Elementary development of the gravitational self-force

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Abstract  The gravitational field of a particle of small mass $m$ moving through curved spacetime, with metric $g_{ab}$, is naturally and easily decomposed into two parts each of which satisfies the perturbed Einstein equations through $O(m)$. One part is an inhomogeneous field $h_{ab}^{S}$ which, near the particle, looks like the Coulomb $m/r$ field with tidal distortion from the local Riemann tensor. This singular field is defined in a neighborhood of the small particle and does not depend upon boundary conditions or upon the behavior of the source in either the past or the future. The other part is a homogeneous field $h_{ab}^{R}$. In a perturbative analysis, the motion of the particle is then best described as being a geodesic in the metric $g_{ab} + h_{ab}^{R}$. This geodesic motion includes all of the effects which might be called radiation reaction and conservative effects as well.

1 Introduction

Newton’s apple hangs in a tree. The force of gravity is balanced by the force from a branch, and the apple is at rest. Later, the apple falls and accelerates downward until it hits the ground.

Einstein’s insight elevates the lowly force of gravity to exalted status as a servant of geometry. Einstein’s apple, being sentient and hanging in a tree, explains its own non-geodesic, non free-fall, accelerated motion as being caused by the force it feels from the branch. When the apple is released by the branch, its subsequent free-fall motion is geodesic and not accelerated. The apple is freed from all forces and does not accelerate until it hits the ground.

These two perspectives have differing explanations and differing descriptions of the motion, but the actual paths through the events of spacetime are the same. New-
ton’s understanding that the gravitational mass is identical to the inertial mass implies that a small object in free-fall moves along a trajectory which is independent of the object’s mass. Einstein’s Equivalence Principle requires that a small object in free-fall moves along a geodesic of spacetime, a trajectory which is independent of the object’s mass. Newton’s free-fall motion and Einstein’s geodesic motion describe a small object as moving along one and the same sequence of events in spacetime.

Thorne and Hartle [11] give a clear and careful description of the motion of a small nearly-Newtonian object through the geometry of spacetime. They conclude that such motion might have a small acceleration, consistent with Newtonian analysis, from the coupling of the internal mass multipole moments of the object with the multipole moments of the external spacetime geometry, which are related to the components of the Riemann tensor in the vicinity of the object [cf. Eqs. (41)–(44)].

If the object orbits a large black hole, then the analysis implies that the motion is geodesic as long as any asphericity of the object, perhaps caused by rotation or tidal distortion, can be ignored. An acceleration larger than allowed by the coupling of the multipole moments is inexplicable in the context of either General Relativity or of Newtonian gravity and must necessarily result from some non-gravitational force.

How does the Thorne-Hartle description meld with the notion that Einstein’s apple orbits a black hole, emits gravitational waves, radiates away energy and angular momentum, and cannot then move along a geodesic of the black hole geometry? Radiation reaction is not a consequence of any asphericity of the apple. Does the apple move along a geodesic? Would the apple, being sentient, describe its own motion as free-fall?

For the moment consider the familiar electromagnetic radiation-reaction force on an accelerating charge $q$ as given below in Eq. (14). A notable feature is that the force is proportional to $q^2$. Consequently this force is often described as resulting from the charge $q$ interacting with its own electromagnetic field, and the force is called the electromagnetic self-force.

Similar language is used with gravitation, but in that case the force is proportional to $m^2$ and the resulting acceleration is proportional to $m$. In general terms, the gravitational self-force is said to be responsible for any aspect of motion that is proportional to the mass $m$ of the object at hand. Yet, with either Newtonian gravity or General Relativity, the motion of an object of small mass $m$ is independent of $m$. Gravitational self-force appears to be an oxymoron.

But, even Newtonian gravity contains a gravitational self-force. One might describe the motion of the Moon about the Earth as free-fall in the Earth’s gravitational field and conclude that

$$ma = m \left( \frac{2 \pi}{T} \right)^2 r = \frac{GMm}{r^2} \tag{1}$$

1 If the acceleration of gravity $g$ differs significantly across a large object, then the center of mass moves responding to some average, over the object, of $g$ which does not necessarily match a free-fall trajectory.
where \( r \) is the radius of the Moon’s orbit, so that the orbital period is

\[
T = \sqrt{\frac{4\pi^2 r^3}{GM}}
\]  

(2)

A more accurate description of the motion includes the influence of the Moon back on the Earth. Then the Moon is in free-fall in the Earth’s gravitational field while the Earth orbits their common center of mass. And the conclusion becomes

\[
m\left(\frac{2\pi}{T}\right)^2 r = \frac{GMm}{r^2(1 + m/M)^2}
\]

\[
T = \sqrt{\frac{4\pi^2 r^3}{GM(1 + m/M)}}.
\]  

(3)

The mass of the Moon has an influence on its own motion in Eq. (3), and this influence could be (although it rarely is) described as a consequence of the Newtonian gravitational self-force. Nevertheless, Newton’s law of gravity still implies that the Moon does not exert a net gravitational force on itself. The acceleration of the Moon is still properly lined up with the gradient of the Earth’s gravitational potential, and the Moon’s motion is described as free-fall or geodesic, depending upon whether one is Newton or Einstein.

To me it seems inappropriate to describe the presence of the \( m/M \) term in Eq. (3) as resulting from the interaction of the Moon with it’s own gravitational field. Rather, the \( m/M \) term arises because the Earth orbits the common center of mass of the Earth-Moon system.

The conundrum of radiation reaction as being consistent with geodesic motion can now be resolved. Einstein’s apple orbiting a black hole must move along a geodesic, but the geometry through which it moves is the black hole metric disturbed by the presence of the apple. Nevertheless, this disturbed metric is a vacuum solution of the Einstein equations in the neighborhood of the apple. If the motion were not geodesic, then the apple could not explain its own motion as being free-fall in a vacuum gravitational field. Such motion would violate Newton’s laws as well as Einstein’s Equivalence Principle.

Throughout this manuscript we focus on the self-force acting on small objects which are otherwise in unconstrained, free-fall motion—this includes the most interesting case of the two body problem in general relativity. This specifically excludes forced motion from, for example, a mass bouncing on the end of a spring. This restricted interest allows us in a general way to avoid many mathematical complications of Green’s functions in curved spacetime and to rely instead on a strongly intuitive perspective which may be backed up with detailed analysis.

The Newtonian self-force problem in this Introduction is expanded upon in Sect. 2 where it becomes clear that careful definitions of coordinates are difficult to come by, and that physics is best described in terms of precisely defined and physically measurable quantities.
Outline

In Sect. 3 we describe Dirac’s classical treatment of radiation reaction in the context of electricity and magnetism in a language which mimics our approach to the gravitational self-force and to an illustrative toy problem in Sect. 4.

Perturbation theory in General Relativity is described in Sect. 5.1, applied to locally inertial coordinates in Sect. 5.2, applied to a neighborhood around a point mass in Sect. 5.3, and used to describe a small object moving through spacetime in Sect. 5.4.

The gravitational self-force is described in Sect. 6, which includes discussions of the conservative and dissipative effects and of some different possible implementations of self-force analyses.

The important and yet very confusing issue of gauge freedom in perturbation theory is raised in Sect. 7. And an example of gauge confusion in action is given in Sect. 8.

An outline of the necessary steps in a self-force calculation is given in Sect. 9, and some recent examples of actual gravitational self-force results are in Sect. 10 and 10.1. Sect. 10.2 describes a possible future approach to self-force calculations which is amenable to a 3+1 numerical implementation in the style of numerical relativity.

Concluding remarks are in Sect. 11.

Notation

The notation matches that used in an earlier review by the author and is described here and again later in context.

Spacetime tensor indices are taken from the first third of the alphabet $a, b, \ldots, h$, indices which are purely spatial in character are taken from the middle third, $i, j, \ldots, q$ and indices from the last third $r, s, \ldots, z$ are associated with particular coordinate components. The operator $\nabla_a$ is the covariant derivative operator compatible with the metric at hand. We often use $x^i = (x, y, z)$ for the spatial coordinates, and $t$ for a timelike coordinate. An overdot, as in $\dot{E}^{ij}$, denotes a time derivative along a timelike worldline. The tensor $\eta_{ab}$ is the flat Minkowski metric $(-1, 1, 1, 1)$, down the diagonal. The tensor $f^{ij}$ is the flat, spatial Cartesian metric $(0, 1, 1, 1)$, down the diagonal. The projection operator onto the two dimensional surface of a constant $r$ two sphere is $\sigma^i = f^i_j - x_i x^j / r^2$. A capitalized index, $A, B, \ldots$ emphasizes that the index is spatial and tangent to such a two sphere. Thus when written as $\sigma_{AB}$ the projection operator is exhibiting its alternative role as the metric of the two-sphere. The tensor $\varepsilon_{ijk}$ is the spatial Levi-Civita tensor, which takes on values of $\pm 1$ depending upon whether the permutation of the indices are even or odd in comparison to $x, y, z$. A representative length scale $\mathcal{R}$ of the geometry in the region of interest in spacetime is the smallest of the radius of curvature, the scale of inhomogeneities, and the time scale for changes along a geodesic. Typically, if the region of interest is
a distance \( r \) away from a massive object \( M \), then \( \mathcal{R}^{-2} \sim M/r^3 \) provides a measure of tidal effects, and \( \mathcal{R} \sim \) an orbital period.

### 2 Newtonian examples of self-force and gauge issues

Newtonian gravity self-force effects appeared in the introduction. Why don’t we discuss these effects in undergraduate classical mechanics? The primary reason is that the Newtonian two-body problem can be solved easily and analytically without mention of the self-force. But in addition, a description of the Newtonian self-force introduces substantial, unavoidable ambiguities which are similar to the relativistic choice of gauge. Only because gauge confusion haunts all of perturbation theory in General Relativity do we now examine the Newtonian self-force using an elementary example made unavoidably confusing.

Consider a smaller mass \( m_1 \) and a larger mass \( m_2 \) in circular orbits of radii \( r_1 \) and \( r_2 \) about their common center of mass, so

\[
m_1 r_1 = m_2 r_2 .
\]  \(4\)

And their separation is

\[
R = r_1 + r_2 = r_1 (1 + m_1/m_2) .
\]  \(5\)

Newton’s law of gravity gives

\[
\frac{m_1 v_1^2}{r_1} = \frac{G m_1 m_2}{(r_1 + r_2)^2} .
\]  \(6\)

The velocity \( v_1 \) of the small object could be measurable by a redshift experiment. For this Newtonian system

\[
v_1^2 = \frac{G m_2 r_1}{r_1 (1 + m_1/m_2)^2} = \frac{G m_2}{r_1 (1 - 2m_1/m_2 + \ldots)} .
\]  \(7\)

Thus we could state that in the limit that \( m_1 \to 0 \), the gravitational self-force decreases the orbital speed \( v_1 \) by a fractional amount \(-m_1/m_2\). But, as an alternative, it is also true that

\[
v_1^2 = \frac{G m_2}{R (1 + m_1/m_2)} = \frac{G m_2}{R} (1 - m_1/m_2 + \ldots) .
\]  \(8\)
Thus we could equally well state that in the limit that $m_1 \to 0$, the gravitational self-force decreases the orbital speed $v_1$ by a fractional amount $-m_1/2m_2$. Which would be correct?

How does the ambiguity arise? In the first treatment, near by the orbit the radius $r_1$ was implicitly held fixed while we took the limit $m_1 \to 0$, and in that limit $R$ approaches $r_1$ from above. In the second treatment the separation $R$ was implicitly held fixed in the limit, and in that case $r_1$ approaches $R$ from below. Which of these is the “correct” way to take the limit? When viewed near by, which is a better description of the size of the orbit $r_1$ or $R$?

In this Newtonian situation there might be some specific reason to make one choice rather than the other and the confusion could be resolved by including the detail of which quantity is being held fixed during the limiting process. But, in General Relativity for a small mass $m_1$ orbiting a much more massive black hole $m_2$ the ambiguity persists. After including self-force effects on the motion of $m_1$, it would be tempting to state that the Schwarzschild coordinate $r$ of $m_1$’s location should be held fixed while $m_1 \to 0$ to reveal the true consequences of the gravitational self-force. However, only the spherical symmetry of the exact Schwarzschild geometry allows for the unambiguous definition of $r$. Whereas the actual perturbed geometry is not spherically symmetric and has no natural $r$ coordinate.

A clear statement of a perturbative gauge choice (cf Sect. 7) that fixes the gauge freedom can provide a mathematically well-defined quantity $r$ on the manifold. But physics has no preferred gauge and has no preferred choice for $r$, just as neither $r_1$ nor $R$ is preferred in this Newtonian example.

Rather than arguing the benefits of one gauge choice over another, it is far better to discard the focus on the radius $r_1$ or the separation $R$ of the orbit, and to consider only quantities that could be determined with clear, unambiguous physical measurements. The orbital frequency $\Omega$ could be determined from the periodicity of the system, and the speed of the less massive component $v_1$ could be measured via a Doppler shift. We now look for a relationship between these two physically measurable quantities.

From the Newtonian analysis above,

$$\Omega^2 = \frac{Gm_2}{r_1(r_1 + r_2)^2} = \frac{Gm_2}{r_1^3(1 + m_1/m_2)^2}$$

so that

$$r_1 = \left[ \frac{Gm_2}{(1 + m_1/m_2)^2} \right]^{1/3} \Omega^{-2/3}$$

and

$$r_1 = \left[ \frac{Gm_2}{(1 + m_1/m_2)^2} \right]^{1/3} \Omega^{-2/3}$$

(9)

and

(10)
Next, it seems appropriate to define a quantity with units of length in terms of the physically measurable $\Omega$, 

$$R_\Omega^3 = \frac{Gm_2}{\Omega^2}. \quad (12)$$

Now the velocity $v_1$ of the orbit and the orbital frequency $\Omega$ are related by 

$$v_1^2 = \frac{Gm_2}{R_\Omega} \left( 1 - \frac{4}{3} \frac{m_1}{m_2} + \ldots \right), \quad (13)$$

and in terms of these measurable quantities it is unambiguous to state that the gravitational self-force changes $v_1$, for a fixed $\Omega$ by a fractional amount $-\frac{2}{3} \frac{m_1}{m_2}$.

This describes the effect of the self-force on two physically measurable observables and thus qualifies as a true, unambiguous self-force effect.

### 3 Classical electromagnetic self-force

The standard expression \[4\] for the electromagnetic radiation reaction force on a charged particle $q$ is 

$$F_{\text{rad}} = \frac{2}{3} q \frac{c^2}{\ddot{v}}. \quad (14)$$

Equation (14) has issues of interpretation, but it does indeed describe the radiation reaction force when applied with care.

Dirac’s \[2\] derivation of Eq. (14) is my favorite and can be described in a way that blends rather well with my preferred description of the self-force and the toy problem described in the next section.

First, Dirac considers the causally interesting retarded electromagnetic field $F_{\text{ret}}^{ab}$ of an accelerating charge. But, he also considers the advanced field $F_{\text{adv}}^{ab}$ and then describes what I call the electromagnetic singular source $S$ field in flat spacetime 

$$F_{ab}^{S} = \frac{1}{2} (F_{ab}^{\text{ret}} + F_{ab}^{\text{adv}}). \quad (15)$$

The field $F_{ab}^{S}$ might also be called the symmetric field, as in “symmetric under reversal of causal structure.” $F_{ab}^{S}$ has unphysical causal features, but it is an exact solution to Maxwell’s equations with a source. In curved spacetime the definition of the singular source $S$ field is more complicated than in the flat-space version of Eq. (15).
Dirac next allows the charge $q$ to be of finite size. Then he presents a subtle analysis using the conservation of the electromagnetic stress-energy tensor in a neighborhood of the charge to show that $F_{ab}^{S}$ exerts no net force on the charge in the limit that the size of the charge is vanishingly small.

Now let $F_{ab}^{\text{act}}$ be the actual, measurable electromagnetic field. Then $F_{ab}^{\text{act}}$ may be separated into two parts

$$F_{ab}^{\text{act}} = F_{ab}^{S} + F_{ab}^{R}$$

where the remainder R-field is defined by

$$F_{ab}^{R} \equiv F_{ab}^{\text{act}} - F_{ab}^{S}.$$  \hspace{1cm} (17)

Both $F_{ab}^{\text{act}}$ and $F_{ab}^{S}$ are solutions to Maxwell’s equations, in the neighborhood of $q$, with identical sources. Thus $F_{ab}^{R}$ is necessarily a vacuum solution of the electromagnetic field equations and is therefore regular in the neighborhood of the particle.

Dirac then states that the radiation reaction force on the charge $q$ moving with four-velocity $u^a$ is

$$F_{b}^{\text{rad}} = qu^a F_{ab}^{R}$$  \hspace{1cm} (18)

and later shows that this is consistent with Eq. (14). In this context $F_{ab}^{R}$ might be called the radiation reaction field, in view of the force it exerts on the charge.

Imagine the situation as viewed by a local observer who moves with the particle and is able to measure and analyze the actual electromagnetic field only in a neighborhood which includes the particle but is substantially smaller than the wavelength of any radiation. The observer is therefore not privy to any information whatsoever about distant boundary conditions, or about the possible existence of electromagnetically active material outside the neighborhood or even about the possibility of electromagnetic radiation either ingoing or outgoing at a great distance.

After considering the motion of the charge, the observer could calculate $F_{ab}^{S}$ and then subtract it from the measured $F_{ab}^{\text{act}}$ to yield $F_{ab}^{R}$. Finally the observer could apply Eq. (18) and conclude that the Lorentz force law correctly describes the electromagnetic contribution to the acceleration of the charge, even though the observer might be completely unaware of the presence of the radiation.

Thus $F_{ab}^{\text{act}}$ is decomposed into two parts. One part $F_{ab}^{S}$ is singular at the point charge, can be identified as the particle’s own electromagnetic field, and exerts no force on the particle itself. The other part $F_{ab}^{R}$ does exert a force on the particle, is a locally source-free solution of Maxwell’s equations and can be locally identified only as an externally generated field of indeterminate origin. A local observer would have no direct information about the source of $F_{ab}^{R}$ and, in particular, could not distinguish the effects of radiation reaction from the effects of boundary conditions.
4 A toy problem with two length scales that creates a challenge for numerical analysis

Binary inspiral of a small black hole into a much larger one presents substantial difficulties to the numerical relativity community. Perhaps the primary difficulty results from having two very different length scales. On the one hand, a very coarse grid size would allow easy resolution of the metric of the large black hole as well as coverage out to the wavezone resulting in the efficient production of gravitational waveforms. On the other hand, a very fine grid size would provide the detailed information about the metric in a neighborhood of the small black hole necessary for tracking the evolution of the binary system and for providing accurate gravitational waveforms.

The following toy problem shares the two length-scale difficulty of binary inspiral. But it is elementary, not complicated by curved spacetime or subtle dynamics, and yet leads to some insight on how the binary inspiral problem might be approached. In addition, its resolution involves some aspects of Dirac’s analysis of electromagnetic radiation reaction as presented in the previous section.

Consider this flat space numerical analysis problem in electrostatics: An object of small radius \( r_0 \) has a spherically symmetric electric charge density \( \rho(r) \) with an associated electrostatic potential \( \phi \). The object is inside an odd shaped grounded, conducting box which is much larger than \( r_0 \). The boundary condition on the potential is that \( \phi = 0 \) on the box. For simplicity assume that the small object is at rest at the origin of coordinates. Thus, there is no radiation and the field equation for \( \phi \) is elliptic. Then

\[
\nabla^2 \phi = -4\pi \rho 
\]

where \( \nabla \) is the usual three-dimensional flat space gradient operator, and \( \nabla^2 \) the Laplace operator. Let \( r \) refer to the displacement from the center of the object at the origin to a general point in the domain within the box.

Here is the goal: Given \( \rho(r) \), numerically determine \( \phi \) as a function of \( r \) everywhere inside the box, subject to the field equation (19) and to the boundary condition that \( \phi = 0 \) on the boundary of the box. Then find the total force on the small object which results from its interaction with \( \phi \).

Here is the difficulty: If the object is much smaller than the box, then the difference in length scales complicates calculating \( \phi \). The object is very small so an accurate analysis would require a very fine grid size. However, the distance from the object to the boundary of the box is large compared to the size of object. Thus a relatively coarse grid size would be desired to speed up the numerical evaluation. The difficulty is exacerbated if we are also interested in the force from \( \phi \) acting back on the object; this requires accurately knowing the value of \( \phi \) inside the small object precisely where \( \phi \) has substantial variability.

We will shortly introduce a variety of versions of the potential under consideration. For clarity, the actual electrostatic potential \( \phi^{\text{act}} \) actually satisfies both the field equation (19) with the actual source and also the relevant boundary conditions.
Thus, $\phi^{\text{act}}$ is the potential which an observer would actually measure for the problem at hand.

### 4.1 An approach which avoids the small length scale

To remove the two-length-scale numerical difficulty we take the following approach: In a neighborhood of the object the potential ought to be approximated by the function $\phi^S$ defined as the usual electrostatic potential of a spherical distribution of charge which for a constant charge density $\rho(r)$ and total charge $q$ is

$$\phi^S(r) = \begin{cases} 
q/2r_0^3(3r_0^2 - r^2) & \text{for } r < r_0 \\
q/r & \text{for } r > r_0
\end{cases}$$

The source field $\phi^S(r)$ is completely determined by local considerations in the neighborhood of the object, and it is chosen carefully to be an elementary solution of

$$\nabla^2 \phi^S = -4\pi \rho.$$  \hfill (21)

Sometimes $\phi^S$ is called the singular field to emphasize the $q/r$ behavior outside but near a small source. Viewed from near by, the actual field $\phi^{\text{act}}$ is approximately $\phi^S$.

Given $\phi^S$, the numerical problem may be reformulated in terms of the field

$$\phi^R \equiv \phi^{\text{act}} - \phi^S$$

which is then a solution of

$$\nabla^2 \phi^R = -\nabla^2 \phi^S - 4\pi \rho = 0,$$  \hfill (22)

where the second equality follows from Eq. (21). The regular field $\phi^R$ is thus a source free solution of the field equation, and is sometimes called the remainder when the subtrahend $\phi^S$ is removed from the actual field $\phi^{\text{act}}$ in Eq. (22).

Viewed from afar, the boundary condition that $\phi^{\text{act}} = 0$ on the box plays an important role and determines the boundary condition that $\phi^R = -\phi^S$ on the box. Thus, rewriting the problem in terms of the analytically known $\phi^S$ and the “to be determined numerically” $\phi^R$ leaves us with the boundary value problem

$$\nabla^2 \phi^R = 0 \quad \text{with the boundary condition that } \phi^R = -\phi^S \text{ on the box.}$$  \hfill (24)

It is important to note that $\phi^R$ is a regular, source-free solution of the field equation.

In this formulation based upon Eq. (24) $\phi^R$ scales as the charge $q$ but has no structure with the length scale of the source $r_0$. The small length scale has been completely removed from the problem. The removal is at the expense of introducing a complicated boundary condition—but at least the boundary condition does not
have an associated small length scale. Once $\phi^R$ has been determined, the actual field $\phi^{\text{act}} = \phi^R + \phi^S$ is easily constructed.

But that’s not all: This formulation has the bonus that it simplifies the calculation of the force on the object from the field. The force is an integral over the volume of the object,

$$ F = -\int \rho(r) \nabla \phi^{\text{act}} \, d^3x. \quad (25) $$

In the original formulation using Eq. (19), the actual field $\phi^{\text{act}}$ in the integral would be dominated by $\phi^S$ which changes dramatically over the length scale of the object, and $\phi^R$ could be easily lost in the noise of the computation. The spherical symmetry of $\phi^S$ and $\rho$ imply that

$$ \int \rho(r) \nabla \phi^S \, d^3x = 0. \quad (26) $$

Then the substitution $\phi^{\text{act}} \rightarrow \phi^S + \phi^R$ in the integral of Eq. (25) leads to the conclusion that

$$ F = -\int \rho(r) \nabla \phi^R \, d^3x. \quad (27) $$

Thus the force acting on the object may be written in terms of only $\phi^R$.

But that’s not all: The field $\phi^R$ does not change significantly over a small length scale, so if the object is extremely small (Think: an approximation to a $\delta$-function.) then an accurate approximation to the force is

$$ F = -q \nabla \phi^R|_{r=0}, \quad (28) $$

when viewed from near by.

Standard jargon calls the force in Eq. (28) the “self-force” because it is necessarily proportional to $q^2$ and apparently results from the object interacting with “its own field.” But, it is important to note that this force clearly depends upon the shape of the box, i.e. the details of the boundary conditions. In my opinion the physics appears more intuitive to have “the object’s own field,” refer only to $\phi^S$ whose local behavior is defined uniquely and independently of any boundary conditions. And $\phi^S$ is also guaranteed to exert no force back on the charge. Then $\phi^R$ is a regular source-free solution to the field equation in the neighborhood of the object and is solely responsible for the force acting on the object. An observer local to the object would know $\rho(r)$, could calculate $\phi^S$ and measure $\phi^{\text{act}}$. Subtracting $\phi^S$ from the actual field $\phi^{\text{act}}$ then results in the regular remainder $\phi^R = \phi^{\text{act}} - \phi^S$. While the force described in Eq. (28) is indeed proportional to $q^2$, it still seems sensible to refer to this as simply “the force” on the object.

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2 Following Dirac’s usage, I prefer to use the word “actual” to refer to the complete, and total field that might be measured at some location. Often in self-force treatises the “retarded field” plays this central role. But, this obscures the fact that, viewed from near by, a local observer unaware of boundary conditions could make no measurement which would reveal just what part of the field is the retarded field. This confusion is increased if the spacetime is not flat, so that the retarded field could be determined only if the entire spacetime geometry were known.
4.2 An alternative that resolves boundary condition issues

The previous resolution of the difficulty of the two length scales caused a change and complication of the boundary conditions. With a slight variation, the problem can be reformulated in a way that brings back the original, natural boundary conditions.

The alternative approach deals with the boundary condition complication by introducing a window function \( W(r) \) which has three properties:

A. \( W(r) = 1 \) in a region which includes at least the entire source \( \rho(r) \), that is all \( r \leq r_0 \).

B. \( W(r) = 0 \) for \( r > r_W \) where \( r_W \) is generally much larger than \( r_0 \) but is restricted so that the entire region \( r < r_W \) is inside the box.

C. \( W(r) \) is \( C^\infty \) and changes only over a long length scale comparable to \( r_W \).

For this alternative approach the field defined by

\[
\Phi^R \equiv \phi^{\text{act}} - W \phi^S
\]

is a solution of

\[
\nabla^2 \Phi^R = -\nabla^2 (W \phi^S) - 4\pi \rho \\
= -\phi^S \nabla^2 W - 2 \nabla W \cdot \nabla \phi^S - W \nabla^2 \phi^S - 4\pi \rho \\
= -\phi^S \nabla^2 W - 2 \nabla W \cdot \nabla \phi^S \equiv S_{\text{eff}},
\]

where \( S_{\text{eff}} \) is the effective source and the third equality follows from Eq. (21) and property (A). The boundary condition is now that \( \Phi^R = 0 \) on the box, which is the natural boundary condition. Thus, rewriting the problem in terms of the analytically known \( \phi^S \) and the to-be-determined-numerically \( \Phi^R \) leaves us with the field equation

\[
\nabla^2 \Phi^R = S_{\text{eff}}
\]

and the natural boundary condition that \( \Phi^R = 0 \) on the box.

It is important to note that the effective source \( S_{\text{eff}} \) defined in Eq. (30) is zero inside the small object where \( W(r) = 1 \) and changes only over a long length scale \( r_W \). Thus the field \( \Phi^R \) is a regular, source-free solution of the field equation inside the object, and outside the object \( \Phi^R \) only changes over a long length scale \( r_W \). And Eqs. (27) and (28) provide the force acting on the object, after \( \phi^R \) is replaced with \( \Phi^R \).

This alternative approach completely removes the small length scale from the problem and leaves the natural boundary condition \( \Phi^R = 0 \) on the box intact.

In applications of this approach to problems in curved spacetime, the singular field \( \phi^S \) is rarely known exactly. In fact, for a \( \delta \)-function source often only a finite number of terms in an asymptotic expansion are available. This limits the differentiability of the source of Eq. (31) which, in turn, limits the differentiability of \( \Phi^R \) at the particle. But the procedure remains quite adequate for solving self-force problems.
This approach to the self-force, which introduces a window function, has now been implemented for a scalar charge in a circular orbit of the Schwarzschild geometry and is discussed below in Sect. 10.2.

5 Perturbation theory

Perturbation theory has had some great successes in General Relativity particularly in the realm of black holes [7, 8, 9, 10] by proving stability [7, 8, 11, 12], analyzing the quasi-normal modes, [13, 14, 15, 16, 17] and calculating the gravitational waves from objects falling in and around black holes [5, 18, 19, 20, 21] to highlight just a few of the earlier accomplishments.

In preparation for the era of gravitational wave astronomy, relativists are now turning their attention to second and higher order perturbation analysis. However, we focus on linear order and give a brief description of this theory.

In Sect. 5.1 we begin with an overview that emphasizes the Bianchi identity’s implication that a perturbing stress-energy tensor $T_{ab}$ must be conserved $\nabla^a T_{ab} = 0$ to have a well formulated perturbation problem. This requires that an object of small size and mass must move along a geodesic.

We use perturbation theory in Sect. 5.2 to describe the geometry in the vicinity of a timelike geodesic $\Gamma$ of a vacuum spacetime. We specifically use a locally inertial and harmonic coordinate system, THZ coordinates introduced by Thorne and Hartle [1], to represent the metric as a perturbation of flat spacetime $g_{ab} = \eta_{ab} + H_{ab}$ in a particularly convenient manner within a neighborhood of the geodesic.

In Sect. 5.3 we put a small mass $m$ down on this same geodesic $\Gamma$ and treat its gravitational field $h_{ab}^S$ as a perturbation of $g_{ab}$.

Finally, in Sect. 5.4, we identify $h_{ab}^S$ as the $S$-field of $m$, the analogue of $F_{ab}^S$ in Sect. 3 and of $\phi^S$ in Sect. 4. In particular $h_{ab}^S$ is a metric perturbation which is singular at the location of $m$, is a solution of the field equation for a $\delta$-function point mass and exerts no force back on the mass $m$ itself.

5.1 Standard perturbation theory in General Relativity

We start with a spacetime metric $g_{ab}$ which is a vacuum solution of the Einstein equations $G_{ab}(g) = 0$. Then we ask, “What is the slight perturbation $h_{ab}$ of the metric created by a small object moving through the spacetime along some worldline $\Gamma$?”

Let $\mathcal{R}$ be a representative length scale of the geometry near the object which is the smallest of the radius of curvature, the scale of inhomogeneities, and the time scale for changes in curvature along the world line of the object. When we say “small object” we imply that the size $d$ of the object is much less than $\mathcal{R}$ and that the mass $m$ is much smaller than $d$. 

As a notational convenience, the Einstein tensor $G_{ab}(g + h)$ for a perturbed metric may be expanded in powers of $h$ as

$$G(g + h) = G(g) + G^{(1)}(g, h) + G^{(2)}(g, h) + \ldots \quad (32)$$

where $G^{(n)}(g, h) = O(h^n)$. The zeroth order term $G(g)$ is zero if $g_{ab}$ is a vacuum solution of the Einstein equations. The first order part is $G^{(1)}(g, h)$, which resembles a linear wave operator on $h_{ab}$ and is equivalent to the operator $-E_{ab}(h)$ given below in Eq. (35). The second order part $G^{(2)}(g, h)$ consists of terms such as “$\nabla h \nabla h$” or “$h \nabla^2 h$,” similar to the Landau-Lifshitz pseudo tensor [22]. The third and higher order terms in the expansion (32) are less familiar.

Next, we assume that the stress-energy tensor of the object $T_{ab}$ is $O(m)$, and that the perturbation in the metric $h_{ab}$ is also $O(m)$. At first perturbative order,

$$G_{ab}(g + h) = 8\pi T_{ab} + O(h^2). \quad (33)$$

We expand $G_{ab}(g + h)$ through first order in $h$ via the symbolic operation

$$G^{(1)}_{ab}(g, h) = \frac{\delta G_{ab}}{\delta g_{cd}} h^{cd} \quad (34)$$

and define the wave operator mentioned above by $E_{ab}(h) \equiv -G^{(1)}_{ab}(g, h)$, so that

$$2E_{ab}(h) = \nabla^2 h_{ab} + \nabla_a \nabla_b h - 2\nabla(\nabla_c h_{bp})c + 2R_{a}^{\ c \ d} h_{cd} + g_{ab}(\nabla^2 h_{cd} - \nabla^2 h), \quad (35)$$

with $h \equiv h_{ab}g^{ab}$. Also $\nabla_a$ and $R_{a}^{\ c \ d}$ are the derivative operator and Riemann tensor of $g_{ab}$. If $h_{ab}$ solves

$$E_{ab}(h) = -8\pi T_{ab}, \quad (36)$$

then Eq. (33) is satisfied.

In an actual project, the biggest technical task is usually solving Eq. (36). As an example, the study of gravitational radiation from an object orbiting a Schwarzschild black hole typically invokes the Regge-Wheeler-Zerilli formalism [7, 8].

With a vacuum-spacetime metric $g_{ab}$ and any symmetric tensor $k_{ab}$, the Bianchi identity implies that

$$\nabla^a E_{ab}(k) = 0. \quad (37)$$

This is easily demonstrated by direct analysis, after starting with Eq. (35). Thus, for a solution of Eq. (36) to exist, it is necessary that the integrability condition

$$\nabla^a T_{ab} = 0 \quad (38)$$

for the stress-energy tensor be satisfied.

If the stress-energy tensor is only approximately conserved $\nabla^a T_{ab} = O(m^2)$ then the solution for $h_{ab}$ might be in error at $O(m^2)$. In some circumstances this might be acceptable, in which case if $T_{ab}$ represents the stress-energy tensor for a particle of
small size, then the particle must move along an approximate geodesic of $g_{ab}$ with an acceleration no larger than $O(m)$. Then the integrability condition is nearly satisfied and $h_{ab}$ can be determined from Eq. (36).

Next, one might be inclined to attempt the analysis of the Einstein equations through second order in the perturbation $h_{ab}$. But, this requires that $T_{ab}$ be conserved, not in the metric $g_{ab}$, but rather in the first order perturbed metric $g_{ab} + h_{ab}$. Thus the worldline of a particle is not geodesic in $g_{ab}$ and its acceleration as measured in $g_{ab}$ is often said to result from the gravitational self-force. After the self-force problem is solved for the $O(m)$ adjustment to the motion of the particle, then the second order field equation from Eq. (32) determines $h_{ab}$ through $O(m^2)$.

As described by Thorne and Kovács [24], this process continues: With the improved metric, solve the dynamical equations for a more accurate worldline and stress-energy tensor. With the improved stress-energy tensor solve the field equations for a more accurate metric perturbation. Repeat.

This alternation of focus between the dynamical equations and the field equations is quite similar to that used in post-Newtonian analyses.

5.2 An application of perturbation theory: locally inertial coordinates

Before dealing with perturbing masses, we first consider vacuum perturbations of a vacuum spacetime and focus on a neighborhood of a timelike geodesic $\Gamma$ where the metric appears as a perturbation $H_{ab}$ of the flat Minkowskii metric $\eta_{ab}$.

This application is simplified by use of a convenient coordinate system described by Thorne and Hartle [1]. It is well known in General Relativity [25], that for a timelike geodesic $\Gamma$ in spacetime there is a class of locally inertial coordinate systems $x^a = (t,x,y,z)$, with $r^2 = x^2 + y^2 + z^2$, which satisfies the following conditions:

A. The geodesic $\Gamma$ is identified with $x = y = z = r = 0$ and $t$ measures the proper time along the worldline.

B. On $\Gamma$, the metric takes the Minkowskii form $g_{ab} = \eta_{ab}$.

C. All first derivatives of $g_{ab}$ vanish on $\Gamma$ so that the Christoffel symbols also vanish on $\Gamma$.

Fermi-normal coordinates [26] provide an example which meets all of these locally inertial criteria.

With a locally inertial coordinate system in hand, it is natural to Taylor expand $g_{ab}$ about $\Gamma$ with

$$g_{ab} = \eta_{ab} + H_{ab} + \ldots$$

where
\( H_{ab} = 2H_{ab} + 3H_{ab} \),
\( 2H_{ab} = \frac{1}{2} x^i x^j \partial_i \partial_j g_{ab} \),
\( 3H_{ab} = \frac{1}{6} x^i x^j x^k \partial_i \partial_j \partial_k g_{ab} \),
(40)

and the partial derivatives are evaluated on \( \Gamma \).

The quantities \( 2H_{ab} \) and \( 3H_{ab} \) scale as \( O\left(\frac{r^2}{R^2}\right) \) and \( O\left(\frac{r^3}{R^3}\right) \) in a small neighborhood of \( \Gamma \), and these may be treated as perturbations of flat spacetime with \( r/R \) being the small parameter. Recall that \( R \) is a length scale of the background geometry. First order perturbation theory is applicable here because \( H_{ab} \) has no \( O\left(\frac{r}{R}\right) \) term but starts at \( O\left(\frac{r^2}{R^2}\right) \). Thus \( 2H_{ab} \) and \( 3H_{ab} \) may be treated as independent perturbations and the first nonlinear term appears at \( O\left(\frac{r^4}{R^4}\right) \). Thus, \( H_{ab} \) is a perturbation which must satisfy the source-free perturbed Einstein equations \( E_{ab}(H) = 0 \).

Thorne and Hartle [1] and Zhang [27] show that a particular choice of locally inertial coordinates leads to a relatively simple expansion of the metric. Initially they introduce spatial, symmetric, trace-free multipole moments of the external spacetime \( \mathcal{E}_{ij}, \mathcal{B}_{ij}, \mathcal{E}_{ijk}, \text{and} \mathcal{B}_{ijk} \) which are functions only of \( t \) and are directly related to the Riemann tensor evaluated on \( \Gamma \) by

\[ \mathcal{E}_{ij} = R_{itj}, \]
(41)
\[ \mathcal{B}_{ij} = \frac{1}{2} \varepsilon_{pqij} R_{pqjt}/2, \]
(42)
\[ \mathcal{E}_{ijk} = \left[ \partial_k R_{itj} \right]^{\text{STF}}, \]
(43)

and
\[ \mathcal{B}_{ijk} = \frac{3}{8} \left[ \varepsilon_{pqij} \partial_k R_{pqjt} \right]^{\text{STF}}. \]
(44)

Here \( ^{\text{STF}} \) means to take the symmetric, tracefree part with respect to the spatial indices, and \( \varepsilon_{ij} \) is the flat, spatial Levi-Civita tensor, which takes on values of \( \pm 1 \) depending upon whether the permutation of the indices are even or odd in comparison to \( x, y, z \). Also, \( \mathcal{E}_{ij} \) and \( \mathcal{B}_{ij} \) are \( O\left(1/R^2\right) \), while \( \mathcal{E}_{ijk} \) and \( \mathcal{B}_{ijk} \) are \( O\left(1/R^3\right) \). All of the above multipole moments are tracefree because the external background geometry is assumed to be a vacuum solution of the Einstein equations.

Spatial STF tensors are closely related to linear combinations of spherical harmonics. For example the STF tensor \( \mathcal{E}_{ij} \) with two spatial indices is related to the \( \ell = 2 \) spherical harmonics \( Y_{2,m} \) by

\[ \mathcal{E}_{ij} x^i x^j = r^2 \sum_{m=-2}^{2} E_{2,m} Y_{2,m}, \]
(45)

with the five independent components of \( \mathcal{E}_{ij} \) being determined by the five independent coefficients \( E_{2,m} \).

Next an infinitesimal coordinate transformation (a perturbative gauge transformation, Sect.1) changes the description of \( H_{ab} \) to a form where the partial deriva-
tives in the Taylor expansion are equivalent to the components of the Riemann tensor and represented by the multipole moments. The result is

\[ 2H_{ab}dx^a dx^b = -\partial_{ij}x^i (d^2 + f_{kl} dx^k dx^l) + \frac{4}{3} \varepsilon_{kpq} \mathcal{R}^{k}_{l} x^p x^l dx^k \]

\[- \frac{20}{21} \left[ \partial_{ij}x^i x^j x^k - \frac{2}{7} r^2 \partial_{ij}x^j \right] dx^k \]

\[ + \frac{5}{21} \left[ x_i \varepsilon_{ipq} \mathcal{R}^{i}_{j} x^p x^k - \frac{1}{5} r^2 \varepsilon_{pqij} \mathcal{R}^{j}_{i} x^p \right] dx^j dx^k + O(r^4/\mathcal{R}^4) \] (46)

and

\[ 3H_{ab}dx^a dx^b = -\frac{1}{3} \partial_{ij}x^i x^j (d^2 + f_{lm} dx^l dx^m) \]

\[ + \frac{2}{3} \varepsilon_{kpq} \mathcal{R}^{k}_{ij} x^p x^l dx^k + O(r^4/\mathcal{R}^4), \] (47)

where \( f_{kl} \) is the flat, spatial Cartesian metric \((0, 1, 1, 1)\), down the diagonal. The overdot represents a time derivative along \( \Gamma \) of, say, \( \partial_{ij} = O(\mathcal{R}^{-2}) \), and then \( \partial_{ij} = O(\mathcal{R}^{-3}) \) because \( \mathcal{R} \) bounds the time scale for variation along \( \Gamma \).

A straightforward evaluation of the Riemann tensor for the metric \( \eta_{ab} + 2H_{ab} + 3H_{ab} \) confirms that the STF multipole moments are related to the Riemann tensor as claimed in Eqs. (41)–(44).

We call the locally inertial coordinates of Thorne, Hartle and Zhang used in Eqs. (46) and (47) THZ coordinates.

If interest is focused only on the lower orders \( O(r^2/\mathcal{R}^2) \) and \( O(r^3/\mathcal{R}^3) \), then THZ coordinates are not unique and freedom is allowed in their construction away from the worldline \( \Gamma \). Given one set of THZ coordinates \( x^i \), a new set defined from \( x^i_{new} = x^i + \lambda^i_{ijklm} x^j x^k x^l x^m \), where \( \lambda^i_{ijklm} = O(1/\mathcal{R}^4) \) is an arbitrary function of proper time on \( \Gamma \), preserves the defining form of the expansion given in Eqs. (46) and (47).

Work in preparation describes a direct, constructive procedure for finding a THZ coordinate system associated with any geodesic of a vacuum solution of the Einstein equations.

5.3 Metric perturbations in the neighborhood of a point mass.

We are now prepared to use perturbation theory to determine \( h^S_{ab} \), the gravitational analogue of \( F^S_{ab} \) in Sect. 3 and of \( \varphi^S \) in Sect. 4.

We consider the perturbative change \( h_{ab} \) in the metric \( g_{ab} \) caused by a point mass \( m \) traveling through spacetime. We look for the solution \( h_{ab} \) to Eq. (56) with the stress-energy tensor \( T^{ab} \) of a point mass

\[ T^{ab} = m \int_{-\infty}^{m} \frac{dt^a dt^b}{\sqrt{-g}} \delta^4(x^a - X^a(s)) ds \] (48)
where $X^\mu(s)$ describes the worldline of $m$ in an arbitrary coordinate system as a function of the proper time $s$ along the worldline.

The integrability condition for Eq. (36) requires the conservation of $T^{ab}$, and we put $m$ down on the geodesic $\Gamma$ of the previous section and limit interest to a neighborhood of $\Gamma$ where $r^4/\mathscr{R}^4$ is considered negligible although $r^3/\mathscr{R}^3$ is not. And we use THZ coordinates. The perturbed metric of Sect. 5.2 is now viewed as the “background” metric, $g_{ab} = \eta_{ab} + H_{ab}$, with $H_{ab}$ given in Eqs. (46) and (47). The stress-energy tensor $T_{ab}$ for a point mass is particularly simple in THZ coordinates and has only one nonzero component

$$T_{tt} = -m\delta^3(x').$$  (49)

For this stress-energy tensor and this background metric, we call the solution to Eq. (36) $h_S^{ab}$, for reasons explained below, and its derivation is given elsewhere [28, 3]. Here we present the results:

$$h_S^{ab} = 0h_{ab}^S + 2h_{ab}^S + 3h_{ab}^S,$$  (50)

where

$$0h_{ab}^S dx^a dx^b = 2\frac{m}{r}(dr^2 + dr^2)$$  (51)

is the Coulomb $m/r$ part of the Schwarzschild metric, and

$$2h_{ab}^S dx^a dx^b = \frac{4m}{r} \delta_{ij} x^i x^j dr^2 - 2\frac{4mr}{3} \epsilon_{pqrs} x^p x^q dr dx^k$$

$$+ \hat{\delta}_{ij} \text{ and } \hat{\mathcal{B}}_{ij} \text{ terms}$$  (52)

are the quadrupole tidal distortions of the Coulomb part. The terms involving $\hat{\delta}_{ij}$ and $\hat{\mathcal{B}}_{ij}$ are more complicated and are not given here. The octupole tidal distortions of the Coulomb field are

$$3h_{ab}^S dx^a dx^b = \frac{m}{3r} \delta_{ijk} x^i x^j x^k \left[ 5dr^2 + d\sigma^2 + 2\sigma_{AB} dx^4 dx^B \right]$$

$$- 2\frac{10m}{9r} \epsilon_{pqrs} x^p x^q x^r dr dx^k.$$  (53)

Recall that $\sigma_{AB}$ is the two dimensional metric on the surface of a constant $r$ two sphere.

The perturbation $h_S^{ab}$ is a solution to Eq. (36) only in a neighborhood of $\Gamma$. The next perturbative-order terms which are not included in $h_S^{ab}$ scale as $mr^3/\mathscr{R}^4$. The operator $E_{ab}$ involves second derivatives, and it follows that for $h_S^{ab}$ given above

$$E_{ab}(h^S) = -8\pi T_{ab} + O(mr/\mathscr{R}^4).$$  (54)

In some circumstances we might wish to introduce a window function $W$ similar to that described in Sect. 4.2 which would multiply all of the terms on the right hand side of Eq. (50). If so, the window function near by $m$ must be restricted by
the condition that
\[ W = 1 + O(\rho^4/R^4) \]  
(55)
in order to preserve the delicate features of \( h_{ab}^S \) in a neighborhood of \( m \), especially the property revealed in Eq. (54). Away from \( m \), it is only necessary that \( W \) vanish in some smooth manner.

The perturbations \( 2h_{ab}^S \) and \( 3h_{ab}^S \) should not be confused with a consequence of Newtonian tides. When a small Newtonian object moves through spacetime, its mass distribution is tidally distorted by the external gravitational field. The extent of this distortion depends upon the size \( d \) of the object itself. For a self-gravitating, non-rotating incompressible fluid, the quadrupole distortion of the matter leads to a change in the Newtonian gravitational potential outside the object which scales as \( \delta U \sim \mathcal{I}_{ij}x^ix^j/r^3 \sim d^5/r^3R^2 \), where \( \mathcal{I}_{ij} \) is the mass quadrupole moment tensor. Such behavior is not at all similar to that of \( 2h_{tt}^S = O(m\rho/R^2) \), and \( 3h_{tt}^S = O(m\rho^2/R^3) \).

The quadrupole distortion revealed in \( 2h_{tt}^S \) is not a consequence of a distortion of the object \( m \) itself, but rather results from the curvature of spacetime acting on the monopole field of \( m \) and has no Newtonian counterpart.

### 5.4 A small object moving through spacetime

As a concrete example we now focus on a small Newtonian object of mass \( m \) and characteristic size \( d \) moving through some given external vacuum spacetime with metric \( g_{ab} \). Naturally, \( m \) is approximately moving along a geodesic \( \Gamma \), and \( g_{ab} \) has a characteristic length and time scale \( \mathcal{R} \) associated with \( \Gamma \). We assume that \( m \) and \( d \) are both much smaller than \( \mathcal{R} \).

In a region comparable to \( d \), the object appears Newtonian, and its gravitational potential can be determined. The structure of the object depends upon details like the density, type of matter, amount of rotation and whether it is stationary or oscillating.

The Newtonian object might have a mass quadrupole moment \( \mathcal{I}_{ij} = O(md^2) \) perhaps sustained by internal stresses in the matter itself. Independent of the cause of the quadrupole moment, the external Newtonian gravitational potential would have a quadrupole part \( \mathcal{I}_{ij}x^ix^j/r^5 \).

The coupling between a mass quadrupole moment of the small object and an external octupole gravitational field \( \varepsilon_{ijk}x^ix^jx^k \) results in the small acceleration of the center of mass, away from free-fall, given by
\[ a_i = \frac{1}{2m} \varepsilon^{ijk} \mathcal{I}_{jk} \]  
(56)
in either the context of Newtonian physics or of General Relativity. This tidal acceleration scales as

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3 A terse but adequate description of perturbative tidal effects on a Newtonian, self-gravitating, non-rotating, incompressible fluid is given on p. 467 of [29].
If our small Newtonian object is actually a nonrotating fluid body then it would naturally be spherically symmetric except for distortion caused by an external tidal field such as $\mathcal{E}_{ij}x^ix^j$. In that case $\mathcal{I}_{ij} = O(d^2/R^2)$ as discussed at the end of Sect. 5.3 [29], and the tidal acceleration then scales as

$$a = O(d^5/mR^5).$$

(58)

We conclude that a Newtonian object in free motion is only allowed an acceleration away from free-fall which is limited as in Eqs. (57) or (58). Any larger acceleration must involve some non-gravitational force.

It is also possible to analyze the situation if we replace the Newtonian object with a small Schwarzschild black hole of mass $m$. In that case it is easiest to turn the perturbation problem inside-out and to consider the Schwarzschild metric as the background with the metric perturbation being caused by $H_{ab}$ given in Eqs. (46) and (47). One boundary condition is that $h_{ab}$ approach $H_{ab}$ for $m \ll r \ll R$. The boundary condition at the event horizon is that $h_{ab}$ be an ingoing wave, or well-behaved in the time independent limit. The time independent problem is well studied; historically in Refs. [7, 8], more recently in the present context in Ref. [28], and with slow time dependence in Refs. [31, 32].

In the time independent limit, the generic quadrupole perturbation of the metric of the Schwarzschild spacetime results in

$$(g^{\text{Schw}}_{ab} + h^{\text{Schw}}_{ab}) \, dx^a dx^b = - \left(1 - \frac{2m}{r}\right) \left[1 - \mathcal{E}_{ij}x^ix^j \left(1 - \frac{2m}{r}\right)\right] \, dr^2 + \frac{4}{3} \mathcal{E}_{kpq} \mathcal{B}^l_{pq} x^k \left(1 - \frac{2m}{r}\right) \, dt \, dx^l + \left(1 - \frac{1}{1 - \frac{2m}{r}} - \mathcal{E}_{ij}x^i x^j\right) \, dr^2 + \left(r^2 - (r^2 - m^2) \mathcal{E}_{ij}x^i x^j\right) \left(d\theta^2 + \sin^2 \theta \, d\phi^2\right).$$

(59)

In this expression $x^i$ represents $x, y$ and $z$ which are related to $r, \theta$ and $\phi$ in the usual way in Cartesian space.

It is elementary to check that if $m = 0$ then this reduces to the time independent limit of Eq. (46). If $\mathcal{E}_{ij}$ and $\mathcal{B}_{ij} = 0$ then this reduces to the Schwarzschild metric. And the terms which are bilinear in $m$ and either $\mathcal{E}_{ij}$ or $\mathcal{B}_{ij}$ are equivalent to the time independent limit of Eq. (52). An expression with similar features holds for the octupole perturbations.

The metric of Eq. (59) represents a Schwarzschild black hole at rest on the geodesic $\Gamma$ in a time-independent external spacetime. And note that there is no black hole quadrupole moment induced by the external quadrupole field as there are no quadrupole $1/r^3$ terms in this metric in the region where $m \ll r \ll R$. The Schwarzschild black hole equivalent of $\mathcal{I}_{ij}$ vanishes. It follows that, in this situation, the black hole has no acceleration away from $\Gamma$.

Time dependence in $\mathcal{E}_{ij}$ slightly changes this situation. In [28], it is argued that with slow time dependence, with a time-scale $O(R)$, the induced quadrupole field

$$a = O(d^2/R^3).$$

(57)
of the Schwarzschild metric in fact scales as $\sim m^5/r^3R^2$, and that the acceleration from coupling with an external octupole field, $\mathcal{E}_{ijk}x^i x^j x^k \sim r^3/R^2$, gives an acceleration

$$a = O(m^4/R^5).$$

This result is consistent with the Newtonian result in Eq. (58) if the size $d$ of the Newtonian object is replaced with the mass $m$ of the black hole.

An elementary approach using dimensional analysis arrives at this same result. Acceleration is a three-vector with a unit of $1/\text{length}$. If the only quantities in play are $m$, $\mathcal{E}_{ij}$, and $\mathcal{E}_{ijk}$. The only combination of these which yields a vector with the units of acceleration is $m^4\mathcal{E}_{ijk}\mathcal{E}_{jkl} = O(m^4/R^5)$.

However, Eric Poisson has pointed out that a combination involving the magnetic multipole moments, such as $m^3\mathcal{E}_{ijk}\mathcal{B}_{kl}\varepsilon_{lji} = O(m^3/R^4)$, might provide a lower order acceleration.

The field $h_{ab}^S$ is now seen to satisfy the requirements desired for a “Singular field:”

A. $h_{ab}^S$ is a solution of the field equation in the vicinity of a $\delta$-function mass source on a geodesic $\Gamma$.

B. $h_{ab}^S$ exerts no force back on its $\delta$-function source as evidenced by the facts that $h_{ab}^S$ is the part of the perturbed Schwarzschild geometry that is linear in $m$, and that the small black hole has acceleration no larger than $O(m^3/R^4)$, while all that is required is that the acceleration be no larger than $O(m^2/R^3)$.

6 Self-force from gravitational perturbation theory

For an overview of the general approach to gravitational self-force problems about to be described, we refer back to the treatment of the electromagnetic self-force in Sect. 3, the toy-problem of Sect. 4, and particularly to the introduction of $h_{ab}^S$ in Sects. 5.3 and 5.4.

At a formal level, we begin with a metric $g_{ab}$ which is a vacuum solution of the Einstein equation and look for an approximate solution for $h_{ab}^\text{act}$ from

$$G(g + h_{ab}^\text{act}) = 8\pi T + O(h^2),$$

with appropriate boundary conditions, where $T_{ab} = O(m)$ is the stress-energy tensor of a point particle $m$.

Initially we assume that $m$ is moving along a geodesic $\Gamma$. In a neighborhood of $\Gamma$, $h_{ab}$ is well approximated by $h_{ab}^S$. Thus we define $h_{ab}^R$ via the replacement

$$h_{ab}^\text{act} = h_{ab}^S + h_{ab}^R,$$

and use the expansion in Eq. (32) and the definition in Eq. (35) to write
\[ G_{ab}(g + h^{\text{act}}) = G_{ab}(g) - E_{ab}(h^{\text{act}}) + O(h^2) \]
\[ = -E_{ab}(h^R) - E_{ab}(h^S) + O(h^2) \] (63)

where we use the assumption that \( G_{ab}(g) = 0 \) and the linearity of the operator \( E_{ab}(h) \).

In Sect. 5.3 the properties of \( h^S_{ab} \) were chosen carefully so that
\[ E_{ab}(h^S) = -8\pi T_{ab} + O(mr/R^4) \text{ in a neighborhood of } m. \] (64)

We can demonstrate this result by letting \( 4h^S_{ab} = O(mr^3/R^4) \) be the next term not included in the expansion (50). The operator \( E_{ab} \) has second order spatial derivatives, and every time derivative brings in an extra factor of \( 1/R \). Thus \( E_{ab}(4h^S_{ab}) = O(mr/R^4) \), and Eq. (64) follows.

Now we define the effective source
\[ 8\pi S_{ab} = 8\pi T_{ab} + E_{ab}(h^S), \]
\[ = O(mr/R^4). \] (65)

Thus \( S_{ab} \) is zero at \( r = 0 \), where it is continuous but not necessarily differentiable. Everywhere else \( S_{ab} \) is \( C^\infty \).

The first perturbative order problem Eq. (61) is now reduced to solving
\[ E_{ab}(h^R) = -8\pi S_{ab}, \] (66)

and then Eq. (62) reconstructs \( h^{\text{act}}_{ab} \). The limited differentiability of \( S_{ab} \) causes no fundamental difficulty for determining \( h^R_{ab} \), and introduces no small length scale either. The resulting \( h^R_{ab} \) will be \( C^2 \) at the location of the point mass, and \( C^\infty \) elsewhere.

At this order of approximation Sect. 5.4 showed that the mass \( m \) moves along a geodesic of the actual metric \( g^{\text{act}}_{ab} \) with \( h^S_{ab} \) removed, i.e. along a geodesic of \( g_{ab} + h^{\text{act}}_{ab} = g_{ab} + h^R_{ab} \). Thus, the gravitational self-force results in geodesic motion not in \( g_{ab} \) but rather in \( g_{ab} + h^R_{ab} \).

Admittedly, \( g_{ab} + h^R_{ab} \) is not truly a vacuum solution of the Einstein equation. But, by construction it is clear that
\[ G_{ab}(g + h^R) = O(mr/R^4). \] (67)

More terms of higher order in \( r/R \) in the expression for \( h^S_{ab} \) would result in a remainder with more powers of \( r/R \) on the right hand side of Eq. (67). But these would not change the first derivatives of \( h^R_{ab} \) on \( \Gamma \) which are all that would appear in the geodesic equation for \( m \). So the expansion for \( h^S_{ab} \) as given in Sect. 5.4 is adequate for our purposes.
6.1 Dissipative and conservative parts

When viewed from near by, the effect of the gravitational self-force on a small mass \( m \) arises as a consequence of the purely local phenomenon of geodesic motion. In the neighborhood of \( m \), it is impossible then to distinguish the dissipative part of the self-force from the conservative part.

Viewed from afar with the usually appropriate boundary conditions, the metric perturbation \( h_{\text{act}}^{\text{ab}} \) is actually the retarded field \( h_{\text{ret}}^{\text{ab}} \) and it is often useful then to distinguish the dissipative effects which remove energy and angular momentum from the conservative effects which might affect, say, the orbital frequency.

In the case that \( h_{\text{act}}^{\text{ab}} = h_{\text{ret}}^{\text{ab}} \), it is natural to define the dissipative part of the regular field as

\[
h_{\text{dis}}^{\text{ab}} = \frac{1}{2} (h_{\text{ret}}^{\text{ab}} - h_{\text{adv}}^{\text{ab}})
\]

(68)

The advanced and the retarded fields are each solutions of the same wave equation with the same \( \delta \)-function source. Thus their difference is a solution of the homogeneous wave equation and is therefore regular at the point mass. And the dissipative effects of the self-force are revealed as geodesic motion in the metric \( g_{\text{ab}} + h_{\text{dis}}^{\text{ab}} \).

In a complementary fashion, the conservative part of the regular field is naturally defined as

\[
h_{\text{con}}^{\text{ab}} = h_{\text{R}}^{\text{ab}} - \frac{1}{2} (h_{\text{ret}}^{\text{ab}} - h_{\text{adv}}^{\text{ab}})
\]

\[
= h_{\text{ret}}^{\text{ab}} - h_{\text{S}}^{\text{ab}} - \frac{1}{2} (h_{\text{ret}}^{\text{ab}} - h_{\text{adv}}^{\text{ab}})
\]

\[
= \frac{1}{2} (h_{\text{ret}}^{\text{ab}} + h_{\text{adv}}^{\text{ab}}) - h_{\text{S}}^{\text{ab}}
\]

(69)

And the conservative effects of the self-force are revealed as geodesic motion in the metric \( g_{\text{ab}} + h_{\text{con}}^{\text{ab}} \).

With these definitions it is natural that

\[
h_{\text{R}}^{\text{ab}} = h_{\text{con}}^{\text{ab}} + h_{\text{dis}}^{\text{ab}}
\]

(70)

This decomposition into conservative and dissipative parts follows an aspect of the procedure that Mino describes \[33\] as a possible method for computing the dissipative effects of gravitational radiation reaction on the Carter constant \[34, 35\] for a small mass orbiting a Kerr black hole.

6.2 Gravitational self-force implementations

When it is actually time to search for some self-force consequences there are a number of different choices to be made.
6.2.1 Field regularization via the effective source

The majority of this review has been leading toward a natural implementation of self-force analysis using the standard 3+1 techniques of numerical relativity. Assume that \( h^{R}_{ab} \) and its first derivatives, and also the position and four-velocity of \( m \) are known at one moment of time.

1. Use the position and four-velocity of \( m \) to analytically determine \( h^{S}_{ab} \).
2. Obtain the effective source \( S_{ab} \) via Eq. \( 65 \).
3. Evolve Eq. \( 66 \) for \( h^{R}_{ab} \) one step forward in time.
4. Move the particle a step forward in time using the geodesic equation for \( g^{ab} + h^{R}_{ab} \).
5. Repeat.

Section 10.2 describes the application of this approach to a scalar field problem and includes figures which reveal some generic characteristics of the source function.

6.2.2 Mode-sum regularization

Mode-sum regularization \([36, 37]\) avoids the singularity of \( h^{act}_{ab} \) and its derivatives on \( \Gamma \) by an initial multipole-moment decomposition, say, into spherical harmonic components \( h^{act/m}_{ab} \). With the assumption that \( h^{S}_{ab} \) is carefully defined away from \( m \) in a fashion that also allows for a decomposition in terms of spherical harmonics \( h^{S/m}_{ab} \), then \( h^{R/m}_{ab} = h^{act/m}_{ab} - h^{S/m}_{ab} \) would be the decomposition of \( h^{R}_{ab} \). The collection of the multipole moments \( h^{S/m}_{ab} \), their derivatives and various of their linear combinations are, together, known as “regularization parameters.” This essentially leads to the mode-sum regularization procedure of Barack and Ori\([36, 37]\) which has been used in nearly all of the self-force calculations to date.

6.2.3 The gravitational self-force actually resulting in acceleration

We have strongly pushed our agenda of treating the gravitational self-force in local terms as geodesic motion through a vacuum spacetime \( g^{ab} + h^{R}_{ab} \). However, when viewed from afar the worldline \( \Gamma \) of \( m \) is indeed accelerated and not a geodesic of the background geometry \( g^{ab} \). This acceleration can be described as a consequence of \( m \) interacting with a spin-2 field \( h^{R}_{ab} \) which leads to the resulting acceleration

\[
u^{b} \nabla_{b} u^{a} = - \left( g^{ab} + u^{a} u^{b} \right) u^{c} u^{d} \left( \nabla_{c} h^{R}_{db} - \frac{1}{2} \nabla_{b} h^{R}_{cd} \right)
\]

(71)

away from the original worldline in the original metric \( g^{ab} \).

Under some circumstances this might be a convenient interpretation. The resulting worldline would be identical to the geodesic of \( g^{ab} + h^{R}_{ab} \) and would correctly incorporate all self-force effects, although the worldline would not be parameterized by the actual proper time. It is important to note that the acceleration of Eq. (71) can-
not be measured with an accelerometer and, by itself, has no actual, direct physical consequence.

In the next section we describe some general consequences of gauge transformations in perturbation theory. Be warned that if Eq. (71) is used to calculate the deviation \( \zeta^a \) of the worldline away from a geodesic in the background metric \( g_{ab} \), then any gauge transformation whose gauge vector \( \xi^a = -\zeta^a \), on the world line, would automatically set the right hand side of Eq. (71) to zero and leave \( m \) on its original geodesic. This possibility certainly confuses the interpretation of the right hand side of Eq. (71). Such a removal of the self-force only works as long as the deviation vector \( \zeta^a \sim O(h) \). If self-force effects accumulate in time, such as from dissipation or orbital precession, then after a long enough time the effects of the self-force will be revealed.

7 Perturbative gauge transformations

In General Relativity, the phrase “choice of gauge” has different possible interpretations depending upon whether one is interested in perturbation theory or, say, numerical relativity. With numerical relativity, “choice of gauge” usually refers to the choice of a specific coordinate system, with the understanding that general covariance implies that the meaning of a calculated quantity might be as ambiguous as the coordinate system in use.

In perturbation theory the “choice of gauge” is more subtle. One considers the difference between the actual metric \( g_{ab}^{\text{act}} \) of a spacetime of interest and an abstract metric \( g_{ab} \) of a given, background spacetime. The difference

\[
h_{ab} = g_{ab}^{\text{act}} - g_{ab} \tag{72}
\]

is assumed to be small. The perturbed Einstein equations govern \( h_{ab} \), and knowing \( h_{ab} \) might provide answers to questions concerning the propagation and emission of gravitational waves, for example.

In this perturbative context “choice of gauge” involves the choice of coordinates, but in a very precise sense [38, 39, 40, 41]. The subtraction in Eq. (72) is ambiguous. The two metrics reside on different manifolds, and there is no unique map from the events on one manifold to those of another. Usually the names of the coordinates are the same on the two manifolds, and this provides an implicit mapping between the manifolds. But this mapping is not unique. For example, the Schwarzschild geometry is spherically symmetric. This allows the Schwarzschild coordinate \( r \) to be defined in terms of the area \( 4\pi r^2 \) of a spherically symmetric two-surface. The perturbed Schwarzschild geometry is not spherically symmetric, and to describe the coordinate \( r \) on the perturbed manifold as the “Schwarzschild \( r \)” does not describe the meaning of \( r \) in any useful manner and is not a perturbative choice of gauge.

In perturbation theory a gauge transformation is an infinitesimal coordinate transformation of the perturbed spacetime.
\(x_{\text{new}}^a = x_{\text{old}}^a + \xi^a, \quad \text{where} \quad \xi^a = O(h), \quad (73)\)

and the coordinates \(x_{\text{new}}^a, x_{\text{old}}^a\), and the coordinates on the abstract manifold are all described by the same names, for example \((t, r, \theta, \phi)\) for perturbations of the Schwarzschild geometry. The transformation of Eq. (73) not only changes the components of a tensor by \(O(h)\), in the usual way, but also changes the mapping between the two manifolds and hence changes the subtraction in Eq. (72). With the transformation (73),

\[
h_{ab}^{\text{new}} = (g_{cd} + h_{cd}^{\text{old}}) \frac{\partial x^c_{\text{old}}}{\partial x^a_{\text{new}}} \frac{\partial x^d_{\text{old}}}{\partial x^b_{\text{new}}} - \left( \frac{\partial g_{ab}}{\partial x^c} \xi^c \right).
\]

(74)

The \(\xi^c\) in the last term accounts for the \(O(h)\) change in the event of the background used in the subtraction. After an expansion, this provides a new description of \(h_{ab}\)

\[
h_{ab}^{\text{new}} = h_{ab}^{\text{old}} - \xi^c g_{cb} \frac{\partial}{\partial x^a} - \xi^d g_{cb} \frac{\partial}{\partial x^d} - \xi^c \frac{\partial g_{ab}}{\partial x^c}.
\]

(75)

through \(O(h)\); the symbol \(\mathcal{L}\) represents the Lie derivative and \(\nabla_a\) is the covariant derivative compatible with \(g_{ab}\). A gauge transformation does not change the actual perturbed manifold, but it does change the coordinate description of the perturbed manifold.

A little clarity is revealed by noting that

\[
E_{ab}(\nabla_c \xi^d) = 0
\]

(76)

for any \(C^2\) vector field \(\xi^a\); and if \(\xi^a\) has limited differentiability or is a distribution, then Eq. (76) holds in a distributional sense [3]. Thus \(-2\nabla(a)\xi^b\) is a homogeneous solution of the linear Eq. (35). It appears as though any \(-2\nabla(a)\xi^b\) may be added to an inhomogeneous solution of Eq. (35) to create a “new” inhomogeneous solution. In fact the new solution is physically indistinguishable from the old—they differ only by a gauge transformation with gauge vector \(\xi^a\).

Generally, the four degrees of gauge freedom contained in the gauge vector \(\xi^a\) are used to impose four convenient conditions on \(h_{ab}\). For perturbations of the Schwarzschild metric, it is common to use the Regge Wheeler metric which sets four independent parts of \(h_{ab}\) to zero; this results in some very convenient algebraic simplifications. The Lorenz gauge requires that \(\nabla_a (h^{ab} - \frac{1}{2} g^{ab} h_c c) = 0\) and is formally attractive but unwieldy in practice [42, 43, 44].

The Bianchi identity implies that there are four relations among the ten components of the Einstein equations. Choosing a gauge helps focus on a self-consistent method for solving a subset of these equations. A physicist might have a favorite for a gauge choice, but Nature has no preference whatsoever.
8 Gauge confusion and the gravitational self-force

If a particular physical consequence of the gravitational self-force requires a particular choice of gauge, then it is unlikely that this physical consequence has any useful interpretation. This was already demonstrated with the example presented in Sect. 2 where the magnitude of the effect of the Newtonian self-force on the period in an extreme-mass-ratio binary depended upon the definition of the variable \( r \).

The quasi-circular orbits of the Schwarzschild geometry provide a fine example which reveals the insidious nature of gauge confusion in self-force analyses. Ref. [45] contains a thorough discussion of this subject and this section has two self-force examples which highlight the confusion that perturbative gauge freedom creates.

It is straightforward to determine the components of the geodesic equation for the metric \( g^{\text{Schw}}_{ab} + h^R_{ab} \). A consequence of these is that the orbital frequency of \( m \) in a quasi-circular orbit about a Schwarzschild black hole of mass \( M \) is given by

\[
\Omega^2 = \frac{M}{r^3} \left( r - \frac{3M}{2r^2} \partial_a u^b \partial_b h^R_{ab} \right) \tag{77}
\]

which can be proven to independent of the gauge choice. Clearly the self-force makes itself known to the orbital frequency through the last term. So we focus on the orbit at radius \( r = 10M \), choose to work in the Lorenz gauge, work hard and successfully evaluate all of the components of the regularized field \( h^R_{ab} \) as well as its radial derivative. Then we calculate the second term in Eq. (77) and determine that \( \Omega \) changes by a specific amount \( \Delta \Omega_{\text{L}} \). We now know the gauge invariant change in the orbital frequency for \( m \) in the orbit at 10M.

Or do we? To check this result we repeat the numerical work but this time use the Regge-Wheeler gauge, and find that the change in \( \Omega \) is \( \Delta \Omega_{\text{R}} \) and

\[
\Delta \Omega_{\text{R}} \neq \Delta \Omega_{\text{L}} ! \tag{78}
\]

What’s going on? For a quasi circular orbit \( \Omega \) can be proven to be independent of gauge, and yet with two different gauges we find two different orbital frequencies for the single orbit at 10M.

When I first discovered this conundrum I was reminded of my experience trying to understand special relativity and believing that apparently paradoxical situations made special relativity logically inconsistent. Eventually the paradoxes vanished when I understood that coordinates named \( t, x, y \) and \( z \) are steeped in ambiguity and that only physical observables are worth calculating and discussing.

The resolution of this self-force confusion is similar. The two evaluations of \( \Omega^2 \) are each correct. But, one is for the orbit at the Schwarzschild radial coordinate \( r = 10M \) in the Lorenz gauge, while the other is at the Schwarzschild radial coordinate \( r = 10M \) in the Regge-Wheeler gauge. These are two distinct orbits. In fact, the gauge vector \( \xi^a \) which transforms from the Lorentz gauge to the Regge-Wheeler gauge has a radial component \( \xi^r \) whose magnitude is just right to make the change in the first term in Eq. (77) balance the change in the second term.
The angular frequency of \( m \) orbiting a black hole is a physical observable and independent of any gauge choice. But the perturbed Schwarzschild geometry is not spherically symmetric and there is then no natural definition for a radial coordinate.

A second example of gauge confusion appears when one attempts to find the self-force effect on the rate of inspiral of a quasi-circular orbit of Schwarzschild. It is natural to find the energy \( E \), \( \Omega \) and \( dE/dt \) all as functions of the radius of a circular orbit and then to use

\[
\frac{d\Omega}{dt} = \frac{dE}{dt} \times \frac{d\Omega/dr}{dE/dr}
\]  

(79)

to determine the rate of change of \( \Omega \). We can find the self force effect on each of these quantities so we can apparently find the self force effect on \( d\Omega/dt \) which is a physical observable and must be gauge invariant.

This situation is subtle. Why do we believe Eq. (79)? With some effort it can be shown that the geodesic equation for \( g_{ab}^{\text{Schw}} + h_{ab}^R \) implies that Eq. (79) holds for a quasi-circular orbit \([45]\). Part of this proof depends upon the \( t \)-component of the geodesic equation which is

\[
\frac{dE}{dt} = -\frac{1}{2u} u^a u^b \partial_a h_{ab}^R
\]  

(80)

and this is a gravitational self-force effect. But, note that the right hand side of Eq. (79) is already first order in \( h_{ab}^R \) from the factor \( dE/dt \). While self-force effects on \( d\Omega/dr \) and \( dE/dr \) can be found, if these are included then second order self force effects on \( dE/dt \) must also be found for a consistent solution.

The end result is that you really can’t see the effect of the conservative part of the self-force on the waveform for quasi-circular orbits using first order perturbation theory.

9 Steps in the analysis of the gravitational self-force

We now highlight the major steps involved in most gravitational self-force calculations.

First the metric perturbation \( h_{ab}^{\text{act}} \) is determined. For a problem in the geometry of the Schwarzschild metric, this involves solving the Regge-Wheeler\([7]\) and the Zerilli\([8]\) equations to determine the actual metric perturbations. The Kerr metric still presents some challenges. The Teukolsky\([46, 10]\) formalism can provide the Weyl scalars but finding the metric perturbations\([47]\) from these is difficult at best, and does not include the non-radiating monopole and dipole perturbations. One possibility for Kerr is to find the metric perturbations directly, perhaps in the Lorenz gauge, but this would likely require a \( 3 + 1 \) approach. Another possibility being discussed\([48]\) is to Fourier transform in \( \phi \), and then use a \( 2 + 1 \) formalism which...
results in an $m$-sum. Rotating black holes continue to be a challenge for self-force calculations.

Next, the singular field $h^S_{ab}$ is identified for the appropriate geodesic in the background spacetime. A general expansion of the singular field is available [28], but it is not elementary to use. Work in progress provides a constructive procedure for the THZ coordinates in the neighborhood of a geodesic, and this would lead to explicit expressions for $h^S_{ab}$ in the natural coordinates of the manifold. However, this procedure is not yet in print, and it is not yet clear how difficult it might be to implement.

Then the perturbation is regularized by subtracting the singular field from the actual field resulting in $h^R_{ab} = h^{\text{act}}_{ab} - h^S_{ab}$. Most applications have taken this step using the mode-sum regularization procedure of Barack and Ori [36, 37]. In this case, a mode-sum decomposition of the singular (or “direct,” cf. footnote 4) field is identified and then removed from the mode-sum decomposition of the actual field. The remainder is essentially the mode-sum decomposition of the regular field. Generally, this mode-sum converges slowly as a power law in the mode index, $l$ or $m$. Although some techniques have been used to speed up this convergence [55]. More recently, “field regularization” (discussed in Sect. 10.2 and in [6]) has been used for scalar field self-force calculations. For this procedure in the gravitational case, Eq. (66) might be used to obtain the regular field $h^R_{ab}$ directly via $3+1$ analysis.

After the determination of $h^R_{ab}$, the effect of the gravitational self-force is then generically described as resulting in geodesic motion for $m$ in the metric $g^{\text{act}}_{ab} + h^R_{ab}$. This appears particularly straightforward to implement using field regularization. Alternatively, the motion might also be described as being accelerated by the gravitational self-force as described in Eq. (71).

At this point, one should be able to answer the original question—whatever that might have been! In fact, the original question should be given careful consideration before proceeding with the above steps. Formulating the question might be as difficult as answering it. It is useful to keep in mind that only physical observables and geometrical invariants can be defined in a manner independent of a choice of coordinates or a choice of perturbative gauge.

My prejudices about the above choices for each step are not well hidden. But, for whatever technique or framework is in use, a self-force calculation should have the focus trained upon a physical observable, not upon the method of analysis.

Self-force calculations unavoidably involve some subtlety. Experience leads me to be wary about putting trust in my own unconfirmed results. Good form requires independent means to check analyses. Comparisons with the previous work of others, with Newtonian and post-Newtonian analyses, or with other related analytic weak-field situations all lend credence to a result.

---

4 Expansions for the somewhat related “direct” field are also available [37, 49, 50, 51, 52, 53, 54], though their use is, similarly, not at all elementary.
10 Applications

Recently, the effect of the gravitational self-force on the orbital frequency of the innermost stable circular orbit of the Schwarzschild geometry has been reported by Barack and Sago [56]. They find that the self-force changes the orbital frequency of the ISCO by $0.4870(\pm 0.0006)m/M$. To date this result is by far the most interesting gravitational self-force problem that has been solved. But it is too recent a result to be described more fully herein.

10.1 Gravitational self-force effects on circular orbits of the Schwarzschild geometry

As an elementary example we consider a small mass $m$ in a circular orbit about the Schwarzschild geometry. Details of this analysis may be found in [45]. The gravitational self-force affects both the orbital frequency $\Omega$ and also the Schwarzschild $t$-component of the four-velocity, $u^t$, which is related to a redshift measurement. The self-force effects on these quantities are known to be independent of the gauge choice for $h_{ab}$, as would be expected because they can each be determined by a physical measurement. However the radius of the orbit depends upon the gauge in use and has no meaning in terms of a physical measurement.

Notwithstanding the above, we define $R_\Omega$ via

$$\Omega^2 = M/R_\Omega^3$$

as a natural radial measure of the orbit which inherits the property of gauge independence from $\Omega$. The quantity $u^t$ can be divided into two parts $u^t = 0u^t + 1u^t$, where each part is separately gauge independent. Further the functional relationships between $\Omega$, $0u^t$ and $R_\Omega$ are identical to their relationships in the geodesic limit,

$$0u^t = [1 - 3(\Omega M)^{2/3}] + O(m^2)$$

and shows no effect from the self-force. The remainder

$$1u^t = u^t - 0u^t$$

is, a true consequence of the self-force, and we plot the numerically determined $1u^t$ as a function of $R_\Omega$ in Fig. 1. The numerical data of Fig. 1 have also been carefully compared with and seen to be in agreement with the numerical results of Sago and Barack, as shown in [57], despite the fact that very different gauges were in use and different numerical methods were employed.

We have derived a post-Newtonian expansion for $1u^t$ based upon the work of others [58, 59]. Our expansion is in powers of $m/R_\Omega$, which is $v^2/c^2$ in the Newtonian limit, and we find
Fig. 1 From [45]. The quantity $1u'$, which is the gauge independent $O(m)$ part of $u'$, is given as a function of $R_{\Omega}$ for circular orbits in the Schwarzschild geometry. Also shown are $u'$ as calculated with Newtonian, 1PN and 2PN analyses in [45] based upon results in [58] and [59]. The 3PN line is based on a numerical determination of the 3PN coefficient in Eq. (84) in [45].

$$1u' = \frac{m}{M} \left[ -\left( \frac{M}{R_{\Omega}} \right) - 2 \left( \frac{M}{R_{\Omega}} \right)^2 - 5 \left( \frac{M}{R_{\Omega}} \right)^3 + \cdots \right],$$

which includes terms of order $v^6/c^6$. Further, with numerical analysis we have fit these results to determine a 3PN parameter of order $v^8/c^8$ and found that the coefficient of the $(M/R_{\Omega})^4$ term is $-27.61 \pm 0.03$.

Work in progress, with Blanchet, Le Tiec, and Whiting, includes a full 3PN determination of the same 3PN coefficient as well as a more precise numerical determination via self-force analyses. The consistency of these two efforts has the possibility of giving greatly increased confidence in the self-force numerical analysis as well as in the post-Newtonian analysis, each of which involves substantial complications.

This self-force result is primarily only of academic interest. But it is consistent with a post-Newtonian expansion and includes an estimate of the previously unknown $O(v^8/c^8)$ coefficient in the expansion. Modest though it might be, this is a result.

### 10.2 Field regularization via the effective source

The ultimate goal of self-force analysis has become the generation of accurate gravitational waveforms from extreme mass-ratio inspiral (EMRI). It would be amusing to “see” numerically the waves emitted by a small black hole in a highly eccentric
orbit about a much larger one and to see the changes in the orbit while the small hole loses energy and angular momentum.

Such a project appears to require a method to solve for the gravitational waves while simultaneously modifying the worldline of the small hole as it responds to the gravitational self-force. The toy problem in Sect. 4 shows how this might be done using the expertise of numerical relativity groups coupled with the self-force community.

Our group is in the early stages of development of infrastructure that any numerical relativity group could use to get gravitational self-force projects up and running with a minimum of effort. We intend to provide the software that will produce the regularized-field source $S_{ab}$, for a small mass $m$ as a function of location and four-velocity. A numerical relativist could then evolve the linear field equation

\begin{align*}
0 &= \nabla^2 \psi - 2 \psi \nabla \cdot \nabla f + \mathcal{W} \psi^5,
\end{align*}

where $f$ is the standard ADM lapse and $\mathcal{W}$ is the self-force. The expression $\nabla \cdot \nabla f$ is the modified flux constraint that controls the gravitational waves.

\[
\begin{figure}
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Comparison of time-domain (TD) and frequency-domain (FD) results for the $l = m = 2$ multipole moment of the scalar field. The regular field is represented by the blue dashed line. Adding this to the $l = m = 2$ multipole moment of the analytically known singular field, $\mathcal{W} \psi^5$, results in the computed, actual field to good agreement. The inset shows near the point charge that $\psi^R$ is very well behaved and that $\psi^R + \mathcal{W} \psi^5$ is indistinguishable from the actual, retarded field $\psi^\text{ret}$, just as it should be.}
\end{figure}


Table 1 From [6]. Summary of scalar field self-force results for a circular orbits at $R = 10M$ and $R = 12M$. The error is determined by a comparison with an accurate frequency-domain calculation [55].

| $R$   | Time-domain | Frequency-domain | error          |
|-------|-------------|------------------|---------------|
|      | $\partial_t \psi^R$ | $10M$ | $3.750211 \times 10^{-5}$ | $3.750227 \times 10^{-5}$ | $0.00043\%$ |
|      | $\partial_r \psi^R$ | $10M$ | $1.380612 \times 10^{-5}$ | $1.378448 \times 10^{-5}$ | $0.157\%$ |
|      | $\partial_t \psi^R$ | $12M$ | $1.747278 \times 10^{-5}$ | $1.747254 \times 10^{-5}$ | $0.00139\%$ |
|      | $\partial_r \psi^R$ | $12M$ | $5.715982 \times 10^{-6}$ | $5.710205 \times 10^{-6}$ | $0.101\%$ |

$E_{ab}(h^R) = -8\pi S_{ab}$

for $h^R_{ab}$, while simultaneously adjusting the worldline according to Eq. (71).

As described in Sect. 6 such a computation of $h^R_{ab}$ would provide not only the effects of the gravitational self-force but also the gravitational wave itself.

Ian Vega [6] has led a first attempt at directly solving for the regularized field and self-force using a well tested problem involving a scalar charge in a circular orbit of the Schwarzschild geometry. This analysis used a multipole decomposition of the source and field. And Vega solved for the multipole components in the time domain using a $1+1$ code. Figure 2 shows the $\ell = m = 2$ mode and compares the accurate frequency domain evaluation of the retarded field $\psi_{\text{ret}}$ to the sum $\psi^R + \psi^S$ as determined using $1+1$ methods with field-regularization as described in Sect. 4.

Table 1 compares the numerical results of regularized fields and forces from the field-regularization approach of [6] with the mode-sum regularization procedure [37, 52] used in [55].

Figure 3 shows an example of the source-function used in a test of this approach with a scalar field. The “double bump” shape far from the charge is a characteristic of any function similar to $\nabla^2 (W/|r - r_0|)$ with a window function $W$ which satisfies the three window properties given in Sect. 4.2.

Figure 4 reveals the $C^0$ nature of the effective source at the location of the particle on a dramatically different scale. It is important to note that limited differentiability of this sort does not introduce a small length scale into the numerical problem, and might be treated via a special stencil in the neighborhood of the charge.

A recent collaboration with Peter Diener, Wolfgang Tichy and Ian Vega [60] looks at the same test problem but involves two distinct $3+1$ codes, which were developed completely independently. One uses pseudo-spectral methods, the other uses a multiblock code with high order matching across block boundaries. With a modest amount of effort these two codes, each developed for generic numerical relativity problems, were modified to accommodate the effective source of the scalar field and are able to determine all components of the effective source with errors less than $1\%$. The future of numerical $3+1$ self-force analysis looks promising.
11 Concluding remarks

Ptolemy was able to model accurately the motion of the planets in terms of epicycles and circles about the Earth. However, the precise choice of which circles and epicycles should be used was debated. Copernicus realized that a much cleaner description resulted from having the motion centered upon the sun. The two competing models were equally able to predict the positions of the planets for the important task of constructing horoscopes. But for understanding the laws of physics, Newton clearly favored the Copernican model.

There appear to be two rather distinct attitudes toward calculating the effects of the gravitational self-force for a mass $m$ orbiting a black hole. Both lead to identical conclusions about physically measurable quantities. If the motion is to be described as accelerating in the black hole geometry, then the acceleration depends upon the perturbative gauge choice and is not related to any acceleration that an observer local to $m$ could actually measure. If the motion is described as geodesic in the spacetime geometry through which $m$ moves, then it is immediately apparent that the only quantities worth calculating are those which are physically measurable, or

![Fig. 3 From [6]. The effective source $S_{\text{eff}}$ on the equatorial plane for a scalar charge in a circular orbit of the Schwarzschild metric. The particle is at $r/M = 10$, $\phi/\pi = 0$, where $S_{\text{eff}}$ appears to have no structure on this scale. The spiky appearance is solely a consequence of the grid resolution of the figure. In fact the source is $C^\infty$ everywhere except at the location of the scalar charge where $S_{\text{eff}}$ appears quite calm on this scale.](image-url)
at least independent of the gauge choice. With this second attitude, one is left with the rather satisfying perspective that the effects of the gravitational self-force are neither more nor less than the result of free-fall in a gravitational field.

In this review, I have eschewed mention of Green’s functions. The asymptotic matching perspective promoted here seems more effective to me at getting to the physics of the gravitational self-force and less likely to lead to mathematical confusion.

The singular field $h_{ab}^S$, which plays a fundamental role, has a reasonably straightforward description in convenient locally inertial coordinates. And it appears nearly immediately in the DW [5] formulation of radiation reaction via the Green’s function $G_{abc'd'}^S$. This Green’s function has odd acausal structure with support on the past and future null cone of the field point and also in the spacelike related region outside these null cones. Such causal structure is consistent with the fact that $h_{ab}^S$ exerts no self-force. Based upon personal conversations, this feature appears problematical to some. However, the integrability condition of the perturbed Einstein equation requires that the worldline of a point source be a geodesic. Geodesic motion is the

\footnote{In fact the singular field was discovered first [28] using matched asymptotic expansions. And the Green’s function appeared only later during an attempt to show consistency with the usual DeWitt-Brehme [61] approach to radiation reaction.}
General Relativistic equivalent of Newtonian no motion, and the singular field is the curved space equivalent of a Coulomb field. Not much is happening at the source or to the singular field. I cannot imagine that such behavior somehow leads to an effect that might be described as acasual.

The S-field $h^S_{ab}$ is defined via an expansion in a neighborhood of the source and does not depend upon boundary conditions, and the restriction to geodesic motion precludes any unexpected behavior of the point mass in either the past or the future. The S-field is precisely the nearly-Newtonian monopole field with minor tidal distortions from the surrounding spacetime geometry.

While orbiting a black hole, Einstein’s apple emits gravitational waves and spirals inward. However, the apple is in free fall and not accelerating. In fact, it is not moving in its locally inertial frame of reference, and is aware of neither its role as the source of any radiation nor of its role acting out the effects of radiation reaction.

S. Chandrasekhar was fond of describing a conversation with the sculptor Henry Moore. In his own words, Chandra “had the occasion to ask Henry Moore how one should view sculptures: from afar or from near by. Moore’s response was that the greatest sculptures can be viewed—indeed should be viewed—from all distances since new aspects of beauty will be revealed at every scale.”[62] The self-force analysis in General Relativity also reveals different aspects when viewed from afar and when viewed from near by. From afar a small black hole dramatically emits gravitational waves while inspiralling toward a much larger black hole. From near by the small hole reveals the quiet simplicity and grace of geodesic motion. Rather than “beauty,” a satisfying sense of physical consistency is “revealed at every scale.”

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