SMOOTH DENSITIES OF THE LAWS OF PERTURBED DIFFUSION PROCESSES

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ABSTRACT. Under some regularity conditions on $b, \sigma$ and $\alpha$, we prove that the following perturbed stochastic differential equation

\begin{equation}
X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + \alpha \sup_{0 \leq s \leq t} X_s, \quad \alpha < 1
\end{equation}

admits smooth densities for all $0 \leq t \leq t_0$, where $t_0 > 0$ is some finite number.

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1. INTRODUCTION

There have been a considerable body of literatures devoted to the study of perturbed stochastic differential equations (SDEs), see [1]-[7], [9], [11]. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$, let $\{B_t\}_{t \geq 0}$ be a one-dimensional standard $\{\mathcal{F}_t\}_{t \geq 0}$-Brownian Motion. Suppose that $\sigma(x), b(x)$ are Lipschitz continuous functions on $\mathbb{R}$. It was proved in [5] that the following perturbed stochastic differential equation:

\begin{equation}
X_t = x + \int_0^t b(X_s)ds + \int_0^t \sigma(X_s)dB_s + \alpha \sup_{0 \leq s \leq t} X_s, \quad \forall \alpha < 1,
\end{equation}

admits a unique solution. If $|\sigma(x)| > 0$, it was shown in [11] that the law of $X_t$ is absolutely continuous with respect to Lebesgue measure, i.e. the law of $X_t$ admits a density for $t > 0$.

There seem no smooth density results for the law of a perturbed diffusion process, this paper aims to partly fill in this gap. The smoothness of densities is a popular topic in stochastic analysis and has been intensively studied for several decades, we refer readers to [8], [10] and references therein. Our approach to proving the smoothness of densities is by Malliavin calculus, so let us first recall some well known results on Malliavin calculus [8] to be used in this paper.

Let $\Omega = C_0(\mathbb{R}_+)$ be the space of continuous functions on $\mathbb{R}_+$ which are zero at zero. Denote by $\mathcal{F}$ the Borel $\sigma$-field on $\Omega$ and $\mathbb{P}$ the Wiener measure, then the canonical
coordinate process \( \{ \omega_t, t \in \mathbb{R}_+ \} \) on \( \Omega \) is a Brownian motion \( B_t \). Define \( \mathcal{F}_t^0 = \sigma(B_s, s \leq t) \) and denote by \( \mathcal{F}_t \) the completion of \( \mathcal{F}_t^0 \) with respect to the \( \mathbb{P} \)-null sets of \( \mathcal{F} \).

Let \( H := L^2(\mathbb{R}_+, \mathcal{B}, \mu) \) where \((\mathbb{R}_+, \mathcal{B})\) is a measurable space with \( \mathcal{B} \) being the Borel \( \sigma \)-field of \( \mathbb{R}_+ \) and \( \mu \) being the Lebesgue measure on \( \mathbb{R}_+ \), we denote the norm of \( H \) by \( \|\cdot\|_H \).

Let \( H \in \mathcal{D} \) be a continuously differentiable function space \( \mathcal{D} \) such that \( H \in \mathcal{S} \) and all of its partial derivatives have polynomial growth. Let \( \mathcal{D} \) be the set of smooth random variables defined by

\[ \mathcal{S} = \{ F = f(W(h_1), \ldots, W(h_n)); h_1, \ldots, h_n \in H, n \geq 1, f \in C^\infty_0(\mathbb{R}^n) \}. \]

Let \( F \in \mathcal{S} \), define its Malliavin derivative \( D_t F \) by

\[ D_t F = \sum_{i=1}^n \partial_i f(W(h_1), \ldots, W(h_n)) h_i(t), \]

and its norm by

\[ \| F \|_{1,2} = [\mathbb{E}(|F|^2) + \mathbb{E}(|D_t F|_H^2)]^{\frac{1}{2}}, \]

where \( ||D_t F||_H^2 = \int_0^\infty |D_t F|^2 \mu(dt) \). Denote by \( \mathbb{D}^{1,2} \) the completion of \( \mathcal{S} \) under the norm \( \|\cdot\|_{1,2} \). We further define the norm

\[ ||F||_{m,2} = \left[ \mathbb{E}(|F|^2) + \sum_{k=1}^m \mathbb{E}(|D^k F|_H^2) \right]^{\frac{1}{2}}. \]

Similarly, \( \mathbb{D}^{m,2} \) denotes the completion of \( \mathcal{S} \) under the norm \( \|\cdot\|_{m,2} \).

We shall use the following two propositions:

**Proposition 1.1** (Proposition 1.2.3 of [8]). Let \( \phi : \mathbb{R}^d \to R \) be a continuously differentiable function with bounded partial derivatives. Suppose that \( F = (F^1, \ldots, F^d) \) is a random vector whose components belong to the space \( \mathbb{D}^{1,2} \). Then \( \phi(F) \in \mathbb{D}^{1,2} \), and

\[ D(\phi(F)) = \sum_{i=1}^d \partial_i \phi(F) D F^i. \]

**Proposition 1.2** (Proposition 2.1.5 of [8]). If \( F \in \mathbb{D}^{\infty,2} \) with \( \mathbb{D}^{\infty,2} = \cap_{m \geq 1} \mathbb{D}^{m,2} \) and \( \|D F\|_H^{-1} \in \cap_{p \geq 1} L^p(\Omega), \) then the density of \( F \) belongs to the infinitely continuously differentiable function space \( C^\infty(\mathbb{R}) \).

Throughout this paper, for a bounded measurable function \( f \), we shall denote

\[ \|f\|_\infty = \sup_{x \in \mathbb{R}} |f(x)|. \]
2. Main Results

Throughout this paper, we need to assume $\alpha < 1$ to guarantee that Eq. (1.1) has a unique solution [5]. Furthermore, it is shown in [11] that

**Theorem 2.1.** ([11, Theorem 3.1]) Let $(X_t)_{t \geq 0}$ be the unique solution to Eq. (1.1). Then $X_t \in D^{1,2}$ for all $t > 0$.

**Theorem 2.2.** ([11, Theorem 3.2]) Assume that $\sigma$ and $b$ are both Lipschitz continuous, and $|\sigma(x)| > 0$ for all $x \in \mathbb{R}$. Then, for $t > 0$, the law of $X_t$ is absolutely continuous with respect to Lebesgue measure.

In this paper, we shall prove the following results about the smoothness of densities:

**Theorem 2.3.** Assume that $b$ is bounded smooth and that $\sigma(x) \equiv \sigma$. If $\alpha < 1$, $t_0 > 0$ and $b$ satisfy

$$\theta(t_0, \alpha, b) < \frac{1}{2},$$

with $\theta(t_0, \alpha, b) := \sqrt{2\|b\|_\infty^2 + 8\alpha^2 + \|b\|_\infty^2 t_0^2 + 4\alpha^2}$, then the law of $X_t$ in (1.1) admits a smooth density for all $t \in (0, t_0]$.

**Theorem 2.4.** Assume that $b$ is bounded smooth, and $\sigma$ is bounded smooth with $\|\sigma\|_\infty < \infty$, $\|\sigma''\|_\infty < \infty$ and $\inf_{x \in \mathbb{R}} |\sigma(x)| > 0$. Let

$$F(y) = \int_x^y \frac{1}{\sigma(u)} du, \quad y \in (-\infty, \infty)$$

and $\tilde{b}(x) = \frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))} - \frac{1}{2}\sigma'(F^{-1}(x))$, then $\tilde{b}$ is bounded smooth with $\|\tilde{b}\|_\infty < \infty$. If $\alpha < 1$, $t_0 > 0$ and $b$ satisfy

$$\theta(t_0, \alpha, \tilde{b}) < \frac{1}{2}$$

with $\theta(t_0, \alpha, \tilde{b}) := \sqrt{2\|\tilde{b}\|_\infty^2 t_0^2 + 8\alpha^2 + \|\tilde{b}\|_\infty^2 t_0^2 + 4\alpha^2}$, then the law of $X_t$ in (1.1) admits a smooth density for all $t \in (0, t_0]$.

**Proofs of Theorems 2.3 and 2.4:** The main idea is to use Proposition 1.2 to prove the two theorems. To verify the conditions in Proposition 1.2, it suffices to prove that $X_t \in D^{m,2}$ for all $m \geq 1$ and $\|DX_t\|_H \geq c > 0$ a.s. for some constant $c > 0$.

Theorem 2.3 immediately follows from Lemmas 3.1 and 3.4 below.

Now we prove Theorem 2.4. Recall $Y_t = \int_x^{X_t} \frac{1}{\sigma(u)} du$ in Lemma 3.5 below, by the condition of $\sigma$, $F$ is a continuous and strictly increasing function with bounded derivative and thus

$$\|DY_t\|_H = \|DF(X_t)\|_H \leq \frac{1}{\inf_{x \in \mathbb{R}} |\sigma(x)|} \|DX_t\|_H.$$ 

Hence, by Lemmas 3.1 and 3.5 below, under the same condition as in Theorem 2.4 we have

$$\|DX_t\|_H \geq \inf_{x \in \mathbb{R}} |\sigma(x)| \cdot \|DY_t\|_H \geq \inf_{x \in \mathbb{R}} |\sigma(x)| \cdot \frac{[1 - 2\theta(t_0, \alpha, \tilde{b})]t}{2(1 + 2\|\tilde{b}\|_\infty^2 t_0^2 + 2\alpha^2)} \quad t \in [0, t_0].$$
Hence, \( X_t \) admits a smooth density for all \( t \in (0, t_0] \).

\[ \square \]

3. Auxiliary Lemmas

It is well known that \( \|DX_t\|_H \) has the following representation [11] for all \( t > 0 \):

\[
\|DX_t\|_H = \left( \int_0^t |D_r X_t|^2 dr \right)^{\frac{1}{2}}
\]

with \( D_r X_t \) satisfying

\[
D_r X_t = \sigma(X_r) + \int_r^t D_r b(X_s) ds + \int_r^t D_r \sigma(X_s) dB_s + \alpha D_r \left( \sup_{0 \leq s \leq t} X_s \right).
\]

We shall often use the following fact ([11], [8])

\[
D_r X_t = 0 \quad \text{if} \quad r > t,
\]

\[
\left\| D \left( \sup_{0 \leq s \leq t} X_s \right) \right\|_H \leq \sup_{0 \leq s \leq t} \|DX_s\|_H,
\]

where

\[
\left\| D \left( \sup_{0 \leq s \leq t} X_s \right) \right\|_H^2 = \int_0^t \left| D_r \left( \sup_{0 \leq s \leq t} X_s \right) \right|^2 dr, \quad \|DX_t\|_H^2 = \int_0^t |D_r X_t|^2 dr.
\]

3.1. \( X_t \) is an element in \( \mathbb{D}^{m,2} \) for all \( t > 0 \) and \( m \geq 1 \).

**Lemma 3.1.** Let \( X_t \) be the solution of the perturbed stochastic differential equation (1.1), and suppose that the coefficients \( b \) and \( \sigma \) are smooth with bounded derivatives of all orders. Then \( X_t \) belongs to \( \mathbb{D}^{m,2} \) for all \( t > 0 \) and all \( m \geq 1 \).

**Proof.** We shall use Picard iteration to prove the lemma. Letting \( X_t^0 = x_0 \) for all \( t > 0 \), define \( X_t^{n+1} \) be the unique, adapted solution to the following equation:

\[
X_t^{n+1} = x_0 + \int_0^t \sigma(X_s^n) dB_s + \int_0^t b(X_s^n) ds + \alpha \max_{0 \leq s \leq t} \left( X_s^{n+1} \right),
\]

which obviously implies

\[
\max_{0 \leq s \leq t} \left( X_s^{n+1} \right) = x_0 + \max_{0 \leq s \leq t} \left( \int_0^t \sigma(X_s^n) dB_s + \int_0^t b(X_s^n) ds \right) + \alpha \max_{0 \leq s \leq t} \left( X_s^{n+1} \right).
\]

Therefore,

\[
\max_{0 \leq s \leq t} \left( X_s^{n+1} \right) = \frac{x_0}{1 - \alpha} + \frac{1}{1 - \alpha} \max_{0 \leq s \leq t} \left( \int_0^t \sigma(X_s^n) dB_s + \int_0^t b(X_s^n) ds \right),
\]
this and (3.4) further gives
\[
X_{i+1} = \frac{x_0}{1 - \alpha} + \int_0^t \sigma(X^n_u)dB_s + \int_0^t b(X^n_u)du
\]
+ \frac{\alpha}{1 - \alpha} \max_{0 \leq s \leq t} \left( \int_0^s \sigma(X^n_u)dB_u + \int_0^s b(X^n_u)du \right).
\]
By the above representation of \(X_{i+1}\) and a standard method [5], for every \(t > 0\) we have
\[
\lim_{n \to \infty} X^n_t = X_t \quad \text{in} \quad L^2(\Omega).
\]
Let \(m \geq 1\), it is standard to check that \(X^n_t \in \mathbb{D}^{m,2}\) for every \(t > 0\) and \(n \geq 1\) [11, Theorem 3.1]. By a similar argument as in [11, Theorem 3.1], we have
\[
\sup_{n \geq 1} \mathbb{E} \left[\|D^k X^n_t\|_{H^\otimes k}^2\right] < \infty, \quad k = 1, ..., m.
\]
Next we prove \(X_t \in \mathbb{D}^{m,2}\) by the argument of [8, Proposition 1.2.3]. Indeed, by (3.6), there exists some subsequence \(D^k X^n_{i,j}\) weakly converges to some \(\alpha_k\) in \(L^2(\Omega, H^\otimes k)\) for \(k = 1, ..., m\). By (3.5) and the remark immediately below [8, Proposition 1.2.2], the projections of \(D^k X^n_{i,j}\) on any Wiener chaos converge in the weak topology of \(L^2(\Omega)\), as \(n_j\) tends to infinity, to those of \(\alpha_k\) for \(k = 1, ..., m\). Hence, \(X_t \in \mathbb{D}^{m,2}\) and \(D^k X_t = \alpha_k\) for \(k = 1, ..., m\). Moreover, for any weakly convergent subsequence the limit must be equal to \(\alpha_1, ..., \alpha_m\) by the same argument as above, and this implies the weak convergence of the whole sequence.

3.2. Additive noise case. If \(\sigma(x) \equiv \sigma\), then Eq. (3.1) reads as
\[
D_r X_t = \sigma + \int_r^t D_r b(X_s)ds + \frac{\alpha D_r}{\sup_{0 \leq s \leq t} X_s}.
\]

Lemma 3.2. Let \(t > 0\) be arbitrary and \(b\) be bounded smooth with \(\|b\|_\infty < \infty\). For all \(0 < t_1 < t_2 \leq t\), we have
\[
\left\| DX_{t_2} \right\|_{H^\otimes k}^2 - \left\| DX_{t_1} \right\|_{H^\otimes k}^2 \leq 2 \left[ \sqrt{2\|b\|_\infty^2(t_2 - t_1)^2 + 8\alpha^2} + \|b\|_\infty^2(t_2 - t_1)^2 + 4\alpha^2 \right] \sup_{0 \leq s \leq t} \left\| DX_s \right\|_{H^\otimes k}^2.
\]

Proof. It is easy to see that
\[
\left\| DX_{t_2} \right\|_{H^\otimes k}^2 - \left\| DX_{t_1} \right\|_{H^\otimes k}^2 = \left| \int_{t_1}^{t_2} (D_r X_{t_2})^2 dr - \int_{0}^{t_1} (D_r X_{t_1})^2 dr \right| \leq I_1 + I_2,
\]
where
\[
I_1 := \int_{t_1}^{t_2} (D_r X_{t_2})^2 dr, \quad I_2 := \int_{0}^{t_1} \left| (D_r X_{t_2})^2 - (D_r X_{t_1})^2 \right| dr.
\]
We claim that
\[
(3.8) \quad \int_0^{t_2} (D_r X_{t_2} - D_r X_{t_1})^2 \, dr \leq 2 \left[ \| b' \|_\infty^2 (t_2 - t_1)^2 + 4\alpha^2 \right] \sup_{0 \leq s \leq t} \| D X_s \|_H^2.
\]
and we will prove it in the last part of this proof.

Let us now estimate $I_1$ and $I_2$ by (3.8). Observe
\[
I_1 = \int_{t_1}^{t_2} (D_r X_{t_2} - D_r X_{t_1})^2 \, dr \leq \int_0^{t_2} (D_r X_{t_2} - D_r X_{t_1})^2 \, dr,
\]
by (3.8) we have
\[
I_1 \leq 2 \left[ \| b' \|_\infty^2 (t_2 - t_1)^2 + 4\alpha^2 \right] \sup_{0 \leq s \leq t} \| D X_s \|_H^2.
\]
Further observe
\[
I_2 \leq \left[ \int_0^{t_1} (D_r X_{t_2} - D_r X_{t_1})^2 \, dr \right]^\frac{1}{2} \left[ \int_0^{t_1} |D_r X_{t_2} + D_r X_{t_1}|^2 \, dr \right]^\frac{1}{2}
\leq \sqrt{2} \left[ \int_0^{t_1} (D_r X_{t_2} - D_r X_{t_1})^2 \, dr \right]^\frac{1}{2} \left[ \int_0^{t_1} |D_r X_{t_2}|^2 + |D_r X_{t_1}|^2 \, dr \right]^\frac{1}{2}
\leq \sqrt{2} \left[ \int_0^{t_1} (D_r X_{t_2} - D_r X_{t_1})^2 \, dr \right]^\frac{1}{2} \left[ \int_0^{t_2} |D_r X_{t_2}|^2 \, dr + \int_0^{t_1} |D_r X_{t_1}|^2 \, dr \right]^\frac{1}{2}
\leq 2 \left[ \int_0^{t_1} (D_r X_{t_2} - D_r X_{t_1})^2 \, dr \right]^\frac{1}{2} \sup_{0 \leq s \leq t} \| D X_s \|_H
\leq 2 \left[ \int_0^{t_2} (D_r X_{t_2} - D_r X_{t_1})^2 \, dr \right]^\frac{1}{2} \sup_{0 \leq s \leq t} \| D X_s \|_H,
\]
this inequality and (3.8) gives
\[
I_2 \leq 2 \sqrt{2} \left[ \| b' \|_\infty^2 (t_2 - t_1)^2 + 4\alpha^2 \right] \sup_{0 \leq s \leq t} \| D X_s \|_H^2.
\]
Combining the estimates of $I_1$ and $I_2$, we immediately get the desired inequality in the lemma.

It remains to prove (3.8). By (3.7), we have
\[
(D_r X_{t_2} - D_r X_{t_1})^2 \leq 2 \left[ \int_{t_1}^{t_2} D_r b(X_s) \, ds \right]^2 + 2\alpha^2 \left| D_r \left( \sup_{0 \leq s \leq t_1} X_s \right) - D_r \left( \sup_{0 \leq s \leq t_2} X_s \right) \right|^2
\leq 2 \left[ \int_{t_1}^{t_2} D_r b(X_s) \, ds \right]^2 + 2\alpha^2 \left| D_r \left( \sup_{0 \leq s \leq t_1} X_s \right) \right|^2 + 4\alpha^2 \left| D_r \left( \sup_{0 \leq s \leq t_2} X_s \right) \right|^2.
\]
By Hölder inequality, (3.2) and Proposition 1.1, we have
\[
\int_0^{t_2} \left| \int_{t_1}^{t_2} D_r b(X_s) ds \right|^2 dr \leq \|b'\|_\infty^2 \int_0^{t_2} (t_2 - t_1) \int_{t_1}^{t_2} |D_r X_s|^2 ds dr \\
= \|b'\|_\infty^2 (t_2 - t_1) \int_{t_1}^{t_2} \sup_{0 \leq s \leq t} |D_r X_s|^2 dr ds \\
\leq \|b'\|_\infty^2 (t_2 - t_1)^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2.
\]
Moreover, by (3.3) and (3.2) we have
\[
\int_0^{t_2} \left| D_r \left( \sup_{0 \leq s \leq t_2} X_s \right) \right|^2 dr \leq \sup_{0 \leq s \leq t_2} \|DX_s\|_H^2 \leq \sup_{0 \leq s \leq t} \|DX_s\|_H^2, \\
\int_0^{t_2} \left| D_r \left( \sup_{0 \leq s \leq t_1} X_s \right) \right|^2 dr = \int_0^{t_1} \left| D_r \left( \sup_{0 \leq s \leq t_1} X_s \right) \right|^2 dr \leq \sup_{0 \leq s \leq t} \|DX_s\|_H^2.
\]
Collecting the above four inequalities, we immediately get the desired (3.8).

\[\textbf{Lemma 3.3.} \text{ Let } b \text{ be bounded smooth with } \|b'\|_\infty < \infty, \text{ we have} \]
\[\sup_{0 \leq s \leq t} \|DX_s\|_H^2 \geq \frac{\sigma^2 t}{2(1 + 2\|b'\|_\infty^2 t^2 + 2a^2)}, \quad t > 0. \tag{3.10}\]

\[\textbf{Proof.} \text{ By (3.7) and using } (a + b)^2 \geq \frac{1}{2} a^2 - b^2, \text{ we have} \]
\[(D_r X_t)^2 \geq \frac{1}{2} \sigma^2 - \left[ \int_r^t D_r b(X_s) ds + \alpha D_r \left( \sup_{0 \leq s \leq t} X_s \right) \right]^2 \\
\geq \frac{1}{2} \sigma^2 - 2 \left( \int_r^t D_r b(X_s) ds \right)^2 - 2\alpha^2 \left[ D_r \left( \sup_{0 \leq s \leq t} X_s \right) \right]^2.
\]
Further observe
\[
\int_0^t \left( \int_r^t D_r b(X_s) ds \right)^2 dr \leq \int_0^t (t - r) \int_r^t |D_r b(X_s)|^2 ds dr \\
\leq \int_0^t (t - r) \|b'\|_\infty^2 \int_r^t |D_r X_s|^2 ds dr \\
\leq \int_0^t \|b'\|_\infty^2 \int_r^t |D_r X_s|^2 ds dr \\
\leq \frac{t\|b'\|_\infty^2}{\sup_{0 \leq s \leq t} \|DX_s\|_H^2} \\
\leq \frac{t^2 \|b'\|_\infty^2}{\sup_{0 \leq s \leq t} \|DX_s\|_H^2}, \tag{3.11}
\]
where the second inequality is by Proposition 1.1. Hence,
\[ \|DX_t\|_H^2 \geq \frac{\sigma^2t}{2} - 2\|b'\|^2 t^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2 - 2\alpha^2 \|D(\sup_{0 \leq s \leq t} X_s)\|_H^2 \]
\[ \geq \frac{\sigma^2t}{2} - 2\|b'\|^2 t^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2 - 2\alpha^2 \|DX_s\|_H^2, \]
where the last inequality is by (3.3).
This clearly implies
\[ \sup_{0 \leq s \leq t} \|DX_s\|_H^2 \geq \frac{\sigma^2t}{2} - 2\|b'\|^2 t^2 \sup_{0 \leq s \leq t} \|DX_s\|_H^2 - 2\alpha^2 \|DX_s\|_H^2, \]
which immediately yields the desired bound. \qed

**Lemma 3.4.** Let \( b \) is bounded smooth with \( \|b'\|_\infty < \infty \) and \( \sigma(x) \equiv \sigma \) with \( \sigma \neq 0 \). If \( \alpha < 1, t_0 > 0 \) and \( b \) satisfy
\[ \theta(t_0, \alpha, b) < 1/2 \]
with \( \theta(r, \alpha, b) := \sqrt{2\|b'\|_\infty^2 + 8\alpha^2 + \|b'\|_\infty^2 + 4\alpha^2} \), then
\[ \|DX_t\|_H^2 \geq \frac{[1 - 2\theta(t_0, \alpha, b)]\sigma^2t}{2(1 + 2\|b'\|_\infty^2 + 2\alpha^2)}, \quad t \in [0, t_0]. \]  

**Proof.** Let \( t \in [0, t_0] \). For all \( 0 \leq t_1 \leq t_2 \leq t \), by Lemma 3.2, we have
\[ \|DX_{t_2}\|_H^2 - \|DX_{t_1}\|_H^2 \leq 2\theta(t_2 - t_1, \alpha, b) \sup_{0 \leq s \leq t} \|DX_s\|_H^2. \]
Hence, for all \( s \in [0, t] \),
\[ \|DX_s\|_H^2 \leq \|DX_t\|_H^2 - \|DX_{t_1}\|_H^2 + \|DX_s\|_H^2 \]
\[ \leq 2\theta(t - s, \alpha, b) \sup_{0 \leq s \leq t} \|DX_s\|_H^2 + \|DX_t\|_H^2, \]
and consequently
\[ \sup_{0 \leq s \leq t} \|DX_s\|_H^2 \leq 2\theta(t - s, \alpha, b) \sup_{0 \leq s \leq t} \|DX_s\|_H^2 + \|DX_t\|_H^2. \]
The above inequality and (3.10) further give
\[ \|DX_t\|_H^2 \geq \frac{[1 - 2\theta(t - s, \alpha, b)]\sigma^2t}{2(1 + 2\|b'\|_\infty^2 + 2\alpha^2)} \]
\[ \geq \frac{[1 - 2\theta(t_0, \alpha, b)]\sigma^2t}{2(1 + 2\|b'\|_\infty^2 + 2\alpha^2)}, \quad t \in [0, t_0]. \]
Combining the above inequality and Lemma 3.3 immediately gives the desired inequality. \qed
3.3. **Multiplicative noise case.** By the condition of $\sigma$, we have $\sup_{x \in \mathbb{R}} \sigma(x) < 0$ or $\inf_{x \in \mathbb{R}} \sigma(x) > 0$. Without loss of generality, we assume that

$$\inf_{x \in \mathbb{R}} \sigma(x) > 0.$$ 

Let us consider the following well known transform

$$F(X_t) = \int_x^{X_t} \frac{1}{\sigma(u)} du,$$

it is easy to see that $F$ is a strictly increasing function with bounded derivative. Hence,

$$\sup_{0 \leq s \leq t} F(X_s) = F\left(\sup_{0 \leq s \leq t} X_s\right).$$

By Itô formula, we have

$$F(X_t) = \int_0^t \left( \frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \int_0^t \frac{1}{\sigma(X_s)} dM_s$$

determined by $M_t = \sup_{0 \leq s \leq t} X_s$. It is easy to see that $M_t$ is an increasing function of $t$ and that $\frac{1}{\sigma(X_s)}$ has a contribution to the related integral only when $X_s = M_s$. Hence,

$$F(X_t) = \int_0^t \left( \frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \int_0^t \frac{1}{\sigma(M_s)} dM_s.$$

Since $M_t$ is a continuous increasing function with respect to $t$, we have

$$F(X_t) = \int_0^t \left( \frac{b(X_s)}{\sigma(X_s)} - \frac{1}{2} \sigma'(X_s) \right) ds + B_t + \alpha \sup_{0 \leq s \leq t} F(X_s).$$

Denote $Y_t = F(X_t)$, it solves the following perturbed SDE:

$$Y_t = \int_0^t \dot{b}(Y_s) ds + B_t + \alpha \sup_{0 \leq s \leq t} Y_s$$

where $\dot{b}(x) = \frac{b(F^{-1}(x))}{\sigma(F^{-1}(x))} - \frac{1}{2} \sigma'(F^{-1}(x))$. Applying Lemma 3.4, we get the following lemma about the dynamics $Y_t$:

**Lemma 3.5.** Assume that $b$ is bounded smooth and that $\sigma$ is bounded smooth with $\|\sigma\|_\infty < \infty$, $\|\sigma''\|_\infty < \infty$ and $\inf_{x \geq 0} |\sigma(x)| > 0$. Then $b$ is bounded smooth. If $\alpha < 1$, $t_0 > 0$ and $b$ satisfy

$$\theta(t_0, \alpha, \tilde{b}) < 1/2$$
with $\theta(r, \alpha, \tilde{b}) := \left[ \sqrt{2\|\tilde{b}\|_\infty^2 r^2 + 8\alpha^2} + \|\tilde{b}'\|_\infty^2 r^2 + 4\alpha^2 \right]$, then

$$\|DY_t\|_H^2 \geq \frac{[1 - 2\theta(t_0, \alpha, \tilde{b})]t}{2(1 + 2\|\tilde{b}'\|_\infty^2 t^2 + 2\alpha^2)}, \quad t \in [0, t_0].$$

(3.20)

**Proof.** It is easy to check that under the conditions in the lemma $\tilde{b}$ is bounded smooth with $\|\tilde{b}\|_\infty < \infty$. Hence, the lemma immediately follows from applying Lemma 3.4 to $Y_t$. □

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