ON TOPOLOGICAL LATTICES AND AN APPLICATION TO FIRST SUBMODULES

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Abstract. We introduce the notion of a (strongly) topological lattice \( \mathcal{L} = (L, \wedge, \vee) \) with respect to a subset \( X \subseteq L \); a prototype is the lattice of (two-sided) ideals of a ring \( R \), which is (strongly) topological with respect to the prime spectrum of \( R \). We investigate and characterize (strongly) topological lattices. Given a non-zero left \( R \)-module \( M \), we introduce and investigate the spectrum \( \text{Spec}^f(M) \) of first submodules of \( M \). We topologize \( \text{Spec}^f(M) \) and investigate the algebraic properties of \( R_M \) by passing to the topological properties of the associated space.

1. Introduction

Yassemi [Yas2001] introduced the notion of second (sub)modules of a given non-zero module over a commutative ring. This notion was studied for modules over arbitrary associative rings by Annin [Ann2002], where a second module was called a coprime module. Moreover, the notion of coprime submodules was investigated by Kazemifard et al. [KNR]. In this paper, we dualize the notion of a coprime submodule to present the spectrum \( \text{Spec}^f(M) \) of first submodules of a given non-zero left module \( M \) over an arbitrary associative, not necessarily commutative, ring \( R \) with unity. We topologize this spectrum to obtain a dual Zariski-like topology, study properties of the resulting topological space and investigate the interplay between the properties of that space and the algebraic properties of \( M \) as an \( R \)-module.

To achieve this goal, we begin in the second section with a more general framework of a topological complete lattice \( \mathcal{L} = (L, \wedge, \vee, 0, 1) \) with respect to a proper subset \( X \subseteq L \). We investigate such lattices and characterize them; moreover, we investigate the irreducibility of the closed subsets of \( X \). In Section 3, we apply the results we obtained in Section 2 to the concrete example \( \mathcal{L}(M) \), the complete lattice of \( R \)-submodules of a given non-zero \( R \)-module \( M \), and \( X = \text{Spec}^f(M) \), the spectrum of \( R \)-submodules of \( M \) which are prime

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as $R$-modules. In Section 4, we obtain several algebraic properties of $R M$ by passing to the topological properties of $\text{Spec}^f(M)$.

2. **Topological Lattices**

Throughout, $\mathcal{L} = (L, \land, \lor, 0, 1)$ is a complete lattice, $X \subseteq L \setminus \{1\}$ is a non-empty subset and $\mathcal{P} = (\mathcal{P}(X), \cap, \cup, \emptyset, X)$ is the complete lattice on the power set of $X$. We define an order-reversing map

$$V : L \rightarrow \mathcal{P}(X), \quad a \mapsto V(a) = \{p \in X \mid a \leq p\}.$$ 

It is clear that $V(0) = X$, $V(1) = \emptyset$ and $V(\lor A) = \bigcap_{a \in A} V(a)$ for every $A \subseteq L$. This means that the image of $V$ contains $X$, $\emptyset$ and is closed under arbitrary intersections. If $\text{Im}(V)$ is also closed under finite unions, then the elements of $V(L)$ can be considered the closed sets of a topology on $X$.

**Definition 2.1.** We say that $\mathcal{L}$ is a *topological $X$-lattice* (or $X$-*top*, for short) iff $V(L)$ is closed under finite unions.

The purpose of this section is to characterize $X$-*top lattices*. Notice that the map $V$ represents the lower adjoint map of a Galois connection between $\mathcal{L}$ and $\mathcal{P}$, where the upper adjoint map is

$$I : \mathcal{P}(X) \rightarrow L, \quad A \mapsto \bigwedge A.$$ 

Since $V, I$ are order reversing and $a \leq I(V(a)), A \subseteq V(I(A))$ hold for all $a \in L, A \in \mathcal{P}(X)$, we conclude that $(V, I)$ is a Galois connection [Gra2011, 3.13] and that

$$V = V \circ I \circ V \quad \text{and} \quad I = I \circ V \circ I.$$ 

(1)

The compositions $I \circ V$ and $V \circ I$ are closure operators [Gra2011, Lemma 32] and the closed elements with respect to this Galois connection are

$$\mathcal{C}(L) = \{a \in L \mid a = I(V(a))\} = \{I(A) \mid A \subseteq X\} = \text{Im}(I)$$

and

$$\mathcal{C}(\mathcal{P}(X)) = \{A \in \mathcal{P}(X) \mid A = V(I(A))\} = \{V(a) \mid a \in L\} = \text{Im}(V).$$

Clearly, $V$ is a bijection between $\mathcal{C}(L)$ and $\mathcal{C}(\mathcal{P}(X))$ with inverse $I$.

**A lattice structure on $\mathcal{C}(L)$.** Note that $X \subseteq \mathcal{C}(L)$, because for every element $p \in X$ we have $I(V(p)) = \bigwedge \{[p], 1 \cap X\} = p$. Moreover, $(\mathcal{C}(L), \wedge, \bigwedge X)$ is a complete lower semilattice because if $Y \subseteq \mathcal{C}(L)$, then for each $y \in Y$ we have $y = I(A_y)$ for some subset $A_y \subseteq X$ and it follows that

$$\bigwedge Y = \bigwedge_{y \in Y} \bigwedge y \subseteq A_y = \bigwedge_{y \in A_y} A_y = I(\bigcup_{y \in Y} A_y) \in \mathcal{C}(L).$$

This makes $\mathcal{C}(L)$ a complete lattice by defining a new join for each subset $Y \subseteq \mathcal{C}(L)$ as

$$\overline{\bigvee} Y := IV(\bigvee Y) = \bigwedge\{c \in \mathcal{C}(L) \mid y \leq c \forall y \in Y\}.$$

Notice that this new join $\overline{\bigvee}$ is usually *different* from the original join $\vee$ of $L$. 
Before we characterize $X$-top lattices, we need to recall the following definition (see for example [AL2013, Definition 1.1.]). An element $p$ in a lower semilattice $(L, \wedge)$ is called irreducible iff for all $a, b \in L$ with $p \leq a, b$:

$$a \wedge b \leq p \quad \Rightarrow \quad a \leq p \text{ or } b \leq p. \quad (2)$$

The element $p$ is called strongly irreducible iff Equation (2) holds for all $a, b \in L$.

**Theorem 2.2.** The following statements are equivalent:

(a) $L$ is an $X$-top lattice;

(b) $V : (\mathcal{C}(L), \wedge, \bar{\vee}) \rightarrow (\mathcal{P}(X), \cap, \cup)$ is an anti-homomorphism of lattices;

(c) every element $p \in X$ is strongly irreducible in $(\mathcal{C}(L), \wedge)$;

(d) $(\mathcal{C}(L), \wedge, \bar{\vee})$ is a distributive lattice and every element $p \in X$ is irreducible in $(\mathcal{C}(L), \wedge)$.

**Proof.** (a) $\Rightarrow$ (b) Suppose that $L$ is $X$-top, i.e. $V(L)$ is closed under finite unions. Let $a, b \in \mathcal{C}(L)$. By assumption, $V(a) \cup V(b) = V(c)$ for some $c \in L$. Hence

$$a \wedge b = I(V(a)) \wedge I(V(b)) = I(V(a) \cup V(b)) = I(V(c))$$

and it follows that $V(a \wedge b) = V(I(V(c))) = V(c) = V(a) \cup V(b)$. Moreover, it is clear that $V(a \bar{\vee} b) = V(a) \cap V(b)$ for all $a, b \in \mathcal{C}(L)$.

(b) $\Rightarrow$ (c) Let $p \in X$ and $a, b \in \mathcal{C}(L)$. If $a \wedge b \leq p$, then $V(p) \subseteq V(a \wedge b) = V(a) \cup V(b)$ whence $p \in V(a)$ or $p \in V(b)$, i.e. $a \leq p$ or $b \leq p$.

(c) $\Rightarrow$ (a) Let $V(a)$ and $V(b)$ be two closed sets. By Equation (11), we can write them as $V(a) = V(a')$ and $V(b) = V(b')$ for some $a', b' \in \mathcal{C}(L)$. Let $p \in V(a' \wedge b')$, whence $a' \wedge b' \leq p$. Since $p$ is strongly irreducible in $\mathcal{C}(L)$, $a' \leq p$ or $b' \leq p$, i.e. $p \in V(a')$ or $p \in V(b')$. Thus $V(a' \wedge b') \subseteq V(a) \cup V(b)$. Since $V(a) \cup V(b) = V(a') \cup V(b') \subseteq V(a' \wedge b')$ always holds, the equality follows.

(d) $\Rightarrow$ (c) holds by [AL2013, Lemma 1.20].

(b + c) $\Rightarrow$ (d) Note that $V : \mathcal{C}(L) \rightarrow \mathcal{P}(X)$ is injective and, by (b), the dual lattice $\mathcal{C}(L)^\circ$ is isomorphic to a sublattice of the distributive lattice $\mathcal{P}$, whence $(\mathcal{C}(L), \wedge, \bar{\vee})$ is distributive as well. On the other hand, every strongly irreducible element is in particular irreducible. \qed

**Example 2.3.** Let $R$ be an associative, not necessarily commutative, ring with unity, $X = \text{Spec}(R)$ be the spectrum of prime ideals of $R$ and $L_2(R)$ the lattice of ideals of $R$. Notice that $\text{Im}(I)$ consists of all ideals that are intersections of prime ideals, i.e. the semiprime ideals of $R$ [Wis1991 2.5]. It is clear that every prime ideal $P$ is strongly irreducible in $L_2(R)$; in particular, $P$ is strongly irreducible in $\text{Im}(I)$ whence $\mathcal{L}_2(R)$ is a $\text{Spec}(R)$-top lattice. The topology on $\text{Spec}(R)$ is the ordinary Zariski topology.

**Definition 2.4.** We say that $L$ is a strongly $X$-top lattice (or strongly $X$-top for short) iff every element of $X$ is strongly irreducible in $(L, \wedge)$.

The proof of the following result is similar to that of Theorem 2.2. If all elements $p \in X$ are strongly irreducible in $(L, \wedge)$, then it follows by Theorem 2.2 that $L$ is an $X$-top lattice.
Moreover, for all \(a, b \in L\) we have
\[
p \in V(a \land b) \Rightarrow [a \land b \leq p \Rightarrow a \leq p \text{ or } b \leq p] \Rightarrow p \in V(a) \cup V(b),
\]
i.e. \(V(a \land b) \subseteq V(a) \cup V(b)\). The reverse inclusion is obvious; this means that \(V(a \land b) = V(a) \cup V(b)\) for all \(a, b \in L\). On the other hand, it is clear that \(V(a \lor b) = V(a) \cap V(b)\) for all \(a, b \in L\).

**Proposition 2.5.** The following statements are equivalent:

(a) \(\mathcal{L}\) is a strongly \(X\)-top lattice;

(b) \(V : \mathcal{L} \to \mathcal{P}\) is an anti-homomorphism of lattices.

**Example 2.6.** Let \(R\) be an arbitrary associative ring with unity and \(X = \text{Spec}(R)\). As mentioned in Example [2.3](#), every prime ideal is strongly irreducible in \(\mathcal{L}_2(R)\). In particular, if \(R\) is commutative (or more generally left duo), then the lattice \(\mathcal{L}(R)\) of left ideals of \(R\) is strongly \(X\)-top. However, if \(\mathcal{L}_2(R) \neq \mathcal{L}(R)\), then \(\mathcal{L}(R)\) might not be strongly \(X\)-top. For example, if \(R\) is a prime ring which is not uniform as a left \(R\)-module, then \(\mathcal{L}(R)\) is not strongly \(X\)-top because \(P = 0\) is a prime ideal and there are non-zero left ideals \(A, B\) of \(R\) with \(A \cap B = 0\). An example of such a ring is given by the full \(n \times n\)-matrix ring \(R = M_n(K)\) over a field \(K\) where \(n \geq 2\).

Recall from [Bou1966](#) that for a non-empty topological space \(X\), a non-empty subset \(\mathcal{A} \subseteq X\) is said to be irreducible in \(X\) iff for all proper closed subsets \(A_1, A_2\) of \(X\) we have
\[
\mathcal{A} \subseteq A_1 \cup A_2 \Rightarrow \mathcal{A} \subseteq A_1 \text{ or } \mathcal{A} \subseteq A_2.
\]

A maximal irreducible subset of \(X\) is called an irreducible component and is necessarily closed.

**Proposition 2.7.** Let \(\emptyset \neq \mathcal{A} \subseteq X\).

1. Let \(\mathcal{L}\) be \(X\)-top. If \(I(\mathcal{A})\) is irreducible in \((\mathcal{C}(L), \land)\), then \(\mathcal{A}\) is an irreducible subset of \(X\).

2. Let \(\mathcal{L}\) be strongly \(X\)-top. The following are equivalent:
   (a) \(I(\mathcal{A})\) is irreducible in \((\mathcal{C}(L), \land)\);
   (b) \(\mathcal{A}\) is an irreducible subset of \(X\);
   (c) \(I(\mathcal{A})\) is (strongly) irreducible in \((L, \land)\).

**Proof.** (1) By our assumption, \(X\) becomes a topological space. Suppose that \(\mathcal{A} \subseteq V(a_1) \cup V(a_2)\) for some \(a_1, a_2 \in L\). Set \(\mathcal{A}_i = V(a_i) \cap \mathcal{A}\) for \(i = 1, 2\), so that \(\mathcal{A} = \mathcal{A}_1 \cup \mathcal{A}_2\). Notice that \(I(\mathcal{A}) = I(\mathcal{A}_1) \land I(\mathcal{A}_2)\), whence \(I(\mathcal{A}) = I(\mathcal{A}_i)\) for some \(i = 1, 2\) as \(I(\mathcal{A})\) is assumed to be irreducible in \(\mathcal{C}(L)\), and it follows that
\[
\mathcal{A} \subseteq V(I(\mathcal{A})) = V(I(\mathcal{A}_i)) \subseteq V(I(V(a_i))) = V(a_i).
\]

(2) Suppose that all elements of \(X\) are strongly irreducible in \((L, \land)\).

\(a\) \Rightarrow \(b\) follows by (1).

\(b\) \Rightarrow \(c\) Let \(\mathcal{A}\) be an irreducible subset of \(X\) and assume that \(a_1 \land a_2 \leq I(\mathcal{A})\) for some \(a_1, a_2 \in L\). It follows that
\[
\mathcal{A} \subseteq V(I(\mathcal{A})) \subseteq V(a_1 \land a_2) = V(a_1) \cup V(a_2).
\]
As \( A \) is irreducible, \( A \subseteq V(a_i) \) for some \( i = 1, 2 \), whence \( I(A) \geq I(V(a_i)) \geq a_i \) showing that \( I(A) \) is strongly irreducible in \((L, \land)\).

\((c) \Rightarrow (a)\) is obvious.

Example 2.8. Let \( R \) be a simple ring and \( X = \text{Spec}(R) = \{0\} \). Clearly, \( \mathcal{L}(R) \) is an \( R \)-top lattice. Notice that \( X \) is irreducible since it is a singleton. However, \( I(X) = 0 \) is irreducible in \((\mathcal{L}(R), \cap)\) if and only if \( R \) is uniform as left \( R \)-module if and only if \( \mathcal{L}(R) \) is strongly \( X \)-top. Thus, every simple ring that is not left uniform can be taken as an example to show that the hypothesis on \( \mathcal{L} \) to be strongly \( X \)-top in \((\mathcal{L}(R), \cap)\) cannot be dropped.

Corollary 2.9. If \( \mathcal{L} \) is \( X \)-top and \( A \subseteq X \) is such that \( I(A) \in X \), then \( A \) is irreducible.

The following result will be needed when dealing with first submodules.

Corollary 2.10. Let \( \mathcal{L} \) be \( X \)-top. If \([x, 1] \subseteq X \) for some \( x \in X \), then \([x, 1]\) is a chain. Moreover, if \([x, 1] \subseteq X \) for every \( x \in X \), then every non-empty subset \( A \subseteq X \) with \( I(A) \in X \) is a chain.

\textbf{Proof.} Let \( x \in X \) be such that \([x, 1] \subseteq X \) and \( a, b \in L \) be such that \( x \leq a, b \). By hypothesis, \( a, b \) and \( c := a \land b \) belong to \( X \). Thus, by Theorem 2.2, \( c \) is strongly irreducible in \((\mathcal{C}(L), \land)\), i.e. \( a = c \) or \( b = c \). Hence, \( a \leq b \) or \( b \leq a \). Assume now that \([x, 1] \subseteq X \) for every \( x \in X \) and let \( A \subseteq X \) be a non-empty subset. If \( I(A) \in X \), then \([I(A), 1]\) is a chain and hence \( A \subseteq [I(A), 1] \) is a chain as well. \( \Box \)

Example 2.11. Let \( R \) be an associative, not necessary commutative, ring with unity and \( X = \text{Max}(R) \) the spectrum of maximal ideals of \( R \). The lattice \( \mathcal{L}_2(R) \) of all ideals of \( R \) is clearly strongly \( X \)-top. If \( R \) has the property that every ideal is contained in a unique maximal ideal (e.g. \( R \) is local), then every closed set, in particular every connected component, is a singleton whereas \( X \) is totally disconnected.

Example 2.12. Let \( X = \text{Max}(R) \) be the spectrum of maximal left ideals of \( R \). The lattice \( \mathcal{L}(R) \) of left ideals of \( R \) is not strongly \( X \)-top (cf. [AL2013, Example 2.12]).

3. First Submodules

Throughout, \( R \) is an associative, not necessarily commutative, ring with unity, \( M \) is a non-zero left \( R \)-module, \( \mathcal{L}(M) = (\text{Sub}(M), \land, +, 0, M) \) is the complete lattice of \( R \)-submodules of \( M \) and \( \mathcal{S}(M) \) is the (possibly empty) class of simple submodules of \( M \). Moreover, \( \mathcal{P} = \{2, 3, 5, 7, \cdots \} \) is the set of all prime positive integers.

\textbf{Prime modules.} Recall from [GW2004] the following definition: \( R^M \) is fully faithful iff every non-zero \( R \)-submodule of \( M \) is faithful. Moreover, call \( R^M \) a prime module iff \( M \) is a non-zero fully faithful \( R / \text{ann}_R(M) \)-module (see [GW2004, p.48]). It is easy to see that \( \text{ann}_R(M) \) is a prime ideal if \( M \) is prime module (see [GW2004, Exercise 3I]). For every prime ideal \( P \) of \( R \), the cyclic left \( R \)-module \( M = R/P \) is a left prime module, because if \( N = I/P \) is any non-zero left \( R \)-submodule of \( M \) with \( I \) a left ideal of \( R \) properly containing \( P \), then \( \text{ann}_R(N) I \subseteq P \), i.e. \( \text{ann}_R(N) \subseteq P = \text{ann}_R(M) \). The class of left prime \( R \)-modules is denoted by \( \mathbb{P} \) and is clearly closed under non-zero submodules.
Prime submodules. We call a proper submodule $N$ of $M$ a prime submodule iff $M/N \in \mathbb{P}$. Taking

$$X = \text{Spec}^p(M) = \{ N \in \mathcal{L}(M) \mid N \text{ is a prime submodule of } M \},$$

one defines $M$ to be a top$^p$-module iff $\mathcal{L}(M)$ is $X$-top (cf. [MMS1998]). There are other choices to topologize certain subsets of $\mathcal{L}(M)$. For instance, one could take $X = \text{Spec}^b(M)$, the class of fully prime submodules [Abu2011-a] or $X = \text{Spec}^c(M)$ the class of fully coprime submodules [Abu2011-b]. Other choices are $X = \text{Spec}^c(M)$ the class of coprime submodules, or $X = \text{Spec}^a(M)$ the class of second submodules [Abu]. For other possible choices for $X$, see the (co)primeness notions in the sense of Bican et al. [BJKN80].

First submodules. In this work, we are interested in the set $X$ of those submodules of $M$ which belong to $\mathbb{P}$, i.e. those which are, as modules, prime. We set

$$\text{Spec}^f(M) := \mathbb{P} \cap \mathcal{L}(M)$$

and call its elements first submodules of $M$. We say that $R M$ is firstless iff $\text{Spec}^f(M) = \emptyset$.

The following proposition can be easily proved and includes some characterizations of first submodules that will be used in the sequel; more characterizations can be derived from [Wij2006] 1.22.

Proposition 3.1. The following are equivalent for a non-zero $R$-submodule $0 \neq F \leq_R M$.

1. $F \leq_R M$ is a first submodule;
2. $\text{ann}_R(F) = \text{ann}_R(H)$ for every non-zero submodule $0 \neq H \leq_R M$;
3. $\text{ann}_R(F) = \text{ann}_R(H)$ for every non-zero fully invariant submodule $0 \neq H \leq^i_R M$;
4. every non-zero fully invariant submodule of $F$ is a first submodule;
5. every non-zero submodule of $F$ is a first submodule;
6. For every $r \in R$ and $f \in F$ we have:

$$rRf = 0 \Rightarrow f = 0 \text{ or } rF = 0.$$  (3)

Recall that one calls $R M$ is colocal (or cocyclic [Wis1991]) iff the intersection of all non-zero submodules of $M$ is non-zero.

Remark 3.2. If $0 \neq F \leq_R M$ is simple, then $F$ is indeed a first $R$-submodule (i.e. $\mathcal{S}(M) \subseteq \text{Spec}^f(M)$). So, if $R M$ has an essential socle (called also atomic [HS2010]), then $\text{Spec}^f(M) \neq \emptyset$.

Example 3.3. Let $0 \neq F \leq_R M$. If $\text{ann}_R(F) \in \text{Max}(R)$, then $R F$ is first in $M$ : if $H \leq_R F$ is such that $\text{ann}_R(H)F \neq 0$, then $\text{ann}_R(F) + \text{ann}_R(H) = R$ whence $H = (\text{ann}_R(F) + \text{ann}_R(H))H = 0$. It follows that if $R$ is a simple ring, then every non-zero $R$-submodule of $M$ is first. In particular, every non-zero subspace of a left vector space over a division ring is first.

Examples 3.4. (1) If $0 \neq F \leq_R M$ has no non-trivial fully invariant $R$-submodules, then $F$ is a first submodule of $M$. For instance, $\mathbb{Q} \leq_{\mathbb{Z}} \mathbb{R}$ is a first submodule since $\mathbb{Q}$ has no non-trivial fully invariant $\mathbb{Z}$-submodules.
(2) A non-zero semisimple submodule of \( M \) need not be first. In case \( R \) is commutative, a semisimple \( R \)-submodule of \( M \) is first if and only if it is non-zero and homogeneous semisimple.

(3) Consider the \( \mathbb{Z} \)-module \( M = \mathbb{Z} \oplus \mathbb{Q} \oplus \mathbb{R} \) and \( F = \mathbb{Z} \oplus \mathbb{Q} \). Every fully invariant \( \mathbb{Z} \)-submodule of \( F \) is of the form \( n\mathbb{Z} \oplus \mathbb{Q} \) for some \( n \in \mathbb{N} \) and indeed \( \text{ann}_\mathbb{Z}(n\mathbb{Z} \oplus \mathbb{Q}) = (0) = \text{ann}_\mathbb{Z}(\mathbb{Z} \oplus \mathbb{Q}) \). It follows that \( F \) is first in \( M \).

(4) Let \( M := \bigoplus_{n \in \mathbb{N}} \mathbb{Z}/n\mathbb{Z} \). The \( \mathbb{Z} \)-submodule \( F := \bigoplus_{n \in A} \mathbb{Z}/n\mathbb{Z} \), where \( A \subseteq P \) is any infinite subset, is not a first submodule since for any \( p \in A \) we have \( p\mathbb{Z} = \text{ann}_\mathbb{Z}(\mathbb{Z}/p\mathbb{Z}) \neq 0 = \text{ann}_\mathbb{Z}(F) \).

(5) The Prüfer \( p \)-group

\[
\mathbb{Z}_{p^\infty} := \{ \frac{n}{p^k} + \mathbb{Z} \cap \mathbb{Z} | n \in \mathbb{Z} \text{ and } k \in \mathbb{N} \}
\]

is not first in \( \mathbb{Q}/\mathbb{Z} \): if \( H \leq \mathbb{Z} \mathbb{Z}_{p^\infty} \), then \( H = \mathbb{Z}\{ \frac{1}{p^k} + \mathbb{Z} \} \) for some \( k \in \mathbb{N} \) (e.g. [Wis1991, 17.13]) whence \( \text{ann}_R(H) \neq 0 = \text{ann}_R(\mathbb{Z}_{p^\infty}) \).

Following [Lam1999, p. 86], we call a (prime) ideal of \( R \) an associated prime of \( M \) if \( p = \text{ann}_R(N) \) for some \( N \in \text{Spec}^p(M) \); the class of associated primes of \( M \) is denoted by \( \text{Ass}(R \mathcal{M}) \). If \( R \) is commutative, then \( p \in \text{Ass}(R \mathcal{M}) \) if and only if \( p \) is prime and \( p = (0 :_R m) \) for some \( 0 \neq m \in M \) (e.g. [Lam1999, Lemma 3.56]).

**Example 3.5.** Let \( R \) be a commutative ring. If \( p \) is an associated prime of \( M \), then \( R/p \hookrightarrow M \) is a first \( R \)-submodule. Notice that we might not have such an embedding if \( R \) is non-commutative (e.g. [Ann2002, Fact 36]).

**Remark 3.6.** If \( F \in \text{Spec}^f(M) \), then \( \text{ann}_R(F) \) is a prime ideal: let \( I, J \in \mathcal{L}_2(R) \) be such that \( IJ \subseteq \text{ann}_R(F) \) and suppose that \( J \not\subseteq \text{ann}_R(F) \), i.e. \( K := JF \neq 0 \). Since \( RF \) is first in \( M \) and \( IK = I(JF) = (IJ)F = 0 \), we conclude that \( IF = 0 \), i.e. \( I \subseteq \text{ann}_R(F) \). Notice that the converse is not true: for example, \( \text{ann}_\mathbb{Z}(\mathbb{Z}/8\mathbb{Z}) = (0) \) is a prime ideal of \( \mathbb{Z} \); however, \( \mathbb{Z}/8\mathbb{Z} \) is not a first \( \mathbb{Z} \)-submodule of \( \mathbb{Z}/8\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z} \) since \( \text{ann}_\mathbb{Z}(0 \oplus \mathbb{Z}/8\mathbb{Z}) = 8\mathbb{Z} \neq (0) \).

4. The Topological Structure of \( \text{Spec}^f(M) \)

Throughout this section, we fix the general setting of Section 3. In particular, \( M \) is a non-zero left \( R \)-module over the associative unital ring \( R \) and \( \mathcal{P} \) is the class of prime \( R \)-module. An \( R \)-submodule \( N \leq_R M \) is said to be (strongly) hollow if \( N \) is (strongly) irreducible in \( \mathcal{L}(M)^\circ = (\text{Sub}(R \mathcal{M}), +, \cap) \). The class of strongly hollow submodules of \( M \) is denoted by \( \mathcal{SH}(M) \). In this section, we give some applications of the results in Section 2 to the dual lattice \( \mathcal{L}(M)^\circ \).

**Top-modules.** Since \( \text{Sub}(R \mathcal{N}) \subseteq \text{Sub}(R \mathcal{M}) \) for every submodule \( N \) of \( M \), we have \( \text{Spec}^f(N) \subseteq \text{Spec}^f(M) \). Hence, in order to use the map \( V \) from the second section, we will use the dual lattice \( \mathcal{L}(M)^\circ \) of \( \mathcal{L}(M) \) and \( X = \text{Spec}^f(M) \). In this case, we have the order-preserving map

\[
V : \text{Sub}(R \mathcal{M}) \longrightarrow \mathcal{P}(X), \quad N \mapsto V(N) = \{ P \in \mathcal{P} | P \subseteq N \}.
\]
The map $V$ forms a Galois connection with the map

$$I : \mathcal{P}(X) \rightarrow \text{Sub}(R M), \quad A \mapsto I(A) = \sum_{P \in A} P.$$ 

As before, we have $V = V \circ I \circ V$ and $I = I \circ V \circ I$. Denote the image of $V$ by $\xi f(M)$. From Section 2, we know that $\xi f(M)$ contains $X, \emptyset$ and is closed under intersections; note that because of considering the dual lattice of $L(M)$ one has

$$\bigcap_{\lambda \in \Lambda} V(N_\lambda) = V \left( \bigcap_{\lambda \in \Lambda} N_\lambda \right).$$

The set $\xi f(M)$ can be described as

$$\xi f(M) = \{ V(I(A)) | A \subseteq \text{Spec}f(M) \}$$

and depends only on those submodules that are of the form $I(A)$ for some subset $A \subseteq \text{Spec}f(M)$. The image of $I$ is

$$\mathcal{I}(M) := \mathcal{C}(L(M)^\circ) = \{ I(A) | A \subseteq \text{Spec}f(M) \}$$

which is the set of closed elements relative to the Galois connection $(V, I)$ and forms an upper subsemilattice $(\mathcal{I}(M), +)$ of $L(M)^\circ$. Note that $\text{Spec}f(M) = \mathbb{P} \cap \text{Sub}(R M) \subseteq \mathcal{I}(M)$.

A lattice structure on $\mathcal{I}(M)$. The upper semilattice of closed elements $(\mathcal{I}(M), +)$ is complete, whence it has a greatest element (which we call the coradical of $M$):

$$\text{Corad}f(M) = I(\text{Spec}f(M)) = \sum_{P \in \text{Spec}f(M)} P.$$

This allows defining a new meet on $\mathcal{I}(M)$ as follows: consider a family $\{C_\lambda\}_{\lambda \in \Lambda}$, where $C_\lambda = I(A_\lambda)$ and $A_\lambda \subseteq \text{Spec}f(M)$ for each $\lambda \in \Lambda$, and define

$$\bigwedge_{\lambda \in \Lambda} C_\lambda = IV(\bigcap_{\lambda \in \Lambda} C_\lambda) = I(\bigcap_{\lambda \in \Lambda} V(C_\lambda))$$

$$= \sum_{P \in \text{Spec}f(M)} \{ I(A) \mid I(A) \leq C_\lambda \ \forall \ \lambda \in \Lambda \}$$

$$= \sum_{F \in \text{Spec}f(M)} \{ F \leq \bigcap_{\lambda \in \Lambda} C_\lambda \}.$$

Notice that this new meet $\bigwedge$ is usually different from the original meet $\cap$.

**Definition 4.1.** We say that $M$ is a $\text{top} f$-module iff $L(M)^\circ$ is $\text{Spec}f(M)$-top, i.e. iff $\xi f(M)$ is closed under finite unions; $\text{strongly top} f$-module iff $L(M)^\circ$ is strongly $\text{Spec}f(M)$-top, i.e. iff every first submodule of $M$ is strongly hollow.

From Theorem 2.2 and Corollary 2.10 we get

**Theorem 4.2.** The following statements are equivalent:
(a) \(M\) is a top\(^f\)-module;
(b) \(V : (\mathcal{I}(M), \wedge, +) \to (\xi^f(M), \cap, \cup)\) is a lattice isomorphism;
(c) every first submodule of \(M\) is strongly hollow in \(\text{Corad}^f(M)\);
(d) \((\mathcal{I}(M), \wedge, +)\) is a distributive lattice and every first submodule of \(M\) is a hollow (uniserial) module.

**Proof.** The equivalence follows from Theorem 2.2. Every \(R\)-submodule of \(P \in \text{Spec}^f(M)\) is also a prime module, hence \([P, 0] \subseteq \text{Spec}^f(M)\) in \(\mathcal{L}(M)^\circ\). Thus, Corollary 2.10 applies and proves that every \(P \in \text{Spec}^f(M)\) is uniserial. □

**Lemma 4.3.** If \(\text{Soc}(R M) \neq 0\), then the following are equivalent:
(a) All isomorphic simple submodules of \(M\) are equal.
(b) \(\text{Soc}(M)\) is a direct sum of non-isomorphic simple modules;
(c) \(\text{Soc}(M)\) is distributive;

**Proof.** (a) ⇒ (b) this is clear.
(b) ⇔ (c) By [Ste2004] Proposition 1.3, \(\text{Soc}(M) = \bigoplus_{\lambda \in \Lambda} E\lambda\) is distributive if and only if \(E\alpha\) and \(E\beta\) are unrelated for all \(\alpha \neq \beta\) in \(\Lambda\); the later means for simple modules that \(\text{Hom}_R(E\alpha, E\beta) = 0\).
(c) ⇒ (a) By [Ste2004] Proposition 1.2, if \(\text{Soc}(M) = \bigoplus_{\lambda \in \Lambda} E\lambda\) \((E\lambda\) is simple for each \(\lambda \in \Lambda\) \) and \(E\alpha\) is unrelated to \(E\beta\) for all \(\alpha \neq \beta\) in \(\Lambda\), then for every submodule \(X \subseteq \bigoplus_{\lambda \in \Lambda} E\lambda\) one has \(X = \bigoplus_{\lambda \in \Lambda} (X \cap E\lambda)\). In particular, if \(X\) is simple, then \(X = E\lambda\) for some \(\lambda \in \Lambda\). □

**Corollary 4.4.** If \(RM\) is a top\(^f\)-module, then \(\text{Soc}(M)\) is a (direct) sum of non-isomorphic simple modules.

**Proof.** This follows from the fact that \(\mathcal{L}(\text{Soc}(M)) = (\text{Sub}(\text{Soc}(M), \cap, +))\) is a sublattice of the distributive lattice \((\mathcal{I}(M), \wedge, +)\), whence is also distributive. This is equivalent, by Lemma 4.3, to the stated property for \(\text{Soc}(M)\). □

**Remark 4.5.** Recall from [Abu2011-b] that \(RM\) has the min-property iff for every simple \(R\)-submodule \(H \leq R M\) we have \(H \nsubseteq H\), where \(H = \sum_{K \in \text{Soc}(M) \setminus \{H\}} K\). By Lemma 4.3 and [Smi2011] Theorem 2.3, \(\text{Soc}(M)\) is distributive if and only if \(RM\) has the min-property.

**Notation.** We set \(\text{Sub}_{\cap}(M) := \{(0 : M I) \mid I \in \mathcal{L}_2(R)\}\), \(\mathcal{X}(L) := \text{Spec}^f(M) \setminus V(L)\) and
\[
\xi^f(M) := \{V(L) \mid L \in \text{Sub}_{\cap}(R M)\}; \quad \xi^f_c(M) := \{V(L) \mid L \in \text{Sub}_{\cap}(R M)\};
\]
\[
\tau^f(M) := \{\mathcal{X}(L) \mid L \in \text{Sub}(R M)\}; \quad \tau^f_c(M) := \{\mathcal{X}(L) \mid L \in \text{Sub}_{\cap}(R M)\};
\]

**Remark 4.6.** Let \(M\) be a strongly top\(^f\)-module.
(a) \(M\) is a top\(^f\)-module: this follows directly from observation that \(\text{Spec}^f(M) \subseteq \mathcal{SH}(M)\) if and only if \(V(L_1) \cup V(L_2) = V(L_1 + L_2)\) for every pair of submodules \(L_1, L_2 \leq R M\).
(b) \(\text{Spec}^f(M)\) has a basis of open sets given by
\[
\{\mathcal{X}^f(H) \mid H \leq R M\text{ is finitely generated}\}\]
Theorem 4.7. \((\text{Spec}^f(M), \tau^f_\ast (M))\) is a topological space.

Proof. It is obvious that \(V(0) = \emptyset, V(M) = \text{Spec}^f(M)\) and that \(\bigcap_{L \in L_\lambda} V(L_\lambda) = V(\bigcap_{L \in L_\lambda} L_\lambda)\) for every subset \(\{L_\lambda\}_A \subseteq \text{Sub}_c(M)\). We show now that for all ideals \(I, \bar{I}\) or \(R\) we have

\[
V((0 :_M I)) \cup V((0 :_M \bar{I})) = V((0 :_M I) + (0 :_M \bar{I})) = V((0 :_M I \cap \bar{I})) = V((0 :_M I \bar{I})). \tag{4}
\]

Indeed, the following inclusions are obvious

\[
V((0 :_M I)) \cup V((0 :_M \bar{I})) \subseteq V((0 :_M I) + (0 :_M \bar{I})) \subseteq V((0 :_M I \cap \bar{I})) \subseteq V((0 :_M I \bar{I})). \tag{5}
\]

On the other hand, let \(F \in V((0 :_M I \bar{I}))\) and suppose that \(F \not\subseteq (0 :_M \bar{I})\). Since \(I(\bar{I}F) = (\bar{I}I)F = 0\) and \(\bar{I}F \neq 0\), we conclude that \(IF = 0\) (recall that \(F\) is a first submodule of \(M\)), i.e. \(F \subseteq (0 :_M I)\). Consequently, \(F \in V((0 :_M I)) \cup V((0 :_M I \bar{I})). \)

Example 4.8. For every non-empty subset \(A \subseteq P\), the \(\mathbb{Z}\)-module \(M := \bigoplus_{p \in A} \mathbb{Z}/p\mathbb{Z}\) (with no repetition) is a top\(^f\)-module: it can be easily seen that \(\text{Spec}^f(M) = \{\mathbb{Z}/p\mathbb{Z} | p \in A\}\) and that \(\mathcal{E}^f(M)\) is closed under finite unions.

Example 4.9. Let \(R = \mathbb{Z}\). The prime \(\mathbb{Z}\)-modules are the torsionfree modules and the Abelian \(p\)-groups, for \(p\) a prime number. If \(M\) is a torsion abelian group, then the first submodules are the \(p\)-subgroups of \(M\). Since \(M = \bigoplus_{p \text{ prime number}} T_p(M)\), where \(T_p(M)\) is the \(p\)-torsion part of \(M\), every submodule of \(M\) is a sum of \(p\)-subgroups, i.e. the lattice \(\text{Im}(I)\) is equal to the whole lattice \(\mathcal{L}(M)\) of subgroups of \(M\). If \(M\) is a top\(^f\)-module, then \(T_p(M)\) has to be uniserial for each \(p\) by Theorem 4.2. On the other hand, if the \(p\)-torsion parts \(T_p(M)\) of \(M\) are uniserial, then for every \(p\)-subgroup \(N\) of \(M\) contained in a sum of subgroups \(H + K\) one has \(N \subseteq T_p(H + K) = T_p(H) + T_p(K)\). Since \(T_p(M)\) is uniserial, \(T_p(H + K) = T_p(H)\) or \(T_p(H + K) = T_p(K)\). Hence \(N \subseteq H\) or \(N \subseteq K\). This shows that a torsion abelian group is a top\(^f\)-module if and only if all its \(p\)-torsion parts are uniserial. For instance, \(\mathbb{Q}/\mathbb{Z}\) is top\(^f\)-module.

Example 4.10. Over a simple ring \(R\), every non-zero left \(R\)-module is prime. Theorem 4.2 shows that the (strongly) top\(^f\)-modules over a simple ring are precisely the non-zero uniserial modules.

Remarks 4.11. Let \(M\) be a top\(^f\)-module, \(H\) a non-zero submodule of \(M\) and set \(\mathcal{X}^f(H) = \text{Spec}^f(M) \setminus V(H)\).

(a) \(\text{Spec}^f(M)\) is a \(T_0\) (Kolmogorov) space.
(b) The closure of any subset \(\mathcal{A} \subseteq \text{Spec}^f(M)\) is \(\bar{\mathcal{A}} = V(I(\mathcal{A}))\).
(c) \(\mathcal{X}(H) = \emptyset\) if and only if \(\text{Corad}^f(M) \subseteq H\).
(d) \(\mathcal{X}(H) = \emptyset\) if and only if \(\text{Corad}^f(M) \subseteq H\).
(e) If \(R_M\) has essential socle, then \(\text{Spec}^f(H) = \emptyset\) if and only if \(H = 0\).
(f) \(\text{Spec}^f(H)\) is a subspace of \(\text{Spec}^f(M)\).
(g) If \(M \simeq N\), then \(\text{Spec}^f(M) \approx \text{Spec}^f(N)\) are homeomorphic and \(\text{Corad}^f(M) \simeq \text{Corad}^f(N)\).
Recall (e.g. [Tug2004, A-TF2007]) that $M$ is said to be a multiplication (comultiplication) module iff every $R$-submodule of $M$ is of the form $IM$ (or $IM$) for some ideal $I$ of $R$, or equivalently iff for every $R$-submodule $H \leq_R M$ we have $H = (H :_R M)M$ ($L = (0 :_M (0 :_R L))$).

**Proposition 4.12.** Let $0 \neq F \leq_R M$.

(a) If $R F$ is comultiplication, then $F$ is first in $M$ if and only if $R F$ is simple.

(b) If $R F$ is multiplication, then $F$ is first in $M$ if and only if $\text{ann}_R(F)$ is a prime ideal.

**Proof.** (a) If $R F$ is simple, then $F$ is first in $M$ by Remark 3.2. On the other hand, let $F$ be first in $M, 0 \neq H \leq_R F$ and consider $I := \text{ann}_R(H)$. Since $F$ is first in $M$, we have $I = \text{ann}_R(F)$ and so $H = (0 :_F (0 :_R H)) = (0 :_F (0 :_R F)) = F$, i.e. $R F$ is simple.

(b) If $F$ is first in $M$, then $\text{ann}_R(F)$ is a prime ideal by Remark 3.6. On the other hand, assume that $\text{ann}_R(F) \in \text{Spec}(R)$. Let $0 \neq H \leq_R F$ and consider $I := \text{ann}_R(H)$. Since $R F$ is multiplication, $H = JF$ for some $J \in \mathcal{L}_2(R)$. Notice that $I J \subseteq \text{ann}_R(F)$, whence $IJ = 0$ since $\text{ann}_R(F)$ is a prime ideal and $J \not\subseteq \text{ann}_R(F)$. Consequently, $R F$ is first. □

**Remark 4.13.** Let $R$ be zero-dimensional (i.e. every prime ideal of $R$ is maximal). It follows by Example 3.3 and Remark 3.6 that

$$
\text{Spec}^f(M) = \{F \leq_R M \mid \text{ann}_R(F) \text{ is prime ideal}\}.
$$

Examples of zero-dimensional rings include biregular rings [Wis1991, 3.18] and left (right) perfect rings.

**Definition 4.14.** Let $0 \neq H \leq_R M$. A maximal element of $V(H)$, if any, is said to be maximal under $H$. A maximal element of $\text{Spec}^f(M)$ is said to be a maximal first submodule of $M$.

**Lemma 4.15.** Let $R M$ have an essential socle and

1. $R$ is zero-dimensional; or
2. every submodule of $R M$ is multiplication.

For every $0 \neq H \leq_R M$, there exists $F \in \text{Spec}^f(M)$ which is maximal under $H$.

**Proof.** Let $0 \neq H \leq_R M$. Since $\text{Soc}(M) \leq_R M$ is essential, $\emptyset \neq S(H) \subseteq V(H)$. Let

$$
F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq F_{n+1} \subseteq \cdots
$$

be an ascending chain in $V(H)$ and set $\overline{F} := \bigcup_{i=1}^{\infty} F_i$. Then we have a descending chain of prime ideals

$$
(0 :_R F_1) \supseteq (0 :_R F_2) \supseteq \cdots \supseteq (0 :_R F_n) \supseteq (0 :_R F_{n+1}) \supseteq \cdots
$$

and it follows that $p := (0 :_R \overline{F}) = \bigcap_{i=1}^{\infty} (0 :_R F_i)$ is a prime ideal. If $R$ is zero-dimensional, then $\overline{F} \in \text{Spec}^f(M)$ by Remark 4.13. On the other hand, if $R \overline{F}$ is multiplication, then $\overline{F} \in V(H)$ by Proposition 4.12(b). In either case, it follows by Zorn’s Lemma that $V(H)$ has a maximal element. □
Example 4.16. Recall from [JST2012, p. 128] that $RM$ is \textit{completely cyclic} (or \textit{fully cyclic} [BW2000]) iff every $R$-submodule of $RM$ is cyclic. If $RM$ is a uniserial module and $R$ is a left (or right) Artinian left duo ring, then $RM$ is completely cyclic by [JST2012, Lemma 13.9], whence every $R$-submodule of $RM$ is multiplication by [Tug2004]; moreover, since $RM$ is cyclic (finitely generated) and $R$ is left Artinian, $RM$ is also Artinian whence $\text{Soc}(M) \leq RM$ is essential.

4.17. Let $M$ be a top$^f$-module and consider $\text{Spec}^f(M)$ with the associated topology. Since the lattice $I(M)$ and the lattice $\xi^f(M)$ of closed subsets are isomorphic, some topological conditions on $\text{Spec}^f(M)$ translate to module theoretical conditions on $M$. Recall from [Bou1966, Bou1998] that a topological space is called \textit{Noetherian} (Artinian) iff every descending (ascending) chain of closed sets is stationary. Therefore, $\text{Spec}^f(M)$ is Noetherian (Artinian) if and only if $M$ satisfies the descending (ascending) chain condition on submodules of the form $I(A)$ for subsets $A \subseteq \text{Spec}^f(M)$. In particular, if $M$ is Noetherian (Artinian), then $\text{Spec}^f(M)$ is Artinian (Noetherian).

Lemma 4.18. Let $M$ be a top$^f$-module, $A \subseteq \text{Spec}^f(M)$ an irreducible subset and $H$ a non-zero submodule of $I(A)$. If $\text{Spec}^f(H) \neq \emptyset$, then $\text{ann}_R(H) = \text{ann}_R(I(A))$.

Proof. Let $P \in \text{Spec}^f(H)$ be a cyclic first submodule. Setting

$$A_0 = \{Q \in A \mid Q \cap P = 0\},$$

we have

$$A \subseteq V(I(A_0)) \cup V(I(A \setminus A_0)).$$

By the irreducibility of $A$ we have that $A$ is contained in one of the two closed sets. Suppose that $A \subseteq V(I(A_0))$, whence $P \subseteq I(A_0)$. As $P$ is cyclic, there is a finite set $\{Q_1, \cdots, Q_n\} \subseteq A_0$ with $P \subseteq Q_1 + \cdots + Q_n$. Since $M$ is a top$^f$-module, the lattice of submodules of the form $I(A)$ is distributive (by Theorem 4.2). Hence

$$P = P \cap (Q_1 + \cdots + Q_n) = P \cap Q_1 + \cdots + P \cap Q_n = 0,$$

since $Q_i \in A_0$ for all $i = 1, \cdots, n$. This is a contradiction to $P$ being non-zero. Hence, $A \subseteq V(I(A \setminus A_0))$ and $P \subseteq I(A) = \sum \{Q \in A \mid Q \cap P \neq 0\}$. This shows that

$$\text{ann}_R(P) \supseteq \text{ann}_R(I(A)) = \bigcap_{Q \cap P \neq 0} \text{ann}_R(Q) = \bigcap_{Q \cap P \neq 0} \text{ann}_R(Q \cap P) = \text{ann}_R(P).$$

Thus $\text{ann}_R(P) = \text{ann}_R(H) = \text{ann}_R(I(A))$. \hfill $\square$

Remark 4.19. Note that if $I(A)$ is a distributive module for a non-empty subset $A$, then $\text{Spec}^f(H) = \emptyset$ if and only if $H = 0$ for all submodules $H \in I(A)$, because if $H$ is non-zero and $C$ is a non-zero cyclic submodule of $H$, then $C \subseteq I(A)$ implies that there are finitely many first submodules $Q_1, \ldots, Q_n$ such that $C \subseteq Q_1 + \cdots + Q_n$. By distributivity, $C = C \cap Q_1 + \cdots + C \cap Q_n$ and since $C \neq 0$, there must be some $i = 1, \cdots, n$ such that $C \cap Q_i \neq 0$. Thus $C \cap Q_i \in \text{Spec}^f(H)$.

Proposition 4.20. Let $M$ be a top$^f$-module and let $\emptyset \neq A \subseteq \text{Spec}^f(M)$. 

(1) If $I(\mathcal{A})$ is a hollow module, then $\mathcal{A}$ is irreducible. The converse holds if $M$ is a strongly top$^t$-module.

(2) The following are equivalent:
   (a) $\mathcal{A}$ is irreducible and $\text{Spec}^t(H) \neq \emptyset$ for any $0 \neq H \subseteq I(\mathcal{A})$.
   (b) $\mathcal{A}$ is irreducible and $I(\mathcal{A})$ is distributive.
   (c) $I(\mathcal{A})$ is a first submodule;
   (d) $I(\mathcal{A})$ is uniserial;
   (e) $\mathcal{A}$ is a chain.

Proof. (1) follows from Proposition 2.7 applied to the dual lattice $\mathcal{L}(M)^\circ$.

(2) (a) $\Rightarrow$ (c). The hypotheses of Lemma 4.18 are fulfilled for any non-zero submodule of $I(\mathcal{A})$. Hence, all non-zero submodules have the same annihilator, which shows that $I(\mathcal{A})$ is a prime module.

(c) $\Rightarrow$ (a) By Corollary 2.9, $\mathcal{A}$ is irreducible. Clearly any non-zero submodule of a prime module is first; so, if $I(\mathcal{A})$ is a first submodule, then any non-zero submodule of it is first as well.

(c) $\Rightarrow$ (e) follows by Corollary 2.10.

(e) $\Rightarrow$ (c) Assume now that $\mathcal{A}$ is a chain; in particular, $I(\mathcal{A}) = \bigcup_{P \in \mathcal{A}} P$. Since for all $Q, P \in \mathcal{A}$ either $Q \subseteq P$ or $P \subseteq Q$ and since $P$ and $Q$ are prime modules, $\text{ann}_R(P) = \text{ann}_R(Q)$. Every cyclic submodule $U = Rm$ of $I(\mathcal{A})$ lies in one of the members of $\mathcal{A}$ and thus has the same annihilator, i.e. $I(\mathcal{A})$ is a prime module or equivalently $I(\mathcal{A}) \in \text{Spec}^t(M)$.

(d) $\iff$ (e) clear.

(a + d) $\Rightarrow$ (b) is clear because a uniserial module is distributive.

(b) $\Rightarrow$ (a) holds by Remark 4.19. □

Remark 4.21. Let $M$ be a top$^t$-module and $\emptyset \neq \mathcal{A} \subseteq \mathcal{S}(M)$. Every non-zero submodule of $I(\mathcal{A}) \subseteq \text{Soc}(M)$ contains a simple (hence first) submodule and so we get as an immediate consequence from Proposition 4.20 that the following statements are equivalent:

(a) $\mathcal{A}$ is irreducible;
(b) $I(\mathcal{A})$ is a first submodule of $M$;
(c) $\mathcal{A} = \{K\}$ as singleton.

Example 4.22. Let $M$ be a top$^t$-module. It follows by Remark 4.21 that $\mathcal{S}(M) \subseteq \text{Spec}^t(M)$ is irreducible if and only if $\text{Soc}(M)$ is a first submodule of $M$ if and only if $M$ contains a single simple $R$-submodule.

Remark 4.23. Let $M$ be a top$^t$-module and $\mathcal{A} \subseteq \text{Spec}^t(M)$ be such that $I(\mathcal{A})$ is a first submodule of $M$. By Theorem 4.2, $I(\mathcal{A})$ is a hollow module (in fact $I(\mathcal{A})$ is moreover a uniserial module). It follows then from Proposition 4.20 (2) that $\mathcal{A}$ is irreducible.

Definition 4.24. We say a top$^t$-module is consistent iff for every $\mathcal{A} \subseteq \text{Spec}^t(M)$ we have: $I(\mathcal{A}) \in \text{Spec}^t(M)$ if (and only if) $\mathcal{A}$ is irreducible.

Remark 4.25. From Proposition 4.20 and Remark 4.19 we see that the following statements are equivalent for a top$^t$-module $M$:
(a) $M$ is a consistent;
(b) $\text{Spec}^f(H) \neq \emptyset$ for every non-zero submodule $H \subseteq I(\mathcal{A})$ and every irreducible subset $\mathcal{A} \subseteq \text{Spec}^f(M);
(c) I(\mathcal{A})$ is distributive for every irreducible subset $\mathcal{A} \subseteq \text{Spec}^f(M)$.

For property (c) we use the obvious fact that uniserial modules are distributive.

**Example 4.26.** Every top$^f$-module with essential socle is consistent. Moreover, every top$^f$-module $M$, for which Corad$^f(M)$ is distributive, is consistent.

**Proposition 4.27.** Let $RM$ be a consistent top$^f$-module with $\text{Spec}^f(M) \neq \emptyset$. The following are equivalent for $\mathcal{A} \subseteq \text{Spec}^f(M)$:

(a) $\mathcal{A}$ is irreducible;
(b) $I(\mathcal{A})$ is a first submodule of $M$;
(c) $0 \neq I(\mathcal{A})$ is a hollow module;
(d) $0 \neq I(\mathcal{A})$ is uniserial;
(e) $\emptyset \neq \mathcal{A}$ is a chain.

**Theorem 4.28.** Let $RM$ be a consistent top$^f$-module with $\text{Spec}^f(M) \neq \emptyset$. The following are equivalent:

(a) $\text{Spec}^f(M)$ is irreducible;
(b) Corad$^f(M)$ is a first submodule of $M$;
(c) $0 \neq \text{Corad}^f(M)$ is hollow (uniserial);
(d) $\text{Spec}^f(M)$ is a chain.

**Notation.** Set

$$\text{Max}(\text{Spec}^f(M)) := \{K \in \text{Spec}^f(M) \mid K \text{ is a maximal first submodule of } M\}. \quad (7)$$

**Proposition 4.29.** Let $RM$ be a consistent top$^f$-module.

(a) We have a bijection

$$\text{Spec}^f(M) \xymatrix{\to \ar@/_1pc/@{<->}^{(8)}\ar@/^-1pc/@{<->} \{\mathcal{A} \mid \mathcal{A} \subseteq \text{Spec}^f(M) \text{ is an irreducible closed subset}\}.}$$

(b) The bijection $(8)$ restricts to a bijection

$$\text{Max}(\text{Spec}^f(M)) \xymatrix{\to \ar@/_1pc/@{<->}^{(8)}\ar@/^-1pc/@{<->} \{\mathcal{A} \mid \mathcal{A} \subseteq \text{Spec}^f(M) \text{ is an irreducible component}\}.}$$

**Proof.** (a) Let $K \in \text{Spec}^f(M)$. Notice that $K = I(V(K))$ and so the closed set $V(K) = \{K\}$ is irreducible (see Proposition 4.27). On the other hand, let $\mathcal{A} \subseteq \text{Spec}^f(M)$ be a closed irreducible subset. Notice that $I(\mathcal{A})$ is first in $M$ by Proposition 4.27 and that $\mathcal{A} = \mathcal{A} = V(I(\mathcal{A}))$. Clearly, the maps $V$ and $I$ are bijective and the result follows.

(b) This follows from (a), the definitions and the fact that $V$ is order preserving. \qed

**Corollary 4.30.** If $RM$ is a consistent top$^f$-module, then $\text{Spec}^f(M)$ is a sober space.
Proof. Let $\mathcal{A} \subseteq \text{Spec}^f(M)$ be an irreducible closed subset. By Proposition 4.29 (1), $\mathcal{A} = V(K)$ for some $K \in \text{Spec}^f(M)$. It follows that $\mathcal{A} = \overline{\mathcal{A}} = V(I(\mathcal{A})) = V(K) = \{K\}$, i.e. $K$ is a generic point for $\mathcal{A}$. If $H$ is a generic point of $\mathcal{A}$, then $V(K) = V(H)$ whence $K = H$. \hfill \Box

Theorem 4.31. Let $RM$ be a top$^f$-module with essential socle.

(a) If $\mathcal{S}(M)$ is finite, then $\text{Spec}^f(M)$ is compact.

(b) If $\mathcal{S}(M)$ is countable, then $\text{Spec}^f(M)$ is countably compact.

Proof. We prove only (a); the proof of (b) is similar. Assume that $\mathcal{S}(M) = \{N_1, \cdots, N_k\}$. Let $\{V(H_\alpha)\}_{\alpha \in I}$ be an arbitrary collections of closed subsets of $\text{Spec}^f(M)$ with $\bigcap_{\alpha \in I} V(H_\alpha) = \emptyset$. Since $\mathcal{S}(M) \subseteq \text{Spec}^f(M)$, we can pick for each $i = 1, \cdots, k$ some $\alpha_i \in I$ such that $N_i \not\subseteq H_{\alpha_i}$. If $\tilde{H} := \bigcap_{i=1}^k H_{\alpha_i} \neq 0$, then there exists a simple $R$-submodule $0 \neq N \subseteq \tilde{H}$ (since $\text{Soc}(\tilde{H}) = \tilde{H} \cap \text{Soc}(M) \neq 0$), a contradiction since $N = N_i \not\subseteq H_{\alpha_i}$ for some $i = 1, \cdots, n$. It follows that $\tilde{H} = 0$, whence $\bigcap_{i=1}^k V(H_{\alpha_i}) = V(\bigcap_{i=1}^k H_{\alpha_i}) = V(0) = \emptyset$. \hfill \Box

Connectedness Properties. Recall (e.g. [Bou1966], [Bou1998]) that a non-empty topological space $X$ is said to be

ultracompact, iff the intersection of any two non-empty closed subsets is non-empty;

irreducible (or hyperconnected), iff $X$ is not the union of two proper closed subsets, or equivalently iff the intersection of any two non-empty open subsets is non-empty;

connected, iff $X$ is not the disjoint union of two proper closed subsets; equivalently, iff the only subsets of $X$ that are clopen (i.e. closed and open) are $\emptyset$ and $X$.

Proposition 4.32. Let $RM$ be a top$^f$-module and assume that every first submodule of $M$ is simple.

(a) $\text{Spec}^f(M)$ is discrete.

(b) $M$ has a unique simple $R$-submodule if and only if $\text{Spec}^f(M)$ is connected.

(c) $RM$ is colocal if and only if $\text{Spec}^f(M)$ is connected and $\text{Soc}(M) \leq_R M$ is essential.

Proof. (a) Notice that $RM$ has the min-property by Corollary 4.4 and Remark 4.5. It follows that for every $K \in \text{Spec}^f(M) = \mathcal{S}(M)$ we have $\{K\} = \mathcal{X}'(\{K\}_c)$ an open set.

(b) (⇒) clear.

(⇐) By (a), $\text{Spec}^f(M)$ is discrete and so $\mathcal{S}(M) = \text{Spec}^f(M)$ has only one point since a discrete connected space cannot contain more than one-point.

(c) follows directly from the definitions and (b). \hfill \Box

Remark 4.33. Let $RM$ be a top$^f$-module with essential socle. Recall that $\mathcal{S}(M) \subseteq \text{Spec}^f(M)$ without any conditions on $RM$. If $\{H\}$ is closed in $\text{Spec}^f(M)$ for some $H \leq_R M$,
then \( \{H\} = V(K) \) for some \( 0 \neq K \leq R M \) and we conclude that \( R H \) is simple; if not, then there exists some simple \( R \)-submodule \( \bar{H} \leq R H \) and we would have \( \{H, \bar{H}\} \subseteq V(K) = \{H\} \), a contradiction. So, \( H \leq R M \) is simple if and only if \( H \) is a first submodule of \( M \) and \( V(H) = \{H\} \) if and only if \( \{H\} \) is closed in \( \text{Spec}^f(M) \). Assume that

Combining Proposition 4.32 and Remark 4.33 we obtain

**Theorem 4.34.** For a top\(^f\)-module \( M \) with essential socle, the following are equivalent:

1. \( \text{Spec}^f(M) = \mathcal{S}(M) \);
2. \( \text{Spec}^f(M) \) is discrete;
3. \( \text{Spec}^f(M) \) is \( T_2 \) (Hausdorff space);
4. \( \text{Spec}^f(M) \) is \( T_1 \) (Fréchet space).

**Proposition 4.35.** Let \( _RM \) be comultiplication.

(a) \( _RM \) is a strongly top\(^f\)-module; in particular, \( _RM \) is a top\(^f\)-module.
(b) \( \mathcal{S}(M) = \text{Spec}^f(M) \), i.e. every first submodule of \( M \) is simple.
(c) \( \text{Spec}^f(M) \) is discrete.

**Proof.** Let \( _RM \) be comultiplication.

(a) This follows directly from the fact that \( \text{Sub}_e(M) = \text{Sub}(M) \), Equation (4) (see Remark 4.6).

(b) This follows from Lemma 4.12 (a) and the fact that all submodules of a comultiplication module are also comultiplication.

(c) This follows from Proposition 4.32 (a).

**Example 4.36.** If \( R \) is a left dual ring \([NY2003]\), then \( _RR \) is a strongly top\(^f\)-module and \( \text{Spec}^f(_RR) = \text{Min}(_RR) \) the set of minimal left ideals of \( R \).

**Example 4.37.** \( \mathbb{Z}_{p^\infty} \) is a comultiplication \( \mathbb{Z} \)-module, whence a strongly top\(^f\)-module. Any \( \mathbb{Z} \)-submodule of \( \mathbb{Z}_{p^\infty} \) is of the form \( \mathbb{Z}(\frac{1}{p^n} + \mathbb{Z}) \) for some \( n \in \mathbb{N} \) and so \( \mathbb{Z}_{p^\infty} \notin \text{Spec}^f(\mathbb{Z}_{p^\infty}) \) since \( \text{ann}_\mathbb{Z}(\mathbb{Z}_{p^\infty}) = 0 \neq \text{ann}_\mathbb{Z}(\mathbb{Z}(\frac{1}{p^n} + \mathbb{Z})) \) for every \( n \in \mathbb{N} \). Moreover, it is evident that

\[
\text{ann}_\mathbb{Z}(\mathbb{Z}(\frac{1}{p^n} + \mathbb{Z})) \supseteq \text{ann}_\mathbb{Z}(\mathbb{Z}(\frac{1}{p^{n+1}} + \mathbb{Z})), \quad \text{whence} \quad \mathbb{Z}(\frac{1}{p^n} + \mathbb{Z}) \notin \text{Spec}^f(\mathbb{Z}_{p^\infty}) \quad \text{if} \ n_1 \leq n_2.
\]

Consequently, \( \text{Spec}^f(\mathbb{Z}_{p^\infty}) = \{\mathbb{Z}(\frac{1}{p^n} + \mathbb{Z})\} = \mathcal{S}(\mathbb{Z}_{p^\infty}) \). Clearly, \( \tau^f(\mathbb{Z}_{p^\infty}) = \{\emptyset, \{\mathbb{Z}(\frac{1}{p^n} + \mathbb{Z})\}\} \) is the trivial topology and is connected.

**Proposition 4.38.** A top\(^f\)-module \( M \) with essential socle is uniform if and only if \( \text{Spec}^f(M) \) is ultraconnected.

**Proof.** (\( \Rightarrow \)) Let \( _RM \) be uniform. For any non-empty closed subsets \( V(K_1), V(K_2) \subseteq \text{Spec}^e(M) \), we have indeed \( H_1 \neq 0 \neq H_2 \) whence \( V(H_1) \cap V(H_2) = V(H_1 \cap H_2) \neq \emptyset \), since \( H_1 \cap H_2 \neq 0 \) by uniformity of \( _RM \) and so it contains by assumption some simple \( R \)-submodule which is indeed first in \( M \).

(\( \Leftarrow \)) Assume that the \( \text{Spec}^f(M) \) is ultraconnected. Let \( H_1 \) and \( H_2 \) be non-zero \( R \)-submodules of \( _RM \). It follows that \( V(H_1) \neq \emptyset \neq V(H_2) \). By assumption, \( V(H_1 \cap H_2) = V(H_1) \cap V(H_2) \neq \emptyset \), hence \( H_1 \cap H_2 \neq 0 \).
TOPOLOGICAL LATTICES

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