REPRESENTATIONS DISTINGUISHED BY PAIRS OF EXCEPTIONAL REPRESENTATIONS AND A CONJECTURE OF SAVIN

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Abstract. We study representations of $\text{GL}_n$ appearing as quotients of a tensor of exceptional representations, in the sense of Kazhdan and Patterson. Such representations are called distinguished. We characterize distinguished principal series representations in terms of their inducing data. In particular, we complete the proof of a conjecture of Savin, relating distinguished spherical representations to the image of the tautological lift from a suitable classical group.

1. Introduction

Let $F$ be a local non-Archimedean field of characteristic different from 2. Let $\tau$ be an admissible representation of $\text{GL}_n(F)$ and let $\theta$ and $\theta'$ be a pair of exceptional representations in the sense of Kazhdan and Patterson [KP84], of the metaplectic double cover $\widetilde{\text{GL}}_n(F)$ of $\text{GL}_n(F)$. The representation $\tau$ is called distinguished if there is a nonzero trilinear form on the space of $\tau \times \theta \times \theta'$, which is $\text{GL}_n(F)$-invariant. Equivalently,

$$\text{Hom}_{\text{GL}_n(F)}(\theta \otimes \theta', \tau^\vee) \neq 0. \quad (1.1)$$

Here $\tau^\vee$ is the representation contragradient to $\tau$.

We study distinguished representations. The main result of this work is the following combinatorial characterization, of irreducible distinguished principal series representations. We say that a character $\eta = \eta_1 \otimes \ldots \otimes \eta_n$ of the diagonal torus satisfies condition $(\star)$ if, up to a permutation of the characters $\eta_i$, there is $0 \leq k \leq \lfloor n/2 \rfloor$ such that

- $\eta_{2i} = \eta_{2i-1}^{-1}$ for $1 \leq i \leq k$,
- $\eta_i^2 = 1$ for $2k + 1 \leq i \leq n$.

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**Theorem 1.** Let $\tau$ be a principal series representation of $\text{GL}_n(F)$, induced from the character $\eta$. If $\tau$ is distinguished, $\eta$ satisfies $(\star)$. Conversely, if $(\star)$ holds and $\tau$ is irreducible, then $\tau$ is distinguished.

The main corollary of this theorem, is the validity of the “only if” part of the following conjecture of Savin [Sav92].

**Theorem 2.** Let $\tau$ be a spherical representation of $\text{GL}_n(F)$, i.e., an irreducible unramified quotient of some principal series representation, with a trivial central character. Then $\tau$ is distinguished if and only if $\tau$ is the lift of a representation of (split) $\text{SO}_{2\lfloor n/2\rfloor}(F)$ if $n$ is even or $\text{Sp}_{2\lfloor n/2\rfloor}(F)$ if $n$ is odd.

Both theorems currently hold under the plausible assumption that for $n > 3$, the exceptional representations do not have Whittaker models. This was proved by Kazhdan and Patterson for local fields of odd residual characteristic ([KPS84] Section I.3, see also [Kab01] Theorem 5.4). The remaining case - a field of characteristic 0 and even residual characteristic, is expected to be completed through progress in the trace formula ([BG92] p. 138, see [FKS90] Lemma 6).

For $n = 3$, Theorem 2 was proved (unconditionally) by Savin [Sav92], who analyzed the dimension of (1.1) for an arbitrary irreducible quotient of a principal series representation of $\text{GL}_3(F)$.

The “if” part of Theorem 2 (for any $n$) was proved by Kable [Kab02]. We briefly describe his approach. Kable used analytic techniques, resembling the methods of [CS80, BFF97]. He started with a “pseudo integral” $\Upsilon$ on $\tau \times \theta \times \theta'$ satisfying the equivariance properties, defined for inducing data $\eta$ in an appropriate cone of convergence. As a function of $\eta$, $\Upsilon$ was a polynomial in the Satake parameters of $\eta$ and $\eta^{-1}$. Using Bernstein’s continuation principle ([Ban98]), $\Upsilon$ was extended to all characters. The key for using this principle is a one-dimensionality result, namely, the space (1.1) is at most one dimensional, for a “large enough” subset of characters. This was proved by Kable [Kab01] (Theorem 6.4) using the theory of derivatives of Bernstein and Zelevinsky [BZ76, BZ77]. The assumption concerning the absence of Whittaker models for $n > 3$ was needed also for his results. We also mention that Kable [Kab01] (Theorem 6.3) already proved that if a principal series representation is distinguished, the character $\eta$ satisfies a combinatorial condition, different from $(\star)$ in that $\eta_{2i}$ and $\eta_{2i-1}^{-1}$ only agree on $(F^*)^2$, that is, $\eta_{2i}^2 = \eta_{2i-1}^{-2}$. The methods used for his proof seem insufficient to deduce the precise result ([Kab02] p. 1602).

Condition $(\star)$ implies that distinguished representations do not enjoy complete hereditary properties, in the sense that induction from
distinguished representations does not exhaust all distinguished representations. We do have upper heredity, given by the following result.

**Theorem 3.** Let $\tau_1$ and $\tau_2$ be a pair of distinguished representations of $\text{GL}_{n_1}(F)$ and $\text{GL}_{n_2}(F)$. The representation $\tau$ parabolically induced from $\tau_1 \otimes \tau_2$ is distinguished.

Theorems 1 and 3 and similar results (e.g. [Mat11, FLO12], see below) motivate the following conjecture, giving the combinatorial characterization of irreducible generic distinguished representations.

**Conjecture 1.** Let $\tau$ be an irreducible generic representation of $\text{GL}_n(F)$. Then $\tau$ is distinguished if and only if $\tau$ is isomorphic to a representation parabolically induced from a representation $\Delta_1 \otimes \ldots \otimes \Delta_m$ of $\text{GL}_{n_1}(F) \times \ldots \times \text{GL}_{n_m}(F)$ with the following properties:

- Each $\Delta_i$ is essentially square integrable,
- There is $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ such that $\Delta_{2i} = \Delta_{2i-1}^\vee$ for $1 \leq i \leq k$,
- The representation $\Delta_i$ is distinguished for $2k + 1 \leq i \leq n$.

In the case of a principal series representation, the conjecture becomes Theorem 1 because for $n = 1$, an irreducible representation $\Delta$ is distinguished if and only if $\Delta^2 = 1$.

The exceptional representations of $\widetilde{\text{GL}}_n(F)$ were first defined and studied by Kazhdan and Patterson [KP84]. Their motivation was global, to study a class of automorphic forms on the metaplectic group. One of the significant applications of their theory was the construction of a Rankin-Selberg integral representation for the symmetric square $L$-function by Bump and Ginzburg [BG92].

Let $k$ be a number field with a ring of adeles $\mathbb{A}$. Let $\pi$ be a cuspidal automorphic representation of $\text{GL}_n(\mathbb{A})$ with a unitary central character. In their seminal work, Bump and Ginzburg [BG92] showed that the only possible poles of the partial $L$-function $L_S(s, \pi, \text{Sym}^2)$ are at $s = 0, 1$ and furthermore, the existence of a pole at $s = 1$ implies the nonvanishing of a period integral of the form

$$\int_{Z' \backslash \text{GL}_n(F) / \text{GL}_n(\mathbb{A})} \rho(m) \varphi(m) \varphi'(m) dm. \quad (1.2)$$

Here $Z'$ is a subgroup of finite index in the center $Z_n(\mathbb{A})$ of $\text{GL}_n(\mathbb{A})$, $\rho$ is a cusp form in the space of $\pi$ and $\varphi$ and $\varphi'$ are automorphic forms corresponding to the global exceptional representations $\theta$ and $\theta'$. Takeda [Tak14] extended the results of [BG92] to the twisted symmetric square $L$-function, but did not consider a period integral.

Now consider an irreducible supercuspidal representation $\tau$ such that the (local) $L$-function $L(s, \tau, \text{Sym}^2)$ has a pole at $s = 0$. As an application of the descent method of Ginzburg, Rallis and Soudry ([GRS97b, GR...], this work, we apply the previous results to obtain the following:

**Theorem 4.** Let $\tau$ be an irreducible supercuspidal representation of $\text{GL}_n(F)$ with a unitary central character. Then $\tau$ is distinguished if and only if $\tau$ is isomorphic to a representation parabolically induced from a representation $\Delta_1 \otimes \ldots \otimes \Delta_m$ of $\text{GL}_{n_1}(F) \times \ldots \times \text{GL}_{n_m}(F)$ with the following properties:

- Each $\Delta_i$ is essentially square integrable,
- There is $0 \leq k \leq \lfloor \frac{n}{2} \rfloor$ such that $\Delta_{2i} = \Delta_{2i-1}^\vee$ for $1 \leq i \leq k$,
- The representation $\Delta_i$ is distinguished for $2k + 1 \leq i \leq n$.

In the case of a principal series representation, the conjecture becomes Theorem 1 because for $n = 1$, an irreducible representation $\Delta$ is distinguished if and only if $\Delta^2 = 1$.
one can globalize $\tau$ to a cuspidal automorphic representation $\pi$ of $GL_n(\mathbb{A})$, such that $L_S(s, \pi, \text{Sym}^2)$ has a pole at $s = 1$ (see the appendix of [PR12]). Therefore (1.2) implies that $\tau$ is distinguished.

This is an example of the analytic approach to the study of distinguished representations. In an ongoing work by Shunsuke Yamana and the author, we develop a global theory of distinguished representations and extend the results of [BG92, Tak14]. In the case of even $n$, we present a novel integral representation for the twisted symmetric square $L$-function. We characterize the pole at $s = 1$ in terms of a period integral similar to (1.2). Furthermore, we determine the irreducible distinguished summands of the discrete spectrum of $GL_n$.

The case of $GL_n$ can be placed in a more general context. The following co-period integral was studied in [Kap].

$$\int_{SO_{2n+1}(k) \backslash SO_{2n+1}(\mathbb{A})} \text{Res}_{s=1/2} E(g; \rho, s) \Phi(g) \Phi'(g) dg.$$ 

Here $E(g; \rho, s)$ is an Eisenstein series corresponding to an element $\rho$, in the space of the representation of $SO_{2n+1}(\mathbb{A})$, parabolically induced from an automorphic cuspidal representation $\pi$ of $GL_n(\mathbb{A})$, $\Phi$ and $\Phi'$ are automorphic forms in the space of the small representation of $SO_{2n+1}(\mathbb{A})$ of Bump, Friedberg and Ginzburg [BFG03]. The nonvanishing of this integral was related to the nonvanishing of a period similar to (1.2). This global result has a local counterpart, showing that a local representation $I(\tau, s)$ of $SO_{2n+1}(F)$, induced from a Siegel parabolic subgroup and $\tau | \text{det} |^s$, is distinguished at $s = 1/2$ whenever $\tau$ is distinguished.

Exceptional representations are related to a broader class of small, or minimal, representations. Perhaps the first example was the Weil representation of $\tilde{Sp}_{2n}$. These representations played a fundamental role in constructions of lifts and Rankin-Selberg integrals. They are extremely useful for applications, mainly because they enjoy the vanishing of a large class of twisted Jacquet modules, or globally phrased, Fourier coefficients [GRS03, Gin06, GJS11]. Minimal or small representations have been studied and used by many authors, including [Vog81, Kaz90, KS90, Sav93, BK94, Sav94, GRS97a, BFG00, GRS01, KPW02, BFG03, JS03, GS05, Sou06, LS08, GRS11].

The term “distinguished” has been used in the following context. Let $\xi$ be a representation of a group $G$ and let $\eta$ be a character of a subgroup $H < G$. Then $\xi$ is said to be $(H, \eta)$-distinguished if $\text{Hom}_H(\xi, \eta) \neq 0$. 

[GRS99a, GRS99b, GRS01, GSS02, JS03, JS04, Sou05, Sou06, GRS11]
There are numerous studies on local and global distinguished representations, including [Jac91, JR92, FJ93, OH06, OS07, OS08, OH09, Jac10, Mat10a, Mat10b, Mat11, FLO12, Mat].

Matringe [Mat10a, Mat10b, Mat11] studied representations of $GL_n(F_0)$, where $F_0$ is a quadratic extension of $F$, which are $(GL_n(F), \eta)$-distinguished. He proved (Mat10b) that an irreducible generic representation $\xi$ is distinguished, if and only if its Rankin-Selberg Asai $L$-function $L(s, \xi, Asai)$ has an exceptional pole at 0. Matringe also proved a combinatorial classification result (Mat11 Theorem 5.2) similar to Conjecture 1, which he used in Mat09 to prove $L(s, \xi, Asai) = L(s, \rho(\xi), Asai)$, where $\rho(\xi)$ is the Langlands parameter associated with $\xi$.

Feigon, Lapid and Offen [FLO12] studied representations distinguished by unitary groups, locally and globally.

One tool, used repeatedly for local analysis in the aforementioned works, is Mackey theory, or a variant on the Geometric Lemma of Bernstein and Zelevinsky [BZ77]. For example, Matringe [Mat11] considered the filtration of a representation parabolically induced from $Q(F_0)$ to $GL_n(F_0)$, as a $GL_n(F)$-module.

In our setting we look at the structure of the $GL_n(F)$-module $\theta \otimes \theta'$ and exploit the properties of exceptional representations. Our analysis of the space (1.1) is based on ideas of Savin [Sav92] and Kable [Kab01]. To study $\theta \otimes \theta'$, we extend the filtration argument in [Sav92] to any $n$. However, Savin used a geometric model for $\theta$ (FKS90), which is not available in general. We utilize the computation of derivatives of $\theta$ in [Kab01], to reduce several problems to questions on modules of the mirabolic subgroup. We develop certain extensions to the functors of Bernstein and Zelevinsky [BZ77] (Section 3), which may be of independent interest.

One delicate point about exceptional representations, is that when $n$ is even, the center $Z_n(F)$ of $GL_n(F)$ is not central in the cover. In turn $\theta$ does not admit a character of $\tilde{Z}_n(F)$. This will require extra care in our inductive argument (see the proof of Proposition 1.5). It is interesting to note that when $n = 2$, $\tilde{Z}_n(F)$ acts by a character on the second derivative of $\theta$. This was proved by Gelbart and Piatetski-Shapiro [GPS80] (Theorem 2.2), and was used by Kable [Kab01] (Theorem 5.3) to compute the second derivative of $\theta$.

The relation between equivariant trilinear forms and $\epsilon$-factors has been studied by Prasad [Pra90].

The rest of this work is organized as follows. Section 2 contains preliminaries and notation. In Section 3 we prove several technical results, concerning representations of the mirabolic subgroup, that will
be used for the computations of Jacquet modules. Our main results on distinguished representations occupy Section 4.

2. Preliminaries

2.1. The groups. Let \( F \) be a local non-Archimedean field of characteristic different from 2. Let \( (, \) be the Hilbert symbol of order 2 of \( F \) and put \( \mu_2 = \{-1, 1\} \). We usually denote by \( \psi \) a fixed nontrivial additive character of \( F \) and then \( \gamma_\psi \) is the normalized Weil factor (\cite{Wei64} Section 14, \( \gamma_F(a, \psi) \) is \( \gamma_F(a, \psi) \) in the notation of \cite{Rao93}, \( \gamma_F(\cdot)^4 = 1 \)).

In the group \( \text{GL}_n \), fix the Borel subgroup of upper triangular invertible matrices \( B_n = T_n \ltimes N_n \), where \( T_n \) is the diagonal torus. If \( m_1, \ldots, m_l \geq 0 \) satisfy \( m_1 + \ldots + m_l = n \), let \( Q_{m_1, \ldots, m_l} = M_{m_1, \ldots, m_l} \ltimes U_{m_1, \ldots, m_l} \) be the standard maximal parabolic subgroup with a Levi part \( M_{m_1, \ldots, m_l} \cong \text{GL}_{m_1} \times \ldots \times \text{GL}_{m_l} \). Let \( Z_n \) be the center of \( \text{GL}_n \). Denote by \( Y_n \) the mirabolic subgroup of \( \text{GL}_n \), that is, the subgroup of elements whose last row is \((0, \ldots, 0, 1)\). Also denote by \( I_n \) the identity matrix of \( \text{GL}_n(F) \). For any \( l \leq n \), \( \text{GL}_l \) is embedded in \( \text{GL}_n \) in the top left corner. For \( d \in \mathbb{N} \) and \( H < T_n(F) \), put \( H^d = \{ h^d : h \in H \} \). Also put \( F^* = (F^*)^d \). If \( x, y \in \text{GL}_n(F) \) and \( Y < \text{GL}_n(F) \), \( xy = xyyx^{-1} \) and \( xY = \{ xy : y \in Y \} \).

Let \( \widetilde{\text{GL}}_n(F) \) be the metaplectic double cover of \( \text{GL}_n(F) \), as constructed by Kazhdan and Patterson \cite{KP84} (with their \( c \) parameter equal to 0). Recall that they defined their cover using an embedding of \( \text{GL}_n(F) \) in \( \text{SL}_{n+1}(F) \), and the cover of \( \text{SL}_{n+1}(F) \) of Matsumoto \cite{Mat69}. We use the block-compatible cocycle of Banks, Levi and Sepanski \cite{BLS99}. If \( n = 2 \), this cocycle coincides with the one given by Kubota \cite{Kub67}. Let \( p : \widetilde{\text{GL}}_n(F) \to \text{GL}_n(F) \) be the natural projection. For any subset \( X \subset \text{GL}_n(F) \), denote \( \widetilde{X} = p^{-1}(X) \). Let \( e \) be 1 if \( n \) is odd, otherwise \( e = 2 \). Then \( \widetilde{Z}_n(F)^e \) is the center of \( \widetilde{\text{GL}}_n(F) \).

Henceforth we exclude the field \( F \) from the notation.

2.2. Representations. Let \( G \) be an \( l \)-group \((\cite{BZ76} 1.1)\). Throughout, representations of \( G \) will be complex and smooth. We let \( \text{Alg} \) denote the category of these representations. If \( \pi \) is a representation of \( G \), \( \pi^\vee \) is the representation contragradient to \( \pi \). The central character of \( \pi \), if exists, is denoted \( \omega_\pi \).

Regular induction is denoted \( \text{Ind} \) while \( \text{ind} \) is the compact induction. When inducing from a parabolic subgroup, induction is always taken to be normalized.
We say that $\pi$ is glued from representations $\pi_1, \ldots, \pi_l$ if $\pi$ has a filtration whose quotients are, after a permutation, $\pi_1, \ldots, \pi_l$. For convenience, we also write $\pi = s.s. \bigoplus_{i=1}^l \pi_i$ and refer to both sides as $G$-modules (of course each $\pi_i$ is a $G$-module, but the right-hand side is not a direct sum). The representations $\pi_i$ might be isomorphic or zero.

If $\pi$ is a representation of a subgroup $H < G$ and $w \in G$, denote by $w^* \pi$ the representation of $wH$ on the space of $\pi$ acting by $w^* \pi(h) = \pi(w^{-1} h)$.

If $\pi$ and $\pi'$ are a pair of genuine representations of $\text{GL}_n$, their (outer) tensor product $\pi \otimes \pi'$ can be regarded as a representation of $\text{GL}_n$ by $g \mapsto \pi(\varphi(g)) \otimes \pi'(\varphi(g))$, where $\varphi: \text{GL}_n \to \text{GL}_n$ is an arbitrary section. The actual choice of $\varphi$ does not matter, hence it will usually be omitted.

2.3. Filtration of induced representations. We recall the increasing filtration of induced representations of Bernstein and Zelevinsky [BZ76] (2.24). Let $G$ be an $l$-group, $H < G$ be a closed subgroup and $\pi$ be a representation of $H$ on a space $E$. Denote the space of the induced representation $\text{ind}_G^H(\pi)$ by $W$. For a compact open subgroup $V < G$, let $W^V$ be the subspace of vectors invariant by $V$. Choose a set of representatives $\Omega_V$ for $H \backslash G / \text{sl} \rightarrow H \text{sl} \rightarrow G / \text{sl} \rightarrow V$. Then $W^V$ is linearly isomorphic with the space of functions $f: \Omega_V \rightarrow E$, such that $f(g)$ is invariant by $H \cap g \mathcal{V}$ for all $g \in \Omega_V$, and $f$ vanishes outside of a finite subset of $\Omega_V$. Then if $V_1 > \ldots > V_l > \ldots$

is a decreasing sequence of compact open subgroups,

$W^{V_1} \subset \ldots \subset W^{V_l} \subset \ldots$

is an increasing filtration of $W$. A similar result holds for $\text{Ind}_G^H(\pi)$, except that the functions $f$ may be nonzero on an infinite number of representatives from $\Omega_V$.

2.4. Jacquet modules. Let $\pi$ be a representation of an $l$-group $G$ on a space $E$. If $U$ is a unipotent subgroup, which is exhausted by its compact subgroups (always the case for $U < \text{GL}_n$), let $E(U) \subset E$ be the subspace generated by the vectors $\pi(u)v - v$ where $u \in U$ and $v \in E$. Put $E_U = E(U) \backslash E$. If $M$ is the normalizer of $U$ in $G$, the following sequence of $M$-modules

$0 \rightarrow E(U) \rightarrow E \rightarrow E_U \rightarrow 0$

is exact. The Jacquet module of $\pi$ with respect to $U$ is $E_U$ and we call $E(U)$ the Jacquet kernel. The normalized Jacquet module $j_U(\pi)$ is defined as in [BZ77] (1.8): it is the representation of $M$ on $E_U$ given
by
\[ j_U(m)(v + E(U)) = \text{mod}_U^{1/2}(m)(\pi(m)v + E(U)). \]

Here \( \text{mod}_U \) is the topological module of \( U \). If \( G = \text{GL}_n \) (or its cover) and \( U \) is the unipotent radical of a parabolic subgroup \( Q \), \( \text{mod}_U = \delta_Q \).

We recall from [BZ76] (2.32-2.33) that for any unipotent subgroups \( U \) and \( V \), \( E(U)V = E(U) \cap E(V) \), and if \( V < UV \), \( E(U) + E(V) = E(UV) \) and \( E_{UV} = (E_V)_U \).

Let \( Q < \text{GL}_n \) be a closed subgroup containing \( N_n \) and let \( E \) be a \( Q \)-module. For \( 0 \leq m \leq n \) and \( b \in \{0,1\} \), define the following functor
\[
\mathcal{L}^{n,m}_b : \text{Alg}(Q \cap Q_{n-m,m}) \to \text{Alg}(Q \cap Q_{n-m,m})
\]
by
\[
\mathcal{L}^{n,m}_b(E) = \begin{cases} 
E(U_{n-m,m}) & b = 0, \\
E_{U_{n-m,m}} & b = 1.
\end{cases}
\]

The functor \( \mathcal{L}^{n,m}_b \) is exact. Note that \( \mathcal{L}^{0,n}_0(E) = 0 \) and \( \mathcal{L}^{1,n}_1(E) = E \), because \( U_{0,0} = U_{0,n} = \{I_n\} \).

Now for \( b = (b_1, \ldots, b_l) \in \{0,1\}^l \), where \( 1 \leq l \leq n - m + 1 \),
\[
\mathcal{L}^{n,m}_b : \text{Alg}(Q \cap Q_{n-m-l+1,1^{l-1},m}) \to \text{Alg}(Q \cap Q_{n-m-l+1,1^{l-1},m})
\]
is given by
\[
\mathcal{L}^{n,m}_b(E) = \mathcal{L}^{n,m+l-1}_{b_l} \cdots \mathcal{L}^{n,m+1}_{b_2} \mathcal{L}^{n,m}_{b_1}(E).
\]

The representation \( \mathcal{L}^{n,1}_b(E) \) is always a \( Q \cap B_n \)-module. For formal reasons, if \( l = 0 \) (i.e., \( b \) is empty), set \( \mathcal{L}^{n,m}_b(E) = E \).

The aforementioned properties of Jacquet modules imply the following lemma and corollary.

**Lemma 2.1.** Let \( \pi \) and \( \pi' \) be representations of \( B_n \) on the spaces \( E \) and \( E' \) (resp.). As \( B_n \)-modules
\[
(E \otimes E')_{U_n} = s.s. \bigoplus_{b \in \{0,1\}^{n-1}} (\mathcal{L}^{n,1}_b(E) \otimes \mathcal{L}^{n,1}_b(E'))_{U_n}.
\]

**Proof of Lemma 2.1.** Since
\[
(E_{U_{n-1,1}} \otimes E'_{U_{n-1,1}})_{U_{n-1,1}} = E_{U_{n-1,1}} \otimes E'_{U_{n-1,1}}
\]
and
\[
(E(U_{n-1,1}) \otimes E'_{U_{n-1,1}})_{U_{n-1,1}} = E(U_{n-1,1})_{U_{n-1,1}} \otimes E'_{U_{n-1,1}} = 0,
\]
the module \( (E \otimes E')_{U_{n-1,1}} \) is glued from
\[
(E(U_{n-1,1}) \otimes E'(U_{n-1,1}))_{U_{n-1,1}} ; \quad E_{U_{n-1,1}} \otimes E'_{U_{n-1,1}}.
\]
Applying the same argument to both representations and using $U_{n-1,1}U_{n-2,2}$ and [BZ76] (2.32-2.33),

$$(E \otimes E')_{U_{n-1,1}U_{n-2,2}} = \text{s.s.} \bigoplus_{(b_1,b_2) \in \{0,1\}^2} (L_{b_1,b_2}^{n,1}(E) \otimes L_{b_1,b_2}^{n,1}(E'))_{U_{n-1,1}U_{n-2,2}}.$$ 

Proceeding up to $U_{1,n-1}$ yields the result. 

\textbf{Corollary 2.2.} Let $\pi$ and $\pi'$ be representations of $\tilde{B}_n$ on the spaces $E$ and $E'$ (resp.). Assume $0 \leq m \leq n$, $0 \leq l \leq n - m + 1$ and $b \in \{0,1\}^l$ are given. Assume $m \geq 1$. As $B_n$-modules

$$\tag{2.1} (L^{n,m}_b(E) \otimes L^{n,m}_b(E'))_{N_n} \text{ s.s. } \bigoplus_c (L^{n,1}_c(E) \otimes L^{n,1}_c(E'))_{N_n},$$

where $c = (c_1, \ldots, c_{m-1}, b_1, \ldots, b_l, c_m, \ldots, c_{n-l})$ varies over $\{0,1\}^{n-l-1}$. When $m = 0$ we have the following special cases:

- $l = 0$: \text{(2.1)} holds where $c$ varies over $\{0,1\}^{n-1}$,
- $l > 0$, $b_1 = 0$: $L^{n,m}_b(E) = L^{n,m}_b(E') = 0$,
- $l > 0$, $b_1 = 1$: \text{(2.1)} holds and $c = (b_2, \ldots, b_l, c_1, \ldots, c_{n-l})$ varies over $\{0,1\}^{n-l}$.

\textbf{Proof of Corollary 2.2.} Assume $m \geq 1$. First apply Lemma 2.1 to deduce

$$(L^{n,m}_b(E) \otimes L^{n,m}_b(E'))_{N_n} = \text{s.s.} \bigoplus_{d \in \{0,1\}^{n-1}} (L^{n,1}_d L^{n,m}_b(E) \otimes L^{n,1}_d L^{n,m}_b(E'))_{N_n}.$$ 

We claim

$$\tag{2.2} L^{n,1}_d L^{n,m}_b(E) = \begin{cases} L^{n,1}_d(E) & (d_m, \ldots, d_{m+l-1}) = b, \\ 0 & \text{otherwise.} \end{cases}$$

To see this we repeatedly apply the following identities, all derived from the definitions and the fact that $U_{n-k,k} \lhd N_n$ for all $k$. For $x, y \in \{0,1\}$ and $i \neq j$,

- $L^{n,i}_x L^{n,j}_y = L^{n,j}_y L^{n,i}_x$.
- $L^{n,i}_x L^{n,i}_y$ equals $L^{n,i}_x$ if $x = y$, otherwise it vanishes.

Equality \text{(2.2)} clearly implies the result.

The remaining cases of $m = 0$ follow from $L^{n,0}_0(E) = 0$ and $L^{n,0}_1(E) = E$. 

\textbf{2.5. Metaplectic tensor.} Irreducible representations of Levi subgroups of classical groups are usually described in terms of the tensor product. Preimages in $\tilde{GL}_n$ of direct factors of Levi subgroups of $GL_n$, do not commute. Hence the tensor construction cannot be extended in a straightforward manner. The metaplectic tensor has been studied
by several authors [FK86, Sun97, Kab01, Mez04, Tak13], in different contexts.

We briefly recall the tensor construction of Kable [Kab01], whose results will be used throughout. For a Levi subgroup \( M < \text{GL}_n \), let \( M^\circ = \{ m \in M : \det m \in F^{*2} \} \). If \( \pi \) is a representation of \( \widetilde{M} \), denote its restriction to \( \widetilde{M}^\circ \) by \( \pi^\circ \).

For \( i = 1, 2 \), let \( \widetilde{M}_i < \text{GL}_{n_i} \) be a standard Levi subgroup, regarded a subgroup of \( M = \widetilde{M}_1 \times \widetilde{M}_2 < \text{GL}_n \), \( n = n_1 + n_2 \). Consider a pair \( \pi_1 \) and \( \pi_2 \) of genuine irreducible admissible representations of \( \widetilde{M}_1 \) and \( \widetilde{M}_2 \). The subgroups \( \widetilde{M}_1^\circ \) and \( \widetilde{M}_2^\circ \) commute in \( \text{GL}_n \), hence the usual tensor \( \pi_1^\circ \otimes \pi_2^\circ \) is defined and may be regarded as a genuine representation of \( p^{-1}(\widetilde{M}_1^\circ \times \widetilde{M}_2^\circ) \). In fact \( \widetilde{M}_1^\circ \) and \( \widetilde{M}_2 \) also commute, the representation \( \pi_1^\circ \otimes \pi_2 \) (and similarly \( \pi_1 \otimes \pi_2^\circ \)) is defined.

For any character \( \omega \) of \( \mathbb{Z}^e_n \) which coincides with \( \omega_{\pi_1}|_{\mathbb{Z}^e_{n_1}} \otimes \omega_{\pi_2}|_{\mathbb{Z}^e_{n_2}} \) on \( \mathbb{Z}^e_n \), Kable [Kab01] defined the metaplectic tensor \( \pi_1 \tilde{\otimes}_\omega \pi_2 \) as an irreducible summand of

\[
\text{ind}_{p^{-1}((M_1^\circ \times M_2^\circ))}(\pi_1^\circ \otimes \pi_2^\circ),
\]

on which \( \mathbb{Z}^e_n \) acts by \( \omega \). The summand might not be unique, but all such summands are isomorphic ([Kab01] Theorem 3.1).

We mention that the definitions of Kable [Kab01] are more general, and include genuine admissible finite length representations, which admit a central character. In particular for genuine admissible finite length indecomposable representations, the tensor was defined as an indecomposable summand of \( (2.3) \). When starting with irreducible representations, the tensor is irreducible ([Kab01] Proposition 3.3).

A more specific description was given in [Kab01] (Corollary 3.1): if \( n_2 \) is even or \( n_1 \) and \( n_2 \) are odd, there is an irreducible summand \( \sigma \subset \pi_2^\circ \) such that

\[
(2.4) \quad \pi_1 \tilde{\otimes}_\omega \pi_2 = \text{ind}_{p^{-1}(M_1^\circ \times M_2^\circ)}(\pi_1 \otimes \sigma).
\]

If \( n_2 \) is even and \( n_1 \) is odd, \( \sigma \) is uniquely determined by the requirement \( \omega = \omega_{\pi_1} \otimes \omega_\sigma \) on \( \mathbb{Z}^e_n \); if both \( n_2 \) and \( n_1 \) are even, \( \sigma \) is arbitrary; otherwise both are odd and \( \sigma = \pi_2^\circ \). The definition for the remaining case of odd \( n_2 \) and even \( n_1 \) is similar with the roles of \( n_1 \) and \( n_2 \) reversed.

By [Kab01] (Theorem 3.1),

\[
(\pi_1 \tilde{\otimes}_\omega \pi_2)^\circ = \begin{cases} 
[F^* : F^{*2}]\pi_1^\circ \otimes \pi_2^\circ & n_1 \text{ and } n_2 \text{ are odd,} \\
\pi_1^\circ \otimes \pi_2^\circ & \text{otherwise.}
\end{cases}
\]

We need a slightly stronger result.
Claim 2.3. The following holds.

\[
(\pi \widetilde{\otimes} \omega \pi_2)|_{p^{-1}(M^0 \times M_2)} = \begin{cases} 
\pi_1^0 \otimes \pi_2 & \text{even } n_2, \\
\pi_1^0 \otimes \bigoplus_{g \in \overline{M}_2^0 \setminus \overline{M}_2} \chi_g \pi_2 & \text{odd } n_1 \text{ and } n_2, \\
\bigoplus_{g \in \overline{M}_1^0 \setminus \overline{M}_1} \sigma \otimes \chi_g \pi_2 & \text{even } n_1, \text{ odd } n_2.
\end{cases}
\]

Here \(\chi_g\) is the character of \(\overline{M}_2^0 \setminus \overline{M}_2\) given by \(\chi_g(x) = (\det x, \det g)\) and \(\sigma\) is an irreducible summand of \(\pi_1^0\).

Remark 2.1. By [Kab01] (Proposition 3.2), this claim implies the result for \((\pi_1 \widetilde{\otimes} \omega \pi_2)^0\).

Proof of Claim 2.3. The assertions follow from (2.4) by Mackey’s theory. For the first two cases, note that the space \((M^0 \times M_2)\setminus (M_1 \times M_2^0)\) is trivial, \(\text{ind}_{\overline{M}_2^0}^{\overline{M}_2}(\sigma) = \pi_2\) when \(n_2\) is even ([Kab01] Proposition 3.2), and when both \(n_1\) and \(n_2\) are odd, \(\sigma = \pi_1^0\) and \(\text{ind}_{\overline{M}_2^0}^{\overline{M}_2}(\sigma) = \Phi_g \chi_g \pi_2\), where the summation is over \(\overline{M}_2^0 \setminus \overline{M}_2\) ([Kab01] Proposition 3.1). For the last case, we have a sum over \(g \in \overline{M}_2^0 \setminus \overline{M}_1\) of representations \(g(\sigma \otimes \pi_2)\), where now \(\sigma \in \pi_1^0\). Since \(m_1 m_2 = (\det m_1, \det m_2)m_2 m_1\) for \(m_i \in \overline{M}_i\), \(g(\sigma \otimes \pi_2) = g(\sigma \otimes \chi_g \pi_2)\) (see [Kab01] p. 748).

The metaplectic tensor was shown by Kable to satisfy several natural properties. For example, it is associative ([Kab01] Proposition 3.5). If \(U_i \subset \text{GL}_{n_i}\) are unipotent subgroups, \(j_{U_1 U_2} (\pi_1 \widetilde{\otimes} \omega \pi_2) = j_{U_1} (\pi_1) \otimes \omega j_{U_2} (\pi_2)\) ([Kab01] Proposition 4.1, here \(j_{U_1} (\pi_1)\) might not be indecomposable). Note that in contrast with the usual tensor, it is not true in general that \((\pi_1 \widetilde{\otimes} \pi_2)(U_1)(U_2) = \pi_1(U_1) \otimes \omega \pi_2(U_2)\). Indeed, the right-hand side might not be defined (e.g., \(\pi_1(U_1)\) does not necessarily admit a central character). This point complicated our proof of the “only if” part of Theorem 4.5 (see Proposition 4.5 in Section 4) and led to the development of some of the technical results of Section 3.

2.6. Exceptional representations. We describe the exceptional representations introduced and studied by Kazhdan and Patterson [KP84]. Recall the construction of principal series representations of \(\text{GL}_n\). Let \(\xi\) be a genuine character of the center \(\mathcal{T}_n^0 \mathbb{Z}_{\omega^n}\) of \(\mathcal{T}_n\). We extend \(\xi\) to a maximal abelian subgroup of \(\mathcal{T}_n\), then induce to a genuine representation \(\rho(\xi)\) of \(\mathcal{T}_n\), which is irreducible and independent of the particular extension. The corresponding principal series representation is then formed by extending \(\rho(\xi)\) trivially on \(N_n\), then inducing to \(\text{GL}_n\).
The character $\xi$ is called exceptional if $\xi(I_{i-1}, x^2, x^{-2}, I_{n-i-1}) = |x|$ for all $1 \leq i \leq n-1$ and $x \in F^*$. In this case the representation $\text{Ind}_{B_n}^{\text{GL}_n}(\rho(\xi))$ has a unique irreducible quotient $\theta$, called an exceptional representation. The representation $\theta$ is admissible.

The exceptional characters $\xi$ are parameterized in the following manner. Let $\chi$ be a character of $F^*$. Let $\gamma : F^* \to \mathbb{C}^*$ be a mapping such that $\gamma(xy) = \gamma(x)\gamma(y)(x, y)^{|n/2|}$ and $\gamma(x^2) = 1$ for all $x, y \in F^*$. We call such a mapping a pseudo-character. Define

$$\xi_{\chi, \gamma}(\zeta s(zI_n)s(t)) = \zeta \gamma(z) \chi(z^n \det t) \delta_{\mu_2}^{1/4}(t), \quad \zeta \in \mu_2, t \in T_n^2, z \in F^{**}.$$  

Here $s : \text{GL}_n \to \overline{\text{GL}}_n$ is the section of [BLS99] (it is a splitting of $T_n^2$). Of course, when $n$ is even, the choice of $\gamma$ is irrelevant. When $n = 1 (4)$, $\gamma$ is simply a square trivial character of $F^*$. If $n = 3 (4)$, $\gamma = \gamma_\psi$ for some nontrivial additive character $\psi$ of $F$. (The value of the cocycle on $(zI_n, z'I_n)$ is $(-1)^{|n/2|}$.) The corresponding exceptional representation will be denoted $\theta_{n, \chi, \gamma}$. Since $\chi\theta_{n, 1, \gamma} = \theta_{n, \chi, \gamma}$, where on the left-hand side we regard $\chi$ as a character of $\overline{\text{GL}}_n$ via $g \mapsto \chi(\det g)$, we will occasionally set $\chi = 1$. The character $\gamma$ will usually be fixed.

The mapping $\zeta s(zI_n) \mapsto \zeta^\gamma(z)$ is a genuine character of $\overline{\text{Z}}_n^e$. This is precisely the central character $\omega_{\theta_{n, 1, \gamma}}$.

One strong and useful property of exceptional representations, is that the Jacquet functor carries them into exceptional representations of Levi subgroups. In particular $j_{N_n}(\theta_{n, \chi, \gamma}) = \xi_{\chi, \gamma}$, in contrast with the case of general principal series representations, whose Jacquet modules with respect to $N_n$ are of length $n!$.

According to [Kab01] (Theorem 5.1),

$$j_{u_{n, 1}, u_{n, 2}}(\theta_{n_1 + n_2, 1, \gamma}) = \delta_{Q_{n_1, n_2}}^{-1/4} \theta_{n_1, 1, \gamma_1} \overline{\theta}_{n_2, 1, \gamma_2},$$

where $\gamma_1$ and $\gamma_2$ are arbitrary (nontrivial). Written without the normalization of $j_{u_{n, 1}, u_{n, 2}}$,

$$(\theta_{n_1 + n_2, 1, \gamma})_{u_{n_1, u_{n, 2}}} = \delta_{Q_{n_1, n_2}}^{1/4} \theta_{n_1, 1, \gamma_1} \overline{\theta}_{n_2, 1, \gamma_2}. \tag{2.6}$$

Note that in the definition of the metaplectic tensor $\pi_1 \overline{\theta}_{\omega} \pi_2$ (see Section 2.5), $\omega$ was a character of $\overline{\text{Z}}_n^e$ which agrees with $\omega_{\pi_1}|_{\overline{\text{Z}}_{n_1}^2} \otimes \omega_{\pi_2}|_{\overline{\text{Z}}_{n_2}^2}$ on $\overline{\text{Z}}_n^e$. The pseudo-character $\gamma$ is regarded here as the character $\zeta s(zI_n) \mapsto \zeta^\gamma(z)$.

Kazhdan and Patterson [KP84] (Section I.3, see also [BG92] p. 145 and [Kab01] Theorem 5.4) proved that for $n \geq 3$, if $|2| = 1$ in $F$, the exceptional representations do not have Whittaker models. For $n = 3,$
Flicker, Kazhdan and Savin \[FKS90\] (Lemma 6) used global methods to extend this result to the case \(|2| = 1\). It is expected that arguments similar to those of \[FKS90\] will be applicable for \(n > 3\) (see \[FKS90\] Lemma 6 and \[BG92\] p. 138).

3. Filtrations of representations induced to \(Y_n\)

In this section we compute certain filtrations of representations induced to the mirabolic subgroup. The results will be utilized in Section 4 for the proof of Theorem 1. Recall the functors \(\Phi^+\) and \(\Psi^+\) of Bernstein and Zelevinsky \[BZ77\]. We define analogous functors \(\Phi^+_x\) and \(\Psi^+_x\), without the normalization. For representations \(\pi_0\) of \(\text{GL}_{n-2}\) and \(\pi\) of \(Y_{n-1}\),

\[
\Psi^+_x : \text{Alg} \text{GL}_{n-2} \to \text{Alg} Y_{n-1}, \quad \Phi^+_x : \text{Alg} Y_{n-1} \to \text{Alg} Y_n,
\]

\[
\Psi^+_x(\pi_0) = \text{ind}_{\text{GL}_{n-2} U_{n-2,1}}^{Y_{n-1}}(\pi_0), \quad \Phi^+_x(\pi) = \text{ind}_{Y_{n-1} U_{n-1,1}} Y_n(\pi \otimes \psi),
\]

where \(\psi\) is a nontrivial additive character of \(F\), considered also as a character of \(U_{n-1,1}\) by \(\psi(u) = \psi(u_{n-1,n})\). In contrast with \[BZ77\], here the induction is not normalized.

For any \(H < \text{GL}_n\), denote \(H^x = H \cap Y_n\).

3.1. \(B^x_n\)-filtration of \(\Phi^+_x \Psi^+_x\). The results of this section are stated for \(Y_n\), but apply also to \(\overline{Y}_n\). Note that \(\Phi^+_x \Psi^+_x(\pi_0) = \Phi^+ \Psi^+ (|\det| \pi_0)\) and if \(\tau\) is a representation of \(Y_n\) such that \(|\det| \pi_0\) is its second derivative, \(\Phi^+ \Psi^+ (|\det| \pi_0)\) is the second quotient appearing in the filtration of \(\tau\), with respect to its derivatives (see \[BZ77\] 3.5). The results here make no assumption on \(\pi_0\) (except being smooth).

A function \(f\) in the space of \(\Phi^+_x(\pi)\) is determined by its restriction to a set of representatives of \(Y_{n-1} U_{n-1,1} \setminus Y_n \cong F^{n-1} - \{0\}\). This isomorphism extends to a topological isomorphism, where \(F^{n-1} - \{0\}\) is regarded as an open subset of \(F^{n-1}\). For \(0 \neq x \in F^{n-1}\), set

\[
k_x = \min\{i : x_i \neq 0\}
\]

\((1 \leq k_x \leq n - 1)\). We choose a set of representatives \(\Omega = \{\ell(x) : 0 \neq x \in F^{n-1}\}\) as in \[Fli90\], with

\[
\ell(x) = \begin{pmatrix}
I_{k_x-1} & 0 & 0 & 0 \\
0 & 0 & I_{n-k_x-1} & 0 \\
0 & x_{k_x} & (x_{k_x+1}, \ldots, x_{n-1}) & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]

There is a compact subset \(\Omega_0 \subset \Omega\) such that \(f|_\Omega\) vanishes outside of \(\Omega_0\). In particular, the image of \(f|_\Omega\) in the space of \(\pi\) is a finite set and
furthermore, there is a constant $c_f$ such that for any $\ell(x) \in \Omega_0$, $|x_i| > c_f$ for some $i$.

We use this description to compute Jacquet modules and kernels of $\Phi_\phi^*(\pi)$. In general if $U < Y_n$ is a unipotent subgroup, according to the Jacquet-Langlands characterization of the kernel of the Jacquet functor (see e.g. [2Z76, 2.33]), $\Phi_\phi^*(\pi)(U)$ is the space of functions $f \in \Phi_\phi^*(\pi)$, for which there is a compact subgroup $\mathcal{N} < U$, such that

\begin{equation}
\int_{\mathcal{N}} f(\ell(x)v) \, dv = 0, \quad \forall 0 \neq x \in F^{n-1}.
\end{equation}

Also note that if $f \in \Phi_\phi^*(\pi)$, for any $u \in U_{n-1,1}$,

\begin{equation}
u \cdot f(\ell(x)) = f(\ell(x)u) = \psi(xu)f(\ell(x)).
\end{equation}

Here and onward, if $x \in F^l$ and $u \in U_{1,1}$, when we write $\psi(xu)$ we refer to $u$ as a column in $F^l$. For example $U_{n-1,1} = \{ (\ell_{n-1} u) \} \cong F^{n-1}$.

For any open $\Omega_0 \subseteq \Omega$, denote by $\Phi_\phi^{+;\Omega_0}(\pi)$ the subspace of $\Phi_\phi^*(\pi)$ consisting of functions $f$, such that the support of $f|_{\Omega_0}$ is contained in $\Omega_0$. Let $\Omega(j) = \{ \ell(x) \in \Omega : k_x \leq j \}$. Then $\Phi_\phi^{+;\Omega(n-1)}(\pi) = \Phi_\phi^*(\pi)$. For each $j$, $\Phi_\phi^{+;\Omega(j)}(\pi)$ is a $Q_j$-module. To see this note that if $y \in Q_j$ and $k_x > j$, $\ell(x)y = y'\ell(x')$, where $y' \in Q_j \cap Y_n U_{n-1,1}$ and $x' \in F^{n-1}$ satisfies $k_{x'} = k_x$. This follows from the computation

$$
\begin{pmatrix}
0 & I_l \\
x & y
\end{pmatrix}
\begin{pmatrix}
a & b \\
0 & d
\end{pmatrix}
= 
\begin{pmatrix}
d & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
I_l & 0 \\
x & \frac{1}{d} by + d
\end{pmatrix}, \quad x, a \in F^l.
$$

In particular we have the following filtration of $B^\phi_n$-modules,

$$0 \subset \Phi_\phi^{+;\Omega(1)}(\pi) \subset \ldots \subset \Phi_\phi^{+;\Omega(n-1)}(\pi) = \Phi_\phi^*(\pi).$$

For formal reasons, put $\Phi_\phi^{+;\Omega(0)}(\pi) = 0$.

**Example 3.1.** If $n = 4$,

$$\Omega(1) = F^* \times F \times F,$$

$$\Omega(2) = F^* \times F \times F \cup F \times F^* \times F,$$

$$\Omega(3) = F^3 - \{0\}.$$

We will need certain generalizations of $\Phi_\phi^*$. Let $2 \leq i \leq j \leq n$. We define functors

$$E_{j,i}, E_{j,i}^{-} : \text{Alg} Q_{n-j,1^{i-1},j-2} \to \text{Alg} Q_{n-j,1^{i}}.$$ 

The functor $E_{j,i}$ will be used to describe the quotients of the aforementioned filtration, see Claim 3.2 below. For a subgroup $X < \text{GL}_i$
put

\[ E_{j,i}(X) = \left\{ \begin{pmatrix} g & u_1 & u_2 \\ b & u_3 \\ x \end{pmatrix} : g \in \text{GL}_{n-j}, b \in B_{j-i}, x \in X \right\}. \]

Let \( \pi_0 \) be a representation of the parabolic subgroup \( Q_{n-j,1+i-j-2} < \text{GL}_{n-2} \). Regarding \( \text{GL}_{n-2} \) as a subgroup of \( Y_{n-1} \), we can extend \( \pi_0 \) trivially on \( U_{n-2,1} \) and form a representation \( \pi_0 \otimes \psi \) of \( E_{j,i}(Y_{i-1}U_{i-1,1}) = Q_{n-j,1+i-j-2} \ltimes U_{n-2,1,1} \), where \( \psi \) is regarded as a character of \( U_{n-1,1} \), as above. Now consider the induced space

\[ \text{ind}_{E_{j,i}(Y_{i-1}U_{i-1,1})}^{E_{j,i}(Y_{i-1}U_{i-1,1})} (\pi_0 \otimes \psi). \]

This is in particular a \( Q^o_{n-j,1+i-j-2} \)-module. Functions in this space are determined by their restriction to \( Y_{i-1}U_{i-1,1} \setminus Y_i \). Choose a set of representatives as above, \( \Omega^{(i)} = \{ \ell(x) \mid 0 \neq x \in F^{i-1} \} \), then \( \Omega^{(i)}(l) = \{ \ell(x) \in \Omega^{(i)} : k_x \leq l \} \). Set

\[ \mathcal{E}_{j,i}(\pi_0) = \{ f \in \text{ind}_{E_{j,i}(Y_{i-1}U_{i-1,1})}^{E_{j,i}(Y_{i-1}U_{i-1,1})} (\pi_0 \otimes \psi) : f|_{\Omega^{(i)}} \text{ vanishes outside } \Omega^{(i)}(1) \}. \]

This is still a \( Q^o_{n,j,1+j} \)-module, because for a representation \( \pi_i \) of \( Y_{i-1} \), \( \Phi^{+,\Omega(1)}_{*}(\pi_i) \) is a \( B^o_{i} \)-module. Note that if \( i \leq n-1 \), then \( Q_{0,1+i-j-2} = Q_{1,1+i-j-1} \) and

\[ E_{n-1,i}(\pi_0) = \mathcal{E}_{n,i}(\pi_0). \]

By the definition, Equality (3.2) applies also to functions in \( E_{j,i}(\pi_0) \).

Further denote by \( \mathcal{E}^-_{j,i}(\pi_0) \) the representation obtained by the above construction, with \( \psi \) replaced by the trivial character. The motivation for defining \( \mathcal{E}^-_{j,i} \) is that by (3.2),

\[ (\mathcal{E}_{j,i}(\pi) \otimes \mathcal{E}_{j,i}(\pi'))_{Y_{n-1,1}} = \mathcal{E}^-_{j,i}(\pi \otimes \pi'). \]

Clearly \( \mathcal{E}_{j,i} \) and \( \mathcal{E}^-_{j,i} \) are exact. The following claims will be applied repeatedly below.

**Claim 3.1.** For \( 0 \leq m \leq \min(1, n-j) \) and \( b \in \{0, 1\} \), as functors

\[ \text{Alg} Q_{n-j,1+j-i-2} \rightarrow \text{Alg} Q^o_{n-j-m,1+i+j-m-i}, \]

\( L_{b}^{n,j+m} \mathcal{E}_{j,i} = \mathcal{E}_{j+m,i}L_{b}^{n-j+i-m-2} \).

**Proof of Claim 3.1.** Assume \( b = 0 \). We need to prove that for a representation \( \pi_0 \),

\[ \mathcal{E}_{j,i}(\pi_0)(U_{n-j-m,j+m}) = \mathcal{E}_{j+m,i}(\pi_0(U_{n-j-m,j+m-2})). \]

First we show that \( \mathcal{E}_{j,i}(\pi_0)(U_{n-j-m,j+m}) \) consists of the functions \( f \in \mathcal{E}_{j,i}(\pi_0) \) such that \( f|_{\Omega^{(i)}} \) is contained in \( \pi_0(U_{n-j-m,j+m-2}) \).
Indeed, let \( f \in \mathcal{E}_{j,i}(\pi_0(U_{n-j-m,j+m})) \). Then by [2.33], there is a compact subgroup \( N < U_{n-j-m,j+m} \) for which (3.1) holds. As a subgroup of \( E_{j,i}(Y_i) \), \( Y_i \) normalizes \( U_{n-l,l} \) for any \( i \leq l \leq n \). Moreover, the last two columns of \( U_{n-j-m,j+m} \) act trivially on the left, because \( \psi \) is trivial on \( U_{n-j-m,j+m} \cap U_{n-1,1} \) whenever \( j \geq 2 \). Thus for all \( 0 \neq x \in F^{i-1} \),

\[
0 = \int_N f(\ell(x)v)\,dv = \int_{(v)N\cap U_{n-j-m,j+m-2}} \pi_0(v)f(\ell(x))\,dv.
\]

It follows that the image of \( f|_{\Omega^j} \) is contained in \( \pi_0(U_{n-j-m,j+m-2}) \).

Conversely, because \( f|_{\Omega^j} \) is compactly supported, one may choose a large enough compact subgroup \( N < U_{n-j-m,j+m} \) such that (3.1) holds and hence \( f \in \mathcal{E}_{j,i}(\pi_0(U_{n-j-m,j+m})) \).

It follows that restriction of \( f \) to a function on \( E_{j+m,i}(Y_i) \) defines an injection into the right-hand side of (3.1). It is also a bijection, as we now explain (this is clear if \( m = 0 \)).

We use the increasing filtration of \( \mathcal{E}_{j+m,i}(\pi_0(U_{n-j-m,j+m-2})) \) (2.24, see Section 2.3). Let \( f_1 \in \mathcal{E}_{j+m,i}(\pi_0(U_{n-j-m,j+m-2})) \). Taking a small enough compact open subgroup \( \mathcal{V}_1 < E_{j+m,i}(Y_i) \), we can regard \( f_1|_{\Omega^j} \) as a locally constant function on \( F^{i-1} - \{0\} \) such that

\[
f_1(\ell(x)) \in \pi_0(U_{n-j-m,j+m-2})E_{j+m,i}(Y_i^{-1}U_{i-1,1}) \cap \ell(x)\mathcal{V}_1, \quad \forall 0 \neq x \in F^{i-1}.
\]

Select a compact open subgroup \( \mathcal{V} < E_{j,i}(Y_i) \) such that each vector \( f_1(\ell(x)) \) is fixed by \( E_{j,i}(Y_i^{-1}U_{i-1,1}) \cap \ell(x)\mathcal{V} \). Since

\[
E_{j,i}(Y_i^{-1}U_{i-1,1}) \cap \mathcal{V} \cong E_{j+m,i}(Y_i^{-1}U_{i-1,1}) \cap \mathcal{V},
\]

we can use \( \mathcal{V} \) to define \( f \in \mathcal{E}_{j,i}(\pi_0) \) with \( f|_{\Omega^j} = f_1|_{\Omega^j} \). It follows that \( f \) belongs to \( \mathcal{E}_{j,i}(\pi_0)(U_{n-j-m,j+m}) \) and is the preimage of \( f_1 \).

Regarding the case of \( b = 1 \), according to the arguments above, if \( \alpha : \pi_0 \to \pi_0(U_{n-j-m,j+m-2}) \) is the natural projection, the mapping \( \mathcal{E}_{j,i}(\pi_0) \to \mathcal{E}_{j+m,i}(\pi_0(U_{n-j-m,j+m-2})) \) given by \( f \to \alpha f|_{E_{j+m,i}(Y_i)} \) is onto and its kernel is precisely \( \mathcal{E}_{j,i}(\pi_0)(U_{n-j-m,j+m}) \).

\[\square\]

Claim 3.2. For \( 1 \leq j \leq n - 1 \), as \( Q^0_{j-1,n-j+1} \)-modules

\[
\frac{\Phi^*_{\Omega(j)}(\Psi^*(\pi_0))}{\Phi^*_{\Omega(j-1)}(\Psi^*(\pi_0))} = \mathcal{E}_{n-j+1,n-j+1}(\pi_0).
\]

Proof of Claim 3.2. Put \( l = n - j + 1 \). First observe that under the embedding \( Y_i < E_{l,i}(Y_i) < Y_n \),

\[
\Omega(j)(1) = \{ \ell(x) : 0 \neq x \in F^{n-1}, k_x = j \}.
\]
Also in general, restriction of a locally constant compactly supported function on $F^* \times F \cup F \times F^*$ to $\{0\} \times F^*$ is a locally constant compactly supported function. Hence restriction $f \mapsto f|_{E_{l,t}(Y)}$ defines a mapping

$$\Phi^{+,\Omega(j)}_\phi \Psi^+_\xi(\pi_0) \to E_{l,f}(\pi_0)$$

whose kernel is $\Phi^{+,\Omega(j-1)}_\phi \Psi^+_\xi(\pi_0)$.

To show this is onto, we argue as in the proof of Claim 3.1. For $f_1 \in E_{l,t}(\pi_0)$, regard $f_1|_{\Omega(0)}$ as a locally constant function, whose support is contained in $\Omega(l)(1)$, and for $0 \neq x \in F^{l-1}$,

$$f_1(\ell(x)) \in \pi_0 \cap Y_{n-l}U_{n-l,1} \cap \ell(\nu) \nu.$$

Here $\nu < Y_n$ is compact open (in particular $\nu \cap E_{l,t}(Y)$ is compact open). Since $\Omega(l)(1)$ is a closed subgroup of $F^{n-l} \times \Omega(l)(1)$, and the latter is open in $\Omega$, there is a locally constant function on $\Omega$, such that its restriction to $\Omega(l)$ agrees with $f_1|_{\Omega(0)}$. Hence we can define

$$f \in \Phi^{+,\Omega(j)}_\phi \Psi^+_\xi(\pi_0)$$

satisfying $f|_{\Omega(l)} = f_1|_{\Omega(l)}$. 



Here is the main result of this section.

**Lemma 3.3.** Let $\pi_0$ be a representation of $GL_{n-2}$, $n \geq 3$, and let $b \in \{0,1\}^{n-2}$. If $b = 0^{n-2}$, set $k = n - 2$; if $b_1 = 1$, set $k = 0$; otherwise let $k$ be the first index such that $b_k = 0$ and $b_{k+1} = 1$. As a $B^o_n$-module

$$L^{n,2}_b(\Phi^{+}\Psi^+_\xi(\pi_0)) = \text{s.s.} \bigoplus_{m=\min(1,k)}^k E_{n,m+2}L^{n-2,m}_{(0^{k-m},b_{k+1},...,b_{n-2})}(\pi_0).$$

**Proof of Lemma 3.3.** We prove the lemma in three steps.

**Claim 3.4.** For $1 \leq k \leq n-2$, as a $Q^o_{n-k-1,1^{k+1}}$-module $L^{n,2}_b(\Phi^{+}\Psi^+_\xi(\pi_0))$ is glued from

$$\Phi^{+,\Omega(n-k-1)}_\phi \Psi^+_\xi(\pi_0),$$

$$E_{k+1,m+2}L^{n-2,m}_{(0^{k-m},b_{k+1},...,b_{n-2})}(\pi_0), \quad 1 \leq m \leq k-1.$$ 

**Claim 3.5.** If $0 \leq k \leq n-3$, as $Q^o_{n-k-2,1^{k+2}}$-modules

$$L^{n,2}_{(b_{k+1},1)}(\Phi^{+}\Psi^+_\xi(\pi_0)) = \text{s.s.} \bigoplus_{m=\min(1,k)}^k E_{k+2,m+2}L^{n-2,m}_{(0^{k-m},1)}(\pi_0).$$

**Claim 3.6.** For $0 \leq k \leq n-4$ and $\min(1,k) \leq m \leq k$, as $B^o_n$-modules

$$L^{n,k+3}_{(b_{k+2},...,b_{n-2})}E_{k+2,m+2}(L^{n-2,m}_{(0^{k-m},1)}(\pi_0)) = E_{n,m+2}L^{n-2,m}_{(0^{k-m},1,b_{k+2},...,b_{n-2})}(\pi_0).$$
The lemma follows from these claims (proved below). Specifically, for $k = n - 2$ apply Claim 3.4, use $\Phi_\phi^{+;\Omega(1)}\Psi_\phi^+(\pi_0) = \mathcal{E}_{n,n}(\pi_0)$ and (3.3) for $1 \leq m \leq n - 3$. If $k = n - 3$, the result is stated in Claim 3.5 (and again use (3.3)). For $k \leq n - 4$ apply Claims 3.5 and 3.6 note that
\[ \mathcal{L}^{n,2}_b = \mathcal{L}^{n,k+3}_{(b,\ldots,b,n-2)}\mathcal{L}^{n,2}_{(0^k,1)} \]
and these functors are exact.

**Proof of Claim 3.4.** By definition
\[ \mathcal{L}^{n,2}_{0^k}\Phi_\phi^+\Psi_\phi^+(\pi_0) = \bigcup_{m=2}^{k+1} \Phi_\phi^{+;\Omega(n-m,m)}(U_{n,m,m}). \]
Let $2 \leq l \leq n-1$. According to Claim 3.2 (with $j = n-l+1$) and the exactness of taking a Jacquet kernel, as a $Q_{n-l,l}^\circ$-module $\Phi_\phi^{+;\Omega(n-l+1)}(U_{n-l,l})$ is glued from
\[ \Phi_\phi^{+;\Omega(n-l)}\Psi_\phi^+(\pi_0)(U_{n-l,l}), \quad \mathcal{E}_{l,l}(\pi_0)(U_{n-l,l}). \]
If $u \in U_{n-l,l} \cap U_{n-1,1}$ and $k_x \leq n - l$, the character appearing on the right-hand side of (3.2) is a nontrivial function of $u$. Hence
\[ \Phi_\phi^{+;\Omega(n-l)}\Psi_\phi^+(\pi_0)(U_{n-l,l}) = \Phi_\phi^{+;\Omega(n-l)}\Psi_\phi^+(\pi_0). \]
Also by Claim 3.1 (with $j = l$ and $m = b = 0$)
\[ \mathcal{E}_{l,l}(\pi_0)(U_{n-l,l}) = \mathcal{E}_{l,l}(\pi_0(U_{n-l,l-2})). \]
Hence $\Phi_\phi^{+;\Omega(n-l+1)}(U_{n-l,l})$ is glued from
\[ \Phi_\phi^{+;\Omega(n-l)}\Psi_\phi^+(\pi_0), \quad \mathcal{E}_{l,l}(\pi_0(U_{n-l,l-2})). \]
The result follows from a repeated application of this observation and Claim 3.1. \qed

**Proof of Claim 3.3.** Since $\mathcal{L}^{n,2}_{(0^k,1)} = \mathcal{L}^{n,k+2}_{1}\mathcal{L}^{n,2}_0$ and these functors are exact, we can apply $\mathcal{L}^{n,k+2}_1$ to the factors computed by Claim 3.4. As explained in the proof of Claim 3.4 (with $l = k + 2$, $2 \leq l \leq n - 1$ because $0 \leq k \leq n - 3$),
\[ \Phi_\phi^{+;\Omega(n-k-2)}\Psi_\phi^+(\pi_0) = \mathcal{L}^{n,k+2}_0\Psi_\phi^{+;\Omega(n-k-2)}\Psi_\phi^+(\pi_0). \]
Hence
\[ \mathcal{L}^{n,k+2}_1\Phi_\phi^{+;\Omega(n-k-2)}\Psi_\phi^+(\pi_0) = 0 \]
and
\[ L_{1}^{n,k+2} \Phi_{\circ}^{\Omega(n-k-1)}(\pi_{0}) = L_{1}^{n,k+2} \left( \Phi_{\circ}^{\Omega(n-k-1)}(\pi_{0}) \right) \]
\[ = L_{1}^{n,k+2} \mathcal{E}_{k+2,m+2}(\pi_{0}) = \mathcal{E}_{k+2,m+2}L_{1}^{n,k}(\pi_{0}). \]

Here the second equality follows from Claim 3.2 and the third from Claim 3.1. Note that for \( k \leq 1 \) this already gives the result, because then \( k = m \).

For \( 1 \leq m \leq k - 1 \), applying Claim 3.1 and using \( L_{1}^{n,k}L_{0}^{n,k,m} = L_{1}^{n,m} \),
\[ L_{1}^{n,k+2} \mathcal{E}_{k+1,m+2}(L_{0}^{n,k,m}(\pi_{0})) = \mathcal{E}_{k+2,m+2}L_{1}^{n,m}(\pi_{0}). \]

The result follows.

Proof of Claim 3.6. This follows from a repeated application of Claim 3.1.

\[ \square \]

3.2. Jacquet modules of \( \mathcal{E}_{n,i}(\pi_{0}) \). Recall the functor
\[ \mathcal{E}_{n,i} : \text{Alg } Q_{1,n-1,i-2} \rightarrow \text{Alg } B_{n}^{-} \]
defined in Section 3.1. Assume \( n \geq 3 \). We compute \( L_{1}^{n,1} \mathcal{E}_{n,i} \).

We describe an operation of a partial Fourier transform on \( \mathcal{E}_{n,i}(\pi_{0}) \). Let \( S(\Omega^{(i)}(1),\pi_{0}) \) be the space of Schwartz-Bruhat functions on \( \Omega^{(i)}(1) \cong F^{*} \times F^{i-2} \) taking values in the space of \( \pi_{0} \). Let \( f \in \mathcal{E}_{n,i}(\pi_{0}) \). Then \( f_{\Delta} = f|_{\Omega^{(i)}(1)} \in S(\Omega^{(i)}(1),\pi_{0}) \) (see Section 2.3). Consider the partial Fourier transform of \( f_{\Delta} \),
\[ \widetilde{f}_{\Delta}(\ell(x)) = \int_{r_{i-2}} f_{\Delta}(\ell(x_{1},y))y^{-1}(x_{2},\ldots,x_{i-1}) dy. \]

Here \( t^{y} \) denotes the transpose of the row \( y \). Then \( \widetilde{f}_{\Delta} \in S(\Omega^{(i)}(1),\pi_{0}) \).

The mapping \( f \mapsto \widetilde{f}_{\Delta} \) is a linear embedding of \( \mathcal{E}_{n,i}(\pi_{0}) \) in \( S(\Omega^{(i)}(1),\pi_{0}) \), whose image is denoted \( \mathcal{E}_{n,i}^{-}(\pi_{0}) \). This embedding is extended to an embedding of \( B_{n}^{-} \)-modules by defining \( b \cdot \widetilde{f}_{\Delta} = \widetilde{(bf)}_{\Delta} \). When \( i = 2 \), the Fourier transform is trivial and \( \widetilde{f}_{\Delta} = f_{\Delta} \).

We will need more explicit formulas, for the \( B_{n}^{-} \) action on \( \mathcal{E}_{n,i}(\pi_{0}) \) in a few cases. First observe that for any \( \ell(x) \in \Omega^{(i)}(1), 1 \leq l \leq i - 2 \) and \( u \in U_{l,1} < \text{GL}_{l+1} < Y_{i} \),
\[ \ell(x)u = u_{0}\ell(x + (0, \sum_{m=1}^{l} x_{m} u_{m}, 0_{i-2-i})). \]
Here \( u_0 \) is the element in \( U_{i-1,1} \), corresponding to the column obtained from \( u \) by removing its first coordinate.

**Example 3.2.** When \( i = n = 4 \),
\[
\begin{pmatrix}
1 & 1 \\
1 & 1 \\
x_1 & x_2 & x_3 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & v_1 & v_2 \\
v_1 & v_2 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & v_3 \\
v_3 & 1 \\
1 & 1 \\
1 & 1
\end{pmatrix}
\begin{pmatrix}
1 \\
1 \\
x_1 & x_2 + x_1v_1 & x_3 + x_1v_2 + x_2v_3 \\
1
\end{pmatrix}.
\]

Now for \( 0 \neq x \in F^{i-1} \), \( u \in U_{n-i+1,1} \cap Y_i \) with \( 1 \leq l \leq i-2 \), \( t = \text{diag}(I_{n-i}, t_1, \ldots, t_{i-1}, 1) \) and \( b \in B_{n-i} \),
\[
(3.6)
\]
\[ u \cdot \overline{f_\Delta}(\ell(x)) = \psi(x_{l+1}(x_1, \ldots, x_l)u) \pi_0(u_0) \overline{f_\Delta}(\ell(x)), \]
\[
(3.7)
\]
\[ t \cdot \overline{f_\Delta}(\ell(x)) = \prod_{j=2}^{i-1} |t_j|^{-1} \pi_0(\text{diag}(I_{n-i}, t_2, \ldots, t_{i-1})) \overline{f_\Delta}(\ell(t_1x_1, t_2^{-1}x_2, \ldots, t_{i-1}^{-1}x_{i-1})), \]
\[
(3.8)
\]
\[ b \cdot \overline{f_\Delta}(\ell(x)) = \pi_0(b) \overline{f_\Delta}(\ell(x)). \]

Here \( u_0 \) is defined as above and if \( i = 2 \), \( \text{diag}(I_{n-i}, t_2, \ldots, t_{i-1}) = I_{n-2} \).

Equality \((3.6)\) follows from \((3.5)\); \((3.8)\) holds because \( Y_i \) commutes with \( B_{n-i} \).

**Lemma 3.7.** Regard \( (\pi_0)_{N_{i-1}} \) as a \( T_{n-2} \)-module. Then as \( T_n \)-modules,
\[
\mathcal{E}_{n,i}(\pi_0)_{N_{i-1}} \cong \text{ind}_{n-i}^{n} \pi_0(\text{diag}(I_{n-i}, (I_{i-1}^{-1}))), \quad w_i = \text{diag}(I_{n-i}, (I_{i-1}^{-1})).
\]

**Remark 3.1.** Note that \( T_n^n = T_{n-1} \) and \( w^n T_{n-2} = \{ \text{diag}(t_1, \ldots, t_{n-1}, 1, t_{n-i+1}, \ldots, t_{n-2}, 1) \} \).

**Proof of Lemma 3.7.** Regard \( N_{i-1} < \text{GL}_{i-1} \) as a subgroup of \( Y_i \) (embedded in \( E_{n,i}(Y_i) \)). Then
\[
\mathcal{E}_{n,i}(\pi_0)_{N_{i-1}} = (\mathcal{E}_{n,i}(\pi_0)_{U_{1n-i,i-1}})_{N_{i-1}}.
\]

Since \( U_{n-1,1} \) acts trivially on \( \mathcal{E}_{n,i}^-(\pi_0) \), \( \mathcal{E}_{n,i}^-(\pi_0)_{U_{1n-i,i-1}} = \mathcal{E}_{n,i}^-(\pi_0)_{U_{1n-i,i}} \).

Observe that
\[
(3.9) \quad \mathcal{E}_{n,i}^-(\pi_0)_{U_{1n-i,i-2}} = \mathcal{E}_{n,i}^-(\pi_0)_{U_{1n-i,i-2}}.
\]

Indeed, similarly to the proof of Claim 3.1 since \( Y_i \) normalizes \( U_{1n-i,i} \) and \( \pi_0 \) is trivial on the last two columns of \( U_{1n-i,i} \) \( (i \geq 2) \),
\[
(\mathcal{E}_{n,i}^-(\pi_0))(U_{1n-i,i}) = \mathcal{E}_{n,i}^-(\pi_0(U_{1n-i,i-2})).
\]

Then \((3.9)\) holds because \( (\pi_0)_{U_{1n-i,i-2}} \) is still a representation of \( Q_{1n-i,i-2} \) and \( \mathcal{E}_{n,i}^- \) is exact.
Put \( \vartheta = (\pi_0)_{U_{i_1,\ldots,i_r}} \), then

\[
E_{n,i}^-(\pi_0)_{N_{i-1}} \cong \tilde{E}_{n,i}^-(\vartheta)_{N_{i-1}}.
\]

Next we claim

\[
E_{n,i}^-(\pi_0)\left(N_{i-1}\right) = \{ \tilde{f}_\triangle \in \tilde{E}_{n,i}^-(\vartheta) : \tilde{f}_\triangle(\ell(x_1, 0^{i-2})) \in \vartheta(N_{i-2}), \forall x_1 \in F^* \}.
\]

This is trivial when \( i = 2 \) (both sides are equal to zero), assume \( i > 2 \). If \( \tilde{f}_\triangle \in (\tilde{E}_{n,i}^-(\vartheta))(N_{i-1}) \), Equality (3.11) implies that for some compact subgroup \( \mathcal{N} < N_{i-1} \),

\[
\int_\mathcal{N} v \cdot \tilde{f}_\triangle(\ell(x_1, 0^{i-2})) \, dv = 0, \quad \forall x_1 \in F^*.
\]

Since (3.6) shows \( v \cdot \tilde{f}_\triangle(\ell(x_1, 0^{i-2})) = \vartheta(v_0)\tilde{f}_\triangle(\ell(x_1, 0^{i-2})) \) and if \( v \) varies in a large compact subgroup of \( N_{i-1} \), \( v_0 \) varies in a large compact subgroup of \( N_{i-2} \), we see that \( \tilde{f}_\triangle \) belongs to the right-hand side of (3.11).

In the other direction, let \( \mathcal{N} < N_{i-1} \) be compact such that (3.11) holds. Let \( m_0 \) be such that the support of \( \tilde{f}_\triangle \) in \( x_1 \) is contained in \( \{ x \in F^* : q^{-m_0} < |x| < q^{m_0} \} \), where \( q \) is the residual characteristic of the field. Select a large enough \( m \), with respect to \( \tilde{f}_\triangle \), \( \mathcal{N} \) and \( m_0 \), such that if \( q^{-m_0} < |x_1| < q^{m_0} \), \( |x_2| \leq q^{-m}, \ldots, |x_{i-1}| \leq q^{-m} \), then

\[
\tilde{f}_\triangle(\ell(x_1, \ldots, x_{i-1})) = \tilde{f}_\triangle(\ell(x_1, 0^{i-2})),
\]

\[
\psi(x_{i+1}(x_1, \ldots, x_i) v_i) = 1, \quad \forall v = v_1 \ldots v_{i-2} \in \mathcal{N}, \quad v_i \in U_{n-i+1,1}, \quad 1 \leq l \leq i - 2.
\]

Note that if \( v \in \mathcal{N} \), we can write uniquely \( v = v_1 \ldots v_{i-2} \) with \( v_i \in U_{n-i+1,1} \), then the coordinates of \( v_i \) are bounded from above by a constant depending on \( \mathcal{N} \) and \( i \). Next take a compact \( \mathcal{N} < \mathcal{N}_1 < N_{i-1} \) such that if \( q^{-m_0} < |x_1| < q^{m_0} \) and \( |x_{i+1}| > q^{-m} \) for some \( 1 \leq l \leq i - 2 \),

\[
\int_\mathcal{N} \psi(x_{i+1} x_1 a) \, da = 0,
\]

where \( a \) varies over any nontrivial coordinate of \( \mathcal{N}_1 \).

We show that for all \( 0 \neq x \in F^{i-1} \),

\[
\int_{N_{i-1}} v \cdot \tilde{f}_\triangle(\ell(x)) \, dv = 0.
\]

Observe that by (3.6), if \( \tilde{f}_\triangle(\ell(x)) = 0 \), \( v \cdot \tilde{f}_\triangle(\ell(x)) = 0 \) for all \( v \in N_{i-1} \). Hence (3.12) holds unless \( q^{-m_0} < |x_1| < q^{m_0} \). If \( |x_2|, \ldots, |x_{i-1}| \leq q^{-m} \), Equality (3.6) implies

\[
v \cdot \tilde{f}_\triangle(\ell(x)) = v \cdot \tilde{f}_\triangle(\ell(x_1, 0^{i-2})), \quad \forall v \in \mathcal{N}.
\]
Hence by (3.11),
\[ \int_N v \cdot \mathcal{F}_\Delta(\ell(x)) \, dv = 0 \]
and (3.12) follows. Otherwise \( |x_{i+1}| > q^{-m} \) for some \( 1 \leq l \leq i - 2 \) and then
\[ \int_{N_t \cap U_{n-i+l,1} \cap U_{n-i+1,1}} v \cdot \mathcal{F}_\Delta(\ell(x)) \, dv = \mathcal{F}_\Delta(\ell(x)) \int_{N_t \cap U_{n-i+l,1} \cap U_{n-i+1,1}} \psi(x_{i+1} x_1 v) \, dv = 0. \]

We conclude that (3.12) holds for all \( x \).

Now consider the function
\[ f_1(t) = t \cdot \mathcal{F}_\Delta(\ell(1, 0^{i-2})) + \vartheta(N_{i-2}), \quad t \in T_n. \]

Equalities (3.7), (3.8) and (3.10) imply that \( \mathcal{F}_\Delta \mapsto f_1 \) is an isomorphism between \( \mathcal{E}_{n, \tau}^e(\vartheta)_{N_{i-1}} \) and \( \text{ind}_{\Gamma_{T_n}^e}^{\Gamma_{T_{n-2}}^e}(\delta_{Y_1}^{-1} u_i(\vartheta_{N_{i-2}})) \). The latter is isomorphic to \( \text{ind}_{\Gamma_{T_n}^e}^{\Gamma_{T_{n-2}}^e}(\delta_{Y_1}^{-1} u_i(\pi_0)_{N_{n-2}})) \). \( \square \)

4. Distinguished representations

4.1. Definitions. Let \( \tau \) be an admissible representation of \( \text{GL}_n \) with a central character \( \omega_\tau \). Let \( \chi \) and \( \chi' \) be characters of \( F^* \) and let \( \gamma \) and \( \gamma' \) be a pair of pseudo-characters (see Section 2.6). We say that \( \tau \) is \( (\chi, \gamma, \chi', \gamma') \)-distinguished if
\[ \text{Hom}_{\text{GL}_n}(\theta_{n, \chi, \gamma} \otimes \theta_{n, \chi', \gamma'}, \tau) \neq 0. \]

Here \( \theta_{n, \chi, \gamma} \otimes \theta_{n, \chi', \gamma'} \) is regarded as a representation of \( \text{GL}_n \), as explained in Section 2.2. Equivalently, the space
\[ \text{Tri}_{\text{GL}_n}(\tau, \theta_{n, \chi, \gamma}, \theta_{n, \chi', \gamma'}) \]
of \( \text{GL}_n \)-invariant trilinear forms on \( \tau \times \theta_{n, \chi, \gamma} \times \theta_{n, \chi', \gamma'} \) is nonzero.

If \( \tau \) is \( (\chi, \gamma, \chi', \gamma') \)-distinguished, in particular \( \omega_\tau^{-1} = \theta_{n, \chi, \gamma} \theta_{n, \chi', \gamma'} \) on \( Z_n^e \), and according to (2.25),
\[ \omega_\tau^{-1}(z I_n) = \chi(z^n) \chi'(z^n) \gamma(z) \gamma'(z), \quad \forall z \in F^{*e}. \]

The next claim implies that the appearance of the pseudo-characters in the definition is redundant and furthermore, if we fix \( \gamma, \gamma' \) will be determined by (4.1).

Claim 4.1. Assume that \( \tau \) is \( (\chi, \gamma_0, \chi', \gamma'_0) \)-distinguished. Then for any \( \gamma \), Equality (4.1) determines a set of pseudo-characters \( \gamma' \) such that \( \tau \) is \( (\chi, \gamma, \chi', \gamma') \)-distinguished.

Proof of Claim 4.1. This is trivial when \( n \) is even. In the odd case let \( \gamma_0 \) be given. Then \( \eta = \gamma / \gamma_0 \) is a square trivial character of \( F^* \). Define \( \gamma' = \eta \gamma_0 \), it is a pseudo-character. Then since \( \eta(z) = \eta^n(z) \) \( (n \) is odd),
\[ \theta_{n, \chi, \gamma} = \theta_{n, \chi, \eta \gamma_0} = \theta_{n, \eta \chi, \gamma_0} = \eta \theta_{n, \chi, \gamma_0} \]
and
\[ \theta_{n,X,\gamma} \otimes \theta_{n,X',\gamma'} = \eta^2 \theta_{n,X,\gamma} \otimes \theta_{n,X',\gamma'} = \theta_{n,X,\gamma} \otimes \theta_{n,X',\gamma'}. \]

This implies that \( \tau \) is \((\chi, \gamma, \chi', \gamma')\)-distinguished. \( \square \)

Additionally, because \( \theta_{n,X,\gamma} = \chi \theta_{n,1,\gamma} \), we see that \( \tau \) is \((\chi, \gamma, \chi', \gamma')\)-distinguished if and only if \( \tau_0 = \chi' \tau \) is \((1, \gamma, 1, \gamma')\)-distinguished. Thus we can simply take \( \chi = \chi' = 1 \).

In light of the observations above, we say that \( \tau \) is distinguished if it is \((1, \gamma, 1, \gamma')\)-distinguished for some \( \gamma \) and \( \gamma' \).

As an example consider the minimal case of \( n = 1 \).

**Claim 4.2.** For \( n = 1 \), \( \tau \) is distinguished if and only if \( \tau^2 = 1 \).

**Proof of Claim 4.2.** Assume that \( \tau \) is distinguished. Then \( \tau^2 = 1 \) follows from (1.1) because \( \gamma^2 \gamma'^2 = \gamma^4 = 1 \) \((\gamma' / \gamma)\) is a square trivial character hence \( \gamma^2 = \gamma^2 \), \( \gamma^4 = 1 \) because \( \gamma(x^4) = \gamma(x^2) \gamma(x^2) (x^2, x^2) \). Conversely, for a fixed \( \gamma \), \( z \mapsto \gamma(z) \gamma'(z) \) is a character of \( F^{*2}/F^* \) and any such character is obtained by varying \( \gamma' \). Hence \( \omega_{\gamma}^{-1} = \gamma \gamma' \) for some \( \gamma' \) and \( \tau \) is distinguished. \( \square \)

The proof implies the following observation, for any \( n \) (already noted in [Kab01] p. 766).

**Corollary 4.3.** If \( \tau \) is distinguished, \( \omega_{\tau}^2 = 1 \).

### 4.2. Heredity.

We now prove the upper heredity of distinguished representations. The proof is based on the following observation: given an invariant form on a representation \( \xi \) of a Levi subgroup, one can construct an invariant form on a representation parabolically induced from \( \xi \) (assuming a certain condition on modulus characters). One complication here, is that we do not have invariancy with respect to a full Levi subgroup. We compensate for this using the properties of the metaplectic tensor product (see Section 2.5).

**Proof of Theorem 5.** Let \( \tau_1 \) and \( \tau_2 \) be a pair of distinguished representations of \( \text{GL}_{n_1} \) and \( \text{GL}_{n_2} \). We have to prove that \( \tau = \text{Ind}_{Q_{n_1,n_2}^{*n_1,n_2}}^{\text{GL}_{n_1} \times \text{GL}_{n_2}^{*n_1,n_2}} (\tau_1 \otimes \tau_2) \) is distinguished, with \( n = n_1 + n_2 \).

We may assume that either \( n_1 \) and \( n_2 \) have the same parity, or \( n_2 \) is even (if \( n_1 \) is even and \( n_2 \) is odd, the argument will be repeated with their roles replaced). Let \( Q_{n_1,n_2}^{*n_1,n_2} = M_{n_1,n_2}^{*n_1,n_2} \ltimes U_{n_1,n_2} \), where

\[ M_{n_1,n_2}^{*n_1,n_2} = \text{GL}_{n_1} \times \text{GL}_{n_2}^{\ominus}, \]

The assumptions on \( \tau_i \) imply that for suitable pairs of pseudo-characters \( (\gamma_i, \gamma'_i) \), \( i = 1, 2 \),

\[ \text{Tri}_{M_{n_1,n_2}^{*n_1,n_2}} (\tau_1 \otimes \tau_2, \theta_{n_1,1,\gamma_1} \otimes (\theta_{n_2,1,\gamma_2})^\ominus, \theta_{n_1,1,\gamma'_1} \otimes (\theta_{n_2,1,\gamma'_2})^\ominus) \neq 0. \]
Let $\sigma$ (resp. $\sigma'$) be an irreducible summand of $\theta_{n_2,\gamma_2}^{\Box}$ (resp. $\theta_{n_2,\gamma_2}^{\Box'}$), such that

$$\text{Tri}_{M_{n_1,n_2}} (\tau_1 \otimes \tau_2; \theta_{n_1,1,\gamma_1} \otimes \sigma, \theta_{n_1,1,\gamma_1'} \otimes \sigma') \neq 0.$$  \hfill (4.2)

If $n_2$ is odd $\sigma = \theta_{n_2,1,\gamma_2}^{\Box}$ and $\sigma' = \theta_{n_2,1,\gamma_2}^{\Box'}$ ([Kab01] Proposition 3.1).

According to (2.4), there exists a pseudo-character $\gamma$ with

$$\theta_{n_1,1,\gamma_1} \otimes \gamma \theta_{n_2,1,\gamma_2} = \text{ind}_{\bar{M}_{n_1,n_2}}^{M_{n_1,n_2}} (\theta_{n_1,1,\gamma_1} \otimes \sigma).$$  \hfill (4.3)

Indeed, this is clear if $n_1$ and $n_2$ have the same parity (e.g., if both are even, any summand $\sigma$ is suitable for any $\gamma$). Regarding the last case of odd $n_1$ and even $n_2$, first note that $\sigma$ admits a character on $\bar{Z}_{n_2}$ ($\bar{Z}_{n_2}$ is abelian and central in $\text{GL}_{n_2}^\Box$), which is trivial on $\text{Gal}(\bar{Z}_{n_2})$ (because $\omega_{\theta_{n_2,1,\gamma_2}^{\Box}}(\text{Gal}(\bar{Z}_{n_2})) = 1$). Hence the mapping $\gamma_3(z) = \sigma(\text{Gal}(z))$ is a pseudo-character. Therefore, in this case $\gamma$ is determined by the condition $\gamma = \gamma_1\gamma_3$ (this is a pseudo-character because here $[n_1/2] + [n_2/2] = [n/2]$).

Similarly, for some $\gamma'$,

$$\theta_{n_1,1,\gamma_1'} \otimes \gamma' \theta_{n_2,1,\gamma_2'} = \text{ind}_{\bar{M}_{n_1,n_2}}^{M_{n_1,n_2}} (\theta_{n_1,1,\gamma_1'} \otimes \sigma').$$

Applying Frobenius reciprocity to (2.6) and using (4.3), we see that $\theta_{n_1,1,\gamma}$ is a subrepresentation of

$$\text{Ind}_{Q_{n_1,n_2}}^{\text{GL}_n} (\delta_{Q_{n_1,n_2}}^{1/4} \text{ind}_{\bar{M}_{n_1,n_2}}^{M_{n_1,n_2}} (\theta_{n_1,1,\gamma_1} \otimes \sigma)),$$  \hfill (4.4)

which is isomorphic to

$$\text{ind}_{Q_{n_1,n_2}}^{\text{GL}_n} (\delta_{Q_{n_1,n_2}}^{1/4} (\theta_{n_1,1,\gamma_1} \otimes \sigma)).$$  \hfill (4.5)

Note that the induction from $\bar{Q}_{n_1,n_2}^*$ is not normalized. The isomorphism is given by $\varphi_0 \mapsto \varphi$ where $\varphi(g) = \varphi_0(g)(1)$ ($\varphi_0$ in (4.4)).

Regard an element $\varphi$ in the space of $\theta_{n,1,\gamma}$ as a function in (4.5). Specifically, $\varphi$ is a function on $\text{GL}_n$, compactly supported modulo $\bar{Q}_{n_1,n_2}^*$, taking values in the space of $\theta_{n_1,1,\gamma_1} \otimes \sigma$, and satisfying for $m \in \bar{M}_{n_1,n_2}^*$, $u \in U_{n_1,n_2}$ and $g \in \text{GL}_n$,

$$\varphi(\text{mug}) = \delta_{Q_{n_1,n_2}}^{1/4} (m)(\theta_{n_1,1,\gamma_1} \otimes \sigma)(m)\varphi(g).$$

Similar properties hold for $\varphi'$ in the space of $\theta_{n,1,\gamma'}$.

Let $L \neq 0$ belong to (4.2) and take $f$ in the space of $\tau$. Then for $q \in Q_{n_1,n_2}^*$,

$$L(f(q), \varphi(q), \varphi'(q)) = \delta_{Q_{n_1,n_2}}(q)L(f(1), \varphi(1), \varphi'(1)).$$
Hence the following integral is (formally) well defined (see e.g. [BZ76, 1.21]),

$$T(f, \varphi, \varphi') = \int_{Q_{n_1,n_2}^* \setminus \text{GL}_n} L(f(g), \varphi(g), \varphi'(g)) \, dg.$$ 

It is absolutely convergent because according to the Iwasawa decomposition, it is equal to

$$\int_K \int_{Q_{n_1,n_2}^* \setminus Q_{n_1,n_2}} L(f(qk), \varphi(qk), \varphi'(qk)) \delta^{-1}_{Q_{n_1,n_2}}(q) \, dq \, dk,$$

where $K$ is the hyperspecial compact open subgroup of $\text{GL}_n$.

Since $T \in \text{Tr}GL_n(\tau, \theta_{n_1,\gamma}, \theta_{n_1,\gamma'})$, it is left to show $T \neq 0$. Assume $L(x, y, y') \neq 0$ for corresponding data $x, y$ and $y'$. Take $f$ supported on $Q_{n_1,n_2} \mathcal{V}$, where $\mathcal{V}$ is a small compact open neighborhood of the identity in $\text{GL}_n$, and

$$f(muv) = \delta^{1/2}_{Q_{n_1,n_2}}(m)(\tau_1 \otimes \tau_2)(m)x, \quad \forall m \in M_{n_1,n_2}, u \in U_{n_1,n_2}, v \in \mathcal{V}.$$ 

Next we show that there is $\varphi$ in the space of $\theta_{n_1,\gamma}$ with

$$\varphi(m) = \begin{cases} \delta^{1/4}_{Q_{n_1,n_2}}(m)(\theta_{n_1,1,\gamma_1} \otimes \sigma)(m)y & m \in \hat{M}_{n_1,n_2}^*, \\ 0 & m \in \hat{M}_{n_1,n_2} - \hat{M}_{n_1,n_2}^*. \end{cases}$$

Indeed, take $\varphi_0$ in the space of $\theta_{n_1,\gamma}$ as a subrepresentation of (4.3), such that $\varphi_0(1) \neq 0$. Let $\beta_y$ be the function in the space of $\text{ind}_{\hat{M}_{n_1,n_2}^*}(\theta_{n_1,1,\gamma_1} \otimes \sigma)$, which vanishes on $\hat{M}_{n_1,n_2} - \hat{M}_{n_1,n_2}^*$ and for $m \in \hat{M}_{n_1,n_2}^*$, $\beta_y(m) = (\theta_{n_1,1,\gamma_1} \otimes \sigma)(m)y$. Because $\theta_{n_1,1,\gamma_1}$ and $\theta_{n_2,1,\gamma_2}$ are irreducible, so is their metaplectic tensor, whence by (4.3), $\text{ind}_{\hat{M}_{n_1,n_2}^*}(\theta_{n_1,1,\gamma_1} \otimes \sigma)$ is irreducible. Hence there is $m_0 \in M_{n_1,n_2}$ such that $\varphi_0(m_0) = \beta_y$. Let $\varphi$ be the image in (4.5) of $m_0 \varphi_0$. Then $\varphi$ satisfies (4.3).

The same argument applies to $\theta_{n_1,\gamma'}$ and denote the corresponding function by $\varphi'$, it satisfies (4.3) with respect to $y'$ (and $\theta_{n_1,1,\gamma'_1} \otimes \sigma'$).

Using (4.3) and $Q_{n_1,n_2} \mathcal{V} \cap K = (Q_{n_1,n_2} \cap K) \mathcal{V}$,

$$T(f, \varphi, \varphi') = \int_{(Q_{n_1,n_2} \cap K) \mathcal{V}} \int_{Q_{n_1,n_2} \setminus Q_{n_1,n_2}^*} L(f(qk), \varphi(qk), \varphi'(qk)) \delta^{-1}_{Q_{n_1,n_2}}(q) \, dq \, dk.$$ 

For a sufficiently small $\mathcal{V}$ (with respect to $\varphi$ and $\varphi'$), the $dk$-integration may be ignored. Using the fact that $Q_{n_1,n_2}^* \setminus Q_{n_1,n_2} = M_{n_1,n_2}^* \setminus M_{n_1,n_2}$ is finite, we obtain (up to a nonzero measure constant)

$$\sum_{m \in M_{n_1,n_2}^* \setminus M_{n_1,n_2}} L(f(m), \varphi(m), \varphi'(m)) \delta^{-1}_{Q_{n_1,n_2}}(m).$$
According to (4.8) this equals \( L(x, y, y') \) which is nonzero. We conclude that \( \tau \) is distinguished. \[\square\]

4.3. **Combinatorial characterization.** We characterize distinguished principal series representations in terms of their inducing data. We start with the case of \( \text{GL}_2 \), which in fact can be reproduced from the results of Savin \cite{Sav92} for \( \text{GL}_3 \). The argument is provided, for completeness and because this case will be used as a base case in the course of proving Theorem 4.

**Claim 4.4.** Let \( \tau \) be a principal series representation of \( \text{GL}_2 \), induced from the character \( \eta_1 \otimes \eta_2 \) of \( T_2 \). Then \( \tau \) is distinguished if and only if \( \eta_1^2 = \eta_2^2 = 1 \) or \( \eta_2 = \eta_1^{-1} \).

**Proof of Claim 4.4.** Put \( \theta = \theta_{2,1,\gamma^\psi} \) and \( \theta' = \theta_{2,1,\gamma_{\psi^{-1}}} \). By the Frobenius reciprocity,

\[
(4.9) \quad \text{Hom}_{\text{GL}_2}(\theta \otimes \theta', \tau^\vee) = \text{Hom}_{T_2}((\theta \otimes \theta')_{N_2}, \delta_{B_2}^{1/2}(\eta_1^{-1} \otimes \eta_2^{-1})).
\]

Assume that \( \eta_1 \otimes \eta_2 \) is of the prescribed form, we prove that \( \tau \) is distinguished. If \( \eta_1^2 = \eta_2^2 = 1 \), then \( \eta_1 \) and \( \eta_2 \) are distinguished and the result follows from Theorem 3. Now assume \( \eta_2 = \eta_1^{-1} \) and \( \eta_1^2 \neq 1 \).

We use the geometric realization of \( \theta \) given in \cite{Sav92, FKS90, Fli90}. Let \( \mathcal{S}(F) \) be the space of Schwartz-Bruhat functions on \( F \). Let \( C_2 \subset \mathcal{S}(F) \) be the subspace of functions \( f \), for which there is a constant \( A_f > 0 \) satisfying \( f(x) = 0 \) for \( |x| > A_f \) and \( f(y^2x) = f(x) \) for all \( x, y \) with \( |y^2x|, |x| < A_f^{-1} \). Also for \( m \in \mathbb{Q} \), let \( C_2^m \) denote the space of functions \( f \) for which \( |\cdot|^m f \in C_2 \). The representation \( \theta \) can be realized on \( C_2^{1/4} \). In particular, the action of \( N_2 \) is given by \( (1 \, \chi) \cdot f(x) = \psi(ux)f(x) \) (\cite{Sav92} p. 372).

Following the arguments of Savin \cite{Sav92} (proof of Proposition 6), one sees that \( (\theta \otimes \theta')_{N_2} \) is embedded in \( C_2^{1/2} \), and under this embedding \( (\theta(N_2) \otimes \theta'(N_2))_{N_2} = \mathcal{S}(F^*) \). If \( f \in C_2^{1/2} \) is the image of an element from \( (\theta \otimes \theta')_{N_2} \), the action of \( T_2 \) is given by

\[
diag(a, b) \cdot f(x) = |ab^{-1}|f(xab^{-1})
\]

(see \cite{Sav92} p. 372 and use \( \gamma_{\psi^{-1}} = \gamma_{\psi}^{-1} \)). According to \cite{Sav92} (Proposition 4), for a character \( \mu \) of \( F^* \) such that \( \mu^2 \neq 1 \), the nontrivial functional on \( \mathcal{S}(F^*) \) given by

\[
f \mapsto \int_{F^*} f(x)\mu(x^{-1})d^* x
\]
extends to \( C_2 \). Since for \( f \in C_2^{1/2}, |.|^{1/2} f \in C_2 \), and \( \eta_2 \neq 1 \), it follows that
\[
f \mapsto \int_{F^*} |x|^{1/2} f(x) \eta_1^{-1}(x^{-1}) d^* x
\]
defines a functional in
\[
\text{Hom}_{T_2}((\theta \otimes \theta')_{N_2}, \delta_{B_2}^{1/2}(\eta_1^{-1} \otimes \eta_1)),
\]
which is nontrivial because it does not vanish on the subspace \((\theta(N_2) \otimes \theta'(N_2))_{N_2} \). Looking at (4.9) we see that \( \tau \) is distinguished.

In the other direction assume that \( \tau \) is distinguished. By virtue of Lemma 2.1 either
\[
\text{Hom}_{T_2}(\theta_{N_2} \otimes \theta'_{N_2}, \delta_{B_2}^{1/2}(\eta_1^{-1} \otimes \eta_2^{-1})) \neq 0
\]
or
\[
\text{Hom}_{T_2}((\theta(N_2) \otimes \theta'(N_2))_{N_2}, \delta_{B_2}^{1/2}(\eta_1^{-1} \otimes \eta_2^{-1})) \neq 0.
\]
In the former case, by (2.6) we have
\[
\text{Hom}_{T_2}((\theta_{1,1,1,1} \otimes \eta_1^{-1} \eta_1^{-1}), \delta_{B_2}^{1/2}(\eta_1^{-1} \otimes \eta_2^{-1})) \neq 0.
\]
In particular when restricting to \( T_2^2 \) we obtain (see Section 2.5)
\[
\text{Hom}_{T_2^2}((\theta_{1,1,1} \otimes \theta_{1,1,1}^2), \delta_{B_2}^{1/2}(\eta_1^{-1} \otimes \eta_2^{-1})) \neq 0,
\]
whence \( \eta_2^2 = \eta_2 \).

In the latter case
\[
\text{Hom}_{T_2}(\mathcal{S}(F^*), \delta_{B_2}^{1/2}(\eta_1^{-1} \otimes \eta_2^{-1})) \neq 0.
\]
The action of \( T_2 \) on \( \mathcal{S}(F^*) \) was given above and immediately implies \( \eta_1^{-1} = \eta_2 \).

\[ \Box \]

**Remark 4.1.** If \( \eta_1 \) is unramified and \( |2| = 1 \), the fact that \( \text{Ind}_{B_2}^{\text{GL}_2}(\eta_1 \otimes \eta_1^{-1}) \) is distinguished follows as a particular case from a result of Kable ([Kab02, Theorem 5.4]).

Let \( \eta = \eta_1 \otimes \ldots \otimes \eta_n \) be a character of \( T_n \). We say that \( \eta \) satisfies condition (\( \dagger \)) if, up to a permutation of the characters \( \eta_i \), there is \( 0 \leq k \leq [n/2] \) such that
\begin{itemize}
  \item \( \eta_{2i} = \eta_{2i-1}^{-1} \) for \( 1 \leq i \leq k \),
  \item \( \eta_i^0 = 1 \) (i.e., \( \eta_i \) is distinguished) for \( 2k + 1 \leq i \leq n \).
\end{itemize}
Claims 4.2 and 4.3 state that for \( n = 1, 2 \), the principal series representation induced from \( \eta \) is distinguished if and only if \( \eta \) satisfies (\( \dagger \)). The main goal of this work is to extend this to irreducible principal series representations of \( \text{GL}_n \), for arbitrary \( n \). Henceforth we proceed under
the mild assumption that for $n > 3$, an exceptional representation does not have a Whittaker model (see Section 2.6).

**Remark 4.2.** This assumption is only needed in Lemma 4.7 below, used by Proposition 4.5: in that lemma we use the fact that the second derivative of an exceptional representation is its highest nonzero derivative ([Kab01] Theorems 5.3 and 5.4), this is true if and only if this representation has no Whittaker model.

**Proposition 4.5.** Let $\eta$ be a character of $T_n$. Assume that for some $c \in \{0,1\}^{n-1}$,

$$
\text{Hom}_{T_n}(\left( L_c^{n,1} \theta_{n,1,\gamma} \otimes L_c^{n,1} \theta_{n,1,\gamma'} \right)_{N_n}, \delta B_n^{1/2} \eta) \neq 0.
$$

(4.10)

Then $\eta$ satisfies ($\ast$).

The proof is given below. Now we prove Theorem 1. Namely, let $\tau$ be a principal series representation of $\text{GL}_n$ induced from $\eta$. If $\tau$ is distinguished, $\eta$ satisfies ($\ast$). Conversely, if $\tau$ is irreducible and $\eta$ satisfies ($\ast$), then $\tau$ is distinguished.

**Proof of Theorem 1** Assume that $\tau$ is distinguished. According to the Frobenius reciprocity, for some $\gamma$ and $\gamma'$,

$$
\text{Hom}_{T_n}(\left( \theta_{n,1,\gamma} \otimes \theta_{n,1,\gamma'} \right)_{N_n}, \delta B_n^{1/2} \eta^{-1}) \neq 0.
$$

(4.11)

Lemma 2.1 implies that the condition of Proposition 4.5 holds and the result follows from the proposition.

In the other direction, if $\eta$ satisfies ($\ast$), then since $\tau$ is irreducible, permuting the inducing data does not change $\tau$. Hence we may assume that $\tau$ is induced from

$$(\eta_1 \otimes \eta_1^{-1}) \otimes \ldots \otimes (\eta_k \otimes \eta_k^{-1}) \otimes \eta_{2k+1} \otimes \ldots \otimes \eta_n,$$

where $0 \leq k \leq \lfloor n/2 \rfloor$ and for $i > 2k$, $\eta_i^2 = 1$. Now Claims 4.2 and 4.4 and Theorem 3 imply that $\tau$ is distinguished.

**Theorem 2** - the characterization of distinguished spherical representations of $\text{GL}_n$, is easily seen to follow from Theorem 1 and the description of the tautological lift (see e.g. [Kab02] Section 6).

**Proof of Theorem 2** The “if” part was proved by Kable [Kab02]. If $\tau$ is the irreducible unramified quotient of $\text{Ind}_{B_n}^{\text{GL}_n}(\eta)$ and $\tau$ is distinguished, for some $\gamma$ and $\gamma'$,

$$
\text{Hom}_{\text{GL}_n}(\theta_{n,1,\gamma} \otimes \theta_{n,1,\gamma'}, \text{Ind}_{B_n}^{\text{GL}_n}(\eta^{-1})) \neq 0.
$$
Hence by Theorem 1 the character $\eta$ satisfies (⋆). Since there are exactly two unramified square trivial characters of $F^*$, we may assume, perhaps after applying a permutation, that $\eta$ takes the form

$$\eta_1 \otimes \ldots \otimes \eta_{[n/2]} \otimes \eta_0 \otimes \eta_{[n/2]} \otimes \ldots \otimes \eta_{1}^{-1}.$$  

Here the character $\eta_0$ appears only when $n$ is odd. Because we assumed $\omega_\tau = 1$, $\eta_0 = 1$. This implies that $\tau$ is the lift of a representation of $SO_{2[n/2]}$ in the even case, or $Sp_{2[n/2]}$ when $n$ is odd ([Kab02] Section 6).

Proof of Proposition 4.3. The idea is to reduce the proof to a computation on $\tilde{Y}_n$ (or $Y_n$)-modules. This is possible if $c_1 = 0$. Then we appeal to the results of Section 3. Specifically, the $\tilde{B}_n^0$-modules $\mathcal{L}^{n,1}_c \theta_{n,1,\gamma}$ and $\mathcal{L}^{n,1}_c \theta_{n,1,\gamma'}$ are described using Lemma 3.3, the Jacquet module of their tensor, which is a representation of $B_n^0$, is analyzed using Lemma 3.7. Then $\eta$ becomes a quotient of a representation induced from $T_{n-2}$ to $T_n$. The passage from $T_n$ to $T_0$ depends on the parity of $n$.

To simplify the notation, denote $\theta = \theta_{n,1,\gamma}$, or $\theta_n$ to clarify the dimension. Similarly, $\theta' = \theta_{n,1,\gamma'}$. The actual pseudo-characters $\gamma$ and $\gamma'$ may vary during the proof, but this will not bare any impact on the arguments.

Our first step is to reduce the proof to the case $c_1 = 0$. To this end we claim the following.

**Lemma 4.6.** Let $b \in \{0, 1\}^{n-2}$, $n \geq 3$. If

$$\text{Hom}_{T_n}(\mathcal{L}^{n,1}_{(1,b)} \theta \otimes \mathcal{L}^{n,1}_{(1,b)} \theta'), \delta_{B_n}^{1/2} \eta_1 \otimes \ldots \otimes \eta_n) \neq 0,$$

then $\eta_m^n = 1$ and for some character $\epsilon$ of $F^*$ such that $\epsilon^2 = 1$,

$$\text{Hom}_{T_{n-1}}(\mathcal{L}^{n-1,1}_b \theta_{n-1} \otimes \mathcal{L}^{n-1,1}_b \theta'_{n-1}), \delta_{B_{n-1}}^{1/2} \epsilon \eta_1 \otimes \ldots \otimes \epsilon \eta_{n-1}) = 0.$$  

**Remark 4.3.** In general $\eta_1 \otimes \ldots \otimes \eta_n$ satisfies (⋆) if and only if $\epsilon \eta_1 \otimes \ldots \otimes \epsilon \eta_n$ does, because $\epsilon^2 = 1$.

Next we state the following lemma.

**Lemma 4.7.** Let $n \geq 3$ and assume Proposition 4.3 holds for all $n_0 \leq n - 2$. Let $\eta$ be a character of $T_n$ such that (4.10) holds with $c_1 = 0$. If $n$ is even, assume $\eta_m^n = 1$. Then $\eta$ satisfies (⋆).

Before proving the lemmas, let us derive the proposition. According to Claim 4.2 and the proof of Claim 4.4, Proposition 4.5 holds for $n = 1, 2$.

Let $n = 3$. If $c_1 = 0$, $\eta$ satisfies (⋆) according to Lemma 4.7. Otherwise $c_1 = 1$, by Lemma 4.6 we get $\eta_3^2 = 1$ and reduce the problem to $n = 2,$
and then either \( \eta_1^2 = \eta_2^2 \) or \( \eta_1^{-1} = \eta_2 \). This completes the proof (in fact, for \( n = 3 \) the proof can also be deduced from \([\text{Sav94}]\)).

Let \( n \geq 3 \) be odd and assume Proposition 4.5 holds for \( n_0 \leq n \). We deduce it for \( n + 1 \).

(1) \( c_1 = 1 \): Apply Lemma 4.6 to \( \eta_1 \otimes \ldots \otimes \eta_{n+1} \) and obtain \( \eta_{n+1}^2 = 1 \) and

\[
\text{Hom}_{T_n}((\mathcal{L}_b^{n,1} \theta_n \otimes \mathcal{L}_b^{n,1} \theta_n')_{N_n}, \delta_B^{1/2} \eta_1 \otimes \ldots \otimes \epsilon_{\eta_n}) \neq 0.
\]

Then we can apply Proposition 4.5 with \( n \) and deduce that \( \eta_1 \otimes \ldots \otimes \eta_n \) satisfies \((*)\) and so does \( \eta_1 \otimes \ldots \otimes \eta_{n+1} \).

(2) \( c_1 = 0 \) and \( \eta_{n+1}^2 = 1 \): the result follows from Lemma 4.7 \((n + 1 \text{ is even})\).

(3) \( c_1 = 0 \) and \( \eta_{n+1}^2 \neq 1 \). Let \( \eta_0 \) be a character of \( F^* \) such that \( \eta_0^2 = 1 \).

According to Theorem 3, the principal series representation induced from the character \( \eta_0 \otimes \eta_1 \otimes \ldots \otimes \eta_{n+1} \) of \( T_{n+2} \) is distinguished. Hence for some \( d \in \{0,1\}^{n+1} \),

\[
\text{Hom}_{T_{n+2}}((\mathcal{L}_d^{n+2,1} \theta_{n+2} \otimes \mathcal{L}_d^{n+2,1} \theta_{n+2}')(N_{n+2}), \delta_B^{1/2} \eta_0 \otimes \ldots \otimes \eta_{n+1}) \neq 0.
\]

Since \( \eta_{n+1}^2 \neq 1 \), Lemma 4.6 implies \( d_1 = 0 \). Therefore we may apply Lemma 4.7 and deduce that \( \eta_0 \otimes \ldots \otimes \eta_{n+1} \) satisfies \((*)\) and so does \( \eta_1 \otimes \ldots \otimes \eta_{n+1} \).

Having established Proposition 4.5 for \( n + 1 \) unconditionally, we handle \( n + 2 \). If \( c_1 = 1 \), we may proceed as in (1) and apply Lemma 4.6. We reduce the proof to a character of \( T_{n+1} \) where the result is now known to hold. Finally if \( c_1 = 0 \) we appeal directly to Lemma 4.7.

**Proof of Lemma 4.6.** Since \( U_{n-1,1} \) acts trivially on \( \theta_{U_{n-1,1}} \) and \( U_{n-l,l} \nparallel N_n \) for all \( l \), \( \mathcal{L}_{n-1}^{n,1}(\theta) = \mathcal{L}_b^{n-1,1} \theta_{U_{n-1,1}} \). Hence the assumption implies

\[
(4.12) \text{Hom}_{T_n}((\mathcal{L}_b^{n-1,1} \theta_{U_{n-1,1}} \otimes \mathcal{L}_b^{n-1,1} \theta_{U_{n-1,1}})_{N_{n-1}}, \delta_B^{1/2} \eta_1 \otimes \ldots \otimes \eta_n) \neq 0.
\]

Put \( T_n^* = \{ \text{diag}(t_1, \ldots, t_{n-1}, t_2^n): t_i \in F^* \} \). Then

\[
(\mathcal{L}_b^{n-1,1} \theta_{U_{n-1,1}})_{|p^{-1}(T_n^* N_{n-1})} = (\mathcal{L}_b^{n-1,1} (\theta_{U_{n-1,1}} | p^{-1}(\text{GL}_{n-1} \times \text{GL}_2^0))))_{|p^{-1}(T_n^* N_{n-1})}.
\]

According to (2.6) and Claim 2.3

\[
\delta_{Q_{n-1,1}}^{1/4} \theta_{U_{n-1,1}} |_{p^{-1}(\text{GL}_{n-1} \times \text{GL}_2^0)} = \xi \otimes \theta_1^D,
\]

where

\[
\xi = \begin{cases} 
\bigoplus_{g \in F^* \setminus F^*} \chi_g \theta_{n-1} & \text{if } n-1 \text{ is odd}, \\
\theta_{n-1} & \text{if } n-1 \text{ is even}.
\end{cases}
\]
Here $\chi_g(x) = (\det x, g)$. The tensor $\xi \otimes \theta_1^g$ commutes with the application of $\mathcal{L}^{n-1,1}_b$ whence
\[
\mathcal{L}^{n-1,1}_b(\theta_{U_{n-1},1}|_{\mathfrak{gl}(\mathfrak{gl}_{n-1} \times \mathfrak{gl}_1^c)}) = \delta^{1/2}_{Q_{n-1,1}} (\mathcal{L}^{n-1,1}_b \xi) \otimes \theta_1^g.
\]
Therefore
\[
(\mathcal{L}^{n-1,1}_b \theta_{U_{n-1},1} \otimes \mathcal{L}^{n-1,1}_b \theta_{U_{n-1},1})|_{T_{n-1}^n} = \delta^{1/2}_{Q_{n-1,1}} (\mathcal{L}^{n-1,1}_b \xi \otimes \mathcal{L}^{n-1,1}_b \xi')|_{B_{n-1}^1} \otimes (\theta_1^g \otimes \theta_1^g)|_{F^{1,2}},
\]
where $\xi'$ is defined as $\xi$, with respect to $\theta_{U_{n-1}}$. Also $\delta^{1/2}_{Q_{n-1,1}} \delta_{B_{n-1}^1} = \delta_{B_{n-1}^1}^{1/2}$.
Plugging these observations into (4.12) yields
\[
\text{Hom}_{T_{n-1}}(\mathcal{L}^{n-1,1}_b \xi \otimes \mathcal{L}^{n-1,1}_b \xi', \delta^{1/2}_{B_{n-1}^1} \eta_1 \otimes \ldots \otimes \eta_{n-1}) = 0.
\]
This implies $\eta_{n-1}^2 = 1$ and
\[
\text{Hom}_{T_{n-1}}(\mathcal{L}^{n-1,1}_b \xi \otimes \mathcal{L}^{n-1,1}_b \xi', \delta^{1/2}_{B_{n-1}^1} \eta_1 \otimes \ldots \otimes \eta_{n-1}) = 0.
\]
Now if $n-1$ is even, the second assertion follows immediately. Assume $n-1$ is odd. Because $\mathcal{L}^{n-1,1}_b$ commutes with finite direct sums and $\chi_g$ is a character of $\mathfrak{gl}_{n-1}$, we reach
\[
\text{Hom}_{T_{n-1}}(\bigoplus_{g,g' \in F^*} \chi_g \chi_{g'}(\mathcal{L}^{n-1,1}_b \theta_{n-1} \otimes \mathcal{L}^{n-1,1}_b \theta_{n-1}'), \delta^{1/2}_{B_{n-1}^1} \eta_1 \otimes \ldots \otimes \eta_{n-1}) = 0.
\]
As a character of $T_{n-1}$, $\chi_g(t) \chi_{g'}(t) = \chi_{gg'}(t) = \prod_{i=1}^{n-1} \epsilon_{gg'}(t_i)$, where $\epsilon_{gg'}(x) = (x, gg')(x \in F^*)$. Hence for some $h \in F^*$,
\[
\text{Hom}_{T_{n-1}}(\mathcal{L}^{n-1,1}_b \theta_{n-1} \otimes \mathcal{L}^{n-1,1}_b \theta_{n-1}'), \delta^{1/2}_{B_{n-1}^1} \epsilon_h \eta_1 \otimes \ldots \otimes \epsilon_h \eta_{n-1}) = 0.
\]
Clearly $c^2 = 1$ and the claim is proved.

\textbf{Proof of Lemma 4.7.} Fix $c \in \{0, 1\}^{n-1}$ with $c_1 = 0$. Kable [Kab01] (Theorem 5.3) showed that the second derivative of $\theta = \theta_{n-1,1}$ is $|\det|^{-1/2} \theta_{n-2,1,1} \varphi^{-1} \eta$, where $\varphi$ is the character with respect to which the derivation is defined, and furthermore, this is the highest nonzero derivative (Kab01) Theorem 5.4). Hence the kernel of the Jacquet functur $\theta|_{\mathfrak{y}^o_{n-1}}(U_{n-1,1})$ is equal, as a $\mathfrak{y}_{n}$-module, to the application of $\Phi^* \Psi^*$ to the second derivative ([BZ77] 3.2 (e) and 3.5). Put $\theta_{n-2} = \theta_{n-2,1,1} \varphi^{-1} \eta$. Thus
\[
\theta|_{\mathfrak{y}^o_{n-1}}(U_{n-1,1}) = \text{ind}_{\mathfrak{y}^o_{n-1} U_{n-1,1}}^{\mathfrak{y}^o_{n-1}} (\delta_{\mathfrak{y}^o_{n-1}}^{1/2} \text{ind}_{\mathfrak{gl}_{n-2}}^{\mathfrak{gl}_{n-2} U_{n-1,1}} (|\det|^{1/2} |\det|^{-1/2} \theta_{n-2}) \varphi^-).\]
Since $\delta_{\mathfrak{y}^o_{n-1}}|_{\mathfrak{gl}_{n-2}} = |\det|$, in the notation of Section 8
\[
\theta|_{\mathfrak{y}^o_{n-1}}(U_{n-1,1}) = \Phi^* \Psi^* (\pi_0), \quad \pi_0 = |\det|^{1/2} \theta_{n-2}.
\]
Then as $\mathfrak{y}_{n}$-modules
\[
L^{n-1}_b(\theta|_{\mathfrak{y}^o_{n-1}}) = L^{n-1}_b \Phi^* \Psi^* (\pi_0), \quad b = (c_2, \ldots, c_{n-1}).
\]
While restriction to $\bar{Y}_n$ enables us to apply the results of Section 3, we seem to be losing the last coordinate of $\bar{T}_n$. We explain how to remedy this. Since the pair of subgroups $\bar{B}_n^{\circ}$ and $\bar{Z}_n^{\circ}$ are commuting in $\bar{G}\bar{L}_n$, one can form the genuine representation $\omega_0\mathcal{L}_c^{n,1}(\theta|\bar{Y}_n)$ of $p^{-1}(B_n^{\circ}Z_n^{\circ})$, which is in fact $\mathcal{L}_c^{n,1}(\theta)|_{p^{-1}(B_n^{\circ}Z_n^{\circ})}$. If $n$ is odd, $p^{-1}(B_n^{\circ}Z_n^{\circ}) = \bar{B}_n$ and we recover $\mathcal{L}_c^{n,1}(\theta)$.

By virtue of Lemma 3.3 applied to the right-hand side of (4.13), if $k$ is defined by the lemma and $b^{k,\gamma} = (b_{k+1}, \ldots, b_{n-2})$, as $\bar{B}_n$-modules

$$\mathcal{L}_c^{n,1}(\theta|\bar{Y}_n) = \text{s.s.} \bigoplus_{m=\min(1,k)}^k \mathcal{E}_{n,m+2}\mathcal{L}_{(0k-m,b^{k,\gamma})}^{n-2,m}(\pi_0).$$

Then as $p^{-1}(B_n^{\circ}Z_n^{\circ})$-modules

$$\omega_0\mathcal{L}_c^{n,1}(\theta|\bar{Y}_n) = \text{s.s.} \bigoplus_{m=\min(1,k)}^k \omega_0\mathcal{E}_{n,m+2}\mathcal{L}_{(0k-m,b^{k,\gamma})}^{n-2,m}(\pi_0).$$

Recall that $\theta' = \theta_{n,1,\gamma'}$. We take the derivative with respect to $\psi^{-1}$ (instead of $\psi$). Put $\theta'_{n-2} = \theta_{n-2,1,\gamma_{\psi_1}^{-1},\gamma'}$, $\pi_0' = |\det|^{1/2}\theta'_{n-2}$, $\Phi_0^+\psi$ is defined using $\psi^{-1}$ and then (4.14) holds for $\theta'$, with $\pi_0'$. Next we see that according to (3.2), for any $\min(1,k) \leq m, \bar{m} \leq k$,

$$\mathcal{E}_{n,m+2}\mathcal{L}_{(0k-m,b^{k,\gamma})}^{n-2,m}(\pi_0) \otimes \mathcal{E}_{n,\bar{m}+2}\mathcal{L}_{(0k-\bar{m},b^{k,\gamma})}^{n-2,\bar{m}}(\pi_0')|_{U_{n-1,1}}$$

$$= \begin{cases} 
\mathcal{E}_{n,m+2}\mathcal{L}_{(0k-m,b^{k,\gamma})}^{n-2,m}(\pi_0) \otimes \mathcal{L}_{(0k-\bar{m},b^{k,\gamma})}^{n-2,\bar{m}}(\pi_0') & m = \bar{m}, \\
0 & m \neq \bar{m}.
\end{cases}$$

Put

$$\Lambda(b) = \mathcal{L}_{(0k-m,b^{k,\gamma})}^{n-2,m}(\pi_0) \otimes \mathcal{L}_{(0k-\bar{m},b^{k,\gamma})}^{n-2,\bar{m}}(\pi_0').$$

This is a representation of $\bar{Q}_{1n-m-2,m}$ which factors through $Q_{1n-m-2,m}$, therefore we will regard it as a representation of the latter.

**Remark 4.4.** If one takes the derivative of $\theta'$ with respect to $\psi_\alpha$, where $\psi_\alpha(x) = \psi(\alpha x)$ ($\alpha \in F^\ast$), instead of $\psi^{-1}$, the isomorphism (4.15) is deduced using

$$\ind_{\bar{Y}_n}^{\mathcal{E}_{n-1}U_{n-1,1}}(\ind_{\mathcal{G}\mathcal{L}_{n-2}U_{n-2,1}}^{\mathcal{E}_{n-1}(F)}(|\det|^{1/2}\theta_{n-2,1,\gamma_{\psi_\alpha}^{-1},\gamma'}) \otimes \psi_\alpha)$$

$$\cong \ind_{\bar{Y}_n}^{\mathcal{E}_{n-1}U_{n-1,1}}(\ind_{\mathcal{G}\mathcal{L}_{n-2}U_{n-2,1}}^{\mathcal{E}_{n-1}(F)}(|\det|^{1/2}\theta_{n-2,1,\gamma_{\psi_\alpha}^{-1},\gamma'}) \otimes \psi^{-1}).$$

To see this note that $(-\alpha^{-1}, x)\gamma_{\psi_\alpha}(x) = \gamma_{\psi^{-1}}(x)$. 

We obtain the following equality of $B_n^e$-modules
\[
(L_c^{n,1}(\theta|_{\overline{Y}_n}) \otimes L_c^{n,1}(\theta'|_{\overline{Y}_n}))_{U_{n-1,1}} = \text{s.s.} \bigoplus_{m=\min(1,k)}^k \mathcal{E}_{n,m+2}(\Lambda(b)).
\]

Put $i = m + 2$, apply Lemma 3.7 to each summand and obtain, as $T_n^e$-modules,
\[
(L_c^{n,1}(\theta|_{\overline{Y}_n}) \otimes L_c^{n,1}(\theta'|_{\overline{Y}_n}))_{N_n} = \text{s.s.} \bigoplus_{m=\min(1,k)}^k \text{ind}_{\psi(T_{n-2})}^{T_n^e} (\delta^{1/2} \psi(\Lambda(b)_{N_{n-2}})).
\]

Here we used $N_n = N_{n-1} \ltimes U_{n-1,1}$. This equality yields the following identity of $T_n^e Z_n^e$-modules,
\[
(\omega_\theta L_c^{n,1}(\theta|_{\overline{Y}_n}) \otimes \omega_\theta L_c^{n,1}(\theta'|_{\overline{Y}_n}))_{N_n} = \text{s.s.} \bigoplus_{m=\min(1,k)}^k \omega_\theta \omega_\theta \psi \text{ind}_{\psi(T_{n-2})}^{T_n^e} (\delta^{1/2} \psi(\Lambda(b)_{N_{n-2}})).
\]

Thus for some $\min(1,k) \leq m \leq k$,
\[
(4.16) \quad \text{Hom}_{T_n^e Z_n^e} (\omega_\theta \omega_\theta \psi \text{ind}_{\psi(T_{n-2})}^{T_n^e} (\delta^{1/2} \psi(\Lambda(b)_{N_{n-2}})), \delta^{1/2} B_n^e \eta_1 \otimes \ldots \otimes \eta_n) \neq 0.
\]

Next we claim,

**Claim 4.8.** For $x \in F^t e$, the element $t_x = \text{diag}(I_{n-i}, x, I_{i-2}, x) \in T_n^e Z_n^e$ acts on
\[
\omega_\theta \omega_\theta \psi \text{ind}_{\psi(T_{n-2})}^{T_n^e} (\delta^{1/2} \psi(\Lambda(b)_{N_{n-2}}))
\]
via $\delta^{1/2}_{B_n^e} (t_x)$.

The claim and (4.16) immediately imply $\eta_{n-i+1}(x) \eta_n(x) = 1$ for all $x \in F^t e$. Now if $n$ is odd, we deduce $\eta_{n-i+1} = \eta_n^{-1}$ (!). In the even case we find $\eta_{n-i+1} = \eta_n^2$ and according to our assumption (!) $\eta_n^2 = 1$, we obtain $\eta_{n-i+1} = 1$.

We proceed to apply the induction hypothesis. After restricting (4.16) to $T_n^e$ and applying Frobenius reciprocity,
\[
(4.17) \quad \text{Hom}_{T_{n-2}} (\delta^{1/2} \psi(\Lambda(b)_{N_{n-2}}), \delta^{1/2} B_n^e \eta_1 \otimes \ldots \otimes \eta_n) \neq 0.
\]

Using
\[
\delta_{B_n^e}(w_i t) = \prod_{j=1}^{n-i} |t_j| \delta_{B_{n-2}}(t) = |\det t| \delta_{B_{n-2}}^{1/2}(w_i t) \delta_{B_{n-2}}(t), \quad \forall t \in T_{n-2},
\]
we obtain
\[
\text{Hom}_{T_{n-2}} (|\det |^{-1} \Lambda(b)_{N_{n-2}}, \delta^{1/2}_{B_n^e} \eta^{(n-2)}) \neq 0, \quad \eta^{(n-2)} = \eta_1 \otimes \ldots \otimes \eta_{n-1} \otimes \eta_{n-i+2} \otimes \ldots \otimes \eta_n.
\]
Note that $2 \leq i \leq n$ ($i = m + 2$). Thus

$$\text{Hom} \left( \left( L_{(0k^{-m},yk^+)}^{n-2,m} \right) \left( \theta_{n-2} \right) \otimes L_{(0k^{-m},yk^+)}^{n-2,m} \left( \theta'_{n-2} \right) \right)_{N_{n-2}}, \delta_{B_{n-2}}^{1/2} \eta^{(n-2)} \right) \neq 0.$$ 

According to Corollary 2.2 as $B_{n-2}$-modules

$$\left( L_{(0k^{-m},yk^+)}^{n-2,m} \right) \left( \theta_{n-2} \right) \otimes L_{(0k^{-m},yk^+)}^{n-2,m} \left( \theta'_{n-2} \right) \right)_{N_{n-2}} \neq \oplus_{c^{(n-2)}} \left( L_{(n-2)}^{n-2,1} \right) \left( \theta_{n-2} \right) \otimes L_{(c^{(n-2)})}^{n,1} \left( \theta'_{n-2} \right) \right)_{N_{n-2}},$$

where $c^{(n-2)}$ varies as described in the corollary. Therefore for some $c^{(n-2)} \in \{0, 1\}^{n-3},$

$$\text{Hom} \left( \left( L_{(n-2)}^{n-2,1} \right) \left( \theta_{n-2} \right) \otimes L_{(c^{(n-2)})}^{n,1} \left( \theta'_{n-2} \right) \right)_{N_{n-2}}, \delta_{B_{n-2}}^{1/2} \eta^{(n-2)} \right) \neq 0.$$ 

This is condition (4.10) for $n-2$ and implies, by Proposition 4.3 which is assumed to hold for $n-2$, that $\eta^{(n-2)}$ satisfies (*) and then so does $\eta$. The proof of the lemma is complete.

**Proof of Claim 4.8.** Let $x \in F^*$ and write $t_x = z_x d_x$ with

$$z_x = x I_n \in Z_n^o, \quad d_x = \text{diag}(x^{-1} I_{n-i}, 1, x^{-1} I_{i-1}, 1).$$

Because $w_{i}^{-1} d_x = \text{diag}(x^{-1} I_{n-i}, 1) \in Z_{n-2}^o, \omega_{\gamma_{n-2}}(g(w_{i}^{-1} d_x)) = \gamma_{\psi}^{-1}(x^{-1}) \gamma(x^{-1}).$

Hence $d_x$ acts on $w_{i}(\Lambda(b)_{N_{n-2}})$ by

$$|x|^{-2+2} \psi^{-1}(x^{-1}) \gamma'(x^{-1}) = |x|^{-n+2} \gamma'(x^{-1}).$$

$(\gamma_{\psi^{-1}} = \gamma_{\psi^{-1}})$ The element $z_x$ acts on $\omega_{\psi} \omega_{\theta} \text{ind}_{L_{n-2}^{(n-2)}} \left( \delta_{Y_{i}^{-1}} w_{i}(\Lambda(b)_{N_{n-2}}) \right)$ by $\omega_{\psi} \omega_{\theta} \left( g(z_x) \right) = \gamma(x) \gamma'(x).$ Since $\delta_{Y_{i}^{-1}}(d_x) = |x|^{-2}$ and

$$\gamma(x) \gamma'(x) \gamma(x^{-1}) = 1$$

$(\gamma(x) \gamma(x^{-1}) = (x, x^{-1})^{2})$, $t_x$ acts on $\omega_{\psi} \omega_{\theta} \text{ind}_{L_{n-2}^{(n-2)}} \left( \delta_{Y_{i}^{-1}} w_{i}(\Lambda(b)_{N_{n-2}}) \right)$ via $|x|^{-n} = \delta_{B_{n}}^{1/2}(t_x).$ \hfill \qed

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