NONEXTENSIVE THEORY OF DARK MATTER AND GAS DENSITY PROFILES

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ABSTRACT

Pronounced core-halo patterns of dark matter and gas density profiles, observed in relaxed galaxies and clusters, were hitherto fitted by empirical power laws. On the other hand, similar features are well known from astrophysical plasma environments, subject to long-range interactions, modeled in the context of a nonextensive entropy generalization. We link nonextensive statistics to the problem of density distributions in large-scale structures and provide fundamentally derived density profiles, representing accurately the characteristics of both dark matter and hot plasma distributions as observed or generated in simulations. The bifurcation of the density distribution into a kinetic dark matter/thermodynamic gas branch turns out to be a natural consequence of the theory and is controlled by a single parameter, $\kappa$, measuring physically the degree of coupling within the system. Consequently, it is proposed to favor nonextensive distributions, derived from the fundamental physical context of entropy generalization and accounting for nonlocality and long-range interactions in gravitationally coupled systems, when modeling observed density profiles of astrophysical structures.

Subject headings: cosmology: theory — dark matter — galaxies: halos — galaxies: structure — plasmas

The analysis of dark matter (DM) and gas density distributions in galaxies and clusters is presently based on a variety of phenomenological models. Following King (1962), relaxed DM halo density profiles are well fitted by empirical power laws (Burkert 1995; Salucci & Burkert 2000) and from $N$-body simulations by $\rho_{\text{DM}} \sim (r/r_c)^{-3}(1 + r/r_c)^{-2}$ (Navarro-Frenk-White [NFW] model), where $r_c$ constitutes a scaling radius, chosen to join the asymptotic $r$-dependence (Navarro et al. 1996, 1997). Subsequently, a number of modifications were proposed (Fukushige & Makino 1997; Moore et al. 1999, 1999) along with criticism of the "universality" of the NFW profile (Jing & Suto 2000; Borriello & Salucci 2001). Comparing the density profiles of DM halos in cold dark matter (CDM) $N$-body simulations, the functional dependence $\rho_{\text{DM}} \sim (r/r_c)^{-\alpha}(1 + r/r_c)^{-1-\alpha}$ (Zhao 1996) was found to provide good fits to all halos, from dwarf galaxies to clusters at any redshift (Ricotti 2003), where $\alpha$ is related to the spectral index of the initial power spectrum of density perturbations. Recently, a universal density profile for dark and luminous matter was suggested (Merritt et al. 2005). Physically, we regard the DM halo as a self-gravitating collisionless system of weakly interacting particles in dynamical equilibrium, scale-invariant from galactic to cluster scales (Burkert 2000; Firmani et al. 2000; Spengel & Steinhard 2000).

On the other hand, the phenomenological $\beta$-model $\rho_{\text{gas}} \sim (1 + r^2/r_c^2)^{-3/2\beta}$ (Cavaliere & Fusco-Femiano 1976), where $r_c$ is the core radius, and/or the double $\beta$-model, a convolution of two $\beta$-models with the aim of resolving the $\beta$-discrepancy (Bahcall & Lubin 1994), provide reasonable representations of the hot gas density distribution in galaxies and clusters (Xue & Wu 2000; Ota & Mitsuda 2004). Physically, $\beta$ corresponds to the ratio of kinetic DM to thermal gas energy, assuming values of $\sim 0.3$.

Since any astrophysical system is subject to long-range gravitational or electromagnetic interactions, the present situation motivates us to introduce nonextensive statistics as a theoretical basis for both DM and hot plasma density profiles, utilized successfully to understand observed core-halo structures in astrophysical plasmas (Leubner 2004; Leubner & Voeroes 2005). In this situation, a single parameter, $\kappa$, characterizes the degree of nonextensivity or coupling within the system. The corresponding derived power-law distributions constitute a particular thermodynamic equilibrium state (Treumann 1999), commonly applied in astrophysical plasma modeling (Leubner & Schupfer 2001; Leubner 2002).

The concept of the Boltzmann-Gibbs-Shannon (BGS) thermostatistics constitutes a powerful tool whenever the physical system is extensive, i.e., whenever the entropy is additive. This situation holds when the range of microscopic interactions and memory are short and the environment is an Euclidean spacetime, i.e., a continuous and differentiable manifold. However, astrophysical systems are generally subject to spatial or temporal long-range interactions, making their behavior nonextensive. A generalization of the BGS entropy for statistical equilibrium was introduced from first principles by Renyi (1955) and Tsallis (1988), suitably extending the standard additivity to nonextensivity. The main theorems of the classical Maxwell-Boltzmann statistics admit profound generalizations within nonextensive statistics, leading to a variety of physical consequences (see, e.g., Tsallis 1995). Those include a reformulation of the classical $N$-body problem (Plastino et al. 1994) and/or the development of nonextensive distributions (Silva et al. 1998; Almeida 2001; Andrade et al. 2002), in which the duality of nonextensive statistics, which we will focus on, was recognized (Karlin et al. 2002). Astrophysical applications (Plastino & Plastino 1993; Kaniadakis et al. 1996; Nakamichi et al. 2002) provided a further manifestation of nonextensivity in nature. For a reformulation in the context of special relativity, see Kaniadakis (2002).

The generalized entropy $S(\kappa)$ characterizing systems subject to long-range interactions and couplings in nonextensive statistics reads (Tsallis 1988; Leubner 2004)

$$S_\kappa = \kappa k_B \left( \sum p_i^{1/\kappa} - 1 \right), \tag{1}$$

where $p_i$ is the probability of the $i$th microstate, $k_B$ is Boltzmann’s constant, and the “entropic index” $\kappa$ denotes a coupling parameter quantifying the degree of nonextensivity (or, equivalently, statistical correlations) within the system. A crucial property of this entropy is the pseudoadditivity for given sub-
systems, in the sense of the factorizability of the microstate probabilities. The transformation $\kappa = 1/(1 - q)$ links the $\kappa$-formalism to the Tsallis $q$-statistics (Leubner 2002). Here $\kappa$ is defined in the interval $-\infty \leq \kappa \leq \infty$, where $\kappa = \infty$ represents the extensive limit of statistical independence and recovers the classical BGS entropy as $S_0 = -k B \sum p_i \ln p_i$. Considering two subsystems $A$ and $B$, the nonextensive characteristics can be illuminated in view of the entropy mixing by $S_\kappa(A + B) = S_\kappa(A) + S_\kappa(B) + S_\kappa(A)S_\kappa(B)/\kappa$, a relation consistent with equation (1), where the last term accounts for the couplings; $\kappa < 0$ leads to an entropy decrease, providing a state of higher order, whereas for $\kappa > 0$ the entropy increases, and the system evolves into disorder. Hence, $\kappa$ can be interpreted as a bifurcation parameter measuring the two statistical realizations of ordering or disordering through correlations. A generalization for multiple subsystems is discussed in Milovanov & Zelenyi (2000).

Since entropy and probability distributions reside physically on the same level, the corresponding generalized energy distributions follow as $f^\kappa(v) = \lambda^\kappa [1 + \kappa v^2(\kappa a^2)]^{-1/\kappa}$ (Silva et al. 1998; Leubner 2004). The superscripts $\pm$ correspond to positive or negative values of $\kappa$; $\lambda^\pm$ are proper normalization constants, and $\sigma$ denotes the velocity dispersion or mean energy of the distribution characterizing their width (variance). Negative values of $\kappa$ are conveniently introduced by changing the sign at appearance, which generates a cutoff at $v = \sqrt{\kappa}\sigma$ (see Leubner 2004 for details).

Upon generalization to a spherical symmetric, self-gravitating, and collisionless N-body system, the corresponding steady state phase-space distribution $f(r, v)$ obeys the Vlasov equation. If the particles (stellar system, galaxies) themselves provide the gravitational potential and if $f(r, v)$ is regarded as the mass distribution, then Poisson’s equation, $\Delta \Phi = 4\pi G\rho_0$, reads

$$\Delta \Phi = 4\pi G \int f\left(\frac{1}{2} v^2 + \Phi\right) dv^3$$

and represents the fundamental equation governing the equilibrium of the system, where $f(\frac{1}{2} v^2 + \Phi) = f(E)$ depends on the energy only. Commonly, the relative particles’ energy $E_r = -1/2v^2 + \Phi$ is introduced, where $\Phi$ is the relative potential $\Phi = -\Phi + \Phi_0$, which is chosen to vanish at the systems boundary and satisfy Poisson’s equation as $\Delta \Phi = -4\pi G\rho_0$.

If $f(E_r)$ resembles the exponential mass distribution function defining the structure of an isothermal self-gravitating sphere of gas, in this case identical to the phase-space density distribution of a collisionless system of particles,

$$f(E_r) = \frac{\rho_0}{(2\pi\sigma^2)^3/2} \exp\left(-\frac{v^2/2 - \Phi}{\sigma^2}\right),$$

then the corresponding density distribution $\rho = \rho_0 \exp(\Psi/\sigma^2)$ is found after integrating over all velocities. Combined with Poisson’s equation (2), the solution governs the structure of the isothermal self-gravitating sphere (Binney & Tremaine 1987). The equilibrium distribution (eq. [3]) can be obtained by extremizing the standard BGS entropy with regard to conservation of mass and energy.

Since equation (3) applies exclusively to a system of independent particles, we now introduce long-range interactions by the generalized entropy functional (eq. [1]). Extremizing equation (1) after replacing the entropy function $f \ln f$ of an uncorrelated ensemble by the generalized functional $-\kappa f(1 - f^{1-1/\kappa})$ and applying Lagrange multipliers (Plastino & Plastino 1993), the resulting distribution function reads

$$f^\kappa(E_r) = B^\pm \left(1 + \frac{\kappa v^2/2 - \varphi}{\sigma^2}\right)^{-\kappa}$$

As previously mentioned, the superscripts refer to the positive or negative intervals of the entropic index $\kappa$, accounting for lower (+) or higher (−) organized states and thus reflecting the accompanying entropy increase or decrease, respectively. If we identify $f^\kappa(E_r)$ again as a mass distribution, the $\kappa$-dependent generalized constants $B^\pm$ assure proper normalization and dimension and differ for positive and negative definite $\kappa$-values as $B^+ = CT(\kappa)[\kappa^{3/2}\Gamma(\kappa - 3/2)]$ and $B^- = CT(\kappa + 5/2)[\kappa^{3/2}\Gamma(\kappa + 1)]$, where $C = \rho_0/(2\pi\sigma^2)^{3/2}$ and $\Gamma$ denotes the standard gamma function (Leubner 2004). The different normalization is caused by the interval corresponding to negative $\kappa$-values, which generates an energy cutoff in equation (4) leading to the constraint $\kappa v^2/2 - \varphi \leq \kappa a^2$ and also restricting the integration limits in velocity space. For $\kappa \rightarrow \infty$, equation (4) approaches the exponential distribution function (eq. [3]) defining the density profile of the isothermal sphere.

After incorporating the sign of $\kappa$ into equation (4), we perform separately for positive and negative definite $\kappa$ the integration of equation (4) over all velocities where $B^\pm$ must be used consistently. The resulting solution provides a modification of the velocity space context introduced by Leubner (2004) for the density evolution of a system in a gravitational potential as

$$\rho^\pm = \rho_0 \left(1 - \frac{\Psi}{\kappa \sigma^2}\right)^{3/2 - \kappa},$$

Analogous to the corresponding tandem character in velocity space, equation (5) generates, in a gravitational potential for finite positive values of $\kappa$, pronounced density tails, whereas for negative $\kappa$-values, the solutions are restricted within the cutoff at $\Psi = \kappa \sigma^2$ and $\kappa = -\infty$.

The duality of equilibria in nonextensive statistics is manifest in two families, the nonextensive thermodynamic equilibria and the equilibria of kinetic equations, and both are related by $\kappa' = 1 - 1/q$ (Karlin et al. 2002). Since $q = 1 - 1/\kappa$ (Leubner 2002), we find for the entropic index $\kappa' = -\kappa$, relating the two families of equilibria, where, with regard to equation (5), $\kappa > 0$ corresponds to stationary states of thermodynamics and $\kappa < 0$ corresponds to kinetic stationary states. The limiting BGS state for $\kappa = \infty$ is therefore characterized by self-duality. The nonextensive parameter $\kappa$ also finds a physical interpretation in terms of the heat capacity of a medium (Almeida 2001). A system with $\kappa > 0$ represents an environment with finite positive heat capacity and vice versa; for $\kappa < 0$, the heat capacity is negative. Negative heat capacity is a typical property of self-gravitating systems (see, e.g., Firmani et al. 2000). Moreover, contrary to thermodynamic systems, where the tendency toward disorganization is accompanied by increasing entropy, self-gravitation tends to result in higher organized structures of decreased entropy. Consequently, the nonextensive bifurcation of the singular isothermal sphere solution into two distributions, $f^\kappa(E_r)$ or $\rho^\kappa$, requires us to identify the density profile (eq. [5]) for positive definite $\kappa$ as the proper distribution of the thermodynamic state of the gas, whereas the negative definite counterpart is associated with the self-gravitating DM distribution. For $\kappa \rightarrow \infty$, both so-
The isothermal sphere solution, denoted as \( \rho = \rho_0 \), is governed by the equation of state of the isothermal sphere, \( \rho \propto r^{-1/2} \). Physically finite values of \( \kappa \) represent long-range interactions and correlations within the system, whereas the transition to \( \kappa = \infty \) in the central curve defines the extensive limit of statistical independence. Since we focus here on the shapes of radial profiles and their physical foundations, normalizations are conveniently applied. As \( r \to 0 \), \( \rho = \rho_0 = 1 \), and the solution meets the physically required condition \( d\rho/dr = 0 \) in the origin.

In Figure 2 we compare one negative (DM) solution to equation (6) (with the NFW model as well as one symmetrically, positive (gas) solution with a single \( \beta \)-model. On small scales, the theoretical DM density distribution is characterized consistently (e.g., Firmani et al. 2000) by a shallow core of finite density as \( r \to 0 \). Despite the gradual tail-like structure of the theoretical profile, the halo characteristics are similar to the NFW profile. Changes in the mean energy or variance \( \sigma \) generate a radial shift of the entire profile, which practically corresponds to variations in the scale radius \( r_s \) of the NFW model.

As a measure of the long-range interactions, the dimensionless second parameter \( \kappa \) of the theory controls the shape and characteristic mean slope of the profile on intermediate and large scales (see also Fig. 1). Thus, both the shallower and steeper slopes (Moore et al. 1998, 1999), interpreted in the nonextensive context as the results of different coupling strengths, are accessible. Recently, an improved fitting formula, converging to a finite density in the center, was introduced and compared with the standard NFW profile as well (Navarro et al. 2004). The accompanying discussion also critically illustrates the potential difficulties arising for simulations of the innermost structure of CDM halos (see also Trott & Melatos 2005). On the other hand, high-resolution rotation curve analyses of galaxies...
are consistent with cored halos only (de Blok et al. 2003; Gentile et al. 2004), supporting the physical results of the nonextensive statistics formulation by observational evidence. The nonextensive gas density distribution follows a single $\beta$-model in the core but deviates with a halo-tail formation. This deviation can be accurately fitted by a double $\beta$-model (a decomposition [as in Xue & Wu 2000] is shown by the dotted lines in Fig. 2), indicating that the nonextensive theory naturally provides a theoretical context in which to solve the $\beta$-discrepancy.

Figure 3 presents the radial dependence of the integrated mass of the DM and gas components for symmetric values of $\kappa = \pm 7$ as compared to the NFW and $\beta$-models, where on observational grounds a 20% central gas density fraction $\rho_0$ is introduced for proper visualization. Consistent with Figure 2, the NFW integrated mass exceeds the nonextensive solution slightly, whereas the integrated $\beta$-model is practically identical to the generalized entropy approach.

The dual nature of the nonextensive theory provides a solution to the problem of DM and gas density distributions of clustered matter in fundamental physics, where both parameters ($\kappa, \sigma$) admit a physical interpretation. The bifurcation of the density distribution into a kinetic DM and thermodynamic gas branch turns out to be a natural consequence of the theory and is controlled by the entropic index $\kappa$, which accounts physically for the nonlocality and long-range interactions of the nonextensive systems. Due to the different correlation properties of clustered structures, particular DM and gas density profiles might be subject to different values of $\kappa$ and $\sigma$, regulating the details of the possibly nonuniversal, mass-dependent profiles. The theory was also found to reproduce accurately the density profiles generated by $N$-body and hydrodynamic simulations, which is a subject that will be addressed elsewhere. In conclusion, the theory is proposed to favor the family of nonextensive distributions, derived from the fundamental physical context of entropy generalization, over empirical models when fitting observed density profiles of astrophysical structures.

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