Dense graphs with scale-free feature

Fei Ma\textsuperscript{a,1} Xiaomin Wang\textsuperscript{a,2} Ping Wang\textsuperscript{h,c,d,3} and Xudong Luo\textsuperscript{e,4}

\textsuperscript{a} School of Electronics Engineering and Computer Science, Peking University, Beijing 100871, China
\textsuperscript{b} National Engineering Research Center for Software Engineering, Peking University, Beijing, China
\textsuperscript{c} School of Software and Microelectronics, Peking University, Beijing 102600, China
\textsuperscript{d} Key Laboratory of High Confidence Software Technologies (PKU), Ministry of Education, Beijing, China
\textsuperscript{e} School of Mathematics and Statistics, Lanzhou University, 730000 Lanzhou, China

Abstract: Complex networks, representing various of complex systems, have attracted more attention from a broad range of science fields in recent years. The both prominent characters, scale-free feature and small-world property, have been extensively observed in a large amount of complex networks. While the authors in Ref \cite{1} had already stated that all scale-free networks are sparse, there exist some real-world networks, for instance, social networks \cite{2}, urban networks \cite{3}, information networks \cite{4}, which are by observation dense. To understand both dynamics and structure on these such networks, recently much effort has been spent and hence many techniques have developed. By contrast, in this paper, we propose a novel framework for generating scale-free graphs with dense feature using two simple yet helpful operations, first-order subdivision and Line-operation, from graph theory. It turns out both analytically and numerically that our instrument is more convenient to implement than those pre-existing methods. From theoretical point of view, our method can be used not only to produce desired scale-free graphs with power-law exponent $1 < \gamma \leq 2$ but also to establish unexpected networked models which disprove some widely known statements, such as “Scale-free networks are ultrasmall” due to Cohen, \textit{et al}, in \cite{5}. Our findings may shed lights on the fundamental understanding of complex networks, in particular, scale-free graphs.

Keywords: Complex network, Dense graph, Scale-free feature.

1 Introduction

Complex networks, usually interpreting diverse complex systems around us, has attracted more attention in the past years. Studied example networks include the Internet and the World Wide Web \cite{6}, scientific citation and collaboration \cite{7}, sexual contract network \cite{8}, metabolic network \cite{9}, and predator-prey chain \cite{10}, to name just a few. It is a convention for one to denote a networked model by a graph $G(V, \mathcal{E})$ which, in the simplest form, is a set of vertices in $V$, representing individual members of model, joined together in pairs by edges in $\mathcal{E}$, indicating relationships between members. With a such representation, many intriguing properties planted in the topological structure of networks have been unveiled, for instance, small-world property \cite{11}, power-law degree distribution (i.e., the so-called scale-free feature) \cite{12}, community structure \cite{13}, self-similarity \cite{14}, assortative mixing \cite{15}. To better understand the generation principles which control or produce the emergence of characters mentioned above, a wide range of tech-

\textsuperscript{1} The author’s E-mail: mafei123987@163.com.
\textsuperscript{2} The author’s E-mail: wmxwm0616@163.com.
\textsuperscript{3} The corresponding author’s E-mail: pwang@pku.edu.cn.
\textsuperscript{4} The author’s E-mail: luoxudong117@163.com.
technical methods have been developed and then used to established a large variety of theoretical models. For example, the well-known WS-model was proposed by Watts et al [11] to try to explain small-world phenomena in various real-world networks using two measures, diameter (or average path length) and clustering coefficient. For probing mechanisms governing the scale-free feature, on the other hand, a great deal of models have been constructed based on various thoughts and there seems to be no a complete consensus in current science community. The most prominent of widely studied networked models is the BA model built by Barabasi et al [12] using two rules, growth and preferential attachment, where newly added vertex tends to connect with higher probability to higher connected vertices. From then on, various requirements may be adopted to the empirical systems that appear to have the scale-free feature.

While a large number of networked models have been generated for modeling real-life networks, the most attractive of them are scale-free networked models. As said previously, from theoretical point of view, the extensive study of these such models triggers the blossom of studying scale-free graphs themselves. In 2003, through measuring the diameter $D$ or average path length $APL$ on scale-free networks with vertex number $|V|$ and degree distribution $P(k) \sim k^{-\gamma}$, $\gamma \in [2,3]$, Cohen et al proved, using analytical arguments, that these networks are small, i.e., $D \sim \ln |V|$ when $\gamma \in (2,3)$, and even ultrasmall, i.e. $D \sim \ln \ln |V|$ when $\gamma = 3$. In fact, the latter is not true for all scale-free graphs with power-law exponent $\gamma = 3$. An ensemble of graphs following this requirement have been completely built, see [17] for more detail. At the same time, such an abnormal phenomenon can also been found in the following scale-free graphs with density structure whose generation is the main topic of this paper.

For a theoretical networked model with an infinity of vertices, it is easy to show by definition of average degree $\langle k \rangle = 2|E|/|V|$ whether this is sparse or not. Sparse models are of $\langle k \rangle \rightarrow \alpha$ in the limit of large graph size and the dense ones are of $\langle k \rangle \neq \alpha$ where $\alpha$ is a constant. Obviously, all scale-free networks with $\gamma$ in the range from 2 to 3 are sparse. In 2011, based on extreme value arguments, Genio et al show both numerically and analytically that the probability for finding a scale-free network with a given $\gamma(\in [0,2])$ is 0 $\llbracket$. Therefore, they demonstrated that all scale-free networks have sparsity structure. As we will show shortly, scale-free graphs with exponent $\gamma = 2$ may be easily constructed using a novel framework proposed in this paper. In addition, other scale-free graphs whose exponents $\gamma$ are in the interval from 1 to 2 are able to completely generated in terms of our framework. These theoretical graphs will be used to disprove the above demonstration in the subsequent discussions.

In practice, the dense scale-free networks have been paid little attention in the whole history of scale-free network studies. The most important of reasons for this can be that ones have gradually accepted some seemingly complete conclusions as pointed in the preceding paragraphs. Recently, according to both many real-world example networks and some fresh instruments in the literature [2-4], this branch begin to become active. Most generally, a simplest method for densifying a sparse graph with a given vertex number $|V|$ is to consecutively add new edges to connect some vertex pairs, which are not connected previously, to satisfy the requirement of density. While such an implementation can easily achieve a desirable dense graph $G'$, some interesting structure properties rooted on initial graph $G$ may be destroyed thoroughly. As an immediate example, the quantities close associated with topological structure of graph $G$ can be first damaged, for instance, degree distribution and diameter. Therefore, on the other hand, one attempts to directly produce some desirable graphs with both density structure and some scientific interest, such as, power-law degree distribution. Lambiotte, et al, in [4] introduced a minimal generative model, named the copying model, for densifying networks $G(V,E)$ in which a new vertex attaches to a randomly selected target vertex and also to each of its neighbors with probability $p$. Based on rate equation approach and some assumptions, they have proven analytically that these dense networked models follow the power-law degree distribution with exponents $\gamma$ meeting the following equation

$$\gamma = 1 + p^{-1} - p^{\gamma-2}. \quad (1)$$
Figure 1: The switching procedure from a sparse scale-free graph into a denser scale-free graph. For a graph $G(V,E)$, its line graph based on Line-operation $f$ can be defined in the following form to be $G'(V',E')$ whose vertex set is $V' = \{ v_e | f(e) = v_e, \forall e \in E \}$ in which the function of mapping $f$ is to transform an edge to a unique vertex. Two vertices $v_{e_i}$ and $v_{e_j}$ in $V'$ are adjacent if the corresponding edges $e_i$ and $e_j$ in $E$ are adjacent in $G(V,E)$. These such edges constitute the edge set $E'$. Here, average degree $\langle k \rangle$ of graph in Fig.1(a) is equal to $4/3$ and average degree $\langle k' \rangle$ of its line graph shown in Fig.1(b) is 2. Apparently, $\langle k' \rangle$ is greater than $\langle k \rangle$, implying that Line-operation $f$ indeed achieves the transformation from sparser graphs into denser ones.

As pointed in [17], in the dense regime, many features of the degree distribution of networked models mentioned above become anomalous. For instance, the distribution does not self-average but appears to slowly converge to a form that is close to, but distinct from, a log-normal in the large graph size limit.

Obviously, how to effectively construct a dense graph with scale-free structure is a challenging and intriguing problem of significant theoretical flavor. In order to address this such issues, we will design a novel framework for densifying sparse graphs. Different from those schemes listed above, our framework is not directly produce a dense graph whose degree distribution has a power-law form but to indirectly generate a desirable graph. In fact, it contains two components as shown shortly.

2 Theory

Here, we develop a theoretical framework for switching a scale-free graph with sparsity structure into a candidate graph with density structure. As mentioned above, we will introduce our framework in two stages. First, a well-studied operation $f$ from graph theory, named Line-operation, is to transform an initial graph $G(V,E)$ to its corresponding line graph, also called edge graph, denoted by $G'(V',E')$. As an illustrative example, Fig.1 shows a such operation $f$ transforming a sparser graph in the left panel into a denser one in the right panel.

Consider a given graph $G(V,E)$ built by a graph generate function

$$G(x) = \sum_{k=0}^{\infty} p_k x^k$$

(2)
here $p_k$ is the fraction of vertices with degree $k$, its average degree is able to be expressed as $\langle k \rangle = G'(1)$. After implementing Line-operation, the line graph $G'(V', E')$ has average degree equal to

$$\langle k' \rangle = \frac{\langle k^2 \rangle - \langle k \rangle}{\langle k \rangle} = G''(1)$$

where $\langle k^i \rangle$ is the $i$-th moment of vertex degrees of graph $G(V, E)$. Armed with the statements, certifying $\langle k' \rangle$ much than $\langle k \rangle$ is to show $G''(1)$ no less than $(G'(1))^2$. More generally, the latter is true for a great number of sparse graphs, seeing Fig.1. Therefore, we may densify the topological structure of a sparse graph using Line-operation plotted in Fig.2.

On the other hand, the Line-operation can drastically damage many properties of initial graph $G$ which are close related to topological structure of the underlying graph, for instance, degree distribution and clustering coefficient, as we will show shortly. The topic of this paper, however, is to construct dense graphs with power-law degree distribution. Therefore, the key is to choose an available graph $G$ with an expected degree distribution that can be conveniently deduced to the power-law form under Line-operation in Fig.2. The selection of these such graphs will be successfully accomplished using the other component of our novel framework.

From now on, let us divert insights into the development of the other component. First, for a given graph $G(V, E)$, one can easily obtain its first-order subdivision graph $G_1(V_1, E_1)$ by inserting one young vertex on each edge in $E$, see Fig.3 for an illustrative example. Such an operation is called the first-order subdivision in the jargon of graph theory. As we will demonstrate later, it is the first-order subdivision that guarantees the construction of seminal graphs that may be proven to satisfy those requirements mentioned above.

By far, both components of our framework have been completely established. Now, our task is to clarify the detailed procedures of running our framework to obtain a dense graph with scale-free feature, as follows

**Step 1** For an arbitrary sparse graph $G(V, E)$ whose degree distribution obeys

$$P(k) \sim k^{-\gamma}, \quad 2 < \gamma \leq 3$$

one should apply the first-order subdivision to each edge in $E$. And then, the resulting graph is denoted by $G_1(V_1, E_1)$.

**Step 2** For the preceding graph $G_1(V_1, E_1)$, one can manipulate Line-operation on each edge in $E_1$. The end graph is regarded as graph $G'_1(V'_1, E'_1)$.

Fig.4 illustrates the skeleton of our framework described here. Below provides a theoretical proof to a fact that graph $G'_1(V'_1, E'_1)$ is not only dense but also scale-free. In addition, with the help of

![Figure 2](image-url)

Figure 2: The diagram of Line-operation for transforming a sparser graph into a denser one.
Figure 3: The diagram of subdivision. Given an arbitrary graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$, if one inserts a new vertex $w$ to every edge $uv \in \mathcal{E}$ then the resulting graph, denoted by $\mathcal{G}_1(\mathcal{V}_1, \mathcal{E}_1)$, is called a first-order subdivision graph of the original graph. To put this another way, such a graph $\mathcal{G}_1(\mathcal{V}_1, \mathcal{E}_1)$ can be obtained from graph $\mathcal{G}(\mathcal{V}, \mathcal{E})$ by replacing every edge $uv \in \mathcal{E}$ by a unique path $uwv$ with length two.

Some other properties of scientific interest will be discussed in the rest of this paper. Interestingly, some of which can be used to disprove some demonstrations that were considered truth in previous studies of complex networks, in particular, scale-free networked graphs.

3 Results

In this section, we analyze the resulting graph $\mathcal{G}_1'(\mathcal{V}_1', \mathcal{E}_1')$ by analytically calculating solutions for some measures associated with topological structure, including degree distribution, average degree, diameter, clustering coefficient, mixing structure and community structure.

Figure 4: The diagram of our framework for transforming a scale-free graph with sparsity structure into its corresponding dense graph with scale-free feature.
3.1 Degree distribution

Implementing the first-order subdivision will divide each edge $uv$ in $E$ into two edges by inserting new vertex $w$. As a result, the one of them will connect old vertex $u$ with degree $k_u$ to degree 2 vertex $w$ and the other connects degree $k_v$ vertex $v$ to young vertex $w$. To make further progress, the end graph $G'_1(V'_1, E'_1)$ has degree distribution in form

$$P'(k) = \sum_{u=0}^{k_u} \sum_{v=0}^{k_v} P^*(u) P^*(v) \delta_{k_u+2, k_v}$$

in which $P^*(u)$ and $P^*(v)$ indicate, respectively, the fraction of vertices with degree $u$ and $v$ where the both vertices are adjacent by edge $uv$ in $E_1$. Symbol $\delta_{i,j}$ is Kronecker delta function. To put it another way, the fraction of vertices with degree $k$ of graph $G'_1(V'_1, E'_1)$ is exactly equivalent to the fraction of edges in $E_1$ in which the degrees of two endvertices sum to $k+2$. As before, these such edges is in essence made of edges in $E_1$ whose one endvertex is an old one with degree $k$ in $E$ and the other is newly added vertex of degree 2. Thus, combining Eq.(4) with Eq.(5) yields

$$P'(k) = k P(k) \sim k^{-\gamma'}, \quad 1 < \gamma' \leq 2.$$ 

This suggests that graph $G'_1(V'_1, E'_1)$ obtained from scale-free graph $G(V, E)$ by using our framework displays a power-law degree distribution. In other words, we indeed generate scale-free graphs with exponent falling into the interval from 1 to 2, clearly implying that the demonstration in Ref[1] is incomplete.

3.2 Average degree

In view of Eqs (3) and (6), it is straightforward to exactly calculate the solution for average degree $\langle k' \rangle$ in terms of

$$\langle k' \rangle = \int_{0}^{k'_{\text{max}}} k P'(k) dk \sim \begin{cases} \frac{1}{2 - \gamma'} k'_{\text{max}}^{2-\gamma'}, & 1 < \gamma' < 2 \\ \zeta(1; k'_{\text{max}}, k'_{\text{min}}), & \gamma' = 2 \end{cases}$$

where $k'_{\text{min}}$ and $k'_{\text{max}}$ represent, respectively, the most minimum and largest degrees of vertices of graph $G'_1(V'_1, E'_1)$, and symbol $\zeta(1; u, v)$ stands for Riemann zeta function with constraints, defined as $\zeta(1; u, v) = \sum_{i=1}^{u} i^{-1}$. As above, the largest degree value $k'_{\text{max}}$ is precisely equivalent to the greatest degree $k_{\text{max}}$ of vertices of graph $G(V, E)$. In general, the value for $k_{\text{max}}$ can be asymptotically expressed with respect to the vertex number $|V|$ as

$$k_{\text{max}} \sim |V|^{1-\gamma}.$$ 

Plugging Eq.(8) into Eq.(7), for an arbitrary $\gamma'$ in range $(1, 2]$, average degree $\langle k' \rangle$ will become infinite in the limit of large graph size. This means that scale-free graph $G'_1(V'_1, E'_1)$ is by definition dense, further indicating that the previous statement, that is, all scale-free networks are sparse, in Ref[1] is not true.

3.3 Diameter

As the simplest and important of both measures for investigating small-world property of complex networks, diameter $D$ is the maximum among distances of all vertex pairs. As stated in Ref[3], scale-free networks are ultrasmall according to the relationship between diameter $D$ and vertex number $|V|$. However, some networked graphs based on our framework will be able to serve as the counterexamples for
the above statement as shown shortly. Now, if we suppose the diameter of an initial graph \( G(V, E) \) be equal to \( D \), then the first-order subdivision will make the diameter \( D_1 \) of graph \( G_1(V_1, E_1) \) at most equivalent to two times \( D \), i.e., \( D_1 \approx 2D \). After that, it is clear to see that the diameter \( D' \) of the end graph \( G'_1(V'_1, E'_1) \) will be approximately identical to diameter \( D_1 \) after applying Line-operation to graph \( G_1(V_1, E_1) \). In another words, the diameter \( D' \) is magnitude of order the diameter \( D \), namely

\[
D' = O(D).
\]

This implies that if the seminal graph \( G(V, E) \) has small diameter, then the diameter of the resulting \( G'_1(V'_1, E'_1) \) will be small. On the contrary, the large diameter of graph \( G(V, E) \) will ensure a larger diameter of graph \( G'_1(V'_1, E'_1) \). As reported in our prior work [16], the growth scale-free networked graph has a very large diameter (\( D = 2^t \), see [16] for a lot). Therefore, we can choose a such networked graph as a seed and then obtain a dense graph with both scale-free and large diameter using our framework. Most obviously, these such graphs have ability to disprove the above statement.

### 3.4 Clustering coefficient

By definition, the clustering coefficient of a graph \( G \) can be written in the following form

\[
C = \frac{3 \times \text{triangle number}}{\text{number of connected triple}}
\]

here a triangle is a cycle \( C_3 \) on three vertices and a connected triple means which a vertex is connected to a pair of other vertices. As described above, using the Line-operation, vertex \( u \) with degree \( k'_u \) of the resulting graph \( G'_1(V'_1, E'_1) \) will be allocated on a clique \( K_{k'_u} \), a subgraph in which all vertex pairs are connected. Hence, the clustering coefficient \( C'_u \) of vertex \( u \) is calculated equal to \((k'_u - 2)/k'_u \). According to Eq.(6), substituting the above consequence into Eq.(10) produces

\[
C' = \sum_{u \in E'_1} C'_u \sim \int^{k'_{\max}}_{k'_{\min}} P'(k)C'_u dk \sim G(1). \tag{11}
\]

In the large graph size limit, \( C' \) will approach the theoretical upper bound, i.e. unity.

Armed with Eqs.(7), (9) and (11), we can assert that when the seminal graph \( G(V, E) \) has scale-free feature and small-world property, the resulting graph \( G'_1(V'_1, E'_1) \) must be both scale-free and small-world. Furthermore, if the power-law exponent \( \gamma \) of the initial graph falls in the range \((2, 3]\), then the end graph has density structure.

### 3.5 Mixing structure

Recent works have shown that in many networks, a number of vertices tend to be connected to other vertices like themselves. To analytically depict a such phenomenon, Newman in Ref[15] introduced a measure \( r \), called Pearson correlation coefficient, in terms of

\[
r = \frac{|E|^{-1} \sum_{e_{ij} \in E} k_i k_j - \left( |E|^{-1} \sum_{e_{ij} \in E} \frac{1}{2}(k_i + k_j) \right)^2}{|E|^{-1} \sum_{e_{ij} \in E} \frac{1}{2}(k_i^2 + k_j^2) - \left( |E|^{-1} \sum_{e_{ij} \in E} \frac{1}{2}(k_i + k_j) \right)^2}. \tag{12}
\]
Figure 5: The diagram of Pearson correlation coefficient for the BA-models. It is obvious to see that for different parameter $m$, Pearson correlation coefficient $r$ of BA-model will always keep negative and approach 0 in the limit of large graph size.

In practice, as stated in the process of calculating clustering coefficient, the resulting graph $G'_1(V'_1, E'_1)$ contains various types of cliques as subgraphs. Its corresponding $r'$ will be greater in comparison with the $r$ of initial $G(V, E)$. In order to make our statements more concrete, we make use of the well-known scale-free graph, the BA-model due to Barabasi et al in [12], as a seed to generate its corresponding dense graphs. At the same time, we obtain solutions for Pearson correlation coefficients for six scale-free graphs by varying the number $m$ of edges originating from each newly added vertex, see Fig.5 and Fig.6 for more details.

3.6 Community structure

As the final topological measure discussed in our work, community structure, with which there is a higher density of edges and between which there is a few, has been a focus of current researches of significant interest, particularly within statistical physics and computer science [13]. Here, we utilize one approach in widest current use, i.e, modularity maximization, to investigate the community structure of the resulting graph $G'_1(V'_1, E'_1)$ and further probe the distribution of community size $s'$ on graph $G'_1(V'_1, E'_1)$. The modularity of a graph $G(V, E)$ is given by

$$Q = \frac{1}{2|E|} \sum_{ij} \left[ A_{ij} - \frac{k_i k_j}{2|E|} \right] \delta_{g_i, g_j}$$

(13)

where $k_i$ is the degree of vertex $i$, $\delta_{i,j}$ is the Kronecker delta as above and $A_{ij}$ is the element of the adjacency matrix of graph $G(V, E)$ which is equal to 1 when vertex $i$ is adjacent to vertex $j$ and 0 otherwise. Before processing the following calculations, let us recall the construction of $G'_1(V'_1, E'_1)$ and can evidently see that the sizes of various types cliques is the same as the degree sequence of initial $G(V, E)$. With the help of two assertions in [19], (Assertion 1 In a maximum modularity clustering of graph $G(V, E)$, none of the cliques $H_1; ..., H_k$ is split. Assertion 2 In a maximum modularity clustering of $G(V, E)$, every cluster contains at most one of the cliques $H_1; ..., H_k$.) we can confirm that the community size $s'$ distribution has a power-law form.
Figure 6: The diagram of Pearson correlation coefficient for the end graphs based on BA-models using our framework. It is clear to the eye that for different parameter \( m \), Pearson correlation coefficient \( r' \) of their corresponding graphs by means of our framework all are positive and tend to the theoretical upper bound, i.e., unity, under the same situation.

\[
P(s') \sim s'^\gamma, \quad 2 < \gamma \leq 3.
\]  

(14)

Surprisingly, such a phenomenon have been discovered in some real-world complex networks, such as, Amazon copurchasing network in [20].

4 Discussion

We have introduced a novel framework for producing a dense graph with scale-free feature from a given sparse graph whose degree distribution obeys a power-law form. From the theoretical point of view, the resulting graphs based on our framework can provide strong proofs to disprove some statements in the literature related to complex networks. In addition, these resulting graphs also display some other interesting topological properties unseen in most of scale-free graphs, such as, much higher clustering efficient shown in Eq. (11) and much stronger assortative structure plotted in Fig.5 and Fig.6. In addition to these merits, the community size distribution of the resulting graph may share an identical form with the degree distribution of its corresponding graph, as said in Eq.(14).

In conclusion, our theory can provide another class of generative method for establishing some desirable graphs of scientific interest. And then, these findings can lead us to test meaningful hypotheses in an evolving networked graph, particularly, scale-free graphs. Most importantly, it allows us to better understand some of the fundamental structure characters of complex networked graphs.

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