The oscillating dark energy: future singularity and coincidence problem

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We consider the oscillating dark energy with periodic equation of state in two equivalent formulations: ideal fluid or scalar-tensor theory. It is shown that such dark energy suggests the natural way for the unification of early-time inflation with late-time acceleration. We demonstrate how it describes the transition from deceleration to acceleration or from non-phantom to phantom era and how it solves the coincidence problem. The occurrence of finite-time future singularity for the oscillating (phantom) universe is also investigated.

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I. INTRODUCTION.

It became clear recently that current universe is expanding with acceleration which is caused by dark energy (for a recent review, see\(^1\)). However, the observational data (for a recent review, see\(^2\)) are far from being complete. It is not even known what is the current value of the dark energy effective equation of state (EoS) parameter \(w\) which lies close to \(-1\): it could be equal to \(-1\) (standard \(\Lambda\)CDM cosmology), a little bit upper than \(-1\) (the quintessence dark energy\(^3\)) or less than \(-1\) (phantom dark energy, see\(^4\) for very incomplete list of related papers). Even less is known about the evolution of this EoS parameter, the origin of dark energy and why currently the energy-density of dark energy is approximately equal to the one of dust matter (coincidence problem). As a result, there are numerous proposals for dark energy which so far does not exist as consistent theory.

The oscillating dark energy has been proposed in refs.\(^5\)\(^6\)\(^7\). It has been shown there that such models very naturally resolve the coincidence problem due to periods of acceleration and that they may be consistent with observations. They also serve as very good candidates for unification of the early time inflation with late time acceleration. It was argued that oscillated dark energy does not lead to Big Rip singularity.

In the present paper the late-time cosmological consequences of dark energy with time-dependent, periodic EoS are further described. We work in two (mathematically equivalent) formulations: ideal fluid EoS description or scalar-tensor theory with some specific potential. The general method to go from one formulation to another is proposed. It is demonstrated for number of examples that such oscillating dark energy may describe the transition from deceleration to acceleration, or from non-phantom to phantom era (if current dark energy is of phantom type). In the last case, the classical universe may end up in the finite-time, future singularity (any of four known types of Big Rip may occur as is shown explicitly). The fact that oscillating dark energy universe may end up in the future singularity was not discussed before. It is proved that the oscillating dark energy may be the key for resolution of the coincidence problem in accordance with earlier findings of refs.\(^5\)\(^6\). Finally, we show that the unification of early-time inflation with late-time acceleration is very natural in such oscillating universe which could be embedded in the cyclic (expanding/contracting) universe.

II. OSCILLATING EQUATION OF STATE OF THE UNIVERSE.

Let us consider the universe filled with the ideal fluid (dark energy) where the equation of state (EoS) parameter \(w\) depends on time \(t\):

\[
p = w(t)\rho \, .
\]
Recent observational data (for review, see ref. 2) admit the possibility of time-dependent $w$ which lies currently close to $-1$.

The conservation of the energy

$$\dot{\rho} + 3H (p + \rho) = 0, \tag{2}$$

and the FRW equation

$$\frac{3}{\kappa^2} H^2 = \rho, \tag{3}$$

give

$$\dot{\rho} + \kappa \sqrt{3} (1 + w(t)) \rho^{3/2} = 0, \tag{4}$$

which can be integrated as

$$\rho = \frac{4}{3\kappa^2 (\int dt (1 + w(t)))^2}. \tag{5}$$

Using Eq. (4), the Hubble rate may be found

$$H = \frac{2}{3\kappa (\int dt (1 + w(t)))}. \tag{6}$$

When $w$ is a constant (not equal to phantom divide value), the standard expression is recovered

$$H = \frac{2}{3(1 + w)(t - t_s)}. \tag{7}$$

If $\int dt (1 + w(t)) = 0$, $H$ diverges, which corresponds to the Big Rip type singularity, which occurs when $t = t_s$. In case $w > -1$, for the expanding universe where $H > 0$, the cosmological time $t$ should be restricted to be $t > t_s$.

The situation where $w(t)$ is time-dependent, periodic function may be of physical interest. Indeed, this can explain easily why the effective value of $w$ is different at different epochs, being currently close to $-1$ (if the period of such function is comparable with the universe age). An illustrative example is given by

$$w = -1 + w_0 \cos \omega t. \tag{8}$$

Here, $w_0, \omega > 0$. (Such EoS has been considered in ref. 3, where it was shown that it may be consistent with observations for some values of parameters.) Then Eq. (3) gives

$$H = \frac{2\omega}{3(w_1 + w_0 \sin \omega t)}. \tag{9}$$

Here $w_1$ is a constant of the integration. When $|w_1| < w_0$, the denominator can vanish, which corresponds to the Big Rip singularity. When $|w_1| > w_0$, there does not occur such a singularity. Since

$$\dot{H} = -\frac{2\omega^2 w_0 \cos \omega t}{3(w_1 + w_0 \sin \omega t)^2}, \tag{10}$$

when $w_0 \cos \omega t < 0$ ($w_0 \cos \omega t > 0$), the universe lives in phantom (non-phantom) phase where $\dot{H} > 0$ ($\dot{H} < 0$). In this case, the energy density $\rho$ is given by

$$\rho(t) = \frac{4\omega^2}{3\kappa^2 (w_1 + w_0 \sin \omega t)^2}, \tag{11}$$

which also oscillates. Hence, $\dot{\rho} > 0$ in phantom phase and $\dot{\rho} < 0$ in non-phantom phase. Hence, kind of energy transfer occurs in such oscillating dark energy universe: the energy-density grows in phantom phase but decreases in non-phantom era. This may suggest the qualitative explanation of the fact of the growth of phantom energy. Moreover, in such oscillating universe the unification of late-time (phantom or quintessence) acceleration with early-time, phantom inflation naturally occurs: indeed, the (phantom or quintessence) acceleration epoch occurs periodically due to the periodic behaviour of the universe EoS.

Since the pressure is given by

$$p = -\frac{1}{\kappa^2} \left(2\dot{H} + 3H^2\right), \tag{12}$$

from the second FRW equation, we find

$$p(t) = \frac{4\omega^2 (w_0 \cos \omega t - 1)}{3\kappa^2 (w_1 + w_0 \sin \omega t)^2}. \tag{13}$$

Then as long as $w_1 + w_0 \sin \omega t > 0$, that is, the universe is expanding ($H > 0$), we find the following EoS for dark energy ideal fluid:

$$w_0^2 = \left(w_1 - \frac{2\omega}{\kappa \sqrt{3\rho}}\right)^2 + \left(1 + \frac{p}{\rho}\right)^2. \tag{14}$$

Another interesting example considered in ref. 4 is

$$w = -1 + \alpha \left(1 + \beta \cos \omega t\right), \quad \alpha > 0, \quad 1 > \beta > 0. \tag{15}$$

Here $w < -1$ and the universe is not in phantom phase. Hence, this model corresponds to oscillating dark energy where the unification of early-time
quintessence inflation with late-time quintessence acceleration occurs. Moreover, if \( \alpha (1 + \beta) > 1 \), \( w \) crosses \( w = 0 \) and we find
\[
H = \frac{2\omega}{3\alpha \{\omega(t - t_0) + \beta \sin \omega t\}} ,
\]
(16)
Here \( t_0 \) is a constant of the integration. In this case, the energy density \( \rho \) is given by
\[
\rho = \frac{4\omega^2}{3\alpha^2 \{\omega(t - t_0) + \beta \sin \omega t\}^2} ,
\]
(17)
which is monotonically decreasing, oscillating function. Since
\[
p = \frac{4\omega^2 \{(\alpha - 1) + \alpha \beta \cos \omega t\}}{3\alpha^2 \{\omega(t - t_0) + \beta \sin \omega t\}^2} ,
\]
(18)
we find the following EoS for ideal fluid:
\[
0 = \frac{p + (1 - \alpha) \rho}{\alpha \beta \rho} - \cos \left(\omega t_0 + \frac{\alpha \kappa}{2\omega} \sqrt{3\rho}\right)
+ \frac{1}{\alpha \rho} \left\{ (\alpha^2 \beta^2 - \alpha^2 + 2\alpha - 1) \rho^2
- \rho^2 - 2(1 - \alpha) \rho p \right\} .
\]
(19)
In a more realistic situation, universe is filled with matter and dark energy. The matter, in general, interacts with the dark energy. In such a case, the total energy density \( \rho_{\text{tot}} \) consists of the contributions from the dark energy and the matter: \( \rho_{\text{tot}} = \rho + \rho_m \). If we define, however, the matter energy density \( \rho_m \) properly, we can also define the matter pressure \( p_m \) and the dark energy pressure \( p \) by
\[
\rho_m \equiv -\rho_m + \frac{\dot{\rho}}{3H}, \quad p \equiv \rho_{\text{tot}} - \rho_m .
\]
(20)
Here \( \rho_{\text{tot}} \) is the total pressure. Hence, the matter and dark energy satisfy the energy conservation laws separately,
\[
\dot{\rho}_m + 3H (\rho_m + p_m) = 0, \quad \dot{\rho} + 3H (\rho + p) = 0 .
\]
(21)
We now consider the case that universe is filled with the oscillating dark energy and the matter ideal fluid with a constant EoS parameter \( w_m \): \( p_m = w_m \rho_m \). Since \( w_m \) is a constant, one finds
\[
\rho_m = \rho_0 a^{-3(1+w_m)} .
\]
(22)
Here \( \rho_0 \) is a constant. FRW equation leads to
\[
\rho = \frac{3}{\kappa^2} H^2 - \rho_0 a^{-3(1+w_m)} .
\]
(23)
Using the conservation law \( \Box \), we find
\[
-3(1 + w(t)) = \frac{d}{dt} \ln \rho
= \frac{\kappa^2 H^2 + 3(1 + w_m) H \rho_0 a^{-3(1+w_m)}}{\kappa^2 H^2 - \rho_0 a^{-3(1+w_m)}} .
\]
(24)
That is,
\[
1 + w(t) = \frac{2}{\kappa^2} \frac{H}{H^2 - \rho_0 a^{-3(1+w_m)}} .
\]
(25)
Note that the denominator of the r.h.s. in \( \Box \) is always positive since the denominator is nothing but energy-density.
Let us consider the transition from non-phantom to the phantom era, where \( H = 0 \). Then \( 1 + w(t) < 0 \) as long as \( w_m > -1 \). Thus, in order that the transition occurs, \( w(t) \) should become less than \( -1 \) before the transition.
As an example, we consider the scale factor
\[
a(t) = a_0 \left( \frac{t}{t_s - t} \right)^{h_0} ,
\]
(26)
with constants \( a_0, t_s, \) and \( h_0 > 0 \). Eq. \( \Box \) gives,
\[
H = \frac{h_0 t_s}{t(t_s - t)} ,
\]
\[
\frac{\dot{a}}{a} = \frac{h_0 t_s (2t - (1 - h_0) t_s)}{t^2 (t_s - t)^2} .
\]
(27)
The transition from non-phantom to phantom era occurs at \( t = t_s/2 \), and if \( h_0 < 1 \), the transition from deceleration to acceleration epoch occurs at \( t = t_s(1 - h_0)/2 \). Using \( \Box \), \( w(t) \) corresponding to \( \Box \) is given by
\[
w(t) = \frac{2}{\kappa^2} \frac{H(t_s/2 - t)}{H(t_s - t)} \left\{ \begin{array}{l}
2h_0 t_s (2t - t_s) \\
\left( \frac{t}{t_s - t} \right)^{-3(1+w_m)h_0} \\
+ (1 + w_m) \rho_0 a_0^{-3(1+w_m)} \left( \frac{t}{t_s - t} \right)^{-3(1+w_m)h_0} \\
\times \left\{ \begin{array}{l}
\frac{3t_s^2}{\kappa^2 t_s^2 (t_s - t)^2} \\
- \rho_0 a_0^{-3(1+w_m)} \left( \frac{t}{t_s - t} \right)^{-3(1+w_m)h_0} \end{array} \right\}^{-1}
\end{array} \right. .
\]
(28)
Now the energy-density is given by
\[ \rho(t) = \frac{3h_0^2 t_s^2}{\kappa^2 t^2 (t_s - t)^2} - \rho_0 a_0^{-3(1+w_m)} \left( \frac{t}{t_s - t} \right)^{-3h_0(1+w_m)} \] (29)

Since the second term in (28) is negative, \( \rho(t) \) is not always positive. The negative energy density is, of course, not physical. When \( t \to 0 \), if \( 3h_0 (1+w_m) > 2 \), the second term dominates and the energy density becomes negative. On the other hand, when \( t \to t_s \), if \( 3h_0 (1+w_m) < -2 \), one finds that \( \rho \) becomes negative, again. Hence, physically allowed region of the parameters could be
\[ -2 \leq 3h_0 (1+w_m) \leq 2 . \] (30)

For simplicity, we consider the case \( 3h_0 (1+w_m) = 2 \). Then the expression (28) reduces to
\[ \rho(t) = \frac{1}{t^2} Q(t), \]
\[ Q(t) = \frac{3h_0^2 t_s^2}{\kappa^2 (t-t_s)^2} - \rho_0 a_0^{-3h_0(1+w_m)} (t_s - t)^2 . \] (31)

Since \( Q(t) \) is monotonically increasing function as long as \( t < t_s \), if
\[ Q(0) = \frac{3h_0^2}{\kappa^2} - \rho_0 a_0^{-3h_0(1+w_m)} t_s^2 \geq 0 , \] (32)
\( \rho(t) \) is always positive.

From the second FRW equation, the pressure may be found:
\[ p = -\frac{1}{\kappa^2} \left( 2\dot{H} + 3H^2 \right) - w_m \rho a_0^{-3(1+w_m)} . \] (33)

Hence,
\[ p(t) = -\frac{2h_0 t_s (2t - t_s) + 3h_0^2 t_s^2}{\kappa^2 t^2 (t_s - t)^2} - \rho_0 a_0^{-3(1+w_m)} \left( \frac{t}{t_s - t} \right)^{-3h_0(1+w_m)} . \] (34)

At the phantom-non-phantom transition point \( t = t_s/2 \), one gets
\[ \rho(t) + p(t) = -\left( 1 + w_m \right) \rho_0 a_0^{-3(1+w_m)} \times \left( \frac{t}{t_s - t} \right)^{-3h_0(1+w_m)}, \] (35)

which is negative, corresponding to \( w(t) < -1 \) there.

As the next example, the following scale factor may be considered:
\[ a = a_0 e^{h_0 t + \frac{1}{\kappa} \sin \omega t}, \] (36)

with constants \( h_0 > h_1 > 0 \) and \( \omega > 0 \). Since
\[ H = h_0 + h_1 \cos \omega t, \] (37)

Here, the universe lives in non-phantom (phantom) era when \( 2\pi n < \omega t < 2\pi n + \pi \) \( (2\pi n - \pi < \omega t < 2\pi n) \), where \( n \) is an integer. The corresponding, time-dependent EoS parameter \( w(t) \) is found to be
\[ w(t) = -1 - \left\{ \frac{2h_1}{\kappa^2} \sin \omega t + (1 + w_m) \rho a_0^{-3(1+w_m)} e^{-3(1+w_m)(h_0 t + \frac{1}{\kappa} \sin \omega t)} \right\} \times \left\{ \frac{3}{\kappa^2} (h_0 + h_1 \cos \omega t)^2 \right\}^{-1} . \] (38)

Now the energy density \( \rho \) and pressure \( p \) are
\[ \rho(t) = \frac{3}{\kappa^2} (h_0 + h_1 \cos \omega t)^2 - \rho_0 a_0^{-3(1+w_m)} e^{-3(1+w_m)(h_0 t + \frac{1}{\kappa} \sin \omega t)} , \]
\[ p(t) = \frac{1}{\kappa^2} \left\{ 2h_1 \omega \sin \omega t - 3(h_0 + h_1 \cos \omega t)^2 \right\} - w_m \rho a_0^{-3(1+w_m)} e^{-3(1+w_m)(h_0 t + \frac{1}{\kappa} \sin \omega t)} . \] (39)

The \( \rho(t) \) (39) becomes negative for large negative \( t \). If we restrict the parameters to satisfy
\[ \frac{3}{\kappa^2} (h_0 - h_1)^2 - \rho_0 a_0^{-3(1+w_m)} \geq 0 , \] (40)

it follows \( \rho(t) \geq 0 \) when \( t \geq 0 \).

It is interesting to understand if the oscillating dark energy may bring the universe evolution to Big Rip singularity. (One should not forget that Big Rip is typically classical effect which seems to disappear with proper account of quantum corrections \[ 3, 8, 10 \].) The classification of future, finite-time singularities is given as \[ 10 \]:

- Type I (“Big Rip”) : For \( t \to t_s \), \( a \to \infty \), \( \rho \to \infty \) and \( |p| \to \infty \)
- Type II (“sudden”) : For \( t \to t_s \), \( a \to a_s \), \( \rho \to \rho_s \) or 0 and \( |p| \to \infty \)
• Type III: For \( t \to t_s, a \to a_s, \rho \to \infty \) and \(|p| \to \infty\).

\[
\begin{align*}
  H \sim h_0 (t_s - t)^{-\alpha}, \quad (h_0 > 0), \tag{41}
\end{align*}
\]

near the singularity \( t \sim t_s \). Then one gets
\[
\begin{align*}
  \dot{H} &\sim \left\{ \begin{array}{ll}
  h_0 (t_s - t)^{-\alpha+1} & (\alpha \neq 0) \\
  0 & (\alpha = 0)
\end{array} \right. \\
  \ln a &\sim \left\{ \begin{array}{ll}
  h_0 (t_s - t)^1 & (\alpha \neq 1) \\
  h_0 \ln (t_s - t) & (\alpha = 1).
\end{array} \right. \tag{42}
\end{align*}
\]

Since \( \rho \propto \sqrt{H} \) and \( p \propto 3H^2 + 2\dot{H} \), in terms of \( \alpha \), the above four types of the singularities could be classified as

Type I: \( \alpha \geq 1 \), Type II: \( 0 > \alpha > -1 \), Type III: \( 1 > \alpha > 0 \), Type IV: \( \alpha < -1 \) but \( \alpha \) is not an integer (for account of Hubble rate dependent terms in inhomogeneous EoS, see \[11\]).

Without the matter, in terms of \( w(t) \), by comparing \([11]\) with \([9]\), we find
\[
1 + w(t) \sim (t_s - t)^{\alpha - 1}. \tag{43}
\]

Thus, all four types of future singularities are realized here for above values of parameter \( \alpha \).

It is interesting to study what happens with the classical future singularity in the presence of matter. For this purpose, we assume Eq.\([11]\). Then from \([25]\), we find
\[
\begin{align*}
  1 + w(t) &\sim -\left( -\frac{2\alpha}{\kappa^2} h_0 (t_s - t)^{-\alpha - 1} + 3 (1 + w_m) \rho_0 a_0 e^{-\frac{h_0 (1 + w_m)}{1 + \alpha}(t_s - t)^{-\alpha + 1}} \right) \\
  &\times \left( \frac{3}{\kappa^2} h_0^2 (t_s - t)^{-2\alpha} - \rho_0 a_0 e^{-\frac{h_0 (1 + w_m)}{1 + \alpha}(t_s - t)^{-\alpha + 1}} \right)^{-1}, \tag{44}
\end{align*}
\]

when \( \alpha \neq 1 \) and for \( \alpha = 1 \)
\[
\begin{align*}
  1 + w(t) &\sim -\left( -\frac{2\alpha}{\kappa^2} h_0 (t_s - t)^{-2} + 3 (1 + w_m) \rho_0 a_0 (t_s - t)^{-h_0 (1 + w_m)} \right) \\
  &\times \left( \frac{3}{\kappa^2} h_0^2 (t_s - t)^{-2} - \rho_0 a_0 (t_s - t)^{-h_0 (1 + w_m)} \right)^{-1}. \tag{45}
\end{align*}
\]

In case \( \alpha > 1 \), which corresponds to Type I singularity, one obtains
\[
1 + w(t) \sim \frac{2\alpha}{3h_0} (t_s - t)^{\alpha - 1} \to 0. \tag{46}
\]

if \( 1 + w_m < 0 \). The case \( 1 + w_m > 0 \) is excluded since the denominator of the r.h.s. in \([25]\) becomes negative.

In case \( \alpha = 1 \), which also corresponds to Type I singularity,
\[
1 + w(t) \sim \left\{ \begin{array}{l}
\frac{-2\alpha h_0 + 3 (1 + w_m) \rho_0 a_0}{3h_0^2 - \rho_0 a_0} \quad \text{if } h_0 (1 + w_m) = 2 \\
\frac{2\alpha}{3h_0} \quad \text{if } h_0 (1 + w_m) = 2.
\end{array} \right. \tag{47}
\]

The case \( h_0 (1 + w_m) > 2 \) is also prohibited since the denominator of the r.h.s. in \([25]\) becomes negative.

In case \( 0 < \alpha < 1 \), which corresponds to Type III singularity, we find
\[
1 + w(t) \sim \frac{2\alpha}{3h_0} (t_s - t)^{\alpha - 1} \to \infty, \tag{48}
\]

and in case \( \alpha = 0 \),
\[
1 + w(t) \sim \frac{3 (1 + w_m) \rho_0 a_0}{\frac{1}{1} h_0^2 - \rho_0 a_0}. \tag{49}
\]

The case \(-1 < \alpha < 0 \), which corresponds to Type II singularity, and the case \( \alpha < -1 \), which corresponds to Type IV, are excluded since the denominator of the r.h.s. in \([25]\) becomes negative. It is interesting that the Type II and IV singularities are excluded with the account of matter, while they occur without matter. Thus, we demonstrated that within oscillating dark energy universe, all four types of future singularities may occur on the classical level. This is our main qualitative result: the oscillating dark energy of general form does not prevent the universe to end up in the future singularity of any from four known types.
III. SCALAR-TENSOR DESCRIPTION AND COINCIDENCE PROBLEM

In this section we will present the oscillating dark energy with time-dependent, explicit EoS in an equivalent, scalar-tensor description following to the method developed in ref.**12**. (Note that such method may be applied to include also the time-dependent bulk viscosity**13** introduced for dark energy description in ref.**14**.) Let us start with the following action

\[
S = \int d^4x \sqrt{-g} \left\{ \frac{1}{2\kappa^2} R - \frac{1}{2} \Omega(\phi) \partial_\mu \phi \partial^\mu \phi - V(\phi) \right\} ,
\]

\[
\Omega(\phi) = -\frac{2}{\kappa^2} f' (\phi) ,
\]

\[
V(\phi) = \frac{1}{\kappa^2} \left( 3f (\phi)^2 + f' (\phi) \right) ,
\]

(50)

with an adequate function \(f(\phi)\). The following solution of FRW equation exists**17**:

\[
\phi = t , \quad H = f(t) .
\]

(51)

Then by comparing (51) with (6), one may identify

\[
f(t) = h(t) .
\]

(52)

For the action (50), the explicit EoS is given by:

\[
p = -\rho - \frac{2}{\kappa^2} f' (f^{-1} \left( \kappa \sqrt{\frac{\rho}{\beta}} \right)) ,
\]

(53)

Thus, we demonstrated how an arbitrary oscillating dark energy may be rewritten explicitly in mathematically-equivalent scalar-tensor form.

Hence, for Eq. (50), the following equivalent description in scalar-tensor form may be constructed:

\[
\Omega(\phi) = \frac{4 \omega^2 w_0 \cos \omega \phi}{3 \alpha \kappa^2 \{ \Omega (\phi - t_0) + \beta \sin \omega \phi \}^2} ,
\]

\[
V(\phi) = \frac{\omega^2 (4 - 2w_0 \cos \omega \phi)}{3 \kappa^2 \{ w_0 + \sin \omega \phi \}^2} .
\]

(54)

For EoS**14**, the equivalent scalar-tensor theory description**12** leads to

\[
\Omega(\phi) = -\frac{4 \omega^2 (1 + \beta \cos \omega \phi)}{3 \alpha \kappa^2 \{ \Omega (\phi - t_0) + \beta \sin \omega \phi \}^2} ,
\]

\[
V(\phi) = \frac{2 \omega^2 (2 - \alpha - \alpha \beta \cos \omega \phi)}{3 \alpha \kappa^2 \{ \omega (\phi - t_0) + \beta \sin \omega \phi \}^2} .
\]

(55)

Comparing (51) with (6), one also finds that in scalar-tensor theory (50), the four types of future singularities could be realized by the choice

\[
f(\phi) \sim f_0 (t_s - \phi)^{-\alpha} = f_0 \varphi^{-\alpha} ,
\]

(56)

with positive constant \(f_0\). Here \(\varphi \equiv t_s - \phi\). In this case, we obtain

\[
\Omega(\phi) \sim -\frac{2 \alpha f_0}{\kappa^2} (t_s - \phi)^{-\alpha - 1} ,
\]

\[
V(\phi) = \frac{1}{\kappa^2} \left( 3f_0^2 (t_s - \phi)^{-2\alpha} + \alpha f_0 (t_s - \phi)^{-\alpha - 1} \right) ,
\]

(57)

One should bear in mind that even FRW equations are the same in EoS or scalar-tensor description, the emerging universes may be different. Indeed, the number and stability of FRW solutions may not coincide, the newton law may be slightly different, etc.

As an extension of (50), the matter with constant EoS parameter \(w = w_m\) may be added into the action (50). By using a single function \(g(t)\), if we choose the scalar potentials in the action (50) with matter as

\[
\Omega(\phi) = -\frac{2}{\kappa^2} g''(\phi) - \frac{w_m + 1}{2} g_0 e^{-3(1+w_m)g(\phi)} ,
\]

\[
V(\phi) = \frac{1}{\kappa^2} \left( 3g'(\phi)^2 + g''(\phi) \right) + \frac{w_m - 1}{2} g_0 e^{-3(1+w_m)g(\phi)} ,
\]

(58)

the following solution of FRW equation may be constructed (see also**15**):

\[
\phi = t , \quad H = g'(t) ,
\]

\[
\left( \alpha = a_0 e^{g(t)} , \quad a_0 \equiv \left( \frac{\rho m_0}{g_0} \right)^{\frac{1}{3(1+w_m)}} \right)
\]

(59)

In the case of scale factor (50),

\[
g(\phi) = \ln \left( \frac{\phi}{t_s - \phi} \right)^{h_0} .
\]

(60)

and in the case of (50),

\[
g(\phi) = \ln \left( h_0 \phi + \frac{h_1}{\omega} \sin \omega \phi \right) .
\]

(61)

Let us now consider a little bit different function

\[
g(\phi) = a \ln \left\{ g_0 t + \frac{\hat{g}_1}{\omega} \cos \omega \phi \right\} .
\]

(62)
Here $\alpha$, $\dot{g}_0$, $\dot{g}_1$, and $\omega$ are positive constants. Then the Hubble rate is given by

$$H = \frac{\alpha (\dot{g}_0 - \dot{g}_1 \sin \omega t)}{\dot{g}_0 t + \frac{2}{3} \omega \cos \omega t}.$$ \hspace{1cm} (63)

If $t > 1/\omega$, the Hubble rate $H$ is always positive and if $\dot{g}_0$ is large enough compared with $\dot{g}_1$ and $\omega$, we find $H < 0$. Taking $\rho$ and $p$ in (1) as total energy density and pressure, $w(t)$ is given by

$$w(t) = -1 + \frac{2}{3\alpha} + \frac{2\dot{g}_1 \omega \cos \omega t}{3\alpha \dot{g}_0^2}.$$ \hspace{1cm} (64)

Then if

$$\frac{\dot{g}_1 \omega}{\dot{g}_0} > |1 - \alpha|,$$ \hspace{1cm} (66)

$w$ oscillates and crosses $w = -1/3$. Hence, the universe experiences the transition from the deceleration to the acceleration epoch.

Having in mind the oscillating nature of dark energy under consideration, it is expected that such type of dark energy may naturally solve the coincidence problem (for recent discussion and list of refs., see [11]). Let us confirm this in both formulations of the oscillating dark energy under consideration. We now separate $\rho_{\text{tot}}$ into the contributions from the dark energy and matter: $\rho_{\text{tot}} = \rho + \rho_m$. If the EoS parameter $w_m$ of the matter is constant as in (22), by using the first FRW equation (3), one obtains

$$r \equiv \frac{\rho}{\rho_m} = -1 + \frac{3H^2 a_0^3 (1 + w_m)}{\kappa^2 \rho_{m0}}.$$ \hspace{1cm} (67)

For Eq. (62), it follows

$$r = -1 + \frac{3\alpha^2 a_0^3 (1 + w_m)}{\kappa^2 \rho_{m0}} (g_0 - g_1 \sin \omega t)$$

$$\times \left( g_0 t + \frac{g_1}{\omega} \cos \omega t \right)^{(1 + w_m)\alpha - 2}.$$ \hspace{1cm} (68)

With the choice $\alpha = 2/3 (1 + w_m)$, the Eq. (68) reduces to

$$r = -1 + \frac{3\alpha^2 a_0^3 (1 + w_m)}{\kappa^2 \rho_{m0}} (g_0 - g_1 \sin \omega t).$$ \hspace{1cm} (69)

Then $r$ has a maximum and minimum as

$$-1 + \frac{3\alpha^2 a_0^3 (1 + w_m)}{\kappa^2 \rho_{m0}} (g_0 - g_1) \leq r \leq -1 + \frac{3\alpha^2 a_0^3 (1 + w_m)}{\kappa^2 \rho_{m0}} (g_0 + g_1).$$ \hspace{1cm} (70)

Hence, if

$$-1 + \left( \frac{3\alpha^2 a_0^3 (1 + w_m)}{\kappa^2 \rho_{m0}} \right) (g_0 \pm g_1) \sim \mathcal{O}(1),$$

the ratio between the energy densities of the dark energy and matter could be always $\mathcal{O}(1)$. Hence, oscillating dark energy may solve the coincidence problem.

In case of (20) or (30), from (29) one gets

$$r = -1 + \frac{3h_0^2 t^2 a_0^3 (1 + w_m)}{\kappa^2 \rho_0 t^2 - 3h_0 (1 + w_m)}.$$ \hspace{1cm} (71)

We now assume (30).

When $t \rightarrow t_x$, if $3h_0 (1 + w_m) \neq -2$, $r$ diverges but if $3h_0 (1 + w_m) = -2$, $r$ is finite:

$$r \rightarrow r_0 \equiv -1 + \frac{3h_0^2 a_0^3 (1 + w_m)}{\kappa^2 \rho_0 t_x^2}.$$ \hspace{1cm} (72)

On the other hand, when $t = 0$, if $3h_0 (1 + w_m) \neq -2$, $r$ diverges but if $3h_0 (1 + w_m) = 2$, $r$ is finite:

$$r \rightarrow r_0.$$ \hspace{1cm} (73)

If we write $t = \gamma t_x$, when $3h_0 (1 + w_m) = 2$, we find

$$r \rightarrow r_1 \equiv -1 + \frac{3h_0^2 a_0^3 (1 + w_m)}{\kappa^2 \rho_0 t_x^2 (1 - \gamma)^4},$$ \hspace{1cm} (74)

In the present universe, $0 < \gamma < 1$ and $\gamma \sim \mathcal{O}(1)$. In order that $r_0 \sim \mathcal{O}(1)$, $3h_0^2 a_0^3 (1 + w_m) / \kappa^2 \rho_0 t_x^2$ should be larger than 1 and $\mathcal{O}(1)$. Then we also find $r_1 \sim \mathcal{O}(1)$. This tells that if the ratio of the energy densities of the matter and the dark energy is $\mathcal{O}(1)$, even in the present universe, the ratio is $\mathcal{O}(1)$, which may solve the coincidence problem.

For the model in (30) or (31), by using (30) one obtains

$$r = -1 + \frac{3h_0^2 a_0^3 (1 + w_m)}{\kappa^2 \rho_0} (h_0 + h_1 \cos \omega t)^2$$

$$\times e^{3(1 + w_m)(h_0 t + \frac{h_1}{\omega} \sin \omega t)},$$ \hspace{1cm} (75)

which increases exponentially with time if $w_m > -1$.

Thus, we confirmed that oscillating dark energy may naturally solve the coincidence problem in both (EoS or scalar-tensor) formulations which is not clear from the beginning because the mathematical equivalence does not mean the physical equivalence.
IV. DISCUSSION

In summary, we presented two equivalent formulations of oscillating dark energy. It is demonstrated that such dark energy may describe very naturally most relevant late-time cosmological phenomena: transition from deceleration to acceleration era and transition from non-phantom to phantom epoch (if the universe currently enters such epoch). If the universe currently lives in phantom era, it is demonstrated explicitly how the oscillating dark energy may bring the evolving universe to the one of four known types of finite-time, future singularity (on classical level). Moreover, oscillating dark energy may suggest nice resolution of the coincidence problem as was suggested in refs. \[1, 2\]. Finally, such dark energy unifies in a clear and natural way the early-time inflation and late-time acceleration whatever (quintessence, cosmological constant or phantom) nature they have. However, in order to check if such unification is realistic one, it is necessary to analyse the details of inflation, especially, the preheating scenario. This lies beyond the scopes of our paper because some work in this direction (for phantom inflation) was already done in \[7\]. It is also interesting that using method \[12\] the oscillating dark energy may be presented as some special form of modified gravity (for recent review, see \[17\]).

One can go further and propose the following cosmological model:

\[
H = h_0 \sin \omega t \left(1 + h_1 \sin m \omega t \right). \tag{76}
\]

Here \(m\) is a positive integer, \(h_0, h_1, \omega\) are positive parameters and it is assumed that \(m \gg 1, h_1 < 1\). When \(2\pi n < \omega t < 2\pi n + \pi (n:\text{integer})\), \(H\) is positive and universe is expanding but when \(2\pi n - \pi < \omega t < 2\pi n (n:\text{integer})\), \(H\) is negative and universe is contracting. Hence, the universe is oscillating with period \(2\pi/\omega\). The conjecture could be that our cyclic universe (compare with \[13\]) lives currently in the expanding era (repeating the acceleration/superacceleration circles each 14-15 billion years) while the transition from accelerating to contracting era occurs with much bigger period of time. Since it is assumed \(m \gg 1\), adiabatically one obtains

\[
\dot{H} = h_0 \omega \left\{ \cos \omega t \left(1 + h_1 \sin m \omega t \right) + h_1 \sin \omega t \cos m \omega t \right\} = m \omega h_0 h_1 \sin \omega t \cos m \omega t. \tag{77}\]

Then in the expanding phase \((\sin \omega > 0)\), if \(2\pi n - \pi/2 < \omega < 2\pi n + \pi/2 (n:\text{integer})\), we find \(\dot{H} > 0\), that is, the universe enjoys the phantom era and if \(2\pi n + \pi/2 < \omega < 2\pi n + 3\pi/2 (n:\text{integer})\), \(\dot{H} < 0\), that is, the universe is in non-phantom era. In the contracting phase \((\sin \omega < 0)\), if \(2\pi n - \pi/2 < \omega < 2\pi n + \pi/2 (n:\text{integer})\), \(\dot{H} < 0\) and if \(2\pi n + \pi/2 < \omega < 2\pi n + 3\pi/2 (n:\text{integer})\), \(\dot{H} > 0\). Hence, the universe iterates phantom and non-phantom eras with the period \(2\pi/m\omega\). There is no problem to construct the EoS and/or scalar-tensor description corresponding to scale factor \(a(t)\). Nevertheless, the existing data cannot confirm/reject cyclic cosmological models at high confidence level so one awaits the next generation observational data which will be able to reconstruct the realistic evolution of the cosmological parameters.

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