Conformal Invariance in Periodic Quantum Chains

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Abstract

We show how conformal invariance predicts the functional form of two-point correlators in one-dimensional periodic quantum systems. Numerical evidence for this functional form in a wide class of models — including long-ranged ones — is given and it is shown how this may be used to significantly speed up calculations of critical exponents.

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Conformal invariance has been shown to be remarkably powerful in predicting the universal long-distance behavior of low-dimensional field theories and statistical systems both at the classical and the quantum level. Independent verification of conformal predictions by finite-size studies was made possible by Cardy’s use of conformal mappings to relate a problem in one geometry to one in a finite-sized geometry. The numerical success of this approach offers a convincing validation of the principle of conformal invariance at critical points.

The main ideas leading to the conformal finite-size approach in two-dimensions may be summarized as follows. Under a conformal mapping $z \mapsto w(z)$, the two-point correlation function of a scalar scaling operator $\phi(z)$ at a critical point will transform covariantly, i.e.

$$
\langle \phi(z_1, \bar{z}_1), \phi(z_2, \bar{z}_2) \rangle = |w'(z_1)|^x |w'(z_2)|^x \langle \phi(w_1, \bar{w}_1), \phi(w_2, \bar{w}_2) \rangle,
$$

where $x$ is the scaling dimension of $\phi$. A mapping of the entire $z$-plane onto the surface of a cylinder of circumference $L$ is achieved by choosing the particular conformal mapping

$$
w = \frac{L}{2\pi} \ln(z)
$$

We can then relate the two-point function of the scalar operator $\phi(z)$ in the infinite plane, to the two-point function of the operator $\phi(w)$, now evaluated in the cylinder geometry. Putting $w = u + iv$, so that $u$ measures distance along the infinite cylinder and $v$ is the periodic coordinate across, we may explicitly calculate the two-point function to be

$$
\langle \phi(u_1, v_1), \phi(u_2, v_2) \rangle = \frac{(2\pi/L)^2x}{[2 \cosh \frac{2\pi}{L}(u_1 - u_2) - 2 \cos \frac{2\pi}{L}(v_1 - v_2)]^x} (1)
$$

For large separation along the cylinder axis, i.e. assuming $u_1 - u_2 \gg L$, the correlator exhibits an exponential decay

$$
\langle \phi(u_1, v_1), \phi(u_2, v_2) \rangle = \left(\frac{2\pi}{L}\right)^2x e^{-\frac{2\pi x}{L}(u_1 - u_2)}
$$

and we may deduce the correlation length

$$
\xi = \frac{L}{2\pi x} (2)
$$
For one-dimensional quantum systems, we now reinterpret the coordinate \( u \) as running along the infinite chain of sites and the width \( L \) in the \( v \) direction as corresponding to the inverse temperature \( \beta \), i.e. the above reviewed 2D results may be applied also to an infinitely long quantum chain at finite temperature. This then opens the possibility of checking the predictions for the critical exponents as obtained by (2), by directly studying the long-distance behavior of the correlation functions.

Analytic results for correlation functions, however, can only be given for a rather select group of models and thus it is at this point that numerical studies and simulations become indispensable tools for the verification of conformal results. Various methods for the numerically challenging problem of calculating correlation functions have been devised over the years, but due to the complexity of the problem, computations are still limited to lattice sizes of about \( N < 100 \). The extrapolation of the critical exponents from these finite-size data usually involves taking the thermodynamic limit by holding the density \( d \) fixed as \( N \to \infty \), as a first step. However, the correlation functions still exhibit an oscillatory behavior. For the one-particle reduced density matrix \( \rho(r) \), this behavior leads to model-specific singularities in the momentum distribution \( n(k) \). Thus, the next step in extracting the long-distance behavior should be to let the average inter-site (or inter-particle) spacing \( 1/d \) go to zero. The oscillations will then be pushed towards the origin. (For small lattice sizes or calculations at fixed density, where this limit is not practical, one can instead consider a suitable averaging procedure, leading to equivalent results.)

There is, however, another interpretation of the two-point function (1). Most of the above numerical studies are performed not on a long chain with free boundary conditions, but on a ring of length \( N \) with periodic boundary conditions. Thus it seems natural, to use the periodic coordinate \( v \) to label the sites and the coordinate \( u \) to measure the time. In this picture, we just ‘roll’ the cylinder in a different way, i.e. we have a ring of length \( L = N \) and infinite extent in time. The equal-time correlator may then be constructed from (1) by choosing \( \Delta u = u_1 - u_2 = 0 \) and defining \( \Delta v = v_1 - v_2 \equiv r \). The two-point function of (1) now reads as
\[ \langle \phi(r), \phi(0) \rangle = \left( \frac{2\pi}{L} \right)^{2x} \left( \frac{1}{\sin \frac{\pi r}{L}} \right)^{2x}, \]  

(3)

where we have suppressed the \( u_1, u_2 \) arguments. Thus we arrive at a prediction, not only of the critical exponents of a given theory, but also of the analytic behavior of the correlation functions for a finite-sized system on a ring.

Before we present numerical evidence for the above functional form, let us examine in what limit this behavior is expected to occur. Because of the non-universal oscillations, we have true agreement only for \( 1/d \rightarrow 0 \). This limit can easily be achieved by plotting the two-point function as a function of \( r/L \) on a fixed scale from \(-1/2\) to \(+1/2\). We then expect the universal law (3) to dominate as \( L \rightarrow \infty \). (Note that for fermionic systems, we use the envelope of the two-point function.)

The consequences of (3) are impressive. First of all, we have a much stronger test of conformal invariance. After all, not only must the critical exponents agree, but also the complete analytic behavior. Let us now test our scaling law for the reduced one-particle density matrix \( \rho(r) \) and the spin-correlation \( \langle S_0 S_r \rangle \) in a variety of models. For the \( s = 1/2 \) Heisenberg antiferromagnet, it has been shown via bosonization techniques\(^\text{14}\) that the singular behavior of \( \langle S_0 S_r \rangle \), i.e.

\[ \langle S_0 S_r \rangle \propto (-1)^r r^{-\eta} \]  

(4)

is described by \( \eta = 1 \). Recently, long standing conjectures of possible logarithmic corrections have been verified by including marginal operators in the above reviewed conformal finite-size analysis\(^\text{15}\). In Fig.\(^1\) we show a comparison of recent Monte-Carlo simulations\(^\text{11}-\text{13}\) and the scaling law (3). Note that the lattice sizes are still too small to completely push the oscillations to the origin. The agreement is, however, quite good.

For hard-core bosons (the \( \delta \)-function gas of Ref.\(^\text{16}\)), the one-particle density matrix \( \rho(r) \) may be either deduced in terms of generalizations of the Painlevé transcendent\(^\text{9,8}\), by application of the quantum inverse scattering method\(^\text{17}\) using the equivalence with the non-linear Schrödinger equation, or by random matrix theory\(^\text{18}\). In all methods the critical
exponent is found to be $\eta = 1/2$. Data taken from Ref. 9 again shows good agreement with the above scaling law in Fig. 2.

Finally, the long-ranged $g/r^2$ models have been shown to have a set of critical exponents compatible with conformal predictions, both for their discrete versions such as the Haldane-Shastry model\textsuperscript{19–21} and for the continuum system\textsuperscript{22–25}. The critical exponents vary continuously with interaction strength $g$ and $\eta_\theta$ is independent of density $d$. In Fig. 3 we plot the results of Ref. 25 for $\rho(r)$ and the scaling curve corresponding to (3). Note that the $\eta_2 = 1$ curve corresponds to the Haldane-Shastry spin-chain. In addition, the continuum data coincides with the discrete data, the first being the low-density limit of the second.

Another important consequence following from (3) is that Monte-Carlo simulations to determine critical exponents can now be restricted to a single calculation for each lattice size $L$. This is done most conveniently by studying the behavior half way around the chain. So, using as an example the spin-correlator in the $s = 1/2$ antiferromagnet,

$$\langle S_0 S_{L/2} \rangle \propto \left( \frac{2\pi}{L} \right)^{\eta}.$$  (5)

Taking data from Kubo et al.\textsuperscript{12}, we show a plot of $\langle S_0 S_{L/2} \rangle$ versus $1/L$ in Fig. 4. A least-squares fit gives $\eta = 0.94787$ which is a reasonable fit for just 5 data points, especially in light of the logarithmic corrections. Note that the errorbars are estimates of the oscillation amplitudes and do not represent the statistical Monte-Carlo errors. Incidentally, this method has been used by the present authors to compute the critical exponents in the $g/r^2$ models to high accuracy.\textsuperscript{25}

Let us close this short note with a speculation. Using our interpretation of equation (1), we may compute the time correlators at a single site by choosing $u_1 - u_2 = t$, $v_1 - v_2 = 0$. Surpressing the $v_1, v_2$ arguments, the result is given by

$$\langle \phi(t), \phi(0) \rangle = \left( \frac{2\pi}{L} \right)^{2x} \left( \frac{1}{\sinh \frac{\pi t}{L}} \right)^{2x}.$$  

By choosing $v_1 - v_2 = L/2$, the same reasoning gives the large distance asymptotics of the time correlator.
\[ \langle \phi(t), \phi(0) \rangle = \left( \frac{2\pi}{L} \right)^{2x} \left( \frac{1}{\cosh \frac{\pi t}{L}} \right)^{2x}. \]

In Fig. 5 we plot the complete behavior of (1) for hard-core bosons in the \((t, r)\)-plane. It will be very interesting to compare these predictions to the calculations for the hard-core bose gas of Ref. 17 and for the \(\sigma\)-model of Ref. 20.
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FIGURES

FIG. 1. The scaled and normalized spin-correlator for $N = 24, 30, 40$. The line is the predicted scaling behavior for $\eta = 1$. Note that even the nearest-neighbor correlations are correctly predicted.

FIG. 2. The lattice version of the one-particle density matrix expansion $\rho(r)$ for hard core bosons is compared to data from a direct calculation.

FIG. 3. The scaled and normalized one-particle density function for the $g/r^2$ model is plotted with the predicted scaling curves. From left to right, the curves correspond to $\eta = 5/4$ ($g = -1/2$ fermionic, $g = 4$ fermionic), $\eta = 1$ ($g = 0$ fermionic, $g = 4$ bosonic), $\eta = 1/2$ ($g = 0$ bosonic), $\eta = 1/4$ ($g = -1/2$ bosonic) and are symmetric about the y-axis.

FIG. 4. We plot the spin-correlator halfway around the lattice, i.e. $\langle S_0 S_{N/2} \rangle$, as a function of the inverse lattice length $1/N$. Data for $N = 6, 12, 24, 32, 40$ is taken from the Monte-Carlo calculation of Kubo et al. The solid curve is a fit for the Luther-Peschel $r^{-\eta}$ result which gives the estimate $\eta = .94787$.

FIG. 5. The scaled correlation function of hard-core bosons ($\eta = 1/2$) in the $(t, z)$-plane. The value at $t = 0, r = L/2$ is normalized to 1.
