Analyticity and Forward Dispersion Relations in Noncommutative Quantum Field Theory

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Abstract

We derive the analytical properties of the elastic forward scattering amplitude of two scalar particles from the axioms of the noncommutative quantum field theory. For the case of only space-space noncommutativity, i.e. $\theta_{0i} = 0$, we prove the dispersion relation which is similar to the one in commutative quantum field theory. The proof in this case is based on the existence of the analog of the usual microcausality condition and uses the Lehmann-Symanzik-Zimmermann (LSZ) or equivalently the Bogoliubov-Medvedev-Polivanov (BMP) reduction formalisms. The existence of the latter formalisms is also shown. We remark on the general noncommutative case, $\theta_{0i} \neq 0$, as well as on the nonforward scattering amplitude and mention their peculiarities.
1 Introduction

The proof of the analytical properties of scattering amplitudes is one of the most remarkable achievements of the axiomatic approach to quantum field theory. The dispersion relations (DR) for the elastic scattering amplitude were derived in the works of Gell-Mann, Goldberger, Thirring, Miyazawa, Nambu and Oehme [1]-[4]. They were rigorously proven in the works of Bogoliubov [5], Oehme [6], Symanzik [7], Bremermann, Oehme, Taylor [8] and Lehmann [9]. The detailed proof of DR was given in the book of Bogoliubov, Medvedev and Polivanov [10].

The implications of the modern ideas of noncommutative geometry [11] in physics have been lately of great interest, though attempts can be traced back as far as 1947 [12]. Plausible new arguments for studying noncommutative quantum field theories (NC QFT) [15, 14, 13] (for a review, see [16]) render the problem of analyticity in such theories actual. However, the task of establishing the analytical properties of noncommutative field theory is highly nontrivial. In passing from a usual space-time manifold to a space on which the coordinate operators do not commute, i.e.

\[ [x_\mu, x_\nu] = i\theta_{\mu\nu}, \]  

where \( \theta_{\mu\nu} \) is an antisymmetric constant matrix of dimension \((\text{length})^2\), the interactions acquire a nonlocal character and at the same time the Lorentz invariance is lost. It is mainly this nonlocal nature which gives rise to a novel behaviour of the NC QFT. For the derivation of dispersion relations, of crucial importance is the microcausality, which is affected by the noncommutativity of space-time. The effect is drastic when time does not commute with the spacial coordinates \( (\theta_{0i} \neq 0) \), in the sense that microcausality is completely lost [17, 18]
(see also [19] for acausal macroscopic effects in scattering). In the case of theories with commutative time ($\theta_{0i} = 0$) microcausality survives, but as a weaker condition than in the commutative case [17] (see eq. (3)). For this reason one may hope that dispersion relations can still be obtained in field theories with only space-space noncommutativity.

The first step in this direction was made by Liao and Sibold [20]. The essential difference between the analytical properties of the scattering amplitude in commutative and noncommutative cases found in their work was related to a specific way of continuation of the scattering amplitude to the complex plane. As a result, in [20] it was concluded that a derivation of the DR was not possible.

In the present work we aim at deriving DR first for forward elastic scattering of two spinless particles with masses $m$ and $M$. In the case of scattering of particles with spin, such as $\pi N$-scattering, our considerations refer to the invariant amplitude of those processes. We prove that if the noncommutativity affects only the space variables $i$, i.e. when $\theta_{0i} = 0$, then the standard DR with $n$ subtractions, analogous to the commutative case, are valid.

In the case of space-space noncommutativity we can choose the coordinates in such a way that only $\theta_{12} = -\theta_{21} \neq 0$. Then the usual condition of local commutativity can be substituted by its analog containing only the $x_0$ and $x_3$ coordinates [17] (see eqs. (2) and (3)).

We show that the above-mentioned noncommutative analog of local commutativity is indeed sufficient for proving the same analytical properties of the forward elastic scattering amplitude as in the commutative case. We admit that, similarly to the commutative case, $^{1}$

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$^{1}$The same case of noncommutativity was considered in [20].
the scattering amplitude is bounded by a polynomial (8). However our proof is valid under a weaker condition (31) than is usually used. Specifically, we substitute the condition of polynomial boundedness on the scattering amplitude by anything less than an exponential growth.

In the general case ($\theta_{0i} \neq 0$), the analyticity issue is rather obscure due to the lack of noncommutative analog of local commutativity. Besides, the existence of reduction formulas, which is the basis for the proof of analyticity, is not clear. Nevertheless, if reduction formulas survive in this case, we come to the conclusion that in the relations which follow from analyticity, the appearance of an additional term is very likely.

We have proven the analyticity of the elastic scattering amplitudes on the basis of Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas [21]. In the end of the paper we show that the same results can alternatively be derived using the Bogoliubov-Medvedev-Polivanov (BMP) approach [10].

In the Appendix the status of the reduction formulas in NC space-space theory is considered.

2 Forward scattering

We shall study the problem of analyticity of forward elastic scattering amplitude in case of noncommutative quantum field theory.

We consider the case when time commutes with the space variables, $\theta_{0i} = 0$, and restrict ourselves to the scattering of two scalar particles with masses $m$ and $M$. 
In the commutative case we admit the condition of local commutativity:

\[ [j(x), j(y)] = 0, \quad \text{if} \quad (x - y)^2 < 0, \]  

(2)

where \( j(x) \) is the current of interacting fields.

The local commutativity condition (2) is an independent axiom and not the consequence of Lorentz invariance. This condition means the absence of infinite speed of any interaction propagation. Lorentz invariance gives us the possibility to write this condition in an invariant form.

In the noncommutative case with \( \theta_{\text{bi}} = 0 \), the Lorentz symmetry \( SO(3,1) \) is broken to \( SO(1,1) \times SO(2) \) [17] and we can choose the coordinates in such a way that only \( \theta_{12} = -\theta_{21} \neq 0 \). In the direction perpendicular to the noncommutative \( (x_1, x_2) \)-plane we admit the existence of the maximal speed of interactions propagation. Then on the same basis as in the usual case, we assume the local commutativity condition (or microcausality) to be [17]

\[ [j(x), j(y)] = 0 \quad \text{if} \quad (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0. \]  

(3)

This condition was shown to be valid [18] for \( x_0 = y_0 \) using the equal-time commutation relations for the cases when \( j(x) \) is any power of field operators with \( \ast \)-product. Due to the remaining \( SO(1,1) \) symmetry, this implies the validity of (3) in the whole region \( (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0. \)

2.1 Analyticity in the framework of LSZ approach

If in the noncommutative case ”in” and ”out” fields can be constructed in the same way as in usual theory then the standard Lehmann-Symanzik-Zimmermann (LSZ) reduction formulas
are valid (see the Appendix) and the scattering amplitude is:

\[ F(E, \vec{q}) = \int d^4x \ e^{i(Ex_0 - \vec{q} \cdot \vec{x})} \tau(x_0) F(x), \]  

(4)

where

\[ F(x) = <M | \left[ j \left( \frac{x}{2} \right), j \left( -\frac{x}{2} \right) \right] | M >, \quad j(x) \equiv (\Box + m^2) \varphi(x). \]

We omit in (4) numerical factors which are irrelevant to the analytical properties of \( F(E, \vec{q}) \).

Eq. (4) is written in the reference frame in which the particle with the mass \( M \) is at rest. \( E \) and \( \vec{q} \) are the energy and momentum of the particle with mass \( m \).

Actually \( F(x) \) contains an additional term:

\[ \delta(x_0) <M \left[ j \left( \frac{x}{2} \right), \frac{\partial}{\partial x_0} \varphi \left( -\frac{x}{2} \right) \right] | M >, \]

which does not change the analytical properties of \( F(E, \vec{q}) \). The contribution of this term in (4) is some polynomial in \( E \). To show this it is sufficient to admit the standard assumption that \( j(x) \) is some polynomial of \( \varphi(x) \) and use equal time commutation relations (see e.g. [8], eq. (2.2) or [22], chapter 18).

Precisely speaking, the matrix element in (4) is an operator-valued generalized function (see, e.g. [23] and [10]). The corresponding questions are not specific to the NC case and that is why we do not dwell on them any further. We only mention that as our proof is valid under a weaker condition than polynomial boundedness of the scattering amplitude (see the condition (31)), we can consider the class of generalized functions to be more general than the tempered distributions.

In order to extend \( F(E, \vec{q}) \) to the upper complex \( E \)-plane \( (\text{Im} E > 0) \) we integrate (4) over \( x_1 \) and \( x_2 \) (similarly as in [20]).
Then, using (3), \( F(E, \vec{q}) \) is represented in the form:

\[
F(E, |\vec{q}|, \vec{e}) = \int_0^\infty e^{i E x_0} d x_0 \int_{-x_0}^{x_0} e^{-i e_3 x_3 \sqrt{E^2 - E_0^2}} \Phi(x_0, x_3) d x_3,
\]

where

\[
\Phi(x_0, x_3) = \int F(x) e^{-i(q_1 x_1 + q_2 x_2)} d x_1 d x_2.
\]

As shown in [17] in space-space noncommutative theory, due to the \( SO(1,1) \) symmetry, \( q_0^2 - q_3^2 = const. \) In the usual (commutative) case one has from the energy-momentum relation

\[
E_0^2 \equiv E^2 - q_3^2 = m^2 + q_1^2 + q_2^2.
\]

In the noncommutative case the energy-momentum relation is altered. However, the explicit expression for \( E_0^2 \) is not essential for our analyticity considerations.

In order to exclude the singularity at \( \sqrt{E^2 - E_0^2} \), we make the substitution

\[
F(E, |\vec{q}|, \vec{e}) \rightarrow \frac{1}{2} (F(E, |\vec{q}|, \vec{e}) + F(E, |\vec{q}|, -\vec{e}) \equiv F(E).
\]

(This is a standard procedure, see [24], chapter 10.) In accordance with (5) and writing \( \vec{q} \) in the form \( \vec{q} = \vec{e}|\vec{q}|, |\vec{e}| = 1 \), we obtain

\[
F(E) = \int_0^\infty e^{i E x_0} d x_0 \int_{-x_0}^{x_0} \cos(e_3 x_3 \sqrt{E^2 - E_0^2}) \Phi(x_0, x_3) d x_3.
\]

A direct extension of \( F(E) \) into the complex \( E \)-plane is impossible since

\[
Im \sqrt{E^2 - E_0^2} > Im E
\]

(see [24], chapter 10).
To overcome this obstacle, following [24], we substitute $F(E)$ by the regularized amplitude $F_\varepsilon(E)$:

$$
F_\varepsilon(E) = \int_0^\infty e^{iE x_0} \, dx_0 \int_{-x_0}^{x_0} \cos \left( e_3 x_3 \sqrt{E^2 - E_0^2} \right) e^{-\varepsilon(x_0^2 + x_3^2)} \Phi(x_0, x_3) \, dx_3.
$$

(7)

$F_\varepsilon(E)$ is an analytical function in the upper half-plane, where the integral in (7) converges.

The main problem is to prove the existence of the analytical function $F(E) = \lim_{\varepsilon \to 0} F_\varepsilon(E)$. To this end we shall use the analytical properties of $F_\varepsilon(E)$. Our goal is to represent $F_\varepsilon(E)$ in the complex $E$-plane as an integral over the real axis only and then take the limit $\varepsilon \to 0$. But it is impossible to do this directly as $F_\varepsilon(E) \not\to 0$ as $E \to \infty$. So first we have to construct a function which would have this property. We admit that there exists a number $n$ such that

$$
F(E) \frac{E^n}{E} \to 0 \quad \text{as} \quad E \to +\infty.
$$

(8)

In the commutative case one takes $n = 2$ in accordance with the Froissart-Martin bound [25, 26, 27], but here we have to admit a more general condition. Evidently, from (8) it follows that

$$
F_\varepsilon(E) \frac{E^n}{E} \to 0 \quad \text{as} \quad E \to +\infty.
$$

Condition (8) is valid also as $E \to -\infty$ since

$$
F(-E + i0) = F^*(E + i0), \quad F_\varepsilon(-E + i0) = F_\varepsilon^*(E + i0).
$$

(9)

Eq. (9) is the standard crossing symmetry condition. We point out that $j(x)$ is a Hermitian operator.
Evidently, the function

\[ \psi_\varepsilon(E) = \frac{F_\varepsilon(E)}{\prod_1^n (E - E_i)}, \quad E_i > E_0 \]

satisfies the condition

\[ \psi_\varepsilon(E) \to 0, \quad E \to \pm \infty. \]  \hspace{1cm} (10)

Thus we can use the Cauchy formula,

\[ \psi_\varepsilon(E) = \frac{1}{2\pi i} \int_C \frac{\psi_\varepsilon(E')}{E' - E} \, dE', \quad \text{Im} E > 0, \]  \hspace{1cm} (11)

where \( C \) consists of the interval \((-R, R)\), excluding \( n \) arbitrarily small semicircles around \( E_i \), and a semicircle in the upper half-plane.

Now we shall demonstrate that, due to the local commutativity condition (3),

\[ \psi_\varepsilon(RE^{i\varphi}) \to 0, \quad \text{if} \quad R \to \infty, \quad 0 < \varphi < \pi. \]  \hspace{1cm} (12)

Indeed, if \(|E| \to \infty\), then

\[ \text{Im} \sqrt{E^2 - E_0^2} \cong \text{Im} E - \text{Im} \frac{E_0^2}{2E}. \]  \hspace{1cm} (13)

Thus

\[ \left| e^{iE x_0 \cos(x_3 \sqrt{E^2 - E_0^2})} \right| \leq e^{-\text{Im} E (x_0 - x_3) \frac{E_0^2 |x_3| \sin \varphi}{E}}. \]  \hspace{1cm} (14)

The first factor on the r.h.s. of (14) is less than unity as \( x_0 > x_3 \). Due to the factor \( \exp(-\varepsilon x_3^2) \), in (7) the integral over \( x_3 \) converges when \( x_0 \to \infty \), so the integration is actually over some finite interval \((-\bar{x}_3(\varepsilon), \bar{x}_3(\varepsilon))\). Thus the second factor tends to unity at any fixed \( \varepsilon \) if \( R \to \infty \). Thus the growing factor in the integrand in eq. (7) disappears as \(|E| \to \infty\). We can thus conclude that condition (12) follows from the condition (10).
Actually in order to prove that condition (12) is the consequence of condition (10), it is sufficient to assume that $\psi_\varepsilon (R e^{i\varphi})$ grows more slowly than any exponent and use the Phragmen-Lindelöf theorem (see e.g. [28]). This is a very weak requirement on the behaviour of the scattering amplitude at infinite energies. Any function, which grows even as $\exp(R^\alpha), \ 0 < \alpha < 1$, satisfies it.

Thus we can put $R = \infty$ in (11). So

$$\psi_\varepsilon (E) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\psi_\varepsilon (E') dE'}{E' - E} - \frac{1}{2} \sum_{i=1}^{n} \frac{F_\varepsilon (E_i)}{(E_i - E) \prod_{i \neq j} (E_i - E_j)}, \quad Im\ E > 0. \quad (15)$$

Eq. (15) is valid at any fixed $\varepsilon$. Now we shall take the limit $\varepsilon \to 0$. First we consider the interval $(m, \infty)$. If $E' > E_0$, we can go to the limit $\varepsilon \to 0$ without any problem as in this interval $\lim_{\varepsilon \to 0} F_\varepsilon (E') = F (E')$ (see eqs. (6) and (7)). In the interval $(m, E_0)$ we can not use (7). But this interval is a physical one and so (4) and (5) coincide in this interval.

Thus

$$F_\varepsilon (E') = \int e^{i E' x_0} \cos (\vec{q} \cdot \vec{x}) \exp(-\varepsilon (x_0^2 + x_3^2)) F(x) d^4 x,$$

and we see that $F_\varepsilon (E') \to F (E')$ as $\varepsilon \to 0$.

We stress that only this interval is specific for the NC case. The interval $(-\infty, -m)$ can be treated similarly in accordance with (9).

The remaining interval can be considered as in the commutative case (see e.g. [24]).

To handle this interval, we shall construct the analytical function in the lower half-plane and then prove that this function is an analytical continuation of $F_\varepsilon (E)$. To this end we use the function

$$\tilde{F} (E, q) = \int d^4 x \ e^{i (E x_0 - \vec{q} \cdot \vec{x})} \tau (-x_0) F(-x). \quad (16)$$
Then we substitute $\tilde{F}(E, \vec{q})$ by

$$
\tilde{F}_\varepsilon(E) = \int d^4x \cos(\vec{q} \cdot \vec{x}) \tau(-x_0) e^{-\varepsilon(x_0^2 + x_3^2)} F(-x).
$$

(17)

Evidently

$$
\tilde{F}_\varepsilon(E - i 0) = F_\varepsilon(-E + i 0) = F^*_\varepsilon(E + i 0).
$$

(18)

The last equality in (18) is eq. (9). To prove the first equality it is sufficient to replace $x$ by $-x$. In (18) we can put $\varepsilon = 0$ and obtain

$$
\tilde{F}(E - i 0) = F^*(E + i 0).
$$

(19)

Similar to eq. (7) we have

$$
\tilde{F}_\varepsilon(E) = \int_0^\infty e^{ix_0} dx_0 \int_{-x_0}^{x_0} \cos(e_3 x_3 \sqrt{E^2 - E_0^2}) e^{-\varepsilon(x_0^2 + x_3^2)} \tilde{\Phi}(x_0, x_3) dx_3. \tag{20}
$$

The function

$$
\tilde{\psi}_\varepsilon(E) = \frac{\tilde{F}_\varepsilon(E)}{\prod_i (E - E_i)}, \quad E_i > E_0
$$

is an analytical function in the lower half-plane and $\tilde{\psi}_\varepsilon(E) \to 0$, $E \to \pm \infty$. We use the same arguments as for the proof of analyticity of $\psi_\varepsilon(E)$ in the upper half-plane. Thus

$$
\frac{1}{2\pi i} \int_{\tilde{C}} \frac{\tilde{\psi}_\varepsilon(E') dE'}{E' - E} = 0, \quad \text{Im} E > 0, \tag{21}
$$

where $\tilde{C}$ consists of the interval $(R, -R)$, excluding $n$ arbitrarily small semicircles around $E_i$ and a semicircle in the lower half-plane.

We shall now sum up the expressions (11) and (21). Using (18) and taking into account that the integral over a semicircle in the lower half-plane tends to zero if $R \to \infty$ for the
same reason as the corresponding integral in the upper half-plane, we obtain that

\[
\psi_\varepsilon (E) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{Im \psi_\varepsilon (E') d E'}{E' - E} + \frac{1}{\pi} \int_{-\infty}^{m} \frac{Im \psi_\varepsilon (E') d E'}{E' - E} - \sum_{i=1}^{n} \frac{Re F_\varepsilon (E_i)}{(E_i - E) \prod_{i \neq j} (E_i - E_j)} + \frac{1}{2\pi i} \int_{-m}^{m} \frac{(\psi_\varepsilon (E') - \tilde{\psi}_\varepsilon (E')) d E'}{E' - E}, \quad Im E > 0. \tag{22}
\]

In the first three terms in (22) we can go to the limit \( \varepsilon \to 0 \). In order to be able to take the corresponding limit in the remaining integral, we shall first obtain in the physical domain the expression for \( F (E, \vec{q}) - \tilde{F} (E, \vec{q}) \), suitable to extension for nonphysical \( E \), i.e. \( -m < E < m \).

From the definitions (4) and (16) it follows that

\[
F (E, \vec{q}) - \tilde{F} (E, \vec{q}) = F_+ (E, \vec{q}) - F_- (E, \vec{q}), \tag{23}
\]

where

\[
F_\pm (E, \vec{q}) = \int d^4 x \, e^{i(E x_0 - \vec{q} \cdot \vec{x})} F_\pm (x), 
\]

\[
F_+ (x) =< M \left| j \left( \frac{x}{2} \right) j \left( -\frac{x}{2} \right) \right| M >, \quad F_- (x) = F_+ (-x). \tag{25}
\]

Assuming that the vectors \( |p, n> \) form a complete set of basis vectors, we have

\[
< M \left| j \left( \frac{x}{2} \right) j \left( -\frac{x}{2} \right) \right| M > = \sum_{n} \sum_{p_n^0} \int d^3 p < M \left| j \left( \frac{x}{2} \right) \right| p, n > < p, n \left| j \left( -\frac{x}{2} \right) \right| M >, \tag{26}
\]

where \( p \) stands for the momentum of the state, \( p_n^0 \) is the energy of the state \( |p, n> \) and \( n \) denotes all other quantum numbers.

Using the equality

\[
< p' \left| j \left( x \right) \right| p > = e^{i(p' - p)a} < p' \left| j \left( x - a \right) \right| p >, \tag{27}
\]

where \( |p> \) and \( |p'> \) are eigenvectors of the operator \( p \), we see that, due to (25) and (26)

\[
F_\pm (E, \vec{q}) = \sum_{n} \sum_{p_n^0} \left| < M \left| j \left( 0 \right) \right| p, n > \right|^2 \delta (p_n^0 - M \mp E), \quad \vec{p} = \mp \vec{q}. \tag{28}
\]
Thus \( F_\pm (E, \vec{q}) \neq 0 \) only if
\[
\sqrt{M_n^2 + \vec{q}^2} = M \pm E, \quad p_n^0 = \sqrt{M_n^2 + \vec{q}^2}.
\] (29)

Let us assume that (as e.g. in the case of \( \pi N \)-scattering) \( M_n \geq M + m \), thus excluding the one-particle intermediate state, \( M \). We can extend the expression (28) for \( E \) in the interval \((-m, m)\). The functions \( F_\pm (E, \vec{q}) \neq 0 \) in this interval if
\[
\sqrt{M_n^2 + E^2 - m^2} = M \pm E,
\]
which is possible only if \( M_n = M \) and \( E = -m^2/2M \).

Thus we see that in the integral under consideration (excluding two points: \( \pm \frac{m^2}{2M} \))
\[
\lim_{\varepsilon \to 0} (\psi_\varepsilon (E) - \tilde{\psi}_\varepsilon (E)) = 0,
\]
as
\[
F_+ (E, \vec{q}) - F_- (E, \vec{q}) = 0.
\]

In order to make the integral over the interval \((-m, m)\) vanish, it is sufficient to substitute \( \psi_\varepsilon (E) \) and \( \tilde{\psi}_\varepsilon (E) \) by the functions:
\[
\Phi_\varepsilon (E) = \frac{E^2 - \frac{m^4}{4M^2}}{(E - E_{n+1})(E - E_{n+2})} \psi_\varepsilon (E), \quad E_{n+1} > E_0, \ E_{n+2} > E_0,
\]
\[
\tilde{\Phi}_\varepsilon (E) = \frac{E^2 - \frac{m^4}{4M^2}}{(E - E_{n+1})(E - E_{n+2})} \tilde{\psi}_\varepsilon (E), \quad E_{n+1} > E_0, \ E_{n+2} > E_0.
\]

Representing \( \Phi_\varepsilon (E) \) by an expression analogous to (22), we see that there exists \( \lim_{\varepsilon \to 0} \Phi_\varepsilon (E) = \Phi (E) \). Moreover \( \tilde{\Phi} (E) = \lim_{\varepsilon \to 0} \tilde{\Phi}_\varepsilon (E) \) is an analytical continuation of \( \Phi (E) \). The function \( \Phi (E) \) and consequently the function \( (E^2 - \frac{m^4}{4M^2}) \Phi (E) \) are analytical in the whole \( E \)-plane.
excluding the cuts $(-\infty, -m), (m, \infty)$. $F(E)$ is an analytical function in the same domain excluding the points $\pm \frac{m^2}{2M}$, where it has poles.

Finally, using (9) and (19), we arrive at the usual expression for $F(E)$:

$$F(E) = \frac{2E^n}{\pi} \int_{m}^{\infty} \frac{Im F(E') dE'}{(E')^{n-1}(E'^2 - E^2)} + \sum_{k=0, \text{even}}^{n-2} C_k E^k + \text{pole terms}, \quad Im E \neq 0. \quad (30)$$

In the limit $Im E \to 0$, (30) becomes the usual dispersion relation.

We can conclude that if the LSZ reduction formulas are valid in NC field theory and the condition of local commutativity can be replaced by the condition (3), the NC forward scattering amplitude has the same analytical properties as in the commutative case.

We would like to point out that our proof of analyticity of the forward scattering amplitude presented above remains still valid if one allows asymptotically a growth of the amplitude $F(E)$ much faster than a polynomially bounded one. Indeed, it is sufficient to assume that there exists $\alpha$, $0 < \alpha < 1$ such that

$$|F(E)| < \exp(E^\alpha), \quad E \to \infty. \quad (31)$$

Consequently, the function

$$\psi_{e}(E) = F_{e}(E) \xi(E), \quad (32)$$

where

$$\xi(E) = \exp \left[ - \left( \sqrt{m^2 - E^2} \right)^\beta \exp(-i \pi \beta) \right], \quad 0 < \beta < 1, \quad \alpha < \beta,$$

satisfies the necessary condition (10).

In fact, for any $\varphi$, $0 < \varphi < \pi$, we have

$$|\xi(E)| < \exp \left[ -|E|^\beta \cos \left( \varphi - \frac{\pi}{2} \right) \right].$$
Then it is easy to see that $\xi(E)$ is an analytical function in the whole $E$-plane with cuts $(m, \infty), (-\infty, -m)$ satisfying the conditions
\[ \xi(-E + i0) = \xi^*(E + i0) = \xi(E - i0). \]
One can check that all the previous steps in the proof go through also for the new function $\psi_{\varepsilon}(E)$.

2.2 Analyticity in the framework of BMP approach

The same results can be obtained on the basis of Bogoliubov-Medvedev-Polivanov (BMP) [10] reduction formulas and by using the analog of Bogoliubov microcausality condition [10, 24] in the NC case. For the forward scattering, the reduction formula is:
\[ F(E, \vec{q}) = \int d^4x e^{i(E x_0 - \vec{q} \cdot \vec{x})} F^c(x), \quad (33) \]
where
\[ F^c(x) = \langle M \left| \frac{\delta^2 S}{\delta \varphi \left( \frac{x}{2} \right) \delta \varphi \left( -\frac{x}{2} \right)} S^* \right| M \rangle. \quad (34) \]
We replace now the role of the local commutativity condition (3) by the modified Bogoliubov microcausality condition:
\[ \frac{\delta}{\delta \varphi(x)} j(y) = 0, \quad \text{if} \quad x_0 < y_0 \quad \text{or} \quad (x_0 - y_0)^2 - (x_3 - y_3)^2 < 0, \quad (35) \]
where
\[ j(x) \equiv i \frac{\delta S}{\delta \varphi(x)} S^*, \quad j(x) = j^*(x) \]
is the current in the BMP axiomatic approach. The condition $x_0 < y_0$ coincides with the corresponding original condition in the commutative case. The condition $(x_0 - y_0)^2 - (x_3 - y_3)^2 < 0$ substitutes the usual $(x - y)^2 < 0$. 

15
To extend $F(E, \vec{q})$ in the upper and lower half-planes, we use

$$F^{ret}(E, \vec{q}) = \int d^4 x e^{i(E x_0 - \vec{q} \cdot \vec{x})} F^{adv}(x)$$

(36)
correspondingly, where

$$F^{ret}(x) = <M \left| \frac{\delta}{\delta \varphi (-\frac{x}{2})} \left( \frac{\delta S}{\delta \varphi \left( \frac{x}{2} \right)} S^* \right) \right| M >, \quad F^{adv}(x) = F^{ret}(-x).$$

In accordance with (35), $F^{ret}(x) = 0$ if $x_0 < 0$ or $x_0^2 < x_3^2$. In this formalism the proof of the analytical properties of $F(E, \vec{q})$ is the same as in LSZ formalism. We only need to show that $F(E, \vec{q}) - F^{ret}(E, \vec{q}) = 0$ at physical energies. This can be done in the same way as (28) has been obtained. It is easy to show that

$$F(E, \vec{q}) - F^{ret}(E, \vec{q}) \sim \delta \left( \sqrt{M_n^2 + \vec{q}^2} + E - M \right) = 0, \quad \text{for} \quad M_n \geq M.$$

We point out that at physical energies the equality of the usual and retarded amplitudes is valid for nonforward scattering as well (see [24]).

3 Comments on the general NC case $\theta_{0i} \neq 0$

In the following we shall briefly consider the general case $\theta_{0i} \neq 0$. Let us assume (though this assumption may be not valid) that the LSZ reduction formula (4) is valid also in this case, but we do not have the local commutativity condition (3). In [18] it was shown that condition (3) is indeed not fulfilled in this case.

The function $F_{\epsilon}(E)$ is an analytical function as before, but, even if the polynomial boundedness condition (8) is valid, we can not exclude the possibility that this function grows exponentially in the whole complex plane. If (8) exists then we can obtain DR for
forward direction, but with an additional term. Indeed, DR follows from a similar relation at finite $R$ by taking the limit $R \to \infty$. Now we can also put $R = \infty$, but we have no arguments that the integral in the complex plane tends to zero if $R \to \infty$. Nevertheless this integral converges as the integral over the real axis converges and $E$ is the physical energy.

We point out that we have to work with a regularized amplitude and only at the end go to the $\varepsilon \to 0$ limit. Although in the commutative case the analytical properties of the scattering amplitude in the absence of local commutativity conditions have been studied [29, 30], in the NC case with $\theta_{0i} \neq 0$ the issue requires further investigation.

4 Conclusions

We have derived the analytical properties for the forward elastic scattering amplitude of two spinless particles on a space-time with space-space noncommutativity ($\theta_{0i} = 0$). Based on the axioms of noncommutative quantum field theory and using as an essential ingredient a microcausality (local commutativity) condition analogous, but weaker than the one in the commutative case, we have proven the existence of forward dispersion relations for the above-mentioned type of noncommutativity. The proof has been given using the Lehmann-Symanzik-Zimmermann (LSZ) formalism and it has been also shown to hold by using the Bogoliubov-Medvedev-Polivanov (BMP) formalism. In both frameworks, the existence of the reduction formulas for the space-space noncommutativity has also been demonstrated.

For the general noncommutative case, $\theta_{0i} \neq 0$, however, the microcausality (local commutativity) condition does not exist anymore and the existence of the reduction formulas is also doubtful.
As for the case of the elastic nonforward scattering amplitude, the number of independent kinematical variables increases, becoming five for the general noncommutativity case $\theta_{0i} \neq 0$ [31], as compared with two in the usual commutative case. Thus the analytical properties of the scattering amplitude in the nonforward case require a special investigation.

Finally, we would like to recall that in the commutative case, the existence of dispersion relations is the key step in extending the analyticity domain from the Lehmann ellipse [9] to the enlarged Lehmann-Martin ellipse [27] and in ultimately deriving the high-energy Froissart-Martin bound [25, 26] on the scattering amplitude from the axioms of quantum field theory. The aim for similar applications lies behind the idea of the present paper (see also [31]), which can be regarded as the first step along that direction. We hope to present further results in a future communication.

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**A Appendix: LSZ and BMP reduction formulas for NC quantum field theory**

**A.1 LSZ (Lehmann-Symanzik-Zimmermann) reduction formulas**

In the following we shall demonstrate that all the steps in deriving the LSZ reduction formulas for the space-space case of noncommutativity are the same as in the commutative case ([21],
see also [22]). To this end, we shall study elastic scattering amplitude of two scalar particles

\[ \langle p_{\text{out}}', q_{\text{out}}' | q_{\text{in}}, p_{\text{in}} \rangle = \langle p_{\text{out}}', q_{\text{out}}' | S | q_{\text{out}}, p_{\text{out}} \rangle. \]  

(37)

However, the consideration given below is quite general.

In order to exclude contribution of the unit operator \((S = 1 + iT)\), we shall put \(p', q' \neq p, q\).

Evidently

\[ |q_{\text{in}}, p_{\text{in}} \rangle = a_{\text{in}}^+(q) a_{\text{in}}^+(p) |0 \rangle = a_{\text{in}}^+(q) |p \rangle. \]

Under the stability of one-particle, we have states \(|p_{\text{in}} \rangle = |p_{\text{out}} \rangle = |p \rangle\). We represent \(a_{\text{in}}^+(q)\) as integral of \(\varphi_{\text{in}}(x)\) and \(\dot{\varphi}_{\text{in}}(x)\). To this end we use the mode decomposition of \(\varphi_{\text{in}}(x)\)

\[ \varphi_{\text{in}}(x) = \int \left( a_{\text{in}}^+(q') f^*_q(x) + a_{\text{in}}(q') f_{q'}(x) \right) d^3 q', \]

\[ f_q(x) = \frac{e^{-iqx}}{(2\pi)^{\frac{3}{2}} \sqrt{2 q_0}}, \]  

(38)

where \(a_{\text{in}}(q')\) is the annihilation operator. From (38) it follows that

\[ \dot{\varphi}_{\text{in}}(x) = i \int q_0 \left( a_{\text{in}}^+(q') f^*_q(x) - a_{\text{in}}(q') f_{q'}(x) \right) d^3 q'. \]  

(39)

Multiplying the combination \(i q_0 \varphi_{\text{in}}(x) + \dot{\varphi}_{\text{in}}(x)\) by \(f_q(x)\), integrating this expression over \(x\) and using (38) and (39), we obtain that

\[ a_{\text{in}}^+(q) = i \int \left( \varphi_{\text{in}}(x) \dot{f}_q(x) - \dot{\varphi}_{\text{in}}(x) f_q(x) \right) d^3 x. \]  

(40)

We substitute \(\varphi_{\text{in}}(x)\) by \(\varphi(x)\), where \(\varphi(x)\) is an interacting field, so that for any states \(|\alpha \rangle\) and \(|\beta \rangle\)

\[ \langle \alpha | \varphi(x) - \varphi_{\text{in}}(x) |\beta \rangle \rightarrow 0 \quad \text{as} \quad x_0 \rightarrow -\infty. \]  

(41)
Thus
\[ a_+^{in} (q) = i \int_{x_0 \to -\infty} (\varphi (x) \dot{f}_q (x) - \dot{\varphi} (x) f_q (x)) \, d^3 x. \] (42)

Now we shall use the general equality
\[ \int \left[ \right]_{x_0 = -\infty}^{x_0 = +\infty} d^3 x = \int \left[ \right]_{x_0 = +\infty}^{x_0 = -\infty} d^3 x - \int \frac{\partial}{\partial x_0} \left[ \right] d^4 x \] (43)

with the integrand
\[ \left[ \right] = < p'_{out}, q'_{out} \mid a_+^{in} (q) \mid p > = i \int < p'_{out}, q'_{out} \mid \psi (x) \mid p > \bigg|_{x_0 = -\infty}^{x_0 = +\infty} d^3 x, \] (44)

where \( \psi (x) \equiv \varphi (x) \dot{f}_q (x) - \dot{\varphi} (x) f_q (x) \).

First we show that the first term in the r.h.s. of (43) is zero. Indeed,
\[ i \int < p'_{out}, q'_{out} \mid \psi (x) \mid p > \bigg|_{x_0 = +\infty}^{x_0 = -\infty} d^3 x = \]
\[ i \int < p'_{out}, q'_{out} \mid \psi_{out} (x) \mid p > \bigg|_{x_0 = +\infty}^{x_0 = -\infty} d^3 x = < p'_{out}, q'_{out} \mid a_+^{out} (q) \mid p >, \]

where we have used (40) for the out-field. But
\[ < p'_{out}, q'_{out} \mid a_+^{out} (q) \mid p > = 0, \]

since \( q \neq p', q' \). For the remaining term, by direct calculations we obtain
\[ \frac{\partial}{\partial x_0} \left[ \varphi (x) \dot{f}_q (x) - \dot{\varphi} (x) f_q (x) \right] = \varphi (x) \ddot{f}_q (x) - \ddot{\varphi} (x) f_q (x) = \]
\[ \varphi (x) \left( \nabla^2 - m^2 \right) f_q (x) - \ddot{\varphi} (x) f_q (x) \] as \( (\Box + m^2) f_q (x) = 0. \)

Using integration by parts we have finally:
\[ < p'_{out}, q'_{out} \mid a_+^{in} (q) \mid p > = i \int < p'_{out}, q'_{out} \mid j (x) \mid p > f_q (x) d^4 x, \]
where \( j(x) \equiv (\Box + m^2) \varphi(x) \).

The following steps are also similar to the commutative case (see e.g. [22]). We represent in the same way \( a_{out}(q') \):

\[
< p_{out}' q_{out}' | = < p_{out}' a_{out}(q').
\]

Performing similar calculations (see (42)) we obtain

\[
a_{out}(q') = -i \int \left( f_{q'}(y) \varphi(y) - f_{q'}^*(y) \dot{\varphi}(y) \right) |_{x_0=+\infty} d^3 y.
\] (45)

Here we have used the equality

\[
\int [ ]|_{x_0=+\infty} d^3 y = \int [ ]|_{x_0=-\infty} d^3 y + \int \frac{\partial}{\partial y_0} [ ] d^4 y,
\] (46)

but in order to show that

\[
\int [ ]|_{x_0=-\infty} d^3 y = 0
\]

we need some preliminary manipulations with the term on the l.h.s of (46). To this end we substitute \( a_{out}(q') j(x) \) by \([a_{out}(q'), j(x)]\). This can be done, since

\[
< p' j(x) a_{out}(q')|q >= 0.
\]

Then we multiply \([a_{out}(q'), j(x)]\) by \( \tau(y_0 - x_0) \) as \( y_0 \to \infty \). Now the first term in the r.h.s. of (46) is equal to zero as \( \tau(y_0 - x_0) = 0 \) when \( y_0 \to -\infty \).

We can also substitute (at \( y_0 = +\infty \)) \( \varphi(y) j(x) \) by \( \tau(y_0 - x_0) \varphi(y) j(x) + \tau(x_0 - y_0) j(x) \varphi(y) = T(\varphi(y) j(x)) \). It remains then to consider the last term in (46). By calculations similar to those made above we obtain

\[
< p_{out}' q_{out}'|q_{in}, p_{in} >= - \int < p' |\psi(x, y)|p > f_q(x) d^4 y d^4 x,
\] (47)
where
\[
\psi(x, y) = f_q^*(y) \tau(y_0 - x_0) [j(y), j(x)] + \delta(y_0 - x_0) \left( -\dot{f}_q^*(y) \varphi(y) + f_q^*(y) \dot{\varphi}(y) \right)
\]
(\text{using } \frac{\partial}{\partial y_0} \tau(y_0 - x_0) = \delta(x_0 - y_0)).

Taking into account that
\[
< p' \mid \psi(x, y) \mid p > = e^{i(p' - p)a} < p' \mid \psi(x - a, y - a) \mid p >,
\]
after translation by \( a = \frac{x + y}{2} \) and trivial calculations, we finally obtain:
\[
< p'_{\text{out}}, q'_{\text{out}} \mid q_{\text{in}}, p_{\text{in}} > = -\delta(p' + q' - p - q)
\times \{ \int f_q^*(\frac{x}{2}) \tau(x_0) < p' \mid \left[ j \left( \frac{x}{2} \right), j \left( -\frac{x}{2} \right) \right] \mid p > f_q^*(\frac{x}{2}) \}.
\]
\[
\text{(48)}
\]

The last term is some polynomial in \( q \) and \( q' \). To show this we admit the standard assumption that \( j(x) \) is some polynomial in \( \varphi(x) \) with \( \ast \)-product, \( \theta_{0i} = 0 \), and use the equal-time commutation relations.

After integrating over the noncommuting variables \( x_1 \) and \( x_2 \) we come to a similar formula with eq. (6). (We recall that \( \theta_{3i} = 0 \), see the Introduction.) Thus the LSZ reduction formulas are valid in NC case.

A.2 BMP (Bogoliubov-Medvedev-Polivanov) reduction formulas

We shall now consider the BMP reduction formulas. In the commutative case, the \( S \)-matrix is represented in the general form:
\[
S = \sum_{n=0}^{\infty} \int f_n(x_1, \ldots, x_n) : \varphi(x_1) \cdots \varphi(x_n) : d^4 x_1 \cdots d^4 x_n,
\]
\[
\text{(49)}
\]
where $f_n(x_1, \cdots x_n)$ are some functions, $\varphi (x) \equiv \varphi_{\text{out}} (x)$ and normal product is used [10, 24].

It is evident that

$$< p', q'| S| q, p > = < p', q'| [S, a^+ (q)]| p >$$ (50)

since

$$< p', q'| a^+ (q) S| p > = 0.$$

We omit here the index "out".

From (38) it follows that

$$[\varphi (x), a^+ (q)] = f_q (x).$$ (51)

Commuting $a^+ (q)$ with $\varphi (x_n)$, $\varphi (x_{n-1})$ and so on, we see that

$$[S, a^+ (q)] = \int \frac{\delta S}{\delta \varphi (x)} f_q (x) \, dx,$$ (52)

where

$$\frac{\delta S}{\delta \varphi (x)} = \sum_{n=0}^{\infty} \sum_{n=0}^{n} \int f_n (x_1, \cdots x_i = x, \cdots x_n)$$

$$\times \varphi (x_1) \cdots \varphi (x_i) \cdots \varphi (x_n); d^4 x_1 \cdots d^4 x_i \cdots d^4 x_n.$$ (53)

The notation $\sim$ means the absence of the corresponding term.

Thus

$$< p', q'| S| q, p > = \int f_q (x) \, < p', q'| \frac{\delta S}{\delta \varphi (x)}| p > \, d^4 x.$$ (54)

The next step is similar. We substitute

$$< p'| a^- (q') \frac{\delta S}{\delta \varphi (x)}| p >$$
by
\[< p'|[a^- (q'), \frac{\delta S}{\delta \varphi (x)}]|p>\]
and then use the analog of (52)
\[[a^- (q'), S] = \int f^*_q (y) \frac{\delta S}{\delta \varphi (y)} d^4 y. \tag{55}\]
In accordance with (54) and (55)
\[< p', q|S|q, p> = \int f^*_q (y) f_q (x) < p'|\frac{\delta^2 S}{\delta \varphi (x) \delta \varphi (y)}|p> d^4 y d^4 x \]
\[\sim \delta (p' + q' - p - q) \int e^{i (p' + q') x} < p'|\frac{\delta^2 S}{\delta \varphi (\frac{x}{2}) \delta \varphi (-\frac{x}{2})} S^*|p> d^4 x, \tag{56}\]
where we have used that \(|p >= S^* S|p >= S^*|p >, \) in accordance with the unitarity of the S-matrix and stability of one-particle state. Eq. (56) is the BMP reduction formula.

If we put
\[j (x) \equiv i \frac{\delta S}{\delta \varphi (x)} S^*, \quad (j (x) = j^* (x)),\]
we can check that
\[\frac{\delta^2 S}{\delta \varphi (x) \delta \varphi (y)} S^* = -T (j (x) j (y)) \tag{57}\]
and so BMP and LSZ reduction formulas coincide. We should point out that (57) follows from the Bogoliubov microcausality condition:
\[\frac{\delta}{\delta \varphi (x)} j (y) = 0, \quad \text{if} \quad x_0 < y_0 \quad \text{or} \quad (x - y)^2 < 0. \tag{58}\]

We turn now towards the noncommutative case. Here it is natural to represent S-matrix by expression (49), but using \(*\)-product and \(f_n (x_1, \cdots x_n, \theta_{12})\) instead of \(f_n (x_1, \cdots x_n)\).

Equality (51) is valid as before since the descriptions of asymptotic fields in the noncommutative and commutative theories are the same when \(\theta_{0i} = 0, \) i.e. time is commutative.
Taking into account that
\[
\int f_n(x_1, \cdots, x_n, \theta_{12}) \star f_q(x_n) \, d^4 x_n = \int f_n(x_1, \cdots, x_n, \theta_{12}) f_q(x_n) \, d^4 x_n
\]
and similar formulas for \( f_q(x_i) \) we come to eq. (54), where \( \frac{\delta S}{\delta \varphi(x)} \) is determined by (53), but with \( \star \)-product.

We note that, using integration by parts, we can define \( S \)-matrix with standard product, but with redefined \( f_n(x_1, \cdots, x_n, \theta_{12}) \).

Eq. (56) is valid on the same basis as eq. (54). However, in equality (58) we have to substitute the condition \((x - y)^2 < 0\) by the condition \((x_0 - y_0)^2 - (x_3 - y_3)^2 < 0\), due to the modified Bogoliubov microcausality condition (35).

References

[1] M. Gell-Mann, M. L. Goldberger and W. E. Thirring, Phys. Rev. 95 (1954) 1612.

[2] M. L. Goldberger, Phys. Rev. 99 (1955) 979.

[3] M. L. Goldberger, H. Miyazawa and R. Oehme, Phys. Rev. 99 (1955) 986.

[4] R. Oehme, Phys. Rev. 100 (1955) 1503; 102 (1956) 1174.

[5] N. N. Bogoliubov, Lecture at International Congress on Theoretical Physics, Seattle, 1956 (unpublished).

[6] R. Oehme, Nuovo Cim. 10 (1958) 1316.

[7] K. Symanzik, Phys. Rev. 105 (1957) 743.
[8] H. J. Bremermann, R. Oehme and J.G. Taylor, *Phys. Rev.* **109** (1958) 2178.

[9] H. Lehmann, *Nuovo Cim.* **10** (1958) 579.

[10] N. N. Bogoliubov, B. V. Medvedev and M. K. Polivanov, *Theory of Dispersion Relations*, Lawrence Radiation Laboratory, Berkeley, California, 1961.

[11] A. Connes, *Noncommutative Geometry*, Academic Press, New York, 1994.

[12] H. S. Snyder, *Phys. Rev.* **71** (1947) 38.

[13] S. Doplicher, K. Fredenhagen and J. E. Roberts, *Phys. Lett.* **B331** (1994) 39; *Comm. Math. Phys.* **172** (1995) 187.

[14] F. Ardalan, H. Arfaei and M. M. Sheikh-Jabbari, *JHEP* **9902** (1999) 016, hep-th/9810072.

[15] N. Seiberg and E. Witten, *JHEP* **9909** (1999) 32, hep-th/9908142 and references therein.

[16] M. R. Douglas and N. A. Nekrasov, *Rev. Mod. Phys.* **73** (2001) 977, hep-th/0106048.

[17] L. Alvarez-Gaumé, J. L. F. Barbon and R. Zwicky, *JHEP* **0105** (2001) 057, hep-th/0103069.

[18] M. Chaichian, K. Nishijima and A. Tureanu, hep-th/0209006, to appear in Phys. Lett. B.

[19] N. Seiberg, L. Susskind and N. Toumbas, *JHEP* **0006** (2000) 044, hep-th/0005015.

[20] Y. Liao and K. Sibold, *Phys. Lett.* **B 549** (2002) 352, hep-th/0209221.
[21] H. Lehmann, K. Symanzik and W. Zimmermann, *Nuovo Cim.* 1 (1955) 1425; 6 (1957) 319.

[22] J. D. Bjorken, S. D. Drell, *Relativistic Quantum Fields*, Mc Graw-Hill Book Company, 1965.

[23] R. Streater and A.S. Wightman, *PCT, Spin and Statistics, and All That*, Benjamin, New York, 1964.

[24] N. N. Bogoliubov and D. V. Shirkov, *Introduction to the Theory of Quantized Fields*, Wiley, New York, 1980, 3rd ed.

[25] M. Froissart, *Phys. Rev.* 123 (1961) 1053.

[26] A. Martin, *Phys. Rev.* 129 (1963) 1432.

[27] A. Martin, *Nuovo Cim.* 42 (1966) 901.

[28] N. N. Meiman, *Zh. Eksp. Teor. Fiz.* 43 (1962) 2247 (translation in *Sov. Phys. JETF* 16 (1963) 1609).

[29] V. Ya. Fainberg and Sh. Yu. Lomsadze, *Kratkie Soobshcheniya po Fizike* 5 (1988) 23 (translated in *Soviet Physics - Lebedev Institute Reports*).

[30] Yu. S. Vernov and M. N. Mnatsakanova, in *Proceedings of the XIV International Seminar on High Energy Physics and Quantum Field Theory*, Protvino, Nauka, 1992, p. 290.

[31] M. Chaichian, C. Montonen and A. Tureanu, hep-th/0305243.