ON THE HERMITIAN CURVATURE OF SYMPLECTIC MANIFOLDS

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Abstract. In this paper we give conditions for the integrability of almost complex structures calibrated by symplectic forms. We show that in the symplectic case Newlander-Nirenberg theorem reduces to $\nabla''N_J = 0$ and we give integrability conditions in terms of the curvature and the Hermitian curvature of the induced metric.

1. Introduction

The interplay between complex and symplectic structures has been recently studied by many authors. Indeed, on any symplectic manifold $(M, \kappa)$ there exists a $\kappa$-calibrated almost complex structure $J$, so that $(M, g, J, \kappa)$ is an almost Kähler manifold.

In the context of almost Kähler geometry it is natural to study the integrability of the almost complex structure.

In [4] Goldberg proved that, if the curvature operator of an almost Kähler manifold $(M, g, J, \kappa)$ commutes with $J$, then $(M, g, J, \kappa)$ is a Kähler manifold. He also conjectured that an Einstein almost Kähler metric on a compact manifold is a Kähler metric.

If the scalar curvature is nonnegative the conjecture has been proved by Sekigawa in [6].

For a survey, other references and results on this topic we refer to [2].

In this paper we show that, if $(M, g, J, \kappa)$ is an almost Kähler manifold satisfying certain properties, then it is Kähler. Namely we give some conditions on the derivative of the Nijenhuis tensor and on the curvature respectively, in order that $J$ is integrable.

In section 2 we start by recalling some facts and fixing some notations.

In section 3 we prove that if the $(0, 1)$-part of the covariant derivative of the Nijenhuis tensor of $J$ vanishes, then $J$ is integrable (theorem 3.3). This result generalizes theorem 2 of [7].

In section 4 we consider three types of curvature tensors on $(M, g, J, \kappa)$.
Namely, the Riemann curvature $R$, the curvature $\tilde{R}$ of the Hermitian connection $\tilde{\nabla}$ and the tensor $R^J$ defined by

$$R^J(X,Y,Z,W) = g(\nabla_X J \nabla_Y Z - \nabla_Y J \nabla_X Z - \nabla_{[X,Y]} J Z, JW).$$

We show that if $R^J$ and $R$ have the same components along certain directions, then $J$ is integrable (theorem 4.2).

Finally we prove that if the bisectional curvature of $g$ and the Hermitian bisectional curvature coincides, then $g$ is a Kähler metric (theorem 4.8).

A key tool in the proof of our results is the existence of generalized normal holomorphic frames (see [7]).

2. Preliminaries

Let $M$ be a $2n$-dimensional (real) manifold.

A symplectic structure on $M$ is a closed non-degenerate 2-form $\kappa$, i.e. $d\kappa = 0$ and $\kappa^n \neq 0$. The pair $(M, \kappa)$ is said to be a symplectic manifold.

An almost complex structure on $M$ is a smooth section $J$ of $\text{End}(TM)$, such that $J^2 = -\text{Id}$.

An almost complex structure $J$ is said to be integrable if the Nijenhuis tensor

$$N_J(X,Y) = [JX, JY] - J[JX, Y] - J[X, JY] - [X, Y]$$

vanishes. In view of the celebrated Newlander-Nirenberg theorem, $J$ is integrable if and only if it is induced by a holomorphic structure.

An almost complex structure $J$ on $M$ induces a natural splitting of the complexified of the tangent bundle. Indeed let $T^{1,0}_J M$, $T^{0,1}_J M$, be the eigenspaces relatively to $i$ and $-i$ respectively; then $TM \otimes \mathbb{C} = T^{1,0}_J M \oplus T^{0,1}_J M$. The sections of $T^{1,0}_J M$, $T^{0,1}_J M$ are called vector fields of type $(1,0)$ and $(0,1)$ respectively.

We have that $T^{1,0}_J M = T^{0,1}_J M$ and that the map $X \mapsto X - iJX$ defines an isomorphism between $TM$ and $T^{1,0}_J M$.

An almost complex structure $J$ is said to be $\kappa$-calibrated if

$$g[x](\cdot, \cdot) = \kappa[x](\cdot, J_{x}\cdot)$$

is a Hermitian metric on $M$. In this case the triple $(g, J, \kappa)$ is said to be an almost Kähler structure on $M$ and $(M, g, J, \kappa)$ an almost Kähler manifold.

Let us denote by $\mathcal{C}_\kappa(M)$ the set of the almost complex structure on $M$ calibrated by $\kappa$. It is known that $\mathcal{C}_\kappa(M)$ is a non-empty and contractible set (see e.g. [1]). Therefore any symplectic manifold admits almost Kähler structures.

Let $(M, \kappa)$ be a symplectic manifold and $J \in \mathcal{C}_\kappa(M)$: if $J$ is integrable the triple $(g, J, \kappa)$ is said to be a Kähler structure on $M$ and $(M, g, J, \kappa)$ a Kähler manifold (see e.g. [5]).
3. SOME INTEGRABILITY CONDITIONS

In this section we give conditions on the covariant derivative of the Nijenhuis tensor in order that the complex structure is integrable.

Let \((M, \kappa, J, \kappa)\) be a \(2n\)-dimensional almost Kähler manifold. We denote by \(\nabla\) the Levi-Civita connection relatively to \(g\) and by \(R\) the curvature tensor of \(g\).

We have the following (see \([3]\) and \([7]\))

**Theorem 3.1** (Generalized normal holomorphic frames). For any point \(o \in M\) there exists a local complex \((1,0)\)-frame, \(\{Z_1, \ldots, Z_n\}\) around \(o\), satisfying the following conditions:

1. \(\nabla_k Z_i(o) = 0,\ 1 \leq k, i \leq n;\)
2. \(\nabla_k Z_i(o)\) is of type \((0,1)\), \(1 \leq k, i \leq n;\)
3. if \(G_{r\pi} := g(Z_r, Z_\pi)\), then: \(G_{r\pi}(o) = \delta_{rs},\ dG_{r\pi}[o] = 0;\)
4. \(\nabla_r \nabla_\tau Z_i(o) = 0,\ 1 \leq r, k, i \leq n,\)

where \(\overline{Z}_i := Z_\tau\) and \(\nabla Z_i Z_j := \nabla_i Z_j.\)

By definition \(\{Z_1, \ldots, Z_n\}\) is said a generalized normal holomorphic frame around \(o\).

We recall that the following fundamental relation holds:

\[
2g((\nabla_X J)Y, Z) = g(N_J(Y, Z), JX).
\]

From the complex extension of \([1]\) it can be easily proved the following (see \([7]\))

**Corollary 3.2.** The covariant derivative along \((0,1)\)-vector fields of the almost complex structure \(J\) is a tensor of complex type \((0,1)\), i.e.

\[
(\nabla_{\overline{Z}_j} J)Z_i = 0
\]

for any \(Z_i, Z_j\) complex vector fields of type \((1,0)\).

Denote by \(\nabla''\) the \((0,1)\)-part of the covariant derivative associated to the Levi-Civita connection, i.e.

\[
\nabla''_W X = \nabla_{W^{0,1}} X,
\]

where \(W^{0,1}\) denotes the natural projection of \(W\) on \(T^{0,1}_J M\). As a first application of the generalized normal holomorphic frames we have the following

**Theorem 3.3.** Assume that

\[
\nabla'' N_J = 0;
\]

then \((M, g, J, \kappa)\) is a Kähler manifold.
Proof. Let $o$ be an arbitrary point in $M$ and let $\{Z_1, \ldots, Z_n\}$ be a generalized normal holomorphic frame around $o$.
Since $N_J(T^{1,0}_o \times T^{1,0}_M) \subset T^{0,1}_o M$ we have
$$Z_\tau g(N_J(Z_i, Z_k), Z_\overline{k}) = 0.$$ Therefore, by the assumption $\nabla''N_J = 0$, we get
$$g(N_J(Z_i, Z_k), \nabla Z_\tau) = 0$$ and consequently
\[ g(N_J(Z_i, Z_k), [Z_\tau, Z_\overline{k}]) = 0. \]
Moreover a direct computation gives
\[ N_J(Z_i, Z_k)(o) = -\frac{1}{4} [Z_i, Z_k](o). \]
Hence (2) and (3) imply $N_J(Z_i, Z_k)(o) = 0$, i.e. $J$ is integrable, so that $(M, g, J, \kappa)$ is a Kähler manifold. \hfill \square

Remark 3.4. Let $B$ the $(2,1)$-tensor on $(M, g, J, \kappa)$ defined by
\[ B(X, Y) = J(\nabla_X J)Y - (\nabla_J X)Y. \]
Then the Nijenhuis tensor of $J$ is the antisymmetric part of $B$. In [7] (theorem 2) it is proved that an almost complex structure $J$ is integrable if and only if $\nabla''B = 0$. Therefore theorem 3.2 generalizes theorem 2 of [7].

Remark 3.5. With respect to a generalized normal holomorphic frame, $\{Z_1, \ldots, Z_n\}$, the components of the curvature tensor $R_{i\overline{j}r\overline{s}}(o)$ and $R_{i\overline{j}r\overline{s}}(o)$ are given by
\[ R_{i\overline{j}r\overline{s}}(o) = -g(\nabla \nabla_i Z_r, Z_\overline{s})(o), \]
\[ R_{i\overline{j}r\overline{s}}(o) = g(\nabla \nabla_i Z_r, Z_\overline{s})(o). \]
In order to study the integrability of a $\kappa$-calibrated almost complex structure $J$ it is useful to introduce the following tensor
$$L(X, Y, Z, W) = g((\nabla_X B)(Y, Z), W),$$
where $B$ is defined by (4).
A first property of the tensor $L$ is given by the following

Lemma 3.6. This identities hold
\[ L_i j r s = L_i j r s, \]
\[ L_i j r s = L_i j r s = L_i j r s = 0, \]
for $1 \leq i, j, r, s \leq n$. 

Proof. Let \( o \) be an arbitrary point in \( M \) and let \( \{ Z_1, \ldots, Z_n \} \) be a generalized normal holomorphic frame around \( o \). By the definition of \( L \) and the properties of generalized normal holomorphic frames, we get

\[
L_{ijr}(o) = g((\nabla^{B}_r)(Z_j, Z_r), Z_\pi)(o)
\]

\[
= g(\nabla^{B}_r(B(Z_j, Z_r)), Z_\pi)(o)
\]

\[
= g(\nabla^{B}_r(J\nabla_j Z_r, Z_\pi)(o) - ig(\nabla^{B}_r(\nabla_j J) Z_r, Z_\pi)(o)
\]

\[
= 2ig(\nabla^{B}_r J\nabla_j Z_r, Z_\pi)(o) + 2g(\nabla^{B}_r J\nabla_j Z_r, Z_\pi)(o)
\]

\[
= 2ig(\nabla^{B}_r J\nabla_j Z_r, Z_\pi)(o) + 2R_{ijr}(o),
\]

i.e.

\[(5)\]

\[
L_{ijr}(o) = 2ig(\nabla^{B}_r J\nabla_j Z_r, Z_\pi)(o) + 2R_{ijr}(o).
\]

By a direct computation we get

\[
L_{ijr}(o) = 2iZ_r g(J\nabla_j Z_r, Z_\pi)(o) + 2iZ_j g(Z_r, J\nabla_j Z_\pi)(o)
\]

\[
+ 2R_{ijr}(o).
\]

Then we have

\[
L_{ijr}(o) = 2iZ_r g(J\nabla_j Z_r, Z_\pi)(o) + 2iZ_j g(Z_r, J\nabla_j Z_\pi)(o)
\]

\[
- 2ig(Z_r, \nabla_j J\nabla_j Z_\pi)(o) + 2R_{ijr}(o)
\]

\[
= - 2Z_r g(\nabla_j Z_r, Z_\pi)(o) + 2Z_j g(Z_r, \nabla_j Z_\pi)(o)
\]

\[
- 2ig(Z_r, \nabla_j J\nabla_j Z_\pi)(o) + 2R_{ijr}(o)
\]

\[
= - 2g(\nabla_j Z_r, Z_\pi)(o) + 2g(Z_r, \nabla_j Z_\pi)(o)
\]

\[
- 2ig(Z_r, \nabla_j J\nabla_j Z_\pi)(o) + 2R_{ijr}(o)
\]

\[
= - 2R_{ijr}(o) + 2R_{ijr}(o)
\]

\[
- 2ig(Z_r, \nabla_j J\nabla_j Z_\pi)(o) + 2R_{ijr}(o)
\]

\[
= - 2R_{ijr}(o) + 2R_{ijr}(o).
\]

Then we obtain

\[
L_{ijr}(o) = - 2ig(Z_r, \nabla_j J\nabla_j Z_\pi)(o) + 2R_{ijr}(o).
\]

Therefore

\[
L_{ijr}(o) = L_{ijr}(o).
\]

The second part of the proof is straightforward. \(\square\)

As a corollary of theorem 3.3 we have the following (see also [3]):

**Proposition 3.7.** Assume that

\[
L(Z_\tau, Z_i, Z_j, Z_\tau) = 0,
\]

for any complex vector fields \( Z_i, Z_j \) of type \((1,0)\); then \((M, g, J, \kappa)\) is a Kähler manifold.
Proof. Let \{Z_1, \ldots, Z_n\} be a generalized normal holomorphic frame around \(o\). A direct computation gives
\[
L(Z_i, Z_j, Z_i, Z_j)(o) = g(\nabla_i Z_j, \nabla_j Z_i)(o);
\]
and hence \(\nabla_i Z_j(o) = 0\).
Therefore \(N_f(Z_i, Z_j)(o) = -\frac{1}{4}[Z_i, Z_j](o) = 0\).
\(\square\)

4. Curvature and Integrability

In this section we give an integrability condition in terms of curvature.
We start defining the following tensor
\[
R^J(X, Y, Z, W) = g(\nabla_X J \nabla_Y Z - \nabla_Y J \nabla_X Z - \nabla_{[X,Y]} J Z, JW).
\]
We have

**Lemma 4.1.** Let \(\{Z_1, \ldots, Z_n\}\) be a generalized normal holomorphic frame around \(o\), then
\[
R^J_{ijr\pi}(o) = -ig(\nabla_i J \nabla_j Z_r, Z_\pi)(o).
\]

**Proof.** By the definition of \(R^J\) it follows that
\[
R^J_{ijr\pi}(o) = -ig(\nabla_i J \nabla_j Z_r, Z_\pi)(o) + ig(\nabla_j J \nabla_i Z_r, Z_\pi)(o)
- g(\nabla_{[Z_r,Z_i]} J Z_r, Z_\pi)(o)
= -ig(\nabla_i J \nabla_j Z_r, Z_\pi)(o) + ig(\nabla_j J \nabla_i Z_r, Z_\pi)(o).
\]
By corollary 3.2 we have \((\nabla_j J) Z_j = 0\). Therefore we obtain
\[
g(\nabla_j J \nabla_i Z_r, Z_\pi)(o) = ig(\nabla_j \nabla_i Z_r, Z_\pi)(o) = 0
\]
and then
\[
R^J_{ijr\pi}(o) = -ig(\nabla_i J \nabla_j Z_r, Z_\pi)(o).
\]
\(\square\)

The previous lemma and equation (5) give us
\[
L^J_{ijr\pi}(o) = -2R^J_{ijr\pi}(o) + 2R^J_{ijr\pi}(o).
\]
Moreover equation (7) and lemma (3.6) imply the following

**Theorem 4.2.** If
\[
R^J(Z_i, Z_j, Z_i, Z_j) = R(Z_i, Z_j, Z_i, Z_j),
\]
for any \(Z_i, Z_j\) of type \((1, 0)\), then \(J\) is integrable.

**Remark 4.3.** The hypothesis of the previous theorem can be replaced by
\[
R^J(JX, Y, X, JY) = R(JX, Y, X, JY)
\]
for any \(X, Y\) real vector fields.
Let \( \tilde{\nabla} \) be the Hermitian connection on \((M, g, J, \kappa)\); \( \tilde{\nabla} \) is the connection defined by
\[
\tilde{\nabla} = \nabla - \frac{1}{2} J \nabla J.
\]
Then \( \tilde{\nabla} \) preserves the metric \( g \), the almost complex structure \( J \) and its torsion is given by the Nijenhuis torsion, namely
\[
\tilde{\nabla} g = 0, \quad \tilde{\nabla} J = 0, \quad T^{\nabla} = \frac{1}{4} N_J.
\]
Let us denote by \( \tilde{R} \) the curvature tensor of \( \tilde{\nabla} \).

From [6] we have the following

**Lemma 4.4 ([6]).** For any \( X, Y, Z, W \) we have the following formula
\[
\tilde{R}(X, Y, Z, W) = \frac{1}{2} R(X, Y, Z, W) + \frac{1}{2} R(X, Y, JZ, JW) - \frac{1}{4} g((\nabla_X J)(\nabla_Y J)Z - (\nabla_Y J)(\nabla_X J)Z, W).
\]

Now we have

**Lemma 4.5.** Let \( \{Z_1, \ldots, Z_n\} \) be a local \((1, 0)\)-frame. Then
\[
\tilde{R}_{i\bar{j},r\bar{s}} = R_{i\bar{j},r\bar{s}} + \frac{1}{4} L_{i\bar{j},r\bar{s}}
\]
for any \( 1 \leq i, j, r, s \leq n \).

**Proof.** Let \( \{Z_1, \ldots, Z_n\} \) be a local \((1, 0)\)-frame; from lemma 4.4 we have
\[
\tilde{R}_{i\bar{j},r\bar{s}} = \frac{1}{2} R_{i\bar{j},r\bar{s}} + \frac{1}{2} R_{i\bar{j},r\bar{s}} - \frac{1}{4} g((\nabla_i J)(\nabla_{\bar{j}} J)Z_r, Z_{\bar{s}}) + \frac{1}{4} g((\nabla_{\bar{i}} J)(\nabla_i J)Z_r, Z_{\bar{s}})
\]
\[
= R_{i\bar{j},r\bar{s}} + \frac{1}{4} g(\nabla_{\bar{i}} J)(\nabla_i J)Z_r, Z_{\bar{s}}).
\]

From corollary 3.2 we have \((\nabla_j J)Z_r; \) then we get
\[
g((\nabla_i J)(\nabla_{\bar{i}} J)Z_r, Z_{\bar{s}}) = 0.
\]

Therefore
\[
\tilde{R}_{i\bar{j},r\bar{s}} = R_{i\bar{j},r\bar{s}} + \frac{1}{4} g((\nabla_{\bar{i}} J)(\nabla_i Z_r, Z_{\bar{s}}) - \frac{1}{4} g((\nabla_{\bar{i}} J)J\nabla_i Z_r, Z_{\bar{s}})
\]
\[
= R_{i\bar{j},r\bar{s}} + \frac{1}{4} g((\nabla_{\bar{i}} J)(\nabla_i Z_r, Z_{\bar{s}}) - \frac{1}{4} g((\nabla_{\bar{i}} J)J\nabla_i Z_r, Z_{\bar{s}})
\]
\[
+ \frac{1}{4} g(\nabla_{\bar{i}} J)(\nabla_i Z_r, Z_{\bar{s}})
\]
\[
= R_{i\bar{j},r\bar{s}} + \frac{1}{4} g((\nabla_{\bar{i}} J)(\nabla_i Z_r, Z_{\bar{s}}) - \frac{1}{4} g((\nabla_{\bar{i}} J)J\nabla_i Z_r, Z_{\bar{s}})
\]
\[
+ \frac{1}{4} g(\nabla_{\bar{i}} J)(\nabla_i Z_r, Z_{\bar{s}})
\]
\[
= R_{i\bar{j},r\bar{s}} + \frac{1}{4} g((\nabla_{\bar{i}} J)(\nabla_i Z_r, Z_{\bar{s}}) - \frac{1}{4} g((\nabla_{\bar{i}} J)J\nabla_i Z_r, Z_{\bar{s}})
\]
\[
+ \frac{1}{4} g(\nabla_{\bar{i}} J)(\nabla_i Z_r, Z_{\bar{s}}).
\]
Let assume now that \( \{Z_1, \ldots, Z_n\} \) is a generalized normal holomorphic frame around a point \( o \). Then equations (9) and (5) imply
\[
\tilde{R}_{ijrst} (o) = R_{ijrst} (o) + \frac{1}{4} L_{ijrst} (o).
\]

Theorem 4.2 and lemma 4.5 imply the following

**Theorem 4.6.** If
\[
\tilde{R}(Z_i, Z_t, Z_j, Z_t) = R(Z_i, Z_t, Z_j, Z_t)
\]
for any \( Z_i, Z_j \ (1,0) \)-fields, then \( (M, g, J, \kappa) \) is a Kähler manifold.

**Remark 4.7.** The hypothesis of theorem 4.6 can be given as
\[
\tilde{R}(X, JX, Y, JY) = R(X, JX, Y, JY)
\]
for any pair of real vector fields \( X, Y \).

The last theorem can be stated in terms of the bisectional curvatures of \( \nabla \) and \( \tilde{\nabla} \).

Let \( p \in M \) and \( v, w \) be two unit vector of \( T_p M \). Then the holomorphic bisectional curvature of the planes \( \sigma_1 = \langle v, Jv \rangle, \sigma_2 = \langle w, Jw \rangle \) is defined by
\[
K[p](\sigma_1, \sigma_2) = R(v, Jv, w, Jw).
\]

We denote by \( \tilde{K}[p](\sigma_1, \sigma_2) \) the Hermitian bisectional curvature (i.e. the bisectional curvature of the Hermitian connection). For any \( p \in M \) let \( P^{1,1}_p (M) \) be the set of \( J \)-invariant planes in \( T_p M \).

Hence we get

**Theorem 4.8.** If
\[
K(p)(\sigma_1, \sigma_2) = \tilde{K}(p)(\sigma_1, \sigma_2)
\]
for any \( p \in M \) and \( \sigma_1, \sigma_2 \in P^{1,1}_p (M) \), then \( J \) is an integrable almost complex structure and therefore \( (M, g, J, \kappa) \) is a Kähler manifold.
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