ENERGY PROPERTIES OF CRITICAL KIRCHHOFF PROBLEMS WITH APPLICATIONS
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Abstract. In this paper we fully characterize the sequentially weakly lower semicontinuity of the parameter-depending energy functional associated with the critical Kirchhoff problem. We also establish sufficient criteria with respect to the parameters for the convexity and validity of the Palais-Smale condition of the same energy functional. We then apply these regularity properties in the study of some elliptic problems involving the critical Kirchhoff term.

1. Introduction

The time-depending state of a stretched string is given by the solution of the nonlocal equation

\[ u_{tt} - \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = h(t, x, u), \quad (t, x) \in (0, +\infty) \times \Omega, \]

proposed first by Kirchhoff [16] in 1883. In (1.1), \( \Omega \subset \mathbb{R}^d \) is an open bounded domain, the solution \( u : (0, \infty) \times \Omega \rightarrow \mathbb{R} \) denotes the displacement of the string, \( h : (0, \infty) \times \Omega \times \mathbb{R} \rightarrow \mathbb{R} \) is a Carathéodory function representing the external force, \( a \) is the initial tension, while \( b \) is related to the intrinsic properties of the string (such as Young’s modulus of the material). Other nonlocal equations similar to (1.1) appear also in biological systems, where \( u \) describes a process depending on its average (over a given set), like population density, see e.g. Chipot and Lovat [5].

Let \( \Omega \subset \mathbb{R}^d \) be an open bounded domain, \( d \geq 4 \). The primary aim of the present paper is to establish basic properties of the energy functional associated with the stationary form of (1.1), involving a critical term and subject to the Dirichlet boundary condition, namely,

\[
\begin{aligned}
- \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u &= |u|^{2^* - 2} u \quad \text{in} \quad \Omega, \\
\quad u &= 0 \quad \text{on} \quad \partial \Omega,
\end{aligned}
\]

where \( a, b > 0 \) and \( 2^* = \frac{2d}{d - 2} \) is the critical Sobolev exponent. In spite of the competing effect of the nonlocal term \( \int_{\Omega} |\nabla u|^2 \, dx \Delta u \) with the critical nonlinearity \( |u|^{2^* - 2} u \) as well as the lack of compactness of the Sobolev embedding \( H^1_0(\Omega) \hookrightarrow L^{2^*}(\Omega) \), there are several contributions concerning existence and multiplicity of solutions for problem \((P_{a,b})\), by using fine arguments both from variational analysis (see e.g. Autuori, Fiscella and Pucci [3], Chen, Kuo and Wu [4], Corrêa and Figueiredo [6], Figueiredo [11], Perera and Zhang[20, 21]) and topological methods (see e.g. Fan [9], Figueiredo and Santos [12]). It is also worth mentioning that the Palais-Smale compactness condition combined with the Lions concentration compactness principle [18] are still the most popular tools to deal with elliptic problems involving critical terms. We note that problem \((P_{a,b})\) is sensitive with respect to the size of the space dimension \( d \). Indeed, different arguments/results are applied/obtained for the lower dimensional case \( d \in \{3, 4\} \) (see e.g. Alves, Corrêa and Figueiredo [1], Deng and Shuai [8], Lei, Liu and Guo [17] and Naimen [19]) and for the higher dimensional case \( d > 4 \) (see Alves, Corrêa and Ma [2], Hebey [13, 14], Yao and Mu [26]); moreover, the parameters \( a \) and \( b \) should satisfy suitable constraints in order to employ the aforementioned principles.

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Moreover, as a formal observation, we notice that

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which will play crucial roles in the lower semicontinuity, validity of the PS-condition and convexity in (1.2) but never achieved except when \( \omega \) being the volume of the unit ball in \( \mathbb{R}^d \). Note that for every \( \omega \) is sharp in (1.2) but never achieved except when \( \Omega = \mathbb{R}^d \), see e.g. Willem [28]. The energy functional \( E_{a,b} : H^1_0(\Omega) \to \mathbb{R} \) associated with problem \( (\mathcal{P}_{a,b}) \) is defined by

\[
E_{a,b}(u) = \frac{a}{2} \|u\|^2 + \frac{b}{4} \|u\|^4 - \frac{1}{2^*} \|u\|_{2^*}^{2^*}.
\]

For a fixed \( d \geq 4 \), we introduce the constants

\[
L_d = \begin{cases} 
\frac{4(d - 4)^{d-4}}{d^{d-2} S_d^d}, & d > 4 \\
\frac{1}{S_4^2}, & d = 4,
\end{cases} \quad \text{PS}_d = \begin{cases} 
\frac{2(d - 4)^{d-4}}{(d - 2)^{d-2} S_d^d}, & d > 4 \\
\frac{1}{S_4^2}, & d = 4,
\end{cases}
\]

and

\[
C_d = \begin{cases} 
\frac{2(d - 4)^{d-4} (d + 2)^{d-2}}{(d - 2)^{d-2} S_d^2}, & d > 4 \\
\frac{3}{S_4^2}, & d = 4,
\end{cases}
\]

which will play crucial roles in the lower semicontinuity, validity of the PS-condition and convexity of \( E_{a,b} \), respectively. Note that for every \( d \geq 4 \), we have

\[
L_d \leq \text{PS}_d \leq C_d.
\]

Moreover, as a formal observation, we notice that

\[
\lim_{d \to 4} L_d = L_4; \quad \lim_{d \to 4} \text{PS}_d = \text{PS}_4; \quad \lim_{d \to 4} C_d = C_4.
\]

Our main result reads as follows:

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^d \) be an open bounded domain \( (d \geq 4) \), \( a, b > 0 \) two fixed numbers, and \( E_{a,b} \) be the energy functional associated with problem \( (\mathcal{P}_{a,b}) \). Then the following statements hold:

(i) \( E_{a,b} \) is sequentially weakly lower semicontinuous on \( H^1_0(\Omega) \) if and only if \( a \frac{d-4}{2} b \geq L_d \);

(ii) \( E_{a,b} \) satisfies the Palais-Smale condition on \( H^1_0(\Omega) \) whenever \( a \frac{d-4}{2} b > \text{PS}_d \);

(iii) \( E_{a,b} \) is convex on \( H^1_0(\Omega) \) whenever \( a \frac{d-4}{2} b \geq C_d \). In addition, \( E_{a,b} \) is strictly convex on \( H^1_0(\Omega) \) whenever \( a \frac{d-4}{2} b > C_d \).
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Figure 1.1. Curves $a \frac{d-4}{2} b = L_d$ and $\text{PS}_d$ and $C_d$ for $d > 4$ (case (i)) and $d = 4$ (case (ii)).

Remark 1.1. (i) By the proof of Theorem 1.1/(i) we observe that the sequentially weakly lower semicontinuity of $\mathcal{E}_{a,b}$ holds on any open domain $\Omega \subseteq \mathbb{R}^d$ (not necessary bounded). However, the optimality of the constant $L_d$ requires that $\Omega \neq \mathbb{R}^d$, see Section 2.

(ii) Note that a similar result as Theorem 1.1/(ii) (with the same assumption $a \frac{d-4}{2} b > \text{PS}_d$) has been proved by Hebey [14] on compact Riemannian manifolds. We provide here a genuinely different proof than in [14] based on the second concentration compactness lemma of Lions [18].

In the sequel, we provide two applications of Theorem 1.1. First, we consider the model Poisson type problem

\[
- \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = |u|^{2^* - 2} u + h(x) \quad \text{in} \quad \Omega,
\]
\[
u = 0 \quad \text{on} \quad \partial \Omega,
\]

where $h \in L^\infty(\Omega)$ is a positive function.

**Theorem 1.2.** Let $\Omega \subseteq \mathbb{R}^d$ be an open bounded domain ($d \geq 4$), $a, b > 0$ be fixed numbers. Then

(i) if $a \frac{d-4}{2} b \geq L_d$, problem $(\mathcal{P}_{a,b}^h)$ has at least a weak solution in $H^1_0(\Omega)$;

(ii) if $a \frac{d-4}{2} b > C_d$, problem $(\mathcal{P}_{a,b}^h)$ has a unique weak solution in $H^1_0(\Omega)$.

As a second application we consider the double-perturbed problem of $(\mathcal{P}_{a,b})$ of the form

\[
- \left( a + b \int_{\Omega} |\nabla u|^2 \right) \Delta u = |u|^{2^* - 2} u + \lambda |u|^{p-2} u + \mu g(x,u) \quad \text{in} \quad \Omega,
\]
\[
u = 0 \quad \text{on} \quad \partial \Omega,
\]

where $a, b, \lambda, \mu$ are positive parameters, $1 < p < 2^*$ and $g : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ is Carathéodory function belonging to the class $\mathcal{A}$ which contains functions $\varphi : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ such that

\[
\sup_{(x,t) \in \Omega \times \mathbb{R}} \frac{|\varphi(x,t)|}{1 + |t|^{q-1}} < +\infty
\]

for some $1 < q < 2^*$.

The single-perturbed problem $(\mathcal{P}_{a,b}^{p,0})$ (i.e., $g \equiv 0$) is of particular interest. Indeed, when $\lambda > 0$ is small enough and $d = 4$, Naimen [19] proved that $(\mathcal{P}_{a,b}^{p,0})$ has a positive solution if and only if $b < S_4^{-2}$; when $d > 4$, there are also some sufficient conditions for guaranteeing the existence of positive solutions for $(\mathcal{P}_{a,b}^{p,0})$. In addition, if $p \in (2,4)$ and $b > S_4^{-2} = L_4 = \text{PS}_4$, one can easily prove that any weak solution $u \in H^1_0(\Omega)$ of $(\mathcal{P}_{a,b}^{p,0})$ fulfills the a priori estimate

\[
\|u\| \leq \left( \frac{\lambda S_4^{-2} - \mu}{bS_4^{-2} - 1} \right) \frac{1}{p}.
\]
The following result is twofold. First, it complements the result of Naïmen [19] (i.e., we consider $b > S_d^{-2}$ for $d = 4$); second, having in our mind the global estimate (1.6), it shows that the weak solutions of the perturbed problem ($\mathcal{P}_{a,b}^p$) by means of any subcritical function will be stable with respect to the $H^1_0$-norm (whenever $\lambda > 0$ is large enough).

**Theorem 1.3.** Let $\Omega \subset \mathbb{R}^d$ be an open bounded domain $(d \geq 4)$, $a, b > 0$ two fixed numbers such that $a \frac{d-4}{2} > S_d^{-2}$ and $p \in (2, 2^*)$. Then there exists $\lambda^* > 0$ such that for each compact interval $[\alpha, \beta] \subset (\lambda^*, +\infty)$, there exists $r > 0$ with the following property: for every $\lambda \in [\alpha, \beta]$, and for every $g \in \mathcal{A}$, there exists $\mu^* > 0$ such that for each $\mu \in [0, \mu^*]$, problem $\mathcal{P}_{a,b}^p(\mu)$ has at least three weak solutions whose norms are less than $r$.

In fact, instead of Theorem 1.3 a slightly more general result will be given in Section 3, replacing the term $u \mapsto |u|^{p-2}u$ by a function $f \in \mathcal{A}$ verifying some mild hypotheses.

2. PROOF OF THEOREM 1.1

Proof of Theorem 1.1(i). We divide the proof into two parts.

**Step 1.** Assume first that $a \frac{d-4}{2} \geq L_d$; we are going to prove that the energy functional $\mathcal{E}_{a,b}$ is sequentially weakly lower semicontinuous on $H^1_0(\Omega)$. To see this, let $u_n \in H^1_0(\Omega)$ be an arbitrary sequence such that $u_n \rightarrow u$ in $H^1_0(\Omega)$. Thus, up to a subsequence, we have for every $p < 2^*$ that $u_n \rightarrow u$ in $L^p(\Omega)$ and $\nabla u_n \rightharpoonup \nabla u$ in $L^2(\Omega)$ as $n \rightarrow \infty$.

By the latter relation, it is clear that

$$\|u_n\|^2 - \|u\|^2 = \|u_n - u\|^2 + 2\int_\Omega \nabla(u_n - u) \nabla u = \|u_n - u\|^2 + o(1), \quad n \rightarrow \infty.$$ 

We also have that

$$\|u_n\|^4 - \|u\|^4 = (\|u_n\|^2 - \|u\|^2)(\|u_n\|^2 + \|u\|^2)$$

$$= (\|u_n - u\|^2 + o(1)) \left(\|u_n - u\|^2 + 2\int_\Omega \nabla(u_n - u) \nabla u + 2\|u\|^2\right)$$

$$= (\|u_n - u\|^2 + o(1)) \left(\|u_n - u\|^2 + 2\|u\|^2 + o(1)\right), \quad n \rightarrow \infty.$$

On the other hand, by the Brézis-Lieb Lemma (see e.g. Willem [28]), one has

$$\|u_n\|_{2^*} - \|u\|_{2^*} = \|u_n - u\|_{2^*} + o(1), \quad n \rightarrow \infty.$$ 

Combining the above estimates, it yields

$$\mathcal{E}_{a,b}(u_n) - \mathcal{E}_{a,b}(u) = \frac{a}{2}\left(\|u_n\|^2 - \|u\|^2\right) + \frac{b}{4}\left(\|u_n\|^4 - \|u\|^4\right) - \frac{1}{2^{*}}\left(\|u_n\|_{2^*}^2 - \|u\|_{2^*}^2\right)$$

$$= \frac{a}{2}\|u_n - u\|^2 + \frac{b}{4}\left(\|u_n - u\|^4 + 2\|u\|^2\|u_n - u\|^2\right) - \frac{1}{2^{*}}\|u_n - u\|_{2^*}^2 + o(1)$$

$$\geq \frac{a}{2}\|u_n - u\|^2 + \frac{b}{4}\|u_n - u\|^4 + S_d^{-\frac{2^{*}}{2^*}}\|u_n - u\|^{2^{*}} + o(1)$$

$$\geq \frac{a}{2}\|u_n - u\|^2 + \frac{b}{4}\|u_n - u\|^4 - S_d^{-\frac{2^{*}}{2^*}}\|u_n - u\|^{2^{*}} + o(1)$$

$$= \|u_n - u\|^2 \left(\frac{a}{2} + \frac{b}{4}\|u_n - u\|^2 - S_d^{-\frac{2^{*}}{2^*}}\|u_n - u\|^{2^{*} - 2}\right) + o(1), \quad n \rightarrow \infty.$$

Let us consider the function $f_d : [0, \infty) \rightarrow \mathbb{R}$ defined by

$$f_d(x) = \frac{a}{2} + \frac{b}{4}x^2 - S_d^{-\frac{2^{*}}{2^*}}x^{2^{*} - 2}, \quad x \geq 0.$$ 

We claim that the function $f_d$ is positive for all $x \geq 0$. 

(2.1)
Case 1: \( d = 4 \). If follows that \( 2^* = 4 \), thus by the hypothesis \( b \geq L_d \) – which is equivalent to \( bS_4^2 \geq 1 \) – it directly follows that

\[
f_4(x) = \frac{a}{2} + \frac{b - S_4^{-2}}{4} x^2 \geq 0, \ \forall x \geq 0.
\]

Case 2: \( d > 4 \). The minimum of the function \( f_d \) is at \( m_d > 0 \), where

\[
m_d = \left( \frac{2^* b}{2(2^* - 2) S_d^2} \right)^{\frac{1}{2 - \frac{4}{d}}}. \tag{2.2}
\]

A simple algebraic computation shows that

\[
a d^{\frac{4}{d - 4}} b \geq L_d \iff f_d(m_d) = \frac{1}{2} \left( a - b - \frac{2}{d - 4} L_d \right) \geq 0,
\]

which proves the claim.

Summing up the above estimates, we have that

\[
\liminf_{n \to \infty} (E_{a,b}(u_n) - E_{a,b}(u)) \geq \liminf_{n \to \infty} \| u_n - u \| f_d(\| u_n - u \|) \geq 0, \tag{2.3}
\]

which proves the sequentially weakly lower semicontinuity of \( E_{a,b} \) on \( H^1_0(\Omega) \).

Step 2. Now, we prove that the constant \( L_d \) in Theorem 1.1 is sharp. Assume the contrary, i.e., \( E_{a,b} \) is still sequentially weakly lower semicontinuous on \( H^1_0(\Omega) \) for some \( a, b > 0 \) with the property that

\[
a d^{\frac{4}{d - 4}} b < L_d. \tag{2.4}
\]

Case 1: \( d = 4 \). Fix a minimizing sequence \( \{ u_n \} \subset H^1_0(\Omega) \) for \( S_4 \) in (1.2); by its boundedness it is clear that there exists \( u \in H^1_0(\Omega) \setminus \{ 0 \} \) such that, up to a subsequence, \( u_n \rightharpoonup u \) in \( H^1_0(\Omega) \). Moreover, the sequentially weakly lower semicontinuity of the norm \( \| \cdot \| \) implies that \( \| u \| \geq \liminf_{n \to \infty} \| u_n \| =: L \) and there exists a subsequence \( \{ u_{n_j} \} \) of \( \{ u_n \} \) such that \( L = \lim_{j \to \infty} \| u_{n_j} \| \); in particular, \( L > 0 \).

By recalling the function \( f_4 \) from (2.1), due to (2.4), it is clear that on \( (x_0, \infty) \) the function \( f_4 \) is decreasing and negative, where

\[
x_0 = \left( \frac{2aS_4^2}{1 - S_4^2 b} \right)^{\frac{1}{2}} \text{ is the unique solution of } f_4(x) = 0, \ x \geq 0.
\]

Figure 2.1. Shape of the function \( x \mapsto f_4(x), \ x \geq 0 \), when (2.4) holds.
Let \( c > 0 \) be such that \( cL \geq c\|u\| > x_0 \). It is also clear that \( \{cu_n\} \) is a minimizing sequence for \( S_4 \) and \( cu_n \rightharpoonup cu \) in \( H_0^1(\Omega) \) as \( j \to \infty \). Consequently, since \( f_4 \) is continuous, we have that
\[
\liminf_{n \to \infty} E_{a,b}(cu_n) \leq \liminf_{j \to \infty} E_{a,b}(cu_{n_j}) \leq \liminf_{j \to \infty} \left( \frac{a}{2} \|cu_{n_j}\|^2 + \frac{b}{4} \|cu_{n_j}\|^4 - \frac{S_4^{-2}}{4} \|cu_{n_j}\|^4 \right) = (cL)^2 f_4(cL). \tag{2.5}
\]
Since \( cL \geq c\|u\| > x_0 \), we have that \( f_4(cL) \leq f_4(\|cu\|) < 0 \), thus by (2.5), we get that
\[
\liminf_{n \to \infty} E_{a,b}(cu_n) \leq \|cu\|^2 f_4(\|cu\|).
\]
On the other hand, by (1.3) we have
\[
\|cu\|^2 f_4(\|cu\|) = \frac{a}{2} \|cu\|^2 + \frac{b}{4} \|cu\|^4 - \frac{S_4^{-2}}{4} \|cu\|^4 \leq \frac{a}{2} \|cu\|^2 + \frac{b}{4} \|cu\|^4 - \frac{1}{4} \int_{\Omega} \|cu\|^4 = E_{a,b}(cu). \tag{2.6}
\]
By the above estimates we have that \( \liminf_{n \to \infty} E_{a,b}(cu_n) \leq E_{a,b}(cu) \). In fact, we have strict inequality in the latter relation; indeed, otherwise we would have \( S_4^{-2}\|cu\|^4 = \|cu\|^4 \), i.e., \( u \) would be an extremal function in (1.2). However, since \( \Omega \neq \mathbb{R}^d \), no extremal function exists in (1.2), see Willem [28, Proposition 1.43]. Thus, we indeed have
\[
\liminf_{n \to \infty} E_{a,b}(cu_n) < E_{a,b}(cu),
\]
which contradicts the sequentially weakly lower semicontinuity of \( E_{a,b} \) on \( H_0^1(\Omega) \). Accordingly, it yields that (2.4) cannot hold whenever the sequentially weakly lower semicontinuity of \( E_{a,b} \) on \( H_0^1(\Omega) \) is assumed, which proves the optimality of the constant \( L_d \) in the case when \( d = 4 \).

Case 2: \( d > 4 \). Since \( 0 < 2^* - 2 < 2 \), it is clear that \( f_d(+\infty) = +\infty \) and the assumption (2.4) together with the equivalence (2.2) ensures that the function \( f_d \) has its global minimum point at \( m_d > 0 \) with \( f_d(m_d) < 0 \).

Consider a minimizing sequence \( \{u_n\} \subset H_0^1(\Omega) \) for \( S_d \), and let \( \{u_{n_j}\} \) be a subsequence of \( \{u_n\} \), \( L > 0 \) and \( u \in H_0^1(\Omega) \) as in Case 1. Let \( c = \frac{m_d}{L} > 0 \). Since \( \|u\| \leq L \) and the minimum has the

**Figure 2.2.** Shape of the function \( x \mapsto f_d(x), \ x \geq 0 \), when \( d > 4 \) and (2.4) holds.
property that \( f_d(m_d) < 0 \), it follows, similarly as before, that

\[
\liminf_{n \to \infty} \mathcal{E}_{a,b}(cu_n) \leq \liminf_{j \to \infty} \mathcal{E}_{a,b}(cu_{n_j})
\]

\[
= \liminf_{j \to \infty} \left\{ \frac{a}{2} \|cu_{n_j}\|^2 + \frac{b}{4} \|cu_{n_j}\|^4 - \frac{S_d}{2^*} \|cu_{n_j}\|^{2^*} \right\}
\]

\[
= \liminf_{j \to \infty} \|cu_{n_j}\|^2 f_d(\|cu_{n_j}\|)
\]

\[
= (cL)^2 f_d(cL) = (cL)^2 f_d(m_d)
\]

\[
\leq \|cu\|^2 f_d(m_d)
\]

\[
\leq \|cu\|^2 f_d(\|cu\|).
\]

Similarly as in (2.6) and using Willem [28, Proposition 1.43], we have that \( \|cu\|^2 f_d(\|cu\|) < \mathcal{E}_{a,b}(cu) \), i.e., \( \mathcal{E}_{a,b} \) is not sequentially weakly lower semicontinuous on \( H_0^1(\Omega) \), a contradiction. \( \square \)

**Proof of Theorem 1.1/(ii).** Let \( \{u_n\} \subset H_0^1(\Omega) \) be a PS-sequence for \( \mathcal{E}_{a,b} \), i.e., for some \( c \in \mathbb{R} \),

\[
\begin{cases}
\mathcal{E}_{a,b}(u_n) \to c \\
\mathcal{E}_{a,b}'(u_n) \to 0
\end{cases} \quad \text{as } n \to \infty.
\]

One can prove that \( \mathcal{E}_{a,b} \) is of class \( C^2 \) on \( H_0^1(\Omega) \); in particular, a direct calculation yields (see also Willem [28, Proposition 1.12]) that

\[
\langle \mathcal{E}_{a,b}'(u), v \rangle = (a + b\|u\|^2) \int_{\Omega} \nabla u \nabla v - \int_{\Omega} |u|^{2^* - 2} uv, \quad \forall u, v \in H_0^1(\Omega).
\]

(2.7)

Note that \( \mathcal{E}_{a,b} \) is coercive on \( H_0^1(\Omega) \); indeed, the claim follows by (1.3) together with the facts that if \( d > 4 \) then \( 4 > 2^* \), while if \( d = 4 \) then \( b > \text{PS}_4 = S_4^{-2} \). In particular, it follows that \( \{u_n\} \) is bounded in \( H_0^1(\Omega) \), thus there exists \( u \in H_0^1(\Omega) \) such that (up to a subsequence),

\[
\begin{align*}
&u_n \to u \quad \text{in } H_0^1(\Omega), \\
&u_n \to u \quad \text{in } L^p(\Omega), \quad p \in [1, 2^*), \\
&u_n \to u \quad \text{a.e. in } \Omega.
\end{align*}
\]

By using the second concentration compactness lemma of Lions [18], there exist an at most countable index set \( J \), a set of points \( \{x_j\}_{j \in J} \subset \Omega \) and two families of positive numbers \( \{\eta_j\}_{j \in J} \), \( \{\nu_j\}_{j \in J} \) such that

\[
\begin{align*}
|\nabla u_n|² \to d\eta &\geq |\nabla u|^2 + \sum_{j \in J} \eta_j \delta_{x_j}, \quad (2.8) \\
|u_n|^{2^*} \to dv &\geq |u|^{2^*} + \sum_{j \in J} \nu_j \delta_{x_j}, \quad (2.9)
\end{align*}
\]

in the sense of measures, where \( \delta_{x_j} \) is the Dirac mass concentrated at \( x_j \) and such that

\[
S_d^n \frac{\eta_j}{\nu_j} \leq \eta_j, \quad \forall j \in J. \quad (10.10)
\]

We are going to prove that the index set \( J \) is empty. Arguing by contradiction, we may assume that there exists a \( j_0 \) such that \( \nu_{j_0} \neq 0 \) at \( x_0 \). For a sufficiently small \( \varepsilon > 0 \) we consider a non-negative cut-off function \( \phi_\varepsilon \) such that

\[
0 \leq \phi_\varepsilon \leq 1 \quad \text{in } \Omega, \\
\phi_\varepsilon \equiv 1 \quad \text{in } B(x_0, \varepsilon), \\
\phi_\varepsilon = 0 \quad \text{in } \Omega \setminus B(x_0, 2\varepsilon), \\
|\nabla \phi_\varepsilon| \leq \frac{2}{\varepsilon}.
\]
where \( B(x_0, r) = \{ x \in \mathbb{R}^d : |x - x_0| < r \} \) for \( r > 0 \). It is clear that the sequence \( \{ u_n \phi_\varepsilon \} \) is bounded in \( H^1_0(\Omega) \), thus
\[
\lim_{n \to \infty} \mathcal{E}_{a,b}'(u_n)(u_n \phi_\varepsilon) = 0.
\]
In particular, by (2.7) it turns out that when \( n \to \infty \), one has
\[
o(1) = \mathcal{E}_{a,b}'(u_n)(u_n \phi_\varepsilon)
= (a + b\|u_n\|^2) \int_\Omega \nabla u_n \nabla (u_n \phi_\varepsilon) - \int_\Omega |u_n|^{2^*} \phi_\varepsilon
= (a + b\|u_n\|^2) \left( \int_\Omega |\nabla u_n|^2 \phi_\varepsilon + \int_\Omega u_n \nabla u_n \nabla \phi_\varepsilon \right) - \int_\Omega |u_n|^{2^*} \phi_\varepsilon.
\]
First, by Hölder’s inequality, there exists \( C > 0 \) (not depending on \( n \)) such that
\[
\left| \int_\Omega u_n \nabla u_n \nabla \phi_\varepsilon \right| = \left| \int_{B(x_0, 2\varepsilon)} u_n \nabla u_n \nabla \phi_\varepsilon \right| \leq \left( \int_{B(x_0, 2\varepsilon)} |\nabla u_n|^2 \right)^{\frac{1}{2}} \left( \int_{B(x_0, 2\varepsilon)} |u_n \nabla \phi_\varepsilon|^2 \right)^{\frac{1}{2}} \\
\leq C \left( \int_{B(x_0, 2\varepsilon)} |u_n \nabla \phi_\varepsilon|^2 \right)^{\frac{1}{2}}.
\]
The Lebesgue dominated convergence theorem implies that
\[
\lim_{n \to \infty} \int_{B(x_0, 2\varepsilon)} |u_n \nabla \phi_\varepsilon|^2 = \int_{B(x_0, 2\varepsilon)} |u \nabla \phi_\varepsilon|^2,
\]
and by
\[
\left( \int_{B(x_0, 2\varepsilon)} |u \nabla \phi_\varepsilon|^2 \right)^{\frac{1}{2}} \leq \left( \int_{B(x_0, 2\varepsilon)} |u|^{2^*} \right)^{\frac{1}{2^*}} \left( \int_{B(x_0, 2\varepsilon)} |\nabla \phi_\varepsilon|^d \right)^{\frac{1}{d}} \\
\leq C \left( \int_{B(x_0, 2\varepsilon)} |u|^{2^*} \right)^{\frac{1}{2^*}},
\]
for some \( C > 0 \), we obtain
\[
\lim_{\varepsilon \to 0, n \to \infty} (a + b\|u_n\|^2) \left| \int_\Omega u_n \nabla u_n \nabla \phi_\varepsilon \right| = 0.
\]
Second, by (2.8) it follows that
\[
\lim_{n \to \infty} (a + b\|u_n\|^2) \int_\Omega |\nabla u_n|^2 \phi_\varepsilon \geq \lim_{n \to \infty} \left[ a \int_{B(x_0, 2\varepsilon)} |\nabla u_n|^2 \phi_\varepsilon + b \left( \int_\Omega |\nabla u_n|^2 \phi_\varepsilon \right)^2 \right] \\
\geq a \int_{B(x_0, 2\varepsilon)} |\nabla u_n|^2 \phi_\varepsilon + b \left( \int_\Omega |\nabla u_n|^2 \phi_\varepsilon \right)^2 + a \eta_{j_0} + b \eta_{j_0}^2,
\]
thus
\[
\lim_{\varepsilon \to 0, n \to \infty} (a + b\|u_n\|^2) \int_\Omega |\nabla u_n|^2 \phi_\varepsilon \geq a \eta_{j_0} + b \eta_{j_0}^2.
\]
Third, by (2.9) one has that
\[
\lim_{\varepsilon \to 0, n \to \infty} \int_\Omega |u_n|^{2^*} \phi_\varepsilon = \lim_{\varepsilon \to 0} \int_\Omega |u|^{2^*} \phi_\varepsilon + \nu_{j_0} = \lim_{\varepsilon \to 0, n \to \infty} \int_{B(x_0, 2\varepsilon)} |u|^2 \phi_\varepsilon + \nu_{j_0} = \nu_{j_0}.
\]
Summing up the above estimates, one obtains
\[
0 \geq a \eta_{j_0} + b \eta_{j_0}^2 - \nu_{j_0} \geq a \eta_{j_0} + b \eta_{j_0}^2 - S_d^{\frac{2^*}{2}} \eta_{j_0}^{\frac{2^*}{2}} \\
= \eta_{j_0} \left( a + b \eta_{j_0} - S_d^{\frac{2^*}{2}} \eta_{j_0}^{\frac{2^*}{2} - 1} \right). \tag{2.11}
\]
Let \( \tilde{f}_d : [0, \infty) \to \mathbb{R} \) be the function defined by
\[
\tilde{f}_d(x) = a + bx - S_d^{-\frac{2}{d}} x^{\frac{2}{d}-1}, \quad x \geq 0.
\]
One can see that the assumption \( a \frac{d-4}{d} b > \mathcal{P}_d \) implies that \( \tilde{f}_d(x) > 0 \) for all \( x \geq 0 \). In particular, it follows that \( a + b\eta_{j_0} - S_d^{-\frac{2}{d}} \eta_{j_0}^{\frac{2}{d}-1} > 0 \), therefore by (2.11) we necessarily have that \( \eta_{j_0} = 0 \), contradicting \( \nu_{j_0} \neq 0 \) and (2.10). The latter fact implies that \( J \) is empty. In particular, by (2.9) and Brezis-Lieb lemma it follows that \( u_n \to u \) in \( L^{2^*}(\Omega) \) as \( n \to \infty \); thus
\[
\lim_{n \to \infty} \int_{\Omega} |u_n|^{2^*-2} u_n (u - u_n) = 0. \tag{2.12}
\]
Since \( \mathcal{E}_{a,b}'(u_n) \to 0 \) as \( n \to \infty \), we have by (2.7) and (2.12) that
\[
0 = \lim_{n \to \infty} \mathcal{E}_{a,b}'(u_n)(u_n - u) = \lim_{n \to \infty} \left( (a + b\|u_n\|^2) \int_{\Omega} \nabla u_n (\nabla u - \nabla u_n) + \int_{\Omega} |u_n|^{2^*-2} u_n (u - u_n) \right)
= \lim_{n \to \infty} \left( (a + b\|u_n\|^2) \int_{\Omega} \nabla u_n \nabla (u - u_n) \right).
\]
By the boundedness of \( \{u_n\} \subset H^1_0(\Omega) \), the latter relation and the fact that \( u_n \to u \) in \( H^1_0(\Omega) \), i.e., \( \int_{\Omega} \nabla u \nabla (u - u_n) \to 0 \) as \( n \to \infty \), we obtain at once that \( \|u_n - u\|^2 \to 0 \) as \( n \to \infty \), which concludes the proof. \( \square \)

**Proof of Theorem 1.1 (iii).** It is well known that the energy functional \( \mathcal{E}_{a,b} : H^1_0(\Omega) \to \mathbb{R} \) is convex if and only if \( \mathcal{E}_{a,b}' \) is monotone, or equivalently,
\[
\langle \mathcal{E}_{a,b}''(u), v \rangle \geq 0, \quad \forall u, v \in H^1_0(\Omega).
\]

By using (2.7), we have
\[
\langle \mathcal{E}_{a,b}''(u), v \rangle = a\|v\|^2 + b\|u\|^2\|v\|^2 + 2b \left( \int_{\Omega} \nabla u \nabla v \right)^2 - (2^*-1) \int_{\Omega} |u|^{2^*-2} v^2.
\]
Moreover, by Hölder and Sobolev inequalities, one can see that
\[
\langle \mathcal{E}_{a,b}''(u), v \rangle \geq a\|v\|^2 + b\|u\|^2\|v\|^2 - (2^*-1) S_d^{-\frac{2}{d}} \|u\|^{2^*-2} \|v\|^2
= \|v\|^2 \left[ a + b\|u\|^2 - (2^*-1) S_d^{-\frac{2}{d}} \|u\|^{2^*-2} \right].
\]

Let us consider the function \( \overline{f}_d : [0, \infty) \to \mathbb{R} \) given by
\[
\overline{f}_d(x) = a + bx^2 - (2^*-1) S_d^{-\frac{2}{d}} x^{2^*-2}, \quad x \geq 0.
\]
We claim that the function \( \overline{f}_d \) is positive on \([0, \infty)\).

**Case 1:** \( d = 4 \). Since \( 2^* = 4 \), the hypothesis \( b \geq C_d \) (which is equivalent to \( b S_d^2 \geq 3 \)) implies that
\[
\overline{f}_d(x) = a + bx^2 - 3S_d^{-2} x^2 = a + x^2 (b - 3S_d^{-2}) \geq 0, \quad \forall x \geq 0.
\]

**Case 2:** \( d > 4 \). The global minimum of the function \( \overline{f}_d \) is at \( m_d > 0 \), where
\[
m_d = \left[ \frac{2b S_d^2}{(2^*-1)(2^*-2)} \right]^{\frac{1}{2^*-2}}.
\]
It turns out that
\[
a \frac{d-4}{d} b \geq C_d \iff \overline{f}_d(m_d) \geq 0,
\]
which proves the claim. The strict convexity of \( \mathcal{E}_{a,b} \) similarly follows whenever \( a \frac{d-4}{d} b > C_d \) is assumed. \( \square \)
3. Applications: proof of Theorems 1.2 & 1.3

Proof of Theorem 1.2. We consider the energy functional associated with problem \((P_{a,b}^h)\), i.e.,
\[
E(u) = E_{a,b}(u) - \int_\Omega h(x)u(x)dx, \quad u \in H^1_0(\Omega).
\]
It is easy to prove that \(E\) belongs to \(C^1(H^1_0(\Omega), \mathbb{R})\) and its critical points are exactly the weak solutions of problem \((P_{a,b}^h)\). Moreover, \(E\) is bounded from below and coercive on \(H^1_0(\Omega)\), i.e., \(E(u) \to +\infty\) whenever \(||u|| \to +\infty\).

(i) If \(a \frac{d-4}{2} b \geq L_d\), by Theorem 1.1/(i) and the fact that \(u \mapsto \int_\Omega h(x)u(x)dx\) is sequentially weakly continuous on \(H^1_0(\Omega)\) (due to the boundedness of \(\Omega\) and the compactness of the embedding \(H^1_0(\Omega) \hookrightarrow L^p(\Omega), p \in [1, 2^*)\)), \(E\) turns to be sequentially weakly lower semicontinuous on \(H^1_0(\Omega)\). Thus the basic result of the calculus of variations implies that \(E\) has a global minimum point \(u \in H^1_0(\Omega)\), see Zeidler [27, Proposition 38.15], which is also a critical point of \(E\).

(ii) If \(a \frac{d-4}{2} b > C_d\), Theorem 1.1/(iii) implies that \(E\) is strictly convex on \(H^1_0(\Omega)\). By Zeidler [27, Theorem 38.C] it follows that \(E\) has at most one minimum/critical point. The inequality (1.5) and (i) conclude the proof. \(\Box\)

For \(f \in A\), let us denote by \(F(x, t) = \int_0^t f(x, s)ds\). We now prove the following result which directly implies Theorem 1.3.

**Theorem 3.1.** Let \(\Omega \subset \mathbb{R}^d\) be an open bounded domain \((d \geq 4)\), let \(f \in A\), and \(a, b > 0\) two fixed numbers such that \(a \frac{d-4}{4} b > PS_d\). Assume also that
\[
\begin{align*}
H_1) & \quad \lim_{t \to 0} \frac{\sup_{x \in \Omega} F(x, t)}{t^2} \leq 0; \\
H_2) & \quad \sup_{u \in H^1_0(\Omega)} \int_\Omega F(x, u) > 0.
\end{align*}
\]
Set
\[
\lambda^* = \inf \left\{ \frac{a \|u\|^2 + b \|u\|^4 - \frac{1}{2^*} \|u\|^{2^*}}{\int_\Omega F(x, u)} : u \in H^1_0(\Omega), \int_\Omega F(x, u) > 0 \right\}.
\]
Then, for each compact interval \([\alpha, \beta] \subset (\lambda^*, +\infty)\), there exists \(r > 0\) with the following property: for every \(\lambda \in [\alpha, \beta]\), and for every \(g \in A\), there exists \(\mu^* > 0\) such that for each \(\mu \in [0, \mu^*]\), the problem \((P_{a,b}^g)\) has at least three weak solutions whose norms are less than \(r\).

*Proof.* Denote by \(J_f : H^1_0(\Omega) \to \mathbb{R}\) the functional defined by
\[
J_f(u) = \int_\Omega F(x, u),
\]
and consider as before the functional
\[
E_{a,b}(u) = \frac{a}{2} \|u\|^2 + \frac{b}{2} \|u\|^4 - \frac{1}{2^*} \|u\|^{2^*},
\]
From Theorem 1.1/(i), \(E_{a,b}\) is sequentially weakly lower semicontinuous (see (1.5)), and if \(\{u_n\}\) weakly converges to \(u\) and \(\liminf_{n \to \infty} E_{a,b}(u_n) \leq E_{a,b}(u)\), then \(\{u_n\}\) has a subsequence strongly convergent to \(u\), see (2.3). Moreover it is of class \(C^1\) on \(H^1_0(\Omega)\). Since \(f\) has a subcritical growth, \(J_f\) is sequentially weakly continuous in \(H^1_0(\Omega)\), of class \(C^1\) too and bounded on bounded sets.

From assumption \(H_1\) it follows that
\[
\limsup_{u \to 0} \frac{J_f(u)}{E_{a,b}(u)} \leq 0,
\]
therefore \(E_{a,b} - \lambda J_f\) has a (strong) local minimum at zero for every \(\lambda > 0\). By Ricceri [24, Theorem C], zero turns out to be a local minimizer of \(E_{a,b} - \lambda J_f\) in the weak topology of \(H^1_0(\Omega)\).
It is also clear that $E_{a,b} - \lambda J_f$ is coercive for every $\lambda$ and, if $\lambda > \lambda^*$, its global minimum is different to zero.

To proceed, fix $[\alpha, \beta] \subset (\lambda^*, +\infty)$ and choose $\sigma > 0$. By the coercivity of $E_{a,b} - \lambda J_f$ it follows that the set $(E_{a,b} - \lambda J_f)^{-1}((\infty, \sigma))$ is compact and metrizable with respect to the weak topology. Also,

$$
\bigcup_{\lambda \in [\alpha, \beta]} (E_{a,b} - \lambda J_f)^{-1}((\infty, \sigma)) \subseteq B_\eta,
$$

for some positive radius $\eta$, where $B_\eta = \{ u \in H^1_0(\Omega) : \| u \| < \eta\}$. Let $c^* = \sup_{B_\eta} E_{a,b} + \beta \sup_{B_\eta} |J_f|$ and let $\tau > \eta$ be such that

$$
\bigcup_{\lambda \in [\alpha, \beta]} (E_{a,b} - \lambda J_f)^{-1}((\infty, c^* + 2)) \subseteq B_\tau.
$$

(3.2)

Let $\lambda \in [\alpha, \beta]$ and fix $g \in A$. Thus, if $J_g : H^1_0(\Omega) \to \mathbb{R}$ is the functional defined by

$$
J_g(u) = \int_{\Omega} G(x, u) \quad \text{where} \quad G(x, t) = \int_0^t g(x, s)ds,
$$

then, $J_g$ is of class $C^1$, with compact derivative. Choose a function $h \in C^1(\mathbb{R})$, bounded, such that $h(t) = t$ for every $t$ such that $|t| \leq \sup_{B_\tau} |J_g|$. Define $\tilde{J}_g = h \circ J_g$. Then, $\tilde{J}_g$ has compact derivative and $\tilde{J}_g(u) = J_g(u)$ for every $u \in B_\tau$.

Applying Ricceri [23, Theorem 4] with $P = E_{a,b} - \lambda J_f$, $Q = \tilde{J}_g$, $\tau$ the weak topology of $H^1_0(\Omega)$, we deduce the existence of some $\delta > 0$ such that for every $u \in [0, \delta]$, $E_{a,b} - \lambda J_f - \mu \tilde{J}_g$ has two local minimizers in the $\tau_{E_{a,b} - \lambda J_f}$ topology (the smallest topology containing both the weak topology and the sets $\{ (E_{a,b} - \lambda J_f)^{-1}((\infty, s)) \}_{s \in \mathbb{R}}$), say $u_1, u_2$, such that

$$
u_1, u_2 \in (E_{a,b} - \lambda J_f)^{-1}((\infty, \sigma)) \subseteq B_\eta \subseteq B_\tau.
$$

(3.3)

Since the topology $\tau_{E_{a,b} - \lambda J_f}$ is weaker than the strong topology, $u_1$ and $u_2$ turn out to be local minimizers of the functional $E_{a,b} - \lambda J_f - \mu \tilde{J}_g$. Define now $\mu^* = \min \left\{ \delta, \frac{1}{\sup_{B_\tau} h} \right\}$. One can see that $E_{a,b} - \lambda J_f - \mu \tilde{J}_g$ satisfies the Palais-Smale condition as in the Theorem 1.1/(ii) (Palais-Smale for $E_{a,b}$), thus from Pucci and Serrin [22, Theorem 1] there exists a critical point of $E_{a,b} - \lambda J_f - \mu \tilde{J}_g$, say $u_3$, such that

$$
(E_{a,b} - \lambda J_f - \mu \tilde{J}_g)(u_3) = \inf_{\gamma \in S} \sup_{t \in [0,1]} \gamma(t),
$$

where

$$
S = \{ \gamma \in C^0([0,1], H^1_0(\Omega)) : \gamma(0) = u_1, \ \gamma(1) = u_2 \}.
$$

In particular, if $\gamma(t) = t u_1 + (1 - t)u_2$, $t \in [0,1]$, then $\gamma \in S$ and

$$
\gamma(t) \in B_\eta, \quad \text{for all} \ t \in [0,1].
$$

Recall that $u_1, u_2 \in B_\eta$, see (3.3). So, by the definition of $c^*$ and $\mu^*$, one has

$$
(E_{a,b} - \lambda J_f - \mu \tilde{J}_g)(u_3) \leq \sup_{t \in [0,1]} (E_{a,b} - \lambda J_f - \mu \tilde{J}_g)(\gamma(t))
$$

$$
\leq c^* + \mu^* \sup_{\mathbb{R}} h \leq c^* + 1.
$$

Therefore,

$$
(E_{a,b} - \lambda J_f)(u_3) \leq c^* + 1 + \mu^* \sup_{\mathbb{R}} h \leq c^* + 2,
$$

and from (3.2) one has

$$
u_3 \in B_\tau.
$$

Accordingly, we conclude that $J_g(u_i) = J_g(u_i), \ i = 1, 2, 3$, so that $u_1, u_2, u_3$ are critical points of $E_{a,b} - \lambda J_f - \mu J_g$, i.e., weak solutions to problem $(P_{a,b}^{I_g})$. \qed
Remark 3.1. We conclude the paper by giving an upper estimate of $\lambda^\ast$ (see (3.1)) when

$$f(x, t) = \alpha(x)h(t),$$

where $\alpha \in L^\infty(\Omega)$ and $h : \mathbb{R} \to \mathbb{R}$ is a continuous function with $H(t_0) > 0$ for some $t_0 > 0$, 

$$\lim_{t \to t_0} \frac{H(t)}{t} = 0$$

and $\text{essinf}_{x \in \Omega} \alpha =: \alpha_0 > 0$; hereafter, $H(t) = \int_0^t h(s)ds$. Assumption $H_1$ is trivially verified. In order to verify $H_2$, we consider the function

$$u_\sigma(x) = \begin{cases} 0 & \text{if } x \in \Omega \setminus B(x_0, R); \\ t_0 (R - |x - x_0|) & \text{if } x \in B(x_0, R) \setminus B(x_0, \sigma R); \\ t_0 & \text{if } x \in B(x_0, \sigma R), \end{cases}$$

where $x_0 \in \Omega$, $\sigma \in (0, 1)$, and $R > 0$ is chosen in such a way that $R < \text{dist}(x_0, \partial \Omega)$. It is clear that

$$\|u_\sigma\|^2 = t_0^2(1 - \sigma)^{-2}(1 - \sigma^d)R^d - 2\omega_d;$$

$$\int_\Omega u_\sigma^2 \geq t_0^2\sigma^d R^d \omega_d;$$

$$\int_\Omega H(u_\sigma) \geq H(t_0)\sigma^d - \max_{|t| \leq t_0} H(t)(1 - \sigma^d) \geq R^d \omega_d.$$

If $\sigma \in (0, 1)$ is close enough to 1, the right-hand side of the last estimate becomes strictly positive; let $\sigma_0 \in (0, 1)$ such a value. In particular, one has that

$$\int_\Omega F(x, u_{\sigma_0}) \geq \alpha_0 \left[ H(t_0)\sigma^d_0 - \max_{|t| \leq t_0} H(t)(1 - \sigma^d_0) \right] R^d \omega_d > 0,$$

which proves the validity of $H_2$. Moreover, by the above estimates, it turns out that

$$\lambda^\ast \leq \frac{\alpha_0^2(1 - \sigma^d_0)^{-2}(1 - \sigma^d_0)}{2} + b(t_0^2(1 - \sigma^d_0)^{-2}(1 - \sigma^d_0))^2 R^d - 2\omega_d/4 - t_0^2\sigma_0^d R^d/2^* =: \bar{\lambda}.$$ 

Therefore, instead of $\lambda^\ast$ in Theorem 3.1, we can use the more explicit value of $\bar{\lambda} > 0$; the same holds for Theorem 1.3 with the choice $\alpha = \alpha_0 = 1$, $h(t) = |t|^{q-2}t$, $t_0 = 1$ and $\sigma_0 = (3/4)^{1/d}$.

Remark 3.2. We conclude the paper by stating that the regularity results from Theorem 1.1 and the applications in Theorems 1.2&1.3 can be extended to compact Riemannian manifolds with suitable modifications. The most sensitive part of the proof is the equivalence from Theorem 1.1/(i), which explores the non-existence of extremal functions in the critical Sobolev embedding; such a situation is precisely described in the paper of Hebey and Vaugon [15]. The non-compact case requires a careful analysis via appropriate group-theoretical arguments as in Farkas and Kristály [10]. We leave the details for interested readers.

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