Evaluating community structure in large network with random walks

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Abstract. Community structure is one of the most important properties of networks. Most community algorithms are not suitable for large networks because of their time consuming. In fact there are lots of networks with millions even billions of nodes. In such case, most algorithms running in time $O(n^2 logn)$ or even larger are not practical. What we need are linear or approximately linear time algorithm. Rising in response to such needs, we propose a quick methods to evaluate community structure in networks and then put forward a local community algorithm with nearly linear time based on random walks. Using our community evaluating measure, we could find some difference results from measures used before, i.e., the Newman Modularity. Our algorithm are effective in small benchmark networks with small less accuracy than more complex algorithms but a great of advantage in time consuming for large networks, especially super large networks.

Keywords: community structure, random walk, modularity

1 Introduction

Networks are important tools to study real systems. Nodes in networks usually organized into relative densely groups called communities or clusters. Community structure have become one of the important directions. With the computer and internet techniques developing, networks we could get become larger and larger. Take the liveJournal online social network and U.S. patent dataset for example. LiveJournal is a free on-line community with almost 10 million members [2]. The U.S. patent dataset is maintained by the National Bureau of Economic Research and includes about 3,923,922 patents [14]. It is reasonable to believe lots of other real networks are larger and will increase quickly in future. There is a great need for developing quick community detection algorithms.

To study community structure in large networks, we should evaluate whether a given network have community structure and how to find them if there are. So far, the most accepted measure to evaluate the community structure is modularity [5] [18]. However, modularity has an intrinsic scale indicting that modules smaller than that scale may not be resolved [10] and finding the partition with maximal modularity is not a trivial thing. Local methods which works independent on the global structure seems more practical in such large datas [1] [3] [4].
One kind of local algorithms divide the whole network into two parts \[3\] \[4\] \[17\], a community \(C\) and the set of nodes with links to \(C\), say \(B\). They usually start from a given node \(s\) and then explore \(B\) and select one or more nodes to merge into \(C\). Such operation is repeated until some terminal condition is satisfied. Another kind of local algorithms work in some different ways. Firstly, they calculate a vector around a given node \(s\). This vector includes information that indicates the tendencies to the \(C\) we are to find. Then sort the vector according the score and a new vector called a support vector is got. Finally, we could take a sweep over this support vector based on some quality and find the community. One of such algorithms is \[1\] which has been used by Leskovec to find lots of interesting phenomenon in real networks \[15\] \[16\].

As we know, random walks have close relationship with community. A random walker from a given position will be 'trapped' in a community with high probability. There are lots of community algorithms proposed inspired by this idea. A vertices similarity measure and a community similarity measure are proposed by Latapy and Pons and then communities are get by a agglomerative procedure \[19\]. Also based on random walks, Zhou define a distance between pairs of nodes and use divisive procedure to detect communities with running time \(O(n^3)\) \[20\] \[27\] \[28\]. Community structure could be related to random walk through the information theoretic approach where the community detecting procedure becomes compressing a description of the probability flow of random walk \[20\]. Delvenne introduce a quality function indicating the persistence of clustering over time and unifies the modularity measures \[5\] \[18\] as well as several definition related with random walk \[7\]. Other community algorithm based on random walk includes \textit{MarkovClusterAlgorithm} (MCL) with running time \(O(nk^2)\) \[23\], methods using signaling process \[12\] and methods of minimizing the matrix distance \[24\]. More detail about these methods could be found in \[9\]. All of these algorithm scale \(O(n^2\log n)\) or higher, and are not practical in super large networks with millions and billions of nodes. In this paper, we give a quick measure to evaluate the community structure and a local methods to find communities based on random walks in nearly linear time, which could be used in very large networks.

We arrange the rest of the paper as follows. In section 2 we propose a modularity measure based on random walks. Then we give a algorithm and test it on benchmarks in section 3. The result of some experiments in very large networks are given in section 4. Finally, we give the conclusion in section 5.

### 2 Random Walk Modularity

As pointed above, the random walk has significant indication of network structure. We consider the following situation, a random walk is terminated when it forms a ring and the corresponding number of steps is called random walk length (RWL for short). A question of interesting is how long the expectation of RWL is for a given network? We test the relationship between the RWL in ER random networks \[8\] where every pair of nodes are linked with probability
$p$ and the planted $l$-partition model [11] which has been used largely to test a community algorithm’s performance.

![Fig. 1. The relationship between random walk length and $p$ in ER random network](image)

In the ER random networks where every pair of nodes are linked with random, the ring are usually formed by two nodes (the backward walk) as the probability more than two nodes form a ring is small when the network size tend to infinite. In every step, the probability a ring is formed by a backward walk is $\frac{1}{d}$. We assume every nodes have the same degree $d$ for the sake of discussion. Let $L_r$ be the expectation of the random walk length, $q_l$ is the probability that a random walker ends with $l$ step, we have the following:

$$L_r = \sum_{l \leq n} q_l l = \frac{1}{d} + 2 * (1 - \frac{1}{d}) * \frac{1}{d} + ... + n * (1 - \frac{1}{d})^{n-1} * \frac{1}{d}$$

$$= d - (d + n)(1 - \frac{1}{d})^n$$

(1)

So when $n \rightarrow \infty$ the expectation of RWL is mainly affected by $d$. When the degree is not constant, the analysis become complex, but there seems a linear relationship between the average random walk length and $p$ see figure 1.

On the other hand, the ARL has inverse correlation with the community structure. The planted $l$-partition benchmark is used to illustrate the relationship between random walk length and community structure. This benchmark has been very popular to test the performance of community algorithm since proposed by Condon and Karp [6] and a special case of planted $l$-partition model is given by Newman [11]. In the Newman model, 128 vertices are partitioned into 4 groups with each group 32 vertices. Every vertex has $z_{in}$ links in the same group, and $z_{out}$ links outside of the group. The average total degree of vertices are fixed to 16. $p$ is the ratio between $z_{out}$ and average degree of each vertex. So the community structure could be controlled by $p$. The average RWL has a close relationship with $p$ see figure 2. This is according with previous idea, a
Fig. 2. Average step length a walker have to pass before he encounters a ring in \( l \)-partition model [11]. In this model, every node has a fixed average degree 16, \( p \) is the ratio between the its out degree and its total degree

random walker could be easily trapped in a community. So the chance that a random walk forms a ring increases in networks with clear community structure. As \( p \) gets larger, the community structure becomes more and more fuzzy. As a result, the random walker has more probability to escape from the ‘trap’ and RWL increases.

From above analysis, we know RWL is mainly affected by the degree sequence and the community structure. Inspired by this, we propose a simple community evaluating measure called Random Walk Modularity which could be calculated in approximate linear time. The definition is as follows:

\[
Q(G) = 1 - \frac{L(G)}{L(G_r)}
\] (2)

\( L(G) \) means the average random length in graph \( G \), and \( L(G_r) \) means the average random length in graph \( G_r \) which is the configure model of \( G \) with the same degree sequence of \( G \). This measure removes the influence of degree and reflects community structure of different networks.

There are some difference performance in real networks between the Random Walk Modularity and the modularity [18] given by Newman which we called Newman Modularity in this paper. Firstly, random network are usually thought to have no community structure, but when \( p \) is small, the Newman modularity and Conductance Modularity (see section 3) can be very large. On the contrary Random Walk Modularity is not affect by \( p \) see figure 3, which is more accord with our intuition. Secondly, some deterministic networks, eg. the ring and the lattice, have a high Newman Modularity value, but whether they have community structure is disputable. In the lattice, every vertex has the same position, so their community structure even if they have does not interest us. The Random Walk Modularity all remove such networks by a low value, see table 4. Further more, for networks with clear community structure like the planted \( l \)-partition
networks, Random Walk Modularity always give high value compared with other networks with fuzzy community structure, see figure 4. Finally, the time to calculate Random Walk Modularity is mainly determined by the average random walk length. Most real networks have average random length less than 10, and all of them less than 20 in our experiment, see section 4. So Random Walk Modularity can be calculated in a nearly linear time which indicate that it has an advantage for large networks.

Fig. 3. Different modularity measures in ER random network, every pair of vertices are linked with probability $p$, $c_m$ means conductance modularity, $n_m$ means Newman modularity, $r_m$ means random walk modularity

Fig. 4. Different modularity measures in planted $l$-partition model, $p$ is the ratio between $z_{out}$ and its average degree, $c_m$ means conductance modularity, $n_m$ means Newman modularity, $r_m$ means random walk modularity
Table 1. Different modularity measures in some deterministic networks. The ring is a one dimensional lattice, includes 1000 nodes. The tree includes 1000 nodes with all vertices having the same number of children, 2 in our experiments. The lattice have two dimensions, each dimension has 100 nodes

| modularity | network | ring  | tree | lattice |
|------------|---------|-------|------|---------|
| RandomWalk | 0.003   | 0.0004| 0.08 |
| Newman     | 0.94    | 0.93  | 0.89 |
| Conductance| 0.97    | 0.93  | 0.9  |

3 Algorithm

In the last section, we propose a quick measure to evaluate the community structure in large networks. Starting a random walk from a given node, one could easily be trapped in a community. The average random walk length has an inverse correlation with the community structure. Owing to the community structure, the average random walk length becomes shorter. Our community algorithm are based on this idea. We perform a series of random walk from a given node \( s_0 \), and ends a walk when this walk forms a ring. At the same time, we assume the tendencies of nodes to the community of \( s_0 \) has a positive correlation with the ring position. Our algorithm is given in the following.

\[ \text{RandomWalkRing}(G, s_0, n) \]
1. Set vector \( P = \phi \)
2. Perform a random walk from \( s_0 \), record each node passed, when encounter a ring ends this walk and record its position \( l \). \( \forall v \) in the random walk trail, let \( P(v) = P(v) + \frac{1}{\text{deg}(v)} \)
3. repeat step2 \( n \) times
4. Order nodes in \( P \) by decreasing value \( P(v) \), get a support vector \( S \)
5. Compute the conductance \( \phi(S_i) \) of the first \( i \) nodes, for \( i \leq |S| \)
6. find the index \( k^* \) at every local optimal of \( \phi(S) \), return \( S_0 = \{ v_i | i \leq k^* \} \)

In the above algorithm, we need a quality function to extract the community from the support vector. We use conductance as the quality function which has been proposed in [1] and has been used by Leskovec [25]. Conductance has been popular to measure community structure in recent years [13][15][16]. Let the volume \( \text{vol}(S) \) of a set \( S \) be the total degree of vertices in it, i.e., \( \text{vol}(S) = \sum_{v \in S} \text{deg}(v) \). The conductance \( \phi(S) \) of a set \( S \) is defined to be the ratio of the number of edges \( e(S, \bar{S}) \) coming out of \( S \) with the minimum of the volume of itself and the volume of its complement \( \bar{S} \), i.e., \( \phi(S) = e(S, \bar{S})/\min\{\text{vol}(S), \text{vol}(\bar{S})\} \). The conductance of the graph is the minimum conductance over all sets and it is extensively studied in computer science, with applications to random walks, spectral or flow based graph partitioning, and combinatorial object constructions. Intuitively, a set of low conductance (smaller than some constant \( \phi_0 \)) can be thought of a nice community. Using this definition, lots of interesting phenomenon have been found. Leskovec finds most
networks seems have a ‘core’ contains a constant faction of the nodes with a
periphery consisting of a large number of relatively small ‘whiskers’ [15].

Let $C_1$ and $C_2$ are two communities, we use the following community similarity measures to evaluate our algorithm, where the planted $l$-partition model is also used.

$$S(C_1, C_2) = \frac{|C_1 \cap C_2|}{\sqrt{|C_1| \ast |C_2|}}.$$  \hspace{1cm} (3)

For each real communities, we find the most similar communities return by our algorithm, see figure 5 for the performance of our algorithm and the algorithm of [1]. Our algorithm has similar performance when $p$ is small, and a little less accuracy when $p$ is larger than 0.25. What we emphasize is that our algorithms is very quick in large networks as pointed before. So such sacrificing of accuracy is inevitable. Something should be noticed that the accuracy of our algorithm is affected by $n$ a lot. Generally, the larger $n$ is, the more the accuracy of our algorithm is. So there is a compromise between the performance and speed. In our experiment we set $n$ to 1000.

Conductance is a local definition and could not give a global knowledge to judge whether a network has good community structure or not. We give another modularity measure which we called Conductance Modularity to differ from previous ones. Let $c$ be a real number between 0 and 1, $f$ is the corresponding fraction of nodes in community with smaller conductance than $c$. then Conductance Modularity is as follows.

$$C(G) = max_{c \in [0,1]} \sqrt{(1 - c) \ast f}$$

As we know, usually a smaller conductance indicates the corresponding community is better. A network with good community structure should have as many as possible nodes in good communities. The Conductance Modularity considers both community’s quality and nodes number, which could give us a intuition whether a network has community structure or not from the point of conductance.
Figure 4 and figure 3 are an comparison among three modularity measures. All of them are sensitive with community structure in planted l-partition model. While Conductance Modularity and Newman Modularity are affected by $p$ in ER random networks a lot, Random Walk Modularity is independent on $p$ and seems more better.

4 Application

We perform our algorithm on 34 networks in a acceptable time including some very large networks. Using our algorithms, we could find more than one communities from each node indicating different level of communities just as Leskovec do. In table 2 all the results are calculated from the first local optimal community for the sake of discussion. As in most case we are more care about the smallest group includes us, which is always more compact and has more influence for us although the conductance is not the optimal in global.

The Random Walk Modularity has a different interpretation about the community structure compared other measures. Before discussion, we give the following classification of network by their corresponding modularity. Random Walk Modularity in ER random networks is or very near to 0. So networks with Random Walk Modularity below 0.05 are thought to have no clear community structure, between 0.05 and 0.1 are thought to have weak community structure, and above 0.1 are thought to have clear community structure. For Newman Modularity and Conductance Modularity, the boundary of clear community structure are set to be 0.3 and 0.5 respectively.

From the point of Newman Modularity and Conductance Modularity, all networks have clear community structure, except livejournal and vikivote networks whose Newman Modularity could not be calculated in acceptable time by the fast greedy algorithm. The road, web, amazon and some collaboration, citation and email networks have high value, while others networks have relative small value. The Newman Modularity and Conductance Modularity are usually consistent with each other, which means when one measure give a high score, the other is always give a high score.

When consider the Random Walk Modularity, the situation is different. Networks are divided into three classes as discussed before. The road, p2p, vikivote and email_ennall networks have no clear community structure, even the road networks have the highest Newman Modularity and Conductance Modularity. Citation_arnetminer and Citation_patents networks has weak community structure. Other networks are thought to have clear community structure. Such difference could be explained by figure 3 and table 1. Those networks with high Newman Modularity and Conductance Modularity but small Random Walk Modularity

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1 We only consider the corresponding undirected graphs for all the networks. Except the citation_arnetminer network [22] [21] is from http://arnetminer.org/citation and the football network is from http://www-personal.umich.edu/~mejn/netdata, all other networks can be found from http://snap.standford.edu
Table 2. The statistics of network. RM means Random Walk Modularity, NM means Newman Modularity found by fast greedy algorithm [5]. CM means Conductance Modularity, AvgC means the average conductance of all communities, ARL means average random walk length, AvgS means the average size of all communities.

| network          | RM  | NM  | CM  | AvgC | ARL  | AvgS  |
|------------------|-----|-----|-----|------|------|-------|
| amazon0302       | 0.25| 0.82| 0.79| 0.18 | 5.86 | 19.95 |
| amazon0312       | 0.31| 0.8 | 0.73| 0.24 | 8.77 | 26.29 |
| amazon0505       | 0.31| 0.76| 0.73| 0.24 | 8.86 | 26.42 |
| amazon0601       | 0.32| 0.74| 0.73| 0.24 | 8.95 | 26.96 |
| cit_arxiv        | 0.06| 0.65| 0.63| 0.42 | 6.63 | 16.8  |
| cit_hepph        | 0.25| 0.56| 0.55| 0.48 | 18.5 | 28.7  |
| cit_hepth        | 0.27| 0.53| 0.59| 0.42 | 18.2 | 31.6  |
| cit_patents      | 0.08| 0.76| 0.59| 0.45 | 8.95 | 18.6  |
| col_arxiv        | 0.34| 0.51| 0.63| 0.39 | 14   | 22.7  |
| col_condmat      | 0.29| 0.64| 0.74| 0.32 | 6.75 | 18.3  |
| col_gqrc         | 0.31| 0.79| 0.83| 0.33 | 5    | 17.7  |
| col_hepph        | 0.42| 0.58| 0.71| 0.33 | 11   | 19.9  |
| col_hepth        | 0.19| 0.69| 0.75| 0.33 | 5.4  | 16.7  |
| email_enron      | 0.18| 0.5 | 0.57| 0.47 | 9.34 | 49.1  |
| email_enall      | 0.01| 0.73| 0.66| 0.46 | 3.98 | 409   |
| football         | 0.17| 0.57| 0.88| 0.14 | 7.2  | 20.9  |
| livejournal      | 0.19| -1  | 0.53| 0.48 | 15.2 | 19.3  |
| p2p4             | 0.007| 0.38| 0.53| 0.56 | 8.2  | 11.9  |
| p2p5             | 0.006| 0.4 | 0.54| 0.59 | 8.1  | 12.6  |
| p2p6             | 0.006| 0.39| 0.54| 0.59 | 8.1  | 12.5  |
| p2p8             | 0.015| 0.46| 0.58| 0.54 | 7.4  | 12.7  |
| p2p9             | 0.014| 0.46| 0.58| 0.54 | 7.3  | 12.5  |
| p2p24            | 0.002| 0.47| 0.62| 0.48 | 5.9  | 11.1  |
| p2p25            | 0.005| 0.49| 0.63| 0.47 | 5.8  | 11.5  |
| p2p30            | 0.005| 0.46| 0.62| 0.46 | 5.8  | 11.3  |
| p2p31            | 0.003| 0.5 | 0.63| 0.46 | 5.7  | 11    |
| roadnet_pa       | 0.04| 0.99| 0.93| 0.087| 3.67 | 26.7  |
| roadnet_pa       | 0.04| 0.99| 0.93| 0.087| 3.68 | 26.9  |
| roadnet_texas     | 0.04| 0.99| 0.93| 0.1  | 3.63 | 26.2  |
| vikivote         | 0.002| -1  | 0.58| 0.5  | 4.89 | 496   |
| web_berkstan      | 0.54| 0.91| 0.65| 0.32 | 9    | 44.4  |
| web_google        | 0.39| 0.92| 0.79| 0.17 | 6.68 | 30.3  |
| web_notredame     | 0.35| 0.93| 0.76| 0.16 | 5    | 88.2  |
| web_stanford      | 0.47| 0.88| 0.65| 0.35 | 7.9  | 40.8  |
maybe networks like lattice or with very small average degree whose community structure are debatable. In all, Random Walk Modularity are more strict to evaluate the community structure.

In table 2 we also give some other properties. As a whole, most networks tend to have small RWL, small conductance and small communities size. The small communities size maybe influenced by our selection of the first optimal conductance. The short AWL is clear, even the most largest networks the live-Journal online social network and U.S. patent dataset only need about 15 and 9 steps to form a ring. The RWL seems have a upper bound by the average degree as analysis before. Owing the influence of community structure, real RWL is always smaller than that value. The results show Random Walk Modularity are independent on conductance, RWL and community size. If a network has high Random Walk Modularity value, we are more believe it has community structure.

5 Conclusion

In this paper, we propose a method to evaluate community structure and a local community algorithm based on random walks with approximately linear running time. Our experiments show the average random walk length are affected by two factors, the average degree of the graph and community structure. Average random walk length are very short in real networks, which is either caused by networks’ sparseness or community structure or both. Such short average random walk guarantees Random Walk Modularity could be calculated in near linear time. We also give a modularity measure from the conductance view, which gives us a profile about a large networks. Usually the Conductance Modularity and Newman Modularity are consistent in our experiment, while Random Walk Modularity could give a different judge. Random Walk Modularity has advantageous both in evaluating the community structure and speed. Networks with high Random Walk Modularity are more believable to have good community structure, while Newman Modularity and Conductance Modularity could also give some ER random network high value. So Random Walk Modularity should be used when we cared about the network community structure without the debatable community.

The running time of random ring algorithm is mainly influenced random walk length and the random walk number. The former are influence by average degree and community structure and is usually small, less than 20 in all network in our experiments. N could be set by user where both accuracy and speed should be considered. Our results show, with some little accuracy sacrifice we could improve the algorithm’s speed a lot. The random ring algorithm could be used on very large networks with millions or billons of nodes.

In the future, we will study the evolution of community structure and explain why networks form different structures. Methods proposed in this paper could help disclosed the large network structure a lot.
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