Einstein-Yang-Mills theory:
I. Asymptotic symmetries

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ABSTRACT Asymptotic symmetries of the Einstein-Yang-Mills system with or without cosmological constant are explicitly worked out in a unified manner. In agreement with a recent conjecture, one finds a Virasoro-Kac-Moody type algebra not only in three dimensions but also in the four dimensional asymptotically flat case.
1 Introduction

Even though the first discussions of asymptotic symmetries dealt with four dimensional general relativity, both at null [1, 2, 3] and at spatial infinity [4], most of the recent work was devoted to three dimensions because of the occurrence of a classical central charge [5] that plays a key role in symmetry based explanations [6] of the entropy of the BTZ black hole [7, 8] and in other aspects of the AdS/CFT correspondence (see e.g. [9], chapter 5).

In recent work [10], Strominger suggested to extend the analysis for gravity in four dimensions at null infinity to include Yang-Mills fields and established a relation to field theoretic soft photon and graviton theorems [11]. During these considerations, the symmetry algebra was argued to be of Virasoro-Kac-Moody type.

In this note, we confirm this conjecture. We start by showing that the residual symmetry algebra of a standard gauge choice adapted to the asymptotic analysis of the Einstein-Yang-Mills system is simply the gauge algebra in one dimension lower. The asymptotic symmetry algebra is then obtained by a further reduction that comes from suitable fall-off
conditions on the remaining fields. Details for various standard cases, including the flat case with asymptotics at null infinity, are provided.

At this stage, one might wonder why the enhancement of the $U(1)$ electromagnetic gauge symmetry has not been discussed in previous detailed investigations of the asymptotic properties of the Einstein-Maxwell system \[12, 13, 14\]. With hindsight, the reason is that the focus was on the modifications of the equations of motions and their solutions due to the presence of the electro-magnetic field which had been included through its field strength. It turns out however that a formulation in terms of gauge potentials is required if one wants to discuss action principles and asymptotic symmetries for both gravitational and Yang-Mills type gauge fields in a unified manner.

With the symmetries under control, the next stage is to work out asymptotic solutions. This should be done, for simplicity first in three, and then in four dimensions, along the lines of the detailed analysis of the Einstein-Maxwell system. Once this is done, the symmetry transformations of the fields characterizing asymptotic solutions can be computed. Then one is ready to work out the holographic current algebra, including potential central extensions. These questions will be addressed elsewhere.

2 Gauge structure of the Einstein-Yang-Mills system

The Einstein-Yang-Mills system in $d$ dimensions is described by the action

$$S = \frac{1}{16\pi G} \int d^d x \sqrt{|g|} \left[ R - 2\Lambda - g_{ij} F_{\mu\nu}^i F_{\mu\nu}^j \right], \quad \Lambda = -\frac{(d-1)(d-2)}{2l^2}, \quad (2.1)$$

where $g_{ij}$ is an invariant non-degenerate metric in a basis $T_i$ of the internal gauge algebra $\mathfrak{g}$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ is the field strength, $A_\mu = A^i_\mu T_i$ and the bracket denotes the Lie bracket in $\mathfrak{g}$, $[T_i, T_j] = f^k_{ij} T_k$.

The complete gauge algebra consists of pairs $(\xi, \epsilon)$ of a vector field $\xi^\mu \partial_\mu$ and an internal gauge parameter $\epsilon^i T_i$. A generating set of gauge symmetries is given by

$$\delta_{(\xi, \epsilon)} g_{\mu\nu} = -\mathcal{L}_\xi g_{\mu\nu}, \quad \delta_{(\xi, \epsilon)} A_\mu = -\mathcal{L}_\xi A_\mu + D_\mu^A \epsilon, \quad (2.2)$$

with $D_\mu^A \epsilon = \partial_\mu \epsilon + [A_\mu, \epsilon]$.

Let the fields be collectively denoted by $\phi^\alpha = (g_{\mu\nu}, A_\mu)$. When the gauge parameters $(\xi, \epsilon)$ depend only on the spacetime coordinates but not on the fields $\phi^\alpha$, one has

$$[\delta_{(\xi_1, \epsilon_1)}, \delta_{(\xi_2, \epsilon_2)}] \phi^\alpha = \delta_{(\hat{\xi}, \hat{\epsilon})} \phi^\alpha, \quad (2.3)$$

with $\hat{\xi} = [\xi_1, \xi_2]$ the Lie bracket for vector fields and $\hat{\epsilon} = \xi_1^\mu \partial_\mu \epsilon_2 - \xi_2^\mu \partial_\mu \epsilon_1 + [\epsilon_1, \epsilon_2]$. The Lie bracket for field independent gauge parameters is given by

$$[(\xi_1, \epsilon_1), (\xi_2, \epsilon_2)] = (\hat{\xi}, \hat{\epsilon}). \quad (2.4)$$
In the case of gauge parameters \((\xi, \epsilon)\) that are field dependent, one finds instead
\[
[\delta_{(\xi_1, \epsilon_1)}, \delta_{(\xi_2, \epsilon_2)}] \phi^\alpha = \delta_{(\hat{\xi}_M, \hat{\epsilon}_M)} \phi^\alpha,
\]
(2.5)
with
\[
\hat{\xi}_M = \hat{\xi} + \delta_{(\xi_1, \epsilon_1)} \xi_2 - \delta_{(\xi_2, \epsilon_2)} \xi_1,
\]
(2.6)
\[
\hat{\epsilon}_M = \hat{\epsilon} + \delta_{(\xi_1, \epsilon_1)} \epsilon_2 - \delta_{(\xi_2, \epsilon_2)} \epsilon_1,
\]
(2.7)
and the Lie (algebroid) bracket for field dependent gauge parameters is thus defined through
\[
[(\xi_1, \epsilon_1), (\xi_2, \epsilon_2)]_M = (\hat{\xi}_M, \hat{\epsilon}_M).
\]
(2.8)

3 Dimensional reduction through gauge fixation

In terms of coordinates \(x^\mu = (u, r, x^A)\), where \(x^A\) are angular variables in \(d - 2\) dimensions, we make the following gauge fixing ansatz for the metric and Yang-Mills potentials:
\[
g_{\mu\nu} = \begin{pmatrix}
e^{2\beta} \frac{V}{r} + g_{CD} U^C U^D & -e^{2\beta} g_{BC} & 0 \\
e^{-2\beta} & 0 & 0 \\
-g_{AC} U^C & 0 & g_{AB}
\end{pmatrix},
\]
(3.1)
\[
A_\mu = (A_u, 0, A_A).
\]

In addition, one imposes the determinant condition \(\det g_{AB} = r^{2(d-2)} \det \bar{\gamma}_{AB}\), with \(\bar{\gamma}_{AB}\) the metric on the unit \(d-2\)-sphere.

As in the purely gravitational four dimensional case [2], these conditions fix the gauge freedom up to some \(r\) independent functions. Indeed, the gauge transformations \(2.2\) that preserve this gauge choice, i.e., the residual gauge symmetries, are determined by gauge parameters that have to satisfy
\[
\mathcal{L}_\xi g_{rr} = 0, \quad \mathcal{L}_\xi g_{rA} = 0, \quad g^{AB} \mathcal{L}_\xi g_{AB} = 0, \quad -\mathcal{L}_\xi A_r + D_r^A \epsilon = 0.
\]
(3.2)

This gives the differential conditions
\[
\partial_r \xi^u = 0, \quad \partial_r \xi^A = \partial_B \xi^u g^{AB} e^{2\beta},
\]
\[
\partial_r (\hat{\xi}^r) = -\frac{1}{d-2} (\bar{D}_B \partial_r \xi^B - \partial_B \xi^u \partial_r U^B), \quad \partial_r \epsilon = \partial_B \xi^u g^{AB} e^{2\beta} A_A,
\]
(3.3)
the general solution of which is
\[
\xi^u = F(u, x^A), \quad \xi^A = Y^A(u, x^B) - \partial_B F \int_r^\infty dr' \left( e^{2\beta} g^{AB} \right),
\]
(3.4)
\[
\hat{\xi}^r = -\frac{r}{d-2} (\bar{D}_B \hat{\xi}^B - \partial_B F U^B), \quad \epsilon = E(u, x^A) - \partial_B F \int_r^\infty dr' (g^{BA} e^{2\beta} A_A),
\]
and involves $d-1+n$ arbitrary $r$-independent functions $F(u,x^A), Y^A(u,x^B), E^i(u,x^A)$.

At this stage, it is sufficient to impose the following fall-off conditions on the components of the metric and the gauge potentials,

$$e^{2\beta} g^{AB} = O(r^{-1-\epsilon}) = e^{2\beta} g^{AB} A_A, \quad U^C e^{2\beta} g^{AB} = o(r^{-1}) \text{ for } d > 3. \tag{3.5}$$

In particular, the first of these conditions guarantee that the integrals for $\xi^A$ and $\epsilon$ in (3.4) are well-defined and that $\lim_{r \to \infty} \xi^A = Y^A, \lim_{r \to \infty} \epsilon = E$.

Consider then the vector fields $\xi^R = F\partial_u + Y^A \partial_A$ and the internal gauge parameter $\epsilon^R = E^i T_i$, equipped with the Lie bracket

$$[(\xi^R_1, \epsilon^R_1), (\xi^R_2, \epsilon^R_2)] = (\hat{\xi}^R, \hat{\epsilon}^R), \tag{3.6}$$

for field independent gauge parameters (2.4) of the Einstein-Yang-Mills system in $d-1$ dimensions. We are now ready to state the main result of this section:

*The Lie algebra of residual gauge parameters* (3.4) *equipped with the Lie bracket $[\cdot, \cdot]_M$ of the $d$ dimensional Einstein-Yang-Mills system is a faithful representation of the Lie algebra of field independent gauge parameters $(\xi^R, \epsilon^R)$ of the $d-1$ dimensional Einstein-Yang-Mills system.*

The proof for the diffeomorphism part is almost exactly the same as in [15], except for the additional $u$ dependence in $Y^A$, which is easily taken into account. We will thus not repeat all details here. First, one needs to check that the result holds for $\hat{\xi}_M^u, \hat{\xi}_M^A, r^{-1} \hat{\xi}_M^r, \hat{\epsilon}_M$ at $r \to \infty$. This is where the fall-off conditions (3.5) are needed. Note however that the fall-off condition on $U^A$ have been considerably relaxed and, in particular, there are no conditions for $d = 3$. The rest of the proof consists in verifying that $\partial_r \hat{\xi}_M^u, \partial_r \hat{\xi}_M^A, \partial_r(r^{-1} \hat{\xi}_M^r), \partial_r \hat{\epsilon}_M$ satisfy equations (3.3) with $(\xi^R, \epsilon^R)$ replaced by $(\hat{\xi}^R, \hat{\epsilon}^R)$.

### 4 Fall-off conditions and asymptotic symmetry structure

Suppose now that in spacetime dimensions 4 or higher, precise fall-off conditions for the metric coefficients and gauge potentials are given by

$$\beta = o(1), \quad U^A = o(1), \quad g_{AB} dx^A dx^B = r^2 \eta_{AB}(x^C) dx^A dx^B + o(r^2), \quad V \frac{r}{l^2} = -\frac{r^2}{l^2} + o(r^2), \quad A_u = o(1), \quad A_B = A_B^0(u, x^C) + o(1). \tag{4.1}$$

In the asymptotically flat $U(1)$ case in four dimensions, these fall-off conditions are consistent with those of [12, 13, 14]. They imply in particular the conditions required in (3.5).
The gauge transformations that preserve these fall-off conditions have to satisfy, in addition to (3.2), the supplementary conditions
\[ L_\xi g_{ur} = o(1), \quad L_\xi g_{uA} = o(r^2), \quad L_\xi g_{AB} = o(r^2), \quad L_\xi g_{uu} = o(r^2), \]
\[ - L_\xi A_u + D_u A_\xi = o(1), \quad - L_\xi A_B + D_B A_\xi = o(1). \]
\[ (4.2) \]
They are equivalent to the following differential equations on the \((\xi_R, \epsilon_R)\)
\[ \partial_u F = \frac{1}{d - 2} \Psi, \quad \partial_u Y^A \gamma_{AB} = \frac{1}{l^2} \partial_B F, \]
\[ L_Y \gamma_{AB} = \frac{2}{d - 2} \psi \gamma_{AB}, \quad \partial_u E = \frac{1}{l^2} \partial_B F A^0_B, \]
\[ (4.3) \]
with \(\Psi = \bar{D}_B Y^B\), the general solution of which is
\[ F = f(x^A) + \frac{1}{d - 2} \int_0^u du' \Psi, \quad Y^A = y^A(x^B) + \frac{1}{l^2} \int_0^u du' (\gamma^{AB} \partial_B F), \]
\[ E = e(x^A) + \frac{1}{l^2} \int_0^u du' (\gamma^{AB} \partial_A F A^0_B). \]
\[ (4.4) \]
Let us denote by \(a_B\) the values of \(A^0_B\) at \(u = 0\) and consider time independent conformal Killing vectors of the \(d - 2\) sphere,
\[ \mathcal{L}_y \gamma_{AB} = \frac{2}{d - 2} \psi \gamma_{AB}, \]
\[ (4.5) \]
with \(\psi = \bar{D}_B y^B\). In addition, in the case of a non-vanishing cosmological constant, the vectors \(\partial^A f = \bar{\gamma}^{AB} \partial_B f\) are also required to be conformal Killing vectors of the \(d - 2\) sphere, as follows by differentiating the third of (4.3) with respect to \(u\) and setting \(u = 0\). A second derivative with respect to \(u\) at \(u = 0\) then implies that \(\partial^A \psi\) are also conformal Killing vectors of the \(d - 2\) sphere. This can be continued for higher order derivatives.

In terms of these quantities, the asymptotic symmetry structure is described through the brackets
\[ \hat{f} = \frac{1}{d - 2} f_1 \psi_2 + y_1^A \partial_A f_2 - (1 \leftrightarrow 2), \quad \hat{y}^A = \frac{1}{l^2} f_1 \partial^A f_2 + y_1^B \partial_B y^A - (1 \leftrightarrow 2), \]
\[ \hat{e} = \frac{1}{l^2} f_1 \partial^A f_2 a_A + y_1^A \partial_A e_2 - (1 \leftrightarrow 2) + [e_1, e_2]. \]
\[ (4.6) \]
On account of the explicit field dependence in \(\hat{e}\), one has to use the Lie algebroid bracket in order to check the Jacobi identity for \(e\) with \(\delta_y a_A = - \mathcal{L}_y a_A + D^a_A e\) in the case of a non-vanishing cosmological constant. More generally, it is implicitly understood that each time an element depends explicitly on the fields, the Lie algebroid bracket has to be used.

By the same reasoning as before, one then shows that the asymptotic symmetry structure is represented at infinity for all values of \(u\) through the Lie algebroid bracket that
involves the dependence on $A_B^0$, and then also in the bulk spacetime through the result of the previous section.

The gauge theory part of the asymptotic symmetry structure consists of elements of the form $(0,0,e)$. It is is a non-abelian ideal that contains an arbitrary $g$-valued function on the $d-2$ sphere. In that sense, it is a generalisation of a loop algebra where the base space is a higher dimensional sphere instead of a circle.

The quotient of the total structure by this ideal is the spacetime part. It can be described by elements of the form $(f,y^A,0)$ with brackets determined by the first line of (4.6). In the case of vanishing cosmological constant, elements of the form $(f,y^A,0)$ form a subalgebra that acts on the gauge theory ideal.

5 Explicit description of asymptotic symmetry structure in particular cases

5.1 Dimensions 4 and higher, anti-de Sitter case

For $d \geq 4$ and $l \neq 0$, the space-time part of the asymptotic symmetry structure is isomorphic to $\mathfrak{so}(d-1,2)$, the algebra of exact Killing vectors of $d$-dimensional anti-de Sitter space, in agreement with the analysis in [16].

Indeed, in the coordinates we are using, the anti-de Sitter metric is given by

$$g_{\mu\nu} = \begin{pmatrix} -\frac{r^2}{l^2} - 1 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & r^2\bar{\gamma}_{AB} \end{pmatrix}.$$  (5.1)

Besides the conditions

$$\bar{\xi}^u = \bar{F}(u,x), \quad \bar{\xi}^A = \bar{Y}^A(u,x) - \frac{1}{r}\partial^A\bar{F}, \quad \bar{\xi}^r = \frac{1}{d-2}(-r\bar{\Psi} + \bar{\Delta}\bar{F}),$$  (5.2)

$$\partial_u\bar{F} = \frac{1}{d-2}\bar{\Psi}, \quad \partial_u\bar{Y}^A = \frac{1}{l^2}\partial^A\bar{F},$$  (5.3)

where $\bar{Y}^A$ and $\partial^A\bar{F}$ are conformal Killing vectors of $\bar{\gamma}_{AB}$, which correspond to an asymptotic Killing vector evaluated for the anti-de Sitter metric, an exact Killing vector $\bar{\xi} = \bar{\xi}^u\partial_u + \bar{\xi}^r\partial_r + \bar{\xi}^A\partial_A$ has also to satisfy the additional conditions

$$\partial_B\bar{F} = -\frac{1}{d-2}\partial_B\bar{\Delta}\bar{F}, \quad \bar{\Psi} = -\frac{1}{d-2}\bar{\Delta}\bar{\Psi}.$$  (5.4)

The latter are automatically satisfied for conformal Killing vectors $\bar{Y}^A, \partial^A\bar{F}$ of the unit $d-2$ sphere.
Even though it is not needed for this proof, one can also check directly that, if \( y^A \) are conformal Killing vectors of the \( d-2 \) sphere, then the requirement that \( \partial^A \psi \) are also conformal Killing vectors is automatically satisfied if \( d \neq 4 \), while for \( d = 4 \), this reduces the local conformal algebra in 2 dimensions to the globally well defined algebra \( so(3,1) \) on the 2 sphere.

### 5.2 Dimensions 5 and higher, flat case

For \( l \to \infty \), the asymptotic symmetry structure of field independent parameters \( (f, y^A, e) \) simplifies. The subalgebra \( (0, y^A, 0) \) of conformal Killing vectors of the \( d-2 \geq 3 \) sphere represents the Lorentz algebra \( so(d-1,1) \). It acts both on the abelian ideal \( (f,0,0) \) of arbitrary functions on the sphere, representing supertranslations, and on the gauge theory ideal.

Stronger fall-off conditions motivated by the Einstein equations of motions have been considered in [17]. They require \( \partial^A F \) to be conformal Killing of the \( d-2 \) sphere. In turn this requires both \( \partial^A f \) and \( \partial^A \psi \) to be conformal Killing vectors. Again, by comparing with the conditions satisfied by exact Killing vectors of Minkowski space-time, the only additional conditions are (5.4), which are automatically satisfied for conformal Killing vectors \( Y^A, \partial^A F \). This shows that the additional conditions reduce super to standard translations so that the spacetime part of the asymptotic structure becomes the Poincaré algebra \( iso(d-1,1) \).

### 5.3 4 dimensional flat case

In 4 dimensions, it is useful to introduce stereographic coordinates \( \zeta = \cot \frac{\theta}{2}e^{i\phi} \) and its complex conjugate, so that \( \tilde{\gamma}_{AB}dx^A dx^B = 2P_S^{-2}d\zeta d\bar{\zeta} \) with \( P_S = \frac{1}{\sqrt{2}}(1 + \zeta \bar{\zeta}) \). The covariant derivative on the 2 surface is then encoded in the operator

\[
\tilde{\partial}\eta^s = P_S^{1-s}\tilde{\partial}(P_S^s\eta^s), \quad \tilde{\bar{\partial}}\eta^s = P_S^{1+s}\partial(P_S^{1-s}\eta^s),
\]

where \( \tilde{\partial}, \tilde{\bar{\partial}} \) raise respectively lower the spin weight \( s \) by one unit and satisfy

\[
[\tilde{\partial}, \tilde{\bar{\partial}}]\eta^s = \frac{s}{2}R_S \eta^s,
\]

with \( R_S = 4P_S^2\tilde{\partial}\tilde{\bar{\partial}} \ln P_S = 2 \).

Let \( \mathcal{Y} = P_S^{-1}y^\zeta \) and \( \mathcal{\bar{Y}} = P_S^{-1}y^{\bar{\zeta}} \) be of spin weights \(-1\) and \(1\) respectively. The conformal Killing equations and the conformal factor then become

\[
\tilde{\partial}\mathcal{Y} = 0 = \bar{\partial}\mathcal{\bar{Y}}, \quad \psi = (\partial\mathcal{Y} + \bar{\partial}\mathcal{\bar{Y}}).
\]

It follows for instance that \( \tilde{\partial}\partial\mathcal{Y} = -\mathcal{Y}, \tilde{\partial}^2\psi = \bar{\partial}^3\mathcal{Y}, \tilde{\partial}\bar{\partial}\psi = -\psi \).
In order to describe the asymptotic symmetry structure there are then two options.

The first is to require well-defined functions on the 2-sphere. This amounts to restricting oneself to the conformal Killing vectors that satisfy $\partial^2 \mathcal{V} = 0 = \overline{\partial}^2 \mathcal{V}$ and require that the functions $f, e^a$ that occur in (4.6) (with $l \to \infty$) can be expanded in spherical harmonics. The asymptotic symmetry algebra is then the semi-direct sum of the globally well-defined $bms_{\text{glob}}^3$ algebra [1, 3] with a globally well-defined “sphere” algebra.

Alternatively [18, 15], one admits Laurent series and expands $y^i \partial_x$ in terms of $l_m = -\zeta^m \partial_x$, $y^i \partial_x$ in terms of $\bar{l}_m$, $f$ in terms of $t_{m,n} = P^{-1}_S \zeta^m \bar{\zeta}^n$ and $e$ in terms of $j_{\tilde{m},n} = \tilde{T}_i \zeta^m \bar{\zeta}^n$. In these terms, the non-vanishing brackets of the asymptotic symmetry algebra become

\[
[l_i, t_{m,n}] = \left( \frac{l+1}{2} - m \right) t_{m+l,n}, \quad [l_m, l_n] = (m-n) l_{m+n}, \quad (5.8)
\]

\[
[\bar{l}_i, t_{m,n}] = \left( \frac{l+1}{2} - n \right) t_{m,n+l}, \quad [\bar{l}_m, \bar{l}_n] = (m-n) \bar{l}_{m+n}, \quad (5.9)
\]

\[
[l_i, j_{\tilde{m},n}] = -n \tilde{j}_{\tilde{l},m+l,n}, \quad [\bar{l}_m, j_{\tilde{m},n}] = -n j_{\tilde{m},n+l}, \quad (5.10)
\]

\[
[j_{\tilde{l},l}, j_{\tilde{m},n}] = f_{ij} \tilde{j}_{\tilde{l},k}^{l+m+p+n}. \quad (5.11)
\]

### 5.4 3 dimensional anti-de Sitter case

On the metric components, we use the same fall-off conditions as in [4]. Note that the determinant condition requires $g_{\phi \phi} = r^2$ and that the fall-off conditions allow for $\ln r$ terms both in $g_{uu}$ and $g_{u \phi}$. The spacetime part of the asymptotic symmetry structure is then described by two copies of the conformal algebra [5], $F \partial_u + Y \partial_\phi = Y^+(x^+) \partial_+ + Y^-(x^-) \partial_-$, where $x^\pm = \frac{y}{r} \pm \phi$.

In order to accommodate the charged and rotating black hole solution [19], the fall off conditions on the gauge potentials can be chosen as $A_+ = O(\ln r)$, while one simultaneously requires $A_- = o(1)$. Alternatively, one could also exchange the role of $+$ and $-$. Requiring $-\mathcal{L}_\zeta A_+ + D^A_+ \epsilon = O(\ln r)$ gives no conditions, while $-\mathcal{L}_\zeta A_- + D_-^A \epsilon = o(1)$ leads to $\partial_- E = 0$. In this case, there is no explicit field dependence and the asymptotic symmetry structure simplifies as compared to the higher dimensional case.

When expanding $Y^+ \partial_+, Y^- \partial_-, E$ in terms of modes, $l_m^\pm = e^{imx^\pm} \partial_x, j^m_i = T_i e^{imx^+}$, the non-vanishing brackets of the asymptotic symmetry algebra are explicitly given by

\[
i[l_m^\pm, l_n^\pm] = (m-n)l_{m+n}^\pm, \quad i[l_m^\pm, j_i^m] = -n j_{i,m+n}^m, \quad i[j_i^m, j_n^m] = if_{ij}^{k} j_{k,m+n}^m. \quad (5.12)
\]

### 5.5 3 dimensional flat case

In this case, the spacetime part of the asymptotic symmetry structure is the $bms_3$ algebra described by $F \partial_u + Y \partial_\phi = [f(\phi) + uy(\phi)] \partial_u + y \partial_\phi$. For the gauge potentials, one
may then choose $A_u = o(1), \ A_\phi = O(\ln r)$. Requiring $-\mathcal{L}_\xi A_u + D_u^A \epsilon = o(1)$ leads to $\partial_u E = 0$, while $-\mathcal{L}_\xi A_\phi + D_\phi^A \epsilon = O(\ln r)$ gives no conditions.

When expanding $F \partial_u + Y \partial_\phi, E$ in terms of modes, $l_m = e^{im\phi} \partial_u + uim\phi e^{im\phi} \partial_u, t_m = e^{im\phi} \partial_u, j^m_i = T_i e^{im\phi}$, the non vanishing brackets of the asymptotic symmetry algebra are given by

$$
\begin{align*}
\{l_m, l_n\} &= (m-n)l_{m+n}, \\
\{l_m, t_n\} &= (m-n)t_{m+n}, \\
\{l_m, j^n_i\} &= -nj^m_i, \\
\{j^m_i, j^n_j\} &= if^k_{ij} j^m_k + n f^k_{ij} j^m_k.
\end{align*}
$$

(5.13)

6 Discussion

The gauge fixing and fall-off conditions that we have considered have been mainly dictated by the desire to yield the usual asymptotic symmetry structure, at least for the spacetime part, while otherwise being as relaxed as possible. As partly already discussed in the text, additional more restrictive conditions motivated by finiteness of associated conserved currents or their integrability for example can further reduce the asymptotic symmetry structure, in particular also in the Yang-Mills part.

On the other hand, one may wonder how far these conditions can be relaxed even further. From a holographic point of view, the role of the gauge fixing conditions considered in section 3 is to fix the dependence in $r$ of the gauge parameters. This can be achieved in various ways. In the Newman-Unti gauge [20] for instance, one can require $g_{ur} = -1$ instead of the determinant condition, leading to another integration function in $\xi^r$, that may or may not be fixed through additional conditions, see e.g. [21]. One may also relax conditions (3.5). For instance at fixed but finite $r$ no such conditions are needed. Even though one will then not get the symmetry structure of the Einstein-Yang-Mills system in one dimension lower, the resulting structure will still be well-defined.

Similarly, except for the fall-off conditions on $g_{AB}$, the role of the other conditions in section 4 is to fix the time dependence of the gauge parameters and thus of the symmetry structure of the dual boundary theory. In the present set-up, the fall-off conditions on $g_{AB}$ are the only ones that constrain the dependence of the symmetry structure, or more precisely of $Y^A, F$, on the spatial coordinates $x^A$. In other words, relaxing this condition leads to “superrotations” that, like supertranslations and the Yang-Mills gauge parameters, have an arbitrary $x^A$ dependence.

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References

[1] H. Bondi, M. G. van der Burg, and A. W. Metzner, “Gravitational waves In general relativity. 7. Waves from axi-symmetric isolated systems,” Proc. Roy. Soc. Lond. A 269 (1962) 21.

[2] R. K. Sachs, “Gravitational waves in general relativity. 8. Waves in asymptotically flat space-time,” Proc. Roy. Soc. Lond. A 270 (1962) 103.

[3] R. K. Sachs, “Asymptotic symmetries in gravitational theory,” Phys. Rev. 128 (1962) 2851–2864.

[4] T. Regge and C. Teitelboim, “Role of surface integrals in the Hamiltonian formulation of general relativity,” Ann. Phys. 88 (1974) 286.

[5] J. D. Brown and M. Henneaux, “Central charges in the canonical realization of asymptotic symmetries: An example from three-dimensional gravity,” Commun. Math. Phys. 104 (1986) 207.

[6] A. Strominger, “Black hole entropy from near-horizon microstates,” JHEP 02 (1998) 009, arXiv:hep-th/9712251.

[7] M. Banados, C. Teitelboim, and J. Zanelli, “The black hole in three-dimensional space-time,” Phys. Rev. Lett. 69 (1992) 1849–1851, hep-th/9204099.

[8] M. Banados, M. Henneaux, C. Teitelboim, and J. Zanelli, “Geometry of the (2+1) black hole,” Phys. Rev. D48 (1993) 1506–1525, gr-qc/9302012.

[9] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri, and Y. Oz, “Large N field theories, string theory and gravity,” Phys. Rept. 323 (2000) 183–386, hep-th/9905111.

[10] A. Strominger, “Asymptotic Symmetries of Yang-Mills Theory,” arXiv:1308.0589 [hep-th].

[11] S. Weinberg, “Infrared photons and gravitons,” Phys. Rev. 140 no. 2B, (Oct, 1965) B516–B524.
[12] A. I. Janis and E. T. Newman, “Structure of gravitational sources,”
*Journal of Mathematical Physics* **6** no. 6, (1965) 902–914.
http://link.aip.org/link/?JMP/6/902/1

[13] M. G. J. van der Burg, “Gravitational Waves in General Relativity X. Asymptotic
Expansions for the Einstein-Maxwell Field,”
*Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences* **310** no. 1501, (1969)
http://rspa.royalsocietypublishing.org/content/310/1501/221.abstract

[14] A. Exton, E. Newman, and R. Penrose, “Conserved quantities in the
Einstein-Maxwell theory,” *J.Math.Phys.* **10** (1969) 1566–1570.

[15] G. Barnich and C. Troessaert, “Aspects of the BMS/CFT correspondence,”
*JHEP* **05** (2010) 062, arXiv:1001.1541 [hep-th].

[16] M. Henneaux, “Asymptotically anti-de Sitter universes in d = 3, 4 and higher
dimensions,” in *Proceedings of the Fourth Marcel Grossmann Meeting on General
Relativity, Rome 1985*, R. Ruffini, ed., pp. 959–966. Elsevier Science Publishers
B.V., 1986.

[17] K. Tanabe, S. Kinoshita, and T. Shiromizu, “Asymptotic flatness at null infinity in
arbitrary dimensions,” *Phys.Rev.* **D84** (2011) 044055
arXiv:1104.0303 [gr-qc].

[18] G. Barnich and C. Troessaert, “Symmetries of asymptotically flat 4 dimensional
spacetimes at null infinity revisited,” *Phys.Rev.Lett.* **105** (2010) 111103
arXiv:0909.2617 [gr-qc].

[19] C. Martinez, C. Teitelboim, and J. Zanelli, “Charged rotating black hole in three
space-time dimensions,” *Phys.Rev.* **D61** (2000) 104013
arXiv:hep-th/9912259 [hep-th].

[20] E. T. Newman and T. W. J. Unti, “Behavior of asymptotically flat empty spaces,”
*Journal of Mathematical Physics* **3** no. 5, (1962) 891–901
http://link.aip.org/link/?JMP/3/891/1

[21] G. Barnich and P.-H. Lambert, “A Note on the Newman-Uni Group and the BMS
Charge Algebra in Terms of Newman-Penrose Coefficients,” *Adv. Math. Phys.* **16**
(2012) 197385, arXiv:1102.0589 [gr-qc]
http://dx.doi.org/10.1155/2012/197385%197385