The mechanical problems on additive manufacturing of viscoelastic solids with integral conditions on a surface increasing in the growth process

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Abstract. Quasistatic mechanical problems on additive manufacturing aging viscoelastic solids are investigated. The processes of piecewise-continuous accretion of such solids are considered. The consideration is carried out in the framework of linear mechanics of growing solids. A theorem about commutativity of the integration over an arbitrary surface increasing in the solid growing process and the time-derived integral operator of viscoelasticity with a limit depending on the solid point is proved. This theorem provides an efficient way to construct on the basis of Saint-Venant principle solutions of nonclassical boundary-value problems for describing the mechanical behaviour of additively formed solids with integral satisfaction of boundary conditions on the surfaces expanding due to the additional material influx to the formed solid. The constructed solutions will retrace the evolution of the stress-strain state of the solids under consideration during and after the processes of their additive formation. An example of applying the proved theorem is given.

1. Introduction

One can find a wide variety of processes of solids additive formation in nature and technology. Many of them can be considered as continuous accretion — when an infinitely thin layer of additional material joins to the surface of the solid being formed every infinitely small time period, — or as piecewise-continuous accretion — when the stages of continuous accretion alternate with pauses during which the additional material does not attach to the solid. In the course of additive processes different factors influence on solids being formed and cause their deformation. The traditional in solid mechanics kinematical description of deformation processes is not suitable for accreted solids. However, as particles of the additional material after its attaching to the accreted solid surface move in the composition of a solid continuum, the enough smooth field of the material particles velocities is uniquely determined in the time-varying space region occupied by the whole growing solid at the current time instant. Therefore, development of the stress-strain state of accreted solids can be described in terms of rates of stress-strain characteristics. Such approach requires knowing the whole history of changing the state of the additional material elements up to their inclusion in the composition of the considered solid.

In the proposed study we follow the indicated approach and consider the situation when the solid being additively formed exhibits the properties of deformation heredity (viscoelasticity)
and aging (weakening the deformation properties over time regardless stresses existing in the solid), and therefore, during pauses in the accreting process as well as after the final cessation of accretion the solid continues to change its stress-strain state. This situation is quite difficult to simulate as rheological manifestations in the deformation response of the material continuously interact with mechanical reactions of the solid on the process of adding new material elements to it running in time.

Remark that different approaches to description of mechanical behavior of additively formed solids deforming in processes of their continuous or piecewise-continuous accretion are being actively developed nowadays in the framework of growing solids mechanics [1–12].

2. Defining relations of the material used

We consider a homogeneously aging isotropic linearly viscoelastic material with the common and constant Poisson’s ratio $\nu = \text{const}$ for elastic and creep strain. The material is described by the following equation of state [13, 14]:

$$\mathcal{H}_{\tau_0(r)} T(r, t) = 2E(r, t) + \frac{2\nu}{1 - 2\nu} 1 \text{ tr } E(r, t)$$

where $T$ and $E$ are the stress and the small strain tensors at the point $r$ at the time instant $t$; $1$ is the unit tensor of the second rank. The linear time-operator $\mathcal{H}_s$ with the real parameter $s \geq 0$ is defined by the expressions

$$\mathcal{H}_s f(t) = \frac{f(t)}{G(t)} - \int_s^t \frac{f(\tau)}{G(\tau)} K(t, \tau) d\tau,$$

$$K(t, \tau) = G(\tau) \frac{\partial \Delta(t, \tau)}{\partial \tau}, \quad \Delta(t, \tau) = \frac{1}{G(\tau)} + \omega(t, \tau).$$

Here $G(t)$ is the elastic shear modulus; $K(t, \tau)$, $\Delta(t, \tau)$, and $\omega(t, \tau)$ are respectively the kernel of creep, the specific strain function, and the creep measure for pure shear ($t \geq \tau \geq 0$). It is accepted by definition that $\omega(\tau, \tau) \equiv 0$.

The quantity $\tau_0(r)$ is the time instant at which nonzero stresses appear at the point corresponding to the radius-vector $r$ in the solid under consideration. The start of timing $t$ is taken be the moment of the material nucleation.

The material defining relations written out above were developed originally for the description of creep processes in concretes. But they are also well suited to describe the analogous processes in rocks, polymers, soils, ice.

3. The investigated problems

The state equation (1) is used in the present work to describe the mechanical behaviour of growing solids which are built up additively by attaching the additional material to the current solid surface. We suppose that the additional material is being loaded in such accreting process simultaneously with its attaching to the solid. In this case, the function $\tau_0(r)$ in (1) is to be determined in the following way.

The original part $V_0$ of the growing solid was initially formed stress-free and loaded then before the accreting process start. In the original part the function $\tau_0(r)$ should be identically equal to the time moment $t_0$ of loading of this part. In the formed due to accreting process additional part $V_A$ of the solid the function $\tau_0(r)$ should coincide with the distribution $\tau_*(r)$ of moments of attaching particles $r$ of the additional material to the solid. So,

$$\tau_0(r) = \begin{cases} t_0, & r \in V_0, \\ \tau_*(r), & r \in V_A. \end{cases}$$
If the accreting process starts at the time instant \( t = t_1 \) then, obviously, \( \tau_s(\mathbf{r}) \geq t_1 \geq t_0 \).

We assume that the considered process of adding the material to the solid can be adequately modelled as a process of piecewise-continuous accretion. This means that the process consists of \( N \) stages of continuous accretion \( t \in [t_{2k-1}, t_{2k}), k = 1, \ldots, N \), during which an infinitely thin layer of the additional material adheres to the growing solid surface every infinitely small time period. Before the first stage of continuous accretion when \( t \in [t_0, t_1) \), between the stages when \( t \in [t_{2k}, t_{2k+1}), k = 1, \ldots, N - 1 \), and after the last stage when \( t \in [t_{2N}, +\infty) \), the additional material influx is absent, the solid does not grow. The part \( \Sigma(t) \) of the growing solid surface to which the additional material continuously inflows at the current time instant \( t \) during any stage of continuous accretion is named the (current) growth surface. It is clear that the growth surface \( \Sigma(t) \) is the \( t \)-level surface of the function \( \tau_s(\mathbf{r}) \).

We investigate mechanical problems for growing solids in quasistatic statement and in the approximation of small strains and displacements. The latter let us consider the time-variable space domain \( V(t) = V_0 \cup V_\ast(t) \) occupied with the whole growing solid to the current time instant \( t \) to be known at any time instant and prescribed by the specific simulated growing process; here the domain \( V_\ast(t) \subseteq V_\Lambda \) is the piece of the additional part already formed to the time instant \( t \). So the growth surface \( \Sigma(t) \) of the growing solid moves in the space in a known manner, and its motion forms the domain \( V_\Lambda \).

We denote for the notation conciseness

\[
g^0(\mathbf{r}, t) = \mathcal{H}_{t_0(\mathbf{r})} g(\mathbf{r}, t) \tag{4}
\]

for arbitrary function \( g(\mathbf{r}, t) \) of solid point \( \mathbf{r} \) and time \( t \), and

\[
h^0(t) = \mathcal{H}_{t_0} h(t) \tag{5}
\]

for arbitrary function of time \( h(t) \) which is not associated with specific points of considered solid.

4. Theorem about commutativity of the time-derived integral operator of viscoelasticity and the integration over an arbitrary surface increasing due to growing the solid

Let the mechanical problem to be solved for a solid growing in the above described manner contain integral conditions (e.g., integral conditions for the force or the moment of force value) on some parts of the solid surface expanding due to the influx of additional material. We can prove the following proposition giving an efficient way to solve the problem in question on the basis of Saint-Venant principle.

**Theorem.** Let \( \Omega_0 \) and \( \Omega_\Lambda \) be two arbitrary surfaces inside or on the boundary of a solid subordinated to the state equation (1) and formed in a process of piecewise-continuous accretion in \( N \) stages of continuous growth \( t \in [t_{2k-1}, t_{2k}), k = 1, \ldots, N \), with arbitrary long pauses \( t_{2k+1} - t_{2k}, k = 1, \ldots, N - 1 \), between the stages. The surface \( \Omega_0 \) lies entirely within the original (existing before accreting) part \( V_0 \) of the solid considered. The surface \( \Omega_\Lambda \) lies entirely in the additional (formed in the accreting process) part \( V_\Lambda \) of the solid and is obtained by motion in space of an arbitrary curve \( \Gamma(t) \), \( t \in [t_1, +\infty) \), which belongs to the current growth surface \( \Sigma(t) \) of the solid at every moment \( t \) of its continuous accreting and is fixed in the pauses between the stages of continuous accretion and after the accreting process end, i.e. on the time periods \( t \in [t_{2k}, t_{2k+1}), k = 1, \ldots, N \), where \( t_{2N+1} = +\infty \). Let \( g(\mathbf{r}, t) \) be an arbitrary function defined at the points \( \mathbf{r} \) of the both surfaces \( \Omega_0 \) and \( \Omega_\Lambda \) for \( t \geq t_0(\mathbf{r}) \). Assume that

\[
g(\mathbf{r}, \tau_s(\mathbf{r})) \equiv 0, \quad \mathbf{r} \in \Omega_\Lambda. \tag{6}
\]
Then, when \( t > t_1 \) the formula

\[
\frac{d}{dt} \left[ \int_{\Omega(t)} g(r, t) \, dS \right] = \int_{\Omega(t)} \frac{\partial g^\circ(r, t)}{\partial t} \, dS
\]

will be fair, where the expanding in time (disconnected in general) surface \( \Omega(t) \) combines the surface \( \Omega_0 \) and that part of the surface \( \Omega_A \), which has already been formed by the time \( t \geq t_0 \):

\[
\Omega(t) = \begin{cases} 
\Omega_0, & t \in [t_0, t_1], \\
\Omega_0 \cup \Omega_*(t), & t \in (t_1, +\infty),
\end{cases} \quad \Omega_*(t) = \{ \Gamma(\tau) \mid t_1 \leq \tau \leq t \} \subseteq \Omega_A. \tag{8}
\]

We are to emphasize that the surfaces \( \Omega_0 \) and \( \Omega_A \) considered in the Theorem may have arbitrary curvatures. Meanwhile the boundaries of these surfaces may not have common points. It is also possible that \( \Omega_0 = \emptyset \). Forming a surface \( \Omega_A \) curves \( \Gamma(t) \) can be unclosed or closed. In the latter case the surface \( \Omega_A \) may “circle” the original part of the solid or form a “tube” enveloping only the material of the additional part of the having been finally formed solid.

5. Proof of the Theorem

Introduce on the surface \( \Omega_A \) the parameterization \( r = r(\xi, \eta) \) which is induced by the ongoing growth program, in the following way. Let us select a certain general parameter \( \eta \in [A, B] \), where \( A \) and \( B \) — are some constants, for all the curves \( \Gamma(t) \), \( t \geq t_1 \), the surface \( \Omega_A \) is composed from. This means that curves \( \Gamma(t) \) form a family of \( \eta \)-lines on the surface \( \Omega_A \). The choice of the parameter \( \eta \) for all the curves \( \Gamma(t) \) is to ensure that through each point of the surface \( \Omega_A \) it goes one and only one line from the second family — the family of lines consisting of all those points on different curves \( \Gamma(t) \), which correspond to the same value \( \eta \).

Let the geometric position of the curve \( \Gamma(t) \) on the surface \( \Omega_A \) at a particular value of time \( t \) be in one-to-one correspondence to a particular value of some quantity \( \xi \), namely the value \( \xi = \Xi(t) \): if at different \( t' \) and \( t'' \) the curves \( \Gamma(t') \) and \( \Gamma(t'') \) geometrically coincide, then \( \Xi(t') = \Xi(t'') \); and vice versa, if at different \( t' \) and \( t'' \) the curves \( \Gamma(t') \) and \( \Gamma(t'') \) do not geometrically coincide, then \( \Xi(t') \neq \Xi(t'') \). The function \( \Xi(t) \) we consider continuous for all \( t \geq t_1 \) and monotonically non-decreasing on the time intervals \( t \in [t_{2k-1}, t_{2k}) \), \( k = 1, \ldots, N \) corresponding to the stages of the continuous growth of the solid. Beyond these intervals, that is, when \( t \in [t_{2k}, t_{2k+1}) \), where \( t_{2N+1} = +\infty \), when the influx of additional material to the solid is absent and therefore the movement of the curve \( \Gamma(t) \) over the surface \( \Omega_A \) is temporarily (when \( k = 1, \ldots, N - 1 \) or ultimately (when \( k = N \)) completed the function \( \Xi(t) \) takes obviously constant values:

\[
\Xi(t) \equiv \xi_k, \quad t \in [t_{2k}, t_{2k+1}),
\]

where \( \xi_k = \Xi(t_{2k}) \). Due to continuity of the function \( \Xi(t) \) the value \( \xi_k \) is the value of the parameter \( \xi \) corresponding to the position of the curve \( \Gamma(t) \) at the end of the \( k \)-th stage of continuous growth.

If the value \( \xi_0 = \Xi(t_1) \) corresponds to the position of the curve \( \Gamma(t) \) at the initial time instant of the process of growing the solid, then due to monotonicity of the function \( \Xi(t) \) we will have

\[
\xi_0 \leq \xi_1 \leq \cdots \leq \xi_{N-1} \leq \xi_N,
\]

and the parameter \( \xi \) over the surface \( \Omega_A \) will vary on the interval \( \xi \in [\xi_0, \xi_N] \). An example of dependence of \( \xi = \Xi(t) \) in the case \( N = 3 \) is illustrated on the graph in figure 1.

Note that a possible non-strict monotonicity (non-decreasing) of the function \( \Xi(t) \) on the intervals of continuous growth \( t \in [t_{2k-1}, t_{2k}) \) and, consequently, possible non-strict signs in
Figure 1. Example of dependence of the parameter $\xi$ on the surface $\Omega_A$ on time $t$ in the case of $N = 3$ stages of continuous accretion of the solid.

chain (9), arise from the possibility of pursuing such a variant of accreting the solid when a region of space that it occupies at the stage of continuous growth is constantly expanding due to the influx of additional material to the part $\Sigma(t)$ of boundary surface of the growing solid — the current surface of growth, — but herewith the trace $\Gamma(t)$ of the moving in space growth surface $\Sigma(t)$ on the surface $\Omega_A$ selected by us inside or on the boundary of the growing solid remains for some time stationary.

The family of lines consisting of points corresponding to the same value of the parameter $\eta$ on different curves $\Gamma(t)$ which was discussed above is obviously the family of $\xi$-lines on the surface $\Omega_A$.

The couple of parameters $(\xi, \eta)$ will be considered as curvilinear coordinates on the surface $\Omega_A$. The convenience of introducing such coordinates is explained for us by the following key fact. Since at each value $t$ the curve $\Gamma(t)$ lies entirely on the current growth surface $\Sigma(t)$, then at the coordinates $(\xi, \eta)$ the time instant $\tau_0(r)$ of occurrence of stresses at the points $r$ of the surface $\Omega_A$ that coincides with time instant $\tau_\star(r)$ of the inclusion of these points in the composition of the growing solid (see (3)) depends only on the coordinate $\xi$:

$$
\tau_0(r) \equiv \tau_0(\xi),
$$

moreover, in accordance with the definition of the function $\Xi(t)$ it is true the identity

$$
\Xi(\tau_0(\xi)) \equiv \xi
$$

for all $\xi \in [\xi_0, \xi_N]$.

By using the curvilinear coordinates $(\xi, \eta)$ the integral over the surface $\Omega_s(t) \subseteq \Omega_A$ of the arbitrary function $f(r, t)$ of the point and time is written as follows:

$$
\int_{\Omega_s(t)} f(r, t) \, dS = \int_A^B d\eta \int_{\xi_0}^{\Xi(t)} f(\xi, \eta, t) J(\xi, \eta) \, d\xi.
$$

Here the value $J(\xi, \eta)$ defines an element of the area of the surface $\Omega_A$ in the curvilinear coordinates $(\xi, \eta)$ and is equal to

$$
J(\xi, \eta) = \left| \frac{\partial \mathbf{r}}{\partial \xi} \times \frac{\partial \mathbf{r}}{\partial \eta} \right|.
$$
By choosing \( f(\mathbf{r}, t) = \frac{\partial g^o(\mathbf{r}, t)}{\partial t} \) and using the rule of differentiation of the integral by parameter, we can transform the inner integral in (12):

\[
\int_{\xi_0}^{\Xi(t)} \frac{\partial g^o(\xi, \eta, t)}{\partial t} J(\xi, \eta) \, d\xi = \frac{\partial}{\partial \tau} \int_{\xi_0}^{\Xi(t)} g^o(\xi, \eta, t) J(\xi, \eta) \, d\xi - g^o(\Xi(t), \eta, t) J(\Xi(t), \eta) \Xi'(t). \tag{13}
\]

Let us reveal the symbolic notation \( (\cdot)^o \) in the integral standing on the right in (13) (see definition (4), (2) and formula (10)):

\[
\int_{\xi_0}^{\Xi(t)} g^o(\xi, \eta, t) J(\xi, \eta) \, d\xi = \int_{\xi_0}^{\Xi(t)} \frac{g(\xi, \eta, \tau)}{G(\tau)} J(\xi, \eta) \, d\xi - \int_{\xi_0}^{\Xi(t)} J(\xi, \eta) \, d\xi \int_{\tau_0(\xi)}^{\tau} \frac{g(\xi, \eta, \tau)}{G(\tau)} K(t, \tau) \, d\tau. \tag{14}
\]

Let us change in (14) the order of integration in the repeated integral (see figure 1 and identity (11)):

\[
\int_{\xi_0}^{\Xi(t)} J(\xi, \eta) \, d\xi \int_{\tau_0(\xi)}^{\tau} \frac{g(\xi, \eta, \tau)}{G(\tau)} K(t, \tau) \, d\tau = \int_{\xi_0}^{\Xi(\tau)} \frac{g(\xi, \eta, \tau)}{G(\tau)} J(\xi, \eta) \, d\xi. \tag{15}
\]

We consider now the term beyond the integrals in (13). We have (see definition (4), (2) and formula (10))

\[
g^o(\Xi(t), \eta, t) = \frac{g(\Xi(t), \eta, \tau)}{G(\tau)} - \int_{\tau_0(\Xi(t))}^{\tau} \frac{g(\Xi(t), \eta, \tau)}{G(\tau)} K(t, \tau) \, d\tau.
\]

On all pieces of the strict monotonicity of the function \( \Xi(t) \) this function has a reverse one, which is the function \( \tau_0(\xi) \) in accordance with identity (11). Therefore at the lower limit in the last integral it will be \( \tau_0(\Xi(t)) = t \) and this integral will be identically equal to zero. Herewith we also have

\[
g(\Xi(t), \eta, t) \big|_{\xi=\Xi(t)} = g(\xi, \eta, t) \big|_{t=\tau_0(\xi)} = g(\mathbf{r}, t) \big|_{t=\tau_0(\mathbf{r})} = 0
\]

in accordance with condition (6) and representation (3).

Thus, on pieces of the strict monotonicity of the function \( \Xi(t) \) the term beyond the integrals in (13) becomes zero. On pieces of the function \( \Xi(t) \) constancy (in particular, in the pauses between the stages of continuous accretion of the solid and after its ultimate growth completing) it will become \( \Xi'(t) = 0 \), therefore this term vanishes as well.

So, from (13)–(15) we get

\[
\int_{\xi_0}^{\Xi(t)} \frac{\partial g^o(\xi, \eta, t)}{\partial t} J(\xi, \eta) \, d\xi = \frac{d}{dt} \int_{\xi_0}^{\Xi(t)} \frac{g(\xi, \eta, t)}{G(\tau)} J(\xi, \eta) \, d\xi - \int_{\tau_0(\xi)}^{\tau} \frac{g(\xi, \eta, \tau)}{G(\tau)} K(t, \tau) \, d\tau \int_{\xi_0}^{\Xi(\tau)} \frac{g(\xi, \eta, \tau)}{G(\tau)} J(\xi, \eta) \, d\xi.
\]

Hence, due to representation (12) we have

\[
\int_{\Omega_s(t)} \frac{\partial g^o(\mathbf{r}, t)}{\partial t} \, dS = \frac{d}{dt} \int_{\Omega_s(t)} \frac{g(\mathbf{r}, t)}{G(\tau)} \, dS - \int_{t_1}^{t} K(t, \tau) \, d\tau \int_{\Omega_s(\tau)} \frac{g(\mathbf{r}, \tau)}{G(\tau)} \, dS. \tag{16}
\]

To complete the proof of formula (7) it remains to present the integral of the function \( f(\mathbf{r}, t) = \frac{\partial g^o(\mathbf{r}, t)}{\partial t} \) over the surface \( \Omega_0 \) supplementing the surface \( \Omega_s(t) \) up to \( \Omega(t) \) in a form similar to (16). As the surface \( \Omega_0 \) is unchanged in time, then the time derivative can be taken outside the sign of the integral over this surface:

\[
\int_{\Omega_0} \frac{\partial g^o(\mathbf{r}, t)}{\partial t} \, dS = \frac{d}{dt} \int_{\Omega_0} g^o(\mathbf{r}, t) \, dS. \tag{17}
\]
We then reveal the symbolic notation \( (\cdot)^0 \) in the integral on the right (see definition (4), (2)) by reversing the order of integration over time and over the surface \( \Omega_0 \) that does not dependent on time:

\[
\int_{\Omega_0} g^0(r, t) \, dS = \int_{\Omega_0} \frac{g(r, t)}{G(t)} \, dS - \int_{\Omega_0} dS \int_{t_0}^t \frac{g(r, \tau)}{G(\tau)} K(t, \tau) \, d\tau
\]

\[
= \int_{\Omega_0} \frac{g(r, t)}{G(t)} \, dS - \int_{t_0}^t K(t, \tau) \, d\tau \int_{\Omega_0} \frac{g(r, \tau)}{G(\tau)} \, dS. \quad (18)
\]

It is taken here into account that \( \tau_0(r) \equiv t_0 \) for \( r \in \Omega_0 \) as the surface \( \Omega_0 \) lies entirely in the originally existing part of the solid, stresses at all points of which arose at the same moment of time \( t = t_0 \) (see (3)).

When \( t > t_1 \) we can split the time integral in (18) into two integrals — from \( t_0 \) to \( t_1 \) and from \( t_1 \) to \( t \), — given that \( \Omega(t) \equiv \Omega_0 \) for \( t \in [t_0, t_1] \) (see (8)):

\[
\int_{\Omega_0} g^0(r, t) \, dS = \int_{\Omega_0} \frac{g(r, t)}{G(t)} \, dS - \int_{t_0}^{t_1} K(t, \tau) \, d\tau \int_{\Omega(\tau)} \frac{g(r, \tau)}{G(\tau)} \, dS - \int_{t_1}^t K(t, \tau) \, d\tau \int_{\Omega(\tau)} \frac{g(r, \tau)}{G(\tau)} \, dS. \quad (19)
\]

As a result, due to (17) we will have

\[
\int_{\Omega_0} \frac{\partial g^0(r, t)}{\partial t} \, dS = \frac{d}{dt} \left[ \int_{\Omega_0} \frac{g(r, t)}{G(t)} \, dS - \int_{t_0}^{t_1} K(t, \tau) \, d\tau \int_{\Omega(\tau)} \frac{g(r, \tau)}{G(\tau)} \, dS - \int_{t_1}^t K(t, \tau) \, d\tau \int_{\Omega(\tau)} \frac{g(r, \tau)}{G(\tau)} \, dS \right]
\]

which together with (16) with regard to (8) and (5) gives the formula (7) being proved.

6. An example of applying the Theorem to solving boundary-value problems for growing solids on the basis of Saint-Venant principle

Applying the above proved Theorem to solving the growing solids mechanics problems with integral force conditions is demonstrated in the paper [15] on the example of modelling additive processes of formation of the relatively long in the axial direction conical solids of rotation of the isotropic homogeneously aging linearly viscoelastic material subordinated to the constitutive equation (1).

It is assumed that in the process of formation of the solid its end surfaces are acted with a loading statically equivalent to the axial central tension–compression force \( P \) which can arbitrarily vary with time \( t \). Forming the solid under consideration is carried out by means of its axisymmetric thickening in the radial direction due to attaching the additional material to the conic side surface. This process is piecewise-continuous in time, i.e. consists of arbitrary number of consecutive stages of continuous accretion alternating with pauses of arbitrary duration. Gradual thickening occurs in such a way that in each time moment the accreted body maintains the shape of a circular truncated cone of fixed length.

The conical side surface moving due to the influx of additional material represents the current growth surface \( \Sigma(t) \) in the considered example. The trace of its passing in the space forms an additional part \( \Lambda \) of the growing cone.

The material used for the cone thickening is supposed to be initially free from stresses, so the initial condition

\[
\mathbf{T}(r, \tau_0(r)) = 0, \quad r \in \Lambda,
\]

should take place in the additional part of the considered growing solid.

With the help of the above proved Theorem we can solve on the basis of the Saint-Venant principle the boundary-value problem for the proposed mechanical model of the conical solid
growth. As a surface $\Omega(t)$ expanding over time $t$ due to the additional material influx it is necessary to consider in this problem the flat surface constituting one end side of the growing cone for every $t \geq t_0$. On this surface the integral boundary conditions for the components of the force $P$ and its moment are to be formulated. In this case the surface $\Omega_A$ is annular, and its forming curves $\Gamma(t)$ are concentric circles. The surface $\Omega_0$ is the disc of the radius being equal to the initial radius of the corresponding cone end.

The condition (20) in the considered example acts as the requirement (6) in the Theorem.

A closed form analytical solution of the described example problem is constructed in [15]. The time evolution of the stress-strain state of the conical solid before the start, during and upon the completion of the process of its additive manufacturing under specified conditions of loading is investigated.

Acknowledgments
This work was supported by the Russian Science Foundation under grant No. 17-19-01257.

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