A DOMAIN CONTAINING ALL ZEROS OF THE PARTIAL THETA FUNCTION

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Abstract. We consider the partial theta function, i.e. the sum of the bi variate series
\( \theta(q, z) := \sum_{j=0}^{\infty} q^{(j+1)/2} z^j \) for \( q \in (0, 1), z \in \mathbb{C} \). We show that for any value of the parameter \( q \in (0, 1) \) all zeros of the function \( \theta(q, \cdot) \) belong to the domain \( \{ \Re z < 0, |\Im z| \leq 132 \} \cup \{ \Re z \geq 0, |z| \leq 18 \} \).

Key words: partial theta function, Jacobi theta function, Jacobi triple product

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1. Introduction

We consider the bivariate series \( \theta(q, z) := \sum_{j=0}^{\infty} q^{(j+1)/2} z^j \) for \( q \in (0, 1), z \in \mathbb{C} \). We regard \( q \) as a parameter and \( z \) as a variable. This series is convergent and defines an entire function called partial theta function. The terminology is explained by the resemblance of this formula with the one for the function
\[ \Theta^*(q, z) := \sum_{j=-\infty}^{\infty} q^{(j+1)/2} z^j, \]
because the latter is connected with the Jacobi theta function
\[ \Theta(q, z) := \sum_{j=-\infty}^{\infty} q^{j^2} z^j \]
by the formula \( \Theta^*(q, z) = \Theta(q^{1/2}, q^{1/2}z) \). The word “partial” reminds that in the formula for \( \theta \) the summation is performed from 0 to \( \infty \), not from \( -\infty \) to \( \infty \).

Studying the function \( \theta \) is motivated by its applications in several domains the most recent of which concerns section-hyperbolic polynomials, i.e. real univariate polynomials of degree \( \geq 2 \) with all roots real and such that when their highest-degree monomial is deleted this gives again a polynomial having only real roots. The relationship between \( \theta \) and such polynomials is explained in [10]. Previous research on section-hyperbolic polynomials was performed in [6] and [11] which in turn was based on classical results of Hardy, Petrovitch and Hutchinson (see [4], [12] and [5]). Other domains in which the partial theta function is used are statistical physics and combinatorics (see [13]), asymptotic analysis (see [2]), Ramanujan-type \( q \)-series (see [14]) and the theory of (mock) modular forms (see [3]); see also [1].

In the present paper we prove the following theorem:

Theorem 1. (1) For \( \Re z \geq 0 \) the function \( \theta(q, \cdot) \) has no zeros outside the closed half-disk \( \{ \Re z \geq 0, |z| \leq 18 \} \), for any \( q \in (0, 1) \).
(2) For $\Re z < 0$ and for any $q \in (0, 1)$ the function $\theta(q, .)$ has no zeros outside the half-strip $\{\Re z < 0, |\Im z| \leq 132\}$.

In order to explain the importance of this theorem we recall in Section 2 certain facts about the zeros of $\theta$. Then we give an example of a value of $q \in (0, 1)$ for which $\theta(q, .)$ has a complex conjugate pair of zeros in the right half-plane. The proof of the theorem is given in Section 3.

2. Properties of the function $\theta$

In the present section we recall some results concerning the function $\theta$. We denote by $\Gamma$ the spectrum of $\theta$, i.e., the set of values of $q$ for which $\theta(q, .)$ has a multiple zero (the notion has been introduced by B. Z. Shapiro in [10]). The following results are proved in [7].

**Theorem 2.** (1) The spectrum $\Gamma$ consists of countably-many values of $q$ denoted by $0 < \tilde{q}_1 < \tilde{q}_2 < \cdots < \tilde{q}_N < \cdots < 1$ with $\lim_{j \to \infty} \tilde{q}_j = 1$.

(2) For $\tilde{q}_N \in \Gamma$ the function $\theta(\tilde{q}_N, .)$ has exactly one multiple real zero $y_N$ which is negative, of multiplicity 2 and is the rightmost of its real zeros.

(3) For $q \in (\tilde{q}_N, \tilde{q}_N + 1)$ (we set $\tilde{q}_0 := 0$) the function $\theta$ has exactly $N$ complex conjugate pairs of zeros (counted with multiplicity). All its other zeros are real negative.

**Remarks 1.** (1) It is proved in [10] that $\tilde{q}_1 = 0.3092\ldots$. Up to 6 decimals the first 12 spectral numbers equal (see [10])

$0.309249, 0.516959, 0.630628, 0.701265, 0.749269, 0.783984,$

$0.810251, 0.830816, 0.847353, 0.860942, 0.872305, 0.881949$.

(2) It is shown in [7] that for $q \in (0, \tilde{q}_1)$ all zeros of $\theta$ are real, negative and distinct. For all $q \in (0, 1)$ it is true that as $q$ increases, the values of the local minima of $\theta$ between two negative zeros increase and the values of its maxima between two negative zeros decrease. It is always the rightmost two negative zeros with a minimum of $\theta$ between them that coalesce to form a double zero of $\theta$ for $q = \tilde{q}_N$ and then a complex conjugate pair for $q = \tilde{q}_N^+$. For any $q \in (0, 1)$ the function $\theta(q, .)$ has infinitely-many negative zeros and no positive ones; $\theta(q, .)$ is increasing for $x > 0$ and tends to $\infty$ as $x \to \infty$; there is no finite accumulation point for the zeros of $\theta(q, .)$.

(3) In [8] the following asymptotic expansions of $\tilde{q}_N$ and $y_N$ are given:

\[
\begin{align*}
\tilde{q}_N & = 1 - \pi/2N + (\log N)/8N^2 + O(1/N^2), \\
y_N & = -e^{\pi}e^{-(\log N)/4N + O(1/N)}.
\end{align*}
\]

(2.1)

The importance of Theorem 2 lies in the fact that while the real zeros of $\theta$ remain all negative for any $q \in (0, 1)$, no information was known about its complex conjugate pairs. It would be interesting to know whether all complex conjugate pairs remain (for all $q \in (\tilde{q}_1, 1]$) within some compact domain in $\mathbb{C}$ (independent of $q$).

**Lemma 1.** The function $\theta(0.73, .)$ has exactly one complex conjugate pair of zeros inside the open half-disk $D := \{|z| < 3, \Re z > 0\}$. 

Proof. Consider the truncation of \( \theta(0.73, \ldots) \) of degree 20 w.r.t. \( x \), i.e. the polynomial 
\[
\theta_{20} := \sum_{j=0}^{20} 0.73^{(j+1)/2} x^j.
\]
One checks numerically (say, using MAPLE) that \( \theta_{20} \) has zeros \( 0.03356612894 \ldots + 2.885381139 \ldots i \). These are the only zeros of \( \theta_{20} \) in the closure of \( \tilde{D} \). Numerical check shows that the modulus of the restriction of \( \theta_{20} \) to the border of \( \tilde{D} \) is everywhere larger than 0.016. On the other hand the sum \( \sum_{j=21}^{\infty} |0.73^{(j+1)/2} x^j| \) is \( \leq \sum_{j=21}^{\infty} 0.73^{j(1+1)/2} 3^j < 3 \times 10^{-22} \). By the Rouché theorem the functions \( \theta(0.73, \ldots) \) and \( \theta_{20} \) have one and the same number of zeros inside the half-disk \( \tilde{D} \). \( \square \)

3. Proof of Theorem 1

As for \( q \in (0, \bar{q}_1) \) all zeros of \( \theta(q, \ldots) \) are negative (see Remarks 1), we prove Theorem 1 only for \( q \in (\bar{q}_1, 1) \).

3.1. The Jacobi theta function. In the proof of Theorem 1 we use the Jacobi theta function \( \Theta(q, z) := \sum_{j=-\infty}^{\infty} q^j z^j \). By the Jacobi triple product one has
\[
\Theta(q, z^2) = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + z^2 q^{2m-1})(1 + z^{-2} q^{2m-1})
\]
from which for the function \( \Theta^*(q, z) := \Theta(q^{1/2}, q^{1/2} z) = \sum_{j=-\infty}^{\infty} q^{j(1+1)/2} z^j \) one deduces the formula
\[
\Theta^*(q, z) = \prod_{m=1}^{\infty} (1 - q^m)(1 + z q^m)(1 + q^{m-1}/z).
\]

Notation 1. We set
\[
s_m := 1 + q^{m-1}/z , \quad t_m := 1 + z q^m , \quad Q := \prod_{m=1}^{\infty} (1 - q^m) , \quad P := \prod_{m=1}^{\infty} t_m \quad \text{and} \quad R := \prod_{m=1}^{\infty} s_m.
\]

Thus \( \Theta^* = Q P R \).

3.2. Proof of part (1). We begin with the observation that for any factor \( s_m \) (see (5.2)) and Notation 1 one has
\[ s_m = 1 + z q^{m-1}/|z|^2 \quad \text{hence} \quad |s_m| \geq \text{Re } s_m \geq 1 \quad \text{for } \text{Re } z \geq 0. \]

Clearly, for any factor \( t_m \) it is true that \( |t_m| \geq \text{Re } t_m \geq 1 \) and \( |t_m| \geq |z q^m| \) for \( \text{Re } z \geq 0. \)

Further in the proof of Theorem 1 we subdivide the interval \((0, 1)\) to which \( q \) belongs into intervals of the form
\[
q \in (1 - 1/(n-1) , 1 - 1/n] , \quad n \in \mathbb{N} , \quad n \geq 3 \quad \text{and}
\]
\[
q \in (\bar{q}_1 , 1/2].
\]

Notation 2. We set \( \theta := \Theta^* - G \), where \( G := \sum_{j=-\infty}^{-1} q^{j(j+1)/2} z^j \), and \( u := 2e^{(r^2/6)} = 10.36133664 \ldots \).
Remark 1. Clearly, for $|z| > 1$ one has $|G| \leq \sum_{j=1}^{\infty} 1/|z|^j = 1/(|z| - 1)$. In particular, for $|z| \geq 18$ (resp. for $|z| \geq u$) one has $|G| \leq 1/17$ (resp. $|G| \leq 1/(u-1)$).

Suppose first that $q \in (1/2, 1)$. We show that for $|z| \geq u$, $\text{Re } z \geq 0$ one has $|\Theta^*| > |G|$ from which part (1) of the theorem follows.

Lemma 2. For $q \leq 1 - 1/n$, $n \in \mathbb{N}$, $n \geq 2$, one has $Q \geq e^{(\pi^2/6)(1-n)}$.

The proof of part (2) is also based on formula (3.2).

\begin{align*}
|P_0| & \geq |z|^n(1-1/(n-1))(n+1)/2 = |z|^n(1-1/(n-1))(n-1)(n+2)/2+1 \\
& \geq |z|^n4^{-(n+2)/2}(1-1/(n-1)) \geq |z|^n4^{-(n+3)/2} = |z|^n2^{-(n+3)};
\end{align*}

we use the inequalities

\begin{align*}
(1-1/(n-1))^{n-1} & \geq 1/4 \\
\text{and } 1-1/(n-1) & \geq 1/2 \text{ which hold true for } n \geq 3.
\end{align*}

Set $P_1 := \prod_{m=n+1}^{\infty} t_m$. Hence we have $|P_1| \geq 1$, $|R| \geq 1$ and

\begin{align*}
|\Theta^*| & = Q|P_0||P_1||R| \\
& \geq e^{(\pi^2/6)(1-n)}|z|^n2^{-(n+3)} \\
& = (e^{(\pi^2/6)/2^3})(|z|/2e^{(\pi^2/6)})^n.
\end{align*}

Obviously, for $|z| \geq u$ one has $|z|/2e^{(\pi^2/6)} \geq 1$. As $e^{(\pi^2/6)/2^3} = 0.64 \ldots > 1(u-1)$, one obtains the inequalities $|\Theta^*| > 1/(u-1) \geq |G|$ which proves part (1) of the theorem for $q \in (1/2, 1)$ (because $u < 18$).

Suppose that $q \in (\tilde{q}_1, 1/2)$. In this case for $|z| \geq 18$ and $\text{Re } z \geq 0$ one has $|t_1| \geq 18q_1$, $|t_m| \geq 1$, $|s_m| \geq 1$ for $m \in \mathbb{N}$ and (by Lemma \ref{lem2}) with $n = 2$ $Q \geq e^{-\pi^2/6}$, so $|\Theta^*| \geq e^{-\pi^2/6}18q_1 > 1 > 1/(|z| - 1) \geq |G|$.

3.3. Proof of part (2). The proof of part (2) is also based on formula \ref{eq2}. We aim to show that for $\text{Re } z < 0$ and $|\text{Im } z| \geq 132$ one has $|\Theta^*| > |G|$. The following technical result is necessary for the estimations and for the understanding of Figure \ref{fig:1}.

Lemma 3. For $x \in [0, 0.683]$ one has $\ln(1-x) \geq -x - x^2$ with equality only for $x = 0$.

Proof. We set $\zeta(x) := \ln(1-x) + x + x^2$, so $\zeta(0) = 0$. As $\zeta' = -1/(1-x) + 1 + 2x = x(1-2x)/(1-x)$ which is nonnegative on $[0, 1/2]$ and positive on $(0, 1/2)$, one has $\zeta(x) > 0$ for $x \in (0, 1/2]$. On $(1/2, 1)$ one has $\zeta' < 0$, so $\zeta$ is decreasing. As $\lim_{x \to 1^{-}} \zeta = -\infty$, $\zeta$ has a single zero on $[1/2, 1)$. Numerical computation shows that this zero is $> 0.683$ which proves the lemma.

To estimate the factor $R$ we use the following lemma:

Lemma 4. For $q \in (1-1/(n-1), 1-1/n]$, $n \geq 2$, and $|\text{Im } z| \geq b \geq 1.5$ one has $|R| \geq e^{-n(b+1)/b^2}$.
Proof. Indeed, the condition $b \geq 1.5$ implies $1/|z| < 0.683$, so one can apply Lemma 3.

\[
\ln |R| = \sum_{m=1}^{\infty} \ln |s_m| \geq \sum_{m=1}^{\infty} \ln(1 - q^{m-1}/|z|) \\
\geq -\sum_{m=1}^{\infty} (q^{m-1}/|z| + q^{2m-2}/|z|^2) = -1/(1 - q)|z| - 1/(1 - q^2)|z|^2 \\
> -(|z| + 1)/(1 - q)|z|^2
\]

which for $q \leq 1 - 1/n$ is $\geq -n(|z| + 1)/|z|^2 \geq -n(b + 1)/b^2$.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure1.png}
\caption{The points $1 + q^m z$.}
\end{figure}

We identify the complex numbers and the points in $\mathbb{R}^2$ representing them. On Fig. 1 we represent the points

\[ z = -a + bi \quad (a > 0) \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad t_0 := 1 + z = 1 - a + bi \]

\[ t_1 := 1 + qz = 1 - qa + qbi \quad \text{and} \quad t_2 := 1 + q^2 z = 1 - q^2 a + q^2 bi \]

The last three of them are situated on the straight line $L$ passing through $1 + z$ and $(1,0)$.

In what follows we assume that $b \geq 0$. The set of zeros of $\theta$ being symmetric w.r.t. the real axis this leads to no loss of generality.

The point $A \in L$ is such that the segment $OA$ is orthogonal to $L$. An easy computation shows that $A = (b^2/(a^2 + b^2), ab/(a^2 + b^2))$ (one has to use the fact that the vector $(-a, b)$ is collinear with $L$). The points $B$ and $C$ belong to $L$. The unit circumference intersects the line $L$ at $(1,0)$ and $D$. Denote by $\Delta$ the length $||[A, (1,0)]||$ of the segment $[A, (1,0)]$. The points $B$ and $C$ are defined such that

\[(3.6) \quad ||[A, B]|| = ||[A, C]|| = 0.317 \Delta \quad \text{and} \quad ||[C, (1,0)]|| = ||[D, B]|| = 0.683 \Delta.\]

The line $L'$ is parallel to $L$. It passes through the points $z$ and $O := (0,0)$. The points $B'$, $A'$, $C'$ and $D'$ belong to $L'$. The lines $BB'$, $AA'$, $CC'$ and $DD'$ are
parallel to the \(x\)-axis. Hence the segments \([D, B], [C, (1, 0)], [D', B']\) and \([C', O]\) are of length 0.683 \(\Delta\) while \([B, A], [A, C], [B', A']\) and \([A', C']\) are of length 0.317 \(\Delta\).

Our aim is to estimate the product \(|P| := \prod_{m=1}^{\infty} |t_m| = \prod_{m=1}^{\infty} |1 + q^m z|\).

Notation 3. (1) We set

\[ (3.7) \quad |P| := \tilde{P} P^\dagger P^\sharp P^\ddagger, \]

where \(\tilde{P}, P^\dagger, P^\sharp\) and \(P^\ddagger\) are the products of the moduli \(|t_m|\) for which the point \(t_m\) belongs to the segment \([1 + z, D], [D, B], [B, C]\) and \([C, (1, 0)]\) respectively.

(2) We denote by \(t_{m_0}, t_{m_0+1}, \ldots\) the points \(t_m\) belonging to the segment \([C, (1, 0)]\) and we set \(c_{m_0} := C, c_{m_0+k} := 1 + (C-1)q^k\). Hence \(|t_{m_0+k}| \geq |c_{m_0+k}|\) with equality only if \(t_{m_0} = C\).

Lemma 5. For \(q \in (1 - 1/(n-1), 1 - 1/n], n \geq 2\), one has \(P^\dagger \geq e^{-1.149489n}\) and \(P^\sharp \geq e^{-1.149489n}\) (where 1.149489 = 0.683 + 0.683^2).

Proof. We notice first that the segment \([C, (1, 0)]\) is of length \(< 0.683\). Hence \(|t_{m_0+k}| \geq |c_{m_0+k}| \geq (1 - 0.683q^k), k = 0, 1, \ldots\), so we can use Lemma 3 to get

\[
\ln P^\dagger \geq \ln(|c_{m_0}| |c_{m_0+1}| \cdots) \geq \ln(\prod_{m=1}^{\infty} (1 - 0.683q^{m-1}))
\]

\[
\geq - \sum_{m=1}^{\infty} (0.683q^m - 1) + 0.683^2 q^{2m-2})
\]

\[
= -0.683/(1-q) - 0.683^2/(1-q^2) > -1.149489/(1-q)
\]

Thus \(P^\dagger \geq e^{-1.149489n}\). Next, the distance between any two consecutive points \(1 + q^m z\) and \(1 + q^{m+1} z\) belonging to \([B, D]\) is greater than the distance between any two such points belonging to \([C, (1, 0)]\). Denote by \(U_1, U_2, \ldots, U_r\) the points \(t_m\) belonging to the segment \([B, D]\), where \(U_1\) (resp. \(U_r\)) is closest to \(B\) (resp. \(D\)). Then \(|U_1| \geq |c_{m_0}|, |U_2| \geq |c_{m_0+1}|, \ldots, |U_r| \geq |c_{m_0+r-1}|\). As \(|c_m| \leq |t_m| < 1\) for \(m \geq m_0\), one has \(P^\sharp \geq |c_{m_0}||c_{m_0+1}| \cdots |c_{m_0+r-1}| > |c_{m_0}| |c_{m_0+1}| \cdots > e^{-1.149489n}\). \(\Box\)

Lemma 6. There are \(\leq \mu_1\) factors \(|t_m|\) in \(P^\bigstar\), where

\[
\mu_1 := \ln \lambda_1 / \ln(1/q) + 1
\]

and

\[
\lambda_1 := (0.634 + 0.683)/0.683 = 1.928\ldots
\]

with \(\ln \lambda_1 = 0.6566\ldots\). Hence

\[ P^\bigstar \geq (b^2/(a^2 + b^2))^{\mu_1/2} = (1/(b^2 + 1))^{\mu_1/2}, \quad \beta := a/b. \]

Proof. Consider the points \(C', A'\) and \(B'\) and the numbers \(zq^{m_1} \in [B', A']\) and \(zq^{m_2} \in [A', C']\) closest to \(B'\) and \(C'\) respectively. The lengths of the segments \([B', A'], [A', C']\) and \([C', O]\) (see (3.8)) imply \(|zq^{m_1}|/|zq^{m_2}| \leq \lambda_1\), i.e. \(m_2 - m_1 \leq \lambda_1/\ln(1/q)\). The number of factors \(|t_m|\) in \(P^\bigstar\) equals \(m_2 - m_1 + 1\) from which one deduces the first claim of the lemma. All factors \(|t_m|\) in \(P^\dagger\) are \(< 1\) and \(\geq ||O,A|| = b/(a^2 + b^2)^{1/2}\) from which the second claim of the lemma follows. \(\Box\)

Remark 2. When \(q \in (1/2, 1)\) and (3.3) holds true, then \(n/(n-1) \leq 1/q < (n-1)/(n-2)\). As for \(x \in (0, 1)\) one has \(x - x^2/2 < \ln(1 + x) < x\) (by the Leibniz criterion for alternating series), one obtains the inequalities
Indeed, if
\[(2n-3)/2(n-1)^2 = 1/(n-1) - 1/2(n-1)^2 < \ln(1/q) < 1/(n-2)\]
hence
\[(\ln \lambda_1)(n-2) + 1 < \mu_1 \leq \mu_1^0 := (\ln \lambda_1)(2(n-1)^2/(2n-3)) + 1 .\]

**Lemma 7.** For \(q \in (\tilde{q}_1, 1)\) and \(b \geq \max(a, 132)\) one has \(|\Theta^*| > |G|\).

**Proof.** Prove first the lemma for \(q \in (1/2, 1)\), see the first line of (3.3). Set again
\[P_0 := \prod_{m=1}^n t_m .\]
Hence
\[|t_m| \geq \text{Im } t_m \text{ and } |P_0| \geq \prod_{m=1}^n bq^m = b^n q^{n(n+1)/2} \geq b^{n^2/2-(n+3)} ,\]
see (3.4) and (3.5). With \(Q, P^t, P^\dagger\) and \(P^\sharp\) defined in Notations (3.1) and (3.3) one has
\[(3.9) |\Theta^*| \geq Q|P_0|P^tP^\dagger P^\sharp R .\]
Indeed, if \(b \geq 132\) and if \(q\) satisfies the first line of conditions (3.3), then
\[(3.10) q^m \geq q^{n} \geq (1-1/(n-1))^{n-1}(1-1/(n-1)) \geq (1/4)(1-1/(n-1)) \geq 1/8 \]
and \(|bq^m| \geq |bq^n| \geq 132/8 > 1\). This means that all factors \(|t_m|\) in \(|P_0|\) are \(> 1\). Moreover, some factors \(|t_m|\) with \(|t_m| > 1\) which are present in \(|\Theta^*|\) (i.e. in \(\tilde{P}\), see (3.7)) might be missing in the right-hand side of (3.9). Recall that each of the factors \(P^t\) and \(P^\sharp\) is minorized by \(e^{-1.149489n}\) and that \(|R| \geq e^{-(b+1)n/b^2}\), see Lemmas (3.2) and (3.4). Recall also that by Lemma (3.5)
\[P^\sharp \geq (1/(\beta^2 + 1))^{\mu_1/2} \geq 2^{-\mu_1/2} \text{ (because } \beta = a/b \leq 1) \]
and that \(\mu_1 \leq \mu_1^0\), see (3.3). Hence the right-hand side of (3.9) is
\[\geq H := e^{(\pi^2/6)(n^3/2)}b^{n^2-2/(n+3)}e^{(-1.149489n)} \times e^{-\ln(2)(\ln \lambda_1)2(n-1)^2/(2n-3)+1}/2 e^{-(b+1)n/b^2} .\]
Taking into account that
\[(3.11) 2(n-1)^2/(2n-3) = n-1/2 + 1/2(n-3) ,\]
we represent the expression \(H\) in the form \(e^{K_1 n+K_0}\), where
\[K_1 := -\pi^2/6 + \ln b - \ln 2 - 2.298978 -(\ln 2)(\ln \lambda_1)/2 - (b+1)/b^2 \]
\[K_0 := \pi^2/6 - 3 \ln 2 + (\ln 2)(\ln \lambda_1)/4 - (\ln 2)/2 - (\ln 2)(\ln \lambda_1)/4(2n-3) .\]
Recall that \(n \geq 3\), see the first line of (3.3). The sum \(K_0\) is minimal for \(n = 3\). For \(b \geq 132\) one has \(K_1 > 0\) and \(K_0|_{n=3} > 0 > -\ln(b-1)\) which implies the inequalities
\[|\Theta^*| \geq Q|P_0|P^tP^\dagger P^\sharp R \geq 1/(b-1) \geq 1/|z| - 1 \geq |G| .\]
Prove the lemma for \(q \in (\tilde{q}_1, 1/2]\). One has \(Q \geq e^{-\pi^2/6}\) (Lemma (3.2) with \(n = 2\)), \(R \geq e^{-2(b+1)/b^2} \geq e^{-2 \times 133/132^2}\) (Lemma (3.4) with \(n = 2\)) and \(P^t \geq e^{-2.298978}\) (Lemma (3.5) with \(n = 2\)).
Lemma 8. For \( q \in (\tilde{q}_1, 1/2] \) the product \( P^1 P^2 \) contains at most two factors.

Proof. Indeed, consider the line \( L' \), see Fig. \( 1 \). One has \( ||[O, C']|| = 0.683 \tilde{A} \) and \( ||[O, D']|| < 4 \times 0.683 \Delta, \) see (\ref{eq:3.10}) and the lines that follow. Hence if the point \( zq^m \) belongs to the segment \([D', C']\), then this is not the case of the point \( zq^{m-2} \), because for \( q \in (\tilde{q}_1, 1/2] \) one has \( q^{-2} \geq 4 \) (but one could possibly have \( zq^{m-1} \in [D', C'] \)).

All factors \( |t_m| \) of the product \( P^1 P^2 \) belong to \([b/(a^2 + b^2)^{1/2}, 1] \), therefore by Lemma \( \ref{lem:8} \) \( P^1 P^2 \geq b^2/(a^2 + b^2) \) which for \( a \leq b \) is \( \geq 1/2 \).

On the other hand, the moduli of the first three factors \( |t_m| \) in \( \tilde{P} \) are not less than respectively

\[
132 \tilde{q}_1 - 1 > 39.814 \quad \text{and} \quad 132 \tilde{q}_1^2 - 1 > 11.619 \quad \text{and} \quad 132 \tilde{q}_1^3 - 1 > 2.902
\]

and the moduli of all other factors \( |t_m| \) in \( \tilde{P} \) (if any) are \( \geq 1 \), so for \( b \geq 132 \)

\[
|\Theta^*| \geq e^{-\pi/6} \times (39.814 \times 11.619 \times 2.902) \times (1/2) \times e^{-2 \times 133/132^2} \geq 12.8 > |G|.
\]

Lemma 9. For \( a \geq b \geq 132 \) one has \( |\Theta^*| > |G| \).

Proof. Suppose first that \( q \in (1/2, 1] \). We define \( n \geq 3 \) from conditions (\ref{eq:3.10}). Recall that the number \( \mu_1 \) was defined in Lemma \( \ref{lem:9} \) and that inequalities (\ref{eq:3.10}) hold true. For \( n \geq 3 \) equality (\ref{eq:3.11}) implies

\[
\mu_1 \leq (\ln \lambda_1)(n-1)^2/(2n-3) + 1 \tag{3.12}
\]

\[
= (\ln \lambda_1)(n-1/2 + 1/2(n-3)) + 1 < (\ln \lambda_1)n + 0.782 ,
\]

because \( 1/2(2n-3) \leq 1/6 \) and \( (\ln \lambda_1)(-1/2 + 1/6) + 1 = 0.7811 \ldots \).

Consider a factor \( t_m \) from \( P_0 := \prod_{n=1}^m t_m \). One has

\[
|t_m|^2 = (aq^m - 1)^2 + b^2 q^{2m} \geq 0.9(a^2 + b^2)q^{2m} ;
\]

this follows from

\[
(a^2 + b^2)q^{2m}/10 - 2aq^m + 1 = (aq^n - 10)^2/10 + (b^2 q^{2m} - 90)/10 \geq 0 ;
\]

the last inequality results from \( b \geq 132 \) and (\ref{eq:3.10}) (remember that if \( q \) satisfies conditions (\ref{eq:3.3}) with \( n \geq 3 \), then the inequality (\ref{eq:3.5}) holds true), so \( b^2 q^{2m} \geq (132/8)^2 > 90 \).

Set \( A := (a^2 + b^2)^{(n-\mu_1)/2} \). Hence

\[
|P_0| \geq (a^2 + b^2)^{n/2} q^{(n+1)/2} (0.9)^{n/2} \geq (a^2 + b^2)^{n/2 - (n+3)} (0.9)^{n/2}
\]

and

\[
P^2 \geq b^{\mu_1}/(a^2 + b^2)^{\mu_1/2} ;
\]

so

\[
|P_0| P^2 \geq Ab^{(\ln \lambda_1)(n-2)+12-(n+3)} (0.9)^{n/2} .
\]
(we use inequalities (3.8)). As \((a^2 + b^2)^{1/2} \geq 132\sqrt{2}\) and as \(n - \mu_1 \geq \omega_1 n - 0.782\), \(\omega_1 := 1 - (\ln \lambda_1)\), see (3.12), one obtains the minoration \(|R_I|P^t \geq e^M\), where

\[M := (\omega_1 n - 0.782) \ln(132\sqrt{2}) + (\ln b)((\ln \lambda_1)(n - 2) + 1)\]

\[-(n + 3) \ln 2 + (n/2) \ln 0.9\,.

To estimate \(P^t\) and \(P^\dagger\) we use Lemma 5. As in the proof of Lemma 7 one can minorize the right-hand side of (3.9) by

\[e^{(\pi^2/6)(1-n)}e^{M}e^{2(-1.149489n)}e^{-(b+1)n/b^2}\,.

This expression is of the form \(e^{L_1 n + L_0}\) with

\[L_1 = -\pi^2/6 + \omega_1 \ln(132\sqrt{2}) + (\ln b)((\ln \lambda_1)\]

\[-\ln 2 + (\ln 0.9)/2 - 2.298978 - (b + 1)/b^2\,

\[L_0 = \pi^2/6 - 0.782 \ln(132\sqrt{2}) + (\ln b)(-2 \ln \lambda_1 + 1) - 3 \ln 2\,.

For \(a \geq b = 132\) one has \(|z| \geq 132\sqrt{2}\), also \(L_1 > 0.3044 > 0\) and \(L_0 = -6.0491\,\ldots\,

For \(a \geq b = 132\), \(n \geq 3\) one has

\[L_1 n + L_0 \geq -5.136 \ldots > -5.224 \ldots = -\ln(132\sqrt{2} - 1)\,,

i.e. \(|\Theta^*| > 1/(|z| - 1) \geq |G|\). The functions \(L_1 n + L_0\) and \(e^{L_1 n + L_0}\) when considered as functions in \(b\) (for \(n \geq 3\) fixed) are increasing while the functions \(-\ln(b - 1)\) and \(1/(b - 1)\) are decreasing. Therefore one has \(|\Theta^*| > 1/(|z| - 1) \geq |G|\) for \(a \geq b \geq 132\), \(n \geq 3\) from which for \(q \in (1/2, 1)\) the lemma follows.

Suppose that \(q \in (q_1, 1/2)\). One deduces from Lemma 5 (as in the proof of Lemma 7) that \(P^t P^\dagger \geq b^2/(a^2 + b^2)\). On the other hand, consider the factors \(t_1\) and \(t_2\). One can apply to them inequality (3.13) with \(m = 1\) and \(2\). Hence \(|t_1| > 1\), \(|t_2| > 1\),

\[\tilde{P} \geq |t_1||t_2| > 0.9(a^2 + b^2)^{1/6}\tilde{q}_1^{16} \quad \text{and} \quad \tilde{P} P^t P^\dagger > 0.9b^2\tilde{q}_1^{16} > 463\,.

As in the proof of Lemma 7 we show that \(R \geq e^{-2 \times 133/132^2}\) and \(P^t \geq e^{-2 \times 298978}\), so finally

\[|\Theta^*| = \tilde{P} P^t P^\dagger Q P^\dagger R > 463 e^{-\pi^2/6} e^{-2 \times 298978} e^{-2 \times 133/132^2} > 8.8 > |G|\,.

\[\square\]

Lemmas 7 and 8 together imply that for \(q \in (0, 1)\) and \(z = -a + bi\), \(b > 132\), \(a > 0\), the function \(\theta(q, \ldots)\) has no zeros. Theorem 1 is proved.

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