TRANSITIVE POINTS VIA FURSTENBERG FAMILY

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Abstract. Let \((X, T)\) be a topological dynamical system and \(\mathcal{F}\) be a Furstenberg family (a collection of subsets of \(\mathbb{Z}_+\) with hereditary upward property). A point \(x \in X\) is called an \(\mathcal{F}\)-transitive one if \(\{n \in \mathbb{Z}_+ : T^n x \in U\} \in \mathcal{F}\) for every nonempty open subset \(U\) of \(X\); the system \((X, T)\) is called \(\mathcal{F}\)-point transitive if there exists some \(\mathcal{F}\)-transitive point. In this paper, we aim to classify transitive systems by \(\mathcal{F}\)-point transitivity. Among other things, it is shown that \((X, T)\) is a weakly mixing E-system (resp. weakly mixing M-system, HY-system) if and only if it is \(\{\text{D-sets}\}\)-point transitive (resp. \(\{\text{central sets}\}\)-point transitive, \(\{\text{weakly thick sets}\}\)-point transitive).

It is shown that every weakly mixing system is \(\mathcal{F}_{ip}\)-point transitive, while we construct an \(\mathcal{F}_{ip}\)-point transitive system which is not weakly mixing. As applications, we show that every transitive system with dense small periodic sets is disjoint from every totally minimal system and a system is \(\Delta^* (\mathcal{F}_{wt})\)-transitive if and only if it is weakly disjoint from every P-system.

1. Introduction

Throughout this paper a topological dynamical system (TDS for short) is a pair \((X, T)\), where \(X\) is a non-vacuous compact metric space with a metric \(d\) and \(T\) is a continuous map from \(X\) to itself. A non-vacuous closed invariant subset \(Y \subset X\) (i.e., \(TY \subset Y\)) defines naturally a subsystem \((Y, T)\) of \((X, T)\).

Let \(\mathbb{Z}\), \(\mathbb{Z}_+\) and \(\mathbb{N}\) denote the sets of the integers, the non-negative integers, and the positive integers, respectively.

Let \((X, T)\) be a TDS, for two open (standing for non-empty open) subsets \(U, V\) of \(X\), set

\[
N(U, V) = \{n \in \mathbb{Z}_+ : T^n U \cap V \neq \emptyset\} = \{n \in \mathbb{Z}_+ : U \cap T^{-n} V \neq \emptyset\}.
\]

We call \(N(U, V)\) the hitting time set of \(U\) and \(V\). Recall that a system \((X, T)\) is called topologically transitive (or just transitive) if for every two open subsets \(U, V\) of \(X\) the hitting time set \(N(U, V)\) is infinite. A system \((X, T)\) is called weakly mixing if the product system \((X \times X, T \times T)\) is transitive; strongly mixing if for every two open subsets
of $X$, the hitting time set $N(U, V)$ is cofinite, i.e., there exists some $N \in \mathbb{N}$ such that $N(U, V) \supset \{N, N + 1, \ldots\}$.

A system $(X, T)$ is called minimal if it contains no proper subsystem. Each point belonging to some minimal subsystem of $(X, T)$ is called a minimal point. Let $(X, T)$ be a transitive system, $(X, T)$ is called a $P$-system if it has dense periodic points; an $M$-system if it has dense minimal points; an $E$-system if it has an invariant measure with full support; a topologically ergodic system if for every two open subsets $U, V$ of $X$, the hitting time set $N(U, V)$ is syndetic (bounded gaps).

1.1. Furstenberg families. Before going on, let us recall some notations related to a family (for more details see [1]). For the set of nonnegative integers $\mathbb{Z}_+$, denote by $\mathcal{P} = \mathcal{P}(\mathbb{Z}_+)$ the collection of all subsets of $\mathbb{Z}_+$. A subset $\mathcal{F}$ of $\mathcal{P}$ is called a Furstenberg family (or just family), if it is hereditary upward, i.e., $F_1 \subset F_2$ and $F_1 \in \mathcal{F}$ imply $F_2 \in \mathcal{F}$. A family $\mathcal{F}$ is called proper if it is a nonempty proper subset of $\mathcal{P}$, i.e., neither empty nor all of $\mathcal{P}$. Any nonempty collection $A$ of subsets of $\mathbb{Z}_+$ naturally generates a family

$$\mathcal{F}(A) = \{F \subset \mathbb{Z}_+ : F \supset A \text{ for some } A \in A\}.$$ 

For a family $\mathcal{F}$, the dual family of $\mathcal{F}$, denoted by $\kappa \mathcal{F}$, is

$$\{F \in \mathcal{P} : F \cap F' \neq \emptyset, \forall F' \in \mathcal{F}\}.$$ 

Sometimes the dual family $\kappa \mathcal{F}$ is also denoted by $\mathcal{F}^*$. Let $\mathcal{F}_{inf}$ be the family of all infinite subsets of $\mathbb{Z}_+$. It is easy to see that its dual family $\kappa \mathcal{F}_{inf}$ is the family of all cofinite subsets, denoted by $\mathcal{F}_{cf}$.

All the families considered in this paper are assumed to be proper and contained in $\mathcal{F}_{inf}$.

Let $F$ be a subset of $\mathbb{Z}_+$, the upper Banach density of $F$ is

$$BD^*(F) = \limsup_{|I| \to \infty} \frac{|F \cap I|}{|I|}$$

where $I$ is taken over all nonempty finite intervals of $\mathbb{Z}_+$ and $| \cdot |$ denote the cardinality of the set. Denote by $\mathcal{F}_{pubd}$ the family of sets with positive upper Banach density.

A subset $F$ of $\mathbb{Z}_+$ is called thick if it contains arbitrarily long runs of positive integers, i.e., for every $n \in \mathbb{N}$ there exists some $a_n \in \mathbb{Z}_+$ such that $[a_n, a_n + n] \subset F$; syndetic if there is $N \in \mathbb{N}$ such that $[n, n + N] \cap F \neq \emptyset$ for every $n \in \mathbb{Z}_+$; piecewise syndetic if it is the intersection of a thick set and a syndetic set. The families of all thick sets, syndetic sets and piecewise syndetic sets are denoted by $\mathcal{F}_t$, $\mathcal{F}_s$ and $\mathcal{F}_{ps}$, respectively. It is easy to see that $\kappa \mathcal{F}_s = \mathcal{F}_t$.

For a sequence $\{p_i\}_{i=1}^\infty$ in $\mathbb{N}$, define the finite sums of $\{p_i\}_{i=1}^\infty$ as

$$FS\{p_i\}_{i=1}^\infty = \\{\sum_{i \in \alpha} p_i : \alpha \text{ is a nonempty finite subset of } \mathbb{N} \}.$$
A subset $F$ of $\mathbb{Z}_+^+$ is called an IP set if there exists a sequence $\{p_i\}_{i=1}^\infty$ in $\mathbb{N}$ such that $FS\{p_i\}_{i=1}^\infty \subset F$. We denote by $F_{ip}$ the family of all IP sets.

For a family $F$, the block family of $F$, denoted by $bF$, is the family consisting of sets $F \subset \mathbb{Z}_+^+$ for which there exists some $F' \in \mathcal{F}$ such that for every finite subset $W$ of $F'$ one has $m + W \subset F$ for some $m \in \mathbb{Z}_+^+$ (see [12]). It is easy to see that $Nb \not\in 0$ (see Lemma 1.1).

Since the difference family of $F$ is $\sum_{a,b \in F} (F' \cap [0,n]) \subset F$, define $\Delta(F)$ as

$$\Delta(F) = \{F \subset \mathbb{Z}_+^+ : \exists F' \in \mathcal{F}, \text{ s.t. } F' - F' \subset F\}$$

and $\Delta^*(F) = \kappa\Delta(F)$.

**Lemma 1.1.** Let $F$ be a family, then $\Delta(F) = \Delta(bF)$.

**Proof.** Since $F \subset bF$, $\Delta(F) \subset \Delta(bF)$. If $F \in \Delta(bF)$, there exists $F_i \in bF$ such that $F_i - F_i \subset F$. Since $F_i \in bF$, there exists $F_2 \in F$ and $\{a_n\}_{n=1}^\infty$ in $\mathbb{Z}_+$ such that $a_n + F_2 \cap [0,n] \subset F_1$. Then $F_2 - F_2 \subset F_1 - F_1$ and $F \in \Delta(F)$. Hence, $\Delta(F) = \Delta(bF)$.

1.2. **Three ways to classify transitive systems.** It is well known that the study of transitive systems and its classification play a big role in topological dynamics. There are several ways to classify transitive systems. The first one was started with Furstenberg by the hitting time sets of two open subsets. Let $F$ be a family, we call $(X,T)$ is $F$-transitive if for every two open subsets $U,V$ of $X$ the hitting time set $N(U,V) \in F$. In his seminal paper [17], Furstenberg showed that a TDS $(X,T)$ is weakly mixing if and only if it is $\{\text{thick sets}\}$-transitive.

The second way is by weak disjointness. Two TDSs are called weakly disjoints if their product system is transitive [17]. Then a system is weakly mixing if and only if it is weakly disjoint from itself.

The third way is by the complexity of open covers which was introduced in [6]. For a TDS $(X,T)$ and a finite open cover $\alpha$ of $X$, let $N(\alpha)$ denote the number of sets in a finite subcover of $\alpha$ with small cardinality. For an infinite set $A = \{a_1 < a_2 < \cdots \} \subset \mathbb{Z}_+$ put

$$c_A(\alpha,n) = N(T^{-a_1} \alpha \vee T^{-a_2} \alpha \vee \cdots \vee T^{-a_n} \alpha).$$

We call $c_A(\alpha,n)$ the complexity function of the cover $\alpha$ along $A$. An open cover $\alpha = \{U_1,U_2,\cdots,U_k\}$ of $X$ is called non-trivial if $U_i$ is not dense in $X$ for each $1 \leq i \leq k$. Let $F$ be a family, we call $(X,T)$ is $F$-scattering if for every $A \in F$ and non-trivial open cover $\alpha$ of $X$ the complexity function $c_A(\alpha,n)$ is unbounded.
Recently, the authors in \cite{1, 3, 6, 9, 14, 15, 18} have successfully classified transitive systems by the above three ways. We summarize the results in the following table:

| Transitive properties | $\mathcal{F}$-transitivity | $\mathcal{F}$-scattering | Weakly disjointness |
|-----------------------|----------------------------|--------------------------|---------------------|
| Transitivity          | $\mathcal{F}_{inf}$        | No?                      | Mild mixing or trivial system |
| Total transitivity    | $\kappa\mathcal{F}_{rs}$  | No?                      | Periodic system |
| Weakly scattering     | ?                          | No?                      | Minimal equi-continuous system |
| Scattering            | $\Delta^*(\mathcal{F}_{ps})$ | $\mathcal{F}_{ps}$ or $\{\mathbb{Z}_+\}$ | Minimal system or M-system |
| Strong scattering     | $\Delta^*(\mathcal{F}_{pubd})$ | $\mathcal{F}_{pubd}$ or $\mathcal{F}_{pud}$ | E-system |
| Extreme scattering    | ?                          | ?                        | Topologically ergodic system |
| Weak mixing           | $\mathcal{F}_{t}$          | ?                        | Itself |
| Mild mixing           | $\Delta^*(\mathcal{F}_{ip})$ | $\mathcal{F}_{ip}$       | Transitive system |
| Full scattering       | ?                          | $\mathcal{F}_{inf}$      | No |
| Strong mixing         | $\mathcal{F}_{cf}$         | No                       | No |

We understand this table in the following way. For example, a system is scattering if and only if it is $\Delta^*(\mathcal{F}_{ps})$-transitive if and only if it is $\mathcal{F}_{ps}$-scattering if and only if it is $\{\mathbb{Z}_+\}$-scattering if and only if it is weakly disjoint from every minimal system if and only if it is weakly disjoint from every M-system.

1.3. A new way to classify transitive systems. In this paper we propose a new way to classify transitive systems. Before going on, we recall some basic notions.

For $x \in X$, denote the orbit of $x$ by $\text{Orb}(x, T) = \{x, Tx, T^2x, \ldots\}$. Let $\omega(x, T)$ be the $\omega$-limit set of $x$, i.e., $\omega(x, T)$ is the limit set of $\text{Orb}(x, T)$. A point $x \in X$ is called a recurrent point if $x \in \omega(x, T)$; a transitive point if $\omega(x, T) = X$. It is easy to see that a system $(X, T)$ is transitive if and only if the set of all transitive points, denoted by $\text{Trans}(X, T)$, is a dense $G_\delta$ subset of $X$ and $(X, T)$ is minimal if and only if $\text{Trans}(X, T) = X$.

Let $(X, T)$ be a TDS, for $x \in X$ and an open subset $U$ of $X$, set

$$N(x, U) = \{n \in \mathbb{Z}_+ : T^n x \in U\}.$$ 

We call $N(x, U)$ the entering time set of $x$ into $U$. Then it is easy to see that a point $x \in X$ is a transitive point if and only if for every open subset $U$ of $X$ the entering time set $N(x, U)$ is infinite. This suggests that we can use the entering time set of a point into an open subset to characterize transitive points. Let $\mathcal{F}$ be a family, a point $x \in X$ is called an $\mathcal{F}$-transitive point if for every open subset $U$ of $X$
the entering time set \( N(x, U) \in \mathcal{F} \). A system \((X, T)\) is called \(\mathcal{F}\)-point transitive if there exists some \(\mathcal{F}\)-transitive point. In this paper, we aim to classify transitive systems by \(\mathcal{F}\)-point transitivity. We summarize our results in the following table:

| Transitive properties | \(\mathcal{F}\)-point transitivity | Transitive properties | \(\mathcal{F}\)-point transitivity |
|-----------------------|-----------------------------------|-----------------------|-----------------------------------|
| Transitive system     | \(b\mathcal{F}_{ip}\)            | Weakly mixing system  | \(?\)                            |
| E-system              | \(\mathcal{F}_{ipubd}\)          | Weakly mixing E-system| \(\mathcal{F}_D\)                |
| M-system              | \(\mathcal{F}_{ps}\)             | Weakly mixing M-system| \(\mathcal{F}_{cen}\)            |
| Transitive system with dense small periodic sets | \(b\mathcal{F}_{wt}\) | HY-system             | \(\mathcal{F}_{wt}\)            |

This paper is organized as follows. In Section 2, we discuss the connection between families and topological dynamics and show some basic properties of \(\mathcal{F}\)-point transitivity. In Sections 3 and 4, we prove the main results which are declared in the above table. It is shown that every weakly mixing system is \(\mathcal{F}_{ip}\)-point transitive, while we construct an \(\mathcal{F}_{ip}\)-point transitive system which is not weakly mixing. In the final section as applications, we discuss disjointness and weak disjointness. We show that every transitive system with dense small periodic sets is disjoint from every totally minimal system and a system is \(\Delta^*(\mathcal{F}_{wt})\)-transitive if and only if it is weakly disjoint from every \(P\)-system.

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2. Preliminary

The idea of using families to describe dynamical properties goes back at least to Gottschalk and Hedlund [10]. It was developed further by Furstenberg [8]. For a systematic study and recent results, see [1], [9], [15] and [10].

Let \((X, T)\) be a TDS and \(\mathcal{F}\) be a family, a point \(x \in X\) is called an \(\mathcal{F}\)-recurrent point if for every neighborhood \(U\) of \(x\) the entering time set \(N(x, U) \in \mathcal{F}\). Denote the set of all \(\mathcal{F}\)-recurrent points by \(\text{Rec}_\mathcal{F}(X, T)\).

It is well known that the following lemmas hold (see, e.g., [8] Theorem 1.15, Theorem 1.17, Theorem 2.17))

**Lemma 2.1.** Let \((X, T)\) be a TDS and \(x \in X\). Then

1. \(x\) is a minimal point if and only if it is an \(\mathcal{F}_s\)-recurrent point.
2. \(x\) is a recurrent point if and only if it is an \(\mathcal{F}_{ip}\)-recurrent point.
Recall that a TDS $(X, T)$ is called \( F \)-transitive if for every two open subsets $U, V$ of $X$ the hitting time set $N(U, V) \in F$; \( F \)-mixing if $(X \times X, T \times T)$ is $F$-transitive.

**Lemma 2.2** ([7] [1]). Let $(X, T)$ be a TDS and $F$ be a family. Then

1. $(X, T)$ is weakly mixing if and only if it is $F_t$-transitive.
2. $(X, T)$ is strongly mixing if and only if it is $F_{cf}$-transitive.
3. $(X, T)$ is $F$-mixing if and only if it is $F$-transitive and weakly mixing.

Recall that a TDS $(X, T)$ is called \( F \)-center if for every open subset $U$ of $X$ the hitting time set $N(U, U) \in F$.

It is well known that a system $(X, T)$ is transitive if and only if there exists some transitive point. It is interesting that how to characterize transitive systems by transitive points via a family.

Let $(X, T)$ be a TDS and $F$ be a family, a point $x \in X$ is called an \( F \)-transitive point if for every open subset $U$ of $X$ the entering time set $N(x, U) \in F$. Denote the set of all $F$-transitive points by $\text{Trans}_{F}(X, T)$. The system $(X, T)$ is called \( F \)-point transitive if there exists some $F$-transitive point.

Though the terminology “$F$-point transitivity” is first introduced in this paper, the idea has appeared in several literatures, such as [13] [16]. We state their results in our way as

**Theorem 2.3.** Let $(X, T)$ be a TDS. Then

1. $(X, T)$ is an $E$-system if and only if it is $F_{pubd}$-point transitive if and only if $\text{Trans}_{F_{pubd}}(X, T) = \text{Trans}(X, T) \neq \emptyset$ ([13]).
2. $(X, T)$ is an $M$-system if and only if it is $F_{ps}$-point transitive if and only if $\text{Trans}_{F_{ps}}(X, T) = \text{Trans}(X, T) \neq \emptyset$ ([16]).

The following remark shows some basic facts about $F$-point transitivity.

**Remark 2.4.** (1). It is easy to see that $x \in X$ is an $F$-transitive point if and only if for every $F \in \kappa F$ one has $\{T^n x : n \in F\}$ is dense in $X$.

2. If $x \in X$ is an $F$-transitive point, then so is $Tx$. Thus, if $(X, T)$ is $F$-point transitive then $\text{Trans}_{F}(X, T)$ is dense in $X$.

3. It should be noticed that $F$-transitivity may differ greatly from $F$-point transitivity. For example, $(X, T)$ is $F_t$-transitive if and only if it is weakly mixing, but if $\text{Trans}_{F_t}(X, T) \neq \emptyset$ then $X$ must be a singleton.

Similarly to $F$-center, we can define $F$-point center. A system $(X, T)$ is called \( F \)-point center if for every open subset $U$ of $X$ there exists $x \in U$ such that the entering time set $N(x, U) \in F$.

**Lemma 2.5.** Let $(X, T)$ be a TDS and $F$ be a family. If $(X, T)$ is transitive and $F$-point center, then $\text{Trans}_{F}(X, T) = \text{Trans}(X, T)$. 

Proof. Let $x$ be a transitive point and $U$ be an open subset of $X$. Since $(X, T)$ is $\mathcal{F}$-point center, there exists $y \in U$ such that $N(y, U) \in \mathcal{F}$. For every finite subset $W$ of $N(y, U)$, by the continuity of $T$, there exists $m \in \mathbb{Z}_+$ such that $m + W \subset N(x, U)$. Then $N(x, U) \in b\mathcal{F}$ and $x$ is a $b\mathcal{F}$-transitive point. □

Remark 2.6. (1). Since every recurrent point is $\mathcal{F}_{ip}$-recurrent, every transitive system is $\mathcal{F}_{ip}$-point center. Thus, a system $(X, T)$ is transitive if and only if it is $b\mathcal{F}_{ip}$-point transitive.

(2). If $(X, T)$ is $\mathcal{F}$-point transitive, then it is $\mathcal{F}$-point center, so by Lemma 2.5 $\text{Trans}_{\mathcal{F}}(X, T) = \text{Trans}(X, T)$. In particular, if $b\mathcal{F} = \mathcal{F}$, then $(X, T)$ is $\mathcal{F}$-point transitive if and only if $\text{Trans}_{\mathcal{F}}(X, T) \neq \emptyset$.

3. Weakly mixing systems

In this section, we consider weakly mixing systems. We should use the following useful lemma:

Lemma 3.1 (Ulam [2]). Let $X$ be a compact metric space without isolated points and $R$ be a dense $G_δ$ subset of $X \times X$. Then there exists a dense $G_δ$ subset $Y$ of $X$ such that for every $x \in Y$, $R(x) = \{y \in X : (x, y) \in R\}$ is a dense $G_δ$ subset of $X$.

Recall that a sequence $F$ in $\mathbb{Z}_+$ is called a Poincaré sequence if for any measure-preserving system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists $0 \neq n \in F$ such that $\mu(A \cap T^{-n}A) > 0$. Let $\mathcal{F}_{Poin}$ denote the family of all Poincaré sequences. It is well known that $\mathcal{F}_{Poin} = \Delta^*(\mathcal{F}_{pulg})$ [8, 19].

Theorem 3.2. Let $(X, T)$ be a TDS. Consider the following conditions:

(1) $(X, T)$ is weakly mixing.
(2) $\text{Trans}_{\mathcal{F}_{ip}}(X, T)$ is residual in $X$.
(3) $(X, T)$ is $\mathcal{F}_{ip}$-point transitive.
(4) $(X, T)$ is $\mathcal{F}_{Poin}$-transitive.

Then (1)$\Rightarrow$(2)$\Rightarrow$(3)$\Rightarrow$(4). In addition, if $(X, T)$ is an E-system, then (4)$\Rightarrow$(1).

Proof. (1)$\Rightarrow$(2) Let $R = \text{Trans}(X \times X, T \times T)$, then $R$ is a dense $G_δ$ subset of $X \times X$. Since $(X, T)$ is transitive, $X$ has no isolated points. By Ulam Lemma, there exists a dense $G_δ$ subset $Y$ of $X$ such that for every $x \in Y$, $R(x)$ is a dense $G_δ$ subset of $X$. Then it suffices to show that $Y \subset \text{Trans}_{\mathcal{F}_{ip}}(X, T)$.

Let $x \in Y$ and $y \in R(x)$, we have $(y, y) \in \overline{\text{Orb}((x, y), T \times T)}$, then by the following claim, $x$ is an $\mathcal{F}_{ip}$-transitive point.

Claim: If $(y, y) \in \overline{\text{Orb}((x, y), T \times T)}$, then for every neighborhood $U$ of $y$ the entering time set $N(x, U)$ is an IP set.
Proof of the Claim: For every neighborhood $U$ of $y$, let $U_1 = U$ then there exists $p_1 \in \mathbb{N}$ such that $T^{p_1}x \in U_1$ and $T^{p_1}y \in U_1$. Let $U_2 = U_1 \cap T^{-p_1}U_1$, then $U_2$ is a neighborhood of $y$, so there exists $p_2 \in \mathbb{N}$ such that

$$T^{p_2}x \in U_2 \text{ and } T^{p_2}y \in U_2.$$ 

Then for every $m \in FS\{p_i\}_{i=1}^{n}$

$$T^m x \in U \text{ and } T^m y \in U.$$ 

We continue inductively. Assume $p_1, p_2, \ldots, p_n$ have been found such that for every $m \in FS\{p_i\}_{i=1}^{n}$

$$T^m x \in U \text{ and } T^m y \in U.$$ 

Let $U_{n+1} = U \cap (\bigcap_{m \in FS\{p_i\}_{i=1}^{n}} T^{-m}U)$, then $U_{n+1}$ is a neighborhood of $y$, so there exists $p_{n+1} \in \mathbb{N}$ such that

$$T^{p_{n+1}}x \in U_{n+1} \text{ and } T^{p_{n+1}}y \in U_{n+1}.$$ 

Then for every $m \in FS\{p_i\}_{i=1}^{n+1}$

$$T^m x \in U \text{ and } T^m y \in U.$$ 

Thus, $FS\{p_i\}_{i=1}^{\infty} \subset N(x, U)$.

(2)$\Rightarrow$(3) is obvious.

(3)$\Rightarrow$(4) follows from the fact that every IP set is a Poincaré sequence (8, p74).

In addition, if $(X, T)$ is an E-system, we show that (4)$\Rightarrow$(1). It is sufficient to show that for every open subsets $U_1, U_2, V$ of $X$, $N(U_1, U_2) \cap N(V, V) \neq \emptyset$. Since $(X, T)$ is an E-system, there exists an invariant measure $\mu$ with full support. Then $(X, B_X, \mu, T)$ is a measure dynamical system and $\mu(V) > 0$. Since $N(U_1, U_2) \in F_{Poin}$, by the definition of Poincaré sequence one has $N(U_1, U_2) \cap N(V, V) \neq \emptyset$. □

Corollary 3.3. Let $(X, T)$ be a minimal system. Then $(X, T)$ is weakly mixing if and only if it is $F_{ip}$-point transitive.

In [14], the authors constructed an $F_{Poin}$-transitive system which is not weakly mixing, we show that:

Proposition 3.4. There exists an $F_{ip}$-point transitive system which is not weakly mixing.

Proof. Since the construction is somewhat long and complicated, we leave it to the appendix. □

3.1. Weakly mixing M-system. In this subsection, we characterize weakly mixing M-systems. To this end, we need the concept of the central set which was first introduced in 8.

Let $(X, T)$ be a TDS, a pair $(x, y) \in X \times X$ is called proximal if there exists a sequence $\{n_i\}_{i=1}^{\infty}$ in $\mathbb{N}$ such that $\lim_{i \to \infty} T^{n_i}x = \lim_{i \to \infty} T^{n_i}y$. A subset $F$ of $\mathbb{Z}_+$ is called a central set, if there exists a system $(X, T)$,
transitive points via Furstenberg family

$x \in X$, a minimal point $y \in X$ and a neighborhood of $U$ of $y$ such that $(x, y)$ is proximal and $N(x, U) \subset F$. Let $F_{cen}$ denote the family of all central sets.

**Lemma 3.5.** [8, Proposition 8.10] $F_{cen} \subset F_{ip} \cap F_{ps}$.

**Theorem 3.6** (Akin-Kolyada [11]). Let $(X, T)$ be a TDS. If $(X, T)$ is weakly mixing, then for every $x \in X$ the proximal cell $\text{Prox}(x) = \{y \in X : (x, y) \text{ is proximal} \}$ is a dense $G_δ$ subset of $X$.

**Theorem 3.7.** Let $(X, T)$ be a TDS. Then the following conditions are equivalent:

1. $(X, T)$ is a weakly mixing M-system.
2. $\text{Trans}_{F_{cen}}(X, T)$ is residual in $X$.
3. $(X, T)$ is $F_{cen}$-point transitive.

**Proof.** (1)⇒(2). Let $\{y_n\}_{n=1}^\infty$ be a sequence of minimal points which is dense in $X$. By Akin-Kolyada Theorem, $\bigcap_{n=1}^\infty \text{Prox}(y_n)$ is a dense $G_δ$ subset of $X$. Then it suffices to show that $\bigcap_{n=1}^\infty \text{Prox}(y_n) \subset \text{Trans}_{F_{cen}}(X, T)$.

Let $x \in \bigcap_{n=1}^\infty \text{Prox}(y_n)$ and $U$ be an open subset of $X$. There exists $n \in \mathbb{N}$ such that $y_n \in U$. Since $(x, y)$ is proximal and $y$ is a minimal point, by the definition of central set, we have $N(x, U) \in F_{cen}$. Thus, $x$ is a $F_{cen}$-transitive point.

(2)⇒(3) is obvious.

(3)⇒(1). By $F_{cen} \subset F_{ps}$ and Theorem 2.3, $(X, T)$ is an M-system. Therefore, by Theorem 3.2 and $F_{cen} \subset F_{ip}$, $(X, T)$ is weakly mixing. □

### 3.2. Weakly mixing E-system

In this subsection, we characterize weakly mixing E-systems. To this end, we need the concept of the D-set which was first introduced in [5].

A subset $F$ of $\mathbb{Z}_+$ is called a $D$-set if there exists a system $(X, T)$, $x \in X$, an $F_{pubd}$-recurrent point $y \in X$ and a neighborhood of $U$ of $y$ such that $(y, y) \in \text{Orb}((x, y), T \times T)$ and $N((x, y), U \times U) \subset F$. Let $F_D$ denote the family of all D-sets.

**Lemma 3.8** ([5]). $F_D \subset F_{ip} \cap F_{pubd}$.

**Theorem 3.9.** Let $(X, T)$ be a TDS. Then the following conditions are equivalent:

1. $(X, T)$ is a weakly mixing E-system.
2. $\text{Trans}_{F_D}(X, T)$ is residual in $X$.
3. $(X, T)$ is $F_D$-point transitive.

**Proof.** (1)⇒(2). It is easy to see that $(X \times X, T \times T)$ is also an E-system. Let $R = \text{Trans}_{pubd}(X \times X, T \times T)$, then by Theorem 2.3, $R$ is a dense $G_δ$ subset of $X \times X$. By Ulam Lemma, there exists a dense $G_δ$
subset $Y$ of $X$ such that for every $x \in Y$, $R(x)$ is a dense $G_δ$ subset of $X$. Then it suffices to show that $Y \subset \text{Trans}_{\mathcal{F}_D}(X,T)$.

Let $x \in Y$ and $U$ be an open subset of $X$. Choose $y \in R(x) \cap U$, then $y$ is $\mathcal{F}_{\text{pubd}}$-recurrent. Since $(x,y)$ is a transitive point in $X \times X$, $(y,y) \in \overline{\text{Orb}((x,y),T \times T)}$. Clearly, $N((x,y),U \times U) \subset N(x,U)$, so by the definition of $D$-set one has $N(x,U) \in \mathcal{F}_D$. Thus, $x$ is an $\mathcal{F}_D$-transitive point.

$(2) \Rightarrow (3)$ is obvious.

$(3) \Rightarrow (1)$. By $\mathcal{F}_D \subset \mathcal{F}_{\text{pubd}}$ and Theorem 2.3 $(X,T)$ is an E-system. Therefore, by Theorem 3.2 and $\mathcal{F}_D \subset \mathcal{F}_{\text{ip}}$, $(X,T)$ is weakly mixing.

Remark 3.10. (1). There exists a weakly mixing system which is not an E-system ([11, 12]).

(2). There exists a weakly mixing E-system which is not an M-system ([5]).

4. Systems with dense small periodic sets

In this section, we characterize transitive systems with dense small periodic sets. Let $(X,T)$ be a TDS, we call $(X,T)$ has dense small periodic sets ([16]) if for every open subset $U$ of $X$ there exists a closed subset $Y$ of $U$ and $k \in \mathbb{N}$ such that $Y$ is invariant for $T^k$ (i.e., $T^kY \subset Y$). Clearly, every P-system has dense small periodic sets. If $(X,T)$ is transitive and has dense small periodic sets, then it is an M-system.

To characterize the system with dense small periodic set, we need a new kind of subsets of $\mathbb{Z}_+$. A subset $F$ of $\mathbb{Z}_+$ is called weakly thick if there exists some $k \in \mathbb{N}$ such that \{\text{\{n \in \mathbb{Z}_+: kn \in F\}\}} is thick. Let $\mathcal{F}_{\text{wt}}$ denote the family of all weakly thick sets.

Lemma 4.1. Let $(X,T)$ be a TDS. Then the following conditions are equivalent:

(1) $(X,T)$ has dense small periodic sets.

(2) For every open subset $U$ of $X$, there exists $k \in \mathbb{N}$ such that $U$ contains a minimal subsystem of $(X,T^k)$.

(3) It is $b\mathcal{F}_{\text{wt}}$-point center.

Proof. (1)⇒(2) follows from the fact that every system constants a minimal subsystem.

(2)⇒(3). Let $U$ be an open subset of $X$, then there exists $k \in \mathbb{N}$ such that $U$ contains a minimal subsystem $Y$ of $(X,T^k)$. Choose a point $y \in Y \subset U$. Then $kU \subset N(y,U)$ and $(X,T)$ is $b\mathcal{F}_{\text{wt}}$-point center since $kU \in b\mathcal{F}_{\text{wt}}$.

(3)⇒(1). Let $U$ be an open subset of $X$. Choose an open subset $V$ of $X$ such that $V \subset U$. Since $(X,T)$ is $b\mathcal{F}_{\text{wt}}$-point center, there exists $x \in V$ such that $N(x,V) \in b\mathcal{F}_{\text{wt}}$. Let $F = N(x,V)$, then there exist two sequences \{\text{\{a_i\}_{i=1}^\infty, \{n_i\}_{i=1}^\infty \in \mathbb{Z}_+ and k \in \mathbb{N} such that $U_{i=1}^\infty (a_i + k[n_i, n_i + 1]) \subset F$}. Without lose of generality, assume that
\( \lim_{i \to \infty} T^{a_i + kn_i}x = y \in \{T^n x : n \in F\} \subset V. \) Let \( Y = \text{Orb}(y, T^k). \) Clearly, \( T^kY \subset Y. \) Then it suffices to show that \( Y \subset U. \)

For every \( m \in \mathbb{N}, \) if \( i > m, \) then \( a_i + k(n_i + m) \in F, \) so

\[
T^{km}y = \lim_{i \to \infty} T^{a_i + k(n_i + m)}x \in \{T^n x : n \in F\} \subset V.
\]

Thus, \( Y = \{T^{km}y : m \in \mathbb{N}\} \subset V \subset U. \) \( \square \)

**Theorem 4.2.** Let \((X, T)\) be a TDS. Then the following conditions are equivalent:

1. \((X, T)\) is transitive and has dense small periodic sets.
2. \(\text{Trans}_{bF_{\text{wt}}}(X, T) = \text{Trans}(X, T) \neq \emptyset.\)
3. \((X, T)\) is \(bF_{\text{wt}}\)-point transitive.

**Proof.** (1)\(\Rightarrow\)(2) follows from Lemma 2.5 and Lemma 4.1.

(2)\(\Rightarrow\)(3) is obvious.

(3)\(\Rightarrow\)(1) also follows from Lemma 4.1 \( \square \)

Denote \( F_{rs} = \{F \subset \mathbb{Z}_+ : \exists k \in \mathbb{N}, \text{s.t. } k\mathbb{N} \subset F\}. \) Let \((X, T)\) be a TDS, a point \( x \in X \) is called a quasi-periodic point if it is \(F_{rs}\)-recurrent.

**Corollary 4.3.** Let \((X, T)\) be an infinite minimal system. Then the following conditions are equivalent:

1. \((X, T)\) has dense small periodic sets.
2. It is an almost one-to-one extension of some adding machine system.
3. It has some quasi-periodic point.
4. It is \(bF_{\text{wt}}\)-point transitive.

**Proof.** (1)\(\Leftrightarrow\)(2)\(\Leftrightarrow\)(3) were proved in [12] and (1)\(\Leftrightarrow\)(4) follows from Theorem 4.2 \( \square \)

Recall that a system \((X, T)\) is called totally transitive if for every \( k \in \mathbb{N}, \) \((X, T^k)\) is transitive. In [16] Huang and Ye showed that a system which is totally transitive and has dense small periodic sets is disjoint from every minimal system. We call such a system HY-system for abbreviation. It is not hard to see that an HY-system is a weakly mixing M-system ([16]).

**Theorem 4.4.** Let \((X, T)\) be a TDS, then the following conditions are equivalent:

1. \((X, T)\) is an HY-system.
2. \(\text{Trans}_{F_{\text{wt}}}(X, T) = \text{Trans}(X, T) \neq \emptyset.\)
3. \((X, T)\) is \(F_{\text{wt}}\)-point transitive.
4. \((X, T^k)\) is an HY-system for every \( k \in \mathbb{N}.\)
5. \((X^k, T^{(k)})\) is an HY-system for every \( k \in \mathbb{N}, \) where \( X^k = X \times X \times \cdots \times X (k \text{ times}) \) and \( T^{(k)} = T \times T \times \cdots \times T (k \text{ times}).\)
Proof. (1)⇒(2). We need to show every transitive point is an $\mathcal{F}_{wt}$-transitive point. Let $x$ be a transitive point and $U$ be an opene subset of $X$. Since $(X, T)$ has dense small periodic sets, there exists a closed subset $Y$ of $U$ and $k \in \mathbb{N}$ such that $T^k Y \subset Y$. Since $(X, T)$ is totally transitive, $x$ is also a transitive point for the system $(X, T^k)$. By the continuity of $T$, it is easy to see that $\{n \in \mathbb{Z}_+: (T^k)^n x \in U\}$ is thick. Thus, $N(x, U) = \{n \in \mathbb{Z}_+: T^n x \in U\}$ is weakly thick, so $x$ is an $\mathcal{F}_{wt}$-transitive point.

(2)⇒(3) is obvious.

(3)⇒(4). We first prove the following Claim.

Claim: If $F$ is weakly thick, then for every $k \in \mathbb{N}$, $F_k = \{m \in \mathbb{Z}_+: mk \in F\}$ is also weakly thick.

Proof of the Claim: If $F$ is weakly thick, then there exists $r \in \mathbb{N}$ such that $F_r = \{m \in \mathbb{Z}_+: rm \in F\}$ is thick. Then for every $k \in \mathbb{N}$,

$$\{m \in \mathbb{Z}_+: rm \in F_k\} = \{m \in \mathbb{Z}_+: krm \in F\} = \{m \in \mathbb{Z}_+: mk \in F_r\}$$

is also thick. Thus, $F_k$ is weakly thick.

To show that $(X, T^k)$ is an HY-system for every $k \in \mathbb{N}$, it suffices to show that $(X, T^k)$ is transitive and has dense small periodic sets for every $k \in \mathbb{N}$. Now fix $k \in \mathbb{N}$. Let $x \in X$ be an $\mathcal{F}_{wt}$-transitive point in $(X, T)$, then by the Claim, $x$ is also an $\mathcal{F}_{wt}$-transitive point in $(X, T^k)$. Thus, $(X, T^k)$ is transitive and by Lemma 4.1 $(X, T^k)$ has dense small periodic sets.

(4)⇒(1) and (5)⇒(1) are obvious.

(1)⇒(5). Since every HY-system is weakly mixing, $(X^k, T^{(k)})$ is totally transitive for every $k \in \mathbb{N}$. Then it suffices to show that $(X^k, T^{(k)})$ has dense small periodic sets for every $k \in \mathbb{N}$.

Now fix $k \in \mathbb{N}$. Let $W$ be an opene subset of $X^k$, then there exist opene subsets $U_1, U_2, \ldots, U_k$ of $X$ such that $U_1 \times U_2 \times \cdots \times U_k \subset W$. Since $(X, T)$ has dense small periodic sets, there exists closed subsets $Y_1, Y_2, \ldots, Y_k$ of $U_1, U_2, \ldots, U_k$ and $n_1, n_2, \ldots, n_k \in \mathbb{N}$ such that $T^{n_i} Y_i \subset Y_i$ for $i = 1, 2, \ldots, k$. Let $Y = Y_1 \times Y_2 \times \cdots \times Y_k$ and $n = n_1 n_2 \cdots n_k$. Then $Y \subset W$ and $(T^{(k)})^n Y \subset Y$. Thus, $(X^k, T^{(k)})$ has dense small periodic sets.

Remark 4.5. (1) There exists a weakly mixing M-system which is not an HY-system (116).

(2) There exists an HY-system without periodic points (116).

5. Applications

In this section as applications, we discuss disjointness and weak disjointness.

5.1. Disjointness. The notion of disjointness of two TDSs was introduced by Furstenberg his seminal paper [7]. Let $(X, T)$ and $(Y, S)$ be
two TDSs. We call $J \subset X \times Y$ is a joining of $X$ and $Y$ if $J$ is a non-empty closed invariant set, and is projected onto $X$ and $Y$ respectively. If each joining is equal to $X \times Y$ then we call $(X, T)$ and $(Y, S)$ are disjoint, denoted by $(X, T) \perp (Y, S)$ or $X \perp Y$. Note that if $(X, T) \perp (Y, S)$ then one of them is minimal, and if $(X, T)$ is minimal then the set of recurrent points of $(Y, S)$ is dense \([15]\).

**Lemma 5.1.** Let $\mathcal{F}$ be a family. If $(X, T)$ is $\mathcal{F}$-point transitive and $(Y, S)$ is minimal with $Rec_{\mathcal{F}}(Y, S) = Y$, then $(X, T) \perp (Y, S)$.

**Proof.** Let $J$ be a joining of $X$ and $Y$, and $x \in Trans_{\mathcal{F}}(X, T)$. There exists some $y \in Y$ such that $(x, y) \in J$. For every open subset $U$ of $X$ and neighborhood $V$ of $y$, we have $N(x, U) \in \mathcal{F}$ and $N(y, V) \in \kappa_{\mathcal{F}}$. Then $Orb((x, y), T \times S) \cap U \times V \neq \emptyset$. Therefore, $X \times \{y\} \subset Orb((x, y), T \times S) \subset J$. By $J$ is $T \times S$-invariant and $Y$ is minimal, we have $J = X \times Y$. Thus, $(X, T) \perp (Y, S)$. \qed

Recall that a system is called distal if there is no proper proximal pairs.

**Theorem 5.2.** [8 Theorem 9.11] A minimal system $(X, T)$ is distal if and only if $Rec_{\mathcal{F}_{ip}}(X, T) = X$.

**Corollary 5.3.** Every $\mathcal{F}_{ip}$-point transitive system is disjoint from every minimal distal system.

Recall that a system $(X, T)$ is called totally minimal if for every $k \in \mathbb{N}$, $(X, T^k)$ is minimal.

**Theorem 5.4.** (1). Every HY-system is disjoint from every minimal system \([16]\).

(2). Every transitive system with dense small periodic sets is disjoint from every totally minimal system.

**Proof.** (1). By Theorem 4.4, every HY-system is $\mathcal{F}_{wt}$-point transitive. Then by Lemma 5.1 it suffices to show that for every minimal system $(X, T)$ we have $Rec_{\mathcal{F}_{wt}}(X, T) = X$.

Let $(X, T)$ be a minimal system, $x \in X$ and $U$ be a neighborhood of $x$. Then it suffices to show that for every $F \in \mathcal{F}_{wt}$, $N(x, U) \cap F \neq \emptyset$. Let $F \in \mathcal{F}_{wt}$, there exists $k \in \mathbb{N}$ such that $\{n \in \mathbb{N} : kn \in F\}$ is thick. Since $x$ is a minimal point in $(X, T)$, $x$ also is a minimal point $(X, T^k)$. Then $\{n \in \mathbb{N} : (T^k)^n x \in U\}$ is syndetic. Therefore, $N(x, U) \cap F \neq \emptyset$.

(2). By Theorem 4.2, every transitive system with dense small periodic sets is $b\mathcal{F}_{wt}$-point transitive. Then by Lemma 5.2 it suffices to show that for every totally minimal system $(X, T)$ we have $Rec_{b\mathcal{F}_{wt}}(X, T) = X$. We first prove the following Claim.

**Claim:** If $F \in b\mathcal{F}_{wt}$, there exists $q \in \mathbb{Z}_+$ such that $-q + F \in \mathcal{F}_{wt}$.

**Proof of the Claim:** Let $F \in b\mathcal{F}_{wt}$, then there exist two sequences $\{a_i\}_{i=1}^{\infty}$, $\{b_i\}_{i=1}^{\infty}$ in $\mathbb{Z}_+$ and $k \in \mathbb{N}$ such that $\bigcup_{i=1}^{\infty}(a_i + k[b_i, b_i + i]) \subset F$. 

Without lose of generality, assume that there exists \( q \in [0, k) \) such that for every \( i \in N \) \( a_i \equiv q \pmod{k} \). Let \( a_i = kc_i + q \) and \( F' = \bigcup_{i=1}^{\infty} k[b_i + c_i, b_i + c_i + i] \), then \( F' \) is weakly thick and \( q + F' \subset F \). Thus, \( -q + F \) is weakly thick.

Let \((X, T)\) be a totally minimal system, \( x \in X \) and \( U \) be a neighborhood of \( x \). Then it suffices to show that for every \( F \in bF_{\text{wt}} \), \( N(x, U) \cap F \neq \emptyset \). Let \( F \in bF_{\text{wt}} \), by the Claim there exists \( q \in \mathbb{Z}_+ \) such that \(-q + F \) is weakly thick. Let \( F' = -q + F \). Since \((X, T)\) is totally minimal, \( N(T^n x, U) \cap F' \neq \emptyset \), then \( N(x, U) \cap F \neq \emptyset \) since \( N(T^n x, U) + p \subset N(x, U) \).

5.2. **Weak disjointness.** Let \((X, T)\) and \((Y, S)\) be two TDSs, they are called **weakly disjoint** if \((X \times Y, T \times S)\) is transitive, denoted by \((X, T) \not\prec (Y, S)\) or \(X \not\prec Y\) \([17]\). It is easy to see that if \((X, T)\) and \((Y, S)\) are weakly disjoint, then both of them are transitive.

**Lemma 5.5.** Let \( F \) be a family. If \((X, T)\) is \( F \)-transitive and \((Y, S)\) is transitive and \( \kappa F \)-center, then \((X, T) \not\prec (Y, S)\).

**Proof.** Let \( U_1, U_2 \) be two open subsets of \( X \) and \( V_1, V_2 \) be two open subsets of \( Y \). Since \((Y, S)\) is transitive, then there exists \( n \in \mathbb{N} \) such that \( V_1 \cap S^{-n} V_2 \neq \emptyset \). Let \( V = V_1 \cap S^{-n} V_2 \). Then it is easy to see that

\[
n + N(U_1, T^{-n} U_2) \cap N(V, V) \subset N(U_1 \times V_1, U_2 \times U_2).
\]

Since \((X, T)\) is \( F \)-transitive and \((Y, S)\) is \( \kappa F \)-center, \( N(U_1, T^{-n} U_2) \cap N(V, V) \neq \emptyset \). Then \( N(U_1 \times V_1, U_2 \times U_2) \neq \emptyset \). Thus, \((X \times Y, T \times S)\) is transitive, i.e., \((X, T) \not\prec (Y, S)\).

**Lemma 5.6.** Let \( F \subset \mathbb{Z}_+ \) be weakly thick, then there exists a \( P \)-system \((X, T)\), a transitive point \( x \in X \) and an open subset \( U \) of \( X \) such that \( N(x, U) \subset F \).

**Proof.** Let \( F \subset \mathbb{Z}_+ \) be a weakly thick set, there exists \( k \in \mathbb{N} \) and a sequence \( \{a_n\}_{n=1}^\infty \) of \( \mathbb{Z}_+ \) such that \( \bigcup_{n=1}^\infty (k[a_n, a_n + n]) \subset F \). Without lose of generality, assume that \( a_{n+1} > a_n + 2n \). Let \( F_0 = \bigcup_{n=1}^\infty (k[a_n, a_n + n]) \). We will construct \( x^{(n)} = 1_{a_n} \in \{0, 1\}^{\mathbb{Z}_+} \) such that \( A_n \subset F_0 \) and \( x = \lim x^{(n)} = 1_A \). Moreover, \( X = \overline{\text{Orb}(x, T)} \) is a \( P \)-system and \( N(x, U) = A \subset F_0 \), where \( U = \{y \in X : y(0) = 1\} \).

To obtain \( x^{(n)} \) we construct a finite word \( B_n \) such that \( x^{(n)} \) begins with \( B_n \) and \( B_n \) appears in \( x^{(n)} \) weakly thick, and in the next step let \( B_{n+1} \) begin with \( B_n \). Let \( \{P_i\}_{i=0}^\infty \) is a partition of \( \mathbb{N} \) and each \( P_i \) is infinite.

**Step 1:** Construct \( x^{(1)} \). Let \( B_1 = 1_{F_0}[0, ka_1+k-1] \) and \( r_1 = |B_1| \) be the length of \( B_1 \). Put \( x^{(1)}[0, r_1 - 1] = B_1 \). For every \( p \in P_1 \), if \( kp \geq r_1 \), let \( l \) be the integer part of \( \frac{kp}{r_1} \), put \( x^{(1)}[ka_p+jr_1, ka_p+(j+1)r_1-1] = B_1 \) for \( j = 0, 1, \ldots, l-1 \) and \( x^{(1)}(i) = 0 \) for other undefined position \( i \). Let \( A_1 \subset \mathbb{Z}_+ \) such that \( 1_{A_1} = x^{(1)} \), then \( A_1 \subset F_0 \).
Step 2: Construct $x^{(2)}$. Let $B_2 = x^{(1)}[0, ka_2 - 1]$ and $r_2 = |B_2|$ be the length of $B_2$. Put $x^{(2)}[0, r_2 - 1] = B_2$. For every $p \in P_2$, if $kp \geq r_2$, let $l$ be the integer part of $\frac{kp}{r_2}$, put $x^{(2)}[ka_p + jr_2, ka_p + (j + 1)r_2 - 1] = B_2$ for $j = 0, 1, \ldots, l - 1$ and $x^{(2)}(i) = x^{(1)}(i)$ for other undefined position $i$. Let $A_2 \subset \mathbb{Z}_+$ such that $1_{A_2} = x^{(2)}$, then $A_1 \subset A_2 \subset F_0$.

Step 3: If $x^{(n)}$ has been constructed, now construct $x^{(n+1)}$. Let $B_{n+1} = x^{(n)}[0, ka_{n+1} - 1]$ and $r_{n+1} = |B_{n+1}|$ be the length of $B_{n+1}$. Put $x^{(n+1)}[0, r_{n+1} - 1] = B_{n+1}$. For every $p \in P_{n+1}$, if $kp \geq r_{n+1}$, let $l$ be the integer part of $\frac{kp}{r_{n+1}}$, put $x^{(n+1)}[ka_p + jr_{n+1}, ka_p + (j + 1)r_{n+1} - 1] = B_{n+1}$ for $j = 0, 1, \ldots, l - 1$ and $x^{(n+1)}(i) = x^{(n)}(i)$ for other undefined position $i$. Let $A_{n+1} \subset \mathbb{Z}_+$ such that $1_{A_{n+1}} = x^{(n+1)}$, then $A_n \subset A_{n+1} \subset F_0$.

In such a way, let $x = \lim x^{(n)} = 1_A$ and $X = \text{Orb}(x, T)$, then $x$ is a recurrent point and $N(x, U) = A \subset F_0$. It is easy to see that $X$ has dense periodic points. This completes the proof. □

**Theorem 5.7.** Let $(X, T)$ be a TDS, then the following conditions are equivalent:

1. $(X, T)$ is $\Delta^*(\mathcal{F}_{wt})$-transitive.
2. $(X, T)$ is weakly disjoint from every transitive and $\Delta(\mathcal{F}_{wt})$-center system.
3. $(X, T)$ is weakly disjoint from every $b\mathcal{F}_{wt}$-point transitive system.
4. $(X, T)$ is weakly disjoint from every $P$-system.

**Proof.** (1)$\Rightarrow$(2) follows from Lemma 5.3.

(2)$\Rightarrow$(3). Let $(Y, S)$ be a $b\mathcal{F}_{wt}$-point transitive system. It suffices to show that $(Y, S)$ is $\Delta(\mathcal{F}_{wt})$-center. Let $U$ be an open subset of $Y$, there exists a $b\mathcal{F}_{wt}$-transitive point $y \in U$. It is easy to see that $N(U, U) = N(y, U) - N(y, U)$. Then $N(U, U) \in \Delta(\mathcal{F}_{wt})$ since $N(y, U) \in b\mathcal{F}_{wt}$.

(3)$\Rightarrow$(4) follows from the fact that every $P$-system is $b\mathcal{F}_{wt}$-point transitive.

(4)$\Rightarrow$(1). Let $U, V$ be two open subsets of $X$. For every $F \in \mathcal{F}_{wt}$, by Lemma 5.6 there exists a $P$-system $(Y, S)$, a transitive point $x \in X$ and an open subset $W$ of $X$ such that $N(x, W) \subset F$. Then $N(W, W) \subset (F - F) \cup \{0\}$. Since $(X, T)$ is weakly disjoint from $(Y, S)$, $N(U, V) \cap N(W, W)$ is infinite, then $N(U, V) \cap (F - F) \neq \emptyset$. Thus, $N(U, V) \in \Delta^*(\mathcal{F}_{wt})$. □

**Appendix A**

In this appendix, as announced before, we construct an $\mathcal{F}_{ip}$-point transitive system which is not weakly mixing.

**Proof of Proposition 3.4.** Let $\Sigma = \{0, 1\}^\mathbb{Z}_+$ and $T$ be the shift map. We will construct a sequence $W_1 \subset W_2 \subset W_3 \cdots \subset \mathbb{Z}_+$ and $W = \bigcup_{n=1}^\infty W_n$. 


Let $x = 1_W$ and $X = \text{Orb}(x, T)$, we want to show that $(X, T)$ is $\mathcal{F}_{ip}$-point transitive but not weakly mixing.

Let $p_0 = 0$ and $p_{2,0} = 0$ for all $i, j \in \mathbb{N}$. Let $W_0 = \{p_0\} = A_0$. Choose $p_1 > 2$ and set

$$W_1 = W_0 \cup \{p_1\}.$$

Arrange $W_2$ as

$$W_1 = A_0 \cup A_1,$$

where $A_1 = \{p_1\}$. Let $k_1 = 1$, then $\min A_i > 3 \max A_j + 2$ for $0 \leq j < i \leq k_1$.

Choose $p_2$ and $p_{1,1}^{(1)}$ are positive integers to be defined later and set

$$W_2 = W_1 \bigcup (W_1 + p_2) \bigcup (W_1 + p_{1,1}^{(1)} - 1).$$

Arrange $W_2$ as

$$W_2 = W_1 \bigcup A_2 \bigcup A_3.$$

Let $k_2 = 3$, choose appropriate $p_2$ and $p_{1,1}^{(1)}$ such that $\min A_i > 3 \max A_j + 2$ for $0 \leq j < i \leq k_2$.

Choose $p_3, p_{1,2}^{(1)}, p_{1,1}^{(2)}$ and $p_{2,1}^{(2)}$ are positive integers to be defined later and set

$$W_3 = W_2 \bigcup (W_2 + p_3) \bigcup (W_1 + p_{1,2}^{(1)} - 1 + FS\{p_{1,1}^{(1)}\}_{i=0}^{1})$$

$$\bigcup (W_2 + p_{1,1}^{(2)} - 1) \bigcup (W_2 + p_{2,1}^{(2)} - 2).$$

Arrange $W_3$ as

$$W_3 = W_2 \bigcup A_4 \bigcup A_5 \bigcup A_6 \bigcup A_7.$$

Let $k_3 = 7$, choose appropriate $p_3, p_{1,2}^{(1)}, p_{1,1}^{(2)}$ and $p_{2,1}^{(2)}$ such that $\min A_i > 3 \max A_j + 2$ for $0 \leq j < i \leq k_3$.

Assume that $W_n$ has been constructed, now construct $W_{n+1}$, choose $p_{n+1}, p_{1,n}^{(1)}, p_{1,n-1}^{(2)}, p_{2,n-1}^{(2)}, \ldots, p_{n,1}^{(n)}$ are positive integers to be defined later and set

$$W_{n+1} = W_n \bigcup (W_n + p_{n+1}) \bigcup (W_1 + p_{1,n}^{(1)} - 1 + FS\{p_{1,i}^{(1)}\}_{i=0}^{n-1})$$

$$\bigcup \bigcup_{j=1}^{2} (W_2 + p_{j,n-1}^{(2)} - j + FS\{p_{j,i}^{(2)}\}_{i=0}^{n-2}) \bigcup \cdots$$

$$\bigcup \bigcup_{j=1}^{n-1} (W_{n-1} + p_{j,2}^{(n-1)} - j + FS\{p_{j,i}^{(n-1)}\}_{i=0}^{1})$$

$$\bigcup \bigcup_{j=1}^{n} (W_n + p_{j,1}^{(n)} - j).$$
Arrange $W_{n+1}$ as

$$W_{n+1} = W_n \bigcup A_{k_n+1} \bigcup A_{k_n+2} \bigcup \cdots \bigcup A_{k_{n+1}},$$

where $A_i$ is $W_n + p_{n+1}$ or in the form of $W_r + p_{j,i}^{(r)} - j + F S \{ p_{j,i}^{(r)} \}_{i=0}^{t-1}$.

Choose appropriate $p_{n+1}, p_{1,n}, p_{2,n-1}, . . . , p_{n,1}$ such that $\min A_i > 3 \max A_j + 2$ for $0 \leq j < i \leq k_{n+1}$.

Let $W = \bigcup_{n=0}^{\infty} W_n$. For every $n \in \mathbb{N}$, let $P_n = F S \{ p_j \}_{j=n+1}^{\infty}, P^{(n)} = F S \{ p_{j,i}^{(n)} \}_{i=1}^{\infty}$ for $i = 1, \ldots , n$. Then for every $n \in \mathbb{N}$, $W_n + P^{(n)} - i \subset W$ for $i = 0, 1, \ldots , n$.

Let $x = 1_W$ and $X = \text{Orb}(x,T)$. First, we show that $(X,T)$ is $\mathcal{F}_{ip}$-point transitive. It suffices to show that $x$ is an $\mathcal{F}_{ip}$-transitive point.

Let $r_n = \max W_n$ and $[W_n] = \{ y \in X : i \in [0,r_n] \}$. Then $\{ [W_n] : n \in \mathbb{N} \}$ is a neighborhood base of $x$. It is easy to see that

$$N(x,[W_n]) \supset \bigcup_{i=0}^{n} (P^{(n)}_i - i).$$

For every open subset $U$ of $X$, there exists $k \in \mathbb{Z}_+$ such that $T^k x \in U$. By the continuity of $T$, there exists $n > k$ such that $T^k ([W_n]) \subset U$, so

$$P^{(n)}_k \subset k + N(x,[W_n]) \subset N(x,U).$$

Thus, $x$ is an $\mathcal{F}_{ip}$-transitive point.

To see that $(X,T)$ is not weakly mixing, it suffices to show that $N([1],[1]) = W - W$ is not thick. In fact, we show that for every $n \geq 1$, if $a \neq b \in W_n - W_n$, then $|a - b| > 2$.

**Claim 1:** If $a \in A_i - A_i$ for some $i \in [k_n + 1, k_{n+1}]$ is not zero, then there exist $1 \leq i_1 < i_2 \leq k_n$ such that $a \in A_{i_2} - A_{i_1}$.

We prove this claim by induction of $n$. It is easy to see that the result holds for $n = 1, 2$. Now assume that the result holds for $1, 2, \ldots, n - 1$. Let $a \in A_i - A_i$ for some $i \in [k_n + 1, k_{n+1}]$. By the construction, every element in $A_i$ has the same part in $\{ p_{n+1}, p_{1,n}^{(1)} - 1, p_{1,n-1}^{(2)} - 1, p_{2,n-1}^{(3)} - 2, \ldots , p_{n,1}^{(n)} - n \}$, then $a \in W_{n-1} - W_{n-1}$, by the assume the proof is complete.

**Claim 2:** For every $n \geq 1$, if $a \neq b \in W_n - W_n$, then $|a - b| > 2$.

Again we prove this claim by induction of $n$. It is easy to see that the result holds for $n = 1, 2$. Now assume that the result holds for $1, 2, \ldots, n - 1$.

Let $a \neq b \in W_n - W_n$. There exist $s_1, s_2, t_1$ and $t_2 \in [0,k_n]$ such that $a \in A_{s_1} - A_{s_2}$ and $b \in A_{t_1} - A_{t_2}$. By the Claim 1, we can assume that $s_1 > s_2$ and $t_1 > t_2$.

1. Case $s_1 > t_1$. Then $a - b \geq \min A_{s_1} - 3 \max A_{s_1 - 1} > 2$.
2. Case $s_1 = t_1$. 


(a) $s_2 = t_2$. Then $a - b \in (A_{s_1} - A_{s_2}) - (A_{t_1} - A_{t_2}) = (A_{s_1} - A_{t_1}) - (A_{s_2} - A_{t_2})$. By the Claim 1, we have $a - b = c - d$ for some $c, d \in W_{n-1} - W_{n-1}$, so $a - b > 2$ by induction.

(b) $s_2 > t_2$. Then $a - b \in (A_{s_1} - A_{s_2}) - (A_{t_1} - A_{t_2}) = (A_{s_1} - A_{t_1}) - (A_{s_2} - A_{t_2})$. By the Claim 1, we have $A_{s_1} - A_{t_1} \subset W_{n-1} - W_{n-1}$. If $s_2 > k_{n-1}$, this turn to the case (1), so $|a - b| > 2$. If $s_2 \leq k_{n-1}$, then $|a - b| > 2$ by induction.

Hence, $(X, T)$ is as required. \hfill $\Box$

References

[1] E. Akin, Recurrence in Topological Dynamical Systems. Families and Ellis Actions, The University Series in Mathematics. Plenum Press, New York, 1997.

[2] E. Akin, Lectures on Cantor and Mycielski sets for dynamical systems, Chapel Hill Ergodic Theory Workshops, 21–79, Contemp. Math., 356, Amer. Math. Soc., Providence, RI., 2004.

[3] E. Akin and E. Glasner, Residual properties and almost equicontinuity, J. Anal. Math. 84 (2001), 243–86.

[4] E. Akin and S. Kolyada, Li-Yorke sensitivity, Nonlinearity, 16 (2003), 1421–1433.

[5] V. Bergelson and T. Downarowicz, Large sets of integers and hierarchy of mixing properties of measure-preserving systems, Colloquium Mathematicum 110 (2008), no. 1, 117-150.

[6] F. Blanchard, B. Host and A. Maass, Topological complexity, Ergod. Theory Dyn. Syst. 20 (2000), 641–62.

[7] H. Furstenberg, Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation, Math. Systems Theory, 1 (1967), 1–49.

[8] H. Furstenberg, Recurrence in ergodic theory and combinatorial number theory, M. B. Porter Lectures. Princeton University Press, Princeton, N.J., 1981.

[9] E. Glasner, Classifying dynamical systems by their recurrence properties, Topol. Methods Nonlinear Anal. 24 (2004), 21–40.

[10] W. H. Gottschalk and G. A. Hedlund, Topological Dynamics, Amer. Math. Soc. Colloquium Publications, Vol.36. Providence, R.I., 1955.

[11] W. He and Z. Zhou, A topologically mixing system with its measure center being a singleton, Acta Math. Sinica, 45(5)(2002), 929-934.

[12] W. Huang, H. Li and X. Ye, Family-independence for topological and measurable dynamics, to appear in Trans. Amer. Math. Soc, http://arxiv.org/abs/0908.0574.

[13] W. Huang, K. K. Park and X. Ye, Dynamical systems disjoint from all minimal systems with zero entropy, Bull. Soc. Math. France, 135 (2007), 259–282.

[14] W. Huang and X. Ye, An explicit scattering, non-weakly mixing example and weak disjointness, Nonlinearity 15, (2002), 849–862.

[15] W. Huang and X. Ye, Topological complexity, return times and weak disjointness, Ergod. Theor. Dynam. Syst., 24 (2004), 825–846.

[16] W. Huang and X. Ye, Dynamical systems disjoint from any minimal system, Trans. Amer. Math. Soc., 357, (2005), no. 2, 669–694.

[17] R. Peleg, Weak disjointness of transformation groups, Proc. Amer. Math. Soc., 33 (1972), 165–170.

[18] S. Shao and X. Ye, $F$-mixing and weakly disjointness. Topology and its applications, 135 (2004), no.1–3, 231–247.
[19] B. Weiss, *Single Orbit Dynamics*, Amer. Math. Soc., Regional Conference Series in Mathematics, No. 95, Providence, RI, 2000.

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