AN ARITHMETICAL APPROACH TO THE CONVERGENCE PROBLEM OF SERIES OF DILATED FUNCTIONS

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Abstract. Given a periodic function $f$, we study the convergence almost everywhere and in norm of the series $\sum_{k} c_k f(kx)$. Let $f(x) = \sum_{m=1}^{\infty} a_m \sin 2\pi mx$ where $\sum_{m=1}^{\infty} a_m^2 d(m) < \infty$ and $d(m) = \sum_{d|m} 1$, and let $f_n(x) = f(nx)$. We show by using a new decomposition of squared sums that for any $K \subset \mathbb{N}$ finite, $\| \sum_{k \in K} c_k f_k \|_{2}^2 \leq (\sum_{m=1}^{\infty} a_m^2 d(m)) \sum_{k \in K} c_k^2 d(k^2)$. If $f^*(x) = \sum_{j=1}^{\infty} \frac{\sin \pi j x}{j^{s+1}}$, $s > 1/2$, by only using elementary Dirichlet convolution calculus, we show that for $0 < \varepsilon < 2s - 1$, $\zeta(2s)^{-1} \| \sum_{k \in K} c_k f_k \|_{2}^2 \leq \frac{d K}{d} \sum_{k \in K} |c_k|^2 \sigma_{1+\varepsilon-2s}(k)$, where $\sigma_\varepsilon(n) = \sum_{d|n} d^\varepsilon$. From this we deduce that if $f \in BV(\mathbb{T}), \langle f, 1 \rangle = 0$ and

$$\sum_{k} c_k^2 \frac{(\log \log k)^4}{(\log \log k)^2} < \infty,$$

then the series $\sum_{k} c_k f_k$ converges almost everywhere. This slightly improves a recent result, depending on a fine analysis of the polydisc: [1], th.3 (n_k = k), where it was assumed that $\sum_{k} c_k^2 (\log \log k)^\gamma$ converges for some $\gamma > 4$. We also show that the same conclusion holds under the arithmetical condition

$$\sum_{k} \frac{c_k^2 (\log \log k)^{2+b}}{\langle (\log \log k)^{2+b} \rangle} < \infty,$$

for some $b > 0$, or if $\sum_{k} c_k^2 d(k^2)(\log \log k)^2 < \infty$. We further prove an important complementary result to Wintner’s famous characterization of mean convergence of series $\sum_{k=0}^{\infty} c_k f_k$.

1. Introduction

Given a periodic function $f$ and an increasing sequence $N = \{n_k, k \geq 1\}$ of positive integers, one can formally define the series $\sum_{k=1}^{\infty} c_k f(n_k x)$ and ask under which conditions this series converges in norm or almost everywhere, for instance for any real coefficient sequence $c = \{c_k, k \geq 1\} \in L^2(\mathbb{N})$. This is one of the oldest and most central problems in the theory of systems of dilated sums. We only briefly outline the kind of results obtained. First studies were made at the beginning of the last century (see Jerosch and Weyl [29] where a.e. convergence is obtained under growth conditions on coefficients and Fourier coefficients of $f$), parallel to similar ones for the trigonometrical system.

This partly explains why until Carleson’s famous proof of Lusin’s hypothesis, the results obtained essentially concerned functions with slowly growing modulus or integral modulus of continuity and/or sequences $N$ verifying the classical Hadamard gap condition: $n_{k+1}/n_k \geq q > 1$ for all $k$. Carleson’s result triggered a new interest, permitting substantial progresses in this direction, under the main impulse of Russian analysts, among them Gaposhkin and later by Berkes. We refer to [4] for more details and references. Then the attention to these problems declined until very recently where there is a renewed activity.

In analogy with parallel questions concerning partial sums $\sum_{k=1}^{n} f(kx)$, $n = 1, 2, \ldots$, strong law of large numbers, studied by Gál, Koksma (see also [5]), and law of the iterated logarithm, central limit theorem, invariance principle, much explored by Erdös, Berkes and Philipp, and Gaposhkin notably, recent works show that the arithmetical nature of the support of the coefficient sequence, as well as the analytic nature of $f$, interact in a complex way in the study of the convergence almost everywhere and in norm of these series. The part of the theory devoted to individual results, namely the search of convergence conditions linking $f$, $N$ and $c$ is, to say the least, barely investigated. Our main concern in this work is precisely the search of individual conditions ensuring

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the almost everywhere convergence of the series \( \sum_{k=1}^{\infty} c_k f(n_k x) \). We propose new approaches for treating these questions. Notice before continuing, that the problem under consideration is a natural continuation of the study of the trigonometrical system, since by Carleson’s result, the series \( \sum_k c_k f(n_k x) \) converges almost everywhere for any trigonometrical polynomial \( f \). And this is in fact a convergence problem that can be put inside the study of the two-indices trigonometrical system with \( \{e_{jk}, j, k \geq 1\} \) where we denote \( e(x) = e^{2\pi i x}, e_n(x) = e(nx), n \geq 1 \). Let \( T = \mathbb{R}/\mathbb{Z} = [0, 1[. \) Let \( f(x) \sim \sum_{j=1}^{\infty} a_j e_j(x) \). Let \( f_n(x) = f(nx), n \in \mathbb{N}. \) We assume throughout that

\[
\langle f, 1 \rangle = 0.
\]

A key preliminary step naturally consists with the search of bounds of \( \| \sum_{k \in K} c_k f_k \|_2 \) integrating in their formulation the arithmetical structure of \( K \). That question has received a satisfactory answer only for specific cases. In this work we propose an approach based on elementary Dirichlet convolution calculus and on a new decomposition of squared sums. Although quite natural in regard of the posed problem, it seems at least to our knowledge, that this direction was not prospected before, apart in the recent works \[37, 6\].

We show that our approach is strong enough to recover and even slightly improve a recent a.e. convergence result \[1\] (Theorem 3) in the case \( N = \mathbb{N} \) without using analysis on the polydisc, see Theorem 3.2

**Notation.** We write \( \log \log x = \log \log (x \vee e^x) \), \( \log \log \log x = \log \log \log (x \vee e^x), x > 0 \).

We begin with stating mean results.

### 2. Mean Results

#### 2.1. General Estimates.

Let \( d(n) \) be the divisor function, namely the number of divisors of \( n \).

**Theorem 2.1.** Assume that \( \sum_{m=1}^{\infty} a_m^2 d(m) < \infty \). Then, for any finite set \( K \) of positive integers,

\[
\| \sum_{k \in K} c_k f_k \|_2^2 \leq \left( \sum_{m=1}^{\infty} a_m^2 d(m) \right) \sum_{k \in K} c_k^2 d(k^2).
\]

The presence of the arithmetical factor \( d(k^2) \) comes from formula (2.3). In \[37\], we recently showed a similar estimate, however restricted to sets \( K \) such that \( K \subset [e^r, e^{r+1}] \) for some integer \( r \). Then,

\[
\| \sum_{k \in K} c_k f_k \|_2^2 \leq \left( \sum_{\nu=1}^{\infty} a_\nu^2 \Delta(\nu) \right) \sum_{k \in K} c_k^2 d(k),
\]

where \( \Delta(\nu) \) is Hooley’s Delta function,

\[
\Delta(\nu) = \sup_{u \in \mathbb{R}} \sum_{d \leq \nu} 1.
\]

This one can be used to prove that under the conditions

\[
A = \sum_{\nu \geq 1} a_\nu^2 \Delta(\nu) < \infty, \quad B = \sum_j c_j^2 d(j)(\log j)^2 < \infty
\]

the series \( \sum_{k=0}^{\infty} c_k f_k(x) \) converges for almost all \( x \). A slightly weaker result was established in \[37\] (see Theorem 1.1). Condition \( A < \infty \) is very weak. As by \[35\],

\[
\frac{1}{x} \sum_{n \leq x} \Delta(n) = O \left( e^{c \sqrt{\log \log x \log \log \log x}} \right)
\]

for a suitable constant \( c > 0 \) (whereas \( \frac{1}{x} \sum_{\nu \leq x} d(\nu) \sim \log x \)), it reduces when the Fourier coefficients are monotonic to

\[
\sum_{\nu \geq 1} a_\nu^2 e^{c \sqrt{\log \log x \log \log \log x}} < \infty.
\]
Theorem 2.1 is deduced from a more general result. Introduce the necessary notation. Let

\[ A_k = \sum_{\nu=1}^{\infty} a_{\nu k}. \]  

Let \( \zeta_h \) denotes the arithmetic function defined by \( \zeta_h(n) = n^h \) for all positive \( n \). In particular \( \zeta_0(n) = 1 \) for all \( n \). Let \( \theta(n) \) denotes the number of squarefree divisors of \( n \). Then \( \theta(n) = 2^\omega(n) \) where \( \omega(n) \) is the prime divisor function, and by Mertens estimate, \( \sum_{k \leq x} 2^\omega(k) = C x \log x + O(x), \) \( x \geq 2 \), where \( C \) is some positive constant [13], p.70).

Given \( K \subset \mathbb{N} \), we note \( F(K) = \{ d \geq 1 : \exists k \in K : d \mid k \} \). If \( K \) is factor closed (\( d \mid k \Rightarrow d \in K \) for all \( k \in K \)), then \( F(K) = K \). These sets are usually termed FC sets (see [14] §3.3, [19]). Typical examples are \( \{1, \ldots, n\} \) or sets of mutually coprime integers. We also note \( SF(K) = \{ d \in F(K) : d \text{ squarefree} \} \). Recall that if \( \psi, \phi \) are arithmetical functions, the Dirichlet convolution \( \psi * \phi \) is defined by \( \psi * \phi(n) = \sum_{d \mid n} \psi(d) \phi(n/d) \).

**Theorem 2.2.** Let \( \psi \) be any arithmetical function taking only positive values.

i) For any finite set \( K \) of positive integers,

\[ \| \sum_{k \in K} c_k f_k \|_2^2 \leq \sum_{d \in F(K)} \left( \sum_{k \in K : d \mid k} c_k^2 \psi\left(\frac{k}{d}\right) \right) \left( \sum_{k \in K : d \mid k} c_k^2 \theta\left(\frac{k}{d}\right) \right). \]

ii) In particular,

\[ \| \sum_{k \in K} c_k f_k \|_2^2 \leq B \sum_{k \in K} c_k^2 \psi \ast \zeta_0(k), \]

where

\[ B = \sup_{d \in F(K)} \left( \sum_{k \in K : d \mid k} A_{\psi \ast \zeta_0}(d) \right) < \infty. \]

Choose for instance \( \psi = \theta \) and note ([30] formula 1.54)) that

\[ \theta \ast \zeta_0(k) = \sum_{d \mid k} \theta(d) = d(k^2). \]

Let \( d \in F(K) \), then

\[ \sum_{k \in K : d \mid k} A_{\psi \ast \zeta_0}(d) \sum_{k \in K : d \mid k} a_{\nu k}^2 = \sum_{m=1}^{\infty} a_m^2 \left( \sum_{j \in K : m \mid j} 1 \right) = \sum_{m=1}^{\infty} a_m^2 \left( \sum_{j \in K : m \mid j} 1 \right) \leq \sum_{m=1}^{\infty} a_m^2 d(m). \]

Since it is true for any \( d \in F(K) \), we deduce that \( B \leq \sum_{m=1}^{\infty} a_m^2 d(m) \), and so Theorem 2.1 follows from Theorem 2.2

**2.2. A generic class.** Next we consider more particularly the class of functions introduced in [29],

\[ f^s(x) = \sum_{j=1}^{\infty} \frac{\sin 2\pi j x}{j^s} \]

where \( s > 1/2 \). Recall that

\[ \langle f_k^s, f_\ell^s \rangle = \zeta(2s) \frac{(k, \ell)^{2s}}{k^s \ell^s}. \]

And so

\[ \| \sum_{k=1}^{n} c_k f_k^s \|_2^2 = \zeta(2s) \sum_{k, \ell=1}^{n} c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s}. \]
This class of functions was recently much studied. We begin with examining the quadratic form appeared above. Some observations are prealably necessary. The question of the reduction of a quadratic form whose coefficients are a function of the greater commun divisor of their indices

\[ X = \sum_{i,j} x_i x_j F((i,j)) \]

was considered long ago by Cesàro [11, 12] in 1885-1886, after the works of Smith [34] and Mansion in 1875-1876 who calculated their determinant (Cesàro also calculated other classes of arithmetical determinants). Later Jacobstahl [25], Carlitz [10] and other authors investigated this problem (see the survey on GCD matrices by Haukkanen, Wang and Sillanpää [19] for more references). In the present case, the reduction takes the following form,

(2.6) \[ \sum_{k,\ell=1}^{n} c_k c_\ell (k, \ell)^{2s}_{j^s \ell^s} = \sum_{i=1}^{n} J_{2s}(i) \left( \sum_{k=1}^{n} c_k k^{-s} 1_{1|k} \right)^2. \]

Here \( J_{2s}(i) = i^{2s} \prod_{p|j}(1 - p^{-2s}) \) is the generalized Euler toitent function (see Section 5). This formula, which is used in [6] (see Lemma 1.1), was already known by Cesàro. Obviously, (2.6) remains true when replacing \{1, \ldots, n\} by a factor-closed set. A dual problem is Möbius inversion of a family of vectors (with Gram matrix \( F((i,j))_{i,j} \)). Recent related works are in Balazard and Saias [3], Brémont [4].

In matrix form, this can be condensed in the following proposition, which generalizes Proposition 2.2 in [4], based on Carlitz Lemma, see also Li’s representation of GCD matrices [28] and [21]. As the proof is elementary and short, we shall give it right after some necessary complementary remarks.

**Proposition 2.3.** Let \( T = (t_{i,j})_{n \times n} \) and \( \hat{T} = (\hat{t}_{i,j})_{n \times n} \) be matrices defined by

(2.7) \[ t_{i,j} = \delta_i \theta_j 1_{1|i}, \quad \hat{t}_{i,j} = \frac{1}{\theta_i \theta_j} 1_{1|i} \mu \left( \frac{1}{\ell} \right), \quad i = 1, \ldots, n, \]

where \( \delta_i, \theta_i, i = 1, \ldots, n \) are real numbers verifying \( \delta_i > 0, \theta_i \neq 0 \) for all \( i \). Then,

(a) \( T \) is invertible and \( T^{-1} = \hat{T} \).

(b) Let \( H_m, m = 1, \ldots, n \) be real numbers defined as follows

(2.8) \[ H_m = \sum_{k|m} \delta^2_k, \quad m = 1, \ldots, n. \]

Then \( ^t T T = A \) where \( A = (\theta_i \theta_j H_{(i,j)})_{n \times n} \). Further \( A \) is positive definite.

(c) Let \( G = (g_i)_{1 \leq i \leq n} \) be vectors in an inner product space such that Gram(\( G \)) = \( A \). Then \( F = ^t \hat{T} G = (f_i)_{1 \leq i \leq n} \) has orthonormal components.

(d) For any reals \( c_i \), we have

(2.9) \[ \sum_{k=1}^{n} c_k g_k = \sum_{i=1}^{n} b_i f_i, \quad \text{where} \quad b_i = \delta_i \left( \sum_{k=1}^{n} c_k \theta_k 1_{1|k} \right), \quad i = 1, \ldots, n. \]

In particular,

\[ \| \sum_{k=1}^{n} c_k g_k \|^2 = \sum_{i=1}^{n} b_i^2. \]

**Remark 2.4.** We recall that positive semi-definite matrices are always Gram matrices (of vectors in an inner product space), hence the existence of \( G \) in (c). Further, a matrix \( B \) is positive definite if and only if there exists a non-singular lower triangular matrix \( L \) such that \( A = L^t L \), see [23], Corollary 7.2.9. Furthermore, by the Möbius inversion formula,

\[ H_m = \sum_{k|m} \delta^2_k, \quad m = 1, \ldots, n \quad \iff \quad \delta^2_k = \sum_{\ell|k} \mu \left( \frac{k}{\ell} \right) H_\ell, \quad k = 1, \ldots, n. \]
Proof of Proposition 2.3. Let \( \mathbf{X} \) be a matrix. Then \( \delta_k = \sqrt{J_2s(k)} \) and

\[
a_{i,j} = \frac{(i,j)^{2s}}{i^s j^s}, \quad b_i = \sqrt{J_2s(i)} \left( \sum_{k=1}^n c_k k^{-s} \mathbf{1}_{i|k} \right).
\]

Hence (2.6). Further, the system

\[
h_i^s = \frac{1}{\sqrt{J_2s(i)}} \sum_{j|i} j^s \mu(j) f_j^s \quad i = 1, \ldots
\]

is orthonormal and

\[
\sum_{i=1}^n c_k f_k^s = \sum_{i=1}^n b_i h_i^s \quad \text{where} \quad b_i = \sqrt{J_2s(i)} \left( \sum_{k=1}^n c_k k^{-s} \mathbf{1}_{i|k} \right), \quad i = 1, \ldots, n, \quad n \geq 1.
\]

This is Bréon’s result, who deduced that the series \( \sum_k c_k f_k^s \) converges in \( L^2(\mathbb{T}) \) if and only if the following uniformity condition holds,

\[
\lim_{n \to \infty} \sup_{N > n} \sum_{i=1}^n J_2s(i) \left( \sum_{k=N+1}^\infty c_k k^{-s} \mathbf{1}_{i|k} \right)^2 = 0.
\]

Notice that by assumption and Cauchy-Schwarz’s inequality, the series \( \sum_{k \geq 1} |c_k| k^{-s} \) is convergent. See [4], Proposition 2.2 and Corollary 2.3. Although satisfactory, (2.12) is however implicit, and it would be desirable to find a more concrete characterization, namely depending more directly on the coefficients \( (c_k)_{k} \). As a (non trivial) application, it is show in [4], that \( L^2(\mathbb{T}) \)-convergence holds if \( |c_k| \leq \delta(\mathbf{X}) \) where \( \delta \) is multiplicative and \( \sum_n \delta^2(n) < \infty \).

Proof of Proposition 2.3. Let \( I \) denote the \( n \times n \) identity matrix.

(a) We have,

\[
\sum_{k=1}^n \mathbf{1}_{i,k} \mathbf{1}_{k,j} = \sum_{k=1}^n \frac{1}{\theta_k} \mathbf{1}_{i|k} \mu(k) \delta_k \mu(j) = \sum_{k=1}^n \mathbf{1}_{i|k} \mu(k) \mathbf{1}_{k|j} = \sum_{k=1}^n \mathbf{1}_{i|k} \mu(k) \mathbf{1}_{j|k} = 0,
\]

if \( i \neq j \), since \( \sum_{\ell=m} \mu(\ell) = 0 \), if \( m \geq 2 \). Hence \( TT = I \), and similarly \( T^* T = I \).

(b) We compute the \( (i,j) \)-th entry of \( ^t T \).

\[
\sum_{k=1}^n \mathbf{1}_{i,k} \mathbf{1}_{k,j} = \mathbf{1}_{i,j} \sum_{k=1}^n \delta_k \theta_j = \mathbf{1}_{i,j} H_{i,j},
\]

by the Möbius inversion formula. We also have \( \det A = (\det T)^2 = \prod_{i=1}^n \delta_i \theta_i \neq 0 \). And if \( X \neq 0 \), then \( ^t X A X = ^t Y Y > 0 \) with \( Y = TX \neq 0 \).

(c) \[
\langle f_i, f_j \rangle = \sum_{k=1}^n \sum_{\ell=1}^n \mathbf{1}_{i,k} \mathbf{1}_{\ell,j} \langle g_k, g_\ell \rangle = \sum_{k=1}^n \sum_{\ell=1}^n \mathbf{1}_{i,k} \left( \sum_{u=1}^n \mathbf{1}_{k,u} \mathbf{1}_{u,\ell} \right) \mathbf{1}_{\ell,j} = \sum_{k=1}^n \mathbf{1}_{i,k} \mathbf{1}_{k,j} = \mathbf{1}_{i=j}.
\]

(d) As \( G = ^t T \), we have \( t_{i,j} = \delta_i \mu \mathbf{1}_{i|k} \)

\[
\sum_{k=1}^n c_k g_k = \sum_{k=1}^n c_k \sum_{i=1}^n \mathbf{1}_{i,k} f_i = \sum_{k=1}^n \mathbf{1}_{i,k} f_i \sum_{k=1}^n c_k \delta_k \mathbf{1}_{i|k} = \sum_{i=1}^n f_i \delta_i \sum_{k=1}^n c_k \mathbf{1}_{i|k} f_i = \sum_{i=1}^n f_i \delta_i.
\]

Another approach was proposed by Hilberdink [20] who has estimated the sums

\[
\sum_{k=1}^N \left( \frac{n_k n_{k+1}}{n_k^2 n_{k+1}^2} \right)^{2s}.
\]
when \( n_k = k \) and obtained optimal bounds in this case. He showed that if \( b_n = n^{-s} \sum d|n \, d^s a_d \), then

\begin{equation}
(2.13) \quad \sum_{n=1}^{N} |b_n|^2 = \sum_{m,n \leq N} a_m \overline{a_n} \frac{(m,n)^{2s}}{m^s n^s} \sum_{k \leq N/[m,n]} \frac{1}{k^{2s}}.
\end{equation}

Here we introduce the symbol \((*)\) to mean that the sum is 0 when the summation index is empty. And this requires some restriction with respect to the original statement, see Proposition 3.1 and after in \([20]\). More precisely, if \( a_n \geq 0 \), the right-term is less than \( \zeta(2s) \sum_{m,n \leq N} a_m \overline{a_n} \frac{(m,n)^{2s}}{m^s n^s} \). And a similar lower bound occurs when \( a_n \geq 0 \).

When the \( n_k \)'s are arbitrary but distinct positive integers, the initial result is due to Gál \([17]\), who showed for \( s = 1 \) that

\begin{equation}
(2.14) \quad \sum_{k,\ell=1}^{N} \frac{(n_k, n_\ell)^{2s}}{n_k n_\ell} \leq C N (\log \log N)^2,
\end{equation}

where \( C \) is an absolute constant, and moreover that estimate is optimal. It follows for this choice of values of \( n_k \) and by taking \( c_k \equiv 1 \) that in this case

\begin{equation}
(2.15) \quad \| \sum_{k=1}^{N} c_k f_{n_k}^1 \|_2^2 \geq C N (\log \log N)^2 \gg \sum_{k=1}^{N} c_k^2.
\end{equation}

This is a famous result and a few explanatory words concerning the proof are necessary. Gál’s proof is based on the observation that the sum in \((2.14)\) will be not maximal unless \( \{n_1, \ldots, n_k\} \) is an FC set, namely \( d|n_j \Rightarrow d = n_i \) for some \( i \). Hence it follows that if the sum is maximal, then the corresponding \( n_i \) are products of powers of at most \( C \log N \) primes.

This result was recently extended in \([1]\) to the case \( 0 < s \leq 1 \) (see also \([9]\) for recent improvements, in the case \( s = 1/2 \) notably) by representing these sums as Poisson integrals on the polydisc and by suitably modifying Gál’s combinatorial argument. When sieving the coefficients \( c_k \) according to their order of magnitude, that estimate can be implemented and then becomes a decisive tool when \( f \) has slowly decreasing Fourier coefficients, typically when \( f = f^1 \). That allowed the authors to establish quite sharp results for the a.e. convergence of series \( \sum_k c_k f_{n_k}^1 \), and in fact by a plain monotonicity argument on the Fourier coefficients, for any \( f \in BV(\mathbb{T}) \). The authors further extended their result to any \( f \in Lip_{1/2}(\mathbb{T}) \). These results are of relevance in the present work.

**Representation using Cauchy measures.** Notice before continuing that

\begin{equation}
\frac{(n_k, n_\ell)^{2s}}{n_k n_\ell} = \prod_p p^{-s|v_p(n_k) - v_p(n_\ell)|}
\end{equation}

where \( v_p(n) \) is the \( p \)-valuation of \( n \) (namely \( p^{v_p(n)} || n \) and \( v_p(n) = 0 \) if \( p \nmid n \)). And from the relation 

\begin{equation}
e^{-|\sigma|} = \int_{\mathbb{R}} e^{i\sigma t} \frac{dt}{\pi(t^2 + 1)},
\end{equation}

follows that

\begin{equation}
e^{-s|v_p(n_k) - v_p(n_\ell)|\log p} = \int_{\mathbb{R}} \frac{1}{p^{v_p(n_k)\sigma t} p^{-v_p(n_\ell)\sigma t} \pi(\sigma^2 + 1)} dt.
\end{equation}

So that

\begin{equation}
(2.16) \quad \sum_{k,\ell=1}^{N} c_k \overline{c_\ell} \frac{(n_k, n_\ell)^{2s}}{n_k n_\ell} = \sum_{k,\ell=1}^{N} c_k \overline{c_\ell} \int_{\mathbb{R}^\infty} \prod_p \left( \frac{1}{p^{v_p(n_k)\sigma t} p^{-v_p(n_\ell)\sigma t} \pi(\sigma^2 + 1)} \right) \frac{dt_p}{\pi(\sigma^2 + 1)}.
\end{equation}

namely, the sum directly expresses as a squared norm with respect to the infinite Cauchy measure.
2.3. A new arithmetical estimate. It turns out that even for this specific class of functions, another much simpler device can be used, based on Dirichlet convolution calculus, which also leads, at least when \( n_k = k \), to slightly sharper convergence results. The basic tool, which we are going to state now, provides a new estimate of \( \| \sum_{k \in K} c_k f_k^s \|_2 \), \( K \) arbitrary. This estimate is of individual type, in the sense that it is expressed by means of the values taken on \( K \) by some elementary arithmetical functions. Let for \( u \in \mathbb{R} \), \( \sigma_u(k) = \sum_{d|k} d^u \). In particular \( \sigma_0 = d, \sigma_1 = \sigma \) is the usual sum-divisor function and \( \sigma_{-\alpha}(n) = n^{-\alpha} \sigma_\alpha(n) \). Let also \( \varphi(n) = \# \{ m \leq n : (m, n) = 1 \} \) be the Euler totient function.

**Theorem 2.6.** Let \( s > 0 \) and \( 0 \leq \tau \leq 2s \). Let also \( \psi_1(u) > 0 \) be non-decreasing. Then for any finite set \( K \) of integers,

\[
\sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^{s} \ell^{s}} \leq \left( \sum_{u \in F(K)} \frac{1}{\psi_1(u) \sigma_\tau(u)} \right) \left( \sum_{\nu \in K} c_\nu^2 \psi_1(\nu) \sigma_{\tau-2s}(\nu) \right).
\]

In particular,

(i) \[
\sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^{s} \ell^{s}} \leq M(K) \left( \sum_{k \in K} |c_k|^2 \sigma_{\tau-2s}(k) \right),
\]

with \( M(K) = \sum_{k \in F(K)} \frac{1}{\sigma_\tau(k)} \).

(ii) \[
\sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^{s} \ell^{s}} \leq \left( \sum_{u \in F(K)} \frac{1}{\sigma_-s(u) \sigma_s(u)} \right) \left( \sum_{\nu \in K} c_\nu^2 \sigma_-s(\nu) \sigma_s(\nu) \right),
\]

where \( \sigma_-s(u) := \max\{\sigma_-s(v), 1 \leq v \leq u\} \).

(iii) Further

\[
\sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k \ell} \leq \frac{\pi^2}{6} \left( \sum_{u \in F(K)} \frac{\varphi(u)}{u^2 \log \log u} \right) \left( \sum_{\nu \in K} c_\nu^2 \sigma_{-1}(\nu) \log \log \nu \right).
\]

The above Theorem essentially concerns the range of values \( 0 < s \leq 1 \), the case \( s > 1 \) being simpler (see [29]). As an immediate consequence, we get

**Corollary 2.7.** Let \( s > 1/2 \), Thus for any set \( K \), by (2.3)

\[
(2s)^{-1} \| \sum_{k \in K} c_k f_k^s \|_2^2 \leq \frac{1 + \varepsilon}{\varepsilon} \left( \sum_{k \in K} |c_k|^2 \sigma_{1-2s}(k) \right).
\]

Indeed, let \( 0 < \varepsilon \leq 2s - 1 \) and take \( \tau = 1 + \varepsilon \). From the obvious inequality \( \sigma_\varepsilon(k) \geq k^\tau \), follows that

\[
M(K) \leq \sum_{k \in F(K)} \frac{1}{k^{1+\varepsilon}} \leq \sum_{k \geq 1} \frac{1}{k^{1+\varepsilon}} \leq \frac{1 + \varepsilon}{\varepsilon}.
\]

So that Corollary [2.7] just follows from assertion (ii) of Theorem 2.6

Hence also,

**Corollary 2.8.** (i) Let \( s > 1/2 \). Assume that the following condition is fulfilled:

For some \( \varepsilon > 0 \), \( \sum_{k \geq 1} |c_k|^2 \sigma_{1+\varepsilon-2s}(k) \). \( \varphi(\nu) \), \( \nu^2 \log \log \nu \) \( < \infty \).

Then the series \( \sum_{k \geq 1} c_k f_k^s \) converges in \( L^2(\mathbb{T}) \).

(ii) Let \( s = 1 \), \( N = \{ n_k, k \geq 1 \} \) be an increasing sequence of positive integers and assume that

\[
\sum_{\nu \in F(N)} \frac{\varphi(\nu)}{\nu^2 \log \log \nu} < \infty.
\]
Then the series $\sum_{k \geq 1} c_k f_{n_k}$ converges in $L^2(T)$, whenever the following condition holds
\[ \sum_{k \geq 1} c_k^2 \sigma_1(n_k) \log \log n_k < \infty. \]

Note that the last condition is extremely weak and nearly optimal. The first part is straightforward, since by Corollary 2.7
\[ \sup_{n,m \geq N} \left\| \sum_{n \leq k \leq m} c_k f_k \right\|_2^2 \leq C(s, \varepsilon) \sum_{k \geq N} |c_k|^2 \sigma_{1+\varepsilon}(k) \rightarrow 0, \]
as $N$ tends to infinity. Hence $\{\sum_{1 \leq k \leq m} c_k f_k^2, m \geq 1\}$ is a Cauchy sequence in $L^2(T)$. As to the second, it follows similarly from assertion (iii) of Theorem 2.6. See also the recent work [2] for a different proof of (i).

**Remark 2.9.** By a plain monotonicity argument on Fourier coefficients, Corollary 2.8 immediately extends with no change to functions $f \sim \sum_{j=1}^{\infty} a_j \sin 2\pi j x$ such that $a_j = O(j^{-s})$, $s > 1/2$.

**Remark 2.10.** Estimates (i)-(iii) also provide sharp bounds to GCD sums indexed on FC sets. Estimate (i) with $s = \tau = 1$ further implies
\[ \sum_{k,\ell \in K} c_k c_{\ell} \frac{(k,\ell)^2}{k\ell} \leq \frac{\pi^2}{6} \left( \sum_{k \in K} \frac{\varphi(k)}{k^2} \right) \left( \sum_{k \in K} |c_k|^2 \sigma_1(k) \right), \]
Concerning the first factor, notice that $\sum_{k=1}^{\infty} \frac{\varphi(k)}{k^2} = \frac{\zeta(s-1)}{\zeta(s)}$ for $s > 2$, and also that $\frac{C_k}{\log \log k} \leq \varphi(k) \leq k$. Recall for later use that by Gronwall’s estimates, (13) p. 119–122,
\begin{equation}
\limsup_{n \to \infty} \frac{\log \left( \frac{\sigma(n)}{n^\alpha} \right)}{\log \log n} = \frac{1}{1-\alpha} \quad (0 < \alpha < 1), \quad \limsup_{n \to \infty} \frac{\sigma(n)}{n \log \log n} = e^\gamma,
\end{equation}
where $\gamma$ is Euler’s constant. The validity of the seemingly slightly stronger inequality $\sigma(n) < e^{\gamma n \log \log n}$ for all $n \geq 5041$, is known to be equivalent to the Riemann Hypothesis. There are useful alternative ways to bound $\sigma_1(n)$ using arithmetical functions. First recall Duncan’s inequality. Let $\omega(n)$ be the prime divisor function (counting the number of prime divisors of $n$). Then $\sigma_1(n) < \frac{\omega_{\text{even}}(n)+10}{6}$. Further, Satyanarayana and Vangipuram showed that if $n$ is odd and $3 \nmid n$, $5 \nmid n$, then
\[ \sigma_1(n) < \left( \frac{P^-(n) + 2\omega(n) + 21/2}{P^-(n) + 3} \right)^{1/2}, \]
where $P^-(n)$ is the smallest prime divisor of $n$. See [12] p.78-79.

**Remark 2.11.** A strengthened form of Theorem-(i) 2.6 involving a more delicate analysis is proved in Section 9.

Before passing to Wintner’s theorem, we shall discuss the problem of estimating the eigenvalues attached to the arithmetical matrix
\[ M(K, s) = \left\{ \frac{(k,\ell)^{2s}}{k^s \ell^s} \right\}_{k,\ell \in K}. \]

### 2.4. Eigenvalues arithmetical estimates.

The recent estimates established in [20, 11, 9] are sharp but are not of arithmetical type. An important and quite challenging question is precisely to know whether it is possible to provide bounds of this type, expressed in a simple way by arithmetical functions. In this direction, the following GCD sum estimate established in [6], (Proposition 1.13) is relevant.

**Proposition 2.12.** Let $0 < s \leq 1$. For any $k \in K$ (letting $K_- = \min K$, $K^+ = \max(K)$),
\[ \sum_{k \in K} \left( \sum_{\ell \in K} k^s \ell^s \right)^{-1} \frac{\sigma_1(k)}{k^s \ell^s} \leq \begin{cases} 2 \left( \log \frac{K^+}{K^-} \right) \sigma_1(k) & \text{if } s = 1, \\ 2s^{k^s-1} \left( \int_{K^-} \frac{du}{u^s} \right) \sigma_{1-2s}(k) & \text{if } 0 < s < 1. \end{cases} \]
Theorem 3.1. prove conditions of mixed type, namely multipliers partly expressed by arithmetical functions. We will observe in [6]. These estimates turn up to be of crucial use in the last part of the proof of (2.20)

Corollary 2.14. Let 0 < s ≤ 1. Let λ(k, s), k ∈ K be the eigenvalues of M(K, s). Then for any k ∈ K,

\[ |λ(k, s) - ζ(2s)| \leq \begin{cases} 2(\log \frac{K_1}{K_2}) \sup_{k \in K} \sigma_{-1}(k) & \text{if } s = 1, \\ 2s(\frac{K_1}{K_2})^{1-s} \sup_{k \in K} \sigma_{1-2s}(k) & \text{if } s < 1. \end{cases} \]

We apply Geršgorin’s theorem stating that the eigenvalues of an arbitrary matrix lie in the union of the closed disks (Geršgorin disks)

\[ (k, \ell)_{2s} \leq 2s(\frac{K_1}{K_2})^{1-s} \sigma_{1-2s}(k), \]

no allows to conclude.

Remark 2.13. These inequalities are actually two-sided if K = [K_-, K^+].

It is easy to derive eigenvalues estimates of M(K, s) for K arbitrary.

Corollary 2.14. Let 0 < s ≤ 1. Let λ(k, s), k ∈ K be the eigenvalues of M(K, s). Then for any k ∈ K,

\[ |λ(k, s) - ζ(2s)| \leq \begin{cases} 2(\log \frac{K_1}{K_2}) \sup_{k \in K} \sigma_{-1}(k) & \text{if } s = 1, \\ 2s(\frac{K_1}{K_2})^{1-s} \sup_{k \in K} \sigma_{1-2s}(k) & \text{if } s < 1. \end{cases} \]

Gronwall’s estimates (2.17) further allow to provide quantitative bounds.

Proof. We apply Geršgorin’s theorem stating that the eigenvalues of an n × n matrix (a_{i,j}) with complex entries lie in the union of the closed disks (Geršgorin disks)

\[ |z - a_{i,i}| \leq \sum_{j=1 \atop j \neq i}^{n} |a_{i,j}| \quad (i = 1, 2, \ldots, n) \]

in the complex plane, see for instance [36]. Hence

\[ |λ(k, s) - ζ(2s)| \leq \sup_{k \in K} \sum_{i,j} \frac{(k, \ell)_{2s}}{k^s \ell^s}. \]

Applying Proposition 2.12 and noticing that when s < 1,

\[ \sum_{i,j} \frac{(k, \ell)_{2s}}{k^s \ell^s} \leq 2s(\frac{K_1}{K_2})^{1-s} \sigma_{1-2s}(k), \]

allows to conclude.

When combined with the classical weighted estimate for quadratic forms:

\[ \text{For any system of complex numbers } \{x_i\} \text{ and } \{α_{i,j}\}, \]

\[ \left| \sum_{1 \leq i, j \leq n \atop i \neq j} x_i x_j α_{i,j} \right| \leq \frac{1}{2} \sum_{i=1}^{n} |x_i|^2 \left( \sum_{i,j} \left( |α_{i,j}| + |α_{j,i}| \right) \right), \]

Proposition 2.12 immediately implies that

\[ \left\| \sum_{k \in K} c_k f_k \right\|_2^2 \leq \begin{cases} 2s(\frac{K_1}{K_2})^{1-s} \sum_{k \in K} \sigma_{1-2s}(k)c_k^2 & \text{if } 1/2 < s \leq 1, \\ 2(\log \frac{K_1}{K_2}) \sum_{k \in K} d(k)c_k^2 & \text{if } s = 1/2, \end{cases} \]

as observed in [6]. These estimates turn up to be of crucial use in the last part of the proof of Theorem 3.2.

3. Almost Everywhere Convergence Results.

We first apply Theorem 2.2 to almost everywhere convergence. We obtain new convergence conditions of mixed type, namely multipliers partly expressed by arithmetical functions. We will prove

Theorem 3.1. Assume that a_m = O(m^{-α}) for some α > 1/2.

i) Let 1/2 < α < 1. Then the series \( \sum_{k \geq 1} c_k f_k \) converges almost everywhere whenever the following condition is satisfied,

\[ \sum_{k \geq 1} c_k^2 (\log k)^{4(1-α)}(\log \log k)^{2(1-α)} d(k^2) < ∞. \]

ii) Let α = 1. Then the same conclusion holds true if the above condition is replaced by

\[ \sum_{k \geq 3} c_k^2 d(k^2)(\log \log k)^2 < ∞. \]
iii) Assume that
\[ a_m = O(m^{-1/2}(\log m)^{(1+h)/2}) \text{ for some } h > 1. \]
Then the same conclusion holds true under the following condition
\[ \sum_{k \geq 3} c_k^2 d(k^2)(\log k)^2(\log \log k)^{1-h} < \infty. \]

These arithmetical conditions are meaningful for coefficient sequences supported by sets of integers \( k \) having few divisors. In [6] Theorem 2.8, we showed that the condition
\[ \sum_{k \geq 1} c_k^2 (\log k)^2 \sigma_{1-2\alpha}(k) < \infty, \]
also implies the convergence almost everywhere of the series \( \sum_{k \geq 1} c_k f_k \). Although not exactly comparable with the condition given in i), this one yields a better condition for coefficient sequences supported by integers with few divisors. A similar remark holds concerning the general condition given in [6] (see Corollary 2.6 and Remark 2.7). The condition given in (ii) has to be compared with the one in Theorems 3.2, 3.3.

As to (ii) and (iii), the non-arithmetical factors of the multipliers are significantly better than those in Theorem 3.2 and Theorem 1.1 in [37], respectively. Recall concerning (ii) that condition (see Theorems 3.7 in [1])
\[ \sum_{k \geq 3} c_k^2 (\log \log k)^{\gamma} < \infty, \]
for \( \gamma < 2 \) is necessary for the convergence almost everywhere of the series \( \sum_{k \geq 1} c_k f_k \).

We further prove the following almost everywhere convergence result concerning the Banach space \( BV(\mathbb{T}) \) of functions with bounded variation.

**Theorem 3.2.** Let \( f \in BV(\mathbb{T}), \langle f, 1 \rangle = 0 \). Assume that
\[ \sum_{k \geq 3} c_k^2 \frac{(\log k)^4}{(\log \log k)^2} < \infty. \]
Then the series \( \sum_{k} c_k f_k \) converges almost everywhere.

This slightly improves Theorem 3 in [1] \( (n_k = k) \), where it was assumed that the series
\[ \sum_{k=1}^{\infty} c_k^2 (\log k)^{\gamma} \]
converges for some \( \gamma > 4 \).

We will also prove the following rather delicate result where multipliers have arithmetical factors.

**Theorem 3.3.** Let \( f \in BV(\mathbb{T}), \langle f, 1 \rangle = 0 \). Assume that for some real \( b > 0 \),
\[ \sum_{k \geq 3} c_k^2 (\log k)^{2+b+\frac{1}{(\log \log k)^{1/3}}}(k) < \infty. \]
Then the series \( \sum_{k} c_k f_k \) converges almost everywhere.

We will derive it, as well as (3.1), directly from Theorem 2.6, thus without using analysis on the polydisc as in [1].

**Remark 3.4.** In spite of the regular decay of its Fourier coefficients, a function \( f \in BV(\mathbb{T}) \) may have very pathological behavior. Jordan [27] gave in 1881 a remarkably simple and elegant construction of a function with bounded variation, having positive jumps on each rational, and being continuous almost everywhere.
Remark 3.5. Theorem 3.2 applies to the case \( s = 1 \) in (2.4) which corresponds to the Fourier expansion of the function \( \langle x \rangle = \frac{x}{2}(1 - 2x), \) \( 0 \leq x \leq 1 \)

\[
\langle x \rangle = \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{n} \quad (0 < x < 1)
\]

(3.3)

the series being discontinuous at \( x = 0 \). It is quite interesting to notice by expanding \( \langle x \rangle \) with respect to the system \( \cos(n + \frac{1}{2})x, \sin(n + \frac{1}{2})x, n = 0, 1, \ldots \) (which is orthogonal and complete over any interval of length \( 2\pi \)), that one also gets (10) p.71

\[
\langle x \rangle = \frac{4}{\pi} \sum_{n=0}^{\infty} \frac{\cos 2\pi(n + 1/2)x}{(2n + 1)^2} \quad (0 \leq x \leq 1),
\]

(3.4)

where this time the series is absolutely and uniformly convergent. Let \( \zeta(x) \) denote the series in the right handside. Further, it is not a complicated task to prove that the series \( \sum c_k \zeta(n_kx) \) converges for almost every \( x \) under the minimal condition \( \sum c_k^2 < \infty \). However \( \zeta(x) \) is 2-periodic whereas \( \langle x \rangle \) is 1-periodic. The study of the system \( \{nx, n \in \mathbb{N}\} \) goes back to Riemann’s work [33]. Davenport [15] [16] much investigated its properties. It is known that this system possesses smoothness properties going at the opposite of those of the trigonometrical system (the series \( \sum c_k \varphi_k(x) \) is never continuous unless the coefficients \( c_k \) all vanish). We refer to Jaffard [25]. However, the a.s. convergence properties of series attached to this system seem to remain relatively close to those of the trigonometrical system, namely to belong close to the domain of applicability of Carleson’s theorem.

3.1. A complement to Wintner’s Theorem. We finally also prove an important complementary result to Wintner’s famous characterization of mean convergence of series \( \sum_{k=0}^{\infty} c_k \varphi_k \). Recall some necessary facts. Let \( f \in L^2(\mathbb{T}) \) with \( \langle f, 1 \rangle = 0 \) and denote \( \tilde{f} = \{f_n, n \geq 0\} \) where we recall that \( f_n(x) = f(nx) \). We say that the system \( \tilde{f} \) is mean convergent if the series \( \sum_{k=0}^{\infty} c_k \varphi_k \) converges in \( L^2(\mathbb{T}) \) for any \( \{c_k, n \geq 0\} \in \ell^2 \). This property is characterized by the following well-known theorem.

Theorem 3.6 (Wintner [39]). The following statements are equivalent:
1. The series \( \sum_{k=1}^{\infty} c_k \varphi_k(x) \) converges in \( L^2(\mathbb{T}) \) for any coefficient sequence \( (c_k)_k \in \ell^2 \).
2. There exists a constant \( c > 0 \) such that for any \( n \geq 1 \) and any reals \( \{c_k, 1 \leq k \leq n\} \) we have

\[
\left\| \sum_{k=1}^{n} c_k \varphi_k \right\|^2 \leq c \sum_{k=1}^{n} c_k^2.
\]

(3.5)

3. The infinite matrix \((\langle f_k, f_l \rangle)_{k,l}\) defines a bounded operator on \( \ell^2 \).
4. The Dirichlet series \( \sum_{n=1}^{\infty} a_n n^{-s} \) and \( \sum_{n=1}^{\infty} b_n n^{-s} \) are regular and bounded in the half-plane \( \Re(s) > 0 \).

Suppose \( \tilde{f} \) is mean convergent. It is natural to ask whether there always exists a class of coefficients \( (c_k)_k \) for which the series \( \sum_{k=1}^{\infty} c_k \varphi_k \) will converge almost everywhere. The theorem below answers this affirmatively by identifying a general class of coefficients.

Recall a useful notion. A sequence of coefficients \( \{c_k, n \geq 0\} \) is called universal if for any orthonormal system \( \Phi \) of functions defined on a bounded interval (and possibly extended periodically over the real line), the series \( \sum_{k=1}^{\infty} c_k \varphi_k \) converges a.e.

Theorem 3.7. Assume that \( \tilde{f} \) is mean convergent. Then the series \( \sum_{k=1}^{\infty} c_k \varphi_k(x) \) converges a.e. for any universal coefficient sequence \( (c_k)_k \).

The paper is organized as follows: in Sections 4 to 8 we prove the results stated before. In Section 9, a strengthened form of Theorem 2.6 is proved.
4. Proof of Theorem 2.2

Let \( \delta \) be the arithmetical function defined by

\[
\delta(n) = \begin{cases} 
1 & \text{if } n = 1, \\
0 & \text{if } n \neq 1.
\end{cases}
\]

Let \( \mu \) denotes the Möbius function and recall that

\[
\sum_{d|n} \mu(d) = \delta(n).
\]

We have

\[
\left\| \sum_{k \in K} c_k f_k \right\|_2^2 = \sum_{k, \ell \in K} c_k c_\ell \sum_{\nu=1}^\infty a_{\nu k} a_{\nu \ell} \delta((k, \ell)).
\]

We decompose the right-hand side according to the values taken by \((k, \ell)\),

\[
\sum_{k, \ell \in K} c_k c_\ell \sum_{\nu=1}^\infty a_{\nu k} a_{\nu \ell} = \sum_{d \in F(K)} S_d,
\]

where

\[
S_d = \sum_{k, \ell \in K \atop (k, \ell) = d} c_k c_\ell \sum_{\nu=1}^\infty a_{\nu k} a_{\nu \ell}.
\]

We claim that

\[
|S_1| \leq \left( \sum_{k \in K} |c_k|^2 \psi(k) \right) \left( \sum_{k \in K} \frac{A_k}{\psi(k)} \right).
\]

Indeed, by (4.1), (4.2),

\[
S_1 = \sum_{k, \ell \in K} c_k c_\ell \sum_{\nu=1}^\infty a_{\nu k} a_{\nu \ell} \delta((k, \ell)) = \sum_{k, \ell \in K} c_k c_\ell \sum_{\nu=1}^\infty a_{\nu k} a_{\nu \ell} \sum_{d|(k, \ell)} \mu(d)
\]

\[
= \sum_{\nu=1}^\infty \sum_{d \in F(K)} \mu(d) \sum_{k, \ell \in K \atop d|(k, \ell)} c_k c_\ell a_{\nu k} a_{\nu \ell} = \sum_{\nu=1}^\infty \sum_{d \in F(K)} \mu(d) \left( \sum_{k \in K \atop d|k} c_k a_{\nu k} \right)^2.
\]

This thus factorizes, and now we can apply Cauchy-Schwarz’s inequality to get

\[
|S_1| \leq \sum_{\nu=1}^\infty \sum_{d \in F(K)} \left( \sum_{k \in K \atop d|k} c_k |\psi(k)| \frac{a_{\nu k}}{\sqrt{\psi(k)}} \right)^2
\]

\[
\leq \left( \sum_{k \in K} |c_k|^2 \psi(k) \right) \sum_{\nu=1}^\infty \sum_{d \in F(K)} \sum_{k \in K \atop d|k} \frac{a_{\nu k}^2}{\psi(k)}
\]

\[
= \left( \sum_{k \in K} |c_k|^2 \psi(k) \right) \left( \sum_{k \in K \atop d|k} \frac{A_k}{\psi(k)} \sum_{d \in F(K)} 1 \right)
\]

\[
= \left( \sum_{k \in K} |c_k|^2 \psi(k) \right) \left( \sum_{k \in K \atop d|k} \frac{A_k}{\psi(k)} \sum_{d \in F(K)} \theta(k) \right),
\]

Now let \( K_d = \frac{1}{d}(d\mathbb{N} \cap K) \). For the sum \( S_d \) defined in (4.4), we have

\[
S_d = \sum_{k, \ell \in K \atop (k, \ell) = d} c_k c_\ell \sum_{\nu=1}^\infty a_{\nu k} a_{\nu \ell} = \sum_{k, \ell \in K_d \atop (\ell', \ell'') = d} c_{k \ell'} c_{\ell' \ell''} \sum_{\nu=1}^\infty a_{\nu k} a_{\nu \ell'}.\]
Thus $S_d$ has just same form than the sum $S_1$ studied before, with $K_d, k', c_{k,d}, a_{k,d}$ in place of $K, k, c_k, a_k$. We deduce from (4.5) that

$$S_d \leq \left( \sum_{k' \in K_d} |c_{k'}|^2 \psi(k') \right) \left( \sum_{k' \in K_d} \frac{A_k}{\psi(k')} \theta(k') \right)$$

Thus

$$S_d \leq \left( \sum_{k \in K} |c_k|^2 \psi\left(\frac{k}{d}\right) \right) \left( \sum_{k \in K} \frac{A_k}{\psi\left(\frac{k}{d}\right)} \theta\left(\frac{k}{d}\right) \right).$$

Using (4.3), we get

$$\left\| \sum_{k \in K} c_k f_k \right\|_2^2 = \sum_{d \in F(K)} S_d \leq \sum_{d \in F(K)} \left( \sum_{k \in K} |c_k|^2 \psi\left(\frac{k}{d}\right) \right) \left( \sum_{k \in K} \frac{A_k}{\psi\left(\frac{k}{d}\right)} \theta\left(\frac{k}{d}\right) \right) \leq \sum_{d \in F(K)} \left( \sup_{d' \in F(K)} \sum_{k \in K} \frac{A_k}{\psi\left(\frac{k}{d}\right)} \theta\left(\frac{k}{d}\right) \right) \left( \sum_{k \in K} |c_k|^2 \psi\left(\frac{k}{d}\right) \right)$$

$$= B \sum_{k \in K} |c_k|^2 \psi \ast \zeta_0(k).$$

5. Proof of Theorem 2.6

Let $\{c_k, k \geq 1\}$ be a sequence of coefficients $\{c_k, k \geq 1\}$ supported by $K$, $(c_k = 0$ if $k \notin K$). Let $\varepsilon > 0$. We recall that the generalized Euler totient function $J_\varepsilon$ is the multiplicative arithmetical function defined by

$$J_\varepsilon(n) = \zeta_\varepsilon \ast \mu(n) = \sum_{d|n} d^\varepsilon \mu\left(\frac{n}{d}\right).$$

By Möbius inversion Theorem,

$$n^\varepsilon = \sum_{d|n} J_\varepsilon(d).$$

Step (1) is as in [1], except that we introduce an arithmetic function $\psi$. It is necessary to display it here. Step (ii) uses basic properties of Dirichlet convolutions.

1. Noticing that if $d|k$ and $k \in K$, then $d \in F(K)$, we have by (5.1)

$$(k, \ell)^\varepsilon = \sum_{d \in F(K)} J_\varepsilon(d) \mathbf{1}_{d|k} \mathbf{1}_{d|\ell}.$$ 

Thus

$$L := \sum_{k, \ell = 1}^n c_k c_\ell d_{k\ell}^{2s} = \sum_{k, \ell \in K} c_k c_\ell \left\{ \sum_{d \in F(K)} J_{2s}(d) \mathbf{1}_{d|k} \mathbf{1}_{d|\ell} \right\}.$$ 

Writing $k = ud, \ell = vd$ and noting that $u, v \in F(K)$, we have

$$L \leq \sum_{u, v \in F(K)} \frac{1}{u^s v^s} \left( \sum_{d \in F(K)} J_{2s}(d) \frac{d^{2s}}{d^2} c_{ud} c_{vd} \right).$$

By the Cauchy-Schwarz inequality,

$$\sum_{d \in F(K)} J_{2s}(d) \frac{d^{2s}}{d^2} c_{ud} c_{vd} \leq \left( \sum_{d \in F(K)} J_{2s}(d) \frac{d^{2s}}{d^2} c_{ud}^2 \right)^{1/2} \left( \sum_{d \in F(K)} J_{2s}(d) \frac{d^{2s}}{d^2} c_{vd}^2 \right)^{1/2}. $$

Hence,

$$L \leq \left[ \sum_{u \in F(K)} \frac{1}{u^s} \left( \sum_{d \in F(K)} J_{2s}(d) \frac{d^{2s}}{d^2} c_{ud}^2 \right)^{1/2} \right]^2.$$
Let $\psi$ be a positive arithmetic function. Writing $\frac{1}{u} = \frac{1}{u^{s/\psi(u)}} \psi(u)^{1/2}$ and applying Cauchy-Schwarz’s inequality again gives,

$$L \leq \left( \sum_{u \in F(K)} \frac{1}{u^s \psi(u)} \right) \left( \sum_{u \in F(K)} \frac{\psi(u)}{u^s} \sum_{d \in F(K)} \frac{J_{2s}(d)}{d^2 s} c_u^{2} \right).$$

Let $F^2(K) = \{ ud : u, d \in F(K) \}$. Then

$$\sum_{u \in F(K)} \frac{\psi(u)}{u^s} \sum_{d \in F(K)} \frac{J_{2s}(d)}{d^2 s} c_u^{2} \leq \sum_{\nu \in F^2(K)} \frac{c_{\nu}^{2}}{\nu^{2s}} \sum_{u \in F(K)} \frac{\psi(u)}{u^s} \frac{J_{2s}(\frac{\nu}{u})}{(\frac{\nu}{u})^2}$$

(5.3)

$$= \sum_{\nu \in K} \frac{c_{\nu}^{2}}{\nu^{2s}} \sum_{u \in F(K)} J_{2s}(\frac{\nu}{u}) u^s \psi(u),$$

since $c_{\nu} = 0$ if $\nu \notin K$. Hence we get

$$L \leq \left( \sum_{u \in F(K)} \frac{1}{u^s \psi(u)} \right) \left( \sum_{\nu \in K} \frac{c_{\nu}^{2}}{\nu^{2s}} \sum_{u \in F(K)} J_{2s}(\frac{\nu}{u}) u^s \psi(u) \right).$$

(5.4)

(2) Choose $\psi(u) = u^{-s} \psi_1(u) \sigma_\tau(u)$ (recalling that $\psi_1(u) > 0$ is non-decreasing). Then,

$$L \leq \left( \sum_{u \in F(K)} \frac{1}{\psi_1(u) \sigma_\tau(u)} \right) \left( \sum_{\nu \in K} \frac{c_{\nu}^{2}}{\nu^{2s}} \sum_{u \in F(K)} J_{2s}(\frac{\nu}{u}) \psi_1(u) \sigma_\tau(u) \right)$$

$$\leq \left( \sum_{u \in F(K)} \frac{1}{\psi_1(u) \sigma_\tau(u)} \right) \left( \sum_{\nu \in K} \frac{c_{\nu}^{2} \psi_1(\nu)}{\nu^{2s}} \sum_{u \in F(K)} J_{2s}(\frac{\nu}{u}) \sigma_\tau(u) \right).$$

As $\nu \in K$,

$$\sum_{u \in F(K)} \frac{J_{2s}(\frac{\nu}{u}) \sigma_\tau(u)}{u \nu} = J_{2s} * \sigma_\tau(\nu).$$

By commutativity and associativity of the Dirichlet convolution,

$$J_{2s} * \sigma_\tau = (\zeta_2 \ast \mu) * (\zeta \ast \zeta_0) = (\zeta_2 \ast \zeta) * (\zeta_0 \ast \mu) = (\zeta_2 \ast \zeta) \ast \delta,$$

since by (4.2), $\zeta_0 \ast \mu = \delta$. Further

$$\zeta_2 \ast \zeta_\tau(n) = \sum_{d|n} d^{2s} \left( \frac{n}{d} \right)^{\tau} = n^{\tau} \sum_{d|n} d^{2s - \tau} = n^{\tau} \sigma_{2s - \tau}(n).$$

Consequently,

$$J_{2s} * \sigma_\tau(\nu) = \sum_{n|\nu} n^{\tau} \sigma_{2s - \tau}(n) \delta \left( \frac{\nu}{n} \right) = n^{\tau} \sigma_{2s - \tau}(\nu).$$

By reporting

$$L \leq \left( \sum_{u \in F(K)} \frac{1}{\psi_1(u) \sigma_\tau(u)} \right) \left( \sum_{\nu \in K} \frac{c_{\nu}^{2} \psi_1(\nu)}{\nu^{2s}} n^{\tau} \sigma_{2s - \tau}(\nu) \right),$$

(5.5)

as claimed. Taking $\psi_1(u) \equiv 1$ gives

$$L \leq \left( \sum_{u \in F(K)} \frac{1}{\sigma_\tau(u)} \right) \left( \sum_{\nu \in K} \frac{c_{\nu}^{2} \sigma_{2s - \tau}(\nu)}{\nu^{2s}} \right),$$

which is (i). Taking now $\psi_1(u) = \sigma_s(u)$, $\tau = s$ gives

$$L \leq \left( \sum_{u \in F(K)} \frac{1}{\sigma_s(u) \sigma_s(u)} \right) \left( \sum_{\nu \in K} \frac{c_{\nu}^{2} \sigma_s(u) \sigma_s(u)}{\nu^{2s}} \right),$$
Lemma 6.1. Since (13), p.10, \( \sigma(n) \varphi(n) > 6n^2/\pi^2 \). This is (iii) and the proof is now complete.

Remark 5.1. Quite similarly, one can also prove that

\[
L \leq \left( \sum_{u \in F(K)} \frac{1}{\sigma(u) \log \log u} \right) \left( \sum_{\nu \in K} c_2 \sigma_1(\nu) \log \log \nu \right)
\]

since (13, p.10) \( \sigma(n) \varphi(n) > 6n^2/\pi^2 \). This is (iii) and the proof is now complete.

6. Proof of Theorem 3.1

Basically, the principle of the proof consists with showing that the studied case belongs to the "domain of attraction" of Carleson's theorem. First, recall for reader's convenience Lemma 8.3.4 from [38].

Lemma 6.1. Let \( \gamma > 1 \), \( 0 < \beta \leq 1 \) and consider a finite collection of random variables \( E = (X_1, \ldots, X_N) \subset L^{\gamma}(\mathbb{P}) \), and reals \( 0 \leq t_1 \leq t_2 \leq \cdots \leq t_N \leq 1 \) such that

\[
\|X_j - X_i\|_{\gamma} \leq (t_j - t_i)^\beta \quad (\forall 1 \leq i \leq j \leq N).
\]

Then, there exists a constant \( K_{\beta, \gamma} \) depending on \( \beta, \gamma \) only, such that

\[
\left\| \sup_{1 \leq i, j \leq N} |X_i - X_j| \right\|_{\gamma} \leq \begin{cases} K_{\beta, \gamma} & \text{if } \beta \gamma > 1, \\ K_{\beta, \gamma} \log N & \text{if } \beta \gamma = 1, \\ K_{\beta, \gamma} \frac{\log \log N}{\log \log \log N} & \text{if } \beta \gamma < 1. \end{cases}
\]

This standard Lemma will be used repeatedly. Let \( \{N_j, j \geq 1\} \) be an increasing sequence of integers to be specified later on. Let \( S_n = \sum_{k=1}^{n} c_k f_k \), \( n \geq 1 \). Put

\[
R_j = \sum_{m=1}^{j} a_m e_m, \quad r_j = \sum_{m=j+1}^{\infty} a_m e_m.
\]

We decompose \( S_n \) as follows: if \( N_j \leq n < N_{j+1} \), then for some \( J = J(j) \) depending on \( j \), the value of which being specified in the course of the proof, we write that

\[
S_n = \sum_{k=1}^{N_j} c_k f_k + \sum_{N_j \leq k \leq n} c_k R_k^J + \sum_{k=1}^{n} c_k r_k^J.
\]

This way to proceed is not new; we refer for instance to Theorem 2.6 in [6] where it is used already in the proof. By Carleson-Hunt's inequality,

\[
\left\| \sup_{N_j \leq u \leq v \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k R_k^J \right| \right\|_2 \leq \sum_{m=1}^{J} |a_m| \left\| \sup_{N_j \leq u \leq v \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k e_k m \right| \right\|_2 \leq C \left( \sum_{m=1}^{J} |a_m| \right) \left( \sum_{N_j \leq k \leq N_{j+1}} c_k^2 \right)^{1/2}.
\]

We will show in (6.2), (6.7) by using Abel summation that the series \( \sum_{m \geq 1} a_m^2 d(m) \) is convergent in each of the considered cases (i)-(iii), and we will estimate the tail \( \sum_{m \geq L} a_m^2 d(m) \). By Theorem 2.4

\[
\left\| \sum_{u \leq k \leq v} c_k r_k^J \right\|_2 \leq \left( \sum_{m \geq J+1} a_m^2 d(m) \right) \sum_{u \leq k \leq v} c_k^2 d(k^2).
\]
By Lemma $6.1$ we deduce
\[
\left\| \sum_{u \leq k \leq v} c_k r_k^J \right\|_2^2 \leq C (\log N_{j+1})^2 \left( \sum_{m=J+1}^{\infty} a_m^2 d(m) \right) \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2).
\]

By combining both estimates we arrive to
\[
\left\| \sum_{u \leq k \leq v} c_k f_k \right\|_2^2 \leq C \left( \sum_{m=1}^{\infty} |a_m| \right)^2 \left( \sum_{N_j \leq k \leq N_{j+1}} c_k^2 \right) + C (\log N_{j+1})^2 \left( \sum_{m=J+1}^{\infty} a_m^2 d(m) \right) \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2).
\] (6.1)

Notice that if $a_m = o(m^{-\alpha})$, $\alpha > 1/2$, we have by applying Abel summation and using the well-known estimate $\sum_{m \leq \ell} d(m) \leq C \ell \log \ell$,
\[
\sum_{m=L+1}^{\infty} a_m^2 d(m) \leq \sum_{m=L+1}^{\infty} \frac{d(m)}{m^{2\alpha}} \leq C \left\{ \sum_{m=L+1}^{\infty} \frac{\log m}{m^{2\alpha}} + \sup_{m>L} m^{1-2\alpha} \log m \right\} \leq C_{\alpha} L^{1-2\alpha} (\log L).
\] (6.2)

Now we give the proof of assertion (i). Let $1/2 < \alpha < 1$. We then deduce from (6.1), (6.2)
\[
\left\| \sum_{u \leq k \leq v} c_k f_k \right\|_2^2 \leq C \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \left( \sum_{m=1}^{\infty} |a_m| \right)^2 + C (\log N_{j+1})^2 \left( \sum_{m=J+1}^{\infty} a_m^2 d(m) \right) \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2).
\] (6.3)

We choose $N_j$ so that
\[
(\log N_j)^{4(1-\alpha)} (\log \log N_j)^{2(1-\alpha)} \sim J^4.
\]

Next choose $J$ so that
\[
\frac{J}{\log J} \sim (\log N_{j+1})^2.
\]

Then $J^{2(1-\alpha)} \sim (\log N_{j+1})^{4(1-\alpha)} (\log \log N_{j+1})^{2(1-\alpha)}$. And we obtain
\[
\left\| \sum_{u \leq k \leq v} c_k f_k \right\|_2^2 \leq C_{\alpha} \sum_{N_j \leq k \leq N_{j+1}} c_k^2 (\log k)^{4(1-\alpha)} (\log \log k)^{2(1-\alpha)} d(k^2).
\]

In view of the assumption made, we deduce
\[
\sum_{N_j \leq u \leq v \leq N_{j+1}} \left\| \sum_{u \leq k \leq v} c_k f_k \right\|_2^2 < \infty.
\] (6.4)

By Tchebycheff’s inequality and by using Theorem 2.1 and (6.2) with $L = 2$,
\[
\lambda \left\{ \sum_{N_j < k \leq N_{j+1}} c_k f_k > j^{-2} \right\} \leq C_{\alpha} j^2 \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2)
\]
\[
\leq C_{\alpha} \frac{J^4}{(\log N_j)^{4(1-\alpha)} (\log \log N_j)^{2(1-\alpha)} \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2)} \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2)(\log N_j)^{4(1-\alpha)} (\log \log N_j)^{2(1-\alpha)}
\]

The assumption made implies that
\[ \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) (\log k)^{4(1-\alpha)} (\log \log k)^{2(1-\alpha)}. \]
The assumption made implies that
\[ \sum_j \lambda \left\{ \sum_{N_j \leq k \leq N_{j+1}} c_k f_k \mid j^{-2} \left| j^{-2}\right. \right\} < \infty. \]
By the Borel-Cantelli lemma, the series \( \sum_j \sum_{N_j \leq k \leq N_{j+1}} c_k f_k \) converges almost everywhere. As by \( (6.4) \), the oscillation of partial sums around this subsequence is almost surely asymptotically tending to 0, this allows to conclude.

We continue with giving the proof of assertion (ii). If \( \alpha = 1, (6.3) \) is slightly modified as follows
\[ \left\| \sup_{N_j \leq u \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| \right\|_2^2 \leq C \left( \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \right) \left\{ (\log J)^2 + (\log N_{j+1})^2 \frac{\log J}{J} \right\}. \]
We choose \( N_j \) so that
\[ \log \log N_j \sim j^2. \]
And we choose \( J \) so that
\[ \frac{J}{\log J} \sim (\log N_{j+1})^2. \]
We deduce
\[ \left\| \sup_{N_j \leq u \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| \right\|_2^2 \leq C \left( \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \right) \left( \log \log N_{j+1} \right)^2 \leq C \sum_{N \leq k \leq N_{j+1}} c_k^2 d(k^2) \left( \log \log k \right)^2. \]
According to the assumption made,
\[ \sum_j \left\| \sup_{N_j \leq u \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| \right\|_2^2 < \infty. \]
By Tchebycheff’s inequality and by using Theorem 2.2
\[ \lambda \left\{ \left| \sum_{N_j \leq k \leq N_{j+1}} c_k f_k \right| > j^{-2} \right\} \leq C j^4 \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \leq C \left( \log \log N_{j+1} \right)^2 \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \left( \log \log k \right)^2 \leq C \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \left( \log \log k \right)^2. \]
By the Borel-Cantelli lemma, the series \( \sum_j \sum_{N_j < u \leq N_{j+1}} c_k f_k \) converges almost everywhere. This along with \( (6.6) \) allows to conclude.

Finally we give the proof of assertion (iii). By using again Abel summation
\[ \sum_{m \geq L} a_m^2 d(m) \leq C \left\{ \sum_{m \geq L} \frac{1}{m (\log m)^h} + \sup_{m \geq L} \frac{1}{(\log m)^h L} \right\} \leq \frac{C h}{(\log L)^{h-1}}. \]
And \( \sum_{m=1}^J |a_m| \leq C h \sqrt{J / (\log J)^{1+h}}. \) Then by \( (6.1) \),
\[ \left\| \sup_{N_j \leq u \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| \right\|_2^2 \leq C h \left( \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \right) \left\{ \frac{J}{(\log J)^{1+h}} + C h (\log N_{j+1})^2 \right\}. \]
We choose \( N_j \) so that
\[
\frac{(\log N_{j+1})^2}{(\log \log N_{j+1})^{h-1}} \sim j^4.
\]
We choose this time \( J \) so that \( \frac{J}{(\log J)^{1+\kappa}} \sim \frac{(\log N_{j+1})^2}{(\log \log N_{j+1})^{h-1}}, \) namely
\[
\frac{J}{(\log J)^2} \sim (\log N_{j+1})^2.
\]
Thus \( \log J \sim \log \log N_{j+1} \) and
\[
\frac{J}{(\log J)^{1+\kappa}} \sim \frac{(\log N_{j+1})^2}{(\log \log N_{j+1})^{h-1}} \sim \frac{(\log N_{j+1})^2}{(\log \log N_{j+1})^{h-1}}.
\]
We deduce
\[
\left\| \sup_{N_j \leq u \leq v \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| \right\|_2^2
\leq \left( \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \right) \left\{ C_h \frac{J}{(\log J)^{1+\kappa}} + C_{h'} \frac{(\log N_j)^2}{(\log \log N_{j+1})^{h-1}} \right\}
\leq C_h \left( \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \right) \frac{(\log N_{j+1})^2}{(\log \log N_{j+1})^{h-1}}
\leq C_h \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \frac{(\log k)^2}{(\log \log k)^{h-1}}.
\]
Using the assumption made, it follows that
\[
\sum_{j \geq 1} \left\| \sup_{N_j \leq u \leq v \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| \right\|_2 < \infty.
\]
We conclude by proceeding exactly as before. Noticing first from estimate \( \ref{est_6.7} \) that \( \sum_{m \geq 1} a_m^2 d(m) \leq C_h \), by Tchebycheff’s inequality and Theorem \( \ref{thm_2.1} \), we get
\[
\lambda \left\{ \left| \sum_{N_j \leq k \leq N_{j+1}} c_k f_k \right| > \frac{1}{j^2} \right\} \leq C_j^4 \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2)
\leq C_h J^4 \frac{(\log \log N_{j+1})^{h-1}}{(\log N_{j+1})^2} \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \frac{(\log k)^2}{(\log \log k)^{h-1}}
\leq C_h \sum_{N_j \leq k \leq N_{j+1}} c_k^2 d(k^2) \frac{(\log k)^2}{(\log \log k)^{h-1}}.
\]
By the Borel-Cantelli lemma, the series \( \sum_j \sum_{N_j \leq u \leq N_{j+1}} c_k f_k \) converges almost everywhere. We conclude as before.

7. Proof of Theorems \( \ref{thm_3.2} \), \( \ref{thm_3.3} \)

Let \( \{N_j, j \geq 1\} \) be an increasing unbounded sequence of positive reals. We write
\[
\sum_{N_j \leq k < N_{j+1}} c_k f_k = \sum_{N_j \leq k < N_{j+1}} c_k R_k^J + \sum_{N_j \leq k < N_{j+1}} c_k r_k^J
\]
where
\[
R_k^J(x) = \sum_{1 \leq \ell < J} \sin \frac{2 \pi \ell x}{\ell}, \quad r_k^J(x) = \sum_{\ell > J} \sin \frac{2 \pi \ell x}{j},
\]
and \( J \) is a real number greater than 1 and defined later on with respect to \( j \).
Proof of Theorem 3.3. Let $b > 0$. We choose $N_j$ so that $\log \log N_j = j^{\beta/b}$ for some $\beta > 2$. As $f \in \text{BV}(T)$, $a_j = O(j^{-1})$, and so

$$\sup_{N_j \leq u \leq v \leq N_{j+1}} | \sum_{u \leq k \leq v} c_k R_k'(x) | \leq C \left( \sum_{\ell=1}^J \sup_{N_j \leq u \leq v \leq N_{j+1}} | \sum_{u \leq k \leq v} c_k \sin 2\pi k \ell x | \right).$$

By using Carleson-Hunt’s maximal inequality

$$\left\| \sup_{N_j \leq u \leq v \leq N_{j+1}} | \sum_{u \leq k \leq v} c_k R_k' | \right\|_2 \leq C (\log J) \left( \sum_{N_j \leq u \leq N_{j+1}} c_k^2 \right)^{1/2}.$$ (7.1)

We now combine our Theorem 2.6 with the $(\varepsilon, 1-\varepsilon)$ argument introduced in [1]. Let $0 < \varepsilon < 1/2$.

From the bound

$$\sum_{j, j > j} \frac{1}{ij} \leq C \min \left( \frac{(k, \ell)}{(k \vee \ell)} \right)^2 \leq C \left( \frac{(k, \ell)}{k \ell} \right)^{1-\varepsilon} \leq C \left( \frac{(k, \ell)}{k \ell} \right)^{1-\varepsilon/2} = C \frac{\langle f \rangle^{1-\varepsilon/2}}{\langle f \rangle} = C \frac{\langle f \rangle^{1-\varepsilon/2}}{\langle f \rangle},$$

we get by applying Theorem 2.6 (i),

$$\left\| \sum_{u \leq k \leq v} c_k R_k' \right\|_2 \leq C \left\| \sum_{u \leq k \leq v} |c_k| \langle f \rangle^{1-\varepsilon/2} \right\|_2 \leq C \frac{\langle f \rangle^{1-\varepsilon/2}}{\langle f \rangle} \left( \sum_{u \leq k \leq v} c_k^2 \right)^{1/2}.$$ (7.2)

By taking $\tau = 1 + \varepsilon$ and using Corollary 2.7 this becomes,

$$\left\| \sum_{u \leq k \leq v} c_k R_k' \right\|_2 \leq \frac{C}{\varepsilon \langle f \rangle} \left( \sum_{u \leq k \leq v} c_k^2 \sigma_{-1+2\varepsilon}(k) \right).$$

By using Lemma 6.1 we obtain

$$\left\| \sup_{N_j \leq u \leq v \leq N_{j+1}} | \sum_{u \leq k \leq v} c_k R_k' | \right\|_2 \leq C (\log J \langle f \rangle^{1-\varepsilon/2}) \left( \sum_{u \leq k \leq v} c_k^2 \right) \sigma_{-1+2\varepsilon}(k).$$ (7.3)

Combining (7.1), (7.3) gives

$$\left\| \sup_{N_j \leq u \leq v \leq N_{j+1}} | \sum_{u \leq k \leq v} c_k f_k | \right\|_2 \leq C (\log J \langle f \rangle^{1-\varepsilon/2}) \left( \sum_{u \leq k \leq v} c_k^2 \right) \sigma_{-1+2\varepsilon}(k) \left( \log J \langle f \rangle^{1-\varepsilon/2} + \frac{C (\log N_{j+1})^2}{\varepsilon \langle f \rangle} \right).$$ (7.4)

Choose $\varepsilon, J$ as follows:

$$\log J = (\log \log N_{j+1})^{1+b}, \quad \varepsilon = \frac{2}{(\log \log N_{j+1})^b}.$$ 

Then $J^\varepsilon = \exp \{ \varepsilon \log J \} = \exp \{ 2 \log \log N_{j+1} \} = (\log N_{j+1})^2$. So that

$$\frac{(\log N_{j+1})^2}{\varepsilon J^\varepsilon} = \frac{(\log \log N_{j+1})^b}{2}$$

We deduce that

$$\left\| \sup_{N_j \leq u \leq v \leq N_{j+1}} | \sum_{u \leq k \leq v} c_k f_k | \right\|_2 \leq C (\log \log N_{j+1})^{2(1+b)} \sum_{N_j \leq k \leq N_{j+1}} c_k^2 \sigma_{-1+2\varepsilon}(k).$$
By using again Lemma 6.1 we obtain

\[ C_b \sum_{N_j \leq k \leq N_{j+1}} c_k^2 (\log \log k)^{(1+b)} \sigma_{-1+\frac{4}{(\log \log k)^b}}(k). \]

By Tchebycheff’s inequality,

\[ \lambda \left\{ \sup_{N_j \leq u \leq v \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| > j^{-3/2} \right\} \leq C_b j^\beta \sum_{N_j \leq k \leq N_{j+1}} c_k^2 (\log \log k)^{(1+b)} \sigma_{-1+\frac{4}{(\log \log k)^b}}(k) \]

\[ \leq \frac{C_b j^\beta (\log \log N_{j+1})^b}{(\log \log N_{j+1})^b} \sum_{N_j \leq k \leq N_{j+1}} c_k^2 \sigma_{-1+\frac{4}{(\log \log k)^b}}(k)(\log \log N_{j+1})^{2(1+b)} \]

\[ \leq C_b \sum_{N_j \leq k \leq N_{j+1}} c_k^2 \sigma_{-1+\frac{4}{(\log \log k)^b}}(k)(\log \log k)^{2+3b}. \]

By the Borel-Cantelli lemma and the assumption made, the series

\[ \sum_j \sup_{N_j \leq u \leq v \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| \]

converges almost everywhere. This allows to conclude.

**Proof of Theorem 5.2.** The main change will be in the treatment of the contribution due to the sum \( R^J \). Let \( \beta > 1 \). Choose \( N_j = e^{\varepsilon J^b} \) with \( B = 2\beta/\delta \) and \( \delta \) is a (small) positive real. From estimate 2.17 follows that,

\[ \sigma_{-1+2\varepsilon}(k) \leq \exp \left\{ \frac{\varphi (\log k)^{2\varepsilon}}{2\varepsilon \log \log k} \right\}, \]

where \( \varphi \) is some positive number. Thus (7.3) with Corollary 2.7 gives

\[ \| \sum_{u \leq k \leq v} c_k f_k \|^2 \leq \frac{C}{J^\beta} \exp \left\{ \frac{\varphi (\log N_{j+1})^{2\varepsilon}}{2\varepsilon \log \log N_{j+1}} \right\} \left( \sum_{u \leq k \leq v} c_k^2 \right). \]

By using again Lemma 6.1 we obtain

\[ \| \sup_{N_j \leq u \leq v \leq N_{j+1}} \sum_{u \leq k \leq v} c_k f_k \|^2 \leq \frac{C}{\varepsilon J^\beta (\log N_{j+1})^2} \exp \left\{ \frac{\varphi (\log N_{j+1})^{2\varepsilon}}{2\varepsilon \log \log N_{j+1}} \right\} \left( \sum_{N_j \leq k \leq N_{j+1}} c_k^2 \right). \]

Choose \( \varepsilon, J \) so that

\[ \varepsilon J^\varepsilon = (\log N_{j+1})^2 \exp \left\{ \frac{\varphi (\log N_{j+1})^{2\varepsilon}}{\varepsilon \log \log N_{j+1}} \right\}, \quad \varepsilon = \frac{\log \log \log N_{j+1}}{2 \log \log N_{j+1}}. \]

We get

\[ \| \sup_{N_j \leq u \leq v \leq N_{j+1}} \sum_{u \leq k \leq v} c_k f_k \|^2 \leq \sum_{N_j \leq k \leq N_{j+1}} c_k^2. \]

We have

\[ \log J = \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} + \frac{\varphi (\log N_{j+1})^{2\varepsilon}}{\varepsilon^2 \log \log N_{j+1}} \]

Further

\[ \frac{1}{\varepsilon} \log \frac{1}{\varepsilon} = \frac{2 \log \log N_{j+1}}{\log \log \log N_{j+1}} \log \left( \frac{2 \log \log N_{j+1}}{\log \log \log N_{j+1}} \right) \sim 2 \log \log N_{j+1}, \]

and

\[ (\log N_{j+1})^{2\varepsilon} = \varepsilon (\log \log N_{j+1}) (\log \log \log N_{j+1}) / (\log \log N_{j+1}) = \log \log N_{j+1}. \]
By the Borel-Cantelli lemma, the series
\[
\log J \sim 2(\log \log N_{j+1}) + \frac{4(\log \log N_{j+1})^2}{(\log \log \log N_{j+1})} + \frac{\rho}{\epsilon^2} \frac{(\log N_{j+1})^2}{(\log \log N_{j+1})^2}
\]
\[
= 2(\log \log N_{j+1}) + \frac{4(\log \log N_{j+1})^2}{(\log \log \log N_{j+1})} + \frac{\rho}{\epsilon^2} \frac{(\log N_{j+1})^2}{(\log \log N_{j+1})^2}
\]
\[
\leq C \left( \frac{\log \log N_{j+1}}{(\log \log N_{j+1})^2} \right).
\]

Now by (7.1),
\[
\| \sup_{N_j \leq u \leq v \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| \|_2^2 \leq C (\log J)^2 \left( \sum_{N_j \leq u \leq N_{j+1}} c_k^2 \right)
\]
\[
\leq C \left( \frac{\log \log N_{j+1}}{(\log \log \log N_{j+1})^2} \right) \left( \sum_{N_j \leq u \leq N_{j+1}} c_k^2 \right)
\]
\[
\leq C \sum_{N_j \leq u \leq N_{j+1}} c_k^2 \left( \frac{\log \log k}{(\log \log \log k)^2} \right).
\]

By combining (7.5),
\[
\| \sup_{N_j \leq u \leq v \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| \|_2^2 \leq C \sum_{N_j \leq u \leq N_{j+1}} c_k^2 \left( \frac{\log \log k}{(\log \log \log k)^2} \right).
\]

By the assumption made, this immediately implies that the series
\[
\sum_j \sup_{N_j \leq u \leq v \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right|^2
\]
converges almost everywhere. And so the oscillation of the partial sum sequence \(\{\sum_{k=1}^N c_k f_k, N \geq 1\}\) around the subsequence \(\{\sum_{k=1}^N c_k f_k, j \geq 1\}\) tends to zero almost everywhere.

By Tchebycheff’s inequality,
\[
\lambda \left\{ \sup_{N_j \leq u \leq v \leq N_{j+1}} \left| \sum_{u \leq k \leq v} c_k f_k \right| > j^{-\beta} \right\} \leq C j^{2\beta} \sum_{N_j \leq k \leq N_{j+1}} c_k^2
\]
\[
\leq C \left( \frac{\log \log N_{j+1}}{(\log \log \log N_{j+1})} \right) \sum_{N_j \leq u \leq N_{j+1}} c_k^2 \log \log k
\]
\[
\leq C \sum_{N_j \leq u \leq N_{j+1}} c_k^2 (\log \log k).
\]

By the Borel-Cantelli lemma, the series
\[
\sum_j \sum_{N_j \leq u \leq N_{j+1}} c_k f_k
\]
converges almost everywhere. We shall prove that the series
\[
\sum_j \sum_{N_j \leq u \leq N_{j+1}} c_k f_k
\]
also converges almost everywhere. This part is more tricky. We begin with a remark concerning the sum related to the component \(R^j\). Recall that
\[
\langle R_k^j, R_{\ell}^j \rangle = \sum_{j, \ell} \frac{1}{1h} = \left( \sum_{1 \leq u \leq j} \frac{1}{u} \right) \left( \sum_{1 \leq \ell \leq j} \frac{1}{u} \right)
\]
\[
= \sum_{1 \leq \ell \leq j} \frac{1}{u} \left( \frac{1}{u} \right)^2
\]
\[
= \frac{1}{k \ell}
\]

The existence of a solution in \(h \) and \(i \) (automatically of the form \(i = u \frac{k}{(k, \ell)}, h = u \frac{\ell}{(k, \ell)}, u \geq 1\)) imposes constraints on the integers \(k, \ell\). These must satisfy the following conditions
\[
\ell \leq u \ell = (k, \ell) h \leq J(k, \ell), \quad k \leq uk = (k, \ell) i \leq J(k, \ell),
\]
that is \((k, \ell) \geq \frac{(k \vee \ell)}{k \ell}\). Consequently, as obviously \((k, \ell) \leq (k \wedge \ell)\), it is necessary to have
\[
\frac{1}{J} (k \vee \ell) \leq (k \wedge \ell).
\]

In this case \(0 \leq \langle R^j_{\ell}, R^k_{\ell} \rangle \leq \zeta(2) \frac{(k, \ell)^2}{k \ell}\). Observe before continuing that in our situation \(J \ll N_{j+1}\) while \(N_j < k, \ell \leq N_{j+1}\).

Let \(h\) and \(H\) be such that \(J^h < N_j \leq J^{h+2} = \ldots \leq J^{h+H-1} \leq N_{j+1} < J^{h+H}\). It follows from the remark previously made and estimate (2.20) that,
\[
\| \sum_{N_j < k \leq N_{j+1}} c_k R^j_{\ell} \|_2^2 \leq \zeta(2) \sum_{N_j < k \leq N_{j+1}} |c_k||c_{\ell}| \frac{(k, \ell)^2}{k \ell}
\]
\[
= \zeta(2) \sum_{\mu = h}^H \sum_{k \leq N_{j+1}} |c_k||c_{\ell}| \frac{(k, \ell)^2}{k \ell}
\]
\[
\leq \zeta(2) \sum_{\mu = h}^H \sum_{j^{h-1} \leq k \leq j^{h+2}} |c_k||c_{\ell}| \frac{(k, \ell)^2}{k \ell}
\]
\[
\leq (4\zeta(2) \log J) \sum_{\mu = h}^H \sum_{j^{h-1} \leq k \leq j^{h+2}} c_k^2 \sigma_1(k)
\]
\[
\leq C \sum_{\mu = h}^H \sum_{j^{h-1} \leq k \leq j^{h+2}} c_k^2 \frac{(\log \log k)^2}{\log \log \log k} \sigma_1(k)
\]

since \(N_j < k \leq N_{j+1}\) and
\[
\log J \sim 4(1 + o(1)) \frac{(\log \log N_{j+1})^2}{(\log \log \log N_{j+1})} \sim 4(1 + o(1)) \frac{(\log \log k)^2}{\log \log \log k}
\]

Therefore
\[
(7.6) \quad \| \sum_{N_j < k \leq N_{j+1}} c_k R^j_{\ell} \|_2^2 \leq C \sum_{J^{-1} N_j < k \leq N_{j+1}} c_k^2 \frac{(\log \log k)^2}{\log \log \log k} \sigma_1(k).
\]

By Tchebycheff’s inequality,
\[
\lambda \left\{ \sum_{N_j < k \leq N_{j+1}} c_k R^j_{\ell} \right\} > j^{-\beta}
\]
\[
\leq C j^{2\beta} \sum_{J^{-1} N_j < k \leq N_{j+1}} c_k^2 \frac{(\log \log k)^2}{\log \log \log k} \sigma_1(k)
\]
\[
\leq C j^{2\beta - B \beta} (\log \log N_{j+1})^\beta \sum_{J^{-1} N_j < k \leq N_{j+1}} c_k^2 \frac{(\log \log k)^2}{\log \log \log k} \sigma_1(k)
\]
\[
\leq C \sum_{J^{-1} N_j < k \leq N_{j+1}} c_k^2 \frac{(\log \log k)^2}{\log \log \log k} \sigma_1(k)
\]

Recall that \(J = J(j)\) is associated with the interval \([N_j, N_{j+1}]\). Now observe that
\[
\frac{N_{j+2}}{J(j + 2)} > N_{j+1} J(j)^2.
\]

Indeed,
\[
J(j)^2 J(j + 2) \sim e^{\frac{a+o(1)}{B}} \frac{e^B}{\log y} + e^{\frac{a+o(1)}{B}} \frac{(j+2)^B}{\log (y(j+2))}.
\]
Thus $J(j)^2J(j+2) \leq e^{Cnj^B}$, whereas for $j$ large

$$\frac{N_{j+2}}{N_{j+1}} = e^{(j+2)eta} - e^{(j+1)eta} \geq e^{(j+2)\beta} / 2 \gg J(j)^2J(j+2).$$

This means that the intervals $[J(j)^{-1}N_j, N_{j+1}J(j)^2]$ $j = 2, 4, 6 \ldots$ are disjoint. The same holds for the sequence of intervals with odd indices. Consequently,

$$\sum_j \lambda \{ \sum_{N_j < k \leq N_{j+1}} |c_kR_k| > j^{-\beta} \} \leq C \sum_k c_k^2 \frac{(\log \log k)^{2+\delta}}{\log \log k\sigma_1(k)} < \infty,$$

by assumption. Hence by the Borel-Cantelli lemma, the series

$$\sum_j \sum_{N_j < k \leq N_{j+1}} c_kR_k^j$$

converges almost everywhere. This allows to conclude.

**Remark 7.1.** It is interesting to notice that from estimates (7.1) and (7.6) and Grönwall’s estimate (??),

$$\| \sup_{N_j \leq u \leq v \leq N_{j+1}} \sum_{u \leq k \leq v} c_kR_k^j \|_2^2 \leq C \sum_{N_j \leq k \leq N_{j+1}} c_k^2 \frac{(\log \log k)^4}{(\log \log k)^2},$$

while

$$\| \sum_{N_j < k \leq N_{j+1}} c_kR_k^j \|_2^2 \leq C \sum_{J(j)^{-1}N_j < k \leq N_{j+1}J(j)^2} c_k^2 \frac{(\log \log k)^3}{\log \log k}.$$

**Remark 7.2.** The following set $B = \{ k, \ell : N < k, \ell \leq M : J(k, \ell) \geq (k \lor \ell) \}$ appeared in the last step of the proof. Concerning the size of such sets, one can show the general bound

$$\#(B) \leq CJM(\log JM)^3,$$

where $C$ is absolute.

**8. Proof of Theorem 5.7**

Changing $f$ for $f/c$ if necessary, we may assume for our purpose that $c = 1$ in (5.5). Let $G_n = (\gamma_{k,\ell})$, where $\gamma_{k,\ell} = \int_{f_n} f_kf_\ell dx$ denotes the Gram matrix of the system $f_1, \ldots, f_n$. As $H_n = I - G_n$ is nonnegative definite, there exist in $\mathbb{R}^n$ vectors $u_1, \ldots, u_n$ with Gram matrix $H_n$, for instance the rows of $H_n^{-1/2}$. Given any bounded interval $Y$, it follows that there exist in $L^2(Y)$ (in fact in any separable Hilbert space), vectors $v_1, \ldots, v_n$ with Gram matrix $I - G_n$. By induction (using isometry), it is plain that if $v_1, \ldots, v_n$ are already chosen with Gram matrix $H_n$, a vector $v_{n+1}$ can be added so that the new system $v_1, \ldots, v_{n+1}$ will have Gram matrix $H_{n+1}$. Consequently there exist $(g_k)$ supported on $Y$ such that $(f_k + g_k)$ is an orthonormal system on $T \times Y$. Thus for any $(c_k)$ universal, the series $\sum_k c_k(f_k + g_k)$ converges a.e. on $T \times Y$, and thereby converges a.e. on $T$. Since $g_k \equiv 0$ on $T$, it follows that $\sum_k c_kf_k$ converges a.e.

**Remark 8.1.** The construction of $(g_k)$ is exactly as in the proof of Schur’s Lemma (32, p. 56).

**9. A strengthened form of Theorem 2.6**

Our goal in this section is to show that the form of the upper bound provided in Theorem 2.6 in fact strongly depends on how the considered GCD quadratic form can be bounded from below. This is a quite intriguing property, expressed in Corollary 9.3 and which we will study more thoroughly elsewhere. We first establish the following stronger estimate.

**Theorem 9.1.** Let $s > 1/2$ and $0 \leq \tau \leq 2s$. For any finite coefficient sequence $\{ c_k, k \in K \}$,

$$\left| \sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} - \sum_{k \in K} c_k^2 \right|$$

is absolute.
\[
\left( \sum_{k \in K} \frac{|c_k|}{k^s} \right)^2 + 2(\zeta(2s) \sum_{k \in K} c_k^2)^{1/2} \left( \sum_{k, \ell \in K^*} |c_k| |c_\ell| \frac{(k, \ell)^{2s}}{k^s \ell^s} \right)^{1/2} + \\
\left( \sum_{u \in F'(K)} \frac{1}{\sigma_r(u)} \right) \left( \sum_{\nu \in K^*} c_\nu \sigma_{-2s}(\nu) \right),
\]

where \( F'(K) = F(K) \setminus \{1\} \) and \( K^* = \{ \ell \in K; \exists k \in K, k < \ell : k|\ell \}. \)

**Remark 9.2.** In general, \( K^* \) is a significantly smaller set than \( K \). If \( K = [a, b] \) with \( b > a > 1 \), then \( K^* \not\subseteq K \), then \( a + 1 \not\in K^* \) simply because \( a \not| a+1 \). The extremal case corresponds to Rudin sets, namely sets of integers none of which divides the least common multiple of the others, in which case \( K^* = \emptyset \).

**Corollary 9.3.** Let \( s > 1/2 \) and \( 0 \leq \tau \leq 2s \). Let \( \rho = \sum_{a \in K} \frac{1}{a^s} \). Assume \( c_k \geq 0, k \in K \) and \( \rho \leq 1/16 \). We have the following alternative. Either

\[
\sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} \leq \frac{1}{\sqrt{\rho}} \left( \sum_{k \in K} c_k \right)^2 + \frac{1}{\rho \sigma_r(u)} \sum_{\nu \in K^*} c_\nu \sigma_{-2s}(\nu),
\]

or

\[
\sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} \leq \frac{1}{1 - 3\sqrt{\rho}} \sum_{\nu \in K^*} c_\nu^{2s}.
\]

**Proof of Theorem 9.1.** From [5.2] and Möbius inversion formula [5.1],

\[
\sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} = \sum_{k, \ell \in K} \frac{c_k c_\ell}{k^s \ell^s} \left( \sum_{d \in F(K)} J_{2s}(d) 1_{d|k} 1_{d|\ell} \right)
\]

\[
= \sum_{\ell \in K} \frac{c_\ell^2}{\ell^{2s}} \left( \sum_{d \in F(K)} J_{2s}(d) 1_{d|\ell} \right) + \sum_{k, \ell \in K} \frac{c_k c_\ell}{k^s \ell^s} \left( \sum_{d \in F(K)} J_{2s}(d) 1_{d|k} 1_{d|\ell} \right)
\]

\[
= \sum_{\ell \in K} c_\ell^2 + \sum_{k, \ell \in K} c_k c_\ell \left( \sum_{d \in F(K)} J_{2s}(d) 1_{d|k} 1_{d|\ell} \right).
\]

We begin with isolating particular values of \( d \) in the sum in brackets. Observe that the case \( d = 1 \) contributes at most for

\[
(9.1) \quad \left( \sum_{k \in K} \frac{|c_k|}{k^s} \right)^2 \leq \zeta(2s) \sum_{k \in K} |c_k|^2,
\]

by using Cauchy-Schwarz’s inequality. Now if \( d = k \) or \( d = \ell \), then \( J_{2s}(d) 1_{d|k} 1_{d|\ell} \) is either equal to

\[
J_{2s}(k) 1_{k|\ell} \quad \text{or} \quad J_{2s}(\ell) 1_{\ell|k}
\]

These cases contribute for

\[
2 \sum_{\ell \in K} \sum_{k \in K} \frac{c_k c_\ell}{k^s \ell^s} J_{2s}(k).
\]

We have

\[
\left| \sum_{\ell \in K} \sum_{k \in K} \frac{c_k c_\ell}{k^s \ell^s} J_{2s}(k) \right| = \left| \sum_{\ell \in K^*} \sum_{k \in K} \frac{c_k c_\ell}{k^s \ell^s} \frac{J_{2s}(k)}{k^{2s}} \right|
\]

\[
\leq \sum_{\ell \in K^*} |c_\ell| \sum_{k \in K} |c_k| \left( \frac{k}{\ell} \right)^s = \sum_{\ell \in K^*} |c_\ell| b_\ell,
\]

with

\[
b_\ell = \sum_{k \in K} |c_k| \left( \frac{k}{\ell} \right)^s.
\]
Operating as in [20] (proof of Proposition 3.1),
\[
\sum_{\ell \in K^s} b^2_\ell = \sum_{\ell \in K^s} \sum_{u,v \in K_{[u,v];\ell}} |c_u| |c_v| u^s v^s \frac{u^s v^s}{\lambda^{2s}} = \sum_{u,v \in K} |c_u| |c_v| u^s v^s \sum_{\lambda \in K, v} \frac{1}{\lambda^{2s}}.
\]
By the Cauchy-Schwarz inequality,
\[
\left| \sum_{\ell \in K^s} c_\ell b_\ell \right| \leq \left( \sum_{\ell \in K^s} c^2_\ell \right)^{1/2} \left( \sum_{\ell \in K^s} b^2_\ell \right)^{1/2} \leq \left( \sum_{\ell \in K^s} c^2_\ell \right)^{1/2} \left( \sum_{u,v \in K} |c_u| |c_v| u^s v^s \sum_{\lambda \in K, v} \frac{1}{\lambda^{2s}} \right)^{1/2}.
\]
(9.3)
We next concentrate on the sum
\[
\Sigma := \sum_{k, \ell \in K} \frac{c_k c_\ell}{k^s \ell^s} \left\{ \sum_{d \in F'} J_{2s}(d) 1_{d | k} 1_{d | \ell} \right\} = \sum_{d \in F'} \frac{J_{2s}(d)}{d^{2s}} \sum_{u,v \in F'} \frac{|c_{uv}| |c_{vd}|}{u^s v^s}.
\]
Writing \( k = ud, \ell = vd \) and noticing that \( u, v \in F'(K) \), we have
\[
\left| \sum_{d \in F'} J_{2s}(d) \sum_{k, \ell \in K} \frac{c_k c_\ell}{k^s \ell^s} 1_{d | k} 1_{d | \ell} \right| \leq \sum_{d \in F'} \frac{J_{2s}(d)}{d^{2s}} \sum_{u,v \in F'} \frac{|c_{uv}| |c_{vd}|}{u^s v^s} = \sum_{u,v \in F'} \frac{1}{u^s v^s} \left( \sum_{d \in F'} \frac{J_{2s}(d)}{d^{2s}} |c_{uv}| |c_{vd}| \right).
\]
By the Cauchy-Schwarz inequality,
\[
\sum_{d \in F'} \frac{J_{2s}(d)}{d^{2s}} |c_{uv}| |c_{vd}| \leq \left( \sum_{d \in F'} \frac{J_{2s}^2(d)}{d^{4s}} \right)^{1/2} \left( \sum_{d \in F'} \frac{1}{d^{2s}} \right)^{1/2}.
\]
Hence,
\[
\Sigma \leq \left[ \sum_{u \in F'} \frac{1}{u^s} \left( \sum_{d \in F'} \frac{J_{2s}(d)}{d^{2s}} c_{ud}^2 \right)^{1/2} \right]^2.
\]
Now we continue as in the proof of Theorem 2.4. Let \( \psi \) be a positive arithmetic function. Writing
\[
\frac{1}{u^s} = \frac{1}{u^{s/2}} \frac{\psi(u)^{1/2}}{u^{s/2}}
\]
and applying Cauchy-Schwarz’s inequality again gives,
\[
\Sigma \leq \left( \sum_{u \in F'} \frac{1}{u^s \psi(u)} \right) \left( \sum_{u \in F'} \frac{\psi(u)}{u^s} \sum_{d \in F'} \frac{J_{2s}(d)}{d^{2s}} c_{ud}^2 \right)^{1/2} \leq \left( \sum_{u \in F'} \frac{1}{u^s \psi(u)} \right) \left( \sum_{u \in F'} \frac{\psi(u)}{u^s} \sum_{d \in F'} \frac{J_{2s}(d)}{d^{2s}} c_{ud} \right).
\]
We note that the first sum is indexed on \( F'(K) \) this time. The second sum was already estimated. We recall that
\[
\sum_{u \in F'(K)} \frac{\psi(u)}{u^s} \sum_{d \in F'} \frac{J_{2s}(d)}{d^{2s}} c_{ud}^2 \leq \sum_{\nu \in K} \frac{c_{\nu}^2}{\nu^{2s}} \sum_{u \in F'} \frac{J_{2s}(\nu/\mu)}{u^s} u^s \psi(u),
\]
and for \( \psi(u) = u^{-s} \sigma_{r}(u) \) with \( r \leq 2s \)
\[
\sum_{u \in F'(K)} \frac{\sigma_{r}(u)}{u^s} u^s \psi(u) = \nu^r \sigma_{2s-r}(\nu).
\]
So that
\[
\Sigma \leq \left( \sum_{u \in F'(K)} \frac{1}{\sigma_{r}(u)} \right) \left( \sum_{\nu \in K} \frac{c_{\nu}^2}{\nu^{2s}} \nu^r \sigma_{2s-r}(\nu) \right).
\]
Finally,

\[(9.4) \quad \Sigma \leq \left( \sum_{u \in F'(K)} \frac{1}{\sigma_\tau(u)} \right) \left( \sum_{\nu \in K} c_\nu^2 \sigma_{\tau-2s}(\nu) \right),\]

By combining (9.1), (9.3), (9.4) we get

\[\left| \sum_{k, \ell \in K} c_k c_\ell \left( \frac{(k, \ell)^{2s}}{k^s \ell^s} - \sum_{\ell \in K} c_\ell^2 \right) \right| \leq \left( \sum_{k \in K} c_k^2 \right)^2 + 2 \left( \sum_{\ell \in K^*} c_\ell^2 \right)^{1/2} \left( \sum_{u, v \in K} c_u c_v \sum_{\lambda \in K^*_F} \frac{1}{\lambda^{2s}} \right)^{1/2} + A \]

(9.5)

\[+ \left( \sum_{u \in F'(K)} \frac{1}{\sigma_\tau(u)} \right) \left( \sum_{\nu \in K} c_\nu^2 \sigma_{\tau-2s}(\nu) \right).\]

**Proof of Corollary 9.3.**

Put

\[
\begin{align*}
A &= \left( \sum_{u \in F'(K)} \frac{1}{\sigma_\tau(u)} \right) \left( \sum_{\nu \in K} c_\nu^2 \sigma_{\tau-2s}(\nu) \right) \\
L &= \sum_{k, \ell \in K} c_k c_\ell \frac{(k, \ell)^{2s}}{k^s \ell^s} \\
\varepsilon &= \sum_{\ell \in K} c_\ell^2.
\end{align*}
\]

By (9.3),

\[
|L - \varepsilon| \leq \left( \sum_{k \in K} c_k^2 \right)^2 + 2 \left( \sum_{\ell \in K^*} c_\ell^2 \right)^{1/2} \left( \sum_{u, v \in K} c_u c_v \sum_{\lambda \in K^*_F} \frac{1}{\lambda^{2s}} \right)^{1/2} + A
\]

\[\leq \left( \sum_{k \in K} c_k^2 \right)^2 + A + (L \varepsilon)^{1/2}.
\]

Let \(0 < h < 1\). Either \((1 - h)L < \varepsilon\). Thus

\[L \leq \frac{1}{1 - h} \sum_{\ell \in K} c_\ell^2.
\]

Or \((1 - h)L \geq \varepsilon\), and so

\[L \leq \left( \sum_{k \in K} c_k^2 \right)^2 + (1 - h)L + 2((1 - h)\rho)^{1/2}L + A.
\]

Thus

\[L(h - 2((1 - h)\rho)^{1/2}) \leq \left( \sum_{k \in K} c_k^2 \right)^2 + A.
\]

Let \(h = 3\sqrt{\rho}\). Then \(h - 2((1 - h)\rho)^{1/2} \geq \sqrt{\rho}\) and \((1 - h) \geq 1/4 \geq \sqrt{\rho}\). Now

\[
\left( \sum_{k \in K} \frac{c_k}{k^s} \right)^2 + A = \left( \sum_{k \in K} c_k^2 \right)^2 + \left( \sum_{u \in F'(K)} \frac{1}{\sigma_\tau(u)} \right) \left( \sum_{\nu \in K} c_\nu^2 \sigma_{\tau-2s}(\nu) \right)
\]

\[\leq \left( \sum_{k \in K} c_k^2 \right)^2 + \left( \sum_{u \in F'(K)} \frac{1}{\sigma_\tau(u)} \right) \left( \sum_{\nu \in K} c_\nu^2 \sigma_{\tau-2s}(\nu) \right)^2
\]

\[\leq \left( \sum_{k \in K} c_k + \sum_{u \in F'(K)} \frac{1}{\sigma_\tau(u)} + \sum_{\nu \in K} c_\nu^2 \sigma_{\tau-2s}(\nu) \right)^2.
\]

Hence

\[L \leq \frac{1}{\sqrt{\rho}} \left( \sum_{k \in K} c_k^2 + \sum_{u \in F'(K)} \frac{1}{\sigma_\tau(u)} + \sum_{\nu \in K} c_\nu^2 \sigma_{\tau-2s}(\nu) \right)^2.
\]
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