On the heterochromatic number of hypergraphs associated to geometric graphs and to matroids

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Abstract

The heterochromatic number $h_c(H)$ of a non-empty hypergraph $H$ is the smallest integer $k$ such that for every colouring of the vertices of $H$ with exactly $k$ colours, there is a hyperedge of $H$ all of whose vertices have different colours. We denote by $\nu(H)$ the number of vertices of $H$ and by $\tau(H)$ the size of the smallest set containing at least two vertices of each hyperedge of $H$. For a complete geometric graph $G$ with $n \geq 3$ vertices let $H = H(G)$ be the hypergraph whose vertices are the edges of $G$ and whose hyperedges are the edge sets of plane spanning trees of $G$. We prove that if $G$ has at most one interior vertex, then $h_c(H) = \nu(H) - \tau(H) + 2$. We also show that $h_c(H) = \nu(H) - \tau(H) + 2$ whenever $H$ is a hypergraph with vertex set and hyperedge set given by the ground set and the bases of a matroid, respectively.

Keywords: Heterochromatic; Geometric Graph; Matroid.

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1 Introduction

Let $k$ be a positive integer and $H$ be a hypergraph with at least $k$ vertices. A $k$-colouring of $H$ is an assignment of colours to the vertices of $H$ that uses exactly $k$ colours. Given a colouring $c$ of a hypergraph $H$, a heterochromatic hyperedge is a hyperedge $e$ of $H$ such that $c$ assigns different colours to different vertices of $e$.

The heterochromatic number $h_c(H)$ of a non-empty hypergraph is the smallest integer $k$ such that $H$ contains a heterochromatic hyperedge for each $k$-colouring of $H$. The heterochromatic number was defined by Arocha et al [1] and is closely related to anti-Ramsey numbers [6] and to the upper chromatic number of mixed hypergraphs [8].

A double transversal of hyperedges of a loopless hypergraph $H$ is a set $T$ of vertices of $H$ such that each hyperedge of $H$ contains at least two vertices in $T$. We denote by $\nu(H)$ and $\tau(H)$, the number of vertices and the size of the smallest double transversal of hyperedges of $H$, respectively.

A general lower bound for the heterochromatic number of loopless non-empty hypergraphs is obtained as follows: Consider a double transversal $T$ of hyperedges of $H$ with $|T| = \tau(H)$. Assign colour 1 to every vertex in $T$ and a different colour to each of the remaining $\nu(H) - \tau(H)$ vertices of $H$. Since $T$ is a double transversal of hyperedges of $H$, there are no heterochromatic edges of $H$ for this $(\nu(H) - \tau(H) + 1)$-colouring of $H$ and therefore $h_c(H) \geq \nu(H) - \tau(H) + 2$.

Jiang and West [4] proved that $h_c(H_n) = (n^2 - 1) + 2$ if $H_n$ is the hypergraph whose vertices are the edges of a complete graph $G$ with $n$ vertices and whose hyperedges are the the edge sets of all spanning trees of $G$. Notice that, in this case, $\nu(H_n) = \binom{n}{2}$, $\tau(H_n) = 2n - 3$ and $h_c(H_n) = \nu(H_n) - \tau(H_n) + 2$.

In this article we study hypergraphs $H$ associated to complete geometric graphs and to matroids for which the general lower bound also give the exact value of $h_c(H)$. For a complete geometric graph $G$, we associate a hypergraph $H(G)$ with vertex set given by the edges of $G$ and whose hyperedges are the sets of edges of non-self intersecting spanning trees of $G$. For a matroid $M$ we consider the hypergraph $H(M)$ whose vertices are the elements of the ground set of $M$ and whose hyperedges are the bases of $M$.

We show that $h_c(H) = \nu(H) - \tau(H) + 2$ if $H$ is the hypergraph associated as above to a complete geometric graph with at most one interior vertex or to a matroid.

2 Geometric graphs

Let $P$ be a set of points in general position in the plane, a geometric graph on $P$ is a graph $G$ with vertex set $P$ drawn in such away that each edge is a straight line segment with ends in $P$.

The complement of a geometric graph $G$ is the geometric graph $G^c$ with the same vertex set $P$ whose edges are all the line segments with ends in $P$ which are not edges of $G$. 
A plane spanning tree of a geometric graph \( G \) is a non-selfintersecting subtree that contains every vertex of \( G \).

A tree \( T \) is a caterpillar if the removal of the endpoints of \( T \) leaves a path called the body of \( T \). Let \( P \) be a set of points in general position in the plane. A plane geometric tree \( R \) with vertex set \( P \) is a geometric caterpillar if \( R \) is a caterpillar such that the entire body of \( R \) lies in the boundary of the convex hull \( CH(P) \) of \( P \) and for each leg \( e \) of \( R \), the straight line containing \( e \) does not intersect \( R \) at any point not in \( e \).

The following results, due to Károlyi et al [5] and to Urrutia-Galicia [7], respectively, give sufficient conditions for the complement of a geometric graph to contain a plane spanning tree.

**Theorem 1.** If the edge set of a complete geometric graph \( G \) is partitioned into two sets, then there exists a plane spanning tree of \( G \), all of whose edges lie in the same part.

**Theorem 2.** Let \( R \) be a plane spanning tree of a complete geometric graph \( G \). The complementary geometric graph \( R^c \) contains a plane spanning tree if and only if \( R \) is neither a star or a geometric caterpillar.

For any set \( P \) of points in general position in the plane, we denote by \( i(P) \) the number of points of \( P \) not lying in the boundary of \( CH(P) \).

Lemma 3 (due to Garcia et al [3]) and Lemma 4 provide double transversals of plane spanning trees in complete geometric graphs \( G \) with \( i(V(G)) \leq 1 \).

**Lemma 3.** Let \( P \) be a set of \( n \geq 3 \) points in convex position in the plane. If \( R \) is a plane spanning tree of the complete geometric graph with vertex set \( P \), then at least two edges of \( R \) lie in the boundary of \( CH(P) \).

**Lemma 4.** Let \( P \) be a set of \( n \) points in general position in the plane such that \( i(P) = 1 \) and \( w \) be the unique point in \( P \) not lying in the boundary of \( CH(P) \). Let \( Q \) be the set of edges of the boundary of \( CH(P) \) together with two edges \( uw \) and \( vw \) such that the angle \( \measuredangle uwv \) is maximal. If \( R \) is a plane spanning tree of the complete geometric graph with vertex set \( P \), then at least two edges of \( R \) lie in \( Q \).

**Proof.** Let \( G \) be the complete geometric graph with vertex set \( P \). If \( n = 4 \), then each spanning tree of \( G \) has 3 edges and \( Q \) contains all but one of the edges of \( G \). We proceed by induction assuming \( n \geq 5 \) and that the result holds for every subset \( P' \) of \( P \) with \( i(P') = 1 \).

Let \( R \) be a plane spanning tree of \( G \). Either \( R \) contains two edges \( ux \) and \( vy \) in \( Q \) which are incident in \( u \) and \( v \), respectively, or \( R \) contains an edge \( e \) incident in \( u \) or in \( v \) which is a diagonal of the boundary of \( CH(P) \).

Suppose \( R \) contains such a diagonal edge \( e \). As \( \measuredangle uwv \) is maximal, then \( e \) cannot intersect the edges \( uw \) and \( wv \) in a point other than \( u \) or \( v \). Let \( P^- \) be the set of points in \( P \) lying on or to the left of \( e \) and \( P^+ \) be the set of points in \( P \) lying on or to the right of \( e \), notice that \( w \) is an interior point of \( P^- \) or an
Proof. A heterochromatic plane spanning tree of $G$ is a colouring of the edges of $G$ with exactly $\binom{n}{2} - n + 2$ colours, then $G$ has a heterochromatic plane spanning tree.

By Theorem 1, the subtree $R$ of $R$ with vertex set $P$ contains two edges in $Q^-$ and by induction, the subtree $R^+$ of $R$ with vertex set $P^+$ contains two edges in $Q^+$. Since $Q^- \cup Q^+ = Q$ and $Q^+ \cap Q^+ = \{e\}$, at least two edges of $R$ lie in $Q$.

We can now prove the main results of this section.

**Theorem 5.** Let $G$ be a complete geometric graph with $n \geq 3$ vertices. If $c$ is a colouring of the edges of $G$ with exactly $\binom{n}{2} - n + 2$ colours, then $G$ has a heterochromatic plane spanning tree.

Proof. Let $X$ be a heterochromatic set with $\binom{n}{2} - n + 2$ edges of $G$ and let $Y = E(G) \setminus X$. Since $|Y| = n - 2$, no spanning tree of $G$ has all edges in $Y$. By Theorem 1, graph $G$ has a plane spanning tree $R$ all of whose edges lie in $X = E(G) \setminus Y$. Clearly $R$ is a heterochromatic tree.

**Theorem 6.** Let $G$ be a complete geometric graph with $n \geq 3$ vertices. If $i(V(G)) = 1$ and $c$ is a colouring of the edges of $G$ with exactly $\binom{n}{2} - n + 1$ colours, then $G$ has a heterochromatic plane spanning tree.

Proof. Let $X$ be a heterochromatic set with $\binom{n}{2} - n + 1$ edges of $G$ and let $Y = E(G) \setminus X$. Since $|Y| = n - 2$, either $Y$ is the set of edges of a plane spanning tree of $G$ or no spanning tree of $G$ has all edges in $Y$. As in the proof of Theorem 5, in the later case, $G$ has a heterochromatic plane spanning tree $R$.

Assume $Y$ is the set of edges of a plane spanning tree $S$ of $G$. By Theorem 2, either $S$ is a geometric caterpillar, $S$ is a star or $S^c$ contains a plane spanning tree $R$. In the latter case, $R$ is a heterochromatic plane spanning tree of $G$.

For the case where $S$ is a geometric caterpillar but not a star let $y = uv$ be an edge in the body of $S$ with $d_S(u) \geq 2$ and $d_S(v) \geq 2$. There is an edge $x \in X$ with $c(x) = c(y)$ since each colour is assigned to an edge in $X$.

Let $S' = (S - y) + x$ and notice that $S'$ is not a star by the choice of $y$. Suppose $S'$ is a geometric caterpillar in which case $x$ must lie in the body of $S'$ since $y$ lies in the body of $S$. This implies that both $S$ and $S'$ are paths whose union is the boundary of $CH(P)$ which is not possible since $i(V(G)) = 1$. Therefore $S'$ is neither a star or a geometric caterpillar.

By Theorem 2, the geometric graph $(S')^c$ contains a plane spanning tree, that is a plane spanning tree $R'$ of $G$ all of whose edges lie in $X' = E(G) \setminus E(S') = (X \setminus \{x\}) \cup \{y\}$. Since $X$ is heterochromatic and $c(x) = c(y)$, the set $X'$ and the tree $R'$ are also heterochromatic.

In an analogous way, one may find a heterochromatic plane spanning tree of $G$ in the case where $S$ is a star.
Proof. By Theorem 5, \( h_c(H) \leq \binom{n}{2} - n + 2 \). For the case where \( i(V(G)) = 0 \), Lemma 3 asserts that the boundary of \( CH(P) \) is a double transversal of hyperedges of \( H \). Hence \( \tau(H) \leq n \), and by the general lower bound, \( h_c(H) \geq \nu(H) - \tau(H) + 2 = \binom{n}{2} - n + 2 \).

If \( i(V(G)) = 1 \), by Theorem 6, \( h_c(H) \leq \binom{n}{2} - n + 1 \). By Lemma 4, \( G \) contains a double transversal of plane spanning trees with \( n + 1 \) edges; therefore \( \tau(H) \leq n + 1 \). By the general lower bound, \( h_c(H) \geq \nu(H) - \tau(H) + 2 = \binom{n}{2} - (n + 1) + 2 = \binom{n}{2} - n + 1 \).

\[ \square \]

3 Matroids

For a simple connected graph \( G \) with at least three vertices, we denote by \( \gamma(G) \) the smallest integer \( k \) for which there is a set of \( k \) edges of \( G \) whose removal brakes \( G \) into three connected components.

J. Arocha and V. Neumann-Lara [2] proved that if \( G \) is a simple connected graph with \( m \geq 2 \) edges and \( c \) is a colouring of the edges of \( G \) with exactly \( m - \gamma(G) + 2 \) colours, then \( G \) has an heterochromatic spanning tree. We generalise this result for matroids.

Theorem 8. Let \( M \) be a matroid with \( m \) elements and rank at least 2 and let \( \tau \) denote the size of a smallest double transversal of bases of \( M \). If \( c \) is a colouring of \( M \) with exactly \( m - \tau + 2 \) colours, then \( M \) has a heterochromatic basis.

Proof. Denote by \( E \) the ground set of \( M \) and let \( X \subseteq E \) be a heterochromatic set with \( m - \tau + 2 \) elements. The complementary set \( Y = E \setminus X \) cannot be a double transversal of bases of \( M \) since \( |Y| = |E \setminus X| = \tau - 2 \). Hence, there is a basis \( R \) of \( M \) that meets \( Y \) in at most one element.

If \( R \) is not heterochromatic, then \( R \) contains an element \( x \in X \) such that \( c(x) = c(y) \), where \( y \neq x \) is the unique element in \( R \cap Y \).

Let \( Z = Y \cup \{x\} \), as \( |Z| = |Y| + 1 = \tau - 1 \), set \( Z \) is not a double transversal of bases of \( M \) either. This implies that there is a basis \( S \) of \( M \) that intersects \( Z \) in at most one element.

Assume \( S \) is not heterochromatic in which case \( |S \cap Z| = 1 \) and \( |S \setminus Z| = |S| - 1 \). Since \( |R \setminus \{x, y\}| = |R| - 2 = |S| - 2 = |S \setminus Z| - 1 \), there is an element \( z \in S \setminus Z \subseteq X \) such that \( (R \setminus \{x, y\}) \cup \{z\} \) is an independent set of \( M \). This implies that either \( x \) or \( y \) must lie in the unique circuit contained in \( R \cup \{z\} \) and therefore either \( (R \cup \{z\}) \setminus \{x\} \) or \( (R \cup \{z\}) \setminus \{y\} \) is a basis of \( M \), a heterochromatic basis of \( M \).

\[ \square \]
As an immediate consequence we obtain the following:

**Corollary 9.** Let $M$ be a matroid with rank $r \geq 2$. If $H = H(M)$ is the hypergraph whose vertices and hyperedges are the elements of the ground set and the bases of $M$, respectively, then $h_c(H) = \nu(H) - \tau(H) + 2$.

4 Final remarks and acknowledgements

We conjecture that if $H$ is the hypergraph associated as in section 2 to any complete geometric graph $G$ with $n \geq 3$ vertices in general position, then $h_c(H) = \nu(H) - \tau(H) + 2$. Moreover, we conjecture that if $G$ is a complete geometric graph with $n \geq 3$ vertices in general position, then there is a double transversal of plane spanning trees of $G$ with $n + i(V(G))$ edges and that if $c$ is a colouring of the edges of $G$ with $\binom{n}{2} - (n + i(V(G))) + 2$ colours, then $G$ has a heterochromatic plane spanning tree.

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