Degeneracy of spectrum of XXX model in the three-magnon sector at the centre of the Brillouin zone

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Abstract. An exact diagonalization of the XXX one-dimensional Heisenberg model of a magnet with the single-node 1/2 reveals a specific degeneracy in the sector of $r = 3$ reversed spins at the centre $k = 0$ of the Brillouin zone. The degeneracy manifests itself by the second power (i.e. the double degeneracy) of a factor of the characteristic polynomial of the related secular matrix. The factor, treated as a polynomial in the variable $x$ being an eigenenergy, is indecomposable over the field $\mathbb{Q}$ of rationals, and its roots correspond to appropriate rigged configurations of strings. The degeneracy is associated with the fact that quasimomenta of the two-string and the one-string are conserved independently each of the other, which is a particular consequence of integrability of the system.

1. Introduction
We report in this note on a specific degeneracy in the energy spectrum of the XXX one-dimensional Heisenberg model of a 1/2-spin magnet with $N$ nodes [1]. This degeneracy arises in the procedure of an immediate diagonalization of the standard nearest neighbour Heisenberg Hamiltonian within the sector of $r = 3$ spin deviations from the ferromagnetic saturation state. Translational degrees of freedom are accounted within the basis of wavelets [2].

The effective secular matrix for an appropriate block of the Hamiltonian for the chain of $N$ nodes involves exact quantum numbers ($r = 3, k = 0$), with $k$ being the total quasimomentum of each eigenstate within the block. Here we are interested in the centre $k = 0$ of the Brillouin zone, i.e. in standing eigensolutions. The size of the secular matrix for this case is such that one can easily calculate the exact characteristic polynomial $w(x)$ for this matrix, with $x$ being the eigenenergy. For $N$ not too large, this polynomial can be presented in a factorised form, with factors being polynomials, which are indecomposable over the field $\mathbb{Q}$ of rationals. These calculations reveal that the polynomial $w$ is of the form

$$w(x) = w_0(x)\varphi(x)^2,$$

where $\varphi(x)$ is a polynomial, indecomposable within the field $\mathbb{Q}$, the second power points out the double degeneracy of each root of $\varphi$, and $w_0$ encompasses the remaining factors of $w$. We aim to explain here the origin of this degeneracy in terms of rigged string configurations which are
used for a complete classification of exact Bethe Ansatz eigenstates [3]. It is worth to point out that this degeneracy cannot be attributed neither to rotational symmetry of the XXX model, nor to translational symmetry of the linear chain, since the corresponding quantum numbers: the total angular momentum \( S \) and its \( z \)-projection \( M = N/2 - r \) (= \( S \), since it corresponds to the highest weight states) as well as the quasimomentum \( k = 0 \) are already accounted within the considered block of the secular matrix.

2. Detail description of degeneracy

Let \( w_{Nrk}(x) \) be the characteristic polynomial of the block of the Heisenberg Hamiltonian, corresponding to the space of all states with specified number \( N \) of magnetic nodes, number \( r \) of spin deviations, and quasimomentum \( k \).

Here we are interested in \( w_{N30}(x) \). Clearly, \( w_{N}(x) \equiv w_{N30h.w.}(x) = w_{N30}(x)/w_{N20}(x) \) is the polynomial corresponding to all highest weight vectors in this space, that is the vectors with the total spin \( S = N/2 - 3 \). We point out that the polynomial \( w_{N}(x) \) has the form (1), and interpret the related degeneracy. Explicit values of the polynomial \( w_{N}(x) \) for a few smallest values of \( N \) are listed in Table 1 (Note that \( N = 6 \) is the smallest value of the number of nodes, where the corresponding eigenstates are "not beyond equator" for \( r = 3 \)).

### Table 1. The characteristic polynomials \( w_{N}(x) \) (cf. Eq. 2) for 6 \( \leq \) \( N \) \( \leq \) 10 and their roots

| \( N \) | \( w_{N}(x) \) | Energy |
|---|---|---|
| 6 | \( x + 6 \) | -6 |
| 7 | \( (x + 5)^2 \) | -5 |
| 8 | \( (x + 6)(x + 4)^2 \) | -6, -4 |
| 9 | \( (x^3 + 17x^2 + 90x + 147)^2 \) | -3.23, -5.52, -8.25 |
| 10 | \( (x + 6)(x^3 + 15x^2 + 69x + 96)^2 \) | -6, -2.64, -4.83, -7.53 |

We thus observe the double degeneracy for a single root for \( N = 7 \) \( (\varphi = x + 5) \) and \( N = 8 \) \( (\varphi = x + 4) \), then for three roots for \( N = 9 \) \( (\varphi = x^3 + 17x^2 + 90x + 147) \) and \( N = 10 \) \( (\varphi = x^3 + 15x^2 + 69x + 96) \), and expect a similar behaviour for greater \( N \) (it is also supported by combinatoric counts of dimensions for appropriate rigged strings).

3. Characterization of degenerated energy levels in Bethe Ansatz string configuration scheme

It is well known that exact eigenstates of BA are classified by rigged string configurations [1, 4]. We follow the notation of paper [3] concerning the assignment of a detail rigging \( L \) to string configuration \( \nu = \{3\} \), \( \{21\} \) and \( \{1^3\} \) – the possible partitions \( \nu + r \) for \( r = 3 \). A typical rigged string configuration for our case has the form

\[
\nu L = \begin{array}{c}
L_2 \\
L_1
\end{array} \begin{array}{c}
N - 6 \\
N - 4
\end{array}
\]

(3)

We recall here shortly that the exact eigenstate \( \nu L \) given by Eq. (3) corresponds to a 2-string and a 1-string (the first and the second row of the Young diagram \( \nu = \{21\} \)), with the rigging
Figure 1. Selected examples for revealing degeneracies in the centre of the Brillouin zone ($k = 0$).
\( \mathcal{L} = \{ L_2, L_1 \} \) subject to quantization rules 0 \( \leq L_2 \leq N - 6 \) and 0 \( \leq L_1 \leq N - 4 \). We keep the convention in which the rigging of an \( l \)-string is inserted to the corresponding row of the Young diagram \( \nu \), whereas the range of rigging \( P_l \) is put to the right of this row (here, \( P_2 = N - 6 \) and \( P_1 = N - 4 \)).

We follow here kinematical interpretation of riggings, given in [3, 5]. In short, intergrability of BA implies that the quasimomentum can be attributed not only to the total eigenstate \( \nu \mathcal{L} \), but also to each string. Namely, a string \( lv \) of the length \( l \) carries the quasimomentum

\[
k_{lv} = (L_{lv} + Q_l) \mod N
\]

where \( v \) is the label of an \( l \)-string (redundant in the case of (3)), \( L_{lv} \) is the rigging of this string, and \( Q_l \) is the number of boxes in the first \( l \) columns of the Young diagram \( \nu \) (\( Q_l \) accounts for some kinematical restrictions for motion of strings – cf. [3]), and the result is taken modulo \( N \), such that \( |k_{lv}| \leq N/2 \). Therefore, each exact BA eigenstate of the string configuration \( \nu = \{ 21 \} \), denoted by its rigging \( \mathcal{L} = \{ L_2, L_1 \} \), can be also uniquely presented in terms of the corresponding quasimomenta \( \{ k_2, k_1 \} \), which satisfy \( k_1 + k_2 = 0 \mod N \). In other words, quasimomenta of both strings become exact quantum numbers separately, and they compensate mutually. The same mechanism works also for the string configuration \( \nu = \{ 1^3 \} \), where each of three \( 1 \)-strings corresponds to a conserved quasimomentum. We denote the rigging by \( \mathcal{L} = \{ L, L', L'' \} \) and the related quasimomenta by \( \{ k, k', k'' \} \) subjected to the restriction \( k + k' + k'' = 0 \mod N \). These rules exactly satisfy the requirement of completeness of BA solutions within our secular matrix. Detail presentation of such solutions for \( N = 7, 8, 9 \) and 10 are given in figure 1.

We observe that our degeneracy arises at \( N = 7 \). For \( N = 7 \) and 8 one has a single doubly degenerated pair, which corresponds to the polynomial \( \varphi = x + 5 \) and \( x + 4 \), respectively. The related quasimomenta \( \{ k_2, k_1 \} \) are either \( \{ 3, -3 \} \) or \( \{-3, 3 \} \) in both cases. For \( N = 9 \) and 10 three such pairs arise, two of them are related to the string configuration \( \nu = \{ 21 \} \) and the third to \( \nu = \{ 1^3 \} \). Now quasimomenta of degenerated pairs are \( \{ k_2, k_1 \} = \{ 3, -3 \}, \{ 3, -3 \} \) and \( \{-4, 4\}, \{ 4, -4 \} \) for \( \nu = \{ 21 \} \), and \( \{ k, k', k'' \} = \{ 3, 3, 3 \}, \{-3, -3, 3 \} \) for \( N = 9 \), and \( \{ 3, 3, 4 \}, \{-4, -3, -3 \} \) for \( N = 10 \). We conclude that the double degeneracy is associated with the *sings* of quasimomenta of individual strings, with the vanishing total quasimomentum.

4. Discussion

Degeneracy introduced in this paper is a symptom of the string structure of Bethe Ansatz solutions, which is present in exact solutions not only for asymptotic case but also for finite \( N \). These states are connected with kinematic independence for different string lengths. The states are distinguishable and energetically degenerated.

One can observe that degenerated solutions in sector \( r = 3 \), \( k = 0 \) begin to appear for \( N = 7 \) – this takes place for configurations of shape \( \{ 21 \} \) and additionally starting with \( N = 9 \) there are degenerated solutions of shape \( \{ 1^3 \} \). For odd number of \( N \) we can collect degenerated solutions in pairs, while for even ones an extra non-degenerated solution of shape \( \{ 21 \} \) exists.

References

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