BRAIDED HOPF ALGEBRAS OBTAINED FROM COQUASITRIANGULAR HOPF ALGEBRAS

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ABSTRACT. Let \((H, \sigma)\) be a coquasitriangular Hopf algebra, not necessarily finite dimensional. Following methods of Doi and Takeuchi, which parallel the constructions of Radford in the case of finite dimensional quasitriangular Hopf algebras, we define \(H_\sigma\), a sub-Hopf algebra of \(H^0\), the finite dual of \(H\). Using the generalized quantum double construction and the theory of Hopf algebras with a projection, we associate to \(H\) a braided Hopf algebra structure in the category of Yetter-Drinfeld modules over \(H_\sigma\). Specializing to \(H = SL_q(N)\), we obtain explicit formulas which endow \(SL_q(N)\) with a braided Hopf algebra structure within the category of left Yetter-Drinfeld modules over \(U^\text{ext}_q(sl_N)^{\text{cop}}\).

1. INTRODUCTION

The Drinfeld double construction plays an important role in the theory of minimal quasitriangular Hopf algebras since the Drinfeld double of a finite dimensional Hopf algebra is minimal quasitriangular. Conversely, any minimal quasitriangular Hopf algebra is a quotient of a Drinfeld double \([15]\). This second statement is a consequence of the following facts. If \((H, R)\) is a quasitriangular Hopf algebra then from an expression for \(R \in H \otimes H\) of minimal length, one can define two finite dimensional sub-Hopf algebras of \(H\), denoted by \(R_\ell\) and \(R_r\), respectively, and there is a quasitriangular Hopf algebra morphism between the Drinfeld double of \(R_\ell\) and \(H\) whose image is \(H_{R_\ell} = R_\ell R_r\), the minimal quasitriangular Hopf algebra associated to \((H, R)\). Note that there is an isomorphism \(R_\ell^{\text{cop}} \cong R_r\) and this isomorphism, together with the evaluation map, gives a skew pairing from \(R_r \otimes R_\ell\) to \(k\).

Now suppose that \((H, \sigma)\) is a coquasitriangular Hopf algebra, not necessarily finite dimensional. The analogues of the Hopf algebras \(R_\ell\) and \(R_r\), denoted by \(H_\ell\) and \(H_r\), are, in general, infinite dimensional sub-Hopf algebras of \(H^0\), the finite dual of \(H\), and \(H_r H_\ell = H_\ell H_r\) is a sub-Hopf algebra of \(H^0\) denoted by \(H_\sigma\). In general, there is no analogue to the isomorphism \(R_\ell^{\text{cop}} \cong R_r\), but the coquasitriangular map \(\sigma\) induces a skew pairing on \(H_r \otimes H_\ell\). Although the classical Drinfeld double is not available in this setting, there exists a generalized quantum double, in the sense of Majid \([13]\) or Doi and Takeuchi \([5]\), for Hopf algebras in duality, so that the skew pairing between \(H_r\) and \(H_\ell\) gives a generalized quantum double, denoted \(D(H_r, H_\ell)\). In general, \(D(H_r, H_\ell)\) has no (co)quasitriangular structure (see Proposition \([23]\) and

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Remark 5.4): however, there is a surjective Hopf algebra morphism from $D(H_r, H_l)$ to $H_\sigma$ given by multiplication in $H^0$. Note that, if $(H, \sigma)$ is finite dimensional then $H_\sigma = H^*_R$, the minimal quasitriangular Hopf algebra associated to $(H^*, R)$, where $R$ is the matrix of $H^*$ obtained from $\sigma$. Thus, in the finite dimensional case, $(H_\sigma, R)$ is quasitriangular. In the infinite dimensional case, neither $H_\sigma$ nor $H^0$ is guaranteed a quasitriangular structure.

Drinfeld [6] observed that for a finite dimensional quasitriangular Hopf algebra $H$, its double $D(H)$ is a Hopf algebra with a projection. Majid [10] proved that the converse of Drinfeld’s result also holds, and computed on $H^*$ the braided Hopf algebra structure associated to this projection [16]. Here, for $(H, \sigma)$ coquasitriangular but not necessarily finite dimensional, using the evaluation pairing, we construct the generalized quantum double $D(H^{\cop}_{\sigma}, H)$. In the finite dimensional case, $H^{\cop}_{\sigma} = (H^*_R)^{\cop}$ has a quasitriangular structure and there is a Hopf algebra morphism from $D(H^{\cop}_{\sigma}, H)$ to $H^{\cop}_{\sigma}$ covering the natural inclusion, so $D(H^{\cop}_{\sigma}, H)$ is a Hopf algebra with a projection. We show that, for $H$ not necessarily finite dimensional, $D(H^{\cop}_{\sigma}, H)$ is still a Hopf algebra with projection, and we describe explicitly the induced structure on $H$ of a braided Hopf algebra in the category of left Yetter-Drinfeld modules over $H^{\cop}_{\sigma}$.

This paper is organized as follows. In Section 2, Preliminaries, we first define pairings and skew pairings on Hopf algebras and give the construction of the generalized quantum double. Coquasitriangular Hopf algebras give important examples of Hopf algebras with a skew pairing. As well, we describe the structure of a Hopf algebra $A$ with projection (see [16]), so that $A$ is isomorphic to a Radford biproduct $B \times K$, where $B$ is a braided Hopf algebra in the category of left Yetter-Drinfeld modules over $K$.

In Section 3, we first work in a rather general context. For $U,V$ bialgebras such that there is an invertible pairing from $U \otimes V$ to $k$, we show that there is a projection from $D(U^{\cop}, V)$ to $U^{\cop}$ covering the natural inclusion if and only if there is a bialgebra morphism $\gamma : V \to U^{\cop}$ satisfying a relation analogous to the coquasitriangularity condition for a skew pairing. (See also [5] and [17].) Now if $U,V$ are Hopf algebras with an invertible pairing and $\gamma$ exists as above, then $V$ has a Hopf algebra structure in the category of left Yetter-Drinfeld modules over $U^{\cop}$. We describe this structure explicitly. For example, if $(U, R)$ is quasitriangular, then such a map $\gamma$ exists. Then, for $H$ finite dimensional we prove that there is a projection $\pi : D(H) = D(H^{*^{\cop}}, H) \to H^{*^{\cop}}$ covering the inclusion $H^{*^{\cop}} \hookrightarrow D(H)$ if and only if $H$ is coquasitriangular (Proposition 3.9). A fortiori, $H$ has a braided Hopf algebra structure. Moreover, this structure allows us to obtain the left version of the transmutation theory for coquasitriangular Hopf algebras (Remark 3.10).

The transmutation theory for coquasitriangular Hopf algebras is due to Majid [11]. Using the dual reconstruction theorem, he associated to any coquasitriangular Hopf algebra $(H, \sigma)$, a braided commutative Hopf algebra $\mathcal{H}$ in the category of right $H$-comodules, called the function algebra braided group associated to $H$. In Section 4.4 we show that $\mathcal{H}$ can be obtained by writing a generalized quantum double as a Radford biproduct using the projection $\pi : D(H, H) \to H$ given by multiplication in $H$. To this projection corresponds a braided Hopf algebra $\mathcal{H}$ in the category
of left $H$-comodules. By considering the co-opposite case, we obtain, after some identification, that $(H_{\sigma}^{\text{cop}})^{\text{cop}} = H_{\sigma}$ as braided Hopf algebras (Proposition 4.2).

In Section 4.2 we describe the “dual case”, and show that $D(H_{\sigma}^{\text{cop}}, H)$ is always a Hopf algebra with a projection. This follows from a more general construction where we take $A$ and $X$ to be sub-Hopf algebras of $H$, $H_{rA}$ and $H_{lX}$ the corresponding sub-Hopf algebras of $H_r$ and $H_l$ respectively, and $H_{X,A}$ the corresponding sub-Hopf algebra of $H_{\sigma}$. Then there is a Hopf algebra projection from $D(H_{X,A}^{\text{cop}}, X)$ to $H_{X,A}^{\text{cop}}$ which covers the inclusion. We stress the fact that this “dual case” give rise to a new braided Hopf algebra structure on $X$ (denoted by $\mathcal{X}$) in the category of left $H_{X,A}^{\text{cop}}$ Yetter-Drinfeld modules. This new construction cannot be viewed as an example of the transmutation theory because the transmutation theory associates to an ordinary coquasitriangular Hopf algebra a braided Hopf algebra in the category of corepresentations over itself. We show that, in fact, the two constructions above are related by a non-canonical braided functor. More exactly, we show that there exists a braided functor $F : \mathcal{M}^X \to H_{X,A}^{\text{cop}} \mathcal{D}$ such that $F(\mathcal{X}) = \mathcal{X}$ (Theorem 4.10). Nevertheless, we think it worthwhile to have the construction of $\mathcal{X}$. This is first of all because $\mathcal{X}$ lies in a category of left Yetter-Drinfeld modules over $H_{X,A}^{\text{cop}}$ which, in general, may not have a quasitriangular structure. (See the example in Section 6). Secondly, this is because evaluation gives a duality between $H_{X,A}$ and $X$ and the associated quantum double, $D(H_{X,A}^{\text{cop}}, X)$, is isomorphic to $\mathcal{X} \times H_{X,A}^{\text{cop}}$.

In Section 5 we present the finite dimensional case in full detail, linking the results of this paper to those of Radford described above. Also, we note that starting with a finite dimensional quasitriangular Hopf algebra $(H, R)$ and taking $A = X = H^*$ then the corresponding braided Hopf algebra $H^*$ is precisely the categorical dual of $H_{\sigma}^{\text{cop}}$, the associated enveloping algebra braided group of $(H_{\text{cop}}^*, R_{21})$ constructed by Majid in [12] (Remarks 5.8).

Finally, in Section 6 we apply the constructions of Section 4.2 to the coquasitriangular Hopf algebra $H := SL_q(N)$. By direct computations we show that $H_l$ and $H_r$ are the Borel-like Hopf algebras $U_q(b_+)$ and $U_q(b_-)$, respectively, associated to $U_q^{\text{ext}}(sl_N)$, and obtain that $H_{\sigma} = U_q^{\text{ext}}(sl_N)$. The general theory above leads to more conceptual proofs for some well known results. Namely, there exist dual parings of the pairs of Hopf algebras $U_q^{\text{ext}}(b_+)$ and $U_q(b_-)$, $U_q^{\text{ext}}(sl_N)$ and $SL_q(N)$, and $U_q^{\text{ext}}(sl_N)$ and $SL_q(N)$, respectively (Corollary 6.11 and Corollary 6.14, and $U_q^{\text{ext}}(sl_N)$ and $U_q(sl_N)$ are factors of generalized quantum doubles (Corollary 6.12 and Remark 6.13). Also, the Hopf algebra structure of $U_q(sl_N)$ is achieved in a natural way (Remark 6.9 and Remark 6.13). The rest of the Section 6 is dedicated to a description of the braided Hopf algebra structure of $SL_q(N)$ in the category of left $U_q^{\text{ext}}(sl_N)^{\text{cop}}$ Yetter-Drinfeld modules. The explicit formulas for this braided Hopf algebra structure can be found in Proposition 6.16 and Theorem 6.17.

Concluding, the general results in this paper have led not only to some properties of the couple $(U_q^{\text{ext}}(sl_N), SL_q(N))$ well known in quantum group theory, but also to some new ones. Namely, $SL_q(N)$ has a braided Hopf algebra structure in the category of left Yetter-Drinfeld modules over $U_q^{\text{ext}}(sl_N)^{\text{cop}}$ which coincides with that of the image of the transmutation object of $SL_q(N)$ through a non-canonical braided functor, and the corresponding Radford biproduct is isomorphic to the generalized quantum double associated to this dual pair.
2. Preliminaries

Throughout, we work over a field $k$ and maps are assumed to be $k$-linear. Any unexplained definitions or notation may be found in [8], [13], [14] or [18].

For $B$ a $k$-bialgebra, we write the comultiplication in $B$ as $\Delta(b) = b_1 \otimes b_2$ for $b \in B$. For $M$ a left $B$-comodule, we write the coaction as $\rho(m) = m_{-1} \otimes m_0$. For $N$ a right $B$-comodule, we will use the subscript bracket notation to differentiate subscripts from those in comultiplication expressions, i.e. we will write $\rho(m) = m_{(0)} \otimes m_{(1)}$ for $m \in M$. For $k$-spaces $M$ and $N$, $\text{tw}$ will denote the usual twist map from $M \otimes N$ to $N \otimes M$.

2.1. Pairings on Hopf algebras and the generalized quantum double. We first recall the definition for two bialgebras or Hopf algebras to be in duality (see [21, Section 1] or [13, Section 1.4]); this notion was first introduced by Takeuchi and called a Hopf pairing.

Throughout this section $U$ and $V$ will denote bialgebras over $k$.

Definition 2.1. A bilinear form $\langle \cdot, \cdot \rangle : U \otimes V \to k$ is called a pairing of $U$ and $V$ if

\begin{align*}
(1) & \quad \langle mn, x \rangle = \langle m, x_1 \rangle \langle n, x_2 \rangle, \\
(2) & \quad \langle m, xy \rangle = \langle m_1, x \rangle \langle m_2, y \rangle, \\
(3) & \quad \langle 1, x \rangle = \varepsilon(x), \quad \langle m, 1 \rangle = \varepsilon(m),
\end{align*}

for all $m, n \in U$ and $x, y \in V$. Then $U$ and $V$ are said to be in duality.

Remarks 2.2. (i) A bilinear form from $U \otimes V$ to $k$ is called a skew pairing if it is a pairing from $U^{\text{op}} \otimes V$ to $k$ or, equivalently, a pairing from $U \otimes V^{\text{op}}$ to $k$.

(ii) The form $\langle \cdot, \cdot \rangle$ is a pairing of $U$ and $V$ if and only if there is a bialgebra morphism $\phi : U \to V^0$, defined by $\phi(u)(v) = \langle u, v \rangle$, if and only if there exists a bialgebra morphism $\psi : V \to U^0$ defined by $\psi(v)(u) = \langle u, v \rangle$. If $U$ and $V$ are Hopf algebras, these maps are Hopf algebra maps.

(iii) If $U$ and $V$ are Hopf algebras and $\langle \cdot, \cdot \rangle$ is a pairing of $U$ and $V$, then this bilinear form is invertible in the convolution algebra $\text{Hom}(U \otimes V, k)$ and its inverse is the bilinear form which maps $u \otimes v$ to $\langle S_U(u), v \rangle = \langle u, S_V(v) \rangle$.

(iv) If $\langle \cdot, \cdot \rangle$ is a (skew) pairing between $U$ and $V$, then there is a (skew) pairing between the sub-bialgebras $\phi(U) \subseteq V^0$ and $\psi(V) \subseteq U^0$ defined by the bilinear form $B : \phi(U) \otimes \psi(V) \to k$,

$$B(\phi(u), \psi(v)) = \langle u, v \rangle.$$

The form $B$ is well-defined since $\phi(u) = \phi(u')$ if and only if $\langle u, - \rangle = \langle u', - \rangle$ and $\psi(v) = \psi(v')$ if and only if $\langle -, v \rangle = \langle -, v' \rangle$. It is straightforward to check that $B$ is a (skew) pairing.

(v) If $\langle \cdot, \cdot \rangle$ is a pairing of the bialgebras $U$ and $V$ and $\mathfrak{U} \subseteq U$, $\mathfrak{V} \subseteq V$ are sub-bialgebras then the restriction of $\langle \cdot, \cdot \rangle$ to $\mathfrak{U} \otimes \mathfrak{V}$ is a pairing of $\mathfrak{U}$ and $\mathfrak{V}$. As above, the bialgebras $\phi(\mathfrak{U})$ and $\psi(\mathfrak{V})$ are also in duality.

Example 2.3. For $H$ a bialgebra, then the evaluation map provides a duality between $H^0$, the finite dual of $H$, and $H$, that is, $\langle f, h \rangle = f(h)$ for all $f \in H^0$ and $h \in H$.

Another class of examples for bialgebras in duality is provided by coquasitriangular bialgebras, also called braided bialgebras. This concept is dual to the idea of quasitriangular bialgebras (see [8] or [13] for the definition). We recall the notion of coquasitriangularity.
Definition 2.4. A bialgebra $H$ is called coquasitriangular (CQT for short) if there exists a convolution invertible $k$-bilinear skew pairing $\sigma : H \otimes H \to k$, i.e., for all $h, h', g \in H$,

\begin{align}
(2.4) \quad \sigma(hh', g) &= \sigma(h, g_1)\sigma(h', g_2) \\
(2.5) \quad \sigma(g, hh') &= \sigma(g_2, h)\sigma(g_1, h') \\
(2.6) \quad \sigma(1, h) &= \sigma(h, 1) = \varepsilon(h),
\end{align}

which also satisfies the coquasitriangular condition,

\begin{equation}
(2.7) \quad \sigma(h_1, h'_1)h_2h'_2 = h'_1h_1\sigma(h_2, h'_2).
\end{equation}

Remarks 2.5. (i) Doi \cite{Doi} showed that for $(H, \sigma)$ a CQT Hopf algebra, one may define an invertible element $v \in H^*$ by

\begin{equation}
(2.8) \quad v(h) = \sigma(h_1, S(h_2)) \text{ with inverse } v^{-1}(h) = \sigma(S^2(h_1), h_2),
\end{equation}

and

\begin{equation}
(2.9) \quad S^2(h) = v^{-1}(h_1)h_2v(h_3), \quad \forall \ h \in H.
\end{equation}

Similarly, the element $u \in H^*$ defined by $u(h) = \sigma(h_2, S(h_1))$, is invertible with inverse $u^{-1}(h) = \sigma(S^2(h_2), h_1)$ and also defines the square of the antipode $S$ of $H$ as a co-inner automorphism of $H$, i.e., $S^2(h) = u(h_1)h_2u^{-1}(h_3)$, for all $h \in H$. In particular, the antipode $S$ is bijective.

(ii) Let $(H, \sigma)$ be a CQT bialgebra. Then $\sigma$ is a convolution invertible pairing between $H$ and $H^{\text{cop}}$ and also between $H^{\text{cop}}$ and $H$.

(iii) Clearly any sub-bialgebra of a CQT bialgebra $(H, \sigma)$ is CQT.

(iv) If $(H, \sigma)$ is CQT, then so are $(H^{\text{cop}}, \sigma^{-1})$ and $(H^{\text{cop}}, \sigma^{-1})$ where, if $H$ is a Hopf algebra, $\sigma^{-1} = \sigma \circ (S_H \otimes Id_H) = \sigma \circ (Id_H \otimes S_H^{-1})$ is the convolution inverse of $\sigma$. \cite[1.2]{Majid}. Moreover, $(H, \sigma_{21}^{-1} = \sigma^{-1} \circ tw)$ is another CQT structure for $H$, so that $(H^{\text{cop}}, \sigma_{21}) := \sigma \circ tw)$ and $(H^{\text{cop}}, \sigma_{21})$ are CQT also.

Let $\mathcal{B}$, $\mathcal{H}$ be bialgebras and $\varphi$ an invertible skew pairing $\varphi : \mathcal{B} \otimes \mathcal{H} \to k$. We now define the bialgebra (Hopf algebra) structure on $\mathcal{B} \otimes \mathcal{H}$ to be studied in this paper. We follow the presentation in \cite{Majid}.

For $A$ a bialgebra, an invertible bilinear form $\tau$ on $A$ is called a unital 2-cocycle if for all $a, b, c \in A$,

\begin{equation}
\tau(a_1, b_1)\tau(a_2b_2, c) = \tau(b_1, c_1)\tau(a, b_2c_2) \text{ and } \tau(a, 1) = \tau(1, a) = \varepsilon(a).
\end{equation}

If $\tau$ is a 2-cocycle on $A$, then we may form $A^\tau$, the bialgebra which has coalgebra structure from $A$ but with the new multiplication

\begin{equation}
\quad a \bullet b = \tau(a_1, b_1)a_2b_2\tau^{-1}(a_3, b_3).
\end{equation}

It is shown in \cite{Majid} that if $\varphi$ is an invertible skew pairing on $\mathcal{B} \otimes \mathcal{H}$, then the bilinear form on $\mathcal{B} \otimes \mathcal{H}$ defined by $\tau(b \otimes h, b' \otimes h') = \varepsilon(b)\varphi(b', h) \varepsilon(h')$ is a unital 2-cocycle. Then we may form the bialgebra $(\mathcal{B} \otimes \mathcal{H})^\tau$. As a coalgebra, $(\mathcal{B} \otimes \mathcal{H})^\tau = \mathcal{B} \otimes \mathcal{H}$, the unit is $1 \otimes 1$, and the multiplication is defined by

\begin{equation}
(2.10) \quad (b \otimes h)(b' \otimes h') = \varphi(b'_1, h_1)\varphi^{-1}(b'_3, h_3)b_3b'_1 \otimes h_2h'.
\end{equation}

For $\mathcal{B}, \mathcal{H}$ Hopf algebras with bijective antipodes, the inverse of $\varphi$ as a skew-pairing on $\mathcal{B} \otimes \mathcal{H}$ is given by $\varphi^{-1}(b, h) = \varphi(S_{\mathcal{B}}(b), h) = \varphi(b, S^{-1}_{\mathcal{H}}(h))$. Then $(\mathcal{B} \otimes \mathcal{H})^\tau$
\( \mathfrak{H} \), also often denoted by \( \mathfrak{B} \otimes \mathfrak{H} \), has antipode given by
\[
S(b \otimes h) = (1 \otimes S_\mathfrak{H}(h)) \cdot (S_\mathfrak{B}(b) \otimes 1)
\]
(2.11)
\[
\varphi(S_\mathfrak{B}(b_1), S_\mathfrak{H}(h_1))\varphi^{-1}(S_\mathfrak{B}(b_2), S_\mathfrak{H}(h_2))S_\mathfrak{B}(b_2) \otimes S_\mathfrak{H}(h_2).
\]

This construction is a special case of the generalized quantum double (see [13 Chapter 7]) for matched pairs of Hopf algebras.

**Definition 2.6.** We denote \((\mathfrak{B} \otimes \mathfrak{H})^\tau\) by \(D(\mathfrak{B}, \mathfrak{H})\) and will refer to this bialgebra (Hopf algebra) with multiplication as in (2.10), coalgebra structure from the tensor product \(\mathfrak{B} \otimes \mathfrak{H}\), and antipode as in (2.11) as a generalized quantum double.

In fact, in many of our constructions, we will begin with a pairing \(\langle , \rangle : U \otimes V \to k\) which is then a skew pairing \(\rho\) from \(U^\text{cop} \otimes V \to k\). Then, continuing to write subscripts in \(U\), and using the fact that \(\langle S_U^{-1}(m), S_V(x) \rangle = (m, x)\), we have that the formulas for the multiplication, comultiplication and antipode in \(D(U^\text{cop}, V)\) are given by
\[
\langle m \otimes x, n \otimes y \rangle = \langle n_3, x_1 \rangle \langle S_U^{-1}(n_1), x_3 \rangle mn_2 \otimes x_2y
\]
(2.12)
\[
\Delta(m \otimes x) = (m_2 \otimes x_1) \otimes (m_1 \otimes y_2)
\]
(2.13)
\[
S(m \otimes x) = \langle m_1, x_3 \rangle \langle S_V^{-1}(x_1), S_U^{-1}(m_2) \otimes S_V(x_2) \rangle
\]
(2.14)

**Example 2.7.** (cf. [13 7.2.5]) If \(H\) is a finite dimensional Hopf algebra, then \(H^*\) and \(H\) are in duality via the evaluation map as mentioned above and the double \(D(H^\text{cop}, H)\) is the usual Drinfeld double, denoted in this case by \(D(H)\).

It is shown in [13] that if \((H, \sigma)\) is CQT, so is the double \((H \otimes H)^\tau = D(H, H)\), where \(\tau\) is the unital 2-cocycle on \(H \otimes H\) defined by the skew-pairing \(\sigma\).

The next proposition shows that this generalizes to \(D(\mathfrak{B}, \mathfrak{H})\) where \(\mathfrak{B}, \mathfrak{H}\) are CQT Hopf algebras.

**Proposition 2.8.** Let \(\mathfrak{B}, \mathfrak{H}\) be Hopf algebras, \(\varphi : \mathfrak{B} \otimes \mathfrak{H} \to k\) a skew pairing, and \(D = D(\mathfrak{B}, \mathfrak{H}) = (\mathfrak{B} \otimes \mathfrak{H})^\tau\) the generalized quantum double of Definition 2.6. Then \(D\) is CQT if and only if \(\mathfrak{B}\) and \(\mathfrak{H}\) are.

**Proof.** Since sub-Hopf algebras of a CQT Hopf algebra are CQT, then if \(D\) is CQT, so are \(\mathfrak{B}\) and \(\mathfrak{H}\).

Now suppose that \((\mathfrak{B}, \sigma_\mathfrak{B})\) and \((\mathfrak{H}, \sigma_\mathfrak{H})\) are CQT. Then we form the CQT Hopf algebra \((A, \sigma) = (\mathfrak{B} \otimes \mathfrak{H}, (\sigma_\mathfrak{B} \otimes \sigma_\mathfrak{H}) \circ (Id_\mathfrak{B} \otimes tw \otimes Id_\mathfrak{H}))\). From the above discussion, \(\tau : A \otimes A \to k\) is a unital 2-cocycle, where \(\tau(b \otimes h, b' \otimes h') = \varepsilon(b)\varphi(b', h)\varepsilon(h')\).

Then by [13] p.61 (2.24)], \(A^\tau\) is CQT via the bilinear form \(\omega\) defined by \(\omega(a, a') = \tau(a_1, a_1)\sigma(a_2, a'_2)\tau^{-1}(a_3, a'_3)\). Specifically, \((D = A^\tau, \omega)\) is coquasitriangular where
\[
\omega(b \otimes h, b' \otimes h') = \varphi(b_1, b_1')\sigma_\mathfrak{B}(b_2, b_2')\sigma_\mathfrak{H}(h_1, h_1')\varphi(S_\mathfrak{B}(b_2'), h_2).
\]

(By using (2.7) for \(\varphi\) generously, one can even check the coquasitriangularity conditions (2.4) to (2.7) for \(\omega\) directly.)

Next we show how the doubles \(D(U^\text{cop}, V)\) and \(D(\psi(U^\text{cop}), \psi(V))\) are related.

**Lemma 2.9.** Let \(U, V\) be Hopf algebras and \(\langle , \rangle\) a pairing of \(U\) and \(V\). Then \(\phi(U) \subseteq V^0\) and \(\psi(V) \subseteq U^0\) are also Hopf algebras with a pairing \(B\) on \(\phi(U) \otimes \psi(V)\) defined by \(B(\phi(u), \psi(v)) = \langle u, v \rangle\). Then \(\phi \otimes \psi : D(U^\text{cop}, V) \to D(\phi(U^\text{cop}), \psi(V))\) is a surjection of Hopf algebras.
Proof. From Remarks 2.2 and the fact that \( \phi \otimes \psi : U^{\text{cop}} \otimes V \to \phi(U^{\text{cop}}) \otimes \psi(V) \) is a Hopf algebra surjection, it remains only to show that \( \phi \otimes \psi \) respects the multiplication in the double. This can be checked by a straightforward computation, or by noting that for \( D(U^{\text{cop}}, V) = (U^{\text{cop}} \otimes V)^r \) and \( D(\phi(U^{\text{cop}}), \psi(V)) = (\phi(U^{\text{cop}}) \otimes \psi(V))^r \) for cocycles \( \tau, \tau' \) as above, then \( \tau(m \otimes x, n \otimes y) = \tau'(\phi(m) \otimes \psi(x), \phi(n) \otimes \psi(y)) \). \( \square \)

2.2. Hopf algebras with projection. Let \( K \) be a bialgebra. Recall that a left Yetter-Drinfeld module over \( K \) is a left \( K \)-module \( M \) which is also a left \( K \)-comodule, such that the following compatibility relation holds. For all \( \kappa \in K \) and \( m \in M \):

\[
\kappa_1 m_{-1} \otimes \kappa_2 \cdot m_0 = (\kappa_1 \cdot m)_{-1} \kappa_2 \otimes (\kappa_1 \cdot m)_0,
\]

where \( K \otimes M \ni \kappa \otimes m \mapsto \kappa \cdot m \in M \) is the left \( K \)-action. The category of left Yetter-Drinfeld modules over \( K \) and \( k \)-linear maps that preserve the \( K \)-action and \( K \)-coaction is denoted by \( \text{KD} \).

The category \( \text{KD} \) is pre-braided. If \( M, N \in \text{KD} \) then \( M \otimes N \) is a left Yetter-Drinfeld module over \( K \) via the structures defined by

\[
\kappa \cdot (m \otimes n) = \kappa_1 \cdot m \otimes \kappa_2 \cdot n
goal: for all \( \kappa \in K, m \in M \) and \( n \in N \). The pre-braiding is given by

\[
e_{M, N}(m \otimes n) = m_{-1} \cdot n \otimes m_0.
\]

If \( K \) is a Hopf algebra then \( e \) is invertible, so \( \text{KD} \) is a braided monoidal category.

The structure of a Hopf algebra with projection was given in [16]. More precisely, if \( K \) and \( A \) are Hopf algebras with Hopf algebra maps \( K \xrightarrow{i} A \) such that \( \pi \circ i = \text{Id}_K \), then there exists a braided Hopf algebra \( B \) in the category of left Yetter-Drinfeld modules \( \text{KD} \) such that \( A \cong B \times K \) as Hopf algebras, where \( B \times K \) denotes Radford’s biproduct between \( B \) and \( K \) (for more details see [16]).

As \( k \)-vector space \( B = \{a \in A \mid a_1 \otimes \pi(a_2) = a \otimes 1\} \). Now, \( B \) is a \( K \)-module subalgebra of \( A \), where \( A \) is a left \( K \)-module algebra via the left adjoint action induced by \( i \), that is \( \kappa \cdot a = i(\kappa_1) a i(S(\kappa_2)) \), for all \( \kappa \in K \) and \( a \in A \). Moreover, \( B \) is an algebra in the braided category \( \text{KD} \) where the left coaction of \( K \) on \( B \) is given for all \( b \in B \) by

\[
\lambda_B(b) = \pi(b_1) \otimes b_2,
\]

Also, as \( k \)-vector space, \( B \) is the image of the \( k \)-linear map \( \Pi : A \to A \) defined for all \( a \in A \) by

\[
\Pi(a) = a_1 i(S(\pi(a_2))).
\]

For all \( a \in A \), we define

\[
\Delta(\Pi(a)) = \Pi(a_1) \otimes \Pi(a_2).
\]

This makes \( B \) into a coalgebra in \( \text{KD} \) and a bialgebra in \( \text{KD} \). The counit of \( B \) is \( \varepsilon = \varepsilon |_B \). Moreover, we have that \( B \) is a braided Hopf algebra in \( \text{KD} \) with antipode \( S \) given by

\[
S(b) = i(\pi(b_1)) S_A(b_2),
\]

where \( S_A \) is the antipode of \( A \).

The Hopf algebra isomorphism \( \chi : B \times K \to A \) is given by

\[
\chi(b \times \kappa) = bi(\kappa),
\]
We form the generalized quantum double $D$. Proposition 3.1. Let $\rho$ be an invertible pairing, so that $\gamma$ is a bialgebra map.

Hence, we have $\gamma$ holds, we compute $(3.1)$. Define $\pi,\gamma$ to be $\pi(m \otimes x) = m\gamma(x)$, for all $m \in U$ and $x \in V$. Then for $m \in U$, we have that $\pi \circ i(m) = \pi(m \otimes 1) = m\gamma(1) = m$. Furthermore, by \[2.4\] with $B = J = U^{\text{cop}}$, $H = V$, $\alpha = Id_U$, $\beta = \gamma$, we have that $\pi$ is a bialgebra morphism.

Conversely, given $\pi$, define $\gamma$ by $\gamma = \pi \circ j$ where $j : V \to D(U^{\text{cop}}, V)$ is defined by $j(x) = 1 \otimes x$. Then, since $\pi, j$ are bialgebra maps, so is $\gamma$. To verify that $(3.1)$ holds, we compute

$$
\gamma(y)m = (\pi \circ j)(y)m = \pi((1 \otimes y)(m \otimes 1)) = \rho^{-1}(m_1, y_3)\rho(m_3, y_1)m_2\gamma(y_2).
$$

(ii) The proof of (ii) is analogous. □

Example 3.2. (cf. \[3.1\]) Let $\sigma$ be an invertible skew pairing on a bialgebra $H$ and form $D(H, H)$. Then the identity map $Id_H$ satisfies $(3.1)$ if and only if $(H, \sigma)$ is CQT if and only if the multiplication map $\pi : D(H, H) \to H$, $\pi(h \otimes l) = hl$ is a bialgebra map.

Example 3.3. In the setting of Proposition $3.1$, if $V$ is quasitriangular via $R = R^1 \otimes R^2 \in V \otimes V$, then the map $\pi : D(U^{\text{cop}}, V) \to V$ defined by $\pi(m \otimes x) =$
\langle m, R^1 \rangle R^2 x \text{ is a bialgebra projection. Here the map } \mu : U^\text{cop} \to V \text{ is given by } \mu(m) = \langle m, R^1 \rangle R^2. \text{ (The details can be found in [5, 2.5].)}

Likewise, if } U \text{ is quasitriangular with the } R\text{-matrix } R = R^1 \otimes R^2 \in U \otimes U, \text{ then } \pi : D(U^\text{cop}, V) \to U^\text{cop} \text{ defined by } \pi(m \otimes x) = \langle R^2, x \rangle m S(R^1) \text{ is a Hopf algebra projection. In this case the map } \gamma : V \to U^\text{cop} \text{ is given by } \gamma(y) = \langle R^2, y \rangle S(R^1).

\text{Remarks 3.4. (i) In later computations we will use that [3.1] is equivalent to}
\begin{equation}
(3.3) \quad \rho(m_1, y_2) \gamma(y_1) m_2 = \rho(m_2, y_1) m_1 \gamma(y_2).
\end{equation}

If } m = \gamma(x), \ x \in V, \text{ then [3.1] becomes}
\begin{equation}
(3.4) \quad \rho(\gamma(x_2), y_2) \gamma(y_1) \gamma(x_1) = \rho(\gamma(x_1), y_1) \gamma(x_2) \gamma(y_2).
\end{equation}

(ii) Note that if the above map } \gamma : V \to U^\text{cop} \text{ is injective then the map } \sigma : V \otimes V \to k \text{ defined by } \sigma (x, y) = \rho(\gamma(x), y) = \langle \gamma(x), y \rangle \text{ gives } V \text{ a CQT structure. The relations } (2.4), (2.5), (2.6) \text{ are easy to check and (2.7) is equivalent to}
\begin{equation}
(3.5) \quad \gamma(x_1) x_2 y_2 = \langle \gamma(x_2), y_2 \rangle y_1 x_1.
\end{equation}

This equation holds if and only if it holds when the injective map } \gamma \text{ is applied to both sides, i.e., when [3.4] holds.

If } U, V \text{ are Hopf algebras with bijective antipodes, then for } \rho \text{ a skew pairing from } U^\text{cop} \otimes V \text{ to } k, \text{ we have } \rho^{-1}(m, x) = \langle S^{-1}_U(m), x \rangle = \langle m, S^{-1}_V(x) \rangle. \text{ In this case, we have the identities below which are useful in the following computations and also provide generalizations of the equations describing the square of the antipode in Remarks 2.5(i).}

\textbf{Proposition 3.5.} Let } U, V \text{ be Hopf algebras in duality and assume that there exists a map } \gamma \text{ as in Proposition 2.7. Then:}

(i) \text{ The map } \vartheta \in V^* \text{ defined by } \vartheta(x) = \langle \gamma(x_1), S_V(x_2) \rangle, \text{ for all } x \in V, \text{ is convolution invertible with } \vartheta^{-1}(x) = \langle \gamma(S_V^2(x_1)), x_2 \rangle. \text{ Moreover, for any } x \in V,
\begin{equation}
(3.5) \quad \gamma(S_V^2(x)) = \vartheta^{-1}(x_1) \gamma(x_2) \vartheta(x_3).
\end{equation}

(ii) \text{ Similarly, the map } \upsilon \in V^* \text{ defined by } \upsilon(x) = \langle \gamma(x_2), S_V(x_1) \rangle, \text{ for all } x \in V, \text{ is convolution invertible with } \upsilon^{-1}(x) = \langle \gamma(S_V^2(x_2)), x_1 \rangle. \text{ In addition, for all } x \in V,
\begin{equation}
(3.6) \quad \gamma(S_V^2(x)) = \upsilon(x_1) \gamma(x_2) \upsilon^{-1}(x_3).
\end{equation}

\textbf{Proof.} We only sketch the proof for (i); the rest of the details are left to the reader. For all } x \in V \text{ we have
\begin{equation}
\vartheta(x_1) \gamma(S_V^2(x_2)) = \langle \gamma(x_3), S_V(x_4) \rangle \gamma(x_1) \gamma(S_V(x_2)) \gamma(S_V^2(x_5)) = \langle \gamma(x_3), S_V(x_4) \rangle \gamma(x_1) S_U^{-1}(\gamma(S_V(x_5)) \gamma(x_2)) = \langle \gamma(x_2), S_V(x_3) \rangle \gamma(x_1) S_U^{-1}(\gamma(x_3) \gamma(S_V(x_4))) = \langle \gamma(x_2), S_V(x_3) \rangle \gamma(x_1) = \gamma(x_1) \vartheta(x_2),
\end{equation}

and, in a similar manner, one can prove that
\begin{equation}
(3.7) \quad \gamma(S_V^2(x_1)) \vartheta^{-1}(x_2) = \vartheta^{-1}(x_1) \gamma(x_2).
\end{equation}

Now, for } x \in V, \text{ using the fact that } \gamma : V \to U^\text{cop} \text{ is a Hopf algebra map, we have}
\begin{equation}
\vartheta(x_1) \vartheta^{-1}(x_2) = \vartheta(x_1) \langle \gamma(S_V^2(x_2)), x_3 \rangle = \vartheta(x_2) \langle \gamma(x_1), x_3 \rangle = \langle \gamma(x_2), S_V(x_3) \rangle \langle \gamma(x_1), x_4 \rangle = \langle \gamma(x_1), S_V(x_2)x_3 \rangle = \varepsilon(x).
\end{equation}
Similarly, using (3.7) we can show that \( \vartheta^{-1}(x_1)\vartheta(x_2) = \varepsilon(x) \), so we are done. \( \square \)

Now suppose that \( U, V \) are Hopf algebras with bijective antipodes and we have a Hopf algebra projection \( \pi \) from \( D(U^{\text{cop}}, V) \) to \( U^{\text{cop}} \) that splits \( i \). Then there exists a Hopf algebra \( B \) in the category of Yetter-Drinfeld modules \( U^{\text{cop}} \text{-} \text{YD} \) such that \( D = D(U^{\text{cop}}, V) \cong B \times U^{\text{cop}} \), a Radford biproduct. From Subsection 2.2 we know that

\[
B = \{ a \in D \mid a_1 \otimes \pi(a_2) = a \otimes 1 \}.
\]

**Proposition 3.6.** The map \( \theta : V \to B \) given by \( \theta(y) = \gamma(S_V^{-1}(y_2)) \otimes y_1 = S_U(\gamma(y_2)) \otimes y_1 \) is a bijection.

**Proof.** Clearly the map \( \theta \) is injective since \( \varepsilon \circ \gamma = \varepsilon \). It remains to show that \( \text{Im}(\theta) = B \). For \( y \in V \), we have \( \theta(y) \in B \) since

\[
\theta(y_1) \otimes \pi(\theta(y)_2) = (S_U(\gamma(y_4)) \otimes y_1) \otimes \pi(S_U(\gamma(y_3)) \otimes y_2)
\]

\[
= (S_U(\gamma(y_4)) \otimes y_1) \otimes (S_V^{-1}(y_2)y_2)
\]

\[
= (S_U(\gamma(y_2)) \otimes y_1) \otimes 1 = \theta(y) \otimes 1.
\]

Conversely, suppose that \( m \otimes y \in B \), i.e., \( (m_2 \otimes y_1) \otimes m_1 \gamma(y_2) = (m \otimes y) \otimes 1 \in D \otimes U^{\text{cop}} \). Then we have

\[
m \otimes y = S_U(1)m \otimes y
\]

\[
= S_U(m_1 \gamma(y_2))m_2 \otimes y_1
\]

\[
= S_U(\gamma(y_2))S_U(m_1)m_2 \otimes y_1
\]

\[
= \varepsilon_U(m)\theta(y).
\]

Similarly, if \( z = \sum_i m_i \otimes y_i \in B \), then \( z = \theta(\sum_i \varepsilon(m_i)y_i) \). \( \square \)

Now we denote by \( \underline{V} \) the vector space \( V \) with the structure of a Yetter-Drinfeld module induced by that of \( B \).

**Proposition 3.7.** The structure of the left \( U^{\text{cop}} \) Yetter-Drinfeld module \( \underline{V} \) is given by the left action and right coaction

\[
m \triangleright y = \langle m_1, S_V^{-1}(y_1) \rangle y_2(m_2, S_V^{-2}(y_2)) = \langle m, S_U^{-1}(S_V^{-1}(y_3)y_1) \rangle y_2; \tag{3.8}
\]

\[
\lambda_{\underline{V}}(y) = \gamma(S_U^{-1}(y_3)y_1) \otimes y_2. \tag{3.9}
\]

**Proof.** For \( m \in U \) and \( y \in \underline{V} \) we have that \( m \triangleright y = \vartheta^{-1}(m \triangleright \vartheta(y)) \) by Section 2.2 and using (2.12), (3.1) and the fact that \( \langle m, v \rangle = \langle S_U^{-1}(m), S_V(v) \rangle \), we compute

\[
i(m_2)\theta(y)i(S_U^{-1}(m_1)) = (m_2 \otimes 1)(\gamma(S_V^{-1}(y_2)) \otimes y_1)(S_U^{-1}(m_1) \otimes 1)
\]

\[
= m_4\gamma(S_V^{-1}(y_4))S_U^{-1}(m_2)(S_U^{-1}(m_3), S_V^{-1}(y_3)) \otimes y_2(S_U^{-1}(m_1), y_1)
\]

\[
= m_4S_U^{-1}(m_3)\gamma(S_V^{-1}(y_3))(S_U^{-1}(m_2), S_V^{-1}(y_4)) \otimes y_2(S_U^{-1}(m_1), y_1)
\]

\[
= \langle S_U^{-1}(m_1), y_1 \rangle \theta(y_2)(S_U^{-1}(m_2), S_V^{-1}(y_3))
\]

\[
= \langle S_U^{-1}(m_1), y_1 \rangle \theta(y_2),
\]

\[
\langle m, S_V^{-1}(y_1)S_V^{-2}(y_3) \rangle \theta(y_2),
\]
and this concludes the proof of the formula for the action. We now compute the coaction.

\[(Id_U \otimes \theta) \circ \lambda_V(y) = (\pi \otimes Id_D) \Delta(\theta(y))\]
\[= (\pi \otimes Id_D) \Delta(y \otimes y_1)\]
\[= (\pi \otimes Id_D)((\gamma(S_V^{-1}(y_4)) \otimes y_1) \otimes (\gamma(S_V^{-1}(y_3)) \otimes y_2))\]
\[= \gamma(S_V^{-1}(y_4)y_1) \otimes \gamma(S_V^{-1}(y_3)) \otimes y_2\]
\[= \gamma(S_V^{-1}(y_3)y_1) \otimes \theta(y_2),\]

for all \(y \in V\), and thus the formula for the coaction is also verified. \(\square\)

We now describe the structure of \(V\) as a Hopf algebra in the category of left Yetter Drinfeld modules over \(U^{\text{cop}}\).

**Proposition 3.8.** The structure of \(V\) as a Hopf algebra in the category \(U^{\text{cop}}_{\text{YD}}\) is given by the formulas:

\[(3.10) \quad x \cdot y = \langle \gamma(y_2), S_V(x_1)x_2 \rangle x_2 y_1;\]

\[(3.11) \quad \Delta(x) = \langle \gamma(S_V(x_4)x_6), S_V^{-1}(x_3)x_1 \rangle x_2 \otimes x_3;\]

\[(3.12) \quad S(x) = \langle \gamma(x_4), x_1 S_V(x_3) \rangle S_V(x_2).\]

The identity in \(V\) is \(\theta^{-1}(1 \otimes 1) = 1\) and the counit is \(\varepsilon = \varepsilon\).

**Proof.** To see (3.10), we compute

\[\theta(x)\theta(y) = (S_U(\gamma(x_2)) \otimes x_1)(S_U(\gamma(y_2)) \otimes y_1)\]
\[= (\Pi \otimes \Pi)(\Delta(\theta(x))) = \Pi(S_U(\gamma(x_4)) \otimes x_1) \otimes \Pi(S_U(\gamma(x_3)) \otimes x_2)\]
\[= \Pi(S_U(\gamma(x_3)) \otimes x_1) \otimes \theta(x_2)\]
\[= \langle \gamma(y_2), x_3 \rangle \langle S_U(\gamma(y_4)), x_1 \rangle S_U(\gamma(y_3) \gamma(x_4)) \otimes x_2 y_1\]
\[= \langle \gamma(y_2), x_3 \rangle \langle \gamma(y_3), S_V(x_1) \rangle \theta(x_2 y_1)\]
\[= \langle \gamma(y_2), S_V(x_1) x_3 \rangle \theta(x_2 y_1).\]

Similarly, to verify (3.11), we compute

\[\Delta(\theta(x)) = (\Pi \otimes \Pi)(\Delta(\theta(x))) = \Pi(S_U(\gamma(x_4)) \otimes x_1) \otimes \Pi(S_U(\gamma(x_3)) \otimes x_2)\]
\[= \Pi(S_U(\gamma(x_3)) \otimes x_1) \otimes \theta(x_2)\]
\[= \langle \gamma(y_2), x_3 \rangle \langle S_U(\gamma(y_4)), x_1 \rangle S_U(\gamma(y_3) \gamma(x_4)) \otimes x_2 y_1\]
\[= \langle \gamma(y_2), S_V(x_1) x_3 \rangle \theta(x_2 y_1).\]

Now we use the equivalent form of (3.3)

\[\theta^{-1}(S_V^{-1}(y_2)) \gamma(S_V^{-1}(y_1)) = \gamma(S_V(y_2)) \theta^{-1}(S_V^{-1}(y_1)),\]

to replace \(\langle S_U^{-1}(\gamma(x_6)), S_V^{-1}(x_5) \rangle \gamma(S_V(x_7))\) by \(\langle S_U^{-1}(\gamma(x_7)), S_V^{-1}(x_6) \rangle \gamma(S_V^{-1}(x_5))\),

and obtain

\[\Delta(\theta(x)) = \langle \gamma(x_12), S_V^{-1}(x_4) \rangle \langle S_U^{-1}(\gamma(x_7)), S_V^{-1}(x_6) \rangle \langle S_U^{-1}(\gamma(x_8)), x_1 \rangle\]
\[\times \langle \gamma(x_10), x_2 \rangle \gamma(S_V^{-1}(x_13) S_V(x_7) x_11) \otimes x_3 \otimes \theta(x_9).\]

From (3.11) we have

\[\langle \gamma(x_12), S_V^{-1}(x_4) \rangle \gamma(S_V^{-1}(x_5)) \gamma(x_{11}) = \langle \gamma(x_{11}), S_V^{-1}(x_5) \rangle \gamma(x_{12}) \gamma(S_V^{-1}(x_4)),\]
so we can conclude that

\[
\Delta(\theta(x)) = \langle \gamma(x_{11}), S^{-1}_V(x_5)\rangle \{S^{-1}_U(\gamma(x_7)), S^{-1}_V(\gamma(x_6))\} \langle \gamma(x_{10}), x_1 \rangle \langle \gamma(x_{10}), x_2 \rangle \\
\times \gamma(S^{-1}_V(x_{13})x_{12}) \gamma(S^{-1}_U(x_4)) \otimes x_3 \otimes \theta(x_9)
\]

\[
= \langle \gamma(x_{10}), S^{-1}_V(x_4)\rangle \langle S^{-1}_U(\gamma(x_6)), S^{-1}_V(\gamma(x_7))\rangle \langle \gamma(x_1) \rangle \\
\times \gamma(x_3) \theta(x_3) \otimes \theta(x_8)
\]

\[
= \langle \gamma(S_V(x_4)x_8), S^{-1}_V(x_3)\rangle \langle \gamma(S_V(x_5)x_7), x_1 \rangle \theta(x_2) \otimes \theta(x_6)
\]

\[
= \langle \gamma(S_V(x_4)x_6), S^{-1}_V(x_3)x_1 \rangle \theta(x_2) \otimes \theta(x_5).
\]

Finally, we verify \textbf{(3.12)}. From \textbf{(2.23)}, we have that

\[
\hat{S}(\theta(x)) = i(\pi(\theta(x_1)))S(\theta(x_2))
\]

\[
= (S_U(\gamma(x_4))\gamma(x_1) \otimes 1)S(S_U(\gamma(x_3)) \otimes x_2)
\]

\[
= (S_U(\gamma(x_4))\gamma(x_1) \otimes 1)(1 \otimes S_V(x_2))(\gamma(x_3) \otimes 1)
\]

\[
= (S^{-1}_U(\gamma(x_7)), S_V(x_2))\langle \gamma(x_5), S_V(x_4)\rangle \gamma(x_3) \gamma(x_6) \otimes S_V(x_3)
\]

\[
= \langle \gamma(x_7), x_2 \rangle \langle \gamma(x_5), S_V(x_4)\rangle S_U(\gamma(x_8)) \gamma(x_1) \gamma(x_6) \otimes S_V(x_3).
\]

By \textbf{(3.4)} we can replace \((\gamma(x_7), x_2)\gamma(x_1)\gamma(x_6)\) by \((\gamma(x_6), x_1)\gamma(x_7)\gamma(x_2)\) and we obtain

\[
\hat{S}(\theta(x)) = \langle \gamma(x_6), x_1 \rangle \langle \gamma(x_5), S_V(x_4)\rangle S_U(\gamma(x_8)) \gamma(x_7) \gamma(x_2) \otimes S_V(x_3)
\]

\[
= \langle \gamma(x_6), x_1 \rangle \langle \gamma(x_5), S_V(x_4)\rangle \gamma(x_2) \otimes S_V(x_3)
\]

\[
= \langle \gamma(x_5), x_1 S_V(x_4)\rangle S_U(\gamma(x_2)) \otimes S_V(x_3)
\]

\[
= \langle \gamma(x_4), x_1 S_V(x_3) \rangle \theta(S_V(x_2)).
\]

The final statement is clear.

We noted in Example 3.3 that if \(V\) is quasitriangular then \(D(UCop, V)\) is a Hopf algebra with a projection. So for \(V = (H, R)\) a finite dimensional quasitriangular Hopf algebra and \(U = H^*\) we obtain Drinfeld’s projection [6], and by an analogue of Propositions 3.7 and 3.8 the structure of \(H^*\) as a braided Hopf algebra in \(H^* \Join \mathcal{YD}\) computed by Majid in [10]. (In fact, it can be proved that this braided Hopf algebra lies in the image of a canonical braided functor from \(H \mathcal{M}\) to \(H^* \Join \mathcal{YD}\).) On the other hand, if \(U = H^*\) is quasitriangular, and \(V = H\), then we have the following.

\textbf{Proposition 3.9.} Let \(H\) be a finite dimensional Hopf algebra. Then there exists a Hopf algebra projection \(\pi : D(H) \rightarrow H^{Cop}\) covering the natural inclusion \(H^{Cop} \hookrightarrow D(H)\) if and only if \(H\) is CQT. Moreover, if this is the case, then \(\pi\) is a quasitriangular morphism.

\textbf{Proof.} Suppose \((H, \sigma)\) is CQT and let \(\{e_i, e^i\}\) be a dual basis for \(H\). Then \(H^*\) is quasitriangular with \(R\)-matrix \(R = \sum_{i,j}^{\mathbb{C}} \sigma(e_i, e_j) e^i \otimes e^j\). Since \(D(H) = D(H^{Cop}, H)\), from Example 3.8 the map \(\pi : D(H) \rightarrow H^{Cop}\) given by

\[
\pi(h^* \otimes h) = \sum_{i,j}^{\mathbb{C}} \sigma(e_i, e_j) e^j(h) h^* \otimes (e^i \circ S) = \sum_{i}^{\mathbb{C}} \sigma(e_i, h) h^* \otimes (e^i \circ S),
\]

for all \(h^* \in H^*\) and \(h \in H\), is a Hopf algebra morphism which covers the inclusion \(H^{Cop} \hookrightarrow D(H)\).
Conversely, if such a morphism exists, then by Proposition 3.3, there exists a Hopf algebra morphism \( \gamma : H \to \mathcal{H}^{\text{cop}} \) satisfying
\[
\langle h_1^*, x_2 \rangle \gamma(x_1) h_2^* = \langle h_2, x_1 \rangle h_1^* \gamma(x_2),
\]
for all \( x \in H \) and \( h^* \in H^* \), where \( \langle \cdot, \cdot \rangle : H^* \otimes H \to k \) is the evaluation map. It is clear that the above condition is equivalent to
\[
\langle \gamma(x_1), y_1 \rangle x_2 y_2 = \langle \gamma(x_2), y_2 \rangle y_1 x_1,
\]
for all \( x, y \in H \). Now, if we define \( \sigma(x, y) = \langle \gamma(x), y \rangle \), for all \( x, y \in H \), then one can easily check that \( \sigma \) defines a CQT structure on \( H \) (see also [17, Theorem 3.3.14]).

Finally, the canonical \( R \)-matrix of \( D(H) \) is \( R = \sum (e \otimes e_i) \otimes (e^i \otimes 1) \). So if \( (H, \sigma) \) is CQT then the above morphism \( \pi \) is quasitriangular since
\[
(\pi \otimes \pi)(R) = \sum_{i,j} \sigma(e_j, S^{-1}(e_i)) e^j \otimes e^i = \sum_{i,j} \sigma^{-1}(e_j, e_i) e^j \otimes e^i,
\]
and, from the dual version of Remarks 2.5 (iv), the last term defines a QT structure for \( H^{\text{cop}} \).

Proof. We only note that from the proof of Proposition 3.3, and Propositions 3.7 and 3.8, we obtain the following which may be viewed as a left version of the transmutation theory for CQT Hopf algebras. Further details will follow in Section 4.1.

**Corollary 3.10.** If \( H \) is a CQT Hopf algebra then \( H \) has a braided Hopf algebra structure, denoted by \( H \), within \( \mathcal{H}^{\text{YD}} \). Namely, \( H \) is a left-right Yetter-Drinfeld module over \( H \) via
\[
x \mapsto x_0(0) \otimes x_{(1)} := x_2 \otimes S^{-1}(x_1)S^{-2}(x_3)
\]
\[
h \cdot x = \sigma(h, S^{-1}(x_1)S^{-2}(x_3)) x_2 = \sigma_2^{-1}(S^{-1}(x_3) x_1, h) x_2,
\]
for all \( h, x \in H \). \( H \) is a Hopf algebra in \( \mathcal{H}^{\text{YD}} \) with the same unit and counit as \( H \) and
\[
x \cdot y = \sigma(S(x_1) x_3, S^{-1}(y_2)) x_2 y_1 = \sigma_2^{-1}(y_2, S(x_1) x_3) x_2 y_1,
\]
\[
\Delta(x) = \sigma(S^{-1}(x_3) x_1, S^{-1}(x_6) x_4) x_2 \otimes x_5 = \sigma_2^{-1}(S(x_4) x_6, S^{-1}(x_3) x_1) x_2 \otimes x_5,
\]
\[
\nabla(x) = \sigma(x_3 S(x_5), S^{-1}(x_4) S(x_2)) x_2 = \sigma_2^{-1}(x_4, x_3 S(x_5)) S(x_2).
\]

**Proof.** We only note that from the proof of Proposition 3.3, the Hopf algebra morphism \( \gamma : H \to \mathcal{H}^{\text{cop}} \) is given in this case by \( \gamma(h) = l_{S^{-1}(h)} \), for all \( h \in H \).

If \( B \) is a bialgebra and \( \mathcal{M} \) is a left Yetter-Drinfeld module over \( B \), then the map \( R_{\mathcal{M}} : \mathcal{M} \otimes \mathcal{M} \to \mathcal{M} \otimes \mathcal{M} \), \( R_{\mathcal{M}}(m \otimes m') = m' \cdot m \) is a solution in \( \text{End}(\mathcal{M}^{\otimes 3}) \) of the quantum Yang-Baxter equation
\[
R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12}.
\]
Above, we have that $V$ is a left Yetter-Drinfeld module over $U^\text{cop}$. Then a solution $R_V \in \text{End}(V \otimes V)$ to the quantum Yang-Baxter equation is given by
\[
R_V(x \otimes y) = \gamma(S_V(y_3)y_1) x \otimes y_2 = (\gamma(S_V^{-1}(y_3)y_1), S_V^{-1}(S_V^{-1}(x_3)x_1)) x_2 \otimes y_2.
\]

4. Coquasitriangular Bialgebras and Generalized Quantum Doubles

4.1. Transmutation theory. As was mentioned in the introduction, the braided reconstruction theorem associates to any CQT Hopf algebra $H$ a braided commutative Hopf algebra $H_\delta$ in the category of right $H$-comodules, $\mathcal{M}^H$. The goal of this subsection is to show that $H_\delta$ can be obtained from the structure of a generalized quantum double with a projection.

Let $X$ and $A$ be sub-Hopf algebras of a CQT Hopf algebra $(H, \sigma)$. Then $\sigma$ induces a skew pairing on $A \otimes X$, still denoted by $\sigma$ and the generalized quantum double $D(A, X)$ is defined. By (2.7), it follows that $XA = AX$, so $XA$ is a sub-Hopf algebra of $H$. From [5, 3.1] the map
\[
\pi' : D(A, X) \to AX = XA, \quad \pi'(a \otimes x) = ax
\]
is a surjective Hopf algebra morphism. Although $D(H, H)$ is a Hopf algebra with projection, we cannot, in general, make the same claim for $D(A, X)$. Nevertheless, $XA$ and $X$ are sub-Hopf algebras of $(H, \sigma)$, so applying the same arguments we find that
\[
\pi : D(AX, X) \to AX = XA, \quad \pi(ax \otimes y) = axy
\]
is a surjective Hopf algebra morphism covering the inclusion map $i : AX \to D(AX, X)$ with the map $\gamma$ from Proposition [3, 1] being the inclusion of $X$ into $AX = XA$. From Propositions [3, 7] and [5, 3.1] for $\gamma : X \to AX$ and $\langle \cdot, \cdot \rangle = \sigma : AX \otimes X \to k$, $X$ has a braided Hopf algebra structure in the braided monoidal category $\frac{XA}{X} \mathcal{Y}D$; this braided Hopf algebra is denoted, as usual, by $X$. The structures are given by
\[
\begin{align*}
xa \triangleright y &= \sigma(xa, S^{-1}(S^{-1}(y_3)y_1))y_2, \\
\lambda_X(y) &= S^{-1}(y_3)y_1 \otimes y_2, \\
x \cdot y &= \sigma(y_2, S(x_1)x_2y_1), \quad 1 = 1, \\
\Delta(x) &= \sigma(S(x_4)x_6, S^{-1}(x_3)x_1)x_2 \otimes x_5, \quad \varepsilon = \varepsilon, \\
\xi(x) &= \sigma(x_4, x_1S(x_3))S(x_2),
\end{align*}
\]
for all $x, y \in X$ and $a \in A$. Next, we show that $X$ lies in the image of a canonical braided functor from $X\mathcal{M}$ to $\frac{XA}{X} \mathcal{Y}D$, so this general context reduces to the case $X = A = H$. We recall some background on CQT Hopf algebras and braided monoidal categories.

For a CQT bialgebra $(\mathcal{B}, \varsigma)$ it is well known that its category of left or right $\mathcal{B}$ comodules has a braided monoidal structure. Namely, if $\mathcal{M}$ and $\mathcal{N}$ are two left (respectively right) $\mathcal{B}$ comodules then
\[
\mathcal{M} \otimes \mathcal{N} \ni m \otimes n \mapsto m_{-1}n_{-1} \otimes m_0 \otimes n_0 \in \mathcal{B} \otimes \mathcal{M} \otimes \mathcal{N}
\]
defines the monoidal structure of $\mathcal{B}\mathcal{M}$ and
\[
c_{\mathcal{M}, \mathcal{N}} : \mathcal{M} \otimes \mathcal{N} \to \mathcal{N} \otimes \mathcal{M}, \quad c_{\mathcal{M}, \mathcal{N}}(m \otimes n) = \varsigma(n_{-1}, m_{-1})n_0 \otimes m_0
\]
defines a braiding on $\mathcal{B}\mathcal{M}$, while
\[
\mathcal{M} \otimes \mathcal{N} \ni m \otimes n \mapsto m_{(0)} \otimes n_{(0)} \otimes m_{(1)}n_{(1)} \in \mathcal{M} \otimes \mathcal{N} \otimes \mathcal{B}
\]
for }
gives the monoidal structure of $\mathcal{M}^B$ and

\[(4.3) \quad c_{\mathcal{M},\mathcal{N}}(m \otimes n) = \varsigma(m_{(1)}, n_{(1)})n_{(0)} \otimes m_{(0)} \]

provides a braided structure on $\mathcal{M}^B$ (see [13, Example 9.4.10] for terminology).

We denote by $(B,c)$ the category of left $B$-comodules with the braiding in $(4.1)$. Similarly, $\mathcal{M}^{(B,c)}$ is the category of right $B$-comodules with the braiding defined by $(4.3)$. One can easily see that $(B^{\text{cop}}, \varsigma_2) = \mathcal{M}^{(B,c)}$, as braided monoidal categories, where $\varsigma_2 \overset{\varsigma}{\rightarrow} tw$ is the CQT structure of $B^{\text{cop}}$ defined in Remarks 2.3.

Secondly, if $(B, \varsigma)$ is a CQT bialgebra and $\mathcal{M}$ a left $B$-comodule then $\mathcal{M}$ is a left Yetter-Drinfeld module over $B$ with the initial comodule structure and with the $B$-action defined by

\[(4.4) \quad b \cdot m := \varsigma(m_{-1}, b)m_0.\]

Thus there is a well defined braided functor $\mathfrak{F}((B,\varsigma)) : (B,\varsigma)\mathcal{M} \rightarrow \mathcal{D}(B,\varsigma)$ where $\mathfrak{F}((B,\varsigma))$ sends a morphism to itself.

**Lemma 4.1.** In the setting above, the braided Hopf algebra $X$ lies in the image of the composite of the canonical functor $(X,\sigma_{z_1})\mathcal{M} \rightarrow (XA,\sigma_{z_1})\mathcal{M}$ and the functor $\mathfrak{F}((XA,\sigma_{z_1}))$.

**Proof.** We have

\[\lambda_X(x) := x_{-1} \otimes x_0 = S^{-1}(x_3)x_1 \otimes x_2.\]

Therefore the $X$-action on $X$ can be rewritten as

\[x \triangleright y = \sigma(x, S^{-1}(y-1))y_0 = \sigma^{-1}(x, y_1)x_0 = \sigma_{z_1}^{-1}(y_1, x)y_0,\]

and this finishes the proof. $\Box$

In general, if $C$ is a braided monoidal category with braiding $c$ then $C^{\text{in}}$ is $C$ as a monoidal category, but with the mirror-reversed braiding $c_{M,N} = c_{N,M}^{-1}$. Note that, if $B \in C$ is a braided Hopf algebra with comultiplication $\Delta$ and bijective antipode $S$ then $B^{\text{cop}}$, the same object $B$, but with the comultiplication and antipode

\[\Delta^{\text{cop}} = c_{B,B}^{-1} \circ \Delta \quad \text{and} \quad S^{\text{cop}} = S^{-1}\]

respectively, and with the other structure morphisms the same as for $B$, is a braided Hopf algebra in the category $C^{\text{in}}$. Now, since the transmutation object $H$ is a braided Hopf algebra in $\mathcal{M}^H$ and not $H\mathcal{M}$ we must apply the above correspondence $X \mapsto X$ to $(H^{\text{cop}}, \sigma_{z_1})$ rather than $(H, \sigma)$ and thus obtain a braided Hopf algebra

\[H^{\text{cop}} \in (H^{\text{cop}}, \sigma_{z_1})\mathcal{M} \equiv \mathcal{M}^{(H, \sigma_{z_1})} \equiv \mathcal{M}^{(H, \sigma)^{\text{in}}}.\]

We can now prove the connection between $H^{\text{cop}}$ and $H$. The structures of $H$ can be found in [13, Example 9.4.10].

**Proposition 4.2.** Let $(H, \sigma)$ be a CQT Hopf algebra. Then $H^{\text{cop}} = H$ as braided Hopf algebras in $\mathcal{M}^{(H, \sigma)}$.

**Proof.** One can easily see that $H^{\text{cop}}$ is an object of $\mathcal{M}^{H}$ via the structure

\[h \mapsto h_{(0)} \otimes h_{(1)} = h_2 \otimes S(h_1)h_3,\]

and that its algebra structure within $\mathcal{M}^{(H, \sigma)}$ is given by

\[h \cdot g = \sigma(S^{-1}(h_3)h_1, g_1)h_2g_2 = \sigma(S(h_1)h_3, S(g_1))h_2g_2,\]

for all $h, g \in H$. Clearly, the unit of $(H^{\text{cop}})^{\text{cop}}$ is the unit of $H$. 

Now, by Proposition 4.3 we have that $c^{-1}$, the inverse of the braiding $c$ of $M^{(H, \sigma)}$, is defined by $c^{-1}_{H}(n \otimes m) = \sigma(n_{(1)}, m_{(1)})m_{(0)} \otimes n_{(0)}$. Therefore, the comultiplication of $(H^{\text{cop}})^{\text{cop}}$ is given by

$$h \mapsto \sigma(S(h_{4})h_{6}, S^{-1}(h_{3})h_{1})c^{-1}(h_{5}, h_{2})$$

$$= \sigma((h_{2})_{(1)}, S^{-1}(h_{1})_{(1)})c^{-1}((h_{2})_{(0)} \otimes (h_{1})_{(0)})$$

$$= \sigma^{1}((h_{2})_{(2)}, (h_{1})_{(2)})\sigma((h_{2})_{(1)}, (h_{1})_{(1)})(h_{1})_{(0)} \otimes (h_{2})_{(0)}$$

$$= h_{1} \otimes h_{2} = \Delta(h),$$

the comultiplication of $H$. The counit of $(H^{\text{cop}})^{\text{cop}}$ is $\varepsilon$, the counit of $H$.

Finally, the antipode of $H^{\text{cop}}$ is defined by $S(h) = \sigma(h_{4}S^{-1}(h_{2}), h_{1})S^{-1}(h_{3})$, so the antipode of $(H^{\text{cop}})^{\text{cop}}$ is given, for all $h \in H$, by

$$S^{-1}(h) = \sigma(S^{2}(h_{3})S(h_{1}), h_{4})S(h_{2}).$$

Indeed, for all $h \in H$ we have

$$(S \circ S^{-1})(h) = \sigma(h_{1}S(h_{6}), S^{-1}(h_{7}))\sigma(S(h_{2})h_{4}, S(h_{5}))h_{3}$$

$$= \sigma(h_{1}, S^{-1}(h_{7}))\sigma(S(h_{7}), S^{-1}(h_{5}))\sigma(h_{2}, h_{3})\sigma(h_{4}, S(h_{5}))h_{3}$$

$$= \sigma(h_{1}, S^{-1}(h_{7}))\sigma(h_{2}, h_{5})v^{-1}(h_{6})h_{3}$$

$$= \sigma(h_{1}, S^{-1}(h_{3}))\sigma(h_{2}, S^{-2}(h_{4}))h_{3}$$

$$= \sigma(h_{1}, S^{-2}(h_{3})S^{-1}(h_{4}))h_{2} = h,$$

as needed. In a similar way we can prove that $S^{-1} \circ S = \text{Id}_{H}$, the details are left to the reader. Comparing the above structures of $(H^{\text{cop}})^{\text{cop}}$ with those of $H$ from Proposition 4.3 we conclude that $(H^{\text{cop}})^{\text{cop}} = H_{\text{cop}}$ as braided Hopf algebras in $M^{(H, \sigma)}$. □

Remark 4.3. From Corollary 3.10 we can deduce the left version of the transmutation theory for a CQT Hopf algebra $(H, \sigma)$ as follows. Observe that there is a braided functor $G : (H, \sigma)M \to H \mathcal{YD}^{H}$ defined by $G(M) = M$, but now viewed as a braided Hopf algebra in $M^{(H, \sigma)}$. This $G$ sends a morphism to itself.

Now, one can easily see that $H_{\text{cop}}$ in Corollary 3.10 lies in the image of the functor $G$. In other words we can associate to $(H, \sigma)$ a Hopf algebra structure in $M^{(H, \sigma)}$, still denoted by $H_{\text{cop}}$. Note that $H_{\text{cop}}$ has the left $H$-comodule structure given by

$$x \mapsto \lambda_{H}(x) = x_{1} \otimes x_{0} = S^{-1}(x_{3})x_{1} \otimes x_{2},$$

for all $x \in H_{\text{cop}}$, and is a braided Hopf algebra in $M^{(H, \sigma)}$ via the structure in Corollary 3.10 (In fact, this $H_{\text{cop}}$ is the braided Hopf algebra in Lemma 4.1 corresponding to $X = (H, \sigma_{21})$. ) We can conclude now that $H_{\text{cop}, \text{cop}}$ is the braided Hopf algebra in $M^{(H, \sigma)}$ associated to $(H, \sigma)$ through the left transmutation theory. By $H_{\text{cop}, \text{cop}}$ we denote $H_{\text{cop}}$ viewed now as a Hopf algebra in $M^{(H, \sigma)}$ via

$$x \circ y := m_{H} \circ c_{H, H}(x, y) = \sigma(y_{-1}, x_{-1})y_{0}x_{0} = \sigma(y_{2}, S(x_{1})x_{3})x_{2}y_{1},$$

$$\Delta_{H_{\text{cop}, \text{cop}}} = c_{H, H}^{-1} \circ \Delta = \Delta,$$

and the other structure morphisms equal those of $H$. 

MARGARET BEATTIE AND DANIEL BULACU
4.2. The “dual” case. Let \((H, \sigma)\) be a CQT bialgebra, so that \(\sigma\) is a pairing from \(H \otimes H^\text{op} \to k\) and \(H^\text{cop} \otimes H \to k\). Then by Remarks \[2,2\] we have bialgebra morphisms
\[
\phi : H^\text{cop} \to H^0 \text{ defined by } \phi(h)(l) = \sigma(h, l)
\]
and
\[
\psi : H^\text{op} \to H^0 \text{ defined by } \psi(h)(l) = \sigma(l, h).
\]
We denote \(r_h := \phi(h) = \sigma(h, -)\) and \(r_I = \text{Im}(\phi)\). Similarly, \(l_h := \psi(h) = \sigma(-, h)\) and \(l_I = \text{Im}(\psi)\). More generally, if \(X\) and \(A\) are sub-bialgebras of a CQT bialgebra \(H\) then we define
\[
H_{r_A} := \phi(A^\text{cop}) = \{r_a = \sigma(a, -) \mid a \in A\}
\]
\[
H_{l_X} := \psi(X^\text{op}) = \{l_x = \sigma(-, x) \mid x \in X\}.
\]

**Proposition 4.4.** Let \((H, \sigma)\) be a CQT bialgebra (Hopf algebra) and \(X, A \subseteq H\) two sub-bialgebras (sub-Hopf algebras). Then

(i) \(H_{l_X}\) and \(H_{r_A}\) are sub-bialgebras (sub-Hopf algebras) of \(H^0\). The structure maps for \(\psi(X^\text{op}) = H_{l_X}\) are given by
\[
(4.5) \quad l_{xy} = l_y l_x; \quad l_1 = \varepsilon; \quad \Delta(l_x) = l_{x_1} \otimes l_{x_2}; \quad \varepsilon(l_x) = \varepsilon(x); \quad S(l_x) = l_{S^{-1}(x)},
\]
and the structure maps for \(\phi(A^\text{cop}) = H_{r_A}\) are given by
\[
(4.6) \quad r_{ab} = r_a r_b; \quad r_1 = \varepsilon; \quad \Delta(r_a) = r_{a_1} \otimes r_{a_2}; \quad \varepsilon(r_a) = \varepsilon(a); \quad S(r_a) = r_{S^{-1}(a)}.
\]
(ii) The bilinear form from \(H_{r_A} \otimes H_{l_X}\) to \(k\) defined for all \(a \in A, x \in X\), by \(r_a \otimes l_x : \sigma(a, x)\), is a skew pairing between \(H_{r_A}\) and \(H_{l_X}\).

(iii) \(H_{r_A} H_{l_X} = H_{l_X} H_{r_A}\) is a sub-bialgebra of \(H^0\) and is a sub-Hopf algebra if \(A, X\) are sub-Hopf algebras of the Hopf algebra \(H\).

**Proof.** (i) If \(X\) is a sub-bialgebra (sub-Hopf algebra) of \(H\), then \(X^\text{op}\) is a sub-bialgebra (sub-Hopf algebra) of \(H^\text{op}\). Since \(\sigma : H \otimes H^\text{op} \to k\) is a duality, by Remarks \[2,2\] \(\psi : X^\text{op} \subseteq H^\text{op} \to H^0\) is a bialgebra morphism and thus \(H_{l_X}\), the image of \(X^\text{op}\) under \(\psi\), is a sub-bialgebra (sub-Hopf algebra) of \(H^0\) with the structure maps given above. The proof for \(H_{r_A}\) is similar, using the map \(\phi\).

(ii) Follows directly from Remarks \[2,2\] since \(\sigma : A^\text{cop} \otimes X^\text{op} \to k\) is a skew pairing.

(iii) We refer to \[5, 3.2 (b)]\. Here it is shown that the map from \(D(H_{r_A}, H_{l_X})\) to \(H^0\) defined by \(a \otimes l_x \to r_a l_x\) is a bialgebra map, and our statement follows. The proof is based on the observation that
\[
(4.7) \quad l_y r_a = \sigma(a_1, S(y_3)) \sigma(a_3, y_1) r_{a_2} l_{y_2} \text{ or, equivalently,}
\]
\[
(4.8) \quad r_a l_y = \sigma(S^{-1}(a_3), y_1) \sigma(a_1, y_3) r_{a_2} l_{y_2},
\]
for any \(y \in X, a \in A\). \(\square\)

We now assume that \(H\) is a Hopf algebra. We denote by \(H_{X,A}\) the Hopf sub-algebra of \(H^0\) equal to \(H_{l_X} H_{r_A} = H_{r_A} H_{l_X}\).

**Proposition 4.5.** Let \((H, \sigma)\) be a CQT Hopf algebra, \(X, A\) and \(H\) sub-Hopf algebras of \(H\), and \(H_{l_X}\) and \(H_{r_A}\) the sub-Hopf algebras of \(H^0\) from Proposition \[4,4\]. Let \(D(H_{r_A}, H_{l_X})\) be the generalized quantum double from the skew pairing in Proposition \[4,4\] (ii) induced by \(\sigma\).

(i) The map \(f : D(H_{r_A}, H_{l_X}) \to H_{X,A}, f(r_a \otimes l_x) = r_a l_x, a \in A, x \in X\), is a surjective Hopf algebra morphism.
(ii) Then the evaluation map \( \langle \cdot \rangle : H_{X,A} \otimes \mathcal{H} \to k \) defined for all \( x \in X \), \( a \in A \) and \( h \in \mathcal{H} \) by

\[
\langle l_x r_a, h \rangle = \sigma(h_1, x) \sigma(a, h_2) = l_x(h_1) r_a(h_2)
\]

provides a duality between the Hopf algebras \( H_{X,A} \) and \( \mathcal{H} \), and therefore between \( D(H_{r_a \epsilon_\mathcal{H}}) \) and \( \mathcal{H} \) via the map \( f \) in (i). In particular, \( \langle \cdot \rangle \) is a skew pairing on \( H_{X,A}^{\text{cop}} \otimes \mathcal{H} \) and \( D(H_{r_a \epsilon_\mathcal{H}}) \).

Proof. Statement (i) follows directly from \([5, 2.4]\) with \( \alpha = \text{Id}_{H_{r_a \epsilon_\mathcal{H}}} \), \( \beta = \text{Id}_{H_{x \epsilon X}} \) and using \((4.7)\). Statement (ii) is immediate.

From now on, we assume that \( (H, \sigma) \) is a CQT Hopf algebra with \( X, A \) and the evaluation pairing \( \langle \cdot \rangle \) as above. Consider the generalized quantum double \( D(H_{X,A}^{\text{cop}}, X) \). From \((2.12)\), the multiplication in \( D(H_{X,A}^{\text{cop}}, X) \) is given by

\[
(l_x r_a \otimes y)(l_{x'} r_b \otimes y') = (l_{x'_{x_1}} r_{b_{y_1}} S^{-1}(y_2)) (l_{x_{x_1}} r_{b_{y_1}} y_1) l_x r_a l_{x_2} r_b \otimes y_2 y',
\]

for all \( x, x', y, y' \in X \), \( a, b \in A \), and since \( D(H_{X,A}^{\text{cop}}, X) = H_{X,A}^{\text{cop}} \otimes X \) as a coalgebra, the comultiplication is given by

\[
\Delta(l_{x} r_{a} \otimes y) = (l_{x_2} r_{a_1} \otimes y_1) \otimes (l_{x_1} r_{a_2} \otimes y_2).
\]

The unit is \( \varepsilon \otimes 1 \) and the counit is defined by \( \varepsilon(l_x r_a \otimes y) = \varepsilon(x) \varepsilon(a) \varepsilon(y) \).

We now prove that there is a Hopf algebra projection from \( D(H_{X,A}^{\text{cop}}, X) \) to \( H_{X,A}^{\text{cop}} \) which covers the canonical inclusion \( i : H_{X,A}^{\text{cop}} \to D(H_{X,A}^{\text{cop}}, X) \).

**Proposition 4.6.** The map \( \pi : D(H_{X,A}^{\text{cop}}, X) \to H_{X,A}^{\text{cop}} \) defined by

\[
\pi(l_x r_a \otimes y) = l_x r_a l_{S^{-1}(y)},
\]

for all \( x, y \in X \), \( a \in A \), is a Hopf algebra morphism such that \( \pi \circ i = \text{Id}_{H_{X,A}^{\text{cop}}} \).

Proof. Define \( \gamma : X \to H_{X,A}^{\text{cop}} \subseteq H_{X,A}^{\text{cop}} \) by \( \gamma(x) = l_{S^{-1}(x)} \). We show that \( \gamma \) is a bialgebra morphism and that \((3.1)\) holds.

For \( x, y \in X \), we have

\[
\gamma(xy) = l_{S^{-1}(xy)} = l_{S^{-1}(y)S^{-1}(x)} = l_{S^{-1}(x)} l_{S^{-1}(y)} = \gamma(x) \gamma(y),
\]

so that \( \gamma \) preserves multiplication. Similarly,

\[
(\gamma \otimes \gamma)(x_1 \otimes x_2) = l_{S^{-1}(x_1)} \otimes l_{S^{-1}(x_2)} = \Delta_{H_{X,A}^{\text{cop}}} (\gamma(x)),
\]

so that comultiplication is preserved. Clearly \( \gamma \) preserves the unit and counit. To verify \((3.1)\), we compute

\[
\gamma(x) l_y = l_{S^{-1}(x)} l_y = l_{y S^{-1}(x)} \overset{(2.4)}{=} \sigma(S^{-1}(x_2), y_2) \sigma(x_1, y_3) l_{y_1 S^{-1}(x_3)}
\]

\[
\overset{(2.7)}{=} \sigma(S^{-1}(x_3), y_1) \sigma(x_1, y_3) l_{S^{-1}(x_2) y_2}
\]

\[
= \langle l_{y_1}, S^{-1}(x_3) \rangle \langle l_{y_3}, x_1 \rangle l_{y_2} \gamma(x_2).
\]

As well we have that

\[
\gamma(x_2) r_a = l_{S^{-1}(x_2)} r_a \overset{(4.7)}{=} \sigma(a_1, S(S^{-1}(x_2))) \sigma(a_3, S^{-1}(x_4)) r_{a_2} l_{S^{-1}(x_3)}
\]

\[
= \langle r_{a_1}, x_2 \rangle \langle r_{a_3}, S^{-1}(x_4) \rangle r_{a_2} l_{S^{-1}(x_3)}.
\]
Combining these equations, we obtain
\[
\gamma(x)l_y r_a = \langle l_y r_a, x_1 \rangle \langle l_y r_a, S^{-1}(x_3) \rangle \langle l_y r_a, x_2 \rangle \\
= \langle l_y r_a, x \rangle \langle l_y r_a, S^{-1}(x) \rangle \langle l_y r_a, x_2 \rangle \gamma(x_2).
\]
Since \( \pi = m_{H_0} \circ (Id \otimes \gamma) \), the statement follows from Proposition 3.4.

We now apply the results of Section 3 to this Radford biproduct.

**Proposition 4.7.** Let \( H, X, A, \gamma, \pi \) be as above. Then \( D(H^\text{cop}_{X,A}, X) \cong B \times H^\text{cop}_{X,A} \) where \( B \) is a Hopf algebra in the category \( H^\text{cop}_{X,A} \). In addition,

(i) \( B = \{ l_{S^{-2}(x)} \otimes x_1 \mid x \in X \} \) and is isomorphic to \( X \) as a \( k \)-space;

(ii) \( X \) is a left \( H^\text{cop}_{X,A} \) Yetter-Drinfeld module with the structure

\[
\begin{align*}
\Delta(x) &= \sigma(S(x_1)x_3, S(x_4)x_6 \otimes x_5), \\
\Delta(x) &= \sigma(S^2(x_3)S(x_1), x_4)S(x_2),
\end{align*}
\]

for all \( x, y \in X \). \( X \) has the same unit and counit as \( \underline{X} \subseteq H \).

**Proof.** Apply the formulas in Proposition 3.8 with \( U = H_{X,A} \) and \( V = X \).

**Remarks 4.9.** (i) From Equations (2.8) and (2.9) in Remarks 2.5 we obtain another formula for the antipode \( S \) of \( X \). For we note that

\[
S(x) = \sigma(S^2(x_3), x_4)S(x_1), x_5)S(x_2)
= \sigma(S(x_1), S(x_4)^{-1}(x_3)S(x_2)
= \sigma(S(x_1), x_4)^{-1}(x_2)S^{-1}(x_3)
= \sigma(S(x_1), S(x_4)^{-1}(x_2)S^{-1}(x_3)
= \sigma(S(x_1), x_5)S(x_2), x_3)S^{-1}(x_4)
= \sigma(S(x_1), S^{-1}(x_2)x_4)S^{-1}(x_3).
\]

(ii) The Hopf algebra isomorphism \( \chi : X \times H^\text{cop}_{X,A} \rightarrow D(H^\text{cop}_{X,A}, X) \) is given by \( \chi(x \otimes l_y r_a) = (l_{S^{-2}(x)} \otimes x_1)(l_y r_a \otimes 1) \), for all \( x, y \in X \) and \( a \in A \).

(iii) One can easily check that, in general, \( X \) is neither quantum commutative nor quantum cocommutative as a bialgebra in \( H^\text{cop}_{X,A} \). But, if \( \sigma \) restricted to \( X \otimes X \) gives a triangular structure on \( X \) (that is, \( \sigma^{-1}(x, y) = \sigma(y, x) \), for all \( x, y \in X \) then \( X \) is quantum commutative, i.e.

\[
x \circ y = (x \downarrow \triangleright y) \circ x_0, \forall x, y \in X.
\]
(iv) Any proper Hopf subalgebra \( X \) of a CQT Hopf algebra \((H, \sigma)\) can be viewed as a braided Hopf algebra in (at least) three different left Yetter-Drinfeld module categories by applying Proposition 4.7 and Theorem 4.8 with different \( A \). Specifically, over \( H^\text{cop}_X \) (take \( A = k \)), over the co-opposite of \( H^\text{cop}_X H_x = H_x H^\text{cop}_X \) (take \( A = X \)), and over the co-opposite of \( H^\text{cop}_X H_r = H_r H^\text{cop}_X \) (take \( A = H \)). Note that \( H^\text{cop}_X \subseteq H^\text{cop}_X H_x \subseteq H^\text{cop}_X H_r \).

We note that the solution to the Yang-Baxter equation from the Yetter Drinfeld module \( X \) comes from the adjoint coaction.

For if \((B, \sigma)\) is a CQT bialgebra and \( M \) a left \( B \)-comodule then we have seen that \( M \) is a left Yetter-Drinfeld module over \( B \) with the initial comodule structure and with the \( B \)-action defined by (1.4). In particular, we get that \( R_M \in \text{End}(M \otimes M) \) given for all \( m, m' \in M \) by

\[
R_M(m \otimes m') = \sigma(m_{-1}, m'_{-1})m_0 \otimes m'_0,
\]

is a solution for the quantum Yang-Baxter equation.

In the present setting, with \((H, \sigma)\) CQT, then \( X \) is a left \( X \)-comodule via \( x \mapsto S(x_1)x_3 \otimes x_2 \). Then \( R_{X,\text{ad}} \in \text{End}(X \otimes X) \) given for all \( x, y \in X \) by

\[
R_{X,\text{ad}}(x \otimes y) = \sigma(S(x_1)x_3, S(y_1)y_3)x_2 \otimes y_2
\]

is a solution for the quantum Yang-Baxter equation. From the discussion at the end of Section 5 writing \( S \) for \( S_X \), we note that

\[
R_X(x \otimes y) = \langle \gamma(S^{-1}(y_3)y_1), S^{-1}(S^{-1}(y_3)x_1))x_2 \otimes y_2
\]

\[
= \langle l_{S^{-1}(S^{-1}(y_3))}, S^{-1}(S^{-1}(S^{-1}(y_1)))x_2 \otimes y_2
\]

\[
= \sigma(S^{-1}(S^{-1}(x_1)y_1)), S^{-1}(S^{-1}(y_1)x_1))x_2 \otimes y_2
\]

\[
= \sigma(S(x_1)x_3, S(y_1)y_3)x_2 \otimes y_2
\]

\[
= R_{X,\text{ad}}(x \otimes y).
\]

Next, we show the connection between our \( X \) and the transmutation theory.

**Theorem 4.10.** Let \((H, \sigma)\) be a CQT Hopf algebra and \( A, X \) sub-Hopf algebras of \( H \). Then there is a braided functor \( F : \mathcal{M}^X \rightarrow H^\text{cop}_{X,A} \mathcal{Y}D \) such that \( F(X) = X \).

**Proof.** We start by constructing the functor \( F \) explicitly. For \( M \) a right \( X \)-comodule, let \( F(M) = M \) as a \( k \)-vector space, endowed with the following structures:

\[
l_x r_a \triangleright m = l_x r_a (S^{-2}(m_1))m_0 \quad \text{and} \quad \lambda_M(m) = l_{S^{-2}m_1}m_0,
\]

for any \( l_x r_a \in H_{X,A} \) and \( m \in M \). From the definitions, \( F(M) \) is both a left \( H^\text{cop}_{X,A} \)-module and comodule. Actually, \( F(M) \) is a Yetter-Drinfeld module over \( H^\text{cop}_{X,A} \). Here, using the co-opposite of the structure maps in Proposition 4.3 we see that relation (2.15) is

\[
\langle l_{x_1} r_{a_2}, S^{-2}(m_1) \rangle l_{x_2} r_{a_1} l_{S^{-2}m_2} \otimes m_0
\]

\[
= \langle l_{x_2} r_{a_1}, S^{-2}(m_2) \rangle l_{S^{-2}(m_1)} \otimes m_0
\]
and this holds since

$$\langle l_x r_a, y_1 \rangle l_x r_a, l y_2 = \sigma(y_1, x_1) \sigma(a_2, y_2) l_x r_a, l y_2$$

(4.8)

$$\sigma(y_1, x_1) \sigma(a_2, y_2) \sigma(S^{-1}(a_3), y_2) \sigma(a_1, y_5) l_x y_4 r_{a_2}$$

(2.5)

$$\sigma(y_1, x_1) \sigma(a_1, y_3) l_y x_2 r_{a_2}$$

(2.7)

$$\sigma(y_2, x_2) \sigma(a_1, y_3) l_x y_1 r_{a_2}$$

(2.7)

$$= \langle l_x r_{a_1}, y_2 \rangle l_y x_1 r_{a_2}$$

for all $x, y \in X$ and $a \in A$. So $\mathbf{F}$ is a well defined functor from $\mathcal{M}^X$ to $\mathcal{M}_X^A$. (\(\mathbf{F}\) sends a morphism to itself.)

We claim that $\mathbf{F}$ has a monoidal structure defined by $\varphi_0 = \text{Id} : k \to \mathbf{F}(k)$ and $\varphi_2(\mathcal{M}, \mathcal{N}) : \mathbf{F}(\mathcal{M}) \otimes \mathbf{F}(\mathcal{N}) \to \mathbf{F}(\mathcal{M} \otimes \mathcal{N})$, the family of natural isomorphisms in $\mathcal{M}_X^A \mathcal{Y}D$ given by

$$\varphi_2(\mathcal{M}, \mathcal{N})(m \otimes n) = \sigma^{-1}(m(1), n(1)) m(0) \otimes n(0),$$

for all $m \in \mathcal{M} \in \mathcal{M}^X$ and $n \in \mathcal{N} \in \mathcal{M}^X$ (for more details see for example [8, XI.4]). To this end, observe that $\varphi_2(\mathcal{M}, \mathcal{N})$ is left $\mathcal{M}_X^A$-linear since

$$\varphi_2(\mathcal{M}, \mathcal{N})(l_x r_a \triangleright (m \otimes n))$$

(2.16)

$$\varphi_2(\mathcal{M}, \mathcal{N})(l_x r_a \triangleright m \otimes l_x r_a \triangleright n)$$

(4.18)

$$= \langle l_x r_a, S^{-2}(n(1)m(1)) \rangle \varphi_2(\mathcal{M}, \mathcal{N})(m(0) \otimes n(0))$$

$$= \langle l_x r_a, S^{-2}(n(2)m(2)) \rangle \sigma^{-1}(m(1), n(1)) m(0) \otimes n(0)$$

(2.7)

$$\varphi_2(\mathcal{M}, \mathcal{N})(l_x r_a \triangleright (m_0 \otimes n_0))$$

(1.24.18)

$$= \sigma^{-1}(m(1), n(1)) l_x r_a \triangleright (m(0) \otimes n(0))$$

$$= l_x r_a \triangleright \varphi_2(\mathcal{M}, \mathcal{N})(m \otimes n),$$

for all $x \in X$, $a \in A$, $m \in \mathcal{M}$ and $n \in \mathcal{N}$. The fact that $\varphi_2(\mathcal{M}, \mathcal{N})$ is left $\mathcal{M}_X^A$-colinear can be proved in a similar manner, the details are left to the reader.

Clearly, $\varphi_2(\mathcal{M}, \mathcal{N})$ is bijective and $\varphi_0$ makes the left and right unit constraints in $\mathcal{M}^X$ and $\mathcal{M}_X^A \mathcal{Y}D$ compatible. So it remains to prove that $\varphi_2$ respects the associativity constraints of the two categories above. It is not hard to see that this fact is equivalent to

$$\sigma(h_1, g_1) \sigma(h_2 g_2, h') = \sigma(g_1, h'_1) \sigma(h, h'_2 g_2),$$

for all $h, h', g \in H$, i.e., $\sigma$ is a 2-cocycle, a well known fact which follows by applying the properties of coquasitriangularity or see [4].

Moreover, we have $\mathbf{F}$ a braided functor, this means

$$\mathbf{F}(\epsilon_{\mathcal{M}, \mathcal{N}}) \circ \varphi_2(\mathcal{M}, \mathcal{N}) = \varphi_2(\mathcal{N}, \mathcal{M}) \circ \mathbf{F}(\epsilon_{\mathcal{M}}, \mathbf{F}(\mathcal{N}),$$

for all $\mathcal{M}, \mathcal{N} \in \mathcal{M}^X$. Indeed, on one hand, by (4.3) we have

$$\mathbf{F}(\epsilon_{\mathcal{M}, \mathcal{N}}) \circ \varphi_2(\mathcal{M}, \mathcal{N})(m \otimes n) = n \otimes m.$$
On the other hand, by (2.17) and (4.18) we compute:
\[
\varphi_2(\mathcal{M}, \mathcal{M}) \circ \mathcal{F}(\mathcal{M}, \mathcal{F}(\mathcal{M}))(m \otimes n) = \varphi_2(\mathcal{M}, \mathcal{M})(S^{-2}(m(1)) \triangleright n \otimes m(0)) = \sigma(S^{-2}(n(1)), S^{-2}(m(1)) \varphi_2(\mathcal{M}, \mathcal{M})(n(0) \otimes m(0))) = n \otimes m,
\]
as needed.

Finally, by Proposition 4.12 and Proposition 4.7 it follows that \( \mathcal{F}(X) = X \) as objects in \( H_{X,A}^{\text{cop}} \mathcal{Y} \mathcal{D} \). Furthermore, since \( \mathcal{F} \) is a braided functor and \( Y \) is a braided Hopf algebra in \( \mathcal{M}_X \) it follows that \( \mathcal{F}(X) \) is a braided Hopf algebra in \( H_{X,A}^{\text{cop}} \mathcal{Y} \mathcal{D} \) with multiplication given by
\[
\mathcal{M}_{\mathcal{F}(X)} : \mathcal{F}(X) \otimes X \xrightarrow{\varphi_2(X, X)} X \otimes \mathcal{F}(X) \xrightarrow{\mathcal{F}(m_\mathcal{F})} \mathcal{F}(X),
\]
the comultiplication defined by
\[
\Delta_{\mathcal{F}(X)} : \mathcal{F}(X) \xrightarrow{\mathcal{F}(\Delta_X)} X \otimes X \xrightarrow{\varphi_2(X, X)} X \otimes \mathcal{F}(X),
\]
and the same antipode as those of \( X \). More precisely, according to the proof of Proposition 4.12 the multiplication \( \bullet \) is given by
\[
x \bullet y = \sigma^{-1}(x_{(1)}, y_{(1)})x_{(0)} \cdot y_{(0)} = \sigma^{-1}(x_{(1)}, y_{(1)})S(y_3)x_{(0)} \cdot y_2 = \sigma(x_{(2)}, S^{-1}(y_4)y_1)S(x_{(1)}, S(y_2))x_{(0)}y_3 = \sigma(x_{(1)}, S^{-1}(y_2))x_{(0)}y_1 = \sigma(S(x_3)S^2(x_1), y_2)x_2y_1,
\]
for all \( x, y \in X \). Similarly, we have
\[
\Delta_{\mathcal{F}(X)}(x) = \sigma((x_{(1)}(1), x_{(2)}(1))(x_{(1)}(0) \otimes x_{(2)}(0)) = \sigma(S(x_1)x_3, S(x_4)x_6)x_2 \otimes x_5,
\]
for all \( x \in X \). Comparing to the structures in Theorem 4.8 we conclude that \( \mathcal{F}(X) = X \) as braided Hopf algebras in \( H_{X,A}^{\text{cop}} \mathcal{Y} \mathcal{D} \), and this finishes the proof. \( \square \)

We end this section with few comments.

Remarks 4.11. Let \( (H, \sigma) \), \( X \) and \( A \) be as above.

(i) One can easily see that the map \( \gamma : A \rightarrow H_{r_A}^{\text{cop}} \subseteq H_{X,A}^{\text{cop}} \) given by \( \gamma(a) = r_a \) is a Hopf algebra morphism. Moreover, for this map \( \gamma \), (3.1) reduces to
\[
r_ar_b = \sigma^{-1}(b_3, a_3)\sigma(b_1, a_1)r_{b_2}r_{a_2}
\]
which holds by (2.7). So on the \( k \)-vector space \( A \) we have a braided Hopf algebra structure, denoted by \( \Delta \), in \( H_{X,A}^{\text{cop}} \mathcal{Y} \mathcal{D} \). Namely, \( \Delta \) is a left Yetter-Drinfeld module over \( H_{X,A}^{\text{cop}} \) via
\[
l_xr_a \triangleright b = (l_xr_a, S^{-1}(b_1)S^{-2}(b_3))b_2, \ \lambda_{\Delta}(a) = r_{S^{-1}(a_3)a_1} \otimes a_2,
\]
and a Hopf algebra with the same unit and counit as $A$ and
\[
\Delta(a) = \sigma(S(a_1)S^{-1}(a_3)a_2) \otimes \sigma(s_3)S^{-1}(a_1) = \sigma^{-1}_2(S(a_3)S^2(a_1), b_2) a_2 b_1,
\]
\[
\Sigma(a) = \sigma(a_4, a_1S(a_3))S(a_2) = \sigma^{-1}_2(S^2(a_3)S(a_1), a_4)S(a_2).
\]
Comparing with the structure of $X$ we conclude that $\Lambda$ can be obtained from $X$ by replacing $(H, \sigma)$ with $(H, \sigma^{-1})$ and interchanging $X$ and $A$. For this, observe that if $\tilde{l}_a$ and $\tilde{r}_a$ are the elements of $H^0$ corresponding to $\tilde{H} = (H, \sigma^{-1})$ then $\tilde{r}_a = l_{S^{-1}(x)}$ and $\tilde{l}_a = r_{S(a)}$, so $\tilde{H}_{A,X} := \tilde{H}_A \tilde{H}_X = H_{R,A}H_{X} = H_{X,A}$, as Hopf algebras. The remaining details are now trivial.

(ii) Although it might seem more natural to try to obtain a braided Hopf algebra structure in $H_{X,A}^{\otimes n}$ on $X \otimes A$, for this we would need a Hopf algebra map $\pi : AX \rightarrow H_{X,A}$ satisfying condition (3.1) in Proposition 3.1. A candidate is $\pi(ax) = r_{l_{S^{-1}(x)}}$ but, in general, it is not well defined. (Take for example $X = A = H = (\text{SL}_q(N), \sigma)$, the CQT Hopf algebra defined in the last section of this paper.)

5. THE FINITE DIMENSIONAL CASE

In this section we discuss how our results thus far relate to those of Radford [13] for finite dimensional Hopf algebras.

Throughout this section, $(H, R)$ is a finite dimensional quasitriangular (QT for short) Hopf algebra, so its dual linear space $H^*$ has a CQT structure given by
\[
\sigma : H^* \otimes H^* \rightarrow k, \quad \sigma(p, q) = p(R^1)q(R^2),
\]
for all $p, q \in H^*$, where $R := R^1 \otimes R^2 \in H \otimes H$. Thus, we can consider the sub-Hopf algebras $H^*_R$ and $H^*_R$ of $H^*$ defined in the previous section. Identifying $H^{**}$ and $H$ via the canonical Hopf algebra isomorphism
\[
\Theta : H \rightarrow H^{**}, \quad \Theta(h)(h^*) = h^*(h),
\]
for all $h \in H$ and $h^* \in H^*$, we will prove that the sub-Hopf algebras $H^*_R$ and $H^*_R$ of $H^{**}$ can be identified with the sub-Hopf algebras $R(r)$ and $R(r)$ of $H$ constructed in [13]. Recall that
\[
R(t) := \{q(R^2)R^1 \mid q \in H^*\} \text{ and } R(v) := \{p(R^1)R^2 \mid p \in H^*\}.
\]
Also, note that, if we write $R = \sum_{i=1}^{m} u_i \otimes v_i \in H \otimes H$, with $m$ as small as possible, then
\[
\{u_1, \ldots, u_m\} \text{ is a basis for } R(t) \text{ and } \{v_1, \ldots, v_m\} \text{ is a basis for } R(v),
\]
respectively. We extend $\{u_1, \ldots, u_m\}$ to a basis $\{u_1, \ldots, u_m, \ldots, u_n\}$ of $H$ and denote by $\{u_i\}_{i=1}^{\infty}$ the dual basis of $H^*$ corresponding to $\{u_i\}_{i=1}^m$. Similarly, we extend $\{v_1, \ldots, v_m\}$ to a basis $\{v_1, \ldots, v_m, \ldots, v_n\}$ of $H$ and denote by $\{v_i\}_{i=1}^{\infty}$ the dual basis of $H^*$ corresponding to $\{v_i\}_{i=1}^m$.

Lemma 5.1. For the above context the following statements hold:

i) The map $\mu_1 : H^*_t \rightarrow R(t)$ defined by $\mu_1(q) = q(R^2)R^1$, for all $q \in H^*$, is well defined and a Hopf algebra isomorphism. Its inverse is given by $\mu_1^{-1}(u_i) = l_{v_i}$, for all $i = 1, \ldots, m$. In particular, $\{l_{v_i}\}_{i=1}^{\infty}$ is a basis for $H^*_t$.

ii) The map $\mu_r : H^*_r \rightarrow R(r)$ given by $\mu_r(p) = p(R^1)R^2$, for all $p \in H^*$, is well defined and a Hopf algebra isomorphism. Its inverse is defined by $\mu_r^{-1}(v_i) = r_{u_i}$, for all $i = 1, \ldots, m$. In particular, $\{r_{u_i}\}_{i=1}^{\infty}$ is a basis for $H^*_r$. 
Proof. We show only i), with the proof of ii) being similar. First observe that
\[ l_v(h^*) = \sigma(h^*, q) = h^*(R^1)q(R^2) = \Theta(q(R^2)R^1)(h^*), \]
for all \( h^* \in H^* \). Thus \( H^*_f = \{ \Theta(q(R^2)R^1) \mid q \in H^* \} = \Theta(R(l)) \cong R(l) \). Clearly \( \mu_l \) is precisely the restriction and corestriction of \( \Theta^{-1} \) at \( \Theta(R(l)) \) and \( R(l) \), respectively. Hence \( \mu_l \) is well defined and a Hopf algebra isomorphism.

Finally, for all \( h^* \in H^* \) we have
\[ l_v(h^*) = \sigma(h^*, v^i) = h^*(R^1)v^i(R^2) = \sum_{j=1}^{m} h^*(u_j)v^i(v_j), \]
and therefore \( l_v = 0 \), for all \( i \in \{ m+1, \ldots, n \} \), and \( l_v(h^*) = h^*(u_i) = \Theta(u_i)(h^*) \), i.e. \( l_v = \Theta(u_i) \), for all \( i = 1, \ldots, m \). This shows that \( \mu_l^{-1}(u_i) = l_v \), for all \( i = 1, \ldots, m \), so the proof is complete. □

From Proposition 4.4, \( \sigma \) gives a pairing on \( H^*_f \otimes H^*_f \), and we consider the generalized quantum double \( D(H^*_f, H^*_f) \). On the other hand, \( H^*_f \) and \( H^*_f \) are finite dimensional Hopf algebras, so we can construct the usual Drinfeld quantum doubles \( D(H^*_f) \) and \( D(H^*_f) \). To demonstrate the connections between these Hopf algebras, we first need the following lemma.

**Lemma 5.2.** Suppose that \( \langle \cdot, \cdot \rangle : U \otimes V \to k \) defines a duality between the finite dimensional Hopf algebras \( U \) and \( V \) such that there exists a Hopf algebra isomorphism \( \Psi : V^* \to U \) with the property
\[ \langle \Psi(v^*), v \rangle = v^*(v), \quad \forall v^* \in V^*, \ v \in V. \]
Then \( D(U^{\text{cop}}, V) \cong D(V) \cong D(U^{\text{op}, \text{cop}})^{\text{op}} \) as Hopf algebras.

**Proof.** As coalgebras
\[ D(U^{\text{cop}}, V) = U^{\text{cop}} \otimes V \quad \Psi^{-1} \otimes \text{Id} \quad V^{\text{cop}} \otimes V = D(V). \]
Thus will be enough to check that \( \Psi \otimes \text{Id} : D(V) \to D(U^{\text{cop}}, V) \) is an algebra morphism. For all \( x^*, y^* \in V^* \) and \( x, y \in V \) we compute
\[
(\Psi \otimes \text{Id})(x^* \otimes x)(y^* \otimes y) \]
\[
= y^*_i (S^{-1}(x_i)) y^*_i (x_i) \Psi(x^*_i y^*_i) \otimes x_2 y \]
\[
= \langle \Psi(y^*_i), x_3 \rangle \langle \Psi(y^*_i), x_1 \rangle \Psi(x^*_i) \otimes x_2 y \]
\[
= (\Psi(x^*) \otimes x) (\Psi(y^*) \otimes y) = (\Psi \otimes \text{Id})(x^* \otimes x) (\Psi \otimes \text{Id})(y^* \otimes y),
\]
as needed. Clearly, \( \Psi \otimes \text{Id} \) respects the units, so \( \Psi \otimes \text{Id} : D(U^{\text{cop}}, V) \to D(V) \) is a Hopf algebra isomorphism.

From [14] Theorem 3 we know that \( D(U^{\text{op}, \text{cop}})^{\text{op}} \cong D(U^*) \) as Hopf algebras. But \( U^* \cong V^{**} \cong V \) as Hopf algebras, and therefore \( D(U^*) \cong D(V) \) as Hopf algebras. We conclude that \( D(U^{\text{op}, \text{cop}})^{\text{op}} \cong D(V) \) as Hopf algebras, and this finishes the proof. □

**Proposition 5.3.** Let \((H, R)\) be a finite dimensional QT Hopf algebra with an \( R \)-matrix \( R = \sum_{i=1}^{m} u_i \otimes v_i \in H \otimes H \), with \( m \) as small as possible. If \( H^*_f \) and \( H^*_f \) are the Hopf subalgebras of \( H^{**} \cong H \) defined above then 
\[ D(H^*_f, H^*_f) \cong D(H^*_f) \cong D((H^*_f)^{\text{op}})^{\text{op}}, \]
as Hopf algebras.
Proof. We apply Lemma 5.2 for \( U = \left(H_r^*\right)^{\text{cop}} \) and \( V = H_r^* \). So we shall prove that there exists a Hopf algebra morphism \( \Psi : \left(H_r^*\right)^* \to \left(H_r^*\right)^{\text{cop}} \) (or, equivalently, from \( \left(H_r^*\right)^{\text{cop}} \) to \( H_r^* \)) satisfying (5.1), this means \( \langle \Psi(\eta), l_q \rangle = \eta(q) \), for all \( \eta \in \left(H_r^*\right)^* \) and \( q \in H^* \). We will use the Hopf algebra isomorphism \( \xi : R_{(l)}^{\text{cop}} \to R_{(r)} \) from Proposition 2(c) defined by \( \xi(\zeta) = \zeta(R^1)R^2 \), for all \( \zeta \in R_{(l)}^{\text{cop}} \). Note that \( \xi^{-1}(u_i) = \tilde{u}^i \), for all \( i = 1, m \), where \( \tilde{u}^i \) is the restriction of \( u^i \) at \( R_{(l)} \).

Now, define \( \Psi : \left(H_r^*\right)^{\text{cop}} \to H_r^* \) as the composition of the following Hopf algebra isomorphisms

\[
\Psi : \left(H_r^*\right)^{\text{cop}} \xrightarrow{\left(\mu_{r^{-1}}\right)^*} R_{(l)}^{\text{cop}} \xrightarrow{\xi} R_{(r)} \xrightarrow{\mu_{r^{-1}}} H_r^*.
\]

Explicitly, we have \( \Psi(\eta) = \sum_{i=1}^{m} \eta \langle l_{v^j}, r_{u^i} \rangle \), for all \( \eta \in \left(H_r^*\right)^* \), and this allows us to compute \( \langle \Psi(\eta), l_q \rangle \) to be

\[
\sum_{i=1}^{m} \eta \langle l_{v^j}, r_{u^i} \rangle = \sum_{i=1}^{m} \eta \langle l_{v^j}, r_{u^i} \rangle = \sum_{i,j=1}^{m} \eta \langle l_{v^j}, r_{u^i} \rangle = \sum_{i,j=1}^{m} \eta \langle l_{v^j}, r_{u^i} \rangle = \eta(q),
\]

for all \( s = 1, m \). Since \( \{l_{v^j} \mid s = 1, m\} \) is a basis for \( H_r^* \) it follows that \( \langle \Psi(\eta), l_q \rangle = \eta(q) \), for all \( \eta \in \left(H_r^*\right)^* \) and \( q \in H^* \), so the proof is finished.

Remark 5.4. Let \( U \) and \( V \) be two Hopf algebras in duality via the bilinear form \( \langle \cdot, \cdot \rangle \). If there is an element \( \sum_{i=1}^{t} u_i \otimes v_i \in U \otimes V \) such that, for all \( u \in U \) and \( v \in V \),

\[
\sum_{i=1}^{t} \langle u_i, v \rangle v_i = v \quad \text{and} \quad \sum_{i=1}^{t} \langle u, v_i \rangle u_i = u,
\]

then \( R = \sum_{i=1}^{t} \sum_{i=1}^{m} (1 \otimes u_i) \otimes (u_i \otimes 1) \) is an \( R \)-matrix for \( D(U^{\text{cop}}, V) \).

Let now \( (H, R) \) be a finite dimensional QT Hopf algebra with \( R = \sum_{i=1}^{m} u_i \otimes v_i \), with \( m \) as small as possible. Then \( \sum_{i=1}^{m} r_{u^i} \otimes l_{v^j} \in H_r^* \otimes H_r^* \) satisfies the conditions in (5.2). Indeed, for all \( p \in H^* \) we have

\[
\sum_{i=1}^{m} \langle r_{u^i}, l_{v^j} \rangle r_{u^i} = \sum_{i,j=1}^{m} p(\langle u_j, v_i \rangle) r_{u^i} = \sum_{i=1}^{m} p(\langle u_i \rangle) \Theta(v_i) = \Theta(p(R^1)R^2) = r_p,
\]

and in a similar manner one can verify that \( \sum_{i=1}^{m} \langle r_{u^i}, l_{v^j} \rangle l_{v^j} = l_q \), for all \( q \in H^* \).

Therefore, \( R = \sum_{i=1}^{m} (1 \otimes l_{v^j}) \otimes (r_{u^i} \otimes 1) \) is an \( R \)-matrix for \( D(H_r^*, H_r^*) \).

On the other hand, a basis on \( \left(H_r^*\right)^* \) can be obtained by using the inverse of the Hopf algebra isomorphism \( \Psi \) constructed in Proposition 5.3. Since \( \Psi^{-1}(r_{u^i}) = \tilde{u}^i \otimes \mu_i \), it follows that \( \{\tilde{u}^i \otimes \mu_i \mid i = 1, m\} \) is a basis of \( \left(H_r^*\right)^* \). Moreover, we can easily see that it is the basis of \( \left(H_r^*\right)^* \) dual to the basis \( \{l_{v^j} \mid i = 1, m\} \) of \( H_r^* \), so \( \mathcal{R} = \sum_{i=1}^{m} (1 \otimes l_{v^j}) \otimes (\tilde{u}^i \otimes \mu_i \otimes 1) \) is an \( R \)-matrix for \( D(H_r^*) \). Thus, we can conclude that the Hopf algebra isomorphism \( \Psi \otimes Id : D(H_r^*) \to D(H_r^*, H_r^*) \) constructed in
Proposition 5.5 is actually a QT Hopf algebra isomorphism, that is, in addition, 
\((\Psi \otimes Id) \otimes (\Psi \otimes Id)) (R) = \mathfrak{R}\). The verification of the details is left to the reader.

We are now able to prove that, for our context, the Hopf algebra surjection from Proposition 4.5 is in fact a QT morphism, and that it can be deduced from the surjective QT morphism \(F : D(R_l) \to H_R := R_q R_r \subseteq R_q R_l\) considered in [5, Theorem 2]. Namely, \(F(\zeta \otimes h) = \xi(\zeta) h\), for all \(\zeta \in R^*_l\) and \(h \in R_l\), where \(\xi(\zeta) = \zeta(R^1)R^2\) is the Hopf algebra isomorphism from \(R^*_{(l)}\) to \(R_{(r)}\) considered in Proposition 5.3.

**Proposition 5.5.** Under the above circumstances and notations we have the following diagram commutative.

\[
\begin{array}{ccc}
D(H^*_r, H^*_l) & \xrightarrow{f} & H^*_r := H^*_r H^*_l = H^*_r \otimes H^*_l \\
\Psi^{-1} \otimes Id & \cong & (\mu^*_r)^{-1} \otimes \mu^*_l \\
D(H^*_l) & \xrightarrow{F} & H_R := R_{(r)} R_{(l)} = R_{(l)} R_{(r)}
\end{array}
\]

Here \((\mu^*_r)^{-1}\) is the transpose of \(\mu^{-1}_r\) and \(\mu^{-1}_r \mu^{-1}_l\) is defined by

\[
\mu^{-1}_r \mu^{-1}_l (p(r^1)^2 q(R^2)) R^1 = r_p l_q, \quad \forall p, q \in H^*,
\]

where \(R = R^1 \otimes R^2 = r^1 \otimes r^2\) is the \(R\) matrix of \(H\).

**Proof.** Straightforward. We only note that \(p(r^1) r^2 q(R^2) R^1 = p'(r^1) r^2 q'(R^2) R^1\) iff \(\Theta^{-1}(r_p) \Theta^{-1}(l_q) = \Theta^{-1}(r_{p'}) \Theta^{-1}(l_{q'})\), iff \(\Theta^{-1}(r_p l_q) = \Theta^{-1}(r_{p'} l_{q'})\), iff \(r_p l_q = r_{p'} l_{q'}\), so \(\mu^{-1}_r \mu^{-1}_l\) is well defined. \(\square\)

**Corollary 5.6.** \(H^*_\sigma\) is QT with \(\mathfrak{R} = \sum_{i=1}^{m} l_{u^i} \otimes r_{u^i}\), and in this way \(f\) becomes a surjective QT Hopf algebra morphism.

Finally, we are able to show that in the finite dimensional case the morphism \(\pi\) in Proposition 4.6 arises from the particular situation described above.

**Corollary 5.7.** Let \((H, R)\) be a finite dimensional QT Hopf algebra with an \(R\) matrix \(R = \sum_{i=1}^{m} u_i \otimes v_i\), \(m\) as small as possible. For any Hopf subalgebra \(X\) of \(H^*\) there exists a surjective Hopf algebra morphism \(\pi : D(H^*_{\sigma^{\text{cop}}}, X) \to H^*_{\sigma^{\text{cop}}}\) covering the canonical inclusion \(i_{H^*_l} : H^*_l \to D(H^*_l), X)\).

**Proof.** From Corollary 5.6 and Example 3.3 such a morphism \(\pi\) exists. Moreover, using the definition of \(\pi\) in Example 3.3 for all \(p, q \in H^*\) and \(h^* \in X\), we have

\[
\pi(l_p r_q \otimes h^*) = \sum_{i=1}^{m} (u_i h^*) l_p r_q l_{S^{i-1}(\omega^i)} = \sum_{i,j=1}^{m} \sum_{i,j=1}^{m} u^i(u_j) h^*(v_j) l_p r_q l_{S^{i-1}(\omega^i)}
\]

\[
= \sum_{i=1}^{m} h^*(v_i) l_p r_q l_{S^{i-1}(\omega^i)} = \sum_{i=1}^{m} h^*(v_i) l_p r_q l_{S^{i-1}(\omega^i)}
\]

\[
= l_p r_q l_{S^{i-1}(\omega^i)}.
\]
Remarks 5.8. (i) Here, the sub-Hopf algebra \( H^* \) of \( H^{**} \cong H \) is the smallest sub-Hopf algebra \( C \) of \( H^{**} \) such that \( \mathcal{R} := (\Theta \otimes \Theta)(R) \in C \otimes C \) and \( \mathcal{R} \) played a crucial role in the definition of \( \pi \). Now, for \( H \) CQT not necessarily finite dimensional with sub-Hopf algebras \( X \) and \( A \) as in Proposition 4.6, we cannot ensure that \( H_{X,A} \) has a quasitriangular structure but we can define the projection \( \pi \) in the same way.

(ii) Applying Corollary 5.7 with \( X = H^* \) we see that \( H^* \) has a braided Hopf algebra structure in the braided category of left Yetter-Drinfeld modules over \( H^{**} \). Identifying \( H^{**} \) with \( H^{**} \) we obtain that \( H^* \) is a braided Hopf algebra in \( H^{**} \), denoted in what follows by \( H^* \). From Proposition 4.7 the structure of \( H^* \) as left \( H^{**} \) Yetter-Drinfeld module is given by

\[
h \cdot \varphi = (S^{-1}(\varphi_1)S^{-2}(\varphi_3))((h)\varphi_2) = S^{-2}(h_2) \hookrightarrow \varphi \leftarrow S^{-1}(h_1),
\]

\[
\lambda_{H^*}(\varphi) = (S^{-1}(\varphi_1)S^{-2}(\varphi_3))(R^2)R^1 \otimes \varphi_2 = R^1 \otimes R^2 \cdot \varphi,
\]

for all \( h \in H_R \) and \( \varphi \in H^* \), and from Theorem 4.8 the structure of \( H^* \) as a bialgebra in \( H^{**} \) is the following. The multiplication is defined by

\[
\varphi_2 \psi = \varphi_3(S(R^1_1))\varphi_1(S^2(R^2_1))\varphi_2(R^2_2)\varphi_1(R^1_1) \hookrightarrow \varphi \leftarrow S^2(R^2_2) \otimes R^2 \rightarrow \psi,
\]

for all \( \varphi, \psi \in H^* \), the unit is \( \varepsilon \), the comultiplication is given by

\[
\Delta_{H^*}(\varphi) = \varphi_1(S(R^1_1))\varphi_3(R^1_2)\varphi_4(S(R^2_3))\varphi_6(R^2_3)\varphi_2 \varphi_5 = R^2_1 \rightarrow \varphi_1 \leftarrow S(R^1_1) \otimes R^2_2 \rightarrow \varphi_2 \leftarrow S(R^2_3),
\]

for all \( \varphi \in H^* \), and the counit is \( \Theta(1) \). Finally, according to the part 1) of Remarks 4.9 \( H^* \) is a braided Hopf algebra with antipode

\[
S_{H^*}(\varphi) = \varphi_1(S(R^1_1))\varphi_4(R^1_3)\varphi_2(S^{-1}(R^2_3))\varphi_3 \circ S^{-1} = (R^1_2 \rightarrow \varphi \leftarrow S(R^1_1)S^{-1}(R^2_3)) \circ S^{-1}.
\]

Using similar arguments to those in Subsection 4.1 one can easily see that \( H^* \) coincides with \( (H^{**})^\star \), the categorical right dual of \( H^{**} \) in \( H^{**} \cdot \mathcal{M} \), viewed as a braided Hopf algebra in \( H^{cop} \cdot H^{cop} \cdot \mathcal{D} \) through a composite of two canonical braided functors. (The structures of \( H^{cop} \), associated enveloping algebra braided group of \( (H^{cop}, R_{21} := R^2 \otimes R^1) \), can be obtained from \cite[Example 9.4.9]{13}, and then the braided structure of \( (H^{cop})^\star \) can be deduced from \cite[or \cite]{11}.)

More precisely, \( H^* = \tilde{\Phi}(H^{cop}, R_{21}) \otimes (H^{cop})^\star \), as braided Hopf algebras in \( H^{cop} \cdot H^{cop} \cdot \mathcal{D} \), where, in general, if \( (H, R) \) is a QT Hopf algebra then

(i) \( \tilde{\Phi}(H, R) : H \mathcal{M} \rightarrow H \mathcal{M} \) is the braided functor which acts as identity on objects and morphisms and for all \( m \in H \mathcal{M} \), \( \tilde{\Phi}(H)(m) = m \) as \( H \)-module, and has the left \( H \)-coaction given by \( \lambda_{\tilde{\Phi}}(m) = R^2 \otimes R^1 \cdot m \), for all \( m \in \mathcal{M} \);

(ii) \( \Phi(H, R) : (H, R) \mathcal{M} \rightarrow (H, R) \mathcal{M} \) is the functor of restriction of scalars which is braided because the inclusion \( H_R \hookrightarrow H \) is a QT Hopf algebra morphism.

The verification of all these details is left to the reader.
6. The Hopf algebras \(SL_q(N)\) and \(U_q^{\text{ext}}(sl_N)\)

In this section we apply the results in Section 4 to the CQT Hopf algebra \(H = SL_q(N)\) introduced in \([21]\). We will show by explicit computation that \(H_\sigma \cong U_q^{\text{ext}}(sl_N)\). The computation should be compared to results of \([5, 3.2]\) and the description of \(U_q^{\text{ext}}(sl_N)\) from \([9, \text{Ch. 8, Theorem 33}]\).

Our approach is to study the image of the map \(\omega\) in \([5, 3.2(b)\)], that is, \(H_\sigma\), a sub-Hopf algebra of \(H^0\). If, instead, we studied the image of the map \(\theta\) in \([5, 3.2(a)]\), then we would be considering a sub-bialgebra of \((H^0)^{\text{cop}}\). Constructing the isomorphism between these two approaches seems to be no simpler computationally than the direct calculations we supply below.

To make this section as self-contained as possible, we first recall the definition of the CQT bialgebra \(M_q(N)\), and outline the construction of \(SL_q(N)\).

For \(V\) a \(k\)-vector space with finite basis \(\{e_1, \ldots, e_N\}\) and any \(0 \neq q \in k\), we associate a solution \(c\) of the Yang-Baxter equation, \(c : V \otimes V \rightarrow V \otimes V\), by

\[
c(e_i \otimes e_j) = q^{\delta_{ij}} e_j \otimes e_i + [i > j](q - q^{-1})e_i \otimes e_j,
\]

for all \(1 \leq i,j \leq N\), where \(\delta_{ij}\) is Kronecker’s symbol and \([i > j]\) is the Heaviside symbol, that is, \([i > j] = 0\) if \(i \leq j\) and \([i > j] = 1\) if \(i > j\) (see for instance \([8, \text{Proposition VIII 1.4}]\)]. By the FRT construction, to any solution \(c\) of the quantum Yang-Baxter equation we can associate a CQT bialgebra, denoted by \(A(c)\), (see \([7, 8]\)), obtained by taking a quotient of the free algebra generated by the family of indeterminates \((x_{ij})_{1 \leq i,j \leq N}\). For the map \(c\) above, \(A(c)\) is denoted by \(M_q(N)\) and has the following structure, cf. \([8, \text{Exercise 10, p. 197}]\). As an algebra \(M_q(N)\) is generated by \(1\) and by \((x_{ij})_{1 \leq i,j \leq N}\). (Note that we write \(x_{ij}\) as \(x_{ij}\) if the meaning is clear.) Multiplication in \(M_q(N)\) is defined by the following relations:

\[
\begin{align*}
(6.1) \quad x_{im}x_{in} &= qx_{in}x_{im}, \forall n < m, \\
(6.2) \quad x_{jm}x_{im} &= qx_{im}x_{jm}, \forall i < j, \\
(6.3) \quad x_{jn}x_{im} &= x_{im}x_{jn}, \forall i < j \text{ and } n < m, \\
(6.4) \quad x_{jm}x_{in} - x_{im}x_{jn} &= (q - q^{-1})x_{im}x_{jn}, \forall i < j \text{ and } n < m.
\end{align*}
\]

The coalgebra structure on the \(x_{ij}\) is that of a comatrix coalgebra, that is

\[
\Delta(x_{ij}) = \sum_{k=1}^{N} x_{ik} \otimes x_{kj}, \varepsilon(x_{ij}) = \delta_{i,j},
\]

for all \(1 \leq i,j \leq N\).

A CQT structure on \(M_q(N)\) is given by the skew pairing \(\sigma' : M_q(N) \otimes M_q(N) \rightarrow k\) defined on generators \(x_{im}, x_{jn}\) by

\[
\sigma'(x_{im}, x_{jn}) = q^{\delta_{ij}} \delta_{m,j} + [j > i](q - q^{-1})\delta_{m,i},
\]

and satisfying (2.3) to (2.7). Complete details of this construction can be found in \([8, \text{Theorem VIII 6.4}]\) or see \([5, (3.3)]\) and \([4]\). This skew pairing is invertible with inverse obtained in the formulas above by replacing \(q\) by \(q^{-1}\).

For \(z \in k\) such that \(z^N = q^{-1}\), we define another skew pairing \(\sigma\) on \(M_q(N) \otimes M_q(N)\) by defining

\[
\sigma(1,-) = \sigma(-,1) = \varepsilon \text{ and } \sigma(x_{im}, x_{jn}) = z\sigma'(x_{im}, x_{jn})
\]
and extending. For example, $\sigma(x_{im}x_{kl}, x_{jn}) = z^2 \sum_{r=1}^{N} \sigma'(x_{im}, x_{jr})\sigma'(x_{kl}, x_{rn})$. Since $\sigma$ satisfies (2.7) when $h, h'$ are generators $x_{ij}$, then by [8, Lemma VIII 6.8], $\sigma$ satisfies (2.4), and $\sigma$ gives $M_q(N)$ a CQT structure. Explicitly, we have that

\begin{equation}
(6.8) \quad \sigma(x_{ii}, x_{ii}) = zq;
\end{equation}

\begin{equation}
(6.9) \quad \sigma(x_{ii}, x_{jj}) = z \text{ for } i \neq j;
\end{equation}

\begin{equation}
(6.10) \quad \sigma(x_{ij}, x_{ji}) = z(q - q^{-1}) \text{ if } i < j;
\end{equation}

\begin{equation}
(6.11) \quad \sigma(x_{ij}, x_{kl}) = 0 \text{ otherwise.}
\end{equation}

The inverse $\sigma^{-1}$ is obtained by replacing $q$ by $q^{-1}$ and $z$ by $z^{-1}$.

The bialgebra $M_q(N)$ does not have a Hopf algebra structure, but it possesses a remarkable grouplike central element

\[ \det_q = \sum_{p \in S_N} (-q)^{-l(p)}x_{1p(1)} \cdots x_{Np(N)}, \]

which allows us to construct $SL_q(N) := M_q(N)/(\det_q - 1)$. Here $S_N$ denotes the group of permutations of order $N$, and $l(p)$ the number of the inversions of $p \in S_N$. As an algebra, $SL_q(N)$ is generated by $(x_{ij})_{1 \leq i,j \leq N}$ with relations (6.1)-(6.4) and

\begin{equation}
(6.12) \quad \sum_{p \in S_N} (-q)^{-l(p)}x_{1p(1)} \cdots x_{Np(N)} = 1.
\end{equation}

The coalgebra structure is the comatrix coalgebra structure. To define the antipode $S$ of $SL_q(N)$, denote $X = (x_{ij})_{1 \leq i,j \leq N}$ and then define $Y_{ij}$ as the generic $(N - 1) \times (N - 1)$ matrix obtained by deleting the $i$th row and the $j$th column of $X$. Then

\begin{equation}
(6.13) \quad S(x_{ij}) = (-q)^{j-i} \det_q(Y_{ji}) = (-q)^{j-i} \sum_{p \in S_{j,i}} (-q)^{-l(p)}x_{1p(1)} \cdots x_{j-1p(j-1)}x_{j+1p(j+1)} \cdots x_{Np(N)},
\end{equation}

where $S_{j,i}$ is the set of bijective maps from $\{1, \ldots, j - 1, j + 1, \ldots, N\}$ to $\{1, \ldots, i - 1, i + 1, \ldots, N\}$. Moreover, for all $1 \leq i, j \leq N$, we have

\begin{equation}
(6.14) \quad S^2(x_{ij}) = q^{2(j-i)}x_{ij}.
\end{equation}

We include the next lemma to provide complete details of the construction.

**Lemma 6.1.** For $\sigma$ the skew pairing defined by (6.8) to (6.11), for all $x \in M_q(N)$,

\begin{equation}
(6.15) \quad \sigma(\det_q, x) = \sigma(x, \det_q) = \varepsilon(x)
\end{equation}

and thus $\sigma$ is well defined on $SL_q(N) \otimes SL_q(N)$.

**Proof.** Since $\det_q$ is a grouplike element, from (2.4) and (2.5), it suffices to check that (6.15) holds on generators. From (6.6) it follows that $\sigma'(x_{im}, x_{jn}) = 0$ unless
\( i \leq m \), and if \( i = m \), then \( \sigma'(x_{im}, x_{jn}) = \eta^q_{i, j} \delta_{n, j} \). Now we compute

\[
\sigma'(\det_q, x_{ij}) = \sum_{p \in S_N} (-q)^{-l(p)} \sigma'(x_{1p(1)} \cdots x_{Np(N)}, x_{ij})
\]

\[
= \sum_{p \in S_N} (-q)^{-l(p)} \sum_{k=1}^{N} \sigma'(x_{1p(1)} \cdots x_{N-1p(N-1), x_{ik}}) \sigma'(x_{Np(N), x_{kj}})
\]

\[
= \sum_{p \in S_N, \ p(N) = N} (-q)^{-l(p)} \sum_{k=1}^{N} \sigma'(x_{1p(1)} \cdots x_{N-1p(N-1), x_{ik}}) \sigma'(x_{NN, x_{kj}})
\]

\[
= q^{\delta_{N, j}} \sum_{p \in S_{N-1}} (-q)^{-l(p)} \sigma'(x_{1p(1)} \cdots x_{N-1p(N-1), x_{ij}}).
\]

By induction, it follows that

\[
\sigma'(\det_q, x_{ij}) = q^N \sum_{k=1}^{N} \delta_{k, j} \delta_{i, k} = q^\varepsilon(x_{ij}),
\]

and hence, \( \sigma(\det_q, x_{ij}) = z^N q^\varepsilon(x_{ij}) = \varepsilon(x_{ij}) \), for all \( 1 \leq i, j \leq N \).

In a similar manner, using the fact that \( \sigma'(x_{im}, x_{jn}) = 0 \) unless \( j \geq n \), and if \( j = n \), then \( \sigma'(x_{im}, x_{jn}) = q^{\delta_{i, j} \delta_{m, i}} \), one can prove that

\[
\sigma(x_{ij}, \det_q) = z^N \sum_{k=1}^{N} \delta_{k, j} \delta_{i, k} = \varepsilon(x_{ij}).
\]

Thus \( \sigma \) is well defined on the quotient \( \mathbb{M}_q(N)/(\det_q - 1) \) and \( SL_q(N) \) has a CQT structure. \( \square \)

We now apply the results of Section 4 to the CQT Hopf algebra \( H = (SL_q(N), \sigma) \). For all \( 1 \leq i, j \leq N \), let us denote \( r_{ij} := r_{x_{ij}} \) and \( l_{ij} := l_{x_{ij}} \). (Note that we will insert commas in these subscripts only for more complicated expressions.)

**Lemma 6.2.** Let \( H = (SL_q(N), \sigma) \).

(6.16) \quad r_{ij}(x_{mn}) = z(q - q^{-1}) \delta_{i,n} \delta_{j,m}, \forall \ i < j; \\
(6.17) \quad l_{ij}(x_{mn}) = z(q - q^{-1}) \delta_{i,n} \delta_{j,m}, \forall \ i > j; \\
(6.18) \quad l_{ii}(x_{mn}) = r_{ii}(x_{mn}) = z^2 q^{\delta_{i,m} \delta_{m,i}}; \\
(6.19) \quad l_{ij} = r_{ji} = 0, \forall \ i < j.

**Proof.** Equations (6.16)-(6.18) follow directly from (6.8) to (6.10). Now suppose that \( i < j \). From (6.11), we have that \( l_{ij} \) and \( r_{ji} \) are 0 on generators \( x_{mn} \). Let \( a, b \in \{x_{mn} \mid 1 \leq m, n \leq N \} \). Since \( \sigma \) is a skew pairing, we have

\[
l_{ij}(ab) = \sigma(ab, x_{ij}) = \sum_{k=1}^{N} l_{ik}(a) l_{kj}(b).
\]

Since \( i < j \) we cannot have both \( i \geq k \) and \( k \geq j \), so it follows that the product \( l_{ik}(a) l_{kj}(b) = 0 \), for all \( k = 1, N \). By mathematical induction it follows that \( l_{ij} = 0 \), for any \( i < j \). The statement for \( r_{ji} \) is proved similarly. \( \square \)

**Corollary 6.3.** The maps \( l_{ii} \) and \( r_{ii} \) are equal grouplike elements of \( H^0 \). Also, denoting \( l_{ii}^{-1} = 1_{S^{-1}(x_{ii})} = S^*(l_{ii}) \), we have

(6.20) \quad l_{ii}^{-1}(x_{mn}) = r_{ii}^{-1}(x_{mn}) = z^{-1} q^{-\delta_{i,m} \delta_{m,i}}.
Proof. The fact that \( l_i \) and \( r_i \) are grouplike follows directly from (6.19). Then, since these are algebra maps equal on generators, they are equal. The rest is immediate. \( \square \)

The next lemma describes the commutation relations for the generators of \( H_r \) and \( H_l \).

**Lemma 6.4.** In \( H_r \) the commutation relations for the generators \( r_{ij} \) are given by:

\[
\begin{align*}
(6.21) \quad r_{im}r_{in} &= qr_{in}r_{im}, \quad \forall \ n < m; \quad r_{jm}r_{im} = qr_{im}r_{jm}, \quad \forall \ i < j; \\
(6.22) \quad r_{jm}r_{in} &= r_{im}r_{jn}; \quad r_{jm}r_{in} - r_{in}r_{jm} = (q - q^{-1})r_{im}r_{jn}, \quad \forall \ i < j, \ n < m.
\end{align*}
\]

In \( H_l \) the commutation relations for the generators are given by:

\[
\begin{align*}
(6.23) \quad l_{in}l_{im} &= q l_{im}l_{in}, \quad \forall \ n < m; \quad l_{in}l_{jm} = q l_{jm}l_{in}, \quad \forall \ i < j; \\
(6.24) \quad l_{im}l_{jn} &= l_{jn}l_{im}; \quad l_{in}l_{jm} - l_{jm}l_{in} = (q - q^{-1})l_{jn}l_{im}, \quad \forall \ i < j \text{ and } n < m.
\end{align*}
\]

As well, for all \( i, j \), we have that

\[
(6.25) \quad l_1 l_2 \cdots l_N = r_1 r_2 \cdots r_N = \varepsilon \quad \text{and} \quad r_{ii} r_{jj} = l_{ii} l_{jj} = l_{jj} l_{ii} = r_{ij} r_{ji}.
\]

**Proof.** Relations (6.21) to (6.24) follow from (6.11) to (6.14) together with (6.12), (6.19), and Corollary 6.3. By the second relation in (6.24) we have that \( l_i l_{jj} - l_{jj} l_i = (q - q^{-1})l_{jj} l_i \) if \( i < j \), and \( l_{ji} l_{ii} - l_{ii} l_{ji} = (q - q^{-1})l_{ij} l_{ji} \) if \( i < j \). By (6.19), these are both 0. Therefore, \( l_i l_{jj} = l_{jj} l_i \), for all \( 1 \leq i, j \leq N \). \( \square \)

Note that (6.22) and (6.24) imply that

\[
(6.26) \quad l_{im}l_{in} = l_{in}l_{im} \quad \text{if } i < j, \ n < m \quad \text{and either } i > m \text{ or } j > n.
\]

We now describe another set of algebra generators for \( H_l \) and \( H_r \).

**Proposition 6.5.** Let \( 1 \leq i \leq N \) and \( 1 \leq s, t \leq N - 1 \) and define:

\[
\begin{align*}
\hat{K}_i &= l_i; \quad \hat{K}_i^{-1} := l_i^{-1}; \\
E_s &= l_{s+1,s+1}^{-1}l_{s+1,s}; \quad F_s := (q - q^{-1})^{-2}r_{s,s+1}^{-1}r_{s,s+1}^{-1}.
\end{align*}
\]

As an algebra \( H_l \) is generated by \( \hat{K}_i, \hat{K}_i^{-1} \), and the \( E_s \), with the following relations.

\[
\begin{align*}
(6.27) \quad \hat{K}_i \hat{K}_j &= \hat{K}_j \hat{K}_i, \quad \hat{K}_i \hat{K}_i^{-1} = \hat{K}_i^{-1} \hat{K}_i = \varepsilon, \quad \hat{K}_1 \hat{K}_2 \cdots \hat{K}_N = \varepsilon; \\
(6.28) \quad \hat{K}_i E_t \hat{K}_i^{-1} &= q^{\delta_{i,t} - \delta_{i,t+1}} E_t; \\
(6.29) \quad E_t E_s &= E_s E_t, \quad \text{if } |s - t| > 1; \\
(6.30) \quad E_s^2 E_t - (q + q^{-1})E_s E_t E_s + E_t E_s^2 = 0, \quad \text{if } |s - t| = 1.
\end{align*}
\]

**Similarly,** \( H_r \) is generated as an algebra by the \( \hat{K}_i \), \( \hat{K}_i^{-1} \), and the \( F_s \), with relations (6.27) and

\[
\begin{align*}
(6.31) \quad \hat{K}_i F_t \hat{K}_i^{-1} &= q^{\delta_{i,t+1} - \delta_{i,t}} F_t; \\
(6.32) \quad F_t F_s &= F_s F_t, \quad \text{if } |s - t| > 1; \\
(6.33) \quad F_s^2 F_t - (q + q^{-1})F_t F_s F_t + F_t F_s^2 = 0, \quad \text{if } |s - t| = 1.
\end{align*}
\]
Case III: If \( i < t \), then the map \( \hat{K}_i \mapsto \hat{K}_i^{-1} \), \( E_s \mapsto F_s \), is a well defined algebra isomorphism from \( H_t \) to \( H_r \).

Since \( \{ l_{ij} | i \geq j \} \) is a set of algebra generators for \( H_t \), it suffices to prove that for any \( 1 \leq i \leq N \) and \( 1 \leq j \leq N - i \) the element \( l_{i+j,i} \) can be written as a linear combination of products of the elements \( \hat{K}_1^{-1}, \cdots, \hat{K}_N^{-1}, E_1, \cdots, E_{N-2} \) to see that these elements generate. For \( 1 \leq i \leq N - 1 \), we have \( l_{i+1,i} = \hat{K}_{i+1} E_i = q^{-1} E_i \hat{K}_{i+1} \). Suppose that \( l_{i+j,i} \) can be written as a linear combination of products of the elements \( \hat{K}_1^{-1}, \cdots, \hat{K}_N^{-1}, E_1, \cdots, E_{N-1} \). By the second relation in (6.24),

\[
l_{i+j,i} = l_{i+j+1,i+j} - l_{i+j+1,i+j} = (q - q^{-1}) l_{i+j+1,i+j} \]

for all \( 1 \leq j \leq N - i - 1 \). Now we compute

\[
l_{i+j+1,i} = (q - q^{-1})^{-1} l_{i+j+1,i+j+1} l_{i+j+1,i+j} = l_{i+j+1,i+j}^{-1} l_{i+j+1,i+j}^{-1}.
\]

We now verify (6.28) case by case.

Case I: If \( i < t \) or \( i > t + 1 \), by (6.24) we have \( l_{i,i} l_{i+1,t} = l_{i+1,t} l_{i,i} \), and therefore

\[
\hat{K}_i E_i \hat{K}_i^{-1} = l_{i+1,t+1} l_{i+1,t} = E_t = q^{\delta_{i,t} - \delta_{i+1,t+1}} E_t.
\]

Case II: If \( i = t \), by (6.25) we have \( l_{i,i} l_{i+1,t} = q l_{i+1,t} l_{i,i} \), so that

\[
\hat{K}_t E_t \hat{K}_t^{-1} = q l_{i+1,t+1} l_{i+1,t} = q E_t = q^{\delta_{i,t} - \delta_{i+1,t+1}} E_t.
\]

Case III: If \( i = t + 1 \), again using (6.24) we have \( l_{i+1,i} l_{i+1,t+1} = q l_{i+1,t+1} l_{i+1,i} \), and

\[
\hat{K}_{i+1} E_t \hat{K}_{i+1}^{-1} = l_{i+1,t+1} l_{i+1,t+1} = q^{-1} E_t = q^{\delta_{i+1,t} - \delta_{i+1,t+1}} E_t.
\]

Now to verify (6.29), we first note that if \( t < s - 1 \), (6.26) implies that the pairs \( l_{i+1,t} \) and \( l_{s+1,s+1}, l_{i+1,s} \), and \( l_{s+1,s} \), and \( l_{s+1,s+1} \) commute and thus

\[
E_t E_r = E_r E_t = E_r E_t = E_r E_t.
\]

If \( t > s + 1 \) then \( s < t - 1 \), so interchanging \( s \) and \( t \) in the argument above we obtain \( E_r E_t = E_t E_r \).

It remains to prove the relation in (6.30). We first compute

\[
E_s E_{s+1} = 1_{s+1,s+1} l_{s+1,s} l_{s+1,s+1} l_{s+1,s+1} l_{s+1,s+2}.
\]

Using (6.24) we have

\[
l_{s+1,s+1} = q^{-1} l_{s+2,s+2} l_{s+1,s+1} l_{s+2,s+2}.
\]

Finally, using (6.20) and (6.24) we get

\[
l_{s+2,s+2} l_{s+2,s+2} l_{s+1,s+1} l_{s+2,s+2} = q^{-1} l_{s+2,s+2} l_{s+1,s+1} l_{s+2,s+2}.
\]
In other words, we have proved that
\begin{equation}
E_s E_{s+1} = q^{-1} E_{s+1} E_s + (q - q^{-1}) l_{s+2,s+2}^{1,s+2},
\end{equation}
for all $1 \leq s \leq N - 2$. Clearly, this is equivalent to

\begin{equation}
E_{s+1} E_s = q E_s E_{s+1} - q(q - q^{-1}) l_{s+2,s+2}^{1,s+2},
\end{equation}

for all $1 \leq s \leq N - 2$. Using the two relations above we obtain
\begin{align*}
E_s^2 E_{s+1} &= q^{-1} E_s E_{s+1} E_s + (q - q^{-1}) E_s l_{s+2,s+2}^{1,s+2}, \\
E_{s+1} E_s^2 &= q E_{s+1} E_s - q(q - q^{-1}) l_{s+2,s+2}^{1,s+2},
\end{align*}
from which we compute
\begin{align*}
E_s^2 E_{s+1} - (q^{-1} + q) E_s E_{s+1} E_s + E_s E_{s+1}^2 &= (q - q^{-1}) [E_s l_{s+2,s+2}^{1,s+2} - q l_{s+2,s+2}^{1,s+2} E_s].
\end{align*}

Now, since
\begin{align*}
E_s l_{s+2,s+2}^{1,s+2} &= l_{s+1,s+1}^{1,s+1} l_{s+2,s+2}^{1,s+2} \\
&= q l_{s+2,s+2}^{1,s+1} l_{s+1,s+1}^{1,s+1} \\
&= l_{s+2,s+2}^{1,s+1} l_{s+1,s+1}^{1,s+1} = q l_{s+2,s+2}^{1,s+1} E_s,
\end{align*}

it follows that $E_s^2 E_{s+1} - (q^{-1} + q) E_s E_{s+1} E_s + E_s E_{s+1}^2 = 0$, for all $1 \leq s \leq N - 2$.
Similarly, again using (6.34) and (6.35) we have
\begin{align*}
E_{s+1} E_s &= q E_{s+1} E_s E_{s+1} - q(q - q^{-1}) E_{s+1} l_{s+2,s+2}^{1,s+2}, \\
E_s E_{s+1}^2 &= q^{-1} E_{s+1} E_s E_{s+1} + (q - q^{-1}) l_{s+2,s+2}^{1,s+2},
\end{align*}
and therefore
\begin{align*}
E_s^2 E_{s+1} - (q + q^{-1}) E_{s+1} E_s E_{s+1} + E_s E_{s+1}^2 &= (q - q^{-1}) [E_s l_{s+2,s+2}^{1,s+2} E_{s+1} - q E_{s+1} l_{s+2,s+2}^{1,s+2}].
\end{align*}

Now, we have
\begin{align*}
q E_{s+1} l_{s+2,s+2}^{1,s+2} &= q^2 l_{s+2,s+2}^{1,s+2} l_{s+2,s+2}^{1,s+2} \\
&= q l_{s+2,s+2}^{1,s+2} l_{s+2,s+2}^{1,s+2} \\
&= l_{s+2,s+2}^{1,s+2} l_{s+2,s+2}^{1,s+2} = l_{s+2,s+2}^{1,s+2} E_{s+1},
\end{align*}

and therefore $E_s^2 E_{s+1} - (q + q^{-1}) E_{s+1} E_s E_{s+1} + E_s E_{s+1}^2 = 0$, for all $1 \leq s \leq N - 2$, and this finishes the proof.

Next we fix some notation:

**Definition 6.6.** For any $x$ and $y$, let $[x, y]_q := qxy - yx$.

(i) For $1 \leq i \leq j \leq N$, define $F_{i,i} := 1$, $F_{i,i+1} := q(1 - q^{-2})^2 F_i$, and
\begin{equation}
F_{i,j} := q(1 - q^{-2})^{j-i+1} [F_{j-1}, F_{j-2}, \cdots, F_{i+1}, F_i]_q \cdot \cdots \cdot [F_{j-1}, F_{j-2}, \cdots, F_{i+1}, F_i]_q q, \quad \text{for } j \geq i + 2,
\end{equation}

i.e., the $F_{i,j}$ are defined inductively by $F_{i,j+1} = (1 - q^{-2}) [F_j, F_{i,j}]_q$.

(ii) Similarly, define $E_{i,j} := 1$, $E_{i+1,j} := q^{-1} E_j$ and,
\begin{equation}
E_{i,j} := q^{-1} (q - 1)^{j-i} [E_j, E_{j+1}, E_{j+2}, \cdots, E_{i-1}]_q, \quad \text{for } i \geq j + 2,
\end{equation}

i.e., the $E_{i,j}$ are defined inductively by $E_{i+1,j} = (q^2 - 1)^{-1} [E_{i,j}, E_i]_q$. 


Now, using Proposition \(6.5\), \(I_{ij}\) and \(r_{ij}\) can be expressed in terms of the new generators \(\hat{K}^{\pm 1}_i\), \(E_s\) and \(F_s\) in the spirit of Serre relations.

**Corollary 6.7.** For \(1 \leq i \leq j \leq N\), we have that \(I_{ij} = E_{ji} \hat{K}_j\), and \(r_{ij} = F_{ij} \hat{K}_i\).

**Proof.** We prove the statement for \(r_{ij}\) by mathematical induction on \(j\); the similar proof for \(I_{ij}\) is left to the reader. If \(j = i\), the statement is clear. If \(j = i + 1\), then the statement follows from Lemma \(6.4\) and the definitions. Now assume that \(r_{ij} = F_{ij} \hat{K}_i\) for \(r_{ij}\) with \(j \geq i + 1\) and we show that it holds for \(r_{ij+1}\). By \(6.22\), we have

\[
r_{ij+1}r_{ij} - r_{ij+1}r_{ij} = (q^{-1})r_{ij+1}r_{ij},
\]

and thus

\[
r_{ij+1} = (q^{-1})^{-1} [r_{ij+1}r_{ij} - r_{ij+1}r_{ij}]
\]

\[
= (q^{-1})^{-1} [qr_{ij+1}r_{ij}^{-1}r_{ij} - q^{-1}r_{ij+1}r_{ij}^{-1}r_{ij+1}]
\]

\[
= (q^{-1})[F_{ij}r_{ij} - q^{-1}r_{ij}F_{ij}]
\]

\[
= (1 - q^{-2})[F_{ij}, r_{ij}]_q
\]

\[
= (1 - q^{-2})[F_{ij}, F_{ij}]_q \hat{K}_i
\]

\[
= (1 - q^{-2})F_{ij+1} \hat{K}_i,
\]

and the proof is complete. \(\square\)

The next proposition describes the comultiplication, counit and antipode for \(H_l\) and \(H_r\).

**Proposition 6.8.** The coalgebra structure \(\Delta\), \(\varepsilon\) and the antipode \(S\) for \(H_l\) are determined by

\[
\Delta(\hat{K}^{\pm 1}_i) = \hat{K}^{\pm 1}_i \otimes \hat{K}^{\pm 1}_i,\quad \varepsilon(\hat{K}^{\pm 1}_i) = 1,
\]

\[
\Delta(E_s) = \varepsilon \otimes E_s + E_s \otimes \hat{K}^{-1}_{s+1} \hat{K}_s,\quad \varepsilon(E_s) = 0,
\]

\[
S(\hat{K}^{\pm 1}_i) = \hat{K}^{\mp 1}_i,\quad S(E_s) = -E_s \hat{K}^{-1}_{s+1} \hat{K}_s,
\]

for \(1 \leq i \leq N\) and \(1 \leq s \leq N - 1\). Similarly, the coalgebra structure \(\Delta\), \(\varepsilon\) and the antipode \(S\) for \(H_r\) are determined by \(6.30\), the first equality in \(6.38\) and

\[
\Delta(F_s) = F_s \otimes \varepsilon + \hat{K}_{s+1} \hat{K}^{-1}_s \otimes F_s,\quad \varepsilon(F_s) = 0,
\]

\[
S(F_s) = -\hat{K}_s \hat{K}^{-1}_{s+1} F_s.
\]

**Proof.** We give the details for \(E_s\) and leave those for \(F_s\) to the reader. Also, we note that the algebra isomorphism \(\hat{K}_i \rightarrow \hat{K}^{-1}_i\), \(E_s \rightarrow F_s\), defined in the proof of Proposition \(6.5\) is actually a Hopf algebra isomorphism between \(H_l\) and \(H_r^{\text{op}}\).

Now, we compute:

\[
\Delta(E_s) = \Delta(I_{s+1,s+1}^{-1} I_{s+1,s}) = (I_{s+1,s+1}^{-1} \otimes I_{s+1,s+1}^{-1}) \sum_{k=s}^{s+1} I_{s+1,k} \otimes I_{k,s}
\]

\[
= (I_{s+1,s+1}^{-1} \otimes I_{s+1,s+1}^{-1})(I_{s+1,s} \otimes I_{s+1,s+1} \otimes I_{s+1,s})
\]

\[
= E_s \otimes \hat{K}^{-1}_{s+1} \hat{K}_s + \varepsilon \otimes E_s.
\]
as needed. Also, \( \varepsilon(E_s) = \varepsilon(I_{s+1,s+1}^{-1} I_{s+1,s}) = \delta_{s+1,s} = 0 \), for all \( 1 \leq s \leq N-1 \). The formula for \( S(E_s) \) is then immediate.

**Remark 6.9.** For \( 1 \leq i \leq N-1 \) we now define \( K_i := \hat{K}_{i+1}^{-1} \hat{K}_i = \hat{K}_i \hat{K}_{i+1}^{-1} \), and then define \( H_i' \) to be the subalgebra of \( H_1 \) generated by \( \{ K_i^{\pm 1}, E_i \mid 1 \leq i \leq N-1 \} \). From Proposition 6.5, one can easily see that the relations between the algebra generators of \( H_i' \) are \( (6.29), (6.30) \) and

\[
K_i^{\pm 1} K_i^{\mp 1} = \varepsilon, \ K_i K_j = K_j K_i, \ K_i E_j K_i^{-1} = q^{a_{ij}} E_j,
\]

for all \( 1 \leq i, j \leq N-1 \), where \( a_{ii} = 2, a_{ij} = -1 \) if \( |i-j| = 1 \), and \( a_{ij} = 0 \) if \( |i-j| > 1 \). Actually, \( H_i' \) is a Hopf subalgebra of \( H_1 \). The induced Hopf algebra structure is given by

\[
\Delta(K_i^{\pm 1}) = K_i^{\pm 1} \otimes K_i^{\mp 1}, \ \varepsilon(K_i^{\pm 1}) = 1,
\]

\[
\Delta(E_i) = \varepsilon \otimes E_i + E_i \otimes K_i, \ \varepsilon(E_i) = 0,
\]

\[
S(K_i^{\pm 1}) = K_i^{\mp 1}, \ S(E_i) = -E_i K_i^{-1},
\]

for all \( 1 \leq i \leq N-1 \). Similarly, define \( H'_s \) to be the subalgebra of \( H_1 \) generated by \( \{ K_i^{\pm 1}, F_i \mid 1 \leq i \leq N-1 \} \). Then its algebra generators satisfy the relations in \( (6.32), (6.33) \), the first two relations in \( (6.41) \) and

\[
K_i F_j K_i^{-1} = q^{-a_{ij}} F_j, \ \forall 1 \leq i, j \leq N-1,
\]

where \( a_{ij} \) are the scalars defined above. Moreover, \( H'_s \) is a Hopf subalgebra of \( H_1 \).

The elements \( K_i^{\pm 1} \) are grouplike elements and

\[
\Delta(F_i) = K_i^{-1} \otimes F_i + F_i \otimes \varepsilon, \ \varepsilon(F_i) = 0, \ S(F_i) = -K_i F_i,
\]

for all \( 1 \leq i \leq N-1 \). Also, the map \( K_i \mapsto K_i^{-1}, E_i \mapsto F_i \) is well defined and a Hopf algebra isomorphism from \( H'_i \) to \( H_i^{\text{cop}} \).

We can now describe the Hopf algebra structure of \( H_\sigma := H_1 H_\tau \).

**Proposition 6.10.** The Hopf algebra \( H_\sigma := H_1 H_\tau \) can be presented as follows. It is the algebra generated by \( \{ \hat{K}_i^{\pm 1} \mid 1 \leq i \leq N \} \cup \{ E_s, F_s \mid 1 \leq s \leq N-1 \} \) with relations \( (6.27)-(6.33) \) and

\[
E_s F_t - F_t E_s = \delta_{s,t}(q - q^{-1})^{-1}[\hat{K}_s \hat{K}_{s+1}^{-1} - \hat{K}_{s}^{-1} \hat{K}_{s+1}],
\]

for all \( 1 \leq s, t \leq N-1 \). The Hopf algebra structure of \( H_\sigma \) is given by the relations in Proposition 6.8. In other words, \( H_\sigma = U_q^{\text{ext}}(sl_N) \), the extended Hopf algebra of \( U_q(sl_N) \) defined in [3, Section 8.5.3].

**Proof.** It remains only to prove \( (6.44) \). It suffices to show first that \( (6.44) \) holds when applied to \( x_{mn} \) for all \( 1 \leq m, n \leq N \), and then to show that if the maps in \( (6.44) \) are equal on \( a \) and \( b \) then they are also equal on \( ab \).

The relations from Lemma 5.2 will be key in the following computations. Also we will sometimes use the notation \( K_s = \hat{K}_s \hat{K}_{s+1}^{-1} \) from Remark 6.9.

Recall that \( \hat{K}_i(x_{mn}) = I_{i_i}(x_{mn}) = z q^\delta_{i,m} \delta_{m,n} \) by \( (6.18) \) and \( \hat{K}_i^{-1}(x_{mn}) = I_{i_i}^{-1}(x_{mn}) = z^{-1} q^{-\delta_{i,m}} \delta_{m,n} \) by \( (6.20) \). Thus,

\[
E_s(x_{mn}) = I_{s+1,s+1}^{-1} I_{s+1,s}(x_{mn}) = \sum_{k=1}^{N} I_{s+1,s+1}^{-1}(x_{mk}) I_{s+1,s}(x_{kn})
\]

\[
= q^{-\delta_{s+1,n}}(q - q^{-1}) \delta_{s+1,n} \delta_{s,m} = (q - q^{-1}) \delta_{s+1,n} \delta_{s,m}.
\]
Similarly, we compute $F_t(x_{mn}) = (q - q^{-1})^{-1}\delta_{t,n}\delta_{t+1,m}$, and therefore

$$E_s F_t(x_{mn}) = \sum_{k=1}^{N} E_s(x_{mk}) F_t(x_{kn}) = (q - q^{-1})\delta_{s,m} F_t(x_{s+1,n}) = \delta_{s,m} \delta_{t,n} \delta_{s,t} = \delta_{s,m} \delta_{s,t} \delta_{m,n},$$

and

$$F_t E_s(x_{mn}) = \sum_{k=1}^{n} F_t(x_{mk}) E_s(x_{kn}) = (q - q^{-1})^{-1} \delta_{t+1,m} E_s(x_{1n}) = \delta_{t+1,m} \delta_{s+1,n} \delta_{s,t} = \delta_{s+1,m} \delta_{s,t} \delta_{m,n}.$$ 

We conclude that

$$(6.45) \quad (E_s F_t - F_t E_s)(x_{mn}) = \delta_{s,t} [\delta_{s,m} - \delta_{s+1,m}] \delta_{m,n}.$$ 

On the other hand, we have

$$\tilde{K}_s \tilde{K}_{s+1}^{-1}(x_{mn}) = \sum_{k=1}^{N} \tilde{K}_s(x_{mk}) \tilde{K}_{s+1}^{-1}(x_{kn}) = z q^{\delta_{s,m} \delta_{m,k} - 1} q^{-\delta_{s+1,m}} \delta_{k,n} = q^{\delta_{s,m} - \delta_{s+1,m}} \delta_{m,n}.$$ 

Similarly, $\tilde{K}_s \tilde{K}_{s+1}^{-1}(x_{mn}) = q^{\delta_{s+1,m} - \delta_{s,m}} \delta_{m,n}$. Thus

$$[\tilde{K}_s \tilde{K}_{s+1}^{-1} - \tilde{K}_s^{-1} \tilde{K}_{s+1}](x_{mn}) = [q^{\delta_{s,m} - \delta_{s+1,m}} - q^{\delta_{s+1,m} - \delta_{s,m}}] \delta_{m,n}.$$ 

Since

$$q^{\delta_{s,m} - \delta_{s+1,m}} - q^{\delta_{s+1,m} - \delta_{s,m}} = \begin{cases} 0 & \text{if } m \not\in \{s, s+1\} \\ q - q^{-1} & \text{if } m = s \\ -(q - q^{-1}) & \text{if } m = s + 1 \end{cases},$$

it follows that $\delta_{s,t} [q - q^{-1}]^{-1}[K_s - K_s^{-1}](x_{mn}) = \delta_{s,t} [\delta_{s,m} - \delta_{s+1,m}] \delta_{m,n}$. Together with (6.44) this proves that (6.41) holds on generators.

Since the $K_s$ are grouplike, the right hand side of (6.44) is $(K_s, K_s^{-1})$-primitive. Also the left hand side is $(K_s, K_s^{-1})$-primitive. This follows from the fact that the matrix $(a_{ij})_{1 \leq i, j \leq N}$ defined in Remark 6.3 is symmetric, and thus

$$K_t E_s \otimes F_t K_s = q^{-a_{st}} E_s K_t^{-1} \otimes q^{a_{st}} K_s F_t = E_s K_t^{-1} \otimes K_s F_t.$$ 

If $s \neq t$, then both sides of (6.44) are 0 on generators $a, b \in SL_q(N)$, and thus, being skew-primitive, on $ab$. Otherwise, both sides are $(K_s, K_s^{-1})$-primitive and thus equal on the product $ab$. The statement then follows by induction. 

From Proposition 6.8 and Proposition 6.10 we conclude that the Hopf algebras $H_t$ and $H_s$ are precisely $U_q(b_+)$ and $U_q(b_-)$, the Borel-like Hopf algebras associated to $U_q^{\text{ext}}(SL_N)$. The fact that $U_q(b_+)^{\text{cop}}$ and $U_q(b_-)$ are Hopf algebras in duality is well known but using the general theory above we are now able to present a more conceptual proof.

**Corollary 6.11.** The pairing $\langle , \rangle : U_q(b_+) \otimes U_q(b_-) \to k$ defined by

$$\langle \tilde{K}_i, \tilde{K}_j \rangle = \langle \tilde{K}_i^{-1}, \tilde{K}_j^{-1} \rangle = z q^{\delta_{i,j}}, \quad \langle \tilde{K}_i, \tilde{K}_j^{-1} \rangle = \langle \tilde{K}_i^{-1}, \tilde{K}_j \rangle = z^{-1} q^{-\delta_{i,j}},$$

$$\langle \tilde{K}_i, E_s \rangle = \langle \tilde{K}_i^{-1}, E_s \rangle = 0, \quad \langle F_s, \tilde{K}_j \rangle = \langle F_s, \tilde{K}_j^{-1} \rangle = 0,$$

$$\langle F_s, E_t \rangle = (q - q^{-1})^{-1} \delta_{s,t},$$
for all $1 \leq i, j \leq N$ and $1 \leq s, t \leq N - 1$, defines a duality between $U_q(b_+)^{\text{cop}}$ and $U_q(b_-)$ as in Proposition 4.4.

Also, $U_q(b_-)$ is self dual, in the sense that there is a duality between $U_q(b_-)$ and itself given for all $1 \leq i, j \leq N$ and $1 \leq s, t \leq N - 1$ by

$$
\langle \hat{K}_i, \hat{K}_j \rangle = \langle \hat{K}_i^{-1}, \hat{K}_j^{-1} \rangle = z^{-1}q^{\delta_{i,j}}, \quad \langle \hat{K}_i, \hat{K}_j^{-1} \rangle = \langle \hat{K}_i^{-1}, \hat{K}_j \rangle = zq^{\delta_{i,j}},
$$

$$
\langle \hat{K}_i, E_s \rangle = \langle \hat{K}_i^{-1}, E_t \rangle = \langle E_s, \hat{K}_i \rangle = \langle E_t, \hat{K}_i^{-1} \rangle = 0, \quad \langle E_s, E_t \rangle = (q - q^{-1})^{-1}\delta_{s,t}.
$$

Proof. The first set of equations follows from Proposition 4.4 and from (6.8) - (6.11). We verify only the third equation of the first set. Note first that $\langle r_{ss}, 1_{ij} \rangle = 0$ unless $i = j$ by (6.8) - (6.11) so that $\langle r_{ss}^{-1}, E_i \rangle = 0$. Now we compute

$$
\langle F_s, E_t \rangle = (q - q^{-1})^{-2}\langle r_{ss}^{-1} r_{s,s+1}, E_t \rangle
$$

$$
= (q - q^{-1})^{-2}\langle [r_{ss}^{-1}, \langle r_{s,s+1}, E_t \rangle + \langle r_{r_{s,s+1}, \hat{K}_t^{-1}\hat{K}_t} \rangle
$$

$$
= (q - q^{-1})^{-2}\langle r_{s,s+1}, l_{t+1,t+1}^{-1}r_{t+1,t} \rangle
$$

$$
= (q - q^{-1})^{-2}\langle r_{s,s+1} \rangle l_{t+1,t}^{-1}(x_{s,s+1})
$$

$$
= (q - q^{-1})^{-2}z^{-1}q^{\delta_{s,t+1}}z(q - q^{-1})^{-1}\delta_{s,t},
$$

as we claimed. The second set of equations follows from the isomorphism between $H_r^{\text{cop}}$ and $H_l$ described at the beginning of the proof of Proposition 6.8. □

Corollary 6.12. The Hopf algebra $U_q^{\text{ext}}(s\lambda_N)$ is isomorphic to a factor of the generalized quantum double $D(U_q(b_+), U_q(b_-)) \equiv D(U_q(b_+)^{\text{cop}}, U_q(b_-))$.

Proof. It is an immediate consequence of Corollary 6.11 and Proposition 4.5. □

Remark 6.13. Further to Remark 6.9 define $H'_p = H'_p[H'_p] = H'_p H'_p$, a Hopf subalgebra of $H_p \equiv U_q^{\text{ext}}(s\lambda_N)$. As expected, $H'_p \equiv U_q(s\lambda_N)$, the Hopf algebra with algebra generators $\{ K_i^{\pm1}, E_i, F_i \mid 1 \leq i \leq N - 1 \}$ and relations from Remark 6.9 as well as

$$
E_i F_j - F_j E_i = \delta_{ij} K_i - K_i^{-1} / q - q^{-1}, \quad \forall 1 \leq i, j \leq N - 1,
$$

from Proposition 6.11. The comultiplication, the counit and the antipode are defined in Remark 6.9. (For more details see [8], VI.7 & VII.9.)

Note that $H'_p$ and $H'_l$ are exactly the Borel-like Hopf algebras associated to $U_q(s\lambda_N)$ and the situation is similar to that of Corollary 6.11 i.e., there is a duality on $H'_p^{\text{cop}} \otimes H'_l$ given by

$$
\langle K_i, K_j \rangle = \langle K_i^{-1}, K_j^{-1} \rangle = q^{a_{ij}}, \quad \langle K_i^{-1}, K_j \rangle = \langle K_i, K_j^{-1} \rangle = q^{-a_{ij}},
$$

$$
\langle K_i, E_s \rangle = \langle K_i^{-1}, E_t \rangle = \langle F_s, K_i \rangle = \langle F_t, K_i^{-1} \rangle = 0, \quad \langle F_s, E_t \rangle = (q - q^{-1})^{-1}\delta_{s,t},
$$

for all $1 \leq i, j \leq N$ and $1 \leq s, t \leq N - 1$. Moreover, since $H'_p^{\text{cop}}$ is isomorphic to $H'_l$, then $H'_p$ is self dual and the generalized quantum double $D(H'_p, H'_l) \equiv D((H'_p)^{\text{cop}}, H'_l)$ Furthermore, one can easily see that the associated generalized quantum double $D(H'_p, H'_l)$ can be identified as a sub-Hopf algebra of $D(H_r, H_l)$.
hence we have a Hopf algebra morphism from $D(H'_1, H'_1)$ to $H_\sigma$. The image of this morphism is $H'_\sigma$, so $U_q(sl_N)$ is also a factor of a generalized quantum double.

**Corollary 6.14.** The evaluation pairings on $U_q^{ext}(sl_N) \otimes SL_q(N)$ and $U_q(sl_N) \otimes SL_q(N)$, are given by

\[\langle \tilde{K}_i, x_{mn} \rangle = z^q \delta_{i,m} \delta_{m,n}, \quad \langle \tilde{K}_i^{-1}, x_{mn} \rangle = z^{-1} q^{-\delta_{i,m}} \delta_{m,n},\]

\[\langle E_s, x_{mn} \rangle = (q - q^{-1}) \delta_{s+1,n} \delta_{s,m}, \quad \langle F_s, x_{mn} \rangle = (q - q^{-1})^{-1} \delta_{s,n} \delta_{s+1,m};\]

and

\[\langle K_i, x_{mn} \rangle = q^\delta_{i,m} - \delta_{i+1,m} \delta_{m,n}, \quad \langle K_i^{-1}, x_{mn} \rangle = q^{\delta_{i+1,m} - \delta_{i,m}} \delta_{m,n},\]

\[\langle E_s, x_{mn} \rangle = (q - q^{-1}) \delta_{s+1,n} \delta_{i,m}, \quad \langle F_s, x_{mn} \rangle = (q - q^{-1})^{-1} \delta_{s,n} \delta_{i+1,m},\]

respectively.

**Proof.** Both $U_q^{ext}(sl_N)$ and $U_q(sl_N)$ are Hopf subalgebras of $SL_q(N)^0$, so the evaluation map gives dual pairings. The formulas for the pairings come directly from Lemma 6.2 or from the proof of Proposition 6.10.

From Theorem 4.8 $SL_q(N)$ is a braided Hopf algebra in the braided category of left Yetter-Drinfeld modules over $U_q^{ext}(sl_N)^{\text{cop}}$. We end this paper by computing the structures of the braided Hopf algebra $SL_q(N)$. First we need the following.

**Lemma 6.15.** For any $1 \leq s \leq N - 1$ and $1 \leq m, n \leq N$ we have

\[1_{s+1,s}(S(x_{mn})) = -z^{-1}(q - q^{-1}) \delta_{s+1,n} \delta_{s,m},\]

\[r_{s,s+1}(S(x_{mn})) = -z^{-1} q^{-2}(q - q^{-1}) \delta_{s,n} \delta_{s+1,m}.\]

**Proof.** As usual, we only prove (6.46); (6.47) can be proved similarly. Recall that $\sigma^{-1}$ is obtained by replacing $q$ by $q^{-1}$ and $z$ by $z^{-1}$ in (6.46) and (6.47). We compute

\[1_{s+1,s}(S(x_{mn})) = \sigma(S(x_{mn}), x_{s+1,s}) = \sigma^{-1}(x_{mn}, x_{s+1,s}) = z^{-1}[q^{-\delta_{m,r}} + \delta_{n,r}] + [s + 1 > m](q - q^{-1}) \delta_{s+1,n} \delta_{s,m}\]

\[= -z^{-1}(q - q^{-1}) \delta_{s+1,n} \delta_{s,m},\]

since $[s + 1 > m] \delta_{s,m} = \delta_{s,m}$, so the proof is complete.

Now we can describe concretely the left Yetter-Drinfeld module structure of $SL_q(N)$ over $U_q^{ext}(sl_N)^{\text{cop}}$.

**Proposition 6.16.** $SL_q(N)$ is a left Yetter-Drinfeld module over $U_q^{ext}(sl_N)^{\text{cop}}$ via the structure

\[\tilde{K}_i \triangleright x_{mn} = q^\delta_{i,m} - \delta_{i+1,m} x_{mn}, \quad \tilde{K}_i^{-1} \triangleright x_{mn} = q^{\delta_{i,m} - \delta_{i+1,m}} x_{mn},\]

\[E_s \triangleright x_{mn} = (1 - q^{-2}) [q^{-1} \delta_{s+1,n} x_{ms} - q^{-1} \delta_{s,m} x_{s+1,m}],\]

\[F_s \triangleright x_{mn} = q(1 - q^{-2})^{-1} [q^\delta_{s,m} - \delta_{s+1,m} x_{ms} - q^{-1} \delta_{s+1,m} x_{ms+1}],\]

\[x_{mn} \mapsto \sum_{j=n}^{N} q^{2(j-n)} E_{j,n} \tilde{K}_j \left( \tilde{K}_m^{-1} \otimes x_{mj} \right) + \sum_{1 \leq i \leq m-1} \sum_{p \in \mathbb{B}_{i,m}} (-q)^{-l(p)+(m-i)} E_{m,p(m)} \cdots E_{i+1,p(i+1)} \tilde{K}_i^{-1} \otimes x_{ij},\]
where the $E_{i,j}$ are from Definition 6.6. Also for $i \leq m-1$, $B_{i,m}$ denotes the set of bijective maps $p: \{i+1, \ldots, m\} \to \{i, \ldots, m-1\}$ such that $p(k) \leq k$, for all $i+1 \leq k \leq m$.

Proof. We apply Proposition 4.17 to this setting. By (6.13) and (6.20) that $\hat{K}_i^{\pm 1} (x_{mn}) = (z q^{\delta_{i,n}})^{\pm 1} \delta_{i,n}$ for $1 \leq i, m, n \leq N$, we have

$$\hat{K}_i^{\pm 1} \triangleright x_{mn} = \sum_{j,k=1}^{N} \hat{K}_i^{\pm 1} (S^{-1} (x_{mj}) S^{-2} (x_{kn})) x_{jk} = \sum_{j,k=1}^{N} \hat{K}_i^{\pm 1} (x_{mj}) \hat{K}_i^{\pm 1} (x_{kn}) x_{jk}$$

$$= q^{\pm (\delta_{i,n} - \delta_{i,m})} x_{mn}.$$ 

Next, using (6.46), we compute

$$E_s (S(x_{mn})) = \sum_{i,j=1}^{N} E_s (S^{-1} (x_{mi}) S^{-2} (x_{jn})) x_{ij}$$

$$= \sum_{i,j=1}^{N} \left[ q^{2(j-n)} \delta_{m,i} E_s (x_{jn}) + q^{2(m-i)+2(j-n)} E_s (S(x_{mi})) \hat{K}_{s+1}^{-1} \hat{K}_s (x_{jn}) \right] x_{ij}$$

$$= q^{-1} (q - q^{-1}) [q^{-1} \delta_{s+1,n} x_{ms} - \delta_{s,m} \sum_{j=1}^{N} q^{2(j-n)} K_s (x_{jn}) x_{s+1j}]$$

$$= (1 - q^{-2}) [q^{-1} \delta_{s+1,n} x_{ms} - \delta_{s,m} \delta_{s+1,n} x_{s+1n}],$$

as claimed. Similarly, one can compute $E_s \triangleright x_{mn}$; the verification of the details is left to the reader.

Finally, from (6.13) and (6.14), we see that the coaction of $U_q^{\text{ext}}(sl_N)^{\text{cop}}$ on $\text{SL}_q(N)$ is defined by

$$x_{mn} \mapsto \sum_{i,j=1}^{N} 1_{S^{-1} (x_{mi}) S^{-2} (x_{jn})} \otimes x_{ij} = \sum_{1 \leq i,j \leq N} q^{2(j-n)+2(m-i)} 1_{j} 1_{i} 1_{S(x_{mi})} \otimes x_{ij}.$$ 

From (6.13) we have

$$1_{S(x_{mi})} = (-q)^{-m} \sum_{p \in S_{i,m}} (-q)^{-l(p)} 1_{f_1 \cdot p(1) \cdot \ldots \cdot f_{i-1} \cdot p(i-1) \cdot f_{i+1} \cdot p(i+1) \cdot \ldots \cdot f_N \cdot p(N)}$$

$$= (-q)^{-m} \sum_{p \in S_{i,m}} (-q)^{-l(p)} 1_{p(N)} \cdot \ldots \cdot 1_{i+1,p(i+1)} 1_{i-1,p(i-1)} \cdot \ldots \cdot 1_{1,p(1)},$$

and this forces $k \geq p(k)$, for any $k \neq i$. Since $p$ is bijective, non-zero summands occur only when
(i) $i = m$ and $p(k) = k$, for all $k \neq i$;
(ii) $i \leq m - 1$ and $p(1) = 1, \ldots, p(i - 1) = i - 1$, $\{p(i + 1), \ldots, p(m)\} = \{i, \ldots, m - 1\}$, $p(m + 1) = m + 1, \ldots, p(N) = N$, and $p(k) \leq k$, for any $k \in \{i + 1, \ldots, m\}$.

In other words, we have proved that $I_{S(x,m)} = I_{mm}^{-1}$, and that for $1 \leq i \leq m - 1$,

$$I_{S(x,m)} = (-q)^{i-m} \sum_{p \in B_{i,m}} (-q)^{-l(p)} l_{N} \cdots l_{m+1,m+1} l_{m,p(m)} \cdots l_{i+1,p(i+1)} l_{i-1,i-1} \cdots l_{11}.$$  

From (6.29), for $i + 1 \leq k \leq m$, $m + 1 \leq t \leq N$ and $1 \leq s \leq i - 1$ we have

$$I_{kp(k)} l_{tt} = l_{tt} I_{kp(k)}$$

and $I_{ss} I_{kp(k)} = I_{kp(k)} I_{ss}$, so that, using (6.29), we have

$$I_{S(x,m)} = (-q)^{i-m} \sum_{p \in B_{i,m}} (-q)^{-l(p)} l_{mp(m)} \cdots l_{i+1,p(i+1)} l_{i}^{-1} \cdots l_{mm}^{-1},$$

for $1 \leq i \leq m - 1$. Now, by Corollary 6.7 we have $I_{kp(k)} = \mathcal{E}_{k,p(k)} \tilde{K}_{k}$, for all $k \in \{i + 1, \ldots, m\}$. In particular, if $i = m - 1$ we then have

$$I_{S(x,m)} = -q^{-1} l_{mp(m)} \tilde{K}_{m-1}^{-1} \tilde{K}_{m}^{-1} = -q^{-1} \mathcal{E}_{m,p(m)} \tilde{K}_{i}^{-1}.$$

Now let $1 \leq i \leq m - 2$. From (6.28) it follows that $\tilde{K}_{s} E_{t} = E_{t} \tilde{K}_{s}$, for all $s > t + 1$, so from the definition of $\mathcal{E}_{i,j}$ we conclude that $\tilde{K}_{m-i} \mathcal{E}_{k,p(k)} = \mathcal{E}_{k,p(k)} \tilde{K}_{m-t}$, for all $0 \leq t \leq m - i - 2$ and $i + 1 \leq k \leq m - t - 1$. Consequently,

$$I_{S(x,m)} = (-q)^{i-m} \sum_{p \in B_{i,m}} (-q)^{-l(p)} \mathcal{E}_{m,p(m)} \cdots \mathcal{E}_{i+1,p(i+1)} \tilde{K}_{i}^{-1},$$

for all $1 \leq i \leq m - 1$. Again using Corollary 6.7 for $l_{jn}$ and the formulas above for $I_{S(x,m)}$ we obtain the coaction in the statement of the theorem.

We now apply the results of Theorem 4.15 to obtain the multiplication, comultiplication, unit, counit and antipode for $SL_{q}(N)$.

**Theorem 6.17.** The structure of $SL_{q}(N)$ as a braided Hopf algebra in the category of left $U_{q}^{\text{ext}}(\mathfrak{sl}_{N})^{\cop}$ Yetter-Drinfeld modules is the following. The unit and counit are as in $SL_{q}(N)$. The multiplication is defined by

$$x_{im} \circ x_{jn} = q^{\delta_{i,n} - \delta_{m,n}} x_{im} x_{jn} + (q - q^{-1})^{2} \delta_{i,n} \sum_{s > n} q^{2(s-i) - \delta_{s,m}} x_{sm} x_{js}\$$

$$-[m > n] q^{\delta_{i,n}(q - q^{-1})} x_{im} x_{jm} - [m > n + 1] (q - q^{-1})^{2} \delta_{i,n} \sum_{n < s \leq m} q^{2(s-i)} x_{ss} x_{jm}.$$
Comultiplication is given by

\[ \Delta(x_{im}) = q^{-2N-1}(q^{\delta_{i,m}} \sum_s q^{2s+\delta_{s,i}+\delta_{s,m}} x_{is} \otimes x_{sm} + (q - q^{-1})[x_{im} \otimes x_{im}^m]
\]

\[ -\delta_{i,m} \sum_{s,t > i} q^{2(s+\delta_{s,i})} x_{ts} \otimes x_{st} + x_{im}^i \otimes x_{im} + (q - q^{-1})^2 \sum_{s > i,m} q^{2s-1} x_{sm} \otimes x_{is}
\]

\[ -[i > m]q^{2i+1} \sum_{s > i} x_{sm} \otimes x_{is} - [m > i]q^{2m+1} \sum_{s > m} x_{sm} \otimes x_{is}
\]

\[ -(q - q^{-1})^3 \sum_{t > s > i,m} q^{2s} x_{st} \otimes x_{it}, \]

where we denoted \( x_{+}^{k} := \sum_{s > k} q^{2s} x_{ss} \). The antipode is determined by

\[ S(x_{im}) = q^{2N+1}[q^{-2m-\delta_{i,m}} S(x_{im}) - (q - q^{-1})\delta_{i,m} S(x_{im}^{-m})], \]

where \( x_{im}^{-m} := \sum_{s > m} q^{-2s} x_{ss} \) and \( S \) is the antipode of \( \text{SL}_q(N) \).

Proof. From Lemma 6.2 we have

\[ I_{ij}(x_{mn}) = zq^{\delta_{m,i}\delta_{m,n}\delta_{i,j}} + [i > j]z(q - q^{-1})\delta_{n,i}\delta_{m,j}, \]

\[ I_{ij}(x_{mn}) = z^{-1}q^{-\delta_{m,i}\delta_{m,n}\delta_{i,j}} - [i > j]z^{-1}(q - q^{-1})\delta_{n,i}\delta_{m,j}, \]

where \( I_{ij} \) denotes the map \( \sigma^{-1}(\cdot, x_{ij}) = I_{j^{-1}(x_{ij})} \).

Now, the multiplication of \( \text{SL}_q(N) \) from Theorem 4.3 can be rewritten as

\[ x \circ y = \sigma^{-1}(x_3, y_2)\sigma(S^2(x_1), y_3)x_2y_1, \]

and together with 6.14 this allows us to compute

\[ x_{im} \circ x_{jn} = \sum_{s,t,u,v=1}^N q^{2(s-i)} \sigma^{-1}(x_{im}, x_{uv})\sigma(x_{is}, x_{vn})x_{st}x_{ju}
\]

\[ = \sum_{s,t,u,v=1}^N q^{2(s-i)}I_{uv}(x_{tm})I_{vn}(x_{is})x_{st}x_{ju}
\]

\[ = \sum_{s,t,u,v=1}^N q^{2(s-i)}(q^{-\delta_{t,u}\delta_{s,v}} - [u > v](q - q^{-1})\delta_{m,u}\delta_{v,t})
\]

\[ \times (q^{\delta_{i,s}\delta_{i,n}} + [v > n](q - q^{-1})\delta_{s,v}\delta_{i,n})x_{st}x_{ju}. \]

Splitting the sum above into four separate sums, we obtain the formula in the statement of the theorem.

The computation of \( \Delta(x_{im}) \) is much more complicated. Firstly, observe that the comultiplication in Theorem 4.3 can be rewritten as

\[ (6.49) \quad \Delta(x) = \sigma(x_1, x_3)\sigma^{-1}(x_2, x_8)\sigma(x_4, x_9)v(x_5)x_3 \otimes x_7, \]
where $v$ is the map $v(y) = \sigma(y_1, S(y_2))$ as in (2.3). Next, we compute

$$v(x_{im}) = \sum_{k=1}^{N} \sigma(x_{ik}, S(x_{km})) = \sum_{k=1}^{N} q^{2(m-k)} l_{km}(x_{ik})$$

$$= \sum_{k=1}^{N} z^{-1} q^{2(m-k)} (q^{-\delta_{i,k}} \delta_{i,k} \delta_{k,m} - [k > m](q - q^{-1}) \delta_{i,m})$$

$$= z^{-1} (q^{-\delta_{i,m}} - q(1 - q^{-2}) \sum_{k > m} (q^{-2})^{k-m} \delta_{i,m})$$

$$= z^{-1} (q^{-1} - q^{-1}(1 - (q^{-2})^{N-m}) \delta_{i,m} = z^{-1} q^{-2(N-m)} \delta_{i,m},$$

for all $1 \leq i, m \leq N$. Therefore, by (6.39) we have

$$\Delta(x_{im}) = \sum_{a, \ldots, h=1}^{N} l_{cf}(x_{ia}) l_{gh}(x_{ab}) l_{hm}(x_{cd}) v(x_{de}) x_{bc} \otimes x_{fg}$$

$$= \sum_{a, b, c, d, f, g, h=1}^{N} q^{-2(N-d)-1} (q^{\delta_{i,a}} \delta_{d,f} + [d > f](q - q^{-1}) \delta_{d,a} \delta_{i,f})$$

$$\times \left( q^{-\delta_{a,b}} \delta_{g,h} - [g > h](q - q^{-1}) \delta_{g,b} \delta_{h,a} \right)$$

$$\times \left( q^{\delta_{b,c}} \delta_{h,m} + [h > m](q - q^{-1}) \delta_{h,d} \delta_{c,m} \right) x_{bc} \otimes x_{fg}.$$

Again, splitting the above sum into eight separate sums, a tedious but straightforward computation yields the formula for $\Delta(x_{im})$ in the statement of the theorem.

Finally, the equation for $S(x_{ij})$ follows easily by the general formula in Theorem [18] and by computations similar to the ones above. 

\[ \square \]

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