An Intuitionistically based Description Logic

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Abstract

This article presents $i\text{ALC}$, an intuitionistic version of the classical description logic $\text{ALC}$, based on the framework for constructive modal logics presented by Simpson [14] and related to description languages, via hybrid logics, by dePaiva [3]. This article corrects and extends the presentation of $i\text{ALC}$ appearing in [4]. It points out the difference between $i\text{ALC}$ and the intuitionistic hybrid logic presented in [3]. Completeness and soundness proofs are provided. A brief discussion on the computational complexity of $i\text{ALC}$ provability is taken. It is worth mentioning that $i\text{ALC}$ is used to formalize legal knowledge [10, 9, 8, 7], and in fact, was specifically designed to this goal.

1 Intuitionistic ALC

The $i\text{ALC}$ logic is based on the framework for intuitionistic modal logic IK proposed in [14 5 13]. These modal logics arise from interpreting the usual possible worlds definitions in an intuitionistic meta-theory. As we will see in the following paragraphs, ideas from [1] were also used, where the framework IHL, for intuitionistic hybrid logics, is introduced. $i\text{ALC}$ concepts are described as:

\[
C, D ::= A | ⊥ | ⊤ | ¬ C | C ∧ D | C ∨ D | C ⊑ D | ∃ R.C | ∀ R.C
\]

where $C, D$ stands for concepts, $A$ for an atomic concept, $R$ for an atomic role. We could have used distinct symbols for subsumption of concepts and the subsumption concept constructor but this would blow-up the calculus presentation. This syntax is more general than standard $\text{ALC}$ since it includes subsumption $⊆$ as a concept-forming operator. We have no use for nested subsumptions, but they do make the system easier to define, so we keep the general rules. Negation could be defined via subsumption, that is, $¬C = C ⊑ ⊥$, but we find it convenient to keep it in the language. The constant $⊤$ could also be omitted since it can be represented as $¬⊥$.

A constructive interpretation of $i\text{ALC}$ is a structure $I$ consisting of a non-empty set $Δ^I$ of entities in which each entity represents a partially defined...
individual; a refinement pre-ordering $\preceq^I$ on $\Delta^I$, i.e., a reflexive and transitive relation; and an interpretation function $\cdot^I$ mapping each role name $R$ to a binary relation $R^I \subseteq \Delta^I \times \Delta^I$ and atomic concept $A$ to a set $A^I \subseteq \Delta^I$ which is closed under refinement, i.e., $x \in A^I$ and $x \preceq^I y$ implies $y \in A^I$. The interpretation $I$ is lifted from atomic concepts to arbitrary concepts via:

\[
\begin{align*}
\top^I &= d_f \Delta^I \\
\bot^I &= d_f \emptyset \\
(\neg C)^I &= d_f \{ x \mid \forall y \in \Delta^I. x \preceq^I y \Rightarrow y \notin C^I \} \\
(C \cap D)^I &= d_f C^I \cap D^I \\
(C \cup D)^I &= d_f C^I \cup D^I \\
(\exists R.C)^I &= d_f \{ x \mid \exists y \in \Delta^I. (x, y) \in R^I \text{ and } y \in C^I \} \\
(\forall R.C)^I &= d_f \{ x \mid \forall y \in \Delta^I. x \preceq^I y \Rightarrow \forall z \in \Delta^I. (y, z) \in R^I \Rightarrow z \in C^I \}
\end{align*}
\]

Following the semantics of IK, the structures $I$ are models for $iALC$ if they satisfy two frame conditions:

**F1** if $w \leq w'$ and $wRv$ then $\exists v'. w'Rv'$ and $v \leq v'$

**F2** if $v \leq v'$ and $vRw$ then $\exists w'. w'Rv'$ and $w \leq w'$

The above conditions are diagrammatically expressed as:

\[
\begin{align*}
&\begin{array}{c}
\xymatrix{w' \ar[r]^R & \ar[l]_{\preceq^I} v'}
\end{array} \quad \text{and} \quad \begin{array}{c}
\xymatrix{w' \ar[r]^R & \ar[l]_{\preceq^I} v'}
\end{array} \\
&\begin{array}{c}
\xymatrix{w \ar[r]^R & \ar[l]_{\preceq^I} v}
\end{array} \quad \text{and} \quad \begin{array}{c}
\xymatrix{w \ar[r]^R & \ar[l]_{\preceq^I} v}
\end{array}
\end{align*}
\]

Our setting simplifies $I MIL$, since $iALC$ satisfies (like classical $ALC$) $\exists R. \bot = \bot$ and $\exists R. (C \cup D) = \exists R. C \cup \exists R. D$.

Building up from the Simpson’s constructive modal logics (called here IML), in $[1]$, it is introduced intuitionistic hybrid logics, denoted by IHL. Hybrid logics add to usual modal logics a new kind of propositional symbols, the nominals, and also the so-called satisfaction operators. A nominal is assumed to be true at exactly one world, so a nominal can be considered the name of a world. If $x$ is a nominal and $X$ is an arbitrary formula, then a new formula $x:X$ called a satisfaction statement can be formed. The satisfaction statement $x:X$ expresses that the formula $X$ is true at one particular world, namely the world denoted by $x$. In hindsight one can see that IML shares with hybrid formalisms the idea of making the possible-world semantics part of the deductive system. While IML makes the relationship between worlds (e.g., $xRy$) part of the deductive system, IHL goes one step further and sees the worlds themselves $x, y$ as part of the deductive system, (as they are now nominals) and the satisfaction relation itself as part of the deductive system, as it is now a syntactic operator, with modality-like properties. In contrast with the above mentioned approaches, ours assign a truth values to some formulas, also called assertions, they are not concepts as in $I MIL$, for example. Below we define the syntax of general assertions ($A$)
and nominal assertions \( (N) \) for ABOX reasoning in \( i\text{ALC} \). Formulas \( (F) \) also includes subsumption of concepts interpreted as propositional statements.

\[
N ::= x : C \mid x : N \\
A ::= N \mid x R y \\
F ::= A \mid C \sqsubseteq C
\]

where \( x \) and \( y \) are nominals, \( R \) is a role symbol and \( C \) is a concept. In particular, this allows \( x : (y : C) \), which is a perfectly valid nominal assertion.

**Definition 1 (outer nominal)** In a nominal assertion \( x : \gamma \), \( x \) is said to be the outer nominal of this assertion. That is, in an assertion of the form \( x : (y : \gamma) \), \( x \) is the outer nominal.

We write \( I, w \models C \) to abbreviate \( w \in C_I \) which means that entity \( w \) satisfies concept \( C \) in the interpretation \( I \). Further, \( I \) is a model of \( C \), written \( I \models C \) iff \( \forall w \in I. I, w \models C \). Finally, \( \models C \) means \( \forall I. I \models C \). All previous notions are extended to sets \( \Phi \) of concepts in the usual universal fashion. Given the hybrid satisfaction statements, the interpretation and semantic satisfaction relation are extended in the expected way. The statement \( I, w \models x : C \) holds, if and only if, \( \forall z. x \sqsupseteq_I z \Rightarrow \forall \vec{z} \sqsupseteq_I \text{Nom}(\Gamma, \delta) \Rightarrow I, \vec{z} \models \delta \) (1)

where \( \vec{z} \) denotes a vector of variables \( z_1, \ldots, z_k \) and \( \text{Nom}(\Gamma, \delta) \) is the vector of all outer nominals occurring in each nominal assertion of \( \Gamma \cup \{ \delta \} \). \( x \) is the only outer nominal of a nominal assertion \( \{ x : \gamma \} \), while a (pure) concept \( \gamma \) has no outer nominal.

A Hilbert calculus for \( i\text{ALC} \) is provided following [13, 14, 5]. It consists of all axioms of intuitionistic propositional logic plus the axioms and rules displayed in Figure 1. The Hilbert calculus implements TBox-reasoning. That is, it decides the semantical relationship \( \Theta, \emptyset \models C \). \( \Theta \) has only formulas as members.

A Sequent Calculus for \( i\text{ALC} \) is also provided. The logical rules of the Sequent Calculus for \( i\text{ALC} \) are presented in Figure 2. The structural rules and

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1. In IHL, this \( w \) is a world and this satisfaction relation is possible world semantics
2. Here we consider only acycled TBox with \( \sqsubseteq \) and \( \equiv \).
3. The reader may want to read Proof Theory books, for example, [15, 2, 12, 6].

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0. all substitution instances of theorems of IPL
1. \( \forall R. (C \sqsubseteq D) \sqsubseteq (\forall R.C \sqsubseteq \forall R.D) \)
2. \( \exists R. (C \sqsubseteq D) \sqsubseteq (\exists R.C \sqsubseteq \exists R.D) \)
3. \( \exists R. (C \sqcup D) \sqsubseteq (\exists R.C \sqcup \exists R.D) \)
4. \( \exists R. \bot \sqsubseteq \bot \)
5. \( (\exists R.C \sqsubseteq \forall R.C) \sqsubseteq \forall R.(C \sqsubseteq D) \)

**MP** If \( C \) and \( C \sqsubseteq D \) are theorems, \( D \) is a theorem too.

**Nec** If \( C \) is a theorem then \( \forall R.C \) is a theorem too.

---

**Figure 1:** The \( i\text{ALC} \) axiomatization

the cut rule are omitted but they are as usual. The \( \delta \) stands for concepts or assertions (\( x: C \) or \( xRy \)), \( \alpha \) and \( \beta \) for concept and \( R \) for role. \( \Delta \) is a set of formulas. In rules \( p-\exists \) and \( p-\forall \), the syntax \( \forall R. \Delta \) means \( \{ \forall R.\alpha \mid \alpha \in \text{concepts}(\Delta) \} \), that is, all concepts in \( \Delta \) are universal quantified with the same role. The assertions in \( \Delta \) are kept unmodified. In the same way, in rule \( p-N \) the addition of the nominal is made only in the concepts of \( \Delta \) (and in \( \delta \) if that is a concept) keeping the assertions unmodified.

The propositional connectives (\( \sqcap, \sqcup, \sqsubseteq \)) rules are as usual, the rule \( \sqcup_{2-r} \) is omitted. The rules are presented without nominals but for each of these rules there is a counterpart with nominals. For example, the rule \( \sqsubseteq-r \) has one similar:

\[
\frac{\Delta, x: \alpha \Rightarrow x: \beta}{\Delta \Rightarrow x: (\alpha \sqsubseteq \beta)} \quad \text{n-}\sqsubseteq-r
\]

The main modification comes for the modal rules, which are now role quantification rules. We must keep the intuitionistic constraints for modal operators. Rule \( \exists-1 \) has the usual condition that \( y \) is not in the conclusion. Concerning the usual condition on the \( \forall-r \) rule, it is not the case in this system, for the interpretation of the a nominal assertion in a sequent is already implicitly universal (Definition 2).

**Theorem 1** The sequent calculus described in Fig. 2 is sound and complete for TBox reasoning, that is \( \Theta, \emptyset \models C \) if and only if \( \Theta \Rightarrow C \) is derivable with the rules of Figure 2.

The completeness of our system is proved relative to the axiomatization of \( i\text{ALC} \), shown in Figure 1. The proof is presented in Section 2.

The soundness of the system is proved directly from the semantics of \( i\text{ALC} \) including the ABOX, that is, including nominals. The semantics of a sequent is defined by the satisfaction relation, as shown in Definition 2. The sequent \( \Theta, \Gamma \Rightarrow \delta \) is valid if and only if \( \Theta, \Gamma \models \gamma \). Soundness is proved by showing
Figure 2: The System SC_{iALC}: logical rules
that each sequent rule preserves the validity of the sequent and that the initial sequent is valid. This proof is presented in Section 3.

We note that although we have here fixed some inaccuracies in the presentation of the $iALC$ semantics in [4], the system presented here is basically the same, excepted that here the propositional rules are presented without nominals. Given that, the soundness of the system proved in [4] can be still considered valid without further problems. Note also that the proof of soundness provides in Section 3 is regarded the full language of $iALC$. It considers nominals and assertion on nominals relationship, that is it concerns ABOX and TBOX. The proof of completeness is for the TBOX only. A proof of completeness for ABOX can be done by the method of canonical models. For the purposes of this article, we choose to show the relative completeness proof with the sake of showing a simpler proof concerning TBOX.

2 The completeness of $SC_iALC$ system

We show the relative completeness of $SC_iALC$ regarding the axiomatic presentation of $iALC$ presented in Figure 1. To prove the completeness of $SC_iALC$ it is sufficient to derive in $SC_iALC$ the axioms 1–5 of $iALC$. It is clear that all substitution instances of IPL theorems can also be proved in $SC_iALC$ using only propositional rules. The MP rule is a derived rule from the $SC_iALC$ using the cut rule. The Nec rule is the $p$-$\forall$ rule in the system with $\Delta$ empty. In the first two proofs below do not use nominals for given better intuition of the reader about the use of rules with and without nominals.

Axiom 1:

\[
\frac{\alpha \Rightarrow \alpha \quad \beta \Rightarrow \beta}{\bar{\Box} \text{-1}}
\]

\[
\frac{\alpha \subseteq \beta, \alpha \Rightarrow \beta}{\bar{p} \text{-3}}
\]

\[
\frac{\forall R. (\alpha \subseteq \beta), \exists R. \alpha \Rightarrow \exists R. \beta}{\bar{\Box} \text{-}\Box -r}
\]

Axiom 2:

\[
\frac{\alpha \Rightarrow \alpha \quad \beta \Rightarrow \beta}{\bar{\Box} \text{-1}}
\]

\[
\frac{\alpha \subseteq \beta, \alpha \Rightarrow \beta}{\bar{p} \text{-}\forall}
\]

\[
\frac{\forall R. (\alpha \subseteq \beta), \forall R. \alpha \Rightarrow \forall R. \beta}{\bar{\Box} \text{-}\Box -r}
\]

Axiom 3:

\[
\frac{x R y, y : \bot \Rightarrow x : \bot}{\bar{3} \text{-1}}
\]

\[
\frac{x : \exists R. \bot \Rightarrow x : \bot}{\bar{\exists} \text{-1}}
\]

\[
\frac{x : (\exists R. \bot \subseteq \bot)}{\bar{\Box} \text{-}\Box -r}
\]

Axiom 4:
3 SOUNDNESS OF SC$_{iALC}$ SYSTEM

\[
x : \exists R.\alpha \Rightarrow x : \exists R.\alpha
\]
\[
x : \exists R.\alpha \Rightarrow x : (\exists R.\alpha \sqcup \exists R.\beta)
\]
\[
\sqcup_{1-r}
\]
\[
x : \exists R.\alpha \Rightarrow x : (\exists R.\alpha \sqcup \exists R.\beta)
\]
\[
\sqcup_{2-r}
\]
\[
x : \exists R.\alpha \Rightarrow x : (\exists R.\alpha \sqcup \exists R.\beta)
\]
\[
\sqcup_{l-1}
\]

Axiom 5:

\[
x Ry, y : \alpha \Rightarrow y : \alpha
\]
\[
x Ry, y : \alpha \Rightarrow xRy
\]
\[
\sqcup_{r}
\]
\[
x Ry, y : \alpha, \exists R.\beta \Rightarrow y : \beta
\]
\[
\sqcup_{\forall l}
\]
\[
x : (\exists R.\alpha \sqsubseteq \forall R.\beta), xRy, y : \alpha \Rightarrow y : \beta
\]
\[
\sqcup_{\forall r}
\]
\[
x : (\exists R.\alpha \sqsubseteq \forall R.\beta) \Rightarrow x : \forall R.\alpha
\]
\[
\sqcup_{r}
\]
\[
x : (\exists R.\alpha \sqsubseteq \forall R.\beta) \Rightarrow x : [\exists R.\alpha \sqsubseteq \forall R.\beta \sqsubseteq \forall R.\alpha]
\]
\[
\sqcup_{r}
\]

3 Soundness of SC$_{iALC}$ system

In this section we prove that.

**Proposition 1** If $\Theta, \Gamma \Rightarrow \delta$ is provable in SC$_{iALC}$ then $\Theta, \Gamma \vdash \gamma$.

**Proof:** We prove that each sequent rule preserves the validity of the sequent and that the initial sequents are valid. The definition of a valid sequent ($\Theta, \Gamma \vdash \gamma$) is presented in Definition 3.

The validity of the axioms is trivial. We first observe that any application of the rules $\sqcup_{r}, \sqcup_{l}, \sqcap_{r}, \sqcap_{l}, \sqcup_{1-r}, \sqcup_{1-l}, \sqcup_{2-r}, \sqcup_{2-l}$ of SC$_{iALC}$ where the sequents do not have any nominal, neither in $\Theta$ nor in $\Gamma$, is sound regarded intuitionistic propositional logic kripke semantics, to which the validity definition above collapses whenever there is no nominal in the sequents. Thus, in this proof we concentrate in the case where there are nominals. We first observe that the nominal version of $\sqcup_{r}$, the validity of the premises includes

\[
\forall \langle Nom(\Gamma), \delta) \rangle \forall \vec{x} \sqsupseteq Nom(\Gamma, \delta), (\mathcal{I}, \vec{z} \vdash \Gamma \Rightarrow \mathcal{I}, \vec{x} \vdash \delta)
\]

This means that $\Gamma$ holds in any worlds $\vec{x} \sqsupseteq \vec{x}$ for the vector $\vec{x}$ of nominals occurring in $\Gamma$. This includes the outer nominal $x_i$ in $\delta$ (if any). In this case the semantics of $\sqcup$ is preserved, since $\vec{x}$ includes $z_i \geq x_i$. With the sake of a more detailed analysis, we consider the following instance:

\[
x : \alpha_1, y : \alpha_2 \Rightarrow x : \beta
\]
\[
\sqcup_{-r}
\]

Consider an iALC structure $\mathcal{I} = \langle U, \preceq, R^2 \ldots, C^2 \rangle$. In this case, for any $\mathcal{I}$ and any $z_1, z_2 \in U^2$ if $z_1 \succeq x^2$, $z_1 \succeq y^2$, such that, $\mathcal{I}, z_i \models \alpha_1$ and $\mathcal{I}, z_i \models \alpha_2$, we have that $\mathcal{I}, z_i \models x : \beta$, since the premise is valid, by hypothesis. In this case, by the semantics of $\sqcup$ we have $\mathcal{I}, z_i \models x : \alpha_1 \sqsubseteq \beta$. The conclusion of the rule is valid too.

The argument shown above for the $\sqcup_{r}$ rule is analogous for the nominal versions of $\sqcup_{-r}$, $\sqcup_{l}, \sqcap_{r}, \sqcap_{l}, \sqcup_{1-r}, \sqcup_{1-l}, \sqcup_{2-r}, \sqcup_{2-l}$. Consider the rule $\forall_{r}$.
\[
\begin{align*}
\Delta, xRy & \Rightarrow y: \alpha & \forall \cdot \\
\Delta & \Rightarrow x: \forall R.\alpha & \forall \cdot
\end{align*}
\]

Since the premise is valid we have that if \( \forall z_x \geq x^T, \forall z_y \geq y^T, (z_x, z_y) \in R^T \) then \( \forall z_y \geq y^T, \mathcal{I}, z_y \models \gamma \). This entails that \( x^T \in (\forall R.\gamma)^T \), for \( x^T \geq x^T \). We observe that by the restriction on the rule application, \( y \) does not occur in \( \Delta \), it only occurs in \( xRy \) and \( y: \alpha \). The truth of these formulas are subsumed by \( \forall R.\gamma \). The conclusion does not need to consider them any more. The conclusion is valid too. Another way to see its soundness is to prove that if \( xRy \Rightarrow y: \alpha \) is valid, then so is \( \Rightarrow x: \forall R.\alpha \). This can be show by the following reasoning:

\[
\forall x^T \forall y^T \exists z_y(x_x \geq x^T \rightarrow (z_y \geq y^T \rightarrow ((z_x, z_y) \in R^T \rightarrow \mathcal{I}, z_y \models y: \alpha)))
\]

that is the same as:

\[
\forall x^T \forall y^T \exists z_y(x_x \geq x^T \rightarrow (z_y \geq y^T \rightarrow ((z_x, z_y) \in R^T \rightarrow \mathcal{I}, y^T \models \alpha)))
\]

Using the fact that \( \forall y^T(y^T \geq y^T) \), we obtain:

\[
\forall x^T \forall z_x(x_x \geq x^T \rightarrow \forall y^T((x_x, y^T) \in R^T \rightarrow \mathcal{I}, y^T \models \alpha))
\]

The above condition states that \( \Rightarrow x: \forall R.\alpha \) is valid.

\[
\forall x^T \forall y^T \exists z_y(x_x \geq x^T \rightarrow (z_y \geq y^T \rightarrow ((z_x, z_y) \in R^T \rightarrow \mathcal{I}, z_y \models \alpha)))
\]

Consider the rule \( \forall \cdot \):

\[
\begin{align*}
\Delta, x: \forall R.\alpha, y: \delta, xRy & \Rightarrow \delta & \forall \cdot \\
\Delta, x: \forall R.\alpha, xRy & \Rightarrow \delta & \forall \cdot
\end{align*}
\]

As in the \( \forall \cdot \) case, we analyze the simplest validity preservation: if \( x: \forall R.\alpha \land xRy \) is valid, then so is \( x: \forall R.\alpha \land y: \alpha \land xRy \). The first condition is:

\[
\forall x^T \forall y^T \exists z_y(x_x \geq x^T \rightarrow \forall z_y(z_y \geq y^T \rightarrow ((\mathcal{I}, z_y \models x: \forall R.\alpha) \land (\mathcal{I}, z_y \models y: \alpha) \land ((z_x, z_y) \in R^T) \rightarrow (\mathcal{I}, z_y \models y: \alpha) \land (\mathcal{I}, z_x \models \alpha))))
\]

Using \( z_y = y^T \), eliminating \( z_x \) from the term, and, using the fact that \( \mathcal{I}, z_y \models y: \alpha \) is valid, iff, \( \mathcal{I}, y^T \models \alpha \), we obtain

\[
\forall x^T \forall y^T \exists z_y(x_x \geq x^T \rightarrow \forall z_y(z_y \geq y^T \rightarrow ((\mathcal{I}, z_y \models x: \forall R.\alpha) \land (\mathcal{I}, z_y \models y: \alpha) \land ((z_x, z_y) \in R^T) \rightarrow (\mathcal{I}, z_y \models \alpha))))
\]

Consider the semantics of \( \exists R.\alpha \):

\[
(\exists R.\alpha)^T =_{df} \{ x \mid \exists y \in U^T (x, y) \in R^T \land y \in \alpha^T \}
\]

and the following rule:
We can see that the premises of the rule entail the conclusion. The premises correspond to the following conditions:

\[
\forall x^I \forall y^I \forall z(x \geq x^I \rightarrow \forall z_y(z_y \geq y^I \rightarrow ((z_x, z_y) \in R^I)))
\]

and

\[
\forall y^I \forall z_y(z_y \geq y^I \rightarrow ((I, z_y \models y : \alpha))
\]

Instantiating in both conditions \(z_y = y^I\) and \(z_x = x^I\), this yields \((x^I, y^I) \in R^I\), such that \(I, y^I \models \alpha\), so \(I, z_x \models x^I : \exists R. \alpha\). Thus, \(\exists R\) is sound. The soundness of \(\exists\) is analogous to \(\forall\).

Finally, it is worth noting that, for each rule, we can derive the soundness of its non-nominal version from the proof of soundness of its nominal version. For instance, the soundness of the nominal version of rule \(\sqcup\) depends on the diamond conditions F1 and F2. The soundness of its non-nominal version, is a consequence of the soundness of the nominal version.

The rules below have their soundness proved as a consequence of the following reasonings in first-order intuitionistic logic that are used for deriving the semantics of the conclusions from the semantics of the premises:

(p-\(\exists\)) \(\forall x(A(x) \land B(x) \rightarrow C(x)) \models \forall x A(x) \land \exists x B(x) \rightarrow \exists x C(x);\)

(p-\(\forall\)) \(A(x) \models B(x))\) implies \(\forall y(R(y, x) \rightarrow A(x)) \models \forall y(R(y, x) \rightarrow B(x));\)

(p-\(N\)) if \(A \models B\) then for every Kripke model \(I\) and world \(x^I\), if \(I, x^I \models A\) then \(I, x^I \models B.\)

\[
\frac{\Delta, \alpha \models \beta}{\forall R. \Delta, \exists R. \alpha \models \exists R. \beta} \quad \text{p-} \exists \quad \frac{\Delta \models \alpha}{\forall R. \Delta \models \forall R. \alpha} \quad \text{p-} \forall \quad \frac{\Delta \models \delta}{x: \Delta \models x: \delta} \quad \text{p-} N
\]

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