The Spaces of Entire Function of Finite Order

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Abstract

This paper is a continuation of the research of the first author. We consider the linear topology space of entire functions of a proximate order and normal type with respect to the proximate order. We obtain the form of continuous linear functional on this space.

Keywords: Entire function; Proximate order; Normal type; Continuous linear functional

Introduction

This paper is a continuation of the research [1] where the linear topology space of entire functions of a proximate order and normal type, less than or equal to $\sigma$, with respect to the proximate order were considered. We introduce the necessary definitions. A function $\rho(r)$, defined on the ray $(0, \infty)$ and satisfying the Lipschitz condition on any segment $[a, b] \subset (0, \infty)$ that satisfies the conditions

$$\lim_{r \to \infty} \rho(r) = \rho \geq 0, \text{ and } \lim_{r \to \infty} r \rho + (r) \ln r = 0$$

This is called a proximate order.

A detailed exposition of the properties of proximate order can be found [2,3]. In this paper we use the notation $V(r) = \rho(r)$. We will assume that $V(r)$ is an increasing function on $(0, \infty)$ and $\lim V(r) = 0$.

We now formulate some simple property of proximate order that we shall need frequently [2].

For $r \to \infty$ and $0 < a \leq k \leq b < \infty$ asymptotic inequality holds uniformly in $k$.

$$(1 - e^{-k})V(r) < V(kr) < (1 + e^{-k})V(r)$$

Let $M_f(r) = \max_{|z|=r} |f(z)|$. If for the entire function $f(z)$ the quantity

$$\sigma_f = \limsup_{r \to \infty} \frac{\log M_f(r)}{V(r)}$$

is different from zero and infinity, then $\rho(r)$ is called a proximate order of the given entire function $f(z)$ and of is called the type of the function $f(z)$ with respect to the proximate order $\rho(r)$.

Let $\rho(r)$ be a proximate order, then $\lim_{r \to \infty} \rho(r) = \rho \geq 0$. A single valued function $f(z)$ of the complex variable $z$ is said to belong to the space $[\rho(r), \mathcal{Y}]$ if

$$f(z) = \sum_{n=0}^{\infty} c_n z^n \in [\rho(r), \mathcal{Y}]$$

A sequence of functions $f_j(z)$ from $[\rho(r), \mathcal{Y}]$ converges in the sense of $[\rho(r), \mathcal{Y}]$ if

(i) it converges uniformly on compacts, (ii) for all $n \geq 1$ such that

$$|f_n(z)| < C(\beta) \exp[\beta V(z)] \text{ for all } z \in \mathbb{C}$$

where $\beta(\beta)$ does not depend on $n$. For a suitable $C(\beta)$, which does not depend on $n$, for all $z$ \n
$$|f_n(z)| < C(\beta) \exp[\beta V(z)] \quad (n \geq 1)$$

The space $[\rho(r), \mathcal{Y}]$ is the linear topology space with sequence topology. Furthermore, a single valued function $f(z)$ of the complex variable $z$ is said to belong to the space $[\rho(r), \mathcal{P}]$ if $f(z)$ has the order less than $\rho(r)$ or equal $\rho(r)$ but in this case type less than or equal $\rho$. A sequence of functions $[f_j(z)]$ from $[\rho(r), \mathcal{P}]$ converges in the sense of $[\rho(r), \mathcal{P}]$ if (i) it converges uniformly on compacts, (ii) for all $\varepsilon > 0$ there exists $r_\varepsilon(n)$ does not depend on $n$ such that

$$|f_n(z)| < \varepsilon |z|^{\rho(n)} \quad (n \geq 1)$$

The space $[\rho(r), \mathcal{P}]$ is also the linear topology space with sequence topology. We introduce the function $\varphi(t)$ defined to be the unique solution of the equation $t = V(r)$. So

$$\varphi(V(t)) = (3)$$

**Theorem 1.1 ([2, Theorem 2, p.42])**

The type of the entire function $f(z) = \sum_{n=0}^{\infty} c_n z^n$ with the proximate order $\rho(r) (\rho > 0)$ is given by the equation

$$\limsup_{n \to \infty} (\frac{c_n}{\varphi(n)})^n \leq 1$$

Let $\rho > 0$.

$$d_\rho = (\frac{\exp(\rho)}{\varphi(\rho)})^\rho n \geq 1, d_\rho = 1$$

For a function $f(z) = \sum_{n=0}^{\infty} c_n z^n \in [\rho(r), \mathcal{P}]$ we associate the function

$$f(z) = \sum_{n=0}^{\infty} b_n z^n \text{ where } b_n = \frac{c_n}{d_\rho} (n \geq 0)$$

It is regular, in any case in the circle $|z| < 1$ [1]. Fact mapping function $f(z)$ of $[\rho(r), \mathcal{P}]$ to the function $F(z)$ as indicated above will be celebrating a record $f(z) - F(z)$.

In [1] it is proved the following two theorems.

**Theorem 1.2**

In order to be a sequence $[f_j(z)]$ of functions from $[\rho(r), \mathcal{P}]$ to converge in the sense of $[\rho(r), \mathcal{P}]$ necessary and sufficient that the sequence $[f_j(z)]$ $(f_0(z) - F_0(z))$ converges uniformly inside the disk $|z| < 1$. 

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Theorem 1.3
Continuous linear functional \(l\) on the space \([\rho(r), p]\) has the form
\[
    l(f) = \sum_{n=0}^{\infty} a_n c_n^* f(z) = \sum_{n=0}^{\infty} c_n z^n,
\]
where the quantities \(a_n\) satisfy
\[
    \limsup_{n \to \infty} \varphi^{-1}(n) \sqrt[n]{|a_n|} = 0
\]
The following is our main result.

Theorem 1.4
Continuous linear functional \(l\) on the space \([\rho(r), F]\) has the form
\[
    l(f) = \sum_{n=0}^{\infty} a_n c_n^* f(z) = \sum_{n=0}^{\infty} c_n z^n,
\]
where the quantities \(a_n\) satisfy
\[
    \limsup_{n \to \infty} \varphi^{-1}(n) \sqrt[n]{|a_n|} = 0
\]

The Space of Entire Functions \([\rho(r), F]\)

We now prove the theorem 1.4. Let \(l(f)\) be a continuous linear functional on the space \([\rho(r), F]\). Set \(l(z^n) = a_n (n \geq 0)\). Let
\[
    f(z) = \sum_{n=0}^{\infty} c_n z^n
\]
as a function in \([\rho(r), F]\). Since the series converges in the sense of \([\rho(r), F]\), then, by continuous of the functional,
\[
    l(f) = \sum_{n=0}^{\infty} c_n l(z^n) = \sum_{n=0}^{\infty} c_n a_n
\]
Hence
\[
    l(f) = \sum_{n=0}^{\infty} a_n c_n^* f(z) = \sum_{n=0}^{\infty} c_n z^n
\]
(10)

Take an arbitrary finite \(p > 0\). Functional \(l(f)\) is, in particular, continuous linear functional on the space \([\rho(r), p]\). By theorem 1.3, the condition
\[
    \limsup_{n \to \infty} \varphi^{-1}(n) \sqrt[n]{|a_n|} < (e \rho p)^{-1/\rho}
\]
But \(p\) is arbitrary, hence,
\[
    \limsup_{n \to \infty} \varphi^{-1}(n) \sqrt[n]{|a_n|} = 0
\]
We now verify that if the condition (9) then the functional (10) is continuous linear functional on the space \([\rho(r), F]\). Let
\[
    f(z) = \sum_{n=0}^{\infty} c_n z^n = [\rho(r), F]
\]
by theorem 1.1, \(\limsup_{n \to \infty} \varphi^{-1}(n) \sqrt[n]{|a_n|} = (e \sigma \rho)^{-1/\rho} < \infty\).
Then
\[
    \limsup_{n \to \infty} \sqrt[n]{|a_n|} = \limsup_{n \to \infty} \varphi^{-1}(n) \sqrt[n]{|a_n|} = 0
\]
And then the series (10) converges.

Let
\[
    f_{k}(z) = \sum_{n=0}^{k} c_n z^n = [\rho(r), F]
\]
then \(f(z) = \sum_{n=0}^{\infty} c_n z^n\) if \(k \to \infty\) and
\[
    \lim_{n \to \infty} \varphi^{-1}(n) \sqrt[n]{|a_n|} = 0\]

Space of Entire Functions \(E_{\rho}(r)\)

We now consider the space of entire functions \(E_{\rho}(r)\) which have a proximate orders less then \(\rho(r)\). A proximate order \(\rho_{p}(r)\) less then \(\rho(r)\) if
\[
    0 < \lim_{r \to \infty} \rho_{p}(r) = \rho_{A} < \lim_{r \to \infty} \rho(r) = \rho_{A}
\]
A sequence of functions \([f(z)]\) from \(E_{\rho}\) converges in the sense of \(E_{\rho}\), if (i) it converges uniformly on compacts, (ii) there exists proximate order \(\rho_{p}(r)\), \(0 < \lim_{r \to \infty} \rho_{p}(r) = \rho_{A} < \lim_{r \to \infty} \rho_{p}(r) = \rho_{A}\) such that
\[
    | f_{r}(z) | = \exp(1/|z|), \quad |z| > r_{p}(\beta)(n \geq 1)
\]
where \(r_{p}(\beta)\) does not depend on \((n, 1)\). The space \(E_{\rho}\) is the linear topology space with sequence topology. A continuous linear functional \(l(f)\) on the space \(E_{\rho}(r)\), has the form (8). Let us find the conditions that satisfy the values \(a_n\). The functional \(l(f)\) is in particular continuous linear functional on the space \([\rho(r), F]\) for all proximate order \(\rho_{p}(r)\), \(0 < \lim_{r \to \infty} \rho_{p}(r) = \rho_{A} < \lim_{r \to \infty} \rho_{p}(r) = \rho_{A}\).

Therefore, by theorem
\[
    \limsup_{n \to \infty} \varphi^{-1}(n) \sqrt[n]{|a_n|} = 0
\]
where \(\varphi(t)\) be defined to be the unique solution of the equation \(t = V(r)\).

From this
\[
    \frac{\log |a_n|}{\varphi(n) n} < 1, \quad n > n_0
\]
So \(\rho_{p}(r)\) is arbitrary less then \(\rho(r)\) that
\[
    \limsup_{n \to \infty} \frac{\log |a_n|}{\varphi(n) n} \leq 1
\]
Contrary, let the condition (13) is true and \(\rho_{p}(r)\) is arbitrary less than \(\rho(r)\). So \(\varphi(n) > \varphi(n_0), n > n_0\), that
\[
    \frac{\log |a_n|}{\varphi(n) n} < 1, \quad n > n_0
\]
Therefor the condition (12) is true and \(l(f)\) is continuous linear functional on the space \([\rho(r), F]\). So \(\rho_{p}(r)\) is arbitrary less then, \(\rho(r)\) that \(l(f)\) is continuous linear functional on the space \(E_{\rho}\).

Theorem 3.1
Continuous linear functional \(l\) on the space \(E_{\rho}(r)\) has the form
\[
    l(f) = \sum_{n=0}^{\infty} a_n c_n^* f(z) = \sum_{n=0}^{\infty} c_n z^n
\]
where the quantities \(a_n\) satisfy
\[
    \limsup_{n \to \infty} \varphi^{-1}(n) \sqrt[n]{|a_n|} = 0
\]
Remark: The case of the spaces \([\rho, F]\) and \(E_{r}\), where \(\rho(r) = \rho > 0\), considered A.F Leon’ev [4].

Conclusion
The linear topology space of entire functions of a proximate order and normal type with respect to the proximate order is considered. We obtain the form of continuous linear functional on this space through our work.

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