On the equivariant Hopf bifurcation in hysteretic networks of van der Pol oscillators

Z Balanov\textsuperscript{1,2}, W Krawcewicz\textsuperscript{2} and D Rachinskii\textsuperscript{3}

\textsuperscript{1} Department of Mathematics and Computer Sciences, Netanya Academic College, 42365, Netanya, Israel
\textsuperscript{2} Department of Mathematical Sciences, University of Texas at Dallas, Richardson,Texas, 75080 US
\textsuperscript{3} Department of Applied Mathematics, University College Cork, Ireland

E-mail: d.rachinskii@ucc.ie

Abstract. The standard approach to study symmetric Hopf bifurcation phenomenon is based on the usage of the equivariant singularity theory developed by M. Golubitsky et al. In this paper, we present the equivariant degree theory based method which is complementary to the equivariant singularity approach. Our method allows systematic study of symmetric Hopf bifurcation problems in non-smooth/non-generic equivariant settings. The exposition is focused on a network of eight identical van der Pol oscillators with hysteresis memory, which are coupled in a cube-like configuration leading to $S_4$-equivariance. The hysteresis memory is the source of non-smoothness and of the presence of an infinite dimensional phase space without local linear structure. Symmetric properties and multiplicity of bifurcating branches of periodic solutions are discussed showing a direct link between the physical properties and the equivariant topology underlying this problem.

1. Introduction

1.1. Subject

Given a family of dynamical systems parameterized by a real parameter $\alpha$, for example,

$$\dot{x} = f(\alpha, x) \quad (\alpha \in \mathbb{R}, \ x \in \mathbb{R}^n),$$

where $f : \mathbb{R} \oplus \mathbb{R}^n \rightarrow \mathbb{R}^n$ is a continuous map, and a curve of trivial stationary solutions $(\alpha, 0)$, the Hopf bifurcation is a phenomenon occurring when $\alpha$ crosses some critical value $\alpha_c$ (for which the linearization admits a purely imaginary eigenvalue) and resulting in appearance of small amplitude periodic solutions near $(\alpha_c, 0)$. Many problems in population dynamics, neural networks, fluid dynamics, solid mechanics, elasticity, chemistry, electrical engineering, etc., lead to studying the Hopf bifurcation in networks of identical oscillators coupled symmetrically. This feature is expressed as the so-called \textit{equivariance} of system (1), i.e. $\mathbb{R}^n$ is a representation of a (compact Lie) group $\Gamma$ and $f(\alpha, \gamma x) = \gamma f(\alpha, x)$ for all $x \in \mathbb{R}^n$, $\gamma \in \Gamma$ and $\alpha \in \mathbb{R}$ (cf. [1–9] and references therein). Typical local problems include: (i) \textit{occurrence of the Hopf bifurcation}, (ii) \textit{classification of symmetry properties of bifurcating branches of periodic solutions}, (iii) \textit{estimation of a minimal number of these branches} (if $\Gamma$ is finite), (iv) \textit{finding their asymptotics}, and (v) \textit{providing the stability analysis}. In many cases, these problems have been effectively studied under certain smoothness conditions.

Published under licence by IOP Publishing Ltd
For the last decades, the interest in the bifurcation phenomena in non-smooth systems with different sources of non-smoothness is growing rapidly (we refer to [10–12] that are in the context relevant to our discussion). In the present paper, we are concerned with the equivariant Hopf bifurcation in systems where the non-smoothness comes from the hysteresis effect in several system components. The system is described by a set of differential equations which are coupled to hysteresis operator relationships between some of the variables. We thus use the mathematical setting provided by the theory of systems with hysteresis memory operators (see [13–17], for a recent survey of applications in several disciplines we refer to [18,19]). We would like to stress that the hysteresis memory operators are intrinsically non-smooth.

Specifically, we discuss the equivariant Hopf bifurcation in networks of van der Pol type oscillators coupled symmetrically, where each oscillator has a hysteretic memory component described by the Preisach operator (see, for example, [13, 14]). To simplify our exposition, we are focused on a concrete electrical network of eight oscillators admitting the octahedral symmetry group (see figures 1, 2). The memory of each oscillator is related to the assumption that the constitutive relationship between the magnetic field and magnetization in the inductance element of each oscillator is hysteretic. In this setting, the above mentioned problems (i), (ii) and (iii) on the equivariant Hopf bifurcation are completely settled.

Figure 1. Symbolic graph representation of eight van der Pol oscillators arranged into a cube-like configuration. Yellow box symbolizes an electronic component used to connect the oscillators. The symmetry group of the cube, which is the octahedral group $S_4$, is the symmetry group of this network of oscillators.

1.2. Methodology.

The standard way to study the equivariant Hopf bifurcation is rooted in the equivariant singularity theory: assuming the system to satisfy several reasonable conditions (for example, $f$ in (1) is smooth and equivariantly generic around the bifurcation point), one can combine the equivariant normal form classification with Center Manifold Theorem/Lyapunov-Schmidt Reduction. For a detailed exposition of this concept and related techniques, we refer to [6, 7] (see also [9,20,21]). Several examples with symmetry groups important from the viewpoint of the present paper, have been considered in [22,23].

(i) Phase space approach. The phase space of a system with the Preisach hysteresis operator includes an infinite dimensional component in the space of states (memory configurations) of the Preisach model. This component is a metric space without a local linear structure, which
Figure 2. A network of identical van der Pol oscillators with octahedral symmetries corresponding to the graph shown in figure 1. The connecting electronic components are marked as rectangles.

is a common situation for dynamical systems with complex hysteresis operators. Therefore, the existing theory of such dynamical systems misses important tools of the smooth theory of ODEs, like invariant manifolds, dimension reduction (including Center Manifold Theorem), and normal forms. Actually, systems with hysteresis operators exhibit dynamical and bifurcation scenarios different from those observed in smooth systems of ODEs (see, for example, [11,24]). By the same reason (lack of local linear structure in the state space and non-smoothness of the hysteresis input-state-output operators), the methods based on the linearization of the evolution operator, such as linear stability analysis, cannot be generally applied to systems with hysteresis.

Several alternative approaches to study stability of solutions (still based on the phase space analysis) developed in the last decades involve: contraction mapping principle [25], energy consideration [17,26,27] and, for scalar non-autonomous differential equations with the Preisach operator, monotonicity argument combined with upper and lower solutions method [28, 29]. A major simplification can be achieved when the evolution operator has an invariant finite-dimensional set in the phase space. For example, the Hopf bifurcation from infinity gives rise to the zero-dimensional invariant set as only one (saturated) state of the hysteresis operator is possible at a moment when the input achieves a sufficiently large value [30]. Another example is related to solutions with piecewise monotone components which do not have extremum points with equal values and do not have degenerate extremum points [31]. However, this simplification breaks down on local periodic problems (being the subject of the present paper). Even for a scalar non-autonomous differential equation coupled with the Preisach operator relationship, the analysis of dynamics in the phase space, such as the one used in [32] to obtain stability
criteria for periodic solutions by invoking the contraction mapping principle and to study simple saddle-node bifurcation of such solutions, is tricky.

(ii) Functional-analytic approach. By the above reasons, in the existence problems, the functional-analytic setting whereby an equivalent fixed point problem is formulated in an appropriate function space, seems to be an appealing alternative to the phase space analysis. Under this approach, a periodic solution is sought in a space of periodic functions, rather than among trajectories of the dynamical systems or their initial values. However, the operator of the fixed point equation, say \( F \), inherits non-smooth features from the dynamical system: it is differentiable on stationary functions, but has points of non-differentiability in arbitrary vicinity of those. In particular, any non-constant periodic function with repeating maximum/minimum inside the period admits a cone of directions for which \( F \) does not have the Gateux derivative. Moreover, the structure of nonlinear terms in “Taylor decomposition” of \( F \) around stationary solutions is rather complicated in nature since these terms involve hysteretic operators. All this makes us believe that, to obtain the existence results related to local Hopf bifurcation in systems with hysteresis, one should analyze topological properties of \( F \).

It should be pointed out that \( F \) is completely continuous that suggests to use homotopy invariants of degree type to study periodic problems (see overview in [24] for results using this general approach). In particular, the existing results on Hopf bifurcation in systems with the Preisach operator, including the model with one van der Pol type oscillator, have been obtained using Leray-Schauder degree (see [33,34]). However, the Leray-Schauder degree based methods meet serious difficulties when applied to study networks of oscillators. In addition, these methods seem to be ineffective to study symmetric properties of bifurcating periodic solutions to systems related to symmetric networks of oscillators (with/without hysteresis).

(iii) Twisted equivariant degree based method. During the last twenty years the equivariant degree theory emerged in non-linear analysis (for the detailed exposition of this theory, including historical remarks, we refer to recent monographs [8,35] and surveys [36,37]; for the prototypal invariants, see [38–40]). In short, the equivariant degree is a topological tool allowing “counting” orbits of solutions to symmetric equations in the same way as the usual Brouwer degree does, but according to their symmetry properties. In particular, the equivariant degree theory has all the attributes allowing its application in non-smooth/non-generic equivariant settings related to equivariant dynamical systems having, in general, infinite dimensional phase spaces with lack of linear structure. In fact, the equivariant degree has different faces reflecting a diversity of symmetric equations related to applications. To study symmetric properties of periodic solutions to dynamical systems, Z. Balanov and W. Krawcewicz introduced in [41] the so-called twisted equivariant degree (see also [8,37]). This is the main technical tool used to prove the results of the present paper.

At first glance, the twisted degree looks like a sophisticated topological tool. However, similarly to the usual Brouwer degree, the twisted degree admits an axiomatic definition (see [8], Chapter 4; see also [36,37,41]), and therefore, can be effectively used outside its theoretical context and foundations. Actually, for its usage one needs no more than basic representation theory, usual Brouwer degree theory, and the so-called Recurrence Formula (see [8], Section 4.6 or [37], Theorem 4.5, Property (T6)) which is a kind of Borsuk-Ulam mod \( p \) Theorem for equivariant maps with one free parameter. Since once done computations can be compiled into a database (that can be conveniently used by means of eTools), the twisted equivariant degree theory becomes as simple as any other symbolic computational package being used for analysis of dynamical systems.
1.3. Overview
After the Introduction the paper is organized as follows. In Section 2, we present the known Hopf bifurcation results related to one van der Pol oscillator (both in cases with and without hysteretic relationship between the magnetic induction and magnetic field being taken into account). In Section 3, we give an informal introduction to the Preisach model. Finally, in Section 4, the basic example of the network of eight van der Pol circuits with Preisach memory coupled symmetrically in cubic configuration is considered (cf. figure 2). In particular, the mathematical model is deduced (see (20)), the corresponding equivariant bifurcation result is presented (see Theorem 1 and Remark 3), and a sketch of the proof of Theorem 1 is outlined. For the equivariant topology (resp. $G$-representations) background, we refer to [42,43] (resp. [44]).

2. Hopf bifurcation in one van der Pol oscillator with hysteretic element
2.1. Van der Pol oscillator without hysteresis
The van der Pol oscillator is one of the simplest non-conservative oscillators with non-linear damping used in electrical engineering. The prototypical electrical circuit consists of an LCR contour and a negative feedback loop, which can be implemented on the basis of a triode, as in the classical van der Pol circuit, or, for instance, by using tunnel diodes as shown on figure 3. Here, the negative feedback element consists of two tunnel diodes connected in parallel, each with a DC power source connected in series to bias the diode. It has a current-voltage characteristic curve of the form

$$j_d = -\sigma_1 u + \sigma_2 u^3 \quad (2)$$

with $\sigma_1 > 0$ and $\sigma_2 \geq 0$.

![Figure 3](image_url)

**Figure 3.** A van der Pol circuit. $F$ denotes an inductor with ferromagnetic core; $R$ an ohmic resistor; $C$ a capacitor; $N_R$ a resistor with negative resistance for small voltages; $E_0$ a DC power source. $j_d, j_c$ denote the current flowing in the different branches of the circuit.

If one ignores hysteresis effects, then the EMF in the inductor can be expressed as

$$E_{\text{ind}} = L \frac{dj}{dt} \quad (3)$$
where \( j = j_c + j_d \) is the current flowing through the resistance and the inductance elements. Applying simple circuit analysis rules results in the system
\[
j' = -\frac{R}{L}j + \frac{1}{L}u, \quad u' = -\frac{1}{C}j + \frac{\sigma_1}{C}u - \frac{\sigma_2}{C}u^3,
\]
which can be rewritten as the classical van der Pol equation. Here \( C, R \) and \( L \) stand for the capacitance, resistance and inductance of the inductor, resp., while \( u \) represents the voltage drop across the resistor and inductor, i.e., \( u = Rj + E_{ind} \); prime denotes the differentiation with respect to time.

In a standard way, using \( \alpha := \sigma_1 \) as a bifurcation parameter, one can easily study the Hopf bifurcation in (4) (cf., for example, [45]). Namely, the eigenvalues of the Jacobian
\[
A = \begin{pmatrix}
-\frac{R}{L} & \frac{1}{L} \\
-\frac{1}{C} & \frac{\alpha}{C}
\end{pmatrix}
\]
of the right hand side are
\[
\xi = \frac{L\alpha - RC \pm \sqrt{(RC - \alpha L)^2 - 4CL(1 - R\alpha)}}{2CL}.
\]
Clearly, they cross the imaginary axis at the points \( \xi = \pm i\frac{1}{\sqrt{\frac{RC}{CL}}} \) for
\[
\alpha = \frac{RC}{L}
\]
provided that
\[
\frac{R^2C}{L} < 1.
\]
Put \( \xi(\alpha) := u(\alpha) + iv(\alpha) \). Then,
\[
\left. \frac{du}{d\alpha} \right|_{\alpha = \frac{RC}{L}} = \frac{1}{2C} > 0.
\]
Thus, condition (8) ensures that the Hopf bifurcation occurs at the critical parameter value (7).

The amplitude of the small cycle of equation (4) near the bifurcation point scales as \( \sqrt{|\alpha|} \) for \( \sigma_2 \neq 0 \). Stability of cycles born via the Hopf bifurcation is determined by the sign of the cubic term (cf., for example, [45]), namely: for \( \sigma_2 > 0 \) (resp. \( \sigma_2 < 0 \)), the Hopf bifurcation is supercritical, i.e., the small cycles are stable (resp. subcritical, i.e., the small cycles are unstable).

### 2.2. Van der Pol oscillator with hysteresis

If the inductance element contains a ferromagnetic core, the ferromagnetic material can introduce a hysteresis relation between the magnetic induction \( B \) and magnetic field \( H \). A typical picture of hysteresis loops is presented in figure 4. Here the instantaneous value of \( B \) depends not only on the value of \( H \) at the same moment, but also on some previous values of \( H \). Hence the constitutive relationship between \( B \) and \( H \) is an operator relationship. While this hysteresis effect can often be neglected, there are specific applications, where it is essential to include it in the model. Examples of such applications include power transformers and dynamics of power electronics circuits in the presence of the ferroresonance phenomenon. Furthermore, hysteresis effect can be the main effect responsible for the energy dissipation. The models, using the Preisach operator to describe the relation between \( B \) and \( H \) have been shown to provide
Figure 4. Hysteresis loops in relationship between the external magnetic field $H$ (horizontal axis) and the magnetic induction $B$ of a ferromagnetic material (vertical axis).

A better agreement between the numerical results and experimental data in [10] (see Section 3 for an informal discussion of the Preisach model and, for example, [13, 14] for a rigorous mathematical treatment of the model).

To be more specific, if the relationship between the magnetization $M$ and the magnetic field $H$ is modeled by the Preisach input-output operator, then

$$B = \nu H + M = \nu H + \Theta[\eta_0]H.$$  \hspace{1cm} (10)

Here $\nu > 0$, the Preisach input-output operator $\Theta[\eta_0]$ maps the variable magnetic field $H(t)$ to the variable magnetization $M(t) = (\Theta[\eta_0]H)(t)$ of the ferromagnetic material, and the infinite-dimensional parameter $\eta_0$ (called the initial state of the Preisach model) describes the physical state of the ferromagnetic material at the initial moment. As the field $H$ (resp. EMF induced in the inductor) is proportional to the current $j$ (resp. rate of change of the magnetic induction $B$), the EMF is now given by

$$E_{\text{ind}} = L \frac{d}{dt}(j + \Theta[\eta_0]j),$$  \hspace{1cm} (11)

where the Preisach density function (see Section 3) is properly rescaled when we pass from (10) to (11). System (4) changes accordingly to the system

$$(j + \Theta[\eta_0]j)' = -\frac{R}{L}j + \frac{1}{L}u, \quad u' = -\frac{1}{C}j + \frac{\sigma_1}{C}u - \frac{\sigma_2}{C}u^3$$  \hspace{1cm} (12)

involving the Preisach operator. System (12) has the zero equilibrium $j = u = 0$. The Hopf bifurcation for system (12) and similar higher order systems with the Preisach operator was studied in [33, 34] (the impact of symmetries was not addressed). The Leray-Schauder degree theory methods used in [33, 34] allowed the authors to localize the Hopf bifurcation point and to show the existence of a global branch of cycles stretching from the equilibrium to infinity. In particular, similarly to the case of ODEs, it was shown that the Hopf bifurcation point is determined by the Jacobian of the right-hand side of the system. This is explained by the asymptotically small Preisach nonlinearity (in a proper sense) compared to the linear term near the equilibrium, which implies that the hysteresis term does not affect the critical value of the
Parameter. For example, if we assume $R = 0$, then system (4) without hysteresis term and system (12) with the Preisach hysteresis term have the Hopf bifurcation at the same critical value $\sigma_1 = 0$ of the parameter $\sigma_1$.

However, the hysteresis effect has a crucial impact on the asymptotics and stability of cycles born via the Hopf bifurcation. In [33, 34], a numerical evidence is given that the Hopf bifurcation in system (12) is supercritical regardless the sign of $\sigma_2$ due to the effect of the Preisach nonlinearity which dominates the cubic term near the equilibrium point. In particular, system (12) exhibits the supercritical Hopf bifurcation for $\sigma_2 = 0$, i.e., when the negative feedback is linear and the Preisach term is the only nonlinearity and the only source of dissipation in the system. The authors are not aware of the literature where the stability of small cycles of system (12) was rigourously proved, however they believe that the stability analysis method proposed in [32] for problems with forced oscillations can be used for this purpose after a proper modification.

The asymptotics of small cycles born via the Hopf bifurcation is also different for systems (4) and (12). The amplitude of the cycle of system (12) near the bifurcation point is asymptotically proportional to $\sigma_1$ rather than $\sqrt{|\sigma_1|}$ (for $\sigma_2 \neq 0$), and the coefficient of proportionality is defined by parameters of the Preisach nonlinearity irrespective of the value of $\sigma_2$.

**Remark 1** (i) Loosely speaking, the asymptotics and stability of small cycles of system (12) are the same as those of the van der Pol type ODE obtained from (4) by replacing the cubic nonlinearity with a non-smooth quadratic term such as $u'|u|$.

(ii) Observe that several aspects of dynamics of oscillators modeled by differential equations (coupled with the Preisach operator relationship) different from (12), have been studied, for example, in [24, 26]. Maxwell’s equations coupled with the Preisach constitutive relation between $B$ and $H$ in spatially distributed models were considered in [16, 46, 47].

### 3. Preisach model

Preisach-type models are at the heart of the hysteresis modeling paradigm. We use this model as the basis of a quantitative description of ferromagnetic hysteresis. For the reader’s convenience, in this section, we present an informal description of the Preisach model allowing us to formulate the main result (see Theorem 1).

#### 3.1. Non-ideal relay

An elementary building block of the Preisach model is the simplest hysteretic transducer called non-ideal relay. This transducer is characterized by two threshold values $\alpha$ and $\beta$, where $\alpha < \beta$. Its output $\eta(t)$ can take one of two values 0 or 1 at any moment, meaning that the relay is either switched off or switched on. The output varies in response to changes of the input $x(t)$.

The dynamics of the relay are described by figure 5. The variable output of the relay, traditionally denoted by

$$\eta(t) = (R_{\alpha, \beta}[t_0, \eta_0]x)(t), \quad t \geq t_0, \quad (13)$$

depends on the input $x(t)$, $t \geq t_0$, and on the initial state $\eta_0$ of the relay. The input $x(t)$ is an arbitrary continuous scalar function, and $\eta_0$ is either 0 or 1. The output $\eta(t)$ has at most a finite number of jumps between the values 0 and 1 on any finite interval $t_0 \leq t \leq t_1$. The output behaves lazily, preferring to be unchanged as long as the phase pair $(x(t), \eta(t))$ belongs to the union of the bold lines in figure 5. The equalities $\eta(t) = 1$ for $x(t) \geq \beta$ and $\eta(t) = 0$ for $x(t) \leq \alpha$ always hold for $t \geq t_0$. In the context of modeling magnetic materials in a one-dimensional magnetic field, a non-ideal relay is related to dynamics of the magnetic moment of an individual domain in the domain structure created by the field in the ferromagnetic material. In this interpretation, the equality $\eta(t) = 1$ (resp. $\eta(t) = 0$) means that the magnetic moment points in the direction of the applied field (resp. in the opposite direction).
3.2. Preisach operator

The main assumption made in the Preisach model is that the system can be thought of as the sum (or superposition, or parallel connection) of a continuum of weighted non-ideal relays $R_{\alpha,\beta}$ with common input $x(t)$, where the weight of each relay is $\mu(\alpha, \beta)$.

The relays contributing to the system can be represented by points on the two-dimensional half-plane $\Pi = \{ (\alpha, \beta) : \beta > \alpha \}$, which is also known as the Preisach plane (see figure 6). The colored area $S = S(t)$ in figure 6 is the set of threshold values $(\alpha, \beta)$ for which the output of the corresponding relay $R_{\alpha,\beta}$ is 1 at a given moment $t$. The output of the remaining relays at the same moment is 0. Denote by $\eta(t) = \eta(t; \alpha, \beta)$ the characteristic function of the set $S(t)$. The output of the Preisach model is then represented by the formula

$$\theta(t) = \int_{\alpha<\beta} \mu(\alpha, \beta)(R_{\alpha,\beta}[t_0, \eta_0(\alpha, \beta)]x(t)) \, d\alpha d\beta$$

$$= \int_{\alpha<\beta} \mu(\alpha, \beta)\eta(t, \alpha, \beta) \, d\alpha d\beta = \int_{S(t)} \mu(\alpha, \beta) d\alpha d\beta,$$

where the Preisach density function $\mu(\alpha, \beta)$ is an integrable non-negative function defined on $\Pi$. As we see from (14), the output $\theta(t)$ depends on both the input and the initial states $\eta_0(\alpha, \beta)$ of all relays. Therefore, we use the notation

$$\theta(t) = (\Theta[\eta_0]x)(t) \quad (\eta_0 = \eta_0(\alpha, \beta), \quad t \geq t_0)$$

for the input-output operator of the Preisach model. The set $S(t)$, its characteristic function $\eta(t; \alpha, \beta)$, and its boundary $L(t)$ (see figure 6) are called a state of the Preisach nonlinearity.

In magnetism, $\theta$ is interpreted as the magnetization of the ferromagnetic material; all the magnetic moments of domains modeled by the relays contribute to $\theta$.

**Remark 2** (i) The evolution of the input defines the evolution of the state $S(t)$ and thus the evolution of the output (14) by a few simple rules. We refer to http://euclid.ucc.ie/hysteresis/node17.htm where an online applet explaining and implementing these evolution rules is available.

(ii) The state $S(t)$ stores the sequence of certain maximum and minimum values of the input over the time interval between the moments $t_0$ and $t$ (the so-called sequence of main extrema or shock values of the input, see [14]). This sequence defines the curve followed by the input-output pair on the diagram in figure 4 on some time interval after the moment $t$ (i.e., from $t$ until the moment when the input reaches one of its past extremum values stored in $S(t)$). There are infinitely many input-output curves passing through each point of this diagram. The evolution rules of $S$ define switches from one to another input-output curve.
4. Main result: model, formulation and discussion

4.1. Model

We are interested in classifying symmetry properties and estimating minimal number of branches of periodic solutions occurring as a result of the Hopf bifurcation in a cube-like network of identical van der Pol oscillators admitting the $S_4$-symmetry group (see figure 1). We give a complete equivariant topological classification of small amplitude periodic solutions bifurcating from the trivial one at Hopf bifurcation points.

An electrical circuit realization of the network we are interested in is shown in figure 2. We make the following assumption.

Assumption A. (a) An individual oscillator at each vertex of the cube has a ferromagnetic core in the inductance element and is described by the same equations (12) as the oscillator shown in figure 3; in particular, a Preisach density function $\mu$ is given and, for any initial state of the Preisach model $\eta_0$, the Preisach input-output operator $\Theta[\eta_0]$ taking the magnetic field to the magnetization of the ferromagnetic material, is defined by (14)—(15).

(b) The couplings between the oscillators are done by means of resistors only. For simplicity, all the wires connecting the oscillators are assumed to have the same resistance $r$.

In order to derive the equations describing the network shown in figure 2, we essentially use Kirchhoff’s current and voltage laws (in short, KCL and KVL, respectively).

Denote by $J_{ik}$ the current in a red wire from the oscillator $i$ to the oscillator $k$, where $i < k$. Let $\tilde{J}_{ik}$ be the current in a green wire from the oscillator $i$ to the oscillator $k$, where $i < k$. According to KCL, the total current which flows into an oscillator from all the connecting red and green wires must be zero. For the oscillator No. 1 this gives

$$J_{12} + J_{14} + \tilde{J}_{12} + \tilde{J}_{14} + \tilde{J}_{15} = 0.$$  \hfill (16)

A similar equation holds for each of the eight oscillators (in these equations a current is taken with the positive sign if it flows from the oscillator and with the negative sign if it flows into the oscillator). Now, consider the square contour consisting of four green wires, which form the upper face of the green cube. By KVL, the sum of voltages along the sides of this square must be zero, i.e., $r\tilde{J}_{12} + r\tilde{J}_{23} + r\tilde{J}_{34} - r\tilde{J}_{14} = 0$ (we put ‘+’ when we go in the direction of the current and ‘−’ otherwise). Hence,}

$$\tilde{J}_{12} + \tilde{J}_{23} + \tilde{J}_{34} - \tilde{J}_{14} = 0.$$  \hfill (17)
Again, a similar equation is valid for each face of the green cube and for each face of the red cube, giving 12 new equations for the currents. Combining all the 20 equations (such as (16) and (17)) together, one can convert them to an equivalent system of fourteen equations in twelve variables $x_{ik} := J_{ik} + \tilde{J}_{ik}$. By direct verification, the rank of this system is equal to twelve, meaning that the system admits only the trivial solution $x_{ik} = 0$ for all the indices $i, k$. Thus, \[ \tilde{J}_{ik} = -J_{ik}. \] (18)

The next step is to consider the rectangular contour formed by the green wire and the red wire connecting the oscillators No. 1 and No. 2 and by the vertical capacitance links of these two oscillators. Summing the voltages across the four sides of this contour, one obtains (see KVL) \[ u_1 - r\tilde{J}_{12} - u_2 + rJ_{12} = 0. \]

A similar relation holds for each edge of the cube, giving us 12 equations \[ \tilde{J}_{ik} - J_{ik} = \frac{u_i - u_k}{r}. \]

Combining these equations with relations (18) yields \[ -\tilde{J}_{ik} = J_{ik} = \frac{u_k - u_i}{2r}. \] (19)

Finally, consider the circuit of the oscillator No. 1 (with the initial state $\eta_1^0$). It is described by the voltage equation and the current equation. The voltage equation is the same as for the decoupled oscillator (it says that the sum of the voltages across the LCR contour is zero):
\[
L \frac{d}{dt}(j_1 + \Theta[\eta_0^1]j_1) + Rj_1 = u_1,
\]
where $j_1$ is the current through the resistor $R$ and the inductance and $\Theta[\eta_0^1]$ is the Preisach input-output operator defined by (14)–(15). The second equation says that the total current through the capacitance is equal to $-Cu'_1$:
\[
Cu_1' = -j_1 + \sigma_1 u_1 - \sigma_2 u_1^3 + J_{15} + J_{14} + J_{12},
\]
which, due to (19), is equivalent to \[ Cu'_1 = -j_1 + \sigma_1 u_1 - \sigma_2 u_1^3 + \frac{1}{2r}(u_2 + u_4 + u_5 - 3u_1). \]

Writing down a similar pair of voltage-current equations for each oscillator and using the vector notation $j = (j_1, \ldots, j_8)$, $u = (u_1, \ldots, u_8)$, $u^3 = (u_1^3, \ldots, u_8^3)$ and the vector hysteresis operator $\Theta[\eta_0]j = (\Theta[\eta_0^1]j_1, \ldots, \Theta[\eta_0^8]j_8)$ with $\eta_0 = (\eta_1^0, \ldots, \eta_8^0)$, we arrive at the system
\[
\frac{d}{dt}(j + \Theta[\eta_0]j) = -Bj + \frac{1}{C}u - \frac{\sigma_1}{C}u^3 + \frac{1}{2r}Bu
\]
with the interaction matrix
\[
B = \begin{pmatrix}
-3 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\
1 & -3 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & -3 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & -3 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & -3 & 1 & 0 & 1 \\
0 & 1 & 0 & 0 & 1 & -3 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & -3 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & -3
\end{pmatrix}.
\]
4.2. Result
For all values of $\sigma_1$, system (20) has a trivial stationary solution with $j = u = 0$. Take $\alpha = \sigma_1$ as a bifurcation parameter. Before formulating the main result related to the $S_4$-equivariant Hopf bifurcation in (20) (see Theorem 1), some remarks are in order.

First, the periodicity of a solution to system (20) means that both the vector-valued functions $j = j(t)$, $u = u(t)$ are periodic and the evolution of the state $S_k(t)$ ($k = 1, \ldots, 8$) of each Preisach model involved is periodic with the same period (cf. figure 6). The periodicity of $S_k(t)$ implies periodicity of the output $\theta_k(t) = (\Theta[\eta^k_0])k(t)$ of each Preisach model. The infinite dimensional parameters $\eta^k_0$ describing the initial states $S_k(t_0)$ of the Preisach operators are a part of the solution as the choice of $\eta^k_0$ should ensure the periodicity of $S(t) = (S_1(t), \ldots, S_8(t))$. Observe also that system (20) is autonomous—every shift of time takes solutions to solutions. Hence, it is appropriate to talk about a cycle representing the time shifts of a periodic solution.

Second, every branch in question contains periodic solutions with an arbitrarily small norm $\sqrt{j_1^2 + \cdots + j_8^2 + u_1^2 + \cdots + u_8^2}$ of the $u, v$-components. For the rigorous definition of the branch of periodic solutions we refer to [8, 48, 49].

Next, the $S_4$-symmetry group of the graph presented in figure 1 determines in a standard way the $S_4$-representation on $M = \mathbb{R}^8$ and system (20) is equivariant with respect to $M \oplus M$. In order to characterize symmetry properties of bifurcating periodic solutions, we adopt the standard terminology from [6]. More precisely, given a periodic solution $(j, u, S)$, the pair $(\gamma, e^{i\tau}) \in S_4 \times S^1$ is called a spatio-temporal symmetry of this solution if $\gamma j(t + \tau) = j(t)$, $\gamma u(t + \tau) = u(t)$, $\gamma S(t + \tau) = S(t)$ for all $t$ (cf. [6], formula (3.2)). If the solution is non-stationary, then the set of all its spatio-temporal symmetries constitutes a twisted (one-folded) subgroup of $S_4 \times S^1$ (cf [6], formula (3.4)). Since $S_4$ is finite, the solution is the so-called discrete rotating wave (cf. [6], Definition 3.2 and Theorem 3.4). As usual (see, for example, [6], p. 93), $M \oplus M$ can be identified with the complexification $M^c = M \otimes C$, so the twisted subgroups can be viewed from the $S_4 \times S^1$-representation $(M^c, S_4 \times S^1)$ given by

$$(\gamma, e^{i\tau})(j \otimes z) := (\gamma(j) \otimes (e^{i\tau}z)) \quad (j \in M, z \in C, (\gamma, e^{i\tau}) \in S_4 \times S^1).$$

Clearly, any subgroup conjugate to a twisted subgroup is twisted as well (meaning that one can speak about “twisted orbit types”). Denote by $\Phi^t(M^c, S_4 \times S^1)$ the set of all twisted orbit types occurring in $(M^c, S_4 \times S^1)$. The set $\Phi^t(M^c, S_4 \times S^1)$ admits a partial order induced by the standard partial order on the set of all conjugacy classes of subgroups in $S_4 \times S^1$ (see, for example, [8], formula (2.1)).

If the Hopf bifurcation takes place for $\alpha = \overline{\alpha}$, then, following the scheme suggested in [8], Sections 9.2 and 9.3, one can associate to system (20) the equivariant bifurcation invariant $\omega_{S_4 \times S^1}(\overline{\alpha})$. This invariant is expressed in terms of the twisted equivariant degree (see [8], Section 5.2) having a direct link to twisted orbit types of branches of discrete rotating waves bifurcating from $\overline{\alpha}$ (in fact, $\omega_{S_4 \times S^1}(\overline{\alpha})$ takes its values in the free abelian $Z$-module generated by conjugacy classes of twisted subgroups). We refer to Remark 3 where it is explained how to extract the information provided by $\omega_{S_4 \times S^1}(\overline{\alpha})$.

Finally, the following physical parameter

$$c = \frac{r}{R} \left(1 - \frac{R^2C}{L}\right),$$

plays an important role in our considerations. For simplicity, to prevent a steady-state/Hopf mode interaction, we assume that

$$c \neq 1, 2, 3.$$  \hspace{1cm} (23)

**Theorem 1** Under the Assumption A, suppose that conditions (8) and (23) are satisfied. Then:
(i) (Occurrence) System (20) with the bifurcation parameter $\alpha := \sigma_1$ has four bifurcation points given by
$$\alpha_i = \frac{RC}{L} + \frac{i}{r} \quad (i = 0, 1, 2, 3);$$

(ii) (Multiplicity and asymptotics) There are at least 1 branch of discrete rotating waves bifurcating from the zero equilibrium at each of the bifurcation points $\alpha = \alpha_0, \alpha_3$, and at least 27 branches of discrete rotating waves bifurcating from the zero equilibrium at each of the bifurcation points $\alpha = \alpha_1, \alpha_2$; the periods of all the bifurcating cycles are asymptotically close to $2\pi\sqrt{LC}/\sqrt{1 - \frac{R^2C}{L}}$ as $\alpha \to \alpha_i$;

(iii) (Equivariant topology/physics interaction) Given values of the parameter $c$ (see (22)), the equivariant bifurcation invariants $\omega_{S_3 \times S^1}(\alpha_i)$, $i = 0, 1, 2, 3$ (see [8], formulae (9.23), (9.30), (9.31) and Definition 9.30), are completely determined in Tables 1–4 (see also Remark 3);

(iv) (Symmetry properties) For each maximal orbit type $(H) \in \Phi^I(2\sigma, S_3 \times S^1)$, there exists a branch of discrete rotating waves with the orbit type $(H)$ bifurcating from the zero equilibrium at some $\alpha_i$ ($i=0,1,2,3$) (in particular, for $\alpha = \alpha_1, \alpha_2$, symmetry groups of the 27 branches provided by (ii) are organized in five orbit types related to the maximal twisted types occurring in the corresponding “central spaces”).

4.3. Comments

Remark 3 How to use the invariant $\omega_{S_3 \times S^1}(\alpha_i)$? Let us explain how to “decode” the information provided by the invariants $\omega_{S_3 \times S^1}(\alpha_i)$ presented in Tables 1–4.

(a) Under the assumptions of Theorem 1, $\omega_{S_3 \times S^1}(\alpha_i) \neq 0$ for each $\alpha_i$ ($i = 0, 1, 2, 3$). Therefore (cf. [8], Theorem 9.28) the Hopf bifurcation takes place for each $\alpha_i$.

(b) To classify symmetry properties and minimal number of the bifurcating branches, observe that the maximal orbit types in $\Phi^I(2\sigma, S_3 \times S^1)$ are: $(S_4)$, $(S_3^c)$, $(Z_4^c)$, $(D_4^c)$, $(D_3)$, $(Z_3^c)$, $(D_3^c)$, $(D_2^c)$ (see [8], Subsection 5.3.4, for the explicit description of the corresponding subgroups). Take, for example, $\omega_{S_3 \times S^1}(\alpha_1) = (D_4^c) + (D_3) + (D_3^c) + (Z_3^c) + (Z_3^c) - (D_1)$ from Table I. It follows from the proof of Theorem 9.28 in [8] and the definition of a twisted orbit type that there exist 3 branches (of discrete rotating waves) of type $(D_4^c)$, 4 branches of type $(D_3)$, 6 branches of type $(D_3^c)$, 6 branches of type $(Z_3^c)$, and 8 branches of type $(Z_3^c)$ bifurcating from $(\alpha_1, 0)$ (altogether, $3 + 4 + 6 + 6 + 8 = 27$ branches).

(c) It is easy to see that statements (i), (ii) and (iv) of Theorem 1 are consequences of statement (iii), meaning that the invariant $\omega_{S_3 \times S^1}(\alpha_i)$ contains the essential information on bifurcating periodic solutions. Observe also that, in general, the invariant of type $\omega_{S_3 \times S^1}(\alpha_i)$ can be used to study twisted rotating waves with non-maximal symmetries (cf. [8], Remarks 10.15–16), however, this goes beyond the scope of the present paper.

Remark 4 (a) Recall that system (20) has a (non-smooth) hysteresis input-state-output operator leading, in particular, to an infinite dimensional phase space without a local linear structure. If the operator $\Theta$ in (20) is zero (i.e., one has a smooth non-hysteretic finite-dimensional system of ODEs), then conclusions (i) and (ii) of Theorem 1 as well as the existence of a branch of periodic solutions with symmetry $H$ for any $\mathbb{C}$-axial isotropy $H$ in the corresponding central subspace (that is parallel to conclusion (iv) of Theorem 1) follow from the Equivariant Hopf Theorem in [6,7] (see also [22]).

(b) Conclusion (iii) of Theorem 1, on the one hand, shows that the twisted equivariant degree based method used in this paper is not sensitive to any smoothness/genericity requirements, and, on the other hand, connects directly the physics (value of $c$) with the equivariant topology (values of $\omega_{S_3 \times S^1}(\alpha_i)$) underlying the problem in question.
Table 1. Branches of periodic solutions at Hopf bifurcation points for 0 < c < 1.

| Bif. point | $\omega_{S_4 \times S^1}(\alpha_i)$ | # Branches |
|------------|-----------------------------------|------------|
| $\alpha_0$ | $-(S_4)$                            | 1          |
| $\alpha_1$ | $(D_1^3) + (D_3) + (D_2^3) + (Z_1^3) - (D_1) - (D_2)$ | 27         |
| $\alpha_2$ | $(D_1^4) - (D_2^3) - (D_3^4) - (D_2^3) + (Z_1^3) - (Z_1^3) + 3(D_1^2)$ | 27         |
| $\alpha_3$ | $(S_4) - 2(D_1^3) - (D_2^3) - (Z_1^3) + (Z_3) + 2(D_1^3) + (Z_3)$ | 1          |

Table 2. Branches of periodic solutions at Hopf bifurcation points for 1 < c < 2.

| Bif. point | $\omega_{S_4 \times S^1}(\alpha_i)$ | # Branches |
|------------|-----------------------------------|------------|
| $\alpha_0$ | $-(S_4)$                            | 1          |
| $\alpha_1$ | $-(D_1^4) - (D_3) - (D_2^3) - (Z_1) - (Z_1^3) + (D_1) + (Z_2)$ | 27         |
| $\alpha_2$ | $(D_1^3) + (D_2^3) + (D_3^4) + (Z_1^3) + (Z_1^3) - (D_1^3) - (Z_2^3)$ | 27         |
| $\alpha_3$ | $(S_4) - 2(D_2^3) - (D_2^3) + 3(D_1^3) - (Z_1)$ | 1          |

Table 3. Branches of periodic solutions at Hopf bifurcation points for 2 < c < 3.

| Bif. point | $\omega_{S_4 \times S^1}(\alpha_i)$ | # Branches |
|------------|-----------------------------------|------------|
| $\alpha_0$ | $-(S_4)$                            | 1          |
| $\alpha_1$ | $-(D_1^4) - (D_3) - (D_2^3) - (Z_1) - (Z_1^3) + (D_1) + (Z_2) + (Z_2)$ | 27         |
| $\alpha_2$ | $-(D_1^4) - (D_2^3) - (D_2^3) - (Z_1) - (Z_1^3) + (D_1^3) + (Z_2)$ | 27         |
| $\alpha_3$ | $(S_4)$                            | 1          |

Table 4. Branches of periodic solutions at Hopf bifurcation points for c > 3.

| Bif. point | $\omega_{S_4 \times S^1}(\alpha_i)$ | # Branches |
|------------|-----------------------------------|------------|
| $\alpha_0$ | $-(S_4)$                            | 1          |
| $\alpha_1$ | $-(D_1^4) - (D_2^3) - (D_2^3) - (Z_1) - (Z_1^3) + (D_1) + (Z_2)$ | 27         |
| $\alpha_2$ | $-(D_2^3) - (D_2^3) - (D_2^3) - (Z_2) - (Z_1^3) + (D_1) + (Z_2)$ | 27         |
| $\alpha_3$ | $(S_4)$                            | 1          |

4.4. Sketch of the proof of Theorem 1

Step 1: Reduction to $(j, u)$-components. As a matter of fact, every periodic solution $(j(t), u(t), S(t))$ to system (20) belongs to the class of periodic solutions $\Sigma = (j(t), u(t), S_{\eta_0}(t))$ with the same $(j, u)$-components. From the physical viewpoint, variables $(j, u)$ are observable, while the internal dynamics of the Preisach memory of the system (represented by the variable $S_{\eta_0}(t)$) is not. Using the monocyclic property of the hysteresis input-output operator $\Theta$ (see, for example, [14]) and choosing properly a representative in each class $\Sigma$, one can replace system (20) by a differential-operator system

$$\frac{d}{dt}(j + \Theta^T j) = -\frac{R}{L} j + \frac{1}{L} u, \quad u' = -\frac{1}{L} j + \frac{1}{C_1} u - \frac{1}{C_2} u + \frac{1}{2C_2} B u$$

(24)

satisfying the following conditions: (i) (24) is a system in $(j, u)$-components only; (ii) any periodic solution to (24) canonically defines a class $\Sigma$ of periodic solutions to (20); (iii) (24) contains an operator $\Theta^T$ (induced by $\Theta$) which is globally Lipschitz continuous in the space $C$ of $T$-periodic continuous $\mathbb{R}$-valued functions, and satisfies the global estimate $\|\Theta^T x\|_\infty \leq K \|x\|^2_\infty$. 

14
Step 2: Inversion of Preisach operator. Following the standard lines (cf. [15,17]), in order to convert system (24) to the fixed-point operator equation in an appropriate functional space, one studies invertibility properties of the map \( \text{Id} + \hat{\Theta}^T \). Using (iii) from Step 1, one can show that the map \( \text{Id} + \hat{\Theta}^T \) takes \( \mathcal{C} \) onto itself, is invertible, and its inverse is globally Lipschitz continuous.

Step 3: Period normalization. As in the case of autonomous ODEs without hysteresis, one may consider the unknown period \( T \) as an additional parameter, and reduce studying \( T \)-periodic solutions of system (24) to looking for \( 2\pi \)-periodic solutions of an equivalent system, say (D). This transformation uses the rate-independence property of the input-output and input-state operators of the Preisach model (see, for example, [14]).

Step 4: Operator reformulation. Let \( \mathcal{E} \) be the space of \( C^1 \)-smooth \( 2\pi \)-periodic \( \mathbb{R}^{16} \)-valued functions equipped with the standard \( C^1 \)-norm and \( G : S_4 \times S^1 \)-action given by \((\gamma, \tau)x(t) := \gamma x(t + \tau) \) \((\gamma \in S_4, e^{i\tau} \in S^1, t \in \mathbb{R} \) and \( x \in \mathcal{E})\). Combining Steps 1 – 3 with the standard Sobolev compact embedding theorems one can replace system (D) with an equivalent operator equation

\[
\bar{\mathfrak{F}}(\alpha, T, x) = 0 \quad (\alpha \in \mathbb{R}, \ T \in (0, \infty), \ x \in \mathcal{E}),
\]

where \( \bar{\mathfrak{F}}(\alpha, T, x) := x - \mathcal{F}(\alpha, T, x) \) and \( \mathcal{F} : \mathbb{R} \times (0, \infty) \times \mathcal{E} \to \mathcal{E} \) is a completely continuous \( G \)-equivariant operator (cf. [8], pp. 263–264): here \( G \) acts trivially on \( \mathbb{R} \times (0, \infty) \).

Step 5: Linearization. Let \( L_\alpha : \mathbb{R}^{16} \to \mathbb{R}^{16} \) be the linearization of the right-hand side of system (20) with \( \alpha := \sigma_1 \). Then, \( \frac{T}{2\pi}L_\alpha \) canonically determines the linearization of \( \bar{\mathfrak{F}} \) at \((\alpha, T, 0)\) (cf. (25)). Recall that \( \bar{\mathfrak{F}} \) has non-differentiability points in any neighborhood of \((\alpha, T, 0)\).

Step 6: Auxiliary function and invariant \( \omega_{S_4 \times S^1}(\alpha) \). Take \( \alpha_0 \in \mathbb{R} \) and \( T_0 \in (0, \infty) \) such that \( \frac{T_0}{2\pi} \) is the purely imaginary eigenvalue of \( L_{\alpha_0} \). Combining Step 5 with the Implicit Function Theorem, one can construct a \( G \)-invariant neighborhood \( \Omega \subset \mathbb{R} \times (0, \infty) \times \mathcal{E} \) of \((\alpha, T, 0)\) and an invariant continuous function \( \eta : \Omega \to \mathcal{E} \) (the so-called “auxiliary function”, cf. [8], Definition 9.17) satisfying the following conditions: (a) any solution to the equation

\[
\bar{\mathfrak{F}}^\eta(\alpha, T, x) = 0 \quad (\bar{\mathfrak{F}}^\eta = (\eta, \bar{\mathfrak{F}}))
\]

is a solution to (25), and (b) \( \bar{\mathfrak{F}}^\eta \) does not have zeroes on \( \partial \Omega \). In particular, the twisted \( G \)-equivariant degree of \( \bar{\mathfrak{F}}^\eta \) on \( \Omega \) is correctly defined. Using this degree as the bifurcation invariant \( \omega_{S_4 \times S^1}(\alpha) \) (cf. [8], Theorem 9.28) one can prove Theorem 1 (cf. Remark 3).

Acknowledgments

This publication has emanated from research conducted with the financial support of Federal Programme ‘Scientists of Innovative Russia’, grant 2009-1.5-507-007, and Russian Foundation for basic Research, grant 10-01-93112.

References

[1] Iudovíc V I 1971 Prikl. Mat. Mek. 35 638–655
[2] Turing A 1952 Phil. Trans. Roy. Soc. B. 237 37–72
[3] Yoshida K 1982 Hiroshima Math. J. 12 321–348
[4] Wu J 1998 Trans. Amer. Math. Soc. 350 4799–4838
[5] Fiedler B 1988 Global Bifurcation of Periodic Solutions with Symmetry (New York: Springer)
[6] Golubitsky M and Stewart I N 2002 The Symmetry Perspective (London: Imperial College Press)
[7] Golubitsky M, Schaeffer D G and Stewart I N 1988 Singularities and Groups in Bifurcation Theory vol 2
[8] Balanov Z, Krawcewicz W and Steinlein H 2006 Applied Equivariant Degree (Springfield: AIMS)
[9] Field M J 2007 Dynamics and Symmetry
[10] Rezaei-Zare A, Sanaye-Pasand M, Mohseni H, Farhangi Sh and Iravani R 2007 IEEE Trans. Power Deliv. 22 919–920
[11] Colombo A, di Bernardo M, Hogan S J and Kowalczyk P 2007 J. Nonlinear Sci. 17 85–108
[12] Logemann H, Ryan E P and Shvartsman I 2008 Nonlinear Anal. Theory Methods & Appl. 69 363–391
[13] Krasnosel’skii M and Pokrovskii A 1989 Systems with Hysteresis (New York: Springer)
[14] Mayrergoiz I D 1991 Mathematical Models of Hysteresis (Springer)
[15] Brokate M and Sprekels J 1996 Hysteresis and Phase Transitions (New York: Springer)
[16] Visintin A 1994 Differential Models of Hysteresis (Berlin: Springer)
[17] Krejčí P 1996 Hysteresis, Convexity and Dissipation in Hyperbolic Equations (Tokyo: Gakkotosho)
[18] 2006 The Science of Hysteresis, three volume set, ed I Mayrergoiz and G Bertotti (Elsevier, Academic Press)
[19] 2009 Hysteresis, ed R Iyer and X Tan (IEEE Control Systems Magazine 1)
[20] Field M J and Swift J W 1994 Nonlinearity 7 385–402
[21] Guo S and Lamb J S W 2008 Proc. of the American Mathematical Society 136 2031-2041
[22] Ashwin P and Podvigina O 2003 Proc. R. Soc. A 459 1801–1827
[23] Dias A P S and Rodrigues A 2009 Nonlinearity 27 627–666
[24] Brokate M, Pokrovskii A, Rachinskii D and Rasskazov O 2006 The Science of Hysteresis vol 1, ed G Bertotti and I Mayrergoiz (Academic Press) chapter 2 pp 127-291
[25] Brokate M, Collings I, Pokrovskii A and Stagnitti F 2000 Z. Anal. Anw. 19 469-487
[26] Krejčí P 2000 Appl. Math. 45 439-468
[27] Krejčí P, Sprekels J and Zheng S 2001 J. Differential Equations 175 88–107
[28] Appelbe B, Flynn D, McNamara H, O’Kane P, Pimenov A, Pokrovskii A, Rachinskii D and Zhezherun A 2009 IEEE Control Systems Magazine 29 44-69
[29] Krejčí P, O’Kane P, Pokrovskii A and Rachinskii D 2010 J. Physics: Conf. Series to appear
[30] Diamond P, Kuznetsov N A and Rachinskii D I 2001 J. Differential Equations 175 1–26
[31] Pokrovskii A, Power T, Rachinskii D and Zhezherun A 2006 J. of Physics: Conf. Series 55 171–190
[32] Pimenov A and Rachinskii D 2009 Discrete and Continuous Dynamical Systems B 11 997–1018
[33] Appelbe B, Rachinskii D and Zhezherun A 2008 Physica B 403 301–304
[34] Kuznetsov N A, Rachinskii D and Zhezherun A 2008 Doklady Math. 78 705–709
[35] Ize J and Vignoli A 2003 Equivariant Degree Theory (W. de Gruyter)
[36] Balanov Z and Krawcewicz W 2008 Handbook of Differential Equations, Ordinary Differential Equations vol. 4, ed F Battelli and M Feckan (Elsevier) pp 1–131
[37] Balanov Z, Krawcewicz W, Rybicki S and Steinlein H J. of Fixed Point Theory Appl. to appear
[38] Dancer E N 1985 Ann. Inst. H. Poincaré Anal. Non Lineaire 2 1–18
[39] Dancer E N and Toland J F 1993 Proc. London Math. Soc. 66 539–567
[40] Fuller F B 1967 American Journal of Mathematics 89 133–148
[41] Balanov Z, Krawcewicz W and Ruan H 2006 Discrete Contin. Dyn. Syst. Ser. A 15 983–1016
[42] Bredon G E 1972 Introduction to Compact Transformation Groups (New York-London: Academic Press)
[43] tom Dieck T 1987 Transformation Groups (Berlin: W. de Gruyter)
[44] Kirillov A A 1976 Elements of the Theory of Representations (Berlin-Heidelberg-New York: Springer)
[45] Kuznetsov Yu A 1995 Elements of Applied Bifurcation Theory (Berlin-New York: Springer)
[46] Krejčí P 1989 Appl. Math. 34 364-374
[47] Eleuteri M, Kopiova J and Krejčí P 2008 Physica B 403 448-450
[48] Krasnosel’skii M 1964 Positive Solutions of Operator Equations (Groningen: P. Noordhoff Ltd)
[49] Krasnosel’skii A M and Rachinskii D I 2002 Doklady Math. 65 344–349