A Study of GD′-Implications, a New Hyper Class of Fuzzy Implications

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Abstract: In this paper, we introduce and study the GD′-operations, which are a hyper class of the known D′-operations. GD′-operations are in fact D′-operations, that are generated not only from the same fuzzy negation. Similar with D′-operations, they are not always fuzzy implications. Nevertheless, some sufficient, but not necessary conditions for a GD′-operation to be a fuzzy implication, will be proved. A study for the satisfaction, or the violation of the basic properties of fuzzy implications, such as the left neutrality property, the exchange principle, the identity principle and the ordering property will also be made. This study also completes the study of the basic properties of D′-implications. At the end, surprisingly an unexpected new result for the connection of the QL-operations and D-operations will be presented.

Keywords: fuzzy implication; D′-implication; fuzzy negation; t-norm; t-conorm

1. Introduction

The transition from classical to fuzzy logic achieved especially from generalizations of classical tautologies. Many definitions of fuzzy connectives, classes of fuzzy implications and properties of them are such generalizations [1–9].

Many applications have been constructed by using fuzzy implications [9–11]. Fuzzy implications are applied in many different areas, such as approximate reasoning, decision-making theories, control theories and expert systems, image processing, fuzzy mathematical morphology, robotics, and others. Mas et al. in [9] (page 1107) remarked that, the necessity of different classes of fuzzy implications is because they are used to representing imprecise knowledge. Thus, different models, or construction methods are always necessary, since they produce new fuzzy implication functions that can be adequate in specific applications.

In this paper, we deal with a generalization from a known already existing class of fuzzy implications, the so called D′-implications [3]. These implications are the generalization of the following classical tautology

\[(p \Rightarrow q) \equiv [(p \lor q) \land q']'. \tag{1}\]

In classical logic, all operations are uniquely defined. Such an operation is the classical negation. On the other hand this uniqueness of an operation does not exist in fuzzy logic. There are several different types of fuzzy negations and other fuzzy connectives in general. Therefore, when a formula of a fuzzy implication contains a fuzzy negation, more than once, why does this negation have to be the same? This is not binding [4,7] and this is the central idea of this paper. Therefore, we introduce a new class of fuzzy implications the generalized D′-implications (shortly GD′-implications) and we study its basic and most common properties. This new class is a hyper class of the existing D′-implications and increases the variety of fuzzy implications, which is necessary and has addressed by many authors [6,8,9].
The paper is organized as follows. Section 2 provides the preliminaries concepts for understanding the article. Section 3 is divided in 5 subsections. In Section 3.1, we introduce GD'-operations. We prove that they are not always fuzzy implications and that they are a hyper class of that of D'-operations. Moreover, a sufficient, but not necessary condition for a GD'-operation to be a fuzzy implication is proved. A connection of GD'-, D'- and QL-operations is also presented and some important results are presented. In the rest of the subsections, we study whether or not, or under what conditions, a GD'-operation satisfies or violates the main properties of a fuzzy implication. The results of these subsections also complete the study of the basic properties of D'-implications, with the corresponding changes of the conditions, assuming the two fuzzy negations in the formula of GD'-implications as one, thus as equal. In Section 4, we discuss the results of the study, and we mention a new result about QL- and D-operations. This result is another unknown connection between these classes of operations. However, this result is not the main purpose of this study, it encourages the necessity of such a study. In Section 5, we discuss the potentially advantages of GD'-implications. Section 6 contains the conclusions of this study.

2. Preliminaries

Definition 1. [1,12–14] A decreasing function N: [0, 1] → [0, 1] is called fuzzy negation, if N(0) = 1 and N(1) = 0. Moreover, a fuzzy negation N is called

(i) strict, if it is continuous and strictly decreasing,

(ii) strong, if it is an involution, i.e.,

\[ N(N(x)) = x, \text{for all } x \in [0, 1], \]

(iii) non-filling, if

\[ N(x) = 1 \Leftrightarrow x = 0 \]

Remark 1. (i) The so called, crisp fuzzy negations (see [5] Remark 2.1) are

\[ N^\alpha(x) = \begin{cases} 0, & \text{if } x \geq \alpha \\ 1, & \text{if } x < \alpha \end{cases}, \text{where } \alpha \in (0, 1] \quad (2) \]

\[ N_\alpha(x) = \begin{cases} 0, & \text{if } x > \alpha \\ 1, & \text{if } x \leq \alpha \end{cases}, \text{where } \alpha \in [0, 1). \quad (3) \]

(ii) We call \( N_C(x) = 1 - x \) the classical fuzzy negation, which is a strong negation. Moreover, there are several types of fuzzy negations, such as \( N_k(x) = 1 - x^2 \) and \( N_R(x) = 1 - \sqrt{x} \) (see [1] Example 1.4.4 and Table 1.6).

Lemma 1. ([1] Lemma 1.4.9). If \( N' \), \( N \) are fuzzy negations, such that \( N' \circ N = Id_{[0,1]} \), then

(i) \( N' \) is a continuous fuzzy negation,

(ii) \( N \) is a strictly decreasing fuzzy negation.

Definition 2. [1,13,14] A function T: [0, 1]^2 → [0, 1] is called a triangular norm (shortly t-norm), if it satisfies, for all \( x, y, z \in [0, 1] \), the following conditions

\[ T(x, y) = T(y, x), \quad (4) \]

\[ T(x, T(y, z)) = T(T(x, y), z), \quad (5) \]

if \( y \leq z \), then \( T(x, y) \leq T(x, z) \), i.e., \( T(x, \cdot) \) is increasing, \( (6) \)
$$T(x, 1) = x. \quad (7)$$

Dually, a function $S : [0, 1] \rightarrow [0, 1]$ is called a triangular conorm (shortly t-conorm) if it satisfies, for all $x, y, z \in [0, 1]$, the above conditions (4), (5), (6), and, additionally,

$$S(x, 0) = x. \quad (8)$$

Tables 1 and 2 list some of the common t-norms and t-conorms, respectively (see [1] Tables 2.1 and 2.2).

Table 1. Basic t-norms.

| Name             | Formula                                      |
|------------------|----------------------------------------------|
| minimum          | $T_M(x, y) = \min\{x, y\}$                   |
| Łukasiewicz      | $T_L(x, y) = \max\{x + y - 1, 0\}$           |
| drastic product  | $T_D(x, y) = \begin{cases} 0, & \text{if } x, y \in [0, 1) \\ \min\{x, y\}, & \text{otherwise} \end{cases}$ |
| nilpotent minimum| $T_{nM}(x, y) = \begin{cases} 0, & \text{if } x + y \leq 1 \\ \min\{x, y\}, & \text{otherwise} \end{cases}$ |

Table 2. Basic t-conorms.

| Name             | Formula                                      |
|------------------|----------------------------------------------|
| maximum          | $S_M(x, y) = \max\{x, y\}$                   |
| probor           | $S_P(x, y) = x + y - x \cdot y$               |
| Łukasiewicz      | $S_L(x, y) = \min\{x + y, 1\}$               |

Definition 3. ([1] Definitions 2.1.2). A t-norm $T$ is called

(i) idempotent, if

$$T(x, x) = x, \text{ for all } x \in [0, 1],$$

(ii) positive, if

$$T(x, y) = 0 \Leftrightarrow x = 0 \text{ or } y = 0.$$

Definition 4. [1,14] A t-norm $T$ is strictly monotone, if $T(x, y) < T(x, z)$, whenever $x > 0$ and $y < z$.

Proposition 1. ([2] Proposition 9). For all $x, y \in [0, 1]$ it is

$$T(x, y) \leq x \leq S(x, y) \text{ and } T(x, y) \leq y \leq S(x, y).$$

Remark 2. By Proposition 1, it follows that

$$S(1, x) = S(x, 1) = 1, x \in [0, 1] \quad (9)$$

and

$$T(0, x) = T(x, 0) = 0, x \in [0, 1]. \quad (10)$$

Definition 5. ([1] Definition 2.3.14). Let $T$ be a t-norm and $N$ be a fuzzy negation. We say that the pair $(T, N)$ satisfies the law of contradiction if

$$T(N(x), x) = 0, x \in [0, 1]. \quad (11)$$
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(iii) \( N \)

(ii) \( N \)

(i) \( N \)

Let \( T \) be any t-norm and \( N \) its natural negation. Then the \( N \)-dual of \( T \) and we denote it by \( S_{T,N} \). In the case that \( N \) is a strong negation, then \( S_{T,N}(x,y) = N(T(N(x), N(y))), x,y \in [0,1] \).

Respectively, we have the following Definition.

Definition 7. Let \( N \) be a strict negation, \( T \) be a t-norm and \( S \) be a t-conorm, such that \( T(x,y) = N^{-1}(S(N(x), N(y))), x,y \in [0,1] \). Then \( T \) is said to be the \( N \)-dual of \( S \) and we denote it by \( T_{S,N} \). In the case that \( N \) is a strong negation, then \( T_{S,N}(x,y) = N(S(N(x), N(y))), x,y \in [0,1] \).

Definition 8. ([1] Definition 2.3.1). Let \( T \) be a T-Norm. A function \( N_T : [0,1] \to [0,1] \) defined as

\[ N_T(x) = \sup \{ y \in [0,1] | T(x,y) = 0 \}, x \in [0,1], \]

is called the natural negation of \( T \) or the negation induced by \( T \).

Remark 3. (i) It is easy to prove that \( N_T \) is a fuzzy negation (see [1] Remark 2.3.2 (i)).

(ii) If \( T(x,y) = 0 \) for some \( x,y \in [0,1] \), then \( y \leq N_T(x) \) (see [1] Remark 2.3.2 (iii)).

Corollary 1. ([1] Corollary 2.3.7). Let \( T \) be any t-norm and \( N_T \) its natural negation. Then the following statements are equivalent:

(i) \( N_T \) is strictly decreasing;

(ii) \( N_T \) is continuous;

(iii) \( N_T \) is strong.

Proposition 2. ([1] Proposition 2.3.4). If a t-norm \( T \) is left-continuous, then for every \( x, y \in [0,1] \) it is

\[ T(x,y) = 0 \iff N_T(x) \geq y \]

Definition 9. ([1,15] By \( \Phi \) we denote the family of all increasing bijections from \([0,1]\) to \([0,1]\). We say that functions \( f,g : [0,1]^n \to [0,1] \) are \( \Phi \)-conjugate, if there exists a \( \phi \in \Phi \), such that \( g = f \phi \), where

\[ f \phi(x_1,x_2,\ldots,x_n) = \phi^{-1}(f(\phi(x_1),\phi(x_2),\ldots,\phi(x_n))), x_1,x_2,\ldots,x_n \in [0,1]. \]

Remark 4. ([1] Propositions 1.4.8, Remarks 2.1.4(vii), and 2.2.5(viii)). It is easy to prove that if \( \phi \in \Phi \) and \( T \) is a t-norm, \( S \) is a t-conorm and \( N \) is a fuzzy negation (respectively, strict, strong), then \( T \phi \) is a t-norm, \( S \phi \) is a t-conorm and \( N \phi \) is a fuzzy negation (respectively, strict, strong).

Definition 10. ([1,12] A function \( I : [0,1]^2 \to [0,1] \) is called a fuzzy implication if it satisfies, for all \( x,x_1,x_2,y,y_1,y_2 \in [0,1] \), the following conditions:

\[ \text{if } x_1 \leq x_2, \text{ then } I(x_1,y) \geq I(x_2,y), \text{ i.e., } I(\cdot,y) \text{ is decreasing}, \]

\[ \text{if } y_1 \leq y_2, \text{ then } I(x,y_1) \leq I(x,y_2), \text{ i.e., } I(x,\cdot) \text{ is increasing}, \]

\[ I(0,0) = 1, \]

\[ I(1,0) = 0. \]

Remark 5. By axioms (13) and (14) we deduce the normality condition

\[ I(0,1) = 1. \]
Moreover, by Definition 10 it is easy to prove the left and right boundary conditions [1]

\[ I(0, y) = 1, y \in [0, 1], \]

\[ I(x, 1) = 1, x \in [0, 1]. \]

**Definition 11.** ([1] Definition 1.3.1). A fuzzy implication \( I \) is said to satisfy

(i) the left neutrality property, if

\[ I(1, y) = y, y \in [0, 1], \]

(ii) the exchange principle, if

\[ I(x, I(y, z)) = I(y, I(x, z)), x, y, z \in [0, 1]. \]

(iii) the identity principle, if

\[ I(x, x) = 1, x \in [0, 1], \]

(iv) the ordering property, if

\[ I(x, y) = 1 \iff x \leq y, x, y \in [0, 1]. \]

**Remark 6.** (i) Properties (20), (21), (22), and (23) are not limited to fuzzy implications, but in any function \( I : [0, 1]^2 \rightarrow [0, 1] \).

(ii) It is proved that, if \( \phi \in \Phi \) and \( I : [0, 1]^2 \rightarrow [0, 1] \) satisfies (12) (respectively (13), (14), (15), (16)), then \( I_\phi \) is also satisfies (12) (respectively, (13), (14), (15), (16)). Moreover, if \( I : [0, 1]^2 \rightarrow [0, 1] \) is a fuzzy implication, then \( I_\phi \) is also a fuzzy implication (see [1] Proposition 1.1.8).

**Lemma 2.** ([1] Lemma 1.4.14). If a function \( I : [0, 1]^2 \rightarrow [0, 1] \) satisfies (12), (14), and (16), then the function \( N_I : [0, 1] \rightarrow [0, 1] \) is a fuzzy negation, where

\[ N_I(x) = I(x, 0), x \in [0, 1]. \]

**Definition 12.** ([1] Definition 1.4.15). Let \( I : [0, 1]^2 \rightarrow [0, 1] \) be a fuzzy implication. The function \( N_I \) defined by Lemma 2 is called the natural negation of \( I \).

**Definition 13.** ([1] Definition 1.6.12). Let \( N \) be a fuzzy negation and \( I \) be a fuzzy implication. A function \( I_N : [0, 1]^2 \rightarrow [0, 1] \) defined by

\[ I_N(x, y) = I(N(y), N(x)), x, y \in [0, 1], \]

is called the \( N \)-reciprocal of \( I \).

**Theorem 1.** ([1] Theorem 1.6.2). If \( N \) is a fuzzy negation and \( I \) is a fuzzy implication, then \( I_N \) is a fuzzy implication.

**Definition 14.** ([1,8,9,12,16]) A function \( I : [0, 1]^2 \rightarrow [0, 1] \) is called a QL-operation if there exist a t-norm \( T \), a t-conorm \( S \) and a fuzzy negation \( N \), such that

\[ I(x, y) = S(N(x), T(x, y)), x, y \in [0, 1]. \]

If \( I \) is a QL-operation generated from the triple \((T, S, N)\), then we will often denote it by \( I_{T,S,N} \).

**Definition 15.** ([1,8,16,17]) A function \( I : [0, 1]^2 \rightarrow [0, 1] \) is called a D-operation if there exist a t-norm \( T \), a t-conorm \( S \) and a fuzzy negation \( N \), such that

\[ I(x, y) = S(T(N(x), N(y)), y), x, y \in [0, 1]. \]

If \( I \) is a D-operation generated from the triple \((T, S, N)\), then we will often denote it by \( I_{T,S,N} \).
Definition 16. ([3] Definition 12). A function $I : [0,1]^2 \rightarrow [0,1]$ is called a $D'$-operation, if there exist a t-norm $T$, a t-conorm $S$ and a fuzzy negation $N$, such that

$$I(x, y) = N(T(S(x, y), N(y))), \quad x, y \in [0,1].$$

If $I$ is a $D'$-operation generated by the triple $(T, S, N)$, then we denote it by $I_{N,T,S}^N$.

Remark 7. (i) [1,16,17] QL- and D-operations are not fuzzy implications in general since (12) or (13) could not hold, respectively. Only if the QL-or D-operation is a fuzzy implication, we will use the term QL- or D-implication.

(ii) $D'$-operations are not fuzzy implications in general since (13) could not hold (see [3] Example 2). Only if the $D'$-operation is a fuzzy implication, we will use the term $D'$-implication (see [3] Proposition 2).

3. The Main Results

3.1. GD'-Implications

As we mentioned before, there are many definitions in fuzzy logic that are generalizations of classical tautologies. Such a generalization are $D'$-implications which are the generalization of the classical tautology (1). Since a fuzzy negation is not uniquely determined and there are many different functions that represent the notion of a fuzzy negation we are leading to the following definition.

Definition 17. A function $I : [0,1]^2 \rightarrow [0,1]$ is called a $GD'$-operation, if there exist a t-conorm $S$, a t-norm $T$ and two fuzzy negations $N', N$, such that

$$I(x, y) = N'(T(S(x, y), N(y))), \quad x, y \in [0,1].$$

If $I$ is a $GD'$-operation generated by the quadruple $(N', T, S, N)$, then we denote it by $I_{N',T,S,N}^{N'}$.

Theorem 2. Let $I_{N',T,S,N}^{N'}$ be a $GD'$-operation, then $I_{N',T,S,N}^{N'}$ satisfies (12), (14), (15), (16), (17), and (19). Furthermore $N_{I_{N',T,S,N}^{N'}} = N'$, where $N_{I_{N',T,S,N}^{N'}}(x) = I_{N',T,S,N}^{N'}(x,0), x \in [0,1]$.

Proof. Let $I_{N',T,S,N}^{N'}$ be a $GD'$-operation, then for $x, y, z \in [0,1]$, if

$$x \leq y \overset{(6)}{\Rightarrow} S(z, x) \leq S(z, y)$$

$$\overset{(4)}{\Rightarrow} S(x, z) \leq S(y, z)$$

$$\overset{(6)}{\Rightarrow} T(N(z), S(x, z)) \leq T(N(z), S(y, z))$$

$$\overset{(4)}{\Rightarrow} T(S(x, z), N(z)) \leq T(S(y, z), N(z))$$

$$\Rightarrow N'(T(S(x, z), N(z))) \geq N'(T(S(y, z), N(z)))$$

$$\Rightarrow I_{N',T,S,N}^{N'}(x, z) \geq I_{N',T,S,N}^{N'}(y, z),$$

which means that $I_{N',T,S,N}^{N'}$ satisfies (12).

$I_{N',T,S,N}^{N'}$ satisfies (14), since

$$I_{N',T,S,N}^{N'}(0,0) = N'(T(S(0,0), N(0))) \overset{(8)}{=} N'(T(0,1)) \overset{(7)}{=} N'(0) = 1.$$

$I_{N',T,S,N}^{N'}$ satisfies (15), since

$$I_{N',T,S,N}^{N'}(1,1) = N'(T(S(1,1), N(1))) \overset{(9)}{=} N'(T(1,0)) \overset{(4)}{=} N'(T(0,1)) \overset{(7)}{=} N'(0) = 1.$$
\[ I_{N', T, S, N} \] satisfies (16), since
\[ I_{N', T, S, N}(1, 0) = N'(T(S(1, 0), N(0))) \overset{8}{=} N'(T(1, 1)) \overset{7}{=} N'(1) = 0. \]

\[ I_{N', T, S, N} \] satisfies (17), since
\[ I_{N', T, S, N}(0, 1) = N'(T(S(0, 1), N(1))) \overset{4}{=} N'(T(S(1, 0), N(1))) \overset{8}{=} N'(T(1, 0)) \overset{7}{=} N'(0) = 1. \]

\[ I_{N', T, S, N} \] satisfies (19), since \( \forall x \in [0, 1] \) it is
\[ I_{N', T, S, N}(x, 1) = N'(T(S(x, 1), N(1))) \overset{9}{=} N'(T(1, 0)) \overset{4}{=} N'(T(0, 1)) \overset{7}{=} N'(0) = 1. \]

Lastly, we have
\[ N_{I_{N', T, S, N}}(x) = I_{N', T, S, N}(x, 0) = N'(T(S(x, 0), N(0))) \overset{8}{=} N'(T(x, 1)) \overset{7}{=} N'(x). \]
for all \( x \in [0, 1] \). \( \square \)

**Remark 8.** Note that if \( N' = N \) the corresponding GD'-operation is a D' -operation. Therefore, GD'-operations sometimes violate (13). The same happens even if we use different negations according to the following Example 1. Therefore, we called them GD'-operations, instead of GD'-implications.

**Example 1.** Consider the quadruple \( (N_C, T_M, S_P, N_K) \). The corresponding GD'-operation is
\[ I_{N_C, T_M, S_P, N_K}(x, y) = N_C(T_M(S_P(x, y), N_K(y))) \]
\[ = 1 - T_M(S_P(x, y), N_K(y)) \]
\[ = 1 - \min\{x + y - x \cdot y, 1 - y^2\} \]
\[ = \max\{1 - x - y + x \cdot y, 1 - 1 + y^2\} \]
\[ = \max\{1 - x - y + x \cdot y^2\}, \]
which is not a fuzzy implication, since
\[ 0.1 \leq 0.2 \Rightarrow I_{N_C, T_M, S_P, N_K}(0.1, 0.1) = 0.81 > 0.72 = I_{N_C, T_M, S_P, N_K}(0.1, 0.2). \]
Thus, \( I_{N_C, T_M, S_P, N_K} \) violates (13).

**Proposition 3.** A function \( I : [0, 1]^2 \to [0, 1] \) is called GD'-implication, if it is a GD'-operation and satisfies (13).

**Proof.** The proof is obvious. \( \square \)

**Example 2.** Consider the quadruple \( (N^1, T_D, S_M, N_0) \). The corresponding GD'-operation, which is a GD'-implication is
\[ I_{N_0,T_L,S_L,N_C}(x,y) = N_0(T_L(S_L(x,y), N_C(y))) \]
\[ = \begin{cases} 
N_0(T_L(S_L(x,y), 0)), & \text{if } y > 0 \\
N_0(T_L(S_L(x,0), 1)), & \text{if } y = 0 
\end{cases} \]

(10)
\[ \begin{cases} 
N_1(0), & \text{if } y > 0 \\
N_1(\min\{\max\{x,0\}, 1\}), & \text{if } y = 0 
\end{cases} \]
\[ = \begin{cases} 
1, & \text{if } y > 0 \\
N_1(\min\{x,1\}), & \text{if } y = 0 
\end{cases} \]
\[ = \begin{cases} 
1, & \text{if } y > 0 \\
N_1(x), & \text{if } y = 0 
\end{cases} \]
\[ = \begin{cases} 
1, & \text{if } x < 1 \text{ or } y > 0 \\
0, & \text{otherwise} 
\end{cases} \]
\[ = I_1(x,y). \]

**Example 3.** Consider the quadruple \((N_0, T_L, S_L, N_C)\). The corresponding GD'-operation, which is a GD'-implication is

\[ I_{N_0,T_L,S_L,N_C}(x,y) = N_0(T_L(S_L(x,y), N_C(y))) \]
\[ = \begin{cases} 
1, & \text{if } T_L(S_L(x,y), N_C(y)) = 0 \\
0, & \text{otherwise} 
\end{cases} \]
\[ = \begin{cases} 
1, & \text{if } \max\{S_L(x,y) + N_C(y) - 1,0\} = 0 \\
0, & \text{otherwise} 
\end{cases} \]
\[ = \begin{cases} 
1, & \text{if } S_L(x,y) + N_C(y) - 1 \leq 0 \\
0, & \text{otherwise} 
\end{cases} \]
\[ = \begin{cases} 
1, & \text{if } \min\{x+y,1\} + 1 - y - 1 \leq 0 \\
0, & \text{otherwise} 
\end{cases} \]
\[ = \begin{cases} 
1, & \text{if } \min\{x+y,1\} \leq y \\
0, & \text{otherwise} 
\end{cases} \]
\[ = \begin{cases} 
1, & \text{if } x = 0 \text{ or } y = 1 \\
0, & \text{otherwise} 
\end{cases} \]
\[ = I_0(x,y). \]

**Remark 9.** According to Examples 2 and 3 there are GD'-implications, that are not D'-implications, since I_0 and I_1 (for their formulas and symbolisms see [1] Proposition 1.1.7) are not D'-implications (see [3] Remark 7 and page 6). So it is obvious that GD'-implications is a hyper class of that of D'-implications. Moreover, it is a non empty set and a new class of fuzzy implications that contains D'-implications.

**Proposition 4.** Let \(I_{N,T,S,N}\) be a GD'-implication and \(N'\) be a non-filling fuzzy negation. Then the pair \((T, N)\) satisfies \((11)\).

**Proof.** If \(I_{N_T,S_T,N}\) is a fuzzy implication then it satisfies \((18)\). Therefore, for all \(y \in [0,1]\) it is

\[ I_{N',T,S,N}(0,y) = 1 \Rightarrow N'(T(S(0,y), N(y))) = 1 \]
\[ \Rightarrow N'(T(y, N(y))) = 1 \]
\[ \Rightarrow N'_{\not\text{fill}} \text{ is non-filling} \]
\[ \Rightarrow T(N(y),y) = 0. \]
Therefore, the pair \((T, N)\) satisfies (11). □

Example 4. Consider the quadruple \((N_C, T_{nM}, S_P, N_C)\). The corresponding GD'-operation (which is a D'-operation) is

\[
I_{N_C, T_{nM}, S_P, N_C}(x, y) = I_{N_C, T_{nM}, S_P}(x, y) = N_C(T_{nM}(S_P(x, y), N_C(y)) = 1 - T_{nM}(x + y - x \cdot y, 1 - y) = \begin{cases} 1 - 0, & \text{if } x + y - x \cdot y + 1 - y \leq 1 \\ 1 - \min\{x + y - x \cdot y, 1 - y\}, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } x \leq x \cdot y \\ \max\{1 - x - y + x \cdot y\}, & \text{otherwise} \end{cases} = \begin{cases} 1, & \text{if } x = 0 \text{ or } y = 1 \\ \max\{1 - x - y + x \cdot y\}, & \text{otherwise} \end{cases},
\]

which is not a fuzzy implication, because it violates (13), since

\[
0 < 0.5 \Rightarrow I_{N_C, T_{nM}, S_P, N_C}(0.1, 0) = 0.9 > 0.5 = I_{N_C, T_{nM}, S_P, N_C}(0.1, 0.5).
\]

Remark 10. (i) Proposition 4 gives a sufficient, but not necessary condition for a GD'-operation \(I_{N', T_{nM}, S_P, N_{C}}\) to be a fuzzy implication, when \(N'\) is a non-filling fuzzy negation. Note that \(N_C\) is a non-filling fuzzy negation, \((T_{nM}, N_C)\) satisfies (11), but \(I_{N_C, T_{nM}, S_P, N_C}\) is not a fuzzy implication (see Example 4).

(ii) By Proposition 4 it is obvious that, if \(N'\) is a non-filling negation and the pair \((T, N)\) does not satisfy the law of contradiction (11), i.e., \(T(N(x), x) \neq 0\) for some \(x \in (0, 1)\), then the obtained \(I_{N', T_{nM}, S_P, N_{C}}\) GD'-operation, for any t-conorm \(S\), is not a fuzzy implication.

(iii) Therefore, by (ii) we deduce that with any quadruple \((N', T, S, N)\), where \(N'\) is a non-filling fuzzy negation, \(T\) is \(T_P\) or \(T_{nM}\), \(S\) is any t-conorm and \(N\) is \(N_C\) or \(N_K\) or \(N_R\), we cannot generate a GD'-implication.

Theorem 3. If \(\phi \in \Phi\) and \(I_{N', T_{nM}, S_P, N_{C}}\) is a GD'-operation (respectively, implication), then \((I_{N', T_{nM}, S_P, N_{C}})\phi\) is a GD'-operation (respectively, implication) and, moreover,

\[
(I_{N', T_{nM}, S_P, N_{C}})\phi = I_{N'_\phi, T_{nM}, S_P, N_{C}}.
\]

Proof. Let \(I_{N', T_{nM}, S_P, N_{C}}\) be a GD'-operation (respectively implication), then \((I_{N', T_{nM}, S_P, N_{C}})\phi\) is a GD'-operation (respectively, implication) according to the Remark 6 (ii). Moreover, for all \(x, y \in [0, 1]\), we deduce that

\[
(I_{N', T_{nM}, S_P, N_{C}})\phi(x, y) = \phi^{-1}(I_{N', T_{nM}, S_P, N_{C}}(\phi(x), \phi(y))) = \phi^{-1}(N'((S(\phi(x), \phi(y)), N(\phi(y)))) = \phi^{-1}(N'(\phi(\phi^{-1}(T(S(\phi(x), \phi(y)), N(\phi(y))))))) = N'_{\phi}(\phi^{-1}(T(\phi(\phi^{-1}(S(\phi(x), \phi(y)), N(\phi(y))))))) = N'_{\phi}(T_{\phi}(\phi^{-1}(S(\phi(x), \phi(y)), N(\phi(y))))) = N'_{\phi}(T_{\phi}(S_{\phi}(x, y), N_{\phi}(y))) = I_{N'_{\phi}, T_{nM}, S_P, N_{C}}(x, y).
\]

□

A connection between D'-, D- and QL-operations has been studied in [3].

Theorem 4. ([3] Theorem 4.11). If \(N\) is a strong negation and \(I_{N', T_{nM}}\) is a D'-operation, then
(ii) \( I_{N,T,S'} = I_{T_{S'},N}^{T_{S'},S_{T_{S'},N}} = (I_{T_{S'},N}^{T_{S'},S_{T_{S'},N}})_N \), where \( I_{T_{S'},N}^{T_{S'},S_{T_{S'},N}} \) is a D-operation and \( (I_{T_{S'},N}^{T_{S'},S_{T_{S'},N}})_N \) is \( N \)-reciprocal of the QL-operation \( I_{T_{S'},N}^{T_{S'},S_{T_{S'},N}} \) and

(ii) moreover, if one of \( I_{N,T,S'} \), \( I_{T_{S'},N}^{T_{S'},S_{T_{S'},N}} \) and \( I_{T_{S'},N}^{T_{S'},S_{T_{S'},N}} \) is a fuzzy implication, then the other two are fuzzy implications, too.

A similar result for GD'-operations is the following:

**Theorem 5.** Let \( I_{N',T,S,N} \) be a GD'-operation generated from a t-norm \( T \), a t-conorm \( S \) and two strict fuzzy negations \( N', N \), such that \((N')^{-1} = N\). Then, \( I_{N',T,S,N} = I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}} \) is a D-operation and \((I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}})_N \) is \( N' \)-reciprocal of the QL-operation \( I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}} \). Moreover, if one of \( I_{N',T,S,N} \), \( I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}} \) and \( I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}} \) is a fuzzy implication, then the other two are fuzzy implications, too.

**Proof.** Let \( I_{N',T,S,N} \) be a GD'-operation. Then

\[
I_{N',T,S,N}(x, y)^{(N')^{-1}} = N'(T(S(x, y), (N')^{-1}(y)))
\]

\[
= N'(T((N')^{-1}(N'(S((N')^{-1}(N'(x)), (N')^{-1}(N'(y))))), (N')^{-1}(y)))
\]

\[
= S_{T,N}^{-1}(T_{S,N}^{-1}(N'(x), N'(y)), y)
\]

\[
= I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}}(x, y)
\]

and

\[
I_{N',T,S,N}(x, y) = I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}}(x, y)
\]

\[
= S_{T,N}^{-1}(T_{S,N}^{-1}(N'(x), N'(y)), y)
\]

\[
= S_{T,N}^{-1}(T_{S,N}^{-1}(N'(x), N'(y)))
\]

\[
= I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}}(N'(y), N'(x))
\]

\[
= I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}}(N'(y), N'(x))
\]

\[
= (I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}})_N(x, y).
\]

Moreover,

\[
(I_{N',T,S,N})^{(N')^{-1}}(x, y) = I_{N',T,S,N}((N')^{-1}(y), (N')^{-1}(x))
\]

\[
= I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}}((N')^{-1}(y), (N')^{-1}(x))
\]

\[
= (I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}})^{(N')^{-1}}(x, y),
\]

and

\[
(I_{N',T,S,N})^{(N')^{-1}}(x, y) = (I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}})^{(N')^{-1}}(x, y)
\]

\[
= I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}}((N')^{-1}(y), (N')^{-1}(x))
\]

\[
= I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}}((N')^{-1}(y), (N')^{-1}(x))
\]

\[
= I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}}((N')^{-1}(x), (N')^{-1}(y))
\]

\[
= I_{T_{S,N}}^{T_{S,N}^{-1}S_{T,N}^{-1}(N')^{-1}}(x, y).
\]
Therefore, by virtue of the aforementioned equations and Theorem 4, if one of $I^{N', T, S, N}_{N'}$, $I^{T, (N')^{-1}, S, (N')^{-1}, N'}_{S, (N')^{-1}, -1}$ and $(I^{T, (N')^{-1}, S, (N')^{-1}, -1})_{(N')^{-1}}$ is a fuzzy implication, then the other two are also fuzzy implications. \(\square\)

**Theorem 6.** By any quadruple $(N', T, S, N)$, where $N'$ is any non-filling fuzzy negation, $S$ is any $t$-conorm, $N$ is any continuous fuzzy negation and $T$ is any idempotent, strict or positive $t$-norm, it cannot be obtained any GD$'$-implication.

**Proof.** The proof is omitted, because it is similar to the proof of [3] Theorem 5, by using Remark 10 (ii). \(\square\)

### 3.2. GD$'$-Implications and the Left Neutrality Property (20)

**Proposition 5.** Let $I^{N', T, S, N}_{N'}$ be a GD$'$-operation. Then $I^{N', T, S, N}_{N'}$ satisfies (20), if, and only if, $N' \circ N = Id_{[0,1]}$.

**Proof.** Let $I^{N', T, S, N}_{N'}$ be a GD$'$-operation. If $N' \circ N = Id_{[0,1]}$ then for all $x \in [0,1]$ it is

$$I^{N', T, S, N}_{N'}(1, x) = N'(T(S(1, x), N(x)))$$

(9) $\Rightarrow$ $N'(T(1, N(x)))$

(4) $\Rightarrow$ $N'(T(N(x), 1))$

(7) $\Rightarrow$ $N'(N(x))$

$\Rightarrow$ $(N' \circ N)(x)$

$\Rightarrow$ $N' \circ N = Id_{[0,1]}$.

Thus, $I^{N', T, S, N}_{N'}$ satisfies (20).

Conversely, if $I^{N', T, S, N}_{N'}$ satisfies (20), then for all $x \in [0,1]$ it is

$$I^{N', T, S, N}_{N'}(1, x) = x \Rightarrow N'(T(S(1, x), N(x))) = x$$

(9) $\Rightarrow$ $N'(T(1, N(x))) = x$

(4) $\Rightarrow$ $N'(T(N(x), 1)) = x$

(7) $\Rightarrow$ $N'(N(x)) = x$

$\Rightarrow$ $(N' \circ N)(x) = x$

$\Rightarrow$ $N' \circ N = Id_{[0,1]}$.

\(\square\)

**Corollary 2.** Let $I^{N', T, S, N}_{N'}$ be a GD$'$-operation, where at least one of $N'$ and $N$ is a crisp fuzzy negation. Then $I^{N', T, S, N}_{N'}$ violates (20).

**Proof.** If at least one of $N'$ and $N$ is a crisp fuzzy negation then $N' \circ N \neq Id_{[0,1]}$. Therefore, if we assume that $I^{N', T, S, N}_{N'}$ satisfies (20), that is a contradiction according to Proposition 5. \(\square\)

According to the Lemma 1 and the Proposition 5 we find the following Corollaries. Their proofs are omitted, since they are obvious.

**Corollary 3.** Let $I^{N', T, S, N}_{N'}$ be a GD$'$-operation, where $N'$ is not a continuous fuzzy negation. Then $I^{N', T, S, N}_{N'}$ violates (20).
Corollary 4. Let $I_{N',T,S,N}$ be a GD$'$-operation, where $N$ is not strictly decreasing. Then $I_{N',T,S,N}$ violates (20).

3.3. GD$'$-Implications and the Exchange Principle (21)

Proposition 6. Let $I_{N',T,S,N}$ be a GD$'$-operation. If $I_{N',T,S,N}$ satisfies (21), then $N' \circ N \circ N' = N'$.

Proof. Since $I_{N',T,S,N}$ satisfies (21), then for all $x \in [0, 1]$ it is

\[
I_{N',T,S,N}(1, I_{N',T,S,N}(x, 0)) = I_{N',T,S,N}(x, I_{N',T,S,N}(1, 0))
\]

(16) $I_{N',T,S,N}(1, N'(T(S(x, 0), N(0)))) = I_{N',T,S,N}(x, 0)$

(8) $I_{N',T,S,N}(1, N'(T(x, 1))) = N'(T(S(x, 0), N(0)))$

(8) $I_{N',T,S,N}(1, N'(T(x, 1))) = N'(T(x, 1))$

(7) $I_{N',T,S,N}(1, N'(x)) = N'(x)$

$N'(T(S(1, N'(x)), N(N'(x)))) = N'(x)$

(9) $N'(T(1, N(N'(x)))) = N'(x)$

(2) $N'(N(N'(x))) = N'(x)$.

\[\square\]

Example 5. Let $S^2_{SS}$ be the Schweizer–Sklar t-conorm (see [1] Example 2.6.15), then for $\lambda = 2$ it is

\[
S^2_{SS} = 1 - \sqrt{\max\{(1 - x)^2 + (1 - y)^2 - 1, 0\}}.
\]

The obtained GD$'$-implication by the quadruple $(N_C, T^2_{SS}, S_P, N_C)$, where $T^2_{SS} = (S^2_{SS}) T^2_{SS}, N_C$ is (see [3] pages 15–16)

\[
I_{N_C, T^2_{SS}, S_P, N_C}(x, y) = I_{N_C, T^2_{SS}, S_P, N_C}(x, y) = 1 - \sqrt{\max\{(1 - y) \cdot [(1 - y) + (1 - y) \cdot (1 - y)^2 - 2 \cdot (1 - y)], 0\}},
\]

which does not satisfy (21), since

\[
I_{N_C, T^2_{SS}, S_P, N_C}(0.7, I_{N_C, T^2_{SS}, S_P, N_C}(0.8, 0.2)) = 0.84515 \neq 0.80071 = I_{N_C, T^2_{SS}, S_P, N_C}(0.8, I_{N_C, T^2_{SS}, S_P, N_C}(0.7, 0.2)).
\]

Remark 11. (i) Proposition 6 gives us the sufficient condition, that if there is an $x_0 \in (0, 1)$, such that $N'(N(N'(x_0))) \neq N'(x_0)$, then $I_{N',T,S,N}$ violates (21).

(ii) Since $N_C$ is a strong negation, we find that $N_C(N_C(N_C(x))) = N_C(x)$, but according to Example 5 $I_{N_C, T^2_{SS}, S_P, N_C}$ violates (21). Thus, Proposition 6 gives us a sufficient, but not necessary condition, for the satisfaction of (21), when we have a GD$'$-operation.

Proposition 6 is similar to Proposition 13 in [4] for $(N', T, N)$-implications and the condition $N' \circ N \circ N' = N'$ coincides in GD$'$-operations and $(N', T, N)$-implications, where $N'$ is the first fuzzy negation in both formulas. This is the reason we present the following Proposition 7, without proof, since it is the same with this of Proposition 15 in [4], but for GD$'$-operations.

Proposition 7. Let $I_{N',T,S,N}$ be a GD$'$-operation, where $N'$ is strictly decreasing with a fixed point. If $N$ does not have any fixed point, or $N'$ and $N$ have different fixed points, then $I_{N',T,S,N}$ violates (21).
3.4. GD'-Implications and the Identity Principle (22)

**Proposition 8.** Let the quadruple \((N', T, S, N)\), where \(N'\) is a non-filling fuzzy negation. If \(N_T\) is strong and the corresponding GD'-operation, \(I_{N', T, S, N}\) satisfies (22), then

\[
S(x, x) \leq (N_T \circ N)(x), \text{ for all } x \in [0, 1].
\]

**Proof.** Consider that \(I_{N', T, S, N}\) satisfies (22). Therefore, for all \(x \in [0, 1]\) it is

\[
I_{N', T, S, N}(x, x) = 1 \Rightarrow N'(T(S(x, x), N(x))) = 1
\]

Remark 3 (ii) \(\Rightarrow\) \(N(x) \leq N_T(S(x, x))\)

\[
\Rightarrow N_T(N(x)) \geq N_T(N_T(S(x, x)))
\]

Remark 2 \(\Rightarrow\) \(N_T(N(x)) \geq S(x, x)\)

\[
\Rightarrow S(x, x) \leq (N_T \circ N)(x).
\]

\[\square\]

**Theorem 7.** Let the quadruple \((N', T, S, N)\), where \(N'\) is a non-filling fuzzy negation. Let \(N_T\) be a strong fuzzy negation and \(T\) be a left continuous t-norm. Then the following statements are equivalent:

(i) \(I_{N', T, S, N}\) satisfies (22).

(ii) \(S(x, x) \leq (N_T \circ N)(x), \text{ for any } x \in [0, 1]\).

**Proof.** (i) \(\Rightarrow\) (ii)

The proof is obvious. Just apply Proposition 8.

(ii) \(\Rightarrow\) (i)

For any \(x \in [0, 1]\) it is

\[
S(x, x) \leq (N_T \circ N)(x) \Rightarrow N_T(N(x)) \geq S(x, x)
\]

Proposition 2 \(\Rightarrow\) \(T(N(x), S(x, x)) = 0\)

\[
\Rightarrow T(S(x, x), N(x)) = 0
\]

(4) \(\Rightarrow\) \(N'(T(S(x, x), N(x))) = N'(0)\)

\[
\Rightarrow N'(T(S(x, x), N(x))) = 1
\]

\[
\Rightarrow I_{N', T, S, N}(x, x) = 1.
\]

\[\square\]

**Remark 12.** Note that, Proposition 8 and Theorem 7 hold when \(N_T\) is strictly decreasing, or continuous, since it is also strong, according to Corollary 1.

3.5. GD'-Implications and the Ordering Property (23)

**Theorem 8.** Let \(I_{N', T, S, N}\) be a GD'-operation, that satisfies (23). Then the fuzzy negation \(N\) is strictly decreasing.
Proof. Assuming that $N$ is not strictly decreasing. Therefore, there are at least two $x_0, y_0 \in [0,1]$, such that $x_0 < y_0$ and $N(x_0) = N(y_0)$. Since $I_{N', T, S, N}$ satisfies (23), it is



That is a contradiction. Thus, $N$ is strictly decreasing. □

**Remark 13.** Theorem 8 ensures that if $N$ is not strictly decreasing, then the corresponding $I_{N', T, S, N}$ violates (23).

**Theorem 9.** Let $T$ be a t-norm and $N_T$ be a strong fuzzy negation. Then $T$ is not positive.

**Proof.** We assume that $T$ is a positive t-norm. Then, for every $x_0 \in (0, 1)$ it is

$$N_T(x_0) = \sup\{y \in [0, 1] | T(x_0, y) = 0\} = 0.$$

Since $N_T$ is strong, it is

$$N_T(N_T(x_0)) = x_0 \Rightarrow N_T(0) = x_0 \Rightarrow 1 = x_0.$$

That is a contradiction. Thus, $T$ is not a positive t-norm. □

It is obvious that the satisfaction of (23) implies the satisfaction of (22). Therefore, we are leading to the following Theorem.

**Corollary 5.** Let $I_{N', T, S, N}$ be a GD'-operation generated from the quadruple $(N', T, S, N)$ and satisfies (23). Moreover, let $N'$ be a non-filling fuzzy negation and $N_T$ be a strong fuzzy negation. Then the following statements hold:

(i) $S(x, x) \leq (N_T \circ N)(x)$, for any $x \in [0, 1]$;

(ii) $T$ is not a positive t-norm.

**Proof.** (i) It is deduced by Proposition 8 and the fact that, the satisfaction of (23) implies the satisfaction of (22).

(ii) It is deduced directly by Theorem 9. □

**Remark 14.** Note that, Theorem 9 and Corollary 5 hold when $N_T$ is strictly decreasing, or continuous, since it is also strong, according to Corollary 1.

4. Results

Many classes of fuzzy implications can be produced by well known generalizations of the notion of implication from classical tautologies. Moreover, there are fuzzy implications, which in their formula contain at least two times a fuzzy negation. Is it binding this fuzzy negation to be the same every time we meet it? The answer is negative. Therefore, in this paper we have introduced a hyper class of the known class of fuzzy implications, the so called D'-implications. This hyper class named GD'-implications, respectively. GD'-implications have the same disadvantage with D'-implications. They sometimes violate (13) (see Remark 8 and Example 1). Therefore, we use the term GD'-operations instead
of implications in general. It has been proved that GD′-operations is a hyper class of D′-operations (see Examples 2 and 3 and Remark 9). Moreover, some sufficient, but not necessary conditions for a GD′-operation to be a fuzzy implication, has been proved (see Proposition 4 and Remark 10). The relation between GD′-, QL- and D-operations, when we use two strict fuzzy negations has been presented and proved (see Theorem 5). Theorem 6 excludes quadruples that do not generate GD′-implications and in Theorem 3 the relation of Φ-conjugation in GD′-operations was studied.

In the following subsections the basic properties (20), (21), (22), and (23) have been studied extensively. The sufficient and necessary condition under a GD′-operation satisfies (20) has been presented and proved (see Proposition 6) and some Corollaries. Moreover, some conditions under a GD′-implication satisfies or violates (21) have been presented and studied in the homonymous subsection. A similar study for the satisfaction of (22) and the satisfaction or violation of (23) has also been made in the homonymous subsections. Note that, if we consider that $N′ = N$ the results of Sections 3.3–3.5, hold for D′-operations. Thus, this work also completes the study of D′-operations for properties (21), (22) and (23). Property (20) for D′-operations was fully studied in [3] (see Proposition 6 and Figure 2 in [3]).

Surprisingly, although this was not the aim and the purpose of this study a new result came out. In [16,18] Mas at al. studied QL- and D-implications, and their relation when they are generated from strong fuzzy negations. Moreover, in [1] page 108, Baczyński and Jayaram addressed that, the reciprocal of a QL-implication, called Dishkant implication (shortly D-implication), while $N$ is (still) a strong negation. By virtue of Theorem 5, $N$ is not only strong negation. Theorem 5 connects QL- and D-operations in the case we use, not only strong, but strict negations, which are not necessarily the same in the two corresponding formulas. Thus, we find that, the reciprocal of a QL-implication, called Dishkant implication (shortly D-implication), while $N$ is (still) a strict negation. Note that strong negations are also strict. We believe that this result came out firstly of the definition of D′-operations in [3] and secondly of this study, where the idea was the possible usage of two different negations in the formula of GD′-operations.

5. Discussion

Fuzzy implication functions play a crucial role to many applicable areas. They are a basic structure of fuzzy theory and they are studied a lot. Although, they are defined by satisfying the axioms (12)–(16), there are other desirable properties for them, such as (20), (21), (22), and (23). These properties, which are arising from classical logic’s tautologies restrict the uncountable great repertoire of fuzzy implications and allow a researcher to choose, depending on the application, a desired fuzzy implication [1,19]. For example, in [20], (23) was desirable in the construction of the used fuzzy implication in the Neuro-Fuzzy inference system (see [20] Equation (6)).

Therefore, what is the benefit of this study? By excluding, all the mathematical results and the improvements on unanswered questions or remarks, we believe that the real benefit of this hyper class is the production variety of fuzzy implications. This variety achieved by using mainly one dimensional functions, the fuzzy negations. Except, the simplicity of a one dimensional function, the possible usage of two different fuzzy negations gives us the benefit of constructing many different fuzzy implications with the same natural negation. How is this possible? Just by stabilizing $N′$ and generating different fuzzy implications. Thus, GD′-implications are a tool, where in applications we can generate not only a big variety of them, but also a variety of them with some desired properties we need.

6. Conclusions

In this paper, a hyper class of fuzzy implications, the so called GD′-implications was introduced and studied. Its characteristic was the possible usage of different negations in its formula. This advantage increases the variety of fuzzy implications’ construction methods and functions. On the other hand they are not always fuzzy implications and it is
an open problem, whether a GD′-operation is a fuzzy implication. In addition, the main properties of fuzzy implications for GD′-operations have been studied and a connection of them with the D- and QL-operation has also presented. The induced and aforementioned results of this connection also establish the necessity of this study. Moreover, this study also completes the study of the basic properties of D′-operations.

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