Intrinsic Instrumental Polarization and High-Precision Pulsar Timing

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ABSTRACT

Radio telescopes are used to accurately measure the time of arrival (ToA) of radio pulses in pulsar timing experiments that target mostly millisecond pulsars (MSPs) due to their high rotational stability. This allows for detailed study of MSPs and forms the basis of experiments to detect gravitational waves. Apart from intrinsic and propagation effects, such as pulse-to-pulse jitter and dispersion variations in the interstellar medium, timing precision is limited in part by the following: polarization purity of the telescope’s orthogonally polarized receptors, the signal-to-noise ratio (S/N) of the pulsar profile, and the polarization fidelity of the system. Using simulations, we present how fundamental limitations in recovering the true polarization reduce the precision of ToA measurements. Any real system will respond differently to each source observed depending on the unique pulsar polarization profile. Using the profiles of known MSPs we quantify the limits of observing system specifications that yield satisfactory ToA measurements, and we place a practical design limit beyond which improvement of the system results in diminishing returns. Our aim is to justify limits for the front-end polarization characteristics of next generation radio telescopes, leading to the Square Kilometre Array (SKA).

Key words: instrumentation: polarimeters — (stars:) pulsars: general — radio continuum: general — techniques: polarimeters

1 INTRODUCTION

Any dual-polarization polarimeter is characterized by a degree of polarization purity, i.e. the cross-polarization between orthogonal feeds, and the extent to which calibration can be used to retrieve accurate polarization information. In this paper we examine how both of these limitations affect pulsar time of arrival (ToA) measurements, especially in the case of millisecond pulsars (MSPs). We do this by simulating ToA measurements through a sampling of the signal-to-noise ratio (S/N), calibration error, and intrinsic polarization leakage parameter space. Intrinsic polarization leakage includes the apparent leakage between orthogonal receptors due to differential receptor gains, in addition to the cross-coupling between receptors, which is typically thought of as ‘polarization leakage’.

Simulations are used, as it is difficult to analytically quantify the effects of calibration error, integration time, and intrinsic polarization leakage in a general form. The ToA measurement error depends on the ability to observe pulsar profiles with high fidelity. By profile, we mean the stable average shape of the radio pulse of a given pulsar, and its polarization properties. We perform our analysis using profiles from the 20 MSPs in Manchester et al. (2013).

A fundamental limit to any ToA measurement is the design of the polarimeter feeds, which is set by the telescope specifications. As pulsar timing is a key science project for Square Kilometre Array (SKA), see Janssen et al. (2015) and Cordes et al. (2004), it is important to consider the science limitations set during the design process. The decimetre wavelength band, where pulsars are typically observed for timing, will be covered by both dishes and aperture arrays. The analysis presented here applies to both telescope types.

For the design of the feeds, we need to consider the capacity of any dish or aperture array to produce data from
which the true polarization of a signal can be recovered. For example, the full polarization description of an incoming signal can never be recovered by a single dipole, no matter how good the calibration procedure is. On more realistic systems, there is a fundamental limitation in recovering the polarization state due to differential gains between orthogonal receptors. These differential gains coupled with noise in the system result in measurement errors which can not be corrected via any currently used calibration procedure. This affects all ToA measurement methods. For total intensity (e.g. Taylor (1992)) timing methods, i.e. the determination of a ToA through cross-correlation of a total power template, this is an effect in addition to calibration error. Techniques such as the invariant interval (Britton 2000) and matrix template matching (van Straten 2006) methods, despite being largely independent of polarization calibration error, are also affected.

The remainder of this introduction covers the relevant Jones and Mueller mathematical formalisms necessary to describe intrinsic polarization leakage. In Section 2 we describe the simulation setup and the strategy for exploring the relevant parameter space, how the simulated observations are generated, and the methods for determining the ToA. Results are presented in Section 3. Discussion of the results and the implications for current and future telescopes are presented in Section 4.

### 1.1 Intrinsic polarization leakage in Jones and Mueller formalism

We are interested in describing the intrinsic polarization leakage of a linear feed, dual-polarization system as this is a typical design for single pixel feeds and phased arrays such as phased-array feeds (PAFs) and aperture arrays (AAs) used for pulsar timing. Intrinsic polarization leakage can be described with the mathematical structure developed for the radio interferometer measurement equation (RIME) presented in Hamaker et al. (1996) and Smirnov (2011a). Additionally, the RIME can be extended to phased arrays where, to first order, the formed beam is a linear combination of the individual element beams; a full description would also include element mutual coupling terms. Jones matrix formalism is useful to frame the RIME in terms of instrumentation and environmental effects on an electromagnetic signal. The Mueller matrix formalism, which is used in our simulations, is useful in interpreting the RIME in terms of detected power, represented by the Stokes parameters of the signal.

In Jones formalism, transformations are applied to an input electromagnetic signal to produce the observed signal. The transformation from the complex electromagnetic sky Jones vector \( e \) to the observed voltage Jones vector \( v \) is

\[
\mathbf{v} = \mathbf{J}_{\text{sys}} \mathbf{e}
\]

where \( \mathbf{J}_{\text{sys}} \) is the total system Jones matrix representation which is constructed out of multiple linear Jones transformations, each of which can have dependence on time, observing frequency, and source direction (Smirnov 2011b). In the scope of this paper we are interested in the effect of intrinsic polarization leakage. The polarization leakage matrix \( \mathbf{D} \) is usually defined as a direction-independent Jones matrix with the direction-dependent polarization leakage components incorporated into the primary beam matrix \( \mathbf{E} \). For this work we are not focusing on the primary beam direction-dependent sensitivity, but only the potentially direction-dependent polarization leakage, thus for phased arrays we are defining \( \mathbf{D} \) to also include the direction-dependent polarization leakage

\[
\mathbf{D} = \begin{pmatrix}
1 & d_{p \rightarrow q}(
u, \theta, \phi) \\
-d_{q \rightarrow p}(
u, \theta, \phi) & 1
\end{pmatrix}
\]

where \((\theta, \phi)\) are reference frame dependent position angles. In Equation 2 \(d_{p \rightarrow q}\) is the intrinsic leakage of feed \(p\) into feed \(q\). In feed design, the off-diagonal terms are minimized. Ideally they are zero. For phased arrays, however, projection effects will cause \( \mathbf{D} \) to vary with observing direction.

We can then define an explicit RIME for our dish and phased array systems as

\[
\mathbf{J}_{\text{sys, dish}} = \mathbf{B}(\nu)\mathbf{G}(t)\mathbf{C}\mathbf{D}(\nu) \quad \text{(3a)}
\]

\[
\mathbf{J}_{\text{sys, PA}} = \sum_{i=1}^{n} \mathbf{W}_i(\nu, \theta, \phi)\mathbf{B}_i(\nu)\mathbf{G}_i(\nu)\mathbf{C}_i\mathbf{D}_i(\nu, \theta, \phi) \quad \text{(3b)}
\]

where \( \mathbf{B} \) is the frequency-dependent bandpass structure, and \( \mathbf{G} \) is the time-dependent electronic gain. \( \mathbf{B} \) and \( \mathbf{G} \) are diagonal matrices. The idealized nominal feed configuration \( \mathbf{C} \) is a coordinate transform from the sky to observing frame. For a single pixel feed, on axis observation, \( \mathbf{D} \) has no direction dependence and can be simplified to \( \mathbf{D}(\nu) \).

For a phased array, the \( \mathbf{J}_{\text{sys}} \) is the weighted sum over all \( n \) elements of the array, where \( \mathbf{W}_i \) are the complex weights to shape the array beam pattern to ‘point’ in a direction. For a PAF, the direction dependence of \( \mathbf{D} \) will be relative to the dish pointing centre. For an aperture array, the direction dependence of \( \mathbf{D} \) will be relative to the direction of boresight, usually zenith.

Jones formalism is useful for understanding the instrumental effects on a signal. For our simulation, however, we use pulsar profiles described as Stokes vectors, for which Mueller matrices are used to perform operations. Using the notation from Hamaker et al. (1996), any Jones matrix \( \mathbf{J} \) can be transformed to a corresponding Mueller matrix \( \mathbf{M} \) by use of the Kronecker product,

\[
\mathbf{M} = \mathbf{S}^{-1} (\mathbf{J} \otimes \mathbf{J}^*) \mathbf{S} \quad \text{(4)}
\]

where \( \mathbf{S} \) and \( \mathbf{S}^{-1} \) are the conversion matrices to transform between Stokes parameters and the brightness coherency vector. For reference they are presented below:

\[
\mathbf{S} = \frac{1}{2}
\begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 1 & i \\
0 & 0 & 1 & -i \\
1 & -1 & 0 & 0
\end{pmatrix}
\]

\[
\mathbf{S}^{-1} =
\begin{pmatrix}
1 & 0 & 0 & 1 \\
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 0 \\
0 & -i & i & 0
\end{pmatrix}
\]

Using equation [4] the feed-error matrix \( \mathbf{D} \) can be converted to a Mueller form \( \mathbf{D}_{\mathbf{M}} \) (Eq. [7]). Each element of which can be directly computed using \( M_{ij} = \frac{1}{2} \text{tr}((\sigma \mathbf{J} \sigma^* \mathbf{J}^i)) \).
\[
D_M = \begin{pmatrix}
1 + \frac{1}{2}(|d_{p\rightarrow q}|^2 + |d_{q\rightarrow p}|^2) & \frac{1}{2}(|d_{p\rightarrow q}|^2 - |d_{q\rightarrow p}|^2) & \text{Re}[d_{p\rightarrow q} - d_{q\rightarrow p}] & \text{Im}[d_{p\rightarrow q} + d_{q\rightarrow p}] \\
\frac{1}{2}(|d_{p\rightarrow q}|^2 - |d_{q\rightarrow p}|^2) & 1 + \frac{1}{2}(|d_{p\rightarrow q}|^2 + |d_{q\rightarrow p}|^2) & \text{Re}[d_{p\rightarrow q} + d_{q\rightarrow p}] & \text{Im}[d_{p\rightarrow q} - d_{q\rightarrow p}] \\
\text{Re}[d_{p\rightarrow q} + d_{q\rightarrow p}] & \text{Re}[d_{p\rightarrow q} - d_{q\rightarrow p}] & 1 - \frac{1}{2}(|d_{p\rightarrow q}|^2 - |d_{q\rightarrow p}|^2) & -\text{Im}[d_{p\rightarrow q} + d_{q\rightarrow p}] \\
\text{Im}[d_{p\rightarrow q} + d_{q\rightarrow p}] & -\text{Im}[d_{p\rightarrow q} - d_{q\rightarrow p}] & -\text{Im}[d_{p\rightarrow q} + d_{q\rightarrow p}] & 1 + \frac{1}{2}(|d_{p\rightarrow q}|^2 + |d_{q\rightarrow p}|^2)
\end{pmatrix}
\]

where \(\sigma_i\) is the \(i^{th}\) Pauli matrix. In this form, the \((i, j)^{th}\) element can be understood as the response of the \(i^{th}\) output to the \(j^{th}\) input.

### 1.2 Polarimeter performance metric

The polarization leakage of a dual-feed receiver is quantified by using cross-polarization ratio (XPR) metrics (IEEE 1998). Carozzi & Woan (2011), a new XPR, the polarimeter intrinsic cross-polarization ratio (IXR), was introduced. XPRs are used as metrics for radio communication feeds where the polarization of both the source and receiver is known. Thus an XPR can vary by choice of coordinate system (Carozzi & Woan 2011). The IXR is an XPR which is invariant under coordinate transform. This makes the IXR ideal for a radio astronomical polarimeter as there is no preferred sky coordinate frame. The IXR in Jones formalism is defined as

\[
\text{IXR}_J \equiv \left(\frac{g_{\text{max}} + g_{\text{min}}}{g_{\text{max}} - g_{\text{min}}}\right)^2
\]

where \(g_{\text{max}}\) and \(g_{\text{min}}\) are the maximum and minimum amplitude gains of the polarimeter when using singular value decomposition (SVD) to decompose the system Jones matrix \(J_{\text{sys}}\). The SVD theorem (Eq. 9) states that any Jones matrix \(J\) can be decomposed into two unitary transforms \(U\) and \(V\) and one diagonal transform matrix \(\Sigma\).

\[
J = U\Sigma V^\dagger = U \begin{pmatrix}
\sigma_{\text{max}} & 0 \\
0 & \sigma_{\text{min}}
\end{pmatrix} V^\dagger
\]

Given a noise input signal \(e\), the sensitivity to change in \(e\) from Equation 11 is measured by the matrix \(\text{cond}_2(J) \equiv \kappa(J) = g_{\text{max}}/g_{\text{min}}\), where \(\sigma_{\text{max}}\) and \(\sigma_{\text{min}}\) are the maximum and minimum singular values. An ill-conditioned matrix, one with a condition number much larger than 1, will cause an increase in the error of \(e\) with respect to the error of \(e\). Conversely, a well-conditioned matrix, one with a condition number close to 1, will transform \(e\) into \(v\) with minimal effect on the error.

Carozzi & Woan (2011) show that by setting the maximum and minimum amplitude gains to be equal to the maximum and minimum singular values (\(\sigma_{\text{max}} = g_{\text{max}}\) and \(\sigma_{\text{min}} = g_{\text{min}}\)), there is always an orthonormal choice of coordinates systems for the sky and the channels that gives \(J'\) from \(J\) such that the feed error matrix takes the form

\[
J' = \frac{g_{\text{max}} + g_{\text{min}}}{2} \begin{pmatrix}
1 & 1/\sqrt{\text{IXR}_J} \\
1/\sqrt{\text{IXR}_J} & 1
\end{pmatrix}
\]

The IXR\(_J\) is in units of power and Equation 10 has components of \(\sqrt{\text{IXR}_J}\) as a Jones matrix acts as an operation on an electric field vector. Intrinsic polarization leakage can be seen as differential gains or ‘canonical’ polarization leakage depending on the basis. The condition number can thus be related back to the intrinsic polarization leakage. That is \(d_{p\rightarrow q} = -d_{q\rightarrow p} = \frac{\kappa(J) + 1}{\kappa(J) - 1}\) with a normalization factor of \(\frac{\kappa(J) + 1}{2\kappa(J) - 1}\). Redefining the IXR\(_J\) in terms of the condition number, Equation 8 becomes

\[
\text{IXR}_J = \left(\frac{\kappa(J) + 1}{\kappa(J) - 1}\right)^2
\]

This is a crucial quantity which represents a fundamental limit in our ability to recover the true signal. This limit is independent of the polarization calibration.

The IXR is conceptually equivalent to the polconversion (Hamaker 2000) and Lorentz boost (Britton 2000) transformations that have been employed in previous works based on quaternions and geometric algebra, respectively. For example, where \(\beta\) is the velocity parameter that describes a Hermitian Jones matrix in Equation 11 of Britton (2000), \(\kappa(J) = e^{\beta^2}\) and IXR\(_J = \text{coth}^2(\beta)\). These equations enable meaningful comparisons between the results presented in this work and the notation employed in some previous studies (e.g. van Straten 2013).

To understand how the condition number, and by extension the IXR, affects an observation, we can look at how the true sky vector \(e\) is determined. To obtain an estimate of the true Jones sky vector \(\hat{e}\) from the observed Jones vector \(v\), the system Jones matrix \(J_{\text{sys}}\) must be determined and inverted (Eq. 12) via calibration

\[
\hat{e} = J_{\text{sys}}^{-1} v
\]

where \(\hat{e}\) is an estimate of \(e\) due to multiple compounding effects. First, in the measurement of \(v\) there is a limit in precision due to noise. Second, \(J_{\text{sys}}\) is not perfectly known, but is an estimate based on modeling and calibration. Finally, the condition of the components of \(J_{\text{sys}}\) determine how the errors in measurement affect the estimation of \(\hat{e}\). For an ill-conditioned \(J_{\text{sys}}\), a small error in \(v\) will result in a large error in estimated sky vector \(\hat{e}\) compared to the true sky vector \(e\). An ill-conditioned \(J_{\text{sys}}\) matrix will lead to a noisy estimate of \(\hat{e}\), no matter how well known \(J_{\text{sys}}\) is, due to the inherent noise in the measurement of \(v\). As the conditioning of the \(J_{\text{sys}}\) improves, so too does \(\hat{e}\) more accurately describe \(e\).

By definition \(\kappa(J) \geq 1\). Ideally there is no intrinsic polarization leakage between feeds, i.e. the matrix is perfectly conditioned \(g_{\text{max}} = g_{\text{min}} \Rightarrow \kappa(J) = 1\) and the IXR\(_J\) \(\rightarrow \infty\). That is, the two receptors are completely orthonormal. If there is intrinsic leakage between the two feeds then \(g_{\text{max}} > g_{\text{min}} \Rightarrow \kappa(J)\) increases and the IXR\(_J\) decreases. In the worst case (e.g. where the two feeds are perfectly coupled, or one receptor’s sensitivity goes to 0), then \(g_{\text{min}} \rightarrow 0 \Rightarrow \kappa(J) \rightarrow \infty\) and the IXR\(_J\) \(\rightarrow 1\). Since the IXR\(_J\) is a measure of feed response, it is common to use decibel (dB) units, \(\text{IXR}_{J,\text{dB}} = 10 \log_{10}(\text{IXR}_J)\).

We can also consider the IXR in terms of Mueller matrices. Carozzi & Woan (2011) connects the IXR to Mueller matrices by showing \(\kappa(M) = \kappa^2(J)\). This relation is used to show the IXR in Mueller formalism is

\[
\text{IXR}_M = \frac{\kappa(M) + 1}{\kappa(M) - 1} = \frac{1 + \text{IXR}_J}{2\sqrt{\text{IXR}_J}}
\]
which provides a useful metric for measuring the impact of instrumental polarization on the Stokes parameters, especially in the case of impure transformations with no corresponding Jones matrix.

An example of the variation in IXR$_J$ across the field of view of a simple dipole element is shown in Figure 1. The IXR$_J$ is maximized (70 dB) in the direction of zenith, but rises when observing away from boresight. The low IXR$_J$ structure ([0°, 180°] and [90°, 270°] axes) is along the 45° line between the two orthogonal receptors. The variation in IXR$_J$ across the field of view also depends on observing frequency.

Carozzi et al. (2009) and Sutinjo & Hall (2013) show the IXR$_J$ can vary between 0 dB and 66 dB across an aperture array depending on pointing direction and element design. For an idealized short dipole the IXR$_J$ varies smoothly over the observable hemisphere. But for many feeds — such as Vivaldi-type, bow-tie, and narrow-band half-wavelength dipoles — sharp intrinsic polarization leakage structures form across the hemisphere. Figures 3 and 5 in Carozzi et al. (2009) show the IXR$_J$ over a hemisphere for short dipoles and Vivaldi-type elements. Figures 2 and 3 in Sutinjo & Hall (2013) show the IXR$_J$ across the field of view (FoV) of an MWA bow-tie element. These published values and maps provide insight into what range of IXR$_J$ to use in our simulations.

1.3 IXR and signal-to-noise ratio

The error on a ToA measurement is, in general, a function of the S/N of a given observation. By S/N, we hereby refer to the peak pulse value in the Stokes $I$ profile to the standard deviation of the off-pulse signal. Under ideal circumstances, the S/N increases with the square root of integration time.

For a polarized source, instrumental intrinsic polarization leakage will result in a lower observed S/N compared to the ideal S/N, i.e. that obtained from a system with no intrinsic polarization leakage, for a given amount of integration. The blue/solid line in Figure 2 shows the fractional observed S/N compared to the ideal S/N as a function of the IXR for J1603–7202. The red/dashed line shows how using the inverse of a poorly conditioned matrix for calibration amplifies the noise in the measured profile. This will be discussed further in Section 2. All the simulated pulsars have a similar response. An effect of a low IXR is the introduction of a differential gain between feeds. As the IXR goes to 0 the receiver system becomes effectively blind to one polarization, and the observed S/N is approximately half (blue/solid line) that of the ideal S/N in the limit IXR → 0. When calibration is applied, not only is there a differential gain effect, but the inversion of the ill-conditioned matrix will significantly degrade the S/N of any profile (red/dashed line). The ideal S/N is achieved only in the limit IXR → ∞, and there is a one-to-one correspondence with integration time. A reference integration time of $\tau_{\text{ref}} = 1$ is defined as the time it would take to build up an ideal S/N of 1000. All integration time values quoted in this paper are a fraction of this reference integration time. The relationship between $\tau_{\text{ref}}$ and ideal S/N $SN_0$ is

$$\tau_{\text{ref}} = \left( \frac{SN_0}{1000} \right)^2$$

2 THE SIMULATIONS

We have performed simulations with the goal to quantify the effect of intrinsic polarization leakage on pulsar ToA measurements. We have sampled the three-dimensional parameter space that includes the IXR$_J$, ideal pulse profile $SN_0$, and calibration error, which covers current telescope measurements and future telescope specifications. For each
point in the sampled parameter space, ‘observed’ profiles are generated for 500 epochs, by modifying a template profile with the appropriate Mueller matrices. For every epoch, we stochastically generate the $J_{\text{sys}}$ Jones matrix, and the corresponding Mueller matrix $M_{\text{sys}}$, for a given intrinsic polarization leakage. The form of this matrix is described in §2.2. The observed profile is then calibrated by multiplying by the inverse of the system Mueller matrix with additive random calibration errors.

A ToA is determined at each of the 500 epochs, using a standard timing method (section 2.3) included in PSRCHIVE (Hotan et al. 2004; van Straten et al. 2012). In a normal pulsar timing experiment, a model would then be fit to the ToA measurements using TEMPO2 (Hobbs et al. 2006) and, the goodness of the model would be measured by the root mean square (rms) of the timing residuals. Since we are using a simple model of an isolated, stable pulsar of constant period, this rms represents the ideal rms for a given set of parameters.

In the following we give further details of the steps of the simulation.

### 2.1 Pulse profiles

We performed our simulations using profiles based on the mean pulse profiles of 20 well studied MSPs from the Parkes Pulsar Timing Array [Manchester et al. 2013]. It is not our intention to reproduce the results of that work. We use these MSPs as they cover a wide range of profile and polarization structures, and perform our simulations to much higher S/Ns than is possible with current experiments.

A high S/N version of each reference profile was generated by applying a rolling Hann filter with a width of ~1% of the profile. This has the effect of removing high-frequency components and reduces the amplitude (up to a few percent) of narrow profile structures. This was done in order to avoid very low uncertainties in our ToA measurements due to the presence of the same high frequency structures in the templates and the simulated noisy data. Though we retain the pulsar names, by applying this filter, the new profiles are approximations to the original profiles and are meant only to be a representative set of profile structures. The low frequency components that remain, may cause our uncertainties to be systematically lower than real timing experiments to these pulsars, however this effect is consistent across our simulations and therefore does not affect our final result.

The ideal profiles are then used to generate imperfect profiles at each epoch using the process described in Section 2.2. These profiles were also used as the template profiles when performing timing. The first and third columns of Figures 9, 10 at the end of this paper are the Stokes parameters of the ideal profiles.

### 2.2 Simulated observations

Our simulated observations produce an estimated sky Stokes vector $\hat{\mathbf{e}}^S$ for a given pulsar per epoch by estimating the system Mueller matrix, including calibration error, intrinsic polarization leakage, and S/N parameters. Figure 5 shows the stages to arrive at an example $\hat{\mathbf{e}}^S$.

The explicit RIMEs defined in Equation 5 can be simplified for the simulations. By using the mean profile we are making the simplification that the bandpass $\mathbf{B}$ and time varying $\mathbf{G}$ system gains have been solved for and applied to the observed signal. Thus, $\mathbf{B}$ and $\mathbf{G}$ are unity diagonal matrices. We include a polarization calibration error term $\Delta \mathbf{J}$ into our system to simulate the effect of imperfect calibration of $\mathbf{B}$ and $\mathbf{G}$. The nominal feed matrix $\mathbf{C}$ is a telescope-specific basis transform, and will not affect the IXR as it is a coordinate independent metric (i.e. $\text{IXR}_C = \infty$). Thus, the polarization leakage matrix $\mathbf{D}$ is the only matrix which will vary in our parameter space. In practice $\mathbf{D}$ is a function of frequency, but as the profile is an average across a frequency band, so too is the IXR a frequency averaged intrinsic polarization leakage.

To generate a simulated observed profile we start by creating an intrinsic polarization leakage Jones matrix representation. We define the polarization leakage to be $1/\sqrt{\text{IXR}_J}$, in decibel units the intrinsic polarization leakage is $-\text{IXR}_{J,\text{dB}}$. We have chosen this definition as polarization leakage is a common concept within the community. The IXR is a measure of both the cross-coupling between receiver feeds, which is typically thought of as ‘polarization leakage’, and the apparent leakage due to differential feed gains and thus is a more complete metric for ‘polarization leakage’ over previous definitions. A higher intrinsic polarization leakage implies the two feeds are more coupled together than a lower intrinsic polarization leakage. To sample a broad range of intrinsic polarization leakage values we sample the space 0 dB to $-30$ dB, where the upper limit of 0 dB is effectively blind to one polarization, such as a single polarization receiver. The intrinsic polarization leakage sample space in IXR notation is $1 \geq 1/\sqrt{\text{IXR}_J} \geq 0.031$. By inverting equation 11 the condition number is related to the IXR $J$ is

$$\kappa(J) = \frac{\sqrt{\text{IXR}_J} + 1}{\text{IXR}_J - 1}$$

For each run of the simulation with a given set of parameters, we construct a system Jones matrix $J_{\text{sys}}$ (Equation 1) by generating a random complex matrix from a normal distribution ($\mu = 0, \sigma = 1$) for $D$. All other matrices in $J_{\text{sys}}$ are replaced with those for the simulated IXR parameter. Without loss of generality we normalize $\Sigma$ using $\sigma_{\text{max}} = g_{\text{max}} = 1$. The normalized condition number becomes $\kappa(J) = g_{\text{min}}$ and the system Jones matrix due to intrinsic polarization leakage becomes

$$J_{\text{sys}} = U\Sigma V^T = U\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{\kappa(J)} \end{pmatrix} V^T$$

We use a random matrix as there is an infinite set of Jones matrices for a given IXR. The decomposition by SVD, and reconstruction steps are to maintain the same scaling as with the calibration error Jones matrix.

For error in the polarization calibration, we use a sample space starting at the ideal perfect calibration (0%) up to 15% calibration error. This calibration error represents the imperfect measurement of $\mathbf{B}$ and $\mathbf{G}$, the inverse of which are applied to the observed signal to integrate the profile in both time and frequency. A calibration error matrix $\Delta \mathbf{J}$ is generated from a random normal complex distribution ($\mu = 0, \sigma = \text{precent error}/100$). The estimated system Jones matrix is $J_{\text{sys}} = J_{\text{sys}} + \Delta \mathbf{J}$. This error parameter space covers
Stokes $I$ peak is unity, by a scalar $S/N_1$ value and adding a system noise Stokes vector $n^S$ which is a set of real random values from a normal distribution ($\mu = 0, \sigma = 1$). The noise component $n^S$ is a direction-independent effect. This is a practical approximation when the system noise is receiver noise dominated or when the sky noise is isotropic on the scale of the beam primary lobe. In the extreme case where all the system noise is direction-dependent in the direction of the source the effect on rms timing will only be a loss in $S/N$ as seen in Figure 2. As we will see in the following sections, the effect of direction-independent noise is to further degrade the timing solutions beyond the loss in $S/N$. For simplicity we are using only direction-independent noise for the simulation.

An ideal $S/N$ is set to be in the range $\sqrt{10^3}$ to $\sqrt{10^7}$ for our simulations, though we note: the observed $S/N$ is reduced as the condition number increases, and thus the actual $S/N$ is a function of IXR as shown in Figure 2. The estimated Stokes vector $\hat{e}^S$ of the observed pulsar is

$$\hat{e}^S = \mathbf{M}^{-1}_{\text{sys}} (\mathbf{S}_1 \times \mathbf{M}_{\text{sys}} e^S + n^S) \quad (16)$$

The inversion of $\mathbf{M}_{\text{sys}}$ is the calibration stage (Eq. 12).

We have simulated two types of calibration. The first is a ‘gain’ calibration where we are only interested in solving for the bandpass and electronic gain of the system, that is $\mathbf{J}_{\text{sys}} = \mathbf{BGCD} + \Delta \mathbf{J} = \mathbf{I} + \Delta \mathbf{J}$, where $\mathbf{I}$ is the identity matrix. This is an ideal calibration solution where the gain terms can be solved for independently of any $\mathbf{D}$ effects, such as if using a known noise reference in a single dish system. The second type of calibration is a ‘full’ calibration where the $\mathbf{D}$ term is included, $\mathbf{J}_{\text{sys}} = \mathbf{BGCD} + \Delta \mathbf{J} = \mathbf{D} + \Delta \mathbf{J}$. This is a more realistic approach, as the IXR will affect any calibration solution. And, when performing timing with an array, complex gain solutions are required to combine signals in phase by using a sky calibrator source or self-calibration. The calibratability of an array is a topic which should be studied in further work. Figure 2 shows the effect of these two methods on observed $S/N$, independent of pulsar and ideal $S/N$. Examples of these types of calibration on the observed profile are shown in Figures 2(d) and 2(e). After a simulated $\hat{e}^S$ is produced, a ToA is then determined with standard pulsar timing software. The effect of these calibration methods on timing will be shown in the following section. The simulation code is available as a git repository.

2.3 Methods for TOA determination

For ToA measurements, three standard methods are used: total intensity (Taylor 1992), invariant interval (Britton 2000), and matrix template matching (van Straten 2006). These methods are included in PAT, from the PSRCHIVE package. For the total intensity method, each observed Stokes $I$ profile is cross-correlated with the ideal template. The invariant interval technique uses all Stokes parameters in the form of a Lorentz 4-vector $(I^2 - Q^2 - U^2 - V^2)^{1/2}$. By including all Stokes parameters complete information is used. However using the invariant form, the $S/N$ decreases, leading to a less precise ToA determination compared to the

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1 https://github.com/griffinfoster/pulsar-polarization-sims
intensity fitting, when the source is highly polarized or intrinsically weak (see van Straten 2013 for more information on this effect). Matrix template matching represents the profile in Jones notation and produces a ToA measurement while simultaneously solving for a calibration solution by transforming the template profile to the observed profile. In practice, the S/N requirements for timing often “force” a pre-processing step of frequency and time averaging. This requires some prior level of polarization calibration, as described in the previous section, rendering the assumptions for matrix template matching no longer strictly valid, by introducing covariances between the Stokes parameters, especially for large polarization leakage.

Figure 4 shows the ToA rms derived from the three timing methods used in simulations of the well studied MSP J1603–7202. There is a strong dependence on polarization calibration error when using the total intensity timing method. Applying the ‘full’ calibration results in better timing solutions except in the high intrinsic polarization leakage region (> -5 dB), compared to using only the ‘gain’ calibration. We will use the ‘full’ calibration simulations for our total intensity results. The timing error, when using the invariant interval method, is effectively the same for both types of calibration.

For the matrix template matching method there is a significant change in the ToA rms at high intrinsic polarization leakage between the two types of calibration. This is due to ‘full’ calibration introducing covariances between Stokes parameters, compared to just ‘gain’ calibration. Simulations using ‘gain’ calibration and timed with matrix template matching produce the best timing results for a given calibration error and intrinsic polarization leakage. We will present matrix template matching results using both types of calibration, but will focus our analysis on the ‘full’ calibration as it represents a more pragmatic result in terms of system calibration, especially in when using an array for timing which requires multiple signals to be combined in phase.

In practice, total intensity and invariant interval are commonly used methods, where as the matrix template matching method is more rarely used due to the difficulties in practical polarization calibration. However, as timing limits are pushed, matrix template matching will become necessary (van Straten 2013).

3 RESULTS

As polarization calibration error is not the focus of these simulations, we can compress the contour plots in Figure 4 into a more concise and useful figure by collapsing the calibration error axis. Figure 5 shows the bound range for each method on a single plot. The narrowness of the bounds for the invariant interval and matrix template matching show their independence from polarization calibration error. Plots of this style for all the simulated MSPs are shown at the end in Figures 9–10. These figures show that as the intrinsic polarization leakage decrease, so too do the timing residuals using all methods.

We have chosen to use the ToA rms noise as the metric for these results. This is a measure of the time of arrival scatter for the simulations. Using the average ToA uncertainty $\sigma_{\text{ToA}}$ underestimates the error. As per Eq. 8.2 of Lorimer (2005) $\sigma_{\text{ToA}} \approx W/\text{SN}_0$, where $W$ is the pulse width and $\text{SN}_0$ the observed S/N. In our simulation results we see the
intrinsically polarization leakage affects timing results beyond a simple reduction of the S/N, producing poorer rms timing residuals compared to the average $\sigma_{\text{ToA}}$ of the same simulated observations. In the limit the IXR$_J \to \infty$, the rms noise will reach $\sigma_{\text{ToA}}$.

We computed the reduced $\chi^2$ solution to measure the goodness of fit for each sample of the parameter space. When computing the $\chi^2$ of the measured time of arrival against the expected time of arrival, the low intrinsic polarization leakage case results in a good fit, as would be expected. As the intrinsic polarization leakage increases the fit degrades. Paradoxically, as the intrinsic polarization leakage increases to the highest values, the reduced $\chi^2$ fit approaches 1. This is because $\sigma_{\text{ToA}}$ grows exponentially large, and the error of the fit is within the limits of the ToA variance.

Although the diverse set of profiles we have simulated produce different results, seen in Figures 9 and 10, there are general trends we observe. All the simulations with the ‘full’ calibration profiles show exponentially increasing rms timing solutions as the intrinsic polarization leakage increases. Using only ‘gain’ calibration produces good timing results in this region, but is not as realistic as the ‘full’ calibration simulations.

Generally, using matrix template matching, with ‘gain’ or ‘full’ calibration, produces better timing solutions compared to other methods at all intrinsic polarization leakage values. For the majority of the ‘gain’ calibrated profiles, matrix template matching produces timing solutions that are only weakly dependent on intrinsic polarization leakage. It is worth noting that, for a few profiles, such as J1022+1001, matrix template matching of the ‘full’ profile combination produces lower rms timing residuals than the ‘gain’ calibrated profile. This profile has highly polarized components, which are distorted by high intrinsic polarization leakage. At high intrinsic polarization leakage, the profiles with ‘full’ calibration result in high rms timing residuals and dependence on polarization calibration error when using matrix template matching. However, at low intrinsic polarization leakage both types of calibration converge to a similar timing solution.

The results from using the invariant interval method are consistent across all profiles. In a few cases, for example J1744–1134 and J1824–2452, the invariant interval method significantly underperforms compared to the other methods. Both these pulsars are highly polarized; therefore, in the invariant interval, the power in Stokes I is almost entirely cancelled out by the other Stokes parameters.

Timing solutions with the total intensity method have a notable dependence on the polarization calibration error. Plotted the bound regions as individual lines for different polarization calibration errors in Figure 6. There is a polarization leakage point (e.g. for PSR J1603–7202 it is around $-4\,\text{dB}$), after which systems with higher polarization calibration errors produce better timing solutions than an ideally calibrated system. This is true for all MSP profiles used in our simulation, though transition points vary between $-3\,\text{dB}$ and $-7\,\text{dB}$. In this region of the parameter space, this counter-intuitive effect comes about because the error-free calibration transformation also has high intrinsic leakage and therefore further reduces the observed S/N of the calibrated profile. Adding polarization calibration error reduces the intrinsic polarization leakage, leading to an increase in the observed S/N. There is potential use for the total intensity method at high intrinsic polarization leakage when only using a ‘gain’ calibration. Needless to say, this regime should be avoided and this sets a maximum limit on the allowable intrinsic polarization leakage to around $-5\,\text{dB}$.

Figure 7 shows the residual timing rms of J1603–7202 as a function of S/N and IXR$_J$ using a calibration error of 5%. For a given integration time, the achievable time of arrival rms noise is dependent on the system polarization.
value, e.g. in simulation of J1603–7202 for $\tau_{\text{int}} = 0.1$ a system with $-25$ dB intrinsic polarization leakage can achieve a ToA rms in the timing residuals of around 300 ns compared to 3000 ns with a $-5$ dB intrinsic polarization leakage system. This variation due to intrinsic polarization leakage can be better seen in Figure 8 which shows the residual rms as a function of integration length, for a range of intrinsic polarization leakage values. The rms will continue to decrease as the integration time increases. The system intrinsic polarization leakage sets the required integration time to achieve a desired rms. In [14] we discuss the importance of Figure 8 in optimizing science return when there is limited available observing time.

4 DISCUSSION

Gravitational wave detection using high precision timing of MSPs constitutes a key science project for the SKA. This imposes a requirement on the polarization specifications. In Cordes et al. (2004), a case is presented which sets the required polarization purity level to $-40$ dB to accomplish the SKA pulsar key science goals. This polarization purity value is different from what we have defined as polarization purity; it is a measure of the final, calibrated Stokes data and not a specification of the front-end design as we have considered here.

Figure 8 shows that for a given intrinsic polarization leakage, a desired timing residual rms can be achieved with sufficient observing time. This, of course, ignores the other systematic effects that are part of a timing observation and does not include the additional sensitivity modulation of the primary beam shape. We have only focused on intrinsic polarization leakage, which is an effect on any feed design. The main issue is that the intrinsic polarization leakage has a strong effect on the required observation time, which is a limited commodity. For our simulation of MSP J1603–7202 in Figure 8 the difference in required observation time to achieve a desired rms noise at $-15$ dB compared to $-30$ dB is a factor of 1.5. With limited available observing time, we would like to set an upper limit on the intrinsic polarization leakage above which it is no longer optimal to be using observation time on a measurement. Carozzi et al. (2009) and Sutinjo & Hall (2013) show the IXR, for typical feeds to be somewhere between 30 dB and 66 dB at boresight. We see that observing with an intrinsic polarization leakage of $-15$ dB implies at least a 50% increase in observing time compared to that of a typical feed. A high intrinsic polarization leakage is not a design issue for a ‘classic’ single pixel dish system, where a low intrinsic polarization leakage can be achieved when observing a source on axis. This is not the case with aperture arrays, PAFs, and multi-beaming with single pixel dishes. In these cases, the source will likely not be located in the optimal intrinsic polarization leakage region of the beam. For aperture arrays, a source will rarely, if ever, be on axis. Returning to the example beams in Carozzi et al. (2009) and Sutinjo & Hall (2013), we see that the polarization leakage values can quickly increase to above $-10$ dB away from zenith. This effectively limits the declination range of sources, depending on the array latitude, for pulsar timing.

From our simulation we see there is an intrinsic polarization leakage lower limit on feed design at which point there is minimal return in terms of reducing the timing residual rms with further engineering investment, for a given integration time. As there is a cost to every incremental improvement in IXR, we would like to present our simulation results in terms scientific return for marginal improvements in engineering specifications. In an effort to create a meaningful engineering intrinsic polarization leakage lower limit, Table 1 lists the fractional improvement in ToA rms noise for different IXR values. We have picked a typical S/N ($\tau_{\text{int}} = 0.01$) for a timing observation. Columns 2, 4, 6, and 8 are the timing residual rms for each MSP at IXR = 10, 20, 30, 40 dB respectively. Columns 3, 5, and 7 are the percentage change in the rms with the changes in IXR. This table shows the diminishing marginal utility of improving IXR for the benefit of decreasing the time of arrival rms noise. There is, on average, a 29% improvement in the rms when improving the IXR from 10 dB to 20 dB, but can vary significantly with profile shape. For example, timing of J0711–6830 is largely unaffected by improvements in IXR, while timing of J1603–7202 improves with each increase in IXR. Going from 20 dB to 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above 30 dB there is a small improvement, but going above
dB provides essentially no improvement. This indicates that for a feed with IXR, dB > 30 dB there is limited fractional improvement in pulsar timing capabilities. It may be worth considering that low intrinsic polarization leakage across the field of view may be preferable to optimizing for minimal intrinsic polarization leakage around boresight.

5 CONCLUSION

On the pathway towards the SKA a number of aperture arrays, dishes, and PAFs are being developed as precursor instruments. As pulsar timing is a key science project, design of these instruments should take into account the intrinsic polarization leakage specification we have presented in this paper.

There is a relative increase in required integration time as a function of the feed IXR, as seen in Figure 8. At high intrinsic polarization leakage this can make the required integration time inefficiently long.

We have shown that there are diminishing returns (Table 6) on building feed systems which have intrinsic polarization leakage below ~30 dB in the direction of observation. Achieving this intrinsic polarization leakage limit should be easily affordable for single pixel dishes on axis, where the leakage is at a minimum. However, aperture arrays, PAFs, and multi-beam systems, where observations are not always made in the direction of minimum leakage, could have limited use for pulsar timing if intrinsic polarization leakage is not taken into account while developing the feed system. The complex aperture array and PAF beams with sharp, frequency dependent structure lead to regions of high intrinsic polarization leakage.

Given the effect of intrinsic polarization leakage on pulsar timing and a costing model for a feed design, a desirable optimization could be to maximize the average IXR across the intended usable field of view for the element and not just in the direction of boresight.

The calibratability of an array has a key effect on timing as we have shown with the idealized ‘gain’ calibration technique against the more realistic ‘full’ calibration. Beyond this work, there is scope for additional work on effect of calibration on timing. Additionally, the matrix template matching method should be extended to account for the covariances between the Stokes parameters induced by poorly conditioned calibration matrices.

Ideally, we can further refine these values on precursor instruments. The MeerKAT, KAT-7, ASKAP, and APERTIF arrays will provide a platform to study dish array polarization effects with multi-beam feeds and PAFs. LOFAR and MWA, though not ideal for pulsar timing experiments due to the low observing frequencies, will be useful to study instrumental polarization in aperture arrays. A study of these array polarization properties is necessary to assure the SKA science specifications can be met for pulsar timing.

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SIMULATION PLOTS
Figure 9. Time of arrival rms noise from simulation of various MSPs. The first and third columns are the Stokes parameters of the ideal profile: I (black/solid), Q (green/dashed), U (red/dotted), and V (cyan/solid). The second and fourth columns show ToA rms noise ($\mu s$) as a function of intrinsic polarization leakage when using total intensity (red/no border), invariant interval (gray/dashed border), and matrix template matching (full calibration: orange/dotted border, gain calibration: blue/dot-dash border) methods on the profile. The width of the lines show the effect polarization calibration error has on the rms, a polarization calibration error from 0% to 15% was used. These simulations use a fractional integration time of 0.01 which would produce an ideal signal-to-noise ratio of 100. Plots continue in Figure 10.
Table 1. Fractional improvement in time of arrival rms noise between two IXR\(_J\) values for the simulated MSPs using the matrix template method. Percent change is computed as \(\Delta_{ToA} = 100(\text{rms}_i - \text{rms}_f)/\text{rms}_i\), where \(\text{rms}_i\) is the initial (lower) IXR\(_J\) ToA rms noise, and \(\text{rms}_f\) is the final (higher) IXR\(_J\) ToA rms noise. These simulations used a fractional integration length of 0.01, which is an ideal SNR of 100. The reported rms values have an uncertainty of ~ 1% set by the number of ToA simulations (\(n = 5000\)). The last rows of the table are the average ToA rms noise percent improvement and minimum/maximum range.
Figure 10. Continuation of Figure 9, see that figure for description.