Coleman-De Luccia tunneling wave function with Non-Minimal coupling

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Abstract. We examine a novel method in Coleman-de Luccia vacuum decay analysis by using tunneling wave function approach with additional non-minimal coupling term. Our approach has an advantage that the wave function analysis needs no time dependence, thus consistent with Wheeler-de Witt equation. In the calculation, we use WKB approximation and Hamilton-Jacobi Equation. We discover that the probability of tunneling is lower than the previous study and the tunneling probability depends on both the false and true vacuum state of the scalar field.

1. Introduction
In quantum field theory, it is possible for a scalar field to possess two minima, in which one is a local minimum and the other is a global minimum, with different energy density. Classically, transition from local minimum (false vacuum) to global minimum (true vacuum) is prohibited, but this phenomenon may occur by quantum tunneling. This phenomenon is called vacuum decay and has been studied by Coleman and Callan [1] and Coleman and de-Luccia [2]. However, the decay rate approach is problematic with the inclusion of gravity in vacuum decay. The problem is caused by Wheeler-de Witt equation [3]:

$$H\Psi = 0,$$

with $H$ is Hamiltonian of the system and $\Psi$ is the wave function of the system. To see the connection between decay rate approach and Wheeler-de Witt equation, consider Euclidean Feynman path integral [1, 4]:

$$Z = \langle x_f | e^{-HT} | x_i \rangle = N \int_{x(0)=x_i}^{x(T)=x_f} [dx] e^{-S_E[x]},$$

which explain vacuum state evolution in Euclidean background, with $S_E$ is the Lagrangian of the system and $T$ is the time interval in Euclidean background. Using completeness relation $\sum_n |n\rangle \langle n|$, equation (2) can be rewritten as

$$Z = \sum_n e^{-E_n T} \phi_n(x_i) \phi^*_n(x_f),$$

where $E_n$ is the energy level of the system.
with $\phi_n(x) = \langle n|\phi \rangle$. With large $T$, the dominating term will be the one with the highest exponential value. The negative exponential of ground state energy will have the highest value than the negative exponential of other energy values. So as $T$ approaching infinity, we can have expression for $E_0$ as:

$$E_0 = -\lim_{T \to \infty} \frac{\ln Z}{T}. \quad (4)$$

Connection between the ground state energy value and the decay rate can be made by evaluating time-dependent wave function:

$$|\Psi(t)\rangle = \sum_n c_n e^{-iE_nt}|n\rangle. \quad (5)$$

With ground state energy dominating the expression in large $t$, the equation become

$$|\Psi(t)\rangle \approx c_0 e^{-iE_0t}|0\rangle. \quad (6)$$

The above equation can then be expressed in coordinate space representation:

$$\Psi(x) \approx c_0 e^{-iE_0t}\psi_0(x). \quad (7)$$

The probability of the state being in the false vacuum is equal to the absolute square of the wave function:

$$P(x_f,t) = |c_0 \psi_0(x_f)|^2 e^{-2Im[E_0]t}. \quad (8)$$

The decay rate then can be expressed as

$$\Gamma = -2Im[E_0]. \quad (9)$$

The value of ground state energy is the eigenvalue of Hamiltonian operator. But as Eq. (1) state, the eigenvalue of $H$ is equal to zero. This problem is avoided by using tunneling approach to find the probability of vacuum tunneling.

The system that is being evaluated here is described by following action:

$$S = \int d^4x \sqrt{-g} \left( -\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) + \frac{R}{2\kappa} - \frac{1}{2} \xi \phi^2 R \right). \quad (10)$$

This is the action of non-minimally coupled scalar field in curved spacetime with $\xi$ is the coupling constant, $R$ is the Ricci scalar, $g^{\mu\nu}$ is the metric tensor, $\kappa = 8\pi G$, and $G$ is the gravitational constant. If the value of $\xi$ is zero, the coupling is said to be minimal and the study of tunneling probability of this system has been evaluated by Kristiano, Lambaga, and Ramadhan [4].

2. Hamiltonian and Wheeler-de Witt Equation

The action for this system as is described as Eq. (10) is studied using homogeneous and isotropic metric and time dependent scalar field as ansatz:

$$dS^2 = -dt^2 + a^2(t)[d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta d\phi^2)], \quad (11)$$

$$\phi = \phi(t). \quad (12)$$

The action then becomes

$$S = 2\pi^2 \int dt a^3 \left[ \frac{1}{2} \dot{\phi}^2 - V + \left( \frac{1}{2\kappa} - \frac{1}{2} \xi \phi^2 \right) \left( \frac{6(1 + a^2 + a\dddot{\phi})}{a^2} \right) \right]. \quad (13)$$
Two equations of motion can then be obtained:

\[ 1 + a'^2 = \frac{a^2}{3(\frac{1}{\kappa} - \xi \phi^2)} \left( \frac{1}{2} \phi'^2 + V(\phi) - 6 \frac{a'}{a^2} \xi \phi \phi' \right), \]  
(14)

\[ \phi'' + \frac{3a'}{a} \phi' + V_{,\phi} + \xi \phi R = 0, \]  
(15)

with prime denotes derivative with respect to time. Hamiltonian can be obtained by using Legendre transformation:

\[ H = \phi' \pi_\phi + a' \pi_a - L, \]  
(16)

with \( \pi_\phi \) is canonic momentum in form of

\[ \pi_\phi = 2 \pi \frac{\partial L}{\partial \phi'}, \quad \pi_a = - \frac{1}{2\pi} \frac{\partial L}{\partial a'}. \]  
(17)

Hamiltonian can then be expressed as

\[ H = 2\pi a^3 \left[ \frac{1}{2} \phi'^2 + V - \left( \frac{1}{\kappa} - \xi \phi^2 \right) \left( \frac{3 (1 + a'^2)}{a^2} \right) + \frac{6a'}{a} \xi \phi \phi' \right]. \]  
(18)

Substituting Eq. (14) into the bracket, Hamiltonian will be equal to zero, consistent with Wheeler-de Witt equation.

To evaluate the Wheeler-de Witt equation, the Hamiltonian needs to be expressed in second quantization form by substituting the canonical momentums with differential operators:

\[ \pi_\phi = -i \partial_\phi, \quad \pi_a = -1 \partial_a. \]  
(19)

Wheeler-de Witt equation is now expressed as

\[ H \Psi = 0 = \left[ - \frac{\partial^2}{4\pi^2 a^5 g} + \frac{\partial^2}{24\pi^2 a f g} - \frac{\xi \phi \partial_a \partial_a}{2\pi^2 a^2 f g} + \left( V - \frac{3f}{a^2} \right) 2\pi^2 a^3 \right] \Psi, \]  
(20)

with

\[ f = \frac{1}{\kappa} - \xi \phi^2, \quad g = 1 + 6 \xi^2 \phi^2 f^{-1}. \]  
(21)

3. WKB Approximation and Hamilton-Jacobi Equation

By applying WKB approximation and using general expression for wave function in form of

\[ \Psi = A(\phi, a) e^{-iB(\phi,a)}, \]  
(22)

the wave equation becomes

\[ (\partial_a B)^2 - \frac{6f}{a^2} (\partial_\phi B)^2 - \frac{12 \xi \phi}{a} \partial_\phi B \partial_a B - 48 \pi^4 a^4 f g \left( V - \frac{3f}{a^2} \right) = 0. \]  
(23)

This equation is Hamilton-Jacobi equation in form of

\[ (\nabla B)^2 + 2U = 0, \]  
(24)
with vector differential operator obeys an internal metric:

\[ g^{ij} = \begin{pmatrix} -\frac{6f}{a^2} & -6\xi \phi \\ -6\xi \phi & 1 \end{pmatrix} \]  \hspace{1cm} (25)

and with

\[ 2U \equiv (12\pi^2)^2 a^2 fg \left( f - \frac{a^2 V}{3} \right). \]  \hspace{1cm} (26)

The solution for the Hamilton-Jacobi equation is:

\[ B = \int dS \sqrt{-2U}. \]
\[ = \int \frac{d\phi^2 - \frac{6f}{a^2} da^2 + 12\xi \phi d\phi da}{-\frac{6f}{a^2} - (6\xi \phi)^2} \sqrt{-2U}. \]  \hspace{1cm} (27)

The integration path is chosen to minimize the value of \( B \). This can be achieved by the means of calculus of variation and results in a differential equation

\[ \phi'' = \left( \frac{-3f}{a^2} + 3\xi \phi' \right) \left( \frac{1}{a} - \frac{\phi'}{a} \right) - (\phi' + 6\xi \phi) \left( f - \frac{2}{3} a^2 V \right) \left( \frac{1}{a} \right) \left( \frac{A}{B} \right) \]
\[ - \left( \frac{6f}{a^2} + 6\xi \phi' \right) \left( \frac{1}{a} - \frac{\phi'}{a} \right) \left( \frac{A}{B} \right) + \left( 36\xi^2 \phi + \frac{3f_{,\phi}}{a^2} \right) \left( \frac{A}{B} \right)^2 \]
\[ + \left( 6\xi \phi + \phi' \right) \left( \frac{6f}{a^2} - 36\xi^2 \phi^2 \right) \left( \frac{A}{B} \right) - \frac{3f_{,\phi}}{a^2} \left( \frac{A}{B} \right) \]
\[ + \left( 6\xi \phi + \phi' \right) \left( \frac{6f}{a^2} + 6\xi \phi^2 \right), \]  \hspace{1cm} (28)

with

\[ A = \phi'^2 - \frac{6f}{a^2} + 12\xi \phi', \]  \hspace{1cm} (29)
\[ B = \frac{6f}{a^2} - (6\xi \phi)^2, \]  \hspace{1cm} (30)

and prime denotes derivative with respect to \( a \). The potential \( V \) is in the form of:

\[ V(\phi) = \frac{\lambda}{8} (\phi^2 - \phi_0^2)^2 + (\epsilon_f - \epsilon_t) \frac{\phi + \phi_0}{2\phi_0} + \epsilon_t. \]  \hspace{1cm} (31)

4. Results and Discussion

Eq. (28) is then solved with following boundary values:

\[ \phi_i = \phi_0, \ \phi_f = -\phi_0, \]  \hspace{1cm} (32)
\[ a_i = \sqrt{\frac{3f(\phi_0)}{\epsilon_f}}, \ a_f = \sqrt{\frac{3f(-\phi_0)}{\epsilon_t}}. \]  \hspace{1cm} (33)
Figure 1. Solution of Eq. 28.

Figure 2. The term inside the square root of Eq. (27).

Figure 3. Plot of tunneling probability vs $\xi$. The red “+”s show the values of tunneling probability and the dotted blue line shows the interpolation of the data.

However, these values will make the value of B in Eq. (28) equal to zero, creating a singularity in the equation. To eliminate the singularity, the value of A, Eq. (29), must also equal zero at
the boundaries, thus creating an additional boundary value:

\[ \phi_i' = -6\xi\phi_0 - \frac{\sqrt{6(f + 6a^2\xi^2\phi_0^2)}}{a}. \]  

(34)

In solving the equation numerically, parameters from Ref. [5] is used:

\[ \phi_0 = (1 \times 10^{14}) GeV, \quad \epsilon_f = (4 \times 10^{56}) GeV^4, \quad \epsilon_t = (1 \times 10^{56}) GeV^4, \]
\[ \lambda = 80, \quad \xi = \frac{1}{6}, \quad \kappa = (1.7 \times 10^{-37}) GeV^{-2}. \]  

(35)

Then the integration path is obtained in form of Fig. 1.

To obtain the probability of tunneling, Eq. (27) must be carried out. Evaluating the absolute square of the ansatz of wave function,

\[ P = |\Psi|^2 \propto e^{-2Im(|B|)}, \]  

(36)

thus for the calculation of tunneling probability, the only contributing part is the imaginary part. To see the part contributing to the probability value, the term inside the square root of Eq. (27) can be plotted as in Fig. 2.

It can be seen that the negative value, which will contribute to imaginary part of the integral, is found at the boundaries of the figure. This result is different with Ref. [4] that found the contributing part only in the fist part of the integration path. After the integration in Eq. (27) has been carried out and only the imaginary part is taken, the tunneling probability is obtained to be \( P \propto e^{-8.17 \times 10^{18}}, \) a result that is much smaller than previous study [4] which obtained the value of \( P \propto e^{-5.0 \times 10^{12}}. \) Further calculation for different values of \( \xi \) is shown in Fig. 3.

As seen in the figure, the exponential term in tunneling probability shows fluctuating increase along with the increase of \( \xi. \) This signifies the decrease of the value of tunneling probability along with the increase of the coupling constant \( \xi. \) The fluctuation may occur due to differences in computational accuracy.

5. Conclusion

The addition of non-minimal coupling in the analysis of Coleman-de Luccia tunneling wave function makes several differences from the original study with minimal coupling. The most notable difference is in the analytic calculation, in which the non-minimal coupling creates nonlinearity in the internal metric and complicates the calculation of integration path. Due to the use of numerical method, the result of this calculation shows significantly lower value of tunneling probability than previous study, with contribution to the tunneling value comes from the parts near the boundaries. The value of the tunneling probability shows increase with the increase of coupling constant. The significant numerical difference is not preceded by significant difference in integration path. This is caused by the use of parameters in Ref. [5] which makes the contribution of numerical part insignificant.

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