R-COACTIONS ON C*-ALGEBRAS

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Abstract. We give the beginnings of the development of a theory of what we call “R-coactions” of a locally compact group on a C*-algebra. These are the coactions taking values in the maximal tensor product, as originally proposed by Raeburn. We show that the theory has some gaps as compared to the more familiar theory of standard coactions. However, we indicate how we needed to develop some of the basic properties of R-coactions as a tool in our program involving the use of coaction functors in the study of the Baum-Connes conjecture.

1. Introduction

Coactions of locally compact groups on C*-algebras have been around for almost 50 years, and the conventions have evolved in various ways. Initially, the theory was developed spatially, using reduced group C*-algebras and reduced crossed products. In the early 1990’s, Raeburn [Rae92] revolutionized the theory, using universal properties and full group C*-algebras and crossed products. There was one aspect of Raeburn’s new conventions, however, that did not seem to catch on: using maximal rather than minimal tensor products. More precisely, it has become customary to refer to Raeburn’s innovation as full coactions, which are homomorphisms \( \delta: A \to M(A \otimes C^*_v(G)) \) satisfying certain conditions, and to the older style as reduced coactions, which are homomorphisms \( \delta: A \to M(A \otimes C^*_r(G)) \). Raeburn himself, on the other hand, proposed that coactions should map into \( M(A \otimes_{\text{max}} C^*_v(G)) \).

In the course of our program to use coactions as an aid in the study of the Baum-Connes conjecture (see, e.g., [KLQ16, KLQ18, KLQ20]), we found at one point that we needed to find a coaction functor appropriate for the study of a certain crossed-product functor introduced by Baum, Guentner, and Willett [BGW16], the construction of which uses maximal tensor products. In [KLQ20], we handled the case where the group \( G \) is discrete, mainly through the use of Fell bundles. In our
efforts to generalize this to arbitrary locally compact $G$, it occurred to us that a homomorphism $\psi: B \rtimes_{\alpha} G \to (B \otimes_{\text{max}} C) \rtimes_{\alpha \otimes \text{max}} G$ occurring in [BGW16] is somehow similar to a coaction $\delta: A \to M(A \otimes C^*(G))$, but with the minimal tensor product $\otimes$ replaced by the maximal one $\otimes_{\text{max}}$. This insight lead to our approach in [KLQ21] where we need to briefly switch to Raeburn-style coactions. We call these “R-coactions” — see Section 3 for the definition and the comparison to the usual “standard” coactions. We chose the “R” to stand for “Raeburn”. We quote from [Qui91, page 36]: “it may prove convenient to return to the maximal $C^*$-tensor product at some other time” — apparently that time is now.

Eventually we noticed that our paper [KLQ21] was becoming overly long, so we decided to separate out the basic development of R-coactions, leading to the current paper. We develop a few new aspects of the theory of R-coactions, particularly the notation of cocycles.

We should emphasize that we were motivated to write this paper in order to have the basic facts concerning R-coactions available. This has in some ways guided our development. However, we feel that R-coactions will be useful elsewhere.

We begin in Section 2 by recalling preliminary facts and definitions concerning coactions of a locally compact group $G$ on a $C^*$-algebra $A$, particularly the basics of crossed-product duality. To prepare for our development of cocycles for R-coactions, among other topics, we review in some detail the corresponding facts concerning standard coactions. We particularly need quite a bit of preparation involving Fischer’s procedure for maximalization of a coaction, and we review this in detail in a separate Section 3.

In Section 4 we develop the basic theory of R-coactions. We were surprised by the number of familiar features of standard coactions that do not seem to carry over well — or at all, in some cases — to R-coactions. For instance, although every R-coaction descends to a standard one, we do not know whether every standard coaction arises in this way, or whether the R-coaction is unique when it exists. Also, there appears to be no reasonable way to normalize an R-coaction. Finally, although it is possible (see [Rae92]) to form the crossed product of an R-coaction, it seems that it must be done on Hilbert space, not more generally in the multiplier algebra of a $C^*$-algebra. In view of these issues, we needed to be quite careful in our use of R-coactions; indeed, in [KLQ21] we only use R-coactions in a limited way, passing to them briefly but then passing back to standard coactions as soon as possible. We close Section 4 with appropriate notions of equivariant homomorphisms and quotients for R-coactions.
In Section 5 we show how to pass naturally from maximal coactions
to R-coactions. This requires going through the entire maximalization
process, additionally we must develop a theory of one-cocycles for R-
coactions. The culmination is Theorem 5.16 which takes a maximal
coaction \((A, \delta)\) and produces an R-coaction \(\delta^R\) on the same
\(C^*\)-algebra \(A\). There is a subtlety: when the maximal coaction is in fact the dual
coaction \(\tilde{\alpha}\) on a full crossed product \(B \rtimes_\alpha G\), there is a much easier way
to define an R-coaction, and in Theorem 5.18 we show that fortunately
the two methods give the same result in this case.

We close in Section 6 with a brief explanation of why we do not try to
include a theory of crossed-product duality for R-coactions, although
Raeburn did prove a duality theorem in \(\text{[Rae92]}\).

2. Preliminaries

Throughout, \(G\) denotes a locally compact group, and \(A, B, \ldots\) de-
ote \(C^*\)-algebras. We write \(\mathcal{K}\) for the algebra \(\mathcal{K}(L^2(G))\) of compact
operators on \(L^2(G)\). We refer to \(\text{[EKQR06, EKQ04, KOQ16c]}\) for our
conventions regarding actions, coactions, \(C^*\)-correspondences, and co-
cycles for coactions. We recall some notation for the convenience of the
reader.

The left and right regular representations of \(G\) are \(\lambda\) and \(\rho\), respec-
tively, the representation of \(C_0(G)\) on \(L^2(G)\) by multiplication opera-
tors is \(M\), and the unitary element \(w_G \in M(C_0(G) \otimes C^*(G))\) is the
strictly continuous map \(w_G: G \rightarrow M(C^*(G))\) given by the canonical
embedding of \(G\). (Note that we must be careful to understand from
the context whether \(M\) is being used for the representation of \(C_0(G)\) or
for a multiplier algebra.) The actions of \(G\) on \(C_0(G)\) by left and right
translations are denoted by \(lt\) and \(rt\), respectively.

Throughout this paper, we work with numerous categories of what
could be called “decorated” \(C^*\)-algebras, \(i.e.,\) the objects are \(C^*\)-algebras
together with some extra structure, and the morphisms are homomor-
phisms preserving the extra structure. To make it interesting, we have
to deal with two sorts of morphisms:

- “classical” morphisms, which are homomorphisms \(\phi: A \rightarrow B\)
between \(C^*\)-algebras, and
- nondegenerate morphisms, which are nondegenerate homomor-
phisms \(\pi: A \rightarrow M(B)\).

We actually need the classical morphisms for our main results, and
the nondegenerate homomorphisms appear in secondary (albeit cru-
cial) roles. We will formalize the various categories using classical
morphisms, and occasionally refer to nondegenerate categories more
informally. In an attempt to reduce confusion, we will not actually use the term “morphism” when the $C^*$-algebras are not carrying any extra structure; in other words, we will write either “$\phi: A \to B$ is a homomorphism”, or “$\phi: A \to M(B)$ is a nondegenerate homomorphism” when $\phi$ is not required to have any other property.

To help prepare for our “$R$-coactions” we review here in detail some fundamental aspects of “ordinary” coactions. First, a bit of parallel theory of actions: A morphism $\phi: (A, \alpha) \to (B, \beta)$ of actions is an $\alpha - \beta$ equivariant homomorphism $\phi: A \to B$, giving the category of actions. A covariant representation of an action $(A, \alpha)$ in a multiplier algebra $M(B)$ is a pair $(\pi, U)$, where $\pi: A \to M(B)$ is a nondegenerate homomorphism, $U: G \to M(B)$ is a strictly continuous unitary representation, and $\pi \circ \alpha_s = \text{Ad} U_s \circ \pi$ for all $s \in G$. A crossed product of an action $(A, \alpha)$ of $G$ is a universal covariant representation $(i_A, i_G): (A, G) \to M(A \rtimes_{\alpha} G)$, meaning that for every covariant representation $(\pi, U): (A, G) \to M(B)$ there is a unique nondegenerate homomorphism $\pi \times U: A \rtimes_{\alpha} G \to M(B)$, called the integrated form of $(\pi, U)$, such that

$$(\pi \times U) \circ i_A = \pi \quad \text{and} \quad (\pi \times U) \circ i_G = U.$$ 

Any two crossed products of $(A, \alpha)$ are uniquely isomorphic, and in practice a choice is made, and the $C^*$-algebra $A \rtimes_{\alpha} G$ is referred to as the crossed product, but the pair $(i_A, i_G)$ must be kept in mind. We use superscripts $i_A^\alpha, i_G^\beta$ if confusion seems possible. The pair $(M, \rho)$ is a universal covariant representation of $(C_0(G), \text{rt})$, and

$$(i_A, i_G)^\alpha: C_0(G) \rtimes_{\text{rt}} G = K.$$ 

If $\phi: (A, \alpha) \to (B, \beta)$ is a morphism of actions, then there is a unique homomorphism $\phi \times G: A \rtimes_{\alpha} G \to B \rtimes_{\beta} G$ such that

$$(\phi \times G)(i_A(a)i_G^\alpha(c)) = i_B \circ \phi(a)i_G^\beta(c) \quad \text{for all} \quad a \in A, c \in C^*(G).$$

In this way, the crossed-product construction is functorial from actions to $C^*$-algebras. Note that we could not just write, e.g., $(\phi \times G) \circ i_A = i_B \circ \phi$, because $\phi$, and consequently $\phi \times G$, could be degenerate.

There is a parallel “nondegenerate category of actions”, and crossed products are functorial between nondegenerate categories as well. However, as we mentioned already, the classical categories are of primary interest in this paper.

Turning to coactions, first let $\delta_G: C^*(G) \to M(C^*(G) \otimes C^*(G))$ be the integrated form of the unitary representation given by $\delta_G(s) = s \otimes s$ for $s \in G$. Recall the following notational device: for any $C^*$-algebras
A, D we define the tilde multiplier algebra as
\[ \tilde{M}(A \otimes D) = \{ m \in M(A \otimes D) : m(1 \otimes D) \cup (1 \otimes D)m \subseteq A \otimes D \}. \]

Note that \( \tilde{M}(A \otimes D) \) is not symmetric in \( A \) and \( D \). A coaction of \( G \) on \( A \) is a nondegenerate faithful homomorphism \( \delta: A \to \tilde{M}(A \otimes C^*(G)) \) such that \((\delta \otimes \text{id}) \circ \delta = (\text{id} \otimes \delta_G) \circ \delta \) and \( \text{span}\{\delta(A)(1 \otimes C^*(G))\} = A \otimes C^*(G) \). A morphism \( \phi: (A, \delta) \to (B, \varepsilon) \) of coactions is a homomorphism \( \phi: A \to B \) that is \( \delta - \varepsilon \) equivariant in the sense that the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\delta} & \tilde{M}(A \otimes C^*(G)) \\
\downarrow & & \downarrow \phi \otimes \text{id} \\
B & \xrightarrow{\varepsilon} & \tilde{M}(B \otimes C^*(G))
\end{array}
\]
commutes, where the right-hand vertical arrow \( \phi \otimes \text{id} \) is the unique extension, whose existence is guaranteed by [EKQR06, Proposition A.6], of \( \phi \) to \( \tilde{M}(A \otimes C^*(G)) \). In this way we get the category of coactions. A covariant representation of a coaction \( (A, \delta) \) in \( M(B) \) is a pair \((\pi, \mu)\) of nondegenerate homomorphisms
\[
\begin{array}{ccc}
A & \xrightarrow{\pi} & M(B) & \xleftarrow{\mu} & C_0(G)
\end{array}
\]
such that \( \text{Ad} \mu \otimes \text{id}(w_G) \circ (\pi \otimes 1) = (\pi \otimes \text{id}) \circ \delta \). The regular representation of a coaction \( (A, \delta) \) in \( M(A \otimes K) \) is given by
\[
((\text{id} \otimes \lambda) \circ \delta, 1 \otimes M).
\]

A crossed product of a coaction \( (A, \delta) \) of \( G \) is a universal covariant representation \( (j_A, j_G): (A, C_0(G)) \to M(A \rtimes_G G) \), meaning that for every covariant representation \((\pi, \mu): (A, C_0(G)) \to M(B)\) there is a unique nondegenerate homomorphism, \( \pi \times \mu: A \rtimes_G G \to M(B) \), called the integrated form of \((\pi, \mu)\), such that
\[
(\pi \times \mu) \circ j_A = \pi \quad \text{and} \quad (\pi \times \mu) \circ j_G = \mu.
\]
Any two crossed products of \((A, \delta)\) are uniquely isomorphic, and in practice a choice is made, and the \( C^* \)-algebra \( A \rtimes_G G \) is referred to as the crossed product, but the pair \((j_A, j_G)\) must be kept in mind. The regular representation of any coaction \((A, \delta)\) is universal, and so whenever it is convenient we are free to identify \( A \rtimes_G G \) with a nondegenerate subalgebra of \( M(A \otimes K) \). We use superscripts \( j_A^\delta, j_G^\delta \) if confusion seems possible.
If \( \phi: (A, \delta) \to (B, \varepsilon) \) is a morphism of coactions, then there is a unique morphism \( \phi \times G: A \rtimes_\delta G \to B \rtimes_\varepsilon G \) such that

\[
(\phi \times G)(j_A(a)j_G^\delta(f)) = j_B \circ \phi(a)j_G^\varepsilon(f) \quad \text{for all } a \in A, f \in C_0(G).
\]

As with actions, the crossed-product construction is functorial from coactions to \( C^* \)-algebras.

We denote the trivial coaction \( a \mapsto a \otimes 1 \) on \( A \) by \( \delta_{\text{triv}} \), and more generally if \( (A, \delta) \) is a coaction and \( B \subseteq M(A) \), then \( \delta \) is trivial on \( B \) if \( \delta(b) = b \otimes 1 \) for all \( b \in B \).

For an action \( (A, \alpha) \), the dual coaction \( \hat{\alpha} \) of \( G \) on \( A \rtimes_\alpha G \) is the integrated form of the covariant representation \((i_A \otimes 1, (i_G \otimes \text{id}) \circ \delta_G)\), and is the unique coaction that is trivial on \( i_A(A) \) and such that the nondegenerate homomorphism \( i_G: C^*(G) \to M(A \rtimes_\alpha G) \) is \( \delta_G - \hat{\alpha} \) equivariant.

The crossed product \( \phi \times G: A \rtimes_\alpha G \to B \rtimes_\varepsilon G \) of a morphism \( \phi: (A, \alpha) \to (B, \beta) \) of actions is \( \hat{\alpha} - \hat{\beta} \) equivariant, and hence gives a morphism \( \phi \times G: (A \rtimes_\alpha G, \hat{\alpha}) \to (B \rtimes_\beta G, \hat{\beta}) \) of coactions, and in this way the crossed product is functorial from actions to coactions.

For a coaction \( (A, \delta) \), the dual action \( \hat{\delta} \) of \( G \) on \( A \rtimes_\delta G \) is defined so that for each \( s \in G \) the automorphism \( \hat{\delta}_s \) is the integrated form of the covariant representation \((j_A, j_G \circ \text{rt}_s)\), and is the unique action that is trivial on \( j_A(A) \) and such that the nondegenerate homomorphism \( j_G: C_0(G) \to M(A \rtimes_\delta G) \) is \( \text{rt} - \hat{\delta} \) equivariant. The crossed product \( \phi \times G: A \rtimes_\delta G \to B \rtimes_\varepsilon G \) of a morphism \( \phi: (A, \delta) \to (B, \varepsilon) \) of actions is \( \hat{\delta} - \hat{\varepsilon} \) equivariant, and hence gives a morphism \( \phi \times G: (A \rtimes_\delta G, \hat{\delta}) \to (B \rtimes_\varepsilon G, \hat{\varepsilon}) \) of actions, and in this way the crossed product is functorial from coactions to actions. For the trivial coaction \( \delta_{\text{triv}} \) on \( \mathbb{C} \), we have

\[
(C \rtimes_{\delta_{\text{triv}}} G, \hat{\delta}_{\text{triv}}, j_G) = (C_0(G), \text{rt}, \text{id}).
\]

Note that crossed products for both actions and coactions are also functorial between the nondegenerate categories: e.g., if \( \phi: A \to M(B) \) is a nondegenerate \( \alpha - \beta \) equivariant homomorphism then the above construction gives a nondegenerate \( \hat{\alpha} - \hat{\beta} \) equivariant homomorphism \( \phi \times G: A \rtimes_\alpha G \to M(B \rtimes_\beta G) \).

The canonical surjection

\[
\Phi = \Phi_A: A \rtimes_\delta G \rtimes_\hat{\delta} G \to A \otimes K
\]

is the integrated form of the covariant representation

\[
((\text{id} \otimes \lambda) \circ \delta \times (1 \otimes M), 1 \otimes \rho).
\]
If $\Phi$ is injective, the coaction $\delta$ is called maximal. The trivial coaction $(\mathbb{C}, \delta_{\text{triv}})$ is maximal, with
\[
\mathbb{C} \rtimes_{\delta_{\text{triv}}} G \rtimes_{\delta_{\text{triv}}} G = K.
\]

A maximalization of a coaction $(A, \delta)$ consists of a maximal coaction $(B, \varepsilon)$ and a surjective $\varepsilon - \delta$ equivariant homomorphism $\psi: B \to A$ such that the crossed product $\psi \rtimes G: B \rtimes_{\varepsilon} G \to A \rtimes_{\delta} G$ is an isomorphism. Sometimes the coaction $(B, \varepsilon)$ itself is referred to as a maximalization.

3. Fischer maximalization process

Our development of R-coactions will depend heavily upon what we call the Fischer maximalization process, which we describe at the end of this section, after quite a lot of preparation.

We frequently need to compute with possibly degenerate homomorphisms. Our main tool for handling this is “$D$-multipliers”, which we review below after introducing “$D$-decorated algebras”.

**Definition 3.1.** Fix a $C^*$-algebra $D$. A $D$-decorated algebra is a pair $(A, \pi)$, where $A$ is a $C^*$-algebra and $\pi: D \to M(A)$ is a nondegenerate homomorphism. If $(A, \pi)$ and $(B, \psi)$ are $D$-decorated algebras then a homomorphism $\phi: A \to B$ is $\pi - \psi$ compatible if
\[
\phi(\pi(d)a) = \psi(d)\phi(a) \quad \text{for all $d \in D, a \in A$},
\]
in which case we say that $\phi: (A, \pi) \to (B, \psi)$ is a morphism of $D$-decorated algebras.

Note that we could not just write $\phi \circ \pi = \psi$, since $\phi$ might be degenerate.

**Remark 3.2.** Warning: we will frequently need the case $D = C_0(G)$, i.e., $C_0(G)$-decorated algebras. This must not be confused with the more familiar “$C_0(G)$-algebras”, where $C_0(G)$ maps into the central multipliers.

**Lemma 3.3.** With the above structure, $D$-decorated algebras and their morphisms form a category.

**Proof.** The only nonobvious thing is that we can compose morphisms: let $\phi: (A, \pi) \to (B, \psi)$ and $\sigma: (B, \psi) \to (C, \eta)$ be morphisms of $D$-decorated algebras. We must show that the composition $\sigma \circ \phi: A \to C$ is a $D$-decorated algebra morphism. This is completely routine, but we give the computation to indicate the style of how these things go. For $d \in D$ and $a \in A$,
\[
(\sigma \circ \phi)(\pi(d)a) = \sigma(\phi(\pi(d)a))
\]
\[\begin{align*}
\sigma(\psi(d)\phi(a)) &= \eta(d)\sigma(\phi(a)) \quad (\text{because } \psi(d) \in C) \\
\eta(d)\sigma \circ \phi(a). & \quad \square
\end{align*}\]

**Remark 3.4.** Note that the category of \(D\)-decorated algebras is **not** a “coslice” (or “comma”) category (see, for example, [Mac98 Section II.6]), i.e., it is not the category of all objects “under” \(D\) in a category of \(C^*\)-algebras, because the morphisms \(\pi\) and \(\phi\) are not of the same type: we require the \(\pi\) to be nondegenerate, but allow it to map into the multiplier algebra \(M(A)\), whereas we require a morphism \(\phi\) to map into \(B\), and allow it to be degenerate.

We now apply some of the results of [DKQ12, Appendix A] to \(D\)-decorated algebras.

**Definition 3.5.** Let \((A, \pi)\) be a \(D\)-decorated algebra. A \(D\)-multiplier of \(A\) is an element \(m \in M(A)\) such that

\[\pi(D)m \cup m\pi(D) \subseteq A,\]

and \(M_D(A)\) denotes the set of all \(D\)-multipliers of \(A\). The **\(D\)-strict topology** on \(M_D(A)\) is generated by the seminorms

\[m \mapsto \|\pi(d)m\| \quad \text{and} \quad m \mapsto \|m\pi(d)\| \quad \text{for } d \in D.\]

**Example 3.6.** For any \(C^*\)-algebras \(A, D\),

\[\tilde{M}(A \otimes D) = M_D(A \otimes D),\]

where the nondegenerate homomorphism is \(d \mapsto 1 \otimes d\).

The following lemma is [DKQ12 Lemma A.4], which in turn is based upon [EKQR06 Proposition A.5].

**Lemma 3.7 ([DKQ12]).** With the notation from **Definition 3.5**,\n
(i) The \(D\)-strict topology on \(M_D(A)\) is stronger than the relative strict topology from \(M(A)\).

(ii) \(M_D(A)\) is a \(C^*\)-subalgebra of \(M(A)\), and multiplication and involution are separately \(D\)-strictly continuous.

(iii) \(M_D(A)\) is the \(D\)-strict completion of \(A\).

(iv) \(M_D(A)\) is an \(M(D)\)-subbimodule of \(M(A)\).

The whole point of \(D\)-multipliers is to extend possibly degenerate homomorphisms. [DKQ12 Lemma A.5] is a quite general result along these lines; we only need the following special case.
Lemma 3.8 ([DKQ12]). Every morphism $\phi: (A, \pi) \to (B, \psi)$ of $D$-decorated algebras extends uniquely to a $D$-strictly continuous homomorphism $\tilde{\phi}: M_D(A) \to M_D(B)$. Moreover, for $d \in D$ and $m \in M_D(A)$,

\[ \phi(\pi(d)m) = \psi(d)\tilde{\phi}(m) \quad \text{and} \quad \phi(m\pi(d)) = \tilde{\phi}(m)\psi(d). \]

Corollary 3.9. The assignments $(A, \pi) \mapsto M_D(A)$ and $\phi \mapsto \tilde{\phi}$ give a functor from $D$-decorated algebras to $C^*$-algebras.

Proof. Given morphisms $\phi: (A, \pi) \to (B, \psi)$ and $\sigma: (B, \psi) \to (C, \rho)$ of $D$-decorated algebras, the composition $\sigma \circ \phi: M_D(A) \to M_D(C)$ is $D$-strictly continuous, and hence coincides with $\sigma \circ \phi$ by the uniqueness clause of Lemma 3.8. \qed

One natural source of $D$-decorated algebras involves Exel’s $C^*$-blends (see [Exe13]). For example, in Lemma 3.14 we have a $C^*$-blend

$(A, C^*(G), i_A, i_A^G, A \rtimes \alpha G)$.

Recall that a $C^*$-blend is a 5-tuple $\mathcal{M} = (A, D, i, \pi, X)$, where $A$, $D$, and $X$ are $C^*$-algebras and $i: A \to M(X)$ and $\pi: D \to M(X)$ are homomorphisms such that

\[ X = \overline{\text{span}}\{i(A)\pi(D)\}. \]

Taking adjoints, we also have $X = \overline{\text{span}}\{\pi(D)i(A)\}$, and it follows quickly that $i$ and $\pi$ are nondegenerate.

Exel defines morphisms between $C^*$-blends. However, for our purposes it will be convenient to slightly embellish the morphisms of [Exe13]: if $\mathcal{N} = (B, E, j, \psi, Y)$ is another $C^*$-blend, and if

$\Phi: X \to Y, \quad \phi: A \to B, \quad \text{and} \quad \gamma: D \to E$

are homomorphisms, we say that the triple $(\Phi, \phi, \gamma)$ is a morphism from $\mathcal{M}$ to $\mathcal{N}$ if

\[ \Phi(i(a)\pi(d)) = j \circ \phi(a)\psi \circ \gamma(d) \quad \text{for all} \quad a \in A, \quad d \in D. \]

Remark 3.10. Note that the definition of $C^*$-blend as above is symmetric in $A$ and $D$. Also, in fact $i$ maps $A$ into the $D$-multiplier algebra $M_D(X)$ (and similarly $\pi: D \to M_A(X)$). For a morphism, most of the time we will have $D = E$ and $\gamma = \text{id}_D$ (symmetrically, we could have $A = B$ and $\phi = \text{id}_A$), in which case we write a morphism simply as $(\Phi, \phi)$, which satisfies

\[ \Phi(i(a)\pi(d)) = j \circ \phi(a)\psi(d) \quad \text{for all} \quad a \in A, \quad d \in D. \]
The following lemma shows that $\Phi$ is a sort of module map.

**Lemma 3.11.** If $(\Phi, \phi, \gamma): (A, D, i, \pi, X) \to (C, E, j, \psi, Y)$ is a morphism of $C^*$-blends as above, then

$$\Phi(i(a)x) = j \circ \phi(a)\Phi(x) \quad \text{for all } a \in A, x \in X.$$  

Similarly, $\Phi(\pi(d)x) = \psi \circ \gamma(d)\Phi(x)$ for $b \in D, x \in X$, and also similarly for products in the opposite order.

In the case $D = E$ and $\gamma = \text{id}_D$, $\Phi: (X, \pi) \to (Y, \psi)$ is a morphism of $D$-decorated algebras.

**Proof.** By linearity and density we can take $x = i(a')\pi(b)$ with $a' \in A, d \in D$. Then

$$\Phi(i(a)x) = \Phi((i(a)i(a')\pi(d)))$$

$$= \Phi(i(aa')\pi(d))$$

$$= \pi \circ \phi(aa')\tau(d)$$

$$= \pi \circ \phi(a)\pi \circ \phi(a')\tau(d)$$

$$= \pi \circ \phi(a)\Phi(i(a')\pi(d))$$

$$= \pi \circ \phi(a)\Phi(x).$$

The other assertions follow by symmetry and taking adjoints, except that the last part is just a special case. \qed

**Lemma 3.12.** Let $\phi: (A, \pi) \to (B, \gamma)$ be a morphism of $D$-decorated algebras. Further let $(\Phi, \phi, \gamma): (A, D, i, \pi, X) \to (B, E, j, \psi, Y)$ be a morphism of $C^*$-blends. Then

$$\Phi: (X, i \circ \pi) \to (Y, \psi \circ \gamma)$$

is also a morphism of $D$-decorated algebras.

**Proof.** Let $d \in D$ and $x \in X$. We must show that

$$\Phi(i \circ \pi(d)x) = \psi \circ \psi(d)\Phi(x).$$

By the Cohen-Hewitt factorization theorem we can take $x = i(a')x'$ with $a' \in A, x' \in X$. Then

$$\Phi(i \circ \pi(d)x) = \Phi(i \circ \pi(d)i(a')x')$$

$$= \Phi(i(\pi(d)a')x')$$

$$= j \circ \phi(\pi(d)a')\Phi(x') \quad \text{(Lemma 3.11)}$$

$$= j(\phi(\pi(d)a')\Phi(x'))$$

$$= j(\psi(d)\phi(a'))\Phi(x')$$

$$= j \circ \psi(d)j \circ \phi(a')\Phi(x').$$
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\[
= j \circ \psi(d) \Phi(i(a')x')
= j \circ \psi(d) \Phi(x).
\]

\[\Box\]

Remark 3.13. Note that if \( \phi \) is nondegenerate then Lemma 3.11 says that \( \Phi: (X, i) \to (Y, j \circ \phi) \) is a morphism of \( D \)-decorated algebras.

Lemma 3.11 implies the following two well-known lemmas:

Lemma 3.14. Let \( \phi: (A, \alpha) \to (B, \beta) \) be a morphism of actions. Then

\[
(\phi \times G)(i_A(a)x) = i_B \circ \phi(a)(\phi \times G)(x) \quad \text{for all } a \in A, x \in A \rtimes_\alpha G.
\]

Also, \( \phi \times G: (A \rtimes_\alpha G, i_\alpha^G) \to (B \rtimes_\beta G, i_\beta^G) \) is a morphism of \( C^*(G) \)-decorated algebras.

Lemma 3.15. Let \( \phi: (A, \delta) \to (B, \varepsilon) \) be a morphism of coactions. Then

\[
(\phi \times G)(j_A(a)x) = j_B \circ \phi(a)(\phi \times G)(x) \quad \text{for all } a \in A, x \in A \rtimes_\delta G.
\]

Also, \( \phi \times G: (A \rtimes_\delta G, j_\delta^G) \to (B \rtimes_\varepsilon G, j_\varepsilon^G) \) is a morphism of \( C_0(G) \)-decorated algebras.

We will need to know that crossed products are compatible with \( D \)-decorated algebra structure, and we record this in the following lemma, which is a routine application of Lemma 3.12.

Lemma 3.16. Let \( \phi: (A, \pi) \to (B, \psi) \) be a morphism of \( D \)-decorated algebras. If \( \phi \) is also equivariant for actions \( \alpha, \beta \), then

\[
\phi \times G: (A \rtimes_\alpha G, i_A \circ \pi) \to (B \rtimes_\varepsilon G, i_B \circ \psi)
\]

is a morphism of \( C^*(G) \)-decorated algebras.

Similarly if \( \phi \) is equivariant for coactions instead of actions.

Equivariant actions. An equivariant action is a triple \((A, \alpha, \mu)\), where \((A, \alpha)\) is an action of \( G \) and \((A, \mu)\) is a \( C_0(G) \)-decorated algebra such that \( \mu \) is rt – \( \alpha \) equivariant.

Note that the categories of actions and of \( C_0(G) \)-decorated algebras combine immediately to form a category of equivariant actions, where a morphism \( \phi: (A, \alpha, \mu) \to (B, \beta, \nu) \) is just a morphism \( \phi: (A, \alpha) \to (B, \beta) \) of actions that is also \( \mu - \nu \) compatible.

We are now ready for the first functor that we want to record with a name.

Lemma 3.17. The assignments \((A, \delta) \mapsto (A \rtimes_\delta G, \hat{\delta}, \hat{j}_G)\) and \( \phi \mapsto \phi \times G \) give a functor CPC from coactions to equivariant actions.
Proof. This follows immediately from Lemma 3.15 since we know already that the crossed product construction is functorial from coactions to actions. \hfill \Box

The name CPC is an acronym for “crossed product by coactions”, and the functor CPC is the first step in the Fischer maximalization process.

Coactions will combine with $D$-decorated algebras in several contexts, so it is efficient to introduce the following abstract concept.

**Definition 3.18.** A $D$-fixed coaction is a triple $(A, \delta, \pi)$, where $(A, \delta)$ is a coaction, $(A, \pi)$ is a $D$-decorated algebra, and

$$\delta \circ \pi = \pi \otimes 1.$$ 

Note that the categories of coactions and of $D$-decorated algebras combine immediately to form a category of $D$-fixed coactions, where a morphism $\phi: (A, \delta, \pi) \rightarrow (B, \varepsilon, \psi)$ is just a morphism $\phi: (A, \delta) \rightarrow (B, \varepsilon)$ of coactions that is also $\pi - \psi$ compatible.

**Remark 3.19.** One special case we will need is for $D = C_0(G)$. Warning: $C_0(G)$-fixed coactions must not be confused with Nilsen’s “$C_0(X)$-coactions” [Nil99, Section 3], which further require $C_0(X)$ to map into the center $ZM(A)$.

**Coaction cocycles.**

**Definition 3.20.** A cocycle for a coaction $(A, \delta)$ is a unitary $U \in M(A \otimes C^*(G))$ such that

\begin{enumerate}
  \item $(\text{id} \otimes \delta_G)(U) = (U \otimes 1)(\delta \otimes \text{id})(U)$ and
  \item $\text{span}\{\text{Ad} U \circ \delta(A)(1 \otimes C^*(G))\} = A \otimes C^*(G)$.
\end{enumerate}

$U$ is also called a $\delta$-cocycle. If $U$ is a cocycle for $(A, \delta)$, then $\text{Ad} U \circ \delta$ is another coaction, called the perturbation of $\delta$ by $U$, and also is said to be exterior equivalent to $\delta$. There is a subtlety here: in Definition 3.20 (ii) it is enough to require only that $\text{Ad} U \circ \delta$ maps $A$ into $\tilde{M}(A \otimes C^*(G))$; it is a theorem that coaction-nondegeneracy of $\text{Ad} U \circ \delta$ follows from that of $\delta$ (e.g., see [KLQ13, Proposition 2.5 and the discussion surrounding it]).

It is still unknown whether cocycle-nondegeneracy is automatic; perhaps similarly, it is unknown whether Definition 3.20 (ii) is redundant.

If $(A, \delta)$ and $(B, \varepsilon)$ are coactions, $U$ is a $\delta$-cocycle, and $\phi: A \rightarrow M(B)$ is a nondegenerate $\delta - \varepsilon$ equivariant homomorphism, then $(\phi \otimes \text{id})(U)$ is an $\varepsilon$-cocycle and $\phi$ is also $\text{Ad} U \circ \delta - \text{Ad}(\phi \otimes \text{id})(U) \circ \varepsilon$ equivariant. If $U$ is a $\delta$-cocycle and $W$ is an $\text{Ad} U \circ \delta$-cocycle, then $WU$ is a $\delta$-cocycle, and of course $\text{Ad} W \circ \text{Ad} U \circ \delta = \text{Ad} WU \circ \delta$. Also, clearly $U^*$ is an
Ad \(U \circ \delta\)-cocycle and \(\text{Ad} U^* \circ \text{Ad} U \circ \delta = \delta\). It follows from the facts recalled in this paragraph that exterior equivalence is an equivalence relation on coactions.

We need a result going back to Nakagami and Takesaki [NT79, Theorem A.1] (see also [LPRS87, Remark 3.2 (2)] and [QR95, Lemma 1.2]): for every nondegenerate homomorphism \(\mu: C_0(G) \to M(A)\) the unitary \(W = (\mu \otimes \text{id})(w_G) \in M(A \otimes C^*(G))\) satisfies

\[(\text{id} \otimes \delta_G)(W) = W_{12}W_{13},\]

where we use the “leg” notation:

\[W_{12} = W \otimes 1 \quad \text{and} \quad W_{13} = (\text{id} \otimes \Sigma)(W \otimes 1),\]

and where \(\Sigma\) is the “flip automorphism” of \(C^*(G) \otimes C^*(G)\) given on elementary tensors by \(\Sigma(x \otimes y) = y \otimes x\); moreover, every such unitary \(W\) arises in this way from a unique \(\mu\). We will say that \(\mu\) and \(W\) are associated to each other.

Here is a quite fundamental special case: \(w_G\) is the unitary associated to the identity automorphism of \(C_0(G)\), and is a cocycle for the trivial coaction \(\delta_{\text{triv}}\) on \(C_0(G)\). Moreover, by commutativity the perturbed coaction \(\text{Ad} w_G \circ \delta_{\text{triv}}\) is again \(\delta_{\text{triv}}\). We combine this with the established theory of coaction cocycles in the following proposition.

**Proposition 3.21.** Let \((A, \delta)\) be a coaction, and let \(W\) be the unitary associated to a nondegenerate homomorphism \(\mu: C_0(G) \to M(A)\). Then \(W\) is a \(\delta\)-cocycle if and only if \((A, \delta, \mu)\) is a \(C_0(G)\)-fixed coaction.

**Proof.** One direction follows immediately from results quoted in the preceding discussion: if \((A, \delta, \mu)\) is a \(C_0(G)\)-fixed coaction, then \(\mu\) is a nondegenerate \(\delta_{\text{triv}} - \delta\) equivariant homomorphism, so \(W\) is a \(\delta\)-cocycle because \(w_G\) is a \(\delta_{\text{triv}}\)-cocycle.

Conversely, assume that \(W\) is a \(\delta\)-cocycle. For \(U = (\mu \otimes \text{id})(w_G)\) the left-hand side of Definition 3.20 (i) becomes

\[(\text{id} \otimes \delta_G) \circ (\mu \otimes \text{id})(w_G) = (\mu \otimes \text{id} \otimes \text{id}) \circ (\text{id} \otimes \delta_G)(w_G) = (\mu \otimes \text{id} \otimes \text{id})(w_G)_{12}(w_G)_{13} = (\mu \otimes \text{id})(w_G)_{12}(\mu \otimes \text{id})(w_G)_{13},\]

while the right-hand side becomes

\[(\mu \otimes \text{id})(w_G)_{12}(\delta \circ \mu \otimes \text{id})(w_G),\]

so Definition 3.20 (i) implies (and in fact is equivalent, upon cancelling the first factor, to)

\[(\mu \otimes \text{id})(w_G)_{13} = (\delta \circ \mu \otimes \text{id})(w_G).\]
Since
\[(\mu \otimes \text{id})(w_G)_{13} = ((\mu \otimes 1) \otimes \text{id})(w_G),\]
we see that Definition 3.20 (i), together with the Nakagami-Takesaki characterization, implies that the two homomorphisms \(\delta \circ \mu\) and \(\mu \otimes 1\) of \(C_0(G)\) coincide, and hence \((A, \delta, \mu)\) is a \(C_0(G)\)-fixed coaction. \(\square\)

Remark 3.22. The converse direction of Proposition 3.21, namely that \(\delta \circ \mu = \mu \otimes 1\) is necessary for \(W\) to be a \(\delta\)-cocycle, seems to not be previously recorded in the literature; at least, we could not find it.

Definition 3.23. In Proposition 3.21, given a \(C_0(G)\)-fixed coaction \((A, \delta, \mu)\), we will refer to \(W\) as the associated cocycle.

Note that we have already been calling \(W\) the unitary associated to \(\mu\), so now \(W\) becomes associated to two things: a homomorphism \(\mu\), and a \(C_0(G)\)-fixed coaction \((A, \delta, \mu)\); these two usages are consistent.

As we mentioned already, if \((A, \delta)\) and \((B, \varepsilon)\) are coactions, \(U\) is a \(\delta\)-cocycle, and \(\phi: A \rightarrow M(B)\) is a nondegenerate \(\delta - \varepsilon\) equivariant homomorphism, then \((\phi \otimes \text{id})(U)\) is an \(\varepsilon\)-cocycle and \(\phi\) is also \(\text{Ad} U \circ \delta = \text{Ad}(\phi \otimes \text{id})(U) \circ \varepsilon\) equivariant. We need a variant of this fact for the particular type of cocycles of interest to us; it is contained in [KLQ16, Lemma 3.6], but in the following lemma we state it formally for convenient reference in the current paper.

Lemma 3.24 ([KLQ16]). If \(\phi: (A, \delta, \mu) \rightarrow (B, \varepsilon, \nu)\) is a morphism of \(C_0(G)\)-fixed coactions, with associated cocycles \(W\) and \(U\), respectively, then \(\phi\) is also \(\text{Ad} W \circ \delta = \text{Ad} U \circ \varepsilon\) equivariant.

We now have a second functor that we want to officially record with a name.

Lemma 3.25. Let \((A, \alpha, \mu)\) be an equivariant action, and let \(W\) be the unitary associated to the homomorphism \(i_A \circ \mu\). Then \(W\) is an \(\tilde{\alpha}\)-cocycle. Put
\[
\tilde{\alpha} = \text{Ad} W \circ \tilde{\alpha}.
\]
Then \((A \rtimes_\alpha G, \tilde{\alpha}, \mu \rtimes G)\) is a \(K\)-fixed coaction.

Moreover, if \(\phi: (A, \alpha, \mu) \rightarrow (B, \beta, \nu)\) is a morphism of equivariant actions, then the homomorphism \(\phi \rtimes G\) gives a morphism
\[
(A \rtimes_\alpha G, \tilde{\alpha}, \mu \rtimes G) \rightarrow (B \rtimes_\beta G, \tilde{\beta}, \nu \rtimes G)
\]
of \(K\)-fixed coactions.

Finally, the assignments \((A, \alpha, \mu) \mapsto (A \rtimes_\alpha G, \tilde{\alpha}, \mu \rtimes G)\) and \(\phi \mapsto \phi \rtimes G\) give a functor \(\text{CPA}\) from equivariant actions to \(K\)-fixed coactions.
The name CPA is an acronym for “crossed product by action”, and the functor CPA is the second step in the Fischer maximalization process.

Proof. Since $(A \rtimes_{\alpha} G, \widehat{\alpha}, i_A \circ \mu)$ is a $C_0(G)$-fixed coaction by definition of $\widehat{\alpha}$, it follows from Proposition 3.21 that $W$ is a $\widehat{\alpha}$-cocycle.

Recall that we identify $C_0(G) \rtimes_{\text{rt}} G = \mathcal{K}$ with $(i_0, i_0^\alpha G) = (M, \rho)$. Then the definition of dual coactions implies that $(K, \widehat{\text{rt}}, M)$ is a $C_0(G)$-fixed coaction, so the associated unitary $U$ is a $\widehat{\text{rt}}$-cocycle. Moreover, the identity $w_G(1 \otimes s) = (\text{rt}_s \otimes \text{id})(w_G)$ implies that in fact $\text{Ad} U \circ \widehat{\text{rt}}$ is the trivial coaction on $K$. The homomorphism $\mu: C_0(G) \to M(A)$ is $\text{rt} - \alpha$ equivariant, so the crossed-product homomorphism $\mu \rtimes G: C_0(G) \rtimes_{\text{rt}} G = \mathcal{K} \to M(A \rtimes_{\alpha} G)$ is $\widehat{\text{rt}} - \widehat{\alpha}$ equivariant. Since the perturbed coaction $\text{Ad} U \circ \widehat{\text{rt}}$ is trivial, it follows that $\widehat{\alpha}$ is trivial on the image $(\mu \rtimes G)(\mathcal{K})$. Thus $(A \rtimes_{\alpha} G, \widehat{\alpha}, \mu \rtimes G)$ is a $\mathcal{K}$-fixed coaction.

Given a morphism $\phi$, we know from Lemmas 3.14 and 3.16 that $\phi \rtimes G$ is both $(i_A \circ \mu) - (i_B \circ \nu)$ and $i_A^\alpha - i_G^\beta$ compatible, and it follows that it is also $(\mu \rtimes G) - (\nu \rtimes G)$ compatible, since $\mu \rtimes G$ is the integrated form of the covariant representation $(i_A \circ \mu, i_G^\alpha)$ of the action $(C_0(G), \text{rt})$, and similarly for $\nu \rtimes G$. Since $(\phi \rtimes G): (A \rtimes_{\alpha} G, \widehat{\alpha}, i_A \circ \mu) \to (B \rtimes_{\beta} G, \widehat{\beta}, i_B \circ \nu)$ is a morphism of $C_0(G)$-fixed coactions, the $(i_A \circ \mu) - (i_B \circ \nu)$ compatibility, combined with Lemma 3.21 also gives $\widehat{\alpha} - \widehat{\beta}$ equivariance, and therefore $\phi \rtimes G$ is a morphism of $\mathcal{K}$-fixed coactions.

Finally, the functoriality is clear, because we know that the crossed-product construction is functorial from actions to coactions. \(\square\)

Relative commutants.

Remark 3.26. In the following we refer to [Fis04] and [KOQ16a] for relative commutants of the compact operators $\mathcal{K} = \mathcal{K}(L^2(G))$. While the first reference seems not to require $G$ to be second countable — i.e., for the Hilbert space $L^2(G)$ to be separable — the second reference does assume this. However, the methods of [KOQ16a] can be extended to the general case using routine methods.

The relative commutant of a $\mathcal{K}$-decorated algebra $(A, \iota)$ is the $C^*$-algebra

$$C(A, \iota) = \{m \in M(A) : \mu(k) = \iota(k)m \in A \text{ for all } k \in \mathcal{K}\}.$$


Proposition 3.27 ([Fis04, Remark 3.1], [KOQ16a, Proposition 3.4]). If \((A, \iota)\) is a \(\mathcal{K}\)-decorated algebra then there is a unique isomorphism 
\(\theta_A: C(A, \iota) \otimes \mathcal{K} \xrightarrow{\sim} A\) such that 
\[\theta(m \otimes k) = mu(k) \quad \text{for } m \in C(A, \iota), k \in \mathcal{K}.
\]

As Fischer observes, \(C(A, \iota)\) can be characterized as the unique closed subset \(Z\) of \(M(A)\) that commutes elementwise with \(\iota(\mathcal{K})\) and satisfies \(\text{span}\{Z\iota(\mathcal{K})\} = A\). Similarly, \(M(C(A, \iota))\) can be characterized as the set of all elements of \(M(A)\) that commute elementwise with \(\iota(\mathcal{K})\).

Lemma 3.28. Let \(\phi: (A, \iota) \rightarrow (B, \jmath)\) be a morphism of \(\mathcal{K}\)-decorated algebras. Then there is a unique homomorphism 
\(C(\phi): C(A, \iota) \rightarrow C(B, \jmath)\)

such that 
\[\phi(\iota(k)m) = \jmath(k)C(\phi)(m) \quad \text{for all } k \in \mathcal{K}, m \in C(A, \iota).
\]

We also have 
\[\phi(mu(k)) = C(\phi)(m)\jmath(k) \quad \text{for all } m \in C(A, \iota), k \in \mathcal{K}.
\]

Finally, in this way we get a functor 
\[(A, \iota) \mapsto C(A, \iota)\]

from \(\mathcal{K}\)-decorated algebras to \(C^*\)-algebras.

Proof. By Lemma 3.8, \(\phi\) extends uniquely to a \(\mathcal{K}\)-strictly continuous homomorphism \(\overline{\phi}: M_\mathcal{K}(A) \rightarrow M_\mathcal{K}(B)\). Moreover, for \(k \in \mathcal{K}\) and \(m \in M_\mathcal{K}(A)\),
\[\phi(\iota(k)m) = \jmath(k)\overline{\phi}(m) \quad \text{and} \quad \phi(mu(k)) = \overline{\phi}(m)\jmath(k).
\]
Thus the restriction
\[C(\phi) = \overline{\phi}|_{C(A, \iota)}\]
is a homomorphism of \(C(A, \iota)\) into \(M_\mathcal{K}(B)\) satisfying (3.1)–(3.2). We want to know that \(C(\phi)\) maps into \(C(B, \jmath)\): since the range of \(\overline{\phi}\) is contained in \(M_\mathcal{K}(B)\), we only need to observe that if \(m \in C(A, \iota)\) and \(k \in \mathcal{K}\), then
\[\phi(m)\jmath(k) = \phi(mu(k)) = \phi(\iota(k)m) = \jmath(k)\phi(m),
\]
so \(\phi(m) \in C(B, \jmath)\). The uniqueness of \(\overline{\phi}\) subject to (3.1) is clear since \(\jmath\) is nondegenerate.

For the functoriality, let \(\phi: (A, \iota) \rightarrow (B, \jmath)\) and \(\psi: (B, \jmath) \rightarrow (C, \eta)\) be morphisms of \(\mathcal{K}\)-decorated algebras. Then \(\psi \circ \phi: (A, \iota) \rightarrow (C, \eta)\) is a morphism such that for all \(m \in C(A, \iota)\) and \(k \in \mathcal{K}\) we have
\[\psi \circ \phi(mu(k)) = \psi(\phi(mu(k))).\]
so \( C(\psi \circ \phi) = C(\psi) \circ C(\phi) \) by uniqueness. Since identity morphisms pose no problem, we are done. □

It is also useful to have a nondegenerate version of this functor:

**Corollary 3.29** ([Fis04]). Let \((A, \iota)\) and \((B, \jmath)\) be \(K\)-decorated algebras, and let \(\phi: A \to M(B)\) be a nondegenerate homomorphism such that

\[
\phi \circ \iota = \jmath.
\]

Then the restriction of (the canonical extension to \(M(A)\) of) \(\phi\) to \(C(A, \iota)\) is a nondegenerate homomorphism to \(M(C(B, \jmath))\). Moreover, we have the following functoriality properties:

- \(C(\text{id}_A) = \text{id}_{C(A, \iota)}\), and
- if \((C, \zeta)\) is another \(K\)-decorated algebra and \(\psi: B \to M(C)\) is a nondegenerate homomorphism such that \(\psi \circ \jmath = \zeta\), then

\[
C(\psi) \circ C(\phi) = C(\psi \circ \phi).
\]

This appears in [Fis04, discussion preceding Remark 3.2], and we omit the routine proof. Our primary use of Corollary 3.29 is the following (see [Fis04, Remark 3.2]): if \((A, \delta, \iota)\) is a \(K\)-fixed coaction, then \(\delta\) restricts to a coaction, which we also denote by \(C(\delta)\), on \(C(A, \iota)\). This uses the identity

\[
C(A \otimes C^*(G), \iota \otimes 1) = C(A, \iota) \otimes C^*(G).
\]

We now have a third functor that we want to officially record with a name.

**Lemma 3.30.** With the above notation, the assignments \((A, \delta, \iota) \mapsto (C(A, \iota), C(\delta))\) and \(\phi \mapsto C(\phi)\) give a functor \(C\) from \(K\)-fixed coactions to coactions.

The name \(C\) stands for “relative commutant”, and the functor \(C\) is the third and final step in the Fischer maximalization process.
Fischer Maximalization Process. The end result of the composition

\[(A, \delta) \xrightarrow{\text{CPC}} \{C(\mathbb{L}, \iota) \otimes \mathcal{K} \xrightarrow{\sim} A \}
\]

is the \textit{maximalization} \((A^m, \delta^m)\) of the coaction \((A, \delta)\). The canonical isomorphism \(\theta_A: C(A, \iota) \otimes \mathcal{K} \xrightarrow{\sim} A\) can be used to construct a \(\delta^m - \delta\) equivariant surjection \(\psi_A: A^m \rightarrow A\) such that

\[\psi_A \rtimes G: A^m \rtimes_{\delta^m} G \rightarrow A \rtimes_{\delta} G\]

is an isomorphism — this surjection \(\psi_A\) is an official part of the maximalization of \((A, \delta)\). Moreover, maximalization is a functor since each of the steps in its construction is functorial. If \(\phi: (A, \delta) \rightarrow (B, \varepsilon)\) is a morphism of coactions, we write \(\phi^m\) for the associated morphism between maximalizations. The surjection \(\psi\) is natural in the sense that if \(\phi: (A, \delta) \rightarrow (B, \varepsilon)\) is a morphism of coactions then the diagram

\[
\begin{CD}
A^m @>\phi^m>> B^m \\
@V\psi_A VV @V\psi_B VV \\
A @>\phi>> B
\end{CD}
\]

commutes (and in fact this determines \(\phi\) uniquely since \(\psi\) is surjective).

4. R-coactions

We begin with the maximal-tensor-product version of Definition 3.5.

\[\tilde{M}(A \otimes_{\max} D) = \{m \in M(A \otimes_{\max} D) : m(1 \otimes D) \cup (1 \otimes D)m \subseteq A \otimes_{\max} D\}.\]

Note that, using the nondegenerate homomorphism

\[d \mapsto 1 \otimes_{\max} d: D \rightarrow M(A \otimes_{\max} D),\]

we have

\[\tilde{M}(A \otimes_{\max} D) = M_D(A \otimes_{\max} D),\]
where the right-hand side denotes the $D$-multipliers, as in Definition 3.5.

We are ready to define $R$-coactions, but first we need an appropriate version of the homomorphism $\delta_G$. Let $\delta_G^\mathbb{R} : C^*(G) \to M(C^*(G) \otimes_{\text{max}} C^*(G))$ be the integrated form of the unitary representation $s \mapsto s \otimes_{\text{max}} s$ for $s \in G$.

In the following definition, and many places elsewhere, we need to combine maximal tensor products and multipliers. This can be delicate, and in particular we think it prudent to explicitly record the following convention. Given nondegenerate homomorphisms $\pi : A \to M(C)$ and $\rho : B \to M(D)$, we denote by $\pi \otimes_{\text{max}} \rho$ the associated homomorphism

$$\pi \otimes_{\text{max}} \rho : A \otimes_{\text{max}} B \to M(C \otimes_{\text{max}} D),$$

which is of course also nondegenerate. Note that, unlike for minimal tensor products, the maximal tensor product $M(C) \otimes_{\text{max}} M(D)$ need not embed faithfully in $M(C \otimes_{\text{max}} D)$, so our $\pi \otimes_{\text{max}} \rho$ does not necessarily coincide with the canonical map

$$A \otimes_{\text{max}} B \to M(C) \otimes_{\text{max}} M(D)$$
followed by an inclusion into $M(C \otimes_{\text{max}} D)$. A particular case of our convention arises when one of the maps $\pi, \rho$ is the identity — as, for example in the following definition.

**Definition 4.2.** An $R$-coaction of a locally compact group $G$ is a pair $(A, \delta)$, where $A$ is a $C^*$-algebra and $\delta$ is an injective nondegenerate homomorphism

$$\delta : A \to \widetilde{M}(A \otimes_{\text{max}} C^*(G))$$
that satisfies the coaction identity

$$(\delta \otimes_{\text{max}} \text{id}) \circ \delta = (\text{id} \otimes_{\text{max}} \delta_G^\mathbb{R}) \circ \delta$$
and is coaction-nondegenerate:

$$\text{span}\{\delta(A)(1 \otimes_{\text{max}} C^*(G))\} = A \otimes_{\text{max}} C^*(G).$$

Note in particular that $\delta_G^\mathbb{R}$ is an $R$-coaction.

This is the style of coaction proposed by Raeburn in [Rae92]. Since then it has become more common to use the minimal tensor product $\otimes$ rather than $\otimes_{\text{max}}$. In this paper, the term “coaction” by itself will mean the usual “standard” coaction; occasionally we may insert the adjective standard to avoid any confusion.

The theory of $R$-coactions is, in a limited way, parallel to that of coactions (for example, see Lemma 4.3 below) — with a few notable omissions (see the discussion following Lemma 4.3). However, we will limit ourselves to only those aspects that we need; we have no reason to recast the entire theory of coactions in terms of $R$-coactions.
Regarding Definition 4.2, note that, as for coactions, coaction-nondegeneracy implies nondegeneracy as a homomorphism into the multiplier algebra. In fact, again as for coactions:

**Lemma 4.3.** Let $\delta : A \rightarrow M(A \otimes_{\max} C^*(G))$ be a homomorphism satisfying

(i) $(\delta \otimes_{\max} \text{id}) \circ \delta = (\text{id} \otimes_{\max} \delta_R) \circ \delta$ and

(ii) $\text{span}(\delta(A)(1 \otimes_{\max} C^*(G))) = A \otimes_{\max} C^*(G)$.

Then $\delta$ is nondegenerate and injective, and also maps into $\tilde{M}(A \otimes_{\max} C^*(G))$, and hence is an $R$-coaction.

Moreover, letting

$$\Upsilon : A \otimes_{\max} C^*(G) \rightarrow A \otimes C^*(G)$$

be the canonical surjection from maximal to minimal tensor products, the homomorphism $\delta^S = \Upsilon \circ \delta$ is a coaction.

**Proof.** First of all, (ii) clearly implies that $\delta$ is nondegenerate and maps into $\tilde{M}(A \otimes_{\max} C^*(G))$. Before showing that $\delta$ is injective, we note that $\delta^S$ is a coaction: the coaction identity is a routine diagram chase, and the coaction-nondegeneracy is obvious. Thus, by [KLQ13, Lemma 2.2], for example, $\delta^S$ is injective, and hence $\delta$ is injective. \hfill $\Box$

**Definition 4.4.** In the notation of Lemma 4.3, we call $\delta^S$ the standardization of the $R$-coaction $\delta$.

In the opposite direction, we do not know whether every coaction is the standardization of some $R$-coaction; it is true for coactions associated with Fell bundles, and in particular when $G$ is discrete or $A$ is the (full) crossed product of an action of $G$, and — crucially for us — it is true for maximal coactions (see Theorem 5.16), but the general case remains elusive.

Buss and Echterhoff prove in [BE15, Theorem 5.1] that when $G$ is discrete, a coaction $\delta$ of $G$ on $A$ lifts to a homomorphism $A \rightarrow A \otimes_{\max} C^*(G)$ if and only if $\delta$ is maximal. Thus, it begins to look as though the property of lifting to an $R$-coaction is unique to maximal coactions; however, if $G$ is nonamenable but $C^*(G)$ is nuclear, then the canonical coaction of $G$ on $C^*_r(G)$ is not maximal, but does lift to a homomorphism into $M(A \otimes_{\max} C^*(G))$ (see [BE15, Remark 5.3]). We thank Alcides Buss for mentioning [BE15, Theorem 5.1 and Remark 5.3] to us.

Additionally, even if we know that a coaction $\delta$ is the standardization of some R-coaction $\varepsilon$, we have no idea whether $\varepsilon$ is unique — we cannot rule out the possibility of adding suitable elements of $\ker \Upsilon$ to
get a different $R$-coaction with standardization $\delta$. We formalize these questions:

**Question 4.5.** Let $(A, \delta)$ be a coaction.

(i) Is there an $R$-coaction $\varepsilon$ such that $\varepsilon^S = \delta$?

(ii) If the answer to (i) is yes, is $\varepsilon$ unique?

A notable example of a construction for coactions that is missing for $R$-coactions is normalization. As explained in [EKQR06, Example A.71 and the surrounding discussion], if $G$ is a nonamenable discrete group then there is no $R$-coaction on $C^*_r(G)$ such that $\lambda_s \mapsto \lambda_s \otimes_{\text{max}} s$ for $s \in G$ (alternatively, this follows from [BE15, Theorem 5.1]). This has numerous negative consequences: for example, we do not know whether every $R$-coaction has a normalization (in the naïve sense of what a normalization would mean here); in particular, if the canonical $R$-coaction $s \mapsto s \otimes_{\text{max}} s$ on $C^*(G)$ has a normalization, it will not in general be the regular representation (as it is for the coaction $s \mapsto s \otimes s$). Additionally, although it is possible to define covariant representations of an $R$-coaction $(A, \delta)$ on Hilbert space, if $(\pi, \mu)$ is a covariant representation then $\pi(a) \mapsto \text{Ad}(\mu \otimes_{\text{max}} \text{id})(w_G)(\pi(a) \otimes_{\text{max}} 1)$ is not necessarily an $R$-coaction on the image $\pi(A)$ (unlike the case of coactions).

For $R$-coactions we adopt the following conventions:

**Definition 4.6.** If $(A, \delta)$ and $(B, \varepsilon)$ are $R$-coactions, a homomorphism $\phi: A \to B$ is $\delta - \varepsilon$ equivariant if the diagram

\[
\begin{array}{ccc}
A \xrightarrow{\delta} & \tilde{M}(A \otimes_{\text{max}} C^*(G)) & \\
\phi \downarrow & & \phi \otimes_{\text{max}} \text{id} \downarrow \\
B \xrightarrow{\varepsilon} & \tilde{M}(B \otimes_{\text{max}} C^*(G)) & 
\end{array}
\]

commutes. We also say that $\phi: (A, \delta) \to (B, \varepsilon)$ is a morphism of $R$-coactions.

Note that the existence of the homomorphism $\phi \otimes_{\text{max}} \text{id}$ is guaranteed by Lemma 3.8.

The existence of a category of $R$-coactions rests upon the following lemma.

**Lemma 4.7.** For a fixed $C^*$-algebra $D$, the assignments $A \mapsto \tilde{M}(A \otimes_{\text{max}} D)$ and $\phi \mapsto \phi \otimes_{\text{max}} \text{id}$ give a functor on the category of $C^*$-algebras.

**Proof.** The assignments $A \mapsto A \otimes_{\text{max}} D$ and $\phi \mapsto \phi \otimes_{\text{max}} \text{id}$ give a functor from $C^*$-algebras to $D$-decorated algebras, and then by applying Corollary 3.9 we can compose to get the desired functor. \qed
Corollary 4.8. With morphisms as in Definition 4.6, we get a category of $R$-coactions.

Proof. With the aid of Lemma 4.7, it is easy to see that morphisms can be composed, and then associativity and identity morphisms are routinely checked. □

Lemma 4.9. A homomorphism that is equivariant for $R$-coactions is also equivariant for the associated standardizations. Consequently, the assignments $\delta \mapsto \delta^S$ give a functor from $R$-coactions to coactions.

Proof. Let $\phi: (A, \delta) \to (B, \varepsilon)$ be a morphism of $R$-coactions. Then the diagram

commutes, because the left-hand rectangle commutes by assumption and $\Upsilon$ is natural. This proves the first part, and then the functoriality follows from a routine computation. □

Remark 4.10. It is natural to wonder whether Lemma 4.9 is also true in the other direction: if $(A, \delta)$ and $(B, \varepsilon)$ are $R$-coactions and $\phi: (A, \delta^S) \to (B, \varepsilon^S)$ is a morphism of coactions, is $\phi$ also $\delta - \varepsilon$ equivariant? We do not know. This is another way in which the theory of $R$-coactions is impoverished in comparison to coactions, and consequently is why we do not want to establish any more of the theory concerning $R$-coactions than we need.

Remark 4.11. We have defined a category of $R$-coactions in which the morphisms are homomorphisms between the $C^*$-algebras themselves. It is possible to define a “nondegenerate” version, using nondegenerate homomorphisms into multiplier algebras. However, since we have in mind no immediate application of this, we eschew it for now.

Definition 4.12. The dual $R$-coaction $R(\alpha)$ on the crossed product $A \rtimes_\alpha G$ of an action $(A, \alpha)$ is defined as the integrated form of the
covariant representation
\[(i_A \otimes_{\text{max}} 1, (i_G \otimes_{\text{max}} \text{id}) \circ \delta^R_G).\]

**Lemma 4.13.** With the above notation, \( R(\alpha) \) really is an \( R \)-coaction. Moreover, the standardization \( R(\alpha)^S \) of the dual \( R \)-coaction is the usual dual coaction \( \hat{\alpha} \).

**Proof.** Both parts follow from routine calculations. \( \square \)

The following lemma shows that the dual \( R \)-coaction is natural:

**Lemma 4.14.** If \( \phi: (A, \alpha) \mapsto (B, \beta) \) is a morphism of actions then the crossed product \( \phi \rtimes G: A \rtimes_{\alpha} G \to B \rtimes_{\beta} G \) is \( R(\alpha) - R(\beta) \) equivariant, and hence gives a morphism \( \phi \rtimes G: (A \rtimes_{\alpha} G, R(\alpha)) \to (B \rtimes_{\beta} G, R(\beta)) \).

In this way the assignments \( (A, \alpha) \mapsto (A \rtimes_{\alpha} G, R(\alpha)) \) give a functor from actions to \( R \)-coactions.

**Proof.** Using functoriality of \( (A, \alpha) \mapsto A \rtimes_{\alpha} G \), the lemma follows once we have verified that \( \phi \rtimes G \) is \( R(\alpha) - R(\beta) \) equivariant, and we check this on generators: for \( a \in A \) we have
\[
R(\beta) \circ (\phi \rtimes G) \circ i_A(a) = R(\beta) \circ i_B \circ \phi(a) \\
= i_B \circ \phi(a) \otimes_{\text{max}} 1 \\
= ((\phi \rtimes G) \otimes_{\text{max}} \text{id})(i_A(a) \otimes_{\text{max}} 1) \\
= ((\phi \rtimes G) \otimes_{\text{max}} \text{id}) \circ R(\alpha) \circ i_A(a),
\]
and for \( s \in G \) we have
\[
R(\beta) \circ (\phi \rtimes G) \circ i_G^\alpha(s) = R(\beta) \circ i_G^\beta(s) \\
= i_G^\beta(s) \otimes_{\text{max}} s \\
= ((\phi \rtimes G) \otimes_{\text{max}} \text{id})(i_G^\alpha(s) \otimes_{\text{max}} s) \\
= ((\phi \rtimes G) \otimes_{\text{max}} \text{id}) \circ R(\alpha) \circ i_G(s). \quad \square
\]

It is important to note that the functor in Lemma 4.14 produces the same \( C^* \)-algebras \( A \rtimes_{\alpha} G \) and morphisms \( \phi \rtimes G \) as for coactions (more precisely, the morphisms involve the same homomorphisms).

**Definition 4.15.** Let \( (A, \delta) \) be an \( R \)-coaction, let \( I \) be an ideal of \( A \), and let \( \pi: A \to A/I \) be the quotient map. We say that \( I \) is \( \delta \)-invariant if
\[
I \subseteq \ker(\pi \otimes_{\text{max}} \text{id}) \circ \delta. \tag{4.1}
\]
Remark 4.16. Note that it would be sensible to consider another, stronger, notion of $\delta$-invariant ideal $I$ of $A$, namely that $\delta$ should restrict to give an $R$-coaction on $I$. The existence of two different types of invariant ideals will cause no confusion; in this paper, for $R$-coactions we will always mean invariance in the sense of Definition 4.15.

We do not know (and fortunately do not need to know) whether an ideal is $\delta$-invariant if and only if it is $\delta^S$-invariant.

The following is a well-known fact in the context of coactions, and we shall need it for $R$-coactions.

Lemma 4.17. Let $(A, \delta)$ be an $R$-coaction. Then an ideal $I$ of $A$ is $\delta$-invariant if and only if there is a $R$-coaction $\varepsilon$ on $A/I$ such that the quotient map $\pi$ is $\delta - \varepsilon$ equivariant. Moreover, in this case we actually have equality in (4.1).

Proof. Similarly to coactions, the condition (4.1) is exactly what is needed for there to exist a homomorphism $\varepsilon$ making the diagram

$$
\begin{array}{c}
A \xrightarrow{\delta} M(A \otimes_{\text{max}} C^*(G)) \\
\pi \downarrow \\
A/I \xrightarrow{-\varepsilon} M(A/I \otimes_{\text{max}} C^*(G))
\end{array}
$$

commute, in which case it is routine to verify that $\varepsilon$ satisfies (the versions for $R$-coactions of) the coaction identity and coaction-nondegeneracy, and hence by Lemma 4.3 is an $R$-coaction.

In particular, $\varepsilon$ is injective, so

$$
I = \ker \pi = \ker (\pi \otimes_{\text{max}} \text{id}) \circ \delta.
$$

\[ \square \]

5. \text{R-ification}

Now we will construct a functor, which we will call \text{R-ification}, from maximal coactions to $R$-coactions. We start with a maximal coaction $(A, \delta)$. Even though $\delta$ itself is maximal, we need to first apply (a slightly embellished version of) the maximalization functor to it in order to have sufficient data to make a functorial choice of $R$-coaction.

In Section 3 we reviewed (for coactions) the details of the maximalization functor as described in [Fis04] and [KOQ16b, Sections 3–4]. In Section 1 we began the process of adapting some of this to the context of $R$-coactions, and we continue this process in the current section.

Among other things, we will need to adapt to $R$-coactions a bit of the theory of cocycles. But first, we need an appropriate version of the abstract $D$-fixed coactions.
Definition 5.1. Fix a $C^*$-algebra $D$. A $D$-fixed $R$-coaction is a triple $(A, \delta, \pi)$, where $(A, \delta)$ is an $R$-coaction and $(A, \pi)$ is a $D$-decorated algebra such that
\[ \delta \circ \pi = \pi \otimes_{\max} 1. \]

Note that the categories of $R$-coactions and of $D$-decorated algebras combine immediately to form a category of $D$-fixed $R$-coactions, where a morphism $\phi: (A, \delta, \pi) \to (B, \varepsilon, \psi)$ is just a morphism $\phi: (A, \delta) \to (B, \varepsilon)$ of $R$-coactions that is also $\pi - \psi$ compatible.

Definition 5.2. A cocycle for an $R$-coaction $(A, \delta)$, also called a $\delta$-cocycle, is a unitary $U \in M(A \otimes_{\max} C^*(G))$ such that
\begin{align*}
& (i) \ (\text{id} \otimes_{\max} \delta_R^G)(U) = (U \otimes_{\max} 1)(\delta \otimes_{\max} \text{id})(U) \quad \text{and} \\
& (ii) \ \overline{\text{span}}\{(1 \otimes_{\max} C^*(G))U\delta(A)\} = A \otimes_{\max} C^*(G).
\end{align*}

Lemma 5.3. If $(A, \delta)$ is an $R$-coaction and $U$ is a $\delta$-cocycle, then $\varepsilon = \text{Ad} U \circ \delta$ is also an $R$-coaction on $A$.

Proof. The proof is a routine modification of the argument for coactions; we include it here for completeness. We will appeal to Lemma 4.3: certainly
\[ \varepsilon: A \to M(A \otimes_{\max} C^*(G)) \]
is a homomorphism. We check hypotheses (i)–(ii) in Lemma 4.3:
\begin{align*}
(\varepsilon \otimes_{\max} \text{id}) \circ \varepsilon &= (\text{Ad} U \circ \delta \otimes_{\max} \text{id}) \circ \text{Ad} U \circ \delta \\
&= \text{Ad}(U \otimes_{\max} 1) \circ (\delta \otimes_{\max} \text{id}) \circ \text{Ad} U \circ \delta \\
&= \text{Ad}(U \otimes_{\max} 1) \circ \text{Ad}(\delta \otimes_{\max} \text{id})(U) \circ (\delta \otimes_{\max} \text{id}) \circ \delta \\
&= \text{Ad}((U \otimes_{\max} 1)(\delta \otimes_{\max} \text{id})(U)) \circ (\text{id} \otimes_{\max} \delta_R^G) \circ \delta \\
&= \text{Ad}((\text{id} \otimes_{\max} \delta_R^G)(U)) \circ (\text{id} \otimes_{\max} \delta_R^G) \circ \delta \\
&= (\text{id} \otimes_{\max} \delta_R^G) \circ \varepsilon,
\end{align*}

and
\[
\text{span}\{\varepsilon(A)(1 \otimes_{\max} C^*(G))\} = \text{span}\{\text{Ad} U \circ \delta(A)(1 \otimes_{\max} C^*(G))\} = \text{Ad} U\left(\text{span}\{\delta(A)(1 \otimes_{\max} C^*(G))\}\right) = \text{Ad} U(A \otimes_{\max} C^*(G)) = A \otimes_{\max} C^*(G). \]

Definition 5.4. With the above notation, we say that $\varepsilon$ is exterior equivalent to $\delta$, and also we say that it is a perturbation of $\delta$. 

Lemma 5.5. *Exterior equivalence is an equivalence relation on R-coactions.*

**Proof.** Trivially every R-coaction is exterior equivalent to itself (using the cocycle 1 (the identity element of the appropriate multiplier algebra). Symmetry and transitivity follow from the following facts: given an R-coaction \((A, \delta)\), if \(U\) is a \(\delta\)-cocycle, then \(U^*\) is an \(\text{Ad} U \circ \delta\)-cocycle, and if \(W\) is an \(\text{Ad} U \circ \delta\)-cocycle, then \(WU\) is a \(\delta\)-cocycle. Both of these follow from routine computations, just as for coactions. □

Note that condition (ii) in Definition 5.2 guarantees that \(\varepsilon\) is coaction-nondegenerate. This is one of those places where we give only a limited development of the theory, to avoid tedious, irrelevant discussions: for coactions, the usual definition of cocycle has the following condition instead of (ii):

\[(ii)^' \quad \text{Ad} U \circ \delta(A)(1 \otimes_{\text{max}} C^*(G)) \subseteq A \otimes_{\text{max}} C^*(G),\]

and then it is remarked that the coaction \(\text{Ad} U \circ \delta\) is automatically nondegenerate because \(\delta\) is. However, we do not want to spend the effort trying to prove the corresponding fact for R-coactions, so we assume (ii) above, which implies (ii)'. This seems to us to be a reasonable compromise, since we will not need to know very many properties of cocycles for R-coactions.

Lemma 5.6. *Let \((A, \delta)\) and \((B, \varepsilon)\) be R-coactions, let \(\phi: A \to M(B)\) be a nondegenerate \(\delta - \varepsilon\) equivariant homomorphism, and let \(U\) be a \(\delta\)-cocycle. Then \((\phi \otimes_{\text{max}} \text{id})(U)\) is an \(\varepsilon\)-cocycle.*

**Proof.** The proof is a routine modification of the argument for coactions; we give it for completeness:

\[
\begin{align*}
\text{id} \otimes_{\text{max}} \delta_G^R \bigl((\phi \otimes_{\text{max}} \text{id})(U)\bigr) \\
= (\phi \otimes_{\text{max}} \text{id} \otimes_{\text{max}} \text{id}) \bigl((\text{id} \otimes_{\text{max}} \delta_G^R)(U)\bigr) \\
= (\phi \otimes_{\text{max}} \text{id} \otimes_{\text{max}} \text{id}) \bigl((U \otimes_{\text{max}} 1)(\delta \otimes_{\text{max}} \text{id})(U)\bigr) \\
= ((\phi \otimes_{\text{max}} \text{id})(U) \otimes_{\text{max}} 1)((\phi \otimes_{\text{max}} \text{id}) \circ \delta \otimes_{\text{max}} \text{id})(U) \\
= ((\phi \otimes_{\text{max}} \text{id})(U) \otimes_{\text{max}} 1)(\varepsilon \otimes_{\text{max}} \text{id})((\phi \otimes_{\text{max}} \text{id})(U)),
\end{align*}
\]

and

\[
\begin{align*}
\text{span}\{1 \otimes_{\text{max}} C^*(G))(\phi \otimes_{\text{max}} \text{id})(U)\varepsilon(B)\} \\
= \text{span}\{1 \otimes_{\text{max}} C^*(G))(\phi \otimes_{\text{max}} \text{id})(U)\varepsilon(\phi(A)B)\} \\
= \text{span}\{1 \otimes_{\text{max}} C^*(G))(\phi \otimes_{\text{max}} \text{id})(U)(\phi \otimes_{\text{max}} \text{id}) \circ \delta(A)\varepsilon(B)\} \\
= \text{span}\{\phi \otimes_{\text{max}} \text{id}\bigl((1 \otimes_{\text{max}} C^*(G))U\delta(A)\bigr)\varepsilon(B)\}
\end{align*}
\]
\[ \text{span}\left\{ (\phi \otimes \text{id})\left(\text{span}\left\{(1 \otimes_{\max} C^*(G))U\delta(A)\right\}\right)\varepsilon(B) \right\} = \text{span}\left\{ (\phi(A) \otimes_{\max} C^*(G))\varepsilon(B) \right\} = \text{span}\left\{ (\phi(A) \otimes_{\max} 1)(1 \otimes_{\max} C^*(G))\varepsilon(B) \right\} = \text{span}\left\{ (\phi(A) \otimes_{\max} 1)\text{span}\left\{(1 \otimes_{\max} C^*(G))\varepsilon(B) \right\} \right\} = \text{span}\left\{ (\phi(A) \otimes_{\max} 1)(B \otimes_{\max} C^*(G)) \right\} = B \otimes_{\max} C^*(G). \]

The following is a version of Lemma 3.21 for R-coactions:

**Corollary 5.7.** If \((A,\delta,\mu)\) is a \(C_0(G)\)-fixed R-coaction, then \(W = (i_A \circ \mu \otimes \text{id})(w_G)\) is a \(\delta\)-cocycle.

**Proof.** This is a routine adaptation of the argument for Proposition 3.21, replacing minimal tensor products by maximal ones. Note that the trivial coaction \(\delta_{\text{triv}}\) on \(C_0(G)\) may be regarded as an R-coaction, \(w_G\) is still a cocycle for \(\delta_{\text{triv}}\), and \(\mu\) is \(\delta_{\text{triv}} - \delta\) equivariant. Thus we can appeal to Lemma 5.6. \(\square\)

**Definition 5.8.** As we did for coactions, in Corollary 5.7 we call \(W\) the associated cocycle.

**Lemma 5.9.** If \(\phi: (A,\delta,\mu) \to (B,\varepsilon,\nu)\) is a morphism of \(C_0(G)\)-fixed R-coactions, with associated cocycles \(W\) and \(U\), respectively, then \(\phi\) is also \(\text{Ad} W \circ \delta - \text{Ad} U \circ \varepsilon\) equivariant.

**Proof.** For coactions, this result is Lemma 3.24 which in turn is contained in [KLQ18, Lemma 3.6]; we cannot merely adapt the proof found there, however, because it relies in part upon the following embedding property of minimal tensor products: if \(D \subseteq B\) then \(D \otimes C^*(G) \subseteq B \otimes C^*(G)\), which does not carry over to maximal tensor products. So, we have to make a completely new proof.

We must show that the diagram
\[
\begin{array}{ccc}
A & \xrightarrow{\text{Ad}(\mu \otimes_{\max} \text{id})(w_G)\circ \delta} & \tilde{M}(A \otimes_{\max} C^*(G)) \\
\phi \downarrow & & \phi \otimes_{\max} \text{id} \downarrow \\
B & \xrightarrow{\text{Ad}(\nu \otimes_{\max} \text{id})(w_G)\circ \varepsilon} & \tilde{M}(B \otimes_{\max} C^*(G))
\end{array}
\]
commutes. It suffices to show that for all \(a \in A\) we have
\[
(\phi \otimes_{\max} \text{id})\left( (\mu \otimes_{\max} \text{id})(w_G)\delta(a) \right) = (\nu \otimes_{\max} \text{id})(w_G)\varepsilon(\phi(a)),
\]
because we can take the adjoint of both sides, then multiply, and use the fact that every element of $A$ can be factored in the form $ab^*$ for some $a, b \in A$.

It further suffices to show that
\begin{equation}
(\phi \otimes_{\max} \text{id})(\mu \otimes_{\max} \text{id})(x)y = (\nu \otimes_{\max} \text{id})(x)(\phi \otimes_{\max} \text{id})(y)
\end{equation}
holds for all $x \in M(C_0(G) \otimes_{\max} C^*(G))$ and $y \in \widetilde{M}(A \otimes_{\max} C^*(G))$ and we will accomplish this in a sequence of steps.

We first show that (5.3) holds for all $x \in C_0(G) \otimes_{\max} C^*(G)$ (note that $\otimes_{\max} = \otimes$ here) and $y \in A \otimes_{\max} C^*(G)$: by linearity, continuity, and density (for the norm topologies), it suffices to consider elementary tensors $x = f \otimes_{\max} c$ and $y = a \otimes_{\max} d$ for $f \in C_0(G)$, $a \in A$, and $c, d \in C^*(G)$:
\[
(\phi \otimes_{\max} \text{id})(\mu \otimes_{\max} \text{id})(f \otimes_{\max} c)(a \otimes_{\max} d)
\]
\[
= (\phi \otimes_{\max} \text{id})(\mu(f)a \otimes_{\max} cd)
\]
\[
= \phi(\mu(f)a) \otimes_{\max} cd
\]
\[
= \nu(f)\phi(a) \otimes_{\max} cd
\]
\[
= (\nu(f) \otimes_{\max} c)(\phi(a) \otimes_{\max} d)
\]
\[
= (\nu \otimes_{\max} \text{id})(f \otimes_{\max} c)(\phi \otimes_{\max} \text{id})(a \otimes_{\max} d).
\]

We will now deduce that (5.3) holds for any $x \in M(C_0(G) \otimes_{\max} C^*(G))$ and $y \in A \otimes_{\max} C^*(G)$, using norm continuity of $\phi \otimes_{\max} \text{id}$: choose a net $(x_i)$ in $C_0(G) \otimes_{\max} C^*(G)$ converging strictly to $x$ in $M(C_0(G) \otimes_{\max} C^*(G))$, and let $y \in A \otimes_{\max} C^*(G)$. Then $(\mu \otimes_{\max} \text{id})(x_i) \to (\mu \otimes_{\max} \text{id})(x)$ strictly in $M(A \otimes_{\max} C^*(G))$ because $\mu$ is nondegenerate, and so
\[
(\mu \otimes_{\max} \text{id})(x_i)y \to (\mu \otimes_{\max} \text{id})(x)y \quad \text{in norm.}
\]
Since $\phi \otimes_{\max} \text{id}$ is norm continuous,
\[
(\phi \otimes_{\max} \text{id})(\mu \otimes_{\max} \text{id})(x_i)y \to (\phi \otimes_{\max} \text{id})(\mu \otimes_{\max} \text{id})(x)y).
\]
For each $i$,
\[
(\phi \otimes_{\max} \text{id})(\mu \otimes_{\max} \text{id})(x_i)y = (\nu \otimes_{\max} \text{id})(x_i)(\phi \otimes_{\max} \text{id})(y).
\]
Since $\nu \otimes_{\max} \text{id}$: $M(C_0(G) \otimes_{\max} C^*(G)) \to M(B \otimes_{\max} C^*(G))$ is strictly continuous,
\[
(\nu \otimes_{\max} \text{id})(x_i) \to (\nu \otimes_{\max} \text{id})(x) \quad \text{strictly.}
\]
Since $(\phi \otimes_{\max} \text{id})(y) \in B \otimes_{\max} C^*(G)$,
\[
(\nu \otimes_{\max} \text{id})(x_i)(\phi \otimes_{\max} \text{id})(y) \to (\nu \otimes_{\max} \text{id})(x)(\phi \otimes_{\max} \text{id})(y)
\]
in norm. Combining all the above, we get (5.3) for all $x \in M(C_0(G) \otimes_{\max} C^*(G))$ and $y \in A \otimes_{\max} C^*(G)$, as desired.

Finally, we take an arbitrary $x \in M(C_0(G) \otimes_{\max} C^*(G))$ and $y \in \tilde{M}(A \otimes_{\max} C^*(G))$. Letting $(e_i)$ be an approximate identity for $C^*(G)$, we have $y(1 \otimes_{\max} e_i) \in A \otimes_{\max} C^*(G)$ for all $i$, so by the above we obtain

$$
(\phi \otimes_{\max} \text{id})(((\mu \otimes_{\max} \text{id})(x)y)(1_{M(B)} \otimes_{\max} e_i)
= (\nu \otimes_{\max} \text{id})(x)(\phi \otimes_{\max} \text{id})(y(1_{M(A)} \otimes_{\max} e_i))
= (\nu \otimes_{\max} \text{id})(x)(\phi \otimes_{\max} \text{id})(y(1_{M(B)} \otimes_{\max} e_i)).
$$

Since $(1_{M(B)} \otimes_{\max} e_i)$ converges $C^*(G)$-strictly to $1$ in $\tilde{M}(B \otimes_{\max} C^*(G))$, we conclude that

$$
(\phi \otimes_{\max} \text{id})((\mu \otimes_{\max} \text{id})(x)y) = (\nu \otimes_{\max} \text{id})(x)(\phi \otimes_{\max} \text{id})(y).
$$

We need the following version for R-coactions of Lemma 3.25, and the proof is almost the same.

**Lemma 5.10.** Let $(A, \alpha, \mu)$ be an equivariant action. Then

$$
W = (i_A \circ \mu \otimes_{\max} \text{id})(w_G)
$$

is an $R(\alpha)$-cocycle. Put

$$
\tilde{R}(\alpha) = \text{Ad} W \circ R(\alpha).
$$

Then $(A \rtimes_{\alpha} G, \tilde{\alpha}, \mu \rtimes G)$ is a $\mathcal{K}$-fixed R-coaction.

Moreover, if $\phi: (A, \alpha, \mu) \rightarrow (B, \beta, \nu)$ is a morphism of equivariant actions, then the homomorphism $\phi \rtimes G$ gives a morphism

$$
(A \rtimes_{\alpha} G, \tilde{\alpha}, \mu \rtimes G) \rightarrow (B \rtimes_{\beta} G, \tilde{\beta}, \nu \rtimes G)
$$
of $\mathcal{K}$-fixed R-coactions.

Further, the assignments $(A, \alpha, \mu) \mapsto (A \rtimes_{\alpha} G, \tilde{\alpha}, \mu \rtimes G)$ and $\phi \mapsto \phi \rtimes G$ give a functor $\text{CPAR}$ from equivariant actions to $\mathcal{K}$-fixed R-coactions.

Finally, we have $\tilde{R}(\alpha)^S = \tilde{\alpha}$.

In the statement of the above lemma, the name “CPAR” is intended to indicate that we have modified the functor CPA so that the output is an R-coaction rather than a standard coaction.

**Proof.** Similarly to the argument for Lemma 3.25, $\mu \rtimes G$ is equivariant for the R-coactions $R(rt)$ and $R(\alpha)$. Since $(M \otimes_{\max} \text{id})(w_G)$ is an $R(rt)$-cocycle, the unitary $W$ is an $R(\alpha)$-cocycle. Since $\text{Ad}(M \otimes_{\max} \text{id})(w_G) \circ R(rt)$ is trivial, the R-coaction $\tilde{R}(\alpha)$ is trivial on $(\mu \rtimes G)(\mathcal{K})$, and so we get a $\mathcal{K}$-fixed R-coaction $(A \rtimes_{\alpha} G, \tilde{R}(\alpha), \mu \rtimes G)$. 
Given a morphism $\phi$, we know already from the proof of Lemma 3.25 that $\phi \times G$ is $(\mu \times G) - (\nu \times G)$ compatible.

Since $(\phi \times G): (A \rtimes G, \hat{\alpha}, i_A \circ \mu) \to (B \rtimes G, \hat{\beta}, i_B \circ \nu)$ is a morphism of $C_0(G)$-fixed coactions, the $(i_A \circ \mu) - (i_B \circ \nu)$ compatibility, combined with Lemma 5.9, also gives $\tilde{R}(\alpha) - \tilde{R}(\beta)$ equivariance, and therefore $\phi \times G$ is a morphism of $K$-fixed $R$-coactions.

The functoriality is clear, because we know that the crossed-product construction is functorial from actions to coactions.

Finally, the last assertion, concerning the standardization, follows from a routine computation on the generators. \hfill \Box

**Lemma 5.11.** If $(A, \delta, \iota)$ is a $K$-fixed $R$-coaction, then $\delta$ restricts to an $R$-coaction $C(\delta)$ on the relative commutant $C(A, \iota)$. Moreover, the assignments $(A, \delta, \iota) \mapsto (C(A, \iota), C(\delta))$ and $\phi \mapsto C(\phi)$ give a functor $C-R$ from $K$-fixed $R$-coactions to $R$-coactions.

Moreover, this functor respects standardizations, i.e.,

$$C(\delta)^S = C(\delta^S).$$

In the above, the name C-R is meant to indicate “commutant of $R$-coaction”.

For the proof, we need to know how tensoring with a fixed $C^*$-algebra affects relative commutants.

**Lemma 5.12.** Let $(A, \iota)$ be a $K$-decorated algebra, and let $D$ be any $C^*$-algebra. Then we have another $K$-decorated algebra $(A \otimes_{\max} D, \iota \otimes_{\max} 1_D)$. Define an isomorphism $\sigma$ by the commutative diagram

$$
\begin{array}{ccc}
A \otimes_{\max} D & \cong & C(A, \iota) \otimes_{\max} K \otimes_{\max} D \\
\cong & \sigma & \cong \\
& \cong & C(A, \iota) \otimes_{\max} D \otimes_{\max} K.
\end{array}
$$

Then there is a unique isomorphism $\tau: C(A \otimes_{\max} D, \iota \otimes_{\max} 1_D) \cong C(A, \iota) \otimes_{\max} D$ such that

$$\sigma(m) = \tau(m) \otimes_{\max} 1_K \quad \text{for all } m \in C(A \otimes_{\max} D, \iota \otimes_{\max} 1_D).$$

When using Lemma 5.12 we will usually suppress the $\tau$ and identify

$$C(A \otimes_{\max} D, \iota \otimes_{\max} 1_D) = C(A, \iota) \otimes_{\max} D.$$
Lemma 5.13. Let \((A, \iota)\) be a \(K\)-decorated algebra and \(B\) a \(C^*\)-algebra, and suppose that \(\sigma: A \xrightarrow{\sim} B \otimes K\) is an isomorphism such that \(\sigma \circ \iota = 1_B \otimes \text{id}_K\). Then there is a unique isomorphism \(\tau: C(A, \iota) \xrightarrow{\sim} B\) such that \(\sigma(m) = \tau(m) \otimes 1_K\) for \(m \in C(A, \iota)\).

Proof. It is an immediate consequence of the category equivalence between \(K\)-decorated algebras and stable \(C^*\)-algebras [KOQ16a, Theorem 4.4] that any isomorphism between \(K\)-decorated algebras restricts to an isomorphism between the relative commutants, and the lemma follows. \(\square\)

Proof of Lemma 5.12. The isomorphism \(\sigma\) takes \(\iota \otimes_{\text{max}} 1_D\) to the homomorphism \(1_A \otimes_{\text{max}} 1_D \otimes_{\text{max}} \text{id}_K\). Consequently, the result follows from Lemma 5.13. \(\square\)

We are now ready to prove that we can restrict \(R\)-coactions to relative commutants.

Proof of Lemma 5.11. For the first part, similarly to the proof for \(S\)-coactions, we appeal to Lemma 3.29. We have a nondegenerate homomorphism
\[
\delta: A \to M(A \otimes_{\text{max}} C^*(G))
\]
such that \(\delta \circ \iota = \iota \otimes_{\text{max}} 1\),
so applying Lemma 3.29 gives
\[
C(\delta): C(A, \iota) \to C(A \otimes_{\text{max}} C^*(G), \iota \otimes_{\text{max}} 1).
\]
By the abstract Lemma 5.12,
\[
C(A \otimes_{\text{max}} C^*(G), \iota \otimes_{\text{max}} 1) = C(A, \iota) \otimes_{\text{max}} C^*(G),
\]
so we have a nondegenerate homomorphism
\[
C(\delta): (A, \iota) \to M(C(A, \iota) \otimes_{\text{max}} C^*(G)),
\]
which, by construction, is the restriction of \(\delta\) to the \(C^*\)-subalgebra \(C(A, \iota)\) of \(M(A)\).

We will show that \((C(A, \iota), C(\delta))\) is an \(R\)-coaction. As we mentioned before, this is done for coactions in [Fis04, Section 3] (see [KOQ16b, Lemma 3.2] for a more detailed proof), so it might seem tempting to just say something along the lines of “it carries over routinely to \(R\)-coactions”. However, maximal tensor products have hidden subtleties, so it is safer to show (at least some of) the details. We essentially follow the strategy of [KOQ16b], but we streamline things a bit with the help of the observations we made at the beginning of this proof.
By Lemma 4.3, it suffices to verify the coaction identity and coaction-
nondegeneracy. For the coaction identity, we have

\[(C(\delta) \otimes_{\text{max}} \text{id}) \circ C(\delta) = C((\delta \otimes_{\text{max}} \text{id}) \circ \delta) = C((\text{id} \otimes_{\text{max}} \delta_G) \circ \delta) \quad \text{(coaction identity for } \delta) = C(\text{id} \otimes_{\text{max}} \delta_G) \circ C(\delta) \quad \text{(Lemma 3.29 again)}.
\]

But since \(C(\text{id} \otimes_{\text{max}} \delta_G)\) is just the restriction of \(\text{id} \otimes_{\text{max}} \delta_G\) to a \(C^*\)-subalgebra of \(M(A \otimes_{\text{max}} C^*(G))\), the above computation tells us that for all \(a \in A\) we have

\[(C(\delta) \otimes_{\text{max}} \text{id}) \circ C(\delta)(a) = (\text{id} \otimes_{\text{max}} \delta_G) \circ C(\delta)(a),\]

proving the coaction identity.

Next, for the coaction-nondegeneracy of \(C(\delta)\), we must show that

\[\text{span}\{C(\delta)(C(A, \iota))(1 \otimes_{\text{max}} C^*(G))\} = C(A, \iota) \otimes_{\text{max}} C^*(G),\]

and we use the following characterization (see [Fis04, Remark 3.1]): for any \(K\)-decorated algebra \((B, j)\), the relative commutant \(C(B, j)\) is the unique norm-closed subset \(Z\) of \(M(B)\) satisfying

(i) \(zj(k) = j(k)z\) for all \(z \in Z, k \in K\), and
(ii) \(\text{span}\{Zj(K)\} = B\).

It therefore suffices to verify (i)–(ii) for the \(K\)-decorated algebra

\[(A \otimes_{\text{max}} C^*(G), \iota \otimes_{\text{max}} 1)\]

and the closed subset

\[Z = \text{span}\{C(\delta)(C(A, \iota))(1 \otimes_{\text{max}} C^*(G))\}.
\]

Obviously \(1 \otimes_{\text{max}} C^*(G)\) commutes with \(\iota(K) \otimes_{\text{max}} 1\), and so does \(C(\delta)(C(A, \iota))\) because for any morphism \(\phi: (B, j) \rightarrow (C, \zeta)\) of \(K\)-decorated algebras we know that \(C(\phi)(C(B, j))\) commutes with \(\zeta(K)\). This implies (i).

For (ii), we note that with the above notation we also have

\[(5.4) \quad \text{span}\{C(\phi)(C(B, j))\zeta(K)\} = \phi(B),\]

so

\[\text{span}\{Z(\iota(K) \otimes_{\text{max}} 1)\} = \text{span}\{C(\delta)(C(A, \iota))(1 \otimes_{\text{max}} C^*(G))\}(\iota(K) \otimes_{\text{max}} 1)\}
\[= \text{span}\{C(\delta)(C(A, \iota))\iota(K) \otimes_{\text{max}} 1\}(1 \otimes_{\text{max}} C^*(G))\}
\[\subseteq \text{span}\{\delta(A)(1 \otimes_{\text{max}} C^*(G))\}
\[= A \otimes_{\text{max}} C^*(G),\]
where the equality at * follows by applying (5.4) to the morphism δ: (A, ϵ) → (A ⊗ max C∗(G), ϵ ⊗ max 1) of K-decorated algebras. Thus we have shown the coaction-nondegeneracy, so we have an R-coaction C(δ) on C(A, ϵ).

Finally, we turn to the functoriality: Since we already know that (A, ϵ) ↦→ C(A, ϵ) is functorial, it only remains to verify that C(φ) is C(δ) − C(ε) equivariant whenever φ: (A, δ, ϵ) → (B, ε, ψ) is a morphism of K-fixed R-coactions, i.e., the diagram

\[
\begin{array}{ccc}
C(A, ϵ) & \xrightarrow{C(δ)} & \tilde{M}(C(A, ϵ) ⊗ max C∗(G)) \\
\downarrow C(φ) & & \downarrow C(φ) ⊗ max id \\
C(B, ψ) & \xrightarrow{C(ε)} & \tilde{M}(C(B, ψ) ⊗ max C∗(G))
\end{array}
\]

commutes. This might seem almost trivial: by assumption, the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{δ} & \tilde{M}(A ⊗ max C∗(G)) \\
\downarrow φ & & \downarrow φ ⊗ max id \\
B & \xrightarrow{ε} & \tilde{M}(B ⊗ max C∗(G))
\end{array}
\]

commutes, and in (5.5) we seem to be restricting to appropriate subalgebras of the multiplier algebras. There is a subtlety, however: the homomorphisms φ and φ ⊗ max id may be degenerate, so we cannot just “extend and restrict” them. Consequently, we are forced to get our hands dirty: let m ∈ C(A, ϵ). We must show that

\[ε(C(φ)(m)) = (C(φ) ⊗ max id)(δ(m)),\]

in which we have dropped the “C” for δ and ε, since they are nondegenerate homomorphisms, and hence for them we are just extending and restricting as usual. It suffices to show that if we take any k ∈ K, then

\[ε(C(φ)(m))(j(k) ⊗ max 1) = (C(φ) ⊗ max id)(δ(m))(j(k) ⊗ max 1).\]

The computation goes as follows:

\[
\begin{align*}
ε(C(φ)(m))(j(k) ⊗ max 1) \\
= ε(C(φ)(m))j(k) & \quad (ε ▽ j = j ⊗ max 1) \\
= ε(φ(mu(k))) & \quad (Equation (3.1)) \\
= φ ⊗ max id(δ(mu(k))) & \quad (φ is equivariant) \\
= φ ⊗ max id(δ(m)(i(k) ⊗ max 1)) & \quad (δ ▽ i = i ⊗ max 1)
\end{align*}
\]
\[ \ast : C(\phi) \otimes_{\text{max}} \text{id}(\delta(m))(\iota(k) \otimes_{\text{max}} 1), \]

where to justify the equality at \( \ast \) we multiply on the right by \( 1 \otimes_{\text{max}} c \) for an arbitrary \( c \in C^*(G) \):

\[
\begin{align*}
\phi \otimes_{\text{max}} \text{id}(\delta(m)(\iota(k) \otimes_{\text{max}} 1))(1 \otimes_{\text{max}} c) &= (\phi \otimes_{\text{max}} \text{id})(\delta(m)(1 \otimes_{\text{max}} c)(\iota(k) \otimes_{\text{max}} 1)) \\
&= (\phi \otimes_{\text{max}} \text{id})(\delta(m)(1 \otimes_{\text{max}} c)) (\iota(k) \otimes_{\text{max}} 1) \\
&= C(\phi) \otimes_{\text{max}} \text{id}(\delta(m))(1 \otimes_{\text{max}} c) (\iota(k) \otimes_{\text{max}} 1) \\
&= C(\phi) \otimes_{\text{max}} \text{id}(\delta(m))(\iota(k) \otimes_{\text{max}} 1)(1 \otimes_{\text{max}} c),
\end{align*}
\]

and where we used Equation (3.1) three times: for

- \( \phi \otimes_{\text{max}} \text{id} \) and the embeddings of \( C^*(G) \) at equality 1,
- \( \phi \otimes_{\text{max}} \text{id} \) and the embeddings of \( K \) at equality 2, and
- \( C(\phi) \otimes_{\text{max}} \text{id} \) and the embeddings of \( C^*(G) \) at equality 3.

For the assertion regarding standardizations, it is tempting to just say something along the following lines: for both R-coactions and coactions, the coaction on the relative commutant \( C(A, \iota) \) is produced via the nondegenerate version of the relative-commutant functor of Corollary 3.29. But this is really just intuition, not a proof. Here are the details, for which we will use functoriality of the relative commutant for nondegenerate homomorphisms. Consider the diagrams

\[ (5.6) \]
\[
\begin{array}{c}
A \xrightarrow{\delta} M(A \otimes_{\text{max}} C^*(G)) \\
\delta^S \xrightarrow{} M(A \otimes C^*(G)) \\
\gamma
\end{array}
\]

and

\[ (5.7) \]
\[
\begin{array}{c}
C(A, \iota) \xrightarrow{C(\delta)} M(C(A, \iota) \otimes_{\text{max}} C^*(G)) \\
C(\delta)^S \xrightarrow{} M(C(A, \iota) \otimes C^*(G)), \\
\gamma
\end{array}
\]

which both commute by definition of standardization. We will show that \( (5.7) \) in fact is the image of \( (5.6) \) under the relative-commutant functor, and in particular \( C(\delta)^S = C(\delta^S) \). The key is to compare the right-hand vertical arrows, and to recall that both \( C(\delta) \) and \( C(\delta^S) \) are
images under the functor $C$ of morphisms that are nondegenerate as $C^*$-homomorphisms. Recall that by Lemma 5.12 we have

$$C(A \otimes_{\max} C^*(G), t \otimes_{\max} 1) = C(A, t) \otimes_{\max} C^*(G),$$

and similarly for $A \otimes C^*(G)$. Then $\delta$ may be regarded as a $\mathcal{K}$-decorated algebra morphism

$$C(A, t) \to M(A \otimes_{\max} C^*(G), t \otimes_{\max} 1),$$

where on the right-hand side we really mean the multiplier algebra $M(A \otimes_{\max} C^*(G))$ together with the nondegenerate homomorphism

$$t \otimes_{\max} 1: \mathcal{K} \to M(A \otimes_{\max} C^*(G)).$$

Similarly, $\Upsilon$ is also a $\mathcal{K}$-decorated algebra morphism:

$$\Upsilon: (A \otimes_{\max} C^*(G), t \otimes_{\max} 1) \to (A \otimes C^*(G), t \otimes 1),$$

which, being a surjective homomorphism, can of course be uniquely extended to the multipliers. Thus (5.6) is really a commutative diagram in the nondegenerate category of $\mathcal{K}$-decorated algebras:

$$\begin{array}{ccc}
(A, t) & \xrightarrow{\delta} & M(A \otimes_{\max} C^*(G), t \otimes_{\max} 1) \\
& \downarrow{\delta^S} & \downarrow{\Upsilon} \\
& & M(A \otimes C^*(G), t \otimes 1).
\end{array}$$

The nondegenerate relative-commutant functor takes this to a commutative diagram

$$\begin{array}{ccc}
C(A, t) & \xrightarrow{C(\delta)} & M(C(A \otimes_{\max} C^*(G), t \otimes_{\max} 1)) \\
& \downarrow{C(\delta^S)} & \downarrow{C(\Upsilon)} \\
& & M(C(A \otimes C^*(G), t \otimes 1)).
\end{array}$$

Identifying the $C^*$-algebras via the canonical isomorphisms

$$C(A \otimes_{\max} C^*(G), t \otimes_{\max} 1) \simeq C(A, t) \otimes_{\max} C^*(G)$$

$$C(A \otimes C^*(G), t \otimes 1) \simeq C(A, t) \otimes C^*(G),$$

we see that the diagonal arrows in the diagrams (5.7) and (5.8) coincide, as desired. \hfill \Box

\textbf{Remark 5.14.} We pause to contrast the equivariance argument in the above proof with the corresponding argument for coactions in the proof of [KLQ18, Lemma 3.8], where we were able to use the $B(G)$-module structure associated with a coaction; we could not use that tactic here because slice maps do not generally separate the points of maximal tensor products.
Definition 5.15. Let \((A, \delta)\) be a maximal coaction. The composition \(C-R \circ \text{CPAR} \circ \text{CPC}\) produces an \(R\)-coaction \((A^m, \eta)\). The canonical surjection \(\psi_A : A^m \to A\) is an isomorphism because \(\delta\) is maximal, and so it transforms \(\eta\) to an \(R\)-coaction on \(A\), which we call the \(R\)-ification of \(\delta\) and denote by \(\delta^R\).

Theorem 5.16. The above \(R\)-ification process \((A, \delta) \mapsto (A, \delta^R)\) is a functor from maximal coactions to \(R\)-coactions, where the morphisms are left unchanged: \(\phi \mapsto \phi\). Moreover, we have \(\delta^{RS} = \delta\).

Proof. The only nonobvious part is the functoriality of the last step: from start to finish we do have a well-defined process converting a maximal coaction \(\delta\) into an \(R\)-coaction \(\delta^R\) on the same \(C^*\)-algebra \(A\). We need to know that if \(\phi : (A, \delta) \to (B, \varepsilon)\) is a morphism of maximal coactions, then the same homomorphism \(\phi : A \to B\) is also \(\delta^R - \varepsilon^R\) equivariant. Following the process up until the penultimate step, i.e., the composition \(C-R \circ \text{CPAR} \circ \text{CPC}\), gives \(R\)-coactions \(\eta\) and \(\zeta\) on \(A^m\) and \(B^m\), respectively, and the morphism \(\phi\) is taken to the maximalization \(\phi^m : A^m \to B^m\), as we discussed above. Since the diagram

\[
\begin{array}{ccc}
A^m & \xrightarrow{\phi^m} & B^m \\
\downarrow{\psi_A} & & \downarrow{\psi_B} \\
A & \xrightarrow{\phi} & B
\end{array}
\]

commutes, we see that \(\phi : A \to B\) is \(\delta^R - \varepsilon^R\) equivariant, and we have shown that \(R\)-ification is functorial.

Finally, for the assertion that \(\delta^{RS} = \delta\), just combine Lemma 5.10, Lemma 5.11, and the fact that the isomorphism \(\psi_A : A^m \to A\) is \(\delta^m - \delta\) equivariant. \(\square\)

Note that, as we mentioned earlier, the existence of an \(R\)-coaction whose standardization is \(\delta\) does not imply that \(\delta\) is maximal. However, for the present we are satisfied with the above \(R\)-ification functor on maximal coactions.

Definition 5.17. An \(R\)-coaction is \textit{maximal} if its standardization is maximal.

Note that we are not just adapting the definition of maximality to \(R\)-coactions: we make no attempt to discuss the crossed product of an \(R\)-coaction (but see [Rae92]), so in particular it would not make sense to ask about injectivity of the canonical surjection \(\Phi : A \rtimes_{\delta} G \rtimes_{\overline{\delta}} G \to A \otimes K\).
If \((B, \alpha)\) is an action, then we can directly produce the dual \(R\)-coaction \(R(\alpha)\) on the crossed product \(B \rtimes_\alpha G\), as in Definition 4.12. We are finally ready for our main result: the direct approach is consistent with the \(R\)-ification of the dual coaction \(\hat{\alpha}\), as in Definition 5.15.

**Theorem 5.18.** Let \((B, \alpha)\) be an action, with dual coaction \((B \rtimes_\alpha G, \hat{\alpha})\). Then the two \(R\)-coactions \(\hat{\alpha}^R\) and \(R(\alpha)\) on \(B \rtimes_\alpha G\) coincide.

**Proof.** Let:

- \((A, \delta) = (B \rtimes_\alpha G, \hat{\alpha})\)
- \((E, \beta) = (A \rtimes_\delta G, \hat{\delta})\)
- \(\tilde{A} = E \rtimes_\beta G = A \rtimes_\delta G \rtimes_\beta G\)
- \(\pi = i_E \circ j_A: A \to M(\tilde{A})\)
- \(\phi = \pi \circ i_B: B \to M(\tilde{A})\)
- \(U = \pi \circ i^\alpha_G: G \to M(\tilde{A})\)
- \(\nu = i_E \circ j_G: C_0(G) \to M(\tilde{A})\)
- \(\iota = j_G \rtimes G: K \to M(\tilde{A})\)
- \(V = i^\beta_G: G \to M(\tilde{A})\)
- \(W = (\nu \otimes_{\max} \text{id})(w_G) \in M(\tilde{A} \otimes_{\max} C^*(G))\).

Note that when working with \(A \otimes_{\max} K \otimes_{\max} C^*(G)\), we can — and in fact will need to — sometimes replace either the first or the second \(\otimes_{\max}\) by \(\otimes\) (and back again), but we cannot replace both by \(\otimes\) at the same time. Of course, any such replacement must be accompanied by the insertion of parentheses around the minimal tensor product.

Since \(\delta\) is maximal and \(K\) is nuclear, we can identify:

\[
\tilde{A} = A \otimes K \quad \quad \pi = (\text{id} \otimes \lambda) \circ \delta \quad \quad \nu = 1 \otimes M \quad \quad V = 1 \otimes \rho.
\]

Further, we can write

\[
\phi(b) = i_B(b) \otimes_{\max} 1 \quad \text{for } b \in B
\]

\[
U_s = i^\alpha_G(s) \otimes_{\max} \lambda_s \quad \text{for } s \in G,
\]

although again we can replace \(\otimes_{\max}\) by \(\otimes\). Also, we have observed previously that

\[
A^m = C(\tilde{A}, \iota) = A \otimes_{\max} 1,
\]

and so by Lemma 5.12,

\[
C(\tilde{A} \otimes_{\max} C^*(G), \iota \otimes_{\max} 1) = A^m \otimes_{\max} C^*(G) = A \otimes_{\max} 1 \otimes_{\max} C^*(G),
\]
and
\[ W = 1 \otimes_{\max} (M \otimes_{\max} \text{id})(w_G) \in M(A \otimes_{\max} K \otimes_{\max} C^*(G)). \]
The R-coaction directly produced from \( \hat{\beta} \) is given on the generators by:
\[
R(\beta) \circ i_E = i_E \otimes_{\max} 1
\]
\[
R(\beta) \circ i^\beta_G(s) = i^\beta_G(s) \otimes_{\max} s.
\]
Thus:
\[
R(\beta) \circ \pi = \pi \otimes_{\max} 1
\]
\[
R(\beta) \circ \phi(b) = i_B(b) \otimes_{\max} 1 \otimes_{\max} 1 \quad \text{for } b \in B
\]
\[
R(\beta)(U_s) = i^\alpha_G(s) \otimes_{\max} \lambda_s \otimes_{\max} 1 \quad \text{for } s \in G
\]
\[
R(\beta) \circ \nu = \nu \otimes_{\max} 1
\]
\[
R(\beta)(V_s) = 1 \otimes_{\max} \rho_s \otimes_{\max} s \quad \text{for } s \in G.
\]
Now we perturb by the R(\( \beta \))-cocycle \( W \), getting an exterior-equivalent R-coaction \( \tilde{R}(\beta) = \text{Ad} W \circ R(\beta) \). For \( b \in B, s \in G, \) and \( f \in C_0(G) \) we have:
\[
\tilde{R}(\beta) \circ \phi(b) = \text{Ad}(1 \otimes_{\max} M \otimes_{\max} \text{id})(w_G)\left( i_B(b) \otimes_{\max} 1 \otimes_{\max} 1 \right)
\]
\[
= i_B(b) \otimes_{\max} 1 \otimes_{\max} 1,
\]
\[
\tilde{R}(\beta)(U_s) = \text{Ad}(1 \otimes_{\max} M \otimes_{\max} \text{id})(w_G)\left( i^\alpha_G(s) \otimes_{\max} \lambda_s \otimes_{\max} 1 \right)
\]
\[
= i^\alpha_G(s) \otimes_{\max} \text{Ad}(M \otimes \text{id})(w_G)(\lambda_s \otimes 1)
\]
\[
\text{(swapping } \otimes_{\max} \text{ for } \otimes \text{ in } M(K \otimes C^*(G)))
\]
\[
= i^\alpha_G(s) \otimes_{\max} (\lambda_s \otimes s)
\]
\[
\text{(since } (\lambda, M) \text{ is a covariant representation of } \delta_G)
\]
\[
= i^\alpha_G(s) \otimes_{\max} \lambda_s \otimes_{\max} s
\]
\[
\text{(because } \lambda_s \otimes s \in M(K \otimes C^*(G)))
\]
\[
\tilde{R}(\beta) \circ \nu(f) = \text{Ad}(1 \otimes_{\max} M \otimes_{\max} \text{id})(w_G)\left( 1 \otimes_{\max} M_f \otimes_{\max} 1 \right)
\]
\[
= 1 \otimes_{\max} M_f \otimes_{\max} 1
\]
\[
\text{((M } \otimes_{\max} \text{id})(w_G) \text{ commutes with } C_0(G) \otimes_{\max} 1),
\]
and
\[
\tilde{R}(\beta)(V_s) = \text{Ad}(1 \otimes_{\max} M \otimes_{\max} \text{id})(w_G)\left( 1 \otimes_{\max} \rho_s \otimes_{\max} 1 \right)
\]
\[
= 1 \otimes_{\max} \text{Ad}(M \otimes \text{id})(w_G)(\rho_s \otimes s)
\]
\[
\text{(swapping } \otimes_{\max} \text{ for } \otimes \text{ again)}
\]
\[
= 1 \otimes_{\max} (\rho_s \otimes 1)
\]
(because \( \text{Ad}(M \otimes \text{id})(w_G) \circ \hat{r} = \delta_{\text{triv}} \))
\[
= 1 \otimes_{\text{max}} \rho_s \otimes_{\text{max}} 1 \quad (\text{swapping } \otimes \text{ back to } \otimes_{\text{max}}).
\]

Now we compute the coaction \( \varepsilon = C(\hat{R}(\beta)) \) on the relative commutant \( A^m = C(\hat{A}, \iota) \). Since \( A \) is generated by \( i_B(B) \) and \( i^\alpha_G(C^*(G)) \), we can compute on the generators: for \( b \in B \) and \( s \in G \) we have
\[
\varepsilon(i_B(b) \otimes 1) = \varepsilon \circ \phi(b) = i_B(b) \otimes_{\text{max}} 1 \otimes_{\text{max}} 1
\]
and
\[
\varepsilon(i^\alpha_G(s) \otimes 1) = \varepsilon((i^\alpha_G(s) \otimes \lambda_s)(1 \otimes \lambda_{s^{-1}})) = \varepsilon(i^\alpha_G(s) \otimes \lambda_s)\varepsilon(1 \otimes \lambda_{s^{-1}}) = \varepsilon(U_s)(1 \otimes_{\text{max}} \lambda_{s^{-1}} \otimes_{\text{max}} 1)
\]
(since \( \varepsilon \) is a \( K \)-fixed R-coaction)
\[
= (i^\alpha_g(s) \otimes_{\text{max}} \lambda_s \otimes_{\text{max}} s)(1 \otimes_{\text{max}} \lambda_{s^{-1}} \otimes_{\text{max}} 1) = i^\alpha_G(s) \otimes_{\text{max}} 1 \otimes_{\text{max}} s,
\]
and therefore
\[
(\psi \otimes_{\text{max}} \text{id}) \circ \varepsilon(i_B(b) \otimes 1) = (\psi \otimes_{\text{max}} \text{id})(i_B(b) \otimes_{\text{max}} 1 \otimes_{\text{max}} 1) = i_B(b) \otimes_{\text{max}} 1 = \delta^R(i_B(b)) = \delta^R \circ \psi(i_B(b) \otimes 1)
\]
and
\[
(\psi \otimes_{\text{max}} \text{id}) \circ \varepsilon(i^\alpha_G(s) \otimes 1) = (\psi \otimes_{\text{max}} \text{id})(i^\alpha_G(s) \otimes_{\text{max}} 1 \otimes_{\text{max}} s) = i^\alpha_G(s) \otimes_{\text{max}} s = \delta^R(i^\alpha_G(s)) = \delta^R \circ \psi(i^\alpha_G(s) \otimes 1).
\]
Thus the diagram
\[
\begin{array}{ccc}
A^m & \overset{\varepsilon}{\longrightarrow} & M(A^m \otimes_{\text{max}} C^*(G)) \\
\downarrow & \simeq & \downarrow \psi \otimes_{\text{max}} \text{id} \\
A & \overset{\delta^R}{\longrightarrow} & M(A \otimes_{\text{max}} C^*(G))
\end{array}
\]
commutes, and so
\[
\delta^R = \hat{\alpha}^R = R(\alpha). \quad \square
\]
We now have two possible interpretations of the notation $\delta^R_G$; the following corollary assures us that they are consistent.

**Corollary 5.19.** The $R$-ification of the canonical coaction $\delta_G$ on $C^*(G)$ coincides with the $R$-coaction $\delta^R_G$ introduced in Definition 4.2.

6. Conclusion

In the proof of Theorem 5.18 we never seemed to need crossed products by $R$-coactions, or even covariant representations of $R$-coactions. But we did need a bit of the theory of cocycles for $R$-coactions, in particular the fact that if $\delta$ is an $R$-coaction and $W$ is a $\delta$-cocycle, then $\text{Ad} W \circ \delta$ is an $R$-coaction.

Although it is possible — as Raeburn does in [Rae92] — to develop the elementary theory of covariant representations of an $R$-coaction $(A, \delta)$ on Hilbert space, and further to define a crossed product $A \rtimes_\delta G$ (using representations on Hilbert space for the universal property), it would be problematic to try to put this in the more general context of covariant representations in multiplier algebras. One specific problem is that, if $(A, \delta^S)$ is the coaction associated to an $R$-coaction $(A, \delta)$, and if $(\pi, \mu)$ is a covariant representation of $\delta^S$ in $M(B)$, we do not know whether there is a corresponding covariant representation of $\delta$. Consequently, we would not have a satisfactory theory of crossed products using representations in multiplier algebras. Moreover, as discussed in [Rae92] and [Qui91], even if crossed products of $R$-coactions are defined (using Hilbert-space representations), they will be naturally isomorphic to the crossed products by the associated coactions. Consequently, we made no attempt to discuss crossed-product duality for $R$-coactions.

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