Asymptotically optimal delay-aware scheduling in wireless networks

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Abstract

In this paper, we investigate a channel allocation problem in networks taking into account the queues of users. Typically, there are less available channels than users, and at each slot the channels are allocated to users in such a way to minimize the total average queues in the network. We show that the problem falls in the framework of Restless Bandit Problems (RBP), for which obtaining the optimal solution is out of reach. This problem is analyzed in this paper using Whittle index approach. First, using the Lagrangian relaxation method, we provide a relaxed problem and show that it can be decomposed into simpler one-dimensional subproblems for which the optimal solution is a threshold-based policy. This allows us to characterize Whittle’s indices for these one-dimensional systems and to develop an index-based heuristic policy for the original scheduling problem. We prove that this heuristic is asymptotically optimal in the infinitely many users regime and provide numerical results that illustrate its remarkably good performance.

I. INTRODUCTION

In wireless networks, the problem of user and channel scheduling has been widely recognized as a mean to improve the performance of the network and to meet the service demands of the users. This problem has been widely studied in the past and several allocation policies have been developed for various contexts, e.g. see \cite{6}, \cite{7}, \cite{8}, \cite{24}, \cite{25}, \cite{16}, \cite{9} and the references therein. In 5G networks, the problem of channel and user scheduling will be receiving particular interest due to the increase of number of devices and users. Furthermore, the applications nowadays do not need high data rates only but they are more delay-sensitive, which implies that minimizing the delay is a main design metric in future networks.

In addition, the emerging of new physical layer techniques such as Massive-MIMO (M-MIMO) introduce new user and channel allocation problems in wireless networks. For example, in a M-MIMO system operating in Time Division Duplex (TDD) mode, a high spatial multiplexing gain can be achieved allowing thus a high number of users to be scheduled in each time-slot. On the other hand, this high multiplexing gain relies on the instantaneous channel state information (CSI) knowledge of the users at the base station. The CSI can be acquired at the base station by decoding the training sequences transmitted by the users. This requires however that the users employ orthogonal sequences. The number of orthogonal sequences is however limited in practice as the length of these sequences is proportional to their number. Typically the number of users in a network is much higher than the number of pilots, which requires to allocate the pilots to the users. This problem is nothing but a channel and user
scheduling in which a user cannot be allocated more than one channel (as allocating two pilots to a given user is useless). In this paper, we are interested in a scheduling problem inspired by the problem of pilot allocation in massive MIMO systems. Furthermore, in massive MIMO, the spatial diversity results in a well known phenomenon called ”channel hardening” \[11\]. This implies that the fluctuation of the instantaneous received useful signal around its fading-averaged value vanishes when the number of antennas increases. In other words, the channel variations (due to fast fading) average out over the antennas and the Signal to Noise Ratio depends only on the large scale fading. This has an impact on the study of wireless systems, that use massive MIMO, as the bit rate of the users will depend mainly on path loss coefficients.

In this paper, we consider the problem of scheduling and channel allocation in a wireless network composed of one Base Station serving a large number of users. The channel allocation in this paper can be seen as a pilot allocation in the context of massive MIMO as discussed above. Furthermore, we assume that the number of channels is limited and each channel can only be allocated to one user at a time. The objective in this case is to find an allocation policy that minimizes the long-run average delay of the users’ packets. We show in this paper that this problem can be cast as a Restless Bandit Problem (RBP), which is a particular Markov Decision Processes (MDP). However, RBPs are PSPACE-Hard (see Papadimitriou et al. \[20\]), and hence their optimal solution is out of reach. One should therefore propose sub-optimal policies when dealing with such problems. In this paper, we approach the considered RBP problem using the Lagrangian relaxation technique, which consists of relaxing the constraint on the available resources. In other words, instead of having the constraint on the number of available channels satisfied in every time slot, we consider that it has to be satisfied on average. This allows to decompose the large relaxed optimization problem into much simpler one-dimensional problems. Based on the optimal solution of the individual relaxed problems, we develop a heuristic for the original (i.e. non-relaxed) optimization problem. This heuristic is known as the Whittle’s index policy (WIP) and we will show that for our particular model an explicit expression of Whittle’s index can be found. WIP has been proposed as a suboptimal policy for many problems in the literature, see for instance \[1\], \[15\]. It has also been shown to perform near optimally in many scenarios and in the particular case of multiclass M/M/1 queues, WIP which simplifies to the $c\mu$-rule is optimal, see Buyukkoc et al. \[3\], and Larranaga \[13\]. In this paper, we will prove that the developed WIP is asymptotically optimal in the many user regime. In more detail, our contributions in this paper are as follows.

- We provide an analysis of the relaxed optimization problem, which leads to obtain an explicit characterization of Whittle’s indices by (i) proving that threshold policies are optimal for each of the relaxed one-dimensional problems, (ii) obtaining the explicit expressions of the steady-state distribution of the system’s states under threshold policies and (iii) proving the indexability property which ensures existence of the indices.
- We provide further characterization of the threshold-based optimal solution of the relaxed optimization problem. This analysis provided in Section VI is very important to evaluate the asymptotic performance of the proposed WIP.
- We show that WIP is asymptotically optimal in the infinitely many users regime, that is, when the number of users in the system as well as the available channels grow large.
Finally, we provide numerical performance results of WIP that corroborate our claims.

A. Related Work

The problem of resource allocation and scheduling in wireless networks has been widely studied in the literature. In [24], [25], [7], [8], [6], throughput optimal schedulers have been derived for single channel, multi-channel and multi user MIMO contexts. The aforementioned work focuses on developing strategies that stabilize the queues of the users using max weight rule. The classical max weight rule is however known to not be delay optimal. To overcome this issue, many works have been developed in the past to take into account the average delay of the traffic of the users (e.g. see [5] and the references therein). Most of the existing work uses Markov Decision Process (MDP) frameworks and develops allocation strategies using Bellman equation (e.g. by using value iteration, policy iteration, etc.). MDP frameworks and Bellman equation suffer from the curse of dimensionality, which leads to complex resource allocation strategies. In [2], [26], the authors try to minimize the average delay of the users’ queues using Markov Decision Process (MDP) and stochastic learning tools. The complexity of the developed solutions is however much higher than the Whittle index policy. Stochastic learning is also used in [4] to deal with the problem of power allocation in an OFDM (Orthogonal Frequency Division Multiplexing) system with the goal to minimize the average delay of the users’ packets in the queues. The developed solution requires high memory and computational complexity as compared to whittle index policy.

Whittle index based policies have also been used/developed in wireless networks to deal with the problem of pilot allocation over Markovian channel models. If a pilot is allocated to a user, its Channel State Information (CSI) can be estimated correctly and the user can hence transmit at a given rate. In [17], [15] a Gilbert-Elliot channel model is considered and the whittle index is derived. It has been shown in [17], [18] that a policy based on Whittle index is asymptotically optimal for their specific problem. [14] extends the problem of pilot allocation to the case where the channel evolves according to a Markovian process between K states instead of two states as in the Gilbert-Elliot model. In the aforementioned papers, the queues of the users were not considered. In fact, the focus was on the channel allocation such that the long term total throughput (or equivalent objective function) is maximized without taking into account the dynamic traffic of the users. In this paper, we consider that the traffic arrival is bursty and that the objective of the user/channel allocation is to minimize the long term average queues of the users.

In [1], a derivation of whittle index values for a simple multiclass M/M/1 model has been considered (where only one user can be served). However, the optimality of obtained whittle index policy has not been proved in [1] and time is assumed to be continuous. [27] considers the problem of project/job scheduling in which an effort is allocated to fixed number of projects. The performance of a Whittle index based policy is analyzed under a continuous time model. In contrast to these two papers, we consider that the time is slotted and that several users can be scheduled at a given time slot and not only one user. We provide an explicit characterization of Whittle indices, develop a Whittle index allocation policy for our problem and prove the asymptotic optimality of the developed policy in the many users regime.

The remainder of the paper is organized as follows. In Section II we formulate the problem under study and we introduce the Lagrangian relaxation. In Section III we prove the optimality of threshold/monotone policies for the
relaxed problem. In Section IV we compute the steady-state distribution. In Section V we characterize Whittle’s indices explicitly and we explain Whittle’s index policy. Section VI provides further characterization of the optimal solution of the relaxed problem. In Sections VII and VIII we prove respectively the local and global asymptotic optimality of WIP. In Section IX we evaluate the performance of WIP numerically. The proofs are provided in the appendices.

II. SYSTEM MODEL AND PROBLEM FORMULATION

A. System model description

We consider a time-slotted system with one base station, $N$ users and $M$ uncorrelated channels with ($N > M$). A channel can be allocated to at most one user, hence only $M$ users will be able to transmit at time slot $t$. We define $K$ different classes of users and we assume that each user in class-$k$, if scheduled, transmits at most $R_k$ packets per time slot. We will refer to $R_k$ as the maximum transmission rate for every user in class $k$ and we assume $\min_k\{R_k\} \geq 2$. We denote by $\gamma_k$ the proportion of class-$k$ users in the system. The terms users and queues will be used interchangeably in the paper. We further denote by $A_k(t) \in \{0, \ldots, R_k - 1\}$ the number of packets that arrive to queue $i$ in class $k$ at time slot $t$. Let $q_{k,i}^\phi(t)$ denote the number of packets in queue $i$ in class $k$. Furthermore, let $s_{k,i}^\phi(\mathbf{q}^\phi(t))$ denote the transmission action under a decision policy $\phi$ and $\mathbf{q}^\phi(t)$ the vector of all queue lengths $(q_{1,1}^\phi(t), \ldots, q_{N,1}^\phi(t), \ldots, q_{1,K}^\phi(t), \ldots, q_{N,K}^\phi(t))$. For the sake of clarity, we define $s_{k,i}^\phi(t) := s_{k,i}^\phi(\mathbf{q}^\phi(t))$. If policy $\phi$ prescribes to schedule user $i$ in class $k$, then $s_{k,i}^\phi(t) = 1$, and $s_{k,i}^\phi(t) = 0$ otherwise. We denote by $L$ the buffer capacity, which is the same for all queues and is greater than all maximum rates, i.e., $L \geq \max_k\{R_k\}$. We further make the following assumption, which is required for the derivations in Sections IV and V.

Assumption 1. Let $R_k$ be the maximum transmission rate of class-$k$ users and $L$ the buffer size. We assume $L \geq 2 \max_k\{R_k\}$.

The number of packets in queue $i$ in class $k$ evolve as follows

$$q_{k,i}^\phi(t + 1) = \min\{(q_{k,i}^\phi(t) - R_k s_{k,i}^\phi(t))^+ + A_k^\phi(t), L\},$$

where $(x)^+ = \max\{x, 0\}$.

![Figure 1: System Model](image)
It is worth mentioning that assumption 1 is useful to simplify the derivations of the stationary distribution and whittle index values in Sections IV and V. This assumption is also realistic as the maximum buffer length $L$ is often much higher than the transmission rate $R_k$. The objective of the present work is to find the scheduling policy $\phi$ that minimizes the average queue length of the users, which results according to Little Law to minimize the average delay.

**B. Problem formulation**

The cost incurred by user $i$ in class $k$, at time $t$ is equal to $a_k q_{k,\phi}^i(t)$ for all $i \in \{1, \ldots, \gamma_k N\}$ where $a_k$ is a predefined weight. One can see that the model described in Section II-A is a Restless Bandit Problem (RBP) [28].

We consider the broad class $\Phi$ of scheduling policies in which a scheduling decision depends on the history of observed queue states and scheduling actions. Our user and channel allocation problem is then to identify a policy $\phi \in \Phi$ that minimizes the infinite horizon expected average queues, subject to the constraint on the number of users selected in each time slot. Given the initial state $q_0 = (q_1^1(0), \ldots, q_{\gamma_1 N}^1(0), \ldots, q_1^K(0), \ldots, q_{\gamma_K N}^K(0))$ the problem can be formulated as follows:

$$\min_{\phi \in \Phi} \mathbb{E} \left[ \limsup_{T \to \infty} \frac{1}{T} \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} a_k q_{k,\phi}^i(t) \mid q_0 \right],$$

(2)

subject to

$$\sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} s_{k,\phi}^i(t) \leq \alpha N, \text{ for all } t,$$

(3)

where $\alpha = M/N$ is the fraction of users that can be scheduled.

**III. RELAXED PROBLEM AND THRESHOLD-BASED POLICY**

As it has been discussed in the introduction of this paper, RBPs are PSPACE-Hard (see Papadimitriou et al. [20]) and therefore one should develop well performing sub-optimal policies. In this paper, the development of our policy is done through several steps. First, We consider a Lagrangian relaxation of our problem and show it can be decomposed into several one-dimensional problems. We then prove that the optimal solution to each of these relaxed problems is a threshold-based policy. We then compute the stationary distribution of the states of the system under the aforementioned threshold policy. This allows us to obtain a closed form expression of the Whittle index values of the relaxed problem and develop a Whittle index-based scheduling policy for the original RBP.

In this section, we first formulate the relaxed problem and prove that its optimal policy is a threshold-based one.

**A. Relaxed Problem and Dual Problem**

The Lagrangian relaxation consists of relaxing the constraint of the available resources. Namely, we consider that the constraint in Equation (3), has to be satisfied on average and not in every decision epoch, that is,

$$\limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} s_{k,\phi}^i(t) \right] \leq \alpha N.$$  

(4)
Note that, contrary to the strict constraint in Equation (3), the relaxed constraint allows the activation of more than $\alpha$ fraction of users in each time slot. If we note $W$ the Lagrangian multiplier for the constrained problem, then the Lagrange function equals to:

$$f(W, \phi) = \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} \left( a_k q_i^{k,\phi}(t) + W s_i^{k,\phi}(t) \right) Q[0] \right] - W \alpha N,$$

where $W$ can be seen as a subsidy for not transmitting. Therefore, the dual problem for a given $W$ is

$$\min_{\phi \in \Phi} f(W, \phi).$$

(5)

**B. Problem Decomposition and Threshold-based Policy**

In this section, we show that the relaxed problem can be decomposed into $N$ one-dimensional subproblems, for which the optimal solution is a threshold-based policy. To do that, we first get rid of the constants that do not depend on $\phi$ and reformulate the problem as follows,

$$\min_{\phi \in \Phi} \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} \left( a_k q_i^{k,\phi}(t) + W s_i^{k,\phi}(t) \right) \mid q(0) \right].$$

(6)

One can see that the solution of this problem is the solution of well known Bellman equation, see Ross [22], namely,

$$\bar{V}(\bar{q}) + \theta = \min_{\bar{s}} \left\{ \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} C_k(q_i^k, s_i^k) + \sum_{\bar{q}} \Pr(q' \mid q, \bar{s}) \bar{V}(q') \right\},$$

(7)

for all $\bar{q} = (q_1^1, \ldots, q_{\gamma_1 N}^1, \ldots, q_1^K, \ldots, q_{\gamma_k N}^K)$, with $q_i^k \in \{1, \ldots, L\}$ the queue length of class-$k$ user $i$, and $\bar{s} = (s_1^1, \ldots, s_{\gamma_1 N}^1, \ldots, s_1^K, \ldots, s_{\gamma_k N}^K)$, with $s_i^k \in \{0, 1\}$ the action taken with respect to user $i$ in class $k$. In Equation (7), $V(\cdot)$ represents the Value Function, $\theta$ is the optimal average cost and $C_k(q_i^k, s_i^k)$ is the holding cost $a_k q_i^k + W s_i^k$.

The optimal decision for each state $q$ can be obtained by minimizing the right hand side of Equation (7). We now show that the problem can be decomposed into $N$ independent subproblems by decomposing $\bar{V}(\cdot)$ into Value Functions for each user $i$ in class $k$, i.e., $V_i^k(\cdot)$. In other words, the optimal decision $\bar{s}$ to Problem (7) is a vector composed of elements $s_i^k$, where each $s_i^k$ is nothing but the optimal decision that solves the individual Bellman equations.

$$V_i^k(q_i^k) + \theta_i^k = \min_{s_i^k} \left\{ C_k(q_i^k, s_i^k) + \sum_{q_i^k} \Pr(q_i^k \mid q_i^k, s_i^k) V_i^k(q_i^k) \right\}. $$

(8)

This is proven in the next proposition.

**Proposition 1.** Let $V_i^k(\cdot)$ be the optimal value function that solves Equation (8), and let $\bar{V}(\cdot)$ be the optimal value function that solves Equation (7) then

$$\bar{V}(\cdot) = \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} V_i^k(\cdot).$$

Proof. See appendix A

In this section we show that the solution to each individual problem (for each user $i$) follows the structure of a threshold policy. For ease of notation, we drop the indices $k$ and $i$ and consider that $V(\cdot)$ is the value function for a given user. We first provide the definition of threshold policies.
The solution of bellman equation \( V(\cdot) \) can be obtained by the well known Value iteration algorithm, which consists in updating \( V_t(\cdot) \) using the following equation

\[
V_{t+1}(q) = \min_s \{C(q, s) + \sum_{q'} Pr(q'|q, s)V_t(q')\} - \theta
\]

We consider that the initial value function \( V_0 \) equal to 0 for any \( q \), (i.e. for all \( q \ V_0(q) = 0 \)). After many iteration \( V_t(\cdot) \) will converge to the unique fixed point of the equation (8) called \( V(\cdot) \). However, value iteration algorithm might have high complexity and it can take time to converge, then we will be limited to give some structural properties of the value function \( V_t(\cdot) \) for any \( t \) and conclude that the optimal policy is a threshold-based one.

For that, we consider the operator \( TO \) such that for each \((q, s) \in \{0, \ldots, L\} \times \{0,1\}\)

\[
(\text{TO}(V))(q, s) = C(q, s) + \sum_{q'} Pr(q'|q, s)V(q') - \theta
\]

We first give some useful definitions and preliminary results before proving the desired result.

**Definition 2.** We say that function \( f \) is \( R \)-convex in \( X = \{0, \ldots, L\} \), if for any \( x \) and \( y \) in \( X \) such that \( x < y \), we have:

\[
f(y + R) - f(x + R) \geq f(y) - f(x)
\]

**Lemma 1.** If for given function \( f \), there exist \( R \) such that for any \( x \in \{0, \ldots, L-1\} \), \( f(x+1+R) - f(x+R) \geq f(x+1) - f(x) \), then \( f \) is \( R \)-convex

**Proof.** Considering \( y \) and \( x \) in \( \{0, \ldots, L-1\} \), with \( y > x \), we have:

\[
f(y + R) - f(x + R) = \sum_{k=x}^{y-1} [f(k+1+R) - f(k+R)] \geq \sum_{k=x}^{y-1} [f(k+1) - f(k)] = f(y) - f(x)
\]

We conclude the result. □

**Definition 3.** Let \( g(q, s) \) be a real valued function defined on \( X \times S \), with \( S = \{0,1\} \), and \( X = \{0, \ldots, L\} \), we say that \( g \) is submodular if \( g(q+1,1) - g(q+1,0) \leq g(q,1) - g(q,0) \) for all \( q \) on \( X \).

**Theorem 1.** \( \min_s \text{TO}(\cdot,s) \) conserves the \( R \)-convexity and increasiness properties, where \( R \) is the maximum rate. In other words, if the input of the operator \( TO \), i.e. a given \( V(\cdot) \), is \( R \)-convex and increasing function in \( q \) then \( \min_s \text{TO}(V)(\cdot,s) \) is \( R \)-convex and increasing function in \( q \).
Proof. Let us consider that the input of $TO(V)(\cdot, s)$, i.e. a given, $V(\cdot)$ is R-convex and increasing in $q$ [19].

For the increasiness property of $\min_s TO(V)(\cdot, s)$, we have by definition that $C(\cdot, s)$ is increasing in $q$. We also have that $V(\cdot)$ is increasing in $q$ and the number of queue state is finite, then $\sum_{q'} Pr(q'|\cdot, s)V(q')$ is increasing function in $q$, see Puterman [21]. Since $\theta$ is a constant, $TO(V)(\cdot, s)$ is increasing in $q$, therefore $\min_s TO(V)(\cdot, s)$ is increasing in $q$.

For R-convexity, we should first prove the following lemma. In the remaining of this section, we drop the indices $k$ and $i$ to simplify the notation (e.g. the queue length of a given user is denoted by $q$).

Lemma 2. If $V(\cdot)$ is R-convex and increasing in $q$, $C(q, s)$ and $\sum_{q'} Pr(q'|q, s)V(q')$ are submodular functions.

Proof. The proof is given in appendix [B] ■

This demonstrates that the function $TO(V)(\cdot, \cdot)$ is submodular since it is the sum of two submodular functions. Let us now show that $\min_s TO(V)(\cdot, s)$ is R-convex. For that, we consider the function $\Delta TO(V)(q) = TO(V)(q, 1) - TO(V)(q, 0)$ which is decreasing in $q$ since $TO(V)(\cdot, \cdot)$ is submodular. Therefore there exists $r \in \mathbb{R} \cup \{+\infty\}$ such that for $q \leq r$, $\Delta TO(V)(q) \geq 0$ and for $q \geq r$, $\Delta TO(V)(q) \leq 0$. In the remaining of the proof, we consider all possible cases of $q$ and $r$.

If $q + R + 1, q + R, q + 1 \geq r$:

$$ \min_s TO(V)(q + 1 + R, s) - \min_s TO(V)(q + R, s) $$
$$ = TO(V)(q + 1 + R, 1) - TO(V)(q + R, 1) $$
$$ = TO(V)(q + 1, 0) - TO(V)(q, 0) $$
$$ \geq TO(V)(q + 1, 1) - TO(V)(q, 1) $$
$$ = \min_s TO(V)(q + 1, s) - \min_s TO(V)(q, s) $$

where the inequality is due to the sub-modularity of $TO(V)(\cdot, \cdot)$.

If $q \leq R \leq q + 1, q + R, q + 1 + R$:

$$ \min_s TO(V)(q + 1 + R, s) - \min_s TO(V)(q + R, s) $$
$$ = TO(V)(q + 1 + R, 1) - TO(V)(q + R, 1) $$
$$ = TO(V)(q + 1, 0) - TO(V)(q, 0) $$
$$ \geq TO(V)(q + 1, 1) - TO(V)(q, 0) $$
$$ = \min_s TO(V)(q + 1, s) - \min_s TO(V)(q, s) $$
if \( q, q + 1 \leq r \leq q + R, q + R + 1 \):

\[
\min_s \text{TO}(V(q + 1 + R, s)) - \min_s \text{TO}(V(q + R, s)) \\
= \text{TO}(V(q + 1 + R, 1)) - \text{TO}(V(q + R, 1)) \\
= \text{TO}(V(q + 1, 0)) - \text{TO}(V(q, 0)) \\
= \min_s \text{TO}(V(q + 1, s)) - \min_s \text{TO}(V(q, s))
\]

if \( q, q + 1, q + R \leq r \leq q + R + 1 \):

\[
\min_s \text{TO}(V(q + 1 + R, s)) - \min_s \text{TO}(V(q + R, s)) \\
= \text{TO}(V(q + 1 + R, 1)) - \text{TO}(V(q + R, 0)) \\
\geq \text{TO}(V(q + 1 + R, 1)) - \text{TO}(V(q + R, 1)) \\
= \text{TO}(V(q + 1, 0)) - \text{TO}(V(q, 0)) \\
= \min_s \text{TO}(V(q + 1, s)) - \min_s \text{TO}(V(q, s))
\]

If \( q, q + 1, q + R; q + R + 1 \leq r \):

\[
\min_s \text{TO}(V(q + 1 + R, s)) - \min_s \text{TO}(V(q + R, s)) \\
= \text{TO}(V(q + 1 + R, 0)) - \text{TO}(V(q + R, 0)) \\
\geq \text{TO}(V(q + 1 + R, 1)) - \text{TO}(V(q + R, 1)) \\
= \text{TO}(V(q + 1, 0)) - \text{TO}(V(q, 0)) \\
= \min_s \text{TO}(V(q + 1, s)) - \min_s \text{TO}(V(q, s))
\]

Using lemma \[\min_s \text{TO}(V(\cdot, s))\] is R-convex in \( q \), i.e. we conclude the R-convexity conservation.

**Remark 1.** We proved in the previous Theorem that if the value function \( V_t \) is increasing and R-convex, then the value function \( V_{t+1} \) in equation (9), which is computed with the operator TO is increasing and R-convex. Thus, as \( V_0 \) is increasing and R-convex, all \( V_t \) are increasing and R-convex, then we conclude that the value function \( V \) will be also R-convex and increasing in \( q \).

**Corollary 1.** The optimal policy \( \phi^* \) of each one-dimensional relaxed subproblem is a threshold-based policy.
Proof. As explained in Definition 1, it is sufficient to prove that the optimal policy $\phi^*$ is monotone in $q$.

We consider $q_1 \leq q_2$. Using lemma 2, $TO(V)$ is submodular. Therefore, we have:

$$(TO(V))(q_1, 1) - (TO(V))(q_1, 0) \geq (TO(V))(q_2, 1) - (TO(V))(q_2, 0)$$

If $\phi^*(q_2) = \arg\min_s (TO(V))(q_2, s) = 0$

Hence

$$(TO(V))(q_1, 1) - (TO(V))(q_1, 0) \geq (TO(V))(q_2, 1) - (TO(V))(q_2, 0) \geq 0$$

Then

$$(TO(V))(q_1, 1) - (TO(V))(q_1, 0) \geq 0$$

which leads to

$$\arg\min_s (TO(V))(q_1, s) = 0$$

i.e.

$$\phi^*(q_1) \leq \phi^*(q_2)$$

If $\phi^*(q_2) = \arg\min_s (TO(V))(q_2, s) = 1$, obviously we have:

$$\phi^*(q_1) \leq \phi^*(q_2)$$

We conclude that the optimal solution is monotone and increasing in $q$, which implies that it is a threshold policy. 

IV. Stationary distribution

We have seen previously that the optimal solution of problem \(\phi^*\) is a threshold-based policy. Let us define $n_k$ as the threshold for users in class $k$, i.e. if the queue length of user $i$ in class $k$ is such that $q_k^i \leq n_k$ then the user will not be scheduled, and else the user will be selected for transmission. The objective of this section is to derive the stationary distribution of the users’ states. This will be useful in the subsequent section in the derivation of a closed form expression of the whittle index values. We assume here that at each queue $i$ in class $k$, packets arrive according to a discrete uniform distribution, that is, $\mathbb{P}(A_k^i(t) = x) = \rho_k$ for all $0 \leq x \leq R_k - 1$ and 0 otherwise, where $\rho_k = 1/R_k$.

For ease of notation, we again drop the indices $k$ and $i$ (e.g. we denote the threshold by $n$ and the queue length by $q$). We denote by $p^n(i, j)$ the transition probability from state $i$ to $j$, by $u$ the stationary distribution under threshold policy $n$, and by $R$ the maximum rate. One can notice that $u$ verifies the full balance equation, i.e.:

$$u(i) = \sum_{j=0}^{L} p^n(j, i)u(j) = \sum_{j=0}^{n} p^n(j, i)u(j) + \sum_{j=n+1}^{L} p^n(j, i)u(j)$$

Definition 4. We define $\pi_i$ as:

$$\pi_i = \begin{cases} 
\rho & \text{if } 0 \leq i \leq R - 1 \\
0 & \text{else} 
\end{cases}$$
Proposition 2. The expressions of \( p^n(j, i) \) are given by:

if \( 0 \leq i < L \) and \( j \leq n \)

\[
p^n(j, i) = \pi_{i-j} = \begin{cases} 
\rho & \text{if } 0 \leq i - j \leq R - 1 \\
0 & \text{else}
\end{cases}
\]

if \( 0 \leq i < L \) and \( n < j \leq L \)

\[
p^n(j, i) = \pi_{i-(j-R)^+} = \begin{cases} 
\rho & \text{if } 0 \leq i - (j - R)^+ \leq R - 1 \\
0 & \text{else}
\end{cases}
\]

if \( i = L \) and \( n < j \leq L \)

\[
p^n(j, L) = (R - L + j)\pi_{L-j} = \begin{cases} 
(R - L + j)\rho & \text{if } 0 \leq L - j \leq R - 1 \\
0 & \text{else}
\end{cases}
\]

if \( i = L \) and \( n < j \leq L \)

\[
p^n(j, L) = 0
\]

Proof. See appendix C.

Proposition 3. The expressions of the stationary distribution

1) \(-1 \leq n < R:\)

\[
u(i) = \begin{cases} 
\rho - (n - i)\rho^2 & \text{if } 0 \leq i \leq n \\
\rho & \text{if } n + 1 \leq i \leq R - 1 \\
(n + R - i)\rho^2 & \text{if } R \leq i \leq n + R
\end{cases}
\]

2) \(R \leq n < L - R:\)

\[
u(i) = \begin{cases} 
\rho - (n - i)\rho^2 & \text{if } n - R + 1 \leq i \leq n \\
(n + R - i)\rho^2 & \text{if } n \leq i \leq n + R - 1
\end{cases}
\]

3) \(L - R \leq n < L:\)

\[
u(i) = \begin{cases} 
\rho^2(R - n + i) & \text{if } n - R + 1 \leq i \leq L - R - 1 \\
(1 - \rho)^{n-i}\rho & \text{if } L - R \leq i \leq n \\
\rho - \rho^2(i - n) & \text{if } n + 1 \leq i \leq L - 1 \\
(1 - \rho)^{n-L+R+1} - \rho(L - 1 - n) & \text{if } i = L
\end{cases}
\]

4) \(n = L:\)

\[
u(i) = \begin{cases} 
0 & \text{if } 0 \leq i \leq L - 1 \\
1 & \text{if } i = L
\end{cases}
\]

Proof. See appendix D.
V. Whittle’s Index

In this section, we provide the derivation of the Whittle indices, which are values that depend on the queue state of the user and its maximum rate. Although this derivation is made using the relaxed problem, it allows us to develop a heuristic for the original problem. It is worth mentioning that the whittle’s index at a given state, say \( n \), represents the Lagrange multiplier for which the optimal decision of the relaxed problem at this state is indifferent (passive and active decision are both optimal). However, Whittle’s index is well defined only if the property of indexability is satisfied. This property requires to establish that as the Lagrange multiplier (or equivalently the subsidy for passivity \( W \)) increases, the collection of states in which the optimal action is passive increases. In this section, we work on a given class \( k \), and we consider its maximum transmission rate is \( R \) with \( \rho = 1/R \). All the obtained results here can be applied for any class. We start the derivation by first reformulating the relaxed problem using the stationary distribution derived in the previous section. Since the solution of the relaxed problem (given a constant \( W \)) is a threshold-based policy, we can reformulate the problem as follows:

\[
\min_{n \in [0, L]} \mathbb{E}[aq^n + Ws^n] = \min_{n \in [0, L]} \left\{ \sum_{q=0}^{L} au^n(q)q - W \sum_{q=0}^{n} u^n(q) \right\} \tag{10}
\]

with \( n \) the threshold, \( u^n \) the stationary distribution under threshold policy \( n \).

The new formulation of the problem turns out to be useful to derive the whittle indices since for any \( W \), we can find the minimizer of the expression in equation (10).

We give first the expression of the mean cost in equation (10) given threshold \( n \) (for all possible values of \( n \)):

if \(-1 \leq n \leq R - 1\):

\[
\sum_{q=0}^{L} au^n(q)q = a\left[ \frac{R-1}{2} + \frac{n(n+1)}{2R} \right]
\]

if \( R \leq n \leq L - R \):

\[
\sum_{q=0}^{L} au^n(q)q = an
\]

if \( L - R + 1 \leq n \leq L - 1 \):

\[
\sum_{q=0}^{L} au^n(q)q = a[n + 1 - R + 2\rho(1 - \rho)^{n-L+R+1} + \rho(L - 1 - n)(n - L)]
\]

if \( n = L \):

\[
\sum_{q=0}^{L} au^n(q)q = a
\]

Second, we give the expression of the passive decision’s average time in equation (10) given threshold \( n \):

if \(-1 \leq n \leq R - 1\):

\[
\sum_{q=0}^{n} u^n(q) = (1 - \frac{n}{2R})(\frac{n+1}{R})
\]

if \( R \leq n \leq L - R \):

\[
\sum_{q=0}^{n} u^n(q) = \frac{1}{2} + \frac{1}{2R}
\]

if \( L - R + 1 \leq n \leq L - 1 \):

\[
\sum_{q=0}^{n} u^n(q) = \frac{\rho^2}{2}(L - 1 - n)(L - n) + 1 - (1 - \rho)^{n-L+R+1}
\]
if \( n = L \):
\[
\sum_{0}^{n} u^n(q) = 1
\]

A. Computation of Whittle’ index values

We first formalize the indexability and the whittle’s index in the following definition.

**Definition 5.** A class is indexable if the set of states in which the passive action is the optimal action (denoted by \( D(W) \)) increases in \( W \), that is, \( W' < W \Rightarrow D(W') \subseteq D(W) \). When the class is indexable, the whittle’s index in state \( n \) is defined as:

\[
W(n) = \min\{W | n \in D(W)\}
\]

In the literature, several works have been conducted to find whittle index values. For example, an interesting iterative algorithm has been provided in [13]. Even though the context of our work here is different from the one considered in [13], we will prove in the sequel that the proposed algorithm in [13] can be adapted to our case up to some modifications (e.g. in our case we have a maximum buffer state L, etc). In addition, further analysis will be provided here to derive a closed form expression of the whittle index values. We will first provide this modified algorithm and then prove that it allows computing the whittle’s index values for our problem.

**Algorithm 1:**

**Step 0:**

\[
W_0 = \inf_{n \in \mathbb{N}} \frac{\sum_{0}^{L} au^n(q)q - \sum_{0}^{L} au^{n-1}(q)q}{\sum_{0}^{n} u^n(q)}
\]

We call \( n_0 \) the largest minimizer of this expression, then we define \( W(k) = W_0 \) for all \( k \leq n_0 \).

**Step j:** We compute:

\[
W_j = \inf_{n \in \mathbb{N} \setminus \{n : \sum_{0}^{n} u^n(q) = \sum_{0}^{n+j} u^{n+j-1}(q)\} \cup \{0, \ldots, n_j-1\}} \frac{\sum_{0}^{L} au^n(q)q - \sum_{0}^{L} au^{n_j-1}(q)q}{\sum_{0}^{n_j} u^n(q) - \sum_{0}^{n_j-1} a u^{n_j-1}(q)}
\]

We denote \( n_j \) the largest minimizer, and we define \( W(k) = W_j \), for all \( n_{j-1} < k \leq n_j \), we go to step \( j + 1 \). We stop when \( n_j = L \).

The whittle index of state \( k \) is given by \( W(k) \).

**Proposition 4.** *Assuming that the optimal solution is a threshold policy, and that \( \sum_{0}^{n} u^n(q) \) is increasing, then the class is indexable. Moreover, if \( \sum_{0}^{L} au^n(q)q \) is increasing in \( n \) and for all \( i \) and \( j \) such that \( i < j \)

\[
\sum_{0}^{i} w^i(q) = \sum_{0}^{j} w^j(q) \Rightarrow \sum_{0}^{L} au^n(q)q < \sum_{0}^{L} au^j(q)q,
\]

then the Whittle’s index values are computed by applying Algorithm 1.

**Proof.** For the proof, see appendix [11].

**Remark 2.** In order to simplify the notation in the sequel, we denote \( \sum_{0}^{L} au^n(q)q \) by \( a_n \) and \( \sum_{0}^{n} u^n(q) \) by \( b_n \).
In order to apply Algorithm 1 that allows to obtain the whittle’s index for each state in our case, we need to prove that the conditions given in proposition 4 are satisfied.

**Theorem 2.** For each k, the class-k is indexable.

**Proof.** According to proposition 4, we need just to prove that \( \sum_{0}^{n} u^{n}(q) \) is increasing. The proof is based on the two following two lemmas.

**Lemma 3.** \( \sum_{0}^{n} u^{n}(q) \) is strictly increasing in \([-1, R - 1]\)

**Proof.** See appendix K

**Lemma 4.** \( \sum_{0}^{n} u^{n}(q) \) is strictly increasing in \( n \in [L - R + 1, L - 1] \)

**Proof.** See appendix L

We have for any \( n \in [R, L - R] \):

\[
\sum_{0}^{n} u^{n}(q) = \sum_{0}^{R-1} u^{R-1}(q) + \sum_{0}^{L-R+1} u^{L-R+1}(q) = \frac{1}{2} + \frac{1}{2R}
\]

Then

\[
\sum_{0}^{R-1} u^{R-1}(q) \leq \sum_{0}^{n} u^{n}(q) \leq \sum_{0}^{L-R+1} u^{L-R+1}(q)
\]

Moreover

\[
\sum_{0}^{L} u^{L}(q) = 1 > 1 - (1 - \rho)^{R} = \sum_{0}^{L-1} u^{L-1}(q)
\]

Therefore combining Lemma 3 and Lemma 4 \( \sum_{0}^{n} u^{n}(q) \) is increasing in \([0, L]\), thus we conclude the indexability of the class.

We prove the two others conditions of Proposition 4 which are the increasiness of \( \sum_{0}^{L} au^{n}(q)q \) in \( n \), and that for all \( i \) and \( j \) such that \( i < j \) \( \sum_{i}^{j} w^{i}(q) = \sum_{i}^{j} w^{j}(q) \implies \sum_{0}^{L} au^{i}(q)q < \sum_{0}^{L} au^{j}(q)q \)

From the expression of \( a_{n} \) when \( n \in [-1, R - 1] \), \( a_{n} \) is clearly increasing in \( n \). For \( n \in [R, L - R - 1] \), \( a_{n} \) is strictly increasing and \( a_{R-1} = a(R - 1) < aR = a_{R} \), which implies that \( a_{n} \) is increasing in \([-1, L - R - 1]\). For \( n \in [L - R, L - 1] \), we give the following lemma

**Lemma 5.** \( \sum_{0}^{L} au^{n}(q)q \) is strictly increasing in \([L - R, L - 1]\).

**Proof.** See appendix N

We have \( \sum_{0}^{L} au^{L-R}(q)q = a(L-R) > a(L-R-1) = \sum_{0}^{L} au^{L-R+1}(q)q \), and \( \sum_{0}^{L} au^{L-1}(q)q = aL - aR(1 - 2(1 - \rho)^{R}) < aL = \sum_{0}^{L} au^{L}(q)q \) (because \( 1 - 2(1 - \rho)^{R} > 1 - 2 \exp(-1) \geq 0 \)), then we conclude that \( a_{n} \) is increasing in \([-1, L]\).

For the second condition (for all \( i \) and \( j \) such that \( i < j \) \( \sum_{i}^{j} w^{i}(q) = \sum_{i}^{j} w^{j}(q) \implies \sum_{0}^{L} au^{i}(q)q < \sum_{0}^{L} au^{j}(q)q \)), the only case when \( \sum_{0}^{L} w^{i}(q) = \sum_{0}^{L} w^{j}(q) \) is when \( i \) and \( j \) are in the set \([R - 1, L - R + 1]\). In this set we have shown
that \( \sum_{0}^{n} u^{n}(q) \) is strictly increasing, then for \( i < j \) and \( (i, j) \in [R - 1, L - R + 1]^{2} \), \( \sum_{0}^{L} au^{i}(q)q < \sum_{0}^{L} au^{j}(q)q \), hence the two conditions are satisfied.

As the indexability is satisfied and the two conditions of proposition 4 are verified, then we can apply Algorithm 1 to get the Whittle’s index for each state. However, the complexity of this algorithm is \( L^{2} \), where \( L \) is the maximum buffer length which could be large in practice. In order to overcome this complexity issue, we will provide further analysis and derive simple expressions of Whittle indices.

We proceed first by giving the following definitions and lemmas.

**Definition 6.** For given increasing threshold policy \( n \), we define \( y^{n} \) as function of the subsidy \( W \), such that

\[
y^{n}(W) = \sum_{0}^{L} au^{n}(q)q - W \sum_{0}^{n} u^{n}(q) = a_{n} - Wb_{n}.
\]

**Lemma 6.** For any state \( (i, j) \in [-1, L]^{2} \), the intersection point’s abscess between \( y^{i}(W) \) and \( y^{j}(W) \) denoted by \( x_{i,j} \) is:

\[
\sum_{0}^{L} au^{i}(q)q - \sum_{0}^{L} au^{j}(q) \sum_{0}^{i} u^{i}(q) - \sum_{0}^{j} u^{j}(q)
\]

**Proof.** See appendix O. 

**Definition 7.** We define for \( 0 \leq n \leq R - 1 \), \( w_{n} = x_{n,n-1} = \frac{\sum_{0}^{L} au^{n}(q)q - \sum_{0}^{L} au^{n-1}(q)}{\sum_{0}^{n} u^{n}(q) - \sum_{0}^{n-1} u^{n-1}(q)} = \frac{a_{n} - a_{n-1}}{b_{n} - b_{n-1}} = \frac{aR_{n}}{R-n} \) (by replacing \( a_{n} \) and \( b_{n} \) by their expressions when \( 0 \leq n \leq R - 1 \)).

**Definition 8.** We define a function \( f \), such that for each \( n \in [0, R - 1] \), \( f(n) = w_{n}\left[\sum_{0}^{L} u^{n}(q)q - \sum_{0}^{n} u^{n}(q)\right] + \sum_{0}^{L} au^{n}(q)q = w_{n}[1 - (1 - \frac{n}{2R})^{\frac{n+1}{n}}] + a(\frac{R-1}{2} + \frac{n(n+1)}{2R}) \) for \( n = R \), \( f(R) = +\infty \), and for \( n = -1 \), \( f(-1) = 0 \).

**Lemma 7.** \( f \) is strictly increasing in \( n \), for \( n \in [-1, R] \).

**Proof.** See appendix P. 

**Lemma 8.** We consider \( e = \frac{5}{2}R - \frac{\sqrt{14R^{2} - 4R}}{2} \). Assume that \( L \geq 2R \), then for all \( n < \min(e, R) \), \( L > \frac{f(n)}{a} \).

**Proof.** See appendix O. 

**Theorem 3.** If there exists an integer \( d \) in \( [\min(e-1, R-1), R-1] \), such that \( \frac{f(d)}{a} < L \leq \frac{f(d+1)}{a} \), the Whittle’s index expressions are:

for \( 0 \leq n \leq d \): \( W(n) = w_{n} = x_{n,n-1} = \frac{aR_{n}}{R-n} \)

for \( d < n \leq L \): \( W(n) = x_{L,d} = a(\frac{L-(\frac{R-1}{2})+\frac{d(d+1)}{R}}{1-(\frac{n}{2R})^{\frac{n+1}{n}}}) \)

**Proof.** We first give a justification of the interval \([\min(e-1, R-1), R-1] \). In fact, for an integer \( d < \min(e-1, R-1) \) we have \( d < e - 1 \) (that is \( d + 1 < e \)). By using Lemma 8, we conclude that \( f(d + 1)/a < L \), which implies that the inequality \( \frac{f(d)}{a} < L \leq \frac{f(d+1)}{a} \) cannot be true for \( d < \min(e-1, R-1) \).

To prove Theorem 3 according to proposition 4, we have to prove that, from \( 0 \leq j \leq d \), the largest minimizer at step
is \( j \) and at step \( d \) is \( L \). In other words, for all \( 0 \leq j \leq d \), we have \( \frac{a_j-a_{j-1}}{b_j-b_{j-1}} < \frac{a_n-a_{j-1}}{b_n-b_{j-1}} \) for all \( n > n_{j-1} + 1 = j \) such that \( b_n \neq b_{j-1} \) and \( \frac{a_n-a_d}{b_n-b_d} \leq \frac{a_n-a_{d+1}}{b_n-b_{d+1}} \) for all \( n \geq n_d + 1 = d + 1 \) such that \( b_n \neq b_d \), with \( n_j \) the largest minimizer at step \( j \).

For the detailed proof see appendix \[R\]

\[B. \text{ Whittle index policy for the original problem}\]

We now consider the original optimization problem \[3\] and propose a simple Whittle index policy. This policy consists simply of allocating the channels to the \( M \) users that have the highest whittle index at time \( t \), denoted by \( WI \), and computed using the simple expressions in Theorem \[3\].

\[\text{Figure 2: Illustration of Whittle’ index Policy}\]

In Figure 2 the normal lines are for \( n \leq R - 1 \), the broken ones for \( R \leq n \leq L - R \), the dotted ones for \( L - R + 1 \leq n \leq L - 1 \), and the line with rounds for \( n = L \). As we observe, the slope of this latter line is very high if we compare it with the other curves. This means that all intersection points between the round line and normal lines are surely smaller enough than all intersection points between the dotted and normal lines, which confirms our whittle index expressions.

\[VI. \text{ Further analysis of the optimal solution of the relaxed problem}\]

In this section, we provide further analysis and give the structure of the optimal solution for the relaxed problem, which will be useful for the proof of optimality of the whittle index policy. As we have seen in section \[III\] for given \( W \), the optimal solution for the dual relaxed problem \[6\] is a threshold-based policy for each user. By using the whittle index expressions defined in section \[V\] we will provide a derivation of the optimal threshold for each class as function of the Lagrange parameter \( W \). In this section we denote by \( W_i^k \) the whittle’s index at state \( i \) in class \( k \) (the user and class indices cannot be dropped here as in the previous sections). We denote by \( l = (l_1, l_2, \ldots, l_K) \) the vector which represents the set of thresholds for each class \( k \). We also consider that for each class, the integer
\(d_k\) (which depends on the maximum rate \(R_k\)), defined in proposition 3, is equal to \(R_k - 1\) (this can be true for \(L > \max_k \frac{f(R_k - 1)}{a}\)) and that \((W \leq W^*_k)\) [4] We denote by \(\mu_k^\star\), the stationary distribution for class \(k\) under threshold \(n\).

**Proposition 5.** For a given \(W\), the optimal threshold vector \(l = (l_1(W), l_2(W), \ldots, l_K(W))\) for the dual problem satisfies:

For each \(k:\)

\[l_k(W) = \max_i \{\arg\max_i \{W^k_i | W^k_i \leq W\}\}\]

or

\[l_k(W) = \max_i \{\arg\max_i \{W^k_i | W^k_i < W\}\}\]

In other word, \(l_k\) is the biggest index among the ones that give the biggest whittle index less than \(W\), or strictly less than \(W\).

We note that the solution can also be a linear combination between the threshold policies \(\max_i \{\arg\max_i \{W^k_i | W^k_i \leq W\}\}\) and \(\max_i \{\arg\max_i \{W^k_i | W^k_i < W\}\}\).

**Proof.** See appendix S. □

Now we give the structure of the optimal solution of the constrained relaxed problem.

**Proposition 6.** The solution for the constrained relaxed problem is of type threshold policy \(l(W^\star)\), with \(l\) the function vector defined in Proposition 5 and \(W^\star\) satisfies \(\alpha = \sum_0^K \gamma_k \sum_0^L u_k^\star (W^\star) (i)\).

**Proof.** See appendix T. □

In conclusion, the solution of the relaxed problem is a threshold policy characterized by its \(W^\star\), which verifies the constraint. To be more precise, there exists \(W^\star\) that satisfies the constraint and gives a threshold policy for all classes except one class for which the optimal solution is a linear combination of two threshold policies. This structure of the optimal policy is very interesting and will be used in the proof of asymptotic optimality.

Moreover, using the stationary distribution derived in Section IV, the optimal cost \(C_{RP,N}^{RP,N}\) can be given by the following expression:

\[C_{RP,N}^{RP,N} = \sum_{k=1}^K \gamma_k N \sum_{i=0}^L a_k l_k(W^\star) (i) i = \sum_{k=1}^K \gamma_k N \sum_{i=0}^L a_k l_k^\star (W^\star) (i) i\]

**VII. LOCAL OPTIMALITY**

In this section, we will show that the performance of whittle index policy is asymptotically locally optimal, that means, for large number of users \(N\) and large number of channels \(M\) (\(\alpha = \frac{M}{N}\) is a constant value), the whittle index policy is locally optimal for each class.

---

1In fact, if there exists \(k\) such that \(W > W_k^L\), then for all queues belonging to this class and for all states the optimal decision is a passive action. This is not realistic if we take into account the fairness, since all these queues can never transmit.
index policy is locally optimal. For that we will compare the average cost obtained by the whittle index policy WI
with the one obtained for the relaxed problem RP.
In fact, if we denote $C_{WI,N}^{N}$ the average cost of whittle index policy, $C_{OP}^{N}$ the optimal cost of the original
problem with instantaneous inequality and $C_{RP,N}^{N}$ the optimal cost of the relaxed problem, we have: $C_{WI,N}^{N} \geq C_{OP}^{N} \geq C_{RP,N}^{N}$.
Hence, in order to show the local asymptotic optimality, we just need to prove that for large $N$, $C_{WI,N}^{N}$ converges
to $C_{RP,N}^{N}$ (this directly implies that $C_{WI,N}^{N}$ will converge to $C_{OP}^{N}$).
For that, we will be in need of the optimal cost expression for the relaxed problem $C_{RP,N}^{N}$ derived in section VI.
First, we denote by $Z_{i}^{k,N}$ the proportion of state $i$ in all class $k$ queues. We have $Z^{N} = (Z_{1}^{N}, \ldots, Z_{K}^{N})$ with $Z_{i}^{k,N} = (Z_{1}^{k,N}, \ldots, Z_{L}^{k,N})$ and $\sum_{i=0}^{L} Z_{i}^{k,N} = \gamma_{k}$ for each class $k$. In order to prove the local optimality, we use
the fluid limit technique that consists of analyzing the evolution of the expectation of $Z^{N}[t]$ under Whittle’s Index
Policy. For that, we define vector $z[1]$ as follows:
$$z[t+1] - z[t]|z[t] = z = \mathbb{E}[Z^{N}[t+1] - Z^{N}[t]|Z^{N}[t] = z]$$
If we denote by $w_{i}^{h}$ the whittle’s index for class $h$ at state $j$ and by $p_{i}^{k}(z)$ the probability that a user is selected randomly among $z_{i}^{k}$ to transmit, one can show easily that [27]:
$$p_{i}^{k}(z) = \min\{z_{i}^{k}, \max(0, \alpha - \sum_{w_{j}^{h} > w_{i}^{k}} z_{j}^{h})\}/z_{i}^{k}$$
We denote by $q_{i,j}^{k,0}$ the probability to pass from state $i$ to state $j$, in a class $k$ queue, if the queue is not scheduled for transmission. We denote by $q_{i,j}^{k,1}$ the probability to pass from state $i$ to state $j$, in a class $k$ queue, if the queue is scheduled for transmission. Then the probability to pass from state $i$ to state $j$, in class $k$, is:
$$q_{i,j}^{k}(z) = p_{i}^{k}(z)q_{i,j}^{k,1} + (1 - p_{i}^{k}(z))q_{i,j}^{k,0}$$
We consider $w^{*}$ is the lagrangian parameter that gives the optimal solution of the relaxed problem, and that $(l_{1}, \ldots, l_{k})$ is the corresponding optimal threshold vector (see Proposition 6). We assume that there exists a given class $m$ such that $w_{i}^{m} = w^{*}$. Since we proved in Proposition 5 that for all $k$ the threshold $l_{k}$ can be $l_{k} < R_{k}$ or $l_{k} = L$, we assume in the sequel that $l_{k} < R_{k}$ for all $k$.
We define $j_{w^{*}}$ the set such that at system state $z \in j_{w^{*}}$, if we use whittle index policy, all users with whittle index value higher than $w^{*}$ are scheduled, the users with whittle index value smaller than $w^{*}$ stay idle, while the users at index value $w^{*}$ are scheduled with certain randomization. Specifically, $j_{w^{*}} = \{z : \sum_{w_{i}^{k} > w^{*}} z_{i}^{k} > \alpha, \sum_{w_{i}^{k} \leq w^{*}} z_{i}^{k} \leq \alpha\}$ We start with $z[0]$ in $j_{w^{*}}$
then
$$z_{i}^{k}(t+1) - z_{i}^{k}(t) = \sum_{j \neq i} q_{i,j}^{k}(z(t)) z_{j}^{k}(t) - \sum_{i \neq j} q_{i,j}^{k}(z(t)) z_{i}^{k}(t)$$
We have the following equality for all $k$ and $t$:
$$\sum_{j=0}^{L} z_{j}^{k}(t) = \gamma_{k}$$
Then we replace in all equations $z^k(t)$ by $\gamma_k - \sum_{j=0, j \neq k}^L z^k_j(t)$.

As $z(t) \in j_w^*$, we can show the following:

1) $k \neq m$:

$$z^k_i(t + 1) = \sum_{j=0}^{I_i-1} (q_{j,i} - q_{k,i})z^k_j(t) + \sum_{j=I_k+1}^L (q_{j,i} - q_{k,i})z^k_j(t) + \gamma_k q_{k,i}$$

2) $k = m$

$$z^m_i(t + 1) = \sum_{j=0}^{I_i-1} (q_{j,i} - q_{m,i})z^m_j(t) + \sum_{j=I_m+1}^L (q_{j,i} - q_{m,i})z^m_j(t) + (1 - \alpha)q_{m,i} + \alpha q_{m,i}$$

Let $g^m_i = \sum_{k=1}^K \gamma_k (1 - \alpha)q_{m,i} + \alpha q_{m,i}$ and $c^m = ((1 - \alpha)q_{m,i} + \alpha q_{m,i}, (1 - \alpha)q_{m,i} + \alpha q_{m,i})$ for each $k \neq m$.

The following linear relation in $j_w^*$ between $z(t+1)$ and $z(t)$ can be obtained:

$$z(t+1) = Qz(t) + C$$

The expression of matrix $Q$ is given in Appendix \[ U \]. The vector solution of the relaxed problem denoted by $z^*$ is the fixed point of the aforementioned linear equation. Moreover $z^* \in j_w^*$, and if we take vector $z(0) = z^* + \epsilon$, then we obtain:

$$z(t) - z^* = Q^t\epsilon$$

The analysis of the above linear system is therefore important to prove the local optimality. We first provide the following lemma.

**Lemma 9.** If for all eigenvalues $\lambda$ of $Q$, $|\lambda| < 1$, then considering neighborhood $\Omega_\sigma(z^*) \subseteq j_w^*$:

1) There exists $t_0$ such that for all $t \geq t_0$, $||z(t) - z^*|| < \sigma (z(t) \in j_w^*)$

2) If we take $z(0)$ such that $||z(0) - z^*|| < \sigma$, $z(t)$ will converge to $z^*$

**Proof.** The proof follows from linear system convergence.

**Proposition 7.** For all eigenvalue $\lambda$ of $Q$, $|\lambda| < 1$

**Proof.** See the proof in appendix \[ U \]

The aforementioned result, combined with Lemma 9, proves the convergence of the fluid limit system (i.e. $z(t+1) = Qz(t) + C$). However, the above result is not enough to prove the local optimality, as we have to show that the stochastic vector $Z(t)$ converges to $z^*$ in probability. For that we introduce the discrete-time version of Kurtz Theorem applied to our problem, see \[ U \].
Proposition 8. There exists a neighborhood $\Omega_\delta(z^*)$ of $z^*$ such that if $Z^N[0] = z[0] = x \in \Omega_\delta(z^*)$, then for any $\mu > 0$ and finite time horizon $T$ there exists positive constants $C_1$ and $C_2$ such that

$$P_x\left( \sup_{0 \leq t < T} ||Z^N(t) - z(t)|| \geq \mu \right) \leq C_1 \exp(-NC_2)$$

where $\delta < \sigma$, and $P_x$ denotes the probability conditioned on the initial state $Z^N[0] = z(0) = x$. Furthermore, $C_1$ and $C_2$ are independent of $x$ and $N$.

According to this Proposition, the system state $Z^N[t]$ behaves very close to the fluid approximation model $z[t]$ when the number of users $N$ is large. Since we have shown the convergence of $z[t]$ to within $\Omega_\sigma(z^*)$, we are ready to establish the local convergence of the system state $Z^N[t]$ to $z^*$.

Lemma 10. If $Z^N[0] = x \in \Omega_\delta(z^*)$, then for any $\mu > 0$ there exists a time $T_0$ such that for each $T > T_0$, there exists positive constants $s_1$ and $s_2$ with,

$$P_x\left( \sup_{T_0 \leq t < T} ||Z^N(t) - z^*|| \geq \mu \right) \leq s_1 \exp(-Ns_2)$$

Proof. See appendix \[V\]

Now we are ready to prove the asymptotic local optimality.

Proposition 9. If the initial state is within $\Omega_\delta(z^*)$, then

$$\lim_{N \to \infty} \frac{C_{\text{WL},N}}{N} = \frac{C_{\text{RP},N}}{N}$$

Proof. See appendix \[W\]

VIII. Global asymptotic optimality

In this section, we will prove that from any initial state $x$, the expected time average cost obtained with the whittle index policy is optimal when $N$ is very large. In contrast of the method used to prove the local optimality, here we work with the steady state distribution of the stochastic process $Z^N(t)$. To ensure that there is a stationary distribution for $Z^N(t)$, we need to show that there exists at least one recurrent state. Since the states evolve according to a finite state markov chain, we just need to prove that there exists a state reachable from any other states.

Lemma 11. The state $z(0) = (z^1(0), \cdots, z^K(0))$, defined for each class $k$ as $z^k(0) = (1,0,\cdots,0)$, is reachable from any initial state using the whittle index policy.

Proof. See appendix \[X\]

This lemma is stronger than proving the existence of recurrent state. Indeed, this allows us to deduce that $Z^N(t)$ evolves in one recurrent aperiodic class, and that there exists a stationary distribution for $Z^N(t)$ denoted by $Z^N(\infty)$. To deduce the aforementioned statement, we still need to check if for fixed $N$, there exists at least one recurrent state within $\Omega_\epsilon(z^*)$, as otherwise $\Omega_\epsilon(z^*)$ will be a transient class. If such state exists, surely $Z^N(t)$ will evolve in
for all states $h$. In addition, it will be useful for the subsequent analysis in this section also to derive the exact expression of $α$. Rewriting the expression of $α$ given in the proof of Proposition 6 we get:

$$α = \sum_{k \neq m} \sum_{i=l_k+1}^{L} γ_k u^k_{l_k} (i) + \sum_{i=l_m+1}^{L} γ_m u^*_m (i) + (1 - θ)γ_m u^m_{l_m-1}(l_m)$$

The relation between the optimal vector $z^*$ and the stationary distribution under optimal threshold is as follows:

For $k \neq m$: $γ^*_h = γ_k u^k_{l_k} (h)$.

For $k = m$: $γ^*_h = γ_m ((1 - θ)u^m_{l_m-1}(h) + (θu^m_{l_m}(h)) = γ_m u^*_m (h)$.

When $h = l_m \leq R_m - 1$, we have $u^m_{l_m-1}(l_m) = ρ_m = 1/R_m$, and $u^m_{l_m}(l_m) = ρ_m = 1/R_m = u^m_{l_m-1}(l_m)$. Then:

$$γ^*_m = γ_m[(1 - θ)ρ_m + θ_mρ_m] = γ_mρ_m$$

Hence:

$$γ_m(1 - θ)u^m_{l_m-1}(l_m) = γ_m(1 - θ)ρ_m = (1 - θ)γ^*_m$$

Therefore:

$$α = \sum_{k \neq m} \sum_{i=l_k+1}^{L} γ^*_k + \sum_{i=l_m+1}^{L} γ^*_m + (1 - θ)γ^*_m$$

In addition, it will be useful for the subsequent analysis in this section also to derive the exact expression of $u^k_{l_k}(h)$, for all states $h$, by applying the results found in section IV when the threshold $l_k$ is strictly less than $R_k$. For $k \neq m$, we have:

$$0 \leq h \leq l_k - 1 : u^k_{l_k}(h) = ρ_k - (l_k - h)ρ_k^2$$

$$l_k \leq h \leq R_k - 1 : u^k_{l_k}(h) = ρ_k$$

$$R_k \leq h \leq l_k + R_k - 1 : u^k_{l_k}(h) = (l_k + R_k - h)ρ_k^2$$

if $k = m$:

$$0 \leq h \leq l_m - 1 : u^*_m(h) = ρ_m - (l_m - 1 - h + θ)ρ_m^2$$

$$l_m \leq h \leq R_m - 1 : u^*_m(h) = ρ_m$$

$$R_m \leq h \leq l_m + R_m - 1 : u^*_m(h) = (l_m + R_m - 1 - h + θ)ρ_m^2$$

Now, we will find a path from 0 to $z^*$ under the Whittle index policy.

**Proposition 10.** Applying Whittle index policy, the steady state $z^*$ is reachable from state $z(0)$.

**Proof.** See appendix Y.

From this proposition, the state $z^*$ is reachable from any state, which means that $z^*$ is a recurrent state. However, as we remark in the demonstration of Proposition 10 the considered actions schedule a proportion of users (i.e. not
an integer value). This is not feasible and unrealistic for some (small) values of \( N \) since the queues are not splittable. In fact for some values of \( N \), the state \( z^* \) may not exist. On the other hand, we can say that for \( N \) large enough, for any \( \epsilon > 0 \), there exists at least one recurrent state within the neighborhood \( \Omega_\epsilon(z^*) \). This will ensure that there is a path to enter a neighborhood \( \Omega_\epsilon(z^*) \) from any initial state. However, it is important to ensure that the time to enter \( \Omega_\epsilon(z^*) \) should not scale up with \( N \). For that, we give the following assumption which will be later justified via numerical studies in Section IX.

**Assumption 2.** We assume that the expected time to enter a neighborhood of \( z^* \) from any initial state \( x \) does not depend on the number of queues \( N \). In other words, for all \( N \) the time to enter a neighborhood \( \Omega_\epsilon(z^*) \) denoted by \( \Gamma^N_x(\epsilon) \) is bounded by a constant \( T_b \).

Now we provide a useful lemma that allows us to demonstrate the global asymptotic optimality.

**Lemma 12.** Under assumption 2 for any \( \epsilon \), we have:

\[
\lim_{N \to +\infty} P(Z^N_N(\infty) \in \Omega_\epsilon(z^*)) = 1
\]

**Proof.** See lemma 6 in [17].

Since we have found a stationary distribution of \( Z^N(t) \) under whittle index policy, the cost can be written as follows:

\[
C^{WI,N} = \sum_{k=1}^{K} \sum_{i=0}^{L} a_k \mathbb{E} \left[ Z^N_k(\infty) \right] iN
\]

**Theorem 4.** Under assumption 2 and for any initial state, we have:

\[
\lim_{N \to +\infty} \frac{C^{WI,N}}{N} = \frac{C^{RP,N}}{N}
\]

**Proof.** See appendix Z.

**IX. Numerical Results**

In this section, we give some numerical results that confirm the asymptotic optimality of the developed Whittle index policy. We consider \( \alpha = 1/2 \), 2 classes \( R_1 = 5 \) and \( R_2 = 10 \), \( L = 50 \); \( \gamma_1 = \gamma_2 = 1/2 \), and \( a_1 = a_2 = a = 1 \). We also consider two given initial states: \( x \) where all queues are in queue state 0 and \( y \) where all queues are in state \( L \).

**A. Verification of assumption 2**

We plot in Figure 3 the evolution of the time needed to enter a neighborhood \( \Omega_\epsilon(z^*) \) (i.e. hitting time of \( \Omega_\epsilon(z^*) \)) with respect to \( N \), given that \( \epsilon \) is small enough.
One can see that for large values of $N$, the hitting time can be considered as constant and does not diverge, and this is true for initial states $x$ and $y$. This implies that the hitting time is bounded for large values of $N$, and consolidates the assumption

**B. Performance of whittle index policy**

We compare the long run expected average cost per user under the whittle index policy, i.e. 

$$\lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{N} C_{WI,N}^{k,i}(Q_k^i(t)) \big| x \right],$$

with the one obtained by applying the Max-Weight policy $MW$. The latter schedules, at time $t$, the $M$ weighted longest queues (equivalently the $M$ highest $a_k Q_k^i(t)$). We also compare the performance of these two policies with the optimal cost per user obtained by using the optimal solution of the relaxed problem, i.e. $C_{RP}^*/N$. The results are plotted in Figures (4.a) and (4.b) respectively for initial states $x$ and $y$ (defined above).
One can see that for large $N$, from any initial state, the cost given by whittle index policy tends to the optimal cost of the relaxed problem, which proves that it converges to the optimal solution of the original problem. One can remark that the optimal cost of the relaxed problem per user is constant and does not depend on $N$ (see section VI). Moreover, we remark that the solution given by $MW$ is suboptimal.

C. Fairness among users

In order to improve the fairness among the users in the network, one can use the developed Whittle index policy in this paper up to some modifications. For example, we introduce in this section the following new policy $\Theta$ which combines the whittle index policy and the fairness. At each time slot $t$, this policy consists of scheduling the users with the highest $W_k(Q_k^i(t)) D_k(Q_k^i(t))$, where $Q_k^i(t)$ is the queue state of user $i$ in class $k$, $W_k$ is the whittle index of state $Q_k^i(t)$ when the transmission rate is $R_k$ and $D_k(Q_k^i(t)) = \sum_{u=1}^{i} a_k Q_k^i(u)$. To evaluate numerically the performance of this policy, we consider two classes of users. We consider the following two costs $C_1$ and $C_2$ incurred respectively by users of class 1 and users of class 2, specifically $C_1 = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{i=1}^{N} C^{WI,N}(Q_1^i(t)) \mid x \right]$ and $C_2 = \lim_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{i=1}^{N} C^{WI,N}(Q_2^i(t)) \mid x \right]$. We plot these costs with respect to $N$ in Figure 5. In
Figure (5.a), the costs are obtained by applying the new policy $\Theta$ while in Figure (5.b) the whittle index policy is applied.

![Figure 5](image)

We conclude that the new policy gives better performance in terms of fairness, since it reduces the gap between the costs of the two classes of users.

X. CONCLUSION

In this paper, we studied the problem of user and channel scheduling under bursty traffic arrivals. At each time, only $M$ channels can be allocated to the users where a user can be allocated one channel at most. In practice, this problem can arise in the context of pilot allocation in cellular networks where a channel represents a pilot sequence assigned to user in order to estimate its CSI at the Base Station. We formulated a Lagrangian relaxation of the optimization problem and provided a characterization of the optimal solution of this relaxed problem. We then developed a simple Whittle’s index policy to allocate the channels to the users and proved its asymptotic local and global optimality when the numbers of users and channels are large enough. This result is of interest as
the developed Whittle’s Index Policy has low complexity and is near optimal for large number of users. We then provided numerical results that corroborate our claims.

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We consider the Bellman Equation \([8]\). By summing the RHS and the LHS of Equation \([8]\), for all \(k\) and \(i\) we obtain

\[
\sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} [V_i^k(q_i^k) + \theta_i^k] = \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} \min_{s_k} \left\{ C_k(q_i^k, s_i^k) + \sum_{q_i'} \Pr(q_i'|q_i^k, s_i^k)V_i^k(q_i') \right\}
\]

\[
= \min_{\bar{s}} \left\{ \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} \left[ C_k(q_i^k, s_i^k) + \sum_{q_i'} \Pr(q_i'|q_i^k, s_i^k)V_i^k(q_i') \right] \right\},
\]

where \(\bar{s} = (s_1^1, \ldots, s_{\gamma_1 N}^1, \ldots, s_1^K, \ldots, s_{\gamma_k N}^K)\).

We also have that

\[
\Pr(q_i^k|\bar{q}, \bar{s}) = \sum_{q_i'} \Pr(q_i'|\bar{q}, \bar{s}, q_i^k)\Pr(q_i^k|\bar{q}, \bar{s}) = \sum_{q_i'} \Pr(q_i'|\bar{q}, \bar{s}, q_i^k)\Pr(q_i^k|q_i^k, s_i^k),
\]

for all \(\bar{q} = (q_1^1, \ldots, q_{\gamma_1 N}^1, \ldots, q_1^K, \ldots, q_{\gamma_k N}^K)\) and \(q^k = (q_1^1, \ldots, q_{\gamma_1 N}^1, \ldots, q_1^K, \ldots, q_{\gamma_k N}^K)\). Since \(\Pr(q_i^k|\bar{q}, \bar{s})\) only depends on the decision taken with respect to user \(i\) in class \(k\) we obtain

\[
\sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} \sum_{q_i'} \Pr(q_i'|q_i^k, s_i^k)V_i^k(q_i')
\]

\[
= \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} \Pr(\bar{q}^k|\bar{q}, \bar{s}) \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} V_i^k(q_i^k).\]
From the previous equations we obtain
\[
\sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} V_i^k(q_i^k) + \sum_{k=1}^{K} \gamma_k N
\]
\[
= \min_s \left[ \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} C_k(q_i^k, s_i^k) + \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} Pr(q_i^k|q_i, s_i) V_i^k(q_i^k) \right]
\]
\[
= \min_s \left[ \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} C(q_i^k, s_i^k) + \sum_{q_i} Pr(q_i|q, s) \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} V_i^k(q_i^k) \right]
\]

Since we found bounded function \( \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} V_i^k(Q_i^k) \) which verifies the Bellman equation (6), and constant \( \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} \theta_i^k \), and according to theorem 2.1 chapter 2 [22], the optimal policy is to find a decision which minimizes the right hand side of the aforementioned equation. This is equivalent to find for each user the decision that minimizes the right hand side of each individual Bellman equation. This concludes the proof.

**Appendix B**

**Proof of Lemma 2**

We first prove that \( C(\cdot, \cdot) \) is submodular. That is, \((C(q + 1, 1) - C(q + 1, 0)) - (C(q, 1) - C(q, 0)) = a(q + 1) + W - a(q + 1) - (aq + W - aq) = 0 \leq 0\). The latter is obtained by substituting the values of \( C(q', s) \) for \( s \in \{0, 1\} \) and \( q' \in \{q, q + 1\} \).

In order to prove that \( \sum_{q'} Pr(q'|q, s)V(q') \) is submodular we distinguish between two cases,

Case 1) \( q < R \), then:
\[
\sum_{q'} Pr(q'|q + 1, 1)V(q') - \sum_{q'} Pr(q'|q + 1, 0)V(q')
\]
\[
= \sum_{q'=0} Pr(A = q')V(q') - \sum_{q'=q+1} Pr(A = q' - q - 1)V(q')
\]
\[
= \sum_{q'=0} Pr(A = q')V(q') - \sum_{q'=q} Pr(A = q' - q) V(q' + 1)
\]
\[
\leq \sum_{q'=0} Pr(A = q')V(q') - \sum_{q'=0} Pr(A = q' - q) V(q')
\]
\[
= \sum_{q'} Pr(q'|q' + 1)V(q') - \sum_{q'} Pr(q'|q, 0)V(q').
\]

The inequality follows from the fact that \( V(\cdot) \) is increasing. This concludes the proof for \( q < R \).

Case 2) \( q \geq R \), then:
\[
\sum_{q'} Pr(q'|q + 1, 1)V(q') - \sum_{q'} Pr(q'|q + 1, 0)V(q')
\]
\[
= \sum_{q'} Pr(A = q' - q - 1 + R)V(q') - \sum_{q'} Pr(A = q' - q - 1)V(q')
\]
\[
= \sum_{q'=q+1-R} Pr(A = q' - q - 1 + R)V(q') - \sum_{q'=q+1} Pr(A = q' - q - 1)V(q')
\]
\[
= \sum_{q'=q} Pr(A = q' - q)V(q' - R + 1) - \sum_{q'=q} Pr(A = q' - q)V(q' + 1).
\]

(11)
Moreover, we have:

\[
\sum_{q'} Pr(q'|q, 1)V(q') - \sum_{q'} Pr(q'|q, 0)V(q')
\]

\[
= \sum_{q'=q} Pr(A = q' - q)V(q' - R) - \sum_{q'=q} Pr(A = q' - q)V(q').
\]

(12)

Subtracting Equation (11) and (12) (i.e., (11)-(12)) we obtain

\[
\sum_{q'=q} Pr(A = q' - q)((V(q' - R + 1) - V(q' - R)) - (V(q' + 1) - V(q'))) \leq 0,
\]

(13)

which follows from the convexity of \( V(\cdot) \). Therefore, \( \sum_{q'} Pr(q'|q, s)V(q') \) is submodular.

**Appendix C**

**Proof of Proposition 2**

When \( i < L \):

1) \( j \leq n \):

Since \( j \leq n \), the optimal decision is to stay idle, that means if \( A \) denotes the number of arrival packets, in the next time slot the number of packets will be \( i = j + A \) with \( A \leq R - 1 \) and then \( A = i - j \). Therefore, the probability to pass from state \( j \) to \( i \) is the probability that \( A = i - j \), which is exactly \( \pi_{i-j} \).

2) \( j > n \):

The optimal decision in this case is to transmit. However at most \( \min(R, j) \) can be transmitted. Taking into account the \( A \) arrival packets, then the new state for the next time slot will be \( i = j - \min(R, j) + A = (j - R)^+ + A \), which implies \( A = i - (j - R)^+ \). This explains that the probability to pass from state \( j \) to \( i \) is the probability that \( A \) is equal to \( i - (j - R)^+ \) which is equal to \( \pi_{i-(j-R)^+} \).

When \( i = L \):

1) \( j \leq n \):

The optimal decision is passive action. Then \( A \) arrival packets are added to the \( j \) packets present in the queue. For the next time slot, the number of packets is \( j + A \). According to equation (11), since we cannot exceed the buffer length \( L \), we reach the state \( L \) if \( j + A \geq L \). Since \( A \leq R - 1 \), then the probability of this event or equivalently the probability to pass from state \( j \) to state \( L \) is

\[
Pr(L - j \leq A \leq R - 1) = \sum_{k=L-j}^{R-1} Pr(A = k) = (R - L + j)\pi_{L-j}.
\]

2) \( j > n \):

The optimal decision is active action, thus we can never reach state \( L \) because the next state is \( (j - R)^+ + A \) which is less than \( (j - R)^+ + (R - 1) \leq (L - R)^+ + R - 1 = (L - R) + R - 1 < L \) (because \( L \geq R \)). Then the probability to pass from \( j \) to \( i \) is 0. We conclude the result.

**Appendix D**

**Proof of Proposition 3**

We prove the four cases separately:
1) First case: $-1 \leq n < R$

$$u(i) = \sum_{j=0}^{n} p^n(j, i)u(j) + \sum_{j=n+1}^{R} p^n(j, i)u(j) + \sum_{j=R+1}^{L} p^n(j, i)u(j)$$

We first provide the following lemma that follows from proposition 2.

**Lemma 13.** when $i < L$:

$$p^n(j, i) = \begin{cases} 
\pi_{i-j} & \text{if } 0 \leq j \leq n \\
\pi_i & \text{if } n+1 \leq j \leq R-1 \\
\pi_{i-(j-R)} & \text{if } R \leq j \leq L 
\end{cases}$$

when $i = L$:

$$p^n(j, i) = \begin{cases} 
0 & \text{if } 0 \leq j \leq n \\
0 & \text{if } n+1 \leq j \leq L 
\end{cases}$$

Using Lemma 13 we have:

if $i < L$

$$u(i) = \sum_{j=0}^{n} \pi_{i-j}u(j) + \sum_{j=n+1}^{R} \pi_iu(j) + \sum_{j=R+1}^{L} \pi_{i-(j-R)}u(j)$$

By definition of $\pi$ given in definition 4 then:

$$u(i) = \sum_{\min(i, n)}^{\min(i, n)} \rho u(j) + \sum_{n+1}^{R} \pi_iu(j) + \sum_{\min(i+R, L)}^{\min(i+R, L)} \rho u(j)$$

In order to prove proposition 3 for this case, according to lemma 13 we will distinguish between five sub-cases:

a) $i = L$

b) $n + R + 1 \leq i \leq L - 1$

c) $n + 1 \leq i \leq R - 1$

d) $0 \leq i \leq n$

e) $R \leq i \leq n + R$

a) Proof of $u(i) = 0$ for $i = L$:

if $i = L$, since $\forall j \ p^n(j, L) = 0$, then

$$u(L) = 0$$

b) Proof of $u(i) = 0$ for $n + R + 1 \leq i \leq L - 1$:

For this case, we prove by strong induction in decreasing order that $u(i) = 0$
In fact we have \( u(L) = 0 \), and for \( n + R < i \leq L \), \( \pi_i = 0 \) because \( i > R - 1 \), \( \min(i,n) = n < i - R + 1 = \max(i - R + 1, 0) \), and \( i + 1 \geq R + 1 \), then:

\[
u(i) = \sum_{i+1}^{\min(i+R,L)} \rho u(j)
\]

we consider by induction that \( \forall k \in [i, L], u(k) = 0 \).

So \( u(i - 1) = \sum_{i}^{\min(i-1+R,L)} \rho u(j) = 0 \).

Hence we conclude the result.

c) Proof of \( u(i) = \rho \) for \( n + 1 \leq i \leq R - 1 \):

We have \( \max(i - R + 1, 0) = 0, \min(i,n) = n, \pi_i = \rho \) (since \( 0 \leq i \leq R - 1 \)), \( \max(i+1,R+1) = R + 1 \) and \( \min(i+R,L) = i + R \) (recall that \( i + R < 2R \leq L \)). This implies,

\[
u(i) = \sum_{0}^{n} \rho u(j) + \sum_{n+1}^{R} \rho u(j) + \sum_{R+1}^{i+R} \rho u(j)
\]

Now, we prove that \( u(i) = \rho \)

We have

\[
u(i) = \sum_{0}^{n} \rho u(j) + \sum_{n+1}^{R} \rho u(j) + \sum_{R+1}^{i+R} \rho u(j)
\]

\[
= \rho \left( \sum_{0}^{n} u(j) + \sum_{n+1}^{R} u(j) + \sum_{R+1}^{i+R} u(j) \right)
\]

\[
u(i) = \sum_{j=0}^{i+R} \rho u(j)
\]

We have \( i + R > n + R \), then \( u(p) = 0 \) for all \( p \in [n + R + 1, i + R] \). We can hence simplify the expression of \( u(i) \) as follows:

\[
u(i) = \rho \sum_{j=0}^{n+R} u(j)
\]

Since we proved that when \( j > n + R \), \( u(j) = 0 \) (sub-case (b)), then \( \sum_{j=0}^{n+R} u(j) = 1 \) (\( \sum_{0}^{L} u(j) = 1 \) because \( u \) is probability distribution), i.e. \( u(i) = \rho \).

This ends the proof of sub-case (c).

We will provide a useful lemma which allows us to prove the proposition for the cases d and e. Before giving this lemma, we will give general expressions of \( u(i) \) for these two cases.

if \( 0 \leq i \leq n \):

\( i \leq n < R \), which implies that \( i - R + 1 \leq 0 \), \( \max(i - R + 1, 0) = 0 \), \( \min(i,n) = i \), \( \pi_i = \rho \) since \( 0 \leq i \leq n < R \), \( \max(i+1,R+1) = R + 1 \), and \( i + R \leq n + R < 2R \leq L \). Therefore \( \min(i+R,L) = i + R \), which implies that,

\[
u(i) = \sum_{0}^{i} \rho u(j) + \sum_{n+1}^{R} \rho u(j) + \sum_{R+1}^{i+R} \rho u(j)
\]
if $R \leq i \leq n + R$:

We have $\max(i - R + 1, 0) = i - R + 1$, $\min(i, n) = n$ (due to $i \geq R > n$), $\pi_i = 0$ (since $i > R - 1$) and $\max(i + 1, R + 1) = i + 1$. Then:

$$u(i) = \sum_{i-R+1}^{n} \rho u(j) + \sum_{i+1}^{\min(i+R,L)} \rho u(j)$$

**Lemma 14.** for $0 \leq k \leq n$:

$$u(n + R - k) + u(n - k) = \rho$$

**Proof.** See appendix [E]

**d)** Proof of $u(i) = \rho - \rho^2(n - i)$ for $0 \leq i \leq n$:

We start by proving by induction that for $k \in [0, n]$ $u(n-k) = \rho - \rho^2k$, we have for $0 \leq k \leq n$, $0 \leq n-k \leq n$, then:

$$u(n - k) = \sum_{0}^{n-k} \rho u(j) + \sum_{n+1}^{R} \rho u(j) + \sum_{R+1}^{n-k+R} \rho u(j)$$

For $k = 0$,

$$u(n - 0) = \rho[\sum_{0}^{n} u(j) + \sum_{n+1}^{R} u(j) + \sum_{R+1}^{n+R} u(j)]$$

$$= \sum_{j=0}^{n+R} \rho u(j)$$

$$= \rho$$

We suppose that the expression is true for some $k$, we prove it for $k + 1$

$$u(n - (k + 1)) = \sum_{0}^{n-k-1} \rho u(j) + \sum_{n+1}^{R} \rho u(j) + \sum_{R+1}^{(n-k-1)+R} \rho u(j)$$

$$= \sum_{0}^{n-k} \rho u(j) + \sum_{n+1}^{R} \rho u(j) + \sum_{R+1}^{(n-k+R)} \rho u(j)$$

$$= u(n - k) - \rho(u(n - k) + u(n + R - k))$$

$$= \rho - k\rho^2 - \rho(u(n - k) + u(n + R - k))$$

Using Lemma [14], $u(n - k) + u(n + R - k) = \rho$, then:

$$u(n - (k + 1)) = \rho - k\rho^2 - \rho(\rho)$$

$$= \rho - k\rho^2 - \rho^2$$

$$= \rho - (k + 1)\rho^2$$
Thus we conclude that for \( k \in [0, n] \) \( u(n - k) = \rho - k\rho^2 \).

For \( i \in [0, n] \), we replace \( k \in [0, n] \) by \( n - i \) (\( n - i \in [0, n] \)), we get:

\[
u(i) = u(n - (n - i)) = \rho - \rho^2(n - i)\]

e) Proof of \( u(i) = \rho^2(n + R - i) \) for \( R \leq i \leq n + R \):

For that we prove that for \( k \in [0, n] \) \( u(n + R - k) = \rho^2 k \).

From the above result in the case (d), we get \( u(n - k) = \rho - k\rho^2 \).

So, according to Lemma \ref{lemma14}:

\[
u(n + R - k) = \rho - u(n - k)
  = \rho - \rho^2 k
  \]

\[
u(n + R - k) = \rho^2 k
  \]

For \( i \in [R, n + R] \), we replace \( k \in [0, n] \) by \( n + R - i \) (\( n + R - i \in [0, n] \)), we get:

\[
u(i) = u(n + R - (n + R - i)) = \rho^2(n + R - i)
  \]

2) Second case: \( R \leq n < L - R \)

\[
u(i) = \sum_{j=0}^{n} p^n(j, i)u(j) + \sum_{j=n+1}^{L} p^n(j, i)u(j)
\]

**Lemma 15.** when \( i < L \):

\[
p^n(j, i) = \begin{cases} 
\pi_{i-j} & \text{if } 0 \leq j \leq n \\
\pi_{i-(j-R)} & \text{if } n + 1 \leq j \leq L 
\end{cases}
\]

when \( i = L \):

\[
p^n(j, i) = \begin{cases} 
0 & \text{if } 0 \leq j \leq n \\
0 & \text{if } n + 1 \leq j \leq L 
\end{cases}
\]

The result of Lemma \ref{lemma15} comes from Proposition \ref{prop2} Using Lemma \ref{lemma15} if \( i < L \)

\[
u(i) = \sum_{j=0}^{n} \pi_{i-j}u(j) + \sum_{j=n+1}^{L} \pi_{i-(j-R)}u(j)
\]

By definition of \( \pi \) given in definition \ref{def4} then:

\[
u(i) = \sum_{\max(i+1-R,0)}^{\min(n,i)} \rho u(j) + \sum_{\max(n+1,i+1)}^{\min(L,i+R)} \rho u(j)
\]

According to Lemma \ref{lemma15} we will distinguish between five sub-cases:

a) \( i = L \)

b) \( 0 \leq i \leq n - R \)

c) \( n + R + 1 \leq i \leq L - 1 \)

d) \( n + 1 - R \leq i \leq n \)
e) \( n + 1 \leq i \leq n + R \)

a) Proof of \( u(i) = 0 \) for \( i = L \):

if \( i = L \), since \( \forall j \) \( p^n(j, L) = 0 \), then:

\[ u(L) = 0 \]

b) Proof of \( u(i) = 0 \) for \( 0 \leq i \leq n - R \):

We prove by induction that for all \( 0 \leq i < n + 1 - R \), \( u(i) = 0 \).

In fact, if \( 0 \leq i < n + 1 - R \), then \( i < n - R < n \), \( \min(n, i) = i \), and \( \min(i + R, L) \leq i + R < n + 1 = \max(n + 1, i + 1) \). Then:

\[ u(i) = \sum_{(i+1-R)^+}^{i} \rho u(j) \]

for \( i = 0 \) \( u(0) = \rho u(0) \) i.e. \( u(0) = 0 \) since \( \rho < 1 \).

if \( u(j) = 0 \) for all \( j \leq i \), then:

\[ u(i + 1) = \sum_{(i+2-R)^+}^{i+1} \rho u(j) \]

\[ = \sum_{(i+2-R)^+}^{i} \rho u(j) + \rho u(i + 1) \]

\[ = 0 + \rho u(i + 1) \]

\[ u(i + 1) = \rho u(i + 1) \]

This implies that \( u(i + 1) = 0 \).

c) Proof of \( u(i) = 0 \) for \( n + R + 1 \leq i \leq L - 1 \):

If \( i \geq n + R + 1 \) then \( (i+1-R)^+ = i + 1 - R > n = \min(n, i) \) and \( \max(n + 1, i + 1) = i + 1 \). This implies

\[ u(i) = \sum_{\min(i+R,L)}^{i+1} \rho u(j) \]

and we have \( u(L) = 0 \).

we suppose now that for all \( k \) between \( i \) and \( L \): \( u(k) = 0 \) then

\[ u(i - 1) = \sum_{\min(i-1+R,L)}^{i} \rho u(j) = 0 \]

We conclude the result.

We will provide a useful lemma which allows us to prove Proposition [3] for the cases (d) and (e). Before giving this lemma, we will give general expressions of \( u(i) \) for these two cases.
if \( n + 1 - R \leq i \leq n \):

We have \( \min(n, i) = i \), \( \max(n + 1, i + 1) = n + 1 \), and \( \min(L, i + R) = i + R \) (since \( i + R \leq n + R < L - R + R = L \)). Then:

\[
u(i) = \sum_{i=1}^{n+1} \rho u(j) + \sum_{n+1}^{i+R} \rho u(j)
\]

We have \( n - R + 1 \geq 0 \) and \( n - R + 1 \geq i - R + 1 \), then \( n - R + 1 \geq (i + 1 - R)^+ \). If \( n - R + 1 = (i + 1 - R)^+ \), then we replace index \((i + 1 - R)^+\) by \( n - R + 1 \) in the expression of \( u(i) \). If \( n - R + 1 > (i + 1 - R)^+ \), we know that for all \( j \) less or equal to \( n - R \), \( u(j) = 0 \). Then, we can simplify the expression of \( u(i) \) as follows:

\[
u(i) = \sum_{n+1}^{i+R} \rho u(j) \]

if \( n + 1 \leq i \leq n + R \):

\( i > R \), then \( \max(i + 1 - R, 0) = i + 1 - R \), \( \min(n, i) = n \) and \( \max(n + 1, i + 1) = i + 1 \). Therefore:

\[
u(i) = \sum_{i+1}^{n+1} \rho u(j) + \sum_{i+1}^{\min(i+1, i+R)} \rho u(j)
\]

We have \( i + R > n + R \), and \( L > n + R \) because \( n < L - R \), then \( \min(L, i + R) > n + R \). Therefore, given that \( u(j) = 0 \) for all \( j \) between \( n + R + 1 \) and \( \min(L, i + R) \), we can simplify the expression of \( u(i) \) as follows:

\[
u(i) = \sum_{i+1}^{n+R} \rho u(j) \]

**Lemma 16.** for \( 0 \leq k \leq R - 1 \),

\[
u(n + R - k) + u(n - k) = \rho
\]

**Proof.** See appendix [F].

Let us now prove the result for cases (d) and (e).

d) Proof of \( u(i) = \rho - (n - i)\rho^2 \) for \( n + 1 - R \leq i \leq n \):
We prove by induction that, for $0 \leq k \leq R - 1$, $u(n - k) = \rho - k\rho^2$.

For $k = 0$:

$$u(n - 0) = \sum_{n+1}^{n+R} \rho[u(j - R) + u(j)]$$

$$= \sum_{n+1-R}^{n} \rho u(j) + \sum_{n+1}^{n+R} \rho u(j)$$

$$= \rho \sum_{n+1-R}^{n+R} u(j)$$

$$u(n) = \rho$$

We suppose that the expression is true for some $k$, we prove it for $k + 1$.

$$u(n - (k + 1)) = \sum_{n+1}^{n-k+1+R} \rho(u(j - R) + u(j))$$

$$= \sum_{n+1}^{n-k+R} \rho(u(j - R) + u(j)) - \rho[u(n - k) + u(n - k + R)]$$

$$= u(n - k) - \rho[u(n - k) + u(n + R - k)]$$

$$= \rho - k\rho^2 - \rho[u(n - k) + u(n + R - k)]$$

Using Lemma [16], $u(n - k) + u(n + R - k) = \rho$, then

$$u(n - (k + 1)) = \rho - k\rho^2 - \rho(\rho)$$

$$= \rho - k\rho^2 - \rho^2$$

$$u(n - (k + 1)) = \rho - (k + 1)\rho^2$$

Thus we conclude that, for $k \in [0, R - 1]$, $u(n - k) = \rho - k\rho^2$.

For $i \in [n + 1 - R, n]$, we replace $k \in [0, R - 1]$ by $n - i$ ($n - i \in [0, R - 1]$) and get:

$$u(i) = u(n - (n - i)) = \rho - (n - i)\rho^2$$

e) Proof of $u(i) = \rho^2(n + R - i)$ for $n + 1 \leq i \leq n + R$

We prove that, for $k \in [0, R - 1]$, $u(n + R - k) = \rho^2k$. From above, we have $u(n - k) = \rho - k\rho^2$, and by using Lemma [16] we have:

$$u(n + R - k) = \rho - u(n - k)$$

$$= \rho - (\rho - \rho^2k)$$

$$u(n + R - k) = \rho^2k$$

For $i \in [n + 1, n + R]$, by replacing $k \in [0, R - 1]$ by $n + R - i$ ($n + R - i \in [0, n]$), we get:

$$u(i) = u(n + R - (n + R - i)) = \rho^2(n + R - i)$$

This ends the proof of the second case.
3) Third case: $L - R \leq n < L$

$$u(i) = \sum_{j=0}^{n} p^n(j,i)u(j) + \sum_{j=n+1}^{L} p^n(j,i)u(j)$$

**Lemma 17.** When $i < L$:

$$p^n(j,i) = \begin{cases} 
\pi_{i-j} & \text{if } 0 \leq j \leq n \\
\pi_{i-(j-R)} & \text{if } n + 1 \leq j \leq L
\end{cases}$$

When $i = L$:

$$p^n(j,L) = \begin{cases} 
(R - L + j)\pi_{L-j} & \text{if } 0 \leq j \leq n \\
0 & \text{if } n + 1 \leq j \leq L
\end{cases}$$

This Lemma comes from Proposition 2.

So using Lemma 17 and by definition of $\pi$:

if $i < L$:

$$u(i) = \sum_{\max(i-R+1,0)}^{\min(i,n)} \rho u(j) + \sum_{\max(n+1,i+1)}^{\min(L,i+R)} \rho u(j)$$

if $i = L$:

$$u(L) = \sum_{j=0}^{n} (R - L + j)\pi_{L-j}u(j)$$

According to Lemma 17, we will distinguish between five cases:

a) $0 \leq i \leq n - R$

b) $n + 1 \leq i \leq L - 1$

c) $n - R + 1 \leq i \leq L - R - 1$

d) $L - R \leq i \leq n$

e) $i = L$

a) Proof of $u(i) = 0$ for $0 \leq i \leq n - R$:

We prove by induction that, for $i \leq n - R$, $u(i) = 0$.

Since $0 \leq i \leq n - R$, then $\min(i,n) = i$, $i + R \leq n < L$ and $\min(L,i+R) = i + R < n + 1 = \max(n+1,i+1)$.

Therefore:

$$u(i) = \sum_{\max(i-R+1,0)}^{i} \rho u(j)$$

for $i = 0$, $u(0) = \rho u(0) = 0$.

We consider that $u(j) = 0$ for all $j$ between $0$ and $i$, we demonstrate that $u(i + 1) = 0$.

$$u(i + 1) = \sum_{\max(i-R+2,0)}^{i+1} \rho u(j)$$

$$= \sum_{\max(i-R+2,0)}^{i} \rho u(j) + \rho u(i + 1)$$

$$= 0 + \rho u(i + 1)$$

$$u(i + 1) = \rho u(i + 1)$$
This implies that:

\[ u(i + 1) = \rho u(i + 1) \]

Hence we prove that, for all \( i \in [0, n - R] \), \( u(i) = 0 \).

We will provide a useful lemma which allows us to prove Proposition 5 for cases (b) and (c). Before giving this lemma, we will give general expressions of \( u(i) \) for these two cases.

if \( n - R + 1 \leq i \leq L - R - 1 \):

\( i < L - R \leq n \), then \( \min(i, n) = i \), \( \max(n + 1, i + 1) = n + 1 \) and \( \min(L, i + R) = i + R \). This implies that,

\[
u(i) = \sum_{(i-R+1)^+}^i \rho u(j) + \sum_{n+1}^{i+R} \rho u(j)\]

We have \( n - R + 1 > 0 \) and \( n - R + 1 > i - R + 1 \), which implies that \( n - R + 1 > (i + 1 - R)^+ \) and \( n - R \geq (i + 1 - R)^+ \). Since \( u(j) = 0 \) for all \( j \) less or equal to \( n - R \), we can simplify the expression of \( u(i) \) as follows:

\[
u(i) = \sum_{n-R+1}^i \rho u(j) + \sum_{n+1}^{i+R} \rho u(j)\]

if \( n + 1 \leq i < L \):

We have \( (i - R + 1)^+ = i - R + 1 \) (as \( i \geq n + 1 > R \)), \( \min(i, n) = n \), \( \max(n + 1, i + 1) = i + 1 \) and \( \min(L, i + R) = L \) (due to \( i + R > n + R \geq L - R + R = L \)). Then:

\[
u(i) = \sum_{i-R+1}^n \rho u(j) + \sum_{i+1}^L \rho u(j)\]

**Lemma 18.** For \( 1 \leq k \leq L - n - 1 \),

\[ u(n - R + k) + u(n + k) = \rho \]

**Proof.** See appendix [C] ■

b) Proof of \( u(i) = \rho - (i - n)\rho^2 \) for \( n + 1 \leq i \leq L - 1 \):

We prove first that, for \( 1 \leq k \leq L - n - 1 \), \( u(n + k) = \rho - k\rho^2 \).
In fact:

\[ u(n + k) = \sum_{n+k-R+1}^{n} \rho u(j) + \sum_{n+k+1}^{L} \rho u(j) \]

\[ = \rho - \left[ \sum_{n-R+1}^{n+k-R} \rho u(j) + \sum_{n+1}^{n+k} \rho u(j) \right] \]

\[ = \rho - \left[ \sum_{1}^{k} \rho (n - R + j) + \sum_{1}^{k} \rho (n + j) \right] \]

\[ = \rho - \left[ \sum_{1}^{k} \rho [u(n - R + j) + u(n + j)] \right] \]

According to Lemma [18] and given that \(0 \leq k \leq L - n - 1\), then for all \(j \in [1, k]\), \(u(n - R + j) + u(n + j) = \rho\), then:

\[ u(n + k) = \rho - \left[ \sum_{1}^{k} \rho^2 \right] \]

\[ u(n + k) = \rho - k \rho^2 \]

Then for \(1 \leq k \leq L - n - 1\), \(u(n + k) = \rho - k \rho^2\).

For \(i \in [n + 1, L - 1]\), we replace \(k \in [1, L - n - 1]\) by \(i - n\) \((i - n \in [1, L - n - 1])\) and get:

\[ u(i) = u(n + (i - n)) = \rho - (i - n) \rho^2 \]

c) Proof of \(u(i) = \rho^2 (R - n + i)\) for \(n - R + 1 \leq i \leq L - R - 1\):

We need to prove that, for \(k \in [1, L - n - 1]\), \(u(n - R + k) = \rho^2 k\)

Given that \(u(n + k) = \rho - \rho^2 k\) which is proved in case (d), and using Lemma [18] then:

\[ u(n - R + k) = \rho - u(n + k) \]

\[ = \rho - (\rho - \rho^2 k) \]

\[ u(n - R + k) = \rho^2 k \]

For \(i \in [n - R + 1, L - R - 1]\), we replace \(k \in [1, L - n - 1]\) by \(R - n + i\) \((R - n + i \in [1, L - n - 1])\) and get:

\[ u(i) = u(n - R + (R - n + i)) = \rho^2 (R - n + i) \]

This ends the proof of case (c).

d) Proof of \(u(i) = (1 - \rho)^{n-i} \rho\) for \(L - R \leq i \leq n\):

if \(L - R \leq i \leq n\), \((i - R + 1)^+ = i - R + 1\) because \(i \geq L - R \geq R\), \(\min(i, n) = i\), \(\max(n+1, i+1) = n+1\) and \(\min(L, i + R) = L\). Then:

\[ u(i) = \sum_{i-R+1}^{i} \rho u(j) + \sum_{n+1}^{L} \rho u(j) \]
We have \( n \geq i \), then \( n - R + 1 \geq i - R + 1 \). If \( n - R + 1 = i - R + 1 \), we replace \( i - R + 1 \) by \( n - R + 1 \) in the expression of \( u(i) \). If \( n - R + 1 > i - R + 1 \), we know that, for all \( j \) less or equal to \( n - R \), \( u(j) = 0 \).

We can then simplify the expression of \( u(i) \) as follows:

\[
u(i) = \sum_{n-R+1}^{i} \rho u(j) + \sum_{n+1}^{L} \rho u(j)
\]

In order to prove the proposition 3 for this case, we prove by induction that \( u(n - k) = (1 - \rho)^k \rho \) for \( 0 \leq k \leq n - L + R \).

For \( k = 0 \):

\[
u(n) = \sum_{n-R+1}^{n} \rho u(j) + \sum_{n+1}^{L} \rho u(j)
= \sum_{n-R+1}^{L} \rho u(j)
\]

\[
u(n) = \rho
\]

We suppose it is true for \( k \), we prove it for \( k + 1 \):

\[
u(n - (k + 1)) = \sum_{n-R+1}^{n-k-1} \rho u(j) + \sum_{n+1}^{L} \rho u(j)
= \sum_{n-R+1}^{n-k} \rho u(j) + \sum_{n+1}^{L} \rho u(j) - \rho u(n - k)
= u(n - k) - \rho u(n - k)
= (1 - \rho)^k \rho - \rho(1 - \rho)^k \rho
= (1 - \rho)^k \rho(1 - \rho)
\]

\[
u(n - (k + 1)) = (1 - \rho)^{k+1} \rho
\]

Thus we conclude that, for \( k \in [0, n - L + R] \), \( u(n - k) = (1 - \rho)^k \rho \).

For \( i \in [L - R, n] \), we replace for \( k \in [0, n - L + R] \) by \( n - i \) (\( n - i \in [0, n - L + R] \)) and get:

\[
u(i) = u(n - (n - i)) = (1 - \rho)^{n-i} \rho
\]

This proves the result.

e) Proof of \( u(i) = (1 - \rho)^{n-L+R+1} - \rho(L - 1 - n) \) for \( i = L \):

\[
u(L) = \sum_{j=0}^{n}(R - L + j)\pi_{L-j}u(j)
= \sum_{j=L-R+1}^{n}(R - L + j)\rho u(j)
\]
We replace \( u(j) \) by its expression when \( j \in [L - R + 1, n] \) (it corresponds to the sub-case (d))

\[
u(L) = \sum_{j=L-R+1}^{n} (R - L + j)[\rho(1 - \rho)^{n-j} \rho]
\]

\[
u(L) = \rho^2 \sum_{k=0}^{n-L+R-1} (R - L - k + n)(1 - \rho)^k
\]

\[
u(L) = (1 - \rho)^{n-L+R+1} - \rho(L - 1 - n)
\]

4) Fourth case: \( n = L \)

\[
u(i) = \sum_{j=0}^{L} p^L(j, i)u(j)
\]

For \( i \leq L - 1 \):

According to Proposition 2 we have:

\[
u(i) = \sum_{j=0}^{L} \pi_{i-j}u(j)
\]

By definition of \( \pi \), we get:

\[
u(i) = \sum_{(i-R+1)^+}^{i} \rho u(j)
\]

We prove by induction that for \( 0 \leq i < L \) \( u(i) = 0 \)

We have \( u(0) = \rho u(0) = 0 \).

We suppose that \( u(j) = 0 \) for all \( 0 \leq j \leq i \), then:

\[
u(i + 1) = \sum_{(i-R+2)^+}^{i+1} \rho u(j)
\]

\[
= \sum_{(i-R+2)^+}^{i} \rho u(j) + \rho u(i + 1)
\]

\[
= 0 + \rho u(i + 1)
\]

\[
u(i + 1) = 0
\]

Then, for all \( i \in [0, L - 1] \), \( u(i) = 0 \).

Since \( \sum_{j=0}^{L} u(j) = 1 \), we have \( u(L) = 1 - \sum_{j=0}^{L-1} u(j) = 1 - 0 = 1 \).

This ends the proof.

**APPENDIX E**

**PROOF OF LEMMA 14**

\[
u(n - k) + u(n + R - k) = \sum_{j=0}^{n-k} \rho u(j) + \sum_{n+1}^{R} \rho u(j) + \sum_{n-k+1}^{n} \rho u(j) + \sum_{n-R-k+1}^{\min(n-k+2R,L)} \rho u(j)
\]

\[
u(n - k) + u(n + R - k) = \rho \sum_{0}^{\min(2R+n-k,L)} u(j)
\]
We know that $R > n$ and $n - k \geq 0$, which implies that $2R + n - k > n + R$ and $n + R < 2R \leq L$. and hence $\min(2R+n-k, L) > n+R$. Therefore, we get rid of all elements $u(j)$ such that $j \in [n+R+1, \min(2R+n-k, L)]$ since for all $j > n + R$, $u(j) = 0$. Moreover $\sum_0^{n+R} u(j) = 1$, consequently:

$$u(k) + u(R + k) = \rho \sum_0^{n+R} u(j) = \rho$$

**APPENDIX F**

**PROOF OF LEMMA 16**

Since $n - R + 1 \leq n - k \leq n$, and $n + 1 \leq n + R - k \leq n + R$, then:

$$u(n - k) + u(n + R - k) = \sum_{n+1-R}^{n-k} \rho u(j) + \sum_{n+1}^{n-k+R} \rho u(j) + \sum_{n-k+1}^{n+R-k} \rho u(j)$$

$$= \rho \sum_{n+1-R}^{n+R} u(j)$$

Given that $u(j) = 0$ for $j \in [0, n - R] \cup [n + R + 1, L]$, then $\sum_{n+1-R}^{n+R} u(j) = 1$. Consequently:

$$u(n - k) + u(n + R - k) = \rho$$

**APPENDIX G**

**PROOF OF LEMMA 18**

Since $n - R + 1 \leq n - R + k \leq L - R - 1$, and $n + 1 \leq n + k \leq L - 1$, then:

$$u(n - R + k) + u(n + k) = \sum_{n-R+1}^{n-R+k} \rho u(j) + \sum_{n+1}^{n+k} \rho u(j) + \sum_{n+k+1}^{L} \rho u(j)$$

$$= \rho \sum_{n-R+1}^{L} u(j)$$

As we have demonstrated that $u(i) = 0$ for $i \in [0, n - R]$, then $\sum_{n-R+1}^{L} u(j) = 1$. Therefore,

$$u(n - R + k) + u(n + k) = \rho$$

**APPENDIX H**

**PROOF OF PROPOSITION 4**

As mentioned previously in the paper, we denote $\sum_0^L au^n(q)q$ by $a_n$ and $\sum_0^n u^n(q)$ by $b_n$. Before proving the proposition, we give two useful lemmas.

**Lemma 19.** Considering $a_{j-1}, a_j, a_{j+1}$ and $b_{j-1}, b_j, b_{j+1}$, such that $b_{j-1} < b_j < b_{j+1}$.
Then: \[
\frac{a_j - a_{j-1}}{b_j - b_{j-1}} = \frac{a_{j+1} - a_j}{b_{j+1} - b_j}
\]

Then:

\[
\frac{a_j - a_{j-1}}{b_j - b_{j-1}} \leq \frac{a_{j+1} - a_{j-1}}{b_{j+1} - b_{j-1}} \leq \frac{a_{j+1} - a_j}{b_{j+1} - b_j}
\]

Then:

\[
\frac{a_j - a_{j-1}}{b_j - b_{j-1}} \geq \frac{a_{j+1} - a_{j-1}}{b_{j+1} - b_{j-1}} \geq \frac{a_{j+1} - a_j}{b_{j+1} - b_j}
\]

Then:

\[
\frac{a_j - a_{j-1}}{b_j - b_{j-1}} \geq \frac{a_{j+1} - a_{j-1}}{b_{j+1} - b_{j-1}} \geq \frac{a_{j+1} - a_j}{b_{j+1} - b_j}
\]

Then:

\[
\frac{a_j - a_{j-1}}{b_j - b_{j-1}} \geq \frac{a_{j+1} - a_{j-1}}{b_{j+1} - b_{j-1}} \geq \frac{a_{j+1} - a_j}{b_{j+1} - b_j}
\]

Proof. See appendix I.

Lemma 20. The largest minimizer at step \( j \) in algorithm 1 satisfies \( n_j = \min\{k : b_k = b_n\} \)

Proof. See appendix II

We start by indexability:

We consider \( W_1 < W_2 \) and prove that the optimal threshold \( n_1 \), when \( W = W_1 \), is less than \( n_2 \) (when \( W = W_2 \)).

In fact if \( n_1 \leq n_2 \) and the threshold is \( n_1 \), all states \([0, n_1]\), for which the optimal decision is passive action, are included in \([0, n_2]\). This implies the desired result \( D(W_1) \subseteq D(W_2) \).

In order to prove that, we just need to prove that \( b_{n_1} \leq b_{n_2} \) since \( n_1 \leq n_2 \) is equivalent to \( b_{n_1} \leq b_{n_2} \) (due to increasiness of \( b_n \)).

We have according to equation (6) and by definition of \( n_1 \) and \( n_2 \):

\[
a_{n_1} - W_1b_{n_1} \leq a_{n_2} - W_1b_{n_2}
\]

\[
a_{n_1} - W_2b_{n_1} \geq a_{n_2} - W_2b_{n_2}
\]

This implies:

\[
W_2(b_{n_1} - b_{n_2}) \leq a_{n_1} - a_{n_2} \leq W_1(b_{n_1} - b_{n_2})
\]

Therefore: \((W_2 - W_1)(b_{n_1} - b_{n_2}) \leq 0\). Since \( W_2 - W_1 > 0 \), hence: \( b_{n_1} \leq b_{n_2} \), then \( n_1 \leq n_2 \).

We conclude the indexability.
For the whittle’s index expressions, we need to demonstrate that, for \( k \in [n_{j-1}, n_j] \), \( W_j = \min\{W, k \in D(W)\} \).

For that, we prove first that for \( W < W_j \) then \( k \notin D(W) \).

When \( k > n_{j-1}, W < W_j \), and \( b_k \neq b_{n_{j-1}} \), then \( W < W_j \leq \frac{a_k - a_{n_{j-1}}}{b_k - b_{n_{j-1}}} \), and \( a_k - b_k W > a_{n_{j-1}} - b_{n_{j-1}} W \).

When \( k > n_{j-1}, W < W_j \) and \( b_k = b_{n_{j-1}} \), then given that \( a_k > a_{n_{j-1}} \) we have \( a_k - b_k W > a_{n_{j-1}} - b_{n_{j-1}} W \).

Hence we proved that, for \( W < W_j \) and \( k > n_{j-1} \), \( a_k - b_k W > a_{n_{j-1}} - b_{n_{j-1}} W \). That means at \( W \) the optimal threshold is \( n_{j-1} \) or even less. Therefore, for \( k \in [n_{j-1}, n_j] \) where \( k \) is necessary strictly higher than the threshold, the optimal action for \( k \) is active action, i.e. \( k \notin D(W) \).

There is still to prove that \( k \in D(W_j) \).

For that, we prove that the threshold is at least \( n_j \) when \( W = W_j \). In other words, for all \( k < n_j \), \( a_k - b_k W_j \geq a_{n_j} - b_{n_j} W_j \). We demonstrate this result by induction in \( j \).

For \( j = 0 \), we have for all \( n \), \( b_n > 0 \), then \( W_0 \) is well defined.

\[
W_0 \leq \frac{a_j - a_{j-1}}{b_k} \quad \forall k \geq 0.
\]

Then for \( 0 \leq k < n_0 \), according to Lemma 20, \( b_k < b_{n_0} \). Thus, by using Lemma 19 (fourth case), we can deduce that \( \frac{a_{n_0} - a_k}{b_{n_0} - b_k} \leq W_0 \). That means, for \( k \in [-1, n_0] \), \( \frac{a_{n_0} - a_k}{b_{n_0} - b_k} \leq W_0 \), which implies that \( a_k - b_k W_0 \geq a_{n_0} - b_{n_0} W_0 \).

We suppose at step \( j \), \( a_k - b_k W_j \geq a_{n_j} - b_{n_j} W_j \) i.e. \( \frac{a_{n_j} - a_k}{b_{n_j} - b_k} \leq W_j \) for \( k < n_j \) (this remains true since \( b_k < b_{n_j} \) according to Lemma 20).

At \( j + 1 \):

When \( n_j \leq k < n_{j+1} \), then if \( b_k \neq b_{n_j} \), \( \frac{a_k - a_{n_j}}{b_k - b_{n_j}} \geq W_{j+1} \). Thus, by using Lemma 19 (fourth case), we get \( \frac{a_{n_{j+1}} - a_k}{b_{n_{j+1}} - b_k} \leq W_{j+1} \). Hence \( b_k = b_{n_j} \), \( \frac{a_{n_{j+1}} - a_k}{b_{n_{j+1}} - b_k} = \frac{a_{n_{j+1}} + 1 - a_{n_{j+1}}}{b_{n_{j+1}} + 1 - b_k} \leq W_{j+1} \) since \( a_k \geq a_{n_{j+1}} \).

When \( k < n_j \), we have \( \frac{a_{n_j} - a_k}{b_{n_j} - b_k} \leq W_j \) (induction assumption). Using the definition of \( n_j \) defined in Algorithm 1, we have \( W_j < \frac{a_{n_{j+1}} - a_{n_j}}{b_{n_{j+1}} - b_{n_j}} \). Then according to Lemma 19 (third case), \( W_j \leq W_{j+1} \). Therefore \( \frac{a_{n_j} - a_k}{b_{n_j} - b_k} \leq W_{j+1} \) and by using again Lemma 19 (first case), \( \frac{a_{n_{j+1}} - a_k}{b_{n_{j+1}} - b_k} \leq W_{j+1} \). Therefore, for all \( k \leq n_{j+1} \), \( a_k - b_k W_{j+1} \geq a_{n_{j+1}} - b_{n_{j+1}} W_{j+1} \).

Thus, we proved by induction that at any step \( j \), for \( k < n_j \), \( a_k - b_k W_j \geq a_{n_j} - b_{n_j} W_j \).

Then when \( W = W_j \), the threshold is at least \( n_j \). This means that for \( k \in [n_{j-1}, n_{j}], k \) is less or equal than the threshold, which implies that the optimal decision at state \( k \) is passive action, i.e. \( k \in D(W_j) \).

As we demonstrated that for \( k \in [n_{j-1}, n_{j}], W < W_j, k \notin D(W) \) and \( k \in D(W_j) \), then \( W_j = \min\{W, k \in D(W)\} \). This concludes the proof.
APPENDIX I

PROOF OF LEMMA [19]

We will just prove the first case. For the other cases, the proof is similar.

First case: \( \frac{a_i - a_{j-1}}{b_{j-1}} \leq \frac{a_{j+1} - a_j}{b_{j+1} - b_{j}} \implies \frac{a_j - a_{j-1}}{b_{j-1}} \leq \frac{a_{j+1} - a_{j-1}}{b_{j+1} - b_{j-1}} \leq \frac{a_{j+1} - a_j}{b_{j+1} - b_j} \).

For the LHS inequality:

\[
\frac{a_{j+1} - a_{j-1}}{b_{j+1} - b_{j-1}} = \frac{a_{j+1} - a_j}{b_{j+1} - b_j} + \frac{a_j - a_{j-1}}{b_{j+1} - b_{j-1}} \\
\geq \frac{(a_j - a_{j-1})(b_{j+1} - b_j)}{(b_j - b_{j-1})(b_{j+1} - b_{j-1})} + \frac{a_j - a_{j-1}}{b_{j+1} - b_{j-1}}
\]

The inequality above comes from the fact that \( b_{j-1} < b_j < b_{j+1} \) and \( \frac{a_j - a_{j-1}}{b_{j-1}} \leq \frac{a_{j+1} - a_j}{b_{j+1} - b_j} \). Then

\[
\frac{a_{j+1} - a_j}{b_{j+1} - b_j} \geq \frac{a_j - a_{j-1}}{b_{j+1} - b_{j-1}} \frac{b_{j+1} - b_j + b_j - b_{j-1}}{b_{j+1} - b_{j-1}} = \frac{a_j - a_{j-1}}{b_{j-1}}
\]

For the RHS inequality:

\[
\frac{a_{j+1} - a_{j-1}}{b_{j+1} - b_{j-1}} = \frac{a_{j+1} - a_j}{b_{j+1} - b_j} + \frac{a_j - a_{j-1}}{b_{j+1} - b_{j-1}} \\
\leq \frac{a_{j+1} - a_j}{b_{j+1} - b_j} + \frac{(a_{j+1} - a_j)(b_j - b_{j+1})}{(b_{j+1} - b_j)(b_{j+1} - b_{j-1})}
\]

where the above inequality comes from the fact that \( b_{j-1} < b_j < b_{j+1} \) and \( \frac{a_{j+1} - a_j}{b_j - b_{j+1}} \leq \frac{a_{j+1} - a_j}{b_{j+1} - b_j} \). Then

\[
\frac{a_{j+1} - a_j}{b_{j+1} - b_j} \leq \frac{a_{j+1} - a_j}{b_{j+1} - b_j} \frac{b_{j+1} - b_j + b_j - b_{j-1}}{b_{j+1} - b_{j-1}} = \frac{a_{j+1} - a_j}{b_{j+1} - b_j}
\]

APPENDIX J

PROOF OF LEMMA [20]

We consider \( i \) such that \( b_i = b_n, \) and we prove that \( n_j \leq i \):

By construction of \( n_j, b_{n_j - 1} \neq b_n \) and \( n_{j-1} < n_j \). Hence, by increasiness of \( b_k, b_n \geq b_{n_{j-1}} \).

Therefore \( b_i = b_n > b_{n_{j-1}}, \) and \( i > n_{j-1} \). Consequently, according to definition of \( n_j \):

\[
\frac{a_{n_j} - a_{n_{j-1}}}{b_{n_j} - b_{n_{j-1}}} \leq \frac{a_i - a_{n_{j-1}}}{b_i - b_{n_{j-1}}}
\]

\[
\frac{a_{n_j} - a_{n_{j-1}}}{b_{n_j} - b_{n_{j-1}}} = \frac{a_i - a_{n_{j-1}}}{b_{n_j} - b_{n_{j-1}}}
\]

This implies that \( a_{n_j} \leq a_i \).

If \( i < n_j, \) as \( b_i = b_n, \) then \( a_i < a_{n_j} \) which contradicts with \( a_{n_j} \leq a_i \).

Therefore \( n_j \leq i \). This concludes the proof.
APPENDIX K

PROOF OF LEMMA

For \( n \in [-1, R - 2] \)

\[
\sum_{0}^{n+1} u^{n+1}(q) - \sum_{0}^{n} u^{n}(q) = \left(1 - \frac{n + 1}{2R}\right)\left(\frac{n + 2}{R}\right) - \left(1 - \frac{n}{2R}\right)\left(\frac{n + 1}{R}\right) = \frac{R - 1 - n}{R^2} > 0
\]

APPENDIX L

PROOF OF LEMMA

We introduce a useful Lemma:

**Lemma 21.** we have the inequality: for all \( x \in [0, 1[ \)

\[ x + \ln(1 - x)(1 - x) > 0 \]

**Proof.** See appendix \[\]

We note that \( R \geq 2 \), then \( \rho \in [0, 1[ \).

We denote the function \( h(n) = \sum_{0}^{n} u^{n}(q) = \frac{\rho^2}{2}(L - 1 - n)(L - n) + 1 - (1 - \rho)^{n-L+R+1} \). We give the first derivative and the second derivative of \( h \):

\[
\begin{align*}
\hat{h}'(n) &= \frac{\rho^2}{2}(-2L + 1 + 2n) - \ln(1 - \rho)(1 - \rho)^{n-L+R+1} \\
\hat{h}''(n) &= \rho^2 - (\ln(1 - \rho))^2(1 - \rho)^{n-L+R+1}
\end{align*}
\]

For \( n \in [L - R + 1, L - 1] \), \( (1 - \rho)^{n-L+R+1} \) is decreasing in \( n \), then

\[ h''(n) \geq \rho^2 - (\ln(1 - \rho))^2(1 - \rho)^2\]

Using lemma 21

\[ \rho > -\ln(1 - \rho)(1 - \rho) \]

then

\[ \rho^2 > (\ln(1 - \rho))^2(1 - \rho)^2 \]

So

\[ h''(n) \geq \rho^2 - (\ln(1 - \rho))^2(1 - \rho)^2 > 0 \]
i.e. $h'$ is strictly increasing function in $n$.

We have $h'(L - R + 1) = \frac{3\rho^2}{2} - \rho - \ln(1 - \rho)(1 - \rho)^2$. In order to prove the positivity of $h'$, we introduce the function

$$r(x) = \frac{3x^2}{2} - x - \ln(1 - x)(1 - x)^2$$

$$r'(x) = 2(x + \ln(1 - x)(1 - x)) > 0$$

(assuming to Lemma 21), which means $r$ is strictly increasing in $[0,1]$. Hence, for all $x \in [0,1]$, $r'(x) > r(0) = 0$.

Then:

$$h'(L - R + 1) = \frac{3\rho^2}{2} - \rho - \ln(1 - \rho)(1 - \rho)^2 > 0$$

Since $h'$ is increasing function in $n$, then:

$$h'(n) \geq h'(L - R + 1) > 0$$

So $h$ is strictly increasing in $n$. This concludes the proof.

**APPENDIX M**

**PROOF OF LEMMA 21**

We consider the function $v(x) = x + \ln(1 - x)(1 - x)$ in $[0,1]$ the first derivative: $v'(x) = -\ln(1 - x) > 0$ for all $x \in [0,1]$, we have $v(0) = 0$, then for all $x \in [0,1]$ $v(x) > v(0) = 0$, which concludes the result.

**APPENDIX N**

**PROOF OF LEMMA 5**

For $n \in [L - R, L - 2]$, we have:

$$\sum_{0}^{L} au^{n+1}(q)q - \sum_{0}^{L} au^{n+1}(q)q = 1 - 2(1 - \rho)^{n-L+1+R} + 2L\rho - 2n\rho - 2\rho$$

If we denote the function $p$ as:

$$p(n) = 1 - 2(1 - \rho)^{n-L+1+R} + 2L\rho - 2n\rho - 2\rho$$

$$p''(n) = -2(\ln(1 - \rho))^2(1 - \rho)^{n-L+1+R}$$

Hence, as $p''(n) \leq 0$, $p$ is concave, that is $p$ is quasi-concave in $[L - R, L - 1]$ , then:

$$p(n) \geq \min(p(L - R), p(L - 1))$$

$$p(L - R) = 1 - 2(1 - \rho) + 2 - 2\rho = 1 > 0$$

$$p(L - 1) = 1 - 2(1 - \rho)^R$$

As $(1 - \rho)^R \leq \exp(-1)$ (with exp the exponential function) for all $R \geq 2$, then:

$$p(L - 1) \geq 1 - 2\exp(-1) > 0$$

Thus $p(n) > 0$ in $[L - R, L - 1]$.

Hence, for $n \in [L - R, L - 2]$

$$\sum_{0}^{L} au^{n+1}(q)q - \sum_{0}^{L} au^{n+1}(q)q > 0$$
APPENDIX O

PROOF OF LEMMA\textsuperscript{6}

At $W = x_{i,j}$, $y'(W) = y'(W)$, i.e.:

$$
\sum_{i=0}^{L} au^i(q)q - W \sum_{i=0}^{L} u^i(q) = \sum_{j=0}^{L} au^j(q)q - W \sum_{j=0}^{L} u^j(q)
$$

Hence

$$
W = \frac{\sum_{i=0}^{L} au^i(q)q - \sum_{j=0}^{L} au^j(q)}{\sum_{i=0}^{L} u^i(q) - \sum_{j=0}^{L} u^j(q)}
$$

APPENDIX P

PROOF OF LEMMA\textsuperscript{7}

We start by giving a useful lemma.

\textbf{Lemma 22.} $w_n$ is strictly increasing in $n \in [0, R - 1]$.

\textit{Proof:} for $n \in [0, R - 2]$:

$$
w_{n+1} - w_n = \frac{aR^2}{(R-n)(R-n-1)} > 0.
$$

Let us first consider the interval $[0, R - 1]$.

We have:

$$f'(n) = (w_n')\left[1 - \left(1 - \frac{n}{2R}\right) \frac{n+1}{R}\right] + w_n\left[1 - \left(1 - \frac{n}{2R}\right) \frac{n+1}{R}\right] + \left[a\left(\frac{R-1}{2} + n(n+1)\right)\right]'$$

First, we deal with the first term $(w_n')\left[1 - \left(1 - \frac{n}{2R}\right) \frac{n+1}{R}\right]$:

According to Lemma \textsuperscript{22}, $(w_n')$ is positive since $w_n$ is increasing in $n$, and $1 - \sum_{i=0}^{n} u^n(q) = 1 - (1 - \frac{n}{2R}) \frac{n+1}{R}$ is strictly positive since $\sum_{i=0}^{n} u^n(q) < 1$ for $n \leq R - 1 < L$. Then, $(w_n')\left[1 - \left(1 - \frac{n}{2R}\right) \frac{n+1}{R}\right] \geq 0$, for $n \in [0, R - 1]$.

For the second term, we have:

$$w_n\left[1 - \left(1 - \frac{n}{2R}\right) \frac{n+1}{R}\right]' = a\frac{2n^2R - 2R^2n + Rn}{(R-n)(2R^2)}$$

For the third term $[a\left(\frac{R-1}{2} + \frac{n(n+1)}{2R}\right)]' = a\frac{2n + 1}{2R}$.

Adding the second term to the third term, we get:

$$w_n\left[1 - \left(1 - \frac{n}{2R}\right) \frac{n+1}{R}\right]' + \left[a\left(\frac{R-1}{2} + \frac{n(n+1)}{2R}\right)\right]' = a\frac{2n^2R - 2R^2n + Rn}{(R-n)(2R^2)} + a\frac{2n + 1}{2R}$$

$$= a\frac{2R(R-n)}{2R} > 0$$

So $f$ is strictly increasing in $[0, R - 1]$

For $n = -1$, $f(-1) = 0 < f(0) = \frac{a(R-1)}{2}$, and $f(R - 1) < +\infty$ then $f$ in strictly increasing in $[-1, R]$. 

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**APPENDIX Q**

**Proof of Lemma**

In all this proof we consider $R \geq 2$.

After computation we get $f(n)/a = \frac{R^3+R^2n-Rn^2-R^2}{2R(R-n)}$ for $n \in [0, R-1]$, $f(n)/a = 0$ for $n = -1$ and $f(n)/a = +\infty$ for $n = R$.

We aim to see, for which integer values of $n \in [-1, R]$, $f(n)/a$ will be strictly less than $2R$, and consequently less than $L$ according to assumption \[\text{assumption}\]. We have $f(n)/a > L$ for $n = R$ and $f(n)/a < L$ for $n = -1$. We still need to compare $f(n)/a$ with $2R$ when $n \in [0, R-1]$.

We denote $t(n)$ as:

$$t(n) = (2R(R-n))(f(n)/a - 2) = -3R^3 + 5R^2n - Rn^2 - R^2$$

We study the sign of the function $t(x) = -3R^3 + 5R^2x - Rx^2 - R^2$ when $x \in [0, +\infty[$.

We start by the monotonicity:

$$t'(x) = 5R^2 - 2Rx$$

$t'(x) = 0 \iff x = \frac{5}{2}R$.

Then for all $x \in [0, \frac{5}{2}R)$, $t$ is strictly increasing in $x$. Since $R - 1 \leq \frac{5}{2}R$, thus for all $x \in [0, R-1)$, $t$ is strictly increasing.

Now we search for which $x \in [0, R-1]$ $t(x) = 0$. We have:

$$t\left(\frac{5}{2}R\right) = \frac{13}{4}R^3 - R^2 > 0$$

$$t(0) = -3R^3 - R^2 < 0$$

Then, there is a unique $e \in [0, \frac{5}{2}R]$ such that $t(e) = 0$. In fact this $e$ is one of the two roots of $t(x)$. As we know $t$ is a polynomial function of degree two, then it has at most two roots. The two roots are: $\frac{5}{2}R - \frac{\sqrt{13R^2 - 4R}}{2}$ and $\frac{5}{2}R + \frac{\sqrt{13R^2 - 4R}}{2}$. Given that $\frac{5}{2}R + \frac{\sqrt{13R^2 - 4R}}{2} > \frac{5}{2}R$, then clearly $e = \frac{5}{2}R - \frac{\sqrt{13R^2 - 4R}}{2}$.

Hence, as $t$ is strictly increasing in $[0, \frac{5}{2}R]$, $t(x) < t(e) = 0$ if $x \in [0, e]$ and $t(x) \geq 0$ if $x \in [e, \frac{5}{2}R]$.

That is, for the integer $n \in [0, e]$, $t(n) < 0$.

Since the sign of $f(n)/a - 2R$ is that of $(2R(R-n))(f(n)/a - 2R)$ for $n \in [0, R-1]$ because $R > n$, that means:

If $n \in [0, e]$ and $n \in [0, R-1]$, $f(n)/a < 2R \leq L$.

If $n = -1$, $f(n)/a = 0 < 2R \leq L$.

In other words, if $-1 \leq n < \min(e, R)$, $f(n)/a < L$.

This ends the proof.
APPENDIX R

PROOF OF PROPOSITION

In order to prove the proposition, we introduce the following useful lemmas.

**Lemma 23.** For any numerical sequence: \(-1 \leq i_{-1} < i_0 < i_1 < \ldots < i_M \leq L\), such that for any \(k \in [0, M-1]\), \(b_{i_{k-1}} < b_i < b_{i_{k+1}}\) and

\[
\frac{a_{i_k} - a_{i_{k-1}}}{b_{i_k} - b_{i_{k-1}}} < \frac{a_{i_{k+1}} - a_{i_k}}{b_{i_{k+1}} - b_{i_k}}
\]

Then for any \(k \in [0, M-1]\), we have for each \(k < s \leq M\):

\[
\frac{a_{i_s} - a_{i_{k-1}}}{b_{i_s} - b_{i_{k-1}}} > \frac{a_{i_k} - a_{i_{k-1}}}{b_{i_k} - b_{i_{k-1}}}
\]

**Proof:** We fix certain \(k \in [0, M-1]\), we prove the result by induction:

for \(s = k + 1\)

\[
\frac{a_{i_{k+1}} - a_{i_{k-1}}}{b_{i_{k+1}} - b_{i_{k-1}}} = \frac{a_{i_{k+1}} - a_{i_{k-1}} - a_{i_k} + a_{i_k}}{b_{i_{k+1}} - b_{i_k}}
\]

\[
= \frac{a_{i_{k+1}} - a_{i_k}}{b_{i_{k+1}} - b_{i_k}} + \frac{a_{i_k} - a_{i_{k-1}}}{b_{i_k} - b_{i_{k-1}}}
\]

\[
> \frac{(a_{i_k} - a_{i_{k-1}})(b_{i_{k+1}} - b_{i_k})}{(b_{i_{k}} - b_{i_{k-1}})(b_{i_{k+1}} - b_{i_{k-1}})} + \frac{(a_{i_k} - a_{i_{k-1}})(b_{i_k} - b_{i_{k-1}})}{(b_{i_{k}} - b_{i_{k-1}})(b_{i_{k+1}} - b_{i_{k-1}})}
\]

where the strict inequality comes from the lemma’s assumptions. We then have:

\[
\frac{a_{i_{k+1}} - a_{i_{k-1}}}{b_{i_{k+1}} - b_{i_{k-1}}} > \frac{a_{i_{k+1}} - a_{i_{k-1}}}{b_{i_{k+1}} - b_{i_{k-1}}} \left[ \frac{b_{i_{k+1}} - b_{i_k}}{b_{i_{k+1}} - b_{i_{k-1}}} + \frac{b_{i_k} - b_{i_{k-1}}}{b_{i_{k+1}} - b_{i_{k-1}}} \right]
\]

\[
= \frac{a_{i_k} - a_{i_{k-1}}}{b_{i_k} - b_{i_{k-1}}}
\]

By induction, we consider that the above inequality is true for certain \(s\) strictly higher than \(k\). The inequality below is then verified for \(s + 1\):

\[
\frac{a_{i_{s+1}} - a_{i_{k-1}}}{b_{i_{s+1}} - b_{i_{k-1}}} = \frac{a_{i_{s+1}} - a_{i_{k-1}} - a_{i_s} + a_{i_s}}{b_{i_{s+1}} - b_{i_{k-1}}}
\]

\[
= \frac{a_{i_{s+1}} - a_{i_s}}{b_{i_{s+1}} - b_{i_{k-1}}} + \frac{a_{i_s} - a_{i_{k-1}}}{b_{i_{s+1}} - b_{i_{k-1}}}
\]

\[
> \frac{(a_{i_k} - a_{i_{k-1}})(b_{i_{s+1}} - b_{i_s})}{(b_{i_k} - b_{i_{k-1}})(b_{i_{s+1}} - b_{i_{k-1}})} + \frac{(a_{i_k} - a_{i_{k-1}})(b_{i_s} - b_{i_{k-1}})}{(b_{i_k} - b_{i_{k-1}})(b_{i_{s+1}} - b_{i_{k-1}})}
\]

\[
= \frac{a_{i_{s+1}} - a_{i_{k-1}}}{b_{i_{s+1}} - b_{i_{k-1}}} \left[ \frac{b_{i_{s+1}} - b_{i_s}}{b_{i_{s+1}} - b_{i_{k-1}}} + \frac{b_{i_s} - b_{i_{k-1}}}{b_{i_{s+1}} - b_{i_{k-1}}} \right]
\]

\[
= \frac{a_{i_{k+1}} - a_{i_{k-1}}}{b_{i_{k+1}} - b_{i_{k-1}}}
\]

So the inequality is also true for \(s + 1\). This concludes the proof of the lemma.

**Lemma 24.** If \(L \leq \frac{f(d+1)}{d}\), then \(\frac{a_{i_{s+1}} - a_{i_s}}{b_{i_{s+1}} - b_{i_s}} \leq \frac{a_{d+1} - a_d}{b_{d+1} - b_d}\)
Proof: This lemma is an immediate application of Lemma \[19\]. In fact when \(L \leq \frac{d+1}{a}\) it implies that \(\frac{a_{L+1} - a_d}{b_{L+1} - b_d} \leq \frac{a_d - a_d}{b_l - b_d}\). Then according to the second case in Lemma \[19\], we have directly:

\[
\frac{a_L - a_L}{b_L - b_d} \leq \frac{a_{L+1} - a_d}{b_{L+1} - b_d}
\]

Lemma 25. The intersection points \(x_{L,R}\) and \(x_{n,R}\) satisfy \(x_{L,R} \leq x_{n,R}\) when \(n \in [L - R + 2, L - 1]\).

Proof: We have:

\[
x_{L,R} = \frac{2(R - 1)}{R - 1} - 2R
\]

\[
x_{n,R} = \frac{n - R}{\left(\frac{\rho}{2}\right)(L - 1 - n)(L - n) + 1 - (1 - \rho)^{n-L+R+1} - \frac{1}{2} - \frac{1}{2R} - 2R}
\]

\[
x_{n,R} - x_{L,R} = \frac{n - R}{\left(\frac{\rho}{2}\right)(L - 1 - n)(L - n) + 1 - (1 - \rho)^{n-L+R+1} - \frac{1}{2} - \frac{1}{2R} - 2R}
\]

\[
= \frac{(n - R)(R - 1) - 2R(R - 1)(\frac{\rho^2}{2}(L - 1 - n)(L - n) + 1 - (1 - \rho)^{n-L+R+1} - \frac{1}{2} - \frac{1}{2R})}{(R - 1)(\frac{\rho^2}{2}(L - 1 - n)(L - n) + 1 - (1 - \rho)^{n-L+R+1} - \frac{1}{2} - \frac{1}{2R})}
\]

The denominator is greater than 0 since \(R > 1\), and \(h(n) = \frac{\rho^2}{2}(L - 1 - n)(L - n) + 1 - (1 - \rho)^{n-L+R+1} > h(L - R + 1) = \frac{1}{2} + \frac{1}{2R}\) for \(n \in [L - R + 2, L - 1]\) (using Lemma \[4\]). We consider the following function (which is equal to the numerator):

\[
p(x) = (x - R)(R - 1) - 2R(L - 1)(\frac{\rho^2}{2}(L - 1 - x)(L - x) + 1 - (1 - \rho)^{x-L+R+1} - \frac{1}{2} - \frac{1}{2R})
\]

The function \(p\) is concave in the interval \([L - R + 1, L - 1]\) as \(p'\) is negative. Then, \(p\) is quasi-concave in this interval and we have that \(p(x) \geq \min(p(L - R + 1), p(L - 1))\) for all \(x \in [L - R + 1, L - 1]\), where

\[
p(L - R + 1) = (L - 2R + 1)(R - 1) \geq 0
\]

and

\[
p(L - 1) = 2R(L - 1)(1 - \rho)^R + R - R^2 \geq 0
\]

where the last inequality is due to the following analysis. First we use the fact that \((1 - \rho)^R \geq 1/4\) for all \(R \geq 2\) then

\[
2R(L - 1)(1 - \rho)^R \geq \frac{2R(L - 1)}{4}
\]

\[
2R(L - 1)(1 - \rho)^R + R - R^2 \geq \frac{2R(L - 1)}{4} + R - R^2
\]

we have \(L - 1 \geq 2R - 1\), then

\[
2R(L - 1)(1 - \rho)^R + R - R^2 \geq \frac{2R(2R - 1)}{4} + R - R^2
\]

\[
\geq R^2 - \frac{R}{2} + R - R^2
\]

\[
\geq R \geq 0
\]
From all the analysis above, we conclude that for all $n \in [L - R + 1, L - 1]$ $p(n) \geq 0$. This is also true for $n \in [L - R + 2, L - 1]$. Hence, the numerator and denominator of $x_{n,R} - x_{L,R}$ are positive, which concludes the proof.

Lemma 26. For any $d \in [0, R - 1]$, $x_{L,d} \leq x_{n,d}$ for any $n \in [L - R + 2, L - 1]$.

Proof:
We start by proving that $\frac{a_L - a_{R-1}}{b_L - b_{R-1}} \leq \frac{a_n - a_{R-1}}{b_n - b_{R-1}}$. We have:

\[
\frac{a_L - a_{R-1}}{b_L - b_{R-1}} = \frac{a_L - a_R}{b_L - b_R} + \frac{a_R - a_{R-1}}{b_L - b_{R-1}}
\]

Since $b_R = b_{R-1}$ (see the expression of average passive time when $n \in [R - 1, L - R + 1]$), then:

\[
\frac{a_L - a_{R-1}}{b_L - b_{R-1}} = \frac{a_L - a_R}{b_L - b_R} + \frac{a_R - a_{R-1}}{b_L - b_{R-1}}
\]

As we have already proved in Lemma 25 that: $\frac{a_L - a_{R-1}}{b_L - b_{R-1}} \leq \frac{a_n - a_{R-1}}{b_n - b_{R-1}}$. Hence:

\[
\frac{a_L - a_{R-1}}{b_L - b_{R-1}} \leq \frac{a_n - a_{R-1}}{b_n - b_{R-1}} + \frac{a_R - a_{R-1}}{b_L - b_{R-1}}
\]

Since $b_L > b_n$, hence:

\[
\frac{a_L - a_{R-1}}{b_L - b_{R-1}} \leq \frac{a_n - a_{R-1}}{b_n - b_{R-1}} + \frac{a_R - a_{R-1}}{b_L - b_{R-1}} = \frac{a_n - a_{R-1}}{b_n - b_{R-1}}
\]

Thus:

\[
\frac{a_L - a_{R-1}}{b_L - b_{R-1}} \leq \frac{a_n - a_{R-1}}{b_n - b_{R-1}}
\]

If $d = R - 1$, the proof is direct result from the inequality above.

If $d < R - 1$:
Given that $\frac{a_L - a_{R-1}}{b_L - b_{R-1}} \leq \frac{a_n - a_{R-1}}{b_n - b_{R-1}}$, then applying lemma 19 fourth case, we deduce:

\[
\frac{a_L - a_n}{b_L - b_n} \leq \frac{a_L - a_{R-1}}{b_L - b_{R-1}} \leq \frac{a_n - a_{R-1}}{b_n - b_{R-1}}
\]

(14)

Now we prove that:

\[
\frac{a_L - a_{R-1}}{b_L - b_{R-1}} \leq \frac{a_L - a_d}{b_L - b_d}
\]

Given that $L \leq f(d + 1)/a$: $\frac{a_L - a_{d+1}}{b_L - b_{d+1}} \leq \frac{a_{d+1} - a_d}{b_{d+1} - b_d}$.

Hence applying lemma 24:

\[
\frac{a_L - a_d}{b_L - b_d} \leq \frac{a_{d+1} - a_d}{b_{d+1} - b_d}
\]

According to Lemma 23 since $w_{d+1} < \cdots < w_{R-1}$, thus:

\[
\frac{a_{d+1} - a_d}{b_{d+1} - b_d} \leq \frac{a_{R-1} - a_d}{b_{R-1} - b_d}
\]

Then:

\[
\frac{a_L - a_d}{b_L - b_d} \leq \frac{a_{R-1} - a_d}{b_{R-1} - b_d}
\]

Given that $\frac{a_L - a_d}{b_L - b_d} \leq \frac{a_{R-1} - a_d}{b_{R-1} - b_d}$ and applying Lemma 19 (fourth case), then:
\[
\frac{a_L - a_{R-1}}{b_L - b_{R-1}} \leq \frac{a_L - a_d}{b_L - b_d} \leq \frac{a_{R-1} - a_d}{b_{R-1} - b_d}
\]  
(15)

Combining [14] and [15] we conclude:

\[
\frac{a_L - a_n}{b_L - b_n} \leq \frac{a_L - a_{R-1}}{b_L - b_{R-1}} \leq \frac{a_{R-1} - a_d}{b_{R-1} - b_d}
\]

Given this result and applying lemma [19] sixth case, we get our result:

\[
\frac{a_n - a_d}{b_n - b_d} \geq \frac{a_L - a_d}{b_L - b_d}
\]

Hence \(x_{n,d} \geq x_{L,d}\).

This concludes the proof.

Now, we can prove the proposition.

Referring to the algorithm 1 that allows us to obtain the whittle indices, we denote by \(j\) the step \(j\) described in the algorithm.

For \(0 \leq j \leq d \leq R - 1\)

We prove that for all \(n \in [j + 1, L]\), \(\frac{a_n - a_{j-1}}{b_n - b_{j-1}} > \frac{a_j - a_{j-1}}{b_j - b_{j-1}}\)

We study four cases:

1) \(n \in [j + 1, R - 1]\):

Using lemma \([22]\) \(w_j < w_{j+1} < \ldots < w_{R-1}\), therefore considering the set of element \(\{j - 1, j, j + 1, \ldots, R - 1\}\),

we can apply lemma \([23]\) since \(\frac{a_k - a_{k-1}}{b_k - b_{k-1}} < \frac{a_k - a_{k-1}}{b_k - b_{k-1}}\) for all \(k \in [j, R - 2]\).

So for all \(n \in [j + 1, R - 1]\), \(\frac{a_n - a_{j-1}}{b_n - b_{j-1}} > \frac{a_j - a_{j-1}}{b_j - b_{j-1}}\)

2) \(n \in [R, L - R + 1]\):

There are two cases:

a) \(j = R - 1\):

We have \(b_n = b_{R-1} = b_j\), then \(a_n > a_j\). Hence, \(\frac{a_n - a_{j-1}}{b_n - b_{j-1}} > \frac{a_j - a_{j-1}}{b_j - b_{j-1}}\)

b) \(j < R - 1\):

\(b_n = b_{R-1}, a_n > a_{R-1}\), then \(\frac{a_n - a_{R-2}}{b_n - b_{R-2}} > \frac{a_{R-1} - a_{R-2}}{b_{R-1} - b_{R-2}}\). Therefore, by considering the set \(\{j - 1, j, j + 1, \ldots, R - 2, n\}\), we have \(\frac{a_n - a_{j-2}}{b_n - b_{j-2}} > \frac{a_{j-1} - a_{j-2}}{b_{j-1} - b_{j-2}}\).

Thus, we can apply Lemma \([23]\) and get \(\frac{a_n - a_{j-1}}{b_n - b_{j-1}} > \frac{a_j - a_{j-1}}{b_j - b_{j-1}}\).

3) \(n \in [L - R + 2, L - 1]\):

Using Lemma \([26]\) we have \(\frac{a_n - a_d}{b_n - b_d} > \frac{a_L - a_d}{b_L - b_d}\).

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Given that \( f(d) < L \), that means \( \frac{a_L-a_d}{b_L-b_d} > \frac{a_d-a_{d-1}}{b_d-b_{d-1}} \).

So considering the set \( \{j-1,a_j,...,d,n \} \), we have \( \frac{a_n-a_d}{b_n-b_d} > \frac{a_d-a_{d-1}}{b_d-b_{d-1}} = w_d > \cdots > w_j \).

Then we can apply Lemma 23 and obtain \( \frac{a_n-a_{j-1}}{b_n-b_{j-1}} > \frac{a_j-a_{j-1}}{b_j-b_{j-1}} \).

4) \( n = L \)

We have \( \frac{a_L-a_d}{b_L-b_d} > \frac{a_d-a_{d-1}}{b_d-b_{d-1}} = w_d > \cdots > w_j \).

Then, applying Lemma 23 \( \frac{a_L-a_{j-1}}{b_L-b_{j-1}} > \frac{a_j-a_{j-1}}{b_j-b_{j-1}} \).

Therefore, the largest minimizer at step \( j \) is \( j \), and \( W(j) = w_j = \frac{a_j-a_{j-1}}{b_j-b_{j-1}} \).

At step \( d+1 \):

The largest minimizer at step \( d \) was \( d \), then in order to prove that the largest minimizer at this step is \( L \), we should prove that for all \( n > d \), we have \( \frac{a_L-a_d}{b_L-b_d} \leq \frac{a_n-a_d}{b_n-b_d} \). We distinguish again between three cases:

1) \( n \in [d+1,R-1] \):

We know that \( w_{d+1} < \cdots < w_{R-1} \). Then, considering the set \( \{d,d+1,...,R-1\} \) and according to Lemma 23 we get \( \frac{a_{d+1}-a_d}{b_{d+1}-b_d} \leq \frac{a_n-a_d}{b_n-b_d} \) for all \( n \in [d+1,R-1] \).

Since \( \frac{a_L-a_d}{b_L-b_d} = \frac{a_{d+1}-a_d}{b_{d+1}-b_d} \) (according to Lemma 24), then \( \frac{a_L-a_d}{b_L-b_d} \leq \frac{a_n-a_d}{b_n-b_d} \) for all \( n \in [d+1,R-1] \).

2) \( n \in [R,L-R+1] \):

a) \( d = R-1 \):

We have \( b_n = b_{R-1} = b_d \). The case where the passive decision average time \( b_n \) is equal to \( b_{n_d} = b_d \) is not included in the computation of whittle indices (recall that \( n_d \) is the largest minimizer at step \( d \) which is \( d \)). This case can be hence skipped.

b) \( d = R-2 \):

\( b_n = b_{R-1} \) and \( a_n > a_{R-1} \), then applying Lemma 24 we have \( \frac{a_n-a_{R-1}-2}{b_n-b_{R-1}-2} > \frac{a_{R-1}-a_{R-2}}{b_{R-1}-b_{R-2}} \geq \frac{a_L-a_d}{b_L-b_d} \), and we conclude the result.

c) \( d < R-2 \):

We have \( \frac{a_n-a_{R-2}}{b_n-b_{R-2}} > \frac{a_{R-1}-a_{R-2}}{b_{R-1}-b_{R-2}} \). Therefore, by considering the set \( \{d,d+1,...,R-2,n\} \), we have \( \frac{a_{d+1}-a_d}{b_{d+1}-b_d} = w_{d+1} < \cdots < w_{R-2} = \frac{a_{R-2}-a_{R-3}}{b_{R-2}-b_{R-3}} < \frac{a_n-a_{R-2}}{b_n-b_{R-2}} \).

Combining Lemma 23 and Lemma 24 we get \( \frac{a_i-a_d}{b_i-b_d} \leq \frac{a_{d+1}-a_d}{b_{d+1}-b_d} < \frac{a_n-a_d}{b_n-b_d} \) for all \( n \in [R,L-R+1] \)

3) \( n \in [L-R+2,L-1] \):

Applying Lemma 26 we have \( \frac{a_L-a_d}{b_L-b_d} < \frac{a_n-a_d}{b_n-b_d} \).

Hence we proved that at step \( d+1 \), the largest minimizer is \( L \). Therefore the whittle’s index for all state \( i \) from
This concludes the proof of the proposition.

APPENDIX S

PROOF OF PROPOSITION 5

In order to prove this proposition we distinguish between two types of classes:

1) Class \( k \) in which \( W \) is different from all \( W_i^k \).
2) Class \( k \) such that there exists a given state \( j \) that satisfies \( W_j^k = W \).

We start first by describing qualitatively the optimal threshold with respect to \( W \). Then we prove the explicit expression for both types of classes:

First type of classes:
For the class \( k \) in which \( W \) is different from all \( W_i^k \), according to the proposition 3, \( W < W_i^k \) = \( W_i^k \) = \( \frac{a_L - a_d}{b_L - b_d} \).

Applying lemma 22, and using the fact that \( W_{R_k - 1}^k < W_{R_k}^k = x_{L,R_k - 1} \) (since \( L > \frac{f(R_k - 1)}{a} \)), then \( W_i^k \) is strictly increasing in \( i \in [0, R_k] \). Therefore, there exists \( l_k \) strictly less than \( R_k \) such that \( W_{l_k - 1}^k \) is strictly less than \( W \) and \( W_{l_k}^k \) strictly higher than \( W \). Due to the indexability of the class, \( D(W_i^k) \subseteq D(W) \) for all states \( i \) less or equal to \( l_k \), which implies that the optimal decision at state \( i \) is passive action. Furthermore, according to the definition of whittle index, for all states \( i \) greater than or equal to \( l_k + 1 \), we have \( W < W_i^k \) and the optimal decision at those states \( i \) is active decision. Hence the optimal threshold is obviously \( l_k \).

Let us prove that \( l_k = \max \{ \text{arg max}_i \{ W_i^k | W_i^k \leq W \} \} = \max \{ \text{arg max}_i \{ W_i^k | W_i^k < W \} \} \).

As \( W_{l_k}^k < W < W_{l_k + 1}^k \), then

\[
l_k \in \{ \text{arg max}_i \{ W_i^k | W_i^k \leq W \} \}
\]

and

\[
l_k \in \{ \text{arg max}_i \{ W_i^k | W_i^k < W \} \}
\]

As \( W_{l_k - 1}^k < W_{l_k}^k \), then there is one index which maximizes

\[
\{ W_i^k | W_i^k \leq W \}
\]

and

\[
\{ W_i^k | W_i^k < W \}
\]

which is \( l_k \). Hence

\[
l_k(W) = l_k = \max \{ \text{arg max}_i \{ W_i^k | W_i^k \leq W \} \} = \max \{ \text{arg max}_i \{ W_i^k | W_i^k < W \} \}
\]

Second type of classes:
For the class \( k \) such that there exists \( j \), \( W_j^k = W \), we distinguish between two cases:
1) \( j \leq R_k - 1 \):
We know that according to Proposition 3, \( W_j^k = u_j^k = x_{j,j-1} \) which is the point for which if \( W = x_{j,j-1} \), we have \( \sum_0^j a_w(q)q - W \sum_1^j w_j(q) = \sum_0^j a_w(q)q - W \sum_0^j w_j(q) \). That means, according to equation (10), for \( W = x_{j,j-1} \), if \( j \) is a minimizer of this equation \( (j) \) is the optimal threshold), then \( j - 1 \) is also a minimizer of this equation. Due to indexability, for all states less or equal than \( j \) the optimal decision is to stay passive. Also, according to definition of whittle index, for all states strictly higher than \( j \) the optimal decision is to be active. Then, \( j \) could be the threshold, so as for \( j - 1 \).

Hence, the optimal threshold can be either \( j \) or \( j - 1 \).

In fact, since \( W_0^k \leq \cdots \leq W_{j-1}^k \leq W_j^k = W \), \( j = \max \{ \arg \max_i W_i^k | W_i^k \leq W \} \), and \( j - 1 = \max \{ \arg \max_i W_i^k | W_i^k < W \} \).

This proves the proposition for this case.

2) If \( j \geq R_k \):

Then \( W_j^k = W_L^k = W_{R_k} = W \), thus according to proposition 3, \( W = x_{L,R_k-1} \). That means, at \( W \), the threshold policy can be either \( L \) or \( R_k - 1 \). \( L \) is the biggest integer such that \( W_j^k = W \), and \( R_k - 1 \) is the biggest integer that verifies the strict inequality, explicitly \( L = \max \{ \arg \max_i W_i^k | W_i^k \leq W \} \) and \( R_k - 1 = \max \{ \arg \max_i W_i^k | W_i^k < W \} \).

### Appendix T

#### Proof of Proposition 5

From optimization theory, it is known that the optimal solution of the dual problem is less or equal than the primal problem’s solution when the constraint is satisfied, i.e.: 

\[
\max \min_{W, \phi} f(W, \phi) \leq \min \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{k=1}^{K} a_k Q_k^k(t) \mid Q(0), \phi \right]
\]  

(16)

As the optimal solution for fixed \( W \) is a threshold policy, we use the steady state form and the expression of the LHS of the inequality becomes:

\[
\max_{W} \min_{\phi} f(W, \phi) = \max_{W} \left\{ \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} \left[ \sum_{q=0}^{L} a_k u_k^i(q)q - W \sum_{q=0}^{L} u_k^i(q) \right] \right\} + W(1 - \alpha)N
\]  

(17)

with \( \phi \) the threshold policy that corresponds to \( l(W) \) computed using Proposition 3 for fixed \( W \). For \( W^* \) that satisfies the constraint with equality (i.e. \( \alpha N = \sum_{k=1}^{K} \gamma_k N \sum_{i=1}^{L} u_k^i(W^*) \)), which is in fact true for all \( N \), and then we can get rid of \( N \), we get exactly the objective function of the primal problem. Therefore, we get a threshold vector \( l(W^*) \) that gives a solution for the primal problem less than the optimal solution for this problem according to inequality (16). Then, surely this solution given by \( l(W^*) \) is the optimal one for the constrained relaxed problem, since it satisfies the constraint and for all policy \( \phi \) that satisfies the constraint and belong to \( \Phi \), we have

\[
f(W^*, l(W^*)) = \sum_{k=1}^{K} \sum_{q=0}^{L} a_k u_k^i(W^*) \sum_{q=0}^{L} a_k u_k^i(W^*) \mid Q(0), l(W^*)
\]  

\[
\leq \min \limsup_{T \to \infty} \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{\gamma_k N} a_k Q_k^k(t) \mid Q(0), \phi \right]
\]  

We deduce that the solution of the relaxed problem is of type threshold-based policy \( l(W^*) \) with \( W^* \) satisfies

\[
\alpha = \sum_{k=1}^{K} \gamma_k \sum_{i=1}^{L} u_k^i(W^*) \mid q(q).
\]
In order to complete the proof of the proposition, we still need to prove that the existence of $W^*$ that satisfies the constraint as this has assumed in the analysis above. In fact, $W^*$ that satisfies the constraint may not exist, since $\alpha$ is a real number that can take any value in $[0, 1]$, and $\sum_0^K \gamma_k \sum_{k=1}^L u_k(W)(i)$ is discrete, since the vector $l(W)$ can only take discrete values in $[0, L]^K$. For that, we first introduce the following useful lemma.

**Lemma 27.** For each class $k$, $\sum_{n=1}^L u_k^n(i)$ is strictly decreasing in $n$, when $n \in [-1, R_k-1] \cup L$.

**Proof.** We have $\sum_{n=1}^n u_k^n(i)$ is strictly increasing in this set (see Lemma 3 and the fact that $1 = \sum_0^L u_k^L(i) > \sum_{k=R_k-1}^{R_k-1} u_k^{|R_k|-1}(i)$, then $\sum_{n=1}^L u_k^n(i) = 1 - \sum_{n=0}^n u_k^n(i)$, is strictly decreasing in $n$.

We define the following order relation in $\mathbb{R}^K$ such that for any two vectors $l^1$ and $l^2$, $l^1 \leq l^2 \iff$ for each element of vector of index $k$, we have $l^1_k \leq l^2_k$. Recall that we have proved in Proposition 5 that for $W_1 = W_2$, $l(W_1) \leq l(W_2)$. Without loss of generality, when $W \in \mathbb{R}^+$, the corresponding set of threshold vectors $l(W)$ is perfectly ordered. Then, by applying Lemma 27, $\sum_0^K \gamma_k \sum_{l_k=1}^{l_k(W)} u_k^l(W)(i)$ is strictly decreasing in $l(W)$, and take discrete values from 1 to 0. Then, there exists $l \geq l'$ such that $\sum_0^K \gamma_k \sum_{l_k=1}^{l_k} u_k^l(i) < \sum_0^K \gamma_k \sum_{l_k=1}^{l_k} u_k^l(i)$. In fact $l$ and $l'$ will be different only in one element denoted by the class index $m$ such that $l_m = l'_m + 1$. We consider $W_2$ and $W_1$ the subsidy that give the threshold vectors $l$ and $l'$ by the function defined in Proposition 5 respectively, then $W_1 < W_2$. According to Proposition 5 for all $k \neq m$, $W_k^l \leq W_k^m$ and $W_k^{l+1} \geq W_k^m$. Hence, when $W$ is between $W_1$ and $W_2$, the optimal threshold for the class $k \neq m$ will not change and is $l_k$. For class $m$, as we have $W_1 \leq W_m^l \leq W_2$ for $W \in [W_1, W_m^l]$, and since $l'$ is the optimal threshold, the total average time of active decision is constant and is equal to $\sum_0^K \gamma_k \sum_{l_k=1}^{l_k} u_k^l(i)$, which is higher than $\alpha$. However, when $W \in [W_m^l, W_2]$, the total average time of active decision is constant and equal to $\sum_0^K \gamma_k \sum_{l_k=1}^{l_k} u_k^l(i)$ which is less than $\alpha$. If we force $W^*$ to be equal to $W_m^l$, the optimal threshold for class $m$ can be either $l_m$ or $l'_m = l_m - 1$, then we can introduce some randomization when the queues of class $m$ are in this state. In other words, we use the threshold policy $l_m$ with probability $\theta$ and $l_m - 1$ with probability $1 - \theta$. The new stationary distribution for this class is then a linear combination of these two policies: $u_m^l = \theta u_m^{l_m} + (1 - \theta) u_m^{l_m-1}$. So in a state strictly less than $l_m$, the queues will not transmit, whereas in a state strictly greater than $l_m$, they will transmit with probability one. If the queues are in state $l_m$, they will transmit with probability $\frac{(1-\theta) u_m^{l_m-1}(l_m)}{u_m^{l_m}(l_m)+(1-\theta) u_m^{l_m-1}(l_m)}$. Since the probability to be in this state $l_m$ is $u_m^l(l_m)$, the proportion of time that the queues will be in active mode is:

$$\alpha = \sum_{k \neq m} \sum_{i=l_k+1}^L \gamma_k u_k^l(i) + \sum_{i=l_m+1}^L \gamma_m u_m^l(i) + (1 - \theta) \gamma_m u_m^{l_m-1}(l_m)$$

When $\theta = 0$, the threshold policy is $l_m - 1$ and the total average time in active mode is higher than $\alpha$. When $\theta = 1$, the threshold policy is $l_m$ and the total average time in active mode is less than $\alpha$.

Given that $\sum_{k \neq m} \sum_{i=l_k+1}^L \gamma_k u_k^l(i) + \sum_{i=l_m+1}^L \gamma_m u_m^l(i) + (1 - \theta) \gamma_m u_m^{l_m-1}(l_m)$ is continuous in $\theta$, then there exists at least one $\theta$ which verifies the equality. Hence, there exist $W^*$ that satisfies the constraint and give us a threshold policy for all classes except one class where the optimal solution is a linear combination of two threshold
APPENDIX U

PROOF OF PROPOSITION[7]

We derive the eigenvalues of $Q$.

The matrix $Q$ is of the form:

\[
\begin{bmatrix}
Q_1 & 0 & \cdots & \cdots & \cdots & \cdots & 0 \\
0 & Q_2 & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
A_1 & A_2 & \cdots & Q_m & \cdots & A_{K-1} & A_K \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \cdots & \cdots & Q_{K-1} & 0 \\
0 & 0 & \cdots & \cdots & \cdots & 0 & Q_K
\end{bmatrix}
\]

The characteristic polynomial of $Q$ is the product of the characteristic polynomial of each matrix $Q_k$:

\[
\chi_Q(\lambda) = \prod_{k=1}^{K} \chi_{Q_k}(\lambda)
\]

1) The case $k \neq m$:

\[Q_k = \]
After computations and some algebraic manipulations, we get, \( \chi_{Q_k}(\lambda) = (-\lambda)^L \)

2) The case \( k = m \):

\[
Q_m =
\]
After computations and some algebraic manipulations, we get: \( \chi_{Q_m}(\lambda) = (-\lambda)^{L-1}(l_m\rho_m - \lambda) \)

For \( k \neq m \) \( Q_k \) has only 0 as eigen value.

For \( k = m \), \( \chi_{Q_m}(\lambda) = 0 \iff \lambda = 0 \) or \( \lambda = l_m\rho_m \), hence \( Q_m \) has two eigen values which are 0 and \( l_m\rho_m < R_m\rho_m = 1 \).

Consequently, in both cases, the norms of all eigen values of the obtained matrix are strictly less than 1.

**APPENDIX V**

**PROOF OF LEMMA 10**

We take \( 0 < \epsilon < \mu \), \( Z^N(t) \) converges to \( z^* \), i.e. there exists \( T_0 \) such that for all \( t \geq T_0 \), \( ||Z^N(t) - z^*|| \leq \epsilon \).

Hence:

\[
P_x\left( \sup_{T_0 \leq t < T} ||Z^N(t) - z^*|| \geq \mu \right) \leq P_x\left( \sup_{T_0 \leq t < T} ||Z^N(t) - z(t)|| + ||Z^N(t) - z^*|| \geq \mu \right)
\]

\[
\leq P_x\left( \sup_{T_0 \leq t < T} ||Z^N(t) - z(t)|| \geq \mu - \epsilon \right)
\]

\[
\leq P_x\left( \sup_{0 \leq t < T} ||Z^N(t) - z(t)|| \geq \mu - \epsilon \right)
\]

Using proposition 8 there exists \( s_1 \) and \( s_2 \) such that:

\[
P_x\left( \sup_{0 \leq t < T} ||Z^N(t) - z(t)|| \geq \mu - \epsilon \right) \leq s_1 \exp(-Ns_2).
\]
Therefore:

\[ P_z \left( \sup_{t_0 \leq t < T} \| Z^N(t) - z^* \| \geq \mu \right) \leq \exp(-Ns_2). \]

**APPENDIX W**

**Proof of Proposition 9**

We recall that \( Z^N(t) \) represents the proportion vector at time \( t \) under the Whittle index policy.

We have

\[ C^{WI,N} - C^{RP,N} = \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{L} C^{WI,N}(Q^k_i(t)) | x \right] - \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{L} C^{RP,N}(Q^k_i(t)) | x \right] - C^{RP,N} \]

Replacing \( C^{RP,N} \) by its expression given in section VI and knowing that \( z^*_i = \gamma_k u_k^i (i) \) (by definition of \( z^* \)), then:

\[ C^{WI,N} - C^{RP,N} = \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{L} a_k Z_i^{k,N}(t)iN | x \right] - \frac{1}{T} \mathbb{E} \left[ \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{L} a_k z_i^{k,*} iN \right] \]

We divide all by \( N \)

\[ C^{WI,N} - C^{RP,N} = \frac{1}{T} \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{L} E(\gamma_k (a_k Z_i^{k,N}(t)i) - a_k z_i^{k,*} i) \]

\[ \leq \frac{1}{T} \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{L} E(\gamma_k (a_k Z_i^{k,N}(t)i) - a_k z_i^{k,*} i) \]

We have the function \( f: z \rightarrow \sum_{i=0}^{L} a_k z_i^{k,*} i \) is liphcitz and continuous, then for an arbitrary small \( \epsilon \), there exists \( \mu \) such that if \( \| z - z^* \| < \mu \), then \( |f(z) - f(z^*)| < \epsilon \).

We denote \( Y_N \) the event \( \sup_{t_0 \leq t < T} \| Z^N(t) - z^* \| \geq \mu \), we proceed to bound the second term:

\[ P_z(Y_N) \frac{1}{T} \sum_{t=0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{L} E(\gamma_k (a_k Z_i^{k,N}(t)i) - a_k z_i^{k,*} i) \]

\[ = P_z(Y_N) \frac{1}{T} \sum_{t=0}^{T-1} \mathbb{E} \left[ \| \sum_{k=1}^{K} \sum_{i=1}^{L} (a_k Z_i^{k,N}(t)i) - a_k z_i^{k,*} i \| Y_N \right] \]

\[ + (1 - P_z(Y_N)) \frac{1}{T} \mathbb{E} \left[ \| \sum_{t=T_0}^{T-1} \sum_{k=1}^{K} \sum_{i=1}^{L} (a_k Z_i^{k,N}(t)i) - a_k z_i^{k,*} i \| Y_N \right] \]

\[ \leq \frac{T - T_0}{T} \frac{L(L+1)}{T} \sum_{k=1}^{K} a_k \gamma_k P_z(Y_N) + (1 - P_z(Y_N)) \epsilon. \]
Lemma 28. \( W_i^k \leq W_{R_k}^{k'} \) for all \( R_k \) and \( R_{k'} \) greater or equal than 2.
Proof:

\[ W_1^k = \frac{R_k}{R_k - 1} \leq 2 \forall R_k \geq 2 \]

\[ W_{R_k - 1}^R_k = R_k(R_k - 1) \geq 2 \forall R_k \geq 2 \]

Lemma 29.

\[ \sum_{k \neq m} \sum_{i=R_k}^{R_k+l_k-1} \gamma_k u_{k}^l(i) + \sum_{i=R_m}^{R_m+l_m-1} \gamma_m u_{m}^l(i) \leq 1 - \alpha. \]

Proof: In fact

\[ 1 - \alpha = \sum_{k \neq m} \sum_{i=R_k}^{l_k} \gamma_k u_{k}^l(i) + \sum_{i=R_m}^{l_m-1} \gamma_m u_{m}^l(i) + \theta \gamma_m u_{m}^l(l_m). \]

For any \( k \in [1, K] \) and for any threshold \( n_k < R_k \), and by replacing \( u_{k}^n \) by its expression given in section IV, we have:

\[ \sum_{i=0}^{n_k} \gamma_k u_{k}^n(i) = \gamma_k \sum_{i=0}^{n_k} (\rho_k - (n_k - i)\rho_k^2) = \gamma_k(n_k + 1)\rho_k - \gamma_k\rho_k^2 \frac{(n_k + 1)n_k}{2} \]

and

\[ \sum_{i=R_k}^{R_k+n_k-1} \gamma_k u_{k}^n(i) = \gamma_k \sum_{i=0}^{n_k} (n_k - i)\rho_k^2 = \gamma_k\rho_k^2 \frac{(n_k + 1)n_k}{2} \]

we have:

\[ \gamma_k(n_k + 1)\rho_k > \gamma_k\rho_k^2(n_k + 1)n_k \]

Hence:

\[ \sum_{i=0}^{n_k} \gamma_k u_{k}^n(i) \geq \sum_{i=R_k}^{R_k+n_k-1} \gamma_k u_{k}^n(i) \]

(18)

That means, for \( k \neq m \):

\[ \sum_{i=0}^{l_k} \sum_{k \neq m} \gamma_k u_{k}^l(i) \geq \sum_{i=R_k}^{R_k+l_k-1} \gamma_k u_{k}^l(i) \]

For \( k = m \):

\[ \sum_{i=0}^{l_m-1} \gamma_m u_{m}^l(i) + \theta \gamma_m u_{m}^l(l_m) = \gamma_m(1 - \theta) \sum_{i=0}^{l_m-1} u_{m}^l(i) + \theta \gamma_m \sum_{i=0}^{l_m} u_{m}^l(i) \]

\[ \geq \gamma_m(1 - \theta) \sum_{i=R_m}^{R_m+l_m-2} u_{m}^l(i) + \theta \gamma_m \sum_{i=R_m}^{R_m+l_m-1} u_{m}^l(i) \]

\[ = \gamma_m(1 - \theta) \sum_{i=R_m}^{R_m+l_m-1} u_{m}^l(i) + \theta \gamma_m \sum_{i=R_m}^{R_m+l_m-1} u_{m}^l(i) \]

\[ = \sum_{i=R_m}^{R_m+l_m-1} \gamma_m u_{m}^l(i) \]
The inequality comes from [18].

Then:
\[
\sum_{k \neq m}^{R_k + l_k - 1} \sum_{i = R_k}^{R_m + l_m - 1} \gamma_k u_k^i (i) + \sum_{i = R_m}^{l_k} \gamma_m u_m^i (i) \leq \sum_{k \neq m}^{l_k} \sum_{i = 0}^{l_m - 1} \gamma_k u_k^i (i) + \sum_{i = 0}^{l_m - 1} \gamma_m u_m^i (i) + \theta \gamma_m u_m^i (l_m) = 1 - \alpha
\]

In the remaining of the proof, we will consider separately the cases \( \alpha \leq \frac{1}{2} \) and \( \alpha > \frac{1}{2} \).

If \( \alpha \leq \frac{1}{2} \), the proof of the desired result consists of 3 steps.

**Step 1:**
We start by state \( z(0) \), for all \( k \neq m \), we will exactly schedule all proportions: \( z_{l_k+1}^k, \ldots, z_{R_k}^k \), and for \( k = m \), we schedule all proportions \( z_{l_m+1}^m, \ldots, z_{R_m}^m \) plus the proportion \( (1 - \theta) z_{l_m}^m \). The sum of these tree proportions is \( \alpha \). We denote these sets of queues by group A. We consider that, after scheduling, all these proportions will be at state \( R_k - 1 \) (depending on each class). For the rest of proportions which is equal to \( 1 - \alpha \), only \( \alpha \) proportion will be at state 1 (we call this group B). The rest which equals to \( 1 - 2\alpha \) (group C) will be at state 0. The queue state proportions vector for class \( k \neq m \) after this step is:
\[
z^k = \left( \frac{\alpha}{\beta}, \frac{1}{R_k}, 0, 0, \ldots, z_{R_k-1}^k = \sum_{i = l_k+1}^{R_k} z_{i}^k, 0, \ldots, 0 \right)
\]

The queue state proportions vector for class \( k = m \):
\[
z^m = \left( \frac{\alpha}{\beta}, \frac{1}{R_m}, 0, 0, \ldots, z_{l_m+1}^m, \sum_{i = l_m+1}^{R_m} z_{i}^m, 0, \ldots, 0 \right)
\]
with \( \sum \alpha_k = \alpha \) and \( \sum \beta_k = 1 - 2\alpha \).

**Step 2:**
Using the Whittle index policy, according to Lemma [28], group A is scheduled again. After scheduling, we consider that group B which is at state 1 goes to state \( R_k + l_k - 1 \) packets are the arrivals at each class-k queue). For group C, the queues stay at state 0 (no arrivals).

But for the \( \alpha \) proportion scheduled (group A), we have for each \( k \):
1) when \( k \neq m \):
   a) For each state \( h \) from \( l_k + 1 \) until \( R_k - 1 \): exactly \( z_{h}^k \) goes to state \( h \) (this is feasible since if a queue at state \( R_k - 1 \) is scheduled, it can go to any other state strictly less than \( R_k \))
   b) For each state \( h \) from \( R_k \) until \( R_k + l_k - 1 \), we will have exactly \( z_{h}^k \) proportion of queues that go to state \( h - (R_k - 1) \), which is strictly less than \( R_k \).
2) When \( k = m \):
   a) for each state from \( l_m + 1 \) until \( R_m + l_m - 1 \), the same analysis done for \( k \neq m \) holds.
b) for \( h = l_m \), \((1 - \theta)z_{l_m}^{m,*}\) will be at state \( l_m \).

Hence after this step the new queue state proportion vector for class \( k \neq m \) is:

\[
(\beta_k, z_1^k = z_{R_k}^k, \ldots, z_h^k = z_{R_k+1}^k, z_{k+1}^k = z_{R_k+1}^k, \ldots, z_{R_k-1}^k = z_{R_k-1}^k, \alpha_k, 0, \ldots, 0)
\]

The queue state proportion vector for class \( k = m \) is:

\[
(\beta_m, z_1^m = z_{R_m}^m, \ldots, z_h^m = z_{R_m+1}^m + (1 - \theta)z_{l_m}^{m,*}, z_{l_m+1}^m = z_{l_m+1}^{m,*}, \ldots, z_{R_m-1}^m = z_{R_m-1}^{m,*}, \alpha_m, 0, \ldots, 0)
\]

Step 3: Under the assumption that \( L \) is very large, we have \( W_L^k = W_R^k \geq W_R^{k'} \) for all \( k \) and \( k' \) and for \( 0 \leq n \leq R_k - 1 \).

That means, we will schedule all the \( \alpha \) queues at state \( R_k \) (i.e. group B), and we can therefore go to any state less than \( R_k - 1 \).

For the remaining \( 1 - 2\alpha \) queues that are in state 0 (i.e. group C), after applying a passive action (no transmission), their states will change to any state less than or equal to \( R_k - 1 \).

For group A (\( \alpha \) proportion of queues), we have for each \( k \):

1) For each state from \( l_k + 1 \) until \( R_k - 1 \); they stay at same state (0 arrivals).

2) For \( h \) from \( R_k \) until \( R_k + l_k - 1 \), the proportion \( z_{l_m}^{k,*} \) goes from state \( h - (R_k - 1) \) to \( h \) after that \( R_k - 1 \) packets arrive.

3) For \( k = m \) and \( h = l_m \): \((1 - \theta)z_{l_m}^{m,*}\) proportion stays at same state (0 arrivals).

So after this step: we will reach the optimal \( z^* \) of the relaxed problem: The queue state proportion vector for class \( k \neq m \) is:

\[
z_{l_m}^{k,*} = (z_0^k, z_1^k, \ldots, z_{l_k+R_k-1}^k, 0, \ldots, 0)
\]

The queue state proportion vector for class \( m \) is:

\[
z_{l_m}^{m,*} = (z_0^m, z_1^m, \ldots, z_{l_m+R_m-1}^m, 0, \ldots, 0)
\]

This implies that we have reached the optimal proportion \( z^* \).

If \( \alpha > \frac{1}{2} \):

Step 1: the same step as we did when \( \alpha \leq \frac{1}{2} \), however all \( 1 - \alpha \) queues (group B) that are not scheduled will be at state 1 since \( 1 - \alpha < \alpha \). Hence the new queue state proportions vector after this step for \( k \neq m \) is:

\[
z_k = (0, z_1^k = \beta_k, 0, 0, \ldots, z_{R_k-1}^k = \sum_{i=l_k+1}^L z_i^{k,*}, 0, \ldots, 0)
\]

For \( k = m \):

\[
z_m = (0, z_1^m = \beta_m, 0, 0, \ldots, \sum_{i=l_m+1}^L z_i^{m,*} + (1 - \theta)z_{l_m}^{m,*}, 0, \ldots, 0)
\]

with \( \sum \beta_k = 1 - \alpha \)

Step 2: The group A is scheduled again, and the \( 1 - \alpha \) proportion of queues at state 1 (group B), which are
not scheduled, will go to state \( R_k \). For \( l_k + 1 \leq h \leq R_k - 1 \), \( z_{h}^{k,*} \) will be at state \( h \), and for \( k \neq m \) and \( l_m + 1 \leq h \leq R_m - 1 \), \( z_{h}^{m,*} \) will be at state \( h \) after scheduling.

For \( R_k \leq h \leq R_k + l_k - 1 \), \( z_{h}^{k,*} \) will be at state \( h - (R_k - 1) \).

When \( k = m \) and \( R_m \leq h \leq R_m + l - 1 \), \( z_{h}^{m,*} \) will be at state \( h - (R_m - 1) \) and \((1 - \theta)z_{l_m}^{m,*}\) will be at state \( l_m \).

Hence after this step, the queue state proportion vector for class \( k \neq m \) is:

\[
(0, z_1^k, \ldots, z_k^k, z_{R_k + l_k}^k, \ldots, z_{R_k - 1}^k, 0, \ldots, 0)
\]

For \( k = m \):

\[
(0, z_1^m, \ldots, z_m^m, z_{R_m + l_m}^m, \ldots, z_{R_m - 1}^m, \beta_{m}, 0, \ldots, 0)
\]

Step3:

Using the Whittle index policy, we schedule \((1 - \alpha)\) proportion of queues at state \( R_k \) (group B), plus proportion \( A \) among the group A. Note that group A is divided into two disjoint proportions \( A_1 \) and \( A_2 \), where \( A_2 \) is defined as the set that contains all proportions \( z_1^k \) till \( z_{l_k}^k \) for each \( k \) minus part from \( z_{l_m}^m \) which is \((1 - \theta)z_{l_m}^{m,*}\). In fact, we have

\[
A_2 = \sum_{k \neq m} \sum_{i=R_k}^{R_k + l_k - 1} \gamma_k \nu^{k}(i) + \sum_{i=R_m}^{R_m + l_m - 1} \gamma_m \nu^{m}(i).
\]

Since we proved that this sum is less than \( 1 - \alpha \) (we can be sure that the whole proportion is not able to be scheduled ). Since \( B = 1 - \alpha < \alpha \) and \( B + A_1 = 1 - A_2 > \alpha \), we just need to schedule further a proportion from \( A_1 \) called \( A_{11} \). In fact we will choose the \( A_{11} \) highest whittle index’s queues among \( A_1 \) such that \( A_{11} + B = \alpha \). We note \( A_{12} = A_1 - A_{11} \). Hence in this step the proportion scheduled is \( A_{11} + B \) and the proportion for which we take a passive decision is \( A_{12} + A_2 \). However we still need to prove that the whittle index of proportions \( A_2 \) is less than that of the \( \alpha \) proportion scheduled states (i.e. B plus \( A_{11} \)).

For the group B at state \( R_k \), \( W_k^k \geq W_n^k \) for all \( k \) and \( k' \) and \( 0 \leq n \leq R_{k'} - 1 \), then the whittle index of all other queues state belonging to either \( A_1 \) or \( A_2 \) are less than the one of queue state belonging to group B.

For \( A_1 \): their states are surely among the states \( l_k + 1, \ldots, R_k - 1 \) for all \( k \), plus the state \( l_m \). Hence, the whittle index of any of these states is higher or equal than \( W^* \), with \( W^* \) is the optimal subsidy for the relaxed problem (following the definition of the optimal threshold vector \( l \), that is also true for \( A_{11} \)).

For the proportion \( A_2 \), the whole proportion is at a state that has an index less or equal than \( W^* \) that is less than the whittle indices of proportion \( A_{11} \). Hence this proves that the whole proportion is not scheduled.

For the proportion \((1 - \alpha)\) (group B) at \( R_k \), the group of queues can go to any state less than \( R_k - 1 \) after scheduling.

In fact, their states will go to all states less than \( l_k \) for each \( k \) according to the optimal proportion vector \( z^* \), except for the state \( l_m \) at class \( m \) for which only \( \theta z_{l_m}^{m,*} \) goes to state \( l_m \).

For \( A_{11} \): the queues in this group will stay at the same states. In fact, for each class \( k \), the states of the queues are all less than \( R_k \). Then by scheduling these queues, the departure will be equal to the queue length. On the other hand, by considering that the number of arrival packets is equal to the previous queue length, one can ensure that the states of the queues in this group remain unchanged.

For \( A_{12} \): Not scheduling the queues in this group implies that they will stay at the same state (since the number of packet arrival is 0).
For $A_2$: This group is not scheduled. The state of the queues in class $k$ will change by adding $R_k - 1$ arrival packets to their previous length.

Consequently, after this step, the new queue state proportion vector: for $k \neq m$:

$$z^{k,*} = (z_0^{k,*}, z_1^{k,*}, ..., z_{k+R_k-1}^{k,*}, 0, ..., 0)$$

for $k = m$:

$$z^{m,*} = (z_0^{m,*}, z_1^{m,*}, ..., z_{m+R_m-1}^{m,*}, 0, ..., 0)$$

which means that we have reached the optimal proportion vector $z^*$.

**APPENDIX Z**

**PROOF OF THEOREM 4**

$$\frac{C^{WI,N}}{N} - \frac{C^{RP,N}}{N} = \sum_{k=1}^{K} \sum_{i=0}^{L} a_k \mathbb{E}[Z_i^{k,N}(\infty)] i - \sum_{k=1}^{K} \sum_{i=0}^{L} a_k z_i^{k,*} i$$

We have the function $f : z \rightarrow \sum_{k=1}^{K} \sum_{i=0}^{L} a_k z_i^{k,*} i$ is lipchitz and continuous, then for an arbitrary small $\epsilon$, there exists $\mu$ such that if $||z - z^*|| < \mu$, then $|f(z) - f(z^*)| < \epsilon$.

We denote $U_N$ the event $\sup ||Z^N(\infty) - z^*|| \geq \mu$, then:

$$|\frac{C^{WI,N}}{N} - \frac{C^{RP,N}}{N}| \leq P(U_N) \mathbb{E}\left[ \sum_{k=1}^{K} \sum_{i=0}^{L} (a_k Z_i^{k,N}(\infty) i - a_k z_i^{k,*} i) ||U_N|| \right] + (1 - P(U_N)) \mathbb{E}\left[ \sum_{k=1}^{K} \sum_{i=0}^{L} (a_k Z_i^{k,N}(\infty) i - a_k z_i^{k,*} i) ||U_N|| \right]$$

$$\leq L(L+1) \sum_{k=1}^{K} a_k \gamma_k P(U_N) + (1 - P(U_N)) \epsilon$$

According to Lemma 12 we have $\lim_{N \to \infty} P(U_N) = 0$, then:

$$\lim_{N \to \infty} |\sum_{k=1}^{K} \sum_{i=0}^{L} a_k \mathbb{E}[Z_i^{k,N}(\infty)] i - \sum_{k=1}^{K} \sum_{i=0}^{L} a_k z_i^{k,*} i| \leq \epsilon$$

This is true for any $\epsilon$ very small. Finally we have:

$$\lim_{N \to \infty} \frac{1}{N}|C^{WI,N} - C^{RP,N}| = 0$$