Mean-variance portfolio selection under partial information with drift uncertainty

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Abstract

This paper studies a mean-variance portfolio selection problem under partial information with drift uncertainty. It is proved that all the contingent claims in this model are attainable in the sense of Xiong and Zhou [11]. Further, we propose a numerical scheme to approximate the optimal portfolio. Malliavin calculus and the strong law of large numbers play important roles in this scheme.

Keywords: Mean-variance portfolio selection, nonlinear filtering, Clark-Ocone formula, Malliavin calculus, partial information, drift uncertainty.

1 Introduction

The mean-variance portfolio selection model pioneered by Markowitz [9] has paved the foundation for modern portfolio theory and has been widely applied in financial economics. Markowitz proposed and solved the model in a single period setting. For half of a century, however, the optimal dynamic mean-variance portfolio selection problem was not solved due to the non-separable structure of the variance minimization problem in the sense of dynamic programming. This difficulty was finally overcome by Li and Ng [5] and Zhou and Li [14],

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who adopted an embedding scheme for multi-period and continuous-time cases, respectively. Since then, many scholars have been devoted in recent years to the study of the dynamic extensions of the Markowitz model, see, for example, Li et al. [6], Lim and Zhou [8], Zhou and Yin [15], Hu and Zhou [4], Bielecki et al. [2], Li and Zhou [7], Chiu and Li [1] in continuous-time settings. These works assume that the driving Brownian motions are completely observable to the investors. In the reality, however, the driving Brownian motions are often not observable to the investors, and the stock prices are the only observable information based on which the investors make decisions. This fact motivates the study of the so-called partial information portfolio selection problem. Xiong and Zhou [11] established the separation principle to separate the filtering and optimization problems for the mean-variance portfolio selection problem with partial information. They also developed analytical and numerical approaches in obtaining the filter as well as solving the related backward stochastic differential equation (BSDE). However, the numerical scheme proposed there is not very efficient. In order to demonstrate the inefficiency of that scheme, we first recall the model and some definitions in [11] for the convenience of the reader.

Assume that \((\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})\) is a complete filtered probability space, which represents the financial market. The filtration \(\{\mathcal{F}_t\}_{t \geq 0}\) satisfies the usual conditions, and \(P\) denotes the probability measure. Xiong and Zhou [11] considered a market consisting of \(d\) stocks and a bond whose price are stochastic processes \(S_i(t), i = 0, 1, 2, \cdots, d\), governed by the following stochastic differential equations (SDEs):

\[
\begin{align*}
    dS_i(t) &= S_i(t) \left( \mu_i(t)dt + \sum_{j=1}^{m} \tilde{\sigma}_{ij}(t) \right) d\tilde{W}_j(t), \quad i = 1, 2, \cdots, d, \\
    dS_0(t) &= S_0(t)\mu_0(t)dt, \quad t \geq 0,
\end{align*}
\]

where \(\tilde{W} := (\tilde{W}_1, \cdots, \tilde{W}_m)^*\) is an \(\mathcal{F}_t\)-adapted standard Brownian motion; \(\mu_i(t), i = 1, 2, \cdots, d\), are the appreciation rate processes of the stocks; \(\mu_0(t)\) is the interest rate process; and the \(d \times m\) matrix-valued process \(\tilde{\Sigma}(t) := (\tilde{\sigma}_{ij}(t))\) is the volatility process. They also assumed that the \(d \times d\) matrix \(A(t) := (a_{ij}(t))\) is of full rank a.s. (almost surely), where

\[
a_{ij}(t) := \sum_{k=1}^{m} \tilde{\sigma}_{ik}(t)\tilde{\sigma}_{jk}(t), \quad i, j = 1, 2, \cdots, d.
\]

Here and hereafter we use \(M^*\) to denote the transpose of a matrix \(M\).

Let

\[
\mathcal{G}_t := \sigma(S_i(s) : s \leq t, i = 0, 1, 2, \cdots, d), \quad t \geq 0.
\]

In this paper, the partial information means that the filtration \(\mathcal{G}_t\), rather than \(\mathcal{F}_t\tilde{W}\) (the filtration generated by \(\tilde{W}\)), is the only information available to the investors at time \(t\), so that he/she has to make decisions based on \(\mathcal{G}_t\) only.
It is easy to show that the quadratic covariation process between \( \log S_i(t) \) and \( \log S_j(t) \) is given by \( \int_0^t a_{ij}(s)ds \). Therefore, the matrix-valued process \( A(t) \) is \( \mathcal{G}_t \)-adapted. Because \( A(t) \) is symmetric and positive definite, it has a square root. Let \( \Sigma(t) := (\sigma_{ij}(t)) \) be a \( \mathcal{G}_t \)-adapted square root of \( A(t) \). Then it is invertible and completely observable.

Denote by \( L^2_\mathcal{G}(0,T;\mathbb{R}^n) \) the set of all \( \mathbb{R}^n \)-valued, \( \mathcal{G}_t \)-adapted processes \( f(t) \) such that \( \mathbb{E}\int_0^T |f(t)|^2dt < \infty \). Then \( L^2_\mathcal{G}(0,T;\mathbb{R}^n) \) becomes a Hilbert space endowed with the norm

\[
\|f\|_{L^2_\mathcal{G}(0,T;\mathbb{R}^n)} := \left( \mathbb{E}\int_0^T |f(t)|^2dt \right)^{\frac{1}{2}}.
\]

Let \( L^2(\Omega, \mathcal{G}_T, P) \) be the set of all \( \mathbb{R} \)-valued, \( \mathcal{G}_T \)-measurable random variables \( X \) such that \( \mathbb{E}(X^2) < \infty \). Similarly \( L^2(\Omega, \mathcal{G}_T, P) \) becomes a Hilbert space endowed with the norm

\[
\|X\|_{L^2(\Omega, \mathcal{G}_T, P)} := \left( \mathbb{E}(X^2) \right)^{\frac{1}{2}}.
\]

Let \( x(t) \) denote the wealth process and \( u_i(t) \) denote the amount invested in the \( i \)th stock, \( i = 1, 2, \cdots, d \), at time \( t \). For a self-financing portfolio \( u(t) := (u_1(t), u_2(t), \cdots, u_d(t)) \), the wealth process satisfies the following wealth equation:

\[
dx(t) = \left( \mu_0(t)x(t) + \sum_{i=1}^{d} (\mu_i(t) - \mu_0(t))u_i(t) \right) dt + \sum_{i=1}^{d} \sum_{j=1}^{m} \tilde{\sigma}_{ij}(t)u_i(t)d\tilde{W}_j(t), \quad t \geq 0. \tag{1.3}
\]

The partial observed mean-variance portfolio selection model is formulated as the following optimization model:

\[
\text{Minimize} \quad \text{Var}(x(T)) = \mathbb{E}(x(T) - \mathbb{E}x(T))^2 \tag{1.4}
\]

subject to

\[
\begin{cases}
\quad u(t) \text{ is self-financing and admissible}, \\
\quad (x(t), u(t)) \text{ satisfies (1.3) with initial wealth } x_0, \\
\quad \mathbb{E}x(T) = z,
\end{cases} \tag{1.5}
\]

where \( x_0, z \in \mathbb{R} \) are given constants.

Let \( \nu(t) := (\nu_1(t), \cdots, \nu_d(t))^* \) be defined by

\[
\nu(t) = \int_0^t \Sigma(s)^{-1}d\log S(s) - \int_0^t \Sigma(s)^{-1}\left( \bar{\mu}(s) - \frac{1}{2} \bar{A}(s) \right)ds, \tag{1.6}
\]

where

\[
\begin{align*}
\bar{\mu}_i(s) &:= \mathbb{E}(\mu_i(s)|\mathcal{G}_s), \quad \log S(s) := (\log S_1(s), \cdots, \log S_1(s))^*, \\
\bar{\mu}(s) &:= (\bar{\mu}_1(s), \cdots, \bar{\mu}_d(s))^*, \text{ and } \bar{A}(s) := (a_{11}(s), \cdots, a_{dd}(s))^*.
\end{align*}
\]
It is a $d$-dimensional $\mathcal{G}_t$-adapted Brownian motion and called the innovation process.

Note that $M^i_t := \int_0^t \sum_{j=1}^m \tilde{\sigma}_{ij}(s) d\tilde{W}_j(s)$ is a martingale with the quadratic covariation process $\langle M^i, M^k \rangle_t = \int_0^t a_{hk}(t) dt$. By the martingale representation theorem, there exists a standard Brownian motion $W := (W_1, \cdots, W_d)$ on $(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\}_{t \geq 0})$ such that

$$
\sum_{j=1}^m \tilde{\sigma}_{ij}(t) d\tilde{W}_j(t) = \sum_{j=1}^m \sigma_{ij}(t) dW_j(t), \quad i = 1, 2, \cdots, d,
$$

(see, Xiong and Zhou [11]). It is proved in [11] that the wealth process $x(t)$ satisfies the following SDE:

$$
dx(t) = \left(\mu_0(t)x(t) + \sum_{i=1}^d (\bar{\mu}_i(t) - \mu_0(t)) u_i(t)\right) dt + \sum_{i,j=1}^d \sigma_{ij}(t) u_i(t) d\nu_j(t), \quad t \geq 0. \quad (1.7)
$$

Further, they constructed a $\mathcal{G}_t$-adapted real-valued process $\rho(t)$ such that $\rho(t)x(t)$ is a $\mathcal{G}_t$-martingale, where $\rho(t)$ satisfies

$$
d\rho(t) = -\rho(t)\mu_0(t) dt - \sum_{j=1}^d \rho(t) \theta_j(t) d\nu_j(t), \quad \rho(0) = 1,
$$

with

$$
\theta_j(t) := \sum_{i=1}^d \sigma_{ij}(t)(\bar{\mu}_i(t) - \mu_0(t)).
$$

Assume $\theta_j$, $j = 1, 2, \cdots, d$, are uniformly bounded, then $E\rho(T)^p < \infty$ for all $p > 1$.

Finally, Xiong and Zhou reduced the optimization problem (1.4) to seeking the optimal solution for following static optimization problem

$$
\min_{v \in \mathbb{H}} E(v - z)^2 \quad (1.8)
$$

subject to constraints

$$
E v = z \quad \text{and} \quad E(\rho(T)v) = x_0. \quad (1.9)
$$

Here $\mathbb{H} := L^2(\Omega, \mathcal{G}_T, P)$.

**Definition 1.1.** A contingent claim $v \in \mathbb{H}$ is called attainable if there is $\Phi(s) \in L^2_{\mathcal{G}}(0, T; \mathbb{R}^d)$ such that

$$
v \rho(T) = E(v \rho(T)) + \int_0^T \Phi(s)^* d\nu(s). \quad (1.10)
$$

Denote the collection of all attainable contingent claims by $\text{AC}(\mathcal{G})$. Then $\text{AC}(\mathcal{G})$ is a subspace of $\mathbb{H}$. Denote by $\mathbb{H}_0$ the closure of $\text{AC}(\mathcal{G})$ under the norm $\| \cdot \|_{L^2(\Omega, \mathcal{G}_T, P)}$. It is a subset of $\mathbb{H}$. 

Definition 1.2. The market is complete if $H_0 = H$.

Xiong and Zhou [11] did not show if the market is complete. We will give an affirmative answer to this question in the subsequent section.

It was shown in [11] that the optimal solution $v$ to the optimization problem (1.8) under constraint (1.9) is given by

$$v = \frac{(z\langle \beta, \beta \rangle_H - x_0\langle \alpha, \beta \rangle_H)\alpha + (-z\langle \alpha, \beta \rangle_H + x_0\langle \alpha, \alpha \rangle_H)\beta}{\langle \alpha, \alpha \rangle_H (\beta, \beta)_H - \langle \alpha, \beta \rangle_H^2} \tag{1.11}$$

where $\alpha, \beta$ are the orthogonal projections on $H_0$ of 1 and $\rho(T)$, respectively.

To replicate $v$ given by (1.11), they need to find a solution of the following BSDE:

$$\begin{cases} dx(t) = \left(x(t)\mu_0(t) + \sum_{j=1}^{d}(\bar{\mu}_j(t) - \mu_0)u_j(t)\right)dt + \sum_{i,j=1}^{d}\sigma_{ij}(t)u_i(t)d\nu_j(t), 0 \leq t \leq T, \\
x(T) = v. \tag{1.12} \end{cases}$$

The clue of finding the numerical solution of (1.12) is as follows. Note that

$$N(t) := \mathbb{E}(v\rho(T)|\mathcal{G}_t) = \mathbb{E}(v\rho(T)) + \sum_{j=1}^{d}\int_{0}^{t}\Phi^j(s)d\nu_j(s), \tag{1.13}$$

can be approximated using the strong law of large numbers (SLLN). The main difficulty of solving the backward stochastic differential equation is in the calculation of $\Phi(t)$. To this end, they first divide $[0, t]$ into $n$ subintervals and approximate the quadratic covariation process

$$A_t := \langle N, \nu \rangle_t = \int_{0}^{t}\Phi(s)ds, \tag{1.14}$$

by the discrete version over the partitioning points. They further divide each subinterval mentioned above into $m$ smaller ones and obtain an approximation of $\Phi(s)$, $s \leq t$. This procedure is not computationally efficient because the double partition increases the error dramatically. In this paper, we use the Clark-Ocone formula, which is based on the Malliavin calculus, to represent $\Phi$. Together with the SLLN we find a more efficient scheme to approximate the solution of BSDE (1.12). The error of our method consists of those from the Euler approximation and those from the SLLN.

As a further demonstration of our numerical method, we will study the mean-variance problem under partial information with drift uncertainty. We will establish the convergence of our numerical approximation of the optimal portfolio. It is clear that the original mean-variance problem under partial information can be regarded as the one under partial information with drift fixed.
Note that (1.11) together with a numerical scheme were obtained in [11] under the completeness assumption. At the end of that paper, they pointed out that how to calculate \( \alpha \) and \( \beta_i \) (the projection of \( \rho_i(T, t) \) on \( \mathbb{H}_0 \)) numerically is still not known if the market is not complete, and left it as an open problem. The first aim of this paper is to improve the results of [11] to prove that the market is indeed complete. We will also give a different numerical scheme which involves the Malliavin calculus to approximate the optimal terminal and the related backward stochastic differential equation.

The rest of the paper is organized as follows. Some preliminary results on filtering and Malliavin calculus are given in Section 2. We also prove that the market is complete in that section. In Section 3, we will establish an efficient numerical scheme to solve the filtering problem based on the Malliavin calculus. A couple of numerical examples are then presented.

## 2 Completeness of the market

In this section, we state some elementary facts about stochastic filtering and Malliavin calculus for the convenience of the reader. We refer the reader to Sections 8.1-8.3 of Kallianpur [3] for more details about the general filtering problem and the stochastic equation of the optimal filter, and the book of Nualart and Nualart [10] about the Malliavin calculus.

Let \( (\Omega, \mathcal{A}, P) \) be a complete probability space and \( \mathcal{F}_t \) is an increasing family of sub \( \sigma \)-fields of \( \mathcal{A} \). The signal process \( h_t(\omega) \) and the observation process \( Z_t(\omega), \ (t \in [0, T]) \) will be assumed to be \( N \)-dimensional processes defined on \( (\Omega, \mathcal{A}, P) \) and further related as follows:

\[
Z_t(\omega) = \int_0^t h_u(\omega)du + W_t(\omega),
\]

where \( W_t \) is an \( N \)-dimensional Wiener process, and \( h_t(\omega) \) is a \( \mathbb{R}^N \)-valued, \( (t, \omega) \)-measurable function satisfying

\[
\int_0^T \mathbb{E}|h_t|^2dt < \infty,
\]

where \( | \cdot | \) denotes the Euclidean norm of \( N \)-dimensional vector. Further, for each \( s \in [0, T] \), the \( \sigma \)-fields:\( \mathcal{F}_s^{h,W} := \sigma\{h_u, W_u, 0 \leq u \leq s\} \) and \( \mathcal{F}_s^{T} := \sigma\{W_v - W_u, s \leq u < v \leq T\} \) are independent. Let \( \{\mathcal{F}_t^Z\}_{0 \leq t \leq T} \) be the filtration generated by \( Z_t \). This filtration is called the observation \( \sigma \)-fields. Let \( v_t := (v_1^t, \cdots, v_N^t) \) be an \( N \)-dimensional \( \mathcal{F}_t^Z \)-adapted innovation process, which is also a \( \mathcal{F}_t^Z \)-adapted Brownian motion.

The following theorem appears in Section 8.3 of [3] (page 208). We state it here for the convenience of the reader.
Theorem 2.1. Under conditions (2.1) to (2.2), every separable, square-integrable $\mathcal{F}_t$-martingale $Y_t$ is sample-continuous and has the representation

$$Y_t - \mathbb{E}(Y_0) = \sum_{i=1}^{N} \int_0^t \Phi_s^i dw_s^i,$$

where

$$\int_0^T \mathbb{E}|\Phi_s|^2 ds < \infty$$

and $\Phi_s := (\Phi_s^1, \ldots, \Phi_s^N)$ is jointly measurable and adapted to $\mathcal{F}_t$.

The next theorem is called the Clark-Ocone formula (see Theorem 6.1.1 of [10]). It expresses in the integrand of the integral representation theorem of a square integrable random variable in terms of the conditional expectation of its Malliavin derivative. Let $B = (B_t)_{t \geq 0}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P)$, where $(\mathcal{F}_t)_{t \geq 0}$ is the natural filtration of $B$ and $\mathcal{F} = \vee_{t \geq 0} \mathcal{F}_t$. Denote by $D$ the Malliavin derivative operator. We define the Sobolev space $D^{1,2}$ of random variables as follows:

$$D^{1,2} = \left\{ F \in L^2(\Omega, \mathcal{F}, P) : \|F\|_{1,2}^2 = \mathbb{E}|F|^2 + \mathbb{E}\left[ \int_0^\infty |D_t F|^2 dt \right] < \infty \right\}.$$

Theorem 2.2 (Clark-Ocone formula). Let $F \in D^{1,2} \cap L^2(\Omega, \mathcal{F}_T, P)$. Then, $F$ admits the following representation

$$F = \mathbb{E}(F) + \int_0^T \mathbb{E}(D_t F | \mathcal{F}_t) dB_t.$$

Our main theoretical contribution of this paper is as follows.

Theorem 2.3. The market is complete.

Proof. Since $\mathbb{H}_0 \subseteq \mathbb{H}$, it suffices to show $\mathbb{H} \subseteq \mathbb{H}_0$. For any $V \in \mathbb{H}$, let $V_n = V \min\{|V|^{-\frac{1}{n}}, 1\}$. Then

$$(V_n - V)^2 = V^2 1_{|V| > 1}(|V|^{-\frac{1}{n}} - 1)^2 \leq V^2.$$

Since $V \in \mathbb{H}$, we have $\mathbb{E}|V|^2 < \infty$ and by the dominated convergent theorem,

$$\lim_{n \to \infty} \|V_n - V\|_{L^2(\Omega, \mathcal{G}_T, P)} = \lim_{n \to \infty} \mathbb{E}[(V_n - V)^2] = \mathbb{E}\left[ \lim_{n \to \infty} V^2 1_{|V| > 1}(|V|^{-\frac{1}{n}} - 1)^2 \right] = 0.$$

Therefore, if we can show $V_n \in AC(\mathcal{G})$, then $V$ is in the closure of $AC(\mathcal{G})$ under the normal $\| \cdot \|_{L^2(\Omega, \mathcal{G}_T, P)}$, namely $V \in \mathbb{H}_0$, and the claim follows.

Notice for any $n \geq 1$

$$\mathbb{E}|V_n|^{2 + \frac{1}{n}} = \mathbb{E}\left[ |V|^{(1 - \frac{1}{n}) \left( 2 + \frac{1}{n} \right)} 1_{|V| > 1} + |V|^{2 + \frac{2}{n}} 1_{|V| \leq 1} \right] \leq \mathbb{E}(|V|^2 + 1) < \infty,$$
so $V_n \in L^{2+\frac{1}{p}}$. By Hölder’s inequality, we have
\[
\mathbb{E}|V_n\rho(T)|^2 \leq \left( \mathbb{E}\left[|V_n|^{2(1+\frac{1}{2n})}\right] \right)^{\frac{2n}{2n+1}} \left( \mathbb{E}\rho(T)^{2(2n+1)} \right)^{\frac{1}{2n+1}} < \infty,
\]
as $\mathbb{E}\rho(T)^p < \infty$, for all $p > 1$. Hence $\mathbb{E}(V_n\rho(T)|\mathcal{G}_t)$ is a square integrable martingale. By Theorem 2.1, we have
\[
\mathbb{E}(V_n\rho(T)|\mathcal{G}_t) - \mathbb{E}(V_n\rho(0)) = \int_0^t \Phi(s)^* d\nu(s).
\]
(2.5)
for some $\Phi(s) \in L^2(0,T;\mathbb{R}^d)$. When $t = T$, since $V_n\rho(T)$ is $\mathcal{G}_T$ adapted, we have
\[
V_n\rho(T) - \mathbb{E}(V_n\rho(0)) = \int_0^T \Phi(s)^* d\nu(s),
\]
(2.6)
which implies $V_n \in AC(\mathcal{G})$.

3 Model under partial information with drift uncertainty

The optimal redeeming problem of stock loans under drift uncertainty has been studied by Xu and Yi [13], where the drift uncertainty means the inherent uncertainty of the trend (bull and bear) of the stock. The borrower does not know the current trend of the stock so that she/he has to make decisions based on incomplete information. They derive the optimal redeeming strategies based on the prediction of the stock trend. In this section, we study a mean-variance problem under drift uncertainty.

For simplicity, we assume there is only one stock in the market (so that $d = 1$) and the risk-free interest rate is $r$. The price process of the stock is denoted by $S_t$, $t \geq 0$, which satisfies the stochastic differential equation (SDE):
\[
dS_t = \mu S_t dt + S_t dW_t,
\]
(3.1)
where $\mu$ is random and independent of the Brownian motion $W$, and it may only takes two possible values $a$ and $b$ that satisfy
\[
\Delta := a - b > 0.
\]
The stock is said to be in its bull trend when $\mu = a$, and in its bear when $\mu = b$.

The information up to time $t$ is given by
\[
\mathcal{G}_t := \sigma(S_s : s \leq t).
\]
The posteriori probability $\pi = (\pi_t)_{t \geq 0}$ is defined as

$$\pi_t := P(\mu = a | G_t). \quad (3.2)$$

Denote by $Y_t$ the wealth process of an agent, and $u_t$ the self-financing portfolio which is $G_t$-adapted. Under the self-financing condition, the wealth process $Y_t$, starting with an initial wealth $y_0 > 0$, satisfies the following wealth equation:

$$dY_t = (\mu u_t + (Y_t - u_t)r) \, dt + u_t dW_t. \quad (3.3)$$

Now the goal is to find the best portfolio $u_t$, which is $G_t$-adapted, to

Minimize $\text{Var}(Y_T)$ \quad (3.4)

subject to

1. $u_t$ is self-financing and admissible,
2. $(Y_t, u_t)$ satisfies (3.3) with initial wealth $Y_0 = y_0$,
3. $EY_T = z$.

(3.5)

Taken as observation, the log-price process $L = (\log S_t)_{t \geq 0}$ satisfies the following SDE

$$dL_t = (\mu - \frac{1}{2}) dt + dW_t. \quad (3.6)$$

Then, the innovation process

$$\nu_t = L_t - \int_0^t (b - \frac{1}{2} + \Delta \pi_s) ds \quad (3.7)$$

is a Brownian motion with respect to the observation filtration $G_t$. By equations (3.6) and (3.7), we have

$$W_t = \nu_t + \int_0^t (b - \mu + \Delta \pi_s) ds. \quad (3.8)$$

Furthermore, $\pi_t$ satisfies the following SDE:

$$d\pi_t = \Delta \pi_t (1 - \pi_t) d\nu_t. \quad (3.9)$$

**Theorem 3.1.** The optimal terminal wealth for the problem (3.4) is

$$v = \frac{z E \rho_T^2 - y_0 E \rho_T + (y_0 - z E \rho_T) \rho_T}{\text{Var}(\rho_T)}, \quad (3.10)$$

where $\rho_T$ is given by

$$\rho_t := \exp \left( - \int_0^t (b - r + \Delta \pi_s) d\nu_s - \int_0^t (r + \frac{1}{2}(b - r + \Delta \pi_s)^2) ds \right). \quad (3.11)$$
Proof. For any self-financing admissible portfolio \( u_t \), the corresponding wealth process \( Y_t \) satisfies the following SDE:

\[
\begin{align*}
    dY_t &= (u_t(b + \Delta \pi_t) + (Y_t - u_t)r) \, dt + u_t \, d\nu_t, \\
    d\pi_t &= \Delta \pi_t(1 - \pi_t) \, d\nu_t, \\
    Y_0 &= y_0, \quad \pi_0 \text{ fixed.}
\end{align*}
\] (3.12)

Applying Itô’s formula to \( \rho_t \), we get

\[
d\rho_t = -r \rho_t dt - (b - r + \Delta \pi_t) \rho_t d\nu_t.
\] (3.13)

Further, applying Itô’s formula to \( Y_t \rho_t \), we have

\[
d(Y_t \rho_t) = (Y_t (r \rho_t - \mu \rho_t) + u_t \rho_t) \, d\nu_t.
\]

Therefore, \( Y_t \rho_t \) is a \( \mathcal{G}_t \)-martingale and we have

\[\mathbb{E}(Y_t \rho_t) = y_0.\]

To find the optimal portfolio, we see the best \( \mathcal{G}_T \)-measurable terminal wealth \( v \) to minimize the variance \( \mathbb{E}(v - z)^2 \) subject to constraints

\[
\begin{align*}
    \mathbb{E}v &= z \quad \text{and} \quad \mathbb{E}(\rho_T v) = y_0.
\end{align*}
\] (3.14)

By the completeness result of Theorem 2.3, the optimal solution (3.10) follows from the formula (1.11).

After we find the optimal terminal wealth, we then seek the portfolio to realize it.

**Theorem 3.2.** The optimal portfolio is given by

\[
u_t = (b - r + \Delta \pi_t)Y_t + \phi_t \eta_t,
\] (3.15)

where \( \eta_t \in L^2_G(0, T; \mathbb{R}^d) \) such that

\[
\mathbb{E}(\theta | \mathcal{G}_t) = \mathbb{E}\theta + \int_0^t \eta_s d\nu_s, \quad \forall t \in [0, T],
\] (3.16)

and \( \theta = \rho_T Y_T \).
Proof. Summarizing the results we obtained in the proof of last theorem, we seek a solution to the following forward-backward SDE:

\[
\begin{align*}
    dY_t &= (u_t (b + \Delta \pi_t) + (Y_t - u_t) r) \, dt + u_t d\nu_t, \quad Y_0 = y_0, \\
    d\pi_t &= \Delta \pi_t (1 - \pi_t) \, d\nu_t, \\
    Y_T &= \frac{z \mathbb{E}\rho_T^2 - y_0 \mathbb{E}\rho_T + (y_0 - z \mathbb{E}\rho_T) \rho_T}{\text{Var}(\rho_T)}, \\
    d\rho_t &= -r \rho_t \, dt - (b - r + \Delta \pi_t) \, \rho_t \, d\nu_t, \quad \rho_0 = 1,
\end{align*}
\]

(3.17)

with \( \pi_0 \) given.

To prove the invertibility of \( \rho_t \), we define \( \Phi_t \) by the following SDE:

\[
\begin{align*}
    d\Phi_t &= (r + (b - r + \Delta \pi_t)^2) \Phi_t \, dt + (b - r + \Delta \pi_t) \Phi_t \, d\nu_t, \\
    \Phi_0 &= 1.
\end{align*}
\]

(3.18)

Apply Itô’s formula to \( \rho_t \Phi_t \), we have

\[
    d(\rho_t \Phi_t) = 0
\]

so that \( \rho_t \Phi_t \equiv \rho_0 \Phi_0 = 1 \). Since \( \rho_t Y_t \) is a martingale, then

\[
    Y_t = \rho_t^{-1} \mathbb{E}(\rho_T Y_T | \mathcal{G}_t) = \rho_t^{-1} \mathbb{E}(\theta | \mathcal{G}_t) = \Phi_t \mathbb{E}(\theta | \mathcal{G}_t).
\]

(3.19)

Finally, first using (3.16) and (3.18) to apply Itô’s formula to \( Y_t \) given by (3.19), and then comparing the result with (3.17), we get the expression (3.15) of the optimal portfolio.

To propose a numerical approximation of the optimal portfolio given by (3.15), the key is the approximation of the integrand \( \eta_t \) in (3.16) which is difficult to calculate directly. We will use the Clark-Ocone formula from Malliavin calculus to get an expression of \( \eta_t \). In fact, it will be the conditional expectation of a Malliavin derivative. Our numerical scheme will be based on this representation.

**Theorem 3.3.** We can represent \( \eta_t \) as \( \mathbb{E}(D_t \theta | \mathcal{G}_t) \) where \( D_t \) is the Malliavin derivative operator. Further,

\[
    D_u \theta = (c_1 + 2c_2 \rho_T) \, D_u \rho_T,
\]

(3.20)

where \( c_1 = \frac{z \mathbb{E}\rho_T^2 - y_0 \mathbb{E}\rho_T}{\text{Var}(\rho_T)} \) and \( c_2 = \frac{y_0 - z \mathbb{E}\rho_T}{\text{Var}(\rho_T)} \) are constants, and \( D_u \rho_T \) is given by

\[
    D_u \rho_T = \rho_T \left[ - \int_u^T \Delta(b - r + \Delta \pi_s) \, D_u \pi_s \, ds - (b - r + \Delta \pi_u) + \int_u^T \Delta D_u \pi_s \, d\nu_s \right],
\]

(3.21)

with

\[
    D_u \pi_s = \Delta \pi_u (1 - \pi_u) \exp \left( \int_u^s \Delta(1 - \pi_r) \, d\nu_r - \frac{1}{2} \int_u^s \Delta(1 - 2 \pi_r) \, dr \right).
\]

(3.22)
Proof. Note that 
\[ \theta = \rho_T Y_T = c_1 \rho_T + c_2 \rho_T^2, \]
so \((3.20)\) follows by applying the Malliavin derivative on both sides.

Recall that 
\[ \rho_T = \exp \left( - \int_0^T [r + \frac{1}{2} (b - r + \Delta \pi_s)^2] ds - \int_0^T (b - r + \Delta \pi_s) d\nu_s \right), \]
a direct calculation yields \((3.21)\).

Recall equation \((3.9)\), 
\[ \pi_s = \pi_0 + \int_0^s \Delta \pi_r (1 - \pi_r) d\nu_r. \]
Applying Malliavin derivative to both sides, we get 
\[ D_u \pi_s = \Delta \pi_u (1 - \pi_u) + \int_u^s \Delta (1 - \pi_r) D_u \pi_r d\nu_r. \] (3.23)
Then, \((3.22)\) follows by solving the linear SDE \((3.23)\). Finally, \((3.20)\) follows from the Clark-Ocone formula given in Section 2. \(\square\)

Based on the last theorem, it is easy to show that 
\[ \eta_u = \mathbb{E} (D_u \theta | G_u) := N_1(u) + \Delta N_2(u) + \Delta N_3(u), \]
with \(N_j(u) = \mathbb{E} (I_j | G_u), j = 1, 2, 3, \) where 
\[ I_1 = -(c_1 \rho_T + 2 c_2 \rho_T^2) (b - r + \Delta \pi_u), \] (3.24)
\[ I_2 = (c_1 \rho_T + 2 c_2 \rho_T^2) \int_u^T D_u \pi_s d\nu_s, \] (3.25)
\[ I_3 = -(c_1 \rho_T + 2 c_2 \rho_T^2) \int_u^T (b - r + \Delta \pi_s) D_u \pi_s ds, \] (3.26)
and \(D_u \pi_s\) is given by \((3.22)\).

To approximate \(\mathbb{E}(\rho_t | G_u)\), we use conditional SLLN such that \(\rho^i_t\) is given by \((3.17)\) with \(\nu_s\) be replaced by \(\nu^i_s\) for \(s \geq u\), where \(\nu^i, i = 1, 2, \cdots\) are independent copies of \(\nu\). More precisely, we define the following processes \(\rho^i(t, t')\) with two time-indices as follows: For \(t \leq t'\), \(\rho^i(t, t') = \rho_t\), and for \(t \geq t'\),
\[ d\rho^i(t, t') = r\rho^i(t, t') dt - (b - r + \Delta \pi^i(t, t')) \rho^i(t, t') d\nu^i_t, \quad \rho^i(t', t') = \rho(t'). \] (3.27)
Let \(\pi^i(t, t')\) be defined similarly.

By conditional SLLN, we can easily prove the following identities.
Proposition 3.1.

\[
N_1(u) = -(b - r + \Delta \pi_u) \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} (c_1 \rho^i(T, u) + 2c_2(\rho^i(T, u))^2),
\]

\[
N_2(u) = \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \left( c_1 \rho^i(T, u) + 2c_2(\rho^i(T, u))^2 \right) \int_u^T D_u \pi^i(s, u) d\nu^i(s),
\]

\[
N_3(u) = \lim_{m \to \infty} \frac{1}{m} \sum_{i=1}^{m} \left( c_1 \rho^i(T, u) + 2c_2(\rho^i(T, u))^2 \right) \int_u^T (b - r + \Delta \pi^i(s, u)) D_u \pi^i(s, u) ds.
\]

In order to approximate \( N_k(u), (k = 1, 2, 3) \), we use Euler’s scheme to approximate the stochastic integrals. For notational simplicity, from now on we assume \( T = 1 \). Then, we discrete the time interval \([0, 1]\) into \( n \) small intervals and let \( \delta = \frac{1}{n} \).

Firstly, we define \( \rho^{i, \delta}(t, t'), \pi^{i, \delta}(t, t'), t, t' \geq 0 \), in two steps.

For \( l \leq k \), let

\[
\pi^{\delta}(l\delta, k\delta) := \pi^{\delta}((l-1)\delta, k\delta) + \Delta \pi^{\delta}((l-1)\delta, k\delta) \left( 1 - \pi^{\delta}((l-1)\delta, k\delta) \right) \left( \nu_{l\delta} - \nu_{(l-1)\delta} \right)
\]

with \( \pi^{\delta}(0, k\delta) := c \) (c is a constant in \([0, 1]\)); and let

\[
\rho^{\delta}(l\delta, k\delta) := \rho^{\delta}((l-1)\delta, k\delta) - r\delta \rho^{\delta}((l-1)\delta, k\delta)
\]

\[
- (b - r + \Delta \pi^{\delta}((l-1)\delta, k\delta)) \rho^{\delta}((l-1)\delta, k\delta) \left( \nu_{l\delta} - \nu_{(l-1)\delta} \right).
\]

Next we need to approximate \( N_i(u) \) by \( N_i^{m, \delta}(k\delta) \), \((i = 1, 2, 3; m \) is related to the SLLN, which will be chosen later). For all \( s \in [u, T], u \in [0, T] \), let \( k = [nu], j = [ns] \). Then \( u \in [k\delta, (k+1)\delta) \) and \( s \in [j\delta, (j+1)\delta) \). We define \( N_i^{m, \delta}(k\delta), (i = 1, 2, 3) \) as follows:

\[
N_1^{m, \delta}(k\delta) = -(b - r + \Delta \pi^\delta(k\delta)) \frac{1}{m} \sum_{i=1}^{m} \left( c_1 \rho^{i, \delta}(T, k\delta) + 2c_2(\rho^{i, \delta}(T, k\delta))^2 \right).
\]

\[
N_2^{m, \delta}(k\delta) = \frac{1}{m} \sum_{i=1}^{m} \left( c_1 \rho^{i, \delta}(T, k\delta) + 2c_2(\rho^{i, \delta}(T, k\delta))^2 \right) S_2^{i, \delta}(T, k\delta),
\]

\[
N_3^{m, \delta}(k\delta) = \frac{1}{m} \sum_{i=1}^{m} \left( c_1 \rho^{i, \delta}(T, k\delta) + 2c_2(\rho^{i, \delta}(T, k\delta))^2 \right) S_3^{i, \delta}(T, k\delta),
\]
where
\[
S_{2}^{i,\delta}(T, k\delta) = \sum_{l=1}^{n-k} D_{k\delta}\pi_{i,\delta}^{(l+k-1)\delta, k\delta} (\nu_{t\delta}^{i} - \nu_{(i-1)\delta}^{i}) ,
\]
\[
S_{3}^{i,\delta}(T, k\delta) = \sum_{l=1}^{n-k} \delta(b - r + \Delta\pi_{i,\delta}^{(l+k-1)\delta, k\delta}) D_{k\delta}\pi_{i,\delta}^{(l+k-1)\delta, k\delta}.
\]

From the above, \(D_{k\delta}\pi_{i,\delta}^{(j\delta, k\delta)}(j = k, \cdots, n-1)\) are still stochastic integrals. By (3.23), we define \(D_{k\delta}\pi_{i,\delta}^{(j\delta, k\delta)}\) only in one step.

Since \(j \geq k\), we let
\[
D_{k\delta}\pi_{i,\delta}^{(j\delta, k\delta)} := D_{k\delta}\pi_{i,\delta}^{(j\delta-1, k\delta)} + \Delta \pi_{i,\delta}^{(j\delta-1, k\delta)} (\nu_{j\delta}^{i} - \nu_{(j-1)\delta}^{i})
\]
with \(D_{k\delta}\pi_{i,\delta}^{(k\delta, k\delta)} = \Delta\pi_{k\delta}(1 - \pi_{k\delta})\).

Finally, we obtain
\[
\eta_{k\delta}^{m} = N_{1}^{m,\delta}(k\delta) + \Delta N_{2}^{m,\delta}(k\delta) + \Delta N_{3}^{m,\delta}(k\delta). \tag{3.30}
\]

**Lemma 3.1.**
\[
\|\eta_{u} - \eta_{k\delta}^{m}\|_{2} \leq C \left( \sqrt{{\delta}} + \frac{1}{\sqrt{m}} \right), \tag{3.31}
\]
where \(C\) is a constant.

**Proof.** Let \(\pi_{u}^{\delta}, \rho_{T}^{\delta}\) be the Euler approximations with step size \(\delta\) starting at time \(t = 0\) at \(\pi_{0}\) and \(\rho_{0}\), respectively. As is known, the Euler approximation is of strong convergence order \(\frac{1}{2}\), i.e.
\[
\mathbb{E}|\rho_{T} - \rho_{T}^{\delta}| \leq C\sqrt{\delta}, \quad \mathbb{E}|\pi_{u} - \pi_{u}^{\delta}| \leq C\sqrt{\delta}.
\]
From the approximation in (3.30), we estimate the error between \(\eta\) and \(\eta_{k\delta}^{m}\) for three parts.
\[
\|\text{err}_{1}\|_{2} = \left\|\mathbb{E}(I_{1}|\mathcal{G}_{u}) - N_{1}^{m,\delta}(k\delta)\right\|_{2}
\leq |b - r| \left\| \mathbb{E}(c_{1}\rho_{T} + 2c_{2}\rho_{T}^{2}|\mathcal{G}_{u}) - \frac{1}{m} \sum_{i=1}^{m} (c_{1}\rho_{i,\delta}^{(T, k\delta)} + 2c_{2}(\rho_{i,\delta}^{(T, k\delta)})^{2}) \right\|_{2}
\leq \Delta \left\| \pi_{u} \mathbb{E}(c_{1}\rho_{T} + 2c_{2}\rho_{T}^{2}|\mathcal{G}_{u}) - \frac{\pi_{k\delta}^{\delta}}{m} \sum_{i=1}^{m} (c_{1}\rho_{i,\delta}^{(T, k\delta)} + 2c_{2}(\rho_{i,\delta}^{(T, k\delta)})^{2}) \right\|_{2}
:= |b - r| J_{1} + \Delta J_{2}, \tag{3.32}
\]
using the Hölder inequality, we have

\begin{align*}
J_1 & \leq \left\| E \left( c_1 \rho_T + 2c_2 \rho_T^2 | \mathcal{G}_u \right) - E \left( c_1 \rho_T^\delta + 2c_2 (\rho_T^\delta)^2 | \mathcal{G}_u \right) \right\|_2 \\
& \quad + \left\| E \left( c_1 \rho_T^\delta + 2c_2 (\rho_T^\delta)^2 | \mathcal{G}_u \right) - \frac{1}{m} \sum_{i=1}^m \left( c_1 \rho^{i,\delta}(T, k\delta) + 2c_2 (\rho^{i,\delta}(T, k\delta))^2 \right) \right\|_2 \\
& \leq C \left( \sqrt{\delta} + \frac{1}{\sqrt{m}} \right), \quad (3.33)
\end{align*}

and

\begin{align*}
J_2 & \leq \left\| (\pi_u - \pi_{k\delta}^\delta) \right\|_4 \left\| E \left( c_1 \rho_T + 2c_2 \rho_T^2 | \mathcal{G}_u \right) \right\|_4 \\
& \quad + \left\| \pi_{k\delta}^\delta \right\|_4 \left\| E \left( c_1 \rho_T + 2c_2 \rho_T^2 | \mathcal{G}_u \right) - \frac{1}{m} \sum_{i=1}^m \left( c_1 \rho^{i,\delta}(T, k\delta) + 2c_2 (\rho^{i,\delta}(T, k\delta))^2 \right) \right\|_4 \\
& \leq C \left( \sqrt{\delta} + \frac{1}{\sqrt{m}} \right). \quad (3.34)
\end{align*}

For the third error, let \( err_3 = E(I_3|\mathcal{G}_u) - N_3^{\pi,\delta}(k\delta) \), then

\begin{align*}
\| err_3 \|_2 & \leq \left\| E \left( (c_1 \rho_T + 2c_2 \rho_T^2) \int_u^T (b - r + \Delta \pi_s) D_u \pi_s ds | \mathcal{G}_u \right) \\
& \quad - E \left( (c_1 \rho_T^\delta + 2c_2 (\rho_T^\delta)^2) S_3^{i,\delta}(T, k\delta) | \mathcal{G}_u \right) \right\|_2 \\
& \quad + \left\| E \left( (c_1 \rho_T^\delta + 2c_2 (\rho_T^\delta)^2) S_3^{i,\delta}(T, k\delta) | \mathcal{G}_u \right) \\
& \quad - \frac{1}{m} \sum_{i=1}^m \left( c_1 \rho^{i,\delta}(T, k\delta) + 2c_2 (\rho^{i,\delta}(T, k\delta))^2 \right) S_3^{i,\delta}(T, k\delta) \right\|_2 \\
& \leq C \left( \sqrt{\delta} + \frac{1}{\sqrt{m}} \right),
\end{align*}

similarly, we have \( \| err_2 \|_2 \leq C \left( \sqrt{\delta} + \frac{1}{\sqrt{m}} \right). \) Finally, the total error is dominated by

\begin{align*}
\left\| \eta_u - \eta_{k\delta}^{\delta,m} \right\|_2 \leq C \left( \sqrt{\delta} + \frac{1}{\sqrt{m}} \right)
\end{align*}

which converges to 0 if we take \( m = n \) (in this case, \( \delta = \frac{1}{m} \)). \( \square \)

The errors in our numerical scheme only consist of the errors from Euler approximation and those from SLLN. From this point of view, the numerical scheme we proposed is more efficient that that by [11]. We use Matlab to simulate the wealth process \( Y_t \) and the admissible process \( u_t \) by adopting the numerical scheme we proposed.
We set the parameters as following: \( n = 1000, \delta = \frac{1}{1000}, m = 1000, r = 0.04, a = 0.04, b = 0.032, \Delta = 0.008, y_0 = 100, a = 0.04, b = 0.032, \Delta = 0.008, \) and let \( z = y_0 \cdot (1 + r + 0.03), \pi_0 = 0.1. \)

The following is the numerical results for the innovation process \( \nu_t \), the wealth process \( y_t \) and the self-finance admissible process.

![Figure 1: particle representation with \( m = 10^3 \).](image)

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