Approximate Dirac solutions of a complex parity–time-symmetric Pöschl–Teller potential in view of spin and pseudospin symmetries

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Abstract
By employing an exponential-type approximation scheme to replace the centrifugal term, we have approximately solved the Dirac equation for a spin-1/2 particle subjected to complex parity–time-symmetric scalar and vector Pöschl–Teller (PT) potentials with arbitrary spin–orbit κ-wave states in view of spin and pseudospin (p-spin) symmetries. The real bound-state energy eigenvalue equation and the corresponding two-spinor components wave function expressible in terms of hypergeometric functions are obtained by means of wave function analysis. The spin-κ Dirac equation and the spin-0 Klein–Gordon equation with complex PT potentials share the same energy spectrum under the choice of \( S(r) = \pm V(r) \) (i.e. exact spin and p-spin symmetries).

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(Some figures may appear in color only in the online journal)

1. Introduction
A large number of potentials with real and complex forms have been studied in various fields of physics. A consistent theory of quantum mechanics (QM) in terms of Hermitian Hamiltonians is built on a complex Hamiltonian that is non-Hermitian, but the energy spectra are real as a consequence of parity–time reflection symmetry (PT-symmetry). The Hamiltonian is said to be PT-symmetric when \([PT, H] = 0\) where \(P\) and \(T\) are the operators of parity (space) and time-reversal (complex conjugation) transformations. For a given potential \(V(x)\), when we make the following transformations: \(P : x \rightarrow -x\) (or \(x \rightarrow a - x\)), \(T : i \rightarrow -i\), \(P : p \rightarrow -p\), \(T : i I \rightarrow -i I\) and \(PT : p \rightarrow p\), if the potential \(V(-x) \rightarrow V^*(x)\) or \(V(a - x) \rightarrow V^*(x)\), then the Hamiltonian is said to be PT-symmetric, where \(x, p\) and \(I\) are the position, momentum and identity operators acting in the Hilbert space. The PT-symmetry does not always lead to a completely real spectrum and there are some cases when part or all of the energy levels are found to be complex (see [1] and references therein).

Several potentials with complex forms have been studied in the context of the PT-symmetric QM by Bender and Boettcher [2, 3]. The main reason for such interest is because of the reality (PT-symmetry is exact) or complexity (PT-symmetry is spontaneously broken) of the energy spectrum of Hermitian or non-Hermitian Hamiltonians [4, 5]. In recent years, the PT-symmetry of the relativistic QM and quantum field theories [6–8] has also been investigated. The PT-symmetry is not a sufficient and necessary condition for the reality of energy spectra. This has been shown by several non-Hermitian PT-symmetric potentials such as the complexified Pöschl–Teller (PT) potential model (see [9] and references therein).

In recent years, complex potentials have become very popular and powerful tools for quantum mechanical calculations. In most introductory courses on QM, traditionally one is taught that the Hamiltonian operator must be Hermitian.
in order that the energy levels be real to describe a closed system [10]. In recent years, there is growing interest in the use of non-Hermitian Hamiltonians with complex potentials [10] with real energy spectra. These complex potentials extend QM to open systems where the overall probability decreases in time, allowing simulations of decay, transport and scattering phenomena. Examples include the modeling of dissipative processes [11], nuclear [12] and chemical [13] reactions, simulation measurements of arrival, traversal or dwell times of quantum particles [14], atomic multi-photon ionization [15], and electron [16] and spin transport [17]. The motivation for the interest in the study of complex potentials is mentioned in [18]. For example, one of the motivations behind the exactly solvable quantum mechanical systems was studying the classical analogue of the PT-symmetric Scarf II model [19]. The complex square-well potential is used by Feshbach et al [20] in the optical model of the nucleus. Complexification [21] was found to work successfully for the harmonic oscillator potential in the second quantized version of field theory through creation and annihilation operators. Rao et al [22] used complexification in their studies of ion-acoustic waves in plasma and Yang [23] used it for developing a complex mechanics. The applicability and validity of the QM with complex potentials are mentioned in [24].

In this work, we will study the PT-symmetry of one of these potentials, the so-called Pöschl–Teller (PT) potential [25], taking the form

\[ V_{PT}(r) = \frac{\alpha^2}{2M} \left[ \frac{B(B - \alpha)}{\sinh^2 \alpha r} - \frac{A(A + \alpha)}{\cosh^2 \alpha r} \right], \]

where the parameter \( \alpha \) is relevant for the range of the potential and \( A > \alpha, B > +\alpha \). The PT potential is unchanged under the transformations \( A \rightarrow -(A + \alpha) \) and \( B \rightarrow -(B - \alpha) \); we may only discuss the case of \( B < A \). The PT potential has been extensively studied in the literature [26]. It is used as the electron–nucleus potential to study the strong field ionization dynamics of a simplified one-dimensional configuration model of a homogenous molecular ion [27].

Making an imaginary coordinate shift [28], \( r \rightarrow x - i x_0 \), where \( x_0 \) is a real parameter, we obtain a non-Hermitian complex PT-symmetric form of the real PT potential (1) as

\[ V_{CPT}(r) = \frac{\alpha^2}{2M} \left[ \frac{B(B - \alpha)}{\sinh^2 \alpha(x - i x_0)} - \frac{A(A + \alpha)}{\cosh^2 \alpha(x - i x_0)} \right], \]

where \( x \in (0, \infty) \) and \( 0 < \alpha x_0 < \pi/2 \). The potential (2) is a special case of the five-parameter exponential-type potential model [29]. Some authors [28, 29] have studied the PT-symmetric PT potential in the context of the s-wave Schrödinger equation. However, the relativistic treatment of such a form has not yet been reported in the literature. In [30], the authors have investigated the bound-state solutions of the s-wave Dirac equation with equally mixed PT potentials in terms of the supersymmetric QM approach and wave function analysis. Using the same method, the pseudospin (p-spin) symmetry solutions of the Dirac equation with PT potential for the spin–orbit quantum number \( \kappa \) have been investigated. One of us studied the exact solution of the one-dimensional Klein–Gordon (KG) equation for the PT-symmetric generalized Woods–Saxon (WS) potential using the Nikiforov–Uvarov method [33]. The reality of positive and negative exact bound states of the s-states is also investigated for different types of complex generalized WS potentials [33].

The purpose of this paper is to obtain the approximate bound-state solution of the Dirac equation with complexified PT-symmetric PT potential for arbitrary spin–orbit quantum number \( \kappa \) under spin and p-spin symmetries by employing wave function analysis. We shall show that if the Hamiltonian (Hermitian or non-Hermitian) has unbroken PT-symmetry, then the energy spectrum is real. To deal with the centrifugal term \( \propto r^{-2} \), we shall use the exponential-type approximation form and employ an imaginary coordinate shift. To this end, we shall first briefly introduce in section 2 the Dirac equation with radial scalar and vector potentials for arbitrary spin–orbit quantum number \( \kappa \) in view of spin and p-spin symmetries. Next, the general Dirac formulae are applied to deal with complex PT-symmetric scalar and vector PT potentials in order to obtain the energy eigenvalues and the corresponding two-component wave functions under the choice of spin and p-spin symmetries alike. Section 3 is devoted to discussions, where we shall consider some particular cases for our solutions such as the spin-0 KG and spinless Schrödinger wave equations. Finally, section 4 presents the summary and concluding remarks.

2. Bound-state solutions

The Dirac equation for a spin-1/2 particle with mass \( M \) moving in the field of an attractive scalar potential \( S(r) \) and a repulsive vector potential \( V(r) \) reads (in relativistic units \( \hbar = c = 1 \))

\[
\begin{align*}
\left[ \gamma_0 \alpha \cdot \beta (M + S(r)) \right] \psi (\vec{r}) &= [E - V(r)] \psi (\vec{r}),
\end{align*}
\]

where \( E \) is the relativistic energy of the system, \( \vec{p} = -i \vec{\nabla} \) is the three-dimensional momentum operator and \( \vec{\alpha} \) and \( \vec{\beta} \) are the 4 \times 4 usual Dirac matrices [34]. One may closely follow the procedure described in equations (13)–(19) of [34] to obtain

\[
\begin{align*}
\left\{ \frac{d^2}{dr^2} - &\frac{\kappa (\kappa + 1)}{r^2} \left[ M + E_{nx} - \Delta (r) \right] \left( M - E_{nx} + \Sigma (r) \right) \\
&+ \frac{\frac{d\Delta(r)}{dr}}{M + E_{nx} - \Delta (r)} \left( \frac{d}{dr} + \frac{\kappa}{r} \right) \right\} F_{nx} (r) = 0, \\
\kappa (\kappa + 1) = \ell (\ell + 1), \quad r \in (0, \infty),
\end{align*}
\]

\[
\begin{align*}
\left\{ \frac{d^2}{dr^2} - &\frac{\kappa (\kappa - 1)}{r^2} \left[ (M + E_{nx} - \Delta (r)) (M - E_{nx} + \Sigma (r)) \right] \\
&+ \frac{\frac{d\Sigma(r)}{dr}}{M - E_{nx} + \Sigma (r)} \left( \frac{d}{dr} - \frac{\kappa}{r} \right) \right\} G_{nx} (r) = 0, \\
\kappa (\kappa - 1) = \tilde{\ell} (\tilde{\ell} + 1), \quad r \in (0, \infty),
\end{align*}
\]
where $\Delta (r) = V(r) - S(r)$ and $\Sigma (r) = V(r) + S(r)$. The orbit–spin quantum number $\kappa$ is related to the orbital quantum numbers $l$ and $\tilde{l}$ for spin and p-spin symmetric models, respectively, as

$$
\kappa = \begin{cases} 
-(l+1) &= -\left( j + \frac{1}{2} \right) \quad (s_{1/2}, p_{3/2}, \text{etc}) \quad j = l + \frac{1}{2}, \\
+1 &= +\left( j + \frac{1}{2} \right) \quad (p_{1/2}, d_{3/2}, \text{etc}) \quad j = l - \frac{1}{2}, \\
+l &= +\left( j + \frac{1}{2} \right) \quad (s_{1/2}, p_{3/2}, \text{etc}) \quad j = l + \frac{1}{2}, \\
+\tilde{l} &= +\left( j + \frac{1}{2} \right) \quad (d_{1/2}, f_{5/2}, \text{etc}) \quad j = l - \frac{1}{2}, \\
\end{cases}
$$

aligned spin ($\kappa < 0$),

unaligned spin ($\kappa > 0$).

Further, $\kappa$ in the quasi-degenerate doublet structure can be expressed in terms of $\tilde{s} = 1/2$ and $\tilde{l}$, the p-spin and pseudo-orbital angular momentum, respectively, as

$$
\kappa = \begin{cases} 
-l &= -\left( j + \frac{1}{2} \right) \quad (s_{1/2}, p_{3/2}, \text{etc}) \quad j = \tilde{l} - \frac{1}{2}, \\
+l &= +\left( j + \frac{1}{2} \right) \quad (d_{1/2}, f_{5/2}, \text{etc}) \quad j = \tilde{l} + \frac{1}{2}, \\
\end{cases}
$$

where $\kappa = \pm 1, \pm 2, \ldots$. For example, the states $(1s_{1/2}, 0d_{3/2})$ and $(1p_{3/2}, 0f_{5/2})$ can be considered as p-spin doublets.

2.1. Spin symmetric limit

Let us consider the exact spin symmetry, $\frac{d\Delta (r)}{dr} = 0$ or $\Delta (r) = C_s = \text{constant}$ [35, 36]. In equation (4), we set the sum potential $\Sigma (r)$ as the PT-symmetric PT potential, i.e.

$$
\Sigma (r) = 2V_{\text{CPT}} = \frac{\alpha^2}{M} \left[ \frac{B(B - \alpha)}{\sinh^2 \alpha r} - \frac{A(A + \alpha)}{\cosh^2 \alpha r} \right],
$$

and by setting $r \rightarrow x - ix_0$ where $x \in (0, \infty)$ is the real part and $x_0$ is a constant real number, we obtain

$$
\frac{d^2}{dx^2} + \beta^2 = \kappa \left( \kappa + 1 \right) - \frac{\alpha^2}{M} \left[ \frac{B(B - \alpha)}{\sinh^2 \alpha (x - ix_0)} \right] M_s, \quad F_{nx} (x) = 0,
$$

where $\kappa = l$ and $-l - 1$ for $\kappa < 0$ and $\kappa > 0$, respectively. Further, $F_{nx} (r) \equiv F_{nx} (x)$ is used. Note that the above second-order differential Schrödinger-type equation can be solved exactly only for the s-wave ($\kappa = -1$) case. This equation cannot be solved analytically for $\kappa \neq 0$ due to the centrifugal term $\kappa \left( \kappa + 1 \right) r^{-2}$. In order to obtain the approximate analytical solution of equation (6a) for the case of non-zero $\kappa$ values, we use an approximation for the centrifugal term similar to the one used in [37, 38]. We take the following approximation for the centrifugal term

$$
\frac{\kappa (\kappa + 1)}{r^2} \approx \kappa (\kappa + 1) \lim_{\alpha \rightarrow 0} 4\alpha^2 \left[ d_0 + \frac{1}{(e^{\alpha r} - e^{-\alpha r})^2} \right] \approx \alpha^2 \kappa (\kappa + 1) \left[ 4d_0 + \frac{1}{\sinh \alpha r} \right].
$$

[34, 37, 38]:

where $d_0 = 1/12$ is a dimensionless constant. Such an approximation is a good approximation for small values of the screening parameter $\alpha$, i.e. $\alpha r \ll 1$. Note that this approximation turns into the conventional approximation introduced by Greene and Aldrich [39] and used by one of us [40] when $d_0 = 0$. In the present relativistic case, the treatment of this term does not raise any problem as in the non-relativistic case [37], since the centrifugal term (7) can be reduced to $l(l+1)/r^2$ for $\kappa < 0$ and $\kappa > 0$ and hence the non-relativistic techniques are still valid here. Therefore, inserting this approximation into equation (6a) and applying the change of variables $r \rightarrow x - ix_0$, we can recast the Schrödinger-type equation for the upper-spinor component as

$$
\frac{d^2}{dx^2} + \beta^2 = \frac{V_2}{\sinh^2 \alpha (x - ix_0)} + \frac{V_1}{\cosh^2 \alpha (x - ix_0)} \quad F_{nx} (x) = 0,
$$

with the identifications:

$$
V_1 = \alpha^2 A(A + \alpha) M_s M, \quad V_2 = B(B - \alpha) M_s M,
$$

$$
E_{nx} = \beta^2 - 4\alpha^2 \kappa (\kappa + 1) d_0,
$$

The deformed hyperbolic functions [41, 42] are defined by

$$
\sinh_q (\alpha x) = \frac{e^{\alpha x} - q e^{-\alpha x}}{2}, \quad \cosh_q (\alpha x) = \frac{e^{\alpha x} + q e^{-\alpha x}}{2}, \quad q_c = e^{2\alpha x_0}.
$$

$$
\sinh_q (2\alpha x) = \frac{e^{2\alpha x} - q_c e^{-2\alpha x}}{2}, \quad \cosh_q (2\alpha x) = \frac{e^{2\alpha x} + q_c e^{-2\alpha x}}{2}, \quad (x \rightarrow 2x),
$$

where $q_c$ is a complex number. Now, we try to explore the real solution for equation (8). By using the above definitions and further making a little algebra, one can easily show the relations

$$
\sinh \alpha (x - ix_0) = \frac{1}{\sqrt{q_c}} \sinh_q (\alpha x), \quad \cosh \alpha (x - ix_0) = \frac{1}{\sqrt{q_c}} \cosh_q (\alpha x),
$$

$$
2 \cosh^2_q (\alpha x) = q_c + \sinh_q (2\alpha x),
$$

$$
\cosh^2_q (\alpha x) - \sinh^2_q (\alpha x) = q_c.
$$
\[ \cosh^2_{q_s}(\alpha x) + \sinh^2_{q_s}(\alpha x) = \sinh_q(2\alpha x), \quad (11d) \]
\[ 4 \sinh^2_{q_s}(\alpha x) \cosh^2_{q_s}(\alpha x) = \cosh^2_q(2\alpha x), \quad (11e) \]
\[ \cosh^2_{q_s}(2\alpha x) - \sinh^2_{q_s}(2\alpha x) = q_c, \quad (x \to 2x), \quad (11f) \]
where \( q = -q_c^2 \). Inserting the relations in equation (11) into equation (8), we can obtain
\[ \left\{ \frac{d^2}{dz^2} + \tilde{E}_{nx} - U_{\text{eff}}(x) \right\} F_{nx}(z) = 0, \quad (12a) \]
where \( U_{\text{eff}}(x) \) is the effective PT-symmetric PT potential. Now, we need to solve equation (12). This can be done by defining a new variable
\[ 2z = 1 - \frac{i}{\sqrt{q}} \sinh_q(2\alpha x), \quad \frac{d}{dz} = -\frac{i\alpha}{\sqrt{q}} \cosh_q(2\alpha x) \frac{d}{dx}, \]
\[ \frac{d^2}{dx^2} = -\frac{\alpha^2}{q} \left( \cosh^2_q(2\alpha x) \frac{d^2}{dz^2} + 2i\sqrt{q} \sinh_q(2\alpha x) \frac{d}{dz}, \quad (13) \right) \]
where \( q_c = i\sqrt{q} \). Furthermore, making use of equation (10b), we can obtain \( \cosh^2_{q_s}(2\alpha x) - \sinh^2_{q_s}(2\alpha x) = q \). Using equation (13), we can rearrange equation (12) in a simpler form:
\[ z(1 - z) \frac{d^2}{dz^2} F_{nx}(z) + \left( \frac{1}{2} - z \right) \frac{d}{dz} F_{nx}(z) \]
\[ -\frac{1}{4\alpha^2} \left[ \tilde{E}_{nx} + \frac{V_i}{z(1 - z)} + \frac{V_2 - V_1}{(1 - 1)} \right] F_{nx}(z) = 0, \quad (14) \]
where \( z \in (-\infty, 0) \). Further, inserting the appropriate ansatz of the wave function
\[ F_{nx}(z) = z^{-\lambda} (1 - z)^{-\eta} f_{nx}(z) \quad (15) \]
into equation (14) leads to the following hypergeometric differential equation [43]:
\[ z(1 - z) \frac{d^2}{dz^2} f_{nx}(z) + \left[ \frac{1}{2} - 2\lambda - (-2\lambda + 2\eta + 1) z \right] \frac{d}{dz} f_{nx}(z) \]
\[ -\left[ (\lambda + \eta)^2 + \left( \frac{\tilde{E}_{nx}}{2\alpha} \right)^2 + \frac{c_1}{z(1 - z)} + \frac{c_2}{(1 - z)} \right] f_{nx}(z) = 0, \quad (16) \]
where
\[ c_1 = -\lambda^2 - \frac{\lambda}{2} + \frac{V_1}{4\alpha^2}, \quad c_2 = -\eta^2 - \frac{\eta}{2} + \frac{V_2 - V_1}{4\alpha^2}. \quad (17) \]
Equation (16) can be reduced to a hypergeometric equation when taking \( c_1 = c_2 = 0 \). In solving (17), we obtain
\[ \lambda = \frac{1}{4} \left( -1 + \sigma \sqrt{1 + \frac{4V_1}{\alpha^2}} \right), \quad (18a) \]
\[ \eta = \frac{1}{4} \left( -1 + \sigma \sqrt{1 + \frac{4V_2}{\alpha^2}} \right), \quad (18b) \]
where \( \sigma = \pm 1 \) and \( \tau = \mp 1 \). Hence, the solution of this equation is simply the hypergeometric function:
\[ f_{nx}(x) = 2F_1 \left( \alpha, b; c; z \right) = \sum_{\gamma=0}^{\infty} \frac{(a)_\gamma(b)_\gamma}{(c)_\gamma} \frac{x^\gamma}{\gamma!} \]
\[ = 2F_1 \left( \frac{\eta}{\alpha}, \frac{1}{\alpha} \right) \frac{\tilde{E}_{nx}}{2\alpha}; -\lambda - \eta - \frac{\tilde{E}_{nx}}{2\alpha}; \frac{1}{2} - 2\lambda; 1 - iq^{-1/2} \sinh_q(2\alpha x) \right), \quad (19) \]
where \( (a)_m = \Gamma(a + m)/\Gamma(a) \) is a Pochhammer symbol. When either \( a \) or \( b \) equals a negative integer \( -n \), the hypergeometric function \( f_{nx}(x) \) can be reduced to a polynomial of degree \( n \) and asymptotically vanishing under certain boundary conditions. This shows that the hypergeometric function given in equation (19) can be finite under the following quantum condition:
\[ -\lambda - \eta + \frac{\tilde{E}_{nx}}{2\alpha} = -n, n = 0, 1, 2, \ldots, \quad (20) \]
from which, together with equations (6b), (9c) and (18), we finally obtain the energy eigenvalue equation for the nuclei in the field of relativistic PT-symmetric PT potential in view of the spin symmetry,
In addition, note that energy spectrum formula (21) looks the same as equation (24) of [45] in the KG equation case with \( S(r) = V(r) \) (exact spin symmetry, \( C_v = 0 \)) real PT potentials after making changes in the PT potential parameters in equation (1), i.e. \( A(A + \alpha) \rightarrow \lambda(\lambda + 1) \) and \( B(B - \alpha) \rightarrow k(k - 1) \). This means that the two forms (real and complex PT-symmetry) of the PT potential possess the same real energy spectrum under spin symmetry.

The corresponding upper-spinor component \( F_{nx}(x) \) is expressible in the form of a hypergeometric function as

\[
F_{nx}(x) = N_{nx} x^{\lambda + \eta} (p_1(x))^{-\lambda} (p_2(x))^{-\eta} \times \, _2F_1 \left( -n, -2(\lambda + \eta) + n; -2\lambda + \frac{1}{2}; \frac{p_1(0)}{2} \right),
\]

(22)

where \( p_1(x) = 1 - e^{-i2\alpha x_0} \sinh_2(2\alpha x) \), \( p_2(x) = 1 + e^{-i2\alpha x_0} \times \sinh_2(2\alpha x) \) and \( N_{nx} \) is the normalization constant. The wave function should vanish under certain asymptotic behavior of the wave function, i.e. \( F_{nx}(x \rightarrow \pm \infty) = 0 \). We can use the following recurrence relation between hypergeometric functions:

\[
\frac{d}{dx} \, _2F_1 \left( a, b; c; z \right) = \left( \frac{ab}{c} \right) \, _2F_1 \left( a + 1, b + 1; c + 1; z \right),
\]

(23)

and express the hypergeometric function in terms of the Jacobi polynomials

\[
2F_1 \left( -n, -2(\lambda + \eta) + n; -2\lambda + \frac{1}{2}; \frac{p_1(0)}{2} \right)
\]

\[
= \frac{(-2\lambda + 1/2)^n}{n!} \int_0^1 t^{n} e^{-i2x_0 \sinh_2(2\alpha x)} dt,
\]

in finding the lower component \( G_{nx}(x) \), which can be obtained from equation (18) of [34] by using equation (22) as

\[
G_{nx}(x) = \frac{1}{(M + E_{nx} - C_v)} \left[ \frac{\kappa}{r} F_{nx}(x) + \frac{dF_{nx}(x)}{dr} \right] + \frac{2n(\alpha - n - 1)}{(1 - 4\lambda)(M + E_{nx} - C_v)} \alpha e^{-i2\alpha x_0} \cosh_2(2\alpha x) \times \, _2F_1 \left( 1 - n, 1 + n - 2(\lambda + \eta); -2\lambda + \frac{3}{2}; \frac{p_1(0)}{2} \right) \times \lambda (p_1(x))^{-\lambda} (p_2(x))^{-\eta} \int_0^1 t^{n} e^{-i2x_0 \sinh_2(2\alpha x)} dt \times (2\alpha x) \left( \frac{\lambda}{p_1(0)} - \frac{\eta}{p_2(0)} \right), \quad \lambda \neq \frac{1}{4}
\]

(24)

From equation (24), we know that in the limit of exact spin symmetry there are only bound positive energy states; otherwise the lower-spinor component \( G_{nx}(x) \) in (24) will diverge if \( E_{nx} = -M \) and \( C_v = 0 \). There are no bound negative energy states in view of the spin symmetry condition.

The two spinors \( F_{nx}(x) \) and \( G_{nx}(x) \) satisfy the regularity boundary conditions for the bound states when \( \beta > 0, \eta < 0 \) and \( \lambda, \eta \in \mathbb{R} \). Considering the limiting case \( \alpha \rightarrow 0 \), we find from equation (21) that the energy eigenvalue approaches a constant value, i.e. \( \lim_{\alpha \rightarrow 0} E_{nx} = M \) or \( \lim_{\alpha \rightarrow 0} E_{nx} = C_v - M \). However, the energy limit \( \lim_{\alpha \rightarrow 0} E_{nx} = C_v - M \) is not physically acceptable since it makes lower component \( G_{nx}(x) \) diverge. Therefore, we choose the energy limit \( \lim_{\alpha \rightarrow 0} E_{nx} = M \) as the physically acceptable one and obtain

\[
\lim_{\alpha \rightarrow 0} G_{nx}(x) = \left( \frac{\kappa}{x - i\lambda_0} \right) \int_{2M - C_v} E_{nx}(x)
\]

(25)

To show the procedure for determining the bound-state energy eigenvalues from equation (21), we take a set of physical parameter values, \( \alpha = 0.35 \text{ fm}^{-1}, A = 8, B = 2, M = 5.0 \text{ fm}^{-1} \) and \( C_v = -0.35 \text{ fm}^{-1} \), to give a numerical example. When \( n = 0 \) and \( \kappa = 1 \) or \( -2 \), equation (21) yields two values for energy; however, we select the positive energy as the physical solution for the finiteness of the lower component of the wave function (24) and (22) for the upper component of the wave function. If we take \( E_{0,1} = 4.320628792 \text{ fm}^{-1} \) as the solution of equation (21), we find that the values of \( \lambda \) and \( \eta \) are \( \lambda = 4.989 398 388 \) and \( \eta = -1.634 030 092 \), which helps calculate the value \( n_{\text{max}} = 3.550 238 160 \). With the same parameter values \( \alpha, A, B, M \) and \( C_v \), the numerical solutions of equation (21) for the other values of \( n \) and \( \kappa \) are presented in table 1. This table shows the spin partners, i.e., the Dirac eigenstates \( 0p_{3/2} \) and \( 0p_{1/2} \). We have used \( \sigma = 1, \tau = -1 \) or \( \sigma = -1, \tau = 1 \) in calculating the spin symmetric spectrum. Further, in figures 1–5, we plot the variation of the spin symmetric energy eigenvalues as a function of the parameters \( \alpha, A, B, M \) and \( C_v \). We find that \( E_{nx} \) becomes more positive (increases) as the parameter values of \( M, B \) and \( C_v \) increase. However, it becomes less positive (decreases) as the parameter values of \( \alpha \) and \( A \) decrease.

### 2.2. p-spin symmetric limit

The p-spin symmetry occurs when the relationship between the vector potential and the scalar potential is given by \( V(r) = -S(r) \) [46]. Further, if

\[
\frac{d}{dr} \left( V(r) + S(r) \right) = \frac{d}{dr} \frac{d}{dr} = 0,
\]

then \( \Sigma(r) = C_{p_s} \) constant, for which the p-spin symmetry is exact in the Dirac equation [47, 48]. Thus, taking the potential difference \( \Delta(r) \) as the PT-symmetric potential, i.e.

\[
\Delta(r) = 2V_{\text{CPT}}(r) = \frac{a^2}{M} \left( \frac{B(B - \alpha)}{\sinh^2 \alpha r} - \frac{A(A + \alpha)}{\cosh^2 \alpha r} \right).
\]
Figure 1. The variation of the energy levels as a function $\alpha$ in view of spin symmetry with parameter values $A = 8$, $B = 2$, $M = 5.0$, $C_s = 0.35$.

Figure 2. The variation of the energy levels as a function $M$ in view of spin symmetry with parameter values $A = 8$, $B = 2$, $C_s = 0.35$, $\alpha = 0.35$.

Figure 3. The variation of the energy levels as a function $A$ in view of spin symmetry with parameter values $B = 2$, $M = 5.0$, $C_s = 0.35$, $\alpha = 0.35$. 
and setting \( r = x - i x_0 \), equation (5) under this symmetry becomes

\[
\begin{aligned}
\frac{d^2}{dx^2} + \tilde{E}_{nx} &= \frac{\tilde{V}_2}{\sinh^2 \alpha(x - ix_0)} + \frac{\tilde{V}_1}{\cosh^2 \alpha(x - ix_0)} \\
\times G_{nk}(x) &= 0, \\
M_{ps} &= M - E_{nk} + C_{ps} \quad \text{and} \quad \tilde{\beta}^2 = E_{nk}^2 - M^2 - C_{ps} (M + E_{nk}),
\end{aligned}
\]

where

\[
\begin{aligned}
\tilde{V}_1 &= \frac{\alpha^2 A(A + \alpha)M_{ps}}{M}, \\
\tilde{V}_2 &= \alpha^2 \left( \kappa (\kappa - 1) + \frac{B(B - \alpha)M_{ps}}{M} \right), \\
\tilde{E}_{nk} &= \tilde{\beta}^2 - 4\alpha^2 \kappa (\kappa - 1)d_0.
\end{aligned}
\]

To avoid repetition, the negative energy solution of equation (5), in the p-spin symmetric limit: \( V(r) = -S(r) \), can be easily obtained directly via the spin symmetric solution through the parametric mappings [49–53]

\[
\begin{aligned}
F_{nk}(x) &\leftrightarrow G_{nk}(x), \quad \kappa \rightarrow -1, \\
V(x) &\rightarrow -V(x), \quad E_{nk} \rightarrow -E_{nk}, \quad C_5 \rightarrow -C_{ps}.
\end{aligned}
\]

In following the previous procedure, one can easily obtain the energy eigenvalue equation for the nuclei in the field of the

**Table 2.** The p-spin symmetric energy levels of a Dirac particle subjected to a complexified PT-symmetric Pöschl–Teller potential for various values of \( n \) and \( \kappa \).

| \( l \) | \( n, \kappa \) | \( (l, j) \) | \( E_{nk} \) (fm\(^{-1}\)) |
|------|-----------------|-------------|-----------------|
| 1    | 1, -1,2         | 1s\(_{1/2} \), 0d\(_{3/2} \) | -5.170 251 165 |
| 2    | 1, -2,3         | 1p\(_{1/2} \), 0f\(_{5/2} \) | -5.055 448 493 |
| 3    | 1, -3,4         | 1d\(_{3/2} \), 0g\(_{9/2} \) | -4.943 195 896 |
| 4    | 1, -4,5         | 1f\(_{7/2} \), 1h\(_{9/2} \) | -4.846 340 118 |
| 1    | 2, -1,2         | 2s\(_{1/2} \), 1d\(_{3/2} \) | -5.000 631 769 |
| 2    | 2, -2,3         | 2p\(_{3/2} \), 1f\(_{5/2} \) | -4.951 890 564 |
| 3    | 2, -3,4         | 2d\(_{5/2} \), 1g\(_{7/2} \) | -4.915 209 098 |
| 4    | 2, -4,5         | 2f\(_{7/2} \), 1h\(_{9/2} \) | -4.900 619 782 |

**Figure 4.** The variation of the energy levels as a function \( B \) in view of spin symmetry with parameter values \( A = 8, M = 5.0, C_i = 0.35, \alpha = 0.35 \).

**Figure 5.** The variation of the energy levels as a function \( c_s \) in view of spin symmetry with parameter values \( A = 8, B = 2, M = 5.0, \alpha = 0.35 \).
The relativistic complex PT-symmetric PT potential (2) under the exact p-spin symmetry limit

\[
M^2 - E_{nx}^2 + C_{ps} (E_{nx} + M) = - \frac{\alpha^2 \kappa (\kappa - 1)}{3} + 4\alpha^2 \\
\times \left[ -n - \frac{1}{2} + \frac{1}{4} \left( \sigma \sqrt{1 - \frac{4(M - E_{nx} + C_{ps})A(A + \alpha)}{M}} \right) \right] \\
+ \tau \sqrt{(2\kappa - 1)^2 - \frac{4(M - E_{nx} + C_{ps})B(B - \alpha)}{M}^2},
\]

where the following two choices of \( \sigma = +1 \) and \( \tau = -1 \) or \( \sigma = -1 \) and \( \tau = +1 \) give the same numerical results as in the spin symmetric case. Hence, equation (28) satisfies the restriction condition

\[
n_{\text{max}} = \frac{1}{4} \sqrt{1 - \frac{4(M - E_{nx} + C_{ps})A(A + \alpha)}{M}} \\
- \frac{1}{4} \sqrt{(2\kappa - 1)^2 - \frac{4(M - E_{nx} + C_{ps})B(B - \alpha)}{M}^2} = \frac{1}{2},
\]

where \( 0 < n < n_{\text{max}} = \nu + \delta, n = 0, 1, 2, \ldots, \nu + \delta \).

The energy spectrum equation (28) is identical to equation (10) of [32] obtained by Akçay for the real PT potential (1) using the conventional approximation scheme [39]. This means that the two forms (real and complex PT-symmetry) of the PT potential possess the same real energy spectrum in the exact p-spin symmetry. On the other hand, the lower-spinor component of the wave functions is found to be

\[
G_{nx} (x) = 2^{\nu+\delta} \tilde{N}_{nx} (p_1(x))^{-\nu} (p_2(x))^{-\delta} \\
\times _2 F_1 \left( -n, -2(\nu + \delta) + \nu; -2\delta + \frac{1}{2}; \frac{p_1(x)}{2} \right),
\]

where

\[
\nu = -\frac{1}{4} \left( 1 - \sqrt{1 - \frac{4(M - E_{nx} + C_{ps})A(A + \alpha)}{M}} \right),
\]

\[
\delta = -\frac{1}{4} \left( 1 + \sqrt{(2\kappa - 1)^2 - \frac{4(M - E_{nx} + C_{ps})B(B - \alpha)}{M}^2} \right),
\]

where \( \tilde{N}_{nx} \) is the normalization constant. The upper-spinor component of the Dirac wave function can be calculated as

\[
F_{nx} (r) = \frac{1}{M - E_{nx} + C_{ps}} \left( \frac{d}{dr} - \frac{\kappa}{r} \right) G_{nx} (r),
\]

where \( E \neq M + C_{ps} \) and with exact p-spin symmetry (\( C_{ps} = 0 \rightarrow S(r) = -V(r) \)), only negative energy states do exist.
Figure 8. The variation of the energy levels as a function $A$ in the presence of p-spin symmetry taking $B = 2$, $M = 5.0$, $C_{ps} = -15$, $\alpha = 0.35$.

Figure 9. The variation of the energy levels as a function $B$ in the presence of p-spin symmetry with parameter values $A = 8$, $M = 5.0$, $C_{ps} = 0$, $\alpha = 0.35$.

The upper component $F_{n\kappa}(r)$ can be obtained as

$$F_{n\kappa}(r) = \frac{2n(n - 2\nu - 2\delta)2^{\nu+\delta}}{(1 - 4\nu)(M - E_{n\kappa} + C_{ps})} \alpha \, e^{-i2\alpha x_0} \cosh_{\nu}(2\alpha x)$$

$$\times 2F_1\left(1 - n, 1 + n - 2(\nu + \delta); -2\nu + \frac{3}{2}; \frac{1}{2} p_1(x)\right)$$

$$\times (p_1(x))^{-\nu}(p_2(x))^{-\delta}$$

$$+ \frac{G_{n\kappa}(x)2^{\nu+\delta}}{(M - E_{n\kappa} + C_{ps})} \left[ \frac{\kappa}{\kappa - 1x_0} + 2\alpha \, e^{-2i\alpha x_0} \cosh_{\nu}(2\alpha x) \right]$$

$$\times \cosh_{\nu}(2\alpha x) \left( \frac{v}{p_1(x) - \frac{\delta}{p_2(x)}}, \nu \neq \frac{1}{4}, \right)$$

(32)

where $E_{n\kappa} \neq M$ when $C_{ps} = 0$. Note that the singularity of the upper-component $F_{n\kappa}(x)$ at $E_{n\kappa} \neq M$ demands us to choose only the negative energy solution for the sake of the normalizability of the two-spinor components of the p-spin wave function.

To generate the binding energy spectrum of equation (28), we take a set of parameter values, $\alpha = 0.35$ fm$^{-1}$, $A = 8$, $B = 2$, $M = 5.0$ fm$^{-1}$ and $C_{ps} = -10.0$ fm$^{-1}$. With the choice of negative energy eigenvalues as the physical solution to equation (32), we present the energy spectrum for the p-spin case in table 2. Further, in figures 6–10, we plot the variation of the p-spin symmetric energy eigenvalues as a function of the parameters $\alpha$, $A$, $B$, $M$ and $C_{ps}$. We find that $E_{n\kappa}$ becomes more negative (more attractive) as the parameter values of $M$, $A$ and $\alpha$ increase. However, it becomes less negative (less attractive) as the parameter values of $C_{ps}$ and $B$ decrease.

3. Discussions

When the $PT$-symmetric $PT$ potential is taken of the form [45]

$$V_{PT}(r) = \frac{\alpha^2}{2M} \left[ \frac{k(k - 1)}{\sinh^2 \alpha(x - ix_0)} - \frac{\lambda(\lambda + 1)}{\cosh^2 \alpha(x - ix_0)} \right],$$

$k > 1$, $\lambda > 1$,

(33)
it remains unchanged under the transformations of $\lambda \to -\lambda - 1$ and $k \to -k + 1$. The potential (33) has the approximate energy spectrum formula in arbitrary $\kappa$-state (in relativistic $\hbar = c = 1$ units)

$$M^2 - E_{\text{re}}^2 + C_s (E_{\text{re}} - M) = -\frac{\alpha^2 \kappa (\kappa + 1)}{3} + 4\alpha^2 \nu^2 \left[ \sqrt{\frac{4(M + E_{\text{re}} - C_s)}{M}} \right]^2,$$

which is identical to equation (42) of [45] found for the real PT potential. Making the replacements $C_s = 0$ and $\kappa (\kappa + 1) = l(l + 1)$, one can readily find out that the Dirac equation and KG equation share the same energy spectrum under the choice of equally mixed radial scalar and vector potentials, i.e., $S(r) = V(r)$ for the PT-symmetric PT potentials as remarked in [54] (see, e.g., equation (24) of [45]). The energy equation of (33) in the Dirac equation under the p-spin symmetry

$$M^2 - E_{\text{re}}^2 + C_{ps} (E_{\text{re}} + M) = -\frac{\alpha^2 \kappa (\kappa - 1)}{3} + 4\alpha^2 \nu^2 \left[ \sqrt{\frac{4(M + E_{\text{re}} + C_{ps})}{M}} \right]^2,$$

which is the same for the solution of the real PT potential in equation (44) of [45]. We observe that the Dirac and KG equations share the same energy spectrum (see equation (36) of [45]) when $S(r) = -V(r), C_{ps} = 0$ and $\kappa (\kappa - 1) = l(l + 1)$ [54].

In the non-relativistic limits

$$\kappa (\kappa + 1) = l(l + 1), C_s = 0, E_{\text{re}} + M \to 2\mu$$

then equation (34) becomes

$$E_{nl} = \frac{\alpha^2 (l + 1 \pm 1)}{2\mu} \left[ -\frac{1}{2} + \frac{1}{4} \sqrt{1 + 8\lambda (\lambda + 1)} \right],$$

where $d_0 = 1/12$.

4. Summary and concluding remarks

In this paper, we have investigated the bound-state solutions of the Dirac equation with PT-symmetric PT potential for any spin–orbit quantum number $\kappa$. By using an approximation scheme to deal with the centrifugal term and making a complex transformation in coordinates, we have obtained the energy eigenvalue equation and the unnormalized upper- and lower-spinor components of the radial wave function expressible in terms of the Jacobi polynomials in view of spin symmetry with any $\kappa$-wave state. It is noted that the complexified PT-symmetric PT potential carries the same real energy eigenvalue solutions as the real PT in the spin-1/2 Dirac and spin-0 KG equations. However, the wave functions of the complexified PT-symmetric PT potential have a different asymptotic behavior than the wave functions of the real PT potential. Furthermore, in obtaining the p-spin symmetric solutions from the spin symmetric ones, we have employed parametric mappings [49–53]. The complexified PT-symmetric potential may have real (complex) energy spectrum when the wave function is normalizable (non-normalizable) in a given range as stated in our recent work [55].

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