EFFECTIVE THEORY FOR THE NON-RELATIVISTIC
THREE-BODY SYSTEM

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We discuss renormalization of the non-relativistic three-body problem with short-range forces. The problem becomes non-perturbative at momenta of the order of the inverse of the two-body scattering length, and an infinite number of graphs must be summed. This summation leads to a cutoff dependence that does not appear in any order in perturbation theory. We argue that this cutoff dependence can be absorbed in a single three-body counterterm and compute the running of the three-body force with the cutoff.

There has been considerable interest recently in applying the successful concept of effective field theory (EFT) to nuclear physics. If the momenta $k$ are small compared to the inverse range of the interaction $1/R$, EFT provides a systematic expansion in powers of $kR$. More complicated are systems where an unnaturally large parameter appears. Specifically, for systems made out of nucleons and of $^4$He atoms, the two-body scattering length $a_2$ is much larger than $R$. In this case the expansion becomes non-perturbative at momenta of the order of $1/a_2$, in the sense that an infinite number of diagrams must be resummed. This resummation generates a new expansion in powers of $kR$ where the full dependence in $ka_2$ is kept. Consequently, the EFT is valid beyond $k \sim 1/a_2$, comprising, in particular, bound states of size $\sim a_2$. While there has been enormous progress in the two-body case, the extension to three-particle systems presents us with a puzzle. Although in some fermionic channels the resummed leading two-body interactions lead to unambiguous and very successful predictions, amplitudes in bosonic systems and other fermionic channels show sensitivity to the ultraviolet (UV) cutoff, as is evidenced in the well known Thomas and Efimov effects. This happens even though each leading-order three-body diagram with resummed two-body interactions is individually UV finite. Below we will argue that the EFT program can be extended to three-boson systems with large $a_2$ by introducing a one-parameter three-body force counterterm at leading order.

The most general Lagrangian involving a non-relativistic boson $\psi$ and invariant under small-velocity Lorentz, parity, and time-reversal transformations can conveniently be written by introducing a dummy field $T$ with quantum
numbers of two bosons,

\[ \mathcal{L} = \psi^\dagger \left( i\partial_0 + \frac{\nabla^2}{2M} \right) \psi + \Delta T^\dagger T - \frac{g}{\sqrt{2}} (T^\dagger \psi \psi + \text{h.c.}) + h T^\dagger T \psi^\dagger \psi + \ldots . \]  

(1)

We consider particle/bound-state scattering. The diagrams contributing to this process in leading order are illustrated in Fig. 1. All diagrams including only two-body interactions are of the same order for \( k \sim 1/a_2 \) and have to be summed. This can be accomplished by solving the equation represented by the second equality in Fig. 1:

\[ a(p) = K(p,k) + \frac{2}{\pi} \int_0^\Lambda dq \ K(p,q) \frac{q^2}{q^2 - k^2 - i\epsilon} a(q), \]  

(2)

with \( k \) (p) the incoming (outgoing) momentum, \( ME = 3k^2/4 - 1/a_2^2 \) the total energy, \( a(p = k) \) the scattering amplitude normalized in such a way that \( a_3 = -a(0) \) is the particle/bound-state scattering length, and

\[ K(p,q) = \frac{4}{3} \left( \frac{1}{a_2^2} + \sqrt{\frac{3}{4} p^2 - ME} \right) \left[ \frac{1}{pq} \ln \left( \frac{q^2 + pq + p^2 - ME}{q^2 - qp + p^2 - ME} \right) + \frac{h}{Mg^2} \right]. \]  

(3)

The parametric dependence of \( a(p) \) on \( k \) is kept implicit. Three nucleons in the spin \( J = 1/2 \) channel obey a pair of integral equations with similar properties. For \( h = 0 \) and \( \Lambda \rightarrow \infty \), the asymptotic solution of Eq. (2) can be obtained analytically. It turns out that the phase of the solution is undetermined. For a finite \( \Lambda \), however, the solution has a well determined phase which in the intermediate region \( 1/a_2 \ll p \ll \Lambda \) is,

\[ a(p) = A \cos \left( s_0 \ln \frac{p}{\Lambda} + \delta \right), \]  

(4)

with \( \delta \) is some dimensionless, cutoff-independent number. Obviously, the limit \( \Lambda \rightarrow \infty \) is not well defined. Numerical solutions of Eq. (2) with \( k = 0 \) for different values of \( \Lambda \) confirm that the behavior of \( a(p) \) in the region \( 1/a_2 \ll p \ll \Lambda \) is...
$p \ll \Lambda$ is given by Eq. (4) and that small differences in the asymptotic phase lead to large differences in the particle/bound-state scattering length.

This cutoff dependence comes from the amplitude in the UV region, where the EFT Lagrangian, Eq. (1), is not to be trusted. In an EFT, the cutoff-dependent contributions from high loop momenta are cancelled by counterterms in the Lagrangian and all uncertainty from the UV behavior of the theory is parametrized by a few local counterterms. Writing

$$h(\Lambda) = 2 M g^2 H(\Lambda)/\Lambda^2$$

and assuming $H(\Lambda) \sim 1$, it is straightforward to see that the term proportional to $H$ in Eq. (2) becomes important only for $p \sim \Lambda$. The asymptotic form, Eq. (4), is still correct in the intermediate region. A finite value of $H$ only changes the values of the amplitude $A$ and the phase $\delta$, which become functions of $H$ as is confirmed numerically. If $H$ is chosen to be a function of $\Lambda$ such as to cancel the explicit $\Lambda$ dependence, we can make the solution of Eq. (2) cutoff independent for all $p \ll \Lambda$. In particular, the on-shell scattering amplitude $a(k)$ with $k \sim 1/a_2$ will be cutoff independent as well. Thus $H(\Lambda)$ must be chosen such that $-s_0 \ln \Lambda + \delta(H(\Lambda)) = -s_0 \ln \Lambda_\star$, where $\Lambda_\star$ is a parameter fixed by experiment or by matching with a microscopic model.

We can get a handle on the form of $H(\Lambda)$ by considering Eq. (2) with two different values of the cutoff $\Lambda$ and $\Lambda' > \Lambda$. Assuming both solutions have the same phase $\cos(s_0 \ln(p/\Lambda_\star))$ even for $p \sim \Lambda'$, we find

$$H(\Lambda) = \frac{\sin(s_0 \ln(\Lambda/\Lambda_\star) - \arctg(1/s_0))}{\sin(s_0 \ln(\Lambda/\Lambda_\star) + \arctg(1/s_0))}. \quad (5)$$

Consequently, with $H(\Lambda)$ chosen like Eq. (5), the on-shell scattering amplitude $a(k)$ for $k \ll \Lambda$ will be $\Lambda$ independent. We also determine $H(\Lambda)$ numerically by finding the value of $H$ that keeps the three-body scattering length $a_3 = -a(0)$ constant for each value of $\Lambda$ varying over a large range. The numerical values for $H(\Lambda)$ agree with Eq. (5) to high accuracy (see Fig. 2(a)). For illustration we used $a_3 = 1.56 a_2$, but we have verified that similar agreement holds for other values of $a_3$. In Fig. 2(b) we show the corresponding $k \cot \delta = ik + a(k)^{-1}$, where $\delta$ is the $S$-wave phase shift for particle/bound-state scattering, for several values of $\Lambda$. As argued above, it is insensitive to $\Lambda$ as long as $k \ll \Lambda$. The effective range, e.g., is predicted as $r_3 = 0.57 a_2$. These arguments hold for the bound-state problem as well. The shallowest bound state has a cutoff-independent binding energy of $B_3 = 1.5/M a_2^2$.

The ratio $a_3/a_2 = 1.56$ is suggested by the values $a_2 = 124.7$ Å and $a_3 = 195$ Å obtained from a phenomenological $^4$He-$^4$He potential giving the correct dimer binding energy. Fig. 2(b) then represents the phase shifts for atom/dimer scattering, with an effective range $r_3 = 71$ Å. Similarly, our result for the shallowest bound state suggests an excited state of the trimer at $B_3 =$
1.2 mK. Because of similar integral equations, our arguments are relevant for three-fermion systems with internal quantum numbers as well.

In conclusion, we have provided evidence that renormalization of the three-body problem with short-range forces requires in general the presence of a leading order one-parameter contact three-body force.

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