Extrinsic geometric flows
on foliated manifolds, I

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Abstract

We study deformations of Riemannian metrics on a given manifold equipped with a codimension-one foliation subject to quantities expressed in terms of its second fundamental form. We prove the local existence and uniqueness theorem and estimate the existence time of solutions for some particular cases. The key step of the solution procedure is to find (from a system of quasilinear PDEs) the principal curvatures of the foliation. Examples for extrinsic Newton transformation flow, extrinsic Ricci flow, and applications to foliations on surfaces are given.

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Introduction

Some of the most striking recent results in differential geometry and topology are related to the Ricci flow (see, among the others, [To]), that is the deformation $g_t$ of a given Riemannian metric $g_0$ on a manifold $M$ subject to the partial differential equation (PDE)

$$\partial_t g_t = -2 \text{Ric}_t,$$

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where $\text{Ric}_t$ is the Ricci tensor on the Riemannian manifold $(M,g_t)$. (Note that there is a – due to Hamilton and based on suggestions of Yau – strategy of using the Ricci flow in proving the famous Poincaré Conjecture and the Thurston Geometrization Conjecture for 3-manifolds in a series of preprints by G. Perelman.) On the other hand, since at least 30 years there is continuous interest in the study of Mean Curvature Flow (MCF), i.e., the variation of immersions $F_0 : \overline{M} \to M$ of manifolds $\overline{M}$ into Riemannian manifolds $(M,g)$ subject to the mean curvature, i.e., to the PDE

$$\partial_t F_t = H_t,$$

where $H_t$ is the mean curvature vector of $F_t(\overline{M})$ Also, the second author \cite{Wa} considered the foliated version of the MCF: foliations of Riemannian manifolds which are invariant under the flow of the mean curvature vector of their leaves. The recent years have seen a growing interest in Geometric Flows $\partial_t g_t = h(b_t)$ of different types.

In the paper we introduce Extrinsic Geometric Flow (EGF) as deformations of Riemannian metrics on a given manifold $M$ equipped with a codimension-one foliation subject to quantities expressed in terms of its second fundamental form (suitably extended to a symmetric tensor field of type $(0,2)$ on $TM$). Authors propose EGF as a tool for studying the question, see \cite{RW3}: Under what conditions on $(M,\mathcal{F},g_0)$ the EGF metrics $g_t$ converge to one for which $\mathcal{F}$ is umbilical, geodesic, or minimal?

Here, just for the readers’ convenience, we provide the following definition from foliation theory, see \cite{CC}.

**Definition 1** A family $\mathcal{F} = \{L_\alpha\}_{\alpha \in A}$ of connected subsets of manifold $M^m$ is said to be an $n$-dimensional foliation, if

1) $\bigcup_{\alpha \in A} L_\alpha = M^m$,
2) $\alpha \neq \beta \Rightarrow L_\alpha \cap L_\beta = \emptyset$,
3) for any point $p \in M$ there exists a $C^r$-chart (local coordinate system) $\varphi_q : U_q \to \mathbb{R}^m$ such that $q \in U_q$, $\varphi_q(q) = 0$, and if $U_q \cap L_\alpha \neq \emptyset$ the connected components of the sets $\varphi(q(U_q \cap L_\alpha))$ are given by equations $x_{n+1} = c_{n+1}, \ldots, x_m = c_m$, where $c_j$’s are constants. The sets $L_\alpha$ are immersed submanifolds of $M$ called leaves of $\mathcal{F}$. The family of all the vectors tangent to the leaves is an integrable subbundle of $TM$ denoted by $T\mathcal{F}$. If $M$ carries a Riemannian structure, $T\mathcal{F}^\perp$ denotes the subbundle of all the vectors orthogonal to the leaves. A foliation $\mathcal{F}$ is said to be orientable (resp., transversely orientable) if the bundle $T\mathcal{F}$ (resp. $T\mathcal{F}^\perp$) is orientable.
Throughout the paper, \((M^{n+1}, g)\) is a Riemannian manifold with a codimension one transversely oriented foliation \(\mathcal{F}\), \(\nabla\) the Levi-Civita connection of \(g\), \(N\) the positively oriented unit normal to \(\mathcal{F}\), \(A : X \in T\mathcal{F} \mapsto -\nabla_X N\) the Weingarten operator of the leaves, which we extend to a \((1,1)\)-tensor field on \(TM\) by \(A(N) = 0\).

Denote \(\hat{\cdot}\) the \(\mathcal{F}\)-component of a vector. The definition \(\hat{S}(X, Y) := S(\hat{X}, \hat{Y})\) of the \(\mathcal{F}\)-truncated \((0,2)\)-tensor \(S\) will be helpful throughout the paper.

Let \(b : T\mathcal{F} \times T\mathcal{F} \to \mathbb{R}\) be the second fundamental form of (the leaves of) \(\mathcal{F}\) with respect to \(N\), and \(\hat{b}\) its extension to the \(\mathcal{F}\)-truncated symmetric \((0,2)\)-tensor field on \(M\). Notice that \(\hat{b}(N, \cdot) = 0\) and \(\hat{b}(X, Y) = g(A(X), Y)\).

The power sums of the principal curvatures \(k_1, \ldots, k_n\) of the leaves of \(\mathcal{F}\) (the eigenvalues of \(A\)) are given by

\[
\tau_j = k_1^j + \ldots + k_n^j = \text{Tr}(A^j) \quad (j \geq 0).
\]

They can be expressed using elementary symmetric functions \(\sigma_0, \ldots, \sigma_n\)

\[
\sigma_j = \sum_{i_1 < \ldots < i_j} k_{i_1} \cdot \ldots \cdot k_{i_j} \quad (0 \leq j \leq n),
\]
called mean curvatures in the literature (see Section 2.2).

**Remark 1** Certainly, the functions \(\tau_{n+i} (i > 0)\), are not independent: they can be expressed as polynomials of \(\hat{\tau} = (\tau_1, \ldots, \tau_n)\), using the Newton formulae, which in matrix form look like

\[
T_n \begin{pmatrix} \sigma_1 \\ \sigma_2 \\ \vdots \\ \sigma_n \end{pmatrix} = \begin{pmatrix} \tau_1 \\ \tau_2 \\ \vdots \\ \tau_n \end{pmatrix}, \quad \text{where } T_n = \begin{pmatrix} 1 & -2 & \ldots & (-1)^{n-1} \\ \tau_1 & -\tau_2 & \ldots & (-1)^n \\ \tau_{n-1} & -\tau_{n-2} & \ldots & (-1)^{n-1} \end{pmatrix}.
\]

Hence \(\sigma\)'s can be expressed by \(\tau\)-s using \(T_n^{-1}\). Moreover, we have the identities

\[
\tau_i = \det \begin{pmatrix} \sigma_1 & 1 & \ldots & 0 \\ 2\sigma_2 & \sigma_1 & \ldots & 0 \\ \vdots & \sigma_{i-1} & \sigma_i & \ldots \\ i\sigma_i & \sigma_{i-1} & \ldots & \sigma_1 \end{pmatrix}, \quad i = 1, 2, \ldots
\]

Both formulae can be used to express \(\tau_{n+i}\)'s as polynomials of \(\tau_1, \ldots, \tau_n\).

We study two types of evolution of Riemannian structures, depending of functions \(f_j (0 \leq j < n)\) in use (at least one of them is not identically zero):

(a) \(f_j \in C^2(M \times \mathbb{R})\), and

(b) \(f_j = \bar{f}_j(\hat{\tau}, \cdot)\), where \(\bar{f}_j \in C^2(\mathbb{R}^{n+1})\).

Sometimes we will assume that \(f_j = \hat{f}_j(\hat{\tau})\) with \(\hat{f}_j \in C^2(\mathbb{R}^n)\).
Definition 2 Given functions $f_j$ of type either (a) or (b), a family $g_t$, $t \in [0, \varepsilon)$, of Riemannian structures on $(M, F)$ will be called an Extrinsic Geometric Flow (EGF), whenever

$$\partial_t g_t = h(b_t), \quad \text{where } h(b_t) = h_t = \sum_{j=0}^{n-1} f_j \hat{b}_j^t.$$  \hspace{1cm} (1)

Here, $\hat{b}_j^t$ are symmetric $(0,2)$-tensor fields on $M$ $g_t$-dual to $(A_t)^j$.

Hence, the EGF is an evolution equation which deforms Riemannian metrics by evolving them along $F$ in the direction of the tensor $h(b_t)$. The EGF preserves $N$ unit and perpendicular to $F$, the $F$-component of the vector does not depend on $t$. One can interpret EGF as integral curves of a vector field $g \mapsto h(b)$, $h(b)$ being the right hand side of (1), on the space $\mathcal{M}^k(M) = C^k(M, S^+_2(M))$ of Riemannian $C^k$-structures on $M$. Here, certainly, $S^+_2(M)$ is the bundle of positive definite symmetric $(0,2)$-tensors on $M$. This vector field may depend on time or not.

The choice of the right hand side in (1) for $h(b_t)$ seems to be natural: the powers $\hat{b}_j$ are the only $(0,2)$-tensors which can be obtained algebraically from the second fundamental form $b$, while $\tau_1, \ldots, \tau_n$ (or, equivalently, $\sigma_1, \ldots, \sigma_n$) generate all the scalar invariants of extrinsic geometry. The powers $\hat{b}_j$ with $j > 1$ are meaningful: for example, the EGF with $h$ produced by

- the extrinsic Ricci curvature tensor $\text{Ric}^\text{ex}(b)$, see (44), depends on $\hat{b}_1, \hat{b}_2$;
- the extrinsic Newton transformation $T_i(b) = \sigma_i g - \sigma_{i-1} \hat{b}_1 + \ldots + (-1)^i \hat{b}_i$ $(0 < i < n)$, see Section 4, depends on all $\hat{b}_j$ ($1 \leq j \leq i$).

In this paper we investigate existence/uniqueness and properties of metrics $g_t$ satisfying (1), discuss in more details some particular cases and study some examples. In Section 1 we collect main results. Section 2 contains auxiliary results. In Section 3 we prove main results. Section 4 is devoted to particular cases and examples. Studying EGF will be continued in a series of papers remaining in preparation (see [RW1], [RW3]).

1 Main results

We shall omit index $t$ for $t$-dependent tensors $A, b, \hat{b}_j$ and functions $\tau_i, \sigma_i$.

The following theorem concerns the EGF of type (a) and is essential in the proof of Theorem 2 (for the EGF of type (b)).
**Theorem 1** Let \((M, g_0)\) be a compact Riemannian manifold with a codimension-1 foliation \(\mathcal{F}\). Given functions \(\hat{f}_j \in C^2(\mathbb{R} \times \mathbb{R})\) there is a unique smooth solution \(g_t\) to (1) of type (a) defined on some positive time interval \([0, \varepsilon)\).

In particular, there exists a unique smooth solution \(g_t, \ t \in [0, \varepsilon)\) to

\[
\partial_t g_t = \sum_{j=0}^{n-1} a_j(t) \hat{b}_j, \quad a_j \in C^2(\mathbb{R}).
\]

Although (1) of type (b) consists of first order non-linear PDEs, the corresponding power sums \(\tau_i (i > 0)\) satisfy the infinite quasilinear system

\[
\partial_t \tau_i + \frac{i}{2} \{\tau_{i-1} N(f_0) + \sum_{j=1}^{n-1} \frac{j f_j}{i+j-1} \tau_{i+j-1} N(\tau_{i+j-1}) + \tau_{i+j-1} N(f_j)\} = 0, \quad (2)
\]

where \(N(f_j) = \sum_{s=1}^{n} f_{j+s} N(\tau_s)\). By Proposition 4 (Section 2.2), the matrix of the \(n\)-truncated system (2) (where \(\tau_{n+i}\)'s are replaced by suitable polynomials of \(\tau_1, \ldots, \tau_n\)) has the form \(\hat{A} + \hat{B}\), where \(\hat{A} = (\hat{A}_{ij})\),

\[
\hat{A}_{ij} = \frac{i}{2} \sum_{m=0}^{n-1} \tau_{i+m-1} f_{m, \tau_j}, \quad \hat{B} = \sum_{m=1}^{n-1} \frac{m}{2} f_m \cdot (B_{n,1})^{m-1}, \quad (3)
\]

and \(B_{n,1}\) is the **generalized companion matrix** (see Section 2.2) to the characteristic polynomial of \(A_N\). Recall that an \(n\)-by-\(n\) matrix is **hyperbolic** (see Section 2.1) if its right eigenvectors are real and span \(\mathbb{R}^n\).

**Example 1** (a) If \(\partial_t g_t = f(\tau, t) \hat{g}_t\) (i.e., \(f_j = \delta_{j0} f\)), then, by (3), \(\hat{B} = 0\) and \(\hat{A} = (\hat{a}_{ij})\), where \(\hat{a}_{ij} = (i/2) \tau_{i-1} f_{\tau_j}(\tau, 0)\). System (2) reduces to

\[
\partial_t \tau_i + \frac{i}{2} \tau_{i-1} \sum_{j=1}^{n} f_{\tau_j}(\tau, t) N(\tau_j) = 0, \quad i = 1, 2, \ldots \quad (4)
\]

The matrix \(\hat{A}\) of \(n\)-truncated system (1) is hyperbolic if for any \(p \in M\)

- either \(2 \text{Tr} \hat{A} = \sum_i i \tau_{i-1} f_{\tau_i} \neq 0\) or \(\hat{A} \equiv 0\) on the \(N\)-curve through \(p\). \((H_1)\)

(b) If \(\partial_t g_t = f(\tau, t) \hat{b}_1\) (i.e., \(f_j = \delta_{j1} f\)) then, again by (3), \(\hat{B} = \frac{1}{2} f(\tau, 0) \text{id}\) and \(\hat{A} = (\hat{a}_{ij})\), where \(\hat{a}_{ij} = (i/2) \tau_{i-1} f_{\tau_j}(\tau, 0)\). System (2) reduces to

\[
\partial_t \tau_i + \frac{1}{2} f(\tau, t) N(\tau_i) + \frac{i}{2} \tau_i \sum_{j=1}^{n} f_{\tau_j}(\tau, t) N(\tau_j) = 0, \quad i = 1, 2, \ldots \quad (5)
\]

The matrix \(\hat{A} + \hat{B}\) of \(n\)-truncated system (5) is hyperbolic if for any \(p \in M\)

- either \(2 \text{Tr} \hat{A} = \sum i \tau_{i-1} f_{\tau_i} \neq 0\) or \(\hat{A} \equiv 0\) on the \(N\)-curve through \(p\). \((H_2)\)
The central result of our paper is the following.

**Theorem 2 (Short time existence)** Let \((M, g_0)\) be a compact Rie mannian manifold with a codimension-one foliation \(\mathcal{F}\) and a unit normal \(N\). If the matrices \(\tilde{A} + \tilde{B}\) and \(\tilde{B}\) of (3) are hyperbolic for all \(p \in M\) and \(t = 0\), then the EGF (1) of type (b) has a unique smooth solution \(g_t\) defined on some positive time interval \([0, \varepsilon)\).

The proof of Theorem 1 follows standard methods of the theory of first order PDEs with one space variable. The proof of Theorem 2 (Section 3.2) consists of several steps.

1) The power sums \(\tau_i\) are recovered on \(M\) (as a unique solution to a quasilinear hyperbolic system of PDEs) for some positive time interval \([0, \varepsilon)\), see Lemmas 1, 2 and 6, and Proposition 4.

2) Given \((\tau_i)\) (of Step 1), the metric \(g_t\) is recovered on \(M\) (as a unique solution to certain quasilinear system of PDEs), see Theorem 1 and Lemma 3.

3) The \(\tau_i\)-s of the \(g_t\)-principal curvatures of \(\mathcal{F}\) (of Step 2) are shown to coincide with \(\tau_i\) (of Step 1), see Theorem 1 and Lemmas 5 and 7.

Let us remark that the solution in Theorem 2 is unique if only \(\tilde{A} + \tilde{B}\) is hyperbolic. For \(f_j = 0\) \((j \geq 2)\), Theorem 2 holds under weaker condition that only the matrix \(\tilde{A}\) is hyperbolic for all \(p \in M\) and \(t = 0\).

Denote by \(L_Z\) the Lie derivative along a vector field \(Z\).

**Corollary 1** Let \((M, g_0)\) be a compact Riemannian manifold with a codimension-1 foliation \(\mathcal{F}\) and a unit normal \(N\). If \(f \in C^2(\mathbb{R}^{n+1})\) and the condition \((H_2)\) is satisfied at \(t = 0\) and any \(p \in M\), then there exists a unique smooth solution \(g_t\), \(t \in [0, \varepsilon)\) to the EGF

\[
\partial_t g_t = f(\vec{\tau}, t) \hat{b}_1, \quad t \in [0, \varepsilon)
\]

for some \(\varepsilon > 0\). Furthermore, \(g_t\) can be determined from the system \(L_Z g_t = 0\) with \(Z_t = \partial_t + \frac{1}{2} f(\vec{\tau}, t) N\), where \(\vec{\tau}\) are the unique smooth solution to (2).

**Corollary 2** Let \((M, g_0)\) be a compact Riemannian manifold with a codimension-1 foliation \(\mathcal{F}\) and a unit normal \(N\). If \(f \in C^2(\mathbb{R}^{n+1})\) and the condition \((H_1)\) is satisfied at \(t = 0\) and any \(p \in M\), then there exists a unique smooth solution \(g_t\) to the EGF

\[
\partial_t g_t = f(\vec{\tau}, t) \hat{g}_t, \quad t \in [0, \varepsilon)
\]

for some \(\varepsilon > 0\). Furthermore, \(\hat{g}_t = \hat{g}_0 \exp(\int_0^t f(\vec{\tau}, t) dt)\), where the power sums \(\vec{\tau}\) are the unique solution to (4).
In the particular case \( f_1 = C = \text{const}, f_j = 0 \) \( (j \neq 1) \), the system (2) (see also (5) for \( f = C \)) is reduced to the linear PDE
\[
\partial_t \tau_i + (C/2)N(\tau_i) = 0.
\]
The above equation can be interpreted on \( M \times \mathbb{R} \) by saying that \( \tau_i \) is constant along the orbits of the vector field \( X = \partial_t + (C/2)N \). If \( (\psi_t) \) denotes the flow of \( (C/2)N \) on \( M \), then the flow \( (\phi_t) \) of \( X \) is given by \( \phi_t(p,s) = (\psi_t(p), t + s) \) for \( p \in M \) and \( s \in \mathbb{R} \), therefore \( \phi_t \) maps the level surface \( M_s = M \times \{s\} \) onto \( M_{t+s} \), in particular, \( M_0 \) onto \( M_t \). This implies

**Corollary 3** If \( f_1 = \text{const} \) and \( f_j = 0 \) for all \( j \neq 1 \) (for the EGF), then for all \( i \) and \( t \), the following equality holds: \( \tau^*_i = \tau^*_i \circ \phi_{-t} \). In particular, if \( \tau^*_i = \text{const} \) for some \( i \), then \( \tau_i = \text{const} \) for all \( t \).

Recall that a compact manifold \( M \) equipped with a codimension-one foliation \( F \) admits a Riemannian structure \( g \) for which all the leaves are minimal \( (\tau_1 = 0 \) in the above terminology) if and only if \( F \) is topologically taut, that is every its leaf meets a loop transverse to the foliation \( [Su] \). The known proofs of existence of such metrics use Hahn-Banach Theorem and do not show how to construct them. Above observations show how to produce a 1-parameter family of metrics with \( \tau_{2j+1} = 0 \) (with fixed \( j \)) starting from one of them.

For multi-index \( \alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_+ \) define \( \vec{\tau}^\alpha := \tau_1^{\alpha_1} \ldots \tau_n^{\alpha_n} \). Set \( J_{m,l} = \{ \alpha \in \mathbb{Z}^n_+ : \sum_j \alpha_j = m, \sum_j j \alpha_j = l \} \).

Recall that a vector field on a manifold \( M \) is **complete**, if any its trajectory \( c : t \mapsto c(t) \) can be extended to the whole range \( \mathbb{R} \) of parameter \( t \). If \( M \) carries a Riemannian structure \( g \) and a vector field \( X \) has bounded length, then the completeness of \( (M, g) \) is sufficient for the completeness of \( X \).

**Proposition 1** Let \( (M, g_0) \) be a Riemannian manifold with a codimension-1 foliation \( F \) and a complete unit normal field \( N \). Given \( c_\alpha \in \mathbb{R} \) \( (\alpha \in J_{m,l}) \) and \( m, l \in \mathbb{N} \), define functions \( f_t = \sum_{\alpha \in J_{m,l}} c_\alpha (\vec{\tau}^t)^\alpha \) (where \( \vec{\tau}^t \) is, as usual, the vector of power sums of principal curvatures of the leaves) and set
\[
T = \infty \quad \text{if} \quad N(f_0) \geq 0 \quad \text{on} \quad M \quad \text{and} \quad T = -\frac{2}{l+1} / \inf_M N(f_0) \quad \text{otherwise}.
\]

Then the EGF \( \partial_t g_t = f_t \hat{b}_1 \), compare (6), has a unique smooth solution on \( M \) for \( t \in [0,T) \) and does not possess one for \( t \in [0,T] \).
We apply EGF to totally umbilical foliations (that is, such that the Weingarten operator $A$ is proportional to the identity at any point, among them totally geodesic foliations appear when $A = 0$) and to foliations on surfaces. First, we shall show that EGF preserve total umbilicity of $F$.

**Proposition 2** Let $(M, g_0)$ be a Riemannian manifold endowed with a codimension-1 totally umbilical foliation $F$. If $g_t$ ($0 \leq t < \varepsilon$) provide the EGF of type (b) on $(M, F)$, then $F$ is totally umbilical for any $g_t$.

**Proposition 3** Let $(M, g_0)$ be a Riemannian manifold, and $F$ a codimension-1 totally umbilical foliation on $M$ with the normal curvature $\lambda_0$ and a complete unit normal field $N$. Set

$$T = \infty \text{ if } N(\psi(\lambda_0)) \geq 0 \text{ on } M, \text{ and } T = -2/\inf_M N(\psi(\lambda_0)) \text{ otherwise},$$

where $\psi(\lambda) = \sum_{j=0}^{n-\lambda} f_j(n\lambda, \ldots, n\lambda^n) \lambda^j$. Then the EGF with $h = \sum_{j=0}^{n-\lambda} f_j(\bar{\tau}) \hat{b}_j$ has a unique smooth solution $g_t$ on $M$ for $t \in [0, T)$, and does not possess one for $t \geq T$. Moreover, $F$ is $g_t$-totally umbilical and $\hat{g}_t = \hat{g}_0 \exp(\int_0^t \psi(\lambda_t) dt)$, where $\lambda_t$ is a unique smooth solution to the PDE

$$\partial_t \lambda_t + \frac{1}{2} N(\psi(\lambda_t)) = 0. \quad (8)$$

## 2 Preliminaries

### 2.1 Hyperbolic quasi-linear PDEs

Here, we recall known results on quasi-linear PDEs.

Let $A = (a_{ij}(x, t, \vec{u}))$ be an $n \times n$ matrix, $\vec{b} = (b_i(x, t, \vec{u}))$ – an $n$-vector. A first order *quasilinear* system of PDEs, $n$ equations in $n$ unknown functions $\vec{u} = (u_1, \ldots, u_n)$ and two variables $x, t \in \mathbb{R}$, has the form

$$\partial_t \vec{u} + A(x, t, \vec{u}) \partial_x \vec{u} = \vec{b}(x, t, \vec{u}). \quad (9)$$

When the coefficient matrix $A$ and the vector $\vec{b}$ are functions of $x$ and $t$ only, the system is just *linear*; if $\vec{b}$ only depends also on $\vec{u}$, the system is said to be *semilinear*. The initial *value problem* for (9) with initial surface $\Pi = \{t = 0\}$ and given smooth data $\vec{u}_0$, $A$ and $\vec{b}$ consists in finding smooth $\vec{u}$ such that (9) and $\vec{u}(x, 0) = \vec{u}_0(x)$ are satisfied.
Definition 3 The system \((9)\) is hyperbolic in the \(t\)-direction at \((x,t,\vec{u})\) (in an appropriate domain of the arguments of \(A\) and \(\vec{b}\)) if the right eigenvectors of \(A\) are real and span \(\mathbb{R}^n\). For a solution \(\vec{u}(x,t)\) to \((9)\), the corresponding eigenvalues \(\lambda_i(x,t,\vec{u})\) are called the characteristic speeds. The system is strictly hyperbolic if \(\lambda_i(x,t,\vec{u})\) are distinct. For the hyperbolic system \((9)\), the vector field \(\partial_t + \lambda_i(x,t,\vec{u})\partial_x\) is called the \(i\)-characteristic field, and its integral curves are called \(i\)-characteristics.

Remark that the hyperbolicity of \(A\) is equivalent to any of the properties:

1. “\(A\) has real eigenvalues \(\lambda_1 \leq \ldots \leq \lambda_n\) and simple elementary divisors” (i.e., \(A\) has no Jordan cells of order greater than one), and
2. “\(A\) is diagonal in some affine basis”.

Hence, the hyperbolic matrix \(A\) can be represented as \(A = RDR^{-1}\), where \(R\) is a nonsingular \(n \times n\) matrix and \(D\) is a diagonal matrix. The columns \(r_i\) of \(R\) are the right eigenvectors of \(A\), whereas the rows of \(R^{-1}\) are left eigenvectors of \(A\). A hyperbolic system reduces to the ODEs for its characteristic fields. Indeed, multiplying \((9)\) by \(r_i^T\) and using \(\frac{d}{dt}\vec{u} = \partial_t\vec{u} + \lambda_i\partial_x\vec{u}\), we get the ODE \(r_i^T \frac{d\vec{u}}{dt} = r_i^T \vec{b}\) along the characteristic \(dx/dt = \lambda_i(x,t,\vec{u})\).

Theorem A [HW] Let the quasi-linear system \((9)\) be such that

1. it is hyperbolic in the \(t\)-direction in \(\Omega = \{|x| \leq a, 0 \leq t \leq s, \|\vec{u}\|_\infty \leq r\}\) for some \(s, r > 0\),
2. the matrix \(A\) and the vector \(\vec{b}\) are \(C^1\)-regular in \(\Omega\).

If the initial condition

\[\vec{u}(0,x) = \vec{u}_0(x), \quad x \in [-a,a]\] (10)

has \(C^1\)-regular \(\vec{u}_0\) in \([-a,a]\) and \(\|\vec{u}_0\|_\infty < r\), then \((9) - (10)\) admit a unique \(C^1\)-regular solution \(\vec{u}(x,t)\) in \(\bar{\Omega} = \{(x,t) : |x| + Kt \leq a, 0 \leq t \leq \varepsilon\}\), with \(K = \max\{|\lambda_i(x,t,\vec{u})| : (x,t,\vec{u}) \in \Omega, 1 \leq i \leq n\}\).

Example 2 For any function \(\psi \in C^1(\mathbb{R})\), we can multiply the equation

\[\partial_t u + \psi(u) \partial_x u = 0\] (11)

across by \(\psi'(u)\), and obtain \(\partial_t \psi + \psi \partial_x \psi = 0\) (the Burgers’ equation). Thus, the behaviour of the solutions to \((11)\) (for \(t\) before the first singular value) is not expected to be much different from that for Burgers’ equation.
2.2 The generalized companion matrix

Let \( P_n = \lambda^n - p_1 \lambda^{n-1} - \ldots - p_{n-1} \lambda - p_n \) be a polynomial over \( \mathbb{R} \) and \( \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \) be the roots of \( P_n \) for \( n > 1 \). Hence, \( p_i = (-1)^{i-1} \sigma_i \), where \( \sigma_i \) are elementary symmetric functions of the roots \( \lambda_i \).

**Definition 4** The generalized companion matrices of \( P_n \) are defined as

\[
\hat{C}_g = \begin{pmatrix}
0 & \frac{c_{n-1}}{c_n} & 0 & \ldots & 0 \\
\frac{c_{n-1}}{c_n} & 0 & \frac{c_{n-2}}{c_n} & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{c_{n-1}}{c_n} & 0 & \ldots & 0 & \frac{c_2}{c_n} \\
c_{n-1} c_{n-1} & c_{n-1} c_{n-2} & \ldots & c_{n-1} c_2 & c_{n-1} p_n \\
\end{pmatrix}
\]

or

\[
\hat{C}_g = \begin{pmatrix}
c_{1} p_1 & c_{2} p_2 & \ldots & c_{n-1} p_{n-1} & c_{n} p_n \\
\frac{c_1}{c_n} & 0 & \ldots & 0 & 0 \\
\frac{c_2}{c_n} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\frac{c_{n-1}}{c_n} & 0 & \ldots & 0 & 0 \\
0 & 0 & \ldots & 0 & \frac{c_n}{c_n} \\
\end{pmatrix},
\]

where \( c_1 = 1 \) and \( c_i \neq 0 \) (\( i > 1 \)) are arbitrary numbers.

Notice that \( \hat{C}_g \) acts on \( \mathbb{R}^n \) as \( \bar{x} \rightarrow \hat{C}_g \bar{x} \), where \( \bar{x}=(x_1, \ldots, x_n) \). Inverting the order of indices, i.e., taking \( (x_n, \ldots, x_1) \), one may describe this action by \( \hat{C}_g \). If all \( c_i \)'s are equal to 1, the matrix \( \hat{C}_g \) reduces to the standard companion matrix \( C_g \) of \( P_n \). The explicit formulae (polynomials) for entries in powers of \( C_g \) and some applications to the theory of the symmetric functions are given in [CL]. The following matrix \( B_{n,1} \) (the generalized companion matrix with \( c_i = \frac{n}{n-i+1} \)) plays the key role in the paper:

\[
B_{n,1} = \begin{pmatrix}
0 & \frac{1}{\sigma_2} & 0 & \ldots & 0 \\
0 & 0 & \frac{1}{\sigma_3} & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \ldots & 0 & \frac{1}{\sigma_n} \\
(-1)^{n-1} \frac{n}{\sigma_n} & (-1)^{n-2} \frac{n}{\sigma_{n-1}} & \ldots & -\frac{n}{\sigma_2} & \sigma_1 \\
\end{pmatrix}.
\]  

**Lemma 1** The generalized companion matrices have the properties

a) the characteristic polynomial of \( \hat{C}_g \) (or \( \hat{C}_g^t \)) is \( P_n \).

b) \( v_j = (1, \frac{c_{n-1}}{c_n-1} \lambda_j, \frac{c_{n-2}}{c_n-2} \lambda_j^2, \ldots, c_n \lambda_j^{n-1}) \) is the eigenvector of \( \hat{C}_g \) for the eigenvalue \( \lambda_j \), resp., \( w_j = (c_n \lambda_j^{n-1}, \ldots, \frac{c_n}{c_n-1} \lambda_j^2, \frac{c_n}{c_n-1} \lambda_j, 1) \) is the eigenvector of \( \hat{C}_g \) for \( \lambda_j \).

c) \( \hat{C}_g V = VD \), where \( V = \{ \frac{c_n}{c_n-1} \lambda_j^{i-1}, 1 \leq i,j \leq n \} \) is a Vandermonde type matrix, and \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \) a diagonal matrix. (If all \( \lambda_i \)'s are distinct, then obviously \( V^{-1} \hat{C}_g V = D \)).

**Proof.** a) We will show by induction for \( n \) that \( \det |\lambda \text{id}_n - \hat{C}_g| = P_n \), hence the eigenvalues \( \lambda_1 \leq \ldots \leq \lambda_n \) of \( \hat{C}_g \) coincide with the roots of \( P_n \).
Expanding by co-factors down the first column, we obtain
\[
\det |\lambda \text{id}_n - \hat{C}_g| = \lambda P_{n-1} - (-1)^{n-1} c_n p_n \prod_{i=1}^{n-1} \left( -\frac{c_i}{c_{i+1}} \right),
\]
where \( P_{n-1} = \lambda^{n-1} - p_1 \lambda^{n-2} - \ldots - p_{n-2} \lambda - p_{n-1} \) (by the induction assumption) is a certain polynomial of degree \( n - 1 \). Notice that \( c_n \prod_{i=1}^{n-1} \frac{c_i}{c_{i+1}} = 1 \). This completes the proof of the claim.

b) The direct computation shows us that
\[
(\lambda_j \text{id}_n - \hat{C}_g) \vec{v}_j = 0, \quad (\lambda_j \text{id}_n - \hat{C}_g) \vec{w}_j = 0.
\]

c) Hence \( \hat{C}_g V = V D \), where \( D = \text{diag}(\lambda_1, \ldots, \lambda_n) \) is a diagonal matrix. If \( \{\lambda_j\} \) are pairwise distinct, then \( \det V \neq 0 \) and obviously \( V^{-1} \hat{C}_g V = D \). □

Consider the infinite system of linear PDEs with functions \( f_j \in C^2(\mathbb{R}^2) \)
\[
\partial_t \tau_i + \frac{i}{2} \sum_{j=1}^{n-1} \frac{j}{i+j-1} f_j(t, x) \partial_x \tau_{i+j-1} = 0, \quad i = 1, 2, \ldots \quad (13)
\]
where \( \tau_i \) \((i \in \mathbb{N})\) are the power sums of smooth functions \( \lambda_i(t, x) \) \((i \leq n)\). Let \( \sigma_j \) \((j \leq n)\) be the elementary symmetric functions of \( \{\lambda_i(t, x)\} \).

**Proposition 4** The matrix of \( n \)-truncated system \((13)\) (where \( \tau_{n+i}'s \) are eliminated using suitable polynomials of \( \tau_1, \ldots, \tau_n \), as described in Remark [7]) is \( \tilde{B} = \sum_{m=1}^{n-1} f_m B_{n,m-1} \), where
\[
B_{n,m} = \frac{m+1}{2} (B_{n,1})^m \quad (14)
\]
and \( B_{n,1} \) is the generalized companion matrix \((12)\). The eigenvalues of \( \tilde{B} \) are
\[
\tilde{\lambda}_j = \frac{1}{2} \sum_{m=1}^{n-1} m f_m \lambda_j^{m-1}.
\]
while the corresponding eigenvectors are given by \( v_i = (1, 2 \lambda_i, 3 \lambda_i^2, \ldots, n \lambda_i^{n-1}) \).

**Proof.** We need to prove \((14)\) only. Replacing \( \partial_x \tau_{n+j} \) in \((13)\) by linear combinations of \( \partial_x \tau_i \) \((i \leq n)\) due to \((16)\) in what follows, we find the \((i, j)\) entry of \( B_{n,m} \)
\[
b^{(n,m)}_{ij} = \begin{cases} 
\frac{i(m+1)}{2(i+m)} \beta_{i+m}^j & \text{if } i + m \leq n, \\
(-1)^{i-j} \frac{i(m+1)}{2j} \beta_{i+m-n, j} & \text{if } i + m > n. 
\end{cases} \quad (15)
\]
In particular, \( B_{n,0} = (1/2) \text{id}_n \). The equality (14) follows directly from
\[
B_{n,m} = \frac{m+1}{m} B_{n,m-1} B_{n,1}.
\]

Notice that the last formulae are true for \( m = 1 \). We shall show that all the \((i,j)\)-entries of the matrices \( \frac{m+1}{m} B_{n,m-1} B_{n,1} \) and \( B_{n,m} \) coincide.

First, let \( i + m - 1 \leq n \). Then,
\[
\sum_{s=1}^{m+1} b_{is}^{(n,m-1)} b_{sj}^{(n,1)} \tag{15}
= \frac{m+1}{m} \sum_{s=1}^{m+1} \sum_{i=1}^{n} \frac{im}{2(i + m - 1)} \delta_{i+m-1}^{s} \delta_{j-1}^{s} i + m - 1 \delta_{i+m-1}^{s} i + m = \frac{i(m+1)}{2(i + m)} \delta_{i+m}^{s} b_{ij}^{(n,m)}. \tag{15}
\]

Now, let \( \tilde{m} = i + m - 1 - n > 0 \). Then
\[
\sum_{s=1}^{m+1} b_{is}^{(n,m-1)} b_{sj}^{(n,1)} \equiv \frac{m+1}{m} \left[ (-1)^{n-j+1} \frac{im}{2(j-1)} \beta_{n,\tilde{m},j-1}^{(n,1)} + \frac{im}{2n} \beta_{n,\tilde{m},n} (-1)^{n-j} \frac{n}{j} \sigma_{n-j+1} \right] \equiv \frac{i(m+1)}{2j} (-1)^{n-j} \beta_{n,\tilde{m}+1,j} \equiv b_{ij}^{(n,m)} \tag{15}\]

By (14) and (12), if \( \lambda_j \)'s are pairwise different, then the matrix \( B_{n,m} (m > 0) \) is hyperbolic. \( \square \)

**Lemma 2** The coefficients \( \beta_{n,m,i} \) of the decomposition
\[
\frac{1}{n + m} \partial_x \tau_{n+m} = \sum_{i=1}^{n} (-1)^{n-i} \frac{1}{i} \beta_{n,m,i} \partial_x \tau_i, \quad m > 0 \tag{16}
\]
satisfy the following recurrence relations:
\[
\beta_{n,1,i} = \sigma_{n-i+1}, \quad \beta_{n,m,i} = \beta_{n+1,m-1,n+1} \sigma_{n-i+1} - \beta_{n+1,m-1,i}, \quad m > 1, \tag{17}
\]
\[
\beta_{n,m,i} = \beta_{n+j,m,i+j} \quad (1 \leq i \leq n, \ m, j > 0). \tag{18}
\]

**Remark 2** In view of (18), relation (17) reduces to
\[
\beta_{n,m,i} = \beta_{n,m-1,n} \sigma_{n-i+1} - \beta_{n,m-1,i-1}, \quad m > 1. \tag{18}
\]
For small values of \( m, m = 1, 2 \), we have from (16)

\[
\frac{1}{n+1} \partial_x \tau_{n+1} = \sum_{i=1}^{n} \frac{(-1)^{n-i}}{i} \sigma_{n-i+1} \partial_x \tau_i, \tag{19}
\]

\[
\frac{1}{n+2} \partial_x \tau_{n+2} = \sum_{i=1}^{n} \frac{(-1)^{n-i}}{i} (\sigma_1 \sigma_{n-i+1} - \sigma_{n-i+2}) \partial_x \tau_i. \tag{20}
\]

By Proposition 4, the last row of \( B_{n,1} \) (resp., of \( B_{n,2} \)) consists of the coefficients at \( \partial_x \tau_i \)'s on the RHS of (19), (resp., of (20)) and so on.

**Proof of Lemma 2.** Let \( m = 1 \). Equality \( \tau_i = \sum_j \lambda_j^i \) yields \( \frac{1}{t} \partial_x \tau_i = \sum_j \lambda_j^{i-1} \partial_x \lambda_j \). Similarly, we find

\[
\sum_{i=1}^{n} \frac{(-1)^{n-i}}{i} \sigma_{n-i+1} \partial_x \tau_i = \sum_j \partial_x \lambda_j \sum_{i=1}^{n} (-1)^{n-i} \sigma_{n-i+1} \lambda_j^{i-1}.
\]

Define the polynomial \( P_n(x) = \lambda^n - \sigma_1(x) \lambda^{n-1} + \ldots + (-1)^n \sigma_n(x) \). Since \( \lambda_j(x) \) are the roots of \( P_n \), we obtain

\[
\frac{1}{n+1} \partial_x \tau_{n+1} = \sum_{i=1}^{n} \frac{(-1)^{n-i}}{i} \sigma_{n-i+1} \partial_x \tau_i
\]

\[
= \sum_j (\lambda_j^n - \sigma_1 \lambda_j^{n-1} + \ldots + (-1)^n \sigma_n) \partial_x \lambda_j = 0
\]

that proves (19). Hence, \( \beta_{n,1,i} = \sigma_{n-i+1} \).

In order to prove the recurrence relation in (17), assume temporarily that \( \lambda_{n+1} = \varepsilon, \bar{n} = n + 1 \) and \( \bar{m} = m - 1 \). Hence, \( \frac{1}{n+m} \partial_x \tau_{n+m} = \frac{1}{n+\bar{m}} \partial_x \tau_{n+\bar{m}} \varepsilon = 0 \). Then we put \( \varepsilon = 0 \) and replace \( \partial_x \tau_{n+1}(x) \) due to (19)

\[
\frac{1}{n+\bar{m}} \partial_x \tau_{n+\bar{m}} = \sum_{i=1}^{\bar{n}} \frac{(-1)^{\bar{n}-i}}{i} \beta_{\bar{n},\bar{m},i} \partial_x \tau_i
\]

\[
= \beta_{n+1,m-1,n+1} \frac{1}{n+1} \partial_x \tau_{n+1} + \sum_{i=1}^{n} \frac{(-1)^{n-i+1}}{i} \beta_{n+1,m-1,i} \partial_x \tau_i
\]

\[
\varepsilon = 0 \sum_{i=1}^{n} \frac{(-1)^{n-i}}{i} (\beta_{n+1,m-1,n+1} \sigma_{n-i+1} - \beta_{n+1,m-1,i}) \partial_x \tau_i
\]

that completes the proof of (17). For \( m = 2 \), we deduce from (17)

\[
\beta_{n,2,i} = \beta_{n+1,1,n+1} \sigma_{n-i+1} - \beta_{n+1,1,i} = \sigma_1 \sigma_{n-i+1} - \sigma_{n-i+2}
\]

that proves (20).
Finally, we prove (18) by induction on $m$. For $m = 1$ we have

$$\beta_{n+j,1,i+j} \overset{(17)1}{=} \sigma_{(n+j)-(i+j)+1} = \sigma_{n-i+1} \overset{(17)1}{=} \beta_{n,1,i}.$$ 

Assuming (18) for $m - 1$ we deduce it for $m$ using (17):

$$\beta_{n+j,m,i+j} \overset{(17)2}{=} \beta_{(n+j)+1,m-1,\sigma_{(n+j)-(i+j)+1}} = \beta_{(n+j)+1,m-1,i+j} - \beta_{n+1,m-1,i} \overset{(17)}{=} \beta_{n,m,i}$$

that completes the proof of (18). \hfill \square

**Example 3** For $f_j = \delta_{j1}$, (13) reduces to the linear system $\partial_t \tau_i + \frac{1}{2} \partial_x \tau_i = 0$, whose solution is a simple wave along $x$-axis: $\tau_i = \tau_i^0(t - 2x)$. Consider slightly more complicated cases.

1. For $f_j = \delta_{j2}$, (13) is reduced to the system

$$\partial_t \tau_i = -\frac{i}{i+1} \partial_x \tau_{i+1}, \quad i = 1, 2, \ldots \tag{21}$$

The $n$-truncated (21) reads as $\partial_t \bar{\tau} + B_{n,1} \partial_x \bar{\tau} = 0$. For $n = 2$, using (19), we have just two PDEs

$$\partial_t \tau_1 = -\frac{1}{2} \partial_x \tau_2, \quad \partial_t \tau_2 = -\frac{2}{3} \partial_x \tau_3 = (\tau_1^2 - \tau_2) \partial_x \tau_1 - \tau_1 \partial_x \tau_2.$$ 

The matrix $B_{2,1} = \begin{pmatrix} 0 & \frac{1}{2} \\ -2\sigma_2 & \sigma_1 \end{pmatrix}$ has the characteristic polynomial $P_2 = \det(\lambda \text{id} - B_{2,1}) = \lambda^2 - \sigma_1 \lambda + \sigma_2$. If $\lambda_1 \neq \lambda_2$, the eigenvectors of $B_{2,1}$ are equal to $\vec{v}_j = (1, 2\lambda_j)$ ($j = 1, 2$). If $\lambda_1 = \lambda_2 \neq 0$, $B_{2,1}$ has one eigenvector only, hence the system (21) is not hyperbolic in the $t$-direction.

For $n = 3$, (21) reduces to the quasilinear system of three PDEs with the matrix

$$B_{3,1} = \begin{pmatrix} 0 & \frac{1}{3} & 0 \\ 0 & 0 & \frac{1}{3} \\ 3\sigma_3 & -\frac{3}{2}\sigma_2 & \sigma_1 \end{pmatrix}$$

The characteristic polynomial of $B_{3,1}$ is $P_3 = \lambda^3 - \sigma_1 \lambda^2 + \sigma_2 \lambda - \sigma_3$, the eigenvalues are $\lambda_j$, and the eigenvectors are $v_j = (1, 2\lambda_j, 3\lambda_j^2)$, $j = 1, 2, 3$. \hfill 14
2. For \( f_j = \delta_{j3} \), (13) reduces to the system

\[
\partial_t \tau_i + \frac{3i}{2(i+2)} \partial_x \tau_{i+2} = 0, \quad i = 1, 2, \ldots
\]  

(22)

The matrix of this \( n \)-truncated system is \( B_{n,2} = \frac{3}{2} (B_{n,1})^2 \).

For \( n = 3 \), (22) reduces to the system of three quasilinear PDEs

\[
\partial_t \tau_1 = -\frac{1}{2} \partial_x \tau_3, \quad \partial_t \tau_2 = -\frac{3}{4} \partial_x \tau_4, \quad \partial_t \tau_3 = -\frac{9}{10} \partial_x \tau_5,
\]

where \( \partial_x \tau_4 \) and \( \partial_x \tau_5 \) should be expressed using (19) and (20). The matrix of this system is

\[
B_{3,2} = \frac{3}{2} (B_{3,1})^2 = \begin{pmatrix}
0 & 0 & \frac{1}{2} \\
3 \sigma_3 & -\frac{3}{2} \sigma_2 & 0 \\
\frac{9}{2} \sigma_1 \sigma_3 & 0 & \frac{9}{2} (\sigma_3 - \sigma_2)
\end{pmatrix}
\]

It has eigenvalues \( \tilde{\lambda}_j = \frac{3}{2} \lambda^2_j \), and the same eigenvectors as for \( B_{3,1} \).

For \( n = 4 \), (22) reduces to the quasilinear system with the matrix

\[
B_{4,2} = \frac{3}{2} (B_{4,1})^2 = \begin{pmatrix}
0 & 0 & \frac{1}{2} & 0 \\
0 & 0 & 0 & \frac{3}{4} \\
-\frac{9}{2} \sigma_4 & \frac{9}{2} \sigma_3 & -\frac{3}{2} \sigma_2 & \frac{9}{2} \sigma_1 \\
-6 \sigma_1 \sigma_4 & 3 (\sigma_1 \sigma_3 - \sigma_4) & 2 (\sigma_3 - \sigma_1 \sigma_2) & \frac{9}{2} (\sigma_1^2 - \sigma_2)
\end{pmatrix}
\]

with eigenvalues \( \tilde{\nu}_j = \frac{3}{2} \lambda^2_j \), and eigenvectors \( v_j = (1, 2 \lambda_j, 3 \lambda^2_j, 4 \lambda^3_j), \ j = 1, 2, 3, 4 \).

This series of examples can be continued as long as one desires.

### 2.3 Biregular foliated coordinates

The coordinate system described in the following lemma (for the proof see [CC], Section 5.1), is called a biregular foliated chart here.

**Lemma 3** Let \((M,\mathcal{F})\) be a differentiable manifold with a codimension-1 foliation and a vector field \(N\) transversal to \(\mathcal{F}\). Then for any \(p \in M\) there is a coordinate system \((x_0, x_1, \ldots, x_n)\) on a neighborhood \(U_p \subset M\) (centered at \(p\)) such that the leaves on \(U_p\) are given by \(\{x_0 = c\}\) (hence the coordinate vector fields \(\partial_i = \partial_{x_i}, \ i \geq 1\), are tangent to leaves), and \(N\) is directed along \(\partial_0 = \partial_{x_0}\) (one may assume \(N = \partial_0\) at \(p\)).
If \((M, \mathcal{F}, g)\) is a foliated Riemannian manifold and \(N\) is a unit normal, then \(g\) has, in biregular foliated coordinates \((x_0, x_1, \ldots, x_n)\), the form
\[ g = g_{00} \, dx_0^2 + \sum_{i,j>0} g_{ij} \, dx_i dx_j. \] (23)

**Lemma 4** For a metric \((23)\) in biregular foliated coordinates of a codimension one foliation \(\mathcal{F}\) on \((M, g)\), one has \(N = \partial_0/\sqrt{g_{00}}\) (the unit normal) and
\[
\Gamma^0_{ij} = -\frac{1}{2} g_{ij,0}/g_{00}, \quad \Gamma^j_{i0} = \frac{1}{2} \sum_s g_{is,0} g^{sj} \quad \text{(the Christoffel symbols)},
\]
\[
b_{ij} = \Gamma^0_{ij}/\sqrt{g_{00}} = -\frac{1}{2} g_{ij,0}/\sqrt{g_{00}} \quad \text{(the second fundamental form)},
\]
\[
A_i^j = -\Gamma^j_{i0}/\sqrt{g_{00}} = \frac{1}{2} \sqrt{g_{00}} \sum_s g_{is,0} g^{sj} \quad \text{(the Weingarten operator)},
\]
\[
(b_m)_{ij} = A^m_{s_2} \cdots A^m_{s_m} A^m_{s_{m-1}} A^m_{s_{m-1}} g_{js_1} \quad \text{(the m-th “power” of } b_{ij})
\]
\[
\tau_m = \left(-\frac{1}{2\sqrt{g_{00}}}\right)^m \sum \{r_u, \{s_i\}\} g_{r_1 s_{2,0}} \cdots g_{r_{m-1} s_{m,0}} g^{s_{1r_1}} \cdots g^{s_{mr_m}}.
\]

In particular,
\[
\tau_1 = -\frac{1}{2\sqrt{g_{00}}} \sum_{r,s} g_{rs,0} g^{rs}, \quad \tau_2 = \frac{1}{4 g_{00}} \sum_{r_1, r_2, s_1, s_2} g_{r_1 s_{2,0}} g_{r_2 s_{1,0}} g^{s_{1r_1}} g^{s_{2r_2}}, \text{ etc.}
\]

**Proof.** All of that is standard and left to a reader. Just for his convenience, let us observe that the formula for \(A\) follows from that for \(b\) and \(A_i^j = \sum_s b_{is} g^{sj}\). Notice that \((A^m)_i^j = \sum_{\{s_i\}} A^j_{s_2} A^j_{s_3} \cdots A^j_{s_{m-1}} A^j_{s_{m-1}} g_{js_1}\). The formulae for \(b_m\) follow from the above and \((A^m)_i^j g_{sj} = g(A^m e_i, e_j) = (b_m)_{ij}\). The formulae for \(\tau\)'s follow directly from the above and the equality \(\tau_m = \text{Tr}(A^m)\). \(\square\)

3 A solution to general case

3.1 Searching for \(\bar{\tau}\)

Let \(g_t\) satisfy \((1)\) and \(N_t\) be the \(g_t\)-unit normal vector field to \(\mathcal{F}\) on \(M\).

It is easy to see that \(\partial_t N_t = 0\), therefore \(N_t = N\) for all \(t\), \(N\) being the unit normal of \(\mathcal{F}\) on \((M, g_0)\). In fact, for any vector field \(X\) tangent to \(\mathcal{F}\) one has \(0 = \partial_t g_t(X, N_t) = h(b_t)(X, N_t) + g_t(X, \partial_t N_t) = g_t(X, \partial_t N_t)\) and similarly \(0 = \partial_t g_t(N_t, N_t) = h(b_t)(N_t, N_t) + 2 g_t(N_t, \partial_t N_t) = 2 g_t(N_t, \partial_t N_t)\).
Let now $\nabla^t$ denote the Levi-Civita connection on $(M, g_t)$ and $\Pi_t = \partial_t \nabla^t$. $\Pi_t$ is a $(1,2)$-tensor field on $M$. Following, for example, [15], one can write for all $t$-independent $X, Y$ and $Z$

$$g_t(\Pi_t(X, Y), Z) = \frac{1}{2}[(\nabla^t_X h_t)(Y, Z) + (\nabla^t_Y h_t)(X, Z) - (\nabla^t_Z h_t)(X, Y)].$$

(24)

Lemma 5 For the EGF of type (b), on the tangent bundle of $\mathcal{F}$ we have

$$\partial_t b(X, Y) = h_t(AX, Y)$$

$$-\frac{1}{2} \sum_{j=0}^{n-1} [N(f_j(\bar{\tau}, t)) g_t(A^j X, Y) + f_j(\bar{\tau}, t) g_t((\nabla^t_N A^j)X, Y)],$$

(25)

$$\partial_t A = -\frac{1}{2} \sum_{j=0}^{n-1} [N(f_j(\bar{\tau}, t)) A^j + f_j(\bar{\tau}, t)\nabla^t_N A^j].$$

(26)

Proof. By definition, $b(X, Y) = g_t(\nabla^t_X Y, N)$, and $h(\cdot, N) = 0$. Using (24) and the identity $h(AX, Y) = h(AY, X)$, we obtain

$$\partial_t b(X, Y) = \partial_t g_t(\nabla^t_X Y, N) = (\partial_t g_t)(\nabla^t_X Y, N) + g_t(\partial_t \nabla^t_X Y, N)$$

$$= (1/2) \left[ (\nabla^t_X h_t)(Y, N) + (\nabla^t_Y h_t)(X, N) - (\nabla^t_N h_t)(X, Y) \right] + h_t(\nabla^t_X Y, N)$$

$$= (1/2) \left[ h_t(AX, Y) + h_t(AY, X) - N(h_t(X, Y)) \right] = h_t(AX, Y)$$

$$-(1/2) \sum_{j=0}^{n-1} [f_j(\bar{\tau}, t) g_t((\nabla^t_N A^j)X, Y) + N(f_j(\bar{\tau}, t)) g_t(A^j X, Y)].$$

This proves (25). Now, (26) follows from (25) and

$$g_t((\partial_t A)X, Y) = g_t(\partial_t(AX), Y) = \partial_t(g_t(AX, Y)) - (\partial_t g_t)(AX, Y)$$

$$= \partial_t b(X, Y) - h_t(AX, Y).$$

□

The following lemma shows that the functions $\tau_1, \ldots, \tau_n$ satisfy the system of quasilinear PDEs, whose matrix can be built using a generalized companion matrix of the characteristic polynomial of $A_N$.

Lemma 6 The power sums $\{\tau_i\}_{i \in \mathbb{N}}$ of the EGF (1) of type (b) satisfy infinite system of quasilinear PDEs (2). The $n$-truncated system (2) has the form $\partial_t \bar{\tau} + (\bar{A} + \bar{B}) \partial_x \bar{\tau} = 0$ with $\bar{A}$ and $\bar{B}$ given by (3).

Proof. Applying $iA^{i-1}$ to both sides of (26), we obtain the PDE

$$i A^{i-1} \partial_i A = -\frac{i}{2} \sum_{j=0}^{n-1} [N(f_j) A^{i+j-1} + f_j A^{i-1} \nabla^t_N A^j].$$

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Taking the trace of both sides of the above equality, and using the identities
\[ i \text{Tr}(A^{i-1} \partial_t A) = \text{Tr}(\partial_t A^i) = \partial_t \tau_i, \]
\[ \frac{i + j - 1}{j} \text{Tr}(A^{i-1} \nabla_N^t A^j) = \text{Tr}(\nabla_N^t A^{i+j-1}) = N(\tau_{i+j-1}), \]
for \( i, j > 0 \), we obtain (2). The second statement of our lemma follows directly from Proposition 4. □

**Remark 3** Lemma 6 implies the variational formula for \( \sigma_m \). Notice that for any differentiable matrix valued function \( A(t) \in \text{Mat}(n \times n) \), \( t \in [0, \varepsilon) \), one has \( \sigma'_m = \sum_{i=0}^{m-1} (-1)^i \sigma_{m-i-1} \text{Tr}(A^i A') \), \( m = 1, 2, \ldots, n \), see [RW2]. In view of \( (i+1) \text{Tr}(A^i A') = \tau'_{i+1} \), we obtain

\[ \partial_t \tau_i = \sum_{i=0}^{n-1} (-1)^i \frac{1}{i+1} \sigma_{m-i-1} \partial_t \tau_{i+1}. \]

From above identity with \( \partial_t \tau_{i+j} \) replaced due to (2) follows the equality

\[ \partial_t \sigma_m = \frac{1}{2} \sum_{i=0}^{m-1} (-1)^i \sigma_{m-i-1} \left\{ N(f_0(\bar{\tau}, t)) \tau_i \right. \]
\[ \left. + \sum_{j=1}^{n-1} \left\{ N(f_j(\bar{\tau}, t)) \tau_{i+j} + \frac{j}{i+j} f_j(\bar{\tau}, t) N(\tau_{i+j}) \right\} \right\}. \tag{27} \]

The next lemma deals with the evolution equation for EGF of type (a).

**Lemma 7** Let \( g_t \) be the solution to the EGF (1) of type (a). Then the Weingarten operator \( A \) of \( F \) with respect to \( g_t \) satisfies

\[ \partial_t A = -\frac{1}{2} \left[ N(f_0) \text{id} + \sum_{j=1}^{n-1} \left( N(f_j) A^j + f_j \cdot \nabla_N^t A^j \right) \right], \tag{28} \]
and \( \tau_i \) (\( i \geq 1 \)) (the power sums of the eigenvalues of \( A \)) satisfy PDEs

\[ \partial_t \tau_i = -\frac{i}{2} \left\{ \tau_{i-1} N(f_0) + \sum_{j=1}^{n-1} \left[ \frac{j}{i+j-1} f_j N(\tau_{i+j-1}) + \tau_{i+j-1} N(f_j) \right] \right\}. \tag{29} \]

The \( n \)-truncated system (29) has the form

\[ \partial_t \bar{\tau} + \left[ \sum_{j=1}^{n-1} j \bar{f}_j (B_{n,1})^{j-1} \right] N(\bar{\tau}) = \bar{a}, \]

where \( \bar{a} = (\hat{a}_1, \ldots, \hat{a}_n) \), \( \hat{a}_i = -\frac{i}{2} \sum_{j=0}^{n-1} N(f_j) \tau_{i+j-1} \) (\( 1 \leq i \leq n \)), and \( B_{n,1} \) is the generalized companion matrix (12).
Proof. The proof of (28) is similar to that of (26). On $T\mathcal{F}$ we obtain
\[
g_t((\partial_t A) X, Y) = -\frac{1}{2} N(h_t(X, Y)) = -\frac{1}{2} [N(\tilde{f}_0) g_t(X, Y)
+ \sum_{j=1}^{n-1} (N(\tilde{f}_j) g_t(A^j X, Y) + \tilde{f}_j g_t(\nabla^j_{X} A^j X, Y))].
\]
Following the lines of the proof of (2) in Lemma 6, we deduce from above the system (29). By Proposition 4, the $n$-truncated (29) has the required form. □

Lemmas 6 and 7 together with Theorem A provide existence and uniqueness results for the symmetric functions $\vec{\tau}$ satisfying conditions following from (1). In particular, this allows to reduce the problems of existence and uniqueness for EGF of type (b) to those for type (a) as we do in the proof of Theorem 2 in the next section.

3.2 Local existence of $g_t$ (Proofs of Theorems 1 and 2)

Given a Riemannian metric $g$ on a foliated manifold $(M, \mathcal{F})$, the symmetric tensor $h(b)$ defined by (11) can be expressed locally in terms of the first partial derivatives of $g$. Therefore, $g \mapsto h(b)$ is a first order partial differential operator. For a $\mathcal{F}$-truncated symmetric $(0, 2)$-tensor $\hat{S}$ on $T\mathcal{F}$,
\[
h(b) = \hat{S}
\]
is a non-linear system of first order PDEs. The particular case of (30) is the Einstein type relation $h(b) = \lambda \hat{g}$ for some function (or constant) $\lambda$ on $M$.

There are several obvious obstructions to the existence of solutions to (30) even at a single point: for example, if $h(b) = \hat{b}_{2j}$ and $\hat{S}$ takes both, positive and negative, values at $p \in M$, then (30) has no solutions at $p$.

Proof of Theorem 1. Take biregular foliated coordinates $(x_0, x_1, \ldots, x_n)$ on $U_q \subset M$ (with center at $q$); see Lemma 3 and the metric (23). Then, $N = \partial_0 / \sqrt{g_{00}}$ is the unit normal to $\mathcal{F}$. Set $\psi_{ab} = g_{ab,0}$. The system (11) (for $f_j$ of type (a)) along a trajectory $\gamma : x \mapsto \gamma(x)$ of $\partial_0$ has the form
\[
\partial_t g_{ij} = F_{ij}(g_{ab}, \psi_{ab}, t, x),
\]
where $F_{ij} := h(b)_{ij}$. In view of symmetry, $\psi_{ab} = \psi_{ba}$ and $F_{ij} = F_{ji}$, we shall assume $1 \leq i \leq j \leq n$ and $1 \leq a \leq b \leq n$. For example, if $f_m = 0$ ($m > 1$) then (31) is the hyperbolic (diagonal) system
\[
\partial_t g_{ij} = f_0(q, t) g_{ij} - \frac{1}{2} g_{00}^{-1/2} f_1(q, t) \psi_{ij},
\]
that completes the proof in this case.

Now let \( f_m \neq 0 \) for some \( m > 1 \) (e.g., general \( f_m \)). We may assume that \( A \partial_j = k_j \partial_j \), \( g(\partial_i, \partial_j) = \delta_{ij} \) \((i, j > 0)\) and \( g_{00} = 1 \) at the point \( q \) for \( t = 0 \). (By Lemma 3 we have \((b_m)_{ij} = (-1/2)^m \psi^m_{ij} \delta_{ij} \) at \( q \) for \( t = 0 \)).

Differentiating (31) with respect to \( x \) and \( t \), we obtain

\[
\partial_0 p_{ij} = \partial_0 F_{ij} + \sum_{a,b} \left[ \frac{\partial F_{ij}}{\partial g_{ab}} \partial_0 g_{ab} + \frac{\partial F_{ij}}{\partial \psi_{ab}} \partial_0 \psi_{ab} \right],
\]

\[
\partial_t p_{ij} = \partial_t F_{ij} + \sum_{a,b} \left[ \frac{\partial F_{ij}}{\partial g_{ab}} \partial_0 g_{ab} + \frac{\partial F_{ij}}{\partial \psi_{ab}} \partial_0 \psi_{ab} \right],
\]

where \( p_{ij} := \partial_t g_{ij} \), \( \psi_{ij} := g_{ij,0} \), \( F_{ij} := h(b)_{ij} \).

As \( g \) is of class \( C^2 \), we conclude that \( \partial_t \psi_{ab} = \partial^2 g_{ab} \partial_2 = \partial_0 p_{ab} \). Hence, (32) together with (31) may be written in the form

\[
\partial_0 g_{ij} = F_{ij}(\{g_{ab}\}, \{\psi_{ab}\}, t, x),
\]

\[
\partial_t \psi_{ij} - \sum_{a,b} \frac{\partial F_{ij}}{\partial \psi_{ab}} \partial_0 \psi_{ab} = \partial_0 F_{ij} + \sum_{a,b} \frac{\partial F_{ij}}{\partial g_{ab}} \psi_{ab},
\]

\[
\partial_t p_{ij} - \sum_{a,b} \frac{\partial F_{ij}}{\partial \psi_{ab}} \partial_0 p_{ab} = \partial_t F_{ij} + \sum_{a,b} \frac{\partial F_{ij}}{\partial g_{ab}} p_{ab}.
\]

The above quasilinear system consists of parts:

(i) our original equation (33) \(_1\),
(ii) the corresponding equation (33) \(_2\) for \( \partial_t A \), and
(iii) the equation (33) \(_3\) for \( \partial^2 t g \) following from the previous ones.

In general, the following \( \frac{1}{2}n(n+1) \times \frac{1}{2}n(n+1) \) matrix is not symmetric:

\[
d_\psi F = \left\{ \frac{\partial F_{ij}}{\partial \psi_{ab}} \right\}, \quad i \leq j, \quad a \leq b.
\]

We claim that it is hyperbolic. If we change the local coordinate system on \( M \), the components \( F_{ij} \) \((i \leq j)\) and \( \psi_{ab} \) \((a \leq b)\) at \( q \) will be transformed by the same tensor low. Notice that the above \( d_\psi F \) is a \((1, 1)\)-tensor on the vector bundle of symmetric \((0, 2)\)-tensors on \( T\mathcal{F} \). Hence, \( d_\psi F(q) \) can be seen as the linear endomorphism of the space of symmetric \((0, 2)\)-tensors on \( T_q\mathcal{F} \).

The hyperbolicity is a pointwise property, so can be considered at any point \( q \in M \) in a special bifoliated chart around \( q \), for example, such that \( g_{ij} = \delta_{ij} \) and \( A_{ij}(q) = k_i \delta_{ij} \) at \( q \). (Indeed, \((k_i)\) are the principal curvatures of \( \mathcal{F} \) at \( q \) for \( t = 0 \)). In this chart, our calculations show that the matrix \( d_\psi F \)
is diagonal, so has real eigenvalues (vectors) at $q$. Indeed, for $t = 0$ one may find at the point $q$:

$$\frac{\partial F_{ij}}{\partial \psi_{ab}} = \sum_{m \geq 1} f_m(q) \mu(m)_{ij} \delta_{\{a,b\}}^{\{i,j\}}, \quad \mu(m)_{ij} = \sum_{\alpha + \beta = m - 1} k_{i}^{\alpha} k_{j}^{\beta}.$$ 

The order of indices of $d_{\psi} F$ is $[1, 1], [1, 2], \ldots, [1, n], [2, 2], \ldots, [n, n]$. For example, for $F = b_2$ (i.e., $f_j = \delta_{j2}$) the above matrix in an orthonormal frame at any point is

$$\frac{\partial (b_2)_{ij}}{\partial \psi_{ab}} = \frac{1}{4} \begin{bmatrix} 2 \psi_{11} & 2 \psi_{12} & 2 \psi_{13} & 0 & 0 & 0 \\ 2 \psi_{12} & \psi_{12} + \psi_{22} & \psi_{23} & \psi_{12} & \psi_{13} & 0 \\ 2 \psi_{13} & \psi_{13} + \psi_{23} & \psi_{33} & 0 & \psi_{12} & \psi_{13} \\ 0 & 2 \psi_{12} & 0 & 2 \psi_{22} & 2 \psi_{23} & 0 \\ 0 & \psi_{13} & \psi_{12} & \psi_{23} & \psi_{22} + \psi_{33} & \psi_{23} \\ 0 & 0 & 2 \psi_{13} & 0 & 2 \psi_{23} & 2 \psi_{33} \end{bmatrix}$$

with the order of indices $[1, 1], [1, 2], [1, 3], [2, 2], [2, 3], [3, 3]$. At $q$ for $t = 0$ (i.e., $\psi_{ab} = 0$, $a \neq b$ and $\psi_{aa} = k_a$) it is diagonal with the elements $\mu(2)_{ab} = k_a + k_b$. Let $A_0 = [\frac{1}{2}, 1, 1, \frac{1}{2}, 1, \frac{1}{2}]$ be the diagonal matrix. Then the matrix $A_1 = A_0 d_{\psi}(b_2)$ is symmetric, and our system (33) for $h(b) = b_2$ is “symmetrizable”: $A_0 \partial_t \psi = A_1 \partial_t \psi = \{\text{free terms}\}$.

Therefore, (33) for the functions $g_{ij}(t, x), p_{ij}(t, x)$ and $\psi_{ij}(t, x)$ with $i \leq j$ is the hyperbolic system, which is “symmetrizable”. Indeed, multiplying $n$ columns (corresponding to $i = j$) of the matrix $d_{\psi} F$ in an orthonormal frame by the factor $\frac{1}{2}$, we obtain the symmetric matrix.

By Theorem A (in Section 2.1), given $q \in M$ there exists a unique solution to (33) which is defined in $U_q$ along the $N$-curve through $q$ for some time interval $[0, \varepsilon_q)$ and satisfies the initial conditions

$$g_{ij}(0, x) = (g_0)_{ij}(x), \quad p_{ij}(0, x) = h(b_0)_{ij}(x), \quad \psi_{ij}(0, x) = (\partial_0 g)_{ij}(x).$$

By Theorem A, see definitions of $K$ and $\Omega$, the value $\varepsilon_q$ continuously depends on $q \in M$. The claim follows from the above and compactness of $M$.  

**Proof of Theorem 2**  Let $A_0$ and $\bar{\tau}^0$ be the values of extended Weingarten operator and power sums of the principal curvatures $k_i$ of (the leaves of) $F$ determined on $(M, \mathcal{F})$ by a given metric $g_0$.

a) **Uniqueness.** Assume that $g_{t}^{(1)}, g_{t}^{(2)}$ are two solutions to (11) with the same initial metric $g_0$. The functions $\bar{\tau}_{t,1}, \bar{\tau}_{t,2}$, corresponding to $g_{t}^{(1)}, g_{t}^{(2)}$,
satisfy (2) and have the same initial value \( \bar{\tau}^0 \). By Lemma 6 and Theorem A, \( \bar{\tau}^{t,1} = \bar{\tau}^{t,2} = \bar{\tau}^t \) on some positive time interval \([0, \varepsilon_1]\). Hence \( g_t^{(1)}, g_t^{(2)} \) satisfy (11) of type (a) with known \( \bar{f}_j(p,t) := f_j(\bar{\tau}^t(p),t) \). By Theorem 11, \( g_t^{(1)} = g_t^{(2)} \) on some positive time interval \([0, \varepsilon_2]\).

b) Existence. By Lemma 6 and Theorem A, there is a unique solution \( \bar{\tau}^t \) to (2) on some positive time interval \([0, \varepsilon^*]\). By Theorem 11, the EGF (1) of type (a) with known functions \( \bar{f}_j(\cdot, t) := f_j(\bar{\tau}^t, t) \) has a unique solution \( g_t^* \) (\( g_0^* = g_0 \)) for \( 0 \leq t < \varepsilon^* \). The Weingarten operator \( \bar{A}_t^* \) \( (A_0^* = A_0) \) of \( (M, \mathcal{F}, g_t^*) \) satisfies (23), hence the power sums of its eigenvalues, \( \bar{\tau}^{t,*} \) \( (\bar{\tau}^{0,*} = \bar{\tau}^0) \), satisfy (29) with the same coefficient functions \( \bar{f}_j \). By Lemma 7 and Theorem A, the solution of this problem is unique, hence \( \bar{\tau}^t = \bar{\tau}^{t,*} \), i.e., \( \bar{\tau}^t \) are power sums of eigenvalues of \( \bar{A}_t^* \). Finally, \( g_t^* \) is a solution to (11) such that \( \bar{\tau}^t \) are power sums of the principal curvatures of the leaves in this metric. \( \square \)

### 3.3 Proofs of Corollaries 1, 2 and Propositions 1–3

In all the proofs below, \( p \) is an arbitrary point of \( M \) and \( \gamma : x \mapsto \gamma(x) \) \( (\gamma(0) = p, x \in \mathbb{R}) \) is the trajectory of \( N, N – \) the unit normal of \( \mathcal{F} \).

**Proof of Corollary 1** By Lemma 6, we have (2), which in our case reduces to the system (5). The last one means the initial value problem in the \((x,t)\)-plane for the vector function \( \bar{\tau}(x,t) = \bar{\tau}(\gamma(x),t) \)

\[
\partial_t \bar{\tau} + \frac{1}{2} f(\bar{\tau}, t) \text{id}_n + \bar{A}(\bar{\tau}, t) \partial_x \bar{\tau} = 0, \quad \bar{\tau}(x,0) = \bar{\tau}^0(\gamma(x)).
\]

(34)

The matrix \( \bar{A} \) is equal to \( \{ \frac{1}{2} \tau_i f_{\tau_i}(\bar{\tau}, t) \}_{i,j} \). Note that rank \( \bar{A} \leq 1 \). By condition (H2), either the function \( \bar{\lambda} = \text{Tr} \bar{A} = \sum_{1 \leq i \leq n} \frac{1}{2} \tau_i f_{\tau_i}(\bar{\tau}, 0) \) (the eigenvalue of \( \bar{A} \)) is non-zero for all \( x \in \mathbb{R} \), or \( \bar{A}(x) \equiv 0 \). Hence (34) is hyperbolic for small enough \( t \). In first case, the eigenvector of \( \bar{A}_{t=0} \) for \( \lambda(x) \) is \( \bar{v}_1 = (f_{\tau_1}, f_{\tau_2}, \ldots, f_{\tau_n}) \), and the kernel of \( \bar{A}_{t=0} \) is spanned by \( n - 1 \) vectors

\[
\bar{v}_2 = (-2 \tau_2, \tau_1, 0, \ldots, 0), \quad \bar{v}_3 = (-3 \tau_3, 0, \tau_1, 0, \ldots, 0), \ldots \bar{v}_n = (-n \tau_n, 0, \ldots, 0, \tau_1).
\]

(If \( \lambda(x) = 0 \) for some \( x \in \mathbb{R} \), and \( f_{\tau_j}(\bar{\tau}, 0) \neq 0 \) for some \( j \), then \( \bar{A} \) is nilpotent and hence (34) is not hyperbolic). By Theorem A, the initial value problem (34) has a unique solution on a domain \([-\delta, \delta] \times [0, \varepsilon'] \) of the \((x,t)\)-plane. Hence there exists \( t_p > 0 \) such that the solution \( \bar{\tau}(\cdot, t) \) to (5) exists and is unique for \( t \in [0, t_p) \) on a neighborhood \( U_p \subset M \) centered at \( p \). By
compactness of $M$, we conclude that there is $\varepsilon > 0$ such that (6) admits a unique solution $\bar{\tau}(p, t)$ on $M$ for $t \in [0, \varepsilon)$.

One may also apply the *method of characteristics* to solve (34) explicitly when $f = f(\bar{\tau})$. For $\bar{\lambda} \neq 0$, the system has two characteristics: $\frac{dx}{dt} = \bar{\lambda} + \frac{f}{2}$ and $\frac{dx}{dt} = \frac{f}{2}$ corresponding to two eigenvalues $\bar{\lambda} + \frac{f}{2}$ (of multiplicity 1) and $\frac{f}{2}$ (of multiplicity $n - 1$). First, let us observe that the function $u := \bar{\tau}_1^T \cdot \bar{\tau} = \sum_{j \leq n} f j \tau_j \tau_j$ is constant along the first family of characteristics:

$$
\frac{d}{dt} u = \partial_t u + (\bar{\lambda} + \frac{f}{2}) N(u) = 0 \iff u = \text{const} \quad \frac{dx}{dt} = \bar{\lambda} + \frac{f}{2}. \tag{35}
$$

Next, let us find a function that is constant along the second family of characteristics. For each $m > 1$, due to the form of $\bar{v}_m$, calculate the sum of the first equation in (34) multiplied by $-m \tau_m$ with the $m$-th equation multiplied by $\tau_1$ (along the trajectories of $\frac{dx}{dt} = \bar{\lambda} + \frac{f}{2}$)

$$
\tau_1 \partial_t \tau_m - m \tau_m \partial_t \tau_1 + \frac{f}{2} \left( \tau_1 N(\tau_m) - m \tau_m N(\tau_1) \right) = 0.
$$

(The terms with $\sum_j f j \tau_j N(\tau_j)$ cancel.) Using $\frac{d}{dt} \tau_m = \partial_t \tau_m + N(\tau_m) \frac{dx}{dt} = \partial_t \tau_m + \frac{1}{2} f N(\tau_m)$, we get (again, along the second family of characteristics)

$$
\tau_1 \frac{d}{dt} \tau_m - m \tau_m \frac{d}{dt} \tau_1 = 0 \quad (m \geq 2). \tag{36}
$$

The complete integral of (36) is

$$
\tau_m = C_m(x) \tau_1^m \quad (m \geq 2), \tag{37}
$$

where $C_m(x) = \tau_m(x, 0)/\tau_1^m(x, 0)$ are known functions. Since the smooth functions $\bar{\tau}(x, t)$ and $f(\bar{\tau}, t)$ exist for $t \in [0, t_p)$, the EGF under consideration exists and is unique on $M \times [0, t_p)$.

Using the basic properties of the Lie derivative of $g_t$ along $N$, one may show that (6) is equivalent to $L_{Z_t} g_t = 0$ with $Z_t = \partial_t + \frac{1}{2} f(\bar{\tau}^T, t) N$. Denote by $\Phi_t$ the flow of $\frac{1}{2} f(\bar{\tau}^T, t) N$. The solution $g_t$ can be defined by $g_t(\Phi_X Y) = g_0(X, Y)$, $g_t(X, N) = 0$ and $g_t(N, N) = 1$ for all $X$ and $Y$ tangent to $F$. The solution exists and is unique as long as the solution to (5) does. If $f(\cdot, t) = \text{const}$, the solution exists (and is unique) for all $t \geq 0$. \hfill \Box

**Proof of Corollary 2.** Denote by $\bar{\tau}(x, t) = (\tau_1^t, \ldots, \tau_n^t)$ the power sums of the principal curvatures of $F$ at the point $\gamma(x)$ in time $t$. By Lemma 6 we
have (2), which in our case reduces to (1). The last one provides the initial value problem in the $(x,t)$-plane

$$\partial_t \bar{\tau} + \bar{A}(\bar{\tau},t) \partial_x \bar{\tau} = 0, \quad \bar{\tau}(x,0) = \bar{\tau}^0(\gamma(x)), \quad (38)$$

where $\bar{A}$ is equal to $\{ \frac{i}{2} \tau_{i-1} f, \tau_j (\bar{\tau},t) \}_{1 \leq i,j \leq n}$. As before, rank $\bar{A} \leq 1$. Consider the function $\tilde{\lambda} = \text{Tr} \bar{A} = \sum_{1 \leq i \leq n} \frac{i}{2} \tau_{i-1} f, \tau_i (\bar{\tau},0)$ (the eigenvalue of $\bar{A}$). By condition $(H_1)$, either $\tilde{\lambda}(x) \neq 0$ for all $x \in \mathbb{R}$, or $\bar{A}(x,0) \equiv 0$. Hence the system (38) is hyperbolic at $(x,0)$. As in the proof of Corollary 1 we conclude that there is $\varepsilon > 0$ such that $\tilde{\tau}(p,t)$ on $M$ exists and is unique for $t \in [0, \varepsilon)$. Certainly, a unique smooth solution to (7) has the required form. □

**Proof of Proposition 1** Notice that

$$\sum_j (\bar{\tau}^\alpha)_{\tau_j \tau_j} = \sum_j (\bar{\tau}^\alpha)_{\tau_j \tau_j} = l \bar{\tau}^\alpha.$$ 

If $f = \sum_{\alpha} c_{\alpha} \bar{\tau}^\alpha$ then $N(f) = \sum_j \sum_{\alpha} c_{\alpha} \bar{\tau}^\alpha N(\tau_j)$ (the derivative of $f$ along $N$).

Define $\bar{f}(x,t) = f_1(\bar{\tau}(\gamma(x)))$ and $\bar{f}_0 = \bar{f}(\cdot,0)$. One has PDEs (34) in the $(x,t)$-plane. The characteristics of the first family, see (35), are lines, and $\bar{f} = \text{const}$ along them. To show this, observe that (by definition of $f_1(\bar{\tau})$)

$$u := \sum_j f_{t,\tau_j}(\bar{\tau}) \tau_j = m \bar{f}(x,t), \quad \tilde{\lambda} := \sum_j \frac{j}{2} f_{t,\tau_j}(\bar{\tau}) \tau_j = \frac{l}{2} \bar{f}(x,t).$$

Since $\bar{f} = u/m$ is constant along the first family of characteristics in the $(x,t)$-plane, these characteristics are lines given by the equation

$$\frac{dx}{dt} = \frac{1}{2} (l+1) \bar{f} \quad \Leftrightarrow \quad x = \xi + \frac{1}{2} (l+1) t \bar{f}(\xi) t.$$ 

Notice that $\bar{f}' = \sum_{\alpha \in J_{n,n}} c_{\alpha} \sum_j (\bar{\tau}^0)_{\tau_j \tau_j} N(\tau_j^0)$. If $\bar{f}_0' > 0$ on $\gamma$, the solution $\bar{f}$ exists for all $t \geq 0$ (see Example 2) and we set $t_p = \infty$. If $\bar{f}_0'$ is negative somewhere on $\gamma$, then $\bar{f}$ exists (and is continuous) for $t \in [0, t_p)$ where $t_p = -2/[(l+1) \min_{x} \bar{f}_0(\gamma(x))]$.

The second family of characteristics, $\frac{dx}{dt} = \frac{y}{2}$, also exists for $t \in [0, t_p)$. To show this, assume the opposite, i.e., there are $t_0 \in (0, t_p)$ and a trajectory $\gamma_1(t)$ of the second family of characteristics that cannot be continued for values $t \geq t_0$. Therefore, the inclination $\bar{f}(\gamma_1(t),t)/2$ of $\gamma_1$ approaches to infinity when $t \to t_0$, a contradiction to continuity of $\bar{f}$ on the strip $t \in [0, t_0]$ in the $(x,t)$-plane. □
Proof of Proposition 2 For the EGF (1) define the function $\psi$ by

$$
\psi(\lambda, t) := \sum_{j=0}^{n-1} f_j(n\lambda, n\lambda^2, \ldots, n\lambda^n; t)\lambda^j.
$$

(39)

Since $F$ is totally umbilical for $t = 0$, $A_0 = \lambda_0 \hat{id}$. Assuming $A = \lambda_t \hat{id}$ for small enough $t$, we see that (26), or equivalently, (2) for $i = 1$, yields the PDE

$$
\partial_t \lambda_t + \frac{1}{2} N(\psi(\lambda_t, t)) = 0,
$$

whose unique solution exists for small enough $t \geq 0$.

From the uniqueness of solution to (26) it follows that $A = \lambda_t \hat{id}$. □

Proof of Proposition 3 Recall that $AX = \lambda_t X$. Theorem A (with $n = 1$) provides the short-time existence and uniqueness of the solution $\lambda_t$ to (8) for $0 \leq t < T$. Furthermore, the EGF is expressed as $\partial_t g_t = \psi(\lambda_t) \hat{g}_t$, and the solution $\hat{g}_t$ has the required form.

Consider the function $\tilde{\lambda}(x, t) = \lambda(\gamma(x), t)$ in the $(x, t)$-plane along the trajectory $\gamma(x)$, $\gamma(0) = p$, of $N$, and set $\tilde{\lambda}_0(x) = \lambda(\gamma(x), 0)$. The equation (8) in this case has the form of a conservation law

$$
\partial_t \tilde{\lambda} + \frac{1}{2} \partial_x(\psi(\tilde{\lambda})) = 0.
$$

(40)

One may show the following: If $\psi'(\tilde{\lambda}_0(x))$, $\tilde{\lambda}_0(x) \in C^1(\mathbb{R})$ and if $\tilde{\lambda}_0(x)$ and $\psi'(\tilde{\lambda}_0(x))$ are either non-decreasing or non-increasing, the problem

$$
\partial_t \tilde{\lambda} + \frac{1}{2} \partial_x(\psi(\tilde{\lambda})) = 0, \quad \tilde{\lambda}(x, 0) = \tilde{\lambda}_0(x), \quad t \geq 0
$$

(41)

has a unique smooth solution defined implicitly by the parametric equations,

$$
\tilde{\lambda}(x, t) = \tilde{\lambda}_0(x), \quad x = \xi + \frac{1}{2} \psi'(\tilde{\lambda}_0(\xi)) t.
$$

(42)

If $\frac{d}{dx} \psi'(\tilde{\lambda}_0(x))$ is negative elsewhere along $\gamma$, then $\tilde{\lambda}(x, t)$ exists for

$$
t < t_p = -2/\inf_{x \in \mathbb{R}} \frac{d}{dx} \psi'(\tilde{\lambda}_0(x)).
$$

For $\psi(\lambda) = \lambda^2$, (40) is reduced to the Burgers’ equation, see Section 2.1. □

4 Applications and Examples

4.1 Extrinsic Ricci and Newton transformation flows

The extrinsic Riemannian curvature tensor $\text{Rm}^{ex}$ of $\mathcal{F}$ is, roughly speaking, the difference of the curvature tensors of $M$ and of the leaves. More precisely,
by the Gauss formula, we have

\[ \text{Rm}^{\text{ex}}(Z, X)Y = g(AX, Y)AZ - g(AZ, Y)AX. \]

In Section 4.1 we study the extrinsic Ricci flow for small dimensions \( n > 1 \),

\[ \partial_t g_t = -2 \text{Ric}^{\text{ex}}(b_t). \]  \hspace{1cm} (43)

The extrinsic Ricci tensor is given by \( \text{Ric}^{\text{ex}}(X, Y) = \text{Tr} \text{Rm}^{\text{ex}}(\cdot, X)Y \), where \( X, Y \in T F \). Hence,

\[ \text{Ric}^{\text{ex}}(b) = \tau_1 \hat{b}_1 - \hat{b}_2, \quad (\text{Ric}^{\text{ex}}(b) = \sigma_2 \hat{g} \quad \text{when} \quad n = 2). \]  \hspace{1cm} (44)

Therefore, \(-2 \text{Ric}^{\text{ex}}(b)\) relates to \( h(b) \) of (11) with \( f_1 = -2 \tau_1, \ f_2 = 2 \) (others \( f_j = 0 \)).

**Lemma 8** The principal curvatures of extrinsic Ricci flow satisfy PDEs

\[ \partial_t k_i = N(k_i(\tau_1 - k_i)), \quad i = 1, \ldots, n \]  \hspace{1cm} (45)

which for a totally umbilical \( F \), i.e., \( k_i = \lambda \), are reduced to the PDE

\[ \partial_t \lambda = 2(n - 1) \lambda N(\lambda). \]  \hspace{1cm} (46)

The extrinsic scalar curvature of the flow (43) is \(-2 \sigma_2\), where

\[ \partial_t \sigma_2 = (\tau_1^2 + \tau_2)N(\tau_1) - \tau_1N(\tau_2) - \frac{2}{3}N(\tau_3). \]  \hspace{1cm} (47)

For \( n = 2 \), (47) reads as \( \partial_t \sigma_2 = \tau_1 \partial_t \sigma_2 \).

**Proof.** By Lemma 3 for the extrinsic Ricci flow (43) we have

\[ \partial_t \tau_i = i \tau_i N(\tau_1) + \tau_1 N(\tau_i) - \frac{2i}{i + 1}N(\tau_{i+1}), \quad i > 0. \]  \hspace{1cm} (48)

Replacing \( \tau_i = \sum_{j=1}^{n}(k_j)^i \) in (48), we obtain (45). Differentiating identity \( 2\sigma_2 = \tau_1^2 - \tau_2 \), we obtain \( 2 \partial_t \sigma_2 = 2 \tau_1 \partial_t \tau_1 - \partial_t \tau_2 \). From this and (48) with \( i = 1, 2 \) it follows (47). \( \square \)
Corollary 4 Let $(M, g_0)$ be a compact Riemannian manifold with a codimension 1 foliation $F$ and a unit normal $N$. Then there exists a unique solution $g_t$, $t \in [0, \varepsilon)$ (for some $\varepsilon > 0$), to the extrinsic Ricci flow \((43)\) in any of the following cases (i) and (ii):

(i) $n = 2$, and $\tau_1 \neq 0$; (in this case, $\hat{g}_t = \hat{g}_0 \exp \left( \int_0^t \sigma_2 \, dt \right)$);

(ii) $n = 3$, and $|\sigma_1|^3 > 27|\sigma_3| > 0$.

Notice that any, either positive or negative definite, operator $A_N$ satisfies inequalities Corollary 1 (ii) if only $A_N$ is not proportional to the identity.

Proof. It is sufficient to show that in conditions of theorem, \((48)\) is hyperbolic in the $t$-direction.

Let $n = 2$. By equality $\tau_3 = \frac{3}{2}(\tau_1 - \frac{1}{2}\tau_2)$, the matrix of 2-truncated system \((48)\) is $C_2 = \begin{pmatrix} -2\tau_1 & 1 \\ -2\tau_2 & \tau_1 \end{pmatrix}$. One can see directly, or applying condition (H1) to $\text{Ric}^\text{ex}(b) = \sigma_2 \hat{g}$, that $C_2$ is strictly hyperbolic if $\tau_1 \neq 0$. (If $\tau_1 \neq 0$, $C_2$ has real eigenvalues $\lambda_1 = 0$, $\lambda_2 = -\tau_1$, and the left eigenvectors $\vec{v}_1 = (-\tau_1, 1)$ and $\vec{v}_2 = (-2\tau_1, 1)$. If $\tau_1 \equiv 0$, i.e., $F$ is minimal foliation, the matrix $C_2$ is nilpotent, hence it is not hyperbolic). By Corollary 2, $\hat{g}_t = \hat{g}_0 \exp \left( \int_0^t \sigma_2 \, dt \right)$, where $\sigma_2$ exists for $0 \leq t < T$.

Remark that $\tau_1^2 - 2\tau_2 = -(k_1 - k_2)^2 = \text{const}$ along the first family of characteristics $\frac{dx}{dt} = 0$, hence the function $k_1 - k_2$ does not depend on $t$ (the flow preserves the property “umbilic free foliation”). Along the second family of characteristics $\frac{dx}{dt} = -\tau_1$ we have $k_1k_2 = (\tau_1^2 - \tau_2)/2 = \text{const}$. Notice that \((45)\) is reduced to two equations for $k_1$ and $k_2$ with equal RHS’s,

$$\partial_t k_i = \partial_x (k_1 k_2), \quad i = 1, 2,$$

hence $\partial_t (k_2 - k_1) = 0$, i.e., $k_2 - k_1 = \psi(x)$ is a known (at $t = 0$) function of $x$. Recall that the system is hyperbolic in the $t$-direction, if $k_1 + k_2 \neq 0$, otherwise, the matrix is nilpotent. For $k_1$ we have a quasilinear PDE $\partial_t k = \partial_x (k^2 + \psi(x) k)$, whose solution exists for small enough $t$.

For $n = 3$, the matrix of 3-truncated system \((48)\) is

$$C_3 = \begin{pmatrix} -2\tau_1 & 1 & 0 \\ -2\tau_2 & -\tau_1 & \frac{4}{3} \\ \tau_1^3 - 3\tau_1 \tau_2 & 3\tau_1 \tau_2 - \frac{4}{3} (\tau_2 - \tau_1^2) & \tau_1 \end{pmatrix}.$$  

Replacing $\tau$-s by $\sigma$-s, see Remark 1, we obtain the characteristic polynomial $P_3 = \lambda^3 + 2\sigma_1 \lambda^2 + \sigma_1^2 \lambda + 4 \sigma_3$. Substituting $\lambda = y - \frac{2}{3} \sigma_1$ into $P_3$ gives

$$P_3 = y^3 + py + q, \quad \text{where} \quad p = -(1/3) \sigma_1^2 \quad \text{and} \quad q = 4 \sigma_3 - (2/27) \sigma_1^3.$$  

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Depending on the sign of the discriminant \( D = (q/2)^2 + (p/3)^3 \), we have

\[
D \begin{cases} 
> 0, & \text{one real and two complex roots,} \\
< 0, & \text{three different real roots,} \\
= 0, & \text{one real root with multiplicity three in the case } p = q = 0, \\
\quad \text{or a single and a double real roots when } (\frac{1}{3}p)^3 = -(\frac{1}{3}q)^3 \neq 0.
\end{cases}
\]

In our case, \( D = \frac{1}{27} \sigma_3(27 \sigma_3 - \sigma_1^3) \). The condition “three different real roots” is \( D < 0 \iff |\sigma_1|^3 > 27 |\sigma_3| > 0 \). \( \square \)

**Corollary 5** Let \((M, g_0)\) be a Riemannian manifold, and \(\mathcal{F}\) a codimension-1 totally umbilical foliation on \(M\) with the normal curvature \(\lambda_0\) and a complete unit normal field \(N\). Set at \(t = 0\)

\[
T = \infty \text{ if } N(\lambda_0^3) \leq 0 \text{ on } M; \quad T = 1/[(n - 1) \sup M N(\lambda_0^2)] \text{ otherwise.}
\]

Then the extrinsic Ricci flow (43) has a unique smooth solution \(g_t\) on \(M\) for \(t \in [0, T)\), and does not possess one for \(t \geq T\).

**Proof.** The function \(\tilde{\lambda}_t(x) = \lambda(\gamma(x), t)\) along the trajectory \(\gamma(x) (\gamma(0) = p)\), of \(N\) satisfies (3) with \(\psi(\lambda) = 4(1 - n)\lambda^2\) and initial value \(\lambda_0(x) = \lambda(\gamma(x), 0)\). Then we apply Proposition 3. \( \square \)

The Newton transformations of the shape operator can be applied successfully to foliated manifolds, see [RW2]. By the Cayley-Hamilton theorem, \(T_n(b) = 0\). The **extrinsic Newton transformation (ENT) flow** is given by

\[
\partial_t g_t = T_s(b_t).
\]

In other words, we have \(h(b)\) in (4) with \(f_j = (-1)^i \sigma_{s-j}\). Notice that \(\text{Tr } T_s(b) = (n - s) \sigma_s\) (the trace). For the ENT flow, (2) reduces to

\[
\partial_t \tau_i + \frac{i}{2} \left\{ N(\sigma_s) \tau_{i-1} + \sum_{j=1}^{s} (-1)^i [N(\sigma_{s-j}) \tau_{i+j-1} + \frac{j}{i+j-1} \sigma_{s-j} N(\tau_{i+j-1})] \right\} = 0.
\]

From (27) in what follows with \(f_j\) replaced by \((-1)^i \sigma_{s-j}\), we also get

\[
\partial_t \sigma_s = -\frac{1}{2} \sum_{i=0}^{s-1} \sigma_{s-i-1} \{ (-1)^i N(\sigma_s) \tau_i \\
+ \sum_{j=1}^{s} (-1)^{i+j} [N(\sigma_{s-j}) \tau_{i+j} + \frac{j}{i+j} \sigma_{s-j} N(\tau_{i+j})] \}.
\]

(50)

For \(s = 1 < n\), we have the linear PDE \(2 \partial_t \sigma_1 = -(n - 1) N(\sigma_1)\) (see also (2) with \(i = 1\) representing the “unidirectional wave motion” \(\sigma_1'(s) = \sigma_0(s - t(n - 1)/2)\) along any \(N\)-curve \(\gamma(s)\). For \(s = 2 < n\), (50) is reduced to the PDE \(2 \partial_t \sigma_2 = [\tau_2 - (n - 1) \tau_1^2] N(\tau_1) + \frac{1}{2} \tau_1 N(\tau_2) - \frac{2}{3} N(\tau_3)\).
4.2 EGF with rotational symmetric metrics

Notice that the EGF preserves rotational symmetric metrics

\[ g_t = dx_0^2 + \varphi_t^2(x_0) ds_n^2, \quad \text{where } ds_n^2 \text{ is a metric of curvature 1.} \tag{51} \]

The \( n \)-parallels \( \{ x_0 = c \} \) compose a (Riemannian) totally umbilical foliation \( \mathcal{F} \) with \( N = \partial_0 \). In this case, \( \lambda_t \) can be found from \( \Box \), and by Proposition \( \Box \) the EGF with generating functions \( f_j = f_j(\vec{\tau}) \) can be reduced to

\[ \partial_t g_t = \psi(\lambda_t) \hat{g}_t, \quad \text{where } \psi_t(\lambda) := \sum_{j=0}^{n-1} f_j(n\lambda, n\lambda^2, \ldots, n\lambda^n) \lambda^j. \tag{52} \]

For simplicity, assume \( n = 1 \), and consider \( (M^2, g_t, \mathcal{F}) \) in biregular foliated coordinates \( (x_0, x_1) \). Hence \( (g_t)_{00} = 1 \) and \( (g_t)_{11} = \varphi_t^2 \).

From (52) we get

\[ \partial_t \varphi_t = (1/2) \psi(\lambda_t) \varphi_t, \quad \text{or } \varphi_t = \varphi_0 e^{(1/2) \int_0^t \psi(\lambda_t) dt}. \tag{53} \]

By Lemma \( \Box \) (for \( b_{11} = g_t(A\partial_1, \partial_1) = \lambda_t \varphi_t^2 \)), we also have \( \lambda_t = -(\varphi_t)_o / \varphi_t^2 \).

The Gaussian curvature of \( M^2 \) is \( K_t = -(\varphi_t)_o / \varphi_t \). For example, if \( \psi(\lambda) = \lambda \), then (53) reduces to the linear PDE \( \partial_t \lambda + \frac{1}{2} N(\lambda) = 0 \) representing the “unidirectional wave motion” along any \( N \)-curve \( \gamma(s) \)

\[ \lambda_t(s) = \lambda_0(s - t/2). \tag{54} \]

In particular, if \( \lambda_0 = C \in \mathbb{R} \), then also \( \lambda_t = C \), and \( \varphi_t = \varphi_0 e^{(t/2) \psi(C)} \).

![Figure 1: Foliation of \( \mathbb{H}^2 \) with constant \( \lambda \neq 0 \).](image_url)

Foliations with \( \lambda = \text{const} \neq 0 \) exist on a hyperbolic plane. Their leaves are horocycles. On the Poincaré 2-disc \( B \) the leaves of such foliations are represented by Euclidean circles tangent \( \partial B \), Figure \( \Box \) Orthogonal trajectories to above foliations form foliations by geodesics (i.e., \( \lambda = 0 \)).
Example 4 Some of rotational symmetric metrics come from surfaces of revolution in Euclidean 3-space. Evolving them by EGF yields deformations of surfaces of revolution foliated by parallels. Revolving a curve \( \gamma_t(x) = (X_t(x), Y_t(x), Z = 0) \), \( x \in I \), around the \( x \)-axis, we get a surface of revolution \( M^2 \subset \mathbb{R}^3 \), \( (x, y) \to (X_t(x), Y_t(x) \cos y, Y_t(x) \sin y) \). Assuming that \( (X'_t(x))^2 + (Y'_t(x))^2 = 1 \) \( x \) is a natural parameter on \( \gamma_t \), we obtain the rotational symmetric metric \( g_t = dx^2 + Y_t^2(x) dy^2 \) with \( |Y_t'(x)| \leq 1 \). One may recover \( \gamma_t \) from \( g_t \) by its curvature \( K \). Applying the flow \( \partial_t g_t = \lambda_t \hat{g}_t \), by (54) we obtain \( \lambda_t(x) = -\frac{2}{x-\lambda/2} \). The rotational symmetric metric \( g_t = dx^2 + ((x - t/2) \sin \beta)^2 dy^2 \) appears on the same cone translated across the \( x \)-axis, \( C_t : y^2 + z^2 = (x - t/2)^2 \tan \beta \).

(ii) Let us find a curve \( Y = Y(X) > 0 \) such that the metric (55) (on the surface of revolution \( M^2 : [f(Y), Y \cos Z, Y \sin Z] \)) has \( \lambda_0 = \text{const} = 1 \). Using \( \lambda_0 = (1/Y(X)) \sin \phi \), where \( \tan \phi = Y'(X) \), we get the ODE \( 1 = \frac{|Y'(X)|}{Y(X) \sqrt{1 + (Y'(X))^2}} \Rightarrow \frac{dY}{dX} = \frac{Y}{\sqrt{4 + Y^2}} \). The solution is \( X = \log \frac{\sqrt{4 + Y^2} - 2}{\sqrt{4 + Y^2} + 2} + \sqrt{4 + Y^2} + C \), where \( C \in \mathbb{R} \). The surface \( M^2 \) looks like a pseudosphere, see Figure 2 but for \( Y \to \infty \) it is asymptotic to the cone \( [Y + C, Y \cos Z, Y \sin Z] \). The Gaussian curvature of \( M^2 \) is \( K = \frac{-1}{(Y^2 + 2)^2} < 0 \), and \( \lim_{Y \to \pm \infty} K = 0 \).

![Figure 2](image-url)

Figure 2: a) Graph of \( X = \log \frac{\sqrt{4 + Y^2} - 2}{\sqrt{4 + Y^2} + 2} + \sqrt{4 + Y^2} \). b) Surface of revolution.
4.3 EGF on foliated surfaces

In this section, \((M^2, g_t)\) is a two-dimensional Riemannian manifold (surface) with a transversally orientable foliation \(F\) (by curves), \(N\) a unit normal to \(F\), and \(\lambda_t\) the geodesic curvature of the leaves with respect to \(N\).

For \(\psi \in C^2(\mathbb{R}^2)\), the EGF \(g_t\) of type (b) on \((M^2, F)\) is a solution to the PDE \(\partial_t g_t = \psi(\lambda_t, t) \hat{g}_t\), where \(\lambda_t\) obeys the PDE \(\partial_t \lambda_t + \frac{1}{2} \psi'_\lambda(\lambda_t, t)N(\lambda_t) = 0\) (see (2) for \(i = 0\)). By Corollary 2, if \(\psi'(\lambda_0) \neq 0\), see the condition \((H_1)\) with \(n = 1\), then there is a unique local smooth solution \(\lambda_t\) for \(0 \leq t < T\) with initial value \(\lambda_0\) determined by \(g_0\), and \(\hat{g}_t = \hat{g}_0 \exp\left(\int_0^t \psi'(\lambda_t, t) \, dt\right)\) holds.

For compact \(M^2\), we have \(\partial_t(d \text{vol}_t) = \frac{1}{2} \text{Tr} h(b_t) d \text{vol}_t\), see [16]. Hence, the volume \(\text{vol}_t := \int_M d \text{vol}_t\) of \(g_t\) satisfies to

\[
\partial_t \text{vol}_t = \frac{1}{2} \int_M \psi(\lambda_t, t) \, d \text{vol}_t.
\]

In order to estimate the time interval due to Proposition 3, suppose that the function \(\psi \in C^2(\mathbb{R})\) does not depend on \(t\). Define \(T = \infty\) if \(N(\psi(\lambda_0)) \geq 0\) on \(M\), and \(T = -2/\inf_M N(\psi(\lambda_0))\) otherwise. Then the EGF \(\partial_t g_t = \psi(\lambda_t) \hat{g}_t\) has a unique smooth solution \(g_t\) on \(M^2\) for \(t \in [0, T]\), and does not possess one for \(t \geq T\). If, in addition to above \((H_1)\), \(\psi'(\lambda_0)N(\lambda_0) \geq 0\) holds, then the solution exists for all \(t \geq 0\) \((T = \infty)\).

**Proposition 5** The Gaussian curvature of the EGF of type (b) on \((M^2, F)\) is given by the formula

\[
K_t = \text{div}(e^{-\int_0^t \psi(\lambda_t, t) \, dt} N^0(N)) + N(\lambda_t) - \lambda_t^2.
\]

**Proof.** Define a self-adjoint \((1, 1)\)-tensor \(h(A) := \sum_{j=0}^{n-1} f_j A^j\) dual to \(h\) of [11]. We use (24) to compute for any \(X \in TF\)

\[
g(\partial_t(\nabla_N^t N), X) = \frac{1}{2} [2(\nabla_N^t h_t)(X, N) - (\nabla_N^t h_t)(N, N)]
\]

\[
= -h(\nabla_N^t N, X) = -\sum_{j=0}^{n-1} f_j g(A^j(\nabla_N^t N), X) = -h(A)(\nabla_N^t N).
\]

Hence we have the general relation \(\partial_t(\nabla_N^t N) = -h(A)(\nabla_N^t N)\), which for \(n = 1\) looks as \(\partial_t(\nabla_N^t N) = -\psi(\lambda_t, t) \nabla_N^t N\). Integrating above yields

\[
\nabla_N^t N = e^{-\int_0^t \psi(\lambda_t, t) \, dt} N^0(N).
\]
The formula (57) for $K = \text{Ric}(N, N)$ is a consequence of (58) and the formula for $\text{div}(\nabla_N N) + \text{div}(H)$ (of a codimension-one foliation), see [Wa1],

$$\text{div}(\nabla_N N) = \text{Ric}(N, N) + \tau_2 - N(\tau_2).$$

(a) Let $\psi = 1$. The solution to $\partial_t g_t = \hat{g}_t$ is $\hat{g}_t = e^t \hat{g}_0$ ($t \geq 0$). From $\partial_t \lambda = 0$ we get $\lambda_t = \lambda_0$. By (57), the Gaussian curvature is $K = e^{-t} \text{div}(\nabla_N N) + N(\lambda_0) - \lambda_0^2$. There is limit $K_\infty = N(\lambda_0) - \lambda_0^2$.

(b) Let $\psi = \lambda$, i.e., $\partial_t g = \hat{b}_1$. Then $\lambda_t(s) = \lambda_0(s + \frac{t}{2})$ along any $N$-curve $\gamma(s)$ in the $(t, s)$-plane. From $\partial_t g_t = \lambda_0(s + \frac{t}{2}) \hat{b}_1$ we get $\hat{g}_t(s) = \hat{g}_0(s) e^{\int_0^s \lambda_0(s + \frac{\xi}{2}) d\xi}$ ($t \in \mathbb{R}$). For compact $M^2$, by (56), we have $\text{vol}_t = \text{const}$.

(c) Let $\psi = \lambda^2$. Then $\partial_t \lambda + \lambda N(\lambda) = 0$ (the inviscid Burgers equation). If $N(\lambda_0) \geq 0$ (for $t = 0$), then the solution $\lambda_t$ exists for all $t \geq 0$.

**Example 5** Let a function $f \in C^2(-1, 1)$ has vertical asymptotes $x = \pm 1$. Consider the foliation $\mathcal{F}$ in the closed strip, whose leaves are $L_+ = \{x = 1\}$ and $L_s(x) = \{(x, f(x) + s), |x| < 1\}$, where $s \in \mathbb{R}$. The normal $N$ at the origin is directed along $y$-axis. The tangent and normal to $\mathcal{F}$ unit vector fields (on the whole strip) are $X = [\cos \alpha(x), \sin \alpha(x)], N = [-\sin \alpha(x), \cos \alpha(x)]$, where $\alpha(x)$ is the angle between the leaves $L_s$ and the $x$-axis at the intersection points. Indeed, $f$ and $\alpha$ are related by

$$f'(x) = \tan \alpha(x) \quad \text{and} \quad \cos \alpha = [1 + (f')^2]^{-1/2}, \quad \sin \alpha = f'[1 + (f')^2]^{-1/2}.$$ 

The curvature of $L_s$ is $\lambda_0(x) = f''(x)[1 + (f'(x))^2]^{-3/2} = \alpha'(x) \cdot |\cos \alpha(x)|$, where $|x| < 1$. The $N$-curves through the critical points of $f$ are vertical, and divide $\Pi$ into sub-strips. Typical foliations in the strip $|x| < 1$ with one vertical trajectory $x = 0$ are the following two:

(i) $f$ has exactly one strong minimum at $x = 0$.

(ii) $f$ is monotone increasing with one critical point $x = 0$.

Taking $f = \frac{1}{10} [e^{x^2/(1-x^2)} - 1]$ or $\alpha(x) = \frac{\pi}{2} x$ for (i), we get the Reeb foliation. For (ii) one may take $f = \tan(\frac{\pi}{2} x)$, or $\alpha(x) = \frac{\pi}{2} x^2$.

Let $\psi = \psi(\lambda)$, where $\lambda_t(x)$ is known for a positive time interval $[0, \varepsilon)$, see Proposition 3. We use $X \in T\mathcal{F}$ and normal $N$ to represent the standard frame $e_1 = \cos \alpha(x)X - \sin \alpha(x)N, e_2 = \sin \alpha(x)X + \cos \alpha(x)N$ in the $(x, y)$-plane. By $\hat{g}_t = \hat{g}_0 e^{\int_0^t \psi(\lambda_t(x)) dt}$, we get $g_t(X, X) = e^{\int_0^t \psi(\lambda_t(x)) dt}$, $g_t(X, N) = 0$. 

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and \(g_t(N,N) = 1\). The \(g_t\)-scalar products of the frame \(\{e_1, e_2\}\) are

\[
E_t = g_t(e_1, e_1) = \sin^2 \alpha + \cos^2 \alpha e^{\int_0^t \psi(\lambda_t(x)) \, dt},
\]

\[
F_t = g_t(e_1, e_2) = \sin \alpha \cos \alpha [e^{\int_0^t \psi(\lambda_t(x)) \, dt} - 1],
\]

\[
G_t = g_t(e_2, e_2) = \cos^2 \alpha + \sin^2 \alpha e^{\int_0^t \psi(\lambda_t(x)) \, dt}.
\]

The Gaussian curvature \(K_t\) of the metric \(g_t = E \, dx^2 + 2F \, dx \, dy + G \, dy^2\) is

\[
K_t = -\frac{1}{2\sqrt{EG-F^2}} \left[ \partial_x \left( \frac{\partial_G - \partial_F}{\sqrt{EG-F^2}} \right) + \partial_y \left( \frac{\partial_E - \partial_F}{\sqrt{EG-F^2}} \right) - \frac{1}{\sqrt{EG-F^2}} \left[ \frac{\partial}{\partial x} \left( \frac{\partial G}{\partial y} \right) - \frac{\partial}{\partial y} \left( \frac{\partial G}{\partial x} \right) \right] \right]
\]

which in our case, when the coefficients of \(g_t\) do not depend on \(y\)-coordinate, reads as

\[
K_t = \frac{1}{2\sqrt{E_tG_t-F_t^2}} \partial_t \left( \frac{\partial G_t}{\partial y} \right). \tag{59}
\]

We have \(E_tG_t = F_t^2 = e^{\int_0^t \psi(\lambda_t(x)) \, dt}\).

The \(N\)-curves satisfy ODEs \(dx/dt = -\sin \alpha(x), dy/dt = \cos \alpha(x)\). From the first of above ODEs for \(N\)-curves, we deduce the implicit formula \(t = -\int_x^{\phi_t(x)} \frac{dx}{\sin \alpha(x)}\) for local diffeomorphisms \(\phi_t(x)\) \((|x| < 1, t \geq 0)\).

Suppose that \(\psi(\lambda) = \lambda\). Since \(\lambda_t(s) = \lambda_0(s + t/2)\) is a simple wave along \(N\)-curves, we have \(\lambda_t(x) = \lambda_0(\phi_t(x/2))\). For example, \(\lambda_t(0) = \lambda_0(0)\) for all \(t \geq 0\). Substitution \(E_t, F_t\) and \(G_t\) into above formula for \(K_t\) yields

\[
K_t = \frac{1}{8}(\cos(2\alpha) - 1) \left[ \int_0^t \frac{\partial^2}{\partial x^2} \lambda_t \, dt + \left( \int_0^t \frac{\partial}{\partial x} \lambda_t \, dt \right)^2 \right] - \left[ \cos(2\alpha)(\alpha')^2 + \frac{1}{2} \sin(2\alpha) \alpha'' \right] \frac{1}{2} \sin(2\alpha) \alpha' \int_0^t \frac{\partial}{\partial x} \lambda_t \, dt \left[ 3 + e^{-\int_0^t \lambda_t \, dt} \right].
\]

Since \(\alpha(0) = 0\) and \(\lambda_t(0) = \lambda_0(0)\), one has \(K_t(0) = -\alpha'(0)^2 \left[ 1 - e^{-\int_0^t \lambda_0 \, dt} \right]\).

For (i), we get \(\alpha'(0) > 0\) and \(\lambda_0(0) > 0\), so \(K_t(0) < 0\) for \(t > 0\). Since \(\lim_{t \to \infty} \phi_t(x) = 0\) for \(|x| \leq 1\), there also exists \(\lim_{t \to \infty} \lambda_t(x) = \lambda_0(0) > 0\). Hence for any \(x \in [-1, 1]\) there is \(t_x > 0\) such that \(K_t(x) < 0\) for \(t > t_x\).

For (ii) with \(\alpha(x) = (\pi/2)x^2\), we have \(\alpha'(0) = 0\) and \(\alpha''(0) = \pi\). Moreover, \(\lambda_0(0) = 0\) and \(\lambda_0'(0) = \pi \neq 0\). Since \(\lim_{t \to \infty} \phi_t(x) = 0\) for all \(0 < x \leq 1\), there also exists \(\lim_{t \to \infty} \lambda_t(x) = \lim_{t \to \infty} \lambda_0(\phi_t(x/2)) = \lambda_0(0) = 0\). By (59), we have \(K_t(0) = 0\) and the series expansion \(K_t(x) = -\frac{\pi}{2} \int_0^t \frac{\partial}{\partial x} \lambda_t \, dt \, x^3 + O_t(x^4)\).

We conclude that there exists \(t_0 > 0\) such that \(K_t(x)\) for \(t > t_0\) changes its sign when we cross the line \(x = 0\).

**Example 6** Consider a foliation \(\mathcal{F}\) by circles \(L_\rho = \{\rho = c\}\) in the ring \(\Omega = \{c_1 \leq \rho \leq c_2\}\) for some \(c_2 > c_1 > 0\) with polar coordinates \((\rho, \theta)\). Then \(X = \partial_\theta\) and \(N = \partial_\rho\) are tangent and normal vector fields to the
foliation. The metric is \( ds^2 = d\rho^2 + G_t(\rho) d\theta^2 \). Notice that \( \lambda_0(\rho) = 1/\rho \).

Since \( \partial_t G_t = \psi(\lambda_t(\rho), t) G_t \), we have \( G_t = \rho^2 \exp(\int_0^t \psi(\lambda_t(\rho), t) \, dt) \). From the formula for Gaussian curvature of Example \[4\] we have \( K_t = -\frac{1}{2\sqrt{G_t}} \partial_{\rho} \left( \frac{\partial_{\rho} G_t}{\sqrt{G_t}} \right) \).

Let \( \psi = \lambda \). Then \( \lambda_t(s) = \lambda_0(s + t/2) \) on the \( N \)-curves. For the foliation by circles \( \rho = c \) we have \( \lambda_t(\rho) = (\rho + t/2)^{-1} \). The Gaussian curvature \( K_t \equiv 0 \).

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