Stability of two-dimensional ion-acoustic wave packets in quantum plasmas

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The nonlinear propagation of two-dimensional (2D) quantum ion-acoustic waves (QIAWs) is studied in a quantum electron-ion plasma. By using a 2D quantum hydrodynamic model and the method of multiple scales, a new set of coupled nonlinear partial differential equations is derived which governs the slow modulation of the 2D QIAW packets. The oblique modulational instability (MI) is then studied by means of a corresponding nonlinear Schrödinger equation derived from the coupled nonlinear partial differential equations. It is shown that the quantum parameter $H \propto \hbar$, associated with the Bohm potential, shifts the MI domains around the $k\theta$-plane, where $k$ is the carrier wave number and $\theta$ is the angle of modulation. In particular, the ion-acoustic wave (IAW), previously known to be stable under parallel modulation in classical plasmas, is shown to be unstable in quantum plasmas. The growth rate of the MI is found to be quenched by the obliqueness of modulation. The modulation of 2D QIAW packets along $k$ is shown to be described by a set of Davey-Stewartson-like equations. The latter can be studied for the 2D wave collapse in dense plasmas. The predicted results, which could be important to look for stable wave propagation in laboratory experiments as well as in dense astrophysical plasmas, thus generalize the theory of MI of IAW propagations both in classical and quantum electron-ion plasmas.

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I. INTRODUCTION

The importance of quantum effects has been recognized over the last few years in view of its remarkable applications in metallic and semiconductor nanostructures (e.g., metallic nanoparticles, metal clusters, thin metal films, nanotubes, quantum well and quantum dots, nano-plasmonic devices etc.) [1, 2, 3, 4, 5, 6, 7, 8] as well as in dense astrophysical environments (e.g., white dwarfs, neutron stars, supernovae etc.) [9, 10].

It is well known that the nonlinear propagation of wave packets in a dispersive plasma medium is generically subject to amplitude modulations due to the carrier wave self-interaction, i.e., a slow variation of the wave packet’s envelope due to nonlinearities. Under certain conditions, the system’s evolution may thus undergo a modulational instability (MI), leading to energy localization via the formation of envelope solitons. Such solitons are governed by a nonlinear Schrödinger (NLS) equation where the nonlinearity at the first stage of the amplitude is balanced by the group dispersion. This mechanism is encountered in various physical contexts, including pulse formation in nonlinear optics, in material science, as well as in plasma physics. A number of works can be found in the literature for the investigation of MI in classical [11, 12, 13, 14, 15, 16, 17] as well as quantum plasmas (see e.g., Refs. [18, 19, 20, 21]). The MI of ion-acoustic waves (IAWs) has been shown to be a general property in a nonlinear dispersive plasma medium, when the modulation is considered obliquely to the direction of propagation of the wave vector $\mathbf{k}$. Experimental observations of the MI of IAWs have been reported by Watanabe [12]. Stable wave propagation from modulational obliqueness, in both classical [13, 14, 16, 17] and quantum plasmas [20], has also been investigated by a number of authors.

On the other hand, in a wide variety of scientific fields (e.g., nonlinear optics, plasma physics, fluid dynamics etc.), certain nonlinear governing equations often exhibit important phenomena other than solitons, such as shocks (wave singularity), self-similar structures, wave collapse (i.e., blow up with growing amplitude or amplitude decay at finite time or finite distance of propagation), as well as the wave radiation emission leading to the onset of chaos. One such important system in this context is the Davey-Stewartson (DS)-like equations [22], where the system has both quadratic as well as cubic nonlinearities. Such equations can appear not only in the field of fluid dynamics as in the case of water wave propagation [22, 23], or in optical communications and information processing in nonlinear optical media [24, 25, 26], but also in plasma physics community [27, 28]. The DS description of collective excitations in Bose-Einstein condensates has also been studied in the context of matter-wave solitons [29, 30].

The purpose of the present work is to investigate, in more detail, the stability and instability criteria for the modulation of quantum IAW (QIAW) packets using two-dimensional (2D) quantum fluid model, and to generalize the previous investigation [11], both from classical and quantum points of view. Here we show that the nonlinear dynamics of QIAWs as well as the static zeroth harmonic field is governed by a new set of coupled nonlinear partial differential equations, which, in particular, reduces to a set of Davey-Stewartson (DS)-like equations for surface water waves [22]. We show that the quantum parameter $H \propto \hbar$, associated with the Bohm poten-
is the ion velocity normalized by the ion-acoustic speed and the inverse of the density normalized by their equilibrium value.

The paper is organized as follows. In Sec. II, we describe the 2D quantum hydrodynamic model, and derive the governing system of equations using the method of multiple scales. In Sec. III the MI of the QIAWs is studied and the condition for the existence of 2D wave collapse is derived. Finally, the Sec. IV is left for concluding the results.

II. FLUID MODEL AND DERIVATION OF THE EVOLUTION EQUATIONS

We consider the nonlinear propagation of QIAWs in a 2D quantum electron-ion plasma. The basic normalized set of equations in a two-component unmagnetized quantum plasma reads

\[
\frac{\partial n_i}{\partial t} + \nabla \cdot (n_i \mathbf{V}) = 0, \quad (1)
\]

\[
\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla) \mathbf{V} = -\nabla \phi, \quad (2)
\]

\[
\nabla^2 \phi - \frac{1}{3} n_i^{-1/3} \nabla n_i + \frac{H^2}{2} \left( \nabla^2 \sqrt{n_i} \right) = 0, \quad (3)
\]

\[
\nabla^2 \phi = n_e - n_i, \quad (4)
\]

where \( \mathbf{V} \equiv (\partial / \partial x, \partial / \partial y) \), \( n_{i(e)} \) is the electron (ion) number density normalized by their equilibrium value \( n_0 \), \( \mathbf{V} \equiv (u, v) \) is the ion velocity normalized by the ion-acoustic speed \( c_s = \sqrt{k_B T_e / m_i} \) with \( k_B \) denoting the Boltzmann constant, \( m_i \) the ion mass, \( T_e = \hbar^2 (3\pi^2 m_i)^{2/3} / 2k_B m_i \) the electron Fermi temperature, and \( h \) the scaled Planck’s constant. Also, \( H = \hbar \omega_{pe} / k_B T_e \), is denoting the ratio of the ‘plasmon energy density’ to the Fermi thermal energy, where \( \omega_{pe} = \sqrt{n_0 e^2 / 4\pi \epsilon_0 m_i} \) is the plasma frequency for the \( i \)-th particle. Moreover, \( \phi \) is the electrostatic potential normalized by \( k_B T_e / e \). The space and time variables are respectively normalized by \( c_s / \omega_{pe} \) and the inverse of \( \omega_{pe} \). The electron pressure gradient \( (\nabla p_e) \) and quantum force in Eq. (3) appear due to the electron degeneracy in a dense plasma with the Fermi distribution function. The former can be given (since the equilibrium pressure is truly three-dimensional) by the following equation of state \( P_e = \frac{1}{5} \frac{m_e V_{Fe}^2}{\sqrt{3}} n_e^{5/3} \) in which the space and the time variables are stretched as

\[
\xi = \epsilon (x - \nu_{te} t), \quad \eta = \epsilon (y - \nu_{te} t), \quad \tau = \epsilon^2 t, \quad (6)
\]

where \( \epsilon \) is a small parameter representing the strength of the wave amplitude and \( \nu_e \equiv (v_{te}, v_{te}) \) is the normalized (by \( c_s \)) group velocity, to be determined later by the compatibility condition. The dynamical variables are expanded as

\[
n_i = 1 + \sum_{n=1}^{\infty} \sum_{l=-\infty}^{\infty} n_{ij}^{(n)}(\xi, \eta, \tau) \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \quad (7)
\]

\[
(u, v) = \sum_{n=1}^{\infty} \sum_{l=-\infty}^{\infty} \left[ u_l^{(n)}(\xi, \eta, \tau), v_l^{(n)}(\xi, \eta, \tau) \right] \times \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \quad (8)
\]

\[
\phi = \sum_{n=1}^{\infty} \sum_{l=-\infty}^{\infty} \phi_l^{(n)}(\xi, \eta, \tau) \exp[i(\mathbf{k} \cdot \mathbf{r} - \omega t)], \quad (9)
\]

where \( n_{ij}^{(n)} \), \( u_l^{(n)} \), \( v_l^{(n)} \) and \( \phi_l^{(n)} \) satisfy the reality condition \( S_{ij} = S_{ij}^{\ast} \) with asterisk denoting the complex conjugate of the corresponding quantity. Notice that in Eqs. (7)- (9) the perturbed states depend on the fast scales via the phase \( (\mathbf{k} \cdot \mathbf{r} - \omega t) \) (where \( \mathbf{k} \) and \( \omega \) respectively denote the wave vector and wave frequency), whereas the slow scales only enter the \( l \)-th harmonic amplitude. We suppose that \( \mathbf{k} \) makes an angle \( \theta \) with the \( x \)-axis, and the modulation is along any direction in the \( xy \)-plane. In a general way, the wave vector \( \mathbf{k} \) is then \( k \equiv (k_x, k_y) = (k \cos \theta, k \sin \theta) \) and the group velocity, \( \nu_e \equiv (v_{te x}, v_{te y}) = (v_{te} \cos \theta, v_{te} \sin \theta) \).

Substituting the expressions from Eqs. (7)- (9) into the Eqs. (1)- (4) and collecting the terms in different powers of \( \epsilon \) we obtain for \( n = 1, l = 1 \) the linear dispersion relation for the normalized wave frequency \( \omega \) and the wave number \( k \) as

\[
\omega^2 = \frac{k^2 \Lambda}{1 + k^2 \Lambda}, \quad (10)
\]

where \( \Lambda = 1/3 + H^2 / 4 \). Note that the wave number \( k \) in Eq. (10) is not very small such that the wave frequency \( \omega \) is sufficiently large to prevent the appearance of harmonic modes of \( k_x, k_y \) as proper modes. These harmonic modes will virtually appear in higher orders of perturbations. Also, since \( k \) is normalized by the inverse of Fermi screening length \( (\lambda_F = c_s / \omega_{pe}) \), the values of \( k \) greater than unity is inadmissible, otherwise the wavelength would become smaller than the screening length. As a result, the collective behaviors of the plasma will disappear. Moreover, we consider the quantum parameter \( H \), to vary in the range \( 0.1 < H \lesssim 0.45 \) such that the ratio \( H / \epsilon \ll 1 \) [an approximate condition for the nonrelativistic quantum hydrodynamic model to be valid] and the coupling parameter, \( g_0 = 2m_e c_s / \hbar (3\pi^2 m_i)^{2/3} \lesssim 1 \) (which corresponds to the density region where the quantum collective and mean field effects become important).

On the other hand, for the second order reduced equations with \( n = 2, l = 1 \), we obtain the following compatibility con-
dions for the group velocity components

\[
\begin{align*}
    v_{gx} &= \frac{\partial \omega}{\partial k_x} = \frac{\omega^3 k_x}{k^2} \left[ \frac{1}{\omega^2} + \frac{H^2}{4\Lambda} - 1 \right], \\
v_{gy} &= \frac{\partial \omega}{\partial k_y} = \frac{\omega^3 k_y}{k^2} \left[ \frac{1}{\omega^2} + \frac{H^2}{4\Lambda} - 1 \right].
\end{align*}
\]

Next, proceeding in the same way as in Refs. [18, 19, 21] and finally considering the equations for \( l = 1 \) and \( n = 3 \), we obtain the following coupled equations for the propagation of modulated QIAWs

\[
\begin{align*}
    \frac{\partial \phi}{\partial \tau} + P_1^q \frac{\partial^2 \phi}{\partial \xi^2} + P_2^q \frac{\partial^2 \phi}{\partial \eta^2} + P_3^q \frac{\partial^2 \phi}{\partial \xi \partial \eta} + Q_1^q |\phi|^2 \phi \\
    + Q_2^q \psi \phi + Q_3^q \bar{\psi} \bar{\phi} + Q_4^q \bar{\phi} \phi + Q_5^q \bar{\chi} \phi + Q_6^q \bar{\phi} \phi = 0, \\
    R_1^q \frac{\partial \psi}{\partial \xi} + R_2^q \frac{\partial \psi}{\partial \eta} + R_3^q \frac{\partial \bar{\psi}}{\partial \xi} + R_4^q \frac{\partial \bar{\psi}}{\partial \eta} \\
    = S_1^q \frac{\partial |\phi|^2}{\partial \eta} + S_2^q \frac{\partial |\phi|^2}{\partial \xi} + S_3^q \frac{\partial \chi}{\partial \eta} + S_4^q \frac{\partial \bar{\chi}}{\partial \eta},
\end{align*}
\]

where \( \phi = \phi_1^{(1)}, \psi = \psi_1^{(2)}, \bar{\phi} = \int \partial \eta \psi_0^{(2)} \partial \eta, \bar{\psi} = \int \partial \eta \psi_0^{(1)} \partial \eta, \phi_0^{(2)} \xi, \chi = \int \partial \eta \psi_0^{(2)} \partial \eta, \bar{\phi}, \bar{\chi}, \) and the coefficients \( P_1^q, P_2^q, R_1^q, S_1^q \) are given in Appendix A, where \( j = 1, \ldots, 6 \) for \( Q_j^q, j = 1, 2, 3 \) for \( P_j^q \)’s and \( j = 1, \ldots, 4 \) for others. The corresponding coefficients \( P_1^q, Q_1^q, R_1^q, S_1^q \) for the propagation of 2D classical IAWs can also be obtained by the similar method as discussed in Appendix B. The superscripts ‘q’ and ‘c’ are used to denote the coefficients corresponding to 2D quantum and 2D classical electron-ion plasmas. Thus, we have obtained a new system of nonlocal nonlinear equations, which describe the slow (and general) modulation of the QIAW packets in 2D quantum plasmas. The coefficients \( P_1^q, P_2^q, R_1^q, S_1^q \) appear due to the wave group dispersion and 2D motion, and \( P_1^q \) for the arbitrary orientations of the carrier wave propagation as well as the modulation of the QIAW packets. The nonlinear coefficients \( Q_1^q \) is due to the carrier wave self-interaction originating from the zeroth harmonic modes (or slow modes), i.e., the ponderomotive force, and \( Q_2^q, Q_3^q \) are for the combined effects of 2D motion and self-interaction. The nonlinear-nonlocal coefficients \( Q_4^q, Q_5^q, Q_6^q \) arise due to the coupling between the dynamical field associated with the first harmonic (with a ‘cascaded’ effect from the second harmonic) and a static field generated due to the mean motion (zeroth harmonic) in the plasma. The appearance of the coefficients \( R_1^q, S_1^q \) can also be explained similarly.

Equations (13) and (14) can be studied, as for example, for the modulation of Stokes wave train (plane wave) with constant amplitude to a small perturbation as well as to look for envelope solitons, and wave collapse, if any. There are, of course, many other physical insights (e.g., dromion solution), that can also be recovered from this system. In particular, looking for the modulation of QIAW packets parallel to the carrier wave vector, one can obtain the DS-like equations [22]

\[
\begin{align*}
    \frac{\partial \phi}{\partial \tau} + P_1^q \frac{\partial^2 \phi}{\partial \xi^2} + P_2^q \frac{\partial^2 \phi}{\partial \eta^2} + Q_1^q |\phi|^2 \phi + Q_2^q \psi \phi = 0, \\
    R_1^q \frac{\partial^2 \psi}{\partial \xi^2} + R_2^q \frac{\partial^2 \psi}{\partial \eta^2} = S_1^q \frac{\partial |\phi|^2}{\partial \xi},
\end{align*}
\]

in 2D quantum plasmas

\[
\begin{align*}
    \frac{\partial \phi}{\partial \tau} + P_1^q \frac{\partial^2 \phi}{\partial \xi^2} + P_2^q \frac{\partial^2 \phi}{\partial \eta^2} + Q_1^q |\phi|^2 \phi + Q_2^q \psi \phi = 0, \\
    R_1^q \frac{\partial^2 \psi}{\partial \xi^2} + R_2^q \frac{\partial^2 \psi}{\partial \eta^2} = S_1^q \frac{\partial |\phi|^2}{\partial \xi},
\end{align*}
\]

where the coefficients are those obtained at \( \theta = 0 \) from the general expressions. Also, \( Q_1^q = Q_1^q + Q_2^q S_1^q/R_1^q \) and \( Q_2^q = Q_2^q - Q_1^q S_1^q/R_1^q \). Similar forms like Eqs. (15) and (16) can also be obtained by transforming the x-axis through an angle \( \theta \), so that the wave vector \( k \) and the modulation direction coincide. Now, our aim is to investigate the MI of QIAWs by considering its modulation along the x-axis through an angle \( \theta \) with the carrier wave vector \( k \). To this end, we disregard the group velocity component of the modulated wave along the y-axis, as well as the \( \gamma \)- and \( \eta \)-dependence of the physical variables. Thus, we obtain from Eqs. (13) and (14) the NLS equation for the oblique modulation of QIAW packets

\[
\begin{align*}
    \frac{\partial \phi}{\partial \tau} + P_1^q \frac{\partial^2 \phi}{\partial \xi^2} + Q_1^q \psi \phi = 0,
\end{align*}
\]

where \( P = P_1^q \) and \( Q = Q_1^q + Q_2^q s_1^q/R_1^q \).

### III. ANALYSIS OF MODULATION INSTABILITY

We consider the MI of a plane wave solution of Eq. (17) for \( \phi \) with constant amplitude \( \phi_0 \). The boundary conditions that \( \phi \to 0 \) as \( \xi \to \infty \) must now be relaxed, because the plane wave train is still unmodulated and the solution is not unique. Thus, we can represent the solution as the monochromatic solution \( \phi = \phi_0 \exp(iQ|\phi_0|^2 \tau) \), where \( \Delta(\tau) = -Q|\phi_0|^2 \) is the nonlinear frequency shift. To study the stability of this solution we modulate the amplitude against linear perturbation as \( \phi = \phi_0 + \phi_0 \cos(K_\xi - \Omega \tau) \exp(iQ|\phi_0|^2 \tau) \), where \( K \) and \( \Omega \) are, respectively, the wave number and the frequency of modulation. We then readily obtain from Eq. (17) the dispersion relation

\[
\Omega^2 = P^2 K^4 \left( 1 - \frac{K^2(\tau)}{K^2} \right),
\]

where \( K_c = \sqrt{2Q/|\phi_0|^2} \) is the critical wave number such that the MI sets in for a wave number \( K < K_c \), i.e. for all wavelengths above the threshold, \( \lambda_c = 2\pi/K_c \). The instability growth rate (letting \( \Omega = i\Gamma \)) is then given by

\[
\Gamma = PK^2 \sqrt{K_c^2} - 1,
\]

The maximum growth rate, \( \Gamma_{\text{max}} = |Q| |\phi_0|^2 \) as equals to the amount of the nonlinear frequency shift \( |\Delta(\tau)| \), is achieved at \( K = K_c/\sqrt{2} \). Clearly, the instability condition depends only on the sign of the product \( PQ \), which, in turn, depends on the angle of modulation \( \theta \) as well as the quantum parameter \( H \). These may be studied numerically, relying on the exact
expressions of $P$ and $Q$. We find that for a fixed value of $H = 0.3$ (i.e., as one enters the density region $n_0 = 1.5295 \times 10^{33} \text{m}^{-3}$), $P > 0$ in $0.6958 \leq \theta < \pi/2$ and $Q < 0$ in $0 < \theta \leq 0.5281$ and $1.138 \leq \theta < \pi/2$ for $0 < k < 1$. Other domains of $k$ and $\theta$ where $P$ and $Q$ change sign depend on the $P = 0$ and $Q = 0$ curves in the $k\theta$-plane. Similarly, considering another value of $H = 0.45$ (where $n_0 = 1.34 \times 10^{32} \text{m}^{-3}$) one can find $P > 0$ in $0.6748 \leq \theta < \pi/2$ and $Q < 0$ in $0 < \theta \leq 0.6$ and $1.137 \leq \theta < \pi/2$ for $0 < k < 1$. As above, $P, Q$ also change their sign in other regions in the $k\theta$-plane.

Figure 1 displays the contour plots of $PQ = 0$ boundary curves against the normalized wave number $k$ and the modulation angle $\theta$. The upper panel corresponds to the quantum case, whereas the lower one is for the classical electron-ion plasmas. The stable ($PQ < 0$) and unstable ($PQ > 0$) regions are separated in the $k\theta$-plane by the boundary curves of $PQ = 0$ as indicated by the white and the gray regions respectively. We have allowed $k$ to vary in a range $0 < k < 1$ (as explained in Sec. II) and $\theta$ to vary between $\theta = 0$ and $\theta = \pi$, so that all plots seem to be $\pi/2$ periodic, and, in fact, symmetric upon reflection with respect to either $\theta = 0$ or $\theta = \pi/2$ lines. We also consider $H$, in the range $0.1 < H \leq 0.45$ such that $g_Q \leq 1$ (as explained before). It is clear from Fig. 1 that the stability and instability domains are quite different in classical and quantum plasmas. Evidently, the quantum parameter $H$ shifts the MI domains in the $k\theta$-plane. It is, of course, hard to find the common instability regions for both classical and quantum cases. Increasing the value of $H (\lesssim 0.45)$ gives no effective change, except in reducing the stable regions around $k \sim 1$.

In particular, considering the modulation of the QIAWs along the carrier wave vector $\mathbf{k}$ (i.e., the case of $\theta = 0$), the QIAW is shown to be modulational unstable (see Fig. 2). This is in contrast to the classical case where the IAWs are known to be stable under parallel modulation [11]. Physically, the QIAW under parallel modulation becomes unstable due to the nonlinear self-interactions originating from the zeroth harmonic modes or slow modes as well as the second harmonic modes (since $P$ and $Q$ are always negative). However, the case of modulation perpendicular to the wave vector $\mathbf{k}$ ($\theta = \pi/2$) gives rise stable wave propagation, an agreement with classical plasmas [11]. Here the wave is stable due to the second harmonic self-generation of the modes, since $P > 0$ and $Q < 0$. Thus, for the MI to occur the second harmonic mode is very essential in order to maintain the same sign of $P$ and $Q$.

On the other hand, the MI growth rate as given by Eq. (19) can be calculated as depicted in Fig. 3. The upper panel (corresponding to QIAWs) shows that the growth rate can be reduced by increasing the obliqueness of modulation giving rise cut-offs at lower wave numbers of modulation. The quantum parameter $H$ has no significant role in reducing such growth rate. In the classical case, the maximum growth rate reduces slightly with a small change of the angle of modulation (see lower panel of Fig. 3).

The DS-like equations (15) and (16) describes the case of general 2D-evolution of slowly varying weakly nonlinear ion-acoustic waves in the quantum regime. Here we start by pointing out a variational principle for the DS-like system. Introdu-
producing the Lagrangian density

$$\mathcal{L} = \frac{1}{2} \left( \phi \frac{\partial \phi}{\partial \tau} - \phi \frac{\partial \phi^*}{\partial \tau} - P_1 \left| \frac{\partial \phi}{\partial \xi} \right|^2 - Q_{11} \left| \frac{\partial \phi}{\partial \eta} \right|^2 + \frac{1}{2} Q_{11} |\phi|^4 \right) + \frac{Q_{12}}{S_1} \left[ \frac{1}{2} R_1 \left( \frac{\partial^2 \phi}{\partial \xi^2} \right)^2 + \frac{1}{2} R_2 \left( \frac{\partial^2 \phi}{\partial \eta \partial \xi} \right)^2 + S_1 \frac{\partial^2 \phi}{\partial \xi^2} \phi \right]$$  \hspace{1cm} (20)

where the action functional is $\mathcal{A} (\phi, \phi^*, u) = \int \mathcal{L} d\xi d\tau$, and $\partial^2 \phi/\partial \eta \partial \xi \equiv \psi$ plays the role of a potential. We obtain Eqs. \((15)\) and \((16)\) by varying $\phi, \phi^*$ and $u$ and minimizing the action as usual. It is straightforward to show that \((15)\) and \((16)\) possesses constants of motion, representing the conservations of the number of high frequency quanta

$$\frac{d}{d\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\phi|^2 d\eta d\xi = 0 \hspace{1cm} (21)$$

and of longitudinal momentum

$$\frac{d}{d\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \phi \frac{\partial \phi^*}{\partial \xi} - \phi^* \frac{\partial \phi}{\partial \xi} \right] d\eta d\xi = 0 \hspace{1cm} (22)$$

as well as transverse momentum

$$\frac{d}{d\tau} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[ \phi \frac{\partial \phi^*}{\partial \eta} - \phi^* \frac{\partial \phi}{\partial \eta} \right] d\eta d\xi = 0. \hspace{1cm} (23)$$

Furthermore, since no explicit time dependence occurs in the Lagrangian, the Hamiltonian $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \mathcal{H} d\eta d\xi$, derived from $\mathcal{L}$ in \((20)\), is also conserved. The existence of a Variational principle makes it possible to use trial functions as a means to obtain approximate solutions. A good review on this topic for wave equations similar to \((15)\) and \((16)\) is given by Ref. \cite{39}. A thorough variational study of \((15)\) and \((16)\) for the general case is beyond the scope of the present paper. However, in order to illustrate the usefulness of the variational technique, we will state a variational result that applies when a cylindrically symmetric collapse is possible. For the general case, Eqs. \((15)\) and \((16)\) do not admit a cylindrically symmetrical collapse. However, for geometries and parameter regimes where the coefficients fulfill either $R_2 \ll R_1$, or $Q_{12} S_1 / R_1 \ll Q_{11}^0$, our system reduces to a 2D NLS equation for which cylindrically symmetrical solutions are possible. Assuming that at least one of the above strong inequalities apply, and that we have cylindrically symmetric initial conditions (to be defined below), the evolution equation for our system can be written

$$i \frac{\partial \phi}{\partial \tau} + \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + Q_{\text{eff}} |\phi|^2 \phi = 0 \hspace{1cm} (24)$$

where we have introduced the radial coordinate $r^2 = \xi^2 + \eta^2 / P_1 \equiv \psi$, and assumed, cylindrical symmetry, i.e. that the spatial dependence is only through this variable. The coefficient $Q_{\text{eff}}$ is either $Q_{11}^0$ (in case the reduction from \((15)\) and \((16)\) was due to the inequality $Q_{12} S_1 / R_1 \ll Q_{11}^0$) or $Q_{\text{eff}} = Q_{11}^0 + Q_{12} S_1 / R_1$ (in case the reduction was due to $R_2 \ll R_1$). For the case of $Q_{\text{eff}} > 0$, collapse is possible. A good estimate for the collapse threshold can be obtained by using the variational formulation, with a trial function of the form $\phi (\tau, r) = F (\tau) \exp (i b (\tau) r^2)$, see e.g. Ref. \cite{41} for mathematical details. The result is that there will be a collapse towards a zero radius of the pulse profile, $f (\tau) \rightarrow 0$ within a finite time, provided the initial profile is sufficiently intense and well localized, as expressed by the collapse condition

$$Q_{\text{eff}} f^2 (\tau = 0) |F^2 (\tau = 0)| \gtrsim 1.35 \hspace{1cm} (25)$$

The accuracy of this collapse condition is well supported by numerical calculations. Naturally, for cases where cylindrical symmetry does not apply, the evolution will be much more complicated than indicated by our simple example.

V. CONCLUSION

We have investigated the nonlinear propagation of QIAW packets in a 2D quantum plasma. A multiple scale technique is used to derive a coupled set of nonlinear partial differential equations, which governs the dynamics of modulated QIAW packets. The set of equations, in particular (i.e., modulation along the direction of the pump carrier wave), is shown to be reducible to a well-known Davey-Stewartson (DS)-like equations \cite{22}. The latter can be studied for the 2D wave collapse in dense plasmas. The oblique MI of QIAWs is then studied by means of a corresponding NLS equation. It is found that the quantum parameter $\tilde{H}$ has the significant role in shifting the instability domains around the $k \tilde{b}$-plane. The case of parallel modulation, which is known to be stable in classical
plasmas \cite{11}, is shown to be unstable in quantum plasmas. In contrast to the classical case, the MI growth rate in quantum plasmas can significantly be reduced by increasing the angle of modulation.

It is to be mentioned that the numerical simulation of the DS equations could furnish more convincing evidence for the process of soliton collapse, where the quantum parameter $\hat{H}$ may play a crucial role in accelerating or decelerating the collapse process. However, it needs extra effort, and could be an open issue to be explored in future studies. The integrability as well as the dromion solution, if there be any, of the system will also be another investigation, but beyond the scope of the present work.

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In the expression for $Q_I^I$, the first (second) term in the square brackets is due to the first and second harmonic modes (the first and zeroth harmonic modes).

\[
Q_I^I = \frac{\omega v_{gx}}{2} \left[ -3 \left( 1 + \frac{2k_x v_{gx}}{\omega} \right) + \frac{\omega^2 \Lambda}{\omega^2 k^2} \right],
\]

\[
Q_2^I = k_x \left( 1 - \frac{3v_{gy}^2}{2} \right),
\]

\[
Q_3^I = -k_y - 3v_{gy} \left( 2v_{gx} k_x + \frac{\omega}{2} \right) + \frac{2v_{gy} \Lambda}{\omega^2 k^2} - \frac{\omega \Lambda}{\omega^2 k^2},
\]

\[
Q_s^I = k_x k_y \left[ \frac{v_{gy} A_0}{\Lambda^2} + \frac{v_{gy} k_x}{\omega^2} \left( \frac{2k^2}{\omega} + \frac{3k_y v_{gy}}{\omega} \right) \right],
\]

\[
R_{1}^I = v_{gx} \left( \frac{3v_{gx}^2}{2} - 1 \right), R_{2}^I = v_{gx} \left( 9v_{gy}^2 - 1 \right),
\]

\[
R_{3}^I = v_{gy} \left( 9v_{gx}^2 - 1 \right), R_{4}^I = v_{gy} \left( 3v_{gy}^2 - 1 \right),
\]

\[
S_{1}^I = v_{gx} \left( A_1 \omega^2 + 3k_x k_y v_{gy} + k_x^2 - k_y^2 \right),
\]

\[
S_{2}^I = v_{gx} \left( A_1 \omega^2 + 3k_x k_y v_{gy} + 3k_y v_{gy} A_0 \right),
\]

\[
S_{3}^I = A_2 v_{gy},
\]

where

\[
\Lambda = \frac{1}{3} + \frac{3}{4} \omega^2 k^2, A_0 = \frac{5}{9} + \frac{H^2 k^2}{4},
\]

\[
\Lambda = \frac{1}{3} + \frac{H^2 k^2}{4},
\]

\[
\lambda_\phi = \frac{\omega^4 (4 + 27H^2 k^2)}{72 \omega^2 A^2} - \frac{72 \Lambda^2 k^4}{72 \omega^2 A^2} (1 - 4 \omega^2) - \omega^2,
\]

\[
\lambda_\epsilon = \frac{k^2}{\omega^4} \left[ k^2 + \lambda_\phi \omega^2 (1 - 4 \omega^2) \right],
\]

\[
\lambda_\varphi = \frac{k^2}{\omega^4} \left( k^2 + \lambda_\phi \omega^2 \right),
\]

\[
\lambda_{s,x} = \frac{k_{s,x} \lambda_{s,x} + k_y^2}{2 \omega^2},
\]

\[
A_1 = \frac{v_{gy} A_0}{\omega^3} + \frac{2k_x^2}{3k_y} \left( v_{gy} k_x + v_{gy} k_y \right),
\]

\[
A_2 = \frac{v_{gy} A_0}{\omega^3} + \frac{k_y^2}{\omega^3} \left( 2k^2 + 3k_y v_{gy} \right).
\]

**APPENDIX B**

The coefficients corresponding to classical electron-ion plasmas can be obtained by considering a 2D fluid model with continuity and momentum equations for cold ions and Boltzmann distributed electrons [see, e.g., the Eqs. (5)-(8) in Ref. [11]]. The normalizations and the corresponding fluid equations for the variables can be recovered by simply replacing the Fermi temperature $T_F$ by the classical temperature $T_c$ with considering the electron pressure as $p_e = k_B T_e$, instead of the Fermi pressure, and disregarding the Bohm potential term proportional to $h$. These coefficients are presented for a general interest of the readers to study the dynamics of ion-acoustic waves from the evolution equations of the forms (13), (14) or (15) in 2D classical electron-ion plasmas.

Thus, the coefficients $P_{I,j}^I, Q_{I,j}^I, R_{I,j}^I, S_{I,j}^I$ to be appeared in equations like (13) and (14) in the case of classical plasmas can be given as follows:

\[
P_{1,2}^I = \frac{1}{2 \omega} \frac{\partial^2 \omega}{\partial k_{x,y}^2} = \frac{\omega^3}{2k^2} \left[ \frac{1 - \omega^2}{\omega^2} + \frac{v_{gx}(x,y)}{\omega} \left( \frac{k^2 v_{gx}(x,y)}{\omega} - 4k_{x,y} \right) \right],
\]

\[
P_5^I = \frac{v_{gy}}{2k} \left( \frac{k^2 v_{gy}}{\omega} - 4 \right) \sin 2\theta,
\]

\[
Q_I^I = \left( k_x \lambda_\omega + k_y \lambda_\varphi \right) - \frac{1}{2} \omega \lambda_\epsilon + \lambda_\varphi - \frac{k_y^2}{\omega^2}
\]

\[
+ k_x A_1 + \frac{\omega}{2} \left( \frac{k^2}{\omega^2} - 1 \right),
\]

where, in the expression for $Q_I^I$, the first (second) term in the square brackets is due to the first and second harmonic modes (the first and zeroth harmonic modes). Moreover,

\[
Q_2^I = \frac{\omega v_{gx}}{2} \left[ -\left( 1 + \frac{2k_x v_{gx}}{\omega} \right) + \frac{\omega^2}{k^2} \right],
\]

\[
Q_3^I = k_x \left( 1 - v_{gy}^2 \right),
\]

\[
Q_4^I = -k_y - v_{gy} \left( 2v_{gx} k_x + \frac{\omega}{2} \right) + \frac{3v_{gy} \omega}{2k^2},
\]

\[
Q_5^I = k_x k_y \left[ \frac{3}{2k} \left( 1 + \frac{2v_{gy} k_x}{\omega} \right) - \frac{\omega}{2k^2} \right],
\]

\[
Q_6^I = \left[ -v_{gy} + k_y \left( \frac{2k^2}{\omega} + v_{gy} \right) \right],
\]
\begin{align*}
R_1^c &= v_{gx} \left( v_{gx}^2 - 1 \right), \\
R_2^c &= v_{gx} \left( 3v_{gx}^2 - 1 \right), \\
R_3^c &= v_{gy} \left( 3v_{gy}^2 - 1 \right), \\
R_4^c &= v_{gy} \left( v_{gy}^2 - 1 \right),
\end{align*}

and

\begin{align*}
S_1^c &= \frac{1}{\omega^2} \left[ v_{gx} \left( A_1 \omega^2 + k_x k_y v_{gx} \right) + k_x^2 - k_y^2 \right], \\
S_2^c &= \frac{1}{\omega^2} \left[ v_{gy} \left( A_1 \omega^2 + k_y^2 v_{gx} - v_{gx} \omega^2 \right) \\
&\quad + k_y \left( k_x + \frac{2v_{gx} k_y}{\omega^2} \right) \right], \\
S_3^c &= \frac{k_x k_y}{\omega^2} \left( v_{gx}^2 - 1 \right), \\
S_4^c &= A_2 v_{gy},
\end{align*}

where

\begin{align*}
\lambda_\phi &= \frac{\omega^4 - 2k^4}{2\omega^2 \left[ k^2 \left( 1 - 4 \omega^2 \right) - \omega^2 \right]}, \\
\lambda_e &= \frac{k^2}{\omega^4} \left[ k^2 + \lambda_\phi \omega^2 \left( 1 - 4 \omega^2 \right) \right], \\
\lambda_i &= \frac{k^2}{\omega^4} \left( k^2 + \lambda_\phi \omega^2 \right), \\
\lambda_u &= k_y \omega^2 \left( \lambda_\phi + \frac{k^2}{2\omega^2} \right), \\
A_1 &= -v_{gx} + \frac{2k_x^2 k_y}{\omega^4} + \frac{k_y}{\omega^2} \left( v_{gx} k_y + v_{gy} k_x \right), \\
A_2 &= -v_{gy} + \frac{k_x}{\omega^4} \left( 2k^2 + \omega k_y v_{gy} \right)
\end{align*}

where \( \omega^2 = k^2 / \left( 1 + k^2 \right) \) is the normalized classical dispersion relation (\( \omega \rightarrow \omega / \omega_{pi} \) and \( k \rightarrow k_{cs} / \omega_{pi} \)). We note that the difference in the dispersion relation as compared to Eq. (10) stems from the use of a classical (i.e. not a Fermi) equation of state.