Abstract. The aim of this work is to study how the asymptotic boundary of a minimal hypersurface in $H^n \times \mathbb{R}$ determines the behavior of the hypersurface at finite points, in several geometric situations.

1. Introduction

In this article we discuss how, in several geometric situations, the shape at infinity of a minimal surface in $H^2 \times \mathbb{R}$ determines the shape of the surface itself.

A beautiful theorem in minimal surfaces theory is the Schoen’s characterization of the catenoid [12]. It can be stated as follows. Let $M \subset \mathbb{R}^3$ be a complete immersed minimal surface with two annular ends. Assume that each end is a graph, then $M$ is a catenoid.

On the other hand, there exists a complete minimal annulus immersed in a slab of $\mathbb{R}^3$ [6].

A characterization of the catenoid in the hyperbolic space, assuming regularity at infinity, was established by G. Levitt and H. Rosenberg in [5]. In a joint work with L. Hauswirth [3], the authors of the present article proved a Schoen type theorem in $H^2 \times \mathbb{R}$, in the class of finite total curvature surfaces.

Our first result is a new Schoen type theorem in $H^2 \times \mathbb{R}$. Namely, we replace Schoen’s assumption each end is a graph with the assumption each end is a vertical graph whose asymptotic boundary is a copy of the asymptotic boundary of $H^2$ (Theorem 2.1).

Our second result is a maximum principle in a vertical (closed) halfspace. Assume that $M$ is a complete minimal surface, possibly with finite boundary, properly immersed in $H^2 \times \mathbb{R}$ and that the boundary of $M$, if any, is contained in the closure of a vertical halfspace $P_+$. Assume further that the points at finite height of the asymptotic boundary of $M$ are contained in the asymptotic boundary of the halfspace $P_+$. Then $M$ is entirely contained in the halfspace $P_+$, unless $M$ is equal to the vertical halfplane $\partial P_+$ (Theorem 3.1).

Then we generalize our results to higher dimensions.

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Theorem 2.1 and Theorem 3.1 in higher dimension are analogous to the 2-dimensional case. In order to generalize Theorem 2.1, we first need to give a characterization of the $n$-catenoid analogous to that of the 2-dimensional case (Theorem 4.2, see also [2]). Moreover in the higher dimensional case, it is worthwhile to state some interesting consequences of our results.

Let $S_\infty$ be a closed set contained in an open slab of $\partial_\infty \mathbb{H}^n \times \mathbb{R}$ with height equal to $\pi/(n-1)$ such that the projection of $S_\infty$ on $\partial_\infty \mathbb{H}^n \times \{0\}$ omits an open subset.

We prove that there is no complete properly immersed minimal hypersurface $M$ whose asymptotic boundary is $S_\infty$ (Theorem 4.5-(2)).

Finally we prove an Asymptotic Theorem (Theorem 4.6), that implies the following non-existence result. There is no horizontal minimal graph over a bounded strictly convex domain, see [9, Equation (3)], given by a positive function $g$ continuous up to the boundary, taking zero boundary value data (Remark 4.1).

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2. A characterization of the catenoid in $\mathbb{H}^2 \times \mathbb{R}$

We are going to prove the characterization of the catenoid presented in the Introduction. Any surface in $\mathbb{H}^2 \times \mathbb{R}$ with constant third coordinate is a complete totally geodesic surface called a slice. For any $s \in \mathbb{R}$, we denote by $\Pi_s$ the slice $\mathbb{H}^2 \times \{s\}$ and we set $\Pi^+_s = \{(p, t) \mid p \in \mathbb{H}^2, t > s\}$ and $\Pi^-_s = \{(p, t) \mid p \in \mathbb{H}^2, t < s\}$. For simplicity $\Pi$ stands for $\Pi_0$.

Lemma 2.1. Let $\Gamma^+$ and $\Gamma^-$ be two Jordan curves in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ which are vertical graphs over $\partial_\infty \mathbb{H}^2 \times \{0\}$ and such that $\Gamma^+ \subset \partial_\infty \Pi^+$ and $\Gamma^- \subset \partial_\infty \Pi^-$. Assume that $\Gamma^-$ is the symmetry of $\Gamma^+$ with respect to $\Pi$.

Let $M \subset \mathbb{H}^2 \times \mathbb{R}$ be an immersed, connected, complete minimal surface with two ends $E^+$ and $E^-$. Assume that each end is a vertical graph and that $\partial_\infty M = \Gamma^+ \cup \Gamma^-$, that is $\partial_\infty E^+ = \Gamma^+$ and $\partial_\infty E^- = \Gamma^-$. Then $M$ is symmetric with respect to $\Pi$. Furthermore, each part $M \cap \Pi^\pm$ is a vertical graph and $M$ is embedded.

Proof. For any $t > 0$ we set $M^+_t = M \cap \Pi^+_t$. We denote by $M^{t+}$ the symmetry of $M^+_t$ with respect to the slice $\Pi_t$. Furthermore, we denote by $t^+$ the highest $t$-coordinate of $\Gamma^+$. Since $\partial_\infty M = \Gamma^+ \cup \Gamma^-$, then $M \cap \Pi^+ = \emptyset$, by the maximum principle. We denote by $E^+$ the end of $M$ whose asymptotic boundary is $\Gamma^+$. As $E^+$ is a vertical graph, there exists $\varepsilon > 0$ such that $M^+_{t^+ - \varepsilon}$ is a vertical graph, then we can start Alexandrov reflection [1].
Theorem 2.1. is any complete geodesic of $H^a$ a copy of $M$ vertical graph and that doing Alexandrov reflection with slices coming from below, one has that $M$ is symmetric with respect to the slice $\Pi$. We use the same notations as in the proof of Definition 2.1. $M$ can show, as in the proof of [12, Theorem 2], that the whole surface $M$ is symmetric with respect to $\Pi$ and each component of $M \setminus \Pi$ is a graph. Therefore we can show, as in the proof of [12, Theorem 2], that the whole surface $M$ is embedded. This completes the proof. □

Definition 2.1. A vertical plane is a complete totally geodesic surface $\gamma \times \mathbb{R}$ where $\gamma$ is any complete geodesic of $\mathbb{H}^2$.

Theorem 2.1. Let $M \subset \mathbb{H}^2 \times \mathbb{R}$ be an immersed, connected, complete minimal surface with two ends. Assume that each end is a vertical graph whose asymptotic boundary is a copy of $\partial_{\infty}\mathbb{H}^2$. Then $M$ is rotational, hence $M$ is a catenoid.

Proof. Up to a vertical translation, we can assume that the asymptotic boundary is symmetric with respect to the slice $\Pi$. We use the same notations as in the proof of Lemma 2.1. We know from Lemma 2.1 that $M$ is symmetric with respect to $\Pi$ and that $M^+_0$ and $M^-_0$ are vertical graphs. Therefore, at any point of $M \cap \Pi$ the tangent plane of $M$ is orthogonal to $\Pi$.

We have $\partial_{\infty}M = \partial_{\infty}\mathbb{H}^2 \times \{t_0, -t_0\}$ for some $t_0 > 0$. Since $M$ is embedded, $M$ separates $\mathbb{H}^2 \times [-t_0, t_0]$ into two connected components. We denote by $U_1$ the component whose asymptotic boundary is $\partial_{\infty}\mathbb{H}^2 \times [-t_0, t_0]$ and by $U_2$ the component such that $\partial_{\infty}U_2 = \partial_{\infty}\mathbb{H}^2 \times \{t_0, -t_0\}$.

Let $q_\infty \in \partial_{\infty}\mathbb{H}^2$ and let $\gamma \subset \mathbb{H}^2$ be an oriented geodesic issuing from $q_\infty$, that is $q_\infty \in \partial_{\infty}\gamma$. Let $q_0 \in \gamma$ be any fixed point.

For any $s \in \mathbb{R}$, we denote by $P_s$ the vertical plane orthogonal to $\gamma$ passing through the point of $\gamma$ whose oriented distance from $q_0$ is $s$. We suppose that $s < 0$ for any point in the geodesic segment $(q_0, q_\infty)$.

For any $s \in \mathbb{R}$, we call $M_s(l)$ the part of $M \setminus P_s$ such that $(q_\infty, t_0), (q_\infty, -t_0) \in \partial_{\infty}M_s(l)$ and let $M^*_s(l)$ be the reflection of $M_s(l)$ about $P_s$. We denote by $M_s(r)$ the other part of $M \setminus P_s$ and by $M^*_s(r)$ its reflection about $P_s$.

It will be clear from the following two Claims, why we can start the Alexandrov reflection principle with respect to the vertical planes $P_s$ and obtain the result.

By assumptions there exists $s_1 < 0$ such that for any $s < s_1$ the part $M_s(l)$ has two connected components and both of them are vertical graphs. We deduce that $\partial M_s(l)$ has two (symmetric) connected components, each one being a vertical graph.

We recall that $\Pi^+ := \{t > 0\}$ and $\Pi^- := \{t < 0\}$.

Claim 1. For any $s < s_1$, we have that $M^*_s(l) \cap \Pi^+$ stays above $M_s(r)$ and $M^*_s(l) \cap \Pi^-$ stays below $M_s(r)$. Consequently $M^*_s(l) \subset U_2$ for any $s < s_1$.

Observe that $M^*_s(l) \cap \Pi^+$ and $M_s(r) \cap \Pi^+$ have same asymptotic boundary and that $\partial (M^*_s(l) \cap \Pi^+) = \partial M_s(r) \cap \Pi^+$. Therefore the asymptotic and finite boundaries of any
lifting up of $M_s^*(l)$ is above the asymptotic and finite boundaries of $M_s(r)$. Hence any lifting up of $M_s^*(l)$ is above $M_s(r)$ by the maximum principle, which ensures that the whole $M_s^*(l) \cap \Pi^+$ stays above $M_s(r)$ for any $s < s_1$, as desired. The proof of the other assertion is analogous. Then, Claim 1 is proved.

We set

$$\sigma = \sup \{ s \in \mathbb{R} \mid M_t^*(l) \cap \Pi^+ \text{ stays above } M_t(r) \cap \Pi^+ \text{ for any } t \in (-\infty, s) \}.$$  

Claim 2. We have $M_s^*(l) = M_s(r)$. Thus, given a geodesic $\gamma \subset \mathbb{H}^2$, there exists a vertical plane $P_{\sigma}$ orthogonal to $\gamma$ such that $M$ is symmetric with respect to $P_{\sigma}$

Note that we also have

$$\sigma = \sup \{ s \in \mathbb{R} \mid M_t^*(l) \subset U_2 \text{ for any } t \in (-\infty, s) \}.$$  

In order to prove Claim 2, we first establish the following fact.

Assertion. For any $s$ such that $M_s^*(l) \cap \Pi \subset U_2$ then $M_s^*(l) \subset U_2$.

As $M$ is symmetric with respect to $\Pi$ the intersection $M \cap \Pi$ is constituted of a finite number of pairwise disjoint Jordan curves $C_1, \ldots, C_k$. Since $M \cap \Pi^+$ is a vertical graph we deduce

$$(C_j \times \mathbb{R}) \cap M = C_j$$  

for any $j = 1, \ldots, k$.

Moreover, since $M$ is connected and is symmetric about $\Pi$, we get that $M \cap \Pi^+$ is connected.

Let $D_j \subset \Pi$ be the Jordan domain bounded by $C_j$, $j = 1, \ldots, k$. Noticing that:

- $(M \cap \Pi^+) \setminus (\overline{D}_j \times \mathbb{R}) \neq \emptyset$,
- $M \cap \Pi^+$ is connected,
- $M \cap (C_j \times \mathbb{R}) = C_j$,
- $\partial_{\infty} M \cap \Pi^+ = \partial_{\infty} \mathbb{H}^2 \times \{t_0\}$,

we get that $(M \cap \Pi^+) \cap (D_j \times \mathbb{R}) = \emptyset$, $j = 1, \ldots, k$. Hence, $D_i \cap D_j = \emptyset$ for any $i \neq j$.

Therefore, $M \cap \Pi^+$ is a vertical graph over $\Pi \setminus \cup D_i$.

This implies that, for any $\varepsilon > 0$, the vertical translation $(M_s^*(l) \cap \Pi^+) + (0, 0, \varepsilon)$ stays above $M$. This proves the Assertion.

Let us continue the proof of Claim 2. The definition of $\sigma$ implies that $M_{\sigma + \varepsilon}^*(l) \cap U_1 \neq \emptyset$, for $\varepsilon$ small enough.

We deduce from the Assertion that $M_{\sigma + \varepsilon}^*(l) \cap \Pi$ is not contained in $U_2$ for any small enough $\varepsilon > 0$. Hence we infer that $M_s^*(l) \cap \Pi$ and $M_s(r) \cap \Pi$ are tangent at an interior or boundary point lying in some Jordan curve $C_j$ contained in $M \cap \Pi$. Since $M_s^*(l) \subset \overline{U}_2$, $M_s(r) \subset \partial U_2$ and the tangent plane of $M$ is vertical along $M \cap \Pi$, we are able to apply the maximum principle (possibly with boundary) to conclude that $M_s^*(l) = M_s(r)$, that is $P_{\sigma}$ is a plane of symmetry of $M$. This proves Claim 2.

For any $\alpha \in (0, \pi/2]$ consider a family of vertical planes making an angle $\alpha$ with $P_{\sigma}$, generated by hyperbolic translations along the horizontal geodesic $P_{\sigma} \cap \Pi$. Now, doing the Alexandrov reflection principle with this family of planes, we find a vertical plane of symmetry of $M$, say $P^{\alpha}$. Hence $M$ is invariant by the rotation of angle $2\alpha$ around
the vertical geodesic $P^\alpha \cap P_\sigma$. Choosing an angle $\alpha$ such that $\pi/\alpha$ is not rational, we find that $M$ is invariant by rotation around the axis $P^\alpha \cap P_\sigma$. This concludes the proof of Theorem 2.1, as desired. □

Remark 2.1. For any integer $n$, there exists a minimal surface in $\mathbb{H}^2 \times \mathbb{R}$ which is a vertical graph, whose asymptotic boundary is a copy of $\partial_\infty \mathbb{H}^2$ and whose finite boundary is constituted of $n$ smooth Jordan curves in the slice $\Pi$, see [10, Theorem 5.1]. In the same article the second and third author asked about the existence of such graphs with two boundary curves in $\Pi$ cutting orthogonally the slice $\Pi$. Theorem 2.1 implies that the answer to this question is negative.

3. Maximum Principle in a vertical halfspace of $\mathbb{H}^2 \times \mathbb{R}$.

In this section we prove some maximum principle in a vertical halfspace. More precisely, we prove that, under some geometric assumptions, the behavior of the asymptotic boundary of $M$ at finite height, determines the behaviour of $M$.

Definition 3.1. We call a vertical halfspace any of the two components of $(\mathbb{H}^2 \times \mathbb{R}) \setminus P$, where $P$ is a vertical plane.

Theorem 3.1. Let $M$ be a complete minimal surface, possibly with finite boundary, properly immersed in $\mathbb{H}^2 \times \mathbb{R}$. Let $P$ be a vertical plane and let $P_+$ be one of the two halfspaces determined by $P$. If $\partial M \subset P_+$ and $\partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \subset \partial_\infty P_+$, then $M \setminus \partial M \subset P_+$, unless $M \subset P$.

For the proof of Theorem 3.1 we need to consider the one parameter family of surfaces $M_d, d > 0$, that have origin in [7, Section 4] and whose geometry is described in [10, Proposition 2.1]. This family of surfaces was already used, for example, in [8, Example 2.1].

First we describe the asymptotic boundary of $M_d$, for $d > 1$. Consider a horizontal geodesic $\gamma$ in $\mathbb{H}^2$, with asymptotic boundary $\{p, q\}$ and let $\alpha$ be the closure of a connected component of $(\partial_\infty \mathbb{H}^2 \times \{0\}) \setminus (\{p, q\} \times \{0\})$. Let

$$H(d) = \int_{\cosh^{-1}(d)}^{+\infty} \frac{du}{\sqrt{\cosh^2 u - d^2}}, \quad d > 1$$

be the positive number defined in (1) of [10]. Notice that $\lim_{d \to 1} H(d) = +\infty$ and $\lim_{d \to +\infty} H(d) = \pi/2$.

Let $\alpha_d$ in $\partial_\infty \mathbb{H}^2 \times \{H(d)\}$ and $\alpha_{-d}$ in $\partial_\infty \mathbb{H}^2 \times \{-H(d)\}$ be the two curves that project vertically onto $\alpha$. Let $L_d, R_d$ be two vertical segments in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$ of height $2H(d)$ such that the curve $L_d \cup \alpha_d \cup R_d \cup \alpha_{-d}$ is a closed simple curve. Then $\partial_\infty M_d = L_d \cup \alpha_d \cup R_d \cup \alpha_{-d}$.

Now we describe the position of $M_d$ in the ambient space, for $d > 1$. Denote by $Q_\gamma$ the halfspace determined by $\gamma \times \mathbb{R}$, whose asymptotic boundary contains the curve $\alpha$. Let $\gamma_d$ be the curve in $Q_\gamma \cap (\mathbb{H}^2 \times \{0\})$ at constant distance $\cosh^{-1}(d)$ from
\( \gamma \). \( M_d \) contains the curve \( \gamma_d \). Denote by \( Z_d \) the closure of the non mean convex side of the cylinder over the curve \( \gamma_d \). Then, \( M_d \) is contained in \( Z_d \) which is contained in \( Q_\gamma \). Notice that any vertical translation of the surface \( M_d \) is contained in \( Z_d \). Moreover, any vertical translation of \( M_d \) is arbitrarily close to \( Q_\gamma \) if \( d \) is sufficiently close to 1.

We observe that in the description above, \( \gamma \) can be any geodesic of \( \mathbb{H}^2 \).

Proof of Theorem 3.1. The proof is an application of the maximum principle between the surface \( M \) and the one parameter family of surfaces \( M_d \).

We choose the geodesic \( \gamma \), in order to construct the \( M_d \)'s, as follows. Let \( \gamma \subset \mathbb{H}^2 \) be any geodesic such that

\begin{itemize}
  \item P1: The halfspace \( Q_\gamma \) is strictly contained in \( (\mathbb{H}^2 \times \mathbb{R}) \setminus P_+ \).
  \item P2: \( \partial_\infty \gamma \cap \partial_\infty P = \emptyset \).
\end{itemize}

Now, notice that

(1) The intersection of \( \partial_\infty M \) with \( \partial_\infty (\mathbb{H}^2 \times \mathbb{R}) \setminus \partial_\infty P_+ \) contains no points at finite height.
(2) The asymptotic boundary of any vertical translation \( M_d \) is contained in the asymptotic boundary of \( Q_\gamma \subset \mathbb{H}^2 \times \mathbb{R} \setminus P_+ \).

We claim that \( M_d \) and \( M \) are disjoint for any \( d > 1 \). Indeed, letting \( p \rightarrow q \) (recall that \( p, q \) are the endpoints of the geodesic \( \gamma \)), one has that \( M_d \) collapses to a vertical segment in \( \partial_\infty \mathbb{H}^2 \times \mathbb{R} \). Suppose that, when \( p \rightarrow q \), the surfaces \( M_d \) always have a nonempty intersection with \( M \). Then, there would exists a point of the asymptotic boundary of \( M \) at finite height in \( \partial_\infty (\mathbb{H}^2 \times \mathbb{R}) \setminus \partial_\infty P_+ \), giving a contradiction with (1). Then, if \( M \cap M_d \neq \emptyset \), we would obtain a last intersection point between \( M \) and some modified \( M_d \) letting \( p \rightarrow q \), contradicting the maximum principle.

Therefore, by the maximum principle, any vertical translation of \( M_d \) and \( M \) are disjoint.

Let \( d \rightarrow 1 \). By the maximum principle, there is no first point of contact between \( M_d \) and \( M \). As we can apply the maximum principle between any vertical translation of \( M_d \) and \( M \), one has that \( M \) is contained in the closed halfspace \( \mathbb{H}^2 \times \mathbb{R} \setminus Q_\gamma \) for any geodesic \( \gamma \) satisfying the properties P1 and P2. Therefore, \( M \) is included in the closure of \( P_+ \).

Now we have one of the following possibilities:

- Some points of the interior of \( M \) touches \( \partial P_+ = P \), then, by the maximum principle, \( M \subset P_+ \).
- \( M \setminus \partial M \) is contained in the halfspace \( P_+ \).

The result is thus proved. \( \square \)

Let us give a definition, before stating some consequences of Theorem 3.1.

Definition 3.2. We say that \( L \subset \partial_\infty (\mathbb{H}^2 \times \mathbb{R}) \) is a line if \( L = \{ p \} \times \mathbb{R} \) for some \( p \in \partial_\infty \mathbb{H}^2 \).

Given vertical lines \( L_1, \ldots, L_k \) in \( \partial_\infty \mathbb{H}^2 \times \mathbb{R} \), we define the set \( P(L_1, \ldots, L_k) \) as follows. Let \( P_i \) the vertical plane such that \( \partial_\infty P_i \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) = L_i \cup L_{i+1} \) (with the convention
that $L_{k+1} = L_1$). Denote by $\tilde{P}_i$ the halfspace determined by the vertical plane $P_i$ such that $\bigcup_j L_j \subset \partial_\infty \tilde{P}_i$. Then, we set $P(L_1, \ldots, L_k) := \cap_i \tilde{P}_i$.

**Corollary 3.1.** Let $M$ be a complete minimal surface, possibly with finite boundary, properly immersed in $\mathbb{H}^2 \times \mathbb{R}$ and let $\Gamma = \partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R})$. Let $L_1, \ldots, L_k$ be vertical lines in $\partial_\infty \mathbb{H}^2 \times \mathbb{R}$. If $\Gamma \subset L_1 \cup \cdots \cup L_k$ and $\partial M \subset P(L_1, \ldots, L_k)$, then $M \setminus \partial M$ is contained in $P(L_1, \ldots, L_k)$, unless $M$ is contained in one of the $P_i$.

**Proof.** By Theorem 3.1, $M$ is contained in every halfspace $\tilde{P}_i$ determined by the vertical plane $P_i$ such that $\bigcup_j L_j \subset \partial_\infty \tilde{P}_i$, unless it is contained in one of the $P_i$. Hence it is contained in $P(L_1, \ldots, L_k)$, by definition, unless it is contained in one of the $P_i$. □

**Corollary 3.2.** Let $M$ be a complete minimal surface properly immersed in $\mathbb{H}^2 \times \mathbb{R}$. Let $P$ be a vertical plane. If $\partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R}) \subset \partial_\infty P$, then $M = P$.

**Proof.** By Theorem 3.1, $M$ is contained in the closure of both halfspaces determined by $P$, hence it is contained in $P$. Then $M = P$ because it is complete. □

**Corollary 3.3.** Let $M$ be a complete minimal surface properly immersed in $\mathbb{H}^2 \times \mathbb{R}$. Suppose that the asymptotic boundary of $M$ is contained in the asymptotic boundary of a totally geodesic plane $S$ of $\mathbb{H}^2 \times \mathbb{R}$. Then $M = S$.

**Proof.** The proof is a simple consequence of the maximum principle and of the previous results. We do it for completeness. First assume that the asymptotic boundary of $M$ is contained in the asymptotic boundary of a slice, say $\{t = 0\}$. Then, for $n$ sufficiently large, the slice $\{t = n\}$ is disjoint from $M$. Now, we translate the slice $\{t = n\}$ down. The first contact point, cannot be interior because of the maximum principle, hence $M$ must stay below the slice $\{t = 0\}$. One can do the same reasoning with slices coming from the bottom, and $M$ must stay above the slice $\{t = 0\}$. Hence $M$ coincides with the slice $\{t = 0\}$. If the asymptotic boundary of $M$ is contained is the asymptotic boundary of a vertical plane, the result follows by Corollary 3.2. □

**Corollary 3.4.** Let $M$ be a complete minimal surface properly immersed in $\mathbb{H}^2 \times \mathbb{R}$. Assume that the projection of the asymptotic boundary of $M$ into $\partial_\infty \mathbb{H}^2$ omits a closed interval $\alpha$ joining two points $p$ and $q$. Let $\gamma$ be the horizontal geodesic in $\mathbb{H}^2$ whose the asymptotic boundary is $\{p, q\}$ and let $Q_\gamma$ be the halfspace determined by $\gamma \times \mathbb{R}$ whose asymptotic boundary contains $\alpha$. Then $M$ is contained in $\mathbb{H}^2 \times \mathbb{R} \setminus Q_\gamma$.

**Proof.** By hypothesis $\partial_\infty M \cap (\partial_\infty \mathbb{H}^2 \times \mathbb{R})$ is contained in the asymptotic boundary of $(\mathbb{H}^2 \times \mathbb{R}) \setminus Q_\gamma$. The result follows by Theorem 3.1 with $P_+ = (\mathbb{H}^2 \times \mathbb{R}) \setminus Q_\gamma$. □

**Remark 3.1.** There exist examples of minimal surfaces with asymptotic boundary equal to two vertical halflines, lines and a curve at finite height, see [7, Equation (32)] and [10, Proposition 2.1 (2)].
4. Some generalizations to $\mathbb{H}^n \times \mathbb{R}$.

Let us recall the construction and the properties of the $n$-catenoids in $\mathbb{H}^n \times \mathbb{R}$, $n \geq 3$, established, by P. Bérard and the second author in [2, Proposition 3.2]. Given any $a > 0$ we denote by $(I_a, f(a, \cdot))$, where $I_a \subset \mathbb{R}$ is an interval, the maximal solution of the following Cauchy problem:

$$\begin{cases}
f_t = (n - 1)(1 + f_t^2) \coth(f), \\
f(0) = a > 0, \\
f_t(0) = 0.
\end{cases}$$

**Theorem 4.1** ([2]). For $a > 0$, the maximal solution $(I_a, f(a, \cdot))$ gives rise to the generating curve $C_a$, parametrized by $t \mapsto (\tanh(f(a,t)), t)$, of a complete minimal rotational hypersurface $C_a$ (n-atenoid) in $\mathbb{H}^n \times \mathbb{R}$, with the following properties.

1. The interval $I_a$ is of the form $I_a = ]-T(a), T(a)[$ where

   $$T(a) = \sinh^{n-1}(a) \int_a^{\infty} \left( \sinh^{2n-2}(u) - \sinh^{2n-2}(a) \right)^{-1/2} du.$$ 

2. $f(a, \cdot)$ is an even function of the second variable.
3. For all $t \in I_a$, $f(a, t) \geq a$.
4. The derivative $f_t(a, \cdot)$ is positive on $]0, T(a)[$, negative on $]-T(a), 0[$.
5. The function $f(a, \cdot)$ is a bijection from $[0, T(a)]$ onto $[a, \infty]$, with inverse function $\lambda(a, \cdot)$ given by

   $$\lambda(a, \rho) = \sinh^{n-1}(a) \int_a^\rho \left( \sinh^{2n-2}(u) - \sinh^{2n-2}(a) \right)^{-1/2} du.$$ 

6. The catenoid $C_a$ has finite vertical height $h_R(a) := 2T(a)$.
7. The function $a \mapsto h_R(a)$ increases from 0 to $\frac{\pi}{(n-1)}$ when $a$ increases from 0 to infinity. Furthermore, given $a \neq b$, the generating catenaries $C_a$ and $C_b$ intersect at exactly two symmetric points.

For later use, we need the following result. Although we believe that the result is classical, we give a proof for the sake of completeness. The reader is referred to [4, chapter VII] or [13, chapter 9, addendum 3] for the proof of the analogous statement in Euclidean space.

**Proposition 4.1.** Let $S \subset \mathbb{H}^n$ be a finite union of connected, closed and embedded $(n-1)$-submanifolds $C_j$, $j = 1, \ldots, k$, such that the bounded domains whose boundary are the $C_j$ are pairwise disjoint. Assume that for any geodesic $\gamma \subset \mathbb{H}^n$, there exists a $(n-1)$-geodesic plane $\pi_\gamma \subset \mathbb{H}^n$ of symmetry of $S$ which is orthogonal to $\gamma$. Then $S$ is a $(n-1)$-geodesic sphere of $\mathbb{H}^n$.

**Proof.** We will proceed the proof by induction on $n \geq 2$.

First assume that $n = 2$. By hypothesis, there exist two geodesics $c_1, c_2 \subset \mathbb{H}^2$ of symmetry of the closed curve $S$ intersecting at some point $p \in \mathbb{H}^2$ and making an angle
\( \alpha \neq 0 \) such that \( \pi/\alpha \) is not rational. For any \( q \in S \), denote by \( C_q \) the circle centered at \( p \) passing through \( q \). Then \( C_q \) is contained in \( S \). Let \( \tilde{q} \neq q \) be points of \( S \). If \( C_q \neq C_{\tilde{q}} \) then the geodesic disks bounded by \( C_q \) and \( C_{\tilde{q}} \) are not disjoint, since they have the same center, which contradicts the hypothesis. Consequently, we get \( C_q = C_{\tilde{q}} \) and we conclude that \( S \) is a circle.

Let \( n \in \mathbb{N}, n \geq 3 \). Assume that the statement holds for \( k = 2, \ldots, n-1 \).

Let \( \pi_0 \subset \mathbb{H}^n \) be a \((n-1)\)-geodesic plane of symmetry of \( S \).

**Claim 1.** \( S \cap \pi_0 \) is a \((n-2)\)-geodesic sphere of \( \pi_0 \).

Indeed, let \( \gamma \subset \pi_0 \) be a geodesic. By hypothesis there exists a \((n-1)\)-geodesic plane \( \pi_\gamma \subset \mathbb{H}^n \) orthogonal to \( \gamma \) which is a plane of symmetry of \( S \). Since \( \pi_\gamma \) is orthogonal to \( \pi_0 \), then \( S \cap \pi_0 \) is symmetric about \( \pi_\gamma \cap \pi_0 \) (which is a \((n-2)\)-geodesic plane of \( \pi_0 \)), see [11, Lemme 3.3.15]. As \( \pi_0 \) is a \((n-1)\) hyperbolic space, \( S \cap \pi_0 \) satisfies the assumptions of the statement in \( \mathbb{H}^{n-1} \).

By the induction hypothesis we deduce that \( S \cap \pi_0 \) is a \((n-2)\)-geodesic sphere of \( \pi_0 \). This proves Claim 1.

Let \( p_0 \in \pi_0 \) and \( \rho_0 > 0 \) be respectively the center and the radius of the \((n-2)\)-geodesic sphere \( S \cap \pi_0 \).

**Claim 2.** Let \( \pi_1 \subset \mathbb{H}^n \) be a \((n-1)\)-geodesic plane of symmetry of \( S \) orthogonal to \( \pi_0 \). Then \( S \cap \pi_1 \) is a \((n-2)\)-geodesic sphere of \( \pi_1 \) with center \( p_0 \) and radius \( \rho_0 \).

Claim 1 yields that \( S \cap \pi_1 \) is a \((n-2)\)-geodesic sphere of \( \pi_1 \). Since \( \pi_0 \) and \( \pi_1 \) are orthogonal, then the geodesic sphere \( S \cap \pi_0 \) is symmetric about \( \pi_1 \). Therefore \( p_0 \in \pi_1 \).

If \( n > 3 \), then \( (S \cap \pi_0) \cap \pi_1 \) is a \((n-3)\)-geodesic sphere with center \( p_0 \) and radius \( \rho_0 \) of \( \pi_0 \cap \pi_1 \) (which is a \((n-2)\) hyperbolic space). If \( n = 3 \), then \( (S \cap \pi_0) \cap \pi_1 \) is constituted of two points whose the distance is \( 2\rho_0 \). In both cases we infer that \( \text{diam}_{\mathbb{H}^n} (S \cap \pi_1) \geq 2\rho_0 \) and then the radius of the geodesic sphere \( S \cap \pi_1 \) is \( \rho_1 \geq \rho_0 \). Analogously we can show that \( \rho_0 \geq \rho_1 \).

We deduce that \( \rho_1 = \rho_0 \), that is \( S \cap \pi_0 \) and \( S \cap \pi_1 \) have both center at \( p_0 \) and radius \( \rho_0 \). This proves Claim 2.

**Claim 3.** Let \( \pi_2 \subset \mathbb{H}^n \) be any \((n-1)\)-geodesic plane of symmetry of \( S \). Then \( S \cap \pi_2 \) is a \((n-2)\)-geodesic sphere of \( \pi_2 \) with center \( p_0 \) and radius \( \rho_0 \).

Since \( S \) is symmetric with respect to \( \pi_0 \) and \( \pi_2 \), \( \pi_0 \) and \( \pi_2 \) are distinct and \( S \) is compact, then the \((n-1)\)-geodesic planes \( \pi_0 \) and \( \pi_2 \) cannot be disjoint.

Then, we find a third \((n-1)\)-geodesic plane \( \pi_3 \) of symmetry of \( S \), orthogonal to both \( \pi_0 \) and \( \pi_2 \). Claim 2 implies that \( S \cap \pi_2 \) is a \((n-2)\)-geodesic sphere of \( \pi_2 \) with center \( p_0 \) and radius \( \rho_0 \). This proves Claim 3.

Now we finish the proof of the Proposition as follows. Let \( p \in S \) and let \( \pi \subset \mathbb{H}^n \) be any \((n-1)\)-geodesic plane passing through \( p \) and \( p_0 \). Let \( \gamma \subset \mathbb{H}^n \) be the geodesic through \( p_0 \) orthogonal to \( \pi \). By Claim 2, there exists a \((n-1)\)-geodesic plane \( \pi_\gamma \) of symmetry of \( S \) and orthogonal to \( \gamma \). Claim 3 ensures that \( p_0 \in \pi_\gamma \), then \( \pi_\gamma = \pi \). Claim 3 yields also that \( S \cap \pi \) is a \((n-2)\)-geodesic sphere of \( \pi \) with center \( p_0 \) and radius \( \rho_0 \),
thus $d_{\overline{\mathbb{H}^n}}(p,p_0) = \rho_0$. This shows that $S$ is the $(n-1)$-geodesic sphere of $\mathbb{H}^n$ of radius $\rho_0$ and center $p_0$. \hfill \Box

Now we establish a characterization of the $n$-catenoid, that is a generalization to higher dimension of Theorem 2.1.

**Theorem 4.2.** Let $M \subset \mathbb{H}^n \times \mathbb{R}$ be an immersed, connected, complete minimal hypersurface with two ends. Assume that each end is a vertical graph whose asymptotic boundary is a copy of $h_{\pi\infty}^{\mathbb{H}^n}$. Then $M$ is a $n$-catenoid.

**Proof.** Up to a vertical translation, we can assume that the asymptotic boundary of $M$ is symmetric with respect to $\Pi := \mathbb{H}^n \times \{0\}$. We set $\Gamma^+ := \partial_{\infty} M \cap \{t > 0\}$ and recall that $\Gamma^+$ is a copy of $\partial_{\infty} \mathbb{H}^n$. As usual we set $M^+ := M \cap \{t > 0\}$.

Next Claim can be shown in the same fashion as in $\mathbb{H}^2 \times \mathbb{R}$ (see Lemma 2.1 and the proof of Claim 2 of Theorem 2.1). For this reason we just state it.

**Claim.** $M$ is symmetric about $\Pi$, and each connected component of $M \setminus \Pi$ is a vertical graph.

Moreover, for any geodesic $\gamma \subset \Pi$ there exists a vertical hyperplane $P_{\gamma} \subset \mathbb{H}^n \times \mathbb{R}$ orthogonal to $\gamma$ which is a $(n-1)$-plane of symmetry of $M$. Therefore, $\pi_{\gamma} := P_{\gamma} \cap \Pi$ is a $(n-1)$-plane of symmetry of $\Sigma := M \cap \Pi$.

Using the result of the Claim we get that $\Sigma$ satisfies the assumptions of Proposition 4.1. Then $\Sigma$ is a $(n-1)$-geodesic sphere of $\Pi$, since $\Pi = \mathbb{H}^n \times \{0\}$.

Let $C \subset \mathbb{H}^n \times \mathbb{R}$ be the catenoid through $\Sigma$ and orthogonal to $\Pi$. We set $C^+ := C \cap \{t > 0\}$.

Both $C^+$ and $M^+$ are vertical along their common finite boundary $\Sigma$, hence they are tangent along $\Sigma$.

Let $t_C$ (resp. $t_M$) the height of the asymptotic boundary of $C^+$ (resp. $M^+$).

Suppose for example that $t_C \leq t_M$. Then, lifting upward and downward $M^+$, we obtain that $M^+$ is above $C^+$. Therefore we deduce that $M^+ = C^+$ by applying the boundary maximum principle. The case $t_M \leq t_C$ is analogous.

We conclude that $M = C$ and the proof is completed. \hfill \Box

In order to establish the generalization in higher dimension of Theorem 3.1, we need to state some existence results, established for $n \geq 3$, in [2, Theorem 3.8], inspired by [10, Proposition 2.1]. In fact, we only need the $d > 1$ case, but we state the whole result for the sake of completeness.

**Theorem 4.3** ([2]). There exists a one parameter family $\{M_d, d > 0\}$ of complete embedded minimal hypersurfaces in $\mathbb{H}^n \times \mathbb{R}$ invariant under hyperbolic translations.

1. If $d > 1$, then $M_d$ consists of the union of two symmetric vertical graphs over the exterior of an equidistant hypersurface in the slice $\mathbb{H}^n \times \{0\}$.

The asymptotic boundary of $M_d$ is topologically an $(n-1)$-sphere which is homologically trivial in $\partial_{\infty} \mathbb{H}^n \times \mathbb{R}$. More precisely, we set for $d > 1$:

$$ S(d) = \cosh(a) \int_{1}^{\infty} (t^{2n-2} - 1)^{-1/2} (\cosh^2(a) t^2 - 1)^{-1/2} dt, \quad \text{where } d =: \cosh^{n-1}(a). $$
Then, the asymptotic boundary of $M_d$ consists of the union of two copies of an hemisphere $S_n^{-1} \times \{0\}$ of $\partial \infty H^n \times \{0\}$ in parallel slices $t = \pm S(d)$, glued with the finite cylinder $\partial S_n^{-1} \times [-S(d), S(d)]$

The vertical height of $M_d$ is $2S(d)$. The height of the family $M_d$ is a decreasing function of $d$ and varies from infinity (when $d \to 1$) to $\pi/(n-1)$ (when $d \to \infty$).

(2) If $d = 1$, then $M_1$ consists of a complete (non-entire) vertical graph over a halfspace in $H^n \times \{0\}$, bounded by a totally geodesic hyperplane $P$. It takes infinite boundary value data on $P$ and constant asymptotic boundary value data. The asymptotic boundary of $M_1$ is the union of a spherical cap $S$ of $\partial \infty H^n \times \{c\}$ with a half vertical cylinder over $\partial S$.

(3) If $d < 1$, then $M_d$ is an entire vertical graph with finite vertical height. Its asymptotic boundary consists of a homologically non-trivial $(n-1)$-sphere in $\partial \infty H^n \times \mathbb{R}$.

The hypersurfaces $M_d$ are the analogous in higher dimension of the surfaces $M_d$ in $H^2 \times \mathbb{R}$. Also, as in $H^2 \times \mathbb{R}$, by (vertical) hyperplane we mean a complete totally geodesic hypersurface $\Pi \times \mathbb{R}$, where $\Pi$ is any totally geodesic hyperplane of $H^n \times \{0\}$. Moreover, we call a vertical halfspace any component of $(H^n \times \mathbb{R}) \setminus P$ where $P$ is a vertical hyperplane. Thus, working with the hypersurfaces $M_d$ exactly in the same way as in Theorem 3.1, we obtain the following result.

Theorem 4.4. Let $M$ be a complete minimal hypersurface properly immersed in $H^n \times \mathbb{R}$, possibly with finite boundary. Let $P$ be a vertical geodesic hyperplane and $P_+$ one of the two halfspaces determined by $P$. If $\partial M \subset \overline{P_+}$ and $\partial \infty M \cap (\partial \infty H^n \times \mathbb{R}) \subset \partial \infty P_+$, then $M \setminus \partial M \subset P_+$, unless $M \subset P$.

Obviously, the analogous in higher dimension of Corollaries 3.1, 3.2, 3.3 hold as well. Part (1) of next Theorem is a generalization in higher dimension of Corollary 3.4, while part (2) was proved, for $n = 2$ by the second and the third authors [10, Corollary 2.2]

Theorem 4.5. Let $S_\infty \subset \partial \infty H^n \times \mathbb{R}$ be a closed set whose the vertical projection on $\partial \infty H^n \times \{0\}$ omits an open subset $U$.

(1) Let $M$ be a complete minimal hypersurface properly immersed in $H^n \times \mathbb{R}$ such that $\partial \infty M = S_\infty$. Let $Q \subset H^n \times \mathbb{R}$ be a vertical halfspace whose asymptotic boundary is contained in $U \times \mathbb{R}$. Then $M$ is contained in $H^n \times \mathbb{R} \setminus \overline{Q}$.

(2) Assume that $S_\infty$ is contained in an open slab whose height is equal to $\frac{\pi}{n-1}$. Then, there is no complete connected properly immersed minimal hypersurface $M$ in $H^n \times \mathbb{R}$ with asymptotic boundary $S_\infty$.

Proof. The first statement is a consequence of Theorem 4.4 and the proof is analogous to that of Corollary 3.4.

Let us prove the second statement. Assume, by contradiction, that there is such a minimal hypersurface $M$ with asymptotic boundary $S_\infty$. Then, up to a vertical
translation, we can assume that $M$ is contained in the slab $S := \{ \varepsilon < t < \frac{\pi}{n-1} - \varepsilon \}$ for some $\varepsilon > 0$, and thus $S_\infty \subset \partial_\infty S$. By assumption, there exists a $(n-1)$-geodesic plane $\pi \subset \mathbb{H}^n \times \{0\}$ such that a component $\pi^+$ of $\mathbb{H}^n \times \{0\} \setminus \pi$ satisfies:

1. $\partial_\infty \pi^+ \subset U$.
2. $M \cap (\pi^+ \times \mathbb{R}) = \emptyset$.

Let $C \subset \mathbb{H}^n \times (0, \frac{\pi}{n-1})$ be any $n$-catenoid such that a component of its asymptotic boundary stays strictly above $\partial_\infty S$ and the other component stays strictly below $\partial_\infty S$. We take a connected and compact piece $K$ of $C$ such that its boundary lies in the boundary of the slab $S$.

Let $q \in M$ be a point and let $q_0 \in \mathbb{H}^n \times \{0\}$ be the vertical projection of $q$. Let $p_\infty \in \partial_\infty \pi^+$ be an asymptotic point. Denote by $\tilde{\gamma} \subset \partial_\infty \mathbb{H}^n \times \{0\}$ the complete geodesic passing through $q_0$ such that $p_\infty \in \partial_\infty \tilde{\gamma}$. We can translate $K$ along $\tilde{\gamma}$ such that the translated $K$ is contained in the halfspace $\pi^+ \times \mathbb{R}$.

Now we come back translating $K$ towards $M$ along $\tilde{\gamma}$. Observe that the boundary of the translated copies of $K$ does not touch $M$. Therefore, doing the translations of $K$ along $\tilde{\gamma}$ we find a first interior point of contact between $M$ and a translated copy of $K$. Hence, $M = C$ by the maximum principle, which leads to a contradiction. This completes the proof. \hfill \Box

Now we state a generalization of the Asymptotic Theorem proved in [10, Theorem 2.1]. Our result establishes some obstruction for the asymptotic boundary of a complete properly immersed minimal hypersurface in $\mathbb{H}^n \times \mathbb{R}$.

**Theorem 4.6 (Asymptotic Theorem).** Let $\Gamma \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$ be a connected $(n-1)$-submanifold with boundary. Let $\text{Pr} : \partial_\infty \mathbb{H}^n \times \mathbb{R} \to \partial_\infty \mathbb{H}^n$ be the projection on the first factor. Assume that:

1. There is some point $q_\infty \in \partial \text{Pr}(\Gamma)$ such that $q_\infty \notin \text{Pr}(\partial \Gamma)$.
2. $\Gamma \subset \partial_\infty \mathbb{H}^n \times (t_0, t_0 + \frac{\pi}{n-1})$ for some real number $t_0$.

Then, there is no properly and completely immersed minimal hypersurface (maybe with finite boundary) $M \subset \mathbb{H}^n \times \mathbb{R}$ such that $\partial_\infty M = \Gamma$ and $M \cup \Gamma$ is a continuous $n$-manifold with boundary.

**Proof.** Assume, by contradiction, that there is such a minimal hypersurface $M$. Since $q_\infty \in \partial \text{Pr}(\Gamma)$ and $q_\infty \notin \text{Pr}(\partial \Gamma)$, there exists a $(n-1)$-geodesic plane $\omega \subset \mathbb{H}^n \times \{0\}$ such that a component $\omega^+$ of $\mathbb{H}^n \times \{0\} \setminus \omega$ satisfies:

1. $q_\infty \in \partial_\infty \omega^+$, $q_\infty \notin \partial_\infty \omega$ and $\partial_\infty \omega^+ \cap \text{Pr}(\partial \Gamma) = \emptyset$.
2. If $M_0$ denotes the component of $M \cap (\omega^+ \times \mathbb{R})$ containing $q_\infty$ in its asymptotic boundary, then
   
   (a) $M_0 \subset \mathbb{H}^n \times (t_0, t_0 + \frac{\pi}{n-1})$ for some real number $t_0$.
   (b) $\partial M_0 \subset \omega \times (t_0+2\varepsilon, t_0-2\varepsilon + \frac{\pi}{n-1})$ for some $\varepsilon > 0$.

Again, since $q_\infty \in \partial \text{Pr}(\Gamma)$ and $q_\infty \notin \text{Pr}(\partial \Gamma)$, there exists a $(n-1)$-geodesic plane $\pi \subset \mathbb{H}^n \times \{0\}$ such that a component $\pi^+$ of $\mathbb{H}^n \times \{0\} \setminus \pi$ satisfies:
Therefore we can find a compact part $K$ of a $n$-catenoid satisfying:

1. $K$ is connected.
2. $K \subset \pi^+ \times (t_0 + \varepsilon, t_0 - \varepsilon + \frac{\pi}{n-1})$.
3. $\partial K \subset \mathbb{H}^n \times \{t_0 + \varepsilon, t_0 - \varepsilon + \frac{\pi}{n-1}\}$.

We deduce consequently that $M_0 \cap K = \emptyset$. Then, considering the horizontal translated copies of $K$ and arguing as in the proof of Theorem 4.5, we get a contradiction with the maximum principle, which concludes the proof. 

The following result is an immediate consequence of Theorem 4.6.

**Corollary 4.1.** Let $S_\infty \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$ be a $(n-1)$-closed continuous submanifold. Considering the halfspace model for $\mathbb{H}^n$, we can assume that $S_\infty \subset \mathbb{R}^{n-1} \times \mathbb{R}$.

If $S_\infty$ is strictly convex in Euclidean sense, then there is no complete connected properly immersed minimal hypersurface $M$ in $\mathbb{H}^n \times \mathbb{R}$, possibly with finite boundary, with asymptotic boundary $S_\infty$ and such that $M \cup S_\infty$ is a continuous $n$-manifold with boundary.

**Remark 4.1.** It follows from Corollary 4.1 that there is no horizontal minimal graph in $\mathbb{H}^n \times \mathbb{R}$, [9, Equation (3)], given by a positive function $g \in C^2(\Omega) \cap C^0(\overline{\Omega})$, where $\Omega \subset \mathbb{R}^{n-1} \times \mathbb{R} \subset \partial_\infty \mathbb{H}^n \times \mathbb{R}$ is a bounded strictly convex domain in Euclidean sense, assuming zero value on $\partial \Omega$.

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Barbara Nelli
Dipartimento Ingegneria e Scienze dell’Informazione e Matematica
Università di L’Aquila
via Vetoio - Loc. Coppito
67010 (L’Aquila)
ITALY
E-mail address: nelli@univaq.it

Ricardo Sa Earp
Departamento de Matemática
Pontifícia Universidade Católica do Rio de Janeiro
Rio de Janeiro
22453-900 RJ
BRAZIL
E-mail address: earp@mat.puc-rio.br

Eric Toubiana
Université Paris Diderot - Paris 7
Institut de Mathématiques de Jussieu, UMR CNRS 7586
UFR de Mathématiques, Case 7012
Bâtiment Chevaleret
75205 Paris Cedex 13
FRANCE
E-mail address: toubiana@math.jussieu.fr