Post-geodesic corrections to the binding energy during the transition to plunge in numerical relativity simulations

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For a relativistic binary with a small mass-ratio (SMR), post-geodesic corrections to the binding energy due to the gravitational self-force are needed to model the evolution of the system. Perhaps surprisingly, first-order self-force predictions have proven to be very accurate when compared to numerical simulations of binaries with comparable masses, suggesting a greater applicability of the SMR approximation than expected. These comparisons and their agreement have been generally restricted to the early adiabatic inspiral, which breaks down as the smaller mass approaches the innermost stable orbit and transitions into a direct plunge. Here we examine the binding energy of nonspinning, quasicircular binary black holes using numerical simulations, with mass ratios ranging from equal mass to 1 : 20. We demonstrate the validity of the SMR approximation during inspiral, and for the first time show that the binding energy of comparable mass systems follows a fractional power expansion in the symmetric mass ratio \( \nu \equiv m_1 m_2/(m_1 + m_2)^2 \), as predicted in the SMR limit. Our results are in agreement with analytic predictions, and indicate that the transition dynamics is important for modeling the last \( \sim 10 \) gravitational wave cycles before merger for comparable mass systems, such as those currently observed by gravitational wave detectors. We find that second-order GSF effects on the transition are small. This analysis provides further evidence for the applicability of perturbative SMR results to gravitational wave modeling of comparable mass binaries.

The direct detection of gravitational waves (GWs) [1–14] has provided a new window onto the universe, revealing populations of binaries composed of black holes and neutron stars [15–17] and enabling tests of relativity in the dynamical and strong-field regime, e.g. [18–24]. Accurate modeling of the fully relativistic two-body problem is crucial for carrying out sensitive GW searches and measuring the parameters of detected binaries. These models provide predictions for the GWs emitted during the inspiral, merger, and post-merger emission from these systems, for example effective one-body models, e.g. [25–33], and phenomenological models, e.g. [34–40].

The construction of such complete models requires input from a number of methods, such as post-Newtonian (PN), gravitational self-force (GSF) and numerical relativity (NR) techniques. Understanding the limits of each approach is crucial for creating accurate models. For example the PN approximation is valid at large separations and slow velocities. It worsens late in the inspiral, and can be a poor approximation for systems with small mass ratios \( q^{-1} \), where \( q = m_1/m_2 \geq 1 \), which spend many cycles at close separations. In contrast, GSF methods based on the small mass ratio (SMR) expansion of the metric about a black hole background is accurate for fully relativistic systems, but is naturally restricted to the SMR region of the parameter space. These methods were originally developed to describe extreme-mass-ratio inspirals (EMRIs) with mass ratios \( q^{-1} \sim 10^{-4}–10^{-6} \), which are promising targets for the LISA mission [41], and have been recently been pushed to second-order (2GSF) for nonspinning binaries [42–45].

Meanwhile, numerical relativity (NR) provides two-body solutions exact up to numerical errors, but is limited in practice due to the computational time required to evolve systems from a large separation, and the additional resolution requirements for systems with very unequal masses. Because NR is the only non-perturbative method, it can be used to assess the validity of the PN and SMR perturbative approximations in the regime of comparable masses and small separations accessible to NR [46]. NR is also used to calibrate effective one-body and phenomenological models, or even to directly build accurate surrogate models which can interpolate waveform predictions between simulations, e.g. [47–53].

Surprisingly, comparisons between NR and SMR approximations of binary black hole systems have shown that the GSF approach is effective at describing even comparable mass ratio systems, e.g. [44, 45, 54–59]. This is true even for equal mass systems, when considering quantities symmetric under the exchange of the components \( m_1 \leftrightarrow m_2 \). Such quantities include the binding energy, GW fluxes and GW phasing, and is achieved by re-expanding the SMR series in the symmetric mass ratio \( \nu \equiv m_1 m_2/(m_1 + m_2)^2 \) rather than \( q^{-1} \), see e.g. [45, 56, 57]. This promising result indicates that GSF methods may be effective at predicting GWs from systems such as those detected by current [60–62] and future ground-based detectors [63–65], and may be key for modeling intermediate-mass-ratio inspirals (IMRIs), \( q^{-1} \sim 10^2–10^4 \), a regime which remains challenging for NR [66–69].

In this study we tackle an important limitation of
previous analysis. In all cases, the agreement between NR and SMR predictions breaks down as the binary approaches the innermost stable circular orbit (ISCO). This is expected: first because higher order SMR coefficients may grow larger at high frequencies, but also because previous SMR predictions expand around an adiabatic inspiral, and must eventually fail as the binary transitions into plunge and merger. Distinguishing between these two effects is crucial for modeling the binary close to merger and is and particularly relevant for intermediate and comparable mass ratio systems, where the transition region can be large.

The transition can be understood as a sort of singular boundary layer in between the slow inspiral through a sequence of circular orbits and a direct plunge with timescale $T \sim M, M$ the total mass. In the SMR approximation, this transition occurs over a region of characteristic size $\sim M \nu^{2/5}$ around the location of the ISCO $r_* = 6M$, and with a dynamical timescale $T \sim M^{-1/5}$. Following the initial description of the transition dynamics [26, 70], a number of studies have refined and generalized the analytic approximations to the dynamics [71–77]. Recently self-force corrections have been incorporated into a generic expansion of transition equations and their solutions [78, 79], which we use here.

In this study we investigate the binding energy $E$ of nonspinning, quasicircular binary black holes at comparable masses. We show that $E$ follows well-behaved SMR inspiral and transition expansions even through ISCO. During the inspiral we recover the geodesic and post-geodesic coefficients as functions of the invariant radius $r_\Omega \equiv M^{1/3} \Omega^{-2/3}$. During the transition, we find that the $E$ follows the expected fractional power expansion in $\nu$ [79], with the coefficients functions of the rescaled transition radius $R_\Omega \equiv \nu^{-2/5} (r_\Omega - r_*)$. This expansion breaks down near merger and towards inspiral as expected, and we estimate the region of validity to be $-2 \gtrsim R_\Omega / M \gtrsim 7$, corresponding to $r_\Omega \sim r_* - 2M \nu^{2/5}$. The leading $O(\nu^{4/5})$ coefficient from the transition fits is in good agreement with predictions. We are able to extract higher-order coefficients up to $O(\nu^{9/5})$, where unknown 2GSF contributions first appear, and we find them to be negligible within the uncertainty of our analysis.

From our analysis we can estimate the regime over which the inspiral and transition SMR expansions accurately predict $E$ using only first-order GSF (1GSF) information. Our results indicate that an SMR expansion of the two-body system can provide accurate predictions for gravitational waves for comparable mass systems up to the final GW cycle before merger, and is consistent with recent success of EMRI surrogate models [51, 52] and recent 2GSF-accurate inspiral waveforms [45] in describing GWs from comparable mass systems.

Conventions: From here we render all quantities dimensionless by setting $G = c = M = 1$. We use $q = m_1 / m_2 \gtrsim 1$ to refer to the mass ratio of simulations, as is conventional in NR. We use an overdot for time derivatives and a prime for derivatives with respect to $r_\Omega$ or $R_\Omega$, depending on the context. Quantities evaluated at ISCO are indicated by $\ast$, e.g. $r_\ast$.

NR simulations – We select a set of high-resolution, nonspinning and quasicircular binary black hole simulations produced with the Spectral Einstein Code (SpEC) [80, 81] with mass ratios ranging from $q = 1$ to $q = 20$. These simulations have low initial eccentricity $e \lesssim 10^{-4}$, a relatively large number of orbital cycles $N_\text{cycles} \sim 20–45$, and in most cases two resolution levels, which allows us to assess numerical uncertainties.

From each simulation we take the extrapolated gravitational wave strain $h = h_+ - i h_\times$ at future null infinity [80, 82], expanded in $(\ell, m)$ modes of spin-weighted spherical harmonics. The strain $h$ is further corrected by applying a translation and boost that minimizes the effect of the center of mass motion present in the simulations [83, 84]. From the strain we define the invariant radius $r_\Omega$ using an orbital frequency $\Omega$ inferred from the $\ell = 2, m = 2$ mode of the gravitational waves, $\phi_{22} = \arg h_{22}, \quad \Omega = \dot{\phi}_{22}/2$. (1)

Although the quantity of interest for our analysis is $E$, only the energy flux $\dot{E}$ is directly accessible from the strain. One approach is to integrate the flux, choosing the integration constant by matching either to the mass of the final black hole or to PN theory early in the inspiral, but we find that this procedure introduces undesired errors in the analysis. Instead we use the gradients $E'(r_\Omega) = \dot{E} / \dot{r}_\Omega$ during inspiral and $E'(R_\Omega) = \dot{E} / \dot{R}_\Omega$ during the transition as our quantities of interest. For the flux we use the standard formula $\dot{E} = \lim_{r \to \infty} r^2 \sum_{\ell, m} |h_{\ell m}|^2$, (2)

and sum over all the gravitational wave modes available in our simulations ($\ell \leq 8$).

One challenge in our analysis is that the strain $h$ exhibits small modulations beyond those expected from quasicircular inspiral, which become particularly noticeable in the derived $\Omega$. While the origin of these modulations is uncertain, during early inspiral they are dominated by residual junk radiation which can reflect from the outer boundary back into the computational domain, and at later times appear to be due to modulations of the center of mass, see e.g. [59]. To mitigate them, we apply a low-pass filter to $\Omega$ during the early inspiral, with a cutoff frequency chosen conservatively high so that the overall chirping of $\Omega$ is not biased. Towards the transition regime the dynamics are fast enough that the filtering can still potentially bias the result. So, for $r < 9.5$ we smooth the modulations with a rolling fit of $E'(R_\Omega)$ to a quadratic over a fiducial window size of $\Delta R_\Omega = \pm 2$. Further details of our simulations and these procedures are given in the Appendix.
**Inspiral expansion** – During the adiabatic inspiral, post-geodesic corrections to $E$ can be calculated from 1GSF corrections to the redshift quantity $z$ [85] via the first law of binary mechanics (FLBM) [86, 87]. The FLBM assumes a helical symmetry with Killing vector field $K = \partial_t + \Omega \partial_\phi$, which is exact only for perfectly circular orbits. This global symmetry provides a powerful connection between the local redshift and the energy and angular momentum of the system. While this symmetry does not hold for dynamical binaries, the FLBM has been found to be surprisingly accurate when comparing analytic predictions to NR, e.g. [55, 88], and we again find excellent agreement. In our comparison we require the $O(\nu)$ corrections to the derivative of the binding energy. The FLBM gives [55, 87]

$$E'(r_\Omega) = E'_{\text{geo}}(r_\Omega) + \nu E'_{\text{FLBM}}(r_\Omega) + O(\nu^2),$$

with

$$E_{\text{FLBM}}(x) = \frac{1}{2} z_{1\text{GSF}}(x) - \frac{x}{3} z'_{1\text{GSF}}(x) - 1 + \sqrt{1 - 3x + \frac{x}{6} (7 - 24x)} (1 - 3x)^{3/2},$$

where $x = r^{-1}_\Omega$. The 1GSF contribution to $z$, $z_{1\text{GSF}}(x)$, has been computed to high precision with multiple codes, e.g. [89]. For the purpose of our inspiral comparison we make use of the fit formula [87]

$$z_{1\text{GSF}}(x) = \frac{2x(1 - 2.18522x + 1.05185x^2)}{1 - 2.43395x + 0.400665x^2 - 5.9991x^3}. \tag{5}$$

Without assuming the relationships derived from the FLBM, 2GSF information is required to compute $O(\nu)$ contributions to $E$ [42], and we also compare to these results.

**Inspiral results** – To compare NR and SMR approximation during the inspiral, we perform a least-square fit of $E'(r_\Omega)$ to an expansion in integer powers of $\nu$,

$$E'(r_\Omega) = \sum_{i=0}^{i_{\text{max}}} E'_i(r_\Omega) \nu^i, \tag{6}$$

Following the approach of [58, 59], we first fit the NR data at fixed $r_\Omega$ values to both a first and second degree polynomial in $\nu$, without reference to the SMR prediction. From this we extract values for the coefficients as a function of $r_\Omega$, and we recover the geodesic prediction $E'_0(r_\Omega)$ from the NR data alone. This is shown in the top panel of Fig. 1, where we plot the difference of $E'_0$ and $E'_{\text{geo}}$, finding remarkable agreement.

Having confirmed the test particle limit, we next fit the remainder $\nu^{-1}[E'(r_\Omega) - E'_{\text{geo}}(r_\Omega)]$ which allows us to extract the $O(\nu)$ and $O(\nu^2)$ coefficients more accurately. Figure 1 shows the fitted coefficients $E'_1$ and $E'_2$. The result is in very good agreement with the first-order prediction from the FLBM, with systematic deviations starting $r_\Omega \lesssim 10$. The NR data agrees better with the FLBM than with the post-adiabatic result [42] for the binding energy based on a 2GSF calculation. This is interesting on its own: it shows the importance of understanding subtle differences in the definitions of energy and orbital frequency that are used when comparing NR and GSF methods, as discussed in [42]. We also find evidence of a small but non-zero $O(\nu^2)$ term during the inspiral.

We have repeated the inspiral analysis using the orbital angular momentum, with $L'$ derived from the angular momentum flux, and comparing to the post-adiabatic approximation up to $O(\nu^2)$ using the FLBM relations [87]. As with the binding energy, a fit to $O(\nu^2)$ provides excellent agreement throughout the inspiral regime, with a breakdown for $r_\Omega \lesssim 10$. We also find that the adiabatic condition for circular orbits is satisfied for each of our extracted coefficients during inspiral, with $E'_i/L'_i = \Omega$ to within the uncertainty in our fits. This includes the coefficients at $O(\nu^2)$. This agreement provides further evidence for the accuracy of the FLBM during inspiral. In the future, it would be interesting to see whether the
FLBM continues to be accurate through order $\mathcal{O}(\nu^2)$ during the inspiral. This would require a calculation of the 2GSF conservative redshift $2_{\text{GSF}}$ [90].

**Transition expansion** — Using an SMR expansion around the Schwarzschild metric, the binding energy and radius of the orbit during the transition take the form [79]

$$E = E_\ast + \Omega_\ast [\nu^{4/5}\xi(\nu, s) + \nu^{5/5}Y(\nu, s)], \quad r = r_\ast + \nu^{2/5}R(\nu, s).$$

where $s \equiv \nu^{1/5}(\tau - \tau_\ast)$ is the transition time parameter. The transition variables can be expanded in fractional powers of $\nu$,

$$\xi = \sum \xi_i \nu^{i/5}, \quad Y = \sum Y_i \nu^{i/5}, \quad R = \sum R_i \nu^{i/5}. \quad (9)$$

The transition equations provide a method for iteratively solving for each of $\xi_i$, $R_i$, and $Y_i$, with their boundary conditions fixed by matching to the adiabatic inspiral at early times $s \to -\infty$. They also take as input the self-force $F_{\nu}$ in the neighborhood of $r_\ast$. For example, angular momentum conservation reveals $\xi = F_{\nu}^3 |_{r_\ast} + \mathcal{O}(\nu^{4/5})$ [79]. For the self-force $F_{\nu}^3 |_{r_\ast}$, we use first-order flux data at the ISCO, taken from [74, 91]. Note that our gauge-invariant $R_0$ differs from $R$ at $\mathcal{O}(\nu^{2/5})$, and throughout we have re-expanded the small parameter $1/q$ in terms of $\nu$, which alters the usual transition expansion at $\mathcal{O}(\nu^{9/5})$. This means that our final fitted transition functions differ from those of [78, 79] beyond the leading order. For our analysis we numerically solve for the leading order terms $R_0(s)$ and $\xi_0(s)$. We give the leading transition equations we use in the Appendix. Note that 2GSF corrections enter at $\mathcal{O}(\nu^{9/5})$.

**Transition results** — For the transition analysis we follow the same method as for the inspiral but we fit the NR data at fixed $R_\Omega$ to the fractional power expansion

$$E'(R_\Omega) = \sum_{i=4} E_{i/5}'(R_\Omega) \nu^{i/5}. \quad (10)$$

Following the expectation from Eq. 7, we set $E_{5/5}' = 0$ in our first fit. We then let $i_{\text{max}} = 8$, which guarantees the stability of the fit, and which is enough to fit the NR data without leaving any structure in the residuals. Figure 2 shows the results of the transition coefficients $E_{4/5}'$ from this fit. We see that the leading SMR result is in excellent agreement for a range of radius corresponding to $2 \gtrsim R_\Omega \gtrsim 7$, confirming the presence of the predicted transition dynamics even at comparable masses.

Having confirmed that the leading SMR prediction can be recovered purely from the NR data, we next fit the residual between $E'$ and the leading analytic prediction, $\nu^{4/5}\Omega_\ast d\xi_0/dR_0$. In principle, this leading subtraction is not accurate through $\mathcal{O}(\nu)$, and can artificially introduce a term at $E_{5/5}'$. As such, we first fit the NR data including this coefficient, and find the result is fully consistent with $E_{5/5}' = 0$, with the remaining coefficients nonvanishing [92]. We then again set $E_{5/5}' = 0$ and fit the scaled residual $[E' - \nu^{4/5}E_{4/5}' \nu^{-2/5}]$. This allows us to extract accurate coefficients using either $i_{\text{max}} = 8$ or $i_{\text{max}} = 9$, with the latter providing an estimate for the $\mathcal{O}(\nu^{9/5})$ term.

Figure 3 shows the resulting higher-order coefficients. One can see a general trend where the coefficients are comparable to $E_{4/5}'(R_\Omega)$ in a region around ISCO but grow towards larger $R_\Omega$, consistent with a possible breakdown of the transition expansion towards inspiral. The coefficient $E_{7/5}'$ is consistently larger than $E_{6/5}'$, and similar in magnitude to $E_{8/5}'$, which is why we require terms up to $E_{9/5}'$ to recover the leading-order result and why the leading result alone is never accurate at these mass ratios. Meanwhile we see that including $E_{9/5}'$ doesn’t introduce any systematic deviation in the lower order coefficients. Including it seems to worsen the fits, possible by overfitting the residual oscillations, and itself is consistent with zero. Our transition expansion fails for $R_\Omega \gtrsim 7$, as is clear from the failure to recover the leading-order result in Fig. 2, and from the blow-up of the subleading coefficients seen in Fig. 3.

**Conclusions** — We have analyzed for the first time the SMR limit from nonspinning, quasicircular NR simulations in the transition region around the ISCO. Our work extends previous analysis of the validity of the SMR approximation at comparable masses, which were restricted to the inspiral region (but see also [52]). We first revisit the applicability of the adiabatic SMR expansion.
during the inspiral, showing that our simulations are in good agreement with SMR predictions augmented with the FLBM. The agreement with the FLBM is limited to $r_\Omega \gtrsim 10$, and we find that the failure to recover the FLBM result can be explained by the onset of transition dynamics at around $r_\Omega \lesssim 10$.

Our analysis shows that using a transition expansion in fractional powers for the binding energy and angular momentum as functions of $R_\Omega \equiv \nu^{-2/5}(r_\Omega - r_{\text{isco}})$, we can fit the NR data and recover the leading-order SMR result [26, 70] in a region of width $2 \gtrsim R_\Omega \gtrsim 7$ around the ISCO. We find that terms up to $O(\nu^{8/5})$ are necessary to recover this result from NR, and we give a prediction for the value of the higher-order coefficients. We also show that the $O(\nu^{3/5})$ contribution is zero to within our uncertainties, suggesting that the 2GSF contributions to the binding energy during transition may be small.

Our results are summarized in Fig. 4, which shows the NR data for $dE/dr_\Omega$, for three binaries with $q = 1$, $q = 5$, and $q = 20$. We compare our raw NR data with the smoothed and filtered data that we fit, along with the $O(\nu)$-accurate FLBM inspiral prediction and the results of our transition fit up to $O(\nu^{3/5})$. This illustrates the failure of the inspiral treatment near ISCO for higher $q$, the narrowing of the transition region with increasing $q$, and the fact that a combination of the two treatments describes the energy accurately until the last cycle before merger in all cases, with only 1GSF information.

This result provides the next step in bridging the SMR and the comparable-mass regions of the parameter space. It provides compelling evidence that one can continue to model two-body dynamics using 1GSF theory, augmented by the FLBM, to less than a cycle before merger. This work points to a number of future directions. The next step would be to extend this analysis to add transition effects in the GW phasing, and extend the results of [58] to merger. SMR predictions in this regime would be enabled by combining transition modeling [78] with 2GSF-accurate fluxes [44] and waveforms [45].

It then is critical to address systems which include spin, first for aligned-spin systems with an SMR expansion around Kerr, but eventually for systems with precessing orbit. Another promising direction would be to go beyond the quasicircular approximation and examine eccentric binaries. These areas represent the frontier of 2GSF calculations, and if achieved could provide a complete, first-principles model for the two-body problem applicable from EMRIs to equal masses.
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Details of the data analysis and error estimates – During inspiral, we use a second-order forward and backward Butterworth filter $B$ to filter $\dot{\Omega}$. To mitigate the impact on the frequency sweep, we find that finding that applying the filter after subtracting a good estimate of the data improves its performance by reducing the overall variation of the data. For this we use the 2GSF prediction for $\dot{\Omega}$ from [45]. Further, we apply the filter to $\dot{\Omega}$ as a function of its index. Since the time step in our simulation is adaptive, the strain data $h$ is not uniformly sampled, but we find that resampling to uniform time steps makes finding an appropriate cutoff frequency more challenging than for the non-uniform sampling. The reason is that with the the denser sampling rate at late times effectively brings the late-time chirp to lower frequencies, so that a uniform cutoff frequency better targets the actual noisy behavior at all times. The cutoff frequencies are chosen as $f_c = a + b(\nu - \nu_{c,20})$, with $a \in \{6, 8, 10\} \times 10^{-4}$ and $b = 1.95 \times 10^{-3}$. The filtered data is then

$$\dot{\Omega}_{\text{filtered},i} = [B * (\dot{\Omega}_{\text{raw},i} - \dot{\Omega}_{\text{SMR},i})] + \dot{\Omega}_{\text{SMR},i}.$$ (11)

Since the filtering process biases the data late in the simulation, and because we have no SMR data beyond ISCO, we choose a cutoff radius $r_{c,20} = 9.5$ beyond which we simply switch to a rolling fit of $E'(\Omega)$ (or $L'(R_0)$ as discussed below) to a quadratic, over a window $\Delta R_0 \in \pm \{1, 2, 3\}$ around the fitted point. For our fiducial analysis, we select the one corresponding to $\Delta R = 2$ and $a = 8 \times 10^{-4}$.

To create our uncertainty bands for our fitted quantities, we vary all of the parameters involved in the filtering and smoothing of our data within the stated ranges, as well as repeating our analysis with simulations at a lower resolution. The error bands are created by taking the envelope of the variation in our fitted parameters, over the different resolutions, and the 1-$\sigma$ errors of our least-squares fits at each frequency point.

Details of the transition formalism – Our analysis requires that we solve the transition equations to leading

| $q$ | Type | $Mf_0$ | $N_{\text{cycles}}$ | $e_0$ | Levs | SXS ID |
|-----|------|--------|----------------------|------|------|--------|
| 1   | SKS  | 0.01233 | 27.96 | 1.355e-4 | 5, 6 | 2513   |
| 1.5 | SKS  | 0.01250 | 28.98 | 5.77e-5  | 2, 3 | 2331   |
| 2   | SHK  | 0.01554 | 20.70 | 2.408e-4 | 2, 3 | 2497   |
| 2.5 | SKS  | 0.01512 | 22.49 | 7.580e-4 | 2, 3 | 2019   |
| 3   | SHK  | 0.01707 | 20.44 | 9.64e-5  | 5   | 2488   |
| 3.5 | SKS  | 0.01477 | 27.76 | 2.65e-4  | 4, 5 | 2483   |
| 4   | SKS  | 0.01600 | 25.67 | 8.702e-4 | 4, 5 | 2488   |
| 4.5 | SKS  | 0.01616 | 27.37 | 8.290e-4 | 4, 5 | 2484   |
| 5   | SKS  | 0.01589 | 29.13 | 2.323e-4 | 4, 5 | 2487   |
| 5.5 | SKS  | 0.01592 | 30.81 | 4.442e-4 | 4, 5 | 2486   |
| 6   | SKS  | 0.01588 | 32.62 | 5.846e-4 | 4, 5 | 2489   |
| 6.5 | SKS  | 0.01599 | 34.43 | 7.263e-4 | 4, 5 | 2488   |
| 7   | SKS  | 0.01577 | 36.16 | 3.615e-4 | 4, 5 | 2491   |
| 7.5 | SKS  | 0.01597 | 38.97 | 3.964e-4 | 5   | 2490   |
| 8   | SKS  | 0.01584 | 39.53 | 7.08e-4  | 5   | 2493   |
| 8.5 | SKS  | 0.01594 | 41.31 | 8.578e-4 | 5   | 2492   |
| 9   | SKS  | 0.01583 | 43.16 | 2.010e-4 | 3, 4 | 2516   |
| 9.5 | SKS  | 0.01585 | 44.93 | 1.384e-4 | 4   | 2494   |
| 14  | SHK  | 0.02292 | 27.70 | 3.814e-4 | 2, 3 | 2480   |
| 15  | SHK  | 0.02317 | 27.94 | 3.692e-4 | 2, 3 | 2477   |
| 20  | SKS  | 0.02321 | 34.38 | 2.506e-4 | 3, 4 | 2516   |

TABLE I. Details of the nonspinning quasicircular SpEC simulations used in this analysis. The subscript zero denotes the reference time (time at which junk radiation has sufficiently decayed).
order, in order to calibrate our fits and compare with the NR results. The quantities \(R_0(s)\) and \(Y_0(s)\) are solved using the the leading order transition equations [26, 70, 73, 79]

\[
\left(\frac{dR_0}{ds}\right)^2 = -\frac{2}{3} \alpha_s R_0^3 - 2 \beta_s \kappa_s s R_0 + \gamma_s Y_0, \tag{12}
\]

\[
\frac{d^2 R_0}{ds^2} = -\alpha_s R_0^2 - \kappa_s \beta_s s, \tag{13}
\]

\[
\frac{dY_0}{ds} = 2 \kappa_s \frac{\beta_s}{\gamma_s} R_0. \tag{14}
\]

The constants in the above equations are given by

\[
\kappa_s \equiv F_0^i |_{r_s}, \quad \alpha_s \equiv \frac{1}{4} \left. \partial^3 V^{\text{geo}} \right|_{r_s}, \quad \gamma_s \equiv \left. \frac{\partial V^{\text{geo}}}{\partial L} \right|_{r_s}, \tag{15}
\]

Common symbols:

\[
\beta_s = -\frac{1}{2} \left( \frac{\partial^2 V^{\text{geo}}}{\partial r \partial L} + \Omega \frac{\partial^2 V^{\text{geo}}}{\partial r \partial E} \right) \bigg|_{r_s}, \tag{16}
\]

where \(V^{\text{geo}}\) is the effective potential of radial geodesic motion about a Schwarzschild black hole.

Further results: Angular momentum - Here we present the analysis of the angular momentum in both the inspiral and transition regimes. These results provide an independent demonstration of the accuracy of the SMR modeling during inspiral and plunge. They are used to confirm that the binaries are described by adiabatically evolving circular orbits through \(O(\nu^2)\) during inspiral, consistent with the FLBM, as discussed in the main text. We also find that this condition fails at the expected order during transition, by finding agreement with the non-circular correction \(Y_0\) introduced by Kesden [73].

We extract the angular momentum flux from our numerical simulations using the standard formula

\[
\dot{L} = \lim_{r \to \infty} \frac{r^2}{16 \pi} \text{Im} \left( \sum_{l,m} m h^{l,m}(h^{l,m})^* \right). \tag{17}
\]

We use the same filtering technique applied to \(\tilde{\Omega}\) as done for the analysis of \(E(\tau)\) during the inspiral, and the same rolling fit to \(dL/dR_0\) as a function of the transition radius \(R\) for \(\tau_\Omega < 9.5\), as done for analysis of the binding energy.

For the analytic comparisons during inspiral, we use the FLBM-derived expansion relations [55, 87]

\[
L'(\tau_\Omega) = L^{\text{geo}}_1(\tau_\Omega) + \nu L^{\text{FLBM}}_1(\tau_\Omega) + \mathcal{O}(\nu^2), \tag{18}
\]

\[
L^{\text{FLBM}}_1(x) = \frac{1}{3 \sqrt{x}} z^{\text{IGSE}}_1(x) + \frac{1}{6 \sqrt{x}} \left( 4 - \frac{5 x}{(1 - 3 x)^{3/2}} \right) \tag{19}
\]

and the redshift factor of Eq. (5). During inspiral we use an expansion in integer powers of \(\nu\) to fit to the NR data,

\[
L'(\tau_\Omega) = \sum_{i=0}^2 L_{i}'(\tau_\Omega) \nu^i. \tag{20}
\]

As before, the fits across simulations at fixed \(\nu\) give a leading coefficient \(L_0'\) in agreement with the geodesic prediction \(L_0^{\text{geo}}\), and so we subtract this and fit the residual \(L'(\tau_\Omega) - L_0'(\tau_\Omega)\) to improve the accuracy of the fitted \(L_1'(\tau_\Omega)\) and \(L_2'(\tau_\Omega)\) coefficients. The results of the inspiral analysis are depicted in Fig. 5.

For the transition analysis, the analytic approximation for \(L'(R)\) depends on the quantity \(\xi\) [70, 78, 79],

\[
L = L_s + \nu^{4/5} \xi(\nu, s). \tag{21}
\]

This allows us to compare \(L'(R_\Omega)\) during the transition to analytic predictions in the same way as for \(E'(R_\Omega)\). We carry out the same analysis as for \(E'(R_\Omega)\), first performing a fit of the form

\[
L'(R_\Omega) = \sum_{i=4}^{i_{\text{max}}} L_{i/5}^i(R_\Omega) \nu^{i/5}, \tag{22}
\]

with \(i_{\text{max}} = 8\) and the only from analytic theory being that \(\xi_4'(R_\Omega) = 0\). The results for the leading coefficient shown in Fig. 6 confirm that \(L_{4/5}^i\) obeys the expected transition dynamics, in agreement with theory.
fix \( L'_{4/5} \) to the analytic prediction, and verify that \( L'_{5/5} \) remains consistent with zero. Finally fixing both \( L'_{4/5} \) and \( L'_{5/5} \), we fit with \( i_{\text{max}} = 8 \) and \( i_{\text{max}} = 9 \) and extract the transition coefficients plotted in Fig. 7. As with our analysis of the binding energy, we see that the higher terms in the expansion are significant and well behaved through \(-2 \leq R_\Omega \leq 7\), and that the \( \mathcal{O}(\nu^{9/5}) \) term is consistent with zero to within our uncertainties. This again confirms the small size of 2GSF contributions to the transition.

Equivalently, we can focus on the deviation from circularity during transition [73], \( Y'(R_\Omega) \equiv E'(R_\Omega) - \Omega L'(R_\Omega) \). The leading prediction for the deviation is

\[
Y'(R) = Y'_0(R_\Omega)\nu^{4/5} + \mathcal{O}(\nu^{7/5}).
\]  

(23)

We fit the NR data during transition to a fractional power series of the form

\[
Y'(R_\Omega) = \sum_{i=6}^{i_{\text{max}}} Y'_i(R_\Omega)\nu^{i/5},
\]  

(24)

once again setting the first correction to the leading-order behavior to zero, \( Y'_{7/5} = 0 \), and letting \( i_{\text{max}} = 8 \). Figure 8 shows the fit coefficient \( Y'_{6/5}(R_\Omega) \) extracted from this procedure. The leading-order prediction provides a good fit to the data over the same range of \( R_\Omega \) values as we find for the energy analysis, \( 2 \lesssim R_\Omega \lesssim 7 \).

If we retain the \( Y'_{7/5} \) term, the near-degeneracy of the first two terms prevents us from achieving this level of agreement with theory for \( Y'_{6/5} \). We can justify setting \( Y'_{7/5} = 0 \) in two ways. Our first comes from considering what our numerical results for \( E'(R_\Omega) \) imply for the transition expansion. The fact that we find that \( E'_{5/5} = 0 \) after subtracting the leading transition prediction from the NR data for \( E'(R_\Omega) \) implies that the first correction to \( R \) vanishes, \( R_1(R_\Omega) = 0 \). This in turn implies that first term in the transition expansion of the radial self-force, \( f'_{\text{SR}} \) of [79], vanishes. Without this term to source \( Y_1 \) in the transition equations, \( Y_1 \) term can be set to zero in a consistent manner. The second way is to subtract the leading-order prediction (23) from \( Y' \) and fit the remainder, \( Y'(R_\Omega) - \nu^{4/5}Y_0/dY_0/dR_\Omega \) to the fractional power expansion in \( \nu \), starting the series with \( Y'_{7/5} \). When doing so we find the fit for \( Y'_{7/5}(R_\Omega) \) vanishes throughout the transition region, while the higher coefficients are nonzero, similar to what occurred for \( E'(R_\Omega) \).

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