Tracking K-essence

Takeshi Chiba

Department of Physics, Kyoto University, Kyoto 606-8502, Japan
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Abstract

We derive a condition for converging a common evolutionary track for k-essence (a scalar field dark energy with non-canonical kinetic terms). For the Lagrangian density $V(\phi)W(X)$ with $X = \dot{\phi}^2/2$, we find tracker solutions with $w_\phi < w_B$ exist if $\Gamma \equiv V''V/(V')^2 > 3/2$. Here $w_\phi(w_B)$ is the equation-of-state of the scalar field (background radiation/matter). Our condition may be useful for examining the existence of the attractor-like behavior in cosmology with k-essence (for example, rolling tachyon).

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I. INTRODUCTION

Scalar fields play important role in cosmology. For example, inflation in the early universe may be caused by inflatons, or an ultra-light scalar field with mass $\sim 10^{-33}$ eV (or quintessence) may cause the universe to accelerate recently.

In the case of inflation models, the conditions for existence of inflationary solutions are conveniently described in terms of the slow-roll parameters without solving the equation of motion directly: $(V'/V)^2/2\kappa^2 < 1$ and $V''/\kappa^2V < 1$ with $\kappa^2 = 8\pi G$. Similarly, some quintessence models admit so called tracker fields which have attractor-like solutions in the sense that a very wide range of initial conditions rapidly converge to a common cosmic evolutionary track. It may be particularly useful to express the condition for the existence of tracking solutions in terms of a simple condition of $V(\phi)$ without having to solve the equation of motion directly. For quintessence, the condition for the existence of tracker solutions with $w_\phi < w_B$ is $\Gamma = V''V/(V')^2 > 1$. Here $w_\phi(w_B)$ is the equation-of-state of quintessence (background radiation or matter).

Usually the quintessence field (or inflaton) is modeled by a scalar field with a canonical kinetic term and a potential term. However, as shown in [3,4], a scalar field with solely kinetic terms can (even without potential terms), albeit they are non-canonical, mimic such a (canonical) quintessence/inflaton field. However, the condition for the existence of tracker solutions for k-essence is not known to date.

In view of recent interest in scalar field cosmology with non-canonical kinetic terms [6–8], in this paper we shall express the condition for tracker solutions in terms of a function of the Lagrangian density.

II. TRACKING K-ESSENCE

A. Basics of K-essence

The action of K-essence minimally coupled with gravity is

$$S = \int d^4x \sqrt{-g} \left( \frac{1}{2\kappa^2} R + p(\phi, X) \right),$$

where $\kappa^2 = 8\pi G$ and $X = -\nabla^\mu \phi \nabla_\mu \phi/2$. The pressure of the scalar field $\phi$, $p_\phi$, is given by $p(\phi, X)$ itself and the energy density $\rho_\phi$ is given by $\rho_\phi = 2X\partial p/\partial X - p$.\[3,4].

The field equations in flat Friedmann-Robertson-Walker spacetime are

$$H^2 := \left( \frac{\dot{a}}{a} \right)^2 = \frac{\kappa^2}{3}(\rho_B + \rho_\phi) = \frac{\kappa^2}{3} \left( \rho_B + 2X\frac{\partial p}{\partial X} - p \right),$$

$$\frac{\ddot{a}}{a} = -\frac{\kappa^2}{6}(\rho_B + 3p_B + \rho_\phi + 3p_\phi) = -\frac{\kappa^2}{6} ((1 + 3w_B)\rho_B + (1 + 3w_\phi)\rho_\phi),$$

1We named such quintessence as “kinetic” quintessence in [3]. However, later the name “k-essence” was coined by the authors of [3] and got popularity. So in this paper we reluctantly use the term k-essence for such kinetically driven quintessence.
\[
\ddot{\phi} \left( \frac{\partial p}{\partial X} + \phi^2 \frac{\partial^2 p}{\partial X^2} \right) + 3H \frac{\partial p}{\partial X} \dot{\phi} + \frac{\partial^2 p}{\partial X \partial \phi} \phi^2 - \frac{\partial p}{\partial \phi} = 0,
\]

where \( \rho_B \) and \( p_B \) are the energy density and the pressure of the background matter and/or radiation, respectively.

In this paper, as a first step toward more general case, we shall derive a tracker condition for a k-essence with the following factorized form of \( p(\phi, X) \):

\[
p(\phi, X) = V(\phi)W(X).
\]

This form of the Lagrangian is suggested for that of tachyon by using the boundary string field theory \([9,10]\). Moreover, any Lagrangian containing only \( \dot{\phi}^2 \) and \( \dot{\phi}^4 \) terms can be recast in the factorized form after field redefinition.

The equation of motion of the scalar field is then written as

\[
\ddot{\phi} (W_X + 2XW_{XX}) + 3HW_X \dot{\phi} + (2XW_X - W) \frac{V'}{V} = 0,
\]

where \( W_X = dW/dX \).

**B. Tracker Equation**

We can express the equation of motion of \( \phi \) in alternative form which may be useful for the following analysis:

\[
\frac{V'}{V^{3/2}} = \pm \frac{\kappa}{2} \sqrt{\frac{(1+w_\phi)W_X}{3\Omega_\phi}} (6 + Ay'),
\]

\[
A = \frac{(XW_X - W)(2XW_{XX} + W)}{XW_X^2 - WW_X - XWW_{XX}} = \frac{1-w_\phi}{c_s^2 - w_\phi},
\]

where \( y = (1+w_\phi)/(1-w_\phi) \) and \( y' = d\ln y/d\ln a \), and minus(plus) sign corresponds to \( \dot{\phi} > 0(<0) \), respectively. \( c_s^2 \) is the speed of sound of k-essence defined by \([11]\)

\[
c_s^2 = \frac{p_X}{p_X + 2Xp_{XX}}.
\]

Note that for quintessence with a canonical kinetic term, \( c_s^2 = 1 \). For a tracker solution \( (w_\phi \simeq \text{const.}) \), we obtain a relation:

\[
\frac{1}{\sqrt{\Omega_\phi}} = \pm \frac{1}{\kappa \sqrt{3(1+w_\phi)W_X}} \frac{V'}{V^{3/2}},
\]

which might be called the k-essential counterpart of the tracker condition \([8]\).

Similar to \([3]\), we define a dimensionless function \( \Gamma \) by \( \Gamma = VV''/V'^2 \). After taking the time derivative of Eq.(\(7\)), we obtain
\[
\Gamma - \frac{3}{2} = -\frac{1}{(1 + w_\phi)(6 + Ay')} \left[ 3(w_\phi - w_B)(1 - \Omega_\phi) + \frac{(1 - w_\phi)^2}{2(c_s^2 - w_\phi)} y' \right. \\
\left. + \frac{2(1 - w_\phi)(c_s^2 - w_\phi)y'' + 2(w_\phi(1 - c_s^2) - (c_s^2)(1 - w_\phi)) y'/H}{(6 + Ay')(c_s^2 - w_\phi)^2} \right],
\]

where \( y'' = d^2 \ln y / d \ln a^2 \). Eq. (11) might be called the k-essential counterpart of the tracker equation. Therefore for the tracker solution (assuming \( \Gamma \simeq \text{const.} \) and \( \Omega_\phi \ll 1 \)) we can write \( w_\phi \) in terms of \( \Gamma \):

\[
w_\phi = \frac{w_B - 2(\Gamma - 3/2)}{2(\Gamma - 3/2) + 1} \simeq \text{const.}
\]

C. Tracking Condition

**Convergence toward the tracker solution.** We now examine the stability of the tracker solution. Consider a solution which is perturbed from the tracker solution with \( w_0 \) (Eq. (12)) by an amount \( \delta \), then the tracker equation Eq. (11) is expanded to lowest order in \( \delta \) to obtain

\[
2\delta'' + 3(1 + w_B - 2w_0)\delta' + 9(c_s^2 - w_0)(1 + w_B)\delta = 0,
\]

where the prime means \( d/d \ln a \). The solution of this equation is

\[
\delta \propto a^\gamma
\]

\[
\gamma = \frac{3}{4}(1 + w_B - 2w_0) \pm \frac{3}{4}\sqrt{1 + w_B - 2w_0}^2 - 8(c_s^2 - w_0)(1 + w_B).
\]

In order for the real part of \( \gamma \) to be negative so that \( \delta \) decays exponentially and the solution approaches the tracker solution, it is required that

\[
w_0 < \frac{1 + w_B}{2} \quad \text{and} \quad w_0 < c_s^2,
\]

where \( c_s^2 \geq 0 \) is assumed for stability against perturbation. Note that for the canonical quintessence (\( c_s^2 = 1 \)) the second requirement in Eq. (16) is automatically satisfied.

From Eq. (12), the above requirements are written in terms of \( \Gamma \):

\[
\Gamma > \frac{3}{2} - \frac{c_s^2 - w_B}{2(c_s^2 + 1)} \quad \text{and} \quad \Gamma > \frac{3}{2} - \frac{1 - w_B}{6 + 2w_B}
\]

or \( \Gamma < 1 \).

\footnote{This assumption is implicit in \([2]\).}
Tracking behavior with $w_\phi < w_B$. In this case, $\Omega_\phi$ increases with increase of time. Then according to Eq.(10), $|V'/V^{3/2}\sqrt{W_X}|$ decreases for a tracker solution. On the other hand, using the equation of motion, we find that

$$
\left(\frac{V'}{V^{3/2}\sqrt{W_X}}\right) = \frac{V'\dot{\phi}}{V^{5/2}\sqrt{W_X}} \left(\Gamma - \frac{3}{2}\right).
$$

Hence, $|V'/V^{3/2}\sqrt{W_X}|$ decreases if $\Gamma > 3/2$. Therefore, tracking behavior with $w_\phi < w_B$ occurs if $\Gamma > 3/2$ and nearly constant. From the stability analysis, this tracker solution is stable if $\Gamma > \max(3/2, 3/2 - (c_s^2 - w_B)/2(c_s^2 + 1))$. In particular for $w_B = 0$, the condition is simply $\Gamma > 3/2$.

Tracking behavior with $w_\phi > w_B$. This is possible for $1 < \Gamma < 3/2$ and nearly constant. However, from the stability analysis, we find that $w_\phi < \min((1 + w_B)/2, c_s^2)$ is additionally required, which demands $\Gamma - 3/2 > \max(-c_s^2 - w_B)/2(c_s^2 + 1), -(1 - w_B)/(6 + 2w_B))$.

Tracking behavior with $w_\phi < -1$. This is possible for $\Gamma < 1$ and nearly constant. From the stability analysis, we find that this solution is stable.

D. Examples

As an example, we consider the universe consisting of matter/radiation and a k-essential scalar field with a power-law model, $V \propto \phi^{-\alpha}$, studied in \[4\]. Since $\Gamma = (\alpha + 1)/\alpha$, if $0 < \alpha < 2$, then tracking behavior with $w_\phi < w_B$ occurs, while if $\alpha < 0$, tracking behavior with $w_\phi < -1$ occurs. From Eq.(12), in this case we have for the tracker solution

$$
w_\phi = (1 + w_B)^{\frac{\alpha}{2}} - 1,
$$

which indeed coincides with the equation-of-state of the attractor solution studied in \[4\] and indeed the solution is stable \[4\].

Another example is the universe consisting of matter/radiation and the rolling tachyon with $W(X) = -\sqrt{1 - 2X}$. The exact classical potential of it has been computed in \[4\] and is given by

$$
V(T) = V_0 (1 + T/T_0) \exp(-T/T_0),
$$

where $T$ is the tachyon field and $V_0$ is the tension of some unstable bosonic D-brane and $T_0$ is a constant of the order of the string scale. In this case we have

$$
\Gamma = 1 - (T/T_0)^{-2}
$$

and $\Gamma$ becomes nearly constant if $T/T_0 \gg 1$. In the limit of large $T/T_0$, we have $\Gamma = 1$ and obtain $w_\phi = -[w_B(1 - \Omega_\phi) + 1]/\Omega_\phi$ if we do not neglect $\Omega_\phi$ in Eq.(12). Since the rolling tachyon respects the weak energy condition, $w_\phi \geq -1$, this implies $\Omega_\phi \geq 1$ during the radiation dominated epoch (the same result holds for the matter dominated epoch), which would be incompatible with the success of the Big-Bang Nucleosynthesis. Thus the above potential does not admit viable tracker solutions, which implies the need of fine-tuning to account for $\Omega_\phi \sim 1$ today in agreement with the recent analysis \[3\].
III. SUMMARY

We have derived a (sufficient) condition for the existence of tracker solutions for the system of matter/radiation and a scalar field with Lagrangian density of the form Eq.(5). Our results are summarized as follows:

- Tracking behavior with $w_\phi < w_B$ occurs if $\Gamma > \max(3/2, 3/2 + (w_B - c_s^2)/2(c_s^2 + 1))$ and nearly constant. In particular for $w_B = 0$, the condition is simply $\Gamma > 3/2$ and nearly constant.

- Tracking behavior with $w_\phi > w_B$ occurs if $3/2 > \Gamma > 3/2 + \max(-(c_s^2 - w_B)/2(c_s^2 + 1), -(1 - w_B)/(6 + 2w_B))$ and nearly constant.

- Tracking behavior with $w_\phi < -1$ occurs if $\Gamma < 1$ and nearly constant.

Interestingly, thanks to the factorizable ansatz Eq.(5), the conditions are similar to those of canonical quintessence.

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