An application of Mirror extensions

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Abstract

In this paper we apply our previous results of mirror extensions to obtain realizations of three modular invariants constructed by A. N. Schellekens by holomorphic conformal nets with central charge equal to 24.
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1 Introduction

Partition functions of chiral rational conformal field theories (RCFT) are modular invariant (cf. [48]). However there are examples of “spurious” modular invariants which do not correspond to any RCFT (cf.[6], [39] and [15]). It is therefore an interesting question to decide which modular invariants can be realized in RCFT. For many interesting modular invariants this question was raised for an example in [38] and more recently in [11]. For results on related questions, see [5], [6], [36], [23],[24] and [26] for a partial list.

In this paper we examine the holomorphic modular invariants with central charge 24 constructed by A. N. Schellekens in [38]. Besides modular invariance, A. N. Schellekens showed that his modular invariants passed an impressive list of checks from tracial identities which strongly suggested that his modular invariants can be realized in chiral RCFT. Some of Schellekens’s modular invariants were constructed using level-rank duality. In [42] we proved a general theorem on mirror extensions (cf. Th. 2.25) which included modular invariants from level-rank duality (cf. §2.6). It is therefore an interesting question to see if mirror extensions can provide chiral RCFT realization of some of Schellekens’s modular invariants. Our main result in this paper is to show that three of Schellekens’s modular invariants can be realized by holomorphic conformal nets (cf. Th. 3.4): these nets are constructed by simple current extensions (cf. §2.4) of mirror extensions. Our results strongly suggest that there should be Vertex Operator Algebras which realize these modular invariants. We expect our methods to apply to other modular invariants in the literature, especially when level-rank duality plays a role.

This paper is organized as follows: after a preliminary section on nets, mirror extensions and simple current extensions, we examine three of Schellekens’s modular invariants in [38], and obtain realization of these invariants as simple current extensions of three mirror extensions. We end with two conjectures about holomorphic conformal nets with central charge 24 which are motivated by [14] and [38], and we hope that these conjectures will stimulate further research.

2 Preliminaries

2.1 Preliminaries on sectors

Given an infinite factor $M$, the sectors of $M$ are given by
\[ \text{Sect}(M) = \text{End}(M)/\text{Inn}(M), \]

namely \( \text{Sect}(M) \) is the quotient of the semigroup of the endomorphisms of $M$ modulo the equivalence relation: \( \rho, \rho' \in \text{End}(M), \rho \sim \rho' \) iff there is a unitary $u \in M$ such that \( \rho'(x) = u\rho(x)u^* \) for all $x \in M$.

\( \text{Sect}(M) \) is a *-semiring (there are an addition, a product and an involution $\rho \to \bar{\rho}$) equivalent to the Connes correspondences (bimodules) on $M$ up to unitary equiva-
lence. If $\rho$ is an element of $\text{End}(M)$ we shall denote by $[\rho]$ its class in $\text{Sect}(M)$. We define $\text{Hom}(\rho, \rho')$ between the objects $\rho, \rho' \in \text{End}(M)$ by

$$\text{Hom}(\rho, \rho') \equiv \{a \in M : a\rho(x) = \rho'(x)a \ \forall x \in M\}.$$  

We use $\langle \lambda, \mu \rangle$ to denote the dimension of $\text{Hom}(\lambda, \mu)$; it can be $\infty$, but it is finite if $\lambda, \mu$ have finite index. See [21] for the definition of index for type $II_1$ case which initiated the subject and [32] for the definition of index in general. Also see §2.3 of [27] for expositions. $\langle \lambda, \mu \rangle$ depends only on $[\lambda]$ and $[\mu]$. Moreover we have if $\nu$ has finite index, then $\langle \nu\lambda, \mu \rangle = \langle \lambda, \nu\mu \rangle$, $\langle \nu\lambda, \mu \rangle = \langle \lambda, \nu\mu \rangle$ which follows from Frobenius duality. $\mu$ is a subsector of $\lambda$ if there is an isometry $v \in M$ such that $\mu(x) = v^*\lambda(x)v, \forall x \in M$. We will also use the following notation: if $\mu$ is a subsector of $\lambda$, we will write as $\mu \prec \lambda$ or $\lambda \succ \mu$. A sector is said to be irreducible if it has only one subsector.

### 2.2 Local nets

By an interval of the circle we mean an open connected non-empty subset $I$ of $S^1$ such that the interior of its complement $I'$ is not empty. We denote by $\mathcal{I}$ the family of all intervals of $S^1$.

A net $\mathcal{A}$ of von Neumann algebras on $S^1$ is a map

$$I \in \mathcal{I} \to \mathcal{A}(I) \subset B(\mathcal{H})$$

from $\mathcal{I}$ to von Neumann algebras on a fixed separable Hilbert space $\mathcal{H}$ that satisfies:

**A. Isotony.** If $I_1 \subset I_2$ belong to $\mathcal{I}$, then

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2).$$

If $E \subset S^1$ is any region, we shall put $\mathcal{A}(E) \equiv \bigvee_{E \supset I} \mathcal{A}(I)$ with $\mathcal{A}(E) = \mathbb{C}$ if $E$ has empty interior (the symbol $\bigvee$ denotes the von Neumann algebra generated).

The net $\mathcal{A}$ is called local if it satisfies:

**B. Locality.** If $I_1, I_2 \in \mathcal{I}$ and $I_1 \cap I_2 = \emptyset$ then

$$[\mathcal{A}(I_1), \mathcal{A}(I_2)] = \{0\},$$

where brackets denote the commutator.

The net $\mathcal{A}$ is called M"obius covariant if in addition satisfies the following properties C,D,E,F:

**C. M"obius covariance.** There exists a non-trivial strongly continuous unitary representation $U$ of the M"obius group $\text{M"ob}$ (isomorphic to $\text{PSU}(1,1)$) on $\mathcal{H}$ such that

$$U(g)\mathcal{A}(I)U(g^*) = \mathcal{A}(gI), \quad g \in \text{M"ob}, \ I \in \mathcal{I}.$$
D. **Positivity of the energy.** The generator of the one-parameter rotation subgroup of $U$ (conformal Hamiltonian), denoted by $L_0$ in the following, is positive.

E. **Existence of the vacuum.** There exists a unit $U$-invariant vector $\Omega \in \mathcal{H}$ (vacuum vector), and $\Omega$ is cyclic for the von Neumann algebra $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I)$.

By the Reeh-Schlieder theorem $\Omega$ is cyclic and separating for every fixed $\mathcal{A}(I)$. The modular objects associated with $(\mathcal{A}(I), \Omega)$ have a geometric meaning

$$\Delta^i_I = U(\Lambda_I(2\pi t)), \quad J_I = U(r_I).$$

Here $\Lambda_I$ is a canonical one-parameter subgroup of $\text{Möb}$ and $U(r_I)$ is an antiunitary acting geometrically on $\mathcal{A}$ as a reflection $r_I$ on $S^1$.

This implies **Haag duality**:

$$\mathcal{A}(I) = \mathcal{A}(I'), \quad I \in \mathcal{I},$$

where $I'$ is the interior of $S^1 \setminus I$.

F. **Irreducibility.** $\bigvee_{I \in \mathcal{I}} \mathcal{A}(I) = B(\mathcal{H})$. Indeed $\mathcal{A}$ is irreducible iff $\Omega$ is the unique $U$-invariant vector (up to scalar multiples). Also $\mathcal{A}$ is irreducible iff the local von Neumann algebras $\mathcal{A}(I)$ are factors. In this case they are either $\mathbb{C}$ or $\text{III}_1$-factors with separable predual in Connes classification of type III factors.

By a **conformal net** (or diffeomorphism covariant net) $\mathcal{A}$ we shall mean a Möbius covariant net such that the following holds:

G. **Conformal covariance.** There exists a projective unitary representation $U$ of $\text{Diff}(S^1)$ on $\mathcal{H}$ extending the unitary representation of $\text{Möb}$ such that for all $I \in \mathcal{I}$ we have

$$U(\varphi) \mathcal{A}(I) U(\varphi)^* = \mathcal{A}(\varphi I), \quad \varphi \in \text{Diff}(S^1),$$

$$U(\varphi) x U(\varphi)^* = x, \quad x \in \mathcal{A}(I), \quad \varphi \in \text{Diff}(I'),$$

where $\text{Diff}(S^1)$ denotes the group of smooth, positively oriented diffeomorphism of $S^1$ and $\text{Diff}(I)$ the subgroup of diffeomorphisms $g$ such that $\varphi(z) = z$ for all $z \in I'$. Note that by Haag duality we have $U(\varphi) \in \mathcal{A}(I), \forall \varphi \in \text{Diff}(I)$. Hence the following definition makes sense:

**Definition 2.1.** If $\mathcal{A}$ is a conformal net, the **Virasoro subnet of $\mathcal{A}$**, denoted by $\text{Vir}_\mathcal{A}$, is defined as follows: for each interval $I \in \mathcal{I}$, $\text{Vir}_\mathcal{A}(I)$ is the von Neumann algebra generated by $U(\varphi) \in \mathcal{A}(I), \forall \varphi \in \text{Diff}(I)$.

A (DHR) representation $\pi$ of $\mathcal{A}$ on a Hilbert space $\mathcal{H}$ is a map $I \in \mathcal{I} \mapsto \pi_I$ that associates to each $I$ a normal representation of $\mathcal{A}(I)$ on $B(\mathcal{H})$ such that

$$\pi_I | \mathcal{A}(I) = \pi_I, \quad I \subset \tilde{I}, \quad I, \tilde{I} \in \mathcal{I}.$$
\[\pi \text{ is said to be Möbius (resp. diffeomorphism) covariant if there is a projective unitary representation } U_\pi \text{ of } \text{ Möbius (resp. } \text{Diff}(S^1)) \text{ on } \mathcal{H} \text{ such that}
\]
\[\pi g I(U(g)xU(g)^*) = U_\pi(g)\pi I(x)U_\pi(g)^*
\]
for all \( I \in \mathcal{I}, \ x \in \mathcal{A}(I) \) and \( g \in \text{ Möbius (resp. } g \in \text{Diff}(S^1)) \).

By definition the irreducible conformal net is in fact an irreducible representation of itself and we will call this representation the \textit{vacuum representation}.

Let \( G \) be a simply connected compact Lie group. By Th. 3.2 of [13], the vacuum positive energy representation of the loop group \( LG \) (cf. [33]) at level \( k \) gives rise to an irreducible conformal net denoted by \( \mathcal{A}_{G_k} \). By Th. 3.3 of [13], every irreducible positive energy representation of the loop group \( LG \) at level \( k \) gives rise to an irreducible covariant representation of \( \mathcal{A}_{G_k} \).

Given an interval \( I \) and a representation \( \pi \) of \( \mathcal{A} \), there is an \textit{endomorphism of } \( \mathcal{A} \text{ localized in } I \) equivalent to \( \pi \); namely \( \rho \) is a representation of \( \mathcal{A} \) on the vacuum Hilbert space \( \mathcal{H} \), unitarily equivalent to \( \pi \), such that \( \rho I = \text{id} \upharpoonright \mathcal{A}(I) \). We now define the statistics. Given the endomorphism \( \rho \) of \( \mathcal{A} \) localized in \( I \in \mathcal{I} \), choose an equivalent endomorphism \( \rho_0 \) localized in an interval \( I_0 \in \mathcal{I} \) with \( I_0 \cap I = \emptyset \) and let \( u \) be a local intertwiner in \( \text{Hom}(\rho, \rho_0) \), namely \( u \in \text{Hom}(\rho_I, \rho_{I_0}) \) with \( I_0 \) following clockwise \( I \) inside \( I \) which is an interval containing both \( I \) and \( I_0 \).

The \textit{statistics operator} \( \epsilon(\rho, \rho) := u^*\rho(u) = u^*\rho_I(u) \) belongs to \( \text{Hom}(\rho_I^2, \rho_0^2) \). We will call \( \epsilon(\rho, \rho) \) the positive or right braiding and \( \tilde{\epsilon}(\rho, \rho) := \epsilon(\rho, \rho)^* \) the negative or left braiding. The \textit{statistics parameter} \( \lambda_\rho \) can be defined in general. In particular, assume \( \rho \) to be localized in \( I \) and \( \rho_I \in \text{End}(\mathcal{A}(I)) \) to be irreducible with a conditional expectation \( E : \mathcal{A}(I) \rightarrow \rho_I(\mathcal{A}(I)) \), then

\[\lambda_\rho := E(\epsilon)
\]
depends only on the sector of \( \rho \). The \textit{statistical dimension} \( d_\rho \) and the \textit{univalence} \( \omega_\rho \) are then defined by

\[d_\rho = |\lambda_\rho|^{-1}, \quad \omega_\rho = \frac{\lambda_\rho}{|\lambda_\rho|}.
\]

The \textit{conformal spin-statistics theorem} (cf. [?]) shows that

\[\omega_\rho = e^{i2\pi L_0(\rho)}
\]
where \( L_0(\rho) \) is the conformal Hamiltonian (the generator of the rotation subgroup) in the representation \( \rho \). The right hand side in the above equality is called the \textit{univalence} of \( \rho \).

Let \( \{[\lambda], \lambda \in \mathcal{L} \} \) be a finite set of all equivalence classes of irreducible, covariant, finite-index representations of an irreducible local conformal net \( \mathcal{A} \). We will denote the conjugate of \([\lambda]\) by \( [\lambda] \) and identity sector (corresponding to the vacuum representation) by \([1]\) if no confusion arises, and let \( \mathcal{N}_\mu^\nu = \langle [\lambda] | [\mu], [\nu] \rangle \). Here \( \langle \mu, \nu \rangle \) denotes the dimension of the space of intertwiners from \( \mu \) to \( \nu \) (denoted by \( \text{Hom}(\mu, \nu) \)). We will denote by \( \{T_\lambda\} \) a basis of isometries in \( \text{Hom}(\nu, \lambda\mu) \). The univalence of \( \lambda \) and
Lemma 2.4. Let $\lambda, \mu$ be (not necessarily irreducible) representations of $\mathcal{A}$. $H(\lambda, \mu) := \varepsilon(\lambda, \mu)\varepsilon(\mu, \lambda)$. We say that $\lambda$ is local with $\mu$ if $H(\lambda, \mu) = 1$.

**Definition 2.3.** Let $\Gamma$ be a set of DHR representations of $\mathcal{A}$. If $\Gamma$ is an abelian group with multiplication given by composition and $d_\lambda = 1, \omega_\lambda = 1, \forall \lambda \in \Gamma$, then $\Gamma$ is called a local system of automorphisms.

The following Lemma will be useful to check if a set is a local system of automorphisms.

**Lemma 2.4.** (1) Assume that $[\mu] = \sum_{1 \leq i \leq n}[\mu_i]$ and $\lambda, \mu_i, i = 1, \ldots, n$ are representations of $\mathcal{A}$. Then $H(\lambda, \mu) = 1$ if and only if $H(\lambda, \mu_i) = 1$ for all $1 \leq i \leq n$;

(2) If $H(\lambda, \mu) = 1$ and $H(\lambda, \nu) = 1$, then $H(\lambda, \mu\nu) = 1$;

(3) If $\lambda_1, \ldots, \lambda_n$ generate a finite abelian group $\Gamma$ under composition, $\omega_{\lambda_i} = 1, 1 \leq i \leq n$, and $H(\lambda_i, \lambda_j) = 1, 1 \leq i, j \leq n$, then $\Gamma$ is a local system of automorphisms.

**Proof** (1) and (2) follows from [35] or Lemma 3.8 of [5]. As for (3), we prove by induction on $n$. If $n = 1$, then $\varepsilon(\lambda_1, \lambda_1) = \omega_{\lambda_1} = 1$ since $\varepsilon(\lambda_1, \lambda_1)$ is a scalar, and it follows that $\omega_{\lambda_1^2} = \varepsilon(\lambda_1, \lambda_1)^2 = 1, \forall i \geq 1$.

Assume that (3) has been proved for $n - 1$. Let $\mu$ be in the abelian group generated by $\lambda_1, \ldots, \lambda_{n-1}$. Since for any integer $k$ $H(\mu, \lambda_k^k) = 1$ by (2) and assumption, by repeatedly applying (2) and monodromy equation, we have $\omega_{\mu^k\lambda_k} = \omega_{\mu^k}\omega_{\lambda_k} = 1$ by induction hypotheses. It follows that (3) is proved. \[ \blacksquare \]

Next we recall some definitions from [25]. Recall that $\mathcal{I}$ denotes the set of intervals of $S^1$. Let $I_1, I_2 \in \mathcal{I}$. We say that $I_1, I_2$ are disjoint if $\bar{I}_1 \cap \bar{I}_2 = \emptyset$, where $\bar{I}$ is the closure of $I$ in $S^1$. When $I_1, I_2$ are disjoint, $I_1 \cup I_2$ is called a 1-disconnected interval in [43]. Denote by $\mathcal{I}_0$ the set of unions of disjoint 2 elements in $\mathcal{I}$. Let $\mathcal{A}$ be an irreducible Möbius covariant net. For $E = I_1 \cup I_2 \in \mathcal{I}_0$, let $I_3 \cup I_4$ be the interior of the complement of $I_1 \cup I_2$ in $S^1$ where $I_3, I_4$ are disjoint intervals. Let

$$ \mathcal{A}(E) := A(I_1) \vee A(I_2), \quad \hat{\mathcal{A}}(E) := (A(I_3) \vee A(I_4))' $$

Note that $\mathcal{A}(E) \subset \hat{\mathcal{A}}(E)$. Recall that a net $\mathcal{A}$ is split if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2)$ is naturally isomorphic to the tensor product of von Neumann algebras $\mathcal{A}(I_1) \otimes \mathcal{A}(I_2)$ for any disjoint intervals $I_1, I_2 \in \mathcal{I}$. $\mathcal{A}$ is strongly additive if $\mathcal{A}(I_1) \vee \mathcal{A}(I_2) = \mathcal{A}(I)$ where $I_1 \cup I_2$ is obtained by removing an interior point from $I$. 

the statistical dimension of (cf. §2 of [19]) will be denoted by $\omega_\lambda$ and $d(\lambda)$ (or $d_\lambda$) respectively. The following equation is called monodromy equation (cf. [34]):

$$ \varepsilon(\mu, \lambda)\varepsilon(\lambda, \mu)T_e = \frac{\omega_\mu}{\omega_\lambda}\omega_\mu T_e $$

(1)

where $\varepsilon(\mu, \lambda)$ is the unitary braiding operator.

We make the following definitions for convenience:

**Definition 2.2.** Let $\lambda, \mu$ be (not necessarily irreducible) representations of $\mathcal{A}$. $H(\lambda, \mu) := \varepsilon(\lambda, \mu)\varepsilon(\mu, \lambda)$. We say that $\lambda$ is local with $\mu$ if $H(\lambda, \mu) = 1$.

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Definition 2.5. [25, 31] A M"obius covariant net $\mathcal{A}$ is said to be completely rational if $\mathcal{A}$ is split, strongly additive, and the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ is finite for some $E \in \mathcal{I}_2$. The value of the index $[\hat{\mathcal{A}}(E) : \mathcal{A}(E)]$ (it is independent of $E$ by Prop. 5 of [25]) is denoted by $\mu_{\mathcal{A}}$ and is called the $\mu$-index of $\mathcal{A}$.

Note that, by results in [31], every irreducible, split, local conformal net with finite $\mu$-index is automatically strongly additive. Also note that if $\mathcal{A}$ is completely rational, then $\mathcal{A}$ has only finitely many irreducible covariant representations by [25].

Definition 2.6. A M"obius net $\mathcal{A}$ is called holomorphic if $\mathcal{A}$ is completely rational and $\mu_{\mathcal{A}} = 1$, i.e., $\mathcal{A}$ has only one irreducible representation which is the vacuum representation.

Let $\mathcal{B}$ be a M"obius (resp. conformal) net. $\mathcal{B}$ is called a M"obius (resp. conformal) extension of $\mathcal{A}$ if there is a map $I \in \mathcal{I} \to \mathcal{A}(I) \subset \mathcal{B}(I)$ that associates to each interval $I \in \mathcal{I}$ a von Neumann subalgebra $\mathcal{A}(I)$ of $\mathcal{B}(I)$, which is isotonic

$$\mathcal{A}(I_1) \subset \mathcal{A}(I_2), I_1 \subset I_2,$$

and M"obius (resp. diffeomorphism) covariant with respect to the representation $U$, namely

$$U(g)\mathcal{A}(I)U(g)^* = \mathcal{A}(g.I)$$

for all $g \in \text{M"ob}$ (resp. $g \in \text{Diff}(S^1)$) and $I \in \mathcal{I}$. $\mathcal{A}$ will be called a M"obius (resp. conformal) subnet of $\mathcal{B}$. Note that by Lemma 13 of [29] for each $I \in \mathcal{I}$ there exists a conditional expectation $E_I : \mathcal{B}(I) \to \mathcal{A}(I)$ such that $E$ preserves the vector state given by the vacuum of $\mathcal{B}$.

Definition 2.7. Let $\mathcal{A}$ be a M"obius covariant net. A M"obius covariant net $\mathcal{B}$ on a Hilbert space $\mathcal{H}$ is an extension of $\mathcal{A}$ if there is a DHR representation $\pi$ of $\mathcal{A}$ on $\mathcal{H}$ such that $\pi(\mathcal{A}) \subset \mathcal{B}$ is a M"obius subnet. The extension is irreducible if $\pi(\mathcal{A}(I)) \cap \mathcal{B}(I) = \mathbb{C}$ for some (and hence all) interval $I$, and is of finite index if $\pi(\mathcal{A}(I)) \subset \mathcal{B}(I)$ has finite index for some (and hence all) interval $I$. The index will be called the index of the inclusion $\pi(\mathcal{A}) \subset \mathcal{B}$ and will be denoted by $[\mathcal{B} : \mathcal{A}]$. If $\pi$ as representation of $\mathcal{A}$ decomposes as $[\pi] = \sum_{\lambda} m_{\lambda}[\lambda]$ where $m_{\lambda}$ are non-negative integers and $\lambda$ are irreducible DHR representations of $\mathcal{A}$, we say that $[\pi] = \sum_{\lambda} m_{\lambda}[\lambda]$ is the spectrum of the extension. For simplicity we will write $\pi(\mathcal{A}) \subset \mathcal{B}$ simply as $\mathcal{A} \subset \mathcal{B}$.

Lemma 2.8. If $\mathcal{A}$ is completely rational, and a M"obius covariant net $\mathcal{B}$ is an irreducible extension of $\mathcal{A}$. Then $\mathcal{A} \subset \mathcal{B}$ has finite index, $\mathcal{B}$ is completely rational and

$$\mu_{\mathcal{A}} = \mu_{\mathcal{B}}[\mathcal{B} : \mathcal{A}]^2.$$  

Proof $\mathcal{A} \subset \mathcal{B}$ has finite index follows from Prop. 2.3 of [23], and the rest follows from Prop. 24 of [25].
Lemma 2.9. If \( \mathcal{A} \) is a conformal net, and a Möbius covariant net \( \mathcal{B} \) is an extension of \( \mathcal{A} \) with index \( [\mathcal{B} : \mathcal{A}] < \infty \). Then \( \mathcal{B} \) is a conformal net.

Proof. Denote by \( \pi \) the vacuum representation of \( \mathcal{B} \). Denote by \( \mathbf{G} \) the universal cover of \( \text{Möb} \). By definition \( g \in \mathbf{G} \to U_\pi(g) \) is a representation of \( \mathbf{G} \) which implements the Möbius covariance of \( \pi \upharpoonright \mathcal{A} \). On the other hand by \( \S 2 \) of [2] there is a representation of \( g \in \mathbf{G} \to V_\pi(g) \) which implements the Möbius covariance of \( \pi \upharpoonright \mathcal{A} \), and \( V_\pi(g) \in \bigvee_{I \in \mathcal{I}} \pi(V_\mathcal{A}(I)) \), where \( V_\mathcal{A} \) is defined in definition 2.1. Since by assumption \( \pi \upharpoonright \mathcal{A} \) has finite index, by Prop. 2.2 of [19] we have \( U_\pi(g) = V_\pi(g), \forall g \in \mathbf{G} \). Hence \( V_\mathcal{A} \subset \mathcal{B} \) verifies the condition in definition 3.1 of [9], and by Prop. 3.7 of [9] the lemma is proved.

The following is Th. 4.9 of [30] (cf. \( \S 2.4 \) of [23]) which is also used in \( \S 4.2 \) of [23]:

Proposition 2.10. Let \( \mathcal{A} \) be a Möbius covariant net, \( \rho \) a DHR representation of \( \mathcal{A} \) localized on a fixed \( I_0 \) with finite statistics, which contains \( \text{id} \) with multiplicity one, i.e., there is (unique up to a phase) isometry \( w \in \text{Hom} (\text{id}, \rho) \). Then there is a Möbius covariant net \( \mathcal{B} \) which is an irreducible extension of \( \mathcal{A} \) if and only if there is an isometry \( w_1 \in \text{Hom}(\rho, \rho^2) \) which solves the following equations:

\[
\begin{align*}
\vspace{-3mm}
  w_1^* w &= w_1^* \rho(w) \in \mathbb{R}_+ \quad (2) \\
  w_1 w_1 &= \rho(w_1) w_1 \quad (3) \\
  \varepsilon(\rho, \rho) w_1 &= w_1 \quad (4)
\end{align*}
\]

Remark 2.11. Let \( \mathcal{A} \subset \mathcal{B} \) be as in Prop.2.10. If \( U \) is an unitary on the vacuum representation space of \( \mathcal{A} \) such that \( \text{Ad}_U \mathcal{A}(I) = \mathcal{A}(I), \forall I \), then it is easy to check that \( (\text{Ad}_U \rho \text{Ad}_U^*, \text{Ad}_U(w), \text{Ad}_U(w)) \) verifies the equations in Prop. 2.10, and determines a Möbius covariant net \( \text{Ad}_U(\mathcal{B}) \). The spectrum of \( \mathcal{A} \subset \text{Ad}_U(\mathcal{B}) \) (cf. definition 2.7) is \( \text{Ad}_U \rho \text{Ad}_U^* \), which may be different from \( \rho \), but \( \text{Ad}_U(\mathcal{B}) \) is isomorphic to \( \mathcal{B} \) by definition.

2.3 Induction

Let \( \mathcal{B} \) be a Möbius covariant net and \( \mathcal{A} \) a subnet. We assume that \( \mathcal{A} \) is strongly additive and \( \mathcal{A} \subset \mathcal{B} \) has finite index. Fix an interval \( I_0 \in \mathcal{I} \) and canonical endomorphism (cf. [30]) \( \gamma \) associated with \( \mathcal{A}(I_0) \subset \mathcal{B}(I_0) \). Then we can choose for each \( I \subset \mathcal{I} \) with \( I \supset I_0 \) a canonical endomorphism \( \gamma_I \) of \( \mathcal{B}(I) \) into \( \mathcal{A}(I) \) in such a way that \( \gamma_I \upharpoonright \mathcal{B}(I_0) = \gamma_{I_0} \) and \( \rho_{I_0} \) is the identity on \( \mathcal{A}(I_1) \) if \( I_1 \in \mathcal{I}_0 \) is disjoint from \( I_0 \), where \( \rho_I \equiv \gamma_I \upharpoonright \mathcal{A}(I) \). Given a DHR endomorphism \( \lambda \) of \( \mathcal{A} \) localized in \( I_0 \), the inductions \( \alpha_\lambda, \alpha_\lambda^- \) of \( \lambda \) are the endomorphisms of \( \mathcal{B}(I_0) \) given by

\[
\alpha_\lambda \equiv \gamma^{-1} \cdot \text{Ad}_\varepsilon(\lambda, \rho) \cdot \lambda \cdot \gamma, \quad \alpha_\lambda^- \equiv \gamma^{-1} \cdot \text{Ad}_{\varv}(\lambda, \rho) \cdot \lambda \cdot \gamma
\]

where \( \varepsilon \) (resp. \( \varv \)) denotes the right braiding (resp. left braiding) (cf. Cor. 3.2 of [3]). In [46] a slightly different endomorphism was introduced and the relation between the two was given in \( \S 2.1 \) of [44].
Note that \( \text{Hom}(\alpha_\lambda, \alpha_\mu) := \{ x \in \mathcal{B}(I_0) | x \alpha_\lambda(y) = \alpha_\mu(y) x, \forall y \in \mathcal{B}(I_0) \} \) and \( \text{Hom}(\lambda, \mu) := \{ x \in \mathcal{A}(I_0) | x \lambda(y) = \mu(y) x, \forall y \in \mathcal{A}(I_0) \} \).

The following is Lemma 3.6 of [5] and Lemma 3.5 of [3]:

**Lemma 2.12.**

\( \text{Hom}(\alpha_\lambda, \alpha_\mu) = \{ T \in \mathcal{B}(I_0) | \gamma(T) \in \text{Hom}(\rho\lambda, \rho\mu) \} \).

As a consequence of Lemma 2.12 we have the following Prop. 3.2 of [3] (Also cf. the proof of Lemma 3.2 of [46]):

**Lemma 2.13.** \([\alpha_\lambda] = [\alpha_\lambda^{-1}] \) iff \( \varepsilon(\lambda, \rho) \varepsilon(\rho, \lambda) = 1 \).

The following follows from Lemma 3.4 and Th. 3.3 of [46] (also cf. [3]):

**Lemma 2.14.**

1. \([\lambda] \to [\alpha_\lambda], [\lambda] \to [\alpha_\lambda^{-1}] \) are ring homomorphisms;
2. \( \langle \lambda, \mu \rangle = \langle \rho, \mu \rangle \).

### 2.4 Local simple current extensions

**Proposition 2.15.**

1. Assume that \( \mathcal{B} \) is a Möbius extension of \( \mathcal{A} \) of finite index with spectrum \( [\pi] = \sum_{\lambda \in \exp} m_\lambda(\lambda) \). Let \( \Gamma := \{ \lambda | \lambda \in \exp \} \). Assume that \( d_\lambda = 1, \forall \lambda \in \exp \). Then \( \Gamma \) is a local system of automorphisms;
2. If \( \Gamma \) is a finite local system of automorphisms of \( \mathcal{A} \), then there is a Möbius extension \( \mathcal{B} \) of \( \mathcal{A} \) with spectrum \( [\pi] = \sum_{\lambda \in \Gamma} [\lambda] \).

**Proof**

Ad (1): By assumption we have \( \alpha_\lambda \succ 1, \forall \lambda \in \Gamma \). By Lemma 3.10 of [5] \( \omega_\lambda = 1 \). Since \( d_\lambda = d_{\alpha_\lambda} = 1 \), it follows that \( [\alpha_\lambda] = [\alpha_\lambda^{-1}] = [1] \). Note that if \( \lambda \in \Gamma \) iff \( [\alpha_\lambda] = [1] \) and it follows that \( \Gamma \) is an abelian group with multiplication given by composition. By Lemma 2.13 and Lemma 2.4 (1) is proved.

(2) It follows from Prop. 5.5 of [35] (also cf. Th. 5.2 of [12]) that there is a Möbius extension \( \mathcal{B} \) of \( \mathcal{A} \) with spectrum \( [\pi] = \sum_{\lambda \in \Gamma} [\lambda] \).

**Remark 2.16.**

1. We will use the notation \( \mathcal{B} = \mathcal{A} \times \Gamma \) for the extension in Prop. 2.15.
2. One can extend the above theorem to a case when \( \mathcal{B} \) is not local but verifies twisted locality. Such extensions have been used for example in [26].

### 2.5 Mirror extensions

In this section we recall the mirror construction as given in §3 of [42]. Let \( \mathcal{B} \) be a completely rational net and \( \mathcal{A} \subset \mathcal{B} \) be a subnet which is also completely rational.

**Definition 2.17.** Define a subnet \( \tilde{\mathcal{A}} \subset \mathcal{B} \) by \( \tilde{\mathcal{A}}(I) := \mathcal{A}(I)' \cap \mathcal{B}(I), \forall I \in \mathcal{I} \).

We note that since \( \mathcal{A} \) is completely rational, it is strongly additive and so we have \( \tilde{\mathcal{A}}(I) = (\cup_{J \in \mathcal{I}} \mathcal{A}(J))' \cap \mathcal{B}(I), \forall I \in \mathcal{I} \). The following lemma then follows directly from the definition:
Lemma 2.18. The restriction of $\tilde{A}$ on the Hilbert space $\bigvee_I \tilde{A}(I)\Omega$ is an irreducible M"obius covariant net.

The net $\tilde{A}$ as in Lemma 2.18 will be called the coset of $A \subset B$. See [45] for a class of cosets from Loop groups.

The following definition generalizes the definition in §3 of [45]:

Definition 2.19. $A \subset B$ is called cofinite if the inclusion $\tilde{A}(I) \vee A(I) \subset B(I)$ has finite index for some interval $I$.

The following is Prop. 3.4 of [42]:

Proposition 2.20. Let $B$ be completely rational, and let $A \subset B$ be a M"obius subnet which is also completely rational. Then $A \subset B$ is cofinite if and only if $\tilde{A}$ is completely rational.

Let $B$ be completely rational, and let $A \subset B$ be a M"obius subnet which is also completely rational. Assume that $A \subset B$ is cofinite. We will use $\sigma, \sigma_j, ...$ (resp. $\lambda, \mu, ...$) to label irreducible DHR representations of $B$ (resp. $A$) localized on a fixed interval $I_0$. Since $\tilde{A}$ is completely rational by Prop. 2.20, $\tilde{A} \otimes A$ is completely rational, and so every irreducible DHR representation $\sigma$ of $B$, when restricting to $\tilde{A} \otimes A$, decomposes as direct sum of representations of $\tilde{A} \otimes A$ of the form $(i, \lambda) \otimes \lambda$ by Lemma 27 of [25]. Here $(i, \lambda)$ is a DHR representation of $\tilde{A}$ which may not be irreducible and we use the tensor notation $(i, \lambda) \otimes \lambda$ to represent a DHR representation of $\tilde{A} \otimes A$ which is localized on $I_0$ and defined by

$$(i, \lambda) \otimes \lambda(x_1 \otimes x_2) = (i, \lambda)(x_1) \otimes \lambda(x_2), \forall x_1 \otimes x_2 \in \tilde{A}(I_0) \otimes A(I_0).$$

We will also identify $\tilde{A}$ and $A$ as subnets of $\tilde{A} \otimes A$ in the natural way. We note that when no confusion arise, we will use 1 to denote the vacuum representation of a net.

Definition 2.21. A M"obius subnet $A \subset B$ is normal if $\tilde{A}(I)' \cap B(I) = A$ for some $I$.

The following is implied by Lemma 3.4 of [36] (also cf. Page 797 of [47]):

Lemma 2.22. Let $B$ be completely rational, and let $A \subset B$ be a M"obius subnet which is also completely rational. Assume that $A \subset B$ is cofinite. Then the following conditions are equivalent:

(1) $A \subset B$ is normal;

(2) $(1, 1)$ is the vacuum representation of $\tilde{A}$ and $(1, \lambda)$ contains $(1, 1)$ if and only if $\lambda = 1$.

The following is part of Proposition 3.7 of [42]:

Proposition 2.23. Let $B$ be completely rational, and let $A \subset B$ be a M"obius subnet which is also completely rational. Assume that $A \subset B$ is cofinite and normal. Then:

(1) Let $\gamma$ be the restriction of the vacuum representation of $B$ to $\tilde{A} \otimes A$. Then $[\gamma] = \sum_{\lambda \in \text{exp}} [(1, \lambda) \otimes \lambda]$ where each $(1, \lambda)$ is irreducible;
(2) Let $\lambda \in \exp$ be as in (1), then $[\alpha_{(1,\lambda)\otimes 1}] = [\alpha_{1\otimes \lambda}]$, and $[\lambda] \to [\alpha_{1\otimes \lambda}]$ is a ring isomorphism where the $\alpha$-induction is with respect to $\tilde{\mathcal{A}} \otimes \mathcal{A} \subset \mathcal{B}$ as in subsection 2.3; Moreover the set exp is closed under fusion:

(3) Let $[\rho] = \sum_{\lambda \in \exp} m_\lambda [\lambda]$ where $m_\lambda = m_{\tilde{\lambda}} \geq 0, \forall \lambda$, and $[(1, \rho)] = \sum_{\lambda \in \exp} m_\lambda [(1, \lambda)]$. Then there exists an unitary element $T_\rho \in \text{Hom}(\alpha_{(1,\rho)\otimes 1}, \alpha_{1\otimes \rho})$ such that

$$\epsilon((1, \rho), (1, \rho)) T_\rho^* \alpha_{1\otimes \rho}(T_\rho^*) = T_\rho^* \alpha_{1\otimes \rho}(T_\rho) \bar{\epsilon}(\rho, \rho);$$

(4) Let $\rho$, $(1, \rho)$ be as in (3). Then

$$\text{Hom}(\rho^n, \rho^m) = \text{Hom}(\alpha_{1\otimes \rho^n}, \alpha_{1\otimes \rho^m}),$$

$$\text{Hom}((1, \rho)^n, (1, \rho)^m) = \text{Hom}(\alpha_{(1,\rho)^n\otimes 1}, \alpha_{(1,\rho)^m\otimes 1}), \forall n, m \in \mathbb{N};$$

Denote by $\Delta_0 := \{ \lambda [\lambda] = \sum_i [\lambda_i], \lambda_i \in \exp \}$. Assume $\mu_i \in \Delta_0, i = 1, \ldots, n$. For each $[\mu_i] = \sum_j [\mu_{ij}]$ choose DHR representations $M(\mu_i)$ of $\tilde{\mathcal{A}}$ such that $[M(\mu_i)] = \sum_j [1, \lambda_j]$. Let $T_i \in \text{Hom}(\alpha_{\mu(\mu_{i})\otimes 1}, \alpha_{1\otimes \mu_i})$ be an unitary element (not necessarily unique up to phase when $\mu_i$ is not irreducible) as given is (3) of Prop. 2.23. Define

$$T_{i_1 i_2 \ldots i_k} := \alpha_{\mu_1 \ldots \mu_{k-1} \otimes 1}(T_{i_1}) \ldots \alpha_{\mu_1 \ldots \mu_{2} \otimes 1}(T_{i_2}) \ldots \alpha_{\mu_1 \otimes 1}(T_{i_k})T_{i_k} \in \text{Hom}(\alpha_{M(\mu_1) \ldots M(\mu_k) \otimes 1}, \alpha_{1\otimes \mu_1 \ldots \mu_k});$$

For each $S \in \text{Hom}(\mu_1 \ldots \mu_{i_k}, \mu_{i_1} \ldots \mu_{i_m})$ we define $M(S) := T_{i_{1 \ldots m}}^* S T_{i_{1 \ldots i_k}}$.

**Lemma 2.24.** Assume that $S_1, T \in \text{Hom}(\lambda, \mu), S_2 \in \text{Hom}(\nu, \lambda)$ where $\lambda, \mu$ are products of elements from $\{\mu_1, \ldots, \mu_n\}$. If $\nu = \mu_1 \ldots \mu_k$ we define $M(M(\mu_i) \ldots M(\mu_k))$ by $M(\nu)$.

Then:

$$M(S_1 S_2) = M(S_1) M(S_2), M(\nu(T)) = M(\nu)(M(T)), M(\epsilon(\mu_i, \mu_j)) = \epsilon(M(\mu_i), M(\mu_j)).$$

**Proof** The first two identities follow directly from definitions. The third follows from (3) of Prop. 2.3.1 of [44], as (3) of Prop. 2.23.

The following is Th. 3.8 of [42]:

**Theorem 2.25.** Let $\mathcal{B}$ be completely rational, and let $\mathcal{A} \subset \mathcal{B}$ be a Möbius subnet which is also completely rational. Assume that $\mathcal{A} \subset \mathcal{B}$ is cofinite and normal, and let $\exp$ be as in (1) of Prop. 2.23. Assume that $\mathcal{A} \subset \mathcal{C}$ is an irreducible Möbius extension of $\mathcal{A}$ with spectrum $[\rho] = \sum_{\lambda \in \exp} m_\lambda [\lambda], m_\lambda \geq 0$. Then there is an irreducible Möbius extension $\tilde{\mathcal{C}}$ of $\tilde{\mathcal{A}}$ with spectrum $[(1, \rho)] = \sum_{\lambda \in \exp} m_\lambda [(1, \lambda)]$. Moreover $\tilde{\mathcal{C}}$ is completely rational.

**Remark 2.26.** Due to (5) of Prop. 3.7 of [42], the extension $\tilde{\mathcal{A}} \subset \tilde{\mathcal{C}}$ as given in Th. 3.4 will be called the mirror or the conjugate of $\mathcal{A} \subset \mathcal{C}$.

By Lemma 2.8 and Th. 2.25 we have:

**Corollary 2.27.** Let $\tilde{\mathcal{C}}$ be the mirror extension as given in Th. 2.25. Then $\frac{\mu_{\tilde{\mathcal{A}}}}{\mu_{\tilde{\mathcal{C}}}} = \frac{\mu_{\mathcal{A}}}{\mu_{\mathcal{C}}}$. 

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The mirror extension $\tilde{A} \subset \tilde{C}$ is constructed as follows: let $(\rho, w, w_1)$ be associated with extension $A \subset C$ as given in Prop. 2.10. Then the extension $\tilde{A} \subset \tilde{C}$ is given by $(M(\rho), M(w), M(w_1))$ where the map $M$ is defined before Lemma 2.24. Let $\mu, \nu \in \Delta_0$. Consider now inductions with respect to $A \subset C$ and $\tilde{A} \subset \tilde{C}$.

**Proposition 2.28.** Assume that $\mu, \nu \in \Delta_0$, $M(\rho) = (1, \rho), M(\mu) = (1, \mu), M(\nu) = (1, \nu)$. Then

$$\langle \alpha_\mu, \alpha_\nu \rangle = \langle \alpha_{M(\mu)}, \alpha_{M(\nu)} \rangle, \langle \alpha_\mu, \alpha^-_\nu \rangle = \langle \alpha_{M(\mu)}, \alpha^-_{M(\nu)} \rangle.$$

**Proof** By Lemma 2.14 we have

$$\langle \alpha_\mu, \alpha_\nu \rangle = \langle \rho_\mu, \nu \rangle, \langle \alpha_{M(\mu)}, \alpha_{M(\nu)} \rangle = \langle M(\rho)M(\mu), M(\nu) \rangle.$$

By Prop. 2.23 we have proved the first equality. By Lemma 2.12 we have

$$\text{Hom}(\alpha_\mu, \alpha^-_\nu) = \{ T \in C(I_0) \mid \gamma(T) \in \text{Hom}(\rho_\mu, \rho_\nu), \varepsilon(\nu, \rho)\varepsilon(\rho, \nu)\gamma(T) = \gamma(T) \}. $$

By [30], $\gamma(C(I_0)) = \{ x \in A(I_0) \mid x = w_1^*\rho(x)w_1 \}$. It follows that $\langle \alpha_\mu^+, \alpha^-_\nu \rangle$ is equal to the dimension of the following vector space

$$\{ T' \in A(I_0) \mid T' \in \text{Hom}(\rho_\mu, \rho_\nu), \varepsilon(\nu, \rho)\varepsilon(\rho, \nu)T' = T', \quad T' = w_1^*\rho(T'w_1) \}.$$

Now apply the map $M$ to the above vector space and use Lemma 2.24 we have that $\langle \alpha_\mu^+, \alpha^-_\nu \rangle$ is equal to the dimension of the following vector space

$$\{ T' \in \tilde{A}(I_0) \mid T' \in \text{Hom}(M(\rho)M(\mu), M(\rho)M(\nu)), \varepsilon(M(\nu), M(\rho))\varepsilon(M(\rho), M(\nu))T' = T', \quad T' = M(w_1)^*M(\rho)(T'M(w_1)) \}.$$

Since $\varepsilon(M(\nu), M(\rho))\varepsilon(M(\rho), M(\nu)) = (\varepsilon(M(\nu), M(\rho))\varepsilon(M(\rho), M(\nu)))^*$, we conclude that $\langle \alpha_\mu^+, \alpha^-_\nu \rangle$ is equal to the dimension of the following vector space

$$\{ T' \in \tilde{A}(I_0) \mid T' \in \text{Hom}(M(\rho)M(\mu), M(\rho)M(\nu)), \varepsilon(M(\nu), M(\rho))\varepsilon(M(\rho), M(\nu))T' = T', \quad T' = M(w_1)^*M(\rho)(T'M(w_1)) \}$$

which is equal to $\langle \alpha_{M(\mu)}, \alpha^-_{M(\nu)} \rangle$ by Lemma 2.12. $lacksquare$

### 2.6 A series of normal extensions

Let $G = SU(n)$. We denote $LG$ the group of smooth maps $f : S^1 \to G$ under pointwise multiplication. The diffeomorphism group of the circle $\text{Diff} S^1$ is naturally a subgroup of $\text{Aut}(LG)$ with the action given by reparametrization. In particular the group of rotations $\text{Rot} S^1 \simeq U(1)$ acts on $LG$. We will be interested in the projective unitary representation $\pi : LG \to U(H)$ that are both irreducible and have positive energy. This means that $\pi$ should extend to $LG \times \text{Rot} S^1$ so that $H = \bigoplus_{n \geq 0} H(n)$, where the $H(n)$ are the eigenspace for the action of $\text{Rot} S^1$, i.e., $r_\theta \xi = \exp(i n \theta)$ for
\( \theta \in H(n) \) and \( \dim H(n) < \infty \) with \( H(0) \neq 0 \). It follows from \([33]\) that for fixed level \( k \) which is a positive integer, there are only finite number of such irreducible representations indexed by the finite set

\[
P_{++}^k = \left\{ \lambda \in P \mid \lambda = \sum_{i=1,\ldots,n-1} \lambda_i \Lambda_i, \lambda_i \geq 0, \sum_{i=1,\ldots,n-1} \lambda_i \leq k \right\}
\]

where \( P \) is the weight lattice of \( SU(n) \) and \( \Lambda_i \) are the fundamental weights. We will write \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \), \( \lambda_0 = k - \sum_{1 \leq i \leq n-1} \lambda_i \) and refer to \( \lambda_0, \ldots, \lambda_{n-1} \) as components of \( \lambda \).

We will use \( k \Lambda_0 \) or simply \( 1 \) to denote the trivial representation of \( SU(n) \).

For \( \lambda, \mu, \nu \in P_{++}^k \), define \( N^\nu_{\lambda\mu} = \sum_{\delta \in P_{++}^k} S_\lambda^{(\delta)} S_\mu^{(\delta)} S_\nu^{(\delta)} / S_{\lambda_0}^{(\delta)} \) where \( S_\lambda^{(\delta)} \) is given by the Kac-Peterson formula:

\[
S_\lambda^{(\delta)} = c \sum_{w \in S_n} \varepsilon_w \exp(iw(\delta) \cdot \lambda) 2\pi / n
\]

where \( \varepsilon_w = \det(w) \) and \( c \) is a normalization constant fixed by the requirement that \( S_\lambda^{(\delta)} \) is an orthonormal system. It is shown in \([22]\) P. 288 that \( N^\nu_{\lambda\mu} \) are non-negative integers. Moreover, define \( Gr(C_k) \) to be the ring whose basis are elements of \( P_{++}^k \) with structure constants \( N^\nu_{\lambda\mu} \). The natural involution \( * \) on \( P_{++}^k \) is defined by \( \lambda \mapsto \lambda^* = \) the conjugate of \( \lambda \) as representation of \( SU(n) \).

We shall also denote \( S_{\lambda_0}^{(\lambda)} \) by \( S_{1 \lambda_0}^{(\lambda)} \). Define \( d_\lambda = \frac{S_{1 \lambda_0}^{(\lambda)}}{S_{1 \lambda_0}^{(\lambda_0)}} \). We shall call \( (S_{\nu}^{(\delta)}) \) the \( S \)-matrix of \( LSU(n) \) at level \( k \).

We shall encounter the \( \mathbb{Z}_n \) group of automorphisms of this set of weights, generated by

\[
J : \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_{n-1}) \rightarrow J(\lambda) = (k - 1 - \lambda_1 - \cdots - \lambda_{n-1}, \lambda_1, \ldots, \lambda_{n-2}).
\]

We will identity \( J \) with \( k \Lambda_1 \) in the following. Define \( \text{col}(\lambda) = \Sigma_i (\lambda_i - 1) i \). The central element \( \exp \frac{2\pi i}{n} \) of \( SU(n) \) acts on representation of \( SU(n) \) labeled by \( \lambda \) as \( \exp(\frac{2\pi i \text{col}(\lambda)}{n}) \), modulo \( n \) \( \text{col}(\lambda) \) will be called the color of \( \lambda \).

The irreducible positive energy representations of \( LSU(n) \) at level \( k \) give rise to an irreducible conformal net \( \mathcal{A}_{SU(n)_k} \) (cf. \([27]\)) and its covariant representations. \( \mathcal{A}_{SU(n)_k} \) is completely rational (cf. \([41]\) and \([43]\)), and \( \mu_{\mathcal{A}_{SU(n)_k}} = \frac{1}{(s_1)^2} \) by \([43]\). We will use \( \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \) to denote irreducible representations of \( \mathcal{A} \) and also the corresponding endomorphism of \( M = \mathcal{A}(I) \).

All the sectors \([\lambda] \) with \( \lambda \) irreducible generate the fusion ring of \( \mathcal{A} \).

For \( \lambda \) irreducible, the univalence \( \omega_\lambda \) is given by an explicit formula (cf. 9.4 of \([PS]\)). Let us first define \( h_\lambda = \frac{c_2(\lambda)}{k+n} \) where \( c_2(\lambda) \) is the value of Casimir operator on representation of \( SU(n) \) labeled by dominant weight \( \lambda \). \( h_\lambda \) is usually called the conformal dimension. Then we have: \( \omega_\lambda = \exp(2\pi ih_\lambda) \). The conformal dimension of
\[ \lambda = (\lambda_1, \ldots, \lambda_{n-1}) \]

is given by

\[
h_\lambda = \frac{1}{2n(k + n)} \sum_{1 \leq i \leq n-1} i(n-i)\lambda_i^2 + \frac{1}{n(k + n)} \sum_{1 \leq j \leq i \leq n-1} j(n-i)\lambda_i\lambda_j + \frac{1}{2(k + n)} \sum_{1 \leq j \leq n-1} j(n-j)\lambda_j
\]

(5)

Let \( G \subset H \) be inclusions of compact simple Lie groups. \( LG \subset LH \) is called a conformal inclusion if the level 1 projective positive energy representations of \( LH \) decompose as a finite number of irreducible projective representations of \( LG \). \( LG \subset LH \) is called a maximal conformal inclusion if there is no proper subgroup \( G' \) of \( H \) containing \( G \) such that \( LG \subset LG' \) is also a conformal inclusion. A list of maximal conformal inclusions can be found in [18].

Let \( H^0 \) be the vacuum representation of \( LH \), i.e., the representation of \( LH \) associated with the trivial representation of \( H \). Then \( H^0 \) decomposes as a direct sum of irreducible projective representation of \( LG \) at level \( K \). \( K \) is called the Dynkin index of the conformal inclusion.

We shall write the conformal inclusion as \( G_K \subset H_1 \). Note that it follows from the definition that \( A_{H_1} \) is an extension of \( A_{G_K} \). We will be interested in the following conformal inclusion:

\[ L(SU(m)_n \times SU(n)_m) \subset L(SU(nm)) \]

In the classification of conformal inclusions in [GNO], the above conformal inclusion corresponds to the Grassmanian \( SU(m+n)/SU(m) \times SU(n) \times U(1) \).

Let \( \Lambda_0 \) be the vacuum representation of \( L(SU(nm)) \) on Hilbert space \( H^0 \). The decomposition of \( \Lambda_0 \) under \( L(SU(m) \times SU(n)) \) is known, see, e.g., [1]. To describe such a decomposition, let us prepare some notation. We use \( \hat{S} \) to denote the \( S \)-matrices of \( SU(m) \), and \( \hat{\tilde{S}} \) to denote the \( S \)-matrices of \( SU(n) \). The level \( n \) (resp. \( m \)) weight of \( LSU(m) \) (resp. \( LSU(n) \)) will be denoted by \( \hat{\lambda} \) (resp. \( \hat{\bar{\lambda}} \)).

We start by describing \( \hat{P}_+^n \) (resp. \( \hat{\bar{P}}_+^m \)), i.e. the highest weights of level \( n \) of \( LSU(m) \) (resp. level \( m \) of \( LSU(n) \)).

\( \hat{P}_+^n \) is the set of weights

\[ \hat{\lambda} = \bar{k}_0\hat{\lambda}_0 + \bar{k}_1\hat{\lambda}_1 + \cdots + \bar{k}_{m-1}\hat{\lambda}_{m-1} \]

where \( \bar{k}_i \) are non-negative integers such that

\[ \sum_{i=0}^{m-1} \bar{k}_i = n \]

and \( \hat{\lambda}_i = \hat{\lambda}_0 + \hat{\omega}_i, \ 1 \leq i \leq m - 1 \), where \( \hat{\omega}_i \) are the fundamental weights of \( SU(m) \).

Instead of \( \hat{\lambda} \) it will be more convenient to use

\[ \hat{\lambda} + \hat{\rho} = \sum_{i=0}^{m-1} k_i\hat{\lambda}_i \]
with \( k_i = \bar{k}_i + 1 \) and \( m \rightarrow 0 \rightarrow \sum k_i = m + n \). Due to the cyclic symmetry of the extended Dynkin diagram of \( SU(m) \), the group \( \mathbb{Z}_m \) acts on \( \tilde{P}_+ \) by

\[
\hat{\Lambda}_i \to \hat{\Lambda}_{(i+\hat{\mu}) \bmod m}, \quad \mu \in \mathbb{Z}_m.
\]

Let \( \Omega_{m,n} = \tilde{P}_+/\mathbb{Z}_m \). Then there is a natural bijection between \( \Omega_{m,n} \) and \( \Omega_{n,m} \) (see §2 of [1]). The idea is to draw a circle and divide it into \( m + n \) arcs of equal length. To each partition \( \sum_{0 \leq i \leq m-1} k_i = m + n \) there corresponds a ”slicing of the pie” into \( m \) successive parts with angles \( 2\pi k_i/(m + n) \), drawn with solid lines. We choose this slicing to be clockwise. The complementary slicing in broken lines (The lines which are not solid) defines a partition of \( m + n \) into \( n \) successive parts, \( \sum_{0 \leq i \leq n-1} l_i = m + n \). We choose the later slicing to be counterclockwise, and it is easy to see that such a slicing corresponds uniquely to an element of \( \Omega_{n,m} \).

We parameterize the bijection by a map

\[
\beta : \tilde{P}_+^m \to \tilde{P}_+^m
\]

as follows. Set

\[
r_j = \sum_{i=j}^m k_i, \quad 1 \leq j \leq m
\]

where \( k_m \equiv k_0 \). The sequence \( (r_1, \ldots, r_m) \) is decreasing, \( m + n = r_1 > r_2 > \cdots > r_m \geq 1 \). Take the complementary sequence \( (\bar{r}_1, \bar{r}_2, \ldots, \bar{r}_n) \) in \( \{1, 2, \ldots, m + n\} \) with \( \bar{r}_1 > \bar{r}_2 > \cdots > \bar{r}_n \). Put

\[
S_j = m + n + \bar{r}_n - \bar{r}_{n-j+1}, \quad 1 \leq j \leq n.
\]

Then \( m + n = s_1 > s_2 > \cdots > s_n \geq 1 \). The map \( \beta \) is defined by

\[
(r_1, \ldots, r_m) \to (s_1, \ldots, s_n).
\]

The following lemma summarizes what we will use:

**Lemma 2.29.** (1) Let \( \check{Q} \) be the root lattice of \( SU(m) \), \( \check{\Lambda}_i \), \( 0 \leq i \leq m-1 \) its fundamental weights and \( \check{Q}_i = (\check{Q} + \check{\Lambda}_i) \cap \check{P}_+^n \). Let \( \Lambda \in \mathbb{Z}_{mn} \) denote a level 1 highest weight of \( SU(mn) \) and \( \check{\lambda} \in \check{Q}_{\Lambda \bmod m} \). Then there exists a unique \( \check{\lambda} \in \check{P}_+^n \) with \( \check{\lambda} = \mu(\check{\lambda}) \) for some unique \( \mu \in \mathbb{Z}_m \) such that \( H_{\check{\lambda}} \otimes H_{\check{\lambda}} \) appears once and only once in \( H_{\Lambda} \). The map \( \check{\lambda} \to \check{\lambda} = \mu(\check{\lambda}) \) is one-to-one. Moreover, \( H_{\Lambda} \), as representations of \( L(SU(m) \times SU(n)) \), is a direct sum of all such \( H_{\check{\lambda}} \otimes H_{\check{\lambda}} \);

(2) \( \mu_{A_{SU(n)m}^1} = \frac{n}{m} \mu_{A_{SU(m)n}^1} \); 

(3) The subnets \( A_{SU(n)m} \subset A_{SU(m)n} \) are normal and cofinite. The set \( \exp \) as in (1) Prop. 2.23 is the elements of \( P_+^{n+m} \) which belong to the root lattice of \( SU(n) \).

**Proof** (1) is Th. 1 of [1]. (2) follows from Th. 4.1 of [43]. (3) is Lemma 4.1 of [42].
3 Schellekens’s modular invariants and their realizations by conformal nets

In this section we examine three modular invariants constructed by A. N. Schelleken in [38] which are based on level-rank duality. These are entries 18, 27, and 40 in the table of [38]. Our goal in this section is to show that they can be realized by conformal nets as an application of mirror extensions in section 2.5. For simplicity in this section we will use $G_k$ to denote the corresponding conformal net $A_{G_k}$ when no confusion arises.

3.1 Three mirror extensions

3.1.1 $\tilde{SU}(10)_2$

$\tilde{SU}(10)_2$ is the simplest nontrivial example of mirror extensions applying to $SU(2)_{10} \subset Spin(5)_1$ and $SU(2)_{10} \times SU(10)_2 \subset SU(20)_1$ in Theorem 2.25. By Cor. 2.27 and Lemma 2.29

$\mu_{\tilde{SU}(10)_2} = 20$.

Consider the induction for $SU(10)_2 \subset \tilde{SU}(10)_2$. By Th. 5.7 of [7] the matrix $Z_{\mu} = \langle \alpha_\lambda, \alpha_\mu^\perp \rangle$ commutes with the $S, T$ matrix of $SU(10)_2$. Such matrices are classified in [16], and it follows that there are 15 irreducible representations of $\tilde{SU}(10)_2$ given as follows: $\alpha_i^j, 0 \leq i \leq 9, \alpha_j^i \sigma, 0 \leq j \leq 4$. The fusion rules are determined by the following relations:

$[\bar{\sigma}] = [\alpha^2_5 \sigma], [\alpha^5_5 \sigma] = [\sigma], [\sigma \bar{\sigma}] = [1] + [\alpha^5_5]$

The restrictions of these representations to $SU(10)_2$ are given as follows:

$[\alpha_i^j] = [J^i(2\Lambda_0)] + [J^i(\Lambda_3 + \Lambda_7)], 0 \leq i \leq 9; [\alpha_i^j \sigma] = [J^i(\Lambda_0 + \Lambda_3)] + [J^i(\Lambda_5 + \Lambda_8)], 0 \leq j \leq 4.$

It follows that modulo integers the conformal dimensions are given as

$h_{\alpha_i^j} = \frac{i(10 - i)}{10}, 0 \leq i \leq 9, h_{\sigma} = \frac{77}{80}, h_{\alpha_j^i \sigma} = \frac{157}{80}, h_{\alpha_5^5 \sigma} = \frac{173}{80} = h_{\alpha_5^5 \sigma}$.

Remark 3.1. The modular tensor category (cf. [40]) from representations of $\tilde{SU}(10)_2$ as given above seems to be unknown before. It will be interesting to understand our construction from a categorical point of view.

The following simple lemma will be used later:

Lemma 3.2. $A_{Spin(n)_1}$ is a completely rational net whose irreducible representations are in one to one correspondence with irreducible representations of $LSpin(n)_1$. When $n$ is odd there are three irreducible representations $1, \mu_0, \mu_1$ with index $1, 1, \sqrt{2}$ respectively and fusion rules $[\mu_1^2] = [1] + [\mu_0]$; when $n = 4k + 2, k \in \mathbb{N}$ the fusion rule is $\mathbb{Z}_4$; when $n = 4k, k \in \mathbb{N}$ the fusion rule is $\mathbb{Z}_2 \times \mathbb{Z}_2$. 
\textbf{Proof} By Th. 3.10 of [8] it is enough to prove that \( \mu_{\mathcal{A}_{\text{Spin}(n)_1}} = 4 \).

When \( n = 5 \) this follows from conformal inclusion \( SU(2)_10 \subset Spin(5)_1 \) and Lemma 2.8. Consider the inclusion \( SO(n) \times U(1) \subset SO(n + 2) \). Note that the fundamental group of \( SO(n) \) is \( \mathbb{Z}_2 \). It follows that loops with even winding numbers in \( LU(1) \) can be lifted to \( LSpin(n) \), and we have a conformal inclusion \( LSpin(n - 2)_1 \times LU(1)_4 \subset LSpin(n)_1 \). Since \( \mu_{\mathcal{A}_{SU(2)_1}} = 4 \) by §3 of [44], and the index of \( \mathcal{A}_{\text{Spin}(n-2)} \times \mathcal{A}_{U(1)_4} \subset \mathcal{A}_{\text{Spin}(n)_1} \) is checked to be 2, by induction one can easily prove the lemma for all odd \( n \). When \( n \) is even we use the conformal inclusion \( \mathcal{A}_{SU(n/2)_1} \times \mathcal{A}_{U(1)_{2n}} \subset \mathcal{A}_{\text{Spin}(n)_1} \) with index \( n/2 \). Note that \( \mu_{\mathcal{A}_{SU(n/2)_1}} = n/2, \mu_{\mathcal{A}_{U(1)_{2n}}} = 2n \) by §3 of [44], and by Lemma 2.8 we have \( \mu_{\mathcal{A}_{\text{Spin}(n)_1}} = 4 \). □

### 3.1.2 \( \widehat{SU}(9)_3 \)

\( \widehat{SU}(9)_3 \) is an extension of \( SU(9)_3 \) by applying Th. 2.25 to \( SU(3)_9 \subset (E_6)_1 \) and \( SU(3)_9 \times SU(9)_3 \subset SU(27)_1 \). By Cor. 2.27 and Lemma 2.29 \( \mu_{\widehat{SU}(9)_3} = 9 \). Recall the branching rules for \( SU(3)_9 \subset (E_6)_1 \) (We use \( l_0 \) to denote the vacuum representation of \( (E_6)_1 \) and \( L_+ \), \( L_- \) the other two irreducible representations of \( (E_6)_1 \)):

\[
[1_0 l] = \sum_{0 \leq i \leq 2} (\lfloor \hat{J}^i(9\Lambda_0) \rfloor + \lfloor \hat{J}^i(\Lambda_0 + 4\Lambda_1 + 4\Lambda_2) \rfloor), \quad [1_+ l] = [1_- l] = \sum_{0 \leq i \leq 2} (\lfloor \hat{J}^i(5\Lambda_0 + 2\Lambda_1 + 2\Lambda_2) \rfloor)
\]

where \( \hat{J} := 9\Lambda_1 \).

Consider inductions with respect to \( \widehat{SU}(9)_3 \subset SU(9)_3 \).

By Th. 2.25 and Lemma 2.29 the vacuum of \( \widehat{SU}(9)_3 \) restricts to representation

\[
\sum_{0 \leq i \leq 2} (\lfloor \hat{J}^{3i}(9\Lambda_0) \rfloor + \lfloor \hat{J}^{3i}(\Lambda_3 + \Lambda_7 + \Lambda_8) \rfloor)
\]

of \( SU(9)_3 \). Since \( J \) is local with the above representation, by Lemma 2.13 \( \alpha_J \) is a DHR representation of \( \widehat{SU}(9)_3 \), and \( [\alpha_J^3] = [1] \). One can determine the remaining irreducible representations of \( SU(9)_3 \) by using [16] as in §3.1.1. Here we give a different approach which will be useful in §3.1.3. We note that \( M(\hat{J}) = \hat{J}^3, M(\hat{J}(5\Lambda_0 + 2\Lambda_1 + 2\Lambda_2)) = \hat{J}^{3i}(4\Lambda_4 + \lambda_6 + \lambda_8), i = 0, 1, 2 \) by Lemma 2.29, where \( M \) is defined as before Lemma 2.24. By Prop. 2.28 we have

\[
\langle \alpha_{\Lambda_4 + \Lambda_6 + \Lambda_8}, \alpha_{\Lambda_4 + \Lambda_6 + \Lambda_8} \rangle = 2.
\]

It follows that there are two irreducible DHR representations \( \tau_1, \tau_2 \) of \( \widehat{SU}(9)_3 \) such that \( \alpha_{\Lambda_4 + \Lambda_6 + \Lambda_8} \bowtie [\tau_1] + [\tau_2] \), and \( \tau_1, \tau_2 \) are the only two irreducible subsectors of \( \alpha_{\Lambda_4 + \Lambda_6 + \Lambda_8} \) which are DHR representations. We have for \( i = 1, 2 \)

\[
\langle \tau_i, \alpha^-_\mu \rangle \leq \langle \alpha_{\Lambda_4 + \Lambda_6 + \Lambda_8}, \alpha^-_\mu \rangle.
\]

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Note that if the color of \( \mu \) is nonzero, then \( \langle \alpha_{\Lambda_4+\Lambda_6+\Lambda_8}, \alpha_\mu^\perp \rangle = 0 \) by Lemma 2.14 since \( \Lambda_4 + \Lambda_6 + \Lambda_8 \) has color 0. If \( \mu \) has color 0, by Lemma 2.29 and Prop. 2.28 we have

\[
\langle \alpha_{\Lambda_4+\Lambda_6+\Lambda_8}, \alpha_\mu^\perp \rangle
\]

is nonzero only when \( \mu = J^{3i}(\Lambda_4 + \Lambda_6 + \Lambda_8), i = 0, 1, 2 \). It follows that

\[
\langle \tau_i, \alpha_\mu \rangle = 1
\]

when \( \mu = J^{3i}(\Lambda_4 + \Lambda_6 + \Lambda_8), i = 0, 1, 2, \) and

\[
\langle \tau_i, \alpha_\mu \rangle = 0
\]

when \( \mu \neq J^{3i}(\Lambda_4 + \Lambda_6 + \Lambda_8), i = 0, 1, 2 \). Hence the restriction of \( \tau_i \) to \( SU(9)_3 \) are given as follows:

\[
[\tau_i |] = \sum_{0 \leq j \leq 2} [J^{3j}(\Lambda_4 + \Lambda_6 + \Lambda_8)]
\]

It follows that the index of \( \tau_i, i = 1, 2 \) is one, and since

\[
[(\alpha_J \tau_i) |] = \sum_{0 \leq j \leq 2} [J^{3j+1}(\Lambda_4 + \Lambda_6 + \Lambda_8)],
\]

it follows that \( [\alpha_J \tau_i] \neq [\tau_i] \). Hence the irreducible representations of \( \tilde{SU}(9)_3 \) are given by

\[
1, \alpha_J, \alpha_J^2, \alpha_J^3 \tau_k, 0 \leq i \leq 2, k = 1, 2.
\]

These representations generate an abelian group of order 9, it must be either \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) or \( \mathbb{Z}_9 \). Note that by Lemma 2.14

\[
\langle \alpha_J, \tau^k \rangle \leq \langle \alpha_J, \alpha^k_{\Lambda_4+\Lambda_6+\Lambda_8} \rangle = 0, \forall k \geq 0
\]

since \( J \) has color 3 while \( \Lambda_4 + \Lambda_6 + \Lambda_8 \) has color 0, it follows that these representations generate an abelian group \( \mathbb{Z}_3 \times \mathbb{Z}_3 \). Modulo integers the conformal dimensions of \( \tau_k, \alpha_J \) are given by

\[
h_{\alpha_J} = \frac{4}{3}, h_{\tau_k} = \frac{7}{3}, h_{\alpha_J^2} = \frac{7}{3}, h_{\alpha_J \tau_k} = \frac{11}{3}, h_{\alpha_J^2 \tau_k} = \frac{14}{3}, k = 1, 2.
\]

### 3.1.3 \( \tilde{SU}(8)_4 \)

From conformal inclusion \( Spin(6)_8 \subset Spin(20)_1 \) and \( Spin(6) \simeq SU(4) \) we obtain conformal inclusion \( SU(4)_8 \subset Spin(20)_1 \). For simplicity we use \((0), (5/4)_1, (5/4)_2, (1/2)\) to denote irreducible representations of \( Spin(20)_1 \) with conformal dimensions 0, 5/4, 5/4, 1/2
These representations restrict to $SU(h)$.

The conformal dimensions modulo integers are as follows:

$$\text{conformal dimension} \frac{1}{4}$$ has color 0, it follows that $Z$ is given by

$$\sum Z$$ possibility of

By Lemma 3.2 and Lemma 2.8

$$\mu Z$$ of

abelian group of order 8 under compositions, so the abelian group is $\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 = \mathbb{Z}_8$. By Lemma 2.14 $\langle \alpha_J, (3/4)\rangle = \langle \alpha_J, (1/2)^2 \rangle = 0$, $k = 1, 2, \forall j \geq 0$ since the restriction of $\alpha_J$ to $SU(8)_4$ has color 4 while the restriction of $(3/4)_k, (1/2)$ to $SU(8)_4$ has color 0, it follows that $\mathbb{Z}_8$ is impossible. Note that the conjugate of $(1/2)$ has conformal dimension 1/2, and it must be $(1/2)$, so $[(1/2)^2] = [1]$. To rule out the possibility of $\mathbb{Z}_2 \times \mathbb{Z}_4$, note that this can only happen when the order of $(3/4)_1$ is 4,
and we must have \([(1/2)] = [(3/4)^2_1], [(3/4)_2] = [(3/4)^2_2]\). By monodromy equation we have
\[
\varepsilon((3/4)_1, (3/4)_1)^2 = 1, \varepsilon((3/4)_1, (1/2))\varepsilon((1/2), (3/4)_1) = -1.
\]

On the other hand by Lemma 4.4 of [35] we have
\[
\varepsilon((3/4)_1, (1/2))\varepsilon((1/2), (3/4)_1) = \varepsilon((3/4)_1, (3/4)_1)^2 = \varepsilon((3/4)_1, (3/4)_1)^4 = 1,
\]
a contradiction. It follows that irreducible representations of \(SU(8)_4\) generate \(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\) under compositions, and we have
\[
[(3/4)_1] = [(3/4)_1], [(1/2)(3/4)_1] = [(3/4)_2].
\]

### 3.2 Further extensions by simple currents

#### 3.2.1 No. 40 of [38]

The modular invariant No. 40 in [38] suggests that we look for simple current extensions of \(\tilde{SU}(9)_2 \times SU(5)_1 \times SO(7)_1\). For simplicity we use \(y^i = \Lambda_i, 0 \leq i \leq 9\) to denote the irreducible representation of \(SU(5)_1\). Note that \(h_{y^2} = 3/5\). We use \((1/2), (7/16)\) to denote the irreducible representations of \(SO(7)_1\) with conformal dimensions \(1/2, 7/16\). Note that the index of \((1/2), (7/16)\) are 1, 2 respectively. By §3.1.1 the conformal dimension of \(u = (\alpha_j, y^2, (1/2))\) is \(h_{\alpha_j} + h_y + 1/2 = 2\). It follows that \(w_i, 0 \leq i \leq 9\) is a local system of automorphisms. By Prop. 2.15 there is a Möbius extension \(\mathcal{D} = (SU(9)_2 \times SU(5)_1 \times SO(7)_1) \ltimes \mathbb{Z}_{10}\) of \(SU(9)_2 \times SU(5)_1 \times SO(7)_1\). By Cor. 2.27 and Lemma 3.2 \(\mu_{\mathcal{D}} = 4\). Consider now the inductions for \(\tilde{SU}(9)_2 \times SU(5)_1 \times SO(7)_1 \subset \mathcal{D}\)

By using formulas for conformal dimensions in §3.1.1 one checks easily that
\[
H((\sigma, y^3, (7/16)), u) = H((1, 1, (1/2)), u) = 1.
\]

By Lemma 2.13 we conclude that \(\alpha_{(\sigma, y^3,(7/16))}, \alpha_{(1,1,(1/2))}\) are DHR representations of \(\mathcal{D}\) with index 2, 1 respectively. Note that by Lemma 2.14
\[
\langle \alpha_{(\sigma, y^3,(7/16))}, \alpha_{(\sigma, y^3,(7/16))} \rangle = \sum_{0 \leq i \leq 9} \langle (\sigma, y^3, (7/16)), (\sigma, y^3, (7/16))u^i \rangle = 2
\]
where in the last step we have used \([\sigma y^5] = [1]\). It follows that \([\alpha_{(\sigma, y^3,(7/16))}] = [\delta_1] + [\delta_2]\).

Since \(\mu_{\mathcal{D}} = 4\), the list of irreducible representations are given by
\[
1, \alpha_{(1,1,(1/2))}, \delta_1, \delta_2.
\]

The conformal dimensions modulo integers are \(h_{\delta_1} = h_{\delta_2} = 1, h_{\alpha_{(1,1,(1/2))}} = 1/2\).

These representations generate an abelian group of order 4. To rule out \(\mathbb{Z}_4\), note
that \([\alpha^2_{1,1,(1/2)}] = [1]\). Without losing generality we assume that \(\delta_1\) has order 4. Then we must have \([\delta^2_1] = [\alpha_{1,1,(1/2)}], [\delta^2_1] = [\delta_2]\). By monodromy equation we have \(\varepsilon(\delta_1, \delta_1) = -1, \varepsilon(\delta_1, \delta_2) = \varepsilon(\delta_2, \delta_1) = 1\). On the other hand by Lemma 4.4 of \([35]\) we have \(\varepsilon(\delta_1, \delta_2) = \varepsilon(\delta_2, \delta_1) = \varepsilon(\delta_1, \delta_1) = -1\), a contradiction. In particular we have \([\delta^2_1] = [1]\).

Hence \(\delta_1\) is a local system of automorphisms, and by Prop. 2.15 we conclude that the there is further extension \(D \ltimes \mathbb{Z}_2\) of \(D\). By Lemma 2.8 we have \(\mu_D \neq 1\), i.e., \(D \ltimes \mathbb{Z}_2\) is holomorphic. The spectrum of \(SU(10)_2 \times SU(5)_1 \times Spin(7)_1 \subset D \ltimes \mathbb{Z}_2\) is given by entry 40 in the table of \([38]\):

\[
\sum_{0 \leq i,j \leq 9} \left[ (J^i, y^{2i}, (1/2)^i) \right] + \left[ (J^i(\Lambda_3 + \Lambda_7), y^{2i}, (1/2)^i) \right] + \left[ (J^i(\Lambda_3 + \Lambda_6), y^{2i+4}, (7/16)) \right]
\]

### 3.2.2 No. 27 of \([38]\)

No. 27 in the table of \([38]\) suggests that we look for simple current extensions of \(\widetilde{SU(9)_3} \times SU(3)_1 \times SU(3)_1\). Label irreducible representations of \(SU(3)_1\) by their conformal dimensions as \(1, (1/3)_1, (1/3)_2\). Denote by \(x_1 = (\alpha_J, (1/3)_1, (1/3)_1), x_2 = (\tau_1, (1/3)_1, (1/3)_2)\). By using formulas for conformal dimensions in §3.1.2 and Lemma 2.4 it is easy to check that the following set \(x^i_1x^j_2, 0 \leq i,j \leq 2\) is a local system of automorphisms. Hence by Prop. 2.15 there is a Möbius extension \(D_1 = (\widetilde{SU(9)_3} \times SU(3)_1 \times SU(3)_1) \ltimes (\mathbb{Z}_3 \times \mathbb{Z}_3)\) of \(SU(9)_3 \times SU(3)_1 \times SU(3)_1\) with spectrum \(\sum_{0 \leq i,j \leq 2} [x^i_1x^j_2]\). By Lemma 2.8 \(\mu_{D_1} = 1\), so \(D_1\) is holomorphic. The spectrum of \(SU(9)_3 \times SU(3)_1 \times SU(3)_1 \subset D_1\) is given by (entry (27) of \([38]\)):

\[
\sum_{0 \leq i,j \leq 9} \left[ (J^i, (1/3)^i_1, (1/3)^i_1) \right] + \left[ (J^i(\Lambda_4 + \Lambda_6 + \Lambda_8), (1/3)^{i-1}_1, (1/3)^{i+1}_1) \right] \\
+ \left[ (J^i(\Lambda_4 + \Lambda_6 + \Lambda_8), (1/3)^{i+1}_1, (1/3)^{i-1}_1) \right] + \left[ (J^i(\Lambda_3 + \Lambda_7 + \Lambda_8), (1/3)^i_1, (1/3)^i_1) \right]
\]

**Remark 3.3.** One can choose other local systems of automorphisms which generate \(\mathbb{Z}_3 \times \mathbb{Z}_3\). For an example such choice is a local system of automorphisms given by \(x^i_1x^j_2, 0 \leq i,j \leq 2\) with \(x^i_1 = (\alpha_J, (1/3)_1, (1/3)_2), x^j_2 = (\tau_1, (1/3)_1, (1/3)_2)\). However by remark 2.11 it is easy to check that the corresponding extension is simply \(\text{AdU}(D_1)\) which is isomorphic to \(D_1\), where \(\text{AdU}\) implements the outer automorphism of the last factor of \(SU(3)_1\). Similar statement holds for other choices of local systems of automorphisms which generate \(\mathbb{Z}_3 \times \mathbb{Z}_3\).

### 3.2.3 No. 18 of \([38]\)

No. 18 in the table of \([38]\) suggests that we look for simple current extensions of \(\widetilde{SU(8)_4} \times SU(2)_1 \times SU(2)_1 \times SU(2)_1\). As before we label the non-vacuum representation \((1/4)\) of \(SU(2)_1\) by its conformal dimension. Set \(z_1 = (\alpha_J, (1/4), 0, 0), z_2 = ((3/4)_1, 0, (1/4), 0), z_3 = ((3/4)_2, 0, 0, (1/4))\). Then by the formulas for conformal dimensions and fusion rules in §3.1.3 one checks easily that \(H(z_i, z_j) = 1, 1 \leq i, j \leq 3\). Hence \(\{z_1, z_2, z_3\}\) generate an abelian group \(\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2\) which is a local system of
automorphisms by Lemma 2.4. By Prop. 2.15 we conclude that there is a Möbius extension \( \mathcal{D}_2 := \langle SU(8)_4 \times SU(2)_1 \times SU(2)_1 \times SU(2)_1 \rangle \rtimes (\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2) \). By Lemma 2.8 we have \( \mu_{\mathcal{D}_2} = 1 \), i.e., \( \mathcal{D}_2 \) is holomorphic. The spectrum of \( SU(8)_4 \times SU(2)_1 \times SU(2)_1 \times SU(2)_1 \subset \mathcal{D}_2 \) is given by (entry (18) of [38]):

\[
\sum_{0 \leq i \leq 7} ([J^i(1/4)^i, 0, 0]) + [J^i(\Lambda_0 + \Lambda_4 + \Lambda_5 + \Lambda_7), (1/4)^i, 0, 0]) \\
+ [(J^i(\Lambda_5 + 2\Lambda_7), J^i_1, (1/4), (1/4))] + [(J^i(2\Lambda_0 + \Lambda_3 + \Lambda_5), (1/4)^i, (1/4), (1/4))] \\
+ [(J^i(\Lambda_0 + \Lambda_3 + \Lambda_6 + \Lambda_7), (1/4)^i, 0, (1/4))] + [(J^i(\Lambda_0 + \Lambda_3 + \Lambda_6 + \Lambda_7), (1/4)^i, (1/4), 0))]
\]

### 3.2.4 The main Theorem

By Lemma 2.9 \( \mathcal{D} \ltimes \mathbb{Z}_2, \mathcal{D}_1, \mathcal{D}_2 \) as constructed in §3.2.1, §3.2.2 and §3.2.3 are in fact conformal nets since they contain conformal subnets with finite index, and in summary we have proved the following:

**Theorem 3.4.** There are holomorphic conformal nets (with central charge 24) which are conformal extensions of \( SU(10)_2 \times SU(5)_1 \times Spin(7)_1, SU(9)_3 \times SU(2)_1 \times SU(2)_1, SU(8)_4 \times SU(2)_1 \times SU(2)_1 \times SU(2)_1 \) with spectrum given by the representations at the end of §3.2.1, §3.2.2 and §3.2.3 respectively.

### 3.3 Two conjectures

The holomorphic conformal net corresponding to \( V^\mathbb{Z} \) of [14] was constructed in [24]. This net can also be constructed using the result of [10] as a simple current \( \mathbb{Z}_2 \) extension of a \( \mathbb{Z}_2 \) orbifold conformal net associated with Leech lattice given in [10]. Our first conjecture is an analogue of the conjecture in [14] for \( V^\mathbb{Z} \):

**Conjecture 3.5.** Up to isomorphism there exists a unique holomorphic conformal net with central charge 24 and no elements of weight one.

Our second conjecture is motivated by the results of [38]:

**Conjecture 3.6.** Up to isomorphism there exists finitely many holomorphic conformal nets with central charge 24.

Note that if one can obtain a theorem like the theorem in §2 of [38] in the setting of conformal nets, then modulo conjecture (3.5) conjecture (3.6) is reduced to show that up to equivalence, there are only finitely many conformal extensions of a given completely rational net, and this should be true in view of the results of [20]. However new methods have to be developed to carry through this idea.

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