EXISTENCE AND ASYMPTOTIC BEHAVIOR OF STANDING WAVES OF THE NONLINEAR HELMHOLTZ EQUATION IN THE PLANE

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Abstract. In this paper we study the semilinear elliptic problem
\[- \Delta u - k^2 u = Q |u|^{p-2} u \quad \text{in } \mathbb{R}^2,\]
where \(k > 0\), \(p \geq 6\) and \(Q\) is a bounded function. We prove the existence of real-valued \(W^{2,p}\)-solutions, both for decaying and for periodic coefficient \(Q\). In addition, a nonlinear far-field relation is derived for these solutions.

1. Introduction and main results

The purpose of this article is to study the existence and the properties of real-valued solutions of the semilinear problem
\[(1) \quad - \Delta u + \lambda u = Q |u|^{p-2} u \quad \text{in } \mathbb{R}^2\]
that vanish at infinity, in the case where \(p > 2\) and \(Q: \mathbb{R}^2 \to \mathbb{R}\) is a bounded function. For \(\lambda \geq 0\), the problem (1) in \(\mathbb{R}^N\) with such superlinear nonlinearities has received a great deal of attention, starting with the celebrated papers by Berestycki and Lions [6, 7] on the case \(N \geq 3\) and by Berestycki, Gallouët and Kavian [5] for the case \(N = 2\). We refer the reader to the monographs [3, 19, 22, 24, 26] and the references therein for a detailed account on the study of such equations. In contrast, much less is known about the case \(\lambda < 0\), due in particular to the fact that the usual variational method in \(H^1(\mathbb{R}^N)\) breaks down, since the solutions of (1), if any, will not decay faster than \(O(|x|^{-\frac{N}{2}})\) as \(|x| \to \infty\) (see [17]). Recent results obtained by T. Weth and the author [11] confirmed nevertheless the existence for \(\lambda < 0\) of nontrivial \(W^{2,p}(\mathbb{R}^N)\)-solutions for the problem (1) in \(\mathbb{R}^N\) with \(N \geq 3\). In a previous paper [10], existence results for (1) in all dimensions \(N \geq 2\) and for more general nonlinearities were obtained by studying a Dirichlet-to-Neumann boundary-value problem, but only nonlinearities having compact support were considered. Let us also mention results concerning complex-valued solutions of (1) with prescribed asymptotic behavior, obtained using contraction mapping arguments, by Gutiérrez [15] in dimension \(N = 3, 4\) and with \(p = 4\), and by Jalade [16] in dimension \(N = 3\) for more general, compactly supported nonlinearities.

Our present goal is to extend the results of [11] to the two-dimensional case and, at the same time, to provide a basis for further study of the planar nonlinear Helmholtz equation. Without loss of generality, we shall focus on the case \(\lambda = -1\) and therefore deal with the problem
\[(2) \quad - \Delta u - u = Q |u|^{p-2} u \quad \text{in } \mathbb{R}^2.\]
As in [11], we shall reformulate (2) as an integral equation, involving the resolvent operator \(\mathcal{R}\) associated to the inhomogeneous Helmholtz equation
\[-\Delta u - u = f \quad \text{in } \mathbb{R}^2\]
and the outgoing radiation condition, which in two dimensions reads as:
\[(3) \quad \nabla u(x) \cdot \frac{x}{|x|} - iu(x) = o(|x|^{-\frac{1}{2}}), \quad \text{as } |x| \to \infty\]
(see [8] Chap. 3.4]). More precisely, we shall look for solutions of (2) that satisfy the fixed-point equation
\[(4) \quad u = \mathcal{R} \left( Q |u|^{p-2} u \right), \quad u \in L^p(\mathbb{R}^2).\]
Here, \(\mathcal{R}\) denotes the real part of the operator \(\mathcal{R}\). For more details concerning the link between (1) and (2) we refer the reader to the introduction of [11].

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Our first main result deals with the regularity and the asymptotic behavior of solutions of the nonlinear problem \( (1) \). There, and in the sequel, \( \mathcal{F} \) denotes the Fourier transform on the space of tempered distributions.

**Theorem 1.1.** Let \( 6 \leq p < \infty \), \( Q \in L^\infty(\mathbb{R}^2) \) and consider a solution \( u \) of \( (1) \). Then, \( u \in W^{2,q}(\mathbb{R}^2) \) for all \( 6 \leq q < \infty \) and it is a strong solution of \( (2) \).

Moreover, if \( p > 6 \), we have \( u \in W^{2,q}(\mathbb{R}^2) \) for all \( 4 < q < \infty \) and

\[
(5) \quad u(x) = \frac{\pi}{2} |x|^{-\frac{1}{2}} \text{Re}[e^{i|x|+\frac{\pi}{4}f_u(|x|)}] + o(|x|^{-\frac{1}{2}}), \quad \text{as } |x| \to \infty,
\]

where \( f_u(x) = \mathcal{F}[Q|u|^{p-2}u](x) \) for \( x \in \mathbb{R}^2 \) with \( |x| = 1 \).

**Remark 1.2.** As for the Helmholtz equation in dimension 3 (see [11, p. 69]), the pointwise expansion \( (5) \) is satisfied for all noncritical exponents \( p \in (6, \infty) \). In the case \( p = 6 \), it holds for radial solutions, under additional assumptions on the function \( Q \). Indeed, assuming \( Q \) to be \( C^1 \), radially symmetric and radially decreasing, we find by [11, Theorem 4], that every radial solution of \( (1) \) satisfies \( |u(x)| \leq C|x|^{-\frac{1}{2}} \).

From Proposition 2.2 below, we then obtain \( (5) \). In general, however, only the following weaker form of \( (5) \) holds (cf. [11, Lemma 4.3]):

\[
\lim_{R \to \infty} \frac{1}{R} \int_{B_R(0)} \left| u(x) - \frac{\pi}{2} |x|^{-\frac{1}{2}} \text{Re}[e^{i|x|+\frac{\pi}{4}f_u(|x|)}] \right|^2 \, dx = 0.
\]

Our second main result concerns the existence of solutions for \( (1) \), and hence for \( (2) \), under two different assumptions on the nonnegative function \( Q \).

**Theorem 1.3.** For \( 6 \leq p < \infty \) and \( Q \in L^\infty(\mathbb{R}^2) \setminus \{ 0 \}, Q \geq 0 \), the following holds.

(a) If \( Q(x) \to 0 \) as \( |x| \to \infty \), the problem \( (1) \) admits a sequence of pairs of solutions \( \pm u_n \in W^{2,q}(\mathbb{R}^2) \), for all \( 6 \leq q < \infty \) if \( p = 6 \) and all \( 4 < q < \infty \) if \( p > 6 \), such that \( \|u_n\|_p \to \infty \) as \( n \to \infty \).

(b) If \( Q \) is \( \mathbb{Z}^2 \)-periodic and \( p > 6 \), then \( (1) \) has a nontrivial solution pair \( \pm u \in W^{2,q}(\mathbb{R}^2) \) for all \( 4 < q < \infty \).

**Remark 1.4.** Theorem 1.2 and 1.3 can be extended to more general nonlinearities, like those studied in [7] for \( N \geq 3 \). Also, replacing the assumption \( Q(x) \to 0 \) as \( |x| \to \infty \) by \( Q \in L^2(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \) in Theorem 1.3(a), one can prove that for every \( p \in (2, \infty) \) the problem \( (1) \) has infinitely many pairs of solutions \( \{ \pm u_n \} \) such that \( Q^+ u_n \in L^p(\mathbb{R}^2) \) for all \( n \) and \( \int_{\mathbb{R}^2} Q|u_n|^p \, dx \to \infty \) as \( n \to \infty \).

The proof of the above results is based on the method developed in the recent paper [11], but we emphasize that these results do not follow from their higher-dimensional counterparts. Indeed, the presence of a logarithmic singularity at 0 in the kernel of the resolvent operator in \( \mathbb{R}^2 \) (cf. [8]) requires new estimates, different from those obtained in [11], and which we believe to be also of independent interest.

The paper is organized as follows. In the next section, we define the resolvent Helmholtz operator \( \mathcal{R} \) and derive \( L^p \)-estimates similar to [15,18]. Next, the asymptotic expansion and the decay of solutions of linear equations are studied and the section concludes with the proof of Theorem 1.1. Section 3 is devoted to the existence proof for solutions of \( (1) \). It starts with the extension of the dual variational method of [11] to \( \mathbb{R}^2 \) and continues with the proof of Theorem 1.3(a), as an application of the symmetric Mountain Pass Theorem. There, an interaction estimate, more involved than in the case \( N \geq 3 \) is used to construct finite-dimensional subspaces of arbitrary dimension on which the quadratic part of the energy functional is positive. Next, the periodic case is studied and a nonvanishing property for the quadratic form associated to the resolvent \( \mathcal{R} \) is derived. As in [11], it constitutes a key ingredient in the proof of the existence of solutions in the periodic case, by which the paper concludes.

2. The planar resolvent and the far-field relation

For a function \( f \) in the Schwartz space \( \mathcal{S}(\mathbb{R}^2) \), the unique solution of the Helmholtz equation \( -\Delta u - u = f \) in \( \mathbb{R}^2 \) which satisfies the radiation condition \( (3) \) is given by the convolution \( u = \Phi * f \), where

\[
\Phi(x) = \frac{i}{4} \mathcal{F}^{-1}_0(|x|), \quad x \in \mathbb{R}^2.
\]
Here, $H^{(1)}_0$ denotes the Hankel function of the first kind of order 0 (see e.g., [8] Chap. 3.4). In view of the asymptotic behavior of $\Phi$ given by

$$
\Phi(x) = \begin{cases} 
\frac{1}{2\sqrt{2\pi}} |x|^{-\frac{3}{2}} e^{i|x|^{1+\frac{1}{2}}} [1 + O(|x|^{-1})] & \text{as } |x| \to \infty, \\
\frac{1}{2\pi} \log \left(\frac{b}{|x|}\right) \left[1 + O\left(\frac{1}{\log |x|}\right)\right] & \text{as } x \to 0,
\end{cases}
$$

(see, e.g., [20] Eq. (5.16.3)), there is a constant $C_0 > 0$ such that

$$
|\Phi(x)| \leq C_0 \min\{1 + |\log |x||, |x|^{-\frac{3}{2}}\} \quad \text{for } x \in \mathbb{R}^2 \setminus \{0\}.
$$

Following ideas of Kenig-Ruiz and Sogge [18, Theorem 2.3] and Gutiérrez [15, Theorem 6], we prove estimates which show that the resolvent Helmholtz operator $f \mapsto \Phi \ast f$, defined for $f \in \mathcal{S}(\mathbb{R}^2)$, has for certain $1 \leq t, q \leq \infty$ a continuous extension $\mathcal{R} : L^t(\mathbb{R}^2) \to L^q(\mathbb{R}^2)$. In the following, given $1 \leq p \leq \infty$, we let $p'$ denote its conjugate exponent.

**Theorem 2.1.** Let $1 \leq t < \frac{4}{q}$ and $4 \leq q \leq \infty$ satisfy $\frac{2}{t} \leq \frac{1}{t} - \frac{1}{q} < 1$. There is a constant $C = C(t, q) > 0$ such that

$$
||\mathcal{R}f||_q \leq C||f||_t \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^2).
$$

In particular, if $6 \leq q < \infty$ then (8) holds with $t = q'$.

In order to prove the above estimates, we consider a decomposition of the fundamental solution $\Phi$ which we shall use also further below (see Theorem 5.1).

Fix $\psi \in \mathcal{S}(\mathbb{R}^2)$ such that the Fourier transform $\hat{\psi} \in C_c^\infty(\mathbb{R}^2)$ is radial, $0 \preceq \hat{\psi} \leq 1$, $\hat{\psi}(\xi) = 1$ for $||\xi|| - 1 \leq \frac{1}{4}$ and $\hat{\psi}(\xi) = 0$ for $||\xi|| - 1 \geq \frac{1}{2}$. Write $\Phi = \Phi_1 + \Phi_2$, where

$$
\Phi_1 := 2\pi(\psi \ast \Phi), \quad \text{and} \quad \Phi_2 = \Phi - \Phi_1.
$$

Since $\psi$ is a Schwartz function, we obtain from (7), making $C_0$ larger if necessary,

$$
||\Phi_1(x)|| \leq C_0 \min\{1 + |\log |x||, |x|^{-\frac{3}{2}}\}, \quad x \in \mathbb{R}^2.
$$

On the other hand, since $\mathcal{F}(\Phi)(\xi) = \frac{1}{(1-i\xi)^{-1}}$ as a tempered distribution (see [12]) and since $\mathcal{F}(\Phi_2) = (1 - \hat{\psi}) \mathcal{F}(\Phi)$, it follows that $\mathcal{F}(\Phi_2) \in C^\infty(\mathbb{R}^2)$ and $\mathcal{F}(\Phi_2)(\xi) = (||\xi||^2 - 1)^{-1}$ for $||\xi|| \geq \frac{1}{2}$. Consequently, $\partial^\gamma_\xi \mathcal{F}(\Phi_2) \in L^1(\mathbb{R}^2)$ for all $\gamma \in \mathbb{N}^2$ such that $||\gamma|| \geq 1$ and this gives $||\Phi_2(x)|| \leq \kappa_s ||x||^{-s}$ for all $s > 0$, with some constant $\kappa_s > 0$. Using also (7) and (10), we obtain, making again $C_0$ larger,

$$
||\Phi_2(x)|| \leq C_0 \min\{1 + |\log |x||, |x|^{-\frac{3}{2}}\}, \quad x \in \mathbb{R}^2 \setminus \{0\}.
$$

**Proof of Theorem 2.1.** The proof is inspired by Theorem 6 in [15]. In the sequel, $C$ will denote a constant, whose value may change from line to line.

Using (11) we see that $\Phi_2 \in L^t(\mathbb{R}^2)$ for all $1 \leq t < \infty$, and therefore Young’s inequality gives for $1 \leq t, q \leq \infty$ such that $0 \leq \frac{1}{t} - \frac{1}{q} < 1$,

$$
||\Phi_2 \ast f||_q \leq ||\Phi_2||_t ||f||_t \leq C||f||_t \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^2).
$$

To estimate the convolution with $\Phi_1$, let us fix a radial, nonnegative function $\eta \in C_c^\infty(\mathbb{R}^2)$ such that $\eta(x) = 1$ if $0 \leq |x| \leq 1$, $\eta(x) = 0$ if $|x| \geq 2$. For $j \in \mathbb{N}$, define $\varphi_j \in C_c^\infty(\mathbb{R}^2)$ by $\varphi_j(x) = \eta(x/2^j) - \eta(x/2^{j-1})$. Let also $\varphi_0 = \eta$. We then have the dyadic decomposition

$$
\Phi_1 = \sum_{j=0}^\infty \Phi_1^j \quad \text{with } \Phi_1^j := \Phi_1 \varphi_j \quad \text{for } j \in \mathbb{N} \cup \{0\}.
$$

Choosing also $\varphi \in \mathcal{S}(\mathbb{R}^2)$ such that its Fourier transform $\hat{\varphi} \in C_c^\infty(\mathbb{R}^2)$ is radial, nonnegative and satisfies $\hat{\varphi}(\xi) = 1$ on $\{\xi : ||\xi|| - 1 \leq \frac{1}{2}\}$ and $\hat{\varphi}(\xi) = 0$ on $\{\xi : ||\xi|| - 1 \geq \frac{3}{4}\}$, we see that $(\Phi_1 \ast \varphi) \ast f = 2\pi \Phi_1 \ast f$ for all $f \in \mathcal{S}(\mathbb{R}^2)$, since supp $\mathcal{F}(\Phi_1) \subset \{\xi : ||\xi|| - 1 \leq \frac{1}{2}\}$. Hence, we look at the decomposition

$$
\Phi_1 \ast \varphi = \sum_{j=0}^\infty Q_j \quad \text{with } Q_j := \Phi_1^j \ast \varphi \quad \text{for } j \in \mathbb{N} \cup \{0\}.
$$
From the decay properties of $\Phi_1$, we see that
\begin{equation}
||Q^j||_{\infty} \leq ||\varphi||_1 ||\Phi_1^j||_{\infty} \leq C 2^{-j/4}
\end{equation}
for all $j \geq 1$,
where $C$ is independent of $j$. On the other hand, Plancherel’s identity and the Stein-Tomas Theorem \[23\] imply for $1 \leq t \leq \frac{6}{7}$ and $f \in \mathcal{S}(\mathbb{R}^2)$,
\begin{align*}
||Q^j * f||_2^2 &= (2\pi)^2 \int_{|\xi| \leq \frac{1}{4}} |\hat{\Phi}_1^j(\xi)|^2 \, d\xi \\
&\leq C ||\varphi||_1^2 ||f||_2^2 \int_{\mathbb{R}^2} |\hat{\Phi}_1^j(x)|^2 \, dx \\
&\leq C ||\varphi||_1^2 ||f||_2^2
\end{align*}
(15)
where we have set $g = \varphi * f$, and $C$ does not depend on $j$. From these two estimates and the Riesz-Thorin theorem \[23\] Theorem V.1.3], it follows that
\[ \|Q^j * f\|_q \leq C 2^{j\left(\frac{2}{q} - \frac{1}{2}\right)} ||f||_r \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^2), \]
1 $\leq t \leq \frac{6}{7}$ and $2 \leq q \leq \frac{4}{3}$. Observe that the exponent is negative if $q > 4$. Since $\Phi_1^j \in L^1(\mathbb{R}^2) \cap L^{\infty}(\mathbb{R}^2)$, we conclude that for $1 \leq t < \frac{6}{7}$ and $4 < q \leq \frac{12}{5}$,
\[ ||\Phi_1 * f||_q = \frac{1}{2\pi} ||(\Phi_1 * \varphi) * f||_q \leq \sum_{j=0}^{\infty} ||Q^j * f||_q \leq C ||f||_r \sum_{j=0}^{\infty} 2^{j\left(\frac{2}{q} - \frac{1}{2}\right)}, \]
and therefore
\begin{equation}
||\Phi_1 * f||_q \leq C ||f||_r.
\end{equation}
(16)
By duality and convexity, this estimate holds for all $1 \leq t < \frac{1}{7}$ and $4 < q \leq \infty$ such that $\frac{1}{7} - \frac{1}{q} > \frac{2}{3}$. Taking into account the estimate (12) for $\Phi_2$, we obtain (13) for all $1 \leq t < \frac{1}{7}$, $4 < q \leq \infty$ such that $\frac{2}{3} < \frac{1}{7} - \frac{1}{q} < 1$.

To conclude the proof, it remains to show that (16) also holds when $\frac{1}{7} - \frac{1}{q} = \frac{2}{3}$. We proceed similarly to \[15\] Theorem 6]. Using real interpolation (see \[23\] Section V.3), it is enough to prove the restricted weak-type estimates
\begin{equation}
||\Phi_1 * f||_{q,\infty} \leq C ||f||_{r,1} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^2)
\end{equation}
(17)
for the endpoints $(t, q) = (\frac{12}{5}, 4)$ and $(\frac{1}{7}, 12)$, where $||\cdot||_{r,s}$ denotes the norm on the Lorentz space $L^{r,s}(\mathbb{R}^2)$. Moreover, by \[23\] Theorem V.3.13 and Theorem V.3.21, it suffices to prove (17) for characteristic functions of measurable sets:
\begin{equation}
\lambda \{ x \in \mathbb{R}^2 : |(\Phi_1 * 1_E)(x)| > \lambda \} \leq C |E|^{\frac{1}{7}} \quad \text{for all } \lambda > 0, |E| < \infty.
\end{equation}
(18)
Here, $|E|$ denotes the measure of $E \subset \mathbb{R}^2$ and $1_E$ its characteristic function. Setting $A := \{ x \in \mathbb{R}^2 : |(\Phi_1 * 1_E)(x)| > \lambda \}$, choosing $\varphi \in \mathcal{S}(\mathbb{R}^2)$ as above and recalling the dyadic decomposition (13), we can write
\[ |A| = \int_{\mathbb{R}^2} dx \leq \frac{1}{\lambda} \int_{\mathbb{R}^2} |(\Phi_1 * 1_E)(x)| 1_A(x) \, dx \leq \frac{1}{\lambda} \sum_{j=0}^{\infty} \int_{\mathbb{R}^2} |(Q^j * 1_E)(x)| 1_A(x) \, dx. \]
From the estimate (15) with $t = \frac{6}{7}$ and by duality, we obtain
\begin{equation}
||Q^j * f||_2 \leq C 2^{\frac{j}{12}} ||f||_4 \quad \text{and} \quad ||Q^j * f||_6 \leq C 2^{\frac{j}{12}} ||f||_2
\end{equation}
(19)
for all $f \in \mathcal{S}(\mathbb{R}^2)$, the constant $C$ being independent of $j$. Moreover, (13) gives
\begin{equation}
||Q^j * f||_{\infty} \leq C 2^{-\frac{j}{12}} ||f||_1, \quad j \geq 1.
\end{equation}
(20)
For $M \in \mathbb{N}_0$, we therefore obtain, using Hölder’s inequality and approximating $1_E$ by Schwartz functions,
\begin{align*}
\sum_{j=0}^{M} \int_{\mathbb{R}^2} |Q^j * 1_E(x)| 1_A(x) \, dx &\leq \sum_{j=0}^{M} ||Q^j * 1_E||_2 ||1_A||_2^j + \sum_{j=M+1}^{\infty} ||Q^j * 1_E||_{\infty} ||A||_1 \\
&\leq C \left\{ 2^{\frac{j}{12}} |E|^{\frac{1}{7}} ||1_A||_2^j + 2^{-\frac{j}{12}} |E| ||A||_1 \right\}.
\end{align*}
Choosing $M \in \mathbb{N}_0$ with $2^{\frac{M + 1}{2}} \leq |E|^{\frac{1}{2+}}|A|^{\frac{1}{2+}}$, the preceding estimates give

$$\lambda |A| \leq \sum_{j=0}^{\infty} \int_{\mathbb{R}^2} |(Q^j \ast 1_E)(x)|1_A(x) \, dx \leq C|E|^{\frac{1}{2+}}|A|^{\frac{1}{2+}},$$

which yields $\lambda |A|^{\frac{1}{2+}} \leq C|E|^{\frac{1}{2+}}$ and shows (15) for the exponents $t = \frac{1}{10}$, $q = 4$.

Similarly, Hölder’s inequality and the estimates (13), (14) give

$$\sum_{j=0}^{\infty} \int_{\mathbb{R}^2} |(Q^j \ast 1_E)(x)|1_A(x) \, dx \leq \sum_{j=0}^{M} \|Q^j \ast 1_E\|_6|A|^{\frac{3}{2+}} + \sum_{j=M+1}^{\infty} \|Q^j \ast 1_E\|_{\infty} |A|^{\frac{1}{2+}} \leq C \left\{ \frac{2^{\frac{M}{2+}}}{2^{\frac{M+1}{2+}}} |A|^{\frac{1}{2+}} + 2^{-\frac{M+1}{2+}} |A| \right\}.$$  

Choosing this time $M \in \mathbb{N}_0$ such that $2^{\frac{M}{2+}} \leq |E|^{\frac{1}{2+}}|A|^{\frac{1}{2+}} \leq 2^{\frac{M+1}{2+}}$, we find that $\lambda |A|^{\frac{1}{2+}} \leq C|E|^{\frac{1}{2+}}$, proving (15) for the exponents $p = \frac{1}{2}$, $q = 12$. The proof is complete.  

We note that the conclusion of Theorem 2.1 is false for $(t, q) = (1, \infty)$. Indeed, if $f \in C_c^\infty(\mathbb{R}^2) \setminus \{0\}$ is nonnegative and $f(x) = 0$ for $|x| \geq 1$, consider the sequence $(f_k)_k$, where $f_k(x) = k^2 f(kx)$, $x \in \mathbb{R}^2$. Then, $\|f_k\|_1 = \|f\|_1$ for all $k$, but for every $x \neq 0$, we find

$$(\Phi \ast f_k)(x) = \int_{B_1(0)} \Phi(x - k^{-1}y) f(y) \, dy \to \Phi(x) \|f\|_1, \quad \text{as } k \to \infty,$$

by the Dominated Convergence Theorem. Hence, $\|\Phi \ast f_k\|_\infty \to \infty$ as $k \to \infty$.

We now turn to the pointwise asymptotic expansion of solutions of the Helmholtz equation and first look at the linear problem $u = \mathcal{R} f$.

**Proposition 2.2.** Let $f \in L^1(\mathbb{R}^2)$ satisfy $|f(x)| \leq \kappa |x|^{-2-\varepsilon}$ for some $\kappa, \varepsilon > 0$. Then

$$\mathcal{R} f(x) = \sqrt{\frac{\pi}{2}} \frac{e^{|x| + \frac{2\pi}{3}}}{|x|^{\frac{1}{2}}} \tilde{f}(\frac{2\pi}{3}) + o(|x|^{-\frac{3}{2}}), \quad \text{as } |x| \to \infty.$$  

**Proof.** Consider first for $x \in \mathbb{R}^2$ with $|x| \geq 2$,

$$I_1(x) = \int_{B_1(x)} \Phi(x - y) f(y) \, dy.$$

From (7), we see that $|\Phi(z)| \leq C_0 (1 + |\log |z||)$ for all $|z| \leq 1$, and we can write

$$|I_1(x)| \leq C_0 \kappa \int_{B_1(x)} (1 + |\log |x - y||)|y|^{-2-\varepsilon} \, dy \leq C_0 \kappa \left( \frac{|x|}{2} \right)^{-2-\varepsilon} \int_{B_1(0)} (1 + |\log |y||) \, dy,$$

where the last integral is finite. In particular, $I_1(x) = o(|x|^{-\frac{3}{2}})$ as $|x| \to \infty$. Next, let $A(x) = \{ y \in \mathbb{R}^2 : |x - y| > 1 \text{ and } |y| \geq \sqrt{|x|} \}$ and consider

$$I_2(x) = \int_{A(x)} \Phi(x - y) f(y) \, dy.$$
The estimate (7) implies \(|\Phi(z)| \leq C_0 |z|^{-\frac{1}{2}}\) for all \(|z| > 1\), and therefore

\[
|I_2(x)| \leq C_0 \int_{A(x)} |x - y|^{-\frac{1}{2}} |f(y)| \, dy
\]

\[
\leq C_0 |x|^{-\frac{3}{2}} \int_{A(x)} \left(1 + |x - y|^{-\frac{1}{2}} |y|^{\frac{1}{2}}\right) |f(y)| \, dy
\]

\[
\leq C_0 |x|^{-\frac{3}{2}} \left(\int_{A(x)} |f(y)| \, dy + \kappa \int_{A(x)} |x - y|^{-\frac{1}{2}} |y|^{-\frac{3}{2} - \varepsilon} \, dy\right).
\]

Since \(f \in L^1(\mathbb{R}^2)\), the first integral on the last line goes to zero uniformly as \(|x| \to \infty\). The same is true for the second integral, since \(A(x) \subset \mathbb{R}^2 \setminus B_1(0)\) and since \(-\frac{1}{2} + (-\frac{3}{2} - \varepsilon) < -2\) (see, e.g., [2] Appendix 2, Lemma 1]). Hence, \(I_2(x) = o(|x|^{-\frac{3}{2}})\) as \(|x| \to \infty\). Concerning the remaining integral

\[
I_3(x) = \int_{D(x)} \Phi(x - y) f(y) \, dy,
\]

where \(D(x) = \{ y \in \mathbb{R}^2 : |x - y| > 1 \text{ and } |y| \leq \sqrt{|x|} \}\), we can write using (6),

\[
I_3(x) = \frac{e^{\pi i x}}{2 \sqrt{2\pi}} \int_{D(x)} e^{i|x-y|} \left(1 + \delta(|x - y|)\right)f(y) \, dy,
\]

where \(\sup_{r \geq 1} |\delta(r)| < \infty\). Furthermore, setting \(\tilde{x} := \frac{x}{|x|}\) for \(x \neq 0\), one finds

\[
| |x - y| - |x| + \tilde{x} \cdot y | \leq |x|^{-1} |y|^2 \quad \text{ for all } x, y \in \mathbb{R}^2 \text{ with } x \neq 0 \text{ and } |y| \leq \frac{|x|}{2}.
\]

Arguing as in [11] Proposition 2.8 and using the estimate \(|f(y)| \leq \kappa |y|^{-2 - \varepsilon}\), we obtain

\[
\left| I_3(x) - \frac{1}{2 \sqrt{2\pi}} \int_{D(x)} e^{-i\tilde{x} \cdot y} f(y) \, dy \right| \leq \tilde{k}|x|^{-\frac{3}{2} - \varepsilon},
\]

for some constant \(\tilde{k} > 0\). Putting together the estimates for \(I_1, I_2\) and \(I_3\) and using the integrability of \(f\), we deduce that

\[
\Re f(x) = I_1(x) + I_2(x) + I_3(x) = \sqrt{\frac{\pi}{2}} \frac{e^{i|x| + i\tilde{x} \cdot \hat{x}}}{|x|^{\frac{1}{2}}} f(\hat{x}) + o(|x|^{-\frac{1}{2}}), \quad \text{as } |x| \to \infty,
\]

and this concludes the proof. \(\square\)

The next result gives an upper bound for the decay of solutions of convolution equations involving a kernel with the asymptotic properties of \(\Phi\). Combined with the regularity result below, it will provide a decay bound for solutions of the nonlinear problem (4).

**Lemma 2.3.** Let \(u, V : \mathbb{R}^2 \to \mathbb{R}\) be measurable functions satisfying \(V \in L^q(\mathbb{R}^2), V u \in L^\tilde{q}(\mathbb{R}^2)\), where \(1 < q, \tilde{q} < \frac{2}{1}\). If \(u = K * (V u)\) and

\[
|K(x)| \leq C_0 \min\{1 + |\log |x||, |x|^{-\frac{1}{2}}\} \quad \text{for } x \neq 0,
\]

then there exists a constant \(C > 0\) such that \(|u(x)| \leq C|x|^{-\frac{1}{2}}\) for all \(x \neq 0\).

**Proof.** Let \(B_R := B_R(0)\) and \(M_R := \mathbb{R}^2 \setminus B_R\) for \(R > 0\), and define \(\tilde{K}(x) = C_0 \min\{1 + |\log |x||, |x|^{-\frac{1}{2}}\}\) for \(x \neq 0\). Hölder’s inequality, then gives

\[
\int_{M_R} \tilde{K}(x - y)|V(y)||y| \, dy \leq C_0 \int_{M_R} |V(y)| \min\{1 + |\log |x - y||, |x - y|^{-\frac{1}{2}}\} \, dy
\]

\[
\leq C_0 \left(\int_{M_R} |V(y)||y| \, dy \right)^{\frac{1}{2}} \left(\int_{B(1)} (1 + |\log |y||)|y|^{\frac{1}{2}} \, dy + \int_{\mathbb{R}^2 \setminus B(1)} |y|^{-\frac{1}{2}} \, dy\right)^{\frac{1}{2}},
\]

\[
= C_0 \left(\int_{B(1)} (1 + |\log |y||)|y| \, dy + \int_{\mathbb{R}^2 \setminus B(1)} |y|^{-\frac{1}{2}} \, dy\right)^{\frac{1}{2}},
\]
which, as $R \to \infty$, tends to 0 uniformly in $x$, since $4 < q' < \infty$. Hence, we may fix $R > 1$ such that

$$\left(22\right) \sup_{x \in \mathbb{R}^2} \int_{M_R} \tilde{K}(x-y)|V(y)|\,dy < \frac{1}{4}.$$ 

The decay estimate on $u$ will follow with the help of an iteration procedure similar to the one of Zemach and Odeh \cite{27}. For $|x| \geq R$ we set

$$u_0(x) = \int_{B_R} K(x-y)V(y)u(y)\,dy, \quad B_0(x) = \int_{M_R} K(x-y)V(y)u(y)\,dy,$$

and define inductively for $k \geq 1$,

$$u_k(x) = \int_{M_R} K(x-y)V(y)u_{k-1}(y)\,dx, \quad B_k(x) = \int_{M_R} K(x-y)V(y)B_{k-1}(y)\,dx.$$ 

Thus, for each $m \in \mathbb{N}$,

$$u = \sum_{k=0}^{m} u_k, \quad \text{uniformly in } M_R.$$ 

Since $V \in L^{\tilde{q}}(\mathbb{R}^2)$ with $1 < \tilde{q} < \frac{1}{4}$, and $|K(x)| \leq \tilde{K}(x)$ for all $x$, we find that $\beta_0 := \sup_{|x| > R} |B_k(x)| < \infty$. Moreover, setting $\beta_k := \sup_{|x| > R} |B_k(x)|$, \eqref{22} yields $\beta_k \leq \frac{1}{4}\beta_{k-1}$, for all $k \geq 1$, and therefore $\beta_k \to 0$, as $k \to \infty$. This gives

$$u = \sum_{k=0}^{\infty} u_k, \quad \text{uniformly in } M_R.$$ 

Moreover, since $u_0 \in L^\infty(\mathbb{R}^2)$ and $V \in L^1(B_R)$, we have $\mu_0 := \sup_{|x| \geq R} |x|^{\frac{1}{2}}|u_0(x)| < \infty$. Setting, for $k \geq 1$,

$$\mu_k := \sup_{|x| \geq R} |x|^{\frac{1}{2}}|u_k(x)|,$$

and noticing that $\tilde{K}(z) = C_0|z|^{-\frac{1}{2}}$ for all $|z| \geq 1$, we obtain

$$|x|^{\frac{1}{2}}|u_k(x)| \leq \mu_{k-1}|x|^{\frac{1}{2}} \int_{M_R} \tilde{K}(x-y)|V(y)|\,|y|^{-\frac{1}{2}}\,dy$$

$$\leq \mu_{k-1}|x|^{\frac{1}{2}} \left[ C_0 \left( \frac{|x|}{2} \right)^{-\frac{1}{2}} \int_{|y| \geq \max(R, \frac{|x|}{4})} |V(y)|\,|y|^{-\frac{1}{2}}\,dy \right]$$

$$+ \left( \frac{|x|}{2} \right)^{-\frac{1}{2}} \int_{|y| \geq \max(R, \frac{|x|}{4})} \tilde{K}(x-y)|V(y)|\,dy$$

$$\leq 2\sqrt{2}\mu_{k-1} \int_{M_R} \tilde{K}(z-y)|V(y)|\,dy,$$

for all $|x| \geq R$, and from \eqref{22}, we deduce that $\mu_k \leq \frac{\mu_{k-1}}{2\sqrt{2}}$. As a consequence,

$$\sup_{|x| \geq R} |x|^{\frac{1}{2}}|u(x)| \leq \sum_{k=0}^{\infty} \mu_k \leq \mu_0 \sum_{k=0}^{\infty} 2^{-\frac{1}{2}} < \infty,$$

and this concludes the proof. \hfill \square

The last preliminary step towards the proof of Theorem \ref{11} consists in establishing regularity properties for the solutions of \eqref{1}. This will also prove the first part of Theorem \ref{11}.

\textbf{Lemma 2.4.} \textit{Let }$6 \leq p < \infty$, $Q \in L^\infty(\mathbb{R}^2)$ and consider a solution $u \in L^p(\mathbb{R}^2)$ of \eqref{1}. Then $u$ is a strong solution of \eqref{23} and it belongs to $W^{2,q}(\mathbb{R}^2)$ for all $6 \leq q < \infty$, if $p = 6$, and all $4 < q < \infty$, if $p > 6$. \textit{Proof.} Since $Q \in L^\infty(\mathbb{R}^2)$ and $6 \leq p < \infty$, we find that $f := Q|u|^{p-2}u$ belongs to $L^p(\mathbb{R}^2)$ and that $1 < p' < \frac{6}{5}$. Hence, Theorem \ref{24} gives $u \in L^6(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$, and since $p - 1 \geq 5$, this means that $f$ belongs to $L^{\frac{5}{4}}(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2)$. Thus, we obtain from elliptic estimates (see [13] Chapter 9) that $u \in W^{2,q}(\mathbb{R}^2)$ for all $6 \leq q < \infty$. This proves the lemma in the case $p = 6$.

Assuming next $p > 6$, we claim that $u \in L^6(\mathbb{R}^2)$ for all $4 < q \leq \infty$. Supposing for the moment that it is true, we find $f = Q|u|^{p-2}u \in L^1(\mathbb{R}^2)$ for all $\frac{p}{p-1} < t \leq \infty$, where $p - 1 > 1$. Hence, we get from elliptic estimates that $u \in W^{2,q}(\mathbb{R}^2)$ for all $4 < q < \infty$ and the lemma is proved.
We now prove the claim. As a consequence of the first step, we find that \( f \in L^p(\mathbb{R}^2) \cap L^\infty(\mathbb{R}^2) \). Applying Theorem 2.1 with \( t = p' < \frac{6}{5} \), and using the fact that \( u \in L^\infty(\mathbb{R}^2) \), we obtain that \( u \in L^q(\mathbb{R}^2) \) for all \( 4 < q \leq \infty \) such that \( \frac{1}{q} \leq \frac{1}{p'} - \frac{2}{5} \). If \( \frac{1}{p'} - \frac{2}{5} \geq \frac{1}{4} \), the claim follows. Otherwise, we iterate the argument and find that \( f \in L^q(\mathbb{R}^2) \) for all \( 4 \leq t \leq \infty \) such that \( \frac{1}{q} \leq \frac{1}{p'} - \frac{2}{5} \). Let \( \frac{1}{t} = (p-1)\left(\frac{1}{p'} - \frac{2}{5}\right) \) and remark that \( r_1 < p' \), since \( p' < \frac{6}{5} \). Applying Theorem 2.1 with \( t = r_1 \), we obtain \( u \in L^{q_1}(\mathbb{R}^2) \) for all \( 4 < q_1 \leq \infty \) such that \( \frac{1}{q_1} \leq \frac{1}{r_1} - \frac{2}{5} \). Iterating the procedure we find at each step \( u \in L^{q_2}(\mathbb{R}^2) \) for all \( 4 < q_2 \leq \infty \) such that \( \frac{1}{q_2} \leq \frac{1}{r_1} - \frac{2}{5} \), where \( r_m \) is given by

\[
r_0 = p', \quad \frac{1}{r_m} = (p - 1) \left( \frac{1}{r_{m-1}} - \frac{2}{5} \right), \quad m \geq 1.
\]

It satisfies \( \frac{1}{r_m} - \frac{1}{r_{m-1}} = 2(p - 1)^m \left( \frac{1}{p} - \frac{1}{5} \right) > 0 \) and therefore \( \frac{1}{r_m} \to \infty \) as \( m \to \infty \). Thus, after finitely many iterations, we obtain \( \frac{1}{r_m} - \frac{2}{5} \geq \frac{1}{4} \) and the claim follows. \( \square \)

With the help of the above results, we can now give the proof of our first main theorem.

**Proof of Theorem 1.3** As remarked above, the regularity was already proved in Lemma 2.4. Restricting to the case where \( p > 6 \), we see that the functions \( Q|u|^{p-2} \) and \( Q|u|^p - u \) belong to \( L^q(\mathbb{R}^2) \), since \( u \in L^q(\mathbb{R}^2) \) for all \( q > 4 \) by Lemma 2.3. Thus, Lemma 2.3 with \( q = \tilde{q} = \frac{6}{5} \), \( K = \text{Re}(\Phi) \) and \( V = Q|u|^{p-2} \) ensures that \( u(x) = O(|x|^{-\frac{6}{5}}) \) as \( |x| \to \infty \). Therefore, \( f(x) = O(|x|^{-\frac{6}{5}}) \) as \( |x| \to \infty \). Since \( \frac{p-2}{p} > 2 \), the expansion (5) follows from Proposition 2.2 after taking real parts. \( \square \)

### 3. Variational setting in the plane and existence of solutions for the nonlinear problem

In order to prove Theorem 1.3, we extend the dual variational method developed in [11] to the space dimension 2. Let therefore \( Q \in L^\infty(\mathbb{R}^2) \backslash \{0\} \) be a nonnegative function and consider for \( 6 \leq p < \infty \) the energy functional

\[
J : L^p(\mathbb{R}^2) \to \mathbb{R}, \quad J(v) = \frac{1}{p'} \int_{\mathbb{R}^2} |v|^{p'} \, dx - \frac{1}{2} \int_{\mathbb{R}^2} Q(x)|v|^p(\mathbf{R}(Q)\nabla v)(x) \, dx,
\]

In this section, we consider, unless explicitly stated, real-valued functions and denote by \( \mathcal{R} \) the real part of the resolvent operator \( \mathcal{R} \). Obviously, \( J(-v) = J(v) \) for all \( v \in L^p(\mathbb{R}^2) \). Moreover, one can show that \( J \) is of class \( C^4 \) and that every critical point of \( J \) corresponds to a solution of (1) in the following way.

A function \( v \in L^p(\mathbb{R}^2) \) satisfies \( J'(v) = 0 \) if and only if it solves the integral equation \( |v|^{p-2}v = Q \nabla \mathbf{R}(Q)\nabla v \). Setting

\[
u = \mathbf{R}(Q)\nabla \mathbf{R}(Q)\nabla v \in L^p(\mathbb{R}^2),
\]

it follows that \( u = \mathbf{R}(Q)|u|^{p-2}u \), i.e., \( u \) solves (1). Note that, by Lemma 2.4, \( u \) is then a strong solution of (2).

As a consequence of Theorem 2.1, the Birman-Schwinger type operator \( K : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2) \) given by

\[
K_v := Q \nabla \mathbf{R}(Q)\nabla v, \quad v \in L^p(\mathbb{R}^2),
\]

and appearing in the quadratic part of \( J \), is continuous for \( 6 \leq p < \infty \) and has compactness properties which will be important in the sequel. More precisely, the operator \( 1_BK : L^p(\mathbb{R}^2) \to L^p(\mathbb{R}^2) \) is compact for every bounded and measurable set \( B \subset \mathbb{R}^2 \), and if, in addition, \( Q(x) \to 0 \) as \( |x| \to \infty \), then \( K \) itself is compact. Here, \( 1_B \) denotes the characteristic function of the set \( B \subset \mathbb{R}^2 \).

To see this, consider a sequence \( (v_n) \) converging weakly to \( 0 \) in \( L^p(\mathbb{R}^2) \) and choose \( R > 0 \) such that \( B \subset B_R := B_R(0) \). Due to the continuity of the resolvent, there holds \( \mathbf{R}(Q)\nabla v_n \to 0 \) in \( L^p(\mathbb{R}^2) \), and elliptic estimates ensure the boundedness of \( \mathbf{R}(Q)\nabla v_n \) in the Sobolev space \( W^{2,p}(B_R) \). Since \( p < \infty \), the embedding \( W^{2,p}(B_R) \hookrightarrow L^p(B_R) \) is compact, and since \( Q \) is bounded and \( B \subset B_R \), we conclude that \( 1_BKv_n = 1_BQ \nabla \mathbf{R}(Q)\nabla v_n \to 0 \), strongly in \( L^p(\mathbb{R}^2) \), and obtain the first compactness property. In the case where \( Q(x) \to 0 \) as \( |x| \to \infty \), the continuity of \( \mathbf{R} \) and the estimate

\[
\| (1 - 1_B)Kv_n \|_p \leq \text{ess sup}_{|x| \geq R} Q(x) \| \mathbf{R}(Q)\nabla v_n \|_p,
\]

is immediate.
which holds for every \( R > 0 \), ensure the convergence \(||(1 - 1_{B_R})Kv_n||_p \to 0\) as \( R \to \infty \), uniformly in \( n \). On the other hand, as we have already seen, \(||1_{B_R}Kv_n||_p \to 0\) as \( n \to \infty \), for every \( R > 0 \). Combining these two facts yields the strong convergence \( Kv_n \to 0 \) in \( L^p(\mathbb{R}^2) \) and hence the compactness of \( K \).

We start by proving the existence of an unbounded sequence of solutions in the case where \( Q(x) \to 0 \) as \( |x| \to \infty \).

**Proof of Theorem 1.3(a).** The result will follow from the symmetric Mountain Pass Theorem (see \([3]\) for the original work, and \([13]\) for the version we use here) applied to the even functional \( J \).

For this, we first show that \( 0 \) is a strict local minimum or, more precisely, that \( J(v) \geq \delta > 0 \) for all \( ||v|| = \rho \), provided \( \rho > 0 \) is small. Indeed, the operator \( K \) being continuous, there exists a constant \( C > 0 \) such that \( ||Kv||_p \leq C||v||_{p'} \) for all \( v \in L^{p'}(\mathbb{R}^2) \). Hence, if \( ||v||_p = \rho > 0 \), we obtain

\[
J(v) = \frac{1}{p'}\rho^{p'} - \frac{1}{2} \int_{\mathbb{R}^2} vKv \, dx \geq \frac{1}{p'}\rho^{p'} - \frac{C}{2}\rho^2 > 0
\]

for all \( \rho > 0 \) small enough, since \( p' < 2 \).

In the next step, we prove for each integer \( m \) the existence of an \( m \)-dimensional subspace \( \mathcal{W}_m \) of \( L^{p'}(\mathbb{R}^2) \) and of a radius \( R_n > 0 \) with the property that \( J(v) \leq 0 \) for all \( v \in \mathcal{W}_m \) such that \( ||v||_{p'} > R_n \).

Let \( m \) be any integer. Since \( Q \in L^{\infty}(\mathbb{R}^2) \setminus \{0\} \) is nonnegative, there is a point \( x_0 \in \mathbb{R}^2 \) of metric density 1 for the set \( \{Q > 0\} \). Hence, there is \( 0 < \delta < 1 \) such that

\[
|B_\delta(x_0) \cap \{Q > 0\}| \geq \frac{1}{2}|B_\delta(x_0)|.
\]

Since \( \text{Re}(\Phi) \) is bounded outside of every neighborhood of zero, by \( [7] \), and since \( \text{Re}(\Phi(x)) \approx \frac{1}{2\pi} \log\left(\frac{2}{|x|}\right) \to +\infty \) as \( |x| \to 0 \), by \( [9] \), we may also assume that \( \delta > 0 \) satisfies for \( \Psi^*\tau := \inf_{B_{\tau}(0)\setminus\{0\}} \text{Re}(\Phi) \) and \( \Psi^*(\tau) := ||\text{Re}(\Phi)||_{L^{\infty}(\mathbb{R}^2 \setminus B_{\tau}(0))} \) the property,

\[
\Psi^*(\tau^m) > (m - 1)\Psi^*\sigma \quad \text{for } \sigma \in (0, \delta).
\]

Let \( \sigma := \frac{\delta}{4m} \), \( \tau := \frac{1}{2}\sigma^m \) and choose \( m \) disjoint open balls \( B^1, \ldots, B^m \subset B_\delta(x_0) \) as follows. By \( [29] \), we can choose \( x_1 \in B_\delta(x_0) \cap \{Q > 0\} \) and \( \tau_1 \in (0, \tau) \) such that \( B^1 := B_{\tau_1}(x_1) \subset B_\delta(x_0) \) and \( |B^1 \cap \{Q > 0\}| > 0 \). Let now \( \omega_1 := (B_\delta(x_0) \cap \{Q > 0\}) \setminus B_{2\tau_1 + \sigma}(x_1) \) and observe that

\[
|\omega_1| \geq \frac{1}{2}|B_\delta(x_0)| - |B_{2\sigma}(x_1)| \geq \left(\frac{1}{2} - \frac{1}{4m}\right) \pi\delta^2 > 0.
\]

Thus we may choose \( x_2 \in \omega_1 \) and \( \tau_2 \in (0, \tau) \) such that \( B^2 := B_{\tau_2}(x_2) \subset B_\delta(x_0) \) and \( |B^2 \cap \{Q > 0\}| > 0 \). Inductively, we let for \( 2 \leq k \leq m - 1, \omega_k := (B_\delta(x_0) \cap \{Q > 0\}) \setminus \bigcup_{i=1}^{k} B_{2\tau_i + \sigma}(x_i) \) and remark that

\[
|\omega_k| \geq \frac{1}{2}|B_\delta(x_0)| - \sum_{i=1}^{k} |B_{2\sigma}(x_i)| \geq \left(\frac{1}{2} - \frac{k}{4m}\right) \pi\delta^2 > 0.
\]

Therefore, we may choose \( x_{k+1} \in \omega_k \) and \( \tau_{k+1} \in (0, \tau) \) such that \( B^{k+1} := B_{\tau_{k+1}}(x_{k+1}) \subset B_\delta(x_0) \) and \( |B^{k+1} \cap \{Q > 0\}| > 0 \). Notice that, by construction,

\[
\text{diam}B^i \leq \sigma^m \quad \text{and} \quad \text{dist}(B^i, B^j) := \inf\{|x-y| : x \in B^i, y \in B^j\} \geq \sigma
\]

for all \( i \neq j \). Let us now fix \( z_1, \ldots, z_m \in C^\infty(\mathbb{R}^2) \) such that \( z_i > 0 \) in \( B^i \) and \( z_i = 0 \) in \( \mathbb{R}^2 \setminus B^i \). We define \( \mathcal{W}_m \) as the subspace spanned by \( z_1, \ldots, z_m \). Writing any \( v \in \mathcal{W}_m \setminus \{0\} \) as \( v = \sum_{i=1}^{m} a_i z_i \) with
Then there exists \( R > 0 \), \( \zeta > 0 \), and \( (x_n)_n \subset \mathbb{R}^2 \) such that, up to a subsequence,

\[
\int_{B_R(x_n)} |v_n|^{p'} \, dx \geq \zeta \quad \text{for all } n.
\]

(27)
Proof. Recall the decomposition of the fundamental solution \( \Phi = \Phi_1 + \Phi_2 \) introduced in (13), and the estimates (11) and (12):

\[
\| \Phi_1(x) \| \leq C_0 (1 + |x|)^{-\frac{1}{2}} \quad \text{and} \quad \| \Phi_2(x) \| \leq C_0 \min \left\{ 1 + |\log |x||, |x|^{-3} \right\}.
\]

The proof of the theorem consists in three claims. We first prove a variant of the conclusion with \( \Phi_2 \) in place of \( \Phi \) and for Schwartz functions. Next, a decay estimate for the convolution with \( \Phi_1 \) outside larger and larger balls is established. It is used in the third step to obtain the conclusion of the theorem for \( \Phi_1 \) in place of \( \Phi \), again for Schwartz functions.

Claim 1: Let \( 2 < p < \infty \), and \( (v_n)_n \subset \mathcal{S}(\mathbb{R}^2) \) bounded in \( L^p(\mathbb{R}^2) \) satisfy

\[
(28) \quad \sup_{y \in \mathbb{R}^2} \int_{B_r(y)} |v_n|^p \, dx \to 0, \quad \text{as } n \to \infty, \quad \text{for all } p > 0.
\]

Then \( \int_{\mathbb{R}^2} v_n(\Phi_2 * v_n) \, dx \to 0 \) as \( n \to \infty \).

For the proof, let \( A_R := \{ x \in \mathbb{R}^2 : \frac{1}{R} \leq |x| \leq R \} \) and \( D_R := \mathbb{R}^2 \setminus A_R \) for \( R > 1 \), and first remark that (11) gives \( \Phi_2 \leq \mathcal{L}'(\mathbb{R}^2) \) for all \( 1 \leq \ell < \infty \). Hence, Young’s inequality implies

\[
(29) \quad \sup_{n \in \mathbb{N}} \int_{\mathbb{R}^2} v_n[(1,M_n \Phi_2) * v_n] \, dx \leq \| \Phi_2 \|_{L^\infty(D_R)} \sup_{n \in \mathbb{N}} \| v_n \|_{L^p(\mathbb{R}^2)}^2 \to 0, \quad \text{as } R \to \infty,
\]

since \( 2 < p < \infty \). Next, decomposing \( \mathbb{R}^2 \) into disjoint squares \( \{ Q_\ell \}_{\ell \in \mathbb{N}} \) of side length \( R \), and considering for each \( \ell \) the square \( Q_\ell \) with the same center as \( Q_\ell \) but with side length \( 3R \), we obtain by an estimate similar to (11) pp. 109-110,

\[
\int_{\mathbb{R}^2} v_n[(1,A_n \Phi_2) * v_n] \, dx \leq \sum_{\ell \in \mathbb{N}} \int_{Q_\ell} \left( \int_{|x-y| < R} |\Phi_2(x-y)| \| v_n(x) \| dy \right) dx
\]

\[
\leq C R^{\frac{1}{4}} (1 + \log R) \sum_{\ell \in \mathbb{N}} \left( \int_{Q_\ell} |v_n(x)|^p \, dx \right)^{\frac{1}{p}}
\]

\[
\leq C R^{\frac{1}{4}} (1 + \log R) \left[ \sup_{\ell \in \mathbb{N}} \int_{Q_\ell} |v_n(x)|^p \, dx \right]^{\frac{1}{p} - 1} \sum_{\ell \in \mathbb{N}} \int_{Q_\ell} |v_n(x)|^p \, dx
\]

\[
\leq C R^{\frac{1}{4}} (1 + \log R) \left[ \sup_{y \in \mathbb{R}^2} \int_{B_{3R}(y)} |v_n(x)|^p \, dx \right]^{\frac{1}{p} - 1} \| v_n \|_{L^p}.
\]

The assumption (28) therefore gives \( \int_{\mathbb{R}^2} v_n[(1,A_n \Phi_2) * v_n] \, dx \to 0 \), as \( n \to \infty \), for every \( R > 0 \). Combining this with (29), the claim follows.

Claim 2: Let \( 6 < p \leq \infty \), \( \lambda_p := \frac{1}{2} - \frac{1}{p} > 0 \) and \( M_R := \mathbb{R}^N \setminus B_R \) for \( R > 0 \). Then there exists a constant \( C > 0 \) such that, for all \( R \geq 1 \) and \( f \in \mathcal{S}(\mathbb{R}^2) \) with \( \| f \|_{L^p} \leq 1 \),

\[
\| [1,M_n \Phi_1] * f \|_p \leq CR^{-\lambda_p} \| f \|_{L^p}.
\]

Since \( \Phi_1 \) is bounded, it suffices to prove the assertion for \( R \geq 4 \). For this, let us replace in the decomposition (13) the function \( \Phi_1 \) by \( P_R := [1,M_n \Phi_1] \). Then,

\[
P_R * \varphi = \sum_{j=\log_2 R} \infty Q^j \quad \text{with} \quad Q^j := (P_R \varphi_j) * \varphi \quad \text{for} \quad j \in \mathbb{N},
\]

using the fact that \( P_R \varphi_j = 0 \) for all \( j \) such that \( 2^{j+1} \leq R \). Since the asymptotic behavior of \( \Phi_1 \) and \( P_R \) are identical, the arguments used there give as in (15),

\[
\| Q^j * f \|_2 \leq C 2^{\frac{j}{2}} \| f \|_2 \quad \text{for all} \ f \in \mathcal{S}(\mathbb{R}^2), \ j \geq \log_2 R
\]

with a constant \( C > 0 \) independent of \( j \) and \( R \). By duality and interpolation

\[
\| Q^j * f \|_3 \leq C 2^{\frac{j}{2}} \| f \|_4 \quad \text{for all} \ f \in \mathcal{S}(\mathbb{R}^2), \ j \geq \log_2 R.
\]

Interpolating this last estimate with the \( L^1-L^\infty \) estimate

\[
\| Q^j * f \|_\infty \leq C 2^{-\frac{j}{2}} \| f \|_1, \ f \in \mathcal{S}(\mathbb{R}^2), \ j \geq \log_2 R,
\]
similar to (14), we obtain for $p \geq 3$,
\[
\|Q^j \ast f\|_p \leq C \|2^{j/2} \|f\|_{p'} = C \|2^{-j\lambda_p} \|f\|_{p'} \quad \text{for all } f \in \mathcal{S}(\mathbb{R}^2), \quad j \geq \lfloor \log_2 R \rfloor.
\]
Since $P_R \ast f = \frac{1}{2^j} (P_R \ast \varphi) \ast f$ for all $f \in \mathcal{S}(\mathbb{R}^2)$ with supp $\hat{f} \subset \{ |\xi| - 1 \leq \frac{1}{4} \}$, and since $\lambda_p > 0$ for $p > 6$, we conclude that for all such $f$ and all $p > 6$,
\[
\|1_{M_R} \Phi_1] \ast f\|_p = \frac{1}{2\pi} \| (P_R \ast \varphi) \ast f \|_p \leq C \| f \|_{p'} \sum_{j = \lfloor \log_2 R \rfloor}^{\infty} 2^{-j\lambda_p} \leq C R^{-\lambda_p} \|f\|_{p'}.
\]

The claim is proved.

**Claim 3:** Let $6 < p \leq \infty$ and suppose that $(v_n)_n \subset \mathcal{S}(\mathbb{R}^2)$ is a bounded sequence in $L^{p'}(\mathbb{R}^2)$ such that (28) holds. Then \( \int_{\mathbb{R}^2} v_n[\Phi_1 \ast v_n] \, dx \to 0 \) as $n \to \infty$.

To prove the claim, fix a radial function $\chi \in \mathcal{S}(\mathbb{R}^2)$ such that $\hat{\chi} \in C_c^\infty(\mathbb{R}^2)$ is radial, $0 \leq \hat{\chi} \leq 1$, $\hat{\chi}(\xi) = 1$ for $|\xi| - 1 \leq \frac{1}{4}$ and $\hat{\chi}(\xi) = 0$ for $|\xi| - 1 \geq \frac{1}{4}$. Moreover, let $w_n := \chi \ast v_n$. We then have $\Phi_1 \ast v_n = \frac{1}{2\pi} \Phi_1 \ast w_n$, since supp $w_n \subset \{ \xi : |\xi| - 1 \leq \frac{1}{2} \}$. Hence, the decomposition
\[
\int_{\mathbb{R}^2} v_n[\Phi_1 \ast v_n] \, dx = \frac{1}{2\pi} \int_{\mathbb{R}^2} v_n[(1_B \ast \Phi_1) \ast w_n] \, dx + \frac{1}{2\pi} \int_{\mathbb{R}^2} v_n[(1_{M_R} \ast \Phi_1) \ast w_n] \, dx
\]
holds for all $n$ and using Claim 2 we obtain
\[
\sup_{n \in \mathbb{N}} \left| \int_{\mathbb{R}^2} v_n[(1_{M_R} \ast \Phi_1) \ast w_n] \, dx \right| \to 0, \quad \text{as } R \to \infty.
\]
Moreover, using the disjoint squares $Q_\ell$ and $Q'_\ell$ of the proof of Claim 1 (cf. also [11, Lemma 4.3]) and the assumption (28), we obtain
\[
\lim_{n \to \infty} \int_{\mathbb{R}^2} v_n[(1_B \ast \Phi_1) \ast w_n] \, dx = 0 \quad \text{for every } R > 0.
\]
Combining this with (30), the claim follows.

Arguing by contradiction, we obtain from Claim 1 and Claim 3 the conclusion of the theorem for sequences $(v_n)_n \subset \mathcal{S}(\mathbb{R}^2)$, bounded in the $L^{p'}$-norm. Since the bilinear form $\int_{\mathbb{R}^2} \varphi \psi \, dx$ is continuous on $L^{p'}(\mathbb{R}^2)$ and since the property (27) is stable under approximation in the $L^{p'}(\mathbb{R}^2)$-norm, the conclusion follows by density.

We end this paper by giving the proof of the part (b) of our second main result, showing the existence of a nontrivial solution pair for (14).

**Proof of Theorem (13)(b).** Consider the set of paths $\Gamma = \{ \gamma \in C([0,1], L^{p'}(\mathbb{R}^2)) : \gamma(0) = 0, J(\gamma(1)) < 0 \}$ and the energy level
\[
e = \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)).
\]
Notice that $\Gamma \neq \emptyset$ and $e > 0$, since 0 is a strict local minimum of $J$ and there are $v \in L^{p'}(\mathbb{R}^2)$ such that $J(v) < 0$. Indeed, the proof of Theorem (13)(a) gives these facts without any decay assumption on $Q$. Using the standard deformation lemma (see [26, Lemma 2.3]), we obtain the existence of a Palais-Smale sequence $(v_n)_n \subset L^{p'}(\mathbb{R}^2)$ such that $J(v_n) \to e$, as $n \to \infty$. By (29), this sequence is bounded and therefore, as $n \to \infty$,
\[
\int_{\mathbb{R}^2} Q^\frac{1}{p} v_n R(Q^\frac{1}{p} v_n) \, dx = \left( \frac{1}{p'} - \frac{1}{2} \right)^{-1} \left( J(v_n) - \frac{1}{p'} J'(v_n) v_n \right) \to \frac{2p' c}{2 - p'} > 0.
\]
Consequently, Theorem (15) gives $R, \zeta > 0$ and $(x_n)_n \subset \mathbb{R}^2$ satisfying (27), up to a subsequence. Making $R$ larger we may assume $x_n \in \mathbb{Z}^2$ for all $n$. Since $Q$ is $\mathbb{Z}^2$-periodic, the functional $J$ is invariant under $\mathbb{Z}^2$-translations and setting $w_n := v_n(-x_n)$ we find that $(w_n)_n$ is also a bounded Palais-Smale sequence for $J$. Going to a subsequence, $w_n \to w \in L^{p'}(\mathbb{R}^2)$, Moreover, if $\varphi \in C_c^\infty(\mathbb{R}^2)$,
\[
\left| \int_{\mathbb{R}^2} \left( |w_n|^{p'-2} w_n - |w_m|^{p'-2} w_m \right) \varphi \, dx \right| \leq \| J'(w_n) - J'(w_m) \|_\varphi \| \varphi \|_{p'} + \| 1_{B} K (w_n - w_m) \|_{p'} \| \varphi \|_{p'} \to 0,
\]
as \( m, n \to \infty \), thanks to the compactness of \( 1_B K \), where \( B \subset \mathbb{R}^2 \) contains \( \text{supp}(\varphi) \). Since \( C_c^\infty(\mathbb{R}^2) \) is dense in \( L^{p'}(\mathbb{R}^2) \), we infer that for all bounded and measurable \( B \subset \mathbb{R}^2 \), \((1_B |w_n|^{p'-2} w_n)_n \) is a Cauchy sequence in \( L^{p'}(\mathbb{R}^2) \) and thus,

\[
1_B |w_n|^{p'-2} w_n \to 1_B |w|^{p'-2} w \quad \text{as} \quad n \to \infty, \quad \text{strongly in} \ L^p(\mathbb{R}^2).
\]

Recalling (27), we see that

\[
\int_{B_R(0)} |w|^{p'} \, dx = \lim_{n \to \infty} \int_{B_R(0)} |w_n|^{p'} \, dx \geq \zeta > 0,
\]

and consequently, \( w \neq 0 \). In addition, for all \( \varphi \in C_c^\infty(\mathbb{R}^2) \), we obtain

\[
J'(w) \varphi = \int_{\mathbb{R}^2} |w|^{p'-2} w \varphi - \int_{\mathbb{R}^2} \varphi K w \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^2} |w_n|^{p'-2} w_n \varphi - \int_{\mathbb{R}^2} \varphi K w_n \, dx = \lim_{n \to \infty} J'(w_n) \varphi = 0,
\]

and we conclude that \( J'(w) = 0 \). Hence, \( w \) is a nontrivial critical point of \( J \) and, applying Lemma 2.4, the theorem follows.

\[ \square \]

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