On the Penrose inequality for dust null shells in the Minkowski spacetime of arbitrary dimension

Marc Mars and Alberto Soria

Facultad de Ciencias, Universidad de Salamanca, Plaza de la Merced s/n, 37008 Salamanca, Spain

E-mail: marc@usal.es and asoriam@usal.es

Received 13 March 2012
Published 23 May 2012
Online at stacks.iop.org/CQG/29/135005

Abstract

A particular, yet relevant, case of the Penrose inequality involves null shells propagating in the Minkowski spacetime. Despite previous claims in the literature, the validity of this inequality remains open. In this paper, we rewrite this inequality in terms of the geometry of the surface obtained by intersecting the past null cone of the original surface $S$ with a constant time hyperplane and the ‘time height’ function of $S$ over this hyperplane. We also specialize to the case when $S$ lies in the past null cone of a point and show the validity of the corresponding inequality in any dimension (in four dimensions this inequality was proved by Tod (1985 Class. Quantum Grav. 2 L65–8). Exploiting properties of convex hypersurfaces in the Euclidean space, we write down the Penrose inequality in the Minkowski spacetime of an arbitrary dimension $n + 2$ as an inequality for two smooth functions on the sphere $S^n$. We finally obtain a sufficient condition for the validity of the Penrose inequality in the four-dimensional Minkowski spacetime and show that this condition is satisfied by a large class of surfaces.

PACS numbers: 04.20.–q, 02.40.Ft, 03.30.+p, 02.40.–k, 04.70.Bw

1. Introduction

The Penrose inequality is a conjecture on spacetimes containing specific types of spacelike codimension-two surfaces, which play the role of quasi-local replacements of black holes. In addition, the spacetime is assumed to satisfy an energy condition and have suitable asymptotic behaviour at infinity. The inequality bounds from below the total mass of the spacetime in terms of the area of the quasi-local black holes (or suitable surfaces defined in terms of them). The Penrose inequality is important because it provides the strengthening of the positive mass theorem and also, perhaps more importantly, because its validity would give rather strong indirect support for the validity of the weak cosmic censorship conjecture [2]. In fact, the Penrose inequality was originally put forward by Penrose [3] as a way of identifying gravitational configurations that could violate the weak cosmic censorship hypothesis. Since then, and given the absence of counterexamples, the emphasis has turned into trying to
prove the conjecture. So far, the inequality has been proved in full generality in the case of asymptotically flat spacetimes satisfying the null convergence condition and containing a time-symmetric asymptotically flat spacelike hypersurface with an inner boundary composed of outermost closed minimal surfaces. The case of spacetime dimension four and connected inner boundary was dealt with by Huisken and Ilmanen [4]. The case of spacetime dimension up to 8 and no assumption on connectedness of the inner boundary is due to Bray [5]. The other general case where the inequality is known to hold is for spherically symmetric spacetimes satisfying the dominant energy condition (in arbitrary spacetime dimension) [6, 7]. Besides this, there are also many partial results of interest (see [8] for a relatively recent review on this topic and [9] for some new developments).

One version of the Penrose inequality deals with asymptotically flat spacetimes with a regular past null infinity $\mathcal{I}^-$ and satisfying the dominant energy condition. Consider a closed, orientable spacelike $S$ surface and recall that $S$ admits two future-directed null normals $l, k$. If $S$ is such that the null expansion along $l$ vanishes (i.e. it is a marginally outer trapped surface), and moreover, the null hypersurface $\Omega_1$ defined by null geodesics starting on $S$ and tangent to $-k$ extends smoothly all the way to $\mathcal{I}^-$, then the Penrose inequality conjectures that the Bondi mass evaluated at the cut between $\Omega_1$ and $\mathcal{I}^-$ is bounded below in terms of the area of $S$. In four dimensions, the inequality reads

$$M_B \geq \sqrt{\frac{|S|}{16\pi}}.$$  

Ludvigsen and Vickers [10] proposed an argument to prove this inequality in the general case. However, this argument made use of an implicit assumption that does not hold in general [11]. Moreover, it is not easy to write down conditions directly on $S$, which ensures that this extra assumption holds true. The Penrose inequality for the Bondi mass is therefore an open and interesting problem.

A particularly simple case of this version of the inequality can be formulated for spacetimes generated by shells of null dust propagating in the Minkowski spacetime. In fact, this situation was the original setup where the Penrose inequality was discussed in his seminal paper [3]. The idea is the following: imagine that an infinitesimally thin shell of matter is sent from past null infinity in the Minkowski spacetime. The matter content of the shell is null dust, i.e. pressureless matter propagating at the speed of light. Assume that the shape of the shell, as seen by an inertial observer in the Minkowski spacetime, is a convex surface sufficiently near $\mathcal{I}^-$. This guarantees that the null hypersurface defined by the motion of the null dust is regular in a neighbourhood of $\mathcal{I}^-$. Of course, this null hypersurface will develop singularities in the future, where incoming null geodesics meet conjugate points. We denote by $\Omega$ the maximal extension of this null hypersurface as a smooth submanifold in the Minkowski spacetime.

The shell modifies the spacetime geometry after it has passed, and since it is collapsing, it will typically generate a trapped surface in its exterior. The spacetime outside the shell is, in general, very complicated (in particular, because the shell produces gravitational waves), but the interior geometry remains unaffected before the shell goes through. Now, the spacetime geometry right after the shell passes can be determined from the interior geometry and the properties of the shell itself by using the junction conditions between spacetimes; see, e.g., [12]. Moreover, the matter distribution of the shell can be prescribed freely (at one instant of time). It turns out that given any closed (i.e. compact and without boundary), spacelike surface embedded in $\Omega$, the energy distribution of the null shell can be arranged so that $S$ is a marginally outer trapped surface with respect to the spacetime geometry generated by the shell. The direction $l$ along which the null expansion vanishes is transverse to $\Omega$. Moreover, the energy density of the shell determines the Bondi mass of the newly generated spacetime at the cut defined by the intersection of $\Omega$ and $\mathcal{I}^-$. Now, the area of $S$ is the same when measured
with the Minkowskian geometry and when observed from the outside spacetime geometry. Moreover, the jump of the null expansion $\theta_l$ across the shell can be computed in terms of the energy density of the shell. Combining these facts, it follows that the Penrose inequality (1) becomes an inequality for (a suitable class of) closed spacelike surfaces in the Minkowski spacetime. The resulting inequality is (see [3, 1, 13] for four spacetime dimensions and [14] for arbitrary dimension)\[ \int_S \theta_l n \eta_s \geq n (\omega_n)^{\frac{1}{n}} |S|^{\frac{n}{n-1}}, \]where $n$ is the dimension of $S$ (i.e. 2 in the spacetime dimension four) and $\omega_n$ is the area of the standard sphere $\mathbb{S}^n$. In this expression, $|S|$ is the area of the surface $S$ and $\theta_l$ is the null expansion of $S$ with respect to the future-directed, outer (i.e. transverse to $\Omega$) null normal $l$ normalized under the condition $\langle l, k \rangle = -2$, where $k$ is the future-directed null normal to $S$ which is tangent to $\Omega$ and which satisfies $\langle \partial_t, k \rangle = -1$. All these expressions refer to the geometry of the Minkowski spacetime; in particular, $\langle \cdot, \cdot \rangle$ denotes scalar product with the Minkowskian metric and $\partial_t$ is a covariantly constant, unit, timelike vector field in the Minkowski spacetime. The only restriction on the surfaces $S$ is that the null hypersurface obtained by sending light orthogonally from them along $-k$ generates a hypersurface that is regular everywhere and extends all the way to infinity. Geometrically, it is clear that this occurs if and only if the intersection of $\Omega$ with the constant hyperplane $\{ t = t_0 \}$ (for $t_0$ sufficiently negative) is a convex surface of the Euclidean space. We call these surfaces $S$ \textit{spacetime convex} in this paper.

Despite the simplicity of the ambient geometry, proving this inequality is still remarkably difficult. The first case that was solved involved surfaces $S$ that lie on a constant time hyperplane $\{ t = t_0 \}$. In this case, Gibbons proved [15, 14] that the inequality reduces to the classic Minkowski inequality relating the total mean curvature and the area of convex surfaces in the Euclidean space (see expression (4) in section 56 of [16]).

The second case refers to surfaces $S$ contained in the past null cone of a point and leads to a non-trivial inequality for functions on the sphere $\mathbb{S}^n$. In spacetime dimension four, its validity was proved by Tod [1] using the Sobolev inequality in $\mathbb{R}^4$ applied to functions with suitable angular dependence. Regarding the general case, Gibbons claimed [14] to have a general proof. However, the argument contains a serious gap and the problem remains open. This gap was first mentioned in [8] without going into the details. In section 2, we discuss in more detail the argument used by Gibbons and show where it fails.

Our main objective in this paper is to express the Penrose inequality in the Minkowski spacetime of arbitrary dimension in terms of the geometry of the convex Euclidean surface obtained by intersecting $\Omega$ with a constant time hyperplane $\{ t = t_0 \}$ together with the time height function $\tau = |l| - t_0$. This is the contents of Theorem 1. By applying a powerful Sobolev-type inequality on the sphere due to Beckner [18], we prove the validity of this inequality in the case when the surface $S$ lies in the past null cone of a point (Theorem 4). This generalizes to arbitrary dimension the result by Tod [1] in spacetime dimension four mentioned above and shows that a conjecture put forward by this author regarding the optimal form of the inequality is in fact true. The geometry of convex, compact hypersurfaces in Euclidean space can be fully described in terms of a single function $h$ on the unit sphere. This function is called the ‘support function’ and plays an important role in this paper. In spacetime dimension four, the support function was already used in [19] in a related but different context. One of our main results is Theorem 2 where we write down the Penrose inequality in Minkowski as an inequality involving two smooth functions on the $n$-dimensional sphere. Inspired by the argument by Ludvigsen and Vickers [10] and simplified later by Bergqvist [11], we are able to prove (Theorem 6) the validity of this inequality in four spacetime dimensions for a large class of surfaces. This class contains a non-empty open set of surfaces. However, when applied to
surfaces lying on the past null cone of a point, the only case covered by this theorem is when \( S \) is a round sphere. Thus, the cases covered by Theorems 4 and 6 are essentially complementary, which indicates that any attempt of proving the Penrose inequality in Minkowski in the general case will probably require a combination of both methods.

The plan of this paper is as follows. In section 2, we discuss Gibbons’ argument [14] and explain in detail where it fails. For the sake of clarity, we use in this section the same notation and conventions as those in [14]. In section 3, we introduce the notation and conventions that we use in this paper. We also recall some well-known facts on the geometry of null hypersurfaces used later. In section 4, we relate the two null expansions of a spacelike surface embedded in a strictly static spacetime. Although applied in this paper only in the case of the Minkowski spacetime, this result is interesting on its own and has potential application for the Penrose inequality for null dust shells propagating in background spacetimes more general than Minkowski. In section 5, we introduce the notions of spacetime convex null hypersurface and spacetime convex surface, which are useful for stating and studying the Penrose inequality in Minkowski and we rewrite this inequality in terms of the geometry of the projected surface obtained by intersecting the outer-directed past null cone \( \Omega \) of \( S \) with a constant time hyperplane \( \{t = t_0\} \). In section 6, we introduce the support function \( h \) for convex hypersurfaces in Euclidean space and rewrite the inequality in terms of \( h \) and \( \tau \) that identifies \( S \) within \( \Omega \). In section 7, we restrict ourselves to the four-dimensional case and exploit properties of two-dimensional endomorphisms in order to simplify the inequality in terms of \( h \) and \( \tau \). The result is stated in Theorem 5. Finally, in section 8, we prove the validity of the inequality for a large class of surfaces in the four-dimensional case. The method of proof is inspired in the flow of surfaces put forward by Ludvigsen and Vickers [10] and simplified and clarified later by Bergqvist [11]. The explicit form we have for the inequality allows us to make the method work for a much larger class of surfaces than those covered by the original argument. In future work, we intend to study whether this extension can be pushed from the Minkowski spacetime discussed here to more general spacetimes with a complete past null infinity.

2. A critical revision of Gibbons’ argument

In this section, we discuss the gap in Gibbons’ attempt [14] to prove the general inequality (2). Following the notation in [14], we will denote by \( T \) the spacelike, spacetime convex surface involved in the inequality. The future-directed null normals are called \( n^\alpha \) and \( l^\alpha \) and are chosen so that \( n^\alpha \) is inwards (i.e. the geodesics tangent to \( n^\alpha \) generate the null hypersurface \( \Omega \) extending to \( I^- \)) and satisfy \( n^\alpha t_\alpha = -1 \) and \( l^\alpha n_\alpha = -1 \), where \( t^\alpha \) is a covariantly constant, unit, timelike vector field in Minkowski and indices are raised and lowered with the Minkowski metric \( \eta_{\alpha\beta} \). Let us denote by \( \nabla_\alpha \) the covariant derivative in the Minkowski spacetime.

The strategy in [14] was to project \( T \) along \( t^\alpha \) onto a constant time hyperplane orthogonal to \( t^\alpha \). The projected surface is denoted by \( \hat{T} \). The main idea was to rewrite (2) in terms of the geometry of \( \hat{T} \) as a hypersurface in Euclidean space. Gibbons finds that, whenever \( \theta_1 > 0 \), the projected surface \( \hat{T} \) has a non-negative mean curvature and its mean curvature (with respect to the outer unit normal tangent to the constant time hyperplane) \( \hat{J} \) reads (see expression (5.11) in [14])

\[
\hat{J} = \sqrt{\frac{2}{\gamma}} \rho + \sqrt{2\gamma} \mu,
\]
where \( \gamma \defeq -t^\alpha l_\alpha \), \( 2\rho \defeq \nabla_a t^a \) is the null expansion of \( t^a \) (hence \( \rho = \frac{1}{2} \theta_l \) when compared with the normalization that we used in (2)) and \( 2\mu \defeq -\nabla_a n^a \) is minus the null expansion along \( n^a \). As a consequence of (3) and properties of the Minkowski spacetime, we have the following:

\[
\int_T \rho \, dA = \frac{1}{4} \int_{\hat{T}} \hat{J} \, d\hat{A},
\]

where \( dA \) and \( d\hat{A} \) are, respectively, the area elements of \( T \) and \( \hat{T} \). The area of \( \hat{T} \) is not smaller than the area of \( T \) and hence inequality (2) would follow from (4) and the Minkowski-type inequality

\[
\int_{\hat{T}} \hat{J} \, d\hat{A} \geq n(\omega_{\hat{S}})^{\frac{1}{4}}|\hat{T}|^{|\frac{1}{4}|}.
\]

In 1994, Trudinger [20] considered this inequality for general mean convex surfaces in Euclidean space (i.e. surfaces with non-negative mean curvature) and gave an argument for its proof using an elliptic method. However, according to Guan and Li [21], this argument turns out to be incomplete and the inequality is still open (in [21], a parabolic argument is proposed that proves the inequality for mean convex \textit{star-shaped domains} in Euclidean space). Nevertheless, the main problem with Gibbons’ argument does not lie in the validity of (5) but on the orthogonal projection leading to (3). The projection is performed as follows. First, extend \( n^a \) to an ingoing null hypersurface \( \mathcal{N} \) by solving the affinely parametrized null geodesic \( n^a \nabla_a n^b = 0 \) with initial data \( n^a \) on \( S \). Similarly, \( l^a \) is extended to a null vector field on the outgoing null hypersurface \( \mathcal{L} \) passing through \( S \) and with tangent vector \( l^a \). These vector fields are then extended to a spacetime neighbourhood of \( S \) by parallel transport along \( t^a \). With this extension, we have \( n^a l_a = -1 \) everywhere. Defining \( \gamma \) on this neighbourhood by \( \gamma \defeq -t^\alpha l_\alpha \), the following vector field can be introduced:

\[
\hat{\nu}^a = \frac{1}{\sqrt{2\gamma}}(t^a - \gamma n^a).
\]

It follows immediately that \( \hat{\nu}^a \) is everywhere normal to \( t^a \). Moreover, this field is orthogonal to \( \hat{T} \) and unit on this projected surface. Gibbons used in [14] that the mean curvature \( J \) of the projected surface \( \hat{T} \) can be expressed as \( J = \nabla_a \hat{\nu}^a |_{\hat{T}} \). However, the definition of mean curvature gives \( \hat{J} = \nabla_a \hat{\nu}^a |_{\hat{T}} - \frac{1}{2} \hat{\nu}^a \nabla_a (\hat{\nu}, \hat{\nu}) |_{\hat{T}} \). Thus, the expression used by Gibbons is only correct provided \( \hat{\nu}^a \nabla_a (\hat{\nu}, \hat{\nu}) |_{\hat{T}} = 0 \). The extension of \( \hat{\nu}^a \) is uniquely fixed by the definition (6) and \textit{a priori} there is no reason why this vector should remain unit in a neighbourhood of \( \hat{T} \) (or, more precisely, that the derivative of its norm should vanish on \( \hat{T} \)). Moreover, substituting (6) into the (correct) expression for \( J \) gives

\[
\hat{J} = \sqrt{\frac{2}{\gamma}} \rho + \frac{\sqrt{2\gamma}}{\rho} \mu + l^a \nabla_a \left( \frac{1}{\sqrt{2\gamma}} \right) - n^a \nabla_a \left( \sqrt{\frac{\gamma}{2}} \right) - \frac{1}{2} \hat{\nu}^a \nabla_a (\hat{\nu}, \hat{\nu}) |_{\hat{T}},
\]

which agrees with (3) only if the last three terms cancel each other. The third term on the right-hand side of (7) is always zero because \( l^a \nabla_a \gamma = l^a \nabla_a (\gamma^a l_\beta) = -l^\beta n^a \nabla_a l_\beta = 0 \), which follows from the fact that \( l^a \) is geodesic and \( \gamma \) is covariantly constant. However, neither \( \hat{\nu}^a \nabla_a (\hat{\nu}, \hat{\nu}) \) nor the derivative of \( \gamma \) along \( n^a \) need to vanish on \( \hat{T} \). Even more, they need not, and in fact do not, cancel out in general. This fact invalidates (3) that in turn spoils the relationship (4) between the left-hand side of the Penrose inequality (2) and the integral of the mean curvature \( J \) of the projected surface \( \hat{T} \). It is possible to derive general expressions for both \( n^a \nabla_a \gamma \) and \( \hat{\nu}^a \nabla_a (\hat{\nu}, \hat{\nu}) \) on \( T \) (or \( \hat{T} \)), which show that such cancellations do not occur. Instead of doing so, we find it more convenient to present an explicit example where the last two terms
in (7) do not cancel each other. For completeness, we also evaluate \( \dot{J}, \rho \) and \( \mu \) explicitly in this example and show that (3) is not valid.

For the example, we consider spherical coordinates \([t, r, \theta, \phi]\) on Minkowski and consider the past null cone of the origin \( p \) defined by the coordinates \([t = 0, r = 0]\). This past null cone \( \Omega_p \) is defined by the equation \( t + r = 0 \). We consider an axially symmetric (with respect to the Killing vector \( \partial_\phi \) ) spacelike surface \( T \) embedded in \( \Omega_p \). The embedding is then given by \([t = -R(\theta), r = R(\theta), \theta, \phi]\), where \( R \) is a smooth, positive function (satisfying suitable regularity properties at the north and south poles, as usual). With the normalization for \( n^a \), \( l^a \) (and choosing \( t^a = (\partial_t)^a \)), a direct calculation gives

\[
\vec{n}|_T = \partial_t - \partial_r,
\]

\[
\vec{l}|_T = \left( \frac{R^2 + (R')^2}{2R^2} \right) \partial_t + \left( \frac{R^2 - (R')^2}{2R^2} \right) \partial_r - \frac{R'}{R} \partial_\theta,
\]

where prime denotes derivative with respect to \( \theta \). We need to determine the vector field \( l^a(y^\beta) \) as a function of the spacetime coordinates \( y^\beta = [t, r, \theta, \phi] \). The condition that \( l^a \) is parallelly propagated along \( l^a \) means that \( l^a \) does not depend on \( t \), i.e. \( l^a(y^\gamma) \), with \( y^\gamma = [r, \theta, \phi] \). The boundary conditions (9) on \( T \) require

\[
\begin{align*}
l^a(r, \theta, \phi)|_{r=R(\theta)} &= \frac{R^2 + (R')^2}{2R^2}, & l^a(r, \theta, \phi)|_{r=R(\theta)} &= \frac{R^2 - (R')^2}{2R^2}, \\
l^a(r, \theta, \phi)|_{r=R(\theta)} &= -\frac{R'}{R}, & l^a(r, \theta, \phi)|_{r=R(\theta)} &= 0.
\end{align*}
\]

Since \( \gamma = -l^a y^a \nabla_a \gamma = -\partial_t l^a \), Thus, in spherical coordinates, \( n^a \nabla_a \gamma |_T = (\partial_t l^a)|_{r=R(\theta)} \). The component \( \beta = 0 \) of the geodesic equation \( l^a \nabla_a l^b = 0 \) takes the explicit form

\[
l^\gamma \partial_\gamma l^\beta + l^\gamma \partial_\beta l^\gamma + l^\beta \partial_\gamma l^\beta = 0.
\]

Evaluating this expression on \( r = R(\theta) \) and using (10), it is now straightforward to obtain

\[
n^a \nabla_a \gamma |_T = -(\partial_t l^a)|_{r=R(\theta)} = \frac{-2(R')^2 (R'R - (R')^2)}{R^3 (R^2 + (R')^2)}.
\]

Applying a similar argument, it follows that the last term of (7) takes the form

\[
\frac{1}{2} \bar{\theta}^2 \nabla_\theta \bar{\theta} |_{r=R(\theta)} = \frac{(R')^2}{R^2 + (R')^2}.
\]

It is clear that (11) and (12) do not cancel each other in general. As mentioned above, we complete the argument by writing down the explicit expressions for \( \gamma, \rho, \mu \) and \( J \). It is a matter of simple calculation to obtain

\[
\begin{align*}
\gamma |_T &= -\langle t, l \rangle |_T = \frac{R^2 + (R')^2}{2R^2}, \\
\rho |_T &= \frac{R^2 + (R')^2 - RR''}{2R^3} - \frac{R' \cos \theta}{2R^2 \sin \theta}, \\
\mu |_T &= \frac{1}{R}, \\
J |_T &= \frac{1}{\sqrt{R^2 + (R')^2}} \left( \frac{2R^2 + 3(R')^2 - RR''}{R^2 + (R')^2} - \frac{R' \cos \theta}{R \sin \theta} \right).
\end{align*}
\]
Substituting these expressions into the right-hand side of (3) gives

$$\sqrt{\frac{2}{y} + 2\sqrt{y}} \mu \bigg|_{y} = \frac{1}{\sqrt{R^2 + (R')^2}} \left( \frac{2R^2 + 2(R')^2}{R^2} - \frac{R' \cos \theta}{R \sin \theta} \right),$$

which is clearly different to the expression for $\tilde{J}_{y}$ in (13). This proves that (3) cannot be correct. If we instead perform the analogous substitution into (7), we find a consistent expression.

### 3. Notation and basic definitions

Throughout this paper, $(M, g)$ denotes an $(n + 2)$-dimensional spacetime, namely an $(n + 2)$-dimensional oriented manifold endowed with a metric $g$ of Lorentzian signature $+$. We always take $n \geq 2$ and assume $(M, g)$ to be time oriented. Tensors in $M$ carry Greek indices and we denote by $\nabla$ the Levi-Civita covariant derivative of $M$. On a manifold with a metric $\gamma$, we denote by $(\cdot, \cdot)_{\gamma}$ the scalar product with this metric. When the scalar product is with the spacetime metric $g$, we simply write $(\cdot, \cdot)$. Our sign convention for the Riemann tensor is $\text{Riem}(X,Y)Z \overset{\text{def}}{=} \left( \nabla_{X} \nabla_{Y} - \nabla_{Y} \nabla_{X} \right)Z$.

The Penrose inequality in the Minkowski spacetime will involve the geometry of null hypersurfaces, namely codimension one, embedded submanifolds with degenerate first fundamental form. Let $\Omega$ be a null hypersurface and $k$ a future-directed vector field tangent to $\Omega$ that is nowhere zero and null. This vector field is defined up to multiplication with a positive function $F : \Omega \rightarrow \mathbb{R}^+$. It is well known (see, e.g., [22]) that given any point $p \in \Omega$, an equivalence relation can be defined on $T_{p}\Omega$ by means of $X \sim Y$ iff $X - Y = \epsilon k$ with $\epsilon \in \mathbb{R}$. The equivalence class of $X \in T_{p}\Omega$ is denoted by $\bar{X}$ and the quotient space by $T_{p}\Omega/k$. The set $T\Omega/k = \bigcup_{p \in \Omega} T_{p}\Omega/k$ is endowed naturally with the structure of a vector bundle over $\Omega$ (with fibres of dimension $n$), which is called quotient bundle.

Given $\bar{X}, \bar{Y} \in T_{p}\Omega/k$, it follows that $\gamma^{\Omega}(\bar{X}, \bar{Y}) \overset{\text{def}}{=} (X, Y)$ is a positive-definite metric on this quotient space. The tensor $K^{\Omega}(\bar{X}, \bar{Y}) \overset{\text{def}}{=} \langle \nabla_{X}k, Y \rangle$ is well defined (i.e. independent of the representatives $X, Y \in T_{p}\Omega$ of $\bar{X}, \bar{Y}$ and of the extension of $Y$ to a neighbourhood of $p$). This tensor is symmetric and plays the role of a second fundamental form on $\Omega$. The Weingarten map, which we denote by $K^{\Omega}$, is the endomorphism obtained from $K^{\Omega}$ by raising one index with the inverse of $\gamma^{\Omega}$. Finally, the trace of $K^{\Omega}$ is the null expansion $\theta_{k}$ of $\Omega$. Under a rescaling $k \rightarrow Fk$, these tensors transform as $K^{\Omega} \rightarrow FK^{\Omega}$, $K^{\Omega} \rightarrow FK^{\Omega}$ and $\theta_{k} \rightarrow F\theta_{k}$.

A derivative of $T\Omega/k$ can be defined via $(\bar{X}) \overset{\text{def}}{=} \nabla_{X}k$. Again this derivative is well defined (i.e. independent of the representative chosen in the definition). Note, however, that it does depend on the choice of $k$. As usual, this derivative is extended to tensors in $T\Omega/k$ with the Leibniz rule. An important property of null hypersurfaces is that the quotient metric $\gamma^{\Omega}$, the quotient extrinsic curvature $K^{\Omega}$ and the ambient geometry $(M, g)$ are related by the following equations (see, e.g., [22]), which are analogue in the null case to the standard Gauss–Codazzi equations for non-degenerate submanifolds:

$$(\gamma^{\Omega})' = 2K^{\Omega},$$

$$(K^{\Omega})' + K^{\Omega} \circ K^{\Omega} + R - QK^{\Omega} = 0 \quad (\text{Riccati equation}),$$

where $K^{\Omega} \circ K^{\Omega}$ is the composition of endomorphisms, $R(\bar{X}) \overset{\text{def}}{=} \text{Riem}(X, k)k$ and $Q$ is defined by $\nabla_{k}k = Qk$ (the integrals curves of $k$ are necessarily null geodesics but the parameter along them need not be affine).

In order to transform this system of equations into a system of ODE for tensor components, let us choose $k$ to be affinely parametrized, i.e. satisfying $\nabla_{k}k = 0$. Let us also select $n$ vector...
fields $X_k (A, B, C = 1, \ldots, n)$ tangent to $\Omega$ satisfying the properties: (i) $[k, X_{\lambda}] = 0$ and (ii) $[k]_{\gamma}, X_{\lambda}[\gamma]$ is a basis of $T_p\Omega$ at one point $p \in \Omega$. Denote by $\alpha_p(\sigma)$ an affinely parametrized null geodesic containing $p$ and with a tangent vector $k$ (for later convenience we do not fix yet the origin of the affine parameter $\sigma$). Then, $\{X_{\lambda}[\alpha_p(\sigma)]\}$ is a basis of $T_{\alpha_p(\sigma)}\Omega/k$ and the tensor coefficients $\gamma_{AB}^{\Omega}(\sigma), K_{AB}^{\Omega}(\sigma)$ of $\gamma_{\lambda\gamma}^{\Omega}(\sigma)$ and $K_{\lambda\gamma}^{\Omega}(\sigma)$ in this basis satisfy the ODE

$$\frac{d(K^{\Omega})_A}{d\sigma} = -(K^{\Omega})_C(K^{\Omega})^C_B - R^A_B,$$

$$\frac{d(\gamma^{\Omega})_{AB}}{d\sigma} = 2(K^{\Omega})_{AB},$$

where $R^A_B$ are defined by $R(\tilde{X}_A) = R^A_B \tilde{X}_A$ and indices are lowered and raised with the metric $(\gamma^{\Omega})_{AB}$ and its inverse $(\gamma^{\Omega})^{AB}$.

4. Relationship between two null curvatures of a spacelike surface in a strictly static spacetime

In this paper, a spacelike surface $S$ is a connected, codimension-two, spacelike, oriented and closed (i.e. compact and without boundary) smooth, embedded submanifold in a spacetime $(M, g)$. Tensors in $S$ will carry Latin capital indices, and the induced metric and connection on $S$ are denoted, respectively, by $\gamma$ and $D$. Our conventions for the second fundamental form and the mean curvature are $K(X, Y) \overset{\text{def}}{=} -(\nabla_X Y)^\bot$ and $H \overset{\text{def}}{=} \kappa K$. Here $X, Y$ are the tangent vectors to $S$, and $\bot$ denotes the normal component to $S$. If $v$ is a vector field orthogonal to $S$, the second fundamental form along $v$ is $K^v(X, Y) \overset{\text{def}}{=} \langle v, K(X, Y) \rangle = -\langle v, \nabla_X Y \rangle$, with $X, Y \in T_pS$.

The normal bundle of $S$ is a Lorentzian vector bundle that admits a null basis $\{l, k\}$ that we always take smooth, future directed and normalized so that $\langle k, l \rangle = -2$. The second fundamental form $K^\bot$ along $l$ is also called null extrinsic curvature and its trace is the null expansion along $l$, denoted by $\theta_l$. The same applies to the null direction $k$. The mean curvature decomposes into the null basis $\{l, k\}$ as $H = \frac{1}{2}(\theta_l l + \theta_k k)$.

Let us now assume that $M$ is strictly static, i.e. it admits a Killing vector field $\xi$ that is everywhere timelike and hypersurface orthogonal. We define a positive function $V$ by $\langle \xi, \xi \rangle = -V^2$. The integrability of $\xi$ implies, locally, the existence of a smooth function $t$, such that $\xi_\sigma = -V^2 \nabla_\sigma t$. The following lemma shows that the null expansions of any spacelike surface $S$ in a strictly static spacetime are not independent to each other. In the case of the Minkowski spacetime, this result was proved in [14].

**Lemma 1** (Relationship between null extrinsic curvatures). Let $(M, g)$ be an $(n + 2)$-dimensional strictly static spacetime with static Killing vector $\xi$. Let $S$ be a spacelike surface in $(M, g)$. With the above notation, we have

$$-\langle \xi, k \rangle K^\bot_{AB} - \langle \xi, l \rangle K^\bot_{AB} - D_A(\hat{V}^2 D_\theta^l) - D_B(\hat{V}^2 D_\theta^l) = 0,$$

where $\hat{V}$ and $\hat{i}$ are, respectively, the restriction of $V$ and $i$ on $S$.

**Proof.** Since the relationship is local, it suffices to work on a suitably small neighbourhood $U_p$ of a point $p \in S$. We choose $U_p$ small enough so that $\xi_\sigma = -V^2 \nabla_\sigma t$ on $U_p$ and work on $U_p$ from now on. Let $\{X_\lambda\}$ be a basis of the tangent space to $S$ on $U_p$. Decomposing $\xi$ into the basis $\{l, k, X_{\lambda}\}$, we find

$$\xi|_S = -\frac{1}{2}(\xi, k)l - \frac{1}{2}(\xi, l)k - \hat{V}^2 X_{\lambda}(\hat{i}),$$

$$8$$
where $X^C$ is the vector field $X^C \overset{\text{def}}{=} \gamma^{CD} X_D$. The Killing equation $\nabla_\alpha \xi_\beta + \nabla_\beta \xi_\alpha = 0$ implies, on $U_\mu$,

$$
\langle X_B, \nabla_X \xi \rangle + \langle X_A, \nabla_X \xi \rangle = 0.
$$

(17)

Let us work out the first term. Inserting the decomposition (16) and using the definition of null extrinsic curvature $K_{AB}^C$ (and similarly for $k$), we have the following:

$$
\langle X_B, \nabla_X \xi \rangle = -\frac{1}{2} \langle \xi, k \rangle K_{AB}^C - \frac{1}{2} \langle \xi, l \rangle K_{AB}^C - \langle X_B, \nabla_X (\hat{V}^2 X_C(i)) \rangle.
$$

(18)

Now, the tangential projection to $S$ of a spacetime covariant derivative coincides with the intrinsic covariant derivative on $S$. More precisely, for any vector fields $X, Y, Z$ tangent to $S$, we have $\langle X, \nabla_Y Z \rangle = \langle X, D_Y Z \rangle$. Thus, the last term in (18) becomes

$$
\langle X_B, \nabla_X (\hat{V}^2 X_C(i)) \rangle = \langle X_B, D_{X_C} (\hat{V}^2 X_C(i)) \rangle = \langle X_B, D_{X_C} X^C(i) \rangle \hat{V}^2 X_C(i)
$$

$$
+ \gamma (\hat{V}^2 X_B(i)) = D_A (\hat{V}^2 D_B h),
$$

where we used $\langle X_B, X^C \rangle = \delta_B^C$ in the third equality and $\langle X_B, D_{X_C} X^C \rangle = -\Gamma_{AB}^C$, where $\Gamma_{AB}^C$ are the connection coefficients of $D$ in the basis $\{X_A\}$. Inserting this expression into (18), we conclude

$$
\langle X_B, \nabla_X \xi \rangle = -\frac{1}{2} \langle \xi, k \rangle K_{AB}^C - \frac{1}{2} \langle \xi, l \rangle K_{AB}^C - D_A (\hat{V}^2 D_B h),
$$

which combined with (17) proves the lemma. □

Corollary 1. Under the same assumptions as in the previous lemma,

$$
\langle \xi, l \rangle \langle \xi, k \rangle = \hat{V}^2 (1 + \hat{V}^2 |D\hat{i}|^2).
$$

(Here, and in the following, $|Df|^2 = \gamma^{AB} f_A f_B$ for any function $f : S \rightarrow \mathbb{R}$).

Proof. Squaring (16), we have the following:

$$
-\hat{V}^2 = \langle \xi, \xi \rangle = -\langle \xi, k \rangle \langle \xi, l \rangle + \hat{V}^2 \gamma^{CD} X_C(i) X_D(i) = -\langle \xi, k \rangle \langle \xi, l \rangle + \hat{V}^2 |D\hat{i}|^2.
$$

□

5. Penrose inequality in the Minkowski spacetime in terms of the geometry of convex surfaces

We will restrict from now on to the $(n + 2)$-dimensional Minkowski spacetime $(\mathcal{M}^{1,n+1}, \eta)$ ($n \geq 2$). Choose a Minkowskian coordinate system $(t, x^a)$ and define $\xi = \partial_t$. Since this Killing vector is unit, we have $V = 1$ in the notation of the previous section. The hyperplanes at constant $t = t_0$ will be denoted by $\Sigma_0$.

As already mentioned, the physical construction leading to the Penrose inequality involves null hypersurfaces which extend smoothly all the way to past null infinity. We introduce the following definition that captures this notion conveniently (recall that a null hypersurface is maximally extended if it cannot be extended to a larger smooth null hypersurface).

Definition 1 (Spacetime convex null hypersurface). Let $\Omega$ be a maximally extended null hypersurface in $(\mathcal{M}^{1,n+1}, \eta)$. $\Omega$ is spacetime convex if there exists $t_0 \in \mathbb{R}$ for which the surface $\Sigma_0 = \Omega \cap \Sigma_{t_0}$ is closed (i.e. smooth, compact and without boundary), connected and convex as a hypersurface of the Euclidean geometry of $\Sigma_{t_0}$. $\Omega$ is called spacetime strictly convex if $\Sigma_0$ is strictly convex, namely with positive principal curvatures at every point.
Figure 1. Schematic figure representing the construction described in the text where the spacetime convex surface $S$ is projected along $\Omega$ onto the constant time hyperplane $\Sigma_{t_0} = \{ t = t_0 \}$. The vectors on the normal bundle of $S$ are normalized so that $\langle k, \xi \rangle = -1$ and $\langle l, k \rangle = -2$. The vector field $m$ is unit, normal and pointing outside the surface $\tilde{S}_0$ within the hyperplane $\Sigma_{t_0}$.

**Remark.** The idea of the definition is, obviously, that if the shape of the null hypersurface at some instant of Minkowskian time is convex, then the past-directed outgoing null geodesics cannot develop caustics and hence the null hypersurface will extend smoothly to past null infinity. It is also clear that if $\Omega \cap \Sigma_{t_0}$ is closed and convex for some $t_0$, the same occurs for all $t \leq t_0$.

Given a spacetime convex null hypersurface $\Omega$, we always normalize the tangent null vector $k$ uniquely by the condition $\langle k, \xi \rangle = -1$. This vector field will also be normal to any spacelike surface embedded in $\Omega$. Since the Penrose inequality involves precisely this type of surfaces, the following definition is useful.

**Definition 2** (Spacetime convex surface). A spacelike surface $S$ embedded in $(\mathcal{M}^{1,n+1}, \eta)$ is called spacetime (strictly) convex if it can be embedded in a spacetime (strictly) convex null hypersurface $\Omega$ of $(\mathcal{M}^{1,n+1}, \eta)$.

It is intuitively obvious (and easy to prove) that a spacelike surface $S$ can be embedded at most in one spacetime convex null hypersurface $\Omega$. Thus, for any such surface, we can define unambiguously a null basis $\{ l, k \}$ of its normal bundle by the conditions that $k$ is tangent to the spacetime convex null hypersurface $\Omega$ containing $S$ and the normalization conditions $\langle k, \xi \rangle = -1, \langle l, k \rangle = -2$. We refer to $l$ as the outgoing null normal and to $k$ as the ingoing null normal. The Penrose inequality (2) involves the null expansion $\theta_l$ with respect to the outer null normal. The idea that we want to explore in this paper is how this inequality can be related to the geometry of a convex hypersurface of the Euclidean space. The most natural convex surface arising in this setup is precisely the surface $\tilde{S}_0 = \Omega \cap \Sigma_{t_0}$ (see Figure 1). On the other hand, any convex surface $\tilde{S}_0 \hookrightarrow \Sigma_{t_0}$ defines uniquely a spacetime convex null hypersurface $\Omega$, and then, any spacelike surface embedded in $\Omega$ is defined uniquely by the ‘time height’ function over $\Sigma_{t_0}$, namely the function $\tau \overset{\text{def}}{=} t|_{\tilde{S}} - t_0$. This function is defined on $S$. However, there is a canonical diffeomorphism $\phi : S \rightarrow \tilde{S}_0$ defined by the condition that $\phi(p)$ lies on the maximally extended null geodesic $\alpha_p$ passing through $p$ and with a tangent vector $k_p$. This diffeomorphism allows us to transfer geometric information from $S$ onto $\tilde{S}_0$ and vice versa. In
particular, we can define \((\phi^{-1})^*(\tau)\). Since no confusion will arise, we still denote this function by \(\tau\). The precise meaning will be clear from the context.

The idea is thus to transform the Penrose inequality (2) into an inequality involving the geometry of \(\hat{S}_0\) as a hypersurface of Euclidean space \((\mathbb{R}^{n+1}, g_E)\) and the time height function \(\tau\). The result is given in the following theorem.

**Theorem 1** (Penrose inequality in Minkowski in terms of Euclidean geometry). Let 
\((\mathcal{M}^{1,n+1}, \eta)\) be the Minkowski spacetime with a selected Minkowskian coordinate system \((t,x^i)\) and \(\xi = \hat{\alpha}\). Let \((S, \gamma)\) be a spacetime convex surface in \((\mathcal{M}^{1,n+1}, \eta)\) and \(\Omega\) the convex null hypersurface containing \(S\). Consider a closed, convex surface \(\hat{S}_0 = \Omega \cap \Sigma_0\), as a hypersurface of Euclidean space \((\mathbb{R}^{n+1}, g_E)\) and let \(\gamma_0\) be its induced metric, \(\eta_{\mathbb{E}}\) its volume form, \(K_0\) its second fundamental form with respect to the outer unit normal and \(\hat{K}_0\) the associated Weingarten map. Then, the Penrose inequality for \(S\) can be rewritten as

\[
\int_{\hat{S}_0} (1 + [(Id - \tau \hat{K}_0)^{-2}]^{2}g_E) \text{tr}[\hat{K}_0 \circ (Id - \tau \hat{K}_0)^{-1}] \Delta[\tau] \eta_{\mathbb{E}} \\
\geq n(\omega_n)^{\frac{1}{2}} \left( \int_{\hat{S}_0} \Delta[\tau] \eta_{\mathbb{E}} \right)^{\frac{1}{2}},
\]

where \(Id\) is the identity endomorphism, \(\tau = t|_{S} - t_0\) and \(\Delta[\tau] \overset{\text{def}}{=} \text{det}(Id - \tau \hat{K}_0)\).

**Proof.** Let us start by relating \(\theta_t\) with \(\theta_\xi\). Taking the trace of (15) with respect to \(\gamma\) (and using \(V = 1, (k, \xi) = -1\)),

\[
\theta_t - (\xi, l) \theta_k - 2\Delta_\gamma \tau = 0,
\]

where \(\Delta_\gamma = D_\gamma D_\gamma\) is the Laplacian of \((S, \gamma)\). Corollary (1) gives \(- (\xi, l) = 1 + |D\tau|^2\) and the above equation becomes

\[
\theta_t + (1 + |D\tau|^2) \theta_k - 2\Delta_\gamma \tau = 0.
\]

Integrating on \(S\), we have the following:

\[
\int_{\hat{S}} \theta_t \eta_{\mathbb{E}} = -\int_{\hat{S}} (1 + |D\tau|^2) \theta_k \eta_{\mathbb{E}},
\]

which gives the desired relationship.

The second step is to use the Ricatti equations on \(\Omega\) in order to relate \(\theta_k\) on \(S\) with the extrinsic geometry of \(\hat{S}_0\). To that aim, we first note that the vector field \(k\) on \(\Omega\) satisfies \(V_k k = 0\) (this is an immediate consequence of the fact that \(\xi\) is covariantly constant and \((\xi, k) = -1\)). Thus, the Ricatti equations on \(\Omega\) take the form (14) provided we have selected \(n\) vector fields \(\{X_\alpha\}\) tangent to \(\Omega\) and satisfying the requirements that (i) \([k, X_\alpha]\) is 0 and (ii) \([k|_p, X_\alpha|_p]\) is a basis of \(T_p \Omega, \forall p \in \Omega\) (more precisely, \([k, X_\alpha]\) is a basis of the tangent space of \(\Omega\) on suitable open subsets; however, this abuse of notation is standard and poses no complications below). Without loss of generality, we take \(\{X_\alpha\}\) tangent to \(S\). Equations (14) still admit the freedom of choosing the initial value of the affine parameter \(\sigma\) on each one of the null geodesics ruling \(\Omega\). It turns out to be convenient to select \(\sigma\) so that \(\sigma = 0\) on \(\hat{S}_0\). This determines \(\sigma\) uniquely as a smooth function \(\sigma : \hat{S}_0 \rightarrow \mathbb{R}\), which assigns to each point \(p \in \Omega\) the value of the affine parameter of the geodesic starting on \(\hat{S}_0\), with a tangent vector \(k\) and passing through \(p\). Given that

\[
k(t) = dt\ (k) = - (\xi, k) = 1,
\]

and \(l|_{\hat{S}_0} = t_0\), it follows that \(\sigma = t|_\partial - t_0\). In particular, \(\sigma|_S = \tau\) (this is the main reason why this choice of the origin of the affine parameter \(\sigma\) is convenient).
A crucial property of the geometry of a null hypersurface $\Omega$ is that, given any point $p \in \Omega$ and any embedded spacelike surface $S_p$ in $\Omega$ passing through $p$, the induced metric $\gamma_{S_p}$ of $S_p$ and the second fundamental form $K^k_{S_p}$ of $S_p$ along the null normal $k|_p$ satisfy $\gamma_{S_p}(X, Y) = \gamma(\tilde{X}, \tilde{Y})$ and $K^k_{S_p}(X, Y) = K^k(\tilde{X}, \tilde{Y})$, where $X, Y \in T_pS_p$ (see, e.g., [22]). In other words, the induced metric and the extrinsic geometry along $k$ of any embedded spacelike surface in $\Omega$ depend only on $p$ and not on the details of how $S_p$ is embedded in $\Omega$. Applying this result on $\tilde{S}_0$, we have, for any point $\tilde{p} \in \tilde{S}_0$,

$$K^\Omega(\tilde{X}_A, \tilde{X}_B)|_{\tilde{p}} = K^k_{\tilde{S}_0}(\tilde{X}_A, \tilde{X}_B)|_{\tilde{p}},$$  

(21)

where $\tilde{X}_A|_{\tilde{p}}$ is defined by the properties (i) $\tilde{X}_A|_{\tilde{p}} \in \tilde{X}_A|_{\tilde{p}}$ and (ii) $\tilde{X}_A|_{\tilde{p}}$ is tangent to $\tilde{S}_0$ at $\tilde{p}$ (it is immediate that these two properties define a unique $\tilde{X}_A$). Now, the Jordan–Brouwer separation theorem (see, e.g., [23]) states that any connected, closed hypersurface of Euclidean space separates $\mathbb{R}^n$ in two subsets, one with compact closure (called interior) and one with non-compact closure (called exterior). Let $m$ be the unit normal of $\tilde{S}_0$ pointing towards the exterior, and denote by $K_0$ the corresponding second fundamental form and by $k_0$ the associated Weingarten map. Let $(k_0)_{AB}$ be the components of $k_0$ in the basis $\{\tilde{X}_A\}$. Since $\Sigma_{\tilde{S}_0}$ is totally geodesic and $(k, m)|_{\tilde{p}} = -1$ (which follows from the fact that $k$ is ingoing, future-directed, null and satisfies $(k, \xi) = -1$), we have

$$K^k_{\tilde{S}_0}(\tilde{X}_A, \tilde{X}_B)|_{\tilde{p}} = -(k_0)_{AB}|_{\tilde{p}}.$$

(22)

Expressions (21) and (22) provides us with the initial data $K^\Omega_{AB}|_{\sigma = 0} = -(k_0)_{AB}$ for the Ricatti equation (14), which in the Minkowski spacetime simplifies to

$$\frac{d(K^\Omega)^A_{AB}}{d\sigma} = -(K^\Omega)^C_{AB}(K^\Omega)^C_B,$$

(23)

$$\frac{d(\gamma^\Omega)_{AB}}{d\sigma} = 2(k^\Omega)_{AB}.$$  

As is well known (and in any case easy to verify), the solution to these equations with the initial data $K^\Omega_{AB}|_{\sigma = 0} = -(k_0)_{AB}$ is

$$(K^\Omega)^A_{AB}|_{\sigma} = -(k_0)^A_{AB}|_{\sigma} \cdot [(Id - \sigma (p)K_0)_{|\sigma(p)}]^{-1}C_B,$$

(24)

$$(\gamma^\Omega)_{AB}|_{\sigma} = (\gamma_0)_{AC}|_{\sigma(p)} \cdot [(Id - \sigma (p)K_0)_{|\sigma(p)}]^2C_B.$$  

(25)

where $\sigma(p)$ is defined as the unique point on $\tilde{S}_0$ lying on the null geodesic $\alpha_p$. Now, the null expansion $\theta$ is related to $K^\Omega$ by

$$\theta = tr_{\gamma}\ K^k = \gamma^{AB}(\nabla_Xk, X_B) = \gamma^{AC}(\nabla_Xk, X_B) = (\gamma^\Omega)^{AB}K^\Omega_{AB} = (K^\Omega)^A_A.$$  

Evaluating (24) on $S$ (i.e. on $\sigma = \tau$) and taking the trace, we find $\theta|_{\sigma} = -(k_0)^A_A \cdot [(Id - \tau K_0)_{|\sigma(p)}]^{-1}C_B$, or equivalently

$$\theta|_\sigma \circ \phi^{-1} = -tr[K_0 \circ (Id - \tau K_0)^{-1}],$$  

where $\phi \equiv \pi|_S$ is the diffeomorphism between $S$ and $\tilde{S}_0$ introduced above. In order to simplify the notation, we will from now on suppress all references to $\phi$ when transferring information from $S$ to $\tilde{S}_0$ via this diffeomorphism.

The remaining steps are to relate the volume forms of $S$ and $\tilde{S}_0$ and to determine $|D\tau|_\gamma$ (which appears in (20)). Both involve the metric $\gamma$ on $S$. Evaluating (25) on $S$ and using $\gamma_{AB} = \gamma^\Omega_{AB}$, we have the following:

$$\gamma_{AB} = (\gamma_0)_{AC}[\{Id - \tau K_0\}^{2}]C_B.$$  

(26)
By construction, $\gamma_0$ is positive definite, and hence invertible. Obviously, this places restrictions on the range of variation of $\tau$ (which clearly come from the fact that $\Omega$ cannot be extended arbitrarily to the future as a smooth hypersurface). The precise range of variation of $\tau$ will be discussed below. Since $\gamma_0$ is positive definite, it follows that $I d - \tau K_0$ is also invertible and
\[
(y^{-1})^{AB} = [(I d - \tau K_0)^{-2}]^A_C (y_0^{-1})^{CB},
\]
which implies, in particular,
\[
|D\tau|^2 = [(I d - \tau K_0)^{-2}]^A_C (y_0^{-1})^{CB} \tau_A \tau_B.
\]
Taking determinants in (26), it follows that the volume forms of $\hat{S}$ and $S_0$ are related by
\[
\eta_S = \Delta[\tau] \eta_{S_0},
\]
where $\Delta[\tau] \defeq \det(I d - \tau K_0)$. Inserting (26), (28) and (29) into (20), we find
\[
\int_S \eta_S = \int_{\hat{S}_0} (1 + [(I d - \tau K_0)^{-2}]^A_C (y_0^{-1})^{CB} \tau_A \tau_B) \det(K_0 \circ (I d - \tau K_0)^{-1}) \Delta[\tau] \eta_{S_0},
\]
and the Penrose inequality (2) becomes (19), as claimed. \hfill \Box

A natural question for Theorem 1 is what is the class of functions $\tau : \hat{S}_0 \to \mathbb{R}$ for which inequality (19) is conjectured. By construction, this amounts to knowing which is the range of variation of $\sigma$ on the range of variation of $\tau$. Since $\hat{S}_0$ is positive definite, it follows that $\gamma_0$ is also invertible and
\[
\gamma = (y^{-1})^{AB} = [(I d - \tau K_0)^{-2}]^A_C (y_0^{-1})^{CB},
\]
which implies, in particular,
\[
|\tau|^2 = [(I d - \tau K_0)^{-2}]^A_C (y_0^{-1})^{CB} \tau_A \tau_B.
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\[
\int_S \eta_S = \int_{\hat{S}_0} (1 + [(I d - \tau K_0)^{-2}]^A_C (y_0^{-1})^{CB} \tau_A \tau_B) \det(K_0 \circ (I d - \tau K_0)^{-1}) \Delta[\tau] \eta_{S_0},
\]
and the Penrose inequality (2) becomes (19), as claimed. \hfill \Box

6. The Penrose inequality in terms of the support function

A remarkable property of convex hypersurfaces embedded in Euclidean space is that a single function determines all of its geometric properties, both intrinsic and extrinsic, in a very neat way. This function is called the support function and is defined as follows [24].

**Definition 3 (Support function).** Let $\hat{S}_0$ be a closed, convex and connected hypersurface embedded in Euclidean space $(\mathbb{R}^{n+1}, g_E)$. Let $x(p)$ be the position vector of $p \in \hat{S}_0$. The support function $h : \hat{S}_0 \to \mathbb{R}$ is defined by $h(p) = (x(p), m(p))_{g_E}$, where $m(p)$ is the unit normal at $p$ pointing towards the exterior of $\hat{S}_0$.

Closed, convex and connected hypersurfaces in $(\mathbb{R}^{n+1}, g_E)$ are always topologically $\mathbb{S}^n$. Moreover, if the surface is strictly convex, the Gauss map $m : \hat{S}_0 \to \mathbb{S}^n$ is a diffeomorphism. We will restrict ourselves to the strictly convex case from now on. This entails no loss of generality for the Penrose inequality because any convex surface $\hat{S}_0$ can be approximated by strictly convex surfaces (e.g. by mean curvature flow [25]). Let us denote by $\hat{\gamma}$ the pullback on $\hat{S}_0$ of the standard metric on the $n$-sphere and $\hat{\nabla}$ the corresponding connection. Then, the
induced metric $\gamma_0$ and the second fundamental form $K_0$ of $\mathcal{S}_0 \hookrightarrow \mathbb{R}^{n+1}$ can be written in terms of the support function as follows (see, e.g., [26, p. 6]):

\[(K_0)_{AB} = \nabla_A \nabla_B h + \mathcal{T}_{AB} h,\]
\[(\gamma_0)_{AB} = (\mathcal{T}^{-1})^{CD} (K_0)_{AC} (K_0)_{BD}.\]

Combining these formulae with Theorem 1, it becomes possible to rewrite the Penrose inequality for dust null shells in Minkowski as an inequality on the sphere involving two smooth functions, namely $\tau$ and $h$. In this section, we obtain the explicit form of this inequality.

To that aim, it is convenient to introduce the endomorphism $B$ obtained by raising one index to $K_0$ with the spherical metric $\mathcal{T}$, i.e., $B^A_B \overset{\text{def}}{=} (\mathcal{T}^{-1})^{AC} (K_0)_{CB}$. It is immediate from (33) that $B$ is the inverse endomorphism of the Weingarten map $K_0$. Since $\mathcal{S}_0$ is diffeomorphic to $\mathbb{S}^n$ via the Gauss map, we can identify both manifolds and we can think of $\mathcal{T}$, $h$, $B$, etc, as objects defined on $\mathbb{S}^n$. This applies in particular to the function $\tau : \mathcal{S}_0 \rightarrow \mathbb{R}$. With this notation, we can now state and prove the following theorem, which gives the Penrose inequality in Minkowski in terms of the support function.

**Theorem 2** (Penrose inequality in Minkowski in terms of the support function). Let $(S, \gamma)$ be a spacetime strictly convex surface in $(\mathcal{M}^{1,n+1}, \eta)$. With the same notation as in Theorem 1, let $h$ be the support function of $\mathcal{S}_0$. Then, the Penrose inequality takes the form

\[
\int_{\mathcal{S}_0} \left[1 + [(B - \tau 1)I]^{-2}C(\mathcal{T}^{-1})^{CB} \mathcal{T}_{AC} \tau_{AB}\right] \text{tr}[(B - \tau 1)I^{-1}] \det(B - \tau 1) I \eta_{\mathbb{S}^n} \geq n (\omega_n)^{\frac{1}{n}} \left(\int_{\mathbb{S}^n} \det(B - \tau 1) I \eta_{\mathbb{S}^n}\right)^\frac{n-1}{n},
\]

where $\mathcal{T}$, $\nabla$ and $\eta_{\mathbb{S}^n}$ are the standard metric, connection and volume form on $\mathbb{S}^n$:

\[B^A_B \overset{\text{def}}{=} (\mathcal{T}^{-1})^{AC} \nabla_C \nabla_B h + \delta^A_B h,
\]

where $h : \mathbb{S}^n \rightarrow \mathbb{R}$ is the support function of $\mathcal{S}_0 \hookrightarrow \mathbb{R}^{n+1}$ and $\tau : \mathbb{S}^n \rightarrow \mathbb{R}$ is the time height function of $S$.

**Proof.** From (33), it follows that $B$ determines the metric $\gamma_0$ via

\[(\gamma_0)_{AB} = B^C_A B^D_B \mathcal{T}_{CD},\]

which implies

\[\eta_{\mathcal{S}_0} = \det(B) \eta_{\mathbb{S}^n}.
\]

Since $B$ is the inverse of $K_0$, we have

\[\Delta[\tau] \eta_{\mathcal{S}_0} = \det(I - \tau K_0) \eta_{\mathcal{S}_0} = \det(I - \tau K_0) \det(B) \eta_{\mathbb{S}^n} = \det(B - \tau I) \eta_{\mathbb{S}^n}.
\]

Similarly,

\[\text{tr}[K_0 \circ (I - \tau K_0)^{-1}] = \text{tr}[B^{-1} \circ (I - \tau B^{-1})^{-1}] = \text{tr}[(I - \tau I)^{-1}]\]

It only remains to calculate $[(I - \tau K_0)^{-1}]^{AC} (\gamma_0^{-1})^{CB}$. From (36) and using again the fact that $B$ is the inverse of $K_0$, we obtain

\[[(I - \tau K_0)^{-1}]^{AC} (\gamma_0^{-1})^{CB} = [(I - \tau K_0)^{-1}]^{AC} (K_0)^B_F (\mathcal{T}^{-1})^{DF} = [(I - \tau K_0)^{-1}]^{AC} (K_0)^B_F (\mathcal{T}^{-1})^{DF}.
\]

Here, in the last equality, we made use of the property that $(K_0)^B_F (\mathcal{T}^{-1})^{DF}$ is symmetric (this follows from (33), which states in particular that this tensor is the inverse of the symmetric
two-covariant tensor \((K_0)_{BD}\). Since \((Id - \tau K_0)^{-1} \circ K_0 = (B - \tau Id)^{-1}\), we have the following:

\[
[(Id - \tau K_0)^{-2} \circ K_0 = [(Id - \tau K_0)^{-1}] \circ (B - \tau Id)^{-1} \circ K_0 = (B - \tau Id)^{-2},
\]

where in the second equality we have used the fact that \((Id - \tau K_0)\) and \((B - \tau Id)\) commute. Using (41) in (40) yields

\[
[(Id - \tau K_0)^{-2}]^A C (\gamma_0^{-1})^CB = [(B - \tau Id)^{-2}]^A C (\gamma^{-1})^CB.
\]

Substituting (38), (39) and (42) into inequality (19) proves the theorem. \(\square\)

In the following section, we discuss the validity of the Penrose inequality for null dust shells in Minkowski when the incoming shell has spherical shape. Following [19], we refer to this situation as the ‘spherical case’ (note however that the incoming shell need not carry a spherically symmetric matter distribution). In other words, we consider the case when the null hypersurface \(\Omega\) is the past null cone of a point in the Minkowski spacetime and \(S\) is any surface embedded in \(\Omega\). The explicit form of this inequality in the spacetime dimension four appeared already in [3] and led to an inequality for positive functions on the sphere. This inequality turned out to be highly non-trivial. Tod in [1] was able to prove the inequality by using suitable functions on \(\mathbb{R}^4\) and using the Sobolev inequality. In this paper, we show that the Penrose inequality for spherical null dust shells in Minkowski holds in any spacetime dimension.

**6.1. Spherically symmetric case**

Let us restrict ourselves to the case when \(\Omega\) is the past null cone of a point (see Figure 2). As a consequence of Theorem 2, the Penrose inequality transforms in this case into an inequality for a single positive function on the sphere. Its validity will follow as a simple consequence of the Beckner inequality [18], which bounds from above the \(L^q\) norm of a function on the sphere in terms of its \(H^2\) norm. Specifically,

**Theorem 3** (Beckner 1993). Let \(F \in C^1(\mathbb{S}^n)\) and denote as before the standard metric, volume form and connection of the \(n\)-dimensional unit sphere by \(\gamma, \eta_{\mathbb{S}^n}\) and \(\nabla\), respectively. Then,

\[
\frac{q-2}{n} \int_{\mathbb{S}^n} |\nabla F|^2 \eta_{\mathbb{S}^n} + \int_{\mathbb{S}^n} |F|^2 \eta_{\mathbb{S}^n} \geq \left( \int_{\mathbb{S}^n} |F|^q \eta_{\mathbb{S}^n} \right)^{\frac{2}{q}},
\]

where \(2 \leq q < \infty\) if \(n = 1\) or \(n = 2\) and \(2 < q \leq \frac{2n}{n-2}\) if \(n \geq 3\).
The following theorem settles the inequality when $\Omega$ is the past null cone of a point.

**Theorem 4** (Penrose inequality on a past null cone). Consider a point $p \in M^{1,n+1}$ ($n \geq 2$) and $\Omega_p$ the past null cone of $p$. Let $S$ be a closed spacelike surface embedded in $\Omega_p$. Then, the Penrose inequality for $S$ reads

$$
\int_{\mathcal{D}} \left( \rho^{n-1} + \rho^{n-3} |\nabla r|_g^2 \right) \eta_g \geq (\omega_n)^{\frac{1}{n}} \left( \int_{\mathcal{D}} r^n \eta_g \right)^{\frac{n}{n-1}},
$$

where $r = t(p) - t|\mathcal{L}|$. Moreover, this inequality holds true as a consequence of Beckner’s theorem.

**Proof.** Select $t_0 = t(p) - 1$. Then, the function $\tau$ is written in terms of $r$ as $\tau = t|S| - t_0 = t|S| - t(p) + 1 = 1 - r$, and $\mathcal{S}_0 = \Omega_p \cap \Sigma_{t_0}$ is the $n$-dimensional unit sphere embedded in the Euclidean space. This surface has support function $h = 1$, which implies $(K_0)_{AB} = \nabla_A \nabla_B h + \nabla_A h \nabla_B h = \nabla AB$ (this simply states the well-known property that the unit sphere has all principal curvatures equal to 1). Then, $B^A_B = (\nabla^{-1})^\mathcal{C} (K_0)_{CB} = B^\mathcal{C}_B$ and $(\mathcal{B} - \tau \mathcal{I}) = (1 - \tau) \mathcal{I} = \tau \mathcal{I}$, from which

$$
1 + [(\mathcal{B} - \tau \mathcal{I})^{-1}]^2 \hat{\gamma} \gamma_{\mathcal{C}} (\nabla^{-1})^\mathcal{C} \mathcal{B} = 1 + \frac{1}{r^2} |\nabla r|^2.
$$

Substituting into (34) yields immediately (44). In order to show that this inequality is a particular case of the Beckner inequality, we define $q = \frac{2n}{n-2}$ that clearly satisfies the bounds $2 \leq q \leq 4$ if $n \geq 3$. Introducing the function $F = r^\frac{n-2}{2}$, (44) becomes

$$
\left( \frac{2}{n-1} \right)^2 \int_{\mathcal{D}} |\nabla F|_g^2 \eta_g + \int_{\mathcal{D}} F^2 \eta_g \geq (\omega_n)^{\frac{1}{n}} \left( \int_{\mathcal{D}} F^n \eta_g \right)^{\frac{n}{n-1}}.
$$

Since $n \geq 2$, then $\frac{n-2}{n-1} \leq (\frac{2}{n-1})^2$ and inequality (45) is a particular case of (43). \(\square\)

**Remark.** As mentioned above, the case $n = 2$ of this theorem was proved by Tod in [1] using the Sobolev inequality in $\mathbb{R}^2$. In a later paper, Tod proved [27] that the factor $(\frac{2}{n-1})^2$ (i.e. 4 when $n = 2$) in front of the gradient in (45) could be improved to $\frac{8}{3}$ by using the Sobolev inequality of $\mathbb{R}^3$ applied to suitable functions. Tod also conjectured that this factor could be improved to 1. We note that Beckner’s inequality implies in particular the validity of this conjecture by Tod.

7. **Penrose inequality in terms of the support function in spacetime dimension four**

The general expression for the Penrose inequality in terms of the support function as written in Theorem 2 involves the inverse of the endomorphism $\mathcal{B} - \tau \mathcal{I}$, where $B^A_B = \nabla^\mathcal{C} \nabla_B h + \delta^\mathcal{C}_B h$ (for notational simplicity, in this section we will lower and raise all indices with the spherical metric $\nabla$ and its inverse). Hence, the explicit form of the inequality in terms of the support function is rather involved. In this section, we restrict ourselves to the spacetime dimension four, where the expressions simplify notably. The reason is that, in this case, the endomorphism $\mathcal{B}$ acts on a two-dimensional vector space where inverses are much simpler to calculate. In fact,
we will exploit the property that any endomorphism $A : V_2 \rightarrow V_2$ acting on a two-dimensional vector space $V_2$ satisfies the identity

$$A^2 = \text{tr}(A)A - \text{det}(A)Id.$$  

(46)

This identity is a direct consequence of the expression of the minimal polynomial in terms of the eigenvalues of $A$ and the fact that these eigenvalues can be expressed in terms of the trace and determinant of the endomorphism (alternatively, (46) can be proved by direct calculation in any basis). A simple consequence of (46) is that whenever $A$ is invertible,

$$A^{-1} = \frac{-1}{\text{det}(A)}A + \frac{\text{tr}(A)}{\text{det}(A)}Id.$$  

(47)

Taking traces in (46) and (47) yields, respectively,

$$\text{det}(A) = \frac{1}{2}[\text{tr}(A)^2 - \text{tr}(A^2)],$$  

(48)

$$\text{tr}(A^{-1}) = \frac{\text{tr}(A)}{\text{det}(A)}.$$  

(49)

Squaring (47) and using (46) and (48), we obtain an expression for $A^{-2}$ that reads

$$A^{-2} = -\frac{\text{tr}(A)}{[\text{det}(A)]^2}A + \frac{2[\text{tr}(A)^2 + \text{tr}(A^2)]}{[\text{det}(A)]^2}Id.$$  

(50)

Of particular interest below is the case when $A$ is of the form $A = A_0 + fId$ for some scalar $f$. Inserting this, respectively, into (48) and (50) gives, after a straightforward calculation,

$$\text{det}(A_0 + fId) = \frac{1}{2}[\text{tr}(A_0)^2 - \text{tr}(A_0^2)] + f\text{tr}(A_0) + f^2,$$  

(51)

$$(A_0 + fId)^{-2} = -\frac{\text{tr}(A_0) + 2f}{[\text{det}(A_0 + fId)]^2}A_0 + \frac{1}{2}\left[\frac{\text{tr}(A_0)^2 + \text{tr}(A_0^2)}{[\text{det}(A_0 + fId)]^2} + 2f\text{tr}(A_0) + f^2\right]Id.$$  

(52)

Having noticed these algebraic identities, we can now write down the specific form of the Penrose inequality in terms of the support function in the case of four spacetime dimensions.

**Theorem 5.** Let $(S, \gamma)$ be a spacetime strictly convex surface in the Minkowski spacetime $(\mathbb{M}^{1,3}, \eta)$. With the same notation as in Theorem 2, the Penrose inequality can be written in the form

$$\int_{S^2} \left(1 + W_1\nabla \tau \nabla \gamma \right)^2 - W_2(\nabla \nabla \gamma \eta)\nabla \tau \nabla \gamma (\gamma \nabla \delta + 2(h - \tau))\eta_{S^2} \geq \sqrt{16\pi \int_{S^2} \left((h - \tau)^2 + (\gamma \nabla \delta)(h - \tau) - \frac{1}{2}(h \gamma \nabla \delta)\right)\eta_{S^2}},$$  

(53)

where $\Delta_\gamma$ is the Laplacian of the unit 2-sphere and

$$W_1 \overset{\text{def}}{=} \frac{(h - \tau)^2 + 2(h - \tau)\gamma \nabla \delta + \frac{1}{2}[(\gamma \nabla \delta)^2 + (\nabla \nabla \gamma \eta)(\nabla \gamma \delta)]}{[(h - \tau)^2 + (h - \tau)\Delta_\gamma + \frac{1}{2}((\gamma \nabla \delta)^2 - (\nabla \nabla \gamma \eta)(\nabla \gamma \delta))]^2},$$  

(54)

$$W_2 \overset{\text{def}}{=} \frac{(h - \tau)^2 + 2(h - \tau)\gamma \nabla \delta + \frac{1}{2}[(\gamma \nabla \delta)^2 - (\nabla \nabla \gamma \eta)(\nabla \gamma \delta)]}{[(h - \tau)^2 + (h - \tau)\Delta_\gamma + \frac{1}{2}((\gamma \nabla \delta)^2 - (\nabla \nabla \gamma \eta)(\nabla \gamma \delta))]^2}.$$  

(55)
\textbf{Proof.} Define the endomorphism \( A^\Delta h = \nabla^4 \nabla^\Delta h \) so that \( \text{tr}(A^\Delta h) = \Delta^\Delta h \) and \( B - \tau I_d = A^\Delta h + (h - \tau)I_d \). Applying identity (51) with \( f = h - \tau \) gives

\[
\det(B - \tau I_d) = (h - \tau)^2 + \Delta^\Delta h(h - \tau) + \frac{1}{2}((\Delta^\Delta h)^2 - (\nabla_c \nabla^\rho h)(\nabla^\rho \nabla^\nu h)). \tag{56}
\]

Using (49), we have

\[
\text{tr}[(B - \tau I_d)^{-1}] \det(B - \tau I_d) = \text{tr}(B - \tau I_d) = \Delta^\Delta h + 2(h - \tau). \tag{57}
\]

We still need to evaluate \((B - \tau I_d)^{-2}\) from (52). Using the definitions of \( W_1 \) and \( W_2 \), it is immediate to check that

\[
(B - \tau I_d)^{-2} = W_1 I_d - W_2 A^\Delta h. \tag{58}
\]

Substituting (56), (57) and (58) into the left-hand side of inequality (34) gives the left-hand side of (53). In particular, we have obtained an explicit formula for the integral of \( \theta_i \) on \( S \):

\[
\int_S \theta_i \eta_S = \int_{S^2} (1 + W_1(\nabla^4 \nabla^\Delta h) \nabla_A \nabla^\nu h)(\Delta^\Delta h + 2(h - \tau)) \eta_{S^2}. \tag{59}
\]

For the right-hand side of (53), we need to calculate \(|S| = \int_{S^2} \det(B - \tau I_d) \eta_{S^2} \). In particular, we need to integrate \((\Delta^\Delta h)^2 - (\nabla_c \nabla^\rho h)(\nabla^\rho \nabla^\nu h)\) on the sphere. We note the following identity:

\[
\nabla_c [(\nabla^4 \nabla^\Delta h)(\nabla^\rho \nabla^\nu h)] - \nabla_c [(\nabla^\rho \nabla^\nu h)(\nabla^\rho \nabla^\nu h)]
= (\nabla^\rho \nabla^\nu h)(\nabla^\rho \nabla^\nu h) + (\nabla^\rho h)(\nabla_c \nabla^\rho \nabla^\nu h)
= (\nabla^\rho \nabla^\nu h)(\nabla^\rho \nabla^\nu h) - (\nabla^\rho \nabla^\nu h)(\nabla^\rho \nabla^\nu h)
= (\nabla^\rho h)(\nabla^\rho h) - \nabla^\rho h \nabla^\rho h, \tag{60}
\]

where in the last equality we have used the definition of the Riemann tensor and the fact that the sphere has constant curvature equal to 1. Integrating (60) and using the fact that the left-hand side of this expression is a divergence, we have the following:

\[
\int_{S^2} ((\Delta^\Delta h)^2 - (\nabla_c \nabla^\rho h)(\nabla^\rho \nabla^\nu h)) \eta_{S^2} = \int_{S^2} |\nabla h|^2 \eta_{S^2} = \int_{S^2} -2(h \Delta^\Delta h) \eta_{S^2}, \tag{61}
\]

where in the last step we have integrated by parts. Summing up,

\[
|S| = \int_{S^2} (h - \tau)^2 + (\Delta^\Delta h)(h - \tau) - \frac{1}{2}h \Delta^\Delta h \eta_{S^2}, \tag{62}
\]

which on being inserted into the right-hand side of (34) gives the right-hand side of (53) (recall that \( \omega_2 = 4\pi \)).

As already mentioned above, it is well known that when the surface \( S \) lies in a hyperplane of the Minkowski spacetime, the Penrose inequality (2) becomes the classic Minkowski inequality for the total mean curvature \( \hat{J} \) of a surface in the Euclidean space. In the case of 3 + 1 dimensions, the Minkowski inequality reads

\[
\int_{\hat{S}_0} \hat{J} \eta_{\hat{S}_0} \geq \sqrt{16\pi |\hat{S}_0|}. \tag{63}
\]

Using the above theorem, we can obtain the explicit form of the Minkowski inequality in terms of the support function. This result is obviously not new, but stated here for later reference.

\textbf{Corollary 2} (The Minkowski inequality in \((\mathbb{R}^3, g_E)\) in terms of the support function). Let \( \hat{S}_0 \) be a spacetime strictly convex surface embedded in a constant time hyperplane of the Minkowski spacetime \((M^{1,3}, \eta)\). Then, the Minkowski inequality (63) in terms of the support function \( h \) of \( \hat{S}_0 \) takes the form

\[
\left( \int_{S^2} h \eta_{S^2} \right) \geq \sqrt{4\pi \int_{S^2} (h^2 + \frac{1}{2}h \Delta^\Delta h) \eta_{S^2}}. \tag{64}
\]
Proof. Without loss of generality, choose \( t_0 \) as the value of \( t \) on the hyperplane where \( \tilde{S}_0 \) lies. This choice implies \( \tau = 0 \) and that \( h \) is the support function of \( \tilde{S}_0 \). Since \( \nabla_A \tau = 0 \), inequality (53) reduces to (64).

Inequality (53) in terms of the support function is still formidable. However, it is completely explicit in terms of two functions on the sphere. In the next section, we prove its validity for a subset of admissible functions \( \{ h, \tau \} \). This subset has not-empty interior (in any reasonable topology); so, the class of surfaces where the inequality is proved is rather large. The proof is inspired in the flow of surfaces put forward by Ludvigsen and Vickers [10] in their attempt to prove the general Penrose inequality in terms of the Bondi mass. As mentioned in the introduction, Bergqvist [11] found a gap in the argument and showed that the method provides a proof only under additional circumstances, which are, in principle, not straightforward to control directly in terms of the initial surface. In our situation, we have very explicit control of the whole flow of surfaces. This allows us, on the one hand, to find sufficient conditions for the validity of the Penrose inequality directly in terms of the geometry of the initial surface and, on the other, to prove the inequality for a much larger class than the one covered by Bergqvist’s argument. In a future work, we intend to study in detail the relationship between the argument here and the proof in [11] in order to see if the argument here admits a generalization to general spacetimes with complete past null infinity.

8. Dragging the surface along its past null cone

The flow put forward by Ludvigsen and Vickers [10] and analysed further by Bergqvist [11] consists in dragging the initial surface \( S \) along its outer-directed past null cone along affinely parametrized null geodesics. The key property that makes this flow useful is the existence of a monotonic quantity, often called the Bergqvist mass. We start by introducing the flow and defining the Bergqvist mass in our context.

We put ourselves in the setting where \( S \) is a spacetime strictly convex surface in the four-dimensional Minkowski spacetime \((\mathcal{M}^{1,3}, \eta)\), \( \Omega \) is the spacetime convex null hypersurface where it lies and \( \tilde{S}_0 = \Omega \cap \Sigma_{t_0} \) is closed. We have introduced in section 5 a smooth function \( \sigma : \Omega \rightarrow \mathbb{R} \), which assigns to every point \( p \in \Omega \) the affine parameter at \( p \) of the null geodesic tangent to the null vector \( k \) starting on \( S \). By construction, \( \sigma \) vanishes on \( \tilde{S}_0 \) and takes the values \( \sigma|_{S} = \tau \). Let us extend \( \tau \) to a function \( \tau : \Omega \rightarrow \mathbb{R} \) by imposing \( k(\tau) = 0 \) and introduce a new smooth function \( \tilde{\lambda} : \Omega \rightarrow \mathbb{R} \) by \( \tilde{\lambda} = \tau - \sigma \). Geometrically, \( \tilde{\lambda} \) is just a reparametrization of the null geodesics ruling \( \Omega \) (with this parameter the tangent vector is \(-k \) and the geodesics start on \( S \)). It is immediate to see that the level sets \( S_{\lambda} = \{ \tilde{\lambda}^{-1}(\lambda), \lambda \geq 0 \} \) of this function define spacetime convex surfaces embedded in \( \Omega \). The collection of \( \{ S_{\lambda} \}, \lambda \in [0, \infty) \), defines a flow starting at \( S = S_0 \). Let us denote by \( g_{S_{\lambda}} \) and \( \eta_{S_{\lambda}} \) the induced metric and volume form of \( S_{\lambda} \), respectively, and by \( \theta_{\lambda}(\lambda) \) the outer null expansion of \( S_{\lambda} \) (with the normalization \( (l, k) = -2 \), as before). Then, the Bergqvist mass is defined by

\[
M_b(\lambda) \overset{\text{def}}{=} \left( \int_{S_{\lambda}} \theta_{\lambda}(\lambda) \eta_{S_{\lambda}} \right) - 8\pi \lambda.
\]

In [11], the derivative of \( M_b \) with respect to \( \lambda \) is calculated using the spin formalism. For the sake of completeness, let us rederive this derivative using a purely tensorial formalism.

Lemma 2 (Bergqvist [11]). With the above definitions, we have

\[
\frac{dM_b(\lambda)}{d\lambda} = -\int_{S_{\lambda}} 2\langle s_{\lambda}, s_{\lambda} \rangle_{\eta_{S_{\lambda}}} \eta_{S_{\lambda}} \leq 0,
\]
where \( s_i \) is the connection one-form of \( S_i \), defined as \( s_i(X) \overset{\text{def}}{=} -\frac{1}{2} \langle k, \nabla_X l \rangle_{\gamma} \) for any vector field \( X \) tangent to \( S_i \).

**Proof.** Since the variation vector of the flow \( \{S_i\} \) is \(-k\), we need to calculate

\[
\frac{dM_b(\lambda)}{d\lambda} = -8\pi \int_{S_i} \delta_{-k}(\theta_1(\lambda)) \eta_{\Omega^2} + \int_{S_i} \theta_1(\lambda)(\delta_{-k}\eta_{\Omega^2})
\]

\[
= -8\pi \int_{S_i} (\delta_{-k}(\theta_1(\lambda)) - \theta_1(\lambda)(\delta_{-k}\eta_{\Omega^2})),
\]

where \( \delta_{-k} \) stands for geometric variation along \(-k\) and we have used the first variation of volume \( \delta_{k}(\eta_{\Omega_i}) = \theta_i(\lambda)\eta_{\Omega_i} \) (see, e.g., [28]) in the second equality. The first variation of the null expansion \( \theta_1 \) is standard and can be found in many places (see, e.g., (2.23) in [29] where we need to set \( \kappa = 0 \) because our null vector \(-k\) is geodesic and affinely parametrized, or Lemma 3.1 in [30] with \( a = 0 \) for the same reason):

\[
\delta_{-k}(\theta_1(\lambda)) = 2\text{Scal}(S_i) + \theta_1(\lambda)\theta_1(\lambda) - 2s_i s_i + 2\text{div}_S s_i,
\]

where \( \text{Scal}(S_i) \) is the scalar curvature of \((S_i, \gamma_i)\). The surfaces \( S_i \) are topologically spheres; so, the Gauss–Bonnet theorem gives

\[
\int_{S_i} \text{Scal}(S_i)\eta_{\Omega_i} = 8\pi.
\]

Inserting (68) into (66) and using the Gauss–Bonnet theorem proves the lemma.

**Theorem 6** (Class of surfaces where the Penrose inequality in \( M^{1,3} \) holds). Let \((S, \gamma)\) be a spacetime strictly convex surface in \( (M^{1,3}, \eta) \). With the same assumptions and notation as in Theorem 5, let \( h \) be the support function of \( S_0 \) as a hypersurface of Euclidean space and \( \tau = t|S_0 - t_0 \). If these two functions satisfy the inequality

\[
4\pi \int_{\Omega^2} ((\Delta \gamma h)^2 + 2h\Delta \gamma h) \eta_{\Omega^2} \geq 4\pi \int_{\Omega^2} u^2 \eta_{\Omega^2} - \left( \int_{\Omega^2} u \eta_{\Omega^2} \right)^2,
\]

where \( u \overset{\text{def}}{=} \Delta \gamma h + 2(h - \tau) \), then the Penrose inequality (2) holds for \( S \).

**Proof.** In analogy with the definition of \( M_b(\lambda) \) (65), we define a function \( D(\lambda) \) by

\[
D(\lambda) \overset{\text{def}}{=} \sqrt{16\pi |S_i|} - 8\pi \lambda.
\]

With this definition, the Penrose inequality (2) for \( S_i \) can be written in the form

\[
M_b(\lambda) \geq D(\lambda).
\]

Our aim is to prove \( M_b(\lambda = 0) \geq D(\lambda = 0) \). Since Bergqvist’s Lemma 2 ensures that \( M_b(\lambda) \) is monotonically decreasing in \( \lambda \), the idea of the proof is to study the monotonicity properties of \( D(\lambda) \) together with the limiting behaviour of both functions when \( \lambda \to \infty \) in order to see if sufficient conditions can be obtained so that \( M_b(\lambda = 0) \geq D(\lambda = 0) \) holds.

Let us start with the limit of \( M_b(\lambda) \) at infinity. We want to exploit the fact that we obtained in (59) a general expression for the total integral of the outer null expansion \( \theta_1 \) on any spacetime convex surface \( S \), in particular for \( S_0 \). We need to determine the support and the time height function of \( S_0 \). Although it is not the only natural possibility, a convenient choice is to fix one hyperplane \( \Sigma_0 \) and project all surfaces \( S_0 \) along \( \Omega \) onto \( \Sigma_0 \). This procedure has the advantage that \( S_0 \) is the same surface for all \( S_i \) and hence that the support function \( h \) is independent of \( \lambda \).

With this choice, the time height function \( \tau_0 \) of \( S_i \) is

\[
\tau_0 = t|S_i| - t_0 = \sigma |S_i| = \tau - \lambda.
\]
Inserting these functions into (59), we find
\[ M_b(\lambda) = \int_{\mathbb{S}^2} \left( \Delta r h + 2(h - \tau) + (\Delta r h + 2(h - \tau + \lambda)) \right) \]
\[ \times \left( W_1(\lambda)(\nabla^2 r) + W_2(\lambda)(\nabla^2 r h) \right) \eta_{\mathbb{S}^2}, \]
where \( W_1(\lambda) \) and \( W_2(\lambda) \) are obtained from (54)–(55) after substituting \( \tau \to \tau + \lambda \). Since \( W_1(\lambda) \) and \( W_2(\lambda) \) vanish at least as \( \lambda^{-2} \) when \( \lambda \to \infty \) the limit of \( M_b(\lambda) \) is simply
\[ \lim_{\lambda \to \infty} M_b(\lambda) = \int_{\mathbb{S}^2} (\Delta r h + 2(h - \tau)) \eta_{\mathbb{S}^2}. \] (71)

Regarding \( D(\lambda) \), we substitute \( \tau \to \tau_1 \) into (62) to obtain
\[ |S_\lambda| = \int_{\mathbb{S}^2} \left( (h - \tau + \lambda)^2 + \Delta r h \left( \frac{h}{2} - \tau + \lambda \right) \right) \eta_{\mathbb{S}^2}, \] (72)
so that
\[ D(\lambda) = \sqrt{16\pi} \int_{\mathbb{S}^2} \left( (h - \tau + \lambda)^2 + \Delta r h \left( \frac{h}{2} - \tau + \lambda \right) \right) \eta_{\mathbb{S}^2} - 8\pi \lambda. \]

It is straightforward to check that the limit of this expression at infinity is
\[ \lim_{\lambda \to \infty} D(\lambda) = \int_{\mathbb{S}^2} (\Delta r h + 2(h - \tau)) \eta_{\mathbb{S}^2}, \]
which coincides with the limit of \( M_b(\lambda) \) obtained in (71). Since \( M_b(\lambda) \) is monotonically decreasing, and \( M_b(\lambda) \) coincides with \( D(\lambda) \) at infinity, a sufficient condition for the validity of \( M_b(\lambda = 0) \geq D(\lambda = 0) \) is that \( D(\lambda) \) is monotonically increasing. From the definition (70), we have the following:
\[ \frac{dD(\lambda)}{d\lambda} = \sqrt{\frac{4\pi}{|S_\lambda|}} \left( \frac{d|S_\lambda|}{d\lambda} - \sqrt{16\pi |S_\lambda|} \right). \] (73)

It only remains to find out under which conditions the right-hand side of (73) is non-negative. Since \( \frac{d|S_\lambda|}{d\lambda} \geq 0 \) (because \( \theta_k \leq 0 \) on \( \Omega \) and \( \frac{d\theta_k}{d\lambda} = -\theta_k(\lambda) \eta_{S_\lambda} \)), this is equivalent to
\[ \left( \frac{d|S_\lambda|}{d\lambda} \right)^2 - 16\pi |S_\lambda| \geq 0. \] (74)

It is now a matter of simple algebra to show that (74) is equivalent to (69) with the definition \( u \overset{\text{def}}{=} \Delta r h + 2(h - \tau) \).

\[ \square \]

Theorem 6 gives a class of spacetime strictly convex surfaces in the Minkowski spacetime for which the Penrose inequality holds. An important question regarding this result is how large is the class of surfaces covered by the theorem. Since inequality (69) is quadratic in \( h, u \) and its derivatives, a natural strategy is to expand these functions in terms of spherical harmonics and to rewrite (69) as an inequality for the coefficients of these expansions.

Let \( r \in \mathbb{N} \cup 0 \) and \( Y_m^r (m = -r, \ldots, r) \) be \( 2r + 1 \) linearly independent eigenfunctions of the spherical Laplacian with eigenvalue \(-r(r + 1)\), i.e. \( \Delta Y_m^r = -r(r + 1)Y_m^r \). Without loss of generality, we assume that they form an orthonormal basis of \( L^2(\mathbb{S}^2) \), i.e. \( \int_{\mathbb{S}^2} Y_m^r Y_{m'}^s \eta_{\mathbb{S}^2} = \delta^{rs} \delta_{ij} \). Any smooth function \( f \) on the sphere can be decomposed into this basis as
\[ f = \sum_{r=0}^{\infty} a_r \cdot Y^r, \]
where here and in the following we use the notation \( a_r \cdot Y^r \overset{\text{def}}{=} \sum_{m=-r}^{r} a_m^r Y_m^r \). Similarly, we write \( a_r^2 \overset{\text{def}}{=} \sum_{m=-r}^{r} (a_m^r)^2 \). The following corollary identifies the class of surfaces covered in Theorem 6 in terms of the spherical harmonic decompositions of \( h \) and \( u \).
Corollary 3. With the notation of Theorem 6, let us expand the functions $h$ and $u$ in terms of spherical harmonics as

$$h = \sum_{r=0}^{\infty} a_r \cdot Y^r, \quad u = \sum_{r=0}^{\infty} b_r \cdot Y^r. \quad (75)$$

If the coefficients satisfy the inequality

$$\sum_{r=2}^{\infty} a_r^2 r(r+1)(r-1)(r+2) \geq \sum_{r=1}^{\infty} b_r^2, \quad (76)$$

then the Penrose inequality holds for the spacetime convex surface $S$ defined by the support function $h$ and the time height function $\tau = h + \frac{1}{2} \Delta_{\gamma} h - \frac{u^2}{2}$.

Proof. The orthogonality relations of the spherical harmonics imply

$$\int_{S^2} (\Delta_{\gamma} h)^2 \eta_{\mathbb{S}^2} = \sum_{r=0}^{\infty} a_r^2 r^2 (r+1)^2, \quad \int_{S^2} h \Delta_{\gamma} h \eta_{\mathbb{S}^2} = -\sum_{r=0}^{\infty} r(r+1) a_r^2,$$

so that the left-hand side of (69) reads

$$4\pi \int_{S^2} (\Delta_{\gamma} h)^2 \eta_{\mathbb{S}^2} = 4\pi \sum_{r=2}^{\infty} a_r^2 r(r-1)(r+1)(r+2). \quad (77)$$

On the other hand, the spherical harmonic decomposition of $u$ implies $\int_{S^2} u \eta_{\mathbb{S}^2} = \sqrt{4\pi} b_0^2$ and

$$4\pi \int_{S^2} u^2 \eta_{\mathbb{S}^2} - \left( \int_{S^2} u \eta_{\mathbb{S}^2} \right)^2 = 4\pi \sum_{r=0}^{\infty} b_r^2 - (\sqrt{4\pi} b_0^2)^2 = 4\pi \sum_{r=1}^{\infty} b_r^2. \quad (78)$$

Using (77) and (78), we obtain (76), as claimed (we note in passing that (77) and (78) imply that both sides in inequality (69) are non-negative).

 Remark 1. In Theorem 6, we have shown that the Penrose inequality in the spherical case holds as a consequence of the Beckner inequality (or as a consequence of the Sobolev inequality in $\mathbb{R}^m$ in the case of four spacetime dimensions [1]). It is interesting to see how does the spherical case fit into the class of functions covered in Theorem 6. It is well known (and easy to prove) that the support function of a sphere is either a constant (if the origin of Euclidean space coincides with the centre of the sphere) or a linear combination of $r=0, 1$ spherical harmonics (when the sphere is displaced from the origin). In either case, the left-hand side of (76) is identically vanishing, so that the inequality can only hold if the right-hand side also vanishes. This forces $u = \text{const.}$ and hence $\tau = \text{const.}$ too. We see that the only ‘spherical case’ included in Theorem 6 is when the surface $S$ itself is spherically symmetric, which is a trivial case. Thus, in some sense, the cases covered by Beckner’s inequality (which is essentially analytic in nature) and the cases covered by the geometric flow used in Theorem 6 are mutually exclusive. This seems to indicate that any attempt of proving the Penrose inequality for spacetime convex surfaces in the general case most likely needs some sort of combination of both ingredients and almost surely a combination of analytic and geometric arguments.

 Remark 2. The other case where the Penrose inequality in Minkowski was known to hold involves surfaces lying in a constant time hyperplane. It is also natural to see how does this case fit into the class of surfaces covered by Theorem 6. In this situation, we have $\tau = 0$, and hence, $u = \Delta_{\gamma} h + 2h$. Inserting this function into (69), this inequality becomes

$$\left( \int_{S^2} h \eta_{\mathbb{S}^2} \right)^2 \geq 4\pi \int_{S^2} \left( h^2 + \frac{1}{2} \Delta_{\gamma} h \right) \eta_{\mathbb{S}^2}, \quad (79)$$

Class. Quantum Grav. 29 (2012) 135005
M Mars and A Soria
which is exactly the Minkowski inequality for two-dimensional Euclidean surfaces in terms of the support function (see formula (64)). Since the Minkowski inequality is true, it follows that the class of surfaces covered by Theorem 6 includes the case of convex surfaces lying on constant time hyperplanes (incidentally, it is immediate to prove directly the validity of (79) by using the spherical harmonic decomposition \( h = \sum_{\ell=0}^{\infty} a_{\ell} \cdot Y_{\ell} \)).

We finish this section, and the paper, with a particular case of Theorem 6 where the inequality (69) can be interpreted nicely in terms of the geometry of the projected surface \( \hat{S}_0 \) and of the height function \( \tau \) of \( S \).

**Corollary 4.** Let \( \hat{S}_0 \) be a strictly convex surface embedded in a hyperplane \( \Sigma_0 \), and let \( \Omega \) the spacetime convex null hypersurface containing \( \hat{S}_0 \). The Penrose inequality holds for any surface \( S \) embedded in \( \Omega \) and defined by a function \( \tau = t|_{\hat{S}_0} - t_0 \) of the form

\[
\tau = 2\alpha \frac{\hat{J}(\hat{S}_0)}{\text{Scal}(\hat{S}_0)} - \beta,
\]

where \( \alpha \in [0, 1] \), \( \beta \in \mathbb{R} \), and \( \hat{J}(\hat{S}_0) \) and \( \text{Scal}(\hat{S}_0) \) are, respectively, the mean curvature and scalar curvature of \( \hat{S}_0 \) as a hypersurface of Euclidean space.

**Proof.** For surfaces in \( \mathbb{R}^3 \), the scalar curvature can be written as \( \text{Scal}(\hat{S}_0) = 2\kappa_1 \kappa_2 \), where \( \kappa_1 \) and \( \kappa_2 \) are the principal curvatures of \( \hat{S}_0 \) (hence positive everywhere since \( \hat{S}_0 \) is strictly convex). Since \( \hat{J} = \kappa_1 + \kappa_2 \), we have the following:

\[
\tau = 2\alpha \frac{\hat{J}(\hat{S}_0)}{\text{Scal}(\hat{S}_0)} - \beta = \alpha \left( \frac{1}{\kappa_1} + \frac{1}{\kappa_2} \right) - \beta = \alpha \text{tr}(\mathbf{K}_0^{-1}) - \beta = \alpha \text{tr}(\mathbf{B}) - \beta = \alpha(\Delta_{\hat{S}}h + 2h) - \beta.
\]

As a consequence, the function \( u = \Delta_{\hat{S}}h + 2(h - \tau) \) takes the form

\[
u = (1 - 2\alpha)(\Delta_{\hat{S}}h + 2h) + 2\beta,
\]

which in terms of the coefficients in the expansion (75) implies

\[
b_r = -(1 - 2\alpha)(r + 2)(r - 1)a_r, \quad r \geq 1.
\]

Inserting this into (76), we find that this inequality becomes

\[
\sum_{r=2}^{\infty} a_r^2 (r + 2)(r - 1)[r(r + 1) - (1 - 2\alpha)^2(r + 2)(r - 1)] \geq 0.
\]

Since \( h \) is basically arbitrary (it is only restricted by the condition that it defines a strictly convex surface), we need to impose that each term of the sum is non-negative. This is achieved only if

\[
(1 - 2\alpha)^2 \leq \frac{r(r + 1)}{(r + 2)(r - 1)} \overset{\text{def}}{=} Z(r), \quad \forall r \geq 2.
\]

Since the sequence \( Z(r) \) is decreasing and its limit is 1, we see that this inequality holds if \( (1 - 2\alpha)^2 \leq 1 \), which is equivalent to \( \alpha \in [0, 1] \}. Since \( \alpha \) is restricted to this range by hypothesis, (80) holds true and the corollary follows.

**Acknowledgments**

MM is grateful to Lars Andersson, Göran Bergqvist, Jan Metzger, Miguel Sánchez and José M M Senovilla for useful discussions. Financial support under the projects FIS2009-07238 (Spanish MEC) and P09-FQM-4496 (Junta de Andalucía and FEDER funds) are acknowledged. AS acknowledges the PhD grant AP2009-0063 (MEC).
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