Fixing the moment of inertia in the Bohr Hamiltonian through Supersymmetric Quantum Mechanics

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Abstract. A well known problem of the Bohr-Mottelson model is the behavior of nuclear moments of inertia. The coefficient of the angular part \((\gamma, \theta_i)\) of the kinetic energy is proportional to \(\beta^2\). It is shown how a SUSYQM treatment of the \(\beta\) vibrations, emerges moments of inertia compatible with the experimental picture. This emergence is the result of the SUSYQM solution of the radial equation. Fitting results for the \(^{162}\)Dy and \(^{238}\)U in ground state, \(\beta_1\) and \(\gamma_1\) bands are also presented.

1. Introduction

Quesne and Tkachuk began to study SUSYQM methods for non-pointlike quantum oscillators [1, 2], that is harmonic oscillators with non-pointlike excitations. In principle, in such an oscillator the Heisenberg uncertainty relations are modified and this guides the modification of the canonical commutation relations. In [3] the equivalence of such an oscillator with a Schroedinger equation of a position dependent mass problem was established. In [4] a Schroedinger equation of the Bohr-Mottelson type was presented for the case of a mass dependent on the \(\beta\) degree of freedom. In this equation the coefficient of the angular part of the kinetic energy is modified as,

\[
\beta^2 \sin^2 \left( \gamma - \frac{2}{3} \pi k \right) \rightarrow \frac{\beta^2}{f^2(\beta)} \sin^2 \left( \gamma - \frac{2}{3} \pi k \right).
\]

\(f(\beta)\) is called the deformation function which shapes the moments of inertia and its form will be determined by SUSYQM mathematical techniques.

2. SUSYQM for the \(\beta\) vibrations

In order to present a SUSYQM discussion of the \(\beta\) vibrations, a phase space of the \(\beta\) degree of freedom with commutation relations

\[
[\beta, p_\beta] = i\hbar f(\beta),
\]

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can be formed, based on [3, 5]. Because of the presence of \( f(\beta) \), in the second quantization procedure the ladder operators will not be as usual but modified in general as,

\[
A^\pm \rightarrow A^\pm(a, \mu, \nu) = \mp \sqrt{f(a; \beta)} \frac{d}{d\beta} \sqrt{f(a; \beta)} + W(\mu, \nu; \beta). \tag{3}
\]

Here a deformed momentum operator is introduced through the deformation function \( f(a; \beta) \) and the superpotential \( W(\mu, \nu; \beta) \) which signals the SUSYQM method. From the parameters \((a, \mu, \nu)\), only \( a \) will remain free. The Hamiltonian corresponding to these ladder operators will of course give good quantum numbers for the stationary states of \( \beta \)-vibrations, characterized by a function \( R(\beta) \).

The principal SUSYQM demand states that the action of the Hamiltonian operator to the ground state shall give zero. The parameter \( \epsilon_0 \) is introduced, which is assumed to be the energy of the ground state, and therefore the SUSYQM method is valid for the Hamiltonian,

\[
A^+(a, \mu, \nu)A^-(a, \mu, \nu) = H - \epsilon_0, \tag{4}
\]

which gives zero eigenvalue for the vacuum. This Hamiltonian shall correspond to the radial equation which is [5],

\[
HR = - \left( \sqrt{f} \frac{d}{d\beta} \sqrt{f} \right)^2 R + 2uR = 2\epsilon R. \tag{5}
\]

This correspondence emerges the equation,

\[
W^2(\mu, \nu; \beta) - f(\beta)W'(\mu, \nu; \beta) + \epsilon_0 = 2u(\beta). \tag{6}
\]

Now, the main result of [3] is that the Schrödinger equation of Eq. (5) is also obtained for the case of a position dependent mass problem, as discussed in [4, 6] with a change in the potential,

\[
u \rightarrow u_{eff} = u + \frac{1}{4}ff'' + \frac{1}{6}(f')^2. \tag{7}
\]

Therefore the energy of the ground state \( \epsilon_0 \) can be determined from \( u_{eff} \). This can be done if the specific potential \( u_{eff}(\beta) \) has known superpotential and deformation function.

### 2.1. Shape invariance and the Davidson potential

The Schrödinger equation is known to be exactly solvable for the Davidson potential [7]. Shape invariance states that a potential gives exact solutions if and only if retains the same functional dependence under the change of its parameters. In Figure 1, shape invariance is shown for the Davidson’s parameter \( \beta_0 \) which is fitted to a specific nucleus, namely the minimum reflects the ground state of the \( \beta \)-vibrations.

We extend the shape invariance condition for the effective potential. This means that \( u_{eff} \) should retain the Davidson behavior for every change in the parameters, with

\[
W^2(\mu, \nu; \beta) - f(\beta)W'(\mu, \nu; \beta) + \epsilon_0 = 2u_{eff}(\beta) = k_1\beta^2 + k_0 + \frac{k_{-1}}{\beta^2}. \tag{8}
\]

In [8], classes of shape invariant potentials have been studied with the identification of their corresponding superpotentials and deformation functions. The superpotential and deformation function for the Davidson case are,

\[
W(\beta) = \frac{\mu}{\beta} + \nu\beta \quad , \quad f(\beta) = 1 + a\beta^2. \tag{9}
\]

With this deformation function, moments of inertia receive a nice correction as can be seen in Figure 2. Parameter \( a \) controls in a certain nucleus the mass dependence on deformation. A theoretical expectation of this parameter is lacking, and a first, numerical picture can be seen in [5] for the \( \gamma \) unstable and axially symmetric prolate nuclei.
3. Energy spectrum

From Ref. [4] the coefficients of the effective potential are,

\[ k_1 = 2 + a^2(12 + \Lambda), \quad k_0 = a(13 + 2\Lambda), \quad k_{-1} = 2 + \Lambda + 2\beta_0^4, \]  

with \( \Lambda = \tau(\tau + 3) \) for \( \gamma \) unstable and \( \Lambda = \frac{L(L+1)-K^2}{3} + (6c)(n_\gamma + 1) \) for the axially symmetric behavior. The parameter \( c \) controls \( \gamma \) stiffness and \( n_\gamma \) is the quantum number for \( \gamma \) vibrations.

With these equations the energy of the ground state is determined from the expressions,

\[ \mu(\mu + 1) = k_{-1}, \quad \nu(\nu - a) = k_1, \quad 2\mu\nu + \mu a - \nu + \epsilon_0 = k_0, \]  

\[ \mu = -\frac{1}{2} \left( 1 + \sqrt{9 + \Lambda + 8\beta_0^4} \right), \quad \nu = \frac{a}{2} \left( 1 + \sqrt{1 + \frac{8 + 4a^2(12 + \Lambda)}{a^2}} \right). \]

From these equations the energy of the ground state \( \epsilon_0 \) is found to be

\[ \epsilon_0 = \frac{19}{4} a + \frac{5}{2} a + \frac{1}{2} \sqrt{a^2 + 4k_1} + \frac{a}{2} \sqrt{1 + 4k_{-1}} + \frac{1}{4} \sqrt{(a^2 + 4k_1)(1 + 4k_{-1}) + a\Lambda}. \]
Table 1. Normalized to the energy of the first excited state, $E(2^+_1)$, energy levels of the ground state band (gsb) and the $\beta_1$ and $\gamma_1$ bands of $^{162}$Dy and $^{238}$U, obtained from the energy spectrum of Eq. (14) using the parameters given in [5], compared to available experimental data [9].

|       | $^{162}$Dy | $^{162}$Dy | $^{238}$U | $^{238}$U | $^{162}$Dy | $^{162}$Dy | $^{238}$U | $^{238}$U |
|-------|-------------|-------------|-----------|-----------|-------------|-------------|-----------|-----------|
|       | gsb         | gsb         | gsb       | gsb       | th          | th          | exp       | th        |
| 0     | 0.00        | 0.00        | 0.00      | 0.00      | 2           | 11.0        | 11.2      | 23.6      |
| 2     | 1.00        | 1.00        | 1.00      | 1.00      | 3           | 11.9        | 12.1      | 24.6      |
| 4     | 3.29        | 3.30        | 3.30      | 3.31      | 4           | 13.2        | 13.3      | 25.9      |
| 6     | 6.80        | 6.80        | 6.84      | 6.86      | 5           | 14.7        | 14.7      | 27.4      |
| 8     | 11.41       | 11.41       | 11.54     | 11.57     | 6           | 16.4        | 16.5      | 29.2      |
| 10    | 17.04       | 17.01       | 17.27     | 17.33     | 7           | 18.5        | 18.5      | 31.2      |
| 12    | 23.57       | 23.49       | 23.97     | 24.06     | 8           | 20.7        | 20.8      | 33.5      |
| 14    | 30.90       | 30.74       | 31.51     | 31.63     | 9           | 23.3        | 23.3      | 36.0      |
| 16    | 38.90       | 38.70       | 39.82     | 39.97     | 10          | 25.9        | 26.0      | 38.8      |
| 18    | 47.58       | 47.28       | 48.78     | 48.98     | 11          | 29.0        | 28.9      | 41.7      |
| 20    | 58.31       | 58.61       | 60.12     | 62.12     | 12          | 31.4        | 32.1      | 44.9      |
| 22    | 68.31       | 68.77       | 70.57     | 72.57     | 13          | 35.5        | 35.5      | 48.3      |
| 24    | 78.71       | 79.44       | 81.24     | 83.24     | 14          | 39.4        | 39.9      | 51.9      |
| 26    | 89.46       | 90.55       | 92.35     | 94.35     | 15          | 43.2        | 43.8      | 55.5      |
| 28    | 100.57      | 102.08      | 103.88    | 105.88    | 16          | 47.1        | 47.7      | 59.4      |
| 30    | 112.10      | 113.99      | 116.88    | 118.88    | 17          | 51.0        | 51.6      | 63.4      |
|       |             |             |           |           | 18          |             |           |           |
|       |             |             |           |           |             |             |           |           |
| $\beta_1$ | $\beta_1$ | $\beta_1$ | $\beta_1$ | $\beta_1$ | 19          |             |           |           |
| 0     | 17.3        | 15.7        | 20.6      | 20.6      | 20          | 77.3        | 76.6      |           |
| 2     | 18.0        | 16.7        | 21.5      | 21.6      | 21          | 82.1        | 81.3      |           |
| 4     | 19.5        | 19.0        | 23.5      | 24.0      | 22          | 87.0        | 86.1      |           |
| 6     | 21.9        | 22.6        | 23.5      | 24.0      | 23          | 91.9        | 91.0      |           |
| 8     | 24.6        | 27.4        | 23.5      | 24.0      | 24          | 97.0        | 96.1      |           |
|       |             |             |           |           | 25          | 102.1       | 101.3     |           |
|       |             |             |           |           | 26          | 107.4       | 106.6     |           |
|       |             |             |           |           | 27          | 112.7       | 112.0     |           |

Actually this is the energy of the ground state band for the $\beta$ vibration, because of the $\Lambda$ dependence in each case. The energy for all the bands, for the $\beta$ vibrations is found to be

$$\epsilon_n = \frac{1}{2}[ k_0 + \frac{1}{2}a(3 + 2\Delta_1 + 2\Delta_2 + \Delta_1\Delta_2) + 2a(2 + \Delta_1 + \Delta_2)n + a^2n^2], \quad n = 0, 1, 2, \ldots, (14)$$

with $\Delta_1 \equiv \sqrt{1 + 4k_1}$, $\Delta_2 \equiv \sqrt{1 + 4k_1^2}$. The ground state band is obtained from $n = 0$, while the quasi-$\beta_1$ band is obtained from $n = 1$, and the quasi-$\beta_2$ band is obtained from $n = 2$.

4. Fitting

In [5] the above energy spectrum was fitted for the $\gamma$-unstable and axially symmetric prolate nuclei. The fitting measure was the Gaussian error,

$$\sigma = \sqrt{\frac{\sum_{i=1}^{n}(E_i(exp) - E_i(th))^2}{(n-1)E(2^+_1)^2}}. \quad (15)$$
In the axially symmetric prolate case the free parameters are $(\beta_0, a, c)$. A comparison to experimental data [9] is presented in Table 1, which shows fitting results for the ground state, $\beta_1$ and $\gamma_1$ bands in the cases of $^{238}$U and $^{162}$Dy. Both the bandheads and the spacings within bands are in general well reproduced. This is particularly true for the ground state and the $\gamma_1$ bands. The deviation in the gsb of $^{162}$Dy reaches 0.6% at $L = 18$, while in the gsb of $^{238}$U it reaches 1.7% at $L = 30$. The experimental levels of the $\gamma_1$ band of $^{162}$Dy (up to $L = 14$) extend over 28.4 energy units, while the corresponding theoretical predictions spread over 28.7 units, the difference being of the order of 1%. Similarly in $^{238}$U the experimental spread of the $\gamma_1$ band (up to $L = 27$) is 89.1 energy units, while the theoretical one is 87.3 units, the difference being of the order of 2%.

On the other hand, the theoretical level spacings within the $\beta_1$ bands are larger than the experimental ones. It is worth mentioning that results for $B(E2)$ transition rates in axially symmetric prolate nuclei [5], reveal a similar behavior. $B(E2)$s are in overall good agreement with the experimental data, apart from the $\beta_1 \rightarrow$ gsb transitions. The theoretical predictions are overestimating these transitions. This behavior along with the $\beta_1$ bands could be seen as an effect of the Davidson potential. It rises too fast in its right part as seen in Fig. 1, thus producing a large gap between the ground state and the $\beta_1$ bands and in addition increasing their interlevel spacing. However, as it has been pointed out in [10], the form of the quadrupole operator should be tested by changing the mass coefficients of the Bohr Hamiltonian.

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