A Return To Neutrino Normalcy

Peter B. Denton

High Energy Theory Group, Physics Department, Brookhaven National Laboratory, Upton, NY 11973, USA
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Understanding the structure of the fermion mixing matrices is an important question in particle physics. The quark mixing matrix is approximately diagonal while the lepton mixing matrix has large off-diagonal elements. Numerous explanations of these structures exist known as flavor symmetries while an alternative explanation known as anarchy posits that the mass matrix is randomly distributed. These approaches often lack robust predictability and falsifiability. In this letter we propose a new set of conditions to test the structure of mass matrices called normalcy based on how close to diagonal the mixing matrix is. The mass ordering and the octant of $\theta_{23}$ represent two of these conditions. We point out that the quark matrix easily satisfies all six normalcy conditions while none of them are known to be fully satisfied for leptons at high significance. All of the conditions that can be tested for leptons suggest that the matrix could satisfy the normalcy conditions and upcoming experiments such as DUNE and T2HK will most likely determine if the lepton mass matrix satisfies all of them or not.

INTRODUCTION

The weak interaction acts in a different basis than the mass bases for both quarks and leptons [1–4]. While for quarks the two bases are very close together, for leptons the bases are quite different; understanding the structure of how these bases are related is an important open question in our understanding of the Standard Model (SM) of particle physics. The quark mixing matrix is perturbative and is elegantly described by the Wolfenstein parameterization [5] which makes it quite clear not only that the matrix is approximately diagonal, but that as one moves away from the diagonal the elements fall off quite rapidly. In contrast, the lepton mixing matrix includes much larger mixing and is clearly not perturbatively diagonal. To this end there has been a large body of work to parameterize the lepton mixing matrix in the context of various symmetry groups, for some useful reviews see refs. [6–8]. Alternatively, it has been suggested that the parameters in the lepton mixing matrix could be randomly drawn from a uniform distribution drawn over the Haar measure for a 3×3 complex unitary matrix; this suggestion goes by the name of anarchy [9–11]. As upcoming neutrino oscillation experiments are expected to measure the lepton mixing matrix with a fair deal of precision, it is interesting to examine exactly what, if anything, can be determined about the fundamental nature of the lepton mixing matrix. Symmetry and anarchy approaches are difficult to either prove or disprove.

In this letter we propose a set of conditions that can be either verified or falsified. These conditions can provide insight into what is driving the large mixing in the lepton sector, which we dub normalcy conditions. These conditions indicate if the neutrino mass eigenstates mix with the charged lepton states in the way we expect or not. In most cases where normalcy can be tested given currently available data, the data prefers the normal case over the non-normal case. Next generation oscillation experiments such as DUNE and T2HK [12, 13] are necessary and should be sufficient to determine if all the normalcy conditions are simultaneously satisfied or not.

DUNE and JUNO are expected to determine the neutrino mass ordering in coming years [12, 14] and there is already a hint from global fits to oscillation data for the normal mass ordering at $\sim 3\sigma$, although this has yet to be confirmed [15–17]. The mass ordering known as “normal” wherein $\Delta m^2_{31} > 0$ provides the motivation for naming the remaining conditions the “normalcy” conditions.

MASS EIGENSTATE DEFINITIONS

Numerous definitions of the neutrino mass eigenstates exist in the literature. In fact, the exact definition used in any given analysis is often not specified. We will define our fiducial mass eigenstate definition, labeled $e$ to be

$$e : \ |U_{e1}| > |U_{e2}| > |U_{e3}|.$$

This definition has the advantage that we know that there is one mass eigenstate that is $\sim 2/3$ electron neutrino, another that is $\sim 1/3$ electron neutrino, and the final one that is $\sim 2\%$ electron neutrino [18–20]. Note that in this definition, $\theta_{12} < 45^\circ$ by definition while the sign of $\Delta m^2_{31}$ is a free parameter\(^1\), however solar neutrino experiments

\(^1\) Neutrino oscillation experiments are sensitive to two mass orderings: normal and inverted, while absolute mass scale experiments are sensitive to three mass hierarchies: normal, inverted, and quasi-degenerate.

\(^2\) Another commonly used definition, not listed here, treats the solar and atmospheric sectors differently, see the appendix.

\(^3\) We use the standard PDG parameterization for the lepton mixing matrix in terms of three angles and one complex phase $U = O_{23}(\theta_{23})U_{13}(\theta_{13}, \delta)O_{12}(\theta_{12})$ [21].
measured the solar mass ordering to be normal: $\Delta m_{21}^2 > 0$ [18].

Throughout this section we will compare other definitions to this one: if different definitions are equivalent then we say that the given normalcy condition is satisfied.

Another obvious choice assigns the mass eigenstates in increasing mass. Labeled $\mathbf{M}$ this definition is

$$\mathbf{M}: \quad m_1 < m_2 < m_3.$$  \hspace{1cm} (2)

This may well be the definition of choice in the future, especially if it is determined that definitions $\mathbf{e}$ and $\mathbf{M}$ are equivalent. The question of whether or not the $\mathbf{M}$ condition is satisfied is the same as whether the atmospheric mass ordering is normal or inverted, hence the term “normalcy”.

We note, however, that while definition $\mathbf{e}$ is preferential to electron neutrinos due to present experimental data, theoretically one should consider the corresponding definition for tau neutrinos equally,

$$\mathbf{\tau}: \quad |U_{\tau 1}| < |U_{\tau 2}| < |U_{\tau 3}|.$$  \hspace{1cm} (3)

This is also perfectly valid definition, although given the small size of the global $\nu_e$ oscillation data set [22–24], it is not practical to take this as a fiducial definition. Nonetheless this provides another normalcy condition; that is, we say that the lepton mixing matrix is normal if the set of $\mathbf{\tau}$ inequalities are satisfied when the mass eigenstates are defined as in eq. 1.

**NORMALCY CONDITIONS**

The $\mathbf{e}$ and $\mathbf{\tau}$ definitions are based on the flavor basis. One could imagine writing down similar definitions in the mass basis,

$$1: \quad |U_{e1}| > |U_{\mu 1}| > |U_{\tau 1}|,$$  \hspace{1cm} (4)

$$3: \quad |U_{e3}| < |U_{\mu 3}| < |U_{\tau 3}|,$$  \hspace{1cm} (5)

however there is no guarantee that either of these definitions uniquely define the three mass eigenstates. Nonetheless, they do contribute two additional normalcy conditions: whether or not $1$ ($3$) is equivalent to $\mathbf{e}$ or not.

Two additional normalcy conditions exist relating the middle row and column of the matrix,

$$2: \quad |U_{\mu 2}| > |U_{e 2}| , \text{ and } |U_{\mu 2}| > |U_{\tau 2}|,$$  \hspace{1cm} (6)

$$\mu: \quad |U_{\mu 2}| > |U_{\mu 1}| , \text{ and } |U_{\mu 2}| > |U_{\mu 3}|.$$  \hspace{1cm} (7)

These sets of inequalities differ from the others which all require all three numbers to be ordered, while these two only require that one number ($|U_{\mu 2}|$) is larger than two other numbers.

![FIG. 1. The orange shaded regions to the left are the allowed normalcy regions assuming $\theta_{13} = 8.61^\circ$ and $\theta_{12} = 33.82^\circ$. The best fit point and $\Delta \chi^2 = 2.3, 6.18$ corresponding approximately to 1, 2$\sigma$ from the nufit 4.1 global fit (normal mass ordering and without Super-K atmospheric data) [15] are shown as a blue plus and dotted contours respectively. The light region in the lower-left corner is where the first inequality in the $\mathbf{\tau}$ condition no longer holds, although this region is quite disfavored by the data.](image)

Next, we examine when each of these conditions are satisfied. The second inequality in definition $3$ is consistent with $\mathbf{e}$ exactly when $\theta_{23} < 45^\circ$; that is the first octant of $\theta_{23}$ is a normalcy condition. For the remaining conditions, the inequalities involving the $\mu$ and $\tau$ rows and 1 and 2 columns don’t have simple exact solutions, but the second inequality (the one that hasn’t been experimentally determined yet) in each case is well approximated by,

$$1: \quad \cos \delta \gtrsim \frac{s_{12} \cos 2\theta_{23}}{2c_{12}s_{13}} \approx -2.2 \cos 2\theta_{23} ,$$  \hspace{1cm} (8)

$$2: \quad \cos \delta \gtrsim \frac{c_{12} \cos 2\theta_{23}}{2s_{12}s_{13}} \approx 5.0 \cos 2\theta_{23} ,$$  \hspace{1cm} (9)

$$\mu: \quad \cos \delta \lesssim \frac{(c_{13}^2 + c_{12}^2 - s_{12}^2 s_{13}^2)s_{23}^2 - c_{12}^2}{c_{12} s_{12} s_{13}} \approx 10.0 - 24.0 s_{23}^2 ,$$  \hspace{1cm} (10)

$$\tau: \quad \cos \delta \lesssim \frac{(c_{13}^2 + c_{12}^2 - s_{12}^2 s_{13}^2)s_{23}^2 - c_{13}^2 + s_{12}^2 s_{13}^2}{c_{12} s_{12} s_{13}} \approx 14.0 - 24.0 s_{23}^2 ,$$  \hspace{1cm} (11)

where we used the fact that the doubly reduced Jarlskog [25, 26] $J_{rr} \equiv s_{12} c_{12} s_{23} c_{23} s_{13} \approx \frac{1}{2} s_{12} c_{12} s_{13}$ for $\theta_{23} \sim 45^\circ$. The exact inequalities are numerically plotted in fig. 1 assuming $\theta_{13} = 8.61^\circ$ and $\theta_{12} = 33.82^\circ$ [15].

We note that conditions $1$, $2$, and $3$ all cross at exactly $\theta_{23} = 45^\circ$ and $\cos \delta = 0$. The $\theta_{23} = 45^\circ$ statement results from the fact that the second inequality in condition $3$ is exactly whether $\theta_{23}$ is less than $45^\circ$ or not.
Then we use the fact that \(\mu-\tau\) symmetry is exactly satisfied when \(\theta_{23} = 45^\circ\) and \(\cos \delta = 0\), as can easily be seen by expanding the square of the norm of the elements,
\[
|U_{\mu 1}|^2 = s_{12}^2 c_{23}^2 + s_{12}^2 s_{23}^2 s_{13}^2 + 2 s_{12} c_{12} s_{23} c_{23} s_{13} \cos \delta ,
\]
\[
|U_{\tau 1}|^2 = s_{12}^2 c_{23}^2 + s_{12}^2 s_{23}^2 s_{13}^2 - 2 s_{12} c_{12} s_{23} c_{23} s_{13} \cos \delta ,
\]
for the first column and,
\[
|U_{\mu 2}|^2 = c_{12}^2 c_{23}^2 + s_{12}^2 s_{23}^2 s_{13}^2 - 2 s_{12} c_{12} s_{23} c_{23} s_{13} \cos \delta ,
\]
\[
|U_{\tau 2}|^2 = c_{12}^2 s_{23}^2 + c_{12}^2 s_{23} c_{13}^2 + 2 s_{12} c_{12} s_{23} c_{23} s_{13} \cos \delta ,
\]
for the second column. In either case the first two terms are equal when \(\theta_{23} = 45^\circ\) and the third terms are equal when \(\cos \delta = 0\).

While there are six different normalcy conditions, in most cases given the current oscillation picture, only two inequalities are required to be satisfied in order to satisfy all the normalcy conditions as can be seen in fig. 1.
\[
|U_{\mu 2}| > |U_{\mu 3}| \quad \text{and} \quad |U_{\mu 1}| > |U_{\tau 1}| ,
\]
(12)
which can be approximated as,
\[
\cos \delta \gtrsim 4.4 s_{23}^2 - 2.2 ,
\]
(13)
\[
\cos \delta \lesssim 10.0 - 24.0 s_{23}^2 .
\]
(14)

To summarize, assuming that the mass eigenstates are defined by definition \(e\) given in eq. 1, the six (convention independent) normalcy conditions are whether or not the following conditions are true:

\[
M : \quad m_1 < m_2 < m_3 ,
\]
\[
\mu : \quad |U_{\mu 2}| > |U_{\mu 1}| \quad \text{and} \quad |U_{\mu 2}| > |U_{\mu 3}| ,
\]
\[
\tau : \quad |U_{\tau 1}| < |U_{\tau 2}| < |U_{\tau 3}| ,
\]
\[
1 : \quad |U_{e 1}| > |U_{\mu 1}| > |U_{\tau 1}| ,
\]
\[
2 : \quad |U_{\mu 2}| > |U_{e 2}| \quad \text{and} \quad |U_{\mu 2}| > |U_{\tau 2}| ,
\]
\[
3 : \quad |U_{\tau 3}| < |U_{\mu 3}| < |U_{\tau 3}| .
\]

To a good approximation this means that if \(\cos \delta \) and \(s_{23}^2\) satisfy the relationships in eqs. 13-14 and the mass ordering is normal then all the normalcy conditions are satisfied. Schematically the six sets of inequalities relating to the lepton mixing matrix including the fiducial definition \(e\) can be seen in fig. 2.

**CURRENT STATUS AND FUTURE PROSPECTS**

Current global fits indicate that some parts of each condition are satisfied, but none of them are completely satisfied. The \(3\sigma\) allowed region for the absolute value of each element in the lepton mixing matrix is [15]
\[
(0.797 \rightarrow 0.842 \quad 0.518 \rightarrow 0.585 \quad 0.143 \rightarrow 0.156)
\]
\[
0.244 \rightarrow 0.496 \quad 0.467 \rightarrow 0.678 \quad 0.646 \rightarrow 0.772
\]
\[
0.287 \rightarrow 0.525 \quad 0.488 \rightarrow 0.693 \quad 0.618 \rightarrow 0.749
\]

Given that we know that \(\Delta m_{21}^2 > 0\) from solar experiments, we know that \(M\) is partially true. If the atmospheric mass ordering is confirmed to be normal then the \(M\) condition is true. For the \(\tau \) row condition, \(\tau\), we see \(|U_{\tau 1}| < |U_{\tau 3}|\), but \(|U_{\tau 2}|\) could be anywhere relative to the other two.

For the mass eigenstate conditions 3 and 1, we see that the electron component of the conditions are known to be satisfied. For condition 3 that means that we know that \(|U_{e 3}| < |U_{\mu 3}|\) and \(|U_{e 3}| < |U_{\tau 3}|\), but we don’t know which of \(|U_{\mu 3}|\) and \(|U_{\tau 3}|\) are larger. Similarly, for condition 1 that means that we know that \(|U_{e 1}| > |U_{\mu 1}|\) and \(|U_{e 1}| > |U_{\tau 1}|\), but we don’t know which of \(|U_{\mu 1}|\) and \(|U_{\tau 1}|\) are larger.

The picture for the middle row and column is less clear as none of the inequalities are known to be true and some are disfavored at \(\sim 2 \sigma\) (\(\Delta \chi^2 \sim 6\) for the \(\mu\) condition).

The poorest measured oscillation parameters are \(\theta_{23}\) and \(\delta\) which govern essentially all of the remaining uncertainty in the normalcy conditions. DUNE and T2HK both expect to measure \(\theta_{23}\) with \(\lesssim 1^\circ\) resolution and \(\delta\) should be measured with \(\sim 10 - 15^\circ\) precision from each experiment alone [12, 13]. It may be the case that some of these inequalities may be close enough together to make a determination on normalcy quite difficult. In this case a high level of precision will be needed to determine which is larger than the other, although such a strong relationship among elements may be interpreted as an indication of a symmetry present in the lepton mixing matrix. However, we expect that DUNE and T2HK, certainly with addition of other oscillation data in a global fit, should be able to determine if these normalcy conditions are satisfied or not.

We briefly comment on the quark sector. The allowed values of the absolute value of the elements of the quark
mixing matrix at 3σ are [21]4
\[
\begin{pmatrix}
0.974 & 0.975 & 0.223 & 0.226 & 0.003 & 0.005 \\
0.206 & 0.230 & 0.946 & 1.048 & 0.040 & 0.045 \\
0.007 & 0.010 & 0.033 & 0.046 & 0.944 & 1.004
\end{pmatrix},
\]
from which we can see that each normalcy definition is clearly satisfied by also noting that the matrix is defined with both up-like and down-like quarks arranged with increasing mass.

DISCUSSION

These seven sets of inequalities (eqs. 1-7) can be combined into six normalcy conditions, most of which are known to be partially satisfied in the lepton sector suggesting that normalcy may be a valid guiding principle for the lepton mass matrix. If all the normalcy conditions are satisfied for leptons as they are known to be for quarks, that may be an indication that the lepton mixing matrix is derived from some symmetry group. A deviation from normalcy does not rule out symmetry groups as an explanation, but rather normalcy could be seen as an indicator as to the origin of lepton mixing. On the other hand, if one or more of the normalcy conditions are violated it may be seen as a hint that anarchy is the correct description of the lepton mixing matrix, although it is no guarantee that symmetry groups do not describe the lepton mixing matrix. In fact, many symmetries such as tri-, bi-, triaxial, lepton mixing matrix. In fact, many symmetries such as tri-, bi-, triaxial, \( \mu, \tau \), and others predict that \( \theta_{23} \) is very close to 45° [28–31]. If \( \theta_{23} = 45° \), we see that \( |U_{\mu2}| < |U_{\mu3}| \) which is non-normal, thus some flavor symmetry models actually predict a deviation from normalcy.

This normalcy tool thus provides a guiding principle that can be easily interpreted and will mostly likely be either confirmed or ruled out by upcoming experiments. In addition to the obvious importance that measuring CP violation in neutrino has, these normalcy conditions highlight the fact that the neutrino mass ordering and the \( \theta_{23} \) octant are two parts in a larger set of normalcy conditions to be determined. Whether or not the remaining normalcy conditions can be determined depends on the precision with which \( \theta_{23} \) and \( \cos \delta \) can be measured.

The best fit region given the global oscillation picture that satisfies all the normalcy conditions is found at \( \theta_{23} \approx 41° \) and \( \cos \delta \approx -0.31 \) for \( \theta_{12} \) and \( \theta_{13} \) fixed (that is, \( \delta \approx 252° \) given current T2K hints for sin \( \delta \leq 0 \) [32]). This point is mildly disfavored from the best fit non-normal point at \( \Delta \chi^2 = 8 \) which is approximately 2.4σ assuming 2 dof’s.

CONCLUSION

In this letter we stated three definitions of neutrino mass eigenstate numbering. One of them (M) is based on masses of the neutrinos while the other two (e and \( \tau \)) are based on the relative components of the electron and tau neutrinos. Whether or not these are equivalent as one would expect provide two normalcy conditions.

Four additional normalcy conditions (1, 2, 3, and \( \mu \)) are based on the relative components of \( \nu_1, \nu_2, \nu_3, \) and \( \nu_\mu \) respectively. Each of the six normalcy conditions is essentially requiring that the more off-diagonal elements are smaller than the more diagonal elements.

A deviation from normalcy may provide a hint that the parameters in the lepton mass matrix are distributed as suggested by anarchy, while a confirmation of normalcy may be a hint that flavor symmetries describe nature, although there is no exact equivalence in either direction. Regardless of what is measured in upcoming neutrino experiment, neither flavor symmetries in general, nor anarchy, can be confirmed and anarchy cannot be refuted. Normalcy offers an advantage over the other approaches in that we will learn something one way or another about the structure of the lepton mixing matrix.

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4 The 3σ allowed values are taken to be three times the 1σ allowed regions. We also note that allowed values in the quark matrix are determined without an assumption of unitarity, while those in the lepton matrix do assume unitarity. The lepton numbers without unitarity can be found in ref. [27] and are even less restrictive.

5 There are seven sets of inequalities (three of which can be used as definitions of the mass eigenstates), one of which (e) is used as the fiducial definition of \( \nu_1, \nu_2, \) and \( \nu_\mu \). The remaining sets of inequalities provide six normalcy conditions.
Appendix: Solar Mass Eigenstate Definition

In solar neutrino analyses $\Delta m_{21}^2 > 0$ is often taken to be a definition. Then the solar mass ordering question is replaced as a question on the octant of the solar angle. That is, whether or not $\theta_{12} < 45^\circ$ (lower octant) or $\theta_{12} > 45^\circ$.

Thus the definition of the mass eigenstates in these analyses is sometimes
\begin{equation}
  m_1 < m_2 \quad \text{and} \quad |U_{e3}| < |U_{e1}| \quad \text{and} \quad |U_{e3}| < |U_{e2}|.
\end{equation}

We prefer to avoid this definition as it requires mixing two different kinds of definitions: one about the relative size of the terms in the electron neutrino row and one about the ordering of the masses.