1. Introduction

In this paper, we study the (small) quantum cohomology ring of the partial flag manifold. We give proofs of the presentation of the ring and of the quantum Giambelli formula for Schubert varieties. These are known results, but our proofs are more natural and direct than the previous ones.

One of our goals is to give evidence of a relationship between universal Schubert polynomials, which give the answer to a degeneracy locus problem, and quantum Schubert polynomials, which appear in quantum cohomology. It has been known that the universal Schubert polynomials specialize to both the ordinary and the quantum Schubert polynomials, but previous reasons for this have been purely algebraic [Fu].

The quantum cohomology ring of a projective manifold $X$ is a deformation of the ordinary cohomology ring of $X$. The classical Schubert calculus, consisting of Giambelli and Pieri-type formulas which give the multiplicative structure of the cohomology ring of the flag manifold, has been used as a tool to solve enumerative problems. Similarly, the entries in the quantum multiplication table count rational curves on a flag manifold of a given multidegree which meet three general Schubert varieties. These numbers can be interpreted as intersection numbers on appropriate moduli spaces of holomorphic maps from the projective line $\mathbb{P}^1$ to the flag manifold.

In order to understand these intersections, various compactifications of the moduli space of maps have been studied, for example the stable maps of Kontsevich. However, in the case of partial flag manifolds, including Grassmannians and complete flag manifolds, there are smooth compactifications called hyperquot schemes, which generalize Grothendieck’s Quot scheme [G]. They have been studied by Ciocan-Fontanine [C-F1] [C-F2], Laumon [Lau], Kim [K], and in [C]. Most of what is known about the quantum cohomology of flag manifolds rely heavily on computations in the cohomology of hyperquot schemes. We obtain our results through a further study of the intersection theory of hyperquot schemes.

For the sake of notation, we first state and prove our results for the case of the complete flag manifold. Most of the statements hold verbatim for the general case of partial flag manifolds, and many of the proofs need only slight modifications. We give ingredients to extend the arguments in the final section of this paper. The exceptions to this are found in sections 9 and 10, whose constructions and results apply only to complete flag manifolds, and whose methods do not generalize. In section 11, we give an alternate approach completely bypasses this special argument. We include sections 9 and 10 because they may be of outside interest as we introduce and study a new set of degeneracy loci on the hyperquot scheme.
Two components of a classical Schubert calculus are a presentation of the cohomology ring and a Giambelli formula, which writes Schubert classes in terms of the generators of the ring. The classical cohomology ring of the flag manifold $F$ has a presentation $\mathbb{Z}[\sigma_i^j]/I$, where the $\sigma_i^j$'s are determined by Chern classes of certain tautological vector bundles on $F$, and the relations are given by the ideal $I$. By the general result of Siebert and Tian in [ST], the quantum cohomology ring of $F$ has presentation $\mathbb{Z}[\sigma_i^j, q_1, \ldots, q_m]/I_q$, where the variables $q_i$ are deformation parameters and $I_q$ is a deformation of the ideal $I$.

The classical Giambelli formula gives Schubert classes as polynomials in the variables $\sigma_i^j$ called Schubert polynomials. Special Schubert classes are those corresponding to Chern classes of the tautological vector bundles. The quantum Giambelli formula is a deformation of the classical formula, giving the Schubert classes as polynomials in the variables $\sigma_i^j$ and $q_k$.

For complete flag manifolds, generators of the ideal $I_q$ were conjectured by Givental and Kim in [GK]. This and the special case of the quantum Giambelli formula were simultaneously proved by Ciocan-Fontanine in [C-F1]. Using these results and combinatorial methods, Fomin, Gelfand, and Postnikov constructed quantum Schubert polynomials, and proved the general case of the quantum Giambelli formula [FGP]. An independent proof of the presentation of the ring was given by Kim in [K]. The methods of [C-F1] and [FGP] were adapted to prove the results for partial flag manifolds [C-F2].

We provide a simplified argument for both of these results, which simultaneously proves the presentation and the quantum Giambelli formula. One of the key points in the proof is to use a certain degeneracy locus formula for flags of bundles, stated in terms of universal Schubert polynomials, and proved by Fulton [Fu]. Our methods of proof are similar to those used by Bertram and Ciocan-Fontanine, including a liberal use of certain maps constructed by Ciocan-Fontanine to understand the boundary of the hyperquot scheme [B][C-F1][C-F2].

In sections 4 and 5, we review the main constructions of [C-F1] which are used in the paper. In section 6, we review the degeneracy locus formula of Fulton as it applies to our situation. In section 7, we prove the results of the paper via a Main Proposition, which is proved in section 8, modulo a lemma stated as Proposition 6. Its general statement for partial flag manifolds is stated and proved in section 11. In the complete flag manifold case, an alternate proof of the lemma relies on the new ideas and constructions found in sections 9 and 10.

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2. Classical cohomology of the flag manifold

Let $F(n)$ denote the complete flag manifold of $\mathbb{C}^n$, which parametrizes flags of subspaces $V_i$:

$$\{0\} = V_0 \subset V_1 \subset V_2 \subset \ldots \subset V_{n-1} \subset V = \mathbb{C}^n$$

with $\text{dim } V_i = i$. Write $V_X = V \otimes \mathcal{O}_X$ for any scheme $X$.

There is a universal sequence of vector bundles on $F(n)$:

$$V_{F(n)} \twoheadrightarrow Q_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow Q_1$$

with rank $Q_i = i$, and each $Q_i \rightarrow Q_{i-1}$ a surjection. Any flag of successive vector bundle quotients of $V_X$:

$$V_X \twoheadrightarrow F_{n-1} \twoheadrightarrow \cdots \twoheadrightarrow F_1$$

with rank $F_i = i$ gives a map $f : X \rightarrow F(n)$ with $F_i = f^*Q_i$ where the $Q_i$’s are the tautological quotient bundles on $F(n)$.

Fix a flag $V_\bullet : V_1 \subset \cdots \subset V_{n-1} \subset V_n$. For any $w$ in the symmetric group $S_n$, define $r_w(q,p)$ to be the number of $i \leq q$ such that $w(i) \leq p$. The corresponding Schubert variety is given by:

$$\Omega_w(V_\bullet) = \{ U_\bullet \in F(n) : \text{rank} U_p(V_q \otimes \mathcal{O}_{F(n)} \rightarrow Q_q) \leq r_w(q,p) \text{ for } 1 \leq p,q \leq n \}.$$  

This is a codimension $l(w)$ subvariety in $F(n)$, where $l(w)$ is the length of the permutation $w$. Its class is independent of choice of flag $V_\bullet$, and we denote this class by $[\Omega_w]$.

Let $w_0 \in S_n$ be the permutation of longest length, with $w_0(i) = n-i+1$ for $1 \leq i \leq n$. For $w \in S_n$, write $w^\vee = w_0w$. We have the following classical results.

**Theorem 1.** The classes $[\Omega_w]$ form an additive basis for $H^*(F)$. Furthermore, for $w \in S_n$, the Schubert classes $[\Omega_w]$ and $[\Omega_{w^\vee}]$ are Poincaré dual.

**Theorem 2.** Let $x_i = c_1(Q_i \rightarrow Q_{i-1})$ for $1 \leq i \leq n$. Then

$$H^*(F(n),\mathbb{Z}) \cong \mathbb{Z}[x_1,\ldots,x_n]/(e_1(n),\ldots,e_n(n))$$

where $e_i(n)$ is the $i$th symmetric polynomial in $x_1,\ldots,x_n$.

The Giambelli problem is to express $[\Omega_w]$ in terms of this presentation. To do this, we give the definition of Schubert polynomials as given by Lascoux and Schutzenberger [L]. For $1 \leq i \leq n-1$, let $\partial_i$ act on $\mathbb{Z}[x_1,\ldots,x_n]$ by

$$\partial_i P = \frac{P(x_1,\ldots,x_n) - P(x_1,\ldots,x_{i-1},x_{i+1},x_i,x_{i+2},\ldots,x_n)}{x_i - x_{i+1}}.$$  

Let $s_i$ be the transposition $(i,i+1)$. For $w \in S_n$, write $w = w_0 \circ s_{i_1} \circ \cdots \circ s_{i_k}$, where $k = \binom{n}{2} - l(w)$. Then the Schubert polynomial associated to $w$ is defined by

$$\mathcal{G}_w(x) = \partial_{i_k} \circ \cdots \circ \partial_{i_1}(x_1^{n-1}x_2^{n-2}\ldots x_{n-1}).$$

The solution to the Giambelli problem was given by Bernstein, Gelfand, and Gelfand [BGG] and Demazure [D], cf. [Mac].

**Theorem 3.** $[\Omega_w] = \mathcal{G}_w(x)$ in $H^*(F(n),\mathbb{Z})$.  


3. Quantum Multiplication Map

Additively, we can view the small quantum cohomology ring as
\[ QH^*(F(n)) = H^*(F(n)) \otimes \mathbb{Z}[q_1, \ldots, q_{n-1}] \].

Then as \( \mathbb{Z}[q] \)-modules, the ordinary cohomology has a canonical injection into the quantum ring given by \([\Omega_w] \to [\Omega_w] \otimes 1\). Indeed, \( QH^*(F(n)) \) is a deformation of \( H^*(F(n)) \) so that the ordinary ring is recovered by setting \( q_i = 0 \).

For \( w \in S_n \), let \( \sigma_w := [\Omega_w] \otimes 1 \) denote the Schubert class in the small quantum cohomology ring, so that substituting \( q_i = 0 \) into \( \sigma_w \) gives \( [\Omega_w] = \mathcal{S}_w(x) \), the Schubert class defined in section 2. (This notation differs from that used by Bertram and Ciocan-Fontanine \cite{B} \cite{C-F1} \cite{C-F2}. We use the variable \( \mu \) instead of \( \sigma \) to distinguish between the class \( \sigma_w \) in quantum cohomology of the flag manifold and the class \( \mu_w \) in the cohomology of the hyperquot scheme, which is defined in section 5.)

Let \( s_i \in S_n \) be the transposition \((i, i + 1)\). We say that a map \( f : P^1 \to F(n) \) has multidegree \( d = (d_1, \ldots, d_{n-1}) \) when \( f^*[\Omega_w] = \sum d_i [\Omega_{w_s} \sigma_i] \), with each \( d_i \) a nonnegative integer. Recall that \( [\Omega_{w_s} \sigma_i] \) is dual to \( [\Omega_{s_i}] \).

The Gromov-Witten number
\[ \langle \Omega_{w_1}, \ldots, \Omega_{w_N} \rangle_d \in \mathbb{Z} \]

is defined as follows. For any \( t_1, \ldots, t_N \in P^1 \) in general position, and \( \Omega_{w_1}, \ldots, \Omega_{w_N} \subset F(n) \) general translates (obtained by choosing general flags), it is the number of holomorphic maps \( f : P^1 \to F(n) \) of multidegree \( d \) satisfying \( f(t_i) \in \Omega_{w_i} \) for \( 1 \leq i \leq N \) if this number is finite, and zero otherwise.

We write \( q^d = q_1^{d_1} q_2^{d_2} \cdots q_{n-1}^{d_{n-1}} \). Define the quantum multiplication map:
\[ \sigma_{w_1} \ast \sigma_{w_2} = \sum_{d, w} q^d \langle \Omega_{w_1}, \Omega_{w_2}, \Omega_w \rangle_d \sigma_{w'} \]  

This product gives \( QH^*(F(n)) \) the structure of a commutative, associative \( \mathbb{Z}[q] \)-algebra. This follows from general associativity results on the big quantum ring which can be specialized to the small quantum ring, see \cite{FP}.

With this defined multiplication, we have:

Proposition 1.
\[ \sigma_{w_1} \ast \cdots \ast \sigma_{w_N} = \sum_{d, w} q^d \langle \Omega_{w_1}, \ldots, \Omega_{w_N}, \Omega_w \rangle_d \sigma_{w'} \]  

This was first proved for Grassmannians \cite{B} by using the Quot scheme. It has been proved for Grassmannians \cite{FP} and for partial flag manifolds \cite{C-F2} by realizing the Gromov-Witten numbers defined in \cite{B} as intersection numbers on \( M_{0,N+1}(F(n), d) \). The validity of this statement of the quantum product is also proved via the hyperquot scheme and results of section 3 in \cite{C-F1}, using the methods of Bertram.

4. The Hyperquot Scheme

In this section, we summarize the constructions and results that we need for our proofs. In particular, we describe the hyperquot scheme, some degeneracy loci, and a
description of the boundary. The proofs of the results of section 4.4 are found in detail in [C-F1].

4.1. **Properties of** $\mathcal{H}Q_d$. Consider the following functor $\mathcal{F}_d$ from the category of schemes to the category of sets. For a scheme $T$, $\mathcal{F}_d(T)$ is defined to be the set of equivalence classes of flagged quotient sheaves

$$V_{P^{1} \times T}^{s} \rightarrow Q_{n-1} \rightarrow \cdots \rightarrow Q_{1}$$

with each $Q_i$ flat over $T$ with Hilbert polynomial $\chi(P^{1} \times (Q_i)_t(m)) = (m + 1)i + d_{n-i}$ on the fibers of $\pi_T : P^{1} \times T \rightarrow T$, so that $Q_i$ is of rank $i$ and relative degree $d_{n-i}$ over $T$, i.e. that $(Q_i)_t$ is of degree $d_{n-i}$ for every $t \in T$. Two such flags $V_{P^{1} \times T}^{s} \rightarrow Q_{n-1}^{1} \rightarrow \cdots \rightarrow Q_{1}^{1}$ and $V_{P^{1} \times T}^{s} \rightarrow Q_{n-1}^{2} \rightarrow \cdots \rightarrow Q_{1}^{2}$ are in the same equivalence class when there exist maps $Q_{i}^{1} \rightarrow Q_{i}^{2}$ so that all squares commute.

The functor $\mathcal{F}_d$ is represented by the projective scheme $\mathcal{H}Q_d = \mathcal{H}Q_d(F(n))$ [C-F1]. Its construction as the fine moduli space of flat families of flagged quotient sheaves over $P^{1}$ has been described by Ciocan-Fontanine following the ideas of Grothendieck and Mumford [C-F1] [G] [M]. It has also been described in a different way by Kim [K], as a closed subscheme of a product of Quot schemes, and independently by Laumon [La]. We have

**Theorem 4.** $\mathcal{H}Q_d(F(n))$ is an irreducible, rational, nonsingular, projective variety of dimension $(n) + 2 \sum d_i$.

Thus, associated to $\mathcal{H}Q_d(F(n))$ is a universal sequence of sheaves on $P^{1} \times \mathcal{H}Q_d$ of successive quotients of sheaves, each of which is flat over $\mathcal{H}Q_d$:

$$V_{P^{1} \times \mathcal{H}Q_d}^{s} \rightarrow B_{n-1} \rightarrow \cdots \rightarrow B_{1}.$$  

In general, the sheaf $B_{i}$ is not locally free. Consider the sheaves $A_{i} := \ker(V_{P^{1} \times \mathcal{H}Q_d}^{s} \rightarrow B_{n-i})$. Each $A_{i}$ is flat over $\mathcal{H}Q_d$, and it is an easy consequence of flatness and the fact that $P^{1}$ is a nonsingular curve that each $A_{i}$ is locally free. Thus, we have the following universal sequence on $P^{1} \times \mathcal{H}Q_d$:

$$A_{i} \hookrightarrow A_{2} \hookrightarrow \cdots \hookrightarrow A_{n-1} \hookrightarrow V_{P^{1} \times \mathcal{H}Q_d}^{s} \rightarrow B_{n-1} \rightarrow \cdots \rightarrow B_{1}.$$  

with $A_{i}$ and $B_{i}$ of rank $i$. Denote the inclusion maps by $\gamma_{i} : A_{i} \hookrightarrow A_{i+1}$ and the surjections by $\pi_{i} : B_{i+1} \to B_{i}$ for each $1 \leq i \leq n - 1$. We set $A_{n} = B_{n} = V_{P^{1} \times \mathcal{H}Q_d}^{s}$ and $A_{0} = B_{0} = 0$. The map $\gamma_{i} : A_{i} \hookrightarrow A_{i+1}$ is an inclusion of sheaves, not an inclusion of bundles.

Consider $\text{Mor}_d(P^{1}, F(n))$, the space of morphisms from $P^{1}$ to $F(n)$ of multidegree $d = (d_{1}, ..., d_{n-1})$. By the universal property of $F(n)$, a morphism $f \in \text{Mor}_d(P^{1}, F(n))$ corresponds to successive quotient bundles

$$V_{P^{1}} \rightarrow Q_{n-1} \rightarrow \cdots \rightarrow Q_{1}$$

with $Q_{i}$ of rank $i$ and degree $d_{n-i}$. Equivalently, by taking kernels, $f$ corresponds to successive subbundles $S_{1} \hookrightarrow \cdots \hookrightarrow S_{n-1} \hookrightarrow V_{P^{1}}$. Dualizing, we see that $\text{Mor}_d(P^{1}, F(n))$ parametrizes successive quotient bundles of $V_{P^{1}}$ of rank $i$ and degree $d_{n-i}$. In this way, the hyperquot scheme $\mathcal{H}Q_d$ is a compactification of $\text{Mor}_d$. 
We define the hyperquot scheme in terms of the dual trivial vector bundle \( V_{P^1}^* \) instead of \( V_{P^1} \) to ensure that the universal subsheaves \( A_i \hookrightarrow V_{P^1}^* \) are locally free. This allows us to apply the degeneracy locus formula of section 6.3 more readily.

4.2. Description and construction of \( \mathcal{U}_e \). In the next two sections, we review the construction of certain schemes which map to various hyperquot schemes \([C-F1]\). These morphisms are used to understand a recursive structure of the boundary of \( \mathcal{H}Q_d \), which is used to understand intersections of degeneracy loci on the hyperquot scheme, to be defined in sections 5 and 9.

Let \( e = (e_1, \ldots, e_{n-1}) \) be a sequence of nonnegative integers satisfying:

1. \( e_i \leq \min(i, d_i) \) for \( 1 \leq i \leq n-1 \),
2. \( e_i - e_{i-1} \leq 1 \) for \( 2 \leq i \leq n-1 \).
3. \( \sum e_i \geq 1 \)

We prove a lemma to be used in section 8.

**Lemma 1.** For \( e \) satisfying (1), (2), and (3), and setting \( e_0 = 0 \),

1. \( \sum e_i \leq \sum e_i (1 + e_i - e_{i-1}) \)
2. \( \sum e_i (1 + e_i - e_{i-1}) \geq 2 \) with equality if and only if \( \sum e_i = 1 \).

**Proof.** This follows from the observation that

\[
\sum e_i (1 + e_i - e_{i-1}) = \sum e_i + \frac{1}{2} [e_1^2 + (e_2 - e_1)^2 + \cdots + (e_{n-1} - e_{n-2})^2 + e_{n-1}^2].
\]

For each such multiindex \( e \), we consider the scheme \( \mathcal{U}_e \) as in \([C-F1]\) as follows. On \( P^1 \times \mathcal{H}Q_{d-e} \), there is the universal sequence

\[
A_{1}^{d-e} \hookrightarrow \cdots \hookrightarrow A_{n-1}^{d-e} \hookrightarrow V_{P^1 \times \mathcal{H}Q_{d-e}}^*.
\]

For \( 1 \leq i \leq n-1 \), let \( X_i \) be the Grassmann bundle of \( e_i \)-dimensional quotients of \( A_i^{d-e} \), and let \( X_e \) be the fiber product of these \( X_i \)'s, with projection map \( \pi : X_e \rightarrow P^1 \times \mathcal{H}Q_{d-e} \). Let \( K_i \) denote the tautological subbundle of the pullback of \( A_i^{d-e} \) over \( X_i \), and \( Q_i \) the corresponding quotient bundle, so that for \( 1 \leq i \leq n-1 \) there is the exact sequence

\[
0 \rightarrow K_i \rightarrow A_i^{d-e} \rightarrow Q_i
\]

with \( K_i \) of rank \( i - e_i \) and \( Q_i \) of rank \( e_i \). Let \( K_i \) and \( Q_i \) also denote the pullbacks of these bundles to \( X_e \) via the natural projections.

Define \( \mathcal{U}_e \) to be the locally closed subscheme of \( X_e \) given by the conditions

1. The composite map \( K_i \rightarrow A_i \rightarrow A_{i+1} \rightarrow Q_{i+1} \) vanishes for \( 1 \leq i \leq n-2 \),
2. \( \text{rank} \ (K_i \rightarrow V_{X_e}^*) = i - e_i \) for \( 1 \leq i \leq n-1 \).

We review an explicit construction of \( \mathcal{U}_e \), as well as the construction of the morphism \( h_e : \mathcal{U}_e \rightarrow \mathcal{H}Q_d \), which we use in later sections. Let \( U \subset X_e \) be the open subscheme given conditions (2). We construct \( \mathcal{U}_e \) inductively as a sequence of Grassmann bundles.

1. Let \( \rho_1 : \mathcal{U}_1 := G^{e_1} (\pi^* A_1^{d-e}) \rightarrow U \) be the Grassmann bundle of \( e_1 \)-dimensional quotients, with universal subbundle \( K_1 \).
2. Consider the Grassmann bundle $\rho_i : U_i := G^e_i(\rho^+\pi^+ A_i^{d-e}/K_{i-1}) \to U_{i-1}$, with universal subbundle $S_i$, where $\rho$ denotes the composition $\rho_{i-1} \circ \cdots \circ \rho_1$. Let $K_i$ be the natural extension of $\rho_i^* K_{i-1}$ by $S_i$.

Then $U_e = U_{n-1}$.

4.3. **Construction of** $\Delta$. Let $\pi : U_e \to \mathbb{P}^1 \times \mathcal{H}Q_{d-e}$ be the projection map. Consider the map

$$\psi := 1 \times \pi : \mathbb{P}^1 \times U_e \to \mathbb{P}^1 \times \mathcal{H}Q_{d-e}.$$ 

Let $\Delta \subset \mathbb{P}^1 \times \mathbb{P}^1$ be the diagonal, and $\bar{\Delta} \subset \mathbb{P}^1 \times U_e$ be $\psi^{-1}(\Delta \times \mathcal{H}Q_{d-e})$. Denote by $p : \mathbb{P}^1 \times U_e \to U_e$ the second projection. For $1 \leq i \leq n-1$, define $\bar{A}_i^e$ to be the kernel of the map

$$p^* \pi^* A_i^{d-e} \to p^* Q_i|\bar{\Delta}.$$ 

This gives a sequence of sheaf injections

$$\bar{A}_1^e \hookrightarrow \cdots \hookrightarrow \bar{A}_{n-1}^e$$

with each $\bar{A}_i^e$ flat over $U_e$, locally free of rank $i$, and of relative degree $-d_i$ on $\mathbb{P}^1 \times U_e$.

Since $\mathcal{H}Q_d$ represents the hyperquot functor, this defines a morphism $h_e : U_e \to \mathcal{H}Q_d$, which satisfies $(1 \times h_e)^* A_i^d = \bar{A}_i^e$ for every $1 \leq i \leq n-1$.

4.4. **The boundary.**

**Theorem 5.** [C-F] Let the multiindex $e = (e_1, \ldots, e_{n-1})$, $U_e$, and $h_e$ be as above. Then

1. $U_e$ is smooth, irreducible, and of dimension

$$\binom{n}{2} + 2|d| + 1 - \sum_{i=1}^{n-1} e_i(1 + e_i - e_{i-1}).$$

The projection map $\pi : U_e \to \mathbb{P}^1 \times \mathcal{H}Q_{d-e}$ is smooth and proper, with irreducible fibers.

2. If $\text{rank}_{(t,x)} B_i^d = n - i + e_i$ for $1 \leq i \leq n-1$ at $(t, x) \in \mathbb{P}^1 \times \mathcal{H}Q_d$, then $x \in h_e(U_e)$.

Part 2 of the theorem implies that the boundary of the hyperquot scheme $\mathcal{H}Q_d \setminus \text{Mor}_d$ is covered by the images of the $U_e$ under the morphisms $h_e$.

For any $t \in \mathbb{P}^1$ and any multiindex $e$ as above, define $U_e(t) = \pi^{-1}(t \times \mathcal{H}Q_{d-e})$.

5. **Schubert varieties on the hyperquot scheme**

We describe subschemes of the hyperquot scheme which are determined by the degeneracy conditions which describe Schubert varieties, whose associated intersection numbers are equal to certain Gromov-Witten invariants.

Consider the evaluation map

$$ev : \mathbb{P}^1 \times \text{Mor}_d \to \text{F}(n)$$

given by $ev(t, f) = f(t)$ and use this to pull back Schubert varieties of $\text{F}(n)$ to $\text{Mor}_d$ in the following manner. Define for any $w \in S_n$,

$$\Omega_w(t) = ev^{-1}(\Omega_w) \cap (\{t\} \times \text{Mor}_d).$$
We wish to extend $\Omega_w(t)$ over the boundary to the entire hyperquot scheme. This can be achieved as a certain degeneracy locus. We fix a flag $V$. On $\mathbf{P}^1 \times \mathcal{H}_d(F(n))$, we have the dualized universal sequence of subsheaves
\[ V_{\mathbf{P}^1 \times \mathcal{H}_d} \rightarrow A_{n-1}^* \rightarrow \cdots \rightarrow A_1^*. \]

Define $\overline{\Omega}_w$ to be the locus where
\[ \text{rank } (V_p \otimes \mathcal{O}_{\mathbf{P}^1 \times \mathcal{H}_d} \rightarrow A_q^*) \leq r_w(q,p), 1 \leq p, q \leq n, \]
with the natural scheme structure given by vanishing of determinants, and define $\overline{\Omega}_w(t)$ to be its restriction to $\{t\} \times \mathcal{H}_d$, viewed as a subscheme of $\mathcal{H}_d$ via the identification $t \times \mathcal{H}_d \cong \mathcal{H}_d$.

**Lemma 2.** We can write
\[ h^{-1}_e(\overline{\Omega}_w(t)) = \pi^{-1}(\mathbf{P}^1 \times \overline{\Omega}_w(t)) \cup \overline{\Omega}_w(t) \]
with $\overline{\Omega}_w(t)$ the degeneracy locus inside $\pi^{-1}(t \times \mathcal{H}_d)$ given by
\[ \text{rank } (V_p \rightarrow K_q^*) \leq r_w(q,p). \]
This equality is scheme-theoretic away from the intersection.

**Lemma 3.** For $w \in S_n$
\[ \text{codim}_{\mathcal{H}_d} \overline{\Omega}_w(t) = l(\overline{w}^e) \]
for a permutation $\overline{w}^e$ as described in Construction 3.5 of [C-F1], which satisfies $l(w) - l(\overline{w}^e) \leq \sum e_i$.

The argument of Ciocan-Fontanine in [C-F1], which uses the constructions in section 4 and Lemmas 2 and 3 gives the following general position result and two corollaries:

**Theorem 6.** 1. For any subvariety $Y$ in $\mathcal{H}_d$, $w \in S_n$, a general translate $\Omega_w(t) \subset \mathcal{F}(n)$, and any $t \in \mathbf{P}^1$, $Y \cap \Omega_w(t)$ is either empty or pure codimension $l(w)$ in $Y$.

2. If $t_1, \ldots, t_N$ are distinct points in $\mathbf{P}^1$, then for general translates of $\Omega_{w_i}$, the intersection $\bigcap_{i=1}^N \Omega_{w_i}(t_i)$ is either empty of pure codimension $\sum_{i=1}^N l(w_i)$ in $\mathcal{H}_d$ and is the Zariski closure of $\bigcap_{i=1}^N \Omega_{w_i}(t_i)$

**Corollary 1.** The class of $\overline{\Omega}_w(t)$ in $H^{2(l(w))}(\mathcal{H}_d)$ is independent of $t \in \mathbf{P}^1$ and flag $V$. Denote this class by $\mu_w(d)$.

The multidegree $d$ is often understood, and in these cases we write $\mu_w = \mu_w(d)$.

**Corollary 2.** If $\sum_{i=1}^N l(w_i) = \dim \mathcal{H}_d$, and $t_1, \ldots, t_N$ are distinct points in $\mathbf{P}^1$, then $\bigcap_{i=1}^N \Omega_{w_i}(t_i) = \bigcap_{i=1}^N \overline{\Omega}_{w_i}(t_i)$, and the number of points in this intersection is the degree of the product $\mu_{w_1} \cdot \mu_{w_2} \cdots \mu_{w_N}$ in $H^*(\mathcal{H}_d, \mathbb{C})$.

Recall our definition of the number $\langle \Omega_{w_1}, \ldots, \Omega_{w_N} \rangle_d$ in section 3. For $t_1, \ldots, t_N \in \mathbf{P}^1$ in general position, and $\Omega_i$ general translates, this number is equal to the number of points of $\bigcap_{i=1}^N \Omega_{w_i}(t_i)$. Corollary 2 shows that we have the following equality
\[ \langle \Omega_{w_1}, \ldots, \Omega_{w_N} \rangle_d = (\mu_{w_1} \cdot \mu_{w_2} \cdots \mu_{w_N})_d \]
for every multidegree $d$. The number on the left is the Gromov-Witten number defined in [1] and the number on the right is the degree of the intersection product $\mu_{w_1} \cdot \mu_{w_2} \cdots \mu_{w_N}$ in the cohomology of the hyperquot scheme $\mathcal{H} \mathcal{Q}_d$.

6. Universal Schubert Polynomials

Universal Schubert polynomials, introduced in [Fu], specialize to all known types of Schubert polynomials, including the classical version defined in section 2 as well as the quantum Schubert polynomials defined in [FGP]. They appear as the answer to a certain degeneracy locus problem which we use for the proofs of our results.

6.1. Definition of $\mathfrak{S}_w(c)$ and $\mathfrak{S}_w(g)$. We give two equivalent formulations of universal Schubert polynomials. For any $w \in S_{n+1}$, the corresponding classical Schubert polynomial can be written

$$\mathfrak{S}_w(x) = \sum a_{k_1, \ldots, k_n} e_{k_1}(1) \cdots e_{k_n}(n)$$

where the sum ranges over all sequences $(k_1, \ldots, k_n)$ with $k_p \leq p$ and $\sum k_i = l(w)$, and $e_k(l)$ is the $k$th elementary symmetric polynomial in $x_1, \ldots, x_l$. For each $w$, the coefficients $a_{k_1, \ldots, k_n}$ are uniquely determined integers.

First, for $1 \leq k \leq l \leq n$, consider independent variables $c_k(l)$. We set $c_0(l) = 1$ and $c_k(l) = 0$ when $k < 0$ or $k > l$. With this notation, the universal Schubert polynomial is defined by

$$\mathfrak{S}_w(c) = \sum a_{k_1, \ldots, k_n} c_{k_1}(1) \cdots c_{k_n}(n).$$

A second description uses independent variables $g_i[j]$, where $i \geq 1, j \geq 0$, and $i + j \leq n + 1$, and each $g_i[j]$ is of degree $j + 1$. Consider the Dynkin diagram for $(A_n)$, mark the vertices $x_1, \ldots, x_n$, the edges $g_1[1], \ldots, g_{n-1}[1]$, and the paths covering the $j + 1$ consecutive vertices $x_1, \ldots, x_{i+j}$ by $g_i[j]$. In this notation, we have $g_i[0] = x_i$. Denote by $E_k(l)(g)$ the sum of all monomials in the paths $g_i[j]$ covering exactly $k$ of the vertices $x_1, \ldots, x_l$, with no vertex covered more than once. If the variables $g_i[*]$ are understood, we may sometimes write $E_k(l)$ for $E_k(l)(g)$. Equivalently, $E_k(l)$ can defined inductively as

$$E_k(l) = E_k(l-1) + \sum_{j=0}^{k} E_{k-j-1}(l-j-1)g_{l-j}[j].$$

Let $G_i$ be the $l \times l$ matrix with entries $g_i[j - i]$ in the $(i, j)$ position for $1 \leq i \leq j \leq l$, $-1$ in the $(i+1, i)$ position, and 0 elsewhere. Then $G_i$ is also equivalent to $E_k(l)$ being the coefficient of $\lambda^k$ in the determinant of $G_l + \lambda I$. The universal Schubert polynomial $\mathfrak{S}_w(g)$ is obtained by performing the substitution $c_k(l) = E_k(l)$ into $\mathfrak{S}_w(c)$.

Via this substitution, the polynomial rings generated by $c_k(\ast)$ and $g_k(\ast)$ are the same, so that the two formulations are equivalent. As we have seen, each $c_k(l)$ can be written as a polynomial in the $g$’s, but each $g_i[j]$ can be written in terms of the $c_k(l)$’s as well. In particular, from (6), we have

$$c_k(l) = c_k(l-1) + \sum_{j=0}^{k-1} c_{k-j-1}(l-j-1)g_{l-j}[j] + g_{l-k}[k].$$
The $g_i[j]$ can be defined inductively via these relations.

6.2. Definition of quantum Schubert polynomials. We review the definition of quantum Schubert polynomials $\mathcal{S}_w(x, q)$ given in [FGT]. Consider the same Dynkin diagram as in section 6.1, with vertices labeled, $x_1, \ldots, x_n$. Let $q_i$ be the (degree 2) path covering $x_i$ and $x_{i+1}$. Then define $e^q_k(l)$ to be the sums of monomials in $x$ and $q$ covering exactly $k$ of the $x_1, \ldots, x_l$. For $w \in S_n$, define $\mathcal{S}_w(x, q)$ to be the polynomial resulting from the substitutions $e^q_k(l)$ for $e_k(l)$ into the decomposition of the ordinary Schubert polynomial $\mathcal{S}_w(x)$ given by [5].

It is easy to see from these descriptions that $e^q_k(l)$ and $\mathcal{S}_w(x, q)$ are the polynomials resulting from the substitution $g_i[1] = q_i$ and $g_i[j] = 0$ for $j \geq 2$ into $E_k(l)(g)$ and $\mathcal{S}_w(g)$, respectively, and that $\mathcal{S}_w(x, 0) = \mathcal{S}_w(x)$, so that the quantum Schubert polynomials specialize further to the classical Schubert polynomials.

6.3. A degeneracy locus formula. The universal Schubert polynomials are the solution to the following degeneracy locus problem.

Theorem 7. Let $X$ be a Cohen-Macaulay scheme. Consider maps of vector bundles

$$V_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_n = E_n \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_1$$

on $X$ where each $V_i$ and $E_i$ is of rank $i$, and the $V_i$ are trivial vector bundles. For each $w \in S_{n+1}$, there is a degeneracy locus

$$\Omega_w = \{ x \in X : \text{rank}_x(V_p \rightarrow E_q) \leq r_w(q, p) \text{ for } 1 \leq p, q \leq n \}$$

where $r_w(q, p)$ is the number of $i \leq q$ such that $w(i) \leq p$. Let $[\Omega_w]$ be the cohomology class associated to $\Omega_w$ with the scheme structure given locally by vanishing of determinants. Assume that $\Omega_w$ is of the expected codimension $l(w)$. Then setting $c_k(l) = c_k(E_l)$ the $k$th chern class of $E_l$, we have $[\Omega_w] = \mathcal{S}_w(c)$.

This is a consequence of the general result, Proposition 3.1 in [F], because all Chern classes of $V_i$ vanish for $1 \leq i \leq n$.

We have the following bundle maps on $\mathbb{P}^1 \times \mathcal{H}_{\mathbb{Q}_d}$:

$$V_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_n = V_n \rightarrow A^*_{n-1} \rightarrow \cdots \rightarrow A^*_1$$

given by the dual sequence to (5), where we have chosen a fixed flag $V_\bullet$, and have denoted by $V_i$ the corresponding rank $i$ vector bundle.

Let $t \in \mathbb{P}^1$. We can apply Theorem 7 to the bundle maps on $\mathcal{H}_{\mathbb{Q}_d} \simeq t \times \mathcal{H}_{\mathbb{Q}_d}$:

$$V_1 \rightarrow V_2 \rightarrow \cdots \rightarrow V_n = V \otimes O_t \times \mathcal{H}_{\mathbb{Q}_d} \rightarrow (A_{n-1})^*_t \rightarrow \cdots \rightarrow (A_1)^*_t$$

given by restricting all bundles to $t \times \mathcal{H}_{\mathbb{Q}_d}$.

For the purpose of notation, set $C_k(l)$ to be the $k$th Chern class of the bundle $(A_l)^*_t$ on $\mathcal{H}_{\mathbb{Q}_d}$:

$$C_k(l) = c_k((A_l)^*_t).$$
Define $Q_i[j]$ to be the classes in $H^*(\mathcal{H}\mathcal{Q}_d)$ defined inductively in terms of the $C_k(l)$ by the relations:

$$C_k(l) = C_k(l-1) + \sum_{j=0}^{k-1} C_{k-j-1}(l-j-1)Q_{l-j}[j] + Q_{l-k}[k].$$

These relations between the classes $C_*(\cdot)$ and $Q_*(\cdot)$ are specializations of the recursive relations described for the variables $c_*(\cdot)$ and $g_*(\cdot)$ in (6).

By the corollary to Theorem 3, we have

**Proposition 2.** Fix any point $t \in \mathbf{P}^1$. For $1 \leq j \leq n-1$, let $(A_j)_t$ be the restriction of the tautological subbundles on $\mathbf{P}^1 \times \mathcal{H}\mathcal{Q}_d$ to $t \times \mathcal{H}\mathcal{Q}_d$, with $Q_i[j]$ classes in $H^*(\mathcal{H}\mathcal{Q}_d)$ defined in terms of the Chern classes of the bundles $(A^*_t)_t$ as in (10). Then the class $\mu_w(d) = \Omega_w(t)$ in $H^*(\mathcal{H}\mathcal{Q}_d)$ is given by the formula

$$\mu_w = \mu_w(d) = \mathcal{S}_w(c_k((A^*_t)_t)) = \mathcal{S}_w(Q).$$

6.4. A geometric interpretation of the variables $g_*(\cdot)$. In this section, we abuse notation by letting $g_i[j]$ denote the specialization of the variable $g_i[j]$ to certainly cohomology classes on a scheme $X$. We consider the setting of flags of vector bundles on $X$:

$$E_n \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_1,$$

and show that, in some sense, the $g_*(\cdot)$ measure how far the maps of vector bundles $E_i \rightarrow E_k$ are from being surjective. We apply Proposition 3 of this section to the proof of Proposition 3 in section 4.

We define some polynomials which we use in this section. For $a < b$, define $E_i(a,b)(g)$ to be the sum of all monomials in paths $g_*(\cdot)$ covering exactly $i$ of the vertices $x_a, \ldots, x_{b}$, no vertex more than once. When the variables (or classes) $g$ are understood, we write $E_i(a,b)$. With this notation, $E_i(j) = E_i(1,j)$ and $E_i(a,b) = 0$ for $i > b - a + 1$.

**Proposition 3.** Let

$$E_n \rightarrow E_{n-1} \rightarrow \cdots \rightarrow E_1$$

be a sequence of vector bundles on a scheme $X$ with rank $E_i = i$. Let $c_k(l) = c_k(E_i)$ and let $g_i[j]$ be defined in terms of the $c_*(\cdot)$ by the relations (10). If $E_i \rightarrow E_k$ is a surjection of vector bundles for some $l \geq k$, then $g_i[j] = 0$ for all $i, j$ satisfying $i < k + 1 \leq i + j \leq l$.

**Proof.** We proceed by induction on $l$. The base case $l = k$ holds trivially. We assume that the result holds for all $l' < l$. If $E_i \rightarrow E_k$ is surjective, then so is $E_{l'} \rightarrow E_k$ for $k \leq l' < l$. In particular, ker($E_{l'} \rightarrow E_k$) is a vector bundle of rank $l' - k$. Furthermore, by the induction hypothesis, we know that $g_i[j] = 0$ for $i < k + 1 \leq i + j < l$, so it suffices to prove the result for $i + j = l$.

We proceed and use the claim:

**Lemma 4.** Assume that the proposition holds for $l' < l$. Then $c_i(\ker(E_i \rightarrow E_k)) = E_i(k + 1, l')(g)$ for $l' \leq l$, where the polynomials $E_i(j)(g)$ are defined by (10).

**Proof.** Denote by $E(k + 1, l')(g)$ the polynomial $\sum_{i=0}^{l'-k} E_i(k + 1, l')(g)$ The $i$th degree component of $c(E_{l'})$ differs from the $i$th degree of the product $c(E_k)E(k + 1, l')$ by exactly all degree $i$ monomials in $g$ containing a path $g_i[j]$ with $i \leq k$ and $k + 1 \leq i + j \leq l'$, $l' \leq l$,
i.e. paths joining $x_i$ for $i \leq k$ to $x_{i+j}$ for $k+1 \leq i+j$. By assumption, these are all zero except possibly when $i+j = l' = l$. But $g_i[j]$ is of degree $j+1 = l-i+1 = l-k+1$ since $i \leq k$, so such $g_i[j]$ do not occur in $E(k+1,l)$. Therefore we have the equality $c(E_k)E(k+1,l') = c(E_{l'})$ for $l' \leq l$ as needed.

With this result, we can complete the proof of the proposition. We need to prove that $g_i[j] = 0$ for $i \leq k, i+j = l$. We use induction on $j$ for $j \geq l-k-1$. Assume that the result holds for $j' < j$. By the lemma, we know that $c(E_k)E(k+1,l) = c(E_l)$. For any $j$ the difference between the degree $j+1$ parts is the sum of all monomials of degree $j$ containing a variable $g_i[j']$ satisfying $i < k+1 \leq i+j' \leq l$. The variable $g_i[j']$ may only occur when $j' \leq j$ since $g_i[j]$ has degree $j+1$. By the first induction hypothesis, $g_i[j'] = 0$ except when $i+j' = l$, and by the second inductive hypothesis, $g_{i-j'}[j'] = 0$ when $j' < j$. Therefore, the only term remaining is $g_{i-j}[j]$, but on the other hand we began with an equality, so the difference $g_{i-j}[j]$ is forced to be zero as well. This concludes the proof.

**Corollary 3.** With the same notation and hypotheses of Proposition 3.

$$c(\ker(E_l \to E_k)) = \sum_{i=0}^{l-k} E_i(k+1,l)(g).$$

7. The results

In this section, we state and assume the Main Proposition, which is proved in section 8. We use the degeneracy locus formula applied to bundles on the hyperquot scheme as in Proposition 3.

For any $w \in S_n$, let $\sigma_w$ be the corresponding Schubert class in the small quantum cohomology ring $QH^*(\mathbb{F}(n))$. Throughout this section, consider the universal sequence of sheaves $8$. Let the $Q_i[j]$ be classes in $H^*(\mathcal{H}_Q)$, defined in terms of the Chern classes of $(\mathcal{A}_t^*)^t$, for any fixed $t \in \mathbb{P}^1$ as in the statement of Proposition 2. Let $\mu_w$ be the classes in the hyperquot schemes as described in section 3.

We have the following formulation of the definition of the quantum product, given by the definition in (1) and Corollary 2 to Theorem 3.

$$\sigma_{w_1} * \ldots * \sigma_{w_N} = \sum_{d,w} q^d(\mu_{w_1} \cdots \mu_{w_N} \cdot \mu_w)d\sigma_w^{\vee}.$$  

(11)

where $(\mu_{w_1} \cdots \mu_{w_N} \cdot \mu_w)d$ is the intersection product of the classes $\mu_s$ in the hyperquot scheme $\mathcal{H}_Q$.  

The proofs of the results rely on two simple lemmas.

**Lemma 5.** Let $f$ be a polynomial with integer coefficients in variables $\nu_w$ for $w \in S_n$. Then

$$f(\sigma_*) = \sum_{d,w} q^d(f(\mu_*) \cdot \mu_w)d\sigma_w^{\vee}$$  

in the quantum cohomology ring of $\mathbb{F}(n)$.  

Proof. Writing \( f \) as a sum of monomials, the result follows immediately from \([1]\) and the additivity of both the quantum cohomology ring and the cohomology of the hyperquot schemes. \( \square \)

The following lemma is proved by an argument of Bertram, cf. \([3]\).

**Lemma 6.** For \( w \in S_n \), \( \sigma_w = \sum_{d,w} q^d (\mu_w \cdot \mu_w') d \sigma_w \) in \( \mathbb{Q}H^*(F(n)) \).

**Proof.** When \( d = 0 \), i.e. \( d_i = 0 \) for all \( i \), \( \mathcal{H}\mathcal{Q}_{d}(F(n)) = F(n) \), and the \( \mu_w \) are the ordinary Schubert cycles on the flag manifold. In this case, \( \mu_w \cdot \mu_w' = 1 \) when \( w' = w \) and zero otherwise.

Therefore, it is enough to show that \( (\mu_w \cdot \mu_w')_{d} = 0 \) whenever \( d \neq 0 \). Recall that \( (\mu_w \cdot \mu_w')_{d} = (\Omega_w, \Omega_w')_{d} \). This Gromov-Witten number is zero since only two points in \( \mathbb{P}^1 \) have been fixed \( \square \)

For the remainder of the paper, we fix some notation. For any multidegree \( d \), let the classes \( Q_i[j] \in H^*(\mathcal{H}\mathcal{Q}_d) \) be defined in terms of the Chern classes of the tautological bundles by \([1]\), where the multidegree \( d \) of \( \mathcal{H}\mathcal{Q}_d \) is understood by the appearance of the \( Q_i \)'s in intersection products \( (\cdots)_d \) on the corresponding hyperquot scheme \( \mathcal{H}\mathcal{Q}_d \). For any polynomial \( P(g) \) in the variables \( g_i[j] \) for \( j \geq 0, i \geq 1 \) and \( i + j \leq n \), denote by \( P(Q) \) the corresponding polynomial in the classes \( Q_i[j] \). Define \( P(x, q) \) to be the polynomial resulting from setting \( g_i[0] = x_i, g_i[1] = q_i, \) and \( g_i[j] = 0 \) for \( j \geq 2 \) in \( P(g) \).

With this notation, we have the following claim.

**Main Proposition.** For any multidegree \( d \), let the classes \( Q_i[j] \in H^*(\mathcal{H}\mathcal{Q}_d) \) be defined in terms of the Chern classes of the tautological bundles by \([1]\). Let \( P(g) \) be any polynomial in \( g_i[j] \). Then for \( w \in S_n \),

\[
P(x, q) = \sum_{d,w} q^d (P(Q) \cdot \mu_w)_{d} \sigma_w \text{ in } \mathbb{Q}H^*(F(n)).
\]

The Main Proposition is proved in the next section, with the exception of a key lemma, whose proof is found in section \([1]\). We conclude this section by proving the presentation of \( \mathbb{Q}H^*(F(n)) \) and the quantum Giambelli formula as two easy corollaries of the Main Proposition.

For any \( 1 \leq i \leq j \leq n \) with \( i + j \leq n \), let \( E_k(l)(g) \) be the polynomial in the variables \( g_i[j] \) as in section \([6.1]\). Recall that from our observations in section \([6.2]\), \( e_{k}^q(l) = E_k(l)(x, q) \), and that \( e_{k}(l) \) is the \( k \)th elementary symmetric polynomial in the variables \( x_1, \ldots, x_l \).

**Theorem 8.** The small quantum cohomology ring of the complete flag manifold has a presentation:

\[
\mathbb{Q}H^*(F(n)) = \mathbb{Z}[x_1, \ldots, x_n, q_1, \ldots, q_{n-1}]/I_q
\]

where \( I_q = (e_{1}^q(n), \ldots, e_{n}^q(n)) \).

**Proof.** By \([7]\), it suffices to produce \( n \) relations, which specialize to the \( n \) relations in \( H^*(F(n)) \) upon setting \( q_1 = \cdots = q_{n-1} = 0 \). Recall that the relations of \( H^*(F(n)) \) are given by \( e_1(n), \ldots, e_n(n) \). Since \( A_n^* = V_n^* \) is a trivial vector bundle, we know that
\[ c_i(A^*_n) = E_i(n)(Q) = 0 \] for every \( i \). For \( 1 \leq i \leq n \), we apply the Main Proposition to the polynomial \( E_i(n)(g) \) to get
\[ 0 = \sum_{d, w} q^d (E_i(n)(Q) \cdot \mu_w) d \sigma_w^\vee = E_i(n)(x, q) = e_i^q(n). \]

It is clear that \( E_i(n)(x, 0) = e_i(n) \), so that \( e_i^q(n) \) specializes to \( e_i(n) \). This concludes the proof of the theorem.

**Theorem 9.** In the small quantum cohomology ring of \( F(n) \), there is the quantum Giamelli formula,
\[ \sigma_w = S_w(x, q). \]

**Proof.** By the degeneracy locus formula, \( \mu_w = S_w(Q) \). Putting this into Lemma 6 and applying the Main Proposition to the polynomial \( S_w(g) \) gives:
\[ \sigma_w = \sum_{d, w'} q^d (S_w(Q) \cdot \mu_w') d \sigma_{w'}^\vee = S_w(x, q). \]

---

**8. Main Proposition**

The Main Proposition asserts that there is some sort of correspondence between classes \( Q_i[j] \) in \( H^*(HQ_d) \) over various \( d \) with elements in \( QH^*(F(n)) \). In particular, it appears that there should be a relationship between \( Q_i[0] \) and \( Q_i[1] \) with the \( x_i \) and the deformation variables \( g_i \) in \( QH^*(F(n)) \), respectively. Moreover, the classes \( Q_i[j] \) with \( j \geq 2 \) should give, in some sense, zero contribution to quantum cohomology.

In this section, we obtain an understanding of some classes \( Q_i[j] \) for a handful of multidegrees \( d \), and state Proposition 6, which computes a particular type of intersection on the hyperquot scheme. Using the structure of the quantum cohomology ring, the cohomology rings of hyperquot schemes, and the correspondence between \( \mathbb{Z}[c] \) and \( \mathbb{Z}[g] \) in (7), we show that this is enough to prove the Main Proposition.

The general argument for the analog of Proposition 6 for partial flag manifolds is postponed to section 11. For complete flag manifolds, we have additional structure given by describing the classes \( Q_i[1] \in H^*(HQ_d) \) for some \( d \) as classes of particular degeneracy loci, and study them geometrically in sections 9 and 10. Proposition 6 follows as a special case of Proposition 9 in section 10.

For any polynomial \( P(g) \) in the variables \( g_i[j] \) for \( j \geq 0, i \geq 1 \) and \( i + j \leq n \), write \( P^k(g) \) for the resulting polynomial after setting \( g_i[j] = 0 \) for \( j \geq k+1 \) in \( P(g) \). Note that \( P^k(g) \) is a polynomial in variables \( g_i[j] \) with \( j \leq k \). With this notation, \( P^0(x, q) = P(x) \) and \( P^k(x, q) = P(x, q) \) for \( k \geq 1 \).

In order to prove the Main Proposition, we prove an auxiliary result:

**Proposition 4.** Let \( P(g) \) be any polynomial in \( g_i[j] \). For \( k \geq 0 \),
\[ P^k(x, q) = \sum_{d, w} q^d (P^k(Q) \cdot \mu_w) d \sigma_w^\vee. \]
The Main Proposition follows as the special case \( k = n \) of Proposition \( \PageIndex{4} \). We use induction on \( k \) to prove Proposition \( \PageIndex{4} \) using Lemma \( \PageIndex{5} \) in almost every step. We prove the result for \( k = 0 \), and then use induction.

For the remainder of the paper, we set \( y_i = Q_i[0] \), a class in \( \mathcal{H} \mathcal{Q}_d \), and define \( |d| = \sum d_i \).

**Proposition 5.** If \( P(g) \) is a polynomial in the variables \( g_i[0] \), then

\[
P(x) = \sum_{d,w} q^d (P(y) \cdot \mu_w) d \sigma_{w^\vee} \quad \text{in} \quad QH^*(\mathcal{F}(n)).
\]

Proof. In each hyperquot scheme \( \mathcal{H} \mathcal{Q}_d \), we can write \( y_i = Q_i[0] = \mu_{s_i} - \mu_{s_{i-1}} \). There is no quantum deformation of divisor classes so that the quantum Giambelli formula for \( w = s_i \) holds, \( \sigma_s = x_1 + \cdots + x_s \), so that \( x_i = \sigma_{s_i} - \sigma_{s_{i-1}} \). Therefore, it suffices to prove the statement for a polynomial in the \( \sigma_s \)'s. The statement follows as a special case of Lemma \( \PageIndex{5} \) where each \( w \) is some \( s_i \).

**Proposition 6.** For \( 1 \leq i \leq n - 1 \), \( (Q_i[1] \cdot \mu_{w_0})_{e_i} = 1 \), where \( w_0 \) is the permutation of longest length in \( S_n \).

It turns out that these \( n - 1 \) intersections are the only ones that must be computed in order to see the effect of the classes \( Q_i[1] \) on quantum cohomology. There is an analogous result for partial flag manifolds in section \( \PageIndex{11} \).

We prove some facts regarding the classes \( Q_i[j] \) for \( j \geq 2 \) in \( H^*(\mathcal{H} \mathcal{Q}_d) \).

**Lemma 7.** If \( d_k = 0 \) for some \( k \) then \( Q_i[j] = 0 \) when \( i \) and \( j \) satisfy:

1. \( i \leq k \) and
2. \( i + j \geq k + 1 \).

Proof. If \( d_k = 0 \), then \( A_k \to V_n = A_n \) is a vector bundle inclusion, or equivalently, \( A_n^* \to A_k^* \) is a surjection. Then a direct application of Proposition \( \PageIndex{3} \) gives the result.

**Proposition 7.**

1. If \( |d| \leq j - 1 \), then \( Q_i[j] = 0 \) in \( H^*(\mathcal{H} \mathcal{Q}_d) \) for every \( 1 \leq i \leq n - j \).
2. If \( j \geq 2 \), then in \( QH^*(\mathcal{F}(n)) \)

\[
\sum_{d,w} q^d (Q_i[j] \cdot \mu_w) d \sigma_{w^\vee} = 0.
\]

Proof. Part 1 of the proposition follows from the fact that if \( Q_i[j] \neq 0 \) for some \( i, j \), then by Lemma \( \PageIndex{6} \), \( d_k \neq 0 \) for \( k = i, \ldots, i + j - 1 \). In particular, \( d_k \geq 1 \) so that \( |d| \geq \sum_{k=i}^{i+j-1} d_k \geq \sum_{k=i}^{i+j-1} 1 \geq j \).

To prove Part 2, we consider two cases:

1. \( |d| \leq j - 1 \). In this case, by part 1, \( Q_i[j] = 0 \) so that \( (Q_i[j] \cdot \mu_w)_d = 0 \) for all \( w \) and \( d \).
2. \( |d| \geq j \). The product \( (Q_i[j] \cdot \mu_w)_d \) can only be nonzero in the correct dimension, when \( (j + 1) + l(w) = \dim(\mathcal{H} \mathcal{Q}_d) = \binom{n}{2} + 2|d| \). Since \( l(w) \leq \binom{j}{2} \) for all \( w \), and \( j \geq 2 \) implies that \( 2j > j + 1 \), this situation can never occur, so that the intersection is zero by dimension considerations.
This concludes the proof of the proposition. □

Part 1 of Proposition 7 is the only direct knowledge of the geometry of the classes \( Q_i[j] \) for \( j \geq 2 \) needed for the results. Besides the proof of Proposition 6, which requires a detailed geometric description of the classes \( Q_i[1] \) and \( \mu_w \), the remainder of the proof of the Main Proposition is given by algebraic manipulations on the classes \( \mu_w = \mathcal{G}_w(Q) \) and \( Q_i[j] \) in \( H^*(\mathcal{H}Q_d) \) for all \( d \), and properties of the quantum cohomology ring.

**Lemma 8.** Let \( k \) be an integer such that Proposition 4 holds for \( k \). Then \( \sigma_w = \mathcal{G}_w(x, q) \) in \( QH^*(\mathcal{F}(n)) \) for \( l(w) \leq k + 2 \).

**Lemma 9.** Let \( k \) be an integer such that Proposition 4 holds for \( k \). Then the class \( Q_i[k+1] \) can be written as a polynomial in those classes \( \mu_w \) with \( l(w) \leq k + 2 \).

We first prove Lemma 8. For \( l(w) \leq k + 2 \), the polynomial \( \mathcal{G}_w(Q) \) involves classes \( Q_i[j] \) where \( j \leq k + 1 \). Therefore, we can write \( \mathcal{G}_w(Q) = \mathcal{G}_w(Q) + \sum_{i \in I} Q_i[k + 1] \) for some sequence \( \{ i \in I \} \), possibly empty, with the \( i_j \in \{1, \ldots, n - k - 1\} \) not necessarily distinct. Then the additivity of the cohomology of the hyperquot scheme, and the fact that \( \mu_w = \mathcal{G}_w(Q) \) gives:

\[
\sigma_w = \sum_{d, w'} q^d(\mathcal{G}_w(Q) \cdot \mu_{w'}) d\sigma_{w'} \dagger
\]
\[
= \sum_{d, w'} q^d(\mathcal{G}_w^k(Q) \cdot \mu_{w'}) d\sigma_{w'} + \sum_{i_j} \sum_{d, w'} q^d(Q_i[k + 1] \cdot \mu_{w'}) d\sigma_{w'}
\]
\[
= \mathcal{G}_w(x, q) + \sum_{(i, k+1) = (i, 1)} q_i
\]
\[
= \mathcal{G}_w(x, q).
\]

The third equality is an immediate application of the property that \( k \) satisfies the statement of Proposition 6, Proposition 8, and Part 2 of Proposition 6. The final equality follows from the definition of the polynomials \( \mathcal{G}_w^k(x, q) \) and \( \mathcal{G}_w(x, q) \). □

We now prove Lemma 9. Let \( \beta_{i,k} \in S_n \) be the permutation with \( \mu_{\beta_{i,k}} = C_{k+2}(i+k+1) \), where the classes \( C_*(\ast) \) are defined by (3). In cycle notation \( \beta_{i,k} = (i i+1 \cdots k + i + 2) \), and \( l(\beta_{i,k}) = k + 2 \). In the notation of \( \mathcal{C}-\mathcal{F} \), \( \beta_{i,k} = \alpha_{k+3,k+i+2} \).

By the recursion in (10) and the fact that \( \mu_{\beta_{i,k}} = \mathcal{G}_{\beta_{i,k}}(Q) \), we can write

\[
Q_i[k+1] + \cdots + Q_i[k+1] = \mu_{\beta_{i,k}} - \mathcal{G}_{\beta_{i,k}}(Q).
\]

By the induction hypothesis, every \( Q_i[j] \) with \( j \leq k \) can be written as a polynomial in \( \mu_w \) with \( l(w) \leq k + 1 \). Therefore, since \( \mathcal{G}_{\beta_{i,k}}(Q) \) is by definition a polynomial in \( Q_i[j] \) with \( j \leq k \), it can be as well. Therefore, by induction on \( i \), we see that each \( Q_i[k+1] \) can be written as a polynomial in \( \mu_w \) with \( l(w) \leq k + 2 \). □

We are ready to complete the inductive proof of Proposition 7. The base case \( k = 0 \) was proved in Proposition 6. Let \( k \) be such that Proposition 6 holds. By Lemma 8, every \( Q_i[j] \) with \( j \leq k + 1 \) can be written as a polynomial in \( \mu_w \) with \( l(w) \leq k + 2 \).
By definition, $P^{k+1}(Q)$ is a polynomial in these $Q_i[j]$, so that it can be written as $P^{k+1}(Q) = \tilde{p}(\mu_*)$ with $l(w) \leq k + 2$.

We apply Lemma 8 to $P^{k+1}(g)$ to get
\[
\sum_{d,w} q^d(P^{k+1}(Q) \cdot \mu_w)\sigma_{w^\vee} = \sum_{d,w} q^d(\tilde{p}(\mu_*) \cdot \mu_w)\sigma_{w^\vee} = \tilde{p}(\sigma_*) = P(x, q)
\]
where the final equality follows from Lemma 8 and the fact that $\tilde{p}(\sigma_*)$ is a polynomial in $\sigma_w$ with $l(w) \leq k + 2$. □

Except for the proof of Proposition 6, this concludes the proof of Proposition 4 and the Main Proposition, and hence of the presentation of $QH^*(\mathbb{P}(n))$ and of the quantum Giambelli formula.

9. More degeneracy loci on the hyperquot scheme

In this section, we define and study certain degeneracy loci on $\mathcal{H}Q_d$ which allow us to geometrically and explicitly understand the classes $Q_i[1]$ in $H^*(\mathcal{H}Q_d)$ for the various hyperquot schemes. These loci are crucial to the proof of Proposition 6, which is what remains to be proved.

Over $\mathbb{P}^1 \times \mathcal{H}Q_d(\mathbb{P}(n))$, there is the universal sequence of sheaves (3). For $1 \leq i \leq n-1$, define $W_i \subset \mathbb{P}^1 \times \mathcal{H}Q_d$ to be locus over which $\text{rank}(A_i \to A_{i+1}) \leq i - 1$, with the natural scheme structure is given by the vanishing of determinants. Then for $p \in \mathbb{P}^1$, define $W_i(p) = W_i \cap (p \times \mathcal{H}Q_d)$. Identifying $p \times \mathcal{H}Q_d$ with $\mathcal{H}Q_d$, we view $W_i(p)$ as a subscheme of $\mathcal{H}Q_d$. We use points $p$ for the loci $W_i(p)$, while we use points $t$ for the loci $\tilde{W}_i(t)$ introduced in section 8.

We prove a general position result in the spirit of the argument of section 8, which shows that $W_i(p)$ is of expected complex codimension two, thus giving a class in the cohomology of the hyperquot scheme. Furthermore, we show that this class is independent of the choice of the point $p$, and that this class is given by $Q_i[1]$.

The following lemma is the analog of Lemma 8 to these degeneracy loci.

Lemma 10. Let $e$ be a multiindex, with $h_e$ and $\pi$ as defined in section 8. Then
\[
\pi^{-1}(W_i(p)) = \pi^{-1}(\mathbb{P}^1 \times W_i(p)) \cup \tilde{W}_i(p)
\]
with $\tilde{W}_i(p)$ the degeneracy locus inside $\pi^{-1}(p \times \mathcal{H}Q_{d-e})$ given by
\[
\text{rank}(K_i \to K_{i+1}) \leq i - 1.
\]
This equality is scheme-theoretic away from the intersection.

Proof. We recall the construction of the map $h_e$ as stated in section 4. For $1 \leq i \leq n-1$, we have $(1 \times h_e)^* A^d = \tilde{A}^e_i$, so that
\[
h_e^{-1}(W_i(p)) = \{ y \in U_e | \text{rank}(p,y) \tilde{A}_i \to \tilde{A}^e_i \leq i - 1 \}.
\]
Outside $\tilde{A}_j$, for $j = i, i+1$, $\psi^* A_{j}^{d-e}$ is isomorphic to $\tilde{A}^e_j$, while over the locus $\tilde{A}_j$, we have $A^e_j = K_j$. These two observations give the lemma. □

In fact we can say more:
Corollary 4. For a multiindex $e = (e_1, \ldots, e_{n-1})$,
1. If $e_i = 0$, then $h_{e_i}^{-1}(W_i(p)) = \pi^{-1}(P^1 \times W_i(p))$ as schemes.
2. If $e_i > 0$, then $h_{e_i}^{-1}(W_i(p)) = \pi^{-1}(P^1 \times W_i(p)) \cup \pi^{-1}(p \times \mathcal{H}_d - e_i)$ as schemes away from the intersection.

Proof. Recall that $\tilde{W}_i(p)$ is defined as the locus in $\pi^{-1}(p \times \mathcal{H}_d - e_i)$ given by the condition $\text{rank} \left( K_i \rightarrow \mathcal{V} \right) \leq i - 1$. In particular, we see that elements in $\tilde{W}_i(p)$ satisfy the condition $\text{rank} \left( K_i \rightarrow \mathcal{V} \right) = i - e_i$ for $1 \leq i \leq n - 1$.

By these definitions, we see that when $e_i = 0$, $\tilde{W}_i(p)$ is empty, giving the first part of the claim.

The locus where $\text{rank} \left( K_i \rightarrow \mathcal{V} \right) = i - e_i$ is clearly contained in the locus where $\text{rank} \left( K_i \rightarrow \mathcal{V} \right) \leq i - 1$. Therefore, $\mathcal{U}_e(p) = \pi^{-1}(p \times \mathcal{H}_d - e_i)$ is contained in $\tilde{W}_i(p)$. This gives the second part of the claim. \qed 

In order to intersect these loci, we need the following theorem about general position, extending the general position results of [C-F1] as stated in Theorem 6.

Theorem 10. (Moving lemma) For $i_1, \ldots, i_M$ in $\{1, \ldots, n\}$, permutations $w_1, \ldots, w_N \in S_n$, general translates of $\Omega_{w_k} \subset F(n)$, and distinct points $t_1, \ldots, t_N, p_1, \ldots, p_M \in P^1$, the intersection

$$
\bigcap_{j=1}^M W_{i_j}(p_j) \cap \bigcap_{k=1}^N \Omega_{w_k}(t_k)
$$

is either empty or has pure codimension $2M + \sum_{k=1}^N l(w_k)$. Here the $i_j$ and the $w_k$ are not necessarily distinct.

Proof. For $M = 0$, this is Theorem 5. We only need to consider the case where $M \geq 1$. The proof is by induction on $d = (d_1, \ldots, d_{n-1})$.

The base case is when $d_1 = \cdots = d_{n-1} = 0$, so that $\mathcal{H}_d(F(n)) = F(n)$. In particular, $A_i \rightarrow A_{i+1}$ is a vector bundle inclusion for each $i$. Thus $W_i = \emptyset$ for all $i$, and hence $W_i(p)$ (which appears in the intersection if $M \geq 1$) is empty so that the entire intersection is empty.

For two multindices, write $f < d$ when $f_j \leq d_j$ for every $1 \leq j \leq n - 1$ and $f_k < d_k$ for some $1 \leq k \leq n - 1$. Assume that the result holds for all such $f$.

Since any $W_i$ is the locus where $A_i \rightarrow A_{i+1}$ drops rank, we must have rank $\left( A_i \rightarrow \mathcal{V} \right) \leq i - 1$ over $W_i$ as well. In particular, it follows from the second part of Theorem 5 that for any $1 \leq i \leq n - 1$ and $p \in P^1$, $W_i(p)$ is contained in the boundary of the hyperquot scheme.

Let $L = \sum_{k=1}^N l(w_k)$. By assumption, $M \geq 1$ so that the intersection (12) is also contained in the boundary. Since $\bigcup_e h_e(\mathcal{U}_e)$ covers the boundary, it suffices to show that for every $0 < e < d$,

$$\text{codim}_{\mathcal{H}_d} \left( h_e(\mathcal{U}_e) \cap \bigcap_{j=1}^M W_{i_j}(p_j) \cap \bigcap_{k=1}^N \Omega_{w_k}(t_k) \right) \geq 2M + L.$$
The map $h_e$ is birational onto its image, so we only need to show, for every $e$, that
\[
\text{codim}_{U_e} \left( \bigcap_{j=1}^{M} h_e^{-1}(W_i_j(p_j)) \cap \bigcap_{k=1}^{N} h_e^{-1}(\Omega_{w_k}(t_k)) \right)
\geq 2M + L - (\dim \mathcal{Q}_d - \dim U_e)
= 2M + L + 1 - \sum e_i(1 + e_i - e_{i-1}).
\]

$\tilde{W}_{i_j}(p_j)$ and $\Omega_{w_k}(t_k)$ are supported on $p_j \times \mathcal{Q}_d - e$ and $t_k \times \mathcal{Q}_d - e$, respectively. By Lemma 2, Lemma 10, and the fact that $p_1, \ldots, p_M, t_1, \ldots, t_N$ are all distinct points in $\mathbf{P}^1$, we see that after a possible renumbering, the only possible nonempty intersections are of the type
\begin{equation}
(13) \quad \pi^{-1} \left( \mathbf{P}^1 \times \bigcap_{j=1}^{M} W_{i_j}(p_j) \cap \bigcap_{k=1}^{N} \Omega_{w_k}(t_k) \right),
\end{equation}
of type
\begin{equation}
(14) \quad \pi^{-1} \left( t_N \times \bigcap_{j=1}^{M} W_{i_j}(p_j) \cap \bigcap_{k=1}^{N-1} \Omega_{w_k}(t_k) \right) \cap \tilde{\Omega}_{w_N}(t_N),
\end{equation}
or of type
\begin{equation}
(15) \quad \pi^{-1} \left( p_M \times \bigcap_{j=1}^{M-1} W_{i_j}(p_j) \cap \bigcap_{k=1}^{N} \Omega_{w_k}(t_k) \right) \cap \tilde{W}_{i_M}(p_M).
\end{equation}

Since $d - e < d$, by the induction hypothesis and the fact that $\pi$ is smooth, intersections of type (13) are of codimension $\geq 2M + L$ in $U_e$.

By Lemma 10 and Lemma 3 and the induction hypothesis, intersections of type (14) are of codimension $\geq 1 + 2M + L - l(w_N) + l(\tilde{w}_N)$ in $U_e$. But we know that $l(w_N) - l(\tilde{w}_N) \leq \sum e_i$, so that such intersections are codimension at least $2M + L - \sum e_i + 1$ in $U_e$.

By Lemma 10, intersections of type (15) are empty unless $e_{i_M} > 0$, in which case $i = i_M$. From Corollary 4, we have $\tilde{W}_{i_M}(p_M) = \pi^{-1}(p_M \times \mathcal{Q}_d)$. Therefore, (15) is either empty or codimension $2(M - 1) + L$ in $U_e(t_M)$, and hence codimension $2(M - 1) + L + 1 = 2M + L - 1$ in $U_e$.

By part 1 of Lemma 4, for any $e$, we have the inequalities
\[
2M + L \geq 2M + L - \sum e_i + 1 > 2M + L + 1 - \sum e_i(1 + e_i - e_{i-1})
\]
so that intersections of types (13) and (14) are empty and $2M + L - 1 \geq 2M + L + 1 - \sum e_i(1 + e_i - e_{i-1})$. By part 2 of Lemma 4, this is an equality if and only if $|e| = \sum e_i = 1$, so that $e = e_i$ for some $i$. Therefore, in the case $|e| \geq 2$, intersections of type (13) are also empty, and when $e = e_i$, we have $2M + L - 1 = 2M + L - (\dim \mathcal{Q}_d - \dim U_e)$, which gives the needed codimension estimate.

In the course of the proof, we actually showed something stronger, that the only nonempty intersection arise when $e = e_i$, from type (15). In particular, we have proven:
Corollary 5. Consider the same hypotheses as in Theorem 10, with \( e_i \) the multiindex with all zeros except a 1 at the \( i \)th position. Then

\[
\left\{ \bigcap_{j=1}^M W_{ij}(p_j) \cap \bigcap_{k=1}^N \Omega_{w_k}(t_k) \right\} = \bigcup_{i_j=1}^{M-n} \left( \bigcap_{l \neq j} \pi^{-1}(P_1 \times W_{il}(p_i)) \cap \bigcap_{k=1}^N \pi^{-1}(P_1 \times \Omega_{w_k}(t_k)) \right).
\]

Corollary 6. The class \([W_i(p)] \in H^4(\mathcal{H}Q_d(F(n)))\) is independent of the choice of the point \( p \in P^1 \).

Proof. By definition, the \( W_i(p) \) are fibers of the morphism \( W_i \subset P^1 \times \mathcal{H}Q_d \to P^1 \). By Theorem 10, the fibers are of complex codimension two. \( W_i \to P^1 \) is in fact a fiber bundle since the automorphism group of \( P^1 \) is transitive. Therefore \([W_i(p)]\) is independent of the point \( p \).

Corollary 7. For any \( p \in P^1 \) and \( 1 \leq i \leq n-1 \), \( W_i(p) \) is an irreducible scheme, with \( h_{e_i}(\mathcal{U}_{e_i}(p)) \subset W_i(p) \) an open subscheme.

Proof. We use the second part of Corollary 4 applied to \( e = e_i \):

\[
h_{e_i}^{-1}(W_i(p)) = \pi^{-1}(P_1 \times W_i(p)) \cup \pi^{-1}(P_1 \times \mathcal{H}Q_{d-e_i}).
\]

Since \( \mathcal{U}_{e_i}(p) = \pi^{-1}(P_1 \times \mathcal{H}Q_{d-e_i}) \) is the preimage via the smooth map \( \pi \) of an irreducible scheme, it is also irreducible, as its image \( h_{e_i}(\mathcal{U}_{e_i}(p)) \). We observe that \( \text{codim}_{e_i} \mathcal{U}_{e_i}(p) = 1 \) and \( \text{codim}_{e_i} \pi^{-1}(P_1 \times W_i(p)) = 2 \). By Corollary 4, \( h_{e_i}^{-1}(W_i(p)) \) is the union of these two subschemes of \( \mathcal{U}_{e_i} \), so that \( W_i(p) \) is irreducible.

Observe that

\[
\dim W_i(p) = \dim \mathcal{H}Q_d(F(n)) - 2 = \dim P_1 \times \mathcal{H}Q_{d-e_i} - 1 = \dim \mathcal{U}_{e_i} - 1 = \dim \mathcal{U}_{e_i}(p).
\]

We know that \( h_{e_i} \) maps \( \mathcal{U}_{e_i}(p) \) isomorphically onto its image. By the definition of \( \mathcal{U}_{e_i}(p) \), it is a locally closed scheme, so that its image \( h_{e_i}(\mathcal{U}_{e_i}(p)) \) is a union of locally closed schemes. But \( W_i(p) \) is irreducible, so has only one component, and therefore \( h_{e_i}(\mathcal{U}_{e_i}(p)) \) is locally closed in \( W_i(p) \). Since \( \dim \mathcal{U}_{e_i}(p) = \dim W_i(p) \), the image \( h_{e_i}(\mathcal{U}_{e_i}(p)) \) is an open subscheme of \( W_i(p) \).

Proposition 8. Let the classes \( Q_i[j] \in H^*(\mathcal{H}Q_d) \) be defined in terms of the Chern classes of the tautological bundles \((A_i)_p\) on \( \mathcal{H}Q_d \) by \( 14 \). Then there is an equality \([W_i(p)] = Q_i[1]\).

Proof. By Theorem 10 and its corollaries, \([W_i(p)]\) is a class independent of the point \( p \) in \( H^4(\mathcal{H}Q_d) \). It is a degeneracy locus of expected codimension two. Since its expected dimension matches its actual dimension, we can apply Porteous’ formula so that \([W_i(p)] = c_2((A_{i+1})_p/(A_i)_p)\) where \((A_i)_p\) is the bundle \( A_i \) restricted to \( p \times \mathcal{H}Q_d \).
By the definition of the classes \( Q \) as in Proposition \( \text{[2]} \), for any \( 1 \leq l \leq n \), the Chern polynomial can be written \( c(A^l_p) = \sum_{k=0}^l (-1)^k E_k(l)Q \). Let \( b_1 = c_1((A_{i+1})_p/(A_i)_p) \) and \( b_2 = c_2((A_{i+1})_p/(A_i)_p) \). Then
\[
b_1 = c_1((A_{i+1})_p) - c_1((A_i)_p) = Q_{i+1}[0]
\]
and the equation
\[
c((A_{i+1})_p) = c((A_{i+1})_p)/(A_i)_p)c((A_i)_p)
\]
implies that
\[
E_2(i + 1)(Q) = c_2((A_{i+1})_p) = c_2((A_i)_p) + c_1((A_i)_p)b_1 + b_2 = E_2(i) + (-E_1(i))(-Q_{i+1}[0]) + b_2.
\]
We also have the recursion in equation (6)
\[
E_2(i + 1)(Q) = E_2(i) + E_1(i)Q_{i+1}[0] + Q_i[1]
\]
so that \( b_2 = Q_i[1] \).

10. Proof of Proposition \( \text{[3]} \)

In this section, we prove a more general statement than Proposition \( \text{[3]} \), which gives a correspondence between the classes \( Q_i[1] \) in the hyperquot schemes \( \mathcal{H}Q_d \) and the deformation variables \( q_i \) in \( QH^*(F(n)) \).

For any multiindex \( c = (c_1, \ldots, c_{n-1}) \), write
\[
Q[1]^c = Q_1[1]^{c_1}Q_2[1]^{c_2} \cdots Q_{n-1}[1]^{c_{n-1}}.
\]

We prove the following proposition.

**Proposition 9.** For any \( c < d \),
\[
(Q[1]^c \cdot \mu_{w_1} \cdots \mu_{w_N})_d = (\mu_{w_1} \cdots \mu_{w_N})_d - c.
\]

The proof is geometric in nature, involving degeneracy loci of types \( \overrightarrow{\mu}_w(t) \) and \( W_i(p) \) on \( \mathcal{H}Q_d(F(n)) \) and an analysis of the boundary. We use induction on \( c \). For \( c = 0 \), this is Lemma \( \text{[3]} \) so we may assume that \( c_m > 0 \) for some \( m \).

Proposition \( \text{[3]} \) states that \( [W_i(p)] = Q_i[1] \) in \( \mathcal{H}Q_d \) for any choice of point \( p \in P^1 \). By the moving lemma, we know that in order to intersect these \( W_i(p) \) it suffices to choose distinct points. Choose \( p_{i,j} \) and \( t_k \) to be distinct points of \( P^1 \) for \( 1 \leq i \leq n-1, 1 \leq j \leq c_i, \) and \( 1 \leq k \leq N \). Define
\[
Y := \bigcap_{(i,j) \neq (m,1)} W_i(p_{i,j}) \cap \bigcap_{k=1}^N \overrightarrow{\mu}_w(t_k)
\]
where \( m \) is such that \( c_m > 0 \).

By the corollary to Theorem \( \text{[4]} \), the left hand side of Proposition \( \text{[3]} \) can be interpreted as the the intersection number \( [Y] \cdot [W_m(p_{m,1})] \) on \( H^*(\mathcal{H}Q_d) \).

The idea of the proof is to compute this intersection on a particular open subscheme of \( W_m(p_{m,1}) \subset \mathcal{H}Q_d \). For the inductive step, we show that this intersection can be further computed on a smaller hyperquot scheme. In order to do this, we use the morphism \( h_{em} : U_{em} \to \mathcal{H}Q_d \) and the open immersion \( \pi : U_{em} \to \mathcal{H}Q_d - e_m \).
By Lemma 10, \( h_{e_m}^{-1}(Y \cap W_m(p_{m,1})) \) is supported on \( \pi^{-1}(p_{m,j} \times \mathcal{H}_d-e_m) \) for \( 1 \leq j \leq c_i \). Set-theoretically, we have
\[
\pi^{-1}(p_{m,j} \times W_m(p_{m,j}')) = \pi^{-1}(p_{m,j'} \times W_m(p_{m,j}))
\]
for any \( j, j' \). Hence, \( h_{e_m}^{-1}(Y \cap W_m(p_{m,1})) \) is supported on \( \mathcal{U}_{e_m}(p_{m,1}) = \pi^{-1}(p_{m,1} \times \mathcal{H}_d-e_m) \) so that the set-theoretic intersection \( Y \cap W_m(p_{m,1}) \) is contained in \( h_{e_m}(\mathcal{U}_{e_m}(p_{m,1})) \).

By Corollary 6, we know that \( h_{e_m}(\mathcal{U}_{e_m}(p_{m,1})) \) is an open subscheme of \( W_m(p_{m,1}) \). Since (set-theoretically) \( Y \cap W_m(p_{m,1}) \subset h_{e_m}(\mathcal{U}_{e_m}(p_{m,1})) \), we have the equality of cycle intersections on \( \mathcal{H}_d \)
\[
[Y] \cdot [W_m(p_{m,1})] = [Y] \cdot (h_{e_m})_*[\mathcal{U}_{e_m}(p_{m,1})] = [(h_{e_m})_*((h_{e_m})^{-1}(Y) \cap \mathcal{U}_{e_m}(p_{m,1}))]
\]
where the second equality comes from the projection formula.

We know that \( h_{e_m} \) maps \( \mathcal{U}_{e_m}(p_{m,1}) = \pi^{-1}(p_{m,1} \times \mathcal{H}_d-e_m) \) isomorphically onto its image, so that the right hand quantity is equal to the intersection number
\[
[(h_{e_m})^{-1}(Y) \cap \mathcal{U}_{e_m}(p_{m,1})] \text{ on } \mathcal{U}_{e_m}.
\]

By Corollary 3 to Theorem 10, we see that this last intersection is
\[
\bigcap_{(i,j) \neq (m,1)} \pi^{-1}(p_{m,1} \times W_i(p_{i,j})) \cap \bigcap_{k=1}^N \pi^{-1}(p_{m,1} \times \Omega_{w_{k}^{-1}}(t_{k})) = \pi^{-1}\left(p_{m,1} \times \bigcap_{(i,j) \neq (m,1)} W_i(p_{i,j}) \cap \bigcap_{k=1}^N \Omega_{w_{k}^{-1}}(t_{k})\right)
\]
We claim that this is a scheme-theoretic equality. By Lemma 3 and Lemma 11, it suffices to show that this does not intersect any of \( \pi^{-1}(t_k \times \Omega_{w_{k}^{-1}}(t_{k})) \) or \( \pi^{-1}(p_{i,j} \times W_i(p_{i,j})) \). Since all of the points \( t_k \) and \( p_{i,j} \) are distinct, the only case to check is that of \( p_{m,1} \). But here, we see that \( \pi^{-1}(p_{m,1} \times (Y \cap W_m(p_{m,1}))) \) is empty by the codimension results of Theorem 10 applied to \( Y \cap W_m(p_{m,1}) \) on \( \mathcal{H}_d-e_m \).

All of the intersection points lie in the image of the open immersion \( \pi : \mathcal{U}_{e_m} \to \mathbb{P}^1 \times \mathcal{H}_d-e_m \). Therefore, via the identification \( p_{m,1} \times \mathcal{H}_d-e_m \cong \mathcal{H}_d-e_m, [Y] \cdot [W_m(p_{m,1})] \) is the length of the zero-dimensional subscheme of \( \mathcal{H}_d-e_m \) given by
\[
\bigcap_{(i,j) \neq (m,1)} W_i(p_{i,j}) \cap \bigcap_{k=1}^N \Omega_{w_{k}^{-1}}(t_{k})
\]
Alternatively, this is the intersection number
\[
\prod_{(i,j) \neq (m,1)} W_i(p_{i,j}) \cdot \prod_{k=1}^N \Omega_{w_{k}^{-1}}(t_{k}) = Q[1]^{c-e_m} \cdot \mu_{w_1} \cdots \mu_{w_N} \text{ on } H^*(\mathcal{H}_d-e_m).
\]
Therefore, we have shown that
\[
(Q[1]^{c} \cdot \mu_{w_1} \cdots \mu_{w_N})_{d-e_m} = (Q[1]^{c-e_m} \cdot \mu_{w_1} \cdots \mu_{w_N})_{d-e_m}.
\]
Since \( c - e_m < c \), the induction hypothesis on \( c \) implies the result. \( \square \)
11. QUANTUM COHOMOLOGY OF PARTIAL FLAG MANIFOLDS

In this section, we give the necessary ingredients to extend the arguments to the quantum cohomology ring of partial flag manifolds. Let $N$ be the set $\{1 \leq n_1 < \ldots, n_m < n_{m+1} = n\}$. Let $F^N$ denote the partial flag variety corresponding to flags of the form:

$$V_1 \subset V_2 \subset \ldots \subset V_m \subset V = \mathbb{C}^n$$

with $\dim V_{m+1-i} = n - n_i$. There is a universal sequence of vector bundles on $F^N$:

$$V_{F^N} \rightarrow Q_m \rightarrow \ldots \rightarrow Q_1$$

with rank $Q_j = n_j$, and each $Q_j \rightarrow Q_{j-1}$ a surjection, where $Q_0 := 0$ and $Q_{m+1} := V_{F^N}$.

11.1. Classical cohomology of $F^N$. We review the ordinary cohomology of $F^N$. Let $S^{(N)} = \{w \in S_n : w(i) < w(i+1) \text{ for } i \notin N\}$. For $1 \leq l \leq m+1$, let $x_{n_j-1+1}, \ldots, x_{n_j}$ be the Chern roots of the bundle $F_l := \ker(Q_j \rightarrow Q_{j-1})$, and let $\sigma^l_i$ be the $i$th Chern class of $F_j$ for $1 \leq i \leq n_j - n_j-1$. Then $e_k(n_l)$, the $k$th elementary symmetric polynomial in $x_1, \ldots, x_{n_j}$, is symmetric in $x_{n_j-1+1}, \ldots, x_{n_j}$ for every $1 \leq j \leq l$, and can thus be written as a polynomial in the $\sigma^l_i$, which we denote by $\tilde{e}_k(l)(\sigma)$ or $\tilde{e}_k(l)$.

Let the polynomials $\mathfrak{S}_w(x)$ be as defined in section [3]. Then it is a fact that for $w \in S^{(N)}$, $\mathfrak{S}_w(x)$ can be written as a polynomial in $\sigma^l_i$, which we write $\mathfrak{S}_w(\sigma)$. For $w \in S^{(N)}$, consider the degeneracy locus

$$\Omega_w(V^\bullet) = \{U \in F^N : \text{rank}_{U_p}(V_p \otimes O_{F^N} \rightarrow Q_q) \leq r_w(q,p) \}

\text{for } 1 \leq p \leq n, q \in N\}.

This is a codimension $l(w)$ subvariety in $F^N$ whose class is independent of choice of flag, and is denoted by $[\Omega_w]$. Let $w^0 \in S^{(N)}$ be the element of longest length, and let $w^\circ = w^0w$. We have the following classical results.

**Theorem 11.** The classes $\{[\Omega_w]\}_{w \in S^{(N)}}$ form an additive basis for $H^*(F)$. Furthermore, for $w \in S^{(N)}$, the Schubert classes $[\Omega_w]$ and $[\Omega_w^\circ]$ are Poincaré dual.

**Theorem 12.** $H^*(F^N, \mathbb{Z}) \cong \mathbb{Z}[\sigma^1_1]/(\tilde{e}_1(m+1), \ldots, \tilde{e}_n(m+1))$.

**Theorem 13.** For $w \in S^{(N)}$, $[\Omega_w] = \mathfrak{S}^{(N)}_w(\sigma)$ in $H^*(F^N)$.

The quantum cohomology ring is defined as in section [3] with deformation variables $q_1, \ldots, q_m$, where $q_i$ is of degree $n_{i+1} - n_i-1$. For $d = (d_1, \ldots, d_l)$, the corresponding hyperquot scheme $H_{Q_d} = H_{Q_d}(F^N)$ parametrizes flat families of successive quotients of $V_{P^1}$ of rank $n - n_i$ and relative degree $d_{m+1-i}$. There is a universal sequence of sheaves:

$$A_1 \hookrightarrow A_2 \hookrightarrow \ldots \hookrightarrow A_m \hookrightarrow V_{P^1} \times H_{Q_d} \rightarrow B_m \rightarrow \ldots \rightarrow B_1$$

where $A_i$ is locally free of rank $n_i$, and $A_i \rightarrow A_{i+1}$ is an injection of sheaves, not bundles. The subschemes $\overline{\Omega}_w$ of $P^1 \times H_{Q_d}$ are defined by the appropriate degeneracy conditions. We denote by $\mu_w$ the class of $\overline{\Omega}_w(t)$, viewed as a subscheme of $H_{Q_d}$. Details can be found in [3-P2].
11.2. Quantum Schubert polynomials. Let \( g_{i}[j], c_{k}(l), \) and \( \mathfrak{S}_w(c) \) be as in section 6.1. Define \( \mathfrak{S}_w^{(N)}(c) \) to be the polynomial resulting from replacing \( c_{i}(j) \) with \( c_{i}(n_{k}) \) for \( n_{k} \leq j < n_{k+1} \).

Define two sets:

\[
\begin{align*}
g_{\sigma}(N) &= \{(i, j) : n_{l-1} + 1 = i \leq j \leq n_{l} \text{ for some } 1 \leq l \leq m + 1\} \\
g_{q}(N) &= \{(i, j) : 1 \leq i \leq j = n_{l} \text{ for some } 1 \leq l \leq m + 1\}.
\end{align*}
\]

Define \( \tilde{E}_{k}(l)(g) \) to be the result upon setting \( g_{i}[j-i] = 0 \) for all \( (i, j) \notin g_{\sigma}(N) \cup g_{q}(N) \) in \( E_{k}(l)(g) \). Define \( \mathfrak{S}_w^{(N)}(g) \) to be the polynomial resulting from the substitution \( c_{k}(n_{l}) = \tilde{E}_{k}(l)(g) \) into \( \mathfrak{S}_w^{(N)}(c) \). Then the polynomial rings generated by the \( c_{k}(n_{l}) \) and the \( g_{i}[j-i] \) are the same, so that each \( c_{k}(n_{l}) \) can be written in terms of the \( g_{i}[j-i] \), and vice versa.

An application of the degeneracy locus formula in Remark 3.8 of [Fu] gives \( \mu_{w} = (\Omega_{w}(t))|_{C} = \mathfrak{S}_{w}^{(N)}(Q) \), where \( C_{k}(n_{l}) = c_{k}(n_{l}, A_{l}^{*}) \) and \( Q_{l}[j-i] \) are defined by the correspondence between \( \mathbb{Z}[c] \) and \( \mathbb{Z}[g] \). As a purely algebraic fact, we have:

**Lemma 11.** Given a sequence of vector bundles on a scheme \( X \)

\[
E_{m+1} \to E_{m} \to \cdots \to E_{1}
\]

with \( E_{l} \) of rank \( n_{l} \) and \( c_{k}(E_{l}) = \tilde{E}_{k}(l)(g) \), we have

\[
c_{j}(\ker(E_{l+1} \to E_{l})) = \begin{cases} 
g_{n_{l-1}+1}[j-1] & \text{for } 1 \leq j < n_{l} - n_{l-1} \\
g_{n_{l+1}-j+1}[j-1] & \text{for } n_{l} - n_{l-1} \leq j \leq n_{l+1} - n_{l-1}
\end{cases}
\]

11.3. The results. As in the case of complete flag manifolds, the presentation of the quantum cohomology ring and quantum Giambelli formula follow from a Main Proposition, as stated in section 6, and Lemmas 3 and 8, whose proofs carry through unchanged. The difference lies in the inductive step in the proof of Proposition 3.

For the remainder of this section, \( P(g) \) denotes a polynomial in variables \( g_{i}[j-i] \), for \( (i, j) \notin g_{\sigma}(N) \cup g_{q}(N) \), with \( P^{\text{h}}(g) \) as in section 8. Let \( D = \sum_{j=1}^{m} d_{j}(n_{j} - n_{j-1}) \) so that \( \dim \mathcal{Q}_{d} = \dim F + D \). Denote by \( P(\sigma, q) \) the polynomial that results from the substitutions

\[
g_{n_{l-1}+1}[j-1] = \sigma_{j}^{l} \text{ when } 1 \leq j \leq n_{l} - n_{l-1},
\]

\[
g_{n_{l+1}-j+1}[n_{l+1} - n_{l-1} - 1] = (-1)^{n_{l+1}-n_{l-1}}q_{l},
\]

for \( 1 \leq l \leq m + 1 \), and all other \( g_{i}[j] = 0 \). Denote \( \tilde{e}_{k}^{P}(l) = \tilde{E}_{k}(l)(\sigma, q) \). With this notation, we have \( \mathfrak{S}_{w}^{(N)}(\sigma, q) \) equal to the quantum Giambelli polynomials as defined in [C-F2]. The results of the paper are as follows:

**Theorem 14.**

\[
QH^{*}(\mathbb{F}^{N}, \mathbb{Z}) \cong \mathbb{Z}[\sigma_{1}^{j}, q_{1}, \ldots, q_{m}]/(\tilde{e}_{1}^{P}(m+1), \ldots, \tilde{e}_{m}^{P}(m+1))
\]

**Theorem 15.** For \( w \in S^{(N)} \), \( \sigma_{w} = \mathfrak{S}_{w}^{(N)}(\sigma, q) \) in \( QH^{*}(\mathbb{F}^{N}) \).
11.4. Proof. We first note that the proofs of the partial flag versions of Lemmas 8 and 9 carry through unchanged, except for the last equality of the proof of Lemma 8, which now reads
\[ G_w^k(\sigma, q) + \sum (-1)^{n_l+1-n_l+1} q_i = G_w^{k+1}(\sigma, q) = G_w(\sigma, q), \]
where the sum is over \((i, k+1) = (n_{j-1} + 1, n_{j+1} - n_j - 1)\).

The Main Proposition follows immediately from Proposition 11, whose inductive step is given by Lemmas 8 and 9 and the following two propositions. The base case is given by the same arguments as in Proposition 5.

**Proposition 10.** Assume that Theorem 4 holds for \( k = n_{l+1} - n_{l-1} - 1 \). Then
\[ (Q_{n_{l-1}+1}[n_{l+1} - n_{l-1} + 1] \cdot \mu_{w^0})e_l = (-1)^{n_{l+1}-n_{l-1}+1}. \]

This is the analog of Proposition 10 in section 8. The proof is found in section 11.5.

**Proposition 11.** For \((i, j) \in g_q(N), 1 \leq l \leq m + 1, \)
\[ \sum_{d, w} q^d(Q_{\delta i - j} \cdot \mu_w)d\sigma_{w^0} = \begin{cases} (-1)^{n_{l+1} - n_{l-1} + 1} q_l & \text{if } (i, j) = (n_{l-1} + 1, n_{l+1}) \\ 0 & \text{otherwise}. \end{cases} \]

Proof. As in the complete flag case, the proof is based on dimension counts and an understanding of bundle maps. Consider \((i, j)\) and \(d\) such that the intersection number \(Q_{\delta i - j} \cdot \mu_w\) in \(HQ_d\) is nonzero. By dimension considerations, this implies that \(j - i + 1 \geq D\). Let \(l, l'\) be the unique integers so that \(j = n_{l+1}\) and \(n_{l'} + 1 \leq i < n_{l'-1}\), which implies that \(l' \leq l + 1\). Then \(d_i \neq 0\) for \(l' + 1 \leq i < l\), so that \(D = \sum d_i(n_{l+1} - n_{l-1}) \geq n_{l+1} + n_l - n_{l'} - n_{l'-1}\).

Since \(D \leq j - i + 1 \leq n_{l+1} - n_{l'}, n_l - n_{l'-1} \leq 0\), each inequality must be an equality, so that \(l' = l - 1, i = n_{l-1} + 1, \) and \(d_i \neq 0\). By dimension considerations, \(d = e_l\) and \(w = w^0\), the permutation of longest length. Since \((w^0)^c = id\), Proposition 11 concludes the proof.

11.5. Proof of Proposition 10. The proof uses some of the constructions and results of [C-F2], including:
\[ h^{-1}(\Omega_w(t)) = \pi^{-1}(\mathbb{P}^1 \times \Omega_w(t)) \cup \Omega_w(t) \]
with \(\Omega_w(t)\) of codimension \(l(\tilde{w}^0)\) in \(U_e(t)\), with \(l(w) - l(\tilde{w}^0) \leq \sum c_i(n_i - n_{i-1})\).

Denote by \(A_j\) the locally free sheaf \(A_j \otimes \mathcal{O}_{\mathbb{P} \times HQ_q}\) for any fixed point \(p \in \mathbb{P}^1\). For the permutations \(\alpha_{i,j} = s_{n_{j-1}+1} \cdots s_{n_j}\) and \(\beta_{i,j} = s_{n_{j-1}+1} \cdots s_{n_j}\) in \(S(N)\), we have \(\mu_{\alpha_{i,j}} = c_i(A_j^*)\) and \(\mu_{\beta_{i,j}} = c_i(-A_j)\), and:
\[ \alpha_{i,j}^e_l = \begin{cases} \alpha_{i,j} & \text{if } j \neq l, \\ \alpha_{i,l}s_{n_l} & \text{if } j = l. \end{cases} \]
\[ \beta_{i,j}^e_l = \begin{cases} \beta_{i,j} & \text{if } j \neq l, \\ id & \text{if } j = l, 1 \leq i \leq n_l - n_{l-1}, \\ \beta_{i,l}s_{n_l} \cdots s_{n_{l+1}} & \text{if } j = l, n_l - n_{l-1} < i. \end{cases} \]

where \(e_l\) is the multiindex with all zeros except a 1 at the \(i\)th position. We have the following two lemmas:
Lemma 12. For $j \geq 1$, $t \neq t' \in \mathbb{P}^1$,
\[ h_{e_t}(U_{e_t}) \cap \overline{\Omega}_{\alpha_{n_l-n_l-1+j,l}}(t) \cap \overline{\Omega}_{w}(t') = \emptyset. \]

Lemma 13. For $j \geq 1$, $t_1 \in \mathbb{P}^1$, $t_2 \neq t' \in \mathbb{P}^1$,
\[ h_{e_{t_1}}(U_{e_{t_1}}) \cap \overline{\Omega}_{\alpha_{j,t+1}}(t_1) \cap \overline{\Omega}_{\alpha_{n_l-n_l-1;j,l}}(t_2) \cap \overline{\Omega}_{w}(t') = \emptyset. \]

Proof. The lemmas follow from the fact that $\dim U_{e_t} = \dim F + n_l - n_l - 1$ and dimension counts based on the following: For any $w \in S$, $\pi^{-1}(\mathbb{P}^1 \times \overline{\Omega}_w(t))$ is codimension $l(w)$ in $U_{e_t}$ and $\overline{\Omega}_w(t)$ is codimension $l(\overline{\omega})$ in $U_{e_{t'}}$. In particular, $\overline{\Omega}_w(t)$ is codimension $F - 1$ in $U_{e_t}(t)$ and hence codimension $dim F$ in $U_{e_{t'}}$. Furthermore, $\overline{\Omega}_{\alpha_{j,t+1}}(t)$ is codimension $l(\overline{\alpha}_{j,t+1}) = i - 1$ in $U_{e_{t'}}(t')$.

Setting $i = n_l - n_l - 1 + j \geq n_l - n_l - 1 + 1$ gives the proof of Lemma 13.

As a consequence of Lemma 13, we have
\begin{equation}
(17) \quad c_{n_l+1-n_l-1}(A_{t+1}^*) = \sum_{j=0}^{n_l-n_l-1} Q_{n_l+1-j}[n_l+1-n_l-1+j]c_{n_l-n_l-1-j}(A_t^*)
+ \sum_{j=1}^{n_l-n_l-1} Q_{n_l+j+1}[n_l+1-n_l-1-j-1]c_{n_l-n_l-1+j}(A_t^*).
\end{equation}

Recall that $c_k(A_t^*) = \mu_{\alpha_k,l}$. Then intersection of the left hand side with $\mu_{w,0}$ in $H^*(\mathcal{H}Q_{e_t})$ is zero by the argument in Lemma 3. For $1 \leq j \leq n_l - n_l - 1$, the intersection of the terms in the first sum with $\mu_{w,0}$ are zero by the assumption on $k$. The intersection of the terms in the final sum with $\mu_{w,0}$ are zero by Lemma 13.

Since $Q_{n_l+1}[n_l+1-n_l-1] = (-1)^{n_l+1-n_l} c_{n_l+1-n_l}(A_{t+1}/A_t)$ by Lemma 13, we have the equality
\begin{equation}
(18) \quad Q_{n_l+1}[n_l+1-n_l-1] = \sum_{i=0}^{n_l+1-n_l} (-1)^i \mu_{\alpha_{n_l+1-n_l-i,l+1}} \mu_{\beta_{i,l}}.
\end{equation}

For $t \neq t'$, by Lemma 6.2 and Proposition 6.3 of [C-F2], the intersection
\[ \overline{\Omega}_{\alpha_{n_l+1-n_l-l+1}}(t) \cap \overline{\Omega}_{\beta_{i,l}}(t) \cap \overline{\Omega}_{\alpha_{n_l-n_l-1,l}}(t) \cap \overline{\Omega}_{w}(t') \]
lies in $h_{e_t}(U_{e_t})$ and is equal to the intersection number
\[ \mu_{\alpha_{n_l+1-n_l-i,l+1}} \cdot \mu_{\beta_{i,l}} \cdot \mu_{\alpha_{n_l-n_l-1,l}} \cdot \mu_{w,0} \]
in $H^*(\mathcal{H}Q_{e_t})$. By Lemma 13, this is zero for $1 \leq i < n_l+1 - n_l$.

Therefore, after substituting (18) into the intersection of (17) with $\mu_{w,0}$, we get the equality
\[ 0 = (Q_{n_l+1}[n_l+1-n_l-1] \cdot \mu_{w,0})_{e_t} + (-1)^{n_l+1-n_l} (\mu_{\beta_{n_l+1-n_l,l}} \cdot \mu_{\alpha_{n_l-n_l-1,l}} \cdot \mu_{w,0})_{e_t} \]
where the last intersection can also be computed as the Gromov-Witten number
\begin{equation}
\langle \Omega_{\beta_{n_{l+1}-n_l},1}, \Omega_{\alpha_{n_l-n_{l-1}},1}, \Omega_{w^0}\rangle_{\mathfrak{e}_l}.
\end{equation}

This is a direct computation on the space of lines on $\mathbb{P}^N$:

\begin{align*}
\Omega_{\beta_{n_{l+1}-n_l},1}(A_{\bullet}) &= \{ L_\bullet : \dim(L_\bullet \cap A) \geq n_{l+1} - n_l \} \\
\Omega_{\alpha_{n_l-n_{l-1}},1}(B_{\bullet}) &= \{ L_\bullet : \dim(L_\bullet \cap B) \geq 1 \} ; \text{ and} \\
\Omega_{w^0}(C_{\bullet}) &= \{ L_\bullet : L_\bullet = C_\bullet \}
\end{align*}

where $\dim L_\bullet = \dim C_\bullet = n_i$, $\dim A = n_{l-1} + 1$, and $\dim B = n_{l+1} - 1$.

A line on $\mathbb{P}^N$ is of the form

\begin{equation}
\{ L_\bullet : L_\bullet = D_i \text{ for } i \neq l, E \subset L_i \subset E' \}
\end{equation}

where $D_i, E,$ and $E'$ are fixed subspaces of dimension $i$, $n_l - 1$, and $n_l + 1$. There is exactly one such line passing through these three Schubert varieties, given by $D_i = C_i$ for $i \neq l$, $E = C_l \cap B$, and $E' = (A, C_l)$. Therefore, the number in the intersection is one, and

\begin{equation}
(Q_{n_{l-1}+1}[n_{l+1} - n_l - 1] \cdot \mu_{w^0})_{\mathfrak{e}_l} = (-1)^{n_{l+1} - n_l} = (-1)^{n_{l+1} - n_l + 1}
\end{equation}

as needed.

This is equivalent to the single computation giving the relations of the quantum cohomology ring of the Grassmannian $\mathbb{P}^N$. Alternatively, by Lemma 6.2 and Proposition 6.3 of [C-F2], the intersection can be computed as the length of the zero-dimensional scheme

\begin{equation}
\overline{\Omega}_{\beta_{n_{l+1}-n_l},1}(t) \cap \overline{\Omega}_{\alpha_{n_l-n_{l-1}},1}(t) \cap \overline{\Omega}_{w^0}(t') \subset \mathfrak{e}_l(U_{\mathfrak{e}_l}),
\end{equation}

or as $\deg([\overline{\Omega}_{\beta_{n_{l+1}-n_l},1}(t)] \cdot \pi^*[t \times \overline{\Omega}_{w^0}(t')])$ in $U_{\mathfrak{e}_l}$. We can use the projection formula, and the construction of $U_{\mathfrak{e}_l}$ via a projective bundle over $\mathbb{P}^N$ to obtain the result.

This concludes the proof of Proposition 14, and hence of the Main Proposition and the results of the paper.

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