CURVATURE BOUNDS FOR REGULARIZED RIEMANNIAN METRICS

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Abstract. We investigate regularization of riemannian metrics by mollification. Assuming both-sided bounds on the Ricci tensor and a lower injectivity radius bound we obtain a uniform estimate on the change of the sectional curvature. Actually, our result holds for any metric with a uniform bound on the $W^{2,p}$-harmonic radius. We also provide a weaker estimate under lower Ricci and injectivity radius bound.

§ 1. Introduction

The goal of this note is to show that regularization by a naive mollification of a riemannian metric satisfying certain geometric conditions can be set up to control the alteration of the sectional curvature.

A riemannian metric on a smooth manifold $M$ is a section in the symmetric $(0,2)$-tensor bundle over $M$. The regularity of this section is then referred to as the regularity of the metric. Under certain geometric limit processes, it is a common phenomenon to lose a controlled level of regularity. Gromov-Hausdorff limits of isometry classes of smooth riemannian metrics that satisfy certain curvature bounds, are a prominent example for this. We will briefly recall fundamental results in this area in § 2.4. In this work we are interested in procedures to regain regularity and at the same time control the curvature of the corresponding metric in comparison to the original one. Various such techniques have been studied with a view towards different goals (e.g. [CG85, Abr88]).

A fundamental tool in deriving and phrasing our results are chart norms, that control the regularity properties of the metric tensor and its derivatives in a given chart. Let $\psi : (B(0,r),0) \to (M,p)$ be a harmonic chart in a smooth, pointed, $n$-dimensional riemannian manifold $(M^n,g,p)$, where we denote by $B(0,r) \subset \mathbb{R}^n$ the open ball of radius $r$ with respect to the euclidean norm. Recall that a chart is called harmonic, if the coordinate-functions of $\psi$ are harmonic with respect to the Laplace-Beltrami operator of $g$. The harmonic chart norm $\|\psi\|_{W^{m,p},r}^{\text{harm}}$ is bounded by $Q \geq 0$, if $Q$ gives control of

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the derivatives of the metric tensor, its inverse and its first $m$ derivatives in the $L^p$-norm. We refer to §2.3 for a more detailed explanation.

Given a collection of charts $\{\psi_i\}$, a partition of unity $\{\rho_i\}$ and a fixed mollifier function $\varphi_t$ for $t \in (0,T]$ we define a mollified riemannian metric $g^{[t]}$, which, given bounds on the harmonic chart norms, has a curvature tensor that can be controlled as follows.

Unless otherwise stated, we will always denote by $M$ a smooth manifold of dimension $n$.

**Theorem A.** Given $p > 2n$ and $r,Q > 0$. Choose $\beta \in (0, 1 - 2n/p)$. Then there is some $T \in (0, r/2]$ such that for any pointed smooth riemannian manifold $(M,g,p)$ and any finite collection of charts with a corresponding partition of unity

$$\{\psi_i: (B(0,r),0) \to (M,p_i)\}_{i \in I}, \quad \bar{\varrho} := \{\varrho_i: M \to [0,1]\}_{i \in I}$$

we have that for any section $\bar{v} \in T^{\otimes 1} \psi(B(0,r))$ with $g$-norm not greater than 1 that

$$\left( \forall i \in I: \|\psi_i\|_{W^{2,p},r} \leq Q \right) \implies R_{g^{[t]}}(x)(\bar{v}) \leq \sup_{B(x,e^T)} R_g(\bar{v}) \leq Ct^\beta$$

for any $t \in (0,T]$, $x \in \bigcap_{i \in I} \psi_i(B(0,r/2))$ and with

$$C_\varrho = C_\varrho\left(n,p,r,Q,\beta,\{\|\varrho_i \circ \psi_i\|_{C^2}\}_{i \in I}, \|\bar{v}\|_g, \# I \right).$$

The new metric $g^{[t]}$ is obtained by applying a mollification operator $P_t$ in the charts $\psi_i$. We can summarize the ansatz of our proof of this statement as follows:

\begin{equation}
R_{g^{[t]}}(x)(\bar{v}) \leq P_t(R_g(\bar{v}))(x) + |R_{g^{[t]}}(x)(\bar{v}) - P_t(R_g(\bar{v}))(x)| \leq \sup_{y \in \psi(B(0,T))} (g) R_g(\bar{v}) + \|R_{g^{[t]}}(\bar{v}) - P_t R_g(\bar{v})\|_{L^\infty}
\end{equation}

where in the last step we used that convolution does not increase the supremum of a function. Hence it suffices to find an $L^\infty$-estimate for the “commutator”

$$R_{g^{[t]}} - P_t R_g \quad \text{"} = [R, P_t]g \text{"}$$

which will be obtained in Lemma 3.5.

Finally, we state our main result on the alteration of the sectional curvature. Define for a riemannian manifold $(M,g)$

$$\overline{K}(g) := \sup K_g(x)(v,w), \quad K(g) := \sup K_g(x)(v,w)$$

where the suprema are taken over all $x \in M$ and $v, w \in T_x M$ such that $\langle v, w \rangle_g > 0.$
Theorem B. Let $p > 2n$, $r, Q > 0$, and $\beta \in (0, 1 - 2n/p)$. Then there is some $T > 0$ such that for any smooth riemannian manifold $(M, g)$ with $\|\{(M, g)\}_{Harm} \leq Q$ there is a locally finite cover of charts

$$\psi_t : (B(0, r), 0) \to (M, p_t),$$

$i \in I$, such that the mollified metrics $g^\epsilon$ have at any $x \in M$ sectional curvature $K_{\epsilon(t)}(x)$ in the interval

$$\tag{1.2} \left[\mathcal{K}(g^\epsilon_{B(x, t)}), \mathcal{K}(g_{B(x, t)})\right] \cdot [1 - C t, 1 + Ct] + [-Ct^\beta, Ct^\beta]$$

for all $t \in (0, T]$, where $C = C(n, Q, r, p)$.

Moreover for any $\beta' \in (0, \beta]$ and $C' \in [0, \infty)$ we have that

$$\tag{1.3} \left(\forall i \in I : \|g_i \circ \psi_t\|_{C^2, \beta'} < C'\right) \implies \|M\|_{C^2, \beta', r} \leq Q'$$

with $Q' = Q'(n, \beta', \beta', Q, C')$.

Corollary C. Let $\iota > 0$, $\kappa \geq 0$. Then there exist $r > 0$ and $T > 0$ such that for any smooth Riemannian manifold $(M, g)$ with

$$\text{inj rad}(M) \geq \iota, \quad \|\text{Ric}\|_{L^\infty} \leq \kappa$$

there exist charts $\psi_t : B(0, r) \to M$ such that the mollified metrics $g^\epsilon$ have at any $x \in M$ sectional curvature $K_{\epsilon(t)}(x)$ in the interval (1.2) for all $t \in (0, T]$, where $C = C(n, \iota, \kappa)$. Moreover the $\|M\|_{C^2, \beta', r}$ is bounded according to (1.3) for any $\beta' \in (0, \beta]$.

Corollary D. Let $\delta \in (0, 1)$ and assume one of the conditions (1.4a) to (1.4d):

$$\tag{1.4a} \#\pi_2(M) < \infty, \quad \text{or}$$

$$\tag{1.4b} K_g \in [\delta, 1] \quad \text{and} \quad \begin{cases} \dim M \text{ is even, or} \\ \delta \geq \frac{1}{4} - \epsilon', \quad \text{where } \epsilon' \approx 10^{-6} \end{cases}$$

$$\tag{1.4c} K_g \leq 1, \text{Ric}_g \geq \delta \quad \text{and} \quad \dim M = 3.$$ 

Then there exist charts $\psi_t : B(0, r) \to M$ such that the mollified metrics $g^\epsilon$ have at any $x \in M$ sectional curvature $K_{\epsilon(t)}(x)$ in the interval (1.2) for all $t \in (0, T]$, where $C = C(n, \kappa)$. Moreover the $\|M\|_{C^2, \beta', r}$ is bounded for any $\beta' \in [0, \beta]$.

As mentioned before, there are other techniques towards a regularization of riemannian metrics. In [Abr88], Abresch constructs a smoothing operator $S_\epsilon$, which satisfies $\|\nabla^m R_{\epsilon, g}\| \leq C \frac{1}{(1 + \epsilon \Lambda)^{m+2}}$, where $C = C(n)$ is a constant and $\Lambda$ is a bound on the sectional curvature. In particular, $S_\epsilon$ preserves isometries of the original metric. Similarly, it is known that given a bound on the sectional curvature, the application of Ricci flow amounts to a regularization of the metric, which gives $\|\nabla^m R_{\epsilon(t)}\| \leq C(n, m, t)$ (cf. [BMOR84, Shi89, Ron96, Kap05]). In contrast to our results, this again
preserves the isometries of the original metric, but just as Abresch’s result requires stronger bounds on the curvature. The crucial point is that both methods do not provide a bound on the difference \( \| R_g - R_{g(t)} \| \) or \( \| R_g - R_{g_0} \| \).

We are motivated by the viewpoint of moduli spaces of riemannian metrics with pinched curvature and regard Theorem B as a result on a controlled perturbation within such a space.

The paper is structured as follows: In § 2 we will recall mollification, chart norms, and known uniform bounds on the regularity of the metric tensor under geometric conditions. The subsequent § 3 gives a proof of the main technical tool, Theorem A, beginning with a local version of its statement. This is then used in § 4 to derive the main result, Theorem B, and its corollaries.

§ 2. Review of mollification and chart norms

We will give a short introduction to Hölder spaces and mollification. In the subsequent two subsections we will explain norm bounds for a riemannian metric that are independent of a distinguished coordinate system, and state the fundamental examples and properties of such systems.

§ 2.1 Hölder spaces

Besides \( L^p \)-classes we will use Hölder spaces of functions \( f : \mathbb{R}^n \to \mathbb{R}^N \). For \( m = 0, 1, \ldots \) define

\[
\nabla^m f : \Omega \to \mathbb{R}^{N \cdot n^m}
\]

(2.1a)

the function of all derivatives of order \( m \) and

\[
\nabla^{\leq m} f : \Omega \to \mathbb{R}^{N \cdot n^0 + \ldots + N \cdot n^m}.
\]

(2.1b)

the collection of all derivatives of order 0 to order \( m \). Further let

\[
|x_1, \ldots, x_n| := \max\{|x_1|, \ldots, |x_n|\}
\]

for \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) (which is in contrast to the euclidean norm \(|0 \cdot x|\)). Recall that the Hölder norm—for \( m = 0, 1, \ldots, \alpha \in [0, 1] \), and a domain \( \Omega \subset \mathbb{R}^n \)—is given by

\[
\|f\|_{C^{m,\alpha}} = \|f\|_{C^{m,\alpha}(\Omega)} := \|f\|_{C^m} + \sum_{i=0}^{m} \|\nabla^i f\|_{\alpha}
\]

(2.2a)

where \( \|f\|_{\alpha} = 0 \) in case \( \alpha = 0 \) and otherwise

\[
\|f\|_{\alpha} := \sup_{\substack{x, y \in \Omega \\ x \neq y \\ x, y \in \Omega}} \frac{|f(x) - f(y)|}{|x - y|^\alpha}.
\]

(2.2c)

Denote by \( C^{m,\alpha}(\Omega) \) the corresponding spaces of functions on a domain \( \Omega \subset \mathbb{R}^n \). If \( \Omega \) is a bounded open set with Lipschitz boundary, Hölder spaces
are connected to Sobolev spaces by Sobolev’s inequality which states for \( k - n/p = r + \alpha, \, p \in (n, \infty] \) and \( r < k \) that
\[
\|f\|_{C^{r,\alpha}} \leq C \|f\|_{W^{k,p}}.
\]
with \( C = C(n, p) \). From \( \|\cdot\|_\alpha \leq \|\cdot\|_1 \) and the mean value theorem we get the elementary estimate
\[
\|f\|_{C^{m,\alpha}} \leq \|f\|_{C^{m+1}}.
\]

§ 2.2 Mollification

The tool for regularization will be mollification, i.e. convolution with a smooth function. Convolution can be defined for any compactly supported function \( f : \mathbb{R}^n \to \mathbb{R} \) and any locally integrable function \( g : \mathbb{R}^n \to \mathbb{R} \) via
\[
f \ast g(x) := \int f(x-h)g(h)dh.
\]
If one of the functions \( f, g \) is multi-valued, i.e. \( \mathbb{R}^n \to \mathbb{R}^N \), the convolution is defined component-wise. It is elementary to see that if in addition \( f \in C^m \), then \( f \ast g \in C^k \) for \( m = 0, 1, \ldots \) [Hör83, Thm. 1.3.1]. Moreover in this case
\[
\partial^I (f \ast g) = (\partial^I f) \ast g
\]
where \( I \) is a multi-index of order \( m \). The key classical tools will be, for \( 1 = \frac{1}{p} + \frac{1}{q} \) and \( p, q \in [1, \infty] \), Hölder’s inequality
\[
\|fg\|_{L^1} \leq \|f\|_{L^p} \|g\|_{L^q},
\]
Young’s convolution inequality
\[
\|f \ast g\|_{L^\infty} \leq \|f\|_{L^p} \|g\|_{L^q}.
\]

For the definition of mollification fix as the mollification kernel a smooth function \( \varphi : \mathbb{R}^n \to [0, 1] \) supported on \([-1, 1]^n\) with \( \varphi(0) = 1 \). Set for \( t > 0 \)
\[
\varphi_t(x) := t^{-n}\varphi(x/t).
\]
Define the mollification operator by
\[
P_t : f \mapsto \varphi_t \ast f.
\]
As a first application of Young’s convolution inequality in conjunction with (2.5) note that
\[
\|\nabla^m P_t f\|_{L^\infty} \leq \|\nabla^m \varphi_t\|_{L^1} \|f\|_{L^\infty}.
\]

§ 2.3 Definition of chart Norms

Hölder classes of riemannian metrics allow to formulate celebrated regularity results in a more concise and little bit stronger fashion. To formulate these results we introduce norms on charts
\[
\psi : (B(0, r), 0) \to (M, g, p)
\]
where \((M, g, p)\) is a pointed, \(n\)-dimensional riemannian manifold, the maps under consideration are pointed maps, i.e. \(\psi(0) = p\), and \(B(p, r)\) denotes the open ball of radius \(r\) around \(p\) in the metric space to which \(p\) belongs—here \(0\) belongs to \(\mathbb{R}^n\) with euclidean distance. In contrast, we will denote a closed ball by \(B[p, r]\). We will mainly use and adapt definitions from [Pet16, 11.3.1-11.3.5].

**Definition 2.1.** For a chart \(\psi: (B(0, r), 0) \to (M, g, p)\) compatible with the smooth atlas of \(M\) we define \(\|\psi\|_{C^{m,\alpha}, r}\) the chart norm of \(\psi\) on the scale of \(r\), \(\|\psi\|_{C^{m,\alpha}, r}^{\text{harm}}\) the harmonic chart norm of \(\psi\) on the scale of \(r\), resp.) as the minimal quantity such that

\[
\|\psi\|_{C^{m,\alpha}, r} \leq Q \quad \left(\|\psi\|_{C^{m,\alpha}, r}^{\text{harm}} \leq Q, \text{ resp.}\right)
\]

whenever the following conditions (with exception of \((2.N_{\text{harm}})\)) are fulfilled

1. for the differentials we have the bounds \(|D\psi| \leq e^Q\) on \(B(0, r)\) and \(|D\psi^{-1}| \leq e^Q\) on \(\psi(B(0, r))\). Equivalently, this condition can be expressed in coordinates on \(\psi\) by

\[
(2.N_0) \quad e^{-2Q} \delta_{kl} v^k v^l \leq g_{kl} \leq e^{2Q} \delta_{kl} v^k v^l
\]

for every vector \(v \in \mathbb{R}\).

2. for the semi-norm from (2.2c) and any \(k = 0, 1, \ldots, m\)

\[
(2.N_{C^{m,\alpha}}) \quad r^{k+\alpha} \|\nabla^k g_.\|_\alpha \leq Q
\]

where \(g_. := \psi^* g\).

3. (for \(\|\psi\|_{C^{m,\alpha}}^{\text{harm}}\)) the chart \(\psi\) is harmonic meaning that each coordinate function \(x_k (k = 1, \ldots, n)\) is harmonic with respect to \(g_.\), i.e. the Laplace-Beltrami operator vanishes

\[
(2.N_{\text{harm}}) \quad \Delta g_. x_k = 0.
\]

We can directly extend this definition by

\[
(2.9a) \quad \|(M, g, p)\|_{C^{m,\alpha}, r} = \inf_{\psi: (B(0, r), 0) \to (M, g, p)} \|\psi\|_{C^{m,\alpha}},
\]

\[
(2.9b) \quad \|(M, g)\|_{C^{m,\alpha}, r} = \inf_{p \in M} \|(M, p)\|_{C^{m,\alpha}, r}
\]

(mutatis mutandis for \(\|(M, g, p)\|_{C^{m,\alpha}}^{\text{harm}, r} \) and \(\|(M, g)\|_{C^{m,\alpha}}^{\text{harm}, r}\)). In the same manner we can introduce the norm bounds on Sobolev scales

\[
\|\psi\|_{W^{m, p}, r} \leq Q, \quad (M, g, p)\|_{W^{m, p}, r} \leq Q, \quad (M, g)\|_{W^{m, p}, r} \leq Q
\]

\[
\left(\|\psi\|_{W^{m, p}, r}^{\text{harm}} \leq Q, \quad (M, g, p)\|_{W^{m, p}, r}^{\text{harm}} \leq Q, \quad (M, g)\|_{W^{m, p}, r}^{\text{harm}} \leq Q, \text{ resp.}\right)
\]

by retaining condition \((2.N_0)\) (as well as \((2.N_{\text{harm}})\) if appropriate) and replacing condition \((2.N_{C^{m,\alpha}})\) by

\[
(2.N_{W^{m, p}}) \quad r^{k-n/p} \|\nabla^k g_.\|_{L^p} \leq Q.
\]
for \( k = 0, 1, \ldots, m \). Finally, let
\[
\mathcal{M}^n(C^{m,\alpha} \leq_r Q), \quad \mathcal{M}^n(W^{m,p} \leq_r Q)
\]
\[
\left( \mathcal{M}^n(C^{m,\alpha} \leq_{r}^{\text{harm}} Q), \quad \mathcal{M}^n(W^{m,p} \leq_{r}^{\text{harm}} Q), \quad \text{resp.} \right)
\]
denote the space of isomorphism class of pointed smooth riemannian manifolds \((M, g, p)\) with
\[
\|(M, g)\|_{C^{m,\alpha}, r} \leq Q, \quad \|(M, g)\|_{W^{m,p}, r} \leq Q, \quad \text{resp.}
\]
\[
\left( \|(M, g)\|_{C^{m,\alpha}, r}^{\text{harm}} \leq Q, \quad \|(M, g)\|_{W^{m,p}, r}^{\text{harm}} \leq Q, \quad \text{resp.} \right).
\]
These spaces are endowed with the Gromov-Hausdorff topology. Note the elementary estimate [Pet16, Prop. 11.3.2 (4)]
\[
(2.10) \quad (2.N_0) \implies e^{-Q} \min\{|xy|, 2r - |0x|\} \leq |\psi(x)\psi(y)|_g \leq e^Q |xy| \leq e^Q |xy|
\]
for all \( x, y \in B(0, r) \) and \(|-|\) the euclidean norm.

Having introduced spaces with a global bound on the metric tensor in local coordinates, one may be inclined to request why we did not assume any regularity assumption on changes of coordinates. The answer is found in standard regularity theory of elliptic equations. The crucial fact can be stated as follows [GT15, Problem 6.1 (a)]: On a bounded open set \( \Omega \) let \( f : \Omega \to \mathbb{R} \) be a \( C^{m,\alpha} \)-solution of \((a^{ij}(x)\partial_i \partial_j + b^i(x)\partial_i + c(x))u = f \) (summation convention) and assume that the coefficients of \( L \) satisfy \( a^{ij}(x)\xi_i \xi_j \geq \lambda |\xi|^2 \) and \( \|\nabla^m a\|_{\alpha}, \|\nabla^m b\|_{\alpha}, \|\nabla^m c\|_{\alpha} \leq \Lambda \). If \( \Omega' \subset \Omega \) with \( \overline{\Omega'} \not\subset \Omega \), then
\[
\|u\|_{C^{m+2,\alpha}} \leq C(\|u\|_{C^{0}} + \|f\|_{C^{m,\alpha}})
\]
on \( \Omega' \) with \( C = C(n, m, \alpha, \lambda, \Lambda, |\Omega' \partial \Omega|_H) \) where \( |\Omega' \partial \Omega|_H \) denotes the Hausdorff distance. If we apply this statement to a transition function \( \psi_i^{-1} \circ \psi_j \) for two charts \( \psi_i, \psi_j \) with \( \|\psi_i\|_{C^{m,\alpha}, r}, \|\psi_j\|_{C^{m,\alpha}, r} \leq Q \) we first notice that by harmonicity \( \|\psi_i^{-1} \circ \psi_j\|_{C^{m,\alpha}} = 0 \) if \( \|\psi_j\|_{C^{m,\alpha}} = 0 \). If we restrict to the domain \( \Omega' := \psi_i^{-1}(\psi_i(B(0, r/2)) \cap \psi_j(B(0, r/2))) \) the above estimate becomes
\[
(2.11) \quad \|\psi_i^{-1} \circ \psi_j\|_{C^{m+2,\alpha}} \leq C\|\psi_i^{-1} \circ \psi_j\|_{C^{0}} \leq C \cdot r/2
\]
where \( C = C(n, m, \alpha, Q, r) \). For non-harmonic chart norms a similar result with one lower degree of regularity can be found in [Tay06].

\section{2.4 Precompactness theorems for riemannian manifolds}

We review some fundamental results. For every \( n \geq 2 \) and \( s, Q > 0 \)
\[
(2.K_{C^{m,\alpha}}) \quad \mathcal{M}^n(C^{m,\alpha} \leq_r Q) \text{ is compact}
\]
with respect to the Gromov-Hausdorff topology. This statement is sometimes called Fundamental theorem of convergence theorem [Pet16, 11.3.5].
Let $t > 0$, $\alpha \in (0,1)$, $p \in (n, \infty)$, and $\underline{\kappa} \in \mathbb{R}$. For all $Q > 0$ there is $r > 0$ such that every pointed riemannian manifold $(M,g,p)$ satisfies

\[(2.K_{\text{Ric}}) \quad \text{injectivity radius}(M) \geq t \text{ and Ric} \geq -\underline{\kappa}^2 \implies (M,g,p) \in \mathcal{M}^n(C^\alpha) \leq_{\text{harm}} Q) \cap \mathcal{M}^n(W^{1,p} \leq_{\text{harm}} Q); \]

see [AC92]. Likewise, we obtain a result that is stronger by one derivative for an absolute Ricci bound: For every $\kappa \geq 0$, $\iota > 0$, $\alpha \in (0,1)$, $p \in (n, \infty)$, and $Q \in (0, \log(2))$ there is an $r > 0$ such that any smooth manifold $(M,g,p)$

\[(2.K_{|\text{Ric}|}) \quad \text{injectivity radius}(M) \geq t \text{ and } \|\text{Ric}\|_{L^\infty} \leq \kappa \implies (M,g,p) \in \mathcal{M}^n(C^{1,\alpha}) \leq_{r} Q) \cap \mathcal{M}^n(W^{2,p} \leq_{r} Q); \]

the result appeared several times, e.g. [And90] and [HH97, Theorem 11], and found its ways into textbooks [Pet16, 11.4.1].

Note that the Hölder parts of $(2.K_{\text{Ric}})$ and $(2.K_{|\text{Ric}|})$ is implied by Sobolev’s inequality (2.3a).

Remark 2.2. Actually, these regularity results can be improved in at least three ways that we don’t use but should have some attention:

(1) It is possible to assume that the charts can not only be chosen with the respective bound on a norm but in such a way that the coordinate functions are harmonic. Actually, this is even crucial for the corresponding proofs.

(2) The exponent $\alpha$ can be improved from $\alpha \in (0,1)$ to 1 by replacing the Hölder scales with Hölder-Zygmund scales $C^s$, $s > 0$. The space $C^s_\ast(\Omega)$ coincides with $C^{m,\alpha}$ if $s = m + \alpha$ and $\alpha \neq 0,1$. Otherwise $C^m \subset C^s(\omega)$. See [Tay07] for the case of lower Ricci bounds and [AKK+04] for the case of absolute Ricci bounds.

(3) The norm bound in case of an absolute bound on Ricci curvature can be generalized to the case of manifolds with boundary [AKK+04].

§ 3. Mollified riemannian curvature tensor

In this section we begin by proving a local version of Theorem A. Recall that $g_\ast = \psi^* g$ and $P_t(g_\ast) = \varphi_t \ast g_\ast$. The results will be formulated using the Sobolev chart norm $\|\cdot\|_{W^{2,p,r}}$ from $(2.N_{W^m,p})$.

**Proposition 3.1.** Given $p > 2n$ and $r, Q > 0$. Choose $\beta \in (1/2, 1 - n/p)$. Then there is some $T \in (0, r/2)$ such that for a pointed smooth riemannian manifold $(M,g,p)$, a chart $\psi: (B(0,r),0) \to (M,p)$, and any section $\vec{v} \in T^3B(0,r)$ with euclidean norm not greater than 1 we have

$$\|\psi\|_{W^{2,p,r}} \leq Q \implies \text{R}P_t(g_\ast)(\psi(x))(\vec{v}) - \sup_{y \in B(x,t)} \text{R}g_\ast(y)(\vec{v}) \leq Ct^\beta$$

for any $t \in (0,T]$, $x \in B(0,r/2)$, and with $C = C(n,p,r,Q,\beta,\|\vec{v}\|_{L^\infty})$ considering $\vec{v}$ as a real valued function by the canonical euclidean identification.
Of more interest will be the following generalization of the above proposition to a convex combination of mollified metrics. Recall that for \( v \in T^* \) the norm \( \|v\|_g \) is defined as \( \|(w \mapsto \langle w, - \rangle)^{-1}v\|_g \).

**Definition 3.2.** Let \( r > 0 \) and \( M \) be a smooth manifold which is covered by a locally finite family of charts \( \{\psi_i\}_I \) with a corresponding partition of unity \( \{g_i\}_I \), i.e.

\[
(3.1a) \quad \{\psi_i : B(0, r) \to M\}_{i \in I}, \quad \{g_i : M \to [0, 1]\}_{i \in I}
\]

such that for each \( i \in I \) the function \( g_i \) is smooth and

\[
(3.1b) \quad \text{supp } g_i \subset \psi_i(B(0, r)), \quad \sum_{j \in I} g_j \bigg|_{\bigcup_{j \in I} \psi_j(B(0, r/2))} = 1.
\]

Define for \( t \in (0, r/2) \) the mollified metrics

\[
(3.1c) \quad g^{[t]} := P_{t,\{\psi_i\}_I,\{g_i\}_I}(g) := \sum_{i \in I} g_i g^{[t,\psi_i]}
\]

where

\[
(3.1d) \quad g^{[t,\psi]} := (\psi_i)^{-1} P_t(\psi_i^* g).
\]

**Theorem A.** Given \( p > 2n \) and \( r, Q > 0 \). Choose \( \beta \in (0, 1 - 2n/p) \). Then there is some \( T \in (0, r/2) \) such that for any pointed smooth riemannian manifold \( (M, g, p) \) and any finite collection of charts with a corresponding partition of unity

\[
\{\psi_1 : (B(0, r), 0) \to (M, p_1)\}_{i \in I}, \quad \bar{g} := \{g_i : M \to [0, 1]\}_{i \in I}
\]

we have that for any section \( \bar{v} \in T^{1,1} \psi(B(0, r)) \) with \( g \)-norm not greater than 1 that

\[
\left( \forall i \in I : \|\psi_i\|_{W^{2,p},r}^{\text{harm}} \leq Q \right) \implies R_{\bar{g}}(x)(\bar{v}) - \sup_{B(x, r^2 T)} R_{\bar{g}}(\bar{v}) \leq C t^\beta
\]

for any \( t \in (0, T) \), \( x \in \bigcap_{i \in I} \psi_i(B(0, r/2)) \) and with

\[
C_g = C_{\bar{g}} \left(n, p, r, Q, \beta, \{\|g_i \circ \psi_i\|_{C^2}\}_{i \in I}, \|\bar{v}\|_g, \|I\| \right).
\]

Throughout the proof, whose ansatz was pointed out in (1.1), we will consider a metric tensor \( g_\cdot = \psi^* g \) with

\[
(3.2a) \quad \|\psi\|_{W^{2,p},r}, \|\psi\|_{C^{1,\alpha},r} \leq Q
\]

such that

\[
(3.2b) \quad \frac{1}{p} + \frac{1}{q} = 1, \quad n < p < \infty, \quad \frac{n}{p} < \alpha < 1, \quad \text{and } \beta \leq \alpha - n/p.
\]

This condition implies \( p, q \in (0, \infty), \alpha \in (0, 1) \). To guarantee this condition without loss of generality, choose \( \alpha \in (1 - 2n/p, 1 - n/p) \), apply Sobolev’s inequality (2.3a), and replace \( Q \) by \( \max\{Q, CQ\} \), where \( C \) is the constant from Sobolev’s inequality.
Lemma 3.3. In local coordinates the Riemannian curvature tensor $R$ of a smooth manifold $(M, g)$ can be written using notation (2.1b) as

$$R = A(g^\cdot, \nabla^2 g^\cdot) + B(\nabla g^\cdot, \nabla^1 g^\cdot)$$

where

- $A$ is bi-linear;
- $B$ is a polynomial of degree not bigger than 4 and without constant terms; and
- the coefficients of $A$ and $B$ depend only on the dimension.

Proof. The standard coordinate definition the riemannian curvature tensor has the form [Tay13, Section C3]

$$\Gamma^i_{kl} = \frac{1}{2} g^{im} (g_{mk,l} + g_{ml,k} - g_{kl,m})$$

$$R^\sigma_{\sigma \mu \nu} = \partial_\mu \Gamma^\sigma_{\nu \sigma} - \partial_\nu \Gamma^\sigma_{\mu \sigma} + \Gamma^\sigma_{\mu \lambda} \Gamma^\lambda_{\nu \sigma} - \Gamma^\sigma_{\nu \lambda} \Gamma^\lambda_{\mu \sigma}$$

where $g^{kl}(x)$ denotes the inverse of the matrix $g_{kl}(x)$, $\partial_k = \frac{\partial}{\partial x_k}$, and $f_{i,j} = \partial_j f_i$. From this we read that

$$R = L_2(g^\cdot, \nabla^2 g^\cdot) + L_1(\nabla g^\cdot, g^\cdot) + Q_1(g^\cdot, \nabla g^\cdot)$$

where

- $L_2$ is linear in $\nabla^2 g_{kl}$ and $g^{kl}$,
- $L_1$ is linear in $\nabla g^{kl}$ as well as in $\nabla g_{kl}$,
- $Q_1$ is quadratic in both $g^{kl}$ and $\nabla g_{kl}$.

Obviously, choose $A = L_2$ and $B = L_1 + Q_1$. \hfill \Box

Lemma 3.4. Let $f, g: \Omega \to \mathbb{R}^N$ and $p \in \mathbb{R}[X_1, \ldots, X_N]$ a polynomial without constant term. Then

$$\|p(f) - p(g)\|_{L^\infty} \leq C \|f - g\|_{L^\infty}$$

where $C = C(p, \|f\|_{C^n}, \|g\|_{C^n})$.

Proof. By the triangle inequality it is sufficient to prove the claim for a monomial $p = X_{i_1} \cdots X_{i_d}$. Then the claim follows from a telescope argument

$$\|f_{i_1} \cdots f_{i_d} - g_{i_1} \cdots g_{i_d}\|_{L^\infty} \leq \|\left(f_{i_1} - g_{i_1}\right) f_{i_2} \cdots f_{i_d}\|_{L^\infty} + \|g_{i_1} (f_{i_2} - g_{i_2}) \cdots f_{i_d}\|_{L^\infty} + \ldots + \|g_{i_1} \cdots (f_{i_d} - g_{i_d})\|_{L^\infty}$$

since each term $g_{i_1} \cdots (f_{i_k} - g_{i_k}) \cdots f_{i_d}$ is $L^\infty$-bounded by $\|f\|_{L^\infty}^{k-1} \cdot \|g\|_{L^\infty}^{n-k-1}$. $\|f - g\|_{L^\infty}$ (for $k = 1, \ldots, d$). \hfill \Box
We now turn to the crucial $L^p$-estimate. It is a consequence of Hölder’s inequality (2.6) for indices 1 and ∞ stating that for any $q \in (1, \infty)$

$$
\|\varphi_t\|_{L^q} = \|t^{-nq}\chi_{[-t,t]}(\varphi(h/t))^q\|_{L^1}^{1/q} \\
\leq \|t^{-nq}\chi_{[-t,t]}\|_{L^1}^{1/q}\|\varphi(h/t)^q\|_{L^\infty}^{1/q} \\
= t^{-n}\|\chi_{[-t,t]}\|_{L^p}^{1/p}\|\varphi(h/t)^q\|_{L^\infty}^{1/q} \\
= (2^n t^{-nq+n})^{1/q} \cdot 1^{1/q} \\
= 2^{n/q} t^{-n/p}.
$$

(3.3)

Recall the definitions of the commutator of two operators $P, Q$

$$
[P, Q]: f \mapsto PQ - QP
$$

and the multiplication operator

$$
T_g: f \mapsto gf.
$$

Finally note that condition (2.N.0) implies that the eigenvalues of $g_\cdot$ are in $[e^{-2Q}, e^{2Q}]$. This implies

(3.4)

$$
det(g_\cdot) \in [e^{-2Qn}, e^{2Qn}].
$$

Lemma 3.5 (Commutator-like estimate). Let $n, p, q, \alpha$ satisfy (3.2b) and $\Omega \subset \mathbb{R}^n$ an open domain. Let $f \in L^p(\mathbb{R}^n)$ and $a(\cdot): \Omega \times [0, T] \to \mathbb{R}$ such that for some constant $C_a$ we have

$$
\|a_t\|_\alpha \leq C_a \quad \text{and} \quad \|a_t - a_0\|_{C^0} \leq C_a t^\alpha
$$

for all $t \in [0, T]$. Then

$$
\|P T_{a_0}(f) - T_{a_t} P_t(f)\|_{L^\infty} \leq 2^{(n+1)/q} C_a \|f\|_{L^p} t^{\alpha-n/p}.
$$

Proof. First note that by Young’s convolution inequality (2.7) and (3.3)

$$
\|[P_t, T_g]f\|_{L^\infty} = \text{ess sup}_x \left| \int (g(x-h) - g(x))f(x-h)\varphi_t(h)dh \right| \\
\leq \text{ess sup}_x \int |g(x-h) - g(x)||f(x-h)|\varphi_t(h)dh \\
\leq \text{ess sup}_x \|g\|_\alpha |h|\alpha |f(x-h)|\varphi_t(h)dh \\
\leq \|g\|_\alpha t^{\alpha} \|P_t(|f|)\|_{L^\infty} \\
= \|g\|_\alpha t^{\alpha} \|f\|_{L^p} \|\varphi_t\|_{L^q} \\
\leq 2^{n/q} t^{\alpha-n/p} \|g\|_\alpha \|f\|_{L^p}
$$

(3.5)
We use this estimate and (2.7) and (3.3) again to prove the Lemma:
\[
\|P_t T_{a_0}(f) - T_{a_0} P_t(f)\|_{L^\infty} \\
\leq \|P_t T_{a_0}(f) - P_t T_{a_0}(f)\|_{L^\infty} + \|P_t T_{a_0}(f) - P_t T_{a_0}(f)\|_{L^p} \\
= \|P_t T_{a_0}(f)\|_{L^\infty} + \|P_t T_{a_0}(f)\|_{L^p} \\
\leq \|P_t T_{a_0}(f)\|_{L^\infty} + \|P_t T_{a_0}(f)\|_{L^p} \\
\leq 2^{n/q} t^{-n/p} |a_t|_\alpha \|f\|_{L^p} + 2^{n/q} t^{-n/p} |a_t|_\alpha \|f\|_{L^p} \\
\leq C_a t^\alpha \|f\|_{L^p} 2^n t^{-n/p} + C_a t^\alpha \|f\|_{L^p} 2^n t^{-n/p} \\
= 2^{(n+1)/q} C_a \|f\|_{L^p} t^{-n/p}. \quad \square
\]

**Lemma 3.6.** Let \( r > 0, m = 0, 1, \ldots, \) and \( \alpha \in (0, 1]. \) Given a function \( f \in C^{m,\alpha}(B(0, r), \mathbb{R}) \) we have
\[
\|f - P_t f\|_{C^m} \leq t^\alpha \|f\|_{C^{m,\alpha}}
\]
on \( B(0, r/2) \) for \( t \in (0, r/2). \)

**Proof.** Let \( k = 0, \ldots, m. \) The lemma follows from
\[
\|\nabla^k (P_t(f) - f)\|_{C^0} = \|P_t(\nabla^k f) - \nabla^k f\|_{C^0} \\
= \sup_{x \in B(0, r/2)} \left| P_t(\nabla^k f) - \int \varphi_t(h) \nabla^k f(x) dh \right| \\
= \sup_{x \in B(0, r/2)} \left| \int \varphi_t(h)(\nabla^k f(x - h) - \nabla^k f(x)) dh \right| \\
\leq \sup_{x \in B(0, r/2)} \left| \int \varphi_t(h) ||h||_\alpha \|\nabla^k f\|_\alpha dh \right| \\
\leq \|\nabla^k f\|_\alpha \int |\varphi_t(h)| t^\alpha dh \\
= t^\alpha \|\nabla^k f\|_\alpha. \quad \square
\]

**Proof of Proposition 3.1.** As discussed above at (3.2a) we can assume without loss of generality \( \|\psi\|_{W^{2,p}, r}, \|\psi\|_{C^{1,\alpha}, r} \leq Q \) with properties (3.2b). By assumption (2.4C_{m,\alpha}) and Lemma 3.6 this implies
\[
(3.6a) \quad \|P_t(g) - g\|_{C^1} \leq C_n t^\alpha r^{-\alpha} Q
\]
for \( t \in (0, r/2) \) and for a dimension-depending constant \( C_n. \) This in conjunction with the bounds (3.4) implies
\[
(3.6b) \quad \det(g), \det(P_t(g)) \in [e^{-2Q^{n-1}}, e^{2Q^{n+1}}].
\]
for sufficiently small \( T. \) This implies that
\[
(3.6c) \quad \|P_t(g)^{-1}\|_{C^{1,\alpha}}, \|g^{-1}\|_{C^{1,\alpha}} \leq C_{Q, r}
\]
for a constant \( C_{Q, r} \) [CDK11, Corollary 16.30]. Let \( M_{kl} \) denote the \((k,l)-\) minor of \( g, \) \( M'_{kl} \) the \((k,l)-\) minor of \( P_t(g), \) and \( M, M' \), the corresponding
matrices of minors. Lastly, we need an estimate on the difference of the inverse of the mollified metric:

\[ \|(P_t(g_{\cdot\cdot\cdot}))-^{-1} - g_{\cdot\cdot\cdot}^{-1}\|_{C^1} \]

\[ \leq \left\| \frac{1}{\det P_t(g_{\cdot\cdot\cdot})} \frac{M_t}{\det(g_{\cdot\cdot\cdot})} - \frac{1}{\det(g_{\cdot\cdot\cdot})} \frac{M_t}{\det(g_{\cdot\cdot\cdot})} \right\|_{C^1} \]

proceeding by using (3.6b) twice—directly and by the standard estimate

\[ \left\| \frac{1}{\det P_t(g_{\cdot\cdot\cdot})} - \frac{1}{\det(g_{\cdot\cdot\cdot})} \right\|_{C^1} + \left\| \frac{1}{\det(g_{\cdot\cdot\cdot})} \frac{M_t}{\det(g_{\cdot\cdot\cdot})} - M_{\cdot\cdot\cdot} \right\|_{C^1} \]

\[ \leq \left( \frac{1}{\det P_t(g_{\cdot\cdot\cdot})} - \frac{1}{\det(g_{\cdot\cdot\cdot})} \right) \left( \frac{e^{-2Qn+1}}{2} \right) + \left\| \frac{1}{\det(g_{\cdot\cdot\cdot})} \frac{M_t}{\det(g_{\cdot\cdot\cdot})} - M_{\cdot\cdot\cdot} \right\|_{C^1} \]

\[ \leq e^{4Qn+2} C_n t^\alpha r_\alpha Q + e^{2Qn+1} C_\alpha t^\alpha r_\alpha Q \]

(3.6d)

\[ \leq C_n,r,\alpha,Q t^\alpha. \]

where in the penultimate step we used (3.6a) and standard product estimates [CDK11, Theorem 16.28] providing suitable constants \( C_n \) and \( C_\alpha \).

We finally can do the ansatz proposed in (1.1):

\[ R_{P_t(g_{\cdot\cdot\cdot})}(x)(\tilde{v}) \leq P_t(R_{\cdot\cdot\cdot}(\tilde{v}))(x) + P_t(R_{\cdot\cdot\cdot})(x)(\tilde{v}) - R_{P_t(g_{\cdot\cdot\cdot})}(x)(\tilde{v}) \]

\[ \leq \sup_{y \in B(x,t)} R_{g_{\cdot\cdot\cdot}}(y)(\tilde{v}) + \|P_t(R_{\cdot\cdot\cdot}) - R_{P_t(g_{\cdot\cdot\cdot})}\|_{L^\infty} \]

(3.7)

It remains to show that \( \|P_t(R_{\cdot\cdot\cdot})(\tilde{v}) - R_{P_t(g_{\cdot\cdot\cdot})}(\tilde{v})\|_{L^\infty} \) vanishes with modulus \( t^\alpha \) as \( t \to 0 \). Set \( b_{\tilde{v}} := B(\nabla g_{\cdot\cdot\cdot}, \nabla \leq 1 g_{\cdot\cdot\cdot}) \) and \( b_{\tilde{v},t} := B(\nabla P_t g_{\cdot\cdot\cdot}, \nabla \leq 1 (P_t g_{\cdot\cdot\cdot})^{-1}) \).

Observe

\[ \|P_t(R_{\cdot\cdot\cdot}) - R_{P_t(g_{\cdot\cdot\cdot})}\|_{L^\infty} \]

\[ \leq \|P_t A(\nabla g_{\cdot\cdot\cdot}, \nabla^2 g_{\cdot\cdot\cdot}) - A((P_t g_{\cdot\cdot\cdot})^{-1}, \nabla^2 P_t g_{\cdot\cdot\cdot})\|_{L^\infty} + \|P_t(b_{\tilde{v}}) - b_{\tilde{v},t}\|_{L^\infty} \]

\[ \leq \|P_t A(\nabla g_{\cdot\cdot\cdot}, \nabla^2 g_{\cdot\cdot\cdot}) - A((P_t g_{\cdot\cdot\cdot})^{-1}, P_t \nabla g_{\cdot\cdot\cdot})\|_{L^\infty} \]

\[ + \|P_t(b_{\tilde{v}}) - b_{\tilde{v}}\|_{L^\infty} + \|b_{\tilde{v}} - b_{\tilde{v},t}\|_{L^\infty} \]

We will give estimates for each summand:

- For the first summand express \( A \) as \( A(x,y) = \sum_{i,j} a_{ij} x_i y_j \). By linearity of convolution \( P_t(A(x,y)) = \sum_{i,j} a_{ij} P_t T_x(y_j) \). On the other hand \( A(x,P_ty) = \sum_{i,j} a_{ij} T_x P_t(y_j) \). This is to say that we are in situation of Lemma 3.5 providing the desired estimate.

- For the second summand Lemma 3.6 gives the desired estimate.

- In case of the third summand Lemma 3.4 is applicable due to (3.6c) and (2.3\( C_{\alpha,\cdot} \)) providing the estimate

\[ \|b_{\tilde{v}} - b_{\tilde{v},t}\|_{L^\infty} \leq C' \left\| (\nabla g_{\cdot\cdot\cdot}, \nabla \leq 1 g_{\cdot\cdot\cdot}) - (\nabla P_t g_{\cdot\cdot\cdot}, \nabla \leq 1 (P_t g_{\cdot\cdot\cdot})^{-1}) \right\|_{L^\infty} \]

where \( C' \) is provided in Lemma 3.4. The estimate we seek follows now from (3.6a) and (3.6d).
Thus (3.7) becomes $R_{P(\varphi)}(x)(\vartheta) \leq \sup_{y \in B(x,t)} R_{g \cdot}(y)(\vartheta) + Ct^\beta$ for a constant $C$ and $\beta \leq \alpha - n/p$ as in the claim of the theorem. \hfill \Box

Proof of Theorem A. For the proof we examine the metric tensor on some chart $\psi$ with $\|\psi\|_{\text{harm}} < Q$, e.g. $\psi = \psi_i$ for one $i \in I$. For any metric tensor $\tilde{g}$ on $M$ representation with respect to $\psi\text{-coordinates}$ is indicated by $\tilde{g}\cdot$ or $\tilde{g}^{-1}$ for the inverse, e.g. $g^{[t]} = \left(g^{[t]}\right)^{-1}$. By abuse of notation we write $\rho \circ \psi = \varphi_i$.

We define

\begin{equation}
(3.8a) \quad g^{\Delta_{t,\psi_i}} := g - g^{[t,\psi_i]}
\end{equation}

Further we agree on the shorthands

\begin{align}
(3.8b) & \quad A_{g}(x,y) := A(x,y)(\vartheta) \\
(3.8c) & \quad b_{g}[t] := B(\nabla g^{[t]}, \nabla g^{[t]})(\vartheta) \\
(3.8d) & \quad b_{g}^{[t,\psi_i]} := B(\nabla g^{[t,\psi_i]}, \nabla g^{[t,\psi_i]})(\vartheta).
\end{align}

and finally

\begin{equation}
(3.8e) \quad C_{g} := \sum_{i \in I} \|\varphi_i \circ \psi_i\|_{C^2}.
\end{equation}

Observe that by (2.11) the estimates (3.6c) and (3.6d) imply

\begin{align}
(3.9a) & \quad \|g^{[t,\psi_i]}\|_{C^{1,\alpha}} \leq C_{n,Q,r,\alpha}, \quad \|g^{[t]}\|_{C^{1,\alpha}} \leq C_{n,Q,r,\alpha,\varphi} \\
(3.9b) & \quad \|g^{\Delta_{t,\psi_i}}\|_{C^{1}} \leq C_{n,r,\alpha,Q^\alpha}, \quad \|g^{\Delta}\|_{C^{1}} \leq C_{n,r,\alpha,Q,C_{\varphi}^\alpha}
\end{align}

for respective constants. Moreover by (2.11) we have the estimate

\begin{equation}
(3.9c) \quad \|\nabla^2 g^{[t,\psi_i]}\|_{L^\infty} = \left\|\nabla^2 (\psi_i^{-1} \circ \psi)^* P_t(\psi_i^* g)\right\|_{L^\infty} \\
\leq C_n \|\psi_i^{-1} \circ \psi\|_{C^2}^2 \|\nabla^2 P_t(\psi_i^* g)\|_{L^\infty} \\
\leq C_n \|\psi_i^{-1} \circ \psi\|_{C^2}^2 \|P_t(\nabla^2 \psi_i^* g)\|_{L^\infty} \\
\leq C_{n,a,Q,r} \|\psi_i^*\|_{L^p} \|\nabla^2 \psi_i^* g\|_{L^p} \\
\leq C_{n,a,Q,r} t^{\alpha - n/p}
\end{equation}

where we used the Sobolev’s inequality (2.3a) in the penultimate step and estimate (3.3) in the last step.
Before we start with ansatz (1.1) we decompose $R_{g^{[t]}}$ into a convex combination of functions. To this end observe

$$R_{g^{[t]}}(\vec{v}) = A_{\vec{v}} \left( g^{[t] \cdot \cdot}, \nabla^2 g^{[t \cdot \cdot]} \right) + b_{\vec{v}}^{[t]}$$

$$= A_{\vec{v}} \left( g^{[t] \cdot \cdot}, \nabla^2 \sum_{i \in I} \partial_i g^{t, \psi_i} \right) + b_{\vec{v}}^{[t]}$$

$$= A_{\vec{v}} \left( g^{[t] \cdot \cdot}, \nabla^2 \sum_{i \in I} \partial_i g - \sum_{i \in I} \nabla^2 \partial_i g^{\Delta t, \psi_i} \right) + b_{\vec{v}}^{[t]}$$

$$= \sum_{i \in I} A_{\vec{v}} \left( g^{[t] \cdot \cdot}, \partial_i \nabla^2 g - \nabla^2 \partial_i g^{\Delta t, \psi_i} \right) + b_{\vec{v}}^{[t]}$$

$$= \sum_{i \in I} A_{\vec{v}} \left( g^{[t] \cdot \cdot}, \partial_i \nabla^2 \left( g^{t, \psi_i} + g^{\Delta t, \psi_i} \right) + \partial_i \nabla^2 g^{\Delta t, \psi_i} \right) + b_{\vec{v}}^{[t]}$$

Thus for a dimension-depending constant $C_n$ and using definition (2.2a)

$$\left\| R_{g^{[t]}}(\vec{v}) - \sum_{i \in I} A_{\vec{v}} \left( g^{[t] \cdot \cdot}, \partial_i \nabla^2 g^{t, \psi_i} \right) - b_{\vec{v}}^{[t]} \right\|_{L^{\infty}}$$

$$\leq C_n \sum_{i \in I} \left\| \partial_i \nabla^2 g^{\Delta t, \psi_i} - \partial_i \nabla^2 g^{\Delta t, \psi_i} \right\|_{L^{\infty}}$$

$$\leq C_n \sum_{i \in I} \left\| \partial_i \nabla^2 g^{t, \psi_i} - \partial_i \nabla^2 g^{\Delta t, \psi_i} \right\|_{L^{\infty}}$$

$$\leq C_{n, Q, r} \sum_{i \in I} \left( e^{4Q} \right) \left\| \partial_i \nabla^2 g^{t, \psi_i} - \partial_i \nabla^2 g^{\Delta t, \psi_i} \right\|_{L^{\infty}}$$

$$\leq C_{n, Q, r} \sum_{i \in I} \left( e^{4Q} \right) \left\| g^{t, \psi_i} \right\|_{L^{\infty}} + 2 \left\| \nabla \partial_i \nabla^2 g^{t, \psi_i} \right\|_{L^{\infty}}$$

$$\leq 2C_{n, Q, r} e^{4Q} C_i \sum_{i \in I} \left\| g^{\Delta t, \psi_i} \right\|_{W^{1, \infty}}$$

where we used (3.9a) and axiom (2.7) in the antepenultimate step, and definition (3.8e) in the last step. As $g^{\Delta t, \psi_i} = \psi_i^*(\psi_i^{-1})^*(\psi_i^* g)^{\Delta t} = (\psi_i \psi_i^{-1})^*(\psi_i^* g)^{\Delta t}$, combining estimate (2.11) and (3.6a) gives

$$\left\| R_{g^{[t]}}(\vec{v}) - \sum_{i \in I} A_{\vec{v}} \left( g^{[t] \cdot \cdot}, \partial_i \nabla^2 g^{t, \psi_i} \right) - b_{\vec{v}}^{[t]} \right\|_{L^{\infty}} \leq C' t^\kappa$$
for a constant $C' = C'(n, Q, r, \{g_i \circ \psi_i\}_{i \in I}, \alpha)$. Finally this gives

$$R_{g^{(t)}}(\vec{v})(x) \leq \sum_{i \in I} A_{\vec{v}} \left( g^{[t]}, \partial \partial g^{[t,\psi]} \right)(x) + b^{[t]}(x) + C't^\alpha$$

$$\leq \sum_{i \in I} g_i \left( A_{\vec{v}} \left( g^{[t]}, \partial \partial g^{[t,\psi]} \right) + b^{[t]} \right)(x) + C't^\alpha$$

(3.10) $$\leq \sup_{|x| < r/2} \left( A_{\vec{v}} \left( g^{[t]}, \partial \partial g^{[t,\psi]} \right) + b^{[t]} \right)(x) + C't^\alpha.$$

This means that it is sufficient to find an estimate

$$\left( A_{\vec{v}} \left( g^{[t]}; \partial \partial g^{[t,\psi]} \right) + b^{[t]} \right)(x) - R_{g}(\vec{v})(x) \leq C''t^\alpha$$

for every $i \in I$.

We are now in the position to do an estimate similar to the one from the proof of Proposition 3.1:

(3.11) $$\left( A_{\vec{v}} \left( g^{[t]}; \partial \partial g^{[t,\psi]} \right) + b^{[t]} \right)(x)$$

$$\leq \sup_{|x| < r/2} \left( A_{\vec{v}} \left( g^{[t]}; \partial \partial g^{[t,\psi]} \right) + b^{[t]} \right)(x)$$

$$\leq R_{g^{[t,\psi]}}(x)(\vec{v}) + \left\| A_{\vec{v}} \left( g^{[t]} - g^{[t,\psi]} \right); \partial \partial g^{[t,\psi]} \right\|_{L^\infty}$$

$$+ \left\| b^{[t]} - b^{[t,\psi]} \right\|_{L^\infty}.$$

The proof is reduced to seeking bounds for the second and third summand.

With regard to the second summand we have for a constant $C_n$ and using definition (2.2a) the estimate

$$\left\| A_{\vec{v}} \left( g^{[t]} - g^{[t,\psi]} \right); \partial \partial g^{[t,\psi]} \right\|_{L^\infty}$$

$$\leq C_n \left\| (\vec{v})^4 \right\|_{L^\infty} \left\| g^{[t]} - g^{[t,\psi]} \right\|_{L^\infty} \left\| \partial \partial g^{[t,\psi]} \right\|_{L^\infty}$$

$$\leq C_n \left( e^{-2Q} + C_{n,r,\alpha,Q} t^\alpha + C_{n,r,\alpha,Q} t^\alpha \right) C_{n,\alpha,Q} t^\alpha t^{n/p}$$

where we used axiom (2.\textit{N}_0) in the penultimate step along with (3.9a) and (3.9c) in the last step. To summarize: the second summand vanishes with modulus $t^{n-n/p}$.

As for the third summand by Lemma 3.4 we have a bound

$$\left\| b^{[t]} - b^{[t,\psi]} \right\|_{L^\infty} \leq C \left( \left\| (\vec{v})^4 \right\|_{L^\infty} \left\| g^{[t]} - g^{[t,\psi]} \right\|_{W^{1,\infty}} \left\| \partial \partial g^{[t]} \right\|_{L^\infty} \right)$$

$$\leq C \left( \left\| g^{[t]} - g^{[t,\psi]} \right\|_{W^{1,\infty}} + \left\| \partial g^{[t]} \right\|_{L^\infty} \right).$$

Note that by axiom (2.\textit{N}_0) $\| \vec{v} \|_{L^\infty} \leq e^{-2Q} \| \vec{v} \|_{g}$, hence $C$ depends only on admissible parameters.
Finally, by (3.9b) we have a bound
\[
\left\| g^{[\ell]} - g^{[t,\psi_i]} \right\|_{W^{1,\infty}} \leq \left\| g^{[\ell]} - g^- \right\|_{W^{1,\infty}} + \left\| g^- - g^{[t,\psi_i]} \right\|_{W^{1,\infty}} \\
\leq C r^\alpha g, \quad C' t^\alpha
\]
and by (2.11) and (3.6a) there is a bound
\[
\left\| \nabla g^{[\ell]} - \nabla g^{[\ell,\psi_i]} \right\|_{L^\infty} \leq \left\| \nabla g^{[\ell]} - \nabla g^- \right\|_{L^\infty} + \left\| \nabla g^- - \nabla g^{[\ell,\psi_i]} \right\|_{L^\infty} \\
\leq C r^\alpha, Q, r^\gamma t^\alpha.
\]
Thus combining (3.10) with (3.11) we obtain
\[
R_{g^{[\ell]}}(x)(\vec{v}) \leq \sup_{|x\psi_i(0)|_g < r/2} R_{g^{[\ell,\psi_i]}}(x)(\vec{v}) + C r^\alpha \\
\leq \sup_{|x\psi_i(0)|_g < r/2} \left\{ R_g(y)(\vec{v}) \mid y \in \bigcup_{i \in I} \psi_i(B(0, t)) \right\} + C t^\beta \\
\leq \sup_{y \in B(x,e^Q t)} R_g(y)(\vec{v}) + C t^\beta
\]
where in the penultimate step we used Proposition 3.1 which is applicable since $R_{g^{[\ell,\psi_i]}}(\vec{v}) = R_{g^{[\ell]}}(\vec{v})$ is coordinate independent; and in the last step the basic estimate (2.10) came to hand. □

§ 4. Consequence for sectional curvature

This section is devoted to the main result of the paper, which is again phrased in terms of Sobolev chart norms $\| \cdot \|_{W^{2,p,r}}$, see (2.11).

**Theorem B.** Let $p > 2n, r, Q > 0$, and $\beta \in (0, 1 - 2n/p)$. Then there is some $T > 0$ such that for any smooth Riemannian manifold $(M, g)$ with $\| (M, g) \|_{\text{harm} W^{2,p,r}} \leq Q$ there is a locally finite cover of charts
\[
\psi_i : (B(0, r), 0) \rightarrow (M, p_i),
\]
i \in I, such that the mollified metrics $g^{[t]}$ have at any $x \in M$ sectional curvature $K_{g^{[t]}}(x)$ in the interval
\[
(1.2) \quad \left[ K\left( g_{|B(\ell,t)} \right), K\left( g_{|B(\ell,t)} \right) \right] \cdot [1 - Ct, 1 + Ct] + [-Ct^\beta, Ct^\beta]
\]
for all $t \in (0, T]$, where $C = C(n, Q, r, p)$.

Moreover for any $\beta' \in [0, \beta]$ and $C' \in [0, \infty)$ we have that
\[
(1.3) \quad \forall i \in I : \| \partial_{\psi_i} \circ \psi_i \|_{C^{2,\beta'}} < C' \quad \Rightarrow \quad \| M \|_{C^{2,\beta',\ell}} \leq Q'
\]
with $Q' = Q'(n, \beta', Q, C')$.

A glance at (2.3) gives immediately:

**Corollary C.** Let $\iota > 0, \kappa > 0$. Then there exist $r > 0$ and $T > 0$ such that for any smooth Riemannian manifold $(M, g)$ with
\[
\text{injrad}(M) \geq \iota, \quad \| \text{Ric} M \|_{L^\infty} \leq \kappa
\]
there exist charts $\psi_i: B(0, r) \to M$ such that the mollified metrics $g^{[t]}$ have at any $x \in M$ sectional curvature $K_{P_t g}(x)$ in the interval (1.2) for all $t \in (0, T]$, where $C = C(n, t, \kappa)$. Moreover the $\|M\|_{C^{2, \beta'}}$ is bounded according to (1.3) for any $\beta' \in [0, \beta]$.

**Corollary D.** Let $\delta \in (0, 1)$ and assume one of the conditions (1.4a) to (1.4d):

(1.4a) $\#\pi_2(M) < \infty$, or

(1.4b) $K_g \in [\delta, 1]$ and $\dim M$ is even, or

(1.4c) $\delta \geq 1/4 - \varepsilon'$, where $\varepsilon' \approx 10^{-6}$

(1.4d) $K_g \leq 1, \text{Ric}_g \geq \delta$ and $\dim M = 3$.

Then there exist charts $\psi_i: B(0, r) \to M$ such that the mollified metrics $g^{[t]}$ have at any $x \in M$ sectional curvature $K_{P_t g}(x)$ in the interval (1.2) for all $t \in (0, T]$, where $C = C(n, \kappa)$. Moreover the $\|M\|_{C^{2, \beta'}}$ is bounded for any $\beta' \in [0, \beta]$.

**Proof.** Each of (1.4a) to (1.4d) implies a uniform positive lower bound on the injectivity radius $[\text{Tus00, 6}]$. □

The first step to prove Theorem B is to introduce the notion of a locally $N$-finite cover: a cover $\{U_I\}_I$ of a space $X$ is **locally $N$-finite** if every point of $x$ is contained in at most $N$ members of $\{U_I\}_I$. By help of this terminology one can reduce Theorem B to the following claim:

**Lemma 4.1.** Let $p > 2n$, $r, Q > 0$, $\beta \in (0, 1 - 2n/p)$ and $N$ be a natural number. Then there is $T' > 0$ such that for any smooth riemannian manifold $M$ with $\|(M, g)\|_{\text{harm}, W^{2, r}, r} \leq Q$ and any

$$\{\psi_i: (B(0, r), 0) \to (M, p_i)\}_{i \in I}, \quad \tilde{\beta} := \{\tilde{g}_i: M \to [0, 1]\}_{i \in I}$$

a cover of $M$ by charts and a corresponding partition of unity with

- $\|\psi_i\|_{\text{harm}, W^{2, r}, r} \leq Q$ (as defined by (2.2$W_m, r$)),
- $\{\psi_i(B(0, r)) \mid i \in I\}$ is locally $N$-finite,
- $M$ is covered by $\{\psi_i(B(0, e^{-Q}r/2))\}_{i \in I}$, and
- $\|\tilde{g}_i \circ \psi_i\|_{C^2} < C_I$ for all $i \in I$ and a constant $C_I$

the mollified metrics $g^{[t]}$ have at any $x \in M$ sectional curvature $K_{P_t g}(x)$ in the interval (1.2) for all $t \in (0, T']$, where $C = C(n, Q, r, p)$.

**Proof of Theorem B using Lemma 4.1.** By [Gro07, Prop. 5.2] every Gromov-Hausdorff compact class $\mathcal{M}$ of (isometry classes of) pointed metric spaces has the following property: for every space $(M, d, p) \in \mathcal{M}$ the maximal number of disjoint closed balls of radius $\varepsilon$ that fit into $B[p, R]$ is bounded by a finite number that depends only on $\varepsilon$ and $R$. By Sobolev’s inequality (2.3a) $\mathcal{M}(W^{2, p} \leq r, Q) \subset \mathcal{M}(C^{1, \alpha} \leq r, CQ)$ for $\alpha = n/p$ and a constant $C = C(n, p)$.
Hence by (2.K$_{C_{\text{m,}r}}$) $\mathcal{M}(W^{2,p} \leq_r Q)$ is precompact in the Gromov-Hausdorff topology and Gromov’s result is applicable.

Now we apply Gromov’s result to $\varepsilon = re^{-2Q}/5$ and $R = 2re^Q$ obtaining a bound $N$. Let $\{B[x_i, \varepsilon]\}_{i \in I}$ be some maximal disjoint system of $\varepsilon$-balls in $(M, g, p) \in \mathcal{M}(W^{2,p} \leq_r Q)$. The balls $\{B(x_i, re^{-2Q}/2)\}_{i \in I}$ cover $M$ due to maximality of the system. On the other hand for any $x \in M$ the cardinality of the set $\{i \in I \mid x \in B(x_i, re^Q)\}$ is bounded by $N$. For each point $x_i$ choose a chart $\psi_i$: $(B(0, r), 0) \to (M, x_i)$ with $\|\psi_i\|_{W^{2,p,r}} \leq Q$. By (2.10) $B(x_i, re^{-2Q}/2) \subset \psi_i(B(0, e^{-Q}r/2))$ and, hence $\{\psi_i(B(0, e^{-Q}r/2))\}_{i \in I}$ is covering. By the same estimate $\psi_i(B(0, r)) \subset B[x_i, re^Q]$ and, thus, $\{\psi_i(B(0, r))\}_{i \in I}$ is $N$-finite.

To find a suitable partition of unity, choose any bump function $b$: $\mathbb{R}^d \to \mathbb{R}$ with $\text{supp} b \subset B(0, r)$ and $\|b\|_{(0, re^{-2Q}/2)} = 1$. Set $p_i := \psi_i(0)$. We define for all $i \in I$:

\begin{align*}
(4.1a) \quad b_i(x) := \begin{cases} b \circ \psi_i^{-1}(x) & \text{if } x \in \psi_i(B(0, r)) \\ 0 & \text{else,} \end{cases} \\
(4.1b) \quad g_i(x) := \frac{1}{\sum_{j \in I} b_j(x)} b_i(x)
\end{align*}

Due to (2.11) there is a uniform $C^{3,\beta'}$-bound on the transition maps $\psi_i^{-1} \circ \psi_j$ for $i, j \in I$. Moreover the support of $b$ is compact $\|b\|_{C^{3,\beta'}}$ is bounded. Hence there is a uniform $C^{3,\beta'}$-bound on $b_i \circ \psi_j = b \circ \psi_i^{-1} \circ \psi_j$ for each $i \in I$. Since the denominator in (4.1b) is at least 1 at each $x \in M$ for at least one $i$ (namely the $i$ for which $|p_i x| \leq e^{-Q}r/2$), $g_i$ is $C^{2,\beta'}$-bounded uniformly in $i \in I$ as well. This puts us in a situation to apply Lemma 4.1.

Due to (2.8c) we have a uniform $C^{2,\beta'}$-bound on $P_T(\psi_i^*g)$ and thus a uniform $C^{3,\beta'}$-bound on the pullback metrics $(\psi_i^{-1} \circ \psi_j)^*P_T(\psi_i^*g)$ for all $i, j \in I$ as well. Together with the bound on the $g_i$’s from last paragraph this implies a uniform $C^{2,\beta'}$-bound on $\psi_i^*g^{[T]}$ for each $i \in I$. Thus the bound (1.3) holds. \hfill \Box

Proof of Lemma 4.1. Apply Theorem A to $\{\psi_i\}_{i \in I}$, $\{g_i\}_{i \in I}$, and $\beta, C_I, e^{16Q}$ as bounds for the last three parameters of $C$—obtaining a mollification of $g$ defined as $g[\cdot]: [0, T] \to \Gamma(\text{Sym}^2_0 M)$ for some $t' > 0$ with $g[0] := g$ and the regularity property

\begin{equation}
R_{g[0]}(p)(\nu) - \sup_{x \in B(p, e^{tQ})} R_g(x)(\nu) \leq C_t t^{\beta}
\end{equation}

for a constant $C_t = C_t(n, p, r, Q, \beta, C_I, e^{3Q})$. For convenience we abbreviate

\begin{align*}
\langle -, - \rangle_t := \langle -, - \rangle_{g[0]}, \\
\| - \|_t := \| - \|_{g[0]}, \\
R_t := R_{g[0]}, \\
K_t := K_{g[0]}.
\end{align*}
We further agree on the following shorthands for intervals: \([a \pm b] := [a - b, a + b], \) \([\pm b] := [0 \pm b],\) and
\[
[f(x)]_{x \in X} := \left[\inf_{x \in X} f(x), \sup_{x \in X} f(x)\right].
\]
Moreover from multi-linearity of the curvature tensor (use e.g. \(R(v, w) = -R(-v, w)\)) we get immediately the reversed version of (4.2)
\[
-R_{g[0]}(p)(\vec{v}) + \inf_{x \in B(p, eQt)} R_g(x)(\vec{v}) \geq -C_1 t^\beta.
\]

We seek a \(T' \in (0, T]\) such that the claim of the theorem holds, i.e.
\[
K_t(p)(v, w) = \frac{R_t(\vec{v})}{\|\vec{v}\|_t\|w\|_t - \langle v, w\rangle_t},
\]
where \(R_t(\vec{v}) = \langle R_t(p)(v, w)w, v\rangle_t\), is in the interval
\[
(4.3) \quad [K_0(x)(v, w)]_{x \in B(p, eQt)} + [\pm C t^\beta]
\]
for all \(t \in (0, T']\), \(p \in M\), and \(v, w \in TM\) linear independent. Since the sectional curvature depends only on the plane spanned by \(v\) and \(w\), we can assume without loss of generality that
\[
\|v\|_0 = \|w\|_0 = 1, \quad \langle v, w\rangle_0 = 0.
\]

Fix some chart \(\psi: (B(0, r), 0) \to (M, p)\) with \(\|\psi\|_{W^{2,p}, p} \leq Q\). We extend \(v\) and \(w\) to sections \(v_\psi, w_\psi\) \(T^{3,1}(B(0, r))\) using the euclidean identification, i.e. by pushing forward the constant vectors \(\psi^*v\) and \(\psi^*w\) along \(\psi\). Again \(\vec{v}_\psi := (v, w, w, (-, v)\rangle_t\).

By (2.N0) we have \(\|v\|_{\text{eucl.}} \cdot \|w\|_{\text{eucl.}} \leq e^{2Q}\). Thus \(\vec{v} \leq e^{16Q}\). By Sobolev’s inequality the entries of the metric tensor are at least Lipschitz with some bound \(C_{n,p,r,Q}\). Hence we have
\[
(||v_\psi||_0 \|w_\psi\|_0 - \langle v_\psi, w_\psi\rangle_0)(\psi(x)) \in \left[1 \pm C_{n,p,r,Q}e^{2Q} \cdot |0 x|\right]
\]
for \(x \in B(0, r)\). By Lemma 3.6 this implies
\[
(||v_\psi||_t \|w_\psi\|_t - \langle v_\psi, w_\psi\rangle_t)(\psi(x)) \in \left[1 \pm C_2 t\right]
\]
for \(x \in B(0, T')\) and \(t \in [0, T]\). Choose \(T'\) so small that \(C_2 t T' \leq \frac{1}{2}\). Hence we can choose a constant \(C_2 > 0\) such that
\[
\frac{1}{||v_\psi||_t \|w_\psi\|_t - \langle v_\psi, w_\psi\rangle_t}(\psi(x)) \in \left[1 \pm C_2 t\right]
\]
and \(1 - C_2 t \geq \frac{1}{2}\) for all \(t \in (0, T]\).

Gathering all estimates above we conclude the proof:
\[
K_t(p)(v, w) = \frac{R_t(p)(\vec{v}_\psi)}{||\vec{v}_\psi||_t \|w_\psi\|_t - \langle v_\psi, w_\psi\rangle_t} \in R_t(p)(\vec{v}_\psi) [1 \pm C_2 t]
\]
\[
\subset \left([-R_0(x)(\vec{v})]_{x \in B(p, eQt)} + [\pm C_1 t^\beta, C_1 t^\beta]\right) [1 \pm C_2 t]
\]
for some new constant $C'$

$$\subset \left( R_0(x)(\bar{v} \right)_{x \in B(p,eQ_t)} \left[ 1 \pm C_2t \right] + \left[ \pm C't^3 \right]$$

$$\subset \left( K_0(x)(v_\psi, w_\psi) \right)_{x \in B(p,eQ_t)} \left[ 1 \pm C_2't^3 \right] + \left[ 1 \pm C'2t \right] + \left[ \pm C't^3 \right].$$

This is contained in the interval (4.3) for some constant $C$. This proves the theorem. □

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