On Shilnikov attractors of three-dimensional flows and maps

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ABSTRACT
We describe scenarios for the emergence of Shilnikov attractors, i.e. strange attractors containing a saddle-focus with two-dimensional unstable manifold, in the case of three-dimensional flows and maps. The presented results are illustrated with various specific examples.

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1. Introduction

The discovery of spiral chaos, i.e. the proof of the existence of a complex structure of orbits in a neighbourhood of a homoclinic loop of a saddle-focus equilibrium state, is rightfully considered one of the most significant achievements of the theory of dynamical systems. This discovery was made by L.P. Shilnikov in his work [45], published in 1965. At that time, in both mathematics and physics, there was practically no more or less suitable concept for explaining such phenomena in models described by finite-dimensional deterministic systems. Therefore, the existence of complex structure (chaos) in a neighbourhood of a homoclinic loop of a saddle-focus equilibrium was absolutely unexpected and completely contradicted the conceptions based on the theory of two-dimensional systems. Moreover, such a significant difference in the orbit structure for a saddle and a saddle-focus was seemed very strange, since, from the topological point of view, the saddle and the saddle-focus are locally equivalent. However, they differ globally. Thus, from the Shilnikov conditions of nondegeneracy for a homoclinic loop of a saddle equilibrium [43,44,47], one can deduce that in its neighbourhood there is a smooth two-dimensional global central invariant manifold, and thus the problem is effectively two-dimensional. Whereas the
saddle-focus does not have such a manifold even locally, and therefore the dynamics here are principally multidimensional.

A saddle-focus differs from a saddle in that, among its leading eigenvalues (those nearest to the imaginary axis), a pair of complex-conjugate ones exists. Then, in the case of three-dimensional systems, saddle-focus equilibria are divided into two different types:

- a saddle-focus $(2,1)$ with two-dimensional stable and one-dimensional unstable manifolds that has eigenvalues $\lambda \pm i\omega, \gamma$, where $\lambda < 0, \gamma > 0, \omega \neq 0$;
- a saddle-focus $(1,2)$ with one-dimensional stable and two-dimensional unstable manifolds that has eigenvalues $\lambda, \gamma \pm i\omega$, where $\lambda < 0, \gamma > 0, \omega \neq 0$.

Back in the papers \[43,44\] Shilnikov showed that bifurcations of systems with a homoclinic loop of a saddle-focus $(2,1)$ with the saddle value

$$\sigma = \lambda + \gamma$$

to be negative do not differ from the case of a saddle – here a unique stable limit cycle is born when the loop spits inward (this is due to the fact that the first return map turns out to be contracting in this case). The case of a three-dimensional saddle-focus $(1,2)$ is obviously reduced to the case of a saddle-focus $(2,1)$ by time reversal. Therefore, here in the case $\sigma > 0$ a unique completely unstable limit cycle is born from a homoclinic loop of a saddle-focus of type $(1,2)$.

However, if the closest to the imaginary axis eigenvalues are complex conjugate ones, i.e. if $\sigma > 0$ in the case of a saddle-focus $(2,1)$ or $\sigma < 0$ in the case of a saddle-focus $(1,2)$, the situation becomes completely different – already the system itself with a homoclinic loop has a complex structure, since, in any neighbourhood of the loop, there are infinitely many saddle periodic orbits \[45\]. In fact, Shilnikov discovered in Shilnikov \[45\] that the existence of a homoclinic loop of such saddle-focus implies chaos. The notion itself did not exist then (in 1965); the Chaos Theory emerged and became popular only 10–20 years later. Moreover, chaos was found in many nonlinear models and it also occurred that strange attractors in models of various origins often have a spiral structure, i.e. the chaotic orbits seem to move near a saddle-focus homoclinic loop.

According to the classification of saddle-foci, we will also distinguish two types of spiral homoclinic attractors of three-dimensional flows:

1. a Figure 8 spiral attractor, when the attractor contains a saddle-focus $(2,1)$ (with $\sigma > 0$) and entirely both its one-dimensional unstable separatrices (composing a homoclinic–8);
2. a Shilnikov attractor, when the attractor contains a saddle-focus $(1,2)$ (with $\sigma < 0$) and entirely its two-dimensional unstable manifold.

The fact that the homoclinic loop to a saddle-focus equilibrium implies chaos is the Shilnikov theorem \[45,48\], see also \[51\], but why is the converse also often true, i.e. why is the observed chaos often spiral? This question was of great interest to Shilnikov. He discovered \[49,50\] that if a system depends on a parameter and evolves as it changes, from a stable (stationary) regime to a chaotic one, then in this way, a saddle-focus equilibrium arises naturally and, moreover, its stable and unstable manifolds can come close enough to each other, so that the creation of a homoclinic loop (and chaos) becomes quite expected.
Shilnikov described the corresponding scenarios in the paper [50] for the case of a 
one-parameter family \( X_\mu : \dot{x} = X(x, \mu) \) of multidimensional systems. In Section 2, we 
discuss the Shilnikov scenario for the three-dimensional case. As one can see, this scenario is 
extremely simple (if to ignore certain fine intermediate details that themselves can be very 
complicated), and therefore, it should come as no surprise that it is often seen in many 
models. 1 We consider some of these models in Section 3 as examples of systems in which 
Shilnikov’s scenario is implemented.

It is interesting to note that spiral chaos was not initially detected in a number of such 
models, the studies of chaotic dynamics ended either at the stage of Feigenbaum-type 
attractors (which are formed inside the Shilnikov whirlpool (see Section 2) as a result of an 
infinite cascade of period-doubling bifurcations with limit cycles) or a little further, when 
the attractor in the Poincaré section becomes ‘one-component’, like the Rössler attractor 
an example of such attractor is shown in Figure 2(e), or the interpretation of the process 
of its occurrence was inadequate etc. Of course, this situation was a consequence of the 
lack of a ‘guiding thread’ – the Shilnikov scenario in this case.

In the work [50], Shilnikov expressed one more important idea that the scenario he 
proposed for flows can be generalized to the case of maps. In this case, the generaliza-
tion is direct: the equilibrium state is replaced by a fixed (or periodic) point, and the local 
and global bifurcations involved in the scenario are replaced by their analogues for maps. 
In Section 4, we give a phenomenological description of such scenarios for the case of 
orientable and nonorientable three-dimensional maps, and in Section 5, we give some 
examples of how such scenarios can be implemented in the case of three-dimensional 
generalized Hénon maps.

2. On Shilnikov scenario for three-dimensional flows

Consider a one-parameter family \( X_\mu : \dot{x} = X(x, \mu) \) of three-dimensional flows such that 
the system \( X_\mu \) has at \( \mu_0 < \mu < \mu_1 \) a stable equilibrium \( O_\mu \) that is the only attractor in some 
absorbing domain \( D \), see Figure 1(a). Let at \( \mu \in (\mu_0, \mu_1) \) and close to \( \mu_1 \) the equilibrium 
\( O_\mu \) have eigenvalues \( \nu_{1,2} = \lambda(\mu) \pm i\omega(\mu), \nu_3 = \lambda_{ss}(\mu) \), where \( \lambda_{ss} < \lambda < 0 \), that is \( O_\mu \) is a 
stable focus.

Assume that at \( \mu = \mu_1 \) a supercritical (soft) Andronov–Hopf bifurcation occurs with 
\( O_\mu \). Then the eigenvalues \( \nu_{1,2} \) evolve as \( \mu \) changes in such a way that \( \lambda(\mu) < 0 \) if \( \mu < \mu_1 \),

![Figure 1](image_url)
\[ \lambda(\mu_1) = 0, \text{ and } \lambda(\mu) > 0 \text{ if } \mu > \mu_1. \] Thus, at \( \mu > \mu_1 \), the equilibrium \( O_{\mu} \) becomes a saddle-focus \((1,2)\) and a stable limit cycle \( L_{\mu} \) is born in its neighbourhood. Accordingly, for all sufficiently small \( \mu > \mu_1 \), the global attractor in \( D \) of the system \( X_{\mu} \) is this asymptotically stable limit cycle \( L_{\mu} \). The type of stability of the cycle \( L_{\mu} \) is determined by its multipliers \( \rho_1(\mu) \) and \( \rho_2(\mu) \), which are initially both positive and less than one. In this case, the unstable manifold \( W^u(O_{\mu}) \) of the equilibrium \( O_{\mu} \) is a two-dimensional disk with the boundary \( L_{\mu} \), and orbits starting from points on \( W^u(O_{\mu}) \) are wound to \( L_{\mu} \) along spirals ‘from the inside’, see Figure 1(b).

We assume that with a further increase in \( \mu \) the multipliers \( \rho_1(\mu) \) and \( \rho_2(\mu) \) of \( L_{\mu} \) become equal at some \( \mu = \mu^* \) and complex conjugate for \( \mu > \mu^* \). This ‘smooth bifurcation’ corresponds to change of type of \( L_{\mu} \), from nodal type for \( \mu < \mu^* \) to focal one for \( \mu > \mu^* \). Then, the two-dimensional unstable manifold \( W^u(O_{\mu}) \) of the equilibrium \( O_{\mu} \) begins to twist onto the cycle \( L_{\mu} \). In this case, \( W^u(O_{\mu}) \) takes a form of roll which is a boundary of the so-called ‘Shilnikov whirlpool’, inside of which all orbits of system \( X_{\mu} \) starting in \( D \) are tightened (generally speaking, except for one orbit – the stable separatrix \( W^{s-}(O_{\mu}) \) of the equilibrium \( O_{\mu} \)), see Figure 1(c).

When \( \mu \) changes, the sizes of the whirlpool increase, the orbits absorption is preserved, but the limit cycle \( L_{\mu} \) may lose its stability. In particular, a strange attractor can appear in its place as a result of a series of various bifurcations. The sequence of these bifurcations can be very diverse: this can be a cascade of period-doubling bifurcations followed by the appearance of a Feigenbaum attractor or, few further, a Hénon-like attractor in the Poincaré map; or a two-dimensional stable invariant torus \( T_{\mu} \) can be born from the cycle, which can then break down, for example, according to the Afraimovich–Shilnikov scenario [1], giving rise to a strange attractor of the ‘torus-chaos’ type, etc. It is worth noting that yet in the paper [50], Shilnikov emphasized that chaotic dynamics inside the whirlpool can appear instantly due to a rigid (subcritical) bifurcation: this can be, for example, subcritical Andronov–Hopf or period-doubling bifurcations with stable limit cycles.
and so forth. In particular, an example of such subcritical period-doubling bifurcation we found in Gonchenko et al. [27] in the case of the Gaspard–Nicolis model of chemical oscillator. We illustrate this result in Section 3.2. These intermediate attractors appearing inside the Shilnikov whirlpool, in turn, can transform with varying in $\mu$, into the Shilnikov attractor, i.e. a strange attractor that contains the saddle-focus $O_\mu$ of type (1,2) and entirely its two-dimensional unstable manifold.

The moment $\mu = \mu_\ell$ of the formation of a homoclinic loop of the saddle-focus $O_\mu$ can be traced by observing the stable separatrix $W^{s+}(O_\mu)$: at $\mu < \mu_\ell$, it leaves $D$ (as in Figure 1(c)), at $\mu > \mu_\ell$, it enters the whirlpool, and, at $\mu = \mu_\ell$, it falls on the unstable manifold $W^u(O_\mu)$ (thus, $W^{s+}(O_\mu)$ entirely belongs to $W^u(O_\mu)$).

With a further change in $\mu$, the separatrix $W^{s+}(O_\mu)$ enters the whirlpool, makes several (maybe quite many) turns there, then can again lie on $W^u(O_\mu)$ forming a multi-round homoclinic loop and etc. The moments of formation of such loops are discrete in the parameter $\mu$, but they are not isolated: for each such value of the parameter $\mu$, those values of $\mu$ are accumulated again that correspond to the formation of double- [14,20], triple-[22], and longer-round homoclinic loops.

Discrete moments with respect to the parameter $\mu$, when the separatrix $W^{s+}(O_\mu)$ forms homoclinic loops, correspond to the existence of Shilnikov homoclinic attractors. At these moments, the equilibrium $O_\mu$ enters the attractor together with its unstable manifold $W^u(O_\mu)$, and typical orbits of the attractor visit any arbitrarily small neighborhood of the point $O_\mu$. The latter property makes it relatively easy to find the moments of formation of Shilnikov homoclinic attractors using, for example, automated plotting of the $\mu$-dependence of the distance of attractor points from the saddle-focus [12,37].

3. Examples of Shilnikov attractors in three-dimensional flows

For a long time, the Shilnikov’s works [45,46,48] on spiral chaos were practically unknown to the mathematical community. They gained world recognition due to a series of quite affordable papers by Arnéodo et al. [7], Arnéodo et al. [8], Arnéodo et al. [9], Arnéodo et al. [10], in which the importance of the Shilnikov homoclinic loop for the chaos theory was emphasized. In particular, in Arneodo et al. [8], geometric illustrations of the Shilnikov’s theorem were given. Also, in this paper, it was presented a simple example of a piecewise linear oscillatory system, in which the existence of a homoclinic loop of saddle-focus (2,1) was analytically established, and strange spiral attractors were found numerically.

In this section, we consider two examples of three-dimensional systems in which the Shilnikov attractors are observed: the first example is the model proposed by Arnéodo, Coullet, and Tresser in Arnéodo et al. [9], and the second example is the Gaspard–Nicolis model of the chemical oscillator from [21].

3.1. Shilnikov attractor in the Arnéodo–Coullet–Tresser system

In the paper, [9], a smooth three-dimensional system

$$\begin{align*}
x' &= y \\
y' &= z \\
z' &= -y - \beta z + \mu x(1 - x),
\end{align*}$$

(1)
Figure 3. (a) $\beta = 0.4, \mu = 0.86311445$, Shilnikov attractor with a homoclinic loop of the saddle-focus $O_2$ of type (1,2); (b) $\beta = 0.01, \mu = 0.02$, illustration of the formation of the Shilnikov whirlpool; (c) $\beta = 0.4, \mu = \mu_5 = 1.6062$, a homoclinic loop of the saddle-focus $O_1$ of type (2,1), here the attractor no longer exists.

was proposed which demonstrates spiral chaos at certain regions of values of the parameters $\beta$ and $\mu$. Let us consider this system in more detail.

Since system (1) has the constant divergence equal to $-\beta$, attractors can exist only for $\beta > 0$. Note that for all values of parameters $\mu > 0$ and $\beta > 0$, system (1) has two equilibria $O_1(0,0,0)$ and $O_2(1,0,0)$. Moreover, the equilibrium $O_1$ for $\beta < \sqrt{3}$ is always a saddle-focus (2,1), while the equilibrium $O_2$ can be both stable and saddle-focus (1,2).

Following [9] we put $\beta = 0.4$. Now our goal is to illustrate the scenario of the emergence of the Shilnikov attractor containing the saddle-focus $O_2$ when the parameter $\mu$ changes. In Figure 2 main stages of the scenario are shown.

For $0 < \mu < \mu_1 = 0.4$, the stable equilibrium $O_2$ is the only attractor of the system. Note that the boundary of its absorbing domain is formed by the two-dimensional stable invariant manifold of the saddle-focus $O_1$. At $\mu = \mu_1$, the equilibrium $O_2$ undergoes a supercritical Andronov–Hopf bifurcation. As a result, for $\mu > \mu_1$, a stable limit cycle $L$ is born in a neighborhood of the equilibrium $O_2$ that becomes a saddle-focus (1,2), see Figure 2(a). The stable limit cycle exists for $\mu_1 < \mu < \mu_2 \approx 0.72$. Starting with $\mu = \mu_2$, the cycle goes through a cascade of period-doubling bifurcations (see Figure 2(b,c) after the first and second period-doubling bifurcations), and, as a result, a strange attractor of Feigenbaum type appears, see Figure 2(d). Then, this attractor is transformed into a Rössler-like attractor (on a two-dimensional Poincaré section, it can be also interpreted as a Hénon-like attractor), see Figure 2(e). With a further increase in the parameter $\mu$ up to $\mu = \mu_\ell \approx 0.86311445$, the attractor changes quite smoothly and is not homoclinic, since it is separated from the saddle-focus $O_2$.

However, as $\mu \to \mu_\ell$, a ‘hole’ around $O_2$ decreases and, finally, disappears at $\mu = \mu_\ell$. In this case, orbits of the attractor can come arbitrarily close to the saddle-focus $O_2$, see Figure 2(f). As we know, this is due to the appearance of a homoclinic orbit to the saddle-focus $O_2$ and, hence, to the creation of the Shilnikov attractor, see Figure 3(a), where the numerically found homoclinic orbit is also shown.

Note that, using system (1), we can trace a formation of the Shilnikov whirlpool – when the cycle $L$ becomes of focal type. An illustration of the moment of occurrence of the Shilnikov whirlpool is shown in Figure 3(b), where, for greater clarity, we take $\beta = 0.01$ (then the divergence of the system $Div = -\beta$ will be small) in order to be able to show how
the unstable two-dimensional manifold of the saddle-focus $O_2$ winds around the stable limit cycle $L$.

With a further increase in the parameter $\mu$, from $\mu = \mu_3$ to $\mu = \mu_4 \approx 0.873$, a phenomenon of complication of the structure of the Shilnikov whirlpool can be observed, see e.g. [36,37]. At $\mu = \mu_4$, a crisis of the attractor occurs: it collides with the two-dimensional stable manifold of the saddle-focus $O_1$, which is a natural boundary of the absorbing domain for the attractor under consideration. Note that the saddle-focus $O_1$ can also have homoclinic loops, see Figure 3(a) where such a loop is shown for $\mu = \mu_5 \approx 1.6062$ (it was also found numerically in Arneodo et al. [9]). However, any attractors do not exist for close values of $\mu$.

3.2. Example of Shilnikov attractor in the Gaspard–Nicolis model of chemical oscillator.

In this section, we consider an example of an interesting system in which the Shilnikov attractor arises almost instantly as a result of a single rigid (subcritical) bifurcation. This scenario begins as usual: a stable equilibrium $O$ undergoes a supercritical Andronov–Hopf bifurcation, as a result, it becomes a saddle-focus (1,2) and a stable limit cycle $L$ is born. This cycle first becomes the focal one, and the Shilnikov whirlpool (with the boundary $W^u(O)$) is formed. However, this limit cycle itself remains stable for a long time (by the parameter) and, independently of it, non-attractive chaotic dynamics (metastable chaos) have time to develop inside the whirlpool. Next, at a certain value of the parameter, the cycle $L$ undergoes a period-doubling bifurcation that is subcritical: a saddle cycle of doubled period merges with $L$. After this, a strange attractor is immediately observed, which at very close values of a parameter becomes homoclinic – the Shilnikov attractor appears.

Below we will show how this happens in the system

$$\begin{align*}
\dot{x} &= x(\beta x - fy - z + g),
\dot{y} &= y(x + sz - \alpha),
\dot{z} &= (x - \alpha z^3 + bz^2 - cz)/\epsilon,
\end{align*}$$

proposed by P. Gaspard and G. Nicolis in the paper [21], in which it was shown that strange attractors containing a saddle-focus (1,2) can exist in the system (2). The corresponding parameter regions where such attractors can be observed have been specified in Gallas [19]. Following [19] we take some parameters to be fixed

$$b = 3, \quad \epsilon = 0.01, \quad f = 0.5, \quad g = 0.6, \quad s = 0.3, \quad c = 4.8, \quad \alpha = 0.7825,$$

and the parameter $\beta$ choose as the governing one.

For $\beta < \beta_{AH} \approx 0.261$, the nonzero equilibrium $O$ of system (2) is stable, see Figure 4(a) ($O$ is the only equilibrium with positive coordinates having a physical sense). For $\beta = \beta_{AH}$ this equilibrium undergoes a supercritical Andronov–Hopf bifurcation. After this, on the interval $\beta \in (\beta_{AH}, \beta_{PD} \approx 0.3817)$, the attractor of the system is the stable limit cycle $l$, see Figure 4(b), and the equilibrium $O$ becomes a saddle-focus (1,2). At $\beta \approx 0.365$ the cycle $l$ becomes focal and a Shilnikov whirlpool is formed, further both multipliers of $l$ become negative and one of them tends to $-1$ as $\beta \to \beta_{PD}$. Simultaneously, a saddle limit cycle $l_s$
of double period approaches the cycle $l$, see Figure 4(b) and merges with $l$. As a result of this subcritical period-doubling bifurcation, at $\beta > \beta_{PD}$ the limit cycle $l$ becomes saddle and, instantly, a strange attractor is observed. With a further increase in the parameter $\beta$, the orbits of this attractor begin to approach closer and closer to the saddle-focus $O$. At $\beta = \beta_h \approx 0.3921$, a homoclinic loop of the saddle-focus $O$ is formed, i.e. a homoclinic Shilnikov attractor arises, Figure 4(c).

4. On Shilnikov scenarios for three-dimensional orientable and nonorientable maps

We discuss now scenarios of the emergence of discrete Shilnikov attractors, i.e. homoclinic attractors containing a fixed (periodic) point that is a saddle-focus with two-dimensional unstable manifold, thus, it is a saddle-focus (1,2) in the case of three-dimensional maps.

We consider one-parameter families $T_\mu$ of three-dimensional diffeomorphisms in two cases, when map $T_\mu$ is orientable (orientation preserving) and when $T_\mu$ is nonorientable (orientation reversing) map. In order not to get involved with the problems of orientability of the ambient manifold, we will assume that in both cases, $T_\mu$ is given in $\mathbb{R}^3$. Then the Jacobian $J(T_\mu)$ of the map $T_\mu$ will be everywhere positive in the orientable case and everywhere negative in the nonorientable case.

4.1. The orientable case

A sketch of the scenario of a typical discrete Shilnikov attractor appearance for one-parameter families $T_\mu$ of three-dimensional orientable maps is illustrated in Figure 5. This scenario starts with a stable fixed point $O_\mu$ that loses the stability at $\mu = \mu_1$ under a supercritical (soft) Andronov–Hopf bifurcation: for $\mu > \mu_1$ the point $O_\mu$ becomes a saddle-focus (1,2) and a stable closed invariant curve $L_\mu$ is born in a neighborhood of $O_\mu$, Figure 5(a,b). Thus, the curve $L_\mu$ becomes the attractor of map $T_\mu$. During this transition, the point $O_\mu$ changes its type from a stable point to a saddle-focus (1,2) point: for $\mu < \mu_1$ it has three multipliers inside the unit circle, for $\mu = \mu_1$, two complex conjugate multipliers of $O_\mu$ fall on the unit circle, and for $\mu > \mu_1$ they go outside it.
Then the next stage of the scenario is connected with changes in $L_{\mu}$. Typically, this happens in the following way. Just after the Andronov–Hopf bifurcation, at small $|\mu - \mu_1|$, the unstable manifold of $O_{\mu}$ is a two-dimensional disk $D_{\mu}$ with the boundary $L_{\mu}$ that has a type of nodal invariant curve. We assume that, at further changing $\mu$, the curve $L_{\mu}$ undergoes a 'smooth bifurcation', when it changes its type from nodal to focal and, thus, the two-dimensional manifold $W^u(O_{\mu})$ begins to wind up on $L_{\mu}$, like a roll. As a result, $W^u(O_{\mu})$ takes the form of a kind of a roll, the Shilnikov whirlpool, inside which all orbits from the absorbing region are drawn, except for orbits of the stable separatrix $W^{s-}(O_{\mu})$, see Figure 5(c).

Then, chaotic dynamics begin to develop in this whirlpool as $\mu$ changes. At first, the attractor is simple, it is the stable invariant curve $L_{\mu}$, and then it loses its stability. This can happen in a variety of ways (for example, in a soft way through a cascade of doubling of invariant curves with their subsequent destruction and the formation of attractors of torus-chaos type, or in a rigid way through a subcritical bifurcation with some of the stable invariant curves, after which chaos can be observed immediately, 'by explosion', etc.). In principle, for the essence of the phenomenological scenario, it does not matter how this happens. The main thing here is that the invariant manifolds $W^u(O_{\mu})$ and $W^{s+}(O_{\mu})$ begin to intersect and a strange homoclinic attractor can arise containing the saddle-focus $O_{\mu}$ and entirely its two-dimensional unstable manifold, see Figure 5(d). We call this attractor a discrete Shilnikov attractor.

It should be noted that there is an almost complete similarity in the main features of this scenario with the corresponding scenario in the case of a flow, see Section 1. However, even here one can see a significant difference. The Shilnikov homoclinic flow attractor exists only for discrete values of the control parameter corresponding to the existence of homoclinic loops of the saddle-focus equilibrium. Whereas the Shilnikov attractor for maps exists on intervals of parameter values for which the invariant manifolds $(W^u(O_{\mu})$ and $W^{s+}(O_{\mu})$) have transversal intersections. Moreover, these intervals can be large enough and their values can reach even those at which the attractor is destroyed and disappears altogether.

There is also one more important feature of Shilnikov discrete attractors, which flow attractors do not have. This feature manifests itself in the case when the stable invariant curve $L_{\mu}$ is resonant. Then, on the curve itself, there are alternating saddle and stable periodic points of the same period, and the formation of the Shilnikov whirlpool occurs due
Figure 6. Discrete attractors in resonant case: (a) a ‘triangular’ discrete Shilnikov attractor; (b) a ‘super-spiral’ period–4 attractor. We found these attractors in the three-dimensional Mirá map $\tilde{x} = y, \tilde{y} = z, \tilde{z} = M_1 + Bx + M_2z - y^2$ for $B = 0.7$ and (a) $M_1 = 0.195, M_2 = -0.26$; (b) $M_1 = 0.35, M_2 = 0.8$. It is important to note that these attractors appear near the 1:3 and 1:4 strong resonances. Therefore they visually demonstrate something like the Z3 and Z4 symmetry. Although the map itself, of course, does not have such symmetries – they appear only locally, namely for attractors.

to the fact that the stable points become foci. Thus, the manifold $W^u(\Omega_\mu)$ is twisted over $L_\mu$ only in some places (near stable points). This also subsequently affects the shape of the emerging Shilnikov attractor. For example, such an attractor can have a characteristic ‘triangular’ or ‘square’ shape in the case of resonances 1:3 and 1:4, when $\Omega_\mu$ has a pair of multipliers $\lambda e^{\pm i\psi}$ with $\psi$ close to $2\pi/3$ and $\pi/2$, respectively, see Figure 6. In addition, the resonant invariant curves of three-dimensional maps can themselves be destroyed in very interesting ways, giving rise to amusing attractors inside the whirlpool, see, for example, Figure 6(b), which shows a ‘superspiral’ attractor containing two orbits of period 4, which are saddle-foci of type (2,1) and (1,2). We use this term because the attractor contains saddle-foci of both types. In the case of three-dimensional flows with symmetries, such attractors can be found in the class of the so-called multispiral attractors, see e.g. [16,38].

4.2. The nonorientable case

A sketch of a typical discrete Shilnikov scenario for one-parameter families $T_\mu$ of three-dimensional nonorientable maps is illustrated in Figure 7. This scenario starts with those $\mu$ at which $T_\mu$ has a nonorientable stable fixed point $\Omega_\mu$ with multipliers $-\lambda, \gamma^{\pm i\psi}$, where $0 < \lambda < \gamma < 1$ and $0 < \psi < \pi$, Figure 7(a). We assume that $\Omega_\mu$ loses the stability at $\mu = \mu_1$ under a supercritical Andronov–Hopf bifurcation: for $\mu > \mu_1$ the point $\Omega_\mu$ becomes a nonorientable saddle-focus (1,2), and a stable closed invariant curve $L_\mu$ is born in a neighborhood of $\Omega_\mu$, Figure 7(b). Thus, the curve $L_\mu$ becomes the attractor of map $T_\mu$. During this transition, the point $\Omega_\mu$ has three multipliers less than one in the absolute value for $\mu < \mu_1$, then, for $\mu = \mu_1$, two of its complex conjugate multipliers fall on the unit circle, and for $\mu > \mu_1$ they go out and the point $\Omega_\mu$ becomes a saddle-focus (1,2) that is nonorientable since its stable multiplier is negative.
Figure 7. A sketch of scenario of the emergence of a nonorientable discrete Shilnikov attractor.

Figure 8. Stages of the emergence of the Shilnikov attractor in map (4) for $B = 0.5$, $A = 1.49$, as $C$ changes (Figure (b) is only schematic): (a) $C = -1.7$, a stable invariant curve $L$; (b) a schematic picture of a Shilnikov whirlpool, which is formed when $W^u(O)$ begins to wound onto $L$ (the following figures (c)–(h) show what happens inside this whirlpool); (c) $C = -1.73$, the curve $L$ is doubled, $L \rightarrow 2L$; (d) $C = -1.76$ – after the second doubling, $2L \rightarrow 4L$; (e) $C = -1.77$ – after the third doubling, $4L \rightarrow 8L$; (f) $C = -1.775$, a chaotic attractor $A2$ containing the saddle curves $4L$ and $2L$; (g) $C = -1.8$, a chaotic attractor $A1$ containing also the saddle curve $L$; (h) $C = -1.82$, a discrete Shilnikov attractor.

The next stage of the scenario is connected with changes in $L_{\mu}$. Typically, this happens in the following way. Just after the Andronov–Hopf bifurcation, at small $|\mu - \mu_1|$, the unstable manifold of $O_\mu$ is a two-dimensional disk $D_\mu$ with the boundary $L_{\mu}$ where the curve $L_{\mu}$ has a type of nonorientable nodal invariant curve. Because the curve $L_{\mu}$ is nonorientable, it can not become of focal type as in the orientable case. Thus, the manifold $W^u(O_\mu)$ can not take a form of whirlpool just over $L_{\mu}$. However, there exists another way to the creation a whirlpool. Namely, at further changing $\mu$, first, the curve $L_{\mu}$ undergoes a doubling bifurcation: the curve $L_{\mu}$ itself becomes saddle and two stable period–2 invariant curves $L_{\mu}^1$ and
$L_\mu^2$ originate from it here $L_\mu^2 = T_\mu(L_\mu^1)$ and $L_\mu^1 = T_\mu(L_\mu^2)$. Such feature of the bifurcation of doubling of an invariant curve is obtained due to the fact that, near the bifurcation, the curve $L_\mu$ possesses two invariant manifolds, strongly stable $W^{ss}$ and central $W^c$, which locally are two-dimensional cylinders, see Figure 7(b), but the map $T_\mu$ in the restriction to $W^c$ changes orientation. Accordingly, after the bifurcation, the curves $L_\mu^1$ and $L_\mu^2$ lie on $W^c$ on opposite sides of $L_\mu$. After this doubling bifurcation, the unstable manifold $W^u(O_\mu)$ immediately corrugates and begins to rush between the curves $L_\mu^1$ and $L_\mu^2$ but still remains inside the cylinder $W^c(L_\mu)$, see Figure 7(c). Note that the curves $L_\mu^1$ and $L_\mu^2$ are both invariant and orientable for $T^2_\mu$, therefore they can become focal ones when changing $\mu$. In this case, the manifold $W^u(O_\mu)$ starts to wind on both $L_\mu^1$ and $L_\mu^2$ and, thus, a nonorientable Shilnikov whirlpool is created, see Figure 7(d).

Then, in this whirlpool, when $\mu$ changes, chaotic dynamics begin to develop. At first, the attractor is simple, it is the stable period–2 invariant curve $(L_\mu^1, L_\mu^2)$, and then it loses its stability. Again, as for the orientable case, this can happen in a variety of ways... The main thing here is that the invariant manifolds $W^u(O_\mu)$ and $W^s(O_\mu)$ can intersect and a strange homoclinic attractor can arise containing the saddle-focus $O_\mu$ and its two-dimensional unstable manifold. In Figure 7(d), it is shown a sketch of manifolds $W^u(O_\mu)$ and $W^s(O_\mu)$ before they crossed. One can imagine (but difficult to draw) what will happen when they intersect. However, it is clear that in this case, the manifold $W^u(O_\mu)$, since the stable multiplier of the point $O_\mu$ is negative, will accumulate towards itself from both sides, and, accordingly, the point $O_\mu$ will reside inside the attractor. In the case of a discrete orientable Shilnikov attractor, its fixed point $O_\mu$ lies on its boundary (since the global piece of $W^u(O_\mu)$ accumulates at $W^u_{loc}(O_\mu)$ only from one side (namely, from the side where $W^s_{loc}(O_\mu)$ is located).

Thus, we see that orientable and nonorientable discrete Shilnikov attractors have different structures. Although they both exist for an open set of parameter values, and it is also important for their geometry whether the invariant curve $L_\mu$ is resonant or not.

### 5. Examples of three-dimensional maps with Shilnikov attractors

In this section, we consider two examples of discrete Shilnikov attractors in the case of three-dimensional orientable and nonorientable maps. Recall that such attractors contain a fixed point $O$ of the saddle-focus type, with eigenvalues $\lambda, \gamma_{1,2} = \gamma e^{\pm i\psi}$, where $|\lambda| < 1, \gamma > 1, 0 < \psi < \pi$. The unstable manifold of $O$ is two-dimensional, and it resides entirely in the attractor.

First, we consider the case of orientable (with $0 < \lambda < 1$) Shilnikov attractor in three-dimensional map of the form

$$\tilde{x} = y, \quad \tilde{y} = z, \quad \tilde{z} = Bx + Cy + Az - y^2,$$

(4)

where $A, B, C$ are parameters. Numerically found example of such attractor is shown in Figure 8(h) for map (4) with the Jacobian $B = 0.5 > 0$.

Figure 8 shows main stages of the development of a discrete Shilnikov attractor in map (4) for $B = 0.5, A = 1.49$ as $C$ changes. The formation of the attractor proceeds in accordance with the general bifurcation scenario described in Gonchenko et al. [24], Gonchenko et al. [26], see Section 4.1:
At the beginning, the attractor of map (4) is a stable fixed point $O(0, 0, 0)$ (it is stable for $-0.99 > C > c_1 = -1.495$).

Then, at $C = c_1$, the point $O$ undergoes a supercritical Andronov–Hopf bifurcation: $O$ becomes a saddle-focus $(1, 2)$ and a stable closed invariant curve $L$ is born in its neighborhood (the curve $L$ is shown in Figure 8(a) at $C = -1.7$).

Next, the unstable manifold of $O$ starts winding onto $L$ and the Shilnikov whirlpool is formed (schematically, the whirlpool is shown in Figure 8(b)).

Further, the dynamics inside the whirlpool become more and more complicated. In the case of map (4), an example of such development is shown in Figure 8(d–h). First, we observe three successive doubling bifurcations of stable invariant curves, $L \rightarrow 2L \rightarrow 4L \rightarrow 8L$, Figure 8(c–e). We did not observe the doubling of $8L$. Instead, a strange attractor appears which has at the beginning a torus-chaos type due to a destruction of the curve $8L$, then it sequentially captures the saddle curves $4L, 2L$ (Figure 8(f)) and $L$ Figure 8(g).

Finally, when homoclinic intersections are created between $W^u(O)$ and $W^s(O)$, a discrete Shilnikov attractor is formed containing the fixed point $O$, Figure 8(h). In Figure 9, the numerically obtained stable separatrix $W^{s+}(O)$ is shown, which confirms the existence of homoclinic intersections of $W^u(O)$ and $W^{s+}(O)$ within the attractor.

Now we consider the case of the appearance of nonorientable Shilnikov attractor again in map (4), but now with the negative Jacobian $B$. (We note that in this map such attractors were found in Karatetskaia et al. [34]). Figure 10 shows the main stages of the development of this attractor when $B = -0.915, A = -2.786$ are fixed and $C$ changes. The stages of the attractor creation follow the general bifurcation scenario described in Karatetskaia et al. [34], Gonchenko et al. [29], see also Section 4.2:

At the beginning, the attractor is the fixed point $O(0, 0, 0)$ that is stable for $4.701 > C > c_1 \approx -2.712$.

Figure 9. The Shilnikov attractor from Figure 8(h) and a part of the stable separatrix $W^{s+}(O)$ are shown in a suitable angle for viewing.
Then, at $C = c_1$, the point $O$ undergoes a supercritical Andronov–Hopf bifurcation and becomes a nonorientable saddle-focus $(1, 2)$ and a stable closed invariant curve $L$ is born (see Figure 10(a) for $C = -2.723$).

Next, the curve $L$ undergoes a component-doubling bifurcation [28]: a pair of stable period–2 curves $L_1$ and $L_2$ are born from $L$ (these curves are shown in Figure 10(b) for $C = -2.733$). With the further change of $C$, the two-dimensional unstable manifold of $O_{\mu}$ begins to wind up on both the curves $L_1$ and $L_2$ and a nonorientable Shilnikov whirlpool is formed, which has a ‘double roll’ shape (it is shown schematically, in Figure 10(c)).

Further, the dynamics inside the whirlpool become more complicated. In particular, several bifurcations of doubling of invariant curves occur, giving rise to strange attractors inside the whirlpool, (see Figure 10(e), where one of such attractors is shown for $C = -2.736$).

Finally, when homoclinic intersections are created between $W^u(O)$ and $W^s(O)$, a nonorientable discrete Shilnikov attractor is formed, (see Figure 10(f) for $C = -2.743$). In Figure 10(g) a piece of $W^s(O)$ is shown, confirming that the attractor contains the point $O$.

6. Conclusion

We note that the Shilnikov’s ideas about the importance of scenarios for the emergence of spiral attractors and, more generally, homoclinic attractors (strange attractors containing either an equilibrium or a periodic orbit and its entire unstable manifold) were implemented in a number of our works. The first one was our work with L.P. Shilnikov.
in which we gave a phenomenological description of scenarios for the emergence of discrete attractors of various types (discrete Lorenz attractors, Figure 8 attractors and Shilnikov attractors), and also gave examples of the implementation of these scenarios in one-parameter families of three-dimensional maps. Now the theory of discrete homoclinic attractors looks quite advanced, and it is, of course, richer than the corresponding theory of three-dimensional flows, see e.g. [25,26,29,31,32]. On the other hand, there is also a certain feedback here: a detailed study of discrete homoclinic attractors also has led to the discovery of new types of attractors of three-dimensional flows (for example, Lorenz attractors with several equilibria [32]). By the way, these studies are still ongoing.

We note that all spiral attractors of three-dimensional flows and maps, including the Shilnikov attractors, are quasiattractors: although they contain nontrivial hyperbolic subsets, arbitrarily small smooth perturbations also lead to the appearance of stable periodic orbits with extremely narrow absorbing domains. Usually, such orbits cannot be detected in standard numerics, and the attractor still seems to be chaotic. Nevertheless, some stable short-period orbits may appear to be visible – the corresponding parameter domains are called stability windows, and their existence in specific models is a well-known phenomenon in nonlinear dynamics [2,23].

However, spiral attractors of four-dimensional flows, as already shown by Turaev and Shilnikov [52], can be robustly chaotic. Actually, in the paper [52], the theory of pseudohyperbolic attractors, which remain chaotic for all small smooth perturbations, was laid down, and a geometric example of an attractor containing a saddle-focus with a three-dimensional stable and one-dimensional unstable manifold was constructed. Recently, an example of such an attractor in a system of four differential equations (in the generalized Lorenz model) was found in Gonchenko et al. [30]. In the case of four-dimensional maps, the corresponding examples of pseudohyperbolic spiral attractors are not yet known. But we are confident in their existence, including for Shilnikov discrete pseudohyperbolic attractors, and intend to consider the corresponding problems in the nearest future.

Notes

1. Their list is quite large; we will indicate only some of the most known three-dimensional models. Thus, spiral chaos was found in radio-electronic devices, such as the Chua circuit [17], the Anishchenko-Astakhov generator [3], in the optical laser systems [4,5,41,53], in chemical systems [6,11], in a certain class of models describing the behaviour of neurons [18], in biophysical experiments [40], in electromechanical systems [15,35], in electrochemical processes [13,39], in nonlinear convection in magnetic fields [42], in mechanical systems [33], etc.

2. Of course, when inside the whirlpool, there are no other attractors – for example, local bifurcations may lag behind global bifurcations and then, it can happen that both the limit cycle $L_{\mu}$ is stable and a homoclinic loop exists.

3. Such type of bifurcation of doubling of invariant curve is called a component-doubling bifurcation, see more details in Gonchenko et al. [28].

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