HECKE OPERATORS AND ANALYTIC LANGLANDS
CORRESPONDENCE FOR CURVES OVER LOCAL FIELDS

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In memory of Isadore Singer

ABSTRACT. We construct analogues of the Hecke operators for the moduli space of $G$-bundles on a curve $X$ over a local field $F$ with parabolic structures at finitely many points. We conjecture that they define commuting compact normal operators on the Hilbert space of half-densities on this moduli space. In the case $F = \mathbb{C}$, we also conjecture that their joint spectrum is in a natural bijection with the set of $^G$-opers on $X$ with real monodromy. This may be viewed as an analytic version of the Langlands correspondence for complex curves. Furthermore, we conjecture an explicit formula relating the eigenvalues of the Hecke operators and the global differential operators studied in our previous paper [EFK1]. Assuming the compactness conjecture, this formula follows from a certain system of differential equations satisfied by the Hecke operators, which we prove in this paper for $G = PGL_n$.

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1. Introduction

The Langlands Program for a curve $X$ over a finite field $F_q$ studies, in the unramified case, the joint spectrum of the commuting Hecke operators acting on the space of $L^2$ functions on the groupoid of $F_q$-points of the stack $\text{Bun}_G$ of $G$-bundles on $X$ with its natural measure. It aims to express this spectrum in terms of the Galois data associated to $X$ and the Langlands dual group $^L G$.

In this paper, which is a continuation of [EFK1], we study an analogue of this problem when $F_q$ is replaced by a local field $F$. We define analogues of Hecke operators on a dense subspace of the Hilbert space $H_G$ of half-densities on $\text{Bun}_G$ and conjecture that they extend to commuting compact normal operators on $H_G$. Investigation of these operators is a hybrid of functional analysis and the study of the algebraic structure of $\text{Bun}_G$.

In the case $F = \mathbb{C}$, these Hecke operators commute with the global holomorphic differential operators on $\text{Bun}_G$ introduced in [BD1], as well as their complex conjugates. This enables us to describe, subject to the Compactness Conjecture formulated below, the joint spectrum of these operators in terms of the $^L G$-opers on $X$ with real monodromy, which play the role of the Galois data appearing in the case of curves over a finite field. We consider this description as an analogue of the Langlands correspondence in the case of complex curves. In the case of $G = \text{SL}_2$, a similar conjecture about eigenfunctions of the differential operators on $\text{Bun}_G$ was proposed earlier by J. Teschner [T].

1.1. Main objects. Let $F$ be a local field, $G$ a split connected reductive group over $F$, $B$ its Borel subgroup, $X$ a smooth projective curve over $F$ and $S \subset X$ a reduced divisor defined over $F$. We denote by $\text{Bun}_G(X, S)$ the moduli stack of pairs $(\mathcal{F}, r_S)$, where $\mathcal{F}$ is a $G$-bundle on $X$ and $r_S$ is a reduction to $B$ of the restriction $\mathcal{F}|_S$ of $\mathcal{F}$ to $S$.

Let $\text{Bun}_G^s(X, S) \subset \text{Bun}_G(X, S)$ be the substack of regularly stable pairs $(\mathcal{F}, r_S)$, i.e. stable pairs whose group of automorphisms is equal to the center $Z(G)$ of $G$.

We will assume throughout this paper that $\text{Bun}_G^s(X, S)$ is open and dense in $\text{Bun}_G(X, S)$, which means that we are considering one of the following cases:

1. the genus of $X$ is greater than 1, and $S$ is arbitrary;
2. $X$ is an elliptic curve and $|S| \geq 1$;
3. $X = \mathbb{P}^1$ and $|S| \geq 3$.

Just as in the case of curves over $F_q$, the theory naturally extends to incorporate reductions to other parabolic subgroups of $G$. We also expect that it extends to wild ramification.
The stack $\text{Bun}_G^{\circ}(X, S)$ is a $Z(G)$-gerbe over a smooth analytic $F$-variety $\text{Bun}_G^{\circ}(X, S)$, which is the corresponding coarse moduli space. For our purposes, $\text{Bun}_G^{\circ}(X, S)$ is a good substitute for $\text{Bun}_G^\ast(X, S)$ because all objects we need (such as line bundles or differential operators) naturally descend from $\text{Bun}_G^{\circ}(X, S)$ to $\text{Bun}_G^{\circ}(X, S)$. 

R.P. Langlands asked in [L] whether it is possible to develop an analytic theory of automorphic functions for complex curves. In [EFK1] and the present paper we are developing such a theory (for curves over $\mathbb{C}$ and other local fields), understood as a spectral theory of commuting Hecke operators acting on the Hilbert space $\mathcal{H}_G(X, S)$ of half-densities on $\text{Bun}_G^\circ(X, S)$ for pairs $(X, S)$ satisfying the above condition, as well as global differential operators on $\text{Bun}_G(X, S)$ when $F = \mathbb{C}$ or $\mathbb{R}$.

To define the Hilbert space $\mathcal{H}_G(X, S)$, we introduce the $\mathbb{R}_{>0}$-line bundle $\Omega^{1/2}_{\text{Bun}}$ of half-densities on $\text{Bun}_G^\circ(X, S)$.

Namely, our local field $F$ is equipped with the norm map $x \mapsto \|x\|$ (the Haar measure on $F$ multiplies by $\|\lambda\|$ under rescaling by $\lambda \in F$). For instance, for $F = \mathbb{R}$ we have $\|x\| = |x|$, for $F = \mathbb{C}$ we have $\|x\| = |x|^2$, and for $F = \mathbb{Q}_p$ we have $\|x\| = p^{-v(x)}$, where $v(x)$ is the $p$-adic valuation of $x$ (see [W]). Using the norm map, we associate to any line bundle $L$ on any smooth algebraic $F$-variety $X$ a complex line bundle $\|L\|$ on the analytic $F$-variety $Y := \mathbb{Y}(F)$ with the structure group $\mathbb{R}_{>0}$. Namely, if the transition functions of $L$ are $\{g_{\alpha\beta}\}$, then the transition functions of $\|L\|$ are $\{\|g_{\alpha\beta}\|\}$ (in particular, if $F = \mathbb{C}$, then these are $\{|g_{\alpha\beta}|^2\}$). (Note that for archimedean fields the line bundle $\|L\|$ has a $C^\infty$-structure.)

It is shown in [BD1], Sect. 4 (see also [LS]) that the canonical line bundle $K_{\text{Bun}}$ over $\text{Bun}_G(X, S)$ has a square root $K_{\text{Bun}}^{1/2}$: The restriction of $K_{\text{Bun}}^{1/2}$ to $\text{Bun}_G^\circ(X, S)$ descends to a line bundle on $\text{Bun}_G^\circ(X, S)$ for which we will use the same notation $K_{\text{Bun}}^{1/2}$. We then set

\begin{equation}
\Omega^{1/2}_{\text{Bun}} := \|K_{\text{Bun}}^{1/2}\|.
\end{equation}

Alternatively, $\Omega^{1/2}_X$ can be defined as the square root of the line bundle $\|K_X\|$ (since the structure group of the latter is $\mathbb{R}_{>0}$, this is well-defined). This shows that line bundle $\Omega^{1/2}_{\text{Bun}}$ does not depend on the choice of $K_{\text{Bun}}^{1/2}$.

Let $V_G(X, S)$ be the space of smooth compactly supported sections of $\Omega^{1/2}_{\text{Bun}}$ over $\text{Bun}_G^\circ(X, S)(F)$. We denote by $\langle \cdot, \cdot \rangle$ the positive-definite Hermitian form on $V_G(X, S)$ given by

$$
\langle v, w \rangle := \int_{\text{Bun}_G^\circ(X, S)(F)} v \cdot \overline{w}, \quad v, w \in V_G(X, S),
$$

and define $\mathcal{H}_G(X, S)$ as the Hilbert space completion of $V_G(G, S)$.

1.2. Hecke operators. From now on, for brevity, we will drop $(X, S)$ in our notation when no confusion could arise (i.e. write $\text{Bun}_G$ for $\text{Bun}_G(X, S)$, $\mathcal{H}_G$ for $\mathcal{H}_G(X, S)$, etc.). We are going to define analogues of the Hecke operators for curves over a local field.

For non-archimedean local fields, these operators were suggested by A. Braverman and one of the authors in [BK]. For $G = PGL_2$, $X = \mathbb{P}^1$, Hecke operators were
studied by M. Kontsevich in \[Ko\], and he also knew that such operators could be defined in general. In his letters to us (2019) he conjectured compactness and the Hilbert–Schmidt property of averages of the Hecke operators over sufficiently many points.

The idea that Hecke operators over \(\mathbb{C}\) could be used to construct an analogue of the Langlands correspondence was suggested by R.P. Langlands \[L\], who sought to construct them in the case when \(G = GL_2\), \(X\) is an elliptic curve, and \(S = \emptyset\) (however, for an elliptic curve \(X\) we can only define Hecke operators if \(S \neq \emptyset\); see \[Fr2\], Sect. 3).

In the case when \(G\) is a torus, the Hecke operators and their spectra on \(H_G\) were completely described in \[Fr2\], Sect. 2. We recall these results for \(G = GL_1\) in Section 2.1 below.

From now on (except in Section 2 below) we will assume that \(G\) is a connected simple algebraic group over \(F\). All of our results and conjectures generalize in a straightforward way to connected semisimple algebraic groups over \(F\).

Let \(P^\vee_+\) be the set of dominant integral coweights of \(G\). For \(\lambda \in P^\vee_+\) we denote by \(Z(\lambda)\) the Hecke correspondence
\[
\overline{q} : Z(\lambda) \to \text{Bun}_G \times \text{Bun}_G \times (X \setminus S)
\]
describing the \(\lambda\)-modifications of stable \(G\)-bundles at points of \(X \setminus S\). More precisely, \(Z(\lambda)\) classifies triples \((\mathcal{F}, \mathcal{P}, x, t)\), where \(\mathcal{F}\) and \(\mathcal{P}\) are principal \(G\)-bundles on \(X\) with a reduction to \(B\) at \(S\), \(x\) is a closed point in \(X \setminus S\), and
\[
t : \mathcal{F}|_{X \setminus S} \sim \to \mathcal{P}|_{X \setminus S}
\]
is an isomorphism satisfying the following condition. Choose a formal coordinate \(z\) at \(x\) and a trivialization of \(\mathcal{F}\) on the formal disc \(D_x := \text{Spec} \mathbb{C}\langle z \rangle\); then the restriction of \(t\) to \(D_x \times x := \text{Spec} \mathbb{C}(\langle z \rangle)\) naturally gives rise to a point in the affine Grassmannian \(\text{Gr} = G((z))/G[[z]]\). The condition is that this point belongs to the closure \(\overline{\text{Gr}}_\lambda\) of the \(G[[z]]\)-orbit \(\text{Gr}_\lambda = G[[z]] \cdot \lambda(z)G[[z]]\) (this condition does not depend on the above choices). Further, denote by \(Z(\lambda)\) the open part of \(\overline{Z}(\lambda)\) satisfying the condition that this point belongs to \(\text{Gr}_\lambda\) itself, and let \(q\) be the restriction of \(\overline{q}\) to \(Z(\lambda)\).

Let
\[
p_{1,2} : \text{Bun}_G \times \text{Bun}_G \times (X \setminus S) \to \text{Bun}_G
\]
\[
p_3 : \text{Bun}_G \times \text{Bun}_G \times (X \setminus S) \to X \setminus S
\]
be the natural projections, \(\overline{q}_i := p_i \circ \overline{q}\), and \(q_i := p_i \circ q\). Thus, the fibers of the morphism \(\overline{q}_2 \times \overline{q}_3\) are isomorphic to \(\text{Gr}_\lambda\) and the fibers of the morphism \(q_2 \times q_3\) are isomorphic to \(\text{Gr}_\lambda\). Let \(K_2\) be the relative canonical line bundle of the morphism \(q_2 \times q_3\).

Let \(\rho\) be the integral weight of \(G\) such that \(\langle \alpha^\vee_i, \rho \rangle = 1\) for all simple coroots \(\alpha^\vee_i\) of \(G\). If the set \(\{(\lambda, \rho), \lambda \in P^\vee_+\}\) only contains integers, then Beilinson and Drinfeld construct in \[BD1\], Sect. 4 a square root \(K_{\text{Bun}}^{1/2}\) of \(K_{\text{Bun}}\) equipped with a trivialization of its fiber at the trivial \(G\)-bundle. If this set contains half-integers, they construct such an object for each choice of a square root \(K_X^{1/2}\) of \(K_X\). In the latter case, we will
assume throughout that a choice of $K_X^{1/2}$ has been made. We will denote by $\gamma$ the isomorphism class of this $K_X^{1/2}$ and by $K_{Bun}^{1/2}$ the corresponding line bundle on $Bun_G$ (equipped with a trivialization of its fiber at the trivial $G$-bundle). The following result is essentially due to Beilinson and Drinfeld [BD1]. In Section 5.4 below we explain how to derive it from formula (241) of [BD1] (see also [BK], Theorem 2.4).

**Theorem 1.1.** There exists a canonical isomorphism

$$a : q_1^*(K_{Bun}^{1/2}) \sim \rightarrow q_3^*(K_{Bun}^{1/2}) \otimes K_2 \otimes q_3^*(K_X^{-(\lambda, \rho)})$$

where $\rho$ is the half-sum of positive roots.

The isomorphism $a$ gives rise to an isomorphism

$$a^2 : q_1^*(K_{Bun}^{1/2}) \sim \rightarrow q_2^*(K_{Bun}^{1/2}) \otimes (K_2)^2 \otimes q_3^*(K_X^{-2(\lambda, \rho)})$$

which does not depend on the choice of $K_X^{1/2}$ and $K_{Bun}^{1/2}$. Using the formula $\|a\| = \|a^2\|^{1/2}$, we then obtain a canonical isomorphism

$$\|a\| : q_1^*(\Omega_{Bun}^{1/2}) \sim \rightarrow q_2^*(\Omega_{Bun}^{1/2}) \otimes \Omega_2 \otimes q_3^*(\Omega_X^{-(\lambda, \rho)}).$$

Here $\Omega_{X}^{1/2}$ (resp. $\Omega_{Bun}^{1/2}$) is defined as the square root of the line bundle $\|K_{X}\|$ (resp. $\|K_{Bun}\|$); since the structure group of the latter is $\mathbb{R}_{>0}$, these square roots are well-defined. Also, $\Omega_2 := \|K_2\|$, the $C^\infty$ line bundle of densities along the fibers of $q_2 \times q_3$.

Thus, $\|a\|$ does not depend on the choice of $K_X^{1/2}$ (and $K_{Bun}^{1/2}$). This implies that our definition of the Hecke operators given below also does not depend on this choice. However, we need to make these choices to describe the algebra of global differential operators on $Bun_G$ (see Section 1.3) and to state the differential equations (1.13) satisfied by the Hecke operators. Hence, for the sake of uniformity of exposition, we have made these choices from the beginning.

Now let

$$U_G(\lambda) := \{ F \in Bun_G^\circ \mid q_2(q_1^{-1}(F)) \subset Bun_G^\circ \}.$$

This is an open subset of $Bun_G^\circ$, which is dense if the dimension of $Bun_G$ is sufficiently large (for instance, for $G = PGL_2$ it is sufficient that $\dim Bun_{PGL_2} > 1$).

Suppose that $U_G(\lambda) \subset Bun_G^\circ$ is dense. Let $V_G(\lambda) \subset V_G$ be the subspace of half-densities $f$ with compact support such that $\text{supp}(f) \subset U_G(\lambda)$. For $P \in Bun_G^\circ(F)$, let

$$Z_{P,x} := (q_2 \times q_3)^{-1}(P \times x), \quad x \in (X \setminus S)(F).$$

According to the above definition of $Z(\lambda)$, the variety $Z_{P,x}$ is isomorphic to the $G[[z]]$-orbit $Gr_x$ in the affine Grassmannian of $G$. Hence it is proper and has rational singularities. The isomorphism (1.4) and the results of Theorem 2.5 and Sect. 2.6 of [BK] imply that for any $f \in V_G(\lambda)$ and $x \in (X \setminus S)(F)$, the restriction of the pull-back $q_1^*(f)$ to $Z_{P,x}$ is a compactly supported measure with values in the line $(\Omega_{Bun}^{1/2})_P \otimes (\Omega_X^{-(\lambda, \rho)})_x$. Therefore the integral

$$(\hat{H}_\lambda(x) \cdot f)(P) := \int_{Z_{P,x}} q_1^*(f),$$

...
is absolutely convergent for all \( f \in V_G(\lambda) \) and \( P \in \text{Bun}_G^0(F) \). In fact, it belongs to \( V_G \otimes (\Omega_X^{-\langle \lambda, \rho \rangle})_x \).

Since \( V_G \subset \mathcal{H}_G \), this integral defines a Hecke operator

\[ \hat{H}_\lambda(x) : V_G(\lambda) \to \mathcal{H}_G \otimes (\Omega_X^{-\langle \lambda, \rho \rangle})_x. \]

As \( x \) varies along \( X \setminus S \), the operators \( \hat{H}_\lambda(x) \) combine into a section of the line bundle \( \Omega_X^{-\langle \lambda, \rho \rangle} \) over \( X \setminus S \) with values in operators \( V_G(\lambda) \to \mathcal{H}_G \). We denote it by \( \hat{H}_\lambda \).

Remark 1.1. We conjecture (and can prove in a number of cases) that the integrals defining \( \hat{H}_\lambda(f) \) are absolutely convergent for all \( f \in V_G \).

Remark 1.2. There are two basic differences between the cases of finite and local fields. In the case of a curve \( X \) over a finite field \( F \), the set \( \text{Bun}_G(F) \) of isomorphism classes \([P]\) of \( G \)-bundles is a countable set with a natural (Tamagawa) measure such that the measure of \([P]\) is equal to \( 1/|\text{Aut}(P)(F)| \). On the other hand, for a curve \( X \) over a local field \( F \), neither \( \text{Bun}_G(F) \) nor the fibers of the Hecke correspondence carry any natural measures. To overcome this problem, we replace the space of functions on \( \text{Bun}_G(F) \) by the space of half-densities and use the isomorphism (1.4) to write down formula (1.6) for the action of Hecke operators.

The second difference is that for a curve over a finite field \( F \) the integrals (1.6) are finite sums and so the corresponding Hecke operators are well-defined on the space of all functions on \( \text{Bun}_G(F) \). On the other hand, for a curve \( X \) over a local field \( F \) there is no obvious non-trivial space of half-densities stable under these Hecke operators. One could try to consider the space of continuous half-densities with respect to the natural topology on \( \text{Bun}_G(F) \) but this space is equal to \( \{0\} \). The reason is that the natural topology on \( \text{Bun}_G(F) \) is not Hausdorff\(^2\).

The absence of such a subspace creates serious analytic difficulties in defining the notion of the spectrum of the Hecke operators because they are initially defined only on a dense subspace \( V_G(\lambda) \) of \( \mathcal{H}_G \) (under our assumption that \( \text{Bun}_G^0(X, S) \) is dense in \( \text{Bun}_G(X, S) \)). However, we expect (and can prove for \( G = \text{PGL}_2, X = \mathbb{P}^1 \) [EFK2]) that these operators extend to bounded operators on \( \mathcal{H}_G \), which are moreover compact and normal, as we state in the following conjecture.

Conjecture 1.2. Suppose that \( \lambda \neq 0 \).

1. For any identification \( (\Omega_X^{1/2})_x \cong \mathbb{C} \), the operators \( \hat{H}_\lambda(x) : V_G(\lambda) \to \mathcal{H}_G \) extend to a family of commuting compact normal operators on \( \mathcal{H}_G \), which we denote by \( H_\lambda(x) \).
2. \( H_\lambda(x)^* = H_{-w_0(\lambda)}(x) \).
3. \( \bigcap_{\lambda, x} \ker H_\lambda(x) = \{0\} \).

Remark 1.3. (i) It is easy to see that

\[ \langle H_\lambda f, g \rangle = \langle f, H_{-w_0(\lambda)} g \rangle \]

\(^2\)For a simply-connected group \( G \), the closures of any two points in \( \text{Bun}_G(F) \) have a non-empty intersection, and therefore the only continuous functions on \( \text{Bun}_G(F) \) are constants.
for all \( f \in V_G(\lambda), g \in V_G(-w_0(\lambda)) \). Therefore part (2) of the conjecture immediately follows from part (1).

(ii) If \( S \neq \emptyset \), we expect that the statement of part (3) with a fixed \( \lambda \) can be obtained from (1) by considering the limit of \( H_\lambda(x) \) when \( x \) tends to a point of \( S \). This will be discussed in more detail in a follow-up paper.

We will refer to Conjecture 1.2 as the Compactness Conjecture. In our next paper [EFK2], we will prove this conjecture in the case \( G = \text{PGL}_2, X = \mathbb{P}^1, |S| > 3 \). From now on, we will assume the validity of the Compactness Conjecture.

Denote by \( \mathcal{H}_G \) the commutative algebra generated by operators \( H_\lambda(x), \lambda \in \mathcal{P}^\vee, x \in (X\setminus S)(F) \), and by Spec(\( \mathcal{H}_G \)) their joint spectrum.

**Corollary 1.3.** We have an orthogonal decomposition

\[
\mathcal{H}_G = \bigoplus_{s \in \text{Spec}(\mathcal{H}_G)} \mathcal{H}_G(s),
\]

where \( \mathcal{H}_G(s), s \in \text{Spec}(\mathcal{H}_G) \), are finite-dimensional joint eigenspaces of \( \mathcal{H}_G \) in \( \mathcal{H}_G \).

Let \( \mathbb{H}_G(x) \) be the subalgebra of \( \mathbb{H}_G \) generated by \( H_\lambda(x), \lambda \in P^\vee(x), x \in (X\setminus S)(F) \).

**Proposition 1.4 ([BK]).** There is an algebra isomorphism \( \mathbb{H}_G(x) \simeq \mathbb{C}[P^\vee] \). In particular, \( H_\lambda(x) \cdot H_\mu(x) = H_{\lambda+\mu}(x) \).

**Proof.** If \( F \) is a non-archimedean field, this is equation (3.4) of [BK], which is proved in Lemmas 3.5 and 3.9 of [BK]. The same proof works for a general local field.

**Remark 1.4.** Note the difference with the case of a curve over \( \mathbb{F}_q \), where the Satake isomorphism naturally identifies the Hecke algebra at a point \( x \) with \( \text{Rep}_L G \). Thus, in this case the product of the Hecke operators corresponding to \( \lambda, \mu \in P^\vee \) is in general not equal to the Hecke operator corresponding to \( \lambda + \mu \); there are correction terms corresponding to lower weights.

**Remark 1.5.** Using an analogous correspondence

\[
Z_r(\lambda) \to \text{Bun}_G \times \text{Bun}_G \times \text{Sym}^r(X\setminus S)
\]

for a positive integer \( r > 1 \), the above construction can be generalized to yield operators \( H_\lambda(D), D \in \text{Sym}^r(X\setminus S)(F) \). We expect that an analogue of the Compactness Conjecture 1.2 holds for them. Moreover, if \( D = \sum \limits_i m_i x_i, x_i \in X(F) \) then \( H_\lambda(D) = \prod \limits_i H_\lambda(x_i)^{m_i} \).

### 1.3. The case \( F = \mathbb{C} \)

At the moment, we have a conjectural description of the spectra \( \text{Spec}(\mathbb{H}_G) \) only if \( F = \mathbb{C} \) and, in some cases, for \( F = \mathbb{R} \). We will now describe our conjecture for \( F = \mathbb{C} \). Note that in this case we have \( \|z\| = |z|^2 \). For simplicity, we only consider here the case when \( S = \emptyset \) (and hence \( g > 1 \)). But all of our results and conjectures have natural generalizations to the case of an arbitrary \( S \). The case of \( G = \text{PGL}_2, X = \mathbb{P}^1, \) and \( |S| > 3 \) will be discussed in detail in our next paper [EFK2].
In our previous paper [EFK1] we studied the action of global holomorphic and anti-holomorphic differential operators on \( \text{Bun}_G \) on a dense subspace of the Hilbert space \( \mathcal{H}_G \) in the case of a simply-connected simple group \( G \). Here we generalize the setup to the case of an arbitrary connected simple \( G \). Then \( \text{Bun}_G \) has finitely many connected components labeled by \( \pi_1(G) \), which we will denote by \( \text{Bun}_G^\beta, \beta \in \pi_1(G) \).

**Definition 1.5.** Let \( D_G^\beta \) be the algebra of global algebraic (hence holomorphic) differential operators acting on the line bundle \( K_{1/2}^{1/2} \) over a connected component \( \text{Bun}_G^\beta \) of \( \text{Bun}_G \). Beilinson and Drinfeld have proved in [BD1] that these algebras are isomorphic to each other. Therefore, we will also use the notation \( D_G \) for \( D_G^\beta \).

On the other hand, let \( \text{Op}_{L^G}(X) \) be the space of (holomorphic) \( L^G \)-opers on \( X \) defined in [BD1, BD2]. As shown in [BD1], Sect. 3.4, \( \text{Op}_{L^G}(X) \) is a union of connected components, each isomorphic to the affine space \( \text{Op}_{L^G_{\text{ad}}}(X) \), where \( L^G_{\text{ad}} = L^G/Z(L^G) \) is the group of adjoint type associated to \( L^G \) (here \( Z(L^G) \) denotes the finite center of \( L^G \)). The group \( H^1(X, Z(L^G)) \) naturally acts on the set of these components by changing the underlying \( L^G \)-bundle, and this action is simply-transitive. As shown in [BD1], Sect. 3.4.2, if the set \( \{\langle \lambda, \rho \rangle, \lambda \in P^+_1 \} \) only contains integers, then there is a canonical choice of a component, and if the set \( \{\langle \lambda, \rho \rangle, \lambda \in P^+_1 \} \) contains half-integers, then there is a canonical choice of a component \( \text{Op}_{L^G}(X) \) for each isomorphism class \( \gamma \) of \( K_{1/2}^{1/2} \).

For the sake of uniformity of notation, in both cases we will denote the canonical component by \( \text{Op}_{L^G}(X) \), with the understanding that in the latter case \( \gamma \) is the isomorphism class of \( K_{1/2}^{1/2} \) we have chosen before Theorem 1.1. The following theorem is proved in [BD1] (Theorem 3.3.2 and Sects. 2.2.5 and 2.7.4).

**Theorem 1.6.** The algebra \( D_G \) is commutative and \( \text{Spec}(D_G) \) is isomorphic to the affine space \( \text{Op}_{L^G}(X) \).

For a given \( \chi \in \text{Op}_{L^G}(X) \), the system of differential equations for the eigenfunctions of this algebra,

\[
P \cdot f = \chi(P)f, \quad P \in D_G,
\]

where \( f \) denotes a local holomorphic section of \( K_{1/2}^{1/2} \), is known as the quantum Hitchin integrable system. Its local solutions are the same as the homomorphisms from the twisted \( D \)-module \( \Delta_X \) on \( \text{Bun}_G \) to \( K_{1/2}^{1/2} \), where

\[
\Delta_X := \mathcal{D}_{\text{Bun}_G} \otimes_{D_G} \mathbb{C}_X
\]

and \( \mathcal{D}_{\text{Bun}_G} \) is the sheaf of (holomorphic) differential operators acting on \( K_{1/2}^{1/2} \) (in (1.8) we consider the diagonal embedding \( D_G \to \mathcal{D}_{\text{Bun}_G} \) taking the sum over all components \( \text{Bun}_G^\beta \) of \( \text{Bun}_G \)).

**Remark 1.6.** If \( S \neq \emptyset \), the algebra of global holomorphic differential operators acting on \( K_{1/2}^{1/2} \) is non-commutative, but it contains a commutative subalgebra whose spectrum is isomorphic to the space of \( L^G \)-opers on \( X \) with regular singularity and
regular unipotent monodromy at the points of \( S \) (see [EFK1], Sect. 6). There is a similar algebra in the case of higher level structures at \( S \) (this is analogous to the wild ramification in the case of curves over \( \mathbb{F}_q \)), with the corresponding \( ^LG \)-opers having irregular singularities at the points of \( S \). We expect that the theory has a generalization to this case as well.

It is shown in [BD1] that \( \Delta\chi \) is a holonomic \( D \)-module. It also has regular singularities (see Theorem 1.12(1) below). Moreover, there is an open substack \( \text{Bun}_{\text{vs}}^G \) of \( \text{very stable bundles} \) (i.e. \( G \)-bundles \( \mathcal{P} \) such that the vector bundle \( \mathfrak{g}_F \otimes K_X \) does not admit non-zero sections taking nilpotent values everywhere on \( X \)) such that the restriction of \( \Delta\chi \) to \( \text{Bun}_{\text{vs}}^G \) is a vector bundle with a projectively flat connection. But its rank grows exponentially with the genus of the curve \( X \) and it has highly non-trivial monodromy around the divisor \( \text{Bun}^G \setminus \text{Bun}_{\text{vs}}^G \). Therefore, it does not make sense to look for individual holomorphic solutions of the system (1.8).

However, as explained in [T] and in [EFK1], Sect. 1.5, it makes sense to couple the system (1.8) to its anti-holomorphic analogue and look for single-valued solutions of the resulting system of differential equations on the locus \( \text{Bun}_{\text{vs}}^G \) of very stable bundles in \( \text{Bun}^G \). These are the automorphic functions of the analytic theory. It is natural to try to interpret them as eigenfunctions of the algebra

\[
\mathcal{A}_G := D_G \otimes \overline{\mathcal{D}}_G.
\]

This is a non-trivial task because elements of \( \mathcal{A}_G \) correspond to unbounded operators on the Hilbert space \( \mathcal{H}_G \). Initially, they are defined on the dense subspace \( V_G \subset \mathcal{H}_G \). In [EFK1] (Definition 1.7) we introduced a Schwartz space \( S(\mathcal{A}_G) \subset \mathcal{H}_G \) and conjectured that the elements of \( \mathcal{A}_G \) can be extended to \( S(\mathcal{A}_G) \) so that their closures form a family of commuting normal operators. Moreover, we conjectured that their joint spectrum is the set \( \text{Op}_{\gamma} L^G(X)^{\mathbb{R}} \) of \( ^LG \)-opers on \( X \) with real monodromy (see below). In the case of \( G = SL_2 \), a similar conjecture was proposed earlier in [T].

In [EFK1] we were able to prove these conjectures in the simplest non-trivial case. However, in the general case it is a daunting task to prove them directly. This is where the integral Hecke operators come in handy. Like differential operators, they are also initially defined on a dense subspace of the Hilbert space \( \mathcal{H}_G \). But unlike the differential operators, we do expect Hecke operators to extend to mutually commuting continuous operators on the entire \( \mathcal{H}_G \), which are moreover normal, compact, and have trivial common kernel. This is the statement of our Compactness Conjecture 1.2. It implies that \( \mathcal{H}_G \) decomposes into a (completed) direct sum of mutually orthogonal finite-dimensional eigenspaces of the Hecke operators.

Next, we would like to say that the algebra \( \mathcal{A}_G \) preserves the subspaces \( \mathcal{H}_G(s) \). To do this, observe that \( \mathcal{A}_G \) acts on the space \( V^\gamma_G \) of distributions on \( \text{Bun}^G \). It follows from the definition of \( \mathcal{H}_G \) that \( \mathcal{H}_G \) is naturally realized as a subspace of \( V^\gamma_G \). Hence we can apply elements of \( \mathcal{A}_G \) to vectors in the eigenspaces \( \mathbb{H}_G(s) \) of the Hecke operators, viewed as distributions.

**Conjecture 1.7.** Every eigenspace \( \mathbb{H}_G(s) \) of the Hecke operators, viewed as a subspace of \( V^\gamma_G \), is an eigenspace of \( \mathcal{A}_G \).
Remark 1.7. (i) There is a weaker version of Conjecture 1.7 in which $V_G$ is replaced by the space of smooth functions with compact support on some open dense set $U \subset \text{Bun}_G$. For practical purposes such a weak version is almost as good but might be easier to prove. For example, we may take $U$ to be the above locus $\text{Bun}_{G_{\text{vs}}}$ of very stable bundles in $\text{Bun}_G$, on which the eigenfunctions of the Hecke operators are smooth (in fact, this is true for any local field $F$; we will explain this in a follow-up paper). Then the conjecture is that these eigenfunctions satisfy the differential equations of the quantum Hitchin system combined with its complex conjugate system (which are smooth on this locus) in the classical sense.

(ii) The algebra $A_G$ acts on both $V_G(\lambda)$ and $V_G$. Using an identification $[BD1]$ of $D_G$ with a quotient of the center at the critical level $[FF, Fr1]$, one can show that the action of $A_G$ commutes with the Hecke operators $\tilde{H}_\lambda(x): V_G(\lambda) \rightarrow V_G$. This property, however, is not sufficient for proving Conjecture 1.7. But in the case $G = \text{PGL}_n$ we can derive it from the system of differential equations in Theorem 1.18. We expect that a similar argument also proves Conjecture 1.7 for a general $G$. □

In the rest of this subsection, we will assume the validity of Conjecture 1.7. It implies the following statement. Let

$$S(A_G) := \bigoplus_{s \in \text{Spec}(H_G)} H_G(s) \subset H_G.$$ 

According to Conjecture 1.7, the algebra $A_G$ acts on it.

Corollary 1.8. The closures of the actions of the elements of $A_G$ on $S(A_G)$ form a family of commuting normal operators, and each $H_G(s)$ is an eigenspace of these normal operators.

Remark 1.8. It is clear that the subspace $S(A_G) \subset H_G$ is contained in the Schwartz space $S(A_G)$ defined in $[EFK1]$. We expect that $S(A_G)$ is a completion of $S(A_G)$ corresponding to a specific growth condition on the coefficients. □

Corollary 1.8 implies that we can define the joint spectrum of the commutative algebra $A_G$. It follows from the definition that $\text{Spec}(A_G)$ is naturally realized as a subset of $\text{Op}_{lG}(X) \times \overline{\text{Op}_{lG}(X)}$.

Definition 1.9. An $^L G$-oper $\chi$ is called an oper with real monodromy (or real oper for short) if the monodromy representation $\pi_1(X, p_0) \rightarrow ^L G(\mathbb{C})$ corresponding to $\chi$ is isomorphic to its complex conjugate.

Denote by $\text{Op}_{lG}(X)_{\mathbb{R}}$ the subset of real $^L G$-opers in $\text{Op}_{lG}(X)$.

Remark 1.9. It follows from the above definition that the image of the monodromy representation corresponding to a real $^L G$-oper is contained in a real form $^L G_{\mathbb{R}}$ of $^L G$. Moreover, this form is inner to the split real form since for every algebraic representation $V$ of $^L G$, the corresponding monodromy $\rho_V$ is isomorphic to $\overline{\rho}_V$. We conjecture that in fact the form $^L G_{\mathbb{R}}$ is the split real form $^L G(\mathbb{R})$; in other words, that real $^L G$-opers are precisely the $^L G$-opers $\chi$ for which the image of the monodromy
representation of $\chi$ in $L^G(\mathbb{C})$ is contained, up to conjugation, in $L^G(\mathbb{R})$. This is known to be the case for $L^G = PGL_2$ and $SL_2$, see [Fa, Gol, GKM].

We can prove that $L^G_\mathbb{R}$ is split in the case when $S \neq \emptyset$. Indeed, the residue of the oper at each parabolic point is regular nilpotent, so the monodromy is regular unipotent. Thus our real form $L^G_\mathbb{R}$ contains a real regular unipotent element, hence a real Borel subgroup (the unique Borel subgroup containing it), hence it is quasi-split. But an inner quasi-split real form must be split. □

For $\chi \in Op^\gamma_{L^G}(X)$, we denote, following Sect. 3.4 of [EFK1], by $\chi^*$ the $L^G$-oper obtained by applying to $\chi$ the Chevalley involution on $L^G$ (it belongs to the same component $Op^\gamma_{L^G}(X)$). Theorem 4.2 of [EFK1] implies the following.

**Theorem 1.10.** As a subset of $Op^\gamma_{L^G}(X) \times \overline{Op^\gamma_{L^G}(X)}$, the joint spectrum $\text{Spec}(A_G)$ consists of pairs $(\chi, \chi^*)$ where $\chi \in Op^\gamma_{L^G}(X)_\mathbb{R}$. Thus, $\text{Spec}(A_G)$ is naturally realized as a subset of $Op^\gamma_{L^G}(X)_\mathbb{R}$.

**Conjecture 1.11.**

1. The set $Op^\gamma_{L^G}(X)_\mathbb{R}$ is discrete.
2. $\text{Spec}(A_G) = Op^\gamma_{L^G}(X)_\mathbb{R}$.

**Remark 1.10.**

(i) In [EFK1], we proved Conjecture 1.11 in the case $G = SL_2, X = \mathbb{P}^1, |S| = 4$, directly, without relying on the Compactness Conjecture.

(ii) Part (1) of Conjecture 1.11 is known in the case when $G = PGL_2$ (see [Fa], Sect. 7).

In [BDI], Sect. 5.1.1, Beilinson and Drinfeld attached to every $\chi \in Op^\gamma_{L^G}(X)$ a $D$-module on $\text{Bun}_G$, which is a Hecke eigensheaf with respect to the flat $L^G$-bundle corresponding to $\chi$; namely,

$$(1.9) \quad \Delta^0_\chi := K_{\text{Bun}}^{-1/2} \otimes \Delta_\chi$$

where $\Delta_\chi$ is given by (1.8) (see also [EFK1], Sect. 3.2, and Remark 4.2 below). The first part of the following statement has been proved in [AGKRRV] (Corollary 11.6.7). The second part has been proved in [FR] (Theorem 11.2.1.2 and Remark 11.2.1.3). (In the case $G = PGL_n$ this also follows from the results of [Ga].)

**Theorem 1.12.**

1. $\Delta^0_\chi$ has regular singularities.
2. $\Delta^0_\chi$ is irreducible on each connected component of $\text{Bun}_G$.

Recall that $\text{Bun}_G$ has connected components $\text{Bun}_G^\beta$ labeled by $\beta \in \pi_1(G)$. Thus, we have a natural direct sum decomposition

$$(1.10) \quad \mathcal{H}_G = \bigoplus_{\beta \in \pi_1(G)} \mathcal{H}_G^\beta.$$

We also have

$$S(A_G) = \bigoplus_{\beta \in \pi_1(G)} S(A_G)_\beta, \quad S(A_G)_\beta := S(A_G) \cap \mathcal{H}_G^\beta.$$
The action of $A_G$ on $S(A_G)$ preserves the direct summands $S(A_G)_\beta$. Hence we can talk about the joint eigenvalues of $A_G$ on each of them and the corresponding multiplicities.

The following statement is proved in Sect. 1.5 of [EFK1] in the case when $G$ is simply-connected, and the proof generalizes in a straightforward way to the case of an arbitrary connected simple Lie group $G$.

**Proposition 1.13.** Suppose that $\chi \in \text{Op}_G^\gamma(X)$ corresponds to an element of $\text{Spec}(A_G)$ and $\Delta_\chi^0$ has regular singularities and is irreducible on each connected component $\text{Bun}_G^\beta$. Then the multiplicity of $\chi$ in $S(A_G)_\beta$ is equal to one for all $\beta \in \pi_1(G)$.

1.4. **The case $G = \text{PGL}_n$.** According to Theorem 1.12, the spectrum of $A_G$ is simple on each $S(A_G)_\beta \subset \mathcal{H}^\beta_{G}$, $\beta \in \pi_1(\text{PGL}_n) = \mathbb{Z}/n\mathbb{Z}$.

For $\beta \in \mathbb{Z}/n\mathbb{Z}$, let $\psi_{\chi,\beta}$ be a non-zero generator of the one-dimensional eigenspace of $A_G$ in $\mathcal{H}^\beta_G$ corresponding to the eigenvalue $\chi$. Let

$$\mathcal{E}_\chi := \text{span}\{\psi_{\chi,\beta}, \beta \in \mathbb{Z}/n\mathbb{Z}\}.$$

Now consider the Hecke operators $H_{\omega_1}(x), x \in X$, and the corresponding operator-valued section $H_{\omega_1}$ of the line bundle $\Omega_X^{-(n-1)/2}$. These operators act from $\mathcal{H}^\beta_G$ to $\mathcal{H}^{\beta+1}_G$. Corollary 1.3 then implies that $H_{\omega_1}(x)$ and $H_{\omega_1}$ preserve the $n$-dimensional subspace $\mathcal{E}_\chi \subset \mathcal{H}_G$. Moreover, we can normalize the vectors $\psi_{\chi,\beta}$ in such a way that $\Psi_1^\chi := \sum_{\beta \in \mathbb{Z}/n\mathbb{Z}} \psi_{\chi,\beta}$ is an eigenvector of $H_{\omega_1}$. Then the vectors

$$\Psi_\epsilon^\chi := \sum_{\beta \in \mathbb{Z}/n\mathbb{Z}} \epsilon^\beta \psi_{\chi,\beta}, \quad \epsilon \in \mu_n$$

(where $\mu_n$ is the group of $n$th roots of unity) are also eigenvectors of $H_{\omega_1}$ and hence form an eigenbasis of $\mathcal{E}_\chi$. If $\Phi_\chi$ is a section of $\Omega_X^{-(n-1)/2}$ representing the eigenvalue of $H_{\omega_1}$ on $\Psi_1^\chi$, then the section of $\Omega_X^{-(n-1)/2}$ representing the eigenvalue of $H_{\omega_1}$ on $\Psi_\epsilon^\chi$ is $\epsilon \Phi_\chi$.

Thus, to each $\chi \in \text{Spec } A_G$ corresponds a collection $\{\Psi_\epsilon^\chi\}$ of eigenvalues of $H_{\omega_1}$, which is naturally a $\mu_n$-torsor. We will now write a conjectural formula for these eigenvalues.

Recall that $\chi$ is an $\text{SL}_n$-oper in $\text{Op}_{\text{SL}_n}^\gamma(X)_\mathbb{R}$. Let $(\mathcal{V}_{\omega_1}, \nabla_\chi)$ be the corresponding holomorphic flat vector bundle on $X$ with $\text{det}(\mathcal{V}_{\omega_1}, \nabla_\chi)$ identified with the trivial flat line bundle. The definition of an oper provides us with an embedding of a line bundle

$$\kappa_{\omega_1} : K_X^{(n-1)/2} \hookrightarrow \mathcal{V}_{\omega_1},$$

and hence an embedding

$$\tilde{\kappa}_{\omega_1} : \mathcal{O}_X \hookrightarrow \mathcal{V}_{\omega_1} \otimes K_X^{-(n-1)/2}.$$ 

Therefore we obtain a section

$$s_{\omega_1} := \tilde{\kappa}_{\omega_1}(1) \in \Gamma(X, \mathcal{V}_{\omega_1} \otimes K_X^{-(n-1)/2}).$$
In the same way, we obtain a section
\[ s_{\omega_{n-1}} \in \Gamma(X, \mathcal{V}_{\omega_{n-1}} \otimes K_X^{-(n-1)/2}) = \Gamma(X, \mathcal{V}_{\omega_1}^* \otimes K_X^{-(n-1)/2}). \]

By our assumption that \( \chi \in \text{Op}_{\text{SL}_n}(X)_{\mathbb{R}} \), the monodromy representations associated to \( \chi \) and \( \overline{\chi} \) are isomorphic. This means that \( (\mathcal{V}_{\omega_1}, \nabla_\chi) \) and \( (\mathcal{V}_{\omega_1}, \overline{\nabla}_\chi) \) are isomorphic as \( C^\infty \) flat vector bundles on \( X \). Hence we have a non-degenerate pairing
\[ h_\chi(\cdot, \cdot) : (\mathcal{V}_{\omega_1}, \nabla_\chi) \otimes (\overline{\mathcal{V}}_{\omega_{n-1}}, \overline{\nabla}_\chi) \rightarrow (\mathcal{O}_X, d) \]
of \( C^\infty \) flat vector bundles on \( X \). Since \( (\mathcal{V}_{\omega_1}, \nabla_\chi) \) and \( (\overline{\mathcal{V}}_{\omega_{n-1}}, \overline{\nabla}_\chi) \) are associated to flat \( SL_n \)-bundles, their determinants are identified with the trivial flat line bundle. We will require that \( h_\chi(\cdot, \cdot) \) induce the canonical pairing on the corresponding determinant line bundles. The set of \( h_\chi(\cdot, \cdot) \) normalized this way is a \( \mu_n \)-torsor, which we denote by \( \{e h_\chi(\cdot, \cdot)\} \).

**Conjecture 1.14.**

\[ \{\Phi_\chi\} = \{e h_\chi(s_{\omega_1}, s_{\omega_{n-1}})\} \]
as \( \mu_n \)-torsors of global sections of the line bundle \( \Omega_X^{-(n-1)/2} \) on \( X \).

We will prove a slightly weaker form of this conjecture (Corollary 1.19), with roots of unity \( \epsilon \) replaced by non-zero complex numbers, by showing that both sides satisfy the same system of differential equations which has a unique solution up to a scalar.

To explain this, we need an alternative description of the component \( \text{Op}_{\text{SL}_n}(X) \).

Consider \( n \)-th order differential operators \( P : K_X^{-(n-1)/2} \rightarrow K_X^{(n+1)/2} \) (here, if \( n \) is even, we use our chosen square root \( K_{1/2}^{\infty} \)) such that

1. \( \text{symb}(P) \in H^0(X, \mathcal{O}_X) \) is equal to 1;
2. The operator \( P - (-1)^n P^* \) has order \( n - 2 \) (here \( P^* : K_X^{-(n-1)/2} \rightarrow K_X^{(n+1)/2} \) is the algebraic adjoint operator, see [BB], Sect. 2.4).

These operators form an affine space, which we denote by \( D_n^\gamma(X) \). Locally,
\[ P = \partial_z^n + v_{n-2}(z)\partial_z^{n-2} + \ldots + v_0(z). \]

The following statement is proved in [BD2], Sect. 2.8.

**Lemma 1.15.** There is a bijection \( \text{Op}_{\text{SL}_n}(X) \simeq D_n^\gamma(X) \)

\[ \chi \in \text{Op}_{\text{SL}_n}(X) \quad \mapsto \quad P_\chi \in D_n^\gamma(X) \]
such that the sections \( s_{\omega_1} \in \Gamma(X, \mathcal{V}_{\omega_1} \otimes K_X^{-(n-1)/2}) \) and \( s_{\omega_{n-1}} \in \Gamma(X, \mathcal{V}_{\omega_{n-1}} \otimes K_X^{-(n-1)/2}) \) satisfy
\[ P_\chi \cdot s_{\omega_1} = 0, \quad P_\chi^* \cdot s_{\omega_{n-1}} = 0, \]
where \( P_\chi^* \) is the algebraic adjoint of \( P_\chi \) (here we use the \( \mathcal{D}_X \)-module structures on \( \mathcal{V}_{\omega_1} \) and \( \mathcal{V}_{\omega_{n-1}} \) corresponding to the oper connection of \( \chi \)).

The flat vector bundle \( (\mathcal{V}_{\omega_1}, \nabla_\chi) \) is known to be irreducible if \( g > 1 \), see [BD1], Sect. 3.1.5(iii). Therefore we obtain
Corollary 1.16. $h_X(s_{\omega_1}, \bar{s}_{\omega_n-1})$ is a unique, up to a scalar, section $\Phi$ of $\Omega_X^{-(n-1)/2}$ which is a solution of the system of differential equations

\begin{equation}
(1.11) \quad P_X \cdot \Phi = 0, \quad \bar{P}_X \cdot \Phi = 0
\end{equation}

On the other hand, let $\mathcal{V}_{\omega_1}^{\text{univ}}$ be the universal vector bundle over $\text{Op}_n^\gamma \times X$ with a partial connection $\nabla^{\text{univ}}$ along $X$, such that

\begin{equation}
(\mathcal{V}_{\omega_1}^{\text{univ}}, \nabla^{\text{univ}})|_{X \times X} = (\mathcal{V}_{\omega_1}, \nabla_X).
\end{equation}

Let $\mathcal{V}_{\omega_1,X} := \pi_*(\mathcal{V}_{\omega_1}^{\text{univ}})$, where $\pi : \text{Op}_n^\gamma \times X \to X$ is the projection and $\pi_*$ is the $\mathcal{O}$-module push-forward functor. The connection $\nabla^{\text{univ}}$ makes $\mathcal{V}_{\omega_1,X}$ into a left $\mathcal{D}_X$-module.

Recall (see Definition 1.5) that we have denoted by $D_{PGL_n}$ the algebra of global holomorphic differential operators acting on each component $\text{Bun}_{PGL_n}^\beta$ of $\text{Bun}_{PGL_n}$. By Theorem 1.16 we have an isomorphism

$$D_{PGL_n} \simeq \text{Fun Op}_{\text{PGL}_n}(\mathcal{X}).$$

Hence $D_{PGL_n}$ naturally acts on $\mathcal{V}_{\omega_1}^{\text{univ}}$ and this action commutes with the action of $\mathcal{D}_X$. Using the above realization of $\text{PGL}_n$-opers as $n$-th order differential operators, we construct the “universal $\text{PGL}_n$-oper” as follows:

Lemma 1.17. There is a unique $n$-th order differential operator

\begin{equation}
(1.12) \quad \sigma : K_X^{-(n-1)/2} \to D_{PGL_n} \otimes K_X^{(n+1)/2}
\end{equation}

satisfying the following property: for any $\chi \in \text{Op}_n^\gamma \times X = \text{Spec } D_{PGL_n}$, applying the corresponding homomorphism $D_{PGL_n} \to \mathbb{C}$ we obtain $P_X$.

The following is one of the main results of this paper, which will be proved in Section 4.

Theorem 1.18. The Hecke operator $\widehat{H}_{\omega_1}$, viewed as an operator-valued section of $\Omega_X^{-(n-1)/2} = K_X^{-(n-1)/2} \otimes K_X^{-(n+1)/2}$, satisfies the system of differential equations

\begin{equation}
(1.13) \quad \sigma \cdot \widehat{H}_{\omega_1} = 0, \quad \bar{\sigma} \cdot \widehat{H}_{\omega_1} = 0.
\end{equation}

Corollary 1.19. Each of the eigenvalues $\Phi_X^\chi$ of the Hecke operator $H_{\omega_1}$ is equal to a scalar multiple of $h_X(s_{\omega_1}, \bar{s}_{\omega_n-1})$.

Proof of Corollary 1.19 from Theorem 1.18. Equations (1.13) imply that the eigenvalues of $H_{\omega_1}$ satisfy equations (1.11). That’s because if $v$ is an eigenvector of $A_{PGL_n}$ with the eigenvalue of $D_{PGL_n}$ corresponding to a holomorphic $SL_n$-oper $\chi$, then according to Theorem 1.10 the eigenvalue of $D_{PGL_n}$ on $v$ corresponds to the anti-holomorphic $SL_n$-oper $\overline{\chi}$. Furthermore, it is clear that the $n$th order operator $K_X^{-(n-1)/2} \to K_X^{(n+1)/2}$ associated to $\overline{\chi}$ is $\bar{P}_X$. Corollary 1.16 then implies Corollary 1.19.

Corollary 1.19 describes the eigenvalues of the Hecke operator $H_{\omega_1}$ for $G = PGL_n$ up to scalar multiples (it is slightly weaker than Conjecture 1.14 which describes these eigenvalues up to multiplication by $n$th roots of unity).
There is a similar conjectural formula for the eigenvalues of the Hecke operators $H_{\omega_i}, i = 2, \ldots, n - 1$, corresponding to the other fundamental coweights of $PGL_n$ can be found in a similar way. Namely, let $(V_{\omega_i}, \nabla_{\chi, \omega_i})$ be the $i$th exterior power of $(V_{\omega_1}, \nabla_{\chi})$. Note that it is dual to $(V_{\omega_{n-1}}, \nabla_{\chi, \omega_{n-1}})$. The oper Borel reduction gives rise to a section

$$s_{\omega_i} \in \Gamma(X, V_{\omega_i} \otimes K_X^{-(n-i)/2}).$$

If $\chi$ is a real $PGL_n$-oper, then we have a non-degenerate pairing

$$h_{\chi, \omega_i} : (V_{\omega_i}, \nabla_{\chi, \omega_i}) \otimes (V_{\omega_{n-1}}, \nabla_{\chi, \omega_{n-1}}) \to (C^\infty X, d).$$

The eigenvalue of the Hecke operator $H_{\omega_i}$ corresponding to $\chi$ is conjecturally equal to a scalar multiple of

$$h_{\chi, \omega_i}(s_{\omega_i}, s_{\omega_{n-1}}).$$

The eigenvalues of all other Hecke operators $H_{\lambda}$ for $G = PGL_n$ can be found from the eigenvalues of $H_{\omega_i}, i = 1, \ldots, n - 1$, using Proposition 1.4.

Moreover, Theorem 1.18 also implies Conjecture 1.7. Indeed, it can be shown that differential equations (1.13) imply a suitable version of commutativity of the algebra $A_G$ with the Hecke operators. This will be discussed in more detail in our next paper [EFK2].

In Section 4 we will derive Theorem 1.18 from the theorem of Beilinson and Drinfeld [BD1] describing the action of the Hecke functor $H_{\omega_i}$ on the left $D$-module $\mathcal{D}_{Bun_{PGL_n}} \otimes K_{Bun}^{-1/2}$ on $Bun_{PGL_n}$. We will also discuss a generalization of these results to an arbitrary simple Lie group $G$.

This shows a deep connection between the geometric/categorical Langlands correspondence and the analytic/function-theoretic one.

1.5. The structure of the paper. The paper is organized as follows. In Section 2 we consider the abelian case, $G = GL_1$, in which one can already see the main ingredients of our construction in the non-abelian case. In Section 3 we formulate some basic results on the compatibility between the natural operations on functions (pullback, push-forward, and integral transforms) and the corresponding functors between categories of twisted $D$-modules. We then use these results in the following sections to relate the Hecke operators and Hecke functors and derive differential equations on the Hecke operators. In Section 4 we prove that the Hecke operators satisfy the system (1.13) of differential equations corresponding to the “universal $SL_n$-oper” (Theorem 1.18), using a Hecke eigensheaf property established in [BD1]. In Section 5 we formulate the analogues of Theorem 1.18 and Corollary 1.19 describing the eigenvalues of the Hecke operators in the case of an arbitrary simple Lie group $G$. We then outline the proof of these results generalizing the argument we used in Section 4 in the case of $PGL_n$.

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2. The abelian case

In this section we consider the case $G = GL_1$. Though the spectral problem here is 
rather simple (standard Fourier analysis on a torus), it provides a useful illustration of 
our general method. Indeed, one can already observe in it all the essential ingredients 
of the picture for a general group $G$. Hence it is instructive to consider it as a blueprint 
for the general construction.

We first recall the results of [Fr2, EFK1] on the eigenvalues of the corresponding 
Hecke operators and the global differential operators. We will then prove that the 
Hecke operators satisfy a system of differential equations, which are analogous to 
equations (1.13). This system will enable us to prove a relation between the eigen-
values of the Hecke operators and the global differential operators and lead to an 
alternative, and simpler, proof of the results of [Fr2, EFK1] describing these eigen-
values. The differential equations follow from a theorem describing the image of the 
sheaf $D$ of differential operators on the Jacobian of the curve $X$ under the action of 
the corresponding Hecke functor. Thus, these differential equations link the geometric/categorical Langlands correspondence and the analytic Langlands correspondence. 
A generalization of this equation, and this link, to $G = PGL_n$ will be presented in 
Section 4.

2.1. The global picture. Let $G = GL_1$. In this case, the role of $Bun_G$ is played 
by the Picard scheme $Pic(X)$, which is a fine moduli space of line bundles on $X$. 
The canonical line bundle on $Pic(X)$ is isomorphic to the trivial line bundle, and we 
will fix such an isomorphism. Hence we have a positive Hermitian inner product on 
the space of smooth functions on $Pic(X)$. Let $L^2(Pic(X))$ be its completion. This is 
our Hilbert space $H_{GL_1}$. The Hecke operators on $H_{GL_1}$ are easy to define as they do 
not involve integration. Their spectrum was described in [Fr2], Sect. 2 (not only for 
$GL_1$ but also for an arbitrary complex torus), whereas the spectrum of the algebra 
of global differential operators was described in [EFK1], Sect. 5. We recall these 
descriptions below.

Denote by $H_p$ the Hecke operator associated to a point $p \in X$ and the defining 
one-dimensional representation of $GL_1$. It acts on $L^2(Pic(X))$ as the pull-back with 
respect to the map sending a line bundle $\mathcal{L} \in Pic(X)$,

\begin{equation}
\mathcal{L} \mapsto \mathcal{L}(p).
\end{equation}

These operators obviously commute with each other for different $p \in X$.

Recall that $Pic(X)$ is a disjoint union of connected components $Pic^n(X), n \in \mathbb{Z}$, 
labeled by the degrees of line bundles. The Hecke operators shift the degree by 1. Let 
us fix a point $p_0 \in X$ once and for all. The map (2.1) with $p = p_0$ identifies $Pic^n(X)$ 
and $Pic^{n+1}(X)$ for all $n \in \mathbb{Z}$. This implies (see [Fr2], Sect. 2.1, for more details) that
the spectral theory of the operators $H_p, p \in X$, on $L^2(\text{Pic}(X))$ is equivalent to the spectral theory of the operators

$$p_0 H_p := H_p \circ H_{p_0}^{-1}, \quad p \in X,$$

acting on

$$\mathcal{H}^0_{GL_1} := L^2(\text{Pic}^0(X)).$$

Note that $p_0 H_{p_0}$ is the identity operator.

Next, the Hecke operators $p_0 H_p$ commute with the algebra $\mathcal{A}_{GL_1}$ of global differential operators on the Jacobian $\text{Pic}^0(X)$, which is generated by the translation vector fields (both holomorphic and anti-holomorphic). Hence they share the same eigenfunctions; namely, the Fourier harmonics on the Jacobian viewed as a real torus. We will now describe a relation between their eigenvalues following [Fr2], Sect. 2.4 and [EFK1], Sect. 5.

To describe the eigenfunctions and eigenvalues of the operators $p_0 H_p$ (following [Fr2], Sect. 2.4), recall that as a real torus,

$$\text{Pic}^0(X) \simeq H^1(X, \mathbb{R})^*/H_1(X, \mathbb{Z}),$$

where the embedding $H_1(X, \mathbb{Z}) \hookrightarrow H^1(X, \mathbb{R})^*$ is defined by sending $\beta \in H_1(X, \mathbb{Z})$ to the linear functional on $H^1(X, \mathbb{R})$

$$\int \beta \cdot \cdot \cdot \omega_c \in H^0(X, \Omega^{1,0}).$$

Given $\gamma \in H^1(X, \mathbb{Z})$, denote by $\varphi_\gamma$ the harmonic representative of the image of $\gamma$ in $H^1(X, \mathbb{R})$. Then

$$\varphi_\gamma = \omega_\gamma + \overline{\omega}_\gamma, \quad \omega_\gamma \in H^0(X, \Omega^{1,0}).$$

Note that these are precisely the smooth one-forms on $X$ whose integrals over all one-cycles in $X$ are integers.

Now view $\varphi_\gamma$ as a linear functional on $H^1(X, \mathbb{R})^*$ and set

$$f_\gamma = e^{2\pi i \varphi_\gamma}, \quad \gamma \in H^1(X, \mathbb{Z}).$$

**Lemma 2.1.** For any $\gamma \in H^1(X, \mathbb{Z})$, $f_\gamma$ is a well-defined function on $\text{Pic}^0(X)$ given by (2.2).

The functions $f_\gamma, \gamma \in H^1(X, \mathbb{Z})$, are the *Fourier harmonics* of the torus $\text{Pic}^0(X)$. They form an orthogonal basis of the Hilbert space $\mathcal{H}^0_{GL_1}$.

The following statement is proved in [Fr2], Theorem 2.4.

**Proposition 2.2.** The function $f_\gamma, \gamma \in H^1(X, \mathbb{Z})$, is an eigenfunction of the Hecke operators $p_0 H_p$. The eigenvalue $F_\gamma(p)$ of $p_0 H_p$ on $f_\gamma$ is given by the formula

$$F_\gamma(p) = \exp \left( \int_{p_0}^p 2\pi i (\omega_\gamma + \overline{\omega}_\gamma) \right)$$

where the integral is taken over any path connecting $p_0$ to $p$ (the integral does not depend on the choice of this path).
Now consider the algebra of global differential operators on \( \text{Pic}^0(X) \),
\[
\mathcal{A}_{GL_1} = D_{GL_1} \otimes \overline{D}_{GL_1},
\]
where \( D_{GL_1} \) (resp., \( \overline{D}_{GL_1} \)) is the algebra of global holomorphic (resp., anti-holomorphic) differential operators on \( \text{Pic}^0(X) \). Since \( \text{Pic}^0(X) \) is compact and has an abelian group structure, we have
\[
D_{GL_1} = \text{Sym}(\Theta \text{Pic}^0(X)), \quad \overline{D}_{GL_1} = \text{Sym}(\overline{\Theta} \text{Pic}^0(X)),
\]
where \( \Theta \text{Pic}^0(X) \) (resp., \( \overline{\Theta} \text{Pic}^0(X) \)) is the commutative Lie algebra of holomorphic (resp., anti-holomorphic) translation vector fields on \( \text{Pic}^0(X) \). We have
\[
\Theta \text{Pic}^0(X) \cong H^0(X, \Omega^1, 0)^*, \quad \overline{\Theta} \text{Pic}^0(X) \cong H^0(X, \Omega^0, 1)^*.
\]
Therefore the eigenvalues of the algebra \( \mathcal{A}_{GL_1} \) on any joint eigenvector are encoded by a pair \((a, b)\), where \( a \in H^0(X, \Omega^1, 0) \) and \( b \in H^0(X, \Omega^0, 1) \).

We associate to \( a \) a holomorphic \( GL_1 \)-oper, i.e. the trivial line bundle on \( X \) with the holomorphic connection \( d + a \). And we associate to \( b \) an anti-holomorphic \( GL_1 \)-oper, i.e. the trivial line bundle on \( X \) with the anti-holomorphic connection \( d + b \).

**Proposition 2.3** ([EFK1], Theorem 5.4).

1. The eigenvalues of \( \mathcal{A}_{GL_1} \) on \( f_\gamma \) are given by the pairs \((a, b) = (2\pi i \omega_\gamma, 2\pi i \overline{\omega}_\gamma)\).
2. The \( GL_1 \)-opers \( d + 2\pi i \omega_\gamma, \gamma \in H^1(X, \mathbb{Z}) \), are all holomorphic \( GL_1 \)-opers on \( X \) with real monodromy (i.e. the monodromy representation takes values in \( GL_1(\mathbb{R}) \subset GL_1(\mathbb{C}) \)).
3. The spectrum of \( \mathcal{A}_{GL_1} \) on \( H^0_{GL_1} \) is in bijection with the set of holomorphic \( GL_1 \)-opers on \( X \) with real monodromy.

Combining Propositions 2.2 and 2.3, we obtain the following relation between the eigenvalues of the Hecke operators and \( \mathcal{A}_{GL_1} \).

**Theorem 2.4.** Let \( F(p) \) and \( (a, b) \) be the eigenvalues of the Hecke operators \( p_0 H_p, p \in X \), and \( \mathcal{A}_{GL_1} \) on a joint eigenfunction in \( L^2(\text{Pic}^0(X)) \), respectively. Then
\[
F(p) = \exp \left( \int_{p_0}^p (a + b) \right)
\]
and \( b = -\overline{a} \).

This is equivalent to the following statement. Denote by \( \partial \) and \( \overline{\partial} \) the holomorphic and anti-holomorphic de Rham differentials, respectively.

**Proposition 2.5.** The function \( F(p) \) is the single-valued solution of the differential equation
\[
dF = (a + b)F
\]
or equivalently, the system
\[
\partial F = aF, \quad \overline{\partial} F = bF
\]
normalized so that \( F(p_0) = 1 \).
Remark 2.1. Theorem 2.4 does not specify the possible values of the one-forms $a$ and $b$ that appear in this relation (which we already know from Propositions 2.2 and 2.3). But these values can be readily obtained from the relation (2.7).

Indeed, single-valuedness of the function $F(p)$ given by (2.7) implies that the integrals of the one-form $a + b$ over all one-cycles in $X$ are integer multiples of $2\pi i$. This is equivalent to

$$a + b = 2\pi i \varphi_\gamma,$$

where $\varphi_\gamma = \omega_\gamma + \bar{\omega}_\gamma$ is the harmonic one-form introduced above, for some $\gamma \in H^1(X, \mathbb{Z})$. On the other hand, the self-adjointness on $L^2(\text{Pic}_0^0(X))$ of the operators of the form $\xi - \bar{\xi}$ and $(\xi + \bar{\xi})/i$, where $\xi$ is any holomorphic translation vector field on $\text{Pic}^0(X)$, implies that

$$b = -\bar{a}$$

Combining formulas (2.10) and (2.11), we obtain that

$$a = 2\pi i \omega_\gamma,$$

$$b = 2\pi i \bar{\omega}_\gamma$$

for some $\gamma \in H^1(X, \mathbb{Z})$.

Therefore, we can derive Propositions 2.2 and 2.3 from Theorem 2.4. Thus, Theorem 2.4 (or equivalently, Proposition 2.5) is the key statement that yields explicit formulas for the eigenvalues of both Hecke operators and the global differential operators. \hfill $\square$

In the rest of this section, we will give an alternative proof of Proposition 2.5 using the action of the Hecke functors $p_0 \hat{H}_p$ (categorical versions of the Hecke operators $p_0 H_p$) on the sheaf $D_{GL_1}$ of differential operators on $\text{Pic}^0(X)$. This will be our blueprint for proving analogous statements for a general group $G$.

2.2. Hecke operators and Hecke functors. Consider the $GL_1$ version of the Hecke correspondence

$$\hat{\text{Hecke}} = \{(L, L(p_0 - p), p) \in \text{Pic}^0(X) \times \text{Pic}^0(X) \times X \}$$

and let $q_1, q_2 : \hat{\text{Hecke}} \to \text{Pic}^0(X)$ be the two projections and $q_3 : \hat{\text{Hecke}} \to X$. We have

$$q_2 \times q_3 : \hat{\text{Hecke}} \xrightarrow{\sim} \text{Pic}^0(X) \times X$$

and with this isomorphism, $q_1$ becomes the map

$$q_1 : \text{Pic}^0(X) \times X \to \text{Pic}^0(X)$$

$$L \mapsto L(p - p_0)$$

Denote by $p_0 \hat{H}_p$ the restriction of $p_0 H_p$ to the dense subspace of $C^\infty$ functions on $\text{Pic}^0(X)$. This is the operator of pulling back a $C^\infty$ function under the map $q_1$ and restricting the result to $\text{Pic}^0(X) \times p$. As $p$ varies along $X$, these operators combine into a single operator

$$p_0 \hat{H} : C^\infty(\text{Pic}^0(X)) \to C^\infty(\text{Pic}^0(X) \times X).$$
We are going to relate it to the \textit{Hecke functor}
\[ p_0 H := q_1^* : \text{Mod}(\mathcal{D}_{\text{Pic}^0(X)}) \rightarrow \text{Mod}(\mathcal{D}_{\text{Pic}^0(X) \times X}), \]
the $D$-module pull-back functor.

2.3. \textbf{Another derivation of the differential equations.} We will now derive the differential equations (2.9) appearing in Proposition 2.5 using Corollary 3.3 from Section 3.1, in which we will take $Z = \text{Pic}^0(X) \times X, Y = \text{Pic}^0(X)$ and let $q_1$ be the map $Z \rightarrow Y$ given by (2.14). Denote the corresponding section $1_{q_1}$ by $1_{q_1}$. The equations (2.9) will follow from Corollary 3.3 and the fact that $1_{q_1}$ satisfies the differential equation of Proposition 2.9 below.

To prove the Proposition 2.9, denote by $D_{GL_1}$ the sheaf of holomorphic differential operators on $\text{Pic}^0(X)$. The algebra
\[ D_{GL_1} = \Gamma(\text{Pic}^0(X), \mathcal{D}_{GL_1}) = \text{Sym}(\Theta \text{Pic}^0(X)) \]
acts by endomorphisms of $D_{GL_1}$ from the right, and this action commutes with the left action of $D_{GL_1}$. We know that $\text{Spec } D_{GL_1} \simeq H^0(X, \Omega^{1,0}) = \text{Op}_{GL_1}(X)$.

Recall that a (holomorphic) $GL_1$-oper on $X$ is a holomorphic connection $\nabla$ on the trivial line bundle on $X$, so we can write
\[ (2.15) \quad \nabla = \nabla_a := d + \mathbf{a}, \]
where $\mathbf{a} \in H^0(X, \Omega^{1,0})$.

The trivial line bundle $\mathcal{V}$ on $\text{Op}_{GL_1}(X) \times X$ is equipped with a partial connection $\nabla^{\text{univ}}$ along $X$, whose restriction to $\nabla_a \times X \subset \text{Op}_{GL_1}(X) \times X$ is the connection $\nabla_a$ on $X$. Thus, $(\mathcal{V}, \nabla^{\text{univ}})$ is the universal $GL_1$-oper on $\text{Op}_{GL_1}(X) \times X$. We can write an explicit formula for $\nabla^{\text{univ}}$.

Let $W$ be a finite-dimensional vector space and $A = \text{Fun } W = \text{Sym } W^*$. The canonical element of $W^* \otimes W$ gives rise to an element of $A \otimes W$. Taking $W = H^0(X, \Omega^{1,0})$, we obtain a holomorphic one-form $\sigma$ with values in $\text{Fun } \text{Op}_{GL_1}(X)$. Then
\[ (2.16) \quad \nabla^{\text{univ}} = d + \sigma. \]

Explicitly, if $\{\omega_i, i = 1, \ldots, g\}$ is a basis of $H^0(X, \Omega^{1,0})$ and $\{b_i, i = 1, \ldots, g\}$ is the dual basis in $H^0(X, \Omega^{1,0})^* \simeq \Theta \text{Pic}^0(X)$, then
\[ (2.17) \quad \sigma = \sum_{i=1}^{g} b_i \omega_i. \]

Let $\mathcal{V}_X := \pi_*(\mathcal{V})$, where $\pi$ is the projection $\text{Op}_{GL_1}(X) \times X \rightarrow X$. The connection $\nabla^{\text{univ}}$ makes $\mathcal{V}_X$ into a left $\mathcal{D}_X$-module.

Moreover, the unit $1 \in \text{Fun } \text{Op}_{GL_1}(X)$ gives rise to a global section of $\mathcal{V}_X$, which we denote by $1_{\mathcal{V}_X}$. In addition, $\mathcal{V}_X$ is equipped with an action of $\text{Fun } \text{Op}_{GL_1}(X) \simeq D_{GL_1}$ which commutes with the action of $\mathcal{D}_X$. 
This allows us to define the following $D$-module on $\text{Pic}^0(X) \times X$:

$$D_{GL_1} \boxtimes V_X.$$ 

The algebra $D_{GL_1}$ acts on it by endomorphisms which commute with the action of the sheaf $D_{\text{Pic}^0(X) \times X}$.

Define its global section $s$ by the formula

$$s := 1 \boxtimes 1_{V_X}.$$ 

The element $(2.17)$ gives rise to a linear operator

$$\sigma : D_{GL_1} \boxtimes V_X \to D_{GL_1} \boxtimes (V_X \otimes K_X).$$

(namely, we interpret the $b_i$ as elements of $\text{Sym}(\Theta \text{Pic}^0(X)) = D_{GL_1}$).

The $D_X$-module structure on $D_{GL_1} \boxtimes V_X$ allows us to define the action of the holomorphic de Rham differential $\partial_X$ along $X$ on sections of $D_{GL_1} \boxtimes V_X$,

$$\partial_X : D_{GL_1} \boxtimes V_X \to D_{GL_1} \boxtimes (V_X \otimes K_X).$$

Formula (2.16) readily implies the following.

**Lemma 2.6.** The section $s$ satisfies the equation

$$\partial_X s = \sigma \cdot s. \tag{2.18}$$

**Corollary 2.7.** The $D$-module $D_{GL_1} \boxtimes V_X$ is isomorphic to $D_{GL_1} \boxtimes \mathcal{O}_X$ with the action of $D_X$ on the second factor modified so that the holomorphic de Rham differential $\partial_X$ acts as follows

$$\partial_X \mapsto 1 \boxtimes \partial_X + \sigma$$

where $\sigma \mapsto \sum_{i=1}^g b_i \boxtimes \omega_i$.

Now recall the $D$-module $q_1^*(D_X)$ on $\text{Pic}^0(X) \times X$ and its section $1_{q_1}$. The algebra $D_{GL_1}$ naturally acts on $q_1^*(D_X)$ by endomorphisms commuting with the $D$-module structure.

**Theorem 2.8.** There is an isomorphism of $D$-modules on $\text{Pic}^0(X) \times X$ equipped with a commuting action of $D_{GL_1}$,

$$q_1^*(D_{GL_1}) \simeq D_{GL_1} \boxtimes V_X, \tag{2.20}$$

under which the section $1_{q_1}$ is mapped to $s$.

**Proof.** By Corollary 2.7, we need to prove that $q_1^*(D_{GL_1}) \simeq D_{GL_1} \boxtimes \mathcal{O}_X$, with the modified action of $D_X$ on the second factor.

Let $A$ be an abelian variety. It comes equipped with the group homomorphism $m : A \times A \to A$. We claim that $m^*(D_A) \simeq D_A \boxtimes \mathcal{O}_A$, with $D_A$ corresponding to the first factor acting on itself from the left, and $D_A$ corresponding to the second factor.
acting on $O_A$ in a modified way so that the corresponding action of the holomorphic de Rham differential is given by the formula
\begin{equation}
\partial A \mapsto 1 \boxtimes \partial A + \sigma_A,
\end{equation}
where
\[\sigma_A = \sum_i b_i \boxtimes \omega_i.\]
Here $\{\omega_i\}$ is a basis in the space $T^* A = H^0(A, K_A)$ and $\{b_i\}$ is the dual basis in the dual vector space $T^*_A$, which we identify with the space $\Theta(A)$ of holomorphic translation vector fields on $A$ (so the $b_i$ can be viewed as global differential operators on $A$).

To see this, we invoke formula (3.2) from the next section which shows that
\begin{equation}
\mathcal{D}_A \mapsto \mathcal{D}_A \times A/\{\mathcal{D}_A \times A \cdot \Theta A \times A/A\}
\end{equation}
where $\Theta A \times A/A$ is the sheaf of vertical vector fields for the morphism $m: A \times A \to A$. It is generated by the global translation vector fields on $A \times A$ of the form $\xi \boxtimes 1 - 1 \boxtimes \xi$, where $\xi \in \Theta$. The quotient (2.22) can be identified with $\mathcal{D}_A \boxtimes O_A$ but then the action of the vector field $1 \boxtimes \xi$ corresponding to the element $\xi$ in the second factor $\mathcal{D}_A$ is given by the action of $\xi \boxtimes 1$ corresponding to the element $\xi$ in first factor $\mathcal{D}_A$, which coincides with the above description.

Now we apply this to the case $A = \text{Pic}^0(X)$ (so $\mathcal{D}_A = \mathcal{D}_{GL_1}$) and observe that
\[\mathcal{D}_{GL_1} \simeq (m^*(\mathcal{D}_{GL_1}))|_{\text{Pic}^0(X) \times X}\]
where $m$ is the multiplication map on $\text{Pic}^0(X)$ and $X$ is embedded into the second factor $\mathcal{D}_A$ by the action of $\xi \boxtimes 1$ corresponding to the element $\xi$ in first factor $\mathcal{D}_A$, which coincides with the above description.

The theorem now follows from the fact that the linear map from
\[H^0(\text{Pic}^0(X), K_{\text{Pic}^0(X)}) \simeq H^0(X, K_X)\]
to $H^0(X, K_X)$ corresponding to the pull-back of a one-form under the Abel-Jacobi map is the identity, so that formula (2.21) becomes (2.19).

Using Theorem 2.8 and Lemma 2.6, we obtain the sought-after differential equation on $1_{q_1}$.

**Proposition 2.9.** $\partial_X 1_{q_1} = \sigma \cdot 1_{q_1}$.

Corollary 3.3, which is proved in Section 3.1 below, gives us a way to produce differential equations on the operator of pull-back of functions from differential equations satisfied by the unit section of the $\mathcal{D}$-module pull-back of the sheaf of differential operators. More precisely, we will use Corollary 3.3 in the setting where the map $p: Z \to Y$ is $q_1 : \text{Pic}^0(X) \times X \to \text{Pic}^0(X)$, so that $p_C^{-1} = p_0 \hat{H}$ and $1_{Z \to Y} = 1_{q_1}$. Combined with Proposition 2.9, Corollary 3.3 gives us the main result of this section:

**Theorem 2.10.** The Hecke operator $p_0 \hat{H}$, viewed as a smooth function on $X$ with values in operators on $C^\infty(\text{Pic}^0(X))$, satisfies the system of differential equations
\begin{equation}
\partial_X \cdot p_0 \hat{H} = \sigma \cdot p_0 \hat{H}, \quad \overline{\partial} X \cdot p_0 \hat{H} = \overline{\sigma} \cdot p_0 \hat{H}.
\end{equation}
This theorem implies Proposition 2.5. Thus, we have obtained an alternative proof of Proposition 2.5 which relies on the Hecke eigensheaf property (Theorem 2.8) of the sheaf $D_{GL_1}$ (in the sense of Remark 2.3 below).

In other words, the differential equation on the eigenvalues of the Hecke operators follows from the description of the action of the Hecke functor on the sheaf $D_{GL_1}$.

**Remark 2.2.** The equations (2.23) yield the following explicit formula for the Hecke operators in terms of the translation vector fields:

\[
 p_0 H_p = \exp \left( \int_{p_0}^p (\sigma + \sigma) \right) = \exp \left( \int_{p_0}^p \left( \sum_{i=1}^g b_i \omega_i + \sum_{i=1}^g \bar{b}_i \bar{\omega}_i \right) \right)
\]

where the $b_i$ and $\bar{b}_i$ are viewed as translation vector fields on $Pic^0(X)$ (see formula (2.17)), whereas the $\omega_i$ and $\bar{\omega}_i$ are one-forms on $X$ which are integrated over a path connecting the points $p_0$ and $p$. Integrality properties of the eigenvalues of the $b_i$ and $\bar{b}_i$ then imply that the exponential of this integral does not depend on the choice of such path. This way we obtain another interpretation of the relation between the eigenvalues from Theorem 2.4. However, the path-independence of (2.24) is not obvious, and this is why we prefer to express this relation in terms of the system (2.23).

**□**

**Remark 2.3.** For $a \in Op_{GL_1}(X) = Spec D_{GL_1}$, let $I_a$ be the corresponding ideal in $D_{GL_1}$. Define the following $D$-module on $Pic^0(X)$:

\[
 \Delta_a := D_{GL_1}/(D_{GL_1} \cdot I_a).
\]

Theorem 2.8 implies that

\[
 q_!^*(\Delta_a) \simeq \Delta_a \boxtimes \mathcal{V}_a
\]

where $\mathcal{V}_a = (\mathcal{O}_X, \nabla_a)$ is the flat line bundle corresponding to $a$ (see formula (2.15)). The last isomorphism expresses the fact that $\Delta_a$ is the restriction to $Pic^0(X)$ of a Hecke eigensheaf on $Pic(X)$ with the eigenvalue $\mathcal{V}_a$. In fact, Theorem 2.8 is equivalent to this statement for all $a \in Op_{GL_1}(X)$. In this sense, $D_{GL_1}$ is the universal Hecke eigensheaf parametrized by all $GL_1$-opers on $X$.

Here’s a categorical interpretation. Recall that a $D$-module version of the Fourier–Mukai transform \([La, R]\) establishes an equivalence between the category of coherent $D$-modules on $Pic^0(X)$ and the category of coherent sheaves on $Loc_{GL_1}$, the moduli space of flat line bundles on $X$ (it is isomorphic to an affine bundle over $Pic^0(X)$). The space $Op_{GL_1}(X)$ can be realized as a subvariety of $Loc_{GL_1}$ (it is the fiber over the point corresponding to the trivial line bundle in $Pic^0(X)$). Let $\text{Coh}(Op_{GL_1}(X))$ be the category of coherent sheaves supported (scheme-theoretically) on $Op_{GL_1}(X) \subset Loc_{GL_1}$. The restriction of the above Fourier–Mukai transform to $\text{Coh}(Op_{GL_1}(X))$ gives rise to an equivalence $E$ between this (abelian) category and the category of $D$-modules $\mathcal{K}$ on $Pic^0(X)$ with finite global presentation, i.e. such that there is an exact sequence

\[
 D_{GL_1}^\oplus \rightarrow D_{GL_1}^\oplus \rightarrow \mathcal{K} \rightarrow 0.
\]
This equivalence $E$ takes an object $\mathcal{F}$ of Coh($\text{Op}_{GL_1}(X)$) to

$$E(\mathcal{F}) := D_{GL_1} \otimes_{D_{GL_1}} F, \quad F := \Gamma(\text{Op}_{GL_1}(X), \mathcal{F}),$$

where we use the fact that $D_{GL_1} \simeq \text{Fun}_{\text{Op}_{GL_1}(X)}$ (see [FT], Sect. 2).

It follows from this definition that $E(\mathcal{O}_a) = \Delta_a$ and $E(\mathcal{O}_{\text{Op}_{GL_1}(X)}) = D_{GL_1}$. □

3. Generalities on $D$-modules and integral transforms

In this section we discuss the compatibility between natural functors on the categories of (twisted) $D$-modules and the corresponding operations on sections of line bundles. Though most of the results of this section are fairly straightforward, we were unable to find them in the literature. We expect that these results are likely to have other applications, so it is worthwhile to record them here.

We note that closely related topics have been discussed in the works by A. D’Agnolo and P. Schapira [DS] and A. Goncharov [Gon], and in fact, we will use a result of [DS] below. Also, in writing this section we have benefited from the advice of P. Schapira.

First, we discuss the pull-back functor (Section 3.1), then the pull-back functor in the setting of twisted differential operators (Section 3.2), then the push-forward functor (Section 3.3), and finally the integral transform functors associated to correspondences (Section 3.4). We will use the results of the last subsection (specifically, Corollary 3.12) to prove Theorem 1.18 in Section 4. Roughly speaking, the differential equations (1.13) on the Hecke operator will follow from certain properties of the corresponding Hecke functor established in [BD1].

Remark 3.1. In the case of a curve over a finite field $\mathbb{F}_q$, one can pass from Hecke eigensheaves to Hecke eigenfunctions using Grothendieck’s faisceaux-fonctions correspondence. Namely, taking the trace of the Frobenius (a topological generator of the Galois group of $\mathbb{F}_q$) on the stalks of a Hecke eigensheaf on $\text{Bun}_G$, we obtain a function on $\text{Bun}_G(\mathbb{F}_q)$. Crucially, this function is a Hecke eigenfunction because Grothendieck’s correspondence is compatible with natural operations on sheaves and functions.

In the case of a curve defined over $\mathbb{C}$, there is no Frobenius as the field of complex numbers is algebraically closed. Instead, we construct Hecke eigenfunctions as single-valued bilinear combinations of sections of the corresponding Hecke eigensheaf $\Delta_\chi$ (see formula (1.8)) and sections of a complex conjugate sheaf, as explained in Section 1.5 of [EFK1] and Section 1.3 above. One could argue that this procedure is what replaces taking the traces of the Frobenius in the case of a curve over $\mathbb{C}$, but the question remains why the resulting function is an eigenfunction of the Hecke operators. The answer is that the results of this section enable us to derive the Hecke eigenfunction property (and to compute the corresponding eigenvalues) from the Hecke eigensheaf property of $\Delta_\chi$, using the cyclicity of the Hecke eigensheaf $\Delta_\chi$ (viewed as a twisted $D$-module on $\text{Bun}_G$) and the cyclicity of the corresponding “Hecke eigenvalues” (viewed as twisted $D$-modules on the curve $X$). Thus, the results established in this section may be viewed as an analogue in the complex case of the compatibility of Grothendieck’s correspondence with natural operations on sheaves and functions. □
3.1. **Pull-back.** For a smooth complex manifold $Y$, denote by $\mathcal{O}_Y$ and $\mathcal{D}_Y$ the sheaves of holomorphic functions and differential operators on $Y$, respectively (in the analytic topology). Let $\text{Mod}(\mathcal{D}_Y)$ be the category of left $\mathcal{D}_Y$-modules.

**Remark 3.2.** In what follows, we can take as $\mathcal{D}_Y$ the sheaf of algebraic differential operators on $Y$ and the category of modules over it. Then we have analogous statements as well. □

Given sheaves of algebras $\mathcal{A}$ and $\mathcal{B}$ on $Y$, an $(\mathcal{A}, \mathcal{B})$-**bimodule** is, by definition, a sheaf of modules over $\mathcal{A} \otimes \mathcal{B}^{\text{opp}}$ on $Y$.

We need to recall some facts about the pull-back functor for $\mathcal{D}$-modules. Let $p : Z \rightarrow Y$ be a submersion of smooth complex manifolds. Denote by $p^{-1}$ the sheaf-theoretic pull-back functor.

The $\mathcal{O}$-module pull-back functor $p^*$ is defined as follows. If $\mathcal{F}$ is an $\mathcal{O}_Y$-module, then the $\mathcal{O}_Z$-module $p^*(\mathcal{F})$ is

$$p^*(\mathcal{F}) := \mathcal{O}_Z \otimes_{p^{-1}(\mathcal{O}_Y)} p^{-1}(\mathcal{F}).$$

If $\mathcal{F}$ is a left $\mathcal{D}_Y$-module, then $p^*(\mathcal{F})$ has a natural structure of left $\mathcal{D}_Z$-module. To explain this, let $\Theta_{Z/Y}$ be the relative tangent sheaf of the morphism $p : Z \rightarrow Y$ (its sections are vertical vector fields on $Z$ with respect to $p$). This is a sheaf of Lie algebras. It acts by commutator on $\mathcal{D}_Z$, and this action preserves the left ideal $(\mathcal{D}_Z \cdot \Theta_{Z/Y}) \subset \mathcal{D}_Z$. Moreover, we have

$$\left(\mathcal{D}_Z/(\mathcal{D}_Z \cdot \Theta_{Z/Y})\right)^{\Theta_{Z/Y}} \simeq p^{-1}(\mathcal{D}_Y)$$

and hence

$$\mathcal{D}_{Z \rightarrow Y} := \mathcal{D}_Z/(\mathcal{D}_Z \cdot \Theta_{Z/Y})$$

is naturally a $(\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y))$-bimodule.

**Remark 3.3.** Since $p$ is a submersion,

$$\mathcal{D}_{Z \rightarrow Y} \simeq p^*(\mathcal{D}_Y) = \mathcal{O}_Z \otimes_{p^{-1}(\mathcal{O}_Y)} p^{-1}(\mathcal{D}_Y)$$

as an $(\mathcal{O}_Z, p^{-1}(\mathcal{D}_Y))$-bimodule, and this is how $\mathcal{D}_{Z \rightarrow Y}$ is usually defined. Defining it by formula (3.2) for such $p$ has the advantage that it makes the $\mathcal{D}_Z$-module structure on it manifest. □

The isomorphism (3.3) implies that the $\mathcal{O}$-module pull-back $p^*(\mathcal{F})$ of a left $\mathcal{D}_Y$-module $\mathcal{F}$ can be written as

$$p^*(\mathcal{F}) \simeq \mathcal{D}_{Z \rightarrow Y} \otimes_{p^{-1}(\mathcal{D}_Y)} p^{-1}(\mathcal{F})$$

and hence the $\mathcal{O}_Z$-module structure on $p^*(\mathcal{F})$ naturally extends to a $\mathcal{D}_Z$-module structure. Thus, we obtain the pull-back functor for $\mathcal{D}$-modules, which we denote in the same way as the $\mathcal{O}$-module pull-back:

$$p^* : \text{Mod}(\mathcal{D}_Y) \rightarrow \text{Mod}(\mathcal{D}_Z).$$
On the other hand, let \( \mathcal{C}_Y^\infty \) and \( \mathcal{C}_Z^\infty \) be the sheaves of \( \mathbb{C} \)-valued \( C^\infty \) functions on the smooth real manifolds underlying \( Y \) and \( Z \), respectively (in the analytic topology). Consider the internal Hom sheaf

\[
\mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)
\]
on \( Z \) in the category of sheaves of vector spaces. By definition, for an open subset \( U \subset Z \),

\[
(3.4) \quad \mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)(U) := \mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty)|_U, \mathcal{C}_Z^\infty|_U)
\]

(as well-known, the presheaf defined by this formula is a sheaf). In particular, we have a special global section \( p_{\mathcal{C}_\infty}^{-1} \) of \( \mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty) \) over \( Z \), which corresponds to the pull-back of smooth functions from \( Y \) to \( Z \). More precisely, for any open \( U \subset Z \), the restriction of \( p_{\mathcal{C}_\infty}^{-1} \) to \( U \) maps \( f \in (p^{-1}(\mathcal{C}_Y^\infty))(U) = C^\infty(p(U)) \) to its pull-back to \( U \) via \( p \) (note that \( p(U) \) is open because \( p \) is a submersion).

**Lemma 3.1.** The sheaf \( \mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty) \) has a natural \( (\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y)) \)-bimodule structure, which is defined as follows: for every open \( U \subset Z \), given \( P \in \mathcal{D}_Z(U), Q \in (p^{-1}(\mathcal{D}_Y))(U) = \mathcal{D}_Y(p(U)) \), and \( \phi \in \mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)(U) \), which is a compatible system \( \{\phi_{U'} \in \mathcal{H}om(C^\infty(p(U')), C^\infty(U'))\} \) for open subsets \( U' \subset U \),

\[
\phi \mapsto P \circ \phi \circ Q,
\]

where \( P \circ \phi \circ Q \) stands for the compatible system \( \{P|_{U'} \circ \phi_{U'} \circ Q|_{p(U')} \mid U' \subset U\} \).

On the other hand, we have the \( (\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y)) \)-bimodule \( \mathcal{D}_{Z \rightarrow Y} \) given by \( (3.2) \). The unit \( 1_Y \in \mathcal{D}_Y \) gives rise to a global section \( 1_{Z \rightarrow Y} \) of \( \mathcal{D}_{Z \rightarrow Y} \).

**Proposition 3.2.** There is a unique injective homomorphism of \( (\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y)) \)-bimodules

\[
\mathcal{D}_{Z \rightarrow Y} \rightarrow \mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty)
\]

sending \( 1_{Z \rightarrow Y} \) to \( p_{\mathcal{C}_\infty}^{-1} \).

**Proof.** Let

\[
A_{Z,Y} := \mathcal{D}_Z \cdot p_{\mathcal{C}_\infty}^{-1}.
\]

For every open \( U \subset Z \), we have

\[
\xi : p_{\mathcal{C}_\infty}^{-1}(f) = 0, \quad \forall \xi \in \Theta_Z/\Theta_Y(U), \quad \forall f \in C^\infty(p(U)).
\]

It follows that \( A_{Z,Y} \) is naturally isomorphic to the right hand side of \( (3.2) \) and hence to \( \mathcal{D}_{Z \rightarrow Y} \) as a left \( \mathcal{D}_Z \)-module. The isomorphism \( (3.1) \) defines the structure of a \( (\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y)) \)-bimodule on both \( A_{Z,Y} \) and \( \mathcal{D}_{Z \rightarrow Y} \). It is clear that the former is compatible with the \( (\mathcal{D}_Z, p^{-1}(\mathcal{D}_Y)) \)-bimodule structure on \( \mathcal{H}om(p^{-1}(\mathcal{C}_Y^\infty), \mathcal{C}_Z^\infty) \). \( \square \)

**Remark 3.4.** Concretely, we can choose a sufficiently fine open covering of \( Z \), so that each neighborhood is isomorphic to the product of two balls \( B_m \subset \mathbb{C}^m \) and \( B_n \subset \mathbb{C}^n \) and the restriction of the map \( p \) to it is isomorphic to the projection \( B_m \times B_n \rightarrow B_m \). Let \( \{y_1, \ldots, y_m\} \) and \( \{x_1, \ldots, x_n\} \) be coordinates on \( B_m \) and \( B_n \). It is clear that the spaces of sections of both \( A_{Z,Y} \) and \( \mathcal{D}_{Z \rightarrow Y} \) are both isomorphic to \( \mathcal{H}ol(y_1, \ldots, y_m, x_1, \ldots, x_n) \otimes \mathbb{C}[\partial_{y_1}, \ldots, \partial_{y_m}] \), where \( \mathcal{H}ol(y_1, \ldots, y_m, x_1, \ldots, x_n) \) denotes the space of holomorphic functions on \( B_m \times B_n \). \( \square \)
Corollary 3.3. Suppose that $P \cdot 1_{Z \to Y} = 0$ for some $P \in \Gamma(Z, \mathcal{D}_Z)$. Then

(3.5) $P \cdot p_{C_\infty}^{-1} = 0, \quad \mathcal{T} \cdot p_{C_\infty}^{-1} = 0.$

3.2. Pull-back in the twisted setting. The results of Section 3.1 can be generalized to twisted differential operators. Namely, let $\mathcal{L}$ be a holomorphic line bundle on $Y$. Recall that in Section 1.1, using the norm map $a \mapsto \|a\| = |a|^2$ from $\mathbb{C}^\times$ to $\mathbb{R}_{>0}$, we associated to $\mathcal{L}$ a $C^\infty$ complex line bundle

$$\|\mathcal{L}\| = |\mathcal{L}|^2$$

with the structure group $\mathbb{R}_{>0}$ on $Y$, viewed as a complex analytic variety. Clearly,

(3.6) $|\mathcal{L}|^2 \simeq (\mathcal{L} \otimes \overline{\mathcal{T}}) \otimes \mathcal{O}_Y.$

In other words, if the transition functions of $\mathcal{L}$ are $\{g_{\alpha\beta}\}$, then the transition functions of $|\mathcal{L}|^2$ are $\{|g_{\alpha\beta}|^2\}$.

Consider the sheaf of (twisted) differential operators acting on $\mathcal{L}$ (see [BB]),

$$\mathcal{D}_{Y,\mathcal{L}} := \mathcal{L} \otimes \mathcal{D}_Y \otimes \mathcal{L}^{-1}.$$

Likewise, we have the sheaf $\mathcal{D}_{Z,p^*\mathcal{L}}$ of differential operators acting on the line bundle $p^*(\mathcal{L})$ on $Z$.

The role of $\mathcal{D}_{Z \to Y}$ is now played by

(3.7) $\mathcal{D}_{Z \to Y,\mathcal{L}} := p^*(\mathcal{L}) \otimes p^*(\mathcal{D}_Y \otimes \mathcal{L}^{-1}) \simeq p^*(\mathcal{L}) \otimes \mathcal{D}_{Z \to Y} \otimes_{p^{-1}(\mathcal{O}_Y)} p^{-1}(\mathcal{L}),$

which is naturally a $(\mathcal{D}_{Z,p^*(\mathcal{L})}, p^{-1}(\mathcal{D}_{Y,\mathcal{L}}))$-bimodule. The unit $1_Y \in \mathcal{D}_Y$ gives rise to a global section $1_{Z \to Y,\mathcal{L}}$ of $\mathcal{D}_{Z \to Y,\mathcal{L}}$.

On the other hand, let $\mathcal{C}_{Y,|\mathcal{L}|^2}$ and $\mathcal{C}_{Z,|p^*(\mathcal{L})|^2}$ be the sheaves of $C^\infty$ sections of the line bundles $|\mathcal{L}|^2$ on $Y$ and $|p^*(\mathcal{L})|^2$ on $Z$, respectively. Then $\mathcal{D}_{Y,\mathcal{L}}$ naturally acts on $\mathcal{C}_{Y,|\mathcal{L}|^2}$ and $\mathcal{D}_{Z,p^*(\mathcal{L})}$ naturally acts on $\mathcal{C}_{Z,|p^*(\mathcal{L})|^2}$.

Consider the internal Hom sheaf $\mathcal{H}om(p^{-1}(\mathcal{C}_{Y,|\mathcal{L}|^2}), \mathcal{C}_{Z,|p^*(\mathcal{L})|^2})$ on $Z$ and its global section $p_{\mathcal{L}}^{-1}$ corresponding to the natural pull-back map

(3.8) $p_{\mathcal{L}}^{-1} : C^\infty(Y, |\mathcal{L}|^2) \to C^\infty(Z, |p^*(\mathcal{L})|^2).$

The sheaf $\mathcal{H}om(p^{-1}(\mathcal{C}_{Y,|\mathcal{L}|^2}), \mathcal{C}_{Z,|p^*(\mathcal{L})|^2})$ has the structure of a $(\mathcal{D}_{Z,p^*(\mathcal{L})}, p^{-1}(\mathcal{D}_{Y,\mathcal{L}}))$-bimodule defined in the same way as in Lemma 3.1.

Proposition 3.4. There is a unique injective homomorphism of $(\mathcal{D}_{Z,p^*(\mathcal{L})}, p^{-1}(\mathcal{D}_{Y,\mathcal{L}}))$-bimodules

$$\mathcal{D}_{Z \to Y,\mathcal{L}} \to \mathcal{H}om(p^{-1}(\mathcal{C}_{Y,|\mathcal{L}|^2}), \mathcal{C}_{Z,|p^*(\mathcal{L})|^2})$$

sending $1_{Z \to Y,\mathcal{L}}$ to $p_{\mathcal{L}}^{-1}$.

Proof. According to the definition (3.7), we have

(3.9) $\mathcal{D}_{Z \to Y,\mathcal{L}} \simeq p^*(\mathcal{L}) \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z \to Y} \otimes_{p^{-1}(\mathcal{O}_Y)} p^{-1}(\mathcal{L}).$
Moreover, isomorphisms (3.2) and (3.11) imply similar isomorphisms in the twisted case (note that $\Theta_{Z/Y}$ naturally embeds into $D_{Z,p^*(\mathcal{L})}$):

\begin{equation}
(3.10) \quad D_{Z \to Y, \mathcal{L}} \simeq D_{Z,p^*(\mathcal{L})}/(D_{Z,p^*(\mathcal{L})} \cdot \Theta_{Z/Y}).
\end{equation}

\begin{equation}
(3.11) \quad p^{-1}(D_{Y, \mathcal{L}}) \simeq (D_{Z,p^*(\mathcal{L})}/(D_{Z,p^*(\mathcal{L})} \cdot \Theta_{Z/Y}))^{\Theta_{Z/Y}}.
\end{equation}

Now we argue in the same way as in the proof of Proposition 3.2. Let

$$A_{Z,Y,\mathcal{L}} := D_{Z,p^*(\mathcal{L})} \cdot p^{-1}.$$

For every open $U \subset Z$, we have

$$\xi \cdot p^{-1}_L(f) = 0, \quad \forall \xi \in \Theta_{Z/Y}(U), \quad \forall f \in C^\infty(p(U), \mathcal{L}).$$

By restricting to sufficiently small open subsets, we obtain from Proposition 3.2 that $A_{Z,Y,\mathcal{L}}$ is naturally isomorphic to the right hand side of (3.10) and hence to $D_{Z \to Y, \mathcal{L}}$ as a left $D_{Z,p^*(\mathcal{L})}$-module. Using the isomorphism (3.11), we obtain a $(D_{Z,p^*(\mathcal{L})}, p^{-1}(D_{Y,\mathcal{L}}))$-bimodule structure on both $A_{Z,Y,\mathcal{L}}$ and $D_{Z \to Y, \mathcal{L}}$ and therefore an isomorphism between $A_{Z,Y,\mathcal{L}}$ and $D_{Z \to Y, \mathcal{L}}$ as $(D_{Z,p^*(\mathcal{L})}, p^{-1}(D_{Y,\mathcal{L}}))$-bimodules, sending $1_{Z \to Y, \mathcal{L}}$ to $p^{-1}_L$. By construction, the former is precisely the $(D_{Z,p^*(\mathcal{L})}, p^{-1}(D_{Y,\mathcal{L}}))$-submodule of $\mathcal{H}om(p^{-1}(C^\infty_{|\mathcal{L}|^2}), C^\infty_{Z,p^*(\mathcal{L})})$ generated by $p^{-1}_L$. \hfill \square

3.3. **Push-forward.** Suppose that $p : Z \to Y$ is a submersion with compact fibers, and denote by $K_{Z/Y}$ the corresponding relative canonical bundle. Then

$$\Omega_{Z/Y} := |K_{Z/Y}|^2$$

is the $C^\infty$ line bundle of relative densities. Let $K_Y$ and $K_Z$ be the canonical line bundles on $Y$ and $Z$, respectively, and $\Omega_Y := |K_Y|^2$ and $\Omega_Z := |K_Z|^2$ the $C^\infty$ line bundles of densities on $Y$ and $Z$, respectively. We have

$$\Omega_{Z/Y} \simeq \Omega_Z \otimes \Omega_Y^{-1}.$$

Denote by $p_*$ the sheaf-theoretic push-forward functor. Let again $\mathcal{L}$ be a line bundle on $Y$. Consider the internal Hom sheaf (where $\otimes$ stands for $\otimes$ or $\hat{\otimes}$)

$$\mathcal{H}om(p_*(C^\infty_{Z,p^*(\mathcal{L})\otimes\Omega_Z}), C^\infty_{Y,|\mathcal{L}|^2\otimes\Omega_Y}) \simeq \mathcal{H}om(p_*(C^\infty_{Z,p^*(\mathcal{L})\otimes\Omega_Z/Y}), C^\infty_{Y,|\mathcal{L}|^2})$$

on $Y$ and its global section $p_\mathcal{L}^*$ corresponding to the integration map

\begin{equation}
(3.12) \quad p_\mathcal{L}^* : C^\infty(Z, |p^*(\mathcal{L})|^2 \otimes \Omega_{Z/Y}) \to C^\infty(Y, |\mathcal{L}|^2)
\end{equation}

Set

$$\mathcal{M} := p^*(\mathcal{L}) \otimes K_{Z/Y},$$

so that

$$|\mathcal{M}|^2 = |p^*(\mathcal{L})|^2 \otimes \Omega_{Z/Y}. $$
Lemma 3.5. The sheaf $\mathcal{H}om(p_*(C^\infty_{Z,M})(\mathcal{L}),C^\infty_{Y,L})$ has a natural $(\mathcal{D}_{Y,L},p_*(\mathcal{D}_{Z,M}))$-bimodule structure, which is defined as follows: for every open $U \subset Y$, given $R \in (p_*(\mathcal{D}_{Z,M}))(U) = \mathcal{D}_{Z,M}(p^{-1}(U))$, $S \in \mathcal{D}_{Y,L}(U)$, and $\phi \in \mathcal{H}om(p_*(C^\infty_{Z,M}),C^\infty_{Y,L})(U)$, which is a compatible system $\{\phi_{U'} \in \mathcal{H}om(C^\infty(p^{-1}(U')),|\mathcal{M}|^2),C^\infty(U',|\mathcal{L}|^2)\}$ for open subsets $U' \subset U$, 
\[ \phi \mapsto S \circ \phi \circ R, \]
where $S \circ \phi \circ R$ stands for the compatible system $\{S|_{U'} \circ \phi_{U'} \circ R|_{p^{-1}(U')} | U' \subset U\}$.

On the other hand, denote by $p^D_*$ the (derived) $D$-module push-forward functor 
\[ p^D_* : D^b(D_Z) \to D^b(D_Y). \]
If $\mathcal{F}$ is a $\mathcal{D}_Z$-module, then it follows from the definition (see [Ka], Sect. 4.6) that
\[ p^D_*(\mathcal{F}) = R^* p_*(\mathcal{D}_{Y \leftarrow Z} \mathcal{L} \mathcal{D}_Z \mathcal{F}), \]
where 
\[ \mathcal{D}_{Y \leftarrow Z} := K_Z \otimes_{\mathcal{O}_Z} \mathcal{D}_{Z \to Y} \otimes_{p^{-1}(\mathcal{O}_Y)} p^{-1}(K_Y) \]
is a sheaf on $Z$ with a natural structure of $(p^{-1}(\mathcal{D}_Y), \mathcal{D}_Z)$-bimodule.

The right action of $\mathcal{D}_Y$ on $K_Y$ gives rise to a canonical isomorphism 
\[ \mathcal{D}^{\text{opp}}_Y \simeq K_Y \otimes_{\mathcal{O}_Y} \mathcal{D}_Y \otimes_{\mathcal{O}_Y} K_Y^{-1}, \]
which yields an identification (see [Ka], Remark 4.18)
\[ \mathcal{D}_{Y \leftarrow Z} \simeq p^{-1}(\mathcal{D}_Y) \otimes_{p^{-1}(\mathcal{O}_Y)} K_{Z/Y}. \]

Let us apply $p^{D,0}_* := H^0 p^D_*$ to the left $\mathcal{D}_Z$-module $\mathcal{D}_Z \otimes \mathcal{M}$. Since it is free, $\mathcal{L}$ in formula (3.13) is the ordinary $\otimes$. Note also that it carries a commuting right $\mathcal{D}_{Z,M}$-module structure and hence is a $(\mathcal{D}_Z, \mathcal{D}_{Z,M})$-bimodule. This implies that the sheaf $p^{D,0}_*(\mathcal{D}_Z \otimes \mathcal{M})$ is a $(\mathcal{D}_Y, p_*(\mathcal{D}_{Z,M}))$-bimodule.

Setting 
\[ \mathcal{D}_{Y \leftarrow Z,M} := \mathcal{D}_{Y \leftarrow Z} \otimes_{\mathcal{O}_Z} \mathcal{M}^{-1} \simeq p^{-1}(\mathcal{D}_Y) \otimes_{p^{-1}(\mathcal{O}_Y)} p^*(\mathcal{L}^{-1}), \]
we obtain an isomorphism 
\[ p^{D,0}_*(\mathcal{D}_Z \otimes \mathcal{M}) \simeq p_*(\mathcal{D}_{Y \leftarrow Z,M}). \]

Define the following $(\mathcal{D}_{Y,L}, p_*(\mathcal{D}_{Z,M}))$-bimodule on $Y$:
\[ \mathcal{D}_{Y \leftarrow Z,M} := \mathcal{L} \otimes_{\mathcal{O}_Y} p^{D,0}_*(\mathcal{D}_Z \otimes \mathcal{M}) \simeq \mathcal{L} \otimes_{\mathcal{O}_Y} p_*(\mathcal{D}_{Y \leftarrow Z,M}). \]

It follows from the definition of the functor $p^{D,0}_*$ that the unit $1_Z \in \mathcal{D}_Z$ gives rise to a global section $1_{Y \leftarrow Z,L}$ of $\mathcal{D}_{Y \leftarrow Z,M}$.
**Proposition 3.6.** There is a unique injective homomorphism of \((\mathcal{D}_{Y,\mathcal{L}}, p_\ast(\mathcal{D}_{Z,M}))\)-bimodules

\[ \tilde{\mathcal{D}}_{Y\leftarrow Z,M} \rightarrow \text{Hom}(p_\ast(\mathcal{C}_{Y,|\mathcal{L}|^2}^\infty), \mathcal{C}_{Y,|\mathcal{L}|^2}^\infty) \]

sending \(1_{Y\leftarrow Z,\mathcal{L}}\) to \(p_1^\mathcal{L}\).

**Proof.** Let us prove the statement in the case \(\mathcal{L} = \mathcal{O}_Y\), so that \(\mathcal{M} = K_{Z/Y}\).

Recall the relative tangent sheaf \(\Theta_{Z/Y}\). The sheaf \(p_\ast(\Theta_{Z/Y})\) naturally acts on \(p_\ast(K_{Z/Y})\) and on \(p_\ast(\mathcal{C}_{Z,\Omega_{Z/Y}}^\infty)\) by Lie derivatives. Hence \(\Theta_{Z/Y}\) embeds into \(p_\ast(\mathcal{D}_{Z,K_{Z/Y}})\) as a subsheaf of Lie algebras, and so it acts on \(p_\ast(\mathcal{D}_{Z,K_{Z/Y}})\) by commutators. For every open \(U \subset Y\), we have

\[ p_1^\mathcal{L}(\xi \cdot f) = 0, \quad \forall \xi \in p_\ast(\Theta_{Z/Y})(U), \quad \forall f \in C^\infty(p^{-1}(U), \Omega_{Z/Y}). \]

It follows that

\[ p_\ast(\mathcal{D}_{Z,K_{Z/Y}})^\text{opp} \cdot p_1^\mathcal{L} \simeq p_\ast(\mathcal{D}_{Z,K_{Z/Y}})/(p_\ast(\Theta_{Z/Y}) \cdot p_\ast(\mathcal{D}_{Z,K_{Z/Y}})). \]

Formula (3.11) implies that

\[ p_\ast(\mathcal{D}_{Z,K_{Z/Y}})/(p_\ast(\Theta_{Z/Y}) \cdot p_\ast(\mathcal{D}_{Z,K_{Z/Y}})) \simeq \tilde{\mathcal{D}}_{Y\leftarrow Z,K_{Z/Y}}. \]

This completes the proof of the proposition for \(\mathcal{L} = \mathcal{O}_Y\), so that \(\mathcal{M} = K_{Y/Z}\). For a general line bundle \(\mathcal{L}\), the statement of the proposition is derived in a similar way (compare with Propositions 3.2 and 3.3). \(\square\)

### 3.4. Integral transforms

Consider a correspondence

\[ \begin{array}{c}
\begin{array}{c}
Z \\
\end{array}
\end{array} \xymatrix{ \begin{array}{c}
Y_1 \\
\ar@<1ex>[r]^{p_1} \\
\ar@<1ex>[r]^{p_2} \\
Y_2 \\
\end{array} \begin{array}{c}
\end{array} } \]

Assume that \(p_1\) and \(p_2\) are submersions with compact fibers. Let \(\mathcal{L}_1\) and \(\mathcal{L}_2\) be line bundles on \(Y_1\) and \(Y_2\), respectively, such that

\[ p_1^\ast(\mathcal{L}_1) \simeq p_2^\ast(\mathcal{L}_2) \otimes_{\mathcal{O}_Z} K_{Z/Y_2}. \]

In what follows, we will fix such an isomorphism.

Consider the Hom sheaf on \(Y_2\),

\[ \mathcal{H}\text{om}(p_{2\ast}p_1^{-1}(\mathcal{C}_{Y_1,\mathcal{L}_1}^\infty), \mathcal{C}_{Y_2,\mathcal{L}_2}^\infty). \]

Combining Lemmas 3.1 and 3.5, we obtain that it is naturally a \((\mathcal{D}_{Y_2,\mathcal{L}_2}, p_{2\ast}p_1^{-1}(\mathcal{D}_{Y_1,\mathcal{L}_1}))\)-bimodule.

Recall that we have a section

\[ p_2^\mathcal{L}_2 \in \mathcal{H}\text{om}(p_{2\ast}(\mathcal{C}_{Z,|\mathcal{L}_2|^2}^\infty \otimes \Omega_{Z/Y_2}), \mathcal{C}_{Y_2,|\mathcal{L}_2|^2}^\infty) = \mathcal{H}\text{om}(p_{2\ast}(\mathcal{C}_{Z,|\mathcal{L}_1|^2}^\infty), \mathcal{C}_{Y_2,|\mathcal{L}_2|^2}^\infty) \]

(here we apply the isomorphism (3.16)). We also have a section

\[ p_1^{-1}\mathcal{L}_1 \in \mathcal{H}\text{om}(p_1^{-1}(\mathcal{C}_{Y_1,|\mathcal{L}_1|^2}^\infty), \mathcal{C}_{Z,|\mathcal{L}_1|^2}^\infty) \]

which gives rise to a section

\[ p_{2\ast}(p_1^{-1}\mathcal{L}_1) \in \mathcal{H}\text{om}(p_{2\ast}p_1^{-1}(\mathcal{C}_{Y_1,|\mathcal{L}_1|^2}^\infty), p_{2\ast}(\mathcal{C}_{Z,|\mathcal{L}_1|^2}^\infty)). \]
Denote by \( H_{Z, Y_1, Y_2}^{\ell_1, \ell_2} \), or \( H_Z \) for short, the composition
\[
H_Z := p_{2*}^{L_2} \circ p_{2*}(p_{1*}^{-1}(L_{1})),
\]
which is a section of the sheaf \( (3.17) \). This is the integral transform associated to the correspondence \( Z \) and the line bundles \( L_1 \) and \( L_2 \).

On the other hand, recall the \((\mathcal{D}_{Z,p^*(L_1)}, p_{1*}^{-1}(\mathcal{D}_{Y_1,L_1}))\)-bimodule \( \mathcal{D}_{Z} \rightarrow Y_1, L_1 \) and the \((\mathcal{D}_{Y,L_2}, p_{2*}(\mathcal{D}_{Z,M_2}))\)-bimodule \( \mathcal{D}_{Y} \leftarrow Z, M_2 \), where
\[
\mathcal{M}_2 = p_{2*}^{L_2}(L_2) \otimes K_{Z/Y_2}.
\]
Formula \((3.16)\) implies that
\[
\mathcal{D}_{Z,p^*(L_1)} \simeq \mathcal{D}_{Z,M_2}.
\]
Therefore we can form the tensor product
\[
(3.18) \quad \mathcal{D}_{Y_2 \leftarrow Z \rightarrow Y_1}^{\ell_1, \ell_2} := \widetilde{\mathcal{D}}_{Y \leftarrow Z, M_2} \otimes_{p_{2*}(\mathcal{D}_{Z,Y_1})} p_{2*}(\mathcal{D}_{Z \rightarrow Y_1, L_1})
\]
which is naturally a \((\mathcal{D}_{Y_2,L_2}, p_{2*}p_{1*}^{-1}(\mathcal{D}_{Y_1,L_1}))\)-bimodule. The canonical sections of \( \widetilde{\mathcal{D}}_{Y \leftarrow Z, M_2} \) and \( \mathcal{D}_{Z \rightarrow Y_1, L_1} \) introduced above give us a global section of \( \mathcal{D}_{Y_2 \leftarrow Z \rightarrow Y_1}^{\ell_1, \ell_2} \), which we denote by \( 1_{Y_2 \leftarrow Z \rightarrow Y_1} \).

Propositions \( 3.4 \) and \( 3.6 \) imply the following.

**Proposition 3.7.** There is a unique homomorphism of \((\mathcal{D}_{Y_2,L_2}, p_{2*}p_{1*}^{-1}(\mathcal{D}_{Y_1,L_1}))\)-bimodules
\[
\mathcal{D}_{Y_2 \leftarrow Z \rightarrow Y_1}^{\ell_1, \ell_2} \rightarrow \mathcal{H}(p_{2*}p_{1*}^{-1}(\mathcal{C}^{\infty}_{Y_2,\ell_1}), \mathcal{C}^{\infty}_{Y_2,\ell_2})
\]
sending \( 1_{Y_2 \leftarrow Z \rightarrow Y_1} \) to \( H_Z \).

This proposition has the following obvious corollary.

**Corollary 3.8.** Suppose that \( P \cdot 1_{Y_2 \leftarrow Z \rightarrow Y_1} = 0 \) for some \( P \in \Gamma(Y_2, \mathcal{D}_{Y_2,L_2}) \). Then
\[
P \cdot H_Z = 0, \quad \overline{P} \cdot H_Z = 0.
\]

In other words, holomorphic differential equations satisfied by the section \( 1_{Y_2 \leftarrow Z \rightarrow Y_1} \) give rise to differential equations on the integral transform \( H_Z \) obtained from the correspondence \( Z \).

Now we want to connect this to the \( D \)-module integral transform functor associated to the correspondence \( Z \). Namely, we have the functor
\[
(3.19) \quad \mathcal{H}_Z^*: \mathcal{D}^b(Y_1) \rightarrow \mathcal{D}^b(Y_2)
\]
\[
(3.20) \quad \mathcal{G} \mapsto p_{2*}^{D}(p_{1*}^!(\mathcal{G})).
\]

Let
\[
(3.21) \quad \mathcal{H}_Z := L_2 \otimes_{\mathcal{O}_{Y_2}} \mathcal{H}_Z^2(\mathcal{D}_{Y_1} \otimes L^{-1}).
\]

This is a \((\mathcal{D}_{Y_2,L_2}, p_{2*}p_{1*}^{-1}(\mathcal{D}_{Y_1,L_1}))\)-bimodule. The unit \( 1_{Y_1} \) gives rise to a global section of \( \mathcal{H}_Z \), which we denote by \( \psi_Z \).

Recall formula \((3.13)\) for the functor \( p_{2*}^{D} \). The next result follows from [DS], Prop. 2.12.
Proposition 3.9 ([DS]). Suppose that the map \( Z \to Y_1 \times Y_2 \) is a closed embedding. Then \( D_{Y_2 \leftarrow Z} \otimes_{\mathcal{O}_Z} D_{Y_2 \to Y_1} \) is concentrated in cohomological degree 0.

This proposition has the following corollary.

Corollary 3.10. If \( Z \to Y_1 \times Y_2 \) is a closed embedding, then there is an isomorphism of \((D_{Y_2}, \mathcal{L}_2, p_2, p_1^{-1}(D_{Y_1}, \mathcal{L}_1))-bimodules\)

\[
(D_{Y_2 \leftarrow Z} \otimes_{\mathcal{O}_Z} D_{Y_2 \to Y_1}) \simeq \mathcal{H}_Z
\]

under which \( 1_{Y_2 \leftarrow Z} \) is mapped to \( \psi_Z \).

Combining Corollaries 3.8 and 3.10 we obtain the following.

Corollary 3.11. Suppose that \( Z \to Y_1 \times Y_2 \) is a closed embedding. If \( P \cdot \psi_Z = 0 \) for some \( P \in \Gamma(Y_2, D_{Y_2, \mathcal{L}_2}) \), then

\[
P \cdot H_Z = 0, \quad \mathcal{P} \cdot H_Z = 0.
\]

4. The case of \( PGL_n \)

In this section we consider the case of \( G = PGL_n \) (so that \( ^tG = SL_n \)) and the Hecke correspondence \( \mathcal{Z}(\omega_1) \) associated to \( \lambda = \omega_1 \), the first fundamental coweight of \( PGL_n \). There are two advantages in this case that we will exploit: (1) \( \mathcal{Z}(\omega_1) = Z(\omega_1) \), i.e. the fibers of \( q_2 \times q_3 \) are isomorphic to a closed \( G[[z]] \)-orbit \( \text{Gr}_{\omega_1} \), which is smooth and compact (in fact, \( \text{Gr}_{\omega_1} \simeq \mathbb{P}^{n-1} \)); and (2) the special section (corresponding to the oper Borel reduction) of the universal oper bundle \( V_{\omega_1} \) satisfies an \( n \)th order differential equation (see Lemma 1.15). Hence we will be able to apply Corollary 3.12 in the case of the Hecke correspondence \( Z(\omega_1) \) to derive the differential equations (1.13) on the Hecke operator \( \hat{H}_{\omega_1} \) and thus prove Theorem 1.18.

Note that we also have the multiplicity one property (see Proposition 1.13 and Theorem 1.12). We will use the notation of Section 1.2.
4.1. **Hecke functor.** We follow the definition of the Hecke functor $\mathcal{H}^\bullet_{\omega_1}$ given in [BD1] (where it is denoted by $T^\bullet_{\omega_1}$), taking into account the fact that in this case the fibers of the morphism $q_2 \times q_3$ are isomorphic to $\text{Gr}_{\omega_1} \simeq \mathbb{P}^{n-1}$ and hence are smooth. The following definition is taken from [BD1], Sect. 5.2.4.

**Definition 4.1** ([BD1]). The Hecke functor

$$\mathcal{H}^\bullet_{\omega_1} : D^b(\mathcal{D}_{\text{Bun}_{PGL_n}}) \to D^b(\mathcal{D}_{\text{Bun}_{PGL_n} \times X})$$

is defined as follows. For a left $D$-module $M$ on $\text{Bun}_{PGL_n}$,

$$\mathcal{H}^\bullet_{\omega_1}(M) := (q_2 \times q_3)^D_*(q_1^*(M)),$$

where $(q_2 \times q_3)^D_*$ denotes the derived direct image functor for $D$-modules.

Denote by $\mathcal{H}^i_{\omega_1}$ the corresponding $i$th cohomology functor. We now recall a theorem of Beilinson and Drinfeld [BD1] describing the action of $\mathcal{H}^i_{\omega_1}$ on a specific $D$-module on $\text{Bun}_{PGL_n}$.

If $n$ is even, then to define this $D$-module, we need to pick a square root $K_1^{1/2}$ of the canonical line bundle on $X$ (as in Section 1.4). To it, one associates a specific square root

$$L = K_1^{1/2}$$

of the canonical line bundle on $\text{Bun}_{PGL_n}$ following [BD1], Sect. 4 (see also [LS]). If $n$ is odd, the construction of $L$ does not require any choices.

Let $\mathcal{D}_{\text{Bun}_{PGL_n}}$ be the sheaf of differential operators on $\text{Bun}_{PGL_n}$. Then

$$\mathcal{D}_{\text{Bun}_{PGL_n}} \otimes L^{-1}$$

is a left $D$-module on $\text{Bun}_{PGL_n}$ equipped with the commuting right action of the algebra

$$D_{PGL_n} = D^\beta_{PGL_n} = \Gamma(\text{Bun}_{PGL_n}, L \otimes \mathcal{D}_{\text{Bun}_{PGL_n}} \otimes L^{-1})$$

Recall from Definition 1.5 that the right hand side does not depend on $\beta$. Moreover, we have

$$D_{PGL_n} \simeq \text{Fun Op}_{\gamma SL_n}(X)$$

according to the result of [BD1] which is quoted in Theorem 1.6 above. Here $\gamma$ denotes the isomorphism class of $K_X^{1/2}$, as in Theorem 1.6.

4.2. **Hecke eigensheaf property.** Recall the left $D$-module $\mathcal{V}^\text{univ}_{\omega_1,X}$ on $X$ obtained from the universal oper bundle. It is equipped with a commuting action of the above algebra $D_{PGL_n}$. Furthermore, by definition of $SL_n$-opers, we have an embedding

$$K^\text{univ}_{\omega_1} : K_X^{(n-1)/2} \hookrightarrow \mathcal{V}^\text{univ}_{\omega_1,X}$$

and hence a section

$$s^\text{univ}_{\omega_1} \in \Gamma(X, K_X^{-(n-1)/2} \otimes \mathcal{V}^\text{univ}_{\omega_1,X}).$$
Recall the $n$th order differential operator (1.12). Lemmas 1.15 and 1.17 imply the following differential equation on $s_{\omega_1}^{univ}$ (this is an analogue of equation (2.18) in the case of $GL_1$):

$$\sigma \cdot s_{\omega_1}^{univ} = 0.$$  

We will now use this equation to derive the system (1.13).

Consider the isomorphism (1.2):

$$a : q_1^*(L^{1/2}) \cong q_2^*(L) \otimes K_2 \otimes q_3^*(K_X^{-(n-1)/2})$$

It gives rise to a section $\psi_{Z(\omega_1)}$ of $(L \otimes K_X^{-(n-1)/2}) \otimes \mathcal{D}_{\text{Bun}_{PGL_n}} \otimes L^{-1})$.

The first part of the following theorem is Theorem 5.2.9 of [BD1]. The second part follows from Theorems 5.4.11, 5.4.12, and Proposition 8.1.5 of [BD1].

**Theorem 4.2 ([BD1]).** $H^i_{\omega_1}(\mathcal{D}_{\text{Bun}_{PGL_n}} \otimes L^{-1}) = 0$ for $i \neq 0$, and

$$H^0_{\omega_1}(\mathcal{D}_{\text{Bun}_{PGL_n}} \otimes L^{-1}) \simeq (\mathcal{D}_{\text{Bun}_{PGL_n}} \otimes L^{-1}) \otimes D_{PGL_n} \omega_{1,X}^{univ}$$

as left $\mathcal{D}_{\text{Bun}_{PGL_n}} \otimes D_X$-modules equipped with a commuting action of $D_{PGL_n}$.

Moreover, the section $\psi_{Z(\omega_1)}$ of

$$(L \otimes K_X^{-(n-1)/2}) \otimes H^0_{\omega_1}(\mathcal{D}_{\text{Bun}_{PGL_n}} \otimes L^{-1}) \simeq (L \otimes \mathcal{D}_{\text{Bun}_{PGL_n}} \otimes L^{-1})_D \otimes (K_X^{-(n-1)/2} \otimes \omega_{1,X}^{univ})$$

coincides with $1 \otimes s_{\omega_1}^{univ}$.

Theorem 4.2 and equation (1.3) immediately imply:

**Corollary 4.3.** The section $\psi_{Z(\omega_1)}$ satisfies

$$\sigma \cdot \psi_{Z(\omega_1)} = 0.$$  

Now we derive Theorem 1.18 from Corollary 4.3 using Corollary 3.12 in the case of the Hecke correspondence $Z(\omega_1)$.

### 4.3. Proof of Theorem 1.18

We are going to derive Theorem 1.18 from Corollary 3.12 with $Z, Y_1, Y_2, L_1, L_2$, and $L'_2$ defined below. Initially, we would like to take $Y_1 = \text{Bun}_{PGL_n}^\circ, \ Y_2 = \text{Bun}_{PGL_n}^\circ \times X$, and $Z = Z(\omega_1)$ (see Section 1.2). However, in order for the morphism $Z \to Y_1 \times Y_2$ to be a closed embedding and the morphism $Z \to Y_2$ to be proper, we need to restrict $Z(\omega_1)$ to open dense subsets on both sides, as we now explain.

Recall from formula (1.3) that we have an open dense subvariety of $\text{Bun}_{PGL_n}^\circ$,

$$U_{PGL_n}(\omega_1) = \{ \mathcal{F} \in \text{Bun}_{PGL_n}^\circ \mid q_2(q_1^{-1}(\mathcal{F})) \subset \text{Bun}_{PGL_n}^\circ \},$$

which is dense by our assumption. It follows from [NR], Lemma 5.9 that there exist open dense subsets

$$Y_1 \subset U_{PGL_n}(\omega_1), \quad Y_2 \subset \text{Bun}_{PGL_n}^\circ \times X.$$  

\[3\text{We thank Tony Pantev for this reference and a helpful discussion. Note that the correspondence considered in [NR] differs from the Hecke correspondence } Z(\omega_1) \text{ in that one of the two bundles is dualized; but since the dual of a stable bundle is stable, we can use Lemma 5.9 of [NR] in our setting.} \]
such that the restriction $Z$ of $Z(\omega_1)$ to $Y_1 \times Y_2$ is a closed embedding and the corresponding map $q_2 \times q_3 : Z \to Y_2$ is proper. Denote by $p_1$ and $p_2$ the maps $q_1$ and $q_2 \times q_3$ restricted to $Y_1 \times Y_2$, respectively. Finally, set $\mathcal{L}_1 = K^1_{\text{Bun}}$ and $\mathcal{L}_2 = K^1_{\text{Bun}} \otimes K_X^{(n-1)/2}$. Thus, we have $|\mathcal{L}_1|^2 = \Omega^1_{\text{Bun}}$ and $|\mathcal{L}_2|^2 = \Omega^1_{\text{Bun}} \otimes \Omega_X^{-(n-1)/2}$. Then we have the isomorphism (3.16) which follows from the isomorphism (1.4) (see also formula (1.2)).

Now we are in the setting of Corollary 3.12 with $\mathcal{L}'_2 := K^1_{\text{Bun}} \otimes K_X^{(n+1)/2}$. Let $H_Z$ be the corresponding integral transform operator. We then obtain from Corollary 3.12 and equation (1.5) that $H_Z$ satisfies the system of differential equations

\begin{equation}
\sigma \cdot H_Z = 0, \quad \sigma^* \cdot H_Z = 0.
\end{equation}

Next, we relate $H_Z$ to our Hecke operator $\hat{H}_{\omega_1}$. Recall from Section 1.2 that the operator $\hat{H}_{\omega_1}$ is defined as the integral transform via the correspondence $Z(\omega_1)$ from the space $V_{\text{PGL}_n}(\omega_1)$ of smooth compactly supported sections of $|\mathcal{L}_1|^2 = \Omega^1_{\text{Bun}}$ on $U_{\text{PGL}_n}(\omega_1)$ to the space $V_{\text{PGL}_n} \otimes \Gamma(X, \Omega_X^{-(n-1)/2})$ of smooth sections of $|\mathcal{L}_2|^2 = \Omega^1_{\text{Bun}} \otimes \Omega_X^{(n+1)/2}$ on $\text{Bun}_{\text{PGL}_n} \times X$ (in fact, the image consists of compactly supported sections). Let $\tilde{H}_{\omega_1}$ be the restriction of $\hat{H}_{\omega_1}$ to the space of smooth sections of $|\mathcal{L}_1|^2$ that are compactly supported on $Y_1 \subset U_{\text{PGL}_n}(\omega_1)$, followed by the restriction to $Y_2$ of the resulting section of $|\mathcal{L}_2|^2$ on $\text{Bun}_{\text{PGL}_n} \times X$. Thus, $\tilde{H}_{\omega_1}$ acts from the space of smooth compactly supported sections of $|\mathcal{L}_1|^2$ on $Y_1$ to the space of smooth sections of $|\mathcal{L}_2|^2$ on $Y_2$.

It follows from the above definition that $\tilde{H}_{\omega_1}$ is the restriction of $H_Z$ to smooth compactly supported sections of $|\mathcal{L}_1|^2$ on $Y_1$. Therefore $\tilde{H}_{\omega_1}$ also satisfies the system (4.7). Since $Y_1$ is dense in $U_{\text{PGL}_n}(\omega_1)$ and $Y_2$ is dense in $\text{Bun}_{\text{PGL}_n} \times X$, this implies that $\tilde{H}_{\omega_1}$ also satisfies this system of equations. Thus, we obtain the system (1.13). This completes the proof of Theorem 1.13. \hfill \Box

Remark 4.1. It is possible to write an explicit formula for the differential operator $\sigma$ similar to formula (2.17) in the case of $\text{GL}_1$. For example, let $G = \text{PGL}_2$, so $^L G = \text{SL}_2$. Then the space $\text{Op}^0_{\text{SL}_2}(X)$ is an affine space over the vector space $H^0(X, K^2_X)$. Let us pick a point $\chi_0 \in \text{Op}^0_{\text{SL}_2}(X)$ and use it to identify $\text{Op}^0_{\text{SL}_2}(X)$ with $H^0(X, K^2_X)$. Let $\{\varphi_i, i = 1, \ldots, 3g-3\}$ be a basis of $H^0(X, K^2_X)$ and $\{F_i, i = 1, \ldots, 3g-3\}$ the set of generators of the polynomial algebra $\text{Fun} \text{Op}^0_{\text{SL}_2}(X)$ dual to this basis, i.e.

$$F_i(\chi_0 + \varphi_j) = \delta_{ij}.$$  

Let $\{D_i, i = 1, \ldots, 3g-3\}$ be the global holomorphic differential operators on $\text{Bun}_{\text{PGL}_2}$ corresponding to the $F_i$ under the isomorphism $\text{Fun} \text{Op}^0_{\text{SL}_2}(X) \simeq D^\gamma_{\text{PGL}_2}$.

By Lemma 1.15, we have an isomorphism $\text{Op}^0_{\text{SL}_2}(X) \simeq D^\gamma_2(X)$, where $D^\gamma_2(X)$ is the space of projective connections on $X$ (corresponding to our choice of $K^2_X$). It sends

$$\chi \in \text{Op}^0_{\text{SL}_2}(X) \mapsto P_\chi \in D^\gamma_2(X).$$
Then we can write
\[(4.8) \quad \sigma = P_{\chi_0} + \sum_{i=1}^{3g-3} D_i \otimes \varphi_i : K_X^{-1/2} \to D_{\text{PGL}_2} \otimes K_X^{3/2}\]
(it is clear that this operator does not depend on the choice of \(\chi_0\)).

Note that locally on \(X\), after choosing a local coordinate \(z\), we can write the second-order differential operator \(P_{\chi_0} \in D^2(X)\) in the form \(P_{\chi_0} = \partial^2_z + v_0(z)\).

This gives a more concrete realization of the equations (1.13). □

Remark 4.2. For \(\chi \in \text{Op}^\gamma_{\text{PGL}_n}(X) = \text{Spec} D_{\text{PGL}_n}\), let \(C_\chi\) be the corresponding one-dimensional \(D_{\text{PGL}_n}\)-module. In [BD1], Sect. 5.1.1, Beilinson and Drinfeld defined the following left \(D\)-module on \(\text{Bun}_{\text{PGL}_n}\):
\[\Delta_\chi^0 := (D_{\text{Bun}_{\text{PGL}_n}} \otimes \mathcal{L}^{-1}) \otimes_{D_{\text{PGL}_n}} C_\chi\]
(this is the \(D\)-module from Theorem 1.12). They derived from Theorem 4.2 that \(H^i_\omega(\Delta_\chi^0) = 0\) for \(i \neq 0\) and
\[H^0_\omega(\Delta_\chi^0) \simeq \Delta_\chi^0 \otimes (\mathcal{V}_{\omega_1}, \nabla_\chi)\]
(see [BD1], Theorem 5.2.6). This means that \(\Delta_\chi^0\) is a Hecke eigensheaf with respect to the flat \(SL_n\)-bundle corresponding to \(\chi\) [BD1]. In this sense, \(D_{\text{PGL}_n} \otimes \mathcal{L}^{-1}\) is a universal Hecke eigensheaf parametrized by the component \(\text{Op}^\gamma_{\text{SL}_n}(X)\) of the space of \(SL_n\)-opers on \(X\).

Recall the equivalence of categories in the abelian case obtained by restriction of the \(D\)-module version of the Fourier–Mukai transform discussed to Remark 2.3. It has a non-abelian analogue (see e.g. [FT]). In the case of \(G = \text{PGL}_n\), on one side we have the category of coherent sheaves on \(\text{Op}^\gamma_{\text{SL}_n}(X)\). On the other side, we have the category of \(D\)-modules \(\mathcal{K}\) on \(\text{Bun}_{\text{PGL}_n}\) with finite global presentation of the form
\[(D_{\text{Bun}_{\text{PGL}_n}} \otimes \mathcal{L}^{-1})\oplus_m \to (D_{\text{Bun}_{\text{PGL}_n}} \otimes \mathcal{L}^{-1})\oplus_r \to \mathcal{K} \to 0.
\]
The equivalence \(E\) takes an object \(\mathcal{F}\) of the former category to
\[E(\mathcal{F}) := (D_{\text{Bun}_{\text{PGL}_n}} \otimes \mathcal{L}^{-1}) \otimes_{D_{\text{PGL}_n}} F, \quad F := \Gamma(\text{Op}^\gamma_{\text{PGL}_n}(X), \mathcal{F}).\]
In particular, \(E(\mathcal{O}_{\text{Op}^\gamma_{\text{PGL}_n}(X)}) = D_{\text{Bun}_{\text{PGL}_n}} \otimes \mathcal{L}^{-1}\) and \(E(\mathcal{O}_\chi) = \Delta_\chi^0\). □

5. General case

In this section we formulate the analogues of Theorem 1.18 and Corollary 1.19 describing the eigenvalues of the Hecke operators in the case of an arbitrary simple Lie group \(G\) and outline their proof following the argument of the previous section.

The case of \(G = \text{PGL}_n\) and \(\lambda = \omega_1\), which we considered in the previous section, differs from the case of a general group \(G\) and dominant integral coweight \(\lambda \in P^\vee_+\) in two ways. First, in the case of \(G = \text{PGL}_n\) and \(\lambda = \omega_1\) we have \(Z(\lambda) = Z(\lambda)\), i.e. the fibers \((\eta_2 \times \eta_3)^{-1}(\mathcal{P}, x)\) of the Hecke correspondence \(Z(\omega_1)\) are isomorphic to
the $\text{PGl}_n[[z]]$-orbit $\text{Gr}_{\omega_1}$ in the affine Grassmannian of $\text{PGl}_n$ which is smooth and compact. But for general $G$ and $\lambda \in P_+^\vee$ these fibers are isomorphic to the closure of the $G[[z]]$-orbit $\text{Gr}_\lambda$ and are singular. Second, in the case of $G = \text{PGl}_n$ and $\lambda = \omega_1$, the canonical section of the universal oper bundle satisfies a scalar differential equation of Lemma 1.15 but for general $G$ and $\lambda$ this is not the case. Hence for general $G$ and $\lambda$ our construction needs some modifications, which we discuss in this section.

Namely, we formulate an analogue (Conjecture 5.1) of Corollary 1.19 describing the eigenvalues of Hecke operators in terms of real $L$-opers and an analogue (Conjecture 5.5) of Theorem 1.18 describing differential equations satisfied by the Hecke operators. Under the irreducibility assumption of Corollary 5.4, Conjecture 5.1 follows from Conjecture 5.5 in the same way as in the case of $\text{PGl}_n$.

We expect that Conjecture 5.5 can be derived from the Hecke eigensheaf property (established in [BD1] and recalled in Section 5.3) by an argument analogous to the one we used in the proof of Theorem 1.18 in Section 4.3 in the case of $\text{PGl}_n$. However, because the morphism $\overline{q}_2 \times \overline{q}_3$ is not smooth in the general case (in the sense of algebraic geometry), deriving Conjecture 5.5 from this result requires additional care. We leave the details to a follow-up paper.

5.1. Eigenvalues of the Hecke operators. Recall from the discussion before Theorem 1.6 that the space $\text{Op}_{L}^G(X)$ of $L$-opers on $X$ has a canonical component $\text{Op}_{L}^G_{\text{ad}}(X)$ isomorphic to the affine space $\text{Op}_{L}^G_{\text{ad}}(X)$. If the set $\{\langle \lambda, \rho \rangle, \lambda \in P_+^\vee\}$ contains half-integers, then to specify $\text{Op}_{L}^G_{\text{ad}}(X)$ we need to choose a square root $K_{1/2}^X$ of the canonical line bundle on $X$; then $\gamma$ denotes the isomorphism class of this $K_{1/2}^X$ which we also use to construct a square root $L = K_{1/2}^X$ of the canonical line bundle on $\text{Bun}_G$ (see the discussion before Theorem 1.1). Otherwise, the component $\text{Op}_{L}^G_{\text{ad}}(X)$ is well-defined without any choices.

For $\lambda \in P_+^\vee$, let $V_\lambda$ be the corresponding irreducible finite-dimensional representation of $L$. Given $\chi \in \text{Op}_{L}^G_{\text{ad}}(X)$, we obtain a flat holomorphic vector bundle $(V_{\chi, \lambda}, \nabla_{\chi, \lambda})$ on $X$. According to [BD1], §3, the vector bundles $V_{\chi, \lambda}$ are isomorphic to each other for all $\chi \in \text{Op}_{L}^G_{\text{ad}}(X)$. Hence we will use the notation $V_\chi$.

The oper Borel reduction gives rise to an embedding

$$\kappa_\lambda : K_{X}^{(\lambda, \rho)} \hookrightarrow V_\lambda$$

and hence

$$\tilde{\kappa}_\lambda : \mathcal{O}_X \hookrightarrow K_{X}^{-(\lambda, \rho)} \otimes V_\lambda.$$ 

Let

$$s_\lambda := \tilde{\kappa}_\lambda(1) \in \Gamma(X, K_{X}^{-(\lambda, \rho)} \otimes V_\lambda).$$

Now suppose that $\chi \in \text{Op}_{L}^G_{\text{ad}}(X)$. Then we have an isomorphism of $C^\infty$ flat bundles

$$(V_{\chi, \lambda}, \nabla_{\chi, \lambda}) \simeq (\tilde{V}_{\chi, \lambda}, \tilde{\nabla}_{\chi, \lambda}).$$

In [EFK3], Proposition 3.39, we derive the statement of Conjecture 5.1 from Conjecture 5.5 without the irreducibility assumption of Corollary 5.4.
and hence a pairing

\[ h_{\chi,\lambda}(\cdot, \cdot) : (V_{\chi, \lambda}, \nabla_{\chi, \lambda}) \otimes (V_{-w_0(\lambda)}, \nabla_{\chi, -w_0(\lambda)}) \to (C^\infty_X, d) \]

as \( V^*_\lambda \simeq V_{-w_0(\lambda)} \). Since \( \langle -w_0(\lambda), \rho \rangle = \langle \lambda, \rho \rangle \), we have

\[ s_{-w_0(\lambda)} \in \Gamma(X, K_X^{(\lambda, \rho)} \otimes \nabla_{-w_0(\lambda)}). \]

Recall that \( \text{Bun}_G \) has connected components \( \text{Bun}^\beta_G, \beta \in \pi_1(G) \), and we have a direct sum decomposition

\[ (5.2) \quad H_G = \bigoplus_{\beta \in \pi_1(G)} H^\beta_G. \]

According to Conjecture 1.11 for each \( \chi \in \text{Op}^\gamma_G(X)_R \) we have a non-zero eigenspace of \( A_G \) in \( H^\beta_G \) for all \( \beta \in \pi_1(G) \) (which is one-dimensional by Proposition 1.13 and Theorem 1.12). We expect that the Hecke operator \( H_\lambda \) preserves the direct sum of these subspaces (see Section 1.4 in the case of \( PGL_n \)). Moreover, one can find out precisely how \( H_\lambda \) permutes different \( H^\beta_G \) by analyzing the action of the center \( Z(G_\lambda) \) on \( V_\lambda \) (which is naturally identified with the group of characters of \( \pi_1(G) \)). As in the case of \( PGL_n \), this implies that the eigenvalues \( \{ \Phi_\chi(\tau) \} \) of \( H_\lambda \) corresponding to \( \chi \) form a torsor over the group \( \mu_{\alpha(G, \lambda)} \) of roots of unity of some order \( \alpha(G, \lambda) \); for example, \( \alpha(PGL_n, \omega_1) = n \) (this is so even for \( G = SO_{4n}/\{ \pm I \} \), when \( \pi_1(G) \simeq \mathbb{Z}_2 \times \mathbb{Z}_2 \)). This is why we describe these eigenvalues up to a scalar.

The following conjecture is an analogue of Corollary 1.19 for a general group \( G \).

**Conjecture 5.1.** For \( \chi \in \text{Op}^\gamma_G(X)_R \), the section \( \Phi_\chi(\tau) \in \Gamma(X, \Omega_X^{(\lambda, \rho)}) \) is equal to

\[ (5.3) \quad \Phi_\chi(\tau) = h_{\chi, \lambda}(s_\lambda, s_{-w_0(\lambda)}) \]

up to a scalar.

Let us choose a system of embeddings of representations of \( L_G \),

\[ (5.4) \quad \iota_{\lambda, \mu} : V_{\lambda + \mu} \hookrightarrow V_\lambda \otimes V_\mu, \quad \lambda, \mu \in P^\vee_+ \]

satisfying \( \iota_{\lambda, \mu} \circ \iota_{\lambda + \mu, \nu} = \iota_{\lambda, \nu} \circ \iota_{\lambda + \mu, \nu} \). These give rise to a system of linear maps

\[ \tilde{\iota}_{\lambda, \mu} : \Gamma(X, K_X^{-(\lambda+\mu, \rho)} \otimes V_{\lambda+\mu}) \to \Gamma(X, K_X^{-(\lambda, \rho)} \otimes V_\lambda) \otimes \Gamma(X, K_X^{-(\mu, \rho)} \otimes V_\mu) \]

**Lemma 5.2.** The sections \( s_\lambda, \lambda \in P^\vee_+ \), satisfy

\[ (5.5) \quad \tilde{\iota}_{\lambda, \mu}(s_{\lambda+\mu}) = s_\lambda \otimes s_\mu, \quad \forall \lambda, \mu \in P^\vee_+. \]

**Remark 5.1.** By Lemma 5.2, the sections \( \Phi_\chi(\tau) \) given by formula (5.3) satisfy

\[ (5.6) \quad \Phi_{\lambda+\mu}(\chi) = \Phi_\chi(\tau) \Phi_\mu(\tau), \quad \lambda, \mu \in P^\vee_+, \]

provided that the pairings \( h_{\chi, \lambda}, \lambda \in P^\vee_+ \), are normalized so that the restriction of \( h_{\chi, \lambda} \otimes h_{\chi, \mu} \) to the image of the embedding obtained from \( \iota_{\lambda, \mu} \) (see formula (5.4)) is equal to \( h_{\chi, \lambda+\mu} \). Formula (5.6) agrees with the relations \( H_{\lambda+\mu} = H_\lambda \cdot H_\mu \) satisfied by the Hecke operators according to Proposition 1.4.

As in the case of \( G = PGL_n \) (see Section 1.4), we expect that this normalization condition fixes the pairings \( h_{\chi, \lambda}, \lambda \in P^\vee_+ \) (and hence the eigenvalues \( \Phi_\chi(\tau) \) of the
Hecke operators) up to a root of unity of order $\alpha(G, \lambda)$. Conjecture 5.1 can then be refined to a statement that two torsors over the corresponding group $\mu_{\alpha(G, \lambda)}$ of roots of unity are equal to each other, similarly to Conjecture 1.14 for $G = PGL_n$. \hfill $\square$

5.2. **Analogue of the system of differential equations.** To prove Conjecture 5.1, we need an analogue of the system of differential equations which appear in Corollary 1.16 and Theorem 1.18 in the case of $PGL_n$ and $\lambda = \omega_1$. In that case, we used the possibility to interpret opers in terms of scalar differential operators of order $n$ (see Lemma 1.15). Such an interpretation is possible if $\lambda$ is such that $V_\lambda$ remains irreducible under a principal $\mathfrak{sl}_2$ subalgebra of $^Lg$. It is known that this happens for $\lambda = \omega_1$ and $\omega_\ell$ if $g$ is of type $A_\ell$, and for $\lambda = \omega_1$ if $g$ is of types $B_\ell$, $C_\ell$ and $G_2$. For $A_\ell$, $B_\ell$, and $C_\ell$, the corresponding scalar differential operators were described in [DrS, BD2]. In the general case, we replace the equations from Corollary 1.16 and Theorem 1.18 with a statement (Lemma 5.3) about the (twisted) $D$-module on $X$ obtained by applying all possible (twisted) differential operators to the canonical section $s_\lambda$.

Namely, let

$$D_{X, -(\lambda, \rho)} := K^{-\langle \lambda, \rho \rangle}_{X} \otimes_{\mathcal{O}_X} D_X \otimes_{\mathcal{O}_X} K^{\langle \lambda, \rho \rangle}_{X}$$

be the sheaf of differential operators acting on the line bundle $K^{-\langle \lambda, \rho \rangle}_{X}$ on $X$. Set

$$\mathcal{V}^{K}_{X, \lambda} := K^{-\langle \lambda, \rho \rangle}_{X} \otimes_{\mathcal{O}_X} \mathcal{V}_{\lambda}.$$  

The oper connection $\nabla_{X, \lambda}$ defines a $D_X$-module structure on $\mathcal{V}_{\lambda}$, and therefore a $D_{X, -(\lambda, \rho)}$-module structure on $\mathcal{V}^{K}_{X, \lambda}$. We will denote this $D_{X, -(\lambda, \rho)}$-module by $\mathcal{V}^{K}_{X, \lambda}$. Recall that we have a canonical section $s_\lambda \in \Gamma(X, \mathcal{V}^{K}_{X, \lambda})$.

**Lemma 5.3.** If the monodromy representation of the flat vector bundle $(\mathcal{V}_{\lambda}, \nabla_{X, \lambda})$ is irreducible\(^5\) then

$$D_{X, -(\lambda, \rho)} \cdot s_\lambda = \mathcal{V}^{K}_{X, \lambda}.$$ \hfill (5.7)

**Proof.** Note that $D_{X, -(\lambda, \rho)} \cdot s_\lambda$ is a non-zero $D_{X, -(\lambda, \rho)}$-submodule of $\mathcal{V}^{K}_{X, \lambda}$. If the monodromy representation of the flat vector bundle $(\mathcal{V}_{\lambda}, \nabla_{X, \lambda})$ is irreducible, then $(\mathcal{V}_{\lambda}, \nabla_{X, \lambda})$ is an irreducible $D_X$-module, and so $\mathcal{V}^{K}_{X, \lambda}$ is an irreducible $D_{X, -(\lambda, \rho)}$-module. Hence (5.7) follows. \hfill $\square$

\(^5\)This case is discussed in detail in [EFK3], Sect. 3.6.

\(^6\)Denote by $M_\chi$ the subgroup of $^Lg$ which is the Zariski closure of the monodromy of $\chi \in \text{Op}_{^Lg}(X)$. In [EFK3], Corollary 3.32, we prove that the subset of those $\chi$ for which $M_\chi = ^Lg$ (and so the flat vector bundles $(\mathcal{V}_{\lambda}, \nabla_{X, \lambda})$ are irreducible for all $\lambda \in P_{^Lg}^\vee$) is dense in $\text{Op}_{^Lg}(X)$. In general, according to [EFK3], Theorem 3.36, $M_\chi$ is a simple subgroup of $^Lg$ whose Lie algebra contains a principal $\mathfrak{sl}_2$ subalgebra of $^Lg$, and in this case $\chi$ is induced from an $M_\chi$-oper $\eta$ (if $\chi$ is a real oper, then so is $\eta$). For such $^Lg$-opers, $(\mathcal{V}_{\lambda}, \nabla_{X, \lambda})$ decomposes into a direct sum of irreducible flat vector bundles according to the decomposition of $V_\lambda$ into irreducible representations of $M_\chi$. 
Let $I_{\lambda, \chi}$ be the left annihilating ideal of $s_\lambda$ in the sheaf $\mathcal{D}_{X, \langle \lambda, \rho \rangle}$. Thus, we have an exact sequence of left $\mathcal{D}_{X, \langle \lambda, \rho \rangle}$-modules

$$0 \to I_{\lambda, \chi} \to \mathcal{D}_{X, \langle \lambda, \rho \rangle} \to \mathcal{V}^{K}_{\chi, \lambda} \to 0$$

Lemma 5.3 has the following immediate corollary.

**Corollary 5.4.** Suppose that the monodromy representation of the flat vector bundle $(\mathcal{V}_\chi, \nabla_{\chi, \lambda})$, where $\chi \in \text{Op}^G_G(X)_R$, is irreducible. Then $h_{\chi, \lambda}(s_\lambda, s_{-w_0(\lambda)})$ is a unique, up to a scalar, non-zero section $\Psi_{\lambda}(\chi)$ of $\Omega_{\chi, \lambda}^{\langle \lambda, \rho \rangle}$ annihilated by the ideals $I_{\lambda, \chi}$ and $I_{-w_0(\lambda), \chi}$.

**Remark 5.2.** For $G = \text{PGL}_n$, $\lambda = \omega_1$, let $I'_{\omega_1, \chi} := K_X^n \otimes I_{\omega_1, \chi}$, which is a left submodule of the $(\mathcal{D}_{X,(n+1)/2}, \mathcal{D}_{X,(-n+1)/2})$-bimodule of differential operators acting from $K_X^{(-n+1)/2}$ to $K_X^{(n+1)/2}$. It has the property that it is generated by a globally defined $n$th order differential operator $P_X$ on $X$ associated to $\chi$ by Lemma 1.15, i.e.

$$I'_{\omega_1, \chi} = \mathcal{D}_{X,(n+1)/2} \cdot P_X.$$ 

Therefore in this case a section annihilated by the ideal $I_{\lambda, \chi}$ is the same as a section satisfying the $n$th order differential equation (1.11).

For a general simple Lie group $G$, the ideal $I_{\lambda, \chi}$, or its twist such as $I'_{\lambda, \chi}$, doesn’t have such a generator. But as the above corollary shows, this is not necessary. What matters is the cyclicity of the $D$-module $\mathcal{V}^{K}_{\chi, \lambda}$ (formula (5.7)).

We note that it is the cyclicity of two types of twisted $D$-modules: $\mathcal{V}^{K}_{\chi, \lambda}$ on $X$ and $\Delta_\chi$ on $\text{Bun}_G$ (see formula (1.8)), that enables us to link the geometric Langlands correspondence and the analytic one (see Remark 3.1).

**Remark 5.3.** Note that a non-zero section $\Psi_{\lambda}(\chi)$ from Corollary 5.4 satisfies

$$(\mathcal{D}_{X, \langle \lambda, \rho \rangle} \otimes \mathcal{D}_{X, \langle \lambda, \rho \rangle}) \cdot \Psi_{\lambda}(\chi) \simeq \mathcal{V}^{K}_{\chi, \lambda} \otimes \mathcal{V}^{K}_{\chi, -w_0(\lambda)}.$$ 

**□**

Corollary 5.4 is an analogue of Corollary 1.16. We are going to formulate a conjectural analogue of Theorem 1.18 (Conjecture 5.5) in a similar way.

Let $\mathcal{V}^{\text{univ}}_{\lambda}$ be the universal vector bundle over $\text{Op}^G_G(X) \times X$ with a partial connection $\nabla^{\text{univ}}$ along $X$, such that

$$(\mathcal{V}^{\text{univ}}_{\lambda}, \nabla^{\text{univ}})|_{X \times X} = (\mathcal{V}_{\lambda}, \nabla_{\chi, \lambda}), \quad \chi \in \text{Op}^G_G(X).$$

Let $\pi : \text{Op}^G_G(X) \times X \to X$ be the projection and set

$$\mathcal{V}^{\text{univ}}_{X, \lambda} := \pi_* (\mathcal{V}^{\text{univ}}_{\lambda}).$$

7 In [EFK3], Proposition 3.39, we prove the statement of Corollary 5.4 in general (without assuming that the monodromy representation of $(\mathcal{V}_{\lambda}, \nabla_{\chi, \lambda})$ is irreducible). In [EFK3], Sect. 3.9, we use this result to describe an analogue of the Langlands functoriality principle in the analytic Langlands correspondence for curves over $\mathbb{C}$.
Then $K_X^{-(\lambda,\rho)} \otimes V_{X,\lambda}^{\text{univ}}$ is naturally a $\mathcal{D}_{X,-(\lambda,\rho)}$-module on $X$, equipped with a commuting action of $\text{Fun Op}^G_\gamma(X) \simeq D_G$.

Moreover, the oper Borel reduction gives rise to an embedding

$$\kappa^{\text{univ}}_\lambda : K_X^{-(\lambda,\rho)} \hookrightarrow V_{X,\lambda}^{\text{univ}}$$

and hence a canonical section

$$s^{\text{univ}}_\lambda \in \Gamma(X, K_X^{-(\lambda,\rho)} \otimes V_{X,\lambda}^{\text{univ}}).$$

Consider the cyclic $D_G \otimes \mathcal{D}_{X,-(\lambda,\rho)}$-module generated by $s^{\text{univ}}_\lambda$:

$$V_{X,\lambda}^{K,\text{univ}} := (D_G \otimes \mathcal{D}_{X,-(\lambda,\rho)}) \cdot s^{\text{univ}}_\lambda.$$ 

Now recall that the Hecke operator $\hat{H}_\lambda$ is a section of $\Omega_X^{-(\lambda,\rho)}$ with values in operators $V_G(\lambda) \to V_G$. Hence we can apply to it the sheaf $\mathcal{D}_{X,-(\lambda,\rho)}$, as well as the algebra $D_G$, through its action on $V_G$. The two actions commute, and they generate a $\mathcal{D}_{X,-(\lambda,\rho)}$-module inside the sheaf of $C^\infty$ sections of $\Omega_X^{-(\lambda,\rho)}$ on $X$ with values in operators $V_G(\lambda) \to V_G$. Let us denote this $\mathcal{D}_{X,-(\lambda,\rho)}$-module by $\langle H_\lambda \rangle$.

Similarly, we can apply to $\hat{H}_\lambda$ the sheaf $\mathcal{D}_{X,-(\lambda,\rho)}$ and the algebra $\overline{D}_G$. Denote the resulting $\mathcal{D}_{X,-(\lambda,\rho)}$-module by $\overline{\langle H_\lambda \rangle}$. The following is an analogue of Theorem 1.18 for a general group $G$.

**Conjecture 5.5.** There are isomorphisms

$$(5.8) \quad \langle H_\lambda \rangle \simeq V_{X,\lambda}^{K,\text{univ}}, \quad \overline{\langle H_\lambda \rangle} \simeq \overline{V}_{X,\lambda}^{K,\text{univ}}$$

of $D_G \otimes \mathcal{D}_{X,-(\lambda,\rho)}$-modules (resp. $\overline{D}_G \otimes \mathcal{D}_{X,-(\lambda,\rho)}$-modules).

In the case $G = PGL_n, \lambda = \omega_1$, this conjecture is equivalent to the statement of Theorem 1.18.

Suppose that the monodromy representation of the flat vector bundle $(V_\chi, \nabla_{X,\lambda})$, where $\chi \in \text{Op}_{\gamma}^G(X)_R$, is irreducible. Then Conjecture 5.5 follows from Conjecture 5.1 and Corollary 5.4 in the same way as Corollary 1.19 follows from Theorem 1.18 and Corollary 1.16 in the case $G = PGL_n, \lambda = \omega_1$.

### 5.3. Hecke eigensheaf property

As in the case of $PGL_n$, we wish to derive Conjecture 5.5 using the formalism of Section 3.4 from the Hecke eigensheaf property established by Beilinson and Drinfeld [BD1].

We start by recalling the definition of the Hecke functor in the general case from [BD1].

We will use the notation of Section 1.2. Consider the Hecke correspondence $\overline{Z}(\lambda)$. The fibers of the morphism $\overline{q}_2 \times \overline{q}_3$ are isomorphic to the closure $\overline{\text{Gr}}_\lambda$ of the $G[[z]]$-orbit $\text{Gr}_{\lambda}$ in the affine Grassmannian of $G$. We have denoted by $Z(\lambda)$ the open dense part of $\overline{Z}(\lambda)$ such that the fibers of $q_2 \times q_3$ restricted to $Z(\lambda)$ are isomorphic to $\text{Gr}_{\lambda}$, and we have denoted by $q_i$ the restriction of the morphism $\overline{q}_i$ to $Z(\lambda)$.

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8This irreducibility assumption has been removed in [EFK3], Proposition 3.39.
The morphism $\mathfrak{f}_2 \times \mathfrak{f}_3$ is smooth (in the sense of algebraic geometry) if and only if $\lambda$ is minuscule. (In this case, we also have $\text{Gr}_\lambda = \text{Gr}_\lambda$.)

The following definition is taken from [BD1], Sect. 5.2.4.

**Definition 5.6 ([BD1]).** Let $M$ be a left $D$-module on $\text{Bun}_G$. Denote by $q_1^\dagger(M)$ the intermediate extension to $Z(\lambda)$ of $q_1^\dagger(M)$. (Locally, we can choose an isomorphism $Z(\lambda) \simeq \text{Bun}_G \times \text{Gr}_\lambda \times X$ so that $q_1$ is the projection on the first factor; then $q_1^\dagger(M)$ can be identified with the exterior tensor product of $q_1^\dagger(M)$, the irreducible $D$-module on $\text{Gr}_\lambda$, and $\mathcal{O}_X$.) The Hecke functor is defined by the following formula:

\[(5.9) \quad H^i_\lambda(M) := (q_2 \times q_3)^D(q_1^\dagger(M)).\]

Now consider the left $D$-module $\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}$, where $\mathcal{L} = K_{\text{Bun}}^{1/2}$. It is equipped with the commuting right action of $D_G = \Gamma(\text{Bun}_G^\beta, \mathcal{L} \otimes \mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \simeq \text{Fun Op}_{\mathcal{L}}^\gamma(X)$ (see [BD1], Sect. 5.1.1).

The isomorphism (1.2) gives rise to a section $\psi_{Z(\lambda)}$ of $(\mathcal{L} \otimes K_X^{-(\lambda, \rho)}) \otimes H^0_\lambda(\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1})$.

The following theorem is due to [BD1] (see the references before Theorem 4.2 above).

**Theorem 5.7 ([BD1]).** $H^i_\lambda(\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) = 0$ for $i \neq 0$, and

\[H^0_\lambda(\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \simeq (\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \underset{D_G}{\boxtimes} V^\text{univ}_{\lambda, X}\]

as left $\mathcal{D}_{\text{Bun}_G} \boxtimes \mathcal{D}_X$-modules equipped with a commuting action of $D_G$. Moreover, the section $\psi_{Z(\lambda)}$ of

\[(\mathcal{L} \otimes K_X^{-(\lambda, \rho)}) \otimes H_\lambda(\mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \simeq (\mathcal{L} \otimes \mathcal{D}_{\text{Bun}_G} \otimes \mathcal{L}^{-1}) \underset{D_G}{\boxtimes} (K_X^{-(\lambda, \rho)} \otimes V^\text{univ}_{\lambda, X})\]

coincides with $1 \boxtimes s^\text{univ}_\lambda$.

Theorem 5.7 implies

**Corollary 5.8.** There is an isomorphism

\[(5.10) \quad (D_G \otimes \mathcal{D}_{X, -(\lambda, \rho)}) \cdot \psi_{Z(\lambda)} \simeq V_{X, \lambda}^{K, \text{univ}}\]

of $D_G \otimes \mathcal{D}_{X, -(\lambda, \rho)}$-modules.

Conjecture 5.5 (and hence Conjecture 5.1 if the monodromy representation of the flat vector bundle $(V_\lambda, \nabla_{\lambda, \chi})$ is irreducible) can be derived from Corollary 5.8 by adapting the results of Section 3.4 to the present situation (similarly to what we did in the proof of Theorem 1.18 in Section 4.3 in the case of $PGL_n$). For non-minuscule $\lambda$, this requires additional care since the morphism $\mathfrak{f}_2 \times \mathfrak{f}_3$ is not smooth in this case (in the sense of algebraic geometry). We leave the details to a follow-up paper.
5.4. **Proof of Theorem 1.1.** In this subsection we derive Theorem 1.1 from a local statement about line bundles on the affine Grassmannian given in formula (241) of [BD1] (it is reproduced in formula (5.13) below). All ingredients are contained in [BD1]. We include the argument here for completeness.

For a point \( x \) of our curve \( X \), let \( F_x \) be the formal completion of the field of rational functions on \( X \) at \( x \) and \( O_x \) its ring of integers. We will also use the notation \( O = \mathbb{C}[[z]], F = \mathbb{C}(z) \). Let \( \text{Aut} \ O \) be the group of automorphisms of \( O \). It naturally acts on the formal loop group \( G(F) \) preserving the subgroup \( G(O) \). Hence we obtain an action of \( \text{Aut} \ O \) on the affine Grassmannian \( \text{Gr} = G(F)/G(O) \), which preserves the \( G(O) \)-orbits \( \text{Gr}_\lambda, \lambda \in P^\vee_+ \).

Consider first the case when the set \( \{ \langle \lambda, \rho \rangle, \lambda \in P^\vee_+ \} \) only contains integers. Let \( \hat{M} \) be the ind-scheme defined in [BD1], Sect. 2.8.3, which parametrizes quadruples \( (x, t_x, F, \gamma_x) \), where \( x \) is a point of our curve \( X \), \( t_x \) is a formal coordinate at \( x \) (so that we can identify \( O_x \) with \( \mathbb{C}[[t_x]] \)), \( F \) is a \( G \)-bundle on \( X \), and \( \gamma_x \) is a trivialization of \( F \) on the disc \( D_x = \text{Spec} \ O_x \). The projection
\[
\hat{M} \rightarrow \text{Bun}_G \times X
\]
\[
(x, t_x, F, \gamma_x) \mapsto (F, x)
\]
is a torsor for the group \( \text{Aut} \ O \ltimes G(O) \), which naturally acts on \( t_x \) and \( \gamma_x \). We use it to construct a functor \( F \) sending a (ind-)scheme \( Y \) with an action of \( \text{Aut} \ O \ltimes G(O) \) to a (ind-)scheme over \( \text{Bun}_G \times X \),
\[
F(Y) = Y := \hat{M} \times_{\text{Aut} \ltimes G(O)} Y
\]
This is a generalization of the Gelfand–Kazhdan functor [GK] from (ind-)schemes equipped with an action of the group \( \text{Aut} \ O \) to (ind-)schemes over \( X \). Applying it to \( \text{Gr}_\lambda \), we obtain the scheme
\[
F(\text{Gr}_\lambda) = \mathcal{G}_{r\lambda} := \hat{M} \times_{\text{Aut} \ltimes G(O)} \text{Gr}_\lambda
\]
over \( \text{Bun}_G \times X \). Denote by \( r \) the projection \( \mathcal{G}_{r\lambda} \rightarrow \text{Bun}_G \times X \).

**Proposition 5.9** ([BD1], Sect. 5.2.2(ii)). There is a natural isomorphism \( Z(\lambda) \simeq \mathcal{G}_{r\lambda} \) under which the projection \( q_2 \times q_3 : Z(\lambda) \rightarrow \text{Bun}_G \times X \) is identified with \( r \).

Next, in [BD1], Sect. 4.6, a local Pfaffian line bundle was defined on \( \text{Gr} \). We will denote it by \( \mathcal{L}_{\text{Gr}} \). According to the construction, \( \mathcal{L}_{\text{Gr}} \) is \( \text{Aut} \ltimes G(O) \)-equivariant, and hence so is its restriction \( \mathcal{L}_{\text{Gr}_\lambda} \) to the \( G(O) \)-orbit \( \text{Gr}_\lambda \). Applying the above functor \( F \) to it, we obtain a line bundle on \( \mathcal{G}_{r\lambda} \), which we will denote by \( \mathcal{L}_{\mathcal{G}_{r\lambda}} \).

In [BD1], the line bundle \( \mathcal{L}_{\text{Gr}_\lambda} \) was described explicitly. To explain this result, we need to define a certain one-dimensional representation of \( \text{Aut} \ltimes G(O) \). Namely, let us assign to every element \( \phi \) of \( \text{Aut} \) the the image of \( z \in \mathbb{C}[[z]] = O \) under \( \phi \). This assignment sets up a bijection between \( \text{Aut} \) and the space of formal power series
\[
\phi(z) = \sum_{n \geq 0} \phi_n z^{n+1} \in \mathbb{C}[[z]],
\]

where $\phi_0$ is invertible. In particular, we obtain a canonical homomorphism $\gamma : \text{Aut} \ O \rightarrow \mathbb{G}_m, \phi(z) \mapsto \phi_0$.

**Definition 5.10.** For $n \in \mathbb{Z}$, let $\sigma(n)$ be the one-dimensional representation of $\text{Aut} \ O$ on which it acts via the composition of $\gamma$ and the character of $\mathbb{G}_m$ raising $\phi_0$ to the power $n$. We extend it to a one-dimensional representation (denoted in the same way) of $\text{Aut} \ O \ltimes G(O)$.

**Theorem 5.11 ([BD1], formula (241)).** There is a canonical isomorphism of $\text{Aut} \ O \ltimes G(O)$-equivariant line bundles on $\text{Gr}_\lambda$,

\begin{equation}
\mathcal{L}_{\text{Gr}_\lambda} \simeq K_{\text{Gr}_\lambda} \otimes \sigma_\lambda,
\end{equation}

where $K_{\text{Gr}_\lambda}$ is the canonical line bundle on $\text{Gr}_\lambda$ and $\sigma_\lambda := \sigma(\langle \lambda, \rho \rangle)$.

We now derive Theorem 1.1 from this result. First, we need the following statement which is proved e.g. in [FB], Sect. 6.4.

**Lemma 5.12.** Under the functor $\mathcal{F}$ introduced above, the representation $\sigma(n)$ of $\text{Aut} \ O \ltimes G(O)$ goes to the line bundle $r^*_X(K^n_X)$, where $r_X$ is the projection $\mathcal{G}r_\lambda \rightarrow \text{Bun}_G \times X \rightarrow X$.

The functor $\mathcal{F}$ also sends $K_{\text{Gr}_\lambda}$ to the relative canonical line bundle of the morphism $r : \mathcal{G}r_\lambda \rightarrow \text{Bun}_G \times X$, which by Proposition 5.9 is the line bundle $K_2$ introduced in Section 1.2. We also have $r_X = q_3$ under the isomorphism of Proposition 5.9. Therefore, Theorem 5.11 and Lemma 5.12 imply that there is a canonical isomorphism

\begin{equation}
\mathcal{L}_{\mathcal{G}r_\lambda} \simeq K_2 \otimes q_3^*(K_2)^{-1}.
\end{equation}

On the other hand, under our current assumption that the set $\{\langle \lambda, \rho \rangle, \lambda \in P^+_\mathfrak{g} \}$ only contains integers, Beilinson–Drinfeld construction in [BD1], Sect. 4.4.1, produces a square root of the canonical line bundle on $\text{Bun}_G$ (normalized by a trivialization of its fiber at the trivial $G$-bundle). We will denote it by $K_{\text{Bun}}^{1/2}$. The results of [BD1], Sects. 4.4.14 and 4.6, imply that under the isomorphism of Proposition 5.9 we have a canonical identification

\begin{equation}
\mathcal{L}_{\mathcal{G}r_\lambda} \simeq q_1^*(K_{\text{Bun}}^{1/2}) \otimes q_2^*(K_{\text{Bun}}^{1/2})^{-1}.
\end{equation}

Combining the isomorphisms (5.14) and (5.15), we obtain the isomorphism (1.2). This completes the proof of Theorem 1.1 under this assumption.

Now suppose that the set $\{\langle \lambda, \rho \rangle, \lambda \in P^+_\mathfrak{g} \}$ contains half-integers. Then we modify the above argument as follows. Define a double cover $\text{Aut}_2 O$ of the group $\text{Aut} O$ as the subgroup of the group $\text{Aut} O \ltimes \mathbb{G}_m$ consisting of pairs $(\phi, w)$, where $\phi$ is given by formula (5.12) and $w^2 = \phi_0$. Define the homomorphism $\gamma_2 : \text{Aut} O \rightarrow \mathbb{G}_m$ by the formula

\begin{equation}
\gamma_2(\phi, w) = w.
\end{equation}

Next, we fix a square root $K_X^{1/2}$ of the canonical line bundle $K_X$ on $X$. As explained in [BD1], Sect. 4.3.16, we can then extend the above functor $\mathcal{F}$ to a functor $\mathcal{F}_2$ from (ind-)schemes $Y$ with an action of $\text{Aut}_2 O \ltimes G(O)$ to (ind-)schemes over $\text{Bun}_G \times X$,
which has the following defining property. Let \( \sigma(m), m \in \frac{1}{2}\mathbb{Z} \), be the one-dimensional representation of \( \text{Aut}_2 \mathcal{O} \times G(O) \) on which \( G(O) \) acts trivially and \( \text{Aut}_2 \mathcal{O} \) acts as the composition of \( \gamma_2 \) given by formula (5.16) and the one-dimensional representation of \( \mathbb{G}_m \) given by \( w \mapsto w^{2m} \). Then \( \mathbf{F}_2 \) sends \( \sigma(m) \) to the line bundle \( r^*_\lambda(K^m_X) \) for all \( m \in \frac{1}{2}\mathbb{Z} \) (here \( K^m_X \) stands for \( (K^{1/2}_X)^{2m} \), where \( K^{1/2}_X \) is the chosen square root of \( K_X \)).

As shown in [BD1], a local Pfaffian line bundle \( \mathcal{L}_{Gr} \) on \( \text{Gr} \) can still be defined in this case, but it is now \( \text{Aut}_2 \mathcal{O} \times G(O) \)-equivariant. Moreover, the isomorphism (5.13) then holds as an isomorphism of \( \text{Aut}_2 \mathcal{O} \times G(O) \)-equivariant line bundles on \( \text{Gr}_{\lambda} \) and we also have the isomorphism (5.15), where \( K^{1/2}_{\text{Bun}} \) denotes the square root of the canonical line bundle on \( \text{Bun}_G \) associated to the above choice of \( K^{1/2}_X \) (see the discussion before Theorem 1.1). Therefore the same argument proves the isomorphism (1.2) in general. This completes the proof of Theorem 1.1.

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