Momentum-energy tensor associated to the quasiparticles in anisotropic superconductors

L. A. Peña Ardila, W. Herrera, and Virgilio Niño
Departamento de física, Universidad Nacional de Colombia, Bogotá, Colombia

From a Lagrangian density for the Bogoliubov de Gennes equations in anisotropic superconductors, we find the momentum-energy tensor associated to the quasiparticles of the system. For this, we make infinitesimal translations on both space and time and we use the Noether’s theorem. We prove that beyond to the usual terms associated to the electron-hole dynamic in electromagnetic potentials, appears terms that involves the pair potential and that are obtained from the coupling of the electron-like and hole-like quasiparticles.

PACS numbers:

I. INTRODUCTION

The Bogoliubov-de Gennes (BdG) equations describes the bahaviour of the elementary excitation of a superconductor system \[1, 2\]. Since there is a Lagrangian density that allows one to derive these equations and through of the Noether’s theorem for infinitesimal transformations leaving the action invariant \[3\], it is possible to find the conservation’s laws associated to the superconductor system. In this paper we build the momentum-energy tensor associated to the quasiparticles of the system. For this, Noether theorem allows us to find both a set of equations and conservation laws from transformations that leave the action invariant. If an infinitesimal transformation of \( p \) parameters that leaves action unchangeable, then there are \( p \) quantities conserved \[3\]. If the motion equation for the fields \( \Phi_k \) is obtained from the principle of least action and moreover the transformations in space, time and the fields leaves the action invariant. The Noether theorem yields:

\[
\begin{align*}
\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} &= 0, \\
\rho &= \mathcal{L} \delta t - \frac{\partial \mathcal{L}}{\partial (\phi_k)} \left( \sum_{j=1}^{3} \partial_{\phi_k} \phi_j \delta x_j + \phi_k \delta t - \delta \Phi_k \right) \\
J_i &= \mathcal{L} \delta x_i - \frac{\partial \mathcal{L}}{\partial (\phi_i \phi_k)} \left( \sum_{j=1}^{3} \partial_{\phi_k} \phi_j \delta x_j + \phi_k \delta t - \delta \Phi_k \right)
\end{align*}
\]

Where \( \rho \) and \( \mathbf{J} \) correspond to the charge and current densities associated to the infinitesimal transformation.

MOMENTUM-ENERGY TENSOR FOR ANISOTROPIC SUPERCONDUCTORS.

The Bogoliubov - de Gennes equations describe the quasiparticles in a superconductor are given by:

\[
i\hbar \frac{\partial}{\partial t} \psi(r, t) = \hat{H} \psi(r, t) = \hat{H}_0(r, t) \psi(r, t) + \int dr \Delta(r', r') \psi(r', t)
\]

With,

\[
\psi(r, t) = \begin{pmatrix} u(r, t) \\
v(r, t) \end{pmatrix}, \quad \hat{H}_0(r, t) = \frac{\hat{\rho}^2(r, t)}{2m} \hat{\sigma}_z + \hat{V}(r, t) \\
\hat{V}(r, t) = (U(r, t) - \mu) \hat{\sigma}_z, \quad \hat{\pi}(r, t) = \begin{pmatrix} \hat{\pi}_r(r, t) & 0 \\
0 & \hat{\pi}_h(r, t) \end{pmatrix} \\
\Delta(r, r') = \begin{pmatrix} 0 & \Delta^*(r, r') \\
\Delta(r, r') & 0 \end{pmatrix}, \quad \hat{\sigma}_z = \begin{pmatrix} 1 & 0 \\
0 & -1 \end{pmatrix}
\]

where \( \psi(r, t) \) is the wavefunction of the quasiparticle with two components electron \( u(r, t) \) and hole \( v(r, t) \), whereas \( \hat{\pi}_r(r, t) = (-i\hbar \nabla - eA) \) and \( \hat{\pi}_h(r, t) = (-i\hbar \nabla + eA) \) depicts the electron and hole momentum respectively; \( A(r, t) \) is the magnetic vector potential, \( U(r, t) \) is the scalar potential that includes both the Hartree-Fock and the external potential, \( \mu \) is the chemical potential and \( \Delta(r, r') \) is the pair potential associated to quasiparticles and it is coupled to the electron and hole components. From the lagrangian density,

\[
\mathcal{L} = \frac{i\hbar}{2} \left[ \psi^+(r, t) \psi(r, t) - \phi^+(r, t) \phi(r, t) \right] - \psi^+(r, t) \hat{V}(r, t) \hat{\sigma}_z \psi(r, t) - \frac{1}{2m} \left[ \hat{P}(r, t) \phi(r, t) \right]^+ \hat{\sigma}_z \hat{P}(r, t) \phi(r, t) + \int dr ' \psi(r, t)^+ \Delta(r, r') \psi(r', t).
\]
Moreover when Eq. [9] is integrated over all space, one obtains:

$$\frac{d}{dt} \int d^3r \rho(r, t) = \frac{dQ(r, t)}{dt} = 0. \quad (7)$$

Where $Q$ is a generalized charge and is itself the conserved quantity.

**TEMPORAL TRANSLATION-CONSERVATION OF THE ENERGY**

Let us do an infinitesimal transformation in time, in this way:

$$t' = t + \delta t. \quad (8)$$

The variations respect to the fields and spatial coordinates are vanished. By plugging the lagrangian density Eq. [8] into Eq. [6], we obtain:

$$\rho_0(r, t) = \frac{1}{2m} [\pi(r) \psi(r, t)]^+ \sigma_z \cdot \pi(r) \psi(r, t)$$

$$+ \psi^+(r, t) \hat{V}(r) \psi(r, t) + \int d\rho^+ (r, t) \Delta(r, r') \psi(r, t) \quad (9)$$

This expression can be interpreted as an energy density that is shown as the sum of the energy density associated with the electron and hole components plus the energy density that couples electrons and holes depending on the pairs potential $\Delta$.

By using Eq. [7] and recalling that this transformation leaves the action invariant, we have $\frac{df}{dt} \int \rho_0 dV = \frac{d}{dt} \langle E \rangle = 0$, which implies that the expectation value of energy is a constant of motion. Now let us examine what happens with the density of current in this case. From Eq. [8] for the current density and again using the lagrangian density Eq. [3], we have

$$J_{0,j} = \frac{1}{2} \left[ (V_j \psi)^+ \hat{H} \psi + (\hat{H} \psi)^+ (V_j \psi) \right]$$

$$= \frac{1}{2} \left[ (-i\hbar \sigma_z \partial_j - eA_j \hat{1})\psi \right]^+ \hat{H} \psi$$

$$+ (\hat{H} \psi)^+ (-i\hbar \sigma_z \partial_j - eA_j \hat{1})\psi \quad j = 1, 2, 3. \quad (10)$$

Where $\frac{\hat{r}}{m} = \hat{v}$ is the velocity of group of the particle. From this equation is observed that $\mathbf{J}$ is an energy flux, because it can be written as the quasiparticle density of charge times its group velocity.

**SPATIAL TRANSLATION-CONSERVATION OF LINEAR MOMENTUM**

Now we can do an infinitesimal translation in coordinates

$$x_i' = x_i + \delta x_i \quad i = 1, 2, 3. \quad (11)$$

By using the Eq. [5] for each spatial component $x$, $y$, and $z$ we get a charge density associated given by:

$$\rho_i = \psi^+ \hat{\pi}_i \psi = \psi^+ (-i\hbar \sigma_z \partial_i - eA_i \sigma_j) \psi \quad (12)$$

In the same way. Using Eq. [6] the current density associated to $\rho_i$ is given by:

$$J_{i,j} = \frac{1}{2} \left[ (\hat{V}_j \psi)^+ \hat{\pi}_i \psi - (\hat{\pi}_i \psi)^+ \hat{V}_j \psi \right]$$

$$= \frac{1}{2} \left[ (-i\hbar \sigma_z \partial_j - eA_j \hat{1})\psi \right]^+ (-i\hbar \sigma_z \partial_i - eA_i \sigma_j) \psi$$

$$+ [(-i\hbar \sigma_z \partial_i - eA_i \hat{1})\psi] (-i\hbar \sigma_z \partial_j - eA_j \hat{1})\psi \quad (13)$$

We obtain that Eq. [12] corresponds to the density of linear momentum, whereas Eq. [15] corresponds to the density of current of momentum, that is proportional to the density of momentum and to the velocity of group of the quasiparticle. If the action is invariant under these translations, one can integrate over all space the density of momentum and it yields

$$\frac{d}{dt} \int \psi^+ \hat{\pi}_i \psi dV = \frac{d}{dt} \langle \pi_i \rangle \quad (15)$$

Therefore the momentum associated to the quasiparticles is conserved. The density of momentum can be written as $\rho_i = \rho_{ei} + \rho_{hi}$, where $\rho_{ei}$ and $\rho_{hi}$ are the momentum density associated to electron and hole components, respectively. So we have

$$\rho_{ei} = u^*(-i\hbar \partial_i - eA_i)u \quad ; \quad \rho_{hi} = v^*(-i\hbar \partial_i + eA_i)v. \quad (16)$$

In the case that action is not invariant under transformation Eq. [12], there will appear sources proportional to the scalar potential gradient, magnetic vector potential and the pairs potential. All the equations we have derived so far from the space-time transformations are summarized in a tensorial form:

$$T_{uv} = \begin{pmatrix} \rho_0 & J_{01} & J_{02} & J_{03} \\ J_{11} & J_{12} & J_{13} & J_{14} \\ J_{21} & J_{22} & J_{23} & J_{24} \\ J_{31} & J_{32} & J_{33} & J_{34} \end{pmatrix}$$

Where the term $T_{00}$ corresponds to the density of energy of the quasiparticle, whose explicit form is given in
Eq. [3], the terms of the form $T_{ij}$ for $i, j = 1, 2, 3$ are related with the density of current of energy in each component that have been calculated in Eq. [10]. Now the terms of $T_{ij}$ represents the density of linear momentum in the direction $i$ given by Eq. [12] and finally the terms of the form $T_{ij}$ corresponds to the component $j$ of the momentum current density is given by Eq. [35]. In conclusion, the tensorial form for the energy and momentum equations written as:

$$\frac{dT_{uv}}{dx_u} = 0 \quad u, v = 0, 1, 2, 3.$$  \hspace{1cm} (17)

When a spatial or temporal infinitesimal transformation does not leave the lagrangian density invariant an associated source appears. In classical mechanics, when a potential does not depend on the position the total force is zero and is a conserved quantity. In contrast, if the potential does depend on the position the quantity is not conserved and the source is the total force. In the next section we will see that the lagrangian Eq. [3] is not invariant if for instance, it contains a potential vector dependence $L(A(x))$.

**EXAMPLE: LORENTZ FORCE AS A SOURCE FROM A LAGRANGIAN DEPENDING ON THE POTENTIAL VECTOR**

A particle with mass $m$ and electric charge $q = -e$ in presence of a magnetic field can be described by:

$$\hat{H} \psi(\mathbf{r}, t) = \left(\hat{p} - e\hat{A}\right)^2 \psi(\mathbf{r}, t) = -i\hbar \frac{\partial \psi(\mathbf{r}, t)}{\partial t}$$  \hspace{1cm} (18)

where $\hat{\mathbf{p}} = -i\hbar \nabla$. A magnetic field is generated by a potential vector $\hat{\mathbf{A}}$. From Eq. [18] one obtains the continuity equation Eq. [15] with,

$$\hat{J} = \frac{1}{2m} \left[ \psi^+ \left( -i\hbar \nabla - q\hat{A} \right) \psi - \psi \left( -i\hbar \nabla + q\hat{A} \right) \psi^+ \right]$$  \hspace{1cm} (19)

The temporal evolution of an operator $\hat{O}$ is described by

$$i\hbar \frac{d}{dt} \langle \hat{O} \rangle = \langle [\hat{O}, \hat{H}] \rangle$$  \hspace{1cm} (20)

By definition the conjugate momentum is written as, $\hat{\pi} = \hat{\mathbf{p}} - e\hat{\mathbf{A}}$ and the velocity is given by:

$$\hat{\mathbf{v}} = \hat{\mathbf{v}} = \frac{\hat{\mathbf{p}} - e\hat{\mathbf{A}}}{\hbar} = \frac{\hat{\pi}}{\hbar} = \left\langle \frac{\hat{\mathbf{p}} - e\hat{\mathbf{A}}}{\hbar} \right\rangle$$  \hspace{1cm} (21)

therefore $\hat{\pi}/m$ plays the role of velocity and the temporal evolution of $\hat{\pi}$ will give rise to the quantum Lorentz force, rather than the temporal evolution of $\hat{\mathbf{p}}$. In fact, since $\hat{\mathbf{v}} = \hat{\mathbf{v}} - e\hat{\mathbf{A}}$, clearly Eq. [22] does not match with the Lorentz force definition. Here repeated indexes sum is considered and $i = 1, 2, 3$ are the coordinates. Let us find the temporal evolution of the potential vectorial $\hat{\mathbf{A}}$:

$$\frac{d}{dt} \langle \hat{\mathbf{A}}^i \rangle = \left\langle \left[ \hat{\mathbf{A}}^i, \hat{H} \right] \right\rangle$$  \hspace{1cm} (23)

$$\frac{i\hbar}{2m} \frac{d}{dt} \langle \hat{\mathbf{A}}^i \rangle = \left\langle \left[ \hat{\mathbf{A}}^i, \left( \hat{p}_j + q\hat{A}_j \right) \right] \right\rangle$$  \hspace{1cm} (24)

$$\frac{i\hbar}{2m} \frac{d}{dt} \langle \hat{\mathbf{A}}^i \rangle = \left\langle \left[ \hat{\mathbf{A}}^i, \left( \hat{p}_j + q\hat{A}_j \right) \right] \right\rangle$$  \hspace{1cm} (25)

we can use the fact that $[\hat{a}, b^2] = [\hat{a}, \hat{b}\hat{b} + \hat{b}\hat{a}]$. Therefore, straightforwardly

$$\frac{i\hbar}{2m} \frac{d}{dt} \langle \hat{\mathbf{A}}^i \rangle = \left\langle \left[ \hat{\mathbf{A}}^i, \left( \hat{p}_j + q\hat{A}_j \right) \right] \right\rangle$$  \hspace{1cm} (26)

Again ones associates $\langle \hat{p}_j + q\hat{A}_j \rangle = m\hat{\mathbf{v}}_j = \hat{\pi}_j$. Therefore

$$\frac{i\hbar}{2m} \frac{d}{dt} \langle \hat{\mathbf{A}}^i \rangle = \frac{1}{2} \left\langle \left[ \hat{\mathbf{A}}^i, \left( \hat{p}_j + q\hat{A}_j \right) \right] \hat{\mathbf{v}}_j + \hat{\mathbf{v}}_j \left[ \hat{\mathbf{A}}^i, \left( \hat{p}_j + q\hat{A}_j \right) \right] \right\rangle$$  \hspace{1cm} (27)

Evidently $[\hat{\mathbf{A}}^i, \hat{\mathbf{A}}^j] = 0$. One ends up with:

$$\frac{i\hbar}{2m} \frac{d}{dt} \langle \hat{\mathbf{A}}^i \rangle = \frac{i\hbar}{2} \left\langle \left[ \hat{\mathbf{A}}^i, \hat{p}_j \right] \hat{\mathbf{v}}_j + \hat{\mathbf{v}}_j \left[ \hat{\mathbf{A}}^i, \hat{p}_j \right] \right\rangle$$  \hspace{1cm} (28)

Easily one shows that $[\hat{\mathbf{A}}^i, \hat{p}_j] = i\hbar \partial_j \hat{\mathbf{A}}^i$. Then,

$$\frac{i\hbar}{2m} \frac{d}{dt} \langle \hat{\mathbf{A}}^i \rangle = \frac{i\hbar}{2} \left( \partial_j \hat{\mathbf{A}}^i \hat{\mathbf{v}}_j + \hat{\mathbf{v}}_j \partial_j \hat{\mathbf{A}}^i \right)$$  \hspace{1cm} (29)

Results from either Eq. [22] and Eq. [26] give us a direct way to calculate the temporal evolution of the conjugate momentum. It means,

$$\frac{d}{dt} \langle \hat{\pi}^i \rangle = \frac{d}{dt} \langle \hat{\mathbf{v}}^i - e\hat{\mathbf{A}}^i \rangle$$  \hspace{1cm} (30)

Plugging results from Eq. [22] and Eq. [29] yields,
\[
\frac{d}{dt} \langle \pi^i \rangle = \frac{q}{2} \left( \partial_\mu A^\mu_j - \partial_j A^\mu \right) \dot{v}_j + \dot{v}_j \left( \partial_\mu A^\mu_j - \partial_j A^\mu \right) \tag{31}
\]

rewriting in terms of the magnetic field,
\[
\frac{d}{dt} \langle \pi^i \rangle = \frac{q}{2} \left( \varepsilon^{ijk} \dot{B}^k \dot{v}_j + \dot{v}_j \dot{B}^k \right) \tag{32}
\]

That yields directly the definition of the Lorentz force:
\[
\frac{d}{dt} \langle \pi \rangle = \frac{q}{2} \left\langle \vec{\dot{v}} \times \vec{B} - \vec{\dot{B}} \times \vec{v} \right\rangle \tag{33}
\]

it has been showed that the Lorentz force is derived from the canonical momentum \( \pi \) rather than its conjugate one \( \dot{p} \).

The current equation Eq. [19] can be written is terms of velocity as:
\[
J = \frac{1}{2m} \left[ \psi^+ \left( -i\hbar \nabla - q\vec{A} \right) \psi - \psi \left( -i\hbar \nabla + q\vec{A} \right) \psi^+ \right] \tag{34}
\]

\[
J = \frac{1}{2m} \left[ \psi^+ \vec{\dot{v}} \psi - \left( \vec{\dot{v}} \psi \right)^+ \psi \right] \tag{35}
\]

and from Eq. [22],
\[
\frac{d}{dt} \langle \dot{p}_i \rangle = \frac{q}{2} \left( \partial_\mu A^\mu_j \dot{v}_j + \dot{v}_j \partial_\mu A^\mu \right) = \int d\psi^+ \left( \partial_\mu A^\mu_j \dot{v}_j + \dot{v}_j \partial_\mu A^\mu \right) \psi \tag{36}
\]

\[
= \int d\psi^+ \left[ \dot{\psi} \psi + \left( \dot{\psi} \psi \right)^+ \right] \partial_\mu A^\mu = \int d\psi^+ \dot{\psi} A^\mu \tag{37}
\]

Analogously, the lagrangian density can be written as:
\[
L = \frac{i}{2} \left[ \psi^+ \partial_i \psi - \partial_i \psi^+ \right] - \hat{H} \tag{38}
\]

and
\[
L_0 = \frac{i}{2} \left[ \psi^+ \partial_i \psi - \partial_i \psi^+ \right] - \left( \hat{\pi} \psi \right)^+ \tag{39}
\]

where the hamiltonian density is defined as \( \hat{H} = \langle \pi \psi \rangle \) \( \langle \pi \psi \rangle \) and \( L_0 \) is the free lagrangian density (lagrangian density with \( \vec{A} = 0 \)).

The lagrangian density previously written is split in this convenient way:
\[
L = L_0 + J \cdot \vec{A} \tag{40}
\]

and \( J \) is given by Eq. [34]. Let us consider couple of cases of interest: whether or not the vectorial potential has a dependence with position.

### Constant vectorial potential

In this case, one has that the lagrangian is invariant under spacial translation and the associated charged is conserved. In other words
\[
\partial_i T^{\psi}_\nu = 0 \tag{41}
\]

where the energy-momentum tensor is written explicitly as
\[
T^{\psi}_\nu = \frac{\delta L}{\delta \partial_\mu \phi_\alpha} - \partial_\mu \phi_\alpha - \delta L^{\psi}_{\mu} \tag{42}
\]

Recalling for the momentum-component:
\[
T^0_i = \frac{1}{2} \partial_i \phi_\psi \partial_\psi - \partial_i \phi_\psi \tag{43}
\]

### Position-dependence of the vector potential

by performing an spacial translation along the \( i \) direction, it yields that
\[
\int d^3 x \left[ L(\chi^i + \delta \chi^i) - L(\chi^i) \right] = - \int d^3 x \partial_\mu T^\mu_i \delta \chi_i \tag{44}
\]

The vector potential is non invariant under spatial translation. It gives:
\[
\left[ L(\chi^i + \delta \chi^i) - L(\chi^i) \right] = \frac{\delta L}{\delta \partial_\mu \phi_\alpha} \partial_\mu \delta \chi_i \tag{45}
\]

From Eq. [39] one knows that \( \delta L/\delta \partial_\mu \phi_\alpha = q J^i \), therefore
\[
\int d^3 x \left[ L(\chi^i + \delta \chi^i) - L(\chi^i) \right] = - \int d^3 x \partial_\mu T^\mu_i \tag{46}
\]

\[
= - \int d^3 x \partial_\mu T^\mu_i \delta \chi_i = \frac{\delta (\hat{p}^i)}{\delta \chi_i} \tag{47}
\]

In conclusion,
\[
\frac{d}{dt} \langle \hat{\pi} \rangle = \frac{d}{dt} \langle \hat{p} - e \hat{A} \rangle = \frac{d}{dt} \langle \hat{p} \rangle - e \frac{d}{dt} \langle \hat{A} \rangle \tag{48}
\]

\[
\frac{d}{dt} \langle \hat{p} \rangle = \frac{q}{2} \left\langle \vec{\dot{v}} \times \vec{B} - \vec{\dot{B}} \times \vec{v} \right\rangle - e \frac{d}{dt} \langle \vec{A} \rangle \tag{49}
\]

And finally:
\[
\frac{d}{dt} \langle \hat{p}^i \rangle = \hat{F}^i_{\text{Lorentz}} - \frac{e}{2} \left( \partial_\mu \dot{A}^\mu_j + \dot{v}_j \partial_\mu \dot{A}^\mu \right) \tag{50}
\]
CONCLUSIONS

From infinitesimal transformations both in space and time we have found the Energy-Momentum tensor associated to quasiparticles for an anisotropic superconductor. In the case where the transformations leave the action invariant, we find that the energy and linear momentum of quasiparticles are the quantities conserved.

Acknowledgments

The author thanks to the Research division of the Universidad Nacional De Colombia for the financial support.

[1] P.G. De Gennes, Superconductivity of Metals and Alloys (Benjamin, New York, 1966).
[2] Quantum Liquids, Bose Condensation and Cooper Pairing in Condensed-Matter Systems, Anthony James Leggett, Oxford Graduate Texts, 2006.
[3] Goldstein, Herbert (1980). Classical Mechanics (2nd ed.). Reading, MA: Addison-Wesley. pp. 588596.
[4] W. Greiner, Quantum mechanics Symmetries 2a, (1995).
[5] Rev. Mod. Phys. 23, 253, 1951