Application of orthogonal coding in conjunction with discrete amplitude modulation

A V Rabin
Saint-Petersburg State University of Aerospace Instrumentation (SUAI), ul. Bolshaya Morskaya, 67, lit. A, St. Petersburg, 190000, Russia
E-mail: alexey.rabin@guap.ru

Abstract. Most works on coding theory are devoted to the analysis of systems in that part of the bandwidth curve, which corresponds to low spectral efficiency or low signal-to-noise ratio. The reason for this factor is the fact that most of the applications using coding theory used in satellite systems are limited in terms of the used power, but with a wide bandwidth. Thus, to date, none of the coding schemes used can provide noise immunity and significant energy gain with high spectral efficiency. The research of ways to increase noise immunity at a fixed transmission rate through the development of special mathematical methods is an urgent task.

1. Introduction
Noise immunity is one of the most important characteristics of modern information transmission systems. There is a contradiction between the requirements for the noise immunity of radio engineering systems, due to the capabilities of countermeasures, and the lack of methods for increasing the noise immunity [1-7] at high spectral efficiency and coding rate.

Discrete amplitude modulation (AM) signals have the form $s_i(t) = A_i \varphi(t)$, where $\varphi(t)$ is some normalized function set on the interval [0, T] and determining the signal shape, $A_i$ is the amplitude of the $i$-th signal, $i = 0, 1, ..., q-1$ [8]. Let us define the amplitude of the $i$-th signal as

$$A_i = \sqrt{E} \sqrt{1 - \frac{4}{q}}.$$

Then $A_0 = \sqrt{E}$, $A_{q-1} = -\sqrt{E}$, and all intermediate values of the amplitude are located with a uniform step in the interval $[-\sqrt{E}, \sqrt{E}]$. The signal AM set is shown in figure 1.

![Signal set of AM signals](image1.png)

Figure 1. Signal set of AM signals.
2. Characteristics of the transmission channel

The minimum distance between signals, as seen in figure 1, is \( \Delta = 2\sqrt{E} / (q-1) \). Let's determine the energy of each signal. It is obvious that the energy of the \( i \)-th signal is equal to \( E_i = A_i^2 = E \left( 1 - 2i / (q-1) \right)^2 \), that is, the energy of the signals takes on different values. The value \( E \) has the meaning of maximum energy. As usual, we will assume that signals are transmitted with equal probability. Let's find the value of the average energy:

\[
\overline{E} = \frac{1}{q} \sum_{i=0}^{q-1} E_i P_i = \frac{1}{q} \sum_{i=0}^{q-1} \left( 1 - \frac{2i}{q-1} \right)^2 = \frac{E}{q} \left( \sum_{i=0}^{q-1} - \frac{4}{(q-1)} \sum_{i=0}^{q-1} i + \frac{4}{(q-1)^2} \sum_{i=0}^{q-1} i^2 \right).
\]

Find the value of the expression \( \frac{1}{q} \sum_{i=0}^{q-1} \left( 1 - \frac{2i}{q-1} \right)^2 \). Using the identities \( \sum_{i=0}^{q-1} i = k(k+1)/2 \) and \( \sum_{i=0}^{q-1} i^2 = k(k+1)(2k+1)/6 \), we find that

\[
\frac{1}{q} \sum_{i=0}^{q-1} \left( 1 - \frac{2i}{q-1} \right)^2 = \frac{1}{q} \left( \sum_{i=0}^{q-1} 1 - \frac{4}{(q-1)} \sum_{i=0}^{q-1} i + \frac{4}{(q-1)^2} \sum_{i=0}^{q-1} i^2 \right) = \frac{1}{q} \left( q - \frac{4q(q-1)}{2} + \frac{4(q-1)(2q-1)}{6} \right) = \frac{1}{3} \frac{q+1}{q-1}.
\]

Therefore,

\[
\overline{E} = \frac{E}{3} \frac{q+1}{q-1}.
\]  

(1)

The decisive regions for \( i = 1, 2, ..., q-2 \) are segments of length \( \Delta \) centered at signal points \( s_1, s_2, ..., s_{q-2} \), that is, \( R_i = [s_i - \Delta/2, s_i + \Delta/2] \). The decision regions for the extreme points \( s_0 \) and \( s_{q-1} \) are infinite half-lines \( R_0 = (-\infty, A_q, s_0 + \Delta/2) \) and \( R_{q-1} = (-\infty, A_1, s_{q-1} - \Delta/2) \).

The signal at the channel output has the form \( r(t) = s(t) + n(t) \), where \( s(t) \in \{ s_i(t) \} \), \( n(t) \) is the additive white Gaussian noise (AWGN) with the power spectral density \( N_0/2 \). Due to the fact that the signal set of AM signals is one-dimensional, we have the following finite-dimensional representation \( r = A + n \),

where \( A \in \{ A_i \} \), \( n \) is the Gaussian random variable with parameters \( \bar{n} = 0 \), \( n^2 = N_0 / 2 \). Let us find the error probability. By the formula of total probability we have

\[
P_e = \sum_{i=0}^{q-1} P_e(i) P_i = \frac{1}{q} \sum_{i=0}^{q-1} P_e(i),
\]

(2)

where \( P_e(i) \) is the probability of transmitting the \( i \)-th signal, \( P_i \) is the probability of transmitting the \( i \)-th signal, \( P_i = 1/q \). Consider first the calculation of \( P_e(i) \) for \( i = 1, 2, ..., q-2 \).

\[
P_e = \Pr[r \notin R_i | i] = \Pr[A_i + n \notin \left( A_i - \frac{\Delta}{2}, A_i + \frac{\Delta}{2} \right)] = \Pr[n \notin \left( -\frac{\Delta}{2}, \frac{\Delta}{2} \right)] =
\]

\[
= \int_{-\infty}^{-\frac{\Delta}{2}} \frac{1}{\sqrt{2\pi}N_0} e^{-x^2/2N_0} dx + \int_{\frac{\Delta}{2}}^{\infty} \frac{1}{\sqrt{2\pi}N_0} e^{-x^2/2N_0} dx = 2Q\left( \frac{\Delta}{\sqrt{2N_0}} \right).
\]

(3)
Find the remaining probabilities $P_e(0)$ and $P_e(q-1)$

$$P_e(0) = \Pr[r \not\in R_0 | 0] = \Pr[A_0 + n \not\in [A_0 - \Delta/2, \infty)] = \Pr[n \not\in [-\Delta/2, \infty)] =$$

$$= \int_{-\infty}^{-\Delta/2} \frac{1}{\sqrt{\pi N_0}} e^{-x^2/N_0} dx = Q\left(\frac{\Delta}{\sqrt{2N_0}}\right).$$

(4)

Similarly, one can show that

$$P_e(q-1) = Q\left(\frac{\Delta}{\sqrt{2N_0}}\right).$$

(5)

Substitution of expressions (3)-(5) in (2), taking into account that $\Delta=2\sqrt{E/(q-1)}$, gives the final expression

$$P_e = \frac{2q-2}{q} Q\left(\frac{2E}{N_0} \frac{1}{q-1}\right).$$

(6)

determining the dependence of the error probability on the maximum signal-to-noise ratio $E/N_0$. Using the quality (1), we obtain an expression that determines the dependence of the error probability on the mean value of the signal-to-noise ratio $E/N_0$:

$$P_e = \frac{2q-2}{q} Q\left(\frac{6E}{N_0} \frac{1}{q^2-1}\right).$$

(7)

When $q=2$, the equalities (6) and (7) turn into $P_e = Q\left(\sqrt{2E/N_0}\right)$, that is, into the formula for the error probability for binary opposite signals; in this case, there is the equality $E=E_0$.

Consider the derivation of the expression for the bit error probability for AM signals. The average signal-to-noise ratio per bit is

$$\left(\frac{E}{N_0}\right)_{\text{bit}} = \frac{1}{\log_2 q} \frac{E}{N_0}.$$

The bit error probability depends on the mapping of messages (signal numbers) to signal points. The preferred display will be such that closely spaced signal points correspond to messages differing in a small number of bits. This mapping for AM signals is achieved using a Gray code. In this case, the blocks of binary data corresponding to adjacent signal points will differ only in one position (see figure 2 – an example for AM-8).

Figure 2. Signal set of AM-8 signals (display according to the Gray code).
Since an erroneous decision regarding the transmitted signal is most likely in favor of adjacent signals, it will result in the error in only one bit. This means that in most cases, the fraction of erroneous binary digits with an erroneous decision is \( \frac{1}{\log_2 q} = \frac{1}{m} \). Hence, the probability of error per bit as a function of signal-to-noise ratio per bit is given by

\[
P_e \approx \frac{1}{\log_2 q} P_e = \frac{1}{\log_2 q} \frac{2q - 2}{q} Q \left( \sqrt{6 \left( \frac{E}{N_0} \right)} \frac{\log_2 q}{q^2 - 1} \right)
\]

The graphs shown in figure 3 give an idea of the dependence of the error probability on the average signal-to-noise ratio and the error probability per bit on the signal-to-noise ratio per bit. It is important to note that the probability of an error \( P_e \) increases sharply with an increase in the volume of the signal alphabet.

**Figure 3.** Probabilities of error \( P_e \), a) and error per bit \( P_e^{\text{bit}} \), b) for AM signals.
3. Increased noise immunity through the use of orthogonal coding

Let us consider the application of orthogonal coding in systems with discrete AM in a channel with AWGN. Let us estimate the decrease in the resulting error probability due to the use of the orthogonal coding proposed by the author.

In [9-10] it is shown that for the formation of orthogonal codes it is required to synthesize square matrices so that their product is a unit matrix multiplied by a monomial characterizing the correcting ability of the code. Previously, matrices with these properties were developed using combinatorial methods, as a result of which only a few orthogonal codes were obtained. Consequently, the problem arose to develop a regular matrix synthesis algorithm for constructing the orthogonal codes. This problem was solved in [9], provided that the elements of the matrices used are polynomials of the first degree. As a result, a class of matrices was synthesized that allow solving practical problems of enhancing noise immunity.

The encoding \( G(D) \) and decoding \( H(D) \) matrices from the variable delay \( D \) must satisfy the relation

\[
G(D) \cdot H(D) = \rho \cdot D^X \cdot I,
\]

where \( I \) is the identity matrix. The multiplier \( \rho \cdot D^X \) indicates an increase in the amplitude of the input signal in \( \rho \) times and that the symbols in the receiver are obtained with a delay of \( X \) clock cycle.

In the process of joint application of orthogonal coding and AM in the framework of this study, we will use matrices \( H(D) \) with polynomials of the first degree in a variable \( D \).

According to the proposed algorithm, at the first step of the synthesis of the order \( n \) matrix \( H(D) \), the first \( m = 2k \) elements of the main diagonal receive values \( 1 + D \), \( m \leq n \), an integer \( m \) will be called the depth of the matrix. Assign values to the following elements 1. At the last step, we assign values \( 1 - D \) to the elements of odd rows to the right and odd columns under the main diagonal; elements of even rows to the right and even columns under the main diagonal are \( 1 + D \) values.

Consider using only orthogonal coding in a channel with AWGN and AM. Due to the complexity of calculating the error probability, we will conduct a simulation of the transmission system, in which we will use the encoding and decoding devices, built on the basis of synthesized matrices (see table 1) [9].

Table 1. Main characteristics of orthogonal codes.

| \( H[\xi, z] \times H(D) \times H(D) \) | Number of modulation positions | Max. number of errors | \( H[\xi, z] \times H(D) \times H(D) \) | Number of modulation positions | Max. number of errors |
|---|---|---|---|---|---|
| 2, 2 \( 4D \) | 9 | 1 | 6, 6 \( 4D \) | 25 | 1 |
| 3, 2 \( 4D \) | 21 | 1 | 7, 6 \( 4D \) | 25 | 1 |
| 4, 2 \( 12D \) | 45 | 5 | 8, 6 \( 12D \) | 73 | 5 |
| 4, 4 \( 4D \) | 21 | 1 | 8, 8 \( 4D \) | 25 | 1 |
| 5, 2 \( 28D \) | 109 | 13 | 12, 10 \( 28D \) | 73 | 5 |
| 5, 4 \( 4D \) | 25 | 5 | 12, 12 \( 4D \) | 25 | 1 |
| 6, 4 \( 12D \) | 61 | 5 | 16, 16 \( 4D \) | 25 | 1 |

Figure 4 shows the graphs of the dependence of the error probability on the signal-to-noise ratio in the channel with AWGN for binary AM and the graphs of the dependence of the error probability on the signal-to-noise ratio obtained as a result of simulation modeling for the orthogonal coding OC-4.
OC-8, OC-16, OC-32 and AM. For each case, according to table 1, a decoding matrix of depth equal to half the order of the matrix was selected.

![Figure 4. Probabilities of bit error in a channel with AWGN for binary AM and for circuits orthogonal matrix-based coding (4 x 4), (8 x 8), (16 x 16) and (32 x 32).](image)

It can be seen from figure 4, at the $10^{-4}$ bit error probability level when using the orthogonal coding OC-32 and AM, the signal-to-noise gain compared to binary AM without coding is 2.6 dB, and at the bit error probability level $10^{-6} – 4.2$ dB [11-12].

4. Conclusions
The novelty of the orthogonal coding method lies in the fact that it can be considered as a kind of reception as a whole of M-ary discrete amplitude modulation signals with an optimal choice of the manipulation code that matches the binary combinations of the source of the AM signal magnitude. The reason for this optimization is the averaging of the error probability over all bits of the M-ary code.

The practical and scientific value of the materials presented in the article lies in the fact that it proposes an orthogonal coding method that provides an energy gain (for a $10^{-6}$ bit error – 4.2 dB) without a significant increase in the complexity of the equipment.

The technical implementation of orthogonal coding is characterized by low complexity: decoding is reduced to calculating a number of dot products and performing a comparison with a zero threshold. For this reason, the proposed method for coding and construction of transmitting and receiving devices can be used in various communication systems.

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