Entanglement and quantum state geometry of a spin system with all-range Ising-type interaction

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Abstract

The evolution of an $N$ spin-$1/2$ system with all-range Ising-type interaction is considered. For this system we study the entanglement of one spin with the rest spins. It is shown that the entanglement depends on the number of spins and the initial state. Also, the geometry of the manifold, which contains entangled states, is obtained. For this case we find the dependence of entanglement on the scalar curvature of the manifold and examine it for different numbers of spins in the system. Finally we show that the transverse magnetic field leads to a change in the manifold topology.

Keywords: the entanglement, the Fubini–Study metric, the quantum state manifold

1. Introduction

Studies of quantum entanglement play a crucial role in the development of quantum mechanics and quantum-information theory [1]. The discovery of quantum correlations allowed solving the Einstein–Podolsky–Rosen (EPR) paradox [2]. In 1982, testing Bell’s inequality [3] for the entangled states of photons, the EPR paradox was experimentally resolved by Aspect et al [4]. Also Bell’s inequality was tested on other systems. For instance, recently, using the electron and nuclear spins of a singled phosphorus atom in silicon, a violation of Bell’s inequality was demonstrated [5]. In [6, 7] it was shown that in the case of three particle quantum correlations are much more noticeable than for two particles. The study of entanglement is closely related to the problem of preparation of entangled states. Preparation of entangled states is required for implementation of many schemes in quantum computation. For example, the
simplest scheme of quantum teleportation of the qubit state requires the preparation of a two-qubit entangled state for the realization of a quantum channel [8].

Nowadays the quantum entanglement of multipartite systems has been widely studied theoretically as well as experimentally [9]. One of the most suitable systems for this purpose is spin systems [10–17]. The most successful realization of the multipartite entangled state was performed on the trapped ion quantum system [18–23]. An efficient method to produce multiparticle entangled states of ions in a trap was proposed in [18]. This method is based on the implementation of the effective spin system using two laser fields. A scheme for realization of such systems with the Ising and Heisenberg types of interactions was proposed in [19]. Also in [20] the authors suggested a scheme for the manipulation of trapped ions using radiation in the radiofrequency or microwave regime.

To better understand the entanglement of quantum systems, the geometry properties of their quantum states are often studied [24–31]. For these reasons the geometry of the quantum state manifold is investigated. This knowledge allows us to obtain information about the location of the states on quantum manifolds and simplify the problem of their manipulation. For example, the fact that the quantum state space of spin-1/2 is represented by the Bloch sphere allowed the simplification of the solution of the quantum brachistochrone problem [32–35]. Also information about the geometry of the quantum state manifold is important for the implementation of quantum computations [36–39]. For example, in [40] it was shown how the geometry of n-qubit quantum space is associated with the realization of quantum computations. In [41–44] it was shown that the problem of finding of the quantum circuit of unitary operators, which provide time-optimal evolution on a system of qubits, is related to the problem of finding the minimal distance between two points on the Riemannian metric.

So, in our previous papers [45, 46] we found the geometry of the state manifolds of two spins with different types of interaction. As a result we obtained the conditions for the preparation of entangled states on these systems. In the present paper, we investigate the evolution (section 2) and entanglement (section 3) of an N spin-1/2 system with all-range Ising-type interaction. Also we calculate the metric of the quantum state manifold, which contains entangled states, and obtain the dependence of their entanglement on the scalar curvature of the manifold (section 4). The influence of the transverse magnetic field on the topology of the manifold is examined in section 5. Conclusions are presented in section 6.

2. The quantum evolution of a spin system with all-range Ising-type interaction

We consider a N spin-1/2 system with all-range interaction described by the Ising Hamiltonian

$$H = \frac{J}{4} \left( \sum_{j=1}^{N} \sigma_j^z \right)^2,$$

(1)

where $J$ is the interaction coupling, $N$ is the number of spins, and $\sigma_j^z$ is the Pauli matrix for spin $j$. This Hamiltonian has $(N/2 + 1)$ eigenvalues in the case of an even number of spins and $(N + 1)/2$ in the case of an odd number of spins. These eigenvalues correspond to eigenvectors in the following way
So, for each eigenvalue $J/4 (N - 2k)^2$ there exist $\binom{N}{k}$ eigenstates, consisting of all possible combinations of $|\uparrow\rangle$ and $|\downarrow\rangle$ states, where $k = 0, \ldots, N/2$ for even $N$ and $k = 0, \ldots, (N - 1)/2$ for odd $N$.

Let us consider the evolution of the spin system having started from the initial state

$$|\psi_I\rangle = |++\ldots+\rangle,$$

where

$$|+\rangle = \cos \frac{\theta}{2} |\uparrow\rangle + \sin \frac{\theta}{2} e^{i\phi} |\downarrow\rangle$$

is the state of the spin-1/2 projected on the positive direction of the unit vector $n = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta)$. Here $\theta$ and $\phi$ are the polar and azimuthal angles, respectively. The action of the evolution operator with Hamiltonian (1) on this state is as follows

$$|\psi(t)\rangle = \exp \left(-\frac{iJ}{4} \sum_{j=1}^{N} \sigma_j^z \right) |\psi_I\rangle = \sum_{k=0}^{N} c_k \sum_{j_1 < j_2 < \ldots < j_k = 1} \sigma_{j_1}^x \sigma_{j_2}^x \ldots \sigma_{j_k}^x |\uparrow\uparrow\ldots\uparrow\rangle,$$

where

$$c_k = \cos^{N-k} \frac{\theta}{2} \sin \frac{\theta}{2} e^{i\phi} \exp \left(-\frac{i\chi}{4} (N - 2k)^2 \right).$$

Here $\chi = Jt$. We set $\hbar = 1$, which means that the energy is measured in frequency units. From the analysis of state (4) we obtain that it is periodic with respect to parameter $\chi \in [0, 2\pi]$.

3. The entanglement of a spin system

In present section we calculate the entanglement of one spin with the rest spins of the system being in the state (4). This allows us to obtain information about the dynamics of the system through the entangled states. For this purpose we use an expression for geometric measure of entanglement [47–50] represented by the mean value of spin [51]

$$E = \frac{1}{2} (1 - \langle |\sigma_j| \rangle),$$

where in the Cartesian coordinate system $\langle |\sigma_j| \rangle = \langle \sigma_j^x \rangle + \langle \sigma_j^y \rangle + \langle \sigma_j^z \rangle k$ is the mean value of spin $j$. It is important to note that in [52] the connections between geometric measure of entanglement and relative entropy of entanglement was studied in detail. So, calculating the mean value of spin with respect to state (4) and substituting it in (6) we obtain the following expression for entanglement
From the analysis of this equation it is clear that for certain \( \theta \) we achieve the maximal entangled states when \( \chi = \pi/2 \) and \( 3\pi/2 \) and the states with minimal value of entanglement \( (E = 0) \) correspond to \( \chi = 0 \) and \( \chi = \pi \). Therefore taking into account that \( \chi \in [0, 2\pi] \) we claim that for specific \( \theta \) during the evolution we can reach two maximal entangled states and two minimal entangled states (see, for example, figure 1). Also, it is easy to see that the maximal value of entanglement that can be achieved during the evolution depends on the angle \( \theta \). In the case of \( \theta = 0 \) or \( \pi \) we obtain that \( E = 0 \). This is because the initial state is directed along the z-axis. In the case of \( \theta = \pi/2 \) we obtain the following expression for the entanglement \[ E = \frac{1}{2} \left( 1 - 2 \sqrt{\cos^2 \theta + \sin^2 \theta \left( \cos^2 \chi + \cos^2 \theta \sin^2 \chi \right)} \right). \] (7)

Then the maximal entangled state with \( E = 1/2 \) is achieved when \( \chi = \pi/2 \) and \( 3\pi/2 \). These conditions correspond to the preparation of the ‘Schrödinger cat’ state \[18, 21\].

It is easy to convince oneself that for an arbitrary \( \theta \), excluding the cases of \( \theta = 0 \), \( \pi/2 \) and \( \pi \), when the number of spins are increased, then the maximal entanglement of the system also increases. For instance, in figure 2 it is shown the properties of the system for the case of \( \theta = \pi/4 \). However, for a certain \( \theta \) and a large number of spins this value becomes a constant \[ E = \frac{1}{2} (1 - |\cos \theta|). \] (9)
These behaviours of the entanglement are caused by the power $N - 1$ in equation (7). Then the entanglement is reached in time.

We can see from (7) that, for different $\phi$, we have the states with the same entanglement. These states are placed on the different manifolds defined by parameters $\theta$ and $\chi$. These manifolds are geometrically similar but each of them contains the states with a specific value of $\phi$. Let us study the geometry of these manifolds.

4. The Fubini–Study metric of the quantum state manifold

This method, which allows us to study the geometrical properties of the quantum state manifold, is based on the examination of their Fubini–Study metric. The Fubini–Study metric is defined by the infinitesimal distance $ds$ between two neighbouring pure quantum states $|\psi(\xi^\mu)\rangle$ and $|\psi(\xi^\mu + d\xi^\mu)\rangle$ [32, 33, 54–59]

$$ds^2 = g_{\mu\nu} d\xi^\mu d\xi^\nu,$$

where $\xi^\mu$ is a set of real parameters that define the state $|\psi(\xi^\mu)\rangle$. The components of the metric tensor $g_{\mu\nu}$ have the form

$$g_{\mu\nu} = \gamma^2 \Re \left( \langle \psi_{\mu} | \psi_{\nu} \rangle - \langle \psi_{\mu} | \psi \rangle \langle \psi | \psi_{\nu} \rangle \right),$$

where $\gamma$ is an arbitrary factor, which is often chosen to have the value of 1, $\sqrt{2}$ or 2 and

$$|\psi_{\mu}\rangle = \frac{\partial}{\partial \xi^\mu} |\psi\rangle.$$ 

Figure 2. Time dependence of entanglement (7) of a spin system for $\theta = \pi/4$ and different numbers of spins $N$: $N = 2$ (solid curve), $N = 3$ (dashed curve), $N = 6$ (dash-dotted curve) and $N = 9$ (dotted curve).
Let us calculate the metric of the manifold that contains state (4) with respect to \( \theta \) and \( \chi \). Using definition (10) with expression (11) we obtain the following components of the metric tensor

\[
\begin{align*}
g_{\theta \theta} &= \gamma^2 \frac{N}{4}, \\
g_{\chi \chi} &= \gamma^2 \frac{N(N - 1) \sin^2 \theta \left[ N - 1 - \left( \frac{3}{2} \sin^2 \theta \right) \right]}{4}, \\
g_{\theta \chi} &= 0.
\end{align*}
\]

(13)

To obtain the metric tensor we calculate the following scalar products

\[
\begin{align*}
\langle \psi | \psi \rangle &= 0, \\
\langle \psi | \psi \rangle &= -i \frac{N}{4} \left[ 1 + (N - 1) \cos^2 \theta \right], \\
\langle \psi | \psi \rangle &= \frac{N}{16} \left[ (N - 1)(N - 2)(N - 3) \cos^4 \theta + (N - 1)(6N - 8) \cos^2 \theta + 3N - 2 \right], \\
\langle \psi | \psi \rangle &= i \frac{N}{4} (N - 1) \sin \theta \cos \theta.
\end{align*}
\]

(14)

To analyse the topology of this manifold let us calculate its scalar curvature. Using the fact that metric (13) is two-parametric and has a diagonal form, we represent the scalar curvature in the form [60]

\[
R = \frac{2g_{\theta \chi}}{g_{\theta \theta} g_{\chi \chi}},
\]

(15)

where

\[
R_{\theta \chi} = - \frac{1}{2} \frac{\partial^2 g_{\chi \chi}}{\partial \theta^2} + \frac{1}{4g_{\chi \chi}} \left( \frac{\partial g_{\chi \chi}}{\partial \theta} \right)^2.
\]

(16)

is the Riemann curvature tensor. Now we substitute the components of metric tensor (13) into (15) and after simplification obtain

\[
R = \frac{16}{\gamma^2 N} \left( 2 - \frac{(2N - 3) \cos^2 \theta + N}{((2N - 3) \cos^2 \theta + 1) \cos^2 \theta} \right).
\]

(17)

We can see that the curvature depends on parameter \( \theta \) and for \( N > 2 \) is a negative in certain area of manifold which satisfies the following condition

\[
\frac{(2N - 3) \cos^2 \theta + N}{((2N - 3) \cos^2 \theta + 1) \cos^2 \theta} > 2.
\]

(18)

From the analysis of (17) we conclude that for \( \theta = \pi/2 \) we obtain the minimum value of curvature

\[
R_{\min} = \frac{16}{\gamma^2 N} (2 - N)
\]

and for \( \theta = 0 \) and \( \pi \) we obtain the maximal value of curvature

\[
R_{\max} = \frac{16}{\gamma^2 N} \left( 2 - \frac{3}{4(N - 1)} \right).
\]

Also the dependence of \( R \) on \( \theta \) is symmetric with respect to this minimum. For example, the dependence of the scalar curvature for some \( N \) is shown in figure 3.
Taking into account all the above and the fact that $\theta \in [0, \pi]$ and $\chi \in [0, 2\pi]$, we conclude that the manifold described by the metric (13) is closed and has a dumbbell-shape structure. It has a concave part with the centerline at $\theta = \pi/2$ and is symmetric with respect to this line. The evolution of the spin system having started from the initial state (3) happens on the circle of radius

$$r = \frac{\gamma}{2} \sqrt{N(N - 1)} \left[ N - 1 - \left( N - \frac{3}{2} \right) \sin^2 \theta \right]^{1/2} \sin \theta.$$

(19)

For certain $\theta$ (except $\theta = 0, \pi$) this radius depends on the number of spins. The larger the number of spins, the greater the circle radius.

We can see that entanglement (7) and Riemann curvature (17) depend on parameter $\theta$. This means that the entanglement can be expressed by the Riemann curvature. For this purpose, from equation (17) we find

$$\cos^2 \theta = -\frac{R\gamma^2N - 24 + 4\sqrt{R\gamma^2N(1 - N) + 32N - 28}}{(R\gamma^2N - 32)(2N - 3)}.$$

(20)

So, using this expression in equation (7) we represent the entanglement as a function of $R$. The dependence of entanglement on Riemann curvature for $\chi = \pi/2$ and for some number of spins is shown in figure 4. So, the maximal value of entanglement corresponds to the minimal value of curvature and vice versa.

Using the above information we can analyse how the states with different entanglement are located on the manifold. The manifolds with different $\phi$ are similar with respect to the distribution of the entangled states. So, each manifold contains closed lines with respect to $\theta$. 

Figure 3. The dependence of scalar curvature on $\theta$ (17). Results are presented for different numbers of spins $N$: $N = 2$ (solid curve), $N = 3$ (dashed curve), $N = 6$ (dash-dotted curve) and $N = 9$ (dotted curve). Here $\gamma = 1$. 

The dependence of scalar curvature on $\theta$ (17). Results are presented for different numbers of spins $N$: $N = 2$ (solid curve), $N = 3$ (dashed curve), $N = 6$ (dash-dotted curve) and $N = 9$ (dotted curve). Here $\gamma = 1$. 

Figure 3.
The behaviour of entanglement on these lines is defined by equation (7). They have a shape of circle of radius (19) which for certain initial states determine the trajectory of evolution. During the period of time \( t = \frac{2\pi}{J} \) the state vector passes through the whole circle and returns in the initial state.

5. Ising model with all-range interaction in the transverse magnetic field

In this section we study the influence of the transverse magnetic field on the geometry of quantum state manifold of the spin system with the following Hamiltonian

\[
H = \frac{J}{4}\left(\sum_{j=1}^{N}\sigma_{z,j}^2 + \frac{h}{2J}\sum_{j=1}^{N}\sigma_{x,j}\right),
\]

(21)

where \( h \) is proportional to the value of the magnetic field. The spectrum of this model was obtained only in the thermodynamic limit in [61]. Also using the semiclassical analysis and frozen-spin approximation the entanglement dynamics of this model was explored in [53].

So, the evolution of the system having started from state (3) can be expressed in the following form

\[
|\psi\rangle = \exp\left\{-i\chi\left(\frac{1}{4}\left(\sum_{j=1}^{N}\sigma_{z,j}^2 + \frac{h}{2J}\sum_{j=1}^{N}\sigma_{x,j}\right)\right)\right\}|\psi_I\rangle.
\]

(22)

Then the Funini–Study metric takes the form
This metric is reduced to the diagonal form using the following transformation of the coordinates

\[
\theta = \theta' + \frac{h}{J} \sin \phi \chi', \quad \chi = \chi'.
\]

Then we obtain

\[
\begin{align*}
g_{\theta' \theta'} &= \gamma^2 N \frac{3}{4}, \\
g_{\chi' \chi'} &= \gamma^2 N \left[ (N-1) \sin^2 \theta \left( N-1 - \left( N - \frac{3}{2} \right) \sin^2 \theta \right) \right. \\
&\quad - 2 \frac{h}{J} (N-1) \sin \theta \cos^2 \theta \cos \phi + \left( \frac{h}{J} \right)^2 \left( 1 - \sin^2 \theta \cos^2 \phi \right) \left. \right], \\
g_{\theta' \chi'} &= - \frac{\gamma^2 h}{4} J N \sin \phi.
\end{align*}
\]

where parameter \( \theta \) is defined by transformation (24). As we can see, in this case the metric tensor depends on the initial parameter \( \phi \) and value of the magnetic field. For \( h \neq 0 \) and \( \phi \neq \pi/2, -\pi/2 \) the topology of the manifold is changed. In this case the metric tensor exists for any \( \theta \). Then \( \theta \) takes the values in the range from 0 to \( 2\pi \). The form of manifold also depends on the periodicity of evolution. When during the some period of time the system achieves the initial state then the manifold has the form of a torus. Otherwise we have a infinity cylinder. In the case \( \phi = \pi/2 \) or \( -\pi/2 \) the manifold has the topology of a sphere.

6. Conclusion

We considered the evolution of an \( N \) spin-1/2 system with all-range Ising-type interaction having started from the state projected on the positive direction of the unit vector. The state that is achieved during such evolution depends on two spherical angles that define the initial state and the period of time of evolution. The value of entanglement of one spin with the rest spins during the evolution was obtained. This value depends on the polar angle determined by the initial state, time of evolution and number of spins (7). It was shown that for a certain polar angle during the period of time \( \pi/(2J) \) and \( 3\pi/(2J) \) the system from the disentangled initial state reaches some maximal entangled states. The value of entanglement depends on this angle as follows: it increases when the angle changes from 0 to \( \pi/2 \) and from \( \pi \) to \( \pi/2 \). So, the maximal possible entanglement is achieved when the initial state is located in the \( xy \)-plane. The achieved states are the ‘Schrödinger cat’ states [18, 21].

The independence of the entanglement of the spin system on the azimuthal angle of the initial state means that there exists a continuous set of states with the same value of entanglement.
So, we studied the geometry of manifolds that contained the entangled states reached during the evolution of the spin system. In turn, these states were placed on different manifolds defined by the polar angle of initial states and the time of evolution. These manifolds were geometrically similar but each of them contained states with a certain value of the azimuthal angle. We analysed the Fubini–Study metric of these manifolds and obtained that it is closed and has a dumbbell-shape structure. Also, we obtained that, for a certain initial state, the evolution happens along the circle of radius depending on the number of spins (19). So, it can be argued that the larger the number of the spins in the system, the greater the circle radius. We found the dependence of entanglement of the system on the scalar curvature of the manifold. As a result we showed that the states are located on the manifold so that the entanglement of the system decreases with the increase in curvature. So, the maximally entangled states are located on the surface with minimal curvature and vice versa.

Finally we studied the Fubini–Study metric of the quantum state manifold when the spin system is placed in the transverse magnetic field. We showed that the topology of the manifold is changed, except the case when the initial state is located in the perpendicular plane to the magnetic field. Then the manifold is a torus for periodic evolution and an infinity cylinder for non-periodic evolution.

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