Three-loop Standard Model effective potential at leading order in strong and top Yukawa couplings

Stephen P. Martin

Department of Physics, Northern Illinois University, DeKalb IL 60115, 
Fermi National Accelerator Laboratory, P.O. Box 500, Batavia IL 60510, and 
Kavli Institute for Theoretical Physics, University of California, Santa Barbara, CA 9310

I find the three-loop contribution to the effective potential for the Standard Model Higgs field, in the approximation that the strong and top Yukawa couplings are large compared to all other couplings, using dimensional regularization with modified minimal subtraction. Checks follow from gauge invariance and renormalization group invariance. I also briefly comment on the special problems posed by Goldstone boson contributions to the effective potential, and on the numerical impact of the result on the relations between the Higgs vacuum expectation value, mass, and self-interaction coupling.

Contents

| Section | Page |
|---------|------|
| I. Introduction | 2 |
| II. The necessary integrals | 3 |
| III. Effective potential in terms of bare quantities | 8 |
| IV. Effective potential in terms of renormalized quantities | 11 |
| V. The Goldstone boson catastrophe | 16 |
| VI. Numerical impact | 17 |
| VII. Outlook | 21 |
| Appendix: Individual diagram contributions | 21 |
| References | 23 |
I. INTRODUCTION

The discovery [1–4] of the Standard Model Higgs boson with a mass near 126 GeV, and so far no other new fundamental physics, implies that a new era of precision analyses of the minimal electroweak symmetry breaking dynamics has begun. The relation between the Higgs field vacuum expectation value (VEV) and the underlying Lagrangian parameters, as well as the issue of vacuum stability, can be analyzed precisely using the effective potential approach [5–7]. At present, the Standard Model Higgs effective potential has been evaluated at two-loop order in [8]. (The extension to more general models, including supersymmetric ones, is given in [9].) An intriguing aspect of the observed Higgs mass is that the resulting potential is in the metastable region near the critical value associated with a very small Higgs-self interaction at very high energy scales. Analyses of the vacuum stability condition before the Higgs discovery were given in refs. [7, 10–20], and some of the more detailed analyses since after the Higgs mass became known are given in [21–27].

The purpose of this paper is to find the three-loop contributions to the effective potential of the Standard Model, in the approximation that the QCD coupling $g_3$ and the top-quark Yukawa coupling $y_t$ are large compared to the Higgs self-interaction $\lambda$ and the electroweak gauge couplings $g$ and $g'$ and the other quark and lepton Yukawa couplings. In this approximation, the three-loop part of the effective potential is proportional to $m_t^4$, multiplied by terms $g_3^4$, $g_3^2 g_t^2$, and $y_t^4$, and up to cubic logarithms. Here $m_t$ is the field-dependent tree-level top quark mass. The effective potential $V_{\text{eff}}(\phi)$ is found as the sum of one-particle-irreducible vacuum Feynman diagrams, using couplings and masses obtained in the presence of a classical background field $\phi$ whose value at the minimum of the effective potential coincides with the Higgs VEV. The effective potential will be calculated in dimensional regularization [28–32] with modified minimal subtraction [33, 34]. The result may be used to improve the accuracy and/or theoretical error estimates for analyses of vacuum stability and Lagrangian parameter determination for the Standard Model.

To establish conventions for the present paper, consider the Higgs Lagrangian

$$\mathcal{L} = -\partial^\mu \Phi^\dagger \partial_\mu \Phi - \Lambda - m^2 \Phi^\dagger \Phi - \lambda (\Phi^\dagger \Phi)^2, \quad (1.1)$$

where $m^2$ is the (negative) Higgs squared mass parameter, and $\lambda$ is the self-coupling in the normalization to be used in this paper, and I use the metric with signature $(-+++)$. The field-independent vacuum energy term $\Lambda$ must be included in order to maintain renormalization scale invariance of the full effective potential and a proper treatment of renormalization group improvement [35–38, 12], but will play no direct role in the present paper. The complex Higgs doublet field is written

$$\Phi(x) = \begin{pmatrix} \frac{1}{\sqrt{2}}[\phi + H(x)] + iG^0(x) \\ G^+(x) \end{pmatrix}, \quad (1.2)$$

where $\phi$ is the real background field, and $H$ is the real Higgs quantum field, while $G^0$ and $G^+ = G^{--}$ are the real neutral and complex charged Goldstone boson fields. The effective potential is then a function of $\phi$, with a minimum that equals the vacuum expectation value.
The rest of this paper is organized as follows. Section II reviews the integrals that are necessary for the calculation. Section III calculates the three-loop effective potential in terms of bare quantities, with individual diagram contributions provided in an Appendix. Section IV performs the re-expression of the effective potential in terms of renormalized quantities, to obtain the form that can be used for phenomenological analyses. Section V comments briefly on the special problems posed by Goldstone boson contributions to the effective potential, and section VI briefly discusses the numerical impact of the three-loop effective potential on the relations between the Higgs VEV, mass, and self-interaction coupling.

II. THE NECESSARY INTEGRALS

In this section, I review the results for Feynman integrals that are necessary for the calculations in the rest of the paper. Euclidean momentum integrals in

\[ d = 4 - 2\epsilon \]

dimensions are written using the notation

\[ \int_p \equiv \int \frac{d^d p}{(2\pi)^d}. \]

Consider first the integrals that depend only on one squared mass scale, to be denoted \( x \) below. In this paper, \( x \) will be the (bare) field-dependent squared mass of the top quark. The one-loop scalar vacuum master integral is

\[ A \equiv \int_p \frac{1}{p^2 + x} = \frac{\Gamma(1 - d/2)}{(4\pi)^{d/2}} x^{d/2 - 1}. \]

For the two-loop integrals with one mass scale that are relevant below, there are two master integrals:

\[ A^2 = \int_p \int_q \frac{1}{(p^2 + x)(q^2 + x)}, \]

\[ B \equiv x \int_p \int_q \frac{1}{(p^2 + x)q^2(p + q)^2} = \frac{\Gamma(3 - d)\Gamma(d/2)}{\Gamma(2 - d/2)} A^2. \]

Here, the term “master integral” is taken to mean one of the minimal set of integrals at a given loop order to which all others can be reduced, with coefficients that are ratios of polynomials in \( d \), by elementary algebra or integration by parts \[39\] identities. Thus, \( B \) and \( A^2 \) are considered distinct two-loop one-scale master integrals by this criterion, but

\[ \int_p \int_q \frac{1}{(p^2 + x)(q^2 + x)(p + q)^2} = \frac{2 - d}{2(d - 3)} \frac{x}{A^2}. \]
is not a master integral.

The three-loop one-scale Feynman integrals encountered below can be reduced by elementary algebra to integrals of the types

\[ C(n_1, n_2, n_3, n_4, n_5, n_6) = \int_p \int_q \int_k \frac{1}{(p-k)^2 + x^{n_1} |p^2 + x|^{n_2} [q^2 + x]^{n_3} ((q+k)^2 + x)^{n_4} [(p+q)^2]^{n_5} [k^2]^{n_6}}, \]  

\[ D(n_1, n_2, n_3, n_4, n_5, n_6) = \int_p \int_q \int_k \frac{1}{(p-k)^2 + x^{n_1} [(q+k)^2 + x]^{n_2} [p^2]^{n_3} [q^2]^{n_4} [p+q]^2]^{n_5} [k^2]^{n_6}}, \]  

\[ E(n_1, n_2, n_3, n_4, n_5, n_6) = \int_p \int_q \int_k \frac{1}{(p-k)^2 + x^{n_1} [(q+k)^2 + x]^{n_2} [p^2]^{n_3} [q^2]^{n_4} [(p+q)^2]^{n_5} [k^2]^{n_6}}, \]  

\[ F(n_1, n_2, n_3, n_4, n_5, n_6) = \int_p \int_q \int_k \frac{1}{(p+q)^2 + x^{n_1} [(p-k)^2 + x]^{n_2} [(p-k)^2]^{n_3} [(q+k)^2]^{n_4} [q^2]^{n_5} [p^2]^{n_6}}, \] 

illustrated in Figure 2.1 and studied in [40–43]. Here the exponents \( n_i \) are integers, which can be positive, negative, or zero. Some of the integrals with some non-positive exponents \( n_i \) vanish trivially due to the dimensional regularization identity \( \int_p 1/(p^2)^n = 0 \). Expressions involving the remaining integrals can then be systematically simplified using the following identities. First, re-labeling the momenta gives the symmetry identities:

\[ C(n_1, n_2, n_3, n_4, n_5, n_6) = C(n_2, n_3, n_4, n_1, n_6, n_5) = C(n_4, n_3, n_2, n_1, n_5, n_6), \]  

\[ D(n_1, n_2, n_3, n_4, n_5, n_6) = D(n_1, n_3, n_2, n_6, n_5, n_4) = D(n_2, n_3, n_1, n_6, n_4, n_5), \]  

\[ E(n_1, n_2, n_3, n_4, n_5, n_6) = E(n_2, n_1, n_3, n_5, n_4, n_6), \]  

\[ F(n_1, n_2, n_3, n_4, n_5, n_6) = F(n_1, n_2, n_6, n_5, n_4, n_3) = F(n_2, n_1, n_5, n_4, n_3, n_6), \]
and others obtained by repeated application of those. Also, dimensional analysis yields:

\begin{align}
0 &= \left[3d/2 - \sum_{j=1}^{6} n_j + x \sum_{j=1}^{4} n_j j^+\right] C(n_1, n_2, n_3, n_4, n_5, n_6), \\
0 &= \left[3d/2 - \sum_{j=1}^{6} n_j + x \sum_{j=1}^{3} n_j j^+\right] D(n_1, n_2, n_3, n_4, n_5, n_6), \\
0 &= \left[3d/2 - \sum_{j=1}^{6} n_j + x \sum_{j=1}^{2} n_j j^+\right] E(n_1, n_2, n_3, n_4, n_5, n_6), \\
0 &= \left[3d/2 - \sum_{j=1}^{6} n_j + x \sum_{j=1}^{2} n_j j^+\right] F(n_1, n_2, n_3, n_4, n_5, n_6).
\end{align}

where here and below the notation for bold-faced raising and lowering operators is the standard one such that, for each integer \( j = 1, \ldots, 6 \), we have \( j^\pm C(\ldots, n_j, \ldots) \equiv C(\ldots, n_j \pm 1, \ldots) \) and similarly for \( D, E, \) and \( F \). Finally, integration by parts \([39]\) gives the identities:

\begin{align}
0 &= \left[d - n_3 - n_4 - 2n_5 + n_3 3^+(2^- - 5^-) + n_4 4^+(1^- - 5^-)\right] C, \\
0 &= \left[d - n_1 - 2n_2 - n_5 + n_1 1^+(6^- - 2^+ + 2x) + 2xn_2 2^+ + n_5 5^+(3^- - 2^-)\right] C,
\end{align}

and

\begin{align}
0 &= \left[d - n_3 - n_4 - 2n_5 + n_3 3^+(2^- - 5^-) + n_4 4^+(6^- - 5^-)\right] D, \\
0 &= \left[d - n_1 - 2n_2 - n_3 + n_1 1^+(6^- - 2^- + 2x) + 2xn_2 2^+ + n_3 3^+(5^- - 2^- + 2x)\right] D, \\
0 &= \left[d - 2n_2 - n_5 - n_6 + 2xn_2 2^+ + n_5 5^+(3^- - 2^-) + n_6 6^+(1^- - 2^-)\right] D,
\end{align}

and

\begin{align}
0 &= \left[d - n_2 - n_5 - 2n_6 + n_2 2^+(1^- - 6^-) + n_5 5^+(4^- - 6^-)\right] E, \\
0 &= \left[d - 2n_3 - n_4 - n_5 + n_4 4^+(1^- - 3^- - x) + n_5 5^+(2^- - 3^- - x)\right] E, \\
0 &= \left[d - 2n_2 - n_5 - n_6 + 2xn_2 2^+ + n_5 5^+(3^- - 2^- + x) + n_6 6^+(1^- - 2^-)\right] E, \\
0 &= \left[d - n_3 - 2n_4 - n_5 + n_3 3^+(1^- - 4^- - x) + n_5 5^+(6^- - 4^-)\right] E, \\
0 &= \left[d - n_1 - 2n_2 - 2n_3 + n_1 1^+(4^- - 3^- + x) + n_2 2^+(3^- - 3^- + x)\right] E, \\
0 &= \left[d - n_1 - 2n_2 - n_3 + n_1 1^+(6^- - 2^- + 2x) + 2xn_2 2^+ + n_3 3^+(5^- - 2^- + x)\right] E, \\
0 &= \left[d - n_1 - 2n_4 - n_6 + n_1 1^+(3^- - 4^- + x) + n_6 6^+(5^- - 4^-)\right] E,
\end{align}

and

\begin{align}
0 &= \left[d - 2n_2 - n_5 - n_6 + 2xn_2 2^+ + n_5 5^+(4^- - 2^- + x) + n_6 6^+(3^- - 2^- + x)\right] F, \\
0 &= \left[d - n_2 - 2n_5 - n_6 + n_2 2^+(4^- - 5^- + x) + n_6 6^+(1^- - 5^- - x)\right] F.
\end{align}
FIG. 2.2: The three-loop master integrals $J, K, L, P, Q, R,$ and $S$ defined in eqs. (2.33)-(2.39). Solid lines represent scalar propagators with squared mass $x$, and dashed line are for massless propagators. All of the integrals in Figure 2.1 can be reduced to linear combinations of these, with coefficients that are ratios of polynomials in the number of spacetime dimensions, $d$.

In equations (2.19)-(2.32), the arguments $(n_1, n_2, n_3, n_4, n_5, n_6)$ are implicit for $C, D, E,$ and $F$, and were omitted for the sake of simplicity.

Repeated applications of the identities in equations (2.11)-(2.32) allows [40, 41] all of the one-scale integrals of the types $C, D, E,$ and $F$ to be reduced (with many redundant checks) to just the seven master integrals depicted in Figure 2.2:

$$J \equiv C(0,1,1,1,0,0) / x = D(1,1,1,0,0,0) / x = A^3 / x,$$  \hspace{1cm} (2.33)

$$K \equiv C(0,1,0,1,1,1) = D(0,1,1,1,0,1) = E(1,1,0,1,1,0) = F(1,1,0,1,0,1),$$  \hspace{1cm} (2.34)

$$L \equiv C(1,1,1,1,0,0),$$  \hspace{1cm} (2.35)

$$P \equiv E(0,1,1,1,0,1),$$  \hspace{1cm} (2.36)

$$Q \equiv E(1,1,1,0,1,0) = F(1,1,0,1,1,0) = A B / x,$$  \hspace{1cm} (2.37)

$$R \equiv x F(0,1,1,1,1,1),$$  \hspace{1cm} (2.38)

$$S \equiv x^2 F(1,1,1,1,1,1).$$  \hspace{1cm} (2.39)

The expressions for $J, K, P, Q,$ and $R$ are known exactly in terms of gamma functions, and for the remaining integrals $L$ and $S$ the results are known as expansions in $\epsilon$. Actually, the integral $S$ will not be needed in the present paper. The integrals $J, L,$ and $P$ are needed here to order $\epsilon^2$, while $K$ and $Q$ are needed to order $\epsilon^1$, and $R$ to order $\epsilon^0$. Writing each of the other master integrals in terms of $A^3 / x$, the expansions to these orders (and beyond) are found from refs. [40, 42]:

$$J = A^3 / x,$$  \hspace{1cm} (2.40)

$$K = \frac{A^3}{x} \left( -\frac{1}{3} - \frac{\epsilon}{6} + \frac{5\epsilon^2}{12} + \left[ \frac{79}{24} - \frac{8\zeta(3)}{3} \right] \epsilon^3 + \left[ \frac{685}{48} + \frac{2\pi^4}{15} - \frac{4\zeta(3)}{3} \right] \epsilon^4 + \ldots \right),$$  \hspace{1cm} (2.41)

$$L = \frac{A^3}{x} \left( -2 - \frac{5}{3} - \frac{1}{2} \epsilon^2 + \frac{103}{12} \epsilon^3 + \left[ 1141/24 - 112\zeta(3)/3 \right] \epsilon^4 + \left[ 9055/48 + 136\pi^4/45 \right] \epsilon^5 + \ldots \right),$$
It follows that, to the order needed below:

\[ I_{xx} = \frac{A^2}{x} \left\{ \frac{2 - d}{2(d - 3)} \left[ \frac{1}{2} - \epsilon \ln(r) + \epsilon^2 \left[ \frac{1}{2} \ln^2(r) + 2 \ln(r) - 4 \right] \right] + \epsilon^3 \left[ \frac{1}{6} \ln^3(r) - \ln^2(r) + 8 \right] + O(\epsilon^4) + O(\epsilon^2) \right\}, \]  

\[ P = \frac{A^3}{x} \left( \frac{\epsilon}{12} + \frac{3\epsilon^2}{8} + \left[ \frac{67}{64} + \frac{\pi^2}{12} \right] \epsilon^3 + \left[ \frac{457}{96} + 3\pi^2/8 - 5\zeta(3)/6 \right] \epsilon^4 \right. \]
\[ + \left[ \frac{2971}{192} + \frac{67\pi^2}{48} + 23\pi^4/360 - 15\zeta(3)/4 \right] \epsilon^5 + \ldots, \]  

\[ Q = \frac{A^3}{x} \left( -\frac{1}{2} - \frac{\epsilon}{2} + \left[ -1 - \frac{\pi^2}{6} \right] \epsilon^2 + \left[ \zeta(3) - 2 - \frac{\pi^2}{6} \right] \epsilon^3 + \left[ \zeta(3) - 4 - \frac{\pi^2}{3} - \frac{\pi^4}{20} \right] \epsilon^4 + \ldots \right), \]
\[ R = \frac{A^3}{x} \left( \frac{1}{3} + \frac{2\epsilon}{3} + \left[ 5/3 + \pi^2/3 \right] \epsilon^2 + \left[ 4 + 2\pi^2/3 - 4\zeta(3)/3 \right] \epsilon^3 + \ldots \right). \]

A particularly useful and systematic compendium of these and many other one-scale multi-loop vacuum integral results can be found in ref. [43].

In the following, we will also need certain integrals that depend on two squared mass scales: \( x \) (which, as above, will be the top-quark squared mass) and \( y \) (which will be either the Higgs or Goldstone bare squared mass). Because of the approximation used in this paper, it is sufficient to have these integrals to first order in \( y \), and to order \( \epsilon^1 \) for two-loop integrals and \( \epsilon^0 \) for three-loop integrals. The two-scale integrals needed are:

\[ I_{xxy} \equiv \int_p \int_q \left( \frac{1}{(p^2 + x)(q^2 + x)(p + q)^2 + y} \right), \]  

\[ I_{x0y} \equiv \int_p \int_q \left( \frac{1}{p^2 + x)(q^2 + x)(p + q)^2 + y} \right), \]
\[ I_{xxxxy} \equiv \int_p \int_q \int_k \left( \frac{1}{(p - k)^2 + x)(q + k)^2 + x)(p - k)^2 + x)(q + k)^2 + x + k^2 + y} \right), \]
\[ I_{xx00y} \equiv \int_p \int_q \int_k \left( \frac{1}{(p - k)^2 + x)(q + k)^2 + x)(p - k)^2 + x)(q + k)^2 + x + k^2 + y} \right). \]

Using integration by parts and dimensional analysis, these integrals are found to obey the differential equations:

\[ y(4x - y)\frac{d}{dy} I_{xxy} = (d - 3)(2x - y)I_{xxy} + (d - 2) \left[ 1 - (y/x)^{d/2 - 1} \right] A^2, \]
\[ (x - y)^2\frac{d}{dy} I_{x0y} = (3 - d)(x - y)I_{x0y} + (1 - d/2)(1 - y/x)(y/x)^{d/2 - 1} A^2, \]
\[ y(4x - y)\frac{d}{dy} I_{xxxxy} = (2dx - 8x + 5y - 3dy/2)I_{xxxxy} + (3d/2 - 4)L + (4 - 2d)AL_{xxy}, \]
\[ 2y(y - x)\frac{d}{dy} I_{xx00y} = (dx - 2x + 3dy - 10y)I_{xx00y} + (2d - 4)AI_{x0y} + (8 - 3d)K. \]
\[ I_{x0y} = \frac{B}{x} \left\{ 1 + r \left( 1 - 2 \epsilon \ln(r) + \epsilon^2 \left[ \ln^2(r) + 2 \ln(r) - 2 - \frac{2 \pi^2}{3} \right] \right) 
+ \epsilon^3 \left[ -\frac{1}{3} \ln^3(r) - \ln^2(r) + (2 + 2 \pi^2/3) \ln(r) - 2 + 2 \pi^2/3 + 4 \zeta(3) \right] + \mathcal{O}(\epsilon^4) \right\} + \mathcal{O}(r^2) \} , \tag{2.55} \]

\[ I_{xxxxy} = \frac{1}{4(4 - d)} \left[ \left( 3d - 8 \right) \frac{L}{x} + \frac{2(d - 2)^2 A^3}{d - 3} x^2 \right] + r \frac{A^3}{x^2} \left\{ \frac{1}{3} + \epsilon \left[ 2/3 - \ln(r) \right] \right\} 
+ \epsilon^2 \left[ \frac{1}{6} \ln^3(r) - \ln^2(r) - \ln(r) + 16 + \frac{14}{3} \zeta(3) \right] 
+ \mathcal{O}(\epsilon^4) \} + \mathcal{O}(r^2) , \tag{2.56} \]

\[ I_{xxx0y} = \frac{(3d - 8)}{d - 2} \left( \frac{K}{x} - 2 \frac{AB}{x^2} + r \frac{A^3}{x^2} \left\{ \frac{1}{3} + \epsilon \left[ \frac{2}{3} \ln(r) \right] + \epsilon^2 \left[ \frac{1}{2} \ln^2(r) - \frac{\pi^2}{3} - \frac{1}{3} \right] \right\} \right) 
+ \epsilon^3 \left[ -\frac{1}{6} \ln^3(r) - \frac{1}{2} + \frac{\pi^2}{2} + \frac{14}{3} \zeta(3) \right] \right\} + \mathcal{O}(\epsilon^4) \} + \mathcal{O}(r^2) , \tag{2.57} \]

where \( r = y/x \). The above expansions for \( I_{xy} \) and \( I_{x0y} \) can also be obtained as special cases of results in \[44\], and the expansion for \( I_{xxxxy} \) can be obtained as a special case of eq. (3.27) in \[45\].

\section*{III. EFFECTIVE POTENTIAL IN TERMS OF BARE QUANTITIES}

Consider the effective potential written in terms of the bare external scalar field \( \phi_B \) and the bare coupling parameters including the Yukawa coupling \( y_{tB} \), the strong coupling \( g_{3B} \), and the Higgs self coupling \( \lambda_B \). This is calculated in \( d = 4 - 2 \epsilon \) dimensions in terms of the bare parameters in the Lagrangian, without including any counterterms. The conversion to \( \overline{\text{MS}} \) parameters will be done in the next section. The expansion in terms of the loop order \( \ell \) reads

\[ V_{\text{eff}} = \sum_{\ell=0}^{\infty} V_B^{(\ell)} , \tag{3.1} \]

where the tree-level potential in this expansion is

\[ V_B^{(0)} = \Lambda_B + \frac{m_B^2}{2} \phi_B^2 + \frac{\lambda_B}{4} \phi_B^4 . \tag{3.2} \]

The bare field-dependent squared masses of the top quark, Higgs scalar \( H^0 \), and the Goldstone bosons \( G^0, G^\pm \) (in Landau gauge), are denoted by

\[ x = \frac{y_{tB}^2 \phi_B^2}{2} , \tag{3.3} \]
\[ x_H = m_B^2 + 3 \lambda_B \phi_B^2 , \tag{3.4} \]
\[ x_G = m_B^2 + \lambda_B \phi_B^2 , \tag{3.5} \]

and the integrals \( A, B, J, K, L, P, Q, \) and \( R \) of the previous section are taken to be functions of \( x \). In order to maximize the generality of results below, and allow more informative checks, they
FIG. 3.1: The 2-loop vacuum Feynman diagrams involving the top quark, neglecting electroweak interactions. Wavy lines are gluons, heavy solid lines are top quarks, lighter solid lines are bottom quarks, and dashed lines are scalars as labeled.

are written in terms of the group theory quantities

\[ C_G = N_c = 3, \]
\[ C_F = \frac{N_c^2 - 1}{2N_c} = \frac{4}{3}, \]
\[ T_F = \frac{1}{2}, \]
\[ N_q = 6, \]

where \( C_G \) is the Casimir invariant and Dynkin index of the \( SU(3)_c \) gauge group, \( C_F, T_F, \) and \( N_c \) are the Casimir invariant, Dynkin index, and dimension of the fundamental representation, and \( N_q \) is the number of quarks in the theory.

The well-known one-loop order top, Higgs, and Goldstone contributions are then:

\[ V_B^{(1)} = \frac{x A}{d} \left[ -4N_c + (x_H/x)^{d/2} + 3(x_G/x)^{d/2} \right]. \]  

(3.10)

Note that even though the aim of this paper is to neglect terms proportional to the Higgs and Goldstone masses in the three-loop order part of the renormalized result, they do need to be included in the one-loop bare contribution. This is because when \( \lambda_B \) is expressed in terms of renormalized couplings, it includes a counter-term proportional to \( y_t^4 \) with no \( \lambda \). The other one-loop contributions involving electroweak vector bosons and lighter fermions are not written here, because after expressing bare quantities in terms of renormalized quantities, they do not affect the determination of the three-loop contribution at leading order in the QCD and top-quark Yukawa couplings.

At two-loop order, the pertinent contributions are from the diagrams shown in Figure 3.1 (here, and below, each figure is taken to represent diagrams with all helicities and mass insertions consistent with the topology shown.) The gluon is treated with an arbitrary gauge-fixing parameter \( \xi \), with propagator

\[ -i[g_{\mu\nu}/p^2 - (1 - \xi)p_{\mu}p_{\nu}/(p^2)^2], \]

(3.11)

where \( \xi = 0 \) for Landau gauge and \( \xi = 1 \) for Feynman gauge. The dependence on the QCD \( \xi \)
cancels in the effective potential, providing a useful check. The combined two-loop order result is

\[
V_B^{(2)} = g_3^2 N_c C_F \frac{(d-1)(d-2)}{(d-3)} A^2 + y_t^2 N_c \left\{ A^2 \left[1 - (x_H/x)^{d/2-1} - 2(x_G/x)^{d/2-1}\right] 
+ (2x - x_H/2)I_{xxH} - x_G I_{xxG}/2 + (x - x_G)I_{x0xG}\right\}. \tag{3.12}
\]

Note that here one must include terms up to linear order in \(\lambda\). All except diagrams 3.2(r) and 3.2(s) are evaluated by first using the identities in equations (2.11)-(2.32) to reduce the result to the six master integrals of Figure 3.1b,c, again because \(\lambda\) of Figure 3.2 is shown in Figure 3.2. All except diagrams 3.2(r) and 3.2(s) are evaluated by first using the identities in equations (2.11)-(2.32) to reduce the result to the six master integrals \(J, K, L, P, Q, R\) of Figure 3.2. For diagrams 3.2(r) and 3.2(s), it is necessary to also make use of the two-scale identity in eq. (2.32). The task of the next section is to re-express these results in terms of renormalized quantities.

The individual diagram contributions, exhibiting the separate dependences on \(\xi\), are shown in the Appendix. The task of the next section is to re-express these results in terms of \(\overline{\text{MS}}\) renormalized quantities.
FIG. 3.2: The 3-loop vacuum Feynman diagrams involving the top quark and strong interactions. Wavy lines are gluons, heavy solid lines are top quarks, lighter solid lines are other quarks as labeled, dashed lines are $H^0, G^0$ and $G^\pm$ as labeled, and dotted lines are ghosts.

IV. EFFECTIVE POTENTIAL IN TERMS OF RENORMALIZED QUANTITIES

The effective potential in the $\overline{\text{MS}}$ renormalization scheme is obtained by re-expressing the bare quantities in terms of renormalized quantities. Write:

$$\phi_B = \mu^{-\epsilon} \phi \sqrt{Z_{\phi}},$$

(4.1)
\[
Z_\phi = 1 + \sum_{\ell=1}^{\infty} \sum_{n=1}^{\ell} \frac{c^{\phi}_{\ell,n}}{(16\pi^2)\epsilon^n},
\]
(4.2)
\[
z_{kB} = \mu^{\rho_{k}\epsilon} \left( z_k + \sum_{\ell=1}^{\infty} \sum_{n=1}^{\ell} \frac{c^{k}_{\ell,n}}{(16\pi^2)\epsilon^n} \right),
\]
(4.3)

where the subscript \( B \) indicates bare quantities, the absence of a subscript \( B \) indicates an \( \overline{\text{MS}} \) renormalized quantity, \( \ell \) is the loop order, and \( k \) is an index that runs over the list of Lagrangian parameters, including \( z_k = \lambda, y_t, g_3, m^2, \Lambda \), with \( \rho_\lambda = 2, \rho_{g_3} = \rho_{y_t} = 1, \rho_{m^2} = 0, \rho_\Lambda = -2. \)

The mass scale \( \mu \) is the (arbitrary) dimensional regularization scale, introduced so that \( \int d^d x V \) is dimensionless, and so that \( g_3 \) and \( y_t \) are also dimensionless, and the field-dependent top-quark mass \( y_t\phi/\sqrt{2} \) has mass dimension 1, for any \( \epsilon \). The regularization scale \( \mu \) is related to the \( \overline{\text{MS}} \) renormalization scale \( Q \) by \( Q^2 = 4\pi e^{-\gamma_E} \mu^2. \)

(4.4)

where \( \gamma_E = 0.5772\ldots \) is the Euler-Mascheroni constant. The counter-term quantities \( c^{\phi}_{\ell,n} \) and \( c^{k}_{\ell,n} \) are polynomials in the \( \overline{\text{MS}} \) renormalized parameters \( z_k \), and are independent of \( \epsilon \) and \( \phi \). They are determined by the requirement that the full effective potential and other physical quantities have no poles in \( \epsilon \) when expressed in terms of \( \overline{\text{MS}} \) quantities.

The \( \overline{\text{MS}} \) beta functions and the scalar anomalous dimension are defined by

\[
\beta_k \equiv \frac{Q d z_k}{dQ} \bigg|_{\epsilon = 0} = Q \frac{d z_k}{dQ} + \epsilon \rho_k z_k,
\]
(4.5)
\[
\gamma \equiv \frac{Q d \ln \phi}{dQ} \bigg|_{\epsilon = 0} = -Q \frac{d \ln \phi}{dQ} + \epsilon = \frac{1}{2} Q \frac{d}{dQ} \ln(Z_\phi).
\]
(4.6)

It is useful to write these as loop expansions:

\[
\beta_k = \sum_{\ell=1}^{\infty} \frac{1}{(16\pi^2)^{\ell-1} \ell!} \beta_k^{(\ell)},
\]
(4.7)
\[
\gamma = \sum_{\ell=1}^{\infty} \frac{1}{(16\pi^2)^{\ell-1} \ell!} \gamma^{(\ell)}.
\]
(4.8)

Now, by using the fact that the bare quantities \( \phi_B \) and \( z_{kB} \) cannot depend on \( Q \) (or \( \mu \)), one obtains the beta functions and anomalous dimension in terms of the simple pole counterterms:

\[
\beta_k^{(\ell)} = 2\ell c^{k}_{\ell,1},
\]
(4.9)
\[
\gamma^{(\ell)} = -\ell c^{\phi}_{\ell,1},
\]
(4.10)

\[\dagger\] As a simplifying notation, in subscripts and superscripts a specific parameter \( z_k \) is used interchangeably with the corresponding index \( k \), so that \( \rho_k \equiv \rho_{z_k} \) and \( c_{\ell,n}^{k} \equiv c_{\ell,n}^{z_k} \) and \( \beta_k \equiv \beta_{z_k} \).
as well as the consistency conditions for higher pole counterterms with $\ell \geq n \geq 2$:

$$2\ell c^k_{\ell,n} = \sum_{\ell' = 1}^{\ell - n + 1} \sum_j \beta_j^{(\ell')} \frac{\partial}{\partial z_j} c^k_{\ell' - \ell,n-1},$$  \hspace{1cm} (4.11)

$$\ell c^\phi_{\ell,n} = \sum_{\ell' = 1}^{\ell - n + 1} \left( -\gamma^{(\ell')} + \frac{1}{2} \sum_j \beta_j^{(\ell')} \frac{\partial}{\partial z_j} \right) c^\phi_{\ell' - \ell,n-1}.$$  \hspace{1cm} (4.12)

The identities

$$\sum_j \rho_j z_j \frac{\partial}{\partial z_j} c^k_{\ell,n} = (2\ell + \rho_k) c^k_{\ell,n},$$  \hspace{1cm} (4.13)

$$\sum_j \rho_j z_j \frac{\partial}{\partial z_j} c^\phi_{\ell,n} = 2\ell c^\phi_{\ell,n}$$  \hspace{1cm} (4.14)

have been used to simplify the preceding expressions. [Note that eq. (2.11) in ref. [46] has a missing factor of $-\rho_k$ on the left side.]

Equations (4.12)-(4.14) allow the coefficients $c^k_{\ell,n}$ and $c^\phi_{\ell,n}$ to be determined from the known results for the beta functions and scalar anomalous dimension. The ones that are needed for this paper are [46, 49, 8, 50] (see also [51, 52]):

$$c^\lambda_{1,1} = -N_c y_t^4 + \lambda g_t^2 (2N_c) + 12\lambda^2 + \ldots,$$  \hspace{1cm} (4.15)

$$c^\lambda_{2,1} = g_3^2 y_t^6 (-2N_c C_F) + y_t^6 (5N_c/2) + \lambda g_3^2 y_t^2 (5N_c C_F) + \lambda y_t^4 (-N_c/4) + \ldots,$$  \hspace{1cm} (4.16)

$$c^\lambda_{2,2} = g_3^2 y_t^4 (6N_c C_F) + y_t^6 (-2N_c^2 - 3N_c/2) + \lambda g_3^2 y_t^2 (-6N_c C_F)$$
$$+ \lambda y_t^4 (3N_c^2 - 21N_c/2) + \ldots,$$  \hspace{1cm} (4.17)

$$c^\lambda_{3,1} = g_3^4 y_t^4 N_c C_F \{ [8\zeta(3) - 109/6] C_G + [131/6 - 16\zeta(3)] C_F + (16 + 10N_q/3) T_F \}$$
$$+ g_3^2 y_t^6 N_c C_F [20\zeta(3) - 19/6] + y_t^8 N_c [-4\zeta(3) + 13/6 - 65N_c/8] + \ldots,$$  \hspace{1cm} (4.18)

$$c^\lambda_{3,2} = g_3^4 y_t^4 N_c C_F (24C_G + 10C_F - 16T_F N_q/3) + g_3^2 y_t^6 N_c C_F (-25 - 9N_c)$$
$$+ y_t^8 N_c (11/4 + 107N_c/12) + \ldots,$$  \hspace{1cm} (4.19)

$$c^\lambda_{3,3} = g_3^4 y_t^4 N_c C_F (-22C_G/3 - 24C_F + 8T_F N_q/3) + g_3^2 y_t^6 N_c C_F (15 + 18N_c)$$
$$+ y_t^8 N_c (-9/4 - N_c - 3N_c^2) + \ldots,$$  \hspace{1cm} (4.20)

$$c^\nu_{1,1} = g_3^2 y_t (-3C_F) + y_t^3 (N_c/2 + 3/4) + \ldots,$$  \hspace{1cm} (4.21)

$$c^\nu_{2,1} = g_3^2 y_t C_F \left( \frac{97}{12} C_G - \frac{3}{4} C_F + \frac{5}{3} T_F N_q \right) + g_3^2 y_t^5 C_F \left( 3 + \frac{5}{4} N_c \right) + y_t^7 \left( \frac{3}{8} - \frac{9}{8} N_c \right) + \ldots,$$  \hspace{1cm} (4.22)

$$c^\nu_{2,2} = g_3^4 y_t C_F \left( \frac{11}{2} C_G + \frac{9}{2} C_F - 2T_F N_q \right) + g_3^2 y_t^3 C_F \left( -3N_c - 9/2 \right)$$
$$+ y_t^5 \left( \frac{3}{8} N_c^2 + \frac{9}{8} N_c + \frac{27}{32} \right) + \ldots,$$  \hspace{1cm} (4.23)

$$c^\lambda_{1,1} = g_3^2 \left( \frac{11}{6} C_G + \frac{2}{3} T_F N_q \right),$$  \hspace{1cm} (4.24)

$$c^\phi_{1,1} = -y_t^2 N_c + \ldots,$$  \hspace{1cm} (4.25)

$$c^\phi_{2,1} = g_3^2 y_t^2 (-5N_c C_F/2) + y_t^4 (9N_c/8) + \ldots.$$  \hspace{1cm} (4.26)
\[ c_{2,2} = g_3^2 y_t^2 (3N_c C_F) + y_t^4 (-3N_c/4) + \ldots \]  

(4.27)

Here the ellipses refer to contributions that are known, but are suppressed by couplings other than \( y_t \) or \( g_3 \) to a sufficient extent that they are not pertinent for this paper.

Now, plugging eqs. (4.1)-(4.3) into the results (3.1), (3.2), (3.10), (3.12) and (3.13) gives the effective potential in terms of renormalized quantities. This can be written in a loop expansion as

\[ V_{\text{eff}} = \sum_{\ell=0}^{\infty} \frac{1}{(16\pi^2)^{\ell}} V^{(\ell)}. \]  

(4.28)

Note that, as a convention, here the loop factors of \( 1/(16\pi^2)^{\ell} \) have been extracted, unlike the corresponding loop expansion in terms of bare parameters, eq. (3.1). In this section, I write

\[ T = y_t^2 \phi^2 / 2, \]  

(4.29)

\[ H = m^2 + 3\lambda \phi^2, \]  

(4.30)

\[ G = m^2 + \lambda \phi^2, \]  

(4.31)

for the \( \overline{\text{MS}} \) field-dependent squared masses of the top quark, Higgs boson \( H \), and Goldstone bosons \( G^0, G^\pm \) respectively, and define

\[ \ln(X) \equiv \ln(X/Q^2) \]  

(4.32)

for \( X = T, H, G \). Retaining terms quadratic in \( \lambda \) and \( m^2 \) in the one-loop part, and linear in \( \lambda \) and \( m^2 \) in the two-loop part, and taking the limit \( \epsilon \to 0 \), now gives

\[ V^{(0)} = \Lambda + \frac{m^2}{2} \phi^2 + \frac{\lambda}{4} \phi^4, \]  

(4.33)

\[ V^{(1)} = -N_c T^2 \left[ \ln(T) - 3/2 \right] + \frac{H^2}{4} \left[ \ln(H) - 3/2 \right] + \frac{3G^2}{4} \left[ \ln(G) - 3/2 \right], \]  

(4.34)

\[ V^{(2)} = g_3^2 N_c C_F T^2 \left[ 6 \ln^2(T) - 16 \ln(T) + 18 \right] + y_t^2 N_c T^2 \left[ -\frac{3}{2} \ln^2(T) + 8 \ln(T) - \frac{23}{2} - \frac{\pi^2}{6} \right] \]  

\[ + y_t^2 N_c T H \left[ \frac{9}{2} + 4 \ln(T) - \frac{3}{2} \ln^2(T) + \{1 - 3 \ln(T)\} \ln(H/T) \right] \]  

\[ + y_t^2 N_c T G \left[ \frac{3}{2} + \frac{\pi^2}{3} + 2 \ln(T) - \frac{3}{2} \ln^2(T) + 3 \{1 - \ln(T)\} \ln(G/T) \right], \]  

(4.35)

which agrees with the relevant limits of ref. [8], and the new result:

\[ V^{(3)} = g_3^4 N_c C_F T^2 \left\{ C_G \left[ -\frac{22}{3} \ln^3(T) + \frac{185}{3} \ln^2(T) + (24\zeta(3) - \frac{1111}{6}) \ln(T) \right] \right. \]  

\[ + \frac{2609}{12} + \frac{44}{45} \pi^4 - \frac{232}{3} \zeta(3) + \frac{16}{3} \ln^2(2) [\pi^2 - \ln^2(2)] - 128 \text{Li}_4(1/2) \left. \right\} \]  

\[ + C_F \left[ -24 \ln^3(T) + 63 \ln^2(T) - (48\zeta(3) + \frac{121}{2}) \ln(T) + \frac{85}{12} - \frac{88}{45} \pi^4 \right], \]
\[ +192\zeta(3) - \frac{32}{3} \ln^2(2)[\pi^2 - \ln^2(2)] + 256\text{Li}_4(1/2)\] 
\[ + T_F N_q \left\{ \frac{8}{3} \ln^3(T) - \frac{52}{3} \ln^2(T) + \frac{142}{3} \ln(T) - \frac{161}{3} - \frac{64}{3} \zeta(3) \right\} \]
\[ + g_s^2 y_t^2 C_F T^2 \left\{ 15 \ln^3(T) - 90 \ln^2(T) + [407/2 + 3\pi^2 + 60\zeta(3)] \ln(T) - 54\zeta(3) \right\} \]
\[ + \frac{2393}{12} - \frac{29}{6} \pi^2 + \frac{31}{15} \pi^4 + \frac{32}{3} \ln^2(2)[\pi^2 - \ln^2(2)] - 256\text{Li}_4(1/2) \]
\[ + y_t^4 N_c T^2 \left\{ -\frac{9}{4} \ln^3(T) + \frac{57}{4} \ln^2(T) + \left[ -\frac{3}{4} \pi^2 - \frac{121}{4} - 12\zeta(3) \right] \ln(T) \right\} \]
\[ + \frac{529}{24} + \frac{23}{12} \pi^2 - \frac{22}{45} \pi^4 + \frac{93}{2} \zeta(3) - \frac{8}{3} \ln^2(2)[\pi^2 - \ln^2(2)] + 64\text{Li}_4(1/2) \]
\[ + y_t^4 N_c^2 T^2 \left\{ \frac{7}{2} \ln^3(T) + [17/4 + 9 \ln(H/T) + 3 \ln(G/T)] \ln^2(T) \right\} \]
\[ + \left[ -\frac{659}{8} - \frac{5}{6} \pi^2 - 6 \ln(H/T) - 6 \ln(G/T) \right] \ln(T) \]
\[ + \frac{4903}{48} + \frac{3}{4} \pi^2 - 64\zeta(3) + \ln(H/T) + 3 \ln(G/T) \right\}. \] (4.36)

Note that poles in \( \epsilon \) are absent from eqs. (4.34)-(4.36); this is a non-trivial check on the calculation, showing agreement between the counter-term quantities \( c_{\ell,n}^k \) and \( c_{\ell,n}^{\phi} \) as extracted from the known beta functions and anomalous dimension in the literature, and as obtained from the diagrams calculated here. This is equivalent to the check of renormalization group scale independence of the effective potential:

\[ 0 = Q \frac{\partial}{\partial Q} V^{(\ell)} + \sum_{\ell' = 1}^{\ell} \left[ \sum_k \beta_k(\ell') \frac{\partial}{\partial z_k} - \gamma(\ell') \frac{\partial}{\partial \phi} \right] V^{(\ell-\ell')}, \] (4.37)

which follows from \( dV_{\text{eff}}/dQ = 0 \). In fact, eq. (4.37) could have been used to infer all of the terms in \( V^{(3)} \) that contain \( \ln(T) \), just from knowledge of the 2-loop effective potential and the beta functions and scalar anomalous dimension. I have checked this.

Plugging the Standard Model group theory constants of eqs. (3.6)-(3.9) into eq. (4.30) gives

\[ V^{(3)} = g_3^4 T^2 \left\{ -184 \ln^3(T) + 868 \ln^2(T) + (32\zeta(3) - \frac{5642}{3}) \ln(T) + \frac{16633}{9} + \frac{176}{135} \pi^4 \right\} \]
\[ + 32\zeta(3) + \frac{64}{9} \ln^2(2)[\pi^2 - \ln^2(2)] - \frac{512}{3} \text{Li}_4(1/2) \}
\[ + g_s^2 y_t^2 T^2 \left\{ 60 \ln^3(T) - 360 \ln^2(T) + [814 + 12\pi^2 + 240\zeta(3)] \ln(T) - \frac{2393}{3} \right\} \]
\[ - \frac{58}{3} \pi^2 + \frac{124}{15} \pi^4 + \frac{128}{3} \ln^2(2)[\pi^2 - \ln^2(2)] - 216\zeta(3) - 1024\text{Li}_4(1/2) \}
\[ + g_t^4 T^2 \left\{ \frac{99}{4} \ln^3(T) + [81 + 81 \ln(H/T) + 27 \ln(G/T)] \ln^2(T) \right\} \]
\[ - \frac{657}{8} - \frac{39}{4} \pi^2 - 36\zeta(3) - 54 \ln(H/T) - 54 \ln(G/T) \ln(T) \]
\[ + \frac{15767}{16} + \frac{25}{2} \pi^2 - \frac{22}{15} \pi^4 - 8\pi^2 \ln^2(2) + 8 \ln^2(2) - \frac{873}{2} \zeta(3) \]
\[ + 192\text{Li}_4(1/2) + 9 \ln(H/T) + 27 \ln(G/T) \} \right\}. \] (4.38)
or, numerically,

\[
V^{(3)} \approx g_3^4 T^2 \left\{ -184 \ln^3(T) + 868 \ln^2(T) - 1842.2 \ln(T) + 1957.3 \right\} \\
+ g_3^2 y_T^2 T^2 \left\{ 60 \ln^2(T) - 360 \ln^2(T) + 1220.9 \ln(T) - 780.3 \right\} \\
+ y_T^4 T^2 \left\{ 24.75 \ln^3(T) + [81 + 81 \ln(H/T) + 27 \ln(G/T)] \ln^2(T) \\
+ [-971.6 - 54 \ln(H/T) - 54 \ln(G/T)] \ln(T) + 504.5 + 9 \ln(H/T) + 27 \ln(G/T) \right\}. \tag{4.39}
\]

Equation (4.38) or (4.39) may be consistently added to the full 2-loop effective potential as given in ref. [8].

V. THE GOLDSTONE BOSON CATASTROPHE

Because of the doubled Goldstone boson propagators in diagrams (r) and (s) of Figure 3.2, the three-loop effective potential has a logarithmic singularity in the limit \( G = m^2 + \lambda \phi^2 = 0 \), which corresponds to \( \phi \) being at the minimum of the tree-level renormalized potential. In fact, the situation becomes progressively worse at higher loop orders, as these diagrams are part of a family that also includes the one-loop Goldstone contributions and the two-loop diagrams (b) and (c) in Figure 3.1 and more generally, \( \ell \)-loop vacuum diagrams consisting of a ring of \( \ell - 1 \) Goldstone boson propagators (all carrying the same momentum) punctuated by \( \ell - 1 \) top (for \( G^0 \)) or top/bottom (for \( G^\pm \)) one-loop sub-diagrams. These diagrams give rise to contributions to the \( \ell \)-loop effective potential of the form

\[
V^{(\ell)} \sim (N_c y_T^2)^{\ell - 1} T^2 (G/T)^{3 - \ell} [\ln(G/T) + \ldots]. \tag{5.1}
\]

The ellipses in eq. (5.1) includes constant terms. At least for \( \ell = 1, 2, 3 \), higher powers of \( \ln(G/T) \) are absent; from eqs. (4.34)-(4.36) we see that at those loop orders one has specifically:

\[
V^{(1)} \sim \frac{3}{4} G^2 \ln(G/T), \tag{5.2}
\]

\[
V^{(2)} \sim -3 N_c y_T^2 T (\ln T - 1) G \ln(G/T), \tag{5.3}
\]

\[
V^{(3)} \sim 3 [N_c y_T^2 T (\ln T - 1)]^2 \ln(G/T). \tag{5.4}
\]

Equation (5.1) means that at 4-loop order and higher, the singularity in \( V_{\text{eff}} \) as \( G \to 0 \) will be power-law, going like \( 1/G^{\ell - 3} \) multiplied by terms constant and logarithmic in \( G \). Moreover, the first derivative of the effective potential with respect to \( \phi \) diverges logarithmically in the \( G = 0 \) limit even at two-loop order, and the second derivative already at one-loop order.

For a generic choice of renormalization scale, at the minimum of the full radiatively corrected effective potential, \( G \) will be small (compared to \( T \)), but non-zero, and there is no true singularity.

\[\text{† This sort of contribution has been noted before in refs. [12, 53] in the context of non-Goldstone scalars with small field-dependent squared masses.}\]
Nevertheless, the numerical effect can be non-trivial and can be quite important if one happens to choose a renormalization scale where $G$ is very close to 0.

The behavior of the effective potential for small $G$ that is illustrated in eqs. (5.1)-(5.4) seems quite troubling. Unlike similar situations where renormalization group improvement has been employed to study the behavior in the presence of small field-dependent masses in toy models, the fact that $G$ can be small in magnitude (and negative) is not associated with any real or apparent near-instability of the vacuum. Rather, small $G$ is just the expected and inevitable result for any spontaneously broken weakly gauged symmetry, even in a clearly stable vacuum. One might even have naively imagined that a particularly *good* choice of renormalization scale would be one that makes $G$ as small as possible (and positive), given that the Goldstone boson masses should be 0 when computed exactly (and the imaginary parts of the effective potential from negative $G$ do not really correspond to any instability in the theory). But, instead, a choice of renormalization scale that makes $G$ very small will actually provoke unphysically large contributions to the perturbatively computed effective potential and especially to its derivatives, and so apparently should be avoided.

It would be interesting to see in explicit detail how renormalization group improvement (or some other resummation or trick) can mitigate this behavior in the Standard Model case. However, I declare this to exceed the scope of the present paper.

**VI. NUMERICAL IMPACT**

A full numerical study is also beyond the scope of this paper, but a few remarks about the practical impact of the results obtained above are in order. Consider, as a template, the central values of model parameters given in ref. [25]:

\begin{align*}
Q &= M_t = 173.35 \text{ GeV}, \\
\lambda(M_t) &= 0.12710, \\
y_t(M_t) &= 0.93697, \\
g_3(M_t) &= 1.1666, \\
m^2(M_t) &= -(93.36 \text{ GeV})^2, \\
g(M_t) &= 0.6483, \\
g'(M_t) &= 0.3587.
\end{align*}

Now, minimizing the (real part of the) full two-loop effective potential of [8], I obtain the Landau gauge $\overline{\text{MS}}$ VEV:

\[ v(M_t)_{2\text{-loop}} = 247.25 \text{ GeV}. \]  

[At this minimum, one has $G = -(30.76 \text{ GeV})^2$, so that the effective potential computed in perturbation theory has an imaginary part due to $\ln(G)$ factors.] If the three-loop contribution found
above in eq. (4.39) is included, I obtain instead

\[ v(M_t)_{3\text{-loop}} = 246.91 \text{ GeV} \] (6.9)

for the same set of Lagrangian parameters. The majority of this shift comes from the \( g_3^3 \) contribution to \( V^{(3)} \); if only those contributions were included, the VEV would be 246.84 GeV. However, beyond the observation that the effect of \( V^{(3)} \) is to reduce the VEV by about 0.34 GeV when all \( \overline{\text{MS}} \) Lagrangian parameters are held fixed, this way of assessing the impact is of somewhat limited interest, because in the real world the Lagrangian parameters \( m^2 \) and \( \lambda \) are not directly accessible.

Another exercise is to consider the relation between the physical Higgs mass \( M_H \) and \( \lambda \). Writing \( V_{\text{eff}} = V^{(0)} + \Delta V \), the minimum of the potential \( v \equiv \phi_{\text{min}} \) is determined by \( \partial V_{\text{eff}} / \partial \phi = 0 \), which allows us to eliminate \( m^2 \) according to

\[ m^2 = -\lambda v^2 - \frac{1}{\phi} \frac{\partial (\Delta V)}{\partial \phi} \bigg|_{\phi=v} \] (6.10)

The pole squared mass of the Higgs boson is determined from

\[ M_H^2 = m^2 + 3\lambda v^2 + \Pi_{HH}(M_H^2). \] (6.11)

where \( \Pi_{HH}(s) \) is the self-energy function of the external momentum squared \( s = -p^2 \). When evaluated at \( s = 0 \), \( \Pi_{HH} \) coincides with the second derivative of the radiative part of the effective potential. Thus we can write:

\[
M_H^2 = m^2 + 3\lambda v^2 + \left[ \frac{\partial^2 (\Delta V)}{\partial \phi^2} \bigg|_{\phi=v} + \left[ \Pi_{HH}(M_H^2) - \Pi_{HH}(0) \right] \right] \] (6.12)

\[ = 2\lambda v^2 + \left[ \left[ -\frac{1}{\phi^2} \frac{\partial (\Delta V)}{\partial \phi} + \frac{\partial^2 (\Delta V)}{\partial \phi^2} \bigg|_{\phi=v} + \left[ \Pi_{HH}(M_H^2) - \Pi_{HH}(0) \right] \right] \right. \] (6.13)

Now if we consider \( M_H^2 \) and \( v \) as fixed inputs, and treat \( \Pi_{HH}(M_H^2) - \Pi_{HH}(0) \) as small, then we can estimate the change in \( \lambda \) coming from inclusion of a new contribution to the effective potential \( \delta V \) (e.g. 3-loop effects) as

\[ \Delta \lambda \approx -\frac{1}{2v^2} \Delta M_H^2 \approx -\frac{1}{2v^2} \left[ \left[ -\frac{1}{\phi^2} \frac{\partial (\Delta V)}{\partial \phi} + \frac{\partial^2 (\Delta V)}{\partial \phi^2} \bigg|_{\phi=v} + \left[ \Pi_{HH}(M_H^2) - \Pi_{HH}(0) \right] \right] \right] \] (6.14)

The neglect of \( \Pi_{HH}(M_H^2) - \Pi_{HH}(0) \) here is not entirely justified, even for diagrams that involve the top mass as the only other scale, because the expansion parameter \( M_H^2 / M_t^2 \approx 0.53 \) is not so small. For some of the diagrams contributing to \( \Pi_{HH}(s) \), the expansion in \( s/M_t^2 \) tends to have powers of the expansion variable with numerical coefficients that are smaller than 1, and the expansion in \( M_H^2 / M_t^2 \) converges fairly quickly. However, terms of first order in \( M_H^2 / M_t^2 \) can be quite significant. Furthermore, diagrams contributing to \( \Pi_{HH}(M_H^2) \) in which the external momentum can be routed through the diagram in such a way as to miss all top-quark propagators will not be approximated well by \( \Pi_{HH}(0) \) at all.
In particular, this is true of some of the diagrams involving the Goldstone bosons, notably the ones obtained from the vacuum diagrams described in the previous section by attaching two external $H^0$ legs. Those contributions are not just wrong, but potentially very large. The naive estimates from eq. (6.14) and eqs. (5.2)-(5.4) for the most singular contribution as $G \to 0$ from each loop order $\ell = 1, 2, 3$ are:

$$\Delta M_H^2 \bigg|_{1\text{-loop}} \sim \frac{6 \lambda^2 \phi^2}{16 \pi^2} \ln(G/T),$$

$$\Delta M_H^2 \bigg|_{2\text{-loop}} \sim -\frac{12 \lambda^2 \phi^2}{(16 \pi^2)^2} \left[ \frac{N_c y_t^2 T (\ln T - 1)}{G} \right],$$

$$\Delta M_H^2 \bigg|_{3\text{-loop}} \sim -\frac{12 \lambda^2 \phi^2}{(16 \pi^2)^3} \left[ \frac{N_c y_t^2 T (\ln T - 1)}{G} \right]^2,$$

and for higher loop orders, using eq. (6.11):

$$\Delta M_H^2 \bigg|_{\ell\text{-loop}} \sim \frac{\lambda^2 \phi^2}{(16 \pi^2)\ell} \left[ \frac{N_c y_t^2 T}{G} \right]^{\ell-1},$$

where the multiplicative numerical factors and logarithms are unknown. These apparent singularities as $G \to 0$ are unphysical nonsense, and they cannot appear in the correct expression for $M_H^2$.

The resolution is that they are canceled by contributions to $\Pi_{HH}(M_H^2 - \Pi_{HH}(0))$, as one can check explicitly at two-loop order.

Since we do not yet have $\Pi_{HH}(s)$ at 3-loop order, we should certainly not attempt to estimate $\Delta \lambda$ (even roughly) using the part of $V^{(3)}$ involving $y_t^4$, because it includes the offensive $\ln(G/T)$ [and $\ln(H/T)$] factors. However, we can still make estimates of the contributions proportional to $g_3^4$ and $g_3^2 y_t^2$, since at three-loop order these are not singular for $G \to 0$. These should be taken only as estimates because, as mentioned above, corrections from $[\Pi_{HH}(M_H^2) - \Pi_{HH}(0)]$ that go like $M_H^2/M_t^2$ can be significant, even when Goldstone boson shenanigans are absent. With this caveat, using eq. (6.14), one obtains with the model parameters listed above:

$$\Delta \lambda \bigg|_{3\text{-loop } g_3^4 \text{ terms}} = -0.000014$$

$$\Delta \lambda \bigg|_{3\text{-loop } g_3^2 y_t^2 \text{ terms}} = -0.000153$$

for a total of $\Delta \lambda = -0.000167$. This can be compared to the theoretical error estimate used in ref. [25] of $\pm 0.00030$, and the parametric error from the uncertainty on the Higgs mass of $0.000206(\Delta M_H/(100 \text{ MeV})).$

It might seem somewhat surprising that the estimated shift in $\lambda$ from the $g_3^4$ contribution to $V^{(3)}$ is so much smaller than the $g_3^2 y_t^2$ effect, given that $g_3 > y_t$ and the numerical coefficients are larger in the $g_3^4$ terms than in the $g_3^2 y_t^2$ terms. This is due to an accidental cancellation. To see how this works, consider a generic contribution to $V_{\text{eff}}$ of the form:

$$\delta V = T^2 \left[ a_0 + a_1 \ln(T) + a_2 \ln^2(T) + a_3 \ln^3(T) \right].$$

(6.21)
From eq. (6.14), one obtains the estimate for the corresponding shift in $M^2_H$:

$$
\Delta M^2_H = 2y^2_t T \left[ (2a_0 + 3a_1 + 2a_2) + (2a_1 + 6a_2 + 6a_3) \ln(T) \right.
+ (2a_2 + 9a_3) \ln^2(T) \left. + 2a_3 \ln^3(T) \right] \tag{6.22}
$$

Having chosen $Q = M_t$, the logarithms are small, $\ln(T) = -0.11315$, and so the largest contribution might, naively, be expected to come from the term that does not have $\ln(T)$ in it, which is proportional to $2a_0 + 3a_1 + 2a_2$. However, for the one-loop contribution,

$$(a_0, a_1, a_2, a_3)_{1\text{-loop}} = \frac{1}{16\pi^2} (9/2, -3, 0, 0), \tag{6.23}$$

so that $2a_0 + 3a_1 + 2a_2$ happens to vanish. At two loops, for the leading order in QCD:

$$(a_0, a_1, a_2, a_3)_{2\text{-loop, } g_3^2} = \frac{g_3^2}{(16\pi^2)^2} (72, -64, 24, 0), \tag{6.24}$$

and again $2a_0 + 3a_1 + 2a_2$ happens to vanish. At three loops, for the leading order in QCD,

$$(a_0, a_1, a_2, a_3)_{3\text{-loop, } g_3^4} = \frac{g_3^4}{(16\pi^2)^3} (1957.3, -1842.2, 868, -184). \tag{6.25}$$

Here the cancellation is not quite complete, but still

$$(2a_0 + 3a_1 + 2a_2)_{3\text{-loop, } g_3^4} = \frac{g_3^4}{(16\pi^2)^3} (124.1), \tag{6.26}$$

which is well over an order of magnitude smaller than either $a_0$ or $a_1$ individually. Furthermore, the $\ln(T)$ term has the opposite sign, and cancels about 40% of this.

In contrast, for the three-loop $g_3^2 y_t^2$ contribution, the individual coefficients are smaller,

$$(a_0, a_1, a_2, a_3)_{3\text{-loop, } g_3^2 y_t^2} = \frac{g_3^2 y_t^2}{(16\pi^2)^3} (-780.3, 1220.9, -360, 60), \tag{6.27}$$

but there is no efficient accidental cancellation in the term independent of $\ln(T)$:

$$(2a_0 + 3a_1 + 2a_2)_{3\text{-loop, } g_3^2 y_t^2} = \frac{g_3^2 y_t^2}{(16\pi^2)^3} (1382.2). \tag{6.28}$$

The preceding discussion points to an amusing fact. Suppose we took the “new” contribution to the effective potential $\delta V$ to consist of only the three-loop $g_3^4$ and $g_3^2 y_t^2$ contributions that do not include $\ln(T)$, on the grounds that the terms that do have $\ln(T)$ were all in principle known before this paper from the 2-loop effective potential and renormalization group invariance, by virtue of
In other words, consider as the “new” contribution:

$$\delta V = \frac{1}{(16\pi^2)^3} T^2 \left[ 1957.3 g_3^4 - 780.3 g_3^2 y_t^2 \right].$$  \hspace{1cm} (6.29)

From that point of view, we would find, instead of eqs. (6.19) and (6.20) above:

$$\Delta \lambda \bigg|_{3\text{-loop } g_3^4 \text{ terms}} = -0.000710$$ \hspace{1cm} (6.30)

$$\Delta \lambda \bigg|_{3\text{-loop } g_3^2 y_t^2 \text{ terms}} = 0.000182$$ \hspace{1cm} (6.31)

for a total of $$\Delta \lambda = -0.000527$$. The difference between this and the value $$\Delta \lambda = -0.000167$$ obtained above is due to the subset of $$V^{(3)}$$ terms dependent on $$\ln(T)$$. Therefore, a well-meaning attempt to include 3-loop effects by using renormalization group invariance to obtain the $$\ln(T)$$ terms in $$V^{(3)}$$ would have produced a spuriously large estimate for the shift in $$\lambda$$, because it does not capture the accidental cancellations present in the more complete calculation. In any case, the shift in $$\lambda$$ should really be calculated using the full $$M_H^2$$ pole squared mass following from the three-loop $$\Pi_{HH}(s)$$. The effective potential found in this paper will allow a partial check of such a calculation through comparison with the three loop $$\Pi_{HH}(0) = \partial^2(\Delta V)/\partial \phi^2|_{\phi=v}$$.

VII. OUTLOOK

The main new result of this paper is eq. (4.36), [or eq. (4.38) or (4.39)], which contains the three-loop contributions to the effective potential in the Standard Model proportional to $$m_t^2$$ and to $$g_3^4$$, $$g_3^2 y_t^2$$, or $$y_t^4$$. In principle, this allows an improved determination of the relation between the MS Lagrangian parameters and the VEV, although in practice one most deal with the fact that $$m_t^2$$ is not directly accessible. The estimates of the numerical impact of the result, described in the previous section, seem to suggest that the effects are not large compared to the present parametric and other theoretical uncertainties, although there is some accidental cancellation at work. While this is not unexpected, it is always a worthwhile goal to, if possible, reduce all theoretical errors far below the level where experimental errors can compete with them, so that all uncertainties can be reliably blamed on experimentalists. Hopefully, the results above are one step in this direction.

Appendix: Individual diagram contributions

The individual contributions to eq. (3.13) from the diagrams in Figure 3.2 are:

$$V^{(3,a)} = g_{3B} N c C_F (C_F - C_G/2) \left\{ \frac{(7-2d)(d-5)(d-2)^3}{(d-4)^2(d-3)^2} J + \frac{2000 - 3656 d + 2643 d^2 - 939 d^3 + 163 d^4 - 11 d^5}{2(d-4)(d-3)(2d-7)} \xi + \frac{(d-4)(d-3)(d-1)}{2(2d-7)} \xi^2 K + \frac{(d-6)(d-3)(d^2 - 7d + 8)}{2(d-4)^2} L \right\},$$ \hspace{1cm} (A.1)
\[ V^{(3.b)} = g_{3B}N_c C_F \left\{ \frac{(d-2)^2 (d-1)^2}{2(3-d)} J + \frac{(d-1)^2 (3d-8)}{2(2d-7)} + \frac{(1-d)(3d^2-17d+24)}{2d-7} \xi \right\} + \frac{(1-d)(d^2-7d+12)}{2(2d-7)} \xi^2 K \right\}, \]  
(A.2)

\[ V^{(3.c)} = g_{3B}N_c C_F C_G \left\{ \frac{(d-2)^3}{(d-3)^2} J + \frac{7d^4 - 67d^3 + 237d^2 - 373d + 226}{2(3-d)(2d-7)} \right\} \xi + \frac{3(d-1)(d-3)^2}{2d-7} \xi + \frac{(d-4)(d-3)(d-1)}{2(2d-7)} \xi^2 K \right\}, \]  
(A.3)

\[ V^{(3.d)} = g_{3B}N_c C_F C_G \left[ \frac{d-3}{4(2d-7)} \right] \left\{ 7d^2 - 19d + 14 + 2(1-d)(3d-10) \xi \right\} \]  
(A.4)

\[ V^{(3.e)} = g_{3B}N_c C_F C_G \left[ \frac{d-3}{2(2d-7)} K \right], \]  
(A.5)

\[ V^{(3.f)} = g_{3B}N_c C_F T_F \left[ \frac{2(5-d)(d-2)^3}{(d-6)(d-4)(d-3)} J + \frac{d^3 - 7d^2 + 6d + 16}{(d-4)(6-d)} L \right], \]  
(A.6)

\[ V^{(3.g)} = g_{3B}N_c C_F T_F \left( N_q - 1 \right) \frac{4(3-d)(d-2)}{2d-7} K, \]  
(A.7)

\[ V^{(3.h)} = g_{3B}y_{3B}^2 N_c C_F \left\{ \frac{(2-d)(d^4 - 8d^3 + 17d^2 + 8d - 44)}{(d-4)^2 (d-3)^2} J + \frac{(3-d)(d^2 + 2d - 16)}{2d-7} L \right\} \]  
(A.8)

\[ V^{(3.i)} = g_{3B}y_{3B}^2 N_c C_F \left\{ \frac{5d^4 - 60d^3 + 238d^2 - 618d + 520}{(4-d)(d-3)(2d-7)} + \frac{(d-6)(d-3)}{2d-7} \xi K \right\}, \]  
(A.9)

\[ V^{(3.j)} = g_{3B}y_{3B}^2 N_c C_F \left\{ \frac{(d-2)(d-1)}{d-3} J + \frac{(1-d)(3d-8)}{2d-7} + \frac{(6-d)(d-3)}{2d-7} \xi K \right\}, \]  
(A.10)

\[ V^{(3.k)} = g_{3B}y_{3B}^2 N_c C_F \left\{ \frac{(2-d)(d-1)(3d-8)}{2d-4 (d-3)} + \frac{2(d-3)(d-2)}{d-4} \xi P \right\} \]  
(A.11)

\[ V^{(3.l)} = g_{3B}y_{3B}^2 N_c C_F \left\{ \frac{2(d-3)(d-2)}{d-4} \xi P, \right\} \]  
(A.12)

\[ V^{(3.m)} = y_{tB}^4 N_c \left\{ \frac{(d-2)^2 (d^2 - 11d + 26)}{4(d-4)^2 (d-3)^2} J + \frac{(3d-8)(d^2 - 4d + 2)}{2d-4 (d-3)(2d-7)} K \right\} \]  
(A.13)

\[ V^{(3.n)} = y_{tB}^4 N_c \left\{ \frac{1}{2(3-d)} J + \frac{d^2 - 12d + 26}{2d-2 (2d-7)} K \right\}, \]  
(A.14)

\[ V^{(3.o)} = y_{tB}^4 N_c \left\{ \frac{4d^2 - 21d + 28}{2d-4 (d-3)} P + \frac{2d-5}{2d-3} Q \right\}; \]  
(A.15)

\[ V^{(3.p)} = y_{tB}^4 N_c \left\{ \frac{3(2-d)}{8} R, \right\} \]  
(A.16)

\[ V^{(3.q)} = y_{tB}^4 N_c \left\{ \frac{d^2 - 10d + 20}{d(d-2)} K + \frac{2d}{d} Q \right\}, \]  
(A.17)
$$V^{(3,r)} = y_{h}^4 N_c^2 \left\{ \frac{2(d-2)}{(d-4)(d-3)} + \frac{d-2}{2} \left( \frac{x_H}{x} \right)^{d/2-2} + \frac{d-2}{2} \left( \frac{x_G}{x} \right)^{d/2-2} \right\} J$$

$$+ \frac{d-2}{d-4} L + 4x \frac{d}{dx_H} \left[ xI_{xxxxH} - AI_{xxxH} \right] \right\}, \quad (A.18)$$

$$V^{(3,s)} = y_{h}^4 N_c^2 \left\{ \frac{d-2}{4} \left( \frac{x_G}{x} \right)^{d/2-2} J + \frac{18 - 7d}{2(d-2)} K + 3Q + x \frac{d}{dx_G} \left[ \frac{x}{2} I_{xx0xG} - AI_{x0xG} \right] \right\}. \quad (A.19)$$

The sums of these contributions gives eq. (3.13). The cancellation of the dependence on the QCD gauge-fixing parameter $\xi$ provides a useful check.

Acknowledgments: This work was supported in part by the National Science Foundation grant number PHY-1068369. This research was supported in part by the National Science Foundation under Grant No. NSF PHY11-25915.

[1] G. Aad et al. [ATLAS Collaboration], “Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC,” Phys. Lett. B 716, 1 (2012) [1207.7214],

[2] S. Chatrchyan et al. [CMS Collaboration], “Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC,” Phys. Lett. B 716, 30 (2012) [1207.7235].

[3] [ATLAS Collaboration], “Combined measurements of the mass and signal strength of the Higgs-like boson with the ATLAS detector using up to 25 fb$^{-1}$ of proton-proton collision data,” ATLAS-CONF-2013-014, March 6, 2013.

[4] [CMS Collaboration], “Combination of standard model Higgs boson searches and measurements of the properties of the new boson with a mass near 125 GeV” CMS-PAS-HIG-12-045, November 16, 2012.

[5] S. R. Coleman and E. J. Weinberg, “Radiative Corrections as the Origin of Spontaneous Symmetry Breaking,” Phys. Rev. D 7, 1888 (1973).

[6] R. Jackiw, “Functional evaluation of the effective potential,” Phys. Rev. D 9, 1686 (1974).

[7] M. Sher, “Electroweak Higgs Potentials and Vacuum Stability,” Phys. Rept. 179, 273 (1989), and references therein.

[8] C. Ford, I. Jack and D.R.T. Jones, “The Standard model effective potential at two loops,” Nucl. Phys. B 387, 373 (1992) [Erratum-ibid. B 504, 551 (1997)] [hep-ph/0111109]. See also C. Ford and D. R. T. Jones, “The Effective potential and the differential equations method for Feynman integrals,” Phys. Lett. B 274, 409 (1992) [Erratum-ibid. B 285, 399 (1992)].

[9] S.P. Martin, “Two loop effective potential for a general renormalizable theory and softly broken supersymmetry,” Phys. Rev. D 65, 116003 (2002) [hep-ph/0111209].

[10] M. Lindner, M. Sher and H. W. Zaglauer, “Probing Vacuum Stability Bounds at the Fermilab Collider,” Phys. Lett. B 228, 139 (1989).

[11] P. B. Arnold and S. Vokos, “Instability of hot electroweak theory: bounds on m(H) and M(t),” Phys. Rev. D 44, 3620 (1991).

[12] C. Ford, D. R. T. Jones, P. W. Stephenson and M. B. Einhorn, “The Effective potential and the renormalization group,” Nucl. Phys. B 395, 17 (1993) [hep-lat/9210033].

[13] J. A. Casas, J. R. Espinosa and M. Quiros, “Improved Higgs mass stability bound in the standard model and implications for supersymmetry,” Phys. Lett. B 342, 171 (1995) [hep-ph/9409458].

[14] J. R. Espinosa and M. Quiros, “Improved metastability bounds on the standard model Higgs mass,”
[15] J. A. Casas, J. R. Espinosa and M. Quiros, “Standard model stability bounds for new physics within LHC reach,” Phys. Lett. B 382 (1996) 374 [hep-ph/9603227].
[16] G. Isidori, G. Ridolfi and A. Strumia, “On the metastability of the standard model vacuum,” Nucl. Phys. B 609, 387 (2001) [hep-ph/0104016].
[17] J. R. Espinosa, G. F. Giudice and A. Riotto, “Cosmological implications of the Higgs mass measurement,” JCAP 0805, 002 (2008) [0710.2484].
[18] N. Arkani-Hamed, S. Dubovsky, L. Senatore and G. Villadoro, “(No) Eternal Inflation and Precision Higgs Physics,” JHEP 0803, 075 (2008) [0801.2399].
[19] F. Bezrukov and M. Shaposhnikov, “Standard Model Higgs boson mass from inflation: Two loop analysis,” JHEP 0907, 089 (2009) [0904.1537].
[20] J. Ellis, J. R. Espinosa, G. F. Giudice, A. Hoecker and A. Riotto, “The Probable Fate of the Standard Model,” Phys. Lett. B 716, 214 (2012) [1207.0980].
[21] F. Bezrukov, M. Y. Kalmykov, B. A. Kniehl and M. Shaposhnikov, “Higgs Boson Mass and New Physics,” JHEP 1210, 140 (2012) [1205.2893].
[22] G. Degrassi, S. Di Vita, J. Elias-Miro, J. R. Espinosa, G. F. Giudice, G. Isidori and A. Strumia, “Higgs mass and vacuum stability in the Standard Model at NNLO,” JHEP 1208, 098 (2012) [1205.6497].
[23] D. Buttazzo, G. Degrassi, P. P. Giardino, G. F. Giudice, F. Sala, A. Salvio and A. Strumia, “Investigating the near-criticality of the Higgs boson,” [1307.3536].
[24] F. Jegerlehner, M. Y. Kalmykov and B. A. Kniehl, “About the EW contribution to the relation between pole and MS-masses of the top-quark in the Standard Model,” [1307.4226].
[25] A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin, “Three-loop Higgs self-coupling beta-function in the Standard Model with complex Yukawa matrices,” [1310.3806].
[26] C. G. Bollini and J. J. Giambiagi, “Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter,” Nuovo Cim. B 12, 20 (1972). C. G. Bollini and J. J. Giambiagi, “Lowest order divergent graphs in nu-dimensional space,” Phys. Lett. B 40, 566 (1972).
[27] C. G. Bollini and J. J. Giambiagi, “Dimensional Renormalization: The Number of Dimensions as a Regularizing Parameter,” Nuovo Cim. B 12, 20 (1972). C. G. Bollini and J. J. Giambiagi, “Lowest order divergent graphs in nu-dimensional space,” Phys. Lett. B 40, 566 (1972).
[28] J. F. Ashmore, “A Method of Gauge Invariant Regularization,” Lett. Nuovo Cim. 4, 289 (1972).
[29] G. M. Cicuta and E. Montaldi, “Analytic renormalization via continuous space dimension,” Lett. Nuovo Cim. 4, 329 (1972).
[30] G. ’t Hooft and M. J. G. Veltman, “Regularization and Renormalization of Gauge Fields,” Nucl. Phys. B 44, 189 (1972).
[31] G. ’t Hooft, “Dimensional regularization and the renormalization group,” Nucl. Phys. B 61, 455 (1973).
[32] W. A. Bardeen, A. J. Buras, D. W. Duke and T. Muta, “Deep Inelastic Scattering Beyond the Leading Order in Asymptotically Free Gauge Theories,” Phys. Rev. D 18, 3998 (1978).
[33] E. Braaten and J. P. Leveille, “Minimal Subtraction and Momentum Subtraction in QCD at Two Loop Order,” Phys. Rev. D 24, 1369 (1981).
[34] H. Yamagishi, “Coupling Constant Flows and Dynamical Symmetry Breaking,” Phys. Rev. D 23, 1880 (1981). H. Yamagishi, “Renormalization Group Analysis of Supersymmetric Mass Hierarchies,” Nucl. Phys. B 216, 508 (1983).
[36] M. B. Einhorn and D. R. T. Jones, “Scale Fixing by Dimensional Transmutation: Supersymmetric Unified Models and the Renormalization Group,” Nucl. Phys. B 211, 29 (1983).
[37] B. M. Kastening, “Renormalization group improvement of the effective potential in massive phi**4 theory,” Phys. Lett. B 283, 287 (1992).
[38] M. Bando, T. Kugo, N. Maekawa and H. Nakano, “Improving the effective potential,” Phys. Lett. B 301, 83 (1993) [hep-ph/9210228]. “Improving the effective potential: Multimass scale case,” Prog. Theor. Phys. 90, 405 (1993) [hep-ph/9210229].
[39] K. G. Chetyrkin and F. V. Tkachov, “Integration by Parts: The Algorithm to Calculate beta Functions in 4 Loops,” Nucl. Phys. B 192, 159 (1981), F. V. Tkachov, “A Theorem on Analytical Calculability of Four Loop Renormalization Group Functions,” Phys. Lett. B 100, 65 (1981).
[40] D. J. Broadhurst, “Three loop on-shell charge renormalization without integration: Lambda-MS (QED) to four loops,” Z. Phys. C 54, 599 (1992).
[41] L. V. Avdeev, “Recurrence relations for three loop prototypes of bubble diagrams with a mass,” Comput. Phys. Commun. 98, 15 (1996) [hep-ph/9512442].
[42] D. J. Broadhurst, “Massive three - loop Feynman diagrams reducible to SC* primitives of algebras of the sixth root of unity,” Eur. Phys. J. C 8, 311 (1999) [hep-th/9803091].
[43] Y. Schroder and A. Vuorinen, “High-precision epsilon expansions of single-mass-scale four-loop vacuum bubbles,” JHEP 0506, 051 (2005) [hep-ph/0503209]. See also M. Steinhauser, “MATAD: A Program package for the computation of MAssive TADpoles,” Comput. Phys. Commun. 134, 335 (2001) [hep-ph/0009029].
[44] N. I. Usyukina and A. I. Davydychev, “New results for two loop off-shell three point diagrams,” Phys. Lett. B 332, 159 (1994) [hep-ph/9402223]. A. I. Davydychev and J. B. Tausk, “A Magic connection between massive and massless diagrams,” Phys. Rev. D 53, 7381 (1996) [hep-ph/9504431].
[45] M. Y. Kalmykov, “About higher order epsilon-expansion of some massive two- and three-loop master-integrals,” Nucl. Phys. B 718, 276 (2005) [hep-ph/0503070].
[46] M. E. Machacek and M. T. Vaughn, “Two Loop Renormalization Group Equations in a General Quantum Field Theory. 1. Wave Function Renormalization,” Nucl. Phys. B 222, 83 (1983).
[47] M. E. Machacek and M. T. Vaughn, “Two Loop Renormalization Group Equations in a General Quantum Field Theory. 2. Yukawa Couplings,” Nucl. Phys. B 236, 221 (1984).
[48] I. Jack and H. Osborn, “General Background Field Calculations With Fermion Fields,” Nucl. Phys. B 249, 472 (1985).
[49] M. E. Machacek and M. T. Vaughn, “Two Loop Renormalization Group Equations in a General Quantum Field Theory. 3. Scalar Quartic Couplings,” Nucl. Phys. B 249, 70 (1985).
[50] K. G. Chetyrkin and M. F. Zoller, “Three-loop β-functions for top-Yukawa and the Higgs self-interaction in the Standard Model,” JHEP 1206, 033 (2012) [1205.2892].
[51] K. G. Chetyrkin and M. F. Zoller, “β-function for the Higgs self-interaction in the Standard Model at three-loop level,” JHEP 1304, 091 (2013) [1303.2890].
[52] A. V. Bednyakov, A. F. Pikelner and V. N. Velizhanin, “Higgs self-coupling beta-function in the Standard Model at three loops,” Nucl. Phys. B 875, 552 (2013) [1303.4364].
[53] M. B. Einhorn and D. R. T. Jones, “The Effective potential, the renormalisation group and vacuum stability,” JHEP 0704, 051 (2007) [hep-ph/0702295].