Witten Genus on String Toric Complete Intersections

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By using the equivariant localization formula of toric varieties. We prove the vanishing of the Witten genus of some string complete intersections in smooth toric varieties.

1 Introduction

Let $M$ be a $4k$ dimensional closed oriented smooth manifold. In Witten (1988), a multiplicative genus is defined by formally applying the equivariant Atiyah-Singer index theorem to a hypothetical Dirac operator in the loop space and we can get the analogue of $\hat{A}$-genus

$$W(M) = \left\langle \hat{A}(TM)Ch(\Theta(T\mathbb{C}M)), [M] \right\rangle,$$

where

$$\Theta(T\mathbb{C}M) = \bigotimes_{m=1}^{\infty} S_{q^m}(T\mathbb{C}M - \mathbb{C}^{4k})$$

is the Witten bundle where $q = e^{2\pi \sqrt{-1} \tau}$ with $Im(\tau) \geq 0$. Also,

$$S_{q^m}(T\mathbb{C}M - \mathbb{C}^{4k}) := \sum_{k=0}^{\infty} (S^k(T\mathbb{C}M - \mathbb{C}^{4k}))(q^m)^k,$$

where $S^k(T\mathbb{C}M - \mathbb{C}^{4k})$ is the k-th symmetric power of the formal difference $T\mathbb{C}M - \mathbb{C}^{4k}$ in the K-theory. To manifest the modular aspects of Witten genus, following Liu (1996), we can also write it as

$$W(M) = \left\langle \prod_{i} \frac{z_i^{\theta'(0,\tau)}}{\theta(z_i,\tau)}, [M] \right\rangle,$$

where $\{\pm 2\pi \sqrt{-1} z_i, 1 \leq i \geq 2k\}$ are the formal Chern roots of the bundle $T\mathbb{C}M$. 

The oriented manifold $M$ is called spin if the second Stiefel-Whitney class $w_2(TM)$ vanishes. Moreover, manifold $M$ is called string if the half first Pontryagin class vanishes. According to the Atiyah-Singer index theorem, when manifold is spin, the Witten genus is an integral expansion in terms of $q$ (cf. Hirzebruch et al. (1992)). It is also well known that if the manifold is string, the Witten genus is a modular form of weight $2k$ over $SL(2, \mathbb{Z})$ with integral Fourier expansion (Zagier (1988)). Analogous to Lichnerowicz’s classical result on $\hat{A}$ genus (cf. Lawson and Michelsohn (1990)), which stated that $\hat{A}$ genus on spin manifold with positive scalar curvature vanishes, Stolz conjectured that the Witten genus on string manifold with positive Ricci curvature vanishes (for the original arguments, cf. Stolz (1996), and for a review cf. Dessai (2009)).

Towards the Stolz’s conjecture, several vanishing results have been discovered. There are basically two types of methods. One is to apply the theorem of Dessai (1994) which is based on Liu et al. (1995) when the manifold admits some nontrivial action of a semi-simple Lie group. The current results via this method include:

1. String homogeneous spaces of compact semi-simple Lie groups. (Dessai (1994); Liu et al. (1995).
2. Total spaces of fiber bundles with fiber $G/H$, with compact semi-simple structure group $G$ Stolz (1996).
3. Generalized string complete intersection in irreducible, compact, Hermitian, symmetric spaces Förster (2007).
4. String manifold with effective torus action such that $\dim T > b_2(M)$ Wiemeler (2017), where $T$ is a compact torus and $b_2(M)$ is the second Betti number of $M$.

Alternatively, sometimes, one can reduce the calculation of Witten genus to calculation of residues. This method was first used by Landweber and Stone Hirzebruch et al. (1992), which is purely computational and more direct. The vanishing results include:

1. String complete intersection in projective space Hirzebruch et al. (1992).
2. String generalized complete intersection in products of projective spaces Chen and Han (2008)
3. String complete intersection in products of Grassmannians and flag manifolds Zhou and Zhuang (2014); Zhuang (2016)

This method also has many applications in elliptic genus, e.g., Gorbounov and Ochanine (2008); Ma and Zhou (2005). In this paper we generalize the result of Chen and Han (2008) to string complete intersections in Toric varieties. We mainly follow the second method in our calculation and also borrow the techniques of equivariant localization in Dessai (2016).

Main Result

Consider a symplectic toric variety $X$ with a set of invariant divisors $\{D_{\rho_j}\}_{1 \leq j \leq r}$ and of Picard number $k$. Choose a basis $\{q_1, \ldots, q_k\}$ of the Picard group to expand all the invariant divisors with integer coefficients

$$D_{\rho_j} = \begin{cases} q_j & 1 \leq i \leq k \\ \sum_{i=1}^k m_{ji}q_i & k+1 \leq j \leq r. \end{cases}$$

Consider a smooth complete intersection $Y \subset X$ as a generic intersection of $s$ hypersurfaces $\{Y_l\}_{1 \leq l \leq s}$. Each $Y_l$ is dual to the cohomology class $\sum_j d_{li}q_i$.

**Theorem 1.1.** When the integer matrix elements $(m_{ji})$ and $(d_{ji})$ satisfy

$$\sum_{j=1}^n d_{ji}d_{jl} - \sum_{j=k+1}^r m_{ji}m_{jl} = 0 \text{ for } i \neq l.$$

and

$$\sum_{j=1}^n d_{ji}^2 - \sum_{j=k+1}^r m_{ji}^2 - 1 = 0,$$

then the complete intersection $Y$ is string and its Witten genus vanishes.

2 Preparations

We will gather all the necessary notions and lemmas in this section.

**Definition 2.1.** $M$ is a $4k$-dimensional compact oriented manifold with a real vector bundle $E$ of rank $2n$, the **Witten class** of $E$ is defined to be

$$\mathcal{W}(E, M) := \hat{A}(E)Ch(\Theta(E \otimes \mathbb{C})).$$
One can check that Whitney product formula holds for Witten class, i.e. for an exact sequence of real vector bundle $0 \rightarrow E \rightarrow F \rightarrow G \rightarrow 0$, we have

$$W(E) \cdot W(G) = W(F).$$

It is customary to choose Chern roots $\{ \pm 2\pi \sqrt{-1}z_i, 1 \leq i \geq 2k\}$, which simplifies the Witten genus to

$$W(M) = \left\langle \prod_i z_i \frac{\theta'(0, \tau)}{\theta(z_i, \tau)}, [M] \right\rangle,$$

where $\theta(x, \tau)$ is the first Jacobi theta function.

**Global Residue Theorem**

First, we recall the residue theorem (Chapter 5 of Griffiths and Harris (1994)). Let $M$ be a compact complex manifold of dimension $s$. Suppose that $D_i$ for $i = 1, ..., m$ are effective divisors, the intersection of which is a finite set of points. Let $D = D_1 + ... + D_m$. Let $\omega$ be a meromorphic $m$–form on $M$ with polar divisor $D$. For each point $P \in D_1 \cap ... \cap D_m$, we may restrict $\omega$ to a neighborhood $U_P$ of $P$ and define the residue at $P$, denoted by $\text{Res}_P \omega$. Then, one has:

**Lemma 2.2.** *(Residue theorem).*

$$\sum_{P \in D_1 \cap ... \cap D_m} \text{Res}_P \omega = 0.$$

**Jacobi Theta Functions**

We collect all the necessary fact of Jacobi theta function here, all of which can be found in Chandrasekharan (1985). The theta functions are defined as follows:

$$\theta(v, \tau) = 2q^{1/8} \sin(\pi v) \prod_{j=1}^{\infty} \left[ (1 - q^j) (1 - e^{2\pi iv} q^j) (1 - e^{-2\pi iv} q^j) \right],$$

$$\theta_1(v, \tau) = 2q^{1/8} \cos(\pi v) \prod_{j=1}^{\infty} \left[ (1 - q^j) (1 + e^{2\pi iv} q^j) (1 + e^{-2\pi iv} q^j) \right],$$
\[ \theta_2(v, \tau) = \prod_{j=1}^{\infty} \left[ (1 - q^j) \left( 1 - e^{2\pi iv q^j - \frac{1}{2}} \right) \left( 1 - e^{-2\pi iv q^j - \frac{1}{2}} \right) \right], \]
\[ \theta_3(v, \tau) = \prod_{j=1}^{\infty} \left[ (1 - q^j) \left( 1 + e^{2\pi iv q^j - \frac{1}{2}} \right) \left( 1 + e^{-2\pi iv q^j - \frac{1}{2}} \right) \right], \]

where \( q = e^{2\pi \sqrt{-1} \tau} \) with \( \text{Im}(\tau) \geq 0 \). We also have the Jacobi identity:
\[ \theta'(0, \tau) = \frac{\partial}{\partial v} \theta(v, \tau)|_{v=0} = \pi \theta_1(0, \tau) \theta_2(0, \tau) \theta_3(0, \tau). \]

\[ \text{(1)} \]

They satisfy the transformation law under the translation on lattice \( \{ \mathbb{Z} + b \mathbb{Z} \} \)
\[ \theta(v + m, \tau) = (-1)^m \theta(v, \tau), \quad \theta(v + n\tau, \tau) = (-1)^n e^{-2\pi iv\tau - \pi in^2 \tau} \theta(v, \tau), \]
\[ \theta_1(v + m, \tau) = (-1)^m \theta_1(v, \tau), \quad \theta_1(v + n\tau, \tau) = e^{-2\pi iv\tau - \pi in^2 \tau} \theta_1(v, \tau), \]
\[ \theta_2(v + m, \tau) = \theta_2(v, \tau), \quad \theta_2(v + n\tau, \tau) = (-1)^n e^{-2\pi iv\tau - \pi in^2 \tau} \theta_2(v, \tau), \]
\[ \theta_3(v + m, \tau) = \theta_3(v, \tau), \quad \theta_3(v + n\tau, \tau) = e^{-2\pi iv\tau - \pi in^2 \tau} \theta_3(v, \tau). \]

**Equivariant Localization and Genera as Residues**

There is a ring morphism between the ordinary cohomology ring \( H^*(X, \mathbb{Q}) \) and the equivariant cohomology ring \( H_T^*(X, \mathbb{Q}) \). \( H_T^*(X, \mathbb{Q}) \) can be thought of as an \( H^*(BT) \)-module. Note also that the inclusion of a fiber \( i_X : X_{\Sigma} \to X \times_G EG \) induces the “Non-equivariant limit” map \( i_X^* : H_T^*(X, \mathbb{Q}) \to H^*(X, \mathbb{Q}) \), which amounts to mapping all \( \lambda_i \) to 0. In the case of toric varieties, there is an equivariant version of Jurkiewicz-Danilov Theorem, see Chapter 12 of Cox et al. (2009)

**Lemma 2.3.** Every \( D_\rho \) has an equivariant counterpart \( (D_\rho)_T \), where \( \rho \in \Sigma(1) \). This map induces the morphism between cohomology and their equivariant counterpart.
\[ D_\rho \longrightarrow (D_\rho)_T = D_\rho - \lambda_\rho \in H_T^2(X, \mathbb{Q}) \]
\[ H^*(X, \mathbb{Q}) \longrightarrow H_T^*(X_{\Sigma}, \mathbb{Q}) = \mathbb{Q}[\langle(D_{\rho_1})_T, ..., (D_{\rho_s})_T\rangle/(I_T + J_T)] \]

where
\[ I_T = \langle (D_{i_1})_T \cdots (D_{i_s})_T | i_j \text{ are distinct and } \rho_{i_1} + \cdots + \rho_{i_s} \text{ is not a cone of } \Sigma \rangle \]
and \( J_T \) is the ideal generated by the linear forms

\[
\sum_{i} \langle m, u_i \rangle (D_{\rho_i})_T
\]

where \( m \) ranges over \( M \) (or equivalently, over some basis for \( M \)).

We can find an explicit way to proceed calculations via the Atiyah-Bott localization. The following formula can be found in Givental (1998)

**Lemma 2.4.** \( f(q_1, ..., q_k, \{ \lambda_i \}) \) is a polynomial in \( H^*_T(X, \mathbb{Q}) \), integrating it over the fundamental class \([X]\) maps it into \( \mathbb{Q}[\lambda_1, ..., \lambda_r] \). Explicitly, we have

\[
\int_X f(q_1, ..., q_k, \{ \lambda_i \}) = \sum_{\alpha} \text{Res}_{\alpha} \frac{f(q_1, ..., q_k, \{ \lambda_i \})}{(D_1)_T \cdot (D_2)_T \cdot ... \cdot (D_r)_T} dq_1 dq_2 ... dq_k. \tag{2}
\]

The symbol \( \text{Res}_{\alpha} \) refers to the residue of the \( k \)-form at the pole by the order set of equations

\[
\sum_{i=1}^{k} p_i m_{ij} = \lambda_s, \ s = 1, ..., k.
\]

### 3 Main Results

From now on, we will abuse the notation to write \( D_\rho \) instead of \([D_\rho]\) as the cohomology classes and divisor classes. We consider a smooth compact toric variety \( X_\Sigma \) corresponding to \( \Sigma \in N_\mathbb{R} \), where \( N \) is a lattice of dimension \( n \) and \( M \) being its dual lattice. Let \( \Sigma(1) \) denote the set of one dimensional cones in \( \Sigma \) and assume that \( |\Sigma(1)| = r \). By Proposition 4.2.1 and 4.2.5 in Cox et al. (2009), we know \( \text{Pic}(X_\Sigma) \) is torsion-free and

\[
0 \longrightarrow M \longrightarrow \mathbb{Z}^r \longrightarrow \text{Pic}(X_\Sigma) \longrightarrow 0.
\]

If we set the Picard number \( k \), we have the relation

\[
\dim X_\Sigma = \dim N = n = r - k.
\]

After we quotient the ideal generated by the linear relations, we can equivalently say that the cohomology ring is multiplicatively generated by a basis
of the Picard group. We denote the basis of Picard group as \( q_1, \ldots, q_k \), and express the invariant divisors as

\[
D_{\rho_i} = \begin{cases} 
q_i & 1 \leq i \leq k \\
\sum_{j=1}^k m_{ij} q_j & k + 1 \leq i \leq r
\end{cases}
\]

Then let’s consider a smooth complete intersection \( Y \) as a generic intersection of \( s \) hypersurfaces \( \{ Y_l \}_{1 \leq l \leq s} \). Each \( Y_l \) is dual to the cohomology class \( \sum_{j} d_{lj} q_j \). In the context of smooth manifold, by the transversality argument, one can always deform the hypersurfaces (not algebraic now) so that they intersect transversally. Thus \( Y \) is submanifold of complex dimension \( r - k - s \). In this section we will consider the Witten genus on \( Y \). From now on, we will use both \( X := X_\Sigma \) to denote the toric variety, when there is no confusion.

### 3.1 Proof of Theorem 1.1

For smooth toric varieties, there is a **generalized Euler sequence** (see Theorem 8.1.6 in Cox et al. (2009)).

\[
0 \rightarrow \mathcal{O}_{X_\Sigma}^{\oplus k} \rightarrow \bigoplus_{\rho \in \Sigma(1)} \mathcal{O}_{X_\Sigma}(D_{\rho}) \rightarrow TX_\Sigma \rightarrow 0,
\]

where \( k \) is the rank of \( \text{Pic}(X_\Sigma) \) or equivalently the rank of \( H^2(X_\Sigma, \mathbb{Z}) \), from which we can calculate the total Chern class and all the multiplicative genera of \( TX_\Sigma \).

The \( \mathcal{O}_{X_\Sigma}(D_{\rho}) \) is the line bundle defined by the divisor \( D_{\rho} \) in the Picard group, of which the first Chern class is \( D_{\rho} \in H^2(X_\Sigma, \mathbb{Z}) \). Then by the multiplicative property of total Chern class, we have

\[
c(TX_\Sigma) = \bigoplus_{\rho \in \Sigma(1)} c(\mathcal{O}_{X_\Sigma}(D_{\rho})) = \prod_{\rho \in \Sigma(1)} (1 + D_{\rho}).
\]

The \( \{D_{\rho}\} \) play the role of Chern roots here, which is not coincidence. For a general multiplicative genus \( \varphi_Q \) corresponding to even power series \( Q(x) \), we observe that

\[
\varphi_Q(TX_\Sigma) = \frac{\varphi_Q\left(\bigoplus_{\rho \in \Sigma(1)} \mathcal{O}(D_{\rho})\right)}{\varphi_Q(\mathcal{O}(X_\Sigma)^{\oplus k})} = \frac{\prod_{\rho \in \Sigma(1)} Q(D_{\rho})}{Q(0)^k},
\]

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In the case of Witten genus, we have
\[ W(T_X) = \prod_{\rho \in \Sigma(1)} D_{\rho} \theta'(0, \tau). \]
\[ W(X) = \langle W(X), [X] \rangle \]

Consider the inclusion map \( \iota : Y \rightarrow X \), we have the adjunction formula:
\[ 0 \rightarrow TY \rightarrow \iota^* TX \rightarrow \iota^* NYX \rightarrow 0, \]
where the normal bundle \( NYX \) is isomorphic to \( \bigoplus_{i=1}^{s} O(\sum_{j} d_{ij} q_{j}) \). By the multiplicative properties of Chern class, we have the following data:
\[ c(TY) = \iota^* \left( \frac{c(TX)}{c(NYX)} \right) = \iota^* \left( \frac{\prod_{i=1}^{k}(1 + q_{i}) \prod_{j=k+1}^{r} (1 + \sum_{i=1}^{k} m_{ji} q_{i})}{\prod_{i=1}^{s} (1 + \sum_{j} d_{ij} q_{j})} \right). \]

Following these we have
\[ w_2(T_Y) \equiv c_1(TY) (\mod 2) \]
\[ = \iota^* \left( \sum_{i=1}^{k} q_{i} + \sum_{j=k+1}^{r} \sum_{i=1}^{k} m_{ji} q_{i} - \sum_{l=1}^{s} \sum_{i=1}^{k} d_{li} q_{i} \right) (\mod 2) \]
and
\[ p_1(T_Y) = \iota^* \left( \sum_{i=1}^{k} q_{i}^2 + \sum_{j=k+1}^{r} \left( \sum_{i=1}^{k} m_{ji} q_{i} \right)^2 - \sum_{l=1}^{s} \left( \sum_{i=1}^{k} d_{li} q_{i} \right)^2 \right). \]

Because we know little about the \( H^4(X, \mathbb{Z}) \), it is difficult to tell whether the above presentation of Pontryagin class vanishes in the general case. Still, we know the complete intersection is string when all the coefficients of \( \{q_i q_j\} \) vanishes. On the other hand, because of the multiplicative property of Witten class, we have
\[ W(T_Y) = \iota^* \left( \frac{W(T_X)}{W((NYX)_R)} \right) \]
\[ = \iota^* \left( \frac{\prod_{i=1}^{k} q_{i} \theta'(0, \tau) \prod_{j=k+1}^{r} (\sum_{t=1}^{k} m_{jt} q_{t}) \theta'(0, \tau)}{\prod_{l=1}^{s} (\sum_{j} d_{lj} q_{j}) \theta'(0, \tau)} \right). \]
Integrate the above formula over the fundamental class of $Y$, we have

$$
\int_Y \mathcal{W}(T_R Y)
= \int_X \left( \frac{\mathcal{W}(T_R X)}{\mathcal{W}((N_Y X)_R)} \right)
= \int_X \left( \frac{\mathcal{W}(T_R X)}{\mathcal{W}((N_Y X)_R)} \right) \sim e(N_Y X)
= \int_X \left( \frac{\prod_{i=1}^k q_i \theta(q_i, \tau)}{\prod_{j=1}^k \theta'(d_j q_j)} \prod_{l=1}^s \theta(\sum_{j=1}^{k+l} d_j q_j, \tau) \cdot \prod_{j=1}^k \theta(q_j, \tau) \theta'(0, \tau) \prod_{l=1}^s \theta(\sum_{j=1}^{k+l} d_j q_j, \tau) \theta'(0, \tau)}{\prod_{l=1}^s \theta(\sum_{j=1}^{k+l} d_j q_j, \tau) \theta'(0, \tau)} \right) \cdot \prod_{j=1}^k \theta(q_j, \tau) \theta'(0, \tau)
$$

where $e(N_Y X)$ in the second line is the Euler class of the normal bundle.

If we take the non-equivariant limit we can get an explicit formula to calculate the integration of $f(p, 0)$ over the fundamental class $[X]$.

### 3.2 Vanishing result for some string complete intersection

What we want to calculate is

$$
\int_Y \mathcal{W}(T_R Y) = \int_X \left( \frac{\mathcal{W}(T_R X)}{\mathcal{W}((N_Y X)_R)} \right)
= \int_X \left( \frac{\prod_{i=1}^k q_i \theta(q_i, \tau)}{\prod_{j=1}^k \theta'(d_j q_j)} \prod_{l=1}^s \theta(\sum_{j=1}^{k+l} d_j q_j, \tau) \cdot \prod_{j=1}^k \theta(q_j, \tau) \theta'(0, \tau) \prod_{l=1}^s \theta(\sum_{j=1}^{k+l} d_j q_j, \tau) \theta'(0, \tau)}{\prod_{l=1}^s \theta(\sum_{j=1}^{k+l} d_j q_j, \tau) \theta'(0, \tau)} \right)
$$

Note that the complete intersection $Y$ is generally not invariant under the action of torus, but remember that $TX_T$ and $N_Y X \cong \mathcal{O}(\sum_{j=1}^k d_j q_j)$ are invariant bundles, we know that the differential form on the right hand side is invariant. With the localization argument before, we can consider an integral of polynomial $\mathcal{W}(q, \{\lambda_i\})$ in $H^*_T(X, \mathbb{Q})$ which under nonequivariant limit gives the integral we want. The choice is not unique. We can choose
\[ \int_Y \mathcal{W}(T_R Y) \]
\[ = i_X \int_X \left( \prod_{i=1}^k (q_i - \lambda_i) \frac{\theta'(0, \tau)}{\theta(q_i - \lambda_i, \tau)} \right) \prod_{j=k+1}^{n+1} \left( \sum_{t=1}^k m_{jt} q_t - \lambda_j \right) \frac{\theta'(0, \tau)}{\theta(\sum_{t=1}^k m_{jt} q_t - \lambda_j, \tau)} \right) \]

Then by Eq. 2,
\[ \int_Y \mathcal{W}(T_R Y) \]
\[ = i_X \sum_{\alpha} \text{res}_\alpha \left( \prod_{i=1}^k (q_i - \lambda_i) \frac{\theta'(0, \tau)}{\theta(q_i - \lambda_i, \tau)} \right) \prod_{j=k+1}^{n+1} \left( \sum_{t=1}^k m_{jt} q_t - \lambda_j \right) \frac{\theta'(0, \tau)}{\theta(\sum_{t=1}^k m_{jt} q_t - \lambda_j, \tau)} \right) \]
\[ = i_X \sum_{\alpha} \text{res}_\alpha \left( \prod_{i=1}^k \frac{\theta(q_i - \lambda_i, \tau)}{\theta'(0, \tau)} \prod_{j=k+1}^{n+1} \frac{\theta(\sum_{t=1}^k m_{jt} q_t - \lambda_j, \tau)}{\theta'(0, \tau)} \right) dq_1 \cdots dq_k \]
\[ = \lim_{\lambda_i \to 0} \sum_{\alpha} \text{res}_\alpha \left( \prod_{j=1}^r f_j(q, \tau, \{\lambda_i\}) \right) dq_1 \cdots dq_k, \]

where
\[ g(q_1, \ldots, q_k, \tau) = \prod_{l=1}^s \frac{\theta(\sum_{j=1}^k d_{lj} q_j, \tau)}{\theta'(0, \tau)} \]

and
\[ f_j(q_1, \ldots, q_k, \tau, \{\lambda_i\}) = \begin{cases} \frac{\theta(q_j - \lambda_j, \tau)}{\theta'(0, \tau)} & \text{for } 1 \leq j \leq k \\ \frac{\theta(\sum_{t=1}^k m_{jt} q_t - \lambda_j, \tau)}{\theta'(0, \tau)} & \text{for } k + 1 \leq j \leq r \end{cases} \]

Let’s analyze how \( g \) and \( f \) transform under the translation over the lattice \( \mathbb{Z} + \mathbb{Z} \tau \). By the transforming laws of theta functions[1], without loss of generality, \( q_1 \to q_1 + 1 \).

\[ g(q_1 + 1, \ldots, q_k, \tau) = (-1)^{d_{11} + \cdots + d_{1k}} g(q_1, \ldots, q_k, \tau), \]
\[ f_j(q_1 + 1, \ldots, q_k, \tau, \{\lambda_i\}) = \begin{cases} -f_1(q_1, \ldots, q_k, \tau, \{\lambda_i\}) & \text{for } j = 1 \\ f_j(q_1, \ldots, q_k, \tau, \{\lambda_i\}) & \text{for } j \neq 1 \text{ and } j \leq k \\ (-1)^{m_{j1}} f_j(q_1, \ldots, q_k, \tau, \{\lambda_i\}) & \text{for } k + 1 \leq j \leq r \end{cases} \]
Then,
\[
\prod_{j=1}^{s} f_j(q_1, \ldots, q_i + 1, \ldots, q_k, \tau) = (-1)\sum_{j=1}^{r} d_{ji} - \sum_{j=k+1}^{r} m_{ji} - 1 \frac{g(q_1, \ldots, q_k, \tau)}{\prod_{j=1}^{s} f_j(q_1, \ldots, q_k, \tau, \{\lambda_i\})}.
\]

On the other hand under the translation \(q_1 \to q_1 + \tau\), we have
\[
g(q_1 + \tau, \ldots, q_k, \tau) = (-1)^{d_1 + \cdots + d_s} e^{-2\pi i \sum_{j=1}^{s} d_{ji} \tau} g(q_1, \ldots, q_k, \tau),
\]
and
\[
f_j(q_1 + \tau, \ldots, q_k, \tau, \{\lambda_i\}) = \begin{cases} (-1)^{e^{-2\pi i (q_1 - \lambda_1) - \pi i \tau} f_1(q, \tau, \{\lambda_i\})} & \text{for } j = 1 \\ f_j(q, \tau, \{\lambda_i\}) & \text{for } i \neq 1 \text{ and } j \leq k \\ (-1)^{m_j} e^{-2\pi i m_j \tau} f_j(q, \tau, \{\lambda_i\}) & k + 1 \leq j \leq r \end{cases}
\]
Then the we have the transforming law
\[
\prod_{j=1}^{s} f_j(q_1, \ldots, q_i + \tau, \ldots, q_k, \tau, \{\lambda_i\}) = (-1)^{\sum_{j=1}^{r} d_{ji} - \sum_{j=k+1}^{r} m_{ji} - 1} 
\times e^{-2\pi i \sum_{j=1}^{s} d_{ji} \tau - \sum_{i=k+1}^{r} m_{ji} \lambda_i} \sum_{\sum_{u=k+1}^{r} m_{ui} \lambda_u}, 
\times e^{-i \sum_{j=1}^{s} d_{ji} \tau - \sum_{i=k+1}^{r} m_{ji} \lambda_i} \tau \times g(q_1, \ldots, q_i, \ldots, q_k, \tau) 
\prod_{j=1}^{s} f_j(q_1, \ldots, q_i, \ldots, q_k, \tau, \{\lambda_i\})
\]
Remember that
\[
w_2(T_{\mathbb{R}}Y) \equiv c_1(TY)(\text{mod } 2)
\]
\[
= \ell^* \left( \sum_{i=1}^{k} q_i \left( 1 + \sum_{j=k+1}^{r} m_{ji} - \sum_{l=1}^{s} d_{li} \right) \right) \pmod{2},
\]
and
\[
p_1(T_{\mathbb{R}}Y) = \ell^* \left( \sum_{i=1}^{k} q_i^2 + \sum_{j=k+1}^{r} \left( \sum_{i=1}^{k} m_{ji} q_i \right)^2 - \sum_{l=1}^{s} \left( \sum_{i=1}^{k} d_{li} q_i \right)^2 \right)
\]
\[
= \ell^* \left( \sum_{i=1}^{k} q_i^2 \left( 1 + \sum_{j=k+1}^{r} m_{ji} - \sum_{l=1}^{s} d_{li} \right) + \sum_{i=1}^{k} \sum_{j=k+1}^{r} q_i q_j \left( \sum_{u} m_{ji} m_{jl} - \sum_{u} d_{mi} d_{ul} \right) \right)
\]
Also notice that, when

\[ 1 + \sum_{j=k+1}^{r} m_{ji}^2 - \sum_{l=1}^{s} d_{li}^2 = 0, \]

\( \omega_2(T_R Y) \) vanishes automatically because

\[ \sum_{j=k+1}^{r} m_{ji}^2 - \sum_{l=1}^{s} d_{li}^2 \equiv \sum_{j=k+1}^{r} m_{ji} - \sum_{l}^{s} d_{li} \pmod{2}. \]

At nonequivariant limit, we take \( \lambda_i = 0 \), assuming

\[ \sum_{j=1}^{n} d_{ji}d_{jl} - \sum_{j=k+1}^{r} m_{ji}m_{jl} = 0 \text{ for } i \neq l. \]

and

\[ \sum_{j=1}^{n} d_{ji}^2 - \sum_{j=k+1}^{r} m_{ji}^2 - 1 = 0, \]

then the complete intersection \( Y \) is string and the function

\[ \frac{g(q, \tau)}{\prod_{j=1}^{r} f_j(q, \tau, 0)} \]

is elliptic over the lattice \( \Gamma := \{ \mathbb{Z} + \mathbb{Z} \tau \} \). Then equivalently, we can regard

\[ \frac{g(q, \tau)}{\prod_{j=1}^{r} f_j(q, \tau, 0)} dq_1 \cdots dq_k \]

as a meromorphic form over the compact torus \( (\mathbb{C}/\Gamma)^s \). Then by the global residue theorem, the Witten genus

\[ \int_{Y} \mathcal{W}(T_R Y) \]

\[ = \sum_{\alpha} \text{res}_{\alpha} \frac{g(q, \tau)}{\prod_{j=1}^{r} f_j(q, \tau, 0)} dq_1 \cdots dq_k, \]

vanishes.
**Corollary 3.1.** In the special case where the toric variety is product of projective spaces of the form $\mathbb{P}^{n_1} \times \mathbb{P}^{n_2} \times \cdots \times \mathbb{P}^{n_t}$, then up to reordering, the matrix $m_{ij}$ can be written as

$$
\begin{pmatrix}
1,1,...,1 & 0 & \ldots & 0 \\
0 & 1,1,...,1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1,...,1
\end{pmatrix}.
$$

The vanishing condition reduces to

$$
\sum_{j=1}^{n} d_{ji} d_{jl} = 0 \text{ for } i \neq l.
$$

and

$$
\sum_{j=1}^{n} d_{ji}^2 - n_i - 1 = 0,
$$

which reproduces the result of Chen and Han (2008).

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