MODIFIED TRACE FROM PIVOTAL HOPF G-COALGEBRA

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ABSTRACT. In a recent paper the authors Beliakova, Blanchet and Gainutdinov have shown that the modified trace on the category $H$-pmod of the projective modules corresponds to the symmetrised integral on the finite dimensional pivotal Hopf algebra $H$. We generalize this fact to the context of $G$-graded categories and Hopf $G$-coalgebra studied by Turaev-Virelizier. We show that the symmetrised $G$-integral on a finite type pivotal Hopf $G$-coalgebra induces a modified trace in the associated $G$-graded category.

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Key words: modified trace, $G$-integral, symmetrised $G$-integral, pivotal Hopf $G$-coalgebra.

1. Introduction

The notion of a modified trace was introduced by N. Geer, J. Kujawa and B. Patureau-Mirand in the article [7]. This is one of the topological tools which can be used first to renormalize the Reshetikhin-Turaev invariant of links. Later F. Costantino, N. Geer and B. Patureau-Mirand used the modified trace to construct a class of invariants of 3-manifolds (CGP invariant) via link surgery presentations (see [4]). The modified trace is also used to construct invariants of 3-manifolds of type Reshetikhin-Turaev from quantum group associated to the super Lie algebra $\mathfrak{sl}(2|1)$ (see [9]) and for constructing the logarithmic invariant of type Hennings (see [2]). In order to construct invariant of 3-manifolds, Hennings proposed a method based on the theory of integral for a finite dimensional Hopf algebra (see [10]). The notion of integral was introduced by Larson and Sweedler in [12] and is studied in the book [14] of Radford. It is known that under some assumption, both the space of modified trace and that of integral are one dimensional (see [6, 14]). A close relation between the modified trace and the integral has been established recently in [1]. The authors proved that a symmetrised integral for a finite dimensional pivotal Hopf algebra gives a modified trace $t$ on $H$-pmod with an explicit formula. We would like to adapt these results to the unrestricted quantum groups at root of unity. They are infinite dimensional Hopf algebra but can be understood as a Hopf $G$-coalgebra organized into a bundles of algebra over a Lie group. For
a finite type Hopf $G$-coalgebra $H = (H_a)_{α ∈ G}$, there exists a family of linear forms on $H_a$, called $G$-integral (see [16]). The aim of this article is to establish a correspondence between the $G$-integral for the finite type unimodular pivotal Hopf $G$-coalgebra $H$ and the modified trace in the associated $G$-graded category $H$-mod. We introduce now these two notions.

**$G$-integral.** Let $H = (\{ H_α, m_α, 1_α \}, Δ, ε, S)$ be a Hopf $G$-coalgebra over a field $k$ (see in Section 2). A right $G$-integral for the Hopf $G$-coalgebra $H$ is a family of $k$-linear forms $μ = (μ_α : H_α → k)_{α ∈ G}$ satisfying

$$(1.1) \quad (μ_α ⊗ \text{Id}_{H_β})Δ_{α,β}(x) = μ_{αβ}(x)1_β \text{ for any } x ∈ H_{αβ}.$$ 

Similarly, a left $G$-integral $μ^l_α ∈ \prod_{α ∈ G} H^*_α$ satisfies

$$(\text{Id}_{H_α} ⊗ μ^l_β)Δ_{α,β}(x) = μ^l_{αβ}(x)1_α \text{ for any } x ∈ H_{αβ}.$$ 

The linear form $μ_1$ is an usual right integral for the Hopf algebra $H_1$ (see e.g [14]). If $H$ is a finite type Hopf $G$-coalgebra, i.e. a Hopf $G$-coalgebra in which $\dim(H_α) < +∞$ for any $α ∈ G$, the space of right (resp. left) $G$-integral is known to be 1-dimensional (see e.g [16]).

A pivotal Hopf $G$-coalgebra is a pair $(H, g)$, where the pivot is the family of $g ∈ \prod_{α ∈ G} H_α$ satisfying $Δ_{α,β}(g_{αβ}) = g_α ⊗ g_β$ for any $α, β ∈ G$, $ε(g_1) = 1_k$, and $S_{α-1}S_α(x) = g_αxg^{-1}_α$ for any $x ∈ H_α$. Note that $g^{-1} = (S_{α-1}(g_{α-1}))_{α ∈ G}$, i.e. $g^{-1}_α = S_{α-1}(g_{α-1})$ (see e.g [16]).

In particular, $g_1$ is a pivotal element for $H_1$ and $g_1$ is invertible with $g^{-1}_1 = S_1(g_1)$, $ε(g_1) = 0$ (see e.g [11]).

The symmetrised right $G$-integral on $(H, g)$ associated with $μ$ is the family $μ = (μ_α)_{α ∈ G} ∈ \prod_{α ∈ G} H^*_α$ defined by

$$μ_α(x) := μ_α(g_αx) \text{ for any } x ∈ H_α.$$ 

Similarly, a symmetrised left $G$-integral on $(H, g)$ is

$$(1.2) \quad μ^l_α(x) := μ^l_α(g_α^{-1}x) \text{ for any } x ∈ H_α.$$ 

In the case $(H, g)$ is unimodular, i.e. $H_1$ is unimodular, we show that the symmetrised $G$-integrals are symmetric linear forms on $H$ and they are non-degenerate (see Proposition 2.6).

**Modified trace.** Let $C$ be a pivotal $k$-linear category [13]. Let $\text{Proj}(C)$ be the tensor ideal of projective objects of $C$. A modified trace on ideal $\text{Proj}(C)$ is a family of $k$-linear forms $t = \{ t_P : \text{End}_C(P) → k \}_{P ∈ \text{Proj}(C)}$ satisfying the cyclicity property and the partial trace property (see in Section 3.2).
Main results. Let \((H, g) = (\{H_\alpha, m_\alpha, 1_\alpha\}, \Delta, \varepsilon, S, g)\) be a finite type unimodular pivotal Hopf \(G\)-coalgebra. If \(t\) is a right (resp. left) modified trace on \(H\)-pmod, it defines a family of linear forms \(\lambda^t = (\lambda^t_\alpha) \in \prod_{\alpha \in G} H^*_\alpha\) by \(\lambda^t_\alpha(h) = t_{H_\alpha}(R_h)\) for \(h \in H_\alpha\), \(H_\alpha\) is a projective object of \(H\)-mod and \(R_h\) is the right multiplication of \(H_\alpha\).

Theorem 1.1. The application \(t \mapsto \lambda^t\) defined above gives a bijection between the space of right (resp. left) modified traces and the space of symmetrised right (resp. left) \(G\)-integrals. Furthermore, \((H, g)\) is \(G\)-unibalanced if and only if the right modified trace is also left.

The paper contains five section. In section 2 we recall some definition and results for a Hopf \(G\)-coalgebra, we also define a pivotal Hopf \(G\)-coalgebra, a symmetrised \(G\)-integral for a pivotal Hopf \(G\)-coalgebra \(H\) and prove that the symmetrised \(G\)-integrals are symmetric non-degenerate forms on \(H\). Section 3 recall some results about modified traces and the proof of Reduction Lemma in the context of \(G\)-graded categories. In section 4 we present the decomposition of tensor product \(H_\alpha \otimes H_\beta\) and the proof of the main theorem. In section 5 we give an application of the main theorem in the case associated to a quantization of the Lie algebra \(\mathfrak{sl}(2)\).

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2. Pivotal Hopf \(G\)-coalgebra

In this section, we recall some facts about Hopf \(G\)-coalgebra. For details see [15] [16]. We then define a pivotal Hopf \(G\)-coalgebra, a symmetrised \(G\)-integral and give some of its properties.

2.1. Pivotal Hopf \(G\)-coalgebra.

2.1.1. Hopf \(G\)-coalgebra.

Definition 2.1. Let \(G\) be a multiplicative group. A \(G\)-coalgebra over a field \(\mathbb{k}\) is a family \(C = \{C_\alpha\}_{\alpha \in G}\) of \(\mathbb{k}\)-spaces endowed with a family \(\Delta = \{\Delta_{\alpha, \beta} : C_\alpha \otimes C_\beta \to C_{\alpha \beta}\}_{\alpha, \beta \in G}\) of \(\mathbb{k}\)-linear maps (the coproduct) and a \(\mathbb{k}\)-linear map \(\varepsilon : C_1 \to \mathbb{k}\) (the counit) such that

\[ (\Delta_{\alpha, \beta} \otimes \text{Id}_{C_\gamma}) \Delta_{\alpha \beta, \gamma} = (\text{Id}_{C_\alpha} \otimes \Delta_{\beta, \gamma}) \Delta_{\alpha, \beta \gamma}. \]

\[ \text{for all } \alpha \in G, \ (\text{Id}_{C_\alpha} \otimes \varepsilon) \Delta_{\alpha, 1} = \text{Id}_{C_\alpha} = (\varepsilon \otimes \text{Id}_{C_\alpha}) \Delta_{1, \alpha}. \]
A Hopf $G$-coalgebra is a $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}, \Delta, \varepsilon$ endowed with a family $S = \{S_\alpha : H_\alpha \to H_{\alpha^{-1}}\}_{\alpha \in G}$ of $k$-linear maps (the antipode) such that

- each $H_\alpha$ is an algebra with product $m_\alpha$ and unit element $1_\alpha \in H_\alpha$,
- $\varepsilon : H_1 \to k$ and $\Delta_{\alpha,\beta} : H_{\alpha\beta} \to H_\alpha \otimes H_\beta$ are algebra homomorphisms for all $\alpha, \beta \in G$,
- for any $\alpha \in G$
  
  $$m_\alpha(S_{\alpha^{-1}} \otimes \text{Id}_{H_\alpha})\Delta_{\alpha^{-1},\alpha} = \varepsilon 1_\alpha = m_\alpha(\text{Id}_{H_\alpha} \otimes S_{\alpha^{-1}})\Delta_{\alpha,\alpha^{-1}}.$$  

The antipode automatically satisfies additional property:

**Lemma 2.2.** Given a Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}, \Delta, \varepsilon, S$, then

1. $S_\alpha(xy) = S_\alpha(y)S_\alpha(x)$ for any $x, y \in H_\alpha$,
2. $S_\alpha(1_\alpha) = 1_{\alpha^{-1}}$,
3. $\Delta_{\beta^{-1},\alpha^{-1}}S_{\alpha\beta} = \tau(S_\alpha \otimes S_\beta)\Delta_{\alpha,\beta}$ where $\tau : H_{\alpha^{-1}} \otimes H_{\beta^{-1}} \to H_{\beta^{-1}} \otimes H_{\alpha^{-1}}$ is the flip switching the two factors of $H_{\alpha^{-1}} \times H_{\beta^{-1}}$,
4. $\varepsilon S_1 = \varepsilon$.

2.1.2. Graphical axioms for Hopf $G$-coalgebras. We will use the diagrams for the structural maps and the identities corresponding to the Hopf $G$-coalgebra $H = \{H_\alpha\}_{\alpha \in G}$. For simplicity we write $\alpha$ instead of $H_\alpha$ in the diagrams. Figure 1 presents the structural maps of the Hopf $G$-coalgebra which are the product, coproduct, unity, counity and the antipode, respectively. Note that these maps are in the category Vect$_k$ of finite dimensional vector spaces over a field $k$.

The identity of the coassociativity and the algebra homomorphism $\Delta_{\alpha,\beta}$ are defined as in Figure 2. The antipode properties are shown
Example 2.3. Let $H$ be a possibly infinite dimensional pivotal Hopf algebra with the pivot $\phi$. Suppose there is a commutative Hopf subalgebra $C$ contained in the center of $H$ (for example $H$ can be the unrestricted quantum group in [3]; another example will be detailed in Section 6). Let $G = \text{Hom}_{\text{Alg}}(C, k)$ be the group of characters on $C$ with multiplication given by $gh = (g \otimes h) \circ \Delta$ for $g, h \in G$ and $g^{-1} = g \circ S|_C$. For $g \in G$ we define $H_g = H \otimes_{g:C\rightarrow k} k = H/I_g$ where $I_g$ is the ideal generated by elements $z - g(z)$ for $z \in C$. Assume $g = g_1 g_2$ for $g_1, g_2 \in G$, then

$$\Delta(z - g(z)) = \Delta(z) - (g_1 \otimes g_2)(\Delta(z))$$
$$= z_1 \otimes z_2 - g_1(z_1) \otimes g_2(z_2)$$
$$= (z_1 - g_1(z_1)) \otimes z_2 + g_1(z_1) \otimes (z_2 - g_2(z_2))$$

where we used the Sweedler’s notation $\Delta(z) = z_1 \otimes z_2$. This implies that $\Delta(I_g) \subset I_{g_1} \otimes H + H \otimes I_{g_2}$. We thus have that a well defined coproduct $\Delta_{g_1, g_2}$ given by the commutative diagram below.
where \( p_g : H \rightarrow H_g \) is the projective morphism. The family \( \{H_g\}_{g \in G} \) with coproduct \( \Delta_{g,h} \) is a \( G \)-coalgebra. It is also a Hopf \( G \)-coalgebra with the family of antipode given by the commutative diagram

\[
\begin{array}{ccc}
H & \xrightarrow{S} & H \\
\downarrow{p_g} & & \downarrow{p_g^{-1}} \\
H_g & \xrightarrow{S_g} & H_{g^{-1}}.
\end{array}
\]

The family \( S_g \) for \( g \in G \) is well defined since \( S(z - g(z)) = S(z) - g(z) = S(z) - g^{-1}(S(z)) \in I_{g^{-1}} \).

We say a Hopf \( G \)-coalgebra \( H \) is of finite type if \( H_\alpha \) is finite dimensional over \( \mathbb{k} \) for all \( \alpha \in G \).

2.1.3. Pivotal structure. We recall that a \( G \)-grouplike element of a Hopf \( G \)-coalgebra \( H \) is a family \( g = (g_\alpha)_{\alpha \in G} \in \prod_\alpha H_\alpha \) such that \( \Delta_{\alpha,\beta}(g_\beta) = g_\alpha \otimes g_\beta \) for any \( \alpha, \beta \in G \) and \( \varepsilon(g_1) = 1_\mathbb{k} \). Note that \( g_1 \) is a grouplike element of the Hopf algebra \( H_1 \). It follows [10] that the set of the \( G \)-grouplike elements of \( H \) is a group and if \( g = (g_\alpha)_{\alpha \in G} \), then \( g^{-1} = (S_\alpha^{-1}(g_\alpha^{-1}))_{\alpha \in G} \).

**Definition 2.4.** A \( G \)-grouplike element \( g \in H \) is called a pivot if \( S_{\alpha^{-1}}S_\alpha(x) = g_\alpha x g_\alpha^{-1} \) for all \( x \in H_\alpha \). The pair \((H, g)\) of a Hopf \( G \)-coalgebra \( H \) and a pivot \( g \) is called a pivotal Hopf \( G \)-coalgebra.

Remark that for a pivotal Hopf \( G \)-coalgebra \( H = (\{H_\alpha\}_{\alpha \in G}, \Delta, \varepsilon, S, g) \), \( H_1 \) is a pivotal Hopf algebra.

**Example 2.5.** Let \( H \) be a Hopf \( G \)-coalgebra as in Example 2.3. Let \( \phi_g \) be the image of \( \phi \) in the quotient \( H_g \). Then \( H \) is a pivotal Hopf \( G \)-coalgebra.

2.2. Symmetrised right and left \( G \)-integrals. Let \( H = (\{H_\alpha\}_{\alpha \in G}, \Delta, \varepsilon, S) \) be a finite type pivotal Hopf \( G \)-coalgebra with right \( G \)-integral \( \mu \).

The symmetrised right \( G \)-integral associated with \( \mu \) is a family \( \tilde{\mu} = (\tilde{\mu}_\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} H_\alpha^* \) defined by \( \tilde{\mu}_\alpha(x) := \mu_\alpha(g_\alpha x) \) for any \( x \in H_\alpha \).

Using the definition of the right \( G \)-integral, see Equation (1.1) and replacing \( x \in H_{\alpha,\beta} \) by \( g_{\alpha,\beta} x \) we get:

\[
(\tilde{\mu}_\alpha \otimes g_{\alpha,\beta}) \Delta_{\alpha,\beta}(x) = \tilde{\mu}_{\alpha,\beta}(x) 1_{\beta}.
\]

Similarly, the symmetrised left \( G \)-integral is defined by \( \tilde{\mu}_\alpha^l(x) := \mu_\alpha^l(g_\alpha^{-1} x) \) for any \( x \in H_\alpha \). Applying (1.2) for \( g_{\alpha,\beta}^{-1} x \), \( x \in H_{\alpha,\beta} \) we get the defining relation for the symmetrised left \( G \)-integral:

\[
(g_{\alpha}^{-1} \otimes \tilde{\mu}_\beta)(\Delta_{\alpha,\beta}(x)) = \tilde{\mu}_{\alpha,\beta}(x) 1_{\alpha}.
\]

The graphical representation for Equality (2.1) is given in Figure 5. The graphical representation of the relation for the left symmetrised
Figure 5. The graphical representation of the relation for the right symmetrised $G$-integral

$G$-integral is similar.

Since the pivot is invertible Equation (2.1) for $\tilde{\mu}$ is equivalent to Equation (1.1) for $\mu$. As the space of right $G$-integrals is one-dimensional, relation (2.1) defines $\tilde{\mu}$ uniquely (up to a scalar). Similarly the symmetrised left $G$-integral $\tilde{\mu}^l$ defined by (2.2) is unique. Note also that the symmetrised $G$-integral for $H_1$ is the one in the sense of [1].

A Hopf $G$-coalgebra $H$ is unimodular if the Hopf algebra $H_1$ is unimodular, this means that the spaces of left and right cointegrals for $H_1$ coincide.

We say that a family of linear forms $\varphi_\alpha \in H_\alpha^*$ for $\alpha \in G$ is symmetric non-degenerate if for any $\alpha \in G$ the associated bilinear forms $(x,y) \mapsto \varphi_\alpha(xy)$, $x,y \in H_\alpha$ is.

Proposition 2.6. Assume $(H,g)$ is unimodular, then the symmetrised right (resp. left) $G$-integral for $(H,g)$ is symmetric and non-degenerate.

Proof. For any $\alpha \in G$, $x,y \in H_\alpha$, by [16] Lemma 7.1 we have

$$\tilde{\mu}_\alpha(xy) = \mu_\alpha(g_\alpha xy) = \mu_\alpha(S_\alpha^{-1}S_\alpha(y)g_\alpha x) = \mu_\alpha(g_\alpha yx) = \tilde{\mu}_\alpha(yx)$$

and by [16] Corollary 3.7 $H_\alpha^*$ is free left module rank one over $H_\alpha$ with basis $\{\mu_\alpha\}$ when the action is defined by

$$(h \rightarrow \mu_\alpha)(x) := \mu_\alpha(xh) \text{ for } h, x \in H_\alpha.$$ 

If $\tilde{\mu}_\alpha(xy) = \mu_\alpha(g_\alpha yx) = (g_\alpha y \rightarrow \mu_\alpha)(x) = 0$ for all $x \in H_\alpha$, then $g_\alpha y \rightarrow \mu_\alpha = 0$. It follows thus $y = 0$.

For the symmetrised left $G$-integral the proof is similar. □

Also note that the spaces of left and right $G$-integrals are not equal in general. We have a lemma.

Lemma 2.7. The left $G$-integral for $H$ can be chosen as $\mu^l_\alpha(x) = \mu_{\alpha^{-1}}(S_\alpha(x))$ for any $x \in H_\alpha$.

Proof. By [1,1] we have

$$(\mu_{\alpha^{-1}} \otimes \text{Id}_{H_{\beta^{-1}}})\Delta_{\alpha^{-1},\beta^{-1}}(S_{\beta\alpha}(x)) = \mu_{(\beta\alpha)^{-1}}(S_{\beta\alpha}(x))1_{\beta^{-1}}$$

for any $x \in H_{\beta\alpha}$. 
Using Lemma 2.2 (3) \( \Delta_{\alpha^{-1}, \beta^{-1}}(S_{\beta\alpha}(x)) = (S_{\alpha} \otimes S_{\beta})\Delta_{\beta, \alpha}^{op}(x) \) we get 
\[
(\mu_{a^{-1}} \circ S_{\alpha} \otimes S_{\beta})\Delta_{\beta, \alpha}^{op}(x) = (S_{\beta} \otimes \mu_{a^{-1}} \circ S_{\alpha})\Delta_{\beta, \alpha}(x) = \mu_{(\beta a)^{-1}}(S_{\beta\alpha}(x))1_{\beta^{-1}}.
\]
Applying \( S_{\beta}^{-1} \) to both sides of the last equality and \( S_{\beta}^{-1}(1_{\beta^{-1}}) = 1_{\beta} \), we obtain that \( (\text{Id}_{H_{\beta}} \otimes \mu_{a^{-1}} \circ S_{\alpha})\Delta_{\beta, \alpha}(x) = (\mu_{(\beta a)^{-1}} \circ S_{\beta\alpha})(x)1_{\beta} \), i.e. \( \mu_{a^{-1}} \circ S_{\alpha} \) satisfies the definition of the left \( G \)-integral. \( \square \)

2.3. \( G \)-unbalanced Hopf algebras. Let \( H = (\{ H_{a} \}_{a \in G}, \Delta, \varepsilon, S) \) be a finite type Hopf \( G \)-coalgebra with right \( G \)-integral \( \mu \). We call a distinguished \( G \)-grouplike of \( H \) (see e.g. [14]) or \( G \)-comodulus of \( H \) a \( G \)-grouplike element \( a = (a_{a})_{a \in G} \in \prod_{a \in G} H_{a} \) satisfying

\[
(\text{Id}_{H_{a}} \otimes \mu_{\beta})\Delta_{\alpha, \beta}(x) = \mu_{a\beta}(x)a_{\alpha} \quad \text{for any } x \in H_{a\beta}.
\]

Note that \( a_{1} \) is the comodulus element of the Hopf algebra \( H_{1} \) (see [1]). By multiplying (2.3) with \( a^{-1} \) and replacing \( x \) by \( a_{a\beta}x \) we have

\[
(\text{Id}_{H_{a}} \otimes \mu_{\beta}(a_{\beta}x))\Delta_{\alpha, \beta}(x) = \mu_{a\beta}(a_{a\beta}x)1_{\alpha}.
\]

where denote by \( \mu_{\beta}(a_{\beta}x) \) the linear map \( x \mapsto \mu_{\beta}(a_{\beta}x) \) for \( x \in H_{\beta} \). This equality implies that \( \mu_{\beta}(a_{\beta}x) \) is a left \( G \)-integral for \( H \), i.e.

\[
\mu_{\beta}(x) = \mu_{\beta}(a_{\beta}x).
\]

This is another choice for left \( G \)-integral from right \( G \)-integral. This choice of the left \( G \)-integral is the same with the one in Lemma 2.7 by following proposition.

**Proposition 2.8.** We have the relation \( \mu_{a^{-1}}(S_{\alpha}(x)) = \mu_{a}(a_{\alpha}x) \) for any \( x \in H_{a} \).

**Proof.** By (2.4) we get

\[
(\text{Id}_{H_{a}} \otimes \mu_{1}(a_{1}x))\Delta_{\alpha, 1}(x) = \mu_{a}(a_{\alpha}x)1_{\alpha} \quad \text{for } x \in H_{a}.
\]

By Lemma 2.7 we get

\[
(\text{Id}_{H_{a}} \otimes \mu_{1} \circ S_{1})\Delta_{\alpha, 1}(x) = (\mu_{a^{-1}} \circ S_{\alpha})(x)1_{\alpha} \quad \text{for } x \in H_{a}.
\]

Furthermore, Proposition 4.7 [1] gives \( \mu_{1}(S_{1}(x)) = \mu_{1}(a_{1}x) \) for \( x \in H_{1} \).

This implies that \( \mu_{a}(a_{\alpha}x)1_{\alpha} = (\mu_{a^{-1}} \circ S_{\alpha})(x)1_{\alpha} \) for all \( x \in H_{a} \), i.e. \( \mu_{a^{-1}}(S_{\alpha}(x)) = \mu_{a}(a_{\alpha}x) \) for any \( x \in H_{a} \). \( \square \)

A finite type pivotal Hopf \( G \)-coalgebra \( (H, g) \) is called \( G \)-unbalanced if its symmetrised right \( G \)-integral is also left, i.e. \( \tilde{\mu}_{a} = \tilde{\mu}_{a}^{l} \) for any \( \alpha \in G \).

**Lemma 2.9.** Assume \( (H, g) \) is an unimodular pivotal Hopf \( G \)-coalgebra. Then \( (H, g) \) is \( G \)-unbalanced if and only if \( a_{\alpha} = g_{\alpha}^{2} \) for any \( \alpha \in G \).

**Proof.** First, we assume that \( a_{\alpha} = g_{\alpha}^{2} \). Applying (2.3) on \( g_{a\beta}x \) we have

\[
(g_{a^{-1}} \otimes \tilde{\mu}_{\beta})\Delta_{\alpha, \beta}(x) = \tilde{\mu}_{a\beta}(x)1_{\alpha}.
\]


This equality states that \( \tilde{\mu}_\beta \) is a symmetrised left \( G \)-integral, i.e. \( \tilde{\mu}_\beta = \tilde{\mu}_\beta^l \). Second, we assume that \((H, g)\) is \( G \)-unibalanced. By applying the equality (2.5) on \( g^{-1}x \) and the \( G \)-unibalanced condition one gets
\[
\mu^l_\alpha(g^{-1}x) = \tilde{\mu}_\alpha(x) = \tilde{\mu}_\alpha(g_\alpha x) = \mu_\alpha(a_\alpha g^{-1}_\alpha x)
\]
for any \( x \in H_\alpha \). The last equality gives
\[
\mu_\alpha \left((a_\alpha g^{-1}_\alpha - g_\alpha) x\right) = 0 \text{ for any } x \in H_\alpha.
\]
By Proposition 2.6, \( \mu_\alpha \) is non-degenerate. Therefore, the above equality holds if and only if \( a_\alpha = g^2_\alpha \). \( \square \)

3. Traces on finite \( G \)-graded categories

In this section we recall some notions and results from [1]. Let \((H, g)\) be a finite type unimodular pivotal Hopf \( G \)-coalgebra. We determine the pivotal structure in pivotal \( G \)-graded category \( H \)-mod. We also prove the Reduction Lemma in the context of \( G \)-graded categories and recall the close relation between a modified trace on \( H_1 \)-pmod and a symmetrised integral for \( H_1 \) [1].

3.1. Cyclic traces. Let \( C \) be a \( k \)-linear category. We call cyclic trace on \( C \) a family of \( k \)-linear maps
\[
t = \{t_P : \text{End}_C(P) \to k\}_{P \in C}
\]
satisfying cyclicity property, i.e. \( t_V(gh) = t_U(hg) \) for \( g \in \text{Hom}_C(U, V) \) and \( h \in \text{Hom}_C(V, U) \) with \( U, V \in C \). We say that a cyclic trace \( t \) is non-degenerate if the pairings
\[
\text{Hom}_C(M, P) \times \text{Hom}_C(P, M) \to k, \ (f, g) \mapsto t_P(fg)
\]
are non-degenerate for all \( P, M \in C \).

For a finite dimensional algebra \( A \), let \( A\)-pmod be the category of projective \( A \)-modules. There is a bijection from the space of cyclic traces on \( A\)-pmod to the space of symmetric linear forms on \( A \):

**Lemma 3.1.** There is an isomorphism of algebras
\[
R : A^{op} \to \text{End}_A(A)
\]
given by
\[
R(h) = R_h, \quad R^{-1}(f) = f(1)
\]
where \( R_h \) denotes the right multiplication with \( h \), i.e. \( R_h(x) = xh \) for any \( x \in A \).

Lemma 3.1 implies that if \( t \) is a cyclic trace on \( A\)-pmod then
\[
\lambda(h) = t_A(R_h)
\]
defines a symmetric linear form on \( A \).
Proposition 3.2. [1, Proposition 2.4] A symmetric linear form $\lambda$ on a finite dimensional algebra $A$ extends uniquely to a family of cyclic traces $\{t_P : \text{End}_A(P) \rightarrow k\}_{P \in A\text{-pmod}}$ which satisfies Equality (3.3).

If $f \in \text{End}_A(P)$, one can find $a_i \in \text{Hom}(A, P), \ b_i \in \text{Hom}(P, A) \ i \in I$ for some finite set $I$ such that $f = \sum_{i \in I} a_i b_i$ (see [1]). Then the cyclicity property of $t$ implies that

$$t_P(f) = \sum_{i \in I} t_A(a_i) = \sum_{i \in I} \lambda(b_i a_i(1)).$$

Furthermore, the non-degeneracy of the form linear $\lambda$ is equivalent to the one of the pairings (3.2) determined by $(t_P)_{P \in A\text{-pmod}}$ in (3.4) (see [1], Theorem 2.6 where a stronger non-degeneracy condition for traces is considered).

3.2. Modified trace in pivotal category. Let $\mathcal{C}$ be a pivotal $k$-linear category [13]. Then $\mathcal{C}$ is a strict monoidal $k$-linear category, with a unit object $I$, equipped with the data for each object $V \in \mathcal{C}$ of its dual object $V^* \in \mathcal{C}$ and of four morphisms

$$\text{ev}_V : V^* \otimes V \rightarrow I, \quad \text{coev}_V : I \rightarrow V \otimes V^*, \quad \text{ev}_V : V \otimes V^* \rightarrow I, \quad \text{coev}_V : I \rightarrow V^* \otimes V$$

such that $(\text{ev}_V, \text{coev}_V)$ and $(\text{ev}_V, \text{ev}_V)$ are dualities which induce the same functor duality which is monoidal. In the category $\mathcal{C}$ there is a family of isomorphisms

$$\Phi = \{\Phi_V = (\text{ev}_V \otimes \text{Id}_{V^*}) (\text{Id}_V \otimes \text{coev}_V) : V \rightarrow V^{**}\}_{V \in \mathcal{C}}$$

which is a monoidal natural isomorphism called the pivotal structure. We recall the notion of a modified trace on ideal in a pivotal category [7, 8]. Given $U, V, W \in \mathcal{C}$ and $f \in \text{End}_\mathcal{C}(W \otimes V)$.

The left partial trace (with respect to $W$) is the map

$$\text{tr}_W^l : \text{Hom}_\mathcal{C}(W \otimes U, W \otimes V) \rightarrow \text{Hom}_\mathcal{C}(U, V)$$

defined for $f \in \text{Hom}_\mathcal{C}(W \otimes U, W \otimes V)$ by

$$\text{tr}_W^l(f) = (\text{ev}_W \otimes \text{Id}_V)(\text{Id}_W \otimes f)(\text{coev}_W \otimes \text{Id}_V) = \begin{pmatrix} W \downarrow f \uparrow & V \downarrow \downarrow \uparrow \downarrow \uparrow \downarrow \end{pmatrix} \in \text{Hom}_\mathcal{C}(U, V).$$

The right partial trace (with respect to $W$) is the map

$$\text{tr}_W^r : \text{Hom}_\mathcal{C}(U \otimes W, V \otimes W) \rightarrow \text{Hom}_\mathcal{C}(U, V)$$
defined for $f \in \text{Hom}_C(U \otimes W, V \otimes W)$ by (3.5)

$$\text{tr}_W^f(f) = (\text{Id}_V \otimes \overrightarrow{\text{ev}}_W)(f \otimes \text{Id}_W)(\text{Id}_U \otimes \overrightarrow{\text{coev}}_W) = \begin{array}{c} V \\ U \end{array} \in \text{Hom}_C(U, V).$$

Let $\text{Proj}(C)$ be the tensor ideal of projective objects of $C$. A left modified trace on $\text{Proj}(C)$ is a cyclic trace $t$ on $\text{Proj}(C)$ satisfying

$$t_{W \otimes P}(f) = t_P(\text{tr}_W^f(f))$$

for any $f \in \text{End}_C(W \otimes P)$ with $P \in \text{Proj}(C)$ and $W \in C$.

A right modified trace on $\text{Proj}(C)$ is a cyclic trace $t$ on $\text{Proj}(C)$ satisfying

$$t_{P \otimes W}(f) = t_P(\text{tr}_W^f(f))$$

for any $f \in \text{End}_C(P \otimes W)$ with $P \in \text{Proj}(C)$ and $W \in C$.

A modified trace on ideal $\text{Proj}(C)$ is a cyclic trace $t$ on $\text{Proj}(C)$ which is both a left and right trace on $\text{Proj}(C)$.

Next we define the category of $H$-mod which is a pivotal $G$-graded category.

### 3.3. Pivotal structure on $H$-mod.

#### 3.3.1. $G$-graded category.

Given a multiplicative group $G$, we call the category $C$ pivotal $G$-graded k-linear if there exists a family of full subcategories $(C_\alpha)_{\alpha \in G}$ of $C$ such that

- $1 \in C_1$.
- $\forall (\alpha, \beta) \in G^2$, $\forall (V, W) \in C_\alpha \times C_\beta$, $\text{Hom}_C(V, W) \neq \{0\} \Rightarrow \alpha = \beta$.
- $\forall V \in C_\alpha$, $\exists \alpha = (\alpha_1, \ldots, \alpha_n) \in G^n$, $\exists V_i \in C_{\alpha_i}$ for $i = 1, \ldots, n$ such that $V \cong V_1 \oplus \ldots \oplus V_n$.
- $\forall (V, W) \in C_{\alpha} \times C_{\beta}$, $V \otimes W \in C_{\alpha \beta}$.
- $\forall \alpha \in G$, $C_\alpha$ does not reduce to null object.

#### 3.3.2. Pivotal structure on $H$-mod. Let $(H, g) = (\{H_\alpha\}_{\alpha \in G}, \Delta, \varepsilon, S, g)$ be a finite type pivotal Hopf $G$-coalgebra, let $C$ be the k-linear category $\bigoplus_{\alpha \in G} C_\alpha$ in which $C_\alpha$ is $H_\alpha$-mod the category of finite dimensional $H_\alpha$-modules. An object $V$ of $C$ is a finite direct sum $V_{\alpha_1} \oplus \ldots \oplus V_{\alpha_n}$ where $V_{\alpha_i} \in C_{\alpha_i}$. Each object $V$ in $H_\alpha$-mod has a dual $V^* = \text{Hom}_k(V, k)$ in $H_{\alpha^{-1}}$-mod with the $H_{\alpha^{-1}}$ action defined by $(hf)(x) = f(S_{\alpha^{-1}}(h)x)$ for $h \in H_{\alpha^{-1}}$, $f \in V^*$ and $x \in V$. The category $C$ is a $G$-graded tensor category, i.e. for $V_{\alpha} \in C_{\alpha}$, $V_{\beta} \in C_{\beta}$, $V_{\alpha} \otimes V_{\beta} \in C_{\alpha \beta}$ and for $\alpha \neq \beta$, $\text{Hom}_C(V_\alpha, V_\beta) = 0$.

Then $C$ is a pivotal category with pivotal structure given by the left
and right duality morphisms as follows. Assume that \( \{v_j \mid j \in J\} \) is a basis of \( V \in H_\alpha\text{-mod} \) and \( \{v^j \mid j \in J\} \) is the dual basis of \( V^* \), then

\[
\begin{align*}
\text{ev}_V: & \quad V^* \otimes V \to k, \quad f \otimes v \mapsto f(v), \\
\text{coev}_V: & \quad k \to V \otimes V^*, \quad 1 \mapsto \sum_{j \in J} v_j \otimes v^j,
\end{align*}
\]

\[
\begin{align*}
\text{coev}_V: & \quad k \to V \otimes V^*, \quad 1 \mapsto \sum_{i \in J} v^i \otimes g^{-1}_\alpha v_i.
\end{align*}
\]

We call \( H\text{-pmod} \) or \( \text{Proj}(\mathcal{C}) \) the ideal of projective \( H \)-modules. As \( \mathcal{C} = \bigoplus_{\alpha \in G} \mathcal{C}_\alpha \), the projective modules of \( \mathcal{C}_\alpha \) are in \( H\text{-pmod} \cap \mathcal{C}_\alpha = H_\alpha\text{-pmod} \).

**Lemma 3.3.** Let \((H, g)\) be a finite type pivotal Hopf \( G \)-coalgebra. Let \( t \) be a cyclic trace on \( H\text{-pmod} \). Let \( V \in H\text{-pmod} \) and \( \varepsilon W \in H_1\text{-mod} \) be endowed with the trivial action \( \rho_{\varepsilon W} = \varepsilon \text{Id}_{\varepsilon W} \). Then

\[
\begin{align*}
\forall f \in \text{End}_{H\text{-mod}}(V \otimes \varepsilon W), \quad & t_{V \otimes \varepsilon W}(f) = t_V(\text{tr}^r_{\varepsilon W}(f)) \\
\forall f \in \text{End}_{H\text{-mod}}(\varepsilon W \otimes V), \quad & t_{\varepsilon W \otimes V}(f) = t_V(\text{tr}^l_{\varepsilon W}(f)).
\end{align*}
\]

**Proof.** Consider a decomposition of \( \text{Id}_{\varepsilon W} \)

\[
\text{Id}_{\varepsilon W} = \sum_{i \in I} e_i \varphi_i \quad \text{where} \quad \varphi_i : \varepsilon W \to k, \quad e_i : k \to \varepsilon W, \quad \varphi_i(e_j) = \delta_{ij}.
\]

By setting \( \tilde{e}_i = \text{Id}_V \otimes e_i : V \to V \otimes \varepsilon W \) and \( \tilde{\varphi}_i = \text{Id}_V \otimes \varphi_i : V \otimes \varepsilon W \to V \) one gets

\[
\text{Id}_{V \otimes \varepsilon W} = \sum_{i \in I} \tilde{e}_i \tilde{\varphi}_i.
\]

For \( f \in \text{End}_{H\text{-mod}}(V \otimes \varepsilon W) \), on the one hand we have

\[
t_{V \otimes \varepsilon W}(f) = \sum_{i \in I} t_{V \otimes \varepsilon W}(f \tilde{e}_i \tilde{\varphi}_i) = \sum_{i \in I} t_V(\tilde{\varphi}_i f \tilde{e}_i) = \sum_{i \in I} t_V(f_{ii})
\]

where \( f_{ii} = \tilde{\varphi}_i f \tilde{e}_i \in \text{End}_{H\text{-mod}}(V) \). In the above calculations, we use Equation (3.11) in the first equality and the cyclicity property in the second equality.

On the other hand, each map \( f \in \text{End}_{H\text{-mod}}(V \otimes \varepsilon W) \) is presented by
graph below

\[
\begin{array}{c}
\begin{array}{c}
V \quad \varepsilon_W \\
\downarrow \quad \downarrow \quad \downarrow \\
\bigoplus_{i,j \in I} f_{ij} \\
\downarrow \quad \downarrow \quad \downarrow \\
V \quad \varepsilon_W
\end{array}
\end{array}
\]

where \( f_{ij} = \tilde{\varphi}_i f \tilde{e}_j \in \text{End}_{H\text{-mod}}(V) \). From this graphical representation implies

\[
t_V(\text{tr}_\varepsilon_W^r(f)) = \sum_{i,j \in I} t_V \left( \begin{array}{c}
\begin{array}{c}
\bigoplus_{i,j \in I} f_{ij} \\
\downarrow \quad \downarrow \quad \downarrow \\
V \quad \varepsilon_W
\end{array}
\end{array} \right) = \sum_{i \in I} t_V(f_{ii}).
\]

Therefore Equality (3.8) holds.

Remark that the pivotal element acts trivially on \( \varepsilon_W \) so the evaluation \( \leftarrow^\varepsilon_W \) in \( H\text{-mod} \) is just the usual evaluation of \( \text{Vect}_k \).

For Equality (3.9) the proof is similar. \( \Box \)

3.3.3. Reduction Lemma. We have a graded version of Reduction Lemma [1, Lemma 3.2]

Lemma 3.4. Let \((H, g)\) be a finite type unimodular pivotal Hopf \( G\)-coalgebra and \( \lambda = (\lambda^\alpha)_{\alpha \in G} \in \prod_{\alpha \in G} H^*_\alpha \) be a family of symmetric linear forms and \( t = (t^\alpha)_{\alpha \in G} \) be the associated cyclic traces. Then \( t \) is a right modified trace on \( H\text{-pmod} \) if and only if for all \( \alpha, \beta \in G \) and for all \( f \in \text{End}_{H\alpha \beta}(H\alpha \otimes H\beta) \)

(3.12)

\[
t_{H\alpha \otimes H\beta}^{\alpha \beta}(f) = t_P^\alpha(\text{tr}_H^\alpha(f)).
\]

Similarly, \( t \) is a left modified trace on \( H\text{-pmod} \) if and only if for all \( f \in \text{End}_{H\alpha \beta}(H\alpha \otimes H\beta) \)

\[
t_{H\alpha \otimes H\beta}^{\beta \alpha}(f) = t_P^\beta(\text{tr}_H^\beta(f)).
\]

Proof. The proof strictly follows the line of Reduction Lemma 3.2 [1]. The necessity is obvious. We now prove the sufficiency of the condition. By Proposition 3.2 for each \( \alpha \in G \) the symmetric linear form \( \lambda^\alpha \) induces the cyclic trace \( \{t_P^\alpha : \text{End}_{H\alpha}(P) \to k\}_{P \in H\alpha\text{-pmod}} \). We then prove that the cyclic trace \( t^\alpha \) satisfies the right partial trace property.
First, let $P \in H_\alpha\text{-pmod}$, $P' \in H_\beta\text{-pmod}$ and $f \in \operatorname{End}_{H_{\alpha\beta}}(P \otimes P')$. Suppose that $\text{Id}_P$ and $\text{Id}_{P'}$ have the decomposition

\begin{equation}
\text{Id}_P = \sum a_i \circ b_i, \quad \text{Id}_{P'} = \sum a_i' \circ b_i'.
\end{equation}

where $a_i : H_\alpha \rightarrow P$, $b_i : P \rightarrow H_\alpha$ and $a_i' : H_\beta \rightarrow P'$, $b_i' : P' \rightarrow H_\beta$.

The modified trace of $f$ is calculated as follows:

\begin{equation}
\mathcal{t}^{\alpha\beta}_{P \otimes P'}(f) = \mathcal{t}^{\alpha\beta}_{P \otimes P'}(f)
= \mathcal{t}^{\alpha\beta}_{H_\alpha \otimes H_\beta}(f)
= \mathcal{t}^{\alpha}_{H_\alpha}(f) = \mathcal{t}^{\alpha}_{P}(f).
\end{equation}

In this calculation, one uses (3.13) in the first equality, in the second equality one uses the cyclicity property of cyclic traces, the third equality thanks to (3.12) and finally one uses the duality morphisms to move $b_i'$ around the loop then applying again (3.13) and the cyclicity property.

Second, let $P \in H_\alpha\text{-pmod}$, $V \in H_\beta\text{-mod}$ and $f \in \operatorname{End}_{H_{\alpha\beta}}(P \otimes V)$. Set $Q = P \otimes V$, note that $Q \in H_{\alpha\beta}\text{-pmod}$ and $P \otimes P^*, Q \otimes Q^* \in H_1\text{-pmod}$. Consider two morphisms $A \in \operatorname{Hom}_{H_{1\text{-mod}}}(P \otimes P^*, Q \otimes Q^*)$ and $B \in \operatorname{Hom}_{H_{1\text{-mod}}}(Q \otimes Q^*, P \otimes P^*)$ are given by

\[ A = \begin{array}{c}
\begin{array}{c}
Q \\
P
\end{array}
\end{array}, \quad B = \begin{array}{c}
\begin{array}{c}
P
\end{array}
\end{array} \]
According to (3.14) one gets
\[ t^1_{P \otimes P^*} (B \circ A) = t^\alpha_P (\text{tr}^r_{P^*} (B \circ A)) = t^\alpha_P \left( \begin{array}{c} P \\ Q \\ V \\ f \\ P \\ \text{Id} \\ \text{Id} \\ \text{Id} \end{array} \right) = t^\alpha_P \left( \begin{array}{c} P \\ V \end{array} \right) = t^\alpha_P (\text{tr}^r_V (f)). \]

In above calculation, one applies the definition of the partial trace in second equality, in the third equality one uses the properties of the pivotal structure. Similarly we also have
\[ t^1_{Q \otimes Q^*} (A \circ B) = t^{\alpha\beta}_Q (\text{tr}^r_{Q^*} (A \circ B)) = t^{\alpha\beta}_Q \left( \begin{array}{c} Q \\ P \\ f \\ \text{Id} \\ \text{Id} \end{array} \right) = t^{\alpha\beta}_Q \left( \begin{array}{c} Q \\ P \end{array} \right) = t^{\alpha\beta}_P (\text{tr}^r_V (f)). \]

Since the cyclicity property \( t^1_{P \otimes P^*} (B \circ A) = t^1_{Q \otimes Q^*} (A \circ B) \), it follows that \( t^{\alpha\beta}_P (f) = t^{\alpha\beta}_Q (f) \).

The proof in the case of the left modified trace is similar. \( \square \)

3.4. Applications of Theorem 1.1. Theorem 1.1 has two immediate consequences when \( G = \{1\} \) or \( H \) is semi-simple. First, in degree 1 the symmetrised \( G \)-integral is also the symmetrised integral of \( H \) and Theorem 1.1 recovers the main theorem of [1] that we recall here:

**Theorem 3.5 ([1]).** Let \( (H, g) \) be a finite dimensional unimodular pivotal Hopf algebra over a field \( k \). Then the space of right (left) modified traces on \( H \)-pmod is equal to the space of symmetrised right (left) integrals, and hence is 1-dimensional. Moreover, the right modified trace on \( H \)-pmod is non-degenerate and determined by the cyclicity property and by
\[ t_H (f) = \mu (gf(1)) \quad \text{for any} \quad f \in \text{End}_H (H). \]
Similarly, the left modified trace is non degenerate and determined by
\[ t_H (f) = \mu' (g^{-1} f(1)) \quad \text{for any} \quad f \in \text{End}_H (H). \]
In particular, $H$ is unibalanced if and only if the right modified trace is also left.

Second, for a finite type unimodular pivotal Hopf $G$-coalgebra $(H, g)$, if $H$ is semi-simple, i.e. $H_\alpha$ is semi-simple for all $\alpha \in G$ then $H$-pmod = $C$. Then the categorical trace generates the space of modified traces on $H$-pmod: for any $f \in \text{End}_C(V)$, the right and left categorical trace are

$$\text{tr}^r_V(f) := \overrightarrow{\text{ev}}_V(f \otimes \text{Id}_V) \overrightarrow{\text{coev}}_V \in k,$$

$$\text{tr}^l_V(f) := \overrightarrow{\text{ev}}_V((\text{Id}_V \otimes f) \overrightarrow{\text{coev}}_V \in k.$$

As a corollary of Theorem 1.1 we then have the proposition.

**Proposition 3.6.** Let $(H, g)$ be a finite type unimodular pivotal Hopf $G$-coalgebra over a field $k$. The right categorical trace $\text{tr}^r_H\alpha$ and its left version $\text{tr}^l_H\alpha$ are non-zero if and only if $H_\alpha$-mod is semi-simple and in this case coincide up to a scalar with the trace maps

$$f \mapsto \tilde{\mu}_\alpha(f(1)) \text{ and } f \mapsto \tilde{\mu}^l_\alpha(f(1))$$

respectively, where $f \in \text{End}_{H_\alpha}(H_\alpha)$.

### 4. Proof of the main theorem

**4.1. Decomposition of tensor products of the regular representations.** We denote by $H_\alpha$ the left $H_\alpha$-module given by the left regular action. Let us denote by $\varepsilon H_\beta$ the vector space underlying $H_\beta$ equipped with the $H_1$-module structure given by

$$h.m = \varepsilon(h)m \text{ for } m \in \varepsilon H_\beta, \ h \in H_1.$$

We will use Sweedler’s notation: $\Delta_{\alpha,\beta}(h) = h_{(1)} \otimes h_{(2)}$ for $h \in H_{\alpha\beta}$, $h_{(1)} \in H_\alpha, h_{(2)} \in H_\beta$.

**Theorem 4.1.** Let $H = (H_\alpha)_{\alpha \in G}$ be a finite type Hopf $G$-coalgebra. Then

1. the map

$$\phi_{\alpha,\beta} : H_{\alpha\beta} \otimes \varepsilon H_\beta \to H_\alpha \otimes H_\beta$$

$$h \otimes m \mapsto h_{(1)} \otimes h_{(2)}m$$

is an isomorphism of $H_{\alpha\beta}$-modules whose inverse is

$$\psi_{\alpha,\beta} : H_\alpha \otimes H_\beta \to H_{\alpha\beta} \otimes \varepsilon H_\beta$$

$$x \otimes y \mapsto x_{(1)} \otimes S_{\beta}^{-1}(x_{(2)})y.$$

2. the map

$$\phi^l_{\alpha,\beta} : \varepsilon H_\alpha \otimes H_{\alpha\beta} \to H_\alpha \otimes H_\beta$$

$$m \otimes h \mapsto h_{(1)}m \otimes h_{(2)}$$

is an isomorphism of $H_{\alpha\beta}$-modules whose inverse is

$$\psi^l_{\alpha,\beta} : H_\alpha \otimes H_\beta \to \varepsilon H_\alpha \otimes H_{\alpha\beta}$$

$$x \otimes y \mapsto S_{\alpha}^{-1}(y_{(1)})x \otimes y_{(2)}.$$
We prove the theorem using graphical calculus with the graphical representations for Hopf $G$-coalgebras given in Section 2.1.2. The maps $\phi_{\alpha,\beta}$ and $\psi_{\alpha,\beta}$ are presented in Figure 6. The graphical representations for $\phi_{\alpha,\beta}^l$ and $\psi_{\alpha,\beta}^l$ are similar.

Proof. In order to prove part (1), we first check that $\phi_{\alpha,\beta}$ is left inverse to $\psi_{\alpha,\beta}$, by computing the composition one gets

$$\psi_{\alpha,\beta} \circ \phi_{\alpha,\beta} = \text{Id}_{H_{\alpha} \otimes H_{\beta}}$$

where the associativity of the product $m_{\beta}$ is used in the first equality, then we use the coassociativity of the coproduct in the second equality, and finally we use the antipode properties in the last equality. Similarly we have $\phi_{\alpha,\beta} \circ \psi_{\alpha,\beta} = \text{Id}_{H_{\alpha} \otimes H_{\beta}}$.

Next we prove the map $\phi_{\alpha,\beta}$ is $H_{\alpha\beta}$-linear by diagrammatic calculus:

$$\phi_{\alpha,\beta}$$

where we used the property of the algebra homomorphism $\Delta_{\alpha,\beta}$ in the second equality and the associativity of multiplication in the third
equality. The map $\psi_{\alpha,\beta}$ is also $H_{\alpha\beta}$-linear by:

\[\psi_{\alpha,\beta}(\alpha \beta \alpha \beta \epsilon) = \psi_{\alpha,\beta}(\alpha \beta \alpha \beta \epsilon) = \psi_{\alpha,\beta}(\alpha \beta \alpha \beta \epsilon) = \psi_{\alpha,\beta}(\alpha \beta \alpha \beta \epsilon)\]

where we used the property of the algebra homomorphism $\Delta_{\alpha,\beta}$ in the first equality, the coassociativity of coproduct and the antipode properties are used in the second equality, the associativity of multiplication and the antipode properties are used in the third equality, and we used the antipode properties in the last equality.

The proof of the part (2) is similar way.

**Proposition 4.2.** Let $H = (H_\alpha)_{\alpha \in G}$ be a finite type pivotal Hopf $G$-coalgebra. Then we have the equalities of linear maps:

1. $\phi_{\alpha,\beta}(1 \alpha \beta \otimes m) = 1_\alpha \otimes m$ for $m \in \varepsilon H_{\beta}$,
2. $(\mu_{\alpha \beta} \otimes \text{Id}_{\beta}) \circ \psi_{\alpha,\beta} = \tilde{\mu}_{\alpha} \otimes g_{\beta} \text{Id}_{\beta}'$ where $\text{Id}_{\beta}' : H_{\beta} \to \varepsilon H_{\beta}$ is the identity map in $\text{Vect}_k$.

**Proof.** The equality (1) holds by the definition of the map $\phi_{\alpha,\beta}$. Part (2) follows from the diagrammatic calculus in $\text{Vect}_k$:

\[\psi_{\alpha,\beta} = \tilde{\mu}_{\alpha} = \tilde{\mu}_{\alpha} \cdot g_{\beta} \cdot \text{Id}_{\beta}'\]

where in the second equality of (4.1) we used the relation of the right symmetrised $G$-integral in Figure 5.

**4.2. Proof of Theorem 1.1.** Let $(H, g)$ be a finite type unimodular pivotal Hopf $G$-coalgebra, $\mathcal{C}$ be the pivotal $G$-graded category of $H$-modules. The existence of modified trace on $\text{Proj}(\mathcal{C})$ follows from: 1) the existence of non-zero integral on $H_1$ 2) the existence of modified trace in $\mathcal{C}_1$ by applying the results of [1] for $H_1$ and 3) the existence of the extension of ambidextrous trace in [8, Theorem 3.6]. Nevertheless we choose to give a direct proof of this fact following the lines of [1]. Furthermore, Theorem 1.1 also gives an explicit formula to compute the modified trace $t$ from the integral and conversely.

**Proof of Theorem 1.1.** First, we show that a right symmetrised $G$-integral provides a modified trace. Suppose that $\tilde{\mu} = (\tilde{\mu}_\alpha)_{\alpha \in G}$ is the right symmetrised $G$-integral for $H$. By Proposition 3.2, the family of the symmetric forms associated with $\tilde{\mu}$ induces the family of cyclic traces.
$t = (t^a)_{a \in G}$ of $H$-pmod. Here $t^a = \{t^a_P : \text{End}_{H_a}(P) \to k\}_{P \in H_a}$-pmod is determined by

$$t^a_H(f) = \widetilde{\mu}_a(f(1_a)) \text{ for } f \in \text{End}_{H_a}(H_a).$$

To show $t$ is a modified trace, it is enough to check

$$t^{a\beta}_{H_a \otimes H_\beta}(f) = t^a_{H_a}(\text{tr}_{H_\beta}^r(f)) \text{ for any } f \in \text{End}_{H_a}(H_a \otimes H_\beta)$$

thanks to Reduction Lemma \ref{3.4}. The value of $t^{a\beta}_{H_a \otimes H_\beta}(f)$ in Equality \ref{4.3} is calculated

$$t^{a\beta}_{H_a \otimes H_\beta}(f) = t^{a\beta}_{H_a \otimes H_\beta}(\phi_{\alpha\beta}, \psi_{\alpha\beta}).$$

In the above calculation, we use Theorem \ref{4.1} in the first equality; the cyclicity property of trace in the second equality; Lemma \ref{3.3} in the third equality; Equation \ref{4.2} in the fourth equality and in the fifth equality we use the two equalities in Proposition \ref{4.2}.

Second, assume that we have a right modified trace, and hence the symmetric form $t^a_P$ on $\text{End}_{H_a}(P)$ for any projective module $P$ and any $a \in G$. In particular for any $a, \beta \in G$ the symmetric forms $t^a_{H_a}(H_a)$ and $t^{a\beta}_{H_a \otimes H_\beta}$ on $\text{End}_{H_a}(H_a \otimes H_\beta)$ satisfy

$$t^{a\beta}_{H_a \otimes H_\beta}(f) = t^a_{H_a}(\text{tr}_{H_\beta}^r(f)) \text{ for any } f \in \text{End}_{H_a}(H_a \otimes H_\beta).$$
Let \( \tilde{\nu}_\alpha(h) = t^\alpha_{H_\alpha}(R_h) \) for \( R_h \in \text{End}_{H_\alpha}(H_\alpha) \) with \( h \in H_\alpha \). Then \( \tilde{\nu}_\alpha(f(1_\alpha)) = t^\alpha_{H_\alpha}(f) \) for \( f \in \text{End}_{H_\alpha}(H_\alpha) \) (see Lemma 3.1). We prove that the family \( \tilde{\nu} = (\tilde{\nu}_\alpha)_{\alpha \in G} \) satisfies the relation of the right symmetrised \( G \)-integral.

Consider the maps \( k = \Delta_{\alpha,\beta} \circ (R_h \otimes \varphi) : H_{\alpha\beta} \otimes \varepsilon H_\beta \to H_\alpha \otimes H_\beta \) for \( h \in H_{\alpha\beta} \) and \( \varphi \in \varepsilon H_\beta^* \). Then \( k \) is a morphism of \( H_{\alpha\beta} \)-modules.

The graphical representation of the map \( k \) is given in Figure 7. Let \( \tilde{f} = k \circ \psi_{\alpha,\beta} : H_\alpha \otimes H_\beta \to H_\alpha \otimes H_\beta \) then \( \tilde{f} \in \text{End}_{H_{\alpha\beta}}(H_\alpha \otimes H_\beta) \). We now calculate the values of the modified trace for \( \tilde{f} \in \text{End}_{H_{\alpha\beta}}(H_\alpha \otimes H_\beta) \) and \( \text{tr}_{H_\beta}(\tilde{f}) \in \text{End}_{H_\alpha}(H_\alpha) \).

Firstly, we have

\[
t^{\alpha\beta}_{H_\alpha \otimes H_\beta}(\tilde{f}) = t^{\alpha\beta}_{H_\alpha \otimes H_\beta}(k \circ \psi_{\alpha,\beta}) = t^{\alpha\beta}_{H_\alpha \otimes \varepsilon H_\beta}(\psi_{\alpha,\beta} \circ k)
\]

\[
= t^{\alpha\beta}_{H_\alpha \otimes H_\beta} \begin{pmatrix}
\alpha \\
\beta \\
\varepsilon \\
R_h \\
\alpha \beta \\
\varepsilon \beta
\end{pmatrix}
= t^{\alpha\beta}_{H_\alpha \otimes \varepsilon H_\beta} \begin{pmatrix}
\alpha \\
\beta \\
\varphi \\
R_h \\
\alpha \beta \\
\varepsilon \beta
\end{pmatrix}
= t^{\alpha\beta}_{H_\alpha \otimes H_\beta} \begin{pmatrix}
\alpha \\
\beta \\
\varphi \\
R_h \\
\alpha \beta \\
\varepsilon \beta
\end{pmatrix}
= t^{\alpha\beta}_{H_\alpha \otimes \varepsilon H_\beta} \begin{pmatrix}
\alpha \\
\beta \\
\varphi \\
R_h \\
\alpha \beta \\
\varepsilon \beta
\end{pmatrix}
= t^{\alpha\beta}_{H_\alpha \beta} \begin{pmatrix}
\alpha \\
\beta \\
\varphi
\end{pmatrix}
= \tilde{\nu}_{\alpha\beta}(h) \varphi(1_\beta).
\]
In the above calculations, we use the cyclicity property in the second equality; the coassociativity of the coproduct in the fourth equality; the antipode properties in the fifth equality and finally we use the partial trace property.

Secondly, we have

\[ t^\alpha_{H_\alpha} (\text{tr}_H^r (\tilde{f})) = t^\alpha_{H_\alpha} (\text{tr}_H^r (k \circ \psi)) \]

where we use the left evaluation \( \overleftarrow{ev} \) with the pivot \( g_\beta \) and the right coevaluation \( \overrightarrow{coev} \) in the second equality and \( \Delta_{\alpha,\beta} (h) = h_{(1)} \otimes h_{(2)}. \)

By Equality (4.4) one has \( t^\alpha_{H_\alpha \otimes H_\beta} (\tilde{f}) = t^\alpha_{H_\alpha} (\text{tr}_H^r (\tilde{f})) \). This equality means that

\[ \tilde{\nu}_{\alpha \beta} (h) \varphi (1_\beta) = \tilde{\nu}_\alpha (h_{(1)}) \varphi (g_\beta h_{(2)}) \text{ for any } \varphi \in \mathcal{E}_{H_\beta}^*, \ h \in H_{\alpha \beta}. \]

This equality holds for any \( \varphi \in \mathcal{E}_{H_\beta}^* \) implies that \( \tilde{\nu}_{\alpha \beta} (h) 1_\beta = \tilde{\nu}_\alpha (h_{(1)}) g_\beta h_{(2)}, \) i.e. \( (\tilde{\nu}_\alpha \otimes g_\beta) \Delta_{\alpha,\beta} (h) = \tilde{\nu}_{\alpha \beta} (h) 1_\beta \) for any \( h \in H_{\alpha \beta}. \) Therefore the family \( \tilde{\nu} = (\tilde{\nu}_\alpha)_{\alpha \in G} \) is the right symmetrised \( G \)-integral for \( H. \)

For the case of the left modified trace the proof is similar. \( \square \)

5. Modified Trace for the \( G \)-Graded Quantum \( \mathfrak{sl}(2) \)

In this section we present the symmetrised \( G \)-integral for the quantization of \( \mathfrak{sl}(2) \) and the modified trace on ideal of projective modules of category of the weight modules over \( \mathcal{U}_q \mathfrak{sl}(2) \). It explains clearly the relation between the symmetrised \( G \)-integral for a pivotal Hopf \( G \)-coalgebra and the modified trace in associated category \( \mathcal{U}_q \mathfrak{g} \mathfrak{sl}(2) \)-mod.
5.1. **Unrestricted quantum \( \overline{U}_q\mathfrak{sl}(2) \).** Let \( \mathcal{U}_q\mathfrak{sl}(2) \) be the \( \mathbb{C} \)-algebra given by generators \( E, F, K, K^{-1} \) and relations:

\[
KK^{-1} = K^{-1}K = 1, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad [E, F] = \frac{K - K^{-1}}{q - q^{-1}}
\]

where \( q = e^{\frac{i\pi}{r}} \) is a \( 2r \)-th root of unity. The algebra \( \mathcal{U}_q\mathfrak{sl}(2) \) is a Hopf algebra where the coproduct, counit and antipode are defined by

\[
\Delta(E) = 1 \otimes E + E \otimes K, \quad \varepsilon(E) = 0, \quad S(E) = -EK^{-1},
\]

\[
\Delta(F) = K^{-1} \otimes F + F \otimes 1, \quad \varepsilon(F) = 0, \quad S(F) = -KF,
\]

\[
\Delta(K) = K \otimes K, \quad \varepsilon(K) = 1, \quad S(K) = K^{-1}.
\]

Let \( \overline{U} := \overline{U}_q\mathfrak{sl}(2) \) be the algebra \( \mathcal{U}_q\mathfrak{sl}(2) \) modulo the relations \( E^r = F^r = 0 \) and \( C = \mathbb{C}[K^{\pm r}] \) be the commutative Hopf subalgebra in the center of \( \overline{U}_q\mathfrak{sl}(2) \). The algebra \( \overline{U} \) is a pivotal Hopf algebra with the pivot \( g = K^{1-r} \). Let \( G = (\mathbb{C}/2\mathbb{Z}, +) \overset{\sim}{\to} \text{Hom}_{\mathcal{U}_0}(C, \mathbb{C}), \alpha \mapsto (K^r \mapsto q^{r\alpha} := e^{\frac{i\pi \alpha}{2}}) \) and let \( \mathcal{U}_\alpha \) be the algebra \( \overline{U}_q\mathfrak{sl}(2) \) modulo the relations \( K^r = q^{r\alpha} \) for \( \alpha \in G \). By applying Example 2.3 it follows that \( \mathcal{U} = \{\mathcal{U}_\alpha\}_{\alpha \in G} \) is the Hopf \( G \)-coalgebra with the coproduct and the antipode are determined by the commutative diagrams:

\[
\begin{array}{ccc}
\overline{U} & \overset{\Delta}{\longrightarrow} & \overline{U} \otimes \overline{U} \\
\downarrow{p_{\pi+\overline{\beta}}} & & \downarrow{p_{\pi} \otimes p_{\overline{\beta}}} \\
\mathcal{U}_{\pi+\overline{\beta}} & \overset{\Delta_{\pi,\overline{\beta}}}{\longrightarrow} & \mathcal{U}_{\pi} \otimes \mathcal{U}_{\overline{\beta}} \\
\downarrow{p_{\pi+\overline{\beta}}} & & \downarrow{p_{\pi} \otimes p_{\overline{\beta}}} \\
\mathcal{U}_{\pi+\overline{\beta}} & \overset{\Delta_{\pi,\overline{\beta}}}{\longrightarrow} & \mathcal{U}_{\pi} \otimes \mathcal{U}_{\overline{\beta}} \\
\end{array}
\]

where \( p_{\pi} : \overline{U} \rightarrow \mathcal{U}_{\pi} \) is the projective morphism from \( \overline{U} \) to \( \mathcal{U}_{\pi} \). The Hopf \( G \)-coalgebra \( \mathcal{U} = \{\mathcal{U}_\pi\}_{\pi \in G} \) has the pivotal structure given by \( g_{\pi} = q^{r\alpha}K \).

For \( \pi = \overline{0} \) the Hopf algebra \( \mathcal{U}_{\overline{0}} \) is called the restricted quantum \( \mathfrak{sl}(2) \), i.e. the algebra \( \mathcal{U}_q\mathfrak{sl}(2) \) modulo the relations \( E^r = F^r = 0 \) and \( K^r = 1 \).

The right \( \overline{0} \)-integral is the usual right integral given by

\[
\mu_{\overline{0}}(E^m F^n K^l) = \eta \delta_{m,r-1} \delta_{n,r-1} \delta_{l,1}
\]

where \( \eta \) is a constant (see e.g [1]). By definition of right \( G \)-integral (1.1) we get

\[
\mu_{\pi}(E^m F^n K^l) = q^{r\alpha} \eta \delta_{m,r-1} \delta_{n,r-1} \delta_{l,1}.
\]

One can show that the Hopf \( G \)-coalgebra \( \{\mathcal{U}_{\pi}\}_{\pi \in G} \) is \( G \)-unibalanced. The symmetrised right \( G \)-integral for \( \{\mathcal{U}_{\pi}\}_{\pi \in G} \) is determined by

\[
\tilde{\mu}_{\pi}(E^m F^n K^l) = \eta \delta_{m,r-1} \delta_{n,r-1} \delta_{l,0}.
\]
5.2. Modified trace. The category $\mathcal{C} = \mathcal{U}\text{-mod}$ is equal to the $G$-graded category of finite dimensional weight modules over $\mathcal{U}q\mathfrak{sl}(2)$ (module in which $K$ has a diagonalizable action). For $\alpha \in \mathbb{C}$ let $V_\alpha$ be a $r$-dimensional highest weight module of highest weight $\alpha + r - 1$ in $\mathcal{C}$ (see [3]). Recall the modified dimension $d(V_\alpha)$ of $V_\alpha$ for $\alpha \in (\mathbb{C}\setminus \mathbb{Z}) \cup r\mathbb{Z}$ was computed:

$$d(V_\alpha) = t_{V_\alpha}(\text{Id}_{V_\alpha}) = d_0 \prod_{k=1}^{r-1} \frac{\{k\}}{\{\alpha + r - k\}} = d_0^r \prod_{k=1}^{r-1} \frac{\{k\}}{\{\alpha + r - k\}}$$

where $t$ is the modified trace on ideal $\text{Proj}(\mathcal{C})$ of projective modules and $d_0$ is a non-zero complex number. In [5] for the analogous unrolled category, it is normalized by $d_0 = (-1)^{r-1}$. We now present the way to compute the modified dimension of $V_\alpha$ using the symmetrised $G$-integral.

By density theorem we have the isomorphism of algebras

$$\mathcal{U}_\alpha \sim \bigoplus_{k \in H_r} \text{End}(V_{\alpha + 2k})$$

where $H_r = \{-r+1, -r+3, \ldots, r-1\}$. Hence we have the isomorphism of left $\mathcal{U}_\alpha$-modules:

$$\mathcal{U}_\alpha \sim \bigoplus_{k \in H_r} \text{End}(V_{\alpha + 2k}) \sim \bigoplus_{k \in H_r} V_{\alpha + 2k} \otimes \epsilon V^*_{\alpha + 2k}.$$ 

Consider the quantum Casimir element of $\mathcal{U}$ defined by

$$\Omega = FE + \frac{Kq + K^{-1}q^{-1}}{\{1\}^2} = EF + \frac{Kq^{-1} + K^{-1}q}{\{1\}^2}.$$ 

For $k \in \mathbb{N}$, by induction one gets

$$\prod_{i=0}^{k-1} \left( \Omega - \frac{q^{-2i-1}K + q^{2i+1}K^{-1}}{\{1\}^2} \right) = E^k F^k.$$ 

Lemma 5.1. For $k \in \mathbb{N}$ then

$$\Omega^k - E^k F^k \in \text{Span}_\mathbb{C}\{E^j F^j K^i \mid j < k, \ i \in \mathbb{Z}\}.$$ 

Proof. The proof is by induction on $k$. Indeed, by (5.3) $\Omega^k - E^k F^k \in \text{Span}_\mathbb{C}\{\Omega^j K^i \mid j < k, \ i \in \mathbb{Z}\}$ which by the induction hypothesis is contained in $\text{Span}_\mathbb{C}\{E^j F^j K^i \mid j < k, \ i \in \mathbb{Z}\}$. 

Following (5.1) we have the corollary.

Corollary 5.2. For all $k \in \{0, \ldots, r-2\}$ we have $\tilde{\mu}_\pi(\Omega^k) = 0$. For $k = r-1$ then $\tilde{\mu}_\pi(\Omega^{r-1}) = \eta$.

Proof. It follows from (5.1) that $\text{Span}_\mathbb{C}\{E^j F^j K^i \mid j < k, \ i \in \mathbb{Z}\}$ is contained in the kernel of $\tilde{\mu}_\pi$ for $k \in \{0, \ldots, r-2\}$. 

□
For $\alpha \in \mathbb{C} \setminus \mathbb{Z}$, $\Omega$ acts on $V_\alpha$ by the scalar $w_\alpha$ which is calculated as follows: Let $v$ be a highest weight vector of $V_\alpha$. The action of $K$ on $v$ defined by $Kv = q^{a+r-1}v$. This implies that $\Omega v = \frac{q^{a+r+g_{\alpha}}}{(1)^2}v$, i.e. $w_\alpha = \frac{q^{a+r+g_{\alpha}}}{(1)^2}$. The elements $w_{\alpha+2k}$, $0 \leq k < r-1$ are distinct as $w_{\alpha+2i} - w_{\alpha+2j} = \frac{(i-j)q^{a+r+i+j}}{(1)^2} \neq 0$ for $i \neq j$.

We consider in $\mathcal{U}_\Sigma$ the element

$$L_\alpha(\Omega) = \frac{\prod_{k=1}^{r-1}(\Omega - w_{\alpha+2k})}{\prod_{k=1}^{r-1}(w_\alpha - w_{\alpha+2k})}.$$ 

This element is the projector on $V_\alpha \otimes \epsilon V_\alpha \simeq \bigoplus_{k=1}^{r}V_{\alpha}$ as $L_\alpha(w_{\alpha+2k}) = \delta_{0,k}$. The value of symmetrised right $G$-integral on $L_\alpha(\Omega)$ is

$$\tilde{\mu}_R(L_\alpha(\Omega)) = \frac{1}{\prod_{k=1}^{r-1}(w_\alpha - w_{\alpha+2k})} \tilde{\mu}_R\left(\prod_{k=1}^{r-1}(\Omega - w_{\alpha+2k})\right).$$

Corollary 5.2 implies that

$$\tilde{\mu}_R\left(\prod_{k=1}^{r-1}(\Omega - w_{\alpha+2k})\right) = \tilde{\mu}_R(\Omega^{r-1}) = \eta.$$

The equality $\prod_{k=1}^{r-1}(w_\alpha - w_{\alpha+2k}) = (-1)^{r-1} \prod_{k=1}^{r-1} \frac{k(\alpha+k)}{(1)^2}$ gives

$$\tilde{\mu}_R(L_\alpha(\Omega)) = (-1)^{r-1} \eta \prod_{k=1}^{r-1} \frac{1}{(1)^2} \frac{\{1\}^{2}}{\{k\}^2} \frac{\{\alpha+k\}}{1^{\alpha+k}}$$

$$= \eta \prod_{k=1}^{r-1} \frac{1}{(1)^2} \frac{\{1\}^{2}}{\{k\}^2} (-1)^{r-1} \prod_{k=1}^{r-1} \frac{\{k\}}{\{\alpha+r-k\}} = \frac{(1)^{2r-2} \eta}{r^3 d_0} r d(V_\alpha)$$

where we used the identity $\prod_{k=1}^{r-1} \frac{k^2}{\{k\}^2} = (-1)^{r-1} r^2$ in the last equality.

It is clear that the coefficient $\frac{(1)^{2r-2} \eta}{r^3 d_0}$ does not depend on $\alpha$. This proves that $\tilde{\mu}_R(L_\alpha(\Omega)) = r d(V_\alpha)$ with the choice $d_0 = \frac{(1)^{2r-2} \eta}{r^3}$ where $\eta = \tilde{\mu}_R(E^{r-1} F^{r-1})$.

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