Scaling approach to itinerant quantum critical points

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Based on phase space arguments, we develop a simple approach to metallic quantum critical points, designed to study the problem without integrating the fermions out of the partition function. The method is applied to the spin-fermion model of a T=0 ferromagnetic transition. Stability criteria for the conduction and the spin fluids are derived by scaling at the tree level. We conclude that anomalous exponents may be generated for the fermion self-energy and the spin-spin correlation functions below $d = 3$, in spite of the spin fluid being above its upper critical dimension.

In this paper, we introduce a simple scaling approach, designed to study a spin-fermion model at a QCP without integrating the fermions out of the partition function. Already at the tree level, it reveals that the critical behavior is controlled by several couplings, rather than by the single fermion-boson coupling constant $g$, expected naively. At a ferromagnetic quantum critical point, the coupling $g$ becomes relevant below one spatial dimension, analogously to the four-boson coupling constant at a ferromagnetic QCP [5]. At the same time, the four-fermion coupling $u_f$, mediated by the bosons, controls the breakdown of the Fermi liquid theory, and becomes relevant below three spatial dimensions.

We will illustrate the idea on the spin-fermion model of a ferromagnetic quantum critical point, comprising three parts: a conduction electron term $S_f$, a boson term $S_b$, describing the critical magnetic modes, and $S_i$, representing interaction between the fermions and the bosons:

$$S = S_f + S_b + S_i$$

$$S_f = \int d\omega \, d^d k \, \bar{\psi}^\dagger_k (i\omega - \xi_k) \psi_k$$

$$S_b = \int d\omega \, d^d q \, \varphi_q \left( \frac{|\omega|}{q} + q^2 \right) \varphi_q$$

$$S_i = g \int d\omega_1 d\omega_2 d^d k_1 d^d k_2 \left[ \bar{\varphi}_{k_1 - k_2} \psi_{k_1}^\dagger \psi_{k_2} + h.c. \right].$$

Here $\psi$ (\varphi) are the fermion (boson) fields, and $\xi_k$ is the quasiparticle energy counted from the chemical potential. For simplicity, we consider a spherical Fermi surface.

We first perform a Benfatto-Galavotti-Shankar [9,10] renormalization at the tree level, removing the high energy degrees of freedom, and retaining only their contribution to the low energy effective action. At each step we eliminate a shell $\Lambda/s \leq \omega \leq \Lambda$, where $\Lambda$ is the cut-off and $s > 1$. The energies and momenta are then rescaled to restore the original cut-offs and, lastly, the fields are rescaled to leave the quadratic part of the action intact.
However, implementation of this program poses two difficulties. The first one is of geometric origin: the fermion momenta are restricted to a thin shell around the Fermi surface, while the boson momenta are confined to a sphere. The second difficulty stems from the different dynamic exponents \( z \) of the fermion and boson fluids: in the Fermi liquid, \( z_f = 1 \), while the magnetic modes are characterized by \( z_b = 3 \). To perform simultaneous mode elimination for the two species, we rescale the energies and the momenta as per

\[
\omega' = s \omega; \quad k_\perp' = s^{1/z_b} k_\perp = s k_\perp; \quad q' = s^{1/z_b} q = s^{1/3} q,
\]

while the fields rescale as per

\[
\psi' = s^{-3/2} \psi; \quad \varphi' = s^{-(d+z_b+2)/2z_b} \varphi
\]

where \( k_\perp \) is defined by \( \xi_k = v_F k_\perp \).

Before proceeding with the scaling of different quantities, let us make several observations. First, notice an asymmetry between our rescaling procedure and momentum conservation in a single boson scattering process. The two fermion momenta \( k_{1,2} = k_F k_{1,2}(1 + \eta_{1,2}) \), where \( |\eta_{1,2}| \ll 1 \), are related by \( k_1 - k_2 = q \). Since, near a ferromagnetic QCP, the boson momenta are much smaller than \( k_F \), to first order in \( \eta_{1,2} \) one finds

\[
\theta^2 [1 + \eta_1 - \eta_2] = \left( \frac{q}{k_F} \right)^2,
\]

where \( \cos \theta \equiv (k_1 \cdot k_2) \), and \( \theta \ll 1 \). From this, two points follow: (i) rescaling of the boson momentum has to be accompanied by rescaling of the scattering angle \( \theta \) between the two fermion momenta, as in the procedure we adopted; (ii) asymptotically close to the Fermi surface \((\eta_{1,2} \rightarrow 0)\), rescaling fermion and boson momenta differently is consistent with momentum conservation.

Now we are in a position to find the scaling properties of various vertices. To obtain the rescaling of \( g \), rewrite \( S_i \) as

\[
S_i = g \int d\omega_1 d\omega_2 dk_\perp d^d q_\perp \int d^d q_\parallel (\varphi_\parallel \psi_{k_1}^\dagger \psi_{k_2} + h.c.).
\]

Using the scaling properties of the fields (3) and of the different components of momenta (4,5), one arrives at the sought scaling relation:

\[
g' = s^{-d+z_b-1/dz_b} g.
\]

We find that \( g \) is irrelevant for \( d > 1 \), similarly to the four-boson coupling constant in the \( \varphi^4 \) theory \((z_b = 3)\).

One may inquire about the relation between the coupling constant \( u_f \) of the boson-mediated four-fermion interaction (Fig. 2 a) and the spin-fermion scattering vertex \( g \). Naively, one would expect \( u_f \) to scale as \( g^2 \) and, hence, become irrelevant in one spatial dimension or less; we will show that this is not the case: at a ferromagnetic
quantum critical point, $u_f$ becomes relevant already in $d < 3$ spatial dimensions, possibly leading to an instability of the Fermi liquid ground state. Notice that, of the three independent momenta ($p_1, p_2, p_3$) in the diagram a) in Fig. 2, only the momentum transfer $q \equiv p_1 - p_3$ is subject to the phase space restriction discussed above, whereas $p \equiv (p_1 + p_3)/2$ and $p_2$ independently span the entire Fermi surface. Hence the four-fermion term in the action can be re-written to make explicit the scaling properties of the various components of momenta:

$$S_{4f} \rightarrow u_f \int \frac{d\omega_1}{s} \frac{d\omega_2}{s} \frac{d\omega_3}{s} \frac{dp_1}{s} \frac{dp_2}{s} \frac{d^{d-1}q}{s^d(d-1)/z_b} \times \left(7\right)$$

$$\times \frac{dp_{1\perp}}{s} d_{1\perp} d_{2\perp} \left| p - q/2 \right|^2$$

$$\times \frac{s^{2/z_b}}{q^2}$$

from which we read off

$$u_f \rightarrow u_f s^{(3-d)/z_b}.$$  

Which means that, already at the tree level, $u_f$ becomes relevant below three spatial dimensions, while $q$ is relevant only below $d = 1$. This indicates that the Fermi liquid state may break down below three dimensions, where the naive $\varphi^4$ theory would be still above its upper critical dimension.

Finally, let us illustrate how one can use (4,5) to find the scaling of the fermion self-energy $\Sigma(\omega)$ in the lowest order in $g$. It can be found by power counting of the diagram b) in figure 2; its contribution to the self energy is

$$\Sigma(\omega) \simeq g^2 \int d\omega d\omega_\perp d^{d-1}q || G_\varphi(\omega - \nu; p - q) G_\varphi(\nu; q).$$

Since $G_\varphi^{-1}(\omega; p) = i\omega - v_F p_\perp$ and $G_\varphi^{-1}(\omega, q) = |\omega|/(q + q^2)$, one finds, again using (3,4,5):

$$\Sigma(s\omega) = \Sigma(\omega) s^{-d/z_b}$$

This is in agreement with previous work [7,8,11], and points to a possible non-Fermi behavior in $d \leq 3$, as previously observed in the context of gauge theories [11] and recently noted in the context of a ferromagnet [12].

To summarize, we introduced an RG scheme in the spirit of the Shankar approach [9], which allows to treat the fermion and the boson degrees of freedom in the spin-fermion models on an equal footing. We showed that, already at the tree level, the boson-mediated four-fermion coupling is relevant below three spatial dimensions, even though the fermion-boson coupling constant $g$ is relevant only above one dimension. Our approach is general and can be applied to the theory of an antiferromagnetic QCP – as well as to other situations where fermions interact with critical modes, e.g. those involving different dimensionalities of fermions and spin fluctuations. The phase space restriction associated with the fermion scattering off the critical modes is the key ingredient of the approach. A one loop RG treatment in $d = 3 - \epsilon$, including transport properties and thermodynamics, would be a natural extension of this work [14].

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