DUAL VARIATIONAL METHODS AND NONVANISHING FOR THE NONLINEAR HELMHOLTZ EQUATION

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Abstract. We set up a dual variational framework to detect real standing wave solutions of the nonlinear Helmholtz equation

\[ -\Delta u - k^2 u = Q(x)|u|^{p-2} u, \quad u \in W^{2,p}(\mathbb{R}^N) \]

with \( N \geq 3, \frac{2(N+1)}{N-1} < p < \frac{2N}{N-2} \) and nonnegative \( Q \in L^\infty(\mathbb{R}^N) \). We prove the existence of nontrivial solutions for periodic \( Q \) as well as in the case where \( Q(x) \to 0 \) as \( |x| \to \infty \). In the periodic case, a key ingredient of the approach is a new nonvanishing theorem related to an associated integral equation. The solutions we study are superpositions of outgoing and incoming waves and are characterized by a nonlinear far field relation.

1. Introduction

Due to their importance in various problems in physics, nonlinear stationary Schrödinger equations of the type

\[ -\Delta u + \lambda u = f(x,u), \quad x \in \mathbb{R}^N \]

have been studied extensively since the pioneering works of Berestycki and Lions [7, 8], Lions [21], Floer and Weinstein [12], Ding and Ni [10] and Rabinowitz [22] from the 1980ies and 1990ies. For superlinear nonlinearities of the form \( f(x,u) = r(x,|u|^2)u \), solutions of (1) correspond to periodic solutions of the time-dependent nonlinear Schrödinger equation

\[ i\partial_t \psi(t,x) = -\Delta \psi(t,x) - f(x,\psi(t,x)), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \]

via the ansatz \( \psi(t,x) = e^{i\lambda t} u(x) \). Moreover, for \( m \geq 0 \) and \( \lambda < m \), the ansatz \( \psi(t,x) = e^{i\sqrt{m^2 - \lambda} t} u(x) \) leads to periodic solutions of the nonlinear Klein-Gordon equation

\[ \frac{\partial^2 \psi}{\partial t^2}(t,x) - \Delta \psi(t,x) + m^2 \psi(t,x) = f(x,\psi(t,x)), \quad (t,x) \in \mathbb{R} \times \mathbb{R}^N. \]

In the present paper we are interested in standing wave solutions to (2) which arise from real-valued solutions \( u \) of (1). We note that real-valued solutions of (1) with the decay property \( u(x) \to 0 \) as \( |x| \to \infty \) have been studied extensively in the case where \( \partial_h f(\cdot,0) \equiv 0 \) on \( \mathbb{R}^N \) and \( \lambda \geq 0 \), see e.g. [14, 25, 27] and the references therein. On the contrary, very little is known in the case \( \lambda < 0 \), where 0 is contained in the essential spectrum of the Schrödinger operator \(-\Delta + \lambda\). In the present paper we are interested in this case, which is relevant for the analysis of standing wave solutions of (2) with large frequencies according to the ansatz above. In this case, it is customary to set \( \lambda = -k^2 \), and (1) is called nonlinear Helmholtz equation (or nonlinear reduced wave equation). By restricting our attention to the important

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class of power type nonlinearities $f(x, u) = Q(x)|u|^{p-2}u$, we are therefore lead to study real-valued solutions of the problem
\begin{equation}
-\Delta u - k^2 u = Q(x)|u|^{p-2}u, \quad u \in W^{2, p}(\mathbb{R}^N).
\end{equation}

One of the very few existence results available for (3) is due to Gutiérrez [16], who studied the special case $N = 3, 4, p = 4, Q \equiv \pm 1$. She proved the existence of small complex solutions of (3) with the additional (finiteness) property
\begin{equation}
\sup_{R > 1} \frac{1}{R} \int_{B_R} |u|^2 \, dx < \infty.
\end{equation}

Here and in the following, $B_R \subset \mathbb{R}^N$ denotes the open ball of radius $R$ centered around the origin. In order to give a more precise description of the solutions obtained by Gutiérrez, we briefly recall some important facts on (3) in the case $N \geq 3, Q \in L^\infty(\mathbb{R}^N)$ and $\frac{2(N+1)}{(N-1)} \leq p \leq \frac{2N}{N-2}$. In this case, a (complex-valued) function $u \in W^{2, p}(\mathbb{R}^N)$ solves (3), (4) if and only if $u \in L^p(\mathbb{R}^N)$ solves
\begin{equation}
\begin{split}
\phi_g(x) &= \mathbb{R}\left(Q(x)|u|^{p-2}u\right) + \varphi_g \quad \text{for some } \varphi_g \in \mathcal{H}.
\end{split}
\end{equation}

Here $\mathbb{R}$ is the resolvent operator given by convolution with the fundamental solution of the linear Helmholtz equation (see Section 2 below), and $\mathcal{H}$ denotes the space of Herglotz wave functions $\varphi_g : \mathbb{R}^N \to \mathbb{C}$ given as
\begin{equation}
\varphi_g(x) = \int_{\mathbb{R}^N} e^{ik(x-\xi)} g(\xi) \, d\sigma(\xi) \quad \text{for some function } g \in L^2(S^{N-1}).
\end{equation}

It follows from the Stein-Tomas-Theorem (see Theorem 2.3 below) and elliptic estimates that such functions satisfy $\varphi_g \in W^{2, p}(\mathbb{R}^N)$ and solve the linear Helmholtz equation $(\Delta + k^2)\varphi_g = 0$ in the strong sense. For a given solution $u$ of (3), (4), the functions $g$ and $\varphi_g$ are uniquely determined by (5), and we will call $\varphi_g$ the Herglotz wave associated to $u$ in the sequel.

In the case $N = 3, 4, p = 4, Q \equiv \pm 1$, it was proved in [16] Theorem 1] that for given small $g \in L^2(S^{N-1})$ the problem (3) admits a unique (complex-valued) solution $u \in W^{2, 4}(\mathbb{R}^N)$ which is also small in the $L^4$-norm. The proof is based on subtle resolvent estimates combined with a contraction mapping argument. In the present paper, we focus on a complementary class of real-valued solutions of (3) which satisfy the integral equation
\begin{equation}
\begin{split}
u &= \mathbb{R}\left(Q(x)|u|^{p-2}u\right), \quad u \in L^p(\mathbb{R}^N),
\end{split}
\end{equation}

where $\mathbb{R}$ denotes the real part of the resolvent operator $\mathbb{R}$. We shall see that these solutions satisfy (5) with
\begin{equation}
g_u(\xi) = -i\left(\frac{k^2}{4}\right)^{\frac{N-2}{2}} 2^{\frac{N-2}{2}} \mathcal{F}(Q|u|^{p-2}u)(k\xi), \quad \xi \in S^{N-1},
\end{equation}

where, here and in the following, $\mathcal{F}$ denotes the Fourier transform (see [22] below). So the associated Herglotz wave $\varphi_{g_u}$ is related in a nonlinear way to the solution $u$ itself. From resolvent estimates which we recall in Section 2, it easily follows that $u \equiv 0$ is an isolated solution of (1) in $L^p(\mathbb{R}^N)$, and thus nontrivial solutions cannot be found by a contraction mapping argument as in [16]. In this paper, we set up a variational framework to find nontrivial solutions of this problem. Before stating our main results, we mention that there is an intimate relationship between the far field behavior of solutions of (3), (4) and their associated Herglotz waves. In particular, we shall see that if $u \in L^p(\mathbb{R}^N)$ solves (6), then not only does it solve...
the problem (8) but it also satisfies the far field relation
\[ u(x) = -2 \left( \frac{2\pi}{k|x|} \right)^{\frac{N}{2}} \text{Re} \left[ e^{ik|x|-i\frac{1}{2}(N-1)x} g_u(\hat{x}) \right] + o(|x|^{-\infty}) \quad \text{as} \quad |x| \to \infty. \]

with \( \hat{x} = \frac{x}{|x|} \) for \( x \in \mathbb{R}^N \setminus \{0\} \) and \( g_u \) as in (7). Note that (8) implies (4). We are now in a position to state our main results which are related to two different types of weight functions \( Q \).

**Theorem 1.1.** Let \( N \geq 3 \), \( \frac{2N+1}{N-1} < p < \frac{2N}{N-1} \), and let \( Q \in L^\infty(\mathbb{R}^N) \), \( Q \geq 0 \), \( Q \neq 0 \) be \( \mathbb{Z}^N \)-periodic. Then problem (3), (8) admits a nontrivial strong solution such that \( u \in W^{2,q}(\mathbb{R}^N) \cap \mathcal{H}^{1,\alpha}(\mathbb{R}^N) \) for all \( q \in [p, \infty) \), \( \alpha \in (0,1) \).

**Theorem 1.2.** Let \( N \geq 3 \), \( \frac{2N+1}{N-1} < p < \frac{2N}{N-1} \), and let \( Q \in L^\infty(\mathbb{R}^N) \), \( Q \geq 0 \), \( Q \neq 0 \) satisfy \( \lim |x| \to \infty Q(x) = 0 \). Then problem (3), (8) admits a sequence of pairs \( \pm u_n \) of solutions such that \( u_n \in W^{2,q}(\mathbb{R}^N) \cap \mathcal{H}^{1,\alpha}(\mathbb{R}^N) \) for all \( q \in [p, \infty) \), \( \alpha \in (0,1) \), and
\[ \|u_n\|_{L^p(\mathbb{R}^N)} \to \infty \quad \text{as} \quad n \to \infty. \]

Up to our knowledge, these results are the first existence results for problem (3), (8) under the given assumptions on \( Q \). The main difficulty of the problem is the lack of a direct variational approach, since the energy functional formally associated to (3) is not well defined on \( W^{2,p}(\mathbb{R}^N) \), and it is not even well defined on nontrivial solutions of (3), (8). In a previous paper by the authors [11], the case of compactly supported \( Q \) (and a more general class of superlinear nonlinearities \( f \) compactly supported in space) has been studied with a variational reduction method. More precisely, in [11] we used a Dirichlet-to-Neumann map associated to the exterior problem for the linear Helmholtz equation to reduce the problem to the existence of nontrivial critical points of an energy functional in \( H^1(B_R(0)) \) for some \( R > 0 \). We then used linking arguments to get existence results. The method of [11] is obviously restricted to (spatially) compactly supported nonlinearities and therefore cannot be used to derive the results of the present paper. On the other hand, in the case of compactly supported \( Q \), multiple existence of solutions can be shown for any \( N \geq 1 \) and for a larger range of exponents, namely for \( p > 2 \) with \( p < \frac{2N}{N-1} \) if \( N \geq 3 \), see [11].

It is natural to ask whether a stronger, pointwise version of the far field relation (8) in the form
\[ u(x) = \frac{2\pi}{k|x|} \left( \frac{N}{2} \right)^{\frac{N}{2}} \text{Re} \left[ e^{ik|x|-i\frac{1}{2}(N-1)x} g_u(\hat{x}) \right] + o(|x|^{-\infty}) \quad \text{as} \quad |x| \to \infty. \]

is available for the solutions given by Theorems 1.1 and 1.2 above. Related to this question, we have the following result.

**Theorem 1.3.** Let \( N = 3, 4 \leq p \leq 6 \) or \( N = 4, \frac{11}{3} < p \leq 4 \), and let \( Q \in L^\infty(\mathbb{R}^N) \). Then every solution \( u \in L^p(\mathbb{R}^N) \) of (2) satisfies (17). In particular, \( u \) has pointwise decay given by \( |u(x)| = O(|x|^{-\infty}) \) as \( |x| \to \infty \).

Note that, for \( N = 3 \), the full range of admissible exponents in Theorem 1.3 is covered by Theorem 1.1 and thus (10) holds for the solution detected in this Theorem. Moreover, for \( N = 3 \), (10) also holds for the solutions detected in Theorem 1.2 unless \( p = 4 \) in which case the question is open. Our proof of Theorem 1.3 is surprisingly involved, and the main step is to show that \( |u|^{p-2} u \in L^1(\mathbb{R}^N) \) under the assumptions of Theorem 1.3. This is done by a new and completely nonlinear bootstrap argument (see Theorem 1.3) which is based on an asymptotic multiplier method.
estimate. It is open whether the restrictions on $N$ and $p$ in Theorem 1.3 are necessary, but we note that for $p < \frac{3N}{N-2}$ the property (10) implies in general that $|u|^p u \not \in L^1(\mathbb{R}^N)$.

Let us now briefly explain our approach and the organization of the paper. In Section 2 we first recall important estimates and characterizations related to the linear (homogeneous and inhomogeneous) Helmholtz equation. Moreover, we derive far field asymptotics within the linear inhomogeneous setting. In Section 3 we then derive a nonvanishing property related to the resolvent which is a key ingredient in the proof of Theorem 1.1. With the help of this nonvanishing property, the problem of lack of compactness of the periodic case will be overcome.

The nonvanishing property can be seen as an analogue of Lions’ local compactness Lemma (see e.g. [21] or [27] Lemma 1.21), and it relies, in particular, on a combination of arguments as in [3] with an asymptotic multiplier estimate inspired by [16]. In Section 4 we set up a dual variational framework for problem 3, which relies on the corresponding integral equation (4). More precisely, we define an energy functional on $L^p(\mathbb{R}^N)$ such that (13) is reformulated as the corresponding Euler-Lagrange equation for $v := Q \hat{v} |u|^{-2} u$. Here $p' := \frac{p}{p-2}$ denotes the conjugate exponent of $p$. This approach is inspired, in particular, by [3] [17]. We also complete the proof of Theorem 1.3 in this section. In Section 5 we then consider the case where the coefficient $Q$ satisfies $Q(x) \to 0$ as $|x| \to \infty$, and we show that in this case the dual energy functional satisfies the Palais-Smale condition. We then apply a variant of the symmetric mountain-pass lemma in order to complete the proof of Theorem 1.2. In Section 6 we consider the case of periodic $Q$. In this case, the dual energy functional also has the mountain-pass geometry, but it does not satisfy the Palais-Smale condition anymore. Nevertheless, the existence of a bounded Palais-Smale sequence can be shown, and it remains to show that a subsequence converges strongly in $L^{p'}_{loc}(\mathbb{R}^N)$ to a nontrivial critical point of the functional. By this we complete the proof of Theorem 1.1. Finally, in the appendix, we add a result, based on standard elliptic estimates, on the Sobolev regularity imposed by the resolvent operator $\mathcal{R}$.

We close this introduction by fixing some notation. Throughout the paper, we let $B_R(x)$ denote the open ball of radius $R$ centered at $x$, and we also set $B_R := B_R(0)$ and $M_R = \mathbb{R}^N \setminus B_R$. As already mentioned, we put $\hat{x} = \frac{x}{|x|}$ for $x \in \mathbb{R}^N \setminus \{0\}$. The symbols $\mathcal{S}$ and $\mathcal{S}'$ respectively denote the Schwartz space and the space of tempered distributions on $\mathbb{R}^N$. For $f \in \mathcal{S}'$, we write $\mathcal{F}(f)$ or $\hat{f}$ to denote the Fourier transform of $f$. Moreover, for matters of simplicity, we sometimes write $\| \cdot \|$ instead of $\| \cdot \|_{L^s(\mathbb{R}^N)}$ for $s \in [1, \infty]$.

Throughout the remainder of the paper, we restrict our attention to the case $k = 1$. This leads to less complicated formulas in the derivations, and the general case follows from the scaling properties of (3) and the linear (homogeneous and inhomogeneous) Helmholtz equation.

2. Resolvent estimates and far field asymptotics

Throughout this section, we regard all function spaces as spaces of complex-valued functions. Let $\varepsilon > 0$. Then the operator $-\Delta - (1 + i \varepsilon) : H^2(\mathbb{R}^N) \subset L^2(\mathbb{R}^N) \to L^2(\mathbb{R}^N)$ is an isomorphism. Moreover, for any $f$ from the Schwartz space $\mathcal{S}$ its inverse is given by

$$\mathcal{R}_\varepsilon f(x) := [-\Delta - (1 + i \varepsilon)]^{-1} f(x) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} e^{ix \cdot \xi} \frac{\hat{f}(\xi)}{\xi^2 - (1 + i \varepsilon)} d\xi.$$
It is well known (see e.g. [13]) that there exists a linear operator \( R : \mathcal{S} \to \mathcal{S} \) given by
\[
(\mathcal{R}f, g) := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^N} [\mathcal{R}x f](x) g(x) \, dx = \int_{\mathbb{R}^N} [\Phi \ast f](x) g(x) \, dx \quad \text{for } f, g \in \mathcal{S}
\]
with
\[
\Phi(x) := (2\pi)^{-\frac{N}{2}} \mathcal{F}^{-1}((|\xi|^2 - 1 - i0)^{-1})(x) = \frac{i}{4} (2\pi|x|)^{\frac{2-N}{2}} H^{(1)}_{\frac{N-2}{2}}(|x|)
\]
for \( x \in \mathbb{R}^N \setminus \{0\} \). Here we use the notation from [13], which also allows us to briefly write
\[
\mathcal{R}f := \mathcal{S}^{-1}((|\xi|^2 - 1 - i0)^{-1}f) \quad \text{for } f \in \mathcal{S}.
\]

For the Hankel function \( H^{(1)}_{\frac{N-2}{2}} \) we have the asymptotic expansions
\[
H^{(1)}_{\frac{N-2}{2}}(s) = \begin{cases} \sqrt{\frac{2}{\pi s}} e^{i(s - \frac{N-2}{2} \pi)} [1 + O(s^{-1})] & \text{as } s \to \infty, \\ -i \frac{\Gamma(\frac{N-2}{2})}{\pi} \left( \frac{2}{s} \right)^{\frac{N-2}{2}} [1 + O(s)] & \text{as } s \to 0^+ \end{cases}
\]
(see e.g. [20] Section 5.4), so there exists a constant \( C_0 > 0 \) such that
\[
|\Phi(x)| \leq C_0 \max(|x|^{2-N}, |x|^{\frac{2-N}{2}}) \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.
\]

Moreover, \( \Phi \) satisfies the equation \( -\Delta \Phi - \Phi = \delta \) together with Sommerfeld’s outgoing radiation condition
\[
|\nabla \Phi(x) - i\Phi(x)\hat{x}| = o(|x|^{-\frac{N-2}{2}}) \quad \text{as } |x| \to \infty.
\]

As a consequence, for \( f \in \mathcal{S} \), the function \( u = \mathcal{R}f \in \mathcal{S}'(\mathbb{R}^N) \) is a solution of the inhomogeneous Helmholtz equation \( -\Delta u - u = f \) satisfying \( |u(x)| = O(|x|^{-\frac{N-2}{2}}) \) as \( |x| \to \infty \) and the outgoing radiation condition [14] with \( u \) in place of \( \Phi \). We also have the following important estimates.

**Theorem 2.1.** (Special case of [13] Theorem 2.3) by Kenig, Ruiz and Sogge

Let \( \frac{2(N+1)}{(N-1)} \leq p \leq \frac{2N}{N-2} \). Then there exists a constant \( C > 0 \) such that
\[
\|\mathcal{R}f\|_{L^p(\mathbb{R}^N)} \leq C\|f\|_{L^{p'}(\mathbb{R}^N)} \quad \text{for all } f \in \mathcal{S}.
\]

As a consequence of this estimate, the operator \( \mathcal{R} \) can be continuously extended as a mapping from \( L^{p'}(\mathbb{R}^N) \) into \( L^p(\mathbb{R}^N) \) for \( \frac{2(N+1)}{(N-1)} \leq p \leq \frac{2N}{N-2} \) such that [15] still holds for \( f \in L^{p'}(\mathbb{R}^N) \).

**Theorem 2.2.** (Limit case of [16] Theorem 7) by Gutiérrez

Let \( \frac{2(N+1)}{(N-1)} \leq p \leq \frac{2N}{N-2} \). Then there exists a constant \( C > 0 \) such that
\[
\sup_{R \geq 1} \frac{1}{R} \int_{B_R} |\mathcal{R}f(x)|^2 \, dx \leq C\|f\|_{L^{p'}(\mathbb{R}^N)}^2 \quad \text{for all } f \in \mathcal{S}.
\]

We also recall the Stein-Tomas-Theorem which is fundamental for the study of the operator \( \mathcal{R} \).

**Theorem 2.3.** (Stein-Tomas-Theorem [26])

Let \( 1 \leq q \leq \frac{2(N+1)}{(N+3)} \). Then there exists a constant \( C(q) > 0 \) such that for every \( u \in L^q(\mathbb{R}^N) \) we have \( u|_{S^{N-1}} \in L^2(S^{N-1}) \) and
\[
\int_{S^{N-1}} |\hat{u}(\theta)|^2 \, d\theta \leq C\|u\|_{L^q(\mathbb{R}^N)}^2.
\]
For general solutions of $-\Delta u - u = f$ in $\mathbb{R}^N$ we will show that some kind of asymptotic expansion holds. Let us first recall the following consequence of results by Agmon.

**Theorem 2.4. (Agmon [1])**
Let $2 \leq p \leq \frac{2N}{N-2}$ and consider a (distributional) solution $w \in L^p(\mathbb{R}^N)$ of

$$
-\Delta w - w = 0 \quad \text{in } \mathbb{R}^N.
$$

Then there is a unique $g \in H^{-\frac{1}{2}}(S^{N-1})$ such that

$$
w(x) = \int_{S^{N-1}} e^{i\xi \cdot x} g(\xi) \, d\sigma(\xi), \quad x \in \mathbb{R}^N.
$$

If, in addition, $\sup_{R>1} \frac{1}{R} \int_{B_R} |w|^2 \, dx < \infty$, then $g \in L^2(S^{N-1})$, and the following asymptotic expansion is valid.

$$
\lim_{R \to \infty} \frac{1}{R} \int_{B_R} |w(x) - \left(\frac{2\pi}{|x|}\right)^{\frac{N-1}{2}} \left[e^{i|x|\cdot \frac{\xi}{|x|}} g(\xi) + e^{-i|x|\cdot \frac{\xi}{|x|}} g(-\xi)\right]|^2 \, dx = 0
$$

**Proof.** The first assertion follows from Theorem 4.1 and Theorem 6.2 (ii)(a) in [2] with $s = \frac{1}{2}$, whereas the second assertion can be deduced from Theorem 4.3 and Theorem 4.5 (ii) in [1].

Remark that for $u \in L^p(\mathbb{R}^N), f \in L^q(\mathbb{R}^N)$ satisfying $-\Delta u - u = f$ in $\mathbb{R}^N$, the function $w = u - \mathcal{R} f$ belongs to $L^p(\mathbb{R}^N)$, by Theorem 2.4 and solves $-\Delta w - w = 0$ in $\mathbb{R}^N$. Hence, we have the following

**Corollary 2.5.** Let $u \in L^p(\mathbb{R}^N)$ be a solution of $-\Delta u - u = f$ in $\mathbb{R}^N$ with $f \in L^q(\mathbb{R}^N)$ and $\frac{2}{p} + \frac{Q}{q} = \frac{N}{N-1}$. Then there is a unique $g \in H^{-\frac{1}{2}}(\mathbb{R}^N)$ such that

$$
u(x) = [\mathcal{R} f](x) + \int_{S^{N-1}} e^{i\xi \cdot x} g(\xi) \, d\sigma(\xi), \quad \text{for a.e. } x \in \mathbb{R}^N.
$$

Moreover, if $\sup_{R>1} \frac{1}{R} \int_{B_R} |u|^2 \, dx < \infty$, then $g \in L^2(S^{N-1})$.

In order to obtain an asymptotic expansion for $u$, it remains to study the asymptotics of $\mathcal{R} f$ where $f \in L^q(\mathbb{R}^N)$. The remainder of this section is therefore devoted to the far field pattern associated with the operator $\mathcal{R}$. The following result is well-known to experts, and it is stated in [1], Theorem 2.5 for the case $N = 3$. Since we could not find reference for the general case $N \geq 3$, we give the proof here for the reader's convenience.

**Proposition 2.6.** Let $f \in C_c^{\infty}(\mathbb{R}^N)$. Then,

$$
[\mathcal{R} f](x) = \sqrt{\frac{\pi}{2}} \frac{e^{i|x|\cdot \frac{(N-3)\pi}{4\pi}}}{|x|^{\frac{N+1}{2}}} f(\hat{x}) + O\left(|x|^{-\frac{N+1}{2}}\right) \quad \text{as } |x| \to \infty.
$$

**Proof.** Choose $R > 0$ such that $\text{supp} \, f \subset B_R$. For $x \in \mathbb{R}^N$ with $|x| \geq 2R$ we can write, using the asymptotics in [12],

$$
[\mathcal{R} f](x) = \gamma_N \int_{B_R} \frac{e^{i|x-y|}}{|x-y|^\frac{N+1}{2}} \left(1 + \delta(|x-y|)\right) f(y) \, dy,
$$

where the function $r \mapsto \delta(r)$ satisfies $\delta_r := \sup_{r \geq 1} |\delta(r)| < \infty$. Furthermore, there exists a constant $\zeta > 0$ such that

$$
||x-y|-|x| \pm \hat{x} \cdot y| \leq \frac{\zeta |y|^2}{|x|} \quad \text{and} \quad ||x-y| - \frac{\hat{x} \cdot y}{|x|}| \leq \frac{\zeta |y|}{|x|^{\frac{N+1}{2}}}
$$
for all \( x, y \in \mathbb{R}^N \) with \( x \neq 0 \) and \( |y| \leq \frac{|x|}{2} \). As a consequence, we may estimate, for \( |x| \geq 2R \),
\[
\left| \int_{B_R} \left( e^{i|x-y|} - e^{i|x|-|x+y|} \right) f(y) dy \right| \leq \int_{B_R} \left| e^{i(|x-y|-|x+y|)} - \frac{1}{|x|} \right| |f(y)| dy \\
\leq \frac{1}{|x|} \int_{B_R} \left| e^{i(|x-y|) - i|x+y|} \right| + \left| \frac{1}{|x-y|} - \frac{1}{|x|} \right| |f(y)| dy \\
\leq \frac{\sqrt{2N} |y|^2 + \sqrt{N} |y|}{|x|} \left| f(y) \right| \leq \frac{\kappa R \| f \|_{L^1(\mathbb{R}^N)}}{|x|}^{(N+1)}
\]
with \( \kappa_R := (2^{N+1}\sqrt{2R^2} + R)\zeta \) and
\[
\left| \int_{B_R} \left| e^{i|x-y|} \right| \delta(|x-y|) f(y) dy \right| \leq \delta_* \int_{B_R} \frac{|f(y)|}{|x-y|^{(N+1)}} \, dy \leq \frac{2^{N+1}\delta_*}{|x|^{N+1}} \| f \|_{L^1(\mathbb{R}^N)}.
\]
Combining these two estimates, we obtain

\[
\left| \hat{\mathcal{A}} f(x) - \gamma_N \frac{e^{i|x|}}{|x|} \left( 2\pi \right)^{\frac{N}{2}} \hat{f}(\tilde{x}) \right| \leq |\gamma_N| \left( \kappa_R + 2 \frac{(N+1)}{(N+1)} \delta_* \right) \| f \|_{L^1(\mathbb{R}^N)} |x|^{-N+1}
\]
for \( |x| \geq 2R \), and the conclusion follows. \( \square \)

The pointwise asymptotic expansion given in Proposition 2.6 does not extend to general functions \( f \in L^p(\mathbb{R}^N) \) in the case where \( \frac{2(N+1)}{(N+1)} \leq p \leq \frac{N}{N-2} \). Nevertheless, we have the following weaker variant of these asymptotics.

**Proposition 2.7.** Let \( \frac{2(N+1)}{(N+1)} \leq p \leq \frac{2N}{N+1} \) and \( f \in L^p(\mathbb{R}^N) \). Then

\[
\lim_{R \to \infty} \frac{1}{R} \int_{B_R} \left| \hat{\mathcal{A}} f(x) - \sqrt{N} \frac{e^{i|x|}}{|x|} \left( 2\pi \right)^{\frac{N}{2}} \hat{f}(\tilde{x}) \right|^2 \, dx = 0.
\]

**Proof.** As a consequence of Theorems 2.1 and 2.3, the integrand in (21) belongs to \( L^2_{\text{loc}}(\mathbb{R}^N) \). Letting \( \kappa_N = \sqrt{N} \frac{e^{i|x|}}{|x|} \gamma_N \left( 2\pi \right)^{\frac{N}{2}} \hat{f}(\tilde{x}) \), it follows that \( w \in L^2(B_1(0)) \) and therefore

\[
\frac{1}{R} \int_{\{ |x| < 1 \}} \left| \hat{\mathcal{A}} f(x) - \kappa_N \frac{e^{i|x|}}{|x|} \hat{f}(\tilde{x}) \right|^2 \, dx = \frac{1}{R} \| w \|_{L^2(B_1(0))}^2 \to 0 \quad \text{as } R \to \infty.
\]
Let now \( \varepsilon > 0 \) be given, and consider \( g \in \mathcal{C}_c^\infty(\mathbb{R}^N) \) such that \( \| g - f \|_{L^p(\mathbb{R}^N)} < \varepsilon \). By Proposition 2.6 we have

\[
M_g := \sup_{|x| \geq 1} |x|^{N+1} \left| \mathcal{A} g(x) - \kappa_N \frac{e^{i|x|}}{|x|} \hat{g}(\tilde{x}) \right|^2 < \infty
\]
and therefore

\[
\frac{1}{R} \int_{\{ 1 \leq |x| < R \}} \left| \mathcal{A} g(x) - \kappa_N \frac{e^{i|x|}}{|x|} \hat{g}(\tilde{x}) \right|^2 \, dx \\
\leq \frac{M_g}{R} \int_{\{ 1 \leq |x| < R \}} |x|^{-(N+1)} \, dx \leq \frac{M_g \omega_N}{R} \to 0,
\]
as \( R \to \infty \). According to Theorem 2.2,

\[
\sup_{R \geq 1} \frac{1}{R} \int_{B_R} \left| \mathcal{A} f(x) - \mathcal{A} g(x) \right|^2 \, dx < C \varepsilon^2
\]
integral

Proof. Using the asymptotic property (12), we can find a constant $\kappa$ and consider for $\varepsilon > 0$. Combining these estimates, we find

$\limsup R \to \infty \frac{1}{R} \int_{B_R} |\mathcal{H}(x) - \kappa_N \frac{e^{i|x|}}{|x|^{-2}} \hat{f}(\hat{x})|^2 \, dx \leq \limsup R \to \infty \frac{3}{R} \int_{B_R} |\mathcal{H}(x) - \mathcal{H}(\hat{x})|^2 \, dx + \limsup R \to \infty \frac{3|\kappa_N|^2}{R} \int_{B_R} \left| \frac{e^{i|x|}}{|x|^{-2}} \left( \hat{f}(\hat{x}) - \hat{g}(\hat{x}) \right) \right|^2 \, dx$.

Since this holds for every $\varepsilon > 0$, (21) follows. $\square$

In the case where the function $f$ is integrable and exhibits some appropriate decay at infinity, a pointwise asymptotic expansion can also be obtained.

Proposition 2.8. Let $f \in L^1(\mathbb{R}^N)$ satisfy $|f(x)| \leq C|x|^{-N-\varepsilon}$ for all $x \neq 0$ for some $C, \varepsilon > 0$. Then,

$$(\mathcal{H}f)(x) = \sqrt{\frac{\pi}{2}} e^{\frac{(N-3)\varepsilon}{4}} \hat{f}(\hat{x}) + o(|x|^{-\frac{N+1}{2}}), \quad \text{as } |x| \to \infty.$$  

Proof. In order to prove the asymptotic expansion for $\mathcal{H}f = \Phi * f$ we split the integral

$$(\Phi * f)(x) = C_N \int_{\mathbb{R}^N} \frac{H_{\frac{N+2}{2}}(|x-y|)}{|x-y|^{\frac{N+2}{2}}} f(y) \, dy,$$

where $C_N = \frac{i}{4}(2\pi)^{-\frac{2N}{2}}$, into three parts and treat each of them separately. Let us first consider for $x \in \mathbb{R}^N$ with $|x| \geq 2$,

$$I_1(x) = \int_{B_1(x)} \frac{H_{\frac{N+2}{2}}(|x-y|)}{|x-y|^{\frac{N+2}{2}}} f(y) \, dy.$$  

Using the asymptotic property (12), we can find a constant $\kappa_1 > 0$ such that $|H_{\frac{N+2}{2}}(|x-y|)| \leq \kappa_1 |x-y|^{-\frac{2N}{2}}$ for all $x, y \in \mathbb{R}^N$ with $|x-y| < 1$. Therefore, using the decay property of $f$, we find

$$|I_1(x)| \leq \kappa_1 \int_{B_1(x)} |x-y|^{2-N} |f(y)| \, dy \leq \kappa_1 \int_{B_1(x)} |x-y|^{2-N} |y|^{-N-\varepsilon} \, dy \leq 2^N \kappa_1 |x|^{-N-\varepsilon}.$$  

Next, we set $A(x) = \{y \in \mathbb{R}^N : |x-y| > 1 \text{ and } |y| \geq \sqrt{|x|}\}$ and consider

$$I_2(x) = \int_{A(x)} \frac{H_{\frac{N+2}{2}}(|x-y|)}{|x-y|^{\frac{N+2}{2}}} f(y) \, dy.$$  

with a constant $C > 0$ independent of $g$. Moreover, by Theorem 2.3 we have

$$\sup_{R \geq 1} \frac{1}{R} \int_{B_R} \left| e^{i|x|} \left( \hat{f}(\hat{x}) - \hat{g}(\hat{x}) \right) \right|^2 \, dx = \sup_{R \geq 1} \frac{1}{R} \int_{0}^{R} \int_{S^{N-1}} |(f-g)(\omega)|^2 \, d\sigma(\omega) \, d\tau \leq C(p)||f-g||_{L^p(\mathbb{R}^N)}^2 < C(p)\varepsilon^2.$$

Since $0, (21)$ follows.
Hence, combining these last two estimates, we obtain
\[
|I_2(x)| \leq \kappa_2 \int_{A(x)} |x - y|^{\frac{1-N}{2}} |f(y)| \, dy
\]
\[
\leq \kappa_2 \left( \frac{|x|}{2} \right)^{\frac{1-N}{2}} \int_{A(x)} \frac{|x - y|^{\frac{N-1}{2}} + |y|^{\frac{N-1}{2}}}{|x - y|^{\frac{N-1}{2}}} |f(y)| \, dy
\]
\[
\leq \kappa_2 \left( \frac{|x|}{2} \right)^{\frac{1-N}{2}} \left( \int_{A(x)} |f(y)| \, dy + C \int_{A(x)} |x - y|^{\frac{1-N}{2}} |y|^\left(\frac{(N+1+2\varepsilon)}{2}\right) \, dy \right).
\]

Since \( f \in L^1(\mathbb{R}^N) \), the first integral on the last line goes to zero uniformly as \( |x| \to \infty \), and using [3], Appendix 2, Lemma 1], we obtain
\[
\int_{A(x)} |x - y|^{\frac{1-N}{2}} |y|^\left(\frac{(N+1+2\varepsilon)}{2}\right) \, dy \to 0, \quad \text{uniformly as } |x| \to \infty.
\]

Hence, \( I_2(x) = o(|x|^{\frac{N-1}{2}}) \) as \( |x| \to \infty \).

In a last step, we study for \( x \in \mathbb{R}^N \) with \( |x| \geq 4 \) the integral
\[
I_3(x) = \int_{D(x)} \frac{H_{\frac{1}{2}}^0(|x - y|)}{|x - y|^{\frac{N-1}{2}}} f(y) \, dy,
\]
where \( D(x) = \{ y \in \mathbb{R}^N : |x - y| > 1 \text{ and } |y| \leq \sqrt{|x|} \} \). We first notice that by (12) we can write, as in the proof of Proposition 2.6,
\[
I_3(x) = \gamma_N \int_{D(x)} \frac{e^{i|x-y|}}{|x - y|^{\frac{N+1}{2}}} \left( 1 + \delta(|x - y|) \right) f(y) \, dy,
\]
where \( \gamma_N = \frac{(2\pi)^{\frac{N}{2}} e^{-\frac{1}{2}}}{2} \) and \( \delta_* := \sup_{r \geq 1} \delta(r) \) < \( \infty \). As a consequence, we obtain as in Proposition 2.6 using (20),
\[
\left| \int_{D(x)} \frac{e^{i|x-y|}}{|x - y|^{\frac{N+1}{2}}} - e^{i|x|} \right| f(y) \, dy \right| \leq \int_{D(x)} \left( \frac{\sqrt{2\zeta}|y|^2}{|x| |x - y|^{\frac{N-1}{2}}} + \frac{\zeta |y|}{|x|^{\frac{N-1}{2}}} \right) |f(y)| \, dy
\]
\[
\leq \left( \frac{2^{\frac{N+1}{2}} \sqrt{2} + 1}{2} \right) \int_{\{ y \in \mathbb{R}^N : |y| < \sqrt{|x|} \}} |y|^{-N-\frac{1}{2}} \, dy
\]
\[
\leq \kappa_3 |x|^{-\frac{N-1}{2} - \frac{1}{12}} \| f \|_1
\]
for some constant \( \kappa_3 > 0 \), and, moreover,
\[
\left| \int_{D(x)} \frac{e^{i|x-y|}}{|x - y|^{\frac{N+1}{2}}} \delta(|x - y|) f(y) \, dy \right| \leq \frac{2^{\frac{N+1}{2}} \zeta}{|x|^{\frac{N-1}{2}}} \| f \|_1.
\]

Combining these last two estimates, we obtain
\[
|I_3(x) - \gamma_N \frac{e^{i|x|}}{|x|^{\frac{N+1}{2}}} \int_{D(x)} f(y) \, dy| \leq (\kappa_3 + \frac{2^{\frac{N+1}{2}} \zeta}{|x|^{\frac{N-1}{2}}} \| f \|_1) |x|^{-\frac{N-1}{2} - \frac{1}{12}}.
\]
and using the fact that,
\[
\left| \int_{\mathbb{R}^N \setminus B_r(x)} e^{-i\xi \cdot y} f(y) \, dy \right| \leq \int_{B_1(x)} |f(y)| \, dy + \int_{\{ |y| \geq \sqrt{|x|} \}} |f(y)| \, dy \to 0,
\]
uniformly as $|x| \to \infty$, we can write
\[
I_3(x) = \gamma_N \frac{e^{i|x|}}{|x|^N} (2\pi)^{\frac{N}{2}} \hat{f}(\hat{x}) + o(|x|^{-(N-1)}), \quad \text{as } |x| \to \infty,
\]
and the claim follows. \qed

As a consequence of Corollary 2.4 and the above expansions, we obtain that in the case where \( \frac{2(N+1)}{N+1} \leq p \leq \frac{2N}{N-2} \) and \( f \in L^p(\mathbb{R}^N) \) is real-valued, there exists for each solution \( u \in L^p(\mathbb{R}^N) \) of
\[
R = \mathcal{R}(f) = \text{Re}(\mathcal{R} f)
\]
a unique \( g \in L^2(S^{N-1}) \) such that
\[
u(x) = \mathcal{R}(f)(x) + \int_{S^{N-1}} e^{ix \cdot \xi} g(\xi) \, d\sigma(\xi) \quad \text{for a.e. } x \in \mathbb{R}^N,
\]
and a direct identification gives
\[
\int_{S^{N-1}} e^{ix \cdot \xi} g(\xi) \, d\sigma(\xi) = -i \text{Im}(\mathcal{R} f)(x) = \frac{1}{2} \left[ [\mathcal{R} f](x) - |\mathcal{R} f|(x) \right].
\]
Comparing the expansions in (18) and (21), we find
\[
g = -\frac{i}{4} (2\pi)^{\frac{2-N}{2}} \hat{f} \in L^2(S^{N-1}).
\]

We conclude this study of the operator \( \mathcal{R} \) by stating a result on the asymptotic decay of solutions of convolution equations in which the kernel has the same asymptotics as the fundamental solution \( \Phi \) above. We shall use this result in Section 4 below in order to obtain a pointwise asymptotic expansion for real-valued solutions of the nonlinear Helmholtz equation.

Lemma 2.9. Let \( V \in L^q(\mathbb{R}^N) \cap L^s(\mathbb{R}^N) \) with \( q < \frac{2N}{N+1} \leq \frac{N}{2} < s \) and consider a measurable function \( u: \mathbb{R}^N \to \mathbb{R} \) satisfying \( Vu \in L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N) \) and
\[
u = K * (Vu),
\]
where \( |K(x)| \leq C_0 \max\{|x|^{-\frac{N+1}{2}}, |x|^{2-N} \} \) for \( x \neq 0 \). Then there exists a constant \( C > 0 \) such that
\[
|u(x)| \leq C|x|^{-\frac{N-1}{2}} \quad \text{for all } x \neq 0.
\]

Proof. The proof is based on an iteration procedure, similar to the one used by Zemach and Odeh in [28] (see also [4]). We start by remarking that Hölder’s inequality gives for \( \sigma \in \left\{ \frac{N-1}{2}, N-2 \right\} \), \( R > 0 \) and \( |x| \geq R \),
\[
\int_{M_R} |V(y)||x-y|^{-\sigma} \, dy \leq \left( \int_{M_R} |V(y)|^q \, dy \right)^\frac{1}{q} \left( \int_{B_1(x)} |x-y|^{-s' \sigma} \, dy \right)^\frac{1}{s'},
\]
\[
+ \left( \int_{M_R} |V(y)|^q \, dy \right)^\frac{1}{q} \left( \int_{\mathbb{R}^N \setminus B_1(x)} |x-y|^{-q' \sigma} \, dy \right)^\frac{1}{q'},
\]
which tends to 0, as $R \to \infty$, uniformly in $x$. Indeed, our assumptions ensure that $s' = \frac{s}{s-1} < \frac{N}{N-2}$ and $q' > \frac{2N}{N-1}$, and therefore $s' \sigma < N$ whereas $q' \sigma > N$. From now on, we choose $R > 1$ such that

$$C_0 \sup_{|x| > R} \int_{M_R} |V(y)| \left( |x-y|^{\frac{1-N}{2}} + |x-y|^{2-N} \right) dy < 2^{-N},$$

Letting

$$u_0(x) = \int_{B_R} K(x-y)V(y)u(y) dy,$$

we infer that for all $|x| \geq 2R$,

$$|u_0(x)| \leq C_0 \int_{B_R} |x-y|^{\frac{1-N}{2}} |V(y)u(y)| dy \leq C_1 |x|^{\frac{1-N}{2}},$$

with $C_1 = 2^{N-1}C_0 \|Vu\|_1$. We now set for $|x| \geq R$,

$$B_0(x) = \int_{M_R} K(x-y)V(y)u(y) dy$$

and define inductively for $k \geq 1$,

$$u_k(x) = \int_{M_R} K(x-y)V(y)u_{k-1}(y) dy,$$

and

$$B_k(x) = \int_{M_R} K(x-y)V(y)B_{k-1}(y) dy.$$

Thus, for each $m \in \mathbb{N}$,

$$u = \sum_{k=0}^{m} u_k + B_m.$$

Setting $\beta_k = \sup_{|x| \geq R} |B_k(x)|$, a similar calculation as above gives $\beta_0 < \infty$, since $Vu \in L^1(\mathbb{R}^N) \cap L^s(\mathbb{R}^N)$. Moreover, using (23), we obtain $\beta_k \leq 2^{-N} \beta_{k-1}$ for all $k \geq 1$, and thus $\beta_m \leq 2^{-mN} \beta_0 \to 0$ as $m \to \infty$, showing that

$$u = \sum_{k=0}^{\infty} u_k$$

holds uniformly in $M_R$. Next, we remark that $\mu_0 := \sup_{|x| \geq R} |x|^{\frac{N-1}{2}} |u_0(x)| < \infty$ (notice that $u_0 \in \mathcal{L}^\infty(\mathbb{R}^N)$). Hence, setting $\mu_k = \sup_{|x| \geq R} |x|^{\frac{N-1}{2}} |u_k(x)|$ for $k \geq 1$, we obtain

$$|x|^{\frac{N-1}{2}} |u_k(x)| \leq \mu_{k-1} C_0 |x|^{\frac{N-1}{2}} \int_{M_R} |V(y)| |y|^{\frac{1-N}{2}} \left( |x-y|^{\frac{1-N}{2}} + |x-y|^{2-N} \right) dy \leq \mu_{k-1} 2^{\frac{N-3}{2}} C_0 \int_{M_R} |V(y)|(1+|y|^{\frac{1-N}{2}}) \left( |x-y|^{\frac{1-N}{2}} + |x-y|^{2-N} \right) dy \leq \mu_{k-1} 2^{\frac{N-3}{2}} C_0 \int_{M_R} |V(y)|(1+2^{\frac{N-3}{2}}) \left( |x-y|^{\frac{1-N}{2}} + |x-y|^{2-N} \right) dy \leq \frac{1}{2} \mu_{k-1},$$

for all $|x| \geq R$, where the triangle inequality and (23) have been used. Iterating the preceding estimate, we obtain $\mu_k \leq 2^{-k} \mu_0$ and consequently,

$$\sup_{|x| \geq R} |x|^{\frac{N-1}{2}} |u(x)| \leq \mu_0 \sum_{k=0}^{\infty} 2^{-k} = 2 \mu_0 < \infty,$$

which concludes the proof. \qed
Theorem 3.1. \textit{(nonvanishing property)}

Let \( N \geq 3 \) and \( \frac{2(N+1)}{N} < p < \frac{2N}{N-2} \). Moreover, let \( (v_n)_n \subset L^p(\mathbb{R}^N) \) be a bounded sequence satisfying \( \limsup_{n \to \infty} \int_{\mathbb{R}^N} |v_n|^p dx > 0 \). Then there exists \( R > 0 \), \( \zeta > 0 \) and a sequence \( (x_n)_n \subset \mathbb{R}^N \) such that, up to a subsequence,

\[
\int_{B_R(x_n)} |v_n|^p dx \geq \zeta \quad \text{for all } n.
\]

The remainder of this section is devoted to the proof of this result. We fix \( \psi \in \mathcal{S} \) such that \( \hat{\psi} \in \mathcal{C}^\infty_0(\mathbb{R}^N) \) is radial, \( 0 \leq \hat{\psi} \leq 1 \), \( \hat{\psi}(\xi) = 1 \) for \( ||\xi|| - 1 | \leq \frac{1}{R} \) and \( \hat{\psi}(\xi) = 0 \) for \( ||\xi|| - 1 | \geq \frac{1}{R} \). We then write \( \Phi = \Phi_1 + \Phi_2 \) with

\[
\Phi_1 := \psi \cdot \Phi, \quad \Phi_2 = \Phi - \Phi_1.
\]

From (13) it then follows, by making \( C_0 > 0 \) larger if necessary, that \( \Phi_1 \in \mathcal{C}^\infty_0(\mathbb{R}^N) \) and

\[
|\Phi_1(x)| \leq C_0(1 + |x|)^{\frac{1-N}{2}} \quad \text{for } x \in \mathbb{R}^N.
\]

This in particular implies that \( |\Phi_2(x)| = |(\Phi - \Phi_1)(x)| \leq 2C_0|x|^{2-N} \) for \( |x| \leq 1 \). Moreover, since \( \Phi_2 = (1 - \tilde{\psi})\Phi \), and \( \Phi_2(\xi) = (||\xi||^2 - 1 - i0)^{-1} \) with the notation of (13), we have \( \hat{\Phi}_2 \in \mathcal{C}^\infty_0(\mathbb{R}^N) \) with \( \hat{\Phi}_2(\xi) = (||\xi||^2 - 1)^{-1} \) for \( ||\xi|| \geq \frac{1}{R} \). This implies that \( \partial^\gamma \hat{\Phi}_2 \in L^1(\mathbb{R}^N) \) for all \( \gamma \in \mathbb{N}^N_0 \) such that \( ||\gamma|| > N - 2 \), which gives \( |\Phi_2(x)| \leq \kappa_s|x|^{-s} \), \( x \in \mathbb{R}^N \) for all \( s > N - 2 \) with some constant \( \kappa_s > 0 \). In particular, by making \( C_0 > 0 \) larger if necessary, we have

\[
|\Phi_2(x)| \leq C_0 \min\{|x|^{2-N}, |x|^{-N}\} \quad \text{for } x \in \mathbb{R}^N \setminus \{0\}.
\]

We first prove a variant of Theorem 3.1 related to \( \Phi_2 \).

Lemma 3.2. Let \( 2 < p < \frac{2N}{N-2} \), and suppose that \( (v_n)_n \subset \mathcal{S} \) is a bounded sequence in \( L^p(\mathbb{R}^N) \) such that

\[
\lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_{\rho}(y)} |v_n|^p dx \right) = 0 \quad \text{for all } \rho > 0.
\]

Then

\[
\int_{\mathbb{R}^N} v_n[\Phi_2 * v_n] dx \to 0 \quad \text{as } n \to \infty.
\]

Proof. Setting \( A_R := \{x \in \mathbb{R}^N : \frac{R}{2} \leq |x| \leq R\} \) and \( D_R := \mathbb{R}^N \setminus A_R \) for \( R > 1 \), we derive from (26) that

\[
\|\Phi_2\|_{L^p(A_R)} \to 0 \quad \text{as } R \to \infty,
\]

since \( 1 < \frac{2}{p} < \frac{N}{N-2} \). Hence, by Young’s inequality,

\[
\sup_{n \in \mathbb{N}} \int_{\mathbb{R}^N} v_n[(1_{D_R} \Phi_2) * v_n] dx \leq \|\Phi_2\|_{L^p(A_R)} \sup_{n \in \mathbb{N}} \|v_n\|_{L^p(\mathbb{R}^N)}^2 \to 0 \quad \text{as } R \to \infty.
\]
Moreover, decomposing $\mathbb{R}^N$ into disjoint $N$-cubes $\{Q_x\}_{x \in \mathbb{N}}$ of side length $R$, and considering for each $\ell$ the $N$-cube $Q_\ell$ with the same center as $Q_x$ but with side length $3R$, we find, similarly as in [16, pp. 109-110],

\[
\left| \int_{\mathbb{R}^N} v_n \left[ \mathbf{1}_{A_R} \Phi_2 \right] * v_n \right| dx \leq \sum_{\ell=1}^{\infty} \int_{Q_\ell} \left( \int_{|x-y|<R} \Phi_2(x-y) \left| v_n(x) \right| \left| v_n(y) \right| dy \right) dx \\
\leq CR^{N-2} \sum_{\ell=1}^{\infty} \int_{Q_\ell} \left( \int_{|x-y|<R} \left| v_n(x) \right| \left| v_n(y) \right| dy \right) dx \\
\leq CR^{N-2} \sum_{\ell=1}^{\infty} \int_{Q_\ell} \left| v_n(x) \right|^{p'} dx \\
\leq CR^{N-2} \sum_{\ell=1}^{\infty} \sup_{y \in \mathbb{R}^N} \int_{B_{3R} \cap \mathbb{N}(y)} \left| v_n(x) \right|^{p'} dx \\
\leq CR^{N-2} \sum_{\ell=1}^{\infty} \sup_{y \in \mathbb{R}^N} \left| v_n(x) \right|^{p'} dx^{1-\frac{2}{p}} 3^N \|v_n\|_{p'},
\]

so by assumption (27) we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} v_n \left[ \mathbf{1}_{A_R} \Phi_2 \right] * v_n \ dx = 0 \quad \text{for every } R > 0.
\]

Combining (30) and (31), we thus obtain (28), as claimed. \hfill \square

We also need the following Lemma which is related to [16] Lemma 1] (and the remarks before and after that lemma).

**Proposition 3.3.** Let $p > \frac{2(N+1)}{(N-1)}$, $\lambda_p := \frac{N-1}{2} - \frac{N+1}{p} > 0$ and $M_R := \mathbb{R}^N \setminus B_R$ for $R > 0$. Then there exists a constant $C > 0$ such that, for $R \geq 1$,

\[
\| [1_{M_R} \Phi_1] * f \|_p \leq CR^{-\lambda_p} \| f \|_{p'} \quad \text{for } f \in \mathcal{S} \text{ with } \mathrm{supp} \ f \subset \{ \xi : |\xi| - 1 \leq \frac{1}{2} \}.
\]

**Proof.** It suffices to prove the assertion for $R \geq 4$. Put $P_R := 1_{M_R} \Phi_1$ for $R \geq 4$, and fix a radial, nonnegative function $\eta \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ such that $\eta(x) = 1$ if $0 \leq |x| \leq 1$, $\eta(x) = 0$ if $|x| \geq 2$. Moreover, for $j \in \mathbb{N}$, define $\varphi_j \in \mathcal{C}_c^\infty(\mathbb{R}^N)$ by $\varphi_j(x) = \eta(x/2^j) - \eta(x/2^{j-1})$. Since $P_R \equiv 0$ on $B_R$, we then have the corresponding dyadic decomposition

\[
P_R = \sum_{j=\lfloor \log_2 R \rfloor}^{\infty} P_j \quad \text{with } P_j(x) := P_R(x) \varphi_j(x) \text{ for } j \in \mathbb{N},
\]

so that

\[
\| P_j \|_\infty \leq C_1 2^{-\frac{(N+1)}{2}} \quad \text{for all } j \geq \lfloor \log_2 R \rfloor
\]

with $C_1 := 2^{\frac{N+1}{2}} C_0$ and $C_0 > 0$ as in (25). In the following, the letter $C$ stands for (possibly different) positive constants independent of $R$. We claim that

\[
\| P_j * f \|_2 \leq C 2^\frac{j}{2} \| f \|_2 \quad \text{for all } j \geq \lfloor \log_2 R \rfloor
\]

for $j \geq \lfloor \log_2 R \rfloor$ and $f \in \mathcal{S}$ with $\mathrm{supp} \ f \subset \{ \xi : |\xi| - 1 \leq \frac{4}{2} \}$. This follows almost exactly as in the proof of [16] Lemma 1], but we repeat the argument for the convenience of the reader. Since $P_j$ is a radial function, we also write $P_j(r)$ in place of $P_j(\xi)$ if $|\xi| = r$ in the following. Setting $q = \frac{2(N+1)}{(N+3)}$, we then have, using
Theorem 2.3
\[
\|P_j * f\|_2^2 = C \int_{|\xi| \leq \frac{1}{2}} |\hat{P}_j(\xi)\hat{f}(\xi)|^2 d\xi = C \int_{\frac{1}{2}}^{\frac{3}{4}} r^{N-1} |\hat{P}_j(r)|^2 \int_{S^{N-1}} |\hat{f}(r\omega)|^2 d\sigma(\omega) dr \\
\leq C \|f\|_q^2 \int_{\frac{1}{2}}^{\frac{3}{4}} |\hat{P}_j(r)|^2 dr \leq C \|f\|_q^2 \int_{\mathbb{R}^N} |P_j(x)|^2 dx \leq C 2^j \|f\|_q^2.
\]
Hence (34) holds. We now fix \( \varphi \in \mathcal{S} \) such that \( \hat{\varphi} \in C^\infty_c(\mathbb{R}^N) \) is radial, nonnegative and such that \( \hat{\varphi}(\xi) \equiv 1 \) on \( \{ \xi : ||\xi| - 1| \leq \frac{1}{2} \} \) and \( \hat{\varphi}(\xi) \equiv 0 \) on \( \{ \xi : ||\xi| - 1| \geq \frac{3}{4} \} \).

We also define \( Q^j := P^j * \varphi \). As a consequence of (34), we then have
\[
\|Q^j * f\|_2 = \|P^j * (\varphi * f)\|_2 \leq C 2^j \|\varphi * f\|_{2(\frac{N+1}{N+1})} \leq C 2^j \|f\|_{2(\frac{N+1}{N+1})}
\]
for all \( f \in \mathcal{S} \) and \( j \geq \lfloor \log_2 R \rfloor \), since \( \text{supp} \hat{\varphi} f = \text{supp} \hat{\varphi} \in \{ \xi : ||\xi| - 1| \leq \frac{3}{4} \} \).

Note that in the last step we used Young’s inequality (replacing \( C \|\varphi\|_1 \) by \( C \)). By duality, this implies that
\[
\|Q^j * f\|_p \leq C 2^j \|f\|_p
\]
and \( j \geq \lfloor \log_2 R \rfloor \). Setting \( r = \frac{2(N+1)}{N} \), so that \( \frac{1}{2} = \frac{1}{2} \left( \frac{1}{2} + \frac{N-1}{2(N+1)} \right) \), we therefore obtain by complex interpolation that
\[
\|Q^j * f\|_{p'} \leq C 2^j \|f\|_{p'}
\]
and \( j \geq \lfloor \log_2 R \rfloor \). By (33) we also have
\[
\|Q^j\|_\infty \leq \|P^j\|_{\infty} \|\varphi\|_1 \leq C_1 \|\varphi\|_1 \left( \frac{1}{2} \right)^{\frac{j(N-1)}{2}}
\]
for \( j \geq \lfloor \log_2 R \rfloor \), so that, by Young’s inequality,
\[
\|Q^j * f\|_\infty \leq C 2^{-\frac{j(N-1)}{2}} \|f\|_1
\]
and \( j \geq \lfloor \log_2 R \rfloor \). Combining (37) and (38) and applying complex interpolation again, we find that
\[
\|Q^j * f\|_p \leq C 2^{j\left( \frac{N+1}{N+1} - \frac{1}{2} \right)} \|f\|_{p'} = C 2^{-j\lambda_p} \|f\|_{p'}
\]
for all \( f \in \mathcal{S} \), \( p \geq r \) and \( j \geq \lfloor \log_2 R \rfloor \). As in the assumption of the proposition, we now restrict our attention to \( p > \frac{2(N+1)}{(N-1)} > r \), so that \( \lambda_p > 0 \). Recalling (32) and using that \( Q^j * f = P^j * f \) for all \( f \in \mathcal{S} \) with \( \text{supp} \hat{f} \subset \{ \xi : ||\xi| - 1| \leq \frac{1}{2} \} \), we then conclude that
\[
\|[1_{M_k \Phi_1} * f]\|_p = \|P_R * f\|_p \leq C \|f\|_p' \sum_{j=\lfloor \log_2 R \rfloor}^{\infty} 2^{-j\lambda_p} \leq C R^{-\lambda_p} \|f\|_{p'}
\]
for all \( f \in \mathcal{S} \) with \( \text{supp} \hat{f} \subset \{ \xi : ||\xi| - 1| \leq \frac{1}{2} \} \), as claimed. \( \square \)

Lemma 3.4. Let \( p > \frac{2(N+1)}{(N-1)} \), and suppose that \( (v_n)_n \subset \mathcal{S} \) is a bounded sequence in \( L^{p'}(\mathbb{R}^N) \) such that
\[
\lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_2(y)} |v_n|^p dx \right) = 0 \quad \text{for all } p > 0.
\]

Then
\[
\int_{\mathbb{R}^N} v_n [\Phi_1 * v_n] dx \to 0 \quad \text{as } n \to \infty.
\]
Proof. Fix a radial function \( \varphi \in \mathcal{S} \) such that \( \hat{\varphi} \in C_c^\infty(\mathbb{R}^N) \) is radial, \( 0 \leq \hat{\varphi} \leq 1 \), and \( \hat{\varphi}(\xi) = 1 \) for \( ||\xi|| - 1 \leq \frac{1}{2} \) and \( \hat{\varphi}(\xi) = 0 \) for \( ||\xi|| - 1 \geq \frac{1}{2} \). Moreover, let \( w_n := \varphi \ast v_n \in \mathcal{S} \). We then have \( \Phi_1 \ast v_n = \Phi_1 \ast w_n \), since \( \Phi_1 \hat{\varphi} = \Phi_1 \) by construction. Hence

\[
\int_{\mathbb{R}^N} v_n[\Phi_1 \ast v_n] \, dx = \int_{\mathbb{R}^N} v_n[\Phi_1 \ast w_n] \, dx
\]

(40)

\[
= \int_{\mathbb{R}^N} v_n[(1_B \Phi_1) \ast w_n] \, dx + \int_{\mathbb{R}^N} v_n[(1_M \Phi_1) \ast w_n] \, dx \quad \text{for every } n.
\]

Since \( \text{supp} \, w_n \subset \{ \xi : ||\xi|| - 1 \leq \frac{1}{2} \} \) for every \( n \in \mathbb{N} \), Proposition 3.3 implies that

\[
\left| \int_{\mathbb{R}^N} v_n[(1_M \Phi_1) \ast w_n] \, dx \right| \leq C \| v_n \|_{P^p} \| w_n \|_{P^1} R^{-\lambda_p} \leq C \| \varphi \|_1 \| v_n \|_{P^p}^2 R^{-\lambda_p}
\]

for every \( n \in \mathbb{N} \), \( R > 4 \), where we used Young’s inequality in the last step. As a consequence

\[
\sup_{n \in \mathbb{N}} \left| \int_{\mathbb{R}^N} v_n[(1_M \Phi_1) \ast w_n] \, dx \right| \to 0 \quad \text{as } R \to \infty.
\]

Moreover, decomposing \( \mathbb{R}^N \) into disjoint \( N \)-cubes \( \{Q_{\ell} \}_{\ell \in \mathbb{N}} \) of side length \( R \), and considering for each \( \ell \) the \( N \)-cube \( Q_{\ell} \) with the same center as \( Q_\ell \) but with side length \( 3R \), we find, arguing slightly differently than in the proof of Lemma 3.2,

\[
\left| \int_{\mathbb{R}^N} v_n[(1_B \Phi_1) \ast w_n] \, dx \right| \leq \sum_{\ell=1}^{\infty} \int_{Q_{\ell}} \left( \int_{B_\ell} |v_n(x)| \, dx \right) \left( \int_{Q_{\ell}} |w_n(y)| \, dy \right) \, dx
\]

\[
\leq \|| \Phi_1 \||_\infty \sum_{\ell=1}^{\infty} \int_{Q_{\ell}} |v_n(x)| \, dx \int_{Q_{\ell}} |w_n(x)| \, dx
\]

\[
\leq \|| \Phi_1 \||_\infty \left( \sum_{\ell=1}^{\infty} \left( \int_{Q_{\ell}} |v_n(x)| \, dx \right)^{\frac{1}{p}} \left( \sum_{\ell=1}^{\infty} \left( \int_{Q_{\ell}} |v_n(x)| \, dx \right)^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \left( \sum_{\ell=1}^{\infty} \left( \int_{Q_{\ell}} |v_n(x)|^{\frac{1}{p'}} \, dx \right)^{\frac{1}{p'}} \left( \sum_{\ell=1}^{\infty} \left( \int_{Q_{\ell}} |v_n(x)|^{\frac{1}{p'}} \, dx \right)^{\frac{1}{p'}} \right)^{\frac{1}{p'}} \right)^{\frac{1}{p'}}
\]

\[
\leq CR^{\frac{1}{p'}} 3N \||w_n\|_{P^p} \left( \sup_{\ell \in \mathbb{N}} \int_{Q_{\ell}} |v_n(x)|^{\frac{1}{p'}} \, dx \right)^{\frac{1}{p'}} \||v_n\|_{P^p}^{\frac{1}{p'}}
\]

\[
\leq CR^{\frac{1}{p'}} 3N \||w_n\|_{P^p} \left( \sup_{y \in \mathbb{R}^N} \int_{B_{3R} \mathbb{R}^N(y)} |v_n(x)|^{\frac{1}{p'}} \, dx \right)^{\frac{1}{p'}} \||v_n\|_{P^p}^{\frac{1}{p'}}
\]

so by assumption we have

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} v_n[(1_B \Phi_1) \ast w_n] \, dx = 0 \quad \text{for every } R > 0.
\]

Combining (40), (41) and (42), we obtain (39), as claimed. \( \Box \)

Proof of Theorem 3.7 (completed). Without loss of generality, we may assume that

\[
v_n \in \mathcal{S} \quad \text{for all } n \in \mathbb{N}.
\]

Indeed, in any case we may replace \( v_n \) by \( \tilde{v}_n \in \mathcal{S} \) with \( \|v_n - \tilde{v}_n\|_{L^{P'}(\mathbb{R}^N)} \leq \frac{1}{n} \) for every \( n \in \mathbb{N} \). Then

\[
\left| \int_{\mathbb{R}^N} [v_n \mathcal{D} \tilde{v}_n - \tilde{v}_n \mathcal{D} v_n] \, dx \right| = \left| \int_{\mathbb{R}^N} (v_n - \tilde{v}_n) \mathcal{D}(v_n + \tilde{v}_n) \, dx \right|
\]
By Lemmas 3.2 and 3.4, we thus find that
\[ M_{\text{Helmholtz equation}}. \]
As before, we set
\[ \lambda \]
in order to derive pointwise asymptotic estimates for the solutions o f the nonlinear
\[ (44) \lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |v_n|^p \, dx \right) = 0 \quad \text{for all } p > 0. \]
By Lemmas 3.2 and 3.3 we thus find that
\[ \int_{\mathbb{R}^N} v_n \nabla v_n \, dx = \int_{\mathbb{R}^N} v_n[\Phi_1 * v_n] \, dx + \int_{\mathbb{R}^N} v_n[\Phi_2 * v_n] \, dx \to 0 \quad \text{as } n \to \infty, \]
contradicting the assumption. Hence (24) is true for some \( \zeta, R > 0 \).
Moreover, (24) holds if and only if it holds for \( \tilde{\Phi} \) as in (15) and thus \( \lim_{n \to \infty} |v_n|^p \, dx \to 0 \).
Suppose by contradiction that (24) does not hold. Then
\[ \lim_{n \to \infty} \left( \sup_{y \in \mathbb{R}^N} \int_{B_r(y)} |v_n|^p \, dx \right) = 0 \quad \text{for all } p > 0. \]
As a consequence of some of the estimates developed in this section we state
\[ \text{Lemma 3.5.} \]
There exists a constant \( C > 0 \) such that
\[ ||\Phi * f||_{L^p(M_{2R^n})} \leq CR^{-\lambda}||f||_{L^p} \]
for every \( R \geq 1 \) and every \( f \in L^p(\mathbb{R}^N) \) with \( f \equiv 0 \) on \( M_R \).
Proof. We first show the estimate holds for \( \Phi_2 \) in place of \( \Phi \). Indeed, for \( R \geq 1 \) and \( f \in L^p(\mathbb{R}^N) \) with \( f \equiv 0 \) on \( M_R \) we have
\[ ||\Phi_2 * f||_{L^p(M_{2R^n})} \leq \left( \int_{|x| \geq 2R} \left( \int_{|y| \leq R} |\Phi_2(x-y)||f(y)| \, dy \right)^p \, dx \right)^{1/p} \]
\[ \leq \left( \int_{\mathbb{R}^N} \left( \int_{|y-x| \geq R} |\Phi_2(x-y)||f(y)| \, dy \right)^p \, dx \right)^{1/p} \]
\[ = ||(1_{M_R} \Phi_2) * f||_p \leq ||1_{M_R} \Phi_2||_{\frac{p}{2}} ||f||_{L^p}. \]
Hence, as a consequence of (20),
\[ ||1_{M_R} \Phi_2||_{\frac{p}{2}} \leq C \left( \int_R^\infty r^{N-1-\frac{2N}{p}} \, dr \right)^{2/p} \leq CR^{\frac{N(p-2)}{p}}. \]
Since \( \frac{N(p-2)}{p} > \lambda_p \), we conclude that the estimate holds for \( \Phi_2 \) in place of \( \Phi \). Hence it remains to prove the estimate for \( \Phi_1 \), whereas, by density, it suffices to consider functions \( f \in \mathcal{C}_c^\infty(B_R) \) only. Fix a radial function \( \varphi \in \mathcal{S} \) such that \( \hat{\varphi} \in \mathcal{C}_c^\infty(\mathbb{R}^N) \) is radial, \( 0 \leq \hat{\varphi} \leq 1, \hat{\varphi}(\xi) = 1 \) for \( ||\xi||_1 \leq \frac{1}{\xi} \) and \( \hat{\varphi}(\xi) = 0 \) for \( ||\xi||_1 \geq \frac{1}{\xi} \).
Moreover, let \( \tilde{f} := \varphi * f \in \mathcal{S} \). We then have \( \Phi_1 * f = \tilde{f} \), hence \( \tilde{f} \) by construction. We now use that
\[ \Phi_1 * f = \Phi_1 * \tilde{f} = [1_{B_{R/2}} \Phi_1] * \tilde{f} + [1_{M_{R/2} \Phi_1}] * \tilde{f}, \]
whereas, setting \( g_R := [1_{B_{R/2}} \Phi_1] * \varphi \), we have
\[ ||[1_{B_{R/2}} \Phi_1] * \tilde{f}||_{L^p(M_{2R^n})} \leq ||g_R * f||_{L^p(M_{2R^n})} \leq ||g_R||_{L^p(M_{2R^n})} \]
\[ = \int_{|x| \geq 2R} \left( \int_{|y| \leq R} |g_R(x-y)||f(y)| \, dy \right)^p \, dx \]
\[ \leq \int_{\mathbb{R}^N} \left( \int_{|y-x| \geq R} |g_R(x-y)||f(y)| \, dy \right)^p \, dx \]
\[ \leq ||1_{M_R} g_R|| ||f||_p \leq \left( ||1_{M_R} g_R||^2 ||f||_p \right)^{p/2}. \]
Using (25) and the fact that $\varphi \in \mathcal{S}$, we may estimate
\[
\|1_{M_R}g\|_2^p \leq C_0 \int_{|x| \geq R} \left( \int_{|y| \leq R/2} |\varphi(x - y)| \, dy \right)^p \, dx
\]
\[
\leq C \int_{|x| \geq R} \left( \int_{|y| \leq R/2} |x - y|^{-m} \, dy \right)^p \, dx \leq C |B_{R/2}|^{\frac{p}{2}} \int_{|x| \geq R} \left( |x| - \frac{R}{2} \right)^{-mp} \, dx
\]
\[
= CR^{(\frac{N-m}{2}+N) - \frac{mp}{2}} \int_{|x| \geq 1} \left( |x| - \frac{1}{2} \right)^{-mp} \, dx = CR^{(\frac{N-m}{2}+N)},
\]
where $m$ may be fixed sufficiently large such that $\frac{(N-m)p}{2} + N \leq -\lambda_p$. It remains to estimate, with the help of Proposition \ref{prop-4.3},
\[
\|1_{M_{R/2}}\Phi_1 \ast f\|_{L^p(\mathbb{R}^N \setminus B_{R})} \leq \|1_{M_{R/2}}\Phi_1 \ast f\|_p \leq CR^{-\lambda_p}\|f\|_p.
\]
Combining these estimates yields the claim. \hfill \Box

4. A DUAL VARIATIONAL FRAMEWORK FOR THE NONLINEAR HELMHOLTZ EQUATION

Throughout this section, we assume that $N \geq 3$, $\frac{2(N+1)}{N-1} \leq p \leq 2^* := \frac{2N}{N-2}$. We write $\Psi := \text{Re } \Phi$ for the real part of the fundamental solution $\Phi$ given in (11). Note that, by Theorem 2.1,
\begin{equation}
\text{(45)} \quad \text{the linear operator } R : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad R(v) := \Psi \ast v \text{ is bounded.}
\end{equation}

Here and in the following, in contrast to the previous sections, all function spaces are assumed to consist of real-valued functions. For $Q \in L^\infty(\mathbb{R}^N)$ nonnegative with $Q \neq 0$, we wish to set up a dual variational framework to study solutions $u \in L^p(\mathbb{R}^N)$ of (6). Note that equation (6) corresponds to the special case $k = 1$ in \ref{eq-4.1}, i.e., to
\begin{equation}
\text{(46)} \quad -\Delta u - u = Q(x)|u|^{p-2}u, \quad x \in \mathbb{R}^N.
\end{equation}

Setting $v = Q^\frac{1}{p} |u|^{p-2}u$, we are thus led to consider the equation
\begin{equation}
\text{(47)} \quad |v|^{p-2}v = Q^\frac{1}{p} [\Psi \ast (Q^\frac{1}{p}v)] \quad \text{in } \mathbb{R}^N
\end{equation}

Before setting up the variational framework for equation (47), we study the so-called Birman-Schwinger operator (see \ref{sec-3}) which appears on the right-hand side of (47).

Lemma 4.1. Let $\frac{2(N+1)}{N-1} \leq p < \frac{2N}{N-2}$ and consider $Q \in L^\infty(\mathbb{R}^N)$, satisfying $Q(x) \geq 0$ for a.e. $x \in \mathbb{R}^N$. Then the Birman-Schwinger operator
\[
K_p : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad K_p(v) := Q^\frac{1}{p}R(Q^\frac{1}{p}v)
\]
is symmetric in the sense that $\int_{\mathbb{R}^N} wK_p(v) \, dx = \int_{\mathbb{R}^N} vK_p(w) \, dx$ for all $v, w \in L^p(\mathbb{R}^N)$. Moreover,
\begin{itemize}
\item[(i)] for any bounded and measurable set $B \subset \mathbb{R}^N$, the operator $1_B K_p$ is compact. Here $1_B$ denotes the characteristic function of the set $B$.
\item[(ii)] If, in addition, $\text{ess sup } Q(x) \to 0$ as $R \to \infty$, then $K_p$ itself is compact.
\end{itemize}

Proof. Note that $K_p$ is a bounded linear operator due to (45) and since $Q \in L^\infty(\mathbb{R}^N)$ by assumption. We start by proving the compactness of $1_B K_p$ for a bounded set $B \subset \mathbb{R}^N$. For this, it is enough to show that $K_p(v_n) \to 0$ in $L^p(B)$ for every sequence $(v_n)_n \subset L^p(\mathbb{R}^N)$ such that $v_n \to 0$.

Let therefore $(v_n)_n \subset L^p(\mathbb{R}^N)$ converge weakly to 0. The boundedness of $R$ implies that $R(Q^\frac{1}{p}v_n) \to 0$ in $L^p(\mathbb{R}^N)$ and, according to Proposition \ref{prop-4.1}, there
Lemma 4.2.

\[ R(Q^T_v n) \in W^{2, p'}(\mathbb{R}^N) \] and for every \( R > 0 \) there is some constant \( \bar{C} > 0 \) such that

\[ \| R(Q^T_v n) \|_{W^{2, p'}(B_R)} \leq \bar{C} \left( \| R(Q^T_v n) \|_{L^p(\mathbb{R}^N)} + \| Q^T_v n \|_{L^p(\mathbb{R}^N)} \right) \]

holds for all \( n \) (here \( B_R \subset \mathbb{R}^N \) denotes the ball of radius \( R \) centered at the origin). Consequently, \((R(Q^T_v n))_n\) is a bounded sequence in \( W^{2, p'}(B_R) \), and from the compact embedding \( W^{2, p'}(B_R) \hookrightarrow L^p(B_R) \), (recall that \( p < \frac{2N}{N+2} \)) we obtain \( K_p(v_n) = Q^T R(Q^T_v n) \rightarrow 0 \) in \( L^p(B_R) \) as \( n \rightarrow \infty \), using the weak convergence and the fact that \( Q \in L^\infty(\mathbb{R}^N) \). The claim follows by choosing \( R > 0 \) large enough for \( B \subset B_R \) to hold.

Let us now assume that \( Q_R := \text{ess sup} \{Q(x) \rightarrow 0 \}, \) as \( R \rightarrow \infty \). In order to prove the compactness of \( K_p \) in this case, we now show that \( K_p(v_n) \rightarrow 0 \) in \( L^p(\mathbb{R}^N) \) for every sequence \((v_n)_n \subset L^p(\mathbb{R}^N)\) such that \( v_n \rightarrow 0 \). Taking such a sequence \((v_n)_n\), we first note that for every \( R > 0 \), \( \|1_{B_R} K_p(v_n)\|_p \rightarrow 0 \) as \( n \rightarrow \infty \), since \( 1_{B_R} K_p \) is compact. Moreover, for all \( n \in \mathbb{N}, R > 0 \),

\[ \int_{\mathbb{R}^N} |(1 - 1_{B_R}) K_p(v_n)|^p dx \leq \| R(Q^T_v n) \|_p \text{ ess sup} Q(x) \leq CQ_R, \]

since \((R(Q^T_v n))_n\) is bounded in \( L^p(\mathbb{R}^N) \). As a consequence,

\[ \limsup_{n \rightarrow \infty} \| K_p(v_n) \|_p \leq \limsup_{n \rightarrow \infty} \|1_{B_R} K_p(v_n)\|_p + \limsup_{n \rightarrow \infty} \|(1 - 1_{B_R}) K_p(v_n)\|_p \leq CQ_R \]

for all \( R > 0 \). Letting \( R \rightarrow \infty \), we obtain \( \| K_p(v_n) \|_p \rightarrow 0 \), as \( n \rightarrow \infty \), and the compactness of \( K_p \) follows.

To show that \( K_p \) is symmetric, we first consider the case where the functions \( f := Q^T v, g := Q^T w \) both belong to \( \mathcal{S} \). Using the definition of \( K_p \) and the properties of the convolution, we then obtain

\[ \int_{\mathbb{R}^N} w K_p(v) dx = \int_{\mathbb{R}^N} g(\Psi * f) dx = \int_{\mathbb{R}^N} f(\Psi * g) dx = \int_{\mathbb{R}^N} v K_p(w) dx. \]

The conclusion now follows by a density argument. \( \square \)

Consider now the energy functional

\[ J(v) = \frac{1}{p'} \int_{\mathbb{R}^N} |v|^p' dx - \frac{1}{2} \int_{\mathbb{R}^N} Q(x) \frac{1}{p'} v(x) R(Q^T_v v(x)) dx \]
\[ = \frac{1}{p'} \|v\|_{p'}^p - \frac{1}{2} \int_{\mathbb{R}^N} v K_p(v) dx \]

for \( v \in L^p(\mathbb{R}^N) \). From the preceding lemma, we deduce that \( J \in \mathcal{C}^1(L^p(\mathbb{R}^N), \mathbb{R}) \) with

\[ J'(v) w = \int_{\mathbb{R}^N} \left(|v|^p - 2 - v - K_p(v)\right) w dx \text{ for all } v, w \in L^p(\mathbb{R}^N). \]

Moreover, the functional \( J \) has the so-called mountain pass geometry.

Lemma 4.2.

(i) There exists \( \delta > 0 \) and \( 0 < \rho < 1 \) such that \( J(v) \geq \delta > 0 \) for all \( v \in L^p(\mathbb{R}^N) \) with \( \|v\|_{p'} = \rho \).

(ii) There is \( v_0 \in L^p(\mathbb{R}^N) \) such that \( \|v_0\|_{p'} > 1 \) and \( J(v_0) < 0 \).

(iii) Every Palais-Smale sequence for \( J \) is bounded in \( L^p(\mathbb{R}^N) \).
Proof. (i) As a consequence of (45) and the assumption \( Q \in L^\infty(\mathbb{R}^N) \), there exists some constant \( C > 0 \) such that \( \| \mathbf{K}_p(v) \|_p \leq C \| v \|_{p'} \) for all \( v \in L^{p'}(\mathbb{R}^N) \). Hence, if \( \| v \|_{p'} = \rho \), we obtain

\[
J(v) = \frac{1}{p'}|v|^{p'} - \frac{1}{2} \int_{\mathbb{R}^N} v \mathbf{K}_p(v) \, dx \geq \frac{1}{p'}\rho^{p'} - \frac{\rho}{2} \| \mathbf{K}_p(v) \|_p \geq \frac{1}{p'}\rho^{p'} - \frac{C}{2} \rho^2 > 0
\]

for \( \rho > 0 \) small enough, since \( p' < 2 \).

(ii) From (12) it follows that there exists \( r > 0 \) such that \( \Psi(x) > 0 \) for all \( x \in B_{2r}(0) \). Moreover, since \( Q \geq 0 \) a.e. on \( \mathbb{R}^N \) and \( Q \neq 0 \), the metric density of the set \( \omega_Q := \{ x \in \mathbb{R}^N : Q(x) > 0 \} \) (see [23, §7.12]) is 1 for almost every point from this set. Consequently, there exists \( x_0 \in \mathbb{R}^N \) and \( 0 < r < R \) such that \( \omega_Q \cap B_r(x_0) \) has positive measure. Choosing \( z \in \mathcal{C}_c^\infty(\mathbb{R}^N) \) with \( \text{supp } z \subset B_r(x_0) \), \( 0 \leq z \leq 1 \) in \( \mathbb{R}^N \) and \( z = 1 \) in \( B_{3r}(x_0) \), the definition of \( \mathbf{K}_p \) then implies

\[
\int_{\mathbb{R}^N} z \mathbf{K}_p z \, dx = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} Q(x)^{\frac{p}{2}} z(x) \Psi(x-y) Q(y)^{\frac{p}{2}} z(y) \, dy \, dx \\
\geq \int_{B_{3r}(x_0)} \int_{B_{3r}(x_0)} \Psi(x-y) Q(x)^{\frac{p}{2}} Q(y)^{\frac{p}{2}} \, dx \, dy > 0.
\]

For \( t > 0 \) we obtain

\[
J(tz) = \frac{t^{p'}}{p'} \int_{\mathbb{R}^N} |z|^{p'} \, dx - \frac{1}{2} t^2 \int_{\mathbb{R}^N} z \mathbf{K}_p z \, dx \\
= t^2 \left( \frac{1}{p'N} \int_{\mathbb{R}^N} |z|^{p'} \, dx - \frac{1}{2} \int_{\mathbb{R}^N} z \mathbf{K}_p z \, dx \right) < 0,
\]

provided \( t \) is large enough.

(iii) Let \((v_n)_n \subset L^{p'}(\mathbb{R}^N)\) be a Palais-Smale sequence, i.e., there holds \( \sup_n |J(v_n)| < \infty \) and \( J'(v_n) \to 0 \) in \( L^p(\mathbb{R}^N)^* \equiv L^p(\mathbb{R}^N) \) as \( n \to \infty \). Since

\[
J(v_n) = \left( \frac{1}{p'} - \frac{1}{2} \right) \| v_n \|_{p'} + \frac{1}{2} J'(v_n) v_n \geq \left( \frac{1}{p'} - \frac{1}{2} \right) \| v_n \|_{p'} - \frac{1}{2} \| J'(v_n) \| \| v_n \|_{p'}
\]

and \( 1 < p' < 2 \) holds, we infer that \((v_n)_n\) is bounded in \( L^{p'}(\mathbb{R}^N) \).

We now show that every critical point of \( J \) is indeed a solution of our original problem. For this, we first note that for \( v \in L^{p'}(\mathbb{R}^N) \), \( J'(v) = 0 \) if and only if it satisfies (47). Setting

\[
(50) \quad u = R(Q^+ v),
\]

we find that \( u \) solves the equation

\[
(51) \quad u = R(Q|u|^{p-2} u).
\]

Remark furthermore, that \( u \neq 0 \) if \( v \neq 0 \), since the condition \( J'(v) = 0 \) implies \( \| v \|_{p'}^p = \int_{\mathbb{R}^N} |Q^+ v u| \, dx \). In the following result we study the regularity of \( u \) and show that it solves (46).

Lemma 4.3. Let \( Q \in L^\infty(\mathbb{R}^N) \), \( \frac{2(N+1)}{N-1} \leq p \leq \frac{2N}{N-2} \) and consider a solution \( u \in L^{p}(\mathbb{R}^N) \) of (51). Then \( u \) belongs to \( W^{2,q}(\mathbb{R}^N) \cap \mathcal{C}^{1,\alpha}(\mathbb{R}^N) \) for all \( p \leq q < \infty \), \( 0 < \alpha < 1 \), and it is a strong solution of (46). Moreover, \( u \) is the real part of a function \( \tilde{u} \) which satisfies Sommerfeld’s outgoing radiation condition

\[
(52) \quad \lim_{R \to \infty} \frac{1}{R} \int_{B_R} |\nabla \tilde{u}(x) - i\tilde{u}(x) \hat{e}|^2 \, dx = 0,
\]
and its far field pattern is given by the nonlinear relation

$$\lim_{R \to \infty} \frac{1}{R} \int_{B_R} \left| u(x) + 2 \left( \frac{2 \pi}{|x|} \right)^{N/2} \operatorname{Re} \left[ e^{i|x| - \frac{|x|^2}{4} + i} \right] u(x) \cdot \hat{g}_u(x) \right|^2 \, dx = 0$$

with $g_u : S^{N-1} \to \mathbb{C}$, $g_u(\xi) = -\frac{1}{4} (2\pi)^{-\frac{N-1}{2}} P(Q|u|^{p-2}u)(\xi)$.

Proof. Since $Q \in L^\infty(\mathbb{R}^N)$ and $\frac{2(N+1)}{(N-1)} \leq p \leq 2^*$, Proposition A.1 gives $u \in W^{2,p'}_{\text{loc}}(\mathbb{R}^N)$ and, for every $x_0 \in \mathbb{R}^N$, there holds

$$\|u\|_{W^{2,p'}(B(x_0))} \leq \tilde{C} \left( \|u\|_{L^p(\mathbb{R}^N)} + \|Q\|_\infty \|u\|_{L^p(\mathbb{R}^N)}^{p-1} \right)$$

with some constant $\tilde{C} > 0$, independent of $x_0$. Setting

$$D_0 := \tilde{C} \left( \|u\|_{L^p(\mathbb{R}^N)} + \|Q\|_\infty \|u\|_{L^p(\mathbb{R}^N)}^{p-1} \right)$$

and using Sobolev’s embedding theorem, we obtain for all $1 \leq q \leq \frac{Np'}{N-2p'}$ and every $x_0 \in \mathbb{R}^N$ the estimate $\|u\|_{L^q(B(x_0))} \leq \kappa_q D_0$, where $\kappa_q > 0$ is independent of $x_0$. In particular: $Q|u|^{p-2}u \in L^{p'}(\mathbb{R}^N) \cap L^2_{\text{loc}}(\mathbb{R}^N)$, where we have set $t_1 := \frac{Np'(p-1)}{2p'}$ (note that $p' < t_1 < \frac{t_1}{p-1} = \frac{Np'}{N-2p'}$) and for every $x_0 \in \mathbb{R}^N$ there holds

$$\|Q|u|^{p-2}u\|_{L^{t_1}(B(x_0))} \leq \|Q\|_\infty \left( \kappa_{t_1} D_0 \right)^{\frac{1}{p-1}}.$$

We now consider a strictly decreasing sequence of radii $(R_m)_{m \geq 0}$ satisfying $R_0 = 2$ and $R_m \geq 1$ for all $m$. Using Proposition A.1 (i), we obtain $u \in W^{2,t_1}_{\text{loc}}(\mathbb{R}^N)$ and, for every $x_0 \in \mathbb{R}^N$, the estimate

$$\|u\|_{W^{2,t_1}(B(x_0))} \leq D \left( \|u\|_{L^2(B(x_0))} + \|Q|u|^{p-2}u\|_{L^{t_1}(B(x_0))} \right) \leq \tilde{D} \left( \kappa_{t_1} D_0 + \|Q\|_\infty \left( \kappa_{t_1} D_0 \right)^{\frac{1}{p-1}} \right) =: D_1,$$

where the constant $\tilde{D}$, and hence also $D_1$, is independent of $x_0$.

If $t_1 \geq \frac{N}{2}$, Sobolev’s embedding theorem gives for each $1 \leq q < \infty$ the existence of a constant $\kappa_q \geq 0$ such that $\|u\|_{L^q(B(x_0))} \leq \kappa_q D_1$ for all $x_0 \in \mathbb{R}^N$. As a consequence, we obtain $\|Q|u|^{p-2}u\|_{L^q(B(x_0))} \leq \|Q\|_\infty \left( \kappa_q D_1 \right)^{\frac{1}{p-1}}$ for all $1 \leq q < \infty$ and every $x_0 \in \mathbb{R}^N$. Proposition A.1 (i) then gives $u \in W^{2,q}_{\text{loc}}(\mathbb{R}^N)$ for all $1 \leq q < \infty$, as well as the estimate

$$\|u\|_{W^{2,q}(B(x_0))} \leq \tilde{D} \left( \kappa_q D_1 + \|Q\|_\infty \left( \kappa_q D_1 \right)^{\frac{1}{p-1}} \right) =: D_2,$$

for all $x_0 \in \mathbb{R}^N$, where $\tilde{D} > 0$ is independent of $x_0$. From Sobolev’s embedding theorem, it follows that $\|u\|_{L^q(B(x_0))} \leq \kappa_q D_2$ for all $x_0 \in \mathbb{R}^N$, where the constant $\kappa_q$ is independent of $x_0$, showing that $u \in L^\infty(\mathbb{R}^N)$. Consequently, Proposition A.1 (ii) gives $u \in W^{2,\sigma}_{\text{loc}}(\mathbb{R}^N)$ for every $p' \leq \sigma < \infty$, and concludes the proof in this case.

If $t_1 < \frac{N}{2}$, we infer from Sobolev’s theorem, that $\|u\|_{L^q(B(x_0))} \leq \kappa_q D_1$ for all $1 \leq q \leq \frac{Nt_1}{N-2t_1}$ and every $x_0 \in \mathbb{R}^N$, where $\kappa_q \geq 0$ is independent of $x_0$. This gives $\|Q|u|^{p-2}u\|_{L^p(B(x_0))} \leq \kappa_q D_1$, and for all $x_0 \in \mathbb{R}^N$ the estimate $\|Q|u|^{p-2}u\|_{L^{t_2}(B(x_0))} \leq \|Q\|_\infty \left( \kappa_q D_1 \right)^{\frac{1}{p-1}}$ holds. Remarking that $t_2 > t_1$, we may use Proposition A.1 (i) again and iterate the procedure. At each step we find some constant $D_m > 0$, independent of $x_0 \in \mathbb{R}^N$ such that the estimate $\|u\|_{W^{2,t_m}(B(x_0))} \leq D_m$ holds for all $x_0 \in \mathbb{R}^N$, where $t_m$ is defined recursively via
Let \( t_0 = p' \), \( t_{k+1} = \frac{N t_k (p' - 1)}{N - 2 t_k}, k \geq 0 \). Since we have \( t_{m+1} \geq \frac{1}{p'} t_m \), as long as \( t_m < \frac{N}{2} \), and since \( t_1 > p' \), we reach after finitely many steps \( t_m \geq \frac{N}{2} \). Since \( R_m \geq 1 \) for all \( m \), applying Proposition \( \text{A.1} \) one more time and arguing as above, we obtain \( u \in W^2_{\operatorname{loc}}(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \) for all \( 1 \leq q < \infty \), and Proposition \( \text{A.1} \) concludes the argument, giving the desired regularity.

Next we observe that \( u = \operatorname{Re}(\hat{u}) \) for \( \hat{u} := \mathcal{R}(Q|u|^{p-2} u) \), and we show that \( \hat{u} \) satisfies \( \text{[52]} \). We already noted that \( \text{[14]} \) implies that

\[
\lim_{R \to \infty} \frac{1}{R} \int_{B_R} \left| \nabla[\mathcal{R}f](x) - i \hat{\omega}[\mathcal{R}f](x) \right|^2 \, dx = 0 \quad \text{for all } f \in \mathcal{F}.
\]

Let \( (f_n)_n \) be a sequence in \( \mathcal{F} \) such that \( \| f_n - Q|u|^{p-2} u \|_{p'} \to 0 \) as \( n \to 0 \). As a consequence of Theorems \( 7 \) and \( 8 \) in \( \text{[10]} \) and since \( \frac{2N}{N+2} \leq p' \leq \frac{2(N+1)}{(N+3)} \), we then have

\[
\sup_{R > 1} \frac{1}{R} \int_{B_R} \left| \nabla[\mathcal{R}(f_n - Q|u|^{p-2} u)](x) - i \hat{\omega}[\mathcal{R}(f_n - Q|u|^{p-2} u)](x) \right|^2 \, dx \\
\leq C \| f_n - Q|u|^{p-2} u \|_{p'}^2 \to 0, \quad \text{as } n \to \infty.
\]

Consequently, \( \text{[54]} \) also holds with \( f = Q|u|^{p-2} u \) and therefore \( \hat{u} = \mathcal{R}(Q|u|^{p-2} u) \) satisfies \( \text{[52]} \).

Finally, the nonlinear relation \( \text{[53]} \) follows from Proposition \( \text{2.7} \) since \( \text{[21]} \) holds for \( \hat{u} = \mathcal{R}f \) with \( f := Q|u|^{p-2} u \in L^p(\mathbb{R}^N) \).

The remainder of this section is devoted to the proof of Theorem \( \text{[13]} \) which we recall in a slightly different formulation.

**Theorem 4.4.** Let \( Q \in L^\infty(\mathbb{R}^N), \frac{(N-1)}{2} < p' = \frac{4N}{N-2} \) and consider a solution \( u \in L^p(\mathbb{R}^N) \) of \( u = R(Q|u|^{p-2} u) \). Then there exists a constant \( C > 0 \) such that

\[
|u(x)| \leq C|x|^\frac{p-2}{2} \quad \text{for all } x \in \mathbb{R}^N \setminus \{0\}.
\]

Moreover, as \( |x| \to \infty \),

\[
u (x) = -2 \left( \frac{2\pi}{|x|} \right)^{\frac{N-1}{2}} \Re \{ e^{i|x|^{-\frac{(N-1)}{4}} g_u(\hat{x})} \} + o(|x|^{-\frac{N-1}{2}}),
\]

with \( g_u \) as in Lemma \( \text{[4.3]} \).

**Proof.** We start by showing that \( f := Q|u|^{p-2} u \in L^1(\mathbb{R}^N) \). Let \( C > 0 \) be fixed large enough such that the assertions of Theorem \( \text{[2.1]} \) and Lemma \( \text{[3.3]} \) hold. Remark that since \( f \) is real-valued, there holds \( R(f) = \operatorname{Re}(\mathcal{R}f) \) and in this case Theorem \( \text{[2.1]} \) and Lemma \( \text{[3.3]} \) hold with \( R \) in place of \( \mathcal{R} \) with the same constants. Consider the nonlinear operator

\[
N : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N), \quad g \mapsto N(g) = Q|g|^{p-2} g.
\]

Since \( u \in L^p(\mathbb{R}^N) \) is a solution of \( \text{[4]} \), we know that \( f \in L^p(\mathbb{R}^N) \) is a fixed point of the operator \( \overline{N} \circ R : L^p(\mathbb{R}^N) \to L^p(\mathbb{R}^N) \). For \( j \in \mathbb{N} \) we put

\[
r_j := 2^j, \quad g_j := 1_{B_{r_j}} f \quad \text{and} \quad h_j := 1_{M_{r_j}} f.
\]

Next, let \( \kappa_0 := 2^{p-2} C^{p-1} \|Q\|_\infty \|f\|_{p'}^{-1} \). Moreover, fix \( j_0 \in \mathbb{N} \) large enough such that

\[
\|h_j\|_{p'} \leq \min \left\{ \frac{1}{1 + \kappa_0 \left( \frac{2^{-(p-1)} \lambda_{p(j)}}{2^{p-2} (1 + \kappa_0) \|Q\|_\infty C^{p-1}} \right)^{1/j}} \right\} \quad \text{for } j \geq j_0.
\]

Here \( \lambda_p = \frac{N-2}{2} - \frac{N+1}{p} \). We then have

\[
\|h_{j+1}\|_{p'} = \|N \circ R(f)\|_{L^p(M_{r_{j+1}})} \leq \|Q\|_\infty \|R(f)\|_{L^p(M_{r_{j+1}})}^{p-1}.
\]
\[
\begin{align*}
&\leq 2^{p-2} \|Q\|_\infty \left( \|R(g_j)\|_{L^p(M_{r_{j+1}})}^{p-1} + \|R(h_j)\|_{L^p(M_{r_{j+1}})}^{p-1} \right) \\
&\leq 2^{p-2} \|Q\|_\infty \left( C^{p-1} - (p-1)\lambda_p \|g_j\|_{p'}^{p-1} + C^{p-1} \|h_j\|_{p'}^{p-1} \right) \\
&\leq \kappa_0 2^{-(p-1)\lambda_p j} + \frac{2^{-(p-1)\lambda_p}}{\kappa_0 + 1} \|h_j\|_{p'} \quad \text{for } j \geq j_0.
\end{align*}
\]

Inductively, it follows that
\[
\|h_j\|_{p'} \leq (\kappa_0 + 1)2^{-(p-1)\lambda_p (j-j_0)}.
\]

From this we obtain
\[
f \in L^{p_1}(\mathbb{R}^N) \quad \text{if } 0 \leq \left(\frac{1}{p'} - \frac{1}{p_1}\right) < (p-1)\frac{\lambda_p}{N}, \text{ i.e. } 0 \leq \left(\frac{1}{p} - \frac{1}{p_1}\right) < (p-1)\frac{\lambda_p}{N}.
\]

Indeed, in this case we have
\[
\|f\|_{L^{p_1}(M_{r_{j_0}})} \leq \|f\|_{L^{p_1}(B_{r_{j+1}} \setminus B_{r_j})} \leq \sum_{j \geq j_0} \|f\|_{L^{p_1}(B_{r_{j+1}} \setminus B_{r_j})} \leq \kappa \sum_{j \geq j_0} 2^{Nj(\frac{1}{p} - \frac{1}{p_1})} 2^{-(p-1)\lambda_p j} = \kappa \sum_{j \geq j_0} 2^{Nj(\frac{1}{p} - \frac{1}{p_1})} 2^{-(p-1)\lambda_p j},
\]

Here and in the following, \(\kappa\) stands for (possibly different) positive constants. Next, we let \(\alpha = (\frac{1}{N-1}) (1 + p \lambda_p)\) and notice that \(\alpha > 1\) since \(p > \frac{(N-1)}{(N-1)}\). Therefore, we may fix \(p_1 \in (p, \infty)\) such that
\[
\left(\frac{1}{p} - \frac{1}{p_1}\right) > (p-1)\frac{\lambda_p}{\alpha N}.
\]

By making \(C > 0\) larger if necessary, we now assume that the assertion of Lemma 3.5 also holds with \(p_1\) in place of \(p\). We let
\[
\kappa_1 := 2^{p-2} \|Q\|_\infty \left( C|B_2 \setminus B_1| \|f\|_{p_1'} \right)^{p-1},
\]

and we fix \(j_1 \in \mathbb{N}\) large enough such that
\[
\|h_j\|_{p'} \leq \min \left\{ \left(\frac{1}{1 + \kappa_1} \right) \left(2^{p-2}(1 + \kappa_1)\|Q\|_\infty C^{p-1}\right)^{\frac{1}{p-1}}, \right\} \quad \text{for } j \geq j_1.
\]

By a similar but somewhat more careful estimate, we have
\[
\begin{align*}
\|h_{j+1}\|_{p'} &= \|N \circ R(f)\|_{L^{p_1'}(M_{r_{j+1}})} \leq \|Q\|_\infty \|R(f)\|_{L^p(M_{r_{j+1}})} \\
&\leq \|Q\|_\infty \left( \sum_{\tau \geq j+1} \|R(g_j)\|_{L^p(B_{r_{\tau+1}} \setminus B_{r_\tau})} + \|R(h_j)\|_{L^p(M_{r_{j+1}})} \right) \\
&\leq \|Q\|_\infty \left( \sum_{\tau \geq j+1} |B_{r_{\tau+1}} \setminus B_{r_\tau}|^{\frac{1}{p}} \|R(g_j)\|_{L^p(M_{r_{j+1}})} + C \|h_j\|_{p'} \right) \\
&\leq \|Q\|_\infty \left( C\|B_2 \setminus B_1\| \sum_{\tau \geq j+1} 2^{N(\frac{1}{p} - \frac{1}{p_1})} \|g_j\|_{L^{p_1'}(M_{r_{j+1}})} + C \|h_j\|_{p'} \right) \\
&= \|Q\|_\infty \left( C\|B_2 \setminus B_1\| \sum_{\tau \geq j+1} 2^{\frac{\tau}{p} \lambda_p} \|g_j\|_{L^{p_1'}(M_{r_{j+1}})} + C \|h_j\|_{p'} \right) \\
&\leq \|Q\|_\infty \|C\|B_2 \setminus B_1\| \left[ 1 - 2^{\frac{\tau}{p} \lambda_p} \|g_j\|_{L^{p_1'}(M_{r_{j+1}})} + C \|h_j\|_{p'} \right] \\
&\leq \|Q\|_\infty \|C\|B_2 \setminus B_1\| \left[ 1 - 2^{\frac{\tau}{p} \lambda_p} \|g_j\|_{L^{p_1'}(M_{r_{j+1}})} + C \|h_j\|_{p'} \right] \\
&\leq \|Q\|_\infty \|C\|B_2 \setminus B_1\| \left[ 1 - 2^{\frac{\tau}{p} \lambda_p} \|g_j\|_{L^{p_1'}(M_{r_{j+1}})} + C \|h_j\|_{p'} \right].
\end{align*}
\]
and hence
\[ \|h_{j+1}\|_{p'} \leq 2^{p-2} \left( \left[ \|Q\|_\infty C \|B_2 \setminus B_1\|_{p,1} \right]^{p-1} 2^{-\left( \frac{1}{p'} - \frac{1}{p} + \lambda_p \right)(p-1)j} + \|Q\|_\infty C^{p-1} \|h_j\|_{p'} \right)^{\frac{1}{p-1}} \]
\[ \leq \kappa (2^{-\left( \frac{1}{p'} - \frac{1}{p} + \lambda_p \right)(p-1)j} + 2^{-\left( \frac{1}{p'} - \frac{1}{p} + \lambda_p \right)(p-1)(j-j_1)} \|h_j\|_{p'} \]

Inductively, we thus find that
\[ \|h_j\|_{p'} \leq (\kappa_1 + 1) 2^{-\left( \frac{1}{p'} - \frac{1}{p} + \lambda_p \right)(p-1)(j-j_1)} \]
for \( j \geq j_1 \).

Next, we fix \( p_2 \in (p_1, \infty) \) such that
\[ \frac{1}{p} - \frac{1}{p_2} = (p-1) \frac{\frac{1}{p} - \frac{1}{p_2} + \lambda_p}{\alpha N}. \]

Then it follows as above that \( f \in L^p_\Phi(\mathbb{R}^N) \). Now we argue inductively, defining \( p_k \in (0, \infty) \) for \( k \in \mathbb{N} \cup \{0\} \) by \( p_0 := p \) and
\[ \frac{1}{p} - \frac{1}{p_k} = (p-1) \frac{\frac{1}{p} - \frac{1}{p_k} + \lambda_p}{\alpha N} \]
for \( k \geq 1 \).

It is easy to see from our choice of \( \alpha \), that \( p_k = (1 + p\lambda_p)p_{k-1} \) holds for all \( k \geq 1 \). Hence the sequence \( (p_k) \) is strictly increasing and \( \lim_{k \to \infty} p_k = \infty \), so that inductively we get \( f \in L^p_\Phi(\mathbb{R}^N) \) for every \( k \in \mathbb{N} \). Choosing now \( k \in \mathbb{N} \) such that \( 0 < \frac{1}{p_k} < \frac{N-1}{2} - \frac{N}{p-1} \), we can find \( j_k \in \mathbb{N} \) and \( \kappa_k > 0 \) such that
\[ \|h_j\|_{p'} \leq (\kappa_k + 1) 2^{-\left( \frac{1}{p'} - \frac{1}{p} + \lambda_p \right)(p-1)(j-j_k)} \]
and therefore
\[ \|f\|_{L^1(M_{j_k})} \leq \sum_{j \geq j_k} \|f\|_{L^1(\mathbb{R}^N \setminus B_{r+j} \setminus B_{r_j})} \leq \sum_{j \geq j_k} |B_{r+j} \setminus B_{r_j}| 2^{-\left( \frac{1}{p} - \frac{1}{p_k} + \lambda_p \right)(j-j_k)} \|h_j\|_{p'} \]
\[ \leq \kappa \sum_{j \geq j_k} 2^{Nj(1 - \frac{1}{p_k})} 2^{-\left( \frac{1}{p} - \frac{1}{p_k} + \lambda_p \right)(j-j_k)} \|h_j\|_{p'} \]
with some \( \kappa > 0 \). It follows that \( f \in L^1(\mathbb{R}^N) \).

In order to prove the first assertion of the lemma, we notice that by Lemma 2.9 with \( K = \Psi \), it suffices to show that \( V = Q|u|^{p-2} \in L^q(\mathbb{R}^N) \) holds for some \( q < \frac{N}{N-1} \). Indeed, from Lemma 2.3 we know that \( u \in L^p(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N) \), and we have just shown that \( f = Vu \in L^1(\mathbb{R}^N) \). Let us therefore choose \( q = \frac{p-1}{p-2} = 1 + \frac{1}{p-2} \), so that
\[ \int_{\mathbb{R}^N} |V(x)|^q dx = \int_{\mathbb{R}^N} |Q(x)|^{1+\frac{N}{p-2}} |u(x)|^{p-1} dx \leq \|Q\|_\infty^{1+\frac{N}{p-2}} \|f\|_1, \]
which proves the result. The second assertion then follows from Proposition 2.8 by remarking that
\[ |f(x)| \leq \|Q\|_\infty |u(x)|^{p-1} \leq C^{p-1} \|Q\|_\infty |x|^{-\frac{N(p-1)}{2}} \]
for \( x \neq 0 \), where \( \frac{(p-1)(1-N)}{2} < -N \). The proof is therefore complete. \( \square \)
5. Existence of solutions in the compact case

We now assume, in addition, that \( Q(x) \to 0 \) as \(|x| \to \infty \). In this case, we shall prove the existence of infinitely many pairs \( \{ \pm u \} \) of critical points for \( J \) using a variant of the symmetric Mountain Pass Theorem [6].

For this purpose, we collect further properties of \( K_p \) and the functional \( J \).

**Lemma 5.1.** For every \( m \in \mathbb{N} \), there exists an \( m \)-dimensional subspace \( \mathscr{W} \subset \mathcal{C}_c^\infty(\mathbb{R}^N) \) with the following properties:

(i) \( \int_{\mathbb{R}^N} v K_p v \, dx > 0 \) for all \( v \in \mathscr{W} \setminus \{0\} \).

(ii) There exists \( R = R(\mathscr{W}) > 0 \) such that \( J(v) \leq 0 \) for every \( v \in \mathscr{W} \) with \( \|v\|_{W^1} \geq R \).

**Proof.** Since \( Q \not\equiv 0 \), there exists a point of density one for the set \( \{ Q > 0 \} \). Without loss of generality, we may assume that \( x_0 = 0 \). Then for \( \delta > 0 \) sufficiently small we have

\[
|Q^{-1}(0) \cap B_\delta(0)| \leq \left( \frac{1}{4m^2} \right)^N |B_\delta(0)|.
\]

Let

\[
\Psi^*(\tau) := \inf_{B_{\tau}(0) \setminus \{0\}} \Psi \quad \text{and} \quad \Psi_*(\tau) := \|\Psi\|_{L^\infty(\mathbb{R}^N \setminus B_{\tau}(0))} \quad \text{for} \quad \tau > 0.
\]

Since \( \Psi \) is bounded outside of every neighborhood of zero and \( \Psi(x)|x|^{N-2} \) tends to a positive constant as \(|x| \to 0 \) by (12), we may fix \( \delta > 0 \) such that (55) holds and that

\[
\Psi^*(\tau) > (m-1)\Psi_*(m\tau) \quad \text{for} \quad \tau \in (0, \delta].
\]

Moreover, it is easy to see that there exists \( m \) disjoint open balls \( B^1, \ldots, B^m \subset B_\delta(0) \) of diameter \( \tau := \frac{\delta}{m} \) such that

\[
\text{dist}(B_i^i, B_j^i) := \inf\{|x-y| : x \in B_i^i, y \in B_j^i\} \geq \frac{\delta}{m}
\]

Since \( |B^i| = \left( \frac{1}{2m^2} \right)^N |B_\delta(0)| \) for \( i = 1, \ldots, m \), we also have

\[
|B^i \cap \{ Q > 0 \}| > 0 \quad \text{for} \quad i = 1, \ldots, m
\]

by (55). We now fix functions \( z_i \in \mathcal{C}_c^\infty(\mathbb{R}^N) \), \( i = 1, \ldots, m \) such that \( z_i > 0 \) in \( B^i \) and \( z_i \equiv 0 \) in \( \mathbb{R}^N \setminus B^i \). Moreover, we let \( \mathscr{W} \) denote the span of \( z_1, \ldots, z_m \). Then any \( v \in \mathscr{W} \setminus \{0\} \) can be written as \( v = \sum_{i=1}^m a_i z_i \) with \( a = (a_1, \ldots, a_m) \in \mathbb{R}^m \setminus \{0\} \), and thus we have

\[
\begin{align*}
\int_{\mathbb{R}^N} v K_p v \, dx &= \sum_{i,j=1}^m a_i a_j \int_{B_i} \int_{B_j} \Psi(x-y)Q(x) \frac{\partial}{\partial x} Q(y) \frac{\partial}{\partial y} z_i(x) z_j(y) \, dx \, dy \\
&\geq \Psi^*(\tau) \sum_{i=1}^m a_i^2 \left( \int_{B^i} Q(x) \frac{\partial}{\partial x} z_i(x) \, dx \right)^2 \\
&\quad - \Psi_*(m\tau) \sum_{i,j=1}^m a_i |a_j| \left( \int_{B^i} Q(x) \frac{\partial}{\partial x} z_i(x) \, dx \right) \left( \int_{B^j} Q(x) \frac{\partial}{\partial x} z_j(x) \, dx \right) \\
&\geq \Psi^*(\tau) \sum_{i=1}^m a_i^2 \left( \int_{B^i} Q(x) \frac{\partial}{\partial x} z_i(x) \, dx \right)^2 \\
&\quad - \frac{\Psi_*(m\tau)}{2} \sum_{i,j=1}^m |a_i|^2 \left( \int_{B^i} Q(x) \frac{\partial}{\partial x} z_i(x) \, dx \right)^2 + a_i^2 \left( \int_{B^j} Q(x) \frac{\partial}{\partial x} z_j(x) \, dx \right)^2 \\
&\quad + \left( \frac{\Psi_*(m\tau)}{2} - \Psi^*(\tau) \right) \sum_{i,j=1}^m a_i |a_j| \left( \int_{B^i} Q(x) \frac{\partial}{\partial x} z_i(x) \, dx \right) \left( \int_{B^j} Q(x) \frac{\partial}{\partial x} z_j(x) \, dx \right).
\end{align*}
\]
Theorem 5.3. As a consequence of (56) and (57). This shows (i). As a consequence of (i) and continuity, we have

\[ m_\gamma := \inf_{v \in \mathcal{W}, \|v\|_{p'} = 1} \int_{\mathbb{R}^N} v K_p v \, dx > 0, \]

Hence

\[ J(v) = \frac{\|v\|_{p'}^p}{p'} - \int_{\mathbb{R}^N} v K_p v \, dx \leq \|v\|_{p'}^p \left( \frac{1}{p'} - \|v\|_{p'}^{2-p'} m_\gamma \right) \quad \text{for } v \in \mathcal{W}. \]

Thus (ii) follows with

\[ R := \left( m_\gamma \right)^{\frac{1}{2-p'}}. \]

Lemma 5.2. \( J \) satisfies the Palais-Smale condition in \( L^{p'}(\mathbb{R}^N) \).\]

Proof. Let \((v_n) \subset L^{p'}(\mathbb{R}^N)\) be a Palais-Smale sequence. According to Lemma 4.2 (iii), \((v_n)\) is bounded in \( L^{p'}(\mathbb{R}^N) \). Hence, up to a subsequence, we may assume \( v_n \rightharpoonup v \in L^{p'}(\mathbb{R}^N) \). From the convexity of the function \( t \mapsto |t|^{p'} \) we obtain

\[ \frac{1}{p'} \|v\|_{p'}^p - \frac{1}{p'} \|v_n\|_{p'}^p \geq \int_{\mathbb{R}^N} |v_n|^{p'-2} v_n (v - v_n) \, dx \]

\[ = J'(v_n)(v - v_n) + \int_{\mathbb{R}^N} v_n K_p (v - v_n) \, dx \to 0 \]

as \( n \to \infty \), taking into account the symmetry and the compactness of the Birman-Schwinger operator \( K_p \) proven in Lemma 4.1. Consequently, \( \|v\|_{p'} \geq \limsup_{n \to \infty} \|v_n\|_{p'} \). On the other hand, the weak convergence \( v_n \rightharpoonup v \) implies \( \|v\|_{p'} \leq \liminf_{n \to \infty} \|v_n\|_{p'} \), which together gives \( \lim_{n \to \infty} \|v_n\|_{p'} = \|v\|_{p'} \), and hence \( v_n \rightharpoonup v \) strongly in \( L^{p'}(\mathbb{R}^N) \), as claimed.

Combining Lemmas 4.2 and 5.2 with the symmetric Mountain Pass Theorem e.g., in the form of [14, Corollary 7.23], we obtain the existence of infinitely many nontrivial pairs \( \{ \pm u \} \) of critical points for \( J \). Summarizing and taking Lemma 4.3 into account, we can thus state the following.

Theorem 5.3. Let \( \frac{2(N+1)}{N-1} \leq p < \frac{2N}{N-2} \) and consider a nonnegative function \( Q \in L^\infty(\mathbb{R}^N), Q \neq 0 \) such that \( Q(x) \to 0 \) as \( |x| \to \infty \). Then problem (46), (55) has infinitely many strong solutions \( u \in W^{2,q}(\mathbb{R}^N) \cap C^{0,\alpha}(\mathbb{R}^N), p \leq q < \infty, 0 < \alpha < 1 \).

6. Existence in the periodic case

In this section, we treat the case where \( Q \in L^\infty(\mathbb{R}^N) \) is 1-periodic on \( \mathbb{R}^N \), i.e., \( Q(x + e_i) = Q(x) \) for all \( x \in \mathbb{R}^N \) and all \( 1 \leq i \leq N \). \( \{ e_1, \ldots, e_N \} \subset \mathbb{R}^N \) denoting the standard basis in \( \mathbb{R}^N \).

We shall prove the existence of solutions using a dual variational approach as before. Considering the dual functional \( J^*: L^{p'}(\mathbb{R}^N) \to \mathbb{R} \) given by (43), we already know that \( J \) is of class \( C^1 \) on \( L^{p'}(\mathbb{R}^N) \) and, according to Lemma 4.2 that it possesses the mountain-pass geometry. However, since \( K_p \) is not compact anymore, the Palais-Smale condition does not hold in general. Nevertheless, we may define a mountain-pass level for \( J \) by setting

\[ c := \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} J(\gamma(t)). \]
Lemma 6.1. There exists a bounded Palais-Smale sequence \((v_n)_n \subset L^p(\mathbb{R}^N)\) for \(J\) at level \(c\).

Proof. Suppose by contradiction, that no Palais-Smale sequence for \(J\) exists at level \(c\). In that case there are \(0 < \varepsilon < \frac{\delta}{2}\) and \(\delta > 0\) such that \(\|J'(v)\| \geq \delta > 0\) for all \(v \in L^p(\mathbb{R}^N)\) satisfying \(|J(v) - c| \leq 2\varepsilon\). According to the deformation Lemma 2.7, we can therefore find a homotopy \(\eta \in C([0,1] \times L^p(\mathbb{R}^N), L^p(\mathbb{R}^N))\) such that

\[
\begin{align*}
(i) & \quad \eta(0,v) = v \text{ for all } v \in L^p(\mathbb{R}^N), \\
(ii) & \quad \eta(t,v) = v \text{ for all } t \in [0,1], v \in L^p(\mathbb{R}^N) \text{ for which } J(v) \notin [c-2\varepsilon, c+2\varepsilon], \\
(iii) & \quad \eta(t,\cdot) \text{ is a homeomorphism of } L^p(\mathbb{R}^N) \text{ for all } t \in [0,1], \\
(iv) & \quad J(\eta(1,v)) \leq c - \varepsilon \text{ for all } v \in L^p(\mathbb{R}^N) \text{ such that } J(v) \leq c + \varepsilon.
\end{align*}
\]

Choosing now \(\gamma \in \Gamma\) such that \(\max_{t \in [0,1]} J(\gamma(t)) \leq c + \varepsilon\) and setting \(\tilde{\gamma}(t) := \eta(1,\gamma(t))\) for all \(t \in [0,1]\), we obtain from (ii) \(\tilde{\gamma}(0) = 0\) and \(\tilde{\gamma}(1) = \gamma(1)\), which in turn implies \(\tilde{\gamma} \in \Gamma\) and therefore \(\max_{t \in [0,1]} J(\tilde{\gamma}(t)) \geq c\). On the other hand, it follows from (iv) that

\[
J(\tilde{\gamma}(t)) = J(\eta(1,\gamma(t))) \leq c - \varepsilon \quad \text{for all } t \in [0,1]
\]

which is a contradiction. Therefore, there must exist some Palais-Smale sequence \((v_n)_n \subset L^p(\mathbb{R}^N)\) at level \(c\) for \(J\). Moreover, by Lemma 4.2 (iii), \((v_n)_n\) is a bounded sequence.

Theorem 6.2. Let \(\frac{2(N+1)}{(N-1)} < p < 2^*\) and consider a nonnegative function \(Q \in L^\infty(\mathbb{R}^N), Q \neq 0\) which is \(\mathbb{Z}^N\)-periodic on \(\mathbb{R}^N\). Then \([46, 53]\) has a nontrivial strong solution \(u \in W^{2,q}(\mathbb{R}^N) \cap C^{1,\alpha}(\mathbb{R}^N), p \leq q < \infty, 0 < \alpha < 1\).

Proof. Let \((v_n)_n\) and \(v\) be as in Lemma 6.1. Since \(J(v_n) \to c > 0\) and \(J'(v_n)v_n \to 0\) as \(n \to \infty\), we find

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} Q^p v_n R(Q^p v_n) \, dx = \frac{2p'}{(2-p')} \lim_{n \to \infty} \left[ J(v_n) - \frac{1}{p} J'(v_n)v_n \right] = \frac{2p'}{(2-p')} c > 0.
\]

Since \(Q \in L^\infty(\mathbb{R}^N)\), the sequence \((Q^p v_n)_n\) is bounded and Theorem 3.1 gives the existence of \(R, \zeta > 0\) and of a sequence \((x_n)_n \subset \mathbb{R}^N\) such that, up to a subsequence,

\[
\int_{B_R(x_n)} |v_n|^p \, dx \geq \zeta \quad \text{for all } n.
\]

Note that we may assume (taking \(R\) larger if necessary) that \(x_n \in \mathbb{Z}^N\) holds for all \(n\). Setting \(w_n(x) = v_n(x + x_n), x \in \mathbb{R}^N\), we find that \((w_n)_n \subset L^p(\mathbb{R}^N)\) is a bounded sequence. Hence, up to a subsequence, \(w_n \to w\) in \(L^p(\mathbb{R}^N)\). Moreover, \(J(w_n) = J(v_n)\) and \(\|J'(w_n)\| = \|J'(v_n)\|\) for all \(n\) by the periodicity of \(Q\) and the translation equivariance of \(R\). Next we show that

\[
1_{B_{R'}} |w_n|^{p'-2} w_n \to 1_{B_{R'}} |w|^{p'-2} w \quad \text{strongly in } L^p(B_{R'}) \text{ for every } R' > 0.
\]

To see this, fix \(\varphi \in \mathcal{C}_c^\infty(B_{R'}) \subset \mathcal{C}_c^\infty(\mathbb{R}^N)\). Then for \(n, m \in \mathbb{N}\) we have

\[
\left| \int_{\mathbb{R}^N} \left[ |w_n|^{p'-2} w_n - |w_m|^{p'-2} w_m \right] \varphi \, dx \right| = \left| J'(w_n)\varphi - J'(w_m)\varphi + \int_{B_{R'}} \varphi K_p(w_n - w_m) \, dx \right|
\]
Proposition A.1. Let the proof. □

Furthermore, ∥(60)∥x. Therefore, Consequently, every r > 0, and also, since (61), we first show that

Proof. To Lemma 4.1, 1 ˜L 2

For this we first assume that

Since C n → |(62)◁|ϕ dx as n → ∞,

which implies w ≠ 0. Next we show that w is a critical point of J. For every ϕ ∈ C∞ c(RN) we have, by (59),

and also, since Kp is a bounded linear operator,

Consequently,

J′(w)ϕ = ∫RN |w|2−2wϕ dx − ∫RN ϕKp(w) dx

= limn→∞ (∫RN |w|2−2wϕ dx − ∫RN ϕKp(wn) dx) = limn→∞ J′(wn)ϕ = 0.

Therefore, w ∈ LpN(RN) is a nontrivial critical point of J and Lemma[33] concludes the proof. □

APPENDIX

Proposition A.1. Let 2(N+1)(N−1) ≤ p ≤ 2N N−2 and f ∈ LpN(RN). Then u := ⨁f ∈ W2,pN(RN) ∩ LpN(RN) is a strong solution of −∆u − u = f in RN. Moreover, for every r > 0, there exists a constant Ĉ > 0 depending only on r, p and N, such that for all x0 ∈ RN,

(60) ∥u∥W2,pN(Br(x0)) ≤ Ĉ(∥u∥LpN(RN) + ∥f∥LpN(RN)).

Furthermore,

(i) if f ∈ LpN(RN) ∩ LqN(RN) and u ∈ LqN(RN) for some q ∈ (1, ∞), then u ∈ W2,qN(RN), and for every r > 0 there exists a constant D > 0 depending only on r, p, q and N, such that

(61) ∥u∥W2,qN(Br(x0)) ≤ D (∥u∥LqN(B2r(x0)) + ∥f∥LqN(B2r(x0)))

for all x0 ∈ RN.

(ii) If f ∈ LpN(RN) ∩ LqN(RN) and u ∈ LqN(RN) for some q ∈ (1, ∞), then u ∈ W2,qN(RN).

Proof. We first show that

(62) −∆u − u = f in distributional sense.

For this we first assume that f ∈ F. In this case, ⨁f ∈ F is given by

(⨁f, ϕ) = limε→0+ ∫RN f(ξ)ϕ(ξ) dξ − 1 − iε for all ϕ ∈ F.
Hence, setting \( u = \mathcal{R}f \), we obtain for every \( \varphi \in \mathcal{S} \):
\[
\langle -\Delta u - u, \varphi \rangle = \langle u, -\Delta \varphi - \varphi \rangle = \langle \mathcal{R}f, \mathcal{F}((|\xi|^2 - 1)\mathcal{F}^{-1}(\varphi)) \rangle
\]
\[
= \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \frac{\hat{f}(\xi)\hat{\varphi}(\xi)(|\xi|^2 - 1)}{|\xi|^2 - 1 - i\varepsilon} \, d\xi = \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \frac{\hat{f}(\xi)\hat{\varphi}(\xi)(|\xi|^2 - 1 - i\varepsilon)}{|\xi|^2 - 1 - i\varepsilon} \, d\xi
\]
\[
= \int_{\mathbb{R}^N} f(x)\varphi(x) \, dx = \langle f, \varphi \rangle.
\]
Here we used the fact that
\[
\lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^N} \frac{i\varepsilon}{|\xi|^2 - 1 - i\varepsilon} g(\xi) \, d\xi = 0 \quad \text{for every } g \in \mathcal{S},
\]
which follows from Lebesgue’s Theorem since \( \frac{i\varepsilon}{|\xi|^2 - 1 - i\varepsilon} \leq 1 \) for every \( \xi \in \mathbb{R}^N, \varepsilon > 0 \) and \( \lim_{\varepsilon \to 0^+} \frac{i\varepsilon}{|\xi|^2 - 1 - i\varepsilon} = 0 \) for \( \xi \in \mathbb{R}^N \) with \( |\xi| \neq 1 \). Hence we have \(-\Delta u - u = f\) in the distributional sense.

Now let \( f \in L^p(\mathbb{R}^N) \) and consider a sequence \((f_n)_n \subset \mathcal{S}\) with \( \|f_n - f\|_{L^p} \to 0 \) as \( n \to \infty \). Then \( u_n := \mathcal{R}f_n \) solves \(-\Delta u_n - u_n = f_n\) in distributional sense, and \( u_n \to u \) in \( L^p(\mathbb{R}^N) \) by Theorem 2.1. Hence Consequently, \(-\Delta u_n - u_n \to f\) and \( u_n \to u \) in \( \mathcal{S}' \) as \( n \to \infty \), so that (62) is true.

We now take \( x_0 \in \mathbb{R}^N \) and \( R > 0 \) and consider the mollification \((u_{\varepsilon})_{\varepsilon > 0}\) of \( u := \mathcal{R}f \), i.e., \( u_{\varepsilon} := \rho_{\varepsilon} \ast u \) where \( \rho_{\varepsilon}(x) = \varepsilon^{-N} \rho(\tfrac{x}{\varepsilon}), x \in \mathbb{R}^N \) for some function \( \rho \in \mathcal{C}^\infty(\mathbb{R}^N) \) satisfying \( \rho(x) \geq 0 \), for all \( x \in \mathbb{R}^N \), \( \text{supp} \rho \subseteq B_1 \) and \( \int_{\mathbb{R}^N} \rho \, dx = 1 \). Since \( u \in L^p(\mathbb{R}^N) \), we obtain \( u_{\varepsilon} \in L^p(B_R(x_0)) \) and consequently, \( u_{\varepsilon} \to u \) in \( L^p(B_R(x_0)) \) as \( \varepsilon \to 0^+ \). Similarly, considering the mollification \((f_{\varepsilon})_{\varepsilon > 0}\) of \( f \), we see that \( f_{\varepsilon} \to f \) in \( L^p(\mathbb{R}^N) \) and therefore also in \( L^{p'}(B_R(x_0)) \), as \( \varepsilon \to 0^+ \). From the properties of the mollification of \( L^p\)-functions and of tempered distributions, with respect to differential operators with constant coefficients (see [24]), we obtain
\[
-\Delta u_{\varepsilon} - u_{\varepsilon} = -\Delta (u \ast \rho_{\varepsilon}) - (u \ast \rho_{\varepsilon}) = (-\Delta u - u) \ast \rho_{\varepsilon} = f \ast \rho_{\varepsilon} = f_{\varepsilon} \quad \text{in } \mathbb{R}^N.
\]
Therefore, the elliptic regularity theory (see [15, Theorem 9.11]) shows the existence, for all \( \varepsilon > 0 \), of some constant \( C > 0 \), depending only on \( r, p, N \), such that
\[
\|u_{\varepsilon}\|_{W^{2,p'}(B_r(x_0))} \leq C \left( \|u_{\varepsilon}\|_{L^{p'}(B_r(x_0))} + \|f_{\varepsilon}\|_{L^p(\mathbb{R}^N)} \right) \quad \text{for all } \varepsilon > 0. \tag{63}
\]
Choosing some sequence \((\varepsilon_n)_n \subset (0, \infty)\) such that \( \varepsilon_n \to 0 \) as \( n \to \infty \) and replacing \( u_{\varepsilon_n} \) by \( u_{\varepsilon_n} - u_{\varepsilon_m} \) in (63) gives that \((u_{\varepsilon_n})_n\) is a Cauchy sequence in \( W^{2,p'}(B_r(x_0)) \) and therefore, there exists \( u \in W^{2,p'}(B_r(x_0)) \) such that \( u_{\varepsilon_n} \to u \) in \( W^{2,p'}(B_r(x_0)) \) as \( n \to \infty \). Since this also implies \( u_{\varepsilon_n} \to u \) in \( L^p(B_r(x_0)) \), we find that \( u = u \) a.e. in \( B_r(x_0) \), and it follows that \( u \in W^{2,p'}(B_r(x_0)) \) and solves the equation \(-\Delta u - u = f\) almost everywhere in \( B_r(x_0) \). Furthermore, (63) gives
\[
\|u\|_{W^{2,p'}(B_r(x_0))} \leq C \left( \|u\|_{L^{p'}(B_r(x_0))} + \|f\|_{L^p(\mathbb{R}^N)} \right)
\]
\[
\leq \bar{C} \left( \|u\|_{L^p(\mathbb{R}^N)} + \|f\|_{L^p(\mathbb{R}^N)} \right),
\]
where \( \bar{C} = C \max \{1, [\omega_N(2r)^N]^\frac{1}{r^2} \} \) and \( \omega_N \) denotes the volume of the unit ball in \( \mathbb{R}^N \). Since \( r > 0 \) and \( x_0 \in \mathbb{R}^N \) were arbitrarily chosen, it follows that \( u \in W^{2,p'}(\mathbb{R}^N) \) is a strong solution of \(-\Delta u - u = f\) and, for every \( r > 0 \), there exists a constant \( \bar{C} > 0 \) depending only on \( r, p, N \) such that (64) holds for all \( x_0 \in \mathbb{R}^N \).

(i) Considering as before the mollifications \((u_{\varepsilon_n})_{n \geq 0}\) of \( u \) and \((f_{\varepsilon_n})_{n \geq 0}\) of \( f \), we obtain from the previous argument that \(-\Delta u_{\varepsilon_n} - u_{\varepsilon_n} = f_{\varepsilon_n}\) on \( \mathbb{R}^N \). Moreover, for all \( x_0 \in \mathbb{R}^N \) and \( r > 0 \), \( u_{\varepsilon_n} \to u \) and \( f_{\varepsilon_n} \to f \) in \( L^p(B_r(x_0)) \) as \( \varepsilon \to 0^+ \). Using again elliptic regularity theory and reasoning as above, we find \( u \in W^{2,q}_{\text{loc}}(\mathbb{R}^N) \) and, for every \( r > 0 \), the existence of some constant \( \bar{D} \), depending only on \( r, p, q \) and \( N \) such that
(ii) As a consequence of (i), there holds $u \in W^{2, q}_0(\mathbb{R}^N)$ and $u$ solves $-\Delta u - u = f$ a.e. on $\mathbb{R}^N$. Considering again the mollifications $(u_\varepsilon)_{\varepsilon > 0}$ of $u$ and $(f_\varepsilon)_{\varepsilon > 0}$ of $f$ we see, using the Calderón-Zygmund estimate (see [15, Corollary 9.10]), that for any sequence $(\varepsilon_n)_n \subset (0, \infty)$ such that $\varepsilon_n \to 0$ as $n \to \infty$, the sequence $(u_{\varepsilon_n})_n$ is a Cauchy sequence in $W^{2, q}(\mathbb{R}^N)$. Since the argument in (i) implies $u_{\varepsilon_n} \to u$ in $W^{2, q}_0(\mathbb{R}^N)$, we conclude that $u \in W^{2, q}(\mathbb{R}^N)$. □

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