On the Gauge Invariance of the $Z$-Boson Mass

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Abstract

The different schemes for the definition of the $Z$ boson mass are analyzed. It is shown that the scheme, defining the mass as pole of the real part of the $Z$ boson propagator and the width as the imaginary part of the propagator at the same point results in the gauge dependent results for these parameters in a two-loop approximation. On the other hand, the scheme, where the mass and width are related to the position of the pole of the propagator in the complex plane leads to the gauge independent result. It is argued that the gauge dependence of mass and width does not contradicts to the gauge invariance of the amplitude.

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1 Introduction

Recently the problem of the definition of $Z$ boson mass and width has been discussed \[\text{[1, 2]}\]. The problem refers to the definition of the physical parameters of unstable particle. The $Z$ boson’s $in$ and $out$ states can not be considered as the eigenvectors of the energy-momentum operator (clearly this is true for any unstable system). Thus we have no unambiguous theoretical prescription for the definition of $Z$ boson mass. The conventional way is to extract the mass and the width from the amplitudes of processes containing the corresponding resonance in the $S$-channel. E.g. the amplitude of the process $e^+e^- \rightarrow f\bar{f}$ near its peak position is often parameterized as \[\text{[2, 3]}\]:

$$A(S) = \frac{\mathcal{R}}{S - M_Z^2 + i\Gamma_Z M_Z} + r(S)$$  \hspace{1cm} (1)

or

$$A(S) = \frac{\mathcal{R}}{S - (M_Z - i\Gamma_Z/2)^2} + r(S),$$  \hspace{1cm} (2)

where $\mathcal{R}$ is the residue, the remainder, less singular part of the amplitude is designated by $r(S)$ and it is assumed that the first term numerically exceeds $r$. The parameterization (1)-(2) may be obtained from the conditions on the Green function of $Z$ boson field. These conditions define the renormalization scheme, and as a consequence, the resonance parameters.

In consideration of the mass and width two schemes are prevalent.

In the framework of the first scheme the resonance parameters are defined as $M_Z^2 = Re S_p, M_Z\Gamma_Z = Im S_p$ \[\text{[2]}\], where $S_p$ is the position of the pole of propagator in a a complex $S$-plane \[\text{[2]}\]:

$$D^{-1}(S_p) = 0$$  \hspace{1cm} (3)

In (2) $D^{-1}$ is the denominator of the exact propagator of the $Z$ boson.

In the another, so called on mass-shell scheme, the variable $S$ remains real and the definition is following \[\text{[3]}\]:

$$Re D^{-1}(M_Z^2) = 0, \quad M_Z\Gamma_Z = Im D^{-1}(M_Z^2)$$  \hspace{1cm} (4)

Of course, many different schemes can be introduced. E.g. the mass can be defined as the peak position of the amplitude, but in this case it would depend on the process under consideration. From the theoretical point of view it is preferable that the mass of an unstable particle, in a full analogy with the case of stable one, is defined without referring to any particular element of the $S$ matrix but the propagator. The condition imposed on the propagator of an unstable particle is a realization of the scheme defining mass and width.

In this paper we investigate the gauge dependence of the resonance parameters, defined in schemes (3) and (4). Note, that the gauge independence of the physical mass and width is the separate requirement, not being the direct consequence of the
gauge independence of amplitude (1)-(2). In other words, from the requirement of
gauge independence of observables (in our case - Z line shape) it does not follow
necessarily that the resonance parameters should be gauge invariant. In discussing
the gauge dependence one expects that since the pole of the amplitude is a physical
quantity, the mass and the width, defined through scheme (3) are gauge independent,
while it is not evident in case of scheme (4). Below we check this argument in the
framework of perturbation theory.

In section 2 we demonstrate that $M_Z$ and $\Gamma_Z$, defined in scheme (3), are gauge
independent in the two-loop approximation while the scheme (4) leads to the gauge
dependence of these parameters (from [2] we have learned that A. Sirlin has arrived
to the same conclusion about the scheme (4). Unfortunately at the present time
the publication [4] is not available to us and hence we are not able to compare our
arguments).

In section 3 it is argued that if we use the scheme (4) or any other scheme with
the gauge dependent $M_Z$ and $\Gamma_Z$, the scattering amplitude is still gauge independent.

We use the bare Lagrangian in which the Higgs field condensate $v$ and the $ZA$
mixing are chosen in the tree approximation [3]. The input parameters are the bare
electromagnetic charge $e_0$ and the bare masses $M_{0i}$. Calculations are performed in $R_\xi$
gauge in the framework of dimensional regularization. We use Feynmann rules from
the review [3].

2 Two-loop Analysis of Gauge Dependence

To investigate the gauge dependence of $M_Z$ and $\Gamma_Z$ defined in the scheme (3) let us
express $S_P$ as follows:

$$S_P = M_{0Z}^2 + e_0^2 \delta M_1^2 + e_0^4 \delta M_2^2 + ..., (5)$$

where the complex numbers $\delta M_i$ do not depend on $e_0$.

It is straightforward to show that the contributions from terms with the gauge
parameters $\alpha_W$, $\alpha_Z$ and $\alpha_A$ are factorized in the expression for $\delta M_i^2$. The most
time-taking part of calculations in $R_\xi$ gauge is that with the gauge parameter $\alpha_W$
[3]. Therefore, in the two-loop approximation where there is no factorization we will
investigate only the sector with $\alpha_W$ to which we refer throughout this paper as $\alpha$.

This parameter appears in the free propagators of $W$, $\chi$ and $c$ (for the denotions
see [3]; $\chi$ and $c$ correspond to the pseudogoldstone and ghost degrees of freedom
appearing in $R_\xi$ gauge):

$$D_{WW}^{\mu\nu}(p) = \frac{1}{p^2 - M_{0W}^2} \left( g^{\mu\nu} - \frac{p^\mu p^\nu}{M_{0W}^2} \right) + \frac{p^\mu p^\nu}{M_{0W}^2 (p^2 - \alpha M_{0W}^2)}$$

(6)

and

$$D_{\chi}(p) = -\frac{1}{p^2 - \alpha M_{0W}^2}, \quad D_{c\pm}(p) = -\frac{1}{p^2 - \alpha M_{0W}^2}$$

(7)
For brevity we will term the denominators with $\alpha$ as $\alpha$-terms.

One loop correction to the $Z$ boson self energy contains terms, linear in $\alpha$ (i.e. integrand contains only one $\alpha$-denominator) as well as $\alpha^2$-terms (the product of two $\alpha$-denominators). Two loop correction contains $\alpha$, $\alpha^2$ and $\alpha^k$, $k > 2$ terms. The simple calculation based on Ward identities demonstrates that $\alpha^k$, $k > 2$ terms cancel trivially, and the technique is exactly the same as in the case of stable particles. Further we will discuss only $\alpha$ and $\alpha^2$-terms which are present in one-loop and two-loop corrections and interfere with each other.

The $\alpha^2$-terms appear only in the diagram Fig.1,a) and all the diagrams contain terms, linear in $\alpha$. Even now two-loop calculations are still tedious.

To proceed further let us note that in the sense of the gauge dependence the crucial difference between the results following from the schemes (3) and (4) is caused by fermion loops, leading to $\text{Im} \delta M^2_1 \neq 0$. Since our goal is to establish the (possible) gauge dependence of $M_Z$ and $\Gamma_Z$, in the gauge independent part of $\delta M^2_1$ we retain the contribution from the lepton sector only. The quark loop contribution gives the similar result.

Summarizing, we consider only $\alpha$ and $\alpha^2$ terms from the "$\alpha$-sector" and in the expression for $\delta M^2_1$ only the contributions from the electron and neutrino loops are taken into account.

After these introductory remarks we are in a position to discuss the gauge dependence of $M_Z$ and $\Gamma_Z$.

Let us consider first the one-loop contribution to $\Pi_{\mu\rho}$ - self energy of the $Z$ boson. To extract parameters $M_Z$ and $\Gamma_Z$ it is enough to consider the coefficient of $g_{\mu\rho}$. Therefore we will follow only the integrals, leading to this tensor structure. The corresponding contribution of $\alpha$ and $\alpha^2$ terms in $\Pi_{\mu\rho}$ is:

$$\Pi_{\mu\rho}(p, \alpha) = -\epsilon^2_0 \frac{(p^2 - M^2_{WZ})(p^2 + M^2_{Z})}{M^2_{W}(M^2_{WZ} - M^2_{Z})} J_{1\mu\rho}$$

}\)
\[ e_0^2 g_{\mu\rho} \left[ \frac{2}{M_{\mu\rho}^2 - M_{\mu\rho}^2} J_1 + 4 (M_{\mu\rho}^2 - p^2) J_2 \right] - e_0^2 \frac{4 (M_{\mu\rho}^2 - p^2)}{M_{\mu\rho}^2} J_{2\mu\rho}, \]  

where \( p \) is the \( Z \) boson momentum and

\[ J_{1\mu\rho} = i \int d^n q \ \frac{q_{\mu} q_{\rho}}{(q^2 - \alpha M_{\mu\rho}^2)(p + q)^2 - \alpha M_{0\mu}^2)}, \]  

\[ J_{2\mu\rho} = i \int d^n q \ \frac{q_{\mu} q_{\rho}}{(q^2 - M_{\mu\rho}^2)(p + q)^2 - \alpha M_{0\mu}^2)}, \]  

\[ J_1 = i \int d^n \frac{1}{q^2 - \alpha M_{\mu\rho}^2}, \]  

\[ J_2 = i \int d^n q \ \frac{1}{(q^2 - M_{\mu\rho}^2)(p + q)^2 - \alpha M_{0\mu}^2)} \]  

In (9)-(12) \( n \) is a space-time dimension. In the one-loop approximation \( \Pi_{\mu\rho} \) can be expressed as

\[ \Pi_{\mu\rho} = g_{\mu\rho} (p^2 - M_{\mu\rho}^2) - e_0^2 \left[ \Pi_{\mu\rho}^{(1)inv}(p^2) + \Pi_{\mu\rho}(\alpha) \right], \]  

where \( \Pi_{\mu\rho}^{(1)inv} = g_{\mu\rho} \Pi_{\mu\rho}^{(1)inv} \) and \( \Pi_{\mu\rho}(\alpha) \) are correspondingly gauge independent and gauge dependent parts. The gauge independence of \( M_Z \) and \( \Gamma_Z \) in one-loop approximation is evident in both schemes, since in the scheme (3), where \( p^2 = S_p \) we have

\[ S_P - M_{\mu\rho}^2 = e_0^2 \Pi_{\mu\rho}^{(1)inv}(M_{\mu\rho}^2), \]  

and in the scheme (4) where \( p^2 = M_{\mu\rho}^2 \), we obtain:

\[ M_Z^2 - M_{\mu\rho}^2 = e_0^2 Re \ \Pi_{\mu\rho}^{(1)inv}(M_{\mu\rho}^2), \]  

\[ M_Z \Gamma_Z = e_0^2 Im \ \Pi_{\mu\rho}^{(1)inv}(M_{\mu\rho}^2) \]  

Let us proceed to the order \( e_0^4 \). The self energy in the scheme (3) can be expressed as

\[ \Pi_{\mu\rho}(S_P, \alpha) = -e_0^4 \left( \Pi_{\mu\rho}^{(1)inv}(M_{\mu\rho}^2) \right)^2 I_{\mu\rho} + O(e_0^6), \]  

where

\[ I_{\mu\rho} = \frac{2 M_{\mu\rho}^2}{M_{\mu\rho}^2 - M_{\mu\rho}^2} J_{1\mu\rho} - \frac{2}{M_{\mu\rho}^2 - M_{\mu\rho}^2} J_{1\mu\rho} - \frac{4}{M_{\mu\rho}^2} J_{2\mu\rho} \]  

and in the scheme (4) \( Re \ \Pi_{\mu\rho} \) can be expressed as

\[ Re \ \Pi_{\mu\rho}(\alpha, p^2)|_{p^2=M_Z^2} = -e_0^4 Re \ \Pi_{\mu\rho}^{(1)inv} Re \ I_{\mu\rho} \]
Figure 2: Two-loop contribution from "$\alpha$ - sector" in $Z$ boson self energy.

Let us remind that in $\Pi^{(1)inv}_{\mu\nu}$ we consider contributions only from the electron and neutrino. Then, to investigate the gauge dependence in order $e^4_0$ it is necessary to consider the two-loop diagrams, Fig. 2 a) and b):

The contribution from $\alpha^2$-terms in $\Pi_{\mu\nu}$ from the diagram Fig. 2 a) at $p^2 = M^2_0$ is equal to:

$$-i e^4_0 \frac{M^4_0}{4 M^2_{0W} (M^2_0 - M^2_{0W})} J_{1\mu\sigma} \int d^4 q \, tr \left[ G(p + q) \Gamma_\sigma G(q) \gamma_\rho \right], \quad (19)$$

where $G(p) = 1/(m_0 - \hat{p})$ is the free electron propagator and

$$\Gamma_\sigma = \gamma_\sigma \left[ -1 + 4 \frac{M^2_0 - M^2_{0W}}{M^2_0} + \gamma_5 \right] \quad (20)$$

is the $Ze^+e^-$ vertex. Let us discuss the cancelation technique for the terms similar to (19) for the diagram Fig. 2 b), containing two $W$ bosons. First of all, we consider the fermion loop contribution multiplied on virtual $W$ boson’s momenta that correspond to $\alpha^2$-terms:

$$\int \frac{d^4 kd^4 q}{(k^2 - \alpha M^2_{0W})((p + k)^2 - \alpha M^2_{0W})} \Gamma_{\sigma\beta\rho}(-k, p + k, -p) \, tr \left[ G(p + q) \Gamma_\mu G(q) \hat{k}(1 - \gamma_5) S(q - k)(\hat{p} + \hat{k}) \right], \quad (21)$$

where $\Gamma_{\sigma\beta\rho}$ is the $ZW^+W^-$ vertex and $S(p) = -1/\hat{p}$ is the free neutrino propagator. Using $\hat{k} = m_0 - G^{-1}(q) + S^{-1}(q - k)$ and $\hat{p} + \hat{k} = m_0 - G^{-1}(p + q) + S^{-1}(q - k)$ for
the trace in (19) we obtain:

$$tr \left[ G(p + q)\Gamma_\mu G(q)(1 + \gamma_5)(\hat{p} + \hat{k}) + m_0^2 G(p + q)\Gamma_\mu G(q)(1 - \gamma_5)S(q - k) + m_0 G(p + q)\Gamma_\mu G(q)(1 - \gamma_5) - m_0 G(p + q)\Gamma_\mu(1 - \gamma_5)S(q - k) - m_0\Gamma_\mu G(q)(1 - \gamma_5)S(q - k) \right]$$

(22)

The terms with the three fermion propagators are canceled at $p^2 = M_{0Z}^2$ in the sum of all the diagrams of type Fig. 2 b). Without this cancelation the gauge dependence of $M_z$ and $\Gamma_Z$ is unavoidable. The terms in (22), containing the product of propagators $G$ and $S$ are canceled by the $\alpha^2$-terms, originated from the diagram Fig. 2 c). Now, the sum of diagrams Fig. 2 a) and b) is

$$ie_0^4 \frac{M_{0Z}^6 J_{1\rho\sigma}}{16M_{0W}^2(M_{0Z}^2 - M_{0W}^2)} \int d^n q \ tr \left[ G(p + q)\Gamma_\mu G(q)\Gamma_\sigma + S(p + q)\gamma_\mu(1 - \gamma_5)S(q)\gamma_\rho(1 - \gamma_5) \right]$$

(23)

It is straightforward to demonstrate that the terms in (23), leading to the tensor structure $g_{\mu\rho}$ can be expressed as:

$$e_0^4 \frac{2M_{0Z}^2}{M_{0W}^2(M_{0Z}^2 - M_{0W}^2)} \left[ \Pi^{(1)inv}_{(e)}(M_{0Z}^2) + \Pi^{(1)inv}_{(\nu)}(M_{0Z}^2) \right] J_{1\mu\rho},$$

(24)

where $\Pi^{(1)inv}_{(e)}$ and $\Pi^{(1)inv}_{(\nu)}$ refer to contributions of electron and neutrino. The $\alpha$-terms is canceled in the sum of the expressions (16) and (24), and it turns out that $S_p$ is gauge invariant in order of $e_0^4$. On the other hand, the sum of the real parts of (18) and (23) still depends on $\alpha$. Therefore, the scheme (3), in distinct with (4), defines the gauge independent parameters $M_Z$ and $\Gamma_Z$.

We have checked the gauge independence of $S_p$ also in the framework of conventional covariant gauge, when the gauge fixing term is $\partial_\mu W_\mu^2/2\alpha$. In this case the diagrams type of Fig. 1 b) and c) are gauge independent. The cancelation of the gauge dependent terms occurs in a full analogy with the case of $R_\xi$ gauge.

A few words about $ZA$ mixing: when we define the mixing scheme by the requirement that Green’s function $\langle ZA \rangle$ has no poles (i.e. the mass-shell for the photon is defined only from $\langle AA \rangle$ and $S_p$ is defined only from $\langle ZZ \rangle$), there arise a new effective vertices but $S_p$ does not change. Note that the same value of this parameter is obtained as a position of pole of the Green’s functions $\langle aa \rangle$, $\langle B_3B_3 \rangle$ and $\langle aB_3 \rangle$, where $a$ and $B_3$ are the original gauge fields of the gauge groups $U(1)$ and $SU(2)$ leading after mixing to $A$ and $Z$.

Note also that if we take into account the quantum corrections in the definition of the Higgs condensate in the bare Lagrangian, i.e. $v = v_0 + \delta v$, then $\langle \phi \rangle = 0$, but the parameter $S_p$ would be the same. Moreover, the replacement $v_0 \rightarrow v$ does not affect the Green’s function which does not contain the Higgs field $\phi$ on external lines.
3 Gauge dependence of $M_Z$ and $\Gamma_Z$ and the Gauge Invariance of the Amplitude

As we have demonstrated in section 2, the parameters $M_Z$ and $\Gamma_Z$ defined in scheme (3) do not depend on the gauge parameter in the two-loop approximation. Since the parameters $M_Z$ and $\Gamma_Z$ defined in the scheme (4) are gauge dependent, it might seem that the scheme (3) is preferable. Below we show that the requirement of the gauge independence as well as the requirement of numerical convergence does not lead in principle to the preference of one particular scheme.

Let us express $M_Z$ and $\Gamma_Z$ in the framework of regularized theory (i.e. $n \neq 4$) as follows:

$$M_Z = M_{0Z} \sum_{k=1}^{\infty} e^{2k}\delta_{1k}, \quad \Gamma_Z = e^2 M_Z^{n-3} \sum_{k=1}^{\infty} e^{2k}\delta_{2k}$$  \hspace{1cm} (25)

It is possible to resolve $M_{0Z}$ and $e^2$ in terms of $M_Z$ and $\Gamma_Z$:

$$M_{0Z} = M_Z \sum_{k=1}^{\infty} \left( \frac{\Gamma_Z}{M_Z} \right)^k \Delta_{1k} = M_Z Z_{M_Z},$$  \hspace{1cm} (26)

and

$$e^2 = \frac{\Gamma_Z}{M_Z^{n-3}} \sum_{k=1}^{\infty} \left( \frac{\Gamma_Z}{M_Z} \right)^k \Delta_{2k} = \Gamma_Z Z_{\Gamma_Z},$$  \hspace{1cm} (27)

where $Z_{M_Z}$ and $Z_{\Gamma_Z}$ are the mass and width renormalization factors.

Clearly, the coefficients in expressions (25) - (27) are gauge independent when calculated in the scheme (3) and depend on a gauge in the case of scheme (4).

If we substitute (26)-(27) in the expression for the physical quantity, e.g. the amplitude $\langle e^+ e^- | f \bar{f} \rangle$ we obtain the ultraviolet finite result which may be parameterized as (1)-(2), or as follows:

$$A(S) = \sum_{k=1}^{\infty} \left( \frac{\Gamma_Z}{M_Z} \right)^k a_k(S)$$  \hspace{1cm} (28)

The gauge independence of $A(S)$ in a scheme (3) is evident since the coefficients $a_k(S)$ do not depend on gauge parameters. In scheme (4) the coefficients $a_k(S)$ contain $\alpha$ explicitly but this does not lead to the gauge dependence of amplitude $A(S)$. Indeed, for this case, in the expression

$$\frac{dA}{d\alpha} = \frac{\partial A}{\partial \alpha} + \frac{\partial A}{\partial \Gamma_Z} \frac{d\Gamma_Z}{d\alpha} + \frac{\partial A}{\partial M_Z} \frac{dM_Z}{d\alpha}$$  \hspace{1cm} (29)

every term is non vanishing but in the sum they cancel since the scheme (4) is connected with the scheme (3) via the renormalization group equations with respect to the parameter $\alpha$. These equations can be obtained from (26)-(27):

$$\frac{dM_Z}{d\alpha} = \beta_{M_Z} M_Z, \quad \frac{d\Gamma_Z}{d\alpha} = \beta_{\Gamma_Z} \Gamma_Z,$$  \hspace{1cm} (30)
where
\[ Z_{M_2} \beta_{M_2} \equiv -\frac{dZ_{M_2}}{d\alpha}, \quad Z_{\Gamma_2} \beta_{\Gamma_2} \equiv -\frac{dZ_{\Gamma_2}}{d\alpha} \] (31)

The similar equations for the other masses \( M_i \) can be easily obtained. It is evident that if we consider the scheme where \( \beta_{M,\Gamma} \neq 0 \), the independence of physical quantities on \( \alpha \) appears in a full analogy with the independence on a renormalization point. If we use (29) and (30), we would obtain from (28) that
\[
\frac{d}{d\alpha} \sum_{k=1}^{N} \left( \frac{\Gamma_Z}{M_Z} \right)^k a_k(S) = O\left( \left( \frac{\Gamma_Z}{M_Z} \right)^{N+1} \right)
\] (32)
in each order of \( \Gamma_Z \). Clearly, the same relations hold for the remaining gauge parameters. Relation similar to (32) with respect to renormalization point \( \mu \) states that (see, e.g. [5]):
\[
\frac{dP_N(\mu, g(\mu))}{d\mu} \equiv \frac{d}{d\mu} \sum_{k=1}^{N} g^k(\mu) P_k = O\left( g(\mu)^{N+1} \right),
\] (33)
where \( P_N \) is a physical quantity calculated up to order \( N \)-th order in renormalized coupling \( g(\mu) \). In any particular order of perturbation theory the response of a physical quantity on \( \mu \to \mu + d\mu \) is of a next order in \( g(\mu) \). Evidently, the exact expression is the renormalization point (and gauge parameter) independent. So, the requirement of the gauge independence of the amplitude \( \langle e^+e^-|f\bar{f} \rangle \) is not sufficient to prefer scheme (3).

Let us consider the problem of numerical analysis. The peak position of the amplitude, \( S_0 \), is defined as the solution of the equation:
\[
\frac{d}{dS} \mid A(S) \mid = 0
\] (34)
It is easy to show that in a scheme (3) we obtain
\[
S_0 = M_Z^2 + \frac{\Gamma_Z^3}{M_Z^2} \sum_{k=0}^{\infty} \left( \frac{\Gamma_Z}{M_Z} \right)^k C_k(M_i),
\] (35)
while the scheme (4) leads to
\[
S_0 = M_Z^2 + \frac{\Gamma_Z^2}{M_Z^2} \sum_{k=0}^{\infty} \left( \frac{\Gamma_Z}{M_Z} \right)^k C_k^*(M_i)
\] (36)
Evidently, if the series (35) and (36) converge fast enough, the scheme (3) is preferable when \( \Gamma_Z \ll M_Z \). But from the theoretical point of view this is rather troublesome to confirm. Indeed, the coefficient functions \( C_k \) and \( C_k^* \) depend on the masses of all the particles of the standard model, so their values can not be estimated with the help of the current data.
Note that there exists other scheme which leads to the same result as in a two-loop approximation as the scheme (3). For example one can define a new scheme by the relation:

\[
\frac{d}{dS} \mid D^{-1}(S) \mid_{S=M_Z^2} = 0
\]  

(37)

The simple calculation shows that \(M_Z\), defined by (37) differs from the result of scheme (3) only in order \(e_6^0\). In other words, in a two-loop approximation the scheme (37) leads to the gauge independent \(M_Z\) and this \(M_Z\) satisfies relation, similar to (3). The results following from these two schemes differ only in three-loop approximation.

4 Discussion

In this paper, based on the gauge independence and numerical analysis we have investigated the schemes defining the mass and the width of \(Z\) boson.

The relations (35) and (36) demonstrate that the schemes (3) and (37) may be preferable. On the other hand, since in a scheme (4) the arbitrary parameter \(\alpha\) remains, the problem of convergence in (36) requires the consideration of the renormalization group equations (30). It may occur that for some numerical values of \(\alpha\) the series (36) would converge faster than the series (35). It is not surprising since from (29)-(30) it is evident that the gauge parameter \(\alpha\) plays the role, analogous of renormalization point and thus, in principle, the variation of \(\alpha\) can improve the convergence of series in \(\Gamma_Z/M_Z\). Hence, the requirement of the numerical convergence can not be used in defining the physical mass and the width of the \(Z\) boson, since this requirement may be in contradiction with requirement of gauge invariance of these parameters. Note also that the result of numerical analysis depends essentially on the process under consideration.

From the requirement of gauge independence the schemes (3) and (37) are preferable. Unfortunately, this principle is necessary but not sufficient and therefore does not leads to the unique choice of a scheme. Indeed, besides (3) and (37), infinitely many schemes for defining the gauge independent parameters can be pointed out. For example let us express the \(Z\) boson propagator as

\[
D = D^{\text{inv}} + D^{*},
\]

(38)

where \(D^{\text{inv}}\) stands for the contribution of physical degrees of freedom (i.e. \(D^{\text{inv}}\) is gauge independent) and \(D^*\) varies from gauge to gauge. Consider now any conditions on real and imaginary parts of \(D^{\text{inv}}\), leading to the ultraviolet renormalization. It is evident that these conditions define two gauge invariant parameters which may be treated as \(M_Z\) and \(\Gamma_Z\).

Despite of this non uniqueness, the schemes (3) and (37) are remarkable. Indeed, \(M_Z\) and \(\Gamma_Z\) defined in scheme (3) are connected with the position of the pole of propagator on the complex plane and in scheme (37) they can be obtained from the
position of the peak of amplitude on real axis. Apparently it is the scheme (3) that defines the physical mass and the width of Z boson, since, from our point of view, only this scheme allows us to formulate the Lehmann-Symanzik-Zimmermann (LSZ) reduction technique \([5]\) for the processes involving Z boson in initial or final states.

Some papers contain attempts to generalize LSZ technique for unstable particles (see e.g [6]) but we think that more thoroughful analysis is necessary. Our recent calculations show that the residue of the amplitude \(\langle e^+ e^- | f \bar{f} \rangle\) in a two-loop approximation is gauge independent at \(S = S_p\). This fact already indicates the incontrovertible preference of scheme (3) since the gauge independence of the residue apparently leads to the gauge independent amplitude for the decay \(Z \rightarrow f \bar{f}\).

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These results were presented at the Conference "Quarks-92", held in Zvenigorod, Russia, 1992. Since then the accuracy in defining of the Z-boson line shape has been substantially improved what require the consideration of the higher order corrections. Therefore, our results are still actual and worth appearing in the Archive. The text of original version is slightly changed.

The preprint [4] is published in: A.Sirlin, *Phys. Rev. Lett* **67**, 2127, (1991).