STRUCTURE OF BINARY SEQUENCES

J. THARRATS

Abstract. The distribution of a given sequence in the set of all sequences with \( n \) ones and \( m = M - n \) zeros are found by relating the problem to the partitions of a natural number in \( m \) natural summands, taking into account the order. The formulas obtained have many applications both in Physics and Mathematics. Examples discussed in the present paper are: non Markovian chains, partition functions of binary alloys and Ising magnets, generalized Kaplansky lemma, generalized Fibonacci numbers and a general expansion of \( \sum_{h=0}^{m} h^r \binom{m}{h}^2 \) in terms of the Stirling numbers of second kind.

1. Introduction

Some combinatorial problems can be solved immediately by referring them to the structure of the \( \left( \binom{m+n}{n} \right) \) sequences of \( m \) zeros and \( n \) ones of length \( N = m+n \), i.e.: knowing the occurrence distribution of a given ordered string of \( r \) 0’s and \( s \) 1’s \( (r+s < N) \). Take, for example the problem of finding the number of subsets of cardinality \( n \) of the set \( X = \{1, 2, \ldots, N\} \) that contain exactly \( h \) times a number \( r \) of consecutive naturals; this is done knowing how many times the string \((1,1,\ldots,r,1)\) has \( h \) recurrences in those sequences. For a non fixed cardinality one makes the sum over \( n \) and obtains how many elements of the power set have the said property. In this case, as one can see in Riordan [3], problem 1(b), p. 14], for \( h = 0, r = 2 \) one obtains the Fibonacci numbers, though Carlitz [3] had generalized to \( h = 0, r > 2 \); (in 1976 Carlitz [4] precises the problem mixing strings of \((00)\) and \((11)\).

Thus, the structure of digit-sequences can have its own place in Combinatorics, introducing problems concerning the sequences themselves; for instance, in the \( \left( \binom{N}{n} \right) \) sequences, the number of sequences that have each pair of ones separated by at least \( k \) zeros. Although the solution is given by Kaplansky lemma [6], the problem becomes trivial by applying the formula which gives the distribution of the \((0,0,\ldots,k,0;1)\) strings, choosing the result which gives exactly \( n \) strings. More generally, we will also find the case in which only \( s \) ones will be separated by at least \( k \) zeros.

In this theory two points of view can be taken: One may decide to bound the sequences with first and last digit or to consider the first digit following the last in cyclic fashion. Since our main interests are physical applications, only the second point of view will be followed. This view, called here “Ising process” gives in the particular case above the corrected Fibonacci numbers. It is also more symmetrical, which explains why our method is so simple and can be developed free of generating functions, in contrast with the method of bounded
sequences in which the particular cases solved by Carlitz give results strongly dependent on the first digit.

This paper was motivated by the solution of non-markovian random walks. As is well-known, the solution of the one-dimensional random walk, with probabilities \( p \) (moving to the right) and \( q = 1 - p \) (to the left), is given by Bernoulli’s distribution:

\[
P(k) = \binom{N}{\frac{N+k}{2}} p^{\frac{N+k}{2}} q^{\frac{N-k}{2}}.
\]

But if transition probabilities are dependent on past states, i.e., if the particle has memory, then the distribution changes radically. In the simple case of dependence on the last state only, the probability is \( \alpha p \) or \( \alpha q \) for particle moving backwards and \( \beta p \) or \( \beta q \) moving forwards \((\alpha + \beta = 1)\). Let \( T_{\tau}^{mn} \) be the number of paths of length \( N \) with \( \tau \) changes in direction of the particle, or “jumps” in \( N \) steps (number of occurrences of \((01)\) or \((10)\)). Then paths like \( ppqqp \) and \( pqpqp \) have different probability because they are affected by the factors \( \alpha^2 \beta^3 \) and \( \alpha^4 \beta \) respectively, and the binomial coefficient in (1) splits in the form

\[
\binom{N}{\frac{N+k}{2}} \rightarrow \sum_{\tau=0}^{N-k} T_{\tau}^{N+k} \binom{N-k}{\frac{N-k}{2}} \alpha^\tau \beta^{N-\tau}.
\]

Thus the formula (1) changes depending on the constant \( \alpha \) which measures the tendency to go forwards \((\alpha < \frac{1}{2})\) or backwards \((\alpha > \frac{1}{2})\). We see that the problem is solved by finding the \( T \)-numbers.

We also will see how \( T \)-numbers give the natural way to find some Thermodynamic Partition Functions, such as in Ising magnets, in contrast with naive ways given by some physicists.

Historically, it seems André [1] was the first to consider such sequences (see Comptet [5]) in order to solve Bertrand’s scrutiny problem. He considered minimal paths in a square lattice where \( y \)-jumps stand for 0’s and \( x \)-jumps for 1’s. This give a simple way to solve Bertrand’s problem counting the number of paths which do not touch the \( y = x \) line. The solution is not invariant by a circular permutation of the sequences and falls in the case of unbounded sequences, for the path \((0010001)\) is allowable (in each step the eventual winner is winning) but the Ising equivalent path \((0100010)\) is not. Processes like this one are also non-markovian because allowed decisions depend of all anterior states, but can be solved by elementary methods; the scrutiny problem can be solved as in [3] or by reducing it to well known random walks avoiding the origin (projecting paths on \( y = -x \)).

Along the way in this paper many combinatorial formulas arise, some very well known but others related to Stirling numbers are new, as far as we know; but what really matters is the natural way in which they arise.

**Notation.** We prefer to preserve the (old fashioned?) symbols for: \( C_n^m = \binom{n}{m} \); \( S_n^m = \text{Stirling numbers of the second kind} \); \( P_n^m = \text{number of partitions of a natural number } n \text{ in } m \text{ natural summands disregarding the order} \); \( M_n^m = \text{number of partitions regarding the order (} M \text{ stands for De Moivre who discovered it to be equal to} \binom{n-1}{m-1} \)).

\[\text{The paper “Analytical Solution of non-Markovian Random Walks” in which we give some applications to crystal lattices, is to appear.}\]
As $P$-partitions can be associated with Young-Sylvester’s tableaux, $M$-partitions too, taking into account the order. For example, the partition of 6 into 3 parts gives the $P_6^3 = 3$ tableaux:

For $M$-partitions, we have the following $M_6^3 = \binom{6}{3} = \frac{3!}{2!1!} + \frac{3!}{1!1!1!} + \frac{3!}{3!1!} = 10$ tableaux:

which we call Young-De Moivre tableaux.

In the unbounded poset of integers, we call $m \wedge n$ the min $(m, n)$ and $m \vee n$ the max $(m, n)$; (no confusion arise with connectors of formal systems).

We will refer always the known combinatorial formulas to the monumental “Ars” by Knuth.

2. $T$-Numbers

Let us consider the set of all sequences with $n$ ones and $m = N - n$ zeros; an element of this set is, for example (for $N = 24, m = 13$):

000001110011010001100111.

(2)

To every sequence we associate a number $\tau$, which is the number of jumps from 0 to 1, or from 1 to 0, including extremes, so in the case of the sequence (2) we have $\tau = 10$. Obviously, the number $\tau$ is even and it is invariant under any circular permutation of the sequence.

The problem here is to find the distribution of $\tau$’s, i.e.: How many $\tau$ equal 2, how many $\tau$ equal 4, etc. in these $\binom{N}{m}$ sequences.

The following table shows the case $N = 7$ and $m = 3$ with the corresponding values of $\tau$:

| $\tau$ | $\tau$ | $\tau$ | $\tau$ | $\tau$ | $\tau$ |
|--------|--------|--------|--------|--------|--------|
| 0000111 | 2      | 0011001 | 4      | 0101010 | 6      | 1000101 | 4      | 1010100 | 6     |
| 0001011 | 4      | 0011010 | 4      | 0101100 | 4      | 1000110 | 4      | 1011000 | 4     |
| 0001101 | 4      | 0011100 | 2      | 0110001 | 4      | 1001001 | 4      | 1100001 | 2     |
| 0001110 | 2      | 0100011 | 4      | 0110010 | 4      | 1001010 | 6      | 1100010 | 4     |
| 0010011 | 4      | 0100101 | 6      | 0110100 | 4      | 1001100 | 4      | 1100100 | 4     |
| 0010101 | 6      | 0100110 | 4      | 0111000 | 2      | 1010001 | 4      | 1101000 | 4     |
| 0010110 | 6      | 0101001 | 6      | 1000011 | 2      | 1010010 | 6      | 1100000 | 2     |

The table gives 7, 21 and 7 sequences ($7 + 21 + 7 = \binom{7}{3}$) for the corresponding $\tau$-values 2, 4 and 6 respectively. This example illustrates the big difficulty in getting these numbers by “brute force” for large values of $N$, (large means here $N > 10$).

\footnote{Knuth uses the Christoffel symbols of first and second kind for Stirling numbers of first and second kind respectively.}
Looking at (4) we see that the number $\tau$ is related by a particular partition of $m$, interconnected by another partition of $n$, both having the same height, i.e., the number $\tau$ is exactly twice the height of these two partitions:

$$h = 5$$

```
00000 111 00 11 0 1 000 11 00 111
```

then, in this case $\tau = 2h = 10$.

We define a “loop” associated to two partitions as the set of $2h$ strings obtained by matching these partitions, the first string regarded as following the last one, In the example we have:

```
00000 | 111 | 00 | 11 | 0 | 1 | 000 | 11 | 00 | 111
```

For a fixed $\tau$, the set of all loops are obtained by matching the partitions of $m$ with the partitions of $n$, both of height $h = \frac{1}{2} \tau$. Fortunately this is done not by ordinary partitions but by partitions regarding the order, giving the numbers:

$$M^h_m = \binom{m-1}{h-1}, \quad M^h_n = \binom{n-1}{h-1}.$$  

The total number of loops is given by:

$$\frac{N}{h} \binom{m-1}{h-1} \binom{n-1}{h-1}.$$  

Now we can get all sequences by making all $N = m + n$ clockwise circular permutations on the digits in each loop, but in the process the $h$ circular permutations displacing an even number of strings belong to different initial loops. Then, the total number of sequences is:

$$\frac{N}{h} \binom{m-1}{h-1} \binom{n-1}{h-1} = \frac{N}{nm} \binom{m}{h} \binom{n}{h}. \quad (3)$$

Finally, taking $2h = \tau$ our formula is:

$$T^m_{\tau} = \frac{\tau}{2\mu} \binom{m}{\frac{1}{2} \tau} \binom{n}{\frac{1}{2} \tau}; \quad \tau = 2, 4, 6, \ldots; \quad \left( \frac{1}{m} + \frac{1}{n} = \frac{1}{\mu} \right). \quad (4)$$

The even index $\tau$ runs from 2 to $2(m \wedge n)$ and $T^m_{\tau}$ is symmetric in the upper indexes $m, n$.

| Table II. (T-numbers) | |
|------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| $h$ | $\tau$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 |
|------------------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| 1 | 2 | 1 | 2 | 1 | 3 | 2 | 1 | 3 | 2 | 1 | 4 | 3 | 2 | 1 |
| 2 | 4 | 2 | 3 | 4 | 5 | 6 | 5 | 6 | 6 | 7 | 7 | 7 | 8 | 8 |
| 3 | 6 | 2 | 5 | 12 | 9 | 21 | 14 | 36 | 32 | 20 | 30 | 24 | 10 | 54 |
| 4 | 8 | 2 | 7 | 24 | 16 | 54 | 30 | 120 | 100 | 50 | 40 | 25 | 1 |
| 5 | 10 | 2 | 9 | 40 | 25 | 120 | 100 | 50 | 40 | 25 | 1 |

The even index $\tau$ runs from 2 to $2(m \wedge n)$ and $T^m_{\tau}$ is symmetric in the upper indexes $m, n$. 
From (4) we have the recurrence formula:
\[ T_{\tau+2}^{mn} = 4 \frac{(m - \frac{1}{2}\tau)(n - \frac{1}{2}\tau)}{\tau(\tau + 2)} T_{\tau}^{mn} \] (5)
and \( T_{2}^{mn} \) = number of clockwise permutations of the sequence
000 \cdots m \cdot 0111 \cdots n \cdot 1 = m + n = N.

In contrast with further generalizations, (5) shows the simplicity of these numbers:
\[ T_{2}^{mn} = N; \]
\[ T_{4}^{mn} = N \cdot \frac{(m - 1)(n - 1)}{1 \cdot 2}; \]
\[ T_{6}^{mn} = N \cdot \frac{(m - 1)(n - 1)}{1 \cdot 2} \cdot \frac{(m - 2)(n - 2)}{2 \cdot 3}; \]
\[ \vdots \] (6)

3. Properties of the Numbers \( T_{\tau}^{mn} \)

At this point we can relate our problem to the probabilities. Since \( \binom{N}{m}^{-1} T_{\tau}^{mn} \) is the probability that a sequence have \( \tau \) jumps, normalisation gives:
\[ \sum_{\tau=2}^{2(m\wedge n)} T_{\tau}^{mn} = \binom{N}{m} \]
i.e.:
\[ \sum_{h=1}^{m\wedge n} h\binom{m}{h} \binom{n}{h} = \frac{mn}{N} \binom{N}{m} \] (7)
\[ \sum_{h=1}^{m} h\binom{m}{h}^2 = \frac{m}{2} \binom{2m}{m} \] (8)

which is a well known formula \(^3\) (Knuth [7, p. 59])

\[^3\text{From} \sum \binom{r}{h} \binom{s}{m-h} = \binom{r+s}{m}, \] (8')

(Knuth [7, p. 58]), with \( m = r = s \), we have also:
\[ \sum_{h=1}^{m} h^2 \binom{m}{h}^2 = \frac{2m}{m} \binom{2m}{m} \] (8'')

A method which we use later, consists of relating these formulas:
\[ \sum_{h=1}^{m} h \binom{m}{h}^2 = \sum_{0}^{m} (m-h) \binom{m}{m-h}^2 = \sum_{0}^{m} (m-h) \binom{m}{h}^2 = m \sum_{0}^{m} \binom{m}{h}^2 - \sum_{1}^{m} h \binom{m}{h}^2, \]
which gives (8'') from (8').
For the ordinary moments of the above distribution we have:

\[
\overline{\tau^r} = \binom{N}{m}^{-1} \sum \tau^r T_{	au mn}^r;
\]

\[
\overline{\tau^r - 1} = \binom{N}{m}^{-1} \sum (2h)^{r-1} \frac{N}{mn} h^r \binom{m}{h} \binom{n}{h} = \binom{N}{m}^{-1} \frac{N}{mn} 2^{r-1} \sum h^r \binom{m}{h} \binom{n}{h}.
\]

(8)

Obviously the probability of jumping is \( p = \frac{2mn}{N(N-1)} \), (probabilities of finding (00) and (11) are \( \frac{m(m-1)}{N(N-1)} \) and \( \frac{n(n-1)}{N(N-1)} \) respectively); then, according to binomial distribution, the probability of jumping exactly \( \tau \) times is:

\[
P_{\tau} = 2 \binom{N}{\tau} p^\tau (1-p)^{N-\tau} \approx \binom{N}{m}^{-1} T_{\tau mn}^r
\]

(9)

(the factor 2 is due to the fact that jumps came by pairs), which shows how the theory of binary sequences would be complicated, even for physical problems, if one follows statistical formulations, because the exact formulas are much more simple than the approximate ones.

As is shown in the following table, statistical approximation is good enough anyway:

| \( \tau \) | 0  | 2  | 4  | 6  | 8  | 10 |
|-----------|----|----|----|----|----|----|
| \( ^{(10)}_5 T_{\tau}^{55} \) | 10 | 80 | 120| 40 | 2  |    |
| Binomial  | 0.15 | 10.66 | 77.71 | 121.42 | 40.65 | 1.46 |

Since both distributions are so close (at least in this example), we can expect a very good approximations identifying its moments. Call \( \overline{k^r} \) and \( \overline{(k)}_r \) the ordinary and factorial moments respectively. We have for a distribution \( p_0, p_1, \ldots, p_{2m\wedge 2n} \):

\[
\overline{k^r} = \sum_{\ell=0}^{2m\wedge 2n\wedge r} \mathbf{S}_r^\ell \overline{(k)}_\ell = \sum_{\ell=0}^{2m\wedge 2n\wedge r} \mathbf{S}_r^\ell G^{(\ell)}(1)
\]

(10)

the \( \mathbf{S}_r^\ell \) being the Stirling numbers of second kind and \( G^{(\ell)}(1) \) the \( \ell \) derivative of the generating function \( G(t) \) at \( t = 1 \); (here we will be interested only in the case \( 2m \wedge 2n \wedge r = r \)). The generating function for Bernouilli’s distribution is \( G(t) = (q + pt)^N \), so we have for (10):

\[
\overline{k^r} = \sum_{\ell=0}^{r} \mathbf{S}_r^\ell (N)_\ell p^\ell
\]

(11)

then, for \( r > 1 \), (for \( r = 1 \) we have the formula (9)):

\[
\sum_{h=1}^{m\wedge n} h^r \binom{m}{h} \binom{n}{h} \approx \binom{N}{m \wedge n} \frac{m^n n^2}{2^{r-2} N(N-1)} \sum_{\ell=0}^{m\wedge n} \mathbf{S}_{r-1}^\ell A_{mn}^\ell
\]

(12)

with

\[
A_{mn}^1 = 1, \quad A_{mn}^\ell = \left( \frac{2mn}{N} \right)^{\ell-1} \frac{(N-2)_{\ell-2}}{(N-1)_{\ell-2}}; \quad (N)_0 = 1.
\]
As we will see in the sequel, approximation (12) is extraordinarily good for \( m = n \), which gives:

\[
\begin{align*}
\sum_{h=0}^{m} h^r \left( \begin{array}{c} m \\ h \end{array} \right)^2 & \approx \binom{2m}{m} \frac{m^3}{2^{r-1}(2m-1)} \left( S_{r-1}^1 + mS_{r-1}^2 + m^2S_{r-1}^3 \right) \frac{2m-2}{2m-1} \\
+ \cdots + m^{r-2}S_{r-1}^{r-1} \frac{2m-2}{2m-1} \cdots \frac{2m-r+2}{2m-1} \right). 
\end{align*}
\]

(13)

For discussion, we write explicitly (13) from \( r = 0 \) to \( r = 5 \):

\[
\begin{align*}
\binom{2m}{m}^{-1} \sum_{h=0}^{m} h^0 \left( \begin{array}{c} m \\ h \end{array} \right)^2 & \approx 1 \quad \text{(13, 0)} \\
\binom{2m}{m}^{-1} \sum_{h=0}^{m} h^1 \left( \begin{array}{c} m \\ h \end{array} \right)^2 & \approx \frac{m}{2} \quad \text{(13, 1)} \\
\binom{2m}{m}^{-1} \sum_{h=0}^{m} h^2 \left( \begin{array}{c} m \\ h \end{array} \right)^2 & \approx \frac{m^3}{2(2m-1)} \quad \text{(13, 2)} \\
\binom{2m}{m}^{-1} \sum_{h=0}^{m} h^3 \left( \begin{array}{c} m \\ h \end{array} \right)^2 & \approx \frac{m^3}{4(2m-1)}(1 + m) \quad \text{(13, 3)} \\
\binom{2m}{m}^{-1} \sum_{h=0}^{m} h^4 \left( \begin{array}{c} m \\ h \end{array} \right)^2 & \approx \frac{m^3}{8(2m-1)} \left( 1 + 3m + m^2 \frac{2m-2}{2m-1} \right) \quad \text{(13, 4)} \\
\binom{2m}{m}^{-1} \sum_{h=0}^{m} h^5 \left( \begin{array}{c} m \\ h \end{array} \right)^2 & \approx \frac{m^3}{16(2m-1)} \left( 1 + 7m + 6m^2 \frac{2m-2}{2m-1} + m^3 \frac{(2m-2)(2m-3)}{(2m-1)^2} \right) \quad \text{(13, 5)}
\end{align*}
\]

We know that formulas (13, 0) and (13, 1) are exact ((7"), (7)) and we will show that (13, 2) and (13, 3) are exact too and also why identity fails after \( r = 3 \).

**Proof.**

\[
\sum_{h=1}^{m} h^2 \left( \begin{array}{c} m \\ h \end{array} \right)^2 = \sum_{h=1}^{m} \left( h \left( \begin{array}{c} m \\ h \end{array} \right) \right)^2 = m^2 \sum_{h=1}^{m} \left( \frac{m-1}{h-1} \right)^2 = m^2 \left( \frac{2m-2}{m-1} \right)
\]

\[
= m^2 \frac{m(2m-m)}{2m(2m-1)} \left( \frac{2m}{m} \right) = \frac{m^3}{2(2m-1)} \left( \frac{2m}{m} \right).
\]

In the general case:

\[
\sum_{h=1}^{m} h^r \left( \begin{array}{c} m \\ h \end{array} \right)^2 = \sum_{h=0}^{m} \left( m-h \right)^r \left( \begin{array}{c} m \\ h \end{array} \right)^2 = \sum_{k=0}^{r} (-1)^k \binom{r}{k} m^{r-k} \sum_{h=0}^{m} h^k \left( \begin{array}{c} m \\ h \end{array} \right)^2
\]

then, we have \( \sum h^r \left( \begin{array}{c} m \\ h \end{array} \right)^2 \) by recurrence only for odd values of \( r \), because when \( k = r = 2 \) the last term cancels with the left hand side, giving only an identity between moments of lower orders. Only the case \( r = 2 \) has escaped from that inconvenient (due to the square in \( \left( \begin{array}{c} m \\ h \end{array} \right) \)) and provides for (13, 3). Thus, for \( r = 3 \), we have:
\[
\binom{2m}{m}^{-1} \sum_{h=0}^{m} h^3 \binom{m}{h}^2 = \left(\binom{2m}{m}^{-1} \sum_{h=0}^{m} \left(\binom{m}{h}^2 - 3m^2 \sum_{i=1}^{m} h \binom{m}{h}^2 + 3m \sum_{i=1}^{m} h^2 \binom{m}{h}^2 - \sum_{h=0}^{m} h^3 \binom{m}{h}^2\right)\right)
\]
i.e.:
\[
\binom{2m}{m}^{-1} \sum_{h=0}^{m} h^3 \binom{m}{h}^2 = \frac{1}{2} \left( m^3 - 3m^2 \frac{m_2}{2} + 3m \frac{m^3}{2(2m-1)} \right) = \frac{m^3(m+1)}{4(2m-1)}.
\]
Recurrence fails for \(\sum h^4 \binom{m}{h}^2\) and so for higher orders. \(\square\)

Formula (13.4) gives the following approximations. For \(m = 2\): \(30/9 \approx 29/9\); for \(r = 3\):
\(11.70 \approx 11.61\); for \(m = 10\): \(827.40 \approx 827.22\); and for \(m = 15\): \(3829.74 \approx 3829.48\). Formula (13.5) gives, for \(m = 10\): \(4895.51 \approx 4891.65\). Then, the distributions \(\binom{2m}{n}^{-1} \tau_{\bar{m}}\) and \(2 \binom{2^m}{r} p^r (1-p)^{N-r}\) have the same variance; higher order moments are almost equal in both distributions.

Let us look for these relations in case \(m \neq n\): Suppose \(m < n\) (the case \(m > n\) is the same); according to (8)’ we have:
\[
\sum_{h=0}^{m} \binom{m}{h} \binom{n}{h} = \sum_{h=0}^{m} \binom{m}{h} \binom{n}{n-h} = \binom{m+n}{n},
\]
then:
\[
\sum_{h=1}^{m \wedge n} h^2 \binom{m}{h} \binom{n}{h} = \sum_{h=1}^{m} h \binom{m}{h} \binom{n}{h} = mn \sum_{h=1}^{m-1} \binom{m-1}{h-1} \binom{n-1}{h-1} = mn \sum_{h=0}^{m-1} \binom{m-1}{h} \binom{n-1}{h} = mn \binom{m+n-2}{m-1} = \frac{m^2 n^2}{N(N-1)} \binom{N}{m}
\]
which is the formula (12) for \(r = 2\). For \(r = 3\), formula (12) gives:
\[
\sum_{h=1}^{m \wedge n} h^3 \binom{m}{h} \binom{n}{h} \approx \binom{N}{m \wedge n} \left( \frac{1}{2} \frac{m^2 n^2}{N(N-1)} + \frac{m^3 n^3}{N^2(N-1)} \right)
\]
which is only approximate; for \(m = 4, n = 2, (N = 6)\) gives \(56 \approx 58.67\). Of course the approximation will be much better for higher \(N\). Then, for \(m \neq n\), only the mean values are equal in the above distributions, though higher order moments are very close in both distributions.

Let us consider the set of all \(2^N\) sequences of length \(N\). One may expect here complete agreement with probabilities, i.e. jumps must follow the binomial distribution with \(p = \frac{1}{2}\). The next proof shows that this is the case:

Proof. From
\[
\sum_{0 \leq c \leq a} \binom{a-c}{j} \binom{b+c}{k} = \binom{a+b+1}{j+k+1}
\]

(Knuth [4, p. 58]), we have \((b = 0, j = k)\):
\[
\sum_{0 \leq c \leq a} \binom{a-c}{k} \binom{c}{k} = \binom{a+1}{2k+1}
\]
Then the exact number of sequences with \(\tau\) jumps ((01) or (10)) in all \(2^N\) sequences is, according formula (3):
\[
\sum_{n=0}^{N} T_{\tau}^{N-n,n} = \frac{N}{h} \sum_{n} \binom{(N-2)-(n-1)}{h-1} \binom{n-1}{h-1} = \frac{N}{h} \binom{N-1}{2h-1} = 2 \binom{N}{2h} = 2 \binom{N}{\tau}.
\]
So, the probability of jumping \(\tau\) times:
\[
\sum_{n} T_{\tau}^{n,n} / 2^N = \frac{N}{\tau} / 2^{N-1} \text{equals that given by Bernouilli's distribution (3) with } p = \frac{1}{2}: 2 \binom{N}{\tau} \left(\frac{1}{2}\right)^{\tau} \left(\frac{1}{2}\right)^{N-\tau}, (\tau = 0, 2, \ldots, N \text{ (or } N - 1)).
\]

**Application to the Ising model of ferromagnetism.** Let us consider a one-dimensional ferromagnet in absence of external field and put \(\nu = J/kT\) \((J = \text{interaction energy of antiparallel spins } (-J \text{ for parallel}); k \text{ is the Boltzmann constant and } T \text{ the absolute temperature})\). The energy of a given distribution of \(N\) spins is \(J(-(N-\tau)+\tau) = J(-N+2\tau)\), because there are \(\tau\) couplings between antiparallel spins. The partition function for a fixed number of \(N-n\) and \(n\) spins up and spins down respectively, is:
\[
Z = \sum_{\tau} T_{\tau}^{N-n,n} e^{-(N+2\tau)\nu}
\]
(this is also the partition function corresponding to binary alloys (ions A and B instead of spins up and down)). Fortunately (for the Physicists) one must take the sum of all configurations which gives:
\[
Z = \sum_{\tau} \sum_{n} T_{\tau}^{N-n,n} e^{-(N+2\tau)\nu} = 2^{N\nu} \sum_{\tau} \binom{N}{\tau} (e^{-2\nu})^\tau = e^{N\nu} ((1+e^{-2\nu})^N + (1-e^{-2\nu})^N) = (2 \cosh \nu)^N + (2 \sinh \nu)^N.
\]
One wonders why some authors give the solution which contains only the hyperbolic cosine (see, for example, [8, ch. 5]; sinh appears in more elaborate discussion [8, ch. 4, p. 121]) though both solutions are equal for \(N \rightarrow \infty\). We think their countings are not accurate enough, for they take the following recurrence: \(Z_N = (2 \cosh \nu) Z_{N-1}\) on grounds that a new spin provides an amount of \(\pm J\) in the energy, i.e. forgetting a border effect. In fact, consider three spins: A up, B down and C up and add a new spin D; the old energy is \(2J - J = J\), and the new energy is 0 for D up or \(4J\) for D down and not \(J \pm 2J\) given by the above rule.

4. Asymptotic Expansions

After \(N = 8\) we can use the asymptotic approximation for \(T_{\tau}^{nn}\), with the help of Stirling’s formula which gives:
\[
\binom{m}{h} \approx 2^{m+1} e^{-\frac{(2h-m)^2}{2m}} \sqrt{2\pi m}
\]
(15) is surprising accurate even for low values of $h$ ($< m$); $m = 10, h = 3$ gives: $120 = \binom{10}{3} \approx 116.1$. (For $h = \frac{1}{2} m$ we have $\sqrt{\pi N} = \frac{2^{N+1} N!}{\sqrt{2\pi N}}$ which is, of course, the Wallis' formula because $\left(\frac{N}{2N+1}\right)^2 = \frac{\pi N}{N+2}$ and since $\pi_2 = \frac{(2^3)^2}{2^2}(\frac{2}{3})^{-2} = 4$, we have:

$$\pi_4 = 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3}; \pi_6 = 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5}; \cdots; \pi_\infty = \pi = 2 \cdot \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \cdots).$$

We can expect a good approximation for $T_{mn}^h$ when $N \geq 10$; in fact the following formula gives, for $N = 10$ a result very close to the Table II:

$$T_{mn}^h \approx h \left(\frac{1}{m} + \frac{1}{n}\right) \frac{2^{m+n+2}}{2\pi \sqrt{mn}} e^{-2h^2\left(\frac{1}{m} + \frac{1}{n}\right) + 4h - \frac{1}{2}(m+n)}$$

or:

$$T_{mn}^h \approx 2N e^{aN} \frac{\pi \mu^{3/2}}{\sqrt{N}} e^{-4\left(\frac{\mu^2}{2N}\right) - h}$$

In terms of $\tau$'s

$$T_{mn}^\tau \approx \tau e^{-\tau^2 \frac{\mu^2}{2N} + 2 \tau + aN}$$

For $N = 10, m = n = 5$: $T_{55}^h = 0.3514 h \exp \left( -4\left(\frac{h^2}{5} - h\right) \right)$ and we have the following results, as is shown in Figure 1.

**Figure 1.** Asymptotic expansion for $T_{55}^h$

$$T_{55}^h = 0.3514 x \exp \left( -4\left(\frac{x^2}{5} - x\right) \right)$$
At the origin, the derivative of (16) is not zero but $2(\pi \mu \sqrt{mn})^{-1}e^{aN}$ and the curve takes, though very small, negative values for $x < 0$.

On considering the set of all $2^N$ sequences, the distribution of jumps, for a large $N$ is, with (14) and (15):

$$\sum_n \mathbf{T}_{\tau}^{N-n,n} = 2^{N+2} \frac{e^{-\frac{(2\tau - N)^2}{2N}}}{\sqrt{2\pi N}}. \quad (18)$$

Here the Gaussian has nothing to do with statistical distributions, because in this context (18) gives fluctuations, respect to the exact number of jumps, due only to the fact that $N$ is not large enough. Of course (18) becomes a normal probability distribution with standard error $\pm \sqrt{N}$ when the sequences of $N$ digits are taken from a table of binary random digits.

5. The Numbers $\mathbf{T}_h^{mn}(00)$ and $\mathbf{T}_h^{mn}(11)$

Generalized Fibonacci Numbers

Let $U$ be an ordered sequence of $r$ zeros and $s$ ones, $|U| = r + s < N$; being $\tau$ the amount of (01)'s or (10)'s in the sequence $S$, our general problem will consist in finding the number of times, $h = h(U)$, a fixed string $U$ is contained in $S$ in the following sense: If $S_i = \alpha_1 \alpha_2 \alpha_3 \cdots \alpha_N$, $(\alpha_j = 0, 1)$, take the $N$ strings:

$$(\alpha_1 \alpha_2 \alpha_3 \cdots \alpha_{r+s}), (\alpha_2 \alpha_3 \alpha_4 \cdots \alpha_{r+s+1}), \ldots, (\alpha_{N-1} \alpha_N \alpha_1 \cdots \alpha_{r+s-2}), (\alpha_N \alpha_1 \alpha_2 \cdots \alpha_{r+s-1}),$$

then $h$ is the number of these strings having the form $U$. For instance, take in (2) $U = (1100)$, then $h = 3$. We call $\mathbf{T}_h^{mn}U$ the number of sequences which have $h$ times the string $U$.

Now, since after any (01) in a sequence $S$ sooner or later another 0 appears (Ising model!), then $h(10) = h(01) = \frac{1}{2}\tau$, i.e.,

$$\mathbf{T}_h^{mn}(01) = \mathbf{T}_h^{mn}(10) = \mathbf{T}_\tau^{mn}. \quad (19)$$

The numbers $\mathbf{T}_h^{mn}(00)$ and $\mathbf{T}_h^{mn}(11)$ are found by a simple consideration: From all $N$ pairs successively taken in $S$, $m = \text{the number of zero digits} = \text{the number} \ h + \frac{1}{2}\tau$, of them having the form (00) or (01), so the $\tau$ corresponding to $S$ is $2(m - h)$, and we have:

$$\mathbf{T}_h^{mn}(00) = \mathbf{T}_2^{mn}(m - h), \quad \text{similarly:} \quad \mathbf{T}_h^{mn}(11) = \mathbf{T}_2^{mn}(n - h) \quad (20)$$

then:

$$\mathbf{T}_h^{mn}(00) = N \frac{m - h}{mn} \binom{m}{h} \binom{n}{m - h}, \quad \mathbf{T}_h^{mn}(11) = N \frac{n - h}{mn} \binom{m}{m - h} \binom{n}{h} \quad (21)$$

which gives the obvious identity: $\mathbf{T}_h^{mn}(00) = \mathbf{T}_h^{mn}(11)$. 

---

**Table IV.**

| $\tau$ | 2   | 4   | 6   | 8   | 10  | 12  |
|--------|-----|-----|-----|-----|-----|-----|
| $\mathbf{T}_\tau^{55}$ | 10  | 80  | 120 | 40  | 2   | 0   |
| Asymptotic | 8.62 | 80.40 | 127.95 | 34.44 | 1.76 | 0.02 |
Normalization. Since the only values of \( h \) for the \( T^m_{h^n}(00) \) are between \( m \) and \( m - (m \wedge n) \), we have, (the same for \( T^m_{h^n}(11) \)):
\[
\sum_{m - h = 0}^{m \wedge n} T^m_{h^n}(00) = \sum_{m - h = 0}^{m \wedge n} T^m_{n(2m - h)} = \sum_{j = 0}^{m \wedge n} T^m_{2j} = \sum_{\tau = 0}^{m \wedge n} T^m_{T} = \binom{N}{m}.
\]

Let \( F^*(N; r, h) \) be the number of subsets of \( X = \{1, 2, 3, \ldots, N\} \) that do contain exactly \( h \) times \( r \) consecutive integers modulo \( N \); as in Berge \[3\] p. 31] let us associate with each subset \( Y \subseteq X \) a “word” \( S = \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_N \), where \( \alpha_i = 0 \) if \( i \not\in Y \) and \( \alpha_i = 1 \) if \( i \in Y \). The mapping between subsets \( Y \) and words \( S \) being bijective, instead of counting subsets, we can count the number of the sequences \( S \) in which the string \( (11 \ldots 1) \) is repeated exactly \( h \) times. We call Generalized corrected Fibonacci numbers the number of these subsets, i.e.:
\[
F^*(N; r, h) = \sum_n T^{N-n,n}_{h^n}(11 \ldots 1)
\]
then:
\[
F^*(N; 2, h) = \sum_n T^{N-n,n}_{h^n}(11) = \frac{\binom{N+h}{N-h}}{(N-n)} \binom{N-n}{h} \binom{N}{h}
\]

Example. \( X = \{1, 2, 3, 4\}, r = 2, h = 2 \). In this case, in \((23)\) the only index is \( n = 3 \), then \( F^*(4; 2, 2) = 4 \). The subsets are: \( \{1, 2, 3\}, \{2, 3, 4\}, \{3, 4, 1\}, \{2, 4, 1\} \). For \( h = 0 \) one obtains the corrected Fibonacci numbers: (see Berge, \[3\] p. 32)]
\[
F^*(N; 2, 0) = F^*_N = \sum_{n=1}^{\lfloor N \rfloor} \frac{N}{N-n} \binom{N-n}{h}
\]

Next, to each of \( \binom{N}{m} = \binom{N}{n} \) sequences we will associate a type, which is the pair of partitions
\[
(\beta^1_1, \beta^2_2, \ldots, \beta^s_s), (\beta'^1_1, \beta'^2_2, \ldots, \beta'^s_s)
\]
with height \( h \), i.e.: \( \sum_i \alpha_i = \sum_i \alpha_i' = h \), corresponding to the tableaux of 0’s and 1’s, respectively, disregarding the order of the blocks, i.e.: in descending order of the length of blocks, as is customary. Two types are equal iff the have both Young’s tableaux equal. For instance, for \( m = 4, n = 3 \), the 35 sequences of the Table have the following four types:
\[
1 = \{(4), (3)\}, \quad 2 = \{(3, 1), (2, 1)\}, \quad 3 = \{(2^2), (2, 1)\}, \quad 4 = \{(2, 1^3), (1^3)\}.
\]
Thus,
\[
\text{type\{0011100\} = 1, \quad \text{type\{0110100\} = type\{0001011\} = 2, \quad etc.}
\]
In total, for a fixed \( m, n \), there are the number
\[
\sum_{h=1}^{m \wedge n} P^h_m P^h_n
\]
of types. The great number of sequences compared to the small number of types is due to the \( \mathcal{M} \)-partitions; in fact, the number of sequences per type is:
\[
\frac{N h! (h - 1)!}{\alpha_1! \alpha_2! \cdots \alpha_s! \alpha'_1! \alpha'_2! \cdots \alpha'!}
\]
because the $\mathbf{M}$-partitions are found making the sum: $h! \sum \frac{1}{\alpha_1! \alpha_2! \cdots \alpha_s!}$ of all partitions with same $h$. This clarifies the meaning of the formulas:

\begin{align*}
T^{mn}_h(01) &= \frac{N}{n} \mathbf{M}^h_m \mathbf{M}^h_n, \\
T^{mn}_h(00) &= \frac{N}{n} \mathbf{M}^{m-h}_m \mathbf{M}^{n-h}_n, \\
T^{mn}_h(11) &= \frac{N}{n} \mathbf{M}^{m-h}_m \mathbf{M}^{n-h}_n.
\end{align*}

(26)

6. The Numbers $T^{mn}_h U$ for $|U| = 3$

The number of formulas can be reduced by some general considerations; for example, if $U'$ is the string obtained by interchanging in $U$ the 0's and the 1's we have: $T^{mn}_h U' = T^{mn}_h U$. Furthermore, since sequences are unbounded, after any string $(01 \cdots 1)$ in the sequence sooner or later another zero appears, then $T^{mn}_h (11 \cdots 10) = T^{mn}_h (011 \cdots 1)$; also, $T^{mn}_h (00 \cdots 1) = T^{mn}_h (10 \cdots 00)$, so $T^{mn}_h (11 \cdots 10) = T^{mn}_h (10 \cdots 00)$, etc. For $U = 3$ we have the following identities

\begin{align*}
T^{mn}_h (001) &= T^{mn}_h (100) = T^{mn}_h (110) = T^{mn}_h (011); \\
T^{mn}_h (101) &= T^{mn}_h (010); \\
T^{mn}_h (111) &= T^{mn}_h (000).
\end{align*}

Thus we only need formulas for the three expressions:

\begin{align*}
T^{mn}_h (001), \quad T^{mn}_h (101), \quad T^{mn}_h (000).
\end{align*}

(27)

Let $P^h_m$ be the number of partitions, disregarding order, of weight $m$ and dimension $h$; as is well known, the process of deleting the first column on all associated Young’s Tableaux gives:

\begin{align*}
P^h_m = \sum_{\ell=1}^{s} P^{\ell}_{m-h} \quad (s = h \wedge (m - h)).
\end{align*}

(28)

Let us try a similar expansion for $\mathbf{M}$-partitions:

\begin{align*}
\mathbf{M}^h_m = \sum_{\ell=1}^{s} a_{\ell} \mathbf{M}^{\ell}_{m-h},
\end{align*}

(29)

formula (30) gives:

\begin{align*}
\sum_{\ell=1}^{s} \binom{h}{\ell} \mathbf{M}^{\ell}_{m-h} = \frac{1}{m-h} \sum_{\ell=1}^{s} \ell \binom{h}{\ell} \binom{m-h}{\ell} = \frac{h}{m} \binom{m}{h} = \mathbf{M}^h_m.
\end{align*}

(30)

Then the coefficients $a_{\ell}$ in the formal expansion (29) are equal to $\binom{h}{\ell}$. We shall prove that this expansion is not only formal which proves the following lemma: Define $c^i_{jk} = \binom{k}{i-j} \binom{i-j}{k}$ for $i \neq j$ and $c^i_{jk} = i \delta_{0k}$, so that (24) has the form: $\sum c^i_{jk} = \mathbf{M}^i_j$. We have,

**Lemma I**(1). The number of $\mathbf{M}$-partitions of $m$ in $h$ parts is $\mathbf{M}^h_m$ but after deleting the first column in all Young-De Moivre Tableaux we have $\mathbf{M}$-partitions of dimensions $\ell = 0, 1, 2, \ldots, h \wedge (m - h)$, which number is $c^m_{h\ell}$. 
In order that the counting makes sense we must define $c^m_{m0} = 1$. Further, it is important to take in consideration the fact that deleting the first column is made for counting purposes only. Consider, for instance, this process for $m = 6, h = 2$; we have the $\binom{5}{1}$ following tableaux:

\[
\begin{aligned}
\begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
\hline
\end{array}
\quad \quad
\begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
\hline
\end{array}
\quad \quad
\begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
\hline
\end{array}
\quad \quad
\begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
\hline
\end{array}
\quad \quad
\begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
\hline
\end{array}
\quad \quad
\begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
\hline
\end{array}
\end{aligned}
\]

then $c^6_{20} = 2$ means that the tableau \begin{array}{cccc}
\hline
& & & \\
& & & \\
& & & \\
\hline
\end{array}
\end{array}
\]
\] shall be counted twice because it has different sites with respect to the first column.

Proof of Lemma I$^{(1)}$. Before deleting the first column, the number of tableaux which will give tableaux of dimension $\ell$ after the cut, is (Figure 2):

\[
\mathcal{N} = \sum \frac{h!}{(h - \ell)! \alpha_2! \alpha_3! \cdots} = \frac{h!}{(h - \ell)!} \sum \frac{1}{\alpha_2! \alpha_3! \cdots}
\]

where the sum is extended over all tableaux. After deleting the first column, these tableaux will give $\mathbf{M}^\ell_{m-h}$ tableaux of dimension $\ell$, which number is:

\[
\ell! \sum \frac{1}{\alpha_2! \alpha_3! \cdots} \quad (= \mathbf{M}^\ell_{m-h})
\]

then:

\[
\mathcal{N} = \frac{h!}{(h - \ell)!} \frac{\mathbf{M}^\ell_{m-h}}{\ell!} = \binom{h}{\ell} \mathbf{M}^\ell_{m-h} = c^m_{h \ell}.
\]
Due to importance of coefficients $c_{jk}$ we give the tables (for matrices of low $k$) in Appendix A. It is immediate to find a recurrence relation:

$$c_{j,k+1} = c_{j,k} \frac{(j-k)(i-j-k)}{k(k+1)}.$$  \hspace{1cm} (31)

Now it is a simple task to find $T_{mn}^{mn}(01)$; as in the case of $T_{mn}^{mn}(01)$ we match tableaux of zeros with tableaux of ones of same height $h$. Since strings (001) are originated from the rows of 0’s tableaux which remain non empty after deleting the first column, here in the first formula (26) instead of $M_{h}^{m}$ we put $c_{h\ell}^{m}$, which gives a number $\ell$ of (001) strings; these $\ell$ strings are a subset of the $h$ strings (01), ($h \geq \ell$), because both (001) and (101) give (01). Then we have an intermediate formula for (001):

The number of sequences in which the string (01) appears $h$ times and the string (001) appears $\ell$ times is given by:

$$T_{h,\ell}^{mn}(01; 001) = \frac{N}{h} c_{h\ell}^{m} M_{h}^{n}$$  \hspace{1cm} (32)

(Note that: $\sum_{\ell} T_{h,\ell}^{mn}(01; 001) = \frac{N}{h} M_{h}^{n} \sum_{\ell} c_{h\ell}^{m}$, which is formula (26)). Thus $T_{\ell}^{mn}(001)$ is found by disregarding $h$ in (32), i.e.:

$$T_{\ell}^{mn}(001) = N \sum_{h} \frac{c_{h\ell}^{m}}{h} M_{h}^{n} = \frac{N}{h} \sum_{h} c_{h\ell}^{m} \left( \begin{array} {c} n \hfill \cr h \hfill \end{array} \right).$$  \hspace{1cm} (33a)

Let us find the exact limits on the sum; obviously we have: $\ell = 0, 1, 2, \ldots, \left[ \frac{m}{2} \right] \wedge n$, and the inferior limit corresponds to $h = \ell$. Condition $h \leq m, n$ must be restricted because in $c_{jk}$ the low index $k$ runs from $k = 0$ to $k = i - j$, then $h \leq m - \ell, n$ and finally:

$$T_{\ell}^{mn}(001) = \frac{N}{n} \sum_{h=\ell}^{(m-\ell) \wedge n} c_{h\ell}^{m} \left( \begin{array} {c} n \hfill \cr h \hfill \end{array} \right)$$

$$= \frac{N}{n} \sum_{h=\ell}^{(m-\ell) \wedge n} \frac{\ell}{m-h} \left( \begin{array} {c} h \hfill \cr \ell \hfill \end{array} \right) \left( \begin{array} {c} m-h \hfill \cr \ell \hfill \end{array} \right) \left( \begin{array} {c} n \hfill \cr h \hfill \end{array} \right).$$  \hspace{1cm} (33b)

In that follows some binomial coefficients will be taken outside the sum by help of the formula: $\binom{a}{b} \binom{c}{d} = \binom{a-c}{b-d} \binom{a}{b}$, then (33) is:

$$T_{\ell}^{mn}(001) = \frac{N}{n} \binom{n}{\ell} \sum_{h=\ell}^{(m-\ell) \wedge n} \left( \begin{array} {c} m-h-1 \hfill \cr \ell-1 \hfill \end{array} \right) \left( \begin{array} {c} n-\ell \hfill \cr h-\ell \hfill \end{array} \right), \quad \left( \begin{array} {c} 0 \hfill \cr 0 \hfill \end{array} \right) = 1.$$  \hspace{1cm} (33c)

Let us take an example which can be checked by hand on the 70 sequences of 4 zeros and 4 ones. From the tables of Appendix A we have:

| $h$ | 0 | 1 | 2 | 3 | 4 |
|-----|---|---|---|---|---|
| 0   | 1 |   |   |   |   |
| 1   |   | 1 |   |   |   |
| 2   |   | 2 | 1 |   |   |
| 3   |   | 3 |   |   |   |
| 4   |   | 4 |   |   |   |

then $T_{h,\ell}^{44}(01; 001) = \frac{8}{4} c_{h\ell}^{4} \left( \begin{array} {c} 4 \hfill \cr h \hfill \end{array} \right) = \left( \begin{array} {c} 4 \hfill \cr h \hfill \end{array} \right)$.
The sum rows and columns gives $T_{\ell}^{44}(01)$ and $T_{\ell}^{44}(001)$ respectively:

| $\ell$ | 0 | 1 | 2 | 3 | 4 | $T_{\ell}^{44}(01)$ |
|------|---|---|---|---|---|-----------------|
| 0    |   |   |   |   |   | 0               |
| 1    | 8 |   |   |   |   | 8               |
| 2    | 24| 12|   |   |   | 36              |
| 3    | 24|   |   |   |   | 24              |
| 4    | 2 |   |   |   |   | 2               |
|      | 2 | 56| 12| 9 | 0 | $T_{\ell}^{44}(001)$ |

Thus: $T_{0}^{44}(001) = 2$, $T_{1}^{44}(001) = 56$, $T_{2}^{44}(001) = 12$. The first corresponds to two sequences: \{01010101\}, \{10101010\}.

Taking in (33b) the sum over $m$, we have also the distribution of the string (001) for all $2^N$ binary sequences of cardinality $N$:

$$T_N^{\ell}(001) = \sum_{m=\ell+1}^{N} \binom{m-\ell+n}{\ell} \binom{m-h}{\ell} \binom{N-m}{h}$$

(34)

$$= \frac{N}{\ell} \sum_m \left( \binom{N-m-1}{\ell-1} \sum_h \binom{m-h-1}{\ell-1} \binom{N-m-\ell}{h-\ell} \right).$$

Turning to the second expression in (27), we can see in Figure 2 that to each tableau of dimension $\ell$ obtained by deleting the first column, there corresponds the bottom $h - \ell$ positions of $h$ which are responsible for the strings (101), then: 

The number of sequences in which the string (01) appears $h$ times and the string (101) appears $\ell$ times is given by:

$$T_{h,\ell}^{mn}(01;101) = \sum_h \binom{m-n}{h} \binom{m-h}{\ell} \binom{N}{n} c_h^{m-n} c_{h-\ell}^{m} \left( \begin{array}{c} n \\ h \end{array} \right).$$

(35)

Moreover:

$$T_{\ell}^{mn}(101) = \sum_h \binom{m-h}{\ell} \binom{N}{h} c_h^{m-n} \left( \begin{array}{c} n \\ h \end{array} \right).$$

(36a)

Here, the index $\ell$ runs from 0 to $m \wedge (n + \delta_{mn} - 1)$ because the extreme cases have the form (for $m > n$, $m < n$ and $m = n$):

\{000 · · · 0101 · · · 1010 · · · 000\}, \{111 · · · 1010 · · · 101 · · · 111\} and \{10101 · · · 1010\}.

In the sum, index $h$ goes from $h = \ell$ to $h = \left\lfloor \frac{m+\ell}{2} \right\rfloor$ (because $h = \ell = m$ gives $c_m^m = 1$ and $m - h = h - \ell$ for the high index), then:
The number of sequences of \( m \) 0’s and \( n \) 1’s which have \( \ell \) strings (101) is:

\[
T_{mn}^{\ell} (101) = \frac{N}{n} \binom{m+\frac{\ell}{2}}{\ell} \sum_{h=\ell}^{n} \frac{h-\ell}{m-h} \left( \frac{h}{\ell} \right) \left( \frac{m-h}{h-\ell} \right) \left( \frac{n}{h} \right)
\]

(36b)

There remains only the last expression in (27); however we must first see how the numbers \( T_{000} \) are related to string (0001) and so we must postpone its derivation until after the case \( |U| = 4 \). To justify this delay and find a way to obtain \( T_{000} \) we will see that, though more complicated than (20), formula (21) for \( T_{00} \) can be found also in terms of strings in which \( |U| = 3 \). Let \( g \) be a fixed number of strings (00), it is easy to see (Figure 2) that:

\[
m - g = g
\]

i.e., there are as many (00) as zeros in the subtableau of length \( \ell \). After deleting the first column of length \( h = m - g \) there appears a number \( c_{m-g,\ell}^{m} \) of such subtableaux, in total an amount of:

\[
c_{m-g,\ell}^{m} \sum_{h=\ell}^{\infty} \frac{h-\ell}{m-h} \left( \frac{h}{\ell} \right) \left( \frac{m-h}{h-\ell} \right) \left( \frac{n}{h} \right)
\]

(36b)

The last derivation is due to:

\[
\sum_{0}^{m-g} \binom{g}{\ell} \binom{m-g}{\ell} = \sum_{0}^{g} \binom{g}{\ell} \binom{m-g}{\ell} = \sum_{0}^{(m-g)\wedge g} \binom{g}{\ell} \binom{m-g}{\ell},
\]

allowing to use formula (37).

7. The Numbers \( T_{\ell}^{mn}(00^{s+2}01) \).

Generalized Kaplansky Lemma

Define:

\[
c_{jk}^{i} = \sum_{f=k}^{j} \binom{j}{f} \binom{f}{k} \binom{i-j-f}{k-i-j} = \binom{j}{k} \sum_{f=k}^{j} \binom{j-k}{f-k} \binom{i-j-f-1}{k-1}
\]

(37a)

and:

\[
c_{j0}^{i} = \begin{cases} \binom{i}{j} & \text{if } i \leq 2j, \\ 0 & \text{if } i > 2j. \end{cases}
\]

(37b)

then,
Lemma I(2). After deleting the first and second columns in all Young-De Moivre tableaux of weight \( m \) and dimension \( h \) one obtains \( M \)-partitions of dimensions \( \ell = 0, 1, 2, \ldots, h \wedge (m - 2h) \) which number is \( c'_{m}^{h} \).

The proof is like I(1), (see Figure 3).

The desired number is, (the sum without indexes is extended of all \( M \)-partitions):

\[
\mathcal{N} = \sum_{\ell'} \frac{h!}{(h - \ell')!(\ell' - \ell)!} \cdot \frac{1}{\alpha_3! \alpha_4! \cdots} = \ell! \sum_{\ell'} \left( \frac{h}{\ell'} \right) \left( \frac{\ell'}{\ell} \right) \sum \frac{1}{\alpha_3! \alpha_4! \cdots}.
\]

on the other hand:

\[
\ell! \sum \frac{1}{\alpha_3! \alpha_4! \cdots} = \ell! \sum \frac{1}{\alpha_3! \alpha_4! \cdots} = M'_{m-h-\ell'}.
\]

Then:

\[
\mathcal{N} = \sum_{\ell' = \ell}^{h \wedge (m - h - \ell)} \left( \frac{h}{\ell'} \right) \left( \frac{\ell'}{\ell} \right) M'_{m-h-\ell'} = c'_{m}^{h} \quad (\ell \neq 0);
\]

the upper limit is due to the number of elements \( m - h - \ell' \) in the subtableau which must be at least equal to \( \ell \). In the case \( \ell = 0 \) the symbol \( M'_{m-h-\ell'} \) is meaningless. In this case however we observe that the condition \( m - h - \ell' = 0 \) gives \( \mathcal{N} = c'_{m}^{h} = \binom{h}{m-h} \).
The number of sequences with \( m \) zeros and \( n \) ones in which the string \((01)\) appears \( h \) times, string \((001)\) appears \( \ell' \) times and \((0001)\) \( \ell \) times is:

\[
T_{h,\ell'}^{mn}(01;001;0001) = \frac{N}{n} \binom{h}{\ell} \binom{h-\ell}{\ell'-\ell} \binom{m-h-\ell'-1}{\ell-1} \binom{n}{h},
\]

applying the binomial coefficients make elementary computations, it is always convenient to give fundamental formulas in terms of \( c \)'s because its matrices can be tabulated independently of particular cases. We have for \( T_{(0001)}'s \):

\[
T_{\ell}^{mn}(0001) = \frac{N}{n} \sum_{h=\ell}^{n} c_{h}^{m} M_{n}^{h}
\]

\[
= \frac{N}{n} \binom{n}{\ell} \sum_{h=\ell}^{n} \sum_{\ell'=h}^{n} \binom{n-\ell'}{h-\ell} \binom{n-\ell'}{\ell'-\ell} \binom{m-h-\ell'-1}{\ell-1}
\]  \( (\ell \neq 0) \),

\[
T_{0}^{mn}(0001) = \frac{N}{n} \sum_{h=\left[\frac{h}{n}\right]}^{n} \binom{n}{m-h} \binom{n-m+h}{2h-m}.
\]

For the general case we define:

\[
c_{j,k}^{(s)i} = \sum_{i-j-f_{1}-f_{2} \cdots - f_{s} \geq k}^{k} \binom{j}{f_{1}} \binom{j}{f_{2}} \cdots \binom{j}{f_{s}} \binom{f_{1}}{i-j-f_{1}-f_{2} - \cdots - f_{s}} \binom{f_{2}}{i-j-f_{1}-f_{2} - \cdots - f_{s}} \cdots \binom{f_{k}}{i-j-f_{1}-f_{2} - \cdots - f_{s}-1}
\]

\[
= \binom{j}{k} \sum_{j \geq f_{1}, f_{2}, \ldots, f_{s} \geq k}^{j} \binom{j-f_{1}}{f_{1}-f_{1}} \binom{j-f_{2}}{f_{2}-f_{2}} \cdots \binom{j-f_{s}}{f_{s}-f_{s}} \binom{j-k}{f_{s}-k}
\]

\[
\text{if } i \leq (s+1)j,
\]

\[
= \sum_{0}^{\sum_{j \geq f_{1}, f_{2}, \ldots, f_{s} \geq k}^{j} \binom{j-f_{1}}{f_{1}-f_{1}} \binom{j-f_{2}}{f_{2}-f_{2}} \cdots \binom{j-f_{s}}{f_{s}-f_{s}} \binom{j}{f_{s}}}
\]

\[
\text{if } i > (s+1)j;
\]

then,

**Lemma I.** After deleting the \( s+1 \) columns in all Young-De Moivre tableaux of weight \( m \) and dimension \( h \) one obtain \( M \)-partitions of dimensions \( \ell = 0,1,2,\ldots,h \wedge (m-(s+1)h) \) which number is \( c_{h,\ell}^{(s)m} \). (This generalizes \( I^{(1)} \) and \( I^{(2)} \).)
So, we have:

\[ T_{mn}^{\ell_1^{n+1} \cdots \ell_s} (01; 001; \cdots; 00 \cdots 01) = \frac{N}{h_c^{(s)m}} M_n^h, \]

\( c_h^{(s)m} \) being the \( c_h^{(s)m} \) without summation on \( \ell', \ell'', \ldots, \ell(s) \). And:

\[ T_{\ell n}^{mn} (00 \cdots 01) = \frac{N}{n} \left( \begin{array}{c} m - n - sn - 1 \\ n - 1 \end{array} \right) = \frac{N}{N - pn} \left( \begin{array}{c} N - pn \\ n \end{array} \right), \]

Formula (41) generalizes Kaplansky’s lemma (for a redefinition of Kaplansky’s lemma see Comtet [5, p. 35]). In particular, taking \( h = \ell' = \ell'' = \cdots = \ell(s) = n \), we have:

\[ g_n(N, p) = \frac{N}{n} \left( \begin{array}{c} m - n - sn - 1 \\ n - 1 \end{array} \right) = \frac{N}{N - pn} \left( \begin{array}{c} N - pn \\ n \end{array} \right), \]

being \( p = s + 2 \) the least length of blocks which separate the ones.

8. The Numbers \( T_{mn}^{n s+2} (00 \cdots 00) \).

FIBONACCI F*(N; r, h) NUMBERS

Our final problem will consist of looking at the weights \( g \) of the subtableaux, instead of the dimension \( \ell \). We will call \( c_h^{(s)mg} \) the number of subtableaux of weight \( g \), which remain after deleting \( s + 1 \) successive columns, starting from the first, in all the Young-De Moivre tableaux of weight \( m \) (\( > g \)) and dimension \( h \).

For \( s = 0 \), in Figure 2 we can see that the weight of the remaining subtableaux is \( m - h \), then \((\Pi(1)):\n
\[ c_h^{mg} = \sum_{\ell=0}^{h} c_{m-g, \ell}^m = \sum_{g=0}^{m-g} \frac{\ell}{g} \binom{m-g}{\ell} \binom{g}{\ell} = \frac{m-g}{m} \binom{m}{g} \]

(42)

so \( c_h^{mg} = c^{mg} \) does not depend on \( h \), i.e.: Counting shall be made on all tableaux of weight \( m \). Though indirectly, it is thanks to numbers \( c_h^{mg} \) that \( T(00) \) was found and we may now write formula (21) as \( \frac{N}{m-g} c^{mg} M_n^{m-g} \).

For \( s = 1 \), it is easy to see that the problem of counting strings \((000)\) is given also by the formula (39) but collecting the terms with the same \( g \), i.e., from (37) we obtain \( c^{ijg} \), taking in the sum only the term \( f = i - j - g \):

\[ c^{ij}_{jk}(g) = \frac{k}{g} \binom{j}{i-j-g} \binom{g}{k}, \quad c^{ij}_{jk} = \sum_k c^{ij}_{jk}(g). \]
We have (see Figure 3):

\[
c_{mg}^m = \sum_{\ell} \sum_{m-h-\ell=g} \binom{h}{\ell} \binom{\ell}{g} M_{m-h-\ell} = \sum_{\ell=0}^{h} \binom{h}{\ell} \binom{h-\ell}{m-g-h-\ell} M_{g} \tag{43a}
\]

\[
= \frac{1}{g} \sum_{\ell=0}^{h\wedge g(m-h-g)} \binom{h}{\ell} \binom{g}{\ell} \binom{h-\ell}{m-g-h-\ell}, \quad (g \neq 0);
\]

\[
c_{m0}^m = c_{m0}^m = \begin{cases} \binom{h}{m} & \text{if } m \leq 2h, \\
0 & \text{if } m > 2h. \end{cases} \tag{43b}
\]

Then we have:

**Lemma II**. After deleting the first and second column in all Young-De Moivre tableaux of weight \(m\) and dimension \(h\) one obtains \(M\)-partitions of weights \(g = 0 \lor (m-2h), 1 \lor (m-2h), \ldots, m-h-1\) which number is \(c_{mg}^m\).

In order to clarify ambiguity, let us make the process for \(m = 7, h = 3\):

\[
c_{73}^3 = 3
\]

\[
c_{72}^3 = 9 \quad (\text{two types})
\]

\[
c_{71}^3 = 3
\]

In \(c_{72}^3\) tableaux, first and fourth give the same position for the subtableau of weight 2, but since the second column is different that will give different matchings; for example (\(n = 9\)),

\[
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 1 & 1 & 1 & 1 & 1
\end{array}
\quad \text{and} \quad
\begin{array}{cccccc}
0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 1 & 1
\end{array}
\]

give different sequences, \(\{00001111011101\} \quad \text{and} \quad \{00001111101101\}\).

Now, at last, the coefficients \(c_{mg}^m\) provide for the numbers \(T(000)\) (and so for \(T(111)\)). In fact, there are as many strings \((000)\) as zeros \((g)\) in the subtableaux of weight \(g\), so that:

\[
T_{mn}^{mn}(000) = \sum_{h} \frac{N}{h} c_{mg}^m M_{n} = \frac{N}{n} \sum_{h} c_{mg}^m \binom{n}{h}. \tag{44}
\]

Since the matrices \(c_{mg}^m \quad (g \text{ fixed})\) can be tabulated very easily, as in Appendix A, it is preferable to work with them; nonetheless if we write explicitly all binomial coefficients, we have:

\[
T_{mn}^{mn}(000) = \frac{N}{n} \sum_{h} \sum_{\ell} \ell \binom{n}{\ell} \binom{n}{h} \binom{n-\ell}{m-h-g-\ell} \binom{n-m+h+g}{2h+g-m}, \quad (g \neq 0) \tag{45a}
\]
The following table gives the full results:

| T  | (111) | (110) | (101) | (100) | (011) | (001) | (001) | (000) |
|----|-------|-------|-------|-------|-------|-------|-------|-------|
| T^53_0 (000) | 48    | 16    | 24    | 0     | 16    | 8     | 0     | 8     |
| T^53_1 (000) | 8     | 40    | 24    | 32    | 40    | 32    | 32    | 24    |
| T^53_2 (000) | 0     | 0     | 0     | 8     | 24    | 0     | 24    | 16    |
| T^53_3 (000) | 0     | 0     | 0     | 0     | 0     | 16    | 0     | 8     |

In general, for \( s \geq 1 \), take \( i - j - f_1 - f_2 - \cdots - f_s = g \) and define:

\[
c_j^{(s)g} = \frac{1}{g} \sum_{k=1}^{g} \sum_{\substack{f_1 \geq f_2 \geq \cdots \geq f_s \geq 0 \atop f_1 + f_2 + \cdots + f_s = i - j - g}} k \binom{j}{f_1} \binom{f_1}{f_2} \cdots \binom{f_s}{g} \binom{g}{k}, \quad (j \geq f_i \geq k; g \neq 0),
\]  

(46a)

and:

\[
c_j^{(s)0} = \sum_{\substack{f_1 \geq f_2 \geq \cdots \geq f_s \geq 0 \atop f_1 + f_2 + \cdots + f_s = i - j}} \binom{j}{f_1} \binom{f_1}{f_2} \cdots \binom{f_{s-1}}{f_s}, \quad (j \geq f_i \geq k).
\]  

(46b)
(From hereon in order to calculate the $c$’s one has to deal with the number of partitions $P_{i-j}$, $P_{i-j-g}$, etc. which complicates the formulas). Then we have:

**Lemma II.** After deleting the first $s+1$ columns in all Young-De Moivre tableaux of weight $m$ and dimension $h$ one obtains $M$-partitions of weights $g = 0 \lor (m - (s + 1)h), 1 \lor (m - (s + 1)h), \ldots, m - h - s$ which number is $c_h^{(s)mg}$. (This generalizes II(1) and II(2).)

This lemma provides for $U = (00 \cdots 0)$ or $U = (11 \cdots 1)$:

$$T_{mn} g(00 \cdots 0) = \sum_{h} N \frac{c_h^{(s)mg}}{h} M_n = \frac{N}{n} \sum_{h} c_h^{(s)mg} \binom{n}{h}, \quad (47a)$$

and:

$$T_{mn} g(11 \cdots 1) = \frac{N}{m} \sum_{h} c_h^{(s)mg} \binom{m}{h}. \quad (47b)$$

So, the generalized Fibonacci numbers are:

$$F^s(N; s, g) = N \sum_{n} \sum_{h} c_h^{(s-2)mg} \frac{N-n}{N-n} \binom{N-n}{h}. \quad (48)$$

For example, the number of subsets of $X = \{1, 2, 3, \ldots, N\}$ that do not contain three consecutive integers modulo $N$ is:

$$F^s(N; 3, 0) = N \sum_{n} \sum_{h=\lceil \frac{n}{2} \rceil} c_h^{n0} \frac{N-n}{N-n} \binom{N-n}{h} = N \sum_{n=1}^{\frac{N-n}{2}} \sum_{h=\lceil \frac{n}{2} \rceil} \left( 1, \frac{N-n}{h} \right) \frac{(N-n)}{N-n}. \quad (49)$$

**Example.** In (49) take $N = 5$, we have:

$$F^s(5; 3, 0) = 5 \left\{ 0 + 0 + \frac{1}{2} \left( \begin{array}{c} 2 \\ 3 - 2 \end{array} \right) \left( \begin{array}{c} 2 \\ 2 \end{array} \right) + \frac{1}{3} \left[ \left( \begin{array}{c} 1 \\ 1 \end{array} \right) \left( \begin{array}{c} 3 \\ 1 \end{array} \right) + \left( \begin{array}{c} 2 \\ 0 \end{array} \right) \left( \begin{array}{c} 3 \\ 2 \end{array} \right) \right] + \frac{1}{4} \left( \begin{array}{c} 1 \\ 0 \end{array} \right) \left( \begin{array}{c} 4 \\ 1 \end{array} \right) \right\}$$

$$= 20.$$  

The twenty subsets are:

$$\{1, 2, 4\}, \{1, 3, 4\}, \{1, 3, 5\}, \{2, 3, 5\}, \{2, 4, 5\},$$
$$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{1, 5\}, \{2, 3\}, \{2, 4\}, \{2, 5\}, \{3, 4\}, \{3, 5\},$$
$$\{1\}, \{2\}, \{3\}, \{4\}, \{5\}.$$  

**APPENDIX A**

The recurrence relation (31) gives the matrices $c_{jk}^i$, we have, $i$ rows, $j$ columns, $k$ fixed):
### $c_{j0}^i$

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 1 |   | 1 |   |   |   |   |   |   |   |    |    |    |
| 2 |   |   | 2 |   |   |   |   |   |   |    |    |    |
| 3 |   |   |   | 3 |   |   |   |   |   |    |    |    |
| 4 |   |   |   |   | 4 |   |   |   |   |    |    |    |
| 5 |   |   |   |   |   | 5 |   |   |   |    |    |    |
| 6 |   |   |   |   |   |   | 6 |   |   |    |    |    |
| 7 |   |   |   |   |   |   |   | 7 |   |    |    |    |
| 8 |   |   |   |   |   |   |   |   | 8 |    |    |    |
| 9 |   |   |   |   |   |   |   |   |   | 9 |    |    |
| 10|   |   |   |   |   |   |   |   |   |    | 10 |    |
| 11|   |   |   |   |   |   |   |   |   |    |    | 11 |
| 12|   |   |   |   |   |   |   |   |   |    |    |    |

### $c_{j1}^i$

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 1 |   |   |   |   |   |   |   |   |   |    |    |    |
| 2 |   |   |   | 1 |   |   |   |   |   |    |    |    |
| 3 |   |   |   | 1 | 2 |   |   |   |   |    |    |    |
| 4 |   |   |   | 1 | 2 | 3 |   |   |   |    |    |    |
| 5 |   |   |   | 1 | 2 | 3 | 4 |   |   |    |    |    |
| 6 |   |   |   | 1 | 2 | 3 | 4 | 5 |   |    |    |    |
| 7 |   |   |   | 1 | 2 | 3 | 4 | 5 | 6 |    |    |    |
| 8 |   |   |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 |    |    |
| 9 |   |   |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |    |
| 10|   |   |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 11|   |   |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 12|   |   |   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 |

### $c_{j2}^i$

|   | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
|---|---|---|---|---|---|---|---|---|---|----|----|----|
| 1 |   |   |   |   |   |   |   |   |   |    |    |    |
| 2 |   |   |   |   |   |   |   |   | 1 |    |    |    |
| 3 |   |   |   |   |   |   |   |   | 2 | 3 |    |    |
| 4 |   |   |   |   |   |   |   | 4 |   | 9 | 12 |    |
| 5 |   |   |   |   |   |   |   | 5 | 12 | 18 | 20 | 15 |
| 6 |   |   |   |   |   |   | 6 | 15 | 24 | 30 | 30 | 21 |
| 7 |   |   |   |   |   | 3 | 6 | 6 | 4 | 9 | 12 | 10 |
| 8 |   |   |   |   | 5 | 12 | 18 | 20 | 15 | 6 | 15 | 24 |
| 9 |   |   | 6 | 15 | 24 | 30 | 30 | 21 | 7 | 18 | 30 | 45 |
| 10|   | 8 | 21 | 36 | 50 | 60 | 63 | 56 | 36 | 8 | 21 | 36 |
| 11| 9 | 24 | 42 | 60 | 75 | 84 | 84 | 72 | 45 |   |   |   |
| 12|   |   |   |   |   |   |   |   |   |   |   |   |   |
\[
c^i_{j3} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
6 & 1 \\
7 & 3 & 4 \\
8 & 6 & 12 & 10 \\
9 & 10 & 24 & 30 & 20 \\
10 & 15 & 40 & 60 & 60 & 35 \\
11 & 21 & 60 & 100 & 120 & 105 & 56 \\
12 & 28 & 84 & 150 & 200 & 210 & 168 & 84
\end{bmatrix}
\]

\[
c^i_{j4} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
6 & 1 \\
7 & 4 & 5 \\
8 & 10 & 20 & 15 \\
9 & 20 & 50 & 60 & 35 \\
10 & 35 & 100 & 150 & 140 & 70
\end{bmatrix}
\]

\[
c^i_{j5} = \begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 \\
6 & 1 \\
7 & 5 & 6 \\
8 & 15 & 30 & 21
\end{bmatrix}
\]

And the finite matrices \(c^i_{jk}\) (\(i\) fixed, \(j\) rows, \(k\) columns) are:
Let us see how the matrices $c'_{jk}$ can be obtained from the matrices $c_jk$. Formula (37) can be written:

$$c'_{jk} = \sum_f c_{jk} (j_f), \quad \text{i.e.,} \quad C'_k = C_k \tilde{P}$$

(1.A)
where \((C'_k)_{ij} = c''_{jk}, (C) = c\), and \((\mathcal{P})_{rs} = \binom{s}{r}\) is the Pascal matrix, so \((r \text{ rows}, s \text{ columns}):\)

\[
(\tilde{\mathcal{P}})_{rs} = \binom{s}{r} = 1 2 3 4 5 6 7 \cdots
\]

\[
\begin{array}{cccccccc}
1 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
2 & 1 & 3 & 6 & 10 & 15 & 21 & 28 \\
3 & 1 & 4 & 10 & 20 & 35 & 56 & 87 \\
4 & 1 & 5 & 16 & 35 & 70 & 126 & 220 \\
5 & 1 & 6 & 21 & 56 & 126 & 252 & 462 \\
\vdots & & & & & & & \\
\end{array}
\]

We have successively \((i \text{ rows}, j \text{ columns}):\)

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
2 & 1 & & & & & & & & & \\
3 & 1 & 2 & & & & & & & & \\
4 & 1 & 2 & 3 & & & & & & & \\
5 & 1 & 2 & 3 & 4 & & & & & & \\
6 & 1 & 2 & 3 & 4 & 5 & & & & & \\
7 & 1 & 2 & 3 & 4 & 5 & 6 & & & & \\
8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & & & \\
9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & & \\
10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\
11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]

\[
\begin{array}{cccccccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\
2 & 1 & & & & & & & & & \\
3 & 1 & & & & & & & & & \\
4 & 1 & 2 & & & & & & & & \\
5 & 1 & 2 & 3 & & & & & & & \\
6 & 1 & 2 & 3 & 4 & & & & & & \\
7 & 1 & 2 & 3 & 4 & 5 & & & & & \\
8 & 1 & 2 & 3 & 4 & 5 & 6 & & & & \\
9 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & & & \\
10 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & & \\
11 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\
12 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
\end{array}
\]
Finally, formula (43) can be written

\[
c'_{kj} = \sum_{f} \binom{j - f}{i - (k + 2f) - (j - k)} \binom{j}{f} \binom{k - 1}{f - 1},
\]

which gives successively (i rows, j columns):

|   | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 |
|---|----|----|----|----|----|----|----|----|----|----|----|
| 1 | 1  |    |    |    |    |    |    |    |    |    |    |
| 2 | 1  | 1  |    |    |    |    |    |    |    |    |    |
| 3 | 2  | 1  |    |    |    |    |    |    |    |    |    |
| 4 | 1  | 3  | 1  |    |    |    |    |    |    |    |    |
| 5 | 3  | 4  | 1  |    |    |    |    |    |    |    |    |
| 6 | 1  | 6  | 5  | 1  |    |    |    |    |    |    |    |
| 7 | 4  | 10 | 6  | 1  |    |    |    |    |    |    |    |
| 8 | 1  | 10 | 15 | 7  | 1  |    |    |    |    |    |    |
| 9 | 5  | 20 | 21 | 8  | 1  |    |    |    |    |    |    |
|10 | 1  | 15 | 35 | 28 | 9  |    |    |    |    |    |    |

\[c'_{i2} = \begin{array}{cccccccccc}
6 & 1 \\
7 & 2 & 3 \\
8 & 3 & 9 & 6 \\
9 & 4 & 15 & 24 & 10 \\
10 & 5 & 21 & 48 & 50 & 15 \\
11 & 6 & 27 & 72 & 120 & 90 & 21 \\
12 & 7 & 33 & 96 & 200 & 255 & 147 & 28
\end{array}
\]

\[c'_{i3} = \begin{array}{cccccccccc}
6 & 1 \\
7 & 3 \\
8 & 1 & 4 \\
9 & 3 & 4 \\
10 & 6 & 16 & 10 \\
11 & 10 & 36 & 50 & 20 \\
\end{array}
\]
\( c'_{2j} = \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & & & & & & & & \\
2 & & & & & & & & \\
3 & & & & & & & & \\
4 & & & & & & & & \\
5 & & & & & & & & \\
6 & & & & & & & & \\
7 & & & & & & & & \\
8 & & & & & & & & \\
9 & & & & & & & & \\
10 & & & & & & & & \\
\end{array} \)

\( c'_{3j} = \begin{array}{cccccccc}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\
1 & & & & & & & & \\
2 & & & & & & & & \\
3 & & & & & & & & \\
4 & & & & & & & & \\
5 & & & & & & & & \\
6 & & & & & & & & \\
7 & & & & & & & & \\
8 & & & & & & & & \\
9 & & & & & & & & \\
10 & & & & & & & & \\
\end{array} \)

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DEPARTMENT OF PHYSICS, UNIVERSITY OF PUERTO RICO AT RÍO PIEDRAS, PO BOX 23343, SAN JUAN PR 00931-3343
E-mail address: tharrats@rrpac.upr.clu.edu