ODD-DIMENSIONAL GKM-MANIFOLDS OF NON-NEGATIVE CURVATURE

CHRISTINE ESCHER, OLIVER GOERTSCHES, AND CATHERINE SEARLE

Abstract. We prove for closed, odd-dimensional GKM$_3$ manifolds of non-negative sectional curvature that both the equivariant and the ordinary rational cohomology split off the cohomology of an odd-dimensional sphere.

1. Introduction

A long-standing problem in Riemannian geometry is the classification of positively and non-negatively curved manifolds. One characteristic shared by many of the known examples is a high degree of symmetry. The Grove Symmetry Program suggests we attempt the classification of such manifolds with the additional hypothesis of “large” symmetries. The eventual goal of this program is to be able to eliminate the hypothesis of symmetries entirely.

A natural first step is to consider the case of abelian symmetries. For the case of positive curvature, results due to Grove and Searle [21], Rong [34] and Fang and Rong [11], and Wilking [38] give us a classification up to diffeomorphism, homeomorphism, or rational homotopy equivalence for a $T^k$-action, provided $k$ equals $\lfloor (n+1)/2 \rfloor$, $(n-1)/2$, or is greater than or equal to $\lfloor (n+4)/4 \rfloor$, respectively. For non-negative curvature, an equivariant diffeomorphism classification for dimensions less than or equal to nine for $T^k$-actions with $k = \lfloor 2n/3 \rfloor$ follows from work of Galaz-García and Searle [14], Galaz-García and Kerin [13], and Escher and Searle [10]. A diffeomorphism classification for dimensions less than or equal to 6 for $T^k$-actions with $k = \lfloor 2n/3 \rfloor - 1$ follows from work of Kleiner [28] and Searle and Yang [35], Galaz-García and Searle [15], and Escher and Searle [9]. Note that all of these results rely heavily on the existence of fixed point sets of “small” codimension.

From this point of view, one may consider GKM$_k$ manifolds as occupying the other end of the spectrum. Note that these are $2n$-dimensional manifolds with a torus action of rank $\leq n$. Moreover, a consequence of the GKM$_k$ condition is that for a $T^m$-action on $M^{2n}$, for all $l \leq \min(k, m) - 1$, there exist codimension $l$ torus subgroups of $T^m$ fixing $2l$-dimensional submanifolds of $M$, and these fixed point sets have an induced torus action such that the components of the fixed point sets themselves are torus manifolds. GKM$_k$ manifolds of both positive and non-negative curvature were studied by Goertsches and Wiemeler in [18, 19], respectively, where they showed that the GKM$_3$, respectively GKM$_4$, condition allows them to classify such manifolds up to real, respectively, rational cohomology type. The notion of GKM manifold was extended to odd dimensions by He [27]. We will call such manifolds odd GKM manifolds. Inspired by the work of [18, 19] and [27], we consider the case of odd GKM$_3$ manifolds of positive and non-negative curvature.

Our main result shows that for odd GKM$_3$ manifolds of non-negative sectional curvature, the cohomology ring splits off the cohomology ring of an odd-dimensional sphere. We note that throughout this article, we only consider rational coefficients.

Main Theorem 1.1. Let $M^{2n+1}$ be a closed, non-negatively curved odd GKM$_3$ manifold, $\bar{\Gamma}$ the GKM$_3$ graph of $M$, and $k$ the number of floating edges at a vertex of $\bar{\Gamma}$. Then $H^*(M^{2n+1})$ splits off the cohomology ring of an odd dimensional sphere, that is,

$$H^*(M) \cong H^*(\bar{\Gamma}, \alpha, \nabla) \otimes H^*(S^{2k+1})$$

where $(\bar{\Gamma}, \alpha, \nabla)$ is an abstract, even-dimensional GKM$_3$ graph associated to $\bar{\Gamma}$.
In the process of proving Theorem 1.1 we also obtain a similar result for the equivariant cohomology of $M$, see Theorem 5.5. For a definition of the floating edges of a GKM graph of an odd dimensional GKM manifold, see Definition 2.28.

Note that the abstract, even-dimensional GKM graph, $\Gamma$, in Theorem 1.1 has two-dimensional faces that contain at most 4 vertices. The rational cohomology of a GKM manifold is completely determined by its corresponding vertex-edge graph, $\Gamma$. Unfortunately, there is no general classification of even-dimensional GKM graphs whose two dimensional faces contain at most 4 vertices.

That is, for the class of closed, non-negatively curved GKM manifolds, it is as yet unknown whether every such GKM graph corresponds to a closed, non-negatively curved GKM manifold. However, if there are no quadrangles as two-dimensional faces of $\Gamma$, that is, if no two-dimensional face in the odd-dimensional GKM graph of $M$ is of the form (4) in Theorem 4.3, then by the main result of [18], $H^*(\Gamma, \alpha, \nabla)$ is isomorphic to the real cohomology ring of a compact rank one symmetric space (CROSS). Since the result in [18] was obtained via a classification of all possible GKM graphs and applying the GKM theorem, the result also holds for rational coefficients. We then obtain the following result:

**Theorem 1.2.** Let $M^{2n+1}$ be a closed, non-negatively curved odd GKM manifold. If no two-dimensional face in the odd-dimensional GKM graph of $M$ is of the form (4) in Theorem 4.3 then $H^*(M^{2n+1})$ is the tensor product of the rational cohomology ring of an odd dimensional sphere, and a CROSS, that is

$$H^*(M) \cong H^*(S^{2k+1}) \otimes H^*(N^{2n-2k})$$

where $N^{2n-2k}$ is a CROSS.

Using [19, Theorem 5.1], we see that if we assume that our manifold is GKM$_4$, then the cohomology ring of the manifold splits as that of an odd-dimensional sphere and a quotient of a non-negatively curved torus manifold.

**Theorem 1.3.** Let $M^{2n+1}$ be a closed, non-negatively curved odd GKM$_4$ manifold. Then $H^*(M^{2n+1})$ is the tensor product of the cohomology ring of an odd dimensional sphere, and the cohomology ring of a (quotient of) a torus manifold, that is

$$H^*(M) \cong H^*(S^{2k+1}) \otimes H^*(N^{2n-2k}/G)$$

where $N$ is a simply-connected, non-negatively curved torus manifold and $G$ is a finite group acting isometrically on $N$.

Finally, to date, only odd GKM graphs without signs have been treated in the literature. Adding the restriction that the manifold admit an invariant almost contact structure then allows us to talk about odd GKM graphs with signs. Using Theorem 1.5 of [19], we can show the following result.

**Theorem 1.4.** Let $M^{2n+1}$ be a closed, non-negatively curved odd GKM$_4$ manifold which admits an invariant almost contact structure that is alternating. Then the rational cohomology ring of $M$ is isomorphic to the tensor product of the rational cohomology ring of an odd-dimensional sphere and the rational cohomology ring of a generalized Bott manifold.

For a definition of an alternating almost contact structure, see Definition 6.4. We also obtain a full rational cohomology classification for positively curved odd GKM$_3$ manifolds, as follows.

**Theorem 1.5.** Let $M^{2n+1}$ be a closed, positively curved odd GKM$_3$ manifold, then $M^{2n+1}$ has the rational cohomology ring of $S^{2n+1}$.

1.1. **Organization.** The paper is organized as follows. We include basic notation and preliminary material in Section 2. In Section 3 we classify the universal covers of closed, non-negatively curved 5-dimensional GKM manifolds. In Section 4 we classify the corresponding graphs of these 5-manifolds. In Section 5 we prove Theorems 1.1 and 1.5. In Section 6 we give the proof of Theorem 1.4.
Acknowledgements. The first and third authors are grateful to the Max Planck Institute for Mathematics in Bonn for its hospitality and financial support during the summer of 2017, where this work was initiated. C. Escher gratefully acknowledges partial support from the Simons Foundation (#585481, C. Escher). C. Searle gratefully acknowledges partial support from grants from the National Science Foundation (#DMS-1611780), as well as from the Simons Foundation (#355508, C. Searle).

2. Preliminaries

In this section we will gather basic results and facts about transformation groups, equivariant cohomology, even- and odd-dimensional GKM and GKM$_k$ theory, as well as results concerning $G$-invariant manifolds of non-negative sectional curvature.

2.1. Transformation Groups. Let $G$ be a compact Lie group acting on a smooth manifold $M$. We denote by $G_x = \{ g \in G : gx = x \}$ the isotropy group at $x \in M$ and by $G(x) = \{ gx : g \in G \}$ the orbit of $x$. Note that $G(x)$ is homeomorphic to $G/G_x$ since $G$ is compact. We denote the orbit space of the $G$-action by $M/G$ and note that if $M$ admits a lower sectional curvature bound and the $G$-action is isometric, then $M/G$ is an Alexandrov space admitting the same lower curvature bound. We will denote the fixed point set of $M$ by $G$ as either $M^G$ or $\text{Fix}(M;G)$, using whichever may be more convenient.

Definition 2.1 (The $k$-skeleton of $M$). For $G = T$, a torus, we define the $k$-skeleton of $M$ to be

$$M_k = \{ p \in M : \text{dim}(T(p)) \leq k \}.$$

We then obtain a $T$-invariant topological stratification of $M$ as follows: $M_0 \subset M_1 \subset \cdots \subset M_{\text{dim}(T)} = M$ on $M$, where the 0-skeleton $M_0$ is exactly the fixed point set $M^T$.

One measurement for the size of a transformation group $G \times M \to M$ is the dimension of its orbit space $M/G$, also called the cohomogeneity of the action. This dimension is clearly constrained by the dimension of the fixed point set $M^G$ of $G$ in $M$. In fact, $\text{dim}(M/G) \geq \text{dim}(M^G) + 1$ for any non-trivial, non-transitive action. In light of this, the fixed point cohomogeneity of an action, denoted by $\text{cohomfix}(M;G)$, is defined by

$$\text{cohomfix}(M;G) = \text{dim}(M/G) - \text{dim}(M^G) - 1 \geq 0.$$

A manifold with fixed point cohomogeneity 0 is also called a $G$-fixed point homogeneous manifold.

We now recall Theorem I.9.1 of Bredon [4] which allows us to lift a group action to a covering space.

Theorem 2.2. [4] Let $G$ be a connected Lie group acting effectively on a connected, locally path-connected space $X$ and let $X'$ be any covering space of $X$. Then there is a covering group $G'$ of $G$ with an effective action of $G'$ on $X'$ covering the given action. Moreover, $G'$ and its action on $X'$ are unique.

The kernel of $G' \to G$ is a subgroup of the group of deck transformations of $X' \to X$. In particular, if $X' \to X$ has finitely many sheets, then so does $G' \to G$. If $G$ has a stationary point in $X$, then $G' = G$ and $\text{Fix}(X';G)$ is the full inverse image of $\text{Fix}(X;G)$.

2.2. Equivariant cohomology. We begin by providing some basic information about equivariant cohomology and equivariantly formal manifolds for actions of tori.

Definition 2.3 (Equivariant Cohomology). Given an action of a torus $T$ on a compact manifold $M$, the equivariant cohomology of the action is defined as

$$H^*_T(M) = H^*(M \times_T ET),$$

where $ET \to BT$ is the classifying bundle of $T$ and $ET$ is a contractible space on which $T$ acts freely.
The equivariant cohomology has the natural structure of an $H^*(BT)$-algebra, via the projection $M \times_T ET \to BT$. Note that $H^*(BT)$ is isomorphic to the ring of rational polynomials on the Lie algebra $t$, in the following sense. Denoting the rational points in $t$, that is, the tensor product of the integer lattice in $t$ with $\mathbb{Q}$, by $t_\mathbb{Q}$, we have $H^*(BT) \cong S(t^*_\mathbb{Q})$, which is a rational polynomial ring in $\dim(T)$ variables.

Given an action of a torus $T$ on $M$, we may compare $H^*_T(M)$ with $H^*_T(M^T)$ using the Borel Localization Theorem (see, for example, Corollary 3.1.8 in Allday and Puppe [1]).

**Theorem 2.4** (Borel Localization Theorem). The restriction map

$$H^*_T(M) \to H^*_T(M^T)$$

is an $H^*(BT)$-module isomorphism modulo $H^*(BT)$-torsion.

Using this localization theorem, it is clear that if $H^*_T(M)$ is actually a free $H^*(BT)$-module, one can hope for a stronger relation between the manifold $M$ and its fixed point set $M^T$. This motivates the following definition.

**Definition 2.5** (Equivariantly Formal). We say that an action of a torus $T$ on $M$ is equivariantly formal if $H^*_T(M)$ is a free $H^*(BT)$-module, where $S(t^*_\mathbb{Q})$ is the symmetric algebra of real-valued polynomials in $t^*_\mathbb{Q}$.

As a compact manifold has finite-dimensional cohomology, it follows from [1, Corollary 4.2.3] that equivariant formality is equivalent to the degeneration of the Leray-Serre spectral sequence of the Borel fibration $M \to M \times_T ET \to BT$ at the $E_2$-term. Moreover, the following are some well-known and important properties of equivariantly formal actions.

**Proposition 2.6.** An action of a torus $T$ on a compact manifold $M$ with $H^\text{odd}(M) = \{0\}$ is automatically equivariantly formal. The converse implication is true provided that the $T$-fixed point set $M^T$ is finite.

The first statement of Proposition 2.6 follows because if $H^\text{odd}(M) = \{0\}$ then the spectral sequence degenerates at the $E_2$-term. The second statement is a consequence of the Borel Localization Theorem, as then $H^*_T(M) \cong H^*(BT) \otimes H^*(M)$ injects into the module $H^*_T(M^T) \cong H^*(BT) \otimes H^*(M^T)$ which vanishes in odd degrees by assumption. The next proposition is Theorem 3.10.4 in [1].

**Proposition 2.7.** For any action of a torus $T$ on a compact manifold $M$, we have $\dim H^*(M^T) \leq \dim H^*(M)$. Equality holds if and only if the action is equivariantly formal.

The Leray-Hirsch theorem implies that for equivariantly formal actions the ordinary cohomology ring is encoded in the equivariant cohomology algebra:

**Proposition 2.8.** For an equivariantly formal action of a torus $T$ on a compact manifold $M$ the natural map $H^*_T(M) \to H^*(M)$ is surjective and induces an isomorphism of $\mathbb{R}$-algebras

$$\frac{H^*_T(M)}{S^+(t_\mathbb{Q}) \cdot H^*_T(M)} \cong H^*(M),$$

where $S^+(t_\mathbb{Q})$ denotes the ideal in $S(t^*_\mathbb{Q})$ generated by polynomials of positive degree.

For equivariantly formal actions, the Borel Localization Theorem 2.4 gives an embedding of $H^*_T(M)$ into $H^*_T(M^T)$. The image of this embedding can be described as follows, using the well-known Chang-Skjelbred Lemma [6] and the version of the same lemma given in Theorem 11.51 of Guillemin and Sternberg [22].

**Chang Skjelbred Lemma 2.9.** [6] If a $T$-action on $M$ is equivariantly formal, then the equivariant cohomology $H^*_T(M)$ only depends on the fixed point set $M^T$ and the 1-skeleton $M_1$:

$$H^*_T(M) \cong \text{im}(H^*_T(M_1) \to H^*_T(M^T)) \cong \bigcap \text{im}(H^*_T(M^K) \to H^*_T(M^T)),$$

where the intersection is taken over all corank-1 subtori $K$ of $T$. 

Moreover, equivariant formality is *inherited* by subtori of the $T$-action; more precisely, we have the following well-known proposition and corollary, see, for example [27].

**Proposition 2.10.** If a $T$-action on $M$ is equivariantly formal, then for any subtorus $K$ of $T$, both the $K$-subaction on $M$ and the induced $T/K$-action on $M^K$ are equivariantly formal.

**Corollary 2.11.** If a $T$-action on $M$ is equivariantly formal, then for any subtorus $K$ of $T$, every connected component of $M^K$ has $T$-fixed points.

2.3. Even-dimensional GKM theory. The class of manifolds, now referred to as *GKM manifolds*, were first discussed in the seminal work by Goresky, Kottwitz, and MacPherson in [20] to study the relation between equivariant cohomology and ordinary cohomology, and are named for them. Theorem 1.2.2, also known as the GKM Theorem, in [20] states that over an appropriate coefficient ring $R$ the equivariant cohomology ring of the equivariantly formal GKM manifold $M$ can be computed by the 1-skeleton of $M$ and the kernel of the isotropy weights of one-dimensional orbits. Motivated by their work, the concepts of GKM manifold and *GKM graph* were introduced by Guillemin and Zara in [23] to build a bridge between combinatorics and geometry.

**Definition 2.12 (GKM Torus Action and Manifold).** We say that the effective action of a torus $T^l = T$, $l \leq n$, on an orientable, compact, connected manifold $M^{2n}$ is GKM, and $M^{2n}$ is called a GKM manifold, if

1. the fixed point set $M^T$ of the action is finite,
2. for every $p \in M^T$, the weights $\alpha_{i,p} \in t_q^*/\{\pm 1\}$, $i = 1, \ldots, n$, of the isotropy representation of $T$ on $T_p M$ are pairwise linearly independent.
3. it is equivariantly formal.

We note that while the original definition in [23] does not include in the definition the requirement that the manifold be equivariantly formal (they list it as a separate hypothesis in their theorems), it is included in the definition of a GKM manifold in most of the literature (see, for example, Goertsches and Wiemeler [18, 19] and Kuroki [30]), so we will include equivariant formality in our definition as well.

By Proposition 2.10 Conditions 1 and 3 imply the vanishing of the odd rational cohomology groups of $M$. Condition 2 is equivalent to the condition that $M_1$, the 1-skeleton of $M$, consists of a disjoint union of $T$-invariant submanifolds, which are either fixed point free, or are $T$-invariant embedded 2-spheres, each of which contains exactly two $T$-fixed points, see Theorem 1.1.1 in [23]. Note that by Corollary 2.11 the definition implies that $M_1$ consists entirely of $T$-invariant embedded 2-spheres.

**Definition 2.13 (GKM$_k$ Torus Action and Manifold).** We say that the effective action of a torus $T$ on an orientable, compact, connected manifold $M^{2n}$ is GKM$_k$, and we call $M^{2n}$ a GKM$_k$ manifold, if

1. $M^{2n}$ is GKM, and
2. for each $p \in M^T$ any set of $k$ weights, $\alpha_{i,p} \in t_q^*/\{\pm 1\}$, $i = 1, \ldots, n$, of the isotropy representation of $T$ on $T_p M$ is linearly independent.

**Remark 2.14.** A GKM$_k$ manifold is GKM$_l$ for all $2 \leq l \leq k$. In particular, a GKM$_2$ manifold is a GKM manifold.

By convention, the $T^1$-action on $S^2$ is considered to be a GKM$_2$ manifold. Also note that the linear independence is well-defined for elements that are only defined up to sign. Condition 2 is equivalent to the $(k - 1)$-skeleton $M_{k-1} = \{p \in M \mid \dim (T_p) \leq k - 1\}$ being a union of $T$-invariant submanifolds, which are either fixed point free or are $(2k - 2)$-dimensional and have $T$-fixed points. Note once again that by Corollary 2.11 the definition implies that $M_{k-1}$ consists entirely of the $(2k - 2)$-dimensional $T$-invariant submanifolds.

We will employ the following conventions when speaking about abstract graphs. Given an abstract graph $\Gamma$, we denote by

- $E(\Gamma)$ its set of oriented edges; and by
• \( V(\Gamma) \) its set of vertices.

We will always assume that both the edge and vertex sets are finite, and we will allow multiple edges between vertices. Edges may not connect a vertex to itself. For an edge \( e \in E(\Gamma) \) we denote by

- \( \bar{e} \) the edge with opposite orientation;
- by \( i(e) \) its initial vertex; and
- by \( t(e) \) its terminal vertex.

For a vertex \( v \in V(\Gamma) \) the set of edges emanating from \( v \) will be denoted by \( E_v \).

In order to motivate the remaining definitions, we consider the following geometric construction of a graph with its weights, \((\Gamma, \alpha)\), obtained from a GKM \( k \)-torus action on a GKM \( k \)-manifold, \( M^{2n} \).

We define \( \Gamma \) to be the quotient by the torus action of the 1-skeleton, \( M_1/T \), considered as a graph. Then \( V(\Gamma) \) corresponds to the (isolated) fixed points of the torus action and \( E(\Gamma) \) corresponds to the 2-spheres fixed by a codimension one subtorus of \( T \), containing two isolated fixed points of \( T \). We label each edge, \( e \), of \( \Gamma \) with the corresponding weight, \( \alpha(e) \), of the isotropy representation considered as an element of \( t_\mathbb{Q}^*/\{ \pm 1 \} \). In this fashion, we obtain a map

\[
\alpha : E(\Gamma) \to t_\mathbb{Q}^*/\{ \pm 1 \}.
\]

Equivalently, given a GKM action of a torus \( T \) on an orientable \( M^{2n} \), the GKM graph of this action is constructed as follows: we have one vertex for each fixed point; every invariant two-sphere contains exactly two fixed points, and we associate to it one edge connecting the corresponding vertices. The axial function associates to each edge, that is, invariant two-sphere, the corresponding weight spaces whose weights vanish on \( H \). We then see that in the above geometric construction of \( \Gamma \), the graph of the GKM \( k \)-manifold \( M \), the image of the 1-skeleton of \( N \) will be a subgraph \( \Gamma_N \subset \Gamma \) and is called a face. In the special case when \( M^{2n} \) is a quasitoric manifold, then \( M/T \) is an \( n \)-dimensional simple polytope, \( P^n \). In this special case, \( \Gamma \) is the 1-skeleton of \( P^n \) and each face of \( P^n \) corresponds to some \( N^{2l}/T \), whose 1-skeleton is \( \Gamma_N \).

We will now give an abstract definition of a GKM graph. Note that we define the GKM graphs abstractly because when we get to our main result, we will be using an abstract definition of a graph and this abstract graph may not correspond to the graph of a manifold. However, before we can define a GKM graph, we define a connection, \( \nabla \), on a graph \( \Gamma \), as in [23].

**Definition 2.16 (Connection).** A connection on a graph \( \Gamma \) is a collection \( \nabla \) of maps \( \nabla_e : E_{i(e)} \to E_{t(e)} \), for each \( e \in E(\Gamma) \), such that

1. \( \nabla_e(e) = \bar{e} \) and
2. \( \nabla_e = (\nabla_{\bar{e}})^{-1} \),

for all \( e \in E(\Gamma) \).

We are now in a position to define an abstract GKM graph.

**Definition 2.17 (GKM graph).** Let \( k \geq 2 \). Then a GKM graph is a triple \((\Gamma, \alpha, \nabla)\), where \( \Gamma \) is an \( n \)-valent graph, \( \alpha : E(\Gamma) \to V/\{ \pm 1 \} \), \( V \) is a rational vector space and \( \nabla \) is a connection on \( \Gamma \), such that

1. for any \( v \in V(\Gamma) \) and any set of \( k \) distinct edges \( e_1, \ldots, e_k \in E_v \), the elements \( \alpha(e_1), \ldots, \alpha(e_k) \) are linearly independent;
Remark 2.18. For geometric GKM graphs, $V$ is always taken to be $t_0^*$, and so, for the rest of this paper, we will do so as well. Often in the literature, the axial function, $\alpha$, is assumed to take values in $t_0^*$ instead of $t_0^*/\{\pm 1\}$ (see, for example, [23]). We call this variant a signed GKM graph. Geometrically, this is motivated by the fact that one often assumes that a GKM manifold admits an invariant almost complex structure.

We now return to the geometric construction of a GKM graph. Proposition 2.3 in [19] tells us that for any GKM manifold we can define a connection, $\nabla$, that then allows us to define the corresponding GKM graph, $(\Gamma, \alpha, \nabla)$. In the GKM case the connection is easier to define than it is in the GKM case, and since we are only concerned with the GKM case in this article, we will explain the construction here. In this setting, for any given distinct edges $e, f$ emanating from a vertex $v$, there is a unique 2-dimensional face, $F$, containing $e$ and $f$. Let $e' \neq \bar{e}$ be the unique edge in $F$, such that $i(e') = t(e)$. Setting $\nabla_{e,f} = e'$, it follows from [19] that $(\Gamma, \alpha, \nabla)$ satisfies the conditions for a GKM graph. In particular, we see that the connection allows us to slide edges along edges inside the corresponding two-dimensional face of the graph.

We now define the concept of an abstract face of a GKM graph. Let $(\Gamma, \alpha, \nabla)$ be a GKM graph, then we say that $\Gamma'$ is an $l$-dimensional face of $(\Gamma, \alpha, \nabla)$ if it is an $l$-valent subgraph, invariant under $\nabla$.

We can now define the equivariant cohomology of a GKM graph, which was first done in Section 1.7 of [23] and was denoted simply by $H(\Gamma, \alpha)$. We add a subscript to distinguish it from the cohomology of a GKM graph defined below in Definition 2.21.

**Definition 2.19 (Equivariant Cohomology of a GKM Graph).** The equivariant cohomology of a GKM graph $(\Gamma, \alpha, \nabla)$ is defined as $H^*_T(\Gamma, \alpha) = \{ (f_v)_{v \in V(\Gamma)} \in \bigoplus_{v \in V(\Gamma)} S(t_0^*) | \alpha(e) \text{ divides } f_i(e) - f_0(e) \text{ for all } e \in E(\Gamma) \}$, where the generators of $S(t_0^*)$ are assigned degree 2. It is naturally an $S(t_0^*)$-algebra.

We recall Theorem 1.2.2, also known as the GKM Theorem, in [20] here.

**Theorem 2.20.** For a GKM action of a torus $T$ on $M$, with GKM graph $\Gamma$, the injection $H^*_T(M) \to H^*_T(\Gamma)$ has as image exactly $H^*_T(\Gamma, \alpha)$. Thus, $H^*_T(M) \cong H^*_T(\Gamma, \alpha)$ as $S(t_0^*)$-algebras.

Motivated by Proposition 2.8 we define

**Definition 2.21 (Cohomology of a GKM Graph).** The cohomology of a GKM graph $(\Gamma, \alpha, \nabla)$ is defined as $H^*(\Gamma, \alpha) = \frac{H^*_T(\Gamma, \alpha)}{S^+(t_0^*)}$. 

**Remark 2.22.** Thus, for a GKM action of a torus $T$ on $M$, we have $H^*(M) \cong H^*(\Gamma)$ by Theorem 2.20 and Proposition 2.8.

### 2.4. Results in positive and non-negative curvature.

We state here results for GKM manifolds of positive and non-negative sectional curvature that we will need for the proofs of Theorem 1.2 and 1.3. The first is a classification result for positively curved GKM$_3$ manifolds from [18].

**Theorem 2.23.** [18] Let $M$ be a closed positively curved orientable GKM$_3$ Riemannian manifold. Then $M$ has the real cohomology ring of a compact rank one symmetric space.
The following theorem is a classification result for non-negatively curved GKM$_4$ manifolds from [19].

**Theorem 2.24.** [19] Let $M$ be a non-negatively curved GKM$_4$ manifold. Then

$$H^*(M) \cong H^*(\tilde{M}/G),$$

where $\tilde{M}$ is a simply-connected, non-negatively curved torus manifold and $G$ is a finite group acting isometrically on $M$.

2.5. Odd-dimensional GKM theory. GKM theory was generalized to torus actions on odd-dimensional manifolds with one-dimensional fixed point set in [27]. The GKM condition is, as in the usual even-dimensional setting, the condition that the 1-skeleton of the action is as simple as possible.

**Definition 2.25 (Odd GKM action).** We say that the action of a torus $T$ on an orientable, compact, connected manifold $M^{2n+1}$ is GKM if

1. the fixed point set $M_T$ of the action is the finite union of circles,
2. the 1-skeleton $M_1 = \{ p \in M \mid \dim (T \cdot p) \leq 1 \}$ is a finite union of three-dimensional $T$-invariant submanifolds, and
3. it is equivariantly formal.

The condition on the dimension on the 1-skeleton is equivalent to demanding that, for each component of $M^T$, the weights of the isotropy representation, considered as elements in $t^*_Q/\{\pm 1\}$, are pairwise linearly independent. Note that in [27], it is neither assumed that the action is equivariantly formal, nor that $M$ is orientable. We include these conditions because in the even-dimensional setting they are almost always part of the definition of a GKM action.

As in classical GKM theory one can associate to an odd GKM manifold an odd GKM graph, from which one can, for equivariantly formal actions, compute the equivariant, as well as the ordinary rational cohomology of the manifold. The main new feature, when comparing to the even-dimensional setting, is that even in the equivariantly formal case, the three-dimensional invariant submanifolds can contain an arbitrary number of fixed point components.

**Remark 2.26.** One should note that in [17] a different generalization of GKM theory to odd dimensions was introduced for so-called Cohen-Macaulay actions. The GKM-type actions considered there are more general than those of [27], as they do not necessarily have fixed points. That is, using the stratification induced by the $M_l$ skeleta, one only has that $M_l \neq \emptyset$ for some $l \geq 0$, rather than $M_0 \neq \emptyset$. On the other hand, the other definition is also more restrictive in terms of the stratification of the $k$-skeleta. Namely, given $N$, a connected component of $M_{l+1} \setminus M_l$, with $M_l \neq \emptyset$, $N$ contains exactly two components of $M_l$. Whereas in the definition given in [27], the number of such components is greater than or equal to 1.

To encode the structure of the 1-skeleton in a graph, which we will call an odd GKM graph, two types of vertices are defined in [27].

**Definition 2.27 (Vertex types).** The odd GKM graph of an odd-dimensional GKM manifold $M$ has two types of vertices:

1. One circle for each circle in the fixed point set; and
2. One square for each invariant three-dimensional submanifold in $M_1$, the 1-skeleton of $M$.

We denote the set of circles by $V \circ$ and the set of squares by $V \square$.

We also have restrictions on how edges are formed, distinguishing between two particular types, and how weights are assigned, as follows.

**Definition 2.28 (Edge and Edge types).** We connect a circle to a square by an edge if the fixed circle is contained in the corresponding three-dimensional submanifold. Further, at any circle in the graph, we distinguish between the following edge types:
(1) a floating edge, that is, an edge connecting to a square of valence 1, and
(2) a grounded edge, that is, an edge connecting to a square of valence $\geq 2$.

Definition 2.29 (Weights of the graph). To a square $s$ we attach a weight $\alpha(s)$ in the following way: any three-dimensional submanifold is fixed by a codimension one subtorus. As a weight, we put the corresponding weight of the isotropy representation at any circle in the three-dimensional submanifold, regarded as an element in $t^*_G/\{\pm 1\}$.

Then $\alpha : V_\square \rightarrow S(t^*_G)$. We denote by $V_\square(s)$ the set of circles connected to $s \in V_\square$, and by $V_\square(c)$ the set of squares connected to $c \in V_\square$.

Note that by definition, any edge connects a circle to a square.

Example 2.30. The odd GKM graph of a $(2n + 1)$-dimensional sphere $S^{2n+1} \subset \mathbb{C}^{n+1}$ with the standard $T^n$-action induced by the standard representation on $n$ of the $n+1$ summands is a pinwheel with $n$ edges terminating in squares (corresponding to fixed 3-spheres), as follows:

In the following proposition we collect a few properties of odd GKM graphs.

Proposition 2.31. Let $M^{2n+1}$ be an odd GKM manifold, then the following hold:

(1) The odd GKM graph is connected
(2) Each circle in the odd GKM graph has valence $n$.
(3) If the total Betti number of $M^{2n+1}$ is $2m$, then there are exactly $m$ circles in the graph. Moreover, each square in the odd GKM graph has valence bounded between 1 and $m$.

Proof. We include a proof for the sake of completeness, noting that the proof follows along the same lines as in the even-dimensional case. The Chang-Skjelbred Lemma 2.9 states that for an equivariantly formal action, the image of the (injective) map $H^*_T(M) \rightarrow H^*_T(M^T)$ is the same as the image of the map $H^*_T(M_1) \rightarrow H^*_T(M^T)$. As $M$ is connected, it follows that the image of $H^0(M_1) \cong H^0_T(M_1) \rightarrow H^0_T(M^T) \cong H^0(M^T)$ is one-dimensional, which implies that $M_1$ is connected. This is equivalent to the GKM graph being connected.

Any edge emanating from a circle corresponds to a weight of the isotropy representation at that circle. Because the codimension of this circle is $2n$, and $T$ acts on the normal space of this circle without fixed vectors, there are precisely $n$ such weights.

The equivariant formality of the action is equivalent to the equality of total Betti numbers $\dim H^*(M) = \dim H^*(M^T)$ by Proposition 2.7. The third claim follows because any circle in $M^T$ contributes 2 to the total Betti number of $M^T$. Since any square must contain a circle fixed by $T$ and can contain at most $m$, the result follows.

We can also introduce a notion of connection as in the even-dimensional setting. The only difference is that we do not specify a single edge along which we transport, but two circles in the same three-dimensional component of the 1-skeleton.

Definition 2.32. A connection on the odd GKM graph of an odd-dimensional GKM manifold is a collection of maps $\nabla_{c_1,c_2,s_0} : V_\square(c_1) \rightarrow V_\square(c_2)$, for every $s_0 \in V_\square$ and $c_1,c_2 \in V_\square(s_0)$, satisfying the following conditions:

(1) $\nabla_{c_1,c_2,s_0}(s_0) = s_0$
The following proposition guarantees the existence of a connection on the odd GKM graph of every odd-dimensional GKM manifold. The proof is completely analogous to the proof of Proposition 2.3 in [19].

**Proposition 2.33.** There exists a connection on the odd GKM graph of every odd-dimensional GKM manifold.

We are now in a position to define an odd GKM manifold.

**Definition 2.34 (Odd GKM manifolds).** An odd-dimensional GKM manifold is called odd GKM, for \( k \geq 2 \), if the following hold.

1. \( M \) is odd-dimensional GKM, and;
2. At any fixed circle, any \( k \) weights of the isotropy representation are linearly independent.

Thus, odd GKM manifolds are the same as odd GKM2 manifolds. For a GKM manifold \( M \), and any \( k - 1 \) weights at a fixed circle, there is a unique \((2k - 1)\)-dimensional submanifold fixed by a codimension \( k - 1 \) subtorus generated by the intersection of the kernels of the \( k - 1 \) weights.

**Definition 2.35 (Face).** We call the subgraph of the GKM graph of \( M \) corresponding to this submanifold a \((k - 1)\)-face of the graph.

**Remark 2.36.** Note that for GKM3-manifolds, for every \( s_0 \in V_\circ \), given \( c_1, c_2 \in V_0(s_0) \), the condition \( \alpha(\nabla_{c_1, c_2, s_0}(s)) = \pm \alpha(s) + c \alpha(s_0) \) alone determines the square \( \nabla_{c_1, c_2, s_0}(s) \) uniquely, for all \( s \in V_\circ(c_1) \).

Theorem 4.6 in [27] tells us how the GKM graph encodes the equivariant cohomology algebra. Note that this theorem is a little more general, as the the non-orientable case is also being treated.

We choose an orientation on every component of \( M^T \), which then allows us to identify its cohomology canonically with \( H^*(S^1) = \mathbb{Q}[\theta]/(\theta^2) \). The inclusion \( M^T \to M \) induces an injection

\[
H^*_T(M) \to H^*_T(M^T) = \bigoplus_{c \in V_0} S(t^*_c) \otimes H^*(S^1),
\]

and the image of this map is described by the following divisibility relations. For any \( s \in V_\circ \) corresponding to a three-dimensional connected submanifold \( N_s \) fixed by the subtorus with Lie algebra \( \ker \alpha(s) \), and \( c_1, \ldots, c_l \in V_0 \), the circles contained in \( N_s \), and

\[
(P_c + Q_c \theta)_{c \in V_0} \in \bigoplus_{c \in V_0} S(t^*_c) \otimes H^*(S^1),
\]

with \( P_c, Q_c \in S(t^*_c) \), satisfies

\[
P_{c_1} \equiv \cdots \equiv P_{c_l} \mod \alpha(s)
\]

and

\[
\sum_{i=1}^{l} \pm Q_{c_i} \equiv 0 \mod \alpha(s).
\]

Here, the signs in the sum are determined as follows. Recall that for a closed manifold \( M \), the fixed point sets of torus actions are closed submanifolds that are orientable if \( M \) is. As \( N_s \) is orientable, the orbit space \( N_s/T \) is orientable as a topological manifold (with boundary) as well. The circles \( c_i \) are boundary components of \( N_s/T \), and if the pre-chosen orientation on each \( c_i \) coincides with induced boundary orientation, with respect to any orientation of \( N_s/T \), then the sign of \( Q_{c_i} \) is +, and if not, then its sign is −.

**Remark 2.37.** It is not possible in general to consistently orient all components of \( M^T \) in such a way that for all \( N_s \) we find an orientation on \( N_s/T \) with the property that the circles in \( N_s \) carry the induced boundary orientation. Consider, for example, \( S^1 \times \mathbb{C}P^2 \) with \( T^2 \) acting trivially on \( S^1 \) and in the standard fashion on \( \mathbb{C}P^2 \).
2.6. **Geometric results in the presence of a lower curvature bound.** We now recall some general results about $G$-manifolds with non-negative and almost non-negative curvature which we will use throughout.

We recall the classification of closed, non-negatively curved $T^1$-fixed point homogeneous manifolds due to Galaz-García [12].

**Theorem 2.38.** [12] Let $M^3$ be a closed, non-negatively curved $T^1$-fixed point homogeneous Riemannian manifold. Then $M$ is diffeomorphic to one of $S^3$, $L_{p,q}$, $S^2 \times S^1$, $S^3 \times S^1$, $\mathbb{R}P^2 \times S^1$, or $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Moreover, an analysis of the isometric circle action yields the following.

1. If $M^3$ has total Betti number equal to 2, the isometric circle action fixes one circle; and
2. If $M^3$ has total Betti number equal to 4, the isometric circle action fixes two circles.

**Remark 2.39.** By Proposition [7], it follows that non-negatively curved $S^1$-fixed point homogeneous 3-manifolds are equivariantly formal.

Observe that the only manifold on this list that is not a rational cohomology sphere is $S^2 \times S^1$. Moreover, it is the only manifold on this list with total Betti number equal to 4.

The following theorem by Spindeler, [37], gives a characterization of non-negatively curved $G$-fixed point homogeneous manifolds.

**Theorem 2.40.** [37] Assume that $G$ acts fixed point homogeneously on a closed, non-negatively curved Riemannian manifold $M$. Let $F$ be a fixed point component of maximal dimension. Then there exists a smooth submanifold $N$ of $M$, without boundary, such that $M$ is diffeomorphic to the normal disk bundles $D(F)$ and $D(N)$ of $F$ and $N$ glued together along their common boundaries;

\[
(2.1) \quad M = D(F) \cup_D D(N).
\]

Further, $N$ is $G$-invariant and contains all singularities of $M$ up to $F$.

**Remark 2.41.** In fact $N$ is actually $\text{Isom}_F(M)$-invariant, by Lemma 3.30 in [37].

Finally, we recall the following Splitting Theorem due to Cheeger and Gromoll [7].

**Theorem 2.42.** [7] Let $M$ be a compact manifold of non-negative Ricci curvature. Then $\pi_1(M)$ contains a finite normal subgroup $\Psi$ such that $\pi_1(M)/\Psi$ is a finite group extended by $\mathbb{Z}^k$, and $\tilde{M}$, the universal covering of $M$, splits isometrically as $\tilde{M} \times \mathbb{R}^k$, where $\tilde{M}$ is compact.

3. **Closed, non-negatively curved 5-dimensional GKM manifolds**

The goal of this section is to classify the universal covers of closed, non-negatively curved 5-dimensional GKM manifolds, in order to facilitate the classification of their corresponding graphs, which will be addressed in Section 4. In fact, we will prove something slightly stronger than what we actually need by only assuming that the $T^2$-action on $M$ is equivariantly formal. As $M^T \neq \emptyset$ by equivariant formality, this in turn implies that there is an $S^1 \subset T^2$ fixing a codimension 2 submanifold in $M^5$, that is, the $T^2$-action is $S^1$-fixed point homogeneous. Note that GKM actions on 5-manifolds always satisfy these requirements. That is, we will prove the following theorem.

**Theorem 3.1.** Let $M^5$ be a closed, non-negatively curved, equivariantly formal 5-dimensional manifold admitting an isometric $T^2$-action. Then $\text{rk}(H_1(M^5;\mathbb{Z})) \leq 1$ and we may classify their universal covers as follows.

1. For $\text{rk}(H_1(M^5;\mathbb{Z})) = 0$, $\tilde{M}^5$ is diffeomorphic to one of $S^5$, $S^3 \times S^2$, or $S^3 \times S^2$, the non-trivial $S^3$ bundle over $S^2$;
2. For $\text{rk}(H_1(M^5;\mathbb{Z})) = 1$, $\tilde{M}^5$ is diffeomorphic to $\mathbb{R} \times M^4$, where $M^4$ is one of $S^4$, $\mathbb{C}P^2$, $S^2 \times S^2$, or $\mathbb{C}P^2 \# \mathbb{C}P^2$.

As a consequence of a result due to Rong [34] and applying the diffeomorphism classification results of Smale [36] and Barden [2], we have the following theorem for the positive curvature case.
Theorem 3.2. [31] Let $M^5$ be a closed, simply-connected, positively curved 5-dimensional manifold admitting an isometric $T^2$-action. Then $M^5$ is diffeomorphic to $S^5$.

Since positively curved manifolds have finite fundamental group, the following corollary is immediate, allowing us to classify the universal covers of positively curved 5-dimensional GKM manifolds as follows.

Corollary 3.3. Let $M^5$ be a closed, positively curved 5-dimensional GKM manifold admitting an isometric $T^2$-action. Then the universal cover of $M^5$ is diffeomorphic to $S^5$.

In order to prove Theorem 3.1 we first need to prove Proposition 3.4 and Lemmas 3.5, 3.6, which follow. Recall that by Theorem 2.40 if $M^5$ admits an isometric $T^2$-action that is $S^1$-fixed point homogeneous, then we may decompose $M^5$ as a union of disk bundles, that is

\[ M^5 = D^2(F^3) \cup E D(N), \]

where $F^3$ is the codimension two fixed point set of some circle subgroup of $T$, and by Remark 2.41 $N$ is a $T^2$-invariant submanifold.

Proposition 3.4. Let $M^5$ be a closed, orientable, equivariantly formal, non-negatively curved 5-dimensional Riemannian manifold with an isometric $T^2$-action. Then the following hold for $b(M^5)$, the total Betti number of $M^5$.

1. $2 \leq b(M^5) \leq 8$; and
2. If $6 \leq b(M^5) \leq 8$, then $F^3$ is diffeomorphic to $S^2 \times S^1$ in the decomposition of $M^5$ in Display 3.1.

Before we begin the proof, we need the following result concerning the dimension of the submanifold at maximal distance from the codimension two fixed point set in $M^{2n+1}$.

Lemma 3.5. Suppose $M^{2n+1}$ is an $S^1$-fixed point homogeneous closed, orientable manifold of non-negative curvature. Let $F$ be a codimension two fixed point set component of $S^1$ and suppose $N$ is the submanifold given in the disk bundle decomposition of Theorem 2.40 as in Display 2.4. Then if $N \cap \text{Fix}(M; S^1) \neq \emptyset$, it follows that $\text{codim}(N)$ is even.

Proof. Recall that by Theorem 2.40 all singularities of the $S^1$-action are contained in $F$ and $N$. Thus if $N \cap \text{Fix}(M; S^1) \neq \emptyset$, any connected component $A$ of $\text{Fix}(M; S^1)$ is contained in $N$. However, fixed point set components of circles are of even codimension and hence $N$ must also be of even codimension in $M$. \qed

We are now in a position to prove Proposition 3.4.

Proof of Proposition 3.4. We consider a circle subgroup $S^1 \subset T^2$ fixing a three-dimensional connected submanifold $F^3$. By Theorem 2.40 $M^5$ decomposes as a union of disk bundles as in Display 3.1, where $N$ is a $T^2$-invariant submanifold, and there are no fixed points of the $S^1$-action outside $F^3 \cup N$. Since $M^5$ is equivariantly formal, the total Betti number of $\text{Fix}(M; S^1)$ equals that of $M^5$. So if $N$ does not contain any $S^1$-fixed points, the lemma is proven, as by Theorem 2.38 the total Betti number of $F^3$ is either 2 or 4.

We now assume that there are $S^1$-fixed points in $N$, which implies by Lemma 3.3 that $N$ is of dimension 1 or 3 only. If $N$ is 1-dimensional, it follows from the classification of 1-manifolds that $N = S^1$, and so, the total Betti number of $M$ is either 4 or 6.

Assume then that $N$ is 3-dimensional. If $N$ is fixed by some circle subgroup of $T^2$, it is totally geodesic, and thus non-negatively curved. Then Theorem 2.38 is applicable, and the total Betti number of $N$ is 2 or 4. Then by Proposition 2.7 the total Betti number of the $S^1$-fixed point set in $N$ is also bounded from above by 4. It follows that the total Betti number of $M$ is bounded by 8, and if $F^3$ is a rational cohomology sphere, then it is bounded by 6.

Suppose then that $N$ is not fixed by any circle subgroup of $T^2$. In this case $N$ is not necessarily totally geodesic, but the $T^2$-action on $N$ is of cohomogeneity one. Since we are assuming that there are $S^1$-fixed points, by the classification of cohomogeneity one $T^2$ manifolds in Mostert 31, and
Neumann [33], it follows that \( N \) must be one of \( S^3, L_{p,q}, S^2 \times S^1, \mathbb{R}P^2 \times S^1, \) or \( S^2 \times S^1 \). Note that in all these cases, the total Betti number of \( N \) is bounded between 2 and 4. Thus, it follows that the total Betti number of \( M \) is bounded by 8, and if \( F^3 \) is a rational cohomology sphere, it is bounded by 6. This proves Part (1).

We now prove Part (2). We will assume that \( 6 < b(M^5) \leq 8 \) and that \( F \) is a rational cohomology sphere, to derive a contradiction. Note first that if \( F \) is a rational cohomology sphere, because of the total Betti number bounds on \( N \), this implies that \( b(M^5) = 6 \) and \( b(N) = 4 \) and hence \( N \) must be \( S^2 \times S^1 \). It follows from the Gysin sequence that the space \( E \), which is the total space of the circle bundle inside the disc bundle \( D^2(F^3) \to F^3 \), has the same rational homology as \( S^1 \times S^3 \). Consider now the Mayer-Vietoris sequence of the decomposition \( F \)

\[
H_3(M^5) \to \mathbb{Q} \to 0 \to H_4(M^5) \to \mathbb{Q} \to \mathbb{Q}^2 \to H_5(M^5) \\
\to 0 \to \mathbb{Q} \to H_2(M^5) \to \mathbb{Q} \to \mathbb{Q} \to H_1(M^5) \to 0.
\]

Exactness at \( H_2(M^5) \), Poincaré duality and the fact that \( b(M^5) = 6 \), imply that \( b_2 = b_3 \in \{1, 2\} \) and hence \( b_1 = b_4 = 4 - 2b_2 \). However, it is clear that exactness is violated for either set of choices for the \( b_i \), \( 1 \leq i \leq 4 \). \( \square \)

With this information, we now obtain the following lemma.

**Lemma 3.6.** Let \( M^5 \) be a closed, orientable, equivariantly formal, non-negatively curved 5-dimensional Riemannian manifold with an isometric \( T^2 \)-action. Then

\[
\text{rk}(H_1(M^5)) \leq 1.
\]

**Proof.** Recall that by Theorem 2.40 \( M^5 \) decomposes as in Display 3.1 with \( F^3 \) the codimension one fixed point set of \( S^1 \). Since the total Betti number of \( M^5 \) must be greater than or equal to 6, \( F^3 = S^2 \times S^1 \) by Propositions 3.4 and 2.38. Moreover, \( N \) has \( T^1 \)-fixed points by Proposition 2.7. Then Lemma 4.5 implies that \( N \) is of dimension 1 or 3.

Suppose now that \( \text{rk}(H_1(M^5)) = k \geq 2 \) in order to derive a contradiction. Recall that \( E \) is a sphere bundle over both \( F \) and \( N \). So, if \( \text{dim}(N) = 1 \), then \( E \) would be an \( S^3 \) bundle over \( S^1 \) and so \( H_3(E) \cong \mathbb{Q} \). However, it follows from the homology Mayer-Vietoris sequence that \( \mathbb{Q}^k \subset H_3(E) \), \( k \geq 2 \), and so \( \text{dim}(N) = 3 \) and \( N \) cannot be covered by \( S^3 \). This then implies that both \( F \) and \( N \) are diffeomorphic to \( S^2 \times S^1 \). Moreover, since \( E \) is a principal \( S^1 \) bundle over \( S^2 \times S^1 \) and \( \mathbb{Q}^k \subset H_3(E) \), \( k \geq 2 \), using the classification of \( S^1 \) bundles over \( S^2 \), this implies that \( E \) is diffeomorphic to \( S^2 \times T^2 \).

In particular, this immediately tells us that \( k = 2 \). Analyzing the homology Mayer-Vietoris sequence of \( M^5 \) and recalling that we are assuming that \( b_1(M^5) = 2 \), this then implies by Poincaré duality that

\[
b_i(M^5) = \begin{cases} 
1 & i = 0, 5, \\
2 & i = 1, 4, \\
& \text{and } b_i(M^5) \geq 2 \text{ for } i = 2, 3.
\end{cases}
\]

But then the total Betti number of \( M^5 \) is greater than or equal to 10, a contradiction, since Proposition 3.4 guarantees that the total Betti number is bounded above by 8. \( \square \)

**Proof of Theorem 3.1** By Lemma 3.6 \( \text{rk}(H_1(M^5)) \leq 1 \) and by Theorem 2.12 \( \mathring{M}^5 \) is either a closed, simply-connected, non-negatively curved manifold or it splits isometrically as the product of \( \mathbb{R}^1 \) and \( M^4 \), a closed, simply-connected, non-negatively curved 4-manifold.

The proof of Part (1) then follows directly from work of Galaz-García and Spindeler [16] (cf. [8]). The proof of Part (2) follows by noting that since the \( T^2 \)-action on \( M^5 \) has non-empty fixed point set, we may apply Theorem 2.2 to lift the \( T^2 \)-action to \( \mathring{M}^5 \). By Theorem 1 of Hano [25], the isometry group of \( \mathbb{R} \times M^4 \) splits as the product of the isometry groups of \( \mathbb{R} \) and of \( M^4 \). Since \( T^2 \) is a compact Lie group, this implies that the \( T^2 \)-action on the \( \mathbb{R} \) factor is trivial and on \( M^4 \) is isotropy-maximal. In particular, \( M^4 \) is then a non-negatively curved torus manifold. The classification of non-negatively curved torus manifolds up to diffeomorphism follows from work of Kleiner [28], Searle and Yang [35], and Galaz-García [12]. \( \square \)
4. The classification of the graphs corresponding to closed, non-negatively curved 5-dimensional GKM manifolds

Our goal in this section is to classify the graphs corresponding to closed, non-negatively curved 5-dimensional GKM manifolds. We first prove that the lower curvature bound imposes severe restrictions on the odd GKM graphs.

**Proposition 4.1.** Let $M^{2n+1}$ be a non-negatively curved GKM manifold, then each square in the GKM graph has valence one or two.

*Proof.* This follows directly from Theorem 2.38. □

The corresponding result in positive curvature follows directly from the classification of positively curved 3-manifolds due to Hamilton [24].

**Proposition 4.2.** Let $M^{2n+1}$ be a positively curved odd GKM manifold, then each square in the GKM graph has valence one.

In the following theorem, we obtain a classification of the underlying graphs of the GKM graphs for closed, non-negatively curved 5-dimensional GKM manifolds.

**Theorem 4.3.** The underlying graphs of GKM graphs corresponding to the non-negatively curved 5-dimensional GKM manifolds are as follows, according to the total Betti number of $M^5$.

1. For the total Betti number equal to 2, we obtain a circle with two edges terminating in squares

2. For the total Betti number equal to 4, we have the following two possibilities

and

3. For the total Betti number equal to 6, we obtain a closed circuit in the form of a triangle

4. For the total Betti number equal to 8, we obtain a closed circuit in the form of a quadrangle
Proof. By Part 1 of Proposition 2.31 we know that the graph is connected. By Part 2 of Proposition 2.31, since $M$ is 5-dimensional, the graph is 2-valent, that is, exactly two edges emanate from each circle and each such edge must connect to a square. By Part 3 of Proposition 2.31 the number of fixed circles equals half the total Betti number. We showed in Proposition 3.4 that the total Betti number for such 5-manifolds is between 2 and 8. In the case where it is 2, $M^5$ is a rational cohomology sphere, the $T$-fixed point set is a single circle, and the graph is necessarily of the described form. If the total Betti number is 4, the connectedness of the graph implies directly that it is of one of the two given shapes.

For the case of total Betti number 6 or 8, Proposition 3.4 tells us that any $F^3$ is diffeomorphic to $S^2 \times S^1$ in the decomposition of $M^5$ in Display 3.1 which has total Betti number 4. This in turn implies by Proposition 2.38 that every square in the graph has valence 2. The connectedness of the graph then directly implies the claim. □

Example 4.4. The standard examples for Theorem 4.3 are the $T^2$-actions on $S^5$, $S^2 \times S^3$, $S^4 \times S^1$, $\mathbb{C}P^2 \times S^1$ and $S^2 \times S^2 \times S^1$.

Note that for positive curvature, using Proposition 4.2, it follows that only the first graph in Theorem 4.3 occurs and we immediately obtain the following theorem.

Theorem 4.5. The unique GKM graph corresponding to the positively curved 5-dimensional GKM manifolds is a circle with two edges terminating in squares.

5. The Proof of the Main Theorem 1.1

In this section we will prove the Main Theorem 1.1 which we need in order to prove Theorems 1.2, 1.3, and 1.5.

5.1. The Proof of the Main Theorem 1.1 Let $M^{2n+1}$ be a closed, non-negatively curved odd GKM$_3$ manifold. As shown in Proposition 4.1, any square in the GKM graph, $\Gamma$, has valence one or two.

Remark 5.1. The graph in Part (1) of Theorem 4.3 contains only floating edges, whereas the first graph of part (2) of Theorem 4.3 contains two floating edges and two grounded edges.

We now show how one can construct a classical (even-dimensional) graph $\Gamma$ from an odd-dimensional GKM graph, $\bar{\Gamma}$, for which the squares in $\bar{\Gamma}$ are only of valence one or two. Let $\Gamma$ be the graph obtained from the odd-dimensional graph, $\bar{\Gamma}$, by

1. replacing a circle by a vertex;
2. replacing a 2-valent square in 2 grounded edges by a single edge, labelled by the weight of the square;
3. deleting all floating edges, together with their squares.
In order to facilitate discussion of the new graph, $\Gamma$, we will denote the application of these changes to $\bar{\Gamma}$, respectively, as follows:

1. $\pi(c) = v$, where $c \in V_0(\bar{\Gamma})$ and $v$ is its image in $\Gamma$;
2. $\pi(s) = e$, for $s \in V_2^2$, where $s$ is the square connected to $c_1$ and $c_2$, and $e$ is its image in $\Gamma$, where $i(e) = \pi(c_1)$ and $t(e) = \pi(c_2)$.

**Example 5.2.** Applying this construction to the odd-dimensional graphs in Theorem 4.3, we see that

1. the image of the graph in Part (1) is a vertex,

   ![Vertex Image](image)

2. the image of the graphs in Part (2) are an interval and a lune, respectively,

   ![Interval and Lune Images](image)

3. the image of the graph in Part (3) is a triangle,

   ![Triangle Image](image)

4. and the image of the graph in Part (4) is a quadrangle.

   ![Quadrangle Image](image)

**Lemma 5.3.** Let $\bar{\Gamma}$ be the GKM$_3$ graph corresponding to the closed, non-negatively curved GKM$_3$ manifold, $M^{2n+1}$. Then the graph $\Gamma$ constructed as outlined above is a classical GKM$_3$ graph.
Proof. We first claim that the graph $\Gamma$ obtained from $\tilde{\Gamma}$ is $m$-valent, for some $m \leq n$. By Proposition 2.31, the odd-dimensional graph $\tilde{\Gamma}$ is connected. So, to prove this claim it suffices to show that given two circles, $c_1, c_2 \in V(\Gamma)$, that are joined by a square, $s_0$, the number of floating edges emanating from $c_1$ and $c_2$ are the same. Let $e, f \in E_{c_1}(\tilde{\Gamma})$, such that $e$ contains the square $s_0$, and $f$ is a floating edge with square $s$. Observe that because of the GKM$_3$ condition, $e$ and $f$ uniquely determine a 2-dimensional face of $\tilde{\Gamma}$. Moreover, due to the linear dependence condition on the connection, it follows that $\nabla_{c_1,c_2,s_0}(s)$ is a square belonging to the same 2-dimensional face as $s$.

Now, Theorem 4.3 tells us that for odd GKM$_3$ graphs corresponding to odd GKM$_3$ manifolds of non-negative curvature, there is only one graph of a 2-dimensional face that has a floating edge; this graph has two floating edges, two circle vertices, and one grounded edge. This then implies that the connection will send a 1-valent square to a 1-valent square and a 2-valent square to a 2-valent square. Equivalently, the connection slides a floating edge to a floating edge, and a grounded edge to a grounded edge. Thus we may conclude that at every circle there are exactly the same number of floating edges and thus all vertices in the graph $\Gamma$ have the same valency.

The weights of the graph $\Gamma$ are still 3-independent. It remains to show that we have a connection on $\Gamma$ satisfying the conditions of Definition 2.17. For that, we observe that we still have a notion of 2-dimensional faces in $\Gamma$. Namely, given two edges $e, f$ attached to some vertex $v$ in $\Gamma$, there is a unique two-dimensional face in the odd-dimensional graph, $\tilde{\Gamma}$, containing the edges corresponding to $e$ and $f$. We see then from Theorem 4.3 and Example 5.2 that the 2-dimensional face in $\tilde{\Gamma}$ does not contain any floating edges, so it survives to a 2-valent subgraph of $\Gamma$, which is a 2-dimensional face of $\Gamma$. This gives a well-defined connection on $\Gamma$. Namely, we slide edges along edges inside these two-dimensional faces in the usual fashion. It is easy to see that one may directly translate the conditions satisfied by the connection on $\tilde{\Gamma}$ to those of Definition 2.32 by making the following substitutions:

$$\nabla_{c_1,c_2,s_0}(s) \mapsto \nabla_e(f),$$

where $e$ is the edge in $\Gamma$ determined by $s_0$, and

$$f = \pi(s).$$

We now prove the following useful lemma about the existence of a “top-dimensional class” for abstract, classical GKM graphs.

Lemma 5.4. For an $n$-valent, classical, abstract GKM graph $(\Gamma, \alpha, \nabla)$, we have $H^{2n}(\Gamma, \alpha) \neq 0$.

Proof. Let $v_0 \in V(\Gamma)$ be arbitrary. We note that an element $(f_v)_{v \in V(\Gamma)} \in \bigoplus_{v \in V(\Gamma)} S(t_v^+)$ with $f_v = 0$ for all $v \neq v_0$ is contained in $H^2_T(\Gamma, \alpha)$ if and only if $f_{v_0}$ is divisible by all $\alpha(e)$, with $e \in E_{v_0}$. In particular, in this situation the degree of the polynomial $f_{v_0}$ is at least $n$.

We can then define a nonzero element in $H^{2n}_T(\Gamma, \alpha)$ by setting

$$g_v = \left\{ \begin{array}{ll} \pm \prod_{e \in E_{v_0}} \alpha(e) \neq 0 & \text{for } v = v_0 \\ 0 & \text{for } v \neq v_0. \end{array} \right.$$ 

Note that by definition, $\deg(g_v) = n$. We now claim that $g_v$ induces a nonzero element $\tilde{g}_v \in H^{2n}(\Gamma, \alpha)$. By Proposition 2.8, if $\tilde{g}_v$ were trivial, then $g_v$ would be of the form $f \cdot (h_v)_{v \in V(\Gamma)}$ with $f \in S^+(t_v^+)$ and $(h_v)_v \in H^2_T(\Gamma, \alpha)$. However, this implies that $h_v = 0$ for all $v \neq v_0$, and so $h_{v_0}$ is divisible by all $\alpha(e)$, with $e \in E_{v_0}$, as mentioned previously. But since $\deg(g_v) = n$, this implies that the degree of $h_{v_0}$ is smaller than $n$, and we obtain a contradiction.

Finally, we can now speak about the equivariant cohomology of the graph $\Gamma$ obtained from $\tilde{\Gamma}$, the odd GKM$_3$ graph obtained from the odd GKM$_3$ manifold, $M^{2n+1}$. In particular for odd GKM$_3$ graphs, we will prove the following result.
Theorem 5.5. Let $M^{2n+1}$ be a closed, non-negatively curved odd GKM manifold. Let $k$ be the number of floating edges at a vertex in the GKM graph $\Gamma$ of the $T$-action on $M$, and $\bar{\Gamma}$ the classical GKM graph associated to $\Gamma$, as constructed above. Then we have

$$H^*_T(M) \cong H^*_T(\Gamma, \alpha) \otimes H^*(S^{2k+1})$$

as $S(t_0^2)$-algebras, where the $S(t_0^2)$-algebra structure on $H^*_T(\Gamma, \alpha) \otimes H^*(S^{2k+1})$ is the tensor product of the standard $S(t_0^2)$-algebra structure on $H^*_T(\Gamma, \alpha)$ and the trivial one on $H^*(S^{2k+1})$. Therefore, we obtain

$$H^*(M) \cong H^*(\Gamma, \alpha) \otimes H^*(S^{2k+1}).$$

Remark 5.6. The proof of Lemma 5.5 tells us that the number of floating edges, $k$, is independent of the vertex.

Proof of Theorem 5.5. By Proposition 4.1, any square in the odd GKM graph of $M$ has valence one or two. We denote by $V_1^2$ and $V_2^2$ the sets of squares with valence one and two, respectively. For a square $s \in V_1^2$ we denote the unique circle connected to $s$ by $c(s)$; for $s \in V_2^2$ we denote the two circles by $c_1(s)$ and $c_2(s)$, with any ordering.

Then in this situation, Displays (2.1) and (2.2) give us the following divisibility relations for $P_c$ and $Q_c$.

\begin{equation}
(5.1) \quad P_{c_1(s)} \equiv P_{c_2(s)} \mod \alpha(s), \quad \text{and} \quad Q_{c_1(s)} \equiv \pm Q_{c_2(s)} \mod \alpha(s) \quad \text{for} \quad s \in V_2^2,
\end{equation}

and

\begin{equation}
(5.2) \quad Q_{c(s)} \equiv 0 \mod \alpha(s) \quad \text{for} \quad s \in V_1^2.
\end{equation}

Then the equivariant cohomology of $M$ is given as follows.

$$H^*_T(M) \cong \{(P_c + Q_c \theta)_{c \in V_2} \mid P_c, Q_c \text{ satisfy the Relations (5.1) and (5.2)}\}.$$
of the $\alpha(s_i), 1 \leq i \leq k$, whereas Part 2 of the definition of a GKM-manifold tells us $\alpha(s')$ must be pairwise linearly independent to each of the $\alpha(s_i), 1 \leq i \leq k$.

Thus $a_c \neq 0$ for all $c \in V_0$. Note that this also implies that $\dim(H^{2k+1}_T(M)) = 1$, since if we had two linearly independent elements $\mu, \omega \in H^{2k+1}_T(M)$, then $\eta_c = \gamma_c \omega_c - \mu_c$, with $\gamma = \alpha_c^t / a_c^2$, vanishes. But this contradicts the fact that for any element of $H^{2k+1}_T(M), a_c \neq 0$ for all $c \in V_0$, and the result holds.

It follows that multiplication with $\omega$ defines an $S(t^*_Q)$-module injection $H_*^{even}(M) \to H_*^{odd}(M)$, that is,

$$\phi : H_*^{even}(M) \to H_*^{odd}(M)$$

$$\psi \mapsto \psi \cdot \omega,$$

with $\omega \in H_*^{2k+1}(M)$, and we claim that $\phi$ is an isomorphism.

It remains to show that $\phi$ is onto. Let $(Q_c \theta)_{c \in V_0} \in H_*^{odd}(M)$. For $c \in V_0$, and $s \in V_0^1$, the polynomial $Q_c$ is divisible by any $\alpha(s)$ with $c(s) = c$, hence we can write $Q_c \theta = P_c \omega_c$ for some polynomial $P_c$. We have to show that $(P_c)_{c \in V_0} \in H_*^{even}(M)$, that is we must verify that the $P_c$ satisfies the divisibility relations for all $c \in V_0$. Let $s' \in V_0^1$ be arbitrary. Then $\alpha(s')$ divides both $Q_{c_1(s')} \pm Q_{c_2(s')}$ and $\omega_{c_1(s')} \pm \omega_{c_2(s')}$, where $\pm$ is the same sign in both expressions. Then we compute

$$Q_{c_1(s')} \theta \pm Q_{c_2(s')} \theta = P_{c_1(s')} \omega_{c_1(s')} \pm P_{c_2(s')} \omega_{c_2(s')}$$

$$= (P_{c_1(s)} - P_{c_2(s)}) \omega_{c_1(s)} + P_{c_2(s)}(\omega_{c_1(s)} \pm \omega_{c_2(s)})$$

and the divisibility assumptions imply that $\alpha(s')$ divides $(P_{c_1(s')} - P_{c_2(s')}) \omega_{c_1(s')}$. Since $s' \in V_0^1$, $\alpha(s')$ does not divide $\omega_{c_1(s')}$. So $\alpha(s')$ has to divide $P_{c_1(s')} - P_{c_2(s')}$, which is precisely the divisibility relation. Thus, we have shown that $(P_c)_{c \in V_0} \in H_*^{even}(M)$, and hence $\phi$ is onto. Thus multiplication by $\omega$ defines an $S(t^*_Q)$-module isomorphism.

We are now in a position to prove the theorem. The isomorphism $\psi : H^*_T(\Gamma, \alpha) \to H_*^{even}(M)$ extends to an $S(t^*_Q)$-algebra isomorphism

$$\Psi : H^*_T(\Gamma, \alpha) \otimes H^*(S^{2k+1}) \to H^T(M)$$

$$\gamma \otimes id + \beta \otimes \mu_{S^{2k+1}} \mapsto \psi(\gamma) + \phi(\beta) \omega,$$

where $\gamma, \beta \in H^*_T(\Gamma, \alpha)$, and $\mu_{S^{2k+1}}$ is the volume form of $S^{2k+1}$, which implies the first statement of the theorem.

The second statement of the theorem then follows immediately from Proposition 2.8 because the $S(t^*_Q)$-module structure on $H^*(\Gamma, \alpha) \otimes H^*(S^{2k+1})$ is just on the first factor. \hfill $\square$

Theorem 1.1 is a direct consequence of Theorem 5.5.

5.2. Some corollaries. We consider some special subcases of Theorem 5.5. Firstly, if the metric on the GKM$_3$ manifold is positively curved, then because of Theorem 3.2 every two-dimensional face of the GKM graph has only one circle. This implies that the GKM graph is the pinwheel depicted in Example 2.30. Theorem 1.3 is immediate, as all edges are floating.

The proof of Theorem 1.2 follows, as indicated in the introduction, from the main theorem of 18. If no two-dimensional face in the odd-dimensional GKM graph of $M$ is of the form (4) in Theorem 4.3 then the associated classical GKM$_3$ graph $\Gamma$ has no quadrangles. Then, the classification of exactly these types of GKM$_3$ graphs implies that the cohomology $H^*(\Gamma, \alpha, \nabla)$ is the cohomology of a compact rank one symmetric space.

Theorem 1.3 follows in the same way using Theorem 5.1 in 19. All that is necessary to apply that theorem to show that the cohomology $H^*(\Gamma, \alpha, \nabla)$ of the associated classical GKM$_4$ graph is that of a finite group quotient of a non-negatively curved torus manifold is that $\Gamma$ is a graph with small three-dimensional faces, see Definition 3.5 in 19.
6. Invariant almost contact structures

The goal of this section is to prove Theorem 1.4 of the Introduction. We begin by recalling the definition of an almost contact structure.

**Definition 6.1 (Almost contact structure).** An almost contact structure \((\phi, \xi, \eta)\) on a \((2n+1)\)-manifold \(M\) consists of a \((1,1)\)-tensor field \(\phi\), a vector field \(\xi\), and a differential one-form \(\eta\) such that

\[
\eta(\xi) = 1 \quad \text{and} \quad \phi^2(X) = -X + \eta(X)\xi,
\]

for any vector field \(X\) on \(M\). Note that the vector field \(\xi\), which is called the Reeb vector field, is uniquely determined by \(\phi\) and \(\eta\), namely at a point \(p\) it is the unique vector \(\xi_p\) such that \(\phi_p(\xi_p) = 0\) and \(\eta_p(\xi_p) = 1\).

If \(N^{2k+1} \subset M^{2n+1}\) is a submanifold such that \(\xi\) is tangent to \(N\), and \(\phi\) restricts to a well-defined tensor field on \(N\), then the almost contact structure on \(M\) restricts to an almost contact structure on \(N\). In this case we call \(N\) an almost contact submanifold.

The following lemma may be well-known, but the authors were unable to find it in a search of the literature, so we include it for the sake of completeness.

**Lemma 6.2.** Let \((\phi, \xi, \eta)\) be an almost contact structure on a manifold \(M\), invariant under the action of a compact Lie group \(G\). Then every connected component of the fixed point set \(M^G\) of the action is an almost contact submanifold.

**Proof.** It is well-known that the components of the fixed point set \(M^G\) of any compact Lie group action are embedded submanifolds of \(M\), see Kobayashi [29]. For every point \(p \in N \subset M^G\), the tangent space of \(N\) is given by

\[
T_pN = (T_pM)^G,
\]

the set of vectors fixed by the isotropy representation of \(G\) at \(p\). The \(G\)-invariance of \(\phi\) and the fact that \(p\) is fixed by \(G\) then implies that \(\phi\) maps \(T_pN\) to itself. For the same reasons, it follows that \(\xi\) is tangent to \(N\). Thus, the connected components of \(M^G\) are almost contact submanifolds. \(\square\)

**Proposition 6.3.** Let \(M^{2n+1}\) be an odd GKM manifold with a \(T\)-invariant almost contact structure \((\phi, \xi, \eta)\). Then the following are true.

1. Every component of the fixed point set of \(T\) is an isolated, closed flow line of \(\xi\).
2. At any fixed point \(p\) of the torus action, the weights of the isotropy representation at \(p\) are well-defined elements of \(t^*_Q\).

**Proof.** The first statement follows directly from Lemma 6.2 and the definition of a GKM action, because every component of the fixed point set is an isolated circle that is an almost contact submanifold.

To prove the second statement, we note that \(\phi\) defines a \(T\)-invariant almost complex structure on \(\ker\eta_p\). Since the almost contact structure is \(T\)-invariant, we have \(\eta_p(dt_p(v)) = \eta_p(v)\) for all \(v \in T_pM\) and \(t \in T\). So at a fixed point \(p\), \(\ker(\eta_p)\) is \(T\)-invariant. Now we have well-defined weights of the isotropy representation at \(p\), because we have a complex \(T\)-representation at \(p\). \(\square\)

By Proposition 6.3 in the presence of a \(T\)-invariant almost contact structure, we have that the weights are well-defined elements of \(t^*_Q\). This then allows us to slightly modify the odd GKM graph, \(\bar{\Gamma}\), of the \(T\)-action, which we call a signed odd GKM graph, as follows. We consider the same underlying graph, \(\bar{\Gamma}\), but now we assign weights to edges, not to squares, that is, for each edge connecting a circle \(c\) to a square \(s\), we assign the corresponding weight of the isotropy representation at the circle \(c\), which is an element in \(t^*_Q\). Regarded modulo \(\pm 1\), this weight is the same as the weight assigned to the square \(s\) in the original odd-dimensional graph, \(\bar{\Gamma}\). If, in addition, the signed weights on the edges emanating from a square sum to 0, we call such a graph alternating. This leads us to make the following definition.
Definition 6.4. If the signed odd GKM graph induced from the invariant almost contact structure on the odd GKM manifold is alternating, then we say that the almost contact structure is alternating.

The connection of a signed odd GKM graph is modified as follows (cf. Definition 2.32). Formally, if we denote the set of edges emanating from a circle \( c \) by \( E(c) \), then the axial function \( \alpha \) is a collection of maps \( E(c) \to t^*_Q \), for all \( c \). The connection can be regarded as a collection of maps \( \nabla_{c_1,c_2,s_0} : E(c_1) \to E(c_2) \), where \( c_1,c_2 \in V_0(s_0) \), and it satisfies that for every edge \( e \in E(c_1) \) there exists a constant \( c \in \mathbb{Z} \) such that

\[
\alpha(\nabla_{c_1,c_2,s_0}(e)) = \alpha(e) + c\alpha(e_0),
\]

where \( e_0 \) is an edge connecting \( c_1 \) or \( c_2 \) with \( s_0 \).

Remark 6.5. If \( M^{2n} \) is a closed, non-negatively curved GKM\(_k\) manifold admitting an invariant almost complex structure, then the associated classical GKM graph is a signed GKM graph (see Remark 2.18). Using the construction of a classical GKM graph from an odd GKM graph whose squares have valence less than or equal to two given at the beginning of Subsection 5.1, we see that from alternating odd GKM graphs we can construct classical signed GKM graphs.

We now restate a result from the proof of Lemma 5.6 in [18]. Note that this result is independent of curvature.

Lemma 6.6. [18] Let \( \Gamma \) be a classical signed GKM\(_3\) graph. Then \( \Gamma \) admits no biangles.

Recall that we denote by \( \Sigma^k \) the orbit space of the linear, effective action of the \( k \)-dimensional torus on \( S^{2k} \). The following corollary is immediate (cf. the proof of Lemma 7.1 in [19]).

Corollary 6.7. [19] Let \( \Gamma \) be a classical signed GKM\(_3\) graph. Then there are no maximal simplices in the GKM graph of \( M \) with the combinatorial type of \( \Sigma^k \).

Before we prove Theorem 1.4, we first recall the definition of a Bott manifold.

Definition 6.8 (Generalized Bott manifold). We say that a manifold \( X \) is a generalized Bott manifold if it is the total space of an iterated \( \mathbb{C}P^n \)-bundle

\[
X = X_k \to X_{k-1} \to \cdots \to X_1 \to X_0 = \{pt\},
\]

where each \( X_i \) is the total space of the projectivization of a Whitney sum of \( n_i + 1 \) complex line bundles over \( X_{i-1} \).

Remark 6.9. Torus manifolds over \( \prod \Delta^{n_i} \), where \( \Delta^{n_i} \) denotes the standard simplex of dimension \( n_i \) admitting an invariant almost complex structure were classified in [5]. They are all diffeomorphic to the so-called generalized Bott manifolds.

We are now ready to prove Theorem 1.4.

Proof of Theorem 1.4. By Corollary 6.7 a signed classical GKM\(_3\) graph has no maximal simplices in the GKM graph of \( M \) with the combinatorial type of \( \Sigma^k \).

Since GKM\(_4\) manifolds are also GKM\(_3\), it follows that the graph contains no maximal simplices with the combinatorial type of \( \Sigma^k \).

We can now argue as in Section 7 of [19] to obtain the result. We briefly outline the proof here for the sake of completeness. The first step is to use Theorem 3.11 of [19] to show that \( \Gamma \) is finitely covered by a graph, \( \tilde{\Gamma} \), which is the vertex-edge graph of a finite product of simplices. One then shows that the quasitoric manifold corresponding to the graph \( \tilde{\Gamma} \) admits an invariant complex structure in Theorem 7.1 of [19]. Applying Theorem 6.4 of [5], then shows us that \( \tilde{\Gamma} \) is the GKM graph of a generalized Bott manifold. Finally, we use Theorem 7.5 of [19] to show that \( \tilde{\Gamma} = \Gamma \). Thus, we may apply the GKM theorem to conclude that the rational cohomology ring of \( M \) is isomorphic to the tensor product of the rational cohomology ring of an odd-dimensional sphere and the rational cohomology ring of a generalized Bott manifold, as desired. \( \square \)
It seems very likely that the graphs corresponding to non-negatively curved odd GKM manifolds admitting an invariant almost contact structure are alternating. We finish with the following conjecture.

**Conjecture 6.10.** Let $M^{2n+1}$ be a closed, non-negatively curved odd-dimensional GKM manifold admitting an invariant almost contact structure. Then the odd GKM graph corresponding to $M^{2n+1}$ is alternating.

**References**

[1] C. Allday, V. Puppe, *Cohomological Methods in Transformation Groups*, Cambridge University Press, 1993.

[2] D. Barden, *Simply connected 5-manifolds*, Ann. of Math., 82 (1965) 365–385.

[3] A. Borel, *Seminar on Transformation Groups*, Ann. Math. Stud. 46, Princeton U. Press (1960).

[4] G. Bredon, *Introduction to compact transformation groups*, Academic Press, 48, New York-London (1972).

[5] S. Choi, M. Masuda, D.Y. Suh, *Quasitoric manifolds over a product of simplices*, Osaka J. Math., 47 (2010) 109–129.

[6] T. Chang and T. Skjelbred, *The topological Schur lemma and related results*, Ann. of Math. 2 no. 100 (1974), 307–321.

[7] J. Cheeger and D. Gromoll, *The splitting theorem for manifolds of nonnegative Ricci curvature*, J. Diff. Geom., 6 (1971), 119–128.

[8] Z. Dong, C. Escher, and C. Searle, *Almost torus manifolds of non-negative curvature*, arXiv preprint arXiv:1811.01493v1 (2018).

[9] C. Escher and C. Searle, *Non-negatively curved 6-dimensional manifolds of almost maximal symmetry rank*, J. Geometric Analysis, doi.org/10.1007/s12220-018-0026-2 (2018).

[10] C. Escher and C. Searle, *Torus actions, maximality, and non-negative curvature*, arXiv:1506.08685v3 (2017).

[11] F. Fang and X. Rong, *Homeomorphism classification of positively curved manifolds with almost maximal symmetry rank*, Jour. of Pure and Appl. Alg., 91 (1994), 137–142.

[12] F. Galaz-García, *Nonnegatively curved fixed point homogeneous manifolds in low dimensions*, Geom. Ded., 157 (2012), 367–396.

[13] F. Galaz-García and M. Kerin, *Cohomogeneity two torus actions on non-negatively curved manifolds of low dimension*, Math. Zeitschrift 276 (1-2) (2014), 133–152.

[14] F. Galaz-García, C. Searle, *Low-dimensional manifolds with non-negative curvature and maximal symmetry rank*, Proc. Amer. Math. Soc., 139 (2011), 2559–2564.

[15] F. Galaz-García and C. Searle, *Nonnegatively curved 5-manifolds with almost maximal symmetry rank*, Geom. Topol., 18 no. 3 (2014), 1397–1435.

[16] F. Galaz-García, W. Spindeler *Nonnegatively curved fixed point homogeneous 5-manifolds*, Ann. Global Anal. Geom. 41 (2012) 253–263, Erratum, Ann. Global Anal. Geom. 45 (2014) 151–153.

[17] O. Goertsches, H. Nozawa, and D. Töben, *Equivariant cohomology of K-contact manifolds*, Math. Ann. 354 (2012), no. 4, 1555–1582.

[18] O. Goertsches and M. Wiemeler, *Positively curved GKM-manifolds*, Int. Math. Res. Not. 22 (2015), 12015–12041.

[19] O. Goertsches and M. Wiemeler, *Non-negatively curved GKM orbifolds*, preprint, arXiv:1802.05871.

[20] M. Goresky, R. Kottwitz, and R. MacPherson, *Equivariant cohomology, Koszul duality, and the localization theorem*, Invent. Math. 131 (1998), no. 1, 25–83.

[21] K. Grove and C. Searle, *Positively curved manifolds with maximal symmetry rank*, Jour. of Pure and Appl. Alg., 91 (1994), 137–142.

[22] V. Guillemin and S. Sternberg, *Supersymmetry and Equivariant de Rham Theory*, Mathematics, Past and Present, Springer-Verlag, Berlin (1999).

[23] V. Guillemin and C. Zara, *1-skeleta, Betti numbers, and equivariant cohomology*, Duke Math. J. 107 no. 2 (2001), 283–349.

[24] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Diff. Geom., 17 no. 2 (1982), 255–306.

[25] J. Hano, *On affine transformations of a Riemannian manifold*, Nagoya Math. J. 9 (1955), 99–109.

[26] A. Hattori and T. Yoshida, *Lifting compact group actions in fiber bundles*, Proc. Japan Acad., 47 (1971), 119–128.

[27] C. He, *Localization of certain odd-dimensional manifolds with torus actions*, arXiv:1608.04392v4 (2016)

[28] B. Kleiner, *Riemannian Four-Manifolds with Nonnegative Curvature and Continuous Symmetry*, PhD Thesis, UC Berkeley (1990).

[29] S. Kobayashi, *Fixed points of isometries*, Nagoya Math. J. 13 (1958), 63-68.

[30] S. Kuroki, *Introduction to GKM theory*, Trends in Math., no. 2 (2009), 113–129.

[31] P. S. Mostert, *On a compact Lie group acting on a manifold*, Ann. of Math., 65 no. 2 (1957), 447–455.

[32] I. Mundet i Riera, *Lifts of smooth group actions to line bundles*, Bull. London Math. Soc. 33 (2001), no. 3, 351–361.
[33] W. D. Neumann, *3-dimensional $G$-manifolds with 2-dimensional orbits*, 1968 Proc. Conf. on Transformation Groups (New Orleans, La., 1967), Springer, New York, 220–222.

[34] X. Rong, *Positively Curved Manifolds with Almost Maximal Symmetry Rank*, Geom. Ded., 95 no. 1 (2002), 157–182.

[35] C. Searle and D. Yang, *On the topology of non-negatively curved simply-connected 4-manifolds with continuous symmetry*, Duke Math. J. 74 no. 2 (1994), 547–556.

[36] S. Smale, *On the structure of 5-manifolds*, Ann. of Math. 75 (1962) 38–46.

[37] W. Spindeler, *$S^1$-actions on 4-manifolds and fixed point homogeneous manifolds of nonnegative curvature*, PhD Thesis, Westfälische Wilhelms-Universität Münster (2014).

[38] B. Wilking, *Torus actions on manifolds of positive sectional curvature*, Acta Math. 191 (2003), 259–297.

(Escher) Department of Mathematics, Oregon State University, Corvallis, Oregon
E-mail address: tine@math.orst.edu

(Goertsches) Fachbereich Mathematik und Informatik der Philipps-Universität Marburg, Germany
E-mail address: goertsch@mathematik.uni-marburg.de

(Searle) Department of Mathematics, Statistics, and Physics, Wichita State University, Wichita, Kansas
E-mail address: searle@math.wichita.edu