On relativistic spin network vertices

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Abstract
Barrett and Crane have proposed a model of simplicial Euclidean quantum gravity in which a central role is played by a class of $Spin(4)$ spin networks called “relativistic spin networks” which satisfy a series of physically motivated constraints. Here a proof is presented that demonstrates that the intertwiner of a vertex of such a spin network is uniquely determined, up to normalization, by the representations on the incident edges and the constraints. Moreover, the constraints, which were formulated for four valent spin networks only, are extended to networks of arbitrary valence, and the generalized relativistic spin networks proposed by Yetter are shown to form the entire solution set (mod normalization) of the extended constraints. Finally, using the extended constraints, the Barrett-Crane model is generalized to arbitrary polyhedral complexes (instead of just simplicial complexes) representing spacetime. It is explained how this model, like the Barrett-Crane model can be derived from BF theory by restricting the sum over histories to ones in which the left handed and right handed areas of any 2-surface are equal. It is known that the solutions of classical Euclidean GR form a branch of the stationary points of the BF action with respect to variations preserving this condition.

1 Introduction

The “Relativistic spin networks” defined by Barrett and Crane (BC) are a fundamental ingredient of their proposal for a simplicial model of quantum
general relativity in four dimensions. In [BaCr97] the space of intertwiners that relativistic spin networks are allowed to carry is defined implicitly by a set of physically motivated constraints. They also exhibit a single solution to these constraints. Soon after Barbieri [Barb97b] gave a partial proof of the uniqueness up to normalization of this solution, which relies on some unproven hypotheses. BC’s constraints apply only to 4-valent spin networks, but their solution to their constraints has been generalized in a natural way to arbitrary valence by Yetter [Yet98], and Barrett [Ba98] has given a very transparent characterization of this extension in the non q deformed case, $q = 1$.

Here a proof (without auxiliary assumptions) will be given showing that the BC solution is the only one, up to normalization, and similarly Yetter’s generalization of the solution (for $q = 1$) is the unique solution up to normalization of a natural generalization of the BC constraints. In addition a physically motivated extension of the BC model to polyhedral complexes is outlined. The BC model can be obtained from a sum over histories quantization of $Spin(4)$ BF theory by restricting the histories to ones that assign equal (suitably defined) left-handed and right-handed areas to any 2-surface. At the classical continuum level such a constrained BF theory does reproduce GR. The solutions of GR form a branch of the stationary points of the $Spin(4)$ BF action with respect to variations that preserve the constraint that the left and right handed areas be equal for all 2-surfaces [Rei98]. This procedure for obtaining the BC model generalizes straightforwardly to complexes of convex 4-polyhedra.

For background information on spin networks see [YLV62, Pen69, RS95, Bae96, Rei94].

2 Definition of relativistic spin networks

I will adopt the following definition of relativistic spin networks, which generalizes BC’s definition from 4-valent vertices to vertices of arbitrary valence.

**Definition 1**: A relativistic spin network is a $Spin(4)$ spin network such that

1. On each edge the left handed spin, $j_L$, and the right handed spin, $j_R$, are equal.
In any expansion of an n-valent vertex into a sum of trivalent trees only trivalent trees with \( j_L = j_R \) on each of the internal (virtual) edges appear.

In general the edges of a spin network carry non-trivial irreducible representations (irreps) of the gauge group, and the vertices carry intertwiners. The intertwiner for a vertex can be any invariant tensor of the product representation \( R \) formed by the product of the irreps carried by the incoming edges and the duals of the irreps on the outgoing edges. The space of invariant tensors of \( R \) will be denoted \( \text{Inv}(R) \). The evaluation of a spin network is a complex number calculated by contracting the intertwiners of the vertices along the edges. An intertwiner carries one index for each incident edge. In the evaluation of a spin network the pair of indices associated with each edge (one index lives at each end) is contracted, leaving ultimately a single complex number. In the BC model histories of the gravitational field determine relativistic spin networks on the boundaries of the 4-simplices that form the simplicial spacetime, and the probability amplitude of each history is the product of the evaluations of these spin networks (times some simple further factors associated with the lower dimensional simplices).

Relativistic spin networks are spin networks of \( \text{Spin}(4) \), the covering group of \( SO(4) \). \( \text{Spin}(4) \) is the product of two \( SU(2) \)s: \( \text{Spin}(4) = SU(2) \times SU(2) \), where the first \( SU(2) \) factor will be called \( SU(2)_L \), the “left handed” subgroup, and the second \( SU(2)_R \), the “right handed” subgroup. This factorization extends to the irreps of \( \text{Spin}(4) \). These are tensor products of an irrep of \( SU(2)_L \) and an irrep of \( SU(2)_R \), and their carrying spaces, i.e. the vector spaces on which they act, are the tensor products of the carrying spaces of the \( SU(2)_L \) and \( SU(2)_R \) irreps.

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\[1\] The dual of a representation \( D \) of a group is the representation \( D^{-1}T \) formed by the transposes of the inverses of the representation matrices of \( D \). If \( D \) is unitary its dual is the complex conjugate representation \( D^* \) formed by complex conjugating each matrix element in the representation matrices of \( D \).

\[2\] Proof: The representation matrices \( D(g_L, g_R) \) of a representation \( D \) of \( SU(2)_L \times SU(2)_R \) may always be written as \( D_L(g_L)D_R(g_R) \) where \( D_L(g_L) = D(g_L, 1) \) and \( D_R(g_R) = D(1, g_R) \). \( D_L \) and \( D_R \) are of course \( SU(2) \) representations. The carrying space \( \mathcal{H} \) of \( D \) can thus be decomposed into \( D_L \) invariant subspaces carrying irreps of \( SU(2)_L \). The fact that \( D_L \) and \( D_R \) commute, and Shur’s lemma, then imply that \( D_R \) acts on the subspaces of \( \mathcal{H} \) carrying the same irrep of \( SU(2)_L \), i.e. eigensubspaces of left handed spin. The commutation of \( D_L \) and \( D_R \) further implies that when \( D_R \) is reduced in
The factorization of the irreps implies that the product $R$ of irreps incident on a spin network vertex also factorizes into a left-handed factor, $R_L$, and a right handed factor, $R_R$, and hence that the intertwiner space factorizes according to

$$\text{Inv}_{\text{Spin}(4)}(R) = \text{Inv}_{\text{Spin}(4)}(R_L \otimes R_R) = \text{Inv}_{\text{SU}(2)}(R_L) \otimes \text{Inv}_{\text{SU}(2)}(R_R)$$

into a tensor product of two $SU(2)$ intertwiner spaces.

$SU(2)$ irreps are determined by their spin, modulo the freedom to change basis in the carrying space. $Spin(4)$ irreps are therefore characterized in the same sense by the spins $(j_L, j_R)$ of their left and right handed factors. To keep the mathematics as concrete as possible it is convenient to fix the bases in the carrying spaces so that the irreps take a standard form completely determined by their spins $(j_L, j_R)$.

We shall adopt as our standard $(j_L, j_R)$ irrep $U(j_L) \otimes U(j_R)^*$, where for each spin $j$, $U^{(j)}$ is a particular, unitary, spin $j$ $SU(2)$ irrep fixed once and for all by some conventions. $U^{(j)}$ is its dual irrep. This choice can be made because, firstly, the compactness of $SU(2)$ implies that its irreps preserve a hermitean inner product, and are thus unitary in orthonormal bases with respect to this inner product, and secondly because the dual of a spin $j$ $SU(2)$ irrep is also a spin $j$ irrep, so that the $Spin(4)$ irrep chosen really has right handed spin $j_R$.

Now let’s consider a spin network vertex. First let’s define some notation. Once the bases are fixed the product representation $R$ formed by the incident these subspaces they decompose into tensor products of $D_L$ and $D_R$ invariant subspaces, and the restriction of $D$ to these are tensor products of irreps of $SU(2)_L$ and $SU(2)_R$. The orbit of any vector in the carrying space of such a product of irreps clearly spans all of the carrying space, so the products are irreducible representations of $SU(2)_L \times SU(2)_R$.

3 Note that the evaluation of a spin network is invariant under changes of basis in the carrying spaces of the irreps on the edges. The contractions of the indices of the intertwiners are all between vector indices of some irrep and corresponding covector indices, i.e. vector indices of the dual of the irrep.

For instance we may choose

$$U^{(j)}_{m,n}(g) = \left( \begin{array}{c} 2j \\ j + m \end{array} \right)^{-\frac{j}{2}} \left( \begin{array}{c} 2j \\ j + n \end{array} \right)^{-\frac{j}{2}} \sum_{\Sigma M_i = m, \Sigma N_i = n} g_{M_1 N_1} \cdots g_{M_2 N_2} \ \forall g \in SU(2).$$

(3)

Here the indices $M_i$ and $N_i$ range over $\{-\frac{1}{2}, \frac{1}{2}\}$.
irreps is completely determined by the incident spins and whether edges are incoming or outgoing. This information can be gathered into two vectors, \( j_L \) and \( j_R \), which we shall, in a slight abuse of language, refer to as the vectors of incident left handed and right handed spins. Each entry of \( j_L \) corresponds to an incident edge and consists of the left handed spin, \( j_L \), if the edge is incoming and \(-j_L\) if the edge is outgoing. The left handed factor of \( R \) is then
\[
R_L = R(j_L) = \bigotimes_n U(j_n).
\]
(4)

(\( n \) numbers the edges and we have defined \( U(-j) \equiv U(j^*) \)). \( j_R \) is defined in complete analogy to \( j_L \), so our conventions for the \( Spin(4) \) irreps on the edges imply that
\[
R = R(j_L) \otimes R(j_R)^*.
\]
If \( \mathcal{H}(j) \) is the carrying space of the \( SU(2) \) representation \( R(j) \), and \( Inv_{SU(2)}(j) \equiv Inv_{SU(2)}(R(j)) \) is its invariant subspace then the carrying space of \( R = R(j_L) \otimes R(j_R)^* \) is \( \mathcal{H}(j_L) \otimes \mathcal{H}(j_R)^* \) and the \( Spin(4) \) intertwiner space is \( Inv_{Spin(4)}(j_L, j_R) = Inv_{SU(2)}(j_L) \otimes Inv_{SU(2)}(j_R)^* \).

The inner product preserved by the unitary representation \( R(j) \) establishes a one to one correspondence between vectors of \( \mathcal{H}(j)^* \) and linear functions \( \mathcal{H}(j) \rightarrow \mathbb{C} \). A tensor \( \Psi \in \mathcal{H}(j_L) \otimes \mathcal{H}(j_R)^* \) can thus be viewed as a linear mapping \( \Psi : \mathcal{H}(j_R) \rightarrow \mathcal{H}(j_L) \). If the tensor \( \Psi \) is an intertwiner it maps \( Inv_{SU(2)}(j_R) \) into \( Inv_{SU(2)}(j_L) \) and the orthogonal complement of \( Inv_{SU(2)}(j_R) \) in \( \mathcal{H}(j_R) \) to zero.

Condition 1 in the definition of relativistic spin networks implies that \( j_L = j_R \) at their vertices. Thus a relativistic intertwiner \( \Phi \) may be viewed as a mapping of \( \mathcal{H}(j_R) \) into itself, that furthermore maps \( Inv_{SU(2)}(j_R) \) into itself and the orthogonal complement of \( Inv_{SU(2)}(j_R) \) to zero. It is the composition of the orthogonal projector \( P \) onto \( Inv_{SU(2)}(j_R) \) and a linear mapping \( X \) of \( Inv_{SU(2)}(j_R) \) to itself: \( \Phi = XP \).

Condition 2 in the definition of relativistic spin networks refers to trivalent tree expansions of spin network vertices. It is well known (see [YLV62], [DPR96]) that for \( SU(2) \) spin networks each trivalent tree graph having the same external edges as a given vertex defines a basis of the intertwiner space \( Inv_{SU(2)}(j) \) of the vertex. (The trivalent tree has oriented edges and a cyclic ordering of the edges incident at each vertex.) Each element of such a “trivalent tree basis” is associated with an assignment \( J \) of (possibly zero) spins to

\[\text{5 Trivalent tree bases exist for spin networks of any group for which the product of two irreps is completely reducible.}\]
the internal edges of the tree (also known as “virtual” edges because they are not present in the actual spin network). The basis element is evaluated by contracting the intertwiners of the trivalent vertices along the internal edges as in a spin network evaluation. This leaves just the intertwiner indices associated with the external edges free. To complete the definition one needs to specify the trivalent intertwiners. The trivalent intertwiner spaces Inv_{SU(2)}(j_1, j_2, j_3) of SU(2) are all one dimensional, so it is sufficient to fix the freedom to multiply the intertwiners by scalar factors. We will choose the trivalent intertwiners to be normalized. Then, if a normalizing factor $\sqrt{2J+1}$ is included for each internal edge, the trivalent tree intertwiners will be normalized. There remains a phase which must be chosen by convention. We will suppose that such a convention has been adopted so that the trivalent tree $T$ and the vector $J$ of spins on the internal edges determine a unique intertwiner $|T, j, J\rangle \in \text{Inv}_{SU(2)}(j)$. Spin(4) trivalent tree bases can be then be constructed from the SU(2) trivalent tree bases: If the trivalent tree $T$ spans a Spin(4) vertex with incident spins $(j_L, j_R)$ then the multiplet

$$\{(T, j_L, J_L) \otimes (T, j_R, J_R)\}_{j_L, J_R}$$

spans the intertwiner space Inv_{Spin(4)}(j_L, j_R). Here $(T, j, J)$ denotes the complex conjugate of the tensor $|T, j, J\rangle$, which, as has been explained, defines a linear function $H(j) : \mathbb{C}$ via the hermitean inner product, thus justifying the Dirac bra notation. The definition of a relativistic spin network implies that the expansion of an intertwiner $\Phi$ of such a spin network on a trivalent tree basis has the form

$$\Phi = \sum_{J} a_{j, j}^{T} |T, j, J\rangle \otimes \langle T, j, J|,$$

i.e. the left and right handed spins are equal on both external and internal edges, for any tree $T$ spanning the vertex.

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6 The Wigner 3-jm symbols, $\begin{pmatrix} j_1 & j_2 & j_3 \\ m_1 & m_2 & m_3 \end{pmatrix}$, with the standard convention [YLV62] that $(-1)^{j_1-j_2+j_3} \begin{pmatrix} j_1 & j_2 & j_3 \\ j_1 & j_2 & j_3 \end{pmatrix}$ is real and non-negative for all $j_1$, $j_2$, and $j_3$, form a basis of trivalent intertwiners consistent with the convention for $U^{(j)}$ of [3]. Aside from being real this basis also has the attractive feature that the intertwiners have simple symmetry properties under permutations of the incident edges: If $j_1 + j_2 + j_3$ is even they are symmetric, if it is odd they are antisymmetric.
3 The solution to the constraints defining relativistic vertices

Now that relativistic spin networks have been defined and explained we are ready to state and prove our result:

**Theorem** The intertwiner of any vertex in a relativistic spin network is uniquely determined, up to a numerical factor, by the irreps on the incident edges. Let \( j = j_L = j_R \) be the common value of the vectors of left and right handed incident spins (as defined in §2) at a vertex, and let \( \mathcal{H}(j) \) be the space of tensors transforming under the product of the SU(2) irreps with these spins, equipped with the Hermitean inner product preserved by SU(2). When the bases in the carrying spaces of the SU(2)_L and SU(2)_R irreps on the edges of the relativistic spin network are chosen so that the SU(2)_L irrep is the dual of the SU(2)_R irrep on each edge incident on the vertex then

1 tensors, like the intertwiner, that transform under the product of the incident Spin(4) irreps live in the tensor product of \( \mathcal{H}(j) \) and its dual, and can thus be viewed as linear mappings of \( \mathcal{H}(j) \) to itself, and

2 the intertwiner is proportional to the orthogonal projector from \( \mathcal{H}(j) \) to the subspace of invariant tensors \( \text{Inv}_{\text{SU}(2)}(j) \subset \mathcal{H}(j) \).

**Proof.** Part 1 has already been established in §2, so only part 2 remains to be proven. (6) shows that an intertwiner, \( \Phi \), of a relativistic spin network is the composition of the projector \( P \) from the space \( \mathcal{H}(j) \) of tensors with spins \( j \) onto the subspace of intertwiners \( \text{Inv}_{\text{SU}(2)}(j) \) and a linear map \( X \) of \( \text{Inv}_{\text{SU}(2)}(j) \) to itself. (6) furthermore requires that \( X \) is diagonal in each of the trivalent tree bases of \( \text{Inv}_{\text{SU}(2)}(j) \). Obviously \( X = cI \) with \( c \in \mathbb{C} \) satisfies this condition, so \( cP \) is a relativistic intertwiner. To establish the theorem it remains to be shown that the set of all trivalent tree bases is rich enough so that this is the only solution to the condition.

Let us first consider four valent vertices. There are three (unoriented) trivalent trees matching the four incident edges,
each of which has one internal, or virtual, edge. Once the orientations of the external edges are fixed to match those of the four valent vertex being expanded there remains the freedom to choose the orientation of the internal edge and the cyclic ordering of the incident edges at the two trivalent vertices. However these choices only affect the signs of the corresponding trivalent tree basis, modulo sign it is determined by the unoriented tree graph.

Let’s number the incident edges 1,2,3,4 clockwise from the top left, and let $G_{ni}$ for $n \in \{1,2,3,4\}$ be the generators acting in $\mathcal{H}(j)$, of the $SU(2)$ irreps on the incident edges. Let’s also define $G_{mn} = (G_m + G_n)_i$. The first of the trivalent trees drawn above corresponds to a pairing of edges 1 and 2, which join at a trivalent vertex. The corresponding basis intertwiners is a contraction of a trivalent intertwiner at this vertex and one at the other vertex of the graph. The invariance of the intertwiner at the first vertex implies that

$$G_{12}^2 \int J = J(J+1) \int J.$$  

The intertwiner basis is thus the eigenbasis of $G_{12}^2$ in $\text{Inv}_{SU(2)}(j)$. Similarly the trivalent tree bases associated with the second and third trees diagonalize $G_{13}^2$ and $G_{14}^2$ respectively.

Since $X$, the restriction of $\Phi$ to $\text{Inv}_{SU(2)}(j)$, is diagonal in all the trivalent tree bases it commutes with $G_{12}^2$, $G_{13}^2$, and $G_{14}^2$. What’s more, since the spectra of these operators ($\{J(J+1)\}$) are non-degenerate, $X$ can be expressed as a function of any one of them. Choosing $G_{12}^2$ we write

$$X = \sum_{q=0}^{d-1} b_q [G_{12}^2]^q.$$  

\footnote{If the edge $n$ is incoming then $[G_{ni}, G_{nj}] = i\epsilon_{ij}^k G_{nk}$. If the edge is outgoing the negatives $-G_{ni}$ satisfy these commutation relations. The generators belonging to distinct edges of course commute.}
where \( d \) is the dimension of \( \text{Inv}_{SU(2)}(j) \). Since the spin \( J \) defined by the eigenvalues \( J(J + 1) \) of \( G_{12}^2 \) can take only \( d \) values a polynomial of degree \( d - 1 \) can reproduce any dependence of \( X \) on \( G_{12}^2 \).

Because \( X \) also commutes with \( G_{12}^2 \)

\[
0 = [G_{13}^2, X] = \sum_{q=0}^{d-1} b_q [G_{13}^2, (G_{12}^2)^q]
\]

This condition implies that \( b_q = 0 \ \forall q \neq 0 \), so that \( X = b_0 \mathbf{1} \). To prove this it is sufficient to consider the matrix elements of (9) between the basis intertwiners \( |J\rangle \) and \( |J-1\rangle \) (which I will denote \( |J\rangle \) and \( |J-1\rangle \) in the following) for all allowed values of \( J \). \( \langle J|[G_{13}^2, (G_{12}^2)^q]|J-1\rangle \) is obviously zero when \( q = 0 \). When \( q \geq 1 \) it equals

\[
\sum_{r=0}^{q-1} \langle J|(G_{12}^2)^r [G_{13}^2, G_{12}^2] (G_{12}^2)^{q-1-r}|J-1\rangle = \beta_J P_q(J),
\]

where \( \beta_J = \langle J|[G_{13}^2, G_{12}^2]|J-1\rangle \), and

\[
P_q(J) = \sum_{r=0}^{q-1} [(J)(J+1)]^r [(J-1)J]^{q-1-r}.
\]

(9) therefore implies

\[
0 = \beta_J \sum_{q=1}^{d-1} b_q P_q(J).
\]
The matrix elements of $[G^2_{13}, G^2_{12}]$ have been worked out explicitly by Levý-Leblond and Levý-Nahas [LL65]. They find

\[ \beta_J = \frac{2j_4 + 1}{\sqrt{4J^2 - 1}} \left\{ [(j_1 + j_2 + 1)^2 - J^2][J^2 - (j_1 - j_2)^2] \right\}^{\frac{1}{2}} \times \left\{ [(j_3 + j_4 + 1)^2 - J^2][J^2 - (j_3 - j_4)^2] \right\}^{\frac{1}{2}} \]  

(13)

The important feature of this expression for us is that it is non-zero for all values of $J$ such that both $J$ and $J - 1$ satisfy the triangle inequalities

\[ |j_1 - j_2| \leq \text{spin} \leq j_1 + j_2 \]  
\[ |j_3 - j_4| \leq \text{spin} \leq j_3 + j_4 \]  

(14) (15)

for the spin on the internal edge of $\begin{array}{c} \hline \hline \end{array}$. It is thus non-zero for all but the smallest of the values of $J$ corresponding to the intertwiners $|J\rangle$ spanning $\text{Inv}_{\text{SU}(2)}(J)$. The condition (12) therefore implies

\[ 0 = \sum_{q=1}^{d-1} b_q P_q(J) \]  

(16)

for all but one of the $d$ allowed values of $J$.

Now note that, firstly, the highest order term in $P_q(J)$ is $J^{2(q-1)}$, and, secondly, that $P_q(-J) = P_q(J)$. It follows that the $P_q$ have the form $P_q(J) = \sum_{p=0}^{d-2} A_{qp} J^{2p}$, where $A$ is a $(d-1) \times (d-1)$ matrix. Moreover, since their leading powers of $J$ are all distinct, the $P_q$ are linearly independent polynomials, implying that $A$ is invertible. From (13) it follows that $\sum_{q=1}^{d-1} b_q A_{qp} = 0 \ \forall p \in \{0, ..., d-2\}$. The invertibility of $A$ then shows that $b_q = 0 \ \forall q \in \{1, ..., d-1\}$. As claimed $X = b_0 1$.

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8 $[G^2_{13}, G^2_{12}] = -4i\epsilon^{ijk}G_{1i}G_{2j}G_{3k}$. It is thus $-4i$ times the squared 3-volume operator associated with the four valent vertex in loop quantum gravity [RS95, AL96b, Barb97a].

9 (13) corresponds to equation (2.17) of [LL65]. Their $J$ is called $j_4$ in our notation and their $l$ is our $J$. Our formula has an extra factor $(2j_4 + 1)$ relative to theirs because, while they are evaluating the matrix element between states of three spins having definite values ($j_4$ and $m_4$) of the magnitude and 3 axis component of the total spin, we are calculating the matrix elements between states of four spins which have total spin zero. In this latter calculation one must sum over the $2j_4 + 1$ possible values of $m_4$. 

10
The result is thus established for four valent vertices. Let’s now consider a vertex of arbitrary valence. Fix a particular trivalent tree graph $T$ expanding the vertex and consider a four valent fragment of the graph consisting of an internal edge $e$, and the four (internal or external) edges attached to it (see Figure 1). The preceding arguments relating to four valent vertices can be applied directly to this fragment and show that the eigenvalue $a^T_{j\mathbf{J}}$ of $X$ on $|T, j, \mathbf{J}\rangle$ is independent of the spin on $e$. Since this holds for any internal edge of $T$ the eigenvalue is independent of all the internal spins, i.e it has a common value $c \in \mathbb{C}$ on each element of the basis $\{|T, j, \mathbf{J}\rangle\}_J$. Hence $X = c\mathbf{1}$ and the relativistic intertwiner is $cP$.\[\]

I close with a few observations

- The BC model has a simple, physically motivated extension to arbitrary polyhedral complexes (as opposed to simplicial complexes) representing spacetime. The generalized BC constraints of definition 1 are equivalent to the requirement that relativistic spin networks define equal left and right handed areas for any surface, including ones cutting through vertices. Here the left and right handed areas are determined from the left and right handed spins on the spin network edges and on the virtual edges of trivalent tree expansions of the vertices using the area operator of loop quantum gravity [RS95, AL96a, FLR96].

In [Rei98] it has been shown that GR is a branch of the theory obtained by restricting $SO(4)$ (or $Spin(4)$) BF theory to histories in which left handed and right handed areas are equal. That is, the solutions of GR

\[\]
form a branch of the stationary points of the $SO(4)$ BF action with respect to variations that respect the constraint that left handed areas equal right handed areas. This provides a motivation of the BC model which may be extended to polyhedral complexes: $Spin(4)$ BF theory is just two non-interacting $SU(2)$ BF theories, corresponding to $SU(2)_L$ and $SU(2)_R$ respectively. Thus Ooguri’s simplicial lattice sum over histories quantization of $SU(2)$ BF theory immediately provides a simplicial quantization of $Spin(4)$ BF theory. The BC model is then obtained from simplicial $Spin(4)$ BF theory by restricting the histories to ones in which left and right handed areas are equal. A history in the $Spin(4)$ BF theory defines a $Spin(4)$ spin network on the boundary of each 4-simplex (or more precisely, on the 1-skeleton of the dual of the boundary seen as a three dimensional simplicial complex), which plays the role of boundary data in the sense that the spins and intertwiners of the spin networks on two neighboring 4-simplices must match in their mutual boundary. The requirement that left and right handed areas be equal then reduces the allowed $Spin(4)$ spin networks to just relativistic spin networks.

Ooguri’s quantization of BF theory is most easily generalized to arbitrary polyhedral complexes in the connection formulation (see [Rei97] for a detailed discussion). In this formulation the boundary data on each 4-cell is a lattice connection (of $Spin(4)$ in our case) defined by the parallel transport matrices across the 2-cells separating the 3-cells of the boundary of the 4-cell. (Equivalently, it is a lattice connection on the 1-skeleton of the dual of the boundary). The amplitude of this connection is a delta distribution with support on flat connections. Clearly the sum over histories yields the same states on the boundary of the spacetime (once infinities stemming from redundancies in the delta functions are factored out) whether simplices or arbitrary convex polyhedral 4-cells form the spacetime complex.

If one transforms the sum over connection boundary data to a sum over spin network boundary data (see [Rei97]) one finds, for polyhedral

\[^{10}\] This model is also known as the Crane-Kauffman-Yetter model because Crane, Kauffman, and Yetter [CrYe97] regulated it by q deforming it and were then able to show that the resulting regulated model really defines a continuum (lattice independent) topological field theory.
complexes as for simplicial complexes, that the amplitudes of each history is just the product of the evaluations of the spin networks on the 4-cells, times a factor \((2j + 1)^2\) for each 2-cell, with \(j = j_L = j_R\) the common value of the spins carried by the spin network edges crossing that 2-cell. Applying the constraint that left areas equal right areas for all surfaces, even ones crossing the vertices of the spin networks, restricts the spin networks on the boundaries of the 4-cells to be relativistic spin networks in the extended sense of our definition 1. This defines the generalization of the BC model to polyhedral complexes.

- To completely determine the BC sum over histories a normalization has to be chosen for the relativistic intertwiners. The four valent relativistic intertwiner given by BC in \([3aCr97]\) seems to be just \(P\). On the other hand, if the sum over histories is to be truly a restriction of the sum over histories for BF theory then the relativistic intertwiner must be normalized in the sense that its contraction on all indices with its complex conjugate must be 1. Thus it must be \(P/\sqrt{d}\) up to a phase, where \(d\) is the dimensionality of \(\text{Inv}_{SU(2)}(j)\).

- Barrett \([Ba98]\) has shown that Yetter’s extension to arbitrary valence of the four valent relativistic intertwiner found by BC is equal to

\[
\int_{SU(2)} dg \prod_{j \in j} U^{(j)}(g)
\]  

(when the conventions fixing the bases in the irrep carrying spaces are adopted). Here \(U^{(j)}(g)\) is the spin \(j\) representation matrix of \(g \in SU(2)\) corresponding to the basis convention, and the normalized Haar measure is used to integrate over the group. (17) is precisely the orthogonal projector \(P\) on \(\text{Inv}_{SU(2)}(j)\). Thus the theorem shows that the unique solution (mod normalization) of our generalization of BC’s constraints is Yetter’s extension of their solution.

- Since \(j_R\) is everywhere equal to \(j_L\) the BC state sum can be viewed as a sum over histories of left handed “fields”, i.e. the \(j_L\) only. Thus the BC model can be viewed, like the model of \([Rei97]\) as a (proposal for) a formulation of quantum GR in terms of “self-dual” variables. (Note that this does not mean that it is a model of only the self-dual sector of
GR, in which the anti-self dual curvature vanishes. It only means that exclusively self-dual variables are used to express the configuration of the gravitational field.)

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