Synchronization of chaotic networks with time-delayed couplings: An analytic study

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Networks of nonlinear units with time-delayed couplings can synchronize to a common chaotic trajectory. Although the delay time may be very large, the units can synchronize completely without time shift. For networks of coupled Bernoulli maps, analytic results are derived for the stability of the chaotic synchronization manifold.

For a single delay time, chaos synchronization is related to the spectral gap of the coupling matrix. For networks with multiple delay times, analytic results are obtained from the theory of polynomials. Finally, the analytic results are compared with networks of iterated tent maps and Lang-Kobayashi equations which imitate the behaviour of networks of semiconductor lasers.

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INTRODUCTION

Chaos synchronization is a phenomenon which is of fundamental scientific interest in nonlinear dynamics and which is being investigated in the context of secure communication and neural activity [11,12]. In particular, networks of nonlinear units which relax to a common chaotic trajectory are the focus of recent research [6,7].

For many applications, the coupling between nonlinear dynamical units is realized by transmitting a function of their internal variables to their neighbours. In many cases, the transmission time is larger than the internal time scales of the units. One example are chaotic lasers which are coupled by their mutual laser beams [5,8–11]. Thus, networks of nonlinear units which are coupled by their time-delayed variables – including time-delayed self-feedback – are a subject of recent research activities [6,12,31].

The theoretical investigations of chaotic networks are mainly based on numerical simulations. However, there exists a powerful method to determine the stability of the synchronization manifold (SM) of an arbitrary network: The master stability function (MSF) [29]. This method connects topology with function. Using the eigenvalues of the connection matrix, a linear equation for a single unit is derived which determines the stability of chaos synchronization for the complete network. For the case of time-delayed couplings, the MSF has been studied for few systems, only [20,25,32].

The MSF is defined as the maximal Lyapunov exponent of linear equations with time-dependent coefficients. In general it is not possible to derive analytic results for these equations because the coefficients are given by the chaotic trajectory of the network. Thus, one has to rely on numerical simulations of the linear system of differential equations (for chaotic flows) or difference equations (for coupled map lattices).

The purpose of this paper is to derive analytic results for chaotic networks with time-delayed couplings. Thus, we concentrate on a coupled map lattice, a network of chaotic iterated Bernoulli maps, which allows an analytic calculation of the MSF. The corresponding linear difference equations have constant coefficients. Therefore, stability of the solution is related to the roots of polynomials. Results from graph theory, control theory and algebra will help to derive analytic statements about chaos synchronization.

The linear stability equations of Bernoulli networks may be considered as an approximation for linear equations with time-dependent couplings. Thus, we compare our analytic results with numerical results for other iterated maps and with simulations of rate equations for coupled semiconductor lasers. In the following section the MSF for chaotic networks with time-delayed couplings is introduced. We concentrate on networks where chaos is generated by the couplings and/or self-feedback, similar to semiconductor lasers. Afterwards, analytic results are derived for Bernoulli networks with single and multiple delay times. The last section compares these results with networks of iterated tent maps and of laser rate equations.

MASTER STABILITY FUNCTION (MSF)

MSF with time delay

One powerful method to analyze the stability of synchronization in networks of coupled systems with identical units is the master stability function (MSF) proposed by Pecora and Carroll [29]. A network of coupled identical dynamical units can be analyzed by linearizing the dynamical equation around the synchronization manifold.

In order to obtain analytic results, we restrict our investigation to coupled map lattices, and we use an identical function $f(x)$, $x \in [0,1]$ for the internal dynamic, the self-feedback and the couplings. We extend the master stability function of [29] for a system without time delay to a system with arbitrarily many different time delays [27,33,35] where each term with delay time $\tau_i$
is weighted with a positive coupling parameter $\sigma_l$ and a positive self-feedback parameter $\eta_l$. The network consists of $N$ units with variables $x_i^t \in [0, 1]$, where $i = 1, ..., N$ is the index of the unit and $t$ is a discrete time step. The system is defined by

$$x_i^t = \eta_0 f(x_{i-1}^t) + \sum_{l=1}^{M} \eta_l f(x_{i-\tau_l}^t) + \sum_{l=1}^{M} \sum_{j=1}^{N} \sigma_l G_{l,ij} f(x_{i-\tau_l}^t)$$  \hspace{1cm} (1)

Without loss of generality we order the coupling terms with ascending delay times, so that $\tau_M$ is the maximum time delay. Each coupling delay time $\tau_l$ has its own coupling matrix, the normalized weighted adjacency matrix $G_l$ with $\sum_l G_{l,ij} = 1$ and $G_{ij} \geq 0$ [24] [31]. Therefore, the coupling is invasive and non-diffusive, it changes the trajectory of the coupled system in comparison to a non-coupled system. We assume that the coupling matrices $G_l$ commute, otherwise the MSF method cannot be applied. The self-feedback of the system is not included in $G_l$, hence $G_{i,ii} = 0$.

Complete zero-lag synchronization $x_1^t = ... = x_N^t = s_t$ is a solution of these equations. The synchronized trajectory is given by

$$s_t = \eta_0 f(s_{t-1}) + \sum_{l=1}^{M} (\eta_l + \sigma_l) f(s_{t-\tau_l})$$  \hspace{1cm} (2)

The stability of the SM is determined by linearizing Eq. (1) in the vicinity of the SM, Eq. (2). With $\delta x_i^t = x_i^t - s_t$ we obtain

$$\delta x_i^t = \eta_0 f'(s_{t-1}) \delta x_{i-1}^t + \sum_{l=1}^{M} \eta_l f'(s_{t-\tau_l}) \delta x_{i-\tau_l}^t + \sum_{l=1}^{M} \sum_{j=1}^{N} \sigma_l G_{l,ij} f'(s_{i-\tau_l}) \delta x_{i-\tau_l}^j$$  \hspace{1cm} (3)

For a network with $N$ nodes we obtain $N$ coupled linear equations with time-dependent coefficients. Since the coupling matrices $G_l$ commute, we can expand the $N$-dimensional perturbation $\delta \vec{x}_k$ to the common eigenvectors $\vec{w}_k$ with eigenvalues $\gamma_{l,k}$, $k = 1, ..., N$, $l = 1, ..., M$ of the matrices $G_l$. For each mode $k$ of the perturbation, with $\delta \vec{x}_k = \xi_{k,t} \vec{w}_k$, we obtain

$$\xi_{k,t} = \sum_{l=0}^{M} (\eta_l + \sigma_l \gamma_{l,k}) f'(s_{t-\tau_l}) \xi_{k,t-\tau_l}$$  \hspace{1cm} (4)

where we have defined $\sigma_0 = 0$ and $\tau_0 = 1$. $\xi_{k,t}$ is the amplitude of the perturbation corresponding to the eigenvalue $\gamma_{l,k}$ of the coupling matrix $G_l$.

To gain analytic results we focus on a chaotic map with constant slope, namely the Bernoulli map which is given by

$$f(x) = (\alpha \, x) \mod 1$$  \hspace{1cm} (5)

and is chaotic for $\alpha > 1$. Since $f'(s_t) = \alpha$ is constant, Eq. (4) becomes

$$\xi_{k,t} = \sum_{l=0}^{M} (\eta_l + \sigma_l \gamma_{l,k}) \alpha \xi_{k,t-\tau_l}$$  \hspace{1cm} (6)

With the ansatz $\xi_{k,t} = z_k^t \xi_{k,0}$, the whole stability problem becomes a problem of solving the polynomial of degree $\tau_M$

$$z_k^\tau = \sum_{l=0}^{M} (\eta_l + \sigma_l \gamma_{l,k}) \alpha z_k^{\tau-\tau_l} = \sum_{l=0}^{M} \beta_{k,l} z_k^{\tau-\tau_l}$$  \hspace{1cm} (7)

with $\beta_{k,l} = (\eta_l + \sigma_l \gamma_{l,k}) \alpha$.

For each eigenvalue $\gamma_{k}$, Eq. (7) yields $\tau_M$ roots which we label with $z_{k,r}$, $r = 1, ..., \tau_M$. Our goal is to find coupling parameters $\beta_{k,l}$ such that:

1. For $\gamma_{0,l} = 1$, $l = 1, ..., M$, there exists at least one $z_{0,r_m}$ and $|z_{0,r_m}| > 1$. This guarantees a chaotic dynamic of the SM.

2. For each $\gamma_k$ ($k > 1$) all roots $z_{k,r}$ lie inside the unit circle $|z_{k,r}| < 1$. This guarantees a stable SM. In this case, the MSF is defined as

$$\lambda = \max_{k>0,r} \ln|z_{k,r}|$$  \hspace{1cm} (8)

Eq. (7) allows an analytic investigation of chaos synchronization. Later we will discuss to which extend Eq. (7) is a good approximation for other iterated maps with time-dependent slopes $f'$ and even for corresponding differential equations with time-dependent Jacobi matrices.

**Concept of local Lyapunov exponents**

The stability of the SM of the complete network is determined by the linear equation (4). This equation yields the Lyapunov exponents of the network parallel ($\gamma_0 = 1$) and perpendicular ($\gamma_k$, $k > 0$) to the SM. As we will see later, it turns out that it is useful to consider the contributions of individual terms in Eq. (4) to the stability of the SM separately. Hence, we define “local Lyapunov exponents” to discuss these contributions. For example, the local Lyapunov exponent is defined as the maximal one of the equation

$$\xi_{k,t} = \eta_0 f'(s_{t-1}) \xi_{k,t-1}$$  \hspace{1cm} (9)

where $s_t$ is the trajectory of the complete network, including time-delayed terms.
For the case of large delay times $\tau_l$ one finds the following result: When the instantaneous Lyapunov exponent is positive, the network cannot synchronize [25]. Note that local Lyapunov exponents are not the Lyapunov exponents of the isolated units, since the network changes the trajectory. Only for the Bernoulli network the linear equations do not depend on the trajectory, in this case the local Lyapunov exponents are identical to the ones of the corresponding isolated units.

Graph spectrum

The stability of the synchronization manifold (SM) is determined by the eigenvalues of the coupling matrices $G_l$, according to the MSF Eq. (5). We consider matrices with unit row sum, $\sum_j G_{l,ij} = 1$. Since we consider self-feedback separately, we have $G_{l,ii} = 0$.

We restrict our discussion to non-negative matrices [36], $G_{l,ij} \geq 0$, and to completely connected graphs. One eigenvalue is unity, $\gamma_{0,l} = 1$, and according to the Perron-Frobenius theorem $\gamma_{0,l}$ is not degenerate and has the largest modulus of all eigenvalues of $G_l$. Therefore, we order the eigenvalues such that $1 = \gamma_{0,l} \geq |\gamma_{1,l}| \geq |\gamma_{2,l}| \geq \ldots \geq |\gamma_{N-1,l}|$.

The eigenvector for the largest eigenvalue $\gamma_0 = 1$ is $\vec{v}_0 = (1, ..., 1)$. It corresponds to a perturbation parallel to the SM. Since we discuss only chaotic networks, the Lyapunov exponent $\lambda_{\text{max}}$ of the mode $\gamma_0$ is positive; the dynamics in the SM, Eq. (2) is chaotic. As we will show later, for large single delay time $\tau$ the spectral gap $\Delta = 1 - |\gamma_1|$ determines the stability of the SM. Complete zero-lag synchronization is possible in the limit of weak chaos, $\lambda_{\text{max}} \rightarrow 0$, if and only if the spectral gap is nonzero, $|\gamma_1| < 1$.

The theory of nonnegative matrices [36] relates the eigenvalue gap to the loop structure of the corresponding graph: $\Delta$ is nonzero if and only if the greatest common divisor of the length of loops of the graph is unity. Hence, for a single large time delay, it is easy to see whether a network can synchronize in the limit of weak chaos. In fact, this result has been extended to networks with multiple delay times $\tau_l$ by a self-consistent physical argument based on mixing information of the chaotic trajectories of the nodes [23].

For some single graphs the eigenvalues are known analytically. Fig. 1 shows the eigenvalue gap for some graphs. Note that a directed triangle and a directed square have zero gaps, but if one connects them, the greatest common divisor of the loops 3 and 4 is unity and the gap is nonzero [36].

\begin{equation}
\sum_{l=0}^{M} \sum_{k=1}^{M} \beta_l z^{\tau_{M}} = \sum_{l=0}^{M} \sum_{k=1}^{M} \beta_l z^{\tau_{M}}
\end{equation}

Note that each $\beta_l$ includes $\gamma_{k,l}$, we omitted the index $k$ in our notation. The network contains $M + 1$ different delay times $\tau_l$, and $\tau_M$ is the largest one. For each mode $k$ of perturbation we have a set of eigenvalues $\gamma_{k,l}$ of the coupling matrices $G_l$ corresponding to the delay times $\tau_l$. Hence, we have to calculate the roots of Eq. (10) with $\beta_l = (\eta_l + \sigma_l \gamma_{k,l}) \alpha$ for each mode of perturbation where $\sigma_l$ is the coupling strength and $\alpha$ is the slope of the Bernoulli map. Eq. (10) includes the local dynamics with $\tau_0 = 1$ and $\sigma_0 = 0$ and self-feedbacks with delay time $\tau_l$. 

Figure 1: Schemata of bidirectionally coupled pair, spectral gap $\Delta = 0$, bidirectionally coupled triangle, $\Delta = \frac{1}{2}$, unidirectional triangle, $\Delta = 0$, square, $\Delta = 0.1215$ and combination of triangle and square, $\Delta = 0.1215$. These are the values for equal coupling weights.

ANALYTIC RESULTS

In this section, we present analytic results for Bernoulli networks. We focus on complete zero-lag synchronization. According to the previous section, the stability of the synchronization manifold (SM) is determined by the roots of the following polynomials

\begin{equation}
\sum_{l=0}^{M} \sum_{k=1}^{M} \beta_l z^{\tau_{M}} = \sum_{l=0}^{M} \sum_{k=1}^{M} \beta_l z^{\tau_{M}}
\end{equation}
and strength \( \eta_l \). The mode parallel to the SM has eigenvalues \( \gamma_{0,l} = 1 \) for all terms of Eq. (10).

Some results can be derived immediately. The polynomial Eq. (10) has \( \tau_M \) roots \( z_r \) and the theorem of Vieta gives \( \prod z_r = |\beta_M| \). Hence, for \( |\beta_M| > 1 \) at least one root is outside of the unit circle. Consequently, all modes \( k \) with \( |(\eta_M + \sigma_M \gamma_{k,M})\alpha| > 1 \) are unstable. If the coupling \( \sigma_M \) and the self-feedback \( \eta_M \) of the largest delay time \( \tau_M \) are such that there exists one eigenvalue \( \gamma_{k,M} \) with \( k > 0 \) and \( |\eta_M + \sigma_M \gamma_{k,M}| > 1/\alpha \), the network cannot synchronize, the SM is unstable.

Eq. (10) denotes the roots of a polynomial \( P(z) \) which can be written as

\[
P(z) = z^{\tau_M} - \sum_{l=0}^{M} (\eta_l + \sigma_l \gamma_{k,l}) \alpha z^{\tau_M - \tau_l}
\] (11)

If the mode \( k \) is unstable, \( P(z) \) has at least one root with \( |z| > 1 \). Let us assume \( \sum_{l=0}^{M} (\eta_l + \sigma_l \gamma_{k,l}) \alpha > 1 \), which gives

\[
P(1) = 1 - \sum_{l=0}^{M} (\eta_l + \sigma_l \gamma_{k,l}) \alpha < 0
\] (12)

Furthermore, on the real axis we have

\[
\lim_{z \to \infty} P(z) \to \infty
\] (13)

Since \( P(z) \) is continuous we can conclude that \( P(z) \) has a root on the real axis with \( z_0 > 1 \). Hence, the mode \( k \) is unstable if

\[
\sum_{l=0}^{M} (\eta_l + \sigma_l \gamma_{k,l}) \alpha > 1
\] (14)

The Gershgorin’s circle theorem [37] states that all roots of \( P(z) \) lie inside a circle with radius \( R \) given by the inequality

\[
R \leq \max \left\{ 1, \sum_{l=0}^{M} |(\eta_l + \sigma_l \gamma_{k,l})\alpha| \right\}
\] (15)

Hence, the mode \( k \) is stable if

\[
\sum_{l=0}^{M} |(\eta_l + \sigma_l \gamma_{k,l})\alpha| < 1
\] (16)

Eqs. (14) and (16) give the parameter regime for chaos. Note that this is valid for all perturbation modes \( k \).

For a chaotic trajectory the perturbation mode parallel to the synchronization manifold, \( \gamma_{0,l} = 1 \) has to be unstable. This is the case for

\[
\sum_{l=0}^{M} (\eta_l + \sigma_l) \alpha > 1
\] (17)

whereas for

\[
\sum_{l=0}^{M} (\eta_l + \sigma_l) \alpha < 1
\] (18)

the perturbation mode for \( k = 0 \) is stable. Hence, the border to a chaotic trajectory is given by

\[
1 = \sum_{l=0}^{M} \beta_l = \sum_{l=0}^{M} (\eta_l + \sigma_l) \alpha
\] (19)

If the sum of couplings is larger than 1 the system is chaotic. Note that this transition to chaos does not depend on the delay times \( \tau_l \) in contrast to the region of synchronization which is very sensitive to the values of the delay times \( \tau_l \). In this work we consider only parameters where the network is chaotic. Another conclusion from this result is the fact that for some networks it is always possible to find coupling parameters and Bernoulli slopes for which stable chaos synchronization is possible. This networks have to have eigenvalues \( k > 0 \) which fulfill Eq. (16) for parameters which ensure (17). This is the case for \( |\gamma_{k,l}| < 1 \) where \( k > 0 \).

Single delay time

For a network with a single delay time \( \tau \), including coupling and self-feedback we have to find the roots of

\[
z^\tau = \beta_0 z^{\tau-1} + \beta_1 = \eta_0 \alpha z^{\tau-1} + (\eta_1 + \sigma_1 \gamma_{k}) \alpha
\] (20)

in the chaotic region

\[
(\eta_0 + \eta_1 + \sigma_1) \alpha > 1
\] (21)

For each mode \( k \), Eq. (20) has \( \tau \) roots \( z_{k,r} \) which define a spectrum of \( \tau \) Lyapunov exponents \( \lambda_{k,r} = \ln|z_{k,r}| \). Fig. 2 shows this spectrum as a function of the parameters \( \beta_0 \) and \( \beta_1 \). For \( \beta_0 + \beta_1 > 0 \) the maximal Lyapunov exponent is positive. Using the Schur-Cohn theorem [38] the region of stability can be calculated numerically. Fig. 3 shows the result for \( \tau = 2, 4 \) and \( \tau \to \infty \). With increasing delay time this region shrinks to the symmetric triangle \( |\beta_1| < 2 - \beta_0 \).

In fact, one can derive the region of stability analytically for \( \tau \to \infty \). In this limit, we have to consider two
cases: a) the Lyapunov exponent is of order 1, b) it is of order $1/\tau$ [39, 40]. We omit the index $k$ of the root $z_{k,r}$ of the perturbation mode $k$ since $\gamma_k$ is included in $\beta_1$. From Eq. (20) we obtain

$$|z_r| = \left| \frac{\beta_1 z_r}{z_r - \beta_0} \right|$$  \hspace{1cm} (22)

For case a) $|z_r|$ diverges for $\tau \to \infty$ in the region of instability, hence we find $z_r = \beta_0$ for $\tau \to \infty$. If $\beta_0 > 1$, the SM is unstable for any perturbation mode with eigenvalue $\gamma_k$. But $\beta_0 > 1$ means that the uncoupled units are chaotic. Thus, we reproduce the result already found in [25]: If the local Lyapunov exponent defined in the previous section is positive, a network cannot be synchronized by time-delayed couplings if the delay time is much larger than the local time scales.

For case b) we write $z_r = e^{\lambda_r} e^{i\phi_r}$ with $\lambda_r = \Lambda_r/\tau$. Eq. (22) gives

$$e^{\Lambda_r} = \left| \frac{\beta_1}{e^{\lambda_r/\tau} e^{i\phi_r} - \beta_0} \right|$$  \hspace{1cm} (23)

For $\tau \to \infty$ we obtain

$$e^{2\Lambda_r} = \frac{|\beta_1|^2}{\beta_0^2 - 2\beta_0 \cos \phi_r + 1}$$  \hspace{1cm} (24)

For $\tau \to \infty$ the phases $\phi_r$ are uniformly distributed on the circle $[0, 2\pi]$ [11, 42], hence the maximal Lyapunov exponent is given for $\phi_r = 0$ which yields the region of stability

$$|\beta_1| < 1 - \beta_0$$  \hspace{1cm} (25)

Consequently, the network can synchronize if for all $k > 0$

$$|\eta_1 + \sigma_k \gamma_k| \alpha < 1 - \eta_0 \alpha$$  \hspace{1cm} (26)

For large delay times and without self-feedback $\eta_1 = 0$, Eq. (26) reduces to

$$|\sigma_1 \gamma_1 \alpha| < 1 - \eta_0 \alpha$$  \hspace{1cm} (27)

where $\gamma_1$ is the eigenvalue with the second largest absolute value, $1 \geq |\gamma_1| \geq |\gamma_2|$. Neither the sign nor the complex phase of the eigenvalue $\gamma_1$ have an influence on the region of stability. This is different for small values of $\tau$ where the region of stability depends on the complex phase of $\gamma_1$. Note that this symmetry for large delay times has recently been proven for general chaotic networks with a continuous dynamics [20]. Eq. (27) has interesting consequences. The network is chaotic if

$$\sigma_1 \alpha > 1 - \eta_0 \alpha$$  \hspace{1cm} (28)

It can synchronize if

$$\sigma_0 \alpha + |\sigma_1 \gamma_1 \alpha| < 1$$  \hspace{1cm} (29)

Thus, if $|\gamma_1| = 1$, Eq. (27) is identical to Eq. (28) and the chaotic network cannot synchronize for any set of parameters. For bipartite networks one finds $\gamma_1 = -1$, hence bi-partite networks cannot synchronize completely, only sublattice or cluster synchronization is possible [24, 31]. According to Fig. 1, a pair, a square without diagonals or any directed ring cannot synchronize completely for large delay times. A sublattice or cluster synchronization however, is possible [23, 24, 31]. On the other hand, a triangle with bidirectional couplings can synchronize completely, since $\gamma_1 = -\frac{1}{2}$.

We exemplify this result by a system consisting of two bidirectionally coupled Bernoulli units with one time delay $\tau$ for the coupling and the same time delay $\tau$ for self-feedback. By adding a self-feedback with delay time $\tau$, one can achieve complete synchronization in a bidirectionally coupled pair. The dynamical equation of unit $i$ is defined as

$$x_i^j = (1-\varepsilon)f(x_i^j) + \varepsilon \kappa f(x_{i-\tau}^j) + \varepsilon (1-\kappa) f(x_{i-\tau}^j)$$  \hspace{1cm} (30)

with $i,j \in \{1,2\}$. According to Eq. (28), the system is chaotic for any parameters $f'(x) = \alpha > 1$ and $\varepsilon, \kappa \in [0,1]$. The pair has the eigenvalue $\gamma_1 = -1$ which gives $\beta_0 = (1-\varepsilon) \alpha$, $\beta_1 = \varepsilon (2\kappa - 1) \alpha$. Eq. (25) yields two synchronization borders for $\tau \to \infty$

$$\kappa^+(\varepsilon) = 1 - \frac{\alpha - 1}{2\alpha \varepsilon}$$
$$\kappa^-(\varepsilon) = \frac{\alpha - 1}{2 \alpha \varepsilon}$$  \hspace{1cm} (31)

For a triangle with corresponding equations we obtain with $\gamma_1 = -\frac{1}{2}$ the coefficients $\beta_0 = (1-\varepsilon) \alpha$, $\beta_1 = \varepsilon \kappa \alpha - \frac{1}{2} \varepsilon (1-\kappa) \alpha = \frac{1}{2} (3\kappa - 1) \alpha$ which gives

$$\kappa^+(\varepsilon) = 1 - \frac{2(\alpha - 1)}{3 \varepsilon \alpha}$$
$$\kappa^-(\varepsilon) = \frac{2 - \varepsilon}{3 \varepsilon \alpha}$$  \hspace{1cm} (32)
Figure 4: Blue dashed line: border to synchronization for a bidirectionally coupled Bernoulli pair with $\alpha = 1.5$, self-feedback, $\tau \to \infty$ and $\varepsilon, \kappa$ notation. Thick line: $\kappa^+$, thin line: $\kappa^-$. Red solid line: border to synchronization for a bidirectionally coupled triangle, same parameters as for the pair.

Both results are shown in Fig. 4. Without self-feedback ($\kappa = 0$), a bidirectionally coupled pair cannot synchronize whereas a triangle synchronizes for $\varepsilon > 2(\alpha - 1)/\alpha$.

Note that the upper boundaries $\kappa^+$ do not depend on the delay time $\tau$, whereas with decreasing delay time the lower boundary moves down, the region of synchronization increases.

From Eq. (24) a relation between the MSF and the maximal Lyapunov exponent in the limit of $\tau \to \infty$ can be derived. We consider a network without self-feedback and with local stability $\beta_0 < 1$. The MSF, i.e. the largest transversal Lyapunov exponent is given by

$$\lambda = \frac{1}{\tau} \ln \left| \frac{\sigma_1 \alpha \gamma_1}{1 - \beta_0} \right|$$

whereas the maximal Lyapunov exponent, which is the Lyapunov exponent parallel to the SM, is

$$\lambda_{\text{max}} = \frac{1}{\tau} \ln \left| \frac{\sigma_1 \alpha}{1 - \beta_0} \right|$$

Hence, one obtains

$$\lambda = \lambda_{\text{max}} + \frac{1}{\tau} \ln |\gamma_1|$$

Thus, the SM is stable if

$$|\gamma_1| < e^{-\lambda_{\text{max}}} \tau$$

This equation has a fundamental meaning. For any network with a stochastic coupling matrix $G$, it relates the eigenvalue gap $1 - |\gamma_1|$ to the synchronizability of the network. If the second largest eigenvalue is smaller than the largest one, $|\gamma_1| < 1$, the SM is stable for sufficiently weak chaos inside of the SM, i.e. in the limit $\lambda_{\text{max}} \to 0$.

We believe that Eq. (36) holds for any chaotic network of the structure defined in the previous section, even for corresponding delay differential equations. In fact, Eq. (36) has been derived for networks with periodic dynamics and our numerical results of laser equations confirm this condition on chaos synchronization (see the following section).

Eq. (36) has been derived in the limit $\tau \to \infty$. For finite $\tau$ one either has to solve Eq. (20) numerically or one can calculate regions of stability with the Schur-Cohn theorem. But even in this case of finite $\tau$ an analytic result is possible. The border of stability is given by $|z_0| = 1$ where $|z_0|$ is the maximal root for the perturbation modes $\kappa$ transversal to the SM. With $z_0 = e^{i\phi_0}$ one can decompose Eq. (20) into the real and imaginary parts

$$\cos(\phi_0, \tau) = \beta_0 [\cos(\phi_0, (\tau - 1)) + \beta_1]$$

$$\sin(\phi_0, \tau) = \beta_0 [\sin(\phi_0, (\tau - 1)) \right]$$

The solution of Eqs. (37) gives boundaries $\beta_1(\phi)$ and $\beta_0(\phi)$. The maximal value of $\beta_0(\phi)$, i.e. the tip of the phase diagram of Fig. 3, is obtained in the limit $\phi \to 0$, which gives

$$1 = \beta_0 + \beta_1$$

$$\tau = \beta_0 (\tau - 1)$$

The first part agrees with Eq. (19), the second gives the result

$$\max \beta_0 = \frac{\tau}{\tau - 1}$$

In the limit $\tau \to \infty$ one can synchronize only for $\beta_0 < 1$, i.e. if the local Lyapunov exponent (see previous section) is negative. But for finite delay times even chaotic units ($\beta_0 > 1$) can synchronize if

$$\beta_0 < \frac{\tau}{\tau - 1}$$

Two delay times

Now consider a network with two delay times $\tau_1$ and $\tau_2$ ($\tau_2 > \tau_1$). Each delay time belongs to a coupling matrix $G_1$ and $G_2$ which have identical eigenvectors, i.e. identical modes of perturbation. The stability of the SM is determined by the polynomials

$$z^{\tau_2} = \beta_0 z^{\tau_2-1} + \beta_1 z^{\tau_2-\tau_1} + \beta_2$$

We consider only networks where $\beta_1$ and $\beta_2$ belong either to a coupling or a self-feedback, therefore we rule out $\eta_i \neq 0$ and $\sigma_i \neq 0$ for the same system. Each mode of perturbation has an eigenvalue $\gamma_{i,1}$ of the coupling matrices of Eq. (1), which gives three possibilities for Eq. (42): a system with self-feedback $\tau_1$ and coupling $\tau_2$, a system with self-feedback $\tau_2$ and coupling $\tau_1$ and...
a system with two couplings $\tau_1$ and $\tau_2$. For these three cases we obtain the equations

$$z^{\tau_2} = \eta_0 \alpha z^{\tau_2-1} + \eta_1 \alpha z^{\tau_2-\tau_1} + \sigma_2 \alpha \gamma_{k,2}$$  \hspace{1cm} (43)

$$z^{\tau_2} = \eta_0 \alpha z^{\tau_2-1} + \sigma_1 \alpha \gamma_{k,1} z^{\tau_2-\tau_1} + \eta_2 \alpha$$  \hspace{1cm} (44)

$$z^{\tau_2} = \eta_0 \alpha z^{\tau_2-1} + \sigma_2 \alpha \gamma_{k,1} z^{\tau_2-\tau_1} + \sigma_2 \alpha \gamma_{k,2}$$  \hspace{1cm} (45)

For simplicity, we start the discussion omitting the local term, $\eta_0 = 0$, and considering a pair of coupled units with $\gamma_0 = 1$, $\gamma_1 = -1$. Hence, Eq. (42) is reduced to

$$z^{\tau_2} = \beta_1 z^{\tau_2-\tau_1} + \beta_2$$  \hspace{1cm} (46)

Let $\mu$ be the greatest common divisor of $\tau_2$ and $\tau_1$. We can substitute $w = z^\mu$ to obtain

$$w^p = \beta_1 w^{p-q} + \beta_2$$  \hspace{1cm} (47)

with $\tau_2 = p \mu$, $\tau_1 = q \mu$ and $(p, q)$ are relatively prime. Now, all roots $z_r$ for $k > 0$ lie inside the unit circle if and only if all roots $w_r$ for $k > 0$ lie in the unit circle, hence only the ratio $\tau_2/\tau_1 = p/q$ determines the stability of Eq. (46). Fig. 6 shows the regions of stability of Eq. (46) for different values of $p$ and $q$ calculated with the Schur-Cohn theorem [38]. As shown before, the system is chaotic if $(\eta_1 + \sigma_1) \alpha + (\eta_2 + \sigma_2) \alpha > 1$. For the mode $\gamma_0 = 1$ this means $\beta_1 + \beta_2 > 1$. For example, the point in region (I) in Fig. 6a) belongs to a chaotic system.

The stability of the SM for a bipartite network, for example a pair of units, is determined by the mode $\gamma_1 = -1$, which changes the sign of $\beta_1$ and/or $\beta_2$ of the point in region (I), depending whether this term belongs to a coupling or to a self-feedback. For self-feedbacks one has $\beta_i = \gamma_i \alpha$ whereas the coupling gives $\beta_i = -\sigma_i \alpha$. For example if $\tau_1$ belongs to a self-feedback, $\beta_1$ is positive, since $\eta$ and $\alpha$ are positive. For $\tau_2$ belonging to a bipartite coupling, $G_2$ has an eigenvalue $\gamma_1 = -1$, so $\beta_2$ is flipped to $-\beta_2$ which gives the point in region (IV) in Fig. 6a) which is stable. When both delay times $\tau_1$ and $\tau_2$ belong to the coupling, $G_1$ and $G_2$ are coupling matrices with $\gamma_1 = -1$ and we obtain the point in region (III) which is stable, as well. In both cases we find complete zero-lag synchronization. But when $\tau_2$ belongs to a self-feedback and $\tau_1$ to a coupling we obtain the point in region (II) which is unstable. In this case we can never achieve synchronization for any parameters $\sigma_1$ and $\sigma_2$ for which the system is chaotic.

Obviously, the symmetries seen in Fig. 6 prevent synchronization for some cases depending on the values of $p$ and $q$. These symmetries of Eq. (42) can be derived as follows. If we change the sign of the roots, $w = -v$, the conditions of stability do not change, but Eq. (47) is changed to the polynomial

$$v^p = \beta_1 (-1)^q v^{p-q} + \beta_2 (-1)^p$$  \hspace{1cm} (48)

Now, we have to consider three cases:

If $q$ is even and $p$ is odd, the phase diagram has the reflection symmetry $\beta_2 \rightarrow -\beta_2$, see Fig. 6a) for $p = 3$ and $q = 2$. Thus, if $\tau_1$ belongs to the self-feedback and $\tau_2$ to the coupling, the system cannot synchronize, since the condition for stability of the SM is identical to the condition for chaos.

If $q$ and $p$ are odd, the phase diagram has the point symmetry $\beta_1 \rightarrow -\beta_1$ and $\beta_2 \rightarrow -\beta_2$. Thus, if both delay times belong to couplings, the system cannot synchronize.

Therefore, the symmetries of the roots of the polynomials Eq. (46) rule out some ratios of the two delay times for which synchronization can occur. For the ratios which are not forbidden by symmetries, synchronization is possible in a limited parameter region, shown by the dark regions of Fig. 7 for a pair of Bernoulli units. From numerical calculations, we observe that this region shrinks to zero when $p$ and $q$ increase and we will later show that zero-lag synchronization is not possible if $\tau_1$ and $\tau_2$ are large with a small difference.

These results are in agreement with [35, 44] where a pair with multiple feedback and multiple couplings with different delay times was analyzed. The time delays which lead to zero lag synchronization follow Eq. (29), $\sum_{i=1}^{N_d} l_i N_{di} + \sum_{j=1}^{M_m} m_j N_{cj} = 0$, where $N_{di}$ are
Hence, there exists one root which is close to 0 of the roots are uniformly distributed in $[0, 2\pi]$. In the limit of $z_0=1$, we use the fact that the phases $\phi$ of the roots are uniformly distributed in $[0, 2\pi]$. Hence, there exists one root which is close to $\phi = n\pi/2$ with some integer $n$. Following this root in the limit \( \tau_2 \to \infty \) we obtain

$$1 = \eta_0 \alpha + \beta_1 e^{-in\pi q} + \beta_2 e^{-in\pi p} \quad \text{(51)}$$

Now the border to chaos

$$1 = \eta_0 \alpha + (\eta_1 + \sigma_1) \alpha + (\eta_2 + \sigma_2) \alpha \quad \text{(52)}$$

can be mapped to Eq. (51), depending on whether $\tau_1$ and $\tau_2$ belong to self-feedback or coupling delays. For example, if $\tau_1$ and $\tau_2$ belong to a coupling, Eq. (51) becomes

$$1 = \sigma_0 \alpha - \sigma_1 \alpha e^{-in\pi q} - \sigma_2 \alpha e^{-in\pi p} \quad \text{(53)}$$

If both $p$ and $q$ are odd, we choose $e^{-in\pi} = -1$ and Eq. (52). As synchronization regions calculated with the Schur-Cohn theorem show, Eq. (51) is the border to synchronization. Hence, synchronization is not possible. This shows that the ratios $\tau_2/\tau_1 = p/q$ for which synchronization is ruled out by symmetry do not depend on the local term in the limit of large delay times $\tau_i \to \infty$.

In fact, the numerical simulations Fig. 6 show that the symmetries of the complete phase diagram are the same as the ones proven before for $r_0 = 0$, although the phase diagram depends on the strength $\eta_0$ of the local term.

We have shown that the symmetries of the stability equation, Eq. (49), rule out some ratios of the delay times. In fact, these results support self-consistent arguments for general chaotic networks. These arguments are based on the fact, that the information of each trajectory of each unit has to mix after multiples of time intervals $\tau$ in order to achieve zero-lag synchronization [23].

Finally, we consider the question to which extend the network is sensitive to detuning the delay times and $\tau_2$.

It turns out, that synchronization is extremely sensitive to a tiny detuning for large delay times. For simplicity, consider the case $\eta_0 = 0$, $\tau_1 = \tau$, $\tau_2 = \tau + \Delta$ with $\tau \to \infty$ where $\Delta$ remains finite. $\tau_1$ belongs to a self-feedback, and $\tau_2$ to the coupling of a bipartite network, for example a pair of Bernoulli units. The boundary to chaos is determined by

$$1 \leq \eta_1 \alpha + \sigma_2 \alpha \quad \text{(54)}$$

and the stability of the synchronization manifold is given by the polynomial for $\gamma_1 = -1$

$$z^\tau = \eta_1 \alpha z^{-\Delta} - \sigma_2 \alpha \quad \text{(55)}$$

The stability is determined by the largest root $z_k = e^{i\phi} e^{A/\tau}$ which gives

$$e^{2\Lambda} = |\eta_1 \alpha e^{-i\phi} \Delta - \sigma_2 \alpha|^2$$
$$= (\eta_1 \alpha \cos \phi \Delta - \sigma_2 \alpha)^2 + \eta_1^2 \alpha^2 \sin^2 \phi \Delta$$
$$= \eta_1^2 \alpha^2 + \sigma_2^2 \alpha^2 - 2 \eta_1 \sigma_2 \alpha^2 \cos \phi \Delta \quad \text{(56)}$$
the SM is determined by the polynomials all matrices.

In the limit \( \tau \to \infty \) the distribution of the phase \( \phi \) of the roots is uniform in \([0, 2\pi]\), hence we always find one root with \( \cos(\phi \Delta) \approx -1 \) which according to (54) gives a positive transverse Lyapunov exponent \( \Lambda/\tau \). Thus, even when the self-feedback delay differs from the coupling delay by a single time step, \( \tau_2 = \tau_1 \pm 1 \), a pair of Bernoulli units cannot synchronize for \( \tau_2 \to \infty \). In fact, Fig. 7 shows that the cross-correlation immediately decreases to zero when the delay times are detuned.

**Multiple delay times**

Some of the results of the previous paragraphs can immediately be extended to a network with \( M \) delay times \([35, 44]\). In particular, we can rule out complete zero-lag synchronization for some networks in the limit of large delay times.

First: If one delay time is much larger than all other ones, we find the following result: If the network without the long delay does not synchronize, this network cannot be synchronized by adding the long delay.

Second: If for a pair of coupled units all delay times are much larger than the time scales of the isolated units, then synchronization is ruled out for specific ratios of the delay times.

Note that we restrict our discussion to networks defined in the previous section: To each delay time \( \tau_l \) there exists a coupling matrix \( G_l \) with constant row sum, and all matrices \( G_l \) have identical eigenvectors with eigenvalues \( \gamma_{k,l} \); \( k = 0, ..., N - 1 \); \( l = 1, ..., M \). The stability of the SM is determined by the polynomials

\[
z^{\tau_M} = \sum_{l=0}^{M-1} \beta_l z^{\tau_M - \tau_l} \quad (57)
\]

**Long delay time \( \tau_M \)**

First we consider a network where the largest delay time \( \tau_M \) is much larger than all other delays \( \tau_l \), i.e. we discuss the limit \( \tau_M \to \infty \) with finite \( \tau_l, l \neq M \). Eq. (57) can be rewritten as

\[
z = \sum_{l=0}^{M-1} \beta_l z^{1-\tau_l} + \beta_M z^{1-\tau_M} \quad (58)
\]

If this equation has a root \( |z| > 1 \) in the limit of \( \tau_M \to \infty \), this root is obviously determined by the first term of Eq. (58). Hence, if the mode \( k \) is unstable for the network without long delay, the last term of Eq. (58) cannot stabilize this perturbation. If, however, the largest root of Eq. (58) approaches the unit circle for \( \tau_M \to \infty \), the long delay has an influence. We rewrite Eq. (57) as

\[
z^{\tau_M} = \frac{\beta_M}{1 - \sum_{l=0}^{M-1} \beta_l z^{-\tau_l}} \quad (59)
\]

In the limit of \( \tau_M \to \infty \), we make the ansatz

\[
z = e^{\Lambda/\tau_M} e^{i\phi} \quad (60)
\]

As before, the Lyapunov exponents of the \( k \)-mode depend only on the modulus of the coupling \( \beta_M = (\eta_M + \sigma_M \gamma_{k,M}) \alpha \). However, the maximal Lyapunov exponent is not given by \( \phi = 0 \). Only if all parameters \( \beta_l \) are positive, we obtain the bound to stability as

\[
|\beta_M| = \left| 1 - \sum_{l=0}^{M-1} \beta_l \right| \quad (61)
\]

In this case all \( \beta_l \) belong to self-feedbacks, only \( \beta_M \) is a coupling.

The stability of the SM is determined by negative or complex eigenvalues \( \gamma_{k,l} \). For this case we find an interference of different phases \( \phi \tau_l \) in Eq. (60) resulting in a non-obvious value \( \phi_0 \) for the maximal Lyapunov exponent \( \lambda_{\text{max}} \). For example, for \( \tau_1 = 3 \) and \( \tau_2 = 300 \) we obtain the phase diagram of Fig. 8. This means, that a pair of units which is chaotic (cross in Fig. 8) can be synchronized if \( \tau_1 \) belongs to a coupling and \( \tau_2 \) to a coupling or a self-feedback. If, however, \( \tau_1 \) belongs to a self-feedback, \( \tau_2 \) cannot synchronize this pair. Note that the situation is different for networks with \( |\gamma_1| < 1 \), e.g. triangles.
have been previously seen in [35, 44]. If we restrict our discussion to a pair of units where each \(\tau_l\) belongs to a self-feedback \(\beta_l = |\beta_1| + |\beta_2|\). The black crosses exemplify a pair of units which is chaotic and can synchronize if \(\tau_1\) belongs to a coupling and \(\tau_2\) to a coupling or a self-feedback, but not if \(\tau_1\) belongs to a self-feedback.

Symmetry

Similarly to the case of two delay times discussed before, the symmetry of the polynomial Eq. (67) rules out synchronization for a pair of units with specific ratios of the delay times \(\tau_l\). This symmetry holds for \(\beta_0 = 0\) for general values of \(\tau_l\), but for \(\beta_0 \neq 0\) only if all values of \(\tau_l\) are large. For simplicity, we discuss the case \(\beta_0 = 0\).

Firstly, we consider the greatest common divisor \(\tau\) of all delay times \(\tau_l\), i.e. we define \(p_l = \tau_l/\tau\) where the integers \(p_l\) are relatively prime. Secondly, we substitute \(w = z^\tau\) in Eq. (67) and obtain

\[
w^{PM} = \sum_{l=0}^{M} \beta_{k,l} w^{PM-p_l} \tag{62}\]

If a root \(z_r\) of Eq. (67) lies inside resp. outside the unit circle, the root \(w_t\) of Eq. (62) lies inside resp. outside as well. Hence, the stability of the perturbation \(k\) can be discussed using Eq. (62).

The system is chaotic, Eq. (62) has at least one root \(|z_r| > 1\) for \(\gamma_0 = 1\), i.e. for

\[
w^{PM} = \sum_{l=0}^{M} (\eta_l + \sigma_l) \alpha w^{PM-p_l} \tag{63}\]

If we restrict our discussion to a pair of units where each \(\tau_l\) either belongs to a self-feedback \(\beta_l = \eta_l \alpha\) or a coupling \(\beta_l = -\sigma_l \alpha\), we can map the stability of the SM to Eq. (63) for specific values of \(p_l\). We find the following result: Synchronization is ruled out if \(p_l\) is odd for a coupling and if \(p_l\) is even for a self-feedback. These results have been previously seen in [35 44].

**COMPARISON WITH OTHER SYSTEMS**

The analytic results of the previous section were obtained for networks of Bernoulli units. For this case, the linear equations describing the stability of the synchronization manifold (SM) have constant coefficients. This fact allows a stability analysis with polynomials of degree \(\tau_M\), the largest delay time.

For networks of general nonlinear units, however, the stability equations have time-dependent coefficients generated by the chaotic trajectory of the SM. For example, for general coupled map lattices the stability of the SM is determined by Eq. (4), where the time-dependent coefficients are given by the dynamics on the SM, Eq. (2).

To our knowledge, these equations cannot be solved analytically. Hence, we compare our analytic calculations of the section “Master stability function” with numerical simulations of the skewed tent map

\[
f: [0, 1] \to [0, 1], \quad f(x) = \begin{cases} \frac{x}{a} & \text{if } x \leq a \\ \frac{1-x}{1-a} & \text{if } x > a \end{cases} \tag{64}\]

with \(0 < a < 1\). We choose \(a = 0.86\) which results in the same Lyapunov exponent for an isolated unit as the one of the Bernoulli map

\[
f: [0, 1] \to [0, 1], \quad f(x) = \left(\frac{3}{2} \mod 1 \right) \tag{65}\]

Coupled map lattices are special dynamical systems. Thus, it will be interesting to compare the analytic results of the section “Master stability function” with systems of nonlinear differential equations. In particular, we consider the Lang-Kobayashi (LK) equations for coupled semiconductor lasers [23,45]. In some cases we even compare our results with experiments on chaotic semiconductor lasers.

For networks of differential equations, the mathematical structure corresponding to Eq. (1) is defined by

\[
\dot{x}_i(t) = \tilde{F} [\tilde{x}_i(t)] + \sum_{l=1}^{M} \eta_l \tilde{H} [\tilde{x}_i(t - \tau_l)] + \sum_{l=1}^{M} \sum_{j=1}^{N} \sigma_{l,j} \tilde{H} [\tilde{x}_j(t - \tau_l)] \tag{66}\]

Now \(\tilde{x}(t)\) is a multidimensional vector. For example, for the LK equations \(\tilde{x}(t)\) contains the real and imaginary parts of the envelope of the electric field and the population inversion of the charge carriers. Details are given in the appendix. The dynamics of the SM is given by

\[
\dot{s}(t) = \tilde{F} [\tilde{s}(t)] + \sum_{l=1}^{M} (\eta_l + \sigma_l) \tilde{H} [\tilde{s}(t - \tau_l)] \tag{67}\]

The stability of the SM is described by linear equations for each mode \(k\) with eigenvalues \(\gamma_{k,l}\) of the coupling
matrix $G_l$. The equations corresponding to Eq. (4) are
\begin{align}
\dot{\xi}_k(t) &= DF[\vec{s}(t)] \xi_k(t) + \\
&+ \sum_{l=1}^{M} (\eta_l + \sigma_l \gamma_{k,l}) DH[\vec{s}(t-\tau_l)] \xi_k(t-\tau_l)
\end{align}
(68)

$DF$ and $DH$ are the Jacobian matrices of $F$ and $H$ respectively evaluated at the SM. The master stability function for the LK equations is defined in the appendix. In the following paragraphs we consider networks with a single delay time, $M = 1$, and with double delays, $M = 2$.

**Networks with a single delay time**

In the previous section, we obtained the analytic result that the eigenvalue gap of the coupling matrix determines the stability of the SM in the limit of large delay times $\tau$, which will be realized in this section by $\tau = 100\text{ ns}$ for the LK equations and $\tau = 100$ for the tent map and the Bernoulli map. The relation $|\gamma_1| < \exp(-\lambda_{\text{max}} \tau)$ of Eq. (36) is the condition for a stable SM; it relates the maximal Lyapunov exponent $\lambda_{\text{max}}$ of the SM to the second largest modulus of the eigenvalues of the coupling matrix $G$.

As a consequence of Eq. (36), a pair of units without self-feedback cannot be synchronized, since $\gamma_1 = -1$. This result agrees with experiments on semiconductor lasers. Two lasers coupled by their mutual laser beams cannot be synchronized with zero time lag, only high correlations with a time shift of $\tau$ have been observed.

With self-feedback, however, the situation is different. Eqs. (26) and (31) show that a pair of units can be synchronized if the local Lyapunov exponent is negative. Again, this result agrees with experiments on semiconductor lasers where the self-feedback is realized by external mirrors.

Thus, qualitatively, the relation Eq. (36) is in agreement with experiments on lasers. Unfortunately, up to now, experiments on larger networks of coupled lasers are not reported. Hence, the quantitative comparison with lasers has to rely on numerical simulations of the LK equations.

Fig. 9 shows the numerically calculated maximal Lyapunov exponent $\lambda_{\text{max}}$ as a function of the coupling parameter $\sigma$ for a network of lasers and as a function of the coupling parameter $\varepsilon$ for a network of tent maps and Bernoulli maps. Self-feedback is suppressed ($\eta = 0$). The parameters for the LK equations are defined in the appendix.

For a triangle with bidirectional couplings, it is $\gamma_1 = -\frac{1}{2}$. Eq. (36) predicts a transition to synchronization for $\lambda_{\text{max}} \tau = -\ln\left(\frac{1}{2}\right) \approx 0.69$, the horizontal dashed line in Fig. 10(a). The numerical results of Fig. 10 give a critical coupling $\varepsilon_c \approx 0.9$ for the tent maps, vertical dashed line in Fig. 10(b), and two critical couplings $\sigma_{c,1} \approx 0.45\text{ ns}^{-1}$ and $\sigma_{c,2} \approx 45\text{ ns}^{-1}$ for the three lasers, vertical dashed line in Fig. 10(b). Fig. 11 shows the cross-correlations obtained from numerical simulations of the corresponding triangle. We find the measured critical couplings of the tent maps and the LK equations to be in good agreement with the predictions of Eq. (36). A more challenging test of the condition Eq. (36) is a network with directed couplings. In this case the eigenvalue $\gamma_1$ is a complex number. For example, for the square with one diagonal of Fig. 4 one can add an additional coupling strength $\rho$ for the diagonal and obtain the eigenvalue gap of Fig. 12. The largest gap is obtained for $\rho = \frac{3}{2}$ with $\gamma_1 = -\frac{1+i\sqrt{11}}{2}$.

Combining this result with Eq. (36) and $\lambda_{\text{max}}(\sigma)$ (Fig. 9a), we obtain the phase diagram of Fig. 13 for the corresponding laser network. Fig. 13 also shows parameters for which complete zero-lag synchronization is
achieved in numerical simulations of the complete laser network. We define complete zero-lag synchronization between the lasers for isochronal cross-correlations larger than 0.99 in between the power drop-outs of the low frequency fluctuations which happen on a time scale of the order of magnitude of $10 \tau$. While it makes sense to speak of complete synchronization for such values of the cross-correlations, the maximal Lyapunov exponent is still slightly positive for correlations around 0.99. Hence, some of the shown points of complete synchronization in Fig. 13 lie outside of the predicted stability border which corresponds to a maximal Lyapunov exponent of exactly zero. The quantitative agreement between the relation Eq. (36) and the phase diagram Fig. 13 of complete zero-lag synchronization is remarkable.

Networks with two delay times

A pair of units without self-feedback cannot be synchronized if its coupling has a single delay time. In the previous section, however, we have shown in agreement with [35, 44] that zero-lag synchronization is possible if the coupling contains two delay times $\tau_1$ and $\tau_2$. Only if the ratio $\frac{\tau_2}{\tau_1}$ is a ratio of odd relatively prime integers, $\frac{\tau_2}{\tau_1} = \frac{p}{q}$, synchronization is excluded. The parameter region of synchronization is largest for small values of $p$ and $q$.

This result is in agreement with recent experiments on semiconductor lasers [53]. For $\frac{\tau_2}{\tau_1} = \frac{2}{1}$ complete synchronization was observed. Cross-correlations were large for $\frac{\tau_2}{\tau_1} \in \{ \frac{3}{2}, \frac{4}{3}, \frac{5}{4} \}$ and low for $\frac{\tau_2}{\tau_1} \in \{ \frac{1}{2}, \frac{2}{3}, \frac{3}{5} \}$.

The analytic result was based on the symmetry of the phase diagrams, Figs. 5 and 6. Here we show that these symmetries can be observed for the MSF of laser networks and tent maps as well. With two delays, Eq. (67) and (68) reduce to

$$\dot{s}(t) = F[s(t)] + \sigma_1 H[s(t - \tau_1)] + \sigma_2 H[s(t - \tau_2)] \quad (69)$$

$$\dot{\xi}_k(t) = D F[s(t)] \dot{\xi}_k(t) +$$

$$+ \sigma_1 \gamma_{k,1} D H[s(t - \tau_1)] \dot{\xi}_k(t - \tau_1) +$$

$$+ \sigma_2 \gamma_{k,2} D H[s(t - \tau_2)] \dot{\xi}_k(t - \tau_2) \quad (70)$$

Because of the invasive nature of the coupling, we obtain a different trajectory $\mathbf{s}(t)$ on the SM for each pair of coupling strengths $(\sigma_1, \sigma_2)$ which in each case gives a different linear stability equation. In order to compare
with the previous section, we fix $\sigma_1$ and $\sigma_2$, vary $\gamma_{k,1}$ and $\gamma_{k,2}$ and calculate the maximal Lyapunov exponent of the linear equation Eq. (70). In this interpretation of $\beta_1 = \sigma_1 \gamma_{k,1}$ and $\beta_2 = \sigma_2 \gamma_{k,2}$, the results are universal in view of the fact that they make a statement about the stability of all modes of every possible network with real $\gamma_{k,1}$ and $\gamma_{k,2}$ for the chosen coupling strengths $(\sigma_1, \sigma_2)$. The point $\gamma_{0,1} = \gamma_{0,2} = 1$ is shared among all networks; it determines the stability of the dynamics on the SM.

Fig. 14 shows the results of the numerical simulations of Eqs. (69) and (70) with $\tau_1 = 20$ ns, $\tau_2 = 40$ ns, $\gamma_{k,1} = \gamma_{k,2} = 45$ ns$^{-1}$ for the LK equations and $\gamma_{1} = 200$, $\gamma_{2} = 400, \epsilon = 0.9, \kappa = 0.5$ for the tent map and the Bernoulli map. As shown in the previous section, the region of stability should have reflection symmetry at the horizontal axis. The stability region for laser networks (Fig. 14k) is in agreement with this symmetry. The corresponding stability regions of the networks of tent maps and Bernoulli maps (Fig. 14b) coupled with two delay times, show this reflection symmetry as well.

In the previous section, we showed that synchronization is sensitive to detuning of the ratio of the delay times, see Fig. 7. Fig. 15 shows that synchronization of two lasers is destroyed if $\tau_1$ and $\tau_2$ differ by about 10 ps, which corresponds to the coherence length of the chaotic lasers. The coupling has a fixed delay time of $\tau_2 = 100$ ns and a strength of $\sigma_2 = 20$ ns$^{-1}$. The self-feedbacks have a delay time of $\tau_1 = \tau_2 + \Delta$ and a strength of $\sigma_2 = 30$ ns$^{-1}$. Thus, in agreement with the analytic results for Bernoulli networks, lasers are sensitive to detuning of the delay times as well [38, 49, 54]. Of course, the detailed structure of the cross-correlations of Fig. 15 depends on the details of the laser dynamics which cannot be predicted by iterated maps.

Chaos synchronization of networks of identical nonlinear units with time-delayed couplings is investigated. Although the units are coupled with long delay times, they synchronize to a common chaotic trajectory without time shift. For rather general networks with multiple delay times the method of the master stability function allows to relate the stability of the synchronization manifold (SM) to the eigenvalue gaps of the coupling matrices.

Fig. 15: Sensitivity of cross-correlations to detuning of the delay times $\tau_2$ and $\tau_1 = \tau_2 + \Delta$ for a pair of Lang-Kobayashi equations.

**SUMMARY**

For a single delay time, Eq. (36) is the most important result. It relates the eigenvalue gap of the coupling matrix of the whole network to the Lyapunov exponent of the trajectory of a single unit with feedback. It is exact for Bernoulli networks with a long delay time. But it compares well with our numerical results for networks of tent maps and LK units, too. Even the phase diagram of a directed network with complex eigenvalues, which has been calculated for lasers, is in good agreement with this fundamental equation. For a pair of units without self-feedback and for any bipartite network the eigenvalue gap is zero, hence these networks cannot synchronize completely in the limit of a large delay time, only cluster synchronization is possible.

For networks with several delay times, we could not find a simple relation for the stability of the SM. However, the theory of polynomials showed some symmetries of the phase diagrams for Bernoulli networks. Our numerical results for tent maps and LK equations showed
these symmetries, as well. For a pair of units coupled by multiple delay times these symmetries have interesting consequences. Only for special ratios of the delay times synchronization is possible in the limit of long delay times, in agreement with self-consistent arguments related to mixing of information [23]. Again, this analytic result is in agreement with numerical simulations of the LK equations and even with experiments on semiconductor lasers [53].

These results show that networks of iterated Bernoulli units have universal properties. On the one hand, we have analytic tools to calculate the stability of chaos synchronization since the linearized equations do not contain the chaotic trajectory. On the other hand, we have either numerical simulations of the linearized difference/differential equations containing the chaotic trajectory as input, or we have direct simulations of the complete network. We have found that there is good agreement between these different systems, sometimes even on a quantitative level.

The Lang-Kobayashi equations and their master stability equations

The Lang-Kobayashi equations in their complex form are

\[ \dot{E}^i(t) = \frac{1+i\alpha}{2} G_N n^i(t) E^i(t) + \sigma_1 E^i(t - \tau_1) e^{-i\omega_0 \tau_1} + \sigma_2 \sum_{j=1}^{N} G_{ij} E^j(t - \tau_2) e^{-i\omega_0 \tau_2} \]

\[ \dot{n}^i(t) = (p - 1) J_{th} - \gamma n^i(t) - \left[ \Gamma + G_N n^i(t) \right] |E^i(t)|^2 \]

where \( E^i(t) \) is the envelope of the complex electric field and \( n^i(t) \) is the renormalized population inversion of the charge carriers of laser \( i \). The used constants are listed in Tab. I.

For the numerical simulation of the equations, we make the ansatz

\[ E^i(t) = R^i(t) + iI^i(t) \]

and thus obtain the real-valued differential equation system

\[ \dot{R}^i(t) = \frac{1}{2} G_N n^i(t) [R^i(t) - \alpha I^i(t)] + \sigma_1 R^i(t - \tau_1) \cos(\omega \tau_1) + \sigma_1 I^i(t - \tau_1) \sin(\omega \tau_1) + \sigma_2 \sum_{j=1}^{N} G_{ij} R^j(t - \tau_2) \cos(\omega \tau_2) + \sigma_2 \sum_{j=1}^{N} G_{ij} I^j(t - \tau_2) \sin(\omega \tau_2) \]

\[ \dot{I}^i(t) = \frac{1}{2} G_N n^i(t) [I^i(t) + \alpha R^i(t)] - \sigma_1 R^i(t - \tau_1) \sin(\omega \tau_1) - \sigma_1 I^i(t - \tau_1) \cos(\omega \tau_1) - \sigma_2 \sum_{j=1}^{N} G_{ij} R^j(t - \tau_2) \sin(\omega \tau_2) - \sigma_2 \sum_{j=1}^{N} G_{ij} I^j(t - \tau_2) \cos(\omega \tau_2) \]

\[ \dot{n}^i(t) = (p - 1) J_{th} - \gamma n^i(t) - \left[ \Gamma + G_N n^i(t) \right] \left\{ [R^i(t)]^2 + [I^i(t)]^2 \right\} \]

We integrate this differential equation system numerically using Heun’s method [54] which is a numerical integration method of the class of Runge-Kutta methods that is particularly suitable for delay-differential equation systems. The used step size for integration is \( \Delta t = 0.1 \) ps. In order to emulate the measurement of cross-correlations with a gigahertz oscilloscope in an experimental setup, on the one hand, we define cross-correlations between two simulated lasers as the cross-correlations between the absolute values of their electric fields. On the other hand, we use a sampling time of 1 ns per data point during which the absolute values of the electric field are averaged. For the precision measurement of the sensitivity of cross-correlations to detuning of the delay times in

| Parameter                         | Value                       |
|----------------------------------|-----------------------------|
| Linewidth enhancement factor     | \( \alpha = 5 \)            |
| Differential optical gain        | \( G_N = 2.142 \times 10^4 \text{s}^{-1} \) |
| Laser frequency                  | \( \omega_0 = \frac{2\pi c}{635 \text{nm}} \) |
| Pump current relative to \( J_{th} \) | \( p = 1.02 \)             |
| Threshold pump current of solitary laser | \( J_{th} = \gamma N_{\text{sol}} \) |
| Carrier decay rate               | \( \gamma = 9.099 \times 10^9 \text{s}^{-1} \) |
| Carrier number of solitary laser  | \( N_{\text{sol}} = 1.707 \times 10^8 \) |
| Cavity decay rate                | \( \Gamma = 0.357 \times 10^{12} \text{s}^{-1} \) |
Fig. 15, we use a sampling time of 10 ps. Desynchronization of the simulated lasers occurs during the power drop-outs of the low frequency fluctuations which happen on a time scale of the order of magnitude of 10 τ. In order to avoid measuring cross-correlations during them, the cross-correlations are calculated ten times from comparatively short time windows of the length of 5 τ, and then the five largest cross-correlations are averaged.

The master stability equations of the Lang-Kobayashi equations are

\[ \dot{\delta}_R(t) = \frac{1}{2} G_N n(t) [\delta_R(t) - \alpha \delta_I(t)] + \]
\[ + \frac{1}{2} G_N \delta_n(t) [R(t) - \alpha I(t)] + \]
\[ + \sigma_1 \delta_R(t - \tau_1) \cos(\omega \tau_1) + \]
\[ + \sigma_1 \delta_I(t - \tau_1) \sin(\omega \tau_1) + \]
\[ + \sigma_2 \gamma_k \delta_R(t - \tau_2) \cos(\omega \tau_2) + \]
\[ + \sigma_2 \gamma_k \delta_I(t - \tau_2) \sin(\omega \tau_2) \]

and

\[ \dot{\delta}_I(t) = \frac{1}{2} G_N n(t) [\delta_I(t) + \alpha \delta_R(t)] + \]
\[ + \frac{1}{2} G_N \delta_n(t) [I(t) + \alpha R(t)] - \]
\[ - \sigma_1 \delta_R(t - \tau_1) \sin(\omega \tau_1) - \]
\[ - \sigma_1 \delta_I(t - \tau_1) \cos(\omega \tau_1) - \]
\[ - \sigma_2 \gamma_k \delta_R(t - \tau_2) \sin(\omega \tau_2) - \]
\[ - \sigma_2 \gamma_k \delta_I(t - \tau_2) \cos(\omega \tau_2) \]

(77)

(78)

which are integrated numerically using Heun’s method, as well. We calculate the maximal Lyapunov exponent using Farmer’s method [40]. For correspondence to our definition of the cross-correlation above, we define the deviation of the absolute value of the electric field as the metric of the separation function. This is legitimate as the Lang-Kobayashi equations form a strongly coupled differential equation system.

The Schur-Cohn theorem defines determinants

\[ \delta_{\nu+1} = \begin{vmatrix} a_n & 0 & \ldots & 0 & a_0 & a_1 & \ldots & a_\nu \\ a_{n-1} & a_n & \ldots & 0 & a_0 & a_1 & \ldots & a_{\nu-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n-\nu} & a_{n-\nu+1} & \ldots & a_n & 0 & 0 & \ldots & 0 \\ a_{n-1} & a_{n-\nu} & \ldots & 0 & a_n & a_0 & \ldots & a_{\nu-1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_1 & a_2 & \ldots & a_{\nu} & a_{\nu+1} & \ldots & a_n & 0 \end{vmatrix} \]

with \( \nu = 0, 1, \ldots, n - 1 \).

Due to the Schur-Cohn theorem, it is \(|x| < 1\) when all determinants are greater than 0.

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