Entangling Power in the Deterministic Quantum Computation with One Qubit

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The deterministic quantum computing with one qubit (DQC1) is a mixed-state quantum computation algorithm that evaluates the normalized trace of a unitary matrix and is more powerful than the classical counterpart. We find that the normalized trace of the unitary matrix can be directly described by the entangling power of the quantum circuit of the DQC1, so the nontrivial DQC1 is always accompanied with the non-vanishing entangling power. In addition, it is shown that the entangling power also determines the intrinsic complexity of this quantum computation algorithm, i.e., the larger entangling power corresponds to higher complexity. Besides, it is also shown that the non-vanishing entangling power does always exist in other similar tasks of DQC1.

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I. INTRODUCTION

Quantum entanglement is employed in most of quantum information processing tasks (QIPTs) including the quantum algorithms and the quantum communications [1]. There is no doubt that quantum entanglement is an important physical resource in quantum information processing. However, quantum entanglement cannot be competent for all the QIPTs [2-5]. Strong evidence has shown that some QIPTs display the quantum advantage, but there does not exist any entanglement in the tasks [6,7], which is also verified in experiment [8]. One such remarkable evidence is the scheme of the deterministic quantum computing with one qubit (DQC1) which accomplishes to evaluate the normalized trace of a unitary matrix by only measuring a control qubit, irrespective of the complexity of the unitary matrix of interests [9]. But DQC1 cannot be performed effectively by only using classical computation [9,10]. So what quantum property leads to the quantum advantage in the DQC1?

Quantum discord was introduced to effectively distinguish the quantum from the classical correlations [11-13]. It is shown that quantum correlation ‘includes’ quantum entanglement but beyond it due to the presence in separable states [14-16]. In recent years, quantum discord has attracted increasing interests especially in the areas such as the evolution in quantum dynamical system [17-19], operational interpretations [20-26] and the quantification [27,28] and the experiments [29, 30]. In particular, it is found that almost every unitary matrix (random unitary) in the DQC1 will lead to the occurrence of quantum discord in the output state, so it is conjectured that quantum discord could be the quantum nature of the DQC1 [10]. However, after all, there exist some general unitary matrices (for example, Hermitian unitary matrices) that will arrive at the output states without any quantum discord. In this sense, it seems that quantum discord should not be the source of the quantum advantage of the DQC1 either, as is first suspected in Ref. [31]. Thus what on earth is the quantum nature of DQC1 is still left open.

In this paper, we find that the trace calculation and the complexity in the DQC1 can be directly related to the entangling power which is defined by the maximal average on the ability to entangle a qubit with a pure state in a given ensemble. We find that the entangling power of the DQC1 circuit can be directly written as the normalized trace of the unitary transformation to be measured. Based on this result, we find not only that in the nontrivial DQC1 is there the entangling power, but also that the intrinsic complexity (which will be given at the end) of the evaluation of the normalized trace is determined by the entangling power. In this sense, we say that the entangling power could be used to signal the quantum nature of the DQC1. As a supplement, we also consider the other QIPTs using the similar DQC1 circuit. One will find that the entangling power is always present if the information of the system can be extracted by the control qubit.
II. THE ENTANGLING POWER WITH THE NORMALIZED TRACE

We begin with a brief introduction of the generalized DQC1 described by the quantum circuit given in Fig. 1 [9]. The initial state can be written as

\[ \rho_0 = \rho_c^c \otimes \frac{1}{2^n}, \]

where \( \rho_c^c = \frac{1}{2} (1_1 + \alpha \sigma_z) \), \( \sigma_x, \sigma_y, \sigma_z \) are Pauli matrices, the subscript ‘c’ denotes the control qubit, the superscript \( \alpha \) means that the density matrix depends on the parameter \( \alpha \) and \( 1_n \) means the identity of \( n \) qubits. It is noted that in the standard DQC1 [7], the initial control qubit is given by \( |0\rangle \). Through the quantum circuit, the state given in Eq. (1) will be transformed into the final state \( \rho_{n+1} \).

\[ \rho_{n+1} = \frac{1}{2^n} \left( \left(0\right) \left(0\right) \otimes 1_n + \left(1\right) \left(1\right) \otimes 1_n \right) + \alpha \left(0\right) \left(1\right) \otimes U_n^\dagger + \alpha \left(1\right) \left(0\right) \otimes U_n \right). \quad (2) \]

Thus if we measure the control qubit in the basis of \( \sigma_x \) and \( \sigma_y \), respectively, one can obtain the corresponding expectations as \( \frac{1}{2} \text{Re}(\text{Tr} U_n) \) and \( -\frac{1}{2} \text{Im}(\text{Tr} U_n) \). In this way, the normalized trace of the unitary matrix \( U_n \) is obtained only by the measurements on a single qubit, irrespective of the complexity of the unitary matrix. This shows the quantum advantage of the DQC1 in the reduction of the computational complexity.

In this task, the initial state \( \rho_0 \) is obviously mixed, so in practical scenario, it has to be prepared by one of its pure-state realizations \( \{p_i, |\phi_i\rangle\} \) such that \( \rho_0 = \sum p_i |\phi_i\rangle \langle \phi_i| \), \( \sum p_i = 1 \). Here \( |\phi_i\rangle \) is normalized but not necessarily orthogonal. An intuitive observation shows that the DQC1 circuit can lead to the entangled final state of \( |\phi_i\rangle \) if any information on \( U_n \) is unknown. Therefore, it is not difficult to ignite the light to relate this kind of entanglement to the mechanism of the quantum advantage of the DQC1. To do so, we would like to introduce the variational concept of entangling power. It is initially defined for a unitary transformation by measuring the average entanglement produced by this unitary transformation on the separable state subject to some kind of distributions [32]. However, in some special QIPTs, not all the quantum separable states are covered, so it is necessary to give an explicit definition well-suited to the given QIPT. So in the DQC1, we would like to consider the entangling power of the controlled unitary transformation \( 1_{n-1} \otimes U_n \) with the assistance of the Hadamard gate \( H \) subject to the initial ensemble \( \frac{1}{\sqrt{n}} \). In other words, we will quantify the ability of the whole DQC1 circuit to entangle the control qubit with the \( n \)-qubit pure state selected from the initial ensemble \( \frac{1}{\sqrt{n}} \). With this aim, we have the following rigid definition.

**Definition.**- The entangling power of the DQC1 circuit is given by

\[ E_p \left( \tilde{U}_n \right) = \max_{\{i_1, i_2, \ldots, i_n\}} \sum q_i E \left[ \tilde{U}_n \left( \rho_c \otimes |\varphi_i\rangle \langle \varphi_i| \right) \tilde{U}_n^\dagger \right] \]

with \( \tilde{U}_n = (1_{n-1} \otimes U_n) \times (H \otimes 1_n) \), \( \frac{1}{\sqrt{2^n}} = \sum q_i |\varphi_i\rangle \langle \varphi_i| \), where \( E[\cdot] \) represents any a good entanglement measure [1].

Here we let \( E[\cdot] = \sqrt{2 \left( 1 - \text{Tr} \rho^2 \right)} \) with \( \rho_r \) the reduced density matrix of the state taken into account [1]. The maximum is taken due to the non-uniqueness of the realization of \( \frac{1}{\sqrt{n}} \). In addition, \( \rho_c \) is not limited to the pure state, which is different from the original definition of the entangling power besides the limited ensemble. Next, we will give our main results on \( E_p \left( \tilde{U}_n \right) \) by two theorems.

**Theorem 1.**- The entangling power, defined in Eq. (3) for the standard DQC1 circuit corresponding to \( \rho_c^c = |0\rangle \langle 0| \), i.e., \( \alpha = 1 \), is given by

\[ E_p^1 \left( \tilde{U}_n \right) = \sqrt{1 - \frac{\text{Tr} U_n^4}{2^n}}. \]

**Proof.** Substitute \( \rho_c^c = |0\rangle \langle 0| \) and any an \( n \)-partite pure state \( |\varphi_i\rangle \) chosen from the ensemble \( \frac{1}{\sqrt{2^n}} = \sum q_i |\varphi_i\rangle \langle \varphi_i| \) into the DQC1 circuit sketched in Fig. 1, the final state after these substitutions can be written as

\[ |\chi\rangle_{n+1} = \frac{1}{\sqrt{2}} \left( |0\rangle \langle \varphi_i| + |1\rangle \langle \varphi_i| U_n \right). \]

The reduced density matrix by tracing out the control qubit is given by

\[ \rho_r^i = \frac{1}{2} |\varphi_i\rangle \langle \varphi_i| + U_n |\varphi_i\rangle \langle \varphi_i| U_n^\dagger. \]

So the entangling power can be expressed as

\[ E_p^1 \left( \tilde{U}_n \right) = \max_{\{i_1, i_2, \ldots, i_n\}} \sum q_i \sqrt{2 \left( 1 - \text{Tr} \left( \rho_r^i \right)^2 \right)} = \max_{\{i_1, i_2, \ldots, i_n\}} \sum q_i \sqrt{1 - |\langle \varphi_i| U_n |\varphi_i\rangle|^2} \]

\[ \leq \sqrt{1 - \sum q_i |\langle \varphi_i| U_n |\varphi_i\rangle|^2} = \sqrt{1 - \frac{\text{Tr} U_n^4}{2^n}}. \]

The inequality comes from the concave property of the entanglement measure \( E[\cdot] \) and the maximum is attained by the realization \( \frac{1}{\sqrt{2^n}} = \sum q_i |\tilde{\phi}_i\rangle \langle \tilde{\phi}_i| \) where \( \tilde{\phi}_i = \frac{1}{\sqrt{n}} \) and \( |\tilde{\phi}_i\rangle = \sum_k e^{i \frac{2\pi k}{n}} |v_k\rangle \) with \( |v_k\rangle \) the eigenvectors of \( U_n \).

The proof is completed.

It is quite interesting that the entangling power for the standard DQC1 is directly described by the normalized computation.
trace of the measured $U_n$. So long as $\text{Tr}U_n \neq 2^n$ which implies $U_n \neq 1_n e^{i\theta}$, the entangling power will not vanish. This means that the DQC1 will demonstrate the quantum advantage. Otherwise, the entangling power will vanish for $U_n = 1_n e^{i\theta}$, but the unitary matrix $U_n$ in this case will be easily evaluated in the classical computation. So it is a trivial case.

**Theorem 2.** The entangling power for the generalized DQC1 circuit corresponding to $\rho^n = \frac{1}{2}(1 + \alpha \sigma_z)$, i.e., $0 < \alpha < 1$, is given by

$$E_p^n(\hat{U}_n) = \alpha E_p^1(\hat{U}_n) = \alpha \sqrt{1 - \frac{|\text{Tr}U_n|}{2^n}}.$$  \hspace{1cm} \text{(9)}

**Proof.** If $0 < \alpha < 1$, the control qubit is obviously a mixed state. Based on the definition of entangling power given in Eq. (3), we can write

$$E_p^n(\hat{U}_n) = \max_{\{q_i, |\varphi_i\rangle\}} \sum_i q_i E[\varphi_i],$$  \hspace{1cm} \text{(10)}

with

$$\varphi_i = (1_{n-1} \oplus U_n)(H \rho^n H^\dagger \otimes |\varphi_i\rangle \langle \varphi_i|)(1_{n-1} \oplus U_n).$$  \hspace{1cm} \text{(11)}

Since $E[\varphi_i]$ denotes the entanglement measure of the state $\varphi_i$, we have

$$E[\varphi_i] = \min_{\{r_j, |\gamma_j\rangle\}} \sum_j r_j E[|\gamma_j\rangle],$$  \hspace{1cm} \text{(12)}

with $q_i = \sum_j r_j |\gamma_j\rangle \langle \gamma_j|$ and the subscript $i$ corresponding to $\varphi_i$. Thus the entangling power can be rewritten as

$$E_p^n(\hat{U}_n) = \max_{\{q_i, |\varphi_i\rangle\}} \sum_i q_i \min_{\{r_j, |\gamma_j\rangle\}} \sum_j r_j E[|\gamma_j\rangle],$$  \hspace{1cm} \text{(13)}

where the exchange of the maximum and minimum is attributed to the independence of realizations $\{r_j, |\gamma_j\rangle\}$ and $\{q_i, |\varphi_i\rangle\}$. Consider the eigendecomposition of $\varphi_i$: $\varphi_i = M \Phi_i^\dagger$, one can obtain that

$$\Phi_i = (1_{n-1} \oplus U_n)(H \otimes |\varphi_i\rangle),$$  \hspace{1cm} \text{(14)}

$$M = \begin{pmatrix}
\frac{1 + \alpha}{2} & 0 \\
0 & \frac{1 - \alpha}{2}
\end{pmatrix}.$$  \hspace{1cm} \text{(15)}

Thus any decomposition of $\varphi_i$ can be given in terms of the eigendecomposition characterized by Eqs. (14) and (15). Let $\varphi_i = \sum_j r_j |\gamma_j\rangle$ be any decomposition with the form of matrix given by $\varphi_i = \Psi W \Psi^\dagger$ where the columns of $\Psi$ correspond to $|\gamma_j\rangle$ and the diagonal entries of the diagonal matrix $W$ correspond to $r_j$. Therefore, we have $\Psi \Psi^\dagger = M \Phi_i^\dagger T$ where $T$ denotes a right unitary matrix with $TT^\dagger = 1_1$. So $|\gamma_j\rangle$ can be given by the eigenvectors as

$$|\gamma_j\rangle = \frac{1}{r_j^{1/2}}(x_j |0\rangle + y_j |1\rangle) U_n |\varphi_i\rangle,$$  \hspace{1cm} \text{(16)}

where

$$x_j = \frac{1}{\sqrt{2}} \left(T_{1j} \sqrt{\frac{1 + \alpha}{2}} + T_{2j} \sqrt{\frac{1 - \alpha}{2}}\right),$$  \hspace{1cm} \text{(17)}

$$y_j = \frac{1}{\sqrt{2}} \left(T_{1j} \sqrt{\frac{1 + \alpha}{2}} - T_{2j} \sqrt{\frac{1 - \alpha}{2}}\right),$$  \hspace{1cm} \text{(18)}

and $r_j = |x_j|^2 + |y_j|^2$ with $|T_{1j}|^2 + |T_{2j}|^2 \leq 1$ and

$$\sum_j |T_{1j}|^2 = \sum_j |T_{2j}|^2 = 1.$$  \hspace{1cm} \text{(19)}

Substitute Eq. (16) into Eq. (13), we can arrive at

$$E_p^n(\hat{U}_n) = \min_{\{r_j, |\gamma_j\rangle\}} \max_{\{q_i, |\varphi_i\rangle\}} \sum_i \sqrt{q_i r_j^2} \left(|x_j|^2 - |y_j|^2 - 2 |x_j| |y_j| |\langle \varphi_i, U_n |\varphi_i\rangle|^2\right)^{1/2},$$  \hspace{1cm} \text{(20)}

Based on Eq. (7) (or Theorem 1), we have

$$E_p^n(\hat{U}_n) = \min_{\{r_j, |\gamma_j\rangle\}} \sum_j 2 |x_j| |y_j| E_p^1(\hat{U}_n).$$  \hspace{1cm} \text{(21)}

Insert Eqs. (17) and (18) into Eq. (21) and use Eq. (19), it follows

$$E_p^n(\hat{U}_n) = \min_{\{r_j, |\gamma_j\rangle\}} \frac{1}{2} \sum_j \left((1 + \alpha) |T_{1j}|^2 - (1 - \alpha) |T_{2j}|^2\right) E_p^1(\hat{U}_n)$$

$$\geq \frac{1}{2} \sum_j \left(|T_{1j}|^2 - (1 - \alpha) |T_{2j}|^2\right) E_p^1(\hat{U}_n)$$

$$\geq \alpha E_p^1(\hat{U}_n) = \alpha \sqrt{1 - \frac{|\text{Tr}U_n|^2}{2^n}}.$$  \hspace{1cm} \text{(22)}

where the minimum is achieved when both $T_{1j}$ and $T_{2j}$ are real or imaginary for all $j$. The proof is completed.

From the above two theorems, one can find the result given in Theorem 2 can be reduced to Theorem 1 for $\alpha = 1$. That is, the entangling power $E_p^n(\hat{U}_n)$ pertains to all the cases of $\alpha$. In addition, the entangling power of the generalized DQC1 including the standard case is directly described by the normalized trace of $U_n$ to be measured. As is the same as the analysis of the standard DQC1, if $U_n \neq 1_n e^{i\theta}$ which is a trivial case, the DQC1 will demonstrate the quantum advantage of the quantum computation, the entangling power will not vanish. In addition, it can be easily found that the entangling
power will increase as $\alpha$ increasing. When $\alpha = 0$, this means that the control qubit is an identity which will extract nothing about the measured $U_n$. In this case, the entangling power vanishes, which is consistent with our expectation.

III. THE COMPLEXITY WITH THE ENTANGLING POWER

In fact, our entangling power is also closely related to the complexity of the DQC1. The complexity of the DQC1 can be given by

$$O(U_n) = nL(\varepsilon),$$

(23)

where $L(\varepsilon)$ is the measurement complexity which describes how many rounds of measurements are necessary to be operated on the control qubit for a given standard deviation $\varepsilon$, and $n$ is the input complexity which denotes the number of qubits needed to be input into the DQC1 circuit. In usual analysis of the complexity of DQC1, only the measurement complexity $L(\varepsilon)$ is considered. So it is stated that the complexity $L(\varepsilon)$ only depends on the accuracy $\varepsilon$ that we expect instead of the scale of the measured $U_n$, because

$$L(\varepsilon) \sim \frac{\ln(1/P_{\varepsilon})}{(\alpha^2\varepsilon^2)},$$

(24)

where $P_{\varepsilon}$ is the probability that the estimate is farther from the true value than $\varepsilon$ [10]. When we consider the practical experiment, the standard deviation should not exceed the true value of the measured quantity. In this sense, the complexity will also depend on the true value of the measured observables to different extents. Thus, instead of the standard deviation $\varepsilon$, it would be reasonable to describe the accuracy in terms of the relative error defined by

$$\epsilon = \left| \frac{\varepsilon}{X} \right| \times 100\%,$$

(25)

with $X$ the true value of some measured quantities. In the DQC1, $\sigma_x$ and $\sigma_y$ will be measured on the control qubit corresponding to the normalized trace of $U_n$ (actually to the real and imaginary parts, respectively). Let our finally expecting relative errors be $\epsilon \geq \max\{\epsilon(\sigma_x), \epsilon(\sigma_y)\}$ with $\epsilon(\sigma_x)$ and $\epsilon(\sigma_y)$ denoting the expecting relative errors for the measurements $\sigma_x$ and $\sigma_y$ such that

$$\frac{\ln[1/P_{\epsilon}(\sigma_x)]}{|\epsilon(\sigma_x)\text{Re}(\frac{U_n}{2})|^2} = \frac{\ln[1/P_{\epsilon}(\sigma_y)]}{|\epsilon(\sigma_y)\text{Im}(\frac{U_n}{2})|^2} \sim \alpha^2 L.$$  

(26)

Substitute Eqs. (24-26) into Eq. (9), the entangling power can be written as

$$E^\alpha_p \left( \tilde{U}_n \right) \sim \sqrt{\alpha^2 - \frac{M}{L}},$$

(27)

with $M = \frac{\ln[1/P_{\epsilon}(\sigma_x)]}{|\epsilon(\sigma_x)\text{Re}(\frac{U_n}{2})|^2} + \frac{\ln[1/P_{\epsilon}(\sigma_y)]}{|\epsilon(\sigma_y)\text{Im}(\frac{U_n}{2})|^2}$. Thus for a given $M$, the measurement complexity $L$ is directly determined by the entangling power $E^p_p \left( \tilde{U}_n \right)$. Since $M$ is determined by the expecting errors and is independent of the scale of the measured $U_n$, we think that the complexity $L$ with the fixed $M$ is the intrinsic complexity. The larger entangling power means the larger intrinsic complexity $L$. Thus one can find that the intrinsic complexity is directly determined by the entangling power.

IV. DQC1-LIKE CIRCUITS IN OTHER TASKS

We would like to generalize the current DQC1 circuit to general QIPTs, by which one will find that the non-trivial tasks with DQC1 circuit are indeed accompanied with non-vanishing entangling power. Suppose this circuit is used to extract the information of some state $\rho_n$ instead of $1_n$ and the control qubit is the general quantum state

$$\rho_c = \frac{1}{2} (1_1 + \mathbf{P} \cdot \sigma)$$

(28)

with $\mathbf{P}$ the polarization vector and $\sigma$ the corresponding vector of Pauli matrices. The final state $\rho_f$ of $\rho_c$ via the circuit can be written as $\rho_f = \text{Tr}_n \tilde{U}_n (H \rho_c H^\dagger \otimes \rho_n) \tilde{U}_n^\dagger$. The linear entropy of $\rho_f$ is given by

$$L(\rho_f) = 1 - \text{Tr} \rho_f^2$$

$$= \frac{1}{2}(1 - P_1^2 - (P_2^2 + P_3^2) |\text{Tr} U \rho_n|^2).$$

(29)

In order to accomplish this extraction of information, $L(\rho_f)$ should include at least the information on the state $\rho_n$ or the unitary operation $U_n$ based on different aims, since one hopes to extract the information by the control qubit. That is, $L(\rho_f)$ should not vanish. In this case, we can obtain the similar theorem as Theorem 1 and Theorem 2.

Theorem 3. Let the entangling power of the DQC1 circuit subject to $\rho_n$ be $E^{P_3}_p \left( \tilde{U}_n, \rho_n \right)$, $E^{P_3}_p \left( \tilde{U}_n, \rho_n \right)$ does not vanish for $L(\rho_f) > 0$;

$$1 - \text{Tr} \left[ U_n \rho_n U_n^\dagger \rho_n \right] \leq E^{P_3}_p \left( \tilde{U}_n, \rho_n \right) \leq \sqrt{1 - |\text{Tr} U \rho_n|^2},$$

(30)

and

$$E^{P}_p \left( \tilde{U}_n, \rho_n \right) = (\lambda_1 - \lambda_2) E^{P}_p \left( \tilde{U}_n, \rho_n \right),$$

(31)

where the superscript $P_3$ means $\mathbf{P} = (0,0,1)^T$, the superscript $P$ is the general case in $\rho_c$ and $\lambda_i$ are the square root of the eigenvalues of the matrix $\rho_\alpha \sigma_z \rho^*_c \sigma_z$ in decreasing order.

Proof. The proof is quite similar to that of Theorem 2. The details are given in the Appendix. One can easily check that Theorem 1 and Theorem 2 are covered in this theorem.  ■
V. CONCLUSIONS

In summary, we have shown that the normalized trace of the unitary transformation \( U_n \) can be directly described by the entangling power of the circuit of the DQC1. In addition, the entangling power also determines the intrinsic complexity of the evaluation of the normalized trace of the measured unitary matrix. In this sense, we think that the entangling power could be the signature of the quantum advantage of the DQC1. Furthermore, we also present a generalization of the current DQC1 circuit to other QIPTs. We find that the entangling power will not vanish if the information can be extracted by the control qubit.

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VII. APPENDIX: BRIEF PROOF OF THEOREM 3

For the general state \( \rho_n \), it is difficult to calculate the exact entangling power. Here we give its bounds and show that the entangling power is always present, if one can extract some information. The second inequality in Eq. (30) is obviously correct, which is similar to Eq. (8). Now we give the brief proof for the first inequality in Eq. (30). According to the definition of our entangling power, we have, for \( \rho_n = \sum_i q_i |\phi_i^\prime\rangle \langle \phi_i^\prime| \),

\[
E_p^\rho\left(U_n, \rho_n\right) = \max_{\{q_i, |\phi_i\rangle\}} \sum_i q_i \sqrt{2 \left(1 - Tr\left(\rho_n U_n |\phi_i\rangle \langle \phi_i|\right)\right)^2}
\geq 1 - \max_{\{q_i, |\phi_i\rangle\}} \sum_i q_i \left|\langle \phi_i| U_n \rho_n |\phi_i\rangle\right|
= 1 - \max_{i} \sum_i \left|\tilde{T}\tilde{U}\right|_{ii}^{1/2}
= 1 - \text{Tr}(U_n \rho_n U_n^\dagger \rho_n),
\]
where the first inequality holds because the purity of $\rho_c^i$ is not more than 1, and we consider the relation between the eigendecomposition $\rho_n = \Phi M \Phi^\dagger$ and the other decompositions similar to those between Eqs. (15) and (16).

One can find that the lower bound of $E_p^{\rho s} \left( \hat{U}_n, \rho_n \right)$ given in Eq. (30) will vanish if $[U_n, \rho_n] = 0$. However, in this case, one can easily prove that $E_p^{\rho s} \left( \hat{U}_n, \rho_n \right)$ will not vanish unless $U_n = e^{i\theta} \mathbf{1}_n$ which leads to zero $L(\rho_f)$. Thus we show that any non-trivial QIPT with DQC1 circuit is accompanied with entangling power.

In order to show that Eq. (31) holds, we have to rewrite the initial control qubit $\rho_c$ given in Eq. (28). Similar to Eqs. (14,15), any decomposition of $\rho_c$ can be related to its eigendecomposition $\rho_c = \Phi' M' \Phi'^\dagger$, where

$$\Phi' = \begin{pmatrix} \cos \frac{\theta}{2} & e^{i\phi} \sin \frac{\theta}{2} \\ e^{-i\phi} \sin \frac{\theta}{2} & -\cos \frac{\theta}{2} \end{pmatrix},$$

$$M' = \frac{1}{2} \begin{pmatrix} 1 + i \Gamma & 0 \\ 0 & 1 - i \Gamma \end{pmatrix},$$

with $\cos \theta = P_3/\Gamma$, $\Gamma = \sqrt{\sum_{k=1}^{3} P_k^2}$. Thus any decomposition of $\rho_c = \Psi W' \Psi'^\dagger$ can be written by $\Psi \sqrt{W'} = \Phi' \sqrt{M'T'}$ with $T'$ the right unitary matrix. Substitute any one possible pure state $[\Psi_{11}, \Psi_{12}, \cdots] T$ and $\rho_n$ into the DQC1 circuit, one will arrive at the final state as

$$|\gamma_j'\rangle = \frac{1}{\sqrt{r_j'}} \left( x_j' |0\rangle |\varphi_i'\rangle + y_j' |1\rangle U_n |\varphi_i'\rangle \right),$$

where

$$x_j' = \frac{1}{\sqrt{2}} \left( T_{1j} (\Phi_{11}' + \Phi_{21}') \sqrt{1 + \frac{1}{\Gamma}} + T_{2j} (\Phi_{12}' + \Phi_{22}') \sqrt{1 - \frac{1}{\Gamma}} \right),$$

$$y_j' = \frac{1}{\sqrt{2}} \left( T_{1j} (\Phi_{11}' - \Phi_{21}') \sqrt{1 + \frac{1}{\Gamma}} + T_{2j} (\Phi_{12}' - \Phi_{22}') \sqrt{1 - \frac{1}{\Gamma}} \right),$$

and $r_j' = |x_j'|^2 + |y_j'|^2$ with $|T_{1j}'|^2 + |T_{2j}'|^2 \leq 1$ and

$$\sum_j |T_{1j}'|^2 = \sum_j |T_{2j}'|^2 = 1.$$

Similar to Eq. (20), the entangling power can be given by

$$E_p^{\rho} \left( \hat{U}_n, \rho_n \right) = \min_{\{r_j', |\gamma_j'\rangle\}} \max_{\{q_i, |\varphi_i\rangle\}} \sum_{i,j} \sqrt{2q_i} (r_j'^2 - |x_j'|^2 - |y_j'|^2) \sum_{i,j} \sqrt{2q_i} (r_j'^2 - |x_j'|^2 - |y_j'|^2) \frac{1}{\sqrt{2}} \sum_{i,j} \sqrt{2q_i} (r_j'^2 - |x_j'|^2 - |y_j'|^2) \frac{1}{\sqrt{2}}$$

Consider Eqs. (36,37), one will have

$$\min_{\{r_j', |\gamma_j'\rangle\}} \sum_j 2 |x_j'| |y_j'| = \min_{\{r_j', |\gamma_j'\rangle\}} \sum_j |T^T \Phi \sigma_z \Phi^T M \Phi^T \sigma_z |j_j$$

$$= \lambda_1 - \lambda_2,$$

where $\lambda_i$ are the square root of the eigenvalues of the matrix $\rho_c \sigma_z \rho_c^* \sigma_z$ in decreasing order. The proof is completed.