Heating Through Phonon Excitation Implied by Collapse Models

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We calculate the rate of heating through phonon excitation implied by the noise postulated in mass-proportional-coupled collapse models, for a general noise power spectrum. For white noise with reduction rate $\lambda$, the phonon heating rate reduces to the standard formula, but for non-white noise with power spectrum $\lambda(\omega)$, the rate $\lambda$ is replaced by $\lambda_{\text{eff}} = \frac{2}{3\pi^2} \int d^3w e^{-\omega_L^2 \vec{w}^2} \lambda(\omega_L(\vec{w}/r_c))$, with $\omega_L(\vec{q})$ the longitudinal acoustic phonon frequency as a function of wave number $\vec{q}$, and with $r_C$ the noise correlation length. Hence if the noise power spectrum is cut off below $\omega_L(|\vec{q}| \sim r_c^{-1})$, the heating rate is sharply reduced.

There is increasing interest in testing wave function collapse models [1], by searching for effects associated with the noise which drives wave function collapse when nonlinearly coupled in the Schrödinger equation. A recent cantilever experiment of Vinate et al. [2] has set noise bounds consistent with the enhanced noise strength [3] needed to make latent image formation a trigger for state vector collapse, and reports a possible noise signal. Various other suggested experiments [4] focus on noise-induced motions or heating of small masses or collections of oscillators, assuming a white noise spectrum. Since recent experiments on gamma ray emission from germanium [5] have shown that with the enhanced noise strength of [3], a white noise spectrum is experimentally ruled out, it becomes important to take the effects of a cutoff in the noise spectrum into account. In this paper we focus on noise-induced heating, motivated by the astute observation of Vinate [6] that since the noise wave number density is peaked near $|\vec{q}| \sim r_c^{-1}$, heating effects will be reduced if the noise spectrum cuts off below the longitudinal acoustic phonon frequency associated with the wave number peak. Our aim is to give a quantitative calculation of this effect; its application to possible experiments involving bulk heating effects will be given elsewhere [7].

Consider a system in initial state $i$ with energy $E_i = h\omega_i$ at time $t = 0$, acted on by a perturbation $V$ which at time $t$ leads to a transition to a state $f$ with energy $E_f = h\omega_f$. Working in the interaction picture, the transition amplitude $c_{fi}(t)$ is given by

$$ c_{fi}(t) = -\frac{i}{\hbar} \int_0^t V_{fi}(t') e^{i\omega_{fi} t'} dt' , $nolip"}
taneous localization (CSL) model,
\[ V = \int d^3z \frac{dW_t(z)}{dt} \mathcal{V}(\vec{z}, \{\vec{x}\}) \right. , \]
\[ \mathcal{V}(\vec{z}, \{\vec{x}\}) = -\frac{\hbar}{mN} \sum_\ell m_\ell g(\vec{z} - \vec{x}_\ell) \left. , \right. \tag{2} \]
where we have followed the notation used in [8]. Here \( \vec{x}_\ell \) are the coordinates of atoms of mass \( m_\ell \), \( g(\vec{x}) \) is a spatial correlation function, conventionally taken as a Gaussian
\[ g(\vec{x}) = (2\pi)^{-3/2} \left( \frac{r_c}{2\pi} \right)^3 e^{-\vec{r}^2/(2r_c^2)} \left( \frac{r_c}{2\pi} \right)^{3/2} \int d\vec{q} e^{-i\vec{q} \cdot \vec{x}} \right. \tag{3} \]
and the non-white noise has expectation \( \mathcal{E} \)
\[ \mathcal{E} \left[ \frac{dW_t(\vec{x})}{dt} \frac{dW_{t'}(\vec{y})}{dt'} \right] = \frac{1}{2\pi} \int_{-\infty}^\infty d\omega \gamma(\omega) e^{-i\omega(t-t')} \delta^3(\vec{x} - \vec{y}) \right. , \tag{4} \]
with \( \gamma(\omega) = \gamma(-\omega) \) related to the reduction rate parameter \( \lambda(\omega) \) by
\[ \gamma(\omega) = 8\pi^{3/2}r_c^3 \lambda(\omega) \left. . \right. \tag{5} \]

We wish now to calculate the expectation \( \mathcal{E}[E(t)] \) of the energy attained by the system at time \( t \), given by
\[ \mathcal{E}[E(t)] = \mathcal{E}\left[ \sum_f \hbar \omega_{fi} |c_{fi}(t)|^2 \right] \right. . \tag{6} \]
Substituting Eqs. (1) – (5), carrying out integrations, and using the formulas \[ 9 \]
\[ \int_0^t dt' e^{i(\omega_{fi} - \omega)t'} = \frac{e^{i(\omega_{fi} - \omega)t} - 1}{i(\omega_{fi} - \omega)} \equiv 2\pi e^{i(\omega_{fi} - \omega)t/2} \delta(t) (\omega_{fi} - \omega) \] ,
\[ [\delta(t)(\omega_{fi} - \omega)]^2 \approx \frac{t}{2\pi} \delta(t)(\omega_{fi} - \omega) \] ,
we find in the large \( t \) limit the formula for the energy gain rate
\[ t^{-1} \mathcal{E}[E(t)] = \frac{r_c^3}{\pi^{3/2}m_N^2} \int d^3q \sum_f e^{-r_c^2 \vec{q}^2} \lambda(\omega_{fi}) \hbar \omega_{fi} \left| \left( \sum_\ell m_\ell e^{i\vec{q} \cdot \vec{x}_\ell} \right) \right|_{fi}^2 . \tag{8} \]

The next step is to evaluate the matrix element appearing in Eq. (8) by introducing phonon physics, following the exposition in the text of Callaway [10]. We consider first the simplest case of a monatomic lattice with all \( m_\ell \) equal to \( m_A \), independent of the index \( \ell \), and write the atom coordinate \( \vec{x}_\ell \) as
\[ \vec{x}_\ell = \vec{R}_\ell + \vec{u}_\ell \] ,
\[ \tag{9} \]
with $\vec{R}_\ell$ the equilibrium lattice coordinate and with $\vec{u}_\ell$ the lattice displacement induced by the noise perturbation. Writing

$$\sum_\ell m_\ell e^{i\vec{q} \cdot \vec{x}_\ell} = m_A \sum_\ell e^{i\vec{q} \cdot \vec{R}_\ell} e^{i\vec{q} \cdot \vec{u}_\ell},$$

we note that since the Gaussian in Eq. (8) restricts the magnitude of $\vec{q}$ to be less than of order of $r_c^{-1}$, with $r_c \sim 10^{-5}$ cm, whereas the magnitude of the lattice displacement is much smaller than $10^{-8}$ cm, the exponent in $e^{i\vec{q} \cdot \vec{u}_\ell}$ is a very small quantity. So we can Taylor expand to write

$$e^{i\vec{q} \cdot \vec{u}_\ell} \simeq 1 + i\vec{q} \cdot \vec{u}_\ell.$$ 

(11)

The leading term 1 does not contribute to energy-changing transitions, so we have reduced the matrix element in Eq. (8) to the simpler form

$$\left(\sum_\ell m_\ell e^{i\vec{q} \cdot \vec{x}_\ell}\right)_{fi} \simeq im_A \left(\sum_\ell e^{i\vec{q} \cdot \vec{R}_\ell} \vec{q} \cdot \vec{u}_\ell\right)_{fi}, \quad f \neq i.$$ 

(12)

The approximation leading to Eq. (12) is a phonon analog of the electric dipole approximation made in electromagnetic radiation rate calculations.

We now substitute the expression [10] for the lattice displacement in terms of phonon creation and annihilation operators,

$$\vec{u}_\ell = \frac{\Omega}{8\pi^3} \left(\frac{hN}{m_A}\right)^{1/2} \sum_j \int \frac{d^3k}{2\omega_j(\vec{k})^{1/2}} \left[\bar{e}^{(j)}(\vec{k}) e^{i\vec{k} \cdot \vec{R}_\ell} a_j(\vec{k}) + \bar{e}^{(j)*}(\vec{k}) e^{-i\vec{k} \cdot \vec{R}_\ell} a_j^\dagger(\vec{k})\right],$$

(13)

where the sum on $j$ runs over the acoustic phonon polarization states, and where $\Omega$ and $N$ are respectively the lattice unit cell volume, and the number of unit cells. Taking the initial state $i$ to be the zero phonon state, only the $a_j^\dagger$ term in Eq. (13) contributes, and we can evaluate the sum over lattice sites $\ell$ in Eq. (12) using the formula [10]

$$\sum_\ell e^{i(\vec{q} - \vec{k}) \cdot \vec{R}_\ell} = \frac{8\pi^3}{\Omega} \delta^3(\vec{q} - \vec{k}).$$

(14)

Carrying out the $\vec{k}$ integration, noting that $\vec{q} \cdot \bar{e}^{(j)}(\vec{q})$ selects the longitudinal phonon with frequency $\omega_L(\vec{q})$, defining $\vec{w} = r_c \vec{q}$, writing $M = Nm_A$ for the total system mass, and assembling all the pieces, we arrive at the answer

$$t^{-1} \mathcal{E}[E(t)] = \frac{h^2 M}{m_A N r_c^2} \frac{1}{2\pi^{3/2}} \int d^3w e^{-\vec{w}^2} \mathcal{W}^2 \lambda(\omega_L(\vec{w}/r_c)) = \frac{3}{4} \frac{\lambda_{\text{eff}} M}{m_A N r_c^2},$$

$$\lambda_{\text{eff}} = \frac{2}{3\pi^{3/2}} \int d^3w e^{-\vec{w}^2} \mathcal{W}^2 \lambda(\omega_L(\vec{w}/r_c)).$$

(15)
In the white noise case, where $\lambda(\omega)$ is a constant $\lambda$, we can pull it outside the $\vec{w}$ integral and use

$$\int d^3 w e^{-\vec{w}^2 \vec{w}^2} = \frac{3}{2} \pi^{3/2}$$

(16)

to get the standard formula [11]

$$t^{-1} \mathcal{E}[E(t)] = \frac{3 \hbar^2 \lambda M}{4 m^2 r_c^2} \omega_L$$

(17)

When the noise spectrum has a cutoff below $\omega_L(\vec{q})$ for $|\vec{q}| \sim r_c^{-1}$, the energy gain rate is sharply reduced.

Although we have derived the result of Eq. (15) for the case of a monatomic lattice and a zero phonon initial state, the result is more general. For a multi-atom unit cell, the same answer holds, with $m_A$ the sum of masses in the unit cell, and with $\omega_L(\vec{q})$ again the longitudinal acoustic phonon frequency. In the multi-atom case the formula of Eq. (15) neglects optical phonon contributions, but these are the “internal excitations” that are neglected in the derivation of the center-of-mass energy gain formula of Eq. (17). When the initial state is constructed from $n$-phonon states, as in a thermal ground state, the $a^\dagger$ term in Eq. (13) contributes a term proportional to $(n+1)\omega_L$ to the energy gain, while the $a$ term in Eq. (13) contributes a corresponding term proportional to $-n\omega_L$ to the energy gain; the sum of the two terms is proportional to $(n+1-n)\omega_L = \omega_L$, so $n$ drops out and the formula of Eq. (15) is recovered. This simplification could have been anticipated from our earlier analysis of the noise-induced energy gain by an oscillator [12], which showed that the rate of energy gain is a constant independent of the number of oscillator quanta that are present.

I wish Andrea Vinante for an email that stimulated this paper, and to thank Angelo Bassi for helpful conversations.

**Added Note**

Apart from updating Ref. [7], the preceding body of this paper is identical to the version posted on arXiv on Jan. 1, 2018. Andrea Vinante has called our attention to a paper by M. Bahrami [13] posted on Jan. 11, with an update on Jan. 14, in which a similar calculation is done. For a monatomic lattice, Bahrami’s result and ours are in agreement. In his Jan. 14 posting, Bahrami gives a formula for the case of a multi-atom unit cell, which he notes disagrees with our statement that this gives the same result as the monatomic case. Bahrami’s multi-atom formula is incorrect, as a result of his using the wrong normalization for the phonon polarization vectors, and does not reduce to the standard formula in the white noise case when $\lambda(\omega)$ is a constant $\lambda$. In this version of our paper, we have added an Appendix giving a brief derivation of the correct result in the multi-atom case.
Later Added Note
Bahrami agrees, and will revise his posting.

Appendix: Brief derivation of the formula for the multi-atom case
In the monatomic case, focusing only on the atomic mass factors and longitudinal phonon polarization vectors, Eqs. (12) and (13) give a factor

\[ m_A^{1/2} e^{(L)\ast}(\vec{k}) \simeq m_A^{1/2} e^{(L)\ast}(\vec{0}) \quad . \tag{18} \]

After the \( \simeq \) sign we have used the fact, noted after Eq. (10), that the correlation length \( r_C \) allows only contributions from phonon wavelengths that are long on a lattice scale, corresponding to \( \vec{k} \simeq \vec{0} \).

In the multi-atom case, focusing only on acoustic phonons,\(^1\) the left-hand side of Eq. (18) is replaced by

\[ m_\kappa^{1/2} e^{(L)\ast}(\vec{k}) \quad , \tag{19} \]

corresponding to Eqs. (1.4.22a,b) of [10], with \( \kappa \) labeling an atom in the multi-atom unit cell.

Referring now to the unnumbered equation in Callaway [10] between his Eqs. (1.1.22) and (1.1.23), which we write (using the fact that for \( \vec{k} = 0 \) the polarization vectors are real numbers; see Callaway Eq. (1.1.21)) as

\[ m_\kappa^{-1/2} e^{(L)\ast}(\vec{0}) = m_\kappa^{-1/2} e^{(L)}(\vec{0}) = \vec{C} \quad , \tag{20} \]

with \( \vec{C} \) a constant, we see that the longitudinal polarization vectors are no longer unit normalized, as in the monatomic case. Instead, the normalization is given in Eq. (1.1.18a) of [10],

\[ \sum_\kappa \bar{e}^{(L)\ast}(\vec{0}) \cdot \bar{e}^{(L)}(\vec{0}) = 1 \quad , \tag{21} \]

which on substituting Eq. (20) gives

\[ |\vec{C}| = \left( \sum_\kappa m_\kappa \right)^{-1/2} \quad , \tag{22} \]

and implies for small \( \vec{k} \)

\[ m_\kappa^{1/2} \vec{k} \cdot \bar{e}^{(L)\ast}(\vec{k}) \simeq m_\kappa |\vec{C}| = m_\kappa \left( \sum_\kappa m_\kappa \right)^{-1/2} \quad . \tag{23} \]

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\(^1\) Optical phonons leave the unit cell center of mass stationary, so obey \( \sum_\kappa m_\kappa^{1/2} \bar{e}^{(s)\ast}(\vec{0}) = 0 \) for any optical phonon mode \( s \). Hence for mass-proportional noise coupling, optical phonons do not contribute to the energy gain rate to leading order in \( a/r_C \), with \( a \) the unit cell dimension.
Recalling Eqs. (11)–(14), summing over \( \kappa \) to get the total contribution to the one-phonon creation amplitude, we have

\[
\sum_{\kappa} m_{\kappa} \left( \sum_{\kappa} m_{\kappa} \right)^{-1/2},
\]

which when squared gives a factor

\[
\sum_{\kappa} m_{\kappa} = m_{\text{cell}},
\]

which is the total atomic mass in the unit cell. Thus the only change from the monatomic to the multi-atomic case is the replacement of \( m_A \) by \( m_{\text{cell}} \), and since \( N m_{\text{cell}} = M \), the total system mass, the monatomic formula of Eq. (15) is unchanged. Heuristically, the reason for this is that, as emphasized by Callaway, for \( \vec{k} = 0 \) acoustic phonons Eq. (20) implies that all “...particles in each unit cell move in parallel with equal amplitudes”, and so behave as a single particle with mass \( m_{\text{cell}} \).

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