Generalizing Random Fourier Features via Generalized Measures

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Abstract

We generalize random Fourier features, that usually require kernel functions to be both stationary and positive definite (PD), to a more general range of non-stationary or/and non-PD kernels, e.g., dot-product kernels on the unit sphere and a linear combination of positive definite kernels. Specifically, we find that the popular neural tangent kernel in two-layer ReLU network, a typical dot-product kernel, is shift-invariant but not positive definite if we consider $\ell^2$-normalized data. By introducing the signed measure, we propose a general framework that covers the above kernels by associating them with specific finite Borel measures, i.e., probability distributions. In this manner, we are able to provide the first random features algorithm to obtain unbiased estimation of these kernels. Experiments on several benchmark datasets verify the effectiveness of our algorithm over the existing methods. Last but not least, our work provides a sufficient and necessary condition, which is also computationally implementable, to solve a long-lasting open question: does any indefinite kernel have a positive decomposition?

1 Introduction

Random Fourier features (RFF) [1] is a powerful technique in kernel-based learning which samples a series of random features from a distribution (obtained by Fourier transform) to approximate the kernel function. It brings promising performance and solid theoretical guarantees on scaling up kernel methods in classification [2,3], nonlinear component analysis [4,5], and neural tangent kernel (NTK) [6,7]. It is noteworthy that random features can be regarded as a class of two-layer neural networks in the lazy regime [9], and thus can be utilized to analyze over-parameterized neural networks in [8,7]. Due to the great success of RFF in machine learning society, Rahimi and Recht won the Test-of-Time Award in NeurIPS 2017 for their seminal work on RFF [1] and Li et al. [3] won the Honorable Mentions (best paper finalist) in ICML 2019 for their unified theoretical analysis on RFF.

The theoretical foundations behind RFF is intuitive: a positive definite (PD) function corresponds to a nonnegative and finite Borel measure, i.e., a probability distribution, via Fourier transform. Then we can sample random features from this distribution so as to approximate this PD function.

**Theorem 1** (Bochner’s Theorem [10]). Let $k : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be a bounded continuous function satisfying the shift-invariant property, i.e., $k(x, x') = k(x - x')$. Then, $k$ is positive definite if and
only if it is the (conjugate) Fourier transform of a nonnegative and finite Borel measure $\mu$

$$k(x - x') = \int_{\mathbb{R}^d} \exp \left( i\omega^\top (x - x') \right) \mu(d\omega) = E_{\omega \sim \mu} \left[ \exp \left( i\omega^\top (x - x') \right) \right].$$

Here $k(\cdot, \cdot)$ is the kernel function throughout this paper. Typically, the kernel in practical uses is real-valued and thus the imaginary part can be discarded, i.e., $k(x - x') = E_{\omega \sim \mu} \cos(\omega^\top (x - x'))$. It is clear that Bochner’s theorem requires the kernel function $k(\cdot, \cdot)$ to be (i) shift-invariant (also called “stationary”) and (ii) positive definite. These two conditions together exclude a series of commonly used kernels including

- Dot-product kernel\(^1\) polynomial kernels \([11]\), arc-cosine kernels \([12]\), NTK \([6]\).
- Indefinite kernels in a reproducing kernel Kreın space (RKKS) \([13]\): (i) a linear combination of positive definite kernels: Delta-Gaussian \([14]\); (ii) conditionally positive definite kernels, e.g., log kernel, power kernel \([15]\).
- Specifically designed but indefinite kernels: Gaussian kernel on manifolds \([16]\), polynomial kernels on spheres \([17]\), TL1 kernel \([18]\), tanh kernel \([11]\), and indefinite kernels with arbitrary types.

Pennington et al. \([17]\) point out that dot-product kernels with $\ell_2$-normalized data, i.e., on the unit sphere, can be shift-invariant but not always PD. Therefore, in this paper, we consider stationary kernels, either positive definite or indefinite, admitting the following integration representation by denoting $z := x - x'$

$$k(x - x') := c \int_\Omega g(\omega, z)\mu(d\omega) = c_+ \int_\Omega g_+(\omega, z)\mu_+(d\omega) - c_- \int_\Omega g_-(\omega, z)\mu_-(d\omega), \quad (1)$$

where the integral region $\Omega$ is chosen as either $\mathbb{R}^d$ or $\mathbb{R}^d \setminus \{0\}$ in this paper and $c, c_+, c_-$ are some nonnegative coefficients. Here $\mu$ is a signed measure \([19]\) which is a generalized measure by allowing it to have negative values. It admits the Jordan decomposition \([20]\), i.e., $\mu := \mu_+ - \mu_-$ where $\mu_+$ and $\mu_-$ are two nonnegative measures with at least one of them being finite. The total mass $\|\mu\|$ is defined as $\|\mu\| := \int_\Omega |\mu(\omega)|d\omega = \|\mu_+\| + \|\mu_-\|$. Specifically, to make sampling process feasible in algorithm implementation, the total mass is required to be finite, i.e., $\|\mu\| < \infty$. The functions $g(\omega, z)$ and $g_\pm(\omega, z)$ are not limited to $\exp(\omega^\top z)$ in Fourier transform (or in the sense of tempered distribution) and can be extended to other formulations. The integration representation in Eq. (1) is general to cover various kernels, e.g., PD kernels, dot-product kernels on the unit sphere, and indefinite kernels. Note that the Fourier transform of non-PD kernels, i.e., the measure $\mu(\cdot)$ cannot be regarded as a nonnegative Borel measure, or even a measure. Analysis of non-PD kernels is often based on RKKS, but we do not know whether a non-PD kernel can be identified with a reproducing kernel in RKKS, which in fact corresponds to a long-lasting open question \([13, 21, 22]\): does any indefinite kernel have a positive decomposition? The introduced signed measure decomposition would be an accessible way to answer this question, and accordingly we make the following contributions:

- In Section 3 by introducing the signed measure, we generalize RFF to a series of kernels that are not positive definite. We provide a sufficient and necessary condition to answer the above open question in RKKS via the measure decomposition technique. Moreover, this condition also guides us how to find a specific positive decomposition in practice, and thus we can devise a sampling strategy to obtain randomized feature maps. To the best of our knowledge, this is the first work to generate unbiased estimation for non-PD kernel approximation by random features.
- In Section 4 we demonstrate the feasibility of our random feature algorithm on several indefinite kernels. We begin with an intuitive example, e.g., a linear combination of positive definite kernels, and then consider dot-product kernels on the unit sphere for kernel approximation. Moreover, we prove that the popular NTK in two-layer ReLU network is shift-invariant but not positive definite if we use $\ell_2$-normalized data, which motivates us to reconsider its induced functional spaces and related properties.

\(^1\)In this case, the Fourier basis functions are spherical harmonics \([11]\).
• In Section 5, we evaluate various non-PD kernels on several typical large-scale datasets in terms of kernel approximation and the subsequent classification task. Our experimental results validate the theoretical claims and demonstrate the effectiveness of the proposed kernel approximation algorithm.

2 Related Works

A series of research focus on non-stationary kernel approximation, e.g., approximating polynomial kernels by Maclaurin expansion [23], the tensor sketch technique [24]; approximating additive kernels [24, 26]; approximating the Gaussian kernel on the unit sphere by the discrete cosine transform [27]. However, the considered kernels in the above works are still positive definite and the designed approximation algorithms are infeasible to indefinite kernels. Regarding to non-PD kernel approximation, we notice that there are four papers on this task: (i) Pennington et al. [17] find that the polynomial kernel on the unit sphere is not PD, and they use a PD kernel (associated with a positive sum of Gaussian distributions) to approximate it; (ii) Liu et al. [28] decompose a (subset of) kernel matrix into two PD kernel matrices, and then learn their respective randomized feature maps by infinite Gaussian mixtures. However, this approach in fact focuses on approximating kernel matrices rather than kernel functions; (iii) Mehrkanoon et al. [29] investigate Nyström approximation for indefinite kernels in spectral learning; (iv) Oglic and Gärtner [30] propose Nyström methods for low-rank approximation of indefinite kernels in RKKS. The first two works are based on RFF and the last two focus on Nyström approximation. Up till now, approximating non-PD kernels by random features cannot ensure unbiased and has not yet been fully investigated. Instead, based on the measure decomposition technique, our work achieves both simplicity and effectiveness by having (i) an unbiased estimator, (ii) incurring no extra parameters.

3 Randomized Feature Map via Signed Measure

In this section, we begin with the concept of signed measures and answer the open question, and then devise the sampling strategy for random features. For simplicity of notation, we denote \( z := \| z \|_2 = \| x - x' \|_2 \) and \( \omega := \| \omega \| \). Moreover, a function \( k(z) \) is called radial if \( k(z) = k(\| z \|) \), where the distance is usually defined by the \( L_2 \)-norm. To notify, the considered stationary kernels in this paper are all radial, and accordingly, their Fourier transforms are also radial, i.e., \( \mu(\omega) = \mu(\omega) \).

3.1 Signed measure

Let \( \mu : \mathcal{A} \rightarrow [0, +\infty] \) be a measure on a set \( \Omega \) satisfying \( \mu(\emptyset) = 0 \) and \( \sigma \)-additivity (i.e., countably additive). We call \( \mu \) a finite measure if \( \mu(\Omega) < +\infty \). Specifically, \( \mu \) is a probability measure if \( \mu(\Omega) = 1 \), and the triple \((\Omega, \mathcal{A}, \mu)\) is referred as the corresponding probability space. Here we consider the signed measure, a generalized version of a measure allowing for negative values.

Definition 1. (Signed measure [19]) Let \( \Omega \) be some set, \( \mathcal{A} \) be a \( \sigma \)-algebra of subsets on \( \Omega \). A signed measure is a function \( \mu : \mathcal{A} \rightarrow (-\infty, +\infty) \) or \((-\infty, +\infty]\) satisfying \( \sigma \)-additivity.

Based on the definition of signed measures, the following theorem shows that any signed measure can be represented by the difference of two nonnegative measures.

Theorem 2. (Jordan decomposition [20]) Let \( \mu \) be a signed measure defined on the \( \sigma \)-algebra \( \mathcal{A} \) as given in Definition 1. There exists two (nonnegative) measures \( \mu_+ \) and \( \mu_- \) (one of them is a finite measure) such that \( \mu = \mu_+ - \mu_- \).

The total mass of \( \mu \) on \( \mathcal{A} \) is defined as \( \| \mu \| = \| \mu_+ \| + \| \mu_- \| \). Note that this decomposition is not unique. It can be characterized by the Hahn decomposition theorem [31]: space \( \Omega \) can be decomposed by \( \Omega = \Omega_+ \cup \Omega_- \) with \( \Omega_+ \cap \Omega_- = \emptyset \). Here, \( \Omega_+ \) is a positive set for \( \mu, \) i.e., \( \mu(S) \geq 0 \) for all subsets of \( S \in \Omega_+ \), while \( \Omega_- \) is a negative set, i.e., \( \mu(S) \leq 0 \) for all subsets of \( S \in \Omega_- \).

3.2 Answer the open question in RKKS

A reproducing kernel Krein space (RKKS) [32, 33] is an inner-product space, that is analogous to a reproducing kernel Hilbert space (RKHS), and can be decomposed into a direct sum \( \mathcal{H}_K = \mathcal{H}_+ \oplus \mathcal{H}_- \) with two RKHSs \( \mathcal{H}_\pm \). The key difference with RKHSs is that the inner products might be negative for RKKSs, i.e., there exists \( f \in \mathcal{H}_K \) such that \( \langle f, f \rangle_{\mathcal{H}_K} < 0 \). RKKS provides a justification to analyze indefinite kernels as it admits positive decomposition [32] such that \( k = k_+ - k_- \), where
Theorem 3 provides an access via Fourier transform to verify whether a (reproducing) indefinite Fourier transform is denoted by the measure \( \mu \) where random features are obtained by

\[
\mu = \sum_{i=1}^{s} (\text{Re}(\varphi_i(x)), \text{Im}(\varphi_i(x)))
\]

while the theoretical gap cannot be ignored. By virtue of measure decomposition of the signed measure, we provide a sufficient and necessary condition in Theorem 3 to answer the longstanding open question in RKKS: does any indefinite kernel have a positive decomposition? Moreover, this condition serves as a guidance for us to find a specific positive decomposition in practice.

**Theorem 3.** Assume that an indefinite kernel is stationary, i.e., \( k(x, x') = k(x - x') \), and its (generalized) Fourier transform is denoted by the measure \( \mu \). Then \( k \) can be identified with a reproducing kernel in RKKS if and only if the total mass of the measure \( \mu \) except for the origin 0 is finite, i.e., \( \| \mu \| < \infty \).

**Proof.** The proof can be found in Supplementary Materials [B].

**Remark:** We provide an explicit sufficient and necessary condition to link the Jordan decomposition of signed measures to positive decomposition in RKKS. We make the following remarks.

(i) Theorem 3 provides an access via Fourier transform to verify whether a (reproducing) indefinite kernel belongs to RKKS or not. The measure decomposition is much easier to be found than positive decomposition in RKKS that cannot be verified in practice. In the next section, we give some examples including a linear combination of PD kernels, dot-product kernels on the unit sphere, to illustrate our condition in practice.

(ii) Theorem 3 also includes some non-squared-integrable kernel functions, e.g., conditionally positive definite kernels \([34]\), of which the standard Fourier transform does not exist. In this case, we need to consider the Fourier transform in Schwartz space \([35]\). For example, Theorem 2.3 in \([36]\) demonstrates that conditionally positive kernels correspond to a positive Borel measure \( \mu \) on \( \mathbb{R}^d \) with an analytic function in Schwartz space.

(iii) In addition to the above-mentioned indefinite kernels, based on Theorem 3 we can also verify various indefinite kernels, e.g., the TL1 kernel \([18]\), the tanh \([11]\), and any distance-based kernel function in the similar way. Specifically, their Fourier transform (the \( d \)-dimensional integration) are still radial, which makes the Fourier transform more easily computed.

### 3.3 Randomized feature map

Based on Theorem 3 we are ready to develop our random feature algorithm for non-PD kernels. By considering \( g(\omega, z) \) to be \( \exp(i\omega^\top z) \), Eq. (1) can be reformulated as follows:

\[
k(x, x') = c_1 \int_{\mathbb{R}^d} \exp(i\omega^\top z) \mu_+(d\omega) - c_2 \int_{\mathbb{R}^d} \exp(i\nu^\top z) \mu_-(d\nu)
\]

\[
= c_1 \| \mu_+ \| E_{\|\omega\|} \left[ \cos(\omega^\top (x - x')) \right] - c_2 \| \mu_- \| E_{\|\nu\|} \left[ \cos(\nu^\top (x - x')) \right]
\]

\[
= c_1 \mu_+ \|k_+(x - x')\| - c_2 \mu_- \|k_-(x - x')\|
\]

where \( \mu_+, \mu_- \) are two finite nonnegative measures and \( \mu_+ := \| \mu_+ \|, \mu_- := \| \mu_- \| \) are their corresponding Borel measures. According to the Bochner’s theorem, these two Borel measures can be associated with two PD kernels \( k_+ \) and \( k_- \), respectively. Therefore, the Monte Carlo sampling is feasible to approximate \( k(x, x') \) by \( k(x, x') = E_{\|\omega\|} k(x, x') \approx \frac{1}{s} \sum_{i=1}^{s} \langle \varphi_i(x), \varphi_i(x') \rangle \)

where \( \varphi_i(x) \) is the explicit feature mapping \( \varphi_i(x) = [\varphi_1(x), \ldots, \varphi_s(x)]^\top \) with \( \varphi_i(x) \) being

\[
\varphi_i(x) = [\sqrt{c_1} \| \mu_+ \| \cos(\omega_i^\top x), \sqrt{c_1} \| \mu_+ \| \sin(\omega_i^\top x), i\sqrt{c_2} \| \mu_- \| \cos(\nu_i^\top x), i\sqrt{c_2} \| \mu_- \| \sin(\nu_i^\top x)]
\]

where random features are obtained by \( \{\omega_i\}_{i=1}^{s} \sim \mu_+ / \| \mu_+ \| \) and \( \{\nu_i\}_{i=1}^{s} \sim \mu_- / \| \mu_- \| \). It can be easily seen from Eqs. (2) and (3) that this estimation is unbiased. The real and imaginary part in \( \varphi_i(x) \) corresponds to \( k_+ \) and \( k_- \), i.e., \( k_+(x, x') \approx \frac{1}{s} \sum_{i=1}^{s} \langle \text{Re}(\varphi_i(x)), \text{Re}(\varphi_i(x')) \rangle \) and \( k_-(x, x') \approx \frac{1}{s} \sum_{i=1}^{s} \langle \text{Im}(\varphi_i(x)), \text{Im}(\varphi_i(x')) \rangle \), respectively. Moreover, in Eq. (3), the imaginary part \( ix \) can be interpreted as rotating the vector \( x \) by 90 degrees. It shows the consistency for the
Algorithm 1: Random features for various indefinite kernels via generalized measures.

**Input:** A kernel function $k(x, x') = k(z)$ with $z := \|x - x'\|_2$, the training data $\{x_i, y_i\}_{i=1}^n$, and the number of random features $s$.

**Output:** Random feature map $\Phi(\cdot) : \mathbb{R}^d \rightarrow \mathbb{R}^s$ such that $k(x, x') \approx \frac{1}{s} \sum_{i=1}^s \langle \varphi_i(x), \varphi_i(x') \rangle$.

1. Obtain the measure $\mu(\cdot)$ of the kernel $k$ via (generalized) Fourier transform.
2. Given $\mu$, let $\mu = \mu_+ - \mu_-$ be the Jordan decomposition with two nonnegative measures $\mu_\pm$ and compute the total mass $\|\mu\| = \|\mu_+\| + \|\mu_-\|$.
3. Sample $\{\omega_i\}_{i=1}^s \sim \mu_+/\|\mu_+\|$ and $\{\nu_i\}_{i=1}^s \sim \mu_-/\|\mu_-\|$.
4. Output the explicit feature mapping $\Phi(x)$ with $\varphi_i(x)$ given in Eq. (3).

RKKS $\mathcal{H}_K$ associated with $k$ as an orthogonal direct sum: $\mathcal{H}_K = \mathcal{H}_+ \oplus \mathcal{H}_-$, where $\mathcal{H}_\pm$ are two RKHSs associated with $k_\pm$. Though we introduce the imaginary unit to the feature mapping, the computed kernel approximation result remains real-valued. The complete random features process is summarized in Algorithm 1. For any given kernel, the required $\mu_\pm$ can be pre-computed, independent of the training data. In this way, our algorithm achieves the same complexity with the standard RFF by $O((ns)^2)$ time and $O(ns)$ memory.

The formulation in Eq. (2), as well as Algorithm 1, is general enough to cover various PD and non-PD kernels. Stationary PD kernels admit Eq. (2) by choosing $c_1 = 1$ and $c_2 = 0$ where we have $\mu = \mu_+$ associated with $\|\mu\| = 1$, i.e., a probability measure. Hence, the Bochner’s theorem can be regarded as a special case of the considered integration representation (2) in this paper. In the next section, we will demonstrate the feasibility of our Algorithm 1 by considering several typical indefinite kernels, including the linear combination of PD kernels, dot-product kernels on the unit sphere, etc.

4 Examples

In this section, we investigate a series of indefinite kernels for a better understanding of our random features algorithm. We begin with an intuitive example, the indefinite linear combination of PD kernels. Then we employ several dot-product kernels on the unit sphere, including the spherical kernel, the arc-cosine kernel, and the NTK kernel on two-layer ReLU network.

Note that by considering these dot-product kernels on the unit $\ell_2$-sphere, we in fact conduct an equivalent pre-processing by normalizing the data to the unit $\ell_2$-norm. This operator is common to avoid the unboundedness of dot-product kernels and it is beneficial to theoretical analysis.

A linear combination of positive definite kernels: Kernels in this class have the formulation $k = \sum_{i=1}^d a_i k_i$, i.e., a linear combination of PD kernels $\{k_i\}_{i=1}^d$ with the corresponding coefficients $a_i \in \mathbb{R}$. This is a typical example of indefinite kernels in RKKS, which admits positive decomposition such that $k = k_+ - k_-$. Then Theorem 3 guides us to find $\mu_\pm$ based on the sign of $a_i$. Hence we explicitly decompose an indefinite kernel in this class into the difference of two PD kernels, i.e., $k = k_+ - k_- := \sum_{i=1}^d \max(0, a_i) k_i - \sum_{i=1}^d \max(0, -a_i) k_i$. Then the corresponding nonnegative measures $\mu_\pm$ can be subsequently obtained due to the additivity of Fourier transform.

We take the Delta-Gaussian kernel $k(x, x') = \exp(-\|x - x'\|^2/2\sigma^2) - \exp(-\|x - x'\|^2/2\tau^2)$ as an example. This kernel admits $a_1 = 1$ and $\|\mu_+\| = \|\mu_-\| = 1$ in Eq. (2), and its random feature mapping is given by Eq. (3) with $\{\omega_i\}_{i=1}^s \sim \mathcal{N}(0, \tau_1^{-2}I_d)$ and $\{\nu_i\}_{i=1}^s \sim \mathcal{N}(0, \tau_2^{-2}I_d)$.

After providing the above simple and intuitive warming-up example, we now discuss some sophisticated indefinite kernels, e.g., dot-product kernels on the unit sphere, and demonstrate the feasibility of our random features algorithm.

Polynomial kernels on the sphere: Pennington et al. point out that a polynomial kernel on the unit sphere is of $k(x, x') = (1 - \|x - x'\|^2/a^2)^p$ for $\alpha \geq 2$ and $p \geq 1$ and $z := \|x - x'\|_2 \in [0, 2]$. This kernel is indefinite since its Fourier transform is not a nonnegative measure as shown in [17].

$$
\mu(\omega) = \mu(\omega) = \sum_{i=0}^p \frac{p!}{(p-i)!} \left( 1 - \frac{4}{a^2} \right)^{p-i} \left( \frac{2}{a^2} \right)^i \frac{d^{2+i} J_{d/2+i}(2\omega)}{d^{2+i} J_{d/2+i}(2\omega)} , \quad \omega \neq 0 ,
$$

which results from the oscillatory behavior of the Bessel function of the first kind $J_{d/2+i}(2\omega)$. We demonstrate $\|\mu\| < \infty$ (see in Supplementary Materials), which makes the integration representation in Eq. (2) ($c_1 = c_2 = 1$) and our random features algorithm feasible. Since $\mu$ is a signed measure,
it can be decomposed into two nonnegative measures by Eq. (1), that is, \( \mu(\omega) = \mu_+(\omega) - \mu_-(\omega) \) with \( \mu_+(\omega) = \max\{0, \mu(\omega)\} \) and \( \mu_-(\omega) = \max\{0, -\mu(\omega)\} \). Then random feature map for this kernel can be also given by Eq. (3) with \( \{\omega_j\}_{j=1}^s \sim \mu_+ / ||\mu_+|| \) and \( \{\omega_j\}_{j=1}^s \sim \mu_- / ||\mu_-|| \). Therefore, Algorithm 1 is suitable for this kernel and the distributions \( \mu_+ / ||\mu_+|| \) and \( \mu_- / ||\mu_-|| \) can be numerically acquired by a set of uniformly generated 10,000 samples in a range of \([-10, 10]\). Based on the above result, we conclude that polynomial kernels on the unit sphere can be associated with RKKS, as shown in Figure 1 with the decomposition of \( \mu \). Compared to [17] using a positive sum of Gaussians to approximate \( \mu(\omega) \), where parameters in Gaussians need to be optimized beforehand, our algorithm achieves both simplicity and effectiveness by having (i) an unbiased estimator, (ii) incurring no extra parameters.

Next we consider the NTK of two-layer ReLU networks on the unit sphere [37]. Since this kernel in fact consists of arc-cosine kernels [12], we combine them together for discussion.

**NTK of Two-layer ReLU networks on the unit sphere**: Bietti and Mairal [37] consider a two-layer ReLU network of the form \( f(x; \theta) = \sqrt{2s} \sum_{j=1}^s \sum_{a_j} \max\{\omega_j x, 0\} \), with the parameter \( \theta = (\omega_1^\top, \cdots, \omega_s^\top, a_1, \cdots, a_s) \) initialized according to \( N(0, 1) \). By formulating ReLU as \( \max\{\omega_j x, 0\} = (\omega_j x)_+ \), we have the following corresponding NTK [37] [9]:

\[
k(x, x') = 2 \langle x^\top x' \rangle E_{w \sim N(0, I)} \left[ 1 \{ \omega^\top x \geq 0 \} 1 \{ \omega^\top x' \geq 0 \} \right] + 2 \E_{w \sim N(0, I)} [ (\omega^\top x)_+ (\omega^\top x')_+] .
\]

Moreover, this kernel can be further represented by \( k(x, x') = \| x \| \| x' \| \kappa (\langle x, x' \rangle / (\| x \| \| x' \|)) \) with \( \kappa(u) := u \kappa_0(u) + \kappa_1(u) \). Here, \( \kappa_0(u) = 1 - \frac{1}{2} \arccos(u) \) corresponds to the zero-order arc-cosine kernel and \( \kappa_1(u) = \frac{1}{2} (u (\pi - \arccos(u)) + \sqrt{1 - u^2}) \) is the first-order arc-cosine kernel. Furthermore, this kernel is proved to be stationary but indefinite by the following theorem.

**Theorem 4.** For any \( x, x' \in S^{d-1} := \{ x \in \mathbb{R}^d : \| x \|_2 = 1 \} \) on the unit \( (d - 1) \)-sphere, the NTK kernel of a two layer ReLU network of the form \( f(x; \theta) = \sqrt{2s} \sum_{j=1}^s \sum_{a_j} \max\{\omega_j x, 0\} \) is shift-invariant, that is,

\[
k(x, x') = k(z) = \frac{2 - z^2}{\pi} \arccos \left( \frac{1}{2} z^2 - 1 \right) + \frac{z}{2\pi} \sqrt{4 - z^2} ,
\]

where \( z := \| x - x' \|_2 \in [0, 2] \). Specifically, the function \( k(z), z \in [0, 2] \) is not positive definite.

**Proof.** The proof can be found in Supplementary Materials [3].

**Remark:** The indefiniteness of NTK on the unit sphere motivates us to scrutinize the approximation performance, functional spaces, and generalization properties of over-parameterized networks in the future, which in return expands the usage scope of indefinite kernels.

Since the above NTK on the unit sphere can be formulated as \( k(x, x') = \langle x, x' \rangle \kappa_0(x, x') + \kappa_1(x, x') \) associated with arc-cosine kernels, we have the direct corollary for arc-cosine kernels as follows:

**Corollary 4.1.** For any \( x, x' \in S^{d-1} \) on the unit \( (d - 1) \)-sphere, denote \( z := \| x - x' \|_2 \), the zero-order arc-cosine kernel \( \kappa_0(x, x') = \frac{1}{2} \arccos(\frac{1}{2} z^2 - 1) \) and the first-order arc-cosine kernel \( \kappa_1(x, x') = \frac{2 - z^2}{2\pi} \arccos \left( \frac{1}{2} z^2 - 1 \right) + \frac{z}{2\pi} \sqrt{4 - z^2} \) are both shift-invariant but indefinite.

**Remark:** (i) Obtaining \( \mu \) via Fourier transform of arc-cosine kernels and the NTK kernel on the unit sphere appears non-trivial due to a \( d \)-dimensional integration of Bessel functions. However, we still manage to obtain \( \mu \) for a zero-order arc-cosine kernel according to Corollary 4.1 refer to Supplementary Materials [3] for details. Hence Algorithm 1 is still suitable for this kernel.

(ii) If we relax the shift-invariant constraint in Eq. (1), its integration representation also covers some dot-product kernels, e.g., arc-cosine kernels and NTK kernels. For example, an arc-cosine kernel admits \( k(x, x') = \int_{\mathbb{R}^d} g(\omega, x, x') \mu(\omega) \) where \( \omega \sim N(0, I_d) \) and \( g(\omega, x, x') = \frac{1}{2} [1 + \text{sign}(\omega^\top x)] \{ 1 + \text{sign}(\omega^\top x') \} \) in its zero-order case and \( g(\omega, x, x') = \max(\omega^\top x, 0) \max(\omega^\top x', 0) \) in its first-order case. That means, in Eq. (1), the function \( g(\omega, z) \) is substituted by \( g(\omega, x, x') \) for arc-cosine kernels. Likewise, we can devise \( g(\omega, x, x') \) for NTK according to Eq. (3). In the above cases, the coefficients are \( c = c_+ = 1 \) and \( c_- = 0 \) and accordingly, we can conduct the sampling strategy on \( g(\omega, x, x') \) for kernel approximation.

\(^{3}\)The behavior of \( k(z) \) with \( z > 2 \) is undefined. Following [17], we set \( k(z) = 0 \) for \( z > 2 \).
5 Experiments

We evaluate the proposed method with SRF (Spherical Random Features) [17] and DIGMM (Double-Infinite Gaussian Mixtures Model) [28] on four representative datasets including letter, ijcnn1, covtype, and the MNIST dataset [41], see in Table 1. The datasets are normalized to [0, 1]^d by a min-max scaling scheme and are provided by the pre-given training/test partition except for the covtype. In this case, we randomly partition the dataset by half as the training and test sets respectively for the covtype. In our experiment, the used indefinite kernels include the polynomial kernel on the sphere \( k(x, x') = (1 - \|x - x'\|^2/a^2)^p \) with \( a = 2 \), \( p = 1 \) in [17], and the Delta-Gaussian kernel \( k(x, x') = \exp(-\|x - x'\|^2/2\tau_1^2) - \exp(-\|x - x'\|^2/2\tau_2^2) \) with \( \tau_1 = 1 \) and \( \tau_2 = 10 \) in [14]. Specically, we also include Random Maclaurin (RM) [23] and Tensor Sketch (TS) [24] for polynomial kernel approximation. Note that the related error bars and standard deviations are obtained by running the experiments for 10 times. All experiments are implemented in MATLAB and carried out on a PC with Intel® i7-8700K CPU (3.70 GHz) and 64 GB RAM. The source code of our implementation will be made public.

Kernel approximation: The relative error \( \|K - \hat{K}\|_F / \|K\|_F \) is chosen to measure the approximation quality where \( K \) and \( \hat{K} \) denote the exact kernel matrix on 1,000 random selected samples and its approximated kernel matrix, respectively. Figure 2 shows the approximation error under two indefinite kernels as a function of the number of random features \( s \). Our method always achieves lower approximation error than the other algorithms on these datasets. A clear look at the case of Delta-Gaussian kernel approximation will find that our approach significantly improves the approximation quality compared to SRF and DIGMM. This larger approximation error of SRF results from two steps: (i) approximating the indefinite kernel by a PD kernel; (ii) approximating this PD kernel by a sum of Gaussians; DIGMM only focuses on approximating a subset of the kernel matrix. Different

Table 1: Benchmark datasets.

| Datasets | \( d \) | #training | #test |
|----------|--------|-----------|-------|
| letter   | 16     | 12,000    | 6,000 |
| ijcnn1   | 22     | 49,990    | 91,701|
| covtype  | 54     | 290,506   | 290,506|
| MNIST    | 784    | 60,000    | 10,000|

Figure 1: The nonnegative Borel measures of the spherical polynomial kernel on the letter dataset.

Figure 2: Comparisons of various algorithms for approximation error across the polynomial kernel on the unit sphere (top) and the Delta-Gaussian kernel (bottom) on four datasets.

https://archive.ics.uci.edu/ml/datasets.html.

https://www.csie.ntu.edu.tw/~cjlin/libsvmtools/datasets/
from these two, our method directly approximates the indefinite kernel function by an unbiased estimator, which incurs no extra loss for kernel approximation.

![Figure 3: Comparisons of various algorithms for classification accuracy with libSVM across the polynomial kernel on the unit sphere (top) and the Delta-Gaussian kernel (down) on four datasets.](image)

**Classification with linear SVM:** The obtained randomized feature map is used for classification by training a linear classifier with liblinear [42]. The balanced parameter in linear SVM is tuned by five-fold cross validation on a grid of points: \( C = [0.01, 0.1, 1, 10, 100] \). The test accuracy of various algorithms are shown in Figure 3. As we expected, higher-dimensional randomized feature map outputs higher classification accuracy. Our method achieves the best performance in most cases.

![Figure 4: Comparisons of computational time to generate randomized feature map across the polynomial kernel on the unit sphere (top) and the Delta-Gaussian kernel (down) on four datasets.](image)

**Computational time:** Figure 4 shows time spent on generating randomized feature map. Admittedly, our method takes a little more time to generate randomized feature maps than SRF [17] as our feature map introduces the extra imaginary part. However, on each dataset, SRF requires to obtain parameters of a sum of Gaussians in advance by an off-line grid search scheme. This extra time cost often takes tens of seconds, which is not included in our reported results.

**6 Conclusion**

We answer the long-lasting open question of indefinite kernels by the introduced measure decomposition technique. Accordingly, we develop a general random features algorithm with unbiased

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5Though learning with non-PD kernels is non-convex, the optimization algorithm in liblinear still converges.
estimation for various kernels that are non-stationary or/and positive definite. Besides, our findings on
the indefiniteness of NTK on the unit sphere encourages us to have better scrutiny on the approxima-
tion performance, functional spaces, and generalization properties in over-parameterized networks in
the future. Moreover, the mathematical technique in this paper, Fourier analysis in Schwartz spaces,
can also be used to study ReLU networks in Fourier domain, where the ReLU activation function is
also non-squared integrable, refer to [43] for details, which expands the usage of indefinite kernels
and Fourier analysis to neural networks.

**Broader Impact**

This is a theoretical paper that investigates the decomposition of signed measures for random features.
This work gives further insight into kernel approximation, and hence might inspire new practical
ideas for kernel-based learning tasks. Our work can be used to speed up kernel methods in large-scale
situations. This is beneficial to the contemporary environment of data analysis. The developed
technique in this paper contributes to fair and non-offensive societal consequences.

**Acknowledgement**

This work was supported in part by the European Research Council under the European Union’s
Horizon 2020 research and innovation program / ERC Advanced Grant E-DUALITY (787960), in part by
the National Natural Science Foundation of China 61977046, in part by the National Key
Research and Development Project (No. 2018AAA0100702). This paper reflects only the authors’
views and the Union is not liable for any use that may be made of the contained information; Research
Council KUL C14/18/068; Flemish Government FWO project GOA4917N; Onderzoeksprogramma
Artificieele Intelligente (AI) Vlaanderen programme.

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A Preliminaries of RKKS

Here we briefly review on the Kreın spaces and the reproducing kernel Kreın space (RKKS). Detailed expositions can be found in book [32]. Most of the readers would be familiar with Hilbert spaces. Kreın spaces share some properties of Hilbert spaces but differ in some key aspects which we shall emphasize as follows.

Kreın spaces are indefinite inner product spaces endowed with a Hilbertian topology.

Definition 2. (Kreın space [32]) An inner product space is a Kreın space if there exist two Hilbert spaces \( \mathcal{H}^+ \) and \( \mathcal{H}^- \) such that
i) \( \forall f \in \mathcal{H}_K, \) it can be decomposed into \( f = f_+ \oplus f_- \), where \( f_+ \in \mathcal{H}_+ \) and \( f_- \in \mathcal{H}_- \), respectively.
ii) \( \forall f, g \in \mathcal{H}_K, \langle f, g \rangle_{\mathcal{H}_K} = \langle f_+ + g_+ \rangle_{\mathcal{H}_+} - \langle f_- + g_- \rangle_{\mathcal{H}_-} \).

Accordingly, the Kreın space \( \mathcal{H}_K \) can be decomposed into a direct sum \( \mathcal{H}_K = \mathcal{H}_+ \oplus \mathcal{H}_- \). Besides, the inner product on \( \mathcal{H}_K \) is non-degenerate, i.e., \( \forall f \in \mathcal{H}_K, \) \( \langle f, g \rangle_{\mathcal{H}_K} = 0 \) for any \( g \in \mathcal{H}_K \), we have \( f = 0 \). From the definition, the decomposition \( \mathcal{H}_K = \mathcal{H}_+ \oplus \mathcal{H}_- \) is not necessarily unique. For a fixed decomposition, the inner product \( \langle f, g \rangle_{\mathcal{H}_K} \) is given accordingly [33, 14]. The key difference from Hilbert spaces is that the inner products might be negative for Kreın spaces, i.e., \( \langle f, f \rangle_{\mathcal{H}_K} \) can be decomposed into a direct sum of two Kreın spaces, the Kreın space \( \mathcal{H}_K \) is a RKKS associated with a unique indefinite reproducing kernel \( k \) such that the reproducing property holds, i.e., \( \forall f \in \mathcal{H}_K, f(x) = \langle f, k(x, \cdot) \rangle_{\mathcal{H}_K} \).

Proposition 1. (positive decomposition [32]) Let \( k : X \times X \to \mathbb{R} \) be a real-valued kernel function. Then there exists an associated reproducing kernel Kreın space identified with a reproducing kernel \( k \) if and only if \( k \) admits a positive decomposition \( k = k_+ - k_- \), where \( k_+ \) and \( k_- \) are two positive definite kernels.

From the definition, this decomposition is not necessarily unique. Typical examples include a wide range of commonly used indefinite kernels, such as a linear combination of PD kernels [44], and conditionally PD kernels [45, 34]. It is important to note that, not every indefinite kernel function admits a representation as a difference between two positive definite kernels.

B Proof of Theorem

Proof. (i) Necessity.
An stationary indefinite kernel associated with RKKS admits the positive decomposition
\[
k(x - x') = k_+(x - x') - k_-(x - x'), \quad \forall x, x' \in X,
\]
where \( k_+ \) and \( k_- \) are two positive definite kernels. According to the Bochner’s theorem, there exist two probability measures \( \mu_+, \mu_- \) such that by denoting \( z := x - x' \),
\[
k(z) = k_+(z) - k_-(z) = \int_\Omega \exp (i\omega^\top z) \mu_+(d\omega) - \int_\Omega \exp (i\omega^\top z) \mu_-(d\omega).
\]
Denote \( \mu := \mu_+ - \mu_- \), it is clear that \( \mu \) is a signed measure, and its total mass is finite since \( \|\mu\| = \|\mu_+\| + \|\mu_-\| = 2 \).

(ii) Sufficiency.
Let \( \Omega := \mathbb{R}^d \) and \( A \) be the smallest \( \sigma \)-algebra containing all open subsets of \( \Omega \), and \( \mu : A \to [-\infty, \infty] \)
\[
\mu(\omega) = \int_{\Omega \setminus \{0\}} \exp (-i\omega^\top z) k(z)dz.
\]
Since we assume that \( \mu \) has total mass except the origin, \( \|\mu\| < \infty \) indicates that \( \mu \) is a finite measure (except the origin). Therefore, \( \mu \) is a signed measure with \( \mu < \infty \). By virtue of Jordan decomposition, there exist two nonnegative finite measures \( \mu_+ \) and \( \mu_- \) such that \( \mu = \mu_+ - \mu_- \). By using the inverse Fourier transform and Plancherel’s theorem [33], we have
\[
k(z) = \int_{\Omega \setminus \{0\}} \exp (i\omega^\top z) \mu(d\omega) = \int_{\Omega \setminus \{0\}} \exp (i\omega^\top z) \mu_+(d\omega) - \int_{\Omega \setminus \{0\}} \exp (i\omega^\top z) \mu_-(d\omega)
= \|\mu_+\| \int_{\Omega \setminus \{0\}} \exp (i\omega^\top z) \tilde{\mu}_+(d\omega) - \|\mu_-\| \int_{\Omega \setminus \{0\}} \exp (i\omega^\top z) \tilde{\mu}_-(d\omega)
= \|\mu_+\| \tilde{k}_+(z) - \|\mu_-\| \tilde{k}_-(z),
\]
where \(\tilde{\mu}_+ := \mu_+ / \| \mu_+ \|\) and \(\tilde{\mu}_- := \mu_- / \| \mu_- \|\) are two nonnegative Borel measures, which correspond to two positive definite kernels \(k_+\) and \(k_-\), respectively. By defining \(k_+ := \| \mu_+ \| k_+\) and \(k_- := \| \mu_- \| k_-\), we have

\[
k(x, x') = k_+(x, x') - k_-(x, x') , \quad \forall x, x' \in X .
\]

This completes the proof.

\[\square\]

**C  Polynomial kernels on the unit sphere with finite total mass**

We consider the asymptotic properties of the Bessel function of the first kind \(J_\alpha(x)\) under the large and small cases to study the \(\| \mu \|\).

**C.1  A small \(\omega\)**

Consider the asymptotic behavior for small \(\omega\). The Bessel function of the first kind is asymptotically equivalent to

\[
J_\alpha(x) \sim \frac{1}{\Gamma(\alpha + 1)} \left(\frac{x}{2}\right)^\alpha , \quad \text{when } 0 < x \ll \sqrt{\alpha + 1} .
\]

In this case, the measure \(\mu\) is formulated as

\[
\mu(\omega) \sim \sum_{p=0}^P \frac{p!}{(p-i)!} \left(1 - 4a^2\right)^{p-i} \left(\frac{2}{a^2}\right)^i \frac{2d/2+i}{\Gamma(d/2+i+1)} , \tag{6}
\]

which can be regarded as a generalized version of a uniform distribution. Therefore, \(\mu\) is absolutely integrable over a finite range \((0, c_1]\), where \(c_1\) is some constant satisfying \(c_1 \ll \sqrt{\frac{d}{2} - 1}\).

**C.2  A large \(\omega\)**

Consider the asymptotic behavior for large \(\omega\). The Bessel function of the first kind is asymptotically equivalent to

\[
J_\alpha(x) \sim \sqrt{\frac{2}{\pi x}} \cos(x - \frac{\pi \alpha}{2} - \frac{\pi}{4}) , \quad \text{when } x \gg |\alpha^2 - \frac{1}{4}| .
\]

The Fourier transform of the polynomial kernel on the sphere, i.e., the measure \(\mu\), is hence given by

\[
\mu(\omega) \sim \frac{1}{\sqrt{\pi \omega}} \left(1 - 4a^2\right)^P \left(\frac{2}{\omega}\right)^{d/2} \cos\left((d + 1) \frac{\pi}{4} - 2\omega\right) , \tag{7}
\]

In this way, we have \(\int_{c_2}^{\infty} |\mu(\omega)| \, d\omega < \infty\) for a large \(\omega\), where \(c\) is some constant satisfying \(c_2 \gg \frac{1}{4} |d^2 - 1|\).

Accordingly, combining Eq. (7) with Eq. (6), we conclude that

\[
\| \mu \| := \int_0^\infty |\mu(\omega)| \, d\omega = \int_{c_2}^{\infty} |\mu(\omega)| \, d\omega + \int_{c_1}^{c_2} |\mu(\omega)| \, d\omega + \int_{0}^{c_1} |\mu(\omega)| \, d\omega < \infty ,
\]

where we use \(\int_{c_1}^{c_2} |\mu(\omega)| \, d\omega\) is finite due to the continuous, bounded Bessel function \(J_\alpha(x)\) on a finite region \([c_1, c_2]\).

**D  Proof of Theorem 4**

To prove Theorem 4, we firstly derive its formulation on the unit sphere and then demonstrate that it is a shift-invariant but not positive definite kernel via completely monotone functions.

**Definition 3.** (Completely monotone \(\mathbb{C}(\mathbb{R})\)) A function \(f\) is called completely monotone on \((0, +\infty)\) if it satisfies \(f \in C^\infty(0, +\infty)\) and

\[
(-1)^r f^{(r)}(x) \geq 0 ,
\]

for all \(r = 0, 1, 2, \cdots\) and all \(x > 0\). Moreover, \(f\) is called completely monotone on \([0, +\infty)\) if it is additionally defined in \(C[0, +\infty)\).
Note that the definition of completely monotone functions can be also restricted to a finite interval, i.e., \( f \) is completely monotone on \([a, b] \subset \mathbb{R}\), see in [17].

Besides, we need the following lemma that demonstrates the connection between positive definite and completely monotone functions for the proof.

**Lemma 1.** *(Schoenberg’s theorem [46]) A function \( f \) is completely monotone on \([0, +\infty)\) if and only if \( f := g(\| \cdot \|_2^2) \) is radial and positive definite function on all \( \mathbb{R}^d \) for every \( d \).

Now let us prove Theorem [3].

**Proof.** By virtue of \( \langle x, x' \rangle = 1 - \frac{1}{2} \| x - x' \|_2^2 \) and \( \| x \|_2 = \| x' \|_2 = 1 \), we have \( \| x - x' \|_2 \in [0, 2] \).

Therefore, the standard NTK of a two-layer ReLU network can be formulated as

\[
k(x, x') = \langle x, x' \rangle \kappa_0(\langle x, x' \rangle) + \kappa_1(\langle x, x' \rangle)
= (1 - \frac{1}{2} \| x - x' \|_2^2) \kappa_0(1 - \frac{1}{2} \| x - x' \|_2^2 + \kappa_1(1 - \frac{1}{2} \| x - x' \|_2^2)
= 2 - \frac{\| x - x' \|_2^2}{\pi} \arccos(\frac{1}{2} \| x - x' \|_2^2 - 1) + \frac{\| x - x' \|_2}{2\pi} \sqrt{1 - \| x - x' \|_2^2}
= \arccos(\frac{z^2 - 1}{2\pi}) + \frac{z}{2\pi} \sqrt{4 - z^2}, z := \| x - x' \|_2 \in [0, 2],
\]

which is shift-invariant.

Next, we prove that \( k(z) \) is not a positive definite kernel, i.e., \( g(\sqrt{z}) := k(z) \) is not a completely monotone function over \([0, \infty)\) by Lemma [1]. In other words, there exist some value \( x \in [0, \infty) \) such that \((-1)^l g^{(l)}(x) < 0\) for some \( l \). To this end, the function \( g \) is given by

\[
g(x) = \frac{2 - x}{\pi} \arccos(\frac{1}{2} x - 1) + \frac{1}{2\pi} \sqrt{4x - x^2}, x \in [0, 4],
\]

and its first-order derivative is

\[
g'(x) = \frac{4 - 2x}{4\pi\sqrt{4x - x^2}} - \frac{2 - x}{2\pi\sqrt{1 - \left(\frac{2}{x} - 1\right)^2}} = \frac{\arccos\left(\frac{2}{x} - 1\right)}{\pi}.
\]

Since \( g'(x) \) is continuous, and \( \lim_{x \to 0} g'(x) = -\infty \) and \( \lim_{x \to 4} g'(x) = \infty \), there exists a constant \( c \) such that \( g'(x) < 0 \) over \((0, c)\) and \( g'(x) > 0 \) over \((c, 4)\). That is to say, \((-1)^l g^{(l)}(x) < 0\) holds for \( x \in (c, 4) \), which violates the definition of completely monotone functions. In this regard, \( g(\sqrt{z}) := k(z) \) is not a completely monotone function over \([0, \infty)\) and thus \( \{k(z), z \in [0, 2]; 0, z > 2\} \) is not positive definite.

### E The measure of the zero-order arc-cosine kernel

In this section, we derive the measure \( \mu \) of the zero-order arc-cosine kernel admitting \( \kappa_0(x, x') = \frac{1}{2} \arccos(\frac{1}{2} z^2 - 1) \). Accordingly, we have

\[
\mu(\omega) = \int \frac{1}{\pi} \arccos(\frac{1}{2} z^2 - 1) e^{i\omega^\top z} dz
:= \mu(\omega) = \int_0^\pi \frac{z}{\pi} \arccos(\frac{1}{2} z^2 - 1)(z/\omega)^{d/2-1} J_{d/2-1}(z\omega) dz,
\]

where \( \kappa_0(z) \) is a radial function, i.e., \( \kappa_0(z) = \kappa_0(\| z \|_2) \) with \( z := \| z \|_2 \), and thus its Fourier transform is also a radial function, i.e., \( \mu(\omega) = \mu(\| \omega \|_2) \) with \( \omega := \| \omega \|_2 \). Obviously, the integrand in Eq. (8) and the integration region are both bounded, and thus we have \( \mu(\omega) < \infty \). Following the proof of \( \| \mu \| < \infty \) for polynomial kernels on the unit sphere in Section C, we can also demonstrate that \( \| \mu \| < \infty \) for the zero-order arc-cosine kernel on the unit sphere.

To compute the integration in Eq. (8), we take the Taylor expansion of \( \arccos(\frac{1}{2} z^2 - 1) \) with \( t \) terms

\[
\arccos(\frac{1}{2} z^2 - 1) = \frac{\pi}{2} - \sum_{j=0}^{t} \frac{(2j)!}{4^j (2j+1)!} \left(\frac{1}{2} z^2 - 1\right)^{2j+1},
\]
and thus the integration in Eq. (8) can be integrated by each term regarding to Bessel functions. Moreover, by virtue of
\[
\frac{d}{dz} \frac{z^v J_v(z\omega)}{d z} = \omega z^v J_{v-1}(z\omega),
\]
the above integral can be computed by parts
\[
\mu(\omega) = \int_0^2 \frac{z}{\pi} \arccos(\frac{1}{2} z^2 - 1) (z/\omega)^{d/2-1} J_{d/2-1}(z\omega) dz
\]
\[
= \frac{1}{2} \left( \frac{1}{\omega} \right)^{d/2-2} \int_0^2 \omega z^{d/2} J_{d/2-1}(z\omega) dz
\]
\[
- \frac{1}{\pi} \left( \frac{1}{\omega} \right)^{d/2-2} \sum_{j=0}^{\infty} \frac{(2j)!}{4^j (j!)^2 (2j+1)} \int_0^2 \left( \frac{1}{2} z^2 - 1 \right)^{2j+1} \omega z^{d/2} J_{d/2-1}(z\omega) dz,
\]
where the first term equals to \( \left( \frac{1}{\omega} \right)^{d/2-1} J_d(2\omega) \) and next we are going to compute the second term by
\[
\int_0^2 \left( \frac{1}{2} z^2 - 1 \right)^{2j+1} \omega z^{d/2} J_{d/2-1}(z\omega) dz = 2^{d/2} J_d(2\omega) - (2j+1) \int_0^2 z^{d/2+1} J_d(z\omega) \left( \frac{1}{2} z^2 - 1 \right) dz
\]
\[
= 2^{d/2} J_d(2\omega) + \frac{2j+1}{\omega} J_{d+1}(2\omega) - \frac{2j+1}{2} \int_0^2 z^{d/2+3} J_d(z\omega) dz,
\]
where \( \int_0^2 z^{d/2+3} J_d(z\omega) dz \) can be computed by parts
\[
\int_0^2 z^{d/2+3} J_d(z\omega) dz = 2^{d/2+3} J_d(2\omega) - \frac{1}{\omega^2} 2^{d+2} J_{d+2}(2\omega).
\]
Incorporating Eqs. (11), (10) into Eq. (9), we have
\[
\mu(\omega) = \left( \frac{1}{\omega} \right)^{d/2-1} J_d(2\omega) - \frac{1}{\pi} \left( \frac{1}{\omega} \right)^{d/2-2} \sum_{j=0}^{\infty} \frac{(2j)!}{4^j (j!)^2 (2j+1)} \left[ (1 - 4(2j+1)) z^{d/2} J_d(z\omega) + \frac{2j+1}{\omega} J_{d+1}(2\omega) + \frac{2j+1}{2} \omega^2 z^{d/2+1} J_{d+2}(2\omega) \right].
\]
Hence, \( \mu \) can be decomposed into two nonnegative measures with \( \mu(\omega) = \mu_+(\omega) - \mu_-(\omega) \), where \( \mu_+(\omega) = \max \{ 0, \mu(\omega) \} \) and \( \mu_-(\omega) = \max \{ 0, -\mu(\omega) \} \). As a consequence, Algorithm II is also suitable for this kernel.