GENERALIZED LORENZ EQUATIONS FOR ACOUSTIC-GRAVITY WAVES IN THE ATMOSPHERE. ATTRACTORS DIMENSION, CONVERGENCE AND HOMOCLINIC TRAJECTORIES

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Abstract. Attractors dimension of Lorenz-Stenflo system is estimated. Convergence criteria are proved. Fishing principle for existence of homoclinic trajectory is applied.

1. Introduction. In [1] there was shown that the low-frequency, short-wavelength acoustic-gravity perturbation in the atmosphere can be described by the following Lorenz-like system

\[
\begin{align*}
\dot{x} &= \sigma(y - x) + \alpha \vartheta \\
\dot{y} &= rx - xz - y \\
\dot{z} &= -bz + xy \\
\dot{\vartheta} &= -\sigma \vartheta - x,
\end{align*}
\]

where \(\sigma\) is the Prandtl number, \(r\) is the Reyleigh number,

\[b = \frac{4k_1^2}{k_1^2 + k_2^2}, \quad \alpha = \frac{4\Omega^2k_2^2}{\kappa^2(k_1^2 + k_2^2)^6}.
\]

Here \(\Omega\) is an angular frequency of earth’s rotation, \(\kappa\) is a thermal diffusion coefficient, \(k_1\) and \(k_2\) are mode parameters [1].

The passage to chaos in system 1 can be another than the Lorenz scenario [2]. Consider Lorenz parameters \(\sigma = 10, b = 8/3,\) and \(\alpha = 1.\) In this case a homoclinic trajectory for \(r > 1.01\) is lacking. For the Lorenz system such a trajectory exists for \(r = 13.926 \ldots [2].\)

In the present paper there are developed the analytical theory of estimation of attractors dimension, the convergence, and the existence of homoclinic trajectories of system 1. For example, this theory implies that for \(\sigma = 10, b = 8/3,\) \(\alpha = 1,\) \(r < 4.47\) and for \(\sigma = 1, b = 1/2, \alpha = 1, \) \(r < 2.25\) any solution of system 1 tends to equilibrium as \(t \to +\infty.\) For \(\sigma = 10, b = 8/3, \alpha = 1, r = 28\) \(\dim L K < 2.9\) and for \(\sigma = 1, b = 1/2, \alpha = 1, r = 10\) \(\dim L K < 3.21\) and homoclinic trajectory exists for

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certain \( r \in [5.70962749609, 5.7096274961] \). Here \( \text{dim}_L K \) is a Lyapunov dimension of attractor \( K \) of system 1. It is known that
\[
\text{dim}_T K \leq \text{dim}_H K \leq \text{dim}_F K \leq \text{dim}_L K,
\]
where \( \text{dim}_T K, \text{dim}_H K, \text{dim}_F K \) are topological, Hausdorff, and fractal dimensions of attractor \( K \), respectively.

Numerical simulation of the passage to chaos in system 1 is shown for \( \alpha = 1, \sigma = 10, b = 8/3 \) in Figs. 1–7 and for \( \alpha = 1, \sigma = 1, b = 1/2, \) in Figs. 8–12.

\[ r = 13.5 \]

\[ r \approx 14.0233835272883418 \]
2. Preliminaries. O. A. Ladyzhenskaya has given the following definition of global $B$-attractor [3] for the differential equation
\[ \frac{dX}{dt} = f(X), \quad X \in \mathbb{R}^n. \]  

**Definition 2.1.** We say that the invariant set $K$ is uniformly globally attractive if for any $\delta > 0$ and bounded set $B \subset \mathbb{R}^n$ there exists $t(\delta, B) > 0$ such that
\[ X(t, B) \subset K(\delta) \quad \forall t \geq t(\delta, B). \]

Here $K(\delta)$ is $\delta$-neighborhood of set $K$, $X(0, B) = B$.

**Definition 2.2.** We say that $K$ is a global $B$-attractor if it is invariant ($X(t, K) = K$, $\forall t$), bounded, closed, uniformly globally attractive set.

Further we consider global $B$-attractors $K$.

J. Kaplan and J. Yorke [4] have given the following definitions of Lyapunov dimension.
Let $F$ be continuously differentiable mapping of open set $U \subset \mathbb{R}^n$, $K \subset U$. Denote by $T_X F$ the Jacobian matrix of $F$ at the point $X$. Let $K \subset U$ be a bounded invariant set: $FK = K$.

Let $\alpha_j(A)$ denote singular values of $n \times n$ matrix $A$ arranged so that $\alpha_1(A) \geq \cdots \geq \alpha_n(A)$.

**Definition 2.3.** A local Lyapunov dimension of the mapping $F$ at the point $X \in K$ is a number

$$\dim_L(F,X) = j + s,$$

where $j$ is a largest integer number from $[0,n]$ such that

$$\alpha_1(T_X F) \cdots \alpha_j(T_X F) \geq 1$$

and a number $s \in [0,1)$ is such that

$$\alpha_1(T_X F) \cdots \alpha_j(T_X F)\alpha_{j+1}^s(T_X F) = 1.$$
By definition, \( \dim_L(F, X) = 0 \) if \( \alpha_1(T_X F) < 1 \) and \( \dim_L(F, X) = n \) if \( \alpha_1(T_X F) \ldots \alpha_n(T_X F) \geq 1 \).

**Definition 2.4.** The local Lyapunov dimension of one-parameter group of mapping \( F^t \) at the point \( X \in K \) is a number

\[
\dim_L X = \limsup_{t \to +\infty} \dim_L(F^t, X).
\]

**Definition 2.5.** The Lyapunov dimension of mappings \( F^t \) on the set \( K \) is a number

\[
\dim_L K = \sup_K \dim_L X.
\]

Consider system 2 with a continuously differentiable function \( f(X) \). Assume that for any initial data \( X_0 \), system 2 has a solution \( X(t, X_0) \) defined for \( t \in [0, +\infty) \). Here \( X(0, X_0) = X_0 \).

Denote by \( F^t(X_0) = X(t, X_0) \) a shift operator with respect to \( X_0 \), and assume that \( F^t K = K, \forall t \in R^1 \).
Let $J(X)$ be the Jacobian matrix of $f(X)$:

$$J(X) = \frac{\partial f(X)}{\partial X},$$

$S$ is a nonsingular matrix, $\lambda_1(X, S) \geq \ldots \geq \lambda_n(X, S)$ are eigenvalues of the matrix

$$\frac{1}{2}(SJ(X)S^{-1} + (SJ(X)S^{-1})^*).$$

Here $*$ is a sign of transposing.

**Theorem 2.6** ([5]–[7]). Suppose that for integer $j \in [1, n]$ and $s \in [0, 1)$ there exist continuously differentiable function $\vartheta(X)$ and nonsingular matrix $S$ such that

$$\lambda_1(X, S) + \cdots + \lambda_j(X, S) + s\lambda_{j+1}(X, S) + \dot{\vartheta}(X) < 0, \quad \forall X \in U. \quad (3)$$

Then $\dim_L K < j + s$.

Here $\dot{\vartheta}(X) = (\text{grad } \vartheta(X))^* f(X)$. 
Consider a certain set $D \subset \mathbb{R}^n$, which is diffeomorphic to a closed ball, whose boundary $\partial D$ is transversal to the vectors $f(x), x \in \partial D$. The set $D$ is positive invariant for solutions $x(t)$ of system 3.

**Theorem 2.7** ([5]–[7]). Suppose that there exist a continuously differentiable function $\vartheta(X)$ and a nonsingular matrix $S$ such that

$$\lambda_1(X,S) + \lambda_2(X,S) + \dot{\vartheta}(X) < 0, \quad \forall X \in U. \quad (4)$$

Then any solution of system 2 with the initial data $X(0) \in D$ tends to equilibrium as $t \to +\infty$.

Consider a differential equation

$$\frac{dX}{dt} = f(X,q), \quad X \in \mathbb{R}^n, \quad q \in \mathbb{R}^m, \quad (5)$$
where \( f(X,q) \) is a smooth vector-function, \( R^n = \{ X \} \) is a phase space of system 5, \( R^m = \{ q \} \) is a parameter space of system 5.

Let \( \gamma(s), s \in [0,1) \) be a smooth path in a space of parameters \( \{ q \} \). Consider the Tricomi problem [8]–[11]: Is there a point \( q_0 \in \gamma(s) \) for which system 5 with \( q_0 \) has a homoclinic trajectory?

Recall that a trajectory \( X(t) \) of system 5 is said to be homoclinic if the following relation

\[
\lim_{t \to +\infty} X(t) = \lim_{t \to -\infty} X(t) = X_0
\]

is satisfied.

Consider system 5 with \( q = \gamma(s) \), and introduce the following notation:

\( X(t,s)^+ \) is a separatrix of the saddle point \( X_0 \): \( \lim_{t \to -\infty} X(t,s)^+ = X_0 \). \( X(s)^+ \) is a point of the first crossing of separatrix \( X(t,s)^+ \) with the closed set \( \Omega \):

\[
X(t,s)^+ \not\in \Omega, \quad t \in (-\infty, T),
\]

\( X(T,s)^+ = X(s)^+ \in \Omega \), Fig. 13.

If such a crossing is lacking, then it is assumed that \( X(s)^+ = \emptyset \). Here \( \emptyset \) is an empty set.

2.1. Fishing principle. [8]–[11] Suppose that for the path \( \gamma(s) \) there is \( (n - 1) \)-dimensional bounded manifold \( \Omega \) with the piecewise smooth edge \( \partial \Omega \) and it has the following properties:

1) for any \( X \in \Omega \setminus \partial \Omega \) and \( s \in [0,1] \) the vector \( f(X, \gamma(s)) \) is transversal to the manifold \( \Omega \),

2) for any \( s \in [0,1] \), \( f(X_0, \gamma(s)) = 0 \) and the point \( X_0 \in \partial \Omega \) is a saddle of system 5,

3) the inclusion \( X(0)^+ \in \Omega \setminus \partial \Omega \) is satisfied (Fig. 13),

4) the relation \( X(1)^+ = \emptyset \) is valid,

5) for any \( s \in [0,1] \) and \( Y \in \partial \Omega \setminus X_0 \) there exists a neighborhood \( U(Y, \delta) = \{ X \mid |X - Y| < \delta \} \) such that \( X(s)^+ \not\in U(Y, \delta) \),

**Theorem 2.8 ([8]–[11]).** If conditions 1)–5) are satisfied, then there exists \( s_0 \in [0,1] \) such that \( X(t,s_0)^+ \) is a homoclinic trajectory of the saddle \( X_0 \) (Fig. 14).

The fishing principle can be interpreted as follows. Figure 13 shows a fisherman at the point \( X_0 \) with the fishing rod \( X(t,s)^+ \). The manifold \( \Omega \) is a lake surface and \( \partial \Omega \) is a shore line.

If \( s = 0 \), a fish has been caught with the fishing rod. Then \( X(t,s)^+, s \in [0,s_0) \) is the path of the fishing rod with the fish to the shore.

By assumption 5), the fish cannot be taken to the shore \( \partial \Omega \) since \( \partial \Omega \) is a forbidden zone.

Therefore only the situation shown in Fig. 14 is possible (i.e., for \( s = s_0 \), the fisherman has caught a fish). This corresponds to a homoclinic trajectory.

Let us now describe the numerical procedure to define on the path \( \gamma(s) \) the point \( \Gamma \), which corresponds to a homoclinic trajectory. Here it is assumed that conditions 1), 2), 5) of Fishing Principle are satisfied.

Consider a sequence of paths \( \gamma_j(s) \subset \{ \gamma_{j-1}(s), s \in [0,1] \} \subset \{ \gamma(s), s \in [0,1] \} \), \( \forall s \in [0,1] \) such that the length \( \{ \gamma_j(s) \} \) tends to zero as \( j \to +\infty \), for \( \gamma_j(0) \) condition 3) is satisfied and for \( \gamma_j(1) \) condition 4) is valid.
Such a sequence can be obtained if the paths $\gamma$ and $\gamma_j$ are divided sequentially by two parts of the same length and if from two paths it is chosen the path, for the ends of which the opposite conditions 3) and 4) are satisfied.

Obviously, the sequence $\gamma_j(s), s \in [0, 1]$ is contracted to the point $\Gamma \in \{\gamma_j(s), s \in [0, 1]\}, \forall j$. This point corresponds to a homoclinic trajectory of system 5.

Consider now system 5 with a saddle point $X_0 \equiv X_0(s), \forall s \in [0, 1]$ and a positively invariant bounded set $D$. Suppose that $X_0 \in D$ and the Jacobian matrix

$$J(X, s) = \frac{\partial f}{\partial X}(X, \gamma(s)), \ X = X_0$$
has real eigenvalues only.
Consider a nonsingular matrix $S$, and introduce the following notation

$$
\lambda_1(X,s,S) \geq \cdots \geq \lambda_n(X,s,S),
$$

where $\lambda_j(X,s,S)$ are eigenvalues of the matrix

$$
SJ(X,s)S^{-1} + (SJ(X,s)S^{-1})^*.
$$

**Theorem 2.9 ([11]).** Suppose that there exist a continuously differentiable function $\vartheta(X,s)$ and a nonsingular matrix $S$ such that for system 5 with $q = \gamma(s)$ the inequality

$$
\lambda_1(X,s,S) + \lambda_2(X,s,S) + \dot{\vartheta}(X,s) < 0, \quad \forall X \in D, \quad \forall s \in [0, 1],
$$

(6)

where $\dot{\vartheta} = (fX, \gamma(s))^*\text{grad}\vartheta(X,s)$, is satisfied.

Then system 3 has no homoclinic trajectories for all $s \in [0, 1]$.

3. Attractors dimension and convergence.

**Lemma 3.1.** Attractor $K$ of system 1 is located in the set

$$
\Omega = \{(z - r)^2 + y^2 \leq \ell^2 r^2\}
$$

$$
\ell = \begin{cases} 
1 & \text{for } b \leq 2 \\
\frac{b}{2\sqrt{b-1}} & \text{for } b \geq 2.
\end{cases}
$$

**Proof.** Consider a function

$$
V(y,z) = \frac{1}{2}(y^2 + (z - r)^2).
$$

For solutions $x(t), z(t), y(t)$ of system 1 and any $\mu \in (0, \min(1,b))$, we have

$$
\dot{V}(y(t), z(t)) + 2\mu V(y(t), z(t))
\leq (\mu - 1)y(t)^2 + (\mu - b)z(t)^2 - 2r(\mu - b/2)z(t) + \mu r^2
\leq (\mu - b)(z(t) - mr)^2 - m^2 r^2(\mu - b) + \mu r^2
\leq (\mu - m(\mu - b/2))r^2 = \frac{b^2 r^2}{4(b - \mu)}.
$$

Here

$$
m = \frac{\mu - b/2}{\mu - b}.
$$

If $b \leq 2$, then we choose $\mu = b/2$. If $b \geq 2$, then we choose $\mu = 1$. In this case we have two differential inequalities

$$
\dot{V} + bV \leq b^2 r^2/2, \quad \dot{V} + 2V \leq br^2/(b - 1).
$$

This implies the assertion of the lemma. \qed

The Jacobian matrix of the right-hand side of system 1 has the form

$$
J(X) = \begin{pmatrix} 
-\sigma & \sigma & 0 & \alpha \\
r - z & -1 & -x & 0 \\
y & x & -b & 0 \\
-1 & 0 & 0 & -\sigma
\end{pmatrix}
$$
For the matrix 

\[ Q(X) = \frac{(SJ(X)S^{-1} + (SJ(X)S^{-1})^*)}{2} \]

and 

\[ S = \begin{pmatrix}
\sqrt{r/\sigma} & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & \sqrt{\alpha r/\sigma}
\end{pmatrix} \]

the relation 

\[ Q(X) = \begin{pmatrix}
-\sigma & \sqrt{r\sigma} - \frac{z}{2} & \frac{y}{\sqrt{r}} & 0 \\
\sqrt{r\sigma} - \frac{z}{2} & -1 & 0 & 0 \\
\frac{y}{\sqrt{r}} & 0 & -b & 0 \\
0 & 0 & 0 & -\sigma
\end{pmatrix} \]

is valid. The positiveness of matrix 

\[ Q + \lambda I > 0 \] (7)

with 

\[ \lambda = \frac{2\sigma + b + 1}{1 - s} \]

implies relation 3 with \( \vartheta(X) \equiv 0 \). Indeed, \( \lambda = -(1 - s)^{-1}\text{Tr}Q \) and \( \lambda + \lambda_4 > 0 \). Consequently we have 3. Inequality 7 is satisfied if 

\[ \text{det}(Q + \lambda I) > 0. \]

The latter inequality is satisfied if 

\[ \frac{r}{\sigma}[(\lambda - \sigma)(\lambda - 1) - r\sigma] - \frac{1}{4}z^2 - \frac{(\lambda - 1)}{4(\lambda - b)}y^2 + rz > 0. \] (8)

By Lemma 3.1 in \( \Omega \) we have 

\[ z^2 + y^2 < 2rz + (\ell^2 - 1)r^2. \]

Consequently inequality 8 is satisfied if 

\[ 2(\sigma + 1) + (2b - 1)s - b \geq 0 \]

\[ r < \frac{2(\lambda - \sigma)(\lambda - 1)}{\sigma(\ell^2 + 1)}. \] (9)

Thus here it is proved the following result.

**Theorem 3.2.** If inequalities 9 are satisfied, then 

\[ \dim_L K < 3 + s. \]

Note that for \( \sigma = 10, b = 8/3, \alpha = 1, r = 28 \) inequalities 9 are satisfied with \( s = 0 \). Then for these values of parameters we have \( \dim_L K < 3 \). For \( \sigma = \alpha = 1, b = 1/2, r = 10 \) inequalities 9 are satisfied with \( s = 0.21 \). Consequently here \( \dim_L K < 3.21 \).

Denote by \( \mu_1(y, z) \geq \mu_2(y, z) \geq \mu_3(y, z) \) the eigenvalues of the matrix 

\[ P(X) = \begin{pmatrix}
-\sigma & \sqrt{r\sigma} - \frac{z}{2} & \frac{y}{\sqrt{r}} & 0 \\
\sqrt{r\sigma} - \frac{z}{2} & -1 & 0 & 0 \\
\frac{y}{\sqrt{r}} & 0 & -b & 0 \\
0 & 0 & 0 & -\sigma
\end{pmatrix}. \]

Note that for the number \( \mu \) the inequality \( \mu_3(y, z) + \mu \geq 0 \) is satisfied if \( P(X) + \mu I \geq 0 \) and for the number \( \nu \) the inequality \( \mu_1(y, z) + \nu \leq 0 \) is satisfied if \( P(X) + \nu I \leq 0 \).
These inequalities are satisfied if
\[
\det \begin{pmatrix}
\beta - \sigma & \sqrt{r\sigma} & \frac{y}{\sqrt{2} \sqrt{\sigma}} \\
\sqrt{r\sigma} & \beta - 1 & \frac{z}{\sqrt{2} \sqrt{\sigma}} \\
\frac{y}{\sqrt{2} \sqrt{\sigma}} & \frac{z}{\sqrt{2} \sqrt{\sigma}} & \beta - b
\end{pmatrix} > 0,
\]
where \(\beta = \mu\) or \(\beta = \nu\).
This inequality is satisfied if
\[
\frac{r}{\sigma} ([\beta - \sigma] (\beta - 1) - r\sigma) - \frac{z^2}{4} - \frac{(\beta - 1)y^2}{4(\beta - b)} + rz \geq 0.
\]
(10)
By Lemma 3.1 in \(\Omega\) we have
\[
z^2 + y^2 \leq 2rz + (\ell^2 - 1)r^2.
\]
Consequently inequality 10 is satisfied in \(\Omega\) if
\[
\mu = \frac{1}{2} \left[ \sigma + 1 + \sqrt{(\sigma - 1)^2 + 2r\sigma(\ell^2 + 1)} \right],
\]
\[
\nu = \frac{1}{2} \left[ \sigma + 1 - \sqrt{(\sigma - 1)^2 + 2r\sigma(\ell^2 + 1)} \right]
\]
\[
\nu \leq \min(b, 2b - 1).
\]
This implies the following relations
\[
\mu_1(y, z) + \mu_2(y, z) = -(\sigma + 1 + b) - \mu_3(y, z) \leq -(\sigma + 1 + b) + \mu
\]
if \(\mu \geq \max(b, 2b - 1),\) and
\[
\mu_1(y, z) - \sigma \leq -\nu - \sigma
\]
if \(\nu \leq \min(b, 2b - 1).\)
Therefore if \(\mu \geq \max(b, 2b - 1), \nu \leq \min(b, 2b - 1), \) \(\sigma > b),\) then in \(\Omega\) \(\lambda_1(y, z) + \lambda_2(y, z) \leq -(\sigma + 1 + b) + \mu.\) If \(\sigma < b,\) then in \(\Omega\) \(\lambda_1(y, z) + \lambda_2(y, z) \leq -\nu - \sigma.\)
In this case by Theorem 2.7 it is valid the following result.

**Theorem 3.3.** Let
\[
\mu \geq \max(b, 2b - 1), \quad \nu \leq \min(b, 2b - 1).
\]
If \(\sigma > b\) and \(\mu - (\sigma + 1 + b) < 0\) or \(\sigma < b\) and \(\sigma + \nu > 0,\) then any solution of system 1 for \(t \to +\infty\) tends to equilibrium.

Consider now relations
\[
\mu_1(y, z) + \mu_2(y, z) + s\mu_3(y, z) = -(\sigma + b + 1) - \\
(1 - s)\mu_3(y, z) \leq -(\sigma + b + 1) + (1 - s)\mu
\]
\[
\mu_1(y, z) + \mu_2(y, z) - s\sigma \leq -(\sigma + b + 1) + \mu - s\sigma.
\]
We have \(\mu > \sigma.\) Consequently for \(\sigma > b, \) \(\mu \geq \max(b, 2b - 1), \nu \leq \min(b, 2b - 1)\)
\[
\lambda_1(y, z) + \lambda_2(y, z) + s\lambda_3(y, z) \leq -(\sigma + b + 1) + \mu - s\sigma.
\]
Then by Theorem 2.6 we have the following result.
Theorem 3.4. Let $\mu \geq \max(b, 2b - 1)$, $\nu \leq \min(b, 2b - 1)$, $\sigma > b$, 
\[
(1 + s)\sigma + b + 1 > \mu. 
\] (11)

Then
\[
\dim L K < 2 + s. 
\]

Note that inequalities 9 for $s = 0$ and 11 for $s = 1$ coincide. For $\sigma = 10$, $b = 8/3$, $\alpha = 1$, $r = 28$ from Theorem 3.4 it follows the estimate:
\[
\dim L K < 2.
\]

From Theorem 3.3 it follows that for $\sigma = 1$, $b = 1/2$, $\alpha = 1$, $r < 2.25$ and for $\sigma = 10$, $b = 8/3$, $\alpha = 1$, $r < 4.47$ any solution of system 1 tends to equilibrium as $t \to +\infty$.

4. Homoclinic trajectories. We now proceed to the existence problem of homoclinic trajectories of system 1. For this purpose we consider a separatrix $x(t)^+, y(t)^+, z(t)^+, \vartheta(t)^+$ of saddle $x = y = z = 0$ of system 1, assuming that $x(t)^+ > 0$, $\forall t \in (-\infty, T]$.

Lemma 4.1. The inequality
\[
\vartheta(t)^+ < 0, \quad t \in (-\infty, T] 
\] (12)
is satisfied.

This result it follows from the last equations of system 1.

Lemma 4.2. Let $\sigma < 1$, $x(T)^+ = 0$, $\dot{x}(T)^+ = 0$. Then
\[
\ddot{x}(T) = \alpha(1 - \sigma)\vartheta(T)^+ < 0.
\]

This inequality results from 12.

Lemma 4.3. Let $\sigma = 1$. Then
\[
x(t) \equiv 0, \quad z(t) = e^{-bt}z(0), \quad \sigma y(t) + \alpha \vartheta(t) \equiv 0, \quad \vartheta(t) = e^{-\sigma t} \vartheta(0)
\]
is a solution of system 1.

Consider a function
\[
V(x, y, z, \vartheta) = x^2 + y^2 + z^2 + \alpha \vartheta^2 - 2(\sigma + r)z.
\]

It can be seen that
\[
\dot{V} < -\lambda V + 2(\sigma + r)^2 b, \quad \lambda = \min(\sigma, 1, \alpha, b).
\]

This implies the following result.

Lemma 4.4. The relation
\[
V(x(t)^+, y(t)^+, z(t)^+, \vartheta(t)^+) < \frac{2(\sigma + r)^2 b}{\lambda}
\]
is satisfied.

Now we apply to system 1 the Fishing principle with
\[
\Omega = \{x = 0, V(0, y, z, \vartheta) \leq \frac{2(\sigma + r)^2 b}{\lambda}, \sigma(y - x) + \alpha \vartheta \leq 0\}.
\]

Obviously, condition 1 of Theorem 2.8 is satisfied. From Lemmas 4.2–4.4 it follows that if $\sigma \leq 1$, then condition 5) of Theorem 2.8 is valid.
Consider now conditions 3) and 4) of Theorem 2.8 for system 1. For this purpose we consider large values of \(r\) and introduce the following transformations
\[
t \rightarrow \frac{t}{\sqrt{r}}, \quad x \rightarrow x\sqrt{r}, \quad y \rightarrow yr, \quad z \rightarrow zr, \quad \vartheta \rightarrow \vartheta\sqrt{r}.
\]
Then system 1 is transformed into the system
\[
\dot{x} = \sigma y - \varepsilon \sigma x + \alpha \varepsilon \vartheta \\
\dot{y} = x - xz - \varepsilon y \\
\dot{z} = xy - \beta \varepsilon z \\
\dot{\vartheta} = -\sigma \varepsilon \vartheta - \varepsilon x.
\]
Here \(\varepsilon = \frac{1}{\sqrt{r}}\) is a small number. Consider the following approximation of solution of system 13
\[
x(t) = x_0(t) + \varepsilon x_1(t), \\
y(t) = y_0(t) + \varepsilon y_1(t), \\
z(t) = z_0(t) + \varepsilon z_1(t), \\
\vartheta(t) = \vartheta_0(t) + \varepsilon \vartheta_1(t).
\]
Here \(x_0(t), y_0(t), z_0(t), \vartheta_0(t)\) is a solution of the system
\[
\dot{x} = \sigma y - \sigma x \\
\dot{y} = x - xz \\
\dot{z} = xy \\
\dot{\vartheta} = 0
\]
\[
\lim_{t \to -\infty} x_0(t) = \lim_{t \to -\infty} y_0(t) = \lim_{t \to -\infty} z_0(t) = \lim_{t \to -\infty} \vartheta_0(t) = 0.
\]
The functions \(x_1(t), y_1(t), z_1(t), \vartheta_1(t)\) satisfy the relations
\[
\dot{x}_1 = \sigma y_1 - \sigma x_0 \\
\dot{y}_1 = x_1 - z_0 x_1 - x_0 z_1 - y_0 \\
\dot{z}_1 = x_0 y_1 + y_0 x_1 - \beta z_0 \\
\dot{\vartheta}_1 = -x_0
\]
\[
\lim_{t \to -\infty} x_1(t) = \lim_{t \to -\infty} y_1(t) = \lim_{t \to -\infty} z_1(t) = \lim_{t \to -\infty} \vartheta_1(t) = 0.
\]
From these relations it follows that for small \(\varepsilon\) the first approximation of system 1 \(x_0 + \varepsilon x_1, y_0 + \varepsilon y_1, z_0 + \varepsilon z_1\) coincides with the first approximation of the Lorenz system, which is considered in [12]. In [12] it is proved that if
\[
\sigma > \frac{2}{3} b + \frac{1}{3},
\]
then there exists a number \(T > 0\) such that \(x_0(T) + \varepsilon x_1(T) = 0, x_0(t) + \varepsilon x_1(t) > 0, \forall t \in (-\infty, T), \dot{x}_0(T) + \varepsilon \dot{x}_1(T) < 0\). This implies that for sufficiently large \(r\) and inequality 14, condition 3) of Theorem 2.8 is satisfied.

Consider now condition 4) of Theorem 2.8. For this purpose we consider
\[
r = 1 + \frac{\alpha}{\sigma^2} + \varepsilon.
\]
Here \(\varepsilon\) is a small positive number. Consider a path \(r(s), \alpha(s), s \in [0, 1]\) such that
\[
r(s) = 1 + \frac{\alpha(s)}{\sigma^2} + \varepsilon, \quad \alpha(0) = 0, \quad \alpha(1) = \alpha, \quad \alpha(s) \leq \alpha.
\]
It is well known [9]–[12] that for $\alpha = 0$ and small $\varepsilon$, $x(t)^+ > 0$, $\forall t \in (-\infty, +\infty)$.

It can be seen that for $\sigma \in [0.5, 1]$, $\alpha > 0$ and small $\varepsilon > 0$, the characteristic polynomial of the saddle $x = y = z = \vartheta = 0$ has three real negative and one positive roots.

This implies by Theorem 2.9 (see also the proof of Theorem 3.3) that for $\sigma \geq b \geq 0.5$ $\sigma \in [0.5, 1]$, small positive $\varepsilon$, and

$$\alpha < b\sigma(\sigma + b + 1)$$

(15)

$x(t)^+ > 0$ for all $t \in (-\infty, +\infty)$.

This implies the following result.

**Theorem 4.5.** Suppose, the inequalities $1 \geq \sigma \geq b \geq 0.5$, 14 and 15 are satisfied. Then there exists a number $r = r(\sigma, b, \alpha) \in (1 + \alpha/\sigma^2, +\infty)$ such that system 1 with $\sigma, b, \alpha, r(\sigma, b, \alpha)$ has a homoclinic trajectory of the saddle $x = y = z = \vartheta = 0$.

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