Quantumness of Product States

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Product states do not violate Bell inequalities. In this work, we investigate the quantumness of product states by violating a certain classical algebraic models. Thus even for product states, statistical predictions of quantum mechanics and classical theories do not agree. An experiment protocol is proposed to reveal the quantumness.

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The physical picture in quantum mechanics dramatically differs from that in classical local realistic theories. For instance, the intriguing nonlocal correlation called entanglement predicted by quantum mechanics has drawn many researchers’ attention, and fruitful applications have been achieved in quantum information science [1]. Another significant property is nonlocality, detected by the violation of Bell inequalities. Gisin theorem [2] states that any two-qubit entangled pure states violates Bell inequality. But in general these two concepts are different from one another, and so far there has been no clear bound that separate the degree of entanglement and nonlocality [3]. Moreover, The experiment tests for excluding local realistic theories have long been suffering from simultaneously closing the loopholes of locality and detection [4, 5].

When the system $\rho$ is in the separable state, i.e., $\rho = \sum p_i \rho_A^i \otimes \rho_B^i$, Bell inequalities will then not be violated, which implies the possibility of simulating the system by a certain hidden-variable local realistic theory. In 2002, Ollivier and Zurek proposed quantum discord [10] as a novel measurement for multi-particle correlations. In their definition, separable states have non-vanishing quantum discord, unless only one component $\rho_A^i \otimes \rho_B^i$ remains. Throughout the paper, this kind of state is referred to as a product state. Consequently, a product state is usually considered as the classical state for quite a long period.

In this work, surprisingly, we show in product states of multi-particle systems, there exists quantumness, defined by the violation of Alicki-Van Ryn (AR) inequality [11]. It seems that quantumness is quite common in physical systems.

In algebraic model [12], observables are elements of a certain $C^*$-algebra, and states are positive normalized functionals $A \rightarrow \langle A \rangle_\rho$ with $\langle A \rangle_\rho$ denoting the mean value of the observable $A$ in the state $\rho$. In a classical algebraic model, any two elements $A$ and $B$ are commutative, then one gets the following implication

$$0 \leq A \leq B \Rightarrow A^2 \leq B^2. \quad (1)$$

The mean value of the observable $\langle A \rangle_\rho$ has two definitions: (i) classically, $\langle A \rangle_\rho$ is defined by $\int A(x) \rho(x) dx$ where $\rho(x)$ is some probability distribution, and (ii) in quantum mechanics $\langle A \rangle_\rho$ is defined by $Tr(\rho A)$ where $\rho$ is the system state. In other words, a classical model must satisfy the AR inequality

$$\langle A \rangle \geq 0, \quad (2)$$
$$\langle B \rangle \geq 0, \quad (3)$$
$$\langle B - A \rangle \geq 0, \quad (4)$$
$$\langle B^2 - A^2 \rangle \geq 0. \quad (5)$$

However, in quantum mechanics there exist noncommutative observables that violate the fourth constraint, namely, one can find positive-definite observables $A$ and $B$ satisfying $\langle A \rangle \geq 0$, $\langle B \rangle \geq 0$, $\langle B - A \rangle \geq 0$, while $\langle B^2 - A^2 \rangle < 0$. This violation is called quantumness. Experimental tests have been performed for the case of one qubit.

For two-qubit product state $|\phi\rangle = |0\rangle \otimes |0\rangle$, one can write down the following positive-definite matrices $A$ and $B$:

$$A =
\begin{pmatrix}
1 + \frac{\sqrt{295}}{40} & 0 & 0 & -\frac{27}{40} \\
0 & 1 + \frac{\sqrt{295}}{40} & -\frac{25}{160} & 0 \\
0 & -\frac{25}{160} & 1 - \frac{\sqrt{295}}{40} & 0 \\
-\frac{27}{40} & 0 & 0 & 1 - \frac{\sqrt{295}}{40}
\end{pmatrix}, \quad (6)
$$

$$B =
\begin{pmatrix}
\frac{3}{2} & 0 & 0 & -\frac{9}{20} \\
0 & \frac{5}{8} & -\frac{5}{8} & 0 \\
0 & -\frac{5}{8} & \frac{5}{8} & 0 \\
-\frac{9}{20} & 0 & 0 & \frac{1}{2}
\end{pmatrix}, \quad (7)
$$

so that

$$\langle \phi | A | \phi \rangle = 1 + \frac{\sqrt{295}}{40} \geq 0, \quad (8)$$
$$\langle \phi | B | \phi \rangle = \frac{3}{2} \geq 0, \quad (9)$$
$$\langle \phi | (B - A) | \phi \rangle = \frac{20 - \sqrt{295}}{40} \geq 0 \quad (10)$$

hold, while

$$\langle \phi | (B^2 - A^2) | \phi \rangle = \frac{65 - 4\sqrt{295}}{80} < 0, \quad (11)$$

indicating the quantumness of this two-qubit product state.

A proposed experiment protocol is to find a system whose Hamiltonian is in the form of $B^2 - A^2$, then measuring the
ground-state energy would reveal the quantumness. Any $4 \times 4$ Hermitian operator can be in the following form

$$\mathcal{H} = \sum_{i,j=0}^{3} \beta_{ij} \sigma_i \otimes \sigma_j,$$

where $\sigma_0$ is unity identity, $\sigma_{1,2,3}$ is Pauli operators, and coefficient $\beta_{ij}$ is defined by $\text{Tr}(\mathcal{H} \sigma_i \otimes \sigma_j)$. If Hamiltonian $H = B^2 - A^2$, for instance, then the system state is

$$\rho = \frac{e^{-H/kT}}{\text{Tr}(e^{-H/kT})} = \frac{e^{-E_n/kT} \ket{n} \bra{n}}{\sum_{n=1}^{4} (e^{-E_n/kT})},$$

with temperature $T$. The last step is justified by the fact that $H$ is diagonal in our two-qubit case. When $T \to 0$, all positive-energy weights $e^{-E_n/kT}$ vanish; and given the ratio of energy of ground and excited states, only the weight for ground state energy remains, hence $\rho \to \ket{00} \bra{00}$, a nondegenerate pure state.

The generalization to arbitrary-qubit systems can be realized by enlarging matrices $A$ and $B$. Firstly, one has to find several groups of positive-definite $2 \times 2$ matrices $A$ and $B$ that violate the inequality [1]. Secondly, insert the matrix elements of $A_2$ inside $A_1$, that is,

$$\begin{pmatrix}
(A_1)_{11} & (A_1)_{12} \\
(A_2)_{11} & (A_2)_{12}
\end{pmatrix}
\begin{pmatrix}
(A_1)_{11} & (A_1)_{12} \\
(A_2)_{11} & (A_2)_{12}
\end{pmatrix}
\begin{pmatrix}
(A_1)_{21} & (A_1)_{22} \\
(A_2)_{21} & (A_2)_{22}
\end{pmatrix}.$$

The rest entries are set to zero. Thus $A_1$ has been enlarged into a $4 \times 4$ matrix $A$; similarly, one can obtain the enlarged $4 \times 4$ matrix $B$, as shown in [7]. Here we suppose the minimal eigenvalue of $B_2^2 - A_2^2$ is less than that of $B_2^2 - A_2^2$ without generality. Thirdly, by repeating the second step, one finally obtains the desired matrices that violate [1].

$$\begin{pmatrix}
(A_1)_{11} & \cdots & (A_1)_{12} \\
\cdots & \cdots & \cdots \\
(A_2)_{11} & (A_2)_{12} \\
(A_2)_{21} & (A_2)_{22}
\end{pmatrix}
\begin{pmatrix}
(A_1)_{11} & \cdots & (A_1)_{12} \\
\cdots & \cdots & \cdots \\
(A_2)_{11} & (A_2)_{12} \\
(A_2)_{21} & (A_2)_{22}
\end{pmatrix}
\begin{pmatrix}
(A_1)_{21} & \cdots & (A_1)_{22} \\
\cdots & \cdots & \cdots \\
(A_2)_{21} & (A_2)_{22}
\end{pmatrix}.$$

Note that the enlarging process has been performed in such a way that the ground state is always $(1, 0, 0, ..., 0)^T$. Generally speaking, $A$ and $B$ should be $2^n \times 2^n$ square matrices, and at least two groups of $A$ and $B$ have to be found so as to acquire the enlarged positive-definite matrices, since degeneracy is permitted for excited states. Additionally, one may find other group of positive-definite matrices whose minimal eigenvalue of $B^2 - A^2$ is even less than $\frac{65 - 4\sqrt{200}}{80}$. Numerical results show that the minimal eigenvalue could be $-0.059$. In our case, we use two groups of $A$ and $B$, namely,

$$A_1 = \begin{pmatrix}
1 + \frac{27\sqrt{295}}{40} & -\frac{25}{42} \\
-\frac{25}{42} & 1 - \frac{27\sqrt{295}}{40}
\end{pmatrix},
B_1 = \begin{pmatrix}
\frac{5}{4} & -\frac{5}{4} \\
-\frac{5}{4} & \frac{5}{4}
\end{pmatrix},$$

and

$$A_2 = \begin{pmatrix}
1 + \frac{27\sqrt{295}}{40} & -\frac{25}{42} \\
-\frac{25}{42} & 1 - \frac{27\sqrt{295}}{40}
\end{pmatrix},
B_2 = \begin{pmatrix}
\frac{3}{20} & \frac{9}{20} \\
\frac{9}{20} & \frac{9}{20}
\end{pmatrix},$$

with respective minimal eigenvalues $\frac{65 - 4\sqrt{200}}{80}$ and $\frac{501 - 20\sqrt{200}}{1600}$ of $B^2 - A^2$.

In summary, we have shown two-qubits product state exhibits quantumness indicating the obvious deviation from classical models. Then an experiment protocol has been proposed so as to reveal the quantumness. The generalization to arbitrary-qubit cases has also been investigated with the conclusion that quantumness is a common property existing in physical systems in product states, contrary to the preceding understanding of what classical states are, which needs further investigation. The quantumness is based on the violation of classical algebraic models, which is intrinsically distinct from local-hidden-variable models.

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