EXISTENCE OF NODAL SOLUTIONS FOR THE SUBLINEAR MOORE-NEHARI DIFFERENTIAL EQUATION

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Abstract. We study the existence of symmetric and asymmetric nodal solutions for the sublinear Moore-Nehari differential equation, \( u'' + h(x, \lambda)|u|^{p-1}u = 0 \) in \((-1, 1)\) with \( u(-1) = u(1) = 0 \), where \( 0 < p < 1 \), \( h(x, \lambda) = 0 \) for \( |x| < \lambda \), \( h(x, \lambda) = 1 \) for \( \lambda \leq |x| \leq 1 \) and \( \lambda \in (0, 1) \) is a parameter. We call a solution \( u \) symmetric if it is even or odd. For an integer \( n \geq 0 \), we call a solution \( u \) an \( n \)-nodal solution if it has exactly \( n \) zeros in \((-1, 1)\). For each integer \( n \geq 0 \) and any \( \lambda \in (0, 1) \), we prove that the equation has a unique \( n \)-nodal symmetric solution with \( u'(-1) > 0 \). For integers \( m, n \geq 0 \), we call a solution \( u \) an \((m, n)\)-solution if it has exactly \( m \) zeros in \((-1, 0)\) and exactly \( n \) zeros in \((0, 1)\). We show the existence of an \((m, n)\)-solution for each \( m, n \) and prove that any \((m, m)\)-solution is symmetric.

1. Introduction and main results. In this paper, we consider the existence of symmetric and asymmetric nodal solutions for the sublinear Moore-Nehari differential equation

\[
 u'' + h(x, \lambda)|u|^{p-1}u = 0 \quad \text{in} \quad (-1, 1), \quad u(-1) = u(1) = 0,
\]

where \( 0 < p < 1 \), \( h(x, \lambda) = 0 \) for \( |x| < \lambda \) and \( h(x, \lambda) = 1 \) for \( \lambda \leq |x| \leq 1 \) and \( \lambda \in (0, 1) \) is a parameter. Note that a solution does not belong to \( C^2(-1, 1) \) because \( h(x, \lambda) \) is not continuous. Since \( h \in L^\infty(-1, 1) \), we consider a solution in a Sobolev space \( W^{2,\infty}(-1, 1) \). Hereafter, we denote by \( W^{m,q}(a,b) \) a Sobolev space which consists of all \( u \in L^q(a,b) \) such that all the distributional derivatives up to order \( m \) lie in \( L^q(a,b) \). We call \( u \) a solution of (1.1) if it belongs to \( W^{2,\infty}(-1, 1) \) and it satisfies the first equation of (1.1) in the distribution sense and \( u(-1) = u(1) = 0 \). Then \( u \) is a solution defined above if and only if

\[
 u \in C^1[-1, 1] \cap C^2([-1, 1] \setminus \{\pm \lambda\}),
\]

and it satisfies the first equation of (1.1) pointwisely in \((-1, 1) \setminus \{\pm \lambda\}\) and \( u(-1) = u(1) = 0 \). We state the result by Moore and Nehari [15]. They introduced (1.1) and obtained the next result.

Theorem 1.1 (Moore and Nehari [15]). Let \( p > 1 \). For some \( \lambda \in (0, 1) \), (1.1) possesses at least three positive solutions: an even positive solution \( u \), a noneven positive solution \( v \) and its reflection \( v(-x) \).

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The theorem above seems to be similar to a result due to Smets, Willem and Su [17], who investigated the Hénon equation
\[-\Delta u = |x|^\lambda u^p, \quad u > 0 \quad \text{in} \ B, \quad u = 0, \quad \text{on} \ \partial B, \quad (1.2)\]
where $B$ is a unit ball in $\mathbb{R}^N$ and $1 < p < \infty$ when $N = 1, 2$ and $1 < p < (N + 2)/(N - 2)$ when $N \geq 3$. To state the result in [17], we define the Rayleigh quotient $R(u)$ and the Nehari manifold $\mathcal{N}$,
\[R(u) := \left( \int_B |\nabla u|^2 \, dx \right) \left( \int_B |x|^\lambda |u|^{p+1} \, dx \right)^{2/(p+1)},\]
\[\mathcal{N} := \{ u \in H_0^1(B) \setminus \{0\} : \int_B (|\nabla u|^2 - |x|^\lambda |u|^{p+1}) \, dx = 0 \} .\]
A minimizer $u$ of $R$ on $\mathcal{N}$ is called a least energy solution, i.e., $u$ satisfies
\[u \in \mathcal{N} \quad \text{and} \quad R(u) = \inf_{v \in \mathcal{N}} R(v).\]
A least energy solution exists and it becomes a positive solution of (1.2). In [17], they proved that if $\lambda > 0$ is sufficiently large, no least energy solution of (1.2) is radial. Therefore (1.2) has both a positive radial solution and a positive nonradial solution.

We see that the graph of $h(x, \lambda)$ with $\lambda \in (0, 1)$ close to 1 is similar to that of $|x|^\lambda$ with $\lambda$ large enough. Therefore a nonradial least energy solution obtained by [17] is like a noneven solution given by [15]. Indeed, in the definition of $R(u)$, we replace $|x|^\lambda$ by $h(x, \lambda)$ and $B$ by the interval $(-1, 1)$. Then the author [5] proved that no least energy solution of (1.1) is even when $\lambda \in (0, 1)$ is close to 1 (see [6, 7, 8] also).

In (1.1), the coefficient function $h(x, \lambda)$ vanishes for $|x| \leq \lambda$. A similar function was investigated by López-Gómez and Rabinowitz [11]. They studied a coefficient function which vanishes in a subinterval of $(-1, 1)$. In [11], they proved the existence of a positive solution and nodal solutions (see also [12, 13, 14]).

Gritsans and Sadyrbaev [3] investigated (1.1) for $p = 3$ and proved that (1.1) has infinitely many sign-changing solutions. Kajikiya, Sim and Tanaka [10] studied the bifurcation problem of positive solutions for (1.1) with $p > 1$. They proved that for fixed $p > 1$, (1.1) has a unique even positive solution $U(x, \lambda)$ for any $\lambda \in (0, 1)$ and a noneven positive solution bifurcates from the curve $U(x, \lambda)$ at a certain $\lambda^* \in (0, 1)$.

In the present paper, we shall find symmetric and asymmetric nodal solutions of (1.1) for $0 < p < 1$.

**Definition 1.2.** We call a solution $u$ of (1.1) symmetric if it is even ($u(-x) = u(x)$) or odd ($u(-x) = -u(x)$). For a nonnegative integer $n$, we call a solution $u$ $n$-nodal if it has exactly $n$-zeros in $(-1, 1)$. We call a solution $u$ $n$-nodal symmetric if it is both $n$-nodal and symmetric.

By definition, a 0-nodal solution is either a positive solution or a negative solution. An $n$-nodal symmetric solution is clearly an even solution if $n$ is even and an odd solution if $n$ is odd.

**Definition 1.3.** For nonnegative integers $m$ and $n$, we call a solution $u$ an $(m, n)$-solution if it has exactly $m$ zeros in the interval $(-1, 0)$ and exactly $n$ zeros in $(0, 1)$. 
If \( m \neq n \), an \((m,n)\)-solution is obviously asymmetric. Consider a \(2m\)-nodal even solution or a \((2m+1)\)-nodal odd solution. These solutions become \((m,m)\)-solutions. Then we have a question whether any \((m,m)\)-solution must be symmetric or whether an asymmetric \((m,m)\)-solution exists. In [9], we have proved the next result.

**Theorem 1.4** (Kajikiya [9]). Let \( p > 1 \). Then for any nonnegative integer \( m \), there exists a \( \lambda_m \in (0,1) \) such that for \( \lambda \in (\lambda_m, 1) \), \((1.1)\) has an \((m,n)\)-solution for each \( 0 \leq n \leq m \) and has an asymmetric \((m,m)\)-solution.

It seems that Theorem 1.4 remains valid for \( p < 1 \). However, there exists a difference between the superlinear case \( p > 1 \) and the sublinear case \( p < 1 \). This claim comes from the uniqueness of positive solutions for the sublinear equation. The uniqueness is also valid for the elliptic equation

\[-\Delta u = f(x,u), \quad u > 0, \quad \text{in} \ \Omega, \quad u = 0, \quad \text{on} \ \partial \Omega. \quad (1.3)\]

Brezis and Oswald [2] proved the next theorem.

**Theorem 1.5** (Brezis and Oswald [2]). Let \( \Omega \) be a bounded smooth domain in \( \mathbb{R}^N \) and let \( f(x,s) \) be a continuous function on \( \Omega \times [0,\infty) \). Suppose that for each \( x \in \Omega \), \( f(x,s)/s \) is strictly decreasing with respect to \( s \in (0,\infty) \) and there is a \( C > 0 \) such that \( f(x,s) \leq C(s+1) \) for \( x \in \Omega \) and \( s > 0 \). Then (i) and (ii) below hold.

(i) \((1.3)\) has at most one solution.

(ii) A solution of \((1.3)\) exists in \( W^{2,q}(\Omega) \) for any \( 1 \leq q < \infty \) if and only if

\[\lambda_1(-\Delta - a_0(x)) < 0 < \lambda_1(-\Delta - a_\infty(x)),\]

where \( \lambda_1(-\Delta - a(x)) \) denotes the first eigenvalue of \(-\Delta - a(x)\) with the zero Dirichlet boundary condition and \( a_0(x), a_\infty(x) \) are defined by

\[a_0(x) := \lim_{s \to 0} f(x,s)/s, \quad a_\infty(x) := \lim_{s \to \infty} f(x,s)/s.\]

Put \( f(x,u) := h(x,\lambda)|u|^{p-1}u \) with \( 0 < p < 1 \). Since \( f(x,u)/u \) is not strictly decreasing with respect to \( u \) for \( |x| < \lambda \), we can not apply Theorem 1.5 to \((1.1)\). However we have the next result.

**Theorem 1.6.** Let \( 0 < p < 1 \) and let \( \lambda \in (0,1) \). Then \((1.1)\) has at most one positive solution and it becomes an even positive solution.

The theorem above ensures the uniqueness of a positive solution. The existence will be given by Theorem 1.7 later on. Theorem 1.6 says that an asymmetric \((0,0)\)-solution does not exist when \( 0 < p < 1 \). However, by Theorem 1.4, an asymmetric \((0,0)\)-solution exists when \( p > 1 \) and \( \lambda \in (0,1) \) is close to 1.

Hereafter, we always assume that \( 0 < p < 1 \). In view of Theorem 1.6, we conjecture that an asymmetric \((m,m)\)-solution does not exist. This is true. We shall show this claim, which is different from Theorem 1.4.

The purpose of the present paper is to prove the existence and uniqueness of \( n \)-nodal symmetric solution for \((1.1)\) and the existence of \( n \)-nodal asymmetric solutions and to show that \((1.1)\) has an \((m,m)\)-solution for \( m \neq n \) but it has no asymmetric \((m,m)\)-solutions for any \( m \geq 0 \). We stress that the equation \((1.1)\) seems simple, however, the structure of the solution space is very complicated.

Note that if \( u \) is an \( n \)-nodal solution, then \(-u\) is also \( n \)-nodal. To uniquely determine it, we introduce the condition \( u'(-1) > 0 \).
Theorem 1.7. Let \( \lambda, p \in (0, 1) \). For any nonnegative integer \( n \), there exists a unique \( n \)-nodal symmetric solution of (1.1) with \( u'(-1) > 0 \).

Theorem 1.8. Let \( 0 < p < 1 \). For each integer \( n \geq 0 \), there exists a \( \delta_n \in (0, 1) \) such that for \( \lambda \in (0, \delta_n) \), an \( n \)-nodal solution of (1.1) with \( u'(-1) > 0 \) is unique and it is symmetric.

The two theorems above are valid for \( p > 1 \) also, which is proved in our paper [9]. We explain the difference between Theorems 1.7 and 1.8. Theorem 1.7 ensures that a solution is unique in a class of all \( n \)-nodal symmetric solutions. However, Theorem 1.8 means that a solution is unique in a class of all \( n \)-nodal solutions when \( \lambda > 0 \) is small enough.

Let us consider an \((m,n)\)-solution. If \( u \) is an \((m,n)\)-solution with \( m \leq n \), we take \( u(-x) \), which is an \((n,m)\)-solution with \( n \geq m \). Therefore it is enough to find an \((m,n)\)-solution with \( m \geq n \).

Theorem 1.9. Let \( 0 < p < 1 \). For each integer \( m \geq 1 \), there exists a \( \lambda_m \in (0, 1) \) such that for \( \lambda \in (\lambda_m, 1) \), (1.1) has an \((m,n)\)-solution for each \( n \) satisfying \( 0 \leq n \leq m - 1 \).

Obviously, an \((m,n)\)-solution with \( m \neq n \) is asymmetric. Therefore the theorem above gives us an \((m+n)\)-nodal asymmetric solution.

In many papers (see [16] and the references cited therein), the authors proved the existence of solutions with prescribed numbers of zeros in \((-1,1)\), that is, the existence of an \( n \)-nodal solution. We emphasize that an asymmetric \((m,n)\)-solution becomes an \((m+n)\)-nodal solution, however, our solution provides more precise information than that of an \((m+n)\)-nodal solution. In the next theorem, we shall show that an asymmetric \((m,m)\)-solution does not exist.

Theorem 1.10. Let \( \lambda, p \in (0, 1) \). For any nonnegative integer \( m \), any \((m,m)\)-solution is symmetric. Therefore it must be either a \( 2m \)-nodal even solution or a \((2m + 1)\)-nodal odd solution, in other word, the set of all \((m,m)\)-solutions consists only of \( \pm \phi_{2m} \) and \( \pm \phi_{2m+1} \), where \( \phi_n \) is a unique \( n \)-nodal symmetric solution satisfying \( \phi_n'(-1) > 0 \).

To prove the theorems above, we organize the present paper into seven sections. In Section 2, we show the Sturm comparison theorem and the Sturm separation theorem for discontinuous coefficient functions. In Section 3, we investigate the properties of solutions for the sublinear Emden-Fowler equation ((1.1) with \( h(x, \lambda) \equiv 1 \)) and study the relation between the energy of a solution \( u \) and the distance between two consecutive zeros of \( u \). In Section 4, we prove the nondegeneracy of any nontrivial solution for the Emden-Fowler equation. In Section 5, we investigate the sublinear Moore-Nehari equation (1.1) and prove Theorems 1.6 and 1.7. In Section 6, using the nondegeneracy of nontrivial solutions, we prove Theorem 1.8. In Section 7, we prove Theorems 1.9 and 1.10 by using lemmas obtained in Sections 3-5.

2. Sturm comparison theorem. In this section, we prove the Sturm comparison theorem for discontinuous coefficient functions. For a finite interval \((a,b)\), we consider the equations

\[
\begin{align*}
  u'' + h(x)u &= 0, \\
  v'' + H(x)v &= 0,
\end{align*}
\]

(2.1) (2.2)
Lemma 2.2 (Sturm separation theorem). Let \( h \in L^1(a,b) \). Let \( u, v \in W^{2,1}(a,b) \) be nontrivial solutions of (2.1) which are linearly independent. Then the zeros of \( u \) separate and are separated by those of \( v \).

In many books (see e.g. [4, p.334]), the Sturm comparison theorem is proved for continuous functions \( h(x) \) and \( H(x) \). However, it remains valid for \( h, H \in L^1(a,b) \). When \( h \in L^1(a,b) \), a solution \( u \) of (2.1) is considered in \( W^{2,1}(a,b) \). We call \( u \) a solution of (2.1) if \( u \in W^{2,1}(a,b) \) and it satisfies (2.1) in the distribution sense. It is known (see e.g. [1, p.206, Remark 8]) that \( W^{1,1}(a,b) \) coincides with the set of absolutely continuous functions on \([a,b]\). Therefore \( u \) is a solution defined above if and only if \( u \in C^1[a,b] \) and \( u' \) is absolutely continuous and \( u'' \) satisfies (2.1) almost everywhere.

Lemma 2.1 (Sturm comparison theorem). Suppose that \( h, H \in L^1(a,b) \) and \( h(x) \leq H(x) \) a.e. in \((a,b)\). Let \( u, v \in W^{2,1}(a,b) \) be nontrivial solutions of (2.1) and (2.2), respectively. Suppose that \( u(a) = u(b) = 0 \) and \( u \neq 0 \) in \((a,b)\). Then either (i) or (ii) below holds:

(i) \( v \) has at least one zero in \((a,b)\),
(ii) \( v \) is a constant multiple of \( u \).

The assertion (i) only holds true if the set of points \( x \) satisfying \( h(x) < H(x) \) has a positive Lebesgue measure.

Proof. We suppose that \( u(x) > 0 \) in \((a,b)\) after replacing \( u \) by \(-u\), if necessary. We assume that the assertion (i) does not hold, i.e., \( v(x) \) has no zeros in \((a,b)\). Then we suppose that \( v(x) > 0 \) in \((a,b)\) because we replace \( v \) by \(-v\), if necessary. We shall show that \( v \) is a constant multiple of \( u \). We define the Wronskian \( w(x) \) by

\[
w(x) := u'(x)v(x) - u(x)v'(x).
\]

Since \( u', v' \in W^{1,1}(a,b) \), they are absolutely continuous and so is \( w(x) \). Therefore \( w \) is differentiable almost everywhere. Differentiating \( w \), we have

\[
w' = u''v - uv'' = (H - h)uv \geq 0.
\]

(2.3)

Hence \( w(x) \) is nondecreasing. Since \( v(x) > 0 \) in \((a,b)\), we have \( v(a), v(b) \geq 0 \) and obtain

\[
w(a) = u'(a)v(a) \geq 0, \quad w(b) = u'(b)v(b) \leq 0.
\]

(2.4)

Therefore \( w(a) \equiv 0 \) and we have

\[
\left( \frac{v}{u} \right)' = \frac{v'u - u'v}{u^2} = -\frac{w(x)}{u^2(x)} = 0.
\]

Consequently, \( v \) is a constant multiple of \( u \). Therefore either (i) or (ii) holds.

Suppose that the set of \( x \) satisfying \( h(x) < H(x) \) has a positive Lebesgue measure. We shall show that \( v \) has a zero in \((a,b)\). Suppose to the contrary that \( v \) has no zeros in \((a,b)\). We assume that \( v(x) > 0 \) in \((a,b)\). By (2.3), \( w(a) < w(b) \). This contradicts (2.4). Therefore \( v(x) \) has at least one zero in \((a,b)\). The proof is complete. \( \square \)

Put \( h(x) = H(x) \). Then Lemma 2.1 readily implies the next result.

Lemma 2.2 (Sturm separation theorem). Let \( h \in L^1(a,b) \). Let \( u, v \in W^{2,1}(a,b) \) be nontrivial solutions of (2.1) which are linearly independent. Then the zeros of \( u \) separate and are separated by those of \( v \).
3. Sublinear Emden-Fowler equation. Observing (1.1), we see that it is piecewise autonomous. In fact, a solution $u$ satisfies $u'' = 0$ in $(-\lambda, \lambda)$, and hence the graph of the solution in this interval is a line segment. The solution $u$ satisfies the sublinear Emden-Fowler equation

$$u'' + |u|^{p-1}u = 0,$$  

in $(-1, -\lambda]$ and in $[\lambda, 1)$. The purpose of this section is to investigate the properties of solutions for (3.1) and to give several lemmas for later use. For a nontrivial solution $u$ of (3.1), we define the energy $E(u)$ by

$$E(u)(x) := \frac{1}{2} u'(x)^2 + \frac{1}{p+1} |u(x)|^{p+1}.$$  

We summarize the properties of solutions for (3.1) in the next lemma.

**Lemma 3.1.** Let $u$ be a nontrivial solution of (3.1). Then we have the following conclusions.

(i) $E(u)(x)$ is a constant.

(ii) $u$ is a periodic solution having zeros.

(iii) $u$ is symmetric. More strictly, $u(x)$ is odd with respect to $x_0$ if $u(x_0) = 0$ and it is even with respect to $x_1$ if $u'(x_1) = 0$.

(iv) If $v$ is any nontrivial solution of (3.1), then $v(x) = \alpha^{2/(p-1)}u(\alpha x - \xi)$ with some $\alpha > 0$ and $\xi \in \mathbb{R}$.

**Proof.** From the direct computation, we have $E(u)'(x) = 0$ for all $x$. Therefore $E(u)(x)$ is a constant independent of $x$. We rewrite (3.1) as a system

$$u' = v, \quad v' = -|u|^{p-1}u.$$  

Since $E(u)$ is constant, the nontrivial solution satisfies

$$\frac{1}{2} v^2 + \frac{1}{p+1} |u|^{p+1} = c,$$  

with a constant $c > 0$. The equation above defines a Jordan curve around the origin in the $(u, v)$-phase plane. Thus $u$ is a periodic solution having zeros. Let $u(x_0) = 0$. We put $v(x) := u(x_0 - x)$ for $x < 0$ and $v(x) := -u(x_0 + x)$ for $x \geq 0$ and define $w(x) := u(x_0 - x)$ for all $x \in \mathbb{R}$. Then $v$ and $w$ are solutions of (3.1), which satisfy $v(0) = w(0) = 0$ and $v'(0) = w'(0)$. Form the uniqueness of solutions for the initial value problem, it follows that $v(x) \equiv w(x)$, that is, $u(x_0 - x) = -u(x_0 + x)$. Hence $u(x)$ is odd with respect to $x_0$. Note that $|u|^{p-1}u$ is not Lipschitz continuous, however, we have the uniqueness of solutions for the initial value problem. This claim will be proved in Lemma 5.2 later on. In the same way, we see that $u(x)$ is even with respect to $x_1$ if $u'(x_1) = 0$.

Let $v$ be any nontrivial solution of (3.1). Choose points $x_0$ and $x_1$ such that $u(x_0) = 0$, $u'(x_0) > 0$, $v(x_1) = 0$ and $v'(x_1) > 0$. Define $\alpha > 0$ by the relation, $\alpha^{(p+1)/(p-1)} u'(x_0) = v'(x_1)$. Put

$$U(x) := \alpha^{2/(p-1)}u(\alpha x + x_0), \quad V(x) := v(x + x_1).$$  

These functions satisfy (3.1). Since $U(0) = V(0) = 0$ and $U'(0) = V'(0)$, they are identically equal to each other. Consequently,

$$v(x) = \alpha^{2/(p-1)}u(\alpha x - \alpha x_1 + x_0) \quad \text{for all } x.$$  

Therefore the assertion (iv) holds. The proof is complete.  

\□
We consider the equation
\[ \phi'' + |\phi|^{p-1}\phi = 0, \quad \phi(x) > 0 \text{ in } (0,1), \quad \phi(0) = \phi(1) = 0. \] (3.3)

**Lemma 3.2.** (3.3) has a unique solution \( \phi \). Furthermore, \( \phi'(1/2) = 0, \phi'(x) > 0 \) in \((0,1/2)\) and \( \phi'(x) < 0 \) in \((1/2,1)\).

**Proof.** We choose a unique solution \( u \) of the equation
\[ u'' + |u|^{p-1}u = 0, \quad u(0) = 0, \quad u'(0) = 1. \]

Let \( T \) be the first zero of \( u \) in \((0,\infty)\). Put \( \phi(x) := T^{2/(p-1)}u(Tx) \). Then \( \phi(x) \) satisfies (3.3). The uniqueness of solutions for (3.3) follows from Lemma 3.1 (iv). The rest of the assertion follows from Lemma 3.1 (iii). The proof is complete. \( \square \)

Let \( u \) be a nontrivial solution of the Emden-Fowler equation (3.1). By Lemma 3.1 (ii) and (iii), \( u \) has all zeros in equal intervals. If \( a \) and \( b \) are two consecutive zeros of \( u \), i.e., \( u(a) = u(b) = 0 \) and \( u(x) \neq 0 \) in \((a,b)\), then \( x = (a+b)/2 \) is a unique critical point in \((a,b)\), that is, \( u'((a+b)/2) = 0 \). This fact ensures the following elementary lemma.

**Lemma 3.3.** Let \( u \) be a nontrivial solution of (3.1) which satisfies \( u'(t) = u(a) = 0 \) and \( u(x) \neq 0 \) in \((a,t)\) or in \((t,a)\). Then \( 2|a-t| \) is the distance between two consecutive zeros of \( u \).

**Definition 3.4.** For a function \( u \), we denote the number of zeros of \( u \) in an interval \((a,b)\) by \( N[u,(a,b)] \).

For a finite interval \((a,b)\), we consider the problem
\[ u'' + |u|^{p-1}u = 0 \text{ in } (a,b), \quad u(a) = u(b) = 0. \] (3.4)

In the same manner as in Lemma 3.2, we can prove that a positive solution of (3.4) is unique. By Lemma 3.1 (ii) and (iii), a nontrivial solution of (3.4) has all zeros in equal intervals. Therefore, if \( u \) is a solution of (3.4) having exactly \( m \) zeros in \((a,b)\), then its \( i \)-th zero is \( x_i = a + i(b-a)/(m+1) \). This fact and the uniqueness of positive solutions for (3.4) with any \((a,b)\) show the next lemma.

**Lemma 3.5.** For each integer \( m \geq 0 \), (3.4) has a unique solution \( u \) satisfying \( N[u,(a,b)] = m \) with \( u'(a) > 0 \).

Recall that a nontrivial solution of (3.1) has all zeros in equal intervals. Since \( E(u)(x) \) is constant, we abbreviate it as \( E(u) \).

**Lemma 3.6.** Let \( u \) be a nontrivial solution of (3.1) and let \( l \) be a distance between two consecutive zeros of \( u \). Then we have
\[ E(u) = c_1 \|u\|_{L^\infty(\mathbb{R})}^{2/(1-p)}, \quad \|u\|_{L^\infty(\mathbb{R})} = c_2 l^{2/(1-p)}, \] (3.5)
where \( c_1 := \phi'(0)^2/2, c_2 := \|\phi\|_{L^\infty(\mathbb{R})}, \|u\|_{L^\infty(\mathbb{R})} \) denotes the \( L^\infty(\mathbb{R}) \)-norm of \( u \) and \( \phi \) is a unique solution of (3.3).

**Proof.** After a suitable translation of \( u \), we can assume that \( u(0) = 0 \) and \( u'(0) > 0 \). By Lemma 3.1 (iv), \( u(x) \) is written as \( u(x) = \alpha^{2/(p-1)}\phi(\alpha x) \) with some \( \alpha > 0 \). Here \( \phi \) is a unique solution of (3.3). Since \( \phi(0) = \phi(1) = 0 \) and \( \phi(x) > 0 \) in \((0,1)\), \( l = 1/\alpha \) is the distance between consecutive zeros of \( u \). Since \( u(0) = 0 \) and \( u'(0) = \alpha^{(p+1)/(p-1)}\phi'(0) \), the energy is computed as
\[ E(u) = \frac{1}{2} u'(0)^2 = \frac{1}{2} \phi'(0)^2 l^{2(1+p)/(1-p)}, \]
which proves the first equation of (3.5). Since
\[ \| u \|_{L^\infty(\mathbb{R})} = \alpha^{2/(1-p)} \| \phi \|_{L^\infty(\mathbb{R})} = \| \phi \|_{L^\infty(\mathbb{R})}^{2/(1-p)}, \]
the second equation of (3.5) follows. The proof is complete.

We denote by \( l(u) \) the distance between consecutive zeros of a nontrivial solution \( u \) of (3.1). Then (3.5) directly proves the lemma below.

**Lemma 3.7.** Let \( u \) and \( v \) be nontrivial solutions of (3.1). Then the following are equivalent: (i) \( E(v) < E(u) \), (ii) \( \|v\|_{L^\infty(\mathbb{R})} < \|u\|_{L^\infty(\mathbb{R})} \), (iii) \( l(v) < l(u) \).

The lemma above readily guarantees the result below.

**Lemma 3.8.** Let \( u \) and \( v \) be nontrivial solutions of (3.1) such that \( E(v) < E(u) \) and \( u(a) = u(b) = 0 \) with some \( a < b \). Then \( v(x) \) has at least one zero in the interval \((a, b)\).

We choose an exponent \( q \) satisfying
\[ 1 < q < 1/(1-p). \]
Hereafter, we fix \( q \) satisfying the inequality above. For later use, we shall prove the next lemma.

**Lemma 3.9.** Let \( q \) be the exponent satisfying (3.6) and let \( u \) be a nontrivial solution of (3.1) in a finite interval \((a, b)\). Then \( |u(x)|^{p-1} \) belongs to \( L^q(a, b) \).

**Proof.** The function \( |u(x)|^{p-1} \) has a singularity at zeros of \( u \). Each zero of \( u \) is a simple zero, i.e., if \( u(x_0) = 0 \), then \( u'(x_0) \neq 0 \). Indeed, this claim follows from the uniqueness of solutions for the initial value problem (see Lemma 5.2 later on). Let \( x_0 \) be a zero of \( u(x) \). Then we have
\[ u(x) = \alpha (x - x_0) + o(|x - x_0|) \quad \text{as} \quad x \to x_0, \]
where \( o(t)/t \to 0 \) as \( t \to 0 \) and \( \alpha := u'(x_0) \neq 0 \). Therefore we have
\[ |u(x)| \geq \frac{|\alpha|}{2}|x - x_0| \quad \text{for} \quad x \in (x_0 - \varepsilon, x_0 + \varepsilon), \]
with a small \( \varepsilon > 0 \), which shows that
\[ |u(x)|^{q(p-1)} \leq (2^{-1}|\alpha||x - x_0|)^{q(p-1)}. \]
The right hand side is integrable because of (3.6). Therefore \( |u|^{p-1} \) belongs to \( L^q(x_0 - \varepsilon, x_0 + \varepsilon) \). Since \( u \) has at most a finite number of zeros in \([a, b] \), \( |u(x)|^{p-1} \) belongs to \( L^q(a, b) \). The proof is complete.

In Lemma 3.6, we used the \( L^\infty(\mathbb{R}) \)-norm of \( u \). In the next lemma, we use the \( L^\infty(a, b) \)-norm to estimate the number of zeros.

**Lemma 3.10.** Let \((a, b)\) be any finite interval and let \( n \geq 1 \) be an integer. If \( u \) satisfies (3.1) in \((a, b)\) and \( \|u\|_{L^\infty(a, b)} < ((b - a)/n\pi)^{2/(1-p)} \), then \( u \) has at least \( n \) zeros in \((a, b)\).

**Proof.** Put \( d := \|u\|_{L^\infty(a, b)} \). Since \( p - 1 < 0 \), we have \( |u(x)|^{p-1} \geq d^{p-1} \) in \((a, b)\) and the strict inequality holds except for the maximum point. We consider the equation
\[ v'' + d^{p-1}v = 0 \quad \text{in} \quad (a, b). \]
Put \( v(x) := \sin M(x-a) \) with \( M := d^{(p-1)/2} \). Then \( v \) is a solution of (3.7) satisfying \( v(a) = 0 \). It has at least \( n \) zeros in \((a, b)\) because \( M = d^{(p-1)/2} > n\pi/(b - a) \) by
assumption. Compare (3.1) with (3.7). Since $|u|^{p-1} \in L^q(a,b)$ by Lemma 3.9, we can apply the Sturm comparison theorem (Lemma 2.1). Since $|u|^{p-1} \geq d^{p-1}$ and $v(a) = 0$, $u$ has at least $n$ zeros in $(a,b)$. The proof is complete. 

4. Nondegenerate. We consider the boundary value problem

$$u'' + |u|^{p-1}u = 0 \quad \text{in } (-1,1), \quad u(-1) = u(1) = 0. \quad (4.1)$$

A nontrivial solution $u$ of (4.1) is called nondegenerate if the linearized operator $(d^2/dx^2) + p|u|^{p-1}$ does not admit zero as an eigenvalue, i.e., the linear problem

$$v'' + p|u|^{p-1}v = 0, \quad v(-1) = v(1) = 0 \quad (4.2)$$

has only the trivial solution $v(x) \equiv 0$. The purpose of this section is to prove the next proposition.

**Proposition 4.1.** Any nontrivial solution of (4.1) is nondegenerate.

We need the proposition above to prove Theorem 1.8. Before proving the proposition, we shall define a solution $v(x)$ of (4.2). Let $q$ satisfy (3.6). Recall that $|u(x)|^{p-1}$ belongs to $L^q(-1,1)$ by Lemma 3.9. Considering this fact, we call $v(x)$ a solution of (4.2) if

$$v \in W^{2,q}(-1,1) \cap W^{1,q}_0(-1,1)$$

and it satisfies the first equation of (4.2) in the distribution sense. We shall prove Proposition 4.1.

**Proof of Proposition 4.1.** Let $u$ be any nontrivial solution of (4.1). We assume that $u'(-1) > 0$ after replacing $u$ by $-u$, if necessary. Suppose that (4.2) has a nontrivial solution $v(x)$. We denotes the $i$-th zeros of $u$ and $v$ in $(-1,1]$ by $x_i$ and $y_i$, respectively. Denote the $i$-th critical point of $u$ in $(-1,1]$ by $t_i$. Then $u'(t_i) = u(x_i) = v(y_i) = 0$ and

$$-1 < t_1 < x_1 < t_2 < x_2 < \cdots. \quad (4.3)$$

Comparing (4.1) with (4.2), using the Sturm comparison theorem (Lemma 2.1) with the inequality $|u(x)|^{p-1} > p|u(x)|^{p-1}$ and noting $u(-1) = v(-1) = 0$, we find that

$$x_i < y_i \quad \text{provided that } y_i \leq 1. \quad (4.4)$$

Suppose that $u$ has exactly $n$ zeros in $(-1,1)$ with an $n \geq 0$. Then $x = 1$ is the $(n+1)$-th zero of $u$, that is, $x_{n+1} = 1$. We shall show that

$$y_{i-1} < t_i < x_i \quad \text{for } 1 \leq i \leq n + 1, \quad (4.5)$$

where we have put $y_0 := -1$. The second inequality of (4.5) follows from (4.3). Before proving the first inequality, we shall show that $u \in W^{3,q}(-1,1)$. Indeed, since $u' \in C^1[-1,1]$ and $|u|^{p-1} \in L^q(-1,1)$, we have

$$u'' = -p|u|^{p-1}u' \in L^q(-1,1).$$

Therefore $u \in W^{3,q}(-1,1)$. We put $U(x) := u'(x) \in W^{2,q}(-1,1)$. Then it satisfies

$$U'' + p|u|^{p-1}U = 0.$$

Therefore $U$ and $v$ satisfy the same differential equation. Since $U(-1) = u'(-1) > 0$ and $v(-1) = 0$, these solutions are linearly independent. The Sturm separation theorem (Lemma 2.2) says that the zeros of $v$ separate and are separated by those of $U$, i.e., $v$ has exactly one zero in $[t_{i-1}, t_i)$ and $U$ has exactly one zero in $(y_{j-1}, y_j)$. By induction, we shall prove the first inequality of (4.5). For $i = 1$, it is written as $y_0 = -1 < t_1$, which is clearly true. Assume that $y_{j-1} < t_j$. We shall show that
We choose an interval $(y_j, t_{j+1})$. Suppose to the contrary that $y_j \geq t_{j+1}$. Then $y_{j-1} < t_j < x_j < t_{j+1} \leq y_j$. Thus $v(x)$ has no zeros in $(t_j, t_{j+1})$, which contradicts the separation theorem. Therefore $y_j < t_{j+1}$. By induction, (5.5) holds for all $1 \leq i \leq n+1$. Since $x_{n+1} = 1$, it holds that

$$x_n < y_n < t_{n+1} = 1.$$  

(4.6)

By assumption, $v(1) = 0$. Since $v(y_n) = v(1) = 0$, $u(x)$ must have a zero in $(y_n, 1)$ by the Sturm comparison theorem. This contradicts (4.6). Consequently, (4.2) has only the trivial solution. The proof is complete.

\section{Sublinear Moore-Nehari equation} In this section, we investigate the sublinear Moore-Nehari equation (1.1) and prove Theorems 1.6 and 1.7. In the same method as in the proof of Lemma 3.1 (iii), one can easily prove the next lemma, which is elementary but important.

\begin{lemma}
Let $u$ be a nontrivial solution of (1.1). Then we have:
\begin{enumerate}[(i)]
\item $u$ is an odd solution if and only if $u(0) = 0$,
\item $u$ is an even solution if and only if $u'(0) = 0$.
\end{enumerate}
\end{lemma}

The lemma above will be used in a later section. We prove Theorem 1.6.

\begin{proof}[Proof of Theorem 1.6] It is enough to show the uniqueness of positive solutions. In fact, the evenness of the unique positive solution follows from Theorem 1.7. Suppose to the contrary that (1.1) has at least two different positive solutions $u$ and $v$. Then $u(x_0) > v(x_0)$ or $u(x_0) < v(x_0)$ at some $x_0$. Assume that the former holds. Then we choose an interval $(a, b)$ such that

$$u(a) = v(a) \geq 0, \quad u(b) = v(b) \geq 0, \quad u(x) > v(x) \quad \text{in} \quad (a, b).$$  

(5.1)

In the case where $a = -1$ (or $b = 1$), it holds that $u(a) = v(a) = 0$ (or $u(b) = v(b) = 0$). Recall that $h(x, \lambda) = 0$ in $(-\lambda, \lambda)$. If $(a, b) \subset (-\lambda, \lambda)$, then the graphs of $u$ and $v$ are line segments in $(a, b)$. This contradicts (5.1). Therefore $(a, b) \not\subset (-\lambda, \lambda)$ and hence $h(x, \lambda) \neq 0$ in $(a, b)$. Multiplying (1.1) by $v$ and integrating it over $(a, b)$, we have

$$\int_a^b h(x, \lambda) u^p v dx = \int_a^b u'(x) v'(x) dx - u'(b) v(b) + u'(a) v(a).$$  

(5.2)

Replacing $u$ and $v$ with each other, we obtain

$$\int_a^b h(x, \lambda) v^p u dx = \int_a^b u'(x) v'(x) dx - v'(b) u(b) + v'(a) u(a).$$  

(5.3)

Subtracting (5.3) from (5.2) and using (5.1), we have

$$\int_a^b h(x, \lambda) uv(u^{p-1} - v^{p-1}) dx = (v'(b) - u'(b)) u(b) + (u'(a) - v'(a)) u(a).$$  

(5.4)

By (5.1), the right hand side of (5.4) is nonnegative. However, the left hand side is negative because $0 < p < 1$ and $h(x, \lambda) \neq 0$ in $(a, b)$. A contradiction occurs. Consequently, a positive solution is unique.

We extend the definition domain of $h(x, \lambda)$ into $\mathbb{R}$ by putting $h(x, \lambda) := 1$ for $|x| \geq \lambda$. We shall show the existence and uniqueness of solutions for the initial value problem

$$u'' + h(x, \lambda)|u|^{p-1}u = 0, \quad u(x_0) = a, \quad u'(x_0) = b,$$  

(5.5)

with any $x_0, a, b \in \mathbb{R}$. Note that $|u|^{p-1}u$ is not Lipschitz continuous at $u = 0$. 

Lemma 5.2. (5.5) has a unique solution defined for \( x \in \mathbb{R} \).

Proof. The local existence follows from the Peano existence theorem (see [4, p.11, Corollary 2.1]). For \( |x| \leq \lambda \), a solution \( u(x) \) is a line segment. For \( |x| \geq \lambda \), \( u(x) \) is a periodic solution having the energy. Therefore \( u(x) \) is defined for all \( x \in \mathbb{R} \). We shall show the uniqueness of solutions. Let \( u(x) \) and \( v(x) \) be solutions of (5.5). It is enough to show that \( u(x) = v(x) \) in \((x_0 - \varepsilon, x_0 + \varepsilon)\) with \( \varepsilon > 0 \). If \( a \neq 0 \), a solution is locally unique because \(|u|^{p-1}u\) is locally Lipschitz continuous in \( \mathbb{R} \setminus \{0\} \).

Let \( a = 0 \). Then \( u(x_0) = v(x_0) = 0 \). Let \(-\lambda < x_0 < \lambda \). Then \( u'' = v'' = 0 \) in a neighborhood of \( x_0 \). By the initial condition, \( u(x) \equiv v(x) \) in a neighborhood of \( x_0 \). Let \( |x_0| \geq \lambda \). Then \( u \) and \( v \) satisfy the Emden-Fowler equation in a neighborhood of \( x_0 \) and

\[
u(x_0) = v(x_0) = 0, \quad u'(x_0) = v'(x_0) = b.\]

If \( b = 0 \), then \( E(u) = E(v) = 0 \) and \( u \equiv v \equiv 0 \). Let \( b > 0 \). If \( b < 0 \), we replace \( u \) and \( v \) by \(-u \) and \(-v \), respectively. We choose an \( \varepsilon > 0 \) so small that

\[
b/2 < u'(x_0), v'(x_0) < 2b, \quad |u(x)| < 1, \quad |v(x)| < 1 \quad \text{in} \quad (x_0 - \varepsilon, x_0 + \varepsilon).\]

Since \( E(u) = E(v) \), we have

\[
\frac{1}{2} u'(x_0)^2 + \frac{1}{p+1}|u(x)|^{p+1} = \frac{1}{2} v'(x_0)^2 + \frac{1}{p+1}|v(x)|^{p+1},
\]

which is rewritten as

\[
u'(x)^2 - v'(x)^2 = \frac{2}{p+1}(|u|^{p+1} - |v|^{p+1}). \quad (5.6)
\]

There exists a constant \( C > 0 \) such that

\[
||u||^{p+1} - |v|^{p+1} \leq C|u - v| \quad \text{for} \quad |u|, |v| \leq 1.
\]

Since \( u'(x) + v'(x) \geq b \) in \((x_0 - \varepsilon, x_0 + \varepsilon)\), we use the inequality above in (5.6) to obtain

\[
b|u' - v'| \leq \frac{2C}{p+1}|u - v|.
\]

Putting \( D := 2C/b(p+1) \), we have \( |u' - v'| \leq D|u - v| \). Integrating it over \((x_0, x)\), we obtain

\[
|u(x) - v(x)| \leq \int_{x_0}^{x} |u'(t) - v'(t)|dt \leq D \int_{x_0}^{x} |u(t) - v(t)|dt,
\]

in \((x_0 - \varepsilon, x + \varepsilon)\). By the Gronwall inequality, \( u \equiv v \) in \((x_0 - \varepsilon, x_0 + \varepsilon)\).

Let \( x_0 = \lambda \). Even if \( x_0 = -\lambda \), the argument below works well. In the interval \([x_0, x_0 + \varepsilon)\), we use the energy and employ the same argument as in (5.6). Then \( u = v \) in \([x_0, x_0 + \varepsilon)\). In \((x_0 - \varepsilon, x_0)\), \( u \) and \( v \) are line segments. Then \( u = v \) in this interval. The proof is complete. \(\Box\)

To prove the existence of nodal solutions, we shall use a shooting method. To this end, we need the continuous dependence of solutions on the initial data. For \( T > 0 \), we define the \( W^{2,\infty}(-T, T) \)-norm by

\[
||u||_{W^{2,\infty}(-T, T)} := ||u||_{L^{\infty}(-T, T)} + ||u'||_{L^{\infty}(-T, T)} + ||u''||_{L^{\infty}(-T, T)},
\]

where \( ||u||_{L^{\infty}(-T, T)} \) is the \( L^{\infty}(-T, T) \)-norm of \( u \). Fix \( x_0 \in \mathbb{R} \). We denote the unique solution of (5.5) by \( u(x, a, b) \).

Lemma 5.3. It holds that for any \( T > 0 \),

\[
||u(\cdot, a, b) - u(\cdot, a_0, b_0)||_{W^{2,\infty}(-T, T)} \to 0 \quad \text{as} \quad (a, b) \to (a_0, b_0).
\]

(5.7)
Proof. We shall show that for any $A > 0$, there exists a constant $C > 0$ such that

$$
||u(\cdot,a,b)||_{W^{2,\infty}(\mathbb{R})} \leq C \quad \text{when} \quad |a| + |b| \leq A.
$$

(5.8)

Let $A > 0$ and suppose that $|a| + |b| \leq A$. Put $u(x) := u(x,a,b)$. Since $u(x)$ is a line segment in $[-\lambda, \lambda]$ and has the energy in $(-\infty, -\lambda]$ and in $[\lambda, \infty)$, it has an a priori $C^1(\mathbb{R})$ estimate, i.e., there is a $C > 0$ such that

$$
|u(x)| + |u'(x)| \leq C \quad \text{for} \quad x \in \mathbb{R}.
$$

Here $C$ depends only on $A$. By (5.5), we have

$$
|u''(x)| = h(x,\lambda)|u|^p \leq C^p.
$$

Hence we have (5.8).

Let $T > |x_0|$, $(a_0, b_0) \in \mathbb{R}^2$ and let $(a_n, b_n)$ be a sequence converging to $(a_0, b_0)$. Put $u_n(x) := u(x,a_n,b_n)$. Then $|a_n| + |b_n| \leq |a_0| + |b_0| + 1$ for $n$ large enough. By (5.8), $\|u_n\|_{W^{2,\infty}(\mathbb{R})}$ is bounded. By the compact embedding, along a subsequence, $u_n$ converges to a limit $u_0$ in $C^1[-T,T]$. Since $u_n$ is a solution of (5.5), it satisfies

$$
u'_n(x) = u'_n(-T) - \int_{-T}^x h(t,\lambda)|u_n(t)|^{p-1}u_n(t)dt.
$$

Letting $n \to \infty$, we have

$$
u'_0(x) = u'_0(-T) - \int_{-T}^x h(t,\lambda)|u_0(t)|^{p-1}u_0(t)dt.
$$

Since the right hand side is of class $W^{1,\infty}(-T,T)$, $u_0$ belongs to $W^{2,\infty}(-T,T)$ and it satisfies the first equation of (5.5). Since $u_n(x_0) = a_n$ and $u'_n(x_0) = b_n$, it follows that $u_0(x_0) = a_0$ and $u'_0(x_0) = b_0$. Thus $u_0(x) = u(x,a_0,b_0)$. By the uniqueness of the limit, $u_n$ itself (without choosing a subsequence) converges to $u_0(x) = u(x,a_0,b_0)$. By (5.5), we have

$$
\|u''_n - u''_0\|_{L^\infty(-T,T)} = \|h(\cdot,\lambda)(|u_n|^{p-1}u_n - |u_0|^{p-1}u_0)\|_{L^\infty(-T,T)} \to 0.
$$

Therefore (5.7) holds. The proof is complete. \qed

Observe Lemma 5.1 (i). To find an odd solution of (1.1), we consider the the boundary value problem

$$
v'' + h(x,\lambda)|v|^{p-1}v = 0, \quad v(0) = v(1) = 0.
$$

(5.9)

To obtain a solution of the problem above, we study the initial value problem,

$$
v'' + h(x,\lambda)|v|^{p-1}v = 0, \quad v(0) = 0, \quad v'(0) = \alpha.
$$

(5.10)

Lemma 5.4. For each $m \geq 1$, there exists an $\varepsilon > 0$ such that the solution of (5.10) with $\alpha \in (0, \varepsilon)$ has at least $m$ zeros in $(\lambda, 1)$.

Proof. Recall that a solution $v$ of (5.10) is defined on $\mathbb{R}$ and $E(v)$ is constant in $[\lambda, \infty)$. Since $v(x) = \alpha x$ in $[0, \lambda]$, it holds that

$$
E(v)(\lambda) = \frac{1}{2}\alpha^2 + \frac{1}{p+1}(\alpha\lambda)^{p+1}.
$$

If $\alpha > 0$ is small enough, then $E(v)(\lambda) < c_1((1 - \lambda)/m)^{2(1+p)/(1-p)}$. By Lemma 3.6, the distance between adjacent zeros of $v$ in $(\lambda, 1)$ is shorter than $(1-\lambda)/m$. Consequently, $v(x)$ has at least $m$ zeros in $(\lambda, 1)$.

We shall show that when $\alpha > 0$ is large enough, the solution $v$ has no zeros.
Lemma 5.5. If \( \alpha > 0 \) is large enough, then the solution \( v \) of (5.10) has no zeros in \((0, 1]\).

Proof. Let \( v(x) \) be a solution of (5.10). Choose \( A > 0 \) so large that if \( \alpha > A \), then
\[
E(v)(\lambda) \geq \alpha^2/2 > A^2/2 > c_1(2(1 - \lambda))^{2(1+p)/(1-p)}. \tag{5.11}
\]
Let \( \alpha > A \). We shall show that \( v(x) \) has no zeros in \((\lambda, 1]\). Suppose to the contrary that it has a zero. Denote the first zero by \( z \leq 1 \). Since \( v(\lambda) > 0 \) and \( v'(\lambda) > 0 \), \( v \) has a unique critical point \( t \) in \((\lambda, z) \), i.e., \( v'(t) = 0 \). By Lemma 3.3, \( l := 2(z-t) \) is the distance between consecutive zeros of \( v \) in \((\lambda, \infty) \). We have \( l = 2(z-t) \leq 2(1-\lambda) \).

This inequality with (3.5) shows
\[
E(v) = c_1(2(1+p)/(1-p) \leq c_1(2(1 - \lambda))^{2(1+p)/(1-p)},
\]
which contradicts (5.11). Therefore \( v \) has no zeros. \( \square \)

We shall show the existence of \( n \)-nodal solution \( v \) of (5.9). The uniqueness of such a solution will be proved later on.

Theorem 5.6. For each integer \( n \geq 0 \), (5.9) possesses a solution \( v \) which has exactly \( n \) zeros in \((0, 1]\) and satisfies \( v'(0) > 0 \).

Proof. The theorem can be proved in a standard shooting method by using Lemmas 5.4 and 5.5, however, we give a proof to make the paper self-contained. Recall that \( N[v, (\alpha, b)] \) stands for the number of zeros of \( v(x) \) in \((a, b) \). We denote the unique solution of (5.10) by \( v(x, \alpha) \). By Lemmas 5.4 and 5.5, \( N[v(\cdot, \alpha), (0, 1)] = 0 \) for \( \alpha > 0 \) large enough and \( N[v(\cdot, \alpha), (0, 1)] \geq 1 \) for \( \alpha \) small. Therefore we can define
\[
\alpha_0 := \inf\{\alpha \in (0, \infty) : N[v(\cdot, \alpha), (0, 1)] = 0\}.
\]
Then \( \alpha_0 > 0 \). We shall show that \( v(x, \alpha_0) \) is a positive solution of (5.9). Put \( v(x) := v(x, \alpha_0) \). Suppose that \( v(x) \) has a zero \( x_0 \) in \((0, 1) \). This is a simple zero, i.e., \( v'(x_0) \neq 0 \). By Lemma 5.3, \( v(x, \alpha) \) has a zero near \( x_0 \) when \( \alpha \) is close to \( \alpha_0 \). This contradicts the definition of \( \alpha_0 \). Thus \( v(x) > 0 \) in \((0, 1) \). We shall show that \( v(1) = 0 \). Suppose to the contrary that \( v(1) > 0 \). Then \( v(x, \alpha) > 0 \) in \((0, 1) \) when \( \alpha \) is close to \( \alpha_0 \). Therefore \( N[v(\cdot, \alpha), (0, 1)] = 0 \) when \( \alpha \) is slightly smaller than \( \alpha_0 \). This contradicts the definition of \( \alpha_0 \). Accordingly, \( v(1, \alpha_0) = 0 \) and \( v(x, \alpha_0) \) is a positive solution of (5.9).

For \( \alpha \) slightly smaller than \( \alpha_0 \), \( v(x, \alpha) \) has exactly one zero in \((0, 1) \) and \( v(-1, \alpha) < 0 \). Then we define
\[
\alpha_1 := \inf\{\alpha \in (0, \infty) : N[v(\cdot, \alpha), (0, 1)] = 1\}.
\]
Since \( \alpha_1 < \alpha_0 \), \( v(x, \alpha_1) \) has at least one zero in \((0, 1) \). If it has two or more zeros, so does \( v(x, \alpha) \) for \( \alpha \) close to \( \alpha_1 \). This contradicts the definition of \( \alpha_1 \). Therefore \( v(x, \alpha_1) \) has exactly one zero in \((0, 1) \). In the same method as in the case \( \alpha_0 \), we can prove that \( v(1, \alpha_1) = 0 \). Repeating the argument above inductively, we can define for each integer \( n \geq 0 \),
\[
\alpha_n := \inf\{\alpha \in (0, \infty) : N[v(\cdot, \alpha), (0, 1)] = n\}.
\]
Then \( v(x, \alpha_n) \) is a solution of (5.9) which has exactly \( n \) zeros in \((0, 1) \) and satisfies \( v'(0) > 0 \). The proof is complete. \( \square \)

In the next theorem, we shall prove the uniqueness of \( n \)-nodal solutions for (5.9).

Theorem 5.7. For each integer \( n \geq 0 \), a solution \( v \) of (5.9) satisfying \( N[v, (0, 1)] = n \) with \( v'(0) > 0 \) is unique.
Theorem 5.7. \( u \) even), we replace \( m \) in (5.9) such that they have exactly \( n \) zeros in \((0, 1)\) and \( u'(0) > v'(0) > 0 \). Since \( h(x, \lambda) = 0 \) in \([0, \lambda)\), we have

\[
u(x) > v(x) > 0, \quad u'(x) = u'(0), \quad v'(x) = v'(0) \quad \text{in} \quad (0, \lambda).
\]

(5.12)

Since \( h(x, \lambda) = 1 \) for \( x \geq \lambda \), \( u \) and \( v \) can be extended as solutions of the Emden-Fowler equation into \([1, \infty)\). Then they have infinitely many zeros in \((\lambda, \infty)\) and no zeros in \((0, \lambda)\) by (5.12). Denote the \( i \)-th zeros of \( u \) and \( v \) in \((\lambda, \infty)\) by \( x_i \) and \( y_i \), respectively. We shall show that

\[y_i < x_i \quad \text{for all} \quad i \geq 1.
\]

(5.13)

We first prove that \( y_1 < x_1 \). Suppose to the contrary that \( y_1 \geq x_1 \). Then \( u \) and \( v \) intersect with each other at some points in \((0, x_1)\). Denote the first intersection point by \( z_1 \). Then \( z_1 > \lambda \) by (5.12). We have \( u(x) > v(x) \) in \((0, z_1)\) and \( u(z_1) = v(z_1) \geq 0 \) \((v(z_1) = 0 \) occurs when \( x_1 = y_1 = z_1 \). Then \( u'(z_1) \leq v'(z_1) \). If \( u'(z_1) = v'(z_1) \), then \( u(x) \equiv v(x) \) from the uniqueness of solutions for the initial value problem. A contradiction occurs. Hence \( u'(z_1) < v'(z_1) \). Using (5.4) with \((a, b) = (0, z_1)\) and employing \( u(0) = 0 \), we have

\[
\int_0^{z_1} h(x, \lambda) u v(u^{p-1} - v^{p-1}) dx = (v'(z_1) - u'(z_1)) u(z_1).
\]

Note that \( z_1 > \lambda \). The right hand side is nonnegative, however, the left hand side is negative. A contradiction occurs. Therefore \( y_1 < x_1 \).

Since \( u'(\lambda) > v'(\lambda) > 0 \) and \( u(\lambda) > v(\lambda) > 0 \), we compute the energy at \( \lambda \) to obtain \( E(u) > E(v) \) in \([\lambda, \infty)\). Using Lemma 3.7 with \((\lambda, \infty)\), we have \( y_{i+1} - y_i < x_{i+1} - x_i \) for each \( i \geq 1 \). This inequality with \( y_1 < x_1 \) shows (5.13).

Since \( u \) and \( v \) have exactly \( n \) zeros in \((0, 1)\) and \( u(1) = v(1) = 0 \), the point \( x = 1 \) is the \((n + 1)\)-th zero. Hence \( x_{n+1} = y_{n+1} = 1 \). This contradicts (5.13). Consequently, we have the uniqueness of solutions.

Combining Theorems 5.6 and 5.7, we obtain the next result.

Theorem 5.8. Let \( \lambda \in (0, 1) \). For each odd integer \( m \geq 1 \), (1.1) has a unique \( m \)-nodal odd solution \( u \) with \( u'(-1) > 0 \).

Proof. Let \( m \geq 1 \) be any odd integer. Write \( m = 2n + 1 \) with \( n \geq 0 \). Let \( v \) be a solution obtained by Theorem 5.6. Define \( u(x) := v(x) \) for \( x \geq 0 \) and \( u(x) := -v(-x) \) for \( x \leq 0 \). Then \( u \) is an odd solution having exactly \( 2n + 1 \) zeros in \((-1, 1)\), i.e., it is an \( m \)-nodal odd solution of (1.1). If \( u'(-1) < 0 \) (that is, \( n \) is even), we replace \( u \) by \(-u \). Therefore \( u'(-1) > 0 \). The uniqueness follows from Theorem 5.7.

In the same way as in Lemma 3.5, we can prove the next lemma.

Lemma 5.9. Let \((a, b)\) be a finite interval. For each integer \( n \geq 0 \), there exists a unique solution \( u(x) \) of (3.1) in \((a, b)\) which satisfies

\[u(a) = u'(b) = 0, \quad u'(a) > 0, \quad N[u, (a, b)] = n.
\]

We shall construct an \( m \)-nodal even solution.

Theorem 5.10. Let \( \lambda \in (0, 1) \). For each even integer \( m \geq 0 \), (1.1) has a unique \( m \)-nodal even solution \( u \) with \( u'(-1) > 0 \).
Proof. Recall that a solution \( u \) is even if and only if \( u'(-\lambda) = 0 \). Let \( m \geq 0 \) be any even integer. Write it as \( m = 2n \) with \( n \geq 0 \).

If \( u \) is an \( m \)-nodal even solution of (1.1) with \( u'(-1) > 0 \), then \( u \) is a solution of (3.1) in \((-1, -\lambda)\) satisfying
\[
  u(-1) = u'(-\lambda) = 0, \quad u'(-1) > 0, \quad N[u, (-1, -\lambda)] = n. \tag{5.14}
\]
Such a solution exists and is unique by Lemma 5.9.

Conversely, if \( u \) is a solution of (3.1) in \((-1, -\lambda)\) which satisfies (5.14), we put \( u(x) := u(-\lambda) \) in \([-\lambda, 0]\) and extend it on \([-1, 1]\) as an even function. Then it becomes an \( m \)-nodal even solution with \( u'(-1) > 0 \). Consequently, (1.1) has a unique \( m \)-nodal even solution with \( u'(-1) > 0 \).

\[\square\]

Proof of Theorem 1.7. The theorem follows from Theorems 5.8 and 5.10. \[\square\]

6. Proof of Theorem 1.8. In this section, we shall prove Theorem 1.8. To this end, we shall give an a priori estimate of solutions.

Lemma 6.1. There exists a constant \( C > 0 \) such that any solution \( u \) of (1.1) satisfies \( \|u\|_{W^{2,\infty}(-1,1)} \leq C \).

Proof. Denote the \( L^q(-1,1) \)-norm of \( u \) by \( \|u\|_q \). Multiplying (1.1) by \( u \) and integrating it over \((-1,1)\), we have
\[
  \int_{-1}^1 u'(x)^2 \, dx = \int_{-1}^1 h|u|^{p+1} \, dx \leq \|u\|^{p+1}_{p+1} \leq C\|u'\|^{p+1}_2,
\]
with some constant \( C > 0 \) independent of \( u \), where we have used the Sobolev embedding. Dividing both sides by \( \|u'\|_2^{p+1} \), we obtain \( \|u'\|_2^{1-p} \leq C \). Because of the Poincaré inequality, \( \|u'\|_2 \) can be the norm of \( H^1_0(-1,1) \). Here \( H^1_0(-1,1) \) is a Sobolev space consisting of functions \( u \) such that \( u, u' \in L^2(-1,1) \) and \( u(-1) = u(1) = 0 \). Thus we have an a priori estimate of \( H^1_0(-1,1) \)-norm of \( u \). Since this space is embedded in \( L^\infty(-1,1) \), we have \( \|u\|_\infty \leq D \) with a \( D > 0 \) independent of \( u \). By (1.1), we have
\[
  \|u''\| \leq \|h|u|^p\|_\infty \leq D^p.
\]
This inequality with \( \|u\|_\infty \leq D \) ensures an a priori estimate of \( \|u'\|_\infty \). The proof is complete. \[\square\]

Using Lemma 6.1 and employing the nondegeneracy of solutions, we prove Theorem 1.8.

Proof of Theorem 1.8. Let \( n \geq 0 \) be any integer. If we would prove the uniqueness of \( n \)-nodal solution with \( u'(-1) > 0 \), its symmetry follows from Theorem 1.7. We shall show the uniqueness. Suppose to the contrary that there exist sequences \( \lambda_k, u_k \) and \( v_k \) such that \( u_k \) and \( v_k \) are different \( n \)-nodal solutions of (1.1) with \( \lambda = \lambda_k, u_k'(-1), v_k'(-1) > 0, 0 < \lambda_k < 1 \) and \( \lambda_k \to 0 \) as \( k \to \infty \). Suppose that \( u_k'(-1) > v_k'(-1) > 0 \). Since \( \lambda_k \to 0 \), we can assume that \( \lambda_k < 1/2 \) for all \( k \). If
\[
  \|u_k\|_{L^\infty(\lambda_k,1)} < \left( (1 - \lambda_k)/(n + 1)\pi \right)^{2/(1-p)},
\]
\( u_k \) has at least \( n+1 \) zeros in \((\lambda_k,1)\) by Lemma 3.10. This contradicts the assumption that \( u_k \) is \( n \)-nodal. Therefore
\[
  \|u_k\|_{L^\infty(\lambda_k,1)} \geq \left( \frac{1 - \lambda_k}{(n + 1)\pi} \right)^{2/(1-p)} \geq \left( \frac{1}{2(n + 1)\pi} \right)^{2/(1-p)},
\]
because $\lambda_k < 1/2$. Denote the right hand side by $c$, which is independent of $k$. Then $c \leq \|u_k\|_{L^\infty(\lambda_k, 1)} \leq \|u_k\|_{L^\infty(-1, 1)}$. Combining this inequality with Lemma 6.1, we have a constant $C > 0$ independent of $k$ such that

$$c \leq \|u_k\|_{L^\infty(-1, 1)} \leq \|u_k\|_{W^{2, \infty}(-1, 1)} \leq C. \quad (6.1)$$

Thus $u_k$ is bounded in $W^{2, \infty}(-1, 1)$. By the compact embedding, along a subsequence, $u_k$ converges to a certain limit $\psi$ in $C^1[-1, 1]$. Since $u_k$ is a solution of (1.1), we have

$$u'_k(x) - u'_k(-1) = - \int_{-1}^{x} h(t, \lambda_k)|u_k(t)|^{p-1}u_k(t)dt.$$  

Since $h(x, \lambda_k) \to 1$ almost everywhere, we use the Lebesgue dominated convergence theorem to obtain

$$\psi'(x) - \psi'(-1) = - \int_{-1}^{x} |\psi(t)|^{p-1}\psi(t)dt.$$ 

Since the right hand side is of class $C^1[-1, 1]$, $\psi$ belongs to $C^2[-1, 1]$ and it satisfies (4.1). By (6.1), $\|\psi\|_{\infty} \geq c$ and $\psi$ is nontrivial. Since $u'_k(-1) > 0$, we have $\psi'(-1) \geq 0$. Since $\psi \neq 0$, we find $\psi'(-1) > 0$.

Let $x_{k,i}$ be the $i$-th zero of $u_k$ in $(-1, 1)$. We shall show that there exists a constant $c_0 > 0$ independent of $k$ and $i$ such that

$$x_{k,i} - x_{k,i-1} \geq c_0 \quad \text{for all } 1 \leq i \leq n + 1, \ k \geq 1, \quad (6.2)$$

where we have put $x_{k,0} := -1$ and $x_{k,n+1} := 1$. Suppose to the contrary that (6.2) is false. Then there exist $u_k$, $s_j$ and $t_j$ such that $u_k$ is a subsequence of $u_k$, $s_j$ and $t_j$ are consecutive zeros of $u_k$, $-1 \leq s_j < t_j \leq 1$ and $t_j - s_j \to 0$ as $j \to \infty$. There exists an $r_j \in (s_j, t_j)$ such that $u_k'(r_j) = 0$. Along a subsequence, $t_j$ converges to a limit $t \in [-1, 1]$. Then $r_j \to t$ and we have $\psi'(t) = \psi(t) = 0$. Therefore $\psi(x) \equiv 0$, which is a contradiction. Accordingly, (6.2) holds.

By (6.2), we find that $\psi(x)$ has exactly $n$ zeros in $(-1, 1)$, that is, it is an $n$-nodal solution of (4.1) satisfying $\psi'(-1) > 0$. Such a solution is unique by Lemma 3.5. The uniqueness of the limit ensures that $u_k$ itself (without choosing a subsequence) converges to $\psi$. In the same method, $v_k$ also converges to $\psi$.

We shall find a contradiction. Put

$$f(t) := |t|^{p-1}t, \quad w_k(x) := (u_k(x) - v_k(x))/||u_k - v_k||_{\infty},$$

where $|| \cdot ||_{\infty}$ stands for the $L^\infty(-1, 1)$-norm. Then $||w_k||_{\infty} = 1$ and $w_k(x)$ fulfills

$$-w''_k(x) = h(x, \lambda_k)f(u_k) - f(v_k)w_k. \quad (6.3)$$

We shall show that the right hand side belongs to $L^q(-1, 1)$ and is bounded in this space, where $q$ is given by (3.6). To do so, we shall prove

$$\left|\frac{f(u_k(x)) - f(v_k(x))}{u_k(x) - v_k(x)}\right| \leq |u_k(x)|^{p-1} + |v_k(x)|^{p-1}, \quad (6.4)$$

for $x \in (-1, 1)$ except for a finite number of $x$. Note that the graphs of $u_k(x)$ and $v_k(x)$ intersect at most a finite time. In fact, if $u_k(z_i) = v_k(z_i)$ with a sequence $z_i$, then $u_k(z_0) = v_k(z_0)$ and $u'_k(z_0) = v'_k(z_0)$ at an accumulation point $z_0$ of $z_i$. Hence $u_k(x) \equiv v_k(x)$, which is a contradiction. Therefore $u_k(x) = v_k(x)$ holds for at most a finite number of $x \in [-1, 1]$. We define

$$A := \{x \in [-1, 1] : u_k(x) = v_k(x)\},$$
Then $A$ is a finite set. We shall prove (6.4) for $x \in [-1, 1] \setminus A$. Fix $x \in [-1, 1] \setminus A$ arbitrarily and put $u := u_k(x)$ and $v := v_k(x)$. If $u = 0$ or $v = 0$, then the right hand side of (6.4) is infinity and so it is valid. Let $u \neq 0$ and $v \neq 0$. If $u, v > 0$ or $u, v < 0$, there exists a $\xi$ by the mean value theorem such that
\[
\frac{f(u) - f(v)}{u - v} = p|\xi|^{p-1},
\]
with $u < \xi < v$ or $v < \xi < u$. Therefore
\[
\left| \frac{f(u) - f(v)}{u - v} \right| \leq p|u|^{p-1} + p|v|^{p-1},
\]
which shows (6.4). If $v < 0 < u$, we have
\[
\frac{f(u) - f(v)}{u - v} = \frac{u^p + |v|^p}{u + |v|} \leq u^{p-1} + |v|^{p-1},
\]
which proves (6.4). Even if $u < 0 < v$, the same method as above is valid. Consequently, we have (6.4).

We shall prove that the right hand side of (6.4) is bounded in $L^q(-1, 1)$. Denote the $i$-th zero of $\psi$ in $(-1, 1)$ by $x_i$. Put $x_0 := -1$ and $x_{n+1} := 1$. Since $\psi$ is periodic, $x_i$ is written as $x_i = -1 + 2i/(n+1)$. Put
\[
I_i := [x_i - \delta, x_i + \delta] \cap [-1, 1] \quad \text{for } 0 \leq i \leq n+1,
\]
where $\delta > 0$ will be determined later on. Define
\[
I := \bigcup_{i=0}^{n+1} I_i, \quad J := [-1, 1] \setminus I. \tag{6.5}
\]
The right hand side of (6.4) converges to $2|\psi(x)|^{p-1}$ uniformly on $J$ as $k \to \infty$. To prove that the right hand side of (6.4) is bounded in $L^q(-1, 1)$, we have only to show that
\[
\int_{I_i} (|u_k|^{q(p-1)} + |v_k|^{q(p-1)})dx \leq C \quad \text{for } 0 \leq i \leq n+1, \quad k \geq k_0, \tag{6.6}
\]
with a certain $k_0$, where $C > 0$ is a constant independent of $i$ and $k$. We deal with $u_k$ only. Denote the $i$-th zero of $u_k$ in $(-1, 1)$ by $x_k,i$. Put $x_k,0 := -1$ and $x_k,n+1 := 1$. Since $u_k$ converges to $\psi$ in $C^1[-1, 1]$, $x_k,i$ converges to $x_i$ as $k \to \infty$. We have
\[
u_k(x) = u_k'(x_k,i)(x - x_k,i) + o(|x - x_k,i|) \quad \text{as } x \to x_k,i,
\]
where $o(|x - x_k,i|)$ is uniform with respect to $k$. More strictly, for any $\varepsilon > 0$, there exists a $\delta > 0$ independent of $k$ and $i$ such that
\[
\left| \frac{u_k(x)}{x - x_k,i} - u_k'(x_k,i) \right| < \varepsilon \quad \text{when } |x - x_k,i| < \delta. \tag{6.7}
\]
We shall show (6.7). Since $u_k'$ uniformly converges, it is equicontinuous, i.e., for any $\varepsilon > 0$ there exists a $\delta > 0$ independent of $k$ such that $|u_k'(t) - u_k'(s)| < \varepsilon$ if $|t - s| < \delta$. Using the equicontinuity in the next identity
\[
\frac{u_k(x)}{x - x_k,i} - u_k'(x_k,i) = \frac{1}{x - x_k,i} \int_{x_k,i}^{x} \{u_k'(t) - u_k'(x_k,i)\}dt,
\]
we obtain (6.7).

Define $a_i := \psi'(x_i)$ for $0 \leq i \leq n+1$. Note that $a_i \neq 0$. Put
\[
a := (1/4) \min_{0 \leq i \leq n+1} |a_i|.
\]
Since \( u_k'(x_k,i) \) converges to \( a_i \) as \( k \to \infty \), we choose an integer \( k_0 \) so large that
\[
|u_k'(x_k,i)| > |a_i|/2 \quad \text{for} \quad k \geq k_0 \quad \text{and} \quad 0 \leq i \leq n + 1.
\]
Let \( 0 < \varepsilon < a \). Then \( \delta \) is determined by (6.7). By this inequality, we have for \( k \geq k_0 \),
\[
\left| \frac{u_k(x)}{x - x_{k,i}} \right| \geq a \quad \text{when} \quad |x - x_{k,i}| < \delta.
\]
This inequality proves that
\[
|u_k(x)|^q(p-1) \leq (a|x - x_{k,i}|)^q(p-1) \quad \text{when} \quad |x - x_{k,i}| < \delta, \quad k \geq k_0.
\]
Since \( x_{k,i} \) converges to \( x_i \), we choose an integer \( k_1(> k_0) \) such that \( |x_{k,i} - x_i| < \delta/2 \) for \( 0 \leq i \leq n + 1 \) when \( k \geq k_1 \). Then
\[
|u_k(x)|^q(p-1) \leq (a|x - x_{k,i}|)^q(p-1) \quad \text{when} \quad |x - x_i| < \delta/2, \quad k \geq k_1. \quad (6.8)
\]
Rewrite \( \delta/2 \) as \( \delta \). Then (6.8) proves (6.6) for \( u_k \).

In the same way as above, we have (6.6) for \( v_k \). By (6.4) and (6.6), the right hand side of (6.3) is bounded in \( L^q(-1,1) \) because \( \|w_k\|_\infty = 1 \). Accordingly, \( w_k \) is bounded in \( W^{2,q}(-1,1) \). By the compact embedding, along a subsequence, \( w_k \) converges to a limit \( w \) in \( C^1[-1,1] \). We shall show that
\[
-w'' = p|\psi|^{p-1}w \quad \text{in} \quad (-1,1), \quad w(-1) = w(1) = 0. \quad (6.9)
\]
We rewrite \( I \) and \( J \) given by (6.5) as \( I(\delta) \) and \( J(\delta) \), respectively. Since \( u_k \) and \( v_k \) satisfy (6.8), one can easily prove that for any \( \varepsilon > 0 \), there exists a \( \delta_0 > 0 \) such that
\[
\|u_k|^{p-1} + |v_k|^{p-1}\|_{L^q(I(\delta))} < \varepsilon \quad \text{for} \quad \delta \in (0, \delta_0). \quad (6.10)
\]
Put
\[
g_k(x) := (f(u_k) - f(v_k))/(u_k - v_k), \quad h_k(x) := h(x, \lambda_k).
\]
We rewrite (6.3) as
\[
-w''_k = h_k(x)g_k(x)w_k.
\]
Let \( \zeta \in C_0^\infty(-1,1) \) be any test function. Then we have
\[
\int_{-1}^{1} w_k'(\zeta')dx = \int_{I(\delta)} h_k g_k w_k \zeta dx + \int_{J(\delta)} h_k g_k w_k \zeta dx.
\]
Define \( q' \) by \( 1/q + 1/q' = 1 \). Give \( \varepsilon > 0 \) arbitrarily. We choose \( \delta > 0 \) so small that (6.10) holds. By (6.4) and (6.10), \( \|g_k\|_{L^{q'}(I(\delta))} < \varepsilon \). Using the Hölder inequality, we have
\[
\int_{I(\delta)} |h_k g_k w_k \zeta| dx \leq \|g_k\|_{L^{q'}(I(\delta))} \|w_k\|_{L^{q'}(I(\delta))} \|\zeta\|_\infty \leq \varepsilon C,
\]
where \( C > 0 \) is independent of \( k \) and \( \varepsilon \). Then we have
\[
\left| \int_{-1}^{1} w_k' \zeta' dx - \int_{J(\delta)} h_k g_k w_k \zeta dx \right| \leq \varepsilon C.
\]
Letting \( k \to \infty \), we obtain
\[
\left| \int_{-1}^{1} w' \zeta' dx - \int_{J(\delta)} p|\psi|^{p-1}w \zeta dx \right| \leq \varepsilon C.
\]
Since \( |\psi|^{p-1} \in L^q(-1,1) \), we let \( \delta \to 0 \) to get
\[
\left| \int_{-1}^{1} w' \zeta' dx - \int_{-1}^{1} p|\psi|^{p-1}w \zeta dx \right| \leq \varepsilon C.
\]
Since $\varepsilon > 0$ is arbitrary, the inequality above implies
\[
\int_{-1}^{1} w' \zeta' dx = \int_{-1}^{1} p|\psi|^{p-1}w \zeta dx.
\]
Thus $w$ satisfies the first equation of (6.9) in the distribution sense. Since $w_k(\pm 1) = 0$, $w$ satisfies $w(-1) = w(1) = 0$. Since $w \in C^1[-1, 1]$, $|\psi|^{p-1}$ lies in $L^p(-1, 1)$ and $w$ satisfies (6.9) in the distribution sense, $w$ belongs to $W^{2, p}(-1, 1)$ and it is a solution of (6.9). By the definition of $w_k$, $\|w\|_\infty = 1$ and $w$ is a nontrivial solution. This contradicts Proposition 4.1. This conflict proves the uniqueness of $n$-nodal solutions for $\lambda > 0$ small. The proof is complete. 

7. Proofs of Theorems 1.9 and 1.10. In this section, we shall prove Theorems 1.9 and 1.10. Consider the initial value problem
\[
u'' + h(x, \lambda)|u|^{p-1}u = 0, \quad u(-1) = 0, \quad u'(-1) = \alpha > 0. \tag{7.1}
\]
We denote a unique solution of (7.1) by $u(x, \alpha)$ and its $i$-th zero in $(-1, \infty)$ by $z_i(\alpha)$. Recall that $u(x, \alpha)$ has infinitely many zeros in $(-1, \infty)$. For each $i$, $z_i(\alpha)$ is continuous with respect to $\alpha$ because of Lemma 3.3.

To prove Theorem 1.9, we need some preparations. Let $m \geq 1$ be any integer. We choose $\lambda_m \in (0, 1)$ close enough to 1 such that
\[
2^{(1+p)(1+3p)/2(1-p)}c_1^{-1/p}(1-\lambda_m)^{1+p} < (p + 1)^{-1} \lambda_m^{p+1} m^{-(1+p)/2(1-p)}, \tag{7.2}
\]
where $c_1$ has been given by Lemma 3.6. Denote a unique $2m$-nodal even solution of (1.1) with $u'(-1) > 0$ by $u_{2m}(x)$. Put $\alpha_m := u'_{2m}(-1)$. Then $u(x, \alpha_m) = u_{2m}(x)$. The equation $z_m(\alpha) = -\lambda$ for an unknown $\alpha > 0$ has a unique solution. In fact, for a solution $\alpha > 0$, $u(x) = u(x, \alpha)$ satisfies the Emden-Fowler equation in $(-1, -\lambda)$ and fulfills
\[
u(-1) = u(-\lambda) = 0, \quad u'(-1) > 0, \quad N[u, (-1, -\lambda)] = m - 1.
\]
By Lemma 3.5, such a solution $u$ is unique. Therefore $z_m(\alpha) = -\lambda$ has a unique solution $\alpha > 0$. Denote it by $\beta_m$, i.e., $z_m(\beta_m) = -\lambda$.

For a solution $u(x)$ of (1.1), $E(u)$ is a constant in $(-\infty, -\lambda]$ and another constant in $[\lambda, \infty]$. We denote them by $E_-(u)$ and $E_+(u)$, respectively, i.e., $E_-(u) := E(u)(-\infty, -\lambda)$ and $E_+(u) := E(u)(\lambda, \infty)$. Since $u(x, \alpha_m)$ is the $2m$-nodal even solution, it has exactly $m$ zeros in $(-1, -\lambda)$ and no zeros in $[-\lambda, \lambda]$. Hence $z_m(\alpha_m) < -\lambda$. Since $z_m(\beta_m) = -\lambda$, Lemma 3.7 in $(-1, -\lambda)$ shows that $E_-(u(\cdot, \alpha_m)) < E_-(u(\cdot, \beta_m))$. This implies that $\alpha_m < \beta_m$. We shall show
\[
z_m(\alpha) < -\lambda \quad \text{for} \ 0 < \alpha < \beta_m. \tag{7.3}
\]
In fact, if $0 < \alpha < \beta_m$, then $E_-(u(\cdot, \alpha)) < E_-(u(\cdot, \beta_m))$. Lemma 3.7 implies that $z_m(\alpha) < z_m(\beta_m) = -\lambda$ and (7.3) holds. We shall prove
\[
z_{m+1}(\alpha) > \lambda \quad \text{for} \ \alpha \geq \alpha_m. \tag{7.4}
\]
Suppose to the contrary that $z_{m+1}(\alpha') \leq -\lambda$ at some $\alpha' \geq \alpha_m$. Since $E_-(u(\cdot, \alpha_m)) \leq E_-(u(\cdot, \alpha'))$, Lemma 3.7 shows that $z_{m+1}(\alpha_m) \leq z_{m+1}(\alpha') \leq -\lambda$. This contradicts the definition of $\alpha_m$. Accordingly, (7.4) holds. Observe that $u(-\lambda, \alpha_m)$ is positive if $m$ is even and negative if $m$ is odd. By (7.3) and (7.4), $u(-\lambda, \alpha) \neq 0$ for $\alpha \in [\alpha_m, \beta_m]$. Therefore we have
\[
u(-\lambda, \alpha) > 0 \quad \text{in} \ [\alpha_m, \beta_m] \quad \text{when} \ m \ \text{is even}, \tag{7.5}
\]
\[
u(-\lambda, \alpha) < 0 \quad \text{in} \ [\alpha_m, \beta_m] \quad \text{when} \ m \ \text{is odd}. \tag{7.6}
\]
We shall show that $u'(-\lambda, \alpha) \neq 0$ for $\alpha \in (\alpha_m, \beta_m)$. Indeed, if $u'(-\lambda, \alpha) = 0$ at some $\alpha \in (\alpha_m, \beta_m)$, then $u'(x, \alpha) = u'(-\lambda, \alpha) = 0$ in $[-\lambda, \lambda]$. Therefore it is an even solution having exactly $m$ zeros in $(-1, -\lambda)$. This contradicts a uniqueness of $2m$-nodal even solution. Therefore $u'(-\lambda, \alpha) \neq 0$ in $(\alpha_m, \beta_m)$. Note that $u'(-\lambda, \beta_m)$ is positive if $m$ is even and negative if $m$ is odd. Hence we have

\[ u'(-\lambda, \alpha) > 0 \quad \text{for } \alpha \in (\alpha_m, \beta_m) \quad \text{when } m \text{ is even}, \]

\[ u'(-\lambda, \alpha) < 0 \quad \text{for } \alpha \in (\alpha_m, \beta_m) \quad \text{when } m \text{ is odd}. \]

By (7.5)-(7.8),

\[ u(x, \alpha) > 0 \quad \text{for } \alpha \in (\alpha_m, \beta_m), \quad x \in [-\lambda, \lambda], \quad \text{when } m \text{ is even}, \]

\[ u(x, \alpha) < 0 \quad \text{for } \alpha \in (\alpha_m, \beta_m), \quad x \in [-\lambda, \lambda], \quad \text{when } m \text{ is odd}. \]

Since $\alpha_m$ and $\beta_m$ depend on $\lambda$, we denote them by $\alpha_m(\lambda)$ and $\beta_m(\lambda)$, respectively. To prove Theorem 1.9, we need the following proposition.

**Proposition 7.1.** Let $m \geq 1$ and let $\lambda_m$ be chosen to satisfy (7.2). Then it holds that

\[ z_{m+1}(\beta_m(\lambda)) > 1 \quad \text{for } \lambda \in (\lambda_m, 1). \]

**Proof.** Let $\lambda \in (\lambda_m, 1)$ and put $u(x) := u(x, \beta_m(\lambda))$. We write $\beta_m$ instead of $\beta_m(\lambda)$. We deal with the case where $m$ is even, however, the method below is still valid for an odd $m$. Since $m$ is even and $u(-\lambda) = 0$, it follows from (7.7) that $u(\lambda), u'(\lambda) > 0$. Put $S := u(\lambda)(> 0)$. Since $u(-\lambda) = 0$ and the graph of $u(x)$ on $[-\lambda, \lambda]$ is a line segment connecting two points $(-\lambda, 0)$ and $(\lambda, S)$, its slope is computed as

\[ u'(\lambda) = u'(-\lambda) = \frac{S}{2\lambda}. \]

The energy $E(u)$ is a constant $E_-$ in $(-\infty, -\lambda]$ and another constant $E_+$ in $[\lambda, \infty)$. Using (7.12), we compute $E_-$ at $-\lambda$ and $E_+$ at $\lambda$ to obtain

\[ E_- = \frac{1}{2} u'(-\lambda)^2 = \frac{S^2}{8\lambda^2}, \]

\[ E_+ = \frac{1}{2} u'(\lambda)^2 + \frac{1}{p+1} |u(\lambda)|^{p+1} = \frac{S^2}{8\lambda^2} + \frac{S^{p+1}}{p+1}. \]

Let $l$ be the distance between adjacent zeros of $u$ in $(-\infty, -\lambda)$. Since $u(-1) = 0$ and $x = -\lambda$ is the $m$-th zero of $u$ in $(-1, \infty)$, we have $l = (1 - \lambda)/m$. Lemma 3.6 proves

\[ E_- = c_1 \left( \frac{1 - \lambda}{m} \right)^{2(1+p)/(1-p)}. \]

Combining the equation above with (7.13), we obtain

\[ S = 2\sqrt{2}\lambda c_1^{1/2}((1 - \lambda)/m)^{(1+p)/(1-p)}. \]

We shall show that $z_{m+1} := z_{m+1}(\beta_m(\lambda))$ is greater than 1. Suppose to the contrary that $z_{m+1} \leq 1$. Since $u'(\lambda), u(\lambda) > 0$, there exists a unique critical point $t$ of $u$ in $(\lambda, z_{m+1})$, that is, $u'(t) = 0$. Denote the distance between adjacent zeros of $u$ in $[\lambda, \infty)$ by $L$. By Lemma 3.3, $L = 2(z_{m+1} - t)$. Then $L = 2(z_{m+1} - t) \leq 2(1 - \lambda)$. By Lemma 3.6, we have

\[ E_+ = c_1 L^{2(1+p)/(1-p)} \leq c_1 (2(1 - \lambda))^{2(1+p)/(1-p)}. \]
It follows from (7.14) that $E_+ \geq S^{p+1}/(p+1)$. This inequality with (7.16) yields
\[
\frac{1}{p+1} S^{p+1} \leq c_1 (2(1 - \lambda))^{2^p/(1-p)}.
\]
(7.17)
Substituting (7.15) into (7.17), we have
\[
(p + 1)^{-1} 2^{3(p+1)/2} S^{p+1} c_1 (p+1)/2 ((1 - \lambda)/m)^{1-p}/2 (1 - \lambda) 1^p.
\]
This inequality contradicts (7.2) because $\lambda \in (\lambda_m, 1)$. This conflict is caused by the assumption that $z_{m+1} \leq 1$. Consequently, $z_{m+1} > 1$. The proof is complete. 

Using Proposition 7.1, we shall prove Theorem 1.9.

**Proof of Theorem 1.9.** Let $m \geq 1$ be any integer and let $\lambda \in (\lambda_m, 1)$. We have already defined $\alpha_m(\lambda)$ and $\beta_m(\lambda)$ before Proposition 7.1. We shall show the existence of an $(m, n)$-solution for each $0 \leq n \leq m - 1$. In what follows, we abbreviate $\alpha_m(\lambda)$ and $\beta_m(\lambda)$ as $\alpha_m$ and $\beta_m$, respectively.

We denote the unique solution of (7.1) by $u(x, \alpha)$ and its $i$-th zero in $(-1, \infty)$ by $z_i(\alpha)$. By (7.3) and (7.11), if $\alpha$ is slightly less than $\beta_m$, then
\[
z_m(\alpha) < -\lambda < 1 < z_{m+1}(\alpha),
\]
and hence $u(x, \alpha)$ has exactly $m$ zeros in $(-1, -\lambda)$ and no zeros in $[-\lambda, 1]$. Then we can define
\[
a_0 := \inf \{ \alpha \in (\alpha_m, \beta_m) : N[u(\cdot, \alpha), (-\lambda, 1)] = 0 \}.
\]
Since $u(x, \alpha_m)$ has exactly $m$ zeros in $(\lambda, 1)$, $u(x, \alpha)$ with $\alpha$ close to $\alpha_m$ has at least $m$ zeros in $(\lambda, 1)$. Therefore $a_0 > \alpha_m$. We shall show that $u(x, a_0)$ has no zeros in $(-\lambda, 1)$. Suppose to the contrary that it has a zero $x_0$. When $\alpha$ is close to $a_0$, $u(x, \alpha)$ has a zero near $x_0$. This contradicts the definition of $a_0$. Therefore $u(x, a_0)$ has no zeros in $(-\lambda, 1)$. By (7.3) and (7.4), $u(x, a_0)$ has exactly $m$ zeros in $(-1, -\lambda)$ and $u(-\lambda, a_0) \neq 0$. Hence we assume that $m$ is even, however, the discussion below is still valid for an odd $m$.

Since $u(x, a_0)$ has no zeros in $[-\lambda, 1]$ and $m$ is even, $u(x, a_0)$ is positive in $[-\lambda, 1)$ by (7.9). We claim that $u(1, a_0) = 0$. Suppose to the contrary that $u(1, a_0) > 0$. When $\alpha$ is close to $a_0$, $u(x, \alpha) > 0$ in $[-\lambda, 1]$. This contradicts the definition of $a_0$. Accordingly, $u(1, a_0) = 0$ and $u(x, a_0)$ becomes an $(m, 0)$-solution.

By the definition of $a_0$, if $\alpha \in (0, a_0)$, $N[u(\cdot, \alpha), (-\lambda, 1)] \geq 1$. If $\alpha$ is slightly smaller than $a_0$, $u(x, \alpha)$ has exactly one zero in $(-\lambda, 1)$ and $u(1, \alpha) < 0$. We define
\[
a_1 := \inf \{ \alpha \in (\alpha_m, \beta_m) : N[u(\cdot, \alpha), (-\lambda, 1)] = 1 \}.
\]
Since $a_1 < a_0$, $u(x, a_1)$ has at least one zero in $(-\lambda, 1)$. If it has two or more zeros, so does $u(x, \alpha)$ with $\alpha$ close to $a_1$. This is a contradiction. Thus $u(x, a_1)$ has exactly one zero in $(-\lambda, 1)$. This zero lies in $(\lambda, 1)$ by (7.9). In the same argument as in $a_0$, we have $u(1, a_1) = 0$. Hence $u(x, a_1)$ is an $(m, 1)$-solution. Repeating the argument above inductively, we can define, for each integer $n \leq m - 1$,
\[
a_n := \inf \{ \alpha \in (\alpha_m, \alpha_m) : N[u(\cdot, \alpha), (-\lambda, 1)] = n \}.
\]
In the same method as above, we can prove that $u(1, a_n) = 0$ and $u(x, a_n)$ has exactly $n$ zeros in $(-\lambda, 1)$. By (7.9), these zeros lie in $(\lambda, 1)$. By (7.3) and (7.4),
By assumption, the first intersection point of $u$ and $v$ is contrary that $u > v(a) > 0$ and $u'(a) = -v'(a) > 0$. Denote the $i$-th zeros of $u$ and $v$ in $(a,\infty)$ by $x_i$ and $y_i$, respectively. Then $y_i < x_i$ for all $i \geq 1$.

**Proof.** We first show that $y_1 < x_1$. Suppose to the contrary that $y_1 \geq x_1$. Denote the first intersection point of $u$ and $v$ by $z_1$. Then $u(x) > v(x) > 0$ in $(a, z_1)$ and $u(z_1) = v(z_1) \geq 0$. Hence $u'(z_1) < v'(z_1)$. We define the Wronskian $w(x)$ by

$$w(x) := u'(x)v(x) - u(x)v'(x).$$

Then

$$w' = w''v - uv'' = uv(|v|^p - |u|^p - 1) > 0$$

in $(a, z_1)$. Thus $w$ is strictly increasing in $(a, z_1)$. By assumption, it holds that

$$w(a) = u'(a)v(a) - u(a)v'(a) > 0.$$

On the other hand, since $u(z_1) = v(z_1) \geq 0$, we have

$$w(z_1) = u(z_1)(u'(z_1) - v'(z_1)) \leq 0.$$

A contradiction occurs. Therefore $y_1 < x_1$.

By assumption, $E(u) > E(v)$. By Lemma 3.7, it holds that $y_{i+1} - y_i < x_{i+1} - x_i$ for any $i$. This inequality with $y_i < x_i$ for all $i \geq 1$.

**Lemma 7.2.** Let $u$ and $v$ satisfy (3.1) in an interval $(a, \infty)$ with an $a \in \mathbb{R}$. Suppose that $u'(a) = v'(a) > 0$ and $u(a) > v(a) \geq 0$. Denote the $i$-th zeros of $u$ and $v$ in $(a, \infty)$ by $x_i$ and $y_i$, respectively. Then $y_i < x_{i+1}$ for all $i \geq 1$.

**Proof.** By assumption, $E(u) > E(v)$. We first show that $y_1 < x_2$. Suppose to the contrary that $y_1 \geq x_2$. Then $x_1 < x_2 \leq y_1$, and so $v(x)$ has no zeros in $(x_1, x_2)$. This contradicts Lemma 3.8. Therefore $y_1 < x_2$.

Since $E(u) > E(v)$, Lemma 3.7 shows that

$$y_{i+1} - y_i < x_{i+1} - x_i = x_{i+2} - x_{i+1},$$

for $i \geq 1$. This inequality with $y_1 < x_2$ proves that $y_i < x_{i+1}$. The proof is complete.

We conclude this paper by proving Theorem 1.10.

**Proof of Theorem 1.10.** Let $m \geq 0$ be any integer and let $u$ be any $(m, m)$-solution. We shall show the symmetry of $u$.

Let $m = 0$. Then $u$ is a positive solution, a negative solution or a 1-nodal odd solution. If $u$ is a negative solution, then $-u$ is a positive solution. If $u$ is a positive solution, then it is an even solution by Theorem 1.6. Therefore $u$ is symmetric.

Let $m \geq 1$. Suppose to the contrary that $u$ is asymmetric. By Lemma 5.1, $u(0) \neq 0$ and $u'(0) \neq 0$. We can assume that $u(0) > 0$ and $u'(0) > 0$. Indeed, if $u$ does not satisfy these conditions, we replace $u$ by $-u(x)$, $u(-x)$ or $-u(-x)$. They are still asymmetric $(m, m)$-solutions of (1.1) and one of them satisfies both $u(0) > 0$ and $u'(0) > 0$. We extend $u$ as a solution of (1.1) into $\mathbb{R}$. Denote the largest zero of $u(x)$ in $(-1, 0)$ by $z$. We divide the proof into two cases: Case 1: $-1 < z < -\lambda$, Case 2: $-\lambda \leq z < 0$. 


We shall show that each case leads to a contradiction. We first deal with Case 1: $-1 < z < \lambda$. Since the graph of $u$ on $[-\lambda, \lambda]$ is a line segment and $u(0), u'(0) > 0$ and $z < \lambda$, we have $0 < u(-\lambda) < u(\lambda)$ and $u'(\lambda) = u'(-\lambda)$. Put $v(x) := u(-x)$. This function satisfies the Emden-Fowler equation in $(\lambda, \infty)$. Observe that $u(\lambda) > v(\lambda) > 0$ and $u'(\lambda) = -v'(\lambda) > 0$. Denote the $i$-th zeros of $u$ and $v$ in $(\lambda, \infty)$ by $x_i$ and $y_i$, respectively. Applying Lemma 7.2 with $a = \lambda$, we have $y_i < x_i$ for all $i \geq 1$. Since $u'(0) > 0$ and $u(0) > 0$, $u$ has no zeros in $[0, \lambda]$. Therefore $u$ has exactly $m$ zeros in $(\lambda, 1)$. Since $u(1) = 0$, the point $x = 1$ is the $(m + 1)$-th zero, i.e., $x_{m+1} = 1$. Hence $y_{m+1} < x_{m+1} = 1$. Therefore $v$ has at least $m + 1$ zeros in $(\lambda, 1)$, or equivalently, $u$ has at least $m + 1$ zeros in $(-1, -\lambda)$. This contradicts the assumption that $u$ is an $(m, m)$-solution.

We consider Case 2: $\lambda \leq z < 0$. Since $u(0), u'(0) > 0$, we have $u(\lambda) > -u(-\lambda) \geq 0$ and $u'(\lambda) = u'(-\lambda) > 0$. We define $v(x) := -u(-x)$. Then $v$ satisfies the Emden-Fowler equation in $(\lambda, \infty)$. Verify that $u'(\lambda) = v'(\lambda) > 0$ and $u(\lambda) > v(\lambda) \geq 0$. Denote the $i$-th zeros of $u$ and $v$ in $(\lambda, \infty)$ by $x_i$ and $y_i$, respectively. Using Lemma 7.3, we have $y_i < x_{i+1}$ for $i \geq 1$. Since $x_{m+1} = 1$, it follows that $y_m < x_{m+1} = 1$. Therefore $v$ has at least $m$ zeros in $(\lambda, 1)$, in other words, $u$ has at least $m$ zeros in $(-1, -\lambda)$. Moreover, $u(z) = 0$ with $-\lambda \leq z < 0$. Thus $u$ has at least $m + 1$ zeros in $(-1, 0)$. A contradiction occurs. Consequently, Cases 1 and 2 lead to contradictions. Therefore any $(m, m)$-solution is symmetric. The proof is complete.

REFERENCES

[1] H. Brezis, Functional Analysis, Sobolev Spaces and Partial Differential Equations, (Universitext), Springer, New York, 2011.
[2] H. Brezis and L. Oswald, Remarks on sublinear elliptic equations, Nonlinear Anal., 10 (1986), 55–64.
[3] A. Gritsans and F. Sadyrbaev, Extension of the example by Moore-Nehari, Tatra Mt. Math. Publ., 63 (2015), 115–127.
[4] P. Hartman, Ordinary Differential Equations, 2nd edition, Birkhäuser, Boston, (1982).
[5] R. Kajikiya, Non-even least energy solutions of the Emden-Fowler equation, Proc. Amer. Math. Soc., 140 (2012), 1353–1362.
[6] R. Kajikiya, Non-radial least energy solutions of the generalized Hénon equation, J. Differential Equations, 252 (2012), 1987–2003.
[7] R. Kajikiya, Non-even positive solutions of the one dimensional $p$-Laplace Emden-Fowler equation, Applied Mathematics Letters, 25 (2012), 1891–1895.
[8] R. Kajikiya, Non-even positive solutions of the Emden-Fowler equations with sign-changing weights, Proc. Roy. Soc. Edinburgh Sect. A, 143 (2013), 631–642.
[9] R. Kajikiya, Symmetric and asymmetric nodal solutions for the Moore-Nehari differential equation, Submitted for publication.
[10] R. Kajikiya, I. Sim and S. Tanaka, Symmetry-breaking bifurcation for the Moore-Nehari differential equation, Nonlinear Differential Equations and Applications, 25 (2018), article 54.
[11] J. López-Gómez and P. H. Rabinowitz, Nodal solutions for a class of degenerate boundary value problems, Adv. Nonlinear Stud., 15 (2015), 253–288.
[12] J. López-Gómez and P. H. Rabinowitz, Nodal solutions for a class of degenerate one dimensional BVP’s, Topol. Methods Nonlinear Anal., 49 (2017), 359–376.
[13] J. López-Gómez, M. Molina-Meyer and P. H. Rabinowitz, Global bifurcation diagrams of one node solutions in a class of degenerate boundary value problems, Discrete Contin. Dyn. Syst. Ser. B, 22 (2017), 923–946.
[14] J. López-Gómez and P. H. Rabinowitz, The structure of the set of 1-node solutions of a class of degenerate BVP’s, J. Differential Equations, 268 (2020), 4691–4732.
[15] R. A. Moore and Z. Nehari, Nonoscillation theorems for a class of nonlinear differential equations, Trans. Amer. Math. Soc., 93 (1959), 30–52.
[16] Y. Naito and S. Tanaka, On the existence of multiple solutions of the boundary value problem for nonlinear second-order differential equations, *Nonlinear Anal.*, **56** (2004), 919–935.

[17] D. Smets, M. Willem and J. Su, Non-radial ground states for the Hénon equation, *Commun. Contemp. Math.*, **4** (2002), 467–480.

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