Manoel Jarra

Exterior algebras in matroid theory

Received: 18 March 2022 / Accepted: 15 August 2022 / Published online: 5 September 2022

Abstract. Ordered blueprints are algebraic objects that generalize monoids and ordered semirings, and $\mathbb{F}_1^\pm$-algebras are ordered blueprints that have an element $\epsilon$ that acts as $-1$. In this work we introduce an analogue of the exterior algebra for $\mathbb{F}_1^\pm$-algebras that provides a new cryptomorphism for matroids. We also show how to recover the usual exterior algebra if the $\mathbb{F}_1^\pm$-algebra comes from a ring, and the Giansiracusa Grassmann algebra if the $\mathbb{F}_1^\pm$-algebra comes from an idempotent semifield.

Contents

1. Introduction .................................... 427
   1.1. Description of results ...................... 428
   1.2. Remark ................................... 429
2. Algebraic background ............................. 429
   2.1. Monoids and $M$-sets ..................... 429
   2.2. Semirings and modules ...................... 431
   2.3. Ordered blueprints ...................... 432
   2.4. Ordered blue modules ..................... 433
   2.5. Tensor product of $B$-modules .......... 434
3. Exterior algebra .................................. 435
   3.1. Notations ................................ 435
   3.2. Construction .............................. 435
4. Matroids ...................................... 440
References ....................................... 442

1. Introduction

It is a classical theme that $d$-dimensional linear subspaces of the vector space $K^n$ over a field $K$ correspond to certain elements of the exterior algebra $\Lambda K^\times$, which are well-defined up to scalar multiples in $K^\times$.

M. Jarra (✉): University of Groningen, Groningen, The Netherlands
IMPA, Rio de Janeiro, Brazil
e-mail: m.zanoelo.jarra@rug.nl

Mathematics Subject Classification: 15A75 · 05E99

https://doi.org/10.1007/s00229-022-01422-x
The combinatorial counterpart of such linear subspaces are matroids. Baker and Bowler streamline in [1] this analogy in a broad sense by the theory of matroids with coefficients in so-called tracts. Fields, semifields and more generally hyperfields, can be seen as examples of tracts.

Jeffrey and Noah Giansiracusa introduce in [3] an exterior algebra for idempotent semifields $S$ and exhibit a ‘cryptomorphic’ description of $S$-matroids in terms of the exterior algebra, in a formal analogy to the description of $K$-matroids, or linear subspaces of $K^n$, in the case of a field $K$.

Somewhat puzzling, however, is that Giansiracusa’s definition of the exterior algebra for idempotent semifields makes explicit use of the idempotency in the sense that for a free module with basis $\{e_1, \ldots, e_n\}$, one has that $e_i \otimes e_i = 0$, for $i = 1, \ldots, n$, are the only defining relations, in contrast to the larger set of relations for fields.

In this paper, we give a unified approach to both the classical theory over fields and Giansiracusa’s theory for idempotent semifields, which is based on Lorscheid’s theory of ordered blueprints (cf. [4]). Both fields and idempotent semifields can be realized as ordered blueprints in terms of faithful functors:

$(-)^{\text{mon}} : \text{Rings} \rightarrow \text{OBlpr}$ and $(-)^{\text{mon}} : \text{IdempSFields} \rightarrow \text{OBlpr}$

which have respective left inverses

$(-)^{\text{hull}} : \text{OBlpr} \rightarrow \text{Rings}$ and $(-)^{\text{idem}} : \text{OBlpr} \rightarrow \text{IdempSFields}$.

We define an exterior algebra for ordered blueprints and show that it recovers both classical exterior algebra over rings as well as Giansiracusa’s exterior algebra over idempotent semifields in terms of these functors. Moreover, we give a cryptomorphic description of matroids over $\mathbb{F}_1^\pm$-algebras, as introduced by Baker and Lorscheid in [2], by using elements of the exterior algebra, which recovers the classical viewpoint on linear subspaces of $K^n$ and Giansiracusa’s interpretation of matroids over idempotent semifields.

1.1. Description of results

Let $B$ be an $\mathbb{F}_1^\pm$-algebra and $n$ be an integer. The exterior algebra $\Lambda B^n = \bigoplus_{i \geq 0} \Lambda^i B^n$ of $B^n$ is a $B$-module whose underlying semigroup is a (typically non-commutative) $B^+$-algebra. This exterior algebra bears properties analogous to the classical exterior algebra.

**Theorem A.** There are $B$-linear isomorphisms of ordered blue $B$-modules

\[
B \xrightarrow{\sim} \Lambda^0 B^n, \quad B^n \xrightarrow{\sim} \Lambda^1 B^n, \quad 0 \xrightarrow{\sim} \Lambda^k B^n \text{ for } k \geq n + 1,
\]

and $(\Lambda B^n)^+$ is generated by $(\Lambda^1 B^n)^+$ as a $B^+$-algebra.

This unifies and generalizes the classical and Giansircusa’s exterior algebra in the following sense:
Theorem B.

(i) Let $R$ be a ring and $B = R_{\text{mon}}$. Then $\Lambda R^n$ is canonically isomorphic to $(\Lambda B^n)^{\text{hull,}+}$ as an $R$-algebra.

(ii) Let $S$ be an idempotent semiring and $B = S_{\text{mon}}$. Then the Giansiracusa exterior algebra $\Lambda S^n$ is canonically isomorphic to $(\Lambda B^n)^{\text{idem,}+}$ as an $S$-algebra.

Let $[n] = \{1, \ldots, n\}$ and $\Gamma = \binom{[n]}{d}$ be the family of $d$-subsets of $[n]$. We define a $B$-matroid of rank $d$ on $[n]$ as the $B \times$-class $[v]$ of an element $v = (v_I)_{I \in \Gamma}$ of $\Lambda^d B^\Gamma$ that satisfies a certain system of relations (see 4.1) and such that $v_I \in B^\times$ for some $I \in \Gamma$. This recovers the aforementioned concepts of matroids in the following sense:

Theorem C. Consider $0 \leq d \leq n$.

(i) Let $K$ be a field and $B = K_{\text{mon}}$. Then the isomorphism $\Lambda K^{\binom{n}{d}} \simeq (\Lambda B^{\binom{[n]}{d}})^{\text{hull,}+}$ induces a bijection between $K$-matroids (as in [1]) and $B$-matroids.

(ii) Let $S$ be an idempotent semifield and $B = S_{\text{mon}}$. Then the isomorphism $\Lambda S^{\binom{n}{d}} \simeq (\Lambda B^{\binom{[n]}{d}})^{\text{idem,}+}$ induces a bijection between tropical Plücker vectors (as in [3]) and Grassmann-Plücker functions with coefficients in $B$ in the sense of this text.

(iii) Let $B$ be an ordered blueprint. Then there is a canonical bijection between $B$-matroids in the sense of [2] and classes of $B$-Plücker vectors (as defined in 4.3).

1.2. Remark

We draw the reader’s attention to the fact that the functors $(-)_{\text{mon}}$, $(-)^{\text{hull,}+}$ and $(-)^{\text{idem,}+}$ play the same role as in Lorscheid’s approach to tropicalization as a base change from a field to the tropical hyperfield in [5]. This indicates that our theory is part of a larger picture that puts classical theory and idempotent analysis on an equal footing.

2. Algebraic background

In this section, we review some background around $\mathbb{F}_1^\pm$-algebras, following [4].

If $\tau$ is a preorder on a set $X$, viewed as a subset of $X \times X$, we will use $x \leq_{\tau} y$ to denote that $(x, y)$ is in $\tau$ and $a \equiv_{\tau} b$ to denote that both $(a, b)$ and $(b, a)$ are in $\tau$. Note that $\equiv_{\tau}$ is an equivalence relation. If the context is clear, we will denote $\equiv_{\tau}$ simply by $\equiv$.

2.1. Monoids and $M$-sets

A monoid is a unital semigroup. A monoid $M$ is called pointed if it has an absorbing element, i.e., an element $0$ such that $0.m = 0 = m.0$ for all $m$ in $M$. The neutral
and the absorbing element (if they exist) are always unique. For the rest of this text, unless otherwise stated, every monoid is supposed to be commutative.

A submonoid of a monoid $M$ is monoid $N$ that is a subset of $M$, contains $1_M$ and whose operation $\cdot_N$ is the restriction of $\cdot_M$. If $M$ is pointed and its absorbing element is in $N$, we say that $N$ is a pointed submonoid of $M$.

If $M$ and $W$ are monoids, a map $f : M \rightarrow W$ is called a morphism of monoids if $f(1_M) = 1_W$ and $f(x) \cdot f(y) = f(x \cdot y)$ for all $x, y$ in $M$. If $M$ and $W$ are pointed and $f$ carries the absorbing element of $M$ to the absorbing element of $W$, we say that $f$ is a morphism of pointed monoids. The category of pointed monoids will be denoted by $\text{Mon}_*$.

A preorder $\tau$ on $M$ is called multiplicative (or additive, depending on the operation of $M$) if, for all elements $m, n$ and $x$ in $M$, one has $mx \leq_{\tau} nx$ whenever $m \leq_{\tau} n$. A congruence is a multiplicative preorder that is symmetric (thus an equivalence relation). If $\tau$ is a multiplicative preorder on $M$, the set $\mathcal{C}_\tau := \{(m, n) \in M \times M | m \equiv \tau n\}$ is a congruence. The quotient set $M/\mathcal{C}_\tau$ is a monoid, with operation $[m][n] = [mn]$, and has an induced multiplicative partial order $\mathcal{\nu} := \{([a], [b]) | a \leq_{\tau} b\}$.

If $M$ is a monoid, an $M$-set is a set $X$ equipped with a map

$$
M \times X \rightarrow X
$$

$$(m, x) \mapsto m \cdot x
$$

that satisfies:

$$(i) \ (m \cdot n) \cdot x = m \cdot (n \cdot x) \quad \text{and} \quad (ii) \ 1 \cdot x = x,
$$

for all $m, n$ in $M$ and $x$ in $X$. An $M$-subset of $X$ is an $M$-set $Z$ such that $Z$ is a subset of $X$ and whose map $M \times Z \rightarrow Z$ is the restriction of the map $M \times X \rightarrow X$.

If $M$ is a pointed monoid with absorbing element 0, a pointed $M$-set is a pointed set $(X, p)$ equipped with a map $M \times X \rightarrow X$ that makes $X$ an $M$-set and satisfies:

$$(iii) \ 0 \cdot x = p \quad \text{and} \quad (iv) \ m \cdot p = p,
$$

for all $m$ in $M$ and $x$ in $X$. A pointed $M$-subset of $(X, p)$ is a pointed $M$-set $(Z, q)$ such that $Z$ is an $M$-subset of $X$ and $p = q$.

If $\{(X_i, p_i) | i \in I\}$ is a family of pointed $M$-sets, its coproduct is given by

$$
\bigvee_{i \in I} X_i := \left( \bigsqcup_{i \in I} X_i \right) / \sim,
$$

where $\sim := \{(p_i, p_j) | i, j \in I\}$, equipped with the $M$-action

$$
M \times \bigvee_{i \in I} X_i \rightarrow \bigvee_{i \in I} X_i
$$

$$(m, [x_j]) \mapsto [m \cdot x_j]
$$

where $[y_j]$ denotes the class, in $\bigvee_{i \in I} X_i$, of $y_j \in X_j$.

If $X_1, X_2$ and $W$ are $M$-sets, a map $\varphi : X_1 \times X_2 \rightarrow W$ is called $M$-bilinear if

$$
\varphi(-, x_2) : X_1 \rightarrow W \quad \text{and} \quad \varphi(x_1, -) : X_2 \rightarrow W
$$

are morphims of $M$-modules for all $x_1$ in $X_1$ and $x_2$ in $X_2$. We denote the set of $M$-bilinear maps $X_1 \times X_2 \rightarrow W$ by $\text{Bil}_M(X_1 \times X_2, W)$. 


We construct the tensor product $X_1 \otimes_M X_2$ of $X_1$ and $X_2$ as the quotient of $X_1 \times X_2$ by the equivalence relation generated by $\{(m.x_1, x_2) \sim (x_1, m.x_2) | x_i \in X_i$ and $m \in M\}$, and denote the class of $(x_1, x_2)$ by $x_1 \otimes x_2$.

The map

$$M \times (X_1 \otimes_M X_2) \longrightarrow X_1 \otimes_M X_2$$

$$(m, x_1 \otimes x_2) \longmapsto (m.x_1) \otimes x_2 = x_1 \otimes (m.x_2)$$

turns $X_1 \otimes_M X_2$ into an $M$-set.

If $\varphi : X_1 \times X_2 \to W$ is an $M$-bilinear map, then

$$\overline{\varphi} : X_1 \otimes_M X_2 \longrightarrow W$$

$$x_1 \otimes x_2 \longmapsto \varphi(x_1, x_2)$$

defines a morphism of $M$-sets.

The universal property of $X_1 \otimes_M X_2$ can be expressed as the fact that $\varphi \mapsto \overline{\varphi}$ is a bijection from $\text{Bil}_M(X_1 \times X_2, W)$ to $\text{Hom}_M(X_1 \otimes_M X_2, W)$.

### 2.2. Semirings and modules

A **semiring** is a triple $(S, +, \cdot)$, where $(S, +)$ is a commutative monoid with identity 0, $(S, \cdot)$ is a (non-necessarily commutative) pointed monoid with identity 1 and absorbing element 0, and satisfying $(a + b).c = (a.c) + (b.c)$ and $c.(a + b) = (c.a) + (c.b)$ for all $a, b$ and $c$ in $S$. We also use $ab$ to denote $a \cdot b$. A semiring is called **commutative** if $(S, \cdot)$ is commutative. If the operations are clear, we use $S$ to denote the semiring $(S, +, \cdot)$. For the rest of this text, unless otherwise stated, every semiring is supposed to be commutative.

If $S_1$ and $S_2$ are semirings, a map $f : S_1 \to S_2$ is called a **morphism of semirings** if it is, at the same time, a morphism of the underlying monoids $f : (S_1, +) \to (S_2, +)$ and $f : (S_1, \cdot) \to (S_2, \cdot)$. The category of semirings will be denoted by $\text{SRings}$.

A preorder $\tau$ on a semiring $S$ is called **additive and multiplicative** if $a + z \leq_{\tau} b + z$ and $az \leq_{\tau} bz$ whenever $(a, b)$ is in $\tau$ and $z$ is in $S$.

An **ordered semiring** is a pair $(S, \leq)$, where $S$ is a semiring and $\leq$ is an additive and multiplicative partial order on $S$. A **morphism of ordered semirings** is a morphism of semirings that is order-preserving.

A **congruence** on a semiring $S$ is an additive and multiplicative preorder that is symmetric (thus an equivalence relation). If $\tau$ is a congruence on a semiring $S$, the quotient set $S/\tau$ is naturally a semiring, with operations defined by $[a] + [b] := [a + b]$ and $[a].[b] := [ab]$.

If $\tau$ is an additive and multiplicative preorder on a semiring $S$, the set $\tau := \{(a, b) \in S \times S | a \equiv_{\tau} b\}$ is a congruence and the quotient semiring $S/\tau$ has an induced additive and multiplicative partial order $\equiv := \{([a], [b]) | a \leq_{\tau} b\}$.

An **$S$-module** is a pointed monoid $(Y, +)$ with identity 0 equipped with a map

$$\lambda : S \times Y \longrightarrow Y$$

$$(s, y) \longmapsto s.y$$
that makes \( Y \) an \((S, \cdot)\)-set and satisfies
\[
(s + r) \cdot y = (s \cdot y) + (r \cdot y) \quad \text{and} \quad s \cdot (y + z) = (s \cdot y) + (s \cdot z)
\]
for all \( s, r \) in \( S \) and \( y, z \) in \( Y \). A map \( f : Y_1 \to Y_2 \) is called a morphism of \( S \)-modules if \( f(y + z) = f(y) + f(z) \) and \( f(s \cdot y) = s \cdot f(y) \), for all \( y, z \) in \( Y_1 \) and \( s \) in \( S \).

If \( Y_1, Y_2 \) and \( Z \) are \( S \)-modules, a map \( \varphi : Y_1 \times Y_2 \to Z \) is called \( S \)-bilinear if
\[
\varphi(-, y_2) : Y_1 \to Z \quad \text{and} \quad \varphi(y_1, -) : Y_2 \to Z
\]
are morphisms of \( S \)-modules for all \( y_1 \) in \( Y_1 \) and \( y_2 \) in \( Y_2 \). We denote the set of \( S \)-bilinear maps \( Y_1 \times Y_2 \to Z \) by \( \text{Bil}_S(Y_1 \times Y_2, Z) \).

We construct the tensor product \( Y_1 \otimes_S Y_2 \) as the quotient of \( S^{Y_1 \times Y_2} \) by the congruence generated by \( \{ e(s, y_1, y_2) \equiv s \cdot e(y_1, y_2) \mid s \in S \text{ and } y_i \in Y_i \} \) (where \( e(h_1, h_2) \) is the element of \( S^{Y_1 \times Y_2} \) that has 1 in the entry corresponding to \( (h_1, h_2) \) and 0 on the others). We denote the class of \( e(s, y_1, y_2) \) by \( y_1 \otimes y_2 \).

If \( \varphi : Y_1 \times Y_2 \to Z \) is an \( S \)-bilinear map, one has a morphism of \( S \)-modules characterized by
\[
\overline{\varphi} : Y_1 \otimes_S Y_2 \to Z \\
y_1 \otimes y_2 \mapsto \varphi(y_1, y_2).
\]
The association \( \varphi \mapsto \overline{\varphi} \) is a bijection from \( \text{Bil}_S(Y_1 \times Y_2, Z) \) to \( \text{Hom}_S(Y_1 \otimes S Y_2, Z) \).

For a number \( n \geq 1 \) and an \( S \)-module \( Y \), we will denote the \( n \)-fold tensor product \( Y \otimes \ldots \otimes Y \) by \( Y \otimes^n \), and define \( Y \otimes^0 \) as \( S \).

For the \( S \)-module \( TY := \bigoplus_{l=0}^{\infty} (Y)^{\otimes l} \), the map
\[
(y_1 \otimes \ldots \otimes y_i, z_1 \otimes \ldots \otimes z_j) \mapsto y_1 \otimes \ldots \otimes y_i \otimes z_1 \otimes \ldots \otimes z_j
\]
extends, by linearity, to a product of \( TY \) that makes it a (typically non-commutative) \( S \)-algebra.

### 2.3. Ordered blueprints

An ordered blueprint is a triple \((B^*, B^+, \leq)\) such that:

(i) \((B^+, \leq)\) is an ordered semiring;

(ii) \(B^*\) is a pointed submonoid of \((B^+, \cdot)\);

(iii) \(B^*\) generates \(B^+\) as a semiring, i.e., every element of \(B^+\) is a finite sum of elements in \(B^*\).

We call \(B^*\) the underlying monoid and \(B^+\) the ambient semiring of \(B\), and use \(B^\times\) to denote the set of invertible elements of \(B^*\). A morphism of ordered blueprints \(\varphi : B \to C\) is an order-preserving morphism \(\varphi : B^+ \to C^+\) of semirings that satisfies \(\varphi(B^*) \subseteq C^*\).

If \(B\) is an ordered blueprint and \(r\) is an additive and multiplicative preorder on \(B^+\) containing \(\leq\), one has the quotient ordered blueprint \(B//r := [B^*, B^+/c_r, \bar{r}]\),
where $c_\tau$ and $\bar{\tau}$ are the congruence and the partial order induced by $\tau$, respectively, and $B^\bullet$ is the multiplicative submonoid of $B^+ / c_\tau$ whose elements are the classes of elements in $B^\bullet$.

For a subset $H$ of $B^+ \times B^+$, we define the preorder generated by $H$ as

$$\langle H \rangle := \bigcap \{\text{additive and multiplicative preorders on } B^+ \text{ containing } H \text{ and } \leq\}. $$

If $H = \{(a_i, b_i) | i \in I\}$, we write $\langle a_i \leq b_i | i \in I\rangle$ to denote $\langle H \rangle$, and $\langle a_i \equiv b_i | i \in I\rangle$ to denote $\langle a_i \leq b_i \text{ and } b_i \leq a_i | i \in I\rangle$.

For an ordered blueprint $B$, let $\bar{\tau} := \langle a \equiv b | a \leq b\rangle$ and define the algebraic hull of $B$ as the ordered blueprint $B^{\text{hull}} := B / \bar{\tau}$. Note that the partial order of $B^{\text{hull}}$ is trivial.

If $(D, \cdot)$ is a pointed monoid with absorbing element $0_D$, one has the semiring-algebra $\tilde{D} := \mathbb{N}[D] / (0 \equiv 1.0_D)$, whose elements can be seen as formal finite sums of non-zero elements of $D$, with operations

$$\sum n_d.d + \sum m_d.d = \sum (n_d + m_d).d,$$

$$\left( \sum n_d.d \right) \left( \sum m_d.d \right) = \sum_{d \in D \setminus \{0_D\}} \left( \sum_{a,b=d} n_a m_b \right) d.$$ 

If $H \subseteq \tilde{D} \times \tilde{D}$, we use $D / \langle H \rangle$ to denote the ordered blueprint $(D, \tilde{D}, \cdot) / \langle H \rangle$.

There are two canonical functors $(-)^\bullet$ and $(-)^+$ that sends an ordered blueprint to its underlying monoid and ambient semiring, respectively.

2.4. Ordered blue modules

Let $B = (B^\bullet, B^+, \leq_B)$ be an ordered blueprint. An ordered blue $B$-module, or simply $B$-module, is a triple $M = (M^\bullet, M^+, \leq_M)$ such that:

(i) $M^+$ is a $B^+$-module;

(ii) $\leq_M$ is an additive partial order on $M^+$ such that $(b_1.m_1) \leq_M (b_2.m_2)$ whenever $b_1 \leq_B b_2$ and $m_1 \leq_M m_2$;

(iii) $M^\bullet$ is a pointed $B^\bullet$-subset of $M^+$, where the map $B^\bullet \times M^+ \rightarrow M^+$ is the restriction of $B^+ \times M^+ \rightarrow M^+$;

(iv) $M^\bullet$ generates $M^+$ as a semigroup, i.e., every element of $M^+$ is a finite sum of elements in $M^\bullet$.

We call $M^\bullet$ the underlying $B^\bullet$-set and $M^+$ the ambient $B^+$-module of $M$. A morphism of $B$-modules $f : M \rightarrow N$ is an order-preserving morphism $f : M^+ \rightarrow N^+$ of $B^+$-modules such that $f(M^\bullet) \subseteq N^\bullet$. The category of $B$-modules will be denoted by $B\text{-Mod}$.

An additive $B$-preorder on $M$ is an additive preorder $\tau$ on the monoid $M^+$ that contains $\leq_M$ and satisfies $b_1.m_1 \leq_\tau b_2.m_2$ whenever $b_1 \leq_B b_2$ and $m_1 \leq_M m_2$. A congruence is a $B$-preorder that is symmetric. If $\tau$ is a $B$-preorder on $M$, the set $c_\tau := \{(m, n) | m \equiv_\tau n\}$ is a congruence. In this case, we have the quotient $B$-module $M / \tau := (\bar{M}^\bullet, M^+/c_\tau, \bar{\tau})$, where $M^+/c_\tau$ is the quotient $B^+$-module,
\[ M^* := \{ [m] | m \in M^* \} \] and \( \bar{\tau} \) is the partial order induced by \( \tau \). For a subset \( L \) of \( M^+ \times M^+ \), we define the preorder generated by \( L \) as
\[
(L) := \bigcap \{ B\text{-preorders on } M^+ \text{ containing } L \text{ and } \leq_M \}.
\]

If \( M \) is a \( B \)-module, let \( g := \langle (x, y) \in M^+ \times M^+ | x \leq_M y \text{ or } y \leq_M x \rangle \). We define the \textit{algebraic hull} of \( M \) as the \( B \)-module \( M^{\text{hull}} := M/g \). Note that the partial order of \( M^{\text{hull}} \) is trivial.

There are two natural functors \((-)^*\) and \((-)^+\) that send an \( B \)-module to its underlying \( B^* \)-set and ambient \( B^+ \)-module, respectively.

The coproduct of a family \( \{ M_i = (M_i^*, M_i^+, \leq_i) | i \in I \} \) of \( B \)-modules is given by \( M = (M^*, M^+, \leq_M) \), where \( M^+ = \bigoplus_{i \in I} M_i^+ \), \( \leq_M = \{(m_i)_{i \in I} \leq (n_i)_{i \in I} \} \) for all \( n_j, \forall j \in I \) and \( M^* \) is the image of the natural map of \( B^* \)-sets
\[ \Gamma : \bigvee_{i \in I} M_i^* \longrightarrow \bigoplus_{i \in I} M_i^+. \]

2.5. Tensor product of \( B \)-modules

A \( B \)-bilinear map \( \varphi : M_1 \times M_2 \rightarrow N \) is a \( B^+ \)-bilinear map \( \varphi : M_1^+ \times M_2^+ \rightarrow N^+ \) such that
\[ \varphi(-, m_2) : M_1 \rightarrow N \quad \text{and} \quad \varphi(m_1, -) : M_2 \rightarrow N \]
are morphims of \( B \)-modules, for all \( m_1 \) in \( M_1^* \) and \( m_2 \) in \( M_2^* \).

One has the natural map of \( B^* \)-sets
\[ \psi : M_1^* \otimes_{B^*} M_2^* \longrightarrow M_1^+ \otimes_{B^+} M_2^+ \]
\[ m_1 \otimes m_2 \longmapsto m_1 \otimes m_2. \]

The \textit{tensor product} of \( M_1 \) and \( M_2 \) is defined as the \( B \)-module
\[ M_1 \otimes_B M_2 := (\text{im } \psi, M_1^+ \otimes_{B^+} M_2^+, =)/\bar{\tau}, \]
where \( \bar{\tau} := \langle (x_1 \otimes y_1 \leq x_2 \otimes y_2) \rangle \) for every relation \( x_1 \otimes y_1 \leq x_2 \otimes y_2 \) in \( M_1^+ \) and \( y_1 \leq y_2 \in M_2^+ \). For \( \alpha \in M_1^+ \otimes_{B^+} M_2^+ \), we denote its class in \( (M_1 \otimes_B M_2)^+ \) again by \( \alpha \).

Let \( \varphi : M_1 \times M_2 \rightarrow N \) be a \( B \)-bilinear map. As \( \varphi \) is \( B^+ \)-bilinear, there exists a morphism
\[ \tilde{\varphi} : M_1^+ \otimes_{B^+} M_2^+ \longrightarrow N^+ \]
\[ m_1 \otimes m_2 \longmapsto \varphi(m_1, m_2) \]
of \( B^+ \)-modules.

By the definition of \( B \)-bilinearity, \( \tilde{\varphi} \) is contained in \( N^* \) and \( \tilde{\varphi}(x_1 \otimes y_1) \leq \tilde{\varphi}(x_2 \otimes y_2) \) for every relation \( x_1 \otimes y_1 \leq x_2 \otimes y_2 \) in \( M_1^+ \otimes_{B^+} M_2^+ \). Thus there exists a morphism
\[ \bar{\varphi} : M_1 \otimes_B M_2 \longrightarrow N \]
\[ m_1 \otimes m_2 \longmapsto \varphi(m_1, m_2) \]
of \( B \)-modules.
Proposition 2.1. The map

\[\Phi : \text{Bil}_B(M_1 \times M_2, N) \rightarrow \text{Hom}_B(M_1 \otimes M_2, N)\]

\[\varphi \mapsto \Phi(\varphi)\]

is a bijection.

Proof. Let \(\varphi\) and \(\theta\) two \(B\)-bilinear maps such that \(\varphi = \theta\). Thus, in particular, \(\varphi(m_1, m_2) = \varphi(m_1 \otimes m_1) = \varphi(m_1 \otimes m_1) = \theta(m_1, m_2)\), for all \(m_1 \in M_1\) and \(m_2 \in M_2\). With this, we obtain the injectivity of \(\Phi\).

Let \(\zeta : M_1 \otimes_B M_2 \rightarrow N\) be a morphism of \(B\)-modules. Define:

\[\tilde{\zeta} : M_1^+ \times M_2^+ \rightarrow N^+\]

\[(m_1, m_2) \mapsto \zeta(m_1 \otimes m_2)\]

By the construction of the tensor product and from the fact that \(\zeta\) is a morphism, one has that \(\tilde{\zeta}\) is \(B\)-bilinear. As \(\Phi(\tilde{\zeta}) = \zeta\), one has that \(\Phi\) is surjective. \(\square\)

3. Exterior algebra

3.1. Notations

For a blueprint \(B\), a \(B\)-module \(M\), a natural number \(n\) and a set \(I\), we will use the following notations: \([e_i]_{i \in [n]} \subseteq (B^*)^n\) for the canonical basis; \(M^{\otimes n}\) for the \(n\)-fold tensor product \(M \otimes \ldots \otimes M\); and \(M^n\) for the product

\[\prod_{i=1}^{n} M = \left(\prod_{i=1}^{n} M^*, \prod_{i=1}^{n} M^+, \leq_{M^n}\right),\]

where \(\leq_{M^n}\) is defined by \((\alpha_1, \ldots, \alpha_n) \leq_{M^n} (\beta_1, \ldots, \beta_n)\) if \(\alpha_j \leq \beta_j\) for all \(j\) in \([n]\).

Definition 3.1. Let \(\mathbb{F}_1^\pm := ([0, 1], \mathbb{N} \oplus \mathbb{N}, \epsilon, \langle 0 \leq 1 + \epsilon \rangle), \) where \(\epsilon^2 = 1\).

For the rest of this text, fix an \(\mathbb{F}_1^\pm\)-algebra \(B = (B^*, B^+, \leq),\) i.e., an ordered blueprint \(B\) equipped with a morphism \(\mathbb{F}_1^\pm \rightarrow B\) or, equivalently, with a distinguished element \(\epsilon\) in \(B^*\) such that \(\epsilon^2 = 1\) and \(0 \leq 1 + \epsilon\).

3.2. Construction

For a \(B\)-module \(M\), let \(TM := \bigoplus_{l=0}^{\infty} M^{\otimes l} = (\text{im } \Theta, TM^+, \leq),\) where \(TM^+ := \bigoplus_{l=0}^{\infty} (M^+)^{\otimes l}\) is the \(B^+\)-tensor algebra and \(\Theta : \bigvee_{l=0}^{\infty} (M^*)^{\otimes l} \rightarrow TM^+\) is the natural map of \(B^*\)-sets.

Note that the usual product of \(TM^+\) restricts to a product for \(\text{im } \Theta\). This operation turns \(TM\) into a (typically non-commutative) \(B\)-algebra.
For $n \in \mathbb{N}$, let
\[\tau_n := \{e_i \otimes e_i = 0 \mid i \in [n]\} \cup \{e_i \otimes e_j = e_j \otimes e_i \mid i, j \in [n], i \neq j\} \subset (B^n) \otimes \mathbb{C}.\]
If $d \leq n$ and $I = \{i_1, \ldots, i_d\} \subset \binom{[n]}{d}$ with $i_1 < \ldots < i_d$, define
\[e_I := e_{i_1} \otimes \ldots \otimes e_{i_d} \in T(B^n) \langle \tau_n \rangle.\]
Let
\[\gamma_{d,n} : B^+ \binom{[n]}{d} \to (T(B^n) \langle \tau_n \rangle)^+. \]
and define $H_{d,n} := \gamma_{d,n} \left(B^* \left(\binom{[n]}{d}\right)\right)$ and $K_{d,n} := \gamma_{d,n} \left(B^* \left(\binom{[n]}{d}\right)\right).$

**Lemma 3.2.** The set $E := \{e_I \mid I \subset [n]\}$ is a $B^+$-basis of $(T(B^n) \langle \tau_n \rangle)^+$.  

**Proof.** We begin by noticing that $E$ clearly is a generator set. Let $b_I, c_I$ in $B^+$, $I$ subset of $[n]$, such that $\sum_{I \subset [n]} b_I e_I = \sum_{I \subset [n]} c_I e_I$. Thus
\[b := \sum_{i_1 < \ldots < i_d} b_{|i_1,\ldots,i_d|} e_{i_1} \otimes \ldots \otimes e_{i_d} \equiv \sum_{i_1 < \ldots < i_d} c_{|i_1,\ldots,i_d|} e_{i_1} \otimes \ldots \otimes e_{i_d} := c. \tag{1}\]

By the definition of $\{\tau_n\}$, there exists two sequences $b = x_0, \ldots, x_m$ and $c = y_0, \ldots, y_w = x_m$ of elements of $T(B^n)^+$ such that $x_{\ell+1} = x_\ell + \alpha$ (resp. $y_{\ell+1} = y_\ell + \alpha$), where $\alpha$ has the form $b_e a \otimes e_a \otimes e_{z_1} \otimes \ldots \otimes e_{z_f}$ for some $b$ in $B^+$ and $a, z_1, \ldots, z_f$ in $[n]$; or $x_{\ell+1} = \rho + \beta_1$ and $x_\ell = \rho + \beta_2$ (resp. $y_{\ell+1} = \rho = \beta_1$ and $y_\ell = \rho + \beta_2$), where $\beta_1$ has the form $b_e e_a \otimes \ldots \otimes e_{i_f} b_2 = \text{sign}(\sigma) b e_{\sigma(i_1)} \otimes \ldots \otimes e_{\sigma(f)}$, for some $b$ in $B^+$, $\rho$ in $T(B^n)^+$, $i_1, \ldots, i_f$ in $[n]$, a permutation $\sigma$ and interpreting $\text{sign}(\sigma)$ as an element of $\mathbb{P}_1^+$ via the identification of $-1$ with $\epsilon$. This implies that there are $A_1, A_2$ in $T(B^n)^+$, both of the form $\sum_{j_1, \ldots, j_f} d_{j_1, \ldots, j_f} e_{j_1} \otimes \ldots \otimes e_{j_f}$, such that for each set $\{j_1, \ldots, j_f\}$ one has (at least) two indexes $u, v$ in $[f]$ satisfying $j_u = j_v$; and there are, for each index sets $\{i_1, \ldots, i_d\}$ present in (1), two permutations $\sigma_{|i_1,\ldots,i_d|}$ and $\delta_{|i_1,\ldots,i_d|}$ satisfying
\[\sum_{i_1 < \ldots < i_d} \text{sign}(\sigma_{|i_1,\ldots,i_d|}) b_{|i_1,\ldots,i_d|} e_{i_{\sigma_{|i_1,\ldots,i_d|}(1)}} \otimes \ldots \otimes e_{i_{\sigma_{|i_1,\ldots,i_d|}(d)}} + A_1 \tag{2}\]
\[= \sum_{i_1 < \ldots < i_d} \text{sign}(\delta_{|i_1,\ldots,i_d|}) c_{|i_1,\ldots,i_d|} e_{i_{\delta_{|i_1,\ldots,i_d|}(1)}} \otimes \ldots \otimes e_{i_{\delta_{|i_1,\ldots,i_d|}(d)}} + A_2.\]

Thus, looking at the coefficient of $e_{i_1} \otimes \ldots \otimes e_{i_d}$ in (1) and looking at the coefficient of $e_{i_{\sigma_{|i_1,\ldots,i_d|}(1)}} \otimes \ldots \otimes e_{i_{\sigma_{|i_1,\ldots,i_d|}(d)}}$ in (2), we conclude that $b_{|i_1,\ldots,i_d|} = c_{|i_1,\ldots,i_d|}$, for each index set $\{i_1, \ldots, i_d\}$ present in (1). \qed
Let \( S_{d,n} \) be the sub-\( B^+ \)-module of \((T(B^n) \oplus \tau_n)\)\(^+\) generated by \( H_{d,n} \). The \( B \)-module \( \bigwedge^d B^n := (H_{d,n}, S_{d,n}, \leq) \), where \( \leq \) is induced from \((T(B^n) \oplus \tau_n), \), is called the \( d \)th exterior power of \( B^n \).

Let \( H_n := \bigcup_{l=0}^n H_{l,n} \) and note that it generates \((T(B^n) \oplus \tau_n)\)\(^+\) as a semigroup. The \( B \)-module \( \bigwedge B^n := (H_n, (T(B^n) \oplus \tau_n)\)\(^+\), \( \leq) \) is called the exterior algebra of \( B^n \).

The operation \((\vec{a}, \vec{b}) \mapsto \vec{a}\vec{b}\) defines a product on \((\bigwedge B^n)\)\(^+\), making it a (typically non-commutative) \( B^+ \)-algebra. For \( x, y \in (\bigwedge B^n)\)\(^+\), we denote by \( x \wedge y \) the product of \( x \) and \( y \). For \( I = \{i_1, \ldots, i_d\} \) with \( i_1 < \ldots < i_d \), we use \( e_I \) to denote the element \( e_{i_1} \wedge \cdots \wedge e_{i_d} \).

**Remark 3.3.** Note that Lemma 3.2, in particular, proves Theorem A.

**Remark 3.4.** For \( n \geq 2 \), the exterior algebra \( \bigwedge B^n \) may not be a \( B \)-algebra because its underlying pointed set \((\bigwedge B^n)\)\(^*\) could not be multiplicatively closed, as we always have \( e_1 + e_2 \) in \((\bigwedge B^n)\)\(^*\) but, if \( 1 + \epsilon \) is not in \( B^* \), \( (e_1 + e_2) \wedge (e_1 + e_2) = (1 + \epsilon) \cdot e_{\{1,2\}} \) is not in \((\bigwedge B^n)\)\(^*\).

**Remark 3.5.** The difference between the exterior algebra \( \bigwedge B^n \) and the \( B \)-algebra \( T(B^n) \oplus \tau_n \) concerns only the underlying pointed \( B \)-set. This occurs because we need sums of elements of \((T(B^n) \oplus \tau_n)\)\(^\bullet\) to define \( B \)-Plücker vectors (cf. 4.3).

But \( T(B^n) \oplus \tau_n \) has the following universal property (similar to the universal property of the ring-theoretic exterior algebra): given a (not necessarily commutative) \( B \)-algebra \( A \) and a morphism of \( B \)-modules \( \varphi : B^n \to A \) satisfying \( \varphi(x) \cdot \varphi(x) = 0 \) and \( \varphi(x) \cdot \varphi(y) = \epsilon \varphi(y) \cdot \varphi(x) \) for all \( x, y \in B^n \), there exists a unique morphism of \( B \)-algebras \( \Phi : T(B^n) \oplus \tau_n \to A \) with \( \Phi(\vec{x}) = \varphi(x) \) for \( x \in B^n \).

**Proposition 3.6.** The natural maps \( \bigwedge^d B^n \to \bigwedge B^n \) induce \( \bigoplus_{d=0}^n \bigwedge^d B^n \cong \bigwedge B^n \).

**Proof.** By Lemma 3.2, one has that \( S_{d,n} \cap \bigcup_{i \neq d} S_{i,n} = \{0\} \). Thus

\[
(\bigwedge B^n)\)\(^+\) = (T(B^n) \oplus \tau_n)\)\(^+\) = \bigoplus_{d=0}^n S_{d,n} = \bigoplus_{d=0}^n \bigwedge B^n = \bigoplus_{d=0}^n \bigwedge^d B^n
\]

and

\[
(\bigwedge B^n)\)\(^\bullet\) = H_n = \bigcup_{l=0}^n H_{l,n} = \bigvee_{l=0}^n H_{l,n} = \left(\bigoplus_{d=0}^n \bigwedge B^n\right)\)\(^\bullet\).
\]

Let \( \leq_1 \) be the partial order of \( \bigoplus_{d=0}^n \bigwedge^d B^n \) and \( \leq_2 \) be the partial order of \( \bigwedge B^n \).

As, for each \( d \), the partial order of \( \bigwedge^d B^n \) is induced from \( \leq_2 \), one has that \( x \leq_1 y \) implies \( x \leq_2 y \).
Let \( x \) and \( y \) in \( (\bigwedge B^n)^+ \) such that \( x \leq_2 y \). Then there exists \( \alpha \) and \( \beta \) in \( \bigoplus_{d=0}^n ((B^n)^+)_{\otimes d} \), whose classes in \( (\bigwedge B^n)^+ \) are \( x \) and \( y \), respectively, and such that \( \alpha \leq \beta \). Writing \( \alpha = \sum \alpha_i \) and \( \beta = \sum \beta_i \), with \( \alpha_d \) and \( \beta_d \) in \( ((B^n)^+)_{\otimes d} \), one has that \( \alpha_i \leq \beta_i \) for all \( i \). Let \( x_i \) and \( y_i \) the classes of \( \alpha_i \) and \( \beta_i \) in \( (\bigwedge B^n)^+ \), respectively. Thus \( x_d \) and \( y_d \) are in \( S_{n,d} \) and satisfy \( x_d \leq_1 y_d \) for all \( d \). Therefore \( x \leq_1 y \). 

If \( R \) is a ring, one has the monomial ordered blueprint associated to \( R \)

\[
R^{\text{mon}} := R^* \lbrack 1.b \leq \sum 1.a_i \mid b = \sum a_i \in R \rbrack,
\]

where \( R^* \) is the multiplicative underlying monoid of \( R \). This is a \( \mathbb{F}_{+1} \)-algebra, with \( \epsilon = 1.(-1)_R \).

The next theorem proves the first item of Theorem B by showing how to recover the usual exterior algebra of rings from our construction of the exterior algebra for \( \mathbb{F}_{+1} \)-algebras.

**Theorem 3.7.** Let \( R \) be a ring and \( \mathcal{B} := R^{\text{mon}} \). Then \( (\bigwedge \mathcal{B}^n)^{\text{hull},+} \simeq \bigwedge R^n \) (the usual exterior algebra) as \( R \)-algebras.

**Proof.** Note that \( \mathcal{B}^{\text{hull},+} \simeq R \). Thus one has the isomorphisms of \( R \)-modules

\[
(\bigwedge \mathcal{B}^n)^{\text{hull},+} \simeq \left( \bigoplus_{d=0}^n (\bigwedge \mathcal{B}^n)^{\text{hull},+} \right) \simeq \left( \bigoplus_{d=0}^n (\mathcal{B}^n)^{\text{hull},+} \right) \simeq \bigoplus_{d=0}^n R^{(n_d)} \simeq \bigwedge R^n
\]

via

\[
(\bigwedge \mathcal{B}^n)^{\text{hull},+} \left[ e_{i_1} \wedge \ldots \wedge e_{i_d} \right] \mapsto e_{i_1} \wedge \ldots \wedge e_{i_d},
\]

where \( [x] \) denotes the class of \( x \in (\bigwedge \mathcal{B}^n)^+ \) in \( (\bigwedge \mathcal{B}^n)^{\text{hull},+} \).

As this map is a morphism of \( R \)-algebras, we have the result. \( \square \)

For an idempotent semifield \( S \), let

\[
\Omega_S := \left\{ 1.b \leq \sum_{i=1}^m 1.a_i \left| \sum_{i=1}^m a_i = b \sum_{i=1}^m a_i \right. \in S, \forall k \in [m] \right\}
\]

and define the monomial ordered blueprint associated to \( S \) as \( S^{\text{mon}} := S^* \lbrack \Omega_S \rbrack \). This is an \( \mathbb{F}_{+1} \)-algebra, with \( \epsilon = 1.1_S \) and the construction above extends to a functor \( (-)^{\text{mon}} : \text{IdempSFields} \to \text{OB\text{\textregistered}}. \)

For an ordered blueprint \( C \), let \( C^{\text{idem}} := C \lbrack 1 \equiv 1 + 1 \rbrack \) be the idempotent ordered blueprint associated to \( C \). This name comes from the fact that \( C^{\text{idem},+} \) is always an idempotent semiring.

The following definition is due to Jeffrey and Noah Giansiracusa (cf. [3, Definition 3.1.2]).
Definition 3.8. Let $S$ be an idempotent semifield. For $n \in \mathbb{N}$, let
$$
\Psi_n := \{ e_i \otimes e_j \equiv e_j \otimes e_i \mid i, j \in [n] \} \cup \{ e_i \otimes e_i \equiv 0 \mid i \in [n] \} \subseteq (S^n) \otimes 2.
$$
The tropical Grassmann algebra of $S^n$ is the graded $S$-algebra
$$
\bigwedge S^n := \text{Sym } S^n / \langle e_i^2 \equiv 0 \mid i \in [n] \rangle \simeq T(S^n) / \langle \Psi_n \rangle.
$$

The tropical Grassmann algebra of $S^n$ is the graded $S$-algebra
$$
\bigwedge S^n := \text{Sym } S^n / \langle e_i^2 \equiv 0 \mid i \in [n] \rangle \simeq T(S^n) / \langle \Psi_n \rangle.
$$

The $d^{th}$-homogeneous direct summand of $\bigwedge S^n$, denoted by $\bigwedge^d S^n$, is called the $d^{th}$ tropical wedge power of $S^n$.

The next theorem proves the second item of Theorem B by showing how to recover the tropical Grassmann algebra from our construction of the exterior algebra and exterior powers for $\mathbb{F}_1^\pm$-algebras.

Theorem 3.9. Let $S$ be an idempotent semifield and $S := S^{\text{mon}}$. Then $(\bigwedge^d S^n)^{\text{idem,+}} \simeq \bigwedge^d S^n$, for all $d$, and $(\bigwedge S^n)^{\text{idem,+}} \simeq \bigwedge S^n$, where $\bigwedge^d S^n$ and $\bigwedge S^n$ denotes the Giansiracusa $d^{th}$ tropical wedge power and tropical Grassmann algebra of $S^n$, respectively.

Proof. To begin with, we prove that $S^{\text{idem,+}} \simeq S$ as semirings. We will denote the class of $\sum n_i . s_i \in S^+$ in $S^{\text{idem,+}}$ by $[\sum n_i . s_i]$. One has the natural map
$$
\varphi: S^{\text{idem,+}} \longrightarrow S
$$
which is a surjective morphism of semirings. Note that $[\sum_{i=1}^m m_i . s_i] = [\sum_{i=1}^m 1 . s_i]$ in $S^{\text{idem,+}}$, for all positive integers $m_i$ and $s_i$ in $S$.

Let $a$, $b$ and $c := a + b$ be elements of $S$. Note that $c + b = c = c + a$ and $c + c = a + c$. Thus, in $S^+$, one has $1 . c \leq 1 . a + 1 . b$ and $1 . a \leq 2 . c$. Analogously, $1 . b \leq 2 . c$. Therefore, $1 . a + 1 . b \leq 2 . c$. Thus, $1 . a = 1 . a + 1 . b$ in $S^{\text{idem,+}}$. As a consequence, we obtain that $\varphi$ is injective.

Next we observe that $(\bigwedge^d S^n)^+ \simeq (S^{\binom{n}{d}})^+ \simeq (S^+)^{\binom{n}{d}}$ as $S^+$-modules, which implies
$$
(\bigwedge^d S^n)^{\text{idem,+}} \simeq (S^{\binom{n}{d}})^{\text{idem,+}} \simeq (S^{\text{idem,+}})^{\binom{n}{d}} \simeq S^{\binom{n}{d}} \simeq \bigwedge^d S^n
$$
as $S$-modules.

Thus $(\bigwedge S^n)^{\text{idem,+}} \simeq \bigoplus_{d=0}^n (\bigwedge^d S^n)^+ \simeq \bigoplus_{d=0}^n (\bigwedge^d S^n) \simeq \bigwedge S^n$, via
$$
(\bigwedge S^n)^{\text{idem,+}} \longrightarrow \bigwedge S^n
$$
$$
[e_1] \longrightarrow e_1.
$$

As this map is a morphism of $S$-algebras, we have the result. \qed
4. Matroids

In this section we will focus on matroids over blueprints and prove Theorem C.

**Definition 4.1.** A Grassmann-Plücker function of rank $d$ on $[n]$ with coefficients in $B$ is a function $\Delta : \binom{[n]}{d} \to B^*$ such that $\Delta(J)$ is in $B^x$ for some $J$ and satisfies the Plücker relations

$$
\sum_{i_k \notin X} \epsilon^k \Delta(X \cup \{i_k\}) \Delta(Y \setminus \{i_k\}) \geq 0
$$

whenever $X \in \binom{[n]}{d-1}$ and $Y = \{i_1, \ldots, i_{d+1}\} \in \binom{[n]}{d+1}$, with $i_1 < \ldots < i_{d+1}$.

We say that two Grassmann-Plücker functions $\Delta$ and $\Delta'$ are equivalent, and write $\Delta \sim \Delta'$, if there is some $a$ in $B^x$ such that $\Delta(I) = a \cdot \Delta'(I)$ for all $I$. A $B$-matroid (of rank $d$ on $[n]$) is an equivalence class of Grassmann-Plücker functions (of rank $d$ on $[n]$) with coefficients in $B$ (cf. [2, Definition 5.1]).

**Remark 4.2.** Besides other applications, the order in blueprint theory is used to describe relations analogous to algebraic equations. In particular, the positive cone $\{b \in B^+ \mid b \geq 0\}$ in Baker and Lorscheid’s theory of $\mathbb{F}^+_1$-algebras plays the same role as of the null set in Baker and Bowler’s theory of tracts, generalizing previous concepts of zero elements and sets (cf. [2, Section 1.2.4] and [1, Section 2.3]).

For $X \in \binom{[n]}{d-1}$ and $Y = \{i_1, \ldots, i_{d+1}\} \in \binom{[n]}{d+1}$ with $i_1 < \ldots < i_{d+1}$, there exists a (unique) morphism of $B$-modules $\varphi_{X,Y} : \bigwedge^n B^n \otimes \bigwedge^n B^n \to B$ that sends $e_I \otimes e_J$ to $\epsilon^k$ if there is $k \in [d+1]$ such that $I = X \cup \{i_k\}$ and $J = Y \setminus \{i_k\}$, and sends $e_I \otimes e_J$ to 0 otherwise.

**Definition 4.3.** A rank $d$ $B$-Plücker vector is a $v \in H_{d,n} \setminus K_{d,n}$ satisfying $\varphi_{X,Y}(v \otimes v) \geq 0$, i.e., a $v = \sum_{I \in \binom{[n]}{d}} v_I e_I \in H_{d,n}$ such that there exists a $J$ with $v_J \in B^x$ and, for all $X \in \binom{[n]}{d-1}$ and $Y \in \binom{[n]}{d+1}$, one has

$$
\sum_{i_k \notin X} \epsilon^k v_{X \cup \{i_k\}} v_{Y \setminus \{i_k\}} \geq 0,
$$

where $Y = \{i_1, \ldots, i_{d+1}\}$ with $i_1 < \ldots < i_{d+1}$.

Note that $B^x$ acts on the set of $B$-Plücker vectors, and more generally on $H_{d,n}$, by usual multiplication. So we can take the equivalence class of a $B$-Plücker vector in $H_{d,n}/B^x$.

**Theorem 4.4.** There is a bijection

$$
\{\text{rank } d \text{ } B\text{-Plücker vectors}\} \longrightarrow \{\text{Grassmann-Plücker functions } \Delta : \binom{[n]}{d} \to B^*\}
$$

$v = \sum_{I \in \binom{[n]}{d}} v_I e_I \quad \longmapsto \quad \Delta : I \longmapsto v_I$

**Proof.** It follows from Lemma 3.2 and the definitions above. \(\Box\)
The next corollaries prove Theorem C.

**Corollary 4.5.** Let $K$ be a field and $B := K^{\text{mon}}$. Then there exists a bijection between $B$-matroids of rank $d$ on $[n]$ and $K$-matroids of rank $d$ on $[n]$.

**Proof.** Due to the description of $K$-matroids via Grassmann-Plücker functions (cf. [1, p. 841]), the bijection is reached by composing the Grassmann-Plücker functions presented in Theorem 4.4 with the functor $(-)_{\text{hull}^+}$ and taking equivalence classes on both sides. □

The following is a reformulation of Definition 4.1.1 from [3].

**Definition 4.6.** Let $S$ be an indempotent semifield. A rank $d$ tropical $S$-Plücker vector is a nonzero $v = \sum v_I e_I \in \bigwedge^d S^n$ satisfying, for all subsets $A \in \binom{[n]}{d+1}$ and $X \in \binom{[n]}{d-1}$,

$$\sum_{i \in A \setminus X} v_{a-i} v_{X+i} = \sum_{i \in A \setminus X, i \neq p} v_{a-i} v_{X+i}$$

in $S$, for all $p \in A \setminus X$.

**Corollary 4.7.** Let $S$ be an idempotent semifield and $B := S^{\text{mon}}$. Then there exists a bijection between rank $d$ $B$-Plücker vectors and rank $d$ tropical $S$-Plücker vectors.

**Proof.** Note that, due to the description of the relations $\Omega_S$ on the construction of $S^{\text{mon}}$, one has that $v = \sum_{I \in \binom{[n]}{d}} v_I e_I$, with $v_I \in B^\bullet$, is a $B$-Plücker vector if and only if $\tilde{v} = \sum_{I \in \binom{[n]}{d}} v_I e_I$, with $v_I \in S$, is a tropical Plücker vector. □

**Corollary 4.8.** There is a bijection between rank $d$ $B$-matroids and classes of rank $d$ $B$-Plücker vectors.

**Proof.** We just need to take the equivalence class (by the action of $B^\times$) on both sides of the bijection presented Theorem 4.4. □

**Acknowledgements** The author thanks Oliver Lorscheid for useful conversations and for his help with preparing this text, and the anonymous referee for his or her careful report.

**Open Access** This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons licence, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons licence and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

**Funding** The present work was carried out with the support of CNPq, National Council for Scientific and Technological Development - Brazil.
References

[1] Baker, M., Bowler, N.: Matroids over partial hyperstructures. Adv. Math. 343, 821–863 (2019). https://doi.org/10.1016/j.aim.2018.12.004

[2] Baker, M., Lorscheid, O.: The moduli space of matroids. Adv. Math. 390, 107883 (2021). https://doi.org/10.1016/j.aim.2021.107883

[3] Giansiracusa, J., Giansiracusa, N.: A Grassmann algebra for matroids. manuscripta math. 156, 187–213 (2018). https://doi.org/10.1007/s00229-017-0958-z

[4] Lorscheid, O.: Blueprints and tropical scheme theory. Lecture notes, version from May 21, (2018). https://oliver.impa.br/notes/2018-Blueprints/lecturenotes.pdf

[5] Lorscheid, O.: Tropical geometry over the tropical hyperfield. Rocky Mountain J. Math. 52(1), 189–222 (2022). https://doi.org/10.1216/rmj.2022.52.189

Publisher’s Note  Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.