Existence of Minkowski space

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Abstract

Minkowski space serves as a framework for the theoretical constructions that deals with manifestations of relativistic effects in physical phenomena. But neither Minkowski himself nor the subsequent developers of the relativity theory have provided a reasonable rationale for such a mathematical structure. In physics, such a rationale should show lower-level statements that determine where this structure is applicable and yield formal premises for proving its existence.

The above failure has apparently been due to the features of the adopted formalism based on the unjustifiably exclusive use of coordinates in the theoretical analysis of physical phenomena, which ignores the necessity of having physical grounds for mathematical concepts. In particular, the use of a coordinate transformation between two inertial reference frames makes the consideration so cumbersome that it appears useless for solving the fundamental problems of physical theory, including the question of whether Minkowski space exists.

In contrast, a straightforward, though not simple, calculation proves that the transformation of the time and the position vector of a physical event between two physical spaces establishes an equivalence relation between pairs made of these variables. This means the existence of the Minkowski space and shows that the premises for its proof are the same as for the coordinate-free derivation of basic effects of the special relativity theory: the use of the Einsteinian time variable and motions of particles able to interact with each other and electromagnetic field over a short spatial range only.

The high degeneracy of free motions of point particles, together with the intricacy of the above mentioned calculation, suggests that a further generalization of Minkowski space is beyond belief, so that the modification or even the abandonment of the concept of spacetime seems quite natural.

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I.  INTRODUCTION

Physics and relativity textbooks[1] (in agreement with the mathematics monographs[7, 8]) present Minkowski space as a four-dimensional vector space where a system of four coordinates $t, x, y, z$ is supposed to represent an inertial reference space with its clock readings $t$ and spatial Cartesian coordinates $x, y, z$ so that the quadratic form

$$c^2t^2 - x^2 - y^2 - z^2$$  \(1\)

is invariant with respect to a changeover from one coordinate system to another. Here and throughout the article $c$ is the speed of light.

The story began with the report[9], where H. Poincaré had presented the change of time variable and Cartesian spatial coordinates that does not alter the appearance of Maxwell’s equations. He had given it the name 'Lorentz transformation' since in the physical interpretation of this change of the independent variables as a motion of a coordinate system H. Poincaré followed the preceding work[10] of H. A. Lorentz, who attempted to explain why the Earth’s motion is not detectable with an aid of optical experiments conducted on the Earth’s surface. Only the part of Maxwell’s equations that does not involve electric charges appears sufficient in Ref. 9 for inferring Lorentz transformation while in order to obtain the associated transformation of an electric charge density H. Poincaré had exploited an implicit assumption that the total charge of a moving charged body (referred to as ‘the electron’) is independent of the state of the body’s motion.

It should be noted that the term 'transformation' in Ref. 9 bears no relation to any specific occurrence among physical bodies but means that, in general, a change of independent variables along with an induced (i.e. appropriately corresponding) change of unknown functions in a set of partial differential equations is expected to modify the appearance of the equations. Thus, one can say that Ref. 9 presents the change of variables so that the transformed equations appear to be the same combination of the transformed electromagnetic field quantities as the source equations with the source electromagnetic field quantities are.

In contrast, in his initial defining article[11] on relativity A. Einstein investigated physical events, such as an interception of small (parts of) physical bodies by light rays/fronts, from the outset. He distinguished a ‘stationary system’ as a coordinate system where Newtonian equations hold and the measurements are based on the use of a measuring rod and Euclidean geometry.
It should be pointed out that A. Einstein described a coordinate system as relations between rigid bodies[11, S. 892] and associated the 'stationary system' with a 'stationary' space[11, S. 897], so his 'stationary system' is a physical object which this article further refers to as 'an inertial reference frame' to distinguish it from the mathematical concept 'coordinate system'[12]. (For the refinement of the concept 'space' see Section II A below.)

With referring to uniform motion in the 'stationary' space A. Einstein also identified a 'moving system' but then, in accordance with his formulation of the relativity principle[14], considered it to be on a par with a 'stationary system' so that relations between the two inertial reference frames appeared mutual[11, S. 903].

In addition, A. Einstein extended the concept of the time variable with an aid of the propagation of light (for the more detailed remark about it see below.) As a result, he obtained the formulas identical to those of Lorentz transformation[15], which, however, connects the variables that relate to a given event in two inertial reference frames. So, in theoretical physics, one uses the word 'transformation' for the change of the description of a frame-independent object/concept owing to a changeover from one inertial reference frame to another.

In the introductory part of his article[11, S. 891-892] A. Einstein has extended the principle of the relativity to electrodynamics and, in §3 of his article, applied it to Maxwell’s equations for the electromagnetic field free of electrical charges (i.e., similarly to Ref. 9, required that they retain their form when one inertial reference frame is replaced by another.) This allowed him to obtain the relation between the sets of the electromagnetic field quantities in two inertial reference frames. Then he assumed it valid in the presence of electric charges and (in contrast to Ref. 9) thereby arrived at the transformation law of an electric charge density and the conclusion that the total charge of a charged body is a frame-independent quantity.[17]

Considering the covariance of Maxwell’s equations with respect to Lorentz transformation as a purely mathematical fact, in his lecture[22] H. Minkowski proposed the technique of those combinations of (as well as relationships between) mechanical and electromagnetic quantities which make it easily perceivable that they preserve their appearance when subjected to Lorentz transformation along with the transformations induced by Lorentz transformation. Apart from these covariant combinations, lately introduced into theoretical physics by Part II of Ref. 23 as 4-tensors, he has also found some 4-scalars (i.e. the invari-
ants), including the kinematic quantity (1) and two electromagnetic quantities[22, S. 68].

For no reason other than the formal similarity between Lorentz transformation and the transformation equations relating two sets of coordinates of a given spatial (Euclidean) point in two Cartesian coordinate systems that differ in the directions of their coordinate axes[24], H. Minkowski has called 4-tuple $x, y, z, t$ a space-time point[26]. He undeniably implied that, similarly to the relation between different 3-tuples $x, y, z$ and one spatial point they can represent, there must be a non-numerical entity that corresponds to a collection of 4-tuples $x, y, z, t$ connected by Lorentz transformations.

An analogy, however, is not proof and may mislead those who rely on its consequences without regard for its premises. In fact, even in H. Minkowski’s time some mathematicians did try to construct Euclidean geometry on the basis of motions of rigid forms[28], but obviously H. Minkowski was hardly interested in their work and was not going to make sure that there was no problem in modifying it for a 4D space equipped with the ‘metric’ (1).

In §6 of his lecture, for events at two space-time points in a given coordinate system H. Minkowski has found the Lorentz transformation to the system where these events appear simultaneous. The reader of that paragraph may think that H. Minkowski has identified his space-time points with physical events[22, S. 69] or, in an attempt at following Ref. 11, at least mapped events into a set of space-time points, though carelessly mapping a set of one origin/structure into that of another does not necessarily result in a one-to-one correspondence.

Actually, in his approach to space and time concepts H. Minkowski was only able to refer to human experience[27, S. 69] while A. Einstein could write about the positions of an elementary physical event because the latter, such as detecting a particle or emitting/absorbing light by an atom, could include interaction with a small part of that association of solids which represents a body of an inertial reference frame.

The physical description/definition of a time moment of an event, i.e. the division of all events into the groups of events observed at the same time, appeared a more involved issue in Ref. 11 since in his way of going beyond Newtonian mechanics A. Einstein could not keep Newtonian concept of time. In order to partition[31, p. 18] a set accurately, mathematics suggests addressing an equivalence relation[31, p. 16] between each two elements of this set. In Ref. 11 A. Einstein has called the required binary relation between two events a synchronization and given a formulation for its required properties equivalent to its symmetry and
transitivity. He proposed the physical realization for the synchronization between the readings of two clocks in different places with a round-trip of a light pulse between these clocks and adjusting their readings so as to maintain a certain relationship between the times of emitting, reflecting and absorbing the light pulse.[32]

Thus, not only the frame-independent concept of space-time points was proposed without proving its existence but even the extension of the time concept within an inertial reference frame appeared based on wishes rather than evidence.

Although A. Einstein almost completely acknowledged the ideas of H. Minkowski[24], the other physicists showed more caution.[33] The authors with no background in usual, non-relativistic, physics accepted H. Minkowski’s ’world’ outright and much more enthusiastically.[37]

The post-Einsteinian generation of physicists got more focused on applications than attentive to foundations. The authors of textbooks on relativity as well as the lecturers of appropriate theoretical physics courses found it easier and more elegant to start their presentations with the idea of spacetime. In line with this education practice, scientific authors got their theoretical constructions developed in Minkowski space at the outset when expected them to apply to the manifestation of relativistic effects.

The reasonable estimation of the validity of a theoretical consideration need addressing its premises, based either on well-tested lower-level theoretical relations or directly on experimental data. However, since the proof of existence of Minkowski space has not yet been published up to the present time, the most important part of premises for each of the many theoretical constructions remains unrecognized.

The next section draws the reader’s attention to the coordinate-free formalism in the special relativity theory and explains why it is necessary for the required proof. The section II C contains the transformation of the time and the position vector of a point physical event between two spaces. Section III exploits this transformation to make the formal proof of the existence of Minkowski space. Its significance for physical theory is discussed in Section IV.
II. COORDINATE-FREE FORMALISM IN RELATIVITY THEORY

A. Physical spaces in mechanics

It is relations between parts of a solid that underlie the group of motions of rigid bodies. The group includes spatial translations $\hat{T}$ and rotations $\hat{R}$. The additive representation of a spatial translation is usually referred to as a spatial vector. One can use rotations to introduce an angle between two vectors etc.

When the orthonormal vectors $e_x, e_y, e_z$ represent the translations along three mutually perpendicular directions, the decomposition

$$\Delta r = \Delta x e_x + \Delta y e_y + \Delta z e_z$$

of a displacement (the change of a position vector $r$) is just what defines Cartesian coordinates of a small body that can manifest itself by interaction with a solid. If one invokes some other properties of physical bodies and their motion one finds that the positions of the small body obey Euclidean geometry so that the set of small parts of a conjectural boundless solid represents a Euclidean space.

The above described physical realization of Euclidean geometry evidently breaks down at sufficiently small scales, where the atomic/molecular structure of any solid is important. In order to extend the validity of a Euclidean space to smaller scales one has no choice but to address Newtonian mechanics of stable charged particles. Since the limits of applicability of Euclidean geometry are not among the topics of this article, the reader may simply accept the assumption that the set of motions of interacting charged particles is rich enough to ensure the existence of angles and other geometric features, including a position vector $r$.

Only a couple of additional comments are required here.

To build Euclidean geometry with motions of charged particles one has no way but to exploit lower-level relations between their trajectories, such as an interception in a collision of two relatively fast particles, or internal properties of a specific trajectory, such as those of a closed orbit of each of two relatively slow oppositely charged particles. Generally, it can result in the constructions of a moving Euclidean space. Then the indispensable step should be such change of $r$ that stops the center of mass of the particles.

Thus, a set of non-relativistic communicating observers is always capable to label the positions of particles by position vectors $r$ defined in its stationary Euclidean space. To
overcome the limitation on the relative velocity of observers, one should simply divide any set of observers into subsets of observers with non-relativistic relative velocities.[44] This means that the relativity theory can in no way avoid dealing with a variety of moving physical spaces or inertial reference frames.

The realization of Euclidean geometry with an aid of rigid bodies also fails over large scales due to the action of gravity. Then the use of Newtonian mechanics for the extension of Euclidean geometry is again possible, especially since motions of two gravitating masses are similar to those of two opposite electric charges. Further extension of this scheme to include relativistic motions is beyond the scope of this article.

B. Inadequacy of coordinate transformation between two frames

One of the consequences of adopting the idea of spacetime, originated from Ref. 22 and generalized in Part II of Ref. 23, is the exclusive use of coordinates in describing physical relationships.

Minkowski space does make it redundant to provide the transformation of the base vectors between two inertial reference frames in addition to the transformation of the time and Cartesian coordinates of a spacetime point, associated with a physical event.[5, §2.9]

Someone may also try to exploit the coordinate transformation

\[
\rho^{(Bb)} = M_{Aa}^{Bb} \rho^{(Aa)}
\]

in proving the existence of Minkowski space itself.

In Eq. (3) the column vector

\[
\rho^{(Ff)} = \begin{pmatrix} c t^{(Ff)} \\ x^{(Ff)} \\ y^{(Ff)} \\ z^{(Ff)} \end{pmatrix}
\]

is made of the time and Cartesian coordinates of a physical event in an inertial reference frame f introduced in a physical space F, the matrix

\[
M_{Aa}^{Bb} = R^{-1}(n_A^{(b)}) I (v_{BA}) R (n_B^{(a)})
\]

7
where

\[
L(v) = \begin{pmatrix}
\gamma & -\gamma v/c & 0 & 0 \\
-\gamma v/c & \gamma & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\]

is the matrix of the historically original Lorentz transformation, now referred to as the special Lorentz transformation[45, p. 41], the matrix

\[
R(\vec{n}) = \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & \sin \theta \cos \phi & \sin \theta \sin \phi & \cos \theta \\
0 & -\sin \phi & \cos \phi & 0 \\
0 & -\cos \theta \cos \phi & -\cos \theta \sin \phi & \sin \theta
\end{pmatrix}
\]

describes such rotation that \( \vec{e}_x = R(\vec{n})\vec{n} \) for

\[
\vec{e}_x = \begin{pmatrix}
0 \\
1 \\
0 \\
0
\end{pmatrix}, \quad \vec{n} = \begin{pmatrix}
0 \\
\sin \theta \cos \phi \\
\sin \theta \sin \phi \\
\cos \theta
\end{pmatrix}, \quad 0 \leq \phi < 2\pi, 0 \leq \theta < \pi.
\]

The column vector \( \vec{n}_G^{(f)} \) in Eq. (4) describes the direction of the velocity \( \vec{v}_G^{(Ff)} = \vec{n}_G^{(f)} v_{FG} \) of an G space in an inertial reference frame f, stationary in a F space.

The physically evident Eq. (4) corresponds to the decomposition[46] of any physically reasonable transformation between two inertial reference frames into the product \( R_1 \mathbb{B}(\vec{v}_1) \) (or \( \mathbb{B}(\vec{v}_2)R_2 \)) where \( R_1 \) (or \( R_2 \)) is the matrix of a rotation while \( \mathbb{B}(\vec{v}) \) is the matrix of the so called pure boost[47], which itself can be decomposed[48] as \( \mathbb{B}(\vec{v}) = R^{-1}(\vec{n})L(v)R(\vec{n}) \) for the velocity \( \vec{v} = v\vec{n} \).

The limiting case \( A=B=F, a=b=f \) entails that \( v_{FF} = 0 \) and

\[
M_{FF}^F = \mathbb{I}.
\]

The symmetry of exchanging two spaces (along with the frames they contain) yields \( v_{FG} = v_{GF} \) and

\[
M_{Gg}^{FF} = M_{FG}^{Gg}.
\]

As soon as one establishes that for every \( \vec{v}_G^{(Aa)} \) and \( \vec{v}_B^{(Gg)} \) there is \( \vec{v}_B^{(Aa)} \) such that

\[
M_{Aa}^{Bb} = M_{Gg}^{Bb} M_{Gg}^{Gg}.
\]
one can conclude that Eq. (3) is an equivalence relation since Eqs. (5)-(7) represent reflexivity, symmetry and transitivity of Eq. (3). On denoting a corresponding equivalence class by \( \rho \), one can finally get Minkowski space as a set of all possible \( \rho \).

Apparently, the idea that \( M_{(Ff)}(Gg) \) represents a 4D pseudo-rotation provides little help in finding \( \vec{v}_B^{(Aa)} \) for the given \( \vec{v}_G^{(Aa)} \) and \( \vec{v}_B^{(Gg)} \) in order to arrive at Eq. (7) without addressing vectors in Minkowski space prematurely (i.e. without being circular.)

The limiting case \( A=B=G \) is more suggestive since it turns \( M_{(Aa)}^{(Gg)} \) into the rotation in the 3D physical space \( G \) such that \( \vec{n}_G^{(b)} = M_{Ga}^{Gb} \vec{n}_G^{(a)} \), and the corresponding equivalence class turns out to be a point of \( G \), which is customarily marked by the Euclidean position vector \( r \). However, the existence of the vector \( r \) needs no reasoning based on rotations presented as transformations of Cartesian coordinates because Euclidean points and true spatial vectors have their own foundation in physics, as indicated in Section II A.

Since in this evidently true limiting case Eq. (7) yields some relationships between the components of \( \vec{n}^f_G \) if different frames, one can only hope that these relationships in conjunction of tricky and ponderous decompositions of the matrices \( M_{(Ff)}^{(Gg)} \) enable one to succeed in establishing Eq. (7).

To avoid any trickery, one should simply abandon the premature use of reference frames and turn to the transformation of the time and the position vector of an event between physical spaces.

### C. Transformation between two physical spaces

Collisions between the particles that can interact over a short range only as well as their interceptions with features of the propagating electromagnetic field, such as rays and plane phase fronts, form a class of events that underlie the special relativity theory. As soon as one accepts that the events can be marked with the time variable proposed by A. Einstein[11, § 1], the principle of relativity along with the principle of the constancy of the speed of light[11, § 2] lead one to the basic manifestations of the special relativity theory: length contraction, time dilation, time retardation, spatial transversal invariance.[50, Sec. III] The coordinate-free description of these effects results in the transformation between physical spaces in the form of relationships between the times and physically essential components of the position vector of a given event.
Let a superscript (F) of a quantity denote that the quantity is defined in a space F. In addition to the Einsteinian time $t^{(F)}$ and the position vector $r^{(F)}$ of a physical event in a space F, one can use the velocity $v^{(F)}_G$ of another space G there.

Then the transformation of the time and the position vector of a physical event between the spaces A and B can be written as[50, Sec. IV]

$$t^{(B)} = \gamma_{AB} \left[ t^{(A)} - \frac{(v_B^{(A)} \cdot r^{(A)})}{c^2} \right],$$

$$-\frac{(r^{(B)} \cdot v_A^{(B)})}{v_{AB}} = \gamma_{AB} \left[ \frac{(r^{(A)} \cdot v_B^{(A)})}{v_{AB}} - v_{AB} t^{(A)} \right],$$

$$r^{(B)} - \frac{(r^{(B)} \cdot v_A^{(B)}) v_A^{(B)}}{v_{AB}^2} \sim r^{(A)} - \frac{(r^{(A)} \cdot v_B^{(A)}) v_B^{(A)}}{v_{AB}^2}.\tag{10}$$

Since due to the symmetry of exchanging two spaces $v_B^{(A)} = v_A^{(B)}$, the additional notation

$$v_{AB} \equiv \left| v_B^{(A)} \right| = \left| v_A^{(B)} \right|, \quad \gamma_{AB} \equiv \frac{1}{\sqrt{1 - v_{AB}^2/c^2}}\tag{11}$$

are used here and hereinafter.

In contrast to the relation “=”, the relation “∼” connects quantities in different spaces. Still, the principle of relativity implies that[50, Sec. III.B]

$$f_1^{(A)} \sim g_1^{(B)} \text{ and } f_2^{(A)} \sim g_2^{(B)} \text{ entail } f_1^{(A)} + f_2^{(A)} \sim g_1^{(B)} + g_2^{(B)}\tag{12}$$

and

$$f_3^{(A)} \sim g_3^{(B)} \text{ and } f_4^{(A)} \sim g_4^{(B)} \text{ entail } \left( f_3^{(A)} \cdot f_4^{(A)} \right) = \left( g_3^{(B)} \cdot g_4^{(B)} \right)\tag{13}$$

for any spatial vectors $f_i^{(A)}$ in the space A and their counterparts $g_i^{(B)}$ in the space B.

It is worth remarking that the transformation rules (9) and (10) differ from the vector-like relationship presented in the literature[51] because the latter actually deals with column vectors made of Cartesian coordinates of true vectors and appears identical to the so called boost coordinate transformation.[53] However, unlike Eq. (4), this transformation implies the special, apparently unphysical, choice of the coordinate systems where the column vectors $-\vec{v}_B^{(A)}$ and $\vec{v}_A^{(B)}$ are equal. Therefore, such a transformation proves to be completely irrelevant to the question under consideration.
In a more compact form, one can write the transformation (8)-(10) as the mapping

\[
\begin{pmatrix}
ct^{(B)} \\
r^{(B)}
\end{pmatrix} \leftrightarrow \mathbf{M}^{(B)}_A \odot \begin{pmatrix}
ct^{(A)} \\
r^{(A)}
\end{pmatrix}
\]

where

\[
\mathbf{M}^{(B)}_A \equiv \begin{pmatrix}
\gamma_{AB} & -\gamma_{AB}v^A_B/c \\
\gamma_{AB}v^B_A/c & 1 - \gamma_{AB}v^B_A \otimes v^A_B/c^2
\end{pmatrix},
\]

the symbol \(\leftrightarrow\) unites the meaning of \(=\) and the meaning of \(\sim\), the symbol \(\odot\) unites the meaning of the usual product of two numbers and the meaning of the dot product of two spatial vectors, the symbol \(\otimes\) denotes the dyadic (outer) product.

As soon as one shows that Eq. (14) is an equivalence relation, one arrives at a spacetime point \(\rho\) as an equivalence class of columns \(\begin{pmatrix} ct \\ r \end{pmatrix}\) that indicate to that spacetime point in their spaces. Then the set of all \(\rho\) makes Minkowski space.

### III. PROOF OF EXISTENCE OF MINKOWSKI SPACE

#### A. What requires a verification

The mapping (14) is evidently reflexive. To show its symmetry, one could resolve Eqs. (8) and (9) with respect to \(t^{(A)}\) and \((v^A_B \cdot r^{(A)})\), taking Eq. (11) into account. However, this action would be redundant since the symmetry of exchanging two spaces is physically evident and should be considered as a premise rather than an inference. Thus, to show that the mapping (14) is an equivalence relation, one need verifying only its transitivity.

Although Eq. (7) may tempt someone to present the transitivity of (14) as

\[
\mathbf{M}^{(B)}_A = \mathbf{M}^{(B)}_G \odot \mathbf{M}^{(G)}_A,
\]

it actually means that

\[
t^{(B)} = \gamma_{BG} \left[ t^{(G)} - \frac{\left( v^G_B \cdot r^{(G)} \right)}{c^2} \right],
\]

\[
-\frac{\left( v^B_G \cdot r^{(B)} \right)}{v_{BG}} = \gamma_{BG} \left[ \frac{\left( v^G_B \cdot r^{(G)} \right)}{v_{BG}} - v_{BG}t^{(G)} \right],
\]
\[ \mathbf{r}^{(B)} - \frac{(\mathbf{r}^{(B)} \cdot \mathbf{v}_G^{(B)}) \mathbf{v}_G^{(B)}}{v_{BG}^2} \sim \mathbf{r}^{(G)} - \frac{(\mathbf{r}^{(G)} \cdot \mathbf{v}_B^{(G)}) \mathbf{v}_B^{(G)}}{v_{BG}^2} \]  

(19)

and

\[ t^{(G)} = \gamma_{AG} \left[ t^{(A)} - \frac{(\mathbf{v}_G^{(A)} \cdot \mathbf{r}^{(A)})}{c^2} \right], \]

(20)

\[ -\frac{(\mathbf{v}_A^{(G)} \cdot \mathbf{r}^{(G)})}{v_{AG}} = \gamma_{AG} \left[ \frac{(\mathbf{v}_G^{(A)} \cdot \mathbf{r}^{(A)})}{v_{AG}} - v_{AG} t^{(A)} \right], \]

(21)

\[ \mathbf{r}^{(G)} - \frac{(\mathbf{r}^{(G)} \cdot \mathbf{v}_A^{(G)}) \mathbf{v}_A^{(G)}}{v_{AG}^2} \sim \mathbf{r}^{(A)} - \frac{(\mathbf{r}^{(A)} \cdot \mathbf{v}_G^{(A)}) \mathbf{v}_G^{(A)}}{v_{AG}^2} \]

(22)

entail Eqs. (8)-(10).

B. Auxiliary relationships for the velocities of spaces

In accordance with the definition of the velocities \( \mathbf{v}_G^{(A)} \) and \( \mathbf{v}_G^{(B)} \), the motion \( \mathbf{r}^{(G)} = 0 \) of a reference point in the space G is observed as the motion \( \mathbf{r}^{(A)} = \mathbf{v}_G^{(A)} t^{(A)} \) in the space A and as the motion \( \mathbf{r}^{(B)} = \mathbf{v}_G^{(B)} t^{(B)} \) in the space B. With applying the transformation (8)-(10) to this set of events, one finds

\[ t^{(B)} = \gamma_{AB} \left[ 1 - \frac{(\mathbf{v}_B^{(B)} \cdot \mathbf{v}_G^{(B)})}{c^2} \right] t^{(A)}, \]

(23)

\[ -\frac{(\mathbf{v}_A^{(B)} \cdot \mathbf{v}_G^{(B)})}{v_{AB}} t^{(B)} = \gamma_{AB} \left[ \frac{(\mathbf{v}_B^{(B)} \cdot \mathbf{v}_G^{(B)})}{v_{AB}} - v_{AB} \right] t^{(A)}, \]

(24)

\[ \left[ \mathbf{v}_G^{(B)} - \frac{(\mathbf{v}_G^{(B)} \cdot \mathbf{v}_A^{(B)}) \mathbf{v}_A^{(B)}}{v_{AB}^2} \right] t^{(B)} \sim \left[ \mathbf{v}_G^{(A)} - \frac{(\mathbf{v}_G^{(A)} \cdot \mathbf{v}_B^{(A)}) \mathbf{v}_B^{(A)}}{v_{AB}^2} \right] t^{(A)}. \]

(25)

Transposing \( A \leftrightarrow B \) in Eq. (23) (which means the use of the transformation inverse to (8)-(10) in the above calculation) yields

\[ t^{(A)} = \gamma_{AB} \left[ 1 - \frac{(\mathbf{v}_A^{(B)} \cdot \mathbf{v}_G^{(B)})}{c^2} \right] t^{(B)}. \]

(26)

Then one can eliminate \( t^{(A)} \) and \( t^{(B)} \) in Eq. (23) and Eq. (26) to obtain

\[ 1 = \gamma_{AB}^2 \left[ 1 - \frac{(\mathbf{v}_B^{(B)} \cdot \mathbf{v}_G^{(B)})}{c^2} \right] \left[ 1 - \frac{(\mathbf{v}_A^{(B)} \cdot \mathbf{v}_G^{(B)})}{c^2} \right]. \]

(27)
With transposing $B \leftrightarrow G$ and $A \leftrightarrow G$ in Eq. (27), one can also find

\[
1 = \gamma_{AG}^2 \left[ 1 - \frac{(v_{B}^{(A)} \cdot v_{G}^{(A)})}{c^2} \right] \left[ 1 - \frac{(v_{B}^{(G)} \cdot v_{A}^{(G)})}{c^2} \right]
\]

(28)

and

\[
1 = \gamma_{BG}^2 \left[ 1 - \frac{(v_{A}^{(B)} \cdot v_{G}^{(B)})}{c^2} \right] \left[ 1 - \frac{(v_{B}^{(G)} \cdot v_{A}^{(G)})}{c^2} \right].
\]

(29)

In a similar manner, one can start from Eq. (23) and Eq. (24) to go to the relationships

\[
\frac{(v_{B}^{(A)} \cdot v_{G}^{(A)})}{v_{AG}^2} \left[ 1 - \frac{(v_{B}^{(G)} \cdot v_{A}^{(G)})}{c^2} \right] = 1 - \frac{(v_{B}^{(G)} \cdot v_{A}^{(G)})}{v_{AG}^2},
\]

(30)

\[
\frac{(v_{B}^{(A)} \cdot v_{G}^{(A)})}{v_{AB}^2} \left[ 1 - \frac{(v_{B}^{(G)} \cdot v_{A}^{(G)})}{c^2} \right] = 1 - \frac{(v_{B}^{(G)} \cdot v_{A}^{(G)})}{v_{AB}^2},
\]

(31)

\[
\frac{(v_{A}^{(B)} \cdot v_{G}^{(B)})}{v_{BG}^2} \left[ 1 - \frac{(v_{B}^{(G)} \cdot v_{A}^{(G)})}{c^2} \right] = 1 - \frac{(v_{B}^{(G)} \cdot v_{A}^{(G)})}{v_{BG}^2}.
\]

(32)

It is easy see that Eq. (23) and Eq. (25) entail

\[
v_{G}^{(B)} = \frac{(v_{G}^{(B)} \cdot v_{A}^{(B)})}{v_{AB}^2} v_{A}^{(B)} - \frac{(v_{G}^{(A)} \cdot v_{B}^{(A)})}{\gamma_{AB} v_{AB}^2} v_{B}^{(A)}
\]

(33)

In addition, transposing $B \leftrightarrow G$ turns Eq. (33) into

\[
v_{B}^{(G)} = \frac{(v_{B}^{(G)} \cdot v_{A}^{(G)})}{v_{AG}^2} v_{A}^{(G)} - \frac{(v_{B}^{(A)} \cdot v_{G}^{(A)})}{\gamma_{AG} v_{AG}^2} v_{G}^{(A)}
\]

(34)

Combining Eqs. (27)-(29) yields

\[
\gamma_{AB} = \gamma_{AG} \gamma_{BG} \left[ 1 - \frac{(v_{B}^{(G)} \cdot v_{A}^{(G)})}{c^2} \right].
\]

(35)

There are the identities possible due to the symmetry of exchanging the spaces: Since transposing any two of $A$ and $B$ and $G$ does not change the r.h.s. of the equation

\[
\gamma_{AG}^{-2} \left[ 1 - \frac{(v_{G}^{(B)} \cdot v_{A}^{(B)})}{c^2} \right] = \frac{1}{\gamma_{AB} \gamma_{AG} \gamma_{BG}}.
\]
it entails the identities
\[
\gamma_{AG}^{-2} \left[ 1 - \frac{\mathbf{v}_G^{(B)} \cdot \mathbf{v}_A^{(B)}}{c^2} \right] = \gamma_{AB}^{-2} \left[ 1 - \frac{\mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)}}{c^2} \right] \tag{36}
\]
and
\[
\gamma_{AG}^{-2} \left[ 1 - \frac{\mathbf{v}_G^{(B)} \cdot \mathbf{v}_A^{(A)}}{c^2} \right] = \gamma_{GB}^{-2} \left[ 1 - \frac{\mathbf{v}_B^{(A)} \cdot \mathbf{v}_G^{(G)}}{c^2} \right]. \tag{37}
\]

Similarly, the equation
\[
\frac{v_{AG}^2 v_{BG}^2 - \left( \mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)} \right)^2}{\gamma_{AB}^2 c^4} = \frac{2}{\gamma_{AB} \gamma_{AG} \gamma_{BG}} \left( \frac{1}{\gamma_{AG}^2} + \frac{1}{\gamma_{AB} \gamma_{BG}} + \frac{1}{\gamma_{BG}^2} \right) + \frac{1}{\gamma_{AB} \gamma_{AG} \gamma_{BG}}
\]
produces
\[
\frac{v_{AB}^2 v_{BG}^2 - \left( \mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)} \right)^2}{\gamma_{AB}^2} = \frac{v_{AB}^2 v_{BG}^2 - \left( \mathbf{v}_G^{(G)} \cdot \mathbf{v}_A^{(G)} \right)^2}{\gamma_{AG}^2} \tag{38}
\]
and
\[
\frac{v_{AB}^2 v_{AG}^2 - \left( \mathbf{v}_G^{(G)} \cdot \mathbf{v}_A^{(G)} \right)^2}{\gamma_{AG}^2} = \frac{v_{AB}^2 v_{AG}^2 - \left( \mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)} \right)^2}{\gamma_{BG}^2}. \tag{39}
\]

C. Auxiliary expression for a longitudinal length

The transformation rule (18) presents a length \( \left( \mathbf{v}_B^{(G)} \cdot \mathbf{r}^{(G)} \right) / v_{BG} \) along the direction of \( \mathbf{v}_B^{(G)} \) in terms of quantities defined in a space G, associated with that direction. However, the calculations in the following sections require to express a length along a given boost direction via quantities defined in an arbitrary third space. In other words, one needs to express the dot product \( \left( \mathbf{v}_B^{(G)} \cdot \mathbf{r}^{(G)} \right) \) in the space A.

To do the required calculation, one cannot but address the decompositions
\[
\mathbf{r}^{(G)} = \mathbf{r}^{(G)} - \frac{\left( \mathbf{r}^{(G)} \cdot \mathbf{v}_A^{(G)} \right) \mathbf{v}_A^{(G)}}{v_{AG}^2} + \frac{\left( \mathbf{r}^{(G)} \cdot \mathbf{v}_A^{(G)} \right) \mathbf{v}_A^{(G)}}{v_{AG}^2}
\]
and
\[
\mathbf{v}_B^{(G)} = \mathbf{v}_B^{(G)} - \frac{\left( \mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)} \right) \mathbf{v}_A^{(G)}}{v_{AG}^2} + \frac{\left( \mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)} \right) \mathbf{v}_A^{(G)}}{v_{AG}^2}
\]
and then apply the transformation rule (21), the relations (22) and (34) with an aid of (13):
Here the identities (28) and (30) are also used.

Thus,  

\[
\left( v_B^{(G)} \cdot r^{(G)} \right) =  
\]

\[
\gamma_{AG} \left[ 1 - \frac{\left( v_B^{(G)} \cdot v_A^{(G)} \right)}{c^2} \right] \left( v_B^{(A)} \cdot r^{(A)} \right)  
\]

\[
- \gamma_{AG} \left[ 1 - \frac{\left( v_B^{(G)} \cdot v_A^{(G)} \right)}{c^2} \right] \left( v_A^{(G)} \cdot r^{(A)} \right)  
\]

\[
= \gamma_{AG} \left( v_B^{(G)} \cdot v_A^{(G)} \right) t^{(A)} - \gamma_{AG} \left( v_A^{(A)} \cdot r^{(A)} \right) + \gamma_{AG} \left[ 1 - \frac{\left( v_B^{(G)} \cdot v_A^{(G)} \right)}{c^2} \right] \left( v_B^{(A)} \cdot r^{(A)} \right) \quad (40)  
\]

Transposing \( B \leftrightarrow A \) and then \( G \leftrightarrow B \) turns this equation into

\[
\left( v_A^{(B)} \cdot r^{(B)} \right) =  
\]

\[
\gamma_{BG} \left( v_A^{(B)} \cdot v_G^{(B)} \right) t^{(G)} - \gamma_{BG} \left( v_G^{(G)} \cdot r^{(G)} \right) + \gamma_{BG} \left[ 1 - \frac{\left( v_A^{(B)} \cdot v_G^{(B)} \right)}{c^2} \right] \left( v_A^{(G)} \cdot r^{(G)} \right) \quad (41)  
\]

Due to the transformation rule (21) one has

\[
\left( v_A^{(G)} \cdot r^{(G)} \right) = - \gamma_{AG} \left[ \left( v_A^{(A)} \cdot r^{(A)} \right) - v_{AG}^2 t^{(A)} \right] \quad (42)  
\]

With transposing \( A \leftrightarrow B \), one can obtain

\[
\left( v_B^{(G)} \cdot r^{(G)} \right) = - \gamma_{BG} \left[ \left( v_B^{(B)} \cdot r^{(B)} \right) - v_{BG}^2 t^{(B)} \right] \quad (43)  
\]

and

\[
\left( v_A^{(G)} \cdot r^{(G)} \right) =  
\]
\[ = \gamma_{BG} \left( v_B^{(G)} \cdot v_A^{(G)} \right) t^{(B)} - \gamma_{BG} \left( v_G^{(B)} \cdot r^{(B)} \right) + \gamma_{BG} \left[ 1 - \left( \frac{v_B^{(G)} \cdot v_A^{(G)}}{c^2} \right) \right] \left( r^{(B)} \cdot v_A^{(G)} \right) \]  

(44)

from Eq. (42) and Eq. (40).

Transposing \( G \leftrightarrow B \) turns Eq. (40) into

\[ \left( v_B^{(G)} \cdot r^{(G)} \right) = \]

\[ = \gamma_{AB} \left( v_B^{(B)} \cdot v_A^{(B)} \right) t^{(A)} + \gamma_{AB} \left[ 1 - \frac{v_G^{(B)} \cdot v_A^{(B)}}{c^2} \right] \left( r^{(A)} \cdot v_A^{(A)} \right) - \gamma_{AB} \left( v_B^{(A)} \cdot r^{(A)} \right) \]  

(45)

D. Transitivity for the time transformation rule

To arrive at (8) one should simply combine (17) with (20) and then apply (40):

\[ t^{(B)} = \gamma_{BG} \left[ \gamma_{AG} \left[ t^{(A)} - \frac{(v_G^{(A)} \cdot r^{(A)})}{c^2} \right] - \frac{(v_B^{(G)} \cdot r^{(G)})}{c^2} \right] = \]

\[ = \gamma_{BG} \gamma_{AG} t^{(A)} - \gamma_{BG} \frac{\gamma_{AG} (v_B^{(G)} \cdot v_A^{(G)}) + (v_B^{(G)} \cdot r^{(G)})}{c^2} = \]

\[ = \gamma_{BG} \gamma_{AG} t^{(A)} - \gamma_{BG} \frac{(v_B^{(G)} \cdot v_A^{(G)})}{c^2} \gamma_{AG} \left[ 1 - \frac{(v_B^{(G)} \cdot v_A^{(G)})}{c^2} \right] \left( r^{(A)} \cdot v_A^{(G)} \right) = \]

\[ = \gamma_{AB} t^{(A)} - \gamma_{AB} \frac{r^{(A)} \cdot v_A^{(G)}}{c^2}. \]

Here Eq. (35) is also used.

E. Transitivity for the longitudinal length transformation rule

To obtain (9) one can combine Eq. (41) with the transformation rules (21) and (20):

\[ \left( v_A^{(B)} \cdot r^{(B)} \right) = \left( v_A^{(G)} \cdot r^{(G)} \right) \gamma_{BG} \left[ 1 - \frac{(v_B^{(B)} \cdot v_A^{(B)})}{c^2} \right] - \gamma_{BG} \left( v_B^{(G)} \cdot r^{(G)} \right) + \]

\[ + \gamma_{BG} (v_A^{(G)} \cdot v_B^{(B)}) t^{(G)} = -\gamma_{AG} \left[ (v_G^{(A)} \cdot r^{(A)}) - v_{AG} t^{(A)} \right] \gamma_{BG} \left[ 1 - \frac{(v_B^{(G)} \cdot v_A^{(G)})}{c^2} \right] - \]
\[-\gamma_{BG} \left[ \gamma_{AG} \left( \mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)} \right) \mathbf{r}^{(A)} - \gamma_{AG} \left( \mathbf{v}_G^{(A)} \cdot \mathbf{r}^{(A)} \right) \right] + \gamma_{AG} \left[ 1 - \frac{\left( \mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)} \right)}{c^2} \right] \left( \mathbf{r}^{(A)} \cdot \mathbf{v}_B^{(A)} \right) + \gamma_{BG} \left( \mathbf{v}_A^{(B)} \cdot \mathbf{v}_G^{(B)} \right) \gamma_{AG} \left[ \mathbf{r}^{(A)} - \mathbf{v}_G^{(A)} \cdot \frac{1}{c^2} \left( \mathbf{v}_G^{(A)} \cdot \mathbf{v}_A^{(A)} \right) \right] \right] = -\gamma_{BG} \gamma_{AG} \left[ 1 - \frac{\left( \mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)} \right)}{c^2} \right] \left( \mathbf{r}^{(A)} \cdot \mathbf{v}_B^{(A)} \right) + \gamma_{BG} \gamma_{AG} \left[ \mathbf{r}^{(A)} \cdot \mathbf{v}_G^{(A)} \right] \left[ 1 - \frac{\left( \mathbf{v}_G^{(A)} \cdot \mathbf{v}_A^{(A)} \right)}{c^2} \right] \gamma_{AG} \left( \mathbf{r}^{(A)} \cdot \mathbf{v}_B^{(A)} \right) + \mathbf{r}^{(A)} \gamma_{AB} v_{AB}^2 \]

The last equation uses Eq. (35) and Eq. (36) rewritten as

\[
\left[ 1 - \frac{\left( \mathbf{v}_G^{(B)} \cdot \mathbf{v}_A^{(B)} \right)}{c^2} \right] v_{AG}^2 + \left( \mathbf{v}_G^{(B)} \cdot \mathbf{v}_A^{(B)} \right) = \left[ 1 - \frac{\left( \mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)} \right)}{c^2} \right] v_{AB}^2 + \left( \mathbf{v}_B^{(G)} \cdot \mathbf{v}_A^{(G)} \right) .
\]

**F. Transitivity for the transversal relation**

It remains to show that Eq. (10) follows the transformations (17)-(19) and (20)-(22).

Let us consider the expression

\[
\mathbf{E}^{(G)} = \mathbf{r}^{(G)} + \tau_A \mathbf{v}_A^{(G)} + \tau_B \mathbf{v}_B^{(G)}
\]

where the numbers \(\tau_A\) and \(\tau_B\) secure that

\[
\left( \mathbf{E}^{(G)} \cdot \mathbf{v}_A^{(G)} \right) = 0 \quad (46)
\]

and

\[
\left( \mathbf{E}^{(G)} \cdot \mathbf{v}_B^{(G)} \right) = 0 . \quad (47)
\]

Solving these equations with respect to \(\tau_A\) and \(\tau_B\) yields

\[
\tau_A = \frac{\left( \mathbf{r}^{(G)} \cdot \mathbf{v}_B^{(G)} \right) \left( \mathbf{v}_A^{(G)} \cdot \mathbf{v}_B^{(G)} \right) - \left( \mathbf{r}^{(G)} \cdot \mathbf{v}_A^{(G)} \right) v_{BG}^2}{v_{AG}^2 v_{BG}^2 - \left( \mathbf{v}_A^{(G)} \cdot \mathbf{v}_B^{(G)} \right)^2},
\]

\[
\tau_B = \frac{\left( \mathbf{r}^{(G)} \cdot \mathbf{v}_A^{(G)} \right) \left( \mathbf{v}_A^{(G)} \cdot \mathbf{v}_B^{(G)} \right) - \left( \mathbf{r}^{(G)} \cdot \mathbf{v}_B^{(G)} \right) v_{AG}^2}{v_{AG}^2 v_{BG}^2 - \left( \mathbf{v}_A^{(G)} \cdot \mathbf{v}_B^{(G)} \right)^2}.
\]

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With an aid of the equations (42), (40), (43), (44) one can re-express $\tau_A$ and $\tau_B$ in terms of the variables defined in the spaces A and B:

$$\tau_A = \frac{\gamma_{BG} \left[ 1 - \left( \frac{v_{G}^{(B)} \cdot v_{A}^{(G)}}{c^2} \right) \right] \left[ \left( v_{G}^{(B)} \cdot r^{(B)} \right) \left( v_{A}^{(B)} \cdot v_{G}^{(B)} \right) - \left( v_{A}^{(B)} \cdot r^{(B)} \right) \right] v_{BG}^2}{v_{AG}^2 v_{BG}^2 - \left( v_{A}^{(G)} \cdot v_{B}^{(G)} \right)^2}$$

(48)

$$\tau_B = \frac{\gamma_{AG} \left[ 1 - \left( \frac{v_{G}^{(A)} \cdot v_{A}^{(G)}}{c^2} \right) \right] \left[ \left( v_{G}^{(A)} \cdot r^{(A)} \right) \left( v_{B}^{(A)} \cdot v_{G}^{(A)} \right) - \left( v_{B}^{(A)} \cdot r^{(A)} \right) \right] v_{AG}^2}{v_{AG}^2 v_{BG}^2 - \left( v_{A}^{(G)} \cdot v_{B}^{(G)} \right)^2}$$

(49)

Here the equations (28), (29), (30), (32) are also used.

Due to Eq. (47) one can write

$$E^{(G)} = r^{(G)} - \frac{\left( r^{(G)} \cdot v_{B}^{(G)} \right) v_{B}^{(G)}}{v_{BG}^2} + \tau_A \left[ v_{A}^{(G)} - \frac{\left( v_{A}^{(G)} \cdot v_{B}^{(G)} \right) v_{B}^{(G)}}{v_{BG}^2} \right] \sim$$

$$\sim r^{(B)} - \frac{\left( r^{(B)} \cdot v_{G}^{(B)} \right) v_{G}^{(B)}}{v_{BG}^2} + \tau_A \frac{\left( v_{A}^{(B)} \cdot v_{G}^{(B)} \right) v_{G}^{(B)}}{v_{BG}^2} =$$

$$= r^{(B)} - \frac{\left( r^{(B)} \cdot v_{G}^{(B)} \right) v_{G}^{(B)}}{v_{BG}^2} + \left\{ \frac{v_{A}^{(B)}}{v_{BG}^2} - \frac{\left( v_{A}^{(B)} \cdot v_{G}^{(B)} \right) v_{G}^{(B)}}{v_{BG}^2} \right\} \times$$

$$\times \frac{\left[ \left( v_{G}^{(B)} \cdot r^{(B)} \right) \left( v_{A}^{(B)} \cdot v_{G}^{(B)} \right) - \left( v_{A}^{(B)} \cdot r^{(B)} \right) v_{BG}^2 \right]}{v_{AB}^2 v_{BG}^2 - \left( v_{A}^{(B)} \cdot v_{G}^{(B)} \right)^2}.$$  

The last equation follows Eq. (48) along with Eq. (38).

The above last expression is perpendicular to $v_{A}^{(B)}$. To make it evident one needs to re-arrange the terms only. This yields

$$E^{(G)} \sim r^{(B)} - \frac{\left( r^{(B)} \cdot v_{A}^{(B)} \right) v_{A}^{(B)}}{v_{AB}^2} + \left\{ \frac{v_{A}^{(B)}}{v_{AB}^2} - \frac{\left( v_{A}^{(B)} \cdot v_{G}^{(B)} \right) v_{G}^{(B)}}{v_{AB}^2} \right\} T_B$$

(50)

where

$$T_B = \frac{\left( v_{A}^{(B)} \cdot v_{G}^{(B)} \right) \left( v_{A}^{(B)} \cdot r^{(B)} \right) - v_{AB}^2 \left( v_{G}^{(B)} \cdot r^{(B)} \right)}{v_{AB}^2 v_{BG}^2 - \left( v_{A}^{(B)} \cdot v_{G}^{(B)} \right)^2}.\quad (51)$$
With transposing $A \leftrightarrow B$ in Eq. (50), one can also obtain

$$E^{(G)} \sim r^{(A)} - \left( \frac{r^{(A)} \cdot v_B^{(A)}}{v_{AB}^2} \right) v_B^{(A)} + \left[ v_G^{(A)} - \left( \frac{v_B^{(A)} \cdot v_G^{(A)}}{v_{AB}^2} \right) v_B^{(A)} \right] T_A \quad (52)$$

where

$$T_A = \left( \frac{v_B^{(A)} \cdot v_G^{(A)}}{v_{AB}^2} \right) \left( v_B^{(A)} \cdot r^{(A)} \right) - v_{AB}^2 \left( v_G^{(A)} \cdot r^{(A)} \right)$$

$$v_{AB}^2 - \left( v_B^{(A)} \cdot v_G^{(A)} \right)^2 \quad (53)$$

Meanwhile, Eq. (45) and the transformation rule (9) (already derived in the previous section), with an aid of Eq. (31), entail

$$\left( v_G^{(B)} \cdot v_A^{(B)} \right) \left( v_A^{(B)} \cdot r^{(B)} \right) - v_{AB}^2 \left( v_G^{(B)} \cdot r^{(B)} \right) =$$

$$= \gamma_{AB} v_{AB}^2 \left\{ \left[ 1 \left( \frac{v_G^{(B)} \cdot v_A^{(B)}}{v_{AB}^2} \right) \right] \left( v_B^{(A)} \cdot r^{(A)} \right) - \left[ 1 - \left( \frac{v_G^{(B)} \cdot v_A^{(B)}}{c^2} \right) \right] \left( v_G^{(A)} \cdot r^{(A)} \right) \right\} =$$

$$= \gamma_{AB} \left[ 1 - \left( \frac{v_G^{(B)} \cdot v_A^{(B)}}{c^2} \right) \right] \left[ \left( v_G^{(A)} \cdot v_B^{(A)} \right) \left( v_B^{(A)} \cdot r^{(A)} \right) - v_{AB}^2 \left( v_G^{(A)} \cdot r^{(A)} \right) \right] .$$

This relationship and Eq. (39) yield the relation

$$\frac{T_B}{T_A} = \gamma_{AB} \left[ 1 - \left( \frac{v_G^{(B)} \cdot v_A^{(B)}}{c^2} \right) \right] \frac{\gamma_{BG}^2}{\gamma_{AG}^2} \quad (54)$$

between the quantities defined by Eq. (51) and Eq. (53). Then, due to Eq. (37) and Eq. (33), Eq. (54) leads us to the relation

$$\left[ \frac{v_G^{(B)}}{v_{AB}^2} - \left( \frac{v_G^{(A)} \cdot v_B^{(A)}}{v_{AB}^2} \right) v_B^{(A)} \right] T_B \sim \left[ v_G^{(A)} - \left( \frac{v_G^{(A)} \cdot v_B^{(A)}}{v_{AB}^2} \right) v_B^{(A)} \right] T_A ,$$

which, due to Eq. (50) and Eq. (52) and the property (12) results in Eq. (10).

**IV. DISCUSSION**

The proceeding section shows that the transitivity of the transformation (8)-(10) is one of the key points that secure the existence of Minkowski space. The transformation not only
needs the conditions discussed in Section II A but also presupposes the existence of particles able to interact with each other and electromagnetic field over a short spatial range only so that the particles’ motions along with the acts of such interaction explicitly or implicitly underlie the basic effects of the relativity theory.[50, Sec. III]

The above remark suggests that the proof of the existence of Minkowski space in Section III is essentially based on properties of free motion of point particles. But such motion is highly degenerate: an infinite number of initial positions is possible for one trajectory of for a given velocity vector at the place of a given event. Apparently, universal external action, such as gravity, can lower the degree of this degeneracy or even remove it completely.

Then the generalization of Minkowski space is hardly possible, except in the case of high symmetry, such as a spherically symmetric action of gravity. Even if one succeeds in generalizing the concept of physical space to allow for an arbitrary and evolving spatial geometry, perceived by some set of observers, in order to arrive at the full spacetime one has no choice but to postulate the transitivity of the transformation between two sets of observers, which apparently imposes an unnecessary and non-physical restriction.

In addition, it is becoming increasingly clear that the idea of spacetime is consistent with observational data only in conjunction with forced assumptions such as the presence of a considerable amount of unidentified dark matter/energy,[54–56] admittedly exotic, and/or various gravity modifications[57].

Evidently, the reasonable, non-exotic, interpretation of observations needs a theoretical approach as less restricted as possible. Thus, modifying or even relinquishing the concept of spacetime seems quite natural.

V. CONCLUSION

The transformation of the Einsteinian time and Cartesian coordinates between two inertial reference frames does not make it possible to find out whether Minkowski space exists, unless one resorts to calculation tricks.

In contrast, a straightforward, though not simple, calculation shows that the transformation of the time and the position vector of a physical event between two physical spaces establishes an equivalence relation between column vectors made of the time and the position vector of a given event in each space. This means the existence of Minkowski space
and shows that the premises for its proof are the same as for the coordinate-free derivation of basic effects of the special relativity theory: use of the Einsteinian time variable and motions of point particles able to interact with each other and electromagnetic field over a short spatial range only.

The high degeneracy of free motions of point particles, together with the intricacy of the above mentioned calculations, suggests that a further generalization of Minkowski space is beyond belief, so that the modification or even the abandonment of the concept of spacetime seems quite natural.

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[1] See, e.g., Section 11.6 in Ref. 2, Sections 7.4 an 7.5 in Ref. 3, Ch. 1 in Ref. 4, §§2.2-2.4 in Ref. 5, p. 374 in Ref. 6.

[2] John David Jackson, *Classical Electrodynamics*, 3rd edition (Wiley, 1998)

[3] Herbert Goldstein, Charles Poole, and John Safko, *Classical Mechanics*, 3rd edition (Addison-Wesley, 2000)

[4] Edwin F. Taylor, John Archibald Wheeler, *Spacetime Physics*, (Freeman, 1992)

[5] Charles W. Misner, Kip S. Thorne, John Archibald Wheeler, *Gravitation*, (Freeman, 1973)

[6] Walter Greiner, *Classical mechanics. Point particles and relativity*, (Springer, 200)

[7] Gregory L. Naber, *The Geometry of Minkowski Spacetime: An Introduction to the Mathematics of the Special Theory of Relativity*, 2nd edition (Springer, 2012)

[8] Joachim Schröter, *Minkowski Space: The Spacetime of Special Relativity*, (De Gruyter, 2017)

[9] H. Poincaré, “Sur la dynamique de l'électron,” Rendiconti del Circolo matematico di Palermo 21, 129-176 (1906)

[10] H. A. Lorentz, “Electromagnetic phenomena in a system moving with any velocity smaller than that of light,” Proceedings of the Royal Netherlands Academy of Arts and Sciences 6, 809-831 (1904)

[11] A. Einstein, “Zur Elektrodynamik der bewegter Körper,” Ann. Phys. 17, 891-921 (1905)

[12] In fact, even in the time of A. Einstein this term has already been used. See, e.g., Ref. 13, p. 33.

[13] J. H. Jeans, *An elementary treatise on theoretical mechanics*, (Ginn&Co., 1907)

[14] See the item 1 at [11, S. 895].
The term 'Lorentz transformation' was accepted by A. Einstein in § 6 of Ref. 16.

A. Einstein, “Principe de relativité et ses conséquences dans la physique modern,” Archives des sciences physiques et naturelles (ser. 4) 29, 5-28, 125-244 (1910)

Here it is also worth noting that the validity domain for Einstein’s approach to the derivation of Lorentz transformation is certainly larger than for those based on Maxwell’s equations or referring to any theory where the electromagnetic quantities appear to be functions of an observer’s time and position in an inertial reference frame. Particularly, due to the quantum properties of the light, Maxwell’s equations as a part of the classical theory are believed to be valid only for the large photon occupation number (see, e.g., §5 in Ref. 18.) So, in contrast to Ref. 9, the presentation in [11, S. 891-892] is applicable for light in the most common phenomena of its generation and propagation, where the average number of photons per mode is typically much less than unity. (See Section 13.1.2 in Ref. 19 for the estimation of the characteristic photon occupation numbers for the thermal optical fields usually encountered in practice.)

Moreover, as early as 1909 Maxwell’s equations have already been perceived as preserving their form with respect to the wider group of transformations than that of Lorentz transformations[20, 21]. So these equations appear unable to serve as a sole premise in the derivation of Lorentz transformation.

V. B. Berestetskii, E. M. Lifshitz and L. P. Pitaevskii, Quantum Electrodynamics. Course of Theoretical Physics, vol. 4, 2nd edition, (Pergamon Press, 1982)

Leonard Mandel and Emil Wolf, Optical coherence and quantum optics, (Cambridge University Press, 1995)

E. Cunningham, “The Principle of Relativity in Electrodynamics and an Extension Thereof,” Proceedings of the London Mathematical Society 8, 77-98 (1910)

H. Bateman, “The Transformation of Electrodynamical Equations,” Proceedings of the London Mathematical Society 8, 223-264 (1910)

Hermann Minkowski, “Die Grundgleichungen für die elektromagnetischen Vorgänge in bewegten Körpern,” Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse, S. 53-111 (Weidmannsche Buchhandlung, 1908)

A. Einstein und M. Grossmann, “Entwurf einer verallgemeinerten Relativitätstheorie und Theorie der Gravitation,” Zeitschrift für Mathematik und Physik 62, 225-244, 245-261, (1913)
This similarity was later acknowledged by A. Einstein: see the end of §8 in Ref. 16 and the of Ref. 25.

A. Einstein, “Die Relativitätstheorie,” Naturforschende Gesellschaft, Zurich, Vierteljahresschrift 56, 1-14 (1911)

In another lecture[27], H. Minkowski used the more emphatic term 'worldpoint’. This lecture makes it clear that, as a non-physicist, H. Minkowski has completely ignored the fact that the actual underlying basis of his theory, the set of Maxwell’s equations, is not a collection of irrefutable experimental/empirical data but a combination of mathematical concepts, each of which has its own validity domain in physics.

Hermann Minkowski, “Raum und Zeit,” Jahresberichte der Deutschen Mathematiker-Vereinigung, S. 75-88 (1909)

See Ref. 29 and Ref. 30.

M. Pieri, “Della geometria elementare come sistema ipotetico deduttivo: Monografia del pinto e del moto,” Memorie della Reale Accademia delle Scienze di Torino (series 2) 49, 173-222 (1900)

J. Hjelmslev, “Neue Begründung der ebenen Geometrie,” Mathematische Annalen 64, 449-474 (1907)

Joseph A. Gallian, Contemporary Abstract Algebra, 7th edition (Brooks/Cole, 2010)

It is worthwhile for the reader to note that this relationship, equivalent to the principle of constancy of the speed of light or the principle of relativity applied to Maxwell’s equations, is among high-level abstractions of macroscopic physics while the ideas of emitting, reflecting and absorbing of light can also be among lower-level generalizations (such as classifications) of microscopic physics, which are more or less directly inferable from experimental data. The resulting disparate definition of time is still in use because up to now there is no fully consistent theoretical description of exchanging a portion of energy between particles and electromagnetic field.

Physicists took the ideas of H. Minkowski as 'a remarkable and instructive graphic representation of the Lorentz transformation’[34, p. 129], 'a four-dimensional method of expressing the results of the Einstein theory of relativity’[35] and viewed 'absolute world' in Ref. 27 as a room for mathematical objects invariant with respect to Lorentz transformations (along with transformations induced by Lorentz ones) where there is 'a natural generalization of the
ordinary vector and tensor calculus for a four-dimensional manifold.\[36, p. 22\]

34. L. Silberstein, *The Theory of Relativity*, (Macmillan and Co., 1914)

35. Richard C. Tolman, *The Theory of the Relativity of Motion*, (University of California Press, 1917)

36. W. Pauli, *The Theory of Relativity*, (Pergamon Press, 1958)

37. One can, for example, read 'Damit gewinnen die Weltgeometrie: die Welt ist eine vierdimensionale affine Mannigfaltigkeit im Since von Kap. I, behaftet mit einer Metrik, die auf einer quadratischen Grundform (\(\rho \cdot \rho\)) mit drei positiven und einer negativen Dimension beruht.'[38, S. 170] or 'Die Welt ist eine vierdimensionale affine Mannigfaltigkeit, ihr Element ist der Weltpunkt.'[39, S. 262] For earlier references see Sections 3.1 and 3.2 in Ref. 40 and Ref. 41. See also the remark in Ref. 42.

38. Hermann Weyl, *Raum · Zeit · Materie: Vorlesungen über allgemeine Relativitätstheorie*, 5. Auflage (Springer,1923)

39. Max Born, *Die Relativitätstheorie Einsteins. Kommentiert und eweitert von Jürgen Ehlers und Markus Pössel*, 7. Auflage (Spinger, 2003)

40. Scott A. Walter, “Minkowski, mathematicians and the mathematical theory of relativity,” in *The Expanding Worlds of General Relativity*, edited by H. Goenner, J. Renn, and T. Sauer (Birkhäuser, 1999), pp. 45-86.

41. Scott A. Walter, “Hermann Minkowski and the scandal of spacetime,” ESI News, 3 (1), 6-8 (2008)

42. Axiomatic descriptions of Minkowski space have been among the conspicuously vacuous manifestations of non-physicists’ activity in the relativity theory; see, e.g., Ref. 43 and references in its Section 1.4. Since such descriptions mingle physically meaningful statements with those that follow the non-physical patterns of the historical axiomatic systems for Euclidean space, they are completely useless for physics.

43. John W. Schultz, *Independent axioms for Minkowski space-time*, (Longman, 1997)

44. Someone may object that the way of partitioning the observers can affect the result. An objection of this kind along with those to Einstein’s extension for the concept of time at p. 5 are the delicate issues, which are beyond the scope of this article.

45. C. Møller, *The Theory of Relativity*, (Oxford University Press, 1955)

46. See. e.g., Eq. (7.13) in Ref. 3.
[47] See, e.g., Eq. (11.98) in Ref. 2.

[48] See Eq. (13) in Ref. 49.

[49] James T. Cushing, “Vector Lorentz Transformations,” Am. J. Phys. 35, 858-862 (1967)

[50] Serge A. Wagner, “Relativity free of coordinates,” <http://arxiv.org/abs/1611.08018>

[51] See, e.g., the footnote 24 at pp. 10 and 11 in Ref. 36 and the solution to Problem 1-3 at pp. 8 and 9 in Ref. 52.

[52] R. Hagedorn, *Relativistic Kinematics. A Guide to Kinematic Problems of High-energy Physics*, (W. A. Benjamin, Inc., 1964)

[53] See Ref. 49 along with, e.g., Eq. (11.98) at p. 547 in Ref. 2.

[54] C. L. Bennet et al., “Nine-year Wilkinson Microwave Anisotropy Probe (WMAP) observation: final maps and results,” ApJS 208, 20 (2013)

[55] Planck Collaboration: P. A. R. Ade et al. “Planck 2013 results – I. Overview of products and scientific results.” A&A 571, A1 (2014)

[56] Planck Collaboration: P. A. R. Ade et al. “Planck 2015 results – XIII. Cosmological parameters.” A&A 594, A13 (2016)

[57] Planck Collaboration: P. A. R. Ade et al. “Planck 2015 results – XIV. Dark energy and modified gravity.” A&A 594, A14 (2016)