PAIRS OF PANTS, POCHHAMMER CURVES AND $L^2$-INVARIANTS

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Abstract. We propose an intuitive interpretation for nontrivial $L^2$-Betti numbers of compact Riemann surfaces in terms of certain loops in embedded pairs of pants. This description uses twisted homology associated to the Hurewicz map of the surface, and it satisfies a sewing property with respect to a large class of pair-of-pants decompositions. Applications to supersymmetric quantum mechanics incorporating Aharonov–Bohm phases are briefly discussed, for both point particles and topological solitons (Abelian and non-Abelian vortices) in two dimensions.

1. Introduction

The theory of $L^2$-invariants was created out of an effort by Atiyah to extend the index theorem for elliptic operators on compact manifolds to certain noncompact situations. In a ground-breaking paper [1], he proposed a $\Gamma$-index for elliptic differential operators on an infinite Galois cover $\tilde{M}$ of a compact manifold $M$ which combined analysis of operators over a fundamental domain for the action of the group of deck transformations $\Gamma$, identified with an open dense subset of $M$, with analysis on the Hilbert space of sequences $\ell^2(\Gamma)$. This machinery, applied to the operator $d + d^*$ on the manifold $\tilde{M}$ with a cocompact $\Gamma$-invariant Riemannian metric, leads to invariants $b^{(2)}_i$ or $b^{(2)}_i$ which generalize the rank of $L^2$-cohomology groups of finite covers; it can be used to great advantage in cases where the $L^2$-cohomology of the covering space is an infinitely generated vector space. These $L^2$-Betti numbers are obtained as Murray–von Neumann (or renormalized) dimensions $\mathcal{N}(\Gamma)$ of certain Hilbert $\mathcal{N}(\Gamma)$-modules, where $\mathcal{N}(\Gamma) := \mathcal{B}(\ell^2(\Gamma))^\Gamma$ is the von Neumann algebra associated to the discrete group $\Gamma$. For example, if $M = \mathbb{T}^n$ is an $n$-torus (the setting for classical Fourier analysis), applying this construction to the universal cover $\tilde{M} = \mathbb{R}^n$ leads to all $L^2$-Betti numbers $b^{(2)}_i(\mathbb{R}^n, \Gamma)$ for $\Gamma = \pi_1(\mathbb{T}^n) \cong \mathbb{Z}^n$ being zero — this follows from multiplicity under finite covers and a Künneth-type formula. Quite often, $L^2$-Betti numbers vanish in situations where the ordinary Betti numbers of the quotient space do not, and they are not necessarily integral real numbers; a question that was open until recently was whether they can ever be irrational. For a layout of the theory of $L^2$-invariants, and in order to appreciate the impact that these ideas have had in mathematics so far, we refer the reader to the textbook [19]. An informal account of some basic notions, tailored to the purposes of the present paper, will be given in section 4 below.

An example leading to nontrivial $L^2$-invariants is provided by the universal cover $\mathbb{D}$ (the disc) of a compact oriented Riemann surface $M = \Sigma$ of genus $g > 1$. This simple but somewhat crucial example was already considered in Atiyah’s original paper, where he illustrated his new theory by showing that

$$b^{(2)}_i(\Sigma, \Gamma) = \begin{cases} 0 & \text{if } i \neq 1, \\ 2g - 2 & \text{if } i = 1. \end{cases}$$

Availing oneself of the standard machinery [19], the argument runs along the following lines. The $L^2$-Betti numbers in degrees $i = 0$ or $2$ must vanish because $\mathbb{D}$ has no compact component, thus any nontrivial $L^2$-Betti numbers must lie in degree $1$. Then one uses that

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the $L^2$-Euler characteristic of a co-compact space equals the ordinary Euler characteristic of the quotient to conclude that

$$(2) \quad b_1(\Sigma; \Gamma) = -\frac{2}{\pi} \sum_{i=0}^{2} (-1)^i b_i(\Sigma; \Gamma) =: -\chi(\Sigma; \Gamma) = -\chi(\Sigma) = 2g - 2.$$ 

As elegant as this argument may be, it throws very little light on the meaning attached to the result (1). In addition, the definitions that have been provided for these invariants (three different versions are proposed in [19], all of which coincide in Atiyah’s example) do not seem to give a hint to their geometrical content. This is in contrast with ordinary Betti numbers in singular homology or de Rham cohomology, for instance.

In this paper, we attempt to remedy this situation, and place the formula (1) on more intuitive ground. More precisely, our immediate interest is in the case where $\tilde{\Sigma}$ is instead of $\Sigma$.

We may alternatively interpret $\alpha$ in terms of a set of 2 objects naturally associated to pair-of-pants decompositions of $\Sigma$ — in a similar spirit to textbook presentations of concrete bases for $H_1(\Sigma; \mathbb{Z}) \cong \mathbb{Z}^{2g}$ in singular homology, or $H^1(\Sigma; \mathbb{R}) \cong \mathbb{R}^{2g}$ in de Rham cohomology (see e.g. [11] for the latter). The relevant objects to consider will emerge as certain equivariant homology classes that we will associate to Pochhammer 1-cycles on pairs of pants, as explained in section 3. We shall illustrate how this sort of interpretation plays a role in the context of gauge theory of charged particles on a surface $\Sigma$ in section 5 after connecting $L^2$-Betti numbers to Witten’s supersymmetric quantum mechanics [31] coupled to local systems on manifolds.

2. Twisted homology of $\alpha$-covers

We start by fixing an Abelian group $A \cong \mathbb{Z}^n$ of rank $n \in \mathbb{N}$. Let $R := \mathbb{Z}[A]$ be the integral group ring of $A$, and let us denote by $F := \text{Quot}(R)$ its field of fractions. In what follows, we shall be interested in the topology of a given space $X$ equipped with a group homomorphism

$$\alpha : \pi_1(X) \to A.$$ 

We may alternatively interpret $\alpha$ in terms of a cohomology class $\tilde{\alpha} \in H^1(X; A)$ via the universal coefficient theorem in cohomology [13], since $\alpha$ factors through the Hurewicz map. A natural choice might be to take $\alpha$ as the Hurewicz map itself modulo torsion and $n = b_1 := \text{rk} H_1(X; \mathbb{Z})$, but we shall also be interested in the general situation.

There is a cover $p_\alpha : \tilde{X} \to X$ associated to each $\alpha$ (or to its cohomological avatar $\tilde{\alpha}$) such that $C_*(\tilde{X})$ is a free $R$-module; we will refer to $p_\alpha$ and $\tilde{X}$ interchangeably as the $\alpha$-cover of $X$. To understand $p_\alpha$ in concrete terms, one could consider a classifying map $\tilde{\alpha} : \tilde{X} \to BA \cong (S^1)^n$ representing $\tilde{\alpha}$, and define $\tilde{X}$ to be the pull-back of the universal cover of the torus $\mathbb{R}^n \to (S^1)^n$ under this map $\tilde{\alpha}$. Note that the covering space $\tilde{X}$ of an $\alpha$-cover is not necessarily connected. We can tensor the coefficients of the chain complex $C_*(\tilde{X})$ with an $A$-module $M$ to obtain the $\pi_1(X)$-equivariant homology via the homomorphism $\alpha$. This construction provides us with homology groups enriched with information pertaining to the homotopy theory of $X$. For a judicious choice of $\alpha$, they have the potential of capturing information on the topology of $X$ that is missed out by the non-equivariant (e.g singular) homology or cohomology groups of the space $X$.

In this paper, we want to explore a context where the field $F$ provides an interesting choice of coefficients. Thus for $i \in \mathbb{N}_0$ we shall consider the $\alpha$-twisted homology groups

$$H_i^\alpha(X) := H_i(C_*(\tilde{X}) \otimes_R F).$$
Observe that, for each $i$, there is an obvious map $\rho_i$ given by the composition
\[(4) \quad \rho_i : H_i(\tilde{X}; \mathbb{Z}) = H_i(C_*(\tilde{X})) \cong H_i(C_*(\hat{X})) \otimes_R R \rightarrow H_i(C_*(\hat{X}) \otimes_R \mathbb{F}) = \mathcal{H}_i^\alpha(X).
\]
More generally, if $K$ is a field of characteristic zero, we write $F_K := \text{Quot}(K[A]) \supset \mathbb{F}$ and define $\mathcal{H}_i^\alpha(X; F_K) := H_i(C_*(\tilde{X}) \otimes_R F_K)$. Since $F_K$ is necessarily projective over $\mathbb{F}$, one has $\mathcal{H}_i^\alpha(X; F_K) \cong \mathcal{H}_i^\alpha(X) \otimes_{\mathbb{F}} F_K$. We shall sometimes refer to the $\alpha$-cover associated to the Hurewicz map of $X$ as its Hurewicz cover.

If the homomorphism $\alpha$ is trivial (i.e. it maps to $0 \in A$), then the corresponding $\alpha$-cover is simply the second-component projection $A \times X \rightarrow X$. In this case, $C_*(\tilde{X}) \cong C_*(X) \otimes_{\mathbb{Z}} R$ and $C_*(\hat{X}) \otimes_R \mathbb{F} \cong C_*(\tilde{X}) \otimes_{\mathbb{Z}} R \otimes_R \mathbb{F} \cong C_*(X) \otimes_{\mathbb{Z}} \mathbb{F}$. Thus for $\alpha = 0$ we simply recover $\mathcal{H}_i^\alpha(X) \cong H_i(X; \mathbb{F})$, the homology of $X$ with untwisted coefficients in $\mathbb{F}$.

By the universal coefficient theorem in homology [13], this equals $H_i(X; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{F}$.

More generally, suppose that $i' : A' \cong \mathbb{Z}^{n'} \rightarrow A$ is an injective group homomorphism for some $0 \leq n' \leq n$, and that $\alpha$ factors as
\[(5) \quad \alpha : \pi_1(X) \xrightarrow{\alpha'} A' \xrightarrow{\iota'} A.
\]
Write $R' := \mathbb{Z}[A']$ and $F' := \text{Quot}(R')$; then $\iota'$ induces a field extension $F' \hookrightarrow F$. If $\alpha$ is trivial, we can choose $F'$ to be the prime field $\mathbb{Q}$, but we will also be interested in the case when $F'$ is a field intermediate between $\mathbb{Q}$ and $\mathbb{F}$.

Let $X'$ denote the corresponding $\alpha'$-cover of $X$. We have
\[C_*(\tilde{X}) \otimes_R \mathbb{F} \cong (C_*(\tilde{X}') \otimes_{R'} R) \otimes_R \mathbb{F} \cong C_*(\tilde{X}) \otimes_{R'} \mathbb{F}' \otimes_{F'} \mathbb{F}.
\]
Taking the homology of this complex, we obtain the following reduction property:
\[(6) \quad \mathcal{H}_i^\alpha(X) = \mathcal{H}_i^{\alpha'}(X) \otimes_{F'} \mathbb{F}.
\]
We will now look at special cases. The first one is $X = S^1$.

**Lemma 1.** For all $i \in \mathbb{N}_0$,
\[\mathcal{H}_i^\alpha(S^1) \cong \begin{cases} H_i(S^1; \mathbb{Q}) \otimes_{\mathbb{Q}} \mathbb{F} & \text{if } \alpha \text{ is trivial,} \\ 0 & \text{otherwise.} \end{cases}
\]

**Proof.** The first case follows directly from the reduction argument above, while the second statement is a computation that we need to carry out.

Consider first the case $A = \mathbb{Z}$ and $\alpha = \text{id}_\mathbb{Z}$. Then $S^1 \cong \mathbb{R}$ is the universal cover of the circle with the translation action of $\mathbb{Z}$, and $C_*(\tilde{S^1})$ is the $A$-equivariant chain complex $C_1 \cong R, C_0 \cong R$. The boundary map is multiplication by $1 - [\gamma]$, where $\gamma$ generates $A$. Since this element of $R \subset \mathbb{F}$ is nontrivial, after tensoring with $\mathbb{F}$ it induces an isomorphism, and we get $H_*^\alpha(S^1) \cong 0$.

The next case is $A = \mathbb{Z}$ and $\alpha$ nontrivial, say the image of $\alpha$ is the subgroup $A' = mA \subset A$ for some $m \geq 2$. We have that $\alpha$ factors as $\pi_1(X) \xrightarrow{\alpha'} A' \subset A$. There is a corresponding field extension $F' \subset \mathbb{F}$, and by the reduction property, we have that $H_*^\alpha(X) \cong H_*^{\alpha'}(X) \otimes_{F'} \mathbb{F}$, which proves the lemma in this case.

Finally, if $\alpha : \pi_1(S^1) \rightarrow A$ is any nontrivial homomorphism, its image is contained in some split summand $A' \cong \mathbb{Z}$. Applying the previous case, we obtain that $H_*^{\alpha'}(S^1) \cong 0$, and using the reduction property we obtain the result. \(\square\)

Suppose that we have a Mayer–Vietoris situation for the union $X = X' \cup X''$. A homomorphism $\alpha : \pi_1(X) \rightarrow A$ restricts to homomorphisms from $\pi_1(X')$ and $\pi_1(X'')$. Note that the choice of basepoints is immaterial, since $A$ is Abelian. There is a a long exact sequence
\[\cdots \rightarrow \mathcal{H}_i^\alpha(X' \cap X'') \rightarrow \mathcal{H}_i^\alpha(X') \oplus \mathcal{H}_i^\alpha(X'') \rightarrow \mathcal{H}_i^\alpha(X) \rightarrow \mathcal{H}_{i-1}^\alpha(X' \cap X'') \rightarrow \cdots
\]
In particular, if we know that the homology of the intersection \(X' \cap X''\) is trivial, \(H_0^\alpha(X) \cong H_0^\alpha(X') \oplus H_0^\alpha(X'')\). This **additivity property** underlies the main application of \(\alpha\)-twisted homology in this paper, which we will make precise in section \[\text{section}\]

The next case we shall consider is the wedge product of two circles: \(X = S^1 \vee S^1\).

**Theorem 2.** Suppose that the homomorphism \(\alpha : \pi_1(S^1 \vee S^1) \to A\) is nontrivial. Then

\[
H_i^\alpha(S^1 \vee S^1) = \begin{cases} F & \text{if } i = 1, \\ 0 & \text{otherwise}. \end{cases}
\]

Moreover, if \(\alpha\) is trivial on one of the wedge factors, the inclusion of this factor induces an isomorphism on \(H_1^\alpha\).

**Proof.** Let us label the two wedge factors \(S_j\) by \(j \in \{1, 2\}\). First consider the case when \(\alpha\) induces a nontrivial homomorphism on each wedge factor. Then \(H_0^\alpha(S_j) = 0\) for \(j = 1, 2\), so, as a first application of the additivity property, the Mayer–Vietoris long exact sequence degenerates to an isomorphism \(H_0^\alpha(S_1 \vee S_2) \to H_0^\alpha(\{\text{pt}\}) \cong F\).

Since \(\alpha\) is nontrivial, it cannot be nontrivial on both wedge components. Assume without loss of generality that it is nontrivial on \(S_2\). Then \(H_0^\alpha(S_2) = 0\), and the Mayer–Vietoris long exact sequence takes the form

\[
0 \to H_0^\alpha(S_1) \to H_0^\alpha(S_1 \vee S_2) \to H_0^\alpha(\{\text{pt}\}) \to H_0^\alpha(S_1) \to H_0^\alpha(S_1 \vee S_2) \to 0.
\]

By reduction and naturality, since the map \(H_0(\{\text{pt}\}; \mathbb{Q}) \to H_0(S_1; \mathbb{Q})\) is an isomorphism, the map \(H_0^\alpha(\{\text{pt}\}) \to H_0^\alpha(S_1)\) is also an isomorphism. It follows that \(H_0^\alpha(S_1 \vee S_2) \cong 0\), and that the map \(H_1^\alpha(S_1) \to H_1^\alpha(S_1 \vee S_2)\) is an isomorphism. Since we know from Lemma \[\text{lemma}\] that \(H_1^\alpha(S_1) \cong F\), this completes the proof of the theorem.

Of course it can happen that neither of the wedge factors evaluate trivially.

**Lemma 3.** The natural map \(\rho_1 : H_1(S_1 \vee S_2; \mathbb{Z}) \to H_1^\alpha(S_1 \vee S_2)\) defined by \[\text{equation}\] is injective.

**Proof.** First assume that both of the fundamental classes of \(S_1\) and \(S_2\) evaluate nontrivially under the map \(\alpha : \pi_1(S_1 \vee S_2) \to A\). We build \(S_1 \vee S_1\) using the following pushout square:

\[
\begin{array}{ccc}
\{x_1\} & \text{II} & \{x_2\} \\
\downarrow & & \downarrow \\
\{x\} & \longrightarrow & S_1 \vee S_2
\end{array}
\]

This induces Mayer–Vietoris sequences for both \(H_*(S_1 \vee S_2; \mathbb{Z})\) and \(H_*^\alpha(S_1 \vee S_2)\), and a natural transformation \(\rho_*\) between them. It follows that the pushout diagram above gives a ladder whose upper row is given by the functors \(X \mapsto H_*(\tilde{X}; \mathbb{Z})\) and whose lower row is given by the functors \(X \mapsto H_*^\alpha(X)\). The condition on \(\alpha\) implies that \(H_1(\tilde{S}_1; \mathbb{Z}) \cong 0\). Using this fact, we see that the ladder includes the following diagram with exact upper row:

\[
\begin{array}{ccc}
0 & \longrightarrow & H_1(S_1 \vee S_2; \mathbb{Z}) \\
\rho \downarrow & & \rho \downarrow \\
H_1^\alpha(S_1 \vee S_2) & \longrightarrow & H_0^\alpha(\{x_1\} \text{ II} \{x_2\}; \mathbb{Z})
\end{array}
\]

To prove the lemma it suffices to prove that \(\rho_0 : H_0(\{p_1\}; \mathbb{Z}) \to H_0^\alpha(\{p_1\})\) is injective. But this map can be identified with the inclusion of \(R = \mathbb{Z}[A]\) into its quotient field.

We also consider the case where one of the fundamental classes \([S_1]\) evaluates trivially under \(\alpha\). Let assume \(\alpha([S_1]) = 0\) and \(\alpha([S_2]) \neq 0\). There is a homotopy automorphism \(h\) of \(S_1 \vee S_2\) such that \(h_*[S_1] = [S_1] + [S_2]\) and \(h_*[S_2] = [S_2]\). Apply the previous case to the composed map \(\alpha \circ h : \pi_1(S_1 \vee S_2) \to A\). Finally, use naturality to conclude that the lemma is also true for an \(\alpha\) of this sort.

\[\square\]
Let us fix a homomorphism $\alpha : \pi_1(S^1 \lor S^1) \to A$. Suppose that $f : S^1 \to S^1 \lor S^1$ is a map such that the composition $\hat{\alpha} := \alpha \circ \pi_1(f) : \pi_1(S^1) \to A$ is the trivial homomorphism. Then $f$ induces a map

$$f_* : \mathbb{F} \cong \mathcal{H}_1^0(S^1) \to \mathcal{H}_1^0(S^1 \lor S^1) \cong \mathbb{F}.$$  

(7)

The following statement is a generalization of the last part of Theorem 2.

**Lemma 4.** Let $p : \widetilde{S}_1^1 \lor \widetilde{S}_2^1 \to S_1^1 \lor S_2^1$ be the covering corresponding to $\alpha$. Let $f : S^1 \to S_1^1 \lor S_2^1$ a map which is nontrivial in homology. Then $(p \circ f)_* : \mathcal{H}_1^0(S^1) \to \mathcal{H}_1^0(S_1^1 \lor S_2^1)$ is an isomorphism.

**Proof.** $f_* : \mathcal{H}_1^0(S^1) \cong \mathbb{F}$, so $f_*$ is an isomorphism if and only if it is nontrivial. Consider the diagram

$$
\begin{array}{ccc}
H_1(S_1^1; \mathbb{Z}) & \xrightarrow{f_*} & H_1(S_1^1 \lor S_2^1; \mathbb{Z}) \\
\rho \downarrow & & \rho \downarrow \\
H_1^0(S^1) & \xrightarrow{f_*} & H_1^0(S_1^1 \lor S_2^1)
\end{array}
$$

By Lemma 3 the right vertical map is injective. Since $f_*([S^1]) \neq 0$, it follows that $(p \circ f)_*([S^1]) \neq 0$. \qed

**Remark 5.** If for instance $\alpha$ is not injective, consider any closed curve $\gamma : S^1 \to S_1^1 \lor S_2^1$ representing a nontrivial homology class in the kernel of $\alpha$. By covering theory, it can be written as $p \circ f$ for some $f : S^1 \to S_1^1 \lor S_2^1$. Since $(p \circ f)_*([S^1]) \neq 0$, certainly $f_*([S^1]) \neq 0$. By Lemma 4 $\gamma$ induces an isomorphism on $\mathcal{H}_1^0$. If $\alpha$ is injective, a commutator of the inclusion of the two circles will be a map $\gamma : S^1 \to S_1^1 \lor S_2^1$ which lifts to $f : S^1 \to S_1^1 \lor S_2^1$. It is easy to check that $f$ is not trivial on homology, so this map $\gamma$ induces an isomorphism on $\mathcal{H}_1^0$.

Our last result in this section is a K"unneth-type formula for twisted homology that will be needed in section 3.2. Let $\mathbb{K} \subset \mathbb{C}$ be a subfield, for instance $\mathbb{K} = \mathbb{Q}$ or $\mathbb{K} = \mathbb{C}$, and $\mathbb{F}_\mathbb{K} := \text{Quot}(\mathbb{K}[A])$ as before.

**Theorem 6** (K"unneth formula). Let $p : X \times Y \to Y$ be the projection, $\alpha : \pi_1(Y) \to A$ and $\alpha' := \alpha \circ \pi_1(p)$. Then

$$\mathcal{H}_1^0(X \times Y; \mathbb{F}_\mathbb{K}) \cong \bigoplus_{j+k=i} H_j(X; \mathbb{Q}) \otimes_{\mathbb{Q}} H_k^0(Y; \mathbb{F}_\mathbb{K}).$$

**Proof.** The Alexander–Whitney chain homotopy equivalence $C_* (\tilde{X} \times \tilde{Y}) \to C_* (\tilde{X}) \otimes \mathbb{Z} C_* (\tilde{Y})$ is natural for pairs of continuous maps. It follows that it is equivariant under $\mathbb{Z}[\pi_1(X \times Y)] \cong \mathbb{Z}[\pi_1(X) \times \pi_1(Y)]$, so that it induces a chain homotopy equivalence

$$C_* (X \times Y) \otimes_{\mathbb{Z}[\pi_1(X \times Y)]} \mathbb{F}_\mathbb{K} \to C_* (\tilde{X}) \otimes_{\mathbb{Z}} C_* (\tilde{Y}) \otimes_{\mathbb{Z}[\pi_1(Y)]} \mathbb{F}_\mathbb{K}.$$  

Since $\pi_1(X)$ acts trivially on $\mathbb{F}_\mathbb{K}$, this last complex equals $C_* (X) \otimes_{\mathbb{Z}} C_* (\tilde{Y}) \otimes_{\mathbb{Z}[\pi_1(Y)]} \mathbb{F}_\mathbb{K}$, which again equals $(C_* (X) \otimes_{\mathbb{Z}} \mathbb{Q}) \otimes_{\mathbb{Q}} (C_* (\tilde{Y}) \otimes_{\mathbb{Z}[\pi_1(Y)]} \mathbb{F}_\mathbb{K})$. Now apply the usual K"unneth formula to this complex. \qed

### 3. Pochhammer Principle for Pair-of-Pants Decompositions

In this section, we will take $\Sigma$ to be any oriented surface with negative Euler characteristic, which we assume to be compact (possibly with boundary). Topologically, we can think of it as a closed surface of genus $g$ with $h$ holes or punctures (realized by removing open discs), with the condition

$$\chi(\Sigma) = 2 - 2g - h < 0.$$
being enforced. In later sections, we shall specialize to the closed case where \( h = 0 \), but not for the time being.

On such a surface \( \Sigma \), we shall consider pair-of-pants decompositions \( \{P_j\}_j \). Each \( P_j \), which we take to be open, is homeomorphic to a 2-sphere with three closed discs removed, and the boundary \( \partial P_j \subset \Sigma \) is a disjoint union of circles. Recall that the Hurewicz cover \( p_\alpha \) of \( \Sigma \) is associated to the Abelianization homomorphism
\[
\alpha : \pi_1(\Sigma) \to A := H_1(\Sigma; \mathbb{Z}).
\]
We can restrict this \( \alpha \) to any pair of pants of the decomposition, and also to any of its boundary circles, but of course the corresponding covers will in general not be Hurewicz for those subsets. We say that a particular pair of pants \( P \subset \Sigma \) is fashionable if all of its boundary circles \( S_k^1 \subset \partial P \) (there are at most three such circles) define nontrivial elements of the 1-homology group \( H_1(\Sigma; \mathbb{Z}) \); and that a pair-of-pants decomposition \( \{P_j\}_j \) of \( \Sigma \) is fashionable if all its \( P_j \) are. For instance, the pair of pants \( P \subset \Sigma \) illustrated in Fig. 2 is fashionable, but the pairs of pants \( P_1 \) and \( P_2 \) in Fig. 3 are not.

The following result justifies considering fashionable pair-of-pants decompositions.

**Theorem 7.** Let \( \Sigma \) be an orientable surface with \( \chi(\Sigma) = 2 - 2g - h < 0 \). Given a fashionable pair-of-pants decomposition \( \{P_j\}_j \) of \( \Sigma \), the inclusions \( i_j : P_j \hookrightarrow \Sigma \) induce an isomorphism
\[
\bigoplus_j H_1^\alpha(P_j) \xrightarrow{\cong} H_1^\alpha(\Sigma).
\]

**Proof.** If \( d := 2g - 2 + h = 1 \), then \( \Sigma \) is either a 2-sphere with three punctures (homeomorphic to the closure of a pair of pants) or a 2-torus with one puncture (the closure of a pair of pants with two of its circle boundaries identified). In the former case there is nothing to show, whereas in the latter case we observe that the interior circle \( S_0^1 \) formed by the identification of the two boundary components is such that \( H_1^\alpha(S_0^1) \cong 0 \) by Lemma 4 and the statement follows.

We claim inductively (on \( d > 1 \)) that, if \( X \subset \Sigma \) is the closure in \( \Sigma \) of a union of pairs of pants in our decomposition, we have an isomorphism
\[
\bigoplus_{P_j \subset X} H_1^\alpha(P_j) \xrightarrow{\cong} H_1^\alpha(X).
\]
This follows from the induction hypothesis, and from the assumption that each \( P_j \) is fashionable: for each boundary circle \( S_{j,k}^1 \subset P_j \), Lemma 4 again implies that \( H_1^\alpha(S_{j,k}^1) \cong 0 \). The well-known fact that any pair-of-pants decomposition of a surface of genus \( g \) with \( h \) punctures has exactly \( d = 2g - 2 + h \) elements is a consequence of the additivity property of \( \chi \) under gluing and \( \chi(P_j) = -1 \), \( \chi(S^1) = 0 \), \( \chi(D) = 1 \).

By duality, the condition that a simple curve on a closed surface \( \Sigma \) represents zero in \( H_1(\Sigma; \mathbb{Z}) \) is equivalent to the condition that the complement of the curve has two components. This criterion is also appropriate to treat the case of a punctured surface \( \Sigma \): one applies it to the closed surface \( \Sigma' \) obtained from \( \Sigma \) by either capping all its punctures by discs or shrinking boundary circles to points.

Let \( P \) be a pair of pants and \( i : P \hookrightarrow \Sigma \) be an inclusion into an oriented surface.

**Lemma 8.** \( P \) is fashionable if and only if the map \( i_* : H_1(P; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z}) \) is injective.

**Proof.** It is clear that if \( i_* \) is injective, then \( P \) is fashionable. We need to prove the opposite implication. Assume that \( P \) is fashionable. Consider the complement \( P^c := \Sigma \setminus P \). Since \( P \) is fashionable, \( P^c \) is connected. There is a decomposition \( \Sigma = P \cup P^c \). Let \( z \in \ker(i_*) \). From the Mayer–Vietoris long exact sequence of the decomposition, it follows that \( P \) has two boundary circles such that \( z = a_1[C_1] + a_2[C_2] \in H_1(P; \mathbb{Z}) \), and such that \( a_1[C_1] + a_1[C_2] = 0 \in H_1(P^c; \mathbb{Z}) \). These homology classes correspond to the homology...
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Classes of two punctures in $H_1(P^c; \mathbb{Z})$. But since $P^c$ is connected, the only relation satisfied by the homology classes of the punctures of $P^c$ is that the sum of all of them is trivial. Since $P^c$ has at least three punctures, we obtain that $a_1 = a_2 = 0$.

One could ask if there exist fashionable pair-of-pants decompositions on a surface $\Sigma$. Actually, they are quite common. At least for closed $\Sigma$, there is always a finite process that makes any given pair-of-pants decomposition fashionable. Let us describe an elementary operation that is useful for this purpose. Suppose that a given pair of pants $P \subset \Sigma$ in a decomposition of $\Sigma$ is unfashionable because (at least) one of its interior boundary circles $S \subset \partial P \subset \Sigma$ is trivial in $H_1(\Sigma; \mathbb{Z})$; a necessary condition is that this particular circle also occurs as boundary of another pair of pants $P'$ of the decomposition. The union $T := \bar{P} \cup S \cup \bar{P'} \subset \Sigma$ is homeomorphic to a 2-sphere with four disc punctures — or equivalently, to one of the T-shirts depicted in Fig. 1. A T-shirt flip on $S$ consists of two steps: first, for each of $P, P'$ we mark one of the two boundary circles disjoint from $S$ as a sleeve; then, we replace $S$ by a new embedded circle $S' \subset T$ which defines two new pairs of pants $P^\wedge, P^\vee \subset T$ with $T = \bar{P^\wedge} \cup S \cup \bar{P^\vee}$, in such a way that the sleeves remain separated by $S'$; see Fig. 1. One gets three types of such flips up to diffeomorphisms of $T$ relative to $\partial T$, depending on the choice of sleeves.

**Proposition 9.** Let $\Sigma$ be an orientable surface of negative Euler characteristic. There is a fashionable pair-of-pants decomposition of $\Sigma$.

**Proof.** If the Euler characteristic of $\Sigma$ is $-1$, $\Sigma$ is either a sphere with four holes or a torus with one hole, and any pair-of-pants decomposition will do. We claim inductively that the theorem is true for surfaces of Euler characteristic greater or equal to $-d$. Let $\Sigma$ be an orientable surface of Euler characteristic $-d$. Pick a pair-of-pants decomposition. If every boundary curve in this pair-of-pants decomposition is non-separating, we are done. So suppose that we can find a simple curve $S$ that separates $\Sigma$ into two surfaces $\Sigma_1$ and $\Sigma_2$ of Euler characteristic each at least $-d + 1$. By induction, we can find a fashionable pair-of-pants decomposition on each $\Sigma_i$. These decompositions combine to a pair-of-pants decomposition with one single separating boundary curve $S$. The two pair of pants adjacent to $S$ form a 2-sphere with four holes. Flip the pair-of-pants decomposition of this 2-sphere (here, any T-shirt flip of $S$ will do). Now all of the boundary curves are non-separating. \[\square\]

For closed surfaces, the complement of the pairs of pants in a pair-of-pants decomposition (i.e. the collection of all boundary circles) is referred to as a marking. Hatcher and Thurston showed in the Appendix of [14] that markings are always related to each other by a finite number of moves of four types (I to IV), up to isotopy. Moves I and II are produced by our T-shirt flips, whereas move III can be performed by composing T-shirt flips with move IV. Combining their result with Proposition [9] we conclude that any given pair-of-pants decomposition of a closed surface can be rendered fashionable by a finite concatenation of T-shirt flips and Hatcher–Thurston type-IV moves.

In the light of Theorem [7] it is natural to ask whether a concrete description of generators for the twisted 1-homology groups of fashionable pairs of pants can be given. This is provided by our next result.
Theorem 10 (Pochhammer principle). Let $P \subset \Sigma$ be a fashionable pair of pants. Let $f : S^1 \to P$ be a curve homotopic to the commutator of two of the boundary circles. Then $f_* : \mathcal{H}_1^\alpha(S^1) \to \mathcal{H}_1^\alpha(\Sigma)$ is an isomorphism.

Proof. Since a pair-of-pants is homotopy equivalent to a wedge $S^1 \vee S^1$, this follows from Remark \[\Box\]

Our discussion motivates the introduction of the following concept.  

Definition 11. Let $P$ be a pair of pants in an oriented surface $\Sigma$. A Pochhammer curve in $P$ is a loop $\lambda : S^1 \to P$ such that $\lambda$ determines the trivial class in $H_1(P; \mathbb{Z})$, but the induced map $\lambda_* : \mathcal{H}_1^\alpha(S^1) \to \mathcal{H}_1^\alpha(\Sigma)$ on $\alpha$-twisted homology of the Hurewicz cover of $\Sigma$ is injective.

![Diagram](image.png)

**Figure 2.** A Pochhammer curve in a fashionable pair of pants $P \hookrightarrow \Sigma$.

Even though this definition has been made regardless of $P$ being fashionable or not, our primary interest is in the case where $P$ is part of a fashionable pair-of-pants decomposition of $\Sigma$. A concrete example of a Pochhammer curve in a fashionably embedded pair of pants $P \hookrightarrow \Sigma$ is depicted in Fig 2. In the model represented on the upper left-hand side, isotopic to the classical planar model below, the curve is approximately a geodesic for the metric induced by the embedding $P \hookrightarrow \mathbb{R}^3$ in Euclidean space suggested by the drawing.

One could wonder what happens if a pair-of-pants decomposition of $\Sigma$ fails to be fashionable. Suppose that two pairs of pants $P_1, P_2$ in the decomposition have a common boundary which is a homologically trivial simple curve $S = \bar{P}_1 \cap \bar{P}_2$; see Fig. 3 for a concrete example. Both inclusions $i_1 : S \hookrightarrow P_1$ and $i_2 : S \hookrightarrow P_2$ induce isomorphisms on $\mathcal{H}_1^\alpha$. It follows that the images of the maps $\mathcal{H}_1^\alpha(P_1) \to \mathcal{H}_1^\alpha(\Sigma)$ and $\mathcal{H}_1^\alpha(P_2) \to \mathcal{H}_1^\alpha(\Sigma)$ agree, so we cannot have that $\mathcal{H}_1^\alpha(\Sigma)$ decomposes as a direct sum of the subspaces given as images of the twisted homology of the individual pairs of pants $P_1, P_2$.

Suppose also that we are given a homologically trivial simple curve $S \subset \Sigma$. The curve will cut $\Sigma$ into two surfaces $\Sigma_1, \Sigma_2$ with common boundary $S$; see again Fig. 3 for an illustration. Let us assume that both $\chi(\Sigma_j)$ are negative. Then, by Proposition \[9\]

\footnote{On the twice-punctured complex plane $\mathbb{C} \setminus \{0, 1\}$, homeomorphic to a pair of pants, the curve depicted on the lower left-hand side of Fig. 2 was used by Ludwig Pochhammer in his study of the Euler B-function [26, p. 507]. In complex analysis, there is some tradition in referring to this type of curves as Pochhammer contours, even though it has been recognized that they appeared in Camille Jordan’s *Cours d’Analyse* [17] prior to Pochhammer’s work (cf. [26, p. 256]).}
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Each $\Sigma_j$ can be given a pair-of-pants decomposition such that all boundary curves in the decomposition, except for $S$, define nontrivial homology classes in $\Sigma$. In general, the image of $H_1^i(S)$ in $H_1^i(\Sigma)$ will be the sum of the classes defined by the two adjacent pairs of pants $P_1, P_2$ sharing $S$ as boundary circle. In the example depicted in Fig. 3, the ranks of the maps $H_1(P_j; \mathbb{Z}) \to H_1(\Sigma; \mathbb{Z})$ are 1 for $j = 1$ and 0 for $j = 2$, so both $P_1$ and $P_2$ will contribute to this sum.

4. $L^2$-Betti numbers and $\alpha$-twisted homology

As we mentioned in the introduction to this paper, there are essentially three versions of the $L^2$-Betti numbers of a covering space with infinite discrete group of deck transformations. In this section, we will provide a short account of the analytic and algebraic definitions, and review an alternative description of the algebraic viewpoint for the case of Abelian covers, which connects to the definition we have given of twisted homology of $\alpha$-covers, without claiming originality. Among the references where the reader may retrieve more detail to fill out gaps in our necessarily brief exposition are [18, 25, 19].

The analytic definition of the $L^2$-Betti numbers is the one directly relevant to the applications in mathematical physics that we are primarily interested in, and which we shall introduce the reader to in section 4 below. Suppose $M$ is a Riemannian manifold, and that there is a normal subgroup $N$ of its fundamental group with discrete infinite quotient $\Gamma := \pi_1(M)/N$; then we consider the corresponding Galois cover $p: \tilde{M} \to M$ with the pulled-back Riemannian structure and the isometric $\Gamma$-action via deck transformations. The metric on $\tilde{M}$ defines $L^2$-inner products on all spaces of tensors, and in particular on the space of differential $i$-forms $\Omega^i(\tilde{M})$. Let

$$H_i^{(2)}(\tilde{M}) := \{ \omega \in \Omega^i(\tilde{M}) : \Delta \omega = 0, \| \omega \|_{L^2} < \infty \},$$

where $\Delta = d d^* + d^* d$ is the Laplace operator associated to the pulled-back metric (which is used to define the adjoint operators). If we denote by $L^2\Omega^i_c(M)$ the completion of the subspace of compactly supported $i$-forms on $M$ with respect to the $L^2$-inner product, there is an isometric inclusion $H_i^{(2)}(\tilde{M}) \hookrightarrow L^2\Omega^i_c(M)$ for each $i$; a theorem of Dodziuk [10] establishes that this map induces an isomorphism between $H_i^{(2)}(\tilde{M})$ and the de Rham $L^2$-cohomology group

$$H_i^{(2)}(\tilde{M}) := \ker(d : \Omega^i_c(\tilde{M}) \to L^2\Omega^{i+1}_c(\tilde{M})) / \text{im}(d : \Omega^i_c(M) \to L^2\Omega^{i+1}_c(M)).$$

Figure 3. Two unfashionable pairs of pants $P_1, P_2 \subset \Sigma := \Sigma_1 \cup S \Sigma_2$.
where the overline denotes completion with respect to the $L^2$-inner product. Now $\mathcal{H}^i_{(2)}(\tilde{M})$ is a Hilbert module over the group von Neumann algebra $\mathcal{N}(\Gamma) := \mathcal{B}(\ell^2(\Gamma))$, whose elements are bounded operators on the Hilbert space of square-integrable sequences $\ell^2(\Gamma)$ which are equivariant for the natural $\Gamma$-action. There is a unique dimension function $\dim_{\mathcal{N}(\Gamma)}$ for such modules, taking values in $[0, \infty]$, called the Murray–von Neumann dimension [22, 19], which extends the obvious trace in $\mathcal{B}(\ell^2(\Gamma))$ and satisfies natural properties. So one can set

$$b^i_{(2)}(\tilde{M}, \Gamma) := \dim_{\mathcal{N}(\Gamma)} \mathcal{H}^i_{(2)}(\tilde{M})$$

and obtain sensible invariants of $\tilde{M}$ as a $\Gamma$-space, which turn out to be homotopy invariant; these are the analytic $L^2$-Betti numbers. Atiyah’s original definition [1] may seem more intuitive, since it does not explicitly use $\mathcal{N}(\Gamma)$: it expresses the quantity above as the limit of an integral over a fundamental domain $F$ for the $\Gamma$-action on $M$,

$$b^i_{(2)}(\tilde{M}, \Gamma) = \lim_{t \to \infty} \int_F \operatorname{tr}_{\Lambda_+ T^*_x \tilde{M}} \left( e^{-t\Delta} \right) \, dx.$$ 

The integrand is the trace of the quadratic form associated to the heat kernel $e^{-t\Delta}$ on $i$-forms, restricted to the diagonal. In the limit $t \to \infty$, this operator can be intuitively thought of as a pointwise projection onto the space of harmonic $k$-forms, but in [22] the projection over global differential $i$-forms is being regularized to a ‘density of projection’ with respect to the $\Gamma$-action as one restricts the integration to the fundamental domain.

Whereas the analytic version (8)–(9) of the $L^2$-Betti numbers provides perhaps the most direct geometric insight, it is arguably the most unwieldy, and the best way to calculate the invariants may be to reinterpret them in terms of either of the other two definitions before embarking in actual computations. This strategy is always possible if the space $M$ in question is a compact manifold, for then all the three definitions are equivalent [19]. Since the algebraic definition does not depend on any (e.g. Riemannian) extra structure, this also shows that the analytic $L^2$-Betti numbers are independent of the metric structure if $M$ is compact. The standard definition of algebraic $L^2$-Betti numbers for a cover $\tilde{M}$ of $M$ with group of deck transformations $\Gamma$ uses again the Murray–von Neumann dimension:

$$b^i_{(2)}(\tilde{M}, \Gamma) := \dim_{\mathcal{N}(\Gamma)} \mathcal{H}^i(\tilde{M}; \mathcal{N}(\Gamma)).$$

Here, $\mathcal{H}^i(\tilde{M}; \mathcal{N}(\Gamma))$ denotes the $i$-th $\Gamma$-equivariant homology group with values in the von Neumann algebra of $\Gamma$, which is understood as the homology of the $\mathcal{N}(\Gamma)$-chain complex

$$C^\text{sing}_*(\tilde{M}) \otimes_{\mathbb{Z}[\Gamma]} \mathcal{N}(\Gamma),$$

where $C^\text{sing}_*(\tilde{M})$ denotes the singular chain complex of right $\mathbb{Z}[\Gamma]$-modules.

There is a body of results relating the algebraic version (10) of the $L^2$-Betti numbers in certain situations to dimensions over ring extensions of $\mathbb{C}[\Gamma]$ alternative to $\mathcal{N}(\Gamma)$. The basic signpost [24, p. 1] is the following square diagram of ring extensions:

$$\begin{align*}
\mathbb{C}[\Gamma] & \longrightarrow \mathcal{N}(\Gamma) \\
\downarrow & \downarrow \\
\mathcal{D}(\Gamma) & \longrightarrow \mathcal{U}(\Gamma)
\end{align*}$$

In the lower row, $\mathcal{U}(\Gamma)$ is the algebra of affiliated operators of $\Gamma$, defined as the Ore localization [24, 25] of $\mathcal{N}(\Gamma)$ with respect to the multiplicative subset of non-zero divisors, whereas $\mathcal{D}(\Gamma)$ stands for the smallest division-closed intermediate ring between $\mathbb{C}[\Gamma]$ and $\mathcal{U}(\Gamma)$. In [18, Thm. 7], Linnell studies the ring $\mathcal{D}(\Gamma)$ for a large class of groups $\Gamma$. In particular, this class contains the free Abelian groups $\Gamma = A := \mathbb{Z}^n$ which play a central
role in our paper. For this particular set of examples, the corners of the diagram above become familiar commutative $\mathbb{C}$-algebras:

\[
\begin{array}{c}
\mathbb{C}[z_1^\pm, \ldots, z_n^\pm] \longrightarrow L^\infty(T^n) \\
\downarrow \quad \downarrow \\
\mathbb{C}(z_1, \ldots, z_n) \longrightarrow L(T^n)
\end{array}
\]

Here, $T^n := (S^1)^n$ is the $n$-torus equipped with its Haar measure, whereas $L(T^n)$ denotes measurable complex-valued functions, identified whenever they agree almost everywhere.

The transition between the two diagrams is to be understood via Fourier transform, which provides an identification of complex Hilbert spaces $\ell^2(\mathbb{Z}^n) \cong L^2(T^n)$. Note that $L^2(T^n)$ is a module over the ring $L^\infty(T^n)$ (acting by multiplication), which models the von Neumann algebra $\mathcal{N}(\mathbb{Z}^n)$. We use the embedding $S^1 \subset \mathbb{C}$ with a standard complex coordinate to describe the group algebra $\mathbb{C}[\mathbb{Z}^n]$ as a space of Laurent polynomials. In this language, the division closure $\mathcal{D}(\mathbb{Z}^n)$ becomes the ordinary quotient field of rational functions $\mathbb{C}(z_1, \ldots, z_n)$ restricted to the real torus $T^n \subset (\mathbb{C}^\times)^n$, acting on the Hilbert $L^\infty(T^n)$-module $L^2(T^n)$ once again by multiplication. We shall abbreviate, according to our previous notation,

\[
\mathbb{C}(z_1, \ldots, z_n) \cong \mathbb{C}(A) = F_C.
\]

The crucial result we want to highlight in this section is that in the case $\Gamma = A := \mathbb{Z}^n$ for $n = \text{rk } H_1(M; \mathbb{Z})$, and letting $\alpha$ stand for the Hurewicz map of $M$, one has:

**Proposition 12.** $b_i^{(2)}(\tilde{M}, A) = \dim_{F_C} \mathcal{H}_i^F(M; F_C) = \dim_{F_C} \mathcal{H}_i^F(M)$.

**Proof.** This statement is a consequence of the general fact (which holds for arbitrary $\Gamma$)

\begin{equation}
(11) \quad b_i^{(2)}(\tilde{M}, \Gamma) = \dim_{\mathcal{U}(\Gamma)} H_i^F(\tilde{M}, \mathcal{U}(\Gamma))
\end{equation}

obtained in Reich’s PhD thesis [23, Prop. 4.2.(ii)], where $\dim_{\mathcal{U}(\Gamma)}$ is the natural extension of the Murray–von Neumann dimension function [22] to arbitrary $\mathcal{U}(\Gamma)$-modules introduced in [24, Prop. 3.2].

To see how this result implies the proposition in the case $\Gamma = A$, consider the singular chain complex $C^{{\text{sing}}}_\ast(\tilde{M})$ of $\mathbb{C}[A]$-modules and extend its coefficients to the division closure $\mathcal{D}(A) \cong \mathbb{C}(A)$ which was described above. Taking the homology of the resulting complex, one obtains a $\mathbb{C}(A)$-vector space at each degree $i$, which we shall assume finite (for simplicity); let us denote by $\beta_i$ the corresponding dimension over $\mathbb{C}(A)$:

\[
H_i(C^{{\text{sing}}}_\ast(\tilde{M}) \otimes_{\mathbb{C}[A]} \mathbb{C}(A)) \cong \mathbb{C}(A)^{\beta_i}.
\]

Now we tensor the complex once again to obtain

\[
(C^{{\text{sing}}}_\ast(\tilde{M}) \otimes_{\mathbb{C}[A]} \mathbb{C}(A)) \otimes_{\mathbb{C}(A)} \mathcal{U}(A) \cong C^{{\text{sing}}}_\ast(\tilde{M}) \otimes_{\mathbb{C}[A]} \mathcal{U}(A).
\]

Since tensoring over a field is an exact functor, we obtain homology groups

\[
H_i(C^{{\text{sing}}}_\ast(\tilde{M}) \otimes_{\mathbb{C}[A]} \mathcal{U}(A)) \cong \mathcal{U}(A)^{\beta_i}.
\]

The basic normalization property $\dim_{\mathcal{U}(A)} \mathcal{U}(A) = 1$ together with (11) imply that $b_i^{(2)}(\tilde{M}, \Gamma) = \beta_i$. But by construction $\beta_i$ is also equal to $\dim_{F_C} \mathcal{H}_i^F(M; F_C) = \dim_{F_C} \mathcal{H}_i^F(M)$.

We now want to revisit Atiyah’s key example (or rather its Abelian version) mentioned in the introduction. Since the Abelian cover $\tilde{\Sigma}$ of a compact Riemann surface is a co-compact space for the action of $\Gamma = H_1(\Sigma; \mathbb{Z})$, by the discussion above its analytic and algebraic $L^2$-Betti numbers agree, and by Proposition 12 they also agree with the $\mathbb{F}$-dimension of the $\alpha$-twisted homology for the Hurewicz map of the surface:

\[
b_i^{(2)}(\tilde{\Sigma}, \Gamma) = b_i^{(2)}(\tilde{\Sigma}, \Gamma) = \dim_{\mathbb{F}} \mathcal{H}_i^F(\Sigma).
\]
In particular, since our discussion in section [3] provided explicit F-bases for $\mathcal{H}_{10}^0(\Sigma)$ in terms of Pochhammer curves, we have now obtained the intuitive interpretation of Atiyah’s result [1] advertized in the introduction. We emphasize that using this type of argument one calculates the nontrivial invariants, in this case $b^1_2(\Sigma, \Gamma)$, without having to rely on vanishing results for the other $L^2$-Betti numbers.

5. Charged Particles on Closed Surfaces

This final section illustrates how the topological considerations of the previous sections can be applied to two-dimensional gauge theory.

5.1. Aharonov–Bohm effect for supersymmetric particles on surfaces.

Let us recall the setup of $N = (2, 2)$ supersymmetric quantum mechanics [31, 15] on a compact orientable surface $\Sigma$ of genus $g > 1$ equipped with a Kähler metric $g_\Sigma$. Classically, this system is described by pairs $(\phi, \psi)$ consisting of ‘bosonic’ paths $\phi : \mathbb{R} \to \Sigma$ together with their ‘fermionic’ complex deformations $\psi \in \Gamma(\mathbb{R}, \phi^* T\Sigma \otimes \mathbb{C})$. The dynamics of these paths is dictated by a variational principle for the Lagrangian $L : TC^\infty(\mathbb{R}, \Sigma) \otimes \mathbb{C} \to \mathbb{R}$ given by

$$L[(\phi, \psi)] := \frac{1}{2} g_\Sigma(\phi, \phi) + i \omega_\Sigma(\bar{\psi}, \nabla_\phi \psi) + g_\Sigma(\bar{\phi} \psi, R\nabla(\phi \psi, \phi \psi)),$$

where $\nabla$ is the Levi–Civita connection of $g_\Sigma$, $\omega_\Sigma$ the associated Kähler form and $R\nabla$ the Riemann curvature tensor of $\nabla$; we are extending $\mathbb{C}$-linearly all covariant tensors to $\Sigma \otimes \mathbb{C}$. This is the simplest example of a supersymmetric sigma-model (with source $\mathbb{R}$ and target $\Sigma$). Its Euler–Lagrange equations are the ‘fuzzy’ geodesic equations

$$\nabla_{\bar{\phi}} \phi = R\nabla(\phi \psi, \phi \psi) \phi, \quad \nabla_{\bar{\phi}} \psi = 0 = \nabla_{\bar{\psi}} \phi.$$

The canonical quantization of this system produces the infinite-dimensional quantum Hilbert space of complex-valued forms $\mathcal{H} = \Omega^*(\Sigma; \mathbb{C})$, with inner product induced by $g_\Sigma$ and the usual inner product on $\mathbb{C}$. Quantum states of a supersymmetric particle are represented by vectors in $\mathcal{H}$, referred to as waveforms, and the supersymmetric viewpoint uses the splitting $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ into forms of even/odd degree which correspond to bosons/fermions, respectively. This Hilbert space provides a super-representation of the 8-dimensional $N = (2, 2)$ supersymmetry Lie super-algebra generated by the Laplacian $\Delta = dd^* + d^*d$, the Dolbeault operators $\partial, \bar{\partial}$ and their adjoints, as well as the Lefschetz $\mathfrak{sl}_2\mathbb{C}$-triple generated by wedging with $\omega_\Sigma$, its adjoint and their commutator [10]. The operator $\Delta$ plays the role of quantum Hamiltonian, and the spectral decomposition $\mathcal{H} = \bigoplus_{E \in \text{Spec}(\Delta)} \mathcal{H}_E$ has special significance. A consequence of $\Delta = (d + d^*)^2$ is that one has $\text{Spec}(\Delta) \subset [0, \infty[$ (this is the spectrum of energies of the model); moreover, whenever $E > 0$, $d + d^*$ restricts to an isomorphism $\mathcal{H}_E^+ \cong \mathcal{H}_E^-$ where $\mathcal{H}_E^\pm := \mathcal{H}_E \cap \mathcal{H}^\pm$. The eigenspace $\mathcal{H}_0$ is the subspace of harmonic forms, and it is referred to as the sector of ground states or vacua (i.e. quantum states of minimal energy). We can model the elements of $\mathcal{H}_0$ as de Rham cohomology classes via standard Hodge theory. Note that the two summands $\mathcal{H}_0^\pm$ are not necessarily isomorphic, and the formal difference between them represents a net balance of unpaired fermions over the whole $\mathcal{H}$. The dimension of this formal difference is $\text{tr}_{\mathcal{H}}(-1)^{\text{deg}} = \chi(\Sigma) = 2 - 2g < 0$.

Now we want to twist the whole picture by introducing $U(1)$-connections $A$ in bundles over $\Sigma$. For now, we can think of $A$ as a background gauge field to which the supersymmetric particle may couple if it has a charge. The simplest possibility is to demand that $A$ be flat, and in this case the interaction occurs non-locally via holonomies along curves in $\Sigma$ (so we can think of the field produced by $A$ as being undetectable locally, but nontrivially threaded over the ‘holes’ of $\Sigma$). This is precisely the setup for the Aharonov–Bohm effect, which is also used to model statistical phases in two-dimensional
systems. To implement this, one constructs a rank-one local system over \( \Sigma \) by specifying a character \( \xi \) in
\[
\mathcal{R} := \text{Hom}(\pi_1(\Sigma), U(1)) \cong U(1)^{2g} = T^{2g}.
\]
Concretely, we assign to \( \xi \) the Hermitian line bundle \( \mathcal{E}_\xi := \tilde{\Sigma} \times_\xi \mathbb{C} \) with its \( \xi \)-twisted flat connection, where \( \tilde{\Sigma} \) is the universal cover of \( \Sigma \) and \( \mathbb{C} \) carries the usual Hermitian inner product. Since the structure group \( U(1) \) is Abelian, the holonomy along a loop will only depend on the homology class of that loop. This means that each local system \( \mathcal{E}_\xi \) will become trivial already on the maximally connected Abelian cover of \( \Sigma \), which coincides with the Hurewicz cover defined in section [2]. Hence we can, and will, replace the universal cover in the argument above by this Hurewicz cover, which will be denoted \( \tilde{\Sigma} \rightarrow \Sigma \) from now on. The quantum Hilbert space \( \mathcal{H} \) that we considered above should now be replaced by the twisted version \( \mathcal{H}^{(\xi)} := \Omega^*(\Sigma; \mathcal{E}_\xi) \), with the untwisted (or uncharged) situation being recovered when \( \xi \) is the trivial representation.

Physically, the holonomies have measurable effects on the charged particle which can be detected in the form of interference patterns; but such measurements are never totally accurate, so it is more satisfactory to work with values of \( \xi \) that are smeared within a (possibly very small) subset in \( \mathcal{R} \), rather than restricted to a precise value. A natural further step is to promote the character \( \xi \) to be an internal quantum number of the system, and consider a quantum master space \( \prod_{\xi \in \mathcal{R}} \mathcal{H}^{(\xi)} \) incorporating all characters; the quantum operator corresponding to \( \xi \) is usually referred to as a Wilson loop. Unfortunately, such a master space would not be a Hilbert space. To remedy this, one needs to regularize by working with ‘wave-packets’ weighted over the representation variety \( \mathcal{R} \). This will be possible because \( \mathcal{R} = T^{2g} \) carries a natural measure — in this case, it is a compact Lie group and the Haar measure can be used for this purpose.

To make this idea more concrete, we lift the twisted waveforms in each \( \mathcal{H}^{(\xi)} \) to the Hurewicz cover \( \tilde{\Sigma} \). Note that \( \tilde{\Sigma} \) has an action of \( \mathbb{Z}^{2g} \), making the Hilbert space \( L^2(\tilde{\Sigma}; \mathcal{E}_\xi) \) (where the \( L^2 \)-inner product is with respect to the pulled-back metric \( p_{\xi}^{-1} g_\Sigma \)) into a module over the corresponding group von Neumann algebra \( \mathcal{N}(\mathbb{Z}^{2g}) \cong L^\infty(\mathbb{T}^{2g}) \).

Given any fundamental domain \( F \subset \Sigma \) of the \( \Gamma \)-action, the decomposition \([1]\)
\[
L^2(\tilde{\Sigma}; \mathcal{E}_\xi) \cong \Omega^*(F; \mathcal{C}) \otimes \ell^2(\Gamma) \cong \Omega^*(\Sigma; \mathcal{C}) \otimes L^2(\mathbb{T}^{2g})
\]
substantiates the interpretation of \( L^2 \)-integrable forms on \( \Sigma \) as wave-packets of forms on \( \Sigma \) valued in line bundles, for a spectrum of twistings parametrized by \( \mathcal{R} \). For example, to obtain wave-packets that are supported on measurable subsets \( M \subset \mathbb{T}^{2g} \) one can project using the character functions \( \chi_M \), which are idempotents in \( \mathcal{N}(\mathbb{T}^{2g}) \). Working with weights supported on a small subset \( M \ni \xi \), one gets wave-packets of waveforms in each degree \( i \) that are localized around \( \xi \). In particular, we obtain wave-packets of ground states from \( L^2 \)-normalisable harmonic forms on the Hurwitz cover. Since the space \( \Sigma \) is no longer compact, the appropriate version of Hodge theory in this setting relates \( \mathcal{N}(\Gamma) \)-modules of harmonic forms to \( L^2 \)-cohomology, as already stated in section [3]. Whenever we localize with respect to a supporting subset \( M \) with nonzero measure, we are treating an infinite number of characters at a time, and so these modules will be infinitely generated \( \mathbb{C} \)-vector spaces. Now we can extract information from them in the form of their \( L^2 \)-Betti numbers with respect to the \( \Gamma \)-action on \( \Sigma \). Since in this case \( \Gamma \) is an Abelian group without torsion, after dividing by \( \text{Vol}(M) \) one always ends up with integer quantities \( b^2_i(\Sigma, \Gamma) \) that have a probabilistic interpretation of renormalized dimensions of spaces of harmonic \( L^2 \)-waveforms of degree \( i \), for characters averaged over \( M \). A more detailed account of these ideas and techniques will be given elsewhere [5].

We can connect this viewpoint on charged supersymmetric particles with the topological discussion in this paper. We have shown in section [4] that we can compute the \( L^2 \)-Betti numbers for a compact surface of genus \( g > 1 \) as the actual dimensions (over \( \mathbb{F} \))
of $\alpha$-twisted homology groups $\mathcal{H}_1^a(\Sigma)$ with respect to the Hurewicz map of $\Sigma$. More precisely, we are now able to assign to any fashionable pair-of-pants decomposition $\{P_j\}_{j=1}^{2g-2}$ of $\Sigma$, decorated by arbitrary Pochhammer loops $\lambda_j : S^1 \to P_j \subset \Sigma$ on each of its pairs-of-pants, a set of $b^1(\Sigma, \Gamma) = 2g - 2$ Pochhammer vectors

$$|\lambda_j \alpha P_j| := \lambda_j|S^1| \in \mathcal{H}_1^a(\Sigma; \mathbb{C}), \quad j = 1, 2, \ldots, 2g - 2.$$  

These vectors provide a useful device to represent states of charged quantum supersymmetric particles on $\Sigma$ by virtue of the following sewing property:

**Proposition 13.** If $J \subset \{1, 2, \ldots, 2g - 2\}$, then

$$\text{span}_{\mathbb{C}} \{ |\lambda_j \alpha P_j| \}_{j \in J} \cong \mathcal{H}_1^a \left( \bigcup_{j \in J} P_j; \mathbb{C} \right) \cong \mathbb{F}_{\mathbb{C}}^{|J|}.$$  

In particular, the set $\{ |\lambda_j \alpha P_j| \}_{j=1}^{2g-2}$ provides an $\mathbb{F}_{\mathbb{C}}$-basis for the space $\mathcal{H}_1^a(\Sigma; \mathbb{F}_{\mathbb{C}})$.

**Proof.** This follows from Theorem 7 and the Pochhammer principle in Theorem 10. \hfill $\square$

Our discussion so far has been concerned with quantum one-particle states of the charged supersymmetric particle. For $k$ identical particles, the standard recipe in non-relativistic quantum mechanics consists of prescribing first the statistics of the particles (bosonic, fermionic) and then constructing their quantum Hilbert space as the $k$-th exterior or symmetric power of the one-particle states, respectively, projecting out from the tensor product. In a supersymmetric system, the natural procedure is to treat waveforms of all degrees at the same time and consider the whole tensor product of one-particle states. One alternative to this orthodoxy is to use configuration spaces, and this allows for the treatment of anyonic particles (this viewpoint has some tradition in the physics of two-dimensional systems [30]). In the discussion above, this would involve replacing $\Sigma$ by $\text{Conf}_k(\Sigma) := (\Sigma^k \setminus \Sigma_\Delta)/\mathfrak{S}_k$, where $\Sigma_\Delta$ is the diagonal and $\mathfrak{S}_k$ the symmetric group permuting the copies in the cartesian product. The noncompactness of these spaces, as well as the intricacy of their fundamental groups $(\mathbb{C}^r \times \mathbb{R})$, causes difficulties; note also that this framework still requires the statistics of the multiparticles to be prescribed by hand.

**5.2. Vortices in gauged linear sigma-models.**

We shall now argue that our discussion extends to the quantization of certain $(1+2)$-dimensional gauge theories — most immediately, to the study of their nonperturbative vacua. The theories that we have in mind are supersymmetric sigma-models with $\mathbb{R} \times \Sigma$ (carrying the Lorentzian product metric $\text{d}t^2 - g_{\Sigma}$) as source, and target any Kähler manifold $(X, J, \omega_X)$ with a holomorphic Hamiltonian action of a compact Lie group $G$. In this paper, we will be confining the discussion to linear gauged sigma-models and shall take $X = \mathbb{C}^r \times \mathbb{R}$ to be the $\mathbb{C}$-vector space of $r \times r$ matrices with the usual Euclidean metric and action of $G = U(r)$ by left-multiplication. At the bosonic level, the variables are a $U(r)$-connection $A$ on a principal $U(r)$-bundle $\mathcal{P}$ over $\Sigma$ and a section $u$ of the associated vector bundle $\mathcal{P} \times_{U(r)} \mathbb{C}^r \times \mathbb{R}$. The Lagrangians contain the standard sigma-model kinetic terms associated to the $L^2$-norms of the time derivatives and time-components of the fields $(A, \phi)$ in the Lorentzian metric, and a term of potential energy $V_{r,\lambda}$ which can be expressed as follows:

$$V_{r,\lambda}([A, u]) := \frac{1}{2} \int_{\Sigma} \left( |F_A|^2 + |d_A \phi|^2 + \frac{\lambda}{4} |\mu_r \circ \phi|^2 \right).$$  

The third summand is the Fayet–Iliopoulos term, which depends on two real parameters $\lambda$ and $\tau$. The latter is associated to the choice of moment map

$$\mu_r(w) := -\frac{1}{2} (\bar{w} \bar{w}^t - \tau 1_r)$$
for the $U(r)$-action; note that we are identifying $u(r)$ and its dual equivariantly. The identity

$$V_{r,\lambda}[(A, u)] = 2\pi \deg(\det u) + \frac{1}{2} \int_{\Sigma} \left( |F_A + (\mu \circ u) \omega_\Sigma|^2 + 2|\partial_A u|^2 + \frac{\lambda - 1}{2} |\mu_r \circ u|^2 \right)$$

identifies the critical value $\lambda = 1$ of the other parameter as the self-dual point. If the topological charge $\deg(\det u) =: k$ is positive, one can express the minima of $V_{r,1}$ by the system of first-order PDEs

$$\partial_A u = 0, \quad F_A + (\mu \circ u) \omega_\Sigma = 0$$

which are a particular case of the vortex equations. More than studying their solutions, it is important to describe the space

$$\mathcal{M}^U_{\Sigma,k} = \{(A, u) : \partial_A u = 0, \, F_A + (\mu \circ u) \omega_\Sigma = 0, \, \deg \det \phi = k \}/\text{Aut}_\Sigma(P)$$

go of gauge-equivalent solutions of a given charge $k > 0$. In the situation where $\tau > \frac{4\pi k}{\text{Vol}(\Sigma)}$, the moduli space (16) has been described in [3] and [4], for example. It is a complex manifold carrying a nontrivial Kähler structure $\omega_{L_2}$, induced by the kinetic term of the bosonic sigma-model, which depends on $\Sigma, g_\Sigma, \tau, r$ and $k$. The corresponding Kähler metrics encode information about inter-vortex interactions: for instance, their geodesic flow approximates the classical dynamics for the bosonic sigma-models for $\lambda$ close to 1 and small velocities [21] — see [25] for the analysis in the case where $k = 2$ and $\Sigma = \mathbb{C}$.

It turns out that the bosonic sigma-models we have just described admit supersymmetric versions at the critical value $\lambda = 1$. There are two ways to twist [32] the Lagrangians with the fermions added to obtain functionals for globally defined fields on $\Sigma$, and they lead to topological field theories that were described in [2]. In this paper, we will be interested in the A-twist that makes use of the vectorial (global) circle $R$-symmetry [16]. In the Lagrangian formulation of the corresponding TQFT, the path integrals of this model localize to the moduli spaces in (16). This justifies the following strategy to study the A-twisted gauged sigma-models at low energies: one replaces the supersymmetric $(1 + 2)$-dimensional field theories by a supersymmetric 1-dimensional sigma-model analogous to the one described in section 5.1, the only difference being that the target is now taken to be the Kähler manifold $\mathcal{M}^U_{\Sigma,k}$ rather than $\Sigma$, and we end up by considering Hilbert spaces of waveforms on vortex moduli spaces. In this context, the extension to twisted waveforms, incorporating holonomies in a moduli space of magnetically charged particles, is in the same spirit as the generalization of magnetic monopoles in 3 + 1 dimensions to dyonic particles that also possess electric charge [23].

In the Abelian case $r = 1$, the space of 1-vortices is

$$\mathcal{M}^U_{1,1} \cong \Sigma$$

as a complex manifold exactly like in section 5.1, but now it is endowed with a nontrivial metric $g_{L_2}$ for each $\tau$, which is distinct from the metric $g_\Sigma$ used to define the $(1 + 2)$-dimensional Lagrangian, or to write down the vortex equations. The metric $g_\Sigma$ should be thought of as the limit $\lim_{\tau \to \Sigma} g_{L_2}$ where the vortex becomes a point particle, whereas the other extreme $\tau \to \frac{4\pi}{\text{Vol}(\Sigma)}$ corresponds to the limit of ‘dissolved vortices’ associated to the Bergman metric on the Riemann surface $\Sigma$, as discussed in [20]. It was established in [3] that the generalization of (17) for any $r$ subject to $\tau \text{Vol}(\Sigma) > 4\pi$ is

$$\mathcal{M}^U_{1,1} \cong \mathbb{P}^{r-1} \times \Sigma.$$ 

Thus for the Abelian model, a 1-vortex is associated with a point on the surface (where the Higgs field $u$ has a simple zero), whereas for the non-Abelian case $r > 1$, apart from these ‘spatial moduli’, there is an extra factor $\mathbb{P}^{r-1}$ in the configuration space which parametrizes internal structures [3].
We can now extend the discussion of Aharonov–Bohm phases to the 1-vortex moduli spaces \( \mathcal{M}^{U(r)}_{\Sigma, 1} \), with a view of understanding ground states of charged single particles in the quantum (1+2)-dimensional gauged linear sigma-models. Consider the \( \alpha \)-cover associated to the homomorphism

\[
\alpha : \pi_1 (\mathcal{M}^{U(r)}_{\Sigma, 1}) \to A := H_1 (\Sigma; \mathbb{Z})
\]  

obtained by composition of the isomorphism in \( \pi \) with the Hurewicz map of \( \Sigma \). Then we obtain the following result.

**Theorem 14.** Let \( \Sigma \) be a compact Riemann surface of genus \( g > 1 \). The \( L^2 \)-Betti numbers of the moduli space of \( U(r) \)-1-vortices for the \( \alpha \)-cover associated to \( \mathcal{M}^{U(r)}_{\Sigma, 1} \) are

\[
b^{(2)}_i \left( \mathcal{M}^{U(r)}_{\Sigma, 1}, H_1 (\Sigma; \mathbb{Z}) \right) = \begin{cases} 
2g - 2 & \text{if } 1 \leq i \leq 2r - 1 \text{ and } i \text{ is odd}, \\
0 & \text{otherwise}.
\end{cases}
\]

Given a pair-of-pants decomposition \( \{ P_j \}_{i=1}^{2g-2} \) decorated with Pochhammer curves \( \lambda_i \), we can associate the following \( \mathbb{F}_C \)-basis for each nontrivial complex twisted homology group \( \mathcal{H}^i_\alpha (\mathcal{M}^{U(r)}_{\Sigma, 1}; \mathbb{F}_C) \):

\[
\psi_{i-1} \otimes | \lambda_j \propto P_j \rangle \quad i = 1, \ldots, 2r - 1 \text{ odd}, \ j = 1, \ldots, 2g - 2,
\]

where \( \psi_i \) are generators of the vector spaces \( H_i (\mathbb{P}^{r-1}; \mathbb{Q}) \cong \mathbb{Q} \), and \( | \lambda_j \propto P_j \rangle \in \mathcal{H}^i_\alpha(\Sigma; \mathbb{F}_C) \) are the Pochhammer vectors defined in \([15]\).

**Proof.** The calculation of \( L^2 \)-Betti numbers is a straightforward application of Theorem 6 whereas the basis described combines the definition of Pochhammer vectors with the fact that the nontrivial homology groups of complex projective spaces are cyclic. \( \square \)

The basis \([20]\) displays a factorization of the ground states in the one-particle sector of the model into spatial components associated to charged supersymmetric particles on \( \Sigma \) (i.e. the Pochhammer vectors in equation \([15]\)), and internal bosonic waveforms associated to closed complex submanifolds of the space of internal structures \( \mathbb{P}^{r-1} \). This type of factorization (obtained here from first principles by quantization of a classical moduli space) is often assumed in discussions of nontrivial vacua in quantum field theories. Note that all these extended states (which include nontrivial internal structures whenever \( r > 1 \)) are fermionic, just like the states of the charged supersymmetric point particle discussed in section \([5, 14]\).

There are many directions in which to generalize the discussion in this section. Most immediately, one is interested in understanding multiparticle states, for which \( k > 1 \); they can be obtained from quantizing moduli spaces of multivortices \([8]\). The simplest example is the Abelian case \( r = 1 \), and one then deals with moduli spaces \([9, 12]\)

\[
\mathcal{M}^{U(1)}_{\Sigma, k} \cong \text{Sym}^k (\Sigma) := \Sigma^k / \mathfrak{S}_k
\]

with Abelian fundamental group isomorphic to \( H_1 (\Sigma; \mathbb{Z}) \). So in this case, once we think of charged particles, we are back to considering twisting line bundles and smearing out waveforms using weights supported on the same character variety \( \mathcal{R} \) as in \([12]\). The calculation of the \( L^2 \)-Betti numbers \([8]\) makes use of recent results on the topology of the universal cover \( \text{Sym}^k (\Sigma) \) obtained in \([8]\). The basic result is

\[
b^{(2)}_i \left( \mathcal{M}^{U(1)}_{\Sigma, k}, H_1 (\Sigma; \mathbb{Z}) \right) = \binom{2g - 2}{k} \delta^l_k,
\]

where \( \delta^l_k \) is the Kronecker delta, and this formula supports the independent interpretation of quantum vortices as fermionic particles given above. We should emphasize that the statistics of the quantum particles, in either of the two arguments, naturally emerges from...
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calculations derived from first principles, in contrast with the arbitrariness in prescribing statistical properties for point particles in quantum mechanics. A direct connection between formula (21) and the traditional approach to quantum multiparticle states of fermions will be provided in [27]. For the non-Abelian models, the picture is more intricate, as the spaces of internal structures coalesce nontrivially according to a fibration of the moduli space $\mathcal{M}^{U(r)}_{\Sigma,k} \to \text{Sym}^k(\Sigma)$ with typical fibres depending on the natural filtration of $\text{Sym}^k(\Sigma)$ by partitions of $k$.

More interesting structure emerges for gauged nonlinear sigma-models, where fundamental fermionic particles of different types coexist and can even merge nontrivially, giving rise to a much richer spectrum of quantum particles [27]. The fact that the moduli spaces are no longer compact in this situation leads to various difficulties, and their fundamental groups may also become non-Abelian [7]. However, the description of the spatial quantum numbers provided by Pochhammer curves in pair-of-pants decompositions remains useful in this context. This language should provide helpful tools to study quantization of moduli spaces of gauge theories on Riemann surfaces more generally.

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References

[1] M.F. Atiyah: Elliptic operators, discrete groups and von Neumann algebras. Astérisque 32–33 (1976) 43–72
[2] J.M. Baptist: Twisting gauged non-linear sigma-models. JHEP 0802 (2008) 096
[3] J.M. Baptist: Non-abelian vortices on compact Riemann surfaces. Commun. Math. Phys. 291 (2009) 799–812
[4] I. Biswas, N.M. Romão: Moduli of vortices and Grassmann manifolds. Commun. Math. Phys. 320 (2013) 1–20
[5] I. Biswas, N.M. Romão: A no-go theorem for nonabelionic statistics in gauged linear sigma-models; (in preparation)
[6] M. Bökstedt, N.M. Romão: On the curvature of vortex moduli spaces. Math. Z. 277 (2014) 549–573
[7] M. Bökstedt, N.M. Romão: Divisor links and fundamental groups of toric vortex moduli; (in preparation)
[8] M. Bökstedt, N.M. Romão, C. Wegner: $L^2$-invariants and supersymmetric quantum mechanics on vortex moduli spaces; (in preparation)
[9] S.B. Bradlow: Vortices in holomorphic line bundles over closed Kähler manifolds. Commun. Math. Phys. 135 (1990) 1–17
[10] J. Dodziuk: De Rham–Hodge theory for $L^2$-cohomology of infinite coverings. Topology 16 (1977) 157–165
[11] W. Fulton: Algebraic Topology: A First Course, Springer-Verlag, 1995
[12] Ó. García-Prada: A direct existence proof for the vortex equations over a compact Riemann surface, Bull. London Math. Soc. 26 (1992) 88–96
[13] A. Hatcher: Algebraic Topology, Cambridge University Press, 2002
[14] A. Hatcher, W. Thurston, A presentation for the mapping class group of a closed orientable surface, Topology 19,(1980), 221–237
[15] K. Hori, S. Katz, A. Klemm, R. Pandharipande, R. Thomas, C. Vafa, R. Vakil, E. Zaslow: Mirror Symmetry, American Mathematical Society, 2003
[16] D. Huybrechts: Complex Geometry: An Introduction, Springer-Verlag, 2005
PAIRS OF PANTS, POCHHAMMER CURVES AND $L^2$-INVARIANTS

[17] C. Jordan: Cours d’Analyse de l’École Polytechnique, vol. III, Gauthier-Villars, 1887
[18] P.A. Linnell: Zero divisors and group von Neumann algebras. Pacific J. Math. 149 (1991) 349–363
[19] W. Lück: $L^2$-Invariants: Theory and Applications to Geometry and K-Theory, Springer-Verlag, 2002
[20] N.S. Manton, N.M. Romão: Vortices and Jacobian varieties. J. Geom. Phys. 61 (2011) 1135–1155
[21] N. Manton, P. Sutcliffe: Topological Solitons, Cambridge University Press, 2004
[22] F.J. Murray, J. von Neumann: On rings of operators II. Trans. Amer. Math. Soc. 41 (1937) 208–248
[23] D.I. Olive, P.C. West (Eds.): Duality and Supersymmetric Theories, Cambridge University Press, 1999
[24] Ø. Ore: Linear equations in non-commutative fields. Ann. Math. 32 (1931) 463–477
[25] H. Reich: Group von Neumann Algebras and Related Algebras. PhD Thesis, University of Göttingen, 1998
[26] L. Pochhammer: Zur Theorie der Euler’schen Integrale. Math. Ann. 35 (1890) 495–526
[27] N.M. Romão, C. Wegner: $L^2$-Betti numbers and particle counting in a gauged nonlinear sigma-model: (in preparation)
[28] D.M.A. Stuart: Dynamics of abelian Higgs vortices in the near Bogomolny regime. Commun. Math. Phys. 159 (1994) 51–91
[29] E.T. Whittaker, G.N. Watson: A Course of Modern Analysis, 4th Edition, Cambridge University Press, 1927
[30] F. Wilczek: Fractional Statistics and Anyon Superconductivity, World Scientific, 1990
[31] E. Witten: Supersymmetry and Morse theory. J. Diff. Geom. 17 (1982) 661–692
[32] E. Witten: Dynamics of quantum field theory (Notes by P. Etingof, L. Jeffrey, D. Kazhdan, J. Morgan and D. Morrison). In: P. Deligne, P. Etingof, D.S. Freed, L.C. Jeffrey, D. Kazhdan, J.W. Morgan, D.S. Morrison, E. Witten (Eds.): Quantum Fields and Strings: A Course for Mathematicians, vol. 2, American Mathematical Society, 1999

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