General Relativity as a Theory of Two Connections

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ABSTRACT

We show in this paper that it is possible to formulate General Relativity in a phase space coordinatized by two $SO(3)$ connections. We analyze first the Husain-Kuchar model and find a two connection description for it. Introducing a suitable scalar constraint in this phase space we get a Hamiltonian formulation of gravity that is close to the Ashtekar one, from which it is derived, but has some interesting features of its own. Among them a possible mechanism for dealing with the degenerate metrics and a neat way of writing the constraints of General Relativity.

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I Introduction

After the introduction by Ashtekar [1] of a new set of variables to describe the gravitational field we have a picture of classical and quantum gravity that is very different from the geometrodynamical point of view that has prevailed during the last decades (see, for example, [2] [3]). The possibility of restating most of the problems faced by the old program in the new language and the insights gained because of the geometrical nature of the Ashtekar variables have shed light on many important issues and have helped to gain a new perspective in space-time Physics. A key ingredient of the Ashtekar approach is the use of a connection as the basic variable to describe the gravitational field (the frame fields appear only as the momenta canonically conjugate to this connection).

From a geometrical point of view, connections and metrics are very different objects. They find also very different uses in Theoretical Physics. Whereas metrics appear mainly in the context of gravity, connections are a key element in the description of the electroweak and strong interactions. There are lots of results about connections that have no analogue in the case of metrics and can be successfully exploited in the Ashtekar formalism. For example, Wilson loops, that were introduced in the context of Yang-Mills theories, are at the root of the loop-variables approach to the quantization of gravity. The fact that gravity can be described in terms of connections is very appealing, as emphasized by Ashtekar [4], because for the first time all four interactions have something in common in their mathematical formulations; all of them can be formulated in a Yang-Mills phase space.

The purpose of this paper is to show another description of gravity that, although close to the Ashtekar formulation, from which it is derived, has some features of its own that may make it useful in order to better understanding gravity and generally covariant theories. For example, the issue of the degenerate metrics can be related, within the framework we are going to work, to the non-degeneracy of the symplectic 2-form that appears in the Hamiltonian description of the theory. An important point in our approach is the fact that the phase space is now different from the usual
one (connection-densitized triad). In fact, the phase space variables will be now two $SO(3)$ connections. As a consequence of this, the symplectic structure will be non-trivial (notice that connections alone cannot be momenta because they have zero density weight). The ultimate goal of this approach is that of taking advantage of the availability of geometrical objects absent (or, at least, not obvious) in the Ashtekar formulation that might eventually allow us to find a set of elementary variables suitable for the quantization of the theory. This idea is in line with the suggestions made by Isham [5] that non-canonical algebras may be the way to quantize gravity and is close, in this sense, to the loop-variables approach. We want to emphasize from the beginning that the final formulation of gravity that we give in the paper is purely in terms of connections but different from the Capovilla-Dell-Jacobson one [6]. The starting point of these authors is a pure connection action but the Hamiltonian formulation that they find essentially coincides with the Ashtekar one because they use the same variables in the phase space.

The paper is structured as follows. In section 2 we show that the Husain-Kuchař model can be interpreted as a theory of two connections. This is an interesting toy model for the study of gravity and diffeomorphism invariant theories in general and it is simple to analyze because it has no dynamics. All the complications associated with the Hamiltonian constraint are absent simply because there is no scalar constraint in the theory. We will give a new action for this model written in terms of two connections and study the Hamiltonian formulation, phase space structure and constraints of the theory. The action that we will use bears some resemblance with the BF-actions studied by Horowitz [7] and other authors as exactly solvable diff-invariant theories.

In section 3 we show that there is a change of coordinates that will allow us to recover the usual Gauss law and vector constraint written in terms of the usual Ashtekar variables. This transformation will be used to write the Hamiltonian constrain in terms of the two connections. In passing we will find an interesting type of canonical transformation in the new phase space.

Section 4 is devoted to the study of gravity in the two-connection formulation.
An important issue that will be considered here is that of the degenerate metrics. The key observation is that the necessary and sufficient condition for the 3-metric to be non-degenerate is the non-degeneracy of the symplectic 2-form. It is this consistency condition for the Hamiltonian formulation that forces us to restrict ourselves to non-degenerate metrics. Taking advantage of this non-degeneracy we will find an appealing way of writing the constraints of general relativity.

We end the paper with the conclusions and a brief discussion of the open questions related to the two connection approach to gravity.

II The Husain-Kuchař Model as a Two Connection Theory

The Husain-Kuchař model \[8\] is a very interesting example of a diffeomorphism invariant theory. At variance with most diff-invariant theories appearing in the literature it has an infinite number of local degrees of freedom (three per space point) and yet it is simpler than general relativity because the scalar constraint (one of the key sources of trouble in gravity) is not present. The constraints of the model are the Gauss law, that generates \(SO(3)\) gauge transformations and the vector constraint that (combined with the Gauss law) generates diffeomorphisms on the ”spatial” slices of the 3+1 decomposition. These constraints are first class in Dirac’s terminology \[9\]. The absence of the Hamiltonian constraint can be interpreted by saying there is no dynamics, or rather, time evolution. In this section we show that the Husain-Kuchař model can be described as a pure connection theory. This is somewhat similar to what Capovilla, Dell, and Jacobson did for gravity \[6\], the key difference is that now we will introduce two connections instead of one.

The Husain-Kuchař action is:

\[
S = \frac{1}{2} \int_M d^4x \tilde{\eta}^{abcd} F^i_{ab} e^j_c e^k_d \epsilon_{ijk} \tag{1}
\]

where \(e^i_a\) are \(SO(3)\) valued frame fields, \(F^i_{ab}\) is the curvature of the \(SO(3)\) connection \(A^i_a\) (given by \(F^i_{ab} = 2\partial_{[a} A^i_{b]} + \epsilon^{ijk}_{ab} A^j_{ak}\)), \(\tilde{\eta}^{abcd}\) is the 4-dimensional Levi-Civita tensor
density (we will use Ashtekar’s notation and represent the density weights with tildes above or below the fields) and \( \epsilon_{ijk} \) is the internal Levi-Civita tensor. In the following we will restrict our attention to space-time 4-manifolds of the form \( M = \Sigma \times \mathbb{R} \) with \( \Sigma \) a compact 3-manifold. Tangent space indices will be represented by Latin letters from the beginning of the alphabet and internal indices with Latin letters from the middle of the alphabet. We will make no distinction between the indices of 4-dimensional and 3-dimensional objects; it will be clear from the context which kind of field we are talking about.

The key observation to pass from (1) to the two-connection action is that a \( SO(3) \) connection \( A_i^a \) has the same index structure as the frame fields \( e_a^i \) (remember that connections, in the adjoint representation, carry always two indices. It is only the fact that \( SU(2) \) connections are antisymmetric in their internal indices and the availability of \( \epsilon_{ijk} \) that allows us to represent both indices as one). One can then conceivably write \( e_a^i \) as the difference of two \( SO(3) \) connections \( \hat{A}_a^i, \hat{\hat{A}}_a^i \). If one further imposes the condition that the theory should be symmetric under the interchange of both connections it is natural to consider the action:

\[
S = \frac{1}{2} \int_M d^4x \ \bar{\eta}^{abcd} \hat{F}_{abc} \hat{F}_{cdi}
\]

where \( \hat{F}_{abc}, \hat{\hat{F}}_{abc} \) are the curvatures of \( \hat{A}_a^i, \hat{\hat{A}}_a^i \) respectively. In order to see that (2) and (1) describe the same theory we write:

\[
\hat{F}_{abc} = \hat{F}_{abc} + 2 \Gamma_{[a}^b \hat{F}_{c]}^{cdi} + \epsilon_{ijk} e_{aj} e_{bk}
\]

Introducing (3) in (2) we get:

\[
S = \frac{1}{2} \int_M d^4x \ \bar{\eta}^{abcd} \left[ \hat{F}_{abc} \hat{F}_{cdi} + 2 \Gamma_{[a}^b \hat{F}_{c]}^{cdi} + \epsilon_{ijk} e_{aj} e_{bk} \right]
\]

\( ^2 \)We may extend the action of \( \nabla_a \) to space-time indices by introducing a torsion free connection \( \Gamma_{ab}^c \). Our results will be independent of this extension.
The first term in the previous expression is topological and thus we can discard it when checking that (2) and (3) give the same dynamics. The covariant derivative in the second term can be integrated by parts and then the Bianchi identity \( \nabla [a \tilde{F}^i_{bc}] = 0 \) tells us that this term does not contribute to the action either. Finally the third term coincides with the one appearing in (2) provided that we make the identifications \( \tilde{A}_a^i \equiv \hat{A}_a^i \) and \( e_{ai} \equiv \tilde{A}_{ai} - \hat{A}_{ai} \). Notice that in order to get the field equations it is equivalent to vary the action with respect to \( \tilde{A}_a^i, e_{ai}^i \) or \( \hat{A}_a^i, \hat{A}_{ai}^i \).

It is interesting to point out that the action (2) becomes a topological invariant when \( \tilde{A}_a^i = \hat{A}_a^i \). The field equations derived from (3) take a very simple form:

\[
\begin{align*}
\nabla [a \tilde{F}^i_{bc}] &= 0 \\
\nabla [a \hat{F}^i_{bc}] &= 0
\end{align*}
\]

when \( \tilde{A}_a^i = \hat{A}_a^i \) they reduce to the Bianchi identities and hence they are identically satisfied (a reflection of the “triviality” of the action when \( \tilde{A}_a^i = \hat{A}_a^i \)). The simple structure of (3) suggests some interesting classes of solutions. For example, if \( \tilde{A}_a^i \) and \( \hat{A}_a^i \) are such that \( \tilde{F}_{ab}^i = \kappa \hat{F}_{ab}^i \) with \( \kappa \) constant then (3) is trivially satisfied as a consequence of the Bianchi identities. This type of solutions is not obvious if one looks at the field equations as derived from (2):

\[
\begin{align*}
\tilde{\eta}^{abcd} \epsilon_i^j \epsilon_k^l (\nabla a e_{bj}) e_{ck} &= 0 \\
\hat{\eta}^{abcd} \epsilon_i^j e_{aj} F_{bck} &= 0
\end{align*}
\]

(Their equivalence with (5) can be easily checked by using (3), the definition of the curvature as the commutator of covariant derivatives and the Bianchi identities).

We will concentrate now on the discussion of the Hamiltonian form of the theory described by the action (2). Although the final description we will get is equivalent to the usual Husain-Kuchař model the phase space will be different now (because instead of the connection and densitized triads the coordinates in the new phase space will be

\[^{3}\text{In (2), } F_{ab}^i \text{ is the curvature of the connection } A_a^i \text{ used to define the covariant derivative } \nabla\]
two $SO(3)$ connections). This may have some interesting consequences when trying to quantize the theory because we may have now the possibility of finding sets of elementary variables different from the connection-densitized triad pair or the loop variables.

In order to perform the 3+1 decomposition we introduce a foliation of $\mathcal{M}$ given by 3-surfaces of constant $t$ (where $t$ is a scalar function defined on $\mathcal{M}$). We need also a vector field $t^a$ satisfying the condition $t^a \partial_a t = 1$. With the aid of $t^a$ we can write $\tilde{\eta}^{abcd}d^4x = 4t^a \tilde{\eta}^{bcd}d^3x dt$ and then (2) becomes:

$$S = \int dt \int d^3x \left[ t^a \eta^{bcd} \tilde{F}_{ab}^i \tilde{F}_{cdi} + t^c \eta^{dab} \tilde{F}_{ab}^i \tilde{F}_{cdi} \right]$$

Using now the identity $t^a F_{abi} = L^a F_{abi} - \nabla_b A^i_a$ we can rewrite (7) in as:

$$S = \int dt \int d^3x \left[ (L_t A^i_a(\tilde{\eta}^{abc} \tilde{F}_{bc}^i)) \tilde{\eta}^{bcd} + (L_t A^i_a(\tilde{\eta}^{abc} \tilde{F}_{bc}^i)) \tilde{\eta}^{bcd} \right]$$

Where $\tilde{\eta}^{abc}$ is the 3-dimensional Levi-Civita tensor density. From (8) we see that the momenta canonically conjugate to $\tilde{\pi}^{i 0}$, $\tilde{\pi}^{i a}$, and $\tilde{\pi}^{i a}$ are:

$$\frac{\delta L}{\delta (L_t A^i_a)} = 0 \quad \frac{\delta L}{\delta (L_t A^i_a)} = \tilde{\eta}^{abc} \tilde{F}_{bc}^i \equiv \tilde{B}^a_i$$

$$\frac{\delta L}{\delta (L_t A^i_a)} = 0 \quad \frac{\delta L}{\delta (L_t A^i_a)} = \tilde{\eta}^{abc} \tilde{F}_{bc}^i \equiv \tilde{B}^a_i$$

Where $L$ denotes the Lagrangian. The non-zero canonical Poisson brackets are:

$$\{ \tilde{A}^i_0(x), \tilde{\pi}^{0 j}(y) \} = \{ \tilde{A}^i_0(x), \tilde{\pi}^{0 j}(y) \} = \delta^{i j} \delta^3(x, y)$$

$$\{ \tilde{A}^i_a(x), \tilde{\pi}^{a j}(y) \} = \{ \tilde{A}^i_a(x), \tilde{\pi}^{a j}(y) \} = \delta^{i j} \delta^3(x, y)$$

$\{ \tilde{F}_{ab}^i(x), \tilde{\pi}^{a b j}(y) \} = \{ \tilde{F}_{ab}^i(x), \tilde{\pi}^{a b j}(y) \} = \delta^{i j} \delta^3(x, y)$

$4L_t$ denotes the Lie derivative along the direction of the vector field $t^a$ and $A^i_0 \equiv t^a A^i_a$
From the definition of the momenta given by (9) we have the following primary constraints:

\[
\begin{align*}
\frac{1}{\pi^0_i} &= 0 \\
\frac{\tilde{\pi}^0_i}{\pi^0_i} &= \frac{\tilde{\pi}^a_i}{\pi^a_i} = 0
\end{align*}
\]

and the Hamiltonian:

\[
H = -\int_\Sigma d^3x \left[ \frac{\pi^0_i}{\pi^0_i} \frac{\frac{1}{\pi^a_i} \frac{\tilde{\pi}^a_i}{\pi^a_i}}{\pi^a_i} + \frac{\frac{2}{\pi^a_i} \frac{\tilde{\pi}^a_i}{\pi^a_i}}{\pi^a_i} \right]
\]

(12)

Following Dirac (see, for example [9]) we introduce a total Hamiltonian:

\[
H_T = H + \int_\Sigma d^3x \left[ u_i^{\frac{1}{\pi^a_i}} + v_i^{\frac{2}{\pi^a_i}} + u_i^{\frac{1}{\pi^a_i} - \frac{2}{\pi^a_i}} + v_i^{\frac{2}{\pi^a_i} - \frac{1}{\pi^a_i}} \right]
\]

(13)

(\(u^i, v^i, u^a_i,\) and \(v^a_i\) are Lagrange multipliers) and impose the conservation in time of the constraints (11) under the dynamics defined by \(H_T\). We find in this way the following secondary constraints:

\[
\begin{align*}
\nabla_a \tilde{B}^a_i &= 0 \\
\nabla_a \tilde{B}^a_i &= 0
\end{align*}
\]

(14)

Finally we must solve the second class constraints of the theory in order to get the Dirac brackets. These Dirac brackets will provide us with the symplectic structure in the final phase space. The outcome of this analysis is the following. The phase space is spanned by the 3-dimensional \(SO(3)\) connections \(\tilde{A}_{a}^i\) and \(\tilde{A}_{a}^i\), the constraint manifold is defined by the conditions (14) and the symplectic 2-form is:

\[
\Omega = 2 \int_\Sigma d^3x \tilde{\eta}^{abc} \epsilon_{ijk} \left[ \tilde{A}_{a}^i(x) - \tilde{A}_{a}^i(x) \right] d\tilde{A}_{b}^j(x) \land d\tilde{A}_{c}^k(x)
\]

(15)

\(\Omega\) is obviously closed; however, it may be degenerate if the determinant of the \(9 \times 9\) matrix \(\omega_{(a_i)(b_j)} = 2 \tilde{\eta}^{abc} \epsilon_{ijk} \left[ \tilde{A}_{c}^k(x) - \tilde{A}_{c}^k(x) \right]\) is zero. We will discuss this issue in section 4. Notice that we have 18 components of \(\tilde{A}_{a}^i\) and \(\tilde{A}_{a}^i\) and six first class
constraints per space point (this will be shown below) so that we have \( \frac{1}{2}(18 - 6 \times 2) = 3 \) degrees of freedom per point. It is interesting to point out, also, that both the symplectic structure (13) and the constraint hypersurface defined by (14) are insensitive to the interchange of the connections. This fact proves to be useful in order to find the constraint functionals that generate the internal \( SO(3) \) transformations and the diffeomorphisms on \( \Sigma \). Before discussing this point we give several identities that the Dirac brackets satisfy\(^5\). They will help in simplifying the computations that follow. The first of them (which can be directly read from (15)) are:

\[
2\epsilon_{ijk}\tilde{\eta}^{abc} [\tilde{A}_a^i(x) - \tilde{A}_a^i(x)] \{ \tilde{A}_b^j(x), \tilde{A}_d^k(y) \} = \delta^c_e \delta^d_f \delta^3(x, y)
\]

\[
\{ \tilde{A}_a^i(x), \tilde{A}_b^j(y) \} = 0
\]

The first of the previous expressions can be written also as:

\[
\{ \tilde{B}_a^i(x), \tilde{A}_b^j(y) \} + \{ \tilde{A}_a^i(x), \tilde{B}_b^j(y) \} = -\delta^a_b \delta^i_j \delta^3(x, y)
\]

Another two useful expressions are:

\[
\epsilon_{jk}^{kl} \eta^{bde} (\tilde{A}_{dk} - \tilde{A}_{dk}) \{ \tilde{B}_a^i(x), \tilde{A}_e^j(y) \} =
\]

\[
= -\eta^{abe} \delta^i_j \partial_c \delta^3(x, y) - \eta^{abc} \epsilon_{ij}^k \tilde{A}_{ce}^k(x) \delta^3(x, y)
\]

The generating functionals of the internal gauge transformations and diffeomorphisms are:

\[
G(N) = -\int_{\Sigma} d^3x N^i [\nabla_b \tilde{B}_i^b + \nabla_{\tilde{b}} \tilde{B}_i^b]
\]

\(^5\)We define the Poisson (Dirac) brackets of two phase space functions \( f \) and \( g \) as \( \{ f, g \} = \Omega^{\alpha\beta} \partial_\alpha f \partial_\beta g \) where \( \Omega^{\alpha\beta} \) are the components of the symplectic 2-form in some coordinate system in the phase space, and \( \Omega^{\alpha\beta} \Omega_{\beta\gamma} = -\delta^\alpha_\gamma \).
\[ D(\vec{N}) = \int_{\Sigma} d^{3}x N^{a} \left[ \hat{A}_{a}^{i} \nabla_{b} B_{i}^{b} + \hat{A}_{a}^{i} \nabla_{b} \tilde{B}_{i}^{b} \right] \] (19)

The first of these expressions is (modulo a numerical factor) the simplest linear combination of (14) symmetric under the interchange of \( \hat{A}_{a}^{i} \) and \( \hat{A}_{a}^{i} \). This symmetry argument, and the fact that the Lagrange multiplier that should appear in the constraint functional generating diffeomorphisms must be a vector field, tells us that the generator of the diffeomorphisms on \( \Sigma \) must have the general structure:

\[ D(\vec{N}) = \int_{\Sigma} d^{3}x N^{a} \left\{ \alpha \left[ \hat{A}_{a}^{i} \nabla_{b} B_{i}^{b} + \hat{A}_{a}^{i} \nabla_{b} \tilde{B}_{i}^{b} \right] + \beta \left[ \hat{A}_{a}^{i} \nabla_{b} B_{i}^{b} + \hat{A}_{a}^{i} \nabla_{b} \tilde{B}_{i}^{b} \right] \right\} \] (20)

A simple computation gives \( \alpha = 1, \beta = 0 \). Notice that \( \nabla_{b} B_{i}^{b} + \nabla_{b} \tilde{B}_{i}^{b} = 0 \) and \( \hat{A}_{a}^{i} \nabla_{b} B_{i}^{b} + \hat{A}_{a}^{i} \nabla_{b} \tilde{B}_{i}^{b} = 0 \) imply \( \nabla_{b} B_{i}^{b} = 0 \) and \( \nabla_{b} \tilde{B}_{i}^{b} = 0 \) if \( \det [\hat{A}_{a}^{i}(x) - \hat{A}_{a}^{i}(x)] \neq 0 \).

We check now that the algebra of \( G(N) \) and \( D(\vec{N}) \) is the usual one; and as a consequence of this we show that the constraints (14) are a first class system. To this end we compute:

\[
\begin{align*}
\{ G(N), \hat{A}_{a}^{i}(x) \} &= -\nabla_{a} N^{i} \\
\{ G(N), \hat{A}_{a}^{i}(x) \} &= -\nabla_{a} N^{i} \\
\{ D(\vec{N}), \hat{A}_{a}^{i}(x) \} &= \mathcal{L}_{\vec{N}} \hat{A}_{a}^{i}(x) \\
\{ D(\vec{N}), \hat{A}_{a}^{i}(x) \} &= \mathcal{L}_{\vec{N}} \hat{A}_{a}^{i}(x) \\
\end{align*}
\] (21)

With the aid of (21) it is straightforward to obtain:

\[
\begin{align*}
\{ G(N), G(M) \} &= -G([N, M]) \\
\{ D(\vec{N}), G(M) \} &= -G(\mathcal{L}_{\vec{N}} N^{i}) \\
\{ D(\vec{N}), D(\vec{M}) \} &= -D([\vec{N}, \vec{M}]) \end{align*}
\] (22)

Where \([N, M]^{i} = e^{i}_{jk} N^{j} M^{k}\) and \([\vec{N}, \vec{M}] = \mathcal{L}_{\vec{N}} \vec{M}\) is the commutator of the vector fields \( \vec{N} \) and \( \vec{M} \).
III   A Coordinate Transformation in The Phase Space

In this section we introduce a convenient change of coordinates that will allow us to pass from the usual connection-triad fields to the two-connection form of the Husain-Kuchař model. This transformation will be used in the next section to show that gravity itself can be described in the new phase space. We will briefly discuss also an interesting type of canonical transformation that naturally appears in this formalism.

We start by introducing the following functions in the \((\tilde{A}_a^i, \tilde{A}_a)^i\) phase space:

\[
\begin{align*}
\tilde{A}_a^i(x) &= \alpha(x)\tilde{A}_a^i(x) + [1 - \alpha(x)]\tilde{A}_a(x) = \tilde{A}_a^i(x) + \alpha(x)[\tilde{A}_a^i(x) - \tilde{A}_a(x)] \\
\tilde{E}_i^a(x) &= \tilde{\eta}^{abc}\epsilon_{ijk}[\tilde{A}_b^j - \tilde{A}_b][\tilde{A}_k^k - \tilde{A}_k]
\end{align*}
\] (23)

In the first equation we define a connection \(\tilde{A}_a^i\) as a linear combination of two connections. This is possible because the coefficients of \(\tilde{A}_a^i\) and \(\tilde{A}_a\) satisfy the condition that their sum is equal to one. The equations (23) define a change of coordinates only when the Jacobian:

\[
\det \begin{bmatrix}
\delta\tilde{A}_a^i(x) & \delta\tilde{A}_a^i(x) \\
\delta\tilde{A}_a^i(y) & \delta\tilde{A}_a^i(y) \\
\delta\tilde{E}_i^a(x) & \delta\tilde{E}_i^a(x) \\
\delta\tilde{A}_a^i(y) & \delta\tilde{A}_a^i(y)
\end{bmatrix}
\] (24)

is different from zero. In our case (24) has the form:

\[
\det \begin{bmatrix}
\alpha I & (1 - \alpha)I \\
-\omega & \omega
\end{bmatrix}
\] (25)

where \(I \equiv \delta_a^b\delta_i^j\delta^3(x, y)\) and \(\omega^{(ai)(bj)} = 2\tilde{\eta}^{abc}\epsilon_{ijk} [\tilde{A}_b^j(x) - \tilde{A}_b][\tilde{A}_k^k(x) - \tilde{A}_k]\delta^3(x, y)\). A straightforward computation tells us that the previous Jacobian is equal to \(\det \omega\). We see
then that the coordinate transformation introduced above is well defined if and only if the symplectic structure $\Omega$ is non-degenerate. It is straightforward to see that $\Omega$ can be written in terms of $A^a_i$ and $\tilde{E}^a_i$ as:

$$\Omega = \int_\Sigma d^3x \, dA^a_i(x) \wedge d\tilde{E}^a_i(x)$$  \hspace{1cm} (26)

We notice now that the scalar field $\alpha(x)$ does not appear in (26). This means that a change in $\alpha(x)$ defines a canonical transformation because it leaves the symplectic structure invariant. It is also worthwhile pointing out that although (24) seems to make sense for degenerate $\tilde{E}^a_i$ ($\tilde{E}^a_i$ is degenerate when $\det [\hat{\Lambda}^a_i(x) - \tilde{\Lambda}^a_i(x)] = 0$) it is only valid when the change of coordinates introduced above is well defined.

With the help of (23) we can write the Gauss law and the diffeomorphism constraint in terms of $\hat{A}^a_i$ and $\tilde{A}^a_i$. Substituting it in $\nabla_a \tilde{E}^a_i = 0$ we get $\nabla_b \tilde{B}^b_i + \nabla_b \hat{B}^b_i = 0$ with no dependence on $\alpha(x)$. This means that the transformation (equivalent to changing $\alpha(x)$ in (23)):

$$\hat{A}^a_i(x) \rightarrow \hat{A}^a_i(x) + \beta(x) \left[ \hat{\Lambda}^a_i(x) - \tilde{\Lambda}^a_i(x) \right]$$
$$\tilde{A}^a_i(x) \rightarrow \tilde{A}^a_i(x) + \beta(x) \left[ \hat{\Lambda}^a_i(x) - \tilde{\Lambda}^a_i(x) \right]$$ \hspace{1cm} (27)

leaves the Gauss law invariant. The generator $P(\beta)$ of this transformations must satisfy:

$$\{ P(\beta), \hat{A}^a_i(x) \} = \beta(x) \left[ \hat{\Lambda}^a_i(x) - \tilde{\Lambda}^a_i(x) \right]$$
$$\{ P(\beta), \tilde{A}^a_i(x) \} = \beta(x) \left[ \hat{\Lambda}^a_i(x) - \tilde{\Lambda}^a_i(x) \right]$$ \hspace{1cm} (28)

We write $P(\beta)$ as:

$$P(\beta) = \int_\Sigma d^3x \, \beta(x) \Phi(\hat{A}, \tilde{A})$$  \hspace{1cm} (29)

(we have ommit the indices and density weights that $\beta$ and $\Phi$ must carry in order to make the integrand a gauge invariant scalar density of weight +1). The invariance of
the Gauss law and the fact that the infinitesimal parameter of the transformation $\beta$ is a scalar function tells us that $\Phi$ carries no $SO(3)$ indices. If we take now the generator of the diffeomorphisms $D(\vec{N})$ we see that under the action of (27) it transforms as:

$$D(\vec{N}) \rightarrow D(\vec{N}) - \frac{2}{3} \int_\Sigma d^3x (N^a \partial_a \beta) \epsilon_{ijk} \tilde{\eta}^{bcd}[\hat{A}_b^i - \hat{A}_b^i][\hat{A}_c^j - \hat{A}_c^j][\hat{A}_d^k - \hat{A}_d^k]$$

(30)

Knowing that $\{D(\vec{N}), P(N)\} = -P(L\vec{N})$ for any functional $P$ we can read directly from the previous expression for $P(\beta)$:

$$P(\beta) = \frac{2}{3} \int_\Sigma d^3x \beta(x) \epsilon_{ijk} \tilde{\eta}^{bcd}[\hat{A}_b^i - \hat{A}_b^i][\hat{A}_c^j - \hat{A}_c^j][\hat{A}_d^k - \hat{A}_d^k]$$

(31)

As we can see $\Phi(\hat{A}, \hat{\vec{A}})$ is a gauge invariant scalar density of weight +1 in agreement to the argument presented above. It is straightforward to check that $P(\beta)$ generates the infinitesimal gauge transformations (27).

In order to understand the origin of this symmetry one can go to the action (1) and perform the following transformation on the 4-dimensional connection and frame fields:

$$A^i_a(x) \rightarrow A^i_a(x) + \epsilon e^i_a(x)$$

$$e^i_a(x) \rightarrow e^i_a(x)$$

(32)

($\epsilon$ is an arbitrary scalar field). Notice that the pull-back of these transformations onto the 3-dimensional slices $\Sigma$ is obtained from (23) by varying $\alpha$. Introducing (32) in (1) we see that the action transforms into:

$$\int_\mathcal{M} d^4x \tilde{\eta}^{abcd} \left[ \Gamma^i_{ab} + 2\epsilon \nabla_b \epsilon^i_a + 2(\partial_a \epsilon) \epsilon^i_b + \epsilon^2 \epsilon^l_m e^i_a e^l_b e^m_c \right] e^i_a e^k_d e_{ijkl}$$

(33)

If $\epsilon$ is a constant then the third term in (33) is zero, the second one is a total divergence and the last one is identically zero. We thus see that in this case the action is invariant under these transformations. Remember that both the Gauss law and

\footnote{I am grateful to A. Ashtekar and M. Varadarajan for discussions on this point}
the diffeomorphism constraints are invariant under (27) when the parameter \(\epsilon\) is a constant. The changes in \(\alpha\) in (23) that are independent of the point are global transformations; the conserved charge associated with this symmetry is the volume of the "spatial" 3-manifold (an obvious fact because there is no dynamics in the model).

IV Two Connection Gravity

In the previous two sections we have shown that the Husain-Kuchař model can be interpreted in terms of two connections. We have seen also that there is a natural way to translate results in the usual phase space \((A^i_a, \tilde{E}^a_i)\) to the \((\hat{A}^i_a, \hat{\tilde{E}}^a_i)\) one. The main result in this section is showing that gravity itself admits an interpretation as a two connection theory. The idea is to work in the phase space of the complexified Husain-Kuchař model and introducing the Hamiltonian constraint "by hand" using equation (23) to translate it to the two connection form. A point that we want to discuss in this section is the role of the degenerate metrics in the Ashtekar formulation of General Relativity. The possibility of working with degenerate metrics is a feature that distinguishes Ashtekar’s connection dynamics from geometrodynamics. This may well be a welcome fact because one could conceivably accommodate things such as topology changes and evolution past some type of singularities in the formalism [4]. One should point out, however, that degenerate metrics can also be a source of trouble, for example when considering the issue of the existence of the ground state of the quantum theory [10] because, as it has been shown by Varadarajan [11] there are classes of degenerate solutions to all the constraints, in the spherically symmetric case, that are everywhere non-singular but have arbitrary negative energy.

In our formulation it is possible to see that the non-degeneracy condition for the symplectic form is equivalent to the condition that the metric is non-degenerate. The non-degeneracy condition for \(\Omega\) is that at each point of \(\Sigma\) the 9 × 9 matrix \(\omega^{(a_i)(b_j)} = 2 \eta^{abc} \epsilon_{ijk} \left( \hat{A}^k_i(x) - \hat{\tilde{A}}^k_i(x) \right)\) must be invertible. Notice that, in principle, this is different from the non-degeneracy condition for the 3 × 3 matrix \(e^i_a = \hat{A}^i_a - \hat{\tilde{A}}^i_a\). We prove now,
however, that both conditions are equivalent. When det $e \neq 0$ the inverse of $\omega^{(a_i)(b_j)}$ is:

$$\omega^{-1}_{(a_i)(b_j)} = \frac{1}{4 \det e} \left( e_a^i e_b^j - 2 e_a^j e_b^i \right)$$  \hspace{1cm} (34)$$

We conclude then that the non-degeneracy of $e_a^i$ implies the non-degeneracy of $\omega^{(a_i)(b_j)}$. In order to prove the converse let us suppose that $e_a^i$ is non-invertible and different from zero (the $e_a^i = 0$ case is trivially dealt with) then there exists an internal vector $v^i \neq 0$ such that $e_a^i v_i = 0$. Let us consider now $v_a^i = \epsilon^{ijk} e_{aj} v_k$. It is straightforward to show that if $e_a^i \neq 0$ then $v_a^i \neq 0$. Now:

$$\omega^{(a_i)(b_j)}v_a^i = 2\tilde{\eta}^{abc} \epsilon_{ijk} \epsilon^{ilm} e_a^i v_m e_c^k =$$

$$= 2\tilde{\eta}^{abc} (e_{aj} v^i e^k_c - e_{ak} v^j_e e^k_c v^i) = 0$$

because $e_a^i v_i = 0$ and the symmetry of $e_{ak} v^k$ in $a$ and $c$. We see that if $\omega^{(a_i)(b_j)}$ is invertible then so is $e_a^i$. In the two connection phase space only non-degenerate triads are allowed by the non-degeneracy property of the symplectic 2-form.

We write now the Hamiltonian constraint in Ashtekar’s description of General Relativity in terms of $(\hat{A}, \hat{\bar{A}})$ by using (23):

$$\eta_{abc} \epsilon^{ijk} \hat{E}_a^i \hat{E}_b^j \hat{B}_c^k = 8(\det e) e^k_c \hat{B}_k^c =$$

$$= 8(\det e) e^k_c \left[ \frac{1}{2} \alpha \hat{B}_k^c + (1 - \alpha) \hat{B}_k^c + \alpha(\alpha - 1) \tilde{\eta}^{cde} e_{klm} e_d^f e^e_m \right]$$

where $\hat{B}_i^a = \tilde{\eta}^{abc} F_{abi}$. The non-degeneracy of $e_a^i$ allows us to write the Hamiltonian constraint as: $e_a^i \hat{B}_i^a = 0$. We have the possibility of choosing for $\alpha$ any value we want; for example $\alpha = 1$. In this case the Hamiltonian constraint reduces to $e_a^i \hat{B}_i^a = 0$. Summarizing, in the new phase space the constraints of General Relativity are:

$$\nabla_a \hat{B}_i^a = 0$$

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8 This was suggested to the author by A. Ashtekar.
\[ \nabla_a \tilde{B}^a_i = 0 \]  
\[ [\tilde{A}^k_e - \overset{\perp}{A}^k_e] \tilde{B}^a_k = 0 \]

Using the fact that:

\[ \nabla_a \tilde{B}^a_i = \nabla_a \tilde{B}^a_i - \nabla_a \tilde{B}^a_i = \epsilon_{ij}^k \{ \tilde{A}^j_a - \overset{\perp}{A}^j_a \} \tilde{B}^a_k \]

(and the analogous expression for \( \nabla_a \tilde{B}^a_i \)) the constraints (35) can be recast in the very neat form:

\[ \epsilon^k_{ij} e_{a_j} \tilde{B}^a_k = 0 \]
\[ \epsilon^k_{ij} e_{a_j} \tilde{B}^a_k = 0 \]
\[ e^k_{a} \tilde{B}^a_k = 0 \]

As we can see the structure of the constraints is very simple. They are either internal vector products of the curvatures and \( e^i_a \) or the internal scalar product of one of the curvatures and \( e^i_a \). All the previous expressions are densities of weight +1. At this point it is necessary to stress that although (36) describes gravity in the two connection phase space we do not have an action (as we had for the Husain-Kuchar model) that leads to the previous Hamiltonian formulation of Gravity.

It is interesting to point out that we cannot make our mechanism work if we take as the starting point the self-dual actions introduced by Samuel, Jacobson, and Smolin [12], [13]. As it has been emphasized throughout the paper the key idea in the two-connection formulation is that for \( SO(3) \) the frame fields can be written as the difference of two connections. In the self-dual action the fields have \( SO(1,3) \) indices. Although the symplectic structure can be still be written as:

\[ \Omega = 2 \int_\Sigma d^3x \, \tilde{\eta}^{abc} e_{bI} d e_{cJ} \wedge d A^{I J}_a = \int_\Sigma d^3x \, d \tilde{E}^a_{I J} \wedge d A^{I J}_a \]

we cannot use \( (A^{I J}_a, e_{bK}) \) as the phase space (it is not even-dimensional!) and it is not straightforward to relate the non-degeneracy of the symplectic structure with the non-degeneracy of \( \tilde{E}^a_{I J} \).

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9 I, J are \( SO(1,3) \) indices, \( A^{I J}_a \) is the self-dual connection and \( \tilde{E}^a_{I J} = \tilde{\eta}^{abc} e_{bI} e_{cJ} \)
The new constraints \((36)\) look very simple, but, are they really so? In the case of the Ashtekar constraints the variables that are used are canonically conjugate so that when quantizing the theory, for example, in the connection representation, the quantum operators that describe the constraints are very simple. In our case it is not clear what set of elementary variables makes the quantization of the theory easy. Only if a suitable set of elementary variables and a representation of them as operators acting in a vector space can be found such that the quantum constraints are simple should we say that a simplification has occurred because of the introduction of \((36)\). Our hope is that the availability of geometric objects in the two-connection phase space that are not obvious in the usual Ashtekar phase space will lead to sets of elementary variables that would allow us to advance in the quantization of gravity.

V Conclusions

We have shown that it is possible to describe gravity (and some other diff-invariant theories in 3+1 dimensions like the Husain-Kuchar model) in a phase space spanned by two different \(SO(3)\) connections. Due to the form of the symplectic structure \(\Omega\), non-degenerate frame fields are excluded by the non degeneracy property of \(\Omega\). This allows us to simplify the Hamiltonian constraint and write it in a form that is very close to the one of the remaining constraints. The relevance of these results relies mainly on the fact that having a new phase space it is conceivable that new systems of elementary variables can be found that will allow us to attack the quantization of gravity from a different perspective. The hope is that, in analogy with what happened with the introduction of the Ashtekar variables and the subsequent introduction of the loop representation (that so many new results concerning the structure of the space-time etc... have given to us) this new phase space description will help in gaining new information about Quantum Gravity.
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