Casimir energy-momentum tensor for the Robin Surfaces in de Sitter Spacetime

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Abstract

The energy-momentum tensor for a massless conformally coupled scalar field in de Sitter spacetime in the presence of a couple curved branes is investigated. We assume that the scalar field satisfies the Robin boundary condition on the surfaces. Static de Sitter space is conformally related to the Rindler space, as a result we can obtain vacuum expectation values of energy-momentum tensor for conformally invariant field in static de Sitter space from the corresponding Rindler counterpart by the conformal transformation.

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1 Introduction

de Sitter (dS) spacetime is the maximally symmetric solution of Einsten’s equation with a positive cosmological constant. Recent astronomical observations of supernovae and cosmic microwave background [1] indicate that the universe is accelerating and can be well approximated by a world with a positive cosmological constant. If the universe would accelerate indefinitely, the standard cosmology leads to an asymptotic dS universe. de Sitter spacetime plays an important role in the inflationary scenario, where an exponentially expanding approximately dS spacetime is employed to solve a number of problems in standard cosmology. The quantum field theory on dS spacetime is also of considerable interest. In particular, the inhomogeneities generated by fluctuations of a quantum field during inflation provide an attractive mechanism for the structure formation in the universe. Another motivation for investigations of dS based quantum theories is related to the recently proposed holographic duality between quantum gravity on dS spacetime and a quantum field theory living on boundary identified with the timelike infinity of dS spacetime [2].

The Casimir effect is regarded as one of the most striking manifestation of vacuum fluctuations in quantum field theory. The presence of reflecting boundaries alters the zero-point modes of a quantized field, and results in the shifts in the vacuum expectation values of quantities quadratic in the field, such as the energy density and stresses. In particular, vacuum forces arise acting on constraining boundaries. The particular features of these forces depend on the nature of the quantum field, the type of spacetime manifold and its dimensionality, the boundary geometries and the specific boundary conditions imposed on the field. Since the original work by Casimir in 1948 [3] many theoretical and experimental works have been done on this problem (see, e.g., [4, 5, 6, 7, 8, 9] and references therein).

The Casimir effect can be viewed as a polarization of vacuum by boundary conditions. Another type of vacuum polarization arises in the case of an external gravitational fields [10, 11]. Casimir stress for parallel plates in the background of static domain wall in four and two dimensions is calculated in [12, 13]. Casimir stress on spherical bubbles immersed in different de Sitter spaces in- and out-side is calculated in [14, 15].

It is well known that the vacuum state for an uniformly accelerated observer, the Fulling–Rindler vacuum [16, 17, 18, 19, 20], turns out to be inequivalent to that for an inertial observer, the familiar Minkowski vacuum. Quantum field theory in accelerated systems contains many of special features produced by a gravitational field avoiding some of the difficulties entailed by renormalization in a curved spacetime. In particular, near the canonical horizon in the gravitational field, a static spacetime may be regarded as a Rindler–like spacetime. Rindler space is conformally related to the de Sitter space and to the Robertson–Walker space with negative spatial curvature. As a result the expectation values of the energy–momentum tensor for a conformally invariant field and for corresponding conformally transformed boundaries on the de Sitter and Robertson–Walker backgrounds can be derived from the corresponding Rindler counterpart by the standard transformation [10]. Vacuum expectation values of the energy-momentum tensor for the conformally coupled Dirichlet and Neumann massless scalar and electromagnetic fields in four dimensional Rindler spacetime was considered by Candelas and Deutsch [21].

In this paper the vacuum expectation value of the energy-momentum tensor are investigated for a massless conformally coupled scalar field obeying the Robin boundary condi-
tions on two curved surfaces in de Sitter spacetime. Robin type conditions are an extension of Dirichlet and Neumann boundary conditions and appear in variety of situations, for example the casimir effect for massless scalar field with Robin boundary conditions on two parallel plates in de Sitter spacetime is calculated in [22], the Robin type boundary condition in domain wall formation is investigated in [23]. In Ref.[24] the vacuum expectation value of the surface energy-momentum tensor is evaluated for a massless scalar field obeying a Robin boundary condition on an infinite plane moving by uniform proper acceleration through Fulling-Rindler vacuum. By using the conformal relation between the Rindler and de Sitter spacetime, in Ref.[25] the vacuum energy-momentum tensor for a scalar field is evaluated in de Sitter spacetime in presence of a curved brane on which the field obeys the Robin boundary condition with coordinate dependent coefficients. Robin boundary conditions naturally arise for scalar and fermion bulk fields in the Randall-Sundrum model [26, 27, 28].

2 Vacuum expectation values for the energy-momentum tensor

We will consider a conformally coupled massless scalar field \( \varphi(x) \) satisfying the following equation

\[
\left( \nabla_{\mu} \nabla^{\mu} + \frac{1}{6} R \right) \varphi(x) = 0,
\]

on the background of de Sitter space-time in static coordinates. In Eq. (1) \( \nabla_{\mu} \) is the operator of the covariant derivative, and \( R \) is the Ricci scalar for the de Sitter space.

\[
R = \frac{12}{\alpha^2}.
\]

The static form of de Sitter space time which is conformally related to the Rindler space time is given by [10]

\[
ds_{\text{dS}}^2 = [1 - (\frac{r^2}{\alpha^2})]dt^2 - [1 - (\frac{r^2}{\alpha^2})]^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta d\phi^2),
\]

To make maximum use of the flat spacetime calculations, first of all let us present the dS line element in the form conformally related to the Rindler metric. With this aim we make the coordinate transformation \( x^i \rightarrow x'^i = (\tau, \xi, y, z) \) [25]

Let us denote the energy–momentum tensor in coordinates \((\tau, \xi, y, z)\) as \( T'_{\mu\nu} \), and the same tensor in coordinates \((t, r, \theta, \phi)\) as \( T_{\mu\nu} \). The tensor \( T_{\mu\nu} \) has the structure

\[
T_{\mu\nu} = \text{diag}(\varepsilon, -p, -p_{\perp}, p_{\perp}).
\]

Our main interest in the present paper is to investigate the vacuum expectation values (VEV’s) of the energy–momentum tensor (EMT) for the field \( \varphi(x) \) in the background of the above de Sitter space time induced by two curved surfaces. We will let the surfaces \( \xi = a \) and \( \xi = b, b > a \) represent the trajectories of these boundaries for corresponding problem in Rindler spacetime. We will consider the case of a scalar field satisfying Robin boundary condition on the surface of the plates, therefore in the Rindler space we have

\[
(A_{R}^{(j)} + B_{R}^{(j)} n_{R}^{l} \nabla_{l}) \varphi|_{\xi=j} = 0, \quad j = a, b \quad n_{R}^{l} = \delta_{1}^{l}, \quad n_{R_{b}}^{l} = -\delta_{2}^{l}
\]
with constant coefficients $A^{(j)}_R$ and $B^{(j)}_R$. In coordinates $x^i$ the boundary $\xi = j$, $j = a, b$ are presented by the surfaces
\[ \sqrt{\alpha^2 - r^2} = j(1 - \frac{r}{\alpha} \cos \theta), \] (6)
in dS spacetime. The boundary condition (5) in de Sitter space is as following
\[ (A^{(j)} + B^{(j)} n^l_j \nabla_l) \varphi|_{S_j} = 0, \quad j = a, b \] (7)
where $S_j$ is given by the surfaces (6).

The presence of boundaries modifies the spectrum of the zero–point fluctuations compared to the case without boundaries. This results in the shift in the VEV’s of the physical quantities, such as vacuum energy density and stresses. This is the well known Casimir effect. It can be shown that for a conformally coupled scalar by using field equation (1) the expression for the energy–momentum tensor can be presented in the form
\[ T_{\mu\nu} = \nabla_\mu \varphi \nabla_\nu \varphi - \frac{1}{6} \left[ \frac{g_{\mu\nu}}{2} \nabla^\rho \nabla^\sigma + \nabla_\mu \nabla_\nu + R_{\mu\nu} \right] \varphi^2, \] (8)
where $R_{\mu\nu}$ is the Ricci tensor. The quantization of a scalar filed on background of metric Eq.(3) is standard. Let $\{ \varphi_\alpha(x), \varphi_\alpha^*(x) \}$ be a complete set of orthonormalized positive and negative frequency solutions to the field equation (1), obeying boundary condition (7). By expanding the field operator over these eigenfunctions, using the standard commutation rules and the definition of the vacuum state for the vacuum expectation values of the energy-momentum tensor one obtains
\[ \langle 0 | T_{\mu\nu}(x) | 0 \rangle = \sum_\alpha T_{\mu\nu}\{\varphi_\alpha, \varphi_\alpha^*\}, \] (9)
where $| 0 \rangle$ is the amplitude for the corresponding vacuum state, and the bilinear form $T_{\mu\nu}\{\varphi, \psi\}$ on the right is determined by the classical energy-momentum tensor (8). Instead of evaluating Eq. (9) directly on background of the curved metric, the vacuum expectation values can be obtained from the corresponding Rindler space time results for a scalar field $\varphi$ by using the conformal properties of the problem under consideration. Under the conformal transformation $g_{\mu\nu} = \Omega^2 \bar{g}_{\mu\nu}$ the $\bar{\varphi}$ field will change by the rule
\[ \varphi(x) = \Omega^{-1} \bar{\varphi}(x), \] (10)
where for metric Eq.(3) the conformal factor is given by $\Omega = \sqrt{\alpha^2 - r^2}/\xi$. Now by comparing boundary conditions (5) and (7) and taking into account Eq.(10), one obtains the relation between the coefficients in the boundary conditions:
\[ \begin{align} A^i &= \frac{1}{\Omega} \left( A^i_R + B^i_R n^l_j \nabla_l \Omega \right), \quad B^j = B^j_R, \quad x \in S_j. \end{align} \] (11)
To evaluate the expression $n^l_j \nabla_l \Omega$ we need the components of the normal to $S_j$ in coordinates $x^i$. They can be found by transforming the components $n^l_a = \frac{\delta^l}{\Omega}$ and $n^l_b = -\frac{\delta^l}{\Omega}$ in coordinates $x^i$:
\[ n^l_j = \left( 0, \pm \frac{j}{\alpha} (\cos \theta - r/\alpha), \mp \frac{j}{\alpha r} \sin \theta, 0 \right), \] (12)
where upper sign refer to \( j = a \) and lower sign refer to \( j = b \). Now it can be easily seen that \( n^i \nabla_i \Omega = \mp \sqrt{\alpha^2 - r^2}/\alpha^2 \) and, hence, the relation between the Robin coefficients in the Rindler and dS problems takes the form

\[
A^i = \frac{j A^i_j}{\sqrt{\alpha^2 - r^2}} \mp \frac{j B^i_j}{\alpha^2}, \quad B^i = B^i_R.
\]

The Casimir effect with boundary conditions (5) on two parallel plates on background of the Rindler spacetime is investigated in Ref. [29] for a scalar field with a Robin boundary condition. The boundary part of vacuum expectation value of energy-momentum tensor is as

\[
(13)
\]

\[
<0 \left| T^k_i \right| 0 >^B = \sum_{j=a,b} <T^k_i >^j + <T^k_i >^{ab}
\]

where (no summation over \( i \)) \( <T^k_i >^{ab} \) is the interference term, and \( <T^k_i >^j \) are induced by the presence of a single plane boundaries located at \( \xi = a \) and \( \xi = b \) in the regions \( \xi > a \) and \( \xi < b \). The VEV for the Fulling-Rindler vacuum without boundaries are given by \( \langle 0_R | T^k_i | 0_R \rangle \). All divergences are contained in this purely Fuling-Rindler part. These divergences can be regularized subtracting the corresponding VEV’s for the Minkowskian vacuum. Using Eq.(8), for the corresponding difference between the VEV’s of the EMT we have [21, 30]

\[
\langle T^k_i \rangle^{(R)}_{\text{sub}} = \langle 0_R | T^k_i | 0_R \rangle - \langle 0_M | T^k_i | 0_M \rangle = -\frac{1}{480 \pi^2 \xi^4} \text{diag}(1,-1/3,-1/3,-1/3).
\]

(15)

The vacuum energy-momentum tensor on static de Sitter space Eq.(3) is obtained by the standard transformation law between conformally related problems (see, for instance, [10]) and has the form

\[
(16)
\]

\[
\langle T^\mu_{\nu} [g_{\alpha \beta}] \rangle = \xi^4 (\alpha^2 - r^2)^{-2} \langle T^\mu_{\nu} [g_{\alpha \beta}] \rangle_{\text{Rindler}} + \frac{1}{960 \pi^2 \alpha^4} \delta^\mu_\nu
\]

The expression on the right hand side of the above formula is the stress-energy tensor for de Sitter space without boundaries. The vacuum energy-momentum tensor (16) in the presence of boundaries has the following form

\[
(17)
\]

\[
\langle T^\mu_{\nu} [g_{\alpha \beta}] \rangle_{\text{ren}} = \langle T^\mu_{\nu} [g_{\alpha \beta}] \rangle^{(0)}_{\text{ren}} + \langle T^\mu_{\nu} [g_{\alpha \beta}] \rangle^{(B)}_{\text{ren}}.
\]

Where the first term on the right is the vacuum energy–momentum tensor for the situation without boundaries (gravitational part), and the second one is due to the presence of boundaries. By taking into account Eqs.(15,16) the first term in Eq.(17) can be rewritten as following

\[
(18)
\]

\[
\langle T^\mu_{\nu} [g_{\alpha \beta}] \rangle^{(0)}_{\text{ren}} = -\frac{1}{480 \pi^2} (\alpha^2 - r^2)^{-2} \text{diag}(1,-1/3,-1/3,-1/3) + \frac{1}{960 \pi^2 \alpha^4} \delta^\mu_\nu.
\]

The boundary part in Eq.(17) is related to the corresponding Rindler spacetime boundary part Eq.(14) by the relation

\[
(19)
\]

\[
\langle T^\mu_{\nu} [g_{\alpha \beta}] \rangle^{(B)}_{\text{ren}} = \xi^4 (\alpha^2 - r^2)^{-2} (\sum_{j=a,b} <T^k_i >^j + <T^k_i >^{ab}).
\]
Now we turn to the interaction forces between the surfaces. The vacuum force acting per unit surface of the curved surface at $S_j$ is determined by the $T^1_1$-component of the vacuum EMT at this point. The gravitational part of the pressure according to Eq.(18) is equal to

$$P_g = -\langle T^1_1 \rangle = \frac{-1}{960\pi^2\alpha^4} - \frac{1}{1440\pi^2}(\alpha^2 - r^2)^{-2}. \quad (20)$$

The first term is the same from both sides of the surfaces, and hence leads to zero effective force. The second term is negative, then this force is attractive always. But for infinitely thin surfaces the second term also is the same from the both sides of the boundaries and, hence, give the zero effective force. The corresponding effective boundary part pressures can be presented as a sum of two terms

$$p^{(j)}_B = p^{(j)}_{B1} + p^{(j)}_{B(int)}, \quad j = a, b. \quad (21)$$

The first term on the right is the pressure for a single plate at $\xi = j$ when the second plate is absent. This term is divergent due to the well known surface divergences in the subtracted VEV’s. The second term on the right of Eq. (21),

$$p^{(j)}_{B(int)} = -\xi^4(\alpha^2 - r^2)^{-2}[\langle T^1_1 \rangle^{(l)} + \langle T^1_1 \rangle^{(ab)}]_{\xi=j} \quad (22)$$

with $j, l = a, b, l \neq j$, is the pressure induced by the presence of the second plate, and can be termed as an interaction force. In dependence of the values for the coefficients in the boundary conditions, the effective pressures (22) can be either positive or negative, leading to repulsive or attractive forces. For Dirichlet or Neumann boundary conditions on both plates the interaction forces are always attractive [20].

### 3 Conclusion

In the present paper we have investigated the Casimir effect for a conformally coupled massless scalar field in the presence of a couple curved branes, on background of the de Sitter spacetimes which is conformally related to the Rindler spacetime. We have assumed that the scalar field satisfies Robin boundary condition on the surfaces. A plane in Rindler spacetime does not correspond to a plane in de Sitter spacetime, but to a curved surface. Likewise, Robin boundary condition in Rindler spacetime corresponds to a different type of boundary condition in de Sitter spacetime, it has the same form of Robin boundary condition but instead of a constant, the coefficient $A^j$ in Eq.(7) is a function of the coordinates. The vacuum expectation values of the energy-momentum tensor are derived from the corresponding Rindler spacetime results by using the conformal properties of the problem. As the boundaries are static in the Rindler coordinates no Rindler quanta are created [20]. In the region between the surfaces the boundary induced part for the vacuum energy-momentum tensor is given by Eq.(19), and the corresponding vacuum forces acting per unit surface of the curved surfaces have the form Eq.(21). These forces are presented as the sums of two terms. The first ones correspond to the forces acting on a single boundary then the second boundary is absent. The vacuum polarization due to the gravitational field, without any boundary conditions is given by Eq.(18), the corresponding gravitational pressure part has the form Eq.(20), the first term in this equation is the same from both sides of the surfaces, and hence leads to zero effective force. The second gravitational term
is negative, then this force is attractive always, in contrast the effective boundary pressures (22) can be either positive or negative, leading to repulsive or attractive forces. Therefore may be in some special cases the attractive gravitational forces can cancel the effective boundary pressures. But for infinitely thin surfaces the second gravitational term also is the same from the both sides of the boundaries and, hence, give the zero effective force. In this case only the boundary part remain, where for Dirichlet or Neumann boundary conditions on both plates the interaction forces are always attractive [20]. But for Robin boundary condition, in dependence of the values for the coefficients in the boundary conditions, the effective pressures can be either positive or negative, leading to repulsive or attractive forces.

References

[1] A. G. Riess et al., Astron. J. 116, 1009 (1998); S. Perlmutter et al., Astrophys. J. 517, 565 (1999); P. de Bernardis et al., Nature 404, 955 (2000); C. L. Bennett et al., Astrophys. J. Suppl. 148, 1 (2003); M. Tegmark et al., astro-ph/0310723.

[2] A. Strominger, JHEP 0110, 034 (2001); 0110, 049 (2001).

[3] H. B. G. Casimir, Proc. K. Ned. Akad. Wet. 51, 793 (1948).

[4] V. M. Mostepanenko and N. N. Trunov, The Casimir Effect and Its Applications (Clarendon, Oxford, 1997).

[5] G. Plunien, B. Muller, and W. Greiner, Phys. Rep. 134, 87 (1986).

[6] S. K. Lamoreaux, Am. J. Phys. 67, 850 (1999).

[7] M. Bordag, U. Mohidden, and V. M. Mostepanenko, Phys. Rep. 353, 1 (2001).

[8] K. Kirsten, Spectral functions in Mathematics and Physics. CRC Press, Boca Raton, 2001.

[9] E. Elizalde, S. D. Odintsov, A. Romeo, A. A. Bytsenko and S. Zerbini, zeta regularization techniques with applications(World Scientific, Singapore, 1994).

[10] N. D. Birrel and P. C. W. Davies, Quantum Fields in Curved Space (Cambridge: Cambridge University Press, 1982).

[11] A. A. Grib, S. G. Mamayev, and V. M. Mostepanenko, Vacuum Quantum Effects in Strong Fields (St. Petersburg, 1994).

[12] M. R. Setare and A. A. Saharian. Int. J. Mod. Phys. A16, 1463(2001).

[13] M. R. Setare and A. H. Rezaeian. Mod. Phys. Lett. A15, 2159(2000).

[14] M. R. Setare and R. Mansouri. Class. Quant. Grav. 18 (2001) 2331.

[15] M. R. Setare .Class. Quant. Grav. 18 (2001) 4823-4830.

[16] S. A. Fulling, Phys. Rev D7, 2850 (1973).
[17] S. A. Fulling, J. Phys. A: Math. Gen. 10, 917 (1977).
[18] W. G. Unruh, Phys. Rev. D14, 870 (1976).
[19] D. G. Boulware, Phys. Rev. D11, 1404 (1975).
[20] R. M. Avagyan, A. A. Saharian, A. H. Yeranyan, Phys. Rev. D66, 085023, (2002).
[21] P. Candelas and D. Deutsch, Proc. Roy. Soc. Lond. A 354, 79 (1977).
[22] M. R. Setare and R. Mansouri. Class. Quant. Grav. 18, 2659, (2001).
[23] M. R. Setare, Int. J. Mod. Phys. A18, 4285, (2003).
[24] A. A. Saharian, M. R. Setare, Class. Quant. Grav. ,21, 5261, (2004).
[25] A. A. Saharian, M. R. Setare, Phys. Lett. B584, 306, (2004).
[26] A. Flachi and D. J. Toms, Nucl. Phys. B610, 144, (2001).
[27] A. A. Saharian, M. R. Setare, Phys. Lett. B552, 119, (2003).
[28] A. A. Saharian, Nucl. Phys. B712, 196, (2005).
[29] A. A. Saharian, R. M. Avagyan, R. S. Davtyan, hep-th/0504189.
[30] P. Candelas and D. Deutsch, Proc. Roy. Soc. Lond. A362, 78 (1978).