A generalization of Hardy’s operator and an asymptotic Müntz-Szász Theorem

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1 Overview

In this note we shall give a novel proof that Hardy’s Operator $H$, defined on $L^2([0,1])$ by the formula,

$$Hf(x) = \frac{1}{x} \int_0^x f(t)dt, \quad x \in (0,1],$$

is bounded. This new proof relies only on algebra together with the observation that the monomial functions are eigenvectors for $H$. Specifically, for each $k \geq 0$,

$$Hx^k = \frac{1}{k+1} x^k. \quad (1.1)$$

Always intrigued by results in analysis whose proofs rely mainly on algebra, the new proof of Hardy’s Inequality prompts the authors to propose the following definition.

**Definition 1.2.** We say that $T$ is a **monomial operator** if $T$ is a bounded operator on $L^2([0,1])$ and there exist a number $m$ and a sequence of scalars $c_0, c_1, c_2 \ldots$ such that for all $k$,

$$Tx^k = c_k x^{k+m}. \quad (1.3)$$

We shall call the number $m$ in (1.3) the **order** of $T$. It can be any complex number with non-negative real part, though in all our examples it will be a natural number. In addition to $H$, a monomial operator of order 0, other examples of monomial operators are the multiplication operator $M_x$ that sends a function $f(x)$ to the function $xf(x)$, and the Volterra operator $V = M_x H$, the operator given by

$$Vf(x) = \int_0^x f(t)dt, \quad x \in (0,1].$$

Both $M_x$ and $V$ are of order 1.

For $0 \leq s \leq 1$ we shall use $L^2[s,1]$ to denote the closed subspace of $L^2[0,1]$ of functions that are 0 on $[0,s]$.

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Definition 1.4. We shall say that a bounded operator \( T : L^2[0, 1] \rightarrow L^2[0, 1] \) is *vanishing preserving* if \( TL^2[s, 1] \subseteq L^2[s, 1] \) for every \( s \) in \([0, 1)\).

In this note we shall prove the following result.

Theorem 1.5. If \( T \) is a monomial operator, then \( T \) is vanishing preserving.

Why might such a theorem be true? If \( T \) is a monomial operator, and \( f \) is a polynomial that vanishes at 0 to some high order \( M \), then \( Tf \) also vanishes to order at least \( M \). So if one thinks of vanishing on \([0, s]\) as an extreme case of vanishing to high order, one might believe that monomial operators preserve this property.

Our proof of Theorem 1.5 relies on a new type of Müntz-Szász Theorem, wherein the monomial sequence is allowed to drift. This may be more interesting than the theorem itself!

2 Hardy’s Inequality

For a continuous function \( f \) on \([0, 1]\) consider the continuous function \( Hf \) on \((0, 1]\) defined by the formula

\[
Hf(x) = \frac{1}{x} \int_0^x f(t) dt, \quad x \in (0, 1].
\]  

(2.1)

Noting that as \( x \to 0, \frac{1}{x} \to \infty \) and \( \int_0^x f(t) dt \to 0 \), the following question arises:

How does \( Hf \) behave near 0?

Invoking the Mean Value Theorem for Integrals yields that for each \( x \in [0, 1] \), there exists \( c \in [0, x] \) such that \( Hf(x) = f(c) \). Thus,

\[
|Hf(x)| \leq \max_{t \in [0,x]} |f(t)|, \quad x \in (0, 1],
\]  

(2.2)

so that in particular, \( Hf \) is bounded near 0. More delicately, if we apply the MVT to the function \( f(x) - f(0) \), we obtain the estimate

\[
|Hf(x) - f(0)| \leq \max_{t \in [0,x]} |f(t) - f(0)|, \quad x \in (0, 1],
\]  

(2.3)

which implies that \( Hf(x) \to f(0) \) as \( x \to 0 \). Therefore, if we agree to extend the definition of \( Hf \) at the point \( x = 0 \) by setting \( Hf(0) = f(0) \), then our observations imply the following proposition.

Proposition 2.4. If \( f \) is a continuous function on \([0, 1]\), then \( Hf \) is a continuous function on \([0, 1]\). Furthermore,

\[
\max_{x \in [0,1]} |Hf(x)| \leq \max_{x \in [0,1]} |f(x)|.
\]  

(2.5)

Hardy [5] was the first to study the local behavior of \( H \) at 0 for functions equipped with norms other than the supremum norm. His result when specialized to \( L^2([0, 1]) \), the Hilbert space of square integrable Lesbesgue measurable functions on \([0, 1]\), is as follows.
Proposition 2.6. (Hardy’s Inequality in \( L^2([0,1]) \)) If \( f \) is a measurable function on \([0,1]\), then \( Hf \) is a measurable function on \([0,1]\), and

\[
\int_0^1 |Hf(x)|^2 dx \leq 4 \int_0^1 |f(x)|^2. \tag{2.7}
\]

A linear operator \( T \) on a normed vector space \( V \) is called \textit{bounded} \(^1\) if there is some constant \( C \) so that

\[
\|Tv\| \leq C \|v\| \quad \forall v \in V. \tag{2.8}
\]

The infimum of all \( C \) for which (2.8) holds is called the norm of \( T \), and written \( \|T\| \). Using this terminology, (2.7) says \( \|H\| \leq 2 \).

Our proof of Proposition 2.6 in Section 4 relies on a new “sum of squares identity” involving the operator \( H \), proved using (1.1).

3 Hilbert Space distance formula

Let \( h_1, \ldots, h_n \) be \( n \) vectors in a hilbert space \( \mathcal{H} \). We may associate to these vectors their \textit{Gram matrix}, i.e., the \( n \times n \) matrix \( G[h_1, \ldots, h_n] \) defined by

\[
[G[h_1, \ldots, h_n]]_{ij} = [\langle h_j, h_i \rangle_{\mathcal{H}}].
\]

An often used application is the following elegant formula for the distance to the span of \( h_1, \ldots, h_n \).

**Theorem 3.1. (Hilbert Space Distance Formula)** If \( \mathcal{H} \) is a Hilbert space, \( h \in \mathcal{H} \), and \( h_1, h_2, \ldots, h_N \in \mathcal{H} \) are linearly independent, then

\[
\text{dist}(h, \text{span}\{h_1, h_2, \ldots, h_N\}) = \sqrt{\frac{\det G[h, h_1, h_2, \ldots, h_N]}{\det G[h_1, h_2, \ldots, h_N]}}. \tag{3.2}
\]

**Proof.** Write \( h = k + m \), where \( k \) is in the span of \( \{h_1, \ldots, h_N\} \) and \( m \) is perpendicular to the span. Then \( \|m\| = \text{dist}(h, \text{span}\{h_1, h_2, \ldots, h_N\}) \).

We can write

\[
det G[h, h_1, h_2, \ldots, h_N] = det G[k, h_1, h_2, \ldots, h_N] + det G[m, h_1, h_2, \ldots, h_N].
\]

Since \( k \) is in the span of \( \{h_1, \ldots, h_N\} \), we have

\[
det G[k, h_1, h_2, \ldots, h_N] = 0.
\]

Moreover, \( G[m, h_1, h_2, \ldots, h_N] \) is a matrix whose first row is \( \langle m, m \rangle, 0, \ldots, 0 \). Therefore

\[
det G[m, h_1, h_2, \ldots, h_N] = \|m\|^2 \det G[h_1, h_2, \ldots, h_N].
\]

Combining these observations, we get (3.2). \( \quad \text{QED} \)

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\(^1\)It is a straightforward exercise to show that a linear operator is bounded if and only if it is continuous.
A Hilbert space proof of Hardy’s Inequality

The key step in our proof is the following lemma.

Lemma 4.1. An Identity for $H$

\[ \|f\|^2 = \|(1-H)f\|^2 + \left| \int_0^1 f(x)dx \right|^2 \quad \forall f \in L^2([0,1]). \]

Proof. It suffices to show for $f$ a polynomial, since the polynomials are dense in $L^2[0,1]$. If

\[ f(x) = \sum_{j=0}^{n} a_j x^j \]

then

\[ \|f\|^2 = \int_0^1 \sum_{i,j=0}^{n} a_i \overline{a_j} x^{i+j} dx = \sum_{i,j=0}^{n} a_i \overline{a_j} \frac{1}{i+j+1}. \]

Likewise, as

\[ (1-H)f(x) = \sum_{j=0}^{n} \frac{j}{j+1} a_j x^j, \]

\[ \|(1-H)f\|^2 = \sum_{i,j=0}^{n} a_i \overline{a_j} \frac{ij}{(i+1)(j+1)(i+j+1)}. \]

Hence

\[ \|f\|^2 - \|(1-H)f\|^2 = \sum_{i,j=0}^{n} a_i \overline{a_j} \left( \frac{1}{i+j+1} \right) - \frac{ij}{(i+1)(j+1)(i+j+1)} \]

\[ = \sum_{i,j=0}^{n} a_i \overline{a_j} \left( \frac{1}{(i+1)(j+1)} \right) \]

\[ = \left| \int_0^1 f(x)dx \right|^2. \]

QED

Proof. We now complete the proof of Proposition 2.6. We want to prove that $\|H\| \leq 2$. By Lemma 4.1, $\|(1-H)\| \leq 1$. Therefore,

\[ \|H\| = \|1-(1-H)\| \leq 1 + 1 - H \| \leq 1 + 1 = 2. \]

QED
5 An asymptotic M"untz-Szász Theorem

Let $S$ be a subset of the non-negative integers. When is the linear span of the monomials \( \{x^n : n \in S\} \) dense? The M"untz-Szász Theorem, proved by M"untz [6] and Szász [7], answers this question in both \( L^2[0,1] \) and \( C[0,1] \), the continuous functions on \([0,1]\). The answer is basically the same in both cases, but the constant function 1 plays a special role in \( C[0,1] \), since it cannot be approximated by any polynomial that vanishes at 0 (which all the other monomials do).

**Theorem 5.1. (M"untz-Szász Theorem)** (i) The linear span of \( \{x^n : n \in S\} \) is dense in \( L^2[0,1] \) if and only if
\[
\sum_{n \in S} \frac{1}{n+1} = \infty. \tag{5.2}
\]
(ii) The linear span of \( \{x^n : n \in S\} \) is dense in \( C[0,1] \) if and only if \( 0 \in S \) and (5.2) holds.

What happens if the approximants come from a set of linear combinations of monomials that is losing as well as gaining members? Fix an increasing sequence \( \{N_n\} \) of natural numbers and for each \( n \) define
\[
S_n = \{n, n+1, \ldots, n+N_n\} \quad \text{and} \quad \mathcal{M}_n = \text{span} \ \{x^n, x^{n+1}, \ldots, x^{n+N_n}\}.
\]
For each \( n \) let \( \rho_n \) denote the fraction of the non-negative integers less than or equal to \( n+N_n \) that do not lie in \( S_n \), i.e.,
\[
\rho_n = \frac{n}{n+N_n+1}.
\]
Finally, with this setup, let
\[
\mathcal{M}_\infty = \{f \in L^2([0,1]) \mid \lim_{n \to \infty} \text{dist}(f, \mathcal{M}_n) = 0\}. \tag{5.3}
\]

We wish to characterize \( \mathcal{M}_\infty \) (Theorem 5.12 below). We shall follow M"untz’s original proof of Theorem 5.1 [6]. His argument involved an ingenious calculation using Theorem 3.1 and the following venerable formula of Cauchy [3].

**Theorem 5.4. (The Cauchy Determinant Formula)** If \( M \) is the \( N \times N \) Cauchy matrix defined by the formula
\[
M = \begin{bmatrix}
\frac{1}{x_1-y_1} & \frac{1}{x_1-y_2} & \cdots & \frac{1}{x_1-y_N} \\
\frac{1}{x_2-y_1} & \frac{1}{x_2-y_2} & \cdots & \frac{1}{x_2-y_N} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{x_N-y_1} & \frac{1}{x_N-y_2} & \cdots & \frac{1}{x_N-y_N}
\end{bmatrix},
\]
where for all \( i \) and \( j, x_i \neq y_j \), then
\[
\det M = \prod_{1 \leq j < i \leq N} (x_i - x_j)(y_j - y_i) \prod_{1 \leq i, j \leq N} (x_i - y_j). 
\]
We need two more auxiliary results.

**Proposition 5.5. (Baby Brodskii-Donoghue Theorem)** Let \( \mathcal{M} \) be a closed subspace of \( L^2[0,1] \) that is invariant under both \( M_x \) and \( V \). Then \( \mathcal{M} = L^2[s,1] \) for some \( s \) between 0 and 1.

**Proof.** Note that the constant function 1 has the unique representation in \( L^2([0,1]) \),

\[
1 = f + g \quad (5.6)
\]

where \( f \perp \mathcal{M} \) and \( g \in \mathcal{M} \). The fact that \( \mathcal{M} \) is \( M_x \) invariant, implies that \( pg \in \mathcal{M} \) whenever \( p \) is a polynomial\(^2\), it follows that \( f \perp pg \) whenever \( p \) is a polynomial. But then \( f\bar{g} \perp p \) for all polynomials which implies that

\[
f\bar{g} = 0 \quad (5.7)
\]

For a Lebesgue measurable set \( E \subseteq [0,1] \) we let \( \chi_E \) denote the characteristic function of \( E \), i.e., the function defined by

\[
\chi_E(x) = \begin{cases} 
0 & \text{if } x \notin E \\
1 & \text{if } x \in E 
\end{cases}
\]

We observe that (5.6) and (5.7) imply that there exists a partition of \([0,1]\) into two measurable sets \( F \) and \( G \) such that \( f = \chi_F \) and \( g = \chi_G \). We define a parameter \( s \in [0,1] \) by setting

\[
s = \sup \{ x \mid g(t) = 0 \text{ for a.e. } t \in [0,x] \}. \quad (5.8)
\]

Notice that with this definition, we have that

\[
Vg(x) = 0 \text{ for a.e. } x \in [0,s] \quad \text{and} \quad Vg(x) > 0 \text{ for a.e. } x \in [s,1]. \quad (5.9)
\]

Since \( g \in \mathcal{M} \), we have \( Vg \in \mathcal{M} \). Also, recall that \( f \in \mathcal{M}^\perp \). Therefore, using (5.9) we see that \( F \subseteq [0,s] \). In light of (5.6), this implies \([s,1] \subseteq G \), which in turn, implies via (5.8) that

\[
f = \chi_{[0,s]} \quad \text{and} \quad g = \chi_{[s,1]}. \quad (5.10)
\]

As \( pf \in \mathcal{M}^\perp \) and \( pg \in \mathcal{M} \) whenever \( p \) is a polynomial, it follows immediately from (5.10) and the fact that the polynomials are dense in both \( L^2([0,s]) \) and \( L^2([s,1]) \), that

\[
L^2([0,s]) \subseteq \mathcal{M}^\perp \quad \text{and} \quad L^2([s,1]) \subseteq \mathcal{M}.
\]

Hence, we have that both \( L^2([s,1]) \supseteq \mathcal{M} \) and \( L^2([s,1]) \subseteq \mathcal{M} \), so that \( L^2([s,1]) = \mathcal{M} \), as was to be proved.

We call Proposition 5.5 the Baby Brodskii-Donoghue Theorem because Brodskii and Donoghue independently proved the far deeper fact that the only closed invariant subspaces of \( V \) are \( L^2[s,1] \) [2, 4]. The operator \( M_x \) has other invariant subspaces. Indeed the ideas in the preceding proof can be adapted to show that the invariant subspaces of \( M_x \) are the spaces \( \{ f \in L^2[0,1] : f(x) = 0 \text{ a.e. on } F \} \), where \( F \) is any measurable subset of \([0,1]\).

\(^2\)Note that it is also true that \( pf \in \mathcal{M}^\perp \) whenever \( p \) is a polynomial, since \( M_x \) is self-adjoint.
Lemma 5.11. If $\mathcal{M}_\infty$ is as in (5.3), then there exists $s \in [0, 1]$ such that $\mathcal{M}_\infty = L^2([s, 1])$.

Proof. Observe first that if

$$p(x) = \sum_{k=n}^{n+N_n} a_k x^k \in \mathcal{M}_n,$$

then

$$M_x p(x) = xp(x) = \sum_{k=n+1}^{n+N_n+1} a_{k-1} x^k \in \mathcal{M}_{n+1}.$$

Hence,

$$M_x \mathcal{M}_\infty \subseteq \mathcal{M}_\infty.$$

Likewise,

$$V \mathcal{M}_\infty \subseteq \mathcal{M}_\infty.$$

Now the result follows from Lemma 5.5. QED

Theorem 5.12. (Asymptotic Müntz-Szász Theorem) Let $S_n = \{n, n+1, \ldots, n+N_n\}$, let $\mathcal{M}_n = \text{span} \{x^n, x^{n+1}, \ldots, x^{n+N_n}\}$, and let $\rho_n = \frac{n}{n+N_n+1}$. If

$$\lim_{n \to \infty} \rho_n = \rho,$$

then

$$\mathcal{M}_\infty = L^2([\rho^2, 1]).$$

Proof. By Lemma 5.11 there exists $s \in [0, 1]$ such that

$$\mathcal{M}_\infty = L^2([s, 1]).$$

Noting that

$$\text{dist}(1, L^2([s, 1])) = \sqrt{s},$$

we see that the theorem will follow if we can show that

$$\text{dist}(1, \mathcal{M}_\infty) = \rho,$$

or equivalently that

$$\lim_{n \to \infty} \text{dist}(1, \mathcal{M}_n) = \rho.$$  \hspace{1cm} (5.13)

Now fix $N + 1$ distinct real numbers $\alpha_0, \alpha_1, \ldots, \alpha_N \in (-\frac{1}{2}, \infty)$. In Theorem 5.4 if for $i, j = 1, 2, \ldots, N$ we let $x_i = \alpha_i + \frac{1}{2}$ and $y_j = -(\alpha_j + \frac{1}{2})$ we obtain that

$$\det G(x^{\alpha_1}, \ldots, x^{\alpha_N}) = \prod_{1 \leq i < j \leq N} (\alpha_i - \alpha_j)^2 \prod_{1 \leq i, j \leq N} (\alpha_i + \alpha_j + 1),$$

Likewise,

$$\det G(x^{\alpha_0}, x^{\alpha_1}, \ldots, x^{\alpha_N}) = \prod_{0 \leq j < i \leq N} (\alpha_i - \alpha_j)^2 \prod_{0 \leq i, j \leq N} (\alpha_i + \alpha_j + 1).$$
Therefore,
\[
\frac{\det G(x^{\alpha_0}, x^{\alpha_1}, \ldots, x^{\alpha_N})}{\det G(x^{\alpha_1}, \ldots, x^{\alpha_N})} = \frac{1}{2\alpha_0 + 1} \frac{\prod_{i=1}^N (\alpha_i - \alpha_0)^2}{\prod_{i=1}^N (\alpha_i + \alpha_0 + 1)^2}
\]
Hence, using Theorem 3.1 we get
\[
dist(1, \mathcal{M}_n)^2 = \frac{\det G(x^0, x^n, \ldots, x^{n+N_n})}{\det G(x^n, \ldots, x^{n+N_n})} = \prod_{i=0}^{N_n} (n + i)^2 \prod_{i=0}^{N_n} (n + i + 1)^2 = (n + N_n + 1)^2 = \rho_n^2.
\]
Equation (5.13) now follows. QED

6 The Bernstein Conundrum: Asymptotic Müntz-Szász Theorem for $C[0,1]$

The $C[0,1]$ Müntz-Szász Theorem can be deduced from the $L^2$ version. What about the asymptotic version? Let $\mathcal{M}_n$ be as in Section 5, and let $\mathcal{M}_{\infty}^{\text{unif}}$ be
\[
\mathcal{M}_{\infty}^{\text{unif}} = \{ f \in C[0,1] \mid \lim_{n \to \infty} \text{dist}(f, \mathcal{M}_n) = 0 \}, \tag{6.1}
\]
where in this section all distances are with respect to the supremum norm\(^3\).

One way to prove the Weierstraß approximation theorem is to use the Bernstein polynomials. For each $n$, these are the $n + 1$ polynomials defined by
\[
b_{k,n}(x) = \binom{n}{k} x^k (1 - x)^{n-k}.
\]
Bernstein proved in 1912 [1] that for every continuous function $f \in C[0,1]$, the polynomials
\[
p_n(x) = \sum_{k=0}^n f\left(\frac{k}{n}\right) b_{k,n}(x) \tag{6.2}
\]
converge uniformly on $[0,1]$ to $f$.

As the lowest order term in $b_{k,n}$ is $x^k$, if $f$ vanished on $[0, \rho_n]$ and one used the Bernstein formula (6.2) to approximate it, the corresponding polynomial $p_{n+N_n+1}$ would lie in the span of $\{x^{n+1}, \ldots, x^{n+N_n+1}\}$ which is in $x\mathcal{M}_n \subseteq \mathcal{M}_{n+1}$. So one immediately gets that $\mathcal{M}_{\infty}^{\text{unif}}$ contains all the continuous functions that vanish on $[0, \rho]$.

This construction seems natural, and could lead one to suspect that $\mathcal{M}_{\infty}^{\text{unif}}$ should be the functions that vanish on $[0, \rho]$. However, Theorem 6.3 shows this is incorrect.

\(^3\)This means $\|f\|_{C[0,1]} = \sup_{0 \leq x \leq 1} |f(x)|$. 8
Theorem 6.3. (Asymptotic Müntz-Szász Theorem, Continuous Case) If \( \lim_{n \to \infty} \rho_n = \rho \), then
\[
\mathcal{M}_{\infty}^{\text{unif}} = \{ f \in C[0, 1] | f = 0 \text{ on } [0, \rho^2] \}.
\]

Proof. As the supremum norm is larger than the \( L^2 \) norm, we have
\[
\mathcal{M}_{\infty}^{\text{unif}} \subseteq \mathcal{M}_\infty \cap C[0, 1] = \{ f \in C[0, 1] | f = 0 \text{ on } [0, \rho^2] \}.
\]

For the reverse inclusion, notice that it follows from Cauchy-Schwarz that the Volterra operator is a bounded linear map from \( L^2[0, 1] \) into \( C[0, 1] \). (Indeed, if \( g \in L^2[0, 1] \), we get that \( Vg \) satisfies a Hölder continuity condition of order \( \frac{1}{2} \).)

Let \( f \) be a \( C^1 \) function that vanishes on \( [0, \rho^2] \). Then \( f = Vg \), where \( g = f' \). By Theorem 5.12, there are polynomials \( p_n \in \mathcal{M}_n \) that converge in \( L^2 \) to \( g \). Then \( Vp_n \) converges in \( C[0, 1] \) to \( f \), so \( f \) is in \( \mathcal{M}_\infty^{\text{unif}} \). As \( \mathcal{M}_{\infty}^{\text{unif}} \) is closed, and the \( C^1 \) functions that vanish on \( [0, \rho^2] \) are dense in the continuous ones, we get
\[
\{ f \in C[0, 1] | f = 0 \text{ on } [0, \rho^2] \} \subseteq \mathcal{M}_{\infty}^{\text{unif}}.
\]

QED

Question 6.4. Can one prove Theorem 6.3 directly using Bernstein approximation?

7 Proof of Theorem 1.5

Proof. Assume that \( T \) is a monomial operator of order \( m \), let \( s \in (0, 1) \), and fix \( f \in L^2([s, 1]) \). We wish to prove that \( Tf \in L^2([s, 1]) \).

Choose an increasing sequence of natural numbers \( \{N_n\} \) such that
\[
\lim_{n \to \infty} \frac{n}{n + N_n} = \sqrt{s}.
\]

By Theorem 5.12, there exists a sequence of polynomials \( \{p_n\} \) where for each \( n \), \( p_n \) has the form
\[
p_n(x) = \sum_{k=n}^{n+N_n} c_k x^k
\]
and such that
\[
p_n \to f \text{ in } L^2([0, 1]).
\]

As \( T \) is bounded,
\[
Tp_n \to Tf \text{ in } L^2([0, 1]).
\]

Also, as \( T \) is a monomial operator of order \( m \), for each \( n \), \( Tp_n \) has the form
\[
Tp_n(x) = x^m \sum_{k=n}^{n+N_n} d_k x^k.
\]
For any $s > 0$, multiplication by $x^m$ is a bounded invertible map from $L^2([s,1])$ to itself. Therefore
\[ \sum_{k=n}^{n+N_n} d_k x^k \]
converges, as $n \to \infty$, to some function $g(x)$, which by Theorem 5.12 is in $L^2([s,1])$. So $Tf = x^m g(x)$, and lies in $L^2([s,1])$. QED

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