Active velocity processes with suprathermal stationary distributions and long time-tails

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When a particle moves through a turbulent medium the incurred external force may change direction at a rate which grows with its speed. Suppose moreover that a thermal bath provides friction which gets weaker for large speeds, enabling high-energy localization. The result is a unifying framework for the emergence of heavy tails in the velocity distribution. The nonequilibrium mechanism is stochastic acceleration, shown to be relevant for understanding the power-law decay in the electron velocity distribution of space plasma or more generally for explaining non-Maxwellian behavior of driven gases. We also find long-time tails in the velocity autocorrelation, indicating persistence at large speeds for a wide range of parameters and implying superdiffusion of the position variable. While the analysis starts from a toy-model for a particle in an active environment, the arguments and conclusions become important for astrophysical and cosmological plasmas when assuming random sources of strong nonequilibria.

Active matter most often refers to swimming bacteria or self-propelled particles [1–5]. It involves the consumption and transport of energy over different scales, from microscopic to macroscopic levels, and in that sense incorporates important life processes, with active motion resulting from biochemical cycles. The bulk of studies on active particles are then set in the overdamped regime as befits realistic biological environments [6–10]. Standard models add a (random, diffusive) switching of the direction of the velocity. We have in mind run-and-tumble models [8, 11–13] or active Brownian [14–17] and active Ornstein-Uhlenbeck processes [18–20]. More decorated versions where anisotropy [21], chirality [22] or external potentials and interactions [23, 24] are introduced belong to that same category, enriching the phenomenology. Also disorder and inhomogenous switching have been considered at both single and many-particle levels [25], and inertia has been added in e.g. [1, 26].

From a wider perspective, active media may refer to environments (possibly massless) where excess dynamical activity creates a nonequilibrium condition. Probes moving in such environment will evolve under a dynamics for which the fluctuation-dissipation relation is violated, allowing net energy transfer from the medium to the probe which may then be dissipated in another thermal
bath. One could say that the particle (probe), while itself passive, is subject to both a thermal and a mechanical environment, such as tagged grains or electrons in agitated matter or plasma. The thermal bath consists of other such particles while the nonequilibrium forcing may be caused by moving walls or by spacetime dependent external force fields more generally. This is different from self-propelled particles, overdamped or underdamped, where the particles themselves are active, often bio-related and carrying their motors so to speak. Nevertheless, from a mathematical point of view there will be similarities as we simplify the external forcing in the way of run-and-tumble processes.

In the present article we consider a class of dynamics where the “tumbling” concerns the particle acceleration. We call these active velocity processes. The idea is that an external force changes direction depending on space and time, and the incurred external force tumbles at a rate growing linearly with the particle speed. We show that such processes with tumbling forces provide a unifying mechanism for producing suprathermal tails in dilute gases and with strong steady temporal autocorrelation.

The general motivation for studying these active velocity processes is to explore activity beyond biological contexts, and to offer relevant and conceptually simple modeling of media with random force fields. While much less studied, they have appeared before in models of velocity resetting, e.g. as first considered for Fermi acceleration [27], or in depot models [1 28] or in models of Taylor dispersion [29]. The idea of an external tumbling force has been used also in [30 31] for the dynamics of a probe in an active gel, with a study of stationary velocity distribution. Our model and type of results differ greatly nevertheless. We heavily exploit the combined effect of high-energy localization (in scattering with thermal bath) and stochastic acceleration (from turbulent external force field). We will see how the addition of a tumbling acceleration is then responsible for interesting nontrivial behavior that is seen in nature, relevant for astrophysical plasmas and in excited granular media, or in general for the dynamical properties of tagged particles in a thermal bath under turbulent external conditions.

**Model:** We restrict ourselves for simplicity of notation to one spatial dimension, as we assume an isotropic medium. These are not serious restrictions and it makes sense to consider the simplest situation as proof of principle. The active velocity model in one dimension for a particle of mass
\[ m = 1 \] with velocity \( v_t \in \mathbb{R} \) at time \( t \) is given by the Langevin equation \((k_B = 1)\)

\[
\begin{align*}
\dot{x}_t &= v_t \\
\dot{v}_t &= -\gamma(v_t) v_t + A \sigma_t + T \gamma'(v_t) + \sqrt{2\gamma(v_t)T} \xi_t
\end{align*}
\]

(1)

where \( \xi_t \) is standard white noise and we use the Itô-convention and \( \gamma'(v) = \frac{d\gamma}{dv} \). The external force has amplitude \( A \geq 0 \) and the tumbler \( \sigma_t = \pm 1 \) is flipping at a rate denoted by \( \alpha(v_t) \), depending on the instantaneous speed. In other words, \( \alpha(v) = \alpha(|v|) > 0 \) is the flipping rate or the frequency of the incurred tumbling force when the particle moves at speed \( |v| \). At the same time, the particle undergoes energy and momentum exchanges with a thermal bath at temperature \( T \geq 0 \). That environment also provides the nonlinear friction with coefficient \( \gamma(v) = \gamma(|v|) \) > 0.

Mathematically, the dynamics (1) defines a Markov process \((v_t, \sigma_t)\) in velocity and tumble variables. The joint probability on velocity \( v_t \in \mathbb{R} \) and force \( \sigma_t = \pm 1 \) has a density \( \rho_{\pm}(v,t) \) for time \( t \). The corresponding differential equation for the probability density is

\[
\partial_t \rho_{\pm}(v,t) = \partial_v \left[ (\gamma(v) v \mp A - \gamma'(v) T) \rho_{\pm}(v,t) \right] + \alpha(v) [\rho_{\pm}(v,t) - \rho_{\mp}(v,t)] \\
+ T \partial^2_{vv} (\gamma(v) \rho_{\pm}(v,t)), \quad v \in \mathbb{R}
\]

(2)

Observe that for \( A = 0 \) (passive case) the Maxwellian

\[
\rho_{A=0}^\pm(v) \propto \exp[-v^2/2T]
\]

(3)

is the stationary (equilibrium) density, independent of the friction \( \gamma(v) \). For \( A \neq 0 \) there is a higher-order equation for \( \rho(v,t) = \rho_{\pm}(v,t) + \rho_{-}(v,t) \) that determines the stationary velocity distribution \( P(v) (= \rho(v,t \to \infty)) \). We want to understand its behavior as \( |v| \to \infty \) when \( A \neq 0 \), and how it depends on the flipping rate \( \alpha(v) \) and the friction \( \gamma(v) \). However, before we continue the formal analysis it is important to discuss the physical input that determines the interesting choices for \( \alpha(v), \gamma(v) \).

In all cases that we consider below we take

\[
\alpha(v) = \nu_0 + \ell^{-1} |v|, \quad \gamma(v) = \gamma_0 \left[ 1 + \left( \frac{|v|}{v_R} \right)^4 \right]^{-1}
\]

(4)

where the parameter \( \ell \) is a length and \( \nu_0 \) is a frequency, both thought to be set by the external random force field and over which its direction changes. The persistence time of the activity-induced acceleration is thus \( \ell/(\nu_0 \ell + |v|) \) when the speed is \( |v| \). In dilute space plasmas charged particles are influenced by a fluctuating electromagnetic (EM) field. We can think of
FIG. 1: Plot of $P(v)$ vs $v$ with fixed $\gamma_0 = 1 = \ell = A = T$ for different values of $\delta$; see (4). Note the transition from a Maxwellian (for $\delta = 1.0$) to power law decay ($\delta = 3.0$) via compressed ($\delta = 1.5$), simple ($\delta = 2.0$) and stretched ($\delta = 2.5$) exponential regimes for increasing $\delta$. Symbols represent data obtained from Monte-Carlo simulations while solid black lines correspond to the $\mu_s(v)$ obtained from evaluating (9) numerically.

changing direction of the EM-forcing at a rate which depends on $|v|$ such as mimicking the Lorentz force. In contrast, the friction $\gamma(v)$ has its origin in the thermal bath. We take the parameterization with $\gamma_0 > 0$ a linear friction constant and $v_R$ a reference speed beyond which the friction starts to decrease. For $\delta > 0$, $\gamma(v)$ decreases with the speed, which means that the scattering cross section for the particle in the thermal environment decreases like $|v|^{1+\delta}$ for large $|v|$. Coulomb scattering gives $\delta = 3$ (Rutherford formula), and one can expect that depending on the material and shape of the particles in inelastic short-range scattering, values with $\delta < 3$ become available. At any event, when $\delta > 1$ high-energy localization takes place as then the friction force $\gamma(v) v$ decays with $K^{(1-\delta)/2}$ when the kinetic energy $K \propto v^2$ grows large. In what follows we often choose $v_0 = 1$ setting a time-scale; taking $v_R = 1$ adds a length scale.

Stationary distribution $P(v)$: We know of no simple analytic solution for the model (1) with the proposed general dependencies (4) for $\alpha(v)$, $\gamma(v)$. The main idea to get a theoretical prediction for large $|v|$ is to exploit that $\alpha(v)$ grows with $|v|$. When $|v| \gg \ell v_0$, we may expect (extra) diffusive behavior induced by the activity. Consider therefore the contribution of the tumbling force only, as in the updating

$$v_{t+\epsilon} = v_t + A \int_t^{t+\epsilon} ds \sigma_s$$

for fixed small $\epsilon$. Note that the tumbling correlations are given by $\langle \sigma_u \sigma_s \rangle = e^{-2\alpha|u-s|}$ where we were allowed to take $\alpha = \alpha(v_t)$ constant for $0 \leq u, s \leq \epsilon$ as $\epsilon$ is taken very small. Therefore we
FIG. 2: Plot of $P(v)$ vs $v$ for $\delta = 3, A = 1 = T$. The power-law decay in the stationary velocity distributions is shown for (a) $\ell = 1$, and (b) $\ell = 1/2$. Various values of $\gamma_0$ following (4) are plotted. The symbols correspond to the data obtained from numerical simulations and the red dashed lines indicate the theoretically predicted algebraic decay.

have the variance $\langle (v_{t+\epsilon} - v_t)^2 \rangle = A^2 \frac{\xi}{\alpha}$.

Moreover, in distribution,

$$\int_{t}^{t+\epsilon} ds \sigma_s \equiv \frac{1}{\alpha} \int_0^{\alpha \epsilon} du \tilde{\sigma}_u = \sqrt{\frac{\epsilon}{\alpha}} \frac{1}{\sqrt{\alpha \epsilon}} \int_0^{\alpha \epsilon} du \tilde{\sigma}_u$$

where the process $\tilde{\sigma}_u$ runs with flip rate equal to one. Hence, whenever $\alpha(v_t) \epsilon \gg 1$ we can apply the central limit theorem to $\frac{1}{\sqrt{\alpha \epsilon}} \int_0^{\alpha \epsilon} du \tilde{\sigma}_u$ and continue from (5) to get

$$v_{t+\epsilon} \simeq v_t + A \sqrt{\frac{\epsilon}{\alpha(v_t)}} Z$$

where $Z$ is a standard normal random variable. That follows the ideas of stochastic acceleration: for large flipping rate $\alpha$, the tumble force can effectively be modeled by a white noise of strength $D_{\text{eff}} = \frac{A^2}{2\alpha}$; see also [32] and the main mechanism goes back to the phenomenon of Taylor dispersion [29, 33–35], from where the general concept of stochastic or turbulent acceleration arises [36–38]. In that regime, the dynamics (1) appears replaceable by a passive Langevin dynamics $v_t$ (in Itô-sense),

$$\dot{v}_t = -\gamma(v_t) v_t + T \gamma'(v_t) + \sqrt{2\gamma(v_t)T} \xi_t^{(1)} + \sqrt{\frac{A^2}{\alpha(v_t)}} \xi_t^{(2)}$$

with two independent white noises $\xi_t^{(1)}$ and $\xi_t^{(2)}$ of zero mean and unit variance. It should be noted that the approximation (8) instead of (1) requires large $|v|$ (as we assume in (1) that $\alpha(v)$ grows with $|v|$) and gets better for not too low $T$ to exclude the zero-$T$ cut-off $|\dot{v}| \leq \gamma_0 |v| + A$; see below
for more on that around Figs.3. The stationary density $\mu_s(v)$ for (8) can be solved exactly,

$$\mu_s(v) \propto \exp - \int_0^{|v|} du \frac{u}{T + A^2/(2\gamma(u)\alpha(u))}$$

for large $|v|$. The argument above can thus be concluded by the statement that for the large $|v| \gg \ell \nu_0$ we can use $P(v) \simeq \mu_s(v)$. Hence, explicit calculations reveal the decay with $|v|$. For example, there exists an intermediate regime $\ell \nu_0 \ll |v| \ll v_R$, supposing large $v_R \gg \ell \nu_0$, where the behavior in $|v|$ is Gaussian or even faster (depending on temperature $T$) as is easily derived from (9). For the asymptotic behavior there is a difference between $\delta = 3$ and $\delta < 3$.

From (9) algebraic decay clearly appears whenever

$$\alpha(u) \gamma(u) \propto 1/u^2$$

for large $|u|$. More specifically, when $\delta = 3$ in (4), then

$$P(v) \sim v^{-2\kappa}, \quad \kappa = \frac{\gamma_0 v_R^3}{\ell A^2}$$

for large $|v|$, meaning $|v| \gg v_R, \ell \nu_0$ and $v^2/2\kappa \gg T$. Note that for $\kappa \leq 1/2$ (or for $\delta > 3$) no stationary distribution exists. Suprathermal velocity distributions [39], where the higher energy tail is overpopulated with respect to a corresponding Maxwellian, have been observed in space plasmas [40, 41] and there go under the name of kappa-distributions [42–44]. Their decay follows a power-law $\sim v^{-2\kappa}$ for large $|v|$. The fact that an effective diffusivity that depends inversely on the speed can produce suprathermal velocity distribution functions was already discussed, e.g. in [38, 42], in the context of highly energetic space plasmas. A general formulation based on a Fokker-Planck equation was e.g. already given in [39] but without obtaining the kappa-distribution (11). Here we see (11) explicitly derived through the presence of a tumbling force.

Continuing with (9), we predict pure exponential decay for $\delta = 2$, compressed exponential for $1 < \delta < 2$, stretched exponential for $2 < \delta < 3$ and Gaussian for $\delta \leq 1$. When $1 < \delta < 3$ in (4), then

$$P(v) \sim \exp \left[ -\frac{\kappa}{b} \left( \frac{v}{v_R} \right)^{2b} \right], \quad b = \frac{3 - \delta}{2}$$

again for large $|v|$. When $\delta = 1$ we recover the Maxwellian (Gaussian) behavior of (3) for large $|v|$ but with effective temperature $T + v_R^2/(2\kappa)$. In order to verify these predictions we have simulated the dynamics (1) using an Euler-discretization scheme; see Figs.1-2 where excellent agreement is found. We conclude that tumbling forces easily model the dynamics of particles in random force fields to produce heavy velocity tails. The flipping of the direction of the external
force over a distance $\ell$ is easily imagined for Fermi-Ulam ping pong \cite{45,46} or even in the case of granular gases under nonequilibrium driving. The literature is vast and various modeling schemes and approximations have been offered. As an example we refer to the experimental results \cite{47,49} in excited granular media. From the standpoint of statistical physics, the emergence of suprathermal tails due to run-and-tumble processes is new and unifies various phenomena.

It is also interesting to inspect where the tumbling fingerprint lies for small $|v|$. For low enough $T$, bimodality appears in the steady state distribution of $v$; see Fig. 3(a). For a fixed low $T$, $P(v)$ undergoes a shape transition from being highly localized near $v = 0$ to a delocalized distribution as $\gamma_0$ is decreased from very large to small values. This is expected on physical grounds as a very large friction in effect makes the particle immobile. As $T$ is increased the thermal noise takes over and the diffusive behavior leads to a broadening of the peaks, which eventually disappear for large enough temperatures. Similarly, when the tumbling variable takes three values: 0, 1, $-1$ and the flipping between any two values $\sigma$ and $\sigma'$ takes place at a rate $\alpha(v)/2$ (with $\alpha(v)$ given by (4)), we obtain a trimodal distribution for small $v$ at sufficiently small $T$ for varying $\gamma_0$; see Fig. 3(b). Those features resemble well the results found in \cite{30,31}. The suprathermal nature of the stationary velocity distribution is not affected, and the physics now includes processes where the external driving force is turned off ($\sigma_t = 0$) at some times or places.

**Steady time-autocorrelation:** One may wonder whether the heavy tails in the velocity distri-
bution are accompanied by long-time tails in the velocity autocorrelation,

\[ c(t) = \langle v_0 v_t \rangle - \langle v_t \rangle \langle v_0 \rangle \]  

We consider the average is in the steady state so that \( \langle v_t \rangle = 0 = \langle v_0 \rangle \). To estimate the time-dependence of (13) we imagine drawing an initial velocity \( v_0 \) from \( \mu_s(v) \sim P(v) \) under \( |v| \gg v_0 \) in (9) and the question is to see at what time \( v_t \) decorrelates with \( v_0 \). If \( |v_0| \ll v_R \) (small initial speed), the friction induces a time-scale \( \gamma_0^{-1} \) with exponentially fast decorrelation. On the other hand, for large speeds \( |v_0| \), the friction is mostly absent and decorrelation happens after another time-scale. For the heuristics we refer to Fig. 4 to observe a persistence in (large) speed. We get a quantitative prediction by reconsidering (1) for cases when friction and thermal effects are negligible and where the updating is given by (5). Clearly, for no matter what \( v_0 > 0 \), when at time \( t \),

\[ \int_0^t ds \sigma_s \in \left[ -\frac{\Delta}{A}v_0, \frac{\Delta}{A}v_0 \right] \]  

then, \( v_0v_t \simeq (1 \pm \Delta) v_0^2 \). where \( \Delta \simeq 1/2 \) is a dimensionless tolerance. Invoking the central limit theorem as in (7), we are thus asked to estimate the probability that \( \sqrt{\frac{t}{\alpha}} Z \in \left[ -\frac{v_0}{\sqrt{A}}, \frac{v_0}{\sqrt{A}} \right] \), which amounts to evaluating the error-function at a value proportional to \( \sqrt{\alpha/t} v_0/A = t^{-1/2} v_0^{3/2}/(\sqrt{A}) \).

We conclude that the event (14) occurs with high probability if \( t \ll v_0^3/(\ell A^2) \). Therefore, we predict that the time autocorrelation behaves as

\[ c(t) \simeq \int_{a(\ell A^2 t)^{1/3}}^\infty dv P(v) v^2 \]
for some $a > 0$, when $P(v)v^2 < 1/v$ decays sufficiently fast. All other contributions decay faster in time.

In the case where $\delta = 3$ we substitute (11) for the stationary distribution $P(v)$ and therefore, asymptotically in time $t$,

$$c(t) \sim t^{1-2\kappa/3}$$

(assuming $\kappa > 3/2$). It turns out that this rough calculation indeed provides a reasonable estimate when $\kappa > 2$, as can be seen in Fig. 5 for a comparison of (16) with Monte Carlo results. The long-time tails are entirely due to the active medium and the low friction at high speeds. Referring again to space plasmas, measuring the time evolution of a specific space plasma parcel is practically very difficult given that the observer (satellite) does not move with the solar wind expansion. Our estimate (16) offers a specific prediction however. Long-time tails have been reported for driven granular fluids in e.g. [50]. As another consequence, by time-integration of $c(t)$, the position is seen to be superdiffusive for $\kappa < 3$ with $\langle (x_t - x_0)^2 \rangle \sim t^{1+f}$ with $f = 2 - 2\kappa/3 > 0$. Such sub-ballistic behavior is not unseen for tracer particles in bio-active media; see e.g. [51].

For $1 < \delta < 3$ when we substitute in (15) the expression (12) for $P(v)$: for large times $t$, we get

$$c(t) \sim \exp \left[-\bar{k} \left(\gamma_0 t\right)^{\frac{3-\delta}{3}}\right]$$

with $\bar{k} \propto \kappa^{\delta/3}/(3 - \delta)$. We see that prediction compared with the simulation in Fig. 5(b) for two values of $\delta = 2, 2.5$; $c(t)$ for lower values of $\delta$ are more difficult to evaluate. In the passive case,
FIG. 6: (a) Plot of $P(\tau)$ vs $\tau$ in semi-log scale for different values of $v^*$ with $\gamma_0 = 2$ and a fixed width $w = 1$. (b) Plot of the corresponding rate $\lambda$ vs $v^*$ for two different values of $\gamma_0$. The dashed line indicates the expected $1/v^*$ behavior. The other parameters are $T = 1 = \ell = A$ here.

$A = 0$ as for [3], we have exponential decay in time, reflecting the dilute nature of the thermal bath. The same happens for $A \neq 0$ (active case) when $\delta = 0$ where friction remains prominent (and constant) even at large speeds.

Along similar lines, we also provide an estimate for the first passage time probability $P(\tau)$ for the particle to remain in a velocity window $[v^* - w, v^* + w]$ up to a time $\tau$. For a purely diffusive particle, the first passage time probability in a bounded region decays exponentially with a decay rate proportional to the diffusion constant [52]. Using Eqs. (5)-(7), i.e., the effective diffusion picture at low $T$ and large $v^*$, and translating the result of Ref. [52] to our case, we expect,

$$P(\tau) \sim \exp[-\lambda \tau], \quad \text{with } \lambda \propto A^2 \ell/(w^2 v^*)$$

for large $v^*$. The average first passage time $\lambda^{-1}$ increases linearly with $v^*$ which is a signature of the trapping in the velocity space discussed before. Note that in that regime the rate $\lambda$ is independent of the linear friction coefficient $\gamma_0$. We measure the first passage time probability using numerical simulations to verify this prediction; Fig. 6(a) shows plots of $P(\tau)$ vs $\tau$ for different values of $v^*$ which clearly shows the exponential decay. The corresponding $\lambda$ are plotted as a function of $v^*$ in Fig. 6(b) – the expected $1/v^*$ behavior is seen as $v^*$ increases.

Conclusions: The main result of this article is that active forces produce suprathermal stationary velocity distributions and long time-tails in the autocorrelation. The activity of the environment is modeled by a tumbling force with a fixed magnitude with a tumbling rate that depends on the speed. The suprathermal distributions range from power laws over exponentials to Maxwellians, and the time autocorrelation ranges from algebraic to exponential. On a more speculative note,
apart from space plasmas the importance for equilibration times in cosmological plasmas may be even much bigger. In light of the derived long time-tails it indeed cannot be excluded that the usual short-time thermal relaxation assumptions in the derivation of the Kompaneets equation (where photons are treated in contact with electrons having a Maxwellian velocity distribution) cannot be withheld; cf. also [53].

Even though we have restricted the analysis to one-dimensional tumble forces, suprathermality will show up for a much larger class of such active processes as the general arguments predict. A natural extension of our work is to higher dimensions, e.g., for active Brownian motions. Active velocity processes give a unified approach for suprathermal velocity tails from modeling with tumbling forces. The result on long-time tails indicates a persistence in the velocity (or the emergence of extra inertia $\sim \kappa^{-1}$ at high speeds), which in point of fact makes contact with an aspect of self-propelled particles. At the same time it widens the scope of standard activity modeling as for active biological media, reaching out to and including astrophysical and possibly cosmological processes.

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