A STOCHASTIC CASCADE MODEL FOR FX DYNAMICS

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ABSTRACT
A time series model for the FX dynamics is presented which takes into account structural peculiarities of the market, namely its heterogeneity and an information flow from long to short time horizons. The model emerges from an analogy between FX dynamics and hydrodynamic turbulence. The heterogeneity of the market is modeled in form of a multiplicative cascade of time scales ranging from several minutes to a few months, analogous to the Kolmogorov cascade in turbulence.

The model reproduces well the important empirical characteristics of FX rates for major currencies, as the heavy-tailed distribution of returns, their change in shape with increasing time interval, and the persistence of volatility.

Keywords: Hydrodynamic turbulence, Kolmogorov cascade, heterogeneous market, information flow, volatility.

1. Information cascade: an analogy with hydrodynamic turbulence

The idea of a heterogeneous market that consists of traders acting on different time horizons was first advanced by Müller et al. who investigated the absolute values of FX returns on different time scales. They observed that the price changes over longer time intervals have a stronger influence on those over shorter time intervals than conversely. This has been interpreted as an information flow from long-term to short-term traders which directly influences the volatility on different time scales.

Though being very different in its physical nature, the motion of a turbulent fluid medium is also governed by a hierarchical process in which energy flows from large to small spatial scales. Interestingly enough, this formal analogy between FX dynamics and turbulence leads to similar statistical behavior of both phenomena. For the FX dynamics, the hierarchical process can be implemented in form of a volatility cascade, which has recently been visualized by means of wavelet techniques.

In the present contribution we take this hypothesis as granted and construct a stochastic multiplicative volatility cascade driving the FX price process. In the stochastic cascade model (SCM), that we present here, the volatility at an instant of time $t$ is assumed to be created by processes on different time scales reflecting
the heterogeneity of the market dynamics. The information flow between traders on
different time horizons is modeled by a directional interaction scheme between the
levels of the cascade corresponding to the different time scales. More precisely, the
flow of information is accounted for by an update of stochastic “transfer” factors
relating the subsequent levels of the cascade.

2. Stochastic cascade model
We denote the price at time \( t \) by \( p_t \) and define the return as \( r_t \equiv \log p_t - \log p_{t-1} \).
In the proposed SCM the returns are described by a stochastic volatility model, i.e.

\[
  r_t = \sigma_t \xi_t,
\]

where \( \xi_t \) are i.i.d. Student random numbers with 3 degrees of freedom. The volatility
\( \sigma_t \) is governed by a hierarchical process that reflects the heterogeneity of the market
on its different time horizons. Each level \( k \) of the cascade corresponds to a time
horizon of duration \( \tau^{(k)} \). The largest horizon \( \tau^{(0)} \) representing the first level at the
top of the cascade is typically of the order of a few months while the level \( m \) at
the bottom of the cascade has the smallest horizon \( \tau^{(m)} \) on which dealers may act
and which is of the order of minutes. For the sake of simplicity we assume that
the horizon ratio between neighboring levels is constant, i.e. \( \tau^{(k)}/\tau^{(k-1)} = p < 1 \) is
independent of \( k \).

We now assign to each level \( k \) of the hierarchy a volatility \( \sigma^{(k)}_t \) which is deter-
mined by the respective volatility on the level \( k-1 \) and a time-dependent random
factor \( a^{(k)}_t \) that will be specified below. Hence the level volatilities are recursively
defined by:

\[
  \sigma^{(k)}_t = a^{(k)}_t \sigma^{(k-1)}_t.
\]

As a toy model we assume that the market dynamics is completely determined by
the behavior of the dealers and not subject to exogenous time-dependent influences.
Then, the volatility \( \sigma^{(0)}_t \) on the largest horizon is time independent. The volatility
on the shortest horizon determines the returns, as given by eq. (1) with \( \sigma_t = \sigma^{(m)}_t \).
Using eq. (2) we find:

\[
  \sigma_t = \sigma^{(0)} \prod_{k=1}^{m} a^{(k)}_t.
\]

3. Updating rule for the volatility factors \( a^{(k)}_t \)
The time-dependence of the factors \( a^{(k)}_t \) results from the following renewal
scheme: At the initial time \( t_0 \) the factors \( a^{(k)}_{t_0} \) are drawn from independent lognormal
distributions \( LN(a_k, \lambda_k^2) \) with mean values \( a_k \) and variances \( \lambda_k^2 \), \( k = 1, \ldots, m \).
At a later time \( t_{n+1} \) the factor \( a^{(1)}_{t_{n+1}} \) at the top of the hierarchy maintains the
corresponding value \( a^{(1)}_n \) at the preceding time \( t_n \) with a probability \( 1 - w^{(1)} \) and
else is drawn from \( LN(a_1, \lambda_1^2) \). In the latter case all the subsequent factors \( a^{(k)}_{t_{n+1}} \),
\( k > 1 \) are also renewed, i.e. they acquire independent random values from the re-
spective distributions \( LN(a_k, \lambda_k^2) \). If \( a^{(1)}_{t_{n+1}} \) coincides with \( a^{(1)}_n \) the factor \( a^{(2)}_{t_{n+1}} \) will
be renewed with a probability \( w^{(2)} \). These rules apply through the whole cascade
down to \( k = m \): A renewal at some level \( k \) entails one at all higher levels \( k' > k \);
if no renewal has taken place up to level \( k \), the coefficient at level \( k + 1 \) will be
renewed with probability $w^{(k+1)}$ (drawn from the distribution $LN(a_{k+1}^2 \lambda_{k+1}^2)$) or, with probability $1 - w^{(k+1)}$, its value remains constant in time, $a_{t_{n+1}}^{(k+1)} = a_{t_n}^{(k+1)}$.

The renewal probabilities $w^{(k)}$ are given by:

$$w^{(k)} = 1 - \frac{(1 - p^{m-k})}{(1 - p^{m-k+1})}, \quad k = 1, \ldots, m.$$  

(4)

According to this renewal scheme the mean life-times of a factor at level $k$ (measured in no. of time steps of length $\tau(m)$) is given by $p^{k-m}$ where $p$ is the scaling factor of the time horizons as defined above.

4. Results

The time series defined by eqs. (1 - 4) have been simulated and the parameters have been adjusted in such a way that the model simultaneously reproduces the empirical return distributions (Fig. 1), the scaling laws for the moments and the autocorrelation function of the volatility (Fig. 2) of USD/CHF FX spot rates. The distributions of returns (cf. Fig. 1) exhibit the well-known heavy tails, which lose weight with increasing time interval. The autocorrelation function of the absolute returns (Fig. 2) shows the characteristic slowly decaying tail. In both cases the SCM simulation (full line) reproduces the observed data (dotted curves) very well. The
scaling behavior of the moments and the cross correlation function of the volatility are presented in an extended version of the paper [5].

To summarize, the SCM is a hierarchical time series model where a net information flow from long to short time horizons is implemented in terms of a random unidirectional action of the volatility at a given time horizon on that at the next shorter one. With only three adjustable parameters this model is able to reproduce different characteristic properties of intra-day FX price series.

Note that the existence of different groups of traders and the volatility updating mechanism are elements similar to those found in market microstructure models. Therefore the SCM may provide a link between market microstructure models and more conventional models.

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