A PRIORI ESTIMATES FOR POSITIVE SOLUTIONS TO
SUBCRITICAL ELLIPTIC PROBLEMS IN A CLASS
OF NON-CONVEX REGIONS

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Abstract. Building on the a priori estimates established in [3], we obtain
a priori estimates for classical solutions to elliptic problems with Dirichlet bound-
ary conditions on regions with convex-starlike boundary. This includes ring-like
regions. Arguments that go back to [4] are used to prove a priori bounds near
the convex part of the boundary. Using that the boundary term in the Pohozaev
identity on the boundary of a star-like region does not change sign, the proof
is concluded.

1. Introduction. In this paper we prove a priori bounds for the positive solutions
to the boundary-value problem:

\[
\begin{aligned}
- \Delta u &= f(u), & & \text{in } \Omega, \\
  u &= 0, & & \text{on } \partial \Omega,
\end{aligned}
\]

where \( \Omega \subset \mathbb{R}^N, N \geq 2 \), is a bounded \( C^2 \) domains with convex-starlike boundary,
including ring-like regions, and \( f : \mathbb{R}^+ \to \mathbb{R}^+ \) is a subcritical nonlinearity.

We will say that a domain \( \Omega \) has a convex-starlike boundary if \( \partial \Omega = \Gamma_1 \cup \Gamma_2 \)
with \( \Gamma_1 \subset \partial \Omega_1 \), for some convex domain \( \Omega_1 \subset \mathbb{R}^N \), and \( n(x) \cdot (x - y) < 0 \) for some
\( y \in \mathbb{R}^N \) and for all \( x \in \Gamma_2 \). Here \( n(x) \) denotes the outward normal to the boundary
\( \partial \Omega \). A particular case appears when \( \Omega = \Omega_1 \setminus \Omega_2 \) with \( \overline{\Omega_2} \subset \Omega_1 \), where \( \Omega_1 \) is convex,
and \( \Omega_2 \) star-like, that is \( n_2(x) \cdot (x - y) > 0 \), for some \( y \in \mathbb{R}^N \), and for all \( x \in \partial \Omega_2 \).

Here \( n_2(x) \) denotes the outward normal to the boundary \( \partial \Omega_2 \). In that case, we
will say that \( \Omega \) is a ring-like domain. Since (1.1) is invariant under translations,
without loss of generality, we may assume \( y = 0 \), (in other words \( \Omega_2 \) to be star-like
with respect to zero).

Let \( \lambda_1, \phi_1 \) stand for the first eigenvalue, first eigenfunction, of the problem
\(- \Delta \phi_1 = \lambda_1 \phi_1 \) in \( \Omega \), \( \phi_1 = 0 \) on \( \partial \Omega \).

Our main result is:

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75465, MINECO, Spain and Grupo de Investigación CADEDIF, UCM.
Theorem 1. Assume that $\Omega \subset \mathbb{R}^N$ is a bounded $C^2$ domain with convex-starlike boundary. If the nonlinearity $f$ is locally Lipschitzian and satisfies:

(H1) There exist constants $C_0 > 0$ and $\beta_0 \in (0, 1)$ such that $\liminf_{s \to +\infty} \frac{\min_{[\beta_0 s, s]} f(s)}{\frac{1}{s} f(s)} \geq C_0$.

(H2) There exists a constant $C_1 > 0$ such that $\limsup_{s \to \infty} \frac{\max_{[0, s]} f(s)}{f(s)} \leq C_1$.

(H3) There exists a constant $C_2 > 0$ and a non-increasing function $H : \mathbb{R}^+ \to \mathbb{R}^+$ such that

\begin{align*}
\text{(H3.1)} & \quad \liminf_{s \to +\infty} \frac{2NF(s) - (N-2)s f(s)}{sf(s)H(s)} \geq C_2 > 0, \\
\text{(H3.2)} & \quad \lim_{s \to +\infty} \frac{f(s)}{s^{2N-1}} \left[ H(s) \right]^{-\frac{2}{N-2}} = 0.
\end{align*}

(H4) $\liminf_{s \to +\infty} \frac{f(s)}{s} > \lambda_1$, where $\lambda_1$ is the first eigenvalue of $-\Delta$ acting on $H^1_0(\Omega)$.

Then there exists a uniform constant $C$, depending only on $\Omega$ and $f$, such that for every $u > 0$, classical solution to (1.1),

$$\|u\|_{L^\infty(\Omega)} \leq C.$$

Unlike results in [4] or [3], we do not assume $f(s)/s^{\frac{N+2}{N-2}}$ to be nonincreasing. Moreover, Theorem 1 extends the results in [4] and [5] in that it allows a wider class of nonlinearities, such as $f(s) = s^{\frac{N+2}{N-2}}/\ln(s+2)^\alpha$, with $\alpha > 2/(N-2)$, see [3, Corollary 2.2]. The reader is referred to Theorem 1.2 in [3] where a slightly more restrictive hypothesis (H1) is used for convex domains ($\beta_0 = 1/2$). In [2] one can see the role of regions with convex external boundaries in proving the uniqueness of positive solutions to semilinear elliptic boundary value problems.
2. Bounds near the convex boundary and Pohozaev identity. Due to \( n(x) < 0 \) for all \( x \in \Gamma_2 \), we can choose \( \varepsilon > 0 \) such that if \( x \in \Gamma_1 \) and \( d(x, \Gamma_2) < \varepsilon \), then \( n(x) \cdot x < 0 \). Let us define \( \Gamma_1' := \Gamma_1 \setminus \{ x \in \partial \Omega : d(x, \Gamma_2) < \varepsilon \} \), and \( \Gamma_2' := \partial \Omega \setminus \Gamma_1' \).

Due to the convexity of \( \Omega_1 \), arguing as in [4] (see also [3] Theorem A.3) one sees that there exists a neighborhood of \( \Gamma_1' \) and a positive constant \( \gamma \) such that for every \( x \) in that neighborhood there exists a set \( K' \) with \( |K'| \geq \gamma \) such that \( u \) is bounded below by \( u(x) \) in \( K' \). Here \( |K'| \) denotes the Lebesgue measure of the set \( K' \). The main idea behind the proof of Theorem 2 is to use the convexity of \( \Omega_1 \) and moving plane arguments. These lead to the following result whose proof may be found in Step 2 in the proof of [4] Theorem 1.1.

**Theorem 2.** Let \( \Omega \subset \mathbb{R}^N \) be a domain with convex-starlike boundary. Assume that the nonlinearity \( f \) is locally Lipschitzian and that satisfies (H4). If \( u \in C^2(\overline{\Omega}) \) satisfies \([1.1]\) and \( u > 0 \) in \( \Omega \), then there exists a constant \( \delta > 0 \) depending only on \( \Omega_1 \) and not on \( f \) or \( u \), and a constant \( C \) depending only on \( \Omega_1 \) and \( f \) but not on \( u \), such that

\[
\max_{\omega_1} u \leq C
\]

where \( \omega_1 := \{ x \in \Omega; d(x, \Gamma_1') < \delta \} \).

Any classical solutions to \([1.1]\) satisfies the following identity known as Pohozaev identity, see [8].

\[
\int_{\Omega} NF(u) - \frac{N - 2}{2} uf(u) = \int_{\partial \Omega} \left( x \cdot \nabla u \frac{\partial u}{\partial n} + \left[ F(u) - \frac{1}{2} |\nabla u|^2 \right] x \cdot n \right) d\sigma, \quad (2.2)
\]

where \( n(x) \) is the outward normal vector to the boundary at \( x \in \partial \Omega \).

Since \( \partial \Omega = \Gamma_1' \cap \Gamma_2' \) is a convex-starlike boundary, for each \( x \in \Gamma_2' \), we have

\[
x = s(x)n(x) + \tau(x), \quad \text{where} \quad s(x) \leq 0, \quad (2.3)
\]

and \( \tau(x) \) is tangential to \( \partial \Omega \). In particular, \((2.3)\) holds for any \( x \in \Gamma_2' \). Moreover, since \( u \) vanishes on \( \partial \Omega \), for any tangential vector \( t(x) \)

\[
t(x) \cdot \nabla u(x) = 0. \quad (2.4)
\]

Hence

\[
|\nabla u(x)|^2 = \left( \frac{\partial u}{\partial n} \right)^2 \quad \text{and} \quad x \cdot \nabla u(x) = s(x)n(x) \cdot \nabla u(x) = s(x) \frac{\partial u}{\partial n}(x). \quad (2.5)
\]

Substituting \( F(u(x)) = 0 \) for all \( x \in \partial \Omega \), and \((2.3), (2.5)\) in \((2.2)\) we have

\[
\int_{\Omega} \left( NF(u) - \frac{N - 2}{2} uf(u) \right) dx = \int_{\Gamma_1} \left[ x \cdot \nabla u \frac{\partial u}{\partial n} - \frac{1}{2} |\nabla u|^2 (x \cdot n) \right] d\sigma
\]

\[
+ \int_{\Gamma_2'} \left[ x \cdot \nabla u \frac{\partial u}{\partial n} - \frac{1}{2} |\nabla u|^2 (x \cdot n) \right] d\sigma
\]

\[
= \int_{\Gamma_1} \left[ x \cdot \nabla u \frac{\partial u}{\partial n} - \frac{1}{2} |\nabla u|^2 (x \cdot n) \right] d\sigma
\]

\[
+ \int_{\Gamma_2'} \frac{s(x)}{2} \left( \frac{\partial u}{\partial n} \right)^2 d\sigma \quad (2.6)
\]
3. Proof of Theorem 1

From now on, throughout this proof $C$ denotes several constants independent of $u$. From Theorem 2 and de Giorgi-Nash type Theorems, see [1] Theorem 14.1,
\[ \|u\|_{C^{0,\alpha}(\omega_{3/4} \setminus \omega_{3/6})} \leq C, \quad \text{for any } \alpha \in (0, 1), \]
where $\omega_t := \{ x \in \Omega : d(x, \Gamma_1^t) < t \}$.

From hypothesis (H3.1), in particular, there exists a constant $C, \delta > 0$ independent of $u$ such that
\[ \|u\|_{C^{1,\alpha}(S)} \leq C. \]

Finally, combining $L^p$ estimates with Schauder boundary estimates, (see [1], [6])
\[ \|u\|_{W^{2,p}(\omega_{3/2})} \leq C, \quad \text{for any } p \in (1, \infty). \]

Consequently, there exists two constants $C, \delta > 0$ independent of $u$ such that
\[ \|u\|_{C^{1,\alpha}(\omega_\delta)} \leq C, \quad \text{for any } \alpha \in (0, 1). \]  \hfill (3.1)

Also, since $s(x) \leq 0$ for all $x \in \Gamma_2$, from (2.6),
\[ \int_{\Omega} \left( NF(u) - \frac{N-2}{2} u f(u) \right) dx \leq C. \]  \hfill (3.2)

Next we prove that also
\[ \int_{\Omega} \left| NF(u) - \frac{N-2}{2} u f(u) \right| dx \leq C. \]  \hfill (3.3)

From hypothesis (H4), there exists a constant $C$ such that if $s > C$ then $f(s) > 0$. From hypothesis (H3.1), in particular, there exists a constant $C$ such that if $s > C$ then $2NF(s) - (N-2)s f(s) > 0$. Splitting the above integral in the set $S = \{ x \in \Omega : |u| \leq C \}$ and its complement $\Omega \setminus S$, since from (3.2) $\int_{\Omega \setminus S} (NF(u) - \frac{N-2}{2} u f(u)) dx \leq C$, then (3.3) holds.

All other arguments work as in [3] proof of Theorems 1.1 and 1.2. We include them here for the sake of completeness.

From hypothesis (H3.1), there exists a constant $C > 0$ and a non-increasing function $H$ such that
\[ \left| 2NF(s) - (N-2)s f(s) \right| \geq C H(s) s |f(s)|, \quad \text{for any } s \text{ big enough}. \]  \hfill (3.4)

Applying this inequality to any positive solution, and integrating on $\Omega$ we obtain that
\[ \int_{\Omega} \left| 2NF(u) - (N-2)u f(u) \right| dx \geq C \int_{\Omega} u |f(u)| H(u) dx - C', \]  \hfill (3.5)

for some constant $C'$ independent on $u$.

From (3.3) and (3.5),
\[ \int_{\Omega} u |f(u)| H(u) dx \leq C. \]  \hfill (3.6)

From hypothesis (H3.2), \( \lim_{s \to +\infty} \frac{|f(s)|^{\frac{1}{2\alpha-1}}}{s \left[ H(s) \right]^{\frac{\alpha-1}{2\alpha}}} = 0. \) Multiplying numerator and denominator by \( |f(s)| H(s)^{\frac{\alpha-1}{2\alpha}} \), we can assert that there exists a constant $C$ such that
\[ |f(s)|^{1+\frac{1}{2\alpha-1}} \left[ H(s) \right]^\frac{\alpha-1}{2\alpha} \leq s |f(s)| H(s) + C, \quad \text{for any } s > 0. \]
Applying this inequality to any positive solution, integrating on $\Omega$, and using (3.6) we obtain that
\[ \int_{\Omega} |f(u)|^{1+\frac{2}{q-1}} H(u)^{\frac{N}{N+q}} \, dx \leq C. \] (3.7)
Consequently, since $H$ is non-increasing,
\[
\int_{\Omega} |f(u(x))|^q \, dx \leq \frac{1}{H(\|u\|_{\infty})^{\frac{N}{N+q}}} \int_{\Omega} |f(u(x))|^{1+\frac{2}{q-1}} H(u)^{\frac{N}{N+q}} |f(u(x))|^{q-1-\frac{2}{q-1}} \, dx 
\leq C \frac{\|f(u(\cdot))\|_{L^q}^{q-1-\frac{1}{q-1}}}{H(\|u\|_{\infty})^{\frac{N}{N+q}}},
\] (3.8)
for any $q > N/2$.

Therefore, from elliptic regularity, (see [3 Lemma 9.17])
\[ \|u\|_{W^{2,q}(\Omega)} \leq C \|\Delta u\|_{L^q(\Omega)} \leq C \frac{\|f(u(\cdot))\|_{L^q(\Omega)}^{1-\frac{1}{q} - \frac{1}{N+1}}}{H(\|u\|_{\infty})^{\frac{N}{N+q}}}. \] (3.9)

Let us restrict $q \in (N/2, N)$. From Sobolev embeddings, for $1/q^* = 1/q - 1/N$ with $q^* > N$ we may write
\[ \|u\|_{W^{1,q^*}(\Omega)} \leq C \|u\|_{W^{2,q}(\Omega)} \leq C \frac{\|f(u(\cdot))\|_{L^q(\Omega)}^{1-\frac{1}{q} - \frac{1}{N+1}}}{H(\|u\|_{\infty})^{\frac{N}{N+q}}}. \] (3.10)

From Morrey’s Theorem, (see [4 Theorem 9.12 and Corollary 9.14]), there exists a constant $C$ only dependent on $\Omega, q$ and $N$ such that
\[ |u(x_1) - u(x_2)| \leq C |x_1 - x_2|^{1-N/q^*} \|u\|_{W^{1,q^*}(\Omega)}, \quad \forall x_1, x_2 \in \Omega. \] (3.11)

Therefore, for all $x \in B(x_1, R) \subset \Omega$
\[ |u(x) - u(x_1)| \leq C R^{2-\frac{N}{q^*}} \|u\|_{W^{2,q}(\Omega)}. \] (3.12)

From now on, we shall argue by contradiction. Let $\{ u_k \}_k$ be a sequence of classical positive solutions to (1.1) and assume that
\[ \lim_{k \to \infty} \|u_k\| = +\infty, \quad \text{where} \quad \|u_k\| := \|u_k\|_{\infty}. \] (3.13)

Let $x_k \in \Omega$ be such that $u_k(x_k) = \max_{\Omega} u_k$. Let us choose $R_k$ such that $B_k = B(x_k, R_k) \subset \Omega$, and
\[ u_k(x) \geq \beta_0 \|u_k\| \quad \text{for any} \quad x \in B(x_k, R_k), \]
and there exists $y_k \in \partial B(x_k, R_k)$ such that
\[ u_k(y_k) = \beta_0 \|u_k\|. \] (3.14)

Let us denote by
\[ m_k := \min_{[\beta_0 \|u_k\|, \|u_k\|]} f, \quad M_k := \max_{[0, \|u_k\|]} f. \]

Therefore
\[ m_k \leq f(u_k(x)) \quad \text{if} \quad x \in B_k, \quad \text{and} \quad f(u_k(x)) \leq M_k \quad \forall x \in \Omega. \] (3.15)
Then, reasoning as in (3.8), we obtain
\[
\int_{\Omega} |f(u_k)|^q \, dx \leq C \frac{M_k^{q-1-\frac{1}{q}}}{H(||u_k||)}^{\frac{N}{N+2}}. \tag{3.16}
\]

From elliptic regularity, see (3.9), we deduce
\[
\|u_k\|_{W^{2,q}(\Omega)} \leq C M_k^{1-\frac{1}{q} - \frac{1}{q(2^* - 1)q}} \left[ H(||u_k||) \right]^{\frac{N}{2(N+2)q}}. \tag{3.17}
\]

Therefore, from Morrey’s Theorem, see (3.12), for any \(x \in B(x_k, R_k)\)
\[
|u_k(x) - u_k(x_k)| \leq C (R_k)^{2-\frac{N}{q}} M_k^{1-\frac{1}{q} - \frac{1}{q(2^* - 1)q}} \left[ H(||u_k||) \right]^{\frac{N}{2(N+2)q}}. \tag{3.18}
\]

Taking \(x = y_k\) in the above inequality and from (3.14) we obtain
\[
C (R_k)^{2-\frac{N}{q}} M_k^{1-\frac{1}{q} - \frac{1}{q(2^* - 1)q}} \left[ H(||u_k||) \right]^{\frac{N}{2(N+2)q}} \geq |u_k(y_k) - u_k(x_k)| = (1 - \beta_0)\|u_k\|, \tag{3.19}
\]

which implies
\[
(R_k)^{2-\frac{N}{q}} \geq \frac{(1 - \beta_0)\|u_k\| \left[ H(||u_k||) \right]^{\frac{N}{2(N+2)q}}}{C M_k^{1-\frac{1}{q} - \frac{1}{q(2^* - 1)q}}}, \tag{3.20}
\]

or equivalently
\[
R_k \geq \left( \frac{(1 - \beta_0)\|u_k\| \left[ H(||u_k||) \right]^{\frac{N}{2(N+2)q}}}{C M_k^{1-\frac{1}{q} - \frac{1}{q(2^* - 1)q}}} \right)^{1/(2-\frac{N}{q})}. \tag{3.21}
\]

This, (3.15), and the assumption that \(H\) is non-increasing imply
\[
\int_{B(x_k, R_k)} u_k |f(u_k)| H(u_k) \, dx \geq \beta_0 \|u_k\| H(||u_k||) m_k \omega (R_k)^N,
\]

where \(\omega = \omega_N\) is the volume of the unit ball in \(\mathbb{R}^N\).
Due to $B(x_k, R_k) \subset \Omega$, substituting inequality (3.21), and rearranging terms, we obtain

$$
\int_{\Omega} u_k |f(u_k)| H(u_k) \, dx \\
\geq \beta_0 \|u_k\| H(\|u_k\|) m_k \omega \left( \frac{(1 - \beta_0)}{C} \|u_k\| \left[ H(\|u_k\|) \right]^{\frac{N}{N + 2 + q}} \right)^{\frac{N}{2 - \frac{q}{2}}}
$$

$$
= C m_k \left( \|u_k\| H(\|u_k\|) \right)^{\frac{N}{N + 2 + q} - \frac{q}{2}} \left[ H(\|u_k\|) \right]^{\frac{N}{2 - \frac{q}{2}}} \frac{1}{\|u_k\|^{\frac{N}{N + 2 + q} - \frac{q}{2}}}
$$

$$
= C m_k \left( \|u_k\|^{1 + \frac{N}{N + 2 + q} - \frac{q}{2}} H(\|u_k\|)^{\frac{N}{2 - \frac{q}{2}}} \right) \frac{1}{\|u_k\|^{\frac{N}{N + 2 + q} - \frac{q}{2}}}
$$

$$
= C \frac{m_k}{M_k} \left( \|u_k\|^{1 + \frac{N}{N + 2 + q} - \frac{q}{2}} H(\|u_k\|)^{\frac{N}{2 - \frac{q}{2}}} \right) \frac{1}{\|u_k\|^{\frac{N}{N + 2 + q} - \frac{q}{2}}}
$$

From hypotheses (H1) and (H2)

$$
\frac{m_k}{M_k} \geq C, \quad \text{for all} \quad k \quad \text{big enough}. \quad (3.22)
$$

Hence, taking again into account hypothesis (H2), and rearranging exponents, we have

$$
\int_{\Omega} u_k |f(u_k)| H(u_k) \, dx \geq C \left( \|u_k\| \left[ H(\|u_k\|) \right]^{\frac{N}{N + 2 + q} - \frac{q}{2}} \right) \frac{1}{\|u_k\|^{\frac{N}{N + 2 + q} - \frac{q}{2}}}
$$

$$
\geq C \left( \|u_k\| \left[ H(\|u_k\|) \right]^{\frac{N}{2 - \frac{q}{2}}} \right) \frac{1}{\|u_k\|^{\frac{N}{N + 2 + q} - \frac{q}{2}}}
$$

$$
\geq C \left( \|u_k\|^{(N + 2)\left[ \frac{N}{2} - \frac{1}{N + 2 + q} \right]} \left[ H(\|u_k\|) \right]^{2\left[ \frac{N}{2} - \frac{1}{N + 2 + q} \right]} \left[ f(\|u_k\|) \right]^{(N - 2)\left[ \frac{N}{2} - \frac{1}{N + 2 + q} \right]} \right)
$$

Finally, from hypothesis (H3.2) we deduce

$$
\int_{\Omega} u_k |f(u_k)| H(u_k) \, dx \geq C \left( \|u_k\|^{2 - \frac{1}{N}} \left[ H(\|u_k\|) \right]^{\frac{2}{N - 1}} \left[ f(\|u_k\|) \right]^{\frac{N}{2 - \frac{1}{N}} - \frac{q}{2}} \right) \rightarrow \infty
$$

as $k \rightarrow \infty$, which contradicts (3.6), ending the proof. \qed
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