Local controllability of the one-dimensional nonlocal Gray–Scott model with moving controls

VÍCTOR HERNÁNDEZ-SANTAMARÍA AND KÉVIN LE BALC’H

Abstract. In this paper, we prove the local controllability to positive constant trajectories of a nonlinear system of two coupled ODE equations, posed in the one-dimensional spatial setting, with nonlocal spatial nonlinearities, and using only one localized control with a moving support. The model we deal with is derived from the well-known nonlinear reaction–diffusion Gray–Scott model when the diffusion coefficient of the first chemical species $d_u$ tends to $0$ and the diffusion coefficient of the second chemical species $d_v$ tends to $+\infty$. The strategy of the proof consists in two main steps. First, we establish the local controllability of the reaction–diffusion ODE–PDE derived from the Gray–Scott model taking $d_u = 0$, and uniformly with respect to the diffusion parameter $d_v \in (1, +\infty)$. In order to do this, we prove the (uniform) null-controllability of the linearized system thanks to an observability estimate obtained through adapted Carleman estimates for ODE–PDE. To pass to the nonlinear system, we use a precise inverse mapping argument and, secondly, we apply the shadow limit $d_v \to +\infty$ to reduce to the initial system.

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1. Introduction

1.1. The Gray–Scott model

The irreversible Gray–Scott model governs the chemical reaction

\[ U + 2V \rightarrow 3V, \]
\[ V \rightarrow P, \]

in a gel reactor, where \( V \) catalyses its own reaction with \( U \) and \( P \) is an inert product. The gel reactor is coupled to a reservoir in which the concentrations of \( U \) and \( V \) are maintained constant.

Let \( T > 0 \). For \( t \) in the time interval \([0, T]\) and \( x \) in the spatial interval \([0, 1]\), i.e., the gel reactor, we denote \( u = u(t, x) \) and \( v(t, x) \) the concentrations of the two chemical species \( U \) and \( V \), \( d_u, d_v > 0 \) their (constant) diffusion coefficients. Then, the pair of coupled reaction–diffusion equations governing theses reactions are

\[
\begin{align*}
\frac{\partial u}{\partial t} - d_u \frac{\partial^2 u}{\partial x^2} &= -uv^2 + F(1 - u) \quad \text{in} \quad (0, T) \times (0, 1), \\
\frac{\partial v}{\partial t} - d_v \frac{\partial^2 v}{\partial x^2} &= uv^2 - (F + k)v \quad \text{in} \quad (0, T) \times (0, 1), \\
\frac{\partial x}{\partial u} &= \frac{\partial x}{\partial v} = 0 \quad \text{on} \quad (0, T) \times \{0, 1\}, \\
(u, v)(0, \cdot) &= (u_0, v_0) \quad \text{in} \quad (0, 1).
\end{align*}
\]

In (3), the cubic terms \( uv^2 \) and \(-uv^2 \) correspond to the chemical reaction (1), seeing \( u \) as a reactant and \( v \) as a product, by the law of mass action. The linear term \( kv \) comes from the chemical reaction (2) at rate \( k > 0 \) and the positive constant \( F > 0 \) denotes the rate at which \( U \) is fed from the reservoir into the reactor (and this same feed process takes \( U \) and \( V \) out in a concentration-dependent way). We impose homogeneous Neumann boundary conditions on \( u \) and \( v \) to guarantee that the environment is closed. See [10] and the references therein for more details on the Gray–Scott model.

Note that the global existence of classical solutions for (3) follows from classical bootstrap argument because the spatial dimension is one. To obtain the global existence
of classical solutions for (3) in all spatial dimension, one can use for instance [25, Theorem 3.1].

The steady states of the local dynamics (i.e., setting \( d_u = d_v = 0 \)) give us the uniform steady-state solutions to (3). The point \((1, 0)\) is a uniform steady state. Moreover, if \( F \geq 4(F + k)^2 \) or \( F = \frac{1}{2}[(1/4) - 2k \pm \sqrt{1/16 - k}] \), there exist two steady states for (6), \((u_+, v_-)\) and \((u_-, v_+)\), characterized as follows

\[
\begin{align*}
  u_\pm &= \frac{1}{2}(1 \pm \sqrt{1 - 4\gamma^2 F}), \\
  v_\pm &= \frac{1}{2\gamma}(1 \mp \sqrt{1 - 4\gamma^2 F}), \\
  & \text{with } \gamma = \frac{F + k}{F}. \quad (4)
\end{align*}
\]

The proof of this fact can be found, for instance, in [23, Section 3].

For some chemical species \( \mathcal{U} \) and \( \mathcal{V} \), one can have \( d_u \ll 1 \ll d_v \). If we send \( d_u \to 0 \) and \( d_v \to +\infty \), we obtain formally the following nonlocal Gray–Scott model

\[
\begin{equation}
\begin{cases}
  \partial_t u = -uw^2 + F(1 - u) & \text{in } (0, T) \times (0, 1), \\
  w' = \left( \int_0^1 u(t, x) \, dx \right) w^2 - (F + k)w & \text{in } (0, T), \\
  u(0, \cdot) = u_0(\cdot) & \text{in } (0, 1), \quad w(0) = w_0.
\end{cases}
\end{equation}
\]

The proof of such claim can be obtained in two different stages by using semigroup theory arguments. First, we can take the limit as \( d_u \to 0 \) by making an adaptation of [4, Theorem 1.4]. This will lead to system (8). Once this is done, we can follow the methodology in [22, Appendix A] to obtain the convergence towards (5). We emphasize that in system (5), the second component \( w = w(t) \) does not depend on the spatial variable.

The main goal of this paper is to study some controllability properties for (5). More precisely, we introduce the following distributed controlled system

\[
\begin{equation}
\begin{cases}
  \partial_t u = -uw^2 + F(1 - u) + h1_{\omega(t)} & \text{in } (0, T) \times (0, 1), \\
  w' = \left( \int_0^1 u(t, x) \, dx \right) w^2 - (F + k)w & \text{in } (0, T), \\
  u(0, \cdot) = u_0(\cdot) & \text{in } (0, 1), \quad w(0) = w_0.
\end{cases}
\end{equation}
\]

In (6), at \((t, x) \in (0, T) \times (0, 1), (u(t, x), w(t))\) is the state while \( h(t, x) \) is the control input, whose support is localized in a moving subset \( \omega(t) \) of \((0, 1)\).

Typically, we shall consider controls sets \( \omega(t) \) determined by the evolution of a given reference \( \omega \) of \((0, 1)\) through a smooth flow \( X(t, x, 0) \). This type of support for the control variable \( h \) is justified by the fact that there is no diffusion in the variable \( u(t, x) \) in (5) so for obtaining controllability, at least in the variable \( u \), one needs to consider moving control support as in the articles [8,9,20,24] for instance.

We make the following hypothesis on the moving subset \( \omega(t) \).

**Assumption 1.1.** There exist a subset \( \omega_0 \subseteq \omega \), two times \( t_1, t_2 \) with \( 0 < t_1 < t_2 < T \) such that

(a) \( \omega_0(t) \neq (0, 1) \) for all \( t \in (0, T) \),
Typically, for every $m \in (0, 1)$, $0 < t_1 < t_2 < T$, the set
\[
\omega_m(t) = \begin{cases} 
(0, m) & t \in (0, t_1), \\
\left(\frac{(1-m)(t-t_1)}{t_2-t_1}, \frac{(1-m)(t-t_1)}{t_2-t_1} + m\right) & t \in (t_1, t_2), \\
(1 - m, m) & t \in (t_2, T),
\end{cases}
\]
satisfies Assumption 1.1.

Let us notice that the crucial hypothesis in Assumption 1.1 is b) which guarantees that the $\omega_0(t)$ spreads the whole interval $(0, 1)$. Very recently, the conservative hypothesis d) in Assumption 1.1 has been removed in the article [16] for spatial dimension $N \geq 2$.

1.2. Main results

The goal of this part is to present the main results of the paper. Among others, we prove the local controllability to constant positive trajectories for (5).

From now and until the end of the paper, we will use the notations
\[
\Omega := (0, 1), \quad Q_T := (0, T) \times (0, 1) \text{ and } \Sigma_T = (0, T) \times \{0, 1\}.
\]

The first main result of the paper is the following one.

**Theorem 1.2.** We suppose that Assumption 1.1 holds. Let $(u_\pm, v_\mp) \in (0, +\infty)^2$ as in (4). Then there exists $\delta > 0$ such that for every $(u_0, w_0) \in L^2(\Omega) \times \mathbb{R}$, verifying
\[
\|(u_0 - u_\pm, w_0 - v_\mp)\|_{L^2(\Omega) \times \mathbb{R}} \leq \delta,
\]
once can find a control $h \in L^2(Q_T)$ such that the solution $(u, w) \in H^1(0, T; L^2(\Omega)) \times L^\infty(0, T)$ of (6) satisfies
\[
u(T, \cdot) = u_\pm, \quad w(T) = v_\mp.
\]

**Remark 1.3.** Let us make some comments on Theorem 1.2.

- For a given control $h \in L^2(Q_T)$, note that the solutions $(u, w)$ to (6) belonging to $H^1(0, T; L^2(\Omega)) \times L^\infty(0, T)$ are necessarily unique by using classical Gronwall’s argument. The existence of such a solution, associated with some specific control $h$, actually comes from the proof of Theorem 1.2.
- From a modeling point of view, Theorem 1.2 states that for diffusion coefficients $0 < d_u << 1 << d_v$ associated with the chemical species $U$ and $V$, by starting from chemical concentrations closed to a chemical equilibrium, there exists a strategy of control, i.e., by adding or withdrawing some chemical product at
some moving place of the gel reactor, such that the chemical components \( U \) and \( V \) exactly reach the chemical equilibrium. This is particularly relevant when \( (u_\pm, v_\mp) \) is an unstable equilibrium of (3); see [23, Section 3].

- Assumption 1.1 is a natural hypothesis for dealing with the controllability of systems of the form (5). Indeed, this ODE-ODE system has a finite speed of propagation so the time \( T > 0 \) is taken sufficiently large such that the initial support of control \( \omega(0) \) spreads the whole interval \((0, 1)\). This is exactly Assumption 1.1, b).

- Let us remark that \( (u_*, v_*) = (1, 0) \) is also a constant stationary state of (3). But (6) is not locally controllable around \((1, 0)\). This comes from the fact that all solutions \((u, w) \in H^1(0, T; L^2(\Omega)) \times L^\infty(0, T) \) to (6), reaching \((1, 0)\) in time \( T \), satisfy necessarily \( w \equiv 0 \) in \((0, T)\). Indeed, setting \( a(t) = \int_0^1 u(t, x) \, dx \in L^\infty(0, T) \), rewriting the second equation, we obtain

\[
 w'(t) = a(t)w^2(t) - (F + k)w(t), \quad w(T) = 0.
\]

So, by the Cauchy–Lipschitz theorem, we obtain that \( w \equiv 0 \) in \((0, T)\).

- As far as we know, Theorem 1.2 is the first result in the literature which deals with the controllability of nonlinear system of coupled ODE equations, posed in the one-dimensional spatial setting, with nonlocal spatial nonlinearities. For results on the controllability of linear and nonlinear parabolic PDEs with spatially nonlocal terms, see [2, 12, 13, 21] and the recent article of the authors [18].

Actually, a by-product of the proof of Theorem 1.2 is a local controllability result for the following reaction–diffusion ODE–PDE model

\[
\begin{aligned}
\partial_t u &= -uv^2 + F(1 - u) + h_1(t) \quad \text{in } (0, T) \times (0, 1), \\
\partial_t v - d_v \partial_{xx} v &= uv^2 - (F + k)v \quad \text{in } (0, T) \times (0, 1), \\
\partial_x v &= 0 \quad \text{on } (0, T) \times \{0, 1\}, \\
(u, v)(0, \cdot) &= (u_0, v_0) \quad \text{in } (0, 1),
\end{aligned}
\]

for \( d_v \in (1, +\infty) \).

**Theorem 1.4.** We suppose that Assumption 1.1 holds. Let \((u_\pm, v_\mp) \in (0, +\infty)^2 \) as in (4). Then there exist \( \delta > 0, C > 0 \) such that for every \( d_v \in (1, +\infty) \), \((u_0, v_0) \in L^2(\Omega) \times H^1(\Omega)\), verifying

\[
\|(u_0 - u_\pm, v_0 - v_\mp)\|_{L^2(\Omega) \times H^1(\Omega)} \leq \delta,
\]

one can find a control \( h \in L^2(Q_T) \) such that the solution \((u, v)\) of (8) satisfies

\[
\|u\|_{H^1(0, T; L^2(\Omega))} + \|v\|_{L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega))} + \|h\|_{L^2(0, T; L^2(\Omega))} \leq C,
\]

and

\[
(u, v)(T, \cdot) = (u_\pm, v_\mp).
\]
Remark 1.5. Let us make some comments on Theorem 1.4.

- For a given control $h \in L^2(Q_T)$, note that the solutions $(u, v) \in H^1(0, T; L^2(\Omega)) \times L^\infty(Q_T)$ to (8) are necessarily unique by using classical Gronwall’s argument. The existence of such a solution, associated with some specific control $h$, actually comes from a precise inverse mapping argument performed in the proof of Theorem 1.4.

- Theorem 1.4 is a uniform local controllability result with respect to the parameter $d_v \to +\infty$. Let us highlight the fact that the local controllability result of (8) still holds for $d_v \in (0, \infty)$, but the uniform bounds stated in (10) are not guaranteed in the limit $d_v \to 0$.

- Taking $v_0 \in H^1(\Omega)$ seems to be purely technical. It actually comes from the regularity assumptions on the trajectory of the linearized system (see Proposition 2.7) that are crucial to deal with the nonlinear term in the inverse mapping argument (see Proposition 3.3). One could probably prove a uniform local controllability result for $(u_0, v_0) \in L^2(\Omega)^2$ by splitting the time interval of control into two parts: first, take $h \equiv 0$ in $(0, T/2)$ and use a (quantitative) regularizing effect of the second equation of (8) to obtain that $v(T/2, \cdot) \in H^1(\Omega)$, then apply Theorem 1.4 in $(T/2, T)$. See for instance [18, Remark 2] for the application of such a technique.

- By adding a diffusion term $-d_u \partial_{xx} u, d_u > 0$, in the first equation of (8), fixing $d_v > 0$ and setting $\omega(t) = \omega$, any arbitrary nonempty open set contained in $(0, 1)$, we can easily adapt the proof of Theorem 1.4 to obtain a local controllability result to positive constant trajectories for the classical reaction–diffusion Gray–Scott model (cf. [1]).

- As for (6), system (8) is not locally controllable around $(1, 0)$, at least for smooth solutions. This comes from the fact that all solutions $(u, v) \in L^\infty((0, T) \times (0, 1)) \times L^\infty(0, T)$ to (6), reaching $(1, 0)$ in time $T$, satisfies necessarily $v \equiv 0$ in $(0, T) \times (0, 1)$. Indeed, setting $a(t, x) = uv - (F + k) \in L^\infty((0, T) \times (0, 1))$, rewriting the second equation, we obtain

$$
\partial_t v - d_v \partial_{xx} v = a(t, x)v, \ v(T) = 0.
$$

So, by backward uniqueness for parabolic equation with homogeneous Neumann boundary condition, we obtain that $v \equiv 0$ in $(0, T) \times (0, 1)$.

### 1.3. Strategy of the proof

In order to prove Theorem 1.2, a natural strategy would be to linearize (6) around $(u_\pm, v_\mp)$ to obtain

$$
\begin{cases}
\partial_t u = (-v^2_\mp - F)u - 2u_\pm v_\mp w + h1_{\omega(t)} & \text{in } (0, T) \times (0, 1), \\
w' = v^2_\mp \left( \int_0^1 u(t, x) \, dx \right) + (2u_\pm v_\mp - (F + k))w & \text{in } (0, T), \\
u(0, \cdot) = u_0(\cdot) \text{ in } (0, 1), \quad w(0) = w_0.
\end{cases}
$$

(12)
Heuristically, (12) seems to be null-controllable because \( h \) controls the first component \( u \) and the nonlocal coupling term \( v_\pm^2 \left( \int_0^1 u(t, x) \, dx \right) \) indirectly controls the second component \( w \). But, as far as we know, the classical tools in the literature are not well adapted to deal with the null-controllability of such a nonlocal system. That is why we follow a different approach to prove Theorem 1.2.

The method we employ for proving Theorem 1.2 is based on two key points.

First, we prove Theorem 1.4. In order to do this, we mainly follow [20] which establish the local controllability to trajectories for a nonlinear system of ODE–PDE in 1-D. However, two main differences appear comparing Theorem 1.4 and [20, Theorem 1.1]. The first one is the uniformity of the local controllability of (8) with respect to the parameter \( d_v \in (1, +\infty) \). The second one is the localization of the control in the ODE equation of (8), instead of the parabolic equation for [20, System (7)], see Sect. 4.3 for a brief discussion about this issue.

We give the main steps of the proof of Theorem 1.4.

- We first linearize (8) around \((u_\pm, v_\pm)\), and this leads us to study the uniform null-controllability of the linearized system satisfied by the variable \((U, V) = (u - u_\pm, v - v_\pm)\). This is done in Sect. 2.1.
- We prove a uniform observability estimate for the adjoint system of the linearized equations obtained in the previous step. This is done thanks to a uniform Carleman estimate, which is inspired by the arguments of [8, 20]. We highlight the fact that the restriction to the one spatial dimensional case appears in this part because Carleman estimates with similar weights for ODE–PDE have only been proved in 1-D when considering homogeneous Neumann boundary conditions. This is done in Sect. 2.2.
- We deduce from the observability estimate and classical duality arguments, the null-controllability of the linearized system with a source term, exponentially decreasing at \( t = T \) (see, for instance, [5, Theorem 2.44]) when the source is equal to zero. We also prove some extra regularity results on the controlled trajectory. This part is actually crucial to pass to the nonlinear result. This type of argument is inspired from [14, Chapter I, Section 4] and [7]. This is done in Sect. 2.3.
- We use a precise inverse mapping argument to deduce from the (global) null-controllability of the linearized system a local null-controllability result for the nonlinear system satisfied by \((U, V) = (u - u_\pm, v - v_\pm)\). Note that the regularity of the nonlinear mapping is obtained thanks to the extra regularity of the linear controlled trajectory proved in the previous step. This is done in Sect. 3.1.

Secondly, we prove Theorem 1.2 by using Theorem 1.4 and the shadow limit method. Roughly, we obtain that the solution \((u_{d_v}, v_{d_v}, h_{d_v})\) of (8) converges in some sense to \((u, w, h)\) the solution of (6) as \( d_v \to +\infty \). This method relies on an adaptation of the arguments presented in [22, Appendix A]. For the use of such a method in the context of control theory, see [19] and [18].
2. Null-controllability of the linearized system

2.1. Change of variable and linearized system

By setting \((U, V) = (u-u_\pm, v-v_\mp)\), where \((u, v)\) is the solution to (8), we obtain that \((U, V)\) satisfies

\[
\begin{align*}
\partial_t U &= a_{11} U + a_{12} V + N_1(U, V) + h_1 \omega(t) & \text{in } (0, T) \times (0, 1), \\
\partial_t V - d_v \partial_{xx} V &= a_{21} U + a_{22} V + N_2(U, V) & \text{in } (0, T) \times (0, 1), \\
\partial_x V &= 0 & \text{on } (0, T) \times \{0, 1\}, \\
(U, V)(0, \cdot) &= (U_0, V_0) & \text{in } (0, 1),
\end{align*}
\]

with

\[
a_{11} = -v_\mp^2 - F, \quad a_{12} = -2u_\pm v_\mp, \quad a_{21} = v_\mp^2, \quad a_{22} = 2u_\pm v_\mp - (F + k),
\]

\[
N(U, V) := \begin{pmatrix} N_1(U, V) \\ N_2(U, V) \end{pmatrix} := \begin{pmatrix} -(UV^2 + 2v_\mp UV + u_\pm V^2) \\ UV^2 + 2v_\mp UV + u_\pm V^2 \end{pmatrix}.
\]

The goal of Sect. 2 is to prove the null-controllability of the linearized system

\[
\begin{align*}
\partial_t U &= a_{11} U + a_{12} V + F_1 + h_1 \omega(t) & \text{in } (0, T) \times (0, 1), \\
\partial_t V - d_v \partial_{xx} V &= a_{21} U + a_{22} V + F_2 & \text{in } (0, T) \times (0, 1), \\
\partial_x V &= 0 & \text{on } (0, T) \times \{0, 1\}, \\
(U, V)(0, \cdot) &= (U_0, V_0) & \text{in } (0, 1),
\end{align*}
\]

where \(F_1, F_2\) are source terms belonging to an appropriate Banach space \(X\) and exponentially decreasing at \(t = T\) (see Proposition 2.7). This would be indeed possible thanks to

\[
a_{21} \neq 0,
\]

using (14) because \(v_\mp \neq 0\). Heuristically, \(h\) directly controls the component \(U\) thanks to the first equation of (16) and \(U\) indirectly controls \(V\) thanks to the coupling term \(a_{21} U\), appearing in the second equation of (16).

Our objective then will be to prove that we can find \(h\), bounded independently of \(d_v \in (1, +\infty)\) such that the solution \((U, V)\) of (16) satisfies \((U, V)(T, \cdot) = 0\). Moreover, we want that the nonlinear quantity \(N(U, V)\) belongs to \(X\) to use at the end an inverse mapping argument to obtain the controllability of (8) around \((u_\pm, v_\mp)\) (see Sect. 3.1).

The following standard proposition, stated without proof, guarantees the well-posedness of (16). It can be established for instance by using Galerkin approximations.

**Proposition 2.1.** For every \((F_1, F_2) \in L^2(Q_T)^2\), \((U_0, V_0) \in L^2(\Omega)^2\), the system (16) admits a unique weak solution \((U, V) \in [H^1(0, T; L^2(\Omega))] \times [L^2(0, T; H^1(\Omega)) \cap H^1(0, T; H^1(\Omega)')] \).
2.2. Observability of the adjoint system

In order to prove the null-controllability of the linearized system (16), we will first prove an observability inequality for the following adjoint system

\[
\begin{cases}
-\partial_t \phi = a_{11} \phi + a_{21} \psi + g_1 & \text{in } Q_T, \\
-\partial_t \psi = d \nu \partial_{xx} \psi + a_{12} \phi + a_{22} \psi + g_2 & \text{in } Q_T, \\
\partial_x \psi = 0 & \text{on } \Sigma_T, \\
(\phi, \psi)(T, \cdot) = (\phi_T, \psi_T) & \text{in } (0, 1).
\end{cases}
\] (18)

where \( g_1, g_2 \in L^2(Q_T) \) are given source terms. More precisely, we will show that there exists a positive constant \( C > 0 \) such that for every \( (\phi_T, \psi_T) \in L^2(\Omega)^2 \), the solution \( (\phi, \psi) \) of (18) satisfies

\[
\| (\phi, \psi)(0, \cdot) \|_{L^2(\Omega)^2} \leq C \left( \iint_{Q_T} \left[ |g_1(t, x)|^2 + |g_2(t, x)|^2 \right] \, dr \, dx + \int_0^T \int_{\omega(t)} |\phi(t, x)|^2 \, dr \, dx \right). (19)
\]

Note that the null-controllability of (16) when \( F_1 = F_2 = 0 \) is a direct consequence of (19), thanks to a duality argument, called Hilbert uniqueness method (see [5, Theorem 2.44]).

For proving such an observability inequality (19), we will use Carleman estimates.

2.2.1. Preliminaries on Carleman inequalities with moving controls

The ideas presented below were mainly developed in [8,9,20]. In particular, the results presented in [20] are at the heart of our methodology.

We begin by recalling that \( \omega \) satisfies Assumption 1.1. Let us consider \( \omega_i : [0, T] \to 2^{(0, 1)}, i = 1, 2 \), subsets of \( \omega \), such that

\[
\overline{\omega}_0 \subset \overset{\circ}{\omega}_1, \quad \overline{\omega}_1 \subset \overset{\circ}{\omega}_2, \quad \text{and } \overline{\omega}_2 \subset \overset{\circ}{\omega}, \quad (20)
\]

where \( \overset{\circ}{\omega}_i, i = 1, 2 \) and \( \overset{\circ}{\omega} \) stand for the relative interiors with respect to \( [0, 1] \times [0, T] \) of \( \overline{\omega}_i \) and \( \overline{\omega} \), respectively.

As usual in the framework of Carleman estimates, the first step is to construct a suitable weight function. The function introduced below allows to obtain a Carleman inequality for parabolic equations coupled to ordinary differential equations, which is exactly the structure of the adjoint system (18).
Lemma 2.2. There exist a positive number \( \tau \in (0, \min\{1, T/2\}) \), a positive constant \( C_0 > 0 \), and a function \( \eta \in C^\infty([0, 1] \times [0, T]) \) such that

\[
\begin{align*}
\eta_x(t, x) &\neq 0 \quad \forall t \in [0, T], \ \forall x \in (0, 1) \setminus \omega_0(t), \\
\eta_t(t, x) &\neq 0 \quad \forall t \in [0, T], \ \forall x \in (0, 1) \setminus \omega_0(t), \\
\eta_t(t, x) &> 0 \quad \forall t \in [0, \tau], \ \forall x \in (0, 1) \setminus \omega_0(t), \\
\eta_t(t, x) &< 0 \quad \forall t \in [T - \tau, T], \ \forall x \in (0, 1) \setminus \omega_0(t), \\
\eta_x(t, 0) &\geq C_0 \quad \forall t \in [0, T], \\
\eta_x(t, 1) &\leq -C_0 \quad \forall t \in [0, T], \\
\min_{(t, x) \in [0, T] \times [0, 1]} \{\eta(t, x)\} &\geq \frac{1}{4} ||\eta||_{L^\infty(Q_T)}.
\end{align*}
\]

The proof of Lemma 2.2 can be obtained as in [8, Lemma 4.3] with the observation of [20, Lemma 1] stating that precise values for the derivative of the weight at the boundary (25)–(26) have a prescribed sign.

Note that the conditions (25)–(26) can be reformulated as

\[
\forall (t, x) \in (0, T) \times \partial \Omega, \ \frac{\partial \eta}{\partial n}(t, x) \leq -C_0,
\]

where \( n = n(x) \) is the outward unit normal vector to the point \( x \in \partial \Omega \). This assumption seems to be crucial for establishing the Carleman estimate stated in Lemma 2.3. One can see in particular the estimate (95) in “Appendix A” for dealing with the boundary terms. Constructing a weight \( \eta \) satisfying the hypotheses of [8, Lemma 4.3] together with (28) in the multidimensional case is actually an interesting open problem.

Now, let us introduce a function \( r \in C^\infty(0, T) \), symmetric with respect to \( t = \frac{T}{2} \) (more precisely, \( r(t) = r(T - t) \) for any \( t \in (0, T) \)) and such that

\[
r(t) = \begin{cases} 
\frac{1}{t} & \text{for } 0 < t \leq \tau/2, \\
\text{strictly decreasing} & \text{for } \frac{\tau}{2} < t < \tau, \\
1 & \text{for } \tau \leq t \leq T/2.
\end{cases}
\]

For any parameter \( \lambda > 0 \), let us define the weights

\[
\alpha(t, x) := r(t)e^{2\lambda||\eta||_\infty} - e^{\lambda \eta(t, x)} \quad \text{and} \quad \xi(t, x) := r(t)e^{\lambda \eta(t, x)}, \quad \forall (t, x) \in Q_T.
\]

We have the following uniform Carleman estimate for the heat equation with homogeneous Neumann boundary conditions.

Lemma 2.3. For any \( 0 < \epsilon \leq 1 \), there exist positive constants \( \lambda_1, s_1 \), and \( C \), depending on \( \omega_1 \), such that for any \( \lambda \geq \lambda_1, s \geq s_1(\lambda), \) the solution \( \psi \) to

\[
\begin{cases} 
-\partial_t \psi - \frac{1}{\epsilon} \partial_{xx} \psi = f \quad \text{in } Q_T, \\
\psi_x = 0 \quad \text{on } \Sigma_T, \\
\psi(T, \cdot) = \psi_T \quad \text{in } (0, 1).
\end{cases}
\]

with \( \psi_T \in L^2(0, 1) \) and \( f \in L^2(0, T; L^2(0, 1)) \) verifies

\[
I(\psi; \epsilon) \leq C \left( \epsilon^2 \int_Q |f|^2 e^{-2s\alpha} \, dx \, dt + s^3 \lambda^4 \int_{\Omega_1(t) \times (0, T)} \xi^3 |\psi|^2 \, dx \, dt \right)
\]  
(31)

where \( I(\psi, \epsilon) \) stands for

\[
I(\psi, \epsilon) := s^{-1} \int_{Q_T} \xi^{-1} (\epsilon^2 |\partial_t \psi|^2 + |\partial_{xx} \psi|^2) e^{-2s\alpha} \, dx \, dt + s \lambda^2 \int_{Q_T} \xi |\partial_x \psi|^2 e^{-2s\alpha} \, dx \, dt + s^3 \lambda^4 \int_{Q_T} \xi^3 |\psi|^2 e^{-2s\alpha} \, dx \, dt + s^3 \lambda^3 \int_0^T (\xi^2 |\psi|^2 e^{-2s\alpha}) \, dt \bigg|_{x=1} + s^3 \lambda^3 \int_0^T (\xi^2 |\partial_x \psi|^2 e^{-2s\alpha}) \, dt \bigg|_{x=0}.
\]

The proof of this result follows the methodology of [20, Appendix A] and pays special attention to the dependency of \( \epsilon \) during the computations. We give a sketch of the proof in “Appendix A”. Note that the important properties of the weights \( \eta \) for obtaining the parabolic Carleman estimate (31) are (21), (25), (26).

We have the following Carleman estimate for ODE, coming from [8, Lemma 4.5].

**Lemma 2.4.** There exist some numbers \( \lambda_1 \geq \lambda_0, s_1 \geq s_0 \) and \( C_1 > 0 \) such that for all \( \lambda \geq \lambda_1, \) all \( s \geq s_1 \) and all \( q \in H^1(0, T; L^2(0, 1)) \), the following holds

\[
I(q) := s \lambda^2 \int_{Q_T} \xi |q|^2 e^{-2s\alpha} \, dx \, dt \leq C_1 \left( \int_{Q_T} |q|^2 e^{-2s\alpha} \, dx \, dt + \lambda^2 \int_{\Omega_1(t) \times (0, T)} (s \xi^2 |q|^2 e^{-2s\alpha} \, dx \, dt \right).
\]  
(32)

Note that the important properties of the weights \( \eta \) for obtaining the ODE Carleman estimate (32) are (22), (23), (24).

**2.2.2. A uniform observability inequality**

Let us introduce the following useful notations

\[
\alpha^*(t) = \min_{x \in [0, 1]} \alpha(t, x), \quad \widehat{\alpha}(t) = \max_{x \in [0, 1]} \alpha(t, x),
\]

\[
\xi^*(t) = \max_{x \in [0, 1]} \xi(t, x), \quad \widehat{\xi}(t) = \min_{x \in [0, 1]} \xi(t, x).
\]  
(33)

We have the following uniform Carleman estimate for the solution to (18).

**Proposition 2.5.** There exist positive constants \( \lambda_2 > 0, s_2 > 0 \) and \( C > 0 \), such that for any \( d_v \geq 1, \lambda \geq \lambda_2, \) s \( \geq s_2(\lambda) \) and any initial data \((\phi_T, \psi_T) \in L^2(\Omega)^2 and \)
source terms \((g_1, g_2) \in L^2(Q_T)^2\), the solution to (18) verifies
\[
\begin{align*}
\int_Q e^{-2s\alpha} |\phi|^2 \, dx \, dt + s^3 \int_Q e^{-2s\alpha} \xi^3 |\psi|^2 \, dx \, dt \\
\leq C \left( \int_{\omega_2(t) \times (0,T)} e^{-4s\alpha + 2s\alpha}(\xi^*)^{8} |\phi|^2 \, dx \, dt \\
+ s^3 \int_Q e^{-2s\alpha} \xi^3 |g_1|^2 \, dx \, dt + \int_Q e^{-2s\alpha} |g_2|^2 \, dx \, dt \right).
\end{align*}
\]

Proof. We divide the proof in several steps. In the following, the positive constants \(C > 0\) vary from line to line and are independent of the parameters \(d_v, \lambda, s\).

\textbf{Step 1: First estimates}

We apply the Carleman estimate (32) to the first equation of (18)
\[
I(\phi) \leq C \left( \int_Q e^{-2s\alpha} (|\phi|^2 + |\psi|^2 + |g_1|^2) \, dx \, dt + \int_{\omega_1(t) \times (0,T)} e^{-2s\alpha} (s\lambda\xi)^2 (|\phi|^2) \, dx \, dt \right).
\]

We apply inequality (31) to the PDE in (18) with \(\epsilon = 1/d_v\)
\[
I(\psi; d_v^{-1}) \leq C \left( d_v^{-2} \int_Q (|\phi|^2 + |\psi|^2 + |g_2|^2) e^{-2s\alpha} \, dx \, dt \right) + s^3 \lambda^4 \int_{\omega_1(t) \times (0,T)} \xi^3 |\psi|^2 \, dx \, dt.
\]

Adding up (36) and (35), we can use the parameters \(\lambda, s\) to absorb all the lower order terms in the right-hand side. More precisely, we get for \(\lambda, s\) sufficiently large
\[
I(\phi) + I(\psi; d_v^{-1})
\leq C \left( \int_{\omega_1 \times (0,T)} e^{-2s\alpha} (s\lambda\xi)^2 |\phi|^2 \, dx \, dt \right) + s^3 \lambda^4 \int_{\omega_1(t) \times (0,T)} e^{-2s\alpha} \xi^3 |\psi|^2 \, dx \, dt \\
+ \int_Q e^{-2s\alpha} \left( |g_1|^2 + |g_2|^2 \right) \, dx \, dt.
\]

\textbf{Step 2: Local estimate for \(\psi\)}

From (20), let us consider a function \(\zeta \in C^\infty([0, T] \times [0, 1])\) verifying
\[
\begin{align*}
0 \leq \zeta \leq 1 \quad \forall (t, x) \in [0, T] \times [0, 1], \\
\zeta(t, x) = 1 \quad \forall t \in [0, T], \ \forall x \in \omega_1(t), \\
\zeta(t, x) = 0 \quad \forall t \in [0, T], \ \forall x \in [0, L] \setminus \omega_2(t).
\end{align*}
\]

We have
\[
\begin{align*}
s^3 \int_{\omega_1(t) \times (0,T)} e^{-2s\alpha} \xi^3 |\psi|^2 \, dx \, dt & \leq s^3 \int_Q e^{-2s\alpha} \xi^3 |\psi|^2 \, dx \, dt \\
& = \frac{1}{d_2} \int_Q e^{-2s\alpha} \xi^3 \zeta(\psi_t - a_{11} \phi - g_1) \, dx \, dt.
\end{align*}
\]
Observe that at this point is crucial to have $a_{21} \neq 0$ by (17).

Integrating by parts in time in the right-hand side yields

$$s^3 \int_{\omega_1(t) \times (0,T)} e^{-2s\alpha \xi^3} |\psi|^2 \, dx \, dt \leq \frac{1}{a_{21}} s^3 \int_{Q_T} (e^{-2s\alpha \xi^3} \xi)_t \psi \phi \, dx \, dt + \frac{1}{a_{21}} s^3 \int_{Q_T} e^{-2s\alpha \xi^3} \xi \psi \phi \, dx \, dt$$

$$- \frac{a_{11}}{a_{21}} s^3 \int_{Q_T} e^{-2s\alpha \xi^3} \xi |\psi|^2 \, dx \, dt - \frac{1}{a_{21}} s^3 \int_{Q_T} e^{-2s\alpha \xi^3} \xi \psi g_1 \, dx \, dt.$$

Using the equation verified by $\psi$ in the second term on the right-hand side of the above equation, we get

$$s^3 \int_{\omega_1(t) \times (0,T)} e^{-2s\alpha \xi^3} |\psi|^2 \, dx \, dt$$

$$\leq \frac{1}{a_{21}} s^3 \int_{Q_T} (e^{-2s\alpha \xi^3} \xi)_t \psi \phi \, dx \, dt - \frac{d_v}{a_{21}} s^3 \int_{Q_T} e^{-2s\alpha \xi^3} \xi \psi_{xx} \phi \, dx \, dt$$

$$- \frac{a_{12}}{a_{21}} s^3 \int_{Q_T} e^{-2s\alpha \xi^3} \xi |\phi|^2 \, dx \, dt - \frac{a_{22}}{a_{21}} s^3 \int_{Q_T} e^{-2s\alpha \xi^3} \xi \psi \phi \, dx \, dt$$

$$- \frac{a_{11}}{a_{21}} s^3 \int_{Q_T} e^{-2s\alpha \xi^3} \xi \psi \phi \, dx \, dt - \frac{1}{a_{21}} s^3 \int_{Q_T} e^{-2s\alpha \xi^3} \xi \psi g_1 \, dx \, dt$$

$$=: \sum_{i=1}^{6} K_i. \quad (38)$$

We now bound each term $K_i$ for $1 \leq i \leq 6$. For the first one, we have

$$K_1 = \frac{1}{a_{21}} s^3 \int_{Q_T} e^{-2s\alpha \xi^3} \xi_t \psi \phi \, dx \, dt + \frac{1}{a_{21}} s^3 \int_{Q_T} (e^{-2s\alpha \xi^3} \xi)_t \xi \psi \phi \, dx \, dt.$$

Using the properties of the function $\xi$ and $|e^{-2s\alpha \xi^3} \xi_t| \leq Cs^2 e^{-2s\alpha \xi^5}$, we get after applying Cauchy–Schwarz and Young inequalities that

$$|K_1| \leq 2\delta^3 s^3 \int_{Q_T} e^{-2s\alpha \xi^3} |\psi|^2 \, dx \, dt + C_\delta s^7 \int_{\omega_1(t) \times (0,T)} e^{-2s\alpha \xi^7} |\phi|^2 \, dx \, dt \quad (39)$$

for any $\delta > 0$.

We can use definitions (33) and Young’s inequality to obtain

$$|K_2| \leq d_v s^{-2} \int_{Q_T} e^{-2s\alpha \xi^3} |\psi_{xx}|^2 \, dx \, dt + C s^8 \int_{\omega_1(t) \times (0,T)} e^{-4s\alpha + 2s\alpha (\xi^3)^2} (\xi^6) |\phi|^2 \, dx \, dt$$

$$\leq d_v s^{-2} \int_{Q_T} e^{-2s\alpha \xi^3} \xi|\psi_{xx}|^2 \, dx \, dt + C s^8 \int_{\omega_1(t) \times (0,T)} e^{-4s\alpha + 2s\alpha (\xi^3)^8} |\phi|^2 \, dx \, dt. \quad (40)$$

Observe that the constant $C > 0$ is uniform with respect to $d_v$ and that also we have introduced a smaller weight accompanying the variable $\psi_{xx}$. In a future step, we will estimate uniformly this new term.
For $3 \leq i \leq 6$, we can bound easily $K_i$. Using Cauchy–Schwarz and Young inequalities, a straightforward computation gives

$$
\sum_{i=3}^{6} |K_i| \leq C_\delta \left( s^3 \int_{Q_T} e^{-2s\alpha \xi^3} |g_1|^2 \, dx \, dt + s^3 \int_{\omega_2(t) \times (0,T)} e^{-2s\alpha \xi^3} |\phi|^2 \, dx \, dt \right)
+ 3\delta s^3 \int_{Q_T} e^{-2s\alpha \xi^3} |\psi|^2 \, dx \, dt
$$

(41)

for all $\delta > 0$.

Putting together (37), (38), (39), (40), and (41), we can take $\delta$ sufficiently small and obtain

$$
I(\phi) + I(\psi; d_v^{-1})
\leq C \left( s^8 \int_{\omega_2 \times (0,T)} (e^{-4s\alpha^* + 2s\alpha} + e^{-2s\alpha})(\xi^*)^8 |\phi|^2 \, dx \, dt 
+ d_v s^{-2} \int_{Q_T} e^{-2s\alpha}(\hat{\xi})^{-2} |\psi_{xx}|^2 \, dx \, dt + \int_{Q_T} e^{-2s\alpha} |g_2|^2 \, dx \, dt 
+ s^3 \int_{Q_T} e^{-2s\alpha \xi^3} |g_1|^2 \, dx \, dt \right)
$$

(42)

for all $\lambda$ and $s$ sufficiently large.

**Step 4: Uniform global estimate of $\psi_{xx}$ and conclusion**

We devote this step to estimate the global term of $\psi_{xx}$ appearing in (42). Notice that this integral has a factor $d_v^2$ so we need to estimate it uniformly with respect to the parameter $d_v$.

From the PDE verified in (18), we have

$$
d_v \int_{Q_T} |\psi_{xx}|^2 e^{-2s\alpha}(\hat{\xi})^{-2} \, dx \, dt
$$

$$
= - \int_{Q_T} \psi_t \psi_{xx} e^{-2s\alpha}(\hat{\xi})^{-2} \, dx \, dt - a_{12} \int_{Q_T} \phi \psi_{xx} e^{-2s\alpha}(\hat{\xi})^{-2} \, dx \, dt
- a_{22} \int_{Q_T} \psi \psi_{xx} e^{-2s\alpha}(\hat{\xi})^{-2} \, dx \, dt
- \int_{Q_T} g_2 \psi_{xx} e^{-2s\alpha}(\hat{\xi})^{-2} \, dx \, dt.
$$

(43)

Integrating by parts in space on the first term on the right-hand side of (43) and using Cauchy–Schwarz and Young inequalities on the other two, we readily get

$$
d_v \int_{Q_T} |\psi_{xx}|^2 e^{-2s\alpha}(\hat{\xi})^{-2} \, dx \, dt
\leq \frac{1}{2} \int_{Q_T} \left( |\psi_x|^2 \right)_t e^{-2s\alpha}(\hat{\xi})^{-2} \, dx \, dt + 3\delta d_v \int_{Q_T} |\psi_{xx}|^2 e^{-2s\alpha}(\hat{\xi})^{-2} \, dx \, dt
+ C_\delta d_v^{-1} \int_{Q_T} |\phi|^2 e^{-2s\alpha}(\hat{\xi})^{-2} \, dx \, dt + C_\delta d_v^{-1} \int_{Q_T} |\psi|^2 e^{-2s\alpha}(\hat{\xi})^{-2} \, dx \, dt
$$
for any $\delta > 0$. Observe that we have put the parameter $d_v^{-1}$ in front of three of the right-hand side terms, however, the first one is still missing it. Further integration by parts in the time variable and then integrating in space yields

\[
d_v \int_Q \left| \psi_{xx} \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt \leq \frac{1}{2} \int_Q \left| \psi_{xx} \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt + 3\delta d_v \int_Q \left| \psi_{xx} \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt
\]

\[
+ C_\delta d_v^{-1} \int_Q \left| \phi \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt + C_\delta d_v^{-1} \int_Q \left| \psi \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt
\]

\[
+ C_\delta d_v^{-1} \int_Q \left| g_2 \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt,
\]

where we have used that the weight functions are $x$-independent and $\psi$ satisfies homogeneous Neumann boundary conditions.

Arguing as we did for obtaining (44) and using that $|e^{-2\alpha (\hat{\xi})^{-2}}| \leq C s^2 e^{-2\alpha}$, we get

\[
d_v \int_Q \left| \psi_{xx} \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt \leq C_\delta d_v^{-1} s^4 \int_Q \left| \psi \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx d\xi + 4\delta d_v \int_Q \left| \psi_{xx} \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt
\]

\[
+ C_\delta d_v^{-1} \int_Q \left| \phi \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt + C_\delta d_v^{-1} \int_Q \left| \psi \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt
\]

\[
+ C_\delta d_v^{-1} \int_Q \left| g_2 \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt,
\]

for any $\delta > 0$. Taking $\delta$ small enough and then multiplying by $s^{-2} d_v$ on both sides of (45), we obtain the uniform estimate

\[
d_v^2 s^{-2} \int_Q \left| \psi_{xx} \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt
\]

\[
\leq C s^2 \int_Q \left| \psi \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt + C s^{-2} \int_Q \left| \phi \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt
\]

\[
+ C s^{-2} \int_Q \left| \psi \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt + C s^{-2} \int_Q \left| g_2 \right|^2 e^{-2\alpha (\hat{\xi})^{-2}} \, dx \, dt.
\]

(46)
To conclude the proof, we use that \( e^{-2\alpha \tilde{\xi}_t} \leq e^{-2\alpha \xi} \), \( s^{-1}(\tilde{\xi})^{-1} \leq C \) and \( \tilde{\xi} \leq \xi \), to obtain from the above estimate (46) that

\[
d_{\ell}^2 s^{-2} \int_{Q_t} |\psi| e^{-2\alpha \tilde{\xi}_t} dxdt \leq C s^2 \int_{Q_t} |\psi| e^{-2\alpha \xi}^2 dxdt + C \int_{Q_t} |\phi|^2 e^{-2\alpha \xi} dxdt \\
+ C \int_{Q_T} |\psi|^2 e^{-2\alpha \xi} dxdt + C \int_{Q_T} |g_2|^2 e^{-2\alpha \xi} dxdt. \tag{47}
\]

To conclude, we plug (47) into (42) and employ the parameter \( s \) to absorb the remaining global terms. The result follows by using (33) to show that \( e^{-2\alpha \xi} \leq e^{-4\alpha s^* + 2\alpha \tilde{\xi}} \).

\[\square\]

We are going to improve the Carleman inequality (34) in the sense that the new weight functions, that we will use, will only vanish as \( t \to T^- \). To this end, let us consider the function

\[
\ell(t) = \begin{cases} 
1 & \text{for } 0 \leq t \leq T/2, \\
\ell(t) & \text{for } T/2 \leq t \leq T,
\end{cases}
\]

where we recall that \( r(t) \) is defined in (29). We introduce the new weight functions

\[
\beta(t, x) := \ell(t)(e^{2\lambda \eta}) \to e^{2\lambda \eta(t, x)}), \quad \gamma(t, x) := \ell(t)e^{\lambda \eta(t, x)}, \\
\hat{\beta}(t) := \max_{x \in [0, 1]} \beta(t, x), \quad \beta^*(t) := \min_{x \in [0, 1]} \beta(t, x), \\
\hat{\gamma}(t) := \min_{x \in [0, 1]} \gamma(t, x), \quad \gamma^*(t) := \max_{x \in [0, 1]} \gamma(t, x).
\]

(48)

Observe that in this case, the corresponding weight functions only blow up as \( t \to T^- \).

We deduce from Proposition 2.5 and energy estimates the following Carleman estimate.

**Proposition 2.6.** There exist \( \lambda, s \) sufficiently large and a positive constant \( C = C(s, \lambda, T) > 0 \) such that for any given \( d_v \geq 1 \), \( (\phi_T, \psi_T) \in L^2(\Omega)^2 \), the solution to (18) verifies

\[
\|\phi(0, \cdot)\|^2_{L^2(\Omega)} + \|\psi(0, \cdot)\|^2_{L^2(\Omega)} + \int_{Q_T} e^{-2\lambda \hat{\eta}} \hat{\gamma} |\phi|^2 dxdt \\
+ \int_{Q_T} e^{-2\alpha \hat{\beta}(\hat{\gamma})^3} |\psi|^2 dxdt \leq C \left( \int_{Q_T} e^{-2\alpha \beta^*(\gamma^*)^3} |g_1|^2 dxdt \\
+ \int_{Q_T} e^{-2\beta^*} |g_2|^2 dxdt + \int_{\omega_2(t) \times (0, T)} e^{-2\beta^*} (\gamma^*)^8 |\phi|^2 dxdt \right). \tag{49}
\]

The proof of this result can be found in “Appendix B”. Increasing the constant \( C \) in (49) if necessary, we can assume that

\[
\forall t \in (0, T), \quad \beta(t) < \frac{3}{2} \beta^*(t). \tag{50}
\]
2.3. Null-controllability despite a source term

Now we proceed to the definition of the spaces where the linear system (16) will be solved. We define the differential operator

\[ L(U, V) = (\partial_t U - a_{11} U - a_{12} V, \partial_t V - d_v \partial_{xx} V - a_{21} U - a_{22} V), \]

and define the space

\[ W_{T, \Omega} := L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; L^2(\Omega)). \]

Let us set

\[
E := \left\{ (U, V, h) \in E_0 : e^{s \hat{\beta}} (\hat{\gamma})^{-1/2} ((L(U, V))_1 - h \mathbf{1}_{\omega(t)}) \in L^2(Q_T), e^{s \hat{\beta}} (\hat{\gamma})^{-3/2} (L(U, V))_2 \in L^2(Q_T) \text{ and } \partial_x V = 0 \text{ on } \Sigma_T \right\},
\]

where

\[
E_0 := \left\{ (U, V, h) : e^{s \beta^* (y^*)^{-3/2}} U \in L^2(Q_T), e^{s \beta^* V} \in L^2(Q_T), e^{(s/2) \beta^* (y^*)^{-4}} \mathbf{1}_{\omega(t)} h \in L^2(Q_T), e^{s \hat{\beta} - (s/2) \beta^* (\hat{\gamma})^{-1/4}} (U, V) \in H^1(0, T; L^2(\Omega)) \times W_{T, \Omega}, e^{(s/2) \beta^* (\hat{\gamma})^{-1/4}} (U, V) \in H^1(0, T; L^2(\Omega)) \times W_{T, \Omega} \right\}.
\]

We endow \( E \) with the following norm

\[
\| (U, V, h) \|_E := \left\| e^{s \beta^* (y^*)^{-3/2}} U \right\|_{L^2(Q_T)} + \left\| e^{s \beta^* V} \right\|_{L^2(Q_T)} + \left\| e^{(s/2) \beta^* (y^*)^{-4}} \mathbf{1}_{\omega(t)} \right\|_{L^2(Q_T)}
\]

\[
+ \left\| e^{s \hat{\beta} - (s/2) \beta^* (\hat{\gamma})^{-1/4}} (U, V) \right\|_{H^1(0, T; L^2(\Omega)) \times W_{T, \Omega}}
\]

\[
+ \left\| e^{(s/2) \beta^* (\hat{\gamma})^{-1/4}} (U, V) \right\|_{H^1(0, T; L^2(\Omega)) \times W_{T, \Omega}}
\]

\[
+ \left\| e^{s \hat{\beta}} (\hat{\gamma})^{-1/2} ((L(U, V))_1 - h \mathbf{1}_{\omega(t)}) \right\|_{L^2(Q_T)} + \left\| e^{s \hat{\beta}} (\hat{\gamma})^{-3/2} (L(U, V))_2 \right\|_{L^2(Q_T)},
\]

that makes \( (E, \| \cdot \|_E) \) a Banach space.

We also introduce

\[
X = \{(F_1, F_2) : e^{s \hat{\beta}} (\hat{\gamma})^{-1/2} F_1 \in L^2(Q_T), e^{s \hat{\beta}} (\hat{\gamma})^{-3/2} F_2 \in L^2(Q_T)\},
\]

\[
Y = L^2(\Omega) \times H^1(\Omega),
\]

\[
G = X \times Y,
\]

endowed with their natural norm.

The goal of this part is to prove the following result.
Proposition 2.7. For every \((F_1, F_2, U_0, V_0) \in G\), there exists a control \(h \in L^2(0, T; L^2(\Omega))\), bounded independently of \(d_v\), such that, if \((U, V)\) is the associated solution to (16), one has \((U, V, h) \in E\). In particular, \((U, V)(T, \cdot) = 0\) holds.

Moreover, there exists a positive constant \(C > 0\) independent of \(d_v\) such that
\[
\|(U, V, h)\|_E \leq C \|(F_1, F_2, U_0, V_0)\|_G.
\] (55)

Proof. Let \(L^*\) be the adjoint operator of \(L\), defined in (51),
\[
L^*(\phi, \psi) = (-\partial_t \phi - a_{11} \phi - a_{21} \psi, -\partial_t \psi - d_v \partial_{xx} \psi - a_{12} \phi - a_{22} \psi).
\]
and let us introduce the space
\[
P_0 := \{ (\phi, \psi) \in C^\infty(\overline{Q_T})^2; \partial_x \psi = 0 \text{ on } \Sigma_T \}.
\]
We now define the following bilinear form, for \((\phi_1, \psi_1), (\phi_2, \psi_2)\), as
\[
a((\phi_1, \psi_1), (\phi_2, \psi_2)) \colon= \iint_{Q_T} e^{-2s\beta^*}(\gamma^*)^3 \left( (L^*(\phi_1, \psi_1))(L^*(\phi_2, \psi_2)) \right) \mathrm{d}x\mathrm{d}t + \iint_{Q_T} e^{-2s\beta^*}(L^*(\phi_1, \psi_1))^2L^*(\phi_2, \psi_2) \mathrm{d}x\mathrm{d}t + \iint_{Q_T} e^{-s\beta^*}(\gamma^*)^3 \mathbf{1}_{\omega(t)}^2 \phi_1 \phi_2 \mathrm{d}x\mathrm{d}t.
\]
The bilinear symmetric positive form \(a\) on \(P_0\) is definite. Indeed if \(a((\phi, \psi), (\phi, \psi)) = 0\) then the right-hand side of the Carleman estimate (49) is equal to 0 then \(\phi = \psi \equiv 0\). So \(a(\cdot, \cdot)\) is a scalar product on \(P_0\). Therefore, we can consider the space \(P\), the completion of \(P_0\) with respect to the norm associated with the scalar product defined by \(a\), denoted by \(\|\cdot\|_P\). This makes \((P, \|\cdot\|_P)\) a Hilbert Space and \(a(\cdot, \cdot)\) is a coercive, continuous, bilinear form on \(P\).

We now introduce the linear form \(l\), for \((\phi, \psi) \in P\),
\[
l((\phi, \psi)) = \iint_{Q_T} F_1 \phi + \iint_{Q_T} F_2 \psi \mathrm{d}x\mathrm{d}t + \int_{\Omega} U_0(x)\phi(0, x) + \int_{\Omega} V_0(x)\psi(0, x) \mathrm{d}x
\]
It is easy to show that \(l\) is a continuous linear form on \(P\) thanks to the Carleman estimate (49),
\[
|l((\phi, \psi))| \leq C \|(F_1, F_2, U_0, V_0)\|_G \|(\phi, \psi)\|_P.
\] (56)

Consequently, by using Lax-Milgram’s lemma, there exists a unique \((\tilde{\phi}, \tilde{\psi}) \in P\) satisfying
\[
\forall (\phi, \psi) \in P, \quad a((\tilde{\phi}, \tilde{\psi}), (\phi, \psi)) = l((\phi, \psi)).
\] (57)

We set
\[
(\tilde{U}, \tilde{V}) = (e^{-2s\beta^*}(\gamma^*)^3(L^*(\tilde{\phi}, \tilde{\psi})), e^{-2s\beta^*}(L^*(\tilde{\phi}, \tilde{\psi})), e^{-s\beta^*}(\gamma^*)^3 \mathbf{1}_{\omega(t)}^2 \tilde{\phi}, e^{-s\beta^*}(\gamma^*)^3 \mathbf{1}_{\omega(t)}^2 \tilde{\phi}).
\] (58)
We easily deduce from (56), (57) with \((\phi, \psi) = (\tilde{\phi}, \tilde{\psi})\) and (58) that
\[
\|e^{s\tilde{\beta}^*}(\gamma^*)^{-3/2}\tilde{U}\|_{L^2(Q_T)} + \|e^{s\tilde{\beta}^*}\tilde{V}\|_{L^2(Q_T)} + \|e^{(s/2)\tilde{\beta}^*}(\gamma^*)^{-4}\mathbf{1}_{\omega(t)}h\|_{L^2(Q_T)} \\
\leq C\| (F_1, F_2, U_0, V_0) \|_G.
\] (59)

Let \((U, V)\) be the weak solution to
\[
\begin{cases}
\partial_t U = a_{11} U + a_{12} V + F_1 + \tilde{h}\mathbf{1}_{\omega(t)} & \text{in } Q_T, \\
\partial_t V - d_v \partial_{xx} V = a_{21} U + a_{22} V + F_2 & \text{in } Q_T, \\
\partial_x V = 0 & \text{on } \Sigma_T, \\
(U, V)(0, \cdot) = (U_0, V_0) & \text{in } (0, 1).
\end{cases}
\] (60)

This means that \((U, V)\) is the solution of (60) defined by transposition, i.e., \((U, V)\) is the unique function satisfying
\[
\int_Q (U, V) \cdot (g_1, g_2) \, dx \, dt \\
= \int_Q F_1 \phi + F_2 \psi \, dx \, dt + \int_Q \tilde{h} \phi \, dx \, dt + \int_{\Omega} U_0(x) \phi(0, x) + !V(0, x) \psi(0, x) \, dx,
\] (61)

for every \((g_1, g_2)\) where \((\phi, \psi)\) is the solution to the adjoint system (18) with \(\phi_T = \psi_T = 0\). But from the variational formulation (57) satisfied by \((\tilde{U}, \tilde{V})\), we have that \((\tilde{U}, \tilde{V})\) also satisfies (61) then by uniqueness we get
\[
(\tilde{U}, \tilde{V}) = (U, V).
\] (62)

It remains to prove the fact that \((U, V, h) \in E\). For some function depending on time \(\rho(t) = e^{s\tilde{\beta}^{-1/2} \gamma^*} - 1/4\) or \(e^{(s/2)\tilde{\beta}^*(\gamma^*)^{-1/4}}\), we introduce
\[
(U^*, V^*) = \rho(t)(U, V).
\] (63)

From (60), an easy computation shows that \((U^*, V^*)\) is the solution to the following system
\[
\begin{cases}
\partial_t U^* = a_{11} U^* + a_{12} V^* + \rho F_1 + \rho h 1_{\omega(t)} + \rho_t U & \text{in } Q_T, \\
\partial_t V^* - d_v \partial_{xx} V^* = a_{21} U^* + a_{22} V^* + \rho F_2 + \rho_t V & \text{in } Q_T, \\
\partial_x V^* = 0 & \text{on } \Sigma_T, \\
(U^*, V^*)(0, \cdot) = \rho(0)(U_0, V_0) & \text{in } (0, 1).
\end{cases}
\] (64)

The goal is to prove that there exists a positive constant \(C > 0\) such that
\[
\|U^*\|_{H^1(0,T;L^2(\Omega))} + \|V^*\|_{W^{1,\infty}(\Omega)} \leq C \| (F_1, F_2, U_0, V_0) \|_G.
\] (65)
For \( \rho = e^{s \beta - (s/2)^* (\gamma)}^{-1/4} \) or \( \rho = e^{(s/2)^* (\gamma)}^{-1/4} \), it is easy to show from (50) that there exists a positive constant \( C > 0 \) such that

\[
|\rho_t| \leq C e^{s \beta^* (\gamma^*) - 3/2} \leq C e^{s \beta^*}.
\]  

(66)

So, from (66) and (59), we deduce that the right-hand sides of the two first equations of (64) belong to \( L^2 \). Then by a parabolic regularity estimate in \( L^2 \) for the second equation (independent of \( d_v \in (1, +\infty) \)), using the fact that \( V_0 \in H^1(\Omega) \), we obtain (65).

From (59), (63), (65), (60) and (62), we have that \( (U, V, h) \in E \) and it satisfies the desired estimate (55). This concludes the proof. \( \square \)

3. Proof of the local controllability results

3.1. A precise inverse mapping argument for the ODE–PDE system

The goal of this section is to prove Theorem 1.4.

Recalling the change of variables performed in Sect. 2.1, we remark that we have reduced our local controllability problem around \( (u_\pm, v_\mp) \) for (8) to a local null-controllability problem for the variable \( (U, V) \) satisfying (13). So, we will use the null-controllability result for the linearized system (16), established in Proposition 2.7 and a precise inverse mapping argument (see 3.1). We emphasize that the regularity assumptions on the trajectory of the linear system, contained in the definition of the space \( E \), are crucial to treat the nonlinear term.

**Theorem 3.1.** (See [11]) Let \( E \) and \( G \) be two Banach spaces and let \( \mathcal{A} \in C^1(E; G) \) with \( \mathcal{A}(0) = 0 \). We assume that \( \mathcal{A}'(0) \) is an isomorphism from \( E \) onto \( G \). More precisely, we assume that there exists \( C_0 > 0 \) such that

\[
\|e\|_E \leq C_0 \|\mathcal{A}'(0)(e)\|_G, \quad \forall e \in E,
\]

and that there exists \( 0 < \delta < C_0^{-1} \) and \( \eta > 0 \),

\[
\|\mathcal{A}(e_1) - \mathcal{A}(e_2) - \mathcal{A}'(0)(e_1 - e_2)\| \leq \delta \|e_1 - e_2\|, \quad \forall e_1, e_2 \in B_\eta(0).
\]

(68)

Then, the equation \( \mathcal{A}(e) = g \) has a solution \( e \in B_\eta(0) \) for all \( \|g\|_G \leq (C_0^{-1} - \delta)^{-1} \eta \).

**Remark 3.2.** By the mean value theorem, it can be shown that for any \( 0 < \delta < C_0^{-1} \), inequality (68) is satisfied for \( \eta \) such that

\[
\|\mathcal{A}'(e) - \mathcal{A}'(0)\| \leq \delta, \quad \forall \|e\| \leq \eta.
\]

In our setting, we use the previous theorem with the space \( E \) defined in (53) and \( G \) defined in (54) and the operator

\[
\mathcal{A}(U, V, h) = (L(U, V) + N(U, V) + (-h_1(-h_{\omega(t)}), 0), (U(0, \cdot), V(0, \cdot))), \forall (U, V, h) \in E.
\]
Recall that the linear term $L$ is defined in (51) and the nonlinear term $N$ is defined in (15). We have

$$\mathcal{A}'(0, 0, 0) \cdot (U, V, h) = (L(U, V) + (-h\mathbf{1}_{\omega(t)}), 0), U(0, \cdot), V(0, \cdot)) \forall (U, V, h) \in E.$$  

We have the following regularity result for the operator $\mathcal{A}$.

**Proposition 3.3.** We have that $\mathcal{A} \in C^1(E; G)$.

**Proof.** First, we remark that the linear terms in the definition of $\mathcal{A}$ are continuous then continuously differentiable because of the definition of the space $E$.

Let us show that the nonlinear term $N(U, V) \in C^1(E; G)$. For increasing positive constants $\overline{C} > 0$, we have that

$$\|N(U, V)\|_G \leq \overline{C} \left\| e^{\lambda \beta} (\hat{\gamma})^{-1/2} (U V^2 + 2v_\pm U V + U_\pm V_\pm^2) \right\|_{L^2(Q_T)} \leq \overline{C} \left( \left\| e^{\lambda \beta} (\hat{\gamma})^{-1/2} U V^2 \right\|_{L^2(Q_T)} + \left\| e^{\lambda \beta} (\hat{\gamma})^{-1/2} U V \right\|_{L^2(Q_T)} + \left\| e^{\lambda \beta} (\hat{\gamma})^{-1/2} V^2 \right\|_{L^2(Q_T)} \right).$$

For the cubic term, we have

$$\left\| e^{\lambda \beta} (\hat{\gamma})^{-1/2} U V^2 \right\|_{L^2(Q_T)} \leq \overline{C} \left\| e^{\lambda \beta - (s/2) \beta^*} (\hat{\gamma})^{-1/4} U \right\|_{L^2(Q_T)} \left\| e^{(s/2) \beta^*} (\hat{\gamma})^{-1/4} V^2 \right\|_{L^2(Q_T)} \leq \overline{C} \left\| e^{\lambda \beta - (s/2) \beta^*} (\hat{\gamma})^{-1/4} U \right\|_{L^2(Q_T)} \left\| e^{(s/2) \beta^*} (\hat{\gamma})^{-1/4} V \right\|_{L^2(Q_T)} \leq \overline{C} \left\| e^{\lambda \beta} (\hat{\gamma})^{-1/2} V^2 \right\|_{L^2(Q_T)} \leq \overline{C} \| (U, V, h) \|^3_E.$$  

Note that we have used the embedding $W_{T, \Omega} \hookrightarrow L^\infty(0, T)$; see (52) for the definition of $W_{T, \Omega}$.

For the two quadratic terms, we easily have, using same type of arguments,

$$\left\| e^{\lambda \beta} (\hat{\gamma})^{-1/2} U V \right\|_{L^2(Q_T)} + \left\| e^{\lambda \beta} (\hat{\gamma})^{-1/2} U_\pm V_\pm \right\|_{L^2(Q_T)} \leq \overline{C} \| (U, V, h) \|^2_E.$$  

This proves that $N(U, V) \in C^1(E; G)$ and concludes the proof. \hfill \Box

**Proof of Theorem 1.2.** An application of Theorem 3.1 gives the existence of $\delta, \eta > 0$, which a priori depend on $d_e$, such that if

$$\|(u_0 - u_\pm, v_0 - v_\pm)\|_{L^2(\Omega) \times H^1(\Omega)} \leq (C_0^{-1} - \delta)^{-1} \eta,$$

there exists a control $h = h(d_e)$ such that the associated solution $(U, V)$ to (13) verifies

$$(U, V)(T, \cdot) = 0 \text{ and } \|(U, V, h)\|_E \leq \eta.$$  

To finish the proof of Theorem 1.4, we must show that the constants $C_0, \eta$ and $\delta$ do not depend on $d_e$. This is actually a direct consequence from the fact that the constant $C_0$ in
which is actually the constant $C$ appearing in Proposition 2.7, does not depend on $d_v$. So from Remark 3.2 we can take $\delta < C_0^{-1}$ and $\eta$ can be chosen smaller than $\delta/\sqrt{C}$, $\sqrt{\delta/C}$, where $C$ is the maximal constant appearing in the proof of Proposition 3.3.

The expected bound (10) follows from $\| (U, V, h) \|_E \leq \eta$ and the definition of the space $E$. □

3.2. The shadow limit $d_v \to +\infty$ to reduce to the ODE-ODE system

The goal of this section is to prove Theorem 1.2. In order to do this, we will use Theorem 1.4 and the following result, which deals with the convergence of the system (8) to the system (5) in the limit $d_v \to +\infty$.

**Proposition 3.4.** Let $(u_{d_v}, v_{d_v}, h_{d_v}) \in H^1(0, T; L^2(\Omega)) \times L^\infty(Q_T) \times L^2(Q_T)$ be a solution of (8) associated with $(u_0, v_0) \in L^2(\Omega) \times H^1(\Omega)$ such that

\begin{align}
    u_{d_v} &\to u \text{ in } H^1(0, T; L^2(\Omega)) \text{ as } d_v \to +\infty, \\
    v_{d_v} &\to v \text{ in } L^\infty(Q_T) \text{ as } d_v \to +\infty, \\
    h_{d_v} &\to h \text{ in } L^2(Q_T) \text{ as } d_v \to +\infty. 
\end{align}

Then, $(u, w) := (u, v)$ is the solution to (6) associated with $h$ and $(u(0, \cdot), w(0)) = (u_0, \int_\Omega v_0)$.

**Proof.** For the purpose of the proof, let us denote 

$$ f(u, v) = -uv^2 + F(1-u), \quad g(u, v) = uv^2 - (F+k)v. $$

Let $(\tilde{u}, \tilde{v})$ be the solution to

\begin{align}
    \partial_t \tilde{u} &= f(u, v) + h1_{\omega(t)} \quad \text{in } Q_T, \\
    \tilde{v}' &= \int_\Omega g(u, v) \, dx \quad \text{in } (0, T), \\
    \tilde{u}(0, \cdot) &= u_0(\cdot) \text{ in } \Omega, \quad \tilde{v}(0) = \int_\Omega v_0, \tag{72}
\end{align}

where $(u, v, h)$ are the limits coming from (69), (70), (71).

Taking the difference between the first equations of systems (8) and (72), we easily obtain that

\begin{align*}
    \partial_t u_{d_v} - \partial_t u &= f(u_{d_v}, v_{d_v}) - f(u, v) + (h_{d_v} - h)1_{\omega(t)} \\
    &= -v_{d_v}^2 u_{d_v} + F(1-u_{d_v}) + v^2 u - F(1-u) + (h_{d_v} - h)1_{\omega(t)} \\
    &= -v^2 (u_{d_v} - u) - u_{d_v} (v_{d_v}^2 - v^2) - F(u_{d_v} - u) + (h_{d_v} - h)1_{\omega(t)}. 
\end{align*}

By using (69), (70) and (71), it is straightforward to see that

$$ \partial_t u_{d_v} - \partial_t \tilde{u} \to 0 \text{ in } L^2(Q_T) \text{ as } d_v \to +\infty. \tag{73} $$
Then, for every \( \chi \in L^2(Q_T) \), using Fubini’s theorem and (73), we have

\[
\int_{Q_T} (u_{d_v}(t, x) - \bar{u}(t, x)) \chi(t, x) \, dt \, dx = \int_0^T \int_0^1 \left( \int_0^t (\partial_t u_{d_v} - \partial_t \bar{u})(s, x) \, ds \right) \chi(t, x) \, dx \, dt
\]

\[
= \int_0^T \int_0^1 (\partial_t u_{d_v} - \partial_t \bar{u})(s, x) \left( \int_s^T \chi(t, x) \, dt \right) \, dx \, ds,
\]

and from (73), one gets

\[
\int_{Q_T} (u_{d_v}(t, x) - \bar{u}(t, x)) \chi(t, x) \, dt \, dx \to 0 \text{ as } d_v \to +\infty,
\]

so

\[
u_{d_v} - \bar{u} \to 0 \quad \text{in } H^1(0, T; L^2(\Omega)) \quad \text{as } d_v \to +\infty. \quad (74)
\]

Therefore, by uniqueness, recalling (69) and (74), we have

\[
u = \bar{u}. \quad (75)
\]

For the second equation, we begin by writing the solution of (72) and (8) as follows

\[
\bar{v}(t) = \int_{\Omega} v_0(x) \, dx + \int_0^t \int_{\Omega} g(u(s, x), v(s, x)) \, dx \, ds,
\]

\[
v_{d_v}(t, \cdot) = e^{t \partial_{xx}} v_0 + \int_0^t e^{(t-s) \partial_{xx}} g(u_{d_v}(s, \cdot), v_{d_v}(s, \cdot)) \, ds. \quad (76)
\]

Taking the difference between (77) and (76) and computing the \( L^2 \)-norm, we get

\[
\|v_{d_v}(t, \cdot) - \bar{v}(t)\|_{L^2(\Omega)} \leq \left\| e^{t \partial_{xx}} \left( v_0 - \int_{\Omega} v_0 \, dx \right) \right\|_{L^2(\Omega)} + \left\| \int_0^t e^{(t-s) \partial_{xx}} \left( g(u_{d_v}(s, \cdot), v_{d_v}(s, \cdot)) - \int_{\Omega} g(u(s, x), v(s, x)) \, dx \right) \, ds \right\|_{L^2(\Omega)},
\]

where \( \{e^{t \partial_{xx}}\}_{t \geq 0} \) stands for the heat semigroup associated with the diffusion parameter \( d_v > 0 \) with homogeneous Neumann boundary conditions on the interval \((0, 1)\). Employing property a. of Lemma C.1 with \( K = \int_{\Omega} g(u_{d_v}(s, x), v_{d_v}(s, x)) - g(u(s, x), v(s, x)) \, dx \), we can easily deduce that

\[
\|v_{d_v}(t, \cdot) - \bar{v}(t)\|_{L^2(\Omega)} \leq \left\| e^{t \partial_{xx}} \left( v_0 - \int_{\Omega} v_0 \, dx \right) \right\|_{L^2(\Omega)} + \left\| \int_0^t e^{(t-s) \partial_{xx}} \left( g(u_{d_v}(s, \cdot), v_{d_v}(s, \cdot)) - \int_{\Omega} g(u_{d_v}(s, x), v_{d_v}(s, x)) \, dx \right) \, ds \right\|_{L^2(\Omega)}
\]

\[
+ \left\| \int_0^t \int_{\Omega} g(u_{d_v}(s, x), v_{d_v}(s, x)) - g(u(s, x), v(s, x)) \, dx \, ds \right\|_{L^2(\Omega)}. \quad (78)
\]

Let us treat the three terms in the right-hand side of (78) separately.
For the first one, we can use property $b.$ in Lemma C.1 with $z_0 = v_0$ to obtain
\[
 t^{1/2} \left\| e^{td_v \partial_{x_x}} \left( v_0 - \int_\Omega v_0 \, dx \right) \right\|_{L^2(\Omega)} \leq C t^{1/2} e^{-\lambda_1d_v t} \leq C (d_v)^{-1/2}(d_v t)^{1/2} e^{-\lambda_1d_v t} \leq C d_v^{-1/2} \to 0, \text{ as } d_v \to +\infty. \tag{79}
\]

For the second term which has zero-mean and noting that we have a uniform bound in $L^2$ for $g(u_{d_v}(s, \cdot), v_{d_v}(s, \cdot)) - \int_\Omega g(u_{d_v}(s, x), v_{d_v}(s, x)) \, dx$, using again $b.$ in Lemma C.1, (108), we get
\[
 \left\| \int_0^t e^{(t-s)d_v \partial_{x_x}} \left( g(u_{d_v}(s, \cdot), v_{d_v}(s, \cdot)) - \int_\Omega g(u_{d_v}(s, x), v_{d_v}(s, x)) \, dx \right) \right\|_{L^2(\Omega)} \leq C \int_0^t e^{-\lambda_1d_v(t-s)} ds \leq C \frac{1}{\lambda_1d_v} (1 - e^{-\lambda_1d_v t}) \leq C \frac{1}{\lambda_1d_v} \to 0 \text{ as } d_v \to +\infty. \tag{80}
\]

For the last one, we write
\[
 \left| \int_0^t \int_\Omega g(u_{d_v}(s, x), v_{d_v}(s, x)) - g(u(s, x), v(s, x)) \, dx \, ds \right| = \left| \int_0^t \int_\Omega v^2_{d_v} u_{d_v} - (F + k) v_{d_v} - v^2 u + (F + k) v \, dx \, ds \right|. \tag{81}
\]

Adding and subtracting $v^2 u_{d_v}$ and rearranging terms, we get
\[
 \left| \int_0^t \int_\Omega g(u_{d_v}(s, x), v_{d_v}(s, x)) - g(u(s, x), v(s, x)) \, dx \, ds \right| = \left| \int_0^t \int_\Omega v^2 (u_{d_v} - u) + \int_0^t \int_\Omega u_{d_v} (v^2_{d_v} - v^2) + (F + k) \int_0^t \int_\Omega (v - v_{d_v}) \right|. \tag{82}
\]

We use triangle inequality and see the behavior of each term in the right-hand side of (81). For the first one, by using (69) and (70), we have
\[
 \lim_{d_v \to \infty} \left| \int_0^t \int_\Omega v^2 (u_{d_v} - u) \right| = \lim_{d_v \to \infty} \left| \int_0^t \int_\Omega v^2 (u_{d_v} - u) \right| = 0. \tag{83}
\]

For the second one, using again (69) and (70) we have
\[
 \left| \int_0^t \int_\Omega u_{d_v} (v^2_{d_v} - v^2) \right| \leq \| u_{d_v} \|_{L^2(Q_T)} \| v^2_{d_v} - v^2 \|_{L^2(Q_T)} \leq C \| v_{d_v} - v \|_{L^\infty(Q_T)} \to 0. \tag{84}
\]

For the last one, from (69), we have
\[
 (F + k) \int_0^t \int_\Omega (v - v_{d_v}) \to 0 \text{ as } d_v \to +\infty. \tag{85}
\]
From (81), (82), (83), (84), we deduce that
\[
\left| \int_0^t \int_\Omega g(u_{d_v}(s,x), v_{d_v}(s,x)) - g(u(s,x), v(s,x)) \, dx \, ds \right| \to 0 \quad \text{as} \quad d_v \to +\infty. \quad (85)
\]
So from (79), (80) and (85), we obtain that
\[
\| v_{d_v} - \tilde{v} \|_{C([\delta,T];L^2(\Omega))} \to 0 \quad \text{as} \quad d_v \to +\infty, \quad (86)
\]
for any $\delta > 0$. So, by uniqueness, recalling (70) and (86), we have
\[
v = \tilde{v}. \quad (87)
\]
Hence, from (72), (87) and (75), we obtain the conclusion of the proof. \(\square\)

Now, we prove Theorem 1.2.

**Proof of Theorem 1.2.** From Theorem 1.4, we deduce that one can find $h_{d_v}$ such that (10) and (11) hold. Using that $H^1(0, 1) \hookrightarrow L^\infty(0, 1) \hookrightarrow L^2(0, 1)$, the first embedding being compact and the second one continuous, we can apply Aubin-Lions Theorem (see [26, Section 8, Corollary 4]) to deduce that $W_{T,\Omega} \hookrightarrow L^\infty(Q_T)$ compactly. So, at least for subsequence, we have (69), (70) and (71). Therefore, the conclusion of Theorem 1.2 follows from Proposition 3.4. Note that (7) is guaranteed by (11) and the continuous embedding $H^1(0, T; L^2(\Omega)) \hookrightarrow C([0, T]; L^2(\Omega))$. \(\square\)

4. Comments and open problems

4.1. Spatial dimension $N > 1$ and Dirichlet boundary conditions

In this paper, we focus on the spatial dimension $N = 1$. This is due to the Carleman estimate for heat equation with homogeneous Neumann boundary conditions from Lemma 2.3. Indeed, this inequality is only proved in 1-D (see [20, Appendix A]). This restriction comes from the properties (25), (26) of the weight $\eta$. Constructing such a weight in the multidimensional case is actually an interesting open problem.

Note that this type of Carleman estimate from Lemma 2.3 is valid in the multidimensional case for homogeneous Dirichlet boundary conditions; see [8, Lemma 4.4]. Therefore, by a small adaptation of the proof of Theorem 1.4, we can also obtain Theorem 1.4 for homogeneous Dirichlet boundary conditions on the component $v$. Unfortunately, by sending $d_v \to +\infty$ in the system (8), $v_{d_v} \to 0$, so we cannot hope to extend Theorem 1.2 by this methodology.

4.2. Other type of nonlinearities

In this paper, we deal with nonlinear integro-differential equations with cubic nonlinearity, coming from the Gray–Scott model. Actually, we crucially use the fact that the cubic term $uv^2$ is linear in $u$. Indeed, by looking at the definition the space $E$ in
(53), we observe that the linear controlled trajectory \((U, V)\), forgetting the weights, satisfy
\[
(U, V) \in H^1(0, T; L^2(\Omega)) \times W_{T, \Omega}.
\]

The extra regularity on the component \(V\) is the crucial point to treat the cubic term \(UV^2\); see the proof of Proposition 3.3. Moreover, we do not know how to obtain extra regularity on the component \(U\) by looking carefully to the proof of Proposition 2.7. This is due to the fact that ODEs do not imply automatically spatial regularity, contrary to parabolic PDEs. To sum up, obtaining Theorem 1.2 and Theorem 1.4 replacing the Gray–Scott nonlinearity by a nonlinearity superlinear in the variable \(u\) is an interesting open problem.

4.3. Control of the second equation

In this paper, Theorem 1.2 (resp. Theorem 1.4) establishes a local controllability results with a control \(h_{1_{\omega(t)}}\) acting on the first equation of (6) (resp. (8)). One may wonder if the result holds true by taking control on the second equation. First, it seems that one can prove the local controllability of (8) in this setting. Indeed, the adequate change of variable performed in [20, Section 3.1] enables us to consider an ODE–PDE system, with a control acting on the ODE and then the same techniques used in this article probably leads to the expected result. However, we do not know if it is possible to get a uniform local controllability result with this strategy. We emphasize that if one get such a result, one immediately deduce the local controllability of (6) with a control acting on the second equation by using the shadow limit \(d_v \to +\infty\); see Sect. 3.2.

4.4. Uniform controllability with respect to \(d_u \to 0\)

In this paper, we have studied control properties for the ODE–PDE system (8) and obtained a uniform controllability result with respect to the diffusion parameter \(d_v \to +\infty\), leading us to the ODE-ODE system (6). An analogous result, where the starting point is (3) (with control on the first equation) and leading us to (8), is in fact an interesting and open question.

On the very recent article [4], the authors study the limit behavior as \(\epsilon \to 0\) of a chemotaxis problem of the form
\[
\begin{align*}
\partial_t u - \Delta u &= -\text{div}(u \nabla f(v)) \quad \text{in} \ (0, T) \times \Omega, \\
\partial_t v - \epsilon \Delta v &= g(u, v) \quad \text{in} \ (0, T) \times \Omega,
\end{align*}
\]
(88)
complemented with homogeneous Neumann boundary conditions and where \(f \in C^2(\mathbb{R})\) and \(g \in C^2(\mathbb{R}^2; \mathbb{R})\) are suitable nonlinear functions. Under various assumptions on the initial data and the growth of the nonlinearities, they are able to establish
a convergence result as $\epsilon \to 0$ (see [4, Theorem 1.4]) towards the system

$$\begin{cases}
\partial_t u - \Delta u = -\text{div}(u \nabla f(v)) & \text{in } (0, T) \times \Omega, \\
\partial_t \tilde{v} = g(u, \tilde{v}) & \text{in } (0, T) \times \Omega.
\end{cases} \quad (89)$$

At first glance, it is reasonable to expect that such a result can be adapted to our case to deduce conditions for which the system (3) converges to the uncontrolled version (i.e., $h \equiv 0$) of (8). Nevertheless, designing a sequence of controls $\{h_\epsilon\}_\epsilon$ for system (3) which is uniformly bounded with respect to $\epsilon$ seems to be a difficult question.

In fact, from classical papers [3, 15] studying the uniform controllability of vanishing viscosity of parabolic-hyperbolic equation, we can deduce that this is not the case when the support of the control $\omega$ is fixed. This opens up the possibility of obtaining uniform bounds by using moving controls. However, the Carleman estimates developed in [8] are not a priori well suited to treat this case. A first approach might be to study a simple 1-D case for a single heat equation using the tools in [24] and see if a moving control with uniform bounds can be built. The answer is far from obvious.

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**A Uniform parabolic Carleman estimate**

In this part, we prove Lemma 2.3. The proof follows the arguments presented in [20, Appendix A] with some remarks coming from [6]. For completeness, we give a sketch of the proof.
We start by giving some useful properties on the weight function $\alpha$ and its derivatives. By simple computations using (30), we have

$$
\alpha_x = -\lambda \xi \eta_x,
\alpha_{xx} = \lambda^2 \xi (\eta_x - \lambda^{-1} \eta_{xx}),
|\alpha_{xx}| \leq C \lambda^2 \xi,
|\alpha_{xxx}| \leq C \lambda^3 \xi,
|\alpha_{xxxx}| \leq C \lambda^4 \xi,
|\alpha_t| \leq C (T + e^{2\lambda \|\eta\|_{\infty}}) \lambda \xi^2,
|\alpha_{xt}| \leq C (T + 1) \lambda^2 \xi^2,
|\alpha_{tt}| \leq C (T + T^2 + e^{2\lambda \|\eta\|_{\infty}}) \lambda^2 \xi^3.
$$

(90)

As usual, we set $w = e^{-s\alpha} \psi$ and compute

$$
w_x = -s \alpha_x w + e^{-s\alpha} \psi_x.
$$

Using the boundary conditions of $\psi$, we deduce $w_x = -s \alpha_x w$ on $(0, T) \times \{0, 1\}$. We introduce the parabolic operator $P = \partial_t + \frac{1}{\epsilon} \partial_{xx}$. Then, we have

$$
e^{-s\alpha} P (e^{s\alpha} w) = P^e w + P^k w,
$$

(91)

where

$$
P^e w = \frac{1}{\epsilon} w_{xx} + s \alpha_t w + \frac{1}{\epsilon} s^2 \alpha_x^2 w,
$$

(92)

$$
P^k w = w_t + \frac{2}{\epsilon} s \alpha_x w_x + \frac{1}{\epsilon} s \alpha_{xx} w.
$$

(93)

We take the $L^2$-norm in both sides of (91), thus obtaining

$$
\|e^{-s\alpha} P (e^{s\alpha} w)\|_{L^2(\Omega_T)}^2 = \|P^e w\|_{L^2(\Omega_T)}^2 + \|P^k w\|_{L^2(\Omega_T)}^2 + 2 \left( P^e w, P^k w \right)_{L^2(\Omega_T)}.
$$

(94)

A very long, but straightforward computation gives that

$$
2 \left( P^e w, P^k w \right)_{L^2(\Omega_T)} = BT + DT_1 + DT_2,
$$

where

$$
BT := \frac{1}{\epsilon} \int_0^T s \alpha_{xt} w_x^2 \, dt \bigg|_0^1 + \frac{2}{\epsilon^2} \int_0^T s \alpha_x w_x^2 \, dt \bigg|_0^1 + \frac{2}{\epsilon^2} \int_0^T s \alpha_{xx} w_x w \, dt \bigg|_0^1
$$

$$
- \frac{1}{\epsilon^2} \int_0^T s \alpha_{xxx} w^2 \, dt \bigg|_0^1 + \frac{2}{\epsilon} \int_0^T s^2 \alpha_t \alpha_x w^2 \, dt \bigg|_0^1 + \frac{2}{\epsilon^2} \int_0^T s^3 \alpha_x^3 w^2 \, dt \bigg|_0^1.
$$
Moreover, using properties (90) together with (25)–(26), it is not difficult to see that

$$DT_1 := -\frac{4}{\varepsilon^2} \int_{Q_T} s\alpha_{xx} w_x^2 \, dx \, dt,$$

$$DT_2 := -\frac{4}{\varepsilon} \int_{Q_T} s^2 \alpha_{tx} \alpha_x w^2 \, dx \, dt + \frac{1}{\varepsilon^2} \int_{Q_T} s\alpha_{xxxx} w_x^2 \, dx \, dt - \int_{Q_T} s\alpha_{tt} w^2 \, dx \, dt$$

$$- \frac{4}{\varepsilon^2} \int_{Q_T} s^3 \alpha_x^2 \alpha_{xx} w_x^2 \, dx \, dt.$$ Using that $w_x = -s\alpha_x w$ in $(0, T) \times [0, 1]$, we can obtain

$$BT = \frac{4}{\varepsilon^2} \int_0^T s^3 \lambda^3 \xi^3 \varepsilon^2 \theta^2 \, dt \left|_{x=1} + \frac{4}{\varepsilon^2} \int_0^T s^3 \lambda^3 \xi^3 \varepsilon^2 \theta^2 \, dt \right|_{x=0}$$
$$- \frac{2 \lambda}{\varepsilon^2} \int_0^T s^2 \lambda^2 \xi^3 (T + e^{2\lambda \varepsilon^2}) w_x^2 \, dt \left|_{x=1} - \frac{2 \lambda}{\varepsilon^2} \int_0^T s^2 \lambda^2 \xi^3 (T + e^{2\lambda \varepsilon^2}) w_x^2 \, dt \right|_{x=0}$$
$$- \frac{2 \lambda}{\varepsilon^2} \int_0^T s^2 \lambda^2 \xi^3 w_x^2 \, dt \left|_{x=1} - \frac{2 \lambda}{\varepsilon^2} \int_0^T s^2 \lambda^2 \xi^3 w_x^2 \, dt \right|_{x=0}$$
$$- \frac{C}{\varepsilon^2} \int_0^T s(T + 1) \lambda^2 \xi^2 w_x^2 \, dt \left|_{x=1} - \frac{C}{\varepsilon^2} \int_0^T s(T + 1) \lambda^2 \xi^2 w_x^2 \, dt \right|_{x=0}$$
$$- \frac{C}{\varepsilon^2} \int_0^T s^3 \lambda^3 \xi^3 w_x^2 \, dt \left|_{x=1} - \frac{C}{\varepsilon^2} \int_0^T s^3 \lambda^3 \xi^3 w_x^2 \, dt \right|_{x=0},$$

where we have used that $0 < \varepsilon \leq 1$ to adjust the powers of $\varepsilon$. Finally, taking $\lambda \geq C$ and $s \geq C(1 + T + e^{2\lambda \varepsilon^2})$, we get

$$BT \geq \frac{C}{\varepsilon^2} \int_0^T s^3 \lambda^3 \xi^3 w_x^2 \, dt \left|_{x=1} + \frac{C}{\varepsilon^2} \int_0^T s^3 \lambda^3 \xi^3 w_x^2 \, dt \right|_{x=0}.$$ (95)

Let us focus now on the distributed terms. Using (90), we can rewrite

$$DT_1 = \frac{4}{\varepsilon^2} \int_{Q_T} s^2 \lambda^2 \xi^2 \eta_x^2 w_x^2 \, dx \, dt + \frac{4}{\varepsilon} \int_{Q_T} s^3 \lambda^2 \xi^2 \eta_{xx} w_x^2 \, dx \, dt,$$
and

$$DT_2 = -\frac{4}{\varepsilon} \int_{Q_T} s^2 \alpha_{tx} \alpha_x w^2 \, dx \, dt + \frac{1}{\varepsilon^2} \int_{Q_T} s\alpha_{xxxx} w_x^2 \, dx \, dt$$
$$- \int_{Q_T} s\alpha_{tt} w^2 \, dx \, dt + \frac{4}{\varepsilon^2} \int_{Q_T} s^3 \lambda^3 \xi^3 \eta_x^2 \eta_{xx} w_x^2 \, dx \, dt + \frac{4}{\varepsilon^2} \int_{Q_T} s^3 \lambda^3 \xi^3 \eta_x^4 w_x^2 \, dx \, dt.$$
Using estimates (90), we can bound by below as follows

\[ DT_1 \geq \frac{4}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 \eta_x^2 w_x^2 \, dx \, dt - \frac{C}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt. \]  

(96)

and

\[ DT_2 \geq \frac{4}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^4} \xi^4 \eta_x^2 w_x^2 \, dx \, dt - \frac{C(T + 1)}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt - \frac{C}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^4} \xi^2 w_x^2 \, dx \, dt \]

\[ - C \left( T + T^2 + e^{2\lambda \eta \|\eta\|_\infty} \right) \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \]

- \frac{C}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^4} \xi^2 w_x^2 \, dx \, dt.

(97)

Collecting estimates (95)–(97) yield

\[ 2 \left( p^{\epsilon}_{e} \omega, p^{\epsilon}_{k} \omega \right)_{L^2(Q_T)} \geq \frac{C}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \bigg|_{x=1} + \frac{C}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \bigg|_{x=0} \]

\[ + \frac{4}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 \eta_x^2 w_x^2 \, dx \, dt + \frac{4}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 \eta_x^2 \eta_x^2 w_x^2 \, dx \, dt \]

\[ - \frac{C}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt - \frac{C(T + 1)}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \]

\[ - \frac{C}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt - C \left( T + T^2 + e^{2\lambda \eta \|\eta\|_\infty} \right) \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \]

\times \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt - \frac{C}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt.

Using property (21) and taking \( \lambda \geq C \) and \( s \geq C(T + T^2) \), we deduce

\[ 2 \left( p^{\epsilon}_{e} \omega, p^{\epsilon}_{k} \omega \right)_{L^2(Q_T)} \geq \frac{C}{\epsilon^2} \int_0^T \int_{\omega_0(t) \times (0, T)} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt + \frac{1}{\epsilon^2} \int_0^T \int_{\omega_0(t) \times (0, T)} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \]

\[ \geq \frac{C}{\epsilon^2} \int_0^T \int_{\omega_0(t) \times (0, T)} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \bigg|_{x=1} + \frac{C}{\epsilon^2} \int_0^T \int_{\omega_0(t) \times (0, T)} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \bigg|_{x=0} + \frac{C}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \]

\[ + \frac{C}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt. \]  

(98)

Combining (98) with (94), we get

\[ C \| e^{-s \alpha} P(e^{s \alpha} w) \|_{L^2(Q_T)}^2 + \frac{C}{\epsilon^2} \int_0^T \int_{\omega_0(t) \times (0, T)} s^{\lambda^4} \xi^2 \eta_x^2 \omega_x^2 \, dx \, dt \]

\[ + \frac{C}{\epsilon^2} \int_0^T \int_{\omega_0(t) \times (0, T)} s^{\lambda^2} \xi^2 \omega_x^2 \, dx \, dt \]

\[ \geq \| P^{\epsilon}_{e} \omega \|_{L^2(Q_T)}^2 + \| P^{\epsilon}_{k} \omega \|_{L^2(Q_T)}^2 + \frac{1}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \bigg|_{x=1} \]

\[ + \frac{1}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt \bigg|_{x=0} \]

\[ + \frac{1}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^2} \xi^2 w_x^2 \, dx \, dt + \frac{1}{\epsilon^2} \int_0^T \int_{Q_T} s^{\lambda^4} \xi^2 \omega_x^2 \, dx \, dt. \]  

(99)

We will add terms corresponding to \( w_{xx} \) and \( w_t \) to the right-hand side of (99). For this, we multiply (92) by \( s^{-1/2} \xi^{-1/2} \) and take the \( L^2 \)-norm, that is,
where we have used that $\epsilon \leq 1$ and $s \geq CT^2$ in the last line. Arguing in the same way and considering (93), it is not difficult to see that
\begin{align*}
\frac{1}{\epsilon^2} s^{-1/2} \|w_{xx}\|^2 L^2(Q_T) &\leq C s^{-1} \int_Q \xi^{-1} |P^e_\epsilon w|^2 \, dx \, dt \\
&+ \frac{C}{\epsilon^2} \int_Q s \lambda^2 \xi w_x^2 \, dx \, dt + \frac{C}{\epsilon^2} \int_Q s \lambda^4 \xi w^2 \, dx \, dt.
\end{align*}
(100)

Using that $s^{-1} \xi^{-1} \leq C$ for all $(t, x) \in Q_T$, we can use (100) and (101) to estimate from below in (99) and obtain
\begin{align*}
C \|e^{-s^2 w} P(e^{s^2 w})\|^2 L^2(Q_T) + \frac{C}{\epsilon^2} \int_{\omega_0(t) \times (0, T)} s \lambda^4 \xi w^3 \, dx \, dt \\
+ \frac{C}{\epsilon^2} \int_{\omega_0(t) \times (0, T)} s \lambda^2 \xi w_x^2 \, dx \, dt \\
\geq \frac{1}{\epsilon^2} \int_0^T s \lambda^3 \xi^3 w^2 \, dt \bigg|_{x=1} + \frac{1}{\epsilon^2} \int_0^T s \lambda^3 \xi^3 w^2 \, dt \bigg|_{x=0} \\
+ \frac{1}{\epsilon^2} \int_Q s \lambda^2 \xi w_x^2 \, dx \, dt + \frac{1}{\epsilon^2} \int_Q s \lambda^4 \xi^3 w^2 \, dx \, dt \\
+ \frac{1}{\epsilon^2} \int_Q s^{-1} \xi^{-1} |w_{xx}|^2 \, dx \, dt + \int_Q s^{-1} \xi^{-1} |w_t|^2 \, dx \, dt. 
\end{align*}
(102)

To conclude the proof, we need to eliminate the local term of $w_x$ in the above equation. For this, consider a function $\zeta \in C^\infty([0, T] \times [0, L])$ verifying
\begin{align*}
0 \leq \zeta \leq 1 &\quad \forall (t, x) \in [0, T] \times [0, L], \\
\zeta(t, x) = 1 &\quad \forall t \in [0, T], \ \forall x \in \omega_0(t), \\
\zeta(t, x) = 0 &\quad \forall t \in [0, T], \ \forall x \in [0, L] \backslash \omega_1(t).
\end{align*}

We have
\begin{align*}
\frac{1}{\epsilon^2} \int_{\omega_0(t) \times (0, T)} s \lambda^2 \xi w_x^2 \, dx \, dt &\leq \frac{1}{\epsilon^2} \int_Q \zeta s \lambda^2 \xi w_x^2 \, dx \, dt \\
&= -\frac{1}{\epsilon^2} \int_0^T s \lambda^2 \xi w_{xx} w_x \, dx \, dt - \frac{1}{\epsilon^2} \int_Q s \lambda^2 \xi w_{xx} \xi \, dx \, dt \\
&- \frac{1}{\epsilon^2} \int_Q s \lambda^2 \xi w_x w_x \xi \, dx \, dt. 
\end{align*}
Using the fact that $w_x = -s\alpha_x w$ on $(0, T) \times \{0, 1\}$ together with properties (25)–(26) and Cauchy–Schwarz and Young inequalities, we get

$$
\frac{C}{\epsilon^2} \iint_{w_0(t) \times (0, T)} s \lambda^2 \xi w_x^2 \leq \frac{1}{2\epsilon^2} \iint_{Q_T} s^{-1} \xi^{-1} |w_{xx}|^2 \, dx \, dt + \frac{1}{2\epsilon^2} \iint_{Q_T} s \lambda^2 \xi w_x^2 \, dx \, dt + \frac{C}{\epsilon^2} \iint_{w_0(t) \times (0, T)} s \lambda^2 \xi w_x^2 \, dx \, dt.
$$

Using the above estimate in (102) and recalling the change of variables $w = e^{-s\alpha_1} \psi$ gives the desired result. This concludes the proof.

5. Proof of a precise observability inequality from the Carleman estimate

The goal of this part is to prove Proposition 2.6.

**Proof.** The proof is by now standard and relies on well-known arguments, we follow here the presentation in [17, Lemma 4.1]. We fix $\lambda, s$ large enough such that the Carleman estimate from Proposition 2.5 holds and such that we have the following estimate

$$
e^{-4s\alpha^* + 2s\tilde{\alpha}} \leq Ce^{-s\alpha} \leq Ce^{-s\alpha^*}.
$$

All the following positive constants $C > 0$ can now depend on $\lambda, s$.

By construction, $\alpha = \beta$ and $\xi = \gamma$ in $(T/2, T) \times (0, 1)$, therefore

$$
\int_{T/2}^T \int_0^1 \gamma |\phi|^2 e^{-2s \beta} \, dx \, dt + \int_{T/2}^T \int_0^1 \gamma^3 |\psi|^2 e^{-2s \beta} \, dx \, dt
\leq \int_{T/2}^T \int_0^1 \xi |\phi|^2 e^{-2s \alpha} \, dx \, dt + \int_{T/2}^T \int_0^1 \xi^3 |\psi|^2 e^{-2s \alpha} \, dx \, dt.
$$

Moreover, from the definition of $\beta^*$ and $\alpha^*$, we readily see that $e^{-s\alpha^*} \leq e^{-s \beta^*}$.

From this fact and noting that $\beta$ (resp. $\alpha$) blows up exponentially as $t \to T^-$ (resp. $t \to 0^+$ and $t \to T^-$) while $\gamma$ (resp. $\xi$) blows up polynomially as $t \to T^-$ (resp. $t \to 0^+$ and $t \to T^-$), we can use (104), (103) and our Carleman inequality (34) to deduce that

$$
\int_{T/2}^T \int_0^1 \gamma |\phi|^2 e^{-2s \beta} \, dx \, dt + \int_{T/2}^T \int_0^1 \gamma^3 |\psi|^2 e^{-2s \beta} \, dx \, dt
\leq C(s, \lambda) \left( \iint_{Q_T} e^{-2s \beta} g_1^2 \, dx \, dt + \iint_{Q_T} e^{-2s \beta} g_2^2 \, dx \, dt + \iint_{\omega_2 \times (0, T)} e^{-s \beta^*} (\gamma^*)^8 |\phi|^2 \, dx \, dt \right).
$$

For the set $(0, T/2) \times (0, 1)$, we will use energy estimates for the system (18). More precisely, consider the function $v \in C^1([0, T])$ such that

$$
v = 1 \text{ in } [0, T/2], \quad v = 0 \text{ in } [3T/4, T/4], \quad |v'| \leq C / T.
$$
Setting \((\tilde{\phi}, \tilde{\psi}) = (v\phi, v\psi)\), it is not difficult to see that the new variables verify the system

\[
\begin{aligned}
-\tilde{\phi}_t &= a_{11}\tilde{\phi} + a_{21}\tilde{\psi} + v g_1 - v'\phi \\
-\tilde{\psi}_t &= d_v \tilde{\psi}_{xx} + a_{12}\tilde{\phi} + a_{22}\tilde{\psi} + v g_2 - v'\psi \\
\partial_x \tilde{\psi} &= 0 \\
(\tilde{\phi}, \tilde{\psi})(T, \cdot) &= (0, 0)
\end{aligned}
\quad \text{in } QT, \quad \text{on } \Sigma_T, \quad \text{in } (0, 1).
\]

From standard energy estimates, we deduce that system (107) verifies

\[
\int_\Omega |\tilde{\phi}(t, x)|^2 \, dx + \int_\Omega |\tilde{\psi}(t, x)|^2 \, dx + d_v \int_t^T \int_\Omega |\tilde{\psi}_x|^2 \, dx \, dt \\
\leq C \left( \int_0^T \int_\Omega |\tilde{\phi}|^2 \, dx \, dt + \int_0^T \int_\Omega |\tilde{\psi}|^2 \, dx \, dt + \int_0^T \int_\Omega |\eta g_1|^2 \, dx \, dt \\
+ \int_0^T \int_\Omega |\eta g_2|^2 \, dx \, dt + \int_0^T \int_\Omega |\eta'\phi|^2 \, dx \, dt + \int_0^T \int_\Omega |\eta'\psi|^2 \, dx \, dt \right), \quad \forall t \in [0, T],
\]

where \(C\) is a positive constant only depending on \(a_{ij}\).

Dropping the third term in the left-hand side of the above expression, we use Gronwall’s inequality to deduce

\[
\|\tilde{\phi}(0, \cdot)\|_{L^2(\Omega)}^2 + \|\tilde{\psi}(0, \cdot)\|_{L^2(\Omega)}^2 + \int_0^T \int_{\Omega_T} |\tilde{\phi}|^2 \, dx \, dt + \int_0^T \int_{\Omega_T} |\tilde{\psi}|^2 \, dx \, dt \\
\leq C \left( \int_0^T \int_{\Omega_T} |\eta g_1|^2 \, dx \, dt + \int_0^T \int_{\Omega_T} |\eta g_2|^2 \, dx \, dt + \int_0^T \int_{\Omega_T} |\eta'\phi|^2 \, dx \, dt \\
+ \int_0^T \int_{\Omega_T} |\eta'\psi|^2 \, dx \, dt \right),
\]

for some constant \(C > 0\) only depending on \(T\) and \(a_{ij}\).

Recalling the definition of \(\eta\), we obtain from the above expression

\[
\|\phi(0, \cdot)\|_{L^2(\Omega)}^2 + \|\psi(0, \cdot)\|_{L^2(\Omega)}^2 + \int_0^{T/2} \int_\Omega |\phi|^2 \, dx \, dt + \int_0^{T/2} \int_\Omega |\psi|^2 \, dx \, dt \\
\leq C \left( \int_0^{3T/4} \int_\Omega |g_1|^2 \, dx \, dt + \int_0^{3T/4} \int_\Omega |g_2|^2 \, dx \, dt \right) + \frac{C}{T^2} \left( \int_{T/2}^{3T/4} \int_\Omega |\phi|^2 \, dx \, dt \\
+ \int_{T/2}^{3T/4} \int_\Omega |\psi|^2 \, dx \, dt \right).
\]

Since the domain of integration in the above integrals is away from the singularity of the weight functions (48) at \(t = T\) (and therefore they are bounded), we can introduce
them in the above inequality as follows

$$\|\phi(0, \cdot)\|_{L^2(\Omega)}^2 + \|\psi(0, \cdot)\|_{L^2(\Omega)}^2 + \int_0^{T/2} \int_{\Omega} e^{-2s\beta} \gamma \left( |\phi|^2 + |\phi_x|^2 \right) \, dx \, dt$$

$$+ \int_0^{T/2} \int_{\Omega} \gamma^3 |\psi|^2 e^{-2s\beta} \, dx \, dt$$

$$\leq C(s, \lambda, T) \left( \int_0^{3T/4} \int_{\Omega} e^{-2s\beta} \gamma^3 |g_1|^2 \, dx \, dt + \int_0^{3T/4} \int_{\Omega} e^{-2s\beta} |g_2|^2 \, dx \, dt \right)$$

$$+ C(s, \lambda, T) \left( \int_{T/2}^T \int_{\Omega} e^{-2s\beta} \gamma^3 |\psi|^2 \, dx \, dt + \int_{T/2}^T \int_{\Omega} e^{-2s\beta} \gamma^3 |\psi|^2 \, dx \, dt \right).$$

Using estimate (105) to bound all the terms on the last line of the above inequality and adding up the resulting expression to (105) yields

$$\|\phi(0, \cdot)\|_{L^2(\Omega)}^2 + \|\psi(0, \cdot)\|_{L^2(\Omega)}^2$$

$$+ \int_0^{T/2} \int_{\Omega} e^{-2s\beta} \gamma \left( |\phi|^2 + |\phi_x|^2 \right) \, dx \, dt$$

$$\leq C \left( \int_{Q_T} e^{-s\beta} \gamma^3 |g_1|^2 \, dx \, dt + \int_{Q_T} e^{-s\beta} |g_2|^2 \, dx \, dt \right)$$

$$+ C \left( \int_{Q_T} e^{-s\beta} \gamma^3 |\psi|^2 \, dx \, dt \right).$$

To conclude, it is enough to use definitions (48) in the above inequality. This ends the proof. □

6. Some properties of the heat semigroup

We recall in the next result some well-known facts about the heat semigroup with a diffusion parameter $d_v > 0$ with homogeneous Neumann boundary conditions on the interval $(0, 1)$, denoted by $\{e^{t d_v \partial_{xx}}\}_{t \geq 0}$. The proof can be found for instance in [22, Lemma A.1].

Lemma C.1. The following properties hold true.

a. For every constant $K \in \mathbb{R}$, we have $e^{t d_v \partial_{xx}} K = K$ for all $t \geq 0$.

b. For every $z_0 \in L^2(\Omega)$, there exists a constant $C > 0$ only depending on $z_0$ such that for every $t \geq 0$,

$$\left\| e^{t d_v \partial_{xx}} \left( z_0 - \int_\Omega z_0 \, dx \right) \right\|_{L^2(\Omega)} \leq C e^{-\lambda_1 d_v t} \| z_0 \|_{L^2(\Omega)},$$

(108)

where $\lambda_1 > 0$ is the first positive eigenvalue of the Neumann Laplacian operator $-\partial_{xx}$ on $(0, 1)$. 
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Víctor Hernández-Santamaría
Instituto de Matemáticas
Universidad Nacional Autónoma de México
Circuito Exterior, C.U.
C.P. 04510 CDMX
Mexico
E-mail: victor.santamaria@im.unam.mx

Kévin Le Balc’h
Institut de Mathématiques de Bordeaux
351 Cours de la Libération
33400 Bordeaux
France
E-mail: kevin.le-balc@math.u-bordeaux.fr

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