UNSTABLE ENTROPY AND UNSTABLE PRESSURE FOR RANDOM DIFFEOMORPHISMS WITH DOMINATION

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Abstract. Let $\mathcal{F}$ be a $C^2$ random dynamical system with $u$-domination. For the unstable foliation, the corresponding unstable metric entropy, unstable topological entropy and unstable pressure via the dynamics of $\mathcal{F}$ on the unstable foliation are introduced and investigated. A version of Shannon-McMillan-Brieman Theorem for unstable metric entropy is given, and a variational principle for unstable pressure (and hence for unstable entropy) is obtained. Moreover, as an application of the variational principle, equilibrium states for the unstable pressure are investigated.

1. Introduction. In differentiable dynamical systems and smooth ergodic theory for both of deterministic and random cases, entropies (including metric entropy and topological entropy), pressures and Lyapunov exponents are the main ingredients which describe the complexity of the orbit structure of the system from different points of view.

In the seminal papers [10] and [11], the so-called Pesin’s entropy formula and dimension formula which relate metric entropy and positive Lyapunov exponents are given for any $C^2$ diffeomorphism $f$ on a closed manifold $M$ with respect to SRB measures and general invariant measures respectively. These formulas tell us that positive exponents have contribution to the metric entropy $h_\mu(f)$, where $\mu$ is an $f$-invariant measure. In another word, $h_\mu(f)$ is determined by the dynamics of $f$ on the unstable foliations since it can be given by $H_\mu(\xi|f\xi)$, where $\xi$ is an increasing partition subordinate to unstable manifolds. An interesting question is: can we introduce an appropriate definition of topological entropy $h_{\text{top}}^u(f)$ via the dynamics of $f$ on the unstable foliations and obtain a version of variational principle relating $h_{\text{top}}^u(f)$ and $h_\mu(f)$?

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Recently, a partial answer to the above question is obtained, and the theory of entropy and pressure along unstable foliations for $C^1$ partially hyperbolic diffeomorphisms is investigated (cf. [6], [19] and [7]). In [6], Hu, Hua and Wu introduce the definitions of unstable metric entropy $h^n_u(f)$ for any invariant measure $\mu$ and unstable topological entropy $h^n_{top}(f)$, give a version of Shannon-McMillan-Breiman Theorem for $h^n_u(f)$, and obtain a variational principle relating $h^n_u(f)$ and $h^n_{top}(f)$.

We point out that in [6] $h^n_u(f)$ is defined via $H_\mu(\sqrt{n-1} f^{-i} \alpha | \eta)$ (where $\alpha$ is a finite measurable partition and $\eta$ is a measurable partition which is subordinate to unstable manifolds) instead of the classical form $H_\mu(\xi | f \xi)$ (where $\xi$ is an increasing partition subordinate to the unstable manifolds) in [11]. The advantage of this type of definition of $h^n_u(f)$ is that a variational principle then is obtained and hence the theory of pressure and related topics in mathematical statistical mechanics can be considered. Actually, lately in [7], Hu, Wu and Zhu generalize the definition of unstable topological entropy to unstable pressure $P^u_\mu(f, \varphi)$ for any continuous function $\varphi$ on $M$, obtain a variational principle for this pressure, and investigate the properties of the so-called $u$-equilibriums. In fact, we observe that for any $C^1$ diffeomorphism $f$ with $u$-domination (see the precise definition in [18]), unstable entropies $h^n_u(f)$, $h^n_{top}(f)$ and unstable pressure $P^u_\mu(f, \varphi)$ can be defined via the dynamics of $f$ on the unstable foliations, and moreover, a sequence of similar results as in [6] and [7] can be obtained with an effort.

The main purpose of this paper is to consider this topic for random dynamical systems (RDSs). In [8] and [14], Kifer and Liu are mainly concerned about i.i.d. (i.e., independent and identically distributed) RDSs generated by applying at each step a transformation chosen randomly from a given family according to some probability distribution. Specifically, for a $C^2$ i.i.d. RDS $\mathcal{F}$, the ergodic theory, which mainly consists of Pesin theory, of $\mathcal{F}$ has been systematically investigated in [8] and [14]. In [1], Arnold considers a more general version of RDSs, when the random choice of transformations is assumed to be only stationary. We are mainly concerned about the general case in this paper. For RDSs, various random versions of Pesin’s entropy formula and dimension formula were thoroughly investigated in [12, 13, 14, 2, 16], etc., in different settings. In these papers, the metric entropy $h_\mu(\mathcal{F})$ is defined by $H_\mu(\xi | \Theta \xi)$, where $\Theta$ is the induced skew product transformation on $\Omega \times M$ ($\Omega$ is a probability space with probability $\mathbf{P}$), $\mu$ is a $\Theta$-invariant measure with marginal measure $\mathbf{P}$ on $\Omega$ and $\xi$ is an increasing partition of $\Omega \times M$ which is subordinate to the random unstable foliations (see Section 2 for notations and definitions). In this paper we will adapt the techniques in [6] and [7] to the random setting and obtain the corresponding results as in [6] and [7].

We will firstly introduce the so-called $u$-domination for $\mathcal{F}$ (see Definition 2.3). Then for the unstable foliation, we define two types of unstable metric entropies $h^n_u(\mathcal{F})$ (via Bowen balls) and $h^n_{top}(\mathcal{F})$ (via conditional entropy) with respect to any $\mathcal{F}$-invariant measure $\mu$ following the methods in [11] and [6] (see Definition 3.1 and Definition 3.3), and show that these two unstable metric entropies coincide with each other when $\mu$ is ergodic (Theorem A). Some properties of unstable metric entropies are given and a version of Shannon-McMillan-Breiman Theorem, in which the unstable metric entropy is expressed as the limit of certain conditional information functions, is obtained (Theorem B). The next work is, for the unstable foliation, to define unstable pressure $P^u_\mu(\mathcal{F}, \phi)$ for each function $\phi$ on $\Omega \times M$ which is continuous in $x \in M$ and measurable in $\omega \in \Omega$ via the dynamics of $\mathcal{F}$ on local unstable manifolds (see Definition 5.2). Then a version of variational principle for
$P^u(F, \phi)$, which states that $P^u(F, \phi)$ is the supremum of the sum of the unstable metric entropy and the integral of $\phi$ taken over all invariant measures of $F$, is obtained (Theorem C). Since $P^u(F, 0) = h_{\text{top}}^u(F)$ (the unstable topological entropy of $F$), we get a variational principle for $h_{\text{top}}^u(F)$ directly (Corollary C.1).

This paper is organized as follows. In Section 2, we give some preliminaries and state our main results. In Section 3, we give the definition of unstable metric entropy via two methods and obtain the equivalence of them under an ergodicity condition. In Section 4, a version of Shannon-McMillan-Breiman Theorem for unstable metric entropy is given. In Section 5, we give the definition of unstable pressure. In Section 6, a variational principle for unstable pressure is obtained. In the last section, i.e. Section 7, as an application of the variational principle, we discuss the so-called $u$-equilibrium states.

2. Preliminaries and Statement of Main Results. Throughout this paper, we let $M$ be a $C^\infty$ compact Riemannian manifold without boundary. Denote by $\mathcal{B}(M)$ the Borel $\sigma$-algebra of $M$. Let $(\Omega, \mathcal{F}, P)$ be a Polish probability space and $\theta$ be an invertible and ergodic measure-preserving transformation on $\Omega$. Though we require that $P$ is ergodic, it is just for simplicity, in fact, corresponding results can be proved for non-ergodic measure $P$ by its ergodic decomposition as Liu and Xie did in [15].

Definition 2.1. A $C^2$ random dynamical system $F$ on $M$ over $(\Omega, \mathcal{F}, P, \theta)$ is defined as a map

$$F: \mathbb{Z} \times \Omega \times M \to M$$

$$(n, \omega, x) \mapsto F(n, \omega)x,$$

which has the following properties:

(i) $F$ is measurable;
(ii) the maps $F(n, \omega): M \to M$ form a cocycle over $\theta$, i.e. they satisfy

$$F(0, \omega) = \text{id},$$

$$F(n + m, \omega) = F(m, \theta^n \omega) \circ F(n, \omega),$$

for all $n, m \in \mathbb{Z}$ and $\omega \in \Omega$;
(iii) the maps $F(n, \omega): M \to M$ are $C^2$ for all $n \in \mathbb{Z}$ and $\omega \in \Omega$.

For each $\omega \in \Omega$, we define

$$f^n_\omega := \begin{cases} F(1, \theta^{n-1} \omega) \circ \cdots \circ F(1, \omega) & \text{if } n > 0, \\ \text{id} & \text{if } n = 0, \\ F(1, \theta^n \omega)^{-1} \circ \cdots \circ F(1, \theta^{-1} \omega)^{-1} & \text{if } n < 0. \end{cases}$$

Associated with $\Omega \times M$, there is a skew product $\Theta$ induced by $F$, i.e.,

$$\Theta: \Omega \times M \to \Omega \times M$$

$$(\omega, x) \mapsto (\theta \omega, F(1, \omega)x).$$

Definition 2.2 (Invariant measure). A measure $\mu$ on $\Omega \times M$ is said to be an $F$-invariant measure, if it is $\Theta$-invariant and has marginal measure $P$ on $\Omega$. In particular, an $F$-invariant measure $\mu$ is said to be ergodic, if it is ergodic with respect to $\Theta$. 
We denote by $\mathcal{M}_P(F)$ the set of all $F$-invariant measures.

In the following part of this paper, for $\mu \in \mathcal{M}_P(F)$ we always consider the $\mu$-completion of $\mathcal{F} \times \mathcal{B}(M)$, which is still denoted by $\mathcal{F} \times \mathcal{B}(M)$ for simplicity.

According to Rokhlin’s paper [17] and Liu and Qian’s monograph[14], for each $\mu \in \mathcal{M}_P(F)$, there exists a family of sample measures $\mu(\cdot): \Omega \times \mathcal{B}(M) \to [0,1]$ of $\mu$ satisfying the following properties:

(i) for all $B \in \mathcal{B}(M)$, $\omega \mapsto \mu(\omega)(B)$ is $\mathcal{F}$-measurable;

(ii) for $P$-a.e. $\omega \in \Omega$, $\mu(\omega): \mathcal{B}(M) \to [0,1]$ is a probability measure on $M$;

(iii) for $A \in \mathcal{F} \times \mathcal{B}(M)$,

$$\mu(A) = \int_{\Omega} \int_{M} 1_A(\omega, x)d\mu(\omega)(x)dP(\omega),$$

where $1_A$ is the characteristic function of $A \subset \Omega \times M$.

**Remark 1.** For $\mu \in \mathcal{M}_P(F)$, it is clear that $f^n(\omega)\mu(\omega) = \mu(\theta^n\omega)$ for all $n \in \mathbb{Z}$ and $P$-a.e. $\omega \in \Omega$.

Throughout this paper, we always assume that the Probability $P$ on $\Omega$ satisfies

$$\int_\Omega (\log^+ |F(1,\omega)|_{C^2} + \log^+ |F(-1,\omega)|_{C^2})dP(\omega) < \infty,$$  \hspace{1cm} (1)

where $|f|_{C^2}$ denotes the usual $C^2$ norm of $f \in \text{Diff}^2(M)$, and $\log^+ a = \max\{\log a, 0\}$.

Similar to the deterministic case, we can define the Lyapunov exponents for $F$. Let $\Lambda$ be the set of all regular points $(\omega, x) \in \Omega \times M$ in the sense of Oseledec. For $(\omega, x) \in \Lambda$, let $\lambda_1(\omega, x) > \cdots > \lambda_r(\omega, x)$ be its distinct Lyapunov exponents of $\mathcal{F}$ with multiplicities $m_j(\omega, x)$ ($1 \leq j \leq r(\omega, x)$). By (1) we know that all the Lyapunov exponents are finite numbers. In fact, because $P$ is ergodic, above numbers are all non-random, i.e., they are independent of $\omega$, thus we simply denote them by $\lambda_j(x)$ and $m_j(x)$ ($1 \leq j \leq r(x)$) respectively.

Let $\mu \in \mathcal{M}_P(F)$, and denote by $\|\cdot\|$ the norm of vectors in the tangent space of $M$. By the Oseledec Multiplicative Ergodic Theorem, we know that $\Lambda$ is $\Theta$-invariant and $\mu(\Lambda) = 1$. For each $(\omega, x) \in \Lambda$, there is a splitting of $T_xM$ as follows

$$T_xM = E_1(\omega, x) \oplus \cdots \oplus E_{r(\omega, x)}(\omega, x)$$

such that for $i = 1, \ldots, r(\omega, x)$, $\dim E_i(\omega, x) = m_i(x)$ and

$$\lim_{n \to \pm \infty} \frac{1}{n} \log \|D_xf^n(x)v\| = \lambda_i(x), \quad \text{for all } v \in E_i(\omega, x) \setminus \{0\}.$$

Let

$$u(x) = \max\{j: \lambda_j(x) > 0\}.$$

For $(\omega, x) \in \Lambda$, we define the set

$$W^u(\omega, x) = \{y \in M: \limsup_{n \to \infty} \frac{1}{n} \log d(f^{-n}(y), f^{-n}(x)) \leq -\lambda_{u(x)}(x)\},$$

where $d(\cdot, \cdot)$ is the metric on $M$ induced by its Riemannian structure. Let

$$E^u(\omega, x) = \bigoplus_{j=1}^{u(x)} E_j(\omega, x), \quad F^u(\omega, x) = \bigoplus_{j=u(x)+1}^{r(x)} E_j(\omega, x).$$

It is clear that both $E^u(\omega, x)$ and $F^u(\omega, x)$ are invariant under the tangent map, i.e. for $n \in \mathbb{Z}$,

$$D_xf^u F^u(\omega, x) = F^u(\theta^n\omega, f^u x) \quad \text{and} \quad D_xf^u E^u(\omega, x) = E^u(\theta^n\omega, f^u x).$$
The following proposition from [2] ensures that $W^u(\omega, x)$ is an immersed submanifold of $M$.

**Proposition 1.** For $(\omega, x) \in \Lambda$, the set $W^u(\omega, x)$ is a $C^{1,1}$ immersed submanifold of $M$ tangent at $x$ to $E^u(\omega, x)$.

We call the collection $\{W^u(\omega, x) : (\omega, x) \in \Lambda\}$ a $W^u$-foliation.

For a measurable partition $\alpha$ of $\Omega \times M$, and $\omega \in \Omega$, define

$$\alpha_\omega(x) := \{y : (\omega, y) \in \alpha(\omega, x)\},$$

where $\alpha(\omega, x)$ is the element of $\alpha$ containing $(\omega, x)$. It is clear that for $x, x' \in M$, either $\alpha_\omega(x) = \alpha_\omega(x')$ or $\alpha_\omega(x) \cap \alpha_\omega(x') = \emptyset$, so $\alpha_\omega := \{\alpha_\omega(x)\}$ is a partition of $M$.

A measurable partition $\xi$ of $\Omega \times M$ with $\xi \geq \sigma_0$, where $\sigma_0$ is the trivial partition $\{\omega\} \times M : \omega \in \Omega$, is said to be subordinate to the $W^u$-foliation, if for $\mu$-a.e. $(\omega, x) \in \Omega \times M$, $\xi_\omega(x) \subset W^u(\omega, x)$ and it contains an open neighborhood of $x$ in $W^u(\omega, x)$.

For each measurable partition $\eta$ subordinate to $W^u$-foliation, there is a canonical system of conditional measures $\{\mu^\eta_{(\omega, x)}\}_{(\omega, x) \in \Omega \times M}$ of $\mu$ by a classical result of Liu and Qian [14]. And $\mu^\eta_{(\omega, x)}$ can be regarded as a measure on $\eta_\omega(x)$, if we identify $\{\omega\} \times \eta_\omega(x)$ with $\eta_\omega(x)$.

We call a measurable partition $\alpha$ of $\Omega \times M$ fiberwise finite if for $\mathcal{P}$-a.e. $\omega \in \Omega$, $\alpha_\omega$ is finite and

$$\int_\Omega K(\omega) d\mathcal{P} < \infty,$$

where $K(\omega)$ is the cardinality of $\alpha_\omega$.

For a fiberwise finite partition $\alpha$ of $\Omega \times M$ and $\omega \in \Omega$, define $\text{diam}(\alpha_\omega)$ as follows

$$\text{diam}(\alpha_\omega) := \max_{A \in \alpha_\omega} \text{diam}(A),$$

where $\text{diam}(A) = \sup_{x, y \in A} d(x, y)$. The variable $\text{diam}(\alpha) : \Omega \to \mathbb{R}^+$ is defined as the diameter of $\alpha$.

Let $\alpha$ be a fiberwise finite partition of $\Omega \times M$ with diameter small enough. By such $\alpha$ we can construct a measurable partition as follows. Define

$$\eta = \{\alpha(\omega, x) \cap (\{\omega\} \times W^u_{\text{loc}}(\omega, x)) : (\omega, x) \in \Omega \times M\},$$

where $W^u_{\text{loc}}(\omega, x)$ is a local unstable manifold at $(\omega, x)$ whose size is greater than the diameter of $\alpha_\omega$. Denote by $\mathcal{P}(\Omega \times M)$ and $\mathcal{P}^u(\Omega \times M)$ the set of all fiberwise finite partitions and the set of all partitions constructed as above respectively.

**Remark 2.** It is easy to check that if for $\mathcal{P}$-a.e. $\omega$, $\mu_\omega(\partial \alpha_\omega) = 0$, where $\partial \alpha_\omega = \bigcup_{A \in \alpha_\omega} \partial A$, then for a measurable partition described as above is a partition subordinate to the $W^u$-foliation.

The following proposition ensures the existence of another class of useful partitions.

**Proposition 2 (Cf. Proposition 3.7, [2]).** Let $\mathcal{F}$ be a $C^2$ RDS. Then there exists a measurable partition $\xi_u$ of $\Omega \times M$ satisfying the following properties:

(i) $\xi_u$ is increasing, i.e., $\Theta^{-1} \xi_u \geq \xi_u$;

(ii) $\xi_u$ is subordinate to the $W^u$-foliation;

(iii) $\bigvee_{n=1}^{\infty} \Theta^{-n} \xi_u = \varepsilon$, where $\varepsilon$ is the partition of $\Omega \times M$ into points, i.e., $\xi_u$ is a generator;
(iv) \( \mathcal{B}(\bigwedge_{n=1}^{+\infty} \Theta^n u) = \mathcal{B}^u(\mathcal{F}) \), where for a measurable partition \( \xi \) of \( \Omega \times M \), \( \mathcal{B}(\xi) \) is the \( \sigma \)-algebra generated by \( \xi \) and \( \mathcal{B}^u(\mathcal{F}) = \{ B \in \mathcal{F} \times \mathcal{B}(M) : (\omega, x) \in B \implies \{ \omega \} \times W^u(\omega, x) \subseteq B \} \).

A partition is called an increasing partition subordinate to \( W^u \)-foliation if it satisfies Proposition 2, and denote by \( Q^u(\Omega \times M) \) the set of all such partitions.

Now we give the definition of RDSs with domination. A random variable \( t : \Omega \rightarrow \mathbb{R} \) is called \( \theta \)-invariant if \( t(\theta \omega) = t(\omega) \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \), and a random variable \( s : \Omega \rightarrow \mathbb{R}^+ \) is called \( \theta \)-tempered, if it satisfies \( \lim_{n \to \pm \infty} \frac{1}{n} \log s(\theta^n \omega) = 0 \) for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \).

**Definition 2.3.** Let \( \mu \in \mathcal{M}_\mathbb{P}(\mathcal{F}) \). A \( C^2 \) RDS \( \mathcal{F} \) is said to have the property of \( u \)-domination, if there exist a \( \theta \)-tempered random variable \( C \) and a \( \theta \)-invariant variable \( \lambda \) on \( \Omega \) satisfying for \( \mathbb{P} \)-a.e. \( \omega \in \Omega \)

\[
C(\omega) > 1 \quad \text{and} \quad 0 < \lambda(\omega) < 1,
\]

and for all \( (\omega, x) \in \Lambda, n \in \mathbb{N}, u \in F^u(\omega, x) \setminus \{0\} \) and \( v \in E^u(\omega, x) \setminus \{0\} \),

\[
\frac{\|DF^n_{\omega,x}u\|}{\|u\|} \leq C(\omega)(\lambda(\omega))^n \frac{\|DF^n_{\omega,x}v\|}{\|v\|}.
\]

**Remark 3.** (i) In fact, \( \lambda(\omega) \) is a.e. constant, since \( \theta \) is \( \mathbb{P} \)-ergodic.

(ii) Note that the number \( \lambda(\omega) \) in the above definition give a uniform lower bound for the minimum positive Lyapunov exponents \( \lambda_u(x) \).

(iii) For fixed \( \omega \in \Omega \), on one hand, it is clear that \( E^u(x) \) is a non-uniformly expanding subspace of \( T_x M \) with respect to \( \mathcal{F} \), since the corresponding Lyapunov exponents are positive; on the other hand, the parameters in the definition of \( u \)-domination are independent of \( x \) and \( C(\omega) \) is \( \theta \)-tempered. In view of the above two points, we know that \( \mathcal{F} \) is uniformly expanding in \( x \) on \( W^u \)-foliation.

**Basic Assumption.** In the remaining of this paper, we always assume that \( \mathcal{F} \) is a \( C^2 \) RDS with \( u \)-domination.

**Example.** Let \( f \) be a \( C^2 \)-diffeomorphism with \( u \)-domination (see the precise definition in [18]), \( \mathcal{U}(f) \) a small neighborhood of \( f \) in \( \text{Diff}^2(M) \) with the \( C^2 \) topology and \( \nu \) a Borel measure on \( \mathcal{U}(f) \). Let \( \Omega = \bigcup_{i=-\infty}^{+\infty} \mathcal{U}(f) \) and \( \theta \) be the standard left shift operator on \( \Omega \). Then we can get a \( C^2 \) RDS \( \mathcal{F} \) over \( (\Omega, \nu^\theta, \theta) \). By a classical discussion of perturbation theory we can see that such \( \mathcal{F} \) satisfies the above basic assumption.

As the deterministic case, we can define two types of the unstable metric entropies along \( W^u \)-foliation for \( \mathcal{F} \). The first one is defined via the average decreasing rate of the measure of the Bowen balls (as in [16]), and we denote it by \( \hat{h}^u_{\mu}(\mathcal{F}) \). The second one is defined via the conditional entropy of \( \mathcal{F} \) along \( W^u \)-foliation (adapted from [6] and [18]), which we denote by \( h^u_{\mu}(\mathcal{F}) \). Their precise definitions are given in Section 3.

In fact, when \( \mu \) is ergodic, the entropies described above are equivalent, i.e., we have the following theorem.

**Theorem A.** Let \( \mu \) be an ergodic measure of \( \mathcal{F} \). Then

\[
\hat{h}^u_{\mu}(\mathcal{F}) = h^u_{\mu}(\mathcal{F}).
\]

For measurable partitions \( \beta \) and \( \gamma \) of \( \Omega \times M \), we can define the conditional information function and conditional entropy of \( \beta \) respect to \( \gamma \) for an invariant
measure $\mu$ of $\Theta$, which are denoted by $I_\mu(\beta|\gamma)$ and $H_\mu(\beta|\gamma)$ respectively. These notations are standard, and more details are described at the end of this section. The conditional entropy of $\mathcal{F}$ for measurable partition $\beta$ with respect to $\gamma$ can also be given, which is denoted by $h_\mu(\mathcal{F}, \beta|\gamma)$ and described in Section 3.

For integers $0 \leq k < j$, denote $\beta_j^k = \Theta^{-j} \beta \lor \Theta^{-(j-1)} \beta \lor \ldots \lor \Theta^{-k} \beta$. A version of Shannon-McMillan-Breimann Theorem in our case is also obtained as follows.

**Theorem B.** Suppose $\mu$ is an ergodic measure of $\mathcal{F}$. For any $\alpha \in \mathcal{P}(\Omega \times M)$, $\eta \in \mathcal{P}^u(\Omega \times M)$, we have

$$\lim_{n \to \infty} \frac{1}{n} I_\mu(\alpha^{n-1}|\eta)(\omega, x) = h_\mu(\mathcal{F}, \alpha|\eta).$$

We can also define unstable topological entropy and unstable pressure for a potential function $\phi$, whose precise definitions are described in Section 5, and we denote them by $h_{\text{top}}^u(\mathcal{F})$ and $P^u(\mathcal{F}, \phi)$ respectively. Naturally, a variational principle is formulated in the following. Denote the set

$$\{ \phi \in L^1(\Omega \times M) : \phi \text{ is measurable in } \omega, \text{ continuous in } x \}$$

by $L^1(\Omega, C(M))$.

**Theorem C.** Let $\phi \in L^1(\Omega, C(M))$. Then we have

$$\sup_{\mu \in \mathcal{M}_p(\mathcal{F})} \left\{ h_\mu^u(\mathcal{F}) + \int_{\Omega \times M} \phi d\mu \right\} = P^u(\mathcal{F}, \phi).$$

A direct corollary of Theorem C is the following variational principle for unstable entropy.

**Corollary C.1.**

$$\sup_{\mu \in \mathcal{M}_p(\mathcal{F})} \left\{ h_\mu^u(\mathcal{F}) \right\} = h_{\text{top}}^u(\mathcal{F}).$$

In the remaining of this section, we give some knowledge on information function, which is slightly modified in our context. Recall that for a measurable partition $\eta$ of a measure space $X$ and a probability measure $\nu$ on $X$, the canonical system of conditional measures for $\nu$ and $\eta$ is a family of probability measures $\{ \nu_x^\eta : x \in X \}$ with $\nu_x^\eta(\eta(x)) = 1$, such that for every measurable set $B \subset X$, $x \mapsto \nu_x^\eta(B)$ is measurable and

$$\nu(B) = \int_X \nu_x^\eta(B)d\nu(x).$$

See e.g. [17] for more details.

**Definition 2.4.** Let $\mu$ be an invariant measure of $(\Omega \times M, \Theta)$, $\alpha$ and $\eta$ be two measurable partitions of $\Omega \times M$. The information function of $\alpha$ with respect to $\mu$ is defined as

$$I_\mu(\alpha)(\omega, x) := -\log \mu(\alpha(\omega, x)),$$

and the entropy of $\alpha$ with respect to $\mu$ is defined as

$$H_\mu(\alpha) := \int_{\Omega \times M} I_\mu(\alpha)(\omega, x)d\mu(\omega, x) = -\int_{\Omega \times M} \log \mu(\alpha(\omega, x))d\mu(\omega, x).$$

The conditional information function of $\alpha$ with respect to $\eta$ is defined as

$$I_\mu(\alpha|\eta)(\omega, x) := -\log \mu_{(\omega,\eta)}(\alpha(\omega, x)).$$
where \( \{\mu_{(\omega,x)}^\eta\}_{(\omega,x)\in\Omega\times M} \) is a canonical system of conditional measures of \( \mu \) with respect to \( \eta \). Then the conditional entropy of \( \alpha \) with respect to \( \eta \) is defined as

\[
H_{\mu}(\alpha|\eta) := \int_{\Omega\times M} I_{\mu}(\alpha|\eta)(\omega, x) d\mu(\omega, x) = -\int_{\Omega\times M} \log \mu_{(\omega,x)}^\eta(\alpha(\omega, x)) d\mu(\omega, x).
\]

For simplicity, sometimes we will use the following notations. For \( \omega \in \Omega \), denote

\[
H_{\mu}(\omega) := \int_M I_{\mu}(\omega)(x) d\mu_{\omega}(x),
\]
and

\[
H_{\mu}(\omega|\eta) := \int_M I_{\mu}(\omega|\eta)(x) d\mu_{\omega}(x).
\]

It is clear that

\[
H_{\mu}(\alpha) = \int_{\Omega} H_{\mu}(\alpha|\eta) d\mathbf{P}(\omega),
\]
and

\[
H_{\mu}(\alpha|\eta) = \int_{\Omega} H_{\mu}(\alpha) d\mathbf{P}(\omega).
\]

The following lemmas are derived from [6] with slight adaption, which are useful for the proofs of our main results.

**Lemma 2.5.** Given \( \mu \in \mathcal{M}_{\mathbf{P}}(\mathcal{F}) \) and let \( \alpha \), \( \beta \), and \( \gamma \) be measurable partitions of \( \Omega \times M \) with \( H_{\mu}(\alpha|\gamma), H_{\mu}(\beta|\gamma) < \infty \).

(i) If \( \alpha \leq \beta \), then \( I_{\mu}(\alpha|\gamma)(\omega, x) \leq I_{\mu}(\beta|\gamma)(\omega, x) \) and \( H_{\mu}(\alpha|\gamma) \leq H_{\mu}(\beta|\gamma) \);

(ii) \( I_{\mu}(\alpha \lor \beta|\gamma)(\omega, x) = I_{\mu}(\alpha|\gamma)(\omega, x) + I_{\mu}(\beta|\gamma)(\omega, x) \) and \( H_{\mu}(\alpha \lor \beta|\gamma) = H_{\mu}(\alpha|\gamma) + H_{\mu}(\beta|\gamma) \);

(iii) \( H_{\mu}(\alpha \lor \beta|\gamma) \leq H_{\mu}(\alpha|\gamma) + H_{\mu}(\beta|\gamma) \);

(iv) if \( \beta \leq \gamma \), then \( H_{\mu}(\alpha|\beta) \geq H_{\mu}(\alpha|\gamma) \).

**Lemma 2.6.** Given \( \mu \in \mathcal{M}_{\mathbf{P}}(\mathcal{F}) \). Let \( \alpha \), \( \beta \) and \( \gamma \) be measurable partitions of \( \Omega \times M \).

(i)

\[
I_{\mu}(\beta_{0}^{n-1}|\gamma)(\omega, x) = I_{\mu}(\beta|\gamma)(\omega, x) + \sum_{i=1}^{n-1} I_{\mu}(\beta|\Theta^i(\beta_{0}^{i-1} \lor \gamma))(\Theta^i(\omega, x)),
\]

hence

\[
H_{\mu}(\beta_{0}^{n-1}|\gamma) = H_{\mu}(\beta|\gamma) + \sum_{i=1}^{n-1} H_{\mu}(\beta|\Theta^i(\beta_{0}^{i-1} \lor \gamma));
\]

(ii)

\[
I_{\mu}(\alpha_{0}^{n-1}|\gamma)(\omega, x) = I_{\mu}(\alpha|\Theta^{n-1})(\Theta^{n-1}(\omega, x)) + \sum_{i=0}^{n-2} I_{\mu}(\alpha|\alpha_{1}^{n-1-i} \lor \Theta^i)(\Theta^i(\omega, x)),
\]

hence

\[
H_{\mu}(\alpha_{0}^{n-1}|\gamma) = H_{\mu}(\alpha|\Theta^{n-1}) + \sum_{i=0}^{n-2} H_{\mu}(\alpha|\alpha_{1}^{n-1-i} \lor \Theta^i).
\]

**Lemma 2.7.** Let \( \alpha \in \mathcal{P}(\Omega \times M) \) and \( \{\zeta_n\} \) be a sequence of increasing measurable partitions of \( \Omega \times M \) with \( \zeta_n \not\succ \zeta \). Then for \( \varphi_{n}(\omega, x) = I_{\mu}(\alpha|\zeta_n)(\omega, x) \), \( \varphi^* := \sup_n \varphi_{n} \in L^1(\mu) \).
Lemma 2.8. Let $\alpha \in \mathcal{P}(\Omega \times M)$ and $\{\zeta_n\}$ be a sequence of increasing measurable partitions of $\Omega \times M$ with $\zeta_n \nrightarrow \zeta$. Then

(i) $\lim_{n \to \infty} I_\mu(\alpha|\zeta_n)(\omega, x) = I_\mu(\alpha|\zeta)(\omega, x)$ for $\mu$-a.e. $(\omega, x)$, and
(ii) $\lim_{n \to \infty} H_\mu(\alpha|\zeta_n) = H_\mu(\alpha|\zeta)$.

3. Unstable Metric Entropy. Given $\mu \in \mathcal{M}_p(F)$. In this section we give the definition of unstable metric entropy along $W^u$-foliation. Firstly, we give the definition using Bowen balls. For any $(\omega, x) \in \Omega \times M$, we always let $d^u$ denote the distance induced by the Riemannian structure of $W^u(\omega, x)$. Let $V^u(F, \omega, x, n, \epsilon)$ denote the $d^u_{\omega,n}$-Bowen ball in $W^u(\omega, x)$ with center $x$ and radius $\epsilon$, i.e.,

$$V^u(F, \omega, x, n, \epsilon) := \{y \in W^u(\omega, x): d^u_{\omega,n}(x, y) < \epsilon, 0 \leq k \leq n - 1\}$$

where

$$d^u_{\omega,n}(x, y) := \max_{0 \leq j \leq n-1} \{d^u(f^j_\omega(x), f^j_\omega(y))\}.$$ 

Definition 3.1. Let $\xi_u \in \mathcal{Q}^u(\Omega \times M)$, i.e., $\xi_u$ is an increasing partition of $\Omega \times M$ subordinate to $W^u$-foliation as in Proposition 2. Now we define the *unstable metric entropy* along $W^u$-foliation as follows,

$$h_u(F, \xi_u) = \int \int \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mu_{\omega, x} V^u(F, \omega, x, n, \epsilon) d\mu_\omega(x) d\mathcal{P}.$$

We denote

$$\lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log \mu_{\omega, x} V(F, \omega, x, n, \epsilon)$$

by $h_\mu(F, \omega, x, \xi_u)$, and denote

$$\int_M h(F, \omega, x, \xi_u) d\mu_\omega(x)$$

by $h_\mu(F, \omega, \xi_u)$. It has been proved in [16] that $h_\mu(F, \omega, x, \xi_u)$ is $\Theta$-invariant and it is independent of the choice of $\xi_u$. Hence we also denote $h^u_\mu(F) = h_\mu(F, \xi_u)$.

Remark 4. In fact, by the following lemma, we can replace “$\limsup$” by “$\lim$” and remove “$\lim_{\epsilon \to 0}$” in Definition 3.1.

Lemma 3.2. Let $\xi_u \in \mathcal{Q}^u(\Omega \times M)$, then we have

$$h_\mu(F, \omega, x, \xi_u) = \lim_{n \to \infty} \frac{1}{n} \log \mu_{\omega, x} V^u(F, \omega, x, n, \epsilon), \quad \mu$-$a.e.(\omega, x).$$

Proof. Denote

$$\underline{h}(F, \omega, x, \epsilon, \xi_u) = \liminf_{n \to \infty} \frac{1}{n} \log \mu_{\omega, x} V^u(F, \omega, x, n, \epsilon)$$

and

$$\overline{h}(F, \omega, x, \epsilon, \xi_u) = \limsup_{n \to \infty} \frac{1}{n} \log \mu_{\omega, x} V^u(F, \omega, x, n, \epsilon).$$

It has been proved in [16] that

$$\lim_{\epsilon \to 0} \underline{h}(F, \omega, x, \epsilon, \xi_u) = \lim_{\epsilon \to 0} \overline{h}(F, \omega, x, \epsilon, \xi_u).$$

Thus we only need to prove both $\overline{h}(F, \omega, x, \epsilon, \xi_u)$ and $\underline{h}(F, \omega, x, \epsilon, \xi_u)$ are independent of $\epsilon$. By Remark 3 (iii), we know that $F$ is uniformly expanding restricted to $W^u$-foliation, so for any $0 < \delta < \epsilon$, there exists $k > 0$ such that

$$V^u(F, \omega, x, k, \epsilon) \subset B^u(\omega, x, \delta),$$

where $B^u(\omega, x, \delta)$ is the $\delta$-neighborhood of $x$ in $W^u(\omega, x)$. Therefore

$$h^u_\mu(F) = \frac{1}{n} \log \mu_{\omega, x} V^u(F, \omega, x, n, \epsilon).$$
for any $x \in M$, where $B^u(\omega, x, \delta)$ is the ball in $W^u(\omega, x)$ centered at $x$ with radius $\delta$. Hence, for all $n > 0$, we have
\[
V^u(\mathcal{F}, \omega, x, n + k, \epsilon) \subset V^u(\mathcal{F}, \omega, x, n, \delta) \subset V^u(\mathcal{F}, \omega, x, n, \epsilon).
\]
Thus, we get
\[
\liminf_{n \to \infty} -\frac{1}{n} \log \mu^u_{(\omega, x)} V^u(\mathcal{F}, \omega, x, n, \delta) = \liminf_{n \to \infty} -\frac{1}{n} \log \mu^u_{(\omega, x)} V^u(\mathcal{F}, \omega, x, n, \epsilon),
\]
and
\[
\limsup_{n \to \infty} -\frac{1}{n} \log \mu^u_{(\omega, x)} V^u(\mathcal{F}, \omega, x, n, \delta) = \limsup_{n \to \infty} -\frac{1}{n} \log \mu^u_{(\omega, x)} V^u(\mathcal{F}, \omega, x, n, \epsilon),
\]
from which Lemma 3.2 follows.

Now we consider the definition using fiberwise finite partitions.

**Definition 3.3.** Given $\mu \in \mathcal{M}_p(\mathcal{F})$. The conditional entropy of $\mathcal{F}$ for a fiberwise finite measurable partition $\alpha$ with respect to $\eta \in \mathcal{P}^n(\Omega \times M)$ is defined as
\[
h_\mu(\mathcal{F}, \alpha|\eta) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha^{n-1}|\eta).
\]
The conditional entropy of $\mathcal{F}$ with respect to $\eta$ is defined as
\[
h_\mu(\mathcal{F}|\eta) = \sup_{\alpha \in \mathcal{P}(\Omega \times M)} h_\mu(\mathcal{F}, \alpha|\eta),
\]
and the **conditional entropy** of $\mathcal{F}$ along $W^u$-foliation is defined as
\[
h^u_\mu(\mathcal{F}) = \sup_{\eta \in \mathcal{P}^n(\Omega \times M)} h_\mu(\mathcal{F}|\eta).
\]

In the following, we give some details on the increasing partitions in $\mathcal{Q}^n(\Omega \times M)$. Denote by $d(\cdot, \cdot)$ the metric on $\Omega$ by which the $\sigma$-algebra $\mathcal{F}$ can be induced. Suppose $\mu \in \mathcal{M}_p(\mathcal{F})$ is ergodic. By Liu and Qian’s argument in [14] or Bahnmüller and Liu’s argument in [2], we can choose a set $\hat{\Lambda} \subset \Lambda$, $(\omega_0, x_0) \in \hat{\Lambda}$ and positive constants $\hat{\epsilon}$, $\hat{\epsilon}$ with
\[
B_{\hat{\Lambda}} := B_{\hat{\Lambda}}((\omega_0, x_0), \hat{\epsilon}\hat{\tau}/2) = \{(\omega, x) \in \Omega \times M : d(\omega, \omega_0) < \hat{\epsilon}\hat{\tau}/2, d(x, x_0) < \hat{\epsilon}\hat{\tau}/2\}
\]
having positive $\mu$ measure such that the following construction of a partition $\xi_u$ satisfies Proposition 2.

For each $r \in [\hat{\tau}/2, \hat{\tau}]$, put
\[
S_{\tau,r} = \bigcup_{(\omega, x) \in B_{\hat{\Lambda}}} S_\tau(\omega, x, r),
\]
where $S_\tau(\omega, x, r) = \{\omega\} \times (W^u_{\text{loc}}(\omega, x) \cap B(x_0, r))$. Then we can define a partition $\hat{\xi}_{u,x_0}$ of $\Omega \times M$ such that
\[
(\hat{\xi}_{u,x_0})(\omega, y) = \begin{cases} S_\tau(\omega, x, r), & y \in W^u_{\text{loc}}(\omega, x) \cap B(x_0, r) \text{ for some } (\omega, x) \in B_{\hat{\Lambda}}, \\ (\Omega \times M) \setminus S_{\tau,r}, & \text{otherwise.} \end{cases}
\]
Next we can choose an appropriate $r \in [\hat{\tau}/2, \hat{\tau}]$ such that
\[
\xi_u = \bigvee_{j=0}^{\infty} \Theta^j \hat{\xi}_{u,x_0}
\]
is subordinate to $W^u$-foliation. We will use the notation $\hat{\xi}_{u,-k} = \bigvee_{j=0}^{k} \Theta^j \hat{\xi}_{u,x_0}$. 
Given $\mu \in \mathcal{M}_p(\mathcal{F})$. Proposition 3 below shows that the two definitions above using Bowen balls and fiberwise finite partitions are equivalent when $\mu \in \mathcal{M}_p(\mathcal{F})$ is ergodic. Firstly, we need some lemmas.

**Lemma 3.4.** Suppose $\mu$ is an ergodic measure and $\alpha \in \mathcal{P}(\Omega \times M)$ is fiberwise finite. For any $\epsilon > 0$, there exists $K > 0$ such that for any $k \geq K$,

$$\limsup_{n \to \infty} H_\mu(\alpha|\hat{\alpha}_n^\alpha \vee (\hat{\xi}_{u,-k})^\alpha_n) \leq \epsilon.$$ 

**Proof.** Denote $S_{-k} = \bigcup_{j=0}^{k} \Theta^j S_{u,x}$. Let $\epsilon > 0$. Since $\mu$ is ergodic, we have $\mu(S_{-k}) \to 1$ as $k \to \infty$. Recall that $\alpha$ is fiberwise finite, hence $\int_{\Omega} K(\omega) d\mathcal{P}(\omega) < \infty$ where $K(\omega)$ is the cardinality of $\alpha(\omega)$. It follows that $\int_{\Omega \times M} K(\omega) d\mu(\omega, x) < \infty$. Thus there exists $K > 0$ such that for any $k \geq K$, we have

$$\int_{(\Omega \times M) \setminus S_{-k}} K(\omega) d\mu(\omega, x) < \epsilon.$$

Then

$$H_\mu(\alpha|\hat{\alpha}_1^\alpha \vee (\hat{\xi}_{u,-k})^\alpha_1) = \int_{S_{-k}} I_\mu(\alpha|\hat{\alpha}_1^\alpha \vee (\hat{\xi}_{u,-k})^\alpha_1) d\mu(\omega, x) + \int_{(\Omega \times M) \setminus S_{-k}} I_\mu(\alpha|\hat{\alpha}_1^\alpha \vee (\hat{\xi}_{u,-k})^\alpha_1) d\mu(\omega, x).$$

For $(\omega, x) \in S_{-k}$, $\alpha_1^\alpha \vee (\hat{\xi}_{u,-k})^\alpha_1(\omega, x) \subset W^u_{loc}(\omega, x)$. Hence for almost every $(\omega, x) \in S_{-k}$, there exists $N = N(\omega, x) > 0$ such that for any $n \geq N$, $\alpha_n^\alpha \vee (\hat{\xi}_{u,-k})^\alpha_1(\omega, x) \subset \alpha(\omega, x)$, which implies that $\log \mu_n^\alpha(\hat{\xi}_{u,-k})_1^\alpha(\omega, x)) = 0$. Lemma 2.7 with $\zeta_n = \alpha_n^\alpha \vee (\hat{\xi}_{u,-k})^\alpha_1$ and Lebesgue's Dominated Convergence Theorem imply that

$$\limsup_{n \to \infty} \int_{S_{-k}} I_\mu(\alpha|\hat{\alpha}_1^\alpha \vee (\hat{\xi}_{u,-k})^\alpha_1) d\mu(\omega, x) = 0.$$

For $(\omega, x) \in (\Omega \times M) \setminus S_{-k}$, we have

$$\int_{\alpha_n^\alpha \vee (\hat{\xi}_{u,-k})_1^\alpha(\omega, x)} - \log \mu_n^\alpha(\hat{\xi}_{u,-k})_1^\alpha(\omega, y) d\mu_n^\alpha(\hat{\xi}_{u,-k})_1^\alpha(\omega, y) \leq K(\omega),$$

which implies that

$$\int_{(\Omega \times M) \setminus S_{-k}} - \log \mu_n^\alpha(\hat{\xi}_{u,-k})_1^\alpha(\omega, x) d\mu(\omega, x) \leq \epsilon.$$

Thus we get what we need.

**Lemma 3.5.** Let $\mu$ be an ergodic measure. Suppose $\eta \in \mathcal{P}^n(\Omega \times M)$ is subordinate to $W^u$-foliation, and $\hat{\xi}_{u,-k}$ is a partition described as above, where $k \in \mathbb{N} \cup \{\infty\}$. Then for almost every $(\omega, x)$, there exists $N = N(\omega, x) > 0$ such that for any $j > N$, we have

$$(\hat{\xi}_{u,-j} \vee \Theta^j \eta)(\Theta^j(\omega, x)) = (\hat{\xi}_{u,-j})(\Theta^j(\omega, x)).$$

Hence, for any partition $\beta$ of $\Omega \times M$ with $H_\mu(\beta|\hat{\xi}_{u,-k}) < \infty$,

$$I_\mu(\beta|\hat{\xi}_{u,-j} \vee \Theta^j \eta)(\Theta^j(\omega, x)) = I_\mu(\beta|\hat{\xi}_{u,-j})(\Theta^j(\omega, x)),$$
which implies that
\[
\lim_{j \to \infty} H_\mu(\beta \hat{\xi}_{u,-k-j} \lor \Theta^j \eta) = H_\mu(\beta |\xi_u). 
\]

Particularly, if we take \(k = \infty\), then the last two equalities become
\[
I_\mu(\beta |\xi_u \lor \Theta^j \eta)(\Theta^j(\omega, x)) = I_\mu(\beta |\xi_u)(\Theta^j(\omega, x)),
\]
and
\[
\lim_{j \to \infty} H_\mu(\beta |\xi_u \lor \Theta^j \eta) = H_\mu(\beta |\xi_u).
\]

**Proof.** Since \(\eta\) is subordinate to \(W^u\), for \(\mu\)-a.e. \((\omega, x)\), there exists \(\rho = \rho(\omega, x) > 0\) such that \(B^u(\omega, x, \rho) \subset \eta_\omega(x)\). Since \(\mu\) is ergodic, for \(\mu\)-a.e. \((\omega, x)\), there are infinitely many \(n > 0\) such that \(\Theta^n(\omega, x) \in S_{u,r}\). Take \(N = N(\omega, x)\) large enough such that
\[
\Theta^n(\omega, x) \in S_{u,r}
\]
and
\[
f_{\theta^N}\eta((\hat{\xi}_{u,x_0})\Theta^n(\beta f_{\theta^N}(x))) \subset B^u(\omega, x, \rho) \subset \eta_\omega(x).
\]
Then we have
\[
f_{\theta^N}^{-j}(\hat{\xi}_{u,x_0})\Theta^n(\beta f_{\theta^N}(x)) \subset \eta_\omega(x)
\]
for any \(j \geq N\). Since
\[
(\hat{\xi}_{u,-k-j})_{\theta^i \omega} = \bigvee_{l=0}^{k+j} f_{\theta^{i-l}}(\hat{\xi}_{u,x_0})_{\theta^{i-l} \omega} \geq f_{\theta^N}(\hat{\xi}_{u,x_0})_{\theta^N \omega},
\]
so we have
\[
f_{\theta^N}^{-j}(\hat{\xi}_{u,-k-j})_{\theta^i \omega}(f_{\omega}(x)) \subset \eta_\omega(x).
\]
Thus we have
\[
(\hat{\xi}_{u,-k-j})_{\theta^i \omega}(f_{\omega}(x)) \subset (f_{\omega}(f_{\omega}(x)),
\]
which implies that
\[
(\hat{\xi}_{u,-k-j})_{\theta^i \omega} \lor f_{\omega}(x))(f_{\omega}(x)) = (\hat{\xi}_{u,-k-j})_{\theta^i \omega}(f_{\omega}(x)).
\]
We get the first equality.

By the definition of information function, it is clearly that
\[
I_\mu(\beta |\xi_{u,-k-j} \lor \Theta^j \eta)(\Theta^j(\omega, x)) = I_\mu(\beta |\xi_{u,-k-j})(\Theta^j(\omega, x)).
\]
Now we get
\[
I_\mu(\beta |\xi_{u,-k-j} \lor \Theta^j \eta)(\Theta^j(\omega, x)) - I_\mu(\beta |\xi_{u,-k-j})(\Theta^j(\omega, x)) = 0
\]
for \(\mu\)-a.e. \((\omega, x)\). Let
\[
\varphi_j = (I_\mu(\beta |\xi_{u,-k-j} \lor \Theta^j \eta) - I_\mu(\beta |\xi_{u,-k-j})) \circ \Theta^j.
\]
Then
\[
\lim_{j \to \infty} \varphi_j(\omega, x) = 0
\]
for \(\mu\)-a.e. \((\omega, x)\). By Fatou’s Lemma, we have
\[
\liminf_{j \to \infty} \int \varphi_j d\mu \geq \int \liminf_{j \to \infty} \varphi_j d\mu = 0,
\]
which means that
\[
\liminf_{j \to \infty} H_\mu(\beta |\xi_{u,-k-j} \lor \Theta^j \eta) \geq \lim_{j \to \infty} H_\mu(\beta |\xi_{u,-k-j}).
\]
By Lemma 2.8 (ii) with $\zeta_j = \xi_{u,-k-j}$ and $\zeta = \xi_u$, we have
\[
\lim_{j \to \infty} H_\mu(\beta|\xi_{u,-k-j}) = H_\mu(\beta|\xi_u).
\]
It is clear that $H_\mu(\beta|\xi_{u,-k-j} \vee \Theta^j \eta) \leq H_\mu(\beta|\xi_{u,-k-j})$ for any $j > 0$. It follows that
\[
\limsup_{j \to \infty} H_\mu(\beta|\xi_{u,-k-j} \vee \Theta^j \eta) \leq \lim_{j \to \infty} H_\mu(\beta|\xi_{u,-k-j}).
\]
Now we get the last equality.

**Lemma 3.6.** (*Cf. [16] and [15]*) Let $\mu$ be ergodic and $\xi_u \in Q^u(\Omega \times M)$. Then for $\mu$-a.e. $(\omega, x) \in \Omega \times M$,
\[
h_\mu(\mathcal{F}, \omega, x, \xi_u) = H_\mu(\xi_u|\Theta \xi_u).
\]

**Proposition 3.** Suppose $\mu$ is ergodic. Let $\xi_u \in Q^u(\Omega \times M)$, $\alpha \in \mathcal{P}(\Omega \times M)$ and $\eta \in \mathcal{P}^u(\Omega \times M)$, then
\[
h_\mu(\mathcal{F}, \omega, x, \xi_u) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta),
\]
for $\mu$-a.e. $(\omega, x) \in \Omega \times M$.

**Proof.** The proof is similar to which in [6]. Firstly, let us show that $h_\mu(\mathcal{F}, \alpha|\eta)$ is independent of $\eta$, i.e., for $\eta_1$ and $\eta_2 \in \mathcal{P}^u(\Omega \times M)$, we have
\[
h_\mu(\mathcal{F}, \alpha|\eta_1) = h_\mu(\mathcal{F}, \alpha|\eta_2).
\]
In fact, by Lemma 2.5, we have
\[
H_\mu(\alpha_0^{n-1}|\eta_1) + H_\mu(\eta_2|\alpha_0^{n-1} \vee \eta_1) = H_\mu(\alpha_0^{n-1}|\eta_2 \vee \eta_1) + H_\mu(\eta_2|\eta_1),
\]
\[
H_\mu(\alpha_0^{n-1}|\eta_2) + H_\mu(\eta_1|\alpha_0^{n-1} \vee \eta_2) = H_\mu(\alpha_0^{n-1}|\eta_1 \vee \eta_2) + H_\mu(\eta_1|\eta_2). \tag{2}
\]
Because of the construction of $\eta_1$ and $\eta_2$, there exist two fiberwise finite partitions $\alpha_1$ and $\alpha_2$ such that $\eta_j(\omega, x) = \alpha_j(\omega, x) \cap W^u_{10}(\omega, x)$, $j = 1, 2$, for all $\omega \in \Omega \times M$. Let $N_1(\omega)$ and $N_2(\omega)$ be the cardinality of $\alpha_1\omega$ and $\alpha_2\omega$, respectively. For any $(\omega, x) \in \Omega \times M$, $\eta(\omega, x)$ intersects at most $N_2(\omega)$ elements of $\alpha_2\omega$, so intersects at most $N_2(\omega)$ elements of $\eta_2$. Thus, we have
\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\eta_2|\alpha_0^{n-1} \vee \eta_1) \leq \lim_{n \to \infty} \frac{1}{n} H_\mu(\eta_2|\eta_1) \leq \lim_{n \to \infty} \frac{1}{n} \int_\Omega N_2(\omega)d\mathcal{P}(\omega) = 0.
\]
Similarly, we have
\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\eta_1|\alpha_0^{n-1} \vee \eta_2) \leq \lim_{n \to \infty} \frac{1}{n} H_\mu(\eta_1|\eta_2) = 0.
\]
Hence we by (2), we get
\[
\limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta_1) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta_2).
\]
Thus $h_\mu(\mathcal{F}, \alpha|\eta)$ is independent of $\eta$.

Then we show that $h_\mu(\mathcal{F}, \alpha|\eta)$ is independent of $\alpha$, that is, for any $\beta$, $\gamma \in \mathcal{P}(\Omega \times M)$,
\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\beta_0^{n-1}|\eta) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\gamma_0^{n-1}|\eta).
\]
In fact, by Lemma 2.5, we have
\[
H_\mu(\gamma_0^{n-1}|\eta) \leq H_\mu(\beta_0^{n-1}|\eta) + H_\mu(\beta_0^{n-1}|\gamma_0^{n-1} \vee \eta), \tag{3}
\]
and similar to the proof of Lemma 2.7 (ii) in [6], we can show that
\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\beta_0^{n-1} \gamma_0^{n-1} \lor \eta) = 0. \tag{4}
\]
By (3) and (4), we have
\[
\limsup_{n \to \infty} \frac{1}{n} H_\mu(\beta_0^{n-1} | \eta) \leq \limsup_{n \to \infty} \frac{1}{n} H_\mu(\gamma_0^{n-1} | \eta).
\]
By the arbitrariness of \( \beta \) and \( \gamma \), we obtain
\[
\limsup_{n \to \infty} \frac{1}{n} H_\mu(\beta_0^{n-1} | \eta) = \limsup_{n \to \infty} \frac{1}{n} H_\mu(\gamma_0^{n-1} | \eta).
\]
Now we start the proof of the proposition. By Lemma 2.6 (i), let \( \gamma = \eta \) and \( \beta = \xi_{u,-k} \), we have that for any \( \eta \in \mathcal{P}_n(\Omega \times M) \), \( n > 0 \),
\[
\frac{1}{n} H_\mu(\xi_{u,-k}^{n-1} | \eta) = \frac{1}{n} H_\mu(\xi_{u,-k} | \eta) + \frac{1}{n} \sum_{j=0}^{n-1} H_\mu(\xi_{u,-k}^{j} | \Theta \xi_{u,-k}^{j+1} \lor \Theta \eta).
\]
By Lemma 3.5, the right side of above equality converges to \( H_\mu(\xi_{u,-k} | \Theta \xi_{u}) \) as \( j \to \infty \). It is clear that each elements of \( \eta_\omega \) intersects at most \( 2^{k+1} \) elements of \( (\xi_{u,-k})_\omega \). So we have
\[
H_\mu(\xi_{u,-k} | \eta) = \int_{\Omega} H_{\mu_\omega}(\xi_{u,-k} | \eta_\omega) d\mathbf{P}(\omega) \leq \int_{\Omega} 2^{k+1} d\mathbf{P}(\omega) = 2^{k+1}.
\]
Hence we have
\[
\limsup_{n \to \infty} \frac{1}{n} H_\mu(\xi_{u,-k} | \eta) = 0.
\]
Thus we get
\[
\lim_{n \to \infty} \frac{1}{n} H_\mu(\xi_{u,-k}^{n-1} | \eta) = H_\mu(\xi_{u,-k} | \Theta \xi_{u}) \leq H_\mu(\xi_{u} | \Theta \xi_{u}). \tag{5}
\]
Replacing \( \gamma \) by \( \xi_{u,-k}^{n-1} \) in Lemma 2.6(ii) and noticing that
\[
\Theta^j(\xi_{u,-k})^{n-1} = (\xi_{u,-k})^{n-j},
\]
we have
\[
H_\mu(\alpha^{n-1} | (\xi_{u,-k})^{n-1}) = H_\mu(\alpha | \xi_{u,-k}^{n-1}) + \sum_{j=0}^{n-2} H_\mu(\alpha | \alpha^{n-1-j} \lor (\xi_{u,-k})^{n-1-j})
\]
\[
= H_\mu(\alpha | \xi_{u,-k}^{n-1}) + \sum_{j=1}^{n-1} H_\mu(\alpha | \alpha^{j} \lor (\xi_{u,-k})^{j})
\]
\[
\leq H_\mu(\alpha) + \sum_{j=1}^{n-1} H_\mu(\alpha | \alpha^{j} \lor (\xi_{u,-k})^{j}).
\]
For any \( \epsilon > 0 \), take \( k > 0 \) as in Lemma 3.4, thus we have
\[
\limsup_{n \to \infty} H_\mu(\alpha | \alpha^{n-1} \lor (\xi_{u,-k})^{n-1}) \leq \epsilon.
\]
Then we get
\[
\limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha^{n-1} | (\xi_{u,-k})^{n-1}) \leq \epsilon. \tag{6}
\]
By Lemma 2.5, we have
\[
H_\mu(\alpha^{n-1} | \eta) \leq H_\mu((\xi_{u,-k})^{n-1} | \eta) + H_\mu(\alpha^{n-1} | (\xi_{u,-k})^{n-1}). \tag{7}
\]
Thus, by (6), (7), then by (5) and Lemma 3.6 we have

\[
\begin{align*}
    h_\mu(F, \alpha|\eta) &= \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) \\
    &\leq \lim_{n \to \infty} \frac{1}{n} H_\mu((\xi_u|\eta)_0^{n-1}|\eta) + \epsilon \\
    &\leq H_\mu(\xi_u|\Theta\xi_u) + \epsilon \\
    &= h_\mu(F, \omega, x, \xi_u) + \epsilon.
\end{align*}
\]

Since \( \epsilon \) is arbitrary, we have

\[
    h_\mu(F, \alpha|\eta) \leq h_\mu(F, \omega, x, \xi_u).
\]

It remains to prove that \( h_\mu(F, \alpha|\eta) \geq h_\mu(F, \omega, x, \xi_u) \). Let \( \xi_u \in \mathcal{Q}(\Omega \times M) \). Since \( \xi_u \) is a generator, we can choose \( N \) large enough such that the measurable partition \( \tilde{\xi} := \bigvee_{j=0}^N \Theta^{-j} \xi_u \) has diameter small enough. It is clear that \( \tilde{\xi} \) still satisfies the condition of Proposition 2, and by Theorem 0.4.2 in [14], we know that \( h_\mu(F, \tilde{\xi}) = h_\mu(F, \xi_u) \). So we only need to prove the above inequality for \( \tilde{\xi} \).

We can find a sequence of partitions \( \alpha_n \in \mathcal{P}(\Omega \times M) \) such that

\[
    B(\alpha_n) \nearrow B(\Theta^{-1} \tilde{\xi}) \quad \text{as} \quad n \to \infty.
\]

So we have

\[
    \lim_{n \to \infty} H_\mu(\alpha_n|\tilde{\xi}) = H_\mu(\Theta^{-1} \tilde{\xi}|\tilde{\xi}).
\]

Thus, \( \sup_{\alpha \in \mathcal{P}(\Omega \times M), \alpha < \Theta^{-1} \tilde{\xi}} H_\mu(\alpha|\tilde{\xi}) = H_\mu(\Theta^{-1} \tilde{\xi}|\tilde{\xi}) \).

For any \( \alpha \in \mathcal{P}(\Omega \times M) \) with \( \alpha < \Theta^{-1} \tilde{\xi} \), we have that for any \( j > 0 \), \( \Theta^j \alpha_0^{j-1} < \Theta^j(\Theta^{-1} \tilde{\xi})_0^{j-1} = \tilde{\xi} \). Then by Lemma 2.6 (i), we have

\[
    H_\mu(\alpha_0^{n-1}|\eta) = H_\mu(\alpha|\eta) + \sum_{j=1}^{n-1} H_\mu(\alpha|\Theta^j(\alpha_0^{j-1} \lor \eta)) \\
    \geq H_\mu(\alpha|\eta) + \sum_{j=1}^{n-1} H_\mu(\alpha|\tilde{\xi} \lor \Theta^j \eta).
\]

Then by Lemma 3.5 we have

\[
    \lim_{j \to \infty} H_\mu(\alpha|\tilde{\xi} \lor \Theta^j \eta) = H_\mu(\alpha|\tilde{\xi}),
\]

which implies that

\[
    \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) \geq \liminf_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta) \geq H_\mu(\alpha|\tilde{\xi}).
\]
So we have
\[
\sup_{\alpha \in \mathcal{P}(\Omega \times M)} h_\mu(\mathcal{F}, \alpha|\eta) \geq \sup_{\alpha \in \mathcal{P}(\Omega \times M), \alpha \in \Theta^{-1}\xi} h_\mu(\mathcal{F}, \alpha|\eta)
\]
\[
= \sup_{\alpha \in \mathcal{P}(\Omega \times M), \alpha \in \Theta^{-1}\xi} \limsup_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta)
\]
\[
\geq \sup_{\alpha \in \mathcal{P}(\Omega \times M), \alpha \in \Theta^{-1}\xi} \liminf_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta)
\]
\[
\geq \sup_{\alpha \in \mathcal{P}(\Omega \times M), \alpha \in \Theta^{-1}\xi} H_\mu(\alpha|\tilde{\xi})
\]
\[
= H_\mu(\Theta^{-1}\xi|\tilde{\xi}).
\]
We have proved that \( h_\mu(\mathcal{F}, \alpha|\eta) \) is independent of \( \alpha \), meaning
\[
h_\mu(\mathcal{F}, \alpha|\eta) = \sup_{\beta \in \mathcal{P}(\Omega \times M)} h_\mu(\mathcal{F}, \beta|\eta)
\]
for any \( \alpha \in \mathcal{P}(\Omega \times M) \), which implies what we need.

**Proof of Theorem A.** This can be obtained directly from Proposition 3.

**Corollary 1.** Suppose \( \mu \) is ergodic, then for any \( \alpha \in \mathcal{P}(\Omega \times M) \) and \( \eta \in \mathcal{P}^u(\Omega \times M) \), we have
\[
h_\mu^u(\mathcal{F}) = h_\mu(\mathcal{F}, \alpha|\eta) = \lim_{n \to \infty} \frac{1}{n} H_\mu(\alpha_0^{n-1}|\eta).
\]

**Proof.** We can get the result directly from Proposition 3.

The following lemmas are useful for the proof of the variational principle in Section 6, whose proofs are completely parallel to those in [6], so we omit them.

**Lemma 3.7.** For any \( \alpha \in \mathcal{P}(\Omega \times M) \) and \( \eta \in \mathcal{P}^u(\Omega \times M) \), the map \( \mu \mapsto H_\mu(\alpha|\eta) \) from \( \mathcal{M}_\mathcal{P}(\mathcal{F}) \) to \( \mathbb{R}^+ \cup \{0\} \) is concave. Moreover, the map \( \mu \mapsto h_\mu^u(\mathcal{F}) \) from \( \mathcal{M}_\mathcal{P}(\mathcal{F}) \) to \( \mathbb{R}^+ \cup \{0\} \) is affine.

**Lemma 3.8.** Let \( \mu \in \mathcal{M}_\mathcal{P}(\mathcal{F}) \) and \( \eta \in \mathcal{P}^u(\Omega \times M) \). Assume that there exists a sequence of partitions \( \{\beta_n\}_{n=1}^\infty \subset \mathcal{P}(\Omega \times M) \) such that \( \beta_1 < \beta_2 < \cdots < \beta_n < \cdots \) and \( \mathcal{B}(\beta_n) \nrightarrow \mathcal{B}(\eta) \), and moreover, \( \mu_\omega(\partial(\beta_n)\omega) = 0 \) for \( n = 1, 2, \cdots \) and \( \mathcal{P} \)-a.e. \( \omega \in \Omega \). Let \( \alpha \in \mathcal{P}(\Omega \times M) \) satisfy \( \mu_\omega(\partial(\alpha)\omega) = 0 \) for \( \mathcal{P} \)-a.e. \( \omega \in \Omega \). Then for \( \mathcal{P} \)-a.e. \( \omega \in \Omega \), the function \( \mu' \mapsto H_{\mu'}(\alpha|\eta) \) is upper semi-continuous at \( \mu \), i.e.,
\[
\limsup_{\mu' \to \mu} H_{\mu'}(\alpha|\eta) \leq H_\mu(\alpha|\eta).
\]
Moreover, the function \( \mu' \mapsto h_{\mu'}^u(\mathcal{F}) \) is upper semi-continuous at \( \mu \), i.e.,
\[
\limsup_{\mu' \to \mu} h_{\mu'}^u(\mathcal{F}) \leq h_\mu^u(\mathcal{F}).
\]

4. **Shannon-McMillan-Breimann Theorem.** In this section, we give a proof of Theorem B. We follow the method in [6] to prove it, via which Hu, Hua and Wu give a version of Shannon-McMillan-Breimann Theorem for unstable metric entropy in deterministic case. Firstly, we need the following lemmas. In this section, we always suppose that \( \mu \in \mathcal{M}_\mathcal{P}(\mathcal{F}) \) is ergodic.

**Lemma 4.1.** Let \( \alpha \in \mathcal{P}(\Omega \times M) \), \( \eta \in \mathcal{P}^u(\Omega \times M) \). Then for any \( \xi \in \mathcal{Q}^u(\Omega \times M) \), we have
\[
h_\mu(\mathcal{F}, \alpha|\eta) \leq \liminf_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}\xi)(\omega, x) \quad \mu \text{-a.e. } (\omega, x).
\]
Proof. For each $\omega \in \Omega$, take $k = k(\omega)$ such that $\text{diam}(\alpha_0^k \vee \xi)_\omega \leq \epsilon$. Then for $n > 0$, we have

$$(\alpha_0^{k+n-1} \vee \xi)_\omega(x) = \bigvee_{j=0}^{n-1} (\Theta^{-j} \alpha_0^k \vee \xi)_\omega(x) \subset V^u(\mathcal{F}, \omega, x, n, \epsilon).$$

By Proposition 3 and Lemma 3.2 we know that

$$h_\mu(\mathcal{F}, \alpha|\eta) = h_\mu(\mathcal{F}, \omega, x, \xi) = \text{lim inf}_{n \to \infty} -\frac{1}{n} \log \mu^{\xi}(\omega, x, n, \epsilon) \leq \lim inf_{n \to \infty} -\frac{1}{n} \log \mu^{\xi}(\omega, x, ((\alpha_0^{k+n-1})_\omega(x))) = \lim inf_{n \to \infty} -\frac{1}{n} \log \mu^{\xi}(\omega, x, ((\alpha_0^{n-1})_\omega(x))) = \lim inf_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(\omega, x).$$

for $\mu$-a.e. $x$.

The following lemmas are counterparts of those in [6], which are completely parallel to the treatment in [6], so we omit their proofs.

**Lemma 4.2.** Let $\eta \in \mathcal{P}^u(\Omega \times M)$ and $\xi \in \mathcal{Q}^u(\Omega \times M)$. Then for $\mu$-a.e. $(\omega, x)$, we have

$$\text{lim inf}_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(\omega, x) = \text{lim inf}_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(\omega, x),$$

and

$$\text{lim sup}_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi)(\omega, x) = \text{lim sup}_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(\omega, x).$$

**Lemma 4.3.** For any $\eta \in \mathcal{P}^u(\Omega \times M)$ and $\xi \in \mathcal{Q}^u(\Omega \times M)$, we have

$$\text{lim}_{n \to \infty} \frac{1}{n} I_\mu(\theta^{-n} \xi|\eta)(\omega, x) = \text{lim}_{n \to \infty} \frac{1}{n} I_\mu(\Theta^{-n} \xi|\xi)(\omega, x) = h_\mu(\mathcal{F}, \omega, x, \xi).$$

**Lemma 4.4.** Let $\alpha \in \mathcal{P}(\Omega \times M)$, $\eta \in \mathcal{P}^u(\Omega \times M)$. Then for $\mu$-a.e. $(\omega, x)$, we have

$$\text{lim}_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\xi_0^{n-1} \vee \eta)(\omega, x) = 0.$$  

Now, we begin prove Theorem B.

**Proof of Theorem B.** By Lemma 4.1 and Lemma 4.2 we can get directly

$$h_\mu(\mathcal{F}, \alpha|\eta) \leq \text{lim inf}_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(\omega, x). \quad (8)$$

By Lemma 2.5, we have

$$I_\mu(\alpha_0^{n-1}|\eta)(\omega, x) \leq I_\mu(\alpha_0^{n-1} \vee \xi_0^{n-1}|\eta)(\omega, x),$$

$$= I_\mu(\xi_0^{n-1}|\eta)(\omega, x) + I_\mu(\alpha_0^{n-1}|\xi_0^{n-1} \vee \eta)(\omega, x).$$

Then by Lemma 4.4, Lemma 4.3, and Proposition 3, we have

$$\text{lim sup}_{n \to \infty} \frac{1}{n} I_\mu(\alpha_0^{n-1}|\eta)(\omega, x) \leq \text{lim sup}_{n \to \infty} \frac{1}{n} I_\mu(\xi_0^{n-1}|\eta)(\omega, x) = h_\mu^u(\mathcal{F}) = h_\mu(\mathcal{F}, \alpha|\eta). \quad (9)$$

Combining (8) and (9), we get the result what we need.
5. Unstable Pressure. In this section, the definition of unstable pressure for a potential function \( \phi \in L^1(\Omega, C(M)) \) is given.

Fix \( \delta > 0 \), for \((\omega, x) \in \Omega \times M\), Let \( W_u^u(\omega, x, \delta) \) be the \( \delta \)-neighborhood of \( x \) in \( W_u^u(\omega, x) \). A subset \( E \) of \( W_u^u(\omega, x, \delta) \) is called an \((\omega, n, \epsilon)\) \( W_u^u \)-separated set if for any \( y_1, y_2 \in E \), we have \( d_{\omega, n}^u(y_1, y_2) > \epsilon \), where \( d_{\omega, n}^u(y_1, y_2) \) is defined by

\[
d_{\omega, n}^u(y_1, y_2) := \max_{0 \leq j \leq n-1} \{ d^u(f^j_\omega(y_1), f^j_\omega(y_2)) \}.
\]

It is easy to check that \( d_{\omega, n}^u \) is measurable in \( \omega \).

Now we can define \( P_u^u(F, \phi, \omega, x, \delta, n, \epsilon) \) as follows,

\[
P_u^u(F, \phi, \omega, x, \delta, n, \epsilon) = \sup \left\{ \sum_{y \in E} \exp((S_n \phi)(y)) : E \text{ is an } (\omega, n, \epsilon) \text{ } W_u^u \text{-separated set of } W_u^u(\omega, x, \delta) \right\},
\]

where \((S_n \phi)(y) = \sum_{j=0}^{n-1} \phi(f^j_\omega(y))\). Then \( P_u^u(F, \phi, \omega, x, \delta) \) is defined as

\[
P_u^u(F, \phi, \omega, x, \delta) = \lim_{\epsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log P_u^u(F, \phi, \omega, x, \delta, n, \epsilon).
\]

Next, we define \( P_u^u(F, \phi, \omega, \delta) = \sup_{x \in M} P_u^u(F, \phi, \omega, x, \delta) \) and

\[
P_u^u(F, \phi, \delta) = \int_{\Omega} P_u^u(F, \phi, \omega, \delta) d\mathcal{P}(\omega).
\]

Because \( \mathcal{P} \) is ergodic, we have the following lemma.

**Lemma 5.1.** \( P_u^u(F, \phi, \delta) = P_u^u(F, \phi, \omega, \delta) \) for \( \mathcal{P} \)-a.e. \( \omega \in \Omega \).

Finally, we can give the definition of unstable pressure for \( F \).

**Definition 5.2.** The unstable pressure for \( F \) is defined as

\[
P_u^u(F, \phi) = \lim_{\delta \to 0} P_u^u(F, \phi, \delta).
\]

**Remark 5.** As what one can do for classical pressure, we can also define the unstable pressure via spanning sets or open covers. We omit the details here.

**Remark 6.** When \( \phi \equiv 0 \), we call the unstable topological pressure \( P_u^u(F, 0) \) the unstable topological entropy of \( F \), and we denote it by \( h_{top}^u(F) \).

For the proof of the variational principle, we need the following lemma.

**Lemma 5.3.**

\[
P_u^u(F, \phi) = P_u^u(F, \phi, \delta) \quad \text{for any } \delta > 0.
\]

**Proof.** It is clear that \( P_u^u(F, \phi) \leq P_u^u(F, \phi, \delta) \) for any \( \delta > 0 \), since the function \( \delta \to P_u^u(F, \phi, \delta) \) is increasing.

Now let \( \delta > 0 \) be fixed. For any \( \rho > 0 \) and each \( \omega \in \Omega \), there exists \( y_\omega \) such that

\[
P_u^u(F, \phi, \omega, \delta) \leq P_u^u(F, \phi, \omega, y_\omega, \delta) + \frac{\rho}{3}.
\]

Take \( \epsilon_0 > 0 \) such that

\[
P_u^u(F, \phi, \omega, y_\omega, \delta) \leq \limsup_{n \to \infty} \frac{1}{n} \log P_u^u(F, \phi, \omega, y_\omega, \delta, n, \epsilon_0) + \frac{\rho}{3}.
\]
Then choose $0 < \delta_1 < \delta$ small enough such that

$$ P^u(\mathcal{F}, \phi) + \frac{\rho}{3} \geq P^u(\mathcal{F}, \phi, \delta_1). \tag{10} $$

There exists a positive number $N = N(\omega)$ which depends on $\delta, \delta_1$ and the Riemannian structure on $\overline{W^u(\omega, y_\omega, \delta)}$ such that

$$ \overline{W^u(\omega, y_\omega, \delta)} \subset \bigcup_{j=1}^{N} \overline{W^u(\omega, y_j, \delta_1)} $$

for some $y_j \in \overline{W^u(\omega, y_\omega, \delta)}$, $j = 1, 2, \ldots, N$. Then we have

$$ P^u(\mathcal{F}, \phi, \omega, \delta) \leq P^u(\mathcal{F}, \phi, \omega, y_\omega, \delta) + \frac{\rho}{3} $$

$$ \leq \limsup_{n \to \infty} \frac{1}{n} \log P^u(\mathcal{F}, \phi, \omega, y_\omega, \delta, n, \epsilon_0) + \frac{2\rho}{3} $$

$$ \leq \limsup_{n \to \infty} \frac{1}{n} \log \left( \sum_{j=1}^{N} P^u(\mathcal{F}, \phi, \omega, y_j, \delta_1, n, \epsilon_0) \right) + \frac{2\rho}{3} $$

$$ \leq \limsup_{n \to \infty} \frac{1}{n} \log N \rho \cdot \mathcal{P}^u(\mathcal{F}, \phi, \omega, y_\omega, \delta_1, n, \epsilon_0) + \frac{2\rho}{3} \text{ for some } 1 \leq l \leq N $$

$$ = \limsup_{n \to \infty} \frac{1}{n} \log \rho \cdot \mathcal{P}^u(\mathcal{F}, \phi, \omega, y_\omega, \delta_1, n, \epsilon_0) + \frac{2\rho}{3} $$

$$ \leq \lim \limsup_{n \to \infty} \frac{1}{n} \log P^u(\mathcal{F}, \phi, \omega, y_\omega, \delta_1, n, \epsilon) + \frac{2\rho}{3} $$

$$ \leq P^u(\mathcal{F}, \phi, \omega, \delta_1) + \frac{2\rho}{3}. $$

Integrating both sides of the above inequality, we get

$$ P^u(\mathcal{F}, \phi, \delta) \leq P^u(\mathcal{F}, \phi, \delta_1) + \frac{2\rho}{3}. $$

Thus, by (10) we have

$$ P^u(\mathcal{F}, \phi, \delta) \leq P^u(\mathcal{F}, \phi) + \rho. $$

Since $\rho$ is arbitrary, we have

$$ P^u(\mathcal{F}, \phi, \delta) \leq P^u(\mathcal{F}, \phi), $$

completing the proof of the lemma.

The following properties of unstable pressure can be obtained directly from its definition. For $\phi \in L^1(\Omega, C(M))$, we define $\|\phi\|_\mathcal{P} := \int_{\Omega} \sup_{x \in M} |\phi(x, \cdot)| \mathcal{P}(\omega).$

**Proposition 4.** Let $\mathcal{F}$ be a $C^2$ RDS with $u$-domination. Then for any $\phi, \psi \in L^1(\Omega, C(M))$ and $c \in L^1(\Omega, \mathcal{P})$, the following properties hold.

(i) If $\phi \leq \psi$, then $P^u(\mathcal{F}, \phi) \leq P^u(\mathcal{F}, \psi)$;

(ii) $P^u(\mathcal{F}, \phi + c) = P^u(\mathcal{F}, \phi) + \int_{\Omega} c \mathcal{P}(\omega)$;

(iii) $\hat{h}_{\text{top}}(f) + \inf \varphi \leq P^u(\mathcal{F}, \phi) \leq \hat{h}_{\text{top}}(f) + \sup \phi$;

(iv) if $P^u(\mathcal{F}, \cdot) < \infty$, then $|P^u(\mathcal{F}, \phi) - P^u(\mathcal{F}, \psi)| \leq \|\phi - \psi\|$;

(v) if $P^u(\mathcal{F}, \cdot) < \infty$, then the map $P^u(\mathcal{F}, \cdot): L^1(\Omega, C(M)) \to \mathbb{R} \cup \{\infty\}$ is convex;

(vi) if $P^u(\mathcal{F}, \phi + \psi \circ \Theta - \psi)$ is convex;

(vii) $P^u(\mathcal{F}, \phi + \psi) \leq P^u(\mathcal{F}, \phi) + P^u(\mathcal{F}, \psi)$. 


6. A Variational Principle. In this section, we give the proof of Theorem C. Firstly, we give the following well-known lemma, which is almost identical to Lemma 1.24 in [3] except that we have removed the condition $s \leq 1$.

**Lemma 6.1.** Suppose $0 \leq p_1, \ldots, p_m \leq 1$, $s = p_1 + \cdots + p_m$ and $a_1, \ldots, a_m \in \mathbb{R}$. Then

$$\sum_{u=1}^{m} p_u (a_u - \log p_u) \leq s \left( \log \sum_{u=1}^{m} e^{a_u} - \log s \right).$$

**Proposition 5.** For $\mu \in \mathcal{M}_p(\mathcal{F})$,

$$h^u_\mu(\mathcal{F}) + \int_{\Omega \times M} \phi d\mu \leq P^u(\mathcal{F}, \phi).$$

**Proof.** Let $\mu = \int_{\mathcal{M}_p(\mathcal{F})} \nu d\tau(\nu)$ be the unique ergodic decomposition where $\mathcal{M}_p(\mathcal{F})$ is the set of ergodic measures in $\mathcal{M}_p(\mathcal{F})$ and $\tau$ is a Borel probability measure such that $\tau(\mathcal{M}_p(\mathcal{F})) = 1$. Since $\mu \mapsto h^u_\mu(\mathcal{F})$ is affine and upper semi-continuous by Lemma 3.7 and 3.8, then so is $\mu \mapsto h^u_\mu(\mathcal{F}) + \int_{\Omega \times M} \phi d\mu$ and hence

$$h^u_\mu(\mathcal{F}) + \int_{\Omega \times M} \phi d\mu = \int_{\mathcal{M}_p(\mathcal{F})} \left( h^u_\mu(\mathcal{F}) + \int_{\Omega \times M} \phi d\mu \right) d\tau(\nu)$$

by a classical result in convex analysis (cf. Fact A.2.10 on p. 356 in [5]). So we only need to prove the proposition for ergodic measures.

We assume $\mu$ is ergodic. Let $\xi \in \mathcal{Q}^u(\Omega \times M)$, that is, a measurable partition of $\Omega \times M$ subordinate to $W^u$-foliation as in Proposition 2. Then we can pick $(\omega, x) \in \Omega \times M$ satisfying

(i) $\mu^\xi_{(\omega,x)}(\xi_\omega(x)) = 1$;

(ii) there exists $B \subset \xi_\omega(x)$ such that

(a) $\mu^\xi_{(\omega,x)}(B) = 1$,

(b) $h_\mu(\mathcal{F}, \omega, \xi) = h_\mu(\mathcal{F}, \omega, y, \xi) = \lim_{n \to \infty} -\frac{1}{n} \log \mu^\xi_{(\omega,y)}(V^u(\mathcal{F}, \omega, y, n, \epsilon))$ for any $y \in B$ and $\epsilon > 0$, according to Lemma 3.2,

(c) $\lim_{n \to \infty} \frac{1}{n}(S_n \phi)(\omega, y) = \int_{\Omega \times M} \phi d\mu$ for any $y \in B$, which can be obtained by using the Birkhoff ergodic theorem on $(\Omega \times M, \Theta)$.

Fix $\rho > 0$. By property (ii) we know that for any $y \in B$, there exists $N(y) = N(y, \epsilon) > 0$ such that if $n \geq N(y)$ then we have

$$\mu^\xi_{(\omega,y)}(V^u(\mathcal{F}, \omega, y, n, \epsilon)) \leq e^{-n(h_\mu(\mathcal{F}, \omega, \xi) - \rho)}$$

and

$$\frac{1}{n}(S_n \phi)(\omega, y) \geq \int_{\Omega \times M} \phi d\mu - \rho. \tag{11}$$

Denote $B_n = \{ y \in B : N(y) \leq n \}$. Then $B = \bigcup_{n=1}^{\infty} B_n$. So we can choose $n > 0$ such that $\mu^\xi_{(\omega,x)}(B_n) > \mu^\xi_{(\omega,x)}(B) - \rho = 1 - \rho$. If $y \in B \subset \xi_\omega(x)$, then

$$\mu^\xi_{(\omega,y)} = \mu^\xi_{(\omega,x)}.$$ So for any $y \in B_n$ we have

$$\mu^\xi_{(\omega,y)}(V^u(\mathcal{F}, \omega, y, n, \epsilon)) \leq e^{-n(h_\mu(\mathcal{F}, \omega, \xi) - \rho)}. \tag{12}$$

Now we can choose $\delta > 0$ such that $W^u(\omega, x, \delta) \supset \xi_\omega(x)$. Let $F$ be an $(\omega, n, \epsilon/2)$ $W^u$-spanning set of $W^u(\omega, x, \delta) \cap B_n$ satisfying

$$W^u(\omega, x, \delta) \cap B_n \subset \bigcup_{z \in F} V^u(\mathcal{F}, \omega, z, n, \epsilon/2),$$
and $V^u(F, \omega, z, n, \epsilon/2) \cap B_n \neq \emptyset$ for any $z \in F$. Then choose an arbitrary point in
$V^u(F, \omega, z, n, \epsilon/2) \cap B_n$, which is denoted by $y(z)$. Then we have
\[
1 - \rho < \mu_{(\omega,x)}^\xi(W^u(\omega, x, \delta) \cap B_n)
\]
\[
\leq \mu_{(\omega,x)}^\xi \left( \bigcup_{z \in F} V^u(F, \omega, z, n, \epsilon/2) \right)
\]
\[
\leq \sum_{z \in F} \mu_{(\omega,x)}^\xi(V^u(F, \omega, z, n, \epsilon/2))
\]
\[
\leq \sum_{z \in F} \mu_{(\omega,x)}^\xi(V^u(F, \omega, y(z), n, \epsilon)). \tag{13}
\]
Using (11), (12) and Lemma 6.1 with
\[
p_i = \mu_{(\omega,x)}^\xi(V^u(F, \omega, y(z), n, \epsilon)) \quad \text{and} \quad a_i = (S_n \phi)(\omega, y(z)),
\]
we have
\[
\sum_{z \in F} \mu_{(\omega,x)}^\xi(V^u(F, \omega, y(z), n, \epsilon)) \left( n \left( \int_{\Omega \times M} \phi d\mu - \rho \right) + n(h_\mu(F, \omega, \xi) - \rho) \right)
\]
\[
\leq \sum_{z \in F} \mu_{(\omega,x)}^\xi(V^u(F, \omega, y(z), n, \epsilon)) \left( (S_n \phi)(y(z)) - \log \mu_{(\omega,x)}^\xi(V^u(F, \omega, y(z), n, \epsilon)) \right)
\]
\[
\leq \left( \sum_{z \in F} \mu_{(\omega,x)}^\xi(V^u(F, \omega, y(z), n, \epsilon)) \right) \left( \log \sum_{z \in F} \exp((S_n \phi)(y(z))) - \log \sum_{z \in F} H_{(\omega,x)}^\xi(V^u(F, \omega, y(z), n, \epsilon)) \right).
\]
Combining (13),
\[
n \left( \int_{\Omega \times M} \phi d\mu - \rho \right) + n(h_\mu(F, \omega, \xi) - \rho)
\]
\[
\leq \log \sum_{z \in F} \exp((S_n \phi)(y(z))) - \log \sum_{z \in F} \mu_{(\omega,x)}^\xi(V^u(F, \omega, y(z), n, \epsilon))
\]
\[
\leq \log \sum_{z \in F} \exp((S_n \phi)(y(z))) - \log(1 - \rho). \tag{14}
\]
Let $\Delta_{\omega, \epsilon} := \sup\{|\phi(\omega, x) - \phi(\omega, y)|: d(x, y) \leq \epsilon\}$. For any $z \in F$, we have
$\exp((S_n \phi)(\omega, y(z))) \leq \exp((S_n \phi)(\omega, z) + n\Delta_{\omega, \epsilon})$.

Dividing by $n$ and taking the limsup on both sides of (14), we have
\[
\int_{\Omega \times M} \phi d\mu + h_\mu(F, \omega, \xi) - 2\rho \leq \limsup_{n \to \infty} \frac{1}{n} \log \sum_{z \in F} \exp((S_n \phi)(\omega, z)) + \Delta_{\omega, \epsilon}.
\]
We can choose a sequence $\{F_n\}$ of such $F$ such that
\[
\limsup_{n \to \infty} \frac{1}{n} \log \sum_{z \in F_n} \exp((S_n \phi)(\omega, z)) \leq P^u(F, \phi, \omega, \delta).
\]

Since $\rho$ is arbitrary, and $\Delta_{\omega, \epsilon} \to 0$ as $\epsilon \to 0$, we have
\[
\int_{\Omega \times M} \phi d\mu + h_\mu(F, \omega, \xi) \leq P^u(F, \phi, \omega, \delta).
\]

Integrating with respect to $\omega$ gives what we need.
Proof of Theorem C. By Proposition 5, we only need to prove that for any $\rho > 0$, there exists $\mu \in \mathcal{M}_P(\mathcal{F})$ such that $h^u_\mu(\mathcal{F}) + \int_{\Omega \times M} \phi d\mu \geq P^u(\mathcal{F}, \phi) - \rho$.

Given $\delta > 0$, by Lemma 5.1 and Birkhoff Ergodic Theorem, we can choose $\omega_0 \in \Omega$ such that

$$P^u(\mathcal{F}, \phi, \delta) = P^u(\mathcal{F}, \phi, \omega_0, \delta),$$

and

$$\lim_{n \to \infty} \frac{1}{n} \sum_{u=0}^{n-1} K(\theta^n \omega_0) = \int_{\Omega} K(\omega) dP(\omega).$$

Then we can choose $x_0 \in M$ such that

$$P^u(\mathcal{F}, \phi, \omega_0, x_0, \delta) \geq P^u(\mathcal{F}, \phi, \omega_0, \delta) - \rho.$$

Take $\epsilon > 0$ small enough. Then let $E_{\omega_0,n}$ be an $(\omega_0, n, \epsilon)$ $W^u$-separated set of $W^u(\omega_0, x_0, \delta)$ such that

$$\log \sum_{y \in E_{\omega_0,n}} \exp((S_n \phi)(\omega_0, y)) \geq \log P^u(\mathcal{F}, \phi, \omega_0, x_0, \delta, n, \epsilon) - 1.$$

Then we construct measures $\nu_n$ with support $\{\omega_0\} \times M$ such that

$$d\nu_n(\omega_0, x) = d\nu_n(\omega_0)(x) d\delta(\omega),$$

where

$$\nu_n := \frac{\sum_{y \in E_{\omega_0,n}} \exp((S_n \phi)(\omega_0, y)) \delta_y}{\sum_{z \in E_{\omega_0,n}} \exp((S_n \phi)(\omega_0, z))}$$

and $\delta$ denotes a Dirac measure. Let

$$\mu_n = \frac{1}{n} \sum_{u=0}^{n-1} \Theta u \nu_n.$$

Then by Lemma 2.1 in [9], there exists a subsequence $\{n_i\}$ such that

$$\lim_{n \to \infty} \mu_{n_i} = \mu.$$

It is easy to check that $\mu \in \mathcal{M}_P(\mathcal{F})$.

We can choose a partition $\eta \in \mathcal{P}^u(\Omega \times M)$ such that $W^u(\omega_0, x_0, \delta) \subset \eta_{\omega_0}(x_0)$ (by shrinking $\delta$ if necessary). That is, $W^u(\omega_0, x_0, \delta)$ is contained in a single element of $\mathcal{R}_{\omega_0}$. Then choose a fiberwise finite partition $\alpha$ of $\Omega \times M$ with sufficiently small diameter such that $\mu(\partial \alpha) = 0$ for $P$-a.e. $\omega$. Let $\alpha^u$ denote the corresponding measurable partition in $\mathcal{P}^u(\Omega \times M)$ constructed via $\alpha$.

Fix $q$, $n \in \mathbb{N}$ with $1 < q \leq n - 1$. Put $a(j) = \left \lfloor \frac{n-j}{q} \right \rfloor$, $j = 0, 1, \cdots, q - 1$, where we denote by $[a]$ the integer part of $a$. Then

$$\bigcup_{u=0}^{n-1} \Theta^{-u} \alpha = \bigcup_{r=0}^{a(j)-1} \Theta^{-r} \alpha_0^{q-1} \cup \bigcup_{t \in T_j} \Theta^{-t} \alpha,$$

where $T_j = \{0, 1, \cdots, j - 1\} \cup \{j + aq(j), \cdots, n - 1\}$. Note that $\text{Card } T_j \leq 2q$. For $P$-a.e. $\omega \in \Omega$, suppose that $\alpha_\omega$ contains $K(\omega)$ elements, moreover, we require that
diam(αω) ≪ ϵ. Then

\[
\log \sum_{y \in E_{ω_0,n}} \exp((S_n φ)(ω_0, y))
\]

\[
= \sum_{y \in E_{ω_0,n}} ν^{(n)}_ω(\{y\}) \left( -\log ν^{(n)}_ω(\{y\}) + (S_n φ)(ω_0, y) \right)
\]

\[= H_{ν_n}(α_n^{q-1}|η) + \int_{Ω×M} (S_n φ) dν_n. \]

Then following the same calculation in [7], we have that

\[
\log \sum_{y \in E_{ω_0,n}} \exp((S_n φ)(ω_0, y))
\]

\[\leq \sum_{t \in T_j} H_{ν_n}(Θ^{-1}α|η) + H_{Θ^j ν_n}(α_0^{q-1}|θ^j η)
\]

\[+ \sum_{r=1}^{a(j)-1} H_{Θ^{r+j} ν_n}(α_0^{q-1}|Θα^n) + \int_{Ω×M} (S_n φ) dν_n
\]

\[\leq 2q \log K_n(ω_0) + H_{Θ^j ν_n}(α_0^{q-1}|θ^j η)
\]

\[+ \sum_{r=1}^{a(j)-1} H_{Θ^{r+j} ν_n}(α_0^{q-1}|Θα^n) + \int_{Ω×M} (S_n φ) dν_n
\]

where \(K_n(ω_0) := \max_{t \in T_j} K(θ^t ω_0).\) We claim that \(\lim_{n \to ∞} \frac{1}{n} \log K_n(ω_0) = 0.\) Indeed, by the choice of \(ω_0,\) we know that

\[\lim_{n \to ∞} \frac{1}{n} n \sum_{u=0}^{n-1} K(θ^u ω_0) = \int_{Ω} K(ω) dP(ω) < ∞ \]

as \(α\) is fiberwise finite. So \(\lim_{n \to ∞} \frac{1}{n} K(θ^n ω_0) = 0,\) from which the claim follows easily.

Summing the inequality above over \(j\) from 0 to \(q - 1\) and dividing by \(n\), by Lemma 3.7 we have

\[\frac{q}{n} \log \sum_{y \in E_{ω_0,n}} \exp((S_n φ)(ω_0, y))
\]

\[\leq 2q^2 \log K_n(ω_0) + \frac{1}{n} \sum_{j=0}^{q-1} H_{Θ^j ν_n}(α_0^{q-1}|θ^j η)
\]

\[+ \frac{1}{n} \sum_{k=0}^{n-1} H_{Θ^k ν_n}(α_0^{q-1}|Θα^n) + \frac{q}{n} \int_{Ω×M} (S_n φ) dν_n
\]

\[\leq 2q^2 \log K_n(ω_0) + \frac{1}{n} \sum_{j=0}^{q-1} H_{Θ^j ν_n}(α_0^{q-1}|θ^j η)
\]

\[+ H_μ_n(α_0^{q-1}|Θα^n) + q \int_{Ω×M} φ dμ_n. \quad (15)\]

Then we can choose a sequence \(\{n_k\}\) such that

(i) \(μ_{n_k} \to μ\) as \(k \to ∞;\)
(ii) the following equality holds
\[ \lim_{k \to \infty} \frac{1}{n_k} \log P^n(F, \phi, \omega_0, x_0, \delta, n_k, \epsilon) = \limsup_{n \to \infty} \frac{1}{n} \log P^n(F, \phi, \omega_0, x_0, \delta, n, \epsilon); \]

(iii) \( \nu_{n_k} \to \nu \) as \( k \to \infty \).

Since \( \mu_\omega(\partial \alpha_\omega) = 0 \) for \( \mathcal{P} \)-a.e. \( \omega \), by Lemma 3.8,
\[ \limsup_{k \to \infty} H_{\mu_{n_k}}(\alpha_0^{q-1}|\Theta \alpha^n) \leq H_\mu(\alpha_0^{q-1}|\Theta \alpha^n). \]

As \( \tilde{\nu}_n \) is supported on \( \{\omega_0\} \times \tilde{W}^u(\omega_0, x_0, \delta) \), for each \( j = 0, \ldots, q-1 \), we can choose \( \alpha, \beta_n \in \mathcal{P}(\Omega \times M) \) such that \( \beta_1 < \beta_2 < \cdots < \beta_n < \cdots \) and \( \mathcal{B}(\beta_n) \supseteq \mathcal{B}(\Theta^j \eta) \), and moreover, \( (\Theta^j \nu)_{\omega}(\partial(\alpha_0^{q-1})_\omega) = 0 \), \( (\Theta^j \nu)_{\omega}(\partial(\beta_n^{q-1})_\omega) = 0 \) for \( \mathcal{P} \)-a.e. \( \omega \in \Omega \).

Then applying Lemma 3.8 we have
\[ \limsup_{k \to \infty} \frac{1}{n_k} \sum_{j=0}^{q-1} H_{\Theta^j \nu_{n_k}}(\alpha_0^{q-1}|\Theta^j \eta) \leq \limsup_{n \to \infty} \frac{1}{n} \sum_{j=0}^{q-1} H_{\Theta^j \nu}(\alpha_0^{q-1}|\Theta^j \eta) = 0. \]

Thus replacing \( n \) by \( n_k \) in (15) and letting \( k \to \infty \), by the above claim and discussions, we get
\[ q \limsup_{n \to \infty} \frac{1}{n} \log P^n(F, \phi, \omega_0, x_0, \delta, n, \epsilon) \leq H_\mu(\alpha_0^{q-1}|\Theta \alpha^n) + q \int_{\Omega \times M} (S_n \phi) d\mu. \]

By Theorem A,
\[ \limsup_{n \to \infty} \frac{1}{n} \log P^n(F, \phi, \omega_0, x_0, \delta, n, \epsilon) \leq \lim_{q \to \infty} \frac{1}{q} H_\mu(\alpha_0^{q-1}|\Theta \alpha^n) + \int_{\Omega \times M} (S_n \phi) d\mu = h_\mu(F) + \int_{\Omega \times M} (S_n \phi) d\mu. \]

Let \( \epsilon \to 0 \), we have \( P^n(F, \phi, \omega_0, x_0, \delta) \leq h^n_\mu(F) + \int_{\Omega \times M} (S_n \phi) d\mu \). Recall that \( P^n(F, \phi) = P^n(F, \phi, \delta) = P^n(F, \phi, \omega_0) \leq P^n(F, \phi, \omega_0, x_0, \delta) + \rho \). The proof of Theorem C is complete.

7. \( u \)-Equilibrium States. In this section, as an application of Theorem C, we consider the \( u \)-equilibrium states, whose definition is given as follows. Firstly, we give a dual proposition of Theorem C.

Proposition 6. If \( h_{\mu_0}^u(F) < \infty \) and \( \mu_0 \in \mathcal{M}_p(F) \), then
\[ h_{\mu_0}^u(F) = \inf_{\phi \in L^1(\Omega, C(M))} \left\{ \int_{\Omega \times M} (S_n \phi) d\mu_0 \right\}. \]

Proof. By Lemma 3.8, we know that \( \mu \mapsto h_\mu^u(F) \) is upper semi-continuous at \( \mu_0 \in \mathcal{M}_p(F) \), then similar to the proof of Theorem 3.1.6 in [4], we can prove Proposition 6, whose details are omitted.

Given \( \phi \in L^1(\Omega, C(M)) \).
Definition 7.1. $\mu \in \mathcal{M}_P(\mathcal{F})$ is said to be a $u$-equilibrium state for $\phi$, if it satisfies
\[
h_u(\mathcal{F}) + \int_{\Omega \times \mathcal{M}} \phi \, d\mu = P_u(\mathcal{F}, \phi).
\]

We denote by $\mathcal{M}_u(\mathcal{F}, \phi)$ the set of all $u$-equilibrium states for $\phi$.

Proposition 7. Let $\phi \in L^1(\Omega, C(M))$, then we have the following properties on $u$-equilibrium states.

(i) $\mathcal{M}_u(\mathcal{F}, \phi)$ is non-empty, and it is convex;
(ii) the extreme points of $\mathcal{M}_u(\mathcal{F}, \phi)$ are precisely the ergodic members of $\mathcal{M}_u(\mathcal{F}, \phi)$;
(iii) $\mathcal{M}_u(\mathcal{F}, \phi)$ is compact and has an ergodic $u$-equilibrium state;
(iv) assume $\phi, \psi \in L^1(\Omega, C(M))$ are cohomologous, i.e. $\phi = \psi + \sigma - \sigma \circ \Theta - c$ for some $c \in L^1(\Omega, P)$ and $\sigma \in L^1(\Omega, C(M))$. Then $\phi$ and $\psi$ have the same $u$-equilibrium states, and
\[
P_u(\mathcal{F}, \phi) = P_u(\mathcal{F}, \psi) - \int_{\Omega} c dP(\omega).
\]

Proof. The proof of this proposition is completely parallel to the treatment in [20], so we omit it here.

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