Consistency of canonical formulation of Horava gravity

Chopin Soo
Department of Physics, National Cheng Kung University, Tainan, Taiwan
E-mail: cpsoo@mail.ncku.edu.tw

Abstract.
Both the non-projectable and projectable version of Horava gravity face serious challenges. In the non-projectable version, the constraint algebra is seemingly inconsistent. The projectable version lacks a local Hamiltonian constraint, thus allowing for an extra graviton mode which can be problematic. A new formulation (based on arXiv:1007.1563) of Horava gravity which is naturally realized as a representation of the master constraint algebra (instead of the Dirac algebra) studied by loop quantum gravity researchers is presented. This formulation yields a consistent canonical theory with first class constraints; and captures the essence of Horava gravity in retaining only spatial diffeomorphisms as the physically relevant non-trivial gauge symmetry. At the same time the local Hamiltonian constraint is equivalently enforced by the master constraint.

1. Introduction
Horava’s theory of gravity [1] attempts to preserve unitarity by relinquishing space-time covariance, and improve renormalizability by including higher order spatial derivatives. Spatial diffeomorphism symmetry is retained at the fundamental level. A related development, in loop quantum gravity, is the application of the master constraint program [2, 3] to non-perturbative quantization of Einstein’s theory. The master constraint algebra has the advantages of having structure constants rather than structure functions, and of spatial diffeomorphisms forming an ideal (thus allowing for the decoupling of the equivalent quantum Hamiltonian constraint from spatial diffeomorphism generators). In metric theories without full space-time covariance, departures of the constraint algebra from the Dirac algebra are to be expected. I present a new formulation (based on Ref. [4]) of Horava theory as a representation of the master constraint algebra,

\[
\{ \int N^i H_i d^3 x, \int N^j H_j d^3 y \}_{\text{P.B.}} = \int (\mathcal{L}_N N^i) H_i d^3 x,
\]

\[
\{ \int N^i H_i d^3 x, M \}_{\text{P.B.}} = 0, \quad \{ M, M \}_{\text{P.B.}} = 0,
\]

which yields a consistent canonical theory with first class constraints. The local Hamiltonian constraint is equivalently implemented by the master constraint \( M := \int_\Sigma \frac{|H(x)|^2}{\sqrt{q(x)}} d^3 x \approx 0 \).

2. Horava gravity with detailed balance
In the canonical mould, the action for Horava gravity is

\[
S = \int \pi^{ij} \dot{q}_{ij} d^3 x dt - \int (NH + N^i H_i) d^3 x dt,
\]

wherein the super-Hamiltonian, with detailed balance [1], is

\[
H = \frac{\kappa^2}{2} G_{ijkl} \pi^{ij} \pi^{kl} + \frac{\sqrt{q}}{2} \int N^i H_i d^3 x,
\]

which is the master constraint algebra.
action is real only if
\[ \text{sgn} \left[ \frac{\delta W_T}{\delta q_{ij}} \right] \]
and \( H_i := 2q_{ij} \nabla_k \pi^{kj} \approx 0 \) is the super-momentum constraint. It is assumed the spacetime metric is of the Arnowitt-Deser-Misner form \([5]\) \( ds^2 = -N^2 (cdt)^2 + q_{ij}(dx^i + N^i cdt)(dx^j + N^j cdt) \). The DeWitt supermetric, with deformation parameter \( \lambda \) which is not necessarily unity, is \( G_{ijkl} = \frac{1}{2}(q_{ik}q_{jl} + q_{jk}q_{il}) - \frac{\lambda}{3\lambda - 1} q_{ij}q_{kl} \) and \( W_T(q_{ij}) \) is (up to 3rd order in spatial derivatives of the metric) the sum of a Chern-Simons action of the spatial affine connection and the spatial Einstein-Hilbert action with cosmological constant i.e. \( W_T = W_{CS} + W_{EHA} = \frac{1}{4\pi c^2} \int e^{ijk} (\Gamma^t_{im} \partial_t \Gamma^m_{kl} + \frac{2}{3} \Gamma^t_{im} \Gamma^m_{jn} \Gamma^n_{kl}) \) \( dx^i + \frac{\lambda}{2} \sqrt{q} (R - 2\Lambda_W) \) \( dx^3 \).

There is a striking feature in the detailed balance condition for Horava-Lifshitz theory: the Hamiltonian constraint can be succinctly expressed as

\[
H = \frac{\kappa^2 G_{ijkl}(\pi^{ij} + i \frac{\delta W_T}{\delta q_{ij}})(\pi^{kl} - i \frac{\delta W_T}{\delta q_{kl}})}{2\sqrt{q}} =: \frac{\kappa^2}{2\sqrt{q}} G_{ijkl} Q^i_j Q^k_l \approx 0; \tag{2}
\]

wherein, as operators in the quantum theory, \( Q^i_j := \hat{\pi}^{ij} \pm i \frac{\delta W_T}{\delta q_{ij}} \) \( e^{\pm \frac{W_T}{2\sqrt{q}} \hat{\pi}^{ij} e^{\pm \frac{W_T}{2\sqrt{q}}}} \). \( Q^i_j \) are hermitian conjugates of each other if \( W_T \) is hermitian (classically real), and are separately hermitian if \( W_T \) is purely anti-hermitian (classically pure imaginary). In both cases, the classical \( H \) remains real. The superspace metric \([7]\), \( \delta S^2 \equiv G^{ijkl} \delta q_{ij} \delta q_{kl} \), has signature \( (\text{sgn} \left[ \frac{1}{3} - \lambda \right], +, +, +, +) \). So the corresponding Wheeler-DeWitt equation \([7, 8]\) comes equipped with an ‘intrinsic time’ \([7]\) for \( \lambda > \frac{1}{3} \). Intriguingly, the emergent speed of light \( c = \frac{\kappa^2 \mu}{4} \sqrt{\frac{\Lambda_W}{\mu - 3\lambda}} \), the cosmological constant \( \Lambda = \frac{3}{2} \Lambda_W \), and Newton’s gravitational constant \( G = \frac{\kappa^2 c^3}{2\pi \sqrt{\pi}} \) can all be phenomenologically positive for \( \lambda > \frac{1}{3} \) only if \( \mu \) is pure imaginary and \( \kappa \) is real. Then \( H \) and the action is real only if \( u^3 \) is pure imaginary. This set of values renders \( W_T \) to be pure imaginary, and thus \( Q^i_j \) become individually hermitian. States \( \Psi_{Q^i_j} = I e^{\pm \frac{W_T}{2\sqrt{q}}} \) with slowly varying \( I \) are semi-classical; and a pure imaginary \( W_T \) leads to real \( \pi^{ij} = \mp i \frac{\delta W_T}{\delta q_{ij}} \) solving the Hamilton-Jacobi equation with \( \pm i W_T \) as Hamilton functions.

3. Departures from the Dirac algebra

Both the projectable and non-projectable version of the theory face serious obstacles. In the projectable version, with the lapse function \( N(t) \) dependent only on time, the theory has an integrated (rather than local) constraint \( \int H \) \( dx^3 = 0 \). This produces a consistent first class constraint algebra. However, the absence of a local constraint \( H(x) = 0 \) implies the theory has an additional degree of freedom which can be pathological and phenomenologically problematic. In the Einstein-Hilbert theory the would-be pathological scalar mode is eliminated precisely by the local Hamiltonian constraint. Non-projectable Horava gravity with space-time dependent lapse function \( N(x) \) is also problematic. \( \pi_N(x) = 0 \) leads to the secondary local constraint \( H(x) = 0 \). But for Horava gravity, \( \{ H(x), \int N H \) \( dx^3 \} \) \( P.B. = (\Delta + \omega) N(x) \), where \( \Delta \) contains spatial derivatives acting on \( N \) and \( \omega \) does not \([9, 10]\). The only consistent solution is \( N = 0 \) \([10]\). Strictly speaking, \( N = 0 \) considered as a special case of a gauge-fixing condition \( N = f \) results in \( \{ N(x) - f, \pi_N(y) \} \) \( P.B. = \delta(x - y) \) with second class constraints \( \pi_N = N = 0 \). All the constraints are formally stable under evolution. But this ‘consistency’, achieved at the cost of vanishing lapse function \( N \), is dubious; moreover the space-time Arnowitt-Deser-Misner metric is degenerate whenever \( N = 0 \).

Dirac’s algorithm for the analysis of constrained canonical systems \([11]\) should reveal the true gauge symmetries of the theory. For Horava gravity, the requirement of vanishing lapse function from the algorithm seems to signal that only 3-dim spatial diffeomorphism symmetry is physically relevant. Such theories cannot obey the Dirac algebra which is the

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1 After its initial discussion in Ref.\([4]\), this particular form of the Hamiltonian constraint has also been investigated in Ref.\([6]\).
hallmark of space-time covariance and the embeddability of hypersurface deformations, and from which Einstein’s geometricdynamics can be uniquely recovered [12, 13]. There can be interesting modifications to the Dirac algebra in theories without full space-time covariance. For example, in ultra-local gravity with \( \hat{H} = \frac{2\kappa}{\sqrt{q}} G_{ijkl} \pi^{ij} \pi^{kl} \), a strongly vanishing commutator, \( \{ \int N H \, d^3x, \int M H \, d^3y \}_\text{P.B.} = 0 \) (even for \( \lambda \neq 1 \)), replaces the usual commutation relation in the Dirac algebra [13]. When a scalar curvature term is added in deformations of Einstein-Hilbert theory with \( \lambda \neq 1 \) and \( H = \left( \frac{2\kappa}{\sqrt{q}} G_{ijkl} \pi^{ij} \pi^{kl} - \frac{\sqrt{q}}{2\kappa} R \right) \), the corresponding Poisson bracket is

\[
\{ \int N H \, d^3x, \int N' H \, d^3y \}_\text{P.B.} = \int (N \nabla^i N' - N' \nabla^i N) [H_i - \frac{2(1 - \lambda)}{3\lambda - 1} \nabla_i \pi] \, d^3x
\]

With \( \lambda \neq 1 \) (the degenerate case [14] of \( \lambda = \frac{1}{3} \) will not be discussed here), the consequent secondary constraint \( Z_i := \nabla_i \pi = 0 \iff \pi = K(t) \sqrt{q} \) arises. The resultant condition for stability of \( Z_i = 0 \) is \( W := [-\nabla^2 + R + 2\kappa^2 q^{-1} (3\lambda - 1) q^2 - \lambda q^{-1}] N = 0 \). The constraint \( H = 0 \) allows us to write \( R = \frac{4\kappa^2}{q} \left( \pi^{ij}_i \pi^{ij}_j - \frac{1}{3(3\lambda - 1)} \pi^2 \right) \), wherein \( \pi^{ij}_i := \pi^{ij} - \frac{1}{3} q^{ij} \pi \) is the traceless part of the momentum. Together with \( \pi = K \sqrt{q} \), the condition on \( N \) becomes

\[
W = [-\nabla^2 + \frac{4\kappa^2}{q} \left( \pi^{ij}_i \pi^{ij}_j \right) + \frac{16\kappa^2 K^2}{3(3\lambda - 1)}] N = 0.
\]

Since \(-\nabla^2\) and \(\frac{4\kappa^2}{q} \pi^{ij}_i \pi^{ij}_j\) are both positive semi-definite operators, \(W = 0\) can have non-vanishing solution for \(N\) only if \(\lambda < \frac{1}{3}\). For \(\lambda < \frac{1}{3}\), the resultant theory (counting the 6 conjugate pairs \((q_{ij}, \pi^{ij})\), and \(H_i = 0\) as first class and \(H = 0\), \(\pi = K \sqrt{q}\) as second class constraints) has \(\frac{1}{2}[12 - 3(2) - 2(1)] = 2\) degrees of freedom. For \(\lambda > \frac{1}{3}\), we are lead to the fact \(N = 0\) is the only solution for \(W = 0\). These conclusions differ from those of Ref. [15]. Although secondary constraints arise, a non-trivial solution exists for \(N\) for \(\lambda < \frac{1}{3}\). However, for \(\lambda \geq \frac{1}{3}\) (and for Horava gravity with local Hamiltonian constraint) only \(N = 0\) is allowed.

4. Horava gravity as a master constraint theory

The Dirac algebra is not the Lie algebra of 4-dim. diffeomorphisms as structure functions are present in the commutator of two Hamiltonian constraints. But \(H_i\) and \(H\) constraints do generate 4-dim. diffeomorphisms on-shell. In a theory which possesses at the fundamental level only 3-dim. diffeomorphisms as gauge symmetry, we expect the constraints to generate, on-shell, only spatial diffeomorphisms. There is a formulation which precisely achieves this goal, and which at the same time gives rise to a condition equivalent to the local constraint \(H(x) = 0\): Horava gravity realized as a master constraint theory. The master constraint \(M := \int_{\Sigma} \frac{[H(x)]^2}{\sqrt{q(x)}} \, d^3x\) is tailored to be invariant under spatial diffeomorphisms (with \(H(x)\) being a scalar density of weight 1). For any real valued \(H(x)\), the integrand is positive-semi-definite; and the master constraint equation, \(M = 0\), is mathematically equivalent to \(H(x) = 0\) on \(\Sigma\) on the Cauchy surface \(\Sigma\). Thus the master constraint equation replaces the infinite number of local restrictions \((H = 0)\) by a single global restriction. The canonical action for Horava gravity can be assumed to be

\[
S = \int \pi^{ij}_i q_{ij} \, d^3x \, dt - \frac{1}{\epsilon_0} \int N(t) M \, dt + \int N_i H_i \, d^3x \, dt,
\]

with \(H\) of the form in Eq.(2); \(\epsilon_0\) has the physical dimension of energy density. The upshot is a consistent simple closed first class constraint algebra (as in Eq.(1)) with structure constants. Such a theory consistently generate equations of motion which are (on-shell) equivalent to spatial diffeomorphisms since \(\{q_{ij}, N(t) M_{\epsilon_0} + \int N^k H_k \, d^3x\}_{\text{M}=0,\epsilon_0, H=0} \approx \{q_{ij}, \int N^k H_k \, d^3x\}_{\text{P.B.}} = \mathcal{L}_N q_{ij}\)
(and similarly for $\pi^{ij}$). This formulation also realizes, in an explicit manner, the claim in Ref. [10] that time-reparamatization symmetry of Horava gravity and freedom in the choice of $N(t)$ is on-shell trivial. Weak observables $O$ should satisfy $\{O, N(t)M + \int N^i H_i d^3x\}|_{M=0\Rightarrow H=0} \approx \{O, \int N^i H_i d^3x\}_{PB} \approx 0$, and are thus invariant with respect to the local gauge symmetry of spatial diffeomorphisms. In such theories, two configurations differing by four-dimensional, rather than spatial, coordinate transformations can be physically inequivalent. It is to be noted our formulation of Horava gravity also requires the lapse function to depend only on $t$.

'Troubles’ in the constraint algebra of Horava gravity with local Hamiltonian constraint are to be expected of a canonical theory of 3-geometry which fundamentally possesses only spatial diffeomorphism invariance. Canonical Horava gravity can be consistently constructed as a master constraint theory, with the local Hamiltonian constraint equivalently enforced by the master constraint. Horava gravity is in fact an explicit, and highly non-trivial representation, of a master constraint theory with spatial diffeomorphism invariance. In not satisfying the Dirac algebra with local Hamiltonian constraint, Horava gravity is in fact more naturally associated with the master constraint theory than Einstein’s General Relativity.

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