Knotted 3-balls in $S^4$

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Abstract
The unknot $U$ in $S^4$ has non-unique smooth spanning 3-balls up to isotopy fixing $U$. Equivalently there are properly embedded non separating 3-balls in $S^1 \times B^3$ not properly isotopic to $\{\ast\} \times B^3$. In the process we construct subgroups of the low-dimensional homotopy groups of $\text{Diff}(S^1 \times B^n \text{ fix } \partial)$ via fiber sequences that relate these diffeomorphism groups to spaces of embeddings $\text{Emb}(S^1, S^1 \times S^n)$.

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1 Introduction

This paper introduces the study of knotted 3-balls in 4-manifolds, in particular the 4-sphere and $S^1 \times B^3$. A standard or linear 3-ball in $S^4$ denotes the intersection of $S^4$ with a 4-dimensional half-space in $\mathbb{R}^5$. We consider $S^4$ to be the unit sphere in $\mathbb{R}^5$. A 4-dimensional half-space $H$ in $\mathbb{R}^5$ is the solution-set to a pair of linear equations $H = \{v \in \mathbb{R}^5 : L_1(v) = 0, L_2(v) \geq 0\}$, where $L_1, L_2 : \mathbb{R}^5 \to \mathbb{R}$ are independent linear functions. A standard 3-ball in $S^4$ is defined as $S^4 \cap H$. Notice that the boundary of a standard 3-ball is a great 2-sphere, i.e. $\partial(S^4 \cap H) = \{p \in S^4 : L_1(p) = L_2(p) = 0\}$. A knotted ball in $S^4$ means a smoothly-embedded 3-ball $\Delta_1 \subset S^4$ whose boundary is a great 2-sphere, moreover one which is not isotopic, keeping the boundary a great 2-sphere, to a standard 3-ball $\Delta_0$.

The requirement that the boundary be constrained throughout the isotopy is necessary since any two embedded $k$-balls in the interior of a connected $n$-manifold are ambiently isotopic provided $k < n$ [6, p. 231, 22]. In the diagram above, the standard ball $\Delta$ is schematically depicted as the solid red geodesic semi-circle. Alternatively, standard 3-balls in $S^4$ could be described as geodesic 3-balls, i.e. apply the exponential map to a ball of radius $\pi/2$ centred at the origin, in a 3-dimensional subspace of a tangent space to $S^4$. This paper works in the smooth category and unless otherwise said, all mappings are smooth.

We say $N$ is a reducing 3-ball in $S^1 \times B^3$ if $N$ is a properly-embedded submanifold, diffeomorphic to $B^3$ such that the complement $(S^1 \times B^3) \setminus N$ is connected. By properly-embedded we mean that $N \cap \partial(S^1 \times B^3) = \partial N$. A reducing 3-ball $\Delta_1$ is knotted if it is not properly isotopic to the linear reducing 3-ball, $\{1\} \times B^3$. All reducing 3-balls are properly homotopic to $\{1\} \times B^3$.

Isotopy classes of reducing 3-balls in $S^1 \times B^3$ admit a monoid structure coming from an operation similar to boundary connect-sum, that we call concatenation. Given a reducing 3-ball, up to an essentially unique isotopy one can assume that it is standard on the boundary, i.e. $\{1\} \times \partial B^3$. Given any two such reducing 3-balls, one glues the two copies of $S^1 \times B^3$ together along $S^1 \times H$ where $H$ is a hemi-sphere in $\partial B^3$. This produces a new 4-manifold canonically diffeomorphic to $S^1 \times B^3$ together with a new reducing 3-ball. We will see in Sections 3 and 4 that this monoid structure has inverses, i.e. it is a group, with the unit being the linear reducing sphere. A less abstract way to describe this monoid structure would be to use an isotopy of $S^1 \times B^3$ to ensure the reducing balls agree with the linear ball, everywhere except perhaps a neighbourhood of some

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point of \{∗\} × B³. One performs the isotopy so that the two points are disjoint. Then the addition operation is simply the process of taking the ball that agrees with both, where the two balls are not linear, and linear where both balls are linear.

Isotopy classes of oriented 3-balls in S⁴ where the embeddings are required to be linear on the boundary also have a monoid structure, defined in essentially the same way, and these monoids are isomorphic. The key point is that the closed complement (the exterior) of the unknotted S² in S⁴ is diffeomorphic to S¹ × B³. We use the convention that if M is a manifold with boundary, then Diff(∂M) denotes the diffeomorphisms of M that are the identity on the boundary. Similarly, we use the notation Diff₀(M) to denote the subgroup of diffeomorphisms homotopic to the identity.

The diffeomorphism group Diff₀(S⁴) (resp. Diff(S¹ × B³ fix ∂)) acts transitively on 3-balls (resp. reducing balls) with common boundary. A diffeomorphism \( f : S¹ × B³ \to S¹ × B³ \) properly homotopic to the identity, gives rise to the 3-ball \( Δ₁ = f(\{1\} × B³) \) which is unknotted if and only if \( f \) is properly isotopic to a map supported in a 4-ball. The group of isotopy-classes of oriented 3-balls that are linear on their boundary is isomorphic to \( π₀(Diff(S¹ × B³ fix ∂)/Diff(B³ fix ∂)) \). See §3 and §4 for more details.

The main result of this paper is a construction of an infinite family of non-trivial elements of \( π₀(Diff(S¹ × B³ fix ∂)/Diff(B³ fix ∂)) \) with explicit constructions of the corresponding knotted 3-balls in \( S¹ × B³ \) and hence \( S⁴ \). The techniques of this paper also construct subgroups of \( π_{n−3}Diff(S¹ × Bⁿ fix ∂) \) whenever \( n ≥ 3 \). Indeed, we compute the homotopy-group \( π_{n−2}Emb(S¹, S¹ × Sⁿ) \) and show that it contains an infinitely-generated free abelian group provided \( n ≥ 3 \). For \( n = 3 \) we describe generators in terms of embeddings of tori \( T : S¹ × S¹ → S¹ × S³ \). Given such a torus \( T \) one constructs the family of embeddings as the map \( S¹ → Emb(S¹, S¹ × S³) \) by \( z ↦ T(z, z) \), i.e. a type of spinning construction. More generally we construct maps \( Sⁿ−2 → Emb(S¹, S¹ × Sⁿ) \) via embedded tori \( S¹ × Sⁿ−2 → S¹ × Sⁿ \), and analogous spinning constructions. We show there is an embedding of \( π_{n−2}Emb(S¹, S¹ × Sⁿ) \) into \( π_{n−3}Diff(S¹ × Bⁿ fix ∂) \) via a pair of fibre bundles that relates the homotopy-types of the underlying spaces. These (families) of diffeomorphisms of \( S¹ × Bⁿ \) are pseudo-isotopic to the identity, similar to the work of Hatcher-Wagoner [14], see Section 3.

Some further consequences of the results in this paper are:

Denote the component of the unknot in \( Emb(S², S⁴) \) by \( Emb_u(S², S⁴) \). We show that \( Emb_u(S², S⁴) \) does not have the homotopy-type of the subspace of linear embeddings. The latter has the homotopy-type of the Stiefel manifold \( V_{5,3} = SO₅/SO₂ \) while the former has a non-finitely-generated fundamental group. See Theorem 6.1.

We show that \( S¹ × B³ \) has, up to isotopy, infinitely many distinct fiberings over \( S¹ \) with fiber \( B³ \). Note that the exterior of the co-dimension 2 unknot in \( S⁴ \), i.e. the compliment of an open tubular neighbourhood of \( S² × \{0\} ⊂ S⁴ \) is diffeomorphic to \( S¹ × B³ \). See Theorem 3.5 and Corollary 3.6. Thus the spanning disc, i.e. the embedding \( D³ → S⁴ \) that has the unknotted \( S² \) in \( S⁴ \) as its boundary is not unique up to isotopy. This is in stark contrast to the unknot \( S¹ × \{0\} ⊂ S³ \) where the spanning disc \( D² → S³ \) is known to be unique up to isotopy, by a combination of Dehn’s Lemma and Alexander’s theorem.

We give a framework for approaching the smooth 4-dimensional Schönflies problem, describing the set of counter-examples as the fixed points of an endomorphism

\[ π₀Emb(B³, S¹ × B³) → π₀Emb(B³, S¹ × B³) \].

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The endomorphism is given by lifting such an embedding to a non-trivial finite-sheeted covering space of $S^1 \times B^3$. See Section 3.7. The non-trivial elements of $\pi_0 \text{Emb}(B^3, S^1 \times B^3)$ we construct in this paper all belong to the kernel of iterates of this endomorphism.

We show that every embedded non-separating $S^n$ in $S^1 \times S^n$ is the fibre of a fibre-bundle $S^1 \times S^n \to S^1$. Moreover, the diffeomorphism group $\text{Diff}(S^1 \times S^n)$ acts transitively on these non-separating $n$-spheres. Analogously, every reducing $B^n$ in $S^1 \times B^n$ is the fiber of a fiber-bundle over $S^1$, and $\text{Diff}(S^1 \times B^n)$ acts transitively on reducing $n$-balls. These results are true for all $n$. See Theorem 3.7 and Theorem 3.5 respectively.

The embedding space $\pi_0 \text{Emb}(B^i, S^{n-i} \times B^i)$ is a group for all $n$, provided $i > 0$, using the concatenation operation, see Lemma 3.4.

The paper begins with the computation of a range of homotopy groups for the embedding spaces $\text{Emb}(S^1, S^1 \times S^n)$ for $n \geq 3$ in Section 3.2. We construct an epimorphism from $\pi_{n-2} \text{Emb}(S^1, S^1 \times S^n)$ to a group $\Lambda_n^{W_0}$ by a simple transversality argument. The group $\Lambda_n^{W_0}$ is isomorphic to a direct-sum of a countable-infinite number of copies of $\mathbb{Z}$ together with perhaps one $\mathbb{Z}_2$ factor, depending on the magnitude of $W_0$ and the parity of $n$. The integer $W_0$ describes the path-component of $\text{Emb}(S^1, S^1 \times S^n)$ one is considering. Using the Embedding or Manifold Calculus we further show this map is an isomorphism. The $n = 3$ case is exceptional, as the map $\pi_1 \text{Emb}(S^1, S^1 \times S^3) \to \Lambda_3^{W_0}$ has infinite-cyclic kernel, which splits, i.e. we construct an isomorphism $\pi_1 \text{Emb}(S^1, S^1 \times S^3) \to \mathbb{Z} \oplus \Lambda_3^{W_0}$. Using a sequence of fibrations we relate the embedding space $\text{Emb}(S^1, S^1 \times S^n)$ to the diffeomorphism group $\text{Diff}(S^1 \times B^n \text{ fix } \partial)$, in Section 3.3. We argue the group $\pi_{n-3} \text{Diff}(S^1 \times B^n \text{ fix } \partial)$ contains a copy of $\Lambda_n^1$. Our subgroup of $\pi_0 \text{Diff}(S^1 \times B^3 \text{ fix } \partial)$ acts freely (but perhaps not transitively) on the isotopy-classes of embeddings $B^3 \to S^1 \times B^3$ that agree with the standard inclusion $\{1\} \times \partial B^3$ on the boundary.

The group $\pi_0 \text{Diff}(S^1 \times B^3 \text{ fix } \partial)$ is abelian, indeed, $\text{Diff}(S^1 \times B^n \text{ fix } \partial)$ is a $(n + 1)$-fold loop space compatible with the group multiplication [4]. Section 3.

Independently, Tadayuki Watanabe has announced the construction of an invariant

$$Z_1 : \pi_1 B\text{Diff}(S^1 \times B^3 \text{ fix } \partial) / \pi_1 B\text{Diff}(B^4 \text{ fix } \partial) \to A_1(R[t^\pm 1])$$

and conjectures that it is non-trivial on some diffeomorphisms created via his graph surgery construction. If so, it could potentially be related to our $W_2$ invariant, defined in Section 2.

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2 Embeddings of circles in \( S^1 \times S^n \)

In this section we describe a range of low-dimensional homotopy groups of \( \text{Emb}(S^1, S^1 \times S^n) \). These results were essentially known to Dax [8], who used a Haefliger-style parametrized double-point elimination process to describe the low-dimensional homotopy-groups of a variety embedding spaces. Given an element of an embedding space \( f \in \text{Emb}(S^1, S^1 \times S^n) \) we will denote the path-component of \( \text{Emb}(S^1, S^1 \times S^n) \) containing \( f \) by \( \text{Emb}_f(S^1, S^1 \times S^n) \).

We begin with the least technical elements in Theorem 2.5, describing for \( n \geq 3 \), three epimorphisms:

\[
W_0 : \pi_0 \text{Emb}(S^1, S^1 \times S^n) \to \mathbb{Z}
\]

\[
W_1 : \pi_1 \text{Emb}_f(S^1, S^1 \times S^n) \to \mathbb{Z}
\]

\[
W_2 : \pi_{n-2} \text{Emb}_f(S^1, S^1 \times S^n) \to \Lambda_n^{W_0}(f)
\]

The epimorphisms \( W_1 \) and \( W_2 \) are defined for all components of the embedding space. \( \Lambda_n^{W_0} \) is defined as a quotient of the Laurent polynomial ring \( \mathbb{Z}[t^{\pm 1}] \). \( \Lambda_n^{W_0} \) contains a free abelian subgroup of infinite-rank, but can also contain 2-torsion.

For this definition we consider the Laurent polynomial ring \( \mathbb{Z}[t^{\pm 1}] \) to be only a group. We define \( \Lambda_n^{W_0} \) to be the quotient group, \( \mathbb{Z}[t^{\pm 1}] \) modulo the subgroup generated by the relations

\[
( t^k + (-1)^n t^{W_0 - 1 - k} = 0 \ \forall k, \ t^0 = 0, t^{-1} = 0 ).
\]

The group \( \Lambda_n^{W_0} \) is the free abelian on the generators

\[
G_{W_0} = \{ k : k \in \mathbb{Z}, k \geq \frac{W_0 - 1}{2}, k \notin \{-1, 0, W_0, W_0 - 1\} \}
\]

with the sole exception when \( n \) is even, \( |W_0| > 2 \) and \( W_0 \) is odd. In this case one has the same generating set \( G_{W_0} \), but \( t_{\frac{W_0 - 1}{2}} \) represents 2-torsion. The remaining \( t^k \) are free generators, i.e. \( \Lambda_n^{W_0} \simeq \mathbb{Z}_2 \oplus \left( \bigoplus_{k : k \geq \frac{W_0 - 1}{2}, t^k \in G_{W_0}} \mathbb{Z} \right) \).

The definitions of the maps \( W_0, W_1 \) and \( W_2 \) will be elementary applications of basic transversality theory.

**Definition 2.1** Let \( \pi : S^1 \times S^n \to S^1 \) be defined as \( \pi(z, v) = z \). Given an embedding \( f : S^1 \to S^1 \times S^n \), define \( W_0(f) = \text{deg}(\pi \circ f) \in \mathbb{Z} \). This is the degree of the map \( \pi \circ f : S^1 \to S^1 \).

\[
\begin{array}{ccc}
S^1 & \xrightarrow{\pi \circ f} & S^1 \\
\downarrow{f} & & \uparrow{\pi} \\
S^1 \times S^n & & \\
\end{array}
\]

The value \( W_0(f) \) only depends on the homotopy-class of \( f \). Provided \( n \geq 3 \), the homotopy-class of \( f \) agrees with the isotopy class, by transversality. Thus

\[
W_0 : \pi_0 \text{Emb}(S^1, S^1 \times S^n) \to \mathbb{Z}
\]

is a bijection. An embedding \( f : S^1 \to S^1 \times S^n \) satisfying \( \pi(f(z)) = z^n \ \forall z \in S^1 \) would have \( W_0(f) = n \).
Definition 2.2 Given $F : S^1 \to \text{Emb}(S^1, S^1 \times S^n)$ we define
$$W_1(F) = \text{deg}(\hat{F}) \in \mathbb{Z}$$
where $\hat{F} : S^1 \to S^1$ is defined as $\hat{F}(z) = \pi(F(z)(1))$, i.e. we consider $F(z) \in \text{Emb}(S^1, S^1 \times S^n)$ and we evaluate it at $1 \in S^1$.

We can consider $W_1$ to be a function
$$W_1 : \pi_1\text{Emb}(S^1, S^1 \times S^n) \to \mathbb{Z}.$$
As a thought experiment, argue that given $[F] \in \pi_1\text{Emb}(S^1, S^1 \times S^n)$ satisfying $W_1(F) = k$, one can assume $\pi(F(z)(1)) = z^k \forall z \in S^1$. More generally, one can show $\text{Emb}(S^1, S^1 \times S^n) \simeq S^1 \times \text{Emb}^*(S^1, S^1 \times S^n)$ where $\text{Emb}^*(S^1, S^1 \times S^n)$ is the subspace of $\text{Emb}(S^1, S^1 \times S^n)$ where $\hat{F}(1) = 1$.

From this perspective, $W_1$ is simply the induced map from the projection onto the first factor, i.e. $\text{Emb}(S^1, S^1 \times S^n) \to S^1$.

The next invariant, $W_2$, has a definition in terms of a map
$$W_2 : \pi_{n-2}\text{Emb}_f^*(S^1, S^1 \times S^n) \to \Lambda_n^{W_0(f)}.$$

Definition 2.3 Let $C_2M$ denote the configuration space of pairs of distinct points in $M$,
$$C_2M = \{(p_1, p_2) \in M^2 : p_1 \neq p_2\}.$$ 

Denote the cocircular pair subspace of $C_2(S^1 \times S^n)$ by $CC = \{(z_1, p_1), (z_2, p_2) \in C_2(S^1 \times S^n) : p_2 = p_1\}$. The cocircular pair subspace is $(n + 2)$-dimensional, having co-dimension $n$ in $C_2(S^1 \times S^n)$.

Given $F : S^{n-2} \to \text{Emb}(S^1, S^1 \times S^n)$, assume the induced map
$$\hat{F} : S^{n-2} \times C_2S^1 \to C_2(S^1 \times S^n)$$
is transverse to $CC$, where $\hat{F}(p, z_1, z_2) = F(p)(z_1, z_2)$. In such a situation we will associate $W_2(F) \in \Lambda_n^{W_0(f)}$.

Our polynomial will be akin to the transverse intersection number of $\hat{F}$ with $CC$, but we include a few additional observations into the definition. The set $\hat{F}^{-1}(CC)$ is $\Sigma_2$ invariant, and $\Sigma_2$ acts freely on $C_2S^1$. The invariant $W_2(F)$ will be a sum of monomials associated to the points of $\hat{F}^{-1}(CC)/\Sigma_2$.

Given a point $p = (v, z_1, z_2) \in \hat{F}^{-1}(CC)$ we associate an element $L_p(F) \in \mathbb{Z}[t^{\pm 1}]$ and define
$$W_2(F) = \sum_{[p] \in \hat{F}^{-1}(CC)/\Sigma_2} L_p(F) \in \Lambda_n^{W_0}.$$

Define $L_p(F) = e^k$, where $e \in \{\pm 1\}$ is the local oriented intersection number of $\hat{F}$ with $CC$ at $p$. Observe the map
$$S^n \times C_2(S^1) \ni (w, z_1, z_2) \mapsto ((z_1, w), (z_2, w)) \in C_2(S^1 \times S^n)$$
is a diffeomorphism between $S^n \times C_2(S^1)$ and $CC$. This is how we give $CC$ its orientation. This map is also $\Sigma_2$-equivariant.

The monomial degree $k$ is computed via a pair of conventions. If $(z_1, z_2) \in C_2S^1$, let $[z_1, z_2]$ denote the counter-clockwise oriented arc in $S^1$ that starts at $z_1$ and ends at $z_2$. Similarly, given a point of $CC$, $(z_1, p_1), (z_2, p_1)$, the cocircular arc connecting them will be denoted $[z_1, z_2] \times \{p_1\}$. When thinking of $S^1 \times S^n$ we will refer to this as the vertical orientation. The monomial degree $k$ is obtained by concatenating $\hat{F}(v, [z_1, z_2])$ with the opposite-oriented cocircular arc in $CC$ associated to $\hat{F}(v, z_1, z_2)$.
Given \((v, z_1, z_2) \in \tilde{F}^{-1}(CC)\) then we also have \((v, z_2, z_1) \in \tilde{F}^{-1}(CC)\) and one can check
\[
\mathcal{L}_{(v,z_2,z_1)}(F) = (-1)^{n+1}t^{W_0-1} \mathcal{L}_{(v,z_1,z_2)}(F).
\]
We use the notation \(\tilde{\cdot}\) to denote the \(\mathbb{Z}\)-linear mapping \(\tilde{\cdot} : \mathbb{Z}[t^\pm 1] \rightarrow \mathbb{Z}[t^\pm 1]\) satisfying \(\tilde{t}^k = t^{-k}\). Thus \(W_2(F)\) is well-defined for \(F\). The relation \(t^0 = 0\) in \(\Lambda_n^{W_0}\) was chosen to ensure \(W_2(F)\) is a homotopy-invariant of \(F\). We use a compactification of configuration spaces to check homotopy-invariance.

Our manifold compactification of \(C_2S^1\) is diffeomorphic to an annulus \(S^1 \times [-1, 1]\). The boundary circles correspond to ‘infinitesimal’ configurations of pairs of points in \(S^1\); one component where the direction vector from \(z_1\) to \(z_2\) agrees with the orientation of \(S^1\), and the other being the reverse.

The Fulton-MacPherson compactified configuration space has the rather simple model of \(M^2\) blown up along its diagonal \(C_2[M] = Bl_{\Delta M}M^2\). Typically this is made formally precise by defining \(C_2[M]\) to be the closure of the graph of a function \([23]\), such as \(\phi : C_2M \rightarrow S^k\) where \(\phi(p, q) = \frac{p-q}{||p-q||}\), assuming \(M \subset \mathbb{R}^{k+1}\). This compactification is functorial under embeddings of manifolds. The inclusion \(C_2M \rightarrow C_2[M]\) is a homotopy-equivalence, i.e. \(C_2[M]\) is diffeomorphic to \(M^2\) remove an open tubular neighbourhood of the diagonal \(\Delta M = \{(p, p) : p \in M\}\). There is a canonically-defined onto smooth map \(C_2[M] \rightarrow M^2\), where the pre-image of \(\Delta M\) is isomorphic to the unit tangent bundle of \(M\).

We will also use a variation of the Fulton-Macpherson compactification where points are equipped with unit tangent vectors,
\[
C'_2[M] = \{(p, q, v, w) : (p, q) \in C_2[M], v \in UT_p M, w \in UT_q M\}.
\]

We now prove the homotopy-invariance of \(W_2(F)\). Consider what happens in a homotopy of \(F\). The boundary of \(I \times S^{n-2} \times C_2[S^1]\) consists of the temporal part \((\partial I) \times S^{n-2} \times C_2[S^1]\) and the annular part \(I \times S^{n-2} \times \partial C_2[S^1]\). The only monomial degrees that run off the annular part of the boundary are \(t^{-1}, t^0, t^{W_0-1}, t^{W_0}\). For example, \(t^0\) runs off the annular part if in our transverse family we have a tangent vector to our knot pointing in the vertical direction, oriented counter-clockwise. Similarly, \(t^{-1}\) can run off the boundary if we produce a tangent vector in the vertical direction, oriented clockwise. The monomials \(t^{W_0-1}\) and \(t^{W_0}\) are symmetric, after re-labelling the points of the domain \((z_1, z_2) \leftrightarrow (z_2, z_1)\).

Thus it makes sense to consider \(W_2(F)\) to be an element of the quotient group \(\Lambda_n^{W_0}\).

In Theorem 2.5 and Section 3 we need the notion of a half-ball.

**Definition 2.4** Let \(H^n = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq 0\}\) and define the half-ball \(HB^n = B^n \cap H^n\). \(HB^n\) is a manifold with corners. As such, it is a stratified space with two co-dimension one strata, the outer boundary \(HB^n \cap \partial B^n\) and the inner boundary \(HB^n \cap \partial H^n\). These two boundaries meet at the corner (co-dimension two) stratum \(\{0\} \times S^{n-2}\).
Theorem 2.5  Let \( f \in \text{Emb}(S^1, S^1 \times S^n) \). Provided \( n \geq 3 \) both \( W_1 \) and \( W_2 \) are epi-morphisms. When \( n = 3 \) the map
\[
W_1 \times W_2 : \pi_1 \text{Emb}_f(S^1, S^1 \times S^3) \to \mathbb{Z} \oplus \Lambda^W_0
\]
is an epimorphism.

Proof  To show \( W_1 \) is an epi-morphism, consider \( S^1 \) to be the group of translations acting on \( S^1 \). To argue that \( W_2 \) is an epi-morphism, we start with the fixed degree \( W_0 \) embedding \( S^1 \to S^1 \times S^n \), depicted in black in Figure 1. Imagine an immersed half-ball \( i : HB^2 \to S^1 \times S^n \) that fails to be an embedding by a single pair of double-points, with the double points being on the outer boundary. We also ask that \( i^{-1}(f) \) coincides with the inner boundary of \( HB^2 \). Construct the immersion so that the double-point is regular, and that the loop in the half-ball that goes between the double-points, once projected to the \( S^1 \) factor of \( S^1 \times S^n \), has degree \( k \), as in Figure 1. The sum of the tangent spaces at the double-point is 2-dimensional, so the orthogonal complement is \((n-1)\)-dimensional, this gives us an \( S^{n-2} \)-parameter family of unit normal vectors. Using a bump function, given a unit normal vector one can perturb one strand of \( \partial i(\partial HB^2) \) at the double-point, creating an embedded circle in \( S^1 \times S^n \). We surger \( f \) along these resolved half-balls, giving us our family
\[
\theta_{f,k} : S^{n-2} \to \text{Emb}_f(S^1, S^1 \times S^n).
\]
One can assume without any loss of generality that the sum of the tangent spaces at the double point are essentially horizontal, i.e. orthogonal to the vertical direction. This allows us to conveniently identify the cocircular points in our family \( \theta_{f,k} : S^{n-2} \to \text{Emb}_f(S^1, S^1 \times S^n) \), giving us \( W_2(\theta_{f,k}) = t^k - t^{k-1} \).

Note that Theorem 2.5 in part appears in the work of Arone-Szymik [1].

We have an involution of \( \text{Emb}(S^1, S^1 \times S^n) \) that negates the \( W_0 \) invariant. One description is the process that sends the embedding \( f \in \text{Emb}(S^1, S^1 \times S^n) \) to \( z \mapsto f(z^{-1}) \). Call this embedding \( \overline{f} \). Then we have \( \overline{\theta_{f,k}} = \theta_{f,-k-1} \), i.e. the \( \theta \) elements are symmetric about \(-1/2\).
One can readily compute $W_2$ for any similar family of embeddings $S^{n-2} \to \text{Emb}_f(S^1, S^1 \times S^n)$ created by surguring $f$ along a singular half-ball and resolving the double-points. For example, if we let $\beta_{f,k} : S^{n-2} \to \text{Emb}_f(S^1, S^1 \times S^n)$ represent the family where the half-ball $i : HB^2 \to S^1 \times S^n$ attaches to $f$ along the same arc, but connects to its upper strand, rather than the depicted lower strand in Figure [1], then $W_2(\beta_{f,k}) = t^{W_0+k} - t^{W_0+k-1}$. We will see in Theorems [2,7] and [2,8] that $W_2$ (together with $W_1$ if $n = 3$) is a complete invariant of $\pi_{n-2} \text{Emb}(S^1, S^1 \times S^n)$, thus $\theta_{f,k} = \beta_{f,k-W_0}$. Readers are encouraged to convince themselves of this directly via a regular isotopy of $i : HB^2 \to S^1 \times S^n$.

Another natural family $i : HB^2 \to S^1 \times S^n$ to consider is one where $i$ is an embedding. As in previous cases we will ask that the outer-boundary of $HB^2$ coincides with the intersection with $f$. Consider the case where the projection of the embedding $i$ to the $S^1$ factor is not onto, i.e. it is kept constrained in an interval in the $S^1$ factor. Then $i$ connects one strand of $f$ to adjacent strands. Let’s say to the $k$-th strand with $k \in \{0, 1, \cdots, W_0 - 1\}$, assuming $W_0 > 0$ and we use the cyclic ordering on strands induced by the parametrization $f$. Call the resolved family of knots $\gamma_{f,k} : S^{n-2} \to \text{Emb}_f(S^1, S^1 \times S^n)$. Given that, we have $W_2(\gamma_{f,k}) = t^k - t^{k-1}$. Recall that $\Lambda^W_n = \mathbb{Z}[t^\pm 1]/(t^{-1}, t^0, t^k + (-1)^k t^{W_0-1-k} \forall k)$. Thus $\{\gamma_{f,k} : k \in \{0, 1, \cdots, W_0 - 1\}\}$ same subspace of $\Lambda^W_n$ as the monomials $\{t, t^2, \cdots, t^{W_0-2}\}$, i.e. all the intermediate monomials that were not killed by the defining relations of $\Lambda^W_n$.

![Figure 2](image)

Figure 2: (a) Vertical torus in $S^3 \times S^1$. Vertical fibres represent trivial element in $\pi_1 \text{Emb}(S^1, S^3 \times S^1)$, in the component with $W_0 = 1$. (b) Sphere linking tube in 4-space.

We give an alternative way to visualize elements of $\pi_1 \text{Emb}_f(S^1, S^1 \times S^3)$ with $W_2 = 0$ by embedded tori, where $f$ denotes the standard generator of $\pi_1(S^1)$, i.e. $W_0 = 1$. This discussion assumes that $W_1 \times W_2$ is an isomorphism. Each generator will be represented by an embedded torus $T \subset S^1 \times S^3$ which contains the curve $\gamma_0 = S^1 \times y_0$. Such a torus $T$ gives rise to an element $z$ of $\pi_1 \text{Emb}(S^1, S^1 \times S^3)$ by fibering $T$ by parametrized smooth circles $\{\gamma_t | t \in [0,1]\}$ with $\gamma_0 = \gamma_1$. Once $\gamma_0$ is chosen, what really matters is which way to go around the torus. To do this and control $W_1$, we choose an oriented simple closed curve $\mu_w \subset T$, homotopically trivial in $S^1 \times S^3$, that intersects $\gamma_0$ transversely once at some point $w = (x_0, y_0) \in S^1 \times S^3$. The homotopy condition implies that that $W_1(z) = 0$ and the orientation informs us that $\gamma_0$ is required to spin
about $T$ so that $w$ follows $\mu_w$ in the oriented direction. Denote by $(T, \mu_w)$ the represented element of $\pi_1\text{Emb}_f(S^1, S^1 \times S^3)$.

Figure 3: (a) Torus representing $\alpha_{f,1}$ with $W_0(f) = 1$, $W_2 = t^2 - t^0 = t^2 \neq 0$  
(b) Torus representing $\alpha_{f,2}$ with $W_0(f) = 1$, $W_2 = t^3 - t^1 = t^3 \neq 0$

The standard vertical torus $T^*$, shown in Figure 1a) represents the trivial element of $\pi_1\text{Emb}(S^1, S^1 \times S^3)$. Figures 2(a) and (b) describe embedded tori corresponding to $t$ and $t^2$ respectively. In our diagrams, $\mu_w \subset \{x_0\} \times S^3$, with $S^3$ being depicted horizontally. In a similar manner we obtain a torus corresponding to $t^n$, $|n| \geq 1$. Each of our tori are constructed by tubing $T^*$ with an unknotted, unlinked 2-sphere as follows. Emanating from the boundary of a small disc on $T^*$ the tube first links the sphere, then goes $n \in \mathbb{N}$ times around the $S^1$ factor before finally connecting to the 2-sphere. Figures 2(a), 2(b) show the projection of $T$ to $S^1 \times (S^2 \times 0)$ where $S^3$ is identified with $S^2 \times [-\infty, \infty]$, where each component of $S^2 \times \{\pm \infty\}$ is identified to a point. By construction $T \subset S^1 \times (S^2 \times 0)$, except for where the tube links the 2-sphere. See Figure 1(b) for a detail. The crossing convention for the tube and sphere informs us that the part of the tube that projects to the right side of the 2-sphere lives a bit in the past (i.e. in $S^1 \times (S^2 \times [-1, 0))$) and the part of the tube on the left lives in the future. By construction, the 2-sphere bounds a 3-ball $B \subset S^1 \times (S^2 \times 0)$ that intersects the tube is a single simple closed curve. By either reversing the way the tube links the 2-sphere, or reversing the orientation on $\mu_w$ we obtain the inverse of the generator. See Figure 3(b).

We describe how to represent composition of generators. First some terminology. Let $p : T^* \to \mu_w$ be the vertical projection. By construction, each generator is obtained by removing a small disc $D$ from $T^*$ and replacing it by a disc $D'$. Further each knotted disc lies in a small neighborhood of a 1-complex which itself lies in a neighborhood of $p^{-1}(\delta)$, for some interval $\delta \subset \mu_w$. Squeezing, expanding or rotating this interval and correspondingly modifying the discs $D$ and $D'$ does not change the based homotopy class of $(T, \mu_w)$ provided that the expanding or rotating is supported away from $w$. The composition of generators $\beta_0$ and $\beta_1$ is represented as follows. First find tori $(T_0, \mu_w), (T_1, \mu_w)$ constructed as above respectively representing $\beta_0, \beta_1$ so that $T_0$ coincides with $T_1$ near $\mu_w$ and the latter having a fixed orientation. Further, assume that each $T_i$ is standard away from a neighborhood of $p^{-1}(\delta_i)$ where $\delta_0 \cap \delta_1 = \emptyset$ and $\delta_0$ proceeds $\delta_1$ when starting at $x \in \mu_w$. To
obtain \((T, \mu_w)\) representing \(\beta_0 \ast \beta_1\), modify \(T^*\) near both \(p^{-1}(\delta_i), i = 0, 1\) according to \(T_0\) and \(T_1\). See Figure 2(a). To see that \(\beta_0 \ast \beta_1\) is homotopic to \(\beta_1 \ast \beta_0\) observe that the two tori representing these classes are isotopic via an isotopy fixing \(\gamma_0 \cup \mu_w\) pointwise. We conclude that any word in the generators is realizable by an embedded torus and \(\pi_1 \text{Emb}(S^1, S^1 \times S^3)\) is abelian.

We return to the problem of determining if our invariants \(W_0, W_1\) and \(W_2\) are complete invariants of the low-dimensional homotopy-groups of \(\text{Emb}(S^1, S^1 \times S^n)\).

The Functor Calculus is a set of principles that can be useful for studying many familiar functors. The formalism was originally observed in the context of pseudo-isotopy embedding spaces. A pseudo-isotopy embedding space \(PE(N, M)\) requires two inputs, typically \(N\) and \(M\) compact manifolds, with \(N\) a proper submanifold of \(M\).

\[
PE(N, M) = \{ f : N \times [0, 1] \to M \times [0, 1] \text{ an embedding with } f_{|N \times \{0\}} \cup \partial N \times [0, 1] = Id_{N \times \{0\}} \cup \partial N \times [0, 1]. \}
\]

In the context of embedding calculus, one is imagining a handle decomposition of \(N\) and asking how the homotopy-type of \(PE(\cdot, M)\) changes, as one builds \(N\) from its handle decomposition.

Tom Goodwillie proved a multiple-disjunction theorem, generalizing the Morlet Disjunction Lemma. Roughly speaking, this is a statement that you can recover more of the homotopy-type of \(PE(N, M)\) if one keeps track of all the disjunctions (handle attachments) one uses in building \(N\) than one might expect by repeated applications of the Morlet theorem, one handle attachment at a time.

One can view the multiple disjunction lemma as describing roughly how the homotopy-type of a pseudo-isotopy embedding space changes as one varies the domain manifold. The ‘difference’ of pseudo-isotopy embedding spaces in this context is the homotopy fibre of restriction maps \(PE(N, M) \to PE(W, M)\) when \(W\) is a submanifold of \(N\). Goodwillie developed a formalism of polynomial functors, loosely analogous to Taylor expansions of functions. Analytic functors are functors that are well-approximated by their Taylor expansions, just as analytic functions are well-approximated by their Taylor expansions. Perhaps the reason this formalism was discovered with pseudo-isotopy embedding spaces was that the rate of convergence in this context is remarkably high.

To give this some additional context, the Smale-Hirsch theorem, in the language of functor calculus, states that the immersion functor \(\text{Imm}(M, N)\) is a linear functor in the domain manifold \(M\),

\[
\text{Figure 4: (a) Torus with } W_2 = t^2 + t^3
\]

\[
\text{(b) All three tori with } W_2 = -t^2
\]
provided $\dim(N) > \dim(M)$. This is due to Smale-Hirsch saying that in this context the space of immersions is homotopy-equivalent to the space of bundle monomorphisms $TM \to TN$.

We will use functor calculus in the context of embedding spaces $\text{Emb}(M, N)$, viewing it as a functor of the domain manifold $M$, with arrows given by inclusion maps of manifolds. Given an $m$-manifold $M$ and an $n$-manifold $N$, the embedding calculus provided a sequence of functions

$$\text{Emb}(M, N) \to T_k\text{Emb}(M, N)$$

that are $k(n - m - 2) + 1 - m$-connected, meaning the induced maps $\pi_j\text{Emb}(M, N) \to \pi_jT_k\text{Emb}(M, N)$ is an isomorphism for $j < k(n - m - 2) + 1 - m$ and an epimorphism for $j = k(n - m - 2) + 1 - m$. This connectivity result [12] is one of two central results for the embedding calculus. The other result that gives the theory utility, is the description of the polynomial functors, in the context of embedding calculus, done by Michael Weiss [28].

We describe some terminology associated with the embedding calculus due to Goodwillie-Weiss-Klein [12]. In our case $m = 1$, so the map

$$\text{Emb}(S^1, S^1 \times S^n) \to T_k\text{Emb}(S^1, S^1 \times S^n)$$

is $k(n - 2)$-connected. There are maps $T_k\text{Emb}(M, N) \to T_{k-1}\text{Emb}(M, N)$ giving commutative diagrams

$$\begin{array}{ccc}
\text{Emb}(M, N) & \longrightarrow & T_k\text{Emb}(M, N) \\
\downarrow & & \downarrow \\
T_{k-1}\text{Emb}(M, N)
\end{array}$$

The space $T_k\text{Emb}(M, N)$ is called the $k$-th stage of the embedding calculus, while the maps $\cdots \to T_k \to T_{k-1} \to \cdots$ are called the Taylor tower. The homotopy-fibers of the maps $T_k \to T_{k-1}$ are called the layers of the embedding calculus. In the loose analogy with function-calculus the layers are analogous to the degree $k$ homogeneous term of a Taylor expansion. The layers are the primary focus of the paper of Weiss [28]. In day-to-day usage of embedding calculus, one makes a choice of model for the Taylor tower that suits the tools one is using. Weiss’s description of the layers allow one to inductively ensure the model one constructs for the Taylor tower has the appropriate homotopy-type. In that regard it is a fundamental theorem in the subject that has no analogue in ‘regular’ calculus, as we are familiar polynomial functions are in that context.

The $k$-th stage $T_k$ has a description as a homotopy-limit over a category of open subsets of $M$. In our case, we choose to use the Taylor tower model described by Dev Sinha [24] as it has the rather convenient form of a multi-relative mapping space. For example, the loop space of a pair $(X, p)$ is a relative mapping space in the sense that it consists of the space of maps of pairs $(S^1, 1) \to (X, p)$, i.e. these are maps of a circle where the subspace $\{1\}$ is sent to the subspace $\{p\}$. A multi-relative mapping space consists of a space of maps from one space to another, where there are restrictions on how multiple subspaces are mapped. Technically, Dev Sinha did not outline exactly this model in his paper [24]. Sinha was considering spaces of embeddings of an interval into a manifold with
boundary, with the embeddings being fixed on the boundary. The techniques of his paper [24] give this result when the domain is a circle.

In Sinha’s model, $T_k\text{Emb}(S^1,M)$ is a certain subspace of the mapping space $C'_k[S^1] \to C'_k[M]$ where $C'_k[M]$ is the Fulton-Macpherson compactified configuration space of $k$ points in $M$, where the points are equipped with unit tangent vectors. There are three conditions on the maps: (1) they stratum-preserving, (2) aligned and (3) $\Sigma_k$-equivariant. Given that $S^1$ is one-dimensional we use the convention that the unit circle $S^1$ has the counter-clockwise orientation, and the unit vectors associated to points in $C'_k[S^1]$ point in the counter-clockwise direction.

(1) The stratum-preserving condition is that if points correspond to a collision stratum of $C'_k[S^1]$ then their image belongs to the same type of collision stratum on $C'_k[M]$. Thus if a point in $C'_k[S^1]$ has points $i$ and $j$ colliding, then the corresponding point of $C'_k[M]$ must also have points $i$ and $j$ colliding.

(2) The aligned condition requires that when points collide, the tangent vectors to the points must coincide with the tangent vector associated to the collision in the Fulton-Macpherson compactification. In $C'_k[S^1]$ we simplify our model assuming that all tangent vectors point counter-clockwise (otherwise we would need a further linearity condition).

(3) Lastly, since there is an action of $\Sigma_k$ on both the domain and target-space in this mapping space, there is an action of $\Sigma_k$ on the entire mapping space. We consider only $\Sigma_k$-equivariant maps.

The map $\text{Emb}(S^1,M) \to T_k$ is called the evaluation map as given a collection of $k$ points in $S^1$, one simply evaluates the embedding and its derivative at those $k$ points, giving a map $\text{Emb}(S^1,M) \to \text{Maps}(C'_kS^1,C'_kM)$. One can check this map extends uniquely to the Fulton-Macpherson compactification and has the three properties of being stratum-preserving, aligned and equivariant.

**Lemma 2.6** The non-trivial homotopy groups of $C_2(S^1 \times S^n)$ in dimensions $n + 1$ and lower are:

\[
\begin{align*}
\pi_1 C_2(S^1 \times S^n) &\simeq \mathbb{Z}^2 \\
\pi_n C_2(S^1 \times S^n) &\simeq \mathbb{Z} \oplus \mathbb{Z}[t^{\pm 1}] \\
\pi_{n+1} C_2(S^1 \times S^n) &\simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2[t^{\pm 1}]
\end{align*}
\]

The boundary of $C_2(S^1 \times S^n)$ can be canonically identified with $S^1 \times S^n \times S^n$ using our preferred trivialization of $T(S^1 \times S^n)$. Thus $\pi_1 \partial C_2(S^1 \times S^n) \simeq \mathbb{Z}$ and $\pi_n \partial C_2(S^1 \times S^n) \simeq \mathbb{Z}^2$. We compute the induced map on the above homotopy groups for the inclusion map $\partial C_2(S^1 \times S^n) \to C_2(S^1 \times S^n)$. To make sense of this map we need a common choice of basepoint. Identify $\partial C_2(S^1 \times S^n)$ with the unit sphere bundle of $S^1 \times S^n$. Our basepoint will be the direction vector pointing in the counter-clockwise direction of $S^1$, based at $(1,*)$ where $* \in S^n$ is any basepoint choice for $S^n$.

The induced map on $\pi_1$ is identified with the diagonal map $\Delta : \mathbb{Z} \to \mathbb{Z}^2$, $\Delta(t) = (t,t)$. The induced map on $\pi_n$ is identified with $\mathbb{Z}^2 \to \mathbb{Z} \oplus \mathbb{Z}[t^{\pm 1}]$, which in matrix form is

\[
\begin{pmatrix}
1 & 0 \\
0 & 1 - t^{-1}
\end{pmatrix}.
\]

The above computation requires a choice of common basepoint in $\partial C_2(S^1 \times S^n)$ and $C_2(S^1 \times S^n)$, and is valid for any such choice.
These isomorphisms follow from the fact that the fibre bundle
\[ Bl_n(S^1 \times S^n) \to C_2[S^1 \times S^n] \to S^1 \times S^n \]
has a section. There are several sections available: (1) using the trivialization of \( T(S^1 \times S^n) \) or (2) using the antipodal map of \( S^1 \) or \( S^n \) or the combination of the two. All of these sections are homotopic. The section (1) is the only choice that allows for a common base-point in \( C_2(S^1 \times S^n) \) and its boundary.

**Theorem 2.7** The components of \( \text{Emb}(S^1, S^1 \times S^3) \) have abstractly isomorphic fundamental groups,
\[ \pi_1 \text{Emb}_f(S^1, S^1 \times S^3) \simeq \mathbb{Z} \oplus \Lambda_3^{W_0} \quad \forall f \in \text{Emb}(S^1, S^1 \times S^3). \]

Stated another way, the epi-morphisms from Theorem 2.5 are isomorphisms when \( n = 3 \).

The paper by Goodwillie-Weiss-Klein [12] tells us the map \( \text{Emb}(S^1, S^1 \times S^3) \to T_2 \text{Emb}(S^1, S^1 \times S^3) \) induces an isomorphism on fundamental groups of all components. The proof of Theorem 2.7 ultimately amounts to us identifying the map
\[ \pi_1 \text{Emb}_f(S^1, S^1 \times S^3) \to \mathbb{Z} \oplus \Lambda_3^{W_0} \]
from Theorem 2.5 with the induced map \( \pi_1 \text{Emb}_f(S^1, S^1 \times S^3) \to \pi_1 T_2 \text{Emb}_f(S^1, S^1 \times S^3) \).

Although all models for the Taylor tower are equivalent, Dev Sinha’s model [24] is a natural framework that allows us to avoid discussion of homotopy limits.

**Proof** (of Theorem 2.7) The model we prefer to use is due to Dev Sinha, it states that \( T_k \text{Emb}(S^1, S^1 \times S^n) \) is the space of stratum-preserving aligned \( \Sigma_k \)-equivariant maps \( C_k[S^1] \to C_k'[S^1 \times S^n] \).

We are interested in the \( k = 2, n = 3 \) case. \( C_k'[S^1] \) is simply an annulus, i.e. diffeomorphic to \( S^1 \times [-1, 1] \). \( C_k'[S^1 \times S^3] \) is diffeomorphic to \( (S^1 \times S^3) \times Bl_1(S^1 \times S^3) \times (S^3)^2 \), given by the map
\[ C_2(S^1 \times S^3) \ni (z_1, p_1), (z_2, p_2) \mapsto (z_1, p_1), (z_1 z_2^{-1}, p_1 p_2^{-1}) \in S^1 \times S^3 \times (S^1 \times S^3) \setminus \{1\} \]
together with the trivialization of \( T(S^1 \times S^3) \). The blow-up \( Bl_1(S^1 \times S^3) \) deformation-retracts to its 3-skeleton \( S^1 \vee S^3 \).

The boundary of \( C_2[S^1 \times S^3] \) is canonically diffeomorphic to the unit tangent bundle of \( S^1 \times S^3 \). Due to the triviality of \( T(S^1 \times S^3) \) we can think of that as \( S^1 \times S^3 \times S^3 \).

The space of equivariant, stratum-preserving aligned maps \( C_k'[S^1] \to C_k'[S^1 \times S^3] \) fibres over the space of equivariant stratum-preserving maps \( C_2[S^1] \to C_2[S^1 \times S^3] \). There is a fairly elementary argument this bundle induces an isomorphism on \( \pi_0 \) and \( \pi_1 \). The idea being, that the fibre of \( C_k'[S^1 \times S^3] \to C_k[S^1 \times S^3] \) is simply a product of two copies of \( S^3 \), which split from \( C_k'[S^1 \times S^3] \). Consider the pair of projections \( P_i : C_2[S^1 \times S^3] \to S^3 \) for \( i = 1, 2 \). The alignment condition allows us to construct homotopies of the two composites
\[ C_2[S^1] \to C_2[S^1 \times S^3] \xrightarrow{P_i} S^3 \text{ for } i \in \{1, 2\}. \]

If we pre-compose the above two composites with a deformation-retraction of \( C_2[S^1] \) to one of its boundary components, this gives us a homotopy between the above composites and maps \( C_2[S^1] \to S^3 \) that are constant on the fibres of \( C_2[S^1] \to C_2'[S^1] \equiv S^1 \). So by general position, the
projection map from $T_2\text{Emb}(S^1, S^1 \times S^3)$ to the equivariant stratum-preserving maps $C_2[S^1] \to C_2[S^1 \times S^3]$ induces an isomorphism on $\pi_0$ and $\pi_1$.

The fundamental group $\pi_1 C_2[S^1 \times S^3]$ is free abelian on two generators, see Lemma 2.6. The natural set of generators are given by the winding numbers of the first and second points of the configurations about the $S^1$ factor of $S^1 \times S^3$. Consider the covering space of $C_2[S^1 \times S^3]$ corresponding to the homomorphism $\pi_1 C_2[S^1 \times S^3] \to \mathbb{Z}$ given by taking the difference between the two winding numbers. We denote this covering space by $\tilde{C}_2[S^1 \times S^3]$. By design, any stratum-preserving map $C_2[S^1] \to C_2[S^1 \times S^3]$ lifts to this covering space, $C_2[S^1] \to \tilde{C}_2[S^1 \times S^3]$. Since $C_2[S^1 \times S^3]$ fibers over $S^1 \times S^3$, this covering space does as well, but the fibre is the universal cover of $\text{Bl}_*(S^1 \times S^3)$, which could be described as $\text{Bl}_{Z \times \{1\}}(\mathbb{R} \times S^3)$, giving

$$\tilde{C}_2[S^1 \times S^3] \cong \text{Bl}_{Z \times \{1\}}(\mathbb{R} \times S^3) \times S^1 \times S^3.$$ 

The general position argument in our previous section tells us, for the purpose of computing $\pi_0$ or $\pi_1$, we can assume our map is null in the rightmost $S^3$ factor. Thus we have reduced the computation of $\pi_0$ and $\pi_1$ to the space of stratum-preserving maps

$$C_2[S^1] \to \text{Bl}_{Z \times \{1\}}(\mathbb{R} \times S^3) \times S^1.$$ 

If we take the degree of the projection to the $S^1$ factor we recover $W_0$. Given a 1-parameter family of such maps, if we restrict to a single point in $C_2[S^1]$ and take the degree of the remaining 1-parameter family, we recover $W_1$.

Consider the projection $C_2[S^1] \to \text{Bl}_{Z \times \{1\}}(\mathbb{R} \times S^3)$. By design, one boundary stratum is in the blow-up sphere corresponding to $k \times \{1\}$, and the other stratum is in the blow-up sphere corresponding to $(k + W_0) \times \{1\}$. The space $\text{Bl}_{Z \times \{1\}}(\mathbb{R} \times S^3)$ has the homotopy-type of an infinite wedge of 3-spheres, perhaps best thought of as the 3-skeleton of

$$(\mathbb{R} \times S^3) \setminus (Z \times \{1\}) \cong (\mathbb{R} \times \{-1\}) \cup \left( \bigcup_{i \in Z} \left\{ \frac{1}{2} + i \right\} \times S^3 \right).$$

To describe the equivariance condition on our lift $C_2[S^1] \to \text{Bl}_{Z \times \{1\}}(\mathbb{R} \times S^3)$, the relevant $\Sigma_2$-action on the target space is induced by the map $(t, p) \mapsto (2j + W_0 - t, p^{-1})$. Choosing $j = 0$ gives us the same convention as in Theorem 2.5. This computation is done by considering the homeomorphism $C_2(S^1 \times S^3) \to (S^1 \times S^3) \times ((S^1 \times S^3) \setminus \{1\})$. The involution of $\tilde{C}_2(S^1 \times S^3)$ in the $\Sigma_2$-action sends $(z_1, p_1), (z_2, p_2)$ to $(z_2, p_2), (z_1, p_1)$. Conjugating this involution by our identification gives us the map $(z_1, p_1), (z_2, p_2) \mapsto (z_1 z_2, p_2 p_1), (z_2^{-1}, p_2^{-1})$, which, on the fibre lifts to the above map.

Homotopy-classes of maps to wedges of spheres, via the Ponyriagin construction, are characterized by their intersection numbers with the points antipodal to the wedge point. Our maps are equivariant, so our framed points in the domain satisfy a symmetry condition. Our space $\text{Bl}_{Z \times \{1\}}(\mathbb{R} \times S^3)$ equivariantly deformation retracts to the above 3-skeleton. The $\Sigma_2$-stabilizer is a single point if $W_0$ is even, and a pair of points $(\frac{W_0}{2}, \pm 1) \subset \{ \frac{W_0}{2} \} \times S^3$ if $W_0$ is odd. Since the $\Sigma_2$-action on the domain is fixed-point free, all this tells us is the degree associated to this intermediate sphere is even. Thus, if $W_2$ is zero, we can equivariantly homotope our map so that its image
is disjoint from the antipodal points to all the wedge points of the $S^3$ factors. We can therefore assume our map $C_2[S^1] \to \mathbb{R} \times \{-1\} \cup (\bigcup_{i \in \mathbb{Z}} \{\frac{1}{2} + i\} \times S^3)$ is homotopic to a map to the interval $[0, W_0] \times \{-1\}$.

**Theorem 2.8** For $n \geq 4$, the first three non-trivial homotopy groups of the embedding space

$$\text{Emb}(S^1, S^1 \times S^n)$$

are $\pi_0 \text{Emb}(S^1, S^1 \times S^n) \simeq \mathbb{Z}$ with the isomorphism given by $W_0$. For any $f \in \text{Emb}(S^1, S^1 \times S^n)$, $\pi_1 \text{Emb}_f(S^1, S^1 \times S^n) \simeq \mathbb{Z}$ with the isomorphism given by $W_1$, and $\pi_{n-2} \text{Emb}_f(S^1, S^1 \times S^n) \simeq \Lambda_n^{W_0}$ with the isomorphism given by $W_2$.

**Proof** The proof roughly mimics Theorem 2.7. Unfortunately, the bundle $C_2(S^1 \times S^n) \to S^1 \times S^n$ is generally not trivial, so we do not have access to quite as simple an argument, but we take some inspiration from it.

The second stage of the Taylor tower has the same homotopy groups $\pi_i$ as $\text{Emb}(S^1, S^1 \times S^n)$ provided $i < 2(n-2)$. As with Theorem 2.7 we need only consider equivariant stratum-preserving maps $C_2[S^1] \to C_2[S^1 \times S^n]$, i.e. the tangent vectors are not relevant for our low-dimensional computation.

The composite with the bundle projection map $C_2[S^1 \times S^n] \to S^1 \times S^n$ factors through the projection to $S^1$, and is given by the $W_1$ invariant. Consider an element of $\pi_{n-2} \text{Emb}(S^1, S^1 \times S^n)$, then by design the map to $S^1 \times S^n$ is null, and the map lifts to the fibre, moreover it lifts to the covering space corresponding to the difference of the winding numbers.

$$C_2[S^1] \to \tilde{C}_2[S^1 \times S^n]$$

As with the proof of Theorem 2.7 the fibre of the map $\tilde{C}_2[S^1 \times S^n] \to S^1 \times S^n$ can be identified with $Bl_{Z \times \{1\}}(\mathbb{R} \times S^n)$, which is similarly identified with $\mathbb{R} \times \{-1\} \cup (\bigcup_{i \in \mathbb{Z}} \{\frac{1}{2} + i\} \times S^n)$.

Here our argument diverges from the proof of Theorem 2.7 While the action of $\Sigma_2$ on $C_2(S^1)$ is free, it has the invariant subspace of antipodal points on $S^1$.

By restricting to the subspace of antipodal points, we get a fibration from the space of stratum-preserving equivariant maps $C_2[S^1] \to C_2[S^1 \times S^n]$ to the space $\text{Maps}^{Z_2}(S^1, C_2[S^1 \times S^n])$ of equivariant maps. This mapping space can be thought of as the space of maps $S^1 \to S^1 \times S^n$ where antipodal points are required to map to distinct points. By a transversality argument, any $k$-dimensional family of maps to the free loop space $L(S^1 \times S^n)$ can be perturbed to have this property, provided $k < n$. Thus through dimension $n-2$, this space has the same homotopy groups as $L(S^1 \times S^n) \simeq L(S^1) \times L(S^n)$, which are the homotopy-groups of $Z \times S^1$. i.e. this recovers our $W_0$ and $W_1$ invariants.

We are considering the fibration from the space of stratum-preserving equivariant maps

$$C_2[S^1] \to C_2[S^1 \times S^n]$$

to the space $\text{Maps}^{Z_2}(S^1, C_2[S^1 \times S^n])$. The fibre is precisely the space of maps of an annulus $S^1 \times [0, 1]$ to $C_2[S^1 \times S^n]$ that restrict to a fixed map on one boundary circle, and which send the other boundary circle to $\partial C_2(S^1 \times S^n)$. We lift this map to the fibre $\mathbb{R} \times \{-1\} \cup (\bigcup_{i \in \mathbb{Z}} \{\frac{1}{2} + i\} \times S^n)$. From this perspective we can see that the $W_2$ invariant is well-defined for an $(n-2)$-parameter family, and there are no further invariants. □
In this section we relate our results on the homotopy groups of \( \text{Emb}(S^1, S^1 \times S^3) \) to the diffeomorphism group \( \text{Diff}(S^1 \times B^n \text{ fix } \partial) \), and describe how \( \text{Diff}(S^1 \times B^n \text{ fix } \partial) \) is related to the space of reducing \( n \)-balls in \( S^1 \times B^n \).

Given a fixed \( f \in \text{Emb}(S^1, S^1 \times S^n) \) and any diffeomorphism \( g \in \text{Diff}(S^1 \times S^n) \) one can form the composite \( g \circ f \in \text{Emb}(S^1, S^1 \times S^n) \). This gives a locally-trivial fibre bundle

\[
\text{Diff}(S^1 \times S^n \text{ fix } f) \to \text{Diff}(S^1 \times S^n) \to \text{Emb}(S^1, S^1 \times S^n)
\]

where the fibre is the diffeomorphisms of \( S^1 \times S^n \) that fix the image of \( f \) pointwise. In much of topology, locally-trivial fibre bundles are required to be onto functions, but this theorem of Palais’s does not say which path-components of \( \text{Emb}(S^1, S^1 \times S^n) \) this bundle is onto, i.e. fibers over some path components may be empty. Given that \( n \geq 3 \) is our default assumption, we know the path components of \( \text{Emb}(S^1, S^1 \times S^n) \) are prescribed by the \( W_0 \) invariant. Diffeomorphisms of \( S^1 \times S^n \) can preserve or reverse the fundamental class of the \( S^1 \) factor, thus this bundle is onto at most two path components of \( \text{Emb}(S^1, S^1 \times S^n) \). We restrict to the subgroup of diffeomorphisms of \( S^1 \times S^n \) that are homotopic to the identity, \( \text{Diff}_0(S^1 \times S^n) \). This is a subgroup of index 8 in \( \text{Diff}(S^1 \times S^n) \), and index two in the subgroup acting trivially on \( H_*(S^1 \times S^n) \). With this terminology we have a locally-trivial fibre bundle

\[
\text{Diff}_0(S^1 \times S^n \text{ fix } f) \to \text{Diff}_0(S^1 \times S^n) \to \text{Emb}_f(S^1, S^1 \times S^n).
\]

We compute the image of the induced map \( \pi_i \text{Diff}_0(S^1 \times S^n) \to \pi_i \text{Emb}_f(S^1, S^1 \times S^n) \) in the range \( 0 \leq i \leq n - 2 \) in Theorem 3.1.

**Theorem 2.9** To each element of \( \pi_1 \text{Emb}(S^1, S^1 \times S^3) \), there is an explicitly constructible embedded torus that represents that element via the spinning construction.

**3 Bundles of embedding spaces and diffeomorphism groups**

In this section we relate our results on the homotopy groups of \( \text{Emb}(S^1, S^1 \times S^3) \) to the diffeomorphism group \( \text{Diff}(S^1 \times B^n \text{ fix } \partial) \), and describe how \( \text{Diff}(S^1 \times B^n \text{ fix } \partial) \) is related to the space of reducing \( n \)-balls in \( S^1 \times B^n \).

Given a fixed \( f \in \text{Emb}(S^1, S^1 \times S^n) \) and any diffeomorphism \( g \in \text{Diff}(S^1 \times S^n) \) one can form the composite \( g \circ f \in \text{Emb}(S^1, S^1 \times S^n) \). This gives a locally-trivial fibre bundle

\[
\text{Diff}(S^1 \times S^n \text{ fix } f) \to \text{Diff}(S^1 \times S^n) \to \text{Emb}(S^1, S^1 \times S^n)
\]

where the fibre is the diffeomorphisms of \( S^1 \times S^n \) that fix the image of \( f \) pointwise. In much of topology, locally-trivial fibre bundles are required to be onto functions, but this theorem of Palais’s does not say which path-components of \( \text{Emb}(S^1, S^1 \times S^n) \) this bundle is onto, i.e. fibers over some path components may be empty. Given that \( n \geq 3 \) is our default assumption, we know the path components of \( \text{Emb}(S^1, S^1 \times S^n) \) are prescribed by the \( W_0 \) invariant. Diffeomorphisms of \( S^1 \times S^n \) can preserve or reverse the fundamental class of the \( S^1 \) factor, thus this bundle is onto at most two path components of \( \text{Emb}(S^1, S^1 \times S^n) \). We restrict to the subgroup of diffeomorphisms of \( S^1 \times S^n \) that are homotopic to the identity, \( \text{Diff}_0(S^1 \times S^n) \). This is a subgroup of index 8 in \( \text{Diff}(S^1 \times S^n) \), and index two in the subgroup acting trivially on \( H_*(S^1 \times S^n) \). With this terminology we have a locally-trivial fibre bundle

\[
\text{Diff}_0(S^1 \times S^n \text{ fix } f) \to \text{Diff}_0(S^1 \times S^n) \to \text{Emb}_f(S^1, S^1 \times S^n).
\]

We compute the image of the induced map \( \pi_i \text{Diff}_0(S^1 \times S^n) \to \pi_i \text{Emb}_f(S^1, S^1 \times S^n) \) in the range \( 0 \leq i \leq n - 2 \) in Theorem 3.1.

**Theorem 3.1** The induced maps in homotopy for the fibre bundle \( \text{Diff}_0(S^1 \times S^n) \to \text{Emb}_f(S^1, S^1 \times S^n) \) are:

- **Trivial on** \( \pi_0 \).
- **When** \( n = 3 \), it is onto the subgroup of \( \pi_1 \text{Emb}_f(S^1, S^1 \times S^n) \) with \( W_2 = 0 \). In particular all \( W_1 \) invariants are realized.
- **When** \( n > 3 \), it is onto \( \pi_1 \text{Emb}_f(S^1, S^1 \times S^n) \), and trivial in \( \pi_{n-2} \text{Emb}_f(S^1, S^1 \times S^n) \).

**Proof** The essential argument is that in the image of the map

\[
\pi_{n-2} \text{Diff}_0(S^1 \times S^n) \to \pi_{n-2} \text{Emb}_f(S^1, S^1 \times S^n)
\]

we can show \( W_2 = 0 \). A map \( S^{n-2} \to \text{Diff}_0(S^1 \times S^n) \) induces a map \( S^{n-2} \to \text{Diff}(C_2[S^1 \times S^n]) \). These diffeomorphisms lift to the covering space \( \tilde{C}_2[S^1 \times S^n] \), corresponding to the difference of the winding numbers. Our invariant \( W_2 \) was defined in terms of intersection numbers with the various wedge \( S^n \) summands, for a lift of families of maps from \( C_2[S^1] \) to the normal covering space we denoted \( \tilde{C}_2[S^1 \times S^n] \). The above lift will allow us to perform the computation.
Consider $C_2[S^1]$ to be an annulus, and the key condition was our maps were required to be stratum-preserving and equivariant. We can identify $C_2[S^1]$ with $S^1 \times [0, 2\pi]$ via the diffeomorphism $(z_1, z_2) \mapsto (z_1, -i \ln(z_2/z_1))$ where we choose the branch of log with $0 < im(\ln) < 2\pi$. The term $-i \ln(z_2/z_1)$ is the polar angle from $z_1$ to $z_2$. Conjugated via this diffeomorphism, the $\Sigma_2$-action is given by the involution $(z, t) \mapsto (ze^{it}, \pi - t)$. The cylinder associated to $C_2[S^1]$ is the $\Sigma_2$-space $D^2 \times [-\pi, \pi]$, where the involution is given by the map $(p, t) \mapsto (pe^{it}, \pi - t)$.

Notice that an equivariant family of maps $S^{n-2} \times C_2[S^1] \to \tilde{C}_2[S^1 \times S^n]$, has $W_2 = 0$ if and only if this map extends over the cylinder.

Given that our embeddings are all induced by the action of diffeomorphisms on $S^1 \times S^n$, we can simply lift our family of diffeomorphisms to $\tilde{C}_2[S^1 \times S^n]$ and apply them to the cylinder bounding the induced map $C_2[S^1] \to C_2[S^1 \times S^n]$ corresponding to the evaluation map of $f$. □

Although it is not immediately obvious, the group $\pi_0 \text{Diff}(S^1 \times B^3 \text{ fix } \partial)$ is abelian. The proof might be described as half an ‘Alexander trick’. Consider $\text{Diff}(S^1 \times B^3 \text{ fix } \partial)$ as the subgroup of the group of diffeomorphisms of $S^1 \times \mathbb{R}^3$ – the subgroup with support contained in $S^1 \times B^3$. Here we use the support of a diffeomorphism $f$ to mean all the points in the domain where $f(p) \neq p$.

Given a linear map $\phi : \mathbb{R}^3 \to \mathbb{R}^3$ such that $\phi(B^3) \subset B^3$, and given a diffeomorphism $f \in \text{Diff}(S^1 \times B^3)$ we form the conjugate $\phi.f = (\text{Id}_{S^1} \times \phi) \circ f \circ (\text{Id}_{S^1} \times \phi)^{-1}$. This is a diffeomorphism of $S^1 \times B^3$ whose support is contained in $S^1 \times \phi(B^3)$. Thus we can isotope any two diffeomorphisms of $S^1 \times B^3$ to have disjoint support, at which point they commute. Similarly we can isotope families of diffeomorphisms of $S^1 \times B^3$ to have disjoint support, and hence $\pi_1 \text{Diff}(S^1 \times B^n \text{ fix } \partial)$ is abelian. More generally, it is known [4] that $\text{Diff}(S^1 \times B^n \text{ fix } \partial)$ is an $(n + 1)$-fold loop space. Indeed the proof is an elaboration of the above argument.

\begin{center}
\begin{tikzpicture}
\node (circ1) at (0,0) [circle, draw, minimum size = 2cm] {
\includegraphics[width=0.5\textwidth]{cylinder.png}
};
\end{tikzpicture}
\end{center}

Support of $\phi.f$ vs. $f$

**Proposition 3.2** The homotopy group $\pi_{n-3} \text{Diff}(S^1 \times B^n \text{ fix } \partial)$ is abelian and of infinite rank, for all $n \geq 3$. It contains a subgroup isomorphic to $\Lambda_n^1$.

**Proof** Let $u \in \text{Emb}(S^1, S^1 \times S^n)$ denote a representative with $W_0 = 1$. We have the restriction fibration

$$\text{Diff}_0(S^1 \times S^n \text{ fix } u) \to \text{Diff}_0(S^1 \times S^n) \to \text{Emb}_u(S^1, S^1 \times S^n)$$
where the fibre is the diffeomorphisms of $S^1 \times S^n$, homotopic to the identity on $S^1 \times S^n$, and which fix the embedding $u$ pointwise. Every such diffeomorphism gives rise to a tubular neighbourhood of $u$, thus $\text{Diff}_0(S^1 \times S^n \text{ fix } u)$ fibres over the space of tubular neighbourhoods of $u$. The space of tubular neighbourhoods has the homotopy-type of the free loop space $LSO_n$, moreover, every element of $LSO_n$ extends canonically to a fibrewise linear diffeomorphism of $S^1 \times S^n$, but only the path-component of the trivial element results in a diffeomorphism that is homotopic to the identity. Denote the identity path-component of $LSO_n$ by $L_0SO_n$. This gives us the product decomposition homotopy-equivalence

$$\text{Diff}_0(S^1 \times S^n \text{ fix } u) \simeq \text{Diff}(S^1 \times B^n \text{ fix } \partial) \times L_0SO_n.$$ 

By Theorem 3.1 our subgroup $\Lambda'_i$ of $\pi_{n-2}\text{Emb}_u(S^1, S^1 \times S^n)$ injects into $\pi_{n-3}\text{Diff}_0(S^1 \times S^n \text{ fix } u)$. Moreover, this subgroup projects faithfully to the $\pi_{n-3}\text{Diff}(S^1 \times B^n \text{ fix } \partial)$ factor. One way to see this is to use a slightly different fibre bundle. Rather than composing elements of $\text{Diff}_0(S^1 \times S^n)$ with $u$, compose them with a fixed tubular neighbourhood of $u$. This gives a fibre sequence

$$\text{Diff}(S^1 \times B^n \text{ fix } \partial) \to \text{Diff}_0(S^1 \times S^n) \to \text{Emb}^f_u(S^1, S^1 \times S^n).$$

From this perspective there are natural inclusions of $L_0SO_n$ in both the total space and the base-space, thus elements of the homotopy of $L_0SO_n$ map trivially to $\text{Diff}(S^1 \times B^n \text{ fix } \partial)$. \[\Box\]

**Theorem 3.3** Any two elements of $\text{Diff}_0(S^1 \times S^n)$, the group of diffeomorphisms homotopic to the identity, commute up to isotopy.

**Proof** Any element of $\text{Diff}_0(S^1 \times S^n)$ is isotopic to one that fixes a neighborhood of $S^1 \times \{y_0\}$ pointwise. Thus commutativity follows as in the proof of the first part of Proposition 3.2. \[\Box\]

Let $\text{Emb}(HB^i, B^n)$ denote the space of smooth embeddings $HB^i \to B^n$ that restricts to the standard inclusion $x \mapsto (x, 0)$ on the outer boundary $HB^i \cap \partial B^i$. Denote the corresponding framed embedding space by $\text{Emb}^f( HB^i, B^n)$. This space consists of pairs $(f, \nu)$ where $f \in \text{Emb}(HB^i, B^n)$ and $\nu$ is a trivialization of the normal bundle to $f$ that restricts to the canonical trivialization on $HB^i \cap \partial B^i$. Both $\text{Emb}(HB^i, B^n)$ and $\text{Emb}^f( HB^i, B^n)$ are contractible spaces, the proofs are analogous to the homotopy classification of collar neighbourhoods. The role these embedding spaces play is as the total spaces of fibre bundles. The half-ball is defined in Definition 2.4.

The first fibre-bundle to consider is $\text{Emb}(HB^i, B^n) \to \text{Emb}_u( B^{i-1}, B^n)$ where $u$ denotes the unknot component of $\text{Emb}( B^{i-1}, B^n)$, i.e. the path-component of the linear embedding. This bundle is in principle useful, but the fibre is embeddings of $HB^i$ into $B^n$ which are fixed on their boundary, which is not a very familiar space. By taking the derivative $\frac{\partial}{\partial x_1}$ along the boundary $B^{i-1}$ we see this space fibers over $\Omega^{i-1}S^{n-1}$ with fibre homotopy-equivalent to $\text{Emb}(B^i, S^{n-i} \times B^i)$. This latter space is the space of embeddings $B^i \to S^{n-i} \times B^i$ which restricts to the inclusion $p \mapsto (\ast, p)$ on $\partial B^i$, where $\ast \in S^{n-i}$ is some preferred point.

Alternatively we can form the bundle

$$\text{Emb}(B^i, S^{n-i} \times B^i) \to \text{Emb}(HB^i, B^n) \to \text{Emb}^+_u( B^{i-1}, B^n)$$

where the base space consists of embeddings $B^{i-1} \to B^n$ together with a normal vector field along the embedding i.e. $\frac{\partial}{\partial x_1}$. The fiber of this bundle is technically the embeddings of $HB^i$ into $B^n$.
which agree with the standard inclusion (and its derivative) along \( \partial HB^i \). This fiber has the same homotopy-type as \( \text{Emb}(B^i, S^{n-i} \times B^i) \).

Similarly, we have the corresponding bundles for the framed embedding spaces.

\[
\text{Emb}^{fr}(B^i, S^{n-i} \times B^i) \to \text{Emb}^{fr}(HB^i, B^n) \to \text{Emb}^{fr}_u(B^{i-1}, B^n)
\]

Given that the total space is contractible, this allows us to describe the unknot component of these embedding spaces as classifying spaces.

**Lemma 3.4**

\[
B\text{Emb}^{fr}(B^i, S^{n-i} \times B^i) \simeq \text{Emb}^{fr}_u(B^{i-1}, B^n)
\]

\[
B\text{Emb}(B^i, S^{n-i} \times B^i) \simeq \text{Emb}^+_u(B^{i-1}, B^n)
\]

We take a moment to unpack some of the underlying geometric ideas involved in the lemma.

There is an isomorphism of homotopy groups

\[
\pi_k \text{Emb}(B^i, S^{n-i} \times B^i) \to \pi_{k+1} \text{Emb}^+_u(B^{i-1}, B^n)
\]

moreover, this map has an explicit geometric description. To do this, we need the exact fibre of the bundle \( \text{Emb}(HB^i, B^n) \to \text{Emb}^+_u(B^{i-1}, B^n) \). This is the space of embeddings of \( HB^i \) into \( B^n \) which agrees with the standard inclusion \( p \mapsto (p, 0) \) and its derivative on the full boundary of \( HB^i \). Denote this space by \( \text{Emb}_\partial(HB^i, B^n) \). Serre’s homotopy-fibre construction tells us that \( \text{Emb}_\partial(HB^i, B^n) \) is homotopy-equivalent to

\[
HF = \{ \alpha : [0, 1] \to \text{Emb}(HB^i, B^n) \text{ s.t. } \alpha(0) = *, \alpha(1) \in \text{Emb}_\partial(HB^i, B^n) \}.
\]

In the above, \( * \) denotes the basepoint of \( \text{Emb}(HB^i, B^n) \), i.e. the standard inclusion \( p \mapsto (p, 0) \).

The homotopy-equivalence between \( HF \to \text{Emb}_\partial(HB^i, B^n) \) is given by associating \( \alpha(1) \) to \( \alpha \). The homotopy-equivalence between \( HF \) and \( \Omega \text{Emb}^+_u(B^{i-1}, B^n) \) is given by associating \( \overline{\alpha} \) to \( \alpha \) where \( \overline{\alpha}(t) = \alpha(t)|_{B^{i-1}} \).

For the sake of those not familiar with these methods we describe the homotopy-equivalence directly. For this we need to adjust our model slightly. We replace the space \( \text{Emb}(HB^i, B^n) \) with the homotopy-equivalent space of embeddings \( H^i \to \mathbb{R}^n \) where the support is constrained to be in \( HB^i \), i.e. the maps are the standard inclusion \( p \mapsto (p, 0) \) outside of \( HB^i \). Similarly, \( \text{Emb}^+_u(B^{i-1}, B^n) \) would be the space of embeddings \( \mathbb{R}^{i-1} \to \mathbb{R}^n \) with a normal unit vector field, the embeddings and normal vector required to be standard outside of \( B^{i-1} \). From this perspective, the fibre \( \text{Emb}(HB^i, B^n) \to \text{Emb}^+_u(B^{i-1}, B^n) \) is the space of embeddings \( H^i \to \mathbb{R}^n \) where the support is not only contained in \( HB^i \), but the embedding and its derivative in the normal direction is required to be standard on \( \partial H^i \). Observe the explicit deformation-retraction of \( \text{Emb}(HB^i, B^n) \) to a point, given by associating to \( f \in \text{Emb}(HB^i, B^n) \) the path (as a function of \( t \)), \( f_t \in \text{Emb}(HB^i, B^n) \) where

\[
f_t(x_1, x_2, \ldots, x_k) = f(x_1 - t, x_2, \ldots, x_k) + (t, 0, \ldots, 0).
\]

Thus the map \( \text{Emb}_\partial(HB^i, B^n) \to \Omega \text{Emb}^+_u(B^{i-1}, B^n) \) in this model is the one that associates to \( f \in \text{Emb}_\partial(HB^i, B^n) \) the path \( f_t \in \text{Emb}^+_u(B^{i-1}, B^n) \) given by

\[
f_t(x_2, \ldots, x_k) = f(1 - t, x_2, \ldots, x_k) + (t, 0, \ldots, 0).
\]
The vector field being \( \frac{\partial f}{\partial x_j} (1 - t, x_2, \cdots, x_k) \). So it would be reasonable to call this map \textit{slicing the embedding}.

We mention a few elementary consequences of Lemma 3.4.

\[
SO_n \simeq \text{Emb}^{fr}_u(B^0, B^n) \simeq B\text{Emb}^{fr}(B^1, S^{n-1} \times B^1).
\]

For embeddings with 1-dimensional domains we have

\[
\text{Emb}^+_{u}(B^1, B^n) \simeq B\text{Emb}(B^2, S^{n-2} \times B^2).
\]

The space \( \text{Emb}^+_{u}(B^1, B^n) \) is a bundle over \( \text{Emb}_u(B^1, B^n) \), and this space is equal to \( \text{Emb}(B^1, B^n) \) when \( n \geq 4 \). The fibre of this bundle is \( \Omega S^{n-2} \), provided \( n \geq 4 \). There is an elementary argument that this bundle is trivial when \( n = 3 \) [4]. Paolo Salvatore uses a more sophisticated technique [23] to show this bundle is trivial when \( n \geq 4 \), giving

\[
\text{Emb}^+_{u}(B^1, B^n) \simeq \text{Emb}(B^1, B^n) \times \Omega S^{n-2}.
\]

The space \( \text{Emb}^+_{u}(B^1, B^n) \) has a concatenation operation, which could also be thought of as an action of the operad \( (n-1) \)-discs. These cubes actions turn the two sets \( \text{Emb}^+_{u}(B^{n-2}, B^n) \) into commutative monoids under concatenation, therefore an abelian group, isomorphic to \( \pi_1 \text{Emb}^+_{u}(B^{n-2}, B^n) \).

The space \( \text{Emb}^+_{u}(B^{n-1}, S^1 \times B^{n-1}) \) has a concatenation operation, which could also be thought of as an action of the operad \( (n-1) \)-discs. The space \( \text{Emb}^+_{u}(B^{n-2}, B^n) \) similarly has a concatenation operation with one less degree of freedom. It can be encoded as an action of the operad \( (n-1) \)-discs. These cubes actions turn the two sets \( \pi_0 \text{Emb}(B^{n-1}, S^1 \times B^{n-1}) \) and \( \pi_1 \text{Emb}^+_{u}(B^{n-2}, B^n) \) into commutative monoids under the concatenation operation, provided \( n \geq 3 \). Moreover, one can see that the concatenation operation and concatenation of loops are the same operation on \( \pi_1 \text{Emb}^+_{u}(B^{n-2}, B^n) \). Thus, the isomorphism \( \pi_0 \text{Emb}(B^{n-1}, S^1 \times B^{n-1}) \simeq \pi_1 \text{Emb}^+_{u}(B^{n-2}, B^n) \) is an isomorphism of groups, which must be abelian. Lastly, notice that \( \text{Emb}^+_{u}(B^{n-2}, B^n) \) fibers over \( \text{Emb}_u(B^{n-2}, B^n) \) with fiber \( \Omega^{n-2} SO_2 \), provided \( n \geq 4 \). So for all \( n \) we have a homotopy-equivalence

\[
\text{Emb}_u(B^{n-2}, B^n) \simeq B\text{Emb}(B^{n-1}, S^1 \times B^{n-1}).
\]

Since the concatenation operation on \( \text{Emb}(B^{n-1}, S^1 \times B^{n-1}) \) turns \( \pi_0 \text{Emb}(B^{n-1}, S^1 \times B^{n-1}) \) into a group, it makes sense to consider the fibre bundle

\[
\text{Diff}(B^{n+1} \text{fix } \partial) \to \text{Diff}(S^1 \times B^n \text{ fix } \partial) \to \text{Emb}(B^n, S^1 \times B^n).
\]

Every embedding \( B^n \to S^1 \times B^n \) that is standard \( p \mapsto (1, p) \) on \( \partial B^n \) is the fibre of some trivial smooth fibre bundle \( S^1 \times B^n \to S^1 \) with fibre \( B^n \), by Lemma 3.4 and isotopy extension. Thus \( \text{Diff}(S^1 \times B^n \text{ fix } \partial) \) acts transitively on \( \text{Emb}(B^n, S^1 \times B^n) \). We record the observation.
Theorem 3.5 The group $\text{Diff}(S^1 \times B^n \text{ fix } \partial)$ acts transitively on $\text{Emb}(B^n, S^1 \times B^n)$. Moreover every reducing ball $B^n \to S^1 \times B^n$ is the fibre of some smooth fibre bundle $S^1 \times B^n \to S^1$.

One could deduce Theorem 3.5 when $n \geq 4$ by the h-cobordism theorem. In dimension $n = 1$ it follows from the classification of surfaces. When $n = 2$ it follows via Alexander’s theorem and an innermost circle argument.

Notice that the fibre bundle $\text{Diff}(B^{n+1} \text{ fix } \partial) \to \text{Diff}(S^1 \times B^n \text{ fix } \partial) \to \text{Emb}(B^n, S^1 \times B^n)$ admits a splitting at the fibre, i.e. a map $\text{Diff}(S^1 \times B^n \text{ fix } \partial) \to \text{Diff}(B^{n+1} \text{ fix } \partial)$ that is a left homotopy-inverse to the fibre inclusion. The key idea to generate the splitting is that if we add a 2-handle to $S^1 \times B^n$ we can create $B^{n+1}$. Since our diffeomorphisms of $S^1 \times B^n$ are the identity on the boundary, we can extend them to be the identity on the 2-handle, and therefore to elements of $\text{Diff}(B^{n+1} \text{ fix } \partial)$. This tells us that the bundle is trivial, i.e.

$$\text{Diff}(S^1 \times B^n \text{ fix } \partial) \simeq \text{Diff}(B^{n+1} \text{ fix } \partial) \times \text{Emb}(B^n, S^1 \times B^n).$$

Key to this argument is that the splitting at the fibre is a left homotopy-inverse to the fibre inclusion, which ultimately boils down to the ball $B^{n+1}$ being contractible.

We saw in the proof of Theorem 3.1 that our subgroup of $\pi_{n-3}\text{Diff}(S^1 \times B^n \text{ fix } \partial)$ isomorphic to $\Lambda_n^{W_0}$ is generated by an isotopy-extension of an $(n-2)$-parameter family of embedded 1-manifolds. The family $S^{n-2} \to \text{Emb}(S^1, S^1 \times B^n)$ had the property that it came from a modification of the vertical fiber $S^1 \times \{\ast\} - \text{a parametrized embedded surgery construction}$, using an immersed half-ball $i(HB^2)$ that intersects the vertical fiber on one edge. On that edge, the half-ball is constant and intersects on the interior boundary of the half-ball. The half-ball is modified via a family of resolutions of the double point. The family is parametrized by $S^{n-2}$ which is to be thought of as the unit sphere in the subspace orthogonal to the two tangent spaces of the boundary of the half ball at the double-point. The resulting family of embeddings $\theta_{S^1 \times \{\ast\}, k}$ was non-trivial provided $k \neq 0$. The $S^{n-3}$-parameter family of diffeomorphisms of $S^1 \times B^n$ fixing the knot $f$ is obtained by the isotopy extension theorem applied to $\theta$.

For an isotopy-extension, the induced family of diffeomorphisms

$$S^{n-3} \to \text{Diff}(S^1 \times B^n \text{ fix } f)$$

will be supported in an arbitrarily small neighbourhood of the support of the isotopy, i.e. the points of $S^1 \times S^n$ in the image of $\theta$, where $\theta$ is not constant. Provided one constructs these diffeomorphisms using a minimalist isotopy extension, the support of these diffeomorphisms would be constrained to be inside an embedded copy of $S^{n-2} \times D^2 \#_0 S^{n-2} \times D^2$, i.e the boundary connect-sum of two copies of $S^{n-2} \times D^2$. The central spheres of each summand link a single strand of the embedding $f$, like handcuffs.

After attaching our 2-handle, in the resulting manifold we can slide the support of this family of diffeomorphisms across the 2-handle. Once the support has returned to $S^1 \times B^n$, we have constructed a new family $S^{n-3} \to \text{Diff}(S^1 \times B^n \text{ fix } \partial)$, corresponding to the $k = 0 \theta$ family, i.e. a trivial family of diffeomorphisms. Thus our family $\Lambda_1 \subset \pi_{n-3}\text{Diff}(S^1 \times B^n \text{ fix } \partial)$ is in the kernel of the splitting map to $\pi_{n-3}\text{Diff}(B^{n+1} \text{ fix } \partial)$.

Corollary 3.6 Under the product decomposition

$$\text{Diff}(S^1 \times B^n \text{ fix } \partial) \simeq \text{Diff}(B^{n+1} \text{ fix } \partial) \times \text{Emb}(B^n, S^1 \times B^n)$$

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our subgroup $\Lambda_1^n$ of $\pi_{n-3}\text{Diff}(S^1 \times B^n \text{ fix } \partial)$ lives in the $\pi_{n-3}\text{Emb}(B^n, S^1 \times B^n)$ factor, i.e. map

$$\pi_{n-3}\text{Diff}(S^1 \times B^n \text{ fix } \partial) \to \pi_{n-3}\text{Emb}(B^n, S^1 \times B^n)$$

 restricts to an isomorphism on $\Lambda_1^n$.

In dimension 3, further the kernel of the map $\pi_1\text{Emb}_f(S^1, S^1 \times S^3) \to \pi_0\text{Diff}(S^1 \times S^3 \text{ fix } S^1 \times \{\ast\})$ when $W_0(f) = 1$ is precisely the subgroup with $W_2 = 0$.

**Theorem 3.7** The group $\text{Diff}(S^1 \times S^n)$ acts transitively on the non-separating $n$-spheres in $S^1 \times S^n$. Moreover, every non-separating $n$-sphere is the fiber of a fibre-bundle $S^1 \times S^n \to S^1$.

**Proof** Provided $n < 3$ this is classical. When $n \geq 3$ observe that complimentary to a non-separating sphere there is an embedding $S^1 \to S^1 \times S^n$ that intersects the sphere precisely once and transversely. Since $\dim(S^1 \times S^n) \geq 4$, we can isotope our embedding to be equal to $S^1 \times \{\ast\}$ and similarly isotopy our non-separating sphere. If we drill a neighbourhood of $S^1 \times \{\ast\}$ out of $S^1 \times S^n$ we have constructed $S^1 \times B^n$, and our non-separating sphere is converted to a reducing ball. The result follows from Theorem 3.5.

### 4 Knotted 3-Balls and the Schönhflies Conjecture

In this section we first detail the close connection between knotted 3-balls in $S^4$, and knotted 3-balls in $S^1 \times B^3$. We further discuss the relation between the Schönflies conjecture and virtual unknotting of 3-balls.

**Definition 4.1** Let $\Delta_0$ be a 3-ball in $S^4$. We say that the 3-ball $\Delta$ is knotted relative to $\Delta_0$ if $\partial \Delta \neq \partial \Delta_0$ and $\Delta$ is not isotopic rel $\partial \Delta$ to $\Delta_0$.

In what follows $\Delta_0$ will denote a standard or linear 3-ball as defined in the introduction. It is to be fixed once and for all. Its boundary, the 2-unknot, will be denoted by $U$. Unless said otherwise, 3-balls in $S^4$ will all have boundary equal to $U$ and knottedness is relative to $\Delta_0$. Relative knottedness is essential for by [6] p. 231, [22] any two smooth embedded 3-balls in the interior of a connected 4-manifold are smoothly ambiently isotopic.

We abuse notation by letting $\Delta_0$ also denote $\{x_0\} \times B^3 \subset S^1 \times B^3$. It’s use should be clear from context.

**Definition 4.2** The properly embedded non separating 3-ball $\Delta \subset S^1 \times B^3$ is knotted if it is not properly isotopic to $\Delta_0$.

**Remark 4.3** Equivalently $\Delta$ is knotted if it is properly homotopic to but not properly isotopic to $\Delta_0$. Since any two non separating 2-spheres in $S^1 \times S^2$ are isotopic [11], [17], we can assume that $\partial \Delta = \partial \Delta_0$. By uniqueness of regular neighborhoods we can further assume that $\Delta$ coincides with $\Delta_0$ near partial $\Delta$ and since $\text{Diff}_0(S^2)$, the diffeomorphisms homotopic to id, is connected they have the same parametrization there. Finally, since $\pi_1(\text{Emb}(S^2, S^1 \times S^2)) = 1$ up to paths in $SO(3)$ and translations in $S^1 \times S^2$, [17] it follows that if $\Delta_2$ and $\Delta_3$ are isotopic and already

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coincide near $\partial \Delta_2$, then there is an isotopy fixing a neighborhood of the boundary pointwise. By [7], since $\text{Diff}(B^3 \text{ fix } \partial)$ is connected, $\Delta_0$ has a unique parametrization up to isotopy.

Since $S^1 \times B^3$ is diffeomorphic to the closed complement of the unknot $U$ in $S^4$ it follows that we can assume, up to isotopy fixing the boundary pointwise, that all 3-balls $\Delta$ with boundary $\Delta_0$ coincide with $\Delta_0$ near $\partial \Delta_0$. Also, that there exists a uniform neighborhood $N(U)$ of $U$ such that $\Delta \cap N(U) = \Delta_0$.

**Notation 4.4** Let $N(U)$ be a fixed regular neighborhood of the unknot $U$ in $S^4$, with $\Delta_0$ isotoped to be properly embedded in $S^4 \setminus \text{int}(N(U))$. Fix a diffeomorphism $\psi : S^4 - \text{int}(N(U)) \to S^1 \times B^3$ such that $\psi(\Delta_0) = \Delta_0$.

The following is immediate.

**Proposition 4.5** $\psi$ induces a 1-1 correspondence between isotopy classes of knotted 3-balls in $S^4$ and knotted 3-balls in $S^1 \times B^3$.

**Remark 4.6** It is important to remember that our correspondence is given by $\psi$, since by Theorem 3.3 $\text{Diff}(S^1 \times B^n \text{ fix } \partial)$ acts transitively on properly embedded 3-balls of $S^1 \times B^3$.

**Theorem 4.7** If $\Delta_0$ and $\Delta_1$ are properly embedded 3-balls in $S^4 \setminus \text{int}(N(U))$ coinciding near their boundaries, then there exists an orientation preserving diffeomorphism $\phi : (S^4, \Delta_0) \to (S^4, \Delta_0)$ fixing $N(U)$ pointwise. Any 3-ball $\Delta_1$ with boundary $U$ restricts to a fiber of a fibration of $S^4 \setminus \text{int}(N(U))$.

**Proof** i) This is [6], [22] applied to 3-balls in $S^4$, together with uniqueness of regular neighborhoods and the fact that $\text{Diff}_0(S^2)$ is connected.

**Theorem 4.8** $\psi$ induces isomorphisms between the following abelian groups which contain $\Lambda^1_3$ as a subgroup.

i) Isotopy classes of 3-balls in $S^4$ with boundary $\Delta_0$

ii) isotopy classes of 3-balls in $S^1 \times B^3$ with boundary $\Delta_0$

iii) $\pi_0(\text{Diff}(S^1 \times B^3 \text{ fix } \partial)/\text{Diff}(B^4 \text{ fix } \partial))$

**Proof** Recall that by $\pi_0(\text{Diff}(S^1 \times B^3 \text{ fix } \partial)/\text{Diff}(B^4 \text{ fix } \partial))$ we mean isotopy classes of diffeomorphisms of $S^1 \times B^3$ fixing a neighborhood of $S^1 \times S^2$ pointwise modulo diffeomorphisms that are supported in a compact 4-ball.

To an element $[\phi] \in \pi_0(\text{Diff}(S^1 \times B^3 \text{ fix } \partial)/\text{Diff}(B^4 \text{ fix } \partial))$ we associate the 3-ball $\Delta = \phi(\Delta_0)$. It’s isotopy class is well defined since $\text{Diff}(B^4 \text{ fix } \partial)(\Delta_0) = \Delta_0$ up to isotopy. If $\Delta \subset S^1 \times B^3$ is a 3-ball with boundary $\Delta_0$, then by Theorem 4.7 there exists a $\phi \in \text{Diff}(S^1 \times B^3 \text{ fix } \partial)$ with $\phi(\Delta_0) = \Delta$. If $\phi'$ is another such diffeomorphism and $\phi_0 = \phi \circ \phi'^{-1}$, then $\phi_0(\Delta_0) = \Delta_0$. By [7] we can assume that after isotopy $\phi_0(\Delta_0) = \phi_0(\Delta_0)$ pointwise, and then also $\phi_0(N(\Delta_0)) = N(\Delta_0)$ pointwise. Thus $\phi$ is equivalent to $\phi'$ modulo $\text{Diff}(B^4 \text{ fix } \partial)$. It follows that there is a 1-1 correspondence between ii) and iii).
Recall that the group structure on 3-balls in $S^1 \times B^3$ is the one induced from $\pi_0(Diff(S^1 \times B^3/\text{fix} \partial)/Diff(B^4/\text{fix} \partial))$, so $\Delta_0$ is the identity and multiplication is given by concatenation. Use $\psi$ to induce the group structure on isotopy classes of 3-balls in $S^4$ with boundary $\Delta_0$ and hence the isomorphism between i) and ii).

Theorem 3.3 implies that these groups are abelian. Theorem 3.6 implies that these groups contain $\Lambda^1_3$ as a subgroup.

**Remark 4.9** Theorem 1.10 of [10] proves the uniqueness of spanning 2-discs in $S^4$, i.e. two discs with the same boundary are isotopic rel boundary. The existence of knotted 3-balls in $S^4$, i.e. the non uniqueness of spanning 3-discs in $S^4$, negatively answers Question 10.13 of [10] for $k = 3$.

**Corollary 4.10** There exist fiberings of the unknot $U \subset S^4$ not isotopic to the linear fibering. The set of isotopy classes of fiberings forms an infinitely generated abelian group and this group contains a subgroup isomorphic to $\Lambda^1_3$.

The four dimensional smooth Schönflies conjecture is that smooth 3-spheres in the 4-sphere are smoothly isotopically standard. In 1959 Mazur [19] showed that such spheres are topologically standard. More generally Brown [3] and Morse [21] showed that locally flat 3-spheres in $S^4$ are topologically standard. The corresponding conjecture is known to be true smoothly for all dimensions not equal to four.

The following is essentially stated in [10] as Remark 10.14.

**Theorem 4.11** The following are equivalent

i) The Schönflies conjecture is true

ii) For every 3-ball $\Delta$ in $S^4$ with $\partial \Delta = \partial \Delta_0$, there exists $n \in \mathbb{N}$ such that the lift of $\Delta$ to the $n$-fold cyclic branched cover of $S^4$, branched over $\partial \Delta$ is isotopic to $\Delta_0 \text{ rel } \partial \Delta$.

iii) For every non separating properly embedded 3-ball $\Delta$ in $S^1 \times B^3$, there exists $n \in \mathbb{N}$ such that the lift of $\Delta$ to the $n$-fold cyclic cover of $S^1 \times B^3$ is isotopically standard.

**Proof** Let $\Sigma_0 \subset S^4$ be an unknotted 3-sphere. If $\Sigma_1 \subset S^4$ is an embedded 3-sphere that coincides with $\Sigma_0$ in a neighborhood of a 3-disc, then $\Sigma_1$ is isotopic to $\Sigma_0$ if and only if there exists an isotopy that also fixes a neighborhood of the 3-disc pointwise.

We now show that i) implies ii). Let $\Delta'_0 \subset S^4$ be the linear 3-ball such that $\Delta_0 \cap \Delta'_0 = \partial \Delta_0$ and $\Sigma_0 = \Delta_0 \cup \Delta'_0$ is a smooth 3-sphere. Let $\Delta$ be a 3-ball in $S^4$ that coincides with $\Delta_0$ near $\partial \Delta_0$. By passing to a sufficiently high odd degree branched cover over $\partial \Delta_0$ we can assume that there are preimages $\tilde{\Delta}_0, \tilde{\Delta}'_0$ of $\Delta_0, \Delta'_0$ such that $\tilde{\Delta}_0$ coincides with $\tilde{\Delta}$ near $\partial \tilde{\Delta}_0$, $\text{int}(\tilde{\Delta}) \cap \tilde{\Delta}'_0 = \emptyset$ and $\Sigma_0 = \tilde{\Delta}'_0 \cup \tilde{\Delta}_0$ is a smooth, necessarily unknotted, 3-sphere in $S^4$. Now $\Sigma_1 = \tilde{\Delta}'_0 \cup \tilde{\Delta}_1$ is another 3-sphere in $S^4$. It follows from the previous paragraph that $\Sigma_1$ is isotopic rel $\tilde{\Delta}'_0$ to $\Sigma_0$ and so $\tilde{\Delta}$ is isotopic to $\tilde{\Delta}_0 \text{ rel } \partial \tilde{\Delta}_0$ and hence is unknotted.

Now we show that ii) implies i). Let $\Sigma_1$ be a 3-sphere in $S^4$. We can assume that it coincides with the unknotted 3-sphere $\Sigma_0$, constructed above, near the 3-disc $\Delta'_0$. Let $\Delta$ be the closed 3-disc in $\Sigma_1$ complementary to $\Delta'_0$. By hypothesis, $\Delta$ becomes unknotted in a $n$-fold branched cover.
of $S^4$ branched over $\partial \Delta_0$, given by $p : S^4 \to S^4$. In this cover let $\tilde{\Delta}$ be a preimage of $\Delta$, $\tilde{\Delta}_0$ be the preimage of $\Delta_0$ that coincides with $\tilde{\Delta}$ near their common boundary, and $E_1, E_2, \ldots, E_n$ be the preimages of $\Delta'_0$ cyclically ordered about $\partial \Delta_0$ with $\Delta_0 \cup \tilde{\Delta}$ lying in the region $W$ bounded by $E_n \cup E_1$ that contains no other $E_i$’s. If $\tilde{\Delta}$ is isotopic to $\Delta_0$ via an isotopy supported in $W$, then by composing with $p$ we obtain an isotopy between $\Delta$ and $\Delta_0$ supported away from $\Delta'_0$ and hence one between $\Sigma_1$ and $\Sigma_0$. Since the isotopy from $\tilde{\Delta}$ to $\Delta_0$ is supported away from $\partial \Delta_0$ it follows that when lifted to the infinite cyclic branched cover the support of the isotopy hits only finitely many preimages of $\Delta'_0$. Thus the original $n$ could have been chosen so that the support of the isotopy is disjoint from some $E_i$, $i \neq 1, n$. Since the region between $E_1$ and $E_i$ (resp. $E_i$ and $E_n$) is a relative product, the isotopy can be modified to be supported in $W$.

The equivalence of ii) and iii) follows from Theorem 4.7.

As an aside, notice that the Schönflies conjecture has an interesting near-algebraic formulation in this language. The process of lifting an embedding $B^3 \to S^1 \times B^3$ to the $j$-sheeted covering space of $S^1 \times B^3$ ($j \geq 2$) gives us an endomorphism of the group $\pi_0 \text{Emb}(B^3, S^1 \times B^3)$

$$\tilde{D} : \pi_0 \text{Emb}(B^3, S^1 \times B^3) \to \pi_0 \text{Emb}(B^3, S^1 \times B^3).$$

Given that every ball, when lifted to a sufficiently-large cover of $S^1 \times B^3$ will be disjoint from a standard (linear) $B^3$, this tells us that $\tilde{D}$ satisfies a weak form of idempotence. Specifically, given any $\alpha \in \pi_0 \text{Emb}(B^3, S^1 \times B^3)$ then for some $k$ we have that

$$\tilde{D}^{(k+1)} \alpha = \tilde{D}^k \alpha.$$

where our exponential notation refers to iterated composition, $\tilde{D}^{(3)} = \tilde{D} \circ \tilde{D} \circ \tilde{D}$, i.e. it is the operation of lifting a ball to the $j^3$-fold cover. Thus all possible counter-examples to the Schönflies conjecture, together with the standard linear ball $B^3 \to S^1 \times B^3$ are the fixed-point set of $\tilde{D}$.

We complete this section with a result that implies that none of the knotted 3-balls arising from $\Lambda_3^1$ give counter examples to the Schönflies conjecture.

**Lemma 4.12** Let $f : S^1 \to S^1 \times S^3$ be the standard vertical embedding with $W_0(f) = 1$. Let $p : S^1 \times S^3 \to S^1 \times S^3$ the m-fold cyclic cover, let $\{\alpha_i | i > 0\}$ denote the generators of $\pi_1(\text{Emb}(f(S^1, S^1 \times S^3)))$ as in Theorem 4.5 and let $\tilde{\alpha}_i$ denote the $p^*$ pull back of $\alpha_i$. Then $\tilde{\alpha}_0 = 1$ if $m$ does not divide $n$ and $\tilde{\alpha}_n = m \alpha_{n/m}$ if $m$ divides $n$.

**Proof** Represent $\alpha_n$ by the torus $T_n$ as in §2. Then $\tilde{\alpha}_n$ is represented by $p^{-1}(T_n) = R_n$. Now $T_n$ is constructed from the standard vertical torus $T^*$ and an unknotted 2-sphere $K$ by removing small discs $D^T, D^K$ from $T^*$ and $K$, and then adding a tube $Y$ that starts at $\partial D^T$, links through $K$, goes $n$ times about the $S^1$ direction before connecting to $\partial D^K$. Therefore $R_n$ is obtained by removing $m$ small discs from $T^*$ and one from each of the preimages of $K$ and then connecting their boundaries by the $m$ preimages $Y_1, \cdots, Y_m$ of $Y$. I.e. $R_n$ is obtained by removing $m$ standard discs from $T^*$ and replacing them by $m$ other ones. Now assume that $m$ divides $n$. Then the sphere $K_i$ that $Y_i$ links is also the sphere to which $Y_i$ connects. Note that if $D^T_i$ (resp. $D^K_i$) is the preimage of $D^T$ (resp. $D^K$) whose boundary is tubed to $Y_i$, then $((T^* \cup K_i) \setminus (D^T_i \cup D^K_i)) \cup Y_i$ is a torus representing $\alpha_{n/m}$. After an isotopy of $R_n$ supported near the $D^T_i$’s we can assume that the
projection $\pi : T^* \to \mu_w$ has the property that $\pi(D_1^T), \ldots, \pi(D_n^T)$ are disjoint intervals. (Recall that $\mu_w \subset T^*$ is an oriented loop intersecting each vertical $S^1$ fiber once.) It follows that $R_n$ represents an element $\beta$ of $\pi_1(\text{Emb}(S^1, S^1 \times S^3))$ which corresponds to the standard vertical circle sweeping around $T^*$ and going over one knotted disc at a time. Since there are $m$ such discs it follows that $\beta = m\alpha_{n/m}$.

Now assume that $m$ does not divide $n$. In that case each tube $Y_i$ links a sphere distinct from the one to which it connects. Again isotope $R_n$ near the $D_i^T$’s so that the projection $\pi : T^* \to \mu_w$ has the property that $\pi(D_1^T), \ldots, \pi(D_n^T)$ are disjoint intervals. Again let $\beta$ be the element represented by $R_n$. Here the discs swept over by $\beta$ can be individually isotoped back to their $D_i^T$’s without intersecting $T^*$. It follows that $\beta = 1$. □

**Corollary 4.13** Let $p : S^1 \times B^3 \to S^1 \times B^3$ be the $m$-fold cyclic cover, let $\phi_n : S^1 \times B^3 \to S^1 \times B^3$ be the diffeomorphism arising from $\alpha_n$, and $\phi_n$ be the $p^*$ pull-back of $\phi_n$. Then, $\phi_n$ is isotopic to $\text{id}$ if $m$ does not divide $n$ and $\phi_n$ is isotopic to $m\phi_{n/m}$ if $m$ divides $n$. □

**Corollary 4.14** Let $p : S^1 \times B^3 \to S^1 \times B^3$ be the $m$-fold cyclic cover, $\Delta_n = \phi_n(\Delta_0), n \neq 0$, and $\tilde{\Delta}_n$ a lift of $\Delta_n$ to the cover. Then $\tilde{\Delta}_n$ is isotopic to $\Delta_0$ if $m$ does not divide $n$ and isotopic to $m\Delta_{n/m}$ otherwise. □

**Definition 4.15** We say that the 3-ball $\Delta \subset S^1 \times B^3$ is **virtually unknotted** if it becomes unknotted after lifting to some finite cover of $S^1 \times B^3$.

**Corollary 4.16** Every knotted 3-ball $\Delta$ arising from an element of $\pi_1(\text{Emb}(S^1, S^1 \times S^3))$ is virtually unknotted. On the other hand, for each $n \in \mathbb{N}, n \neq 0$, the knotted 3-ball $\Delta_n$ remains knotted after lifting to the $n$-fold cyclic cover.

**Remark 4.17** By [14] all knotted $n$-balls in $S^1 \times B^n, n \geq 6$, become unknotted after lifting to the 2-fold cover of $S^1 \times B^n$.

Lemma 4.12 and its corollaries give another proof of the following special case of Corollary 3.6.

**Corollary 4.18** The subgroup $\Lambda_3$ of $\pi_0(\text{Diff}(S^1 \times B^3))$ lives in the $\pi_0(\text{Emb}(B^3, S^1 \times B^3))$ factor. In particular, no nontrivial element of $\Lambda_3$ corresponds to an element of $\text{Diff}(B^4 \text{ fix } \partial)$ and distinct non trivial elements give rise to distinct knotted 3-balls in $S^4$.

**Proof** The group $\Lambda_3$ is free abelian, thus if $\alpha \in \text{Diff}(S^1 \times B^3 \text{ fix } \partial)$ represents a non trivial element of $\Lambda_3$, then the isotopy class of $\alpha$ is of infinite order. On the other hand if $\alpha$ is supported in a 4-ball, and $\tilde{\alpha}$ is the pull back of $\alpha$ to the $m$-fold cyclic cover of $S^1 \times B^3$, then $\tilde{\alpha}$ is conjugate to $\alpha^m$. Corollary 4.14 now implies that $\alpha^m$ is isotopically trivial for $m$ sufficiently large, which is a contradiction. □
5 Knotted 3-balls in $S^1 \times B^3$ and $S^4$

In this section we describe how to construct the knotted 3-balls $\{\Delta_i \mid i > 0 \}$ in $S^1 \times B^3$ arising from the generators $\{\alpha_i \mid i > 0 \}$ of $\Lambda^3 \subset \pi_1(\text{Emb}_f(S^1, S^1 \times S^3))$ where $f$ is the standard generator of $\pi_1(S^1)$ with $W_0(f) = 1$. The general 3-ball arising from $\Lambda^3$ is then obtained by suitable concatenation.

Recall that $\alpha \in \pi_1(\text{Emb}_f(S^1, S^1 \times S^3))$ gives rise to a diffeomorphism $\phi \in \text{Diff}(S^1 \times B^3 \text{fix } \partial)$ by first representing $\alpha$ by a loop $\{\gamma_i\}$ of embeddings with $\gamma_0 = \gamma_1 = S^1 \times y_0$, next extending the loop to a path $\phi_i$ of diffeomorphisms of $S^1 \times S^3$ so that $\phi_0 = \text{id}$ and $\phi_1|N(S^1 \times y_0) = \text{id}$. Here we are assuming that $W_0(f) = 1, W_1(\alpha) = 0$ and $w = (x_0, y_0)$ is a point of $\gamma_0$. We then obtain $\phi$ by restricting to $S^1 \times S^3 \setminus \text{int}(N(S^1 \times y_0)) = S^1 \times B^3$. The proper isotopy class of $\phi$ is independent of all choices. Let $\phi_i$ denote a diffeomorphism induced from $\alpha_i$. The knotted 3-balls $\Delta_i = \phi_i(\Delta_0), i = 1, 2$ are shown in Figure 5.

![Figure 5: (a) Knotted 3-ball $\Delta_1$ arising from $\alpha_{f,1}$. (b) Knotted 3-ball $\Delta_2$ arising from $\alpha_{f,2}$](image)

**Definition 5.1** Let $M$ be a properly embedded 3-manifold in the 4-manifold $V$. We say that $N$ is obtained from $M$ by **embedded surgery** if there is a sequence $M = M_0, M_1, M_2, \cdots, M_n = N$ such that $M_i$ is obtained from the regular neighborhood $N(M_{i-1})$ by first attaching a single 4-dimensional handle embedded in $V$ and then restricting to some relative boundary components.

We decipher Figure 5a). In the case at hand $M_0 = \Delta_0$, $V = S^1 \times B^3$ and each handle is a 0-framed 2-handle. Here $N(M_i)$ is a product and one keeps the relative boundary component of the modified 4-manifold that is changed by the 2-handle. Now identify $B^3$ as $D^2 \times [-1, 1]$, let $V_s = S^1 \times D^2 \times s$ and let $D_0 = \Delta_0 \cap V_0$. Our $\Delta_1$ is obtained from $\Delta_0$ by embedded surgery along two 2-handles $\sigma$ and $\tau$. Let $C(\sigma)$ and $C(\tau)$ denote their cores. Figure 5a) shows the projection of $\Delta_0$, $C(\sigma)$ and $C(\tau)$ to $V_0$. Here, $C(\sigma) \subset V_0$ and is a standardly looking 2-disc with $\partial C(\sigma) \subset D_0$. The projection $C'(\tau)$ of $C(\tau)$ to $V_0$ is the shaded immersed disc with a single ribbon arc of self intersection. The projection $C(\tau) \to C'(\tau)$ is pointwise $\leq 2$ to 1 and $C'(\tau) \cap D_0$ is a connected arc. $C(\tau) \cap V_0$ is the union of two circles and an arc, all of which are indicated by thick lines. Obtain $M_1$ by doing the embedded surgery corresponding to $\sigma$. Finally, $M_2 = \Delta_1$ is obtained by doing the embedded surgery corresponding to $\tau$. Since $\sigma \cap \tau = \emptyset$, this order can be reversed. Note that...
$\Delta_0 \cap (C(\sigma) \cup C(\tau))$ is the Hopf link in $\Delta_0$ and so $\Delta_1$ is obtained from $\Delta_0$ by 0-surgery along the Hopf link. We therefore directly see that $\Delta_1$ is a 3-ball.

We next decipher Figure 5b). Again $M_0 = \Delta_0$ and here $\Delta_2$ is obtained by sequential embedded surgeries along the four 2-handles $\sigma_1, \sigma_2, \tau_1, \tau_2$. The projections of $\Delta_0$ and the cores of the four 2-handles to $V_0$ are shown in the figure. Again, $C(\sigma_1) \cup C(\tau_2) \subset V_0$ with $\partial C(\sigma_1) \cup \partial C(\tau_2) \subset D_0$. Here $C(\sigma_1) \subset \text{int}(C(\sigma_2))$ and $C(\tau_2) \subset \text{int}(C(\tau_1))$. Let $C'(\tau_i), i = 1, 2$ denote the projections of $C(\tau_i)$ to $V_0$. Here $C'(\tau_2)$ is a shaded immersed 2-disc that intersects $D_0$ in a single arc and its qualitative description is like that of $C'(\tau)$ above. $C'(\tau_1)$ is also a 2-disc, though it has two ribbon arcs of self intersection. We obtain $\Delta_2$ from $\Delta_0$ by embedded surgery as follows. First obtain $M_1$ by doing the surgery corresponding to $\sigma_1$ and then obtain $M_2$ by doing the surgery corresponding to $\sigma_2$. Note that the embedded surgery along $\sigma_1$ creates the space needed to embed the 2-handle $\sigma_2$. Next, obtain $M_3$ by doing the surgery corresponding to $\tau_2$ and finally obtain $M_4 = \Delta_2$ by doing the surgery corresponding to $\tau_1$. Again the $\tau_2$ surgery creates the space needed to embed the handle $\tau_1$. Note that $\Delta_0 \cap (C(\sigma_1) \cup C(\tau_2) \cup C(\tau_1) \cup C(\tau_2))$ is the split union of two Hopf links and $\Delta_2$ is obtained by 0-surgery along this link. It follows that $\Delta_2$ is a 3-ball.

By comparing Figure 5 with Figure 2 we see how to inductively construct all the 3-balls $\Delta_i, i \in \mathbb{N}$. The 3-ball $\Delta_\alpha$ corresponding to the general element $\alpha \in \pi_1(\text{Emb}_f(S^1 \times S^3))$ is obtained as follows. First remove a finite number of pairwise disjoint vertical $S^1 \times B^3$s from the interior of $S^1 \times B^3$ and appropriately replacing them with conjugate copies of $\Delta_i$‘s, an operation that we call concatenation. This completes the construction of the isotopy classes of knotted 3-balls in $S^1 \times B^3$ arising from the various elements of $\pi_1(\text{Emb}_f(S^1 \times S^3))$. Using the embedding $\psi$ of $S^1 \times B^3$ into $S^4$ we obtain the corresponding knotted $B^3$s in $S^4$.

The proof that $\Delta_1$ arises from $\alpha_1$ is contained in Figure 6 with the general case following similarly. Let $\Sigma_0 = \{x_0\} \times S^3$ and $\gamma_t, t \in I$ a fibration of the torus $T_1$ representing $\alpha_1$ with $\gamma_0 = \gamma_1 = S^1 \times \{y_0\}$. Recall that $T_1$ is obtained from the vertical torus $T^*$ by removing a small disc and replacing it by a knotted one. The $\gamma_t$ fibers are vertical except for when they run over the knotted 2-disc. We need to keep track of $\Sigma_0$ as it gets isotoped under an isotopy extension of $\alpha_1$. We let $\Sigma_t$ denote the 3-sphere at time $t$, though abusing notation, our $\Sigma_1$ will be obtained by doing an additional isotopy to the $\Sigma_t$ realized at the end of the isotopy extension. As before, view $S^3 = S^2 \times [-\infty, \infty]/\sim$ and let $Y_s$ denote $S^1 \times S^2 \times \{s\}$. Let $S_0 = \Sigma_0 \cap Y_0$ and let $S_t$ denote the pushed forward $S_0$ under the isotopy extension. Do not confuse $S_t$ with $\Sigma_t \cap Y_0$. We leave it to the reader to keep track of the rest of $\Sigma_t$. The various subfigures of Figure 6 show the projection of $S_1$ to $Y_0$ for various times between and including 0 and 1. They also show the corresponding $\gamma_1$. Figure 6a) shows the initial position of $S_0$ and $\gamma_0$. In Figure 6b) $\gamma_1$ has just poked into $\Sigma_0$, so $S_1$ is a slightly isotoped $S_0$. The open blue and red arcs in $\gamma_t$ indicate points that are respectively in the past and the future. The intersection of the closures of the red and blue arcs consists of two points $p, q$. Recall that $T_1$ is built by tubing $T^*$ and an unknotted 2-sphere $K$. Figure 6c) shows a $\gamma_t$ that has progressed up the back of the tube and through most of the back of $K$. From the perspective of the viewer, the union of $p, q$ and the red and blue arcs lies in front of $S_t$. Figure 6d) shows a $\gamma_t$ that has already swept across $K$ and is heading down the front of the tube. In both Figures 6c) and 6d) $S_t \subset Y_0$. In Figure 6d) two parallel sub arcs of $\gamma_t$ link two parallel subdiscs of $S_t$ following the crossing convention of Figure 1. Figure 6e) shows $\gamma_1 = \gamma_0$. Here $S_t$ no longer fully lies in $Y_0$. The crossing informs what parts of the local tubes linking the parallel subdiscs of $S_t$ lie in the past and future. Finally perform the isotopy of Figure 6c) to 6d) to obtain $S_1$. preprint
6 More Applications

In the next proposition we list some consequences of Proposition 3.2 and Lemma 3.4. We list the consequences in dimension four, although as we see in the proof, all these statements have high-dimensional analogues.

**Theorem 6.1**

1. \( \pi_0 \text{Emb}(B^2, S^2 \times B^2) \simeq \mathbb{Z} \), and this is an isomorphism under the concatenation operation. The group \( \pi_1 \text{Emb}(B^2, S^2 \times B^2) \) is free abelian group of rank two. More generally, 
   \[
   \pi_k \text{Emb}(B^2, S^2 \times B^2) \simeq \pi_{k+1} \text{Emb}(B^1, B^4) \times \pi_k \Omega^2 S^2.
   \]

2. \( \pi_0 \text{Emb}(B^3, S^1 \times B^3) \) is an abelian group with the concatenation operation. Moreover it contains an infinitely-generated subgroup isomorphic to \( \Lambda^1_3 \).

3. \( \text{Emb}_u(B^2, B^4) \) is connected, with \( \pi_1 \text{Emb}_u(B^2, B^4) \) containing a subgroup isomorphic to \( \Lambda^1_n \).
(4) \( \Emb_n(S^2, S^4) \) is connected, with \( \pi_1 \Emb_n(S^2, S^4) \) containing a subgroup isomorphic to \( \Lambda^1_n \).

Proof (1) As we have seen, when \( n \geq 4 \),
\[
\Emb(B^1, B^n) \times \Omega S^{n-2} \simeq B\Emb(B^2, S^{n-2} \times B^2).
\]
The first non-trivial homotopy group of \( \Emb(B^1, B^n) \) is known to be \( \pi_{2n-6} \Emb(B^1, B^n) \simeq \mathbb{Z} \), generated by the Haefliger trefoil \[5\]. In \[5\] the space \( \Emb(B^1, B^n) \) is denoted \( K_{n,1} \). The first non-trivial homotopy group of \( \Omega S^{n-2} \) is \( \pi_{n-3} \Omega S^{n-2} \equiv \pi_{n-2} S^{n-2} \simeq \mathbb{Z} \).

(2) See Corollary 3.6

(3) As we observed earlier, when \( n \geq 4 \) \( \Emb_n(B^{n-2}, B^n) \simeq B\Emb(B^{n-1}, S^1 \times B^{n-1}) \). Given that \( \pi_{n-4} \Emb(B^{n-1}, S^1 \times B^{n-1}) \) contains the subgroup \( \Lambda^1_n \) for all \( n \geq 4 \), we conclude
\[
\pi_{n-3} \Emb_n(B^{n-2}, B^n) \simeq \pi_{n-4} \Emb(B^{n-1}, S^1 \times B^{n-1})
\]
contains the same infinitely-generated abelian group \( \Lambda^1_n \).

(4) There is a homotopy-equivalence \[5\]
\[
\Emb(S^j, S^n) \simeq SO_{n+1} \times SO_{n-j} \Emb(B^j, B^n).
\]
The simplest way to think of this is to consider elements of \( \Emb(B^j, B^n) \) as smooth embeddings \( \mathbb{R}^j \to \mathbb{R}^n \) that restricts to the standard inclusion \( x \to (x,0) \) outside of the ball \( B_j \). One can conjugate such embeddings via a stereographic projection map, converting the embeddings \( \mathbb{R}^j \to \mathbb{R}^n \) to embeddings \( S^j \to S^n \) that are standard on a hemi-sphere. One can then post-compose such an embedding with an isometry of \( S^n \). We are in the fortunate circumstance where the homotopy-fibre of the map \( SO_{n+1} \times \Emb(B^j, B^n) \to \Emb(S^j, S^n) \) can be identified, and it is the orbits of the \( SO_{n-j} \)-action, acting diagonally on the product.

Thus \( \Emb_n(S^2, S^4) \) is a bundle over \( SO_5/ SO_2 \) with fibre \( \Emb_n(B^2, B^4) \). \( \pi_2 SO_5/ SO_2 \simeq \mathbb{Z} \) and this group maps isomorphically to the subgroup of index two in \( \pi_1 SO_2 \), which maps to zero in \( \pi_1 \Emb_n(B^2, B^4) \), so our map \( \pi_1 \Emb_n(B^2, B^4) \to \pi_1 \Emb_n(S^2, S^4) \) is injective.

Remark 6.2 The first sentence of Conclusion (1) is Theorem 10.4 of \[10\]. The proof here is different, generalizable and arguably more direct.

Allen Hatcher’s proof of the Smale Conjecture \[15\] together with his and Ivanov’s work on spaces of incompressible surfaces \[16\] has as the consequence that the component of the unknot in the embedding space \( \Emb(S^1, S^3) \) has the homotopy-type of the subspace of great circles, i.e. the unit tangent bundle \( UT S^3 \simeq S^3 \times S^2 \). From the perspective of the homotopy-equivalence \( \Emb(S^1, S^3) \simeq SO_4 \times SO_2 \Emb(B^1, B^3) \) this is equivalent to saying the unknot component of \( \Emb(B^1, B^3) \) is contractible, \( \Emb_n(B^1, B^3) \simeq \{\ast\} \).

In dimension four we do not know the full homotopy-type of \( \Diff(S^4) \), although there is the recent progress of Watanabe \[27\] where he shows the rational homotopy groups of \( \Diff(S^4) \) do not agree with those of \( O_5 \). In this regard, this paper asserts the the analogy to Hatcher and Ivanov’s spaces of incompressible surfaces results \[16\] are also false in dimension 4, in particular contrast with the theorem \( \Emb(B^2, S^1 \times B^2) \simeq \{\ast\} \).
Hatcher and Wagoner \[14\] (see Cor. 5.5) have computed the mapping class group of \(S^1 \times B^n\) for a range of \(n\). Specifically
\[
\pi_0 \text{Diff}(S^1 \times B^n \text{ fix } \partial) \simeq \Gamma^{n+1} \oplus \Gamma^{n+2} \oplus \bigoplus_\infty \mathbb{Z}_2
\]
provided \(n \geq 6\). The Hatcher-Wagoner diffeomorphisms survive the map \(\pi_0 \text{Diff}(S^1 \times B^n \text{ fix } \partial) \to \pi_0 \text{Diff}(S^1 \times S^n)\), while our diffeomorphisms are in the kernel of this map.

7 Conjectures and Questions

**Conjecture 7.1** There exist knotted 3-spheres in \(S^1 \times S^3\). I.e. there exist non separating 3-spheres not isotopic to \(x_0 \times S^3\).

**Remarks 7.2**

i) Theorem 3.7 implies that a positive solution to Conjecture 7.1 is equivalent to \(\pi_0(\text{Diff}_0(S^1 \times S^3)/\text{Diff}(B^4 \text{ fix } \partial)) \neq 1\). In particular, there exist non isotopic \(S^3\)-bundles over \(S^1\). Here the subscript 0 denotes diffeomorphisms homotopic to \(\text{id}\). Note that if \(\phi \in \text{Diff}_0(S^1 \times S^3)\) and \(\phi(\Sigma_0)\) is isotopic to \(\Sigma_0\), then after a further isotopy we can assume additionally \(\phi|N(\Sigma_0) \cup N(S^1 \times y_0)) = \text{id}\) and hence the support of \(\phi\) lies in a 4-ball.

ii) Conjecture 7.1 is equivalent to the conjecture that there exists elements of \(\pi_0(\text{Diff}(S^1 \times B^3 \text{ fix } \partial)/\text{Diff}(B^4 \text{ fix } \partial))\) that do not arise from the image of \(\pi_1(\text{Emb}_f(S^1, S^1 \times S^3))\) as in Corollary 3.6.

iii) Conjecture 7.1 is equivalent to the conjecture that there exists an element of \(\text{Diff}(S^1 \times S^3)\) that acts as the identity on homology but is not isotopic to a map that fixes \(S^1 \vee S^3\) pointwise.

iv) The Schoenflies conjecture is true if and only if every non separating 3-sphere in \(S^1 \times S^3\) becomes isotopically standard after lifting to a sufficiently high finite degree cyclic cover. This was known to Barry Mazur \[20\] and rediscovered in conversations between the second author and Toby Colding.

We thank Maggie Miller for bringing to our attention the following question.

**Question 7.3** If \(\Delta_1\) is a knotted 3-ball in \(S^4\) and \(T_0, T_1\) are respectively obtained from \(\Delta_0, \Delta_1\) by attaching a small 3-dimensional 1-handle \(h\), then is \(T_1\) knotted, i.e. is not isotopic rel \(\partial T_1\) to \(T_0\)? Does \(\Delta_1\) become unknotted after finitely many such stabilizations?

**Conjecture 7.4** For each \(g \geq 0\) there exists 3-dimensional genus-\(g\) handlebodies \(V_0, V_1 \subset S^4\) such that \(\partial V_0 = \partial V_1\) and the set of \(V_0\)-compressible simple closed curves in \(\partial V_0\) coincides with that of \(V_1\), but \(V_1\) is not isotopic to \(V_0\) via an isotopy that fixes \(\partial V_1\).

**Question 7.5**

i) Determine \(\pi_0(\text{Diff}(S^1 \times B^n \text{ fix } \partial)/\text{Diff}(B^{n+1} \text{ fix } \partial))\) for \(n = 4, 5\). In particular, do all elements become trivial after lifting to the 2-fold cover of \(S^1 \times B^n\).
Remark 7.6 As already noted Hatcher has determined such groups for \( n \geq 6 \) and all its elements become trivial passing to 2-fold covers. On the other hand, this paper shows that for \( n \in \mathbb{N} \) there exist elements that remain non trivial when lifting to the \( n \)-fold cover.

The following is a restatement of a special case of Lemma 3.4, where as explained there, the isomorphism is given by slicing the embedding. Here notation is as in Theorem 6.1.

Theorem 7.7 \( \pi_0 \operatorname{Emb}(B^3, S^1 \times B^3) \simeq \pi_1 \operatorname{Emb}_u(B^2, B^4) \).

Problem 7.8 Use this to prove or disprove the Schönflies conjecture, e.g. to prove the conjecture show that loops in \( \operatorname{Emb}_u(B^2, B^4) \) are homotopically trivial where at discrete times of the homotopy one can do geometric moves corresponding to passing to finite sheeted coverings. Such a move might change the homotopy class and may arise from certain reimbeddings of the loop.

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