ALGEBRAIC APPROXIMATION PRESERVING DIMENSION

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Abstract. We prove that each semialgebraic subset of $\mathbb{R}^n$ of positive codimension can be locally approximated of any order by means of an algebraic set of the same dimension. As a consequence of previous results, algebraic approximation preserving dimension holds also for semianalytic sets.

1. Introduction

If $A$ and $B$ are two closed subanalytic subsets of $\mathbb{R}^n$, the Hausdorff distance between their intersections with the sphere of radius $r$ centered at a common point $P$ can be used to “measure” how near the two sets are at $P$. We say that $A$ and $B$ are $s$–equivalent (at $P$) if the previous distance tends to 0 more rapidly than $r^s$ (if so, we write $A \sim_s B$).

In the papers [FFW1], [FFW2] and [FFW3] we addressed the question of the existence of an algebraic representative $Y$ in the class of $s$–equivalence of a given subanalytic set $A$ at a fixed point $P$. In this case we also say that $Y$ $s$-approximates $A$.

The answer to the previous question is in general negative for subanalytic sets (see [FFW2]).

On the other hand in [FFW1] it was proved that, for any real number $s \geq 1$ and for any closed semialgebraic set $A \subset \mathbb{R}^n$ of codimension $\geq 1$, there exists an algebraic subset $Y$ of $\mathbb{R}^n$ such that $A \sim_s Y$. The proof of the latter result consists in finding equations for $Y$ starting from the polynomials appearing in a presentation of $A$. For instance if $A = \{(x, y) \in \mathbb{R}^n \mid f(x, y) = 0, h(x, y) \geq 0\}$ with $f, h \in \mathbb{R}[x, y]$, then $A$ can be $s$-approximated by the algebraic set $Y = \{(x, y) \in \mathbb{R}^n \mid (f^2 - h^m)(x, y) = 0\}$ for any odd integer $m$ sufficiently large. This procedure does not guarantee that $Y$ has the same dimension as $A$ at $P$ as the following trivial example shows.

Let $A$ be the positive $x_3$-axis in $\mathbb{R}^3$ presented as $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 + x_2^2 = 0, x_3 \geq 0\}$. Then according to the previous procedure, for any sufficiently large odd integer $m$, $A$ is $s$-approximated at the origin $O$ by the algebraic set $Y = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid (x_1^2 + x_2^2)^2 - x_3^m = 0\}$, whose germ at $O$ has dimension 2. However we can also $s$-approximate $A$ at $O$ by the 1-dimensional algebraic set $W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1^2 - x_3^m = 0, x_2 = 0\}$ for any sufficiently large odd integer $m$. This algebraic set can be obtained by a similar construction as before but starting from the different presentation $A = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 = 0, x_2 = 0, x_3 \geq 0\}$.

In [FFW3] we proved that, for any $s \geq 1$, any closed semianalytic subset $A \subset \mathbb{R}^n$ is $s$-equivalent to a semialgebraic set $Y \subset \mathbb{R}^n$ having the same local dimension as $A$. However the arguments used in the proof of this latter result do not guarantee that, even if $A$ is analytic, it can be approximated by means of an algebraic one of the same dimension.

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In this paper we prove in Theorem 4.1 that any semialgebraic set of codimension $\geq 1$ is $s$-equivalent to an algebraic one of the same dimension. Using the mentioned result of [FFW3], we obtain (Corollary 4.3) that any semianalytic set of codimension $\geq 1$ can be $s$-approximated by an algebraic one preserving the local dimension. The proof of Theorem 4.1 works provided that the semialgebraic set is described by means of a suitable presentation, as in the previous example. Therefore Section 3 is devoted to introduce the notion of “regular presentation” and to prove that one can reduce to work with regularly presented sets.

2. Basic properties of $s$-equivalence

In this section we recall the definition and some basic properties of $s$-equivalence of subanalytic sets at a common point which, without loss of genericity, we can assume to be the origin $O$ of $\mathbb{R}^n$. We refer the reader to [FFW2] for the proofs of these results.

If $A$ and $B$ are non-empty compact subsets of $\mathbb{R}^n$, let $\delta(A, B) = \sup_{x \in B} d(x, A)$. Thus, denoting by $D(A, B)$ the classical Hausdorff distance between the two sets, we have that $D(A, B) = \max\{\delta(A, B), \delta(B, A)\}$.

**Definition 2.1.** Let $A$ and $B$ be closed subanalytic subsets of $\mathbb{R}^n$ with $O \in A \cap B$. Let $s$ be a real number $\geq 1$. Denote by $S_r$ the sphere of radius $r$ centered at the origin.

(a) We say that $A \leq_s B$ if one of the following conditions holds:

(i) $O$ is isolated in $A$,

(ii) $O$ is non-isolated both in $A$ and in $B$ and

$$\lim_{r \to 0} \frac{\delta(B \cap S_r, A \cap S_r)}{r^s} = 0.$$ 

(b) We say that $A$ and $B$ are $s$–equivalent (and we will write $A \sim_s B$) if $A \leq_s B$ and $B \leq_s A$.

Observe that, if $A \subseteq B$, then $A \leq_s B$ for any $s \geq 1$. It is easy to check that $\leq_s$ is transitive and that $\sim_s$ is an equivalence relation.

The following result shows the behavior of $s$-equivalence with respect to the union of sets:

**Proposition 2.2.** Let $A$, $A'$, $B$ and $B'$ be closed subanalytic subsets of $\mathbb{R}^n$.

(1) If $A \leq_s B$ and $A' \leq_s B'$, then $A \cup A' \leq_s B \cup B'$.

(2) If $A \sim_s B$ and $A' \sim_s B'$, then $A \cup A' \sim_s B \cup B'$.

A useful tool to test the $s$-equivalence of two subanalytic sets is introduced in the following definition:

**Definition 2.3.** Let $A$ be a closed subanalytic subset of $\mathbb{R}^n$, $O \in A$. For any real $\sigma > 1$, we will call horn-neighbourhood with center $A$ and exponent $\sigma$ the set

$$\mathcal{H}(A, \sigma) = \{x \in \mathbb{R}^n \mid d(x, A) < \|x\|^\sigma\}.$$ 

**Remark 2.4.** If $A$ is a closed semialgebraic subset of $\mathbb{R}^n$ and $\sigma$ is a rational number, then $\mathcal{H}(A, \sigma)$ is semialgebraic. Moreover if $O$ is isolated in $A$, then $\mathcal{H}(A, \sigma)$ is empty near $O$. 

Proposition 2.5. Let $A, B$ be closed subanalytic subsets of $\mathbb{R}^n$ with $O \in A \cap B$ and let $s \geq 1$. Then $A \subseteq_s B$ if and only if there exist real constants $R > 0$ and $\sigma > s$ such that
\[ (A \setminus \{O\}) \cap B(O, R) \subseteq \mathcal{H}(B, \sigma) \]
where $B(O, R)$ denotes the open ball centered at $O$ of radius $R$.

The following technical result shows that it is possible to modify a subanalytic set by means of a suitable horn-neighborhood producing a new subanalytic set $s$-equivalent to the original one:

Lemma 2.6. Let $X \subset Y \subset \mathbb{R}^n$ be closed subanalytic sets such that $O \in X$ and let $s \geq 1$. Then:

1. for any $\sigma > s$ we have $Y \sim_s Y \cup \mathcal{H}(X, \sigma)$;
2. if $Y \setminus X = Y$, there exists $\sigma > s$ such that $Y \setminus \mathcal{H}(X, \sigma) \sim_s Y$.

Another essential tool will be the following version of Lojasiewicz’ inequality, proved in [FFW3]; henceforth for any map $f: \mathbb{R}^n \rightarrow \mathbb{R}^n$ we will denote by $V(f)$ the zero-set $f^{-1}(O)$.

Proposition 2.7. Let $A$ be a compact subanalytic subset of $\mathbb{R}^n$. Assume $f$ and $g$ are subanalytic functions defined on $A$ such that $f$ is continuous, $V(f) \subseteq V(g)$, $g$ is continuous at the points of $V(g)$ and such that $\sup |g| < 1$. Then there exists a positive constant $\alpha$ such that $|g|^\alpha \leq |f|$ on $A$ and $|g|^\alpha < |f|$ on $A \setminus V(f)$.

3. Presentations of semialgebraic sets

This section is devoted to the first crucial step in our strategy, that is reducing ourselves to prove the main theorem for semialgebraic sets suitably presented.

Definition 3.1. Let $A$ be a closed semialgebraic subset of $\mathbb{R}^n$ with $\dim O A = d > 0$. We will say that $A$ admits a good presentation if

(a) the Zariski closure $\overline{A}$ of $A$ is irreducible
(b) there exist generators $f_1, \ldots, f_p$ of the ideal $I(\overline{A}) \subseteq \mathbb{R}[x_1, \ldots, x_n]$ and $h_1, \ldots, h_q$ polynomial functions such that
\[ A = \{ x \in \mathbb{R}^n \mid f_i(x) = 0, h_j(x) \geq 0, i = 1, \ldots, p, j = 1, \ldots, q \} \]
(c) $h_i(O) = 0$ and $\dim_O(V(h_i) \cap V(f)) < d$, for each $i$, where $f = (f_1, \ldots, f_p)$.

Lemma 3.2. Let $A$ be a closed semialgebraic subset of $\mathbb{R}^n$ with $\dim O A = d > 0$. Then there exist closed semialgebraic sets $\Gamma_1, \ldots, \Gamma_r, \Gamma'$ such that

1. $A = (\bigcup_{i=1}^r \Gamma_i) \cup \Gamma'$
2. for each $i$, $\dim O \Gamma_i = d$, and $\dim O \Gamma' < d$
3. for each $i$, $\Gamma_i$ admits a good presentation.

Proof. Arguing as in [FFW3] Lemma 3.2 in the semialgebraic setting, there exist semialgebraic sets $\Gamma_1, \ldots, \Gamma_r, \Gamma'$ fulfilling conditions (1) and (2) of the thesis and such that, for each $i$, $\Gamma_i$ admits a presentation satisfying conditions (a) and (b) of Definition 3.1. In order to achieve also condition (c) it suffices to drop from the presentation of each $\Gamma_i$ all the inequalities $h_j(x) \geq 0$ such that $h_j$ vanishes identically on $\Gamma_i$. □

Since we are interested in preserving dimension, we will reduce ourselves to work with a set presented by as many polynomial equations as its codimension and with the critical locus of the associated polynomial map nowhere dense.
Notation 3.3. Let $\Omega$ be an open subset of $\mathbb{R}^n$. For any smooth $\varphi: \Omega \to \mathbb{R}^p$, denote $\Sigma_r(\varphi) = \{ x \in \Omega \mid \text{rk } d_x \varphi < r \}$ and $\Sigma(\varphi) = \Sigma_p(\varphi)$.

Definition 3.4. Let $A$ be a closed semialgebraic subset of $\mathbb{R}^n$ with $\dim_A A = d > 0$. We will say that $A$ admits a regular presentation if there exist a polynomial map $F: \mathbb{R}^n \to \mathbb{R}^{n-d}$ and polynomial functions $h_1, \ldots, h_q$ such that

(a) $A = \{ x \in \mathbb{R}^n \mid F(x) = 0, h_j(x) \geq 0, j = 1, \ldots, q \}$,

(b) $\dim_A (\Sigma(F) \cap A) < d$

(c) $h_i(\bar{x}) = 0$ and $\dim_A (V(h_i) \cap A) < d$, for each $i$.

A useful tool to pass from a good presentation to a regular one will be the following result (for a proof see for instance [BCR Proposition 7.7.1]):

Lemma 3.5. Let $A$ be a closed semialgebraic subset of $\mathbb{R}^n$ and let $h, g$ polynomial functions on $\mathbb{R}^n$. Then there exist polynomial functions $\varphi, \psi$ with $\varphi > 0$ and $\psi \geq 0$ such that

1. $\text{sign}(\varphi h + \psi g) = \text{sign}(h)$ on $A$

2. $V(\psi) \subseteq V(h) \cap A^Z$.

Proposition 3.6. Let $A$ be a closed semialgebraic subset of $\mathbb{R}^n$ with $\dim_A A = d > 0$ which admits a good presentation. Let $s > 1$. Then there exists a closed semialgebraic subset $\tilde{A}$ of $\mathbb{R}^n$ with $\dim_{\tilde{A}} \tilde{A} = d > 0$ such that

1. $\tilde{A}$ admits a regular presentation

2. $\tilde{A} \sim_s A$.

Proof. By hypothesis, we have that $A = \{ x \in \mathbb{R}^n \mid f(x) = 0, h_j(x) \geq 0, j = 1, \ldots, q \}$ with $f = (f_1, \ldots, f_p)$ such that $V(f)$ is irreducible, $V(f) = \overline{A^Z}$ and $f_1, \ldots, f_p$ generate the ideal $I(V(f))$. In particular $\dim_A (\Sigma_{n-d}(f) \cap V(f)) < d$ (see for instance [BCR] Definition 3.3.3). If $p = n - d$, we have the thesis with $\tilde{A} = A$; thus let $p > n - d$.

Denote by $\Pi$ the set of surjective linear maps from $\mathbb{R}^p$ to $\mathbb{R}^{n-d}$ and consider the smooth map $\Phi: (\mathbb{R}^n - V(f)) \times \Pi \to \mathbb{R}^{n-d}$ defined by $\Phi(x, \pi) = (\pi \circ f)(x)$ for all $x \in \mathbb{R}^n - V(f)$ and $\pi \in \Pi$.

The map $\Phi$ is transverse to $\{0\}$: namely the partial Jacobian matrix of $\Phi$ with respect to the variables in $\Pi$ (considered as an open subset of $\mathbb{R}^{p(n-d)}$) is the $(n - d) \times p(n-d)$ matrix

$$
\begin{bmatrix}
  f(x) & O & O & \ldots & O \\
  O & f(x) & O & \ldots & O \\
  \vdots & & & & \\
  O & O & O & \ldots & f(x)
\end{bmatrix}
$$

thus, for all $x \in \mathbb{R}^n - V(f)$ and for all $\pi \in \Pi$ the Jacobian matrix of $\Phi$ has rank $n - d$.

As a consequence, by a well-known result of singularity theory (see for instance [BK] Lemma 3.2), we have that the map $\Phi_x: \mathbb{R}^n - V(f) \to \mathbb{R}^{n-d}$ defined by $\Phi_x(x) = \Phi(x, \pi) = (\pi \circ f)(x)$ is transverse to $\{0\}$ for all $\pi$ outside a set $\Gamma \subset \Pi$ of measure zero and hence $\pi \circ f$ is a submersion on $V(\pi \circ f) \setminus V(f)$ for all such $\pi$. 

Let \( x \in V(f) \) be a point at which \( f \) has rank \( n - d \). Then there is an open dense set \( U \subseteq \Pi \) such that for all \( \pi \in U \) the map \( \pi \circ f \) is a submersion at \( x \), and hence off some subvariety of \( V(f) \) of dimension less than \( d \).

Thus, if we choose \( \pi_0 \in (\Pi \setminus \Pi) \cap U \), the map \( F = \pi_0 \circ f : \mathbb{R}^n \to \mathbb{R}^{n-d} \) satisfies the following properties:

- \( \dim_\Pi V(F) = \dim_\Pi V(f) = d \),
- \( \Sigma(F) \cap V(F) \subseteq V(\varphi) \subseteq V(F), \)
- \( \dim_\Pi (\Sigma(F) \cap V(F)) < d \).

We want to show that there exist polynomials \( h_i' \) such that

- \( A = \{ x \in \mathbb{R}^n \mid f(x) = \varphi, h_i'(x) \geq 0, i = 1, \ldots, q \} \)
- \( \dim_\Pi (V(F) \cap \bigcup_{i=1}^q V(h_i')) < d \).

Namely for each \( i \in \{1, \ldots, q\} \) denote by \( W_i \) the union of the irreducible components \( Y \) of \( V(f) \) such that \( \dim_\Pi (V(h_i) \cap Y) < d \); let also \( T_i = \left(V(F) \setminus \bigcup_{i=1}^q V(h_i') \right) \). Note that \( V(f) \subseteq W_i \).

If we apply Lemma 3.5 choosing \( h = h_i \) and \( g = \|f\|^2 \) on \( W_i \), then there exist \( \varphi, \psi \) with \( \varphi > 0 \) and \( \psi \geq 0 \) such that the function \( h_i' = \varphi h_i + \psi \|f\|^2 \) has the same sign as \( h_i \) on \( W_i \) and \( V(\psi) \subseteq \left(V(h_i) \cap W_i \right) \). Then

- \( V(h_i') \cap W_i = V(h_i) \cap W_i \)
- \( \dim_\Pi (V(h_i') \cap V(F)) < d \) for any \( i \) and

\[
A = \{ x \in \mathbb{R}^n \mid f(x) = \varphi, h_i'(x) \geq 0, i = 1, \ldots, q \}.
\]

For each \( m \in \mathbb{N} \) denote

\[
\widetilde{A}_m = \{ x \in \mathbb{R}^n \mid F(x) = 0, \|x\|^{2m} - \|f(x)\|^2 \geq 0, h_i'(x) \geq 0, i = 1, \ldots, q \}.
\]

Since \( A \subseteq \widetilde{A}_m \subseteq V(F) \), then \( \dim_\Pi \widetilde{A}_m = d \).

We claim that there exists \( m \) such that \( \widetilde{A}_m \sim_s A \). Since \( A \subseteq \widetilde{A}_m \), we trivially have that \( A \leq \widetilde{A}_m \). Thus it is sufficient to prove that there exists \( m \) such that \( \widetilde{A}_m \leq \widetilde{A}_m \).

Namely, let \( \Lambda = \{ x \in \mathbb{R}^n \mid h_i'(x) \geq 0, i = 1, \ldots, q \} \). Since \( V(\|f\|) \cap \Lambda = V(\varphi, \pi) \cap \Lambda \), by Proposition 2.7 there exist a rational number \( t \) and a real number \( R > 0 \) such that

\[
d(x, A)^+ < \|f(x)\| \quad \forall x \in (\Lambda \setminus V(f)) \cap B(O, R) = (\Lambda \setminus A) \cap B(O, R).
\]

Let \( m > st \). Then \( d(x, A)^{+} < \|f(x)\| \quad \forall x \in (\Lambda \setminus V(f)) \cap B(O, R) \). This implies that \( (\widetilde{A}_m \setminus \{0\}) \cap B(O, R) \subseteq \mathcal{H}(A, \frac{2m}{d}) \) and hence, by Proposition 2.5, \( \widetilde{A}_m \leq \widetilde{A}_m \).

Up to increasing \( m \), we can also assume that \( \dim_\Pi (V(F) \cap V(\|x\|^{2m} - \|f(x)\|^2)) < d \)

and hence that \( 3.1 \) is a regular presentation of \( \widetilde{A}_m \).

It is thus sufficient to choose \( m \) as above and \( A = \widetilde{A}_m \).

\[\square\]

4. Main result

Since \( s \)-equivalence depends only on the germs at \( O \), we are allowed to identify a subanalytic set with a realization of its germ at the origin in a suitable ball \( B(O, R) \). Henceforth we will even omit to explicitly indicate the intersection of our sets with \( B(O, R) \); in particular, given two sets \( U \) and \( U' \), when we write that \( U \subseteq U' \) we mean that \( U \cap B(O, R) \subseteq U' \) for a suitable real constant \( R > 0 \).
Theorem 4.1. For any real number \( s \geq 1 \) and for any closed semialgebraic set \( A \subset \mathbb{R}^n \) of codimension \( \geq 1 \) with \( O \in A \), there exists an algebraic subset \( S \) of \( \mathbb{R}^n \) such that \( A \sim_s S \) and \( \dim_O S = \dim_O A \).

Proof. We will prove the thesis by induction on \( d = \dim_O A \).

If \( d = 0 \) the result holds trivially. So let \( d \geq 1 \) and assume that the result holds for all semialgebraic sets of dimension less that \( d \).

By Lemma 3.2 there exist closed semialgebraic sets \( \Gamma_1, \ldots, \Gamma_r, \Gamma' \) such that

1. \( A = (\bigcup_{i=1}^r \Gamma_i) \cup \Gamma' \)
2. for each \( i \), \( \dim \Gamma_i = d \) and \( \Gamma_i \) admits a good presentation
3. \( \dim_O \Gamma' < d \).

By Proposition 2.2, by Proposition 3.6 and by the inductive hypothesis we can assume that \( A \) is described by means of a regular presentation as

\[
A = \{ x \in \mathbb{R}^n \mid F_i(x) = O, h_j(x) \geq 0, j = 1, \ldots, q \}
\]

with \( F_0 = (f_1, \ldots, f_{n-d}) \). We can assume \( q \geq 1 \), because otherwise there is nothing to prove.

We will use the following notation:

- \( Z_i = \bigcup_{i=1}^r \Gamma_i \cap F_i(x) \) for \( i = 0, \ldots, q-1 \), and \( Z_q = \emptyset \),
- \( X = (\Sigma(F_0) \cup Z_0) \cap A \),
- \( \tilde{f} = (f_2, \ldots, f_{n-d}) : \mathbb{R}^n \to \mathbb{R}^{n-d-1} \) and \( V = V(\tilde{f}) \),
- \( \Lambda_i = \{ x \in \mathbb{R}^n \mid h_j(x) \geq 0, j = i+1, \ldots, q \} \) for any \( i = 0, \ldots, q-1 \), and \( \Lambda_q = \mathbb{R}^n \).

Since the presentation of \( A \) is regular, we have that

\[
\dim_O (\Sigma(F_0) \cap A) < d \quad \text{and} \quad \dim_O (Z_0 \cap A) < d.
\]

Let \( X_1 = X \cap A \setminus X \) and \( X_2 = X \setminus X_1 \). Since \( A = A \setminus X_1 \cup X_2 \) and \( \dim_O X_2 < d \), by the inductive hypothesis it suffices to prove the thesis for \( A \setminus X_1 \).

In other words we can assume that \( A = A \setminus X \). As a consequence Lemma 2.6 shows that there exists a rational number \( \sigma > s \) such that, if \( K = \mathbb{R}^n \setminus H(X, \sigma) \) then \( A \cap K \sim_s A \).

Let \( g_0 = f_1 \). We will recursively construct polynomial functions \( g_1, \ldots, g_q \) such that, if \( F_i = (g_1, g_2, \ldots, g_{n-d}) \), then for any \( i = 0, \ldots, q \) the semialgebraic subset

\[
A_i = \{ x \in \mathbb{R}^n \mid F_i(x) = 0, h_j(x) \geq 0, j = i+1, \ldots, q \} = V(g_i) \cap V \cap \Lambda_i
\]

satisfies the following properties

P1(i): \( A_i \sim_i A_{i-1} \) and \( A_i \cap K \sim_i A_{i-1} \cap K \) if \( i \in \{1, \ldots, q\} \)

P2(i): \( A_i \cap K \sim_0 A_0 \) if \( i = 0 \)

P3(i): \( \Sigma(F_i) \cap A_i \cap K \subseteq \{ O \} \)

As proved above, the set \( A_0 = A \) satisfies the properties P1(0), P2(0) and P3(0). Thus assume that \( 0 \leq i \leq q-1 \), assume that we have already constructed \( A_i \) fulfilling the three previous properties and let us construct \( g_{i+1} \) in such a way that \( A_{i+1} \) satisfies properties P1(i+1), P2(i+1) and P3(i+1).

For any positive integer \( m \) let \( g_{i+1} = g_i^2 - h_{i+1}^m \).
We want to see that there exists \( m_s \in \mathbb{N} \) such that for any odd integer number \( m \geq m_s \) the semialgebraic set \( A_{i+1} = V(g_{i+1}) \cap V \cap \Lambda_{i+2} \) satisfies properties P1(i+1), P2(i+1) and P3(i+1).

Properties P2(i) and P3(i) evidently guarantee that \((A_i \cap K) \cap (\Sigma(F_i) \cup Z_i) \subseteq \{O\}\) for all \( i \geq 0 \).

We will need to strengthen this fact as follows

**Claim:** There exists \( \beta > s \) such that \( \mathcal{H}(A_{i} \cap K, \beta) \cap (\Sigma(F_i) \cup Z_i) \subseteq \{O\}\).

Namely, let \( \phi : \Sigma(F_i) \cup Z_i \to \mathbb{R} \) be the function defined by \( \phi(x) = d(x, A_i \cap K) \) for every \( x \in \Sigma(F_i) \cup Z_i \). The function \( \phi \) is semialgebraic, continuous and, by the previous properties P2(i) and P3(i), \( V(\phi) = A_i \cap K \cap (\Sigma(F_i) \cup Z_i) \subseteq \{O\} \). Hence by Proposition 2.7 there exists a rational positive number \( \beta \) such that \( d(x, A_i \cap K) > \|x\|^\beta \) for all \( x \in (\Sigma(F_i) \cup Z_i) \backslash \{O\} \); evidently we can assume that \( \beta > s \). By definition of horn-neighborhood no \( x \not\in O \) can lie in \( \mathcal{H}(A_i \cap K, \beta) \cap (\Sigma(F_i) \cup Z_i) \) which proves the Claim.

In particular for each \( j = i + 1, \ldots, q \) we have that

\[
(4.1) \quad h_j |B(x, \|x\|^\beta)| > 0 \quad \forall x \in A_i \cap K \backslash \{O\}.
\]

**Property P1(i+1).** Consider the set \( E = \mathbb{R}^n \setminus \mathcal{H}(A_i \cap K, \beta) \).

Evidently the closed semialgebraic set \( W = (V \cap A_{i+1} \cap K \cap E) \cap \{h_{i+1} \geq 0\} \) fulfills the condition

\[
V(g_i) \cap W = (A_i \cap K) \cap E = \{O\}.
\]

Thus by Proposition 2.8 there exists \( m_1 \in \mathbb{N} \) such that for any integer number \( m \geq m_1 \) we have \( g_i(x)^2 \geq h_{i+1}(x)^m \) for all \( x \in W \) and \( g_i(x)^2 > h_{i+1}(x)^m \) for all \( x \in W \backslash \{O\} \).

If we take \( m \) an odd integer \( \geq m_1 \), by construction \( g_{i+1} = g_i^2 - h_{i+1}^m \) is strictly positive on \( W \setminus \{O\} \) and on \( \{h_{i+1} < 0\} \), hence \( g_{i+1} \) is strictly positive on \( (V \cap A_{i+1} \cap K \cap E) \setminus \{O\} \). Since \( A_{i+1} = V(g_{i+1}) \cap V \cap A_{i+1} \), it follows that

\[
A_{i+1} \cap K \subseteq (\mathbb{R}^n \setminus E) \cup \{O\} = \mathcal{H}(A_i \cap K, \beta) \cup \{O\}
\]

and therefore, by Proposition 2.5 that

\[
A_{i+1} \cap K \leq_s A_i \cap K.
\]

We want now to prove that \( A_i \cap K \leq_s A_{i+1} \cap K \).

Consider the set \( B_i = V \cap A_i \supseteq A_i \).

Assume at first that \( B_i \cap K \setminus \{O\} \) is connected and denote by \( d_g \) the geodesic distance on \( B_i \cap K \); denote also by \( B_g(x_0, r) = \{ y \in B_i \cap K \mid d_g(y, x_0) < r \} \) the geodesic ball centered at \( x_0 \in B_i \cap K \).

By [L1], up to working in a suitable Euclidean ball \( B(O, R) \), there exist constants \( C > 0 \) and \( 0 < \alpha \leq 1 \) such that for any \( y_1, y_2 \in B_i \cap K \cap B(O, R) \) we have that

\[
\|y_1 - y_2\| \leq d_g(y_1, y_2) \leq C\|y_1 - y_2\|^\alpha.
\]

Therefore we have

\[
B_g(x_0, r) \subseteq B(x_0, r) \cap B_i \cap K \subseteq B_g(x_0, C^\alpha) \quad \forall x_0 \in B_i \cap K \cap B(O, R)
\]

for \( r \) small enough. Up to decreasing \( R \) and \( \alpha \) if necessary, we can assume that \( C = 1 \).

Property P3(iii) implies that, for any \( x \in A_i \cap K \), we have that \( \dim_x(A_i \cap K) = d \) and, since \( \text{rk} d_x(f) = n - d - 1 \), that \( \dim_x(B_i \cap K) = d + 1 \). Hence \((B_i \cap K) \setminus (A_i \cap K) = B_i \cap K \).
Thus Lemma 2.6 assures that there exists a closed semialgebraic subset $K' \subseteq B_i \cap K$ such that

$$A_i \cap K' = A_i \cap K \cap K' = \{O\} \quad \text{and} \quad B_i \cap K \sim \frac{s + \beta}{\alpha} K'.$$

Evidently $V(g_i) \cap K' = V(g_i) \cap B_i \cap K' = A_i \cap K' = \{O\}$. Thus by Proposition 2.7 there exists $m_2 \in \mathbb{N}$ such that for any integer number $m \geq m_2$ we have $g_i(x)^2 \geq h_{i+1}(x)^m$ for all $x \in K'$ and $g_i(x)^2 > h_{i+1}(x)^m$ for all $x \in K' \setminus \{O\}$.

If we take $m$ an integer $\geq m_2$, by construction $g_{i+1} = g_i^2 - h_{i+1}^m$ is strictly positive on $K' \setminus \{O\}$.

Let $x \in A_i \cap K \setminus \{O\}$. As $h_{i+1}(x) > 0$, then $g_{i+1}(x) < 0$. Since $B_i \cap K \sim \frac{s + \beta}{\alpha} K'$, there exist $\eta > \frac{s + \beta}{\alpha}$ and $z \in K'$ such that $\|x - z\| < \|x\|^\eta$ (and we can assume that $z \neq O$).

As $g_{i+1}$ is strictly positive on $K' \setminus \{O\}$, $g_{i+1}(z) > 0$. Since $z \in B(x, \|x\|^\eta)$, then $z \in B_{g_i}(x, \|x\|^{\eta_0})$. So, by the Intermediate Value Theorem on $B_{g_i}(x, \|x\|^{\eta_0})$, there exists $w \in B_{g_i}(x, \|x\|^{\eta_0}) \subseteq B(x, \|x\|^{\eta_0}) \cap B_i \cap K$ such that $g_{i+1}(w) = 0$.

Moreover, as $\eta \alpha > \beta$, by (4.3) one has in particular that $h_j(w) > 0$ for any $j = i + 2, \ldots, q$, which means that $w \in A_{i+1} \cap K$; hence $x \in \mathcal{H}(A_{i+1} \cap K, \eta_0)$.

We have thus proved that $A_i \cap K \setminus \{O\} \subseteq \mathcal{H}(A_{i+1} \cap K, \eta_0)$ and therefore, since $\eta_0 > s$, that

$$A_i \cap K \leq_s A_{i+1} \cap K$$

by Proposition 2.5.

In the general case, if $B_i \cap K \setminus \{O\}$ is not connected, it is sufficient to perform the previous argument on each connected component $\Delta$ of $B_i \cap K \setminus \{O\}$, find an odd integer number $m_\Delta$ as above, and take $m_2 = \max m_\Delta$.

Hence, if we let $M = \max\{m_1, m_2\}$, then for any odd $m \geq M$ we have

$$A_{i+1} \cap K \sim_s A_i \cap K .$$

(4.3)

In order to conclude the proof that $A_{i+1}$ satisfies property P1(i+1) observe that evidently

$$A_{i+1} \supseteq A_{i+1} \cap K \sim_s A_i \cap K \sim_s A \cap K \sim_s A$$

and thus in particular $A_{i+1} \geq_s A$.

For any $\sigma'$ with $\sigma > \sigma' > s$ we have that $(A_{i+1} \cap \overline{\mathcal{H}(X, \sigma)}) \setminus \{O\} \subseteq \mathcal{H}(X, \sigma')$; hence $A_{i+1} \cap \overline{\mathcal{H}(X, \sigma)} \leq_s X \subseteq A$ and thus $A_{i+1} \cap \overline{\mathcal{H}(X, \sigma)} \leq_s A$. Moreover from (4.2) $A_{i+1} \cap K \leq_s A$. Since

$$A_{i+1} = (A_{i+1} \cap K) \cup (A_{i+1} \cap \overline{\mathcal{H}(X, \sigma)}) ,$$

by Proposition 2.2 we have $A_{i+1} \leq_s A$ and thus $A_{i+1} \sim_s A$. By the inductive hypothesis we also get that

$$A_{i+1} \sim_s A_i$$

and so P1(i+1) is proved.

Property P2(i+1). By (4.2) and the previous Claim, we have that

$$A_{i+1} \cap K \cap (\Sigma(F_i) \cup \mathcal{Z}_i) \subseteq \{O\} .$$

Thus $h_j$ does not vanish on $A_{i+1} \cap K \setminus \{O\}$ for any $j \geq i + 1$, which in particular proves that $A_{i+1}$ satisfies property P2(i+1).
Property P3(i+1). In order to prove P3(i+1) consider the Jacobian matrix of $F_{i+1} = (g_{i+1}, f_2, \ldots, f_{n-d})$, i.e.

$$
\begin{pmatrix}
2g_i \nabla g_i - m \ h_{i+1}^{m-1} \nabla h_{i+1} \\
\nabla f_2 \\
\vdots \\
\nabla f_{n-d}
\end{pmatrix}.
$$

Evaluating it on the points of $A_{i+1}$ we get the matrix

$$
\begin{pmatrix}
h_{i+1}^{\frac{m}{2}}(2\nabla g_i - m \ h_{i+1}^{\frac{m-1}{2}} \nabla h_{i+1}) \\
\nabla f_2 \\
\vdots \\
\nabla f_{n-d}
\end{pmatrix}.
$$

Since, as seen above, $h_{i+1}$ does not vanish on $A_{i+1} \cap K \setminus \{O\}$,

$$
\Sigma(F_{i+1}) \cap A_{i+1} \cap K = \{x \in A_{i+1} \cap K \mid (2\nabla g_i - m \ h_{i+1}^{\frac{m-1}{2}} \nabla h_{i+1}) \wedge \nabla f_2 \wedge \ldots \wedge \nabla f_{n-d} = 0\}.
$$

If we let $\varphi = 4\|\nabla g_i \wedge \nabla f_2 \wedge \ldots \wedge \nabla f_{n-d}\|^2$ and $\psi = m^2\|\nabla h_{i+1} \wedge \nabla f_2 \wedge \ldots \wedge \nabla f_{n-d}\|^2$
we have that

$$
\Sigma(F_{i+1}) \cap A_{i+1} \cap K = \{x \in A_{i+1} \cap K \mid \varphi(x) = |h_{i+1}(x)|^{m-2}\psi(x)\}.
$$

Since $V(\varphi) = \Sigma(F_i)$ and by (4.3) $V(\varphi) \cap A_{i+1} \cap K \subseteq \{O\}$, by Proposition 2.7 there
exists $\lambda$ such that $\varphi(x) \geq \|x\|^\lambda$ on $A_{i+1} \cap K$. For the same reason there exists $\mu$ such that $|h_{i+1}(x)|^\mu \leq \|x\|$ on $A_{i+1} \cap K$. Moreover there exists a constant $N$ such that $\psi \leq N$ on $A_{i+1} \cap K$.

If $m > \lambda \mu + 2$, then $\Sigma(F_{i+1}) \cap A_{i+1} \cap K \subseteq \{O\}$. Namely, if by contradiction there exists a
sequence of points $x_\nu \in A_{i+1} \cap K$ converging to $O$ such that $\varphi(x_\nu) = |h_{i+1}(x_\nu)|^{m-2}\psi(x_\nu)$, then

$$
\|x_\nu\|^\lambda \mu \leq N^\mu \|x_\nu\|^{m-2}
$$

which is a contradiction.

Let $m_3$ be an integer such that $m_3 > \lambda \mu + 2$. Thus for any odd integer $m \geq m_3$ we have that $A_{i+1}$ satisfies property P3(i+1). In particular dim$_O(A_{i+1} \cap K) = d$.

Finally, if we let $m_s = \max\{M, m_3\}$, then for any odd integer $m \geq m_s$ we have that
$A_{i+1}$ satisfies all the properties P1(i+1), P2(i+1) and P3(i+1).

At the end of the recursive construction, observe that the set $A_q$ is algebraic, $A_q \sim_A A$, $A_q \cap K \sim_A A \cap K \sim_A A$ and dim$_O(A_q \cap K) = d$. Moreover

$$
\overline{A_q \cap K} = \overline{A_q} \quad \overline{A_q \cap K} \subseteq A_q \cap K \subseteq A_q \cap K = \overline{A_q \cap K}.
$$

Thus $S = \overline{A_q \cap K}$ satisfies the thesis. $\square$

The previous theorem allows us to strengthen the following result on approximation preserving dimension which can be found in [PPW3].

**Theorem 4.2.** Let $A$ be a closed semianalytic subset of $\mathbb{R}^n$ with $O \in A$. Then for any $s \geq 1$ there exists a closed semialgebraic set $S \subseteq \mathbb{R}^n$ such that $A \sim_A S$ and dim$_O S = \text{dim}_O A$. 
From Theorem 4.1 and from Theorem 4.2 we immediately obtain:

**Corollary 4.3.** For any real number $s \geq 1$ and for any closed semianalytic set $A \subset \mathbb{R}^n$ of codimension $\geq 1$ with $O \in A$, there exists an algebraic subset $S$ of $\mathbb{R}^n$ such that $A \sim_s S$ and $\dim_O S = \dim_O A$.

**Example 4.4.** If $A = \{(x, y, z) \in \mathbb{R}^3 \mid z = 0, x \geq 0, y \geq 0\}$ and $s \geq 1$, the approximation technique described in the proof of Theorem 4.1 yields a surface defined by $(z^2 - x^m)^2 - y^p = 0$ for suitable odd integers $m$ and $p$; the shape of such a surface is represented in Figure 1.

![Figure 1. Algebraic approximation of a quadrant](image-url)