Critical collapse in the axion–dilaton system in diverse dimensions

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Abstract
We study the gravitational collapse of the axion–dilaton system suggested by type-IIB string theory in dimensions ranging from 4 to 10. We extend previous analysis concerning the role played by the global $SL(2, R)$ symmetry and we evaluate the Choptuik exponents in the elliptic case.

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1. Introduction

It is nearly 20 years since Choptuik discovered critical phenomena in gravitational collapse, and revolutionized the field of numerical relativity [1, 2]. In this paper, we revisit the self-similar, spherical gravitational collapse studied in the past in four dimensions in [3, 4], and extend their analysis to higher dimensions and to other implementations of self-similarity. Our work is motivated by the AdS/CFT correspondence [5–8]. We are still far from a holographic description of black hole formation in type-IIB string theory. Some exploratory work was done in [9, 10]. In the context of type IIB (for reviews of string theory and duality see [11, 12]) and AdS/CFT, one would like to consider collapse on spaces which approach asymptotically $AdS_5 \times S^5$. With the bosonic fields in the theory, a natural system to be considered involves the axion–dilaton and the self-dual 5-form field. One can show that in five dimensions the simplest dynamical set to be studied involves just the Einstein axion–dilaton system with a cosmological constant. This poses an apparent problem in considering self-similar collapse, because Einstein spaces do not admit homothetic vector fields. However, in the context of critical gravitational collapse, we are considering the collapse of matter to form small-mass black holes and we only need to consider a small spacetime region close to which the singularity forms. This should be independent of the asymptotic structure of the spacetime where the collapse takes place. There is numerical evidence in asymptotically AdS spacetimes showing that this is the case [13]. Hence, we will eliminate the cosmological constant and analyse self-similar critical collapse in dimensions 4–10. We are interested in the critical solution and the corresponding Choptuik
exponent. Our analysis follows closely [3, 4], and the physical and geometrical interpretation is essentially the same, so we will not review their work in this paper.

In section 2, we write the action and the equations of motion of the axion–dilaton system, and the conditions that follow from the requirement of continuous self-similarity. We found three possible conditions and associated ansätze for the matter fields. We called them the elliptic, hyperbolic and parabolic cases associated with the three different classes of $SL(2, R)$ transformations that can be used to compensate for a scaling transformation in spacetime. The analysis in [3, 4] corresponds to the four-dimensional elliptic case, but their methods can easily be adapted to the two other cases as well as to other dimensions. We do not strive to obtain very precise numerical results given that we have no practical applications for them. We are interested in exploring the existence of critical solutions and the values of their Choptuik exponents as a function of the spacetime dimension. For a massless scalar field, this can be seen in [14, 15], the shape of the plot of our exponents as a function of dimension is quite analogous to the results of Oren and Sorkin in [15]; however, more precise numerical work would be necessary before a detailed comparison is possible. More details as well as the results for the hyperbolic and parabolic cases will appear in a subsequent publication [16].

2. The axion–dilaton system

We will consider the $d$-dimensional spherical collapse of an axion–dilaton $(a, \Phi)$ system. Both fields can be combined into a single complex scalar field $\tau \equiv a + i e^{-\Phi}$. The action for the model is

$$S = \frac{1}{16\pi G} \int d^d x \sqrt{-g} \left( R - \frac{1}{2} \frac{\partial_a \tau \partial^a \bar{\tau}}{(\text{Im} \tau)^2} \right)$$

where $R$ is the scalar curvature. The equations of motion are

$$R_{ab} - \frac{1}{4(\text{Im} \tau)^2} (\partial_a \tau \partial_b \bar{\tau} + \partial_a \bar{\tau} \partial_b \tau) = 0$$

$$\nabla^a \nabla_a \tau + \frac{i \nabla^a \tau \nabla_a \tau}{\text{Im} \tau} = 0.$$  

The theory is classically invariant under $SL(2, R)$ transformations:

$$\tau \to \frac{a \tau + b}{c \tau + d},$$

where $(a, b, c, d) \in R, ad - bc = 1$ and $g_{ab}$ does not transform. This group is supposed to be broken into its integer subgroup $SL(2, Z)$ as a consequence of electric–magnetic duality in string theory [11, 12].

Following [3], the spherically symmetric metric can be written as

$$ds^2 = (1 + u(t, r))(-b(t, r)^2 dr^2 + dr^2) + r^2 d\Omega_{d-2}.$$  

The time coordinate is chosen so that spherical collapse on the time axis first occurs at $t = 0$; hence, the collapse takes place for $t < 0$. We can still implement time redefinitions for (5); hence, as in [3], we set $b(t, 0) = 1, t < 0$ and regularity for $t < 0$ implies that $u(t, 0) = 0, t < 0$.

Continuous self-similarity means the existence of a homothetic Killing vector $\xi$ generating global scale transformations:

$$\mathcal{L}_\xi g_{ab} = 2g_{ab}.$$  

In spherical coordinates, $\xi = t \partial / \partial t + r \partial / \partial r$. Defining the scale-invariant variable $z = -r/t$, self-similarity of the metric means that the unknown functions $u(t, r)$ and $b(t, r)$ are just
the functions of \( z \). The next question to address is the transformation of \( \tau(t, r) \) under scale transformations. Since the action is \( SL(2, R) \)-invariant, we can compensate a scale transformation of the coordinates \((t, r)\) by an \( SL(2, R) \) transformation. Thus, if we change variables to \((t, z)\), we obtain a differential condition for \( \tau(t, z) \):

\[
\frac{\partial}{\partial t} \tau(t, z) = \alpha_0 + \alpha_1 \tau + \alpha_2 \tau^2
\]

with \( \alpha_{0,1,2} \) being the real numbers. The quadratic polynomial on the right-hand side has two roots, which can be two complex conjugate numbers, two distinct real numbers, or a double real root. They correspond to compensating the scaling transformation with, respectively, an elliptic, a hyperbolic, or a parabolic transformation in \( SL(2, R) \). By straightforward manipulations and redefinitions, the three cases yield the following ansätze for \( \tau(t, z) \). In the elliptic case,

\[
\tau(t, r) = \frac{1 - (-t)^w f(z)}{1 + (-t)^w f(z)},
\]

under a scaling transformation \( t \to \lambda t \), \( \tau(t, r) \) changes by an \( SL(2, R) \) rotation. In the hyperbolic case,

\[
\tau(t, r) = \frac{1 - (-t)^w f(z)}{1 + (-t)^w f(z)},
\]

and under a scaling transformation \( t \to \lambda t \), \( \tau(t, r) \) changes by an \( SL(2, R) \) boost. Using an \( SL(2, R) \)-transformation, we can transform \( \tau(t, r) \to -(-t)^w f(z) \). Finally, in the parabolic case, a scaling transformation can be compensated by a translation. After some simple manipulations and redefinitions, the three cases yield the following ansätze for \( \tau(t, z) \). In the elliptic case,

\[
\tau(t, r) = f(z) + \omega \log(-t), \quad \omega \in \mathbb{R}.
\]

We have written \(-t\) throughout the previous equations because the collapse will take place for \( t < 0 \). We can also extend the solutions for \( t > 0 \); hence, the correct ansätze for any value of \( t \) is to replace \(-t \to |t|\). In all these expressions, \( f(z) \) is an arbitrary complex function and \( \omega \) is an arbitrary real parameter to be fixed by requiring the critical solution to be regular.

3. Equations of motion

We collect in this section the equations of motion for the three ansätze, although later we will only present results for the elliptic case. Since we are implementing spherical symmetry, the gravitational degrees of freedom do not propagate, and there are no gravitational waves. Hence, \( u(z) \) and \( b(z) \) should be expressed in terms of \( f(z) \). Common to all three cases, it is simple to show that

\[
u(z) = -\frac{z b'(z)}{(d - 3) b(z)}.
\]

This follows from the Einstein equation for the angular variables. The other equations of motion for the self-similar solution involve \( b(z) \) and \( f(z) \). In fact, the equation for \( b \) is a first-order linear inhomogeneous equation for \( b(z)^2 \) whose initial condition is determined by \( b(0) = 1 \) together with the initial conditions for \( f(z) \) and \( f'(z) \), which are determined by requiring smoothness of the critical solution. This determines not only the initial conditions but also the possible value of \( \omega \).

The equations of motion are quite complicated, and their derivation is rather tedious but straightforward. The equations in the elliptic case are given by

\[
0 = b' + \frac{2(z b^2 - z^2)}{(d - 2) b(-1 + |f|^2)^2} f \bar{f} - \frac{2i \omega (b^2 - z^2)}{(d - 2) b(-1 + |f|^2)^2} (f \bar{f} - \bar{f} f')
\]
It is useful to note that the equations are invariant under a global redefinition of the phase of $f(z)$. We can choose the phase of $f(z)$ to any convenient value at a particular $z$ that we will choose to be the origin.

In the hyperbolic case, the equations of motion are quite similar, but now they are invariant under a constant scaling $f \rightarrow \lambda f$; hence, we can choose $|f(z)|$ or its real or imaginary part as we wish at a particular value of $z$:

$$
0 = b' = \frac{2z(b^2 - z^2)}{(d-2)b^2(f - \bar{f})^2} f^2 \bar{f}^2 + \frac{2\omega(b^2 - z^2)}{(d-2)b^2(f - \bar{f})^2} (f \bar{f}^2 + \bar{f} f^2) + \frac{2\omega^2 z |f|^2}{(d-2)b^2(f - \bar{f})^2},
$$

$$
0 = -f'' = \frac{2z(b^2 + z^2)}{(d-2)b^2(f - \bar{f})^2} f^2 \bar{f}^2 + \frac{2}{f - \bar{f}} \left( \frac{1}{f} + \frac{\omega(b^2 + z^2)}{(d-2)b^2(f - \bar{f})^2} \right) f \bar{f}^2,
$$

$$
0 = f'' - \frac{2z(b^2 + z^2)}{(d-2)b^2(f - \bar{f})^2} f^2 \bar{f}^2 + \frac{2}{f - \bar{f}} \left( \frac{1}{f} + \frac{\omega(b^2 + z^2)}{(d-2)b^2(f - \bar{f})^2} \right) f \bar{f}^2,
$$

Finally, in the parabolic ansatz, the equations of motion are

$$
0 = b' = \frac{2z(b^2 - z^2)}{(d-2)b^2(f - \bar{f})^2} f^2 \bar{f}^2 + \frac{2\omega(b^2 - z^2)}{(d-2)b^2(f - \bar{f})^2} (f \bar{f}^2 + \bar{f} f^2) + \frac{2\omega^2 z |f|^2}{(d-2)b^2(f - \bar{f})^2},
$$

$$
0 = -f'' = \frac{2z(b^2 + z^2)}{(d-2)b^2(f - \bar{f})^2} f^2 \bar{f}^2 + \frac{2}{f - \bar{f}} \left( \frac{1}{f} + \frac{\omega(b^2 + z^2)}{(d-2)b^2(f - \bar{f})^2} \right) f \bar{f}^2,
$$

$$
0 = f'' - \frac{2z(b^2 + z^2)}{(d-2)b^2(f - \bar{f})^2} f^2 \bar{f}^2 + \frac{2}{f - \bar{f}} \left( \frac{1}{f} + \frac{\omega(b^2 + z^2)}{(d-2)b^2(f - \bar{f})^2} \right) f \bar{f}^2.
$$

They are invariant under arbitrary shifts of $f(z)$ by a real number, and thus, we can choose its real part as we wish at a particularly convenient point.
Table 1. Parameters determining the critical solution for dimensions 4–10, with dimension increasing from top to bottom.

| \( \omega \) | \( z_+ \) | \(| f(0) | \) | \(| f(z_+) | \) |
|-----------|---------|--------|--------|
| 1.176     | 2.609   | 0.892  | 0.364  |
| 1.297     | 2.674   | 0.903  | 0.392  |
| 1.469     | 2.771   | 0.910  | 0.397  |
| 1.610     | 2.822   | 0.914  | 0.404  |
| 1.721     | 2.871   | 0.918  | 0.413  |
| 1.791     | 2.929   | 0.923  | 0.427  |
| 1.852     | 2.998   | 0.928  | 0.439  |

4. Properties of the critical solutions in the elliptic case and their Choptuik exponents

We briefly present some of our results in the study of the critical solution in the elliptic case. The analysis follows closely the arguments in [3, 4]. In all three nonlinear systems (12)–(14), we have five singular points, \( z = \pm 0 \) represents the axis \( r = 0 \) and regularity is imposed. The point \( z = \infty \) represents the surface \( t = 0 \). Away from \( r = 0 \), there is nothing special in this surface; hence, regularity is also imposed. The simplest way to see that there is no problem at \( z = \infty \) is through a change of variables and a redefinition of the fields \( f(z) \) and \( b(z) \) [3]. The singularities \( b(z_\pm) = \pm z_\pm \) are those surfaces where the homothetic Killing vector becomes null. They correspond to the backward (resp. forward) light cone of the spacetime origin. For \( b(z_+) = z_+ \), the solution should be smooth across this surface. The forward cone \( b(z_-) = -z_- \) of the singularity represents the Cauchy horizon and we should not require more than continuity of \( f, b \) across this surface. We require smoothness of the spacetime below the forward cone and then extend the space by continuity inside it. In the elliptic case, it is convenient to write \( f(z) = f_m(z) e^{if(z)} \), requiring that regularity at both the origin and \( z_+ \) leaves only four parameters to be determined: \(| f(0) |, w, z_+, | f(z_+) | \). To find them we proceed by integrating from close to the origin towards \( z_+ \) and also from a little below \( z_+ \) towards the origin, and match the results at an intermediate point, say \( z = 1 \). By matching the functions and the relevant first derivatives, we can determine the unknown parameters in the critical solution. We do this for every dimension between \( d = 4 \) and \( d = 10 \). The results are shown in table 1. We did not optimize the numerical precision for these computations. In determining the analogous values of \( z_- \) and \( f(z_-) \) more precision is required because the solution is quite flat inside the forward light cone. The results will appear in [16].

We compute the Choptuik exponents following the methods in [2, 4, 17, 18]. Given the critical solution, we can perturb it to find the critical exponent \( \gamma \) as follows. Let \( h \) be any function determining the critical solution: \( b \) or \( f \). Next, consider small perturbations around the critical solution \( h(z, t) = h_{ss}(z) + \epsilon |t|^{-\kappa} h_{pert}(z) \).

where \( h_{ss}(z) \) is the critical solution, \( \epsilon \) is a small number, \( \kappa \) is a constant and \( h_{pert}(z) \) depends only on \( z \). Substituting \( h(z, t) \) into the full equations of motion (2) and (3) and keeping first-order terms in \( \epsilon \) gives a set of linear equations for the perturbation and an eigenvalue equation for \( \kappa \). This eigenvalue equation can in principle have a number of possible solutions for \( \kappa \). The solution with the largest value of \( \text{Re}(\kappa) \) will be responsible for the fastest growing perturbation in (15) and is called the ‘most relevant mode’. The critical exponent is given by

\[
\gamma = \frac{1}{\text{Re}(\kappa)}.
\]
We also require that the perturbations are smooth at the singularities of the original equations. This determines $h_{\text{pert}}(z)$ up to some rescalings (which is fine given that the equations are linear) and also the most relevant value for $\text{Re}(\kappa)$. In fact, we take $\kappa$ to be real. The values we obtain for the Choptuik exponents in dimensions from 4 to 10 appear in table 2. The numerical accuracy decreases as the dimension increases. The third significant digit in table 2 should not be trusted, specially in dimensions 9 and 10. More powerful numerical methods are necessary to obtain more accurate values for the exponents.

We have preliminary results in the hyperbolic and parabolic cases indicating that there are also critical solutions in diverse dimensions with Choptuik exponents different from those found for the elliptic case [16].

As mentioned in the introduction, the $SL(2, R)$ symmetry of the classical type-IIB string theory is supposed to break into $SL(2, \mathbb{Z})$ once quantum effects are taken into account. This raises the very interesting possibility that the critical solution in the quantum case will not be continuous self-similar but rather it will have discrete self-similarity as for the massless scalar field first analysed by Choptuik [1]. In other words, we want to know if there are elements $\Gamma \in SL(2, \mathbb{Z})$, such that

$$\tau (e^{a\Delta t}, e^{b\Delta r}) = \frac{a\tau + b}{c\tau + d}, \quad \Gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}),$$

with $\Delta r$ being the corresponding echo parameter. To settle this question, one would have to do the full numerical integration of Einstein’s equation without assuming continuous self-similarity.

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