STRONG CONVERGENCE THEOREM OF CESÀRO MEANS WITH RESPECT TO THE WALSH SYSTEM

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Abstract. We prove that Cesàro means of one-dimensional Walsh-Fourier series are uniformly bounded operators in the martingale Hardy space $H_p$ for $0 < p < 1/(1 + \alpha)$.

1. Introduction. The definitions and notations used in this introduction can be found in the next section. It is well-known (see, e.g., [11, p.125]) that Walsh-Paley system is not a Schauder basis in the space $L^1(G)$. Moreover, there is a function $F$ in the dyadic Hardy space $H_1(G)$, such that the partial sums of the Walsh-Fourier series of $F$ are not bounded in the $L^1$-norm. However, in Simon [19] the following estimation was obtained: for all $F \in H_1(G)$

$$
\frac{1}{\log n} \sum_{k=1}^{n} \frac{\|S_k F\|_1}{k} \leq c \|F\|_{H_1}, \quad (n = 2, 3, \ldots),
$$

where $S_k F$ denotes the $k$-th partial sum of the Walsh-Fourier series of $F$ (For the trigonometric analogue see in Smith [21], for the Vilenkin system in Gát [6], for a more general, so-called Vilenkin-like system in Blahota [1]). Simon [16] (see also [27] and [34]) proved that there exists an absolute constant $c_p$, depending only on $p$, such that

$$
1 \log[p] n \sum_{k=1}^{n} \frac{\|S_k F\|_p^p}{k^{2-p}} \leq c_p \|F\|_{H_p}^p, \quad (0 < p \leq 1, \ n = 2, 3, \ldots),
$$

for all $F \in H_p$, where $[p]$ denotes integer part of $p$.

In [25] it was proven that sequence $\{1/k^{2-p}\}_{k=1}^{\infty}$ $(0 < p < 1)$ in (1) is given exactly.

Weisz [35] considered the norm convergence of Fejér means of Walsh-Fourier series and proved that

$$
\|\sigma_n F\|_{H_p} \leq c_p \|F\|_{H_p}, \quad F \in H_p, \quad (1/2 < p < \infty, \ n = 1, 2, 3, \ldots),
$$

where the constant $c_p > 0$ depends only on $p$.

Inequality (2) immediately implies that

$$
\frac{1}{n^{2p-1}} \sum_{k=1}^{n} \frac{\|\sigma_k F\|_p^p}{k^{2-2p}} \leq c_p \|F\|_{H_p}^p, \quad (1/2 < p < \infty).
$$

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If (2) also hold, for $0 < p \leq 1/2$, then we would have

$$
(3) \quad \frac{1}{\log^{1/2+p} n} \sum_{k=1}^{n} \|\sigma_k F\|_{H^p}^p \leq c_p \|F\|_{H^p}^p, \quad (0 < p \leq 1/2, \ n = 2, 3, \ldots),
$$

but in [22] it was proven that the assumption $p > 1/2$ is essential. In particular, there was proven that there exists a martingale $F \in H^p$ ($0 < p \leq 1/2$), such that $\sup_n \|\sigma_n F\|_p = +\infty$.

However, in [26] (see also [3]) it was proven that (3) holds, though (2) is not true for $0 < p \leq 1/2$.

The weak-type $(1,1)$ inequality for the maximal operator of Fejér means $\sigma^*$ can be found in Schipp [14] (see also [13]). Fujii [5] and Simon [18] verified that $\sigma^*$ is bounded from $H_1$ to $L_1$. Weisz [30] generalized this result and proved the boundedness of $\sigma^*$ from the space $H^p$ to the space $L^p$ for $p > 1/2$. Simon [17] gave a counterexample, which shows that boundedness does not hold for $0 < p < 1/2$. The counterexample for $p = 1/2$ is due to Goginava [8] (see also [4]). Weisz [31] proved that $\sigma^*$ is bounded from the Hardy space $H_{1/2}$ to the space $L_{1/2,\infty}$. In [23, 24] it was proven that the maximal operators $\tilde{\sigma}_p^*$ defined by

$$
(4) \quad \tilde{\sigma}_p^* F := \sup_{n \in \mathbb{N}} \frac{|\sigma_n F|}{n^{1/p-2} \log^{2[1/2+p]} n}, \quad (0 < p \leq 1/2, \ n = 2, 3, \ldots)
$$

is bounded from the Hardy space $H^p$ to the space $L^p$, where $F \in H^p$ and $[1/2 + p]$ denotes integer part of $1/2 + p$. Moreover, there was also shown that sequence $\{n^{1/p-2} \log^{2[1/2+p]} n : n = 2, 3, \ldots\}$ in (4) can not be improved.

Weisz [33] proved that the maximal operator $\sigma^{\alpha,*}$ ($0 < \alpha < 1$) of the Cesàro means of Walsh system is bounded from the martingale space $H^p$ to the space $L^p$ for $p > 1/ (1 + \alpha)$. Goginava [9] gave a counterexample, which shows that the boundedness does not hold for $0 < p \leq 1/ (1 + \alpha)$.

Recently, Weisz and Simon [20] show that the maximal operator $\sigma^{\alpha,*}$ is bounded from the Hardy space $H_{1/(1+\alpha)}$ to the space $L_{1/(1+\alpha),\infty}$. An analogical result for Walsh-Kaczmarz system was proven in [7].

In [10] Goginava investigated the behaviour of Cesàro means of Walsh-Fourier series in detail. For some approximation properties of the two dimensional case see paper of Nagy [12].

The main aim of this paper is to generalize estimate (3) for Cesàro means, when $0 < p < 1/ (1 + \alpha)$. We also consider the weighted maximal operator of $(C, \alpha)$ means and proved some new $(H^p, L^p)$-type inequalities for it.

We note that the case $p = 1/ (1 + \alpha)$ was considered in [2].

2. Definitions and Notations. Let $\mathbb{N}_+$ denote the set of the positive integers, $\mathbb{N} := \mathbb{N}_+ \cup \{0\}$. Denote by $\mathbb{Z}_2$ the discrete cyclic group of order 2, that is $\mathbb{Z}_2 := \{0, 1\}$, where the group operation is the modulo 2 addition and every subset is open. The Haar measure on $\mathbb{Z}_2$ is given so that the measure of a singleton is 1/2.
Define the group $G$ as the complete direct product of the group $\mathbb{Z}_2$ with the product of the discrete topologies of $\mathbb{Z}_2$'s. The elements of $G$ are represented by sequences

$$x := (x_0, x_1, \ldots, x_k, \ldots) \quad (x_k = 0, 1).$$

It is easy to give a base for the neighborhood of $G$

$$I_0(x) := G, \quad I_n(x) := \{y \in G \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\} \quad (x \in G, \ n \in \mathbb{N}).$$

Denote $I_n := I_n(0)$ and $\overline{I}_n := G \setminus I_n$. Let

$$e_n := (0, \ldots, 0, x_n = 1, 0, \ldots) \in G \quad (n \in \mathbb{N}).$$

Denote

$$I^{k,l}_M := \begin{cases} I_M(0, \ldots, 0, x_k = 1, 0, \ldots, x_{l+1}, \ldots, x_{M-1}), & k < l < M, \\ I_M(0, \ldots, 0, x_k = 1, 0, \ldots, 0), & l = M. \end{cases}$$

It is evident

$$\overline{I}_M = \left( \bigcup_{k=0}^{M-2} \bigcup_{l=k+1}^{M-1} I^{k,l}_M \right) \bigcup \left( \bigcup_{k=0}^{M-1} I^k_M \right).$$

If $n \in \mathbb{N}$, then every $n$ can be uniquely expressed as $n = \sum_{j=0}^{\infty} n_j 2^j$, where $n_j \in \mathbb{Z}_2$ ($j \in \mathbb{N}$) and only finite number of $n_j$’s differ from zero, that is, $n$ is expressed in the number system of base 2. Let $|n| := \max\{j \in \mathbb{N}, n_j \neq 0\}$, that is $2^{|n|} \leq n \leq 2^{|n|+1}$.

The norm (or quasi-norm) of the space $L_p(G)$ is defined by

$$\|f\|_p := \left( \int_G |f|^p d\mu \right)^{1/p}, \quad (0 < p < \infty).$$

The space $L_{p,\infty}(G)$ consists of all measurable functions $f$, for which

$$\|f\|_{L_{p,\infty}(G)} := \sup_{\lambda > 0} \lambda \mu(f > \lambda)^{1/p} < \infty.$$

Next, we introduce on $G$ an orthonormal system which is called the **Walsh system**. At first, define the functions $r_k(x) : G \rightarrow \mathbb{C}$, the so-called Rademacher functions as

$$r_k(x) := (-1)^{x_k} \quad (x \in G, \ k \in \mathbb{N}).$$

Now, define the Walsh system $w := (w_n : n \in \mathbb{N})$ on $G$ as:

$$w_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \quad (n \in \mathbb{N}).$$

The Walsh system is orthonormal and complete in $L_2(G)$ (see, e.g., [28]).
If \( f \in L_1(G) \), then we can establish Fourier coefficients, partial sums of the Fourier series, Fejér means, Dirichlet and Fejér kernels in the usual manner:

\[
\hat{f}(n) := \int_G f w_n d\mu, \quad (n \in \mathbb{N}),
\]

\[
S_n f := \sum_{k=0}^{n-1} \hat{f}(k) w_k, \quad (n \in \mathbb{N}_+),
\]

\[
\sigma_n f := \frac{1}{n} \sum_{k=1}^{n} S_k f, \quad (n \in \mathbb{N}_+),
\]

\[
D_n := \sum_{k=0}^{n-1} w_k, \quad (n \in \mathbb{N}_+),
\]

\[
K_n := \frac{1}{n} \sum_{k=1}^{n} D_k, \quad (n \in \mathbb{N}_+),
\]

respectively. Recall that (see e.g., [15])

\[
D_{2^n}(x) = \begin{cases} 2^n, & \text{if } x \in I_n, \\ 0, & \text{if } x \notin I_n. \end{cases}
\]

The Cesàro means ((C, \( \alpha \))-means) are defined as

\[
\sigma^n_{\alpha} f := \frac{1}{A^n_{\alpha}} \sum_{k=1}^{n} A^{\alpha-1}_{n-k} S_k f,
\]

where

\[
A^0_{\alpha} := 1, \quad A^n_{\alpha} := \frac{(\alpha + 1) \cdots (\alpha + n)}{n!} \quad \alpha \neq -1, -2, \ldots.
\]

It is well known that

\[
A^n_{\alpha} = \sum_{k=0}^{n} A^{\alpha-1}_{n-k}, \quad A^n_{\alpha} - A^{\alpha-1}_{n-1} = A^{\alpha-1}_{n}, \quad A^n_{\alpha} \sim n^\alpha
\]

and

\[
\sup_n \int_G |K^n_{\alpha}| d\mu \leq c < \infty,
\]

where \( K^n_{\alpha} \) is \( n \)-th Cesàro kernel.

The \( \sigma \)-algebra generated by the intervals \( \{I_n(x) : x \in G\} \) will be denoted by \( F_n (n \in \mathbb{N}) \). Denote by \( F = (F_n, n \in \mathbb{N}) \) the martingale with respect to \( F_n (n \in \mathbb{N}) \) (for details see, e.g., [29]).

The maximal function of a martingale \( F \) is defined by

\[
F^* := \sup_{n \in \mathbb{N}} |F_n|.
\]

In the case \( f \in L_1(G) \), the maximal functions are also be given by
\[ f^*(x) = \sup_{n \in \mathbb{N}} \frac{1}{\mu(I_n(x))} \left| \int_{\mu(x)} f(u) \, d\mu(u) \right|. \]

For \(0 < p < \infty\), the Hardy martingale spaces \(H_p(G)\) consist of all martingales such that

\[ \|F\|_{H_p} := \|F^*\|_p < \infty. \]

A bounded measurable function \(a\) is a \(p\)-atom, if there exist a dyadic interval \(I\) such that

\[ \int_I a \, d\mu = 0, \quad \|a\|_{\infty} \leq \mu(I)^{-1/p}, \quad \text{supp}(a) \subset I. \]

It is easy to check that for every martingale \(F = (F_n, n \in \mathbb{N})\) and for every \(k \in \mathbb{N}\) the limit

\[ \hat{F}(k) := \lim_{n \to \infty} \int G F_n w_k d\mu \]

exists and it is called the \(k\)-th Walsh-Fourier coefficients of \(F\).

Denote by \(A_n\) the \(\sigma\)-algebra generated by the sets \(I_n(x) (x \in G, n \in \mathbb{N})\). If \(F := (S_{2^n} f : n \in \mathbb{N})\) is the regular martingale generated by \(f \in L_1(G)\), then

\[ \hat{F}(k) = \int_G f w_k d\mu =: \hat{f}(k), \quad k \in \mathbb{N}. \]

For \(0 < \alpha \leq 1\), let consider maximal operators

\[ \sigma^{\alpha,*} F := \sup_{n \in \mathbb{N}} \left| \sigma_n^\alpha F \right|, \quad \sigma^\alpha_p F := \sup_{n \in \mathbb{N}} \frac{\left| \sigma_n^\alpha F \right|}{(n+1)^{1/p-1-\alpha}}, \quad 0 < p < 1/(1+\alpha). \]

For the martingale

\[ F = \sum_{n=0}^{\infty} (F_n - F_{n-1}) \]

the conjugate transforms are defined as

\[ \sim F(t) := \sum_{n=0}^{\infty} r_n(t) (F_n - F_{n-1}), \]

where \(t \in G\) is fixed. Note that \(\sim F(0) = F\).

As it is well-known (see, e.g., [29])

\[ \left\| \sim F(t) \right\|_{H_p} = \|F\|_{H_p}, \quad \|F\|_{H_p}^p \sim \int_G \left\| \sim F(t) \right\|_p^p dt, \quad \sigma_m^\alpha F(t) = \sigma_m^\alpha \sim F(t). \]
3. Formulation of main results.

THEOREM 1. a) Let $0 < \alpha < 1$ and $0 < p < 1/(1 + \alpha)$. Then there exists absolute constant $c_{\alpha, p}$, depending on $\alpha$ and $p$, such that for all $F \in H_p(G)$

$$\left\| \sigma^*_{\alpha, p} F \right\|_p \leq c_{\alpha, p} \| F \|_{H_p}.$$ 

b) Let $0 < \alpha < 1$, $0 < p < 1/(1 + \alpha)$ and $\varphi : \mathbb{N}_+ \to [1, \infty)$ be a nondecreasing function satisfying the condition

$$\lim_{n \to \infty} n^{1/p - 1 - \alpha} \varphi(n) = \infty.$$ 

Then the maximal operator

$$\sup_{n \in \mathbb{N}} \frac{|\sigma^*_{\alpha} f|}{\varphi(n)}$$ 

is not bounded from the Hardy space $H_p(G)$ to the space $L_p(G)$.

THEOREM 2. Let $0 < \alpha < 1$ and $0 < p < 1/(1 + \alpha)$. Then there exists absolute constant $c_{\alpha, p}$, depending on $\alpha$ and $p$, such that for all $F \in H_p$

$$\sum_{m=1}^{\infty} \left\| \sigma^*_{m} F \right\|_{H_p}^p \leq c_{\alpha, p} \| F \|_{H_p}^p.$$ 

4. Auxiliary Propositions. The dyadic Hardy martingale spaces $H_p(G)$ have an atomic characterization, when $0 < p \leq 1$:

LEMMA 1 (Weisz [32]). A martingale $F = (F_n, n \in \mathbb{N})$ is in $H_p (0 < p \leq 1)$ if and only if there exists a sequence $(a_k, k \in \mathbb{N})$ of p-atoms and a sequence $(\mu_k, k \in \mathbb{N})$ of a real numbers, such that for every $n \in \mathbb{N}$

$$(12) \quad \sum_{k=0}^{\infty} \mu_k S_{2^k} a_k = F_n,$$

$$\sum_{k=0}^{\infty} |\mu_k|^p < \infty.$$ 

Moreover,

$$\| F \|_{H_p} \sim \inf \left( \sum_{k=0}^{\infty} |\mu_k|^p \right)^{1/p},$$

where the infimum is taken over all decompositions of $F$ of the form (12).

By using Lemma 1 we can easily proved the following:
LEMMA 2 (Weisz [29]). Suppose that an operator $T$ is $\sigma$-linear and for some $0 < p \leq 1$
\[
\int_I |Ta|^p \, d\mu \leq c_p < \infty,
\]
for every $p$-atom $a$, where $I$ denote the support of the atom. If $T$ is bounded from $L_\infty$ to $L_\infty$, then
\[
\|Tf\|_p \leq c_p \|f\|_{H_p}.
\]
To prove our main results we also need the following estimations:
LEMMA 3 ([2]). Let $0 < \alpha < 1$ and $n > 2^M$. Then
\[
\int_{I_M} |K_\alpha^n (x + t)| \, d\mu(t) \leq \frac{c_\alpha 2^{\alpha l + k}}{n^\alpha 2^M},
\]
for $x \in I_{l+1} (e_k + e_l)$, $(k = 0, \ldots, M - 2, l = k + 1, \ldots, M - 1)$ and
\[
\int_{I_M} |K_\alpha^n (x + t)| \, d\mu(t) \leq \frac{c_\alpha 2^k}{2^M},
\]
for $x \in I_M (e_k)$, $(k = 0, \ldots, M - 1)$.

5. Proof of Theorems.
PROOF OF THEOREM 1. Since $\sigma_n$ is bounded from $L_\infty$ to $L_\infty$ (the boundedness follows from (8)) according to Lemma 2 the proof of Theorem 1 will be complete if we show
\[
\sup_{I_M} \int_{I_M} |\sigma_n \ast a|^p \, d\mu < \infty,
\]
where the supremum is taken over all $p$-atoms $a$. We may assume that $a$ is an arbitrary $p$-atom, with support $I$, $\mu(I) = 2^{-M}$ and $I = I_M$. It is easy to see that $\sigma_n^n (a) = 0$, when $n \leq 2^M$. Therefore, we can suppose that $n > 2^M$.
Let $x \in I_M$. Since $\|a\|_\infty \leq c 2^{M/p}$ we obtain
\[
|\sigma_n^n a (x)| \leq \int_{I_M} |a(t)| |K_\alpha^n (x + t)| \, d\mu(t)
\leq \|a(x)\|_\infty \int_{I_M} |K_\alpha^n (x + t)| \, d\mu(t)
\leq c_\alpha 2^{M/p} \int_{I_M} |K_\alpha^n (x + t)| \, d\mu(t).
\]
Let $x \in I_M^{k,l}$, $0 \leq k < l < M$. Then from Lemma 3 we get
\[
|\sigma_n^n a (x)| \leq \frac{c_\alpha, p 2^{M(1/p - 1)} 2^{d+k}}{n^\alpha}.
\]
Let $x \in I^k_M$, $0 \leq k < M$. Then from Lemma 3 we have
\begin{equation}
|\sigma_n^\alpha a(x)| \leq c_{\alpha,p} 2^{M(1/p - 1) + k}.
\end{equation}

By combining (5), (13) and (14) we obtain
\[
\int_{I_M} \sup_{n \in \mathbb{N}} \left| \frac{\sigma_n^\alpha a(x)}{n^{1/p - 1 - \alpha}} \right|^p d\mu(x) \\
= \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \sum_{x_j=0}^1 \int_{I_M} \sup_{n > 2^M} \left| \frac{\sigma_n^\alpha a(x)}{n^{1/p - 1 - \alpha}} \right|^p d\mu(x) \\
+ \sum_{k=0}^{M-1} \int_{I_M} \sup_{n > 2^M} \left| \frac{\sigma_n^\alpha a(x)}{n^{1/p - 1 - \alpha}} \right|^p d\mu(x) \\
\leq \frac{1}{2M(1-(1+\alpha)p)} \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \sum_{x_j=0}^1 \int_{I_M} \sup_{n > 2^M} \left| \sigma_n^\alpha a(x) \right|^p d\mu(x) \\
+ \frac{1}{2M(1-(1+\alpha)p)} \sum_{k=0}^{M-1} \int_{I_M} \sup_{n > 2^M} \left| \sigma_n^\alpha a(x) \right|^p d\mu(x) \\
\leq \frac{c_{\alpha,p}}{2M(1-(1+\alpha)p)} \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{2M(1-p)2(\alpha l+k)p}{2Mp} \\
+ \frac{c_{\alpha,p}}{2M(1-(1+\alpha)p)} \frac{1}{2M} \sum_{k=0}^{M-1} 2M(1-p)+pk \\
\leq \frac{c_{\alpha,p}}{2M(1-(1+\alpha)p)} \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{2k^p}{2M(1-\alpha p)} \\
+ \frac{c_{\alpha,p}}{2M(1-(1+\alpha)p)} \sum_{k=0}^{M-1} \frac{2pk}{2M} \leq c_{\alpha,p} < \infty.
\]

It is easy to show that under condition (11), there exists a sequence of positive integers $\{n_k, k \in \mathbb{N}_+\}$, such that
\[
\lim_{k \to \infty} \frac{(2^{2n_k} + 2)^{1/p - 1 - \alpha}}{\varphi(2^{2n_k} + 2)} = \infty.
\]

Let
\[
f_{n_k} = D_{2^{2n_k+1}} - D_{2^{2n_k}}.
\]

It is evident
\[
\tilde{f}_{n_k}(i) = \begin{cases} 
1, & \text{if } i = 2^{2n_k}, \ldots, 2^{2n_k+1} - 1, \\
0, & \text{otherwise.}
\end{cases}
\]
Then we can write

\[
Snfk = \begin{cases} 
D_i - D_{2n_k}, & \text{if } i = 2^{2n_k} + 1, \ldots, 2^{2n_k+1} - 1, \\
fn_k, & \text{if } i \geq 2^{2n_k}+1, \\
0, & \text{otherwise}.
\end{cases}
\]  

From (6) we get

\[
\|fn_k\|_p = \|f^*_nk\|_p = \|D_{2n_k+1} - D_{2n_k}\|_p \leq c2^{2n_k(1-1/p)}.
\]  

Since \(A_{0}^{\alpha-1} = 1\), by (15) we can write

\[
\frac{\sigma_{2n_k+1}^{\alpha} fn_k}{\varphi(2^{2n_k} + 1)} = \frac{1}{\varphi(2^{2n_k} + 1) A_{2^{2n_k}+1}^{\alpha}} \left[ \sum_{j=1}^{2^{2n_k}+1} A_{2^{2n_k}+1-j}^{\alpha-1} S_j fn_k \right] \\
= \frac{1}{\varphi(2^{2n_k} + 1) A_{2^{2n_k}+1}^{\alpha}} \left[ \sum_{j=2^{2n_k}+1}^{2^{2n_k}+1} A_{2^{2n_k}+1-j}^{\alpha-1} S_j fn_k \right] \\
\geq \frac{1}{\varphi(2^{2n_k} + 1) A_{2^{2n_k}+1}^{\alpha}} \left[ A_{0}^{\alpha-1} (D_{2^{2n_k}+1} - D_{2^{2n_k}}) \right] \\
\geq \frac{1}{\varphi(2^{2n_k} + 1) (2^{2n_k} + 1)^{\alpha}} \cdot c.
\]

From (16) we have

\[
\frac{c/(\varphi(2^{2n_k} + 1) (2^{2n_k} + 1)^{\alpha}) \cdot \mu \{ x : \|\sigma^{\alpha,*}_k f\| \geq c/(\varphi(2^{2n_k} + 1) (2^{2n_k} + 1)^{\alpha}) \}^{1/p}}{\|fn_k\|_p} \cong \frac{c}{\varphi(2^{2n_k} + 1) (2^{2n_k} + 1)^{\alpha}} 2^{2n_k(1-1/p)} \geq \frac{c}{\varphi(2^{2n_k} + 1)} (1/p-1-\alpha) \rightarrow \infty, \text{ as } k \rightarrow \infty.
\]

Theorem 1 is proven.

**Proof of Theorem 2.** Suppose that

\[
\sum_{m=1}^{\infty} \|\sigma_{m}^{\alpha} F\|_p^p \leq \|F\|_{H_p}^p.
\]
Then by using (10) we have

\[ \sum_{m=1}^{\infty} \frac{\|\sigma_m^a F\|_{H_p}^p}{m^{2-(1+\alpha)p}} = \sum_{m=1}^{\infty} \frac{\int_G \|\sigma_m^\alpha F\|_{H_p}^p dt}{m^{2-(1+\alpha)p}} \leq \int_G \sum_{m=1}^{n} \frac{\|\sigma_m^\alpha F\|_{H_p}^p dt}{m^{2-(1+\alpha)p}} \]

\[ \leq \int_G \|\tilde{F}\|_{H_p}^p dt \sim \int_G \|F\|_{H_p}^p dt = \|F\|_{H_p}. \]

According to Theorem 1 and (17) the proof of Theorem 2 will be complete, if we show

\[ \sum_{m=1}^{\infty} \frac{\|\sigma_m^a\|_{p}^{p}}{m^{2-(1+\alpha)p}} \leq c_\alpha < \infty, \]

for every \( p \)-atom \( a \). Analogously to first part of Theorem 1 we can assume that \( n > 2^M \) and \( a \) be an arbitrary \( p \)-atom, with support \( I \), \( \mu(I) = 2^{-M} \) and \( I = I_M \).

Let \( x \in I_M \). Since \( \sigma_n \) is bounded from \( L_\infty \) to \( L_\infty \) (the boundedness follows from (8)) and \( \|a\|_\infty \leq c2^M/p \) we obtain

\[ \int_{I_M} |\sigma_m^a|^p d\mu \leq \int_{I_M} \|K_m\|_1^p \|a\|_\infty^p d\mu \]

\[ \leq c_{_\alpha,p} \int_{I_M} \|a\|_\infty^p d\mu \leq c_{_\alpha,p} < \infty. \]

Hence

\[ \sum_{m=2^M+1}^{\infty} \frac{\int_{I_M} |\sigma_m^a|^p d\mu}{m^{2-(1+\alpha)p}} \leq c_{_\alpha,p} \sum_{m=2^M+1}^{\infty} \frac{1}{m^{2-(1+\alpha)p}} \]

\[ \leq \frac{c_{_\alpha,p}}{2^{M(1-(1+\alpha)p)}} \leq c_{_\alpha,p} < \infty. \]

By combining (5), (13) and (14) analogously to first part of Theorem 1 we can write

\[ \sum_{m=2^M+1}^{\infty} \frac{\int_{I_M} |\sigma_m^a|^p d\mu}{m^{2-(1+\alpha)p}} = \sum_{m=2^M+1}^{\infty} \left( \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{1}{m^{2-(1+\alpha)p}} \int_{I_M} |\sigma_m^a|^p d\mu + \sum_{k=0}^{M-1} \frac{\int_{I_M} |\sigma_m^a|^p d\mu}{m^{2-(1+\alpha)p}} \right) \]

\[ \leq \sum_{m=2^M+1}^{\infty} \left( c_{_\alpha,p} \frac{2^{M(1-p)}}{m^{2-p}} \sum_{k=0}^{M-2} \sum_{l=k+1}^{M-1} \frac{2^{p(\alpha l+k)}}{2^l} + c_{_\alpha,p} \frac{2^{M(1-p)}}{m^{2-(1+\alpha)p}} \sum_{k=0}^{M-1} \frac{2^{pk}}{2^M} \right) \]

\[ < c_{_\alpha,p} 2^{M(1-p)} \sum_{m=2^M+1}^{\infty} \frac{1}{m^{2-p}} + c_{_\alpha,p} \sum_{m=2^M+1}^{\infty} \frac{1}{m^{2-(1+\alpha)p}} \leq c_{_\alpha,p} < \infty, \]

which completes the proof of Theorem 2. \( \square \)
REFERENCES

[1] I. Blahota, On a norm inequality with respect to Vilkenin-like systems, Acta Math. Hungar. 89 (2000), no. 1-2, 15–27.
[2] I. Blahota and G. Tephnadze, On the \((C, \alpha)\)-means with respect to the Walsh system, to appear in Analysis Mathematica.
[3] I. Blahota and G. Tephnadze, Strong convergence theorem for Vilkenin-Fejér means, to appear in Publicationes Mathematicae Debrecen.
[4] I. Blahota, G. Gát and U. Goginava, Maximal operators of Fejér means of Vilkenin-Fourier series, J. Inequal. Pure Appl. Math. 7 (2006), no. 4, Article 149, 7 pp. (electronic).
[5] N. J. Fujii, A maximal inequality for \(H^1\)-functions on a generalized Walsh-Paley group, Proc. Amer. Math. Soc. 77 (1979), no. 1, 111–116.
[6] G. Gát, Investigations of certain operators with respect to the Vilken system, Acta Math. Hungar. 61 (1993), no. 1-2, 131–149.
[7] G. Gát and U. Goginava, A weak type inequality for the maximal operator of \((C, \alpha)\)-means of Fourier series with respect to the Walsh-Kaczmarz system, Acta Math. Hungar. 125 (2009), no. 1-2, 65–83.
[8] U. Goginava, Maximal operators of Fejér means of double Walsh-Fourier series, Acta Math. Hungar. 115 (2007), no. 4, 333–340.
[9] U. Goginava, The maximal operator of the \((C, \alpha)\) means of the Walsh-Fourier series, Ann. Univ. Sci. Budapest. Sect. Comput. 26 (2006), 127–135.
[10] U. Goginava, On the approximation properties of Cesàro means of negative order of Walsh-Fourier series, J. Approx. Theory 115 (2002), no. 1, 9–20.
[11] B. Golubov, A. Efimov and V. Skvortsoy, Walsh series and transformations, Dordrecht, Boston, London, 1991. Kluwer Acad. publ, 1991.
[12] K. Nagy, Approximation by Cesàro means of negative order of Walsh-Kaczmarz-Fourier series, East J. Approx. 16 (2010), no. 3, 297–311.
[13] J. Pál and P. Simon, On a generalization of the concept of derivative, Acta Math. Acad. Sci. Hungar. 29 (1977), no. 1-2, 155–164.
[14] F. Schipp, Certain rearrangements of series in the Walsh system, (Russian) Mat. Zametki 18 (1975), no. 2, 193–201.
[15] F. Schipp, W. R. Wade, P. Simon and J. Pál, Walsh series, An Introduction to Dyadic Harmonic Analysis, Akadémiai Kiadó, (Budapest-Adam Hilger (Bristol-New-York)), 1990.
[16] P. Simon, Strong convergence theorem for Vilkenin-Fourier series, J. Math. Anal. Appl. 245 (2000), no. 1, 52–68.
[17] P. Simon, Cesaro summability with respect to two-parameter Walsh systems, Monatsh. Math. 131 (2000), no. 4, 321–334.
[18] P. Simon, Investigations with respect to the Vilken system, Ann. Univ. Sci. Budapest. Eötvös Sect. Math. 27 (1984), 87–101 (1985).
[19] P. Simon, Strong convergence of certain means with respect to the Walsh-Fourier series, Acta Math. Hungar. 49 (1987), no. 3-4, 425–431.
[20] P. Simon and F. Weisz, Weak inequalities for Cesàro and Riesz summability of Walsh-Fourier series, J. Approx. Theory 151 (2008), no. 1, 1–19.
[21] B. Smith, A strong convergence theorem for \(H^1(T)\). Banach spaces, harmonic analysis, and probability theory, 169–173, Lecture Notes in Math., 995, Springer, Berlin-New York, 1983.
[22] G. Tephnadze, Fejér means of Vilkenin-Fourier series, Studia Sci. Math. Hungar. 49 (2012), no. 1, 79–90.
[23] G. Tephnadze, On the maximal operators of Vilkenin-Fejér means, Turkish J. Math. 37 (2013), no. 2, 308–318.
[24] G. Tephnadze, On the maximal operators of Vilkenin-Fejér means on Hardy spaces, Math. Inequal. Appl.
[25] G. Tephnadze, A note on the Fourier coefficients and partial sums of Vilenkin-Fourier series, Acta Math. Acad. Paedagog. Nyházi. (N.S.) 28 (2012), no. 2, 167–176.

[26] G. Tephnadze, Strong convergence theorems for Walsh-Fejér means, Acta Math. Hungar. 142 (2014), no. 1, 244–259.

[27] G. Tephnadze, On the partial sums of Vilenkin-Fourier series, J. Contemp. Math. Anal. 49 (2014), no. 1, 23–32.

[28] N. Ya. Vilenkin, On a class of complete orthonormal systems, Amer. Math. Soc. Transl. (2) 28 (1963), 1–35.

[29] F. Weisz, Martingale Hardy spaces and their applications in Fourier analysis, Lecture Notes in Math. 1568, Springer, Berlin, 1568, Springer-Verlag, Berlin, 1994.

[30] F. Weisz, Cesàro summability of one- and two-dimensional Walsh-Fourier series, Anal. Math. 22 (1996), no. 3, 229–242.

[31] F. Weisz, Weak type inequalities for the Walsh and bounded Ceselski systems, Anal. Math. 30 (2004), no. 2, 147–160.

[32] F. Weisz, Hardy spaces and Cesàro means of two-dimensional Fourier series, Approximation theory and function series (Budapest, 1995), 353–367, Bolyai Soc. Math. Stud., 5, János Bolyai Math. Soc., Budapest, 1996.

[33] F. Weisz, (C, α) summability of Walsh-Fourier series, Anal. Math. 27 (2001), no. 2, 141–155.

[34] F. Weisz, Strong convergence theorems for two-parameter Walsh-Fourier and trigonometric-Fourier series, Studia Math. 117 (1996), no. 2, 173–194.

[35] F. Weisz, Summability of multi-dimensional Fourier series and Hardy spaces, Mathematics and its Applications, 541. Kluwer Academic Publishers, Dordrecht, 2002.

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