CYCLE FOR INTEGRATION FOR ZONAL SPHERICAL FUNCTION OF TYPE $A_n$

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Abstract. Integral of a certain multivalued form over cycle $\Delta$ provides zonal spherical function of type $A_n$. This paper is devoted to quantum group analysis and verification of monodromy properties of the distinguished cycle $\Delta$. Zonal spherical function is a particular conformal block of $WA_n$-algebra.

0.0 Notations.

$\alpha_1, \alpha_2, \ldots, \alpha_n$ - simple roots of root system of type $A_n$
$\mathcal{R}_+$ - set of positive roots
$\mathcal{R}$ - root system of type $A_n$
$\delta = \frac{1}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha$ - halfsum of positive roots
$k$ - complex parameter (‘halfmultiplicity’ of a root)

$$\rho = \frac{k}{2} \sum_{\alpha \in \mathcal{R}_+} \alpha$$

$x = -\frac{1}{k}$
$q = \exp\left(\frac{2\pi i}{x}\right)$
$\eta_1, \eta_2, \ldots, \eta_n$ are fundamental weights: $(\eta_i, \alpha_j) = \delta_{ij}$
$h_1, h_2, \ldots, h_{n+1}$ are weights of the vector representation of $sl(n+1)$
$h_1 = \eta_1$
$h_2 = h_1 - \alpha_1, h_3 = h_1 - \alpha_1 - \alpha_2, \ldots, h_{n+1} = h_1 - \alpha_1 - \ldots - \alpha_n$

$$h_1 + h_2 + \ldots + h_{n+1} = 0$$

$R$ - $R$-matrix
$\Delta$ - distinguished cycle for integration for zonal spherical function of type $A_n$
$\Delta$ - comultiplication in quantum group
0. Introduction

Let $G$ be a connected real semisimple Lie group with finite center, $K$ - its maximal compact subgroup, $G/K$- Riemannian symmetric space. Let $T = T^c$, $g \in G$ be a continuous unitary representation of $G$ acting in a Hilbert space $H$, which contains a spherical vector $\xi$, i.e. $K\xi = \xi$ and assume that $(\xi, \xi) = 1$. $\lambda$ is a parameter defining this representation. Then the function $\phi_{\lambda}(g) = (T^c_\lambda \xi, \xi)$ is called zonal spherical function [3]. In particular, $\phi_{\lambda}(e) = 1$ and it is right and left $K$-invariant. Zonal spherical function is a common eigenfunction of Laplace-Casimir operators:

$$\mathcal{L}\phi_{\lambda}(g) = \gamma(\mathcal{L})(i\lambda)\phi_{\lambda}(g)$$

where $\gamma(\mathcal{L})(i\lambda)$ is a homomorphism of Laplace-Casimir operators into complex numbers. Using Cartan decomposition $G = KAK$ zonal spherical function $\phi_{\lambda}(g)$ is considered as a function on $A$. Let also $a = Lie(A)$, then $\lambda \in a^*$. Restriction of zonal spherical function to $A$ is a common eigenfunction of radial parts $\hat{\mathcal{L}}$ of Laplace-Casimir operators $\mathcal{L}$:

$$\hat{\mathcal{L}}(\phi_{\lambda}(a)) = \gamma(\hat{\mathcal{L}})(i\lambda)\phi_{\lambda}(a)$$

where $\gamma$ is Harish-Chandra homomorphism, $a \in A$. Among others the operator of second order plays the predominant role:

$$\hat{\mathcal{L}}_2 = H^2_1 + \ldots + H^2_r + \sum_{\alpha \in \Phi^+} m_\alpha \frac{e^\alpha + e^{-\alpha}}{e^\alpha - e^{-\alpha}} H_\alpha$$

where $m_\alpha$ is a multiplicity of restricted root $\alpha$ [6]. It turns out that one can consider $m_\alpha$ as parameters, but the condition $m_\alpha = m_{w_\alpha}$ is required. So there are as many independent parameters as orbits of the Weyl group in restricted root system. One may start with second order differential operator with generic parameters, then recover the whole system of differential operators (radial parts)[41,20]. This system turns out to be holonomic, locally it has $|W|$-dimensional space of solutions, where $|W|$ is the cardinality of the Weyl group $W$, cf. corollary 3.9 of [20]. Among those solutions there is a distinguished one which corresponds to zonal spherical function. It is characterized by analyticity at unity cf. theorem 6.9 [20].

We restrict ourselves to the case of root system of type $A_n$. For the second order differential operator we use

$$\hat{\mathcal{L}}_2 = \sum_i \left( z_i \frac{\partial}{\partial z_i} \right)^2 - k \sum_{i<j} \frac{z_j + z_i}{z_j - z_i} \left( z_i \frac{\partial}{\partial z_i} - z_j \frac{\partial}{\partial z_j} \right) .$$
In [17] we provided an integral representation for the solutions of system (1), in [18] we described cycle \( \Delta \) for integration for zonal spherical function and obtained an explicit version of Harish-Chandra decomposition.

This paper is devoted to quantum group analysis and verification of monodromy properties of distinguished cycle \( \Delta \) (cycle \( \Delta \) is recalled in 2.1 below). Cycle \( \Delta \) is a contour for integration for zonal spherical function of type \( A_n \) of a suitable multivalued form. The form is of type considered in refs. [8,11] and thus all the machinery including quantum groups and R-matrices can be applied. We obtained the form using very simple principle: there is only one line which passes through two given points. Zonal spherical functions for \( SL(n, \mathbb{C}) \) are calculated by I. Gelfand and M. Naimark and using the same method can be calculated for \( SL(n, \mathbb{R}) \) (see [1]). This provides us with the form for parameters \( k = 1 \) and \( k = \frac{1}{2} \), where \( k \) is a half-multiplicity of restricted root. Now use the principle and extend powers of factors linearly on \( k \). The obtained form has several advantages: cycle for integration for zonal spherical function is real and compact; the number of variables of integration is independent on parameter \( \lambda \) (appearance of the flag manifold); there is no complicated meromorphic factor.

Note: in refs. [15,54] the system (1) is proved to be related to a particular case of trigonometric Knizhnik-Zamolodchikov equation. In particular, this implies that solutions to the system (1) can be obtained from the solutions of Knizhnik-Zamolodchikov equations by symmetrization procedure. Solutions to Knizhnik-Zamolodchikov equations are given by certain multidimensional integrals, whose integrand has the standard part times complicated meromorphic factor. This complicated meromorphic factor becomes even more complicated after symmetrization. We would like to emphasize that in this particular case this unpleasant meromorphic factor is not needed, cf. [17], theorem 6.3.

Knizhnik-Zamolodchikov equations are originated in WZNW theory. Reduction from WZNW to \( WA_n \)-algebras is well discussed in the literature [75,63] (quantum Drinfeld-Sokolov reduction). Results of [17] (absence of meromorphic factor) imply that solutions to the system (1) (which is isomorphic to Calogero-Sutherland model) are provided by the conformal blocks of \( WA_n \)-algebra.

Quantum group approach assumes the following. With the multivalued form one associates tensor product of Verma modules over quantum group. Homology of certain type of discriminantal configuration [8,71] are described by the singular vectors of the tensor product of irreducible highest weight modules. Half-monodromy (=braiding) is given by the R-
matrix (PR, where P is a permutation). Universal R-matrix is provided by the Drinfeld’s double [26].

Here is the organization of the paper. In sections 1-3 we recall the multivalued form, distinguished cycle $\Delta$, and normalization constant of ref. [16]. In section 4 we recall the version of quantum group used in refs. [8, 11] for the explicit version of Kohno’s theorem: half-monodromy=R-matrix [28]. In section 5 we encode the distinguished cycle $\Delta$ as an element of the corresponding tensor product and check the monodromy properties. Cycle $\Delta$ has the meaning of q-antisymmetric tensor, cf. theorem 5.7 below, and corresponds to a particular conformal block, fig. 10. Finally, we discuss the properties of the tensor product with vector representation.

1. MULTIVALUED FORM AND DISCRIMINANTAL CONFIGURATION

Consider the following set of variables:

$$z_l, \quad l = 1, \ldots , n + 1, \quad t_{ij}, \quad i = 1, \ldots , j, \quad j = 1, \ldots , n.$$ Variables $z_l$ have meaning of arguments, while variables $t_{ij}$ are variables of integration.

It is convenient to organize variables $z_l$, $t_{ij}$ in the form of a pattern, cf. fig 1. The idea of such an organization is borrowed from Gelfand-Zetlin patterns [7].

$$
\begin{array}{cccc}
z_1 & z_2 & \cdots & \cdots & z_{n+1} \\
t_{1,n} & t_{2,n} & \cdots & t_{n,n} \\
\cdots & \cdots & \cdots \\
t_{1,2} & t_{2,2} \\
t_{1,1}
\end{array}
$$

**Figure 1.** Variables organized in a pattern

**Definition 1.1.** Consider the following multivalued form $\omega(z, t)$:
\[ \omega(z, t) := \prod_{i=1}^{n+1} z_{i1}^{\lambda_i + \frac{k}{2}} \prod_{i_1 > i_2} (z_{i1} - z_{i2})^{1-2k} \times \prod_{i=1}^{n+1} \prod_{i=1}^{n} \prod_{i_1 = 1}^{n-1} (z_{i} - t_{i, n})^{k-1} \times \prod_{j=1}^{n} \prod_{i_1 = 1}^{n-1} \prod_{i_1 > i_2} ((t_{i_1, j} - t_{i_2, j+1})^{k-1} \times \prod_{j=1}^{n} \prod_{i=1}^{n} \prod_{i_1 = 1}^{n} (t_{i_1, j} - t_{i_2, j})^{2-2k} \times \prod_{i=1}^{n} \prod_{j=1}^{n} (t_{i_1, j}^{\lambda_{i_1, j+2} - \lambda_{i_1, j+1} - k} \ dt_{i_1 11} dt_{i_1 12} dt_{i_2 22} \ldots dt_{nn}) \]

Here \( k \) is a complex parameter - ‘halfmultiplicity’ of a root, \( \lambda_1, \ldots, \lambda_{n+1} \) are complex parameters subject to the homogeneity condition:

\[ \lambda_1 + \lambda_2 + \ldots + \lambda_{n+1} = 0 \]

Before proceeding further we would like to make a convention.

**Convention 1.2.** A complex number \( z \) can be represented as \( z = re^{i\alpha} \), where \( r, \alpha \) are real numbers, \( r \geq 0 \). \( r \) is called absolute value of \( z \), while \( \alpha \) is called the phase of \( z \). When we say that the phase of a complex number \( z \) is equal to 0, we mean that \( \alpha = 0 \), or the number itself is real and nonnegative.

**1.3 Configuration.**

Let \( m = \frac{n(n+1)}{2} \). Consider \( (n + 1 + m) \)-dimensional complex space \( \mathbb{C}^{n+1+m} \) with coordinates \( z_1, z_2, \ldots, z_{n+1}, t_{11}, t_{12}, t_{22}, \ldots, t_{nn} \). Let’s delete the following hyperplanes:

\[ t_{i_1, j} - t_{i_2, j} = 0 \quad i_1 < i_2, \ j = 1, \ldots, n \]
\[ t_{i_1, j} - t_{i_2, j+1} = 0 \quad j = 1, \ldots, n-1 \]
\[ z_{i_1} - t_{i_2, n} = 0 \quad i_1 = 1, \ldots, n+1; \ i_2 = 1, \ldots, n \]
\[ t_{i_1} = 0 \quad i_1 = 1, \ldots, j; \ j = 1, \ldots, n \]
\[ z_{i} - z_{j} = 0 \quad i < j \]
Denote the complement of $\mathbb{C}^{n+1}$ to the union of above hyperplanes by $U_{n+1+m}$.

Denote by $\text{Loc}_g$ the trivial 1-dimensional bundle over $U_{n+1+m}$ with the integrable connection $\nabla_g$ with the connection form

$$\sum_{i} (k - 1) \frac{d(t_{jn} - z_{i})}{t_{jn} - z_{i}} + \sum_{i} (1 - 2k) \frac{d(z_{i} - z_{j})}{z_{i} - z_{j}} + \sum_{i} \left( \lambda_1 + \frac{k n}{2} \right) \frac{dz_{i}}{z_{i}} + \sum_{i} (2 - 2k) \frac{d(t_{i1,j} - t_{i2,j})}{t_{i1,j} - t_{i2,j}} + \sum_{i} (\lambda_{n+2-j} - \lambda_{n+1-j} - k) \frac{dt_{ij}}{t_{ij}}$$

Denote by $S_{\mu}$ the local system of horizontal sections of $\nabla_{\mu}$. Consider the projection on the first $n+1$ coordinates $\mathbb{C}^{n+1+m} \rightarrow \mathbb{C}^{n+1}$. For $z = (z_1, z_2, \ldots, z_{n+1})$ such that $z_i \neq z_j$ for all $i < j$ set

$$U(z) = \left\{ (\hat{z}, t) \in U_{n+1+m} | \hat{z} = z \right\}.$$ 

Restrictions of $\text{Loc}_{\mu}, S_{\mu}$ to $U(z)$ are denoted by $\text{Loc}_{\mu}(z), S_{\mu}(z)$. Denote by $S^{*}$ the dual local system and consider homology of $U(z)$ with coefficients in $S^{*}$ (extended by $!$ as explained in ref. [8]). Configuration is preserved under the action of the product of symmetric groups :

$$\Sigma = S_n \times S_{n-1} \times \ldots \times S_2,$$

where $S_j$ permutes $t_{ij}, t_{2j}, \ldots, t_{jj}$. Then one considers the antiinvariant part of the homology group with respect to the action of $\Sigma$: $H_{\ast,m}(U, S^{*})^{-}$.

Remarkably, in order to calculate the cohomology group of the local system of the complement to finite set of hyperlanes in the nonresonance case one can use finite-dimensional complex of hypergeometric forms in the spirit of Arnold-Orlik-Solomon, cf. [12,67].

2. The distinguished cycle $\Delta$

Assume that $z_1, z_2, \ldots, z_{n+1}$ are real and

$$0 < z_1 < z_2 < \ldots < z_{n+1}.$$ 

Definition 2.1. Define cycle $\Delta = \Delta(z)$ by the following inequalities:

$$t_{i,j+1} \leq t_{ij} \leq t_{i+1,j+1}$$

and $z_i \leq t_{in} \leq z_{i+1}$ cf. [16], definition 2.1.

Define form $\omega_{\Delta}(z,t)$ as:
\[
\omega_{\Delta}(z, t) := \prod_{i=1}^{n+1} \frac{z_i + \frac{\lambda_i z_{i+1}}{z_{i+1}}}{z_i + \frac{\lambda_i}{z_{i+1}}} \prod_{i_1 > i_2} (z_{i_1} - z_{i_2})^{1-2k} \\
\times \prod_{i \leq l} (z_l - t_{i,n})^{k-1} \prod_{i > l} (t_{i,n} - z_l)^{k-1} \\
\times \prod_{j=1}^{n-1} \prod_{i_1 > i_2} (t_{i_1,j} - t_{i_2,j+1})^{k-1} \prod_{i_2 \geq t_1} (t_{i_2,j+1} - t_{i_1,j})^{k-1} \\
\times \prod_{j=1}^{n} \prod_{i_1 > i_2} (t_{i_1,j} - t_{i_2,j})^{2-2k} \\
\times \prod_{j=1}^{n} \prod_{i_1 > i_2} t_{i_1,j}^{\lambda_{n-j+2} - \lambda_{n-j+1} - k} \\
\int_{\Delta(z)} dt_{11} dt_{12} dt_{22} \ldots dt_{nn}
\]

It is assumed that phases of factors in the formula for \(\omega_{\Delta}\) are equal to zero if \(k\) and \(\lambda_1, \lambda_2, \ldots, \lambda_{n+1}\) are real. In other words we choose the section of the local system to be positive over \(\Delta\) if \(\lambda, k\) are real. This geometric definition of cycle \(\Delta\) is justified by theorem 5.7 below, i.e. \(\Delta\) is really an element of \(H_{\mathfrak{t},m}(U, S^*)\). This geometric definition is motivated by the classical calculations of Gelfand and Naimark of zonal spherical function for \(SL(n, \mathbb{C})\).

3. Analytic considerations

Let

\[\lambda_{n-j+2} - \lambda_{n-j+1} = k = 0\]

for \(j = 1, \ldots, n\) and \(\lambda_1 + \frac{\lambda_n}{2} = 0\), i.e. we kill an affine part.

Then in these hypotheses

\[
\int_{\Delta(z)} \omega_{\Delta}(z, t) = \frac{\Gamma(k)\Gamma(k) \ldots \Gamma(k)^{n+1}}{\Gamma(k)\Gamma(2k) \ldots \Gamma((n+1)k)}
\]

cf. ref. [16] theorem 1.5 and remark 1.6.

Following the classical work of I.M. Gelfand and M.A. Naimark cf. [23], let

\[
\tau_{ij} = \prod_{i_1 \neq i} (t_{i_1,j-1} - t_{ij}) / \prod_{i_1 \neq i} (t_{i_1,j} - t_{ij})
\]
Note that \( j \sum_{i=1}^{j-1} \tau_{ij} = 1 \), and

\[
D(\tau_1, \ldots, \tau_{j-1}, \tau_j) = \prod_{1 \leq i < k \leq j} (t_{i,j-1} - t_{k,j-1}) \prod_{1 \leq i < p \leq j} (t_{ij} - t_{pj})
\]

[see [23] for the details]. Let also

\[
\tau_{i,n+1} = \frac{n}{\prod_{i_1 \neq i} (z_{i_1} - z_i)}
\]

One has \( \sum_{i=1}^{n+1} \tau_{i,n+1} = 1 \) and

\[
D(\tau_{1,n+1}, \ldots, \tau_{n,n+1}) = \frac{n}{\prod_{1 \leq i < k \leq n} (t_{i,n-1} - t_{k,n}) \prod_{1 \leq i < p \leq n+1} (z_i - z_p)}
\]

In variables \( \tau_{ij} \quad i = 1, \ldots, j-1, \quad j = 2, \ldots, n+1 \) integral \( \int_{\Delta} \omega_{\Delta} \) is written as:

\[
\int_{\Delta} \omega_{\Delta} = \int \prod_{j=1}^{n+1} (\tau_{ij} \tau_{2j} \ldots \tau_{j-1,j} (1 - \tau_{ij} - \ldots - \tau_{j-1,j}))^{k-1} \times d\tau_2 d\tau_3 d\tau_{23} \ldots d\tau_{1,n+1} \ldots d\tau_{n,n+1} .
\]

Remarkably in variables \( \tau_{ij} \) the integration is being performed over one-dimensional simplex times two-dimensional simplex times so on, times \( n \)-dimensional simplex. Here one-dimensional simplex corresponds to a line in two-dimensional plane, two-dimensional simplex corresponds to two-dimensional plane in three-dimensional plane and so on. Thus the integral remembers the flag manifold.

So using Dirichlet’s formula one gets

\[
\int_{\Delta} \omega_{\Delta} = \frac{\Gamma(k) \Gamma(k)^2 \ldots \Gamma(k)^{n+1}}{\Gamma(k) \Gamma(2k) \ldots \Gamma((n+1)k)}
\]

The constant does not depend on \( z_i \) at all and surely remains the same under analytic continuation. This is nontrivial since form \( \omega = \omega(z,t) \) and cycle \( \Delta = \Delta(z) \) do depend on \( z = (z_1, z_2, \ldots, z_{n+1}) \).

Remark 3.1. In view of sections 4 and 5 below, killing of affine part corresponds to erasing of the first factor in the tensor product of Verma modules.
4. Quantum group

Quantum groups are introduced by Drinfeld [26], Jimbo [27], Kulish, Reshetikhin, Sklyanin [37]. We briefly recall the necessary material from [8,11] and refer directly to these references for more details. See also [29].

4.1 Root system.

Let $\mathbb{R}^{n+2}$ be Euclidean $(n+2)$-dimensional vector space with inner product $(.,.)$ and with $g_0, g_1, \ldots, g_{n+1}$ as the orthonormal basis. Let’s realize simple roots of root system of type $A_n$ as $\alpha_i = g_i - g_{i+1}$ for $i = 1, \ldots, n$. Set also $\alpha_0 = g_0 - g_1$.

In particular, one has

$$(\alpha_i, \alpha_i) = 2$$

$$(\alpha_i, \alpha_j) = 0 \quad \text{for} \quad |i - j| > 1$$

$$(\alpha_i, \alpha_j) = -1 \quad \text{for} \quad |i - j| = 1$$

Set also

$$\alpha^\vee = \frac{2\alpha}{(\alpha, \alpha)}.$$

4.2. Quantum group $U_q(sl(n+2))$.

Consider $\mathbb{C}$-algebra with generators $e_i, f_i,$ and $K^\pm_i, K^\mp_i,$ subject to the relations:

$$K^\pm_j e_i = q^{(\alpha_i, \alpha_j)} e_i K^\pm_j$$

$$K^\pm_j f_i = q^{-\frac{(\alpha_i, \alpha_j)}{4}} f_i K^\pm_j$$

$$[e_i, f_j] = (K_i - K_i^{-1})\delta_{ij}$$

$$K^{\pm+}_i K^{\pm}_{-1} K^{\pm}_i = K^{\pm}_i K^{\pm}_{-1} K^{\pm}_i = 1$$

Comultiplication is defined by the rule

$$\Delta(K^{\pm+}_i) = K^{\pm+}_i \otimes K^{\pm+}_i$$

$$\Delta(f_i) = f_i \otimes K^{\pm+}_i + K^{\pm+}_i \otimes f_i$$

$$\Delta(e_i) = e_i \otimes K^{\pm+}_i + K^{\pm+}_i \otimes e_i$$
The following are the quantum Serre’s relations:

\[ f_i^2 f_{i+1} - (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) f_i f_{i+1} f_i + f_{i+1} f_i^2 = 0 \]

\[ f_{i+1}^2 f_i - (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) f_{i+1} f_i f_{i+1} + f_i f_{i+1}^2 = 0 \]

\[ f_i f_j = f_j f_i \quad \text{for} \quad |i - j| \neq 1 \]

\[ e_i^2 e_{i+1} - (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) e_i e_{i+1} e_i + e_{i+1} e_i^2 = 0 \]

\[ e_{i+1}^2 e_i - (q^{\frac{1}{2}} + q^{-\frac{1}{2}}) e_{i+1} e_i e_{i+1} + e_i e_{i+1}^2 = 0 \]

\[ e_i e_j = e_j e_i \quad \text{for} \quad |i - j| \neq 1 \]

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\[ e_i e_j = e_j e_i \quad \text{for} \quad |i - j| \neq 1 \]

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\[ e_i e_j = e_j e_i \quad \text{for} \quad |i - j| \neq 1 \]

\[ e_i e_j = e_j e_i \quad \text{for} \quad |i - j| \neq 1 \]
Example 1. \[ S : v \mapsto v^* \]

Example 2. \[ S : f_1 v \mapsto (q^{(\Lambda, \alpha_1)} - q^{-(\Lambda, \alpha_1)}) (f_1 v)^* , \]

see also fig. 2.

Let \( L(\Lambda) = M(\Lambda)/\text{Ker}S \)
be irreducible module with highest weight \( \Lambda \).

4.5. R-matrix. R-matrix is defined by the following expression:

\[ R = \sum_{\mu} q^{\Omega_0 + \frac{1}{2} (\mu \otimes 1 - 1 \otimes \mu) + d(\mu)} \Omega_{\mu} \]

cf. [26]. Here \( \Omega_0 \) is the element corresponding to the inner product \((, .)\);
for \( \mu = l_0 \alpha_0 + l_1 \alpha_1 + \ldots + l_n \alpha_n \),
where \( l_0, l_1, \ldots, l_n \) are nonnegative integers, \( d(\mu) \in \mathbb{C} \) is a constant defined as follows: represent \( \mu = \sum l_i \alpha_i \) as a sum of simple roots with repetitions \( \mu = \alpha_{i_1} + \alpha_{i_2} + \ldots + \alpha_{i_n} \). Then

\[ d(\mu) = - \sum_{p \leq q} \frac{(\alpha_{i_p}, \alpha_{i_q})}{4}. \]

\( \Omega_{\mu} \) is a canonical element cf. [26].

\( R \) defines a linear operator
\[ R : M \otimes M' \mapsto M \otimes M'. \]

The following diagram is commutative:

\[
\begin{array}{ccl}
M \otimes M' & \xrightarrow{R} & M \otimes M' \\
\downarrow S & & \downarrow S \\
M^* \otimes M'^* & \xrightarrow{R^*} & M^* \otimes M'^*
\end{array}
\]

cf. theorem 7.6.8 of ref.[11]. \( R \) induces a homomorphism of irreducible highest weight modules:

\[ R : L(\Lambda(1)) \otimes L(\Lambda(2)) \mapsto L(\Lambda(1)) \otimes L(\Lambda(2)) \]

which will be also denoted by \( R \).

Denote by \( P \) the transposition of two factors in the tensor product:
\[ P : M \otimes M' \mapsto M' \otimes M. \]
5. Quantum group and cycle $\Delta$

5.1 Data. We are going to check the monodromy properties of cycle $\Delta$ using quantum group argument, cf. [8, 11, 29].

Take now a different indexation of variables $t_{ij}$. Namely, we are going to use $\{t_i^{(j)} | j = 1, \ldots, n; i = j, \ldots, n\}$ cf.[15]. Set also $z_0 = 0$ (affine configuration). Consider the following multivalued form:

$$
\Omega(z, t) = \prod (z_0 - z_i)^{\Lambda(0)} \prod (z_1 - z_j)^{\Lambda(1)} \prod (z_1 - t_i^{(j)})^{(\Lambda(0), -\alpha_j)} \prod (z_0 - t_i^{(j)})^{(\Lambda(0), -\alpha_j)} \prod (t_i^{(j)} - t_i^{(j')})^{(\alpha_j, -\alpha_j')} dt_1^{(1)} \ldots dt_n^{(n)}
$$

Integrals of forms of this type are considered in refs. [8,11, ...]. Now we would like to specialize $\Lambda(0)$, $\Lambda(1)$, \ldots, $\Lambda(n+1)$, $\alpha_1$, $\alpha_2$, \ldots, $\alpha_n$ as follows.

Recall that $\mathbb{R}^{n+2}$ is an $(n + 2)$-dimensional Euclidean vector space with $g_0, g_1, \ldots, g_{n+1}$ as the orthonormal basis. For $i = 1, \ldots, n+1$ set

$$
\Lambda(i) = \Lambda = g_1 - g_0
$$

i.e. to each variable $z_i, i = 1, \ldots, n + 1$ assign the same vector $g_1 - g_0$.

Recall that simple roots of root system of type $A_n$ are realized as follows:

$$
\alpha_i = g_i - g_{i+1}, \quad \text{for} \quad i = 1, \ldots, n.
$$

Remark 5.2. Note: the projection of $\Lambda(i)$ on the span of $\alpha_i, i = 1, \ldots, n$ is exactly the first fundamental weight, i.e. $(\Lambda(i), \alpha_j) = \delta_{ij}$, where $\delta_{ij}$ is a Kronecker’s delta, but $(\Lambda(i), \Lambda(i)) = 2$. This is not very important since it changes only the power of $\prod (z_i - z_j)$ before the integral.

Let $R$ denotes the root system of type $A_n$ with simple roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ as before. $R_+$ denotes the set of positive roots. Let $\delta$ be half the sum of positive roots:

$$
\delta = \frac{1}{2} \sum_{\alpha \in R_+} \alpha
$$

$$
\rho = \frac{k}{2} \sum_{\alpha \in R_+} \alpha
$$

Set also $\varpi = -\frac{1}{2}$. Let $\lambda$ belongs to the span of $\alpha_1, \ldots, \alpha_n$. Set $\Lambda(0) = \varpi \lambda - \delta$, so that

$$
\frac{\Lambda(0)}{\varpi} = \lambda + \rho
$$
Let also
\[ \overline{\lambda} = \kappa \lambda \]

The multivalued form \( \omega(z, t) \) of section 1 differs from \( \Omega(z, t) \) in the above setting only by some meromorphic factor which does not contribute to the monodromy and thus can be omitted for the purposes of this section.

Set \( q = \exp\left(\frac{2\pi i}{\kappa}\right) \).

Remark 5.3. Note: our form is slightly different from the form of ref. [15], in particular, in setting of [15] cycle \( \Delta \) will not serve as a cycle for integration for zonal spherical function (also we do not have the complicated meromorphic factor). For example,
\[
\int_{z_1}^{z_2} (t - z_1)^{-k}(z_2 - t)^{-k} dt = (z_2 - z_1)^{1-2k} \frac{\Gamma(1-k)\Gamma(1-k)}{\Gamma(2-2k)}
\]
In particular, if \( z_2 \) goes around \( z_1 \) counterclockwise then it earns the factor \( \exp(2\pi i(-2k)) \) and thus the cycle is not preserved under the monodromy.

5.4. The most trivial example. Before proceeding further we want to consider the most trivial example. Namely, consider
\[
(z_1 - z_2)^{(\Lambda(1), \Lambda(2))}
\]
If \( z_2 \) goes around \( z_1 \) counterclockwise then this function earns the factor \( \exp\left(\frac{2\pi i}{\kappa}(\Lambda(1), \Lambda(2))\right) \). If \( z_2 \) goes halfway around \( z_1 \) then the function earns the factor \( \exp\left(\frac{\pi i}{\kappa}(\Lambda(1), \Lambda(2))\right) = q^{\frac{(\Lambda(1), \Lambda(2))}{2}} \). At the same time consider \( v_1 \otimes v_2 \) tensor product of highest weight vectors of modules of the corresponding quantum group of weights \( \Lambda(1) \) and \( \Lambda(2) \) correspondingly. Let \( R \) be the \( R \)-matrix, then
\[
R(v_1 \otimes v_2) = q^{\frac{\Omega_0}{2}} v_1 \otimes v_2,
\]
where \( \Omega_0 \) is the canonical element corresponding to inner product, i.e.
\[
q^{\frac{\Omega_0}{2}} v_1 \otimes v_2 = q^{\frac{(\Lambda(1), \Lambda(2))}{2}} v_1 \otimes v_2
\]
in agreement with the above considerations.
5.5. Case of root system of type $A_1$ ($n = 1$).

Consider the tensor product of three dual Verma modules over $U_q(sl(3))$ (simple roots $\alpha_0, \alpha_1$)

$$M(\Lambda(0))^* \otimes M(\Lambda(1))^* \otimes M(\Lambda(2))^*.$$

Let $v_0 \otimes v_1 \otimes v_2$ be the tensor product of highest weight vectors. Then cycle $\Delta$ is encoded as:

$$v_\Delta = -q^{-(\Lambda(1), -\alpha_1)} v_0^* \otimes (f_1 v_1)^* \otimes v_2^* + q^{-(\Lambda(2), -\alpha_1)} v_0^* \otimes v_1^* \otimes (f_1 v_2)^* =$$

$$-q^{-\frac{1}{2}} v_0^* \otimes (f_1 v_1)^* \otimes v_2^* + q^{\frac{1}{2}} v_0^* \otimes v_1^* \otimes (f_1 v_2)^*.$$

Here $\ast$ means dual with respect to the contravariant form $S$.

Consider the action of $R$-matrix on the second and third component of tensor product

$$R: M(\Lambda(1))^* \otimes M(\Lambda(2))^* \mapsto M(\Lambda(1))^* \otimes M(\Lambda(2))^*.$$

Recall that $P$ denotes permutation of factors:

$$P: M(\Lambda(1))^* \otimes M(\Lambda(2))^* \mapsto M(\Lambda(2))^* \otimes M(\Lambda(1))^*.$$

Now one can utilize formulas of example 1 of [8].

Then

$$PR: v_\Delta \mapsto (-1)v_\Delta$$

And so

$$(PR)^2: v_\Delta \mapsto v_\Delta.$$
Figure 2. Chains and quantum group cf. [8,11,29]. Phase is chosen to be zero at the point marked with $+$.  

Figure 3. Decomposition with the help of quantum group cf. [8,11,29].

---

2The choice of comultiplication is dictated by the choice that the phase is equal to zero at the point marked with $+$.  

---
5.6. Case of root system of type $A_n$.

**Theorem 5.7.** Let $v_0, v_1, \ldots, v_{n+1}$ be the highest weight vectors of irreducible highest weight modules with highest weights $\Lambda(0), \Lambda(1) = \Lambda(2) = \ldots = \Lambda(n+1) = \Lambda$ correspondingly. Here $\Lambda(0), \Lambda$ are as in section 5.1. Then cycle $\Delta$ is encoded as:

1. 
   
   $v_\Delta = \sum_{w \in S_{n+1}} (-1)^{l(w)} q^{\frac{1}{4}[n(n+1)]^2 - 2l(w)]} v_0^* \otimes (f_{w(1)} - f_{w(1)} - f_{w(1)} - f_{w(1)})^* \otimes \ldots $ 
   
   $\otimes (f_{w(n+1)} - f_{w(n+1)} - f_{w(n+1)} - f_{w(n+1)})^* $ 

2. 
   
   $v_\Delta = \sum_{w \in S_{n+1}} (-1)^{l(w)} q^{\frac{1}{4}[n(n+1)]^2 - 2l(w)]} \frac{(q^2 - q^{-1})^{\frac{n(n+1)}{2}}}{(q^2 - q^{-1})^{\frac{n(n+1)}{2}}} $ 
   
   $v_0 \otimes f_{w(1)} - f_{w(1)} - f_{w(1)} - f_{w(1)} \otimes \ldots $  
   
   $\otimes f_{w(n+1)} - f_{w(n+1)} - f_{w(n+1)} - f_{w(n+1)} \otimes f_{w(n+1)} - f_{w(n+1)} - f_{w(n+1)} - f_{w(n+1)}$ 

In particular, $e_i v_\Delta = 0$ for $i = 0, 1, \ldots, n$, this implies that cycle $\Delta$ defines correctly an element of $H^*_m(U, S^*)$ and the use of the word ‘cycle’ is justified.

To prove the theorem we repeatedly use figures 3 and 2. Also for the above theorem the theorem 2.6 and remark 2.7 of [17] are helpful: the number of arrows which are ‘to the left’ is equal to the length of the corresponding element of the Weyl group (also reproduced below in theorem 5.3). In fact using elementary decomposition fig. 3 one gets that each arrow to the right brings the factor $q^4$, while each arrow to the left brings the factor $(-1)q^4$. The number of variables $t^{(j)}_i$ is equal to $\frac{n(n+1)}{2}$. The key point is that all the ‘wrong’ diagrams (cf. fig. 5) cancel each other because of the phase argument cf. fig. 6. Where by the ‘wrong’ diagrams we mean the diagrams in which there are two arrows with the same target.

Vice versa, this theorem might be considered as quantum group explanation of theorem 2.6 of [17] (numbers of arrows which are to the left $= l(w)$).
5.8. **Diagrams.** The notion of a diagram was introduced in [38] in the context of trigonometric Knizhnik-Zamolodchikov equation, and later similar notion was introduced in [53] in the context of multidimensional determinants and discriminants. We will use the notion of a diagram in the form of [53], in particular, we borrow the very convenient graphical notation (see figs. 1,2,3 of [53]).

Fix some positive integer $n$. Consider the set of $\frac{n(n+1)}{2}$ points, indexed by pairs of integers $\{(i,j) : i = 1, \ldots, j, j = 1, \ldots, n\}$. It is helpful to organize the points in the form of a pattern, so that points are divided in $n$ rows, $j$th row is formed by points $\{(i,j) : i = 1, \ldots, j\}$; point $(i,j)$ is located under and between points $(i,j+1)$ and $(i+1,j+1)$ (fig. 4a).

![Figure 4a](image)

**Figure 4a.** $n = 4$. Points $(i,j)$ organized in a pattern; $j$ is the number of the row, $i$ is the number in the row

Now mark with a cross one point in each row. Let $\{(i_j,j)\}$ be the subset of marked points (fig. 4b).
Finally, draw an arrow for each point \((i, j)\) with the source in this point \((i, j)\) and target \(\text{tar}(i, j)\) in the next \(j+1\)th row defined as:

\[
\text{tar}(i, j) = \begin{cases} 
(i, j + 1), & \text{if } i < i_{j+1} \\
(i + 1, j + 1), & \text{if } i \geq i_{j+1}
\end{cases}
\]

If \(\text{tar}(i, j) = (i, j + 1)\), then the arrow is called to the \textbf{left}, if \(\text{tar}(i, j) = (i + 1, j + 1)\), then the arrow is called to the \textbf{right}.

Note: neither arrow has a marked point as its target.

In this way one obtains fig 4c.

**Definition 5.9.** A triple

\[
(\{(i, j)\}, \{(i_{j+1}, j)\}, \text{tar})
\]

consisting of:

- set of points \(\{(i, j)\mid i = 1, \ldots, j \; ; \; j = 1, \ldots, n\}\),
- set of marked points \(\{(i_{j+1}, j)\mid j = 1, \ldots, n\}\)
- and function \(\text{tar}\) defined above will be called a \textbf{diagram}.

**Remark 5.10.** One can see that a diagram is determined by the set of marked points.

\[\text{Recall that } \{(i_{j+1}, j)\} \text{ is the set of marked points and } (i_{j+1}, j + 1) \text{ is the only marked point in } j + 1\text{th row.}\]
Figure 4c. Example of a diagram.

Definition 5.11. We describe the correspondence between diagrams and elements of Symmetric group as follows. Consider a diagram as an oriented graph and forget orientation. For \( i = 1, \ldots, n \) define \( w(i) \) as the number of points in the connected component of the point \((i, n)\).

Symmetric group \( S_n \) has standard generators \( \sigma_1, \sigma_2, \ldots, \sigma_{n-1} \), where \( \sigma_i \) permutes \( i \) and \( i + 1 \).

Definition 5.12. The length \( l(w) \) of an element \( w \in S_n \) is the minimal integer \( p \geq 0 \), s.t. \( w \) admits a presentation

\[
w = \sigma_{i_1} \sigma_{i_2} \ldots \sigma_{i_p}.
\]

Any presentation of \( w \) as a product of \( p = l(w) \) generators is called a reduced presentation.

Theorem 5.13. Let a diagram

\[
(\{(i, j) | i = 1, \ldots, j, j = 1, \ldots, n\}, \{(i_j, j) | j = 1, \ldots, n\}, \text{tar})
\]

corresponds to an element \( w \in S_n \). Then the length \( l(w) \) of an element \( w \) is equal to:

\[
l(w) = \sum_{j=1}^{n} (i_j - 1)
\]

In other words, \( l(w) \) is equal to the number of arrows which are to the left in the diagram corresponding to the element \( w \). cf. theorem 2.6 of [17].
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Theorem 5.14. Let $\Lambda(1) = \Lambda(2) = \Lambda$ as in theorem 5.7. Also, let $v_1, v_2$ be the highest weight vectors of Verma modules $M(\Lambda(1)), M(\Lambda(2))$, correspondingly. Let $M(\Lambda(1))^*, M(\Lambda(2))^*$ be dual Verma modules. Let $PR$ be the braiding:

$$PR: M(\Lambda(1))^* \otimes M(\Lambda(2))^* \to M(\Lambda(2))^* \otimes M(\Lambda(1))^*.$$ 

Then for $i > j$ one has:

$$PR ((f_if_i-1 ... f_1v_1)^* \otimes (f_jf_j-1 ... f_1v_2)^*) =$$

$$q^{\frac{1}{2}}(f_jf_j-1 ... f_1v_2)^* \otimes (f_if_i-1 ... f_1v_1)^* +$$

$$q^{\frac{1}{2}}(q^{\frac{1}{2}} - q^{-\frac{1}{2}})(f_if_i-1 ... f_1v_2)^* \otimes (f_jf_j-1 ... f_1v_1)^*$$

For $i < j$ one has:

$$PR ((f_if_i-1 ... f_1v_1)^* \otimes (f_jf_j-1 ... f_1v_2)^*) =$$

$$q^{\frac{1}{2}}(f_jf_j-1 ... f_1v_2)^* \otimes (f_if_i-1 ... f_1v_1)^*$$
Figure 6. Phase argument: decomposition into ordered chains- total sum is identical zero chain

Proposition is easily proved by contour manipulations, see fig. 7 as elementary, but typical example. All calculations are up to the kernel of contravariant form $S$. 

- $q^{-\frac{1}{4}} \left( q^{-\frac{1}{4}} f_1 \tilde{f}_1 + q^{\frac{1}{4}} f_1 \tilde{f}_1 - q^{-\frac{1}{4}} \tilde{f}_1 f_1 + q^{\frac{1}{4}} \tilde{f}_1 f_1 \right) +$
- $q^{\frac{1}{4}} \left( q^{\frac{1}{4}} f_1 \tilde{f}_1 + q^{-\frac{1}{4}} f_1 \tilde{f}_1 + q^{-\frac{1}{4}} \tilde{f}_1 f_1 + q^{\frac{1}{4}} \tilde{f}_1 f_1 \right) = 0$
Corollary 5.15. Suppose $z_1(t), z_2(t), \ldots, z_{n+1}(t), \ t \in [0,1]$ are closed loops on a complex plane, i.e. $z_1(0) = z_1(1), z_2(0) = z_2(1), \ldots, z_{n+1}(0) = z_{n+1}(1)$, such that $z_i(t) \neq z_j(t)$ for $i \neq j$. Let also $\text{Re}(z_i(t)) > 0$ for each $i = 1, \ldots, n+1$. Then the homological class of the cycle $\Delta$ is preserved under the monodromy along paths $z_i(t)$.

In fact such monodromy can be produced as composition of even number of elementary braidings as in theorem 5.14, each of them gives the factor $-1$. Braiding of $z_1$ with $z_0 = 0$ is forbidden by hypotheses.

5.16. Using $R$-matrix for tensor product of vector representations. $R$-matrix for the tensor product of vector representations of $\mathfrak{sl}(n+1)$ reads as:

$$ R = q^{-1/2(n+1)} \left\{ \sum_{i \neq j} E_{ii} \otimes E_{jj} + q^\frac{1}{2} \sum_i E_{ii} \otimes E_{ii} + (q^\frac{1}{2} - q^{-\frac{1}{2}}) \sum_{i<j} E_{ij} \otimes E_{ji} \right\} $$

cf. [26, 27]. Vector representation of $\mathfrak{sl}(n+1)$ has natural basis $e_1, e_2, \ldots, e_{n+1}$ with highest weight vector $e_1$, $E_{ij}$ are matrix units:

$$ E_{ij} e_k = \delta_{kj} e_i, $$

where $\delta_{kj}$ is a Kronecker’s delta. In our case the interesting part of $R$-matrix can be easily obtained from the above one by multiplying by $q^{-n(n+2)/2}$:
\[ \hat{R} = q^{\frac{1}{2}} \left\{ \sum_{i \neq j} E_{ii} \otimes E_{jj} + q^{\frac{1}{2}} \sum_i E_{ii} \otimes E_{ii} + \left( q^{\frac{1}{2}} - q^{-\frac{1}{2}} \right) \sum_{i<j} E_{ij} \otimes E_{ji} \right\}. \]

Then one immediately verifies that:

\[ \hat{R}(e_k \otimes e_l) = q^{\frac{1}{2}} e_k \otimes e_l \quad \text{if} \quad k < l \]

and

\[ \hat{R}(e_k \otimes e_l) = q^{\frac{1}{2}} e_k \otimes e_l + q^{\frac{1}{2}} (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) e_l \otimes e_k \quad \text{if} \quad l < k \]

Also,

\[ \hat{R}(e_k \otimes e_k) = q e_k \otimes e_k \]

Let \( P \) be the transposition, i.e.

\[ Pe_i \otimes e_j = e_j \otimes e_i \]

Then one immediately obtains that for \( k < l \)

\[ P \hat{R}(q^{\frac{1}{2}} e_k \otimes e_l - q^{-\frac{1}{2}} e_l \otimes e_k) = (-1)(q^{\frac{1}{2}} e_k \otimes e_l - q^{-\frac{1}{2}} e_l \otimes e_k) \]

i.e.

\[ q^{\frac{1}{2}} e_k \otimes e_l - q^{-\frac{1}{2}} e_l \otimes e_k \]

is an eigenvector of \( P \hat{R} \) with eigenvalue \(-1\) !

5.17 Tensor product with vector representation. As it is shown by Lusztig and Rosso \([46,47]\) representation theory of \( U_q(g) \) for generic values of \( q \) is the same as in the classical case \( q = 1 \). The tensor product of finite-dimensional representations of \( sl(n+1) \) is governed by the Littlewood-Richardson rule, cf. \([45]\), which is in turn a consequence of the theory of characters and symmetric functions. The rule is essentially simple for the tensor product with vector representation, namely, we should add one box to the Young diagram so that in result we obtain again a Young diagram and if the column with \( n+1 \) boxes appears it should be removed (as corresponding to trivial representation). The tensor product with vector representation is multiplicity free. For example, the tensor product of two vector representations decomposes as:

\[ L(\eta_1) \otimes L(\eta_1) = L(2\eta_1) \oplus L(\eta_2) \]
Figure 8. Tensor product of two vector representations, Littlewood-Richardson rule

where \( L(2\eta_1), L(\eta_2) \) denotes the analog of the symmetric tensor, antisymmetric tensor, see fig. 8.

Let \( P_{2\eta_1}, P_{\eta_2} \) denotes the corresponding projectors. The PR-matrix is given as follows:

\[
PR = q^{-\frac{n^2}{2(n+1)}}(q^2 P_{2\eta_1} - q^{-1} P_{\eta_2}).
\]

So one can see that on antisymmetric tensors it acts as \(-q^{-\frac{n^2}{2(n+1)}}\).

Multiplying by \( q^{\frac{n^2}{2(n+1)}} \) as in previous section we get the coefficient of \( P_{\eta_2} \) is \(-1\).

Now consider the product of finite-dimensional representation of weight \( \lambda \) with vector representation: \( L(\lambda) \otimes L(\eta_1) \). And assume that we add the box to the \( s \)-th row of a Young diagram corresponding to \( \lambda \). In particular, we assume that this is a correct operation. This means that we have variables of integration \( \{t_i|i = 1, \ldots, s-1\} \) and the integral:

\[
\int \prod_{i=1}^{s-1} t_i^{(\lambda_i-1)/\alpha_i} \prod_{j<i} (t_i - t_j)^{(-\alpha_i-\alpha_j)/\alpha_i} \prod_{i=1}^{s-1} (z - t_i)^{(\eta_1-1)/\alpha_i} \frac{dt_1 dt_2 \ldots dt_{s-1}}{t_1 t_2 \ldots t_{s-1}}
\]

The natural domain of integration for asymptotic solution is:

\[0 \leq t_{s-1} \leq t_{s-2} \leq \ldots \leq t_1 \leq z.\]

The leading asymptotic is equal to

\[z^{(\lambda_1 - \sum_{i=2}^{s-1} \alpha_i)/(s-1)}.\]

The integral is easily taken using the formula for Euler’s beta function:

\[
\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha + \beta)}
\]
Figure 9. Cycle for integration for screened vertex fundamental operator: contour for $t_i$ starts and ends at $t_{i-1}, i = 2, 3, \ldots$; contour for $t_1$ starts and ends at $z_1$. In particular, all 'internal' contours are movable.

and the leading asymptotic coefficient is equal to:

$$
\prod_{p=1}^{s-1} \frac{\Gamma\left(\frac{(\lambda, -\sum_{i=p+1}^{i=p} \alpha_i)}{\kappa} \right) - \frac{(p-1)}{\kappa}}{\Gamma\left(\frac{(\lambda, -\sum_{i=p+1}^{i=p} \alpha_i)}{\kappa} + 1 - \frac{p}{\kappa}\right)}
$$

The above cycle for integration can be encoded as singular vector of tensor product of two irreducible modules with highest weights $\lambda$ and $\eta_1$ as follows:

$$
(-1)^{s-1} q^{\frac{1}{4}(s-2)+\frac{1}{4}(\lambda, \sum_{i=1}^{s-1} -\alpha_i)} (f_{s-1} \ldots f_1 v_0)^* \otimes v_1^* + 
q^{\frac{1}{4}(s-1)} v_0^* \otimes (f_{s-1} \ldots f_1 v_1)^* + 
\sum_{p=1}^{s-2} (-1)^{s-p-1} q^{\frac{1}{4}(s-p-2)+\frac{1}{4}(\lambda, \sum_{i=p+1}^{i=p} -\alpha_i)} (f_{s-1} f_{s-2} \ldots f_{p+1} v_0)^* \otimes (f_p f_{p-1} \ldots f_1 v_1)^*
$$
With the help of comultiplication $\Delta$ this might be iterated (in principle) and cycles for asymptotic solutions from ref. [17] can be encoded as singular vectors of the tensor product of irreducible highest weight modules over quantum group:

$$L(\Lambda(0)) \otimes L(\Lambda(1)) \otimes \ldots \otimes L(\Lambda(n+1)).$$

In the case of generic $\Lambda(0)$ one can use Bernstein-Gelfand-Gelfand results on the category $\mathcal{O}$ cf. [78], see also Kostant [77] (and the theory of Knizhnik-Zamolodchikov equations, parameter $q$ is assumed to be generic). In particular for generic $\Lambda(0)$ tensor product $L(\Lambda(0)) \otimes L(\eta_1)$ decomposes as:

$$L(\Lambda(0)) \otimes L(\eta_1) = L(\Lambda(0) + h_1) \oplus L(\Lambda(0) + h_2) \oplus \ldots \oplus L(\Lambda(0) + h_{n+1})$$

The space of singular vectors of weight $\Lambda(0)$ of the tensor product

$$L(\Lambda(0)) \otimes L(\eta_1) \otimes \ldots \otimes L(\eta_1)$$

with $n + 1$ factors $L(\eta_1)$ is $(n + 1)!$-dimensional.

Remark 5.18. The fact that the cycle $\Delta$ corresponds to $q$-antisymmetric tensors is actually clear. Consider tensor product of $n + 1$ vector representations. So there is the only way to get the variables of integration as needed: namely, to add box under box, so that the Young diagram will be the column of $n + 1$ boxes, which corresponds to antisymmetric tensors, see fig. 10. One could also find different contours for integration giving the same homological class as cycle $\Delta$ (fig. 11), but cycle $\Delta$ is especially convenient for obtaining Harish-Chandra decomposition, it has geometric origin in harmonic analysis and there is certain parallelism with Gelfand-Tsetlin patterns (see also [73, 40], [74]).
Concluding remarks. The distinguished cycle $\Delta$ serves as a contour for integration for zonal spherical function of type $A_n$. It goes back to the classical calculation of Gelfand and Naimark of zonal spherical function for $SL(n, \mathbb{C})$, originates in the so-called elliptic coordinates and provides a materialization of the flag manifold. Zonal spherical function is a particular conformal block.

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Figure 11. Another contour for integration for zonal spherical function.

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