On the Computational Complexity of the Forcing Chromatic Number

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Abstract

We consider vertex colorings of graphs in which adjacent vertices have distinct colors. A graph is \( s \)-chromatic if it is colorable in \( s \) colors and any coloring of it uses at least \( s \) colors. The forcing chromatic number \( F_\chi(G) \) of an \( s \)-chromatic graph \( G \) is the smallest number of vertices which must be colored so that, with the restriction that \( s \) colors are used, every remaining vertex has its color determined uniquely. We estimate the computational complexity of \( F_\chi(G) \) relating it to the complexity class US introduced by Blass and Gurevich. We prove that recognizing if \( F_\chi(G) \leq 2 \) is US-hard with respect to polynomial-time many-one reductions. Moreover, this problem is coNP-hard even under the promises that \( F_\chi(G) \leq 3 \) and \( G \) is 3-chromatic. On the other hand, recognizing if \( F_\chi(G) \leq k \), for each constant \( k \), is reducible to a problem in US via a disjunctive truth-table reduction.

Similar results are obtained also for forcing variants of the clique and the domination numbers of a graph.

1 Introduction

The vertex set of a graph \( G \) will be denoted by \( V(G) \). An \( s \)-coloring of \( G \) is a map from \( V(G) \) to \( \{1, 2, \ldots, s\} \). A coloring \( c \) is proper if \( c(u) \neq c(v) \) for any adjacent vertices \( u \) and \( v \). A graph \( G \) is \( s \)-colorable if it has a proper \( s \)-coloring. The minimum \( s \) for which \( G \) is \( s \)-colorable is called the chromatic number of \( G \) and denoted by \( \chi(G) \). If \( \chi(G) = s \), then \( G \) is called \( s \)-chromatic.

A partial coloring of \( G \) is any map from a subset of \( V(G) \) to the set of positive integers. Suppose that \( G \) is \( s \)-chromatic. Let \( c \) be a proper \( s \)-coloring and \( p \) be a partial coloring of \( G \). We say that \( p \) forces \( c \) if \( c \) is a unique extension of \( p \) to a proper \( s \)-coloring. The domain of \( p \) will be called a defining set for \( c \). We call

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$D \subseteq V(G)$ a forcing set in $G$ if this set is defining for some proper $s$-coloring of $G$. The minimum cardinality of a forcing set is called the forcing chromatic number of $G$ and denoted by $F_\chi(G)$. This graph invariant was introduced by Harary in [21]. Here we study its computational complexity.

To establish the hardness of computing $F_\chi(G)$, we focus on the respective slice decision problems which are defined for each non-negative integer $k$ as follows:

\textbf{FORCE}_\chi(k)

\textit{Given}: a graph $G$.

\textit{Decide if}: $F_\chi(G) \leq k$.

The cases of $k = 0$ and $k = 1$ are tractable. It is clear that $F_\chi(G) = 0$ iff $\chi(G) = 1$, that is, $G$ is empty. Furthermore, $F_\chi(G) = 1$ iff $\chi(G) = 2$ and $G$ is connected, that is, $G$ is a connected bipartite graph. Thus, we can pay attention only to $k \geq 2$. Since there is a simple reduction of $\text{FORCE}_\chi(k)$ to $\text{FORCE}_\chi(k + 1)$ (see Lemma 3.2), it would suffice to show that even $\text{FORCE}_\chi(2)$ is computationally hard. This is indeed the case.

Let 3\text{COL} denote the set of 3-colorable graphs and U3\text{COL} the set of those graphs in 3\text{COL} having a unique, up to renaming colors, proper 3-coloring. First of all, note that a hardness result for $\text{FORCE}_\chi(2)$ is easily derivable from two simple observations:

1. If $F_\chi(G) \leq 2$, then $G \in \text{3COL}$;
2. If $G \in \text{U3COL}$, then $F_\chi(G) \leq 2$.

The set 3\text{COL} was shown to be NP-complete at the early stage of the NP-completeness theory in [40, 17] by reduction from SAT, the set of satisfiable Boolean formulas. It will be benefittable to use a well-known stronger fact: There is a polynomial-time many-one reduction $p$ from SAT to 3\text{COL} which is \textit{parsimonious}, that is, any Boolean formula $\Phi$ has exactly as many satisfying assignments to variables as the graph $p(\Phi)$ has proper 3-colorings (colorings obtainable from one another by renaming colors are not distinguished). In particular, if $\Phi$ has a unique satisfying assignment, then $p(\Phi) \in \text{U3COL}$ and hence $F_\chi(p(\Phi)) = 2$, while if $\Phi$ is unsatisfiable, then $p(\Phi) \notin \text{3COL}$ and hence $F_\chi(p(\Phi)) > 2$.

Valiant and Vazirani [42] designed a polynomial-time computable randomized transformation $r$ of the set of Boolean formulas such that, if $\Phi$ is a satisfiable formula, then with a non-negligible probability the formula $r(\Phi)$ has a unique satisfying assignment, while if $\Phi$ is unsatisfiable, then $r(\Phi)$ is surely unsatisfiable. Combining $r$ with the parsimonious reduction $p$ of SAT to 3\text{COL}, we arrive at the conclusion that $\text{FORCE}_\chi(2)$ is NP-hard with respect to randomized polynomial-time many-one reductions. As a consequence, the forcing chromatic number is not computable in polynomial time unless any problem in NP is solvable by a polynomial-time Monte Carlo algorithm with one-sided error.

We aim at determining the computational complexity of $F_\chi(G)$ more precisely. Our first result establishes the hardness of $\text{FORCE}_\chi(2)$ with respect to \textit{deterministic} polynomial-time many-one reductions. The latter reducibility concept will be default in what follows. The complexity class US, introduced by Blass and Gurevich [4], consists of languages $L$ for which there is a polynomial-time nondeterministic Turing
Force χ(2)

Figure 1: Location of the slice decision problems for $F_\chi(G)$ in the hierarchy of complexity classes.

machine $N$ such that a word $x$ belongs to $L$ iff $N$ on input $x$ has exactly one accepting computation path. Denote the set of Boolean formulas with exactly one satisfying assignment by USAT. This set is complete for US. As easily seen, U3COL belongs to US and, by the parsimonious reduction from SAT to 3COL, U3COL is another US-complete set. By the Valiant-Vazirani reduction, the US-hardness under polynomial-time many-one reductions implies the NP-hardness under randomized reductions and hence the former hardness concept should be considered stronger. It is known that US includes coNP [4] and this inclusion is proper unless the polynomial time hierarchy collapses [38]. Thus, the US-hardness implies also the coNP-hardness. We prove that the problem Force χ(2) is US-hard.

Note that this result is equivalent to the reducibility of U3COL to Force χ(2). Such a reduction would follow from the naive hypothesis, which may be suggested by (1), that a 3-chromatic $G$ is in U3COL iff $F_\chi(G) = 2$. It should be stressed that the latter is far from being true in view of Lemma 2.7.3 below.

On the other hand, we are able to estimate the complexity of each Force χ($k$) from above by putting this family of problems in a complexity class which is a natural extension of US. We show that, for each $k \geq 2$, the problem Force χ($k$) is reducible to a set in US via a polynomial-time disjunctive truth-table reduction (dtt-reduction for brevity, see Section 2 for definitions). This improves on the straightforward inclusion of Force χ($k$) in $\Sigma^P_2$.

Denote the class of decision problems reducible to US under dtt-reductions by $P^{US}_{dtt}$. As shown by Chang, Kadin, and Rohatgi [5], $P^{US}_{dtt}$ is strictly larger than US unless the polynomial time hierarchy collapses to its third level. The position of the problems under consideration in the hierarchy of complexity classes is shown in Figure 1, where $P^{NP[log \ n]}$ denotes the class of decision problems solvable by polynomial-time Turing machines with logarithmic number of queries to an NP oracle. The latter class coincides with the class of problems polynomial-time truth-table reducible to NP, see [26]. Another relation of $P^{US}_{dtt}$ to known complexity classes is $P^{US}_{dtt} \subseteq C=\Pi P \subseteq PP$, where a language $L$ is in PP (resp. $C=\Pi P$) if it is recognizable.
by a nondeterministic Turing machine $M$ with the following acceptance criterion: an input word $w$ is in $L$ iff at least (resp. precisely) a half of the computing paths of $M$ on $w$ are accepting. This inclusion follows from the facts that $US \subseteq C_P$ and that $C_P$ is closed under dtt-reductions (see [27, Theorem 9.9]).

In a recent paper [25], Hatami and Maserrat obtain a result that, in a sense, is complementary to our work and by this reason is also shown in Figure 1. Let $\text{FORCE}_\chi(\ast) = \{(x,k) : F_\chi(x) \leq k\}$. The authors of [25] prove that the recognition of membership in $\text{FORCE}_\chi(\ast)$ is a $\Sigma^p_2$-complete problem. Note that [25] and our paper use different techniques and, moreover, the two approaches apparently cannot be used in place of one another.

Our next result gives a finer information about the hardness of $\text{FORCE}_\chi(2)$. Note that, if $\chi(G) = 2$, then $F_\chi(G)$ is equal to the number of connected components of $G$. It turns out that the knowledge that $\chi(G) = 3$ does not help in computing $F_\chi(G)$. Moreover, it is hard to recognize whether or not $F_\chi(G) = 2$ even if it is known that $F_\chi(G) \leq 3$. Stating these strengthenings, we relax our hardness concept from the US-hardness to the coNP-hardness (as already mentioned, the former implies the latter but the converse is not true unless the polynomial time hierarchy collapses). Thus, we prove that the problem $\text{FORCE}_\chi(2)$ is coNP-hard even under the promises that $G \in 3\text{COL}$ and $F_\chi(G) \leq 3$ (see Figure 2). Note that the Valiant-Vazirani reduction implies no kind of a hardness result for the promise version of $\text{FORCE}_\chi(2)$.

In fact, many other graph characteristics also have natural forcing variants. Recall that a clique in a graph is a set of pairwise adjacent vertices. The maximum number of vertices of $G$ in a clique is denoted by $\omega(G)$ and called the clique number of $G$. A clique is optimal if it consists of $\omega(G)$ vertices. A set of vertices is called forcing if it is included in a unique optimal clique. We denote the minimum cardinality of a forcing set by $F_\omega(G)$ and call it the forcing clique number of $G$.

Furthermore, we say that a vertex of a graph $G$ dominates itself and any adjacent vertex. A set $D \subseteq V(G)$ is called dominating if every vertex of $G$ is dominated by a vertex in $D$. The domination number of $G$, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of $G$. Similarly to the above, a forcing set of vertices

![Figure 2: The class $\{G : F_\chi(G) \leq 2\}$ and surroundings.](image-url)
is one included in a unique optimal dominating set. The minimum cardinality of a forcing set is denoted by \( F_\gamma(G) \) and called the forcing domination number of \( G \). This graph invariant is introduced and studied by Chartrand, Gavlas, Vandell, and Harary [6].

For the forcing clique and domination numbers we consider the respective slice decision problems \( \text{Force}_\omega(k) \) and \( \text{Force}_\gamma(k) \) and show the same relation of them to the class US that we have for the forcing chromatic number. Actually, the dtt-reducibility to US is proved for all the three numbers by a uniform argument. However, the US-hardness with respect to many-one reductions for \( \omega \) and \( \gamma \) is proved differently than for \( \chi \). The case of \( \omega \) and \( \gamma \) seems combinatorially simpler because of the following equivalence: A graph \( G \) has a unique optimal clique iff \( F_\omega(G) = 0 \) and similarly with \( \gamma \). The study of unique optimum (UO) problems was initiated by Papadimitriou [36]. Due to the US-hardness of the UO CLIQUE and UO DOMINATING SET problems, we are able to show the US-hardness of \( \text{Force}_\omega(k) \) and \( \text{Force}_\gamma(k) \) using only well-known standard reductions, whereas for \( \text{Force}_\chi(k) \) we use somewhat more elaborate reductions involving graph products.

Overview of previous related work

**Forcing chromatic number of particular graphs.** Let \( K_n \) (resp. \( C_n, P_n \)) denote the complete graph (resp. the cycle, the path) on \( n \) vertices. As a simple exercise, we have \( F_\chi(C_{2m+1}) = m + 1 \). Mahmoodian, Naserasr, and Zaker [34] compute the forcing chromatic number of several Cartesian products:

\[
F_\chi(C_{2m+1} \times K_2) = m + 1, \quad F_\chi(C_{m} \times K_n) = m(n-3) \quad \text{and} \quad F_\chi(P_{n} \times K_n) = m(n-3)+2 \text{ if } n \geq 6, \quad \text{and} \quad F_\chi(K_{m} \times K_{n}) = m(n-m) \text{ if } n \geq m^2.
\]

Mahdian, Mahmoodian, Naserasr, and Harary [33] determine a few missing values: \( F_\chi(C_{m} \times K_3) = \lfloor m/2 \rfloor + 1 \) and, if \( m \) even, \( F_\chi(C_{m} \times K_5) = 2m \). On the other hand, the asymptotics of \( F_\chi(K_{n} \times K_{n}) \) are unknown (a problem having arisen in research on Latin squares, see below). The best lower and upper bounds \( 4(n-2)/3 \leq F_\chi(K_n \times K_n) \leq n^2/4 \) are obtained, respectively, in [28] for \( n \geq 8 \) and [14] for all \( n \). Our results show that no general approach for efficient computing the forcing chromatic number is possible unless \( \text{NP} = \text{P} \) (and even \( \text{US} = \text{P} \)).

**Latin squares and the complexity of recognizing a forcing coloring.** A Latin square of order \( n \) is an \( n \times n \) matrix with entries in \( \{1, 2, \ldots, n\} \) such that every row and column contains all the \( n \) numbers. In a partial Latin square some entries may be empty and every number occurs in any row or column at most once. A partial Latin square is called a critical set if it can be completed to a Latin square in a unique way. Colbourn, Colbourn, and Stinson [11] proved that recognition if a given partial Latin square \( L \) is a critical set is coNP-hard. Moreover, the problem is coNP-complete even if one extension of \( L \) to a Latin square is known. The result of [11] is strengthened in [16].

As it is observed in [34], there is a natural one-to-one correspondence between Latin squares of order \( n \) and proper \( n \)-colorings of the Cartesian square \( K_n \times K_n \)}
which matches critical sets and forcing colorings.\textsuperscript{1} It follows that it is coNP-hard to recognize if a given partial coloring $p$ of a graph is forcing. Moreover, even if this problem is restricted to graphs $K_n \times K_n$ and one extension of $p$ to a proper $n$-coloring is given, the problem is coNP-complete (see also Proposition 2.8 below).

**Complexity of the forcing matching number**  Given a graph $G$ with a perfect matching, Harary, Klein, and Živković [23] define its forcing matching number as the minimum size of a forcing set of edges, where the latter is a set contained in a unique perfect matching. Denote this number by $F_\nu(G)$. Let $\text{FORCE}_\nu(k)$ be the problem of recognition, given $G$, if $F_\nu(G) \leq k$ and let $\text{FORCE}_\nu(*)$ be the variant of the same problem with $k$ given as a part of an input. From the polynomial-time solvability of the perfect matching problem, it easily follows that each $\text{FORCE}_\nu(k)$ is polynomial-time solvable and that $\text{FORCE}_\nu(*)$ is in NP. Afshani, Hatami, and Mahmoodian [3], using an earlier result by Adams, Mahdian, and Mahmoodian [1], prove that the latter problem is NP-complete.

**Variety of combinatorial forcing numbers.** Critical sets are studied since the seventies (Nelder [35]). The forcing chromatic, domination, and matching numbers attracted attention of researchers in the nineties. In fact, a number of other problems in diverse areas of combinatorics have a similar forcing nature. Defining sets in block designs (Gray [18]) have a rather rich bibliography. Other graph invariants whose forcing versions have appeared in the literature are the orientation number (Chartrand, Harary, Schultz, and Wall [7]), the geodetic number, the hull number, the dimension of a graph (Chartrand and Zhang, respectively, [8, 9, 10]), and this list is still inexhaustive. As a general concept, the combinatorial forcing was presented by Harary in a series of talks, e.g. [21].

**Unique colorability.** The concept of a uniquely colorable graph was introduced by Harary, Hedetniemi, and Robinson [22]. Complexity-theoretic concepts related to this combinatorial phenomenon were introduced by Valiant and Vazirani [42] and Blass and Gurevich [4]. However, the complexity-theorists prefer to deal with USAT, the uniqueness version of the archetypical SATisfiability problem. The results established for USAT, as US- and coNP-hardness under many-one reductions and NP-hardness under randomized reductions, carry through for U3COL by means of the parsimonious many-one reduction of SAT to 3COL. A direct, purely combinatorial way to show the coNP-hardness of U3COL is given by a result of Greenwell and Lovász ([19], see also [29, Theorem 8.5]). They prove, in particular, that if a connected graph $G$ is not 3-colorable then the categorical product $G \cdot K_3$ is a uniquely colorable graph. On the other hand, if $G$ is 3-colorable then $G \cdot K_3$ can be colored in two different ways. The latter follows from a simple observation: any proper 3-coloring of one of the factors efficiently induces a proper 3-coloring of the product $G_1 \cdot G_2$. By the NP-completeness of 3COL, we arrive at the conclusion that

\textsuperscript{1}Such a correspondence is also well known for edge colorings of the complete bipartite graph $K_{n,n}$.
the problem of recognizing, given a graph \( H \) and its proper 3-coloring, whether or not \( H \) is uniquely 3-colorable, is coNP-complete. Later this complexity-theoretic fact was observed by Dailey [15] who uses an identical combinatorial argument.

**Organization of the paper**

In Section 2 we define the categorical and the Cartesian graph products, which will play an important role in our proofs, and prove some preliminary lemmas about forcing sets in product graphs. We also state a few basic bounds for \( F_{\chi}(G) \) and determine the complexity of recognition if a given set of vertices in a graph is forcing (the latter result is not used in the sequel but is worth being noticed). The hardness of \( \text{FORCE}_{\chi}(k) \) is established in Section 3. A closer look at \( \text{FORCE}_{\chi}(2) \) is taken in Section 4. In Section 5, using a combinatorial result of Hajiabolhassan, Mehrabadi, Tusserkani, and Zaker [20], we analyze the complexity of a related graph invariant, namely, the largest cardinality of an inclusion-minimal forcing set. Before taking into consideration the forcing clique and domination numbers, we suggest a general setting for forcing combinatorial numbers in Section 6. It is built upon the standard formal concept of an NP optimization problem. We benefit from this formal framework in some proofs. Section 7 is devoted to the forcing clique and domination numbers. The dtt-reducibility of \( \text{FORCE}_{\pi}(k) \) to US for \( \pi \in \{\chi, \omega, \gamma\} \) is shown in Section 8. Section 9 contains a concluding discussion and some open questions.

**2 Background**

**2.1 Basics of complexity theory**

We suppose that the discrete structures under consideration are encoded by binary words. For example, graphs are naturally representible by their adjacency matrices. The set of all binary words is denoted by \( \{0,1\}^* \). A decision problem is identified with a language, i.e., a subset of \( \{0,1\}^* \), consisting of all yes-instances. A many-one reduction of a problem \( X \) to a problem \( Y \) is a map \( r : \{0,1\}^* \rightarrow \{0,1\}^* \) such that \( x \in X \) iff \( r(x) \in Y \). If \( r(x) \) is computable in time bounded by a polynomial in the length of \( x \), the reduction is called polynomial-time. We write \( X \leq_{P}^{m} Y \) to say that there is a polynomial-time many-one reduction from \( X \) to \( Y \).

Let \( C \) be a class of decision problems (or languages). A problem \( Z \) is called \( C \)-hard if any \( X \) in \( C \) is \( \leq_{P}^{m} \)-reducible to \( Z \). A problem \( Z \) is called \( C \)-complete if \( Z \) is \( C \)-hard and belongs to \( C \). USAT and U3COL are examples of US-complete problems.

A disjunctive truth-table reduction (or dtt-reduction) of a language \( X \) to a language \( Y \) is a transformation which takes any word \( x \) to a set of words \( y_1, \ldots, y_m \) so that \( x \in X \) iff \( y_i \in Y \) for at least one \( i \leq m \). We write \( X \leq_{\text{dtt}}^{P} Y \) to say that there is such a polynomial-time reduction from \( X \) to \( Y \).

If \( C \) is a class of languages and \( \leq \) is a reducibility, then \( C \leq X \) means that \( Y \leq X \) for all \( Y \) in \( C \) (i.e., \( X \) is \( C \)-hard under \( \leq \)) and \( X \leq C \) means that \( X \leq Y \) for some \( Y \) in \( C \).
The formal framework of promise problems is developed in [39]. Let $Y$ and $Q$ be languages. Whenever referring to a decision problem $Y$ under the promise $Q$, we mean that membership in $Y$ is to be decided only for inputs in $Q$. A reduction $r$ of an ordinary decision problem $X$ to a problem $Y$ under the promise $Q$ is a usual many-one reduction form $X$ to $Y$ with the additional requirement that $r(x) \in Q$ for all $x$. This definition allows us to extend the notion of $C$-hardness to promise problems.

A polynomial-time computable function $h : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^*$ is called an AND$_2$ function for a language $Z$ if for any pair $x, y$ we have both $x$ and $y$ in $Z$ iff $h(x, y)$ is in $Z$. Such an $h$ is an OR$_2$ function for $Z$ if we have at least one of $x$ and $y$ in $Z$ iff $h(x, y)$ is in $Z$.

### 2.2 Graph products

Let $E(G)$ denote the set of edges of a graph $G$. Given two graphs $G_1$ and $G_2$, we define a product graph on the vertex set $V(G_1) \times V(G_2)$ in two ways. Vertices $(u_1, u_2)$ and $(v_1, v_2)$ are adjacent in the Cartesian product $G_1 \times G_2$ if either $u_1 = v_1$ and $\{u_2, v_2\} \in E(G_2)$ or $u_2 = v_2$ and $\{u_1, v_1\} \in E(G_1)$. They are adjacent in the categorical product $G_1 \cdot G_2$ if both $\{u_1, v_1\} \in E(G_1)$ and $\{u_2, v_2\} \in E(G_2)$.

A set $V(G_1) \times \{v\}$ for $v \in V(G_2)$ will be called a $G_1$-layer of $v$ and a set $\{u\} \times V(G_2)$ for $u \in V(G_1)$ will be called a $G_2$-layer of $u$.

**Lemma 2.1** (Sabidussi, see [29, Theorem 8.1]) $\chi(G \times H) = \max\{\chi(G), \chi(H)\}$.

If $c$ is a proper coloring of $G$, it is easy to see that $c^*(x, y) = c(x)$ is a proper coloring of the categorical product $G \cdot H$. We will say that $c$ induces $c^*$. Similarly, any proper coloring of $H$ induces a proper coloring of $G \cdot H$. This implies the following well-known fact.

**Lemma 2.2** $\chi(G \cdot H) \leq \min\{\chi(G), \chi(H)\}$.

The next proposition shows that the Cartesian and the categorical products are, respectively, AND$_2$ and OR$_2$ functions for 3COL (see [31] for an exposition of AND and OR functions).

**Proposition 2.3** (1) $G \times H \in 3COL$ iff both $G$ and $H$ are in 3COL.

(2) $G \cdot H \in 3COL$ iff at least one of $G$ and $H$ is in 3COL.

**Proof.** Item 1 is straightforward by Lemma 2.1. However, Item 2 does not follow solely from Lemma 2.2 because the equality $\chi(G \cdot H) = \min\{\chi(G), \chi(H)\}$ is still an unproven conjecture made by Hedetniemi. Luckily, the conjecture is known to be true for 4-chromatic graphs (El-Zahar and Sauer, see [29, Section 8.2]), and this suffices for our claim.
2.3 Preliminary lemmas

Lemma 2.4 (Greenwell-Lovász [19]) Let $G$ be a connected graph with $\chi(G) > n$. Then $G \cdot K_n$ is uniquely $n$-colorable.

The proof can be found also in [29, Theorem 8.5]. We will use not only Lemma 2.4 itself but also a component of its proof stated as the next lemma. We call a partial coloring injective if different colored vertices receive different colors.

Lemma 2.5 Let $G$ be a connected graph and $p$ be an injective coloring of a $K_n$-layer of $G \cdot K_n$. Then $p$ forces the $K_n$-induced coloring of $G \cdot K_n$.

Proof. Suppose that $p$ is an injective coloring of a $K_n$-layer of $u \in V(G)$ and that a proper $n$-coloring $c$ of $G \cdot K_n$ extends $p$. Let $v \in V(G)$ be adjacent to $u$. Inferably, $c(v, i) = c(u, i)$ for any $i \in V(K_n)$. Since $G$ is connected, $c$ is forced by $p$ to be monochrome on each $G$-layer.

Lemma 2.6 Any proper 3-coloring of $K_3 \cdot K_3$ is induced by one of the two factors $K_3$.

Proof. Let $V(K_3) = \{1, 2, 3\}$ and denote $L_i = \{(i, 1), (i, 2), (i, 3)\}$. Suppose that $c$ is a coloring of $K_3 \cdot K_3$. Consider it on one of the layers, say, $L_2$. If $|c(L_2)| = 3$, then $c$ is induced by the second factor on the account of Lemma 2.5. Assume that $|c(L_2)| = 2$, for example, $c(2, 1) = 1$, $c(2, 2) = 2$, and $c(2, 3) = 2$. Then $c(1, 2) = c(3, 3) = 3$ is forced, contradictory to the fact that these vertices are adjacent. There remains the possibility that $|c(L_2)| = 1$, for example, $c(2, 1) = c(2, 2) = c(2, 3) = 2$. Color 2 can occur neither in $L_1$ nor in $L_3$. Assume that one of these layers has both colors 1 and 3, say, $c(1, 1) = 1$ and $c(1, 2) = 3$. This forces $c(3, 3) = 2$, a contradiction. Thus, $|c(L)| = 1$ is possible only if $L_1$, $L_2$, and $L_3$ are all monochrome and have pairwise distinct colors. This is the coloring induced by the first factor.

2.4 A few basic bounds

We call two $s$-colorings equivalent if they are obtainable from one another by permutation of colors. Proper $s$-colorings of a graph $G$ are equivalent if they determine the same partition of $V(G)$ into $s$ independent sets. Let $N_\chi(G)$ denote the number of such partitions for $s = \chi(G)$. Thus, $N_\chi(G)$ is equal to the number of inequivalent proper $s$-colorings of $s$-chromatic $G$, while the total number of such colorings is equal to $\chi(G)!N_\chi(G)$. A graph $G$ is uniquely colorable if $N_\chi(G) = 1$. In particular, $G \in U3COL$ iff $\chi(G) = 3$ and $N_\chi(G) = 1$.

Lemma 2.7

(.1) $\chi(G) - 1 \leq F_\chi(G) \leq \log_2 N_\chi(G) + \log_2(\chi(G)!)$.

(.2) If $N_\chi(G) = 1$, then $F_\chi(G) = \chi(G) - 1$.  

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\begin{enumerate}[(.3)]
\item For any \( k \), there is a 3-chromatic graph \( G_k \) on \( 4k + 2 \) vertices with \( F_\chi(G_k) = 2 \) and \( N_\chi(G_k) = 2^{k-1} + 1 \).
\end{enumerate}

The lower bound in Item 1 is sharp by Item 2. The upper bound is sharp because, for example, \( F_\chi(mK_2) = m \) while \( N_\chi(mK_2) = 2^{m-1} \), where \( mK_2 \) denotes the graph consisting of \( m \) isolated edges. Item 3 shows that the converse of Item 2 is false. It is also worth noting that both the lower and the upper bounds in Item 1 are computationally hard, even if one is content with finding an approximate value. The NP-hardness of approximation of the chromatic number is established in [32]. A hardness result for approximate computing \( \log_2 N_\chi(G) \), even for 3-chromatic graphs, is obtained in [43].

\textbf{Proof.} Item 2 and the lower bound in Item 1 are obvious. To prove the upper bound in Item 1, we have to show that a \( k \)-chromatic graph \( G \) has a forcing set of at most \( l = \lfloor \log_2(k!N_\chi(G)) \rfloor \) vertices \( v_1, v_2, \ldots \). Let \( C_1 \) be the set of all \( k!N_\chi(G) \) proper \( k \)-colorings of \( G \). We choose vertices \( v_1, v_2, \ldots \) one by one as follows. Let \( i \geq 1 \). Assume that the preceding \( i-1 \) vertices have been chosen and that a set of colorings \( C_i \) has been defined and has at least 2 elements. We set \( v_i \) to be an arbitrary vertex such that not all colorings in \( C_i \) coincide at \( v_i \). Furthermore, we assign \( v_i \) a color \( p(v_i) \) occurring in the multiset \( \{ c(v_i) : c \in C_i \} \) least frequently. Finally, we define \( C_{i+1} = \{ c \in C_i : c(v_i) = p(v_i) \} \). Note that \( |C_{i+1}| \leq |C_i|/2 \). We eventually have \( |C_{i+1}| = 1 \) for some \( i \leq l = \lfloor \log_2 |C_1| \rfloor \). For this \( i \), denote the single coloring in \( C_{i+1} \) by \( c \). By construction, \( v_1, \ldots, v_i \) is defining for \( c \).

To prove Item 3, consider \( H = K_3 \times K_2 \). This graph has two inequivalent colorings \( c_1 \) and \( c_2 \) shown in Figure 3. Let \( u, v, w \in V(H) \) be as in Figure 3. Note that a partial coloring \( p_1(u) \neq p_1(v) \) forces \( c_1 \) or its equivalent and that \( p_2(v) \neq p_2(w) \) forces \( c_2 \).

Let \( G_k \) consist of \( k \) copies of \( H \) with all \( u \) and all \( v \) identified, that is, \( G_k \) has \( 4k + 2 \) vertices. Since the set \( \{ u, v \} \) stays forcing in \( G_k \), we have \( F_\chi(G_k) = 2 \). If \( u \) and \( v \) are assigned the same color, we are free to assign each copy of \( w \) any of the two remaining colors. It follows that \( N_\chi(G_k) = 2^{k-1} + 1 \).

\end{proof}

2.5 \quad Complexity of a forcing set recognition

\textbf{Proposition 2.8} The problem of recognizing, given a graph \( G \) and two vertices \( u, v \in V(G) \), whether or not \( \{ u, v \} \) is a forcing set is \( US \)-complete.

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3.png}
\caption{Proper 3-colorings of \( K_3 \times K_2 \).}
\end{figure}
Proof. Suppose that \( \chi(G) \geq 3 \) for else the question is efficiently decidable. Set \( p(u) = 1 \) and \( p(v) = 2 \). Obviously, \( \{u, v\} \) is a forcing set iff \( p \) forces a proper 3-coloring of \( G \). Verification of the latter condition is clearly in US.

We now describe a reduction \( R \) of the US-complete problem U3COL to the problem under consideration. If an input graph \( G \) is empty, let \( R(G) \) be an arbitrary graph with two vertices which are not a forcing set. Otherwise, let \( u \) and \( v \) be the lexicographically first pair of adjacent vertices in \( G \). We set \( R(G) = (G, u, v) \). It is not hard to see that \( G \in \text{U3COL} \) iff \( R(G) \) consists of a graph and a 2-vertex forcing set in it.

\[ \Box \]

3 Complexity of \( F_{\chi}(G) \): A lower bound

Theorem 3.1 For each \( k \geq 2 \), the problem \( \text{FORCE}_{\chi}(k) \) is US-hard. Moreover, this holds true even if we consider only connected graphs.

We first observe that the family of problems \( \text{FORCE}_{\chi}(k) \) is linearly ordered with respect to the \( \leq^{\text{P}}_{m} \)-reducibility. A simple reduction showing this does not preserve connectedness of graphs. However, if we restrict ourselves to connected graphs, we are able to show that \( \text{FORCE}_{\chi}(2) \) remains the minimum element in this order. We then prove that \( \text{FORCE}_{\chi}(2) \) is US-hard (even for connected graphs).

Lemma 3.2 \( \text{FORCE}_{\chi}(k) \leq^{\text{P}}_{m} \text{FORCE}_{\chi}(k + 1) \).

Proof. Given a non-empty graph \( G \), we add one isolated vertex to it. Denoting the result by \( G + K_1 \), it is enough to notice that \( F_{\chi}(G + K_1) = F_{\chi}(G) + 1 \).

Lemma 3.3 Let \( k \geq 2 \). Then \( \text{FORCE}_{\chi}(2) \) reduces to \( \text{FORCE}_{\chi}(k) \) even if we consider the problems only for connected graphs.

Proof. Let \( G \) be a graph on \( n \) vertices and \( m \leq n \). Writing \( H = G \oplus mK_2 \), we mean that

- \( V(H) = \{v_1, \ldots, v_n\} \cup \bigcup_{i=1}^{m}\{a_i, b_i\} \),
- \( H \) induces on \( \{v_1, \ldots, v_n\} \) a graph isomorphic to \( G \),
- \( \{v_i, a_i\} \) and \( \{a_i, b_i\} \) for all \( i \leq m \) are edges of \( H \), and
- \( H \) has no other edges.

Suppose that \( \chi(G) \geq 3 \) and \( H = G \oplus mK_2 \). Let us check that \( F_{\chi}(H) \leq 2 + m \) if \( F_{\chi}(G) = 2 \) and \( F_{\chi}(H) \geq 3 + m \) if \( F_{\chi}(G) \geq 3 \). This will give us the following reduction of \( \text{FORCE}_{\chi}(2) \) to \( \text{FORCE}_{\chi}(k) \): Given \( G \), construct an \( H = G \oplus (k - 2)K_2 \).

Let \( F_{\chi}(G) = 2 \). Note that \( G \) must be 3-chromatic. To show that \( F_{\chi}(H) \leq 2 + m \), we construct a forcing set of \( 2 + m \) vertices. We will identify \( G \) and its copy spanned by \( \{v_1, \ldots, v_n\} \) in \( H \). Let \( v_j \) and \( v_l \) be two vertices such that a partial coloring
Lemma 3.4 \textsc{Force}_\chi(2) is US-hard even if restricted to connected graphs.

To prove the lemma, we describe a reduction from U3COL. Note that U3COL remains US-complete when restricted to connected graphs and that our reduction will preserve connectedness. Since the class of 2-colorable graphs is tractable and can be excluded from consideration, the desired reduction is given by the following lemma.

Lemma 3.5 Suppose that $\chi(G) \geq 3$. Then $G \in \text{U3COL}$ iff $F_\chi(G \times K_3) = 2$.

Proof. Case 1: $G \in \text{U3COL}$. We have to show that $F_\chi(G \times K_3) = 2$.

Fix arbitrary $u, v \in V(G)$ whose colors in the proper 3-coloring of $G$ are different, for example, $u$ and $v$ can be any adjacent vertices of $G$. Let $V(K_3) = \{1, 2, 3\}$. Assign $p(u, 1) = 1$ and $p(v, 2) = 2$ and check that $p$ forces a proper 3-coloring of $G \times K_3$. Assume that $c$ is a proper 3-coloring of $G \times K_3$ consistent with $p$. Since $c$ on each $G$-layer coincides with the 3-coloring of $G$ up to permutation of colors, we easily infer that $c(v, 1) = c(u, 2) = 3$ (see Figure 4). This implies $c(u, 3) = 2$ and $c(v, 3) = 1$. Thus, in each $G$-layer we have two vertices with distinct colors, which determines colors of all the other vertices. As easily seen, the coloring obtained is really proper.

Case 2: $G \in \text{3COL} \setminus \text{U3COL}$. We have to check that $F_\chi(G \times K_3) \geq 3$. 

Figure 4: Proof of Lemma 3.5 (Case 1).
Given a partial coloring $p$ of two vertices $a, b \in V(G \times K_3)$, we have to show that it is not forcing. The cases that $p(a) = p(b)$ or that $a$ and $b$ are in the same $G$- or $K_3$-layer are easy. Without loss of generality we therefore suppose that $p(a) = 1$, $p(b) = 2$, $a = (u, 1)$, and $b = (v, 2)$, where $u$ and $v$ are distinct vertices of $G$. Define two partial colorings of $G$ by $c_1(u) = c_1(v) = 1$ and by $c_2(u) = 1, c_2(v) = 3$.

Subcase 2.1: Both $c_1$ and $c_2$ extend to proper $3$-colorings of $G$. Denote the extensions by $e_1$ and $e_2$ respectively. Denote the three $G$-layers of $G \times K_3$ by $G_1, G_2, G_3$ and consider $e_1, e_2$ on $G_1$. For each $i = 1, 2$, $e_i$ and $p$ agree and have a common extension to a proper coloring of $G \times K_3$. Thus, $p$ is not forcing.

Subcase 2.2: Only $c_1$ extends to a proper $3$-coloring of $G$. Since $G$ is not uniquely colorable, there must be at least two extensions, $e_1$ and $e_2$, of $c_1$ to proper $3$-colorings of $G_1$. As in the preceding case, $e_1$ and $e_2$ each agree with $p$ and together with $p$ extend two distinct colorings of $G \times K_3$.

Subcase 2.3: Only $c_2$ extends to a proper coloring of $G$. This case is completely similar to Subcase 2.2.

Case 3: $G \notin 3\text{COL}$. We have $\chi(G \times K_3) \geq 4$ by Lemma 2.1 and $F_\chi(G \times K_3) \geq 3$ by Lemma 2.7.1.

Theorem 3.1 immediately follows from Lemmas 3.4 and 3.3.

4 Hardness of $\text{FORCE}_\chi(2)$: A closer look

Theorem 4.1 The problem $\text{FORCE}_\chi(2)$ is coNP-hard even under the promises that $F_\chi(G) \leq 3$ and $\chi(G) \leq 3$ and even if an input graph $G$ is given together with its proper $3$-coloring.

Let us for a while omit the promise that $F_\chi(G) \leq 3$. Then the theorem is provable by combining the Greenwell-Lovász reduction of coNP to US (Lemma 2.4) and our reduction of US to $\text{FORCE}_\chi(2)$ (Lemma 3.5). Doing so, we easily deduce the following:

- If $\chi(G) > 3$, then $G \cdot K_3$ is uniquely $3$-colorable and hence $F_\chi((G \cdot K_3) \times K_3) = 2$.
- If $\chi(G) = 3$, then $G \cdot K_3$ is $3$-chromatic because it contains an odd cycle (this is an easy particular case of the aforementioned Hedetniemi conjecture).

Moreover, $G \cdot K_3$ has two induced $3$-colorings and hence $F_\chi((G \cdot K_3) \times K_3) \geq 3$.

To obtain Theorem 4.1 (without the promise $F_\chi(G) \leq 3$), it now suffices to make the following observation.

Lemma 4.2 $\chi((G \cdot K_3) \times K_3) = 3$ for any graph $G$. Moreover, a proper $3$-coloring is efficiently obtainable from the explicit product representation of $(G \cdot K_3) \times K_3$.

Proof. By Lemma 2.2, we have $\chi(G \cdot K_3) \leq 3$ and hence, by Lemma 2.1, $\chi((G \cdot K_3) \times K_3) = 3$. Let $V(K_3) = \{1, 2, 3\}$ and denote $V_{i,j} = \{(v, i, j) : v \in V(G)\}$. It is not hard to see that $V_{1,1} \cup V_{2,2} \cup V_{3,3}$, $V_{1,2} \cup V_{2,3} \cup V_{3,1}$, and $V_{1,3} \cup V_{2,1} \cup V_{3,2}$ is a partition of $V((G \cdot K_3) \times K_3)$ into independent sets. □
Remark 4.3 The known facts about graph factorizations [29, Chapters 4 and 5] imply a nontrivial strengthening of Lemma 4.2 under certain, rather general conditions. Namely, if $G$ is a connected nonbipartite graph and no two vertices of $G$ have the same neighborhood, then we do not need to assume that $H = (G \cdot K_3) \times K_3$ is explicitly represented as a product graph because the product structure is efficiently reconstructible from the isomorphism type of $H$. We thank Wilfried Imrich for this observation.

To obtain the full version of Theorem 4.1, we only slightly modify the reduction: Before transforming $G$ in $(G \cdot K_3) \times K_3$, we add to $G$ a triangle with one vertex in $V(G)$ and two vertices new. Provided $\chi(G) \geq 3$, this does not change $\chi(G)$ and hence the modified transformation is an equally good reduction. The strengthening (the promise $F_\chi(G) \leq 3$) is given by the following lemma.

Lemma 4.4 If a graph $G$ is connected and contains a triangle, then $F_\chi((G \cdot K_3) \times K_3) \leq 3$.

Proof. Let $v$ be a vertex of a triangle $T$ in $G$. Consider the product $H = G \cdot K_3$ and a partial coloring $p(v, 1) = 1$, $p(v, 2) = 2$. We claim that $p$ forces the $K_3$-induced coloring of $H$. Obviously, the latter is an extension of $p$. To show that no other extension is possible, assume that $c$ is a proper 3-coloring of $H$ compatible with $p$ and consider the restriction of $c$ on $T \cdot K_3$. By Lemma 2.6, $c$ on $T \cdot K_3$ coincides with the coloring induced by the second factor. In particular, $c(v, 3) = 3$. Our claim now follows from the connectedness of $G$ by Lemma 2.5.

Now, let $H_1, H_2, H_3$ denote the three $H$-layers in $H \times K_3$ and, for each $i = 1, 2, 3$, let $G_{i1}, G_{i2}, G_{i3}$ denote the three $G$-layers in $H_i$. Let $p(v, 1, 1) = 1$, $p(v, 2, 1) = 2$ be
the forcing partial coloring for $H_1$ as described above (see Figure 5). Thus, $p$ forces coloring the whole $G_{1j}$ in color $j$ for each $j = 1, 2, 3$. Suppose that $c$ is a proper 3-coloring of $H \times K_3$ that agrees with $p$. From the product structure of $H \times K_3$ we see that, for each $j$, in $G_{2j}$ there cannot occur color $j$. Let $T^2$ denote the copy of $T \cdot K_3$ in $H_2$. By Lemma 2.6, $c$ on $T^2$ is induced by the second factor and hence $c(v, 1, 2)$, $c(v, 2, 2)$, and $c(v, 3, 2)$ are pairwise distinct. By Lemma 2.5, each of $G_{21}$, $G_{22}$, and $G_{23}$ is monochrome and these layers receive pairwise distinct colors. We already know that $c(G_{2j}) \neq j$. Thus, when we define $p$ in the third point by $p(x) = 3$ for an arbitrary $x \in G_{21}$, this forces $c(G_{21}) = 3$, $c(G_{22}) = 1$, and $c(G_{23}) = 2$. Since every vertex in $G_{3j}$ is in triangle with its clones in $G_{1j}$ and $G_{2j}$, $c$ is uniquely extrapolated on $H_3$.

The proof of Theorem 4.1 is complete.

5 Maximum size of a minimal forcing set

Another related invariant of a graph $G$ is the largest cardinality of an inclusion-minimal forcing set in $G$. We will denote this number by $F^*_\chi(G)$. A complexity analysis of $F^*_\chi(G)$ is easier owing to the characterization of uniquely colorable graphs obtained in [20].

Lemma 5.1 (Hajiabolhassan, Mehrabadi, Tusserkani, and Zaker [20]) A connected graph $G$ is uniquely 3-colorable iff $F^*_\chi(G) = 2$.

Theorem 5.2 The problem of deciding, given a graph $G$ and its proper 3-coloring, whether or not $F^*_\chi(G) \leq 2$ is $\text{coNP}$-complete.

Proof. The problem is in $\text{coNP}$ because a no-instance of it has a certificate consisting of a proper 3-coloring $c$ of $G$ and a 3-vertex set $A \subset V(G)$ such that no 2-element subset $B$ of $A$ is defining for $c$. The latter fact, for each $B$, is certified by another proper 3-coloring $c_B$ that agrees with $c$ on $B$ but differs from $c$ somewhere outside $B$.

The completeness is proved by reduction from the decision problem whether or not $\chi(G) > 3$. The latter problem is $\text{coNP}$-complete even if restricted to connected graphs with $\chi(G) \geq 3$. Let $G$ be a such graph. Our reduction just transforms $G$ into the categorical product $G \cdot K_3$. If $\chi(G) > 3$, then $G \cdot K_3$ is uniquely 3-colorable by Lemma 2.4 and, by Lemma 5.1, we have $F^*_\chi(G \cdot K_3) = 2$. If $\chi(G) = 3$, then $G \cdot K_3$ has at least two inequivalent proper 3-colorings, namely, those induced by the two factors. By Lemma 5.1, we have $F^*_\chi(G \cdot K_3) \geq 3$.

6 General setting

In fact, many other graph characteristics also have natural forcing variants. Taking those into consideration, it will be convenient to use the formal concept of an NP optimization problem (see e.g. [12]).
Let $\{0,1\}^*$ denote the set of binary strings. The length of a string $w \in \{0,1\}^*$ is denoted by $|w|$. We will use notation $[n] = \{1, 2, \ldots, n\}$.

An NP optimization problem $\pi = (\text{opt}_{\pi}, I_{\pi}, \text{sol}_{\pi}, v_{\pi})$ (where subscript $\pi$ may be omitted) consists of the following components.

- $\text{opt} \in \{\text{max}, \text{min}\}$ is a type of the problem.
- $I \subseteq \{0, 1\}^*$ is the polynomial-time decidable set of instances of $\pi$.
- Given $x \in I$, we have $\text{sol}(x) \subseteq \{0, 1\}^*$, the set of feasible solutions of $\pi$ on instance $x$. We suppose that all $y \in \text{sol}(x)$ have the same length that depends only on $|x|$ and is bounded by $|x|^{O(1)}$. Given $x$ and $y$, it is decidable in polynomial time whether $y \in \text{sol}(x)$.
- $v : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \mathbb{N}$ is a polynomial-time computable objective function taking on positive integer values. If $y \in \text{sol}(x)$, then $v(x, y)$ is called the value of $y$.

The problem is, given an instance $x$, to compute the optimum value

$$\pi(x) = \text{opt}_{y \in \text{sol}(x)} v(x, y).$$

Such a problem is called polynomially bounded if $v(x, y) = |x|^{O(1)}$ for all $x \in I$ and $y \in \text{sol}(x)$.

Any $y \in \text{sol}(x)$ whose value is optimum is called an optimum solution of $\pi$ on instance $x$. Let $\text{optsol}(x)$ denote the set of all such $y$. Given an NP optimization problem $\pi$, we define

$$\text{UO}_{\pi} = \{ x : |\text{optsol}(x)| = 1 \}.$$

**Example 6.1** The problem of computing the chromating number of a graph is expressible as a quadruple $\chi = (\text{min}, I, \text{sol}, v)$ as follows. A graph $G$ with vertex set $V(G) = \{v_1, \ldots, v_n\}$ is represented by its adjacency matrix written down row after row as a binary string $x$ of length $n^2$. A feasible solution, that is a proper coloring $c : V(G) \rightarrow [n]$, is represented by a binary string $y = c(v_1)\ldots c(v_n)$ of length $n^2$, where a color $i$ is encoded by string $0^{i-1}10^{n-i}$. The value $v(x, y)$ is equal to the actual number of colors occurring in $y$.

**Example 6.2** For the problem of computing the clique number it is natural to fix the following representation. An instance graph $G$ is encoded as above. A feasible solution, which is a subset of $V(G)$, is encoded by its characteristic binary string of length $n$. The problem of computing the domination number is represented in the same way.

Given a non-empty set $U \subseteq \{0,1\}^l$, we define $\text{force}(U)$ to be the minimum cardinality of a set $S \subseteq [l]$ such that there is exactly one string in $U$ with 1 at every position from $S$. With each NP optimization problem $\pi$ we associate its forcing number $F_{\pi}$, an integer-valued function of instances of $\pi$ defined by

$$F_{\pi}(x) = \text{force}(\text{optsol}(x)).$$
Let $\text{FORCE}_\pi(k) = \{x : F_\pi(x) \leq k\}$. It is easy to check that, if $\chi$, $\omega$, and $\gamma$ are represented as in Examples 6.1 and 6.2, then $F_\chi$, $F_\omega$, and $F_\gamma$ are precisely those graph invariants introduced in Section 1.

Note that $\text{force}(U) = 0$ iff $U$ is a singleton. It follows that for $\pi \in \{\omega, \gamma\}$ we have

$$x \in UO_\pi \iff F_\pi(x) = 0.$$  

This will be the starting point of our analysis of decision problems $\text{FORCE}_\omega(k)$ and $\text{FORCE}_\gamma(k)$ in the next section.

7 Hardness of $\text{FORCE}_\omega(k)$ and $\text{FORCE}_\gamma(k)$

The results stated here are based on known reducibilities between several optimization problems related to our work. Since this material is dispersed through the literature with proofs sometimes skipped, for the reader’s convenience we outline some details. We first introduce some reducibility concepts for NP optimizations problems.

Let $\pi$ and $\varpi$ be NP optimization problems of the same type. Let $f : \{0, 1\}^* \rightarrow \{0, 1\}^*$ and $g : \{0, 1\}^* \times \{0, 1\}^* \rightarrow \{0, 1\}^*$ be polynomial-time computable functions such that for every $x \in I_\pi$ we have $f(x) \in I_\varpi$ and for every $y \in \text{sol}_\varpi(f(x))$ we have $g(x, y) \in \text{sol}_\pi(x)$. Such a pair $(f, g)$ is said to be an $S$-reduction from $\pi$ to $\varpi$ if for every $x \in I_\pi$ we have

$$\text{opt}_\pi(x) = \text{opt}_\varpi(f(x))$$

and, in addition, for every $y \in \text{sol}_\varpi(f(x))$ we have

$$v_\pi(x, g(x, y)) = v_\varpi(f(x), y).$$

We call an $S$-reduction $(f, g)$ a parsimonious reduction from $\pi$ to $\varpi$ if, for any $x \in I_\pi$, $g(x, \cdot)$ is a one-to-one map from $\text{sol}_\varpi(f(x))$ onto $\text{sol}_\pi(x)$. If only a weaker condition is met, namely, that $g(x, \cdot)$ is a one-to-one correspondence between the optimum solutions of $\varpi$ on instance $f(x)$ and the optimum solutions of $\pi$ on instance $x$, then $(f, g)$ will be called a weakly parsimonious reduction from $\pi$ to $\varpi$.

Given a Boolean formula $\Phi$ in the conjunctive normal form (CNF), let $\sigma(\Phi)$ denote the maximum number of clauses of $\Phi$ satisfiable by the same truth assignment to the variables. By $\sigma_3$ we denote the restriction of $\sigma$ to 3CNF formulas (those having at most 3 literals per clause). We regard $\sigma$ and $\sigma_3$ as NP optimization problems. Both problems belong to the class MAX NP introduced by Papadimitriou and Yannakakis. Crescenzi, Fiorini, and Silvestri [13], who introduced the notion of an $S$-reduction, proved that every problem in MAX NP is $S$- reducible to $\omega$, the maximum clique problem. We need a somewhat stronger fact about $\sigma_3$.

Lemma 7.1 There is a parsimonious reduction from $\sigma_3$ to $\omega$.

Proof. Let $\phi$ be a disjunctive clause and $X$ be the set of variables occurring in $\phi$. Let $D(\phi)$ denote the set of all conjunctions that contain every variable from $X$ or
its negation and imply \( \phi \). Note that \( \phi \) is logically equivalent to the disjunction of all \( \psi \) in \( D(\phi) \).

Given a 3CNF formula \( \Phi \), we construct a graph \( G \) as follows. Let \( V(G) \) be the union of \( D(\phi) \) over all clauses \( \phi \) of \( \Phi \). We join \( \psi_1 \) and \( \psi_2 \) in \( V(G) \) by an edge if these conjunctions are consistent, i.e., no variable occurring in \( \psi_1 \) occurs in \( \psi_2 \) with negation and vice versa.

**Lemma 7.2 (Thierauf [41, Section 3.2.3])** The decision problem \( U_{\text{O}_\omega} \) is US-hard.

**Proof.** Denote the restrictions of SAT and USAT to 3CNF formulas by 3SA\( T \) and U3SA\( T \) respectively. Since there is a parsimonious \( \leq_P \)-reduction from SAT to 3SAT (see e.g. [37]), U3SAT is US-complete. We now show that U3SAT \( \leq_m U_{\text{O}_\omega} \).

Given a 3CNF formula \( \Phi \), let \( G \) be the graph constructed from \( \Phi \) by the reduction of Lemma 7.1. Let \( m \) denote the number of clauses in \( \Phi \) and \( H = G + 2K_{m-1} \), the disjoint union of \( G \) and two copies of \( K_{m-1} \).

If \( \Phi \in U3\text{SA}_T \), then \( \omega(H) = \omega(G) = m \) and \( H \in U_{\text{O}_\omega} \) because \( G \in U_{\text{O}_\omega} \).

If \( \Phi \in \text{SAT} \setminus U3\text{SA}_T \), then \( \omega(H) = \omega(G) = m \) and \( H \notin U_{\text{O}_\omega} \) because \( G \notin U_{\text{O}_\omega} \).

If \( \Phi \in \text{SAT} \), then \( \omega(H) \leq m - 1 \) and \( H \notin U_{\text{O}_\omega} \) having at least two optimal cliques.

Thus, \( \Phi \in U3\text{SA}_T \) iff \( H \in U_{\text{O}_\omega} \).

**Lemma 7.3** \( U_{\text{O}_\omega} \leq_P U_{\text{O}_\gamma} \).

**Proof.** Recall that a vertex cover of a graph \( G \) is a set \( S \subseteq V(G) \) such that every edge of \( G \) is incident to a vertex in \( S \). The vertex cover number of \( G \) is defined to be the minimum cardinality of a vertex cover of \( G \) and denoted by \( \tau(G) \). It is easy to see and well known that \( S \subseteq V(G) \) is a clique in \( G \) iff \( V(G) \setminus S \) is a vertex cover in the graph complementary to \( G \). It follows that

\[
U_{\text{O}_\omega} \leq_P U_{\text{O}_\tau}.
\]  

We now show that

\[
U_{\text{O}_\tau} \leq_P U_{\text{O}_\gamma}
\]  

by regarding \( \tau \) and \( \gamma \) as NP minimization problems and designing a weakly parsimonious reduction from \( \tau \) to \( \gamma \). Recall that, given a set \( X \) and a system of its subsets \( \mathcal{Y} = \{Y_1, \ldots, Y_n\} \), a subsystem \( \{Y_{i_1}, \ldots, Y_{i_k}\} \) is called a set cover if \( X = \bigcup_{j=1}^{k} Y_{i_j} \). We compose two known reductions between minimization problems, Reduction A from the minimum vertex cover to the minimum set cover ([2, Theorem 10.11]) and Reduction B from the minimum set cover to the minimum domination number (an adaptation of [30, Theorem A.1]).

**Reduction A.** Given a graph \( G \) and its vertex \( v \), let \( I(v) \) denote the set of the edges of \( G \) incident to \( v \). Consider the set \( X = E(G) \) and the system of its subsets \( \mathcal{Y} = \{I(v) : v \in V(G)\} \). Then \( S \subseteq V(G) \) is an optimal vertex cover for \( G \) iff \( \{I(v) : v \in S\} \) is an optimal set cover for \( (X, \mathcal{Y}) \).
Reduction B. Given a set \( X = \{x_1, \ldots, x_m\} \) and a system of sets \( Y = \{Y_1, \ldots, Y_n\} \) such that \( X = \bigcup_{j=1}^n Y_j \), we construct a graph \( H \) as follows. \( V(H) \) contains each element \( x_i \) in duplicate, namely, \( x_i \) itself and its clone \( x'_i \). There is no edge between these \( 2m \) vertices. Other vertices of \( H \) are indices \( 1, \ldots, n \), with all possible \( \binom{n}{2} \) edges between them. If and only if \( x_i \in Y_j \), both \( x_i \) and \( x'_i \) are adjacent to \( j \). There are no more vertices and edges.

Observe that any optimal dominating set \( D \subseteq V(H) \) is included in \([n]\). Indeed, if \( D \) contains both \( x_i \) and \( x'_i \), then it can be reduced by replacing these two vertices by only one vertex \( j \) such that \( x_i \in Y_j \). If \( D \) contains exactly one of \( x_i \) and \( x'_i \), say \( x_i \), then it should contain some \( j \) such that \( x_i \in Y_j \) to dominate \( x'_i \). But then \( D \) can be reduced just by removing \( x_i \).

It is also clear that a set \( D \subseteq [n] \) is dominating in \( H \) iff \( \{Y_j : j \in D\} \) is a set cover for \((X, Y)\). Thus, there is a one-to-one correspondence between optimal set covers for \((X, Y)\) and optimal dominating sets in \( H \). 

Thus, \( \text{UO}_\omega \) and \( \text{UO}_\gamma \) are both US-hard.

**Lemma 7.4** Let \( \pi \in \{\omega, \gamma\} \). Then \( \text{FORCE}_\pi(k) \leq_m \text{FORCE}_\pi(k+1) \) for any \( k \geq 0 \).

**Proof.** Given a graph \( G \), we have to construct a graph \( H \) such that \( F_\pi(G) \leq k \) iff \( F_\pi(H) \leq k + 1 \). It suffices to ensure that

\[
F_\pi(H) = F_\pi(G) + 1. \tag{5}
\]

Let \( \pi = \omega \). Let \( H \) be the result of adding to \( G \) two new vertices \( u \) and \( v \) and the edges \( \{w, u\} \) and \( \{w, v\} \) for all \( w \in V(G) \). Any optimal clique in \( H \) consists of an optimal clique in \( G \) and of either \( u \) or \( v \). Hence any forcing set in \( H \) consists of a forcing set in \( G \) and of either \( u \) or \( v \) (we use the terminology of Section 1). This implies (5).

If \( \pi = \gamma \), we obtain \( H \) from \( G \) by adding a new isolated edge. 

Putting it all together, we make the following conclusion.

**Theorem 7.5** Let \( \pi \in \{\omega, \gamma\} \). Then

\[
\text{US} \leq_m \text{UO}_\pi = \text{FORCE}_\pi(0) \leq_m \text{FORCE}_\pi(k) \leq_m \text{FORCE}_\pi(k+1)
\]

for any \( k \geq 0 \).

### 8 Complexity of \( \text{FORCE}_\pi(k) \): An upper bound

We first state a simple general property of the class US.

**Lemma 8.1** Every US-complete set has an AND\(_2\) function.\(^2\)

\(^2\)In fact, a stronger fact is true: every US-complete set has an AND function of unbounded arity.
Proof. It suffices to prove the lemma for any particular US-complete set, for example, USAT. Given two Boolean formulas $\Phi$ and $\Psi$, rename the variables in $\Psi$ so the formulas become over disjoint sets of variables and consider the conjunction $\Phi \land \Psi$. As easily seen, the conjunction is in USAT iff both $\Phi$ and $\Psi$ are in USAT.

In Section 6 with a non-empty set $U \subseteq \{0, 1\}^l$ we associated the number $\text{force}(U)$. Additionally, let us put $\text{force}(\emptyset) = \infty$.

Theorem 8.2 Let $\pi$ be a polynomially bounded NP optimization problem. Then $\text{Force}_\pi(k) \leq P_{\text{det}} \text{ US}$ for each $k \geq 0$.

Proof. We will assume that $\pi$ is a minimization problem (the case of maximization problems is quite similar). Suppose that $v(x, y) \leq \mid x \mid^c$ for a constant $c$. Given $1 \leq m \leq \mid x \mid^c$, we define

$$\text{sol}^m(x) = \{ y \in \text{sol}(x) : v(x, y) = m \}$$

and

$$F^m_{\pi}(x) = \text{force}(\text{sol}^m(x)).$$

In particular, $F^m_{\pi}(x) = F_{\pi}(x)$ if $m = \pi(x)$.

Let $k$ be a fixed integer. Notice that

$$\text{Force}_\pi(x) \leq k \iff \bigvee_{m=1}^{\mid x \mid^c} (F^m_{\pi}(x) \leq k \land \pi(x) \geq m)$$

(6)

(actually, only a disjunction member where $m = \pi(x)$ can be true). The set of pairs $(x, m)$ with $\pi(x) \geq m$ is in coNP and hence in US. Let us now show that the set of $(x, m)$ with $F^m_{\pi}(x) \leq k$ is dtt-reducible to US.

Recall that $\text{sol}(x) \subseteq \{0, 1\}^{l(x)}$, where $l(x) \leq \mid x \mid^d$ for a constant $d$. Define $T$ to be the set of quadruples $(x, m, l, D)$ such that $m$ and $l$ are positive integers, $D \subseteq [l]$, and there is a unique $y \in \text{sol}^m(x)$ of length $l$ with all 1’s in positions from $D$. It is easy to see that $T$ is in US and

$$F^m_{\pi}(x) \leq k \iff \bigvee_{l, D : l \leq \mid x \mid^d} (x, m, l, D) \in T.$$  

Combining this equivalence with (6), we conclude that $F_{\pi}(x) \leq k$ iff there are numbers $m \leq \mid x \mid^c$ and $l \leq \mid x \mid^d$ and a set $D \subseteq [l]$ of size at most $k$ such that

$$(x, m, l, D) \in T \land \pi(x) \geq m.$$  

By Lemma 8.1, this conjunction is expressible as a proposition about membership of the quadruple $(x, m, l, D)$ in a US-complete set. Thus, the condition $F_{\pi}(x) \leq k$ is equivalent to a disjunction of less than $\mid x \mid^{c+d(k+1)}$ propositions each verifiable in US.

Corollary 8.3 Let $\pi \in \{\chi, \omega, \gamma\}$. Then $\text{Force}_\pi(k) \leq P_{\text{det}} \text{ US}$ for each $k \geq 0$.

Remark 8.4 Using (3) and Theorem 8.2, we can easily show that for the vertex cover number $\tau$ we also have

$$\text{US} \leq P_{\text{m}} \text{ UO}_\tau = \text{Force}_\tau(0) \leq P_{\text{m}} \text{ Force}_\tau(k) \leq P_{\text{m}} \text{ Force}_\tau(k+1) \leq P_{\text{det}} \text{ US}.$$
9 Concluding discussion and open questions

1. We have considered forcing versions of the three most popular graph invariants: the chromatic, the clique, and the domination numbers \((F_\chi, F_\omega, \text{ and } F_\gamma)\) respectively. We have shown that the slice decision problems for each of \(F_\chi, F_\omega,\) and \(F_\gamma\) are as hard as US under the many-one reducibility and as easy as US under the dtt-reducibility. The latter upper bound is actually true for the forcing variant of any polynomially bounded NP optimization problem. The lower bound in the case of \(F_\omega\) and \(F_\gamma\) is provable by using standard reductions on the account of a close connection with the unique optimum problems \(\text{UO}_\omega\) and \(\text{UO}_\gamma\). However, in the case of \(F_\chi\) we use somewhat more elaborate reductions involving graph products. We point out two simple reasons for the distinction between \(F_\chi\) and \(F_\omega, F_\gamma\). First, unlike the case of \(\omega\) and \(\gamma\), the unique colorability of a graph is apparently inexpressible in terms of \(F_\chi\) (cf. Lemma 2.7.3). Second, we currently do not know any reductions between \(F_\chi, F_\omega,\) and \(F_\gamma\) as optimization problems that would allow us to relate their complexities (cf. further discussion).

2. We have shown that the slice decision problems \(\text{FORCE}_\pi(k)\) for \(\pi \in \{\chi, \omega, \gamma\}\) are close to each other in the complexity hierarchy. Furthermore, let \(\text{FORCE}_\pi(*) = \{(x, k) : F_\pi(x) \leq k\}\). Hamed Hatami (personal communication) has recently shown that, like \(\text{FORCE}_\chi(*)\), the decision problem \(\text{FORCE}_\omega(*)\) is \(\Sigma^P_2\)-complete. Consequently, the problems of computing \(F_\chi\) and \(F_\omega\) are polynomial-time Turing equivalent. It would be also interesting to compare the complexities of \(F_\chi, F_\omega,\) and \(F_\gamma\) using weaker reducibility concepts for optimization problems (as is well known, the similarity of decision versions does not imply the similarity of the underlying optimization problems; For example, the decision versions of \(\chi, \omega,\) and \(\gamma\) are all NP-complete and parsimoniously equivalent but have pairwise different parametrized complexities and the corresponding optimization problems have pairwise different approximation complexities).

3. Is \(\text{FORCE}_\pi(k)\) NP-hard under \(\leq_m^P\)-reductions for any \(\pi\) under consideration and constant \(k\)? It should be noted that the affirmative answer would settle a long-standing open problem if \(C=\text{P}\) contains NP in the affirmative.

4. Let \(\text{UCOL}\) be the set of all uniquely colorable graphs (with no restriction on the chromatic number). Is it true that \(\text{UCOL} \leq_m^P \text{FORCE}_\chi(2)\)? It is not hard to show that \(\text{UCOL}\) is US-hard.

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