Multiband Weighting of X-Ray Polarization Data

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Abstract

An optimal estimate for Stokes parameters is derived for the situation in X-ray astronomy where the instrument has a modulation factor that varies significantly with energy but the signals are very weak or mildly polarized. For such sources, the band of analysis may be broadened in order to obtain a significant polarization measurement. Optimal estimators are provided for the cases of binned and unbinned data and applied to data such as might be obtained for faint or weakly polarized sources observed using the Imaging X-ray Polarimetry Explorer. For a sample situation, the improvement in the minimum detectable polarization is 6%–7% using a count-weighted rms of the modulation factor, when compared to a count-weighted average. Improving the modulation factor, such as when using a neural network approach to Imaging X-ray Polarimetry Explorer event tracks, can provide additional improvement up to 10%–15%. The actual improvement depends on the spectral shape and the details of the instrument response functions.

1. Introduction

X-ray polarimeters are still at an early stage of development where measurements are signal limited across the entire band of sensitivity. Current and planned instruments generally provide sinusoidal signals with fractional half-amplitudes of \( \mu \) for 100% (linear) polarized input, where \( \mu \) is the modulation factor. Often, \( p \) and \( \mu \) are significantly less than unity, providing weak signals. The minimum detectable polarization (MDP) is frequently used to indicate how low an instrument’s sensitivity might be; when the instrument gives \( p > \text{MDP} \), the signal should be significant at high confidence. Most calculations of MDP select a single value for \( \mu \) in order to determine the MDP even though \( \mu \) may vary significantly with energy within the instrument’s bandpass. Here, I examine how one may account for such variation in an optimal fashion.

X-ray polarimeters that use the photoelectric effect, such as the Imaging X-ray Polarimetry Explorer (IXPE; Weisskopf et al. 2016; O’Dell et al. 2018), or practically any other method including scattering (e.g., X-Calibur; Beilicke et al. 2014; Kislat et al. 2018), provide a signal that is probabilistically related to the input polarization direction. A histogram of event phase angles, \( \psi \), then has a characteristic instrument-dependent modulation factor, \( \mu \). Ground-based calibration using sources of known polarization angle are used to determine how \( \mu \) depends on energy \( E \). Polarimeters are usually characterized by their MDP, which depends on \( \mu \) and is also related to the exposure time, \( T \), and the count rates the instrument records for both the source \( R_S \) and background \( R_B \).

\[
\text{MDP}_{99} = \frac{4.292}{\mu R_S} \left[ \frac{R_S + R_B}{T} \right]^{1/2},
\]

and the level of confidence, which we will take to be 99% (Weisskopf et al. 2010). When background is negligible, then

\[
\text{MDP}_{99} = \frac{4.292}{\mu N^{1/2}},
\]

where \( N = TR_S \) is the expected number of counts in the observation.

In general, the modulation factor depends on energy. It is defined as the ratio of the semiamplitude of the event histogram to the average over phase for fully polarized light. Thus, MDP_{99} can be readily computed for a small energy range, but it is more difficult to determine what value of \( \mu \) to use when \( \mu(E) \) is a strong function of \( E \). One approach would be to use a count-weighted average of \( \mu \):

\[
\bar{\mu} = \frac{\sum \mu(E_j) C_j}{N},
\]

where \( C_j \) are the counts observed in energy bin \( j \) and \( N = \sum C_j \) (Elsner et al. 2012). Here, I compute optimal estimates of the polarization Stokes parameters for cases where \( \mu \) can vary significantly across the energy range of the detector as long as the Stokes parameters are constant or vary slowly across the entire range.

For high signal observations of strongly polarized sources, the data can be divided into energy bands over which the assumption of \( \mu = \text{constant} \) is appropriate. In this analysis, I will concentrate on the case when the signal is weak, so that one requires a large bandpass in order to detect a polarization signal. As circular polarization is not yet feasibly detectable, I will assume that Stokes \( V \) is zero. Furthermore, for simplicity, only the case where the background is negligible and the exposure for each angular bin is the same are considered. Kislat et al. (2015) provided a method to determine weights for the case where an instrument must be rotated to sample the Stokes parameters fully and the sampling is not uniform.
2. Weighting of Binned Data

2.1. Assumptions

Let \( i \) designate one of \( n \) angular phase bins of width \( \Delta \psi = 2\pi/n \) about \( \psi_i = i \Delta \psi \) (for \( i = 0, \ldots, n - 1 \)) and \( j \) indicate the energy bin of width \( \Delta E \) about \( E_j \) (for \( j = 1, \ldots, J \)). The instrument provides counts \( C_{ij} \) each with uncertainty \( \sigma_{ij} \). I assume that the background is negligible for this analysis. Also, for the sake of simplicity so that we might gain some intuition about the solution, I will assume, for now, that we have large signals in every bin so that we may use \( \chi^2 \) statistics. To make formulae a little simpler, I will set \( \sigma_{ij} \) to \( \sigma \) for a few of the derivations but then generalize in Section 2.7.

Since we are interested in how to combine data from different energy channels, I also assume that the Stokes parameters \( Q \) and \( U \) are not functions of energy. Otherwise, we would not combine the channels at all. I will not use any formalism involving energy to channel redistribution matrix files (aka RMFs), but instead assume that the energy response functions are narrow compared to the energy bands. This last point is not really restrictive because all we need in the end is a model that describes the counts in each phase-averaged energy bin.

2.2. The Model and Fit Statistic

The expected number of counts in bin \( i, j \) is

\[
\lambda_{ij} = [1 + \mu_j (q \cos 2\psi_i + u \sin 2\psi_i)] f_i A_j T \Delta E \Delta \psi,
\]

where \( f_i \) is the source flux in suitable units (say, photons/cm\(^2\)/s/keV per radian of rotation), \( \mu_j \) is the modulation factor at energy \( E_j \), \( A_j \) is the system effective area, \( T \) is the exposure time, and \( q \) and \( u \) are the fractional Stokes parameters, given by \( Q = P \cos \phi_0 - qI \) and \( U = P \sin \phi_0 - uI \), respectively. The polarization fraction is then \( p \equiv P/I = (q^2 + u^2)^{1/2} \), and the source phase angle is \( \phi_0 = \tan^{-1} u/q \).

The electric vector position angle (EVPA) is \( \psi = \phi_0/2 \).

We form the fit statistic from the data and the model as

\[
\chi^2 = \sum_i \sum_j \frac{(C_{ij} - \lambda_{ij})^2}{\sigma_{ij}^2}.
\]

To make the solution a little easier to read, I define \( \alpha_j = f_i A_j T \Delta E \Delta \psi \). The phase angles are chosen uniformly.

\[
\sum_i s_i = \sum_i c_i = 0
\]

\[
\sum_i s_i c_i = 0
\]

\[
\sum_i s_i^2 = \sum_i c_i^2 = n/2.
\]

2.3. Solving for the Flux (Stokes I)

Setting the derivative of \( \chi^2 \) with respect to \( \alpha_j \) to zero yields

\[
\sum_i C_{ij} = n \Delta \alpha_j + \Delta \mu_j^2 \alpha_j \sum_i (q^2 c_i^2 + u^2 s_i^2)
\]

\[
= n \Delta \alpha_j + \frac{n}{2} \Delta \mu_j^2 p^2 \alpha_j,
\]

where \( \Delta' = \Delta \psi \Delta E \). For this analysis, \( \mu_j p \ll 1 \), so we drop the second term on the right-hand side of Equation (11), obtaining

\[
\hat{\alpha}_j = \frac{\sum_i C_{ij}}{2\pi \Delta E} = \frac{C_j}{2\pi \Delta E},
\]

where \( C_j = \sum_i C_{ij} \) is the expected number of counts in channel \( j \). We also have

\[
\hat{f}_j = \frac{C_j}{2\pi T A_j \Delta E}.
\]

Note that the units are appropriate; the factor of \( 1/2\pi \) comes in because the flux was defined per radian of polarization phase space.

2.4. Solving for the Polarization (Stokes Q and U)

Similarly proceeding, one may derive the best estimates of \( q \) and \( u \):

\[
\hat{q} = \frac{\sum_i C_{ij} \sum_j \beta_j C_{ij}}{\Delta \sum_i \beta_j^2 C_{ij}^2} = \frac{\sum_i \beta_j C_{ij} \sum_j C_{ij}}{\sum_i \beta_j^2 C_{ij}^2} = \sum_i \beta_j c_i
\]

\[
\hat{u} = \frac{n \sum_i s_i \sum_j \beta_j C_{ij}}{\Delta \sum_i \beta_j^2 C_{ij}^2} = \frac{2 \sum_i \beta_j C_{ij} \sum_j s_i C_{ij}}{\sum_j \beta_j^2 C_{ij}^2} = \sum_i \beta_j s_i
\]

where the definition of \( C_j \) and Equation (9) are used and

\[
\hat{x}_i = \frac{2 \sum_i \beta_j C_{ij} C_{ij}}{\sum_j \beta_j^2 C_{ij}^2}.
\]

Also,

\[
\tan(\hat{\phi}_0) = \frac{\sum_j \beta_j C_j \sum_i s_i C_{ij}}{\sum_j \beta_j C_j \sum_i c_i C_{ij}} = \frac{\sum_i s_i x_i}{\sum_i c_i x_i}
\]

\[
\hat{p} = \left[ \left( \sum_i c_i x_i \right)^2 + \left( \sum_i s_i x_i \right)^2 \right]^{1/2} = |f_2(x)|.
\]
where $f(x)$ is the second term of the Fourier transform of $x$, (i.e., the $2\theta$ modulation portion of the phase-binned counts weighted by a channel-dependent term). Note that $p\mu_j C_j$ is the maximum modulated signal in channel $j$, and $\mu_j C_j$ is the maximum modulated signal when $p = 1$.

The best estimates of $q$ and $u$ for a single energy channel $j$ are obtained from Equations (16) and (17) by removing the summation over $j$, giving

$$\hat{q}_j = \frac{2 \sum c_j C_j}{\mu_j C_j^2},$$

$$\hat{u}_j = \frac{2 \sum s_j C_j}{\mu_j C_j^2}.$$  

We can check these formulae by modeling with $C_j = C_j/n(1 + \eta \mu_j \cos 2\psi_j)$ as the response of the detector to a signal that is 100% polarized at an EVPA $\varphi = \varphi_0 = 0$. Substituting the model into Equations (21) and (22) gives $\hat{q}_j = \eta$ and $\hat{u}_j = 0$, as expected, using Equations (8) and (9).

### 2.5. Bandpass Weighting

Solving Equations (21) and (22) for $\sum c_j C_j$ and $\sum s_j C_j$, respectively, then Equations (16) and (17) can be rewritten as

$$\hat{q} = \sum_j W_j \hat{q}_j,$$

$$\hat{u} = \sum_j W_j \hat{u}_j,$$

where

$$W_j = \frac{\mu_j^2 C_j^2}{\sum_i \mu_i^2 C_i^2}.$$  

Because $\sum W_j = 1$, the $W_j$ values can be identified as weights to apply to the individual bandpass values in order to obtain optimal, global values, as long as the polarization angle and fraction do not depend significantly on energy. The relative weight of channel $j$ compared to channel $k$ is just $(\mu_j/\mu_k)^2(C_j/C_k)^2$, the ratio of the variances of the total counts in the two channels weighted by the square of the relevant modulation factors.

### 2.6. Stokes Parameter Uncertainties, Binned Case, Constant Uncertainties

The diagonal terms of the Fisher matrix give the parameter uncertainties for $f$, $q$, and $u$, as the off-diagonal derivatives of Equation (6) are all zero or very small when $p\mu \ll 1$. Because of the assumption that the bandpasses are arranged to have equal variances, $\sigma_j^2$, $C_j = n \sigma^2 = N/J$, giving

$$\sigma_{\delta f}^2 = \frac{2}{\partial f^2} = \frac{2}{(TA_j)^2 \partial f^2} \approx \frac{\sigma^2}{(TA_j \Delta T)^2 n} = \frac{C_j}{(2\pi TA_j \Delta E)^2},$$

$$\sigma_{\delta q}^2 = \frac{2}{\partial q^2} = \frac{2}{(TA_j)^2 \partial q^2} \approx \frac{2n \sigma^2}{\sum \mu_j^2 C_j^2} = \frac{2J}{N \sum \mu_j^2}.$$  

In order to relate the Stokes parameters’ uncertainties to MDP$_{99}$, consider the case of a single channel, where $J = 1$ and $\mu_j = \mu$, so

$$\sigma_{\delta q}^2 = \sigma_{\delta u}^2 = \frac{2}{\mu^2 N}.$$  

MDP$_{99}$ is then just scaled to the uncertainty in $q$ or $u$:

$$\text{MDP}_{99} = \frac{4.292}{\mu N^{0.72}} = 3.035 \sigma_q = 3.035 \sigma_u.$$  

Due to the symmetry between $q$ and $u$ in the absence of a signal, we can determine the MDP for uniformly weighted polimeter counts when there are several bandpasses with differing modulation channels:

$$\text{MDP}_{99} = 3.035 \sigma_q = \frac{4.292 J}{N \sum \mu_j^2} = \frac{4.292}{(\mu N)^{0.72}}.$$  

Equation (31) indicates that the optimal modulation factor to use for a multiband data set with constant counts per energy bin is the rms of the set of $\mu_j$ values, not the mean.

### 2.7. Using Data Uncertainties, Binned Case

While commonly practiced, it is not always appropriate to set spectral bins based on the observed counts in order to obtain a constant number of counts per spectral bin. One reason to avoid such an approach is that the bins may end up too wide to satisfy the requirement that the modulation factor be constant across the bin. So, now we return to the general case where uncertainties depend on the energy bin $j$. With statistical weighting by $1/\sigma_{\delta q}^2$, the weighted sums over $c_i$, $s_i$, and $c_is_i$ are no longer identically equal to zero as the uncertainties track the counts per bin, which will correlate with $\psi_j$ for $p\mu_j > 0$. However, when considering the special case of weakly modulated signals, there can be simple solutions.

The best values of $\alpha_j$, $q$, and $u$ are found by setting $\nabla \chi^2 = 0$. Using Equation (5) for $\chi^2$ gives

$$\Delta \chi_j^2 \sum_i (1 + \hat{q}_j \mu_j c_i + \hat{u}_j \mu_j s_i)^2 \sigma_{\delta q}^2$$

$$= \sum_i C_i (1 + \hat{q}_j \mu_j c_i + \hat{u}_j \mu_j s_i)^2 \sigma_{\delta q}^2, \quad j = 1 .. J$$  

$$\sum_j \mu_j^2 \alpha_j \sum_i (\hat{q}_j c_i + \hat{u}_j s_i)^2 / \sigma_{\delta u}^2$$

$$= \sum_j \mu_j^2 \alpha_j \sum_i (C_i / \Delta' - \hat{\alpha}_j) c_i / \sigma_{\delta q}^2,$$

$$\sum_j \mu_j^2 \alpha_j \sum_i (\hat{q}_j c_i + \hat{u}_j s_i)^2 / \sigma_{\delta u}^2$$

$$= \sum_j \mu_j^2 \alpha_j \sum_i (C_i / \Delta - \hat{\alpha}_j) s_i / \sigma_{\delta u}^2.$$(33)

(34)

These equations will not be tractable in general, but we can derive approximate solutions for $p\mu_j \ll 1$, and we may drop...
the \(c_{ij}\) terms that are small compared to the \(c_i^2\) and \(s_i^2\) terms:

\[
\hat{c}_j = \frac{1}{\Delta^t} \sum_i C_{ij}/\sigma_{ij}^2 = \frac{C_j}{\Delta^t} 
\]

\[
\hat{q} = \sum_i \mu_j C_j \sum_i C_i (C_i - \hat{C}_j)/\sigma_{ij}^2 = \frac{\sum_j \mu_j C_j^2 \sum_i C_i^2/\sigma_{ij}^2}{\sum_j \mu_j^2 C_j^2 / \sigma_{ij}^2} 
\]

\[
\hat{u} = \frac{\sum_j \mu_j C_j \sum_i C_i (C_i - \hat{C}_j)/\sigma_{ij}^2}{\sum_j \mu_j^2 C_j^2 / \sigma_{ij}^2} 
\]

where the quantity \(\hat{C}_j\) is a variance-weighted average of the counts in channel \(j\). When \(\sigma_{ij}^2 = C_{ij}\), then \(\hat{C}_j = n/(\sum_i C_{ij}^{-1})\).

The single-channel estimates of the \(q\) and \(u\) terms are

\[
\hat{q}_j = \frac{\sum_i c_i (C_i - \hat{C}_j)/\sigma_{ij}^2}{\mu_j C_j / \sum_i c_i^2 / \sigma_{ij}^2} 
\]

\[
\hat{u}_j = \frac{\sum_i s_i (C_i - \hat{C}_j)/\sigma_{ij}^2}{\mu_j C_j / \sum_i s_i^2 / \sigma_{ij}^2} 
\]

Again, we can solve Equations (38) and (39) for the terms that sum over \(C_{ij}/\sigma_{ij}^2\) in the numerators to obtain estimates of \(q\) and \(u\) in terms of single-band estimates

\[
\hat{q} = \sum w_j^q \hat{q}_j 
\]

\[
\hat{u} = \sum w_j^u \hat{u}_j 
\]

where

\[
w_j^q = \frac{\mu_j^2 C_j \sum_i c_i^2 / \sigma_{ij}^2}{\sum_k \mu_k^2 C_k \sum_i c_i^2 / \sigma_{ik}^2} 
\]

\[
w_j^u = \frac{\mu_j^2 C_j \sum_i s_i^2 / \sigma_{ij}^2}{\sum_k \mu_k^2 C_k \sum_i s_i^2 / \sigma_{ik}^2} 
\]

and, as before, \(\sum w_j^q = \sum w_j^u = 1\). These are the weights to be applied to single-band results to combine them optimally.

2.8. Stokes Parameter Uncertainties, Binned Case

Again, uncertainties in the derived values can be estimated under the assumption \(\mu_j \ll 1\) so that cross-terms in the Fisher matrix are negligible. Then,

\[
\sigma_{q_j}^2 = \frac{2}{\Delta^t} \sum_i 1/\sigma_{ij}^2 
\]

\[
\sigma_q^2 = \frac{2}{\Delta^t} \sum_i 1/\sigma_{ij}^2 
\]

\[
\sigma_u^2 = \frac{2}{\Delta^t} \sum_i 1/\sigma_{ij}^2 
\]

For weak modulations, the counts in channel \(j\) are approximately independent of phase bin, so we may approximate \(\sigma_{q_j}^2\) by \(\sigma_{q_j}^2/n\), where \(\sigma_{q_j}^2 \approx C_j\) is the variance in total counts in channel \(j\), \(C_j\). Now \(\bar{C}_j = C_j/n\), giving

\[
\sigma_q^2 = \frac{n}{\sum_j \mu_j^2 C_j^2 / \sigma_{ij}^2} = \frac{2}{\sum_j \mu_j^2 C_j^2 / \sigma_{ij}^2} \approx \frac{2}{\sum_j \mu_j^2 C_j} 
\]

\[
\sigma_u^2 = \frac{n}{\sum_j \mu_j^2 C_j^2 / \sigma_{ij}^2} = \frac{2}{\sum_j \mu_j^2 C_j^2 / \sigma_{ij}^2} \approx \frac{2}{\sum_j \mu_j^2 C_j} 
\]

Following the computation of MDP\(_{99}\) as in Section 2.6, MDP\(_{99}\) in this case is

\[
MDP_{99} = 3.035 \sigma_q = \frac{4.292}{(\sum_j \mu_j^2 C_j^2 / \sigma_{ij}^2)^{1/2}} 
\]

Equation (50) indicates that the modulation factor to use in Equation (2) that is optimal for a multiband data set is the count-weighted mean of the set of \(\mu_j\) values, \(\sum_j \mu_j^2 C_j/N\), not the count-weighted mean (Equation (3)). The choice to use Equation (49) or Equation (50) depends on whether the variances are dominated by the total counts in the energy band; if there is significant non-Poisson variance, then Equation (49) will be somewhat more accurate.

3. Weighting of Unbinned Data

In general, an unbinned likelihood analysis is more appropriate than the use of \(\chi^2\) statistics for fitting X-ray event data owing to the Poisson nature of the counting statistics. Furthermore, binning details often involves user discretion. Though normally requiring numerical methods to determine model parameters, the assumption of weak modulations makes it possible to obtain good estimates.

3.1. Likelihood Formulation

Assume that there are \(N\) events, with energies and instrument phases \((E_i, \psi_i)\). At energy \(E_i\), the modulation factor is \(\mu_i\), the instrument effective area is \(A_i\), and the intrinsic source flux is \(f_i = f(E_i)\) based on the spectral model of the source. The event density in a differential energy-phase element \(dEd\psi\) about \((E, \psi)\) is

\[
\lambda(E, \psi) = [1 + \mu(E) (q \cos 2\psi + u \sin 2\psi)] f_E A_E T dE d\psi 
\]

and the log-likelihood for a Poisson probability distribution of events, \(S = -2 \ln L\), is

\[
S = -2 \sum_i \ln \lambda(E_i, \psi_i) + 2T \int f_E A_E dE
\]

\[
\times \int_0^{2\pi} [1 + \mu(E) (q \cos 2\psi + u \sin 2\psi)] d\psi 
\]

\[
= -2 \sum_i \ln f_i - 2 \sum_i \ln(1 + \mu_i \cos 2\psi_i + u \mu_i \sin 2\psi_i)
\]

\[
+ 4\pi T \int f_E A_E dE + \text{constant.} 
\]

Note that Equation (53) has two terms that depend only on the parameters of \(f_E\), and the remaining interesting term
depends only on the parameters of \( q \) and \( u \). If there are no parameters in common, in the case where \( q \) and \( u \) do not depend on the source flux and we are not fitting (yet) for parameters of the spectral model, then the log-likelihood for the polarization parameters is merely

\[
S(q, u) = -2 \sum_i \ln(1 + q \mu_i \cos 2\psi_i + u \mu_i \sin 2\psi_i). \tag{54}
\]

### 3.2. Estimating \( q \) and \( u \) for the Unbinned Case

We can define \( c_i = \mu_i \cos 2\psi_i \) and \( s_i = \mu_i \sin 2\psi_i \) (unlike the case for binned data). Setting \( \partial S/\partial \hat{q} = 0 \) and \( \partial S/\partial \hat{u} = 0 \) to find the best estimates of \( q \) and \( u \) gives

\[
0 = \sum_i \frac{c_i}{1 + \hat{q}c_i + \hat{u}s_i} = \sum_i w_i c_i, \tag{55}
\]

\[
0 = \sum_i \frac{s_i}{1 + \hat{q}c_i + \hat{u}s_i} = \sum_i w_i s_i, \tag{56}
\]

where \( w_i \equiv (1 + \hat{q}c_i + \hat{u}s_i)^{-1} \). It can be shown that \( \sum w_i = N \). These two equations apply under quite general circumstances but require numerical solution. However, for \( \hat{q} \ll 1 \) and \( \hat{u} \ll 1 \), then \( w_i \approx 1 - \hat{q}c_i - \hat{u}s_i \), so Equations (55) and (56) become

\[
\sum c_i = \hat{q} \sum c_i s_i + \hat{u} \sum c_i^2 \tag{57}
\]

\[
\sum s_i = \hat{u} \sum s_i^2 + \hat{q} \sum c_i s_i, \tag{58}
\]

where all sums are implicitly over \( i \). If the source is polarized, then the phase angles are not random, so the sums over the sine or sine-cosine terms are not identically zero. Fortunately, the pair of linear equations can be solved, giving

\[
\hat{q} = \frac{\sum c_i \sum s_i^2 - \sum s_i \sum c_i s_i}{\sum s_i^2 \sum c_i^2 - (\sum c_i s_i)^2} \tag{59}
\]

\[
\hat{u} = \frac{\sum s_i \sum c_i^2 - \sum c_i \sum s_i c_i}{\sum s_i^2 \sum c_i^2 - (\sum c_i s_i)^2}. \tag{60}
\]

In keeping with the assumed approximations of small \( q \) and \( u \), we expect the sine-cosine terms to be small compared to the sums over \( s_i^2 \) or \( c_i^2 \), giving

\[
\hat{q} \approx \frac{\sum c_i}{\sum s_i^2} \tag{61}
\]

\[
\hat{u} \approx \frac{\sum s_i}{\sum c_i^2}. \tag{62}
\]

It should be simple to compute these sums for all the events in an observation in order to obtain a first estimate of the sizes of the fractional Stokes parameters, \( q \) and \( u \). These estimates provide some accounting for the variation of the modulation factor by using it as a weight on each phase term. Furthermore, the solution indicates a simple way to estimate the dependences of \( q \) and \( u \) on \( E \) by merely creating the sums over suitably small energy ranges so one may obtain a nonparametric estimate of the \((q, u)\) energy dependence, assuming that the values are small but measurable.

The RMF does not enter this analysis, making it relatively easy to examine a data set in a preliminary fashion before attempting detailed fitting that might require an iterative fitting method that accounts for energy redistribution. Technically, assigning a value of \( \mu \) to an event requires knowledge of the event’s true energy, which is uncertain because of the probabilistic mapping between the detector signal and energy. However, as long as the modulation factor varies slowly compared to the detector response function, this approach may still be useful and give a valid impression as to how to devise a physical model.

### 3.3. Uncertainties, Unbinned Case

Finally, the uncertainties on \( q \) and \( u \) are estimated from the second derivatives of Equation (54):

\[
\frac{\partial^2 S}{\partial q^2} = 2 \sum w_i^2 c_i^2 \tag{63}
\]

\[
\frac{\partial^2 S}{\partial u^2} = 2 \sum w_i^2 s_i^2 \tag{64}
\]

\[
\frac{\partial^2 S}{\partial q \partial u} = 2 \sum w_i^2 c_i s_i, \tag{65}
\]

and using \( w_i \approx 1 \) and \( \sum w_i^2 c_i s_i \approx 0 \), one obtains simple estimates for the Stokes uncertainties,

\[
\sigma_q^2 \approx \frac{2}{\sum c_i^2} \approx \frac{1}{\sum c_i^2} \tag{66}
\]

\[
\sigma_u^2 \approx \frac{2}{\sum s_i^2} \approx \frac{1}{\sum s_i^2}. \tag{67}
\]

Remember that the \( c_i \) and \( s_i \) values have the modulation factor included in this situation, so the denominators are not \( N/2 \). To estimate the denominators, we note that

\[
\sum c_i^2 + \sum s_i^2 = \sum \mu_i^2 = N \mu^2, \tag{68}
\]
where $\bar{\mu}^2$ is the count-weighted average of $\mu^2$. Furthermore, $c_i$ and $s_i$ should be similarly distributed when $p \ll 1$, giving

$$\sum c_i^2 \approx \sum s_i^2 \approx \frac{N}{2} \bar{\mu}^2.$$  \hspace{1cm} (69)

Finally, as in Section 2.7, we make the connection to MDP99:

$$\text{MDP}_{99} = 3.035\sigma_q \approx \frac{3.035}{\left( \sum c_i^2 \right)^{1/2}} \approx \frac{4.292}{\left( \mu^2 N \right)^{1/2}}.$$ \hspace{1cm} (70)

As in the binned case, the modulation factor to use in Equation (2) that is optimal for the unbinned data set is the count-weighted mean of the set of $\mu_i$ values, not the count-weighted mean (Equation (3)).

### 4. Example Application and Summary

Armed with an appropriate weighting scheme, it is useful to determine how much the MDP may improve using it. To determine the effect of optimal binning, the results of an observation can be computed using assumed functions for the spectral model, effective area, and $\mu(E)$. It is important to get the dependences of these functions on energy approximately correct; the normalizations are not relevant when just selecting a total number of events for each simulated observation.

Figure 1 shows assumed models of the effective area, normalized to 1 at the maximum value, and of $\mu(E)$, both for an instrument like IXPE (O’Dell et al. 2018). The standard modulation curve for IXPE uses a moment method with ellipticity exclusion (Bellazzini et al. 2003); the data are from A. Di Marco et al. (2021, in preparation; see also Weisskopf et al. 2016). The effective area is approximate (as in Elsner et al. 2012)—only the shape is important for this analysis. For a given total number of counts, $N$, two versions of MDP99 were computed using the area and modulation curves for each of two assumed spectral models. The spectral model is a power law with spectral index $\Gamma = 0$ or 2, where $f_E \propto E^{-\Gamma}$. For case 1, MDP99 is determined using a count-weighted modulation factor given by Equation (3). For case 2, MDP99 is given by Equation (50). Thus,

$$\text{MDP}_{99} = \frac{4.292 \left[ \int A(E) E^{-\Gamma} dE \right]^{1/2}}{\left[ N \int \mu(E)^X A(E) E^{-\Gamma} dE \right]^{1/2}},$$ \hspace{1cm} (71)

where $X = 1$ for case 1 and $X = 2$ for case 2 and the integral in the numerator normalizes the spectrum so that there are $N$ expected counts. For this exercise, $N = 10^5$ after the ellipticity exclusion, achievable with IXPE for sources with 2–8 keV fluxes of $\sim 10^{-11}$ erg cm$^{-2}$ s$^{-1}$ in about 10 days (O’Dell et al. 2018). For case 1, MDP99 came out to 3.61% and 4.52% for $\Gamma = 0$ and 2, respectively. For the optimum weighting of case 2, MDP99 values were 3.41% and 4.21%. Thus, the improvement to the MDPs were 6.0% and 7.2% for the two different spectral slopes. It is important to note that the fractional improvement is independent of $N$.

One can obtain further improvement in the MDP by increasing the modulation factor. Peirson et al. (2021) show that an approach to measuring IXPE tracks using convolutional neural networks can increase $\mu(E)$, as shown in Figure 1, and thereby improve the subsequent estimate of the MDP. The actual improvement in MDP in this approach depends somewhat on a weighting parameter, $\lambda$ (see Peirson et al. 2021). Using Equation (71) with the $\lambda = 2$ modulation factor curve from Peirson et al. (2021) and assuming $N = 10^5$ can yield further reductions of MDP99 by 15% and 8% for the two spectral shapes to 2.97% and 3.89%, respectively. For $\lambda = 1$, the improvement is less than a few percent. Peirson et al. (2021) point out that the track uncertainty weight method has no ellipticity exclusion, thus starting with more events than with the moments method, but does reduce the effective number of events by 15%–20%.

Clearly, there is an advantage to optimal weighting using the count-weighted rms of the modulation factor. The actual improvement will depend on the details of the shape of the effective area and modulation factor curves, as well as the actual shape of the spectrum. However, this weighting can be generally applied for the best results.

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**References**

Beilicke, M., Kislat, F., Zajczyk, A., et al. 2014, JAI, 3, 1440008
Bellazzini, R., Angelini, F., Baldini, L., et al. 2003, Proc. SPIE, 4843, 383
Elsner, R. F., O’Dell, S. L., & Weisskopf, M. C. 2012, Proc. SPIE, 8443, 84434N
Kislat, F., Abarr, Q., Behegthipour, B., et al. 2018, JATIS, 4, 1
Kislat, F., Clark, B., Beilicke, M., & Krawczynski, H. 2015, ApJ, 68, 45
O’Dell, S. L., Baldini, L., Bellazzini, R., et al. 2018, Proc. SPIE, 10699, 106991X
Peirson, A., Romani, R., Marshall, H., Steiner, J., & Baldini, L. 2021, NIMPA, 986, 164740
Weisskopf, M. C., Elsner, R. F., & O’Dell, S. L. 2010, Proc. SPIE, 7732, 77320E
Weisskopf, M. C., Ramsey, B., O’Dell, S., et al. 2016, Proc. SPIE, 9905, 990517