Fermionic dual of one-dimensional bosonic particles with derivative delta function potential

B. Basu-Mallick\textsuperscript{1} and Tanaya Bhattacharyya\textsuperscript{1},

\textsuperscript{1}Theory Group, Saha Institute of Nuclear Physics, 1/AF Bidhan Nagar, Kolkata 700 064, India

\textsuperscript{2}Department of Physics, Syamaprasad College, 5/B, R. Das Gupta Road, Kolkata 700 026, India

Abstract

We investigate the boson-fermion duality relation for the case of quantum integrable derivative $\delta$-function bose gas. In particular, we find out a dual fermionic system with nonvanishing zero-range interaction for the simplest case of two bosonic particles with derivative $\delta$-function interaction. The coupling constant of this dual fermionic system becomes inversely proportional to the product of the coupling constant of its bosonic counterpart and the centre-of-mass momentum of the corresponding eigenfunction.

\textsuperscript{*} e-mail address: bireswar.basumallick@saha.ac.in
\textsuperscript{†} e-mail address: tanaya.bhattacharya@saha.ac.in
1 Introduction

As is well known, bosonic and fermionic theories in one spatial dimension can often be related to each other through duality transformations. Such fermion-boson mappings for quantum integrable many-body systems play an important role in the study of their exact solutions. A significant advancement in this direction has been done by establishing the equivalence between one dimensional system of hard-core bosons and spinless free fermions [1]. Even though this equivalence between hard-core bosons and spinless free fermions was initially established only for the ground state of these systems, it can be shown that this type of fermi-bose mapping holds true even for the excited states of impenetrable bosons as well as for all eigenstates of a quantum integrable bose gas interacting through two-body $\delta$-function potential with arbitrary strength of the coupling constant [2,3]. Indeed, it has been established that a novel fermionic model with non-vanishing zero range interaction can be mapped to the above mentioned bose gas interacting through $\delta$-function potential, where the coupling constants of these two models are reciprocal to each other [3]. This fermion-boson equivalence immediately leads to the construction of exact eigenfunctions of the dual fermionic model from the known eigenfunctions of the $\delta$-function bose gas.

In this context it may be noted that, there exists another quantum integrable bosonic system interacting through derivative $\delta$-function potential with Hamiltonian given by

$$H = -\sum_{j=1}^{N} \frac{\partial^2}{\partial x_j^2} + 2i\xi \sum_{l<m} \delta(x_l - x_m)\left(\frac{\partial}{\partial x_l} + \frac{\partial}{\partial x_m}\right),$$

which can be obtained by projecting a certain type of derivative nonlinear Schrödinger (DNLS) field model to its the $N$-particle subspace [4-8]. Classical and quantum versions of such DNLS model have found applications in different areas of physics like circularly polarised nonlinear Alfven waves in plasma [9,10], quantum properties of optical solitons in fibers [11] and in some chiral Luttinger liquids obtained from Chern-Simons model defined in two dimensions [12,13]. Furthermore, it has been established that the ranges of coupling constant $\xi$, which allow the formation of bound states for the exactly solvable Hamiltonian (1.1), are determined through the Farey sequence in number theory [14]. Therefore, it should be interesting to explore whether their exists an exactly solvable fermionic model which is dual to this bosonic system with derivative delta-function potential. In Section 2 of this paper, we construct such a fermionic system with non-vanishing zero range interaction for the simplest case of $N = 2$. For the sake of convenience, we treat these systems in the infinite volume limit, i.e., when all particle coordinates can vary within the full range of the real axis. It turns out that, for $N = 2$ case, the coupling constant appearing in the Hamiltonian of the dual fermionic system becomes inversely proportional to the coupling constant of the bosonic Hamiltonian and also to the centre-of-mass momentum of the corresponding eigenfunction. In Section 3, we attempt to find out the dual fermionic model associated with the derivative $\delta$-function bose gas (1.1) for higher number of particles. Section 4 is the concluding section.
2 Fermion-Boson duality for a system of two particles

Before dealing with the case of derivative $\delta$-function potential, let us briefly discuss how the bosonic two particle system with $\delta$-function potential can be mapped to a fermionic system with short-range interaction [3]. At first, the Hamiltonian of the bosonic two particle system with $\delta$-function potential is reduced to a single particle Hamiltonian depending only on the relative coordinate. Although such a Hamiltonian preserves the continuity of the wavefunction, it induces a discontinuity in the spatial derivative of the wavefunction. It turns out that the fermionic counterpart of this bosonic wavefunction behaves in exactly opposite way; it has a discontinuity in the wavefunction itself but not in its spatial derivative. By applying the method of self-adjoint extension, it is possible to construct a short-range potential which induces this type of discontinuity in the wavefunction and which, therefore, yields the Hamiltonian of the dual fermionic system. In the following, we shall use this approach to find out the dual fermionic system corresponding to the bosonic two particle system with derivative $\delta$-function potential.

For $N=2$, the Hamiltonian (1.1) containing derivative $\delta$-function interaction may be written as

$$H = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + 2i\xi \delta(x_1 - x_2) \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right). \quad (2.1)$$

Let us denote the bosonic wavefunction corresponding to this two-particle system by $\psi_+(x_1, x_2)$, which satisfies the condition $\psi_+(x_1, x_2) = \psi_+(x_2, x_1)$. With the help of the relative and centre of mass coordinates, i.e. $x = x_2 - x_1$ and $X = \frac{1}{2}(x_1 + x_2)$ respectively, the Schrödinger equation $H \psi_+ = E \psi_+$ can be expressed as

$$-2\frac{\partial^2 \psi_+}{\partial x^2} - \frac{1}{2} \frac{\partial^2 \psi_+}{\partial X^2} + 2i\xi \delta(x_1 - x_2) \frac{\partial \psi_+}{\partial X} = E \psi_+. \quad (2.2)$$

It may be observed that the Hamiltonian (2.1) commutes with the total momentum operator $P$ given by

$$P = -i \frac{\partial}{\partial x_1} - i \frac{\partial}{\partial x_2}. \quad (2.3)$$

Consequently, $\psi_+$ might be chosen as a simultaneous eigenstate of the Hamiltonian (2.1) and the momentum operator (2.3). So we write $\psi_+$ in the form

$$\psi_+ = e^{iKX} \phi_+(x), \quad (2.4)$$

where $\phi_+(x) = \phi_+(-x)$ and $P \psi_+ = K \psi_+$. Substituting the expression of $\psi_+$ (2.4) in Eq.(2.2), we obtain

$$-\frac{\partial^2 \phi_+}{\partial x^2} - K\xi \delta(x) \phi_+ (x) = E_r \phi_+ (x) \quad (2.5)$$

where $E_r = \frac{E}{2} - \frac{K^2}{4}$. Note that the above equation can be written in the form $H_r \phi_+ (x) = E_r \phi_+ (x)$, where $H_r$ is given by

$$H_r = -\frac{\partial^2}{\partial x^2} - K\xi \delta(x). \quad (2.6)$$
It is interesting to observe that, in contrast to the case of two particles interacting through \( \delta \)-function potential \([3]\), coupling constant of the Hamiltonian (2.6) now depends on the centre-of-mass momentum of the two-particle wavefunction. In the following, it will be shown that the fermionic dual of \( H_r \) also explicitly depends on this centre-of-mass momentum.

By integrating Eq. (2.5) within a small interval \([-\epsilon, \epsilon]\) and taking \( \epsilon \to 0 \) limit, we obtain
\[
\left. \frac{\partial \phi_+}{\partial x} \right|_{x \to +0} - \left. \frac{\partial \phi_+}{\partial x} \right|_{x \to -0} = -K\xi \phi_+ \bigg|_{x=0}. 
\]
(2.7)

Applying the symmetry relation \( \phi_+(-x) = \phi_+(x) \), it is easy to see that while the spatial derivative of \( \phi_+(x) \) changes its sign around \( x = 0 \), the wavefunction itself is continuous around this point:
\[
\phi_+(0_+) = \phi_+(0_-); \quad \phi_+'(0_+) = -\phi_+'(0_-). 
\]
(2.8)

where \( \phi_+(0_+) = \lim_{a \to 0} \phi_+(\pm a) \) and \( \phi_+'(0_+) = \lim_{a \to 0} \phi_+'(\pm a) \). Now, by simultaneously using Eqs. (2.7) and (2.8), we get the boundary conditions on the bosonic wavefunction as
\[
\phi_+'(0_+) = -\phi_+'(0_-) = -\frac{K\xi}{2} \phi_+(0_+) = -\frac{K\xi}{2} \phi_+(0_-). 
\]
(2.9)

Next, we define another wavefunction depending on \( x \) as
\[
\phi_-(x) = [\theta(x) - \theta(-x)] \phi_+(x), 
\]
(2.10)

where \( \theta(x) = 1 \) when \( x > 0 \) and \( \theta(x) = 0 \) when \( x < 0 \). It is easy to see that \( \phi_-(x) \) satisfies the symmetry relation: \( \phi_-(x) = -\phi_-(x) \). Hence, if we consider a state like \( \psi_-(x) = e^{ikX} \phi_-(x) \), it will represent a two particle fermionic wavefunction. Using Eqs. (2.9) and (2.10), we find that the wavefunction \( \phi_-(x) \) and its spatial derivative satisfy the following conditions around \( x = 0 \):
\[
-\phi_-(0_+) = \phi_-(0_-) = \frac{2}{K\xi} \phi_-'(0_+) = \frac{2}{K\xi} \phi_-'(0_-). 
\]
(2.11)

Hence, the fermionic wavefunction \( \phi_-(x) \) is discontinuous around \( x = 0 \), while its spatial derivative is continuous around this point.

Even though in elementary quantum mechanics one does not usually encounter potentials leading to the discontinuity of the wavefunctions, by using the method of self-adjoint extension it is possible to show that short-range potentials can generate discontinuity for both the wavefunction and its spatial derivative [15-19]. Denoting the wavefunction associated with such short-range potential by \( \phi(x) \) and using notations like before, we can express the discontinuity around the point \( x = 0 \) as
\[
\phi_+'(0_+) + \alpha \phi_+'(0_-) = -\beta \phi(0_-), \quad \phi(0_+) + \gamma \phi(0_-) = -\delta \phi_-'(0_-), 
\]
(2.12)

where \( \alpha, \beta, \gamma \) and \( \delta \) are arbitrary real numbers satisfying the constraint
\[
\alpha \gamma - \beta \delta = 1. 
\]
(2.13)
The short range potential which gives rise to the discontinuity (2.12) may be expressed in the form [19]

\[ \chi(x; \alpha, \beta, \gamma, \delta) = \lim_{a \to +0} \left[ u_\delta(x + a) + u_0\delta(x) + u_\delta(x - a) \right] \]  

(2.14)

where

\[ u_+(a) = -\frac{1}{a} + \frac{\alpha - 1}{\delta}, \quad u_-(a) = -\frac{1}{a} + \frac{\gamma - 1}{\delta}, \quad u_0(a) = \frac{1 - \alpha\gamma}{\beta a^2}. \]  

(2.15)

Note that if we choose the values of the free parameters as \( \alpha = -1, \beta = 0, \gamma = -1, \) and \( \delta = \frac{4}{K\xi}, \) and also assume that \( \phi(x) \) satisfies the fermionic exchange relation \( \phi(-x) = -\phi(x), \) then the general form of discontinuity relation (2.12) reduces to Eq.(2.11). Consequently, by substituting the above mentioned values of the parameters in Eq.(2.14), we obtain the explicit form of short range potential which introduces the discontinuity (2.11) in the fermionic wavefunction:

\[ \epsilon(x; \frac{1}{K\xi}) \equiv \chi(x; -1, 0, -1, \frac{4}{K\xi}) = \lim_{a \to +0} \left( -\frac{K\xi}{2} - \frac{1}{a} \right) \left\{ \delta(x + a) + \delta(x - a) \right\}. \]  

(2.16)

Thus the fermionic dual to the bosonic Schrödinger equation (2.5) is given by

\[ -\frac{\partial^2 \phi_-(x)}{\partial x^2} + \epsilon \left( x; \frac{1}{K\xi} \right) \phi_-(x) = E_r \phi_-(x). \]  

(2.17)

Now if we construct a two particle fermionic wavefunction like \( \psi_-(x) = e^{iKX} \phi_-(x) \), it will evidently satisfy Schrödinger equation of the form

\[ -\frac{\partial^2 \psi_-}{\partial x_1^2} - \frac{\partial^2 \psi_-}{\partial x_2^2} + 2\epsilon (x_2 - x_1; \frac{1}{K\xi}) \psi_- = E \psi_- . \]  

(2.18)

Hence the two-particle fermionic system, which is dual to the two-particle bosonic system (2.1) with derivative \( \delta \)-function interaction, is described by the Hamiltonian

\[ H = -\frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} + 2\epsilon \left( x_2 - x_1; \frac{1}{K\xi} \right). \]  

(2.19)

It is interesting to observe that, in contrast to the case of \( \delta \)-function interaction, the fermionic Hamiltonian (2.19) explicitly depends on the centre-of-mass momentum. More precisely, the coupling constant of this fermionic Hamiltonian becomes inversely proportional to the product of the coupling constant of the corresponding bosonic system and its centre-of-mass momentum. Thus, for the present case, the boson-fermion duality can be established only if we fix the centre-of-mass momentum of the two-particle bosonic system. In this context it may be noted that, the binding energy of the bosonic bound states does not have any momentum dependence for the case of \( \delta \)-function interaction [20]. This result is consistent with the fact that the Hamiltonian of the corresponding
dual fermionic system does not depend of the variable $K$ [3]. On the other hand, for the case of derivative $\delta$-function interaction, the binding energy of the bosonic bound state depends on the total momentum of the system in a nontrivial way [14]. Therefore, the appearance of variable $K$ in the dual Hamiltonian (2.19) is rather natural, which merely ensures that the binding energy of two-particle fermionic bound state would also depend on the centre-of-mass momentum.

3 Boson-fermion duality for higher number of particles

Here our aim is to generalize the treatment of the earlier section to find out the dual fermionic model of the derivative $\delta$-function bose gas (1.1) for $N \geq 3$. For the case of $\delta$-function bose gas, such a generalization is quite straightforward because this many-body problem can be effectively reduced to a set of two-body problems, whose relative coordinates play the key role. Consequently, the boundary value problem associated with the $\delta$-function potential can be written through a set of equations, each of which depends on the partial derivative of the wavefunction with respect to only one relative coordinate [21,3]. Therefore, by applying again the method of self adjoint extension to the case of short range potential depending on a single relative coordinate, it is possible find out the dual fermionic model for this many-body system. However, it has been found recently that derivative $\delta$-function bose gas exhibits a few nontrivial features for $N \geq 3$ [14]. For example, while the chirality property of the corresponding classical solitons is preserved in the quantum theory for $N = 2$, this chirality property is broken in the quantum theory for $N \geq 3$. Therefore, it is important to check whether the boundary value problem associated with derivative $\delta$-function bose gas can in general be written through a set of equations, each of which would depend on the partial derivative of only one relative coordinate. For the sake of convenience, in the following we shall investigate this problem for derivative $\delta$-function bose gas containing three particles.

For the case $N = 3$, the Hamiltonian (1.1) may be written as

$$H = -\nabla^2 + V(x_{12}) \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right) + V(x_{23}) \left( \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right) + V(x_{13}) \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_3} \right),$$

(3.1)

where $\nabla^2 \equiv \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$ and $V(x_{ij}) \equiv 2i\xi\delta(x_i - x_j)$. It is evident that this Hamiltonian commutes with the total momentum operator given by

$$P = -i \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} \right).$$

(3.2)

Therefore, the bosonic wavefunction $\psi_+(x_1, x_2, x_3)$ of three-body derivative $\delta$-function bose gas can be chosen as a simultaneous eigenstate of the Hamiltonian (3.1) and the
momentum operator (3.2). Hence we write $\psi_+(x_1, x_2, x_3)$ in a product form like

$$
\psi_+(x_1, x_2, x_3) = e^{iK(x_1 + x_2 + x_3)} \phi_+,
$$

(3.3)

where $\phi_+$ depends only on the relative coordinates associated with three particles. Since $\phi_+$ remains invariant under the simultaneous translation of all coordinates, we obtain $P\phi_+ = 0$. Thus, due to Eq. (3.3) it follows that, $\psi_+$ would be an eigenstate of the total momentum operator with eigenvalue $3K$. Substituting the form (3.3) of $\psi_+$ to the Schrödinger equation given by $H\psi_+ = E\psi_+$, we obtain

$$
-\nabla^2 \phi_+ + 2iK\{V(x_{12}) + V(x_{23}) + V(x_{13})\} \phi_+ \\
- \left\{ V(x_{12}) \frac{\partial}{\partial x_3} + V(x_{23}) \frac{\partial}{\partial x_1} + V(x_{13}) \frac{\partial}{\partial x_2} \right\} \phi_+ = (E - 3K^2) \phi_+.
$$

(3.4)

Let us now consider the situation where the first and second particles move towards each other such that their centre of mass remains in a fixed position, and the third particle remains stationary far away from these two particles. Hence, we make a transformation of the coordinates like

$$
\{x_1, x_2, x_3\} \rightarrow \{x = x_1 - x_2, \ X = x_1 + x_2, \ x_3\},
$$

and try to find the boundary condition on $\phi_+$ following from Eq. (3.4) when only the coordinate $x$ varies around $x = 0$ and the other two coordinates $X$ and $x_3$ have some fixed values. Under the above mentioned change of coordinates, the $\nabla^2 \phi_+$ term in Eq. (3.4) can be expressed as a sum of three terms $\frac{\partial^2 \phi_+}{\partial x^2}$, $\frac{\partial^2 \phi_+}{\partial X^2}$ and $\frac{\partial^2 \phi_+}{\partial x_3^2}$ with different coefficients. Since only the partial derivative of $\phi_+$ with respect to $x$ has a discontinuity around $x = 0$, the term $\frac{\partial^2 \phi_+}{\partial x_3}$ will only give a non-zero contribution if we integrate $\nabla^2 \phi_+$ within a small interval $[-\epsilon, +\epsilon]$ and finally take $\epsilon \rightarrow 0$ limit. Furthermore, when the potential energy terms in Eq. (3.4) are integrated with respect to $x$ within the small interval specified above, only those terms would contribute which have a $\delta$-function type or derivative $\delta$-function type singularity at $x = 0$. Consequently, by integrating (3.4), we obtain

$$
\lim_{\epsilon \rightarrow 0} \left\{ -2 \int_{-\epsilon}^{\epsilon} \frac{\partial^2 \phi_+}{\partial x^2} \, dx + 2iK \int_{-\epsilon}^{\epsilon} V(x_{12}) \phi_+ \, dx - \int_{-\epsilon}^{\epsilon} V(x_{12}) \frac{\partial \phi_+}{\partial x_3} \, dx \right\} = 0,
$$

which reduces finally to

$$
\frac{\partial \phi_+}{\partial x} \bigg|_{x=+\epsilon} - \frac{\partial \phi_+}{\partial x} \bigg|_{x=-\epsilon} = -2\xi K \phi_+ \bigg|_{x=0} - i\xi \frac{\partial \phi_+}{\partial x_3} \bigg|_{x=0}.
$$

(3.5)

It is worth noting that, in contrast to $N = 2$ case, this boundary condition is not expressed only through the partial derivative with respect to the relative coordinate $x$. In fact, due to the presence of the last term in the r.h.s. of Eq. (3.5), the form of this equation differs from that of Eq. (2.7) which has been derived in the earlier section.
However, we can still proceed with the boundary condition (3.5) in the same way as has been done with the boundary condition (2.7) for $N = 2$ case, provided it is possible to choose $\phi_+$ in some particular form satisfying the condition

$$
\frac{\partial \phi_+}{\partial x_3} \big|_{x=0} = \lambda \phi_+ \big|_{x=0}, \tag{3.6}
$$

where $\lambda$ is an arbitrary constant. In this context it should be noted that, the exact $N$-particle eigenfunction for derivative $\delta$-function bose gas can be obtained through the Bethe ansatz [4,5,14]. In the region $x_1 < x_2 < \cdots < x_N$, such $N$-particle eigenfunctions may be written as

$$
\psi_N(x_1, x_2, \cdots, x_N) = \sum_\omega \left( \prod_{l<m} \frac{A(k_{\omega(m)}, k_{\omega(l)})}{A(k_m, k_l)} \right) \exp \{ ik_1 x_1 + \cdots + k_N x_N \}, \tag{3.7}
$$

where $k_m$s are all distinct wave numbers, $(\omega(1), \omega(2), \cdots, \omega(N))$ represents a permutation of the numbers $1, 2, \ldots, N$ and $\sum_\omega$ implies summing over all such permutations. In the expression (3.7), the ‘matching coefficient’ $A(k_m, k_l)$ is given by

$$
A(k_m, k_l) = \frac{(k_m - k_l) + i\xi (k_m + k_l)}{(k_m - k_l)}.
$$

So, our next aim is to verify whether the condition (3.6) is satisfied for the 3-particle Bethe wavefunction associated with the derivative $\delta$-function bose gas. It is easy to check that, for the case $N = 3$, the Bethe wavefunction (3.7) within the region $x_1 < x_2 < x_3$ can be written in the form of Eq.(3.3), where $K = \frac{k_1 + k_2 + k_3}{3},$

$$
\phi_+ = \sum_{i=1}^6 \phi_i, \tag{3.8}
$$

and $\phi_i$s’ are explicitly given by

$$
\phi_1 \equiv \phi_1(k_1, k_2, k_3) = e^{i \{ x_1(2k_1-k_2-k_3)+x_2(2k_2-k_1-k_3)+x_3(2k_3-k_1-k_2) \}},
$$

$$
\phi_2 = \frac{A(k_1, k_2)}{A(k_2, k_1)} \phi_1(k_1, k_2, k_3),
$$

$$
\phi_3 = \frac{A(k_1, k_2)A(k_1, k_3)}{A(k_2, k_1)A(k_3, k_1)} \phi_1(k_2, k_3, k_1),
$$

$$
\phi_4 = \frac{(A(k_2, k_3)}{A(k_3, k_2)} \phi_1(k_1, k_3, k_2),
$$

$$
\phi_5 = \frac{A(k_1, k_3)A(k_2, k_3)}{A(k_3, k_1)A(k_2, k_2)} \phi_1(k_3, k_1, k_2),
$$

$$
\phi_6 = \frac{A(k_2, k_3)A(k_1, k_3)A(k_1, k_2)}{A(k_3, k_2)A(k_3, k_1)A(k_2, k_1)} \phi_1(k_3, k_2, k_1).
$$

On the plane $x = 0$, these $\phi_i$s’ satisfy relations like

$$
\phi_2|_{x=0} = \frac{A(k_1, k_2)}{A(k_2, k_1)} \phi_1|_{x=0}, \quad \phi_6|_{x=0} = \frac{A(k_2, k_3)}{A(k_3, k_2)} \phi_3|_{x=0}, \quad \phi_5|_{x=0} = \frac{A(k_1, k_3)}{A(k_3, k_1)} \phi_4|_{x=0}. \tag{3.9}
$$
By using Eqs. (3.8) and (3.9), we obtain
\[ \phi_+|_{x=0} = \frac{2(k_1 - k_2)}{(k_1 - k_2) - i\xi(k_1 + k_2)} \phi_1|_{x=0} + \frac{2(k_2 - k_3)}{(k_2 - k_3) - i\xi(k_2 + k_3)} \phi_3|_{x=0} \]
\[ + \frac{2(k_1 - k_3)}{(k_1 - k_3) - i\xi(k_1 + k_3)} \phi_4|_{x=0}. \] (3.10)

Similarly, by differentiating \(\phi_+\) in Eq. (3.8) with respect to \(x_3\) and using Eq. (3.9), we get
\[ \frac{\partial \phi_+}{\partial x_3}|_{x=0} = \frac{2\rho_1(k_1 - k_2)}{(k_1 - k_2) - i\xi(k_1 + k_2)} \phi_1|_{x=0} + \frac{2\rho_2(k_2 - k_3)}{(k_2 - k_3) - i\xi(k_2 + k_3)} \phi_3|_{x=0} \]
\[ + \frac{2\rho_3(k_1 - k_3)}{(k_1 - k_3) - i\xi(k_1 + k_3)} \phi_4|_{x=0}, \] (3.11)

where \(\rho_1 = -\frac{i}{3}(k_1 + k_2 - 2k_3)\), \(\rho_2 = -\frac{i}{3}(k_2 + k_3 - 2k_1)\) and \(\rho_3 = -\frac{i}{3}(k_1 + k_3 - 2k_2)\).

Comparing the r.h.s. of Eq. (3.10) with that of Eq. (3.11), we find that the condition (3.6) can only be satisfied if \(\rho_1 = \rho_2 = \rho_3\), i.e., \(k_1 = k_2 = k_3\). However, as is well known, all \(k_i\)’s take distinct values in the Bethe ansatz eigenfunction (3.7). Hence, these Bethe ansatz eigenfunctions do not satisfy the condition (3.6) for \(N = 3\). This in turn implies that the boundary condition (3.5) can not be expressed only through the partial derivative with respect to the relative coordinate \(x\). Therefore it is clear that, in contrast to what we have found for \(N = 2\) case, the dual fermionic system associated with derivative \(\delta\)-function boson gas containing three particles cannot be constructed by applying the method of self-adjoint extension to any short-range potential which depends on only one relative coordinate. Proceeding in a similar way, one can arrive at the same conclusion for the case of derivative \(\delta\)-function Bose gas containing more than three particles.

## 4 Concluding Remarks

In this article, we attempt to construct the dual fermionic system associated with the quantum integrable derivative \(\delta\)-function boson gas. At first, we consider the simplest case of two bosonic particles interacting through derivative \(\delta\)-function interaction. Similar to the case of \(\delta\)-function potential, the Hamiltonian of this two particle system can be reduced to a single particle Hamiltonian depending only on the relative coordinate. While this Hamiltonian preserves the continuity of the wavefunction, it induces a discontinuity in the spatial derivative of the wavefunction. On the other hand, the fermionic counterpart of this bosonic wavefunction exhibits a discontinuity in the wavefunction itself but not in its spatial derivative. By applying the method of self-adjoint extension, we construct a short-range potential which induces this type of discontinuity in the wavefunction and yields the Hamiltonian of the dual fermionic system. In contrast to the case of \(\delta\)-function boson gas, the coupling constant of this dual fermionic system becomes explicitly dependent on the centre-of-mass momentum of the bosonic wavefunction.
Next we try to find out the dual fermionic model of the derivative $\delta$-function bose gas for higher number of particles. Focussing on the case of derivative $\delta$-function bose gas with three particles, we find that the boundary condition on the corresponding eigenfunction can not be written through a set of equations, each of which depends on the partial derivative with respect to only one relative coordinate. The same conclusion can be drawn for the case of derivative $\delta$-function bose gas containing more than three particles. Therefore, the dual fermionic system associated with derivative $\delta$-function bose gas containing three or higher numbers of particles cannot be constructed in a similar way as we have done for the case of two particles. To find out such dual fermionic system associated with higher numbers of particles, it seems to be necessary to study the self-adjoint extension of short-range potentials which depend on more than one variable.

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