SPIN$^c$ PREQUANTIZATION AND SYMPLECTIC CUTTING

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Abstract. We define spin$^c$ prequantization of a symplectic manifold to be a spin$^c$ structure and a connection which are compatible with the symplectic form. We describe the cutting of an $S^1$-equivariant spin$^c$ prequantization. The cutting process involves a choice of a spin$^c$ prequantization for the complex plane. We prove that the cutting is possible if and only if the moment map level set along which the cutting is done is compatible with this choice.

1. Introduction

Given a compact even-dimensional oriented Riemannian manifold $M$, endowed with a spin$^c$ structure, one can construct an associated Dirac operator $D^+$ acting on smooth sections of a certain (complex) vector bundle over $M$. The spin$^c$ quantization of $M$ with respect to the above structure is defined to be

$$Q(M) = \ker(D^+) - \coker(D^+).$$

This is a virtual vector space, and in the presence of a $G$-action, it is a virtual representation of the group $G$. Spin$^c$ quantization generalizes the concept of Kähler and almost-complex quantization (see [4], especially Lemma 2.7 and Remark 2.9) and in some sense it is a ‘better behaved’ quantization (see [3]).

Quantization was originally defined as a process that associates a Hilbert space to a symplectic manifold (and self-adjoint operators to smooth real valued functions on the manifold). Therefore, one of our goals in this paper is to relate spin$^c$ quantization to symplectic geometry. This can be achieved by defining a spin$^c$ prequantization of a symplectic manifold to be a spin$^c$ structure and a connection on its determinant line bundle which are compatible with the symplectic form (in a certain sense). This definition is analogous to the definition of prequantization in the context of geometric quantization (see [5] and references therein). Our definition is different but equivalent to the one in [4]. It is important to mention that in the equivariant setting, a spin$^c$ prequantization for a symplectic manifold $(M, \omega)$ determines a moment map $\Phi: M \to \mathfrak{g}^*$, and hence the action $G \circlearrowleft (M, \omega)$ is Hamiltonian.

The cutting construction was originally introduced by E. Lerman in [2] for symplectic manifolds equipped with a Hamiltonian circle action. In [3] we explained how one can cut a given $S^1$-equivariant spin$^c$ structure on an oriented Riemannian manifold. Here we extend this construction and describe how to cut a given $S^1$-equivariant spin$^c$ prequantization. This cutting process involves two choices: a choice of an equivariant spin$^c$ prequantization for the complex plane $\mathbb{C}$, and a choice of a level set $\Phi^{-1}(\alpha)$ along which the cutting is done. Our main theorem (Theorem 3.1) reveals a quite interesting fact: Those two choices must be compatible (in a certain sense) in order to make the cutting construction possible. In fact, each one of the two choices determines the other (once we assume that cutting is possible),
so in fact only one choice is to be made. This theorem also explains the ‘mysterious’ freedom one has when choosing a spin\(^c\) structure on \(C\) in the first step of the cutting construction: it is just the freedom of choosing a ‘cutting point’ \(\alpha \in \mathfrak{g}^*\) (or a level set of the moment map along which the cutting is done). Since by our theorem, \(\alpha\) can never be a weight, we see why spin\(^c\) quantization must be additive under cutting (a result already obtained in [3]).

This paper is organized as follows. In Section 2 we review the definitions of the spin groups, spin and spin\(^c\) structures and define the concept of spin\(^c\) prequantization. As an example we will use later, we construct a prequantization for the complex plane. For technical reasons, we chose to define spin\(^c\) prequantization for manifold endowed with closed two-forms (which may not be symplectic). In Section 3 we describe the cutting process in steps and obtain our main theorem relating the spin\(^c\) prequantization for \(C\) with the level set used for cutting. In the last sections we discuss a couple of examples.

Throughout this paper, all spaces are assumed to be smooth manifolds, and all maps and actions are assumed to be smooth. The principal action in a principal bundle will be always a right action. A real vector bundle \(E\), equipped with a fiberwise inner product will be called a Riemannian vector bundle. If the fibers are also oriented, then its bundle of oriented orthonormal frames will be denoted by \(\text{SOF}(E)\). For an oriented Riemannian manifold \(M\), we will simply write \(\text{SOF}(M)\), instead of \(\text{SOF}(TM)\).

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### 2. Spin\(^c\) Prequantization

**2.1. Spin\(^c\) Structures.**

In this section we recall the definition and basic properties of the spin and spin\(^c\) groups. Then we give the definition of a spin\(^c\) structure on a manifold, which is essential for defining spin\(^c\) prequantization.

**Definition 2.1.** Let \(V\) be a finite dimensional vector space over \(\mathbb{K} = \mathbb{R}\) or \(\mathbb{C}\), equipped with a symmetric bilinear form \(B : V \times V \to \mathbb{K}\). Define the Clifford algebra \(\text{Cl}(V, B)\) to be the quotient \(T(V)/I(V, B)\) where \(T(V)\) is the tensor algebra of \(V\), and \(I(V, B)\) is the ideal generated by \(\{v \otimes v - B(v, v) \cdot 1 : v \in V\}\).

**Remark 2.1.** If \(v_1, \ldots, v_n\) is an orthogonal basis for \(V\), then \(\text{Cl}(V, B)\) is the algebra generated by \(v_1, \ldots, v_n\), subject to the relations \(v_i^2 = B(v_i, v_i) \cdot 1\) and \(v_i v_j = -v_j v_i\) for \(i \neq j\).

Also note that \(V\) is a vector subspace of \(\text{Cl}(V, B)\).

**Definition 2.2.** If \(V = \mathbb{R}^k\) and \(B\) is minus the standard inner product on \(V\), then \(\text{Cl}(V, B)\) is the algebra generated by \(v_1, \ldots, v_n\), subject to the relations \(v_i^2 = B(v_i, v_i) \cdot 1\) and \(v_i v_j = -v_j v_i\) for \(i \neq j\).

Also note that \(V\) is a vector subspace of \(\text{Cl}(V, B)\).

**Definition 2.2.** If \(V = \mathbb{R}^k\) and \(B\) is minus the standard inner product on \(V\), then define the following objects:

1. \(C_k = \text{Cl}(V, B)\), and \(C_k^c = \text{Cl}(V, B) \otimes \mathbb{C}\).

   Those are finite dimensional algebras over \(\mathbb{R}\) and \(\mathbb{C}\), respectively.

2. The spin group

\[
\text{Spin}(k) = \{v_1 v_2 \ldots v_l : v_i \in \mathbb{R}^k, \|v_i\| = 1 \text{ and } 0 \leq l \text{ is even}\} \subset C_k
\]
(3) The spin\(^c\) group

\[ Spin^c(k) = (Spin(k) \times U(1)) / K \]

where \( U(1) \subset \mathbb{C} \) is the unit circle, and \( K = \{(1, 1), (-1, -1)\} \).

**Remark 2.2.**

(1) Equivalently, one can define

\[ Spin^c(k) = \{ c \cdot v_1 \cdots v_l : v_i \in \mathbb{R}^k, ||v_i|| = 1, 0 \leq l \text{ is even, and } c \in U(1) \} \subset C_k \]

(2) The group \( Spin(k) \) is connected for \( k \geq 2 \).

**Proposition 2.1.**

(1) There is a linear map \( C_k \to C_k \), \( x \mapsto x^t \) characterized by \( (v_1 \ldots v_l)^t = v_l \ldots v_1 \) for all \( v_1, \ldots, v_l \in \mathbb{R}^k \).

(2) For each \( x \in Spin(k) \) and \( y \in \mathbb{R}^k \), we have \( xyx^t \in \mathbb{R}^k \).

(3) For each \( x \in Spin(k) \), the map \( \lambda(x) : \mathbb{R}^k \to \mathbb{R}^k \), \( y \mapsto xyx^t \) is in \( SO(k) \), and \( \lambda : Spin(k) \to SO(k) \) is a double covering for \( k \geq 1 \). It is a universal covering map for \( k \geq 3 \).

For the proof, see page 16 in [I].

**Definition 2.3.** Let \( M \) be a manifold, and \( Q \) a principal \( SO(k) \)-bundle on \( M \). A spin\(^c\) structure on \( Q \) is a principal \( Spin^c(k) \)-bundle \( P \to M \), together with a map \( \Lambda : P \to Q \) such that the following diagram commutes.

\[
\begin{array}{ccc}
P \times Spin^c(k) & \longrightarrow & P \\
\downarrow_{\Lambda \times \lambda^c} & & \downarrow_{\Lambda} \\
Q \times SO(k) & \longrightarrow & Q \\
\end{array}
\]

Here, the maps corresponding to the horizontal arrows are the principal actions, and \( \lambda^c : Spin^c(k) \to SO(k) \) is given by \([x, z] \mapsto \lambda(x)\), where \( \lambda : Spin(k) \to SO(k) \) is the double covering.

**Remark 2.3.**

(1) A spin\(^c\) structure on an oriented Riemannian vector bundle \( E \) is a spin\(^c\) structure on the associated bundle of oriented orthonormal frames, \( SOF(E) \).

(2) A spin\(^c\) structure on an oriented Riemannian manifold is a spin\(^c\) structure on its tangent bundle.

2.2. **Equivariant spin\(^c\) structures.**

**Definition 2.4.** Let \( G, H \) be Lie groups. A \( G \)-equivariant principal \( H \)-bundle is a principal \( H \)-bundle \( \pi : Q \to M \) together with left \( G \)-actions on \( Q \) and \( M \), such that:

(1) \( \pi(g \cdot q) = g \cdot \pi(q) \) for all \( g \in G, q \in Q \)

(i.e., \( G \) acts on the fiber bundle \( \pi : Q \to M \)).

(2) \( (g \cdot q) \cdot h = g \cdot (q \cdot h) \) for all \( g \in G, q \in Q, h \in H \)

(i.e., the actions of \( G \) and \( H \) commute).
Remark 2.4. It is convenient to think of a $G$-equivariant principal $H$-bundle in terms of the following commuting diagram (the horizontal arrows correspond to the $G$ and $H$ actions).

\[
\begin{array}{ccc}
G \times Q & \longrightarrow & Q \\
\downarrow_{td \times \pi} & & \downarrow_{\pi} \\
G \times M & \longrightarrow & M
\end{array}
\]

Definition 2.5. Let $\pi : E \to M$ be a fiberwise oriented Riemannian vector bundle, and let $G$ be a Lie group. A $G$-equivariant structure on $E$ is an action of $G$ on the vector bundle, that preserves the orientations and the inner products of the fibers. We will say that $E$ is a $G$-equivariant oriented Riemannian vector bundle.

Remark 2.5.

(1) A $G$-equivariant oriented Riemannian vector bundle $E$ over a manifold $M$, naturally turns $SOF(E)$ into a $G$-equivariant principal $SO(k)$-bundle, where $k = \text{rank}(E)$.

(2) If a Lie group $G$ acts on an oriented Riemannian manifold $M$, by orientation preserving isometries, then the frame bundle $SOF(M)$ becomes a $G$-equivariant principal $SO(m)$-bundle, where $m = \text{dim}(M)$.

Definition 2.6. Let $\pi : Q \to M$ be a $G$-equivariant principal $SO(k)$-bundle. A $G$-equivariant spin$^c$ structure on $Q$ is a spin$^c$ structure $\Lambda : P \to Q$ on $Q$, together with a left action of $G$ on $P$, such that

(1) $\Lambda(g \cdot p) = g \cdot \Lambda(p)$ for all $p \in P$, $g \in G$ (i.e., $G$ acts on the bundle $P \to Q$).

(2) $g \cdot (p \cdot x) = (g \cdot p) \cdot x$ for all $g \in G$, $p \in P$, $x \in Spin(k)$

(i.e., the actions of $G$ and $Spin^c(k)$ on $P$ commute).

Remark 2.6.

(1) It is convenient to think of a $G$-equivariant spin$^c$ structure in terms of the following commuting diagram (where the horizontal arrows correspond to the principal and the $G$-actions).

\[
\begin{array}{ccc}
G \times P & \longrightarrow & P \\
\downarrow_{td \times \Lambda} & & \downarrow_{\Lambda} \\
G \times Q & \longrightarrow & Q \\
\downarrow_{td \times \pi} & & \downarrow_{\pi} \\
G \times M & \longrightarrow & M
\end{array}
\]

(2) Note that in a $G$-equivariant spin$^c$ structure, the bundle $P \to M$ is a $G$-equivariant principal $Spin^c(k)$-bundle.

2.3. The definition of spin$^c$ prequantization.

In this section we define the concept of a $G$-equivariant Spin$^c$ prequantization. This
Proof.

For each $\xi \in \mathfrak{g}$ where $\xi \in \text{SOF}(\mathfrak{g})$ (on which a Lie group $G$ acts by orientation preserving isometries, and let $P \to SOF(M) \to M$ be a $G$-equivariant spin$^c$ structure on $M$.

Assume that $\theta: TP \to u(1) \cong i\mathbb{R}$ is a $G$-invariant and Spin$^c(m)$-invariant connection 1-form on the principal $S^1$-bundle $\pi: P \to SOF(M)$, for which

$$\theta(\zeta_P): P \to u(1)$$

is a constant function for any $\zeta \in \mathfrak{spin}(m)$.

For each $\xi \in \mathfrak{g} = \text{Lie}(G)$ define a map

$$\phi^\xi: P \to \mathbb{R} \quad \text{where} \quad \phi^\xi = -i \cdot (\xi_P \theta)$$

where $\xi_P$ is the vector field on $P$ generated by $\xi$.

Then

1. For any $\xi \in \mathfrak{g}$, the map $\phi^\xi$ is Spin$^c(2m)$-invariant, i.e., $\phi^\xi = \pi^* (\Phi^\xi)$ where $\Phi^\xi: M \to \mathbb{R}$ is a smooth function.
2. For any $\xi \in \mathfrak{g}$, we have $d\Phi^\xi = \iota_{\xi_M} \omega$, where $\omega$ is a real two-form on $M$, determined by the equation $d\theta = \pi^* (-i \cdot \omega)$.
3. The map

$$\Phi: M \to \mathfrak{g}^* \quad \text{where} \quad \Phi(m)\xi = \Phi^\xi(m)$$

is $G$-equivariant.

Proof.

1. This follows from the fact that $\theta$ is Spin$^c(m)$-invariant, and that the $G$ and Spin$^c(m)$-actions on $P$ commute.
2. For any $\eta = (\zeta, b) \in \mathfrak{spin}(m) = \mathfrak{spin}(n) \oplus u(1)$, we have

$$\iota_{\eta_P} \theta = \theta(\eta_P) = \theta(\zeta_P) + \theta(b_P) = \theta(\zeta_P) + b .$$

Since $\theta(\zeta_P)$ is constant by assumption, we get that

$$\iota_{\eta_P} d\theta = L_{\eta_P} \theta - dt_{\eta_P} \theta = 0 .$$

This implies that $d\theta$ is horizontal, and hence $\omega$ is well defined by the equation $d\theta = \pi^* (-i \cdot \omega)$.

Now, observe that

$$\pi^* d\Phi^\xi = d (\pi^* \Phi^\xi) = d\phi^\xi = -i d\iota_{\xi_P} \theta = -i [L_{\xi_P} \theta - \iota_{\xi_P} d\theta] = \iota_{\xi_P} (\pi^* \omega) = \pi^* (\iota_{\xi_M} \omega)$$

and since $\pi^*$ is injective, we get $d\Phi^\xi = \iota_{\xi_M} \omega$ as needed.
3. If $g \in G$, $m \in M$, $\xi \in \mathfrak{g}$ and $p \in \pi^{-1}(m)$, then

$$\Phi^\xi(g \cdot m) = \phi^\xi(g \cdot p) = -i (\iota_{\xi_M} \theta) (g \cdot p) = -i (\theta_{g_P}(\xi_P)_{|g_P}) = -i (\theta_{g_P}(g \cdot (\text{Ad}_{g^{-1}}(\xi))_P)_{|g_P}) = -i (\theta_{g_P}(g \cdot (\text{Ad}_{g^{-1}}(\xi))_P)_{|g_P}) = -i (\theta_{g_P}(\xi_P, g \cdot (\text{Ad}_{g^{-1}}(\xi))_P)_{|g_P}) = \phi^{\text{Ad}_{g^{-1}}(\xi)}(p) = \Phi^{\text{Ad}_{g^{-1}}(\xi)}(m)$$
and we ended up with $\Phi^\xi(g \cdot m) = \Phi^{Ad_{-i}\xi}(m)$, which means that $\Phi$ is $G$-equivariant.

\[\square\]

The above claim suggests a compatibility condition between a given two-form and a spin$^c$ structure on our manifold. We will work with two-forms that are closed, but not necessarily nondegenerate. The compatibility condition is formulated in the following definition.

**Definition 2.7.** Let a Lie group $G$ act on a compact $m$-dimensional manifold $M$, and let $\omega$ be a $G$-invariant closed two-form (i.e., $g^*\omega = \omega$ for any $g \in G$). A $G$-equivariant spin$^c$ prequantization for $M$ is a $G$-equivariant spin$^c$ structure $\pi: P \to SOF(M) \to M$ (with respect to an invariant Riemannian metric and orientation), and a $G$ and $Spin^c(m)$-invariant connection $\theta \in \Omega^1(P; u(1))$ on $P \to SOF(M)$, such that

$$\theta(\zeta_P) = 0 \quad \text{for any } \zeta \in \text{spin}(m)$$

and

$$d\theta = \pi^*(-i \cdot \omega).$$

**Remark 2.7.** By the above claim, the action $G \circ (M, \omega)$ is Hamiltonian, with a moment map $\Phi: M \to \mathfrak{g}^*$ satisfying

$$\pi^*(\Phi^\xi) = -i \cdot i_{\xi^*}(\theta) \quad \text{for any } \xi \in \mathfrak{g}.$$

**Remark 2.8.** A $G$-invariant connection 1-form $\theta$ on the $G$-equivariant principal $Spin^c(m)$-bundle $P \to M$ induces a connection 1-form $\tilde{\theta}$ on the principal $S^1$-bundle $P \to SOF(M)$ as follows.

Recall the determinant map

$$\det: Spin^c(n) \to U(1), \quad [A, z] \mapsto z^2.$$  

This map induces a map on the Lie algebras

$$\det_*: \text{spin}^c(n) = \text{spin}(n) \oplus u(1) \to u(1) \simeq i\mathbb{R}, \quad (A, z) \mapsto 2z.$$

This means that the map $\frac{1}{2}\det_*: \text{spin}^c(m) \to u(1)$ is just the projection onto the $u(1)$ component.

The composition $\frac{1}{2}\det_* \circ \theta$ will then be a connection 1-form on $P \to SOF(M)$, which is $G$-invariant, and for which $\theta(\zeta_P) = \frac{1}{2}\det_*(\zeta) = 0$ for any $\zeta \in \text{spin}(m)$.

**Remark 2.9.** The condition $\theta(\zeta_P) = 0$ could have been omitted, since our main theorem can be proved without it. However, this condition is necessary to obtain a discreet condition on the prequantizable closed two forms. See the example in Section 4.

In the following claim, $M$ is an oriented Riemannian $m$-dimensional manifold on which $G$ acts by orientation preserving isometries.

**Claim 2.2.** Let $P \to SOF(M) \to M$ be a $G$-equivariant spin$^c$ structure on $M$. Let $P_{\det} = P/\text{Spin}(m)$ and $q: P \to P_{\det}$ the quotient map. Let $\theta: TP \to u(1)$ be a connection 1-form on the $G$-equivariant principal $U(1)$-bundle $P \to SOF(M)$.

Then $\theta = \frac{1}{2} q^*\tilde{\theta}$ for some connection one form $\tilde{\theta}$ on the $G$-equivariant principal $U(1)$ bundle $P_{\det} \to M$ if and only if $\theta$ is $Spin^c(m)$-invariant and $\theta(\zeta_P) = 0$ for all $\zeta \in \text{spin}(m)$.
Here is the relevant diagram.

\[
\begin{array}{ccc}
P & \xrightarrow{\theta} & P_{\text{det}} \\
\downarrow & & \downarrow \\
SOF(M) & \longrightarrow & M
\end{array}
\]

Note that this is not a pullback diagram. The pullback of \(P_{\text{det}}\) under the projection \(SOF(M) \to M\) is the square of the principal \(U(1)\) bundle \(P \to SOF(M)\).

**Proof of Claim 2.2** Assume that \(\theta = \frac{1}{2} q^*(\overline{\theta})\). Then for any \(g \in Spin^c(m)\): \(P \to P\), write \(g = [A, z] \) with \(A \in Spin(m)\) and \(z \in U(1)\). Since \(\theta\) is \(U(1)\)-invariant, we have

\[
g^* \theta = [A, 1]^* [1, z]^* \theta = [A, 1]^* \theta = \frac{1}{2} [A, 1]^* q^* \overline{\theta} = \frac{1}{2} q^* \overline{\theta} = \theta,
\]

and so \(\theta\) is \(Spin^c(m)\)-invariant. If \(g \in \text{spin}(m)\) then \(q_*(\zeta_P) = 0\), which implies \(\theta(\zeta_P) = 0\).

Conversely, assume that \(\theta\) is \(Spin^c(m)\)-invariant with \(\theta(\zeta_P) = 0\) for all \(g \in \text{spin}(m)\). Define a 1-form \(TP_{\text{det}} \to u(1)\) by

\[
\overline{\theta}(q_* v) = 2 \theta (v) \quad \text{for} \quad v \in TP.
\]

This will be well defined, since if \(q_* v = q_* v'\) for \(v \in T_x P\) and \(v' \in T_x P_{\text{det}}\) where \(g \in Spin(m)\), then \(q_*(v - v') = 0\), which implies that \(v - v' g^{-1} = \zeta_P\) for some \(g \in \text{spin}(m)\). The fact that \(\theta(\zeta_P) = 0\) will imply that \(\theta(v) = \theta(v')\). Smoothness and \(G\)-invariance of \(\overline{\theta}\) are straight forward.

We also need to check that \(\overline{\theta}\) is vertical (i.e., that \(\overline{\theta}(\xi_{P_{\text{det}}}) = \xi\) for \(\xi \in u(1)\)). Note that \(Spin^c(m)/Spin(m)\) is isomorphic to \(U(1)\) via the isomorphism taking the class of \([A, z] \in Spin^c(m)\) to \(z^2 \in U(1)\). This will imply that \(q_*(\xi_P) = 2 \xi_{P_{\text{det}}},\) from which we can conclude that \(\overline{\theta}\) is vertical. \(\square\)

2.4. **\(Spin^c\) prequantizations for \(\mathbb{C}\).**

For the purpose of cutting, we will need to choose an \(S^1\)-equivariant \(Spin^c\) prequantization on the complex plane. The \(S^1\)-action on \(\mathbb{C}\) is given by

\[(a, z) \mapsto a^{-1} \cdot z \quad , \quad a \in S^1 , \ z \in \mathbb{C} .\]

We take the standard orientation and Riemannian structure on \(\mathbb{C}\) and choose our two-form to be

\[\omega_{\mathbb{C}} = 2 \cdot dx \wedge dy = -i \cdot dz \wedge d\bar{z} .\]

For each odd integer \(\ell \in \mathbb{Z}\) we will define an \(S^1\)-equivariant \(Spin^c\) prequantization for \(S^1 \cap (\mathbb{C}, \omega_{\mathbb{C}})\). The prequantization will be denoted as \((P_C^\ell, \theta_{\mathbb{C}})\), and defined as follows.

Let \(P_C^\ell = \mathbb{C} \times Spin^c(2)\) be the trivial principal \(Spin^c(2)\)-principal bundle over \(\mathbb{C}\) with the non-trivial \(S^1\)-action

\[S^1 \times P_C^\ell \to P_C^\ell , \quad (e^{i\varphi} z , [a, w]) \mapsto (e^{-i\varphi} , [z e^{-i\varphi/2} \cdot a , e^{-i\varphi/2} \cdot w])\]

where \(x_{\varphi} = \cos \varphi + \sin \varphi \cdot e_1 e_2 \in Spin(2)\). Note that since \(\ell \in \mathbb{Z}\) is odd, this action is well defined. Next we define a connection

\[\theta_{\mathbb{C}} : TP_C^\ell \to spin^c(2) = spin(2) \oplus u(1) .\]
Denote by $\pi_1: P^E_\mathbb{C} \to \mathbb{C}$ and $\pi_2: P^E_\mathbb{C} \to Spin^c(2)$ the projections, and by $\theta^R$ the right-invariant Maurer-Cartan form on $Spin^c(2)$. Then set

$$\theta_\mathbb{C}: TP^E_\mathbb{C} \to Spin^c(2) \quad , \quad \theta_\mathbb{C} = \pi_2(\theta^R) + \frac{1}{2} \pi_1^\ast(\bar{z} \, dz - z \, d\bar{z}) .$$

Note that $\pi_1^\ast(\bar{z} \, dz - z \, d\bar{z})$ takes values in $i \mathbb{R} = u(1) \subset \mathfrak{spin}^c(2)$, and that the connection $\theta_\mathbb{C}$ does not depend on $\ell$.

Finally, let

$$\tilde{\theta}_\mathbb{C} = \frac{1}{2} det_* \circ \theta_\mathbb{C} .$$

**Claim 2.3.** For any odd $\ell \in \mathbb{Z}$, the pair $(P^E_\ell, \tilde{\theta}_\mathbb{C})$ is an $S^1$-equivariant spin$^c$ prequantization for $(\mathbb{C}, \omega_\mathbb{C})$.

**Proof.** The 1-form $\theta_\mathbb{C}$ (and hence $\tilde{\theta}_\mathbb{C}$) is $S^1$-invariant, since $\bar{z} \, dz - z \, d\bar{z}$ is an $S^1$-invariant 1-form on $\mathbb{C}$, and since the group $Spin^c(2)$ is abelian. The 1-form $\tilde{\theta}_\mathbb{C}$ is given by

$$\tilde{\theta}_\mathbb{C} = \frac{1}{2} det_* \circ \theta_\mathbb{C} = \frac{1}{2} det_* \circ \pi_2^\ast(\theta^R) + \frac{1}{2} \pi_1^\ast(\bar{z} \, dz - z \, d\bar{z})$$

and therefore

$$d(\tilde{\theta}_\mathbb{C}) = 0 + \frac{1}{2} \pi_1^\ast(d\bar{z} \wedge dz - dz \wedge d\bar{z}) = \pi_1^\ast(-dz \wedge d\bar{z}) = \pi_1^\ast(-i \cdot \omega_\mathbb{C})$$

as needed. Finally, by Remark 2.8 we have $\tilde{\theta}_\mathbb{C}(\zeta P^E_\ell) = 0$ for all $\zeta \in \mathfrak{spin}(2)$.

\[\square\]

3. Cutting of a Spin$^c$ Prequantization

The process cutting consists of several steps: Taking the product, restricting and taking the quotient of spin$^c$ structures. We start by discussing those constructions independently.

3.1. The product of two Spin$^c$ Prequantizations.

Let a Lie group $G$ act by orientation preserving isometries on two oriented Riemannian manifolds $M$ and $N$, of dimensions $m$ and $n$, respectively. Given two equivariant spin$^c$ structures $P_M, P_N$ on $M, N$, we can take their ‘product’ as follows. First, note that $P_M \times P_N$ is a $G$-equivariant principal $Spin^c(m) \times Spin^c(n)$-bundle on $M \times N$. Second, observe that $Spin^c(m)$ and $Spin^c(n)$ embed naturally as subgroups of $Spin^c(m+n)$, and thus give rise to a homomorphism

$$Spin^c(m) \times Spin^c(n) \to Spin^c(m+n) \quad , \quad (x, y) \mapsto x \cdot y .$$

This homomorphism is used to define a principal $Spin^c(m+n)$-bundle on $M \times N$, denoted $P_{M \times N}$, as a fiber bundle associated to $P_M \times P_N$.

In the following claim, $\theta^L$ is the left invariant Maurer-Cartan 1-form on the group $Spin^c(m+n)$, and $\omega_M, \omega_N$ are closed $G$-invariant two forms on $M, N$.

**Claim 3.1.** Let $(P_M, \theta_M)$ and $(P_N, \theta_N)$ be two $G$-equivariant spin$^c$ prequantizations for $(M, \omega_M)$ and $(N, \omega_N)$, respectively.

Let

$$P_{M \times N} = (P_M \times P_N) \times_{Spin^c(m) \times Spin^c(n)} Spin^c(m+n)$$

and

$$\theta_{M \times N} = \theta_M + \theta_N + \frac{1}{2} det_* \circ \theta^L \in \Omega^1(P_{M \times N}; u(1)) .$$
Then \((P_{M\times N}, \theta_{M\times N})\) is a \(G\)-equivariant spin\(^c\) prequantization for \((M\times N, \omega_M \oplus \omega_N)\), called the product of \((P_M, \theta_M)\) and \((P_N, \theta_N)\).

**Remark 3.1.**

1. More specifically, the connection \(\theta_{M\times N}\) is given by

\[
\theta_{M\times N}(q_*(u, v, \xi^L)) = \theta_M(u) + \theta_N(v) + \frac{1}{2} \det_\ast(\xi)
\]

where \(u \in TP_M, ~ v \in TP_N, ~ \xi \in \text{spin}^c(m + n)\) and

\[
q: P_M \times P_N \times \text{spin}^c(m + n) \to P_{M\times N}
\]

is the quotient map. This is well defined since \(\theta_M\) and \(\theta_N\) are spin\(^c\)-invariant.

2. The \(G\)-action on \(M \times N\) can be taken to be either the diagonal action

\[
g \cdot (x, y) = (g \cdot x, g \cdot y)
\]

or the ‘\(M\)-action’

\[
g \cdot (x, y) = (g \cdot x, y)
\]

and \((P_{M\times N}, \theta_{M\times N})\) will be a \(G\)-equivariant prequantization with respect to any of those actions.

3. The map \(P_{M\times N} \to SOF(M \times N)\) is the natural one induced from \(P_M \to SOF(M)\) and \(P_N \to SOF(N)\), using the fact that

\[
SOF(M \times N) \cong (SOF(M) \times SOF(N)) \times_{SO(m) \times SO(n)} SO(m + n)
\]

**Proof.** The connection \(\theta_{M\times N}\) is \(G\) and \(\text{spin}^c(m + n)\)-invariant, since \(\theta_M\) and \(\theta_N\) have the same invariance properties. Moreover, since \(d\theta_L = 0\), we get that

\[
d(\theta_{M\times N}) = d(\theta_M) + d(\theta_N) = \pi^*(-i \cdot \omega_M \oplus \omega_N)
\]

as needed, where \(\pi: P_{M\times N} \to M \times N\) is the projection.

Finally, \(\theta_{M\times N}(\zeta_{P_{M\times N}}) = 0\) for all \(\zeta \in \text{spin}^c(m + n)\) since \(\frac{1}{2} \det_\ast(\zeta) = 0\). \(\square\)

### 3.2. Restricting a \text{spin}^c prequantization.

Assume that a Lie group \(G\) acts on an \(m\) dimensional oriented Riemannian manifold \(M\) by orientation preserving isometries. Let \(Z \subset M\) be a \(G\)-invariant co-oriented submanifold of co-dimension 1. Then there is a natural map

\[
i: SOF(Z) \to SOF(M) \quad \text{and} \quad i(f)(a_1, a_2, \ldots, a_{m-1}) + a_m \cdot v_p
\]

where \(f : \mathbb{R}^m \sim T_pZ\) is a frame in \(SOF(Z)\), and \(v \in \Gamma(TM)\) is the vector field on \(Z\) of positive unit vectors orthogonal to \(TZ\).

A \(G\)-equivariant spin\(^c\) structure \(P\) on \(M\) can be restricted to \(Z\), by setting

\[
P_Z = i^*(P),
\]

i.e., \(P_Z\) is the pullback under \(i\) of the circle bundle \(P \to SOF(M)\). The relevant diagram is

\[
\begin{array}{ccc}
P_Z = i^*(P) & \overset{i'}{\longrightarrow} & P \\
\text{SOF}(Z) & \overset{i}{\longrightarrow} & \text{SOF}(M) \\
\downarrow & & \downarrow \\
Z & \longrightarrow & M
\end{array}
\]
The principal action on $P_Z \to Z$ comes from the natural inclusion $\text{Spin}^c(m-1) \hookrightarrow \text{Spin}^c(m)$, and the $G$-action on $P_Z$ is induced from the one on $P$.

Furthermore, if a connection 1-form $\theta$ is given on the circle bundle $P \to \text{SOF}(M)$, we can restrict it to a connection 1-form $\theta_Z$ on $P_Z \to \text{SOF}(Z)$ by letting

$$\theta_Z = (i')^* \theta.$$ 

**Claim 3.2.** Let $(P, \theta)$ be a $G$-equivariant spin$^c$ prequantization for $(M, \omega)$ (for a closed $G$-invariant two form $\omega$), and $Z \subset M$ a co-oriented $G$-invariant submanifold of co-dimension 1. Then the pair $(P_Z, \theta_Z)$ is a $G$-equivariant spin$^c$ prequantization for $(Z, \omega|_Z)$.

**Proof.**

\begin{align*}
d(\theta_Z) &= (i')^*(d\theta) = (i')^*\pi^*(-i \cdot \omega) = \pi^*(-i \cdot \omega|_Z)
\end{align*}

as needed, and

$$\theta_Z(\zeta_{P_Z}) = \theta(\zeta_P) = 0$$

for all $\zeta \in \text{spin}(m-1)$. $\square$

### 3.3. Quotients of spin$^c$ prequantization.

Here is a general fact about connections on principal bundles and their quotients.

**Claim 3.3.** Let $H, K, G$ be three Lie groups, and $P \to X$ an $H$-equivariant and $K$-equivariant principal $G$-bundle. Assume that $H$ acts freely on $X$, and that the $H$ and $K$-actions on $P$ commute (i.e., $h \cdot (k \cdot y) = k \cdot (h \cdot y)$ for all $h \in H$, $k \in K$, $y \in P$), then:

1. $\pi: P/H \to X/H$ is a $K$-equivariant principal $G$-bundle.
2. If $\theta: TP \to \mathfrak{g}$ is a connection 1-form, and $q: P \to P/H$ is the quotient map, then $\theta = q^*(\bar{\theta})$ for some connection 1-form $\bar{\theta}: T(P/H) \to \mathfrak{g}$ if and only if $\theta$ is $H$-invariant, and $\theta(\xi_P) = 0$ for all $\xi \in \mathfrak{h}$.

**Proof.**

1. The surjection $P/H \to M/H$, induced from $\pi: P \to M$, and the right $G$-action on those quotient spaces are well defined since the left $H$-action commutes with the right $G$-action on $P$, and with the projection $\pi$.

To show that $P/H \to X/H$ is a principal $G$-bundle, it suffices to check that $G$ acts freely on $P/H$. Indeed, if $[p] \in P/H$, $g \in G$ and $[p] \cdot g = [p]$, then this implies

$$[p \cdot g] = [p] \quad \Rightarrow \quad p \cdot g = h \cdot p$$

for some $h \in H$, which implies

$$\pi(p \cdot g) = \pi(h \cdot p) \quad \Rightarrow \quad \pi(p) = h \cdot \pi(p).$$

But $H \circ X$ freely, and so $h = id$. Then $p \cdot g = p$, and since $P \circ G$ freely, we conclude that $g = id$, as needed.

It is easy to check that the $K$-action descends to $P/H \to X/H$, since it commutes with the $H$ and the $G$-actions.

2. First assume that $\theta = q^*(\bar{\theta})$. If $h \in H$ acts on $P$, then

$$h \cdot \theta = h^* (q^* \bar{\theta}) = (q \circ h)^* \bar{\theta} = q^* \bar{\theta} = \theta$$

and so $\theta$ is $H$-invariant. Also, if $\xi \in \mathfrak{h}$, then clearly $q_*(\xi_P) = 0$, and hence

$$\theta(\xi_P) = (q^* \bar{\theta})(\xi_P) = 0,$$

as needed.
Conversely, assume that $\theta$ is $H$-invariant and that $\theta(\xi_P) = 0$ for all $\xi \in \mathfrak{h}$. For any $v \in TP$ define
\[
\tilde{\theta}(q_*v) = \theta(v).
\]
This is well defined: If $v \in T_yP$ and $v' \in T_{y'}P$ such that $q_*(v) = q_*(v')$, then $y' = h \cdot y$ for some $h \in H$, and we get that
\[
\theta_{y'}(v') = \theta_{h \cdot y}(v') = h^*(\theta_y((h^{-1})_*v')) = \theta_y((h^{-1})_*v').
\]
Now observe that
\[
q^*(v - (h^{-1})_*v') = q_*(v) - q_*(v') = 0,
\]
and so $v - (h^{-1})_*v' = \xi_P|_x$ (for some $\xi \in \mathfrak{h}$) is in the vertical bundle of $P \to P/H$. By assumption, $\theta(\xi_P) = 0$ and therefore $\theta_y(v) = \theta_{y'}(v')$, and $\tilde{\theta}$ is well defined.

The map $\tilde{\theta} : T(P/H) \to \mathfrak{g}$ is a 1-form. Smoothness is implied from the definition of the smooth structure on $P/H$. Also $\tilde{\theta}$ is vertical and $G$-equivariant because $\theta$ is.

Now assume that $Z$ is an $n$-dimensional oriented Riemannian manifold, and $S^1$ acts freely on $Z$ by isometries. Let $P \to \text{SOF}(Z) \to Z$ be a $G$ and $S^1$-equivariant spin$^c$ structure on $Z$. We would like to explain how one can get a $G$-equivariant spin$^c$ structure on $Z/S^1$, induced from the given one on $Z$.

Denote by $\frac{\partial}{\partial \varphi} \in \text{Lie}(S^1) \simeq i\mathbb{R}$ the generator, and by $\left(\frac{\partial}{\partial \varphi}\right)_Z$ the corresponding vector field on $Z$. Define the normal bundle
\[
V = \left[\left(\frac{\partial}{\partial \varphi}\right)_Z\right] \subseteq TZ
\]
and an embedding $\eta : \text{SOF}(V) \to \text{SOF}(Z)$ as follows. If $f : \mathbb{R}^{n-1} \xrightarrow{\sim} V_x$ is a frame in $\text{SOF}(V)$, then $\eta(f) : \mathbb{R}^n \xrightarrow{\sim} T_xZ$ will be given by $\eta(f)e_i = f(e_i)$ for $i = 1, \ldots, n-1$, and $\eta(f)e_n$ is the unit vector in the direction of $\left(\frac{\partial}{\partial \varphi}\right)_{Z,x}$.

\[
\begin{array}{ccc}
\eta^*(P) & \xrightarrow{\eta'} & P \\
\downarrow & & \downarrow \\
\text{SOF}(V) & \xrightarrow{\eta} & \text{SOF}(Z) \\
\downarrow & & \downarrow \\
Z & \xrightarrow{} & Z
\end{array}
\]

To get a spin$^c$ structure on $Z/S^1$, first consider the equivariant spin$^c$ structure on the vector bundle $V$
\[
\eta^*(P) \to \text{SOF}(V) \to Z.
\]
Once we take the quotient by the circle action, we get the quotient spin$^c$ structure on $Z/S^1$, denoted by $\bar{P}$:
\[
\bar{P} = \eta^*(P)/S^1 \to \text{SOF}(V)/S^1 \cong \text{SOF}(Z/S^1) \to Z/S^1.
\]
If an $S^1$ and $\text{Spin}^c(m)$-invariant connection 1-form $\theta$ is given on the principal circle bundle $P \to \text{SOF}(Z)$, then $(\eta^*)^*\theta$ is a connection 1-form on the principal circle bundle $\eta^*(P) \to \text{SOF}(V)$.

The previous claim tells us exactly when the above connection 1-form will descend to a connection 1-form on the quotient bundle $\bar{P} \to \text{SOF}(Z/S^1)$. The following proposition summarizes the above construction and relates it to $\text{spin}^c$ prequantization.

**Proposition 3.1.** Assume that the following data is given:

1. An $n$-dimensional Riemannian oriented manifold $Z$.
2. A real closed 2-form $\omega$ on $Z$.
3. Actions of a Lie group $G$ and $S^1$ on $Z$, by orientation preserving and $\omega$-invariant isometries.
4. A $G$ and $S^1$-equivariant $\text{spin}^c$ prequantization $(P, \theta)$ on $Z$. Assume that the actions of $G$ and $S^1$ on $P$ and $Z$ commute with each other. Also assume that the action $S^1 \lor Z$ is free.

Then, using the above notation, we have that:

1. $\theta' = (\eta^*)^*\theta$ is a connection 1-form on the principal circle bundle $\pi: \eta^*(P) \to \text{SOF}(V)$, satisfying

\[ d\theta' = \pi^*(-i \cdot \omega), \]

and

\[ \theta'(\zeta_{\eta^*(P)}) = 0 \quad \text{for all} \quad \zeta \in \text{spin}(m-1). \]

2. If $\left(\frac{\partial}{\partial \varphi}\right)_{\eta^*(P)}$ is the vector field generated by the action $S^1 \lor \eta^*(P)$, and $q: \eta^*(P) \to \bar{P} = \eta^*(P)/S^1$ is the quotient map, then $\theta' = q^*(\bar{\theta})$ for some connection 1-form $\bar{\theta}$ on $\bar{P} \to \text{SOF}(Z/S^1)$ if and only if

\[ \theta' \left[ \left(\frac{\partial}{\partial \varphi}\right)_{\eta^*(P)} \right] = 0. \]

Moreover, in this case, $(\bar{P}, \bar{\theta})$ is a $G$-equivariant $\text{spin}^c$ prequantization for $G \lor (Z/S^1, \bar{\omega})$ (where $\omega = q^*(\bar{\omega})$).

**Proof.**

1. We have

\[ d\theta' = (\eta^*)^*d\theta = (\eta^*)^* \circ \pi^*(-i \cdot \omega) = \pi^*(-i \cdot \omega) \]

and

\[ \theta'(\zeta_{\eta^*(P)}) = \theta(\zeta_P) = 0 \]

as needed.

2. The fact that $\theta' = q^*(\bar{\theta})$ if and only if

\[ \theta' \left[ \left(\frac{\partial}{\partial \varphi}\right)_{\eta^*(P)} \right] = 0 \]

follows directly from Claim 3.3 since $\theta'$ is $S^1$-invariant, and $\frac{\partial}{\partial \varphi}$ is a generator.

Finally, $(\bar{P}, \bar{\theta})$ is a prequantization, since
\[ q^*(d\bar{\theta}) = d\theta' = \pi^*(-i \cdot \omega) = q^*\bar{\pi}^*(-i \cdot \bar{\omega}) \Rightarrow d\bar{\theta} = \bar{\pi}^*(-i \cdot \bar{\omega}) \]

where \( \bar{\pi}: \eta^*(P)/S^1 \rightarrow Z/S^1 \) is the projection. Clearly, since all our objects are \( G \)-invariant, and all the actions commute, \((\bar{P}, \bar{\theta})\) is a \( G \)-equivariant prequantization. \( \square \)

**Remark 3.2.** When the condition in part (2) of the above proposition holds, we will say that the prequantization \((P, \theta)\) for \( G \odot (Z, \omega) \) descends to the prequantization \((\bar{P}, \bar{\theta})\) for \( G \odot (Z/S^1, \bar{\omega}) \).

### 3.4. The cutting of a prequantization.

In [2], Lerman describes a cutting construction for symplectic manifolds \((M, \omega)\), endowed with a Hamiltonian circle action and a moment map \( \Phi: M \rightarrow u(1)^* \), which goes as follows. If \( \omega_C = -i \cdot dz \wedge d\bar{z} \), then \((M \times \mathbb{C}, \omega \oplus \omega_C)\) is a symplectic manifold. The action

\[ S^1 \times (M \times \mathbb{C}) \rightarrow M \times \mathbb{C}, \quad (a, (m, z)) \mapsto (a \cdot m, a^{-1} \cdot z) \]

is Hamiltonian with moment map \( \Phi(m, z) = \Phi(m) - |z|^2 \).

If \( \alpha \in u(1)^* \) and \( S^1 \) acts freely on \( Z = \Phi^{-1}(\alpha) \), then \( \alpha \) is a regular value of \( \Phi \), and the (positive) cut space is defined by

\[ M^+_\text{cut} = \Phi^{-1}(\alpha)/S^1 = \{ (m, z) \in M \times \mathbb{C} : \Phi(m) - |z|^2 = \alpha \} \]

This is a symplectic manifold, with the symplectic form \( \omega^+_\text{cut} \) obtained by reduction, and \( S^1 \) acts on \( M^+_\text{cut} \) by \( a \cdot [m, z] = [a \cdot m, z] \). If \( M \) is also Riemannian oriented manifold, so is the cut space (but the natural inclusion \( M^+_\text{cut} \hookrightarrow M \) is not an isometry).

Assume that the following is given:

1. An \( m \)-dimensional oriented Riemannian manifold.
2. A closed real two-form \( \omega \) on \( M \).
3. An action of \( S^1 \) on \( M \) by \( \omega \)-invariant isometries.
4. An \( S^1 \)-equivariant spin\(^c\) prequantization \((P, \theta) = (P_M, \theta_M)\) for \((M, \omega)\).

Recall that the action \( S^1 \odot (M, \omega) \) is Hamiltonian, with moment map \( \Phi: M \rightarrow u(1)^* \) determined by the equation

\[ \pi^*(\Phi^\xi) = -i \cdot \iota_{\xi_P}(\theta) \quad , \quad \xi \in u(1) \]

where \( \pi: P \rightarrow M \) is the projection, and \( \xi_P \) is the vector field on \( P \) generated by the \( S^1 \)-action (see Remark 2.7).

We want to cut the given spin\(^c\) prequantization. For that we choose \( \alpha \in u(1)^* \) and set \( Z = \Phi^{-1}(\alpha) \). We assume that \( S^1 \) acts on \( Z \) freely, and that \( \alpha \) is a regular value of \( \Phi \) (however, we do not assume that \( \omega \) is nondegenerate). Our goal is to get a condition on \( \alpha \) such that cutting along \( Z = \Phi^{-1}(\alpha) \) is possible (i.e., such that a spin\(^c\) prequantization on the cut space is obtained).

We proceed according to the following steps.
Step 1: Let $S^1$ act on the complex plane via
\[(a, z) \mapsto a^{-1} \cdot z, \quad a \in S^1, \quad z \in \mathbb{C} .\]
This action preserves the standard Riemannian structure and orientation, and the two form $\omega_C = -i \cdot dz \wedge d\bar{z}$.

Fix an odd integer $\ell$, and consider the $S^1$-equivariant spin$^c$ prequantization $(P^\ell_C, \tilde{\theta}_C)$ for $S^1 \circ (\mathbb{C}, \omega_C)$ defined in [2,4].

Step 2: Using Claim 3.1 we obtain an $S^1$-equivariant spin$^c$ prequantization $(P_{M \times \mathbb{C}, \theta_{M \times \mathbb{C}}})$ for $S^1 \circ (M \times \mathbb{C}, \omega \oplus \omega_C)$.

Step 3: Denote
\[\tilde{Z} = \{(m, z) : \Phi(m) - |z|^2 = \alpha\} \subset M \times \mathbb{C} .\]
This is an $S^1$-invariant submanifold of codimension 1. By Claim 3.2 we get an $S^1$-equivariant spin$^c$ prequantization $(P_{\tilde{Z}}, \theta_{\tilde{Z}})$ for $(\tilde{Z}, \omega_{\tilde{Z}})$, where $\omega_{\tilde{Z}}$ is the restriction of $\omega \oplus \omega_C$ to $\tilde{Z}$.

Step 4: By Remark 3.3 the pair $(P_{\tilde{Z}}, \theta_{\tilde{Z}})$ is an $S^1$-equivariant prequantization with respect to both the anti-diagonal and the ‘M-action’ (in which $S^1$ acts on the $M$ component via the given action, and on the $\mathbb{C}$ component trivially).

Using the terminology introduced in Remark 3.2 we state our main theorem, which enable us to complete the process and get an equivariant prequantization on the (positive) cut space.

Theorem 3.1. The $S^1$-equivariant spin$^c$ prequantization $(P_{\tilde{Z}}, \theta_{\tilde{Z}})$ descends to an $S^1$-equivariant spin$^c$ prequantization on $(\tilde{Z}/S^1 = M^+_\text{cut}, \omega^+_\text{cut})$ if and only if
\[\alpha = \frac{\ell}{2} \in u(1)^* = \mathbb{R} .\]

Proof. By Proposition 3.1 $(P_{\tilde{Z}}, \theta_{\tilde{Z}})$ will descend to a prequantization on the cut space, if and only if
\[\theta'_{\tilde{Z}} \left[ \frac{\partial}{\partial \varphi} \right]_{\eta^*(P_{\tilde{Z}})} = 0 .\]
This is the same as requiring that $\theta_{\tilde{Z}}$, when restricted to $\eta^*(P_{\tilde{Z}})$, vanishes:
\[\theta_{\tilde{Z}} \left[ \frac{\partial}{\partial \varphi} \right]_{P_{\tilde{Z}}} \bigg|_{\eta^*(P)} = 0 ,\]
which is equivalent to
\[\theta_{M \times \mathbb{C}} \left[ \frac{\partial}{\partial \varphi} \right]_{P_{M \times \mathbb{C}}} = 0 \text{ on } \eta^*(P_{\tilde{Z}}) .\]
Now using the formula for $\theta_{M \times \mathbb{C}}$, we get that
\[\theta_M \left( \frac{\partial}{\partial \varphi} \right)_{P_M} + \theta_{\mathbb{C}} \left( \frac{\partial}{\partial \varphi} \right)_{P_{\mathbb{C}}} = 0 .\]
It is not hard to show that at a point \((z, [A, w]) \in P^\ell_C = \mathbb{C} \times Spin^c(2)\), we have
\[
\left( \frac{\partial}{\partial \varphi} \right)_{P^\ell_C} = i \cdot \left[ \bar{z} \frac{\partial}{\partial \bar{z}} - z \frac{\partial}{\partial z} \right] + \nu|_{[A, w]}
\]
where \(\nu|_{[A, w]}\) is the vector field on \(Spin^c(2)\) generated by the element
\[
\nu = -\frac{1}{2} e_1 e_2 - \frac{i}{2} \ell \in \text{spin}^c(2).
\]
Therefore one computes that
\[
\theta_C \left( \left( \frac{\partial}{\partial \varphi} \right)_{P^\ell_C} \right) = -i \cdot \left( |z|^2 + \frac{\ell}{2} \right)
\]
On the other hand, by the condition defining our moment map, we have that
\[
\theta_M \left( \left( \frac{\partial}{\partial \varphi} \right)_{P_M} \right) = i \cdot \pi^* \left( \Phi \frac{\partial}{\partial \varphi} \right)
\]
where \(\pi: P \to M\) is the projection.
Combining the above we see that \((P, \theta)\) descends to an \(S^1\)-equivariant spin\(^c\) prequantization on \((\tilde{Z} / S^1, \omega^\ell)\) if and only if \((\pi^* (\Phi_{\theta/\partial \varphi}))\):
\[
\pi^* \left( \Phi \frac{\partial}{\partial \varphi} \right) - |z|^2 - \frac{\ell}{2} = 0.
\]
But on the manifold \(\tilde{Z}\) we have \(\Phi(m) - |z|^2 = \alpha\). and hence the last equality is equivalent to
\[
\alpha - \frac{\ell}{2} = 0,
\]
as needed.

\[\square\]

**Remark 3.3.** We can also construct a spin\(^c\) prequantization for the negative cut space \((M_{\text{cut}}^-, \omega_{\text{cut}}^-)\) as follows. Recall that \(M_{\text{cut}}^-\) is defined as the quotient
\[
\{(m, z) \in M \times \mathbb{C} : \Phi(m) + |z|^2 = \alpha \}\, / S^1,
\]
where the \(S^1\)-action on \(M \times \mathbb{C}\) is taken to be the diagonal action, and \(\omega_{\text{cut}}^-\) is defined as before by reduction. The two form on \(\mathbb{C}\) is taken to be \(i dz \wedge d\bar{z}\), and the spin\(^c\) prequantization for \(\mathbb{C}\) is defined using the connection
\[
\theta_{\mathbb{C}} = \pi^* (\theta^R) - \frac{1}{2} (\bar{z} dz - zd\bar{z}).
\]
The \(S^1\)-action on \(P^\ell_C\) will be given by
\[
S^1 \times P^\ell_C \to P^\ell_C, \quad (e^{i \varphi}, (z, [a, w])) \mapsto (e^{i \varphi}, [x_{\varphi/2} \cdot a, e^{-i \varphi} / 2 \cdot w])
\]
(see \[2.4\]).

Other than that, the construction is carried out as for the positive cut space, and we can prove a theorem that will assert that \(\alpha = \ell/2\), if the cutting is to be done along the level set \(\Phi^{-1}(\alpha)\) of the moment map.

**4. An example - The two sphere**

In this section we discuss in detail spin\(^c\) prequantizations and cutting for the two-sphere.
4.1. Prequantizations for the two-sphere. The two-sphere will be thought of as a submanifold of $\mathbb{R}^3$:

$$S^2 = \{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1 \}$$

with the outward orientation and natural Riemannian structure induced from the inner product in $\mathbb{R}^3$. Fix a real number $c$, and let $\omega = c \cdot A$, where $A$ is the area form on the two-sphere

$$A = j^*(x \, dy \wedge dz + y \, dz \wedge dx + z \, dx \wedge dy),$$

and where $j : S^2 \hookrightarrow \mathbb{R}^3$ is the inclusion. Note that $\omega$ is a symplectic form if and only if $c \neq 0$.

For any real $\varphi$ define

$$C_\varphi = \begin{pmatrix}
\cos \varphi & -\sin \varphi & 0 \\
\sin \varphi & \cos \varphi & 0 \\
0 & 0 & 1
\end{pmatrix},$$

and let $S^1$ act on $S^2$ via rotations around the $z$-axis, i.e.,

$$(e^{i\varphi}, v) \mapsto C_\varphi \cdot v, \quad v \in S^2.$$

In Section 7 of [3], we constructed all $S^1$-equivariant spin$^c$-structures over the $S^1$-manifold $S^2$ (up to equivalence). Let us review the main ingredients here.

First, the trivial spin$^c$ structure $P_0$ is given by the following diagram.

$$\begin{array}{ccc}
S^1 \times Spin^c(3) & \longrightarrow & P_0 = Spin^c(3) \\
\downarrow & & \downarrow \\
SO(3) & \leftarrow & Spin^c(3) \times Spin^c(2)
\end{array}$$

$$\begin{array}{ccc}
S^1 \times SO(3) & \longrightarrow & SO(3) \\
\downarrow & & \downarrow \\
S^1 \times S^2 & \longrightarrow & S^2
\end{array}$$

In this diagram we use the fact that the frame bundle of $S^2$ is isomorphic to $SO(3)$. The projection $\pi$ is given by $A \mapsto A \cdot x$.

where $x = (0, 0, 1)$ is the north pole, and the map $\Lambda$ is the obvious one.

The horizontal maps describe the $S^1$ and the principal actions: $S^1$ and $SO(3)$ act on $SO(3)$ by left and right multiplication by $C_\varphi$, respectively. The principal action of $Spin^c(2)$ on $Spin^c(3)$ is just right multiplication, and the $S^1$ action on $Spin^c(3)$ is given by

$$(e^{i\varphi}, [A, z]) \mapsto [x_{\varphi/2} \cdot A, \, e^{i\varphi} \cdot z]$$

where $x_{\varphi/2} = \cos \varphi + \sin \varphi \cdot e_1 e_2 \in Spin(3)$. We can turn this spin$^c$ structure into a spin$^c$ prequantization as follows. Let $\omega_0 = 0$ the zero two form on $S^2$, and consider the 1-form

$$\theta_0 = \frac{1}{2} \det_s \circ \theta^R : TSpin^c(3) \to \mathfrak{u}(1) = i\mathbb{R}$$

where $\theta^R$ is the right-invariant Maurer-Cartan form on $Spin^c(3)$ and the map $\det$ was defined in [2,3]. Clearly, $(P_0, \theta_0)$ is an $S^1$-equivariant spin$^c$ prequantization for $(S^2, \omega_0)$.

Next, we construct all $S^1$-equivariant line bundles over $S^2$. 

Claim 4.1. Given a pair of integers \((k, n)\), define an \(S^1\)-equivariant complex Hermitian line bundle \(L_{k,n}\) as follows:

1. As a complex line bundle,
\[
L_{k,n} = S^3 \times_{S^1} \mathbb{C}
\]
where \(S^1\) acts on \(\mathbb{C}\) with weight \(n\) and on \(S^3 \subset \mathbb{C}^2\) by
\[
S^1 \times S^3 \to S^3, \quad (a, (z, w)) \mapsto (az, aw).
\]

2. The circle group \(S^1\) acts on \(L_{k,n}\) by
\[
S^1 \times L_{k,n} \to L_{k,n}, \quad (e^{i\varphi}, ([z, w], u)) \mapsto [(e^{i\varphi/2}z, e^{-i\varphi/2}w), e^{i(n+2k)\varphi/2}u].
\]

Then every equivariant line bundle over \(S^2\) is equivariantly isomorphic to \(L_{k,n}\) for some integers \(k, n\).

For the proof, see Claim 7.1 in [3] (where slightly different notation is used).

To get all \(\text{Spin}^c\) structures on \(S^2\), we need to twist \(P_0\) with the \(U(1)\)-bundle \(U(L_{k,n})\) associated to \(L_{k,n}\) for some \(k, n \in \mathbb{Z}\). Thus define
\[
P_{k,n} = P_0 \times_{U(1)} U(L_{k,n}).
\]

The principal \(\text{Spin}^c(2)\)-action is given coming from the action on \(P_0\), and the left \(S^1\)-action in induced from the diagonal action.

We now define a connection
\[
\theta_n : TP_{k,n} \to i\mathbb{R}
\]
on the \(U(1)\) bundle \(P_{k,n} \to SO(3) = SOF(S^2)\), which will not depend on \(k\), as follows:
\[
\theta_n = \theta_0 + \frac{n}{2} \left(-\bar{z} dz + z d\bar{z} - \bar{w} dw + w d\bar{w}\right) + u^{-1} du
\]
where \((z, w) \in S^3 \subset \mathbb{C}^2\) are coordinates on \(S^3\) and \(u^{-1} du\) is the Maurer-Cartan form on the \(S^1\) component of \(U(L_{k,n}) = S^3 \times_{S^1} S^1\).

One can compute
\[
d\theta_n = n(dz \wedge d\bar{z} + dw \wedge d\bar{w}) = \pi^*(-in/2 \cdot A)
\]
and hence if we define \(\omega_n = \frac{\pi}{2} \cdot A\) then \((P_{k,n}, \theta_n)\) is a \(\text{Spin}^c\) prequantization for \((S^2, \omega_n)\).

Let \(P_{\text{det}}\) be the \(U(1)\)-bundle associated to the determinant line bundle of a \(\text{Spin}^c\) structure. We proved in Section 7 of [3], that the determinant line bundle of any \(\text{Spin}^c\) structure on the two-sphere is isomorphic to \(L_{2k+1, 2n}\), and hence has a square root (as a non-equivariant line bundle). Using this fact and the construction of \((P_{k,n}, \theta_n)\) above, we prove:

Claim 4.2. The \(S^1\)-manifold \((S^2, \omega = c \cdot A)\) is \(\text{spin}^c\)-prequantizable (i.e., admits an \(S^1\)-equivariant \(\text{spin}^c\) prequantization) if and only if \(2c \in \mathbb{Z}\).

Proof. Assume that \((P, \theta)\) is a \(\text{spin}^c\)-prequantization for \((S^2, \omega)\). Then, by Claim 2.2, \(\theta = \frac{1}{2}q^*\widehat{\theta}\) for some connection 1-form \(\widehat{\theta}\) on the principal \(U(1)\)-bundle \(p : P_{\text{det}} \to S^2\), where \(q : P \to P/\text{Spin}(2) = P_{\text{det}}\) is the quotient map. Since \((P, \theta)\) is a \(\text{spin}^c\)-prequantization, we have
\[
d\theta = \pi^*(-i \cdot \omega) \quad \Rightarrow \quad q^* \left(\frac{1}{2} d\theta\right) = q^* p^* (-i \cdot \omega) \quad \Rightarrow \quad \frac{1}{2} d\theta = p^* (-i \cdot \omega)
\]
which implies 
\[ d\tilde{\theta} = p^*(-2i \cdot \omega). \]
This means that \([-2i \cdot \omega]\) is the curvature class of the determinant line bundle of \(P\). According to the above remark, \(P_{\text{det}}\) is a square, and hence the class 
\[ \frac{1}{2} [-2i \cdot \omega] = [-i \cdot \omega] \]
is a curvature class of a line bundle over \(S^2\). This forces \([−2i \cdot \omega]\) to be integral (Weyl’s theorem - page 172 in [1]), i.e.,
\[ \int_{S^2} \omega \in 2\pi \mathbb{Z} \Rightarrow 2c \in \mathbb{Z} \]
and the conclusion follows.
Conversely, assume that \(2c \in \mathbb{Z}\). Then, as mentioned above, \((P_{k,2c}, \theta_{2c})\) (for any \(k \in \mathbb{Z}\)) is a spin\(^c\) prequantization for \((S^2, c \cdot A)\) as needed. □

Let us now compute the moment map
\[ \Phi : S^2 \to u(1)^* = \mathbb{R} \]
for \((S^2, n/2 \cdot A)\) (for \(n \in \mathbb{Z}\)) determined by the prequantization \((P_{k,n}, \theta_n)\). Recall that 
\[ \theta_n = \theta_0 + \frac{n}{2} (-\bar{z} dz + z d\bar{z} - \bar{w} dw + w d\bar{w}) + u^{-1} du. \]
It is straightforward to show that the vector field, generated by the left \(S^1\)-action on \(P_{k,n}\) is 
\[ \left( \frac{\partial}{\partial \varphi} \right)_{P_{k,n}} = \frac{i}{2} \frac{\partial}{\partial v} - \frac{i}{2} \left( -\bar{z} \frac{\partial}{\partial z} + z \frac{\partial}{\partial \bar{z}} + \bar{w} \frac{\partial}{\partial w} - w \frac{\partial}{\partial \bar{w}} \right) + \frac{i}{2} (n + 2k) \frac{\partial}{\partial u} \]
where \(\frac{\partial}{\partial \varphi}\) is the vector field on \(P_0\) generated by the \(S^1\)-action.

Now compute 
\[ \theta_n \left( \left( \frac{\partial}{\partial \varphi} \right)_{P_{k,n}} \right) = \frac{i}{2} - \frac{in}{4} (-\bar{z}z + z(-\bar{z}) - \bar{w}w + w\bar{w}) + \frac{i}{2} (n + 2k) = \]
\[ = \frac{i}{2} [n(|z|^2 - |w|^2) + n + 2k + 1] \]
and thus \(\Phi\) is given by
\[ \Phi([z, w]) = -i \cdot \theta_n \left( \left( \frac{\partial}{\partial \varphi} \right)_{P_{k,n}} \right) = \frac{n}{2} (|z|^2 - |w|^2 + 1) + k + \frac{1}{2} \]

Remark 4.1. Observe that for \([z, w] \in S^2 = \mathbb{C}P^1\), the quantity \(|z|^2 - |w|^2\) represents the third coordinate \(x_3\) (i.e., the height) on the unit sphere (this is part of the Hopf-fibration). Since \(-1 \leq x_3 \leq 1\), we have (for \(n \geq 0\)):
\[ k + \frac{1}{2} \leq \Phi \leq n + k + \frac{1}{2} \]
and hence the image of the moment map is the closed interval 
\[ \left[k + \frac{1}{2}, n + k + \frac{1}{2}\right] \]
if $n \geq 0$ or
\[
\left[ n + k + \frac{1}{2}, k + 1 \right]
\]
if $n \leq 0$.

4.2. Cutting a prequantization on the two-sphere. Fix an $S^1$-equivariant spin$^c$-prequantization $(P_{k,n}, \theta_n)$ for $(S^2, \omega_n)$, where $\omega_n = \frac{n}{2} \cdot A$ (A is the area form on the two-sphere) and $n \neq 0$.

The corresponding moment map, as computed above, is
\[
\Phi : S^2 \to \mathbb{R}, \quad \Phi([z, w]) = \frac{n}{2} \left( |z|^2 - |w|^2 + 1 \right) + k + \frac{1}{2}
\]

We would like to cut this prequantization along a level set $\Phi^{-1}(\alpha)$ of the moment map. By Theorem 3.1 we must have $\alpha = \frac{\ell}{2}$, for some odd integer $\ell$, and the cutting has to be done using the spin$^c$ structure $(P_{\ell C}, \theta_C)$ on $(C, \omega_C)$ (see §2.4).

In [3, Section 7] we performed the cutting construction for the two-sphere in the case where $\ell = 1$. In this case we showed that the spin$^c$ structures obtained for the cut spaces are
\[
(P_{k,n})^+_{cut} = P_{0,k+n}, \quad (P_{k,n})^-_{cut} = P_{k,-k}.
\]
The computations in [3] can be modified for an arbitrary $\ell$ to get
\[
(P_{k,n})^+_{cut} = P_{(\ell-1)/2,k+n-(\ell-1)/2}, \quad (P_{k,n})^-_{cut} = P_{k,-k+(\ell-1)/2}.
\]

Recall that the cut spaces obtained in this case are symplectomorphic to two-spheres (if $\ell/2$ is strictly between $k + \frac{1}{2}$ and $n + k + \frac{1}{2}$). Using this identification we have:

Claim 4.3. If the symplectic manifold $(S^2, \omega_n)$, endowed with the Hamiltonian $S^1$-action
\[
(e^{i\phi}, v) \mapsto C_{\phi} \cdot v
\]
and the above moment map $\Phi$ is being cut along the level set $\Phi^{-1}(\ell/2)$, then the reduced two-forms on the cut spaces are
\[
\omega^+_{cut} = \omega_{k+n+(1-\ell)/2} \quad \text{and} \quad \omega^-_{cut} = \omega_{-k+(1-\ell)/2}.
\]
Here we assume that $\ell/2$ is strictly between $k + \frac{1}{2}$ and $n + k + \frac{1}{2}$.

Proof. Let us concentrate on the positive cut space. We will use cylindrical coordinates $(\phi, h)$ to describe the point
\[
(x, y, z) = (\sqrt{1-h^2} \cos \phi, \sqrt{1-h^2} \sin \phi, h)
\]
on the unit sphere $S^2$. The positive cut space is obtained by reduction. The relevant diagram is
\[
\begin{array}{c}
\hat{Z} \xrightarrow{i} S^2 \times C \\
p \downarrow \\
\hat{Z}/S^1 \cong S^2
\end{array}
\]
Recall that
\[
\tilde{Z} = \{ (\phi, h, u) \in S^2 \times \mathbb{C} : \Phi(\phi, h) - |u|^2 = \ell/2 \}
\]
and that the two-form on \( S^2 \times \mathbb{C} \) is
\[
\omega_n + \omega_C = \frac{n}{2} \cdot A - i \, du \wedge d\bar{u} .
\]

The map \( p \) is given by
\[
((\phi, h), u = re^{-i\alpha}) \mapsto (\phi + \alpha, \frac{2n}{2n + 2k + 1 - \ell} (h - 1) + 1).
\]

The pullback of the area form on \( S^2 \) via \( p \) is
\[
A' = (d\phi + d\alpha) \wedge \frac{2n}{2n + 2k + 1 - \ell} \, dh = \frac{2n}{2n + 2k + 1 - \ell} (d\phi \wedge dh - \frac{2i}{n} du \wedge d\bar{u}) ,
\]
and thus the pullback of \( \omega_{k+n+(1-\ell)/2} \) via \( p \) is
\[
\frac{k + n + (1 - \ell)/2}{2} \cdot A' = \frac{n}{2} A - i \, du \wedge d\bar{u} = \omega_n + \omega_C
\]
as needed.

A similar proof is obtained for the negative cut space. \( \square \)

To complete the cutting, we need to find out what are the corresponding connections \( \theta^\pm = (\theta_n)^\pm_{\text{cut}} \) on \( (P_{k,n})^\pm_{\text{cut}} \). Instead of going through the cutting process of a connection, we proceed as follows (for the positive cut space).

We know that \( ((S^2)^+_{\text{cut}}, \theta^+_{\text{cut}}) \) must be a spin$^c$ prequantization for
\[
((S^2)^+_{\text{cut}}, \omega^+_{\text{cut}}) = (S^2, \omega_{k+n+(1-\ell)/2}) .
\]
This means that
\[
d\theta^+ = d\theta_{k+n+(1-\ell)/2}
\]
which implies that
\[
\theta^+ - \theta_{k+n+(1-\ell)/2} = \pi^* \beta
\]
for some closed one-form \( \beta \in \Omega^1(S^2; \mathfrak{u}(1)) \). But then \( \beta = df \) is also exact since \( S^2 \) is simply connected. We conclude that
\[
\theta^+ = \theta_{k+n+(1-\ell)/2} + d(\pi^* (f)) ,
\]
thus, the bundle \( ((P_{k,n})^+_{\text{cut}}, \theta^+) \) is gauge equivalent to \( ((P_{k,n})^+_{\text{cut}}, \theta_{k+n+(1-\ell)/2}) \).

A similar argument can be carried out for the negative cut space. We summarize:

**The cutting of \( (S^2, \omega_n) \) along the level set \( \Phi^{-1}(\ell/2) \) yields two spin$^c$ prequantizations:**

\[
(P_{-k+(\ell-1)/2}, \theta_{-k+(\ell-1)/2}) \quad \text{for} \quad ((S^2)^-_{\text{cut}} = S^2, \omega_{-k+(\ell-1)/2})
\]
and
\[
(P_{(\ell-1)/2,k+n+(1-\ell)/2}, \theta_{k+n+(1-\ell)/2}) \quad \text{for} \quad ((S^2)^+_{\text{cut}} = S^2, \omega_{k+n+(1-\ell)/2}) .
\]
In this section we construct a spin\(^c\) prequantization for the complex projective space \(\mathbb{C}P^n\) (with the standard Riemannian structure coming from the Kähler structure). For \(n = 1\) we have shown that a two form \(\omega\) on \(\mathbb{C}P^1 \cong S^2\) is spin\(^c\) prequantizable if and only if \(\frac{1}{2\pi} \omega\) is integral (i.e., \(\int_{\mathbb{C}P^1} \frac{1}{2\pi} \omega \in \mathbb{Z}\) - see Claim 4.2). This is not true in general. We will prove that for an even \(n\), if \((\mathbb{C}P^n, \omega)\) is spin\(^c\) prequantizable then \(\frac{1}{2\pi} \omega\) will not be integral. This is an important difference between spin\(^c\) prequantization and the geometric prequantization scheme of Kostant and Souriau (an excellent reference for geometric quantization is [5]).

From now on, fix a positive integer \(n\). Points in \(\mathbb{C}P^n\) will be written as \([v]\), where \(v \in S^{2n+1} \subset \mathbb{C}^{n+1}\). The Fubini-Study form \(\omega_{FS}\) on \(\mathbb{C}P^n\) will be normalized (as in [6, page 261]) so that \(\int_{\mathbb{C}P^1} \omega_{FS} = 1\) (where \(\mathbb{C}P^1\) is naturally embedded into \(\mathbb{C}P^n\)).

We describe our construction in steps. For simplicity, we discuss the non-equivariant case (where the acting group \(G\) is the trivial group), but our results will apply to the equivariant case as well. Also, \(|\cdot|\) will denote the determinant of a matrix.

**Step 1 - Constructing a Spin\(^c\) structure.**

The group \(SU(n + 1)\) acts transitively on \(\mathbb{C}P^n\) via

\[
SU(n + 1) \times \mathbb{C}P^n \to \mathbb{C}P^n, \quad (A, [v]) \mapsto [A \cdot v].
\]

Let \(p = e_{n+1} \in \mathbb{C}^{n+1}\) denote the unit vector \((0, \ldots, 0, 1)\). The stabilizer of \(p\) under the \(SU(n + 1)\)-action is

\[
H = S(U(n) \times U(1)) = \left\{ \begin{pmatrix} B & 0 \\ 0 & |B|^{-1} \end{pmatrix} : B \in U(n) \right\} \subset SU(n + 1)
\]

and so \(\mathbb{C}P^n \cong SU(n + 1)/H\) via

\[
[A] \mapsto [A \cdot p].
\]

The tangent space \(T_p \mathbb{C}P^n\) can be identified with \(\mathbb{C}^n\) and then the isotropy representation is given by

\[
\sigma : H \to U(n), \quad \sigma \left( \begin{pmatrix} B & 0 \\ 0 & |B|^{-1} \end{pmatrix} \right) = |B| \cdot B.
\]

The frame bundle of \(\mathbb{C}P^n\) can then be described as an associated bundle (using \(U(n) \subset SO(2n)\)):

\[
SOF(\mathbb{C}P^n) = SU(n + 1) \times_{\sigma} SO(2n).
\]

The map

\[
f : U(n) \to SO(2n) \times S^1, \quad A \mapsto (A, |A|)
\]

has a lift \(F : U(n) \to Spin^c(2n)\) (see [1] page 27) for an explicit formula for \(F\). Using that, we define

\[
P = SU(n + 1) \times_{\tilde{\sigma}} Spin^c(2n)
\]

where \(\tilde{\sigma} = F \circ \sigma : H \to Spin^c(2n)\).

Thus we get a spin\(^c\) structure \(P \to SOF(\mathbb{C}P^n) \to \mathbb{C}P^n\) on the n-dimensional complex projective space.
Step 2 - Constructing a connection on $P \to SOF(\mathbb{C}P^n)$.

Let $\theta^R : TSU(n+1) \to su(n+1)$ be the right-invariant Maurer-Cartan form, and define

$$\chi : su(n+1) \to \mathfrak{h} = lie(H), \quad \left( \begin{array}{cc} A & \ast \\ \ast & -tr(A) \end{array} \right) \mapsto \left( \begin{array}{cc} A & 0 \\ 0 & -tr(A) \end{array} \right).$$

Since $\chi$ is an equivariant map under the adjoint action of $H$, we conclude that $\chi \circ \theta^R : TSU(n+1) \to \mathfrak{h}$ is a connection 1-form on the (right-) principal $H$-bundle $SU(n+1) \to \mathbb{C}P^n = SU(n+1)/H$.

This induces a connection 1-form on the principal $Spin^c(2n)$-bundle $P \to \mathbb{C}P^n$:

$$\hat{\theta} : TP \to spin^c(2n).$$

After composing $\hat{\theta}$ with the projection

$$\frac{1}{2} det_* : spin^c(2n) = spin(2n) \oplus u(1) \to u(1) = i\mathbb{R}$$

We get a connection 1-form $\theta = \frac{1}{2} det_* \circ \hat{\theta}$ on the principal $U(1)$-bundle $P \to SOF(\mathbb{C}P^n)$.

In fact, here is an explicit formula for the connection $\theta$:

If $\xi = \left( \begin{array}{cc} A & \ast \\ \ast & -tr(A) \end{array} \right) \in su(n+1), \zeta \in spin^c(2n)$, $\xi^R$ and $\zeta^L$ are the corresponding vector fields on $SU(n+1)$ and $Spin^c(2n)$, and

$$q : SU(n+1) \times Spin^c(2n) \to P$$

is the quotient map, then a direct computation gives

$$\theta(q_*(\xi^R + \zeta^L)) = \frac{n+1}{2} \cdot tr(A) + \frac{1}{2} det_*(\zeta).$$

Note that if $\zeta \in spin(2n)$, then $\theta(q_*(\zeta^L)) = 0$.

Step 3 - Computing the curvature of $\theta$.

Using the formula

$$d\theta(V, W) = V \theta(W) - W \theta(V) - \theta([V, W])$$

for any two vector fields $V, W$ on $P$, we can compute the curvature $d\theta$ of the connection $\theta$. We obtain the following:

If $\xi_1, \xi_2 \in su(n+1), \zeta_1, \zeta_2 \in spin^c(2n)$, and

$$[\xi_1, \xi_2] = \left( \begin{array}{cc} X & \ast \\ \ast & \ast \end{array} \right) \in su(n+1)$$

then we have

$$d\theta(q_*(\xi^R_1 + \zeta^L_1), q_*(\xi^R_2 + \zeta^L_2)) = -\frac{n+1}{2} \cdot tr(X).$$

Let $\omega$ be the real two form on $\mathbb{C}P^n$ for which

$$d\theta = \pi^*(-i \cdot \omega).$$

In fact

$$\omega = -\frac{n+1}{2} \cdot 2\pi \omega_{FS}$$
where $\omega_{FS}$ is the Fubini-Study form. To see this, it is enough, by $SU(n + 1)$-invariance of $\omega$ and $\omega_{FS}$, to show the above equality at one point (for instance, at $[p] \in \mathbb{C}P^n$).

Recall that the cohomology class of $\omega_{FS}$ generates the integral cohomology of $\mathbb{C}P^n$, i.e., $\int_{\mathbb{C}P^n} \omega_{FS} = 1$. This immediately implies that our two form $\omega$ is integral if and only if $n$ is odd, and we have:

$(P, \theta)$ is a spin$^c$ prequantization for $(\mathbb{C}P^n, \omega)$.

Remark 5.1. It is not hard to conclude, that a spin$^c$ prequantizable two form $\omega$ on $\mathbb{C}P^n$ is integral if and only if $n$ is odd. In fact, Proposition D.43 in [7], together with Claim 2.2 imply the following:

For an odd $n$, a two-form $\omega$ on $\mathbb{C}P^n$ is spin$^c$ prequantizable if and only if $\frac{1}{2\pi} \omega$ is integral, i.e., $\left[ \frac{1}{2\pi} \omega \right] \in \mathbb{Z}[\omega_{FS}]$.

For an even $n$, a two-form $\omega$ on $\mathbb{C}P^n$ is spin$^c$ prequantizable if and only if $\left[ \frac{1}{2\pi} \omega \right] \in (\mathbb{Z} + \frac{1}{2}) [\omega_{FS}]$.

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