Generalized Heun and Lamé’s equations: factorization

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Abstract

This paper addresses new results on the factorization of the general Heun’s operator, extending the investigations performed in previous works [Applied Mathematics and Computation 141 (2003), 177 - 184 and 189 (2007), 816 - 820]. Both generalized Heun and generalized Lamé equations are considered.

Key words: Factorization, Heun’s differential equation, Lamé’s differential equation, polynomials.

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1 Introduction

Fuchsian differential’s equations of second order and their confluent forms, built for instance from the Ince techniques \cite{11} play a major role in many partial differential equations of mathematical physics as Laplace, Helmholtz or Schrödinger’s equations.

Without loss of generality the \( k + 3 \) regular single singularities can be located at \( x = 0, x = 1, x = a_i, \ i = 1, k \) and \( x = \infty \) and after an appropriate change of function one of the indices at each finite singularity can be shifted to zero allowing to write the Fuchsian equation as \cite{2,3}:

\footnote{Correspondence should be addressed to: norbert.hounkonnou@cipma.uac.bj, with copy to hounkonnou@yahoo.fr}
\[ y''(x) + \left( \frac{\gamma}{x} + \frac{\delta}{x-1} + \sum_{i=1}^{k} \frac{\epsilon_i}{x-a_i} \right) y'(x) + \frac{\alpha \beta x^k + \sum_{i=1}^{k} \rho_i x^{k-i}}{x(x-1) \prod_{i=1}^{k} (x-a_i)} y(x) = 0. \]

(1)

The two indices at each singularity \( a_i \) are \((0, 1 - \epsilon_i)\) and \((0, 1 - \gamma), (0, 1 - \delta)\), \((\alpha, \beta)\) at \( x = 0 \), \( x = 1 \), and \( x = \infty \), respectively, taking into account the Fuchsian relation:

\[ \alpha + \beta + 1 = \gamma + \delta + \sum_{i=1}^{k} \epsilon_i. \]

(2)

\( \epsilon_i \equiv 0, (i = \overline{1, k}) \), gives the hypergeometric equation and \( k = 1 \) the Heun’s equation \([7]\). The character reducible or irreducible of linear ordinary differential equation (O.D.E) is important in relation with the monodromy group \([2]\). We call reducible a second order O.D.E which admits a non trivial solution satisfying a first order linear equation, and otherwise irreducible. This definition is used by many authors \([4, 5, 6]\) with or without supplementary conditions. It allows to give an easy criterion of reducibility of an equation with polynomial coefficients, from a factorized approach in the following way.

Rewriting (1) in a polynomial form with \( D = \frac{d}{dx} \),

\[ \mathcal{H}_k[y(x)] \equiv \left[ Q_{k+2}(x)D^2 + Q_{k+1}D + Q_k \right] y(x) = \left[ L(x)D + M(x) \right] \left[ \bar{L}(x)D + \bar{M}(x) \right] y(x), \]

where \( Q_j \) are polynomials of degree \( j \), possible (or not possible) identification of the polynomials \( L, \bar{L}, M, \bar{M} \) with polynomials \( Q_{k+2}, Q_{k+1}, Q_k \) will give all possible cases of reducibility and, a contrario, criteria of irreducibility for the equation \( \mathcal{H}_k[y(x)] = 0 \).

This was recently done for the Heun’s equation \([8, 9]\)

\[ y''(x) + \left[ \frac{\gamma}{x} + \frac{\delta}{x-1} + \frac{\epsilon}{x-a} \right] y'(x) + \frac{\alpha \beta x - q}{x(x-1)(x-a)} y(x) = 0 \]

(4)

and also for the four confluent Heun’s equations \([10]\). For the Heun’s equation, this factorization is possible and generate 6 non trivial situations in accordance with the known \( F^j \)-homotopic transformations \([11]\). R. S Maier \([12]\) noticed that for the Lamé equation \((k = 1 \text{ with } \gamma = \delta = \epsilon = 1/2)\)

\[ y''(x) + \frac{1}{2} \left( \frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-a} \right) y'(x) - \frac{l(l+1)x + 4q}{4x(x-1)(x-a)} y(x) = 0, \]

(5)
the factorization reduces only to the two cases \( l = 1, \ l = \frac{1}{2} \). He also mentioned that for the general Lamé equation with \( k + 3 \) singularities, the factorization is not known.

Even if some algorithms exist in order to factorize any ODE with rational function coefficients \([13] [14] [15] [16]\), factorization of an arbitrary \( k \)-Heun equation is probably untractable, except for using numerical tools.

The aim of this work is to investigate the case \( k = 2 \) for both generalized Heun and generalized Lamé equations \([17]\) which are also important in lattice statistics \([18]\). These equations have invested a large number of applications in physics. See \([19]\) (and references therein) for a nice review on physical applications of these equations. To mention a few, retrieved from the indicated works, where for instance Heun’s equations as well as their solutions impose their usefulness, it is worthy of attention to outline their importance in the description of quasi-modes of near extremal black branes, in the hyperspherical harmonics with applications in three-body systems, in the elaboration of a method of calculation of propagators for the case of a massive spin 3/2 field for arbitrary space-time dimensions and mass, in parametric resonance after inflation, as well as in the separation of variables for the Schrödinger equation in a large number of problems, typically for the radial coordinate, and in non-linear formulations involving Painleve type equations. A number of traditional equations of mathematical physics, as for instance the Lame, spheroidal wave, and Mathieu equations, are also particular cases of Heun equations.

Expansion of the equation (3) gives:

\[
\mathcal{H}_k[y(x)] = (LD + M) \left( \bar{L}D + \bar{M} \right) y(x) \\
= \left\{ L\bar{L}D^2 + \left[ L \left( \bar{L}' + \bar{M} \right) + M\bar{L} \right] D + \left( L\bar{M}' + M\bar{M} \right) \right\} y(x)
\]

(6)

and generate the 3 basic relations:

\[
\begin{align*}
Q_{k+2} &= LL, \\
Q_{k+1} &= L \left( \bar{L}' + \bar{M} \right) + M\bar{L}, \\
Q_k &= LM' + M\bar{M}.
\end{align*}
\]

(7)

Computation and properties of the polynomials \( M \equiv M(x), \ \bar{M} \equiv \bar{M}(x), \ L \equiv L(x), \ \bar{L} \equiv \bar{L}(x) \) are explicitly given in several situations.
2 Preliminary remarks

1) $Q_{k+2}(x)$ contains $k + 2$ linear and different factors which are all present in the product $L(x)\bar{L}(x)$ in several ways. For instance, if $j$ factors appear in $L$ and $k + 2 - j$ in $\bar{L}$, the number of decomposition is given by the classical binomial result:

$$\sum_{j=0}^{k+2} \binom{k+2}{j} = 2^{k+2},$$

excluding the extremal cases

$$L(x) = Q_{k+2}(x), \bar{L} = 1; Q_k = 0,$$
$$\mathcal{H}_k = LD(D + M)$$

and

$$L(x) = 1, \bar{L} = Q_{k+2}(x), M = 0 \text{ (integrable case $\bar{M} = Q_{k+1} - Q'_{k+2}$, }$$
$$Q_k = \bar{M}'), \mathcal{H}_k = D(LD + \bar{M})$$

which are irrelevant or trivial in this study.

Selecting a pair $L, \bar{L}$ inside the $2^{k+2} - 2$ possibilities, computation of the polynomials $M(x)$ and $\bar{M}(x)$ of degrees $m$ and $\bar{m}$, respectively, is done in 2 steps.

i) The coefficients in $Q_{k+1}(x)$ and $Q_{k+2}(x)$ allow to linearly compute the coefficients of $M(x)$ and $\bar{M}(x)$, of number $m + \bar{m} \leq k$ from the second and third relations in (7).

ii) The third relation in (7) gives the value of the accessory parameters $\rho_i$, $(i = 1, k)$ and the product $\alpha \beta$, using the polynomials $M$ and $\bar{M}$ computed in the first step.

2) The general $k$–Lamé’s equation is a $k$–Heun’s equation with $\gamma = \delta = \epsilon = 1/2$, $(i = T, k)$. The indices at each finite singularity are equal to $(0, 1/2)$ and the Fuchsian relation (2) gives $\alpha + \beta = k/2$. This equation takes the form

$$\mathcal{L}_k[y] \equiv Q_{k+2}y'' + Q_{k+1}y' + \frac{1}{4} \left( \alpha \beta x^k + \sum_{i=1}^{k} \rho_i x^{k-i} \right) y = 0$$

with now

$$Q_{k+1} = \frac{1}{2}(Q_{k+2})' = \frac{1}{2} [L\bar{L}]',$$

where

$$Q_{k+2} = x(x-1) \prod_{j=1}^{k} (x - a_j) = L\bar{L}.$$
The property (12) and second relation in (7) easily give:

\[
\frac{1}{2} \left( \frac{L}{L^\prime} \right)' = \frac{\bar{L} \tilde{M} + M \bar{L}}{L^2}. \tag{14}
\]

The number of situations to be investigated in the decomposition of \(L\bar{L}\) can be simplified from some symmetry between polynomials \(M, \bar{M}\) and \(M^*, \bar{M}^*\) corresponding to the decomposition \(L\bar{L}\), permuting \(L\) and \(\bar{L}\) giving the factorization:

\[
\mathcal{L}_k[y] = [\bar{L}D + M^*] [LD + \bar{M}^*] y(x). \tag{15}
\]

A similar relation to (14) is

\[
\frac{1}{2} \left( \frac{\bar{L}}{\bar{L}^\prime} \right)' = \frac{\bar{L}M^* + M^* \bar{L}}{L^2}. \tag{16}
\]

We deduce from (14) and (16) that

\[
L \left( \bar{M} + M^* \right) + \bar{L} \left( M + \bar{M}^* \right) = 0. \tag{17}
\]

Equality (17) is satisfied if, for example, we have 
\(M = -\bar{M}^*, \ \bar{M} = -M^*\).

3 Lamé’s equation, \((k = 1)\)

Let us consider equation (4) with \(\gamma = \delta = \epsilon = \frac{1}{2}\). Even if this case has been already solved in [7], as a subcase of the full Heun factorization, we easily recover the polynomials \(M\) and \(\bar{M}\) from the procedure described in section 2 for the factorization \(L = x(x-1), \ \bar{L} = x-a\) as example. Unknown polynomials \(M\) and \(\bar{M}\) must be as:

\[
M = Ax + B, \ \bar{M} = \bar{A} \tag{18}
\]

with

\[
Q_3(x) = L\bar{L} = x^3 - x^2(a + 1) + xa, \tag{19}
\]

\[
Q_2(x) = L(\bar{L}' + \bar{M}) + M\bar{L} = \frac{3}{2}x^2 - x(a + 1) + \frac{a}{2} \tag{20}
\]

\[
Q_1(x) = M\bar{M}. \tag{21}
\]

We easily conclude that
\[
A = 1, \quad B = -\frac{1}{2}, \quad \bar{A} = -\frac{1}{2}.
\]

Next the third relation in (7) gives:

\[
\alpha \beta = -\frac{1}{2}, \quad q = -\frac{1}{4}
\]

and the Fuschian relation yields

\[
\alpha + \beta = \frac{1}{2}.
\]

From section 2, \(M^* = -\bar{M}\) and \(\bar{M}^* = -M\) already solve the factorization problem with \(L = x - a, \bar{L} = x(x - 1)\).

A surprising result, consequence of a remark of Maier [12], is that for \(L = x\), or \((x - 1)\), or \((x - a)\), \((\bar{L} = (x - 1)(x - a), x(x - a), x(x - 1))\), the polynomials \(M\) are the same. From the symmetry property \(M^* = -M, \bar{M}^* = -M\), this result also works for the 3 other factorizations: \(L = (x - 1)(x - a), x(x - a), x(x - 1); \bar{L} = x, (x - 1), (x - a)\). All these results can be seen in table 1, presented in the same way as in Ref [9].

4 Factorization of Lame’s equation \(\mathcal{L}_2[y], k = 2\)

Equation (1) with \(k = 2, a_1 = a, a_2 = b\) and \(\gamma = \delta = \epsilon_1 = \epsilon_2 = \frac{1}{2}\) is the Lamé equation

\[
\mathcal{L}_2[y] = Q_4(x)y'' + Q_3(x)y' + Q_2(x)y = 0,
\]

where

\[
\begin{cases}
Q_4(x) &= x(x - 1)(x - a)(x - b) \\
&= x^4 - x^3(1 + a + b) + x^2(a + b + ab) - abx,
Q_3(x) &= \frac{1}{2}Q_4'(x) = \frac{1}{2} \left[ 4x^3 - 3x^2(1 + a + b) + 2x(a + b + ab) - ab \right], \\
Q_2(x) &= \alpha \beta x^2 + \rho_1 x + \rho_2,
\alpha + \beta &= 1.
\end{cases}
\]

With a first choice \(L = x(x - 1), \bar{L} = (x - a)(x - b)\), \(M\) and \(\bar{M}\) are first degree polynomials.
Table 1. Factorization of the simple Heun’s operator $\mathcal{H}_1 = \mathcal{L}_1[y]$

$$\mathcal{H}_1 = x(x-1)(x-a)D^2 + [\gamma(x-1)(x-a) + \delta x(x-a) + \epsilon x(x-1)] D + (\alpha \beta x - q)I_d$$

$$= (LD + M)(\bar{L}D + \bar{M}), \quad \gamma = \delta = \epsilon = \frac{1}{2}. \quad (25)$$

| $L$      | $\bar{L}$ | $M$      | $\bar{M}$ | $\alpha$ | $\beta$ | $q$  |
|----------|-----------|----------|-----------|----------|---------|------|
| $x$      | $(x-1)(x-a)$ | $\frac{1}{2}$ | $-x + \frac{a+1}{2}$ | $\frac{3}{2}$ | $-1$ | $-\frac{1}{2} \left(\frac{a+1}{2}\right)$ |
| $x-1$    | $x(x-a)$  | $\frac{1}{2}$ | $-x + \frac{a}{2}$ | $\frac{3}{2}$ | $-1$ | $-\left(1 + \frac{a}{4}\right)$ |
| $x - a$  | $x(x-1)$  | $\frac{1}{2}$ | $-x + \frac{1}{2}$ | $\frac{3}{2}$ | $-1$ | $-\left(a + \frac{1}{4}\right)$ |
| $(x-1)(x-a)$ | $x$      | $x - \frac{a+1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $1$ | $-\frac{1}{4}(a+1)$ |
| $x(x-a)$ | $x-1$    | $x - \frac{a}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $1$ | $-\frac{1}{4}a$ |
| $x(x-1)$ | $x - a$  | $x - \frac{1}{2}$ | $-\frac{1}{2}$ | $-\frac{1}{2}$ | $1$ | $-\frac{1}{4}$ |
The four equations satisfied by $A$, $B$, $\bar{A}$, $\bar{B}$ are given by identification of second relation in (7) with the second relation in (27). The solutions are

\[
A = 1, \quad B = -\frac{1}{2}, \quad \bar{A} = -1, \quad \bar{B} = \frac{a + b}{2}. \tag{29}
\]

Comparison of third relation in (7) with the third relation in (27) yields

\[
\alpha \beta = -2 \tag{30}
\]

and also $\rho_1$ and $\rho_2$:

\[
\rho_1 = \frac{a + b + 3}{2}, \tag{31}
\]
\[
\rho_2 = -\frac{a + b}{4}. \tag{32}
\]

These results as well as those corresponding to the other choices of $L$ and $\bar{L}$ are reproduced in tables 2 to 4 (14 situations) for the cases $\gamma = \delta = \epsilon_1 = \epsilon_2 = \frac{1}{2}$ as in the case $k = 1$, in which appear essentially 2 situations [12]; the case $k = 2$ generates only 3 situations with $\alpha \beta = -2$, $-3/4$, $-15/4$.

5 Factorization of the generalized Heun’s equation $\mathcal{H}_2[y]$, $k = 2$

The first extension of Heun’s equation with $k = 2$ can be written as ($y(x) \equiv y$)

\[
y'' + \left[\frac{\gamma}{x} + \frac{\delta}{x - 1} + \frac{\epsilon_1}{x - a} + \frac{\epsilon_2}{x - b}\right]y' + \frac{\alpha \beta x^2 + \rho_1 x + \rho_2}{x(x - 1)(x - a)(x - b)} y = 0 \tag{33}
\]

with the parameters linked by the Fuchs’s condition $\alpha + \beta + 1 = \gamma + \delta + \epsilon_1 + \epsilon_2$ or in polynomial way

\[
\mathcal{H}_2[y] = Q_4(x) y'' + Q_3(x) y' + Q_2(x) y = 0, \tag{34}
\]

where

\[
Q_4(x) = x(x - 1)(x - a)(x - b) = x^4 - x^3(1 + a + b) + x^2(a + b + ab) -xab; \tag{35}
\]
\[ Q_3(x) = \gamma(x-1)(x-a)(x-b) + \delta x(x-a)(x-b); \]
\[ + \epsilon_1 x(x-1)(x-b) + \epsilon_2 x(x-1)(x-a) \quad (36) \]
\[ Q_2(x) = \alpha \beta x^2 + \rho_1 x + \rho_2. \quad (37) \]

The factorized form is

\[ H_2[y] = (LD + M \bar{L}D + \bar{M}) = L\bar{L}D^2 + (L\bar{L} + L\bar{M} + M\bar{L}) D + (LM' + M\bar{M}). \quad (38) \]

Let us consider a peculiar simple situation

\[ L = x(x-1), \quad \bar{L} = (x-a)(x-b). \quad (39) \]

The third relation in (7): \( L\bar{M}'+M\bar{M} = Q_2(x) \) obliges \( M \) and \( \bar{M} \) to be of degree 1. Let \( M = xA + B, \bar{M} = x\bar{A} + \bar{B} \). The unknown \( A, B, \bar{A}, \bar{B} \) are given using also the second relation in (7). From \( Q_2(x) = Ax(x-1) + (xA + B)(x\bar{A} + \bar{B}) \), we get

\[
\begin{cases}
\bar{A} + A\bar{A} = \alpha \beta, \\
-\bar{A} + BA + B\bar{A} = \rho_1.
\end{cases}
\quad (40)
\]

As in the Lamé’s case, some kind of symmetry appears when permuting \( L \) and \( \bar{L} \) in the factorization, allowing to reduce by a factor 2 the number of factorizations. Use of the adjoint operator \( H^* \) of \( H \) is now useful. Indeed, the Lagrange adjoint \( H^* \) of \( H \) is given by [1,21]

\[ H^* = Q_{k+2}^* D^2 + Q_{k+1}^* D + Q_k^* \quad (41) \]

with

\[ Q_{k+2}^* = Q_{k+2}, \quad Q_{k+1}^* = 2Q_{k+2}^* - Q_{k+1}, \quad Q_k^* = Q_{k+2}^* - Q_{k+1}^* + Q_k \quad (42) \]

It is not very easy to write in general \( H^* \) in the form of the equation (41) with star parameters \( \alpha^*, \beta^*, \rho_i^* \), because the corresponding equations are quadratic. But from the factorized form of \( H \), computation of polynomials \( M^* \) and \( \bar{M}^* \) are trivial from the relations:

\[ H = (LD + M)(\bar{L}D + \bar{M}), \quad H^* = (\bar{L}D + \bar{M})^*(LD + M)^* \quad (43) \]

and, from equation (42), we get
\[ \mathcal{H}^* = (-LD + L' - \bar{M})(-LD + L' - M) \]
\[ = (LD + M^*)(LD + M^*) \] (44)

with

\[ \bar{M}^* = \bar{M} - L', \quad M^* = M - L' \] (45)

generalizing the relation (15).

In tables 5 to 7, we give, in the 14 situations, the values of \( M, \bar{M}, \alpha, \beta, \rho_1 \) and \( \rho_2 \) as well as those corresponding the other choices of \( L \) and \( \bar{L} \). Of course, the symmetry in the factorization of \( \mathcal{H}_2 \) is not so rich as in the case of \( \mathcal{L}_2 \). Nevertheless, a quick sight in the tables allows to classify families having the same representation for \( \rho_1 \) and \( \rho_2 \) in terms of \( A, \bar{A}, B, \bar{B} \). See tables 5 - 7. The symmetry between \( L \) and \( \bar{L} \) can also be observed in all tables.

6 Concluding remarks

Some important features deserve to be pointed out.

1. In each case both solutions \( y_i(x), \; (i = 1, 2) \) of the factorized equations are easily obtained solving first the equation

\[ \bar{L}y'_i(x) + \bar{M}y_i(x) = 0. \]

In all Lamé’s cases, these two solutions are given by

\[ y_1(x) = \sqrt{\bar{L}(x)}, \quad y_2(x) = \sqrt{\bar{L}(x)} \int \frac{dx}{\bar{L}(x)\sqrt{L(x)\bar{L}(x)}}. \] (49)

We recover, of course, for \( k = 1 \), the 8 "pseudo polynomials" of Lamé \[22\]

\[ y(x) = \sqrt{\bar{L}(x)}P_N(x), \; (\text{first type}), \] (50)

\( P_N(x) \) being the corresponding Lamé’s polynomials but now with appropriate parameters \( \alpha, \beta \) and \( \rho_i \).

2. This approach does not give a new help in the search of a peculiar solution of a generalized Heun’s equation. But it gives a global approach of all solutions for arbitrary parameters \( (\epsilon_i, \tau_i) \). This factorization defines in some way a parameter space splitted into many domains defined by "factor parameters" given in the several tables.
(3) The factorization becomes more and more difficult to handle with the increase of the number of $k + 2$ singularities. Even with $k = 3$, there are already 32 different situations to investigate, what is irrelevant to be presented as it implies very long and complicated analytical expressions for the polynomials $L, M, \bar{L}, \bar{M}$. A Maple code is now under investigation for symbolic computation for arbitrary number $k$ of singularities.

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Table 2. Factorization of the Lame’s operator $\mathcal{L}_2[y]$

$$\mathcal{L}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha \beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(\bar{L}D + \bar{M})y. \quad M = x + B, \quad \bar{M} = -x + \bar{B}.$$  

| $L$                  | $\bar{L}$               | $B$          | $\bar{B}$ | $\alpha$ | $\beta$ | $\rho_1$     | $\rho_2$     |
|----------------------|-------------------------|--------------|------------|-----------|---------|--------------|--------------|
| $x(x - 1)$           | $(x - a)(x - b)$         | $-\frac{1}{2}$ | $\frac{1}{2}(a + b)$ | $-1$      | $2$     | $\frac{1}{2}(a + b + 3)$ | $-\frac{1}{4}(a + b)$ |
| $(x - a)(x - b)$     | $x(x - 1)$              | $-\frac{1}{2}(a + b)$ | $\frac{1}{2}$ | $2$       | $-1$    | $\frac{1}{2}(3a + 3b + 1)$ | $-\frac{1}{4}(a + b + ab)$ |
| $x(x - a)$           | $(x - 1)(x - b)$         | $-\frac{1}{2}a$ | $\frac{1}{2}(b + 1)$ | $-1$      | $2$     | $\frac{1}{2}(3a + b + 1)$ | $-\frac{1}{4}(a + ab)$ |
| $(x - 1)(x - b)$     | $x(x - a)$              | $-\frac{1}{2}(b + 1)$ | $\frac{1}{2}a$ | $2$       | $-1$    | $\frac{1}{2}(a + 3b + 3)$ | $-\frac{1}{4}(a + 4b + ab)$ |
| $x(x - b)$           | $(x - 1)(x - a)$         | $-\frac{1}{2}b$ | $\frac{1}{2}(a + 1)$ | $-1$      | $2$     | $\frac{1}{2}(a + 3b + 1)$ | $-\frac{1}{4}(b + ab)$ |
| $(x - 1)(x - a)$     | $x(x - b)$              | $-\frac{1}{2}(a + 1)$ | $\frac{1}{2}b$ | $2$       | $-1$    | $\frac{1}{2}(3a + b + 3)$ | $-\frac{1}{4}(4a + b + ab)$ |
Table 3. Factorization of the Lame’s operator $\mathcal{L}_2[y]$ (continuation).

\[ \mathcal{L}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha\beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(\bar{L}D + \bar{M})y. \]

\[ M = \frac{3}{2}x^2 + Bx + C, \quad \bar{M} = -\frac{1}{2}, \quad \alpha = \frac{3}{2}, \quad \beta = -\frac{1}{2}. \]

| $L$                     | $\bar{L}$ | $B$       | $C$       | $\rho_1$   | $\rho_2$   |
|-------------------------|-----------|-----------|-----------|------------|------------|
| $(x - 1)(x - a)(x - b)$ | $x$       | $-(a + b + 1)$ | $\frac{1}{2}(a + b + ab)$ | $\frac{1}{2}(a + b + 1)$ | $-\frac{1}{2}(a + b + ab)$ |
| $x(x - a)(x - b)$      | $x - 1$   | $-(a + b)$   | $\frac{1}{2}ab$     | $\frac{1}{2}(a + b)$     | $-\frac{1}{2}ab$     |
| $x(x - 1)(x - b)$      | $x - a$   | $-(b + 1)$   | $\frac{1}{2}b$      | $\frac{1}{2}(b + 1)$     | $-\frac{1}{2}b$      |
| $x(x - a)(x - 1)$      | $x - b$   | $-(a + 1)$   | $\frac{1}{2}a$      | $\frac{1}{2}(a + 1)$     | $-\frac{1}{2}a$      |
Table 4. Factorization of the Lame’s operator $L_2[y]$ (end).

\[
L_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha \beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(\bar{L}D + \bar{M})y.
\]
\[
\bar{M} = -\frac{3}{2} x^2 + \bar{B}x + \bar{C}, \quad M = \frac{1}{2}, \quad \alpha = -\frac{3}{2}, \quad \beta = \frac{5}{2}.
\]

| L        | \(\bar{L}\)                  | \(\bar{B}\) | \(\bar{C}\) | \(\rho_1\)     | \(\rho_2\)     |
|----------|-------------------------------|--------------|--------------|-----------------|-----------------|
| \(x\)   | \((x - 1)(x - a)(x - b)\)    | \(a + b + 1\)| \(-\frac{1}{2}(a + b + ab)\) | \(\frac{3}{2}(a + b + 1)\) | \(-\frac{1}{4}(a + b + ab)\) |
| \(x - 1\)| \(x(x - a)(x - b)\)           | \(a + b\)   | \(-\frac{1}{2}ab\)            | \(\frac{3}{2}(2 + a + b)\) | \(-\frac{1}{4}(4a + 4b + ab)\) |
| \(x - a\)| \(x(x - 1)(x - b)\)           | \(b + 1\)   | \(-\frac{1}{2}b\)             | \(\frac{3}{2}(2a + b + 1)\) | \(-\frac{1}{4}(4a + b + 4ab)\) |
| \(x - b\)| \(x(x - a)(x - 1)\)           | \(a + 1\)   | \(-\frac{1}{2}a\)             | \(\frac{3}{2}(a + 2b + 1)\) | \(-\frac{1}{4}(a + 4b + 4ab)\) |
Table 5. Factorization of the Heun’s operator $H_2[y]$

$$H_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha \beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(\bar{L}D + \bar{M})y. \quad M = Ax + B, \quad \bar{M} = \bar{A}x + \bar{B}. \quad (46)$$

| $L$          | $\bar{L}$          | $A$    | $\bar{A}$ | $B$    | $\bar{B}$ | $\alpha$ | $\beta$ |
|--------------|---------------------|--------|------------|--------|-----------|----------|---------|
| $x(x - 1)$   | $(x - a)(x - b)$    | $\delta + \gamma$ | $-2 + \epsilon_1 + \epsilon_2$ | $-\gamma$ | $a - \epsilon_2 a + b - \epsilon_1 b$ | $-2 + \epsilon_1 + \epsilon_2$ | $\gamma + \delta + 1$ |
| $(x - a)(x - b)$ | $x(x - 1)$ | $\epsilon_1 + \epsilon_2$ | $-2 + \gamma + \delta$ | $-\epsilon_2 a - \epsilon_1 b$ | $1 - \gamma$ | $\epsilon_1 + \epsilon_2 + 1$ | $-2 + \gamma + \delta$ |
| $x(x - a)$   | $(x - 1)(x - b)$   | $\epsilon_1 + \gamma$ | $\delta - 2 + \epsilon_2$ | $-\gamma a$ | $1 - \epsilon_2 - \delta b + b$ | $\delta - 2 + \epsilon_2$ | $\gamma + \epsilon_1 + 1$ |
| $(x - 1)(x - b)$ | $x(x - a)$ | $\delta + \epsilon_2$ | $-2 + \epsilon_1 + \gamma$ | $-\delta b - \epsilon_2$ | $a - \gamma a$ | $1 + \delta + \epsilon_2$ | $-2 + \epsilon_1 + \gamma$ |
| $x(x - b)$   | $(x - 1)(x - a)$   | $\gamma + \epsilon_2$ | $\delta - 2 + \epsilon_1$ | $-\gamma b$ | $1 - \epsilon_1 - \delta a + a$ | $\delta - 2 + \epsilon_1$ | $1 + \gamma + \epsilon_2$ |
| $(x - 1)(x - a)$ | $x(x - b)$ | $\epsilon_1 + \delta$ | $\epsilon_2 - 2 + \gamma$ | $-\delta a - \epsilon_1$ | $b - \gamma b$ | $1 + \epsilon_1 + \delta$ | $\epsilon_2 - 2 + \gamma$ |

| $L$          | $\bar{L}$          | $\rho_1$ | $\rho_2$ |
|--------------|---------------------|----------|----------|
| $x(x - 1)$   | $(x - a)(x - b)$    | $-\bar{A} + \bar{B} \bar{A} + \bar{A} \bar{B}$ | $\bar{B} \bar{B}$ |
| $(x - a)(x - b)$ | $x(x - 1)$ | $-(a + b)\bar{A} + \bar{B} \bar{A} + \bar{A} \bar{B}$ | $ab \bar{A} + B \bar{B}$ |
| $x(x - a)$   | $(x - 1)(x - b)$   | $-a \bar{A} + \bar{B} \bar{A} + \bar{A} \bar{B}$ | $B \bar{B}$ |
| $(x - 1)(x - b)$ | $x(x - a)$ | $-(1 + b)\bar{A} + \bar{B} \bar{A} + \bar{A} \bar{B}$ | $b \bar{A} + B \bar{B}$ |
| $x(x - b)$   | $(x - 1)(x - a)$   | $-b \bar{A} + \bar{B} \bar{A} + \bar{A} \bar{B}$ | $B \bar{B}$ |
| $(x - 1)(x - a)$ | $x(x - b)$ | $-(1 + a)\bar{A} + \bar{B} \bar{A} + \bar{A} \bar{B}$ | $a \bar{A} + B \bar{B}$ |
Table 6. Factorization of the Heun’s operator $\mathcal{H}_2[y]$ (continuation).

\[
\mathcal{H}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha \beta x^2 + \rho_1 x + \rho_2)y = 0 = (LD + M)(LD + \bar{M})y. \quad M = Ax^2 + Bx + C, \quad \bar{M} = \bar{A}.
\]

| $L$ | $\bar{L}$ | $A$ | $B$ | $C$ | $\bar{A}$ | $\alpha$ | $\beta$ | $\rho_1$ | $\rho_2$ |
|-----|------|-----|-----|-----|--------|----------|--------|-------|-------|
| $(x - 1)(x - a)(x - b)$ | $x$ | $\epsilon_1 + \delta + \epsilon_2$ | $-\delta a - \delta b - \epsilon_2 - \epsilon_2 a - \epsilon_1 - \epsilon_1 b$ | $\epsilon_2 a + \epsilon_1 b + \delta a b$ | $-1 + \gamma$ | $\epsilon_1 + \delta + \epsilon_2$ | $-1 + \gamma$ | $BA$ | $CA$ |
| $x(x - a)(x - b)$ | $x - 1$ | $\epsilon_1 + \gamma + \epsilon_2$ | $-\epsilon_2 a - \gamma a - \epsilon_1 b - \gamma b$ | $\gamma a b$ | $-1 + \delta$ | $\epsilon_1 + \gamma + \epsilon_2$ | $-1 + \delta$ | $BA$ | $CA$ |
| $x(x - 1)(x - b)$ | $x - a$ | $\delta + \gamma + \epsilon_2$ | $-\epsilon_2 - \gamma a - \gamma b - \delta b$ | $\gamma b$ | $-1 + \epsilon_1$ | $\delta + \gamma + \epsilon_2$ | $-1 + \epsilon_1$ | $BA$ | $CA$ |
| $x(x - a)(x - 1)$ | $x - b$ | $\epsilon_1 + \delta + \gamma$ | $-\gamma a - \delta a - \gamma - \epsilon_1$ | $\gamma a$ | $-1 + \epsilon_2$ | $\epsilon_1 + \delta + \gamma$ | $-1 + \epsilon_2$ | $BA$ | $CA$ |
Table 7. Factorization of the Heun’s operator $\mathcal{H}_2[y]$ (end).

$$\mathcal{H}_2[y] = Q_4(x)y'' + Q_3(x)y' + (\alpha \beta x^2 + \rho_1 x + \rho_2) y = 0 = (L \bar{D} + M)(\bar{L}D + \bar{M})y. \quad \bar{M} = \bar{A}x^2 + \bar{B}x + \bar{C}, \quad M = A. \quad (48)$$

| $L$  | $\bar{L}$ | $A$  | $\bar{A}$ | $\bar{B}$ | $\bar{C}$  |
|------|-----------|------|-----------|-----------|------------|
| $x$  | $(x-1)(x-a)(x-b)$ | $\gamma$ | $-3 + \epsilon_1 + \delta + \epsilon_2$ | $2a + 2b + 2 - \delta a - \delta b - \epsilon_2 - \epsilon_2 a - \epsilon_1 - \epsilon_1 b$ | $-ab - a - b + \epsilon_2 a + \epsilon_1 b + \delta ab$ |
| $x - 1$ | $x(x-a)(x-b)$ | $\delta$ | $\gamma - 3 + \epsilon_1 + \epsilon_2$ | $2b - \epsilon_1 b - \gamma b + 2a - \gamma a - \epsilon_2 a$ | $-ab + \gamma ab$ |
| $x - a$ | $x(x-1)(x-b)$ | $\epsilon_1$ | $-3 + \delta + \gamma + \epsilon_2$ | $2b - \delta b - \gamma b + 2 - \gamma - \epsilon_2$ | $-b + \gamma b$ |
| $x - b$ | $x(x-a)(x-1)$ | $\epsilon_2$ | $-3 + \epsilon_1 + \gamma + \delta$ | $2 - \epsilon_1 - \gamma + 2a - \gamma a - \delta a$ | $-a + \gamma a$ |

| $L$  | $\bar{L}$ | $\alpha$  | $\beta$ | $\rho_1$ | $\rho_2$  |
|------|-----------|-----------|---------|---------|-----------|
| $x$  | $(x-1)(x-a)(x-b)$ | $-3 + \epsilon_1 + \delta + \epsilon_2$ | $2 + \gamma$ | $\bar{B} + A\bar{B}$ | $A\bar{C}$ |
| $x - 1$ | $x(x-a)(x-b)$ | $\gamma - 3 + \epsilon_1 + \epsilon_2$ | $2 + \delta$ | $-2\bar{A} + \bar{B} + AB$ | $-\bar{B} + A\bar{C}$ |
| $x - a$ | $x(x-1)(x-b)$ | $-3 + \delta + \gamma + \epsilon_2$ | $2 + \epsilon_1$ | $-2a\bar{A} + \bar{B} + A\bar{B}$ | $-a\bar{B} + A\bar{C}$ |
| $x - b$ | $x(x-a)(x-1)$ | $-3 + \epsilon_1 + \gamma + \delta$ | $2 + \epsilon_2$ | $-2b\bar{A} + \bar{B} + A\bar{B}$ | $-b\bar{B} + A\bar{C}$ |