Hamiltonian Stability and Index of Minimal Lagrangian Surfaces of the Complex Projective Plane

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Abstract. We show that the Clifford torus and the totally geodesic real projective plane $\mathbb{RP}^2$ in the complex projective plane $\mathbb{CP}^2$ are the unique Hamiltonian stable minimal Lagrangian compact surfaces of $\mathbb{CP}^2$ with genus $g \leq 4$, when the surface is orientable, and with Euler characteristic $\chi \geq -1$, when the surface is nonorientable. Also we characterize $\mathbb{RP}^2$ in $\mathbb{CP}^2$ as the least possible index minimal Lagrangian compact nonorientable surface of $\mathbb{CP}^2$.

1. Introduction

The second variation operator of minimal submanifolds of Riemannian manifolds (the Jacobi operator) carries the information about the stability properties of the submanifold when it is thought as a critical point for the volume functional. When the ambient Riemannian manifold is the complex projective space $\mathbb{CP}^n$, Lawson and Simons [LS] characterized the complex submanifolds as the unique stable minimal submanifolds of $\mathbb{CP}^n$. In particular, minimal Lagrangian submanifolds of $\mathbb{CP}^n$ are unstable. In [O1] Oh introduced the notion of Hamiltonian stability for minimal Lagrangian submanifolds of $\mathbb{CP}^n$ (or more generally of any Kähler manifold), as those ones such that the second variation of volume is nonnegative for Hamiltonian deformations of $\mathbb{CP}^n$. He proved that the Clifford torus in $\mathbb{CP}^n$ is Hamiltonian stable and conjectured that it is also volume minimizing under Hamiltonian deformations of $\mathbb{CP}^n$. B. Kleiner had proved that the totally geodesic Lagrangian real projective space $\mathbb{RP}^n$ in $\mathbb{CP}^n$ is volume minimizing under Hamiltonian deformations.

In [U] Urbano (see also [H] Theorem B) characterized the Clifford torus as the unique Hamiltonian stable minimal Lagrangian torus in $\mathbb{CP}^2$ and got a lower bound for the index (the number of negative eigenvalues of the Jacobi operator) of the minimal Lagrangian compact orientable surfaces in $\mathbb{CP}^2$, proving that the index is at least 2 and it is 2 only for the Clifford torus. In this paper the author continues studying these problems, proving, among others, the following results:

The Clifford torus is the unique Hamiltonian stable minimal Lagrangian compact orientable surface of $\mathbb{CP}^2$ with genus $g \leq 4$.

The totally geodesic real projective plane $\mathbb{RP}^2$ is the unique Hamiltonian stable minimal Lagrangian compact nonorientable surface of $\mathbb{CP}^2$ with Euler characteristic $\chi \geq -1$.

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The index of a minimal Lagrangian compact nonorientable surface of $\mathbb{CP}^2$ is at least 3 and it is 3 only for the totally geodesic real projective plane $\mathbb{RP}^2$.

To prove these results we need to have a control of the index of the minimal Lagrangian Klein bottles of $\mathbb{CP}^2$ which admit a one-parameter group of isometries. Following the ideas of [CU], in section 5 we describe explicitly those minimal surfaces and estimate their index.

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2. Preliminaries

Let $\mathbb{CP}^2$ be the complex projective plane with its canonical Fubini-Study metric $\langle \cdot, \cdot \rangle$ of constant holomorphic sectional curvature 4. Then

$$\mathbb{CP}^2 = \{ \Pi(z) = [z] / z \in S^5 \},$$

where $\Pi : S^5 \to \mathbb{CP}^2$ is the Hopf projection, being $S^5$ the unit sphere in the complex Euclidean space $\mathbb{C}^3$. The complex structure $J$ of $\mathbb{C}^3$ induces via $\Pi$ the canonical complex structure $J$ on $\mathbb{CP}^2$ (we will denote both by $J$). The Kähler two form in $\mathbb{CP}^2$ is defined by $\omega(\cdot, \cdot) = \langle J \cdot, \cdot \rangle$.

An immersion $\Phi : \Sigma \to \mathbb{CP}^2$ of a surface $\Sigma$ is called Lagrangian if $\Phi^* \omega = 0$. This means that the complex structure $J$ defines a bundle isomorphism from the tangent bundle to $\Sigma$ onto the normal bundle to $\Phi$, allowing us to identify the sections on the normal bundle $\Gamma(T^\perp \Sigma)$ with the 1-forms on $\Sigma$ by

$$\Gamma(T^\perp \Sigma) \equiv \Omega^1(\Sigma)$$

$$\xi \equiv \alpha$$

being $\alpha$ the 1-form on $\Sigma$ defined by $\alpha(v) = \omega(v, \xi)$ for any $v$ tangent to $\Sigma$, and where $\Omega^p(\Sigma)$, $p = 0, 1, 2$, denotes the space of $p$-forms on $\Sigma$.

From now on our Lagrangian surface will be minimal and compact. Among them, we would like to bring out the Clifford torus

$$T = \{ [z] \in \mathbb{CP}^2 \mid |z|^2 = \frac{1}{3}, i = 1, 2, 3 \},$$

and the totally geodesic Lagrangian real projective plane

$$\mathbb{RP}^2 = \{ [z] \in \mathbb{CP}^2 \mid z_i = \bar{z}_i, i = 1, 2, 3 \},$$

whose $2 : 1$ oriented covering provides the totally geodesic Lagrangian immersion $S^2 \to \mathbb{CP}^2$ of the unit sphere. An important property of these surfaces (see for instance [EGT]), which will be use in the paper, is that $\mathbb{RP}^2 \subset \mathbb{CP}^2$ is the unique minimal Lagrangian projective plane immersed in $\mathbb{CP}^2$ and hence $S^2 \to \mathbb{CP}^2$ is the unique minimal Lagrangian sphere immersed in $\mathbb{CP}^2$.

Using the identification (2.1), if $L : \Gamma(T^\perp \Sigma) \to \Gamma(T^\perp \Sigma)$ denotes the Jacobi operator of the second variation of the area, Oh proved (in [O1]) that $L$ is given by

$$L : \Omega^1(\Sigma) \to \Omega^1(\Sigma)$$

$$L = \Delta + 6I,$$

where $I$ is the identity map and, in general, $\Delta = \delta d + d\delta$ is the Laplacian on $\Sigma$ acting on $p$-forms, $p = 0, 1, 2$, being $\delta$ the codifferential operator of the exterior.
differential \( d \). Hence, the index of \( \Phi \), that we will denote by \( \text{Ind}(\Sigma) \), is the number of eigenvalues (counted with multiplicity) of \( \Delta : \Omega^1(\Sigma) \to \Omega^1(\Sigma) \) less than 6.

To study the Jacobi operator, we consider the Hodge decomposition

\[
\Omega^1(\Sigma) = \mathcal{H}(\Sigma) \oplus d\Omega^0(\Sigma) \oplus \delta\Omega^2(\Sigma),
\]

which allows to write, in a unique way, any 1-form \( \alpha \) as \( \alpha = \alpha_0 + df + \beta \), being \( \alpha_0 \) a harmonic 1-form, \( f \) a real function and \( \beta \) a 2-form on \( \Sigma \). The space of harmonic 1-forms, \( \mathcal{H}(\Sigma) \), is the kernel of \( \Delta \) and its dimension is the first Betti number of \( \Sigma \): \( \beta_1(\Sigma) \). As \( \Delta \) commutes with \( d \) and \( \delta \), the positive eigenvalues of \( \Delta : \Omega^1(\Sigma) \to \Omega^1(\Sigma) \) are the positive eigenvalues of \( \Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma) \) joint to the positive eigenvalues of \( \Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma) \). Hence

\[
(2.2) \quad \text{Ind}(\Sigma) = \beta_1(\Sigma) + \text{Ind}_0(\Sigma) + \text{Ind}_1(\Sigma),
\]

where \( \text{Ind}_0(\Sigma) \) is the number of positive eigenvalues (counted with multiplicity) of \( \Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma) \) less than 6 and \( \text{Ind}_1(\Sigma) \) is the number of positive eigenvalues (counted with multiplicity) of \( \Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma) \) less than 6. When \( \Sigma \) is \( \mathbb{RP}^2 \subset \mathbb{CP}^2 \), \( \text{Ind}_0(\Sigma) = 0 \), \( \text{Ind}_1(\Sigma) = 3 \) and hence \( \text{Ind}(\Sigma) = 3 \).

If the compact surface \( \Sigma \) is orientable, the star operator \( \star : \Omega^0(\Sigma) \to \Omega^2(\Sigma) \) says us that the eigenvalues of \( \Delta \) acting on \( \Omega^0(\Sigma) \) or on \( \Omega^2(\Sigma) \) are the same, and so \( \text{Ind}_0(\Sigma) = \text{Ind}_1(\Sigma) \). Hence if \( \Sigma \) is a minimal Lagrangian compact orientable surface of \( \mathbb{CP}^2 \) of genus \( g \), then

\[
(2.3) \quad \text{Ind}(\Sigma) = 2g + 2\text{Ind}_0(\Sigma).
\]

When \( \Sigma \) is the totally geodesic Lagrangian two-sphere in \( \mathbb{CP}^2 \), \( \text{Ind}_0(\Sigma) = 3 \) and \( \text{Ind}(\Sigma) = 6 \).

The variational vector fields of the Hamiltonian deformations of the Lagrangian surface \( \Sigma \) are the normal components of the Hamiltonian vector fields on \( \mathbb{CP}^2 \). If \( X = J\nabla F \), for certain smooth function \( F : \mathbb{CP}^2 \to \mathbb{R} \), is a Hamiltonian vector field on \( \mathbb{CP}^2 \), the 1-form associated to the normal component of \( X \), under the identification (2.1), is \( df \circ \Phi \). So our minimal Lagrangian compact surface \( \Sigma \) is Hamiltonian stable if the first positive eigenvalue of \( \Delta \) acting on \( \Omega^0(\Sigma) \) is at least 6. But it is well-known that 6 is always an eigenvalue of \( \Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma) \) (see proof of Theorem 3.3). Hence \( \Sigma \) is Hamiltonian stable if the first positive eigenvalue of \( \Delta \) acting on \( \Omega^0(\Sigma) \) is 6. In this setting it is natural to call to \( \text{Ind}_0(\Sigma) \) the Hamiltonian index of \( \Sigma \).

Let \( \Sigma \) be a nonorientable Riemannian compact surface and \( \pi : \tilde{\Sigma} \to \Sigma \) the 2 : 1 orientable Riemannian covering. If \( \tau : \Sigma \to \tilde{\Sigma} \) is the change of sheet involution, then the spaces of forms on \( \tilde{\Sigma} \) can be decompose in the following way:

\[
\Omega^i(\tilde{\Sigma}) = \Omega^i_+(\tilde{\Sigma}) \oplus \Omega^i_-\hspace{1pt}(\tilde{\Sigma}), \quad i = 0, 1, 2,
\]

where

\[
\Omega^i_{\pm}\hspace{1pt}(\tilde{\Sigma}) = \{ \alpha \in \Omega^i(\tilde{\Sigma}) / \tau^*\alpha = \pm\alpha \}.
\]

Also the space of harmonic 1-forms on \( \tilde{\Sigma} \) decomposes into two subspaces \( \mathcal{H}(\tilde{\Sigma}) = \mathcal{H}_+(\tilde{\Sigma}) \oplus \mathcal{H}_-(\tilde{\Sigma}) \), where again \( \mathcal{H}_+\hspace{1pt}(\tilde{\Sigma}) = \{ \alpha \in \mathcal{H}(\tilde{\Sigma}) / \tau^*\alpha = \pm\alpha \} \). In this way we obtain

\[
\Omega^i_+(\tilde{\Sigma}) = \mathcal{H}_+(\tilde{\Sigma}) \oplus d\Omega^i_+(\tilde{\Sigma}) \oplus \delta\Omega^2_+(\tilde{\Sigma}).
\]

As \( \pi \circ \tau = \pi \), the map \( \alpha \in \Omega^i(\Sigma) \mapsto \pi^*\alpha \in \Omega^i(\tilde{\Sigma}) \) allows to identify \( \mathcal{H}(\Sigma) \equiv \mathcal{H}_+(\tilde{\Sigma}) \) and \( \Omega^i(\Sigma) \equiv \Omega^i_+(\tilde{\Sigma}) \), \( i = 0, 1, 2 \). Also, as \( \Sigma \) is nonorientable, \( \star\tau^* = -\tau^*\star \), and so \( \star \)
identifies $\Omega^0(\tilde{\Sigma}) \equiv \Omega^2_+(\tilde{\Sigma})$. Hence we obtain the identification

$$\Omega^2(\Sigma) \equiv \Omega^0(\tilde{\Sigma})$$

$$\alpha \equiv f$$

where $\pi^*\alpha = f\omega_0$, being $\omega_0$ the volume 2-form on $\tilde{\Sigma}$.

Now, let $\Phi : \Sigma \to \mathbb{CP}^2$ be a minimal Lagrangian immersion of a compact nonorientable surface $\Sigma$ and $\Phi \circ \pi : \tilde{\Sigma} \to \mathbb{CP}^2$ the corresponding minimal Lagrangian immersion of its 2:1 orientable covering $\tilde{\Sigma}$. As $\Sigma$ is nonorientable, the eigenvalues of $\Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma)$ are positives, and hence, taking into account the above remarks, $\text{Ind}_1(\Sigma)$ is the number of eigenvalues (counted with multiplicity) of $\Delta : \Omega^2(\Sigma) \to \Omega^2(\Sigma)$ less than 6. Also, as $\text{Ind}_0(\Sigma)$ is the number of positive eigenvalues (counted with multiplicity) of $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ less than 6, we obtain that

$$(2.4) \quad \text{Ind}_0(\Sigma) + \text{Ind}_1(\Sigma) = \text{Ind}_0(\tilde{\Sigma}),$$

and hence from (2.3)

$$2\text{Ind}(\Sigma) = \text{Ind}(\tilde{\Sigma}).$$

3. PROOF OF THE RESULTS

**Theorem 3.1.** Let $\Phi : \Sigma \to \mathbb{CP}^2$ be a Hamiltonian stable minimal Lagrangian immersion of a compact orientable surface of genus $g$. If $g \leq 4$ then $\Phi$ is an embedding and $\Phi(\Sigma)$ is the Clifford torus.

**Proof:** As $\Sigma$ is Hamiltonian stable, the first positive eigenvalue of $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ is 6. Now we use a well-known argument. From the Brill-Noether theory, we can get a nonconstant meromorphic map $\phi : \Sigma \to \mathbb{S}^2$ of degree $d \leq 1 + \left[\frac{g+1}{2}\right]$, where $\left[\cdot\right]$ stands for integer part. Then there exists a Moebius transformation $F : \mathbb{S}^2 \to \mathbb{S}^2$ such that $\int_{\Sigma} |F \circ \phi|^2 = 0$, and so

$$\int_{\Sigma} |\nabla(F \circ \phi)|^2 \geq 6 \int_{\Sigma} |F \circ \phi|^2 = 6 \text{Area}(\Sigma).$$

But $\int_{\Sigma} |\nabla(F \circ \phi)|^2 = 8\pi \text{degree}(F \circ \phi) = 8\pi \text{degree}(\phi) \leq 8\pi(1 + \left[\frac{g+1}{2}\right])$. Hence we obtain that $3\text{Area}(\Sigma) \leq 4\pi(1 + \left[\frac{g+1}{2}\right])$.

On the other hand, Montiel and Urbano ([MU] Corollary 6) proved that $\text{Area}(\Sigma) \geq 2\pi \mu$ ($\mu$ being the maximum multiplicity of the immersion $\Phi$) and that the equality holds if and only if the surface is the totally geodesic two-sphere. So we obtain that

$$3\mu \leq 2(1 + \left[\frac{g+1}{2}\right]),$$

and the equality implies that the surface is the totally geodesic Lagrangian two-sphere.

Using [EGT], Lemma 4.6, we know that $\Phi$ is not an embedding (i.e. $\mu \geq 2$) when $g \geq 2$. So in this case $2 < \left[\frac{g+1}{2}\right]$, which is a contradiction when $g = 2, 3, 4$. If $g = 0$, our surface is the totally geodesic Lagrangian two-sphere, which is Hamiltonian unstable. Hence the surface must be a torus and using [U], Corollary 2, we conclude that it is the Clifford torus.

**Remark 3.2.** If $\lambda_1$ is the first positive eigenvalue of $\Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma)$ and $g \geq 2$ then the above reasoning proves that $\lambda_1 < 2(1 + \left[\frac{g+1}{2}\right])$. Hence if $g = 2$ we obtain that $\lambda_1 < 4$.
Theorem 3.3. Let \( \Phi : \Sigma \rightarrow \mathbb{CP}^2 \) be a minimal Lagrangian immersion of a compact nonorientable surface \( \Sigma \) with Euler characteristic \( \chi(\Sigma) \geq -1 \). Then

1. If \( \Sigma \) is a projective plane with a handle (\( \chi(\Sigma) = -1 \)) then \( \Sigma \) is Hamiltonian unstable.
2. If \( \Sigma \) is a Klein bottle (\( \chi(\Sigma) = 0 \)) then \( \text{Ind}_0(\Sigma) \geq 2 \). In particular \( \Sigma \) is Hamiltonian unstable.

As consequence, if \( \Phi \) is Hamiltonian stable then \( \Phi \) is an embedding and \( \Phi(\Sigma) \) is \( \mathbb{RP}^2 \).

Proof: We will denote also by \( \langle , \rangle \) the Euclidean metric in \( \mathbb{C}^3 \). In the Lie algebra \( \text{so}(6) \) of the isometry group of \( S^5 \), we consider the subspace \( \text{so}^+(6) = \{ A \in \text{so}(6) / AJ = JA \text{ and } \text{Trace} AJ = 0 \} \), which is the real representation of the Lie algebra \( \text{su}(3) \). Then for any \( A \in \text{so}^+(6) \), the function on the sphere \( p \in S^5 \mapsto \langle Ap, Jp \rangle \in \mathbb{R} \) can be projected to \( \mathbb{CP}^2 \), defining a map

\[
F_A : \mathbb{CP}^2 \rightarrow \mathbb{R}
\]

\[
F_A(\Pi(p)) = \langle Ap, Jp \rangle
\]

First we compute the gradient of \( F_A \). If \( v \) is any vector tangent to \( \mathbb{CP}^2 \) at \( \Pi(p) \), then

\[
v \cdot F_A = 2\langle Av^*, Jp \rangle,
\]

being \( v^* \) the horizontal lifting of \( v \) to \( T_p S^5 \). So

\[
(\nabla F_A)_{\Pi(p)} = -2(d\Pi)_p(AJp + F_A(p)p),
\]

for any \( \Pi(p) \in \mathbb{CP}^2 \). Taking derivatives again and using that \( \Pi : S^5 \rightarrow \mathbb{CP}^2 \) is a Riemannian submersion, one has that the Hessian of \( F_A \) is given by

\[
(\nabla^2 F_A)(v, w) = -2F_A\langle v, w \rangle + 2\langle Av^*, Jw^* \rangle,
\]

for any vectors \( v, w \in T_{\Pi(p)} \mathbb{CP}^2 \).

Now, if \( \Phi : \Sigma \rightarrow \mathbb{CP}^2 \) is a minimal Lagrangian immersion of a compact surface \( \Sigma \) and \( f_A : \Sigma \rightarrow \mathbb{R} \) is defined by \( f_A = F_A \circ \Phi \), then by decomposition

\[
\nabla F_A = \nabla f_A + \xi
\]

in its tangential and normal components and taking into account (3.1) we deduce

\[
(\nabla^2 f_A)(v, w) = -2f_A\langle v, w \rangle + 2\langle Av^*, Jw^* \rangle + \langle \sigma(v, w), \xi \rangle,
\]

which implies that \( \Delta f_A + 6f_A = 0 \). So we have defined a linear map

\[
H : \text{so}^+(6) \rightarrow V_6 = \{ f / \Delta f + 6f = 0 \}
\]

\[
A \mapsto f_A,
\]

and hence the multiplicity of the eigenvalue 6 satisfies \( m(6) \geq 8 - \dim \text{Ker} H \). If \( A \in \text{Ker} H \), then \( f_A = 0 \) and so \( \nabla f_A = 0 \), which implies that \( \nabla F_A = \xi \). Using (3.1), the tangent vector field \( J\xi \) satisfies

\[
\langle \nabla_v J\xi, v \rangle = 2\langle Av^*, v^* \rangle = 0,
\]

which means that \( J\xi \) is a Killing field on \( \Sigma \). If \( J\xi = 0 \), then \( \nabla F_A \) vanishes identically on the points of the surface, which implies, looking at the expression of \( \nabla F_A \), that \( A = 0 \). Hence \( \dim \text{Ker} H \leq \dim \{ \text{Killing fields on } \Sigma \} \). Finally we get that

\[
m(6) \geq 8 - \dim \{ \text{Killing fields on } \Sigma \}.
\]

In that follows we will use the following Nadirashvili’s result.
**Theorem A** [N] Let \( \Sigma \) be a compact nonorientable surface with Euler characteristic \( \chi(\Sigma) \leq 0 \) and \( \langle , \rangle \) any Riemannian metric on \( \Sigma \). Then the multiplicity of the \( i \)-th eigenvalue of the Laplacian \( \Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma) \) satisfies \( m(\lambda_i) \leq 3 + 2i - 2\chi(\Sigma) \).

First suppose that \( \Sigma \) is a Hamiltonian stable projective plane with a handle, i.e. \( \chi(\Sigma) = -1 \). Then the first positive eigenvalue of \( \Delta : \Omega^0(\Sigma) \to \Omega^0(\Sigma) \) is 6. But Theorem A says that \( m(6) = m(\lambda_1) \leq 7 \). So (3.2) implies that there exists a non-trivial Killing vector field in our surface, which is impossible. This proves part 1.

Suppose now that \( \Sigma \) is a Hamiltonian stable Klein bottle, i.e. \( \chi(\Sigma) = 0 \). Again the first positive eigenvalue of \( \Sigma \) is \( \lambda_1 = 6 \) and Theorem A says that \( m(6) = m(\lambda_2) \leq 5 \). From (3.2), \( \dim \{ \text{Killing fields on } \Sigma \} \geq 3 \), which is impossible.

Suppose now that \( \Sigma \) is a Klein bottle with \( \text{Ind}_0(\Sigma) = 1 \). Then \( \lambda_1 < 6 \), the multiplicity of \( \lambda_1 \) is \( m(\lambda_1) = 1 \) and \( \lambda_2 = 6 \). Using again Theorem A, \( m(6) = m(\lambda_2) \leq 7 \). From (3.2), our Klein bottle admits a nontrivial Killing field. Proposition 5.1 (see section 5) says that \( \Sigma \) is congruent to some finite Riemannian covering of \( K_{n,m} \) for certain integers \( n, m \). Then, from Proposition 5.2, we have that \( \text{Ind}_0(\Sigma) = \text{Ind}_0(K_{n,m}) \geq 6 \), which is a contradiction. This proves part 2.

Finally, if \( \Sigma \) is Hamiltonian stable, then \( \Sigma \) is a projective plane, i.e. \( \chi(\Sigma) = 1 \), and hence our surface is \( \mathbb{R}P^2 \). This finishes the proof. \( \text{q.e.d.} \)

**Theorem 3.4.** Let \( \Phi : \Sigma \to \mathbb{CP}^2 \) be a minimal Lagrangian immersion of a Klein bottle or a projective plane with a handle. Then \( \text{Ind}_1(\Sigma) \geq 1 \).

**Proof:** Let \( \pi : \tilde{\Sigma} \to \Sigma \) be the 2 : 1 orientable Riemannian covering of \( \Sigma \) and \( \tau : \tilde{\Sigma} \to \tilde{\Sigma} \) the change of sheet involution. If \( \text{Ind}_1(\Sigma) = 0 \) then, taking into account the last remarks of section 2, the first eigenvalue \( \lambda_1 \) of \( \Delta : \Omega^0_-(\tilde{\Sigma}) \to \Omega^0_-(\tilde{\Sigma}) \) satisfies \( \lambda_1 \geq 6 \). Hence
\[
\int_{\tilde{\Sigma}} |\nabla f|^2 \geq 6 \int_{\tilde{\Sigma}} f^2, \quad \forall f \in C^\infty(\tilde{\Sigma}) \quad \text{such that} \quad f \circ \tau = -f.
\]

From Theorem 1 in [RS], we can get a nonconstant meromorphic map \( \phi : \tilde{\Sigma} \to S^2 \) satisfying \( \phi \circ \tau = -\phi \) of degree \( d \leq 1 + g \), where \( g \) is the genus of the compact orientable surface \( \tilde{\Sigma} \). Hence we obtain
\[
\int_{\tilde{\Sigma}} |\nabla \phi|^2 \geq 6 \int_{\tilde{\Sigma}} |\phi|^2 = 6 \text{Area}(\tilde{\Sigma}).
\]

But \( \int_{\Sigma} |\nabla \phi|^2 = 8\pi \text{degree}(\phi) \leq 8\pi(1 + g) \). So we get that \( 3\text{Area}(\tilde{\Sigma}) \leq 4\pi(1 + g) \). Now, as \( \text{Area}(\tilde{\Sigma}) = 2\text{Area}(\Sigma) \), using again Corollary 6 in [MU] as in the proof of Theorem 3.1, we have that \( 3\mu \leq 1 + g \) (\( \mu \) being the maximum multiplicity of \( \Phi \)) and the equality implies that \( \Sigma \) is \( \mathbb{R}P^2 \).

As our surface is either a Klein bottle or a projective plane with a handle, we have that \( g = 1 \) or 2 and that the equality (in the above inequality) is not attained, i.e. \( 3\mu < 1 + g \). This is a contradiction and the proof is finished. \( \text{q.e.d.} \)

**Remark 3.5.** In this nonorientable case, we cannot use that the maximum multiplicity \( \mu \) of \( \Phi \) satisfies \( \mu \geq 2 \) when the Euler characteristic \( \chi(\Sigma) \leq 0 \). The author only knows that \( \Phi \) is not an embedding when \( \Sigma \) is a Klein bottle, i.e. \( \chi(\Sigma) = 0 \). (See [M], Theorem 2).
Corollary 3.6. Let $\Phi : \Sigma \to \mathbb{CP}^2$ be minimal Lagrangian immersion of a compact nonorientable surface $\Sigma$. Then $\text{Ind}(\Sigma) \geq 3$ and the equality holds if and only if $\Phi$ is an embedding and $\Phi(\Sigma)$ is $\mathbb{RP}^2$.

Proof: If $\Sigma$ is a projective plane, then as we mentioned in section 2, $\Phi$ is an embedding and $\Phi(\Sigma)$ is $\mathbb{RP}^2$, whose index is 3.

If $\Sigma$ is a compact nonorientable surface with Euler characteristic $\chi(\Sigma) \leq 0$, then $\beta_1(\Sigma) = 1 - \chi(\Sigma)$. So, from (2.2) $\text{Ind}(\Sigma) \geq \beta_1(\Sigma) \geq 4$ when $\chi(\Sigma) \leq -3$. If $\Sigma$ is either a Klein bottle or a projective plane with a handle (i.e. $\chi(\Sigma) = 0$ or $-1$), Theorems 3.3 and 3.4 joint with (2.2) say again that $\text{Ind}(\Sigma) \geq 4$. Finally, if $\Sigma$ is a projective plane with 2 handles, i.e. $\chi(\Sigma) = -2$, and $\Sigma$ is its $2:1$ orientable covering, then the genus of $\bar{\Sigma}$ is 3, and Theorem 3.1 joint with (2.4) say that

$$\text{Ind}(\Sigma) = 3 + \text{Ind}_0(\Sigma) + \text{Ind}_1(\Sigma) = 3 + \text{Ind}_0(\bar{\Sigma}) \geq 4.$$  

This finishes the proof. q.e.d.

4. AREA MINIMIZING SURFACES IN THEIR HAMILTONIAN ISOTOPY CLASSES.

A Lagrangian immersion $\Phi : \Sigma \to \mathbb{CP}^2$ is called Hamiltonian minimal [O2] if it is a critical point of the area functional for Hamiltonian deformations. The corresponding Euler-Lagrange equation says that $\Sigma$ is Hamiltonian minimal if and only if $\text{div} JH = 0$, where $H$ is the mean curvature vector of $\Sigma$ and $\text{div}$ stands for the divergence operator on $\Sigma$.

Suppose now that $\Phi : \Sigma \to \mathbb{CP}^2$ is a minimal Lagrangian immersion of a compact surface $\Sigma$ and that in its Hamiltonian isotopy class there exists a minimizer $\bar{\Sigma}$ for the area. Then, in particular, $\bar{\Sigma}$ is a Hamiltonian minimal Lagrangian surface, and so $\text{div} J\bar{H} = 0$, being $\bar{H}$ the mean curvature vector of $\bar{\Sigma}$. But, using Theorem I in [O3], the deRham cohomology class defined by the mean curvature vector $\alpha(v) = \omega(v, H)$ is invariant under Hamiltonian isotopies. As $\Sigma$ is minimal, this class must vanish, and hence there exists a smooth function $f : \bar{\Sigma} \to \mathbb{R}$ such that $\nabla f = J\bar{H}$. Now, $\text{div} J\bar{H} = 0$ implies that $f$ is a harmonic function and so it is constant, which means that $\bar{\Sigma}$ is also minimal. Moreover, as $\bar{\Sigma}$ is also Hamiltonian stable, our previous results say that:

The Clifford torus of $\mathbb{CP}^2$ is the unique area minimizing surface in its Hamiltonian isotopy class, provided that someone existed.

There exists no area minimizing surfaces in the Hamiltonian isotopy class of a minimal Lagrangian compact orientable surface of $\mathbb{CP}^2$ of genus 2,3 or 4.

There exists no area minimizing surfaces in the Hamiltonian isotopy class of a minimal Lagrangian Klein bottle or a minimal Lagrangian projective plane with a handle of $\mathbb{CP}^2$.

5. MINIMAL LAGRANGIAN KLEIN BOTTLES OF $\mathbb{CP}^2$ ADMITTING A ONE-PARAMETER GROUP OF ISOMETRIES.

In this section we are going to describe the minimal Lagrangian Klein bottles of $\mathbb{CP}^2$ admitting a one-parameter group of isometries, estimating also their Hamiltonian index. To understand it, we need to give a short introduction to the elliptic Jacobi functions. We will follow the notation and the results of [CU].
Given $p \in [0, 1[$, let \( \text{dn}(x, p) = \text{dn}(x) \), \( \text{cn}(x, p) = \text{cn}(x) \) and \( \text{sn}(x, p) = \text{sn}(x) \) be the elementary Jacobi elliptic functions with modulus \( p \). Then, the following properties are well known:

\[
(5.1) \quad \text{sn}^2(x) + \text{cn}^2(x) = 1, \quad \text{dn}^2(x) + p^2 \text{sn}^2(x) = 1, \quad \forall x \in \mathbb{R}
\]

and

\[
(5.2) \quad \frac{d}{dx} \text{dn}(x) = -p^2 \text{sn}(x) \text{cn}(x),
\]

\[
\frac{d}{dx} \text{cn}(x) = -\text{sn}(x) \text{dn}(x),
\]

\[
\frac{d}{dx} \text{sn}(x) = \text{cn}(x) \text{dn}(x).
\]

Also, if

\[
K(p) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - p^2 \sin^2 \theta}}
\]

is the complete elliptic integral of the first kind, then the elliptic functions have the following symmetry and periodicity properties:

\[
(5.3) \quad \text{dn}(x + 2K) = \text{dn}(x), \quad \text{dn}(K - x) = \text{dn}(K + x),
\]

\[
\text{cn}(x + 2K) = -\text{cn}(x), \quad \text{cn}(K - x) = -\text{cn}(K + x),
\]

\[
\text{sn}(x + 2K) = -\text{sn}(x), \quad \text{sn}(K - x) = \text{sn}(K + x).
\]

In particular all of them are periodic of period \( 4K \) and \( \text{dn} \), \( \text{cn} \) are even, i.e. \( \text{dn}(-x) = \text{dn}(x), \text{cn}(-x) = \text{cn}(x) \), meanwhile \( \text{sn} \) is odd, i.e. \( \text{sn}(-x) = -\text{sn}(x) \). Moreover \( \text{dn} \) is a positive function, \( \text{cn}(x) \) vanishes for \( x = (2k + 1)K, k \in \mathbb{Z} \) and \( \text{sn}(x) \) vanishes for \( x = 2kK, k \in \mathbb{Z} \).

Although in [CU] the authors classified the (non-totally geodesic) minimal Lagrangian immersions in \( \mathbb{CP}^2 \) of simply-connected surfaces invariants by a one-parameter group of isometries of \( \mathbb{CP}^2 \), in fact they only used that condition in order to have a non-trivial Killing field on the surface. So, really, they classified the minimal Lagrangian immersions in \( \mathbb{CP}^2 \) of simply-connected surfaces admitting a Killing field.

Let \( \Phi : (\mathbb{R}^2, g) \to \mathbb{CP}^2 \) be a minimal Lagrangian isometric immersion such that \( (\mathbb{R}^2, g) \) admits a Killing field. Then (see [CU]) the metric \( g \) can be written as \( g = e^{2u(x)}(dx^2 + dy^2) \), where \( u \) is a solution of the following problem

\[
(5.4) \quad u''(x) + e^{2u(x)} - e^{-4u(x)} = 0, \quad e^{2u(0)} = b \in [1, \infty], \quad u'(0) = 0.
\]

We will denote the metric by \( g_b \). The solutions of (5.4) are given by

\[
(5.5) \quad e^{2u(x)} = b(1 - q^2 \text{sn}^2(rx, p)),
\]

where

\[
q^2 = 1 - \frac{1 + \sqrt{1 + 8b^2}}{4b^2}, \quad p^2 = b - \frac{1 - \sqrt{1 + 8b^2}}{4b^2}, \quad p^2 = \frac{bq^2}{\sqrt{r^2}}.
\]

Hence the solutions \( u(x) \) of (5.4) are periodic functions with period \( 2K/r \) and satisfy \( u(-x) = u(x), \forall x \in \mathbb{R} \). The only constant solution of (5.4) corresponds to \( b = 1 \) and the associated minimal Lagrangian immersion is the universal covering of the Clifford torus.

On the other hand, in [CU], Theorem 4.1, the minimal Lagrangian immersions of \( (\mathbb{R}^2, g_b) \) into \( \mathbb{CP}^2 \) were explicitly given. Using a reasoning like in [CU], Theorem
4.2, it is not difficult to prove that the minimal Lagrangian immersion \((\mathbb{R}^2, g_b) \to \mathbb{C}P^2\) corresponding to the initial condition \(b\) is the universal covering of a minimal Lagrangian Klein bottle in \(\mathbb{C}P^2\) if and only if the number \(\frac{1+\sqrt{1+8b^2}}{4b}\), which belongs to the interval \([0, 1]\), satisfies the two following conditions:

1. \(\frac{1+\sqrt{1+8b^2}}{4b}\) is a rational number, and
2. if \(\frac{1+\sqrt{1+8b^2}}{4b} = \frac{m}{n}\) with \(m, n \in \mathbb{Z}\), and \((m, n) = 1\), then \(n\) is odd and \(m + 2n = 6\).

We note that \(0 < m < n\) and that \((2n + m)/3\) is an even integer, and \((n + 2m)/3\) and \((n - m)/3\) are odd integers. Moreover, in such cases, the corresponding group \(G_{n, m}\) of transformations of \(\mathbb{R}^2\) is generated by

\[(x, y) \mapsto (x + 4K/r, y), \quad (x, y) \mapsto (-x, y + \sqrt{2mb\pi}/3)\]

and the corresponding minimal Lagrangian immersion is given by

\[\Psi_{n,m}(x, y) = \begin{pmatrix} \lambda \text{dn}(rx)e^{\frac{i(n+m)y}{2m}} + \mu \text{cn}(rx)e^{\frac{i(n+2y)}{2m}} + \nu \text{sn}(rx)e^{\frac{-i(n-y)}{2m}} \end{pmatrix},\]

where

\[\lambda^2 = \frac{n}{2n + m}, \quad \mu^2 = \frac{n + m}{2n + m}, \quad \nu^2 = \frac{n + m}{n + 2m}, \quad r^2 = \frac{n(n + 2m)}{2b^2m^2},\]

and where the modulus of the elliptic Jacobi functions is given by \(p^2 = (n^2 - m^2)/n(n + 2m)\).

If \(K_{n,m} = \mathbb{R}^2/G_{n,m}\) is the associated Klein bottle and \(P : \mathbb{R}^2 \to K_{n,m}\) the projection, then the induced immersion

\[\Phi_{n,m} : K_{n,m} \to \mathbb{C}P^2, \quad P(x, y) \mapsto \Psi_{n,m}(x, y)\]

defines a minimal Lagrangian immersion of the Klein bottle \(K_{n,m}\) in \(\mathbb{C}P^2\).

We can summarize the above reasoning in the following result.

**Proposition 5.1.** Let \(\Phi : \Sigma \to \mathbb{C}P^2\) be a minimal Lagrangian immersion of a Klein bottle \(\Sigma\) admitting a one-parameter group of isometries. Then \(\Phi\) is congruent to some finite Riemannian covering of \(\Phi_{n,m} : K_{n,m} \to \mathbb{C}P^2\) with \(n\) and \(m\) integers such that \(0 < m < n\), \((m, n) = 1\), \(n\) is odd and \(2n + m = 6\), where \(6\) stands for the positive integer multiples of \(6\).

Now we use a similar method to use by Haskins in [H], Theorem E, in order to estimate the Hamiltonian index of the Klein bottles \(K_{n,m}\). Following the proof of Theorem 3.3, the eigenspace of \(\Delta : \Omega^0(K_{n,m}) \to \Omega^0(K_{n,m})\) corresponding to the eigenvalue \(6\) has at least dimension \(7\), and the functions \(\{g_i : K_{n,m} \to \mathbb{R}, 1 \leq i \leq 7\}\) defined by

\[g_1(P(x, y)) = e^{2n(x)} - (1 + 2b^2)/3b^2,\]

\[g_2(P(x, y)) = (\text{dn} \cdot \text{cn})(rx)\cos\left(\frac{3n+4m}{\sqrt{2mb}}y\right),\]

\[g_3(P(x, y)) = (\text{dn} \cdot \text{cn})(rx)\sin\left(\frac{2n+3m}{\sqrt{2mb}}y\right),\]

\[g_4(P(x, y)) = (\text{dn} \cdot \text{sn})(nx)\cos\left(\frac{2n+3m}{\sqrt{2mb}}y\right),\]

\[g_5(P(x, y)) = (\text{dn} \cdot \text{sn})(nx)\sin\left(\frac{2n+3m}{\sqrt{2mb}}y\right),\]

\[g_6(P(x, y)) = (\text{cn} \cdot \text{sn})(nx)\cos\left(\frac{2n+3m}{\sqrt{2mb}}y\right),\]

\[g_7(P(x, y)) = (\text{cn} \cdot \text{sn})(nx)\sin\left(\frac{2n+3m}{\sqrt{2mb}}y\right),\]

are a basis of such \(7\)-dimensional subspace.

Now suppose that \(6 = \lambda_j\), for some positive integer \(j\). Then the Courant nodal Theorem (see [Ch]) says that the number of nodal sets \(n_i\) of the eigenfunction \(g_i,\)
1 \leq i \leq 7$, satisfies $n_i \leq j + 1$. So to estimate $j$ we are going to compute the number of nodal sets of $g_i, 1 \leq i \leq 7$. To do that we will determine the set of zeroes of $f_i = g_i \circ P : D \to \mathbb{R}, 1 \leq i \leq 7$ on the fundamental domain

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \leq x \leq 4K/r, 0 \leq y \leq \sqrt{2}mb\pi/3\}$$

of the Klein bottle $K_{n,m}$.

It is clear that there exists $a \in ]0, K/r[$ such that

$$f_1^{-1}(0) = \{(x, y) \in D \mid x = a, \frac{2K}{r} - a, \frac{2K}{r} + a, \frac{4K}{r} - a; 1 \leq y \leq \sqrt{2}mb\pi/3\}.$$

Hence the number of nodal sets of $g_1$ is $n_1 = 3$.

The zeroes of the function $(dncn)(rx)$ in the interval $[0, 4K/r]$ are $K/r$ and $3K/r$. So

$$f_2^{-1}(0) = \{(x, y) \in D \mid x = K/r, 3K/r; 0 \leq y \leq \sqrt{2}mb\pi/3\} \cup \{(x, y) \in D \mid 0 \leq x \leq 4K/r; y = \frac{\sqrt{2}mb\pi}{2n + m}k, k = 0, 1, \ldots, (2n + m)/3\}.$$

Hence the number of nodal sets of $g_2$ is $n_2 = 2(2n + m)/3$. A similar reasoning proves that the numbers of nodal sets of the function $g_3$ is $n_3 = 2(2n + 3)/3$.

In a similar way, the zeroes of the function $(dncn)(rx)$ in the interval $[0, 4K/r]$ are $0, 2K/r$ and $4K/r$. So

$$f_4^{-1}(0) = \{(x, y) \in D \mid x = 0, 2K/r, 4K/r; 0 \leq y \leq \sqrt{2}mb\pi/3\} \cup \{(x, y) \in D \mid 0 \leq x \leq 4K/r; y = \frac{\sqrt{2}mb\pi}{n + 2m}k, k = 0, 1, \ldots, (n + 2m)/3\}.$$

Hence the number of nodal sets of $g_4$ (and of $g_5$) is $n_4 = n_5 = 2(n + 2m)/3$.

Finally, the zeroes of the function $(cnsn)(rx)$ in the interval $[0, 4K/r]$ are $0, K/r, 2K/r, 3K/r$ and $4K/r$. So

$$f_6^{-1}(0) = \{(x, y) \in D \mid x = 0, K/r, 2K/r, 3K/r, 4K/r; 0 \leq y \leq \sqrt{2}mb\pi/3\} \cup \{(x, y) \in D \mid 0 \leq x \leq 4K/r; y = \frac{\sqrt{2}mb\pi}{n - m}k, k = 0, 1, \ldots, (n - m)/3\}.$$

Hence the number of nodal sets of $g_6$ (and of $g_7$) is $n_6 = n_7 = 4(n - m)/3$.

Hence, as $2n + m > n + 2m$ and $2n + m > 2(n - m)$ we have obtained that if $6 = \lambda_j$ then

$$j + 1 \geq 2(2n + m)/3 \geq 8.$$

Using that fact joint with Theorem 3.4 and (2.2) we finally obtain the following result.

**Proposition 5.2.** Let $\Phi_{n,m} : K_{n,m} \to \mathbb{CP}^2$ be the minimal Lagrangian immersion of the Klein bottle $K_{n,m}$ with $n, m$ integers satisfying $0 < m < n, (n,m) = 1, n \text{ odd}$ and $2n + m = 6$. Then

$$\text{Ind}_0(K_{n,m}) \geq \frac{2(2n + m)}{3} - 2 \geq 6, \quad \text{Ind}(K_{n,m}) \geq \frac{2(2n + m)}{3} \geq 8.$$

To finalize, we are going to study other interesting properties of the Klein bottles family $\{K_{n,m}\}$.

**Proposition 5.3.** Let $\Phi_{n,m} : K_{n,m} \to \mathbb{CP}^2$ be the minimal Lagrangian immersion of the Klein bottle $K_{n,m}$ with $n, m$ integers satisfying $0 < m < n, (n,m) = 1, n \text{ odd}$ and $2n + m = 6$. Then
(1) The first eigenvalue of $\Delta : \Omega^0(K_{n,m}) \to \Omega^0(K_{n,m})$ satisfies
$$\lambda_1(K_{n,m}) < 2 - \frac{1}{2b^3} = 2 - \frac{m^2}{n(n + m)},$$

(2) The area of the Klein bottle $K_{n,m}$ is given by
$$A(K_{n,m}) = \frac{4\pi \sqrt{n}}{3 \sqrt{n + 2m}}((n + 2m)E - mK),$$
where $E = \int_0^{\pi/2} \sqrt{1 - p^2 \sin^2 \theta} d\theta$ is the complete elliptic integral of the second kind.

Proof: Let $f : K_{n,m} \to \mathbb{R}$ be the function defined by $f(P(x,y)) = \text{sn}(rx)$. Then from (5.3) and (5.5) we have that
$$\int_{K_{n,m}} f dA = \int_0^{4K/r} \int_0^{\sqrt{2m} \pi/3} \text{sn}(rx)e^{2u(x)} dydx$$
$$= \frac{\sqrt{2} \sqrt{m} \pi}{3} \int_0^{4K/r} (\text{sn}(rx) - q^2 \text{sn}^3(rx))dx = 0.$$
Hence, if $\lambda_1$ is the first eigenvalue of $\Delta : \Omega^0(K_{n,m}) \to \Omega^0(K_{n,m})$,
$$- \int_{K_{n,m}} f \Delta f dA \geq \lambda_1 \int_{K_{n,m}} f^2 dA.$$
But using (5.1),(5.2),(5.3) and (5.5) we have
$$\Delta f(P(x,y)) = e^{-2u(x)} \frac{d^2}{dx^2} \text{sn}(rx) = -2f(P(x,y)) + \frac{e^{-2u(x)}}{2b^2} f(P(x,y)),$$
and so, using that $e^{-2u(x)} \geq 1/b$, we have that
$$\lambda_1 \int_{K_{n,m}} f^2 dA < 2 \int_{K_{n,m}} f^2 dA - \frac{1}{2b^3} \int_{K_{n,m}} f^2 dA,$$
which proves (1).

On the other hand, from (5.1) and (5.5) it follows that
$$A(K_{n,m}) = \int_0^{4K/r} \int_0^{\sqrt{2m} \pi/3} e^{2u(x)} dydx = \frac{\sqrt{2} \sqrt{m} \pi}{3} \int_0^{4K/r} (b - r^2 + r^2 \text{dn}^2(rx))dx.$$
If $E(u) = \int_0^u \text{dn}^2(y) dy$, then it is known that $E(u + 2K) = E(u) + 2E$, $\forall u \in \mathbb{R}$. Using this property in the above expression we obtain (2).

q.e.d.

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