Research on numerical solution of one-sided monotonic function dichotomy

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Abstract. Aiming at the numerical solution problem of given interval dichotomy, the concept of one-sided monotonic function is proposed, and the unique existence of the solution is proved when the two ends of the function take different signs. Therefore, this significantly relaxes the discrimination conditions and effectively expands the numerical solution range of the dichotomy method. On this basis, it is proved that the one-sided monotonic function is a monotonic function in a relatively small interval containing the zero point. The introduction of the concept of one-sided monotonic function provides a theoretical basis for further expanding the application range of the dichotomy numerical solution, and has certain theoretical and practical significance for dealing with scientific and engineering problems as well as related numerical calculation problems.

1. Introduction

In scientific research and engineering calculation, we often encounter the problem of numerical solution of equations. Iterative method is one of the most commonly used methods to find numerical solutions of equations. The main idea is to use a certain recurrence relation to make a predictable approximate root more accurate until iterated the root satisfying the accuracy[1-5].

As early as the 1970s, a large number of researches on the solution of nonlinear equations had been done theoretically and numerically[6-8]. The subsequent numerical calculation methods can be divided into two categories: the iterative algorithm without derivative calculation and with derivative calculation. Iterative algorithms that do not need to calculate the derivative of the solution function are: dichotomy and simple iterative method. On the basis of dichotomy, Wood, Martin, Rayskin and others extended its dimension. The original dichotomy convergence is still guaranteed[9-10]. Bachrathy, Stepan described a generalized version of dichotomy. The algorithm can find multiple solutions in a certain interval and reduce the complexity of operation, but the astrigency can’t be guaranteed[11]. Solanki, Thapliyal and others proposed an algorithm to combine the dichotomy with Newton Raphson, which greatly improved the speed of the solution[12-13]. Aiming at the numerical solution problem of given interval of dichotomy, Tanakan and Chhabra designed an improved method. This method first predicts the optimal interval of the numerical solution of the function, and then uses the dichotomy method to find the numerical solution of the function according to the tolerable error, which effectively reduces the number of iterations[14-15]. Iterative algorithms that need to calculate the derivative of the solution function are: classical Newton iteration and secant method. According to a semi-local convergence analysis of Newton’s method on a Banach space, Argyros and Hilout improved the error bounds, order of convergence, and simplify the sufficient convergence conditions[16]. Obadah and Hashim derived two new iterative methods based on the Taylor series
expansion and Halley’s method, and constructed a method with a more efficient index. The new methods have sixth order of convergence[17-20]. Based on the Newton iterative search algorithm for multi-node co-location, Yanli and Yao proposed an improved Newton iterative search algorithm for vibration source location. This method can effectively solve the problem of initial value selection of Newton iteration method and improve the localization accuracy of vibration source[21].

In summary, among these iterative methods, Newton iterative method and secant method have high convergence. However, it is difficult to calculate the derivative value of function reference point and select the initial value, and these factors will affect the accuracy of the final result[22]. The dichotomy is the simplest numerical solution method, which only requires the function to be continuous in the single root interval. In the dichotomy, it is hard to determine single root interval of the function. For this, the concept of one-sided monotonic function is proposed in this paper, which provides a theoretical basis for further expanding the use of dichotomy method.

2. Numerical methods
The commonly used numerical methods are Newton iteration method, secant method and dichotomy method.

2.1. Newton iteration method
The basic idea of this method is to linearize the nonlinear function gradually, so that the nonlinear equation is approximately transformed into solving linear equation. Newton's iteration method formula is:

\[ x_{k+1} = x_k - \frac{f(x_k)}{f'(x_k)} \quad (k = 0,1,2 \ldots) \quad (1) \]

The advantage of equation (2.1) is that it convergence speed is fast, but the disadvantage is that it needs a large amount of calculation, and each iteration needs to calculate the values of \( f(x) \) and \( f'(x) \), also requires \( f'(x) \neq 0 \).

2.2. Secant method
This method is a root-finding method without derivation calculation, and the operation process is relatively simple. Secant method iteration formula is:

\[ x_{k+1} = x_k - \frac{f(x_k)}{f(x_k) - f(x_{k-1})} (x_k - x_{k-1}) \quad (k = 1,2,\ldots) \quad (2) \]

Equation (2.2) has super-linear convergence, but the initial value selected must be close to the numerical solution of the equation, otherwise it may lead to divergence. This has some limitations in the practical application.

2.3. Dichotomy method
The main idea of dichotomy is to shrink half of the interval where the zero point of \( f \) is located, so that the two ends of the interval are gradually approach to the zero point of the function to obtain the approximate value. This method requires that the interval of the function should be single root interval.

**Definition 1. (Single root interval)** If the equation \( f(x) = 0 \) has only one root in the interval \([a, b]\), then the interval \([a, b]\) is called the single root interval of \( f(x) \).

Dichotomy method has the advantages of simplicity, reliability, good convergence and controllable error. Use \( x^* \) to represent the root of \( f(x) = 0 \), and \( x_k \) to represent the numerical solution obtained after \( k \) iterations of dichotomy. The error is:

\[ |x^* - x_k| \leq \frac{b - a}{2^{k+1}} \quad (k = 1,2,\ldots) \quad (3) \]

For the given error, the number of iterations can be obtained from equation (2.3) directly.
3. One-sided monotonic function

Definition 2. If \( f(x) \) satisfies:

1. Continuous in the interval \([a, b]\);
2. The signs of the two endpoints are opposite, \( f(a) \cdot f(b) < 0 \);
3. \( f(x) \) is a monotonic function on one side starting from zero.

Then \( f(x) \) is called one-sided monotonic function.

According to the definition, monotonic function is a special case of one-sided monotonic function. If a function is always zero in a certain interval, it is called a singularity function, which does not belong to the category discussed in this paper. There are four forms of one-sided monotonic functions: monotonically increasing in the upper right, monotonically increasing in the bottom left, monotonically decreasing in the lower right, and monotonically decreasing in the upper left. As shown in Figure 1.

![Figure 1. Schematic diagram of one-sided monotonic function.](a) Monotonically increasing in the upper right (b) Monotonically increasing in the bottom left (c) Monotonically decreasing in the lower right (d) Monotonically decreasing in the upper left

The theoretical proof of the existence of a unique solution for one-sided monotonic functions is given below.

Theorem 1. (One-sided monotonic function unique solution). Given that \( f(x) \) is a one-sided monotonic function in the interval \([a, b]\), then \( f(x) \) exists a unique solution in the interval \((a, b)\).

Proof. According to the definition of one-sided monotonic function, there exists at least one solution \( x_0 \in (a, b) \), which satisfies one of the following four cases:

1. \( f(a) < 0, f(b) > 0, f(x) \) monotonically increases in the upper right of the solution \( x_0 \) (see figure 1 (a));
2. \( f(a) < 0, f(b) > 0, f(x) \) monotonically increases in the bottom left of the solution \( x_0 \) (see figure 1 (b));
3. \( f(a) > 0, f(b) < 0, f(x) \) monotonically decreases in the lower right of the solution \( x_0 \) (see figure 1 (c));
4. \( f(a) > 0, f(b) < 0, f(x) \) monotonically decreases in the upper left of the solution \( x_0 \) (see figure 1 (d)).

The following is to take case (1) as an example, and the others can be deduced by analogy.
We prove the uniqueness by the counter evidence method. Suppose \( f(x) \) has two solutions in the interval \((a, b)\), denoted by \( x_1, x_2 \), that is, \( f(x_1) = f(x_2) = 0 \). Let's assume \( a < x_1 < x_2 < b \). Because \( x_1 < \frac{x_1 + x_2}{2} < x_2 \) and \( f(x) \) increases monotonically in the interval \((x_1, x_2)\), \( f\left(\frac{x_1 + x_2}{2}\right) > f(x_1) = 0 \). Meanwhile the function satisfies \( f\left(\frac{x_1 + x_2}{2}\right) < f(x_2) = 0 \). The value of \( f\left(\frac{x_1 + x_2}{2}\right) \) creates a contradiction. Therefore, it is impossible for \( f(x) \) to have two or more solutions in the interval \((a, b)\).

The proof is complete.

In the following, we prove that a one-sided monotonic function can find a relatively small interval near its solution, and the function is monotonic in this interval. For this, the zero point theorem is introduced firstly.

**Lemma. (Zero point theorem).** Assume that the function \( f(x) \) is continuous in the closed interval \([a, b]\), and the signs of \( f(a) \) and \( f(b) \) are opposite, that is \( f(a) \cdot f(b) \leq 0 \). Then there is at least one zero point of \( f(x) \) in the open interval \((a, b)\). That is, there is at least a point \( \xi \in (a, b) \), so that \( f(\xi) = 0 \).

**Theorem 2. (Monotonicity of one-sided monotonic function).** It is known that \( f(x) \) is a one-sided monotonic function in the interval \((a, b)\), and its solution is \( x_0 \in (a, b) \). There exists \( \Delta x > 0 \), so that \( f(x) \) is a monotonic function in the interval \((x_0 - \Delta x, x_0 + \Delta x) \subset (a, b)\).

**Proof.** From Theorem 1, \( f(x) \) has a unique solution \( x_0 \) in the interval \((a, b)\). Suppose \( f(a) < 0, f(b) > 0, f(x) \) increases monotonically in the upper right of solution \( x_0 \).

In the following, we prove the existence of \( \Delta x > 0 \) such that \( f(x) \) increases monotonously in the interval \((x_0 - \Delta x, x_0)\). Since \( f(x) \) increases monotonically in the interval \((x_0, x_0 + \Delta x)\), it is only necessary to prove that \( f(x) \) increases monotonously in the interval \((x_0 - \Delta x, x_0)\).

The following is proved by the counter evidence method. Suppose there exists \( \delta \) which satisfies \( 0 < \delta < \Delta x \), such that \( f(x) \) does not satisfy monotonic increasing in the interval \((x_0 - \delta, x_0)\), that is, \( f(x_0 - \delta) > f(x_0) = 0 \). Because \( f(x) \) is continuous in the interval \([a, x_0 - \delta]\), and \( f(a) < 0, f(x_0 - \delta) > 0 \), according to the Zero Point Theorem, \( f(x) \) has at least one solution in the interval \([a, x_0 - \delta]\). Therefore, there are at least two solutions in the interval \((a, b)\), which contradicts that \( f(x) \) has a unique solution \( x_0 \) in \((a, b)\). That is, there exists \( \Delta x > 0 \), which makes \( f(x) \) increases monotonically in the interval \((x_0 - \Delta x, x_0)\). Furthermore, \( f(x) \) increases monotonically in the interval \((x_0 - \Delta x, x_0 + \Delta x) \subset (a, b)\). The proof is complete.

Theorem 2 shows that one-sided monotonic function is monotonic function in a relatively small interval, so there exists a unique solution. In practical application, it is difficult to find this small interval, so Theorem 1 is more practical.

**4. Examples of dichotomy application**

In the practical application of oilfield, the sampling time point \( 0 < t_1 < t_2 < t_3 \), the corresponding submergence depth \( 0 < h_1 < h_2 < h_3 \), satisfies:

\[
\frac{h_3 - h_2}{h_2 - h_1} = \frac{x^{t_3} - x^{t_2}}{x^{t_2} - x^{t_1}}
\]

The characteristic function of oil well pumping model is as follows:

\[
f(x) = (h_2 - h_1) \sum_{i=0}^{t_3-t_1-1} x^i - (h_3 - h_1) \sum_{i=0}^{t_2-t_1-1} x^i
\]

In the interval \((0,1)\), \( f(x) \) is a monotonically increasing function in the upper right. Therefore, function \( f(x) \) in equation (4.2) has a unique solution in \( x \in (0,1) \).

(1) For \( x \in (0,1) \), \( f(0) = h_2 - h_3 < 0 \). Next, we prove that \( f(1) > 0 \).

\[
f(1) = \frac{(h_2 - h_1)(t_3 - t_2) - (h_3 - h_1)(t_2 - t_1)}{x^{t_1} - x^{t_2}} \left[ \frac{x^{t_3} - x^{t_2}}{t_3 - t_2} - \frac{x^{t_2} - x^{t_1}}{t_2 - t_1} \right]
\]
Let $F(t) = x^t \ (t \geq 0)$, then $F''(t) = x^t \ln^2(x) > 0$. That is, when $t \geq 0$, $F(t)\ increases monotonically. According to Lagrange Mean Value Theorem, there exists $0 < t_1 < t_2 < \xi_2 < t_3$, such that

$$\frac{x^{t_3} - x^{t_2}}{t_3 - t_2} - \frac{x^{t_2} - x^{t_1}}{t_2 - t_1} = F''(\xi_2) - F''(\xi_1) > 0$$ \hspace{1cm} (7)

And because of $\frac{(h_2 - h_1)(t_3 - t_2)(t_2 - t_1)}{x^{t_3} - x^{t_2}} > 0$, so $f(1) > 0$.

(2) The following proves that when $0 < x < 1$ and $f(x) \geq 0$, $f(x)$ increases monotonically.

Let $H_1 = h_2 - h_1$, $H_2 = h_3 - h_1$, $T_1 = t_2 - t_1$, $T_2 = t_3 - t_1$. For $\Delta x > 0$, $x \in (0,1)$, $x + \Delta x \in (0,1)$, when $f(x) \geq 0$, there are:

$$f(x + \Delta x) = H_1 \sum_{i=0}^{T_2-1} (x + \Delta x)^i - H_2 \sum_{i=0}^{T_1-1} (x + \Delta x)^i$$

$$= H_1 \sum_{i=T_1}^{T_2-1} (x + \Delta x)^i - (H_2 - H_1) \sum_{i=0}^{T_1-1} (x + \Delta x)^i$$

$$> \left[\frac{x + \Delta x}{x}\right]^{T_1} f(x) > f(x) \hspace{1cm} (8)$$

In summary, equation (4.2) is a one-sided monotonic function in the interval $(0,1)$. According to Theorem 1, $f(x)$ has a unique solution $x_0 \in (0,1)$. Therefore, the numerical solution of the characteristic function of oil well pumping model $f(x)$ can be obtained by dichotomy method. figure 2 is the schematic diagram of equation (4.2).

![Figure 2. Schematic diagram of Characteristic function of oil well pumping model.](image)

### 5. Conclusion

In this paper, the problem of determining the single root interval in the numerical solution of dichotomy is analyzed. The concept of one-sided monotonic function is proposed, and the theorems of one-sided monotonic unique solution (Theorem 1) and monotonicity of one-sided monotonic function (Theorem 2) are given. This significantly relaxes the discrimination conditions and provides the basis for determining the single root interval of function and expanding the application range of the dichotomy method.
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