BOUNDED ORBITS OF CERTAIN DIAGONALIZABLE FLOWS ON
$SL_n(\mathbb{R})/SL_n(\mathbb{Z})$

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Abstract. We prove that the set of points that have bounded orbits under certain
diagonalizable flows is a hyperplane absolute winning subset of $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$.

1. Introduction

1.1. Statement of main results. Let $G$ be a connected Lie group, $\Gamma$ a nonuniform lattice
in $G$, and $F = \{g_t : t \in \mathbb{R}\}$ a one-parameter subgroup of $G$ with noncompact closure. We
are interested in the dynamical properties of the action of $F$ on the homogeneous space
$G/\Gamma$ by left translations. Specifically, we will focus on the study of the set
$$E(F) := \{\Lambda \in G/\Gamma : F\Lambda \text{ is bounded in } G/\Gamma\}$$
in this paper. In certain important cases, it turns out that $E(F)$ has zero Haar measure
(For example when $G$ is semisimple without compact factors and $\Gamma$ is irreducible, this
follows from Moore’s ergodicity theorem). If $F$ is Ad-unipotent, $E(F)$ is even smaller.
In this case, by Ratner’s Theorems, $E(F)$ is contained in a countable union of proper
submanifolds, hence has Hausdorff dimension $< \dim G$. When $F$ is Ad-semisimple, the
situation is quite different. Motivated by the work of Dani (cf. [8], [9]), Margulis proposed
a conjecture in his 1990’s ICM report [15], which was settled in a subsequent work of
Kleinbock and Margulis [13]. In that work, they proved: if the flow $(G/\Gamma, F)$ has the
so-called property (Q), then the set $E(F)$ is thick, i.e. for any nonempty open subset $V$ of
$G/\Gamma$ the set $E(F) \cap V$ is of Hausdorff dimension equal to the dimension of the underlying
space $G/\Gamma$. In particular, when $F$ is Ad-semisimple, the flow $(G/\Gamma, F)$ always has property
(Q).

Given countably many Ad-semisimple $F_n$, it is natural to ask whether the set of points
$\Lambda$ such that all the orbits $F_n\Lambda$ are bounded is still thick. This is natural from both the
dynamical point of view and its relation to number theory. This is proved to be true for
$G = SL_2(\mathbb{R})$ and $\Gamma = SL_2(\mathbb{Z})$ in [14], and for $G = SL_3(\mathbb{R})$ and $\Gamma = SL_3(\mathbb{Z})$ in [3]. Note that
this set is the intersection $\bigcap_n E(F_n)$. A powerful tool for studying intersection properties
of different sets is a type of game introduced by Schmidt in [18], which is called Schmidt’s
$(\alpha, \beta)$-game. The game can be played on any metric space, and it defines a class of so-called
$\alpha$-winning sets ($0 < \alpha < 1$). When the metric space is a Riemannian manifold, $\alpha$-winning
sets are thick and stable with respect to countable intersections. In this paper, we will use
a variant of Schmidt’s $(\alpha, \beta)$-game, i.e., the hyperplane absolute game introduced in [6]
and [14]. This game has the advantage that it can be naturally defined on a differential
manifold without picking a Riemannian metric while the hyperplane absolute winning
(abbreviated as HAW) sets also enjoy the thickness and countable intersection properties.
See Section 2 for details. Note that, in both [14] and [3], the authors prove their results
by showing that $E(F)$ is HAW in the corresponding case. In fact, the following conjecture
is proposed in [3].

Conjecture 1.1. [3 Conjecture 7.1] Let $G$ be a Lie group, $\Gamma$ a lattice in $G$, and $F$ a
one-parameter Ad-diagonalizable subgroup of $G$. Then the set $E(F)$ is HAW on $G/\Gamma$. 

In this paper, we restrict ourselves to the case
\[ G = SL_n(\mathbb{R}), \quad \Gamma = SL_n(\mathbb{Z}). \]
Our main theorem is the following, verifying the above conjecture for certain class of \( F \).

**Theorem 1.2.** Let \( F \) be a one-parameter subgroup of \( G \) satisfying the following property, it is diagonalizable and the eigenvalues of \( g_1 \) (denoted by \( \lambda_1, \ldots, \lambda_n \)) satisfy:

\[ \# \{ i : |\lambda_i| < 1 \} = 1 \quad \text{and} \quad \# \{ i : |\lambda_i| = \max_{1 \leq j \leq n} |\lambda_j| \} \geq n - 2. \]  
(1.1)

Then the set \( E(F) \) is HAW on \( G/\Gamma \).

We also prove the following theorem verifying [3, Conjecture 7.2] for \( F \) satisfying (1.1).

**Theorem 1.3.** Let \( F \) be a one-parameter subgroup of \( G \) satisfying (1.1), and \( F^+ = \{ g_t \in F : t \geq 0 \} \). Let \( H(F^+) \) denote the expanding horospherical subgroup of \( F^+ \) which is defined as

\[ H(F^+) = \left\{ h \in G : \lim_{t \to +\infty} g_t^{-1} h g_t = e \right\}. \]  
(1.2)

Then for any \( \Lambda \in G/\Gamma \), the set

\[ \{ h \in H(F^+) : h \Lambda \in E(F^+) \} \]

is HAW on \( H(F^+) \).

**1.2. Connection to number theory.** To begin, let us define a \( d \)-weight \( r \) to be a \( d \)-tuple \( r = (r_1, \ldots, r_d) \) in \( \mathbb{R}^d \) such that each \( r_i \) is positive and their sum equals 1. Due to work of Dani [8] and Kleinbock [12], we know that for a \( d \)-weight \( r \) there is a close relation between the set of \( r \)-badly approximable vectors (abbreviated as \( \text{Bad}(r) \)) and bounded orbits of certain flow corresponding to \( r \) in \( SL_{d+1}(\mathbb{R})/SL_{d+1}(\mathbb{Z}) \). We will not present the explicit definition of \( \text{Bad}(r) \) here. But we remark that they are natural generalizations of the classical badly approximable numbers. Recently, there is a rapid progress on the study of intersection properties of the sets \( \text{Bad}(r) \) for different weight \( r \), for example, see [4, 1, 2, 5, 17, 11]. Concerning the winning properties of such sets, Schmidt proved that \( \text{Bad}_d \) (abbreviation for \( \text{Bad}(\frac{1}{d}, \ldots, \frac{1}{d}) \)) is winning for his game for any \( d \in \mathbb{N} \). They are also proved to be HAW in [6]. Recently, An [2] proved that \( \text{Bad}(r) \) are winning sets for Schmidt’s game for any 2-weight \( r \). The HAW property is also established for such sets by Nesharim and Simmons [17]. To this end, we want to highlight the following theorem proved in [11], since it motivates the results of this paper.

**Theorem 1.4.** (Cf. [11] Theorem 1.4) Let a \( d \)-weight \( r = (r_1, \ldots, r_d) \) satisfy

\[ \sum_{i=1}^d r_i = 1 \quad \text{and} \quad r_1 = \ldots = r_{d-1} \geq r_d \geq 0. \]  
(1.3)

Then \( \text{Bad}(r) \) is HAW.

**Remark 1.5.** Whether \( \text{Bad}(r) \) is winning (\( \alpha \)-winning or HAW) for general weight \( r \) is a challenging open problem proposed by Kleinbock [12].

**1.3. Structure of the paper.** For sake of convenience, from now on we will assume

\[ G = SL_{d+1}(\mathbb{R}), \quad \Gamma = SL_{d+1}(\mathbb{R}). \]

That is, the number \( n \) in the title of the paper is replaced by \( d + 1 \).

The paper is organized as follows. In Section 2 we recall some basics of certain Schmidt games, namely the hyperplane absolute game and the hyperplane potential game. In Section 3.1 we state Theorem 3.1 and then convert it to the Diophantine setting using...
Lemma 3.1. Note that Theorem 3.1, whose proof forms the most technical part of this paper, can be regarded as a special case of Theorem 1.3. In the rest of Section 3, we turn to the study of pairs \((B, P)\), where \(B\) is a closed ball in \(\mathbb{R}^{2d-1}\) and \(P\) is a rational vector in \(\mathbb{Q}^d\). We manage to attach a rational hyperplane and a rational line in \(\mathbb{R}^d\) to the pair \((B, P)\). Section 5 is the core of this paper, in which Theorem 3.1 is proved using the information of the pairs \((B, P)\) and some subdivisions prepared in Section 3 and 4. In the last section, Theorem 1.2 and Theorem 1.3 are deduced from Theorem 3.1.

2. Schmidt games

In this section, we will recall definitions of certain Schmidt games, namely, the hyperplane absolute game and the hyperplane potential game. They are both variants of the \((\alpha, \beta)\)-game introduced by Schmidt in \([18]\). Since we don’t make a direct use of the \((\alpha, \beta)\)-game in this paper, we omit its definition here and refer the interested reader to \([18, 19]\).

2.1. Hyperplane absolute game. The hyperplane absolute game was introduced in \([6]\). It is played on an Euclidean space \(\mathbb{R}^d\). Given a hyperplane \(L\) and a \(\delta > 0\), we denote by \(L(\delta)\) the \(\delta\)-neighborhood of \(L\), i.e.,

\[
L(\delta) := \{x \in \mathbb{R}^d : \text{dist}(x, L) < \delta\}.
\]

For \(\beta \in (0, \frac{1}{3})\), the \(\beta\)-hyperplane absolute game is defined as follows. Bob starts by choosing a closed ball \(B_0 \subset \mathbb{R}^d\) of radius \(\rho_0\). In the \(i\)-th turn, Bob chooses a closed ball \(B_i\) with radius \(\rho_i\), and then Alice chooses a hyperplane neighborhood \(L_i(\delta_i)\) with \(\delta_i \leq \beta \rho_i\). Then in the \((i + 1)\)-th turn, Bob chooses a closed ball \(B_{i+1} \subset B_i \setminus \bigcup_{j=0}^{i-1} B_j \setminus L_i(\delta_i)\) of radius \(\rho_{i+1} \geq \beta \rho_i\). By this process there is a nested sequence of closed balls

\[
B_0 \supseteq B_1 \supseteq B_2 \supseteq \ldots.
\]

We say that a subset \(S \subset \mathbb{R}^d\) is \(\beta\)-hyperplane absolute winning (\(\beta\)-HAW for short) if no matter how Bob plays, Alice can ensure that

\[
\bigcap_{i=0}^{\infty} B_i \cap S \neq \emptyset.
\]

We say \(S\) is hyperplane absolute winning (HAW for short) if it is \(\beta\)-HAW for any \(\beta \in (0, \frac{1}{3})\).

We have the following lemma collecting the basic properties of \(\beta\)-HAW subsets and HAW subsets of \(\mathbb{R}^d\) (\([6, 14, 11]\)):

**Lemma 2.1.**  
(1) A HAW subset is always \(\frac{1}{3}\)-winning.  
(2) Given \(\beta, \beta' \in (0, \frac{1}{3})\), if \(\beta \geq \beta'\), then any \(\beta'\)-HAW set is \(\beta\)-HAW.  
(3) A countable intersection of HAW sets is again HAW.  
(4) Let \(\varphi : \mathbb{R}^d \to \mathbb{R}^d\) be a \(C^1\) diffeomorphism. If \(S\) is a HAW set, then so is \(\varphi(S)\).

The notion of HAW was extended to subsets of \(C^1\) manifolds in \([14]\). This is done in two steps. First, one defines the hyperplane absolute game on an open subset \(W \subset \mathbb{R}^d\). It is defined just as the hyperplane absolute game on \(\mathbb{R}^d\), except for requiring that Bob’s first move \(B_0\) be contained in \(W\). Now, let \(M\) be a \(d\)-dimensional \(C^1\) manifold, and let \(\{(U_\alpha, \phi_\alpha)\}\) be a \(C^1\) atlas on \(M\). A subset \(S \subset M\) is said to be HAW on \(M\) if for each \(\alpha\), \(\phi_\alpha(S \cap U_\alpha)\) is HAW on \(\phi_\alpha(U_\alpha)\). The definition is independent of the choice of atlas by the property (4) listed above. We have the following lemma that collects the basic properties of HAW subsets of a \(C^1\) manifold (cf. \([14]\)).
Lemma 2.2.  
(1) HAW subsets of a $C^1$ manifold are thick.
(2) A countable intersection of HAW subsets of a $C^1$ manifold is again HAW.
(3) Let $\phi : M \to N$ be a diffeomorphism between $C^1$ manifolds, and let $S \subset M$ be a HAW subset of $M$. Then $\phi(S)$ is a HAW subset of $N$.
(4) Let $M$ be a $C^1$ manifold with an open cover $\{U_\alpha\}$. Then, a subset $S \subset M$ is HAW on $M$ if and only if $S \cap U_\alpha$ is HAW on $U_\alpha$ for each $\alpha$.
(5) Let $M, N$ be $C^1$ manifolds, and let $S \subset M$ be a HAW subset of $M$. Then $S \times N$ is a HAW subset of $M \times N$.

2.2. Hyperplane potential game. Being introduced in [10], the hyperplane potential game also defines a class of subsets of $\mathbb{R}^d$ called hyperplane potential winning (HPW for short) sets. The following lemma allows one to prove the HAW property of a set $S \subset \mathbb{R}^d$ by showing that it is winning for the hyperplane potential game. And this is exactly the game we will use in this paper.

Lemma 2.3. (cf. [10, Theorem C.8]) A subset $S$ of $\mathbb{R}^d$ is HPW if and only if it is HAW.

The hyperplane potential game involves two parameters $\beta \in (0, 1)$ and $\gamma > 0$. Bob starts the game by choosing a closed ball $B_0 \subset \mathbb{R}^d$ of radius $\rho_0$. In the $i$-th turn, Bob chooses a closed ball $B_i$ of radius $\rho_i$, and then Alice chooses a countable family of hyperplane neighborhoods $\{L^{(i,k)}_{i,k} : k \in \mathbb{N}\}$ such that
\[
\sum_{k=1}^\infty \delta^{(i,k)}_{i,k} \leq (\beta \rho_i)^\gamma.
\]
Then in the $(i+1)$-th turn, Bob chooses a closed ball $B_{i+1} \subset B_i$ of radius $\rho_{i+1} \geq \beta \rho_i$. By this process there is a nested sequence of closed balls
\[
B_0 \supseteq B_1 \supseteq B_2 \supseteq \ldots.
\]
We say a subset $S \subset \mathbb{R}^d$ is $(\beta, \gamma)$-hyperplane potential winning ($(\beta, \gamma)$-HPW for short) if no matter how Bob plays, Alice can ensure that
\[
\bigcap_{i=0}^\infty B_i \cap \left(S \cup \bigcup_{i=0}^\infty \bigcup_{k=1}^\infty L^{(i,k)}_{i,k}\right) \neq \emptyset.
\]
We say $S$ is hyperplane potential winning (HPW for short) if it is $(\beta, \gamma)$-HPW for any $\beta \in (0, 1)$ and $\gamma > 0$.

3. Converting to the Diophantine setting

Fix $d \geq 2$. Recall that we have assumed
\[
G = \text{SL}_{d+1}(\mathbb{R}), \Gamma = \text{SL}_{d+1}(\mathbb{R})
\]
to simplify notations. That is, the number $n$ in the title of the paper is replaced by $d+1$. Let
\[
\pi : G \to G/\Gamma
\]
be the natural projection.

We will fix a $d$-weight $r$ satisfying (1.5) until the last section. For simplicity, sometimes we also write $\lambda = r_1 = \cdots = r_{d-1}$, $\mu = r_d$. Both Theorem 1.2 and Theorem 1.3 will be deduced from the following theorem.

Theorem 3.1. Let $r$ be a weight satisfying (1.1). Denote
\[
F_r := \{g_t = \text{diag}(e^{r_1 t}, e^{r_2 t}, \cdots, e^{r_d t}, e^{-t}) : t \in \mathbb{R}\}, \quad F_r^+ := \{g_t \in F_r : t \geq 0\},
\]
and

\[ U := \left\{ u_{x,y,z} : x, z \in \mathbb{R}^{d-1}, y \in \mathbb{R} \right\}, \quad \text{where } u_{x,y,z} := \begin{pmatrix} 1d & z & x \\ 1 & y & 1 \end{pmatrix} \in G. \quad (3.1) \]

Then the set \( U \cap \pi^{-1}(E(F_r^+)) \) is HAW on \( U \).

**Remark 3.2.** If \( r \) satisfies \( r_1 > r_d \) in advance, then the expanding horospherical subgroup \( H(F_r^+) \) defined as in \([1,2]\) coincides with the group \( U \) given in \( (3.1) \). Thus in this case, Theorem 3.1 can be regarded as special case of the Theorem 1.3 with \( \Lambda = \Gamma \).

### 3.1. Diophantine characterization.

For technical reasons, we will prove Theorem 3.1 by applying the diffeomorphism \( \mathbb{R}^{2d-1} \to U \) defined as

\[(x, y, z) \mapsto u_{x,y,z}^{-1}.\]

**Remark 3.3.** In view of Lemma 2.2(3), if we can prove the set

\[ \left\{ (x, y, z) \in \mathbb{R}^{2d-1} : F_r^+ u_{x,y,z}^{-1} \Gamma \text{ is bounded in } G/\Gamma \right\} \]

in HAW on \( \mathbb{R}^{2d-1} \), then Theorem 3.1 will follows.

A rational vector \( P \in \mathbb{Q}^d \) will be always written in the following reduced form:

\[ P = \left( \frac{p}{q}, \frac{r}{q} \right), \quad \text{with } q > 0 \text{ and } p = (p_1, \ldots, p_{d-1}) \text{ satifying } \gcd(p_1, \ldots, p_{d-1}, r, q) = 1. \]

Such a form is unique, thus we may write the denominator of \( P \) as a function \( q(P) \).

We need the following Diophantine characterization of the boundedness of \( F_r^+ u_{x,y,z}^{-1} \Gamma \) in \( G/\Gamma \). For \( \epsilon > 0 \) and a rational vector \( P = \left( \frac{p}{q}, \frac{r}{q} \right) \in \mathbb{Q}^d \) written in its reduced form, we denote

\[ \Delta_\epsilon(P) := \left\{ (x, y, z) \in \mathbb{R}^{d-1} \times \mathbb{R} \times \mathbb{R}^{d-1} : \left| y - \frac{r}{q} \right| < \frac{\epsilon}{q^{1+\mu}}, \right. \]

\[ \left. \left\| x - \frac{p}{q} - \left( y - \frac{r}{q} \right) z \right\|_{\infty} < \frac{\epsilon}{q^{1+\lambda}} \right\}, \]

where \( \| \cdot \|_{\infty} \) means the maximal norm on \( \mathbb{R}^{d-1} \), that is, for \( x = (x_1, \ldots, x_{d-1}) \), \( \|x\|_{\infty} = \max\{|x_1|, \ldots, |x_d|\} \). Then we set

\[ S_\epsilon(r) := \mathbb{R}^{2d-1} \setminus \bigcup_{P \in \mathbb{Q}^d} \Delta_\epsilon(P) \]

and

\[ S(r) := \bigcup_{\epsilon > 0} S_\epsilon(r). \]

The following lemma allows us to convert our problem to the Diophantine setting. For the proof, one can refer to \([1,2]\) (see also \([3, \text{Lemma 3.2}]\)).

**Lemma 3.4.** (cf. \([1,2] \text{ Theorem 2.5}\)) The orbit \( F_r^+ u_{x,y,z}^{-1} \Gamma \) is bounded if and only if \( (x, y, z) \in S(r) \), that is, there is \( \epsilon = \epsilon(x, y, z) > 0 \) such that

\[ \max\left\{ q^\mu |qy - r|, q^\lambda \|qx - p - (gy - r)z\|_{\infty} \right\} \geq \epsilon, \quad \forall P = \left( \frac{p}{q}, \frac{r}{q} \right) \in \mathbb{Q}^d. \]
3.2. Attaching hyperplanes. Let \( \mathcal{B} \) denote the set of closed balls in \( \mathbb{R}^{d-1} \) with radius smaller than \( 1/d \). We shall introduce a function 
\[
\mathbf{a}^+ : \mathcal{B} \times \mathbb{Q}^d \to \mathbb{Z}^d
\]
below which enables us to define a linear function on \( \mathbb{R}^d \) that depends on the pair of a closed ball \( B \in \mathcal{B} \) and \( P = (\frac{p}{q}, \frac{r}{q}) \in \mathbb{Q}^d \):
\[
F_{B,P}(\mathbf{w}) = \mathbf{a}^+(B, P) \cdot \mathbf{w} - \mathbf{a}^+(B, P) \cdot \left( \frac{p}{q}, \frac{r}{q} \right), \quad \mathbf{w} \in \mathbb{R}^d.
\] (3.2)
We also write for simplicity
\[
\mathcal{C}(B, P) = \mathbf{a}^+(B, P) \cdot \left( \frac{p}{q}, \frac{r}{q} \right).
\] (3.3)
Finally we can define a hyperplane attached to the pair \( (B, P) \) to be
\[
\mathcal{H}_{B,P} := \{ \mathbf{w} \in \mathbb{R}^d : F_{B,P}(\mathbf{w}) = 0 \}.
\]
Now let us define the function \( \mathbf{a}^+ : \mathcal{B} \times \mathbb{Q}^d \to \mathbb{Z}^d \). We shall need the following lemma:

**Lemma 3.5.** Let \( \mathbf{z} \in \mathbb{R}^{d-1} \). For any \( P = (\frac{p}{q}, \frac{r}{q}) \in \mathbb{Q}^d \), there exists \( (a, b) \in \mathbb{Z}^d \) with \( (a, b) \neq (0, 0) \) such that \( a \cdot \mathbf{p} + br \in q\mathbb{Z} \) and \( |a|_\infty \leq q^{\lambda}, |b + z \cdot a| \leq q^\mu \).

**Proof.** By Minkowski’s linear forms theorem (cf. [7] Chapter III, Theorem III), there exist \( a \in \mathbb{Z}^{d-1}, b, c \in \mathbb{Z} \) which are not all zero, such that
\[
|a \cdot \mathbf{p} + br + cq| < 1, \quad |a|_\infty \leq q^{\lambda}, \quad |b + z \cdot a| \leq q^\mu.
\]
Since \( a \cdot \mathbf{p} + br + cq \in \mathbb{Z} \), it must be 0 by the first inequality above. Assume that \( a = 0 \) and \( b = 0 \). Then it follows from \( a \cdot \mathbf{p} + br + cq = 0 \) and \( q \neq 0 \) that \( c = 0 \), which is a contradiction. Thus \( (a, b) \neq (0, 0) \). The lemma follows. \( \square \)

Now let us consider the following set
\[
\mathcal{A}_{B,P} := \{ (a, b) \in \mathbb{Z}^d : (a, b) \neq (0, 0), a \cdot \mathbf{p} + br \in q\mathbb{Z}, |a|_\infty \leq q^{\lambda}, |b + z_B \cdot a| \leq q^\mu + \rho(B)^{\frac{d}{2}} \},
\]
where \( z_B \) is the \( z \)-coordinate of the center of \( B \) and \( \rho(B) \) is the radius of \( B \). It follows from Lemma 3.5 that \( \mathcal{A}_{B,P} \neq \emptyset \). We choose and fix
\[
\mathbf{a}^+(B, P) = (a(B, P), b(B, P)) \in \mathcal{A}_{B,P}
\]
such that
\[
\xi(B, P) := \max \left\{ |a(B, P)|_\infty, |b(B, P) + z_B \cdot a(B, P)| \right\}
= \min \left\{ \max \left\{ |a|_\infty, |b + z_B \cdot a| \right\} : (a, b) \in \mathcal{A}_{B,P} \right\}.
\] (3.4)
This completes the definition of the function \( \mathbf{a}^+ \). Then we define the **height of \( P \) with respect to \( B \)**
\[
H_B(P) := q(P) \xi(B, P).
\]

**Remark 3.6.** From its definition, one can see that the height function \( H_B(P) \) is not canonically defined, i.e. it may depend on a choice. But we have the following lemma controlling the size of \( H_B(P) \).

**Lemma 3.7.** For any \( (B, P) \in \mathcal{B} \times \mathbb{Q}^d \), we have
\[
q(P) \leq H_B(P) \leq q(P)^{1+\lambda}.
\] (3.5)
Attaching lines.

3.3. Proof. Write \( q(P) \) simply as \( q \), the first inequality is clear from the definition. By Lemma 3.5, \( J_{B,P} \) contains a vector \((a_0, b_0)\) with \( \|a_0\|_{\infty} \leq q^\lambda \) and \( |b_0 + z_B \cdot a_0| \leq q^\mu \). Thus, it follows from (3.4) that

\[
\max \left\{ \|a(B, P)\|_{\infty}, |b(B, P) + z_B \cdot a(B, P)| \right\} \leq \max \left\{ \|a_0\|_{\infty}, |b_0 + z_B \cdot a_0| \right\} \leq \max \{q^\lambda, q^\mu\} = q^\lambda.
\]

The second inequality follows.

Remark 3.8. It follows from the definition of \( a^+(B, P) \) that \( C(B, P) \in \mathbb{Z} \), thus the coefficients of \( F_{B,P} \) belong to \( \mathbb{Z} \).

3.3. Attaching lines. We shall define another function

\[ v^+: \mathcal{B} \times \mathbb{Q}^d \to \mathbb{Q}^d \]

in this subsection. The function \( v^+(\ast, P) \) takes values in the lattice \( \Lambda_P \) which is defined as follows:

\[ \Lambda_P = \mathbb{Z}^d + \mathbb{Z} \left[ \begin{pmatrix} p \quad r \\ q \quad q \end{pmatrix} \right], \text{ where } P = \begin{pmatrix} p \quad r \\ q \quad q \end{pmatrix}. \]

The line attached to the pair \((B, P)\) is defined to be

\[ \mathcal{L}_{B,P} := \left\{ w \in \mathbb{R}^d : w - \begin{pmatrix} p \\ q \end{pmatrix} = tv^+(B, P), \ t \in \mathbb{R} \right\}. \]

The definition of the function \( v^+ \) is given in the following lemma.

Lemma 3.9. For any \((B, P) \in \mathcal{B} \times \mathbb{Q}^d\), there exists a non-zero vector

\[ v^+(B, P) = (v(B, P), u(B, P)) \in \Lambda_P \]

with \( v(B, P) \in \mathbb{R}^{d-1}, u(B, P) \in \mathbb{R} \) such that

\[ \|v(B, P) - u(B, P)z_B\|_{\infty} \leq 2d q(P)^{-\lambda}, \ |u(B, P)| \leq 2d \xi(B, P) q(P)^{-\lambda-\mu}. \quad (3.6) \]

Proof. Write \( q(P) \) simply as \( q \). It is easy to check that \( d(\Lambda_P) = 1/q \), where \( d(\Lambda_P) \) denotes the covolume of the lattice \( \Lambda_P \). We will make use of the vector \( a^+(B, P) \) constructed in the previous subsection. For simplicity, write \( a^+(B, P), a(B, P), a_i(B, P), b(B, P), \xi(B, P), z_B \) as \( a^+, a, a_i, b, \xi, z \) respectively.

We have the following two distinct cases:

1. **Case** \( |a_k| q^{-\lambda} = \max(\|a_1| q^{-\lambda}, \ldots, |a_{d-1}| q^{-\lambda}, |b + z \cdot a|q^{-\lambda-\mu}) \), where \( 1 \leq k \leq d - 1 \).

   Then it is obvious that \( a_k \neq 0 \). Consider the convex body

   \[ \Sigma_k := \left\{ w = (w_1, \ldots, w_d) \in \mathbb{R}^d : |w_i - z_i w_d| \leq q^{-\lambda}, i \neq k, d; \right\}
   \]

   \[ |w_d| \leq \xi q^{-\lambda-\mu}; \ |a^+ \cdot w| < 1 \} \right\}. \]

   A direct computation shows that

   \[ 2^{-d}\text{Vol}(\Sigma_k) = |a_k|^{-1} \xi q^{-\lambda-\mu} \prod_{i \neq k, d} q^{-\lambda} = |a_k|^{-1} \xi q^{-1} \geq q^{-1}. \]
Hence there is a non-zero $\Lambda P$-lattice point $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_d)$ in $\Sigma_k$. Moreover, since $|a^+ \cdot \bar{w}| < 1$ implies $a^+ \cdot \bar{w} = 0$, we have

\[
|\bar{w}_k - z_k \bar{w}_d| = |a_k|^{-1} \left| \sum_{i \neq k,d} a_i (\bar{w}_i - z_i \bar{w}_d) + (b + z \cdot a) \bar{w}_d \right|
\]

\[
\leq |a_k|^{-1} \left( \sum_{i \neq k,d} |a_i| |\bar{w}_i - z_i \bar{w}_d| + |b + z \cdot a||\bar{w}_d| \right)
\]

\[
\leq |a_k|^{-1} \left( \sum_{i \neq k,d} |a_i| q^{-\lambda} + |b + z \cdot a| q^{-\lambda - \mu} \right)
\]

\[
\leq (d - 1) q^{-\lambda}.
\]

(2) Case $|b + z \cdot a| q^{-\lambda - \mu} = \max(|a_1| q^{-\lambda}, \ldots, |a_{d-1}| q^{-\lambda}, |b + z \cdot a| q^{-\lambda - \mu})$.

Then we consider the convex body

\[
\Sigma_d := \left\{ \bar{w} = (w_1, \ldots, w_d) \in \mathbb{R}^d : |w_i - z_i w_d| \leq 2q^{-\lambda}, i \neq d; \ |a^+ \cdot \bar{w}| < 1 \right\}.
\]

A direct computation shows that

\[
2^{-d} \text{Vol}(\Sigma_d) = |b + z \cdot a|^{-1} \prod_{i \neq d} 2q^{-\lambda} \geq 2^{d-1}(q^\mu + 1)q^{-1} \geq q^{-1}.
\]

Thus there is a non-zero $\Lambda P$-lattice point $\bar{w} = (\bar{w}_1, \ldots, \bar{w}_d)$ in $\Sigma_k$. Similarly we have

\[
|\bar{w}_d| = |b + z \cdot a|^{-1} \left| \sum_{i \neq d} a_i (\bar{w}_i - z_i \bar{w}_d) \right|
\]

\[
\leq |b + z \cdot a|^{-1} \left( \sum_{i \neq d} |a_i| |\bar{w}_i - z_i \bar{w}_d| \right)
\]

\[
\leq |b + z \cdot a|^{-1} \left( \sum_{i \neq d} 2|a_i| q^{-\lambda} \right)
\]

\[
\leq 2(d - 1) q^{-\lambda - \mu}.
\]

In each case above we set $v^+(B, P) = \bar{w}$ and this completes the proof. \qed

Remark 3.10. Let $\Pi_{B,P}$ denote the subset of $\mathbb{R}^d$ defined by the inequalities given in (3.6). Note that the volume of $\Pi_{B,P}$ may be smaller than $1/q$, so the above lemma does not follow directly from Minkowski’s linear forms Theorem.

4. Some subdivisions

As aforementioned, we will use the hyperplane potential game in establishing Theorem 3.1. This section is devoted to some preparations for playing hyperplane potential game on $U$ defined in (3.1). Hence we will fix $\beta \in (0, 1)$ and $\gamma > 0$, and a closed ball $B_0 \in \mathcal{B}$ in this section. We are going to define subfamilies $\mathcal{B}_n(n \geq 0)$ of $\mathcal{B}$ and decompositions of $\mathbb{Q}^d$ with respect to the $\beta, \gamma$ and $B_0$ given.

Firstly, denote

\[
\kappa := \max_{(x,y,z) \in B_0} \max \{ ||x||_\infty, ||y||, ||z||_\infty \} + 1
\]

Then choose a positive number $R$ satisfying

\[
R \geq \max \{ 4\beta^{-1}, 10^4 d^6 \kappa^4 \}, \quad \text{and} \quad (R^\gamma - 1)^{-1} \leq \left( \frac{\beta^2}{3} \right)^\gamma,
\]

(4.1)
and set
\[ \epsilon = 10^{-2}d^{-6}k^{-2}R^{-20d^2}\rho_0. \] (4.2)

Let \( B_0 = \{ B_0 \} \). For \( n \geq 1 \), let \( B_n \) be the subfamily of \( B \) defined by
\[ B_n := \{ B \subset B_0 : \beta R^{-n}\rho_0 < \rho(B) \leq R^{-n}\rho_0 \}. \]

In view of (4.1), the families \( B_n \) are mutually disjoint.

Let \( n \geq 0 \), and fix a closed ball \( B \in B_n \) in this paragraph. We define
\[ V_B := \{ P \in \mathbb{Q}^d : H_n \leq H_B(P) \leq 2H_{n+1} \} \]
where
\[ H_n = 2d^2\epsilon\kappa\rho_0^{-1}R^{n+1}. \]

It follows from (4.5) that if \( P \in V_B \), then
\[ H_{n+1} \leq q(P) \leq 2H_{n+1}. \]

We shall also need the following subdivisions of \( V_B \):
\[ V_{B,1} := \left\{ P \in V_B : H_n^{-1} \leq q(P) \leq H_n^{-1}R^{10d^2} \right\} \]
\[ V_{B,k} := \left\{ P \in V_B : H_n^{-1}R^{10d^2+(2k-4)d} \leq q(P) \leq H_n^{-1}R^{10d^2+(2k-2)d} \right\}, \quad k \geq 2. \]

One can show an important inequality here: for \( P \in V_{B,k}, k \geq 2, \)
\[ \frac{\xi(B,P)}{q(P)^\lambda} = \frac{H_B(P)}{q(P)^{1+\lambda}} \leq \frac{2H_{n+1}}{H_n R^{(1+\lambda)(10d^2+(2k-4)d)}} \leq 2R^{-8d^2-2kd+1}. \] (4.3)

Now we define a subfamily \( B'_n \) of \( B_n \) inductively as follows. Let \( B'_0 = \{ B_0 \} \). If \( n \geq 1 \) and \( B'_{n-1} \) has been defined, we let
\[ B'_{n} := \left\{ B \in B_n : B \subset B' \text{ for some } B' \in B'_{n-1}, \text{ and } B \cap \bigcup_{P \in V_B} \Delta_t(P) = \emptyset \right\} \]

The following lemma plays an important role in the proof of Theorem 3.1.

**Lemma 4.1.** Let \( n \geq 0, B \in B'_n \). Then for any \( P \in \mathbb{Q}^d \) with \( q(P)^{1+\lambda} \leq 2H_{n+1} \), we have \( \Delta_t(P) \cap B = \emptyset \).

**Proof.** Note that \( 2H_1 < 1 \), hence we may assume that \( n \geq 1 \). We denote \( B_n = B, \) and let \( B_n \subset \cdots \subset B_0 \) be such that \( B_k \in B'_k \). Assume the contrary that the conclusion of the lemma is not true. Then there exists \( P = (\frac{p}{q}, \frac{r}{s}) \in \mathbb{Q}^d \) with \( q^{1+\lambda} \leq 2H_{n+1} \) such that \( \Delta_t(P) \cap B_k \neq \emptyset \) for every \( 1 \leq k \leq n \). It then follows from the definition of \( B_k \) that \( P \notin V_{B_k} \), that is,
\[ H_{B_k}(P) \notin [H_k, 2H_{k+1}], \quad \forall 1 \leq k \leq n. \] (4.4)

Let \( 1 \leq n_0 \leq n \) be such that
\[ 2H_{n_0} < q^{1+\lambda} \leq 2H_{n_0+1}. \] (4.5)

We claim that
\[ H_{B_k}(P) < H_k, \quad \forall 1 \leq k \leq n_0. \] (4.6)

We prove the above claim inductively as follows. Since \( H_{B_{n_0}}(P) \leq q^{1+\lambda} \leq 2H_{n_0+1} \), it follows from (4.4) that (4.6) holds for \( k = n_0 \). Suppose that \( 1 \leq k \leq n_0 - 1 \) and (4.6) holds if \( k \) is replaced by \( k + 1 \). We prove that
\[ H_{B_k}(P) \leq 2H_{B_{k+1}}(P). \] (4.7)
Denote \( a^+(B_i, P) = (a_i, b_i), z_{B_i} = z_i (i = k, k + 1) \). We claim that
\[
a^+(B_{k+1}, P) \in \mathcal{A}(B_k, P).
\] (4.8)
Since \( a^+(B_{k+1}, P) \in \mathcal{A}(B_{k+1}, P) \), it is clear that
\[
(a_{k+1}, b_{k+1}) \neq (0, 0), \quad a_{k+1} \cdot p + b_{k+1} r + c_{k+1} q = 0, \quad \|a_{k+1}\|_\infty \leq q^k.
\]
On the other hand, it follows from (4.6) and the induction hypothesis that
\[
|b_{k+1} + z_k \cdot a_{k+1}| \leq |b_{k+1} + z_{k+1} \cdot a_{k+1}| + |(z_k - z_{k+1}) \cdot a_{k+1}|
\leq |b_{k+1} + z_{k+1} \cdot a_{k+1}| + d \|a_{k+1}\|_\infty \rho(B_k)
\leq q^\mu + \rho(B_{k+1})^\frac{1}{2} + dq^{-1} H_{B_{k+1}}(P) \rho(B_k)
\leq q^\mu + (\beta R)^{-\frac{1}{2}} + d H_{B_{k+1}}(P) \rho(B_k)
\leq q^\mu + 2^\frac{1}{2} \rho(B_k)^\frac{1}{2} + d(2d^2 \epsilon k R)^\frac{1}{2} \rho(B_k)^\frac{1}{2}
\leq q^\mu + \rho(B_k)^\frac{1}{2}.
\]
This proves our claim (4.8). It then follows from (4.8) and (3.3) that
\[
H_{B_k}(P) = q \max\{\|a_k\|_\infty, |b_k + z_k \cdot a_k|\}
\leq q \max\{\|a_{k+1}\|_\infty, |b_k + z_k \cdot a_{k+1}|\}
\leq q \max\{\|a_{k+1}\|_\infty, |b_k + z_{k+1} \cdot a_k| + d \|a_{k+1}\|_\infty \rho(B_k)\}
\leq 2q \max\{\|a_{k+1}\|_\infty, |b_k + z_{k+1} \cdot a_{k+1}|\}
= 2H_{B_{k+1}}(P).
\]
Thus (4.7) holds. It follows from (4.7) and the induction hypothesis that \( H_{B_k}(P) \leq 2H_{B_{k+1}}(P) \). By (4.1), we have \( H_{B_k}(P) < H_k \). Thus the claim (4.6) follows. This means that \( H_{B_k}(P) < H_1 < 1 \), a contradiction. This completes the proof. \( \square \)

5. Proof of Theorem 3.1

At first, we prove the following proposition which plays a key role in the proof of Theorem 3.1.

**Proposition 5.1.** Fix \( \beta \in (0, 1) \) and \( \gamma > 0 \), and a closed ball \( B_0 \in \mathcal{B} \) as in Subsection 4. Let \( R \) be a positive number satisfying (4.1) and \( \epsilon \) given by (4.2). For \( n \geq 0, B \in \mathcal{B}_n \) and \( k \geq 1 \), consider the set
\[
\mathcal{E}_{B,k,\epsilon} = \{(B', P) \in \mathcal{B} \times \mathbb{Q}^d : B' \in \mathcal{B}_{n+k}, B' \subset B, P \in V_{B',k}, \text{ and } \Delta_\epsilon(P) \cap B \neq \emptyset\}.
\]
Then there exists an affine hyperplane \( E_k(B) \subset \mathbb{R}^{2^d-1} \) such that for any \( (B', P) \in \mathcal{E}_{B,k,\epsilon} \), we have
\[
\Delta_\epsilon(P) \cap B' \subset E_k(B)(R^{-(n+k)} \rho_0).
\]

We shall need the following two lemmas.

**Lemma 5.2.** Let \( (B_1, P_1), (B_2, P_2) \in \mathcal{E}_{B,k,\epsilon} \), and \( F_{B_2, P_2} \) be the function defined in (4.2), then one has
\[
|F_{B_2, P_2}(P_1)| \leq 30d^4 \kappa^2 \epsilon q_{-1}^{-1} R^{e_{k+k+1}}
\]
with
\[
e_k = \begin{cases}
10d^2, & k = 1; \\
2d, & k > 1.
\end{cases}
\]
Lemma 5.3. For any \((B_1, P_1), (B_2, P_2) \in \mathcal{C}_{B,k,e}\), we have \(F_{B_2,P_2}(P_1) = 0\).

**Proof.** For simplicity, we write the objects \(a^+(B_j, P_j), \nu^+(B_j, P_j), \xi_{B_j, P_j}, F_{B_j, P_j}, \mathcal{L}_{B_j, P_j}, \mathcal{H}_{B_j, P_j} (j = 1, 2)\) as \(a_j^+, v_j^+, \xi_j, F_j, \mathcal{L}_j, \mathcal{H}_j\) respectively. We are divided into three cases:
(1) **Case** \( k = 1 \). Then by Lemma 5.2 we have
\[
q_1 |F_2(P_1)| \leq 30d^4 \epsilon R_c^{e_k+k+1} < 1.
\]
As \( q_1 |F_2(P_1)| \in \mathbb{Z} \), we have \( F_2(P_1) = 0 \).

(2) **Case** \( k \geq 2 \) and \( \mathcal{L}_1 \parallel \mathcal{H}_2 \), that is
\[
a_2^+ \cdot v_1^+ = 0.
\] (5.1)
Assume the contrary that \( F_{B_2,P_2}(P_1) \neq 0 \). Write \( v_1^+ = (v_1, u_1) = (v_1,1, \ldots, v_{d-1,1}, u_1) \).

We claim that
\[
q_1 |F_2(P_1)v_{i,1}|, \; q_1 |F_2(P_1)u_1| \in \mathbb{Z} \text{ for each } 1 \leq i \leq d - 1.
\] (5.2)
Indeed, since \( v_1^+ \in \Lambda_{P_1} \setminus \{0\} \), we can write
\[
v_1^+ = c \left( \frac{p_1}{q_1}, \frac{r_1}{q_1} \right) + c,
\] (5.3)
where \( c \in \mathbb{Z} \), \( c \in \mathbb{Z}^d \). Combining (5.1) and (5.3), we get
\[
c F_2(P_1) \in \mathbb{Z}.
\] (5.4)
According to (5.3), \( q_1 v_{i,1}, q_1 u_1 \in c\mathbb{Z} + q_1\mathbb{Z} \). Then the claim (5.2) follows directly from (5.4).

Note that \( v_1^+ \neq 0 \). It follows from (5.2) that
\[
q_1 |F_2(P_1)| \left( \sum_{1 \leq i \leq d - 1} |v_{i,1}| + |u_1| \right) \geq 1.
\]
However, according to Lemma 5.2 and (4.3), we have
\[
q_1 |F_2(P_1)| \left( \sum_{1 \leq i \leq d - 1} |v_{i,1}| + |u_1| \right)
\leq 30d^4 \epsilon R_c^{e_k+k+1} \left( 2d(d - 1) q_1^{-\lambda} + 2d \xi_1 q_1^{-\lambda-\mu} \right)
\leq 60d^6 \epsilon R_c^{e_k+k+1} \xi_1 q_1^{-\lambda}
\leq 120d^6 \epsilon R_c^{2d+k+1} R^{-8d^2-2kd+1}
< 1
\]
which leads to a contradiction.

(3) **Case** \( k \geq 2 \) and \( \mathcal{L}_1 \) intersects \( \mathcal{H}_2 \).
Assume the contrary that \( F_{B_2,P_5}(P_1) \neq 0 \). Let
\[
P_0 = \frac{P_0^+}{q_0} = \left( \frac{p_0}{q_0}, \frac{r_0}{q_0} \right)
\]
be their intersection. Write
\[
\frac{P_0^+}{q_0} = \frac{P_1^+}{q_1} + t_0 v_1^+.
\]
Then \( \left( \frac{p_0^+}{q_0}, t_0 \right)^T \) is the solution of the following linear equations
\[
\begin{pmatrix}
q_1 I_d & -q_1 v_1^{+T} \\
a_2^+ & 0
\end{pmatrix}
\begin{pmatrix}
w \\
t
\end{pmatrix}
= \begin{pmatrix}
p_1^+ \\
C_2
\end{pmatrix}
\]
where $v_1^+T, p_1^+T$ means the transport of $v_1^+, p_1^+$, and $C_2 = C(B_2, P_2)$ is defined in \(3.3\). Let $M$ be the following matrix
\[
\begin{pmatrix}
q_1I_d & -q_1v_1^{+T} \\
\alpha_2 & C_2
\end{pmatrix}
\]
and $M_i(1 \leq i \leq d + 2)$ be the matrix obtained by deleting the $i$-th column of $M$. In view of the fact that $v_1^+ \in \Lambda P_1$, a simple computation immediately implies
\[
\det(M_i) \in q_1^{d-1} \mathbb{Z} \quad (1 \leq i \leq d + 2).
\]

By Cramer’s rule,
\[
\begin{pmatrix}
p_0^+ \\
q_0
\end{pmatrix}
\begin{pmatrix}
t_0
\end{pmatrix}
= \begin{pmatrix}
\det(M_1) & \cdots & \det(M_{d+1}) \\
\det(M_{d+2}) & \cdots & \det(M_{d+2})
\end{pmatrix}.
\]

Hence
\[
|t_0| = \frac{\det(M_{d+1})}{\det(M_{d+2})} = \frac{|F_2(P_1)|}{|\alpha_2 \cdot v_1^+|}.
\]

In view of (5.5) and (5.6), we have
\[
q_0 \leq q_1^{-d+1}\det(M_{d+2}) = q_1|\alpha_2 \cdot v_1^+|.
\]

It is clear that
\[
|\alpha_2 \cdot v_1^+| \leq \sum_{1 \leq i \leq d-1} \left|\varepsilon_2 \cdot 2dq_1^{-\lambda}\right| + \left|q_2^d \cdot 2d\xi q_1^{-1-\lambda-\mu}\right|
\]
\[
\leq 2d(d - 1)R^{e_k}\varepsilon_1 q_1^{-\lambda} + 2dR^{e_k}\varepsilon_1 q_1^{-\lambda}
\]
\[
\leq 2d^2R^{e_k}\xi q_1^{-\lambda}
\]
\[
\leq 2d^2R^{-8d^2-(2k-2)d+1}.
\]

It follows that
\[
\frac{q_0}{q_1} \leq |\alpha_2 \cdot v_1^+|\leq 2d^2R^{-8d^2-(2k-2)d+1}
\]
\[
\leq \frac{1}{2}.
\]

Combine the inequalities (5.8), (5.9), (1.3) and the obvious estimate $\lambda \geq 1/d$, we have
\[
q_0^{1+\lambda} \leq q_1^{1+\lambda}|\alpha_2 \cdot v_1^+|^{1+\lambda}
\]
\[
\leq q_1^{1+\lambda}(2d^2R^{e_k}\xi q_1^{-\lambda})^{1+\lambda}
\]
\[
\leq 4d^4R^{4d}(\xi q_1^{-\lambda})H_B_1(P_1)
\]
\[
< 4d^4R^{4d}R^{-8d^2-2k+1}H_H_n
\]
\[
\leq 8d^4R^{-4d-k+1}H_n
\]
\[
\leq H_n.
\]

Note that
\[
\left\|\frac{p_1}{q_1} - \frac{p_0}{q_0} - \left(\frac{r_1}{q_1} - \frac{r_0}{q_0}\right)z_{B_1}\right\|_{\infty} = |t_0|\|v_1 - z_{B_1}u_1\|_{\infty} \leq \frac{2d|t_0|}{q_1}
\]
and
\[
\left|\frac{r_1}{q_1} - \frac{r_0}{q_0}\right| = |t_0u_1| \leq \frac{2d|t_0|\xi_1}{q_1^{1+\mu}}.
\]
We claim that
\[ \Delta_\varepsilon(P_1) \cap B \subset \Delta_\varepsilon(P_0). \] (5.12)

In view of Lemma 4.11 and 5.11, (5.12) will contradict the assumption that
\[ B \in \mathcal{B}_r. \] It remains to prove (5.12). Indeed, for \( (x, y, z) \in \Delta_\varepsilon(P_1) \cap B \), by (5.10),
3.7 and 5.8 we have
\[
q_0^{1+\mu} \left| y - \frac{r_0}{q_0} \right| \leq q_0^{1+\mu} \left| y - \frac{r_1}{q_1} \right| + q_0^{1+\mu} \left| \frac{r_1}{q_1} - \frac{r_0}{q_0} \right|
\]
\[
\leq q_0^{1+\mu} \frac{\varepsilon}{q_1^{1+\mu}} + q_0^{-2} 2d |\varepsilon| q_1 \frac{\xi_1}{q_1}
\]
\[
\leq \frac{\varepsilon}{2} + 2d q_1 |F_2(P_1)| \frac{\xi_1}{q_1}
\]
\[
\leq \frac{\varepsilon}{2} + 60 d^5 \kappa^2 R^{2d+k+2-8d^2-2kd}\varepsilon
\]
\[
\leq \varepsilon.
\]

and
\[
q_0^{1+\lambda} \left| x - \frac{p_0}{q_0} - \left( y - \frac{r_0}{q_0} \right) z \right| \infty
\]
\[
\leq q_0^{1+\lambda} \left| x - \frac{p_1}{q_1} - \left( y - \frac{r_1}{q_1} \right) z \right| \infty + q_0^{1+\lambda} \left| \frac{p_1}{q_1} - \frac{p_0}{q_0} - \left( \frac{r_1}{q_1} - \frac{r_0}{q_0} \right) z \right| \infty
\]
\[
\leq q_0^{1+\lambda} \frac{\varepsilon}{q_1^{1+\lambda}} + q_0^{1+\lambda} \frac{2d |\varepsilon| q_1}{q_1^{1+\lambda}} + q_0^{-2} d |\varepsilon| q_1 \frac{r_1}{q_1} - \frac{r_0}{q_0} \left| z - z_{B_1} \right| \infty
\]
\[
\leq \frac{\varepsilon}{2} + 2d |q_1 F_2(P_1)| \frac{q_1^{\lambda}}{q_1^{\lambda}} + 2d^2 |q_1 F_2(P_1)| \frac{q_1^{\lambda}}{q_1^{\lambda}} \cdot 2R^{-n} q_0 |\xi_1| q_1^{1+\mu}
\]
\[
\leq \frac{\varepsilon}{2} + 120 d^5 \kappa^2 \varepsilon R^{2d+k+1} \cdot 2d^2 R^{-8d-(2k-2)+1} R^k
\]
\[
\leq \varepsilon.
\]

\[ \square \]

**Proof of Proposition 5.1** Choose \((B'_0, P_0) \in \mathcal{C}_{B,k,\varepsilon}\) such that
\[ q_0 = q(P_0) = \min \left\{ q(P) : \exists \text{ closed ball } B' \text{ with } (B', P) \in \mathcal{C}_{B,k,\varepsilon} \right\}. \]

Consider the attached hyperplane in \(\mathbb{R}^{2d-1}\)
\[ \mathcal{H}_{B'_0, P_0} = \left\{ (x, y, z) \in \mathbb{R}^{2d-1} : a_0 \cdot x + b_0 y - C = 0 \right\} \]
where \(a_0^+ = (a_0, b_0)\) and \(C = C(B'_0, P_0)\) are given in Subsection 3.2. We claim that \(\mathcal{H}_{B'_0, P_0}\)
is the \(E_k(B)\) that we need. In other words, for any \((B', P) \in \mathcal{C}_{B,k,\varepsilon},\)
\[ \Delta_\varepsilon(P) \cap B' \subset \mathcal{H}_{B'_0, P_0}^{(R^{-(n+k)} \rho_0)}. \]
Indeed, we have proved in Lemma 5.3 that $P \in \mathcal{H}_{B_{0},r_{0}}$ for $(B', P) \in \mathcal{G}_{B,k,\epsilon}$. Hence for any $(x, y, z) \in \Delta_{\epsilon}(P) \cap B'$, we have

$$|a_{0} \cdot x + b_{0}y - C| = |a_{0} \cdot \left( x - \frac{P}{q} \right) + b_{0} \left( y - \frac{r}{q} \right)|$$

$$\leq (d - 1)\|a_{0}\|_{\infty} \left\| x - \frac{P}{q} - \left( y - \frac{r}{q} \right) z \right\|_{\infty} + |b_{0} + \beta \cdot a_{0}| \left\| y - \frac{r}{q} \right\|_{\infty}$$

$$\leq (d - 1)q_{0}^{\lambda} \frac{\epsilon}{q^{1+\lambda}} + 2q_{0}^{\mu} \frac{\epsilon}{q^{1+\mu}}$$

$$\leq (d + 1) \frac{\epsilon}{q_{0}}.$$ 

Denote the width of this thicken hyperplane as $\omega$, then

$$\omega \leq \frac{(d + 1)\epsilon}{q_{0} \max\{\|a_{0}\|_{\infty}, |b_{0}|\}}$$

$$\leq \frac{(d + 1)\epsilon}{1 + (d - 1)\kappa} \max\{\|a_{0}\|_{\infty}, |b_{0} + z \cdot a_{0}|\}$$

$$\leq \frac{(d + 1)(1 + (d - 1)\kappa)\epsilon}{H_{n+k}}$$

$$\leq R^{-(\alpha + \kappa)\rho_{0}}$$

which finishes the proof. ∎

**Proof of Theorem 3.1**. In view of Remark 3.3, Lemma 2.3 and Lemma 3.4, to prove Theorem 3.1 it suffices to show that the set $S(\mathbf{r})$ is $(\beta, \gamma)$-HPW for any $\beta \in (0, 1)$ and $\gamma > 0$. Fix $\beta \in (0, 1)$ and $\gamma > 0$ from now on. Bob starts the $(\beta, \gamma)$-hyperplane potential game on $\mathbb{R}^{d-1}$ with target set $S(\mathbf{r})$ by choosing a closed ball $B_{0} \subset \mathbb{R}^{d-1}$ of radius $\rho_{0}$. As discussed in [3, Remark 2.4], without loss of generality we may assume that Bob will play so that $\rho_{0} \leq 1/d$ and $\rho_{i} := \rho(B_{i}) \to 0$, where $B_{i}$ is the ball chosen by Bob at the $i$-th turn. Now we have fixed $\beta \in (0, 1)$ and $\gamma > 0$, and the closed ball $B_{0} \subset \mathcal{B}$ as in Proposition 5.1. Let $R$ be a positive number satisfying (4.1) and $\kappa$ be the constant given by (4.2). Write $i_{n}$ to be the smallest nonnegative integer with $B_{i_{n}} \subset \mathcal{B}_{n}$. Let $\mathcal{N}$ denote the set of all $n \in \mathbb{N}$ with $B_{i_{n}} \subset \mathcal{B}_{n}$.

Let Alice play according to the strategy as follows. At the $i$-th stage, if $i = i_{n}$ for some $n \in \mathcal{N}$, then Alice chooses the family of hyperplane neighborhoods $\{E_{k}(B_{i_{n}}) : k \in \mathbb{N}\}$, where the hyperplane $E_{k}(B_{i_{n}})$ is given by Proposition 5.1. Otherwise, Alice makes an empty move. Since $B_{i_{n}} \subset \mathcal{B}_{n}$, it follows that $\rho_{i_{n}} > \beta R^{-n}\rho_{0}$. Hence Alice’s move is legal as we have

$$\sum_{k=1}^{\infty} (3R^{-(\alpha + \kappa)\rho_{0}})^{\gamma} = (3R^{-n}\rho_{0})^{\gamma}(R^{\gamma} - 1)^{-1} \leq (\beta \rho_{i_{n}})^{\gamma}.$$ 

We claim that this is a winning strategy for Alice, that is, the point $\mathbf{x}_{\infty} = \bigcap_{i=0}^{\infty} B_{i}$ lies in the set

$$S(\mathbf{r}) \cup \bigcup_{n \in \mathcal{N}} \bigcup_{k=1}^{\infty} E_{k}(B_{i_{n}})^{(3R^{-(\alpha + \kappa)\rho_{0}})}.$$ 

To prove our claim, we are divided into two different cases:

1. **Case** $\mathcal{N} = \mathbb{N} \cup \{0\}$. For any $P \in \mathbb{Q}^{d}$, there is $n$ such that $q^{1+\lambda} \leq 2H_{\rho_{i_{n}}+1}$. Since $n \in \mathcal{N}$, we have $B_{i_{n}} \subset \mathcal{B}_{n}$. Then we have $\Delta_{\epsilon}(P) \cap B_{i_{n}} = \emptyset$ by Lemma 4.1. Thus it follows from the definition of $S(\mathbf{r})$ that $\mathbf{x}_{\infty} \subset S_{\epsilon}(\mathbf{r}) \subset S(\mathbf{r})$. Hence Alice wins.
(2) **Case** \( \mathcal{N} \neq \mathbb{N} \cup \{0\} \). Let \( n \) be the smallest integer with \( n \notin \mathcal{N} \). Then we have 
\[ B_{n} \notin \mathcal{B} \text{ and } B_{n-1} \in \mathcal{B}_{n-1} \text{ as } n - 1 \in \mathcal{N}. \]
By the definition of \( \mathcal{B} \), there exists \( P \in V_{B_{n},k} \) with \( 1 \leq k \leq n \) and \( \Delta \) such that \( \Delta \setminus B_{n} \neq \emptyset \). By Proposition 5.1, we have \( \Delta \setminus B_{n} \subseteq E_{k}(B_{n-1})^{(R-n\rho_{0})} \). In view of \( \rho_{n} \leq R^{-n}\rho_{0} \), it follows that \( x_{\infty} \in B_{n} \subseteq E_{k}(B_{n-1})^{(3R-n\rho_{0})} \). Hence Alice wins.

This completes the proof of Theorem 3.1 \( \Box \)

6. **Proof of main theorems**

In this section, we deduce Theorem 1.2 and Theorem 1.3 from Theorem 3.1. Indeed, the argument presented here is similar to the argument presented in [3, Section 6]. For sake of completeness, we reproduce the proof in our setting here.

**Proof of Theorem 1.2**. The proof is divided into three steps:

**Step 1.** We show that it suffices to prove the set \( E(F^{+}) \) is HAW on \( G/\Gamma \). Indeed, by applying the following diffeomorphism
\[
\tau : G/\Gamma \to G/\Gamma, \tau(g\Gamma) = (gT)^{-1}\Gamma
\]
to the set \( E(F^{+}) \), we can see that the set \( E(F^{-}) \) is also HAW if \( E(F^{+}) \) does, where \( F^{-} \) denotes the subgroup \( \{e\} \cup (F \setminus F^{+}) \). Hence, in view of the intersection stability of HAW sets, \( E(F) \) will be HAW if \( E(F^{+}) \) does.

**Step 2.** We show that it suffices to prove the theorem for \( F^{+} = F_{r}^{+} \), which was defined in Theorem 3.1. Indeed, by the real Jordan decomposition (cf. [16, Proposition 4.3.3]), for any one-parameter diagonalizable subsemigroup \( F^{+} \), there are one-parameter subsemigroups \( F_{i}^{+} = \{ g_{t}^{(i)} : t > 0 \} \) such that \( F_{1}^{+} \) is \( \mathbb{R} \)-diagonalizable, \( F_{2}^{+} \) has compact closure, and \( g_{t} = g_{t}^{(1)} g_{t}^{(2)} \) with \( g_{t}^{(1)} \) commuting with \( g_{t}^{(2)} \). It is obvious that \( E(F^{+}) = E(F_{r}^{+}) \). Hence we are reduced to consider the case that \( F \) satisfying (1.1) and \( \mathbb{R} \)-diagonalizable, which is equivalent to say that there exists \( g' \in G \) and \( r \) satisfying (1.3) such that \( F^{+} = g'F_{r}^{+} g'^{-1} \). Note that in this case we have \( E(F^{+}) = g' E(F_{r}^{+}) \). Hence our statement follows from (3) of Lemma 2.2.

**Step 3.** We prove the theorem for \( F_{r}^{+} \). In view of Lemma 2.2, we have to prove that for any \( \Lambda \in G/\Gamma \), there is an open neighborhood \( \Omega \) of \( \Lambda \) in \( G/\Gamma \) such that \( \Omega 
 \cap E(F_{r}^{+}) \) is HAW on \( \Omega \). Let
\[
P = \left\{ g \in G : g = \begin{pmatrix} T & 0 \\ N & T' \end{pmatrix}, T \in GL_{d-1}(\mathbb{R}), N \in M_{2 \times (d-1)}(\mathbb{R}), T' \text{ is lower triangular} \right\}.
\]
It’s not hard to check that for any \( g \in P \), the set \( \{g, gg_{t}^{-1} : t > 0\} \) is bounded in \( G \). By the Bruhat decomposition, the set \( PU \) is Zariski open in \( G \) and the multiplication map \( P \times U \to PU \) is a diffeomorphism.

According to the Borel density theorem, the set \( \pi^{-1}(\Lambda) \) is Zariski dense in \( G \). Hence, \( \pi^{-1}(\Lambda) \cap PU \neq \emptyset \), that is, there exists \( p_{0} \in P \) and \( u_{0} \in U \) such that \( \Lambda = p_{0} u_{0} \Gamma \).

Let \( \Omega_{P} \) and \( \Omega_{U} \) be open neighborhoods of \( p_{0} \) and \( u_{0} \) in \( P \) and \( U \) respectively, which are small enough such that the map \( \phi : \Omega_{P} \times \Omega_{U} \to G/\Gamma, \phi(p, u) = pu\Gamma \) is a diffeomorphism onto an open subset \( \Omega \) in \( G/\Gamma \). In view of Lemma 2.2(4), it suffices to prove that the set
\[
\phi^{-1}(E(F_{r}^{+}) \cap \Omega) = \{ (p, u) \in \Omega_{P} \times \Omega_{U} : pu\Gamma \in E(F_{r}^{+}) \}
\]
is HAW on \( \Omega_{P} \times \Omega_{U} \). By the definition of \( P \), we have that \( pu\Gamma \in E(F_{r}^{+}) \) if and only if \( u\Gamma \in E(F_{r}^{+}) \). It follows that the set (6.1) is equal to
\[
\Omega_{P} \times \{ u \in \Omega_{U} : u\Gamma \in E(F_{r}^{+}) \}.
\]
Then it from Theorem 3.1 and (5) of Lemma 2.2 that the set \( E(F_{r}^{+}) \) is HAW. \( \Box \)
Proof of Theorem 1.3. We will prove the theorem only for \( F^+ = F^+_r \) with \( r \) satisfying (1.3) here, since the proof for general \( F^+ \) satisfying (1.1) follows along the same lines as Step 2 of the proof of Theorem 1.2 and will be omitted. There are two subcases.

(1) Case \( r_1 > r_d \). Then it is easy to check that \( H(F^+) \) is equal to \( U \). We need to prove that for any \( \Lambda \in G/\Gamma \), the set \( u \in U \) such that \( u\Lambda \in E(F^+) \) is HAW on \( U \).

In view of Lemma 2.2, it suffices to prove that for any \( u_0 \in U \), there is an open neighborhood \( \Omega \) of \( u_0 \) in \( U \) such that the set

\[
\{ u \in \Omega : u\Lambda \in E(F^+) \}
\]

is HAW on \( \Omega \). Similar to the proof of Theorem 1.2, the Bruhat decomposition and the Borel density theorem imply that \( \pi^{-1}(\Lambda) \cap u_0^{-1}PU \neq \emptyset \). Choose \( g_0 \in \pi^{-1}(\Lambda) \cap u_0^{-1}PU \). Then \( \Lambda = g_0\Gamma \) and \( u_0g_0 \in PU \). Let \( \Omega_1 \) be an open neighborhood of \( u_0 \) in \( U \) with \( \Omega_1g_0 \subset PU \). Then there are smooth maps \( \phi : \Omega_1 \to P \) and \( \psi : \Omega_1 \to U \) such that

\[
u g_0 = \phi(u)\psi(u) \quad \forall u \in \Omega_1
\]

We claim that

the tangent map \( (d\psi)_{u_0} \) is a linear isomorphism. (6.4)

The set (6.2) is HAW follows from our claim. Indeed, assuming (6.4), we can find a neighborhood \( \Omega \subset \Omega_1 \) such that \( \psi \) is a diffeomorphism when restricted on \( \Omega \). Note that for \( u \in \Omega \), the set

\[
F^+_r u\Lambda = F^+_r u g_0 \Gamma = F^+_r \phi(u)\psi(u) \Lambda
\]

is bounded if and only if \( F^+_r \psi(u)\Lambda \) is bounded. Hence, in view of Theorem 3.1 and Lemma 2.2, we prove that the set (6.2) is HAW modulo Claim (6.4).

Let’s turn to the proof of Claim (6.2). Write \( p_1 = \phi(u) \), \( u_1 = \psi(u) \), then it follows from (6.3) that

\[
dr_{g_0}(Y) = dr_{u_1} \circ (d\phi)_{u_0}(Y) + dl_{p_1} \circ (d\psi)_{u_0}(Y), \quad \forall Y \in T_{u_0}U.
\]

If \( (d\psi)_{u_0}(Y) = 0 \), then the left-hand side of (6.5) belongs to \( T_{u_0g_0}(Uu_0g_0) \) and right-hand side belongs to \( T_{u_0g_0}(Pu_0g_0) \), thus \( Y = 0 \). This proves Claim (6.2).

(2) Case \( r_1 = r_d = \frac{1}{2} \). In this case, the expanding horospherical subgroup \( H \) coincide with the subgroup \( U_0 \) defined as

\[
U_0 := \{ u_x : x \in \mathbb{R}^d \}, \quad \text{where} \quad u_x = \begin{pmatrix} I_d & x \\ 0 & 1 \end{pmatrix} \in G.
\]

In view of the correspondence presented in [8, Theorem 2.20], the set \( \{ x \in \mathbb{R}^d : u_x\Gamma \in E(F^+) \} \) coincides with the set of badly approximable vectors \( \text{Bad}_d \), which is proved to be HAW already in [8]. Then we omit the remaining part of the proof here, since it is similar to the proof of the above case \( r_1 > r_d \).

\[\square\]

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