On admissible memory kernels for random unitary qubit evolution

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We analyze random unitary evolution of the qubit within memory kernel approach. We provide sufficient conditions which guarantee that the corresponding memory kernel generates physically legitimate quantum evolution. Interestingly, we are able to recover several well known examples and generate new classes of nontrivial qubit evolution. Surprisingly, it turns out that a class of quantum evolution with memory kernel generated by our approach gives rise to vanishing non-Markovianity measure based on the distinguishability of quantum states.

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I. INTRODUCTION

Dynamics of open quantum systems plays important role in the analysis of various phenomena like dissipation, decoherence and dephasing [1, 2]. The usual approach to the dynamics of an open quantum system consists of applying the Born-Markov approximation [1] which leads to the dynamics of an open quantum system consists of decoherence and dephasing [1, 2]. The usual approach role in the analysis of various phenomena like dissipation, semigroup defined as follows

\[ L \]

where \( \rho \) is the density matrix of the system investigated and \( L \) the time-independent generator of the dynamical semigroup defined as follows

\[ L[\rho] = -i[H, \rho] + \frac{1}{2} \sum_\alpha \left( [V_\alpha, [V_\alpha^\dagger, \rho]] + [V_\alpha, V_\alpha^\dagger] \right), \tag{2} \]

where \( H \) denotes the effective system Hamiltonian, and \( V_\alpha \) represent noise operators [3, 4]. We call [2] the GKSL form (Gorini-Kossakowski-Sudarshan-Lindblad). The solution to (1) defines Markovian semigroup

\[ \dot{\rho}_t = L \rho_t, \tag{1} \]

where \( \rho_t \) is the density matrix of the system investigated and \( L \) the time-independent generator of the dynamical semigroup defined as follows

\[ L[\rho] = -i[H, \rho] + \frac{1}{2} \sum_\alpha \left( [V_\alpha, [V_\alpha^\dagger, \rho]] + [V_\alpha, V_\alpha^\dagger] \right), \tag{2} \]

where \( \rho \) is an initial state. The dynamical map \( \Lambda_t = e^{tL} \) is completely positive and trace-preserving (CPTP) [1, 2]. Born-Markov approximation assumes weak interaction and a separation of time scales between the system and the environment. Such approach works perfectly for many quantum optical systems [6–8]. When the above assumption are no longer legitimate the description based on (11) is not satisfactory. Recent technological progress and modern laboratory techniques call for more refine approach which takes into account memory effects which are completely neglected in the description based on Markovian semigroup. In recent years we faced an intense activity in the field of non-Markovian quantum evolution (see the recent review [9], the collection of papers in [10] and recent comparative analysis in [11]).

There are basically two approaches which generalize the standard Markovian master equation [1]: time-local approach replaces \( L \) by time-dependent generator \( L_t \). Interestingly, if for all \( t \) the time-dependent generator has the standard GKSL form [8], then \( \Lambda_t = \mathcal{T} \exp \left( \int_0^t L_u du \right) \) defines so called divisible dynamical map [12, 13] which is often considered as the generalization of Markovianity (see [14] for generalization of the notion of divisibility). The second approach is based on the non-local Nakajima-Zwanzig [15] (see also [16])

\[ \dot{\rho}_t = \int_0^t K_{t-\tau} \rho_\tau d\tau, \tag{4} \]

in which quantum memory effects are taken into account through the introduction of the memory kernel \( K_t \). It means that the rate of change of the state \( \rho_t \) at time \( t \) depends on its history (starting at \( t = 0 \)). The Markovian master Eq. (1) is reobtained when \( K_t = 2 \delta(t)L \). The time-dependent kernel is usually referred to as the generator of the non-Markovian master equation. Equation (4) applies to a variety of situations (see eg. [17]). Because of the convolution structure of (4) the time-local approach is often called time-convolutionless [1, 18, 19]. The structure and the properties of (4) were carefully analyzed in [20–22]. In particular the generalization of Markovian evolution to so called semi-Markov was investigated within memory kernel approach by Budini [21] and Breuer and Vacchini [22] (see also discussion in [23]).

In the present paper we study random unitary evolution of the qubit within memory kernel approach. In particular we address the following problem: what is the structure of the corresponding memory kernel \( K_t \) which leads to the legitimate CPTP dynamical map \( \Lambda_t \). The paper has the following structure: in the next section we recall basic facts about random unitary evolution and in Section III we formulate the sufficient condition for \( K_t \) which guaranties legitimate physical evolution. In Section IV we examine the issue of Markovianity. Surprisingly, it turns out that a subclass of quantum evolution with memory kernel generated by our approach gives rise to vanishing non-Markovianity measure based on the distinguishability of quantum states [30]. Section V illustrates our approach by several examples. Final conclusions are collected in Section VI.
II. RANDOM UNITARY QUBIT EVOLUTION

A quantum channel $\mathcal{E} : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is called random unitary if its Kraus representation is given by

$$\mathcal{E}[X] = \sum_k p_k U_k X U_k^\dagger,$$

where $U_k$ is a collection of unitary operators and $p_k$ stands for a probability distribution. In this paper we consider a random unitary dynamical map $\Lambda_t$ defined by

$$\Lambda_t[\rho] = \sum_{\alpha=0}^3 p_\alpha(t) \sigma_\alpha \rho \sigma_\alpha,$$

where $\sigma_\alpha$ are Pauli matrices with $\sigma_0 = I_2$. Initial condition $\Lambda_{t=0} = I$ implies $p_\alpha(0) = \delta_{0}\alpha$. Recently a time-local description based on the following master equation was analyzed \[31]\[32]

$$\dot{\Lambda}_t = L_t \Lambda_t,$$

where $L_t$ is a time-local generator defined by

$$L_t[\rho] = \sum_{k=1}^3 \gamma_k(t) (\sigma_k \rho \sigma_k - \rho),$$

with time-dependent decoherence rates $\gamma_k(t)$. One asks the following question: what are the conditions for $\gamma_k(t)$ which guarantee that the solution $\Lambda_t = \exp(\int_0^t L_t d\tau)$ provides a legitimate dynamical map? Note, that the solution defines random unitary evolution with $p_\alpha(t)$ given by

$$p_\alpha(t) = \frac{1}{4} \sum_{\beta=0}^3 H_{\alpha\beta} \lambda_\beta(t),$$

where $H_{\alpha\beta}$ is the Hadamard matrix

$$H = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{pmatrix},$$

and $\lambda_\beta(t)$ are time-dependent eigenvalues of $\Lambda_t$

$$\Lambda_t[\sigma_\alpha] = \lambda_\alpha(t) \sigma_\alpha,$$

read as follows $\Lambda_0(t) = 1$ and

$$\lambda_1(t) = \exp(-2[\Gamma_2(t) + \Gamma_3(t)]),$$
$$\lambda_2(t) = \exp(-2[\Gamma_1(t) + \Gamma_3(t)]),$$
$$\lambda_3(t) = \exp(-2[\Gamma_1(t) + \Gamma_2(t)]),$$

with $\Gamma_k(t) = \int_0^t \gamma_k(\tau) d\tau$. Now, the map \[40\] is CP iff $p_\alpha(t) \geq 0$ which is equivalent to the following set of conditions for $\lambda_\beta$ \[31]\[32]

$$1 + \lambda_1(t) + \lambda_2(t) + \lambda_3(t) \geq 0,$$

and

$$\lambda_1(t) + \lambda_2(t) \leq 1 + \lambda_3(t),$$
$$\lambda_3(t) + \lambda_1(t) \leq 1 + \lambda_2(t),$$
$$\lambda_2(t) + \lambda_3(t) \leq 1 + \lambda_1(t).$$

III. CONSTRUCTION OF LEGITIMATE MEMORY KERNELS

In this paper we analyze non-local description based on the following memory kernel equation

$$\dot{\Lambda}_t = \int_0^t K_{t-\tau} \Lambda_{\tau} d\tau,$$

with

$$K_t[\rho] = \sum_{i=1}^3 k_i(t) (\sigma_i \rho \sigma_i - \rho),$$

where $k_i(t)$ ($i = 1, 2, 3$) represent non-trivial memory effects. Note, that equation \[14\] considerably simplifies after perming the Laplace transform

$$\dot{\Lambda}_s = \frac{1}{s-K_s},$$

where $\Lambda_s := \int_0^\infty e^{-s t} \Lambda(t) d\tau$ and similarly for $K_s$. The question we address is: what are the conditions for $k_i(t)$ which guarantee that the solution $\Lambda_t$ provides a legitimate dynamical map?

Denoting by $\kappa_\alpha(t)$ the eigenvalues of $K_t$ $K_t[\sigma_\alpha] = \kappa_\alpha(t) \sigma_\alpha$, \[18\] equation \[15\] gives rise to the following set of equations

$$\dot{\lambda}_i(t) = \int_0^t \kappa_i(t-\tau) \lambda_i(\tau) d\tau, \quad i = 1, 2, 3.$$

Note, that $\kappa_0(t) = 0$ and hence $\lambda_0(t) = 1 = \text{const.}$ In terms of the Laplace transforms $\tilde{\lambda}_i(s)$ and $\tilde{\kappa}_i(s)$ one finds

$$\tilde{\lambda}_i(s) = \frac{1}{s - \tilde{\kappa}_i(s)}.$$

In terms of $\tilde{\lambda}_i(s)$ conditions \[18\] may be equivalently reformulated as follows:

$$\frac{1}{s} + \tilde{\lambda}_1(s) + \tilde{\lambda}_2(s) + \tilde{\lambda}_3(s) \quad \text{is CM},$$

and

$$\frac{1}{s} + \tilde{\lambda}_3(s) - \tilde{\lambda}_1(s) - \tilde{\lambda}_2(s) \quad \text{is CM},$$
$$\frac{1}{s} + \tilde{\lambda}_2(s) - \tilde{\lambda}_1(s) - \tilde{\lambda}_3(s) \quad \text{is CM},$$
$$\frac{1}{s} + \tilde{\lambda}_1(s) - \tilde{\lambda}_3(s) - \tilde{\lambda}_2(s) \quad \text{is CM},$$

where CM stands for a completely monotone function \[34\], i.e. a smooth function $f : [0, \infty) \to \mathbb{R}$ satisfying the following condition:

$$(-1)^n \frac{d^n}{ds^n} f(s) \geq 0, \quad s \geq 0, \quad n = 0, 1, 2, \ldots$$

The equivalence of \[14\] and \[22\] follows from the following
Theorem 1 (Bernstein’s Theorem) A function \( f: [0, \infty) \to \mathbb{R} \) is completely monotone on \([0, \infty)\) if and only if it is the Laplace transform of a finite non-negative Borel measure \( \mu \) on \([0, \infty)\), i.e. \( f \) is of the form

\[
    f(s) = \int_0^\infty e^{-st} d\mu(t). \tag{24}
\]

Note that initial condition \( p_0(0) = 1 \) and \( p_k(0) = 0 \) for \( k = 1, 2, 3 \) is equivalent to \( \Lambda_k(0) = 1 \) due to

Theorem 2 (Initial Value Theorem) Let \( \tilde{f}(s) \) be the Laplace transform of \( f(t) \). Then the following relation is true:

\[
    \lim_{t \to 0} f(t) = \lim_{s \to \infty} s \tilde{f}(s) \tag{25}
\]

it is equivalent to

\[
    \lim_{s \to \infty} s \tilde{\lambda}_k(s) = 1, \tag{26}
\]

for \( k = 1, 2, 3 \). This way we proved

Theorem 3 The map \( \tilde{\Lambda}_s \) represented by the following spectral decomposition

\[
    \tilde{\Lambda}_s[\rho] = \frac{1}{2} \sum_{\alpha=0}^{3} \tilde{\lambda}_\alpha(s) \sigma_\alpha \text{tr}[\sigma_\alpha \rho], \tag{27}
\]

with \( \tilde{\lambda}_0(s) = 1/s \), defines the Laplace transform of the legitimate map \( \Lambda_1 \) if and only if conditions \([21], [22]\) and \([26]\) are satisfied.

It is worth to emphasise that there are not many analytical tools for dealing with CM functions, which is due to the fact that an infinite set of conditions \((23)\) must be verified. Nevertheless, we found an important class of CM functions giving rise to CPTP dynamics with a straightforward interpretation. To present them let us first observe that CM functions have the following two properties, which will not be proved:

Property 1 Let \( f \) and \( g \) be arbitrary completely monotone functions. Then

1. \( f \cdot g \) is CM,
2. \( \alpha f + \beta g \) is CM for any \( \alpha, \beta > 0 \),

Property 2 If \( s_0 \geq 0 \) then \( s + s_0 \) is CM.

We are now ready to prove our main result:

Theorem 4 Let \( W(s) \) be a function such that \( \frac{1}{s W(s)} \) is CM. Then the functions

\[
    \tilde{k}_k(s) = -\frac{s}{ak W(s) - 1}, \quad k = 1, 2, 3, \tag{28}
\]

with \( a_1, a_2, a_3 > 0 \) such that

\[
    \frac{1}{s} \left( 4 - \frac{1}{W(s)} \left[ \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right] \right) \text{ is CM}, \tag{29}
\]

and

\[
    \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \geq \frac{1}{a_1}, \quad \frac{1}{a_2} + \frac{1}{a_3} \geq \frac{1}{a_1}, \quad \frac{1}{a_3} + \frac{1}{a_1} \geq \frac{1}{a_2}, \tag{30}
\]

define a legitimate memory kernel

\[
    \tilde{K}_k[\rho] = \frac{1}{2} \sum_{k=1}^{3} \tilde{k}_\alpha(s) \sigma_k \text{tr}[\sigma_k \rho], \tag{31}
\]

i.e. the corresponding \( \tilde{\lambda}_k(s) \) satisfy \([21], [22]\) and \([26]\).

Proof: note that formula \([26]\) implies

\[
    \tilde{\lambda}_k(s) = \frac{1}{s} \left( 1 - \frac{1}{a_k W(s)} \right), \tag{32}
\]

and hence

\[
    \frac{1}{s} + \tilde{\lambda}_3(s) - \tilde{\lambda}_1(s) - \tilde{\lambda}(s) = \frac{1}{s} W(s) \left( \frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_3} \right), \tag{33}
\]

which proves that \( \frac{1}{s} + \tilde{\lambda}_3(s) - \tilde{\lambda}_1(s) - \tilde{\lambda}_2(s) \) is CM due to the fact that \( \frac{1}{s} W(s) \) is CM. Similarly one proves the remaining conditions \([14]\).

Note, that since \( \frac{1}{s} W(s) \) is CM hence due to the Bernstein theorem it is the Laplace transform of a positive function. Hence

\[
    W(s) = \frac{1}{f(s)}, \tag{34}
\]

where \( \tilde{f}(s) \) is the Laplace transform of \( f(t) \) satisfying \( \int_0^t f(\tau) d\tau \geq 0 \) for all \( t \geq 0 \). One finds

\[
    \tilde{k}_k(s) = -s \tilde{f}(s) \frac{a_k}{a_k - \tilde{f}(s)} . \tag{35}
\]

Note, that condition \([29]\) implies

\[
    \left( \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right) \int_0^t f(\tau) d\tau \leq 4 . \tag{36}
\]

Hence to summarize: our class is characterized by a single function \( f(t) \) and three numbers \( a_1, a_2, a_3 > 0 \) such that \( F(t) = \int_0^t f(\tau) d\tau \geq 0 \) and conditions \([34]\) and \([36]\) hold. One finds for \( p_\alpha(t) \):

\[
    p_1(t) = \frac{1}{4} \left( \frac{1}{a_2} + \frac{1}{a_3} - \frac{1}{a_1} \right) F(t) , \tag{37}
\]

\[
    p_2(t) = \frac{1}{4} \left( \frac{1}{a_3} + \frac{1}{a_1} - \frac{1}{a_2} \right) F(t) , \tag{38}
\]

\[
    p_3(t) = \frac{1}{4} \left( \frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_3} \right) F(t) , \tag{39}
\]
and \( p_0(t) = 1 - p_1(t) - p_2(t) - p_3(t) \). In particular taking \( a_1 = a_2 = a \) and \( a_3 = \infty \) one finds

\[
\tilde{k}_1(s) = \tilde{k}_2(s) = \frac{-sf(s)}{a - f(s)} , \quad \tilde{k}_3(s) = 0 , \tag{38}
\]

and hence

\[
\tilde{k}_1(s) = \tilde{k}_2(s) = 0 , \quad \tilde{k}_3(s) = \frac{1}{2a} s f(s) , \tag{39}
\]
gives rise to the legitimate memory kernel

\[
K_i[\rho] = k_3(t)(\sigma_3 \rho \sigma_3 - \rho) , \tag{40}
\]

with arbitrary \( f(t) \) and \( a > 0 \) satisfying additional condition

\[
0 \leq F(t) := \int_0^t f(\tau) d\tau \leq 2a , \tag{41}
\]

for all \( t \geq 0 \). The corresponding solution reads

\[
p_0(t) = 1 - \frac{1}{2a} F(t) , \tag{42}
p_1(t) = p_2(t) = 0 , \tag{42}
p_3(t) = \frac{1}{2a} F(t) .
\]

This approach resembles very much the semi-Markov construction \[23, 28\]: for any \( f(t) \geq 0 \) satisfying \( \int_0^\infty f(t) dt \leq 1 \) the memory kernel \[41\] with

\[
\tilde{k}_3(s) = \frac{s f(s)}{1 - f(s)} , \tag{43}
\]
gives rise to CPTP evolution. In this case one finds

\[
p_0(t) = \frac{1}{2} [1 + \lambda_1(t)] , \tag{44}
p_1(t) = p_2(t) = 0 , \tag{44}
p_3(t) = \frac{1}{2} [1 - \lambda_1(t)] , \tag{44}
\]

where

\[
\lambda_1(t) = \lambda_2(s) = \frac{\tilde{f}(s) + 1}{f(s) - 1} . \tag{45}
\]

It is therefore clear that our approach goes beyond the semi-Markov construction.

Let us recall that Markovian semigroup generated by

\[
L[\rho] = \frac{1}{2} \sum_{k=1}^{3} \gamma_k [\sigma_k \rho \sigma_k - \rho] , \tag{46}
\]

the corresponding Bloch equation reads

\[
\dot{x}_k(t) = -\frac{1}{T_k} x_k(t) , \tag{47}
\]

where \( x_k := \text{tr}[\rho \sigma_k] \) and the relaxation times are defined via

\[
T_1 = \frac{1}{\gamma_2 + \gamma_3} , \quad T_2 = \frac{1}{\gamma_3 + \gamma_1} , \quad T_3 = \frac{1}{\gamma_1 + \gamma_2} . \tag{48}
\]

It is well known \[4\] that complete positivity is equivalent to the following set of conditions upon \( T_k \):

\[
\frac{1}{T_1} + \frac{1}{T_2} \geq \frac{1}{T_3} , \quad \frac{1}{T_2} + \frac{1}{T_3} \geq \frac{1}{T_1} , \tag{49}
\]

\[
\frac{1}{T_3} + \frac{1}{T_1} \geq \frac{1}{T_2} ,
\]

It is therefore clear that condition \[40\] is an analog of \[49\]. Note that condition \[30\] means that there exist \( b_1, b_2, b_3 > 0 \) such that

\[
\frac{1}{2a_1} = \frac{1}{b_1} + \frac{1}{b_3} , \tag{50}
\]

\[
\frac{1}{2a_2} = \frac{1}{b_3} + \frac{1}{b_1} , \tag{50}
\]

\[
\frac{1}{2a_3} = \frac{1}{b_1} + \frac{1}{b_2} .
\]

Now, it terms of \( b_1, b_2, b_3 \) our result may be reformulated as follows

**Corollary 1** For any \( b_1, b_2, b_3 > 0 \) and the function \( f(t) \) satisfying

\[
0 \leq F(t) := \int_0^t f(\tau) d\tau \leq \left( \frac{1}{b_1} + \frac{1}{b_2} + \frac{1}{b_3} \right)^{-1} , \tag{51}
\]

and

\[
\lim_{s \to \infty} \tilde{f}(s) = 0 , \tag{52}
\]

the memory kernel defined by

\[
\tilde{k}_k(s) = -\frac{s f(s)}{a_k f(s)} , \tag{53}
\]

defines legitimate quantum evolution. Moreover one has

\[
p_k(t) = \frac{1}{b_k} F(t) , \tag{54}
\]

and \( p_0(1) = 1 - p_1(t) - p_2(t) - p_3(t) \).

Let us observe that it is in general very hard to invert formula \[45\] to the time domain. Now, we provide a family of \( W(s) \) which enables one to easily compute \( \kappa_i(t) \) and hence the memory kernel in the time domain.

**Theorem 5** Let \( W(s) \) be a polynomial

\[
W(s) = (s + z_1) \cdots (s + z_n) , \tag{55}
\]
with \( z_i > 0 \). If \( a_1, a_2, a_2 \) satisfy \((30)\) and

\[
\prod_{i=1}^{n} z_i \geq \frac{1}{4} \left( \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right),
\]

(56)

then \( \kappa_i(t) \) defined via \((28)\) define a legitimate memory kernel.

Proof: it is clear that it is enough to prove \((21)\).

**Lemma 1** One has the following decomposition

\[
\frac{1}{s \prod_{i=1}^{n} (s + z_i)} = A \left( \frac{1}{s} \sum_{i=1}^{n} \prod_{j=i+1}^{n} (s + z_j) \right),
\]

(57)

where

\[
A = \frac{1}{\prod_{i=1}^{n} z_i}.
\]

(58)

For the proof see Appendix. Now we show that condition \((21)\) holds. According to \((57)\) one has

\[
\frac{1}{s} + \tilde{\lambda}_1(s) + \tilde{\lambda}_2(s) + \tilde{\lambda}_3(s)
= \frac{1}{s} \left( 4 - \left[ \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right] W(s) \right).
\]

\[
= \frac{1}{s} \left( 4 - \left[ \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right. \prod_{i=1}^{n} (s + z_i) \right)
\]

(59)

\[
+ \left[ \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right. \prod_{i=1}^{n} \sum_{j=i+1}^{n} \prod_{k=j+1}^{n} (s + z_j).
\]

A second term in \((59)\) is CM due to the fact that it is a sum of CM functions. Hence, if condition \((56)\) is satisfied then \((21)\) holds.

Note, that

\[
\tilde{\kappa}_i(s) = - \frac{s}{a_k W(s) - 1} = - \frac{1}{a_k} \frac{s}{(s - s_1)(s - s_m)},
\]

(60)

where \( \{s_1, \ldots, s_m\} \) are roots of the polynomial \( a_k W(s) - 1 \). It is therefore clear that formula \((60)\) may be easily inverted to the time domain.

**Remark 1** Note, that \( W(s) \) defined in \((55)\) implies that \( W(s) \) is CM and hence \( \frac{1}{s W(s)} \) is CM as well.

**IV. CHECKING FOR NON-MARKOVIANITY**

Let us recall that according to \((30)\) the evolution represented by \( \Lambda_t \) is non-Markovian if the condition

\[
\frac{d}{dt} ||\Lambda_t[\rho_1 - \rho_2]||_{tr} \leq 0,
\]

(61)

is violated for some initial states \( \rho_1 \) and \( \rho_2 \). One defines \((30)\) well known non-Markovianity measure

\[
\mathcal{N}_{\text{BLP}}[\Lambda_t] = \sup_{\rho_1, \rho_2} \int \frac{d}{dt} ||\Lambda_t[\rho_1 - \rho_2]||_{tr} dt,
\]

(62)

where the integral is performed over the region where \( \frac{d}{dt} ||\Lambda_t[\rho_1 - \rho_2]||_{tr} > 0 \). Now, it was proved \((31)\) that for random unitary qubit evolution if all eigenvalues \( \lambda_k(t) \geq 0 \), then \((61)\) is equivalent to

\[
\frac{d}{dt} \lambda_k(t) \leq 0; \quad k = 1, 2, 3.
\]

(63)

**Proposition 1** For \( W(s) = (s + z_1) \ldots (s + z_n) \) and \( a_1, a_2, a_3 \) satisfying \((30)\) together with

\[
\prod_{i=1}^{n} z_i \geq \frac{1}{a_k}, \quad k = 1, 2, 3,
\]

(64)

the corresponding memory kernel gives rise to the dynamical map \( \Lambda_t \) such that \( \mathcal{N}_{\text{BLP}}[\Lambda_t] = 0 \).

Proof: let us observe that \((64)\) implies \((60)\) and hence it proves that the evolution is CPTP. Indeed, one has from \((64)\)

\[
3 \prod_{i=1}^{n} z_i \geq \left( \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \right),
\]

(65)

and hence \((56)\) follows. Now, using Lemma \(1\) one finds that condition \((64)\) guarantees that \( \tilde{\lambda}_k(s) \) is CM which is equivalent to \( \lambda_k(t) \geq 0 \). Now, \( \frac{d}{dt} \lambda_k(t) \leq 0 \) if and only if \( 1 - s \lambda_k(s) \) is CM and hence taking into account \((20)\) it is equivalent to the requirement that \( -\tilde{\kappa}_k(s) \lambda_k(s) \) is CM. One has therefore

\[
- \tilde{\kappa}_k(s) \lambda_k(s) = \frac{1}{a_k W(s)},
\]

(66)

which ends the proof since \( \frac{1}{s W(s)} \) is CM and \( a_k > 0 \) \( \square \)

**Remark 2** It was shown \((14), (33)\) that BLP condition \((67)\) is equivalent to so called P-divisibility. It means that

\[
\Lambda_t = V_{t,s} \Lambda_s,
\]

(67)

and for any \( t > s \) the propagator \( V_{t,s} \) is positive (but not necessarily completely positive).

Interestingly, our construction provides a class of legitimate random unitary qubit evolution generated by the non-trivial memory kernel but still satisfying BLP condition \((61)\). It is clear that to violate \((61)\) one needs a more refine construction such that \( \frac{1}{W(s)} \) is not CM but \( \frac{1}{s \prod_{i=1}^{n} (s + z_i)} \) is already CM. It deserves further analysis.

**V. EXAMPLES**

**Example 1** Consider the simplest case with polynomial of degree one

\[
W(s) = s + z,
\]

(68)
with $z > 0$. One finds

$$\tilde{\kappa}_k(s) = -\frac{s}{a_k(s + z) - 1},$$

and the inverse Laplace transform gives

$$\kappa_k(t) = -\frac{1}{z}\left(\delta(t) - \left[z - \frac{1}{a_k}\right]e^{-\left[z - \frac{1}{a_k}\right]t}\right).$$

Note, that if $a_k = 1/z$, then the dynamics is purely local. One easily finds

$$\lambda_k(t) = 1 - \frac{1}{z a_k}(1 - e^{-z t}),$$

and finally the solution for $p_k(t)$ is defined by (79) with

$$F(t) = \frac{1}{z}(1 - e^{-z t}) \ .$$

Note, that condition (77) implies the following relation between $z$ and $a_1, a_2, a_3$:

$$4z \geq \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} ,$$

which guarantees that $p_0(t) \geq 0$. In the symmetric case $a_1 = a_2 = a_3 = a$ one finds $p_1(t) = p_2(t) = p_3(t) =: p(t)$ with

$$p(t) = \frac{1}{4za}[1 - e^{-z t}],$$

and $p_0(t) = 1 - 3p(t)$ with $4za \geq 3$. One finds that asymptotically

$$p_0(t) \to 1 - \frac{3}{4za} .$$

Note that for $za > 1$ one has asymptotically $p_0(\infty) < 1/4$. This property cannot be reproduced within the local approach with regular generators $L_1$. Indeed, it follows from (8) (see also [31] for more details)

$$p_0(t) = \frac{1}{4}\left[1 + \lambda_1(t) + \lambda_2(t) + \lambda_3(t)\right],$$

and hence, using (14), one finds

$$p_0(t) \geq \frac{1}{4}.$$

This example shows that local and memory kernel approaches may lead to essentially different evolution.

Example 2 Consider now the same polynomial $W(s) = s + z$ but let $z = 2c > 0$. Moreover

$$a_1 = a_2 = \frac{1}{c}, \ a_3 = \frac{1}{2c} .$$

One finds

$$\tilde{\kappa}_1(s) = \tilde{\kappa}_2(s) = -\frac{s c}{s + c}, \ \tilde{\kappa}_3(s) = -2c,$$

and hence

$$\kappa_1(t) = \kappa_2(t) = -c\delta(t) + c^2 e^{-ct}, \ \kappa_3(t) = -2c\delta(t).$$

Finally, one finds the following formula for the memory kernel

$$K_1[\rho] = \frac{c}{2}[\delta(t)[\sigma_1 \rho \sigma_1 + \sigma_2 \rho \sigma_2 - 2\rho]$$

$$- \frac{c^2}{2} e^{-ct}[\sigma_3 \rho \sigma_3 - \rho].$$

One has

$$\lambda_1(t) = \lambda_2(t) = \frac{1}{2}(1 + e^{-2ct}), \ \lambda_3(t) = e^{-2ct} .$$

Interestingly, this evolution reproduces time-local description with

$$\gamma_1(t) = \gamma_2(t) = \frac{c}{2}, \ \gamma_3(t) = -\frac{c}{2} \tanh(ct) .$$

as discussed in [32]. It was shown [33] that $\Lambda_i$ is a convex combination of two Markovian semigroups $\Lambda_1^{(1)}$ and $\Lambda_1^{(2)}$ generated by

$$L_k[\rho] = \frac{c}{2} [\rho \sigma_k - \rho]; \ k = 1, 2 .$$

that is,

$$\Lambda_i = \frac{1}{2} \left(e^{tL_1} + e^{tL_2}\right) .$$

This simple example shows that convex combination of Markovian semigroups leads to the quantum evolution displaying essential memory effects.

Example 3 Consider now the polynomial of degree two

$$W(s) = (s + c_1)(s + c_2),$$

with $c_2 > c_1 > 0$. Our construction gives rise to the legitimate memory kernel if condition (39) holds and

$$4c_1c_2 \geq \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3}. $$

One finds

$$\tilde{\kappa}_k(s) = -\frac{1}{a_k}\frac{s}{(s + c_1)(s + c_2) - \frac{1}{a_k}}$$

$$= -\frac{1}{a_k}\frac{s}{(s + s_1)(s + s_2)},$$

with

$$s_1 + s_2 = c_1 + c_2, \ s_1s_2 = c_1c_2 - \frac{1}{a_k}. $$

Hence the solution has the form (77) with the function $F(t)$ given by

$$F(t) = \frac{1}{c_2 - c_1}\left(\frac{1}{c_1}[1 - e^{-c_1t}] - \frac{1}{c_2}[1 - e^{-c_2t}]\right).$$
Example 4 Let

\[ W(s) = s^2 + \omega^2. \]  

(87)

Note that \( \frac{1}{s} W(s) \) is CM since

\[ \frac{1}{s} \frac{1}{W(s)} = \frac{1}{\omega} \left( \frac{\omega}{s^2 + \omega^2} \right), \]

is the Laplace transform of \( \int_0^t \sin(\omega t) d\tau \) which is positive for all \( t \geq 0 \). Condition (29) implies

\[ \frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} \leq 2\omega^2. \]

(88)

The corresponding eigenvalues of the memory kernel read

\[ \kappa_i(t) = -\frac{1}{a_i} \cos \left( \sqrt{\frac{\omega^2 - 1}{a_i}} t \right), \]

(89)

for \( \omega^2 \geq 1/a_i \), and

\[ \kappa_i(t) = -\frac{1}{a_i} \cosh \left( \sqrt{\frac{1}{a_i} - \omega^2} t \right), \]

(90)

for \( \omega^2 < 1/a_i \). Moreover one finds

\[ \lambda_k(t) = 1 + \frac{1}{a_1 a_3} [\cos(\omega t) - 1], \]

(91)

and hence

\[ p_1(t) = \frac{1}{4\omega^2} \left( \frac{1}{a_2} + \frac{1}{a_3} - \frac{1}{a_1} \right) [1 - \cos(\omega t)], \]

\[ p_2(t) = \frac{1}{4\omega^2} \left( \frac{1}{a_3} + \frac{1}{a_1} - \frac{1}{a_2} \right) [1 - \cos(\omega t)], \]

(92)

\[ p_3(t) = \frac{1}{4\omega^2} \left( \frac{1}{a_1} + \frac{1}{a_2} - \frac{1}{a_3} \right) [1 - \cos(\omega t)], \]

together with \( p_0(t) = 1 - p_1(t) - p_2(t) - p_3(t) \). In particular taking

\[ a_1 = a_2 = \frac{1}{\omega^2}, \quad a_3 = \infty, \]

(93)

one finds

\[ \kappa_1(t) = \kappa_2(t) = -\omega^2, \quad \kappa_3(t) = 0, \]

(94)

and hence

\[ k_1(t) = k_2(t) = 0, \quad k_3(t) = \frac{\omega^2}{2}, \]

(95)

which proves that the constant (time independent)

\[ K_t[\rho] = \frac{k}{2}(\sigma_3 \rho \sigma_3 - \rho), \]

(96)

provides a legitimate memory kernel for arbitrary \( k = \omega^2 > 0 \).

VI. CONCLUSIONS

We analyzed random unitary evolution of the qubit within memory kernel approach. Our main result formulated in Theorem 4 allows to construct legitimate memory kernel leading to CPTP dynamical map. The power of this method is based on the fact that 1) it allows to reconstruct well known examples of legitimate qubit evolution, 2) the structure of polynomials \( W_k(s) \) enables one to perform the inverse Laplace transform and to find the formula for the kernel in the time domain. The mathematical analysis heavily uses the notion of completely monotone functions. These functions are not commonly used in theoretical physics and the knowledge about their properties is rather limited. There are not known effective methods allowing to check whether a given function is CM. We stress that Theorem 4 provides only a sufficient condition and further analysis is needed to cover physically interesting cases which do not fit assumptions of the Theorem. Interestingly, it turns out that quantum evolution with memory kernel generated by our approach gives rise to vanishing non-Markovianity measure based on the distinguishability of quantum states [31]. It shows that evolution satisfying non-local master equation does not necessarily lead to non-Markovian evolution. It would be also interesting to analyze the relation between semi-Markov evolution and the one governed by our approach in more details.

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Appendix: proof of Lemma 1

Let us observe that (57) may be represented in the following form

\[ \frac{1}{s \prod_{i=1}^n (s + z_i)} = \frac{1}{n} \prod_{i=1}^n (s + z_i) - s \left( \prod_{i=1}^n (s + z_i) + z_1 \prod_{i=2}^n (s + z_i) + \ldots + \prod_{j=1}^{n-2} z_j (s + z_n) + \prod_{j=1}^{n-1} z_j \right), \]

(97)
therefore to prove the Lemma it suffices to show that

\[
\prod_{i=1}^{n} z_i = \prod_{i=1}^{n} (s + z_i) - s \left( \prod_{i=2}^{n} (s + z_i) + \prod_{i=3}^{n} (s + z_i) + \ldots + \prod_{j=1}^{n-2} z_j (s + z_n) + \prod_{j=1}^{n-1} z_j \right).
\] (98)

We will prove this by induction. For \( n = 1 \) it is clear that LHS=RHS=z_1. We assume that (98) is true for \( n \) and prove it is also true for \( (n+1) \). LHS may be written as

\[
\text{LHS} = \prod_{i=1}^{n} z_i \cdot z_{n+1},
\] (99)

while RHS reads

\[
\text{RHS} = \prod_{i=1}^{n} (s + z_i)(s + z_{n+1}) - s \left( \prod_{i=2}^{n} (s + z_i)(s + z_{n+1}) + \prod_{i=3}^{n} (s + z_i)(s + z_{n+1}) + \ldots + \prod_{j=1}^{n-2} z_j (s + z_n) + \prod_{j=1}^{n-1} z_j \cdot z_n \right) =
\]

\[
= (s + z_{n+1}) \left( \prod_{i=1}^{n} (s + z_i) - s \left( \prod_{i=2}^{n} (s + z_i) + \prod_{i=3}^{n} (s + z_i) + \ldots + \prod_{j=1}^{n-2} z_j (s + z_n) + \prod_{j=1}^{n-1} z_j \right) \right) -
\]

\[
- s \prod_{j=1}^{n-1} z_j z_n = s \prod_{i=1}^{n} z_i + z_{n+1} \prod_{i=1}^{n} z_i - s \prod_{i=1}^{n} z_i.
\] (100)

which proves that RHS=LHS. \( \Box \)

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