COUNTING OCCURRENCES OF 3412 IN AN INVOLUTION

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Abstract
We study the generating function for the number of involutions on \( n \) letters containing exactly \( r \geq 0 \) occurrences of 3412. It is shown that finding this function for a given \( r \) amounts to a routine check of all involutions on \( 2r + 1 \) letters.

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1. Introduction

Permutations. Let \( \pi \in S_n \) and \( \tau \in S_m \) be two permutations. An occurrence of \( \tau \) in \( \pi \) is a subsequence \( 1 \leq i_1 < i_2 < \cdots < i_m \leq n \) such that \( \pi(i_1), \ldots, \pi(i_m) \) is order-isomorphic to \( \tau \); in such a context, \( \tau \) is usually called a pattern.

In the last decade much attention has been paid to the problem of finding the numbers \( s_{r\sigma}(n) \) for a fixed \( r \geq 0 \) and a given pattern \( \tau \) (see [1, 2, 3, 5, 6, 7, 18, 20, 23, 25, 26, 27, 28, 29]). Most of the authors consider only the case \( r = 0 \), thus studying permutations avoiding a given pattern. Only a few papers consider the case \( r > 0 \), usually restricting themselves to patterns of length 3. Using two simple involutions (reverse and complement) on \( S_n \) it is immediate that with respect to being equidistributed, the six patterns of length three fall into the two classes \{123, 321\} and \{132, 213, 231, 312\}. Noonan [22] proved that \( s_{123}(n) = \frac{3}{n} \binom{2n}{n} \). A general approach to the problem was suggested by Noonan and Zeilberger [23]; they gave another proof of Noonan’s result, and conjectured that \( s_{123}(n) = \frac{3}{2n(2n-1)} \binom{2n-1}{n-1} + \frac{1}{n-1} \binom{2n}{n} \) and \( s_{132}(n) = \frac{3}{n-3} \binom{2n-3}{n-3} \). The first conjecture was proved by Fulmek [10] and the latter conjecture was proved by Bóna in [6]. A conjecture of Noonan and Zeilberger states that \( s_{r\sigma}(n) \) is \( P \)-recursive in \( n \) for any \( r \) and \( \tau \). It was proved by Bóna [11] for \( \sigma = 132 \). Mansour and Vainshtein [20] suggested a new approach to this problem in the case \( \sigma = 132 \), which allows one to get an explicit expression for \( s_{132}(n) \) for any given \( r \). More precisely, they presented an algorithm that computes the generating function \( \sum_{n \geq 0} s_{132}(n)x^n \) for any \( r \geq 0 \). To get the result for a given \( r \), the algorithm performs certain routine checks for each element of the symmetric group \( S_{2r} \). The algorithm has been implemented in C, and yields explicit results for \( 1 \leq r \leq 6 \).

Involutions. An involution \( \pi \) is a permutation such that \( \pi = \pi^{-1} \); let \( \mathcal{I}_n \) denote the set of all \( n \)-involutions. An even (resp. odd) involution \( \pi \) is an involution such that the number of occurrences of the pattern 21 in \( \pi \) is an even (resp. odd) number.

Several authors have given enumerations of sets of involutions which avoid certain patterns. In [24] Regev provided an asymptotic formula for \( \mathcal{I}_n(12\ldots k) \) and showed that \( \mathcal{I}_n(1234) \) is enumerated by the \( n \)th Motzkin number \( M_n = \sum_{i \geq 0} \binom{n/2}{i} \binom{n}{i} \). In [11] Gessel enumerated \( \mathcal{I}_n(12\ldots k) \). In [13] Gouyou–Beauchamps gave an entirely bijective proof of some very nice exact formulas for \( \mathcal{I}_n(12345) \) and \( \mathcal{I}_n(123456) \).
Pattern-avoiding involutions have also been related with other combinatorial objects. Gire [12] has established a one-to-one correspondence between 1-2 trees having \( n \) edges and \( S_n(321,3412) \) (this is the set of \( n \)-permutations avoiding patterns 321 and 231, except that the latter is allowed when it is a subsequence of the pattern 3412). On the other hand, Guibert [14] has established bijections between 1-2 trees having \( n \) edges and each of the sets \( S_n(231,4132), I_n(3412), \) and \( I_n(4321) \) (and therefore with \( I_n(1234) \), by transposing the corresponding Young tableaux obtained by applying the Robinson-Schensted algorithm). Also, Guibert [14] has established a bijection between \( I_n(2143) \) and \( I_n(1243) \). More recently, Guibert, Pergola, and Pinzani [15] have given a one-to-one correspondence between 1-2 trees having \( n \) edges and \( I_n(2143) \). It follows that all these sets are enumerated by the \( n \)th Motzkin number \( M_n \). It remains an open problem to prove the conjecture of Guibert (in [14]) that \( I_n(1432) \) is also enumerated by the \( n \)th Motzkin number \( M_n \). This conjecture proved by Jaggard [16]. Recently, Egge [8] gave enumerations and generating functions for the number of involutions in \( I_n(3412) \) which avoid various sets of additional patterns. Egge and Mansour [9] refined Egge’s results by studying generating functions for the number of even and odd involutions in \( I_n(3412) \), and they studied the generating functions for the number of even and odd involutions in \( I_n \) containing the pattern 3412 exactly once. For example, they proved the following theorem.

**Theorem 1.1.** (see [9] Proposition 7.7) The generating function for the number of involutions in \( I_n \) which contain the pattern 3412 exactly once is given by

\[
\frac{2x - 1}{2x^2(1 - x)} + \frac{1 - 2x - 2x^2}{2x^2\sqrt{1 - 2x - 3x^2}}.
\]

| \( r \) \n | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| 0 | 1 | 1 | 2 | 4 | 9 | 21 | 51 | 127 | 323 | 835 | 2188 | 5798 | 15511 |
| 1 | 0 | 0 | 0 | 0 | 1 | 5 | 20 | 70 | 231 | 735 | 2289 | 7029 | 21384 |
| 2 | 0 | 0 | 0 | 0 | 0 | 1 | 7 | 37 | 165 | 671 | 2563 | 9375 |  |
| 3 | 0 | 0 | 0 | 0 | 0 | 1 | 4 | 17 | 63 | 236 | 877 | 3270 |  |
| 4 | 0 | 0 | 0 | 0 | 0 | 2 | 12 | 56 | 220 | 803 | 2783 | 9364 |  |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 14 | 80 | 383 | 1658 | 6690 |  |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 11 | 51 | 212 | 856 | 3402 |

Table 1. First values for the sequence \( I_{3412}(n) \) where \( 0 \leq n \leq 12 \) and \( 0 \leq r \leq 6 \)

In this paper, we suggest a new approach to study the number of involutions which contain the pattern 3412 exactly \( r \) times, namely \( I_{3412}(n) \), which allows one to get an explicit expression for \( I_{3412}(n) \) for any given \( r \) (see Table 1). More precisely, we present an algorithm that computes the generating function \( I_r(x) = \sum_{n \geq 0} I_{3412}(n) x^n \) for any \( r \geq 0 \). To get the result for a given \( r \), the algorithm performs certain routine checks for each element of the involutions \( I_{2r+1} \). The algorithm has been implemented in C, and yields explicit results for \( 0 \leq r \leq 6 \).

2. Recall definitions and preliminary results

2.1. The case \( r \geq 1 \). Fix \( r \geq 1 \). To any \( \pi \in I_n \) we assign a bipartite graph \( G_\pi \) in the following way. The vertices in one part of \( G_\pi \), denoted \( V_1 \), are the entries of \( \pi \), and the vertices of the second part, denoted \( V_2 \), are the occurrences of 3412 in \( \pi \). Entry \( i \in V_1 \) is connected by an edge to occurrence \( j \in V_2 \) if \( i \) enters \( j \). For example, let \( \pi = 823137651112910414 \), then \( \pi \) contains 2 occurrences of 3412, and the graph \( G_\pi \) is presented on Figure 1.
Let \( \tilde{G} \) be an arbitrary connected component of \( G_\pi \), and let \( \tilde{V} \) be its vertex set. We denote \( \tilde{V}_1 = \tilde{V} \cap V_1, \tilde{V}_4 = \tilde{V} \cap V_4, t_1 = |\tilde{V}_1|, t_4 = |\tilde{V}_4|. \)

**Lemma 2.1.** For any connected component \( \tilde{G} \) of \( G_\pi \) one has \( t_1 \leq 2t_4 + 2 \).

**Proof.** Assume to the contrary that the above statement is not true. Consider the smallest \( n \) for which there exists \( \pi \in \mathcal{I}_n \) such that for some connected component \( \tilde{G} \) of \( G_\pi \) one has \( t_1 > 2t_4 + 2 \). \((*)\)

Evidently, \( \tilde{G} \) contains more than one vertex, since otherwise \( t_1 = 1 \) and \( t_4 = 0 \), which contradicts \((*)\). Let \( l \) be the number of leaves in \( \tilde{G} \) (recall that a leaf is a vertex of degree 1). Clearly, all the leaves belong to \( \tilde{V}_1 \); the degree of any other vertex in \( \tilde{V}_1 \) is at least 2, while the degree of any vertex in \( \tilde{V}_4 \) equals 4. Calculating the number of edges in \( \tilde{G} \) by two different ways, we get \( l + 2(t_1 - l) \leq 4t_4 \), which together with \((*)\) gives \( l > 4 \), so there exist at least five leaves in \( \tilde{V}_1 \). In the case \( r = 1 \) there only one occurrence of length 4, so there no five leaves in \( V_1 \), therefore we can assume that \( r \geq 2 \).

Let \( a \in \tilde{V}_4 \) and let us consider the following cases corresponding to the number of leaves in \( V_1 \) incident to \( a \):

(i) If \( a \) incident to one leave \( x \in V_1 \) exactly, then the graph \( \tilde{G}' \) which obtained from \( \tilde{G} \) by deleting \( x \) and \( a \) satisfies \( t_1' + 1 = t_1 \) and \( t_4' + 1 = t_4 \), hence by \((*)\) we have that \( t_1' > 2t_4' + 3 \), a contradiction to the minimality of \( n \).

(ii) If \( a \) incident to two leaves \( x, y \in V_1 \) exactly, then the graph \( \tilde{G}' \) which obtained from \( \tilde{G} \) by deleting \( x, y \) and \( a \) satisfies \( t_1' + 2 = t_1 \) and \( t_4' + 1 = t_4 \), hence by \((*)\) we have that \( t_1' > 2t_4' + 2 \), a contradiction to the minimality of \( n \).

(iii) If there exist four leaves incident to \( a \), then the graph \( \tilde{G} \) does not connect component, a contradiction.

(iv) By Cases(i)-(iii) we can assume that every vertex in \( V_4 \) incident to three leaves exactly. Consider \( a, b \in V_4 \) with the leaves \( a^1, a^2, a^3 \in V_1 \) and \( b^1, b^2, b^3 \in V_1 \) such that there exists a vertex \( u \in V_1 \) incident to \( a \) and \( b \). Therefore, if we consider all the involutions \( \pi \) of length at most 7 with two occurrences \( a \) and \( b \) of 3412 and the corresponding graph \( G_\pi \) is connected

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**Figure 1.** Graph \( G_\pi \) for \( \pi = 823137651112910414 \)
component, then we see that each involution has two occurrences \( x_1 x_2 x_3 x_4 \) and \( y_1 y_2 y_3 y_4 \) of 3412 such that \( \{|x_1, x_2, x_3, x_4| \cap \{y_1, y_2, y_3, y_4|\} = 2 \), a contradiction for that every vertex in \( V_4 \) incident to three leaves exactly (more precisely, in \( I_k \), \( k = 0, 1, 2, \ldots, 5 \), there no involutions which contain 3412 exactly twice; in \( I_6 \) there exists one involution which contain 3412 exactly twice, namely 351624; and in \( I_7 \) we have 7 involutions which contain 3412 exactly twice, namely 1462735, 3516247, 3517264, 3614725, 36175 24, 4261735, and 4631725. Each of these involutions has two occurrences \( x_1 x_2 x_3 x_4 \) and \( y_1 y_2 y_3 y_4 \) of 3412 such that there \( \{|x_1, x_2, x_3, x_4| \cap \{y_1, y_2, y_3, y_4|\} = 2 \).

Hence, using Cases (i)-(iv) we get the desired result. \( \square \)

Denote by \( G^1_\pi \) the connected component of \( G_\pi \) containing entry 1. Let \( \pi(i_1), \ldots, \pi(i_s) \) be the entries of \( \pi \) belonging to \( G^1_\pi \), and let \( \sigma = \sigma_\pi \in \mathcal{I}_s \) be the corresponding involution (\( \sigma_\pi \) is an involution for any involution \( \pi \)). We say that \( \pi(i_1), \ldots, \pi(i_s) \) is the kernel of \( \pi \) and denote it \( \ker \pi; \sigma \) is called the shape of the kernel, or the kernel shape, \( s \) is called the size of the kernel, and the number of occurrences of 3412 in \( \ker \pi \) is called the capacity of the kernel. For example, for \( \pi = 82313765\ 111129\ 10\ 414 \) as above, the kernel equals 8 13 1 4, its shape is 3412, the size equals 4, and the capacity equals 1.

The following statement is implied immediately by Lemma 2.1.

**Theorem 2.2.** Fix \( r \geq 1 \). Let \( \pi \in \mathcal{I}_n \) contain exactly \( r \) occurrences of 3412, then the size of the kernel of \( \pi \) is at most \( 2r + 2 \).

We say that \( \rho \) is a kernel involution if it is the kernel shape for some involution \( \pi \). Evidently \( \rho \) is a kernel involution if and only if \( \sigma_\pi = \rho \) is an involution.

Let \( \rho \in \mathcal{I}_n \) be an arbitrary kernel involution. We denote by \( \mathcal{I}(\rho) \) the set of all the involutions of all possible sizes whose kernel shape equals \( \rho \). For any \( \pi \in \mathcal{I}(\rho) \) we define the kernel cell decomposition as follows. The number of cells in the decomposition equals \( s \times s \). Let ker \( \pi = \pi(i_1), \ldots, \pi(i_s) \); the cell \( C_{ij} = C_{ij}(\pi) \) for \( 1 \leq i \leq s \) and \( 1 \leq j \leq s \) is defined by

\[
C_{ij}(\pi) = \{\pi(j) \mid i < j < i+j+1, \pi(i^{\rho^{-1}(m)}) < \pi(j) < \pi(i^{\rho^{-1}(m+1)})\}
\]

where \( i+j+1 = n+1 \) and \( \alpha_{n+1} = n+1 \) for any \( \alpha \in \mathcal{I}_n \). If \( \pi \) coincides with \( \rho \) itself, then all the cells in the decomposition are empty. An arbitrary permutation in \( \mathcal{I}(\rho) \) is obtained by filling in some of the cells in the cell decomposition. A cell \( C \) is called infeasible if the existence of an entry \( a \in C \) would imply an occurrence of 3412 that contains \( a \), two other entries \( x, y \in \ker \pi \), and another entry \( z \) of \( \pi \). Clearly, all infeasible cells are empty for any \( \pi \in \mathcal{I}(\rho) \). All the remaining cells are called feasible; a feasible cell may, or may not, be empty. The set of all feasible cells is distributed into three subsets:

(i) **Free-cells**; a free-cell is a feasible cell may, or may not, be contain an occurrence of the pattern 12, or contain an occurrence of the pattern 21;

(ii) **Diagonal-decreasing-cells**; a diagonal-decreasing-cell is a feasible cell \( C_{ij} = \{\pi_{j_1}, \ldots, \pi_{j_d}\} \) such that \( \pi_{j_1} > \cdots > \pi_{j_d} \) where \( j_1 < \cdots < j_d \);

(iii) **Decreasing-cells**; a decreasing-cell is a feasible cell \( C_{ij} = \{\pi_{j_1}, \ldots, \pi_{j_d}\} \) with \( i \neq j \) such that \( \pi_{j_1} > \cdots > \pi_{j_d} \) where \( j_1 < \cdots < j_d \).

Consider the involution \( \pi = 82313765\ 111129\ 10\ 414 \). The kernel of \( \pi \) equals 8 13 1 4, its shape is 3412. The cell decomposition of \( \pi \) contains four feasible cells: \( C_{11} = \{2, 3\}, C_{22} = \{7, 6, 5\}, C_{33} = \{11, 12, 9, 10\}, \) and \( C_{44} = \{14\} \), see Figure 2. All the other cells are infeasible; for example, \( C_{21} \) is infeasible, since if \( a \in C_{21} \), then \( a \pi'_{i_2} \pi'_{i_3} \pi'_{i_4} \) is an occurrence of 3412 for any \( \pi' \) whose kernel is of shape 3412. The cells \( C_{11}, C_{33}, \) and \( C_{44} \) are free-cells, and \( C_{22} \) is diagonal-decreasing-cell.
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Figure 2. Kernel cell decomposition for \( \pi \in I(3412) \).

Given a cell \( C_{ij} \) in the kernel cell decomposition, all the kernel entries can be positioned with respect to \( C_{ij} \). We say that \( x = \pi(i_k) \in \ker \pi \) lies below \( C_{ij} \) if \( \rho(k) < i \), and above \( C_{ij} \) if \( \rho(k) \geq i \). Similarly, \( x \) lies to the left of \( C_{ij} \) if \( k < j \), and to the right of \( C_{ij} \) if \( k \geq j \). As usual, we say that \( x \) lies to the southwest of \( C_{ij} \) if it lies below \( C_{ij} \) and to the left of it; the other three directions, northwest, southeast, and northeast, are defined similarly.

The following statement plays a crucial role in our considerations.

**Lemma 2.3.** Let \( \pi \in I(\rho) \).

(i) \( C_{ij} \) is infeasible cell if and only if \( C_{ji} \) is infeasible cell.

(ii) \( C_{ij} \) is feasible cell with \( i \neq j \) if and only if \( C_{ij} \) and \( C_{ji} \) are decreasing-cells.

(iii) Let \( C_{ii} \) be a feasible cell; if there exists northwest or southeast of \( C_{ii} \) an occurrence of the pattern 12 in the kernel shape \( \rho \), then \( C_{ii} \) is diagonal-decreasing-cell, otherwise \( C_{ii} \) is free-cell.

**Proof.** (i) Holds immediately by the fact that \( \pi \) is an involution.

(ii) Let \( i \neq j \). Using the fact that \( \pi \) is an involution we get that \( C_{ij} \) contains an occurrence of the pattern 12 if and only if \( C_{ji} \) contains an occurrence of the pattern 12. Thus, if \( C_{ij} \) contains an occurrence of the pattern 12, namely \( xy \), then \( C_{ji} \) contains an occurrence \( uv \) of the pattern 12, so \( xyuv \) is an occurrence of the pattern 3412, a contradiction. Hence, \( C_{ij} \) is feasible cell if and only if \( C_{ij} \) and \( C_{ji} \) are decreasing-cells.

(iii) Let \( C_{ii} \) be a feasible cell such that there exists northwest of \( C_{ii} \) an occurrence of the pattern 12 in the kernel shape \( \rho \), namely \( xy \). So, if \( C_{ii} \) contains an occurrence of the pattern 12 in \( C_{ii} \), say \( uv \), then \( xyuv \) is an occurrence of the pattern 3412. Hence, \( C_{ii} \) is diagonal-decreasing-cell. \( \square \)

Similarly as the arguments in the proof of Lemma 2.3 we have the following result.

**Lemma 2.4.** Let \( a < b \).

(i) If \( C_{ia} \) and \( C_{ib} \) are decreasing-cells then every entry of \( C_{ia} \) is greater than every entry of \( C_{ib} \).

(ii) If \( C_{ai} \) and \( C_{bi} \) are decreasing-cells then the entries of \( C_{ai} \) are lie northwest of the entries of \( C_{bi} \).

(iii) If \( C_{ib} \) is decreasing-cell, then all the cells \( C_{ij} \), with \( i \neq j \), which are lie northeast of \( C_{ab} \) and \( C_{ba} \) are infeasible cells.

As a consequence of Lemmas 2.3 and 2.4 we get the following result. Let us define a partial order \( \prec \) on the set of all free-cells by saying that \( C_{ml} \prec C_{ml'} \) if \( m \leq m' \) and \( l \leq l' \). Similarly, we define a partial order \( \prec' \) and \( \prec'' \) on the set of all diagonal-decreasing-cells and decreasing-cells, respectively.

**Lemma 2.5.** \( \prec, \prec', \) and \( \prec'' \) are linear orders.

Lemmas 2.3 and 2.4 yield immediately the following two results.

**Theorem 2.6.** Let \( \tilde{G} \) be a connected component of \( G_\pi \) distinct from \( G_\pi^1 \). Then all the vertices in \( \tilde{V}_1 \) belong to the same feasible cell in the kernel cell decomposition of \( \pi \).
Let $F(\rho)$ (respectively; $DD(\rho)$ and $D(\rho)$) be the set of all free-cells (respectively; diagonal-decreasing-cells and decreasing-cells which lie above the diagonal) in the kernel cell decomposition corresponding to involutions in $I(\rho)$ and let $f(\rho) = |F(\rho)|$ (respectively; $dd(\rho) = |DD(\rho)|$ and $d(\rho) = |D(\rho)|$). We remark that, by Lemma 2.3 and Lemma 2.4, we get that $d(\rho)$ is a nonnegative integer number. We denote the cells in $F(\rho)$ by $C^1, \ldots, C^{f(\rho)}$ (respectively; $DD^1, \ldots, DD^{dd(\rho)}$ and $D^1, \ldots, D^{d(\rho)}$) in such a way that $C^i \prec C^j$ (respectively; $DD^i \prec DD^j$ and $D^i \prec D^j$) whenever $i < j$.

**Theorem 2.7.** For any given sequence $\alpha_1, \ldots, \alpha_{f(\rho)}$ of arbitrary involutions, and two sequences $\beta_1, \ldots, \beta_{dd(\rho)}$ and $\gamma_1, \ldots, \gamma_{d(\rho)}$ of arbitrary decreasing involutions (a decreasing involution of length $n$ is the involution $n(n-1)\ldots1$), there exists $\pi \in I(\rho)$ such that the content of the free-cell $C^i$ is order-isomorphic to $\alpha_i$, the content of the diagonal-decreasing-cell $DD^j$ is $\beta_j$, and the content of the decreasing-cell $D^j$ is $\gamma_j$.

2.2. The case $r = 0$. First of all, let us describe the cell block decomposition of an involution $\pi \in I_n(3412)$ as follows.

**Proposition 2.8.** (see [8, Proposition 2.8]) Let $\pi \in I_n(3412)$. Then one of the following holds:

(i) $\pi = (1, \alpha)$ where $(\alpha_1, \ldots, \alpha_{n-1} - 1) \in I_{n-1}(3412)$,

(ii) there exists $t$, $2 \leq t \leq n$, such that $\pi = (t, \alpha, 1, \beta)$ where $(\alpha_1, \ldots, \alpha_{t-2} - 1) \in I_{t-2}(3412)$ and $(\beta_1 - t, \ldots, \beta_{n-t} - t) \in I_{n-t}(3412)$.

Thus, the kernel cell decomposition of $\pi \in I_n(3412)$ can be defined as follows. There are two kernel shapes $\rho^1 = 1$ and $\rho^2 = 21$. In the case $\rho^1$ there exists only one cell which is free-cell, and in the case $\rho^2$ there exist four cells: two are infeasible cells ($C_{21}$ and $C_{12}$), and the others are free-cells ($C_{11}$ and $C_{22}$), see Figure 3.

![Figure 3. Kernel cell decomposition for $\pi \in I(1) \cup I(21)$.](image)

3. Main Theorem and explicit results

Let $\rho$ be a kernel involution, and let $s(\rho)$, $c(\rho)$, $f(\rho)$, $dd(\rho)$, and $d(\rho)$ be the size of $\rho$, the capacity of $\rho$, and the number of free-cells, diagonal-decreasing-cells, decreasing-cells in the cell decomposition associated with $\rho$, respectively. Denote by $K$ the set of all kernel involutions, and by $K_i$ the set of all kernel shapes for involutions in $\mathcal{I}_i$. The main result of this note can be formulated as follows.

**Theorem 3.1.** For any $r \geq 1$,

$$
\mathcal{I}_r(x) = \sum_{\rho \in K_{2r+2}} \left( \frac{x^{s(\rho)}}{(1-x^2)\rho(1-x)^{dd(\rho)}} \sum_{r_1 + \ldots + r_{f(\rho)} = r - c(\rho)} f(\rho) \prod_{j=1}^{f(\rho)} \mathcal{I}_{r_j}(x) \right),
$$

where $r_j \geq 0$ for $1 \leq j \leq f(\rho)$.

**Proof.** For any $\rho \in K$, denote by $\mathcal{I}_r^\rho(x)$ the generating function for the number of involutions in $\pi \in I_n \cap I(\rho)$ containing exactly $r$ occurrences of $3412$. Evidently, $\mathcal{I}_r(x) = \sum_{\rho \in K} \mathcal{I}_r^\rho(x)$. To find $\mathcal{I}_r^\rho(x)$, recall that the kernel of any $\pi$ as above contains exactly $c(\rho)$ occurrences of $3412$. The remaining $r-c(\rho)$ occurrences of $3412$ are distributed between the free-cells of the kernel cell decomposition of $\pi$. By Theorem 2.7 each occurrence of $3412$ belongs entirely to one free-cell. Besides, it follows from
Theorem 3.1 that occurrences of 3412 in different free-cells do not influence one another. Also, by Lemmas 2.8 and 2.9 the diagonal-decreasing-cells do not influence one another, and for decreasing cell \(C_{ij}\) we have that \(|C_{ij}| = |C_{ji}|\) where \(C_{ij}\) contains arbitrary decreasing involution. Therefore,

\[
\mathcal{I}_r^x(x) = \frac{x^s(\rho)}{(1-x^2)^{d(\rho)}(1-x)^{dd(\rho)}} \sum_{r_1 + \cdots + r_{f(\rho)} = r} f(\rho) \prod_{j=1}^{f(\rho)} \mathcal{I}_{r_j}(x),
\]

and we get the expression similar to (3.1) with the only difference that the outer sum is over all \(\rho \in K\). However, if \(\rho \in K_t\) for \(t > 2r + 2\), then by Theorem 2.2, \(c(\rho) > r\), and hence \(\mathcal{I}_r^x(x) \equiv 0\). \(\square\)

Theorem 3.1 provides a finite algorithm for finding \(\mathcal{I}_r(x)\) for any given \(r \geq 0\), since we have to consider all involutions in \(\mathcal{I}_{2r+2}\), and to perform certain routine operations with all shapes found so far. Moreover, the amount of search can be decreased substantially due to the following proposition.

Proposition 3.2. Let \(\psi^0 = 21\), \(\psi^1 = 3412\), \(\psi^2 = 351624\), and

\[
\psi^r = 3 5 1 7 2 9 4 11 \cdots (2r + 1) (2r - 4) (2r + 2) (2r - 2) (2r)
\]

for all \(r \geq 3\). Then the only kernel involution of capacity \(r \geq 0\) and size \(2r + 2\) is \(\psi^r\). Its contribution to \(\mathcal{I}_r(x)\) equals \(\frac{x^{r+2}}{(1-x^2)^{r+2}} \mathcal{I}_0(x)\).

This proposition is proved easily by induction, similarly to Lemma 1. The feasible cells in the corresponding cell decomposition is: \(C_{ii}\) is free-cell if \(i = 1, 3, 5, \ldots, 2r+1, 2r+2\), \(C_{ii}\) is diagonal-decreasing-cell if \(i = 2, 4, \ldots, 2r\), and all the other cells are infeasible cells, hence the contribution to \(\mathcal{I}_r(x)\) is as described. By the above proposition, it suffices to search only involutions in \(\mathcal{I}_{2r+1}\). Below we present several explicit calculations.

3.1. The case \(r = 0\). Let us start from the case \(r = 0\). Observe that (3.1) remains valid for \(r = 0\), provided the left hand side is replaced by \(\mathcal{I}_r(x) - 1\); subtracting 1 here accounts for the empty involution. Also, by Subsection 2.2 we have only two shapes \(\rho^1 = 1\) and \(\rho^2 = 21\) with \(s(\rho^1) = 1\), \(c(\rho^1) = 0\), \(f(\rho^1) = 1\), \(dd(\rho^1) = d(\rho^1) = 0\), \(s(\rho^2) = 2\), \(c(\rho^2) = 0\), \(f(\rho^2) = 2\), and \(dd(\rho^2) = d(\rho^2) = 0\). Therefore, we get

\[
\mathcal{I}_0(x) - 1 = x \mathcal{I}_0(x) + x^2 \mathcal{I}_0^2(x).
\]

Corollary 3.3. (see [14] Rem. 4.28) The generating function for the number of involutions which avoid 3412 is given by

\[
\mathcal{I}_0(x) = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2}.
\]

3.2. The case \(r = 1\). Let now \(r = 1\). The involutions in \(\mathcal{I}_4\) are exhibited only one kernel shape distinct from \(\rho^1\) and \(\rho^2\) which is \(\rho^3 = 3412\), whose contribution equals \(\frac{x^3}{1-x} \mathcal{I}_0^3(x)\) (see Figure 4).

![Figure 4. Kernel cell decomposition for \(\pi \in \mathcal{I}(1) \cup \mathcal{I}(21) \cup \mathcal{I}(3412)\).]

Therefore, (3.1) amounts to \(\mathcal{I}_1(x) = x \mathcal{I}_1(x) + 2x^2 \mathcal{I}_0(x) \mathcal{I}_1(x) + \frac{x^4}{1-x} \mathcal{I}_0^3(x)\), and we get the following result.
Corollary 3.4. The generating function for the number of involutions which contain 3412 exactly once is given by

$$I_1(x) = -\frac{1 - 2x}{2x^2(1 - x)} + \frac{1 - 2x - 2x^2}{2x^2} \sqrt{1 - 2x - 3x^2}^{-1}.$$

3.3. The case $r = 2$. Let $r = 2$. We have to check the kernel shapes of involutions in $I_6$. Exhaustive search adds one new shape to the previous list; this is $\rho^4 = 351624$. Calculation of the parameters $s$, $c$, $f$, $d$, $dd$ is straightforward, and we get

Corollary 3.5. The generating function for the number of involutions which contain 3412 exactly twice is given by

$$I_2(x) = \frac{1 - 2x}{2x^2(1 - x)} - \frac{1 - 6x + 8x^2 + 8x^3 - 15x^4 - 2x^5 + 4x^6}{2x^2(1 - x)^2} \sqrt{1 - 2x - 3x^2}^{-3}.$$

3.4. The cases $r = 3, 4, 5, 6, 7$. Let $r = 3, 4, 5, 6$; exhaustive search in $I_8$, $I_{10}$, $I_{12}$, and $I_{14}$ reveals 2, 5, 12, 25, 48, and 100 new kernel shapes, respectively, and we get

Corollary 3.6. Let $r = 3, 4, 5, 6, 7$. Then the generating function for the number of involutions which contain 3412 exactly $r$ times is given by

$$I_r(x) = \frac{1}{2x^2} F_r(x) + \frac{1}{2x^2} G_r(x) \sqrt{1 - 2x - 3x^2}^{1-2r},$$

where

- $(1 - x^2)F_3(x) = -(1 - 2x)(1 + x + x^2)$,
- $(1 - x^2)F_4(x) = -1 + 3x + 4x^2 - 8x^3 - 2x^4$,
- $(1 - x^2)F_5(x) = 3 - 7x - 7x^2 + 12x^3 + 6x^4$,
- $(1 - x^2)^2F_6(x) = -5 + 9x + 21x^2 - 25x^3 - 34x^4 + 16x^5 + 24x^6 - 2x^7 - 2x^8$,
- $(1 - x^2)^2F_7(x) = 7 - 11x - 28x^2 + 20x^3 + 54x^4 - 2x^5 - 46x^6 + 2x^8$,

and

- $(1 - x^2)G_3(x) = 1 - 8x + 18x^2 + x^3 - 29x^4 + 12x^5 + 14x^6 - 2x^7 - 18x^8$,
- $(1 - x)^4G_4(x) = 1 - 14x + 71x^2 - 124x^3 + 166x^4 + 874x^5 - 624x^6 - 1332x^7 + 1909x^8 + 426x^9 - 1585x^{10} + 292x^{11} + 400x^{12} - 126x^{13}$,
- $(1 - x)^4G_5(x) = -3 + 46x - 267x^2 + 627x^3 + 134x^4 - 3321x^5 + 3954x^6 + 5214x^7 - 11775x^8 - 2186x^9 + 14525x^{10} - 1701x^{11} - 8824x^{12} + 1537x^{13} + 2594x^{14} - 216x^{15} - 324x^{16}$,
- $(1 - x)^6G_6(x) = 5 - 94x + 712x^2 - 2582x^3 + 3124x^4 + 8364x^5 - 31620x^6 + 15464x^7 + 77508x^8 - 107098x^9 + 76814x^{10} + 214160x^{11} + 5782x^{12} - 231050x^{13} + 62700x^{14} + 146176x^{15} - 65653x^{16} - 50328x^{17} + 29646x^{18} + 6462x^{19} - 5346x^{20} + 486x^{21}$,
- $(1 - x)^6G_7(x) = -7 + 144x - 1210x^2 + 5020x^3 - 8206x^4 - 12180x^5 + 69464x^6 - 54210x^7 - 181468x^8 + 315366x^9 + 239852x^{10} - 779338x^{11} - 124766x^{12} + 1226006x^{13} - 168810x^{14} - 1272344x^{15} + 418555x^{16} + 813368x^{17} - 373802x^{18} - 279554x^{19} + 153648x^{20} + 37188x^{21} - 23166x^{22} + 486x^{23}$.
4. Further results

As an easy consequence of Theorem 3.1 and Corollary 3.3 we get the following result.

**Corollary 4.1.** Let \( r \geq 0 \), then \( I_r(x) \) is a rational function in the variables \( x \) and \( \sqrt{1 - 2x - 3x^2} \).

Another direction would be to match the approach of this note with the previous results on even (odd) permutations which contain 132 exactly \( r \) times (see [19]). We denote the generating function for the number of even (resp. odd) involutions in \( I_n \) which contain \( r \) occurrences of 3412 by \( E_r(x) \) (resp. \( O_r(x) \)) (see Table 2).

| \( r \) \( \backslash \) \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | 1 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 1 | 0 | 0 | 0 | 0 | 1 | 5 | 14 | 30 | 82 | 1213 |
| 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 11 | 95 | 439 | 1463 | 4407 |
| 3 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 4 | 11 | 29 | 104 | 396 | 1486 |
| 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 14 | 108 | 321 | 1612 | 4782 |
| 5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 6 | 60 | 275 | 878 | 2247 |
| 6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 21 | 122 | 446 | 1504 |

Table 2. Number of even involutions in \( I_n \) which contain 3412 exactly \( r \) times where \( 0 \leq n \leq 12 \) and \( 0 \leq r \leq 6 \).

We define \( N_r(x) = E_r(x) - O_r(x) \) for any \( r \), that is,

\[
N_r(x) = \sum_{n \geq 0} \sum_{\pi \in I_n \text{ contains 3412 exactly } r \text{ times}} (-1)^{21(\pi)} x^n,
\]

where \( 21(\pi) \) is the number of occurrences of the pattern 21 in \( \pi \). Using the arguments in the proof of Theorem 3.1 we get the following result.

**Theorem 4.2.** For any \( r \geq 0 \),

\[
N_r(x) - \delta_{r,0} = \sum_{\rho \in K_{2r+2}} \left[ (-1)^{21(\rho)} x^{s(\rho)} (1 + x)^{d(\rho)} (1 + x^2)^{dd(\rho)} \right] \sum_{r_1 + \cdots + r_{f(\rho)} = r} \prod_{j=1}^{f(\rho)} N_{r_j}(x),
\]

where \( r_j \geq 0 \) for \( 1 \leq j \leq f(\rho) \), \( \delta_{0,0} = 1 \), and \( \delta_{r,0} = 0 \) for all \( r \geq 1 \).

As an easy consequence of Theorem 4.2 we get the following result.

**Corollary 4.3.** Let \( r \geq 0 \), then \( N_r(x) \) is a rational function in the variables \( x \) and \( \sqrt{1 - 2x + 5x^2} \).

Clearly,

\[
E_r(x) = \frac{1}{2}(I_r(x) + N_r(x)) \quad \text{and} \quad O_r(x) = \frac{1}{2}(I_r(x) - N_r(x)),
\]

for all \( r \geq 0 \). Hence, Corollary 4.3 and Corollary 4.3 give the following result.

**Corollary 4.4.** For any \( r \geq 0 \), the generating functions \( E_r(x) \) and \( O_r(x) \) are rational functions in the variables \( x \), \( \sqrt{1 - 2x - 3x^2} \), and \( \sqrt{1 - 2x + 5x^2} \).
Hence, Theorem 4.2 and Theorem 3.1 provide a finite algorithm for finding \( \mathcal{I}_r(x) \), \( N_r(x) \), \( E_r(x) \), and \( O_r(x) \) for any given \( r \geq 0 \), since we have to consider all involutions in \( \mathcal{I}_{2r+2} \), and to perform certain routine operations with all shapes found so far. Moreover, the amount of search can be decreased substantially due to the following proposition which holds as easily consequence of Proposition 3.1.

**Proposition 4.5.** Let \( \psi^0 = 21, \psi^1 = 3412, \psi^2 = 351624, \) and

\[
\psi^r = 3 \cdot 5 \cdot 1 \cdot 7 \cdot 2 \cdot 9 \cdot 4 \cdot 11 \cdots (2r + 1) \cdot (2r - 4) \cdot (2r + 2) \cdot (2r - 2) \cdot (2r)
\]

for all \( r \geq 3 \). Then the only kernel involution of capacity \( r \geq 0 \) and size \( 2r + 2 \) is \( \psi^r \). Its contribution to \( N_r(x) \) equals \((-1)^{r+1} \frac{2^{r+1}(1+x)^r}{(1+2x)^r} N_0^{r+2}(x) \).

By the above proposition, it suffices to search only involutions in \( \mathcal{I}_{2r+1} \). Similarly as our calculations for \( \mathcal{I}_r(x) \) where \( r = 0, 1, 2, 3, 4, 5, 6, 7 \) with using Theorem 4.2 we get the following result.

**Corollary 4.6.** Let \( r = 0, 1, 2, 3, 4, 5, 6, 7 \). Then the generating function \( N_r(x) \) is given by

\[
N_r(x) = \frac{1}{2x^2} P_r(x) + \frac{1}{2x^2} Q_r(x) \sqrt{1 - 2x + 5x^2} \frac{1 - 2r}{1 - 2r},
\]

where

\[
\begin{align*}
P_0(x) &= x - 1, \\
(x^2 + 1)P_1(x) &= (x + 1)(2x^2 - 2x + 1), \\
(x^2 + 1)^2P_2(x) &= (x - 1)(2x^2 - 2x + 1)(1 + x^2), \\
(x^2 + 1)^3P_3(x) &= (2x^2 - 2x + 1)(x^6 + x^5 + 3x^4 - 2x^2 + x^2 + 1), \\
(x^2 + 1)^4P_4(x) &= (1 - x^2)(2x^8 - 2x^7 + 10x^6 + x^5 + 15x^4 - 12x^3 + 12x^2 - 3x + 1), \\
(x^2 + 1)^5P_5(x) &= -4x^{13} - 14x^{11} + 5x^{10} - 33x^9 + 29x^8 - 16x^7 + 34x^6 - 42x^5 + 14x^4 + 6x^3 \\
&-15x^2 + 7x - 3, \\
(x^2 + 1)^6P_6(x) &= -6x^{16} + 6x^{15} - 44x^{14} + 58x^{13} - 128x^{12} + 163x^{11} - 195x^{10} + 271x^9 - 221x^8 \\
&+188x^7 - 170x^6 + 160x^5 - 82x^4 + 27x^3 + 9x^2 - 9x + 5, \\
(x^2 + 1)^7P_7(x) &= -14x^{18} + 12x^{17} - 80x^{16} + 94x^{15} - 176x^{14} + 212x^{13} - 188x^{12} + 247x^{11} \\
&-157x^{10} + 35x^9 - 51x^8 + 28x^7 + 62x^6 - 142x^5 + 102x^4 - 49x^3 - 3x^2 + 11x - 7,
\end{align*}
\]

and

\[
\begin{align*}
Q_0(x) &= 1, \\
(x^2 + 1)Q_1(x) &= (x^2 - 1)(4x^2 - 2x + 1), \\
(x^2 + 1)^2Q_2(x) &= (22x^6 - 58x^5 + 69x^4 - 48x^3 + 22x^2 - 6x + 1)(1 + x^2), \\
(x^2 + 1)^3Q_3(x) &= (x - 1)(100x^{12} - 18x^{11} + 323x^{10} - 507x^9 + 491x^8 - 182x^7 + 52x^6 - 14x^5 \\
&+46x^4 - 34x^3 + 19x^2 - 5x + 1), \\
(x^2 + 1)^4Q_4(x) &= -(1 + x)(650x^{16} - 1880x^{15} + 5992x^{14} - 9143x^{13} + 13671x^{12} - 19666x^{11} \\
&+26060x^{10} - 28683x^9 + 24771x^8 - 16778x^7 + 9158x^6 - 3969x^5 + 1385x^4 - 374x^3 \\
&+78x^2 - 11x + 1), \\
(x^2 + 1)^5Q_5(x) &= (1 - x)(5000x^{21} - 4650x^{20} + 4264x^{19} - 25585x^{18} + 76987x^{17} - 95269x^{16} \\
&+127936x^{15} - 140244x^{14} + 169896x^{13} - 159580x^{12} + 119898x^{11} - 51878x^{10} \\
&-842x^9 + 26302x^8 - 26778x^7 + 17822x^6 - 8604x^5 + 3270x^4 - 940x^3 + 209x^2 \\
&-31x + 3),
\end{align*}
\]
For example, by Corollary 3.3 and Corollary 4.6 for $r=0$ we have that the generating function for the number even involutions which avoid 3412 is given by

$$E_0(x) = \frac{\sqrt{1 - 2x + 5x^2 - \sqrt{1 - 2x - 3x^2}}}{4x^2},$$

and the generating function for the number of odd involution which avoid 3412 is given by

$$O_0(x) = \frac{2 - 2x - \sqrt{1 - 2x + 5x^2 - \sqrt{1 - 2x - 3x^2}}}{4x^2}.$$

For example, by Corollary 3.3 and Corollary 4.6 for $r=0$ we have that the generating function for the number even involutions which avoid 3412 is given by

$$(x^2 + 1)^6Q_6(x) = -43750x^{27} + 133750x^{26} - 542500x^{25} + 1329674x^{24} - 2984612x^{23} + 537899x^{22} - 859039x^{21} + 1223699x^{20} - 15828644x^{19} + 18229621x^{18} - 19177696x^{17} + 18659837x^{16} - 17024788x^{15} + 14266232x^{14} - 10700428x^{13} + 6908636x^{12} - 3700402x^{11} + 1527142x^{10} - 395516x^9 - 27686x^8 + 10158x^7 - 6867x^6 + 30486x^5 - 10181x^4 + 2580x^3 - 493x^2 + 64x - 5,$

$$Q_7(x) = -481250x^{31} + 1658750x^{30} - 5844500x^{29} + 14332172x^{28} - 29824134x^{27} + 52203592x^{26} - 78380980x^{25} + 104774831x^{24} - 124983968x^{23} + 132048678x^{22} - 122776812x^{21} + 101431782x^{20} - 72478438x^{19} + 39230434x^{18} - 4374004x^{17} - 25236483x^{16} + 42491126x^{15} - 44337242x^{14} + 34831062x^{13} - 21298364x^{12} + 9941638x^{11} - 311120x^{10} + 199166x^9 + 519349x^8 - 425180x^7 + 212566x^6 - 78950x^5 + 22882x^4 - 5138x^3 + 874x^2 - 102x + 7.$
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