ROTATIONAL SYMMETRY OF SELF-SIMILAR SOLUTIONS TO THE RICCI FLOW

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Abstract. Let \((M, g)\) be a three-dimensional steady gradient Ricci soliton which is non-flat and \(\kappa\)-noncollapsed. We prove that \((M, g)\) is isometric to the Bryant soliton up to scaling. This solves a problem mentioned in Perelman’s first paper \([20]\).

1. Introduction

Self-similar solutions play a central role in the study of the Ricci flow, and have been studied extensively in connection with singularity formation; see e.g. the work of R. Hamilton \([12]\) and G. Perelman \([20]\), \([21]\), \([22]\). There are three basic types of self-similar solutions, which are referred to as shrinking solitons; steady solitons; and expanding solitons. A steady Ricci soliton \((M, g)\) is characterized by the fact that \(2\, \text{Ric} = \mathcal{L}_X(g)\) for some vector field \(X\). If the vector field \(X\) is the gradient of a function, we say that \((M, g)\) is a steady gradient Ricci soliton.

The simplest example of a steady Ricci soliton is the cigar soliton in dimension 2, which was found by Hamilton (cf. \([12]\)). R. Bryant \([3]\) has discovered a steady Ricci soliton in dimension 3, which is rotationally symmetric. Moreover, Bryant showed that there are no other complete steady Ricci solitons in dimension 3 which are rotationally symmetric. While additional examples are known in higher dimensions (see e.g. \([16]\)), the Bryant soliton is so far the only known example of a non-flat steady Ricci soliton in dimension 3. It is an interesting question whether any three-dimensional steady Ricci soliton is necessarily rotationally symmetric. Perelman mentions the uniqueness problem for steady Ricci solitons in his first paper (see \([20]\), page 32, lines 8-9), without however indicating a strategy for a possible proof.

In this paper, we prove the uniqueness of the Bryant soliton under a noncollapsing assumption, as proposed by Perelman:

**Theorem 1.1.** Let \((M, g)\) be a three-dimensional complete steady gradient Ricci soliton which is non-flat and \(\kappa\)-noncollapsed. Then \((M, g)\) is rotationally symmetric, and is therefore isometric to the Bryant soliton up to scaling.

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We note that several authors have obtained uniqueness results for the Bryant soliton and its higher dimensional counterparts under various additional assumptions. We refer to [4], [5], [6], and [8] for details.

We now outline the main steps involved in the proof of Theorem 1.1. Let \((M,g)\) be a three-dimensional complete steady gradient Ricci soliton which is non-flat and \(\kappa\)-noncollapsed. We may write \(\text{Ric} = D^2 f\) for some real-valued function \(f\). For abbreviation, we put \(X = \nabla f\). Moreover, we denote by \(\Phi_t\) the one-parameter group of diffeomorphisms generated by the vector field \(-X\). We may assume without loss of generality that \(R + |\nabla f|^2 = 1\).

In Section 2, we analyze the asymptotic geometry of \((M,g)\). The local version of the Hamilton-Ivey pinching estimate established by B.L. Chen [7] implies that \((M,g)\) has positive sectional curvature. It then follows from work of Perelman [20] that the flow \((M,g(t))\) is asymptotic to a family of shrinking cylinders near infinity. This fact plays a fundamental role in our analysis. We next show that the restriction of the scalar curvature to the level surface \(\{f = r\}\) satisfies \(R = \frac{1}{r} + O(r^{-\frac{5}{4}})\). As a consequence, the intrinsic Gaussian curvature of the level surface \(\{f = r\}\) equals \(\frac{1}{2r} + O(r^{-\frac{5}{4}})\). This can be viewed as a refined roundness estimate for the level surface \(\{f = r\}\).

In Section 3, we construct a collection of approximate Killing vector fields near infinity. More precisely, we construct three vector fields \(U_1, U_2, U_3\) such that

\[
\sum_{a=1}^{3} U_a \otimes U_a = r \left( e_1 \otimes e_1 + e_2 \otimes e_2 + O(r^{-\frac{1}{2}}) \right),
\]

where \(\{e_1, e_2\}\) is a local orthonormal frame on the level set \(\{f = r\}\).

In Section 4, we consider a vector field \(W\) which satisfies the elliptic equation \(\Delta W + D_X W = 0\). We then consider the Lie derivative \(h = \mathcal{L}_W (g)\). This tensor turns out to satisfy the equation

\[
(1) \quad \Delta_L h + \mathcal{L}_X (h) = 0.
\]

Here, \(\Delta_L\) denotes the Lichnerowicz Laplacian; that is,

\[
\Delta_L h_{ik} = \Delta h_{ik} + 2 R_{ijkl} h^{jl} - \text{Ric}^j_k h_{kl} - \text{Ric}^j_k h_{il}.
\]

In Section 5, we assume that a vector field \(Q\) satisfying \(|Q| \leq O(r^{-\frac{1}{2} - 2\varepsilon})\) is given. We then construct a vector field \(V\) such that \(\Delta V + D_X V = Q\) and \(|V| \leq O(r^{-\frac{1}{2} - \varepsilon})\). In order to construct the vector field \(V\), we solve the Dirichlet problem on a sequence of domains which exhaust \(M\). In order to be able to pass to the limit, we need uniform estimates for solutions of the equation \(\Delta V + D_X V = Q\). These estimates are established using a delicate blow-down analysis; see Proposition 5.4 below.

In Section 6, we consider a symmetric \((0,2)\)-tensor \(h\) which solves the equation (1) and satisfies \(|h| \leq O(r^{-\varepsilon})\) at infinity. Note that such a tensor \(h\)
need not vanish identically. Indeed, the Ricci tensor of \((M, g)\) is a non-trivial solution of the equation (1), which falls off like \(r^{-1}\) at infinity. However, we are able to show that any solution of (1) with \(|h| \leq O(r^{-\varepsilon})\) is of the form \(h = \lambda \text{Ric}\) for some constant \(\lambda \in \mathbb{R}\); see Theorem 6.3 below. The proof of Theorem 6.3 again relies on a parabolic blow-down argument. We also use an inequality due to G. Anderson and B. Chow [1] for solutions of the parabolic Lichnerowicz equation. Related ideas were used in earlier work of M. Gursky [10] and R. Hamilton [11].

Finally, in Section 7 we establish a crucial symmetry principle. To explain this, suppose that \(U\) is a vector field on \((M, g)\) such that \(|\mathcal{L}_U(g)| \leq O(r^{-2\varepsilon})\) and \(|\Delta U + D_X U| \leq O(r^{-1/2} - 2\varepsilon)\) for some small constant \(\varepsilon > 0\). Using the results in Section 5, we can find a vector field \(V\) such that \(\Delta V + D_X V = \Delta U + D_X U\) and \(|V| \leq O(r^{-1/2} - \varepsilon)\). Therefore, the vector field \(W = U - V\) satisfies \(\Delta W + D_X W = 0\). Consequently, the Lie derivative \(h = \mathcal{L}_W(g)\) is a solution of the equation (1). Moreover, we show that \(|h| \leq O(r^{-\varepsilon})\) at infinity. Thus, \(h = \lambda \text{Ric}\) for some constant \(\lambda \in \mathbb{R}\). From this, we deduce that the vector field \(\hat{U} := W - \frac{1}{2} \lambda X\) is a Killing vector field. Moreover, the Killing vector field \(\hat{U}\) agrees with the original vector field \(U\) up to terms of order \(O(r^{1/2} - \varepsilon})\).

Applying this symmetry principle to the approximate Killing vector fields \(U_1, U_2, U_3\) constructed in Section 8 we obtain three exact Killing vector fields \(\hat{U}_1, \hat{U}_2, \hat{U}_3\) on \((M, g)\) with the property that \((\hat{U}_a, X) = 0\) and

\[
\sum_{a=1}^{3} \hat{U}_a \otimes \hat{U}_a = r (e_1 \otimes e_1 + e_2 \otimes e_2 + O(r^{-\varepsilon})),
\]

where \(\{e_1, e_2\}\) is a local orthonormal frame on the level surface \(\{f = r\}\). In particular, at each point sufficiently far out at infinity, the span of the vector fields \(\hat{U}_1, \hat{U}_2, \hat{U}_3\) is two-dimensional.

Finally, let us mention some related results. Our method of proof is inspired in part by the beautiful work of L. Simon and B. Solomon on the uniqueness of minimal hypersurfaces in \(\mathbb{R}^{n+1}\) which are asymptotic to a given cone at infinity (cf. [23], [24]). X.J. Wang [27] has obtained a uniqueness theorem for convex translating solutions to the mean curvature flow in \(\mathbb{R}^3\). The argument in [27] is quite different from ours and relies in a crucial way on a classical theorem of Bernstein (cf. [14]). Finally, the uniqueness problem for the Bryant soliton shares some common features with the black hole uniqueness theorems in general relativity (see e.g. [13], [15]).

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2. The asymptotic geometry of \((M,g)\)

Throughout this paper, we assume that \((M,g)\) is a three-dimensional complete steady gradient Ricci soliton which is \(\kappa\)-noncollapsed and non-flat. It follows from Theorem 1.3 in [28] that \((M,g)\) has positive scalar curvature (see also [5], Proposition 2.2). It is well known that the sum \(R + |\nabla f|^2\) is constant. By scaling, we may assume that \(R + |\nabla f|^2 = 1\). Since \(R \geq 0\), it follows that \(|\nabla f|^2 \leq 1\). Hence, if we denote by \(\Phi_t\) the flow generated by the vector field \(-X\), then \(\Phi_t\) is defined for all \(t \in \mathbb{R}\), and the metrics \(\Phi^*_t(g)\) evolve by the Ricci flow.

**Proposition 2.1.** The manifold \((M,g)\) has bounded curvature, and the sectional curvature is strictly positive.

**Proof.** It follows from a result of Chen that \((M,g)\) has nonnegative sectional curvature (see [7], Corollary 2.4). Since \(R + |\nabla f|^2 \leq 1\), we conclude that \((M,g)\) has bounded curvature. It remains to show that \((M,g)\) has positive sectional curvature. Suppose this is false. Then the manifold \((M,g)\) locally splits as a product, and the universal cover of \((M,g)\) is isometric to the cigar soliton crossed with a line. This contradicts our assumption that \((M,g)\) is \(\kappa\)-noncollapsed.

We next analyze the asymptotic geometry of \((M,g)\) near infinity. We will frequently use the identity

\[
-\langle X, \nabla R \rangle = \Delta R + 2|Ric|^2.
\]

This identity is a consequence of the evolution equation for the scalar curvature under the Ricci flow (cf. [2], Section 2.4).

The following result is a direct consequence of Perelman’s compactness theorem for ancient \(\kappa\)-solutions:

**Proposition 2.2** (G. Perelman [20], [21]). Let \(p_m\) be a sequence of points going to infinity. Then \(|\langle X, \nabla R \rangle| \leq O(1) R^2\) at the point \(p_m\). Moreover, if \(d(p_0,p_m)^2 R(p_m) \to \infty\), then we have \(|\nabla R| \leq o(1) R^2\) and \(|\langle X, \nabla R \rangle + R^2| \leq o(1) R^2\) at the point \(p_m\).

**Proof.** It follows from results in Section 1.5 of [21] that \(|\Delta R| \leq O(1) R^2\). Using [2], we conclude that \(|\langle X, \nabla R \rangle| \leq O(1) R^2\). This proves the first statement.

We now describe the proof of the second statement. To that end, we assume that \(d(p_0,p_m) R(p_m)^2 \to \infty\). Let us consider the rescaled flows

\[
\hat{g}^{(m)}(t) = r_m^{-1} \Phi^{*}_{r_m t}(g),
\]

where \(r_m = R(p_m)^{-1}\). It follows from Perelman’s compactness theorem for ancient \(\kappa\)-solutions that the flows \((M, \hat{g}^{(m)}(t), p_m)\), \(t \in (-\infty, 0]\), converge in the Cheeger-Gromov sense to a non-flat ancient \(\kappa\)-solution \((\overline{M}, \overline{g}(t))\), \(t \in (-\infty, 0]\) (see [20], Theorem 11.7). By Theorem 5.35 in [19], the manifold \((\overline{M}, \overline{g}(0))\) splits off a line. By the strict maximum principle, the limit flow...
(\(\overline{M}, \overline{g}(t)\)), \(t \in (-\infty, 0]\), is isometric to a product of a two-dimensional ancient \(\kappa\)-solution with a line. By Theorem 11.3 in [20], the universal cover of \((\overline{M}, \overline{g}(t))\) is a round cylinder for each \(t \in (-\infty, 0]\). From this, we deduce that \(|\nabla R| \leq o(1) R^\frac{3}{2}\), \(|\Delta R| \leq o(1) R^2\), and \(2|Ric|^2 = (1 + o(1)) R^2\) at the point \(p_m\). Using (2), we conclude that \(-\langle X, \nabla R \rangle = \Delta R + 2|Ric|^2 = (1 + o(1)) R^2\).

**Corollary 2.3.** The scalar curvature converges to 0 at infinity.

**Proof.** Suppose this is false. Then we can find a sequence of points \(p_m\) going to infinity such that \(\liminf_{m \to \infty} R(p_m) > 0\). Using Proposition 2.2 we obtain \(|\langle X, \nabla R \rangle + R^2| \leq o(1)\) and \(|\nabla R| \leq o(1)\) at the point \(p_m\). Since \(|X| \leq 1\), it follows that \(|\langle X, \nabla R \rangle| \leq o(1)\) at the point \(p_m\). Putting these facts together, we conclude that \(R(p_m) = o(1)\), contrary to our assumption.

By Corollary [2.3] we can find a point \(p_0 \in M\) such that \(R(p_0) = \sup_M R\). At the point \(p_0\), we have

\[
0 = \partial_i R = \partial_i^2 f \partial_j f.
\]

By Proposition [2.1] the Hessian of \(f\) is positive definite at each point in \(M\). Consequently, the point \(p_0\) is a critical point of \(f\). Moreover, we can find positive constants \(c_1\) and \(c_2\) such that

\[
c_1 d(p_0, p) \leq f(p) \leq c_2 d(p_0, p)
\]

outside a compact set (see also [9], Proposition 2.3). Without loss of generality, we may assume that \(\inf_M f \geq 1\).

**Proposition 2.4** (H. Guo [9]). The scalar curvature satisfies \(f R = 1 + o(1)\) as \(p \to \infty\).

**Proof.** Using Corollary 2.3 and the identity \(R + |\nabla f|^2 = 1\), we obtain \(|\nabla f|^2 \to 1\) as \(p \to \infty\). In particular, we have \(|\nabla f|^2 \geq \frac{1}{2}\) outside a compact set. Using Proposition 2.2 we obtain \(-\langle X, \nabla R \rangle \leq C R^2\), hence

\[
\langle X, \nabla \left(\frac{1}{R} - 2C f\right)\rangle \leq C (1 - 2|Ric|^2) \leq 0
\]

outside a compact set. Integrating this inequality along the integral curves of \(X\) gives

\[
\sup_M \left(\frac{1}{R} - 2C f\right) < \infty.
\]

Consequently, \(\inf_M f R > 0\). In particular, we have \(d(p_0, p)^2 R(p) \to \infty\) at infinity. Using Proposition 2.2 again, we conclude that

\[
|\langle X, \nabla R \rangle + R^2| \leq o(1) R^2
\]

near infinity. Since \(1 - |\nabla f|^2 = R \to 0\) at infinity, we conclude that

\[
\langle X, \nabla \left(\frac{1}{R} - f\right)\rangle = 1 - |\nabla f|^2 - \frac{1}{R^2} (\langle X, \nabla R \rangle + R^2) = o(1).
\]
Integrating this inequality along the integral curves of $X$, we obtain
\[
\frac{1}{R} = (1 + o(1)) f,
\]
as claimed.

Using work of Perelman \[20\], we can determine the asymptotic geometry of $(M, g)$ near infinity:

**Proposition 2.5 (cf. \[20\]).** Let $p_m$ be a sequence of marked points going to infinity. Consider the rescaled metrics
\[
\hat{g}^{(m)}(t) = r_m^{-1} \Phi_{r_m}^*(g),
\]
where $r_m = f(p_m)$. As $m \to \infty$, the flows $(M, \hat{g}^{(m)}(t), p_m)$ converge in the Cheeger-Gromov sense to a family of shrinking cylinders $(S^2 \times \mathbb{R}, \overline{g}(t))$, $t \in (0, 1)$. The metric $\overline{g}(t)$ is given by
\[
(3) \quad \overline{g}(t) = (2 - 2t) g_{S^2} + dz \otimes dz,
\]
where $g_{S^2}$ denotes the standard metric on $S^2$ with constant Gaussian curvature $1$. Furthermore, the rescaled vector fields $r_m^{-1/2} X$ converge in $C^\infty_{loc}$ to the axial vector field $\frac{\partial}{\partial z}$ on $S^2 \times \mathbb{R}$.

**Proof.** It follows from Proposition 2.4 that the flows $(M, \hat{g}^{(m)}(t), p_m)$, $t \in (-\infty, 1)$, converge in the Cheeger-Gromov sense to a non-flat ancient $\kappa$-solution $(\overline{M}, \overline{g}(t))$, $t \in (-\infty, 1)$. By Theorem 5.35 in \[19\], the limit flow $(\overline{M}, \overline{g}(t))$ is isometric to a product of a two-dimensional ancient $\kappa$-solution with a line (see \[20\], Theorem 11.7). Note that $M$ is homeomorphic to $\mathbb{R}^3$ and in particular does not contain an embedded $\mathbb{R}P^2$. Consequently, $\overline{M}$ cannot contain an embedded $\mathbb{R}P^2$. By Theorem 11.3 in \[20\], we conclude that $(\overline{M}, \overline{g}(t))$ is a family of round cylinders, i.e. $\overline{M} = S^2 \times \mathbb{R}$ and $\overline{g}(t) = (2 - 2t) g_{S^2} + dz \otimes dz$ for each $t \in (-\infty, 1)$.

It remains to analyze the limit of the rescaled vector fields $\hat{X}^{(m)} = r_m^{1/2} X$. Using the identity $1 - |X| = O(r^{-1})$, we obtain
\[
\limsup_{m \to \infty} \sup_{r_m - \delta \leq f \leq r_m + \delta} \left| 1 - |\hat{X}^{(m)}|_{\hat{g}^{(m)}(0)} \right| = 0
\]
for any given $\delta \in (0, 1)$. Moreover, we have $|D^l X| \leq C |D^{l-1} \text{Ric}| = O(r^{\frac{4l+1}{2l}})$ for all $l \geq 1$. This implies
\[
\limsup_{m \to \infty} \sup_{r_m - \delta \leq f \leq r_m + \delta} |D^l_{\hat{g}^{(m)}(0)} \hat{X}^{(m)}|_{\hat{g}^{(m)}(0)} = 0
\]
for any given $\delta \in (0, 1)$ and $l \geq 1$. Hence, after passing to a subsequence, the vector fields $\hat{X}^{(m)}$ converge in $C^\infty_{loc}$ to a vector field $\overline{X}$ on the limit manifold $(S^2 \times \mathbb{R}, \overline{g}(0))$. The limiting vector field $\overline{X}$ is parallel with respect to the metric $\overline{g}(0)$, and we have $|\overline{X}|_{\overline{g}(0)} = 1$. Thus, $\overline{X}$ can be identified with the
axial vector field $\frac{\partial}{\partial z}$ on $S^2 \times \mathbb{R}$.

In the remainder of this section, we establish a roundness estimate for the level surfaces $\{f = r\}$. The proof of this estimate requires several lemmata.

**Lemma 2.6.** On the level surface $\{f = r\}$, we have

$$2 \text{Ric}(\nabla f, \nabla f) = -\langle X, \nabla R \rangle = O(r^{-2}).$$

**Proof.** The identity (2) implies that $2 \text{Ric}(\nabla f, \nabla f) = -\langle X, \nabla R \rangle = \Delta R + 2 |\text{Ric}|^2 = O(r^{-2}).$

**Lemma 2.7.** The mean curvature of the level surface $\{f = r\}$ equals $1 + o(1)\frac{r}{r}$. 

**Proof.** The mean curvature of the level surface $\{f = r\}$ is given by

$$H = \frac{1}{|\nabla f|} R - \frac{1}{|\nabla f|^3} \text{Ric}(\nabla f, \nabla f).$$

Hence, the assertion follows from Proposition 2.4 and Lemma 2.6.

**Lemma 2.8.** The tensor $T = 2 \text{Ric} - R g + R df \otimes df$ satisfies $|T| \leq O(r^{-\frac{3}{2}})$ and $|DT| \leq O(r^{-2})$.

**Proof.** In dimension 3, the Riemann curvature tensor can be written in the form

$$R_{ijkl} = \text{Ric}_{ik} g_{jl} - \text{Ric}_{il} g_{jk} - \text{Ric}_{jk} g_{il} + \text{Ric}_{jl} g_{ik} - \frac{1}{2} R (g_{ik} g_{jl} - g_{il} g_{jk}).$$

This implies

$$D_i \text{Ric}_{jk} - D_j \text{Ric}_{ik} = R_{ijkl} D^k f$$

$$= \text{Ric}_{ik} D_j f - \text{Ric}_{jk} D_i f$$

$$- \frac{1}{2} (D_j R + R D_j f) g_{ik} + \frac{1}{2} (D_i R + R D_i f) g_{jk},$$

hence

$$2 (D_i \text{Ric}_{jk} - D_j \text{Ric}_{ik}) D^j f$$

$$= T_{ik} |\nabla f|^2 - \langle \nabla R, \nabla f \rangle g_{ik} + R^2 D_i f D_k f$$

$$+ D_i R D_k f + D_k R D_i f. \tag{4}$$

By Shi’s estimate, the covariant derivatives of the curvature tensor are bounded by $O(r^{-\frac{3}{2}})$. Consequently, the identity (4) implies that $|T| \leq O(r^{-\frac{3}{2}})$. Moreover, if we differentiate (4), we obtain $|DT| \leq O(r^{-2})$. 

Lemma 2.9. We have
\[ |\langle X, \nabla R \rangle + \Delta_\Sigma R + R^2| \leq O(r^{-\frac{5}{2}}), \]
where \( \Delta_\Sigma \) denotes the Laplacian on the level surface \( \{ f = r \} \).

**Proof.** Differentiating the identity (2), we obtain
\[ -(D^2 R)(X, X) - \langle DX X, \nabla R \rangle = \langle X, \nabla (\Delta R + 2 |\text{Ric}|^2) \rangle. \]
Since \( \nabla R = -2 DX X \), it follows that
\[ -(D^2 R)(X, X) = -\frac{1}{2} |\nabla R|^2 + \langle X, \nabla (\Delta R + 2 |\text{Ric}|^2) \rangle. \]
Using Shi’s estimates, we obtain \( |\nabla R|^2 \leq O(r^{-3}) \) and \( |\nabla (\Delta R + 2 |\text{Ric}|^2)| \leq O(r^{-\frac{5}{2}}) \). Consequently, we have
\[ |(D^2 R)(X, X)| \leq O(r^{-\frac{5}{2}}). \]
Moreover, it follows from Lemma 2.6 and Lemma 2.7 that
\[ |H \langle X, \nabla R \rangle| \leq O(r^{-3}). \]
Combining (5) and (6) gives
\[ |\Delta R - \Delta_\Sigma R| \leq O(r^{-\frac{5}{2}}). \]
Combining this inequality with (2), we obtain
\[ |\Delta_\Sigma R + \langle X, \nabla R \rangle + 2 |\text{Ric}|^2| \leq O(r^{-\frac{5}{2}}). \]
On the other hand, it follows from Lemma 2.8 that
\[ 2 |\text{Ric}| = |R (g - df \otimes df)| + O(r^{-\frac{3}{2}}) = \sqrt{2} R + O(r^{-\frac{3}{2}}). \]
Putting these facts together, the assertion follows.

We next establish a Poincaré-type inequality for the restriction of the scalar curvature to a level surface \( \{ f = r \} \). Our argument uses the Kazdan-Warner identity (cf. [17]), and is inspired in part by work of M. Struwe on the Calabi flow on the two-sphere (cf. [25], p. 263). In the sequel, we denote by \( \mu(r) \) the mean value of the scalar curvature over the level surface \( \{ f = r \} \), so that
\[ \int_{\{ f = r \}} (R - \mu(r)) = 0. \]
Note that \( \mu(r) = \frac{1+o(1)}{r} \) by Proposition 2.4.

Lemma 2.10. We have
\[ \int_{\{ f = r \}} |\nabla_\Sigma R|^2 \geq \frac{2}{r} \left( \int_{\{ f = r \}} (R - \mu(r))^2 \right) - O(r^{-4}) \]
if \( r \) is sufficiently large.
Proof. Let us fix $r$ sufficiently large. Let $0 = \nu_0 < \nu_1 \leq \nu_2 \leq \nu_3 \leq \ldots$ denote the eigenvalues of the Laplace operator on the level surface $\{ f = r \}$, and let $\psi_0, \psi_1, \psi_2, \psi_3, \ldots$ denote the associated eigenfunctions. We assume that the eigenfunctions are normalized so that $\int_{\{f=r\}} \psi_j^2 = 1$ for each $j$.

When $r$ is large, the surface $\{ f = r \}$ equipped with the rescaled metric $\frac{1}{r^2} g$ is $C^\infty$ close to the standard two-sphere with constant Gaussian curvature 1. Consequently, $\nu_1 = \frac{1+o(1)}{r}$, $\nu_2 = \frac{1+o(1)}{r}$, $\nu_3 = \frac{1+o(1)}{r}$, and $\nu_4 = \frac{3+o(1)}{r}$.

Let $K$ denote the intrinsic Gaussian curvature of the level surface $\{ f = r \}$. Using the Gauss equations, we obtain

$$R - \frac{2}{|\nabla f|^2} \text{Ric}(\nabla f, \nabla f) = 2 R(e_1, e_2, e_1, e_2) = 2K + O(r^{-2}).$$

Using Lemma 2.6, we conclude that $|2K - R| \leq O(r^{-2})$, hence

$$\left( \int_{\{f=r\}} (2K - R)^2 \right)^{\frac{1}{2}} \leq O(r^{-\frac{3}{2}}).$$

On the other hand, it follows from the Kazdan-Warner identity (see [17], Theorem 8.8) that

$$\sum_{j=1}^{3} \left| \int_{\{f=r\}} (2K - \mu(r)) \psi_j \right| \leq o(1) \left( \int_{\{f=r\}} (2K - \mu(r))^2 \right)^{\frac{1}{2}}.$$

Putting these facts together, we obtain

$$\sum_{j=1}^{3} \left| \int_{\{f=r\}} (R - \mu(r)) \psi_j \right| \leq o(1) \left( \int_{\{f=r\}} (R - \mu(r))^2 \right)^{\frac{1}{2}} + O(r^{-\frac{3}{2}}).$$

Thus, we conclude that

$$\int_{\{f=r\}} |\nabla^\Sigma R|^2 - \nu_4 \int_{\{f=r\}} (R - \mu(r))^2$$

$$= \sum_{j=1}^{\infty} (\nu_j - \nu_4) \left( \int_{\{f=r\}} (R - \mu(r)) \psi_j \right)^2$$

$$\geq -\frac{3}{r} \sum_{j=1}^{3} \left( \int_{\{f=r\}} (R - \mu(r)) \psi_j \right)^2$$

$$\geq -o(r^{-1}) \left( \int_{\{f=r\}} (R - \mu(r))^2 \right) - O(r^{-4}).$$

Since $\nu_4 = \frac{3+o(1)}{r}$, the assertion follows.

We now prove an important roundness estimate.

Proposition 2.11. We have

$$\int_{\{f=r\}} (R - \mu(r))^2 \leq O(r^{-2})$$
if \( r \) is sufficiently large.

**Proof.** By definition of \( \mu(r) \), we have \( \int_{\{f=r\}} (R - \mu(r)) = 0 \). This implies

\[
\frac{d}{dr} \left( \int_{\{f=r\}} (R - \mu(r))^2 \right) = 2 \int_{\{f=r\}} (R - \mu(r)) \left( \frac{\langle X, \nabla R \rangle}{|X|^2} - \mu'(r) \right) + \int_{\{f=r\}} \frac{H}{|X|} (R - \mu(r))^2
\]

\[
= 2 \int_{\{f=r\}} (R - \mu(r)) \left( \frac{\langle X, \nabla R \rangle}{|X|^2} + \mu(r)^2 \right) + \int_{\{f=r\}} \frac{H}{|X|} (R - \mu(r))^2
\]

\[
= 2 \int_{\{f=r\}} |\nabla^\Sigma R|^2 - \int_{\{f=r\}} \left( 2R + 2\mu(r) - \frac{H}{|X|} \right) (R - \mu(r))^2
\]

\[
+ 2 \int_{\{f=r\}} (R - \mu(r)) \left( \frac{\langle X, \nabla R \rangle}{|X|^2} + \Delta \Sigma R + R^2 \right).
\]

It follows from Lemma 2.10 that

\[
\int_{\{f=r\}} |\nabla^\Sigma R|^2 \geq \frac{2}{r} \left( \int_{\{f=r\}} (R - \mu(r))^2 \right) - O(r^{-4}).
\]

Moreover, we have \( 2R + 2\mu(r) - \frac{H}{|X|} = \frac{3+o(1)}{r} \). Finally, we have

\[
\left| \frac{\langle X, \nabla R \rangle}{|X|^2} + \Delta \Sigma R + R^2 \right| \leq O(r^{-\frac{3}{2}})
\]

by Lemma 2.9. Putting these facts together, we obtain

\[
\frac{d}{dr} \left( \int_{\{f=r\}} (R - \mu(r))^2 \right) \geq \frac{1 - o(1)}{r} \int_{\{f=r\}} (R - \mu(r))^2
\]

\[- O(r^{-\frac{3}{2}}) \int_{\{f=r\}} |R - \mu(r)|
\]

\[- O(r^{-4}).
\]

Using Young’s inequality, we conclude that

\[
\frac{d}{dr} \left( \int_{\{f=r\}} (R - \mu(r))^2 \right) \geq -O(r^{-4}) \text{vol}(\{f=r\}) - O(r^{-4})
\]

\[\geq -O(r^{-3}).\]

Clearly,

\[
\int_{\{f=r\}} (R - \mu(r))^2 \to 0
\]

as \( r \to \infty \). Putting these facts together, we obtain

\[
\int_{\{f=r\}} (R - \mu(r))^2 \leq O(r^{-2}),
\]

as claimed.
Corollary 2.12. We have
\[
\sup_{\{f = r\}} |R - \mu| \leq O(r^{-\frac{5}{4}}),
\sup_{\{f = r\}} |\nabla^\Sigma R| \leq O(r^{-\frac{7}{4}}),
\sup_{\{f = r\}} |\Delta^\Sigma R| \leq O(r^{-\frac{9}{4}}).
\]

Proof. By Proposition 2.11 we have
\[
\int_{\{f = r\}} (R - \mu(r))^2 \leq O(r^{-2}).
\]
Moreover, it follows from Shi’s estimates that
\[
\sup_{\{f = r\}} |D^\Sigma_2 R| \leq O(r^{-\frac{5}{4} + 2}).
\]
Hence, the assertion follows from standard interpolation inequalities (see e.g. [11], Corollary 12.7).

With the aid of Corollary 2.12 we can improve Proposition 2.14 as follows:

Proposition 2.13. We have $|\nabla R| \leq O(r^{-\frac{3}{4}})$ and $f R = 1 + O(r^{-\frac{1}{4}})$.

Proof. Using the estimates $|\nabla^\Sigma R| \leq O(r^{-\frac{5}{4}})$ and $|\langle X, \nabla R \rangle| \leq O(r^{-2})$, we obtain $|\nabla R| \leq O(r^{-\frac{7}{4}})$. This proves the first statement.

We now describe the proof of the second statement. By Corollary 2.12 we have $|\Delta^\Sigma R| \leq O(r^{-\frac{9}{4}})$. Hence, Lemma 2.9 implies
\[
|\langle X, \nabla R \rangle + R^2| \leq O(r^{-\frac{9}{4}}).
\]
From this, we deduce that
\[
\left\langle X, \nabla \left(\frac{1}{R} - f\right) \right\rangle = 1 - |\nabla f|^2 - \frac{1}{R^2} \left(\langle X, \nabla R \rangle + R^2\right) = O(r^{-\frac{1}{4}}).
\]
Integrating this relation along the integral curves of $X$ gives
\[
\frac{1}{R} - f = O(r^{\frac{1}{2}}).
\]
From this, the assertion follows.

Corollary 2.14. The principal curvatures of the level surface $\{f = r\}$ are given by $\frac{1}{2r} + O(r^{-\frac{5}{4}})$. Moreover, the intrinsic Gaussian curvature of the level surface $\{f = r\}$ is given by $\frac{1}{2r} + O(r^{-\frac{5}{4}})$. 
3. Existence of approximate Killing vector fields near infinity

In this section, we shall construct a collection of approximate Killing vector fields near infinity. It is easy to see that the level surfaces of \( f \) are diffeomorphic to \( S^2 \). Hence, we can find a family of diffeomorphisms \( F_r : S^2 \to \{ f = r \} \subset M \) such that \( \frac{\partial}{\partial r} F_r = \frac{X}{|X|} \). We define a metric \( \gamma_r \) on \( S^2 \) by \( \gamma_r = \frac{1}{2} F_r^* (g) \).

**Proposition 3.1.** We have

\[
\left\| \frac{d}{dr} \gamma_r \right\|_{C^l(S^2, \gamma_r)} \leq O(r^{-\frac{9}{8}})
\]

for each \( l \geq 0 \).

**Proof.** By Corollary \[2,14\] the principal curvatures of the level surface \( \{ f = r \} \) are given by \( \frac{1}{r} + O(r^{-\frac{5}{4}}) \). Moreover, the normal velocity of the flow \( F_r : S^2 \to M \) is \( \frac{1}{|X|} = 1 + O(r^{-1}) \). This implies

\[
\sup_{S^2} \left| \frac{d}{dr} F_r^* (g) - \frac{1}{r} F_r^* (g) \right| \leq O(r^{-\frac{5}{4}}).
\]

From this, we deduce that

\[
\sup_{S^2} \left| \frac{d}{dr} \gamma_r \right| \leq O(r^{-\frac{5}{4}}).
\]

Using the estimate \( \sup_{\{ f = r \}} |D^l \text{Ric}| \leq O(r^{-\frac{l+2}{2}}) \), we conclude that the manifold \( (S^2, \gamma_r) \) has bounded curvature, and all the derivatives of the curvature are bounded as well. Using the inequality

\[
\sup_{\{ r - \sqrt{r} \leq f \leq r + \sqrt{r} \}} \left| D^l \left( \mathcal{L}_{\frac{X}{|X|^2}} (g) \right) \right| \leq O(r^{-\frac{l+2}{2}}),
\]

we obtain

\[
\left\| F_r^* \left( \mathcal{L}_{\frac{X}{|X|^2}} (g) \right) \right\|_{C^l(S^2, \gamma_r)} \leq O(1).
\]

Since

\[
\frac{d}{dr} \gamma_r + \frac{1}{r} \gamma_r = \frac{1}{2r} F_r^* \left( \mathcal{L}_{\frac{X}{|X|^2}} (g) \right),
\]

we conclude that

\[
\left\| \frac{d}{dr} \gamma_r \right\|_{C^l(S^2, \gamma_r)} \leq O(r^{-1})
\]

for each \( l \geq 0 \). Using \[7\], \[8\], and standard interpolation inequalities, the assertion follows.

By Proposition A.5 in \[2\], the metrics \( \gamma_r \) converge in \( C^\infty \) to a smooth metric \( \overline{\gamma} \) as \( r \to \infty \). By Corollary \[2,14\] the Gaussian curvature of the
metric $\gamma_r$ is $1 + O(r^{-\frac{1}{2}})$. Consequently, the limit metric $\overline{\gamma}$ must have constant Gaussian curvature 1. Moreover, Proposition 3.1 implies that

$$\left\| \frac{d}{dr} \gamma_r \right\|_{C^1(S^2, \overline{\gamma})} \leq O(r^{-\frac{1}{2}}),$$

hence

$$\left\| \gamma_r - \overline{\gamma} \right\|_{C^1(S^2, \overline{\gamma})} \leq O(r^{-\frac{1}{2}})$$

for each $l \geq 0$.

Let $\overline{U}_1, \overline{U}_2, \overline{U}_3$ be three Killing vector fields on the round sphere $(S^2, \overline{\gamma})$ such that

$$\sum_{a=1}^{3} \overline{U}_a \otimes \overline{U}_a = \frac{1}{2} \left( e_1 \otimes e_1 + e_2 \otimes e_2 \right),$$

where $\{e_1, e_2\}$ is a local orthonormal frame on $(S^2, \overline{\gamma})$. Using (10), we obtain

$$\left\| \mathcal{L}_{\overline{U}_a} (\gamma_r) \right\|_{C^1(S^2, \overline{\gamma})} = \left\| \mathcal{L}_{\overline{U}_a} (\gamma_r - \overline{\gamma}) \right\|_{C^1(S^2, \overline{\gamma})} \leq O(r^{-\frac{1}{2}}).$$

We can find three vector fields $U_1, U_2, U_3$ on $M$ with the property that the vector field $U_a$ is tangential to the level set $\{f = r\}$, and $F_r^* U_a = \overline{U}_a$ for $r$ sufficiently large. Clearly, $[U_a, \overline{X}] = 0$ outside a compact set. This implies

$$[U_a, X] = U_a(|X|^2) \frac{X}{|X|^2}$$

outside a compact set. Since $U_a(|X|^2) = - \langle U_a, \nabla R \rangle = O(r^{-1})$, we conclude that $|[U_a, X]| \leq O(r^{-1})$. Moreover, the inequality $\|\overline{U}_a\|_{C^1(S^2, \overline{\gamma})} \leq O(1)$ gives $\sup_{\{f = r\}} |D_{\overline{U}_a} U_a| \leq O(r^{\frac{1}{2}})$ for each $l \geq 0$. Since $[U_a, \overline{X}] = 0$, we conclude that $\sup_{\{f = r\}} |D_j U_a| \leq O(r^{-\frac{1}{4}})$ for each $l \geq 0$.

**Proposition 3.2.** The vector fields $U_1, U_2, U_3$ on $(M, g)$ satisfy $|\mathcal{L}_{U_a}(g)| \leq O(r^{-\frac{1}{2}})$ and $|\Delta U_a + D_X U_a| \leq O(r^{-\frac{1}{4}})$. Moreover, we have

$$\sum_{a=1}^{3} U_a \otimes U_a = r (e_1 \otimes e_1 + e_2 \otimes e_2 + O(r^{-\frac{1}{4}})),$$

where $\{e_1, e_2\}$ is a local orthonormal frame on the level set $\{f = r\}$.

**Proof.** Let $\{e_1, e_2\}$ be a local orthonormal frame on the level surface $\{f = r\}$. Using (12) and (13), we obtain

$$\langle D_{e_i} U_a, e_j \rangle + \langle D_{e_j} U_a, e_i \rangle = O(r^{-\frac{1}{2}}),$$

$$\langle D_X U_a, e_j \rangle + \langle D_{e_j} U_a, X \rangle = \langle D_{U_a} X, e_j \rangle - \langle U_a, D_{e_j} X \rangle - \langle [U_a, X], e_j \rangle = 0,$$

$$\langle D_X U_a, X \rangle = \langle D_{U_a} X, X \rangle - \langle [U_a, X], X \rangle = -\frac{1}{2} U_a(|X|^2) = O(r^{-1}).$$
Therefore, the tensor $h_a = \mathcal{L}_{U_a}(g)$ satisfies
\[
\sup_{\{f=r\}} |h_a| \leq O(r^{-\frac{1}{8}}).
\]
Moreover, we have
\[
\sup_{\{f=r\}} |D^l h_a| \leq O(r^{-\frac{l}{2}})
\]
for each $l \geq 0$. Thus, standard interpolation inequalities imply that
\[
\sup_{\{f=r\}} |Dh_a| \leq O(r^{-\frac{9}{16}}).
\]
On the other hand, we have
\[
\text{div}(h_a) - \frac{1}{2} \nabla (\text{tr} h_a) = \Delta U_a + \text{Ric}(U_a).
\]
Putting these facts together, obtain
\[
\sup_{\{f=r\}} |\Delta U_a + \text{Ric}(U_a)| \leq O(r^{-\frac{9}{16}}).
\]
Using the estimate $|\text{Ric}(U_a) - D_X U_a| = ||[U_a, X]|| \leq O(r^{-1})$, we conclude that
\[
\sup_{\{f=r\}} |\Delta U_a + D_X U_a| \leq O(r^{-\frac{9}{16}}).
\]
Finally, the identity
\[
\sum_{a=1}^{3} U_a \otimes U_a = r (e_1 \otimes e_1 + e_2 \otimes e_2 + O(r^{-\frac{1}{8}}))
\]
follows immediately from (11).

Note that it is enough to define the vector fields $U_1, U_2, U_3$ outside of a compact region. Since we are only interested in the asymptotic behavior near infinity, we can extend the vector fields $U_1, U_2, U_3$ in an arbitrary way into the interior.

4. A PDE for the Lie derivative of a vector field

Let us fix a small number $\varepsilon > 0$. For example, $\varepsilon = \frac{1}{100}$ will work. In this section, we consider a vector field $W$ satisfying $\Delta W + D_X W = 0$. Our goal is to derive an elliptic equation for the Lie derivative $\mathcal{L}_W(g)$.

**Theorem 4.1.** Suppose that $W$ is a vector field satisfying $\Delta W + D_X W = 0$. Then the Lie derivative $\mathcal{L}_W(g)$ satisfies
\[
\Delta_L(\mathcal{L}_W(g)) + \mathcal{L}_X(\mathcal{L}_W(g)) = 0.
\]
Proof. Let \( g(s) \) be a smooth one-parameter family of metrics with \( g(0) = g \). It follows from Proposition 2.3.7 in [26] that

\[
\frac{\partial}{\partial s} \text{Ric}_{g(s)} \bigg|_{s=0} = -\frac{1}{2} \Delta h + \frac{1}{2} \mathcal{L}_Z(g),
\]

where \( h = \frac{\partial}{\partial s} g(s) \big|_{s=0} \) and

\[
Z = \text{div} h - \frac{1}{2} \nabla (\text{tr} h).
\]

Let us apply the formula (14) to the family of metrics obtained by pulling back \( g \) under the one-parameter group of diffeomorphisms generated by \( W \). This gives

\[
\mathcal{L}_W(\text{Ric}) = -\frac{1}{2} \Delta h + \frac{1}{2} \mathcal{L}_Z(g),
\]

where \( h = \mathcal{L}_W(g) \) and

\[
Z = \text{div} h - \frac{1}{2} \nabla (\text{tr} h) = \Delta W + \text{Ric}(W).
\]

Using the relation \( \Delta W + D_X W = 0 \), we obtain

\[
Z = \Delta W + D_W X = -[X, W].
\]

Substituting this identity into (15), we conclude that

\[
\Delta_L(\mathcal{L}_W(g)) = -2 \mathcal{L}_W(\text{Ric}) + \mathcal{L}_Z(g)
= -\mathcal{L}_W(\mathcal{L}_X(g)) - \mathcal{L}_{[X, W]}(g)
= -\mathcal{L}_X(\mathcal{L}_W(g)).
\]

This completes the proof.

Applying Theorem 4.1 to the vector field \( X \) gives the following result:

**Proposition 4.2.** The vector field \( X \) satisfies \( \Delta X + D_X X = 0 \). Moreover, the Ricci tensor satisfies

\[
\Delta_L(\text{Ric}) + \mathcal{L}_X(\text{Ric}) = 0.
\]

**Proof.** Let \( h = \mathcal{L}_X(g) = 2 \text{Ric} \). The contracted second Bianchi identity implies that

\[
0 = \text{div} h - \frac{1}{2} \nabla (\text{tr} h) = \Delta X + \text{Ric}(X) = \Delta X + D_X X.
\]

Using Theorem 4.1 we obtain

\[
\Delta_L h + \mathcal{L}_X(h) = 0,
\]

as claimed.

We note that the identity \( \Delta_L(\text{Ric}) + \mathcal{L}_X(\text{Ric}) = 0 \) can alternatively be derived from the evolution equation for the Ricci tensor under the Ricci flow (see e.g. [2], Section 2.4).
5. An elliptic PDE for vector fields

Throughout this section, we fix a smooth vector field \( Q \) on \( M \) such that \(|Q| \leq O(r^{-\frac{1}{2}-2\varepsilon})\). Our goal is to construct a vector field \( V \) on \( M \) such that \( \Delta V + DXV = Q \) and \(|V| \leq O(r^{\frac{1}{2}-\varepsilon})\). We first establish some auxiliary results.

**Lemma 5.1.** Let us consider the one-parameter family of shrinking cylinders \( (S^2 \times \mathbb{R}, g(t)) \), \( t \in (0, 1) \), where \( g(t) \) is given by (3). Suppose that \( V(t), t \in (0, 1), \) is a one-parameter family of vector fields satisfying the parabolic equation

\[
\frac{\partial}{\partial t} V(t) = \Delta g(t) V(t) + \text{Ric}_{g(t)}(V(t)).
\]

Moreover, we assume that \( V(t) \) is invariant under translations along the axis of the cylinder, and

\[
|V(t)|_{g(t)} \leq 1
\]

for all \( t \in (0, \frac{1}{2}] \). Then

\[
\inf_{\lambda \in \mathbb{R}} \sup_{S^2 \times \mathbb{R}} \left| V(t) - \lambda \frac{\partial}{\partial z} \right|_{g(t)} \leq L(1 - t)^{\frac{1}{2}}
\]

for all \( t \in \left[ \frac{1}{2}, 1 \right] \), where \( L \) is a positive constant.

**Proof.** Since \( V(t) \) is invariant under translations along the axis of the cylinder, we may write

\[ V(t) = \xi(t) + \eta(t) \frac{\partial}{\partial z} \]

for \( t \in (0, 1) \), where \( \xi(t) \) is a vector field on \( S^2 \) and \( \eta(t) \) is a real-valued function on \( S^2 \). The parabolic equation (16) is equivalent to the following system of equations \( \xi(t) \) and \( \eta(t) \):

\[
\frac{\partial}{\partial t} \xi(t) = \frac{1}{2 - 2t} (\Delta_{S^2} \xi(t) + \xi(t)),
\]

\[
\frac{\partial}{\partial t} \eta(t) = \frac{1}{2 - 2t} \Delta_{S^2} \eta(t).
\]

Moreover, the assumption (17) implies

\[
\sup_{S^2} |\xi(t)|_{g_{S^2}} \leq L_1,
\]

\[
\sup_{S^2} |\eta(t)| \leq L_1
\]

for each \( t \in (0, \frac{1}{2}] \), where \( L_1 \) is a positive constant.

Consider now the operator \( \xi \mapsto -\Delta_{S^2} \xi - \xi \), acting on vector fields on \( S^2 \). It follows from Proposition A.1 that the first eigenvalue of this operator is nonnegative. Using (18) and (20), we conclude that

\[
\sup_{S^2} |\xi(t)|_{g_{S^2}} \leq L_2
\]
for all \( t \in \left[ \frac{1}{2}, 1 \right) \), where \( L_2 \) is a positive constant. Similarly, using (19) and (21), we can show that
\[
\inf_{\lambda \in \mathbb{R}} \sup_{S^2} |\eta(t) - \lambda| \leq L_3 (1 - t)
\]
for each \( t \in \left[ \frac{1}{2}, 1 \right) \), where \( L_3 \) is a positive constant. Combining (22) and (23), the assertion follows.

**Lemma 5.2.** Let \( V \) be a smooth vector field satisfying \( \Delta V + D_X V = Q \) in the region \( \{ f \leq \rho \} \). Then
\[
\sup_{\{ f \leq \rho \}} |V| \leq \sup_{\{ f = \rho \}} |V| + B \rho^{\frac{1}{2} - 2\varepsilon}
\]
for some uniform constant \( B \geq 1 \).

**Proof.** It follows from Kato’s inequality that
\[
\Delta(|V|^2) + \langle X, \nabla(|V|^2) \rangle = 2 |DV|^2 + 2 \langle V, Q \rangle \geq 2 |\nabla|V||^2 - 2 |Q||V|.
\]
This implies
\[
\Delta(|V|) + \langle X, \nabla|V| \rangle \geq -|Q|
\]
when \( V \neq 0 \). Moreover, using the identity \( \Delta f + |\nabla f|^2 = 1 \) and the inequality \( f \geq 1 \), we obtain
\[
\Delta(f^{\frac{1}{2} - 2\varepsilon}) + \langle X, \nabla(f^{\frac{1}{2} - 2\varepsilon}) \rangle
= \left( \frac{1}{2} - 2\varepsilon \right) f^{-\frac{1}{2} - 2\varepsilon} (\Delta f + |\nabla f|^2) - \left( \frac{1}{4} - 4\varepsilon^2 \right) f^{-\frac{3}{2} - 2\varepsilon} |\nabla f|^2
\geq \left( \frac{1}{2} - 2\varepsilon \right) f^{-\frac{1}{2} - 2\varepsilon} - \left( \frac{1}{4} - 4\varepsilon^2 \right) f^{-\frac{3}{2} - 2\varepsilon}
= \left( \frac{1}{2} - 2\varepsilon \right)^2 f^{-\frac{1}{2} - 2\varepsilon}.
\]
By assumption, we can find a constant \( B \geq 1 \) such that
\[
|Q| < \left( \frac{1}{2} - 2\varepsilon \right)^2 B f^{-\frac{1}{2} - 2\varepsilon}.
\]
Putting these facts together, we obtain
\[
\Delta(|V| + B f^{\frac{1}{2} - 2\varepsilon}) + \langle X, \nabla(|V| + B f^{\frac{1}{2} - 2\varepsilon}) \rangle > 0
\]
when \( V \neq 0 \). By the maximum principle, the function \( |V| + B f^{\frac{1}{2} - 2\varepsilon} \) attains its maximum on the boundary; that is,
\[
\sup_{\{ f \leq \rho \}} (|V| + B f^{\frac{1}{2} - 2\varepsilon}) \leq \sup_{\{ f = \rho \}} |V| + B \rho^{\frac{1}{2} - 2\varepsilon}.
\]
From this, the assertion follows.
In the following, we consider a sequence of real numbers $\rho_m \to \infty$. Given any integer $m$, there exists a unique vector field $V^{(m)}$ such that

$$\Delta V^{(m)} + DX V^{(m)} = Q$$

in the region $\{f \leq \rho_m\}$ and $V^{(m)} = 0$ on the boundary $\{f = \rho_m\}$. Moreover, we define

$$A^{(m)}(r) = \inf_{\lambda \in \mathbb{R}} \sup_{\{f = r\}} |V^{(m)} - \lambda X|$$

for $r \leq \rho_m$.

**Lemma 5.3.** Let us fix a real number $\tau \in (0, \frac{1}{2})$ such that $\tau^{-\varepsilon} > 2L$, where $L$ is the constant in Lemma 5.1. Then we can find a real number $\rho_0$ and a positive integer $m_0$ such that

$$2\tau^{-\frac{1}{2} + \varepsilon} A^{(m)}(\tau r) \leq A^{(m)}(r) + r^{\frac{1}{2} - \varepsilon}$$

for all $r \in [\rho_0, \rho_m]$ and all $m \geq m_0$.

**Proof.** Suppose that the assertion is false. After passing to a subsequence, we can find a sequence of real numbers $r_m \leq \rho_m$ such that $r_m \to \infty$ and

$$A^{(m)}(r_m) + r_m^{\frac{1}{2} - \varepsilon} \leq 2\tau^{-\frac{1}{2} + \varepsilon} A^{(m)}(\tau r_m)$$

for all $m$. For each $m$, we choose a real number $\lambda_m$ such that

$$\sup_{\{f = r_m\}} |V^{(m)} - \lambda_m X| = A^{(m)}(r_m).$$

The vector field $V^{(m)} - \lambda_m X$ satisfies the equation

$$\Delta (V^{(m)} - \lambda_m X) + DX (V^{(m)} - \lambda_m X) = Q.$$

Using Lemma 5.2, we obtain

$$\sup_{\{f \leq r_m\}} |V^{(m)} - \lambda_m X| \leq \sup_{\{f = r_m\}} |V^{(m)} - \lambda_m X| + Br_m^{\frac{1}{2} - 2\varepsilon}$$

$$\leq A^{(m)}(r_m) + r_m^{\frac{1}{2} - \varepsilon}$$

if $m$ is sufficiently large. Therefore, the vector field

$$\tilde{V}^{(m)} = \frac{1}{A^{(m)}(r_m) + r_m^{\frac{1}{2} - \varepsilon}} (V^{(m)} - \lambda_m X)$$

satisfies

$$\sup_{\{f \leq r_m\}} |\tilde{V}^{(m)}| \leq 1$$

if $m$ is sufficiently large. We next define

$$\tilde{g}^{(m)}(t) = r_m^{-1} \Phi_{11}(g)$$

and

$$\tilde{V}^{(m)}(t) = r_m^{\frac{1}{2}} \Phi_{11}(\tilde{V}^{(m)}).$$
Since \((M, g)\) is a steady Ricci soliton, the metrics \(\hat{g}^{(m)}(t)\) form a solution to the Ricci flow. Moreover, the vector fields \(\hat{V}^{(m)}(t)\) satisfy the parabolic equation

\[
\frac{\partial}{\partial t} \hat{V}^{(m)}(t) = \Delta \hat{g}^{(m)}(t) \hat{V}^{(m)}(t) + \text{Ric}_{\hat{g}^{(m)}(t)}(\hat{V}^{(m)}(t)) - \hat{Q}^{(m)}(t),
\]

where

\[
\hat{Q}^{(m)}(t) = \frac{r_{m}^{\frac{2}{3}}}{A^{(m)}(r_{m}) + r_{m}^{\frac{1}{3} - \varepsilon} \Phi^{*}_{r_{m}t}(Q)}.
\]

The inequality (24) implies that

\[
\limsup_{m \to \infty} \sup_{t \in [\delta, 1 - \delta]} \sup_{r_{m} - \delta^{-1} \sqrt{r_{m}} \leq f \leq r_{m} + \delta^{-1} \sqrt{r_{m}}} |\hat{V}^{(m)}(t)|_{\hat{g}^{(m)}(t)} < \infty
\]

for any given \(\delta \in (0, \frac{1}{2})\). Moreover, using the estimate \(|Q| \leq O(r^{-\frac{1}{2} - 2\varepsilon})\), we obtain

\[
\limsup_{m \to \infty} \sup_{t \in [\delta, 1 - \delta]} \sup_{r_{m} - \delta^{-1} \sqrt{r_{m}} \leq f \leq r_{m} + \delta^{-1} \sqrt{r_{m}}} |\hat{Q}^{(m)}(t)|_{\hat{g}^{(m)}(t)} = 0
\]

for any given \(\delta \in (0, \frac{1}{2})\).

We now pass to the limit as \(m \to \infty\). To that end, we choose a sequence of marked points \(p_{m} \in M\) such that \(f(p_{m}) = r_{m}\). The sequence \((M, \hat{g}^{(m)}(t), p_{m})\) converges in the Cheeger-Gromov sense to a one-parameter family of shrinking cylinders \((S^{2} \times \mathbb{R}, \overline{g}(t))\), \(t \in (0, 1)\), where \(\overline{g}(t)\) is given by (3). The rescaled vector fields \(r_{m}^{-\varepsilon} \hat{X}\) converge to the axial vector field \(\frac{\partial}{\partial z}\) on \(S^{2} \times \mathbb{R}\). Finally, after passing to a subsequence, the vector fields \(\hat{V}^{(m)}(t)\) converge in \(C_{loc}^{0}\) to a one-parameter family of vector fields \(\overline{V}(t)\), \(t \in (0, 1)\), which satisfy the parabolic equation

\[
\frac{\partial}{\partial t} \overline{V}(t) = \Delta_{\overline{g}(t)} \overline{V}(t) + \text{Ric}_{\overline{g}(t)}(\overline{V}(t)).
\]

(The convergence in \(C_{loc}^{0}\) follows from the Arzela-Ascoli theorem together with standard interior estimates for linear parabolic equations; see e.g. [18, Theorem 7.22].) Using the identity

\[
\Phi^{*}_{r_{m}^{-\varepsilon}}(\hat{V}^{(m)}(t)) = \hat{V}^{(m)}(t) + \frac{s}{\sqrt{r_{m}}},
\]

we conclude that \(\Psi_{s}^{*}(\overline{V}(t)) = \overline{V}(t)\), where \(\Psi_{s} : S^{2} \times \mathbb{R} \to S^{2} \times \mathbb{R}\) denotes the flow generated by the axial vector field \(-\frac{\partial}{\partial z}\). Hence, \(\overline{V}(t)\) is invariant under translations along the axis of the cylinder. Using the estimate (24), we obtain

\[
|\overline{V}(t)|_{\overline{g}(t)} \leq 1
\]

for all \(t \in (0, \frac{1}{2}]\). Using Lemma 5.1, we conclude that

\[
\inf_{\lambda \in \mathbb{R}} \sup_{S^{2} \times \mathbb{R}} |\overline{V}(t) - \lambda \frac{\partial}{\partial z}|_{\overline{g}(t)} \leq L(1 - t)^{\frac{1}{2}}
\]

(25)
for all $t \in \left[ \frac{1}{2}, 1 \right)$. On the other hand, we have
\[
\inf_{\lambda \in \mathbb{R}} \sup_{(f = r\tau m)} \left| \hat{V}(m)(1 - \tau) - \lambda r_m \partial \frac{\partial}{\partial z} X \right|_{g(m)(1 - \tau)} = \inf_{\lambda \in \mathbb{R}} \sup_{(f = r\tau m)} |\hat{V}(m) - \lambda X|_g
\]
\[
= \frac{1}{A(m)(r_m) + r_m^{1 - \epsilon}} \inf_{\lambda \in \mathbb{R}} \sup_{(f = r\tau m)} |V(m) - \lambda X|_g
\]
\[
\geq \frac{1}{2} \tau^{\frac{1}{2} - \epsilon}. \tag{26}
\]
Passing to the limit as $m \to \infty$ gives
\[
\inf_{\lambda \in \mathbb{R}} \sup_{S^2 \times \mathbb{R}} \left| \nabla(1 - \tau) - \lambda \frac{\partial}{\partial z}\right|_{\mathbb{S}^1(1 - \tau)} \geq \frac{1}{2} \tau^{\frac{1}{2} - \epsilon}. \tag{26}
\]
Since $\tau^{-\epsilon} > 2L$, the inequalities (25) and (26) are in contradiction. This completes the proof of Lemma 5.3.

**Proposition 5.4.** There exists a sequence of real numbers $\lambda_m$ such that
\[
\sup_{m} \sup_{\{f \leq \rho_m\}} f^{-\frac{1}{2} + \epsilon} |V(m) - \lambda_m X| < \infty.
\]

**Proof.** Let us fix a real number $\tau \in (0, \frac{1}{2})$ so that $\tau^{-\epsilon} > 2L$, where $L$ is the constant in Lemma 5.1. By Lemma 5.3, we can find a real number $\rho_0$ and a positive integer $m_0$ such that
\[
2 \tau^{-\frac{1}{2} + \epsilon} A(m)(\tau r) \leq A(m)(r) + r^{\frac{1}{2} - \epsilon}
\]
for all $r \in [\rho_0, \rho_m]$ and all $m \geq m_0$. Moreover, Lemma 5.2 implies that
\[
\sup_{\rho_0 \leq r \leq \rho_m} A(m)(r) \leq \sup_{\{f \leq \rho_m\}} |V(m)| \leq B \rho_m^{\frac{1}{2} - 2\epsilon}.
\]
If we iterate the inequality (27), we obtain
\[
\sup_{m \geq m_0} \sup_{\rho_0 \leq r \leq \rho_m} r^{-\frac{1}{2} + \epsilon} A(m)(r) < \infty. \tag{28}
\]
In the next step, we fix a real number $\rho_1 > \rho_0$ such that $\sup_{\{f = \rho_1\}} |X| \geq \frac{1}{2}$. We can find a sequence of real numbers $\lambda_m$ such that
\[
\sup_{\{f = \rho_1\}} |V(m) - \lambda_m X| = A(m)(\rho_1)
\]
for each $m$. Applying Lemma 5.2 to the vector field $V(m) - \lambda X$, we obtain
\[
\sup_{\{f = \rho_1\}} |V(m) - \lambda X| \leq \sup_{\{f = \rho_1\}} |V(m) - \lambda X| + B r^{\frac{1}{2} - 2\epsilon}
\]
\[
\leq \sup_{\{f = \rho_1\}} |V(m) - \lambda X| + B \rho_m^{\frac{1}{2} - 2\epsilon}.
\]
for all \( r \in [\rho_1, \rho_m] \) and all \( \lambda \in \mathbb{R} \). This implies

\[
\sup_{\{f=r\}} |V^{(m)}(r) - \lambda_m X| \\
\leq \sup_{\{f=r\}} |V^{(m)}(r) - \lambda X| + |\lambda - \lambda_m| \\
\leq \sup_{\{f=r\}} |V^{(m)}(r) - \lambda X| + 2 \sup_{\{f=\rho_1\}} |\lambda X - \lambda_m X| \\
\leq \sup_{\{f=r\}} |V^{(m)}(r) - \lambda X| + 2 \sup_{\{f=\rho_1\}} |V^{(m)}(r) - \lambda_m X| + 2 \sup_{\{f=\rho_1\}} |V^{(m)}(r) - \lambda X| \\
\leq 3 \sup_{\{f=r\}} |V^{(m)}(r) - \lambda X| + 2 A^{(m)}(\rho_1) + 2B r^{\frac{1}{2} - 2\varepsilon}
\]

for all \( r \in [\rho_1, \rho_m] \) and all \( \lambda \in \mathbb{R} \). Taking the infimum over \( \lambda \in \mathbb{R} \) gives

\[
\sup_{\{f=r\}} |V^{(m)}(r) - \lambda_m X| \leq 3 A^{(m)}(r) + 2 A^{(m)}(\rho_1) + 2B r^{\frac{1}{2} - 2\varepsilon}
\]

for all \( r \in [\rho_1, \rho_m] \). Consequently, the inequality (28) implies

\[
\sup_{m \geq m_0} \sup_{\rho_1 \leq r \leq \rho_m} \sup_{\{f=r\}} r^{-\frac{1}{2} + \varepsilon} |V^{(m)}(r) - \lambda_m X| < \infty,
\]

hence

\[
\sup_{m \geq m_0} \sup_{\rho_1 \leq r \leq \rho_m} f^{-\frac{1}{2} + \varepsilon} |V^{(m)}(r) - \lambda_m X| < \infty.
\]

Using Lemma 5.2 we conclude that

\[
\sup_{m \geq m_0} \sup_{\{f=\rho_1\}} |V^{(m)}(r) - \lambda_m X| < \infty.
\]

Putting these facts together, the assertion follows.

**Theorem 5.5.** There exists a smooth vector field \( V \) such that \( \Delta V + DXV = Q \) and \( |V| \leq O\left(r^{\frac{1}{2} - \varepsilon}\right) \). Moreover, \( |DV| \leq O\left(r^{-\varepsilon}\right) \).

**Proof.** By Proposition 5.4, we can find a sequence of real numbers \( \lambda_m \) such that

\[
\sup_{m} \sup_{\{f=\rho_m\}} f^{-\frac{1}{2} + \varepsilon} |V^{(m)}(r) - \lambda_m X| < \infty.
\]

Moreover, the vector field \( V^{(m)}(r) - \lambda_m X \) solves the equation

\[
\Delta (V^{(m)}(r) - \lambda_m X) + DX (V^{(m)}(r) - \lambda_m X) = Q
\]

in the region \( \{ f \leq \rho_m \} \). Hence, after passing to a subsequence if necessary, the vector fields \( V^{(m)}(r) - \lambda_m X \) converge to a smooth vector field \( V \) satisfying \( \Delta V + DXV = Q \) and \( |V| \leq O\left(r^{\frac{1}{2} - \varepsilon}\right) \).

It remains to show that \( |DV| \leq O\left(r^{-\varepsilon}\right) \). In order to prove this, we use the standard interior regularity theory for parabolic equations. Consider a sequence \( r_m \to \infty \), and let

\[
\hat{g}^{(m)}(t) = r_m^{-1} \Phi_{r_m t}(g)
\]
for \( t \in [-\frac{1}{2}, 0] \). Moreover, we define
\[
\hat{V}^{(m)}(t) = \Phi_{r_m t}^*(V)
\]
and
\[
\hat{Q}^{(m)}(t) = r_m \Phi_{r_m t}^*(Q)
\]
for \( t \in [-\frac{1}{2}, 0] \). The vector fields \( \hat{V}^{(m)}(t) \) satisfy the parabolic equation
\[
\frac{\partial}{\partial t} \hat{V}^{(m)}(t) = \Delta_{\hat{g}^{(m)}}(t) \hat{V}^{(m)}(t) + \text{Ric}_{\hat{g}^{(m)}}(t) (\hat{V}^{(m)}(t)) - \hat{Q}^{(m)}(t).
\]
Moreover, since \( |Q| \leq O(r^{-\frac{1}{2}} - 2\varepsilon) \), we have
\[
\sup_{t \in [-\frac{1}{2}, 0]} \sup_{r_m - \sqrt{r_m} \leq f \leq r_m + \sqrt{r_m}} |\hat{Q}^{(m)}(t)|_{\hat{g}^{(m)}}(t) \leq O(r_m^{-2\varepsilon}).
\]
Using standard interior estimates for parabolic equations, we obtain
\[
\sup_{\{f = r_m\}} |DV^{(m)}(0)|_{\hat{g}^{(m)}}(0) \leq C \sup_{t \in [-\frac{1}{2}, 0]} \sup_{r_m - \sqrt{r_m} \leq f \leq r_m + \sqrt{r_m}} |\hat{V}^{(m)}(t)|_{\hat{g}^{(m)}}(t)
\]
\[
+ C \sup_{t \in [-\frac{1}{2}, 0]} \sup_{r_m - \sqrt{r_m} \leq f \leq r_m + \sqrt{r_m}} |\hat{Q}^{(m)}(t)|_{\hat{g}^{(m)}}(t)
\]
\[
\leq O(r_m^{-\varepsilon}).
\]
From this, we deduce that
\[
\sup_{\{f = r_m\}} |DV| \leq O(r_m^{-\varepsilon}),
\]
as claimed.

6. Analysis of the Lichnerowicz Equation

**Lemma 6.1.** Let us consider the shrinking cylinders \((S^2 \times \mathbb{R}, \overline{g}(t))\), \( t \in (0, 1) \), where \( \overline{g}(t) \) is given by (3). Suppose that \( \overline{h}(t), t \in (0, 1) \), is a one-parameter family of \((0,2)\)-tensors satisfying the parabolic Lichnerowicz equation
\[
\frac{\partial}{\partial t} \overline{h}(t) = \Delta_{L, \overline{g}} \overline{h}(t).
\]
Moreover, we assume that \( \overline{h}(t) \) is invariant under translations along the axis of the cylinder, and
\[
|\overline{h}(t)|_{\overline{g}(t)} \leq (1 - t)^{-1}
\]
for all \( t \in (0, \frac{1}{2}] \). Then we have
\[
\inf_{\lambda \in \mathbb{R}} \sup_{S^2 \times \mathbb{R}} |\overline{h}(t) - \lambda \text{Ric}_{\overline{g}(t)}|_{\overline{g}(t)} \leq N
\]
for all \( t \in [\frac{1}{2}, 1) \), where \( N \) is a positive constant.
Proof. Since \( \overline{h}(t) \) is invariant under translations along the axis of the cylinder, we may write
\[
\overline{h}(t) = \chi(t) + dz \otimes \sigma(t) + \sigma(t) \otimes dz + \beta(t) \, dz \otimes dz
\]
for \( t \in (0, 1) \), where \( \chi(t) \) is a symmetric \((0,2)\) tensor on \( S^2 \), \( \sigma(t) \) is a one-form on \( S^2 \), and \( \beta(t) \) is a real-valued function on \( S^2 \). The parabolic Lichnerowicz equation (29) is equivalent to the following system of equations for \( \chi(t) \), \( \sigma(t) \), and \( \beta(t) \):
\[
\frac{\partial}{\partial t} \chi(t) = \frac{1}{2 - 2t} (\Delta_{S^2} \chi(t) - 4 \, \overline{\chi}(t)),
\]
\[
\frac{\partial}{\partial t} \sigma(t) = \frac{1}{2 - 2t} (\Delta_{S^2} \sigma(t) - \sigma(t)),
\]
\[
\frac{\partial}{\partial t} \beta(t) = \frac{1}{2 - 2t} \Delta_{S^2} \beta(t).
\]
Here, \( \overline{\chi}(t) \) denotes the trace-free part of \( \chi(t) \) with respect to the standard metric on \( S^2 \). Moreover, the assumption (30) implies
\[
\sup_{S^2} |\chi(t)|_{g_{S^2}} \leq N_1,
\]
\[
\sup_{S^2} |\sigma(t)|_{g_{S^2}} \leq N_1,
\]
\[
\sup_{S^2} |\beta(t)| \leq N_1
\]
for each \( t \in (0, \frac{1}{2}] \), where \( N_1 \) is a positive constant.

Let us consider the operator \( \chi \mapsto -\Delta_{S^2} \chi + 4 \, \overline{\chi} \), acting on symmetric \((0,2)\)-tensors on \( S^2 \). The first eigenvalue of this operator is equal to 0, and the associated eigenspace is spanned by \( g_{S^2} \). Moreover, all other eigenvalues are at least 2 (cf. Proposition A.2 below). Hence, it follows from (31) and (34) that
\[
\inf_{\lambda \in \mathbb{R}} \sup_{S^2} |\chi(t) - \lambda g_{S^2}|_{g_{S^2}} \leq N_2 (1 - t)
\]
for all \( t \in [\frac{1}{2}, 1) \), where \( N_2 \) is a positive constant. We next consider the operator \( \sigma \mapsto -\Delta_{S^2} \sigma + \sigma \), acting on one-forms on \( S^2 \). By Proposition A.1, the first eigenvalue of this operator is at least 2. Using (32) and (35), we deduce that
\[
\sup_{S^2} |\sigma(t)|_{g_{S^2}} \leq N_3 (1 - t)
\]
for all \( t \in [\frac{1}{2}, 1) \), where \( N_3 \) is a positive constant. Finally, using (33) and (36), we obtain
\[
\sup_{S^2} |\beta(t)| \leq N_4
\]
for all \( t \in [\frac{1}{2}, 1) \), where \( N_4 \) is a positive constant. Combining (37), (38), and (39), the assertion follows.
In the following, we study the equation $\Delta_L h + \mathcal{L}_X(h) = 0$ on $(M,g)$.

**Lemma 6.2.** Let $h$ be a solution of the Lichnerowicz-type equation

$$\Delta_L h + \mathcal{L}_X(h) = 0$$

on the region $\{f \leq \rho\}$. Then

$$\sup_{\{f \leq \rho\}} f |h| \leq B \rho \sup_{\{f = \rho\}} |h|,$$

where $B$ is a positive constant that does not depend on $\rho$.

**Proof.** By a result of Anderson and Chow [1], we have

$$\Delta \left( \frac{|h|^2}{R^2} \right) + \left\langle X + 2 \frac{\nabla R}{R}, \nabla \left( \frac{|h|^2}{R^2} \right) \right\rangle \geq 0.$$

Applying the maximum principle, we obtain

$$\sup_{\{f \leq \rho\}} \frac{|h|}{R} \leq \sup_{\{f = \rho\}} \frac{|h|}{R}.$$

Since $\sup_M f R < \infty$ and $\inf_M f R > 0$, the assertion follows.

**Theorem 6.3.** Let $h$ be a solution of the Lichnerowicz-type equation

$$\Delta_L h + \mathcal{L}_X(h) = 0$$

such that $|h| \leq O(r^{-\varepsilon})$. Then $h = \lambda \text{Ric}$ for some constant $\lambda \in \mathbb{R}$.

**Proof.** Let

$$A(r) = \inf_{\lambda \in \mathbb{R}} \sup_{\{f = r\}} |h - \lambda \text{Ric}|.$$

Clearly, $A(r) \leq \sup_{\{f = r\}} |h| \leq O(r^{-\varepsilon})$. We consider two cases:

**Case 1:** Suppose that there exists a sequence of real numbers $r_m \to \infty$ such that $A(r_m) = 0$ for all $m$. For each $m$, we choose a real number $\lambda_m$ such that

$$\sup_{\{f = r_m\}} |h - \lambda_m \text{Ric}| = A(r_m) = 0.$$

Applying Lemma 6.2 to the tensor $h - \lambda_m \text{Ric}$, we obtain

$$\sup_{\{f \leq r_m\}} f |h - \lambda_m \text{Ric}| \leq B r_m \sup_{\{f = r_m\}} |h - \lambda_m \text{Ric}| = 0.$$

Therefore, we have $h - \lambda_m \text{Ric} = 0$ in the region $\{f \leq r_m\}$. Consequently, the sequence $\lambda_m$ is constant and $h$ is a constant multiple of the Ricci tensor.

**Case 2:** Suppose now that $A(r) > 0$ when $r$ is sufficiently large. We fix a real number $\tau \in (0, \frac{1}{2})$ such that $\tau^{-\varepsilon} > 2N B$, where $N$ is the constant in Lemma 6.2 and $B$ is the constant in Lemma 6.2. Since $A(r) \leq O(r^{-\varepsilon})$, we can find a sequence of real numbers $r_m \to \infty$ such that

$$A(r_m) \leq 2 \tau^\varepsilon A(\tau r_m)$$
for all $m$. For each $m$, we choose a real number $\lambda_m$ such that
\[
\sup_{\{f=r_m\}} |h - \lambda_m \text{Ric}| = A(r_m).
\]
The tensor
\[
\tilde{h}^{(m)}(t) = \frac{1}{A(r_m)} (h - \lambda_m \text{Ric})
\]
satisfies the Lichnerowicz-type equation
\[
\Delta_L \tilde{h}^{(m)} + \mathcal{L}_X (\tilde{h}^{(m)}) = 0.
\]
Using Lemma 6.2, we obtain
\[
\sup_{\{f=r\}} |\tilde{h}^{(m)}(t)| \leq \frac{B \cdot r_m}{r} \sup_{\{f=r_m\}} |\tilde{h}^{(m)}(t)| = \frac{B \cdot r_m}{r} \sup_{\{f=r_m\}} |h - \lambda_m \text{Ric}| = \frac{B \cdot r_m}{r}
\]
for $r \leq r_m$.

We now define
\[
g^{(m)}(t) = r_m^{-1} \Phi_{r_m t}^* (g)
\]
and
\[
h^{(m)}(t) = r_m^{-1} \Phi_{r_m t}^* (\tilde{h}^{(m)}).
\]
Since $(M, g)$ is a steady Ricci soliton, the metrics $g^{(m)}(t)$ evolve by the Ricci flow. Moreover, the tensors $h^{(m)}(t)$ satisfy the parabolic Lichnerowicz equation
\[
\frac{\partial}{\partial t} h^{(m)}(t) = \Delta_L \cdot g^{(m)}(t) \cdot h^{(m)}(t).
\]
It follows from (40) that
\[
\limsup_{m \to \infty} \sup_{t \in [0,1]} \sup_{f \in [1-\delta,1]} \sup_{r \leq f \leq r + \delta} |h^{(m)}(t) - g^{(m)}(t)| < \infty
\]
for any given $\delta \in (0, \frac{1}{2})$.

We next take the limit as $m \to \infty$. As above, we choose a sequence of marked points $p_m \in M$ such that $f(p_m) = r_m$. The sequence $(M, g^{(m)}(t), p_m)$ converges in the Cheeger-Gromov sense to a one-parameter family of shrinking cylinders $(S^2 \times \mathbb{R}, \mathcal{G}(t))$, $t \in [0, 1)$, where $\mathcal{G}(t)$ is given by (3). The rescaled vector fields $r_m^2 X$ converge to the axial vector field $\frac{\partial}{\partial z}$ on $S^2 \times \mathbb{R}$. Finally, after passing to a subsequence, the tensors $h^{(m)}(t)$ converge in $C^\infty_{\text{loc}}$ to a one-parameter family of tensor fields $\mathcal{H}(t)$, $t \in (0, 1)$, which satisfy the parabolic Lichnerowicz equation
\[
\frac{\partial}{\partial t} \mathcal{H}(t) = \Delta_L \cdot \mathcal{G}(t) \cdot \mathcal{H}(t).
\]
Using the identity
\[
\Phi_{r_m t}^* (\tilde{h}^{(m)}(t)) = \hat{h}^{(m)}(t + \frac{s}{r_m} \sqrt{r_m}),
\]
we obtain $\Psi_s^* (\mathcal{H}(t)) = \mathcal{H}(t)$, where $\Psi_s : S^2 \times \mathbb{R} \to S^2 \times \mathbb{R}$ denotes the flow generated by the axial vector field $-\frac{\partial}{\partial z}$. In other words, $\mathcal{H}(t)$ is invariant.
under translations along the axis of the cylinder. Moreover, the estimate (40) implies
\[ |\bar{h}(t)|_{g(t)} \leq B \left( 1 - t \right)^{-1} \]
for all \( t \in (0, \frac{1}{2}] \). Using Lemma 6.1, we conclude that
\[ \inf_{\lambda \in \mathbb{R}} \sup_{S^2 \times \mathbb{R}} \left| \bar{h}(t) - \lambda \operatorname{Ric}_{g(t)} \right|_{g(t)} \leq N B \]
for all \( t \in \left[ \frac{1}{2}, 1 \right) \). On the other hand, we have
\[ \inf_{\lambda \in \mathbb{R}} \sup_{\{ f = \tau r_m \}} |\hat{h}(t) - \lambda \operatorname{Ric}_{g}|_g \]
\[ = \frac{1}{A(r_m)} \inf_{\lambda \in \mathbb{R}} \sup_{\{ f = \tau r_m \}} |h - \lambda \operatorname{Ric}_g|_g \]
\[ = \frac{A(\tau r_m)}{A(r_m)} \geq \frac{1}{2} \tau^{-\varepsilon}. \]
Taking the limit as \( m \to \infty \) gives
\[ \inf_{\lambda \in \mathbb{R}} \sup_{S^2 \times \mathbb{R}} \left| \bar{h}(1 - \tau) - \lambda \operatorname{Ric}_{g(1-\tau)} \right|_{g(1-\tau)} \geq \frac{1}{2} \tau^{-\varepsilon}. \]
Since \( \tau^{-\varepsilon} > 2NB \), the inequalities (41) and (42) are in contradiction. This completes the proof of Theorem 6.3.

7. Proof of Theorem 1.1

Combining Theorems 4.1, 5.5, and 6.3, we obtain the following symmetry principle:

**Theorem 7.1.** Suppose that \( U \) is a vector field on \((M, g)\) such that \( |\mathcal{L}_U(g)| \leq O(r^{-2\varepsilon}) \) and \( |\Delta U + D_X U| \leq O(r^{-\frac{3}{2} - 2\varepsilon}) \) for some small constant \( \varepsilon > 0 \). Then there exists a vector field \( \bar{U} \) on \((M, g)\) such that \( \mathcal{L}_{\bar{U}}(g) = 0 \), \( [\bar{U}, X] = 0 \), \( \langle \bar{U}, X \rangle = 0 \), and \( |\bar{U} - U| \leq O(r^{-\frac{1}{2} - \varepsilon}) \).

**Proof.** By Theorem 5.5, we can find a smooth vector field \( V \) such that
\[ \Delta V + D_X V = \Delta U + D_X U \]
and \( |V| \leq O(r^{-\frac{1}{2} - \varepsilon}) \). Moreover, the covariant derivative of \( V \) satisfies \( |DV| \leq O(r^{-\varepsilon}) \). We now define \( W = U - V \) and \( h = \mathcal{L}_W(g) \). Since \( W \) satisfies the equation \( \Delta W + D_X W = 0 \), Theorem 4.1 implies that the tensor \( h \) satisfies the Lichnerowicz-type equation
\[ \Delta_L h + \mathcal{L}_X(h) = 0. \]
Moreover, $|h| \leq O(r^{-\epsilon})$. Hence, it follows from Theorem 6.3 that $h = \lambda \text{Ric}$ for some constant $\lambda \in \mathbb{R}$. Therefore, the vector field $\hat{U} := U - V - \frac{1}{2} \lambda X$ is a Killing vector field. The relation $\mathcal{L}_U(g) = 0$ implies that $\Delta \hat{U} + \text{Ric}(\hat{U}) = 0$. On the other hand, we have $\Delta \hat{U} + D_X \hat{U} = 0$ by definition of $V$. Thus, we conclude that $[\hat{U}, X] = \text{Ric}(\hat{U}) - D_X \hat{U} = 0$. Finally, since $\hat{U}$ is a Killing vector field, we have

$$D^2(\mathcal{L}_{\hat{U}}(f)) = \mathcal{L}_{\hat{U}}(D^2 f) = \frac{1}{2} \mathcal{L}_{\hat{U}}(\mathcal{L}_X(g)) = \frac{1}{2} \mathcal{L}_X(\mathcal{L}_{\hat{U}}(g)) = 0.$$ 

Consequently, the function $\mathcal{L}_{\hat{U}}(f) = \langle \hat{U}, X \rangle$ is constant. Since $X$ vanishes at the point where $f$ attains its minimum, we conclude that the function $\langle \hat{U}, X \rangle$ vanishes identically. This completes the proof of Theorem 7.1.

If we apply Theorem 7.1 to the vector fields $U_1, U_2, U_3$ constructed in Proposition 3.2, we can draw the following conclusion:

**Corollary 7.2.** We can find vector fields $\hat{U}_1, \hat{U}_2, \hat{U}_3$ on $(M, g)$ such that $\mathcal{L}_{\hat{U}_a}(g) = 0$, $[\hat{U}_a, X] = 0$, and $\langle \hat{U}_a, X \rangle = 0$. Moreover, we have

$$\sum_{a=1}^3 \hat{U}_a \otimes \hat{U}_a = r (e_1 \otimes e_1 + e_2 \otimes e_2 + O(r^{-\epsilon})), $$

where $\{e_1, e_2\}$ is a local orthonormal frame on the level set $\{f = r\}$.

In particular, we have $\text{span}\{\hat{U}_1, \hat{U}_2, \hat{U}_3\} = \text{span}\{e_1, e_2\}$ at each point in $M$ which is sufficiently far out near infinity. This shows that $(M, g)$ is exactly rotationally symmetric near infinity. From this, Theorem 1.1 follows easily.

**Appendix A. The eigenvalues of some elliptic operators on $S^2$**

In this section, we collect some well-known results concerning the eigenvalues of certain elliptic operators on $S^2$. In the following, $g_{S^2}$ will denote the standard metric on $S^2$ with constant Gaussian curvature 1.

**Proposition A.1.** Let $\sigma$ be a one-form on $S^2$ satisfying

$$\Delta_{S^2} \sigma + \mu \sigma = 0,$$

where $\Delta_{S^2}$ denotes the rough Laplacian and $\mu \in (-\infty, 1)$ is a constant. Then $\sigma = 0$.

**Proof.** We can find a real-valued function $\alpha$ and a two-form $\omega$ such that $\sigma = d\alpha + d^*\omega$. Using the Bochner formula for one-forms, we obtain

$$0 = \Delta_{S^2} \sigma + \mu \sigma = -dd^* \sigma - d^* d\sigma + (\mu + 1) \sigma = -dd^* d\alpha - d^* dd^* \omega + (\mu + 1) (d\alpha + d^* \omega) = d(\Delta_{S^2} \alpha + (\mu + 1) \alpha) + d^* (\Delta_{S^2} \omega + (\mu + 1) \omega).$$
Consequently, the function $\Delta_{S^2}\alpha + (\mu + 1)\alpha$ is constant, and the two-form $\Delta_{S^2}\omega + (\mu + 1)\omega$ is a constant multiple of the volume form. Since $\mu + 1 < 2$, we conclude that $\alpha$ is constant and $\omega$ is a constant multiple of the volume form. Thus, $\sigma = 0$, as claimed.

**Proposition A.2.** Let $\chi$ be a symmetric $(0,2)$-tensor on $S^2$ satisfying

$$\Delta_{S^2}\chi - 4\hat{\chi} + \mu\chi = 0,$$

where $\hat{\chi}$ denotes the trace-free part of $\chi$ and $\mu \in (-\infty, 2)$ is a constant. Then $\chi$ is a constant multiple of $g_{S^2}$.

**Proof.** The trace of $\chi$ satisfies

$$\Delta_{S^2}(\text{tr } \chi) + \mu (\text{tr } \chi) = 0.$$

Since $\mu < 2$, we conclude that $\text{tr } \chi$ is constant. Moreover, the trace-free part of $\chi$ satisfies

$$\Delta_{S^2}\hat{\chi} + (\mu - 4)\hat{\chi} = 0.$$

Since $\mu - 4 < 0$, it follows that $\hat{\chi} = 0$. Putting these facts together, the assertion follows.

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