Leading-order spin-orbit and spin(1)-spin(2) radiation-reaction Hamiltonians

Han Wang, Jan Steinhoff, Jing Zeng, and Gerhard Schäfer

Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität, Max-Wien-Platz 1, 07743 Jena, Germany, EU
Centro Multidisciplinar de Astrofísica — CENTRA, Departamento de Física, Instituto Superior Técnico — IST, Universidade Técnica de Lisboa, Avenida Rovisco Pais 1, 1049-001 Lisboa, Portugal, EU
Shanghai Astronomical Observatory, NanDan Road 80, Shanghai, 200030, China

(Dated: January 31, 2012)

In the present paper, the leading-order post-Newtonian spin-orbit and spin(1)-spin(2) radiation-reaction Hamiltonians are calculated. We utilize the canonical formalism of Arnowitt, Deser, and Misner (ADM), which has shown to be valuable for this kind of calculation. The results are valid for arbitrary many objects. The energy loss is then computed and compared to well-known results for the energy flux as a check.

PACS numbers: 04.25.Nx, 04.20.Fy, 04.25.-g, 97.80.-d
Keywords: post-Newtonian approximation; ADM canonical formalism; Gravitational radiation reaction; Binaries and multiple stars; Spinning bodies

I. INTRODUCTION

Gravitational radiation reaction is a problem of great interest in the detection of gravitational waves. For second and third generations of gravitational wave detectors, a leading candidate source is the radiation-reaction induced inspiral and merger of two compact objects like black holes or neutron stars. Moreover, the effects of spins are important for the emission of gravitational waves from such systems. Thus in order to develop highly accurate theoretical templates for gravitational wave detectors, one must study the gravitational radiation reaction from compact binary systems with spin effects.

In the present paper, the leading-order post-Newtonian (PN) spin-orbit and spin(1)-spin(2) radiation-reaction equations of motion are 2.5PN orders weaker than the corresponding conservative ones. Recently, the contributions of the spin-dependent Hamiltonians derived in the present paper to the equations of motion are 2.5PN orders weaker than the corresponding leading-order conservative ones. The energy loss is then computed and compared to well-known results for the energy flux as a check.

The conservative leading-order (PN) spin interactions for self-gravitating objects were derived some time ago and recently [12] (see also [1, 13, 14]). This extension is valid to linear order in the single spins of the objects, which not only includes spin-orbit but also spin(1)-spin(2) interactions. The remarkable structure of the extended ADM formalism of the inclusion of the matter into the canonical field momentum [see Eq. (2.6)] is passing an excellent test in the present paper. For Hamiltonians of higher orders in spins see [13, 15–18].

Energy and angular momentum flux relevant for the PN order in question has been well known (see [19], for the next-to-leading-order calculation see [20, 21]). Based on these results, secular equations of motion for the orbital elements corresponding to the leading-order spin-orbit and spin(1)-spin(2) radiation-reaction equations of motion were obtained in [22–24]. The general equations of motion at this order were calculated in [25–27] within the harmonic gauge. The Hamiltonians calculated in the present paper provide a compact expression which contains these general equations of motion (but within a different gauge). And most importantly, the results in the present paper are valid for arbitrary many object systems. The derived Hamiltonians are then applied to the calculation of the energy loss of a binary system, which is then compared with the well-known energy flux as a check.
For the leading-order spin(1)-spin(1) radiation-reaction level calculations see, e.g., [23, 33, 34]. However, only recently the conservative next-to-leading-order spin effects could be treated, starting with the spin-orbit equations of motion in harmonic gauge [35] (with some extensions and misprints corrected in [36]). A corresponding conservative Hamiltonian in the ADM gauge was obtained in [37]. The complete next-to-leading-order spin(1)-spin(2) conservative Hamiltonian was first given in [38]. Other derivations of the conservative next-to-leading-order spin-orbit and spin(1)-spin(2) dynamics can be found in [39-43] and a generalization to arbitrary many objects succeeded in [44]. Notice that the results given in the present paper are already valid for arbitrary many objects. Also the conservative next-to-leading-order spin(1)-spin(1) interaction of black hole and/or neutron star binaries was derived recently [16-18, 45, 46]. The latter requires a modeling of the spin-induced quadrupole deformation, see [33, 47]. Very recently, the conservative spin-dependent part of the post-Newtonian Hamiltonian was extended even to next-to-next-to-leading order for both the spin-orbit [48] and the spin(1)-spin(2) [49] cases. A potential for the spin(1)-spin(2) case was simultaneously calculated within an effective field theory approach [50]. Notice that the conservative next-to-next-to-leading-order spin(1)-spin(2) Hamiltonian and the spin-orbit radiation-reaction Hamiltonian derived in the present paper are both of the order 4PN for maximally rotating objects. However, not all spin-dependent Hamiltonians up to 4PN for maximally rotating objects are known yet. We will in most cases use the phrase ”formal n-th PN order” to represent our counting of PN orders in the present paper. This gives PN orders different from the maximally rotating objects are known yet. We will in most cases use the phrase ”formal n-th PN order” to represent our counting of PN orders in the present paper. This gives PN orders different from the maximally rotating case, which we also occasionally refer to in the present paper (for a more detailed discussion see, e.g., Appendix A of [1]). But one should be aware that the spins are in fact further (independent) expansion variables. Spin effects were also considered within the post-Minkowskian approximation [51, 52].

The paper is organized as follows. First the ADM formalism is reviewed in Sec. II. Then formal expressions for the radiation-reaction Hamiltonians in question are derived in Sec. III. Integrals appearing in these formal expressions are performed in Sec. IV. In Sec. V, the derived Hamiltonians are applied to the calculation of the energy loss, which is then compared with the energy flux. Finally, conclusions are given in Sec. VI.

Our units are such that \( c = 1 \), but for the Newtonian gravitational constant \( G \) no convention will be used. This allows an easy transition to the different conventions for \( G \) used in [2] and [1]. For the signature of spacetime, we choose \(+2\). Latin indices from the beginning of the alphabet, such as \( a, b \), label the individual objects. Greek indices run over 0, 1, 2, 3. Latin indices from the middle of the alphabet run over 1, 2, 3. Round brackets around an index denote a local basis, while round brackets around a number denote the formal order in \( c^{-1} \), as in [1, 14].

A 3-vector \( x^i \) is also denoted by \( \mathbf{x} \). Square brackets denote index antisymmetrization and round brackets index symmetrization, i.e., \( a^{(\mu \nu \rho \sigma)} = \frac{1}{4!}(a^{\mu \nu \rho \sigma} + a^{\mu \nu \sigma \rho} + a^{\nu \mu \rho \sigma} + a^{\nu \mu \sigma \rho}) \).

II. THE ADM FORMALISM

In this section, we provide a short overview of the ADM canonical formalism after gauge fixing [53], see also [54, 55]. The Hamiltonian is given by the ADM energy expressed in terms of certain canonical variables, which also requires a (at least approximate) solution of the field constraints.

The constraints of the gravitational field read

\[
\frac{1}{16\pi G}\sqrt{\gamma} \left[ \gamma R + \frac{1}{2} (\gamma_{ij}\pi^{ij})^2 - \gamma_{ij}\gamma_{kl}\pi^{ik}\pi^{jl} \right] = H_{\text{matter}},
\]

(2.1)

\[
-\frac{1}{8\pi G}\gamma_{ij}\pi^{jk} = H_{i}^{\text{matter}},
\]

(2.2)

with the definitions

\[
\pi^{ij} = -\sqrt{\gamma}(\gamma^{ik}\gamma^{jl} - \gamma^{ij}\gamma^{kl})K_{kl},
\]

(2.3)

\[
H_{i}^{\text{matter}} = \sqrt{\gamma}T_{i\mu}n^\mu n^\nu,
\]

(2.4)

\[
H_{i} = -\sqrt{\gamma}T_{i\mu}n^\nu.
\]

(2.5)

They are as certain projections of the Einstein field equations with respect to a timelike unit 4-vector \( n^\mu \) with components \( n_\mu = (-N, 0, 0, 0) \) or \( N^\mu = (1, -N^j)/N \).

Here, \( \gamma_{ij} \) is the induced three-dimensional metric of the hypersurfaces orthogonal to \( n_\mu \), \( \gamma \) its determinant, \( R \) the three-dimensional Ricci scalar, \( K_{ij} = -(\frac{1}{2}\gamma_{ij,0} - N_{(ij)})/N \) the extrinsic curvature, \( N \) the lapse function, \( N^i \) the shift vector, \( \sqrt{\gamma}T_{i\mu}n^\nu \) the stress-energy tensor density of the matter system, and semicolon denotes the three-dimensional covariant derivative. Partial coordinate derivatives \( \partial_i \) are also indicated by commas.

For nonspinning objects, \( \pi_{ij}^{\text{can}} \) is the canonical momentum conjugate to \( \gamma_{ij} \) before gauge fixing. For spinning objects, the canonical field momentum has to be adapted, see [1, 12]. We write

\[
\pi_{ij}^{\text{can}} = \pi_{ij} + \pi_{ij}^{\text{matter}},
\]

(2.6)

where \( \pi_{ij}^{\text{matter}} \) contains spin-corrections. Throughout this paper we use the ADM transverse-traceless (TT) gauge, which is defined by:

\[
\partial_j(\gamma_{ij} - \frac{1}{4}\delta_{ij}\gamma_{kk}) = 0,
\]

(2.7)

\[
\pi_{ij}^{\text{can}} = 0.
\]

(2.8)

Here, \( \delta_{ij} \) is the Kronecker delta. And one has the decompositions:

\[
\gamma_{ij} = \left( 1 + \frac{\phi}{8} \right)^4 \delta_{ij} + h_{ij}^{TT},
\]

(2.9)

\[
\pi_{ij}^{\text{can}} = \pi_{ij}^{TT} + \pi_{ij}^{\text{can}},
\]

(2.10)
The canonical field variables after gauge fixing are 
\[ \pi^i_{\text{can}} = \frac{2}{3} \delta_{ij} V^k \text{can, } \delta_{ij} \] 
\[ \pi^i_{\text{can, } j} + \pi^j_{\text{can, } i} = \frac{1}{2} \delta_{ij} \pi^k \text{can, } k - \frac{1}{2} \Delta^{-1} \pi^k \text{can, } i, j, k. \] (2.11)

It holds that:
\[ \pi^i_{\text{can, } j} = \Delta^{-1} \pi^i_{\text{can, } j}, \] (2.12)

The canonical field variables after gauge fixing are \( h_{ij}^{TT} \) and \( \pi^i_{\text{can}} \).

In order to obtain the ADM Hamiltonian, the four field constraints must be solved for the four variables \( \phi \) and \( \pi^i_{\text{can}} \) in terms of \( h_{ij}^{TT} \), \( \pi^i_{\text{can}} \) and canonical matter variables, which enter through the stress-energy tensor via the source terms \( H^{\text{matter}} \) and \( H^{\text{int}} \) (for the linear order source terms in spin, see [1, 12, 13] and also [14]). The canonical matter variables are the canonical position \( z^j_{\text{can}} \), momentum \( P_{ai} \), and spin-tensor \( S_{a(i)(j)} \) of the \( a \)-th object. An analytic solution for \( \phi \) and \( \pi^i_{\text{can}} \), however, can in general only be given in some approximation scheme. The ADM Hamiltonian is then given by:
\[ H_{\text{ADM}} = -\frac{1}{16\pi G} \int d^3x \Delta \phi \left[ \pi^i_{\text{can}} + P_{ai}, S_{a(i), h_{ij}^{TT}, \pi^i_{\text{can}}}, \right], \] (2.17)

where \( S_{a(i)} = \frac{1}{2} \epsilon_{ijk} S_{a(j)(k)} \) and \( \epsilon_{ijk} \) is the completely antisymmetric Levi-Civita symbol. \( H_{\text{ADM}} \) is the ADM energy expressed in terms of the canonical variables mentioned above. The Poisson brackets read:
\[ \{ h_{ij}^{TT}(x), \pi^k_{\text{can}}(x) \} = 16G A_{ij}^{TTk} \delta(x - x'), \] (2.18)
\[ \{ \pi^j_{\text{can}}, P_{ai} \} = \delta_{ij}, \] (2.19)
\[ \{ S_{a(i), S_{a(j)}} \} = \epsilon_{ijk} S_{a(k)}, \] (2.20)

all others are zero.

III. RADIATION-REACTION HAMILTONIANS UP TO FORMAL 3.5PN LEVEL

In this section, we generalize the derivation of the radiation-reaction Hamiltonians up to the formal 3.5PN level performed in [2] so it becomes applicable to the spinning case.

A. Interaction Hamiltonian and wave equation

We split the ADM Hamiltonian \( H_{\text{ADM}} \) into matter, field, and interaction parts, i.e.,
\[ H_{\text{ADM}} = H^{\text{matter}} + H^{\text{field}} + H^{\text{int}}, \] (3.1)

where the matter part \( H^{\text{matter}} \) is independent of the (truly dynamical) canonical field variables \( h_{ij}^{TT} \) and \( \pi_{\text{can}} \), the field part \( H^{\text{field}} \) is independent of the canonical matter variables and reads explicitly:
\[ H^{\text{field}} = \frac{1}{16\pi G} \int d^3x \left[ \frac{1}{4} (h_{ij}^{TT})^2 + (\pi^i_{\text{can}})^2 \right], \] (3.2)

and the interaction part \( H^{\text{int}} \) depends on both canonical matter and field variables. The interaction Hamiltonian up to and including the formal 3.5PN level reads ([1], see also [2]):
\[ H^{\text{int}} = \frac{1}{16\pi G} \int d^3x \left( \left( B_{(4)ij} + B_{(6)ij} \right) h_{ij}^{TT} + 2\pi G \delta_{\text{matter}} h_{ij}^{TT} \right) \]
\[ -2\pi G \phi \left( h_{ij}^{TT} \right)^2 - \frac{1}{4} \phi \left( h_{ij}^{TT,k} \right)^2 \]
\[ + 2(V_{(3)}^{\phi(3)j} - \pi^j_{\text{can}} \pi^j_{\text{can}}) \], (3.3)

where
\[ B_{(4)ij} = 16\pi G \delta \left( \int d^3x H^{\text{matter}} \right) \]
\[ \delta h_{ij}^{TT} \]
\[ \frac{1}{8} \phi_{(2), i} \phi_{(2), j}, \] (3.4)

and \( V_{(3)}^{\phi(3)j} \) is a field quantity which will be discussed in Sec. IV. \( B_{(6)ij} \) is given by a similar expression [see (5.14) in [1]]. For comparison with [2], notice that \( 2h_{ij}^{TTk} \delta \left( V_{(3)}^{\phi(3)j} \right) = -\delta_{kl}^{TTij} \left( \phi_{(2), j} \right) \left( \phi_{(3)} \right) \) \( \left( \pi^j_{\text{can}} \right) \) is another field quantity which will be discussed later. Further, in [2] the quantity \( A_{(6)ij} = 2B_{(4)ij} \) is used in this paper.

The equations of motion for the canonical field variables follow from the ADM Hamiltonian by virtue of the Poisson brackets (2.18) as:
\[ \frac{1}{16\pi G} h_{ij}^{TT} = \delta_{kl}^{TTij} \delta H_{\text{ADM}} \]
\[ \frac{1}{16\pi G} \pi^j_{\text{can}} = -\delta_{kl}^{TTij} \delta H_{\text{ADM}} \] (3.5)
(3.6)

Here the dot over a variable denotes the partial time derivative \( \partial_t \equiv \frac{\partial}{\partial t} \). For quantities not depending on the
hypersurface coordinate \( x \), this is to be understood as the ordinary time derivative. In terms of the interaction Hamiltonian \( H^{\text{int}} \), the field equations read

\[
\frac{1}{16\pi G} \Box^\text{TT}_{ij} = \delta^\text{TT}_{kl} \left[ 2 \frac{\delta H^{\text{int}}}{\delta h_{kl}^\text{TT}} - \frac{\partial}{\partial t} \frac{\delta H^{\text{int}}}{\delta \pi_{\text{can}}^\text{TT}} \right], \tag{3.7}
\]

\[
\frac{1}{16\pi G} \pi^\text{TT}_{ij} = \frac{1}{2} \left[ \frac{1}{16\pi G} \pi^\text{TT}_{ij} - \delta^\text{TT}_{kl} \delta \frac{H^{\text{int}}}{\delta \pi_{\text{can}}^\text{TT}} \right], \tag{3.8}
\]

with \( \Box = \Delta - \partial_t^2 \). To arrive at these expressions, the explicit form of \( H^{\text{field}} \) is used as in Eq. (3.2). Notice that it is easier to implement the boundary condition of no incoming gravitational radiation for a wave equation like Eq. (3.7) than for a system of first-order differential equations like Eqs. (3.5) and (3.6). Inserting the 3.5PN-accurate interaction Hamiltonian [Eq. (3.3)], one gets

\[
\Box^\text{TT}_{ij} = \delta^\text{TT}_{kl} \left[ 2B_{(4)kl} + 2B_{(6)kl} - 8\pi G H^{\text{matter}}_{(2)} h_{kl}^\text{TT} \right.

+ \left( \phi_{(2)} h_{kl,m}^\text{TT} \right)_m - 2 \frac{\partial}{\partial t} \left( V^k_{(3)} \phi_{(2),l} \right) \right], \tag{3.9}
\]

\[
\pi^\text{TT}_{ij} = \frac{1}{2} \delta^\text{TT}_{kl} \left[ V^k_{(3)} \phi_{(2),l} - \pi^\text{TT}_{kl} \right], \tag{3.10}
\]

with the definition:

\[
B_{(6)ij} = \dot{B}_{(6)ij} + \pi^\text{TT}_{ij}. \tag{3.11}
\]

One can get alternative expressions for \( B_{(4)ij} \) and \( B_{(6)ij} \) in terms of \( T_{ij} = \sqrt{\gamma} T_{ij} \) by comparing the wave equation for \( h_{ij}^\text{TT} \) with the Einstein equations (see [1]), e.g.,

\[
B_{(4)ij} = -8\pi G T_{(4)ij} - \frac{1}{8} \phi_{(2),i} \phi_{(2),j}. \tag{3.12}
\]

This should agree with Eq. (3.4) after the TT-projection.

### B. Near-zone expansion

At the considered order, aspect like tail effects play no role (see e.g., [56]). We may therefore solve the wave equation for \( h_{ij}^\text{TT} \) by an order-by-order evaluation of the retarded solution. Further, the field solution is only needed in the near-zone.

In order to discuss the near-zone expansion, we write the wave equation for \( h_{ij}^\text{TT} \) schematically as:

\[
\Box h_{ij}^\text{TT} = -8\pi G \delta_{kl}^\text{TT} S_{kl}. \tag{3.13}
\]

The near-zone expansion of the retarded solution to this equation corresponds to a series in \( e^{-1} \) entering through the retarded time \( t_{\text{ret}} = t - e^{-1} |x - x'| \), reading

\[
h_{ij}^\text{TT} = -8\pi G \delta^\text{TT}_{kl} \left[ L_0 S_{kl} - L_1 \dot{S}_{kl} + L_2 \ddot{S}_{kl} + \ldots \right], \tag{3.14}
\]

where the TT-projector was pulled in front of the retarded solution and the integral operator \( L_n \) is defined by

\[
(L_n f)(x,t) = -\frac{1}{4\pi n!} \int d^3x' |x - x'|^{-n} f(x',t). \tag{3.15}
\]

Notice that \( L_{2n} = \Delta^{-1} - n \) for \( n \in \mathbb{N} \), in particular \( L_0 = \Delta^{-1} \).

Using the PN-expanded source of the wave equation from Eq. (3.9) one may arrange the near-zone expansion by PN orders as:

\[
h_{ij}^\text{TT} = h_{(4)ij}^\text{TT} + h_{(5)ij}^\text{TT} + h_{(7)ij}^\text{TT} + \ldots \tag{3.16}
\]

It is important that only a finite number of terms from the near-zone expansion [Eq. (3.14)] contribute to a specific PN order due to the increasing number of time derivatives therein. Therefore, one obtains

\[
h_{ij}^\text{TT} = 2\delta_{kl}^\text{TT} \Delta^{-1} B_{(4)kl}, \tag{3.17}
\]

\[
\Pi_{ij} = -2\delta_{kl}^\text{TT} L_1 B_{(6)kl}, \tag{3.18}
\]

\[
\Pi_{5ij} = 8\pi G \delta_{kl}^\text{TT} L_1 \left( h_{kl}^\text{TT} H^{\text{matter}}_{(2)} \right), \tag{3.19}
\]

\[
\Pi_{4ij} = -2\delta_{kl}^\text{TT} L_3 B_{(4)kl}, \tag{3.20}
\]

\[
Q_{ij} = -8\pi G \delta_{kl}^\text{TT} \Delta^{-1} \left( h_{(5)kl}^\text{TT} H^{\text{matter}}_{(2)} \right). \tag{3.21}
\]

Notice that the application of \( L_1 \) to a total divergence like \( (\phi_{(2)} h_{kl,m}^\text{TT})_m \) leads to a vanishing result. It will become apparent in the next section that \( h_{(6)ij}^\text{TT} \) is not needed in the present paper (but it contributes to the conservative 3PN Hamiltonian). The definitions \( P_{ij} = \Pi_{ij} + \Pi_{2ij}, P_{3ij} = \Pi_{3ij}, \) and \( R_{ij} = \Pi_{4ij} \) were used in [2].

An application of the operator \( L_1 \) obviously leads to a field depending on time only (i.e., not depending on \( x \)). This allows an easy calculation of the (regularized) TT-projections in Eqs. (3.20) – (3.23) by means of the formula

\[
\delta_{kl}^\text{TT} A_{kl}(t) = -\frac{1}{2} A_{ij}^\text{STF}(t), \tag{3.24}
\]

valid for an arbitrary \( \chi \)-independent function \( A_{kl}(t) \) (see [2]). Here \( A_{ij}^\text{STF} \) denotes the symmetric trace-free part,

\[
A_{ij}^\text{STF} = \frac{1}{2} (A_{ij} + A_{ji}) - \frac{1}{3} \delta_{ij} A_{kk}. \tag{3.25}
\]

Further, \( h_{(5)ij,k}^\text{TT} \) is a function of time only, \( h_{(5)ij,k}^\text{TT} = 0 \). As a consequence of these simplifications, we finally have

\[
\chi_{(4)ij} = -\frac{1}{5\pi} \int d^3x B_{(4)ij}^\text{STF}, \tag{3.26}
\]
\[ \Pi_{1ij} = \frac{1}{5\pi} \int d^3x B_{(6)ij}^\text{STF}, \quad (3.29) \]
\[ \Pi_{2ij} = -\frac{4G}{5} \int d^3x h_{(4)ij}^T H_{\text{matter}}^\text{matter}, \quad (3.30) \]
\[ \Pi_{3ij} = -\frac{1}{5\pi} \int d^3x \left( V_{(3)}^i \phi_{(2),j} \right)^{\text{STF}}, \quad (3.31) \]
\[ \Pi_{4ij} = \frac{1}{12\pi} \delta_{kl}^\text{TTij} \int d^3x \left| \mathbf{x} - \mathbf{x}' \right|^2 B_{(4)kl}(\mathbf{x}', t), \quad (3.32) \]
\[ Q_{ij} = \frac{1}{2} \hat{h}_{(5)ij}^T \delta_{ij}^\text{TTij} \phi_{(2)}, \quad (3.33) \] where the PN-expanded Hamilton constraint in the form \( \Delta \phi_{(2)} = -16\pi G H_{(2)}^\text{matter} \) was used to arrive at the last equation.

### C. Radiation-reaction Hamiltonians

The dissipation through emission of gravitational radiation enters the PN-expansion via \( h_{(5)ij}^T \) and \( h_{(7)ij}^T \), which are antisymmetric under time reversal. The parts of the Hamiltonian linear in \( h_{(5)ij}^T \) or \( h_{(7)ij}^T \) thus give the radiation-reaction Hamiltonians at the considered order. Notice that \( H^\text{field} \) does not contribute to the matter equations of motion, so we only need to consider \( H^\text{int} \). The radiation-reaction Hamiltonians are thus given by:

\[ H_{2.5\text{PN}}^\text{int} = \frac{1}{16\pi G} \int d^3x B_{(4)ij}^T h_{(5)ij}^T, \quad (3.34) \]
\[ H_{3.3\text{PN}}^\text{int} = \frac{1}{16\pi G} \int d^3x \left[ B_{(4)ij}^T h_{(7)ij}^T + V_{(3)}^i \phi_{(2),j} \right] h_{(5)ij}^T \\
+ \left( B_{(6)ij} - 4\pi G H_{\text{matter}}^\text{matter} h_{(5)ij}^T \right) h_{(5)ij}^T \right] \\
- \frac{1}{16\pi G} \int d^3x h_{(5)ij}^T \pi_{ij}^\text{TTij}, \quad (3.35) \] where we used \( h_{(5)ij,k}=0 \), with Eqs. (3.11) and (3.10).

Equation (3.30) reads explicitly:

\[ \pi_{ij}^\text{TTij} = \frac{1}{2} h_{(5)ij}^T. \quad (3.36) \]

The last term in Eq. (3.35) corresponds to a canonical transformation and could be dropped, but we keep it for now.

One has to be aware of a subtlety here. The matter variables entering the Hamiltonian via the solution for \( h_{ij}^T \) play a special role as they may not be treated as dynamical (i.e., phase space) variables. Otherwise, the matter equations of motion resulting from the Hamiltonian would in general be wrong (at the conservative level one can use a Routhian to avoid this problem, see [8]). Instead these nondynamical matter variables entering through \( h_{ij}^T \) are treated as functions depending explicitly on time only. This introduces an explicit time-dependence into the radiation-reaction Hamiltonians, which is a very natural description of a dissipative system via canonical methods.

In order to distinguish the nondynamical matter variables from the dynamical ones, we attach a prime to their object label as in, e.g., \( P_{ij}' \) or \( P_{ij}' \), and also talk of primed and unprimed variables for short. Further, we introduce an explicit time derivative \( \partial^\text{ex}_x \), which only acts on the primed variables (The partial and ordinary time derivatives act on both primed and unprimed variables here). A superscript \( a \to a' \) is attached to a field to denote that its solution should be expressed in terms of the primed variables. This denotes an exchange of all object labels by labels with a prime, not just of label \( a \). Thus \( h_{(5)ij}' \) and \( h_{(7)ij}' \) in Eqs. (3.34) and (3.35) should better be denoted by \( h_{(5)ij}' \to a \to a' \) and \( h_{(7)ij}' \to a \to a' \) from now on.

After the equations of motion have been obtained from the Hamiltonian, one may identify primed and unprimed variables (e.g., the objects \( 1 \) and \( 1' \)), which in general requires another application of regularization techniques.

The formulas for the radiation-reaction Hamiltonians Eqs. (3.34) and (3.35) can be simplified further. First, Eq. (3.34) may be written as:

\[ H_{2.5\text{PN}}^\text{int} \equiv \frac{1}{16\pi G} h_{(5)ij}' \to a \to a' \int d^3x B_{(4)ij}^\text{STF}, \quad (3.37) \]

where the \( x \)-independent \( h_{(5)ij}' \) was pulled in front of the integral and \( B_{(4)ij} \) is contracted with the symmetric trace-free \( h_{(5)ij}^T \). As explained previously, \( h_{(5)ij}^T \) must be replaced by \( h_{(5)ij}^T \to a \to a' \). The remaining integral in Eq. (3.37) is identical up to a prefactor to the definition of \( \chi_{(4)ij} \), cf. (3.28). Finally we obtain, inserting Eq. (3.18),

\[ H_{2.5\text{PN}}^\text{int} = \frac{5}{16G} \chi_{(4)ij}' \chi_{(4)ij}, \quad (3.38) \]

which is a well-known result (see [2] and references therein). The problem was reduced to the calculation of \( \chi_{(4)ij} \) via (3.28). Remember that \( \chi_{(4)ij}' \) in this Hamiltonian is explicitly time-dependent.

We proceed with a simplification of the individual parts of Eq. (3.35). Analogous to the simplification of \( H_{2.5\text{PN}}^\text{int} \) given in the last paragraph we have

\[ \frac{1}{16\pi G} \int d^3x h_{(5)ij}' \to a \to a' B_{(6)ij} = \frac{5}{16G} \chi_{(4)ij}' \Pi_{1ij}, \quad (3.39) \]
\[ \frac{1}{16\pi G} \int d^3x h_{(5)ij}' \to a \to a' V_{(3)}^i \phi_{(2),j} = -\frac{5}{16G} \chi_{(4)ij}' \Pi_{3ij}, \quad (3.40) \] where Eqs. (3.29) and (3.31) were used. We may further write

\[ -\frac{1}{4} \int d^3x h_{(5)ij}' \to a \to a' h_{(4)ij}' \to a \to a' H_{\text{matter}}^{(2)} = \frac{5}{16G} \chi_{(4)ij}' \tilde{\Pi}_{2ij}, \quad (3.41) \]

with the definition

\[ \tilde{\Pi}_{2ij} = -\frac{4G}{5} \int d^3x h_{(5)ij}' \to a \to a' H_{\text{matter}}^{(2)}. \quad (3.42) \]
The notation $\tilde{\Pi}_{2ij}$ was chosen because of the similarity to $\Pi_{2ij}$, cf. Equation (3.30). If the self-interaction contributions to the integral in (3.30) vanish, then $\tilde{\Pi}_{2ij}$ can be obtained from $\Pi_{2ij}$ by a relabeling of objects only. For the spin-dependent part of $\tilde{\Pi}_{2ij}$, this will turn out to be possible. The integral over $B(4)_{ij} h_{TT}^{ij}$ in Eq. (3.35) splits into the following five parts, cf. Equation (3.19),

$$\frac{1}{16\pi G} \int d^3 x \tilde{\Pi}^{a\rightarrow a'}_{1ij} B(4)_{ij} = \frac{5}{16G} \Pi^{\rightarrow a'\chi}_{1ij} (\chi(4)_{ij}), \quad (3.43)$$

$$\frac{1}{16\pi G} \int d^3 x \tilde{\Pi}^{a\rightarrow a'}_{2ij} B(4)_{ij} = \frac{5}{16G} \Pi^{\rightarrow a'\chi}_{2ij} (\chi(4)_{ij}), \quad (3.44)$$

$$\frac{1}{16\pi G} \int d^3 x \tilde{\Pi}^{a\rightarrow a'}_{3ij} B(4)_{ij} = \frac{5}{16G} \Pi^{\rightarrow a'\chi}_{3ij} (\chi(4)_{ij}), \quad (3.45)$$

$$\frac{1}{16\pi G} \int d^3 x Q^{a\rightarrow a'}_{ij} B(4)_{ij} = \chi(4)_{ij} (Q_{ij}^\prime + Q_{ij}^\prime) \quad (3.46)$$

Notice that here $\Pi_{1ij}, \Pi_{2ij}, \Pi_{3ij}$ are independent of $x$. The relations Eqs. (3.33) and (3.18) were used in the last integral. The last two integrals were each split into two parts using Eq. (3.12) and the following definitions:

$$R' = -\frac{1}{2} \int d^3 x \mathcal{T}_{(4)ij} \Pi^{\rightarrow a'\chi}_{3ij}, \quad (3.48)$$

$$R'' = -\frac{1}{128 \pi G} \int d^3 x \phi(2)_{ij} \Pi^{\rightarrow a'\chi}_{3ij}, \quad (3.49)$$

$$Q'_{ij} = -\frac{1}{4} \int d^3 x \mathcal{T}_{(4)ij} \delta^{TTkl} \phi^{\rightarrow a'\chi}_{ij}, \quad (3.50)$$

$$Q''_{ij} = -\frac{1}{256 \pi G} \int d^3 x \phi(2)_{ij} \delta^{TTkl} \phi^{\rightarrow a'\chi}_{ij} \quad (3.51)$$

The fact that the explicit time derivative $\partial^i x$ only acts on primed variables was used in (3.46) to pull it in front of the whole expression. Finally, it holds that

$$-\frac{1}{16\pi G} \int d^3 x h_{TT}^{ij} a\rightarrow a' \pi_{(5)ij}^{(\text{matter})} = -\chi(4)_{ij} O_{ij},$$

with the definition

$$O_{ij} = \frac{1}{16\pi G} \int d^3 x \pi_{ij}^{(\text{matter})}. \quad (3.53)$$

Summing up the contributions from Eqs. (3.39) – (3.41), (3.43) – (3.47), and the total time derivative of Eq. (3.52), one gets:

$$H^\text{int}_{3.5PN} = \frac{5}{16G} \left[ \chi(4)_{ij} (\Pi^{\rightarrow a'\chi}_{1ij} + \Pi^{\rightarrow a'\chi}_{2ij} + \Pi^{\rightarrow a'\chi}_{3ij}) + \chi(4)_{ij} (\Pi_{1ij} + \Pi_{2ij}) - \Pi^{\rightarrow a'\chi}_{3ij} \Pi_{3ij} \right] + \chi(4)_{ij} (Q_{ij}^\prime + Q_{ij}^\prime) + (\partial^i x)^3 (R' + R'') \quad (3.54)$$

This agrees with [2] (with misprints corrected in [11]). It should be noted that no time derivatives are present in Eq. (3.3), so all time derivatives in Eqs. (3.38) and (3.54) are introduced by above insertions. Indeed, all these time derivatives should be understood as abbreviations and be performed before the equations of motions are derived from the Hamiltonians. However, for time derivatives of primed variables it is irrelevant at which stage they are eliminated (These are actually all time derivatives except the one acting on $O_{ij}$). One should be aware that an insertion of equations of motion leads to a recombination of PN orders, e.g., inserting the IPN conservative part of the equations of motion leads to 3.5PN contributions from $H^\text{int}_{2.5PN}$, cf. Equation (3.38). Further, one should notice that $\Pi_{2ij}, R', R'', Q', Q''$ depend on both primed and unprimed variables by virtue of their definitions.

### IV. Calculation of the Hamiltonians

Up to formal 3.5PN order, the interaction Hamiltonian is given by Eqs. (3.38) and (3.54). The quantities entering these expressions must be calculated by solving the integrals appearing in their definitions [see Eqs. (3.17), (3.28–3.32), (3.42), (3.48–3.51), and (3.53)]. The leading-order source terms in the pole-dipole case entering these integrals read

$$\mathcal{H}^{\text{matter}}_{(2)} = \sum_a m_a \delta_a, \quad (4.1)$$

$$\mathcal{T}_{(4)ij} = \sum_a \frac{1}{m_a} \left[ P_{ai} P_{aj} \delta_a + P_a (S_{ai}) (\delta_{b}) \delta_b \right], \quad (4.2)$$

$$B(4)_{ij} = -8 \pi G T(4)_{ij} + \frac{1}{8} \phi(2)_{ij} \phi(2)_{ij}, \quad (4.3)$$

see [1] for more details. Here, $m_a$ ($a = 1, 2, \ldots$) are the masses and $\delta_a = \delta(x - z_a)$. $\phi(2)$ is proportional to the Newtonian potential of point-masses, namely:

$$\phi(2) = -16 \pi G \Delta^{-1} \mathcal{H}^{\text{matter}}(2) = 4G \sum_a \frac{m_a}{r_a}, \quad (4.4)$$

where $r_a = |x - z_a|$. Notice that $\phi(2)$ is independent of the spins. The expression for $B(4)_{ij}$ was derived in [1]:

...
\[ B_{(6)i} = 16\pi G \sum_a \left[ \frac{\mathbf{P}_a^2}{4m_a^2} P_{ai} \delta_a + \frac{5}{8m_a} P_{ai} P_{aj} \phi(2) \delta_a + \frac{\mathbf{P}_a^2}{4m_a^2} P_{ai} S_{a(j)(k)} \delta_a \delta_{a,k} \right. \]
\[ + \frac{5}{8m_a} P_{ai} S_{a(j)(k)} \left( \phi(2) \delta_a \right)_k + \frac{2}{8m_a} P_{ai} S_{a(j)(k)} \phi(2) \delta_a - \frac{1}{8m_a} P_{ai} S_{a(k)(j)} \delta_a \delta_{a,k} \]
\[ + \frac{1}{2} S_{a(k)(i)} \left( V_{(3),k} + V_{(3),i} \right) \delta_a \left( \right) \]
\[ + \frac{1}{2} \phi_{(1)}(4) \phi(ij) + \frac{3}{8} \phi_{(2)}(4) \phi(ij) + \frac{5}{64} \phi_{(2)}(2) \phi(ij) \delta_a + \frac{2}{\mathbf{G}} \left( \hat{\pi}^k_{(3)} - \pi^k_{(3)} \right) + 2 \hat{\pi}^{ij}_{(3)} V^{k}_{(3)} + \frac{1}{2} \hat{z}^{ij}_{(3)} \hat{z}^{k}_{(3)} \right]. \]

The field quantities entering Eq. (4.5) are equal to:
\[ \phi_{(1)}(4) = 2G \sum_a \left[ \frac{\mathbf{P}_a^2}{m_a r_a} + \frac{P_{ai} S_{a(j)(i)}}{m_a} \left( \frac{1}{m_a} \right), \right], \quad (4.6a) \]
\[ \phi_{(2)}(4) = -2G^2 \sum_a \left[ \frac{m_a m_b}{r_{ab}} \right], \quad (4.6b) \]
\[ \pi^i_{(3)} = G \sum_a \left[ 2 \frac{P_{ai}}{r_a} + S_{a(i)(j)} \left( \frac{1}{m_a} \right), \right], \quad (4.6c) \]
\[ V^i_{(3)} = G \sum_a \left[ 2 \frac{P_{ai}}{r_a} - \frac{1}{P_{aj}} r_{ai} + S_{a(i)(j)} \left( \frac{1}{m_a} \right), \right], \quad (4.6d) \]
\[ \pi^{ij}_{(3)} = G \sum_a \left[ 2P_{ai} \left( \frac{1}{m_a} \right) + 2P_{aj} \left( \frac{1}{m_a} \right) \right] - \delta^{ij} P_{ak} \left( \frac{1}{m_a} \right) \left( \right)_k - \frac{1}{2} \left( \right)_{ik} \]
\[ - S_{a(k)(i)} \left( \frac{1}{m_a} \right) \left( \right)_{ki} \left( \right), \quad (4.6e) \]

where \( r_{ab} = |\mathbf{r}_a - \mathbf{r}_b| \). Notice that for nonspinning systems the result in [2] is reproduced. Further notice that \( \pi^i_{(3),i} \) does not depend on spin. Finally, the spin correction to the field momentum is given by:
\[ \pi_{(3)}^{ij} \text{matter} = - \sum \frac{4\pi G}{m_a} P_{ak} P_{ai} S_{a(j)(k)} \delta_a, \quad (4.7) \]
to the required order.

A. Spin-dependent part of \( h_{TT}^{ij} \)

The explicit solutions for the point-mass, i.e., spin-independent, contributions to \( h_{TT}^{ij} \) can be found in [2, 8, 11, 57] (but notice that [2] contains some misprints). The spin part of \( h_{TT}^{ij} \), arising from the spin-dependent source terms in Eq. (4.2) via Eqs. (4.3) and (3.17), has been computed in [14] and reads:
\[ h_{TT}^{(ij)} = G \sum_a \frac{m_a P_{ai} S_{a(k)(i)}}{m_a} \left[ 4 \delta_{k(i)} \delta_{j} - 2 \delta_{k(i)} \delta_{j} \right] \frac{1}{r_a} \]
\[ + \left( \delta_{k(i)} \delta_{j} \delta_{k} \right) \frac{1}{r_a} \right], \quad (4.8) \]
where we use the superscript “spin” to denote the spin-dependent part of a quantity from now on. In order to obtain the spin contributions to the radiation-reaction Hamiltonian up to formal 3.5PN order, we also need to compute the spin part of \( h_{TT}^{(ij)} \) and \( h_{TT}^{(ij)} \). \( h_{TT}^{(ij)} \) is easy to compute. From Eqs. (3.28), (4.3), and (4.2), we have
\[ \lambda_{(4)}^{ij} = - \frac{8G}{5} \sum_a \left[ \frac{P_{ai} S_{a(k)(i)}}{m_a} \int d^3x \delta_{k} \delta_{a} \right]_{STF} = 0, \quad (4.9) \]
and thus also \( h_{TT}^{(ij)} = 0 \) [see Eq. (3.18)]. There is no spin contribution to the 2.5PN \( h_{TT}^{ij} \), which is the reason why the leading-order source terms (4.2) are not sufficient to derive the leading-order radiation-reaction Hamiltonian. \( h_{TT}^{(ij)} \) would be more difficult to derive, but it is not needed in our calculation of the leading-order radiation-reaction Hamiltonian with spins, so we will not discuss it in the present paper.

Analogous to Eq. (3.19), we decompose the solution for \( h_{TT}^{(ij)} \) into several parts,
\[ h_{TT}^{(ij)} = \hat{h}_{TT}^{(ij)} + \hat{h}_{TT}^{(ij)} + \hat{h}_{TT}^{(ij)} + \hat{h}_{TT}^{(ij)} \],
where the following definitions are used:
\[ \Pi_{TT}^{(1)} = \frac{1}{5\pi} \int d^3x B_{TT}^{STF}, \quad (4.11) \]
\[ \Pi_{TT}^{(2)} = - \frac{4G}{5} \int d^3x h_{TT}^{(ij)} H_{(2)}^{matter}, \quad (4.12) \]
\[ \Pi_{TT}^{(3)} = - \frac{1}{5\pi} \int d^3x \left( V^{ij} \phi_{(2),ij} \right)_{STF}, \quad (4.13) \]
\[ \Pi_{TT}^{(4)} = - \frac{2G}{3} \delta_{kli} \int d^3x \left| \textbf{x} - \textbf{x}' \right|^2 \phi_{(4),kl}(\textbf{x}', t), \quad (4.14) \]
and obviously \( Q_{TT}^{(ij)} = 0 \), cf. Equations (3.29) – (3.33) and (4.3). These integrals yield the results:
\[ \Pi_{1ij}^{\text{spin}} = \frac{4G^2}{5} \sum_a \sum_{b \neq a} \left\{ \frac{1}{r_{ab}^3} \left[ 3(n_{ab} \cdot P_b) n_{ab}^k (n_{ab}^k S_{a(i)(k)} + n_{ab}^i S_{a(j)(k)}) - 3P_{ab}(n_{ab}^i S_{a(i)(k)} + n_{ab}^k S_{a(j)(k)}) \right] - 3n_{ab}^k (P_{bS} S_{a(i)(k)} + P_{bS} S_{a(j)(k)}) + 4(3n_{ab}^i n_{ab}^k - \delta_{ij}) n_{ab}^k P_{at} S_{a(k)(l)}) + \frac{mb}{r_{ab}^3} \left[ P_{ak}(n_{ab}^i S_{a(j)(k)} + n_{ab}^k S_{a(i)(k)}) \right] + 4n_{ab}^k (P_{ajS} S_{a(i)(k)} + P_{as} S_{a(j)(k)}) \right\} - \frac{S_{a(k)(l)}}{r_{ab}^3} \left[ (3n_{ab}^i n_{ab}^j - \delta_{ij}) S_{b(k)(l)} \right] \right\}, \]

\[ \Pi_{2ij}^{\text{spin}} = -\frac{4G^2}{5} \sum_a \sum_{b \neq a} \left\{ \frac{m_b}{r_{ab}^3} \left[ -2P_{ak}(n_{ab}^i S_{a(i)(k)} + n_{ab}^k S_{a(j)(k)}) + n_{ab}^b (P_{aiS} S_{a(i)(k)} + P_{aj} S_{a(j)(k)}) \right] \right\}, \]

\[ \Pi_{3ij}^{\text{spin}} = \frac{4G^2}{5} \sum_a \sum_{b \neq a} \left\{ \frac{m_b}{r_{ab}^3} \left[ n_{ab}^k (n_{ab}^i S_{a(i)(k)} + n_{ab}^k S_{a(j)(k)}) + n_{ab}^b (P_{aiS} S_{a(i)(k)} + P_{aj} S_{a(j)(k)}) \right] \right\}, \]

\[ \Pi_{4ij}^{\text{spin}} = \frac{4G}{15} \sum_a \frac{r_{ab}}{m_a} \left[ P_{ak}(n_{ab}^i S_{a(i)(k)} + n_{ab}^k S_{a(j)(k)}) - 2n_{ab}^k (P_{ajS} S_{a(i)(k)} + P_{as} S_{a(j)(k)}) + \delta_{ij} P_{at} S_{a(k)(l)}) \right] \right\}, \]

where \(n_a = (x - \hat{z}_a)/r_a\) and \(n_b = (\hat{z}_a - \hat{z}_b)/r_{ab}\). Notice that it holds

\[ \Pi_{1ij}^{\text{spin}} + \Pi_{2ij}^{\text{spin}} + \Pi_{3ij}^{\text{spin}} = -\frac{4G}{5} r_{ij}^{\text{spin}} \right\}, \]

at the considered PN order, where \(r_{ij}\) is a multipole moment of the far-zone expansion of \(h_{ij}^{TT}\) and can be expressed as a double time derivative of a very compact expression, see Eqs. (6.15) and (6.18) in [1].

**B. Derivation of spin contributions to 2.5PN and 3.5PN interaction Hamiltonians**

When taking into account the fact that Eq. (4.9) tells us that \(\chi^{(4)ij} = 0\), we immediately see that the formal 2.5PN order interaction Hamiltonian Eq. (3.38),

\[ H_{2.5PN}^{\text{int}} = \frac{5}{16G} \chi^{(4)ij} \chi^{(4)ij}, \]

has only the well-known point-mass contribution [57, 58]:

\[ \chi^{(4)ij} = \frac{4G}{15} \sum_a \left\{ \frac{2}{r_{ma}} (P_a^2 \delta_{ij} - 3P_{ai} P_{aj}) \right\}, \]

\[ \chi^{(4)ij} = \frac{G}{15} \sum_a \left\{ \frac{1}{r_{ma}^3} \left[ (3n_{ab} \cdot P_b) n_{ab}^k (n_{ab}^k S_{a(i)(k)} + n_{ab}^i S_{a(j)(k)}) - 3P_{ab}(n_{ab}^i S_{a(i)(k)} + n_{ab}^k S_{a(j)(k)}) \right] - 3n_{ab}^k (P_{bS} S_{a(i)(k)} + P_{bS} S_{a(j)(k)}) + 4(3n_{ab}^i n_{ab}^k - \delta_{ij}) n_{ab}^k P_{at} S_{a(k)(l)}) + \frac{mb}{r_{ab}^3} \left[ P_{ak}(n_{ab}^i S_{a(j)(k)} + n_{ab}^k S_{a(i)(k)}) \right] + 4n_{ab}^k (P_{ajS} S_{a(i)(k)} + P_{as} S_{a(j)(k)}) \right\} - \frac{S_{a(k)(l)}}{r_{ab}^3} \left[ (3n_{ab}^i n_{ab}^j - \delta_{ij}) S_{b(k)(l)} \right] \right\}, \]

but no direct spin contribution. However, indirect spin-contributions arise from Eq. (4.20) via the time derivative therein and first appear at the formal 3.5PN level [after taking into account the leading-order conservative spin-orbit and spin-(1)-spin(2) equations of motion [28–30], provided in this paper by Eqs. (5.2) and (5.3) later on].

The spin part of the formal 3.5PN order interaction Hamiltonian Eq. (3.54) can be written as:

\[ H_{3.5PN}^{\text{int}} = \frac{5}{16G} \chi^{(4)ij} \chi^{(4)ij}, \]

\[ \chi^{(4)ij} = \frac{4G}{15} \sum_a \left\{ \frac{2}{r_{ma}} (P_a^2 \delta_{ij} - 3P_{ai} P_{aj}) \right\}, \]

\[ \chi^{(4)ij} = \frac{G}{15} \sum_a \left\{ \frac{1}{r_{ma}^3} \left[ (3n_{ab} \cdot P_b) n_{ab}^k (n_{ab}^k S_{a(i)(k)} + n_{ab}^i S_{a(j)(k)}) - 3P_{ab}(n_{ab}^i S_{a(i)(k)} + n_{ab}^k S_{a(j)(k)}) \right] - 3n_{ab}^k (P_{bS} S_{a(i)(k)} + P_{bS} S_{a(j)(k)}) + 4(3n_{ab}^i n_{ab}^k - \delta_{ij}) n_{ab}^k P_{at} S_{a(k)(l)}) + \frac{mb}{r_{ab}^3} \left[ P_{ak}(n_{ab}^i S_{a(j)(k)} + n_{ab}^k S_{a(i)(k)}) \right] + 4n_{ab}^k (P_{ajS} S_{a(i)(k)} + P_{as} S_{a(j)(k)}) \right\} - \frac{S_{a(k)(l)}}{r_{ab}^3} \left[ (3n_{ab}^i n_{ab}^j - \delta_{ij}) S_{b(k)(l)} \right] \right\}, \]

\[ \chi^{(4)ij} = \frac{4G}{15} \sum_a \left\{ \frac{2}{r_{ma}} (P_a^2 \delta_{ij} - 3P_{ai} P_{aj}) \right\}, \]

\[ \chi^{(4)ij} = \frac{G}{15} \sum_a \left\{ \frac{1}{r_{ma}^3} \left[ (3n_{ab} \cdot P_b) n_{ab}^k (n_{ab}^k S_{a(i)(k)} + n_{ab}^i S_{a(j)(k)}) - 3P_{ab}(n_{ab}^i S_{a(i)(k)} + n_{ab}^k S_{a(j)(k)}) \right] - 3n_{ab}^k (P_{bS} S_{a(i)(k)} + P_{bS} S_{a(j)(k)}) + 4(3n_{ab}^i n_{ab}^k - \delta_{ij}) n_{ab}^k P_{at} S_{a(k)(l)}) + \frac{mb}{r_{ab}^3} \left[ P_{ak}(n_{ab}^i S_{a(j)(k)} + n_{ab}^k S_{a(i)(k)}) \right] + 4n_{ab}^k (P_{ajS} S_{a(i)(k)} + P_{as} S_{a(j)(k)}) \right\} - \frac{S_{a(k)(l)}}{r_{ab}^3} \left[ (3n_{ab}^i n_{ab}^j - \delta_{ij}) S_{b(k)(l)} \right] \right\}, \]
using Eqs. (3.42), (3.48) – (3.51), and (3.53). We also split $R_{\text{spin}}$ into three parts,

\[
R_{\text{spin}} = R_1^{\text{spin}} + R_2^{\text{spin}} + R_3^{\text{spin}},
\]

\[
R_1^{\text{spin}} = \frac{1}{2} \int d^3x \, \mathcal{I}_{(4)ij}^{\text{spin}, a \rightarrow a'},
\]

\[
R_2^{\text{spin}} = \frac{1}{2} \int d^3x \, \mathcal{I}_{(4)ij}^{\text{spin}, a' \rightarrow a},
\]

\[
R_3^{\text{spin}} = -\frac{1}{2} \int d^3x \, \mathcal{I}_{(4)ij}^{\text{spin}, a \rightarrow a'}.
\]

Here, PM denotes the point-mass parts of a function. The quantities entering above integrals will be all given in the present paper, except for $\mathcal{I}_{(4)ij}^{\text{spin}, a \rightarrow a'}$, which can be read from Eq. (36) in [2] using $R_{ij} = \partial^2 \mathcal{I}_{(4)ij}$. The results of the above integrations read as follows:

\[
\mathcal{I}_{(4)ij}^{\text{spin}, a \rightarrow a'} = -\frac{4G^2}{5} \sum_{a,a'} \frac{m_a}{m_a} \frac{1}{r_{aa'}} 
\left[ 2P_{a'}(n_{a'a}^i S_{a'}(j)(k) + n_{a'a}^j S_{a'}(i)(k)) - n_{a'a}^b (P_{a'} S_{a'}(j)(k) + P_{a'} S_{a'}(i)(k)) \right]
\]

\[
- 3(n_{a'a} \cdot P_{a'}) n_{a'a} (n_{a'a}^i S_{a'}(j)(k) + n_{a'a}^j S_{a'}(i)(k)) - (\delta_{ij} + 3n_{a'a}^i n_{a'a}^j) n_{a'a} (P_{a'} S_{a'}(j)(k) + P_{a'} S_{a'}(i)(k))
\]

\[
+ \frac{G}{4} \sum_{a,a'} \frac{m_{a'}}{m_a} \frac{1}{r_{aa'}} 
\left[ 2P_{a'}(n_{a'a}^i S_{a'}(j)(k) + n_{a'a}^j S_{a'}(i)(k)) - n_{a'a} (P_{a} S_{a}(j)(k) + P_{a} S_{a}(i)(k)) \right]
\]

\[
- 3(n_{a'a} \cdot P_{a'}) n_{a'a} (n_{a'a}^i S_{a}(j)(k) + n_{a'a}^j S_{a}(i)(k)) - (\delta_{ij} + 3n_{a'a}^i n_{a'a}^j) n_{a'a} (P_{a} S_{a}(j)(k) + P_{a} S_{a}(i)(k))
\]

\[
+ \frac{G}{7} \sum_{a,b,a',b'} \frac{m_a}{m_a} \frac{1}{r_{ab}} 
\left[ 17n_{a'b'}^i P_{a'} - \frac{2r_{a'a'}}{r_{a'b'}} \left[ 17(n_{a'b'} \cdot P_{a'}) n_{a'b'}^i n_{a'a'}^j + 7n_{a'a'}^j P_{a'} \right] \right]
\]

\[
+ \frac{r_{a'a'}}{r_{a'b'}} \left[ 17(n_{a'b'} \cdot P_{a'}) n_{a'b'}^i n_{a'a'}^j + 7n_{a'a'}^j P_{a'} \right]
\]

\[
+ \frac{G}{4G} \sum_{a,a'} \frac{r_{aa'}}{m_a} \frac{1}{r_{aa'}} 
(n_{a'a}^i P_{a'}) - 2(n_{a'a} \cdot P_{a'}) n_{a'a}^i P_{a'} - (n_{a'a} \cdot P_{a'}) P_{a'} \right] \right]
\]

\[
\mathcal{I}_{(4)ij}^{\text{spin}, a' \rightarrow a} = \frac{4G^2}{5} \sum_{a,a'} \frac{m_a}{m_a} \frac{1}{r_{aa'}} 
\left[ 3P_{a'}(n_{a'a}^i S_{a'}(j)(k) - (P_{a'} \cdot P_{a}) S_{a'}(j)(k) - P_{a'} S_{a'}(j)(k) + P_{a} S_{a}(j)(k)) \right]
\]

\[
\mathcal{I}_{(4)ij}^{\text{spin}, a \rightarrow a'} = 2G^2 \sum_{a,a'} \frac{m_a m_{a'}}{m_a} \frac{1}{r_{aa'}} 
\left[(n_{a'a}^i P_{a'} - 2(n_{a'a} \cdot P_{a'}) n_{a'a}^i) - (n_{a'a} \cdot n_{a'a} n_{a'a}^i) P_{a'} \right]
\]

\[
O_{ij}^{\text{spin}} = \frac{1}{8\pi G^2} \sum_{a} P_{ak}(P_{a} S_{a}(j)(k) + P_{a} S_{a}(i)(k)).
\]

The term in Eq. (4.34) containing $17n_{a'a'}^i P_{a'}$ actually cancels if the sums over $a'$ and $b'$ are performed and may therefore be dropped.

Notice that $\Pi_{(4)ij}^{\text{spin}, a \rightarrow a'}$, $\Pi_{(4)ij}^{\text{spin}, a' \rightarrow a}$, and $Q_{ij}^{\text{spin}}$ are given by almost identical expressions, cf. Equations (4.16), (4.32), and (4.33). This is not accidental, but due to similarities of their defining integrals. With the source mass density given by Eq. (4.1), we obtain from Eq. (4.12):

\[
\Pi_{(4)ij}^{\text{spin}} = \frac{4G}{5} \sum_{a} m_a h_{(4)ij}^{\text{spin}, a \rightarrow a'} \bigg|_{x = \hat{z}_a}.
\]

Similarly, Eq. (4.23) leads to

\[
\Pi_{(4)ij}^{\text{spin}} = \frac{4G}{5} \sum_{a} m_a (h_{(4)ij}^{\text{spin}, a \rightarrow a'} \bigg|_{x = \hat{z}_a}.
\]

Notice that in this expression no regularization is needed for taking $x = \hat{z}_a$, as primed and unprimed objects are not identified yet. In contrast to that there may be contributions from Hadamard regularization in Eq. (4.39). However, for the spin-dependent part, no such contributions appear (in contrast to the nonspinning case in [2]), which explains the great similarity between $\Pi_{(4)ij}^{\text{spin}, a \rightarrow a'}$ and $\Pi_{(4)ij}^{\text{spin}, a' \rightarrow a}$. Further, insertion of Eq. (4.4) into Eq. (4.24) leads to:

\[
Q_{ij}^{\text{spin}} = \frac{1}{4} \int d^3x \, h_{(4)ij}^{\text{spin}}(H_{(2)}^{\text{matter}} a \rightarrow a')
\]

\[
= \frac{1}{4} \sum_{a} m_a h_{(4)ij}^{\text{spin}} \bigg|_{x = \hat{z}_a}.
\]

after performing several partial integrations and using Eqs. (3.3) and (3.17). Here also no regularization is needed. The similarity to Eqs. (4.23) or (4.40) is ob-
vious. The difference is simply an overall factor and a mutual exchange of primed and unprimed variables.

V. ENERGY LOSS OF A BINARY SYSTEM

A. Derivation of the energy loss from the Hamiltonian

The instantaneous (near-zone) energy loss of a two-body system due to gravitational radiation can be written in the form (see, e.g., [2, 11]):

$$\mathcal{L}_{\text{inst}}^{3.5\text{PN}} = -\partial_\tau (\mathcal{H}_{2.5\text{PN}}^{\text{int}} + \mathcal{H}_{3.5\text{PN}}^{\text{int}}).$$

(5.1)

Notice that this energy loss is gauge-dependent in contrast to the energy flux at infinity.

We substitute Eqs. (4.20) and (4.22) into Eq. (5.1) (For the point-mass part of $\mathcal{H}_{3.5\text{PN}}^{\text{int}}$ this was already done in [2]). After that, we need to eliminate the time derivatives in Eq. (5.1) using the leading-order spin-orbit, spin(1)-spin(2), and Newtonian equations of motion derived from the corresponding Hamiltonians [see, e.g., Eqs. (7.28) and (7.29) in [14]],

$$\dot{z}_1^i = \frac{\dot{p}_1^i}{m_1} - \frac{G n_1^1}{r_1^2} (3m_2 S_{(1)(j)} + 4m_1 S_{(2)(j)}),$$

(5.2a)

$$\dot{z}_2^i = (1 = 2),$$

(5.2b)

Note that because the 2.5PN order Hamiltonian does not have spin contributions as we discussed in Sec. IV, we do not include the 1PN point-mass terms because substituting them into the 2.5PN Hamiltonian only produces point-mass terms at 3.5PN order, while substituting them into the 3.5PN Hamiltonian only produces 4.5PN terms which is beyond the scope of this paper.

At this point, we no longer need to distinguish the difference between the primed and unprimed variables. Using the Hadamard regularization method, we remove the singularities produced by the limit $\dot{z}_1 \to \dot{z}_1$ and $\dot{z}_2 \to \dot{z}_2$ and obtain an expression of the energy loss in terms of $\dot{z}_{1(2)}$ and $\dot{p}_{1(2)}$. By realizing that $\dot{z}_a \equiv v_a$, we may use

$$p_1^i = m_1 v_1^i = \frac{G n_1^i}{r_1^2} (3m_2 S_{(1)(j)} + 4m_1 S_{(2)(j)}) ,$$

(5.4a)

to express the particle momenta $p_a$ in terms of the particle coordinate velocities $v_a$, which can be easily obtained from Eq. (5.2). Here, $r = r_{12}$, and $n = n_{12}$. Note we do not include the 1PN point-mass terms in this expression for the reason described above.

To put the energy loss into a more convenient form, we rewrite the individual masses $m_1, m_2$ into the total mass of the system $M \equiv m_1 + m_2$, the reduced mass $\mu \equiv m_1 m_2 / M$, and the symmetric mass-ratio parameter $\eta \equiv \mu / M$ using the relations (assuming $m_1 \geq m_2$):

$$m_1 = \frac{\mu}{2\eta} \left(1 + \sqrt{1 - 4\eta^2}\right) ,$$

(5.5a)

$$m_2 = \frac{\mu}{2\eta} \left(1 - \sqrt{1 - 4\eta^2}\right) .$$

(5.5b)

We also transform the individual coordinate velocities of each particle into the center of mass frame using the relations:
\[
\begin{align*}
v_1 &= \frac{2\eta v}{1 + \sqrt{1 - 4\eta}} + \frac{G}{4r^2} \left[ (n \times S_1) \left( -1 + \sqrt{1 - 4\eta} \right) + (n \times S_2) \left( 1 + \sqrt{1 - 4\eta} \right) \right], \\
v_2 &= \frac{-2\eta v}{1 - \sqrt{1 - 4\eta}} + \frac{G}{4r^2} \left[ (n \times S_1) \left( -1 + \sqrt{1 - 4\eta} \right) + (n \times S_2) \left( 1 + \sqrt{1 - 4\eta} \right) \right],
\end{align*}
\]

where \( v = v_1 - v_2 \) is the relative velocity, \( S_a \) is the individual spin. Notice that here we do not include the 1PN point-mass terms (see, e.g., Eq. (3.13) in [11]) because the 1PN corrections of \( \nu_a \) can only produce 3.5PN terms in the flux when substituted into \( L_{\leq 2.5\text{PN}} \), which is independent of spins, therefore the 1PN point-mass terms in \( \nu_a \) do not contribute any spin-dependent terms at the formal 3.5PN order.

After eliminating the coordinate velocity \( v_1 \) and \( v_2 \) by means of Eq. (5.6), the spin-orbit and spin(1)-spin(2) (\( S_1 S_2 \)) part of the instantaneous energy loss \( L_{\leq 3.5\text{PN}} \) can be written as:

\[
\begin{align*}
L_{\text{inst,SO}} &\leq 3.5\text{PN} = -\frac{G^2 M^2 \eta^2}{15r^5} \left\{ 74 \frac{G^2 M^2}{r^2} + 420(n \cdot v)^4 - 510(n \cdot v)^2 v^2 + 66v^4 + \frac{GM}{r} \left( 54(n \cdot v)^2 + 22v^2 \right) \right\}, \\
L_{\text{inst,}\ S_1 S_2} &\leq 3.5\text{PN} = \frac{2G^2 M \eta}{15r^5} \left\{ (S_1 \cdot S_2) \left[ 12 \frac{G^2 M^2}{r^2} - 120(n \cdot v)^2 v^2 + 24v^4 + \frac{GM}{r} \left( 192v^2 - 348(n \cdot v)^2 \right) \right] \\
&\quad + (S_1 \cdot v)(S_2 \cdot v) \left[ 184 \frac{GM}{r} - 450(n \cdot v)^2 + 138v^2 \right] \\
&\quad + [(n \cdot v)(n \cdot S_2)(S_1 \cdot v) + (n \cdot v)(n \cdot S_1)(S_2 \cdot v)] \left[ -546 \frac{GM}{r} + 1785(n \cdot v)^2 - 1005v^2 \right] \right\}.
\end{align*}
\]

with \( v = \langle v \rangle \), \( S \equiv S_1 + S_2 \), \( \xi \equiv (m_2/m_1)S_1 + (m_1/m_2)S_2 \) are the spin variables, and \( \mathbf{L}_N \equiv r\mathbf{n} \times \mathbf{v} \) is the Newtonian orbital angular momentum per reduced mass.

**B. Comparison with other results**

References [25, 26] recently computed, using the method of direct integration of the relaxed Einstein equations [59, 60], the leading-order spin-orbit and spin(1)-spin(2) equations of motion and the corresponding energy loss in harmonic coordinates. In this subsection, we shall prove that our result is actually equivalent to the results in [25, 26].

In order to compare the instantaneous energy loss, we first need to find the transformation between our ADM canonical variables \( (\mathbf{z}_a, \nu_a \equiv \mathbf{z}_a, S_a) \) and the “harmonic coordinate” variables \( (\mathbf{y}_a, V_a \equiv \mathbf{y}_a, S_a^{\text{WW}}) \). Because the quantity we are comparing is the energy loss \( L_{\text{inst}}^{\leq 3.5\text{PN}} \) at formal 3.5PN order, which is only one formal order higher than the leading-order energy loss \( L_{\text{inst}}^{\leq 2.5\text{PN}} \) caused by the quadrupole radiation of point-masses, the coordinate transformation we are looking for needs to be accurate up to formal 1PN order.

It is well known that for the point-mass case the ADM coordinates are equivalent to the harmonic coordinates at 1PN order in that they result in identical equations of
motion. In addition, the spin-dependent part of the formal 1PN accurate transformation \( z_a(y_a, V_a, S_{a}^{\text{WW}}) \) can be derived from the well-known transformation between different spin supplementary conditions (SSC) (for details, see, e.g., [25]). Namely, for a specific SSC parameter \( k \), which is used to fix the center of mass of the particle, we impose the condition:

\[
S_{a}^{10} - k S_a^{ij} v_a^j = 0, \tag{5.8}
\]

where \( k \) typically has the value 1, 1/2, or 0. The relation between the center of mass for each value of \( k \) can be written as:

\[
(x_{a}^{i})^{(k')} = (x_{a}^{i})^{(k)} + \frac{k - k'}{m A} S_{a}^{ij} v_a^j. \tag{5.9}
\]

It is straightforward to show that at formal 1PN order the SSC in our calculation leads to \( k = 1/2 \), which is identical to the one used in references [25, 26]. Therefore, we have:

\[
z_a(y_a, V_a, S_{a}^{\text{WW}}) = y_a, \tag{5.10}
\]

\[
v_a(y_a, V_a, S_{a}^{\text{WW}}) \equiv \dot{z}_a = V_a. \tag{5.11}
\]

Reference [37] has shown that the difference between the spin parameters \( S_a \) used in the ADM formalism and the ones used in the harmonic coordinates calculations is of formal 2PN order. In other words, the transformation

\[
S_a(y_a, V_a, S_{a}^{\text{WW}}) = S_{a}^{\text{WW}}, \tag{5.12}
\]

can be used in this paper.

From Eqs. (5.10) - (5.12) we know that our ADM canonical variables are actually equivalent to the harmonic gauge ones at the considered PN order. Now we are not comparing with the harmonic gauge energy loss given in [25, 26, 60] directly, but with the far-zone energy flux, which was shown to agree with the former (up to an nonphysical total time derivative). When comparing our result Eqs. (5.7) to the far-zone flux \( [\mathcal{L}_{\leq 3.5\text{PN}}]_{\text{far-zone}} \), for the purpose of this paper, only the parts

\[
[\mathcal{L}_{\leq 3.5\text{PN}}]_{\text{inst}}^{\text{far-zone}} = [\mathcal{L}_{\leq 2.5\text{PN}} + \mathcal{L}_{\leq 3.5\text{PN}}^{\text{SO}} + \mathcal{L}^{S,S}_{\leq 3.5\text{PN}}]_{\text{inst}}, \tag{5.13a}
\]

\[
[\mathcal{L}_{\leq 3.5\text{PN}}]_{\text{far-zone}} = [\mathcal{L}_{\leq 2.5\text{PN}} + \mathcal{L}_{\leq 3.5\text{PN}}^{\text{SO}} + \mathcal{L}^{S,S}_{\leq 3.5\text{PN}}]_{\text{far-zone}}, \tag{5.13b}
\]

are relevant to this paper, where for the instantaneous energy loss in ADM coordinates we substitute Eq. (5.7) and for the far-zone flux we substitute the expressions computed in [19] in harmonic gauge,

\[
[\mathcal{L}_{\leq 2.5\text{PN}}]_{\text{far-zone}} = \frac{8}{15} \frac{G^{3}M^{3} r^{2}}{r^{6}} \left[ (-11 \mathbf{n} \cdot \mathbf{v})^{2} + 12 v^{2} \right], \tag{5.14a}
\]

\[
\mathcal{L}_{\leq 3.5\text{PN}}^{\text{SO}}_{\text{far-zone}} = \frac{4}{15} \frac{G^{3}M^{3} r^{2}}{r^{6}} \left\{ -171 (\mathbf{n} \cdot \mathbf{v}) (\mathbf{n} \cdot \mathbf{S}_{2}) (\mathbf{S}_{1} \cdot \mathbf{v}) - 171 (\mathbf{n} \cdot \mathbf{v}) (\mathbf{n} \cdot \mathbf{S}_{1}) (\mathbf{S}_{2} \cdot \mathbf{v}) + 71 (\mathbf{S}_{1} \cdot \mathbf{v}) (\mathbf{S}_{2} \cdot \mathbf{v}) + (\mathbf{n} \cdot \mathbf{S}_{1}) (\mathbf{n} \cdot \mathbf{S}_{2}) \left[ 807 (\mathbf{n} \cdot \mathbf{v})^{2} - 504 v^{2} \right] + (\mathbf{S}_{1} \cdot \mathbf{S}_{2}) \left[ -165 (\mathbf{n} \cdot \mathbf{v})^{2} + 141 v^{2} \right] \right\}. \tag{5.14c}
\]

It should be noted that the sources (on the right-hand side of these equations) are evaluated at the retarded time with respect to the flux (on the left-hand side), which is not explicitly denoted here. In contrast to the instantaneous near-zone energy loss, these results are actually gauge-independent at the considered PN order, i.e., one gets exactly the same result from Eq. (6.22) in [1] within the ADM gauge. We already showed this in [1] for \( \mathcal{L}_{\leq 3.5\text{PN}}^{\text{SO}}_{\text{far-zone}} \), and we confirmed this for \( [\mathcal{L}_{\leq 2.5\text{PN}}]_{\text{far-zone}} \) and \( \mathcal{L}_{\leq 3.5\text{PN}}^{S,S}_{\text{far-zone}} \), too.

It has been shown in [11] that the spin-independent result \( [\mathcal{L}_{\leq 2.5\text{PN}}]_{\text{inst}} + [\mathcal{L}_{\leq 3.5\text{PN}}]_{\text{inst}} \) and the spin-orbit part of formal 3.5PN order \( \mathcal{L}_{\leq 3.5\text{PN}}^{SO}_{\text{inst}} \) agree with the results computed in harmonic coordinates up to a total time derivative, which is a pure gauge effect and vanishes after orbital average (see, e.g., [27] and [61]).

For spin-dependent instantaneous energy loss, it is possible to write the difference between Eqs. (5.13a) and (5.13b) as a total time derivative using the identities in Appendix A, which has already been presented in Ap-
pendix F of [25] and Appendix A of [26]. Taking into account the leading-order point-mass and spin contributions it holds that

\[ \left[ \mathcal{L}_{\leq 2.5PN} + \mathcal{L}_{\leq 3.5PN}^S + \mathcal{L}_{\leq 3.5PN}^{S_1 S_2} \right]_{\text{inst}} \]

\[ E_{3.5PN}^S = \frac{G^2 M^2 r^2}{r^4} (\mathbf{n} \cdot \mathbf{v}) \left[ (\mathbf{\hat{n}} \cdot \mathbf{S}) \left( \frac{28 GM}{15 r} + 8(\mathbf{n} \cdot \mathbf{v})^2 - \frac{32}{15} v^2 \right) \right. \]
\[ \left. + (\mathbf{\hat{n}} \cdot \mathbf{\xi}') \left( - \frac{2}{15} \frac{GM}{r} + 4(\mathbf{n} \cdot \mathbf{v})^2 - \frac{22}{5} v^2 \right) \right], \quad (5.16b) \]

\[ E_{3.5PN}^{S_1 S_2} = \frac{G^2 M^2 r^2}{r^4} \left[ \frac{(\mathbf{n} \cdot \mathbf{S}_1)(\mathbf{n} \cdot \mathbf{S}_2)}{(\mathbf{n} \cdot \mathbf{v})} \frac{GM}{r} \left( \frac{84(\mathbf{n} \cdot \mathbf{v})^2 - 52v^2 - \frac{44}{5} GM}{r} \right) \right. \]
\[ \left. + \frac{GM}{r} \left( \frac{\mathbf{n} \cdot \mathbf{S}_1}{(\mathbf{S}_2 \cdot \mathbf{v})} + \frac{\mathbf{n} \cdot \mathbf{S}_2}{(\mathbf{S}_1 \cdot \mathbf{v})} \right) \left( -22(\mathbf{n} \cdot \mathbf{v})^2 + \frac{38}{5} v^2 + \frac{22}{5} GM \right) \right] \]
\[ + \frac{16}{5} \frac{GM}{r} \left( \mathbf{S}_1 \cdot \mathbf{v} \right) \left( \mathbf{S}_2 \cdot \mathbf{v} \right) (\mathbf{n} \cdot \mathbf{v}) + \frac{16}{5} \frac{GM}{r} v^2 (\mathbf{n} \cdot \mathbf{v}) (\mathbf{S}_1 \cdot \mathbf{S}_2) \]. \quad (5.16c)

Note that even though the energy loss at 2.5PN order is spin-independent, it does need to be taken into account when comparing the spin-dependent energy losses because of the spin-dependent terms in Eqs. (A1a) and (A1b), which are of formal 1PN order.

It should be noted that the total time derivative on the right-hand side of Eq. (5.15) vanishes to the order \( n^2 \) when averaged over time. This means that the time average of near-zone energy loss and far-zone energy flux agree. Equations (5.16a) – (5.16c) should thus be interpreted as (gauge-dependent) energies that temporarily leave the near-zone, but never reach the far-zone and instead move back into the near-zone at a later time. Therefore, they have in average no effect on the near-zone energy loss.

VI. CONCLUSIONS AND OUTLOOK

Based on developments in [1] the leading-order PN spin-orbit and spin(1)-spin(2) radiation-reaction Hamiltonians were calculated. Corresponding equations of motion were already derived for the binary case in [25–27]. The Hamiltonians given in the present paper are even valid for arbitrary many spinning compact objects and present the dynamics in a compact form. The derivation was performed within the ADM canonical formalism [53], which was extended from point-masses to linear order in the single spin of the objects in [1, 12–14]. The calculation of the needed integrals and their regularization is analogous to calculations for nonspinning objects within the ADM formalism (see, e.g., [2, 8, 62]). In particular, we applied the Hadamard finite part and Riesz-formula based regularizations in the present paper (for the latter see also [63]). Some integrals were checked using Riesz kernels in arbitrary dimensions (see also [64]).

The leading-order spin-orbit and spin(1)-spin(2) energy loss was computed in the present paper from the explicit time derivative of the interaction Hamiltonian. This was compared to well-known results for the corresponding energy flux [19] as a check [In [1], the leading-order spin-orbit energy flux was rederived from the wave equation (3.9)]. This also proofs agreement with the energy loss obtained in the harmonic gauge [25, 26, 60] and thus provides an important check of the ADM canonical formalism for spinning objects, which was derived only very recently [1, 12–14]. Notice that the interaction Hamiltonian in the form of Eq. (3.3) also gives essential contributions to the next-to-next-to-leading-order conservative Hamiltonians [48, 49].

The spin-orbit radiation-reaction Hamiltonian derived in the present paper, which is at 3.5PN when counted in a formal way, is actually of the order 4PN for maximally rotating objects (see also Appendix A of [1]). A derivation of all spin-dependent 4PN Hamiltonians for maximally rotating objects should be envisaged in the future. The most complicated Hamiltonian at this level is the conservative next-to-next-to-leading-order spin(1)-spin(2) one, but it has already been derived very recently (see [49], and also [50] for a corresponding potential). Notice that all Hamiltonians for maximally rotating black holes are known to 3.5PN order [48].
Further, the leading-order spin-orbit and spin(1)-spin(2) radiation-reaction equations of motion can be obtained from the Hamiltonians derived in the present paper and compared with the results from [25–27] in the future. Primed and unprimed variables must be identified in the equations of motion, which requires further application of regularization techniques. Finally, one may transform the general equations of motion into secular equations of motion for the orbital elements, which has already been derived in [22–24] using energy and angular momentum balance.

ACKNOWLEDGMENTS

We thank P. Jaranowski for sharing his insight in the calculation of the 3.5PN point-mass Hamiltonian. J.S. is further grateful to M. Tessmer for useful discussions. This work is supported by the Deutsche Forschungsgemeinschaft (DFG) through SFB/TR7 “Gravitational Wave Astronomy,” project STE 2017/1-1, and GRK 1523, and by the FCT (Portugal) through PTDC project CTEAST/098034/2008.

Appendix A: Total time derivatives

The identities for total time derivatives needed to compare the instantaneous near-zone energy loss and the far-zone energy flux were already provided in Appendix F of [25] and Appendix A of [26]. They read:

\[
\frac{d}{dt} \left( \frac{v^{2s-2} \hat{p}^2}{r^q} \right) = \frac{v^{2s-2} \hat{p}^2 - 1}{r^{q+1}} \left\{ p \dot{v}^2 - (p + q)v^2 - 2s^2 \frac{GM}{r} - \frac{p}{2} \frac{Gv^2}{r^2} \hat{L}_N \cdot (4S + 3\xi) \right. \\
+ \frac{6sGv^2}{\mu r^3} \left[ \hat{r} (\mathbf{S}_1 \cdot \mathbf{S}_2) + (\mathbf{v} \cdot \mathbf{S}_1) (\mathbf{n} \cdot \mathbf{S}_2) + (\mathbf{v} \cdot \mathbf{S}_1) (\mathbf{n} \cdot \mathbf{S}_2) - 5\hat{r} (\mathbf{n} \cdot \mathbf{S}_1) (\mathbf{n} \cdot \mathbf{S}_2) \right] \right\},
\]

\[
\frac{d}{dt} \left( \frac{v^{2s} \hat{p}}{r^{q+1}} \hat{L}_N \right) = \frac{v^{2s-2} \hat{p}^2}{r^{q+1}} \left\{ p \dot{v}^2 - (p + q)v^2 - 2s^2 \frac{GM}{r} - \frac{p}{2} \frac{Gv^2}{r^2} \hat{L}_N \right. \\
+ \frac{p Gv^2}{2r^3} \hat{L}_N \cdot (4S + 3\xi) \right\} \hat{L}_N - \left( \frac{3Gv^2}{\mu r^2} \right) \left[ (\mathbf{n} \cdot \mathbf{S}_1) (\mathbf{n} \cdot \mathbf{S}_2) + (\mathbf{n} \cdot \mathbf{S}_1) (\mathbf{n} \cdot \mathbf{S}_2) \right] \hat{L}_N \\
+ \frac{6sGv^2}{\mu r^3} \left[ \hat{r} (\mathbf{S}_1 \cdot \mathbf{S}_2) + (\mathbf{v} \cdot \mathbf{S}_1) (\mathbf{n} \cdot \mathbf{S}_2) + (\mathbf{v} \cdot \mathbf{S}_2) (\mathbf{n} \cdot \mathbf{S}_1) - 5\hat{r} (\mathbf{n} \cdot \mathbf{S}_1) (\mathbf{n} \cdot \mathbf{S}_2) \right] \hat{L}_N \\
\left. - \frac{3p Gv^2}{\mu r^2} \left[ (\mathbf{n} \cdot \mathbf{S}_1) (\mathbf{n} \cdot \mathbf{S}_2) - 3(\mathbf{n} \cdot \mathbf{S}_1) (\mathbf{n} \cdot \mathbf{S}_2) \right] \hat{L}_N \right\},
\]

\[
\frac{d}{dt} \left( \frac{v^{2s} \hat{p}}{r^{q+1}} x^i x^j \right) = \frac{v^{2s-2} \hat{p}^2}{r^{q+1}} \left\{ p \dot{v}^2 - (p + q)v^2 - 2s^2 \frac{GM}{r} - \frac{p}{2} \frac{Gv^2}{r^2} \right\} \left[ x^i x^j + 2v^2 \hat{r} x^i x^j \right],
\]

\[
\frac{d}{dt} \left( \frac{v^{2s} \hat{p}}{r^{q+1}} v^i v^j \right) = \frac{v^{2s-2} \hat{p}^2}{r^{q+1}} \left\{ p \dot{v}^2 - (p + q)v^2 - 2s^2 \frac{GM}{r} - \frac{p}{2} \frac{Gv^2}{r^2} \right\} \left[ v^i v^j + \frac{v^2}{r^2} \right],
\]

\[
\frac{d}{dt} \left( \frac{v^{2s} \hat{p}}{r^{q+1}} v^i v^j \right) = \frac{v^{2s-2} \hat{p}^2}{r^{q+1}} \left\{ p \dot{v}^2 - (p + q)v^2 - 2s^2 \frac{GM}{r} - \frac{p}{2} \frac{Gv^2}{r^2} \right\} \left[ v^i v^j + \frac{v^2}{r^2} \hat{r} \left( v^i v^j - \frac{GM}{r} n^i n^j \right) \right],
\]

where \( \hat{r} \equiv (\mathbf{n} \cdot \mathbf{v}) \), and \( s, p, q \) are non-negative integers.

\footnote{There was a misprint in Eq. (A1) of [26], we made appropriate changes in the expression here and marked its position by \[\color{red}{\text{[....]}}\].}
[1] J. Steinhoff and H. Wang, “Canonical formulation of gravitating spinning objects at 3.5 post-Newtonian order,” Phys. Rev. D 81 (2010) 024022, arXiv:0910.1008 [gr-qc].

[2] P. Jaranowski and G. Schäfer, “Radiative 3.5 post-Newtonian ADM Hamiltonian for many-body point-mass systems,” Phys. Rev. D 55 (1997) 4712–4722.

[3] C. W. F. Everitt et al., “Gravity Probe B: Final results of a space experiment to test general relativity,” Phys. Rev. Lett. 106 (2011) 221101, arXiv:1105.3456 [gr-qc].

[4] R. P. Breton et al., “Relativistic spin precession in the double pulsar,” Science 321 (2008) 104–107, arXiv:0807.2644 [astro-ph].

[5] M. Pitkin, S. Reid, S. Rowan, and J. Hough, “Gravitational wave detection by interferometry (ground and space),” Living Rev. Relativity 14 (2011) 5, arXiv:1102.3355 [astro-ph.IM].

http://www.livingreviews.org/lrr-2011-5.

[6] B. S. Sathyaprakash and B. F. Schutz, “Physics, astrophysics and cosmology with gravitational waves,” Living Rev. Relativity 12 (2009) 2, arXiv:0903.0338 [gr-qc]. http://www.livingreviews.org/lrr-2009-2.

[7] J. E. McClintock et al., “Measuring the spins of accreting black holes,” Class. Quant. Grav. 28 (2011) 114009, arXiv:1101.0811 [astro-ph.HE].

[8] P. Jaranowski and G. Schäfer, “Third post-Newtonian higher order ADM Hamiltonian for two-body point-mass systems,” Phys. Rev. D 57 (1998) 7274–7291, arXiv:gr-qc/9712075.

[9] T. Damour, P. Jaranowski, and G. Schäfer, “Dimensional regularization of the gravitational interaction of point masses,” Phys. Lett. B 513 (2001) 147–155, arXiv:gr-qc/0105038.

[10] T. Ledvinka, G. Schäfer, and J. Bičák, “Relativistic closed-form Hamiltonian for many-body gravitating systems in the post-Minkowskian approximation,” Phys. Rev. Lett. 100 (2008) 251101, arXiv:0807.0214 [gr-qc].

[11] C. Kónigsdörffer, G. Faye, and G. Schäfer, “The binary black-hole dynamics at the third-and-a-half post-Newtonian order in the ADM-formalism,” Phys. Rev. D 68 (2003) 044004, arXiv:gr-qc/0305048.

[12] J. Steinhoff and G. Schäfer, “Canonical formulation of self-gravitating spinning-object systems,” Europhys. Lett. 87 (2009) 50004, arXiv:0907.1967 [gr-qc].

[13] J. Steinhoff, “Canonical formulation of spin in general relativity,” Ann. Phys. (Berlin) 523 (2011) 296–353, arXiv:1106.4203 [gr-qc].

[14] J. Steinhoff, G. Schäfer, and S. Herlt, “ADM canonical formalism for gravitating spinning objects,” Phys. Rev. D 77 (2008) 104018, arXiv:0805.3136 [gr-qc].

[15] S. Herlt and G. Schäfer, “Higher-order-in-spin interaction Hamiltonians for binary black holes from source terms of Kerr geometry in approximate ADM coordinates,” Phys. Rev. D 77 (2008) 104001, arXiv:0712.1515 [gr-qc].

[16] S. Herlt and G. Schäfer, “Higher-order-in-spin interaction Hamiltonians for binary black holes from Poicare invariance,” Phys. Rev. D 78 (2008) 124004, arXiv:0809.2208 [gr-qc].

[17] J. Steinhoff, S. Herlt, and G. Schäfer, “Spin-squared Hamiltonian of next-to-leading order gravitational interaction,” Phys. Rev. D 78 (2008) 101503(R), arXiv:0809.2200 [gr-qc].

[18] S. Herlt, J. Steinhoff, and G. Schäfer, “The reduced Hamiltonian for next-to-leading order spin-squared dynamics of general compact binaries,” Class. Quant. Grav. 27 (2010) 135007, arXiv:1002.2093 [gr-qc].

[19] L. E. Kidder, “Coalescing binary systems of compact objects to (post)5/2-Newtonian order. V. Spin effects,” Phys. Rev. D 52 (1995) 821–847, arXiv:gr-qc/9506022.

[20] L. Blanchet, A. Buonanno, and G. Faye, “Higher-order spin effects in the dynamics of compact binaries. II. Radiation field,” Phys. Rev. D 74 (2006) 104034, arXiv:gr-qc/0605140.

L. Blanchet, A. Buonanno, and G. Faye, “Erratum: Higher-order spin effects in the dynamics of compact binaries. II. Radiation field,” Phys. Rev. D 75 (2007) 049903(E).

L. Blanchet, A. Buonanno, and G. Faye, “Erratum: Higher-order spin effects in the dynamics of compact binaries. II. Radiation field,” Phys. Rev. D 81 (2010) 089901(E).

[21] R. A. Porto, A. Ross, and I. Z. Rothstein, “Spin induced multipole moments for the gravitational wave flux from binary inspirals to third post-Newtonian order,” JCAP (2011) no. 3, 009, arXiv:1007.1312 [gr-qc].

[22] L. Á. Gergely, Z. I. Perjés, and M. Vasúth, “Spin effects in gravitational radiation backreaction. III: Compact binaries with two spinning components,” Phys. Rev. D 58 (1998) 124001, arXiv:gr-qc/9808063.

L. Á. Gergely, “Spin effects in radiating compact binaries,” Phys. Rev. D 61 (1999) 024035, arXiv:gr-qc/9911082.

[23] L. Á. Gergely, “Second post-Newtonian radiative evolution of the relative orientations of angular momenta in spinning compact binaries,” Phys. Rev. D 62 (2000) 024007, arXiv:gr-qc/0003037.

[24] C. M. Will, “Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations. III. Radiation reaction for binary systems with spinning bodies,” Phys. Rev. D 71 (2005) 084027, arXiv:gr-qc/0502039.

[25] H. Wang and C. M. Will, “Post-Newtonian gravitational radiation and equations of motion via direct integration of the relaxed Einstein equations. IV. Radiation reaction for binary systems with spin-spin coupling,” Phys. Rev. D 75 (2007) 064017, arXiv:gr-qc/0701047.

[26] J. Zeng and C. M. Will, “Application of energy and angular momentum balance to gravitational radiation reaction for binary systems with spin-orbit coupling,” Gen. Relativ. Gravit. 39 (2007) 1661–1673, arXiv:0704.2720 [gr-qc].

[27] B. M. Barker and R. F. O’Connell, “Gravitational two-body problem with arbitrary masses, spins, and quadrupole moments,” Phys. Rev. D 12 (1975) 329–335.

[28] P. D. Eath, “Interaction of two black holes in the slow-motion limit,” Phys. Rev. D 12 (1975) 2183–2199.
[30] B. M. Barker and R. F. O’Connell, “The gravitational interaction: Spin, rotation, and quantum effects—a review,” Gen. Relativ. Gravit. 11 (1979) 149–175.

[31] K. S. Thorne and J. H. Hartle, “Laws of motion and precession for black holes and other bodies,” Phys. Rev. D 31 (1985) 1815–1837.

[32] K. S. Thorne, “Multipole expansions of gravitational radiation,” Rev. Mod. Phys. 52 (1980) 299–339.

[33] E. Poisson, “Gravitational waves from inspiraling compact binaries: The quadrupole-moment term,” Phys. Rev. D 57 (1998) 5287–5290, arXiv:gr-qc/9709032.

[34] L. Á. Gergely and Z. Keresztes, “Gravitational waves from inspiraling compact binaries: Contribution of the quadrupole-monopole interaction,” Phys. Rev. D 67 (2003) 024020, arXiv:gr-qc/0211027.

[35] T. Damour, A. Ohashi, and B. J. Owen, “Gravitational field and equations of motion of spinning compact binaries to 2.5 post-Newtonian order,” Phys. Rev. D 63 (2001) 044006, arXiv:gr-qc/0010014.

[36] G. Faye, L. Blanchet, and A. Buonanno, “Higher-order spin effects in the dynamics of compact binaries. I. Equations of motion,” Phys. Rev. D 74 (2006) 104033, arXiv:gr-qc/0605139.

[37] T. Damour, P. Jaranowski, and G. Schäfer, “Hamiltonian of two spinning compact bodies with next-to-leading order gravitational spin-orbit coupling,” Phys. Rev. D 77 (2008) 064032, arXiv:0711.1048 [gr-qc].

[38] J. Steinhoff, S. Hergt, and G. Schäfer, “Next-to-leading order gravitational spin(1)-spin(2) dynamics in Hamiltonian form,” Phys. Rev. D 77 (2008) 081501(R), arXiv:0712.1716 [gr-qc].

[39] D. L. Perrodin, “Subleading spin-orbit correction to the J. Hartung and J. Steinhoff, “Next-to-leading order gravitational radiation,” Rev. Mod. Phys. 52 (1980) 299–339.

[40] J. Steinhoff, S. Hergt, and G. Schäfer, “Next-to-leading order gravitational spin(1)-spin(2) Hamiltonian for n gravitating spinning compact objects,” Phys. Rev. D 83 (2011) 044008, arXiv:1011.1179 [gr-qc].

[41] R. A. Porto and I. Z. Rothstein, “Next to leading order spin(1)spin(1) effects in the motion of inspiralling compact binaries,” Phys. Rev. D 78 (2008) 044013, arXiv:0804.0260 [gr-qc].

[42] R. A. Porto and I. Z. Rothstein, “Erratum: Next to leading order spin(1)spin(1) effects in the motion of inspiralling compact binaries,” Phys. Rev. D 81 (2010) 029905(E).

[43] J. Steinhoff and G. Schäfer, “Comment on two recent papers regarding next-to-leading order spin-spin effects in gravitational interaction,” Phys. Rev. D 80 (2009) 088501, arXiv:0903.4772 [gr-qc].

[44] W. G. Laarakkers and E. Poisson, “Quadrupole moments of rotating neutron stars,” Astrophys. J. 512 (1999) 282–287, arXiv:gr-qc/9709033.

[45] J. Hartung and J. Steinhoff, “Next-to-next-to-leading order post-Newtonian spin-orbit Hamiltonian for self-gravitating binaries,” Ann. Phys. (Berlin) 523 (2011) 783–790, arXiv:1104.3079 [gr-qc].

[46] J. Hartung and J. Steinhoff, “Next-to-next-to-leading order post-Newtonian spin(1)-spin(2) Hamiltonian for self-gravitating binaries,” Ann. Phys. (Berlin) 523 (2011) 919–924, arXiv:1107.4294 [gr-qc].

[47] M. Levi, “Binary dynamics from spin1-spin2 coupling at fourth post-Newtonian order,” arXiv:1107.4322 [gr-qc].

[48] H. Goenner, U. Gralewski, and K. Westpfahl, “Gravitative Selbstkräfte und Strahlungsverluste klassischer Spinteilchen (erste Nähierung),” Z. Phys. 207 (1967) 186–208.

[49] F. Bennewitz and K. Westpfahl, “Selbstwechselwirkung von Gravitationsfeldern schnell bewegter Pol-Dipolquellen,” Commun. math. Phys. 23 (1971) 296–318.

[50] R. L. Arnowitt, S. Deser, and C. W. Misner, “The dynamics of general relativity,” in Gravitation: An Introduction to Current Research, L. Witten, ed., pp. 227–265. John Wiley, New York, 1962.

[51] R. L. Arnowitt, S. Deser, and C. W. Misner, “Republication of: The dynamics of general relativity,” Gen. Relativ. Gravit. 40 (2008) 1997–2027, arXiv:gr-qc/0405109.

[52] T. Regge and C. Teitelboim, “Role of surface integrals in the Hamiltonian formulation of general relativity,” Ann. Phys. (N.Y.) 88 (1974) 286–318.

[53] B. S. DeWitt, “Quantum theory of gravity. I. The canonical theory,” Phys. Rev. 160 (1967) 1113–1148.

[54] L. Blanchet and T. Damour, “Tail-transported temporal correlations in the dynamics of agravitating system,” Phys. Rev. D 37 (1988) 1410–1435.

[55] G. Schäfer, “The gravitational quadrupole radiation-reaction force and the canonical formalism of ADM,” Ann. Phys. (N.Y.) 161 (1985) 81–100.

[56] G. Schäfer, “The ADM Hamiltonian at the postlinear approximation,” Gen. Relativ. Gravit. 18 (1986) 255–270.

[57] M. Levi, “Next-to-leading order gravitational spin-spin coupling in an effective field theory approach,” Phys. Rev. D 82 (2010) 104004, arXiv:1006.4139 [gr-qc].

[58] R. A. Porto and I. Z. Rothstein, “Spin(1)spin(2) effects in the motion of inspiralling compact binaries at third order in the post-Newtonian expansion,” Phys. Rev. D 78 (2008) 044012, arXiv:0802.0720 [gr-qc].

[59] R. A. Porto and I. Z. Rothstein, “Erratum: Spin(1)spin(2) effects in the motion of inspiralling compact binaries at third order in the post-Newtonian expansion,” Phys. Rev. D 81 (2010) 029904(E).

[60] M. Levi, “Next-to-leading order gravitational spin1-spin2 coupling with Kaluza-Klein reduction,” Phys. Rev. D 82 (2010) 064029, arXiv:0802.1508 [gr-qc].

[61] J. Hartung and J. Steinhoff, “Next-to-next-to-leading order spin-orbit and spin(a)-spin(b) Hamiltonians for n gravitating spinning compact objects,” Phys. Rev. D 83 (2011) 044008, arXiv:1011.1179 [gr-qc].
post-Newtonian order,” *Phys. Rev. D* **65** (2002) 104008, [arXiv:gr-qc/0201001](https://arxiv.org/abs/gr-qc/0201001).

[61] B. R. Iyer and C. M. Will, “Post-Newtonian gravitational radiation reaction for two-body systems,” *Phys. Rev. Lett.* **70** (1993) 113–116.

[62] P. Jaranowski, “Technicalities in the calculation of the 3rd post-Newtonian dynamics,” in *Mathematics of Gravitation, Part II: Gravitational Wave Detection*, A. Królak, ed., pp. 55–63. Banach Center Publications, Vol. 41, Part II, Warszawa, 1997.

[63] M. Riesz, “L’intégrale de Riemann-Liouville et le problème de Cauchy,” *Acta Math.* **81** (1949) 1–222. M. Riesz, “Erratum: L’intégrale de Riemann-Liouville et le problème de Cauchy,” *Acta Math.* **81** (1949) 223(E).

[64] T. Damour, P. Jaranowski, and G. Schäfer, “Dimensional regularization of the gravitational interaction of point masses in the ADM formalism,” in *Proceedings of the 11th Marcel Grossmann Meeting on General Relativity*, H. Kleinert, R. T. Jantzen, and R. Ruffini, eds., p. 2490. World Scientific, Singapore, 2008. [arXiv:0804.2386 [gr-qc]].