UNPERTURBED WEAKLY REDUCIBLE NON-MINIMAL BRIDGE POSITIONS

JUNG HOON LEE

Abstract. A bridge position of a knot is said to be perturbed if there exists a cancelling pair of bridge disks. Motivated by the examples of knots admitting unperturbed strongly irreducible non-minimal bridge positions due to Jang-Kobayashi-Ozawa-Takao, we derive examples of unperturbed weakly reducible non-minimal bridge positions. Also, a bridge version of Gordon’s Conjecture is proposed: the connected sum of unperturbed bridge positions is unperturbed.

1. Introduction

Suppose that $S^3$ is decomposed into two 3-balls by an embedded sphere $S$. A knot $K$ is in $n$-bridge position with respect to $S$ if $K$ intersects each of the 3-balls in a collection of $n$ parallel arcs. The original concept of bridge position, the bridge number, was first introduced by Schubert in 1954 [11]. Thereafter it is generalized to the notion of bridge splitting for (a 3-manifold, a link) pair.

For any $n$-bridge position, we can always give a perturbation to get a perturbed $(n + 1)$-bridge position. Conversely, from a perturbed bridge position we obtain a lower index bridge position. A bridge position is unperturbed if it is not perturbed. It is a fundamental problem to detect whether a given bridge position is unperturbed or not. The unknot has a unique 1-bridge position and every $n$-bridge position $(n > 1)$ of the unknot is perturbed [5]. Non-minimal bridge positions of 2-bridge knots [7], torus knots [8] are perturbed. Zupan showed that if $K$ is an mp-small knot and every non-minimal bridge position of $K$ is perturbed, then every non-minimal bridge position of a $(p, q)$-cable of $K$ is also perturbed [14]. Concerning 2-cables, the author showed that if every non-minimal bridge position of a knot $K$ is perturbed, then every non-minimal bridge position of a $(2, 2q)$-cable link of $K$ is perturbed, without the assumption of mp-smallness of $K$ [4].

On the other hand, there exist knots admitting unperturbed non-minimal bridge positions [3], [9]. All the examples in [3] and [9] are strongly irreducible bridge positions. Weakly reducible bridge positions are the opposites of more complicated strongly irreducible ones, so simpler. For weakly reducible bridge positions, one can ask whether unperturbed non-minimal bridge positions can be attained. We show that there exist unperturbed weakly reducible non-minimal bridge positions by taking the connected sum operation on the knots due to Jang et al.

Theorem 1.1. There exist unperturbed weakly reducible non-minimal bridge positions.

\[ \text{2020 Mathematics Subject Classification.} \ 57K10. \]
\[ \text{Key words and phrases.} \ \text{unperturbed bridge position, weak reducibility, Gordon’s Conjecture.} \]
The unperturbedness is shown by the method of 2-fold branched covering and Gordon’s Conjecture. Can we prove it directly without taking a 2-fold branched covering? This raises the following conjecture.

**Conjecture 1.2** (A bridge version of Gordon’s Conjecture). The connected sum of two unperturbed bridge positions is unperturbed.

The presented examples of knots for Theorem [11, 13] are composite knots. We have the following question.

**Question 1.3.** Does there exist a prime knot admitting an unperturbed weakly reducible non-minimal bridge position?

2. Bridge positions

Let $B$ be a 3-ball. A trivial tangle is a collection of disjoint properly embedded arcs $b_1, \ldots, b_n$ in $B$ such that each $b_i$ cobounds a disk $D_i$ with an arc in $\partial B$ and $D_i \cap b_j = \emptyset$ for all $j \neq i$. Suppose that a 2-sphere $S$ decomposes $S^3$ into two 3-balls $B$ and $C$. Let $K$ be a knot. If $B \cap K$ and $C \cap K$ are trivial tangles, each consisting of $n$ arcs, then we say that $K$ is in $n$-bridge position with respect to $S$. Each arc of the trivial tangles $B \cap K$ and $C \cap K$ is called a bridge. A bridge $b_i$ of the trivial tangle, say $B \cap K = \{b_1, \ldots, b_n\}$, cobounds a bridge disk $D_i$ with an arc in $S$ such that $D_i \cap b_j = \emptyset$ for all $j \neq i$ by definition. By standard cut-and-paste argument, $D_i$’s ($i = 1, \ldots, n$) can be taken to be pairwise disjoint. A collection $\{D_1, \ldots, D_n\}$ of $n$ disjoint bridge disks is called a complete bridge disk system. If $K$ is in bridge position, we have a decomposition of the pair $(S^3, K)$ into $(B, B \cap K)$ and $(C, C \cap K)$. But when it is clear from the context, we will simply use the notation $B \cup_S C$ to indicate the bridge position.

For an $n$-bridge position $B \cup_S C$, we can perturb a small neighborhood of a point $p$ of $K \cap S$ so that it becomes an $(n + 1)$-bridge position having bridge disks $D \subset B$ and $E \subset C$ with $D \cap E = p$. Such an operation is called a perturbation, and a bridge position isotopic to one obtained by a perturbation is said to be perturbed. Each of $D$ and $E$ is a cancelling disk and $(D, E)$ is a cancelling pair. Conversely, a perturbation can be reversed to give a lower index bridge position. A bridge position is unperturbed if it is not perturbed.

A disk $D$ properly embedded in $B$ or $C$ with $D \cap K = \emptyset$ is a compressing disk if $\partial D$ does not bound a disk in $S - K$. A bridge position $B \cup_S C$ is weakly reducible if there exist compressing disks $D \subset B$ and $E \subset C$ such that $D \cap E = \emptyset$. Otherwise, it is strongly irreducible. It is easy to see that if an $n$-bridge position ($n \geq 3$) is perturbed, then it is weakly reducible. Note that a 2-bridge position of the unknot is perturbed and strongly irreducible.

The bridge number $b(K)$ of a knot $K$ is the minimum of $\{n \mid K \text{ admits an } n\text{-bridge position}\}$. For a connected sum $K_1 \# K_2$ of two knots $K_1$ and $K_2$, $b(K_1 \# K_2) = b(K_1) + b(K_2) - 1$ [11, 12]. For a $(p, q)$-torus knot $K_{p,q}$, $b(K_{p,q}) = \min\{ |p|, |q| \}$ [11, 13].

3. Heegaard splittings

In this section, we briefly review basic notions and facts about Heegaard splittings. Connections between Heegaard splittings and bridge positions, via 2-fold branched coverings, will be discussed in the subsequent sections.

For a closed 3-manifold $M$, a Heegaard splitting $V \cup_F W$ is a decomposition of $M$ into two handlebodies $V$ and $W$ of the same genus. The common boundary $F$ of $V$ and $W$ is called the Heegaard surface of $V \cup_F W$. A Heegaard splitting $V \cup_F W$ is stabilized if there exist disks $D \subset V$ and $E \subset W$ such that $|D \cap E| = 1$, and $(D, E)$ is called a cancelling pair. Otherwise,
it is *unstabilized*. If there exist compressing disks $D \subset V$ and $E \subset W$ such that $\partial D = \partial E$ ($D \cap E = \emptyset$ respectively), then the Heegaard splitting is said to be *reducible* (weakly reducible respectively). A Heegaard splitting is *irreducible* (strongly irreducible respectively) if it is not reducible (weakly reducible respectively). It is immediate that a reducible Heegaard splitting is weakly reducible, by slightly pushing one of $D$ and $E$ with $\partial D = \partial E$ to be apart from the other.

Suppose that $V \cup_F W$ is stabilized with a cancelling pair $(D, E)$ and the genus of $F$ is at least two. Then we can see that $V \cup_F W$ is reducible, hence weakly reducible, by band summing two copies of $D$ along $\partial E$ and band summing two copies of $E$ along $\partial D$. As a contrapositive, we have the following.

**Proposition 3.1.** If a Heegaard splitting of genus $g \geq 2$ is strongly irreducible, then it is unstabilized.

### 4. 2-Fold branched coverings

Let $B \cup_S C$ be an $n$-bridge position of a knot $K$. Let $\{D_1, ..., D_n\}$ be a complete bridge disk system for $B \cap K$. Cut $B$ along $\bigcup_{i=1}^n D_i$. Let $B'$ be the resulting 3-ball and let $D'_i_{+}$ and $D'_i_{-}$ denote the two scars of $D_i$ on $\partial B'$. Let $B''$ be a copy of $B'$ and similarly let $D''_{i+}$ and $D''_{i-}$ denote the two scars of $D_i$ on $\partial B''$. Glue $B'$ and $B''$ along $D'_{i\pm}$ and $D''_{i\mp}$ for each $i$. The resulting manifold is a genus $n - 1$ handlebody $V$. There is an involution of $V$ fixing $B \cap K$ such that the quotient map induced by the involution is a 2-fold covering $p_1 : V \to B$ branched along $B \cap K$. Similarly, we can take a 2-fold covering $p_2 : W \to C$ branched along $C \cap K$, where $W$ is a genus $n - 1$ handlebody. Hence we have a 2-fold branched covering map $p$ from a genus $n - 1$ Heegaard splitting $V \cup_F W$ to $B \cup_S C$, branched along the knot $K$ in $n$-bridge position.

Suppose $B \cup_S C$ is perturbed, so it admits a cancelling pair $(D, E)$. The preimages $p^{-1}(D)$ and $p^{-1}(E)$ are disks in $V$ and $W$ respectively that intersect at one point, so $V \cup_F W$ is stabilized. As a contrapositive, we have the following.

**Proposition 4.1.** Suppose that $p : V \cup_F W \to B \cup_S C$ is a 2-fold covering branched along a knot $K$ in bridge position with respect to $S$. If $V \cup_F W$ is unstabilized, then $B \cup_S C$ is unperturbed.

The converse of Proposition 4.1 does not hold. There is a relevant discussion in [2] Section 1. Let $K_{p,q}$ be a $(p,q)$-torus knot with $0 < p < q$. A $p$-bridge position $B \cup_S C$ of $K_{p,q}$ is unperturbed since $b(K_{p,q}) = p$.

A 2-fold covering of $S^3$ branched along $K_{p,q}$ is a small Seifert fibered manifold $M$. It is known that an irreducible Heegaard splitting of a Seifert fibered manifold is either vertical or horizontal [5]. The genus of a vertical splitting of $M$ is at most two. The genus of a horizontal splitting is always an even number. Refer to [5] for more details.

The 2-fold branched covering $V \cup_F W$ of $B \cup_S C$ is of genus $p - 1$. So for example, if $(p, q) = (4, 5)$, then $V \cup_F W$ is a reducible Heegaard splitting of $M$. Since $M$ is an irreducible manifold, $V \cup_F W$ is stabilized. Therefore, if $(p, q) = (4, 5)$, then $B \cup_S C$ is unperturbed and $V \cup_F W$ is stabilized.

### 5. Connected sums

Let $B_1 \cup_{S_1} C_1$ and $B_2 \cup_{S_2} C_2$ be bridge positions of knots $K_1$ and $K_2$ respectively. Let $p_i : V_i \cup_F W_i \to B_i \cup_{S_i} C_i$ ($i = 1, 2$) be 2-fold branched coverings explained in Section 4. See
The connected sum of $B_1 \cup_{S_1} C_1$ and $B_2 \cup_{S_2} C_2$ is defined as follows. Take a small open ball neighborhood $N_i$ at a point of $K_i \cap S_i$. Glue $B_1 - N_1$ and $B_2 - N_2$ along $B_1 \cap \partial N_1$ and $B_2 \cap \partial N_2$ so that $K_1 \cap (B_1 \cap \partial N_1)$ is identified with $K_2 \cap (B_2 \cap \partial N_2)$. Similarly, glue $C_1 - N_1$ and $C_2 - N_2$ along $C_1 \cap \partial N_1$ and $C_2 \cap \partial N_2$ so that $K_1 \cap (C_1 \cap \partial N_1)$ is identified with $K_2 \cap (C_2 \cap \partial N_2)$. The result is a bridge position $(B_1 \natural B_2) \cup_{S_1 \# S_2} (C_1 \natural C_2)$ of $K_1 \natural K_2$.

(1) $(B_1 \cup_{S_1} C_1) \# (B_2 \cup_{S_2} C_2) = (B_1 \natural B_2) \cup_{S_1 \# S_2} (C_1 \natural C_2)$.

Now we consider the connected sum of $M_1 = V_1 \cup_{F_1} W_1$ and $M_2 = V_2 \cup_{F_2} W_2$. Since we want the connected sum to be compatible with the branched covering map, take $p_1^{-1}(N_1)$ and $p_2^{-1}(N_2)$, which are open 3-balls. Glue $V_1 - p_1^{-1}(N_1)$ and $V_2 - p_2^{-1}(N_2)$ along $V_1 \cap \partial (p_1^{-1}(N_1))$ and $V_2 \cap \partial (p_2^{-1}(N_2))$. Similarly, glue $W_1 - p_1^{-1}(N_1)$ and $W_2 - p_2^{-1}(N_2)$ along $W_1 \cap \partial (p_1^{-1}(N_1))$ and $W_2 \cap \partial (p_2^{-1}(N_2))$. The result is a Heegaard splitting $(V_1 \natural V_2) \cup_{F_1 \# F_2} (W_1 \natural W_2)$ of $M_1 \natural M_2$.

(2) $(V_1 \cup_{F_1} W_1) \# (V_2 \cup_{F_2} W_2) = (V_1 \natural V_2) \cup_{F_1 \# F_2} (W_1 \natural W_2)$.

Since $V_i - p_i^{-1}(N_i)$ ($i = 1, 2$) 2-fold branched covers $B_i - N_i$, the handlebody $V_i \natural V_2$ 2-fold branched covers $B_1 \natural B_2$. Similarly, since $W_i - p_i^{-1}(N_i)$ ($i = 1, 2$) 2-fold branched covers $C_i - N_i$, the handlebody $W_1 \natural W_2$ 2-fold branched covers $C_1 \natural C_2$. So $(V_1 \natural V_2) \cup_{F_1 \# F_2} (W_1 \natural W_2)$ 2-fold branched covers $(B_1 \natural B_2) \cup_{S_1 \# S_2} (C_1 \natural C_2)$. See Figure 2 Then by (1) and (2), we have the following lemma.

**Lemma 5.1.** There is a 2-fold branched covering $p : (V_1 \cup_{F_1} W_1) \# (V_2 \cup_{F_2} W_2) \rightarrow (B_1 \cup_{S_1} C_1) \# (B_2 \cup_{S_2} C_2)$.

In other words, by carefully choosing the 3-balls, a connected sum of 2-fold branched coverings is a 2-fold branched covering of a connected sum.

### 6. A Bridge Version of Gordon’s Conjecture

**Conjecture 6.1** (Gordon’s Conjecture). The connected sum of two unstabilized Heegaard splittings is unstabilized.
Gordon’s Conjecture is proved by Bachman [1] and independently by Qiu and Scharlemann [10]. Bachman used the notion of critical surface. The proof in [10] is a constructive combinatorial proof.

We proposed a bridge version of Gordon’s Conjecture in the introduction. Compared to the case of Heegaard splittings, a difficulty that may arise by the presence of a knot is the following. Suppose that a connected sum of two bridge positions is perturbed. When we obtain, from a cancelling pair for the connected sum, subdisks in a summand, there can be two subarcs that intersect at two points. See Figure 3. In Figure 3, $S$ is a bridge sphere and $P$ is a decomposing sphere for the connected sum. On the other hand, it might be a helpful fact that every properly embedded disk in a 3-ball is separating.

7. Proof of Theorem 1.1

Let $K_1$ be a knot admitting an $n_1$-bridge position $B_1 \cup S_1 C_1$ with $n_1 > b(K_1)$ whose 2-fold branched covering $V_1 \cup F_1 W_1$ is an unstabilized Heegaard splitting.

Claim 1. There are infinitely many examples for $K_1$.

Proof. There are infinitely many knots in [3], each of which admits a $(2k + 5)$-bridge position for any integer $k \geq 0$. Let $K_1$ denote one of them, and $B_1 \cup S_1 C_1$ be a $(2k + 5)$-bridge position of $K_1$ with $2k + 5 > b(K_1)$. It is shown in [3] that the 2-fold branched covering $V_1 \cup F_1 W_1$ of $B_1 \cup S_1 C_1$ is strongly irreducible. By Proposition 3.1, $V_1 \cup F_1 W_1$ is unstabilized.

Let $K_2$ be a knot admitting an $n_2$-bridge position $B_2 \cup S_2 C_2$ whose 2-fold branched covering $V_2 \cup F_2 W_2$ is unstabilized. There are also infinitely many examples for $K_2$. Then $(V_1 \cup F_1 \cup V_2 \cup F_2 \cup W_1 \cup W_2)$ is an unstabilized Heegaard splitting.
$W_1 \# (V_2 \cup F_2, W_2)$ is unstabilized by Gordon’s Conjecture. There exists a 2-fold branched covering $p : (V_1 \cup F_1, W_1) \# (V_2 \cup F_2, W_2) \to (B_1 \cup S_1, C_1) \# (B_2 \cup S_2, C_2)$ by Lemma 5.1. By Proposition 4.1, $(B_1 \cup S_1, C_1) \# (B_2 \cup S_2, C_2)$ is unperturbed. It is weakly reducible because it is obtained by a connected sum. The bridge number $b(K_1 \# K_2)$ is $b(K_1) + b(K_2) - 1$ and $(B_1 \cup S_1, C_1) \# (B_2 \cup S_2, C_2)$ is an $(n_1 + n_2 - 1)$-bridge position of $K_1 \# K_2$, where $n_1 + n_2 - 1 > b(K_1) + b(K_2) - 1$.

References

[1] D. Bachman, Connected sums of unstabilized Heegaard splittings are unstabilized, Geom. Topol. 12 (2008), no. 4, 2327–2378.
[2] H. N. Howards and J. Schultens, Thin position for knots and 3-manifolds, Topology Appl. 155 (2008), no. 13, 1371–1381.
[3] Y. Jang, T. Kobayashi, M. Ozawa, and K. Takao, A knot with destabilized bridge spheres of arbitrarily high bridge number, J. Lond. Math. Soc. (2) 93 (2016), no. 2, 379–396.
[4] J. H. Lee, Non-minimal bridge position of 2-cable links, accepted in Michigan Mathematical Journal.
[5] Y. Moriah and J. Schultens, Irreducible Heegaard splittings of Seifert fibered spaces are either vertical or horizontal, Topology 37 (1998), no. 5, 1089–1112.
[6] J.-P. Otal, Présentations en ponts du nœud trivial, C. R. Acad. Sci. Paris Sér. I Math. 294 (1982), no. 16, 553–556.
[7] J.-P. Otal, Présentations en ponts des nœuds rationnels, Low-dimensional topology (Chelwood Gate, 1982), 143–160, London Math. Soc. Lecture Note Ser., 95, Cambridge Univ. Press, Cambridge, 1985.
[8] M. Ozawa, Nonminimal bridge positions of torus knots are stabilized, Math. Proc. Cambridge Philos. Soc. 151 (2011), no. 2, 307–317.
[9] M. Ozawa and K. Takao, A locally minimal, but not globally minimal, bridge position of a knot, Math. Proc. Cambridge Philos. Soc. 155 (2013), no. 1, 181–190.
[10] R. Qiu and M. Scharlemann, A proof of the Gordon Conjecture, Adv. Math. 222 (2009), no. 6, 2085–2106.
[11] H. Schubert, Über eine numerische Knoteninvariante, Math. Z. 61 (1954), 245–288.
[12] J. Schultens, Additivity of bridge numbers of knots, Math. Proc. Cambridge Philos. Soc. 135 (2003), no. 3, 539–544.
[13] J. Schultens, Bridge numbers of torus knots, Math. Proc. Cambridge Philos. Soc. 143 (2007), no. 3, 621–625.
[14] A. Zupan, Properties of knots preserved by cabling, Comm. Anal. Geom. 19 (2011), no. 3, 541–562.

J. H. Lee

Department of Mathematics and Institute of Pure and Applied Mathematics, Jeonbuk National University, Jeonju 54896, Korea

Email address: junghoon@jbnu.ac.kr