Nonlinear Spectral Duality

Francesco Tudisco · Dong Zhang

Received: date / Accepted: date

Abstract Nonlinear eigenvalue problems for pairs of homogeneous convex functions are particular nonlinear constrained optimization problems that arise in a variety of settings, including graph mining, machine learning, and network science. By considering different notions of duality transforms from both classical and recent convex geometry theory, in this work we show that one can move from the primal to the dual nonlinear eigenvalue formulation maintaining the spectrum, the variational spectrum as well as the corresponding multiplicities unchanged. These nonlinear spectral duality properties can be used to transform the original optimization problem into various alternative and possibly more treatable dual problems. We illustrate the use of nonlinear spectral duality in a variety of example settings involving optimization problems on graphs, nonlinear Laplacians, and distances between convex bodies.

Keywords Norm duality · Legendre transform · Polarity transform · Homogeneous functions · Nonlinear eigenproblems · Graph Laplacian

Mathematics Subject Classification (2020) 90C46 · 52A41 · 47J10 · 49N15 · 90C27 · 05C50
1 Introduction and motivation

The critical values and critical points of the ratio of convex homogeneous functions \( f(x)/g(x) \) define (sometimes only a part of) the nonlinear spectrum of the functions pair \((f, g)\). This type of nonlinear eigenvalue problem appears in a wide range of applications. Examples include graph-based machine learning, where the spectral properties of different notions of nonlinear graph and hypergraph Laplacian operators play a central role in unsupervised and semi-supervised classification algorithms [8,9,13,21,24,48,49]; the approximation of matrix and tensor norms [25,27,44]; the solution of the Gross-Pitaevskii equation in quantum chemistry [12,47,54]; the identification and analysis of relevant mesoscopic structures in complex networks, such as central nodes, communities and core-periphery [7,32,50,53]; the optimization of polynomials and generalized polynomials on the unit sphere [26,27,57].

A number of complications arise when moving from the classical matrix eigenvalue problem to the nonlinear one, starting from the fact that the number of eigenvalues and eigenvectors is no longer bounded by the space dimension. However, in most cases one can use the Lusternik-Schnirelmann theory combined with the Krasnoselski genus to define a sequence of variational eigenvalues by means of a Courant-Fisher-like minmax characterization. This subset of variational eigenvalues has very useful properties in most application settings. However, unlike the linear case, evaluating, computing, or approximating the variational eigenvalues is in general a very challenging problem in the nonlinear case, which boils down to a nonsmooth optimization problem for pairs of homogeneous convex functions.

In this paper, we focus on the family of function pairs \((f, g)\) that, on top of being homogeneous and convex, are nonnegative and thus have a linear kernel. These properties are very common in a range of applications, as we will further detail in Section 7.

For this type of functions, we define three duality transforms obtained by adapting the norm duality, the Fenchel’s convex conjugate (i.e. Legendre transform) and the polarity transform (or \(A\)-transform) [2,3]. Thus, we provide three main results showing that the variational spectrum as well as its multiplicities are invariant under these duality transforms. These novel theoretical properties have a number of useful implications as they allow us to move from a given nonlinear eigenvalue problem to several new dual problems which, depending on the particular setting, may result in a more treatable optimization problem or may reveal useful properties that are difficult to observe and to prove using the primal eigenvalue formulation.

For example, if \( f = g = \| \cdot \| \) are norms, convergence guarantees for the fixed point iteration method to compute \( \max f(x)/g(x) \) may be obtained using the dual pair, while the same method may fail to converge for the primal problem [25]. Similarly, a variety of established algorithms for nonlinear eigenproblems such as the inverse iterations [34,35], the family of RatioDCA methods [31,54], the MBO energy landscape and active set search methods for graph total variation [7,18,32], or the continuous gradient-flow approach...
can be directly transferred to the dual eigenvalue equations. The resulting dual iteration or dual flow can be used to solve the optimization of the primal eigenvalue problem and may behave better in practice. Several more specific application settings where nonlinear spectral duality may be used are illustrated in Section 7. Some of the example settings there discussed contain new results we obtain as a consequence of our spectral duality theory.

Our work is based upon and directly complements the recent paper [38], where the authors provide preliminary results on nonlinear spectral duality. Although the theorems in [38] work for norm duality and convex conjugate, no investigation on multiplicities and variational eigenvalues is carried out there and, moreover, they require additional positivity assumptions on the associated functions.

The rest of the paper is structured as follows: In Section 2 we introduce the class of functions of interest and the associated notions of spectrum and variational spectrum. In Section 3 we introduce the notion of norm-like dual for the class of one-homogeneous functions of interest and we review several preliminary properties for this duality operator. Then, in Section 4 we present our main result, showing the spectral invariance for one-homogeneous functions under norm-like duality. In Section 5 we then move on to the class of $p$-homogeneous functions, for $p \geq 1$. We introduce the Legendre and polarity duality mappings and we extend the nonlinear spectral duality theorem to these two alternative notions of duality. Finally, in Section 7 we illustrate a number of example problems from graph theory, network science, and convex geometry, where the new spectral duality theory can be used to provide new insight.

1.1 Notation

We deal with real finite-dimensional spaces, thus we will equivalently write $x^\top y$ or $\langle x, y \rangle$ to denote the Euclidean scalar product. For an operator (or a function) $f$, we let $f^{-1}(y) = \{ x : f(x) = y \}$ denote the preimage of $f$ at $y$ and we equivalently write $\text{Ker}(f)$ and $f^{-1}(0)$ to denote the set $\{ x : f(x) = 0 \}$. We do not differentiate between a matrix $A \in \mathbb{R}^{n \times m}$ and the corresponding linear map $A : \mathbb{R}^m \to \mathbb{R}^n$. For a set $S$, we write $\text{cone}(S) := \{ \lambda v : \lambda > 0, v \in S \}$.

2 Convex homogeneous functions and their spectrum

Consider two real valued functions $f, g : \mathbb{R}^n \to \mathbb{R}$ and suppose they are differentiable. The critical points and critical values of the ratio $r(x) = f(x)/g(x)$, i.e. the pairs $(\lambda, x^*)$ such that $\nabla r(x^*) = 0$ and $r(x^*) = \lambda$, define what we call (nonlinear) spectrum of the function pair $(f, g)$. This is because, $\nabla r(x^*) = 0$ if and only if $x^*$ is such that

$$\nabla f(x^*) = \lambda \nabla g(x^*) .$$
This definition still makes sense without the differentiability assumption. In that case, we can consider Clarke’s sub-differential $\partial$ to show that if $0 \in \partial f(x^*)$ then $0 \in \partial f(x^*) - \lambda \partial g(x^*)$. However, the reverse implication is in general not true without assuming the functions to be differentiable. Overall, we have

**Definition 1** Given $f, g : \mathbb{R}^n \to \mathbb{R}$, we call $(\lambda, x)$ an eigenpair for the function pair $(f, g)$ if

$$0 \in \partial f(x) - \lambda \partial g(x)$$

where $\partial$ denotes Clarke’s generalized derivative [10].

In the linear setting, eigenvectors are defined up to scale. The same fundamental property holds when $f$ and $g$ are homogeneous functions. Recall that a function $f : \mathbb{R}^n \to \mathbb{R}$ is (positively) $p$-homogeneous if $f(\lambda x) = \lambda^p f(x)$ for all $x \in \mathbb{R}^n$ and all $\lambda \in \mathbb{R}, \lambda > 0$. We call $p$ the homogeneity degree of $f$. For the special cases $p = 1$ and $p = 0$ we equivalently say that $f$ is one-homogeneous and scale-invariant, respectively. In particular, in this work, we will focus on the class of homogeneous functions that are convex and have a linear kernel. This type of functions appears frequently in a large number of applications, some of which are discussed in Sections 1 and 7. Precisely, we define

**Definition 2** For $p \geq 1$, let $CH_p^\pm(\mathbb{R}^n)$ denote the collection of all positively $p$-homogeneous functions $f : \mathbb{R}^n \to \mathbb{R}$ with the following properties:

1. $f$ is convex and nonnegative, i.e. $f(x) \geq 0$ for all $x \in \mathbb{R}^n$;
2. $\text{Ker}(f)$ is a linear subspace of $\mathbb{R}^n$.

We remark that properties 1 and 2 above imply that any $f \in CH_p^\pm(\mathbb{R}^n)$ is such that $f(x + z) = f(x)$ for any $z \in \text{Ker}(f)$ and $x \in \mathbb{R}^n$. A possible proof of this property is as follows. Assume the contrary holds: $f(x + z) > f(x)$ for some $x \in \mathbb{R}^n$ and $z \in \text{Ker}(f)$. Fix such $x$ and $z$, and let $\delta = f(x + z) - f(x) > 0$. By the convexity of $f$, for any $t \geq 0$, $\frac{1}{1 + t} f(x + (1 + t)z) + \frac{t}{1 + t} f(x) \geq f(x + z)$, which is equivalent to

$$f(x + (1 + t)z) \geq f(x + z) + t(f(x + z) - f(x)) = f(x + z) + t\delta. \quad (1)$$

Since $\text{Ker}(f)$ is a vector space, $z \in \text{Ker}(f)$ implies $(1 + t)z \in \text{Ker}(f)$. Then, it follows from $(1 + t)z \in \text{Ker}(f)$ and the convexity and $p$-homogeneity of $f$ that

$$\frac{1}{2} f(x) = \frac{f(x) + f((1 + t)z)}{2} \geq f\left(\frac{x + (1 + t)z}{2}\right) = \frac{1}{2^p} f(x + (1 + t)z)$$

which yields $2^{p-1} f(x) > f(x + (1 + t)z)$. Together with (1), we obtain $2^{p-1} f(x) > f(x + z) + t\delta$ for any $t > 0$, but it is impossible, because the right-hand-side tends to $\infty$ when we take $t \to +\infty$.

In general, there can be infinitely many eigenvalues for a function pair, unless $f$ and $g$ are quadratic, in which case the corresponding eigenpairs are standard linear eigenvalue problems. One remarkable properties of the spectrum of homogeneous function pairs is that, when $f$ and $g$ are homogeneous with the same homogeneity degree and $g(\mathbb{R}^n \setminus \{0\}) \subseteq \mathbb{R} \setminus \{0\}$, similarly to the linear...
eigenvalue problem case, we can identify a set of \( n \) variational eigenvalues for the function pair \((f, g)\) via the Lusternik-Schnirelmann theory. In fact, in that case the ratio \( r(x) = f(x)/g(x) : \mathbb{R}^n \setminus 0 \to \mathbb{R} \) is scale invariant and one has that \( 0 \in \partial r(x) \) implies that the pair \((r(x), x)\) is an eigenpair for \((f, g)\). Hence, a set of \( n \) eigenvalues for \((f, g)\) can be identified via the following variational characterization:

\[
\lambda_k = \lambda_k(f, g) = \inf_{\text{genus}(S) \geq k} \sup_{x \in S} r(x), \quad k = 1, \ldots, n,
\]

where \( \text{genus}(S) \) denotes the Krasnoselski’s genus of the closed, symmetric set \( S \) (see e.g. [40]), whose precise defintion is recalled below.

**Definition 3 (Krasnosellskii genus)** Let \( A \) be the class of closed symmetric subsets of \( \mathbb{R}^n \), \( A = \{ S \subseteq \mathbb{R}^n : S \text{ closed, } S = -S \} \). For any \( S \in A \), let \( C_k(S) = \{ \varphi : S \to \mathbb{R}^k \setminus \{0\}, \text{continuous, s.t. } \varphi(x) = -\varphi(-x) \} \). The Krasnosellskii genus of \( S \) is the number defined as

\[
\text{genus}(S) = \begin{cases} 
\inf\{k \in \mathbb{N} : \exists \varphi \in C_k(S)\} & \text{if there exists no such } k \\
\infty & \text{if } A = \emptyset \\
0 & \text{if } A = \emptyset
\end{cases}
\]

This definition of variational eigenvalues (2) is a generalization of the Courant-Fisher min-max characterization of the eigenvalues \( Ax = \lambda Bx \) of the pair of symmetric matrices \((A, B)\). In fact, the Krasnosellskii genus is a homeomorphism-invariant generalization to symmetric sets of the notion of dimension. In particular, \( \text{genus}(S) \geq k \) for any linear subspace \( S \subseteq \mathbb{R}^n \) of dimension greater than \( k \). Thus, Courant-Fisher’s characterization is retrieved from (2) when \( S \) is any linear subspace, the genus is replaced by the dimension of \( S \) and \((f, g)\) are the quadratic functions \( f(x) = x^\top Ax \) and \( g(x) = x^\top Bx \). In particular, note that \( \lambda_n(f, g) = \max_{x \neq 0} r(x) \), \( \lambda_1(f, g) = \min_x r(x) \) and that, since \( f^{-1}(0) \) is linear, the smallest nonzero eigenvalue of \((f, g)\) always coincides with the smallest nonzero variational eigenvalue, i.e.,

\[
\lambda_{d_f+1}(f, g) = \min\{\lambda \text{ eigenvalue of } (f, g) : \lambda > 0\}
\]

where \( d_f = \dim f^{-1}(0) \).

**Remark 1 (On the use of the Lusternik-Schnirelmann category index)** The Krasnoselski’s genus is arguably the most popular index function in the context of variational eigenvalues for nonlinear function pairs. However, when \( r \) is not even, this index cannot be used and other set measures may be required. One possibility is to use the original Lusternik-Schnirelmann category index \( \text{cat}(S) \). However, since \( \mathbb{R}^n \setminus \{0\} \) is homotopy equivalent to \( S^{n-1} \) and \( r = f/g \) is zero-homogeneous on \( \mathbb{R}^n \setminus \{0\} \), it follows from \( \text{cat}(S^{n-1}) = 2 \) that the original Lusternik-Schnirelmann category can only characterize the minimum and maximum eigenvalues in general (in contrast, \( \text{genus}(S^{n-1}) = n \) means that the genus can be used to characterize \( n \) variational eigenvalues when \( r \) is even).
To characterize more variational eigenvalues for $r$ not even, we need to add further assumptions on $r$. For example, if $r : (\mathbb{R}^{n_1}\setminus\{0\}) \times \cdots \times (\mathbb{R}^{n_m}\setminus\{0\}) \to \mathbb{R}$ is a locally Lipschitz function which is zero-homogeneous on each component, that is, $r(t_1x^1, \ldots, t_mx^m) = r(x^1, \ldots, x^m)$ for any $t_i > 0$ and $x^i \in \mathbb{R}^{n_i}$, $i = 1, \ldots, m$, we may use the Lusternik-Schnirelmann category to define $m+1$ eigenvalues of $(f,g)$, as $(\mathbb{R}^{n_1}\setminus\{0\}) \times \cdots \times (\mathbb{R}^{n_m}\setminus\{0\})$ is homotopy equivalent to $S^{n_1-1} \times \cdots \times S^{n_m-1}$ whose category is $m+1$. We emphasize that all the theorems of this paper hold unchanged if genus is replaced by cat. We omit the required straightforward adjustments to the corresponding proofs for the sake of brevity.

In the next sections, we will consider three notions of duality transforms for functions in $CH^p_+(\mathbb{R}^n)$: the norm duality, the Legendre transform and the polarity transform [2,3]. To ensure that the class of functions $CH^p_+(\mathbb{R}^n)$ is closed under such transforms, we make a small modification to these dual operations by composing them with the orthogonal projection onto $\text{Ker}(f)^\perp$, as we will detail later. If one wants to study classes of convex and homogeneous functions where $\text{Ker}(f)$ can be nonlinear and can take the value $+\infty$, one should instead use the standard versions of these dual operations. It is quite interesting that most of the results we present in this paper still hold in a certain sense if we use the standard versions of the three transforms, as we will briefly discuss in Section 6.

3 Norm-like duality

Any norm $\| \cdot \|$ on $\mathbb{R}^n$ is a convex, one-homogeneous, nonnegative function and admits a duality transform by means of which one defines the dual norm

$\| x \|^* := \sup \{ \langle y, x \rangle : \| y \| \leq 1 \}$. The dual norm inherits many properties from the original norm $\| \cdot \|$ and moving from one norm to the other can be of help in many applications. For a review of properties, we refer to [6,10,16,46,55,56]. A similar dual operator $D$ can be defined for general nonnegative one-homogeneous convex functions in $CH^p_+(\mathbb{R}^n)$, as we discuss below. Our main result shows that the considered norm-like duality transform preserves the eigenpairs of any nonnegative homogeneous function pair in $CH^p_+(\mathbb{R}^n)$, as well as the corresponding multiplicities, and their variational eigenvalues.

On $CH^p_+(\mathbb{R}^n)$, consider the dual operator $D : CH^p_+(\mathbb{R}^n) \to CH^p_+(\mathbb{R}^n)$ defined by

$Df(x) := \sup \{ \langle y, x \rangle : f(y) \leq 1 \text{ and } y \perp \text{Ker}(f) \}$

for any $f \in CH^p_+(\mathbb{R}^n)$. It is worth noting that one should be careful with the notation above, as $Df(x)$ denotes the dual of $f$ at $x$, which implicitly depends on the variable $x$ itself.

Note that this dual operator is essentially a composition of the “standard” norm dual operator $f^*(x) = \sup \{ \langle y, x \rangle : f(y) \leq 1 \}$ and a projection onto the
orthogonal complement of $\ker(f)$. In other words, if $P$ denotes the orthogonal projection onto $\ker(f)$, then it is easy to see that it holds
\[
\mathcal{D}f(x) = f^*(x - Px).
\] (3)

We use the “modified” dual $\mathcal{D}f$ instead of the standard norm dual $f^*$ because we want to work on the function space $CH_1^+(\mathbb{R}^n)$ and we want $CH_1^+(\mathbb{R}^n)$ to be closed under the dual operation. However, $f^*(x) = +\infty$ for $x \notin (\ker f)^\perp$ and $f \in CH_1^+(\mathbb{R}^n)$.

A number of useful properties follow directly from the above definition of $\mathcal{D}$, we discuss some of them in the following.

**Proposition 1** For any $f \in CH_1^+(\mathbb{R}^n)$ it holds $\ker(f) = \ker(\mathcal{D}f)$, $\mathcal{D}\mathcal{D}f = f$ and, in particular, $\mathcal{D}f \in CH_1^+(\mathbb{R}^n)$.

**Proof** Clearly, $x \notin f^{-1}(0)$ if and only if there exists $y \perp f^{-1}(0)$ such that $\langle y, x \rangle > 0$. This means that $f(x) > 0 \Leftrightarrow \mathcal{D}f(x) > 0$ for any given $x$, which implies $f^{-1}(0) = (\mathcal{D}f)^{-1}(0)$.

By definition, $\mathcal{D}\mathcal{D}f(x) = \mathcal{D}f^*(x - Px) = f^{**}(x - Px) = f(x - Px) = f(x)$ for any $x \in \mathbb{R}^n$, where we used the well-known identity $f^{**} = f$. So, $\mathcal{D}\mathcal{D}f = f$.

For any $z \in (\mathcal{D}f)^{-1}(0) = f^{-1}(0)$, $Pz = z$, and $\mathcal{D}f(x + z) = f^*(x + z - P(x + z)) = f^*(x - Pz) = \mathcal{D}f(x)$. Therefore, $\mathcal{D}f \in CH_1^+(\mathbb{R}^n)$.

**Proposition 2** Let $S \subseteq \mathbb{R}^n$ be a bounded set and let $\text{conv}(S)$ be its convex hull. Suppose $0$ is in the relative interior of $\text{conv}(S)$, and consider the support function $f_S(x) := \sup_{v \in S} x^\top v$. Then
\[
\mathcal{D}f_S(x) = \inf \left\{ \sum_i \alpha_i : f_S \left( \sum_i \alpha_i v_i - x \right) = 0 \text{ where } \alpha_i \geq 0, v_i \in S \right\}.
\]

**Proof** Note that if $f(y) \leq 1$ and $v \in S$, then $\langle y, v \rangle \leq 1$. Hence, $\mathcal{D}f(v) \leq 1$, $\forall v \in S$. Let
\[
\mathcal{F} = \left\{ f' \in CH_1^+(\mathbb{R}^n) : f'^{-1}(0) = f^{-1}(0) \text{ and } f'(v) \leq 1, \forall v \in S \right\}.
\]

Then, by Proposition 1, we obtain $\mathcal{D}f \in \mathcal{F}$. For any $f' \in \mathcal{F}$, it is clear that $S \subset (f^{-1}(0))^\perp = (f'^{-1}(0))^\perp = (\mathcal{D}f')^{-1}(0)^\perp$, and thus
\[
f(y) = \sup_{v \in S} \langle y, v \rangle \leq \sup_{v \perp f'^{-1}(0), f'(v) \leq 1} \langle y, v \rangle = \mathcal{D}f'(y)
\]
which implies $\mathcal{D}f \geq f'$. That is, $\mathcal{D}f$ is the largest function in $\mathcal{F}$.

Consider the function $\tilde{f} : x \mapsto \inf \{ \sum_i \alpha_i : f(\sum \alpha_i v_i - x) = 0 \text{ for some } \alpha_i \geq 0, v_i \in S \}$. Clearly, $\tilde{f} \in CH_1^+(\mathbb{R}^n)$, $\tilde{f}(v) \leq 1$, $\forall v \in S$, and $\tilde{f}^{-1}(0) = f^{-1}(0)$. That is, $\mathcal{D}f = \tilde{f}$.

For any $f' \in \mathcal{F}$, $f'(x) \leq \sum \alpha_i f'(v_i) \leq \sum \alpha_i$ whenever $x - \sum \alpha_i v_i \in f^{-1}(0)$. Taking the infimum, we get $f'(x) \leq \tilde{f}(x)$. In consequence, we have proved that $\tilde{f}$ is also the largest function in $\mathcal{F}$. The proof of $\mathcal{D}f = \tilde{f}$ is then completed.
Let \( f_S \) be defined as in the proposition above. Clearly one has \( f_S(x) = \sup_{x \in \operatorname{conv}(S)} \langle x, v \rangle \), thus we may assume without loss of generality that \( S \) is convex. In that case, if we assume \( S \) centrally symmetric, then \( f_S \) defines a semi-norm and

\[
\mathcal{D} f_S(x) = \inf \left\{ \sum |\alpha_i| : x - \sum \alpha_i v_i \perp \operatorname{span}(S) \text{ where } v_i \in S \right\}.
\]

In addition, given a norm \( \| \cdot \| \) and a subset \( S \subset \{ v : \|v\| = 1 \} \) with \( \operatorname{conv}((-S) \cup S) = \{ v : \|v\| \leq 1 \} \), we have \( \|x\| = \inf \{ \sum |\alpha_i| : \sum \alpha_i v_i = x, v_i \in S \} \). For example, we can take \( S \) as the set of the extreme points of the unit ball \( \{ v : \|v\| \leq 1 \} \), and this implies the known identity \( \|A\|_{\ell^2,\ell^2} = \inf \{ \sum |\alpha_i| : A = \sum \alpha_i U_i \text{ with } U_i \text{ unitary} \} \), for a square matrix \( A \).

Finally, we remark that, given a norm \( \| \cdot \| \) on \( \mathbb{R}^n \) and a linear subspace \( X \) of \( \mathbb{R}^n \), the map \( x \mapsto \inf \{ \|z\| : z - x \perp X \} \) defines a semi-norm on \( \mathbb{R}^n \). In other terms, \( [x] \mapsto \inf \{ \|y\| : y - x \in X \} \) defines a norm on the quotient space \( \mathbb{R}^n/X \) (we refer to Gromov’s norm for this basic construction [29]).

### 3.1 Linear transformation of homogeneous functions

Given a matrix \( A \in \mathbb{R}^{m \times n} \), i.e. a linear map from \( \mathbb{R}^n \) to \( \mathbb{R}^m \), let \( \mathcal{M}_A : CH^+_1(\mathbb{R}^n) \to CH^+_1(\mathbb{R}^m) \) be defined as

\[
\mathcal{M}_A f(x) := f(A^\top x), \quad \forall f \in CH^+_1(\mathbb{R}^n), \quad \forall x \in \mathbb{R}^m
\]

where \( A^\top \) denotes the transpose of \( A \). Let \( P_A \) denote the orthogonal projection onto \( \operatorname{Ran} A \). As \( \mathbb{R}^m = \operatorname{Ran} A \oplus \operatorname{Ker} A^\top \) we can uniquely define the operator \( \mathcal{P}_A : CH^+_1(\mathbb{R}^n) \to CH^+_1(\mathbb{R}^m) \) as the composition of the so-called infimal postcomposition \( A \triangleright f \) (see e.g. [5]) and the orthogonal projection \( P_A \). Precisely, we set

\[
\mathcal{P}_A f(x) := A \triangleright f(P_A x)
\]

where

\[
A \triangleright f(x) := \inf_{y : Ay = x} f(y).
\]

We use this slightly modified version of the infimal postcomposition because \( A \triangleright f(x) = +\infty \) for \( x \not\in \operatorname{Ran} A \).

**Proposition 3** Given \( f \in CH^+_1(\mathbb{R}^n) \), if \( \operatorname{Ker} f \subseteq \operatorname{Ker} A \), then \( \mathcal{P}_A = \mathcal{D} \mathcal{M}_A \mathcal{D} \).

In particular, if \( f \) is positive (i.e. \( f(x) > 0 \) whenever \( x \neq 0 \)) then \( \mathcal{P}_A = \mathcal{D} \mathcal{M}_A \mathcal{D} \) holds for any matrix \( A \).

**Proof** Keeping the assumption \( f^{-1}(0) \subseteq \operatorname{Ker}(A) \) in mind, we have

\[
\mathcal{P}_A \mathcal{D} f(x) = \inf_{y \in A^{-1}(x) \cup \operatorname{Ker}(A) : f(y) \leq 1} \sup \langle y, u \rangle = \sup_{u \perp \operatorname{Ker}(A) : f(y) \leq 1} \inf \langle y, u \rangle
\]

\[
= \sup_{u \perp \operatorname{Ker}(A) : f(u) \leq 1} \langle Ay, v \rangle = \sup_{v \perp (f \circ A^\top)^{-1}(0) : f(A^\top v) \leq 1} \langle x, v \rangle = \mathcal{D} \mathcal{M}_A f(x).
\]
In the above equalities, we should note that the condition \( f^{-1}(0) \subset \text{Ker}(A) \) implies \( \text{Ker}(A^\top) = (f \circ A^\top)^{-1}(0) \). In fact, \( A^\top z = 0 \Rightarrow f(A^\top z) = 0 \Rightarrow AA^\top z = 0 \Rightarrow A^\top z = 0 \) which means \( A^\top = 0 \) \( \Leftrightarrow f(A^\top z) = 0 \). Then, the second equality from below is proved.

Replacing \( f \) by \( Df \), we have \( \mathcal{P}_A f = \mathcal{P}_ADf = D\mathcal{M}_A Df \). \( \square \)

Before moving on, we collect in the next remark an interesting geometric interpretation of \( D, \mathcal{M}_A \) and \( \mathcal{P}_A \).

Remark 2 Consider a convex body \( K \) in \( \mathbb{R}^n \), it is well-known that the Minkowski functional of \( K \) equals the support function of its dual convex body \( K^\circ \). The dual operator transforms the Minkowski functional of \( K \) to its support function, while \( \mathcal{P}_A \) maps the Minkowski functional of \( K \) to the Minkowski functional of \( A(K) \times \text{Ker}(A) \), and \( \mathcal{M}_A \) maps the support function of \( K \) to the support function of \( A(K) \). If \( A \) is further assumed to be a projection, then \( \mathcal{M}_A \) maps the Minkowski functional of \( K \) to the Minkowski functional of \( K \cap \text{Ker}(A)^\perp \), while \( \mathcal{P}_A \) transforms the support function of \( K \) to the support function of \( K \cap \text{Ker}(A)^\perp \).

Note that, as a consequence of Proposition 3, if \( f \in CH_1^+(\mathbb{R}^n) \) is positive, \( n = m \) and \( A \) is an invertible matrix, we have \( D\mathcal{M}_A Df(x) = f(A^{-1}x) \), and therefore, \( D\mathcal{M}_A Df(x) = \mathcal{M}_A f(x) \) whenever \( A \) is an orthogonal matrix. Moreover, for a general \( f \in CH_1^+(\mathbb{R}^n) \), we have the identities \( \mathcal{M}_A Df = D\mathcal{P}_A f \) and \( D\mathcal{M}_A f = \mathcal{P}_A Df \). The equality \( \mathcal{M}_A Df = D\mathcal{P}_A f \) means that “the section of the dual equals the dual of the projection”, which is a useful observation with direct implications in convex geometry. On the other hand, the equality \( D\mathcal{M}_A f = \mathcal{P}_A Df \) has a similar geometrical meaning, and it has an interesting additional consequence, which we summarize in the following proposition.

**Proposition 4** Let \( \| \cdot \| \) be a monotonic norm on \( \mathbb{R}^d \), i.e., \( \| (t_1, \cdots, t_d) \| = \| (|t_1|, \cdots, |t_d|) \| \) for any \( (t_1, \cdots, t_d) \in \mathbb{R}^d \). Let \( g_i \in CH_1^+(\mathbb{R}^{n_i}) \) be positive-definite, and let \( A_i: \mathbb{R}^n \to \mathbb{R}^{n_i} \) be a linear map, i.e., \( A_i \in \mathbb{R}^{n_i \times n} \), \( i = 1, \cdots, d \). Denote by \( g(x) = \| (g_1(A_1 x), \cdots, g_d(A_d x)) \| \). Then

\[
Dg(x) = \inf_{\sum_{i=1}^d A_i^\top x_i = x} \left( \sum_{i=1}^d \mathcal{D}g_i(x_1), \cdots, \mathcal{D}g_d(x_d) \right),
\]

where \( \| \cdot \|_* \) is the dual norm induced by \( \| \cdot \| \).

Note that, by letting \( g_1, \cdots, g_d \) be norms, we immediately obtain Theorem 6 in [25], which has implications in the design of converging iterations for general matrix norm computations.
Proof Let $\tilde{g}(x_1, \ldots, x_d) = \|(g_1(x_1), \ldots, g_d(x_d))\|$, $\forall (x_1, \ldots, x_d) \in \mathbb{R}^{n_1} \times \cdots \times \mathbb{R}^{n_d}$. Then,

$$
\|\langle Dg_1(x_1), \ldots, Dg_d(x_d) \rangle \| = \sup_{\|\langle t_1, \ldots, t_d \rangle \| \leq 1} \sum_{i=1}^{d} t_i \langle x_i, y_i \rangle \\
= \sup_{\|\langle t_1, \ldots, t_d \rangle \| \leq 1} \sum_{i=1}^{d} |t_i| \sup_{g_i(y_i) \leq 1} \langle x_i, y_i \rangle \\
= \sup_{\|\langle t_1, \ldots, t_d \rangle \| \leq 1} \sum_{i=1}^{d} \sup_{g_i(y_i) \leq |t_i|} \langle x_i, y_i \rangle \\
= \sup_{\|\langle g_1(y_1), \ldots, g_d(y_d) \rangle \| \leq 1} \sum_{i=1}^{d} \langle x_i, y_i \rangle \\
= \sup_{\tilde{g}(y_1, \ldots, y_d) \leq 1} \langle (x_1, \ldots, x_d), (y_1, \ldots, y_d) \rangle \\
= \langle D\tilde{g}(x_1, \ldots, x_d) \rangle.
$$

Note that $\tilde{g}(x) = \tilde{g}(A^\top x)$, where $A := [A_1^\top, \ldots, A_d^\top] \in \mathbb{R}^{n \times (n_1 + \cdots + n_d)}$. The proof is then completed by the identity $D\tilde{g} = DM_A\tilde{g} = P_A D\tilde{g}$. \hfill \Box

4 Main results: spectral invariance for norm-like duality

We state here our main theorem showing that nonzero eigenvalues of function pairs, as well as their multiplicities and their variational eigenvalues (2), are invariant under the norm-like duality and suitable combinations of $M_A$ and $P_A$, for any matrix $A$. The relatively long proofs of this theorem and its main corollary cover the entire section.

Throughout the remainder of this paper, the ‘eigenspace’ of $\lambda$ with respect to the function pair $(f, g)$ is the set $S_\lambda(f, g)$ defined by

$$S_\lambda(f, g) = \{ x : 0 \in \partial f(x) - \lambda \partial g(x) \}.$$

Note that when $f$ and $g$ are even functions, $S_\lambda(f, g)$ is a symmetric set. In this case, we define the multiplicity of the eigenvalue $\lambda$ for $(f, g)$ as

$$\text{mult}_{f,g}(\lambda) = \text{genus}(S_\lambda(f, g)).$$

The following main spectral invariance theorem holds.

**Theorem 1** Let $f, g \in CH_1^+(\mathbb{R}^n)$. Then

1. The nonzero eigenvalues of $(f, g)$ and $(Dg, Df)$ coincide.
P2. If $f$ and $g$ are even functions, then $\text{mult}_{\partial f}(\lambda) = \text{mult}_{\partial g}(\lambda)$, for any nonzero eigenvalue $\lambda$ of $(f, g)$.

P3. If $f$ and $g$ are even functions, then the variational eigenvalues of $(f, g)$ and $(\partial g, \partial f)$ coincide exactly, up to reordering. Precisely, it holds

$$\lambda_k(f, g) = \lambda_k - d_f + d_g(\partial g, \partial f), \quad k = d_f - d_g + 1, \ldots, n - d_g$$

where $d_{fg} := \dim f^{-1}(0) \cap g^{-1}(0)$, $d_f := \dim f^{-1}(0)$ and $d_g := \dim g^{-1}(0)$.

Moreover, combining the norm-like duality operator $D$ with $M_A$ and $P_A$ for a matrix $A$, we obtain the following main consequence of the theorem above.

**Corollary 1** Let $f \in CH^+_1(\mathbb{R}^m)$, $g \in CH^+_1(\mathbb{R}^n)$ and $A \in \mathbb{R}^{n \times m}$. Then, the nonzero eigenvalues of $(M_A^+, f, g)$, $(\partial g, D M_A^+, f)$, $(M_A^+ D g, D f)$, $(f, P_A g)$ and $(M_A^+, f, M_A^+, P_A g)$ coincide. Moreover, if $f$ and $g$ are even functions, then the multiplicities of the nonzero eigenvalues coincide and the nonzero variational eigenvalues of all these function pairs coincide exactly, up to reordering.

We subdivide the relatively long proof of the main results above into several separate parts, as well as a number of smaller preliminary results that are of independent interest.

First, we prove that nonzero eigenvalues are preserved under $D$.

**Proof (Proof of Theorem 1, point P1)** For an eigenpair $(\lambda, x)$ of $(f, g)$ with $\lambda \neq 0$ and $x \neq 0$, it is easy to see that $f(x) = 0 \iff g(x) = 0$, and in this case, we have $D f(x) = 0$, $\partial g(x) = 0$, and $0 \in \partial D f(x) \cap \partial D g(x)$ which implies $0 \in \partial D g(x) - \lambda \partial D f(x)$. Hence, $(\lambda, x)$ is also an eigenpair of $(\partial g, D f)$. In fact, from this proof, we obtain that if $f^{-1}(0) \cap g^{-1}(0) \neq \{0\}$, then the spectra of $(f, g)$ and $(\partial g, D f)$ are $\mathbb{R}$. Therefore, without loss of generality, we assume that $f^{-1}(0) \cap g^{-1}(0) = \{0\}$, $g(x) = 1$ and $f(x) = \lambda \neq 0$. Thus, there exists $u \in \partial g(x)$ such that $\lambda u = \partial f(x)$. Clearly, $u \neq 0$. It follows from the fact $\partial g(x) \subset (g^{-1}(0))^\perp = ((\partial g)^{-1}(0))^\perp$ that $\partial g(u) \neq 0$. Moreover, we have $\langle u, x \rangle = g(x) = 1$ by Euler’s identity, and $\langle u, x' - x \rangle = g(x') - g(x) = g(x') - 1$, $\forall x' \in \mathbb{R}^n$ by the definition of the subgradient. Accordingly, $\partial g(u) = 1$, and for any $u' \in \mathbb{R}^n$, $\langle u' - u, x \rangle = \langle u', x \rangle - 1 \leq D g(u') - 1 = D g(u) - D g(u)$, which implies that $x \in \partial D g(u)$. By $f(x)/\lambda = 1$ and $\lambda u \in \partial f(x) = \partial f(x)/\lambda$, we similarly derive that $x/\lambda \in \partial D f(\lambda u) = \partial D f(u)$ according to the zero-homogeneity of $\partial f$ and $\partial D f$. As a consequence, $0 = x - \lambda \cdot x/\lambda \subset \partial D g(u) - \lambda \partial D f(u)$, i.e., $(\lambda, u)$ is an eigenpair of $(\partial g, D f)$. The converse also holds. And since $\partial D f$ and $\partial D g$ are scaling invariant, we indeed obtain that $\forall u \in \text{cone}(\partial f(x)) \cap \text{cone}(\partial g(x))$, $(\lambda, u)$ is an eigenpair of $(\partial g, D f)$.

Then, we move on to studying their multiplicities. To this end, we first observe that the genus of a compact set grows under the action of the subgradient of even functions. Here and throughout, we say a function is $C^1$-smooth if it has continuous gradient on $\mathbb{R}^n \setminus \{0\}$. 

Lemma 1 Let \( g \in CH^1_+ (\mathbb{R}^n) \) be an even function. Then, the Krasnoselskii genus of a compact subset \( S \) is smaller than or equal to that of the subset \( \partial g(S) := \bigcup_{x \in S} \partial g(x) \).

Proof The proof is based on the deformation nondecreasing property and the continuity of the Krasnoselskii genus. We divide the proof into two steps:

Step 1. Suppose that \( g \) is \( C^1 \)-smooth on \( \mathbb{R}^n \setminus \{0\} \). Since the vector field induced by \( \partial g : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}^n \setminus \{0\} \) is continuous, for any compact subset \( S \subset \mathbb{R}^n \setminus \{0\} \) with genus\((S) = k \), the map \( x \mapsto \partial g(x) \) is continuous and if \( g \) is even, then \( \partial g \) is odd, i.e., \( \partial g(-x) = -\partial g(x) \), \( \forall x \in \mathbb{R}^n \). Therefore, by the deformation nondecreasing property, \( \partial g(S) \) is a subset of \( \mathbb{R}^n \) with genus\((\partial g(S)) \geq k \). That is, for an even, convex and smooth function \( g \), we have genus\((\partial g(S)) \geq \text{genus}(S) \).

Step 2. Suppose that \( g \) is not \( C^1 \)-smooth on \( \mathbb{R}^n \setminus \{0\} \).

In this case, we take the Moreau-Yosida approximation of \( g \), which is defined by

\[
g_\alpha(x) = \inf_{y \in \mathbb{R}^n} g(y) + \frac{1}{2\alpha} \|y - x\|^2_2, \quad \alpha > 0,
\]

where we use the \( l^2 \)-norm \( \| \cdot \|_2 \). It is known that \( g_\alpha \) is \( C^1 \)-smooth and convex. In fact, for sufficiently small \( \epsilon > 0 \), the size of the \( \epsilon \)-neighborhood of \( \partial g(S) \) equals \( \text{genus}(\partial g(S)) \), and for sufficiently small \( \alpha \), \( \partial g_\alpha(S) \) lies in the \( \epsilon \)-neighborhood of \( \partial g(S) \). Therefore, \( \text{genus}(\partial g(S)) \geq \text{genus}(\partial g_\alpha(S)) \), which is larger than or equal to \( \text{genus}(S) \) by Step 1. \( \square \)

Next, we show that for smooth functions the subgradient maps the eigenspace of \( \lambda \) as an eigenvalue of \((f, g)\) into the eigenspace of \( \lambda \) as an eigenvalue of the dual pair \((\mathcal{D}g, \mathcal{D}f)\).

Lemma 2 Let \( f, g \in CH^1_+ (\mathbb{R}^n) \) and let \( \lambda \) be an eigenvalue of \((f, g)\). If \( g \) is differentiable on \( \mathbb{R}^n \setminus \{0\} \), then \( \partial g(S_\lambda(f, g)) \subset S_\lambda(\mathcal{D}g, \mathcal{D}f) \). Similarly, if \( f \) is differentiable, then \( \partial f(S_\lambda(f, g)) \subset S_\lambda(\mathcal{D}g, \mathcal{D}f) \).

Proof By point P1 of Theorem 1 we have that

\[
\emptyset \neq \bigcup_{x \in S_\lambda(f, g)} \text{cone}(\partial f(x)) \cap \text{cone}(\partial g(x)) \subset S_\lambda(\mathcal{D}g, \mathcal{D}f)
\]

for any eigenvalue \( \lambda \) of \((f, g)\). If \( g \) is derivable at any eigenvector \( x \in S_\lambda(f, g) \), then \( \partial g(x) \subset \text{cone}(\partial f(x)) \cap \partial g(x) \). Thus,

\[
\partial g(S_\lambda(f, g)) := \bigcup_{x \in S_\lambda(f, g)} \partial g(x) \subset S_\lambda(\mathcal{D}g, \mathcal{D}f).
\]

The proof of \( \partial f(S_\lambda(f, g)) \subset S_\lambda(\mathcal{D}g, \mathcal{D}f) \) is similar. \( \square \)

Finally, we need the following two technical properties.

Lemma 3 Let \( f, g \in CH^1_+ (\mathbb{R}^n) \) and let \( \lambda \) be an eigenvalue of the function pair \((f, g)\). It holds
1. The map \( x \mapsto \text{cone}(\partial f(x)) \cap \partial g(x) \) is upper semi-continuous, i.e., \( \forall x, \forall \epsilon > 0 \), there exists \( \delta > 0 \) such that for any \( y \in B_\delta(x) \), \( \text{cone}(\partial f(y)) \cap \partial g(y) \subset B_\epsilon(\text{cone}(\partial f(x)) \cap \partial g(x)) \), where \( B_r(S) \) is the \( \epsilon \)-neighborhood of a subset \( S \).

2. For any \( x \in S_\lambda(f, g) \), and for any \( \epsilon > 0 \), there exists an even, \( C^1 \)-smooth function \( g_\epsilon \in CH^+(\mathbb{R}^n) \) with \( g_\epsilon^{-1}(0) = g^{-1}(0) \) and \( \delta > 0 \) such that \( \partial g_\epsilon(B_\delta(x)) \subset B_\epsilon(\text{cone}(\partial f(x)) \cap \partial g(x)) \).

**Proof** Point 1 follows directly from the upper semi-continuity of \( \partial f \) and \( \partial g \). Let us discuss point 2. We only need to deal with the case that \( g \) is positive-definite. For any \( v \in \partial g(x) \), \( \langle x, v \rangle = g(x) > 0 \). Then, by a standard argument in linear algebra, there exists a positive-definite matrix \( A \) such that \( Ax = v \). Then, we take \( g_\epsilon(y) = \sqrt{\langle x, Ay \rangle} \). It is clear that \( g_\epsilon \) is smooth, positive-definite and convex and one-homogeneous. And it is not difficult to check that \( \partial g_\epsilon(x) = Ax = v \). Now, suppose that the vector \( v \) lies in \( \text{cone}(\partial f(x)) \cap \partial g(x) \).

By the above discussion, we immediately obtain that \( \forall \epsilon > 0, \exists \delta > 0 \) such that \( \partial g_\epsilon(B_\delta(x)) \subset B_\epsilon(\text{cone}(\partial f(x)) \cap \partial g(x)) \).

**Proof (Proof of Theorem 2 point P2)**

From Lemmas 1 and 2, we have \( \text{genus}(S_\lambda(f, g)) \leq \text{genus}(S_\lambda(Dg, Df)) \) if \( f \) or \( g \) is differentiable. Conversely, if \( Df \) or \( Dg \) is differentiable, \( \text{genus}(S_\lambda(f, g)) \geq \text{genus}(S_\lambda(Dg, Df)) \). Thus, we obtain that the multiplicity of \( \lambda \) as an eigenvalue of \( (f, g) \) coincides with the multiplicity of \( \lambda \) as an eigenvalue of \( (Dg, Df) \). Next, we prove that the same property holds without the differentiability condition.

Let \( S_\lambda(f, g) = S_\lambda(f, g) \cap \{ x : \| x \|_2 = 1 \} \) be the ‘unit sphere’ of the eigenspace corresponding to \( \lambda \). Then, the multiplicity of \( \lambda \) coincides with \( \text{genus}(S_\lambda(f, g)) \). Fix an \( \epsilon > 0 \) such that

\[
\text{genus} \left( \bigcup_{x \in S_\lambda(f, g)} \text{cone}(\partial f(x)) \cap \partial g(x) \right) = \text{genus} \left( \bigcup_{x \in S_\lambda(f, g)} \text{cone}(\partial f(x)) \cap \partial g(x) \right).
\]

Take \( \epsilon' < \frac{1}{2} \epsilon \). Due to Lemma 3, we can consider a family of open sets \( \{ B_\delta(x) : x \in S_\lambda(f, g) \} \) and the corresponding smooth family \( \{ g_\epsilon : x \in S_\lambda(f, g) \} \) such that for any \( y \in B_{2\delta}(x) \), we have \( \text{cone}(\partial f(y)) \cap \partial g(y) \subset B_{\epsilon'}(\text{cone}(\partial f(x)) \cap \partial g(x)) \), for a sufficiently small \( \delta \).

Since \( S_\lambda(f, g) \) is compact and \( \{ B_\delta(x) : x \in S_\lambda(f, g) \} \) induces an open cover of \( S_\lambda(f, g) \), we can take a finite subfamily \( \{ B_\delta(x_i) \} \) of \( \{ B_\delta(x) : x \in S_\lambda(f, g) \} \) such that the centers \( \{ x_i \} \) of these open balls are distributed centrally symmetrically in \( \mathbb{R}^n \), and \( \partial g_\epsilon(B_\delta(x_i)) \subset B_{\epsilon'}(\text{cone}(\partial f(x_i)) \cap \partial g(x_i)) \), where we simply write \( g_\epsilon \) as \( g \). Then, there exist partitions of unity \( \{ \psi_i \} \) subordinate to the open cover \( \{ B_\delta(x_i) \} \), i.e., \( \text{supp}(\psi_i) \subset B_\delta(x_i) \), \( \psi_i \geq 0 \), \( \sum_i \psi_i = 1 \) and \( \psi_i = \psi_j \) whenever \( x_i = -x_j \).

For example, we can simply take

\[
\psi_i(y) = \frac{\max \{ 0, \delta_i - \| y - x_i \|_2 \} }{\sum_j \max \{ 0, \delta_j - \| y - x_j \|_2 \} }, \quad \forall y \in \mathbb{R}^n.
\]

Taking \( \Psi(x) = \sum_i \psi_i(x) \partial g_i(x) \), then \( \Psi \) is a continuous map.
Given $x \in \tilde{S}_\lambda(f,g)$, let $I(x) = \{i : x \in B_{\delta_i}(x_i)\}$ be the index set of $x$. Note that $\psi_i(x) > 0$ implies $x \in B_{\delta_i}(x_i)$, and thus it holds $\Psi(x) = \sum_{i \in I(x)} \psi_i(x)\partial g_i(x)$ and $\partial g_i(x) \in B_\varepsilon(\text{cone}(\partial f(x_i)) \cap \partial g(x_i))$, whenever $x \in B_{\delta_i}(x_i)$. Moreover, there exists a bijection $\tau : I(x) \to I(-x)$ such that $x_i = -x_{\tau(i)}$, which implies and $\psi_i(x) = \psi_{\tau(i)}(-x)$ and $\partial g_i(x) = -\partial g_{\tau(i)}(-x)$. This implies that

$$
\Psi(-x) = \sum_{i \in I(-x)} \psi_i(-x)\partial g_i(-x) = \sum_{i \in I(x)} \psi_{\tau(i)}(-x)\partial g_{\tau(i)}(-x)
= \sum_{i \in I(x)} -\psi_i(x)\partial g_i(x) = -\Psi(x).
$$

Let $i(x) = \arg\max\{\delta_i : i \in I(x)\}$. Then, for any $i \in I(x)$, $x_i \in B_{\delta_i}(x) \subseteq B_{\lambda}(\delta_i, x_i(x_i)) = B_{\delta_i + \delta_{\lambda}}(x_i(x_i)) \subset B_{2\lambda}(x_i(x_i)).$ Thus, $\forall i \in I(x)$, $\partial g(x_i) \in B_{2\varepsilon}(\text{cone}(\partial f(x_i)) \cap \partial g(x_i(x_i))).$ Therefore, $\partial g_i(x) \in B_{2\varepsilon}(\text{cone}(\partial f(x_i)) \cap \partial g(x_i(x_i)))$ for any $i \in I(x)$. Consequently, we have

$$
\Psi(x) = \sum_{i \in I(x)} \psi_i(x)\partial g_i(x) \in B_{2\varepsilon}(\text{cone}(\partial f(x_i)) \cap \partial g(x_i(x_i)))
\subseteq B_{\varepsilon} \left( \bigcup_{x \in \tilde{S}_\lambda(f,g)} \text{cone}(\partial f(x)) \cap \partial g(x) \right)
$$

which implies that $\Psi(\tilde{S}_\lambda(f,g)) \subset B_{\varepsilon} \left( \bigcup_{x \in \tilde{S}_\lambda(f,g)} \text{cone}(\partial f(x)) \cap \partial g(x) \right)$. Thus,

$$
genus(\tilde{S}_\lambda(f,g)) \leq \text{genus}(\Psi(\tilde{S}_\lambda(f,g)))
\leq \text{genus} \left( \bigcup_{x \in \tilde{S}_\lambda(f,g)} \text{cone}(\partial f(x)) \cap \partial g(x) \right)
= \text{genus} \left( \bigcup_{x \in \tilde{S}_\lambda(f,g)} \text{cone}(\partial f(x)) \cap \partial g(x) \right)
$$

where the first inequality is due to the fact that $\Psi$ is odd continuous, the second inequality is based on the nondecreasing property of the genus, and the last equality follows from the continuity of the genus.

In summary, we have proved that for any $f, g \in CH^+_1(\mathbb{R}^n)$ and any $(\lambda, x)$ eigenpair of $(f, g)$ there always holds

$$
genus(S_\lambda(Dg, Df)) = \text{genus} \left( \bigcup_{x \in S_\lambda(f,g)} \text{cone}(\partial f(x)) \cap \partial g(x) \right) \geq \text{genus}(S_\lambda(f,g)).\quad \Box$$
Thus, we have
\[ x = \lambda_1(f, g) = \cdots = \lambda_{d_f - d_g}(f, g) \leq \cdots \leq \lambda_n - d_g(f, g) \quad \text{and} \quad 0 = \lambda_1(Dg, Df) = \cdots = \lambda_{d_f - d_g}(Dg, Df) \leq \cdots \leq \lambda_n - d_g(Dg, Df). \]
Without loss of generality, we may assume that \( f^{-1}(0) \cap g^{-1}(0) = \{0\} \), and in this case, we shall prove that \( \lambda_k - d_g(Dg, Df) \leq \lambda_k(f, g), \quad k = d_f + 1, \ldots, n - d_g. \)
For any subset \( S \subset g^{-1}(1) \) realizing \( \lambda_k(f, g) \) with \( \text{genus}(S) \geq k \), i.e., a set such that \( \sup_{x \in S} f(x)/g(x) = \lambda_k(f, g) \), we have \( \lambda_k(f, g) \geq f(x)/g(x) = f(x), \forall x \in S. \) Let \( S \) be the unit sphere in the linear subspace \( g^{-1}(0) \) centered at the origin 0. Let \( W = \partial g(S) + S \) be the geometric join of \( \partial g(S) \) and \( S \), i.e.,
\[ W = \{ tu + (1 - t)y : u \in \partial g(S), y \in S, 0 \leq t \leq 1 \}. \]
Since \( \partial g(S) := \cup_{x \in S} \partial g(x) \subset (g^{-1}(0))^\perp \) is orthogonal to the sphere \( S \) in the linear subspace \( g^{-1}(0) \), it holds \( \text{genus}(W) = \text{genus}(\partial g(S)) + \text{genus}(S) \).

For any \( y \in W \cap (f^{-1}(0))^\perp \), there exist \( 0 \leq t \leq 1, u \in \partial g(S) \) and \( -v \in (1 - t)S \), such that \( y = tu - v. \) And there exists \( x \in S \) such that \( u = \partial g(x) \). Therefore, \( x \in \partial Dg(u) = \partial Dg(tu), \quad Dg(tu) = tDg(u) = t \) and \( Dg(v) = 0 \). Thus, we have
\[
Df(tu - v) = \sup_{z \in f^{-1}(0)} \frac{\langle tu - v, z \rangle}{f(z)} = \sup_{z \neq 0} \frac{\langle tu - v, z \rangle}{f(z)} \geq \frac{\langle tu - v, x \rangle}{f(x)} \geq \frac{\langle tu - v, x \rangle}{f(x)} \geq \frac{t}{\lambda_k(f, g)}.
\]
and \( Dg(tu - v) = Dg(tu) = t. \) This implies that \( Dg(tu - v)/Df(tu - v) \leq \lambda_k(f, g) \). Hence \( \text{genus}(W \cap (f^{-1}(0))^\perp) \geq \text{genus}(\partial g(S)) + \text{genus}(S) - \dim f^{-1}(0) \geq \text{genus}(S) + \dim g^{-1}(0) - \dim f^{-1}(0) \geq k + d_g - d_f \)
in which we used the claim \( \text{genus}(\partial g(S)) \geq \text{genus}(S) \). Thus, for \( k = d_f + 1, \ldots, n - d_g \) we obtain \( \lambda_{k + d_g - d_f}(Dg, Df) \leq \lambda_k(f, g) \). Analogously, for \( k' = d_g + 1, \ldots, n - d_f, \) we have \( \lambda_{k' + d_f - d_g}(f, g) \leq \lambda_k(Dg, Df) \). Substituting \( k' = k + d_f - d_g \) into the latter inequality, we get \( \lambda_k(f, g) \leq \lambda_{k + d_g - d_f}(Dg, Df) \), and therefore, we derive \( \lambda_{k + d_g - d_f}(Dg, Df) = \lambda_k(f, g), \quad k = d_f + 1, \ldots, n - d_g. \)

We now move on to the proof of Corollary 1. We need one more preliminary lemma.

**Lemma 4** For \( g \) and \( A \) as in the statement of Corollary 1, define \( g_{\text{Ker}(A)}(x) = \inf_{x \in x + \text{Ker}(A)} g(x) = M_{A^T} P_A g(x) \) and let \( S = \{ x \in \mathbb{R}^n : g(x) = g_{\text{Ker}(A)}(x) \} \).
Then, \( x \) is an eigenvector corresponding to a nonzero eigenvalue of \( (M_{A^T} f, g) \) only if \( x \in S \).

**Proof** If \( x \notin S \), we shall prove that \( \partial g(x) \cap \text{Ker}(A)^\perp = \emptyset \). Otherwise, there exists \( v \in \partial g(x) \) such that \( v \perp \text{Ker}(A) \). Then taking \( y \in x + \text{Ker}(A) \) such that \( g(y) = x + \inf_{x \in x + \text{Ker}(A)} g(x) \), we have \( 0 > g(y) - g(x) \geq \langle v, y - x \rangle = 0 \) which leads to a contradiction. Thus, we have shown that \( \partial g(x) \cap \text{Ker}(A)^\perp = \emptyset \). On the
other hand, $\partial_x f(Ax) = A^T \partial f(Ax) \subset \text{Range}(A^T) = \ker(A)\perp$. This implies that, for any $\lambda \neq 0$, $\partial_x f(Ax) \cap \lambda \partial g(x) \subset \ker(A)\perp \cap \lambda \partial g(x) = \emptyset$, which means that $x$ is not an eigenvector of any nonzero eigenvalue of $(M_{A^\top}, f, g)$. The proof is completed.

\begin{proof} \textbf{(Proof of Corollary \textit{[4]})} \end{proof}

We organize the proof as illustrated by the diagram below

\[
\begin{array}{c}
(M_{A^\top}, f, M_{A^\top} P_A g) \xrightarrow{?} \overset{\text{Thm} \text{[1]}}{(M_{A^\top}, f, g)} \xrightarrow{?} \overset{\text{Thm} \text{[1]}}{(Dg, D(M_{A^\top} f))} \\
(M_{A^\top} Dg, Df) \xrightarrow{?} \overset{\text{Thm} \text{[1]}}{(f, D(M_{A^\top} Dg))}
\end{array}
\]

Here, ‘$\equiv$’ denotes ‘spectral equivalence’, i.e., the thesis holds for the two nonlinear eigenvalue problems connected by ‘$\equiv$’. Note that $M_{A^\top} f = f \circ A \in CH^+_n(\mathbb{R}^n)$ and $g \in CH^+_n(\mathbb{R}^n)$. Thus, by Theorem \textit{[4]} the thesis holds for $(Dg, D(M_{A^\top} f))$ and $(M_{A^\top} f, g)$. The same is true for $(M_{A^\top} Dg, Df)$ and $(f, D(M_{A^\top} Dg))$. In the remainder of the proof, we will show that the two relations marked with a ‘?’ hold.

We first prove that the set of nonzero eigenvalues of $(M_{A^\top}, f, g)$ coincides with the set of nonzero eigenvalues of $(M_{A^\top} Dg, Df)$. For an eigenpair $(\lambda, x)$ of $(M_{A^\top}, f, g)$ with $g(x) = 1$, we have $0 \in \partial_x f(Ax) - \lambda \partial g(x) = A^\top \partial f(Ax) - \lambda \partial g(x)$. Hence, there exists $u \in \partial g(x)$ such that $\lambda u = A^\top v$ for some $v \in \partial f(Ax)$. Thus, $Ax/\lambda \in \partial Df(v)$ and $x \in \partial Dg(u) = \partial Dg(A^\top v/\lambda) = \partial Dg(A^\top v)$. Therefore, $Ax \in A\partial Dg(A^\top v) = \partial_x Dg(A^\top v) = \partial_x M_{A^\top} Dg(v)$, which implies $Ax \in \partial_x M_{A^\top} Dg(v) \cap \lambda \partial Df(v)$ and $(\lambda, v)$ is an eigeneof $(M_{A^\top} Dg, Df)$. Since $(M_{A^\top} Df, Dg) = (M_{A^\top}, f, g)$, the converse also holds. In summary, we have shown that

\[
\emptyset \neq \bigcup_{x \in S_\lambda(M_{A^\top}, f, g)} \partial f(Ax) \cap (A^\top)^{-1}(\lambda \partial g(x)) \subset S_\lambda(M_{A^\top} Dg, Df).
\]

Together with Lemma \textit{[1] this shows that also the multiplicity is maintained.}

Next, we show that $(M_{A^\top}, f, M_{A^\top} P_A g)$ and $(M_{A^\top} f, g)$ have the same nonzero eigenvalues. By the definitions of the operators $M_{A^\top}$ and $P_A$, we have

\[
M_{A^\top} P_A g(x) = \inf_{y \in A^{-1}(Ax)} g(y) = \inf_{z \in \ker(A)} g(x + z).
\]

Let $g_{\ker(A)}$ and $S$ be as in Lemma \textit{[4]}. For any $x \in S$, we have

\[
\partial_x f(Ax) \cap \lambda \partial_x g(x) = A^\top \partial f(Ax) \cap \lambda \partial g(x) \\
= A^\top \partial f(Ax) \cap \ker(A)^\perp \cap \lambda \partial g(x) \\
= \partial_x f(Ax) \cap \lambda \partial g_{\ker(A)}(x)
\]

where we used the fact $\partial g_{\ker(A)}(x) = \partial g(x) \cap \ker(A)^\perp$. In addition, for any $x$,

\[
\partial_x f(Ax) \cap \lambda \partial g_{\ker(A)}(x) = \partial_x f(Ax_{\ker}) \cap \lambda \partial g_{\ker(A)}(x_{\ker}) = \partial_x f(Ax_{\ker}) \cap \lambda \partial g(x_{\ker})
\]
where \( x_{\ker} \in S \cap (x + \text{Ker}(A)) \). Hence, together with Lemma 3 for \( \lambda \neq 0 \), we further obtain

\[
\partial_x f(Ax) \cap \lambda \partial_x g(x) \neq \emptyset \implies \partial_x f(Ax) \cap \lambda \partial g_{\ker(A)}(x) \neq \emptyset
\]

\[
\partial_x f(Ax) \cap \lambda \partial g_{\ker(A)}(x) \neq \emptyset \implies \partial_x f(Ax_{\ker}) \cap \lambda \partial g(x_{\ker}) \neq \emptyset
\]

implying that \( \lambda \) is a nonzero eigenvalue of \((M_{A^\top} f, g)\) if and only if \( \lambda \) is a nonzero eigenvalue of \((M_{A^\top} f, M_{A^\top} \mathcal{P} A g)\), with the same multiplicity.

Finally, we need to show the variational eigenvalues are preserved. For any subset \( S \subset g^{-1}(1) \) realizing \( \lambda_k(M_{A^\top} f, g) \) with \( \text{genus}(S) \geq k \), we have \( \lambda_k(M_{A^\top} f, g) \geq f(Ax)/g(x) = f(Ax), \forall x \in S \). Let \( S \) be the unit sphere in the linear subspace \( \text{Ker}(A^\top) \) centered at the origin 0. Let \( \xi : \mathbb{R}^n \to \text{Ker}(A^\top) \) be a linear map induced by \( (A^\top)^{-1}(x) \cap \text{Ker}(A^\top)^\perp \). Clearly, \( \xi \) is an odd continuous map. Define the geometric join

\[
W := \xi(\partial g(S) \cap \text{Ker}(A)^\perp) \ast S.
\]

For any \( y \in W \), there exist \( 0 \leq t \leq 1 \), \( u \in \xi(\partial g(S) \cap \text{Ker}(A)^\perp) \) and \(-v \in (1-t)S\), such that \( y = tu - v \). Thus, \( A^\top u \in \partial g(S) \cap \text{Ker}(A)^\perp \). So, there exists \( x \in S \) such that \( A^\top u \in \partial g(x) \). Therefore, \( x \in \partial \text{D} g(A^\top u) = \partial \text{D} g(A^\top u) \), \( \text{D} g(A^\top u) = t \text{D} g(A^\top u) = t \) and \( \text{D} g(A^\top v) = 0 \). Note that \( \langle u, Ax \rangle = \langle A^\top u, x \rangle = g(x) = 1 \), which implies \( x \not\in \text{Ker}(A) \). Then, we have

\[
\text{D} f(tu - v) = \sup_{z \neq 0} \frac{\langle tu - v, z \rangle}{f(z)} \geq \frac{\langle tu - v, Ax \rangle}{f(Ax)} = \frac{\langle t A^\top u - A^\top v, x \rangle}{f(Ax)} \geq \frac{t}{\lambda_k(M_{A^\top} f, g)}
\]

and \( \text{D} g(A^\top(tu - v)) = \text{D} g(A^\top u) = t \). Accordingly, we obtain

\[
\frac{\text{D} g(A^\top(tu - v))}{\text{D} f(tu - v)} \leq \lambda_k(M_{A^\top} f, g)
\]

and then

\[
\sup_{y \in W} \frac{\text{D} g(A^\top y)}{\text{D} f(y)} \leq \lambda_k(M_{A^\top} f, g).
\]

Let \( d_A = \dim \text{Ker}(A) \) and \( d_{A^\top} = \dim \text{Ker}(A^\top) \). We estimate the Krasnoselskii genus of \( W \) as

\[
\text{genus}(W) = \text{genus}(\xi(\partial g(S) \cap \text{Ker}(A)^\perp)) + \text{genus}(S) \\
\geq \text{genus}(\partial g(S) \cap \text{Ker}(A)^\perp) + \dim \text{Ker}(A^\top) \\
\geq \text{genus}(S) - d_A + d_{A^\top} \geq k - d_A + d_{A^\top}
\]

where the first equality uses the fact that \( \xi(\partial g(S) \cap \text{Ker}(A)^\perp) \subset \text{Ker}(A^\top)^\perp \) and \( S \) is the unit sphere of the linear subspace \( \text{Ker}(A^\top) \). Therefore, we obtain that

\[
\lambda_k - d_A + d_{A^\top}(M_A \text{D} g, \text{D} f) \leq \lambda_k(M_{A^\top} f, g).
\]

(4)
As the converse holds by a similar argument, we conclude that the identity holds in \( \mathbb{3} \).

To conclude, we prove that \( \lambda_k(\mathcal{M}_A^+ f, g) = \lambda_{k-d_A}(\mathcal{M}_A^+ f, \mathcal{M}_A^+ PAg) \). Let again \( S \) be defined as in Lemma \( \mathbb{3} \). We know that \( \text{genus}(S) = n - \text{dim Ker}(A) \). For any \( W \) with \( \text{genus}(W) > \text{dim Ker}(A) \), \( \text{genus}(W \cap S) \geq \text{genus}(W) - \text{dim Ker}(A) \). It is not difficult to check that

\[
\lambda_k(\mathcal{M}_A^+ f, g) = \inf_{\text{genus}(W) \geq k} \sup_{x \in W} \frac{f(Ax)}{g(x)} \geq \inf_{\text{genus}(W) \geq k} \sup_{x \in W \cap S} \frac{f(Ax)}{g(x)}
\]

\[
= \inf_{\text{genus}(W') \geq k-d_A, W' \cap S \subset W'} \sup_{x \in W'} \frac{f(Ax)}{g(x)}
\]

\[
= \inf_{\text{genus}(W') \geq k-d_A, W' \cap \text{Ker}(A) = \emptyset} \sup_{x \in W'} \frac{f(Ax)}{g_{\text{Ker}(A)}(x)}
\]

\[
= \lambda_{k-d_A}(\mathcal{M}_A^+ f, \mathcal{M}_A^+ PAg).
\]

On the other hand, for any \( W \) realizing \( \lambda_{k-d_A}(\mathcal{M}_A^+ f, \mathcal{M}_A^+ PAg) \), there is an eigenvector in \( W \), and every nontrivial eigenvector lies in \( S \). Fix such a subset \( W' \), consider a family of subsets defined by \( \{ (W \cap S) * (rS) \}_{r>1} \), where \( rS \) is the sphere with radius \( r \) in the linear subspace \( \text{Ker}(A) \) centered at the origin \( 0 \). It is easy to check that \( \text{genus}(W \cap S) * (rS) = \text{genus}(W \cap S) + \text{genus}(rS) \geq k - d_A + d_A = k \) for sufficiently large \( r \). And one can verify that

\[
\lim_{r \to +\infty} \sup_{x \in (W \cap S) * (rS)} \frac{f(Ax)}{g(x)} = \sup_{x \in W \cap S} \frac{f(Ax)}{g(x)}
\]

which implies \( \lambda_k(\mathcal{M}_A^+ f, g) \leq \lambda_{k-d_A}(\mathcal{M}_A^+ f, \mathcal{M}_A^+ PAg) \). Consequently, the proof of \( \lambda_k(\mathcal{M}_A^+ f, g) = \lambda_{k-d_A}(\mathcal{M}_A^+ f, \mathcal{M}_A^+ PAg) \) is completed and we can conclude. \( \square \)

5 Legendre and Polarity transforms

In this section, we use the Legendre and the Polarity transform to provide nonlinear spectral duality results for function pairs in \( CH_p^+ \) with \( p \geq 1 \), and not just \( p = 1 \).

First, we recall the notion of the two transforms for general functions. The Legendre transform of a function \( f : \mathbb{R}^n \to \mathbb{R} \) is defined as

\[
\hat{L} f(x) = \sup_{y \in \mathbb{R}^n} \langle x, y \rangle - f(y) = \inf \{ s \in \mathbb{R} : \langle x, y \rangle \leq f(y) + s, \forall y \in \mathbb{R}^n \},
\]

and the Polarity transform of a function \( f : \mathbb{R}^n \to [0, +\infty) \) is defined as

\[
\hat{A} f(x) = \sup_{y : f(y) > 0} \frac{\langle x, y \rangle - 1}{f(y)} = \inf \{ c \in \mathbb{R} : \langle x, y \rangle \leq cf(y) + 1, \forall y \in \mathbb{R}^n \}.
\]

Similar to the norm-like dual, we now consider a modified version of the two transforms that is better suited for the function family \( CH_p^+(\mathbb{R}^n) \). Precisely, we
define the Legendre and the Polarity transforms of a function \( f \in CH^+_p(\mathbb{R}^n) \) respectively as

\[
\mathcal{L} f(x) = \sup_{y \perp f^{-1}(0)} \langle x, y \rangle - f(y) = \inf \{ s \in \mathbb{R} : \langle x, y \rangle \leq f(y) + s, \forall y \in (f^{-1}(0))^\perp \}
\]

\[
\mathcal{A} f(x) = \sup_{y \perp f^{-1}(0)} \frac{(x, y) - 1}{f(y)} = \inf \{ c \in \mathbb{R} : \langle x, y \rangle \leq cf(y) + 1, \forall y \in (f^{-1}(0))^\perp \}.
\]

Just like the norm dual operator, we note that \( \mathcal{L} f(x) = \hat{\mathcal{L}} f(x - Px) \) and \( \mathcal{A} f(x) = \hat{\mathcal{A}} f(x - Px) \), where \( P \) denotes the orthogonal projection onto \( \text{Ker}(f) \). We emphasize that, as for the norm-like duality, we use these modified transforms instead of the standard ones because \( \hat{\mathcal{L}} f(x) = +\infty \) for \( x \notin (\text{Ker}(f))^\perp \) and \( f \in CH^+_n(\mathbb{R}^n) \). Nonetheless, it is quite surprising that several of the results of the main theorems in this paper still hold in a certain sense if we use the standard concepts of infimal postcomposition, norm dual, Legendre transform and Polarity transform, instead of our modified versions. For the sake of clarity, we postpone this observation to the discussion in Section 6.

The next two Theorems 2 and 3 show spectral invariance under the two duality transforms for pairs of convex \( p \)-homogeneous functions \( f \in CH^+_p(\mathbb{R}^n) \) and \( g \in CH^+_q(\mathbb{R}^n) \), with \( p, q \geq 1 \). Then, in Theorem 4 we will present our main result of this section, which corresponds to the Legendre and polarity transforms’ version of the norm-like duality Theorem 1 and Corollary 1 from the previous section. In particular, Theorem 4 fully characterizes the spectral duality equivalence under the action of \( \mathcal{L}, \mathcal{A}, \mathcal{P}_A \) and \( \mathcal{M}_A \), for a function pair \( f \in CH^+_p(\mathbb{R}^n) \) and \( g \in CH^+_q(\mathbb{R}^n) \), with \( p, q \geq 1 \). Here, and in the rest of the section, for a \( p > 1 \) we let \( p^{\ast} \) be its Hölder conjugate exponent \( 1/p + 1/p^{\ast} = 1 \).

**Theorem 2** For any \( f \in CH^+_p(\mathbb{R}^n) \) and \( g \in CH^+_q(\mathbb{R}^n) \) with some \( p, q > 1 \), the nonzero eigenvalues of \( (f, g) \) and \( (\mathcal{L} g, \mathcal{L} f) \) coincide up to a power factor. Precisely, for any eigenpair \((\lambda, x)\) of \((f, g)\) with \( \lambda \neq 0 \) and \( g(x) \neq 0 \), and for any \( u \in \text{cone}(\partial f(x)) \cap \partial g(x) \), \((\lambda^{p^{\ast} - 1}, u)\) is an eigenpair of \((\mathcal{L} g, \mathcal{L} f)\).

**Proof** It is known that \( \partial f \) is homogeneous of degree \((p - 1)\), and \( \partial g \) is homogeneous of degree \((q - 1)\). Since \((\lambda, x)\) is an eigenpair of \((f, g)\) with \( \lambda \neq 0 \) and \( g(x) \neq 0 \), the inclusion relation \( 0 \in \partial f(x) - \lambda \partial g(x) \) implies that \( 0 = \langle x, \partial f(x) \rangle - \lambda \langle x, \partial g(x) \rangle = pf(x) - \lambda qg(x) \). Thus, \( f(x) > 0 \) and \( \lambda = pf(x)/qg(x) > 0 \). Moreover, there exists \( u \in \partial g(x) \) such that \( \lambda u \in \partial f(x) \), which implies \( u \in \text{cone}(\partial f(x)) \cap \partial g(x) \neq \emptyset \). And for any \( u \in \text{cone}(\partial f(x)) \cap \partial g(x) \), there exists \( v \in \partial f(x) \) and \( \mu \geq 0 \) such that \( u = \mu v \). If follows from \( \langle u, x \rangle = qg(x) \neq 0 \) that \( u \neq 0 \) and hence \( \mu > 0 \). One on hand, \( \langle \lambda \mu v, x \rangle = \lambda \mu pf(x) \), and on the other hand, \( \langle \lambda \mu v, x \rangle = \langle \lambda u, x \rangle = pf(x) \neq 0 \). Thus, \( \lambda \mu = 1 \) and \( v = \lambda u \in \partial f(x) \). Consequently, by the property of Legendre transform, \( x \in \partial \mathcal{L} g(u) \) and \( x \in \partial \mathcal{L} f(\lambda u) = \lambda^{p^{\ast} - 1} \partial \mathcal{L} f(u) \). This implies

\[
x \in \partial \mathcal{L} g(u) \cap \lambda^{p^{\ast} - 1} \partial \mathcal{L} f(u) \neq \emptyset
\]

which means that \((\lambda^{p^{\ast} - 1}, u)\) is an eigenpair of \((\mathcal{L} g, \mathcal{L} f)\). \( \square \)
Theorem 3 For \( f \in CH^+_p(\mathbb{R}^n) \) and \( g \in CH^+_q(\mathbb{R}^n) \), the nonzero eigenvalues of \((f, g)\) and \((Ag, Af)\) coincide up to a scaling factor. Precisely, for any eigenpair \((\lambda, x)\) of \((f, g)\) with \( \lambda \neq 0 \) and \( g(x) \neq 0 \), and for any \( u \in \text{cone}(\partial f(x)) \cap \partial g(x) \), \((\alpha \lambda, u)\) is an eigenpair of \((Ag, Af)\), with \( \alpha = \left(\frac{q}{p}\right)^{p-2} \frac{(q-1)^{q-1}}{(p-1)^{p-1}} \).

Proof Let \((\lambda, x)\) be an eigenpair of \((f, g)\) with \( \lambda \neq 0 \) and \( x \neq 0 \). It is easy to see that \( f(x) = 0 \Leftrightarrow g(x) = 0 \), and in this case, we have \( Af(x) = 0 \), \( Ag(x) = 0 \), and \( 0 \in \partial Af(x) \cap \partial Ag(x) \) which implies \( 0 \in \partial Ag(x) - \lambda \partial f(x) \). Hence, \((\lambda, x)\) is also an eigenpair of \((Ag, Af)\).

In fact, from this proof, we obtain that if \( f^{-1}(0) \cap g^{-1}(0) \neq \{0\} \), then the spectra of \((f, g)\) and \((Ag, Af)\) are \( \mathbb{R} \). Therefore, without loss of generality, we assume that \( f^{-1}(0) \cap g^{-1}(0) = \{0\} \), \( g(x) = 1 \) and \( f(x) = q\lambda/p \neq 0 \). Thus, there exists \( u \in \partial g(x) \) such that \( \lambda u \in \partial f(x) \).

Clearly, \( u \neq 0 \). It follows from the fact \( \partial g(x) \subset (g^{-1}(0))' = ((Ag)^{-1}(0))' \) that \( Ag(u) \neq 0 \). Moreover, we have \( \langle u, x' \rangle = g(x') - g(x') - 1 = q(1) - (q - 1) = (q - 1)^{q-1} \) + 1. Accordingly, \( Ag(u) = (q - 1)^{q-1} \), and for any \( u' \in \mathbb{R}^n \),

\[
\langle u' - u, (q - 1)^{q-1} x \rangle = (q - 1)^{q} \langle u' - u, (q - 1)^{-1} x \rangle
\]

\[
= (q - 1)^{q} \langle u', (q - 1)^{-1} x \rangle - q(q - 1)^{-1}
\]

\[
\leq (q - 1)^{q} (Ag(u')) \leq (q - 1)^{q-1} (q - 1)^{-1} - 1 - q(q - 1)^{-1}
\]

\[
= (q - 1)^{q} (q - 1)^{-1 - 1} (q - 1)^{-1}
\]

\[
= Ag(u') - (q - 1)^{q-1} = Ag(u') - Ag(u)
\]

which implies that \((q - 1)^{q-1} x \in \partial Ag(u)\). We can similarly derive that \( Af(u) = \frac{1}{q} \left(\frac{q-1}{p}\right)^{p-1} \) and \( \left(\frac{q}{p}\right)^{q-1} \frac{1}{q} x \in \partial Af(u)\). Therefore, \((\alpha \lambda, u)\) is an eigenpair of \((Ag, Af)\). \( \square \)

Remark 3 Although Artstein-Avidan and Rubinstein \( \textbf{3} \) introduce a polar subdifferential map which possesses very nice properties for the polarity transform, it is still necessary to use the usual subdifferential in Theorem \( \textbf{3} \).

Before presenting our main and final result of this section, we need a number of relevant preliminary observations and results. First, we show in the next Proposition \( \textbf{3} \) that both Legendre and polarity transforms are directly related to the norm-like transform of Section \( \textbf{3} \). Then, in Proposition \( \textbf{3} \) we show how the eigenpairs of \((f, g)\) change when \( f \) and \( g \) are raised to some power. These two results will allow us to work on the spectral duality for Legendre and polarity transforms for \( p \)-homogeneous convex functions by means of the results previously shown for the case of norm-like duality for one-homogeneous convex functions.

Proposition 5 Given \( p \geq 1 \) and \( r \geq 1/p \), for any nonnegative \( p \)-homogeneous function \( f \), \( f \) is convex if and only if \( f' \) is convex. And, if \( f \) is nonnegative
For Proposition 6 as presented in [1], as for the final statement, note that if \( f \) is convex, \( p \geq 1 \) then we have
\[
\mathcal{D} f = \lim_{p \to 1^+} (\mathcal{L} f_p)^{\frac{1}{p}} = \lim_{p \to 1^+} (A f_p)^{\frac{1}{p}} = \lim_{p \to 1^+} A f_p.
\]

**Proof** The first argument is equivalent to the statement that for any nonnegative one-homogeneous function \( f \), and \( p \geq 1 \), \( f \) is convex \( \iff f^p \) is convex. To show this property, first note that the direction that the convexity of \( f \) implies the convexity of \( f^p \) is easy since \( t \mapsto t^p \) is increasing and convex on \([0, \infty)\). We now show that the convexity of \( f^p \) implies the convexity of \( f \). For any \( x, y \) with \( f(x), f(y) > 0 \), letting \( C = tf(x) + (1-t)f(y) \), we have
\[
\frac{f^p(tx + (1-t)y)}{C^p} = f^p\left(\frac{tf(x)}{C}x + \frac{(1-t)f(y)}{C}y\right) \\
\leq \frac{tf(x)}{C}f^p\left(\frac{x}{f(x)}\right) + \frac{(1-t)f(y)}{C}f^p\left(\frac{y}{f(y)}\right) \\
= \frac{tf(x)}{C} + \frac{(1-t)f(y)}{C} = 1
\]
which yields \( f(tx + (1-t)y) \leq C \). As the case of \( f(x)f(y) = 0 \) is straightforward, we obtain the convexity of \( f \). The equalities shown in (5) are presented in [11]. As for the final statement, note that if \( f_p \) Gamma-converges to \( f \), then \( f_p^{1/p} \) also Gamma-converges to \( f \). And then, by the property of Gamma-convergence, \( \mathcal{D} f_p^{1/p}(x) \) converges to \( \mathcal{D} f \) as \( p \) tends to 1. Thus, by (5),
\[
(\mathcal{L} f_p)^{\frac{1}{p}} = \left(\frac{p-1}{p^p}(\mathcal{D} f_p)^{\frac{1}{p}}\right)^{\frac{1}{p}} = \frac{(p-1)^{\frac{1}{p}}}{p} D f_p^{\frac{1}{p}} \to D f
\]
and
\[
(A f_p)^{\frac{1}{p}} = \left(\frac{(p-1)^{p-1}}{p^p}(\mathcal{D} f_p)^{\frac{1}{p}}\right)^{\frac{1}{p}} = \frac{(p-1)^{\frac{p-1}{p}}}{p} D f_p^{\frac{1}{p}} \to D f
\]
as \( p \) tends to 1. Clearly, \( A f_p \to D f, p \to 1^+ \).

**Proposition 6** For \( \lambda \neq 0 \), \( (\lambda, x) \) is an eigenpair of \( (f, g) \) if and only if \( (\frac{af^{p-1}(x)}{bg^{q-1}(x)}, \lambda) \) is an eigenpair of \( (af^p, bg^q) \). Moreover, the eigenpairs of \( (f, g) \) and \( (af^p, bg^q) \) have a completely equivalent one-to-one correspondence.

**Proof** If \( (\mu, x) \) is an eigenpair of \( (af^p, bg^q) \) where \( \mu \neq 0 \), then \( f(x) > 0 \) and \( g(x) > 0 \), and
\[
0 \in \partial a f^p(x) - \mu \partial bg^q(x) = af^{p-1}(x) \partial f(x) - \mu bg^{q-1}(x) \partial g(x) \\
= af^{p-1}(x) \left( \partial f(x) - \frac{\mu bg^{q-1}(x)}{af^{p-1}(x)} \partial g(x) \right)
\]
which implies that \( \frac{apf^{p-1}(x)}{bpf^{q-1}(x)} \) is an eigenvalue of \((f,g)\). Conversely, it is easy to see that if \((\lambda, x)\) is an eigenpair of \((f,g)\) with \(\lambda \neq 0\), then \(\left( \frac{apf^{p-1}(x)}{bpf^{q-1}(x)} \right, \lambda, x)\) is an eigenpair of \((af^p, bg^q)\).

In addition, it is clear that \((0,x)\) is an eigenpair of \((f,g)\) if and only if \((0,x)\) is an eigenpair of \((af^p, bg^q)\). \(\square\)

Finally, we point out that we need to be careful with the case \(p \neq q\) when dealing with multiplicities and variational eigenvalues. In that case, in fact, \(r = f/g\) is not scale-invariant and the eigenvalues of \((f,g)\) and their multiplicities have degenerate properties. Precisely,

**Lemma 5** Given \(f \in CH^+_p(\mathbb{R}^n)\) and \(g \in CH^+_q(\mathbb{R}^n)\) with \(p, q \geq 1\), for any \(\lambda \geq 0\) and \(t > 0\), there holds \(x \in S_\lambda(f,g)\) if and only if \(tx \in S_{t\lambda}(f,g)\). Moreover, if \(p \neq q\), \(f\) and \(g\) are even, and \((f,g)\) has a nonzero eigenvalue, then the function \(\lambda \mapsto mult_{f,g}(\lambda)\) is constant on \((0, +\infty)\).

**Proof** By the definition of the eigenspace \(S_\lambda\), and the homogeneity of \(\partial f\) and \(\partial g\), we have

\[
x \in S_\lambda(f,g) \iff 0 \in \partial f(x) - \lambda \partial g(x)
\]

\[
\iff 0 \in t^{p-1}\partial f(tx) - \lambda t^{p-1} \partial g(tx)
\]

\[
\iff 0 \in \partial f(tx) - \lambda t^{p-q} \partial g(tx)
\]

\[
\iff tx \in S_{t^{p-q}\lambda}(f,g).
\]

If \(p \neq q\) and \((f,g)\) has a positive eigenvalue \(\hat{\lambda} > 0\), then for any \(t > 0\), \(t^{p-q}\hat{\lambda}\) is also an eigenvalue of \((f,g)\), that is, all positive numbers are eigenvalues of \((f,g)\). Note that for any \(t > 0\), the map \(\varphi_t : \mathbb{R}^n \to \mathbb{R}^n\) defined by \(\varphi_t(x) = tx\) is an odd homeomorphism. Then, for any \(\lambda > 0\), it follows from \(\varphi_t(S_\lambda(f,g)) = S_{t^{p-q}\lambda}(f,g)\) and the homeomorphism-invariance of Krasnoselskii genus that

\[
mult_{f,g}(\lambda) = genus(S_\lambda(f,g)) = genus(S_{t^{p-q}\lambda}(f,g)) = mult_{f,g}(t^{p-q}\lambda),
\]

By the arbitrariness of \(\lambda > 0\) and \(t > 0\), the multiplicity function \(mult_{f,g}(\lambda)\) is independent of \(\lambda > 0\). \(\square\)

Thus, when \(p \neq q\) the (variational) eigenvalues of \((f,g)\) change when the corresponding eigenvector is scaled and their multiplicities are constant. To overcome this issue and have a meaningful definition of variational eigenvalues also for the \(p \neq q\) case, it is useful to restrict the variational eigenvalues to suitable centrally symmetric convex surfaces. In particular, we note that for \(p = q\) we have \(f(x)/g(x) = f(x/g(x)^{1/p})\) and \(g(x)/g(x)^{1/p} = 1\) for all \(x \neq 0\). Thus, we can recast \((2)\) as

\[
\lambda_k(f,g) = \inf_{S \subseteq g^{-1}(1), \text{genus}(S) \geq k} \sup_{x \in S} r(x) = \inf_{S \subseteq g^{-1}(1), \text{genus}(S) \geq k} \sup_{x \in S} f(x) \quad (6)
\]

i.e., for \(p = q\) the \(k\)-th variational eigenvalue equals the \(k\)-th min-max critical value of \(f\) restricted to the centrally symmetric convex hypersurface \(g^{-1}(1)\).
Nonlinear Spectral Duality

By constraining the eigenvalues to \( g^{-1}(1) \), the next theorem provides the Legendre and polarity transforms’ version of Theorem and Corollary i.e., it presents the overall spectral duality equivalence between Frenchel duality, polarity transform and linear transformations.

**Theorem 4** For any \( f \in CH^+_p(\mathbb{R}^m) \), \( g \in CH^+_q(\mathbb{R}^n) \), and linear map \( A : \mathbb{R}^n \rightarrow \mathbb{R}^m \), the strong equivalence relations illustrated in the following diagram hold:

\[
\begin{align*}
(\mathcal{LM}_A^*P_A g, \mathcal{LM}_A^* f) & \iff (\mathcal{MA}^*P_A f, \mathcal{MA}^* g) \iff (\mathcal{AM}_A^*P_A g, \mathcal{AM}_A^* f) \\
(\mathcal{L}g, \mathcal{LM}_A^* f) & \iff (\mathcal{MA}^* f, g) \iff (A g, \mathcal{MA}^* f) \\
(\mathcal{LP}_A g, \mathcal{L} f) & \iff (f, P_A g) \iff (A P_A g, A f)
\end{align*}
\]

where the strong equivalence notation \((f, g) \iff (f', g')\) indicates that for the two pairs \((f, g)\) and \((f', g')\), the nonzero eigenvalues and the nonzero variational eigenvalues restricted to \( g^{-1}(1) \) as in (6) coincide up to some scaling or power factors, and the corresponding multiplicities (when \( p = q \)) coincide exactly.

**Proof** Theorems and imply that the nonzero spectra of \((f, g)\), \((\mathcal{L}g, \mathcal{L} f)\) and \((A g, A f)\) coincide up to some scaling or power factors. For any \( f \in CH^+_p(\mathbb{R}^n) \), let \( \tilde{f} = f^\frac{p}{p-1} \). Then, \( \tilde{f} \in CH^+_1(\mathbb{R}^n) \) and in Proposition implies that

\[
\mathcal{L} f = l_p(D \tilde{f})^{p-1} \quad \text{and} \quad A f = a_p(D \tilde{f})^p
\]

where \( l_p = \frac{p-1}{p} \) and \( a_p = \left(\frac{p-1}{p}\right)^{p-1} \) are constants. Then, by Proposition the eigenvalue problems of \((f, g)\), \((\mathcal{L}g, \mathcal{L} f)\) and \((A g, A f)\) can be equivalently reduced to that of \((\tilde{f}, \tilde{g})\) and \((D \tilde{g}, D \tilde{f})\) up to some scaling factors. It follows from Theorem that the spectra of \((\tilde{f}, \tilde{g})\) and \((D \tilde{g}, D \tilde{f})\) coincide exactly, and hence the eigenvalue problems of \((f, g)\), \((\mathcal{L}g, \mathcal{L} f)\) and \((A g, A f)\) are strongly equivalent.

Moreover, according to Theorem and Corollary we have the following strong equivalences regarding norm-like duality:

\[
\begin{align*}
(\mathcal{MA}^* \tilde{f}, \mathcal{MA}^* P_A \tilde{g}) & \iff (DM_{A^*}^*P_A \tilde{g}, DM_{A^*}^* \tilde{f}) \\
(\mathcal{MA}^* \tilde{f}, \tilde{g}) & \iff (D \tilde{g}, DM_{A^*}^* \tilde{f}) \\
(f, P_A \tilde{g}) & \iff (D \tilde{P}_A \tilde{g}, D \tilde{f})
\end{align*}
\]
Therefore, by Propositions 5 and 6 the strong equivalences among the nine eigenvalue problems shown in the diagram in the statement are established.

\[ \square \]

Remark 4 According to Proposition 3 we can replace \( \mathcal{P}_A \) by \( \mathcal{D} \mathcal{M}_A \mathcal{D} \) in Theorem 4 if we have the additional assumption that \( \text{Ker} \ A \supset f^{-1}(0) \).

6 Spectral duality for standard duality transforms

While in many applications (see also next Section 7) it is useful to consider eigenvalue problems with function pairs that have a linear kernel and whose dual is not infinity, in the field of convex analysis or convex geometry it is frequent to use the standard version of the definitions of duality and infimal post-composition \( \mathcal{P}_A f(x) := A \triangleright f(x) \). Note that if we use the latter in place of \( \mathcal{P}_A \), we can for example remove the condition \( f^{-1}(0) \subset \text{Ker}(A) \) in Proposition 3 that is, for any \( f \in \text{CH}_p^+(\mathbb{R}^n) \), \( \forall x \in \mathbb{R}^n \), we have \( \mathcal{P}_A f(x) = \mathcal{D} \mathcal{M}_A \mathcal{D} f(x) \), where \( \mathcal{D} \) denotes the standard norm dual operator \( \mathcal{D} f(x) = f^*(x) = \sup_{f(y) \leq 1} (x, y) \).

Let \( \text{CVH}_p^+(\mathbb{R}^n) \) be the collection of all convex, positively \( p \)-homogeneous functions from \( \mathbb{R}^n \) to \([0, +\infty]\). Clearly, \( \text{CH}_p^+(\mathbb{R}^n) \subset \text{CVH}_p^+(\mathbb{R}^n) \). It is known that \( \mathcal{D} : \text{CVH}_p^+(\mathbb{R}^n) \to \text{CVH}_p^+(\mathbb{R}^n) \) and \( \mathcal{A} : \text{CVH}_p^+(\mathbb{R}^n) \to \text{CVH}_p^+(\mathbb{R}^n) \) are bijections, whereas \( \mathcal{L} : \text{CVH}_p^+(\mathbb{R}^n) \to \text{CVH}_p^+(\mathbb{R}^n) \) is a bijection when \( p > 1 \). A straightforward modification of the proofs of Theorems 1, 2 and 3 leads to the following results.

Theorem 5 For any nonconstant \( f, g \in \text{CVH}_p^+(\mathbb{R}^n) \), the nonzero eigenvalues of \( (f, g) \) and \( (\hat{D}g, \hat{D}f) \) coincide. Precisely, for any eigenpair \( (\lambda, x) \) of \( (f, g) \) with \( \lambda \neq 0 \) and \( g(x) \neq 0 \), \( \forall u \in \text{cone}(\partial f(x)) \cap \text{cone}(\partial g(x)) \), \( (\lambda, u) \) is an eigenpair of \( (\hat{D}g, \hat{D}f) \). Moreover, if \( f \) and \( g \) are even functions, then the variational eigenvalues 2 of \( (f, g) \) and \( (\hat{D}g, \hat{D}f) \) as well as their multiplicities coincide exactly.

Theorem 6 Given \( p, q > 1 \), for any functions \( f \in \text{CVH}_p^+(\mathbb{R}^n) \) and \( g \in \text{CVH}_q^+(\mathbb{R}^n) \), for any eigenpair \( (\lambda, x) \) of \( (f, g) \) with \( \lambda \neq 0 \) and \( g(x) \neq 0 \), and for any \( u \in \text{cone}(\partial f(x)) \cap \partial g(x) \), \( (\lambda^{p-1}, u) \) is an eigenpair of \( (\hat{L}g, \hat{L}f) \).

Theorem 7 For any functions \( f \in \text{CVH}_p^+(\mathbb{R}^n) \) and \( g \in \text{CVH}_q^+(\mathbb{R}^n) \), for any eigenpair \( (\lambda, x) \) of \( (f, g) \) with \( \lambda \neq 0 \) and \( g(x) \neq 0 \), and for any \( u \in \text{cone}(\partial f(x)) \cap \partial g(x) \), \( (\lambda^{\frac{2}{p}}(\lambda^{\frac{1}{q}} - 1)^{-1}q^{-1}, u) \) is an eigenpair of \( (\hat{A}g, \hat{A}f) \).

7 Example applications

We devote this final section to discussing a number of problems where discrete nonlinear eigenvalue problems and the nonlinear spectral duality properties developed in the previous sections can be used in application settings from graph and hypergraph optimization and convex geometry.
7.1 Nonlinear Laplacians on graphs

Let $A \in \mathbb{R}^{n \times m}$ and consider the functions pair $(\|Ax\|_a, \|x\|_b)$, where $\| \cdot \|_a$ and $\| \cdot \|_b$ are vector norms. Nonlinear eigenvalue problems for this type of convex one-homogeneous functions are among the best studied problems in nonlinear spectral theory and arise in a broad range of application settings, including inverse problems in imaging \[11,21,28\], graph clustering, unsupervised and supervised learning \[9,31,39,49\], community and core-periphery detection in networks \[18,50,53\], graph and hypergraph matching \[44\]. Note that this type of eigenvalue problems are directly connected with generalized operator matrix norms, which coincide with the largest (variational) eigenvalue

$$\lambda_m(f, g) = \max_{x \neq 0} \frac{\|Ax\|_a}{\|x\|_b} =: \|A\|_{a,b}.$$ 

Note that, if $f = \| \cdot \|_a$ and $g = \| \cdot \|_b$, then this type of eigenvalue problem coincides with the eigenvalue problem for the functions pair $(M_{1,1}^T f, g)$. Based on this observation, in this subsection we review several example eigenvalue problems with a direct application to combinatorial optimization problems on finite graphs and discuss what are the various corresponding dual forms. In particular, we will show that several famous nonlinear graph eigenvalue equations can be recast in various different forms, which has the potential to unleash a variety of new results both from the theoretical and the computational points of view. In fact, established algorithms for the solution of these eigenvalue problems, such as the inverse iteration \[34,35\], the family of ratioDCA methods \[31,53\], or the continuous flow approaches \[10,22\], can be directly transferred to their dual versions and may exhibit improved convergence properties. Moreover, new relations between the graph and the nonlinear eigenpair may be shown. Some of the graph theoretic results presented next are known and properly referenced, others are new and are accompanied by proofs and additional details.

Before proceeding, we briefly recall some useful graph notation and terminology. A finite undirected graph $G = (V, E, w)$ is the pair of vertex (or node) set $V = \{1, \ldots, n\}$ and edge set $E \subseteq V \times V$, which we equip with a positive weight function $E \ni ij \mapsto w_{ij} > 0$. Any such graph is uniquely represented by the incidence matrix $K = (\kappa_{e,u}) \in \mathbb{R}^{E \times V}$, which maps any $x \in \mathbb{R}^V$ into the vector with entries $(Kx)_e = \sum_u \kappa_{e,u}x_u = \pm w_{ij}(x_i - x_j)$, where $e = ij \in E$ is the edge connecting nodes $i$ and $j$. Note that the choice of the sign in $(Kx)_e$ is arbitrary but fixed. Different norms of $Kx$ correspond to different energies on $G$. For example, $\|Kx\|_1 = \sum_{ij \in E} w_{ij}|x_i - x_j|$ is the graph total variation, $\|Kx\|_2^2 = \sum_{ij \in E} w_{ij}(x_i - x_j)^2$ the electric potential, $\|Kx\|_\infty = \max_{ij \in E} w_{ij}|x_i - x_j|$ the graph node-wise variation.

7.1.1 $(1,1)$-Laplacian: Cheeger constant

Let $G = (V, E, w)$ be a weighted graph and consider the nonlinear eigenvalue problem

$$0 \in \partial \sum_{ij \in E} w_{ij}|x_i - x_j| - \lambda \partial \sum_{i \in V} |x_i|. \tag{7}$$
Note that, if \( A = K \) is the incidence matrix of \( G \), then (7) coincides with the eigenvalue problem for the functions pair \((\mathcal{M}_A^\top f, g)\), where \( f = \| \cdot \|_1 \) and \( g = \| \cdot \|_1 \) are the standard \( l^1 \)-norms on \( \mathbb{R}^E \) and \( \mathbb{R}^V \), respectively. Thus, by Corollary 1 it follows that (7) is equivalent to the following alternative eigenvalue problems

\[
0 \in \partial \| x \|_\infty - \lambda \partial \inf_{y \in \mathbb{R}^E} \sum_{i \leq j} w_{ij} |y_i - y_j| - \lambda \partial \| y \|_\infty
\]

which correspond to the eigenvalue problems for the pairs \((\mathcal{D}g, \mathcal{D}_M A^\top f), (\mathcal{M}_A Dg, Df)\), and \((\mathcal{M}_A^\top f, \mathcal{M}_A^\top P_A g)\), respectively. All the above nonlinear eigenvalue problems have the same nonzero eigenvalues (with the same corresponding multiplicities).

The eigenvalue problem (7) is known as the 1-Laplacian eigenvalue problem on \( G \). This is one of the key objects of nonlinear spectral graph theory and many useful properties of the 1-Laplacian are known. For example, when the graph is connected, the smallest positive eigenvalue of (7) coincides with the Cheeger isoperimetric constant of \( G \) \([9,15]\). Moreover, when the graph is a tree, each variational eigenvalue of (7) coincides with the \( k \)-th Cheeger constant \([19,49]\). Precisely, let

\[
h_k(G) := \min_{\text{disjoint subsets } V_1, \ldots, V_k \subseteq V} \max_{1 \leq i \leq k} \frac{\text{vol}(\text{cut}(V_i))}{\text{vol}(V_i)},
\]

where \( \text{vol}(V_i) = |V_i| \) and \( \text{vol}(\text{cut}(V_i)) = \sum_{u \in V_i, v \notin V_i} w_{uv} \) are the (weighted) volumes of \( V_i \) and its cut set, respectively. Then, if \( \lambda_k \) is the \( k \)-th variational eigenvalue of the 1-Laplacian (7), it holds \( \lambda_2 = h_2(G) \) and \( \lambda_k = h_k(G) \) for \( k > 2 \) if \( G \) is a tree. More in general, we have \( \lambda_m \leq h_k(G) \leq \lambda_k \) for a generic graph \( G \), where \( m \) is the largest number of nodal domains of any eigenvector of \( \lambda_k \) \([49]\). By Theorem 1 and Corollary 1 the same fundamental graph theoretic properties hold for the variational eigenvalues of each of the nonlinear eigenvalue problems (8)–(11).

7.1.2 (\( \infty, \infty \))-Laplacian: graph’s diameter

Let \( G = (V, E, w) \) be a weighted graph and consider the so-called \( \infty \)-Laplacian eigenvalue problem:

\[
0 \in \partial \max_{i,j \in E} \ |w_{ij} (x_i - x_j)| - \lambda \partial \max_{i \in V} \ |x_i|.
\]
Let \( A = K = (\kappa_{e,i}) \) be the incidence matrix of the graph, and let \( f := \| \cdot \|_\infty \) and \( g := \| \cdot \|_\infty \) be the standard unweighted \( l_\infty \)-norms on \( \mathbb{R}^E \) and \( \mathbb{R}^V \), respectively. Then, the nonlinear eigenvalue problem \( 0 \in \partial_x \| Ax \|_\infty - \lambda \partial_x \| x \|_\infty \), i.e., the eigenvalue problem for the functions pair \( (M_A, f, g) \). By Corollary 1, we obtain several new eigenvalue problems equivalent to the graph \( \infty \)-Laplacian:

\[
0 \in \partial \| x \|_1 - \lambda \partial \inf_{y : K^\top y = x} \| y \|_1 \quad (14)
\]

\[
0 \in \partial \sum_{i \in V} \sum_{e \in E} \kappa_{e,i} y_e - \lambda \partial \inf_{x : K x = y} \| x \|_\infty \quad (15)
\]

\[
0 \in \partial \| y \|_\infty - \lambda \partial \inf_{x : K x = y} \| x \|_\infty \quad (16)
\]

\[
0 \in \partial \max_{i,j \in E} |w_{ij}| x_i - x_j | - \lambda \partial \left\| x - \frac{\max_i x_i + \min_i x_i}{2} \right\|_\infty \quad (17)
\]

where \( \mathbf{1} \) denotes the vector of all ones. We emphasize that the formulation in (15) corresponds to a form of 1-Laplacian eigenvalue problem on the dual graph, i.e., the eigenvalue problem for the functions pair \( f(x) = \| K^\top y \|_1 \) and \( g(x) = \| y \|_1 \).

When the graph is connected, the variational eigenvalues of (13) are related to the graph diameter. More precisely, define a ball \( B = B_r(v) \subseteq V \) centered in \( v \in V \) and of radius \( r = \operatorname{radius}(B) \) as the set \( B_r(v) = \{ u \in V : \operatorname{dist}(u, v) \leq r \} \) where \( \operatorname{dist} \) is the shortest path distance on \( G \). Two such balls \( B = B_r(v) \) and \( B' = B_r(v') \) are disjoint if \( \operatorname{dist}(v, v') \geq r + r' \). With this notation, it holds [20]

\[
\lambda_k \leq \min_{\text{disjoint balls } B_1, \ldots, B_k \subseteq V} \max_{1 \leq i \leq k} \frac{1}{\operatorname{radius}(B_i)}
\]

where \( \lambda_k \) is the \( k \)-th variational eigenvalue of (13). In particular, note that the smallest nonzero variational eigenvalue coincides with \( 2 / \operatorname{diam}(G) \), where \( \operatorname{diam}(G) := \max_{i,j \in V} \operatorname{dist}(i,j) \), and \( \operatorname{dist} \) represents the shortest path distance on \( G \). More precisely, if \( G \) has \( k \) connected components, \( G_1, \ldots, G_k \), then the smallest positive variational eigenvalue coincides with

\[
\min_{i=1, \ldots, k} \frac{2}{\operatorname{diam}(G_i)} = \frac{2}{\max_{i=1, \ldots, k} \operatorname{diam}(G_i)}.
\]

By Corollary 1, all the above properties transfer directly to the nonlinear spectrum of any of the eigenvalue problems (14)–(17).

Remark 5 The cycle graph \( C_n \) is the only graph which is dual to itself, i.e., is such that \( K = K^\top \). If we work on a cycle graph, the 1-Laplacian eigenvalue problem (7) is equivalent to the \( \infty \)-Laplacian eigenvalue problem (13), via the spectral duality equivalence shown in (15). In particular, their \( k \)-th variational eigenvalues coincide, and they are bounded by the \( k \)-th Cheeger constant \( h_k(G) \) which is consistent with the reciprocal of the largest radius of any ball in any set of \( k \) pairwise disjoint balls inside the cycle graph.
It is interesting to note that in a Euclidean space, a ball $B$ of radius $r$ satisfies $\frac{\text{vol}(\partial B)}{\text{vol}(B)} \sim \frac{1}{r}$, where $\partial B$ is the boundary of $B$ and $\sim$ denotes here that the two quantities are proportional. As cut is the graph analogue of the boundary, an interesting open question is whether or not $h_k(G) \sim \frac{1}{r_k(G)}$ for a generic graph $G$, where $r_k(G)$ denotes the largest radius of any ball in any set of $k$ pairwise disjoint balls in the graph.

7.1.3 $(1, \infty)$-Laplacian: maxcut and mincut

Let $G = (V, E, w)$ be a weighted graph. Consider the eigenvalue problem for the functions pair $(M_A^T f, g)$ with $f(x) = \|x\|_1$, $g(x) = \|x\|_\infty$ and $A = K$, namely

$$0 \in \partial \sum_{ij \in E} w_{ij} |x_i - x_j| - \lambda \|x\|_\infty. \quad (18)$$

By our spectral duality principle in Theorem 1 and Corollary 1, (18) is equivalent to

$$0 \in \partial \|x\|_1 - \lambda \partial \inf_{y \in \mathbb{R}^n} \sum_{j<i} w_{ij} y_{ij} - \sum_{j>i} w_{ij} y_{ij} - \lambda \|y\|_\infty \quad (19)$$

$$0 \in \partial \|y\|_1 - \lambda \partial \inf_{x \in \mathbb{R}^n} \sum_{j<i} w_{ij} x_{ij} - \sum_{j>i} w_{ij} x_{ij} - \lambda \|x\|_\infty \quad (20)$$

$$0 \in \partial \sum_{ij \in E} w_{ij} |x_i - x_j| - \lambda \|x\|_\infty - \lambda \|x - \max_i x_i + \min_i x_i\|_\infty. \quad (21)$$

It is shown in [37, Section 4.2] that the smallest nonzero variational eigenvalue and the largest variational eigenvalue of (18) coincide with the mincut and the maxcut values of $G$, respectively defined as

$$\text{mincut}(G) = \min_{S \subseteq V} \text{vol}(\text{cut}(S)) \quad \text{and} \quad \text{maxcut}(G) = \max_{S \subseteq V} \text{vol}(\text{cut}(S)).$$

We also remark that (a) maxcut($G$) is actually equivalent to the largest eigenvalue for the pair $(\|Kx\|_p, \|x\|_\infty)$, for any $1 \leq p < \infty$, see Example 3.1 and Section 4.2 in [37]; and (b) when $w_{ij}$ in (18) is replaced by the modularity weights $m_{ij} := \frac{d_i d_j}{\sum_k d_k} - w_{ij}$, $d_i = \sum_j w_{ij}$, the largest eigenvalue of (18) corresponds to the leading community in $G$, see Theorem 3.7 in [53]. Due to the nonlinear spectral duality principle, the same properties hold for each of the nonlinear eigenvalue problems in (19)–(22). Moreover, the following relation holds for their $k$-th variational eigenvalue

**Theorem 8** Let $\lambda_k$ be the $k$-th variational eigenvalue of the eigenvalue problem (18). Then

$$\lambda_k \leq \min_{V_1, \ldots, V_k \text{ form a partition of } V} \text{maxcut}(G[V_1, \cdots, V_k]).$$
where maxcut\( (G[V_1, \ldots, V_k]) := 2 \max_{S \subseteq \{1, \ldots, k\}} \sum_{i < j} w_{V_i, V_j} \) denotes the max-cut value of the graph \( G[V_1, \ldots, V_k] \), formed by \( k \) vertices corresponding to the \( k \) sets \( V_1, \ldots, V_k \), with edge weights \( w_{V_i, V_j} = \sum_{a \in V_i, b \in V_j} w_{ab} \), \( V_i \neq V_j \).

**Proof** For any partition \((V_1, \ldots, V_k)\) of \( V \), denote by \( 1_{V_i} \) the indicator vector of \( V_i \). Then \( 1_{V_1}, \ldots, 1_{V_k} \) are linearly independent. Thus genus(span(\( 1_{V_1}, \ldots, 1_{V_k} \))) = \( k \) and we have

\[
\lambda_k(\mathcal{M}A_f, g) \leq \max_{x \in \text{span}(1_{V_1}, \ldots, 1_{V_k})} \frac{\sum_{\{i,j\} \in E} w_{ij} |x_i - x_j|}{\|x\|_{\infty}}
\]

\[
= \max_{(t_1, \ldots, t_k) \in \mathbb{R}^k \setminus \{0\}} \frac{\max_{1 \leq j < k} \sum_{i=1}^k w_{V_i, V_j} |t_i - t_j|}{\|t\|_1}
\]

\[
= 2 \max_{S \subseteq \{1, \ldots, k\}} \sum_{i \in S, j \in V \setminus S} w_{V_i, V_j},
\]

where the last equality follows from Theorem 4.1 in [37]. □

7.1.4 \((\infty, 1)\)-Laplacian: graph’s inscribed ball

Let \( G = (V, E, w) \) be a weighted graph. Consider the eigenvalue problem for the functions pair \( f(x) = \|Kx\|_{\infty} \) and \( g(x) = \|x\|_1 \), namely

\[
0 \in \partial \max_{ij \in E} w_{ij} |x_i - x_j| - \lambda \partial \|x\|_1.
\]

Then, by Theorem 11 and Corollary 11, (23) is equivalent to

\[
0 \in \partial \|x\|_\infty - \lambda \partial \sum_{ij < \infty} \inf_{y, y_i = x_i, y_j} \|y\|_1 \leq \infty
\]

\[
0 \in \partial \max_{i \in V} \sum_{j < i} w_{ij} y_{ij} - \sum_{j > i} w_{ij} y_{ij} - \lambda \partial \|y\|_1
\]

\[
0 \in \partial \|y\|_\infty - \lambda \partial \inf_{x:Kx = y} \|x\|_1
\]

\[
0 \in \partial \max_{ij \in E} w_{ij} |x_i - x_j| - \lambda \partial \|x - \frac{\max_i x_i + \min_i x_i}{2} 1\|_1
\]

Moreover, the following result holds for the variational eigenvalue of all the above eigenvalue problems.

**Theorem 9** Let \( \lambda_k \) be the \( k \)-th variational eigenvalue of the eigenvalue equation (23). It holds

\[
\lambda_k \leq \min_{\text{disjoint balls } B_1, \ldots, B_k \subset V} \max_{1 \leq i \leq k} \frac{1}{\text{size}(B_i)}
\]

where, for \( B = B_r(v) \) we let size\( (B) = \sum_{i=0}^r (r - i) |\{u \in V : \text{dist}(u, v) = i\}| \).
For a \( x \in \mathbb{R}^n \) and a ball \( B \) with radius \( r \) and centered at the vertex \( v \), define \( x^B \in \mathbb{R}^n \) by \( (x^B)_i = \max\{r - \text{dist}(v, i), 0\} \). Then, for any \( k \) disjoint balls \( B_1, \ldots, B_k \subset V \), \( x^{B_1}, \ldots, x^{B_k} \) are linearly independent. Thus \( \text{genus}(\text{span}(x^{B_1}, \ldots, x^{B_k})) \geq k \) and we have

\[
\lambda_k \leq \max_{x \in \text{span}(x^{B_1}, \ldots, x^{B_k})} \frac{\max_{(i,j) \in E} |x_i - x_j|}{\|x\|_1} \\
\leq \max_{1 \leq \ell \leq k} \frac{\max_{(i,j) \in E} |x^{B_i}_\ell - x^{B_j}_\ell|}{\|x^{B_\ell}_\ell\|_1} = \max_{1 \leq \ell \leq k} \frac{1}{\text{size}(B_\ell)}.
\]

where the second inequality follows from the fact that the \( x^{B_j} \) have disjoint support. By taking the minimum over all possible disjoint balls we obtain (28).

\[ \square \]

Note that, as a consequence of the above theorem we obtain that the smallest eigenvalue is at most the reciprocal of the size of the largest ball inscribed in the graph.

### 7.1.5 Hypergraphs and core-periphery detection

On top of combinatorial problems on graphs, nonlinear eigenvalue problems appear in a variety of hypergraph mining settings where the optimization of suitable discrete functions is required. Nonlinearity is particularly important when we deal with a hypergraph, as the presence of higher-order node interactions naturally leads to nonlinear eigenvalue equations and the corresponding nonlinear operators. Examples include submodular and diffusion-inspired hypergraph Laplacians [14,41], tensor-based Laplacians [27,33], and game-theoretic homogeneous Laplacians [24]. To provide a concrete example, we consider here the core-periphery detection problem on hypergraphs, as formulated in [52].

Consider a hypergraph \( H = (V,E,w) \) made by a set of vertices \( V = \{1, \ldots, n\} \), hyperedges \( E = \{e_1, \ldots, e_m\} \) and the weight function \( w : E \to \mathbb{R}_+ \). Here, unlike the graph case, each \( e \in E \) contains an arbitrary number of nodes. The core-periphery detection problem consists of identifying the optimal subdivision of \( V \) into a core set highly connected with the rest of \( H \) and a periphery set, connected only (or mostly) to the core.

It is shown in [52] that this combinatorial problem on \( H \) boils down to the norm-constrained optimization problem,

\[
\max_{\ell \in E} w_e \|x_e\|_q \quad \text{s.t.} \quad \|x\|_p = 1.
\]

Clearly, if \( f(x) = \sum_{e \in E} w_e \|x_e\|_q \) and \( g(x) = \|x\|_p \), the above problem coincides with the largest eigenvalue of the nonlinear eigenvalue problem for the functions pair \((f,g)\). Now, we shall write down the dual eigenvalue problem, i.e., the eigenvalue problem for the function pair \((D g, D f)\).
For $g$ we have $Dg(x) = \|x\|_{p',*}$, where $1/p + 1/p' = 1$. As for $f$, note that

$$f(x) = \left\|\left(\|x\|_{e_1}, \ldots, \|x\|_{e_m}\right)\right\|_{1,w},$$

where $\|\cdot\|_{1,w}$ indicates the weighted $l^1$-norm on $\mathbb{R}^E$. Then, by Proposition 4 we have

$$Df(x) = \inf_{\sum_{e \in E} y_e = x} \left\|\left(\frac{\|y_{e_1}\|_{q^*}}{w_{e_1}}, \ldots, \frac{\|y_{e_m}\|_{q^*}}{w_{e_m}}\right)\right\|_{\infty} = \max_{\sum_{e \in E} y_e = x} \left\|\frac{y_e}{w_e}\right\|_{q^*}$$

where $y_e$ denotes a vector in $\mathbb{R}^V$ with the support in $e$.

Moreover, using Corollary 1 we can obtain additional equivalent formulations. For $e \in E$, let $A_e : \mathbb{R}^V \to \mathbb{R}^V$ be a matrix defined as $(A_e x)_i = x_i$ if $i \in e$ and $(A_e x)_i = 0$ if $i \notin e$. Clearly, $\|A_e x\|_q = \|x\|_q$. Thus, we can write $f$ as $f = \tilde{f} \circ A$, i.e., $f(x) = \tilde{f}(Ax)$, where $\tilde{f} : \mathbb{R}^{nm} \to [0, +\infty)$ is the norm defined as

$$\tilde{f}(y_1, \ldots, y_m) = \left\|\left(\|y_1\|_q, \ldots, \|y_m\|_q\right)\right\|_{1,w}$$

with $y_e \in \mathbb{R}^n$ and $A : \mathbb{R}^n \to \mathbb{R}^{nm}$ defined as $Ax = (A_{e_1} x, \ldots, A_{e_m} x)$. Thus, we immediately see that

$$D\tilde{f}(y_1, \ldots, y_m) = \left\|\left(\frac{\|y_1\|_{q^*}}{w_{e_1}}, \ldots, \frac{\|y_m\|_{q^*}}{w_{e_m}}\right)\right\|_{\infty}$$

for any vector $(y_1, \ldots, y_m)$ of dimension $n \times m$. By Corollary 1 the largest eigenvalue of the dual eigenvalue problem $(Dg \circ A^T, D\tilde{f})$, i.e.,

$$0 \in \partial \left\{ \sum_{i=1}^m A_{e_i} y_i \|p'\| - \lambda \partial \left( \left\|\left(\frac{\|y_1\|_{q^*}}{w_{e_1}}, \ldots, \frac{\|y_m\|_{q^*}}{w_{e_m}}\right)\right\|_{\infty} \right) \right\}$$

coincides with the core-periphery eigenvalue problem (20) for $(f, g)$.

### 7.2 Distance between convex bodies

The Banach-Mazur distance is a key quantity in convex geometry and functional analysis, which has led to noteworthy progress in both those areas, see e.g. [36]. Here, we focus on the Banach-Mazur distance in its multiplicative form between two centrally symmetric convex bodies $K$ and $L$, centered at the origin point 0 in $\mathbb{R}^n$. This distance is defined as

$$d(K, L) = \inf\{r \geq 1 : \exists A \in GL(\mathbb{R}^n) \text{ s.t. } L \subset AK \subset rL\},$$

(30)

where $GL(\mathbb{R}^n)$ is the general linear group. By translating our spectral duality properties into the language of convex geometry we can immediately obtain properties about this distance between convex bodies via the eigenvalue problem for function pairs.

For two convex bodies $K$ and $L$ containing the origin as an interior point, there exists some scaling constant $\lambda > 1$ such that the two convex surfaces $\partial K$ and $\lambda \partial L$ are tangent to each other at some point, where $\partial K$ and $\partial L$ are the
boundary surfaces of the bodies $K$ and $L$, respectively. Here, we say that two convex surfaces are tangent at $a$ if they have a common supporting hyperplane at $a$. Let

$$ST(K, L) = \{\lambda > 0 : \partial K \text{ and } \lambda \partial L \text{ are tangent}\}.$$ 

A first key observation is that $ST(K, L)$ is a compact subset of $(0, +\infty)$ and it coincides with the set of all the nonzero eigenvalues of the function pair $(\| \cdot \|_K, \| \cdot \|_L)$, where $\| \cdot \|_K$ is the Minkowski functional norm of $K$, i.e., the norm such that $K = \{ x \in \mathbb{R}^n : \| x \|_K \leq 1 \}$. Also, it is easy to see that $ST(L, K) = \{ \lambda^{-1} : \lambda \in ST(K, L) \}$. For such a pair of convex bodies $K$ and $L$, we can still use (30) to define their simple Banach-Mazur distance and use our spectral duality to introduce new distances and observe new identities.

Precisely, let

$$\lambda_{\text{max}}(K, L) = \max \{ \lambda : \lambda \in ST(K, L) \} \quad \text{and} \quad \lambda_{\text{min}}(K, L) = \min \{ \lambda : \lambda \in ST(K, L) \}.$$ 

Corollary 1 implies that $ST(A^\top L^*, K^*) = ST(AK, L)$ for any $n \times n$ invertible matrix $A$. Thus, we have the following new representation of the Banach-Mazur distance

$$d(K, L) = \inf_{A \in GL(\mathbb{R}^n)} \frac{\lambda_{\text{max}}(AK, L)}{\lambda_{\text{min}}(AK, L)} = \min_{A \in SL(\mathbb{R}^n)} \lambda_{\text{max}}(AK, L) \lambda_{\text{min}}(AK, L).$$

where $SL(\mathbb{R}^n)$ indicates the special linear group, i.e., the set of matrices with determinants equal to one.

From this formulation, we immediately obtain $d(K, L) = d(K^*, L^*)$, which generalizes the known equality for symmetric convex bodies to the nonsymmetric case. In fact, by Corollary 1 $\lambda_{\text{max}}(A^\top L^*, K^*) = \lambda_{\text{max}}(AK, L)$ and $\lambda_{\text{min}}(A^\top L^*, K^*) = \lambda_{\text{min}}(AK, L)$, and thus,

$$d(L^*, K^*) = \inf_{A \in GL(\mathbb{R}^n)} \frac{\lambda_{\text{max}}(A^\top L^*, K^*)}{\lambda_{\text{min}}(A^\top L^*, K^*)} = \inf_{A \in GL(\mathbb{R}^n)} \frac{\lambda_{\text{max}}(AK, L)}{\lambda_{\text{min}}(AK, L)} = d(K, L).$$

Using a similar argument we can obtain a similar result for other distances. In particular, consider the distance defined by

$$\hat{d}(K, L) = \inf \left\{ r \geq 1 : \frac{1}{r} L \subset K \subset rL \right\}.$$ 

This distance is used for studying floating and illumination bodies [43], and is equivalent to the Goldman-Iwahori metric introduced for Bruhat-Tits buildings [30]. We have

$$\hat{d}(K, L) = \max \{ \lambda_{\text{max}}(K, L), \frac{1}{\lambda_{\text{min}}(K, L)} \} = \max \{ \lambda_{\text{max}}(K, L), \lambda_{\text{max}}(L, K) \}$$

and from this new representation, we can easily obtain the duality identity $\hat{d}(K^*, L^*) = \hat{d}(K, L)$ via Theorem 1 and the discussion above.
References

1. Artstein-Avidan, S., Milman, V.D.: The concept of duality in convex analysis and the characterization of the legendre transform. Annals of mathematics 169(2), 661–674 (2009)
2. Artstein-Avidan, S., Milman, V.D.: Hidden structures in the class of convex functions and a new duality transform. Journal of the European Mathematical Society (JEMS) 13, 975–1004 (2011)
3. Artstein-Avidan, S., Rubinstein, Y.A.: Differential analysis of polarity: Polar hamilton-jacobi, conservation laws, and monge ampère equations. Journal d’Analyse Mathématique 132, 133–156 (2017)
4. Ballmann, W.: Der satz von lyusternik und schnirelmann. Bonn. Math. Schr. 102 (1978)
5. Bauschke, H.H., Combettes, P.L.: Convex Analysis and Monotone Operator Theory in Hilbert Spaces. Springer-Verlag (2011)
6. Boyd, S.: Convex Optimization. Cambridge University Press (2004)
7. Boyd, Z.M., Bae, E., Tai, X.C., Bertozzi, A.L.: Simplified energy landscape for modularity using total variation. SIAM Journal on Applied Mathematics 78(5), 2439–2464 (2018)
8. Bresson, X., Tai, X.C., Chan, T.F., Salam, A.: Multi-class transductive learning based on $\ell^4$ relaxations of Cheeger cut and Mumford-Shah-Potts model. J. Math. Imaging Vis. 49(1), 191–201 (2014). DOI 10.1007/s10851-013-0452-5
9. Bühlner, T., Hein, M.: Spectral clustering based on the graph p-Laplacian. In: International Conference on Machine Learning, p. 81–88 (2009). DOI 10.1145/1553374.1553385
10. Burger, M., Gilboa, G., Moeller, M., Eckardt, L., Cremers, D.: Spectral decompositions using one-homogeneous functionals. SIAM Journal on Imaging Sciences 9, 1374–1408 (2016)
11. Cai, Y., Zhang, L.H., Bai, Z., Li, R.C.: On an eigenvector-dependent nonlinear eigenvalue problem. SIAM Journal on Matrix Analysis and Applications 39(3), 1360–1382 (2018)
12. Calder, J.: The game theoretic p-Laplacian and semi-supervised learning with few labels. Nonlinearity 32(1), 301–330 (2018). DOI 10.1088/1361-6544/aae949
13. Chan, T.H.H., Louis, A., Tang, Z.G., Zhang, C.: Spectral properties of hypergraph Laplacian and approximation algorithms. Journal of the ACM (JACM) 65(3), 1–48 (2018)
14. Chung, F.R.: Spectral graph theory, vol. 92. American Mathematical Soc. (1997)
15. Clarke, F.H.: Optimization and Nonsmooth Analysis. Wiley New York (1983)
16. CORNEA, O., Lupton, G., Oprea, J., Taur´e, D.: Lusternik-schnirelmann category. Mathematical Surveys and Monographs 103 (2003)
17. Cristofari, A., Rinaldi, F., Tudisco, F.: Total variation based community detection using a nonlinear optimization approach. SIAM Journal on Applied Mathematics 80(3), 1392–1419 (2020)
18. Deidda, P., Putti, M., Tudisco, F.: Nodal domain count for the generalized graph p-laplacian. arXiv:2201.01248 (2022)
19. Deidda, P., Putti, M., Tudisco, F., Zhang, D.: The graph $\infty$-laplacian eigenproblem and its nodal domains (in preparation)
20. Elmoataz, A., Toutain, M., Tenbrinck, D.: On the $p$-Laplacian and $\infty$-Laplacian on graphs with applications in image and data processing. SIAM Journal on Imaging Science 8, 2412–2451 (2015). DOI 10.1137/15M1022793
21. Feld, T., Aujol, J.F., Gilboa, G., Papadakis, N.: Rayleigh quotient minimization for absolutely one-homogeneous functionals. Inverse Problems 35 (2019)
22. Fernández-Ternero, D., Macas-Virgós, E., Vilches, J.A.: Lusternik–schnirelmann category of simplicial complexes and finite spaces. Topology Appl. 194, 37–50 (2015)
23. Flores, M., Calder, J., Lerman, G.: Analysis and algorithms for $\ell_p$-based semi-supervised learning on graphs. Applied and Computational Harmonic Analysis 60, 77–122 (2022). DOI 10.1016/j.acha.2022.01.004
25. Gautier, A., Hein, M., Tudisco, F.: The global convergence of the nonlinear power method for mixed-subordinate matrix norms. Journal of Scientific Computing 88, 21 (2021)
26. Gautier, A., Nguyen, Q.N., Hein, M.: Globally optimal training of generalized polynomial neural networks with nonlinear spectral methods. Advances in Neural Information Processing Systems 29 (2016)
27. Gautier, A., Tudisco, F., Hein, M.: A unifying Perron–Frobenius theorem for nonnegative tensors via multihomogeneous maps. SIAM Journal on Matrix Analysis and Applications 40(3), 1206–1231 (2019)
28. Gilboa, G.: Nonlinear Eigenproblems in Image Processing and Computer Vision. Springer
29. Gromov, M.: Volume and bounded cohomology. Inst Hautes Études Sci. Publ. Math. 1982(56), 5–99 (1983)
30. Haettel, T.: Injective metrics on buildings and symmetric spaces. Bulletin of the London Mathematical Society (2022)
31. Hein, M., Setzer, S.: Beyond spectral clustering - tight relaxations of balanced graph cuts. In: Advances in Neural Information Processing Systems, vol. 24 (2011)
32. Hu, H., Laurent, T., Porter, M.A., Bertozzi, A.L.: A method based on total variation for network modularity optimization using the mbo scheme. SIAM Journal on Applied Mathematics 73(6), 2224–2246 (2013)
33. Hu, S., Qi, L.: The Laplacian of a uniform hypergraph. Journal of Combinatorial Optimization 29(2), 331–366 (2015)
34. Hynd, R., Lindgren, E.: Approximation of the least Rayleigh quotient for degree p homogeneous functionals. Journal of Functional Analysis 272, 4873–4918 (2017)
35. Jost, J., Zhang, D.: Discrete-to-continuous extensions: Lovász extension, optimizations and eigenvalue problems. arXiv:2106.03189v2 (2021)
36. Krasnosel’ski, M.A.: Topological methods in the theory of nonlinear integral equations. MacMillan (1964)
37. Li, P., Milenikov, O.: Submodular hypergraphs: p-Laplacians, Cheeger inequalities and spectral clustering. In: International Conference on Machine Learning, pp. 3014–3023. PMLR (2018)
38. Lusternik, L., Schnirelmann, L.: Méthodes topologiques dans les problèmes variationnels, Hermann, Paris (1934)
39. Prokopchik, K., Benson, A.R., Tudisco, F.: Nonlinear feature diffusion on hypergraphs. In: International Conference on Machine Learning (2022)
40. Saad, Y., Chelikowsky, J.R., Shontz, S.M.: Numerical methods for electronic structure calculations of materials. SIAM review 52(1), 3–54 (2010)
41. Slepčev, D., Thorpe, M.: Analysis of p-laplacian regularization in semisupervised learning. SIAM Journal on Mathematical Analysis 51(3), 2085–2120 (2019). DOI 10.1137/17M115222X
42. Rockafellar, R.T.: Convex Analysis. Princeton University Press (1970)
43. Saad, Y., Chelikowsky, J.R., Shontz, S.M.: Numerical methods for electronic structure calculations of materials. SIAM review 52(1), 3–54 (2010)
44. Slepčev, D., Thorpe, M.: Analysis of p-laplacian regularization in semisupervised learning. SIAM Journal on Mathematical Analysis 51(3), 2085–2120 (2019). DOI 10.1137/17M115222X
45. Tudisco, F., Hein, M.: A nodal domain theorem and a higher-order Cheeger inequality for the graph p-Laplacian. EMS Journal of Spectral Theory 8, 883–908 (2018). DOI 10.4171/357/216
50. Tudisco, F., Higham, D.J.: A nonlinear spectral method for core-periphery detection in networks. SIAM J. Mathematics of Data Science 1, 269–292 (2019)
51. Tudisco, F., Higham, D.J.: Node and edge eigenvector centrality for hypergraphs. Communications Physics 4(201) (2021)
52. Tudisco, F., Higham, D.J.: Core-periphery detection in hypergraphs. SIAM Journal on Mathematics of Data Science to appear (2022)
53. Tudisco, F., Mercado, P., Hein, M.: Community detection in networks via nonlinear modularity eigenvectors. SIAM Journal on Applied Mathematics 78, 2393–2419 (2018)
54. Upadhyaya, P., Jarlebring, E., Rubensson, E.H.: A density matrix approach to the convergence of the self-consistent field iteration. Numerical Algebra, Control and Optimization 11(1), 99–115 (2021)
55. Yosida, K.: Functional Analysis. Springer Berlin Heidelberg (1974)
56. Zeidler, E.: Nonlinear functional analysis and its applications III: Variational methods and optimization, 2nd edn. Springer (2013)
57. Zhou, G., Caccetta, L., Teo, K.L., Wu, S.Y.: Nonnegative polynomial optimization over unit spheres and convex programming relaxations. SIAM Journal on Optimization 22(3), 987–1008 (2012)