INDECOMPOSABLE PURE-INJECTIVE OBJECTS IN STABLE CATEGORIES OF GORENSTEIN-PROJECTIVE MODULES OVER GORENSTEIN ORDERS

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Abstract. We give a result of Auslander–Ringel–Tachikawa type for Gorenstein-projective modules over a complete Gorenstein order. In particular, we prove that a complete Gorenstein order is of finite Cohen–Macaulay representation type if and only if every indecomposable pure-injective object in the stable category of Gorenstein-projective modules is compact.

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1. Introduction

Ringel and Tachikawa [RT74] proved that if an Artin algebra $A$ is of finite representation type, then every $A$-module is a direct sum of finitely generated $A$-modules. The converse implication also holds due to Auslander [Aus76], where he further proved that an Artin algebra $A$ is of finite representation type if and only if every indecomposable $A$-module is finitely generated. This equivalence, which characterizes infinite representation type by the existence of infinitely generated indecomposable modules, makes us pay attention to the class of large (i.e., possibly infinitely generated) indecomposable modules. However, this class is in general too huge to talk about representation theory, because it can be a proper class even for a finite-dimensional algebra ([Sim05, Theorem 3.1]). This problem can be avoided if we focus on the indecomposable pure-injective modules.

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Pure-injectivity originally appeared as *algebraic compactness*, which is a property of modules over a ring to have solutions for systems of linear equations; see [Fuc70, Chapter VII] and [JL89, Chapter 7] for example. Ziegler discovered in his model theoretic work [Zie84] that the isomorphism classes of indecomposable algebraically compact modules form a small set endowed with a natural topology. This topological space is called the *Ziegler spectrum* of the ring.

It is well known that every finitely generated modules over an Artin algebra $A$ is algebraically compact (i.e., pure-injective), and hence the Ziegler spectrum contains, up to isomorphism, all indecomposable finitely generated modules. In this sense, the Ziegler spectrum is a reasonable place where we can discuss representation theory of algebras including large indecomposable modules. Furthermore, giving another proof for the result of Auslander–Ringel–Tachikawa, model theory of modules provides a refined statement:

**Theorem 1.1.** Let $A$ be an Artin algebra. The following conditions are equivalent.

1. $A$ is of finite representation type.
2. Every right $A$-module is a direct sum of finitely generated right $A$-modules.
3. Every indecomposable pure-injective right $A$-module is finitely generated.

If “pure-injective” is removed from (1), this theorem is precisely due to Auslander–Ringel–Tachikawa. One can find a proof of Theorem 1.1 in [Pre09, §5.3.4], where topological data of the Ziegler spectrum plays an important role.

Among indecomposable large pure-injective modules, *generic* modules have been well studied. They are infinitely generated indecomposable modules of finite endolength, and this notion was introduced by Crawley-Boevey [CB91]. His work characterizes when finite-dimensional algebra over algebraically closed fields have tame representation type in terms of generic modules. Furthermore, generic modules are closely related to tubes in Auslander–Reiten quivers; see Krause [Kra98] and Crawley-Boevey [CB98].

Ringel [Rin00, Rin11] introduced the notion of *Auslander–Reiten quilts* for 1-domestic special biserial algebras. This compactifies and sews connected components of Auslander–Reiten quivers of such algebras by adding infinitely generated indecomposable pure-injective modules. In consequence, we obtain topological pictures, which yield a better understanding of relationship among the connected components. Puninski [Pun18] explained that a similar procedure is possible for a plane curve singularity of type $A_{\infty}$, classifying all indecomposable pure-injective “infinitely generated Cohen–Macaulay modules” up to isomorphism, where he considered a (possibly infinitely generated) module to be Cohen–Macaulay if it does not admit any nonzero morphism from the residue field. He and Los [LP19] further classified such indecomposable pure-injective infinitely generated modules for a plane curve singularity of type $D_{\infty}$. Sadly, Puninski passed away before that paper was published.

For the singularities considered in [Pun18] and [LP19], the Auslander–Reiten quivers of finitely generated Cohen–Macaulay (CM) modules had been well known and they are of countable CM representation type. This fact might have been sufficient for Puninski to start his attempt, while there was no result ensuring the existence of indecomposable infinitely generated CM modules. We want to dispel this uncertain situation to continue the interesting research of infinitely generated CM representations, under a firm foundation. Towards this purpose, we prove the following result, which is a Gorenstein-projective analogue to Theorem 1.1.

**Theorem 1.2.** Let $R$ be a complete CM local ring and $A$ a Gorenstein $R$-order. The following conditions are equivalent:

1. $A$ is of finite CM representation type.
2. Every Gorenstein-projective right $A$-module is a direct sum of finitely generated Gorenstein-projective right $A$-modules.
3. Every indecomposable pure-injective object in the stable category $\overline{\text{GProj} \ A}$ of Gorenstein-projective right $A$-modules is compact.

Here, an $R$-order means a module-finite $R$-algebra which is a nonzero maximal CM $R$-module. The Gorenstein $R$-order $A$ is an $R$-order such that $\text{Hom}_{R}(A, \omega_{R})$ is projective as a right $A$-module, where $\omega_{R}$ stands for a canonical module of $R$. 

Let us first compare the above theorem with existing results. The equivalence \((1) \iff (2)\) in Theorem 1.2 is a partial generalization of [Bel11, Theorem 4.20] due to Beligiannis, whose result treats a complete Gorenstein local ring. It is also possible to deduce the equivalence \((1) \iff (2)\) from recent work by Psaroudakis and Rump [PR22, Corollary 5.14] where they extended Chen’s result [Che08, Main theorem] to a wide class of Iwanaga–Gorenstein rings satisfying some conditions. The result of Chen shows that an Iwanaga–Gorenstein Artin algebra has up to isomorphism only finitely many indecomposable Gorenstein-projective modules if and only if the condition \((2)\) holds. The Gorenstein \(R\)-order \(A\) in Theorem 1.2 is Iwanaga–Gorenstein, but in general an \(R\)-order being Iwanaga–Gorenstein may not be Gorenstein in our sense. Considering Gorenstein \(R\)-orders is important for our purpose so that we can identify the finitely generated Gorenstein-projective \(A\)-modules with the maximal CM \(A\)-modules; see (2.24). If \(R\) in Theorem 1.2 is artinian, the Gorenstein \(R\)-order \(A\) is nothing but a quasi-Frobenius \(R\)-algebra, so Theorem 1.2 follows from Theorem 1.1 and Krause’s [Kra00, Proposition 1.16]. In the case where \(R\) is not artinian, Theorem 1.2(3) is actually a new characterization for \(A\) to have finite CM representation type (even if \(R = A\)).

We next explain an outline of the proof of Theorem 1.2. Our approach for the implication \((1) \Rightarrow (2)\) of Theorem 1.2 is similar to Beligiannis’s idea, but we adopt a slightly different way, which would be simpler and closer to a classical argument; cf. [Pre09, Theorems 4.5.4 and 4.5.7 and Proposition 4.5.6]. The implication \((2) \Rightarrow (3)\) is almost trivial once we know what the compact objects in \(\text{GProj} \ A\); see Theorem 2.3. A difficult part is to prove the implication \((3) \Rightarrow (1)\). In general, the Ziegler spectrum of a ring is quasi-compact as a topological space ([Pre09, Corollary 5.123]), and this fact is used (in the proof of Theorem 1.1 by [Pre09, Theorem 5.3.40]) to find a infinitely generated indecomposable pure-injective module over an Artin algebra of infinite representation type. However, there is no reason that the Ziegler spectrum of the stable category \(\text{GProj} \ A\) is quasi-compact. We handle this problem by using Propositions 3.3 and A.4 and Theorem A.2. It is also non-trivial that the condition \((3)\) makes \(A\) have at most an isolated singularity. This is done in Proposition 4.1.

Lastly we make a remark which relates the present work to balanced big CM modules.

**Remark 1.3.** Recall that a module \(M\) over a commutative noetherian local ring \((R, m)\) is called balanced big CM if every system of parameters of \(R\) is an \(M\)-regular sequence ([BH98, §8.5]). Let \(R\) and \(A\) be as in Theorem 1.2 and define a balanced big CM right \(A\)-module as a right \(A\)-module which is balanced big CM over \(R\). The author of the present paper will show in his subsequent work [Nak] that each of \((1)–(3)\) in Theorem 1.2 is further equivalent to the following condition:

\[(\ast)\] Every indecomposable pure-injective balanced big CM right \(A\)-module is finitely generated.

In other words, \(A\) has infinite CM representation type if and only if there exists an indecomposable pure-injective balanced big CM \(A\)-module that is infinitely generated. Theorem 1.2 is one of crucial steps to give this characterization. Moreover, we can regarded \((\ast)\) as a CM analogue to Theorem 1.2(3).

The present paper will not focus on balanced big CM modules, but we refer to some recent results about balanced big CM modules over orders. For a complete CM local ring \((R, m)\), Bahlekeh, Fotouhi, and Salarian [BFS19, Theorems 6.7] showed that \(R\) is of finite CM representation type if and only if every balanced big CM module having an \(m\)-primary cohomological annihilator (see [BFS19, p. 1675]) decomposes into a direct sums of finitely generated ones (i.e., small CM modules). Using this fact, they also recovered the equivalence \((1) \iff (2)\) of Theorem 1.2 for a complete Gorenstein local ring due to Beligiannis; see [BFS19, Theorems 6.8]. On the other hand, Psaroudakis and Rump proved [PR22, Theorem 5.18] that an \(R\)-order \(A\) for a complete regular local ring \(R\) is of finite CM representation type if and only if every accessible balanced big CM \(A\)-module (see [PR22, p. 20 and Definition 5.6]) decomposes into a direct sum of finitely generated ones.

2. Preliminaries

In this section, we collect various facts to prove the main theorem (Theorem 1.2).
Throughout this paper, a ring $A$ will always mean an associative ring with identity, and an $A$-module will mean a right $A$-module unless otherwise specified. Left $A$-modules are interpreted as right modules over the opposite ring $A^{op}$.

2.1. Gorenstein-projective modules. Let $A$ be a ring. We denote by $\text{Proj} A$ (resp. $\text{proj} A$) the category of projective (resp. finitely generated projective) $A$-modules. A complex $X = (\cdots \to X^1 \to X^0 \to 0 \to \cdots)$ of projective $A$-modules is said to be \textit{totally acyclic} if it is acyclic (i.e., $H^i X = 0$ for all $i \in \mathbb{Z}$) and the complex $\text{Hom}_A(X, P)$ is acyclic for any projective $A$-module $P$. An $A$-module $M$ is said to be \textit{Gorenstein-projective} if there exists a totally acyclic complex $X$ of projective $A$-modules such that $M = Z^0 X := \text{Ker}(X^0 \to X^1)$. If this is the case, $X$ is called a \textit{complete resolution} of $M$.

Assume that $A$ is a (two-sided) coherent ring and $M$ is a Gorenstein-projective $A$-module that is finitely presented. Then $M$ admits a complete resolution by finitely generated projective $A$-modules (see [EJ11 Proposition 10.2.6] and its preceding paragraphs). Denote by $G\text{Proj} A$ (resp. $G\text{proj} A$) the category of Gorenstein-projective (resp. finitely generated Gorenstein-projective) $A$-modules. It is well known that $G\text{Proj} A$ (resp. $G\text{proj} A$) is a Frobenius category whose projective-injective objects are the projective (resp. finitely generated projective) $A$-modules; see, e.g., [Che11 Proposition 3.1] and p. 207] or [Per16 Corollary 11.2.6]. Hence the stable category $G\text{Proj} A$ (resp. $G\text{proj} A$) of $G\text{Proj} A$ (resp. $G\text{proj} A$) are triangulated ([Hap88 Chapter I, Theorem 2.6]). Note that $G\text{Proj} A$ (resp. $G\text{proj} A$) is the stable category associated with $G\text{Proj} A$ (resp. $G\text{proj} A$) in the sense of [Hap88 Chapter I, §2.2].

Since every morphism from a finitely presented $A$-module to a projective $A$-module factors through a finitely generated projective $A$-module, there is a fully faithful functor $G\text{Proj} A \to G\text{Proj} A$ given by $M \mapsto M$. Each (distinguished) triangle in $G\text{Proj} A$ (resp. $G\text{proj} A$) essentially arise from a short exact sequence in $G\text{Proj} A$ (resp. $G\text{proj} A$) ([Hap88 SS2.5 and 2.7]), and so the canonical functor $G\text{Proj} A \to G\text{Proj} A$ is triangulated ([Hap88 §2.8]). Then we can regard $G\text{Proj} A$ as a triangulated full subcategory of $G\text{Proj} A$.

Denote by $K_{\text{ac}}(\text{Proj} A)$ (resp. $K_{\text{ac}}(\text{proj} A)$) the homotopy category of totally acyclic complexes of projective (resp. finitely generated projective) $A$-modules. It is well known that taking complete resolutions by projective $A$-modules yields a triangulated equivalence $G\text{Proj} A \xrightarrow{\sim} K_{\text{ac}}(\text{Proj} A)$ whose quasi-inverse is induced by $Z^0$ (cf. [Buc89 Theorem 4.4.1(1)] and [Kra05b Proposition 7.2]). This triangulated equivalence restricts to a triangulated equivalence $G\text{proj} A \xrightarrow{\sim} K_{\text{ac}}(\text{proj} A)$ as $A$ is coherent.

Denote by $K_{\text{ac}}(\text{Proj} A)$ (resp. $K_{\text{ac}}(\text{proj} A)$) the homotopy category of acyclic complexes of projective (resp. finitely generated projective) $A$-modules. Assume that $A$ is Iwanaga–Gorenstein, i.e., $A$ is a (two-sided) noetherian ring such that it has finite injective dimension as a left and ring $A$-module. This assumption implies that every projective $A$-module has finite injective dimension ([EJ11 Theorem 9.1.10]), so we have $K_{\text{ac}}(\text{Proj} A) = K_{\text{ac}}(\text{Proj} A)$ and $K_{\text{ac}}(\text{proj} A) = K_{\text{ac}}(\text{proj} A)$ by a standard argument (cf. [Kra05b Example 7.15], [MS11 Corollary 4.28], and [Che11 Lemma 4.3]). Consequently, using the canonical functors $G\text{proj} A \xrightarrow{\sim} G\text{Proj} A$ and $K_{\text{ac}}(\text{proj} A) \xrightarrow{\sim} K_{\text{ac}}(\text{Proj} A)$, we obtain the following quasi-commutative diagram:

\[
\begin{array}{ccc}
  K_{\text{ac}}(\text{Proj} A) & \xrightarrow{\sim} & G\text{Proj} A \\
  \downarrow & & \downarrow \\
  K_{\text{ac}}(\text{proj} A) & \xrightarrow{\sim} & G\text{proj} A
\end{array}
\]

(2.1)

**Remark 2.2.** By Jørgensen [Jor05 Theorem 2.4], the homotopy category $K(\text{Proj} A)$ is compactly generated if $A$ is a coherent ring and every flat right $A$-module has finite projective dimension. More generally, Neeman [Nee08 Theorem 1.1] shows that $K(\text{Proj} A)$ is compactly generated whenever $A$ is a left coherent ring. It is well known to experts that the compact generation of $K(\text{Proj} A)$ implies that $K(\text{Proj} A)$ is compactly generated; see [Mur07 Corollary 5.17].

When $A$ is an Iwanaga–Gorenstein ring, every flat right $A$-module has finite projective dimension by [EJ11 Proposition 9.1.2]. Hence the compact generation of $K(\text{Proj} A)$ can be deduced from [Jor05 Theorem 2.4] (see [Che11 Lemma 4.2]). Moreover, in this case, $K_{\text{ac}}(\text{Proj} A)$ is triangulated equivalent
to $\text{GProj}A$ as explained above. Therefore $\text{GProj}A$ is compactly generated. This fact is shown in [Che11] Theorem 4.1; more precisely, it shows the following:

**Theorem 2.3.** Let $A$ be an Iwanaga–Gorenstein ring. The triangulated category $\text{GProj}A$ is compactly generated, and each compact object in $\text{GProj}A$ is a direct summand of some object in the essential image of the canonical functor $\text{GProj}A \rightarrow \text{GProj}A$.

For completing the proof of the main theorem, the reader may assume that all rings in this paper are Iwanaga–Gorenstein, although we will not do this so that we can see what is essential in each result.

2.2. Pure-injectivity and functors commuting with small direct products. A morphism $f : M \rightarrow N$ of right modules over a ring $A$ is said to be a pure monomorphism if the induced morphism $L \otimes_A f$ is a monomorphism for every left $A$-module $L$, or equivalently, if $f$ induces a short exact sequence $0 \rightarrow \text{Hom}_A(F, M) \rightarrow \text{Hom}_A(F, N) \rightarrow \text{Hom}_A(F, \text{Coker } f) \rightarrow 0$ for every finitely presented $A$-module $F$ (see [JL89] Theorem 6.4]). An $A$-module $P$ is said to be pure-injective if the functor $\text{Hom}_A(-, P)$ sends every pure monomorphism $M \rightarrow N$ of $A$-modules to an epimorphism $\text{Hom}_A(N, P) \rightarrow \text{Hom}_A(M, P)$. Pure-injectivity of objects in a compactly generated triangulated category is defined analogously ([Bel00], [Kra00]; see Appendix A.1).

Let $\mathcal{A}$ be an additive category, and assume that $\mathcal{A}$ admits small direct sums and small direct products, i.e., for any family $(X_i)_{i \in I}$ of objects $X_i$ in $\mathcal{A}$ with $I$ a small set, the direct sum $\bigoplus_{i \in I} X_i$ and the direct product $\prod_{i \in I} X_i$ exist in $\mathcal{A}$. Given an object $X \in \mathcal{A}$ and a small set $I$, we write $X^{(I)} := \bigoplus_{i \in I} X_i$ and $X^I := \prod_{i \in I} X_i$, setting $X_i := X$ for each $i \in I$. The summation morphism $X^{(I)} \rightarrow X$ is defined to be the morphism induced by the identity $X_i := X \rightarrow X$ for each $i \in I$.

A module $M$ over a ring is pure-injective if and only if the summation morphism $M^{(I)} \rightarrow M$ factors through the canonical morphism $M^{(I)} \rightarrow M^I$ (see [JL89] Theorem 7.1]). The same characterization of pure-injectivity is available in a compactly generated triangulated category; see Theorem A.1.

Let $\mathcal{A}$ be an additive category with small direct sums and small direct products, i.e., $\mathcal{A}$ is an additive category that admits small direct sums and small direct products. Following [SS20] Definition 5.1(1)], we say that an object $X \in \mathcal{A}$ is pure-injective if, for any small set $I$, the summation morphism $X^{(I)} \rightarrow X$ factors through the canonical morphism $X^{(I)} \rightarrow X^I$.

**Proposition 2.4.** Let $\mathcal{A}$ and $\mathcal{B}$ be additive categories with small direct sums and small direct products. Let $G : \mathcal{A} \rightarrow \mathcal{B}$ be an additive functor that commutes with small direct products. If $X$ is a pure-injective object in $\mathcal{A}$, then $GX$ is pure-injective in $\mathcal{B}$.

**Proof.** Let $X$ be a pure-injective object in $\mathcal{A}$. Let $I$ be a small set, $c : X^{(I)} \rightarrow X^I$ the canonical morphism, and $s : X^I \rightarrow X$ the summation morphism. Since $X$ is pure-injective, there exists a morphism $t : X^I \rightarrow X$ making the following diagram commutative:

$$
\begin{array}{ccc}
X^{(I)} & \xrightarrow{c} & X^I \\
\downarrow{s} & & \downarrow{t} \\
X & & 
\end{array}
$$

Since $G$ commutes with small direct products, we have $G(X^{(I)}) = G(X)^I$. For each $j \in I$, let $e_j : X_j \rightarrow \bigoplus_{i \in I} X_i = X^{(I)}$ be the canonical injection, where $X_i := X$. Let $e : G(X)^I \rightarrow G(X^{(I)})$ be the morphism induced by $G(e_j) : GX_j \rightarrow G(X^{(I)})$ for each $i \in I$. Then we have the following commutative diagram:

$$
\begin{array}{ccc}
(GX)^I & \xrightarrow{e} & G(X^{(I)}) \\
\downarrow{G(s)} & & \downarrow{G(c)} \\
GX & & G(X)^I \\
\end{array}
\xrightarrow{G(t)} (GX)^I
$$
By construction, the composition $G(e) \cdot e$ is the canonical morphism $G(X)^{(I)} \to G(X)^{I}$, and the composition $G(x) \cdot e$ is the summation morphism $G(X)^{(I)} \to GX$. Then the above diagram shows that $GX$ is pure-injective in $B$. 

**Corollary 2.5.** Let $A$ and $B$ be additive categories with small direct sums and small direct products. Let $G : A \to B$ be a right adjoint to some additive functor $B \to A$. If $X$ is a pure-injective object in $A$, then $GX$ is pure-injective in $B$.

**Proof.** Since $G$ is a right adjoint, it commutes with small direct products, so the corollary follows from Proposition 3.4. 

We will use this corollary in the proof of Proposition 3.6.

### 2.3. Injective modules and injective dimension over Noether algebras.

Let $R$ be a commutative noetherian ring. Let $A$ be a Noether $R$-algebra $A$, that is, a ring $A$ together with a ring homomorphism $\varphi : R \to A$ such that the image $\varphi(R)$ is contained in the center of $A$ and $A$ is finitely generated as an $R$-module. We call $\varphi$ the structure map of $A$. We denote by $\text{Mod} A$ (resp. $\text{mod} A$) the category of right $A$-modules (resp. finitely generated right $A$-modules). Given a prime ideal $\mathfrak{p}$ of $R$, we say that $M \in \text{Mod} A$ is $\mathfrak{p}$-local if the canonical $A$-homomorphism $M \to M_{\mathfrak{p}} := M \otimes_R R_{\mathfrak{p}}$ is bijective. Moreover, given an ideal $\mathfrak{a}$ of $R$, we say that $M \in \text{Mod} A$ is $\mathfrak{a}$-torsion if the canonical $A$-homomorphism $\Gamma_{\mathfrak{a}}M := \lim_{\to n \geq 1} \text{Hom}_A(R/\mathfrak{a}, M) \to M$ is bijective.

We denote by $\text{Spec} A$ the set of (two-sided) prime ideals of $A$. For each $P \in \text{Spec} A$, $\mathfrak{p} := \varphi^{-1}(P)$ is a prime ideal of $R$, $P_{\mathfrak{p}}$ is a maximal ideal of the Noether $R_{\mathfrak{p}}$-algebra $A_{\mathfrak{p}}$, and $A_{\mathfrak{p}}/P_{\mathfrak{p}}$ decomposes into a finite direct sum of copies of a simple (right) $A_{\mathfrak{p}}$-module $S_\mathfrak{a}(P)$ (see (2.10)). We denote by $I_{\mathfrak{a}}(P)$ the injective envelope of $S_\mathfrak{a}(P)$ in $\text{Mod} A$. Then $I_{\mathfrak{a}}(P)$ is indecomposable as an $A$-module, and it is $\mathfrak{p}$-local and $\mathfrak{a}$-torsion. Moreover, there is a canonical bijection between $\text{Spec} A$ and the set of isomorphism classes of indecomposable injective $A$-modules given by $P \mapsto I_{\mathfrak{a}}(P)$. See [KN22, §§2.3–2.4] for more details.

Since $A$ is noetherian, every injective $A$-module $I$ into a direct sum of indecomposable injective $A$-modules. By the above-mentioned bijection, we have

$$I \cong \bigoplus_{P \in \text{Spec} A} I_{\mathfrak{a}}(P)(C_{\mathfrak{a}})$$

for some family of sets $\{C_{\mathfrak{a}}\}_{P \in \text{Spec} A}$. Then we have

$$\Gamma_{\mathfrak{a}}I \cong \bigoplus_{P \in \text{Spec} A, \mathfrak{a} \subseteq \varphi^{-1}(P)} I_{\mathfrak{a}}(P)(C_{\mathfrak{a}})$$

for an ideal $\mathfrak{a}$ of $R$; see [KN22, Remark 3.4(1)]. If $R$ is local and $\mathfrak{m}$ is the maximal ideal of $R$, then

$$\Gamma_{\mathfrak{m}}I \cong \bigoplus_{P \in \text{Max } A} I_{\mathfrak{m}}(P)(C_{\mathfrak{a}}),$$

where $\text{Max } A$ denotes the set of maximal ideals of $\text{Spec } A$; see [KN22, Lemma 2.12]. Note that $\text{Max } A$ is a finite set when $R$ is local; see [KN22, Propositions 2.14(2) and 2.15].

We also have an isomorphism

$$I \otimes R S^{-1} R \cong \bigoplus_{P \in \text{Spec } A, \varphi^{-1}(P) \cap S = \emptyset} I_{\mathfrak{a}}(P)(C_{\mathfrak{a}})$$

for a multiplicatively closed subset $S$ of $R$; see [KN22, Remark 3.4(2)]. Hence if $S$ is the multiplicatively closed subset generated by an element $x \in R$, then

$$I_{x} \cong \bigoplus_{P \in \text{Spec } A, x \not\in \varphi^{-1}(P)} I_{\mathfrak{a}}(P)(C_{\mathfrak{a}}),$$

where $M_x := M \otimes R S^{-1} R$ for an $A$-module $M$. 

Remark 2.9. Let \((R, \mathfrak{m})\) be a commutative noetherian local ring and \(A\) a Noether \(R\)-algebra. Denote by \(\mathrm{rad} A\) the Jacobson radical of \(A\). Let \(M \in \text{Mod} A\) and let \(f : M \to E_A(M)\) be an injective envelope in \(\text{Mod} A\). We observe that the induced map

\[
\text{Hom}_A(A/\mathrm{rad} A, f) : \text{Hom}_A(A/\mathrm{rad} A, M) \to \text{Hom}_A(A/\mathrm{rad} A, E_A(M))
\]

is an isomorphism. To see this, recall that \(A/\mathrm{rad} A\) decomposes as

\[
(2.10) \quad A/\mathrm{rad} A \cong \bigoplus_{P \in \text{Max } A} S_A(P)^{n_P}
\]

for some integers \(n_P \geq 0\); see [KN22 Proposition 2.19]. Thus it suffices to show that the inclusion \(\text{Hom}_A(S_A(P), f) : \text{Hom}_A(S_A(P), M) \to \text{Hom}_A(S_A(P), E_A(M))\) is surjective for each \(P \in \text{Max } A\). However this is obvious because \(f\) is an essential extension and \(S_A(P)\) is a simple \(A\)-module.

Let \(M\) be an \(A\)-module and let \(E = (0 \to E^0 \to E^1 \to \cdots)\) be a minimal injective resolution of \(M\). By the above observation, we see that the complex \(\text{Hom}_A(A/\mathrm{rad} A, E)\) has zero differential. Moreover, if we write \(E^i = \bigoplus_{P \in \text{Spec } A} I_A(P)^{(C^i_P)}\) for each \(i \geq 0\), then

\[
\text{Hom}_A(A/\mathrm{rad} A, E^i) \cong \text{Hom}_A(A/\mathrm{rad} A, \Gamma_\mathfrak{m} E^i) \cong \bigoplus_{P \in \text{Max } A} I_A(P)^{(C^i_P)}
\]

in \(\text{Mod } A\), where the first isomorphism holds since \(A/\mathrm{rad} A\) is \(\mathfrak{m}\)-torsion (see [KN22 Lemma 2.12 and Remark 2.23]), the second follows from (2.7), and the last follows from [KN22 Proposition 4.8(2)]. Therefore we see that \(\text{Hom}_A(A/\mathrm{rad} A, E^i) = 0\) if and only if \(\Gamma_\mathfrak{m} E^i = 0\).

The isomorphism \(\text{Hom}_A(A/\mathrm{rad} A, E^i) \cong \bigoplus_{P \in \text{Max } A} S_A(P)^{(C^i_P)}\) can be also deduced from [GN02, Lemma 5.1(2)]; its proof implicitly uses the fact that \(\text{Hom}_A(A/\mathrm{rad} A, E)\) has zero differential.

Given an \(A\)-module \(M\), we denote by \(\text{id}_A M\) the injective dimension of \(M\).

Lemma 2.11. Let \(R\) be a commutative noetherian local ring and \(A\) a Noether \(R\)-algebra. Let \(M\) be a nonzero finitely generated \(A\)-module. Then \(\text{id}_A M = \sup \{i \mid \text{Ext}^i_A(A/\mathrm{rad} A, M) \neq 0\}\).

Proof. See [GN02 Corollary 3.5(3)]. \(\square\)

2.4. Local cohomology of modules over Noether algebras. Let \(R\) be a commutative noetherian ring \(R\) and denote by \(\text{D}(R)\) the unbounded derived category of \(R\)-modules. Let \(a\) be an ideal of \(R\) and consider the right derived functor \(R\Gamma_a : \text{D}(R) \to \text{D}(R)\) of the \(a\)-torsion functor \(\Gamma_a : \text{Mod } R \to \text{Mod } R\). The functor \(R\Gamma_a : \text{D}(R) \to \text{Mod } R\) is denoted by \(H^a\) and called the \(i\)th local cohomology functor with respect to \(a\).

Let \(x = x_1, \ldots, x_n\) be a system of generators of \(a\). For each \(x_i\), consider the complex \((R \to R_{x_i})\) concentrated in degrees 0 and 1, where the morphism \(R \to R_{x_i}\) is the localization map. The (extended) Čech complex with respect to \(x\) is defined as \(\hat{C}(x) := \bigotimes_{i=1}^n (R \to R_{x_i})\). For every \(X \in \text{D}(R)\), there is a natural isomorphism

\[
(2.12) \quad R\Gamma_a X \cong \hat{C}(x) \otimes_R X
\]

in \(\text{D}(R);\) see [Lip02 Proposition 3.1.2]. In particular, given an \(R\)-module \(M\), we have \(H^a M = 0\) for every \(i > n\) ([Lip02 Corollary 3.1.4]). If \(a\) and \(b\) are ideals of \(R\) defining the same Zariski closed subset, we have \(\Gamma_a = \Gamma_b\) as functors \(\text{Mod } R \to \text{Mod } R\); see [BS13 Exercise 1.1.3] and [Mat89 p. 3].

Remark 2.13. Let \((R, \mathfrak{m})\) be a commutative noetherian local ring and \(d := \dim R\), where \(\dim R\) stands for the Krull dimension of \(R\). Let \(x = \{x_1, \ldots, x_d\}\) be a system of parameters of \(R\), i.e., \((x_1, \ldots, x_d)\) is of finite length as an \(R\)-module; see [Mat89 §14]. It is elementary that the Zariski closed subset defined by the ideal \((x_1, \ldots, x_d)\) is just \(\{\mathfrak{m}\}\), so that \(\Gamma_\mathfrak{m} = \Gamma(\{x_1, \ldots, x_d\})\). Thus we have
Remark 2.16. Let $(\hat{\cdot})$ be the residue field. Note that has local endomorphism ring by [Pre09, Theorem 4.3.43] (see also the proof of [LW12, Theorem 1.8]).

Lemma 2.17. Let $A$ be a Noether $\Lambda$-algebra. Then every projective $\Lambda$-module is flat as a left and right $\Lambda$-module.

Proof. By (2.12) and (2.13), there is a natural isomorphism in the category of complexes of $\Lambda$-modules (see [Mat89, p. 63, (4)] or [AM69, Proposition 10.16]). Thus we can regard $\Lambda$ as a Noether $\Lambda$-algebra. Now, let $R$ be a commutative noetherian ring and $A$ an ideal of $R$. Given a module $M$ over a Noether $R$-algebra $A$, we say that $M$ is a $\alpha$-complete if the canonical map $M \to \Lambda^\alpha M := \lim_{\kappa \to \alpha} M/\alpha^\kappa M$ is bijective. Hence $\hat{\Lambda}$ is flat over $R$ (see [Mat89, Theorem 8.8]).

Remark 2.16. Let $(\hat{\cdot})$ be a commutative noetherian local ring and $A$ a Noether $R$-algebra. Let $k$ be the residue field $R/m$. By tensor-hom adjunction, $M^\vee := \text{Hom}_R(M, E(k))$ is a pure-injective left (resp. right) $A$-module for any right (resp. left) $A$-module $M$.

Assume $R$ is complete. Notice that then every finitely generated $A$-module is $\alpha$-complete. Hence, for every finitely generated right $A$-module $M$, the canonical $A$-homomorphism $M \to M^{\vee \vee}$ is an isomorphism by Matlis duality ([BH98, Theorem 3.2.13]). Therefore $M \cong \text{Hom}_R(M^\vee, E(R/m))$ is pure-injective as a right $A$-module. It follows that every indecomposable finitely generated $A$-module has local endomorphism ring by [Pre09, Theorem 4.3.43] (see also the proof of [LW12, Theorem 1.8]). As a consequence, $\text{mod} A$ is a Krull–Schmidt category (see [Kra15, §4]).

Lemma 2.17. Let $R$ be a commutative noetherian complete local ring and $A$ a Noether $R$-algebra. Then every projective $A$-module is a direct sum of indecomposable (finitely generated) projective right $A$-modules.

Proof. Let $F$ be an arbitrary projective $A$-module. Then $F$ is a direct summand of some free $A$-module $A^{(I)}$. By Remark 2.16, $A$ decomposes into a finite direct sum $\bigoplus_{1 \leq i \leq n} P_i$ of indecomposable projective $A$-modules $P_i$ with local endomorphism ring. Thus $A^{(I)} = \bigoplus_{1 \leq i \leq n} P_i^{(I)}$. It follows from [War69, Theorem 1] (see also [Pre09, Theorem E.1.26]) that $F$ is isomorphic to a direct sum of copies of $P_i$ for various $i$ with $1 \leq i \leq n$.

In fact, if $A$ is as in Lemma 2.17, there is an isomorphism $A \cong \bigoplus_{P \in \text{Max} A} T_A(P)^{n_P}$ with $T_A(P) := I_{A^{(P)}}(P)^\vee$, where $n_P$ is the number appearing in (2.15); see [KN22, Proposition 5.2].
2.6. Complexes of flat modules. Let $A$ be a left coherent ring for a while. Then $K(\text{Proj} A)$ is compactly generated (Remark 2.2). Since the inclusion functor $i : K(\text{Proj} A) \to K(\text{Flat} A)$ commutes with small direct sums, it has a right adjoint

$$\gamma : K(\text{Flat} A) \to K(\text{Proj} A)$$

by the Brown representation theorem; see, e.g., [Kra10] Theorem 5.1.1. It follows from [Nee08] Facts 2.14 and a standard argument of Bousfield localization (see, e.g., [Kra10] Proposition 4.9.1) that the counit morphism $i \gamma X \to X$ for each $X \in K(\text{Flat} A)$ induces a triangle

$$(2.18) \quad i \gamma X \to X \to Y \to \Sigma i \gamma X$$

in $K(\text{Flat} A)$ such that $Y$ is a pure acyclic complex of flat right $A$-modules. Recall that a complex $Z$ of right $A$-modules is said to be pure acyclic if $Z \otimes_A L$ is acyclic for any left $A$-module $L$. If $X$ is a complex $X$ of flat right $A$-modules, then it is pure acyclic if and only if all the cycle modules $Z^i X = \text{Ker}(X^i \to X^{i+1})$ are flat. Moreover, $\gamma X = 0$ (i.e., $i \gamma X = 0$) if and only if $X$ is pure acyclic by (2.18); see [Kra10] Propositions 4.9.1, 4.10.1, and 4.12.1.

Now let $R$ be a commutative noetherian ring and $A$ Noether $R$-algebra with structure map $\varphi : R \to A$. Let $p \in \text{Spec} R$. We write $R_p := A^p(R_p)$ and $\hat{A}_p := A^p(A_p)$. Recall that the canonical ring homomorphism $R \to R_p \to \hat{R}_p$ is flat, and there is a natural isomorphism $\hat{R}_p \otimes_R A_p \cong \hat{A}_p$. Moreover, $\hat{A}_p$ is a Noether $\hat{R}_p$-algebra with structure map $A^p(\varphi \otimes R_p) : R_p \to \hat{A}_p$.

Let $\psi : A \to \hat{A}_p$ be the canonical ring homomorphism. We remark that $\hat{A}_p$ is flat as a left and right $A$-module. Consider the triangulated functor

$$- \otimes_A \hat{A}_p : K_{\text{ac}}(\text{Proj} A) \to K_{\text{ac}}(\text{Proj} \hat{A}_p).$$

This commutes with small direct sums, and $K_{\text{ac}}(\text{Proj} \hat{A}_p)$ is compactly generated (Remark 2.2). Hence the functor has a right adjoint

$$\rho : K_{\text{ac}}(\text{Proj} \hat{A}_p) \to K_{\text{ac}}(\text{Proj} A)$$

by the Brown representation theorem.

**Lemma 2.19.** Let $X$ and $Y$ be acyclic complexes of finitely generated projective $\hat{A}_p$-modules. Then the canonical homomorphism

$$\text{Hom}_{K(\text{Proj} \hat{A}_p)}(X, Y) \to \text{Hom}_{K(\text{Proj} A)}(\rho X, \rho Y)$$

is bijective.

To prove this lemma, we make a remark.

**Remark 2.20.** Since $\hat{A}_p$ is flat as a left and right $A$-module, we have the following triangulated functors:

$$\psi^* = - \otimes_A \hat{A}_p : K_{\text{ac}}(\text{Flat} A) \to K_{\text{ac}}(\text{Flat} \hat{A}_p),$$

$$\psi_* = \text{Hom}_{\hat{A}_p}(\hat{A}_p, -) : K_{\text{ac}}(\text{Flat} \hat{A}_p) \to K_{\text{ac}}(\text{Flat} A).$$

Notice that the scalar restriction functor $\psi_*$ is a right adjoint to the scalar extension functor $\psi^*$. Let $Y \in K_{\text{ac}}(\text{Proj} A) \subseteq K_{\text{ac}}(\text{Flat} A)$ and $Z \in K_{\text{ac}}(\text{Proj} \hat{A}_p) \subseteq K_{\text{ac}}(\text{Flat} \hat{A}_p)$. Then we have natural isomorphisms

$$\text{Hom}_{K(\text{Proj} \hat{A}_p)}(Y \otimes_A \hat{A}_p, Z) = \text{Hom}_{K(\text{Flat} \hat{A}_p)}(\psi^* Y, Z)$$

$$\cong \text{Hom}_{K(\text{Flat} A)}(Y, \psi_* Z)$$

$$= \text{Hom}_{K(\text{Proj} A)}(Y, \gamma \psi_* Z),$$

where $\gamma \psi_* Z \in K_{\text{ac}}(\text{Proj} A)$ by (2.18). By these isomorphisms, we see that the composition

$$K_{\text{ac}}(\text{Proj} \hat{A}_p) \xrightarrow{\psi^*} K_{\text{ac}}(\text{Flat} \hat{A}_p) \xrightarrow{\psi_*} K_{\text{ac}}(\text{Flat} A) \xrightarrow{\gamma} K_{\text{ac}}(\text{Proj} A)$$

is bijective.
is a right adjoint to the functor $- \otimes_A \hat{A}_p : K_{sc}(\text{Proj} A) \to K_{sc}(\text{Proj} \hat{A}_p)$. Hence we may interpret the above composition as $\rho$.

Let us further observe that the canonical homomorphism

$$\text{Hom}_{K(\text{Proj} \hat{A}_p)}(X, Y) = \text{Hom}_{K(\text{Flat} \hat{A}_p)}(X, Y) \to \text{Hom}_{K(\text{Proj} A)}(\psi_* X, \psi_* Y)$$

is bijective for $X, Y \in K(\text{proj} \hat{A}_p)$. It suffices to check that $\text{Hom}_{\hat{A}_p}(M, N) = \text{Hom}_A(M, N)$ for $M, N \in \text{proj} \hat{A}_p$. Since $M$ and $N$ are $p$-local $A$-modules, it easily follows that $\text{Hom}_{\hat{A}_p}(M, N) = \text{Hom}_A(M, N)$ (see, e.g., [KN22, Remark 2.13]). In addition, since $M$ and $N$ are finitely generated as $\hat{A}_p$-modules, these are $p$-complete. Then we have $\text{Hom}_{\hat{A}_p}(M, N) = \text{Hom}_A(M, N)$; see [KN22, Proposition A.14] and its proof. Therefore the desired equality $\text{Hom}_{\hat{A}_p}(M, N) = \text{Hom}_A(M, N)$ follows.

**Proof of Lemma 2.19.** By Remark 2.20, it suffices to show that the canonical homomorphism

$$\text{Hom}_{K(\text{Proj} \hat{A}_p)}(X, Y) \to \text{Hom}_{K(\text{Proj} A)}(\gamma \psi_* X, \gamma \psi_* Y)$$

is bijective. Since $X, Y$ are complexes of finitely generated projective $\hat{A}_p$-modules, the canonical homomorphism $\text{Hom}_{K(\text{Proj} \hat{A}_p)}(X, Y) \to \text{Hom}_{K(\text{Proj} A)}(\psi_* X, \psi_* Y)$ is bijective by the remark again. Hence it remains to show that the canonical homomorphism

$$\text{Hom}_{K(\text{Proj} A)}(\gamma \psi_* X, \gamma \psi_* Y) \cong \text{Hom}_{K(\text{Flat} A)}(\gamma \psi_* X, \psi_* Y).$$

The composition of this isomorphism with (2.21) is the homomorphism

$$\text{Hom}_{K(\text{Proj} A)}(\psi_* X, \psi_* Y) \to \text{Hom}_{K(\text{Flat} A)}(\gamma \psi_* X, \psi_* Y)$$

induced by the counit morphism $\gamma \psi_* X \to \psi_* X$ (see, e.g., [KS05, (1.5.6)]). Thus it suffice to show that (2.22) is bijective. Since the mapping cone of the counit morphism $\gamma \psi_* X \to \psi_* X$ is pure acyclic by (2.16), the proof will be completed once we show $\text{Hom}_{K(\text{Flat} A)}(Z, \psi_* Y) = 0$ for every pure acyclic complex $Z$ of flat $A$-modules. The complex $Y$ consists of finitely generated projective $\hat{A}_p$-modules, so $\psi_* Y$ is a complex of flat cotorsion $A$-modules; see [KN22, Proposition 5.2 and Theorem 5.6]. Then we have $\text{Hom}_{K(\text{Flat} A)}(Z, \psi_* Y) = 0$ by [BCIE20, Theorem 1.3].

The last equality in the proof can be also deduced by a similar argument to [NT20, Remark 3.2] (see also [KN22, Proposition A.9]). Lemma 2.14 will be used in the proof of Proposition 3.6 below.

For the reader’s sake, let us mention that $K(\text{Proj} A)$ is well generated for any ring $A$ by Neeman [Nee08, Theorem 1.1], so the functor $\gamma : K(\text{Flat} A) \to K(\text{Proj} A)$ and the triangle (2.16) for each $X \in K(\text{Flat} A)$ exist. Moreover, by [BCIE20, Theorem 1.3] and (2.18), the canonical homomorphism $\text{Hom}_{K(\text{Flat} A)}(X, Y) \to \text{Hom}_{K(\text{Proj} A)}(\gamma X, Y)$ is bijective whenever $X$ and $Y$ are complexes of flat cotorsion modules. In fact, $\gamma$ induces a triangulated equivalence from the homotopy category of complexes of flat cotorsion modules to the homotopy category of complexes of projective modules; see [Nee08, Theorem 1.2] and [NT20, Remark A.9].

### 2.7. Gorenstein orders and maximal Cohen–Macaulay modules

Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. The **depth** of an $R$-module $M$ is defined as

$$\text{depth}_R M := \text{inf}\{i \mid \text{Ext}_R^i(k, M) \neq 0\}.$$

By definition, $\{i \mid \text{Ext}_R^i(k, M) \neq 0\} = \emptyset$ if and only if $\text{depth}_R M = \infty$. Moreover, we have $\text{depth}_R M = \text{inf}\{i \mid \text{Ext}_R^i(k, M) \neq 0\}$ (see [ILL+07, Theorems 9.1]). Since $\text{H}_m M = 0$ for $i > \dim R$ (Remark 2.14), it holds that $\text{depth}_R M \leq \dim R$ whenever $\text{depth}_R M < \infty$. In particular, the inequality $\text{depth}_R M \leq \dim R$ holds for any non-zero finitely generated $R$-module $M$ (see [ILL+07, Definition 1.46 and Theorem 9.3]).
As in the introduction, we abbreviate Cohen–Macaulay to CM. A **maximal CM $R$-module** is a finitely generated $R$-module $M$ such that
\[
\text{depth}_R M \geq \dim R.
\]
Traditionally, maximal CM modules are assumed to be non-zero, but our definition allows the zero module to be maximal CM. This convention is often used in the literature. If a maximal Cohen–Macaulay $R$-module is non-zero, we call it **small CM**. The terminology comes from Hochster’s “small CM modules conjecture” [Hoc75, Conjecture (6) in p. 10]. Our usage of maximal CM modules and small CM modules is the same as Holm [Hol17].

Let $(R, \mathfrak{m})$ be a CM local ring with a canonical module, that is, $R$ is a commutative noetherian local ring being small CM as an $R$-module and there exists a small $R$-module $\omega_R$ such that $\omega_R$ is a dualizing complex for $R$ (cf. [BH98, Definition 3.3.1] and [Har66, Chapter V, Proposition 3.4]). The injective dimension of $\omega_R$ is $d := \dim R$, and if $I$ is a minimal injective resolution of $\omega_R$, then $I^d \cong E_R(k)$; see, e.g., [BH98, Theorem 3.1.17] and [Har66, Chapter V, Proposition 6.1]. A canonical module is unique up to isomorphism ([BH98, Theorem 3.3.4].)

An **$R$-order** is a Noether $R$-algebra $A$ such that $A$ is small CM as an $R$-module. Given an $R$-order $A$, we call a finitely generated $A$-module **maximal CM** (resp. **small CM**) as an $R$-module. We denote by $\text{CM} A$ the category of maximal $CM$ $A$-modules. There is an exact duality
\[
\text{Hom}_R(-, \omega_R) : (\text{CM} A)^{\text{op}} \cong \text{CM} A^{\text{op}}.
\]
See [Yos90, Corollary 1.13] and [IW14, p. 539].

Let $A$ be an $R$-order, and set $\omega_A := \text{Hom}_R(A, \omega_R)$, which is an $(A, A)$-bimodule whose injective dimension is $d$ as a left and right $A$-module. We have
\[
\text{CM} A = \{ M \in \text{mod } A | \text{Ext}^i_R(M, \omega_R) = 0 \text{ for all } i > 0 \} = \{ M \in \text{mod } A | \text{Ext}^i_A(M, \omega_A) = 0 \text{ for all } i > 0 \},
\]
where the first equality follows from local duality [BH98 Theorem 3.5.8] (see also [BH98, Theorem 3.3.7]). Given an $A$-module $M$, we denote by $\text{add}_A(M)$ the full subcategory of $\text{Mod } A$ formed by summands of finite direct sums of copies of $M$. An $R$-order $A$ is called **Gorenstein** if it satisfies one of the following equivalent conditions ([IW14, Lemma 2.15]):

(i) $\omega_A$ is projective as a right $A$-module.
(ii) $\omega_A$ is projective as a left $A$-module.
(iii) $\text{add}_A(\omega_A) = \text{add}_A(A)$.
(iv) $\text{add}_A^0(\omega_A) = \text{add}_A^0(A^{\text{op}})$.

By definition, the injective dimension of a Gorenstein $R$-order $A$ is $d$ as a left and right $A$-module. In particular, $A$ is Iwanaga–Gorenstein ([EJ11 Definition 9.1.1]). Moreover, $A_p$ is a Gorenstein $R_p$-order for each $p \in \text{Spec } R$, and $\widehat{A}_p$ is a Gorenstein $\widehat{R}_p$-order; these facts follow from [Mat89, Theorems 7.11 and 17.3] and [BH98, Corollary 2.1.8 and Theorem 3.3.5]. Note that $R$ is Gorenstein as an $R$-order if and only if $R$ is a Gorenstein local ring (i.e., a commutative noetherian local ring of finite injective dimension), or equivalently, $R$ is isomorphic to $\omega_R$ ([BH98, Theorem 3.3.7]). In general, an $R$-order $A$ being Iwanaga–Gorenstein may not imply that $A$ is Gorenstein as an $R$-order. Indeed, when $R$ is artinian, an $R$-order is Gorenstein if and only if $A$ is self-injective, that is, $A$ is a quasi-Frobenius $R$-algebra.

If $A$ is a Gorenstein $R$-order, then
\[
\text{CM} A = \{ M \in \text{mod } A | \text{Ext}^i_A(M, A) = 0 \text{ for all } i > 0 \} = \text{Gproj } A
\]
by ([ii]) and [EJ11, Corollary 11.5.3].

**Remark 2.25.** Let $A$ be an $R$-order, and interpret $\text{CM} A$ as an exact category in a canonical way. Then $A$ and $\omega_A$ are projective objects and an injective object in $\text{CM} A$, respectively. If $A$ is Gorenstein, then we have $\text{CM} A = \text{Gproj } A$ as observed above, so the projective objects and the injective objects in $\text{CM} A$ coincide since this holds for the Frobenius category $\text{Gproj } A$. 
Let $A$ be an $R$-order. Given $M,N \in \text{CM} A$, write $P(M,N)$ for the set of morphisms factoring through finitely generated projective $A$-modules. We denote by $\text{CM} A$ the stable category of $CM A$, that is, the objects $\text{CM} A$ are those of $CM A$, and the hom-set $\text{Hom}_{\text{CM} A}(M,N)$ for $M,N \in \text{CM} A$ is defined as $\text{Hom}_A(M,N)/P(M,N)$. If $A$ is a Gorenstein $R$-order, then $CM A = \text{Gproj} A$, so $CM A = \underline{\text{Gproj}} A$ by definition.

Let $A$ be an $R$-order and $M \in \text{mod} A$. Consider a projective resolution $\cdots \to P^{-1} \to P^0 \to M \to 0$ by finitely generated projective $A$-modules $P^i$. Set $\Omega^i M := M$, $\Omega^1 M := \text{Ker}(P^0 \to M)$, and $\Omega^i M := \text{Ker}(P^{i-1} \to P^{-i+1})$ for $i > 0$. Then $\Omega^i M$ is called an $i$th syzygy of $M$, which depends on only $M$, up to projective summands. Since $A$ is small CM as an $R$-module, we have $\text{depth}_R P \geq d$ for every finitely generated projective $A$-module $P$. Thus a standard argument yields $\text{depth}_R \Omega^i M \geq d$ (i.e., $\Omega^i M \in \text{CM} A$) for every $i \geq d$; cf. [Yos90] Proposition 1.16.

**Remark 2.26.** Assume that $A$ is a Gorenstein $R$-order and let $M,N \in \text{CM} A$. Take an epimorphism $f : P \to N$ in $\text{mod} A$ with $P$ projective. Consider the exact sequence $0 \to \Omega^1 N \to P \xrightarrow{f} N \to 0$. This induces an exact sequence

$$0 \to \text{Hom}_A(M,\Omega^1 N) \to \text{Hom}_A(M,P) \to \text{Hom}_A(M,N) \to \text{Ext}^1_A(M,\Omega^1 N) \to 0$$

by (2.23). Since any morphism $F \to N$ from a finitely generated projective $A$-module $F$ factors through $f$, the image of the morphism $\text{Hom}_A(M,P) \to \text{Hom}_A(M,N)$ coincides with $P(M,N)$. Hence there is a canonical isomorphism

$$\text{Hom}_{\text{CM} A}(M,N) \cong \text{Ext}^1_A(M,\Omega^1 N)$$

of $R$-modules. This isomorphism will be used in the proof of Proposition 3.3.

Let $A$ be an $R$-order $A$. We have $\text{id}_A A \geq \dim R$ by (2.15) and [BH98] Theorem 3.5.7(b)]. Hence it also follows that $\text{gld} A \geq \dim R$, where $\text{gld} A$ denotes the global dimension of $A$. Recall that $A$ is called non-singular if $\text{gld} A = \dim R$, or equivalently, if $\text{proj} A = \text{CM} A$ ([IW14] Definition 1.6 and Proposition 2.17]). In particular, every non-singular $R$-order is Gorenstein. Moreover, an $R$-order $A$ is non-singular if and only if $A$ is non-singular as an $R$-order; see, e.g., [IW14] Proposition 2.26. For convenience, we say that $A$ is singular if $\text{gld} A > \dim R$, or equivalently, if $\text{proj} A \subseteq \text{CM} A$.

We remark that an $R$-order $A$ is non-singular if and only if so is $\text{A}^{\text{op}}$. Indeed, given a non-singular $R$-order $A$, we see from (2.23) that every $M \in \text{CM} A^{\text{op}}$ is an injective object in $\text{CM} A^{\text{op}}$. Since $\text{A}^{\text{op}}$ is Gorenstein by definition, we have $\text{CM} A^{\text{op}} = \text{Gproj} A^{\text{op}}$, so the projective objects and the injective objects coincide (Remark 2.25). Then it follows that $\text{proj} A^{\text{op}} = \text{CM} A^{\text{op}}$.

2.8. Auslander–Reiten sequences. Let $C$ be an additive category. A morphism $f : L \to M$ in $C$ is called left almost split if $f$ is not a split monomorphism and given any morphism $a : L \to X$ that is not a split monomorphism, there is a morphism $b : M \to X$ such that $a = bf$. Dually, a morphism $g : M \to N$ in $C$ is called right almost split if $g$ is not a split epimorphism and given any morphism $a : X \to N$ that is not a split epimorphism, there is a morphism $b : X \to M$ such that $a = gb$.

Let $(R, m)$ be a complete CM local ring. Then $R$ admits a canonical module ([BH98, Corollary 3.3.8]). Let $A$ be an $R$-order. Recall that mod $A$ is a Krull–Schmidt category (Remark 2.16). A non-split exact sequence $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ in CM $A$ is called an Auslander–Reiten sequence, or an AR-sequence, if $f$ is left almost split and $N$ is indecomposable, or equivalently, if $g$ is right almost split and $L$ is indecomposable (cf. [ARS97] Chapter V, Proposition 1.14 and [LW12] Exercise 13.32]). If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is an AR-sequence in CM $A$, then we refer to it as an AR-sequence starting from $L$ or an AR-sequence ending in $N$.

**Lemma 2.27.** Let $R$ be a complete CM local ring and $A$ an $R$-order. The following conditions are equivalent.

1. proj $A_p = CM A_p$ for every $p \in \text{Spec} R$ with $p \neq m$.
2. Every indecomposable small CM $A$-module $L$ which is not an injective object in CM $A$ has an AR-sequence starting from $L$.
(3) Every indecomposable small CM $A$-module $N$ which is not a projective object in $\text{CM}A$ has an AR-sequence ending in $N$.

Proof. The equivalence $(1) \Rightarrow (3)$ is essentially shown in [ARS7] Theorem 2.1(a)]. See Remark 2.28(2) below.

Suppose $(3)$ holds. By the equivalence $(1) \Rightarrow (3)$ $A_p$ is non-singular for every $p \not= m$, and hence $A_p^{op}$ is non-singular for every $p \not= m$. Thus $\text{proj} A_p^{op} = \text{CM} A_p^{op}$ for every $p \in \text{Spec} R$ with $p \not= m$, that is, $(1)$ holds for the $R$-order $A^{op}$. Hence $(3)$ holds for $A^{op}$. Then $(2)$ holds for $A$ by the duality $(2.23)$. We have shown $(3) \Rightarrow (2)$. Conversely, suppose $(2)$ holds. Then $(3)$ holds for $A^{op}$ by the duality $(2.23)$. Hence $A^{op}$ is non-singular for $p \in \text{Spec} R$ with $p \not= m$, or equivalently, $A_p$ is non-singular for $p \in \text{Spec} R$ with $p \not= m$. Thus $(1)$ holds for $A$. Then $(3)$ holds for $A$. We have shown $(2) \Rightarrow (3)$.

Remark 2.28. (1) Let $(R, m)$ and $(S, n)$ be commutative noetherian local rings such that $\dim R = \dim S$. Let $f : S \to R$ be a ring homomorphism by which $R$ is a Noether $S$-algebra. Then $R/nR$ is an artinian local ring, so the Zariski closed subset of $\text{Spec} R$ defined by $nR$ is just $\{ m \}$. Hence $\Gamma_{nR} = \Gamma_m$ as functors $\text{Mod} R \to \text{Mod} R$. Let $M$ be an $R$-module $M$, which we can also regard as an $S$-module. Then we have a natural isomorphism $H_n^i M \cong H_n^{iR} M = H^i M$ of $S$-modules for every $i \geq 0$ by the independence theorem of local cohomology; see [BST13, 4.2.1]. It follows from this fact that a finitely generated $R$-module is maximal CM if and only if it is maximal CM over $S$. In fact, the isomorphism $H^i_n M \cong H^i_{nR} M$ can be recovered by $(2.15)$.

(2) [ARS7] Theorem 2.1(a)] treats an order over a complete Gorenstein local ring, but the complete Gorenstein local ring can be replaced by a complete CM local ring. Instead of modifying the proof of [ARS7] Theorem 2.1(a)], we apply a standard technique using trivial extensions. For details about trivial extensions, see, e.g., [LL107] Remark 11.40 and Theorem 11.42. Recall also that every Gorenstein local ring is CM ([BH98, Proposition 3.1.20]).

First, let $R$ be a CM local ring with a canonical module $\omega_R$, $S$ the trivial extension of $R$ by $\omega_R$, and $f : S \to R$ the canonical ring homomorphism which is surjective. It is well known that $S$ is a Gorenstein local ring with $\dim R = \dim S$. Hence, by $(1)$, a finitely generated $S$-module is maximal CM if and only if it is maximal CM over $R$. Next, let $A$ be an $R$-order. Since $f$ is surjective, we have $A_p = A_{f^{-1}(p)}$ for every $p \in \text{Spec} R$, and $A_q = 0$ if $q \in \text{Spec} S$ and $\{ p \in \text{Spec} R \mid f^{-1}(p) = q \} = \emptyset$. Finally, assume $R$ is complete. Then $S$ is complete because there is also a canonical ring homomorphism $R \to S$ by which we can regard $S$ as a finitely generated $R$-module. Then $S$ is a complete Gorenstein local ring and $A$ is an $S$-order. It is easily seen from the above observation that each condition of Lemma 2.27 for the $R$-order $A$ is equivalent to that of this lemma for the $S$-order $A$.

Let $R$ be a CM local ring with a canonical module. An $R$-order $A$ is said to have at most an isolated singularity if, for any $p \in \text{Spec} R$ with $p \not= m$, $A_p$ is non-singular. If, conversely, there exists $p \in \text{Spec} R$ with $p \not= m$ such that $A_p$ is singular, then $A$ is said to have a non-isolated singularity. We say that $\text{CM} A$ has AR-sequences if Lemma 2.27 holds, or equivalently, if Lemma 2.28 holds. The next theorem is well known to experts.

Theorem 2.29. Let $R$ be a complete CM local ring and $A$ an $R$-order. The following conditions are equivalent:

(1) $A$ has at most an isolated singularity.

(2) For all $M, N \in \text{CM} A$, $\text{Hom}_{\text{CM} A} (M, N)$ is of finite length as an $R$-module.

(3) $\text{CM} A$ has AR-sequences.

Proof. The equivalence $(1) \iff (3)$ follows from Lemma 2.27. The equivalence $(1) \iff (2)$ follows from [Aus78, Chapter I, Lemma 7.6(d)].

---

1. This does not mean that the full generality of the independence theorem is covered by $(2.15)$.  
2. “Equidimensional” in the sense of [ARS7] (see [Aus78, p. 80]) is different from that of [Mai89] and the same meaning as “cocoquidimensional” in the sense of [BW14].
Let $R$ be a complete CM local ring and $A$ an $R$-order. We say that $A$ is of \textit{finite CM representation type} if there are, up to isomorphism, only finitely many indecomposable small CM $A$-modules. The next theorem is also well known to experts (cf. \cite{Yos90} Theorem 4.22).

**Theorem 2.30.** Let $R$ be a complete CM local ring and $A$ an $R$-order. If $A$ is of finite CM representation type, then $A$ has at most an isolated singularity.

**Proof.** This is proved in \cite{Aus86} §10, Theorem] assuming $R$ is regular, but the same proof works in our setup. \hfill $\Box$

Lemma 2.24 and Theorem 2.29 are also proved in \cite{Aus86} assuming $R$ is regular. We remark that if $R$ is a regular local ring and $A$ is an $R$-order, then a finitely generated $A$-module is maximal CM if and only if it is free over $R$ (cf. \cite{Yos90} Proposition 1.9)). In particular, $A$ is free over $R$ in this case. It is often but not always possible to replace a given $R$-order $A$ over a complete CM local ring $R$ by an order over a complete regular local ring; see the remark below.

**Remark 2.31.** Let $R$ be a commutative noetherian complete local ring, and assume that $R$ contains a field or $\dim R$ is an integral domain. Then there exists a subring $S$ of $R$ such that $S$ is a complete regular local ring and the injection $f : S \hookrightarrow R$ makes $R$ a Noether $S$-algebra, where $\dim R = \dim S$; see \cite{Mat89} Remark in p. 215 and Theorem 29.4(ii) and \cite{BH98} Corollary A.8].

Let $f$ be as above. Assume that $R$ is CM and let $A$ be an $R$-order. Using $f$, we naturally regard $A$ as a Noether $S$-algebra, and then $A$ is small as an $S$-module by Remark 2.28(1). Hence $A$ is an $S$-order. Notice from Remark 2.28(1) that the maximal CM modules over the $R$-order $A$ coincide with the maximal CM modules over the $S$-order $A$.

In general, there exists a complete Gorenstein local ring such that it cannot be an order over any complete regular local ring. Such an example can be found in \cite{Mat89} Remark in p. 226).

**Remark 2.32.** Let $R$ be a CM local ring with a canonical module and $A$ a Gorenstein $R$-order. Then $\mathcal{CM} A = \mathcal{Gproj} A$ (Remark 2.25), so the stable category $\mathcal{CM} A$ is triangulated and we have the canonical functor $\mathcal{CM} A \to \mathcal{Gproj} A$. If $0 \to L \to M \to N \to 0$ is an exact sequence in $\mathcal{CM} R$, we have a triangle in $\mathcal{CM} A$ of the form $L \to M \to N \to \Sigma L$. Moreover, if the sequence $0 \to L \to M \to N \to 0$ is an AR-sequence, the triangle $L \to M \to N \to \Sigma L$ is an AR-triangle. See Section A.3 for the definition of AR-triangles.

For a compactly generated triangulated category $T$, we denote by $T^c$ the full subcategory of compact objects of $T$.

**Theorem 2.33.** Let $R$ be a complete CM local ring and $A$ a Gorenstein $R$-order. The following statements hold:

1. $\mathcal{GProj} A$ is compactly generated and the canonical functor $\mathcal{CM} A = \mathcal{Gproj} A \to \mathcal{GProj} A$ induces a triangulated equivalence $\mathcal{CM} A \cong (\mathcal{GProj} A)^c$.  
2. Assume that $A$ has at most an isolated singularity. Then every indecomposable object $L \in (\mathcal{GProj} A)^c$ has an AR-triangle starting from $L$.

**Proof.** Since $A$ is Iwanaga–Gorenstein, $\mathcal{GProj} A$ is compactly generated, and we have the fully faithful triangulated functor $\mathcal{CM} A = \mathcal{Gproj} A \hookrightarrow (\mathcal{GProj} A)^c \subseteq \mathcal{GProj} A$. Moreover, given $M \in (\mathcal{GProj} A)^c$, there exists $N \in \mathcal{CM} A$ such that $N \cong M \oplus L$ in $(\mathcal{GProj} A)^c$ by some $L \in (\mathcal{GProj} A)^c$. See Theorem 2.3. We want to show that $L$ belongs to $\mathcal{CM} A$, up to isomorphism. Note that $N$ decomposes into a finite direct sum $\bigoplus_{1 \leq i \leq n} N_i$ of indecomposable objects $N_i \in \mathcal{CM} A$ with local endomorphism ring; see Remark 2.24 below. Thus $\bigoplus_{1 \leq i \leq n} N_i \cong M \oplus L \in \mathcal{GProj} A$. Since $\mathcal{GProj} A$ is a triangulated category with small direct sums, any idempotent morphism in $\mathcal{GProj} A$ splits (see \cite{Nee01} Proposition 1.6.8]). Then we can easily deduce from \cite{LW12} Lemma 1.2] that $M$ is in $(\mathcal{GProj} A)^c$ isomorphic to $\bigoplus_{j \in J} N_j$ for some subset $J \subseteq \{1, \ldots, n\}$. We have shown (1).

(2) follows from (1) Theorem 2.29 and Remark 2.32. \hfill $\Box$
**Remark 2.34.** Let $A$ be as in Theorem 2.33. For an indecomposable module $M \in \text{CM} A$, $\text{End}_A(M)$ is a (possibly noncommutative) local ring by Remark 2.13. Furthermore, the canonical ring homomorphism $\text{End}_A(M) \to \text{End}_{\text{CM} A}(M)$ is surjective, so $\text{End}_{\text{CM} A}(M)$ is a local ring whenever $M$ is not a projective $A$-module. Since the objects of $\text{CM} A$ are those of $\text{CM} A$ and the canonical functor $\text{CM} A \to \text{CM} A$ is additive, given a nonzero object $N \in \text{CM} A$, it can be in $\text{CM} A$ decomposed as a finite direct sum $N = \oplus_{1 \leq i \leq n} N_i$ of indecomposable modules $N_i$ in $\text{CM} A$, and then we have $N = \oplus_{1 \leq i \leq n} N_i$ in $\text{CM} A$, where each $N_i \in \text{CM} A$ has local endomorphism ring. In particular, $\text{CM} A$ is a Krull–Schmidt category.

3. PROOF OF THE MAIN THEOREM

We start with the following lemma.

**Lemma 3.1.** Let $R$ be a commutative noetherian local ring and $A$ a Noether $R$-algebra. Let $M$ be a nonzero finitely generated $A$-module. Assume that $\text{Ext}^n_A(A/\text{rad} A, M) = 0$ for some $n \geq \dim R$. Then $\text{id}_A M \leq n - 1$.

**Proof.** Let $E = (0 \to E^0 \to E^1 \to \cdots)$ be a minimal injective resolution of $M$ over $A$. We have a canonical isomorphism $R\Gamma_m M \cong \Gamma_m E$ in $D(R)$ by (2.13). Since $H^i_m M = 0$ for every $i > d := \dim R$ (Remark 2.13), the sequence

$$
\Gamma_m E^d \to \Gamma_m E^{d+1} \to \Gamma_m E^{d+2} \to \cdots
$$

is exact, where each $\Gamma_m E^i$ is injective by (2.7). Furthermore, $\Gamma_m E^n = 0$ as $\text{Hom}_A(A/\text{rad} A, E^n) = \text{Ext}^n_A(A/\text{rad} A, M) = 0$ by assumption and Remark 2.9. Since $n \geq d$, we obtain the exact sequence

$$
0 \to \Gamma_m E^{n+1} \to \Gamma_m E^{n+2} \to \cdots,
$$

which splits in $\text{Mod} A$. Thus the induced complex

$$
0 \to \text{Hom}_A(A/\text{rad} A, \Gamma_m E^{n+1}) \to \text{Hom}_A(A/\text{rad} A, \Gamma_m E^{n+2}) \to \cdots
$$

splits as well. However, this complex has zero differential, because it is a truncated complex of $\text{Hom}_A(A/\text{rad} A, \Gamma_m E) \cong \text{Hom}_A(A/\text{rad} A, E)$; see Remark 2.9. It follows that $\text{Ext}^i_A(A/\text{rad} A, M) = \text{Hom}_A(A/\text{rad} A, E^n) = 0$ for every $i > n$. Since $\text{Ext}^i_A(A/\text{rad} A, M) = 0$ by assumption, we have

$$
\sup \{i \mid \text{Ext}^i_A(A/\text{rad} A, M) \neq 0 \} \leq n - 1.
$$

Then $\text{id}_A M \leq n - 1$ by Lemma 2.11.

**Remark 3.2.** Let $R$, $A$, $M$, and $n$ be as in Lemma 3.1. Then $n$ cannot be zero. Indeed, if $n$ is zero, then $d$ must be zero, so $A$ is an Artin $R$-algebra; in this case the equality $\text{Ext}^n_A(A/\text{rad} A, M) = 0$ implies $M = 0$ (see Remark 2.9), but this contradicts the assumption that $M$ is nonzero.

On the other hand, if $A$ is projective as an $R$-module, then $R$ is CM and $n$ is greater than $\dim R$. This fact follows from the validity of “Bass conjecture” (see [BH98, Theorem 3.1.17, Corollary 9.6.2, and Remark 9.6.4(ii)]) because every injective right $A$-module is an injective $R$-module by the standard isomorphism $\text{Hom}_A(\sim \otimes_R A, I) \cong \text{Hom}_R(\sim, \text{Hom}_A(A, I))$. When $A$ is not projective as an $R$-module, it is unclear if the same fact on $R$ and $n$ holds or not; cf. [GN02, Question 3.11].

The author obtained the following result thanks to a suggestion by Ryo Takahashi.

**Proposition 3.3.** Let $R$ be a CM local ring with a canonical module and $A$ a Gorenstein $R$-order. Set $d := \dim R$. Then $\text{Hom}_{\text{CM} A}(\Omega^d(A/\text{rad} A), N) \neq 0$ for every nonzero object $N \in \text{CM} A$.

**Proof.** Let $0 \neq N \in \text{CM} A$, and suppose that $\text{Hom}_{\text{CM} A}(\Omega^d(A/\text{rad} A), N) = 0$. By Remark 2.26, we have $\text{Ext}^i_A(\Omega^d(A/\text{rad} A), \Omega^1 N) = 0$. In addition, there are isomorphisms

$$
\text{Ext}^i_A(\Omega^d(A/\text{rad} A), \Omega^1 N) \cong \text{Ext}^i_A(\Omega^{d-1}(A/\text{rad} A), \Omega^1 N) \cong \cdots \cong \text{Ext}^{d+1}_A(A/\text{rad} A, \Omega^1 N).
$$

Thus $\text{Ext}^{d+1}_A(A/\text{rad} A, \Omega^1 N) = 0$. This implies that $\Omega^1 N$ has finite injective dimension by Lemma 3.1. Then $\Omega^1 N$ has finite projective dimension since $A$ is Iwanaga–Gorenstein (see [EJ11, Theorem 9.1.10]). Thus $N$ also has finite projective dimension. Since $N \in \text{CM} A = \text{Gproj} A$ by (2.23), it follows that $N$ is projective, i.e., $N = 0$ in $\text{CM} A$. This is a contradiction. □
Lemma 3.4. Let $R$ be a CM local ring with a canonical module and $A$ a Gorenstein $R$-order. Assume that $M \in \text{GProj} A$ is pure-injective in $\text{Mod} A$. Then $M$ is pure-injective in $\text{GProj} A$.

In particular, if $R$ is complete, then every object of $\text{CM} \ A$ is pure-injective in $\text{GProj} A$.

Proof. Since triangulated category $\text{GProj} A$ is compactly generated (Theorem 2.3), it has small direct sums and small direct products (see [Kra00, Lemma 1.5]). Let $I$ be a small set and let $\phi: M(I) \to M^I$ be the canonical morphism in $\text{GProj} A$. It directly follows from the definition of $\text{GProj} A$ that the canonical functor $G: \text{GProj} A \to \text{GProj} A$ commutes with small direct sums; see also (2.1). By the definition of $\text{GProj} A$ again, there exist $N \in \text{GProj} A$ and a morphism $f: M(I) = \bigoplus_{i \in I} M_i \to N$ in $\text{GProj} A$ such that $G(f) = \phi$, where $M_i := M$. For each $j \in I$, we have the canonical injection $\varepsilon_j: M_j \to \bigoplus_{i \in I} M_i$ and the canonical projection $\pi_j: \prod_{i \in I} M_i \to M_j$ in $\text{GProj} A$. Clearly, the canonical injection $e_j: M_i \to \bigoplus_{i \in I} M_i$ in $\text{GProj} A$ satisfies $G(e_j) = \varepsilon_j$. Moreover, there exists a morphism $p_j: N \to M_j$ in $\text{GProj} A$ such that $G(p_j) = \pi_j$. By definition, the composition

$$M_j \xrightarrow{\varepsilon_j} \bigoplus_{i \in I} M_i \xrightarrow{\phi} \prod_{i \in I} M_i \xrightarrow{\pi_j} M_j$$

is the identity $\text{id}_{M_j}: M_j \to M_j$ in $\text{GProj} A$, and so $G(p_j f e_j) = \text{proj} \phi \text{proj} e_j = \text{id}_{M_j}$. Then there exist morphisms $M_j \xrightarrow{g_j} Q_j \xrightarrow{h_j} M_j$ in $\text{GProj} A$ such that $Q_j$ is a pure-injective $A$-module and $p_j f e_j + h_j g_j$ equals the identity $\text{id}_{M_j}: M_j \to M_j$ in $\text{GProj} A$.

Now, denote by $(g_i)_{i \in I}$ the morphism $\bigoplus_{i \in I} M_i \to \bigoplus_{i \in I} Q_i$ induced by $g_i: M_i \to Q_i$ for each $i \in I$. The composition

$$M_j \xrightarrow{e_j} \bigoplus_{i \in I} M_i \xrightarrow{\left( \begin{array}{c} f \\ (g_i)_{i \in I} \end{array} \right)} N \oplus \bigoplus_{i \in I} Q_i \xrightarrow{\left( \begin{array}{c} p_j \\ h_j \end{array} \right)} M_j$$

in $\text{GProj} A$ is $\text{id}_{M_j}$ as $p_j f e_j + h_j g_j = \text{id}_{M_j}$. Then it easily follows that the second morphism of (3.5) is a pure monomorphism in $\text{Mod} A$. Since $M$ is pure-injective in $\text{Mod} A$, $\text{Hom}_A(-, M)$ sends the pure monomorphism to a surjection. Therefore the summation morphism $s: \bigoplus_{i \in I} M_i \to M$ in $\text{GProj} A$ can be written as the composition of the second morphism of (3.5) and some morphism $u: N \oplus \bigoplus_{i \in I} Q_i \to M$. Send $s$, $u$, and (3.5) to $\text{GProj} A$ by the canonical functor $G: \text{GProj} A \to \text{GProj} A$. Then we obtain the following commutative diagram:

$$\begin{array}{ccc}
M_j & \xrightarrow{\varepsilon_j} & \bigoplus_{i \in I} M_i \\
& & \xrightarrow{\phi} \\
& & \prod_{i \in I} M_i \\
\downarrow{G(s)} & & \downarrow{G(u)} \\
M & & M
\end{array}$$

Since $G(s)$ is the summation morphism $\bigoplus_{i \in I} M_i \to M$ in $\text{GProj} A$, the above diagram along with Theorem [A.1] shows that $M$ is pure-injective in $\text{GProj} A$.

The second claim of the lemma follows from the first claim and Remark 2.16. □

Proposition 3.6. Let $R$ be a complete CM local ring and $A$ a Gorenstein $R$-order. Assume that $A$ has a non-isolated singularity. Then there exists a non-compact indecomposable pure-injective object in $\text{GProj} A$.

Proof. By assumption, there exists a non-maximal prime ideal $\mathfrak{p} \in \text{Spec} R$ such that $A_{\mathfrak{p}}$ is singular as an $R_{\mathfrak{p}}$-order, or equivalently, $\widehat{A}_{\mathfrak{p}}$ is singular as an $\widehat{R}_{\mathfrak{p}}$-order. Hence there exists an indecomposable small CM module $M$ over $A_{\mathfrak{p}}$ such that $M \notin \text{proj} A_{\mathfrak{p}}$. Since $\text{End}_{\text{CM} \ A_{\mathfrak{p}}}(M) = \text{End}_{\text{GProj} \ A_{\mathfrak{p}}}(M)$ is a local ring (Remark 2.3), $M$ is indecomposable in $\text{GProj} \ A_{\mathfrak{p}}$. Furthermore, $M$ is pure-injective in $\text{GProj} \ A_{\mathfrak{p}}$ by Remark 2.16 and Lemma 3.4. Let $X \in K_{ac}(\text{proj} \ A_{\mathfrak{p}})$ be a complete resolution of $M$. By the triangulated equivalence $\text{GProj} \ A_{\mathfrak{p}} \cong K_{ac}(\text{proj} \ A_{\mathfrak{p}})$, $X$ is pure-injective in $K_{ac}(\text{proj} \ A_{\mathfrak{p}})$. Then the functor $\rho: K_{ac}(\text{proj} \ A_{\mathfrak{p}}) \to K_{ac}(\text{proj} A)$ defined in Section 2.10 sends $X$ to a pure-injective object $\rho X$.
in $K_{ac}(\text{Proj } A)$; see Corollary 2.5. Since we have the triangulated equivalence $K_{ac}(\text{Proj } A) \cong G\text{Proj } A$, it remains to show that $\rho X$ is non-compact and indecomposable.

By Lemma 2.19 there is a natural isomorphism

$$\text{End}_{K(\text{Proj } A)}(X) \cong \text{End}_{K(\text{Proj } A)}(\sigma X)$$

of rings, and $\text{End}_{G\text{Proj } A}(M) \cong \text{End}_{K(\text{Proj } A)}(X)$ is a local ring. Thus $\sigma X$ is indecomposable in $K_{ac}(\text{Proj } A)$. Moreover, $\text{End}_{K(\text{Proj } A)}(X)$ is a nonzero $R_p$-module and the functor $\rho$ is $R$-linear. Therefore $\text{End}_{K(\text{Proj } A)}(\sigma X)$ is a nonzero $R_p$-module. Then $\sigma X$ cannot be compact in $K_{ac}(\text{Proj } A)$ by Nakayama’s lemma, because for every $Y \in K_{ac}(\text{Proj } A)^c$, $\text{End}_{K(\text{Proj } A)}(Y)$ is finitely generated as an $R$-module by the equivalence $K_{ac}(\text{Proj } A)^c \cong CM_A$; see (2.11) and Theorem 2.33(1).

We are now ready to prove the main theorem.

**Proof of Theorem 1.2.** We first show the implication $(1) \Rightarrow (2)$. Suppose $(1)$ holds. Thus there are only finitely many indecomposable small $CM_A$-modules $M_1, \ldots, M_n$ up to isomorphism. Consider a sequence

$$(3.7) \quad X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} X_2 \xrightarrow{f_3} \ldots$$

of indecomposable compact objects in $G\text{Proj } A$, and suppose that each $f_i$ is not an isomorphism. Since $CM_A \cong (G\text{Proj } A)^c$ by Theorem 2.33(1) without loss of generality, we may assume that there exists an infinite set $S$ of positive integers $i$ such that $M_i = X_i$ in $G\text{Proj } A$ for all $i \in S$; see also Remark 2.34. Since $E := \text{End}_{CM_A}(M_1)$ is a local ring and $A$ has at most an isolated singularity by assumption, $E$ is of finite length as an $R$-module; see Theorem 2.29. Hence $E$ is an artinian local ring.

Let $J$ be the maximal ideal of $E$. Then $J$ consists of the non-isomorphisms from $M_i$ to $M_i$. Moreover, there exists an integer $t \geq 1$ with $J^t = 0$. Since $S$ is an infinite set, we can take some integers $s_1, s_2, \ldots, s_t \in S$ with $0 \leq s_1 < s_2 < \cdots < s_t < s_{t+1}$. For each $1 \leq j \leq t$, let $g_j$ be the composition of all morphisms between $X_{s_j}$ and $X_{s_{j+1}}$ in (3.7). Then we obtain the following sequence

$$X_{s_1} \xrightarrow{g_1} X_{s_2} \xrightarrow{g_2} X_{s_3} \xrightarrow{g_3} \cdots \xrightarrow{g_t} X_{s_{t+1}}.$$

By the definition of $S$, we have $X_{s_i} = M_i$ for each $1 \leq j \leq t$. Furthermore, every $g_j$ is not an isomorphism because for every $i \geq 0$, $f_i$ in (3.7) is not an isomorphism and $X_i$ is indecomposable. As a consequence, we have $g_j \in J$ for every $1 \leq j \leq t$. Thus $g_1 \cdots g_t \in J^t = 0$, and then

$$g_1 \cdots g_t = f_{s_{t+1}} \cdots \cdots f_{s_1}$$

is the zero map. Therefore, by [Kra00] Theorem 2.10 or [Bel09] Theorem 9.3, each object $Y \in G\text{Proj } A$ has a decomposition $Y = \bigoplus_{\lambda \in \Lambda} Y_\lambda$ in $G\text{Proj } A$ with $Y_\lambda \in CM_A \cong G\text{Proj } A^c$ for each $\lambda \in \Lambda$. Then there exist projective $A$-modules $P$ and $Q$ such that $Y \oplus P \cong \bigoplus_{\lambda \in \Lambda} Y_\lambda \oplus Q$ in $G\text{Proj } A$. It follows from Lemma 2.17 and [Var69] Theorem 1] that $Y$ is isomorphic to a direct sum of finitely generated Gorenstein-projective $A$-modules. Thus $(2)$ holds.

The implication $(2) \Rightarrow (3)$ follows from Theorem 2.33(1).

Finally, we prove the implication $(3) \Rightarrow (1)$. Suppose $(3)$ holds. Then $A$ has at most an isolated singularity by Proposition 3.3. Set $\mathcal{T} := G\text{Proj } A$ and $d := \dim R$. By the triangulated equivalence $CM_A \cong \mathcal{T}^c$, we may regard $F := \text{Hom}_{CM_A}(\Omega_d(A/\rad A), -)$ as a coherent functor $\mathcal{T} \to \text{Ab}$; see Appendix A.2. Now, suppose that $CM_A$ has infinitely many indecomposable objects up to isomorphism; then so does $CM_A$ (see Remark 2.33). Thus we see from Remark 2.16 and Lemma 3.4 that $\mathcal{T}^c$ contains, up to isomorphism, infinitely many indecomposable pure-injective objects in $\mathcal{T}$. By Proposition 3.3, the isomorphism classes of all such objects belong to the open set $(F)$ of the Ziegler spectrum $Z_{\mathcal{T}}$ (see Appendix A.2). Therefore, $(F)$ contains, up to isomorphism, infinitely many objects in $\mathcal{T}$. Then there exists a non-compact indecomposable pure-injective object in $\mathcal{T} = G\text{Proj } A$ by Theorem 2.32 and Proposition A.2 but this is a contradiction. Hence $A$ is of finite $CM$ representation type, that is, $(1)$ holds.
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### Appendix A. The Ziegler spectrum

In this appendix we use the Ziegler spectrum of a compactly generated category to lay the groundwork for the proof of (3) ⇒ (1) in Theorem 1.2. Namely we provide a sufficient condition for the existence of a non-compact indecomposable pure-injective object in a compactly generated triangulated category.

Our approach is inspired by Prest’s observation that Theorem 1.3 can be proved using various topological properties of the Ziegler spectrum of an Artin algebra [Pre09, §5.3.4]. For example, the proof of (3) ⇒ (1) of Theorem 1.3 can be viewed as a consequence of the fact that the Ziegler spectrum is quasi-compact and that every finitely generated module corresponds to an isolated point of the spectrum. This point of view is also explained in [Lak] [1.7] in the case of a finite-dimensional algebra. The crucial difference between the Ziegler spectrum of a module category and that of a compactly generated triangulated category is that the latter is not, in general, quasi-compact (see [Pre09, Theorem 17.3.22]). In this appendix, we will adapt Prest’s argument to our setting by working within a quasi-compact open set.

#### A.1. The points of the Ziegler spectrum

Let $\mathcal{T}$ be a compactly generated triangulated category with the full subcategory of compact objects of $\mathcal{T}$ denoted by $\mathcal{T}^c$ (see [Kra10, §5.3]). The category $((\mathcal{T}^c)^{op}, \text{Ab})$ of additive contravariant functors from $\mathcal{T}^c$ to the category $\text{Ab}$ of abelian groups is a locally coherent Grothendieck category (see, for example, [Pre09] Theorems 10.1.3 and 16.1.14, [Kra00] §1.2, and [Kra02, Lemma 1.7]). It follows that the isomorphism classes of indecomposable injective objects of $((\mathcal{T}^c)^{op}, \text{Ab})$ form a small set (Kra97 §3). Moreover, by [Kra00] Lemma 1.7, every injective object $E$ in $((\mathcal{T}^c)^{op}, \text{Ab})$ uniquely determines an object $X \in \mathcal{T}$ (up to isomorphism) such that $E \cong \text{Hom}_{\mathcal{T}}(-, X)|_{\mathcal{T}^c}$.

The following theorem characterises the objects of $\mathcal{T}$ that correspond to the injective objects of $((\mathcal{T}^c)^{op}, \text{Ab})$ in this way. A triangle $X \to Y \to Z \to \Sigma X$ in $\mathcal{T}$ is called pure if the sequence

$$0 \to \text{Hom}_{\mathcal{T}}(C, X) \to \text{Hom}_{\mathcal{T}}(C, Y) \to \text{Hom}_{\mathcal{T}}(C, Z) \to 0$$

is exact for every $C \in \mathcal{T}^c$. An object $X \in \mathcal{T}$ is called pure-injective if every pure triangle of the form $X \to Y \to Z \to \Sigma X$ is a split triangle. Recall that there is a unique morphism $X^{(i)} \to X$ induced by the identity $X_i := X \to X$ for each $i \in I$; we refer to this as the summation morphism.

**Theorem A.1** ([Kra00] Theorem 1.8]). The following statements are equivalent for $X \in \mathcal{T}$.

1. $X$ is a pure-injective object of $\mathcal{T}$.
2. $\text{Hom}_{\mathcal{T}}(-, X)|_{\mathcal{T}^c}$ is an injective object of $((\mathcal{T}^c)^{op}, \text{Ab})$.
3. For every small set $I$, the summation morphism $X^{(i)} \to X$ factors through the canonical morphism $X^{(i)} \to X^I$.

A consequence of this theorem is that the isomorphism classes of indecomposable pure-injective objects in $\mathcal{T}$ form a small set, which is called the Ziegler spectrum of $\mathcal{T}$. We denote this set by $Zg_{\mathcal{T}}$.

#### A.2. The topology on $Zg_{\mathcal{T}}$

In concurrent papers, Krause [Kra97] and Herzog [Her97] defined the (Ziegler) spectrum of a locally coherent Grothendieck category $\mathcal{G}$. The space is given by the set $\text{Spec} \mathcal{G}$ of the isomorphism classes of indecomposable injective objects in $\mathcal{G}$ and the open subsets parametrised by the Serre subcategories of the full subcategory $\text{fp}(\mathcal{G})$ of finitely presented objects in $\mathcal{G}$; indeed, a typical open set is one of the form

$$\{E \in \text{Spec} \mathcal{G} \mid \text{Hom}_{\mathcal{G}}(S, E) \neq 0 \text{ for some } S \in \mathcal{S}\}$$
where \( S \) is a Serre subcategory of \( \text{fp}(G) \). Moreover, there is a basis of quasi-compact open subsets given by those of the form

\[
\{ E \in \text{Spec}(G) \mid \text{Hom}_G(S, E) \neq 0 \}
\]

where \( S \) is an object of \( \text{fp}(G) \) (see [Kra97 Corollary 4.6] or [Her97 Corollary 3.5]).

If we take \( G = ((T^c)\text{op}, \text{Ab}) \), then it follows from Theorem A.1 (along with [Kra01 Corollary 1.9]) that the set \( Z_{G_T} \) carries a topology, whose basis of open subsets correspond to the Serre subcategories of \( \text{fp}((T^c)\text{op}, \text{Ab}) \).

Following [Kra02], we will view the topology on \( Z_{G_T} \) in terms of Serre subcategories of a more convenient category of covariant functors. An additive functor \( F: T \to \text{Ab} \) is called coherent if there exist a morphism \( f: X \to Y \) in \( T^c \) and an exact sequence

\[
\text{Hom}_T(Y, -) \xrightarrow{\text{Hom}_T(f, -)} \text{Hom}_T(X, -) \to F \to 0.
\]

We denote the category of coherent functors and natural transformations by \( \text{Coh}(T) \). By [Kra02 Lemma 7.2], there is an equivalence of categories between \( \text{fp}((T^c)\text{op}, \text{Ab})\text{op} \) and \( \text{Coh}(T) \) induced by the assignment taking a functor \( S: (T^c)\text{op} \to \text{Ab} \) to the coherent functor \( F: T \to \text{Ab} \) where \( F(Z) := \text{Hom}(S, \text{Hom}_T((-), Z)) \) for every \( Z \in T \). This equivalence yields the following characterisation of the topology on \( Z_{G_T} \).

**Theorem A.2** ([Kra02 Fundamental correspondences]). There is an order-preserving bijection

\[
\{ \text{open sets of } Z_{G_T} \} \xrightarrow{\sim} \{ \text{Serre subcategories of } \text{Coh}(T) \}
\]

given by \( O \mapsto S_O := \{ F \in \text{Coh}(T) \mid F(X) = 0 \text{ for all } X \in Z_{G_T} \setminus O \} \). Moreover, the sets

\[
(F) := \{ X \in Z_{G_T} \mid F(X) \neq 0 \}
\]

form a basis of quasi-compact open sets.

**A.3. Auslander–Reiten triangles.** Next we outline the connection between Auslander-Reiten triangles in \( T^c \) and the isolated points of \( Z_{G_T} \), i.e., those \( X \in Z_{G_T} \) such that \( \{ X \} \) is an open subset.

A triangle

\[
L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} \Sigma L
\]

in a triangulated category \( C \) is called an Auslander–Reiten triangle, or an AR-triangle (starting from \( L \)), if \( f \) is left almost split and \( N \) has a local endomorphism ring. Equivalently, we could require that \( g \) is right almost split and \( L \) has a local endomorphism ring or, alternatively, that \( f \) is left almost split and \( g \) is right almost split (see [Bel04 Lemma 3.2]).

**Lemma A.3.** Suppose \( L \in Z_{G_T} \) is compact as an object of \( T \) and that there is an AR-triangle

\[
L \xrightarrow{f} M \xrightarrow{g} N \xrightarrow{h} \Sigma L
\]

in \( T^c \). Then \( L \) is isolated in \( Z_{G_T} \).

**Proof.** In [Kra05a Proposition 3.2] it is shown that the triangle is also an AR-triangle in \( T \). We define a coherent functor \( G_L \) by the following exact sequence

\[
\text{Hom}_T(M, -) \xrightarrow{\text{Hom}_T(f, -)} \text{Hom}_T(L, -) \to G_L \to 0.
\]

It follows from the definitions that, given an indecomposable object \( X \in T^c \), we have \( G_L(X) \neq 0 \) if and only if \( X \approx L \). In other words \( G_L = \{ L \} \).

**A.4. The existence of non-compact indecomposable pure-injective objects.** We are now ready to connect properties of the Ziegler spectrum with the existence of non-compact indecomposable pure-injective objects.

**Proposition A.4.** Let \( F: T \to \text{Ab} \) be a coherent functor. Assume the following conditions hold:

1. Every \( L \in (F) \) with \( L \) compact as an object of \( T \) admits an AR-triangle in \( T^c \) starting from \( L \).
2. The open set \( (F) \) contains, up to isomorphism, infinitely many objects in \( T^c \).
Then there exists a non-compact indecomposable pure-injective object in $\mathcal{T}$.

**Proof.** Suppose that $(F)$ consists, up to isomorphism, only objects in $\mathcal{T}^c$. By assumption, the open set $(F)$ is non-empty. Moreover every $L \in (F)$ admits an AR-triangle $L \xrightarrow{f} M \rightarrow N \rightarrow \Sigma L$ in $\mathcal{T}^c$. Then every $L \in (F)$ is isolated by Lemma A.3. Hence the open set $(F)$ can be written as the infinite (disjoint) union $\bigcup_{L \in (F)} \{ L \}$ of open sets $\{ L \}$, but this is impossible since $(F)$ is quasi-compact by Theorem A.2. Therefore $(F)$ must contain a point $M \in (F)$ which is not compact as an object in $\mathcal{T}$. □

**References**

[AM69] M. F. Atiyah and I. G. Macdonald, *Introduction to commutative algebra*, Addison-Wesley Publishing Co., Reading, Mass.-London-Don Mills, Ont., Reading, Mass.-London-Don Mills, Ont., 1969.

[AR87] Maurice Auslander and Idun Reiten, *Almost split sequences for Cohen-Macaulay-modules*, Math. Ann. 277 (1987), no. 2, 345–349.

[ARS97] Maurice Auslander, Idun Reiten, and Sverre O. Smalø, *Representation theory of Artin algebras*, Cambridge Studies in Advanced Mathematics, vol. 36, Cambridge University Press, Cambridge, 1997.

[Aus76] Maurice Auslander, *Large modules over artin algebras*, Algebra, topology, and category theory: A collection of papers in honor of Samuel Eilenberg, Academic Press, New York, 1976, pp. 1–17.

[Aus78] ———, *Functors and morphisms determined by objects*, Representation theory of algebras (Proc. Conf., Temple Univ., Philadelphia, Pa., 1976), Lecture Notes in Pure Appl. Math., vol. 37, 1978, pp. 1–244.

[Aus86] ———, *Isolated singularities and existence of almost split sequences*, Representation theory II (Ottawa, Ont., 1984), Lecture Notes in Math., vol. 1178, Springer, Berlin, 1986, pp. 194–242.

[BCIE20] Silvana Bazzoni, Manuel Cortés-Izurdiaga, and Sergio Estrada, *Periodic modules and acyclic complexes*, Algebr. Represent. Theory 23 (2020), no. 5, 1861–1883.

[Bel00] Apostolos Beligiannis, *Relative homological algebra and purity in triangulated categories*, J. Algebra 227 (2000), no. 1, 268–361.

[Bel04] ———, *Auslander-Reiten triangles, Ziegler spectra and Gorenstein rings*, K-Theory 32 (2004), no. 1, 1–82.

[Bel11] ———, *On algebras of finite Cohen-Macaulay type*, Adv. Math. 226 (2011), no. 2, 1973–2019.

[BFS19] Abdolnaser Bahlekeh, Fahimeh Sadat Fotouhi, and Shokrollah Salarian, *Representation-theoretic properties of balanced big Cohen-Macaulay modules*, Math. Z. 293 (2019), no. 3–4, 1673–1709.

[BH98] Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, revised ed., Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1998.

[BS13] M. P. Brodmann and R. Y. Sharp, *Local cohomology: An algebraic introduction with geometric applications*, second ed., Cambridge Studies in Advanced Mathematics, vol. 136, Cambridge University Press, Cambridge, 2013.

[Buc86] Ragnar-Olaf Buchweitz, *Maximal Cohen-Macaulay modules and Tate-cohomology over Gorenstein rings*, unpublished manuscript, available at [http://hdl.handle.net/1807/16682](http://hdl.handle.net/1807/16682).

[CB91] William Crawley-Boevey, *Tame algebras and generic modules*, Proc. London Math. Soc. (3) 63 (1991), no. 2, 241–265.

[CB98] ———, *Tame algebras and generic modules*, in *The Ziegler spectrum of a locally coherent Grothendieck category*, Proc. London Math. Soc. (3) 74 (1997), no. 3, 503–558.
INDECOMPOSABLE PURE-INJECTIVE OBJECTS IN STABLE CATEGORIES

[Hol75] Melvin Hochster, *Topics in the homological theory of modules over commutative rings*, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics, No. 24, Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I., 1975.

[Hol17] Henrik Holm, *The structure of balanced big Cohen-Macaulay modules over Cohen-Macaulay rings*, Glasg. Math. J. 59 (2017), no. 3, 549–561.

[ILL+07] Srikanth B. Iyengar, Graham J. Leuschke, Anton Leykin, Claudia Miller, Ezra Miller, Anurag K. Singh, and Uli Walther, *Twenty-four hours of local cohomology*, Graduate Studies in Mathematics, vol. 87, American Mathematical Society, Providence, RI, 2007.

[IW14] Osamu Iyama and Michael Wemyss, *Maximal modifications and Auslander–Reiten duality for non-isolated singularities*, Invent. Math. 197 (2014), no. 3, 521–586.

[JL89] Christian U. Jensen and Helmut Lenzing, *Model theoretic algebra: with particular emphasis on fields, rings, modules*, Algebra, Logic and Applications, vol. 2, Gordon and Breach Science Publishers, New York, 1989.

[Jun05] Peter Jørgensen, *The homotopy category of complexes of projective modules*, Adv. Math. 193 (2005), no. 1, 223–232.

[KN22] Ryo Kanda and Tsutomu Nakamura, *Flat cotorsion modules over Noether algebras*, Doc. Math. 27 (2022), 1101–1167.

[Kra97] Henning Krause, *The spectrum of a locally coherent category*, J. Pure Appl. Algebra 114 (1997), no. 3, 259–271.

[Kra98] Henning Krause, *Generic modules over Artin algebras*, Proc. London Math. Soc. (3) 76 (1998), no. 2, 276–306.

[Kra00] Henning Krause, *Smashing subcategories and the telescope conjecture—an algebraic approach*, Invent. Math. 139 (2000), no. 1, 99–133.

[Kra02] Henning Krause, *Coherent functors in stable homotopy theory*, Fund. Math. 173 (2002), no. 1, 33–56.

[Kra05a] Henning Krause, *Auslander–Reiten triangles and a theorem of Zimmermann*, Bull. London Math. Soc. 37 (2005), no. 3, 361–372.

[Kra05b] Henning Krause, *The stable derived category of a Noetherian scheme*, Compos. Math. 141 (2005), no. 5, 1128–1162.

[Kra10] Henning Krause, *Localization theory for triangulated categories*, Triangulated categories, London Math. Soc. Lecture Note Ser., vol. 375, Cambridge Univ. Press, Cambridge, 2010, pp. 161–235.

[Kra15] Henning Krause, *Krull-Schmidt categories and projective covers*, Expo. Math. 33 (2015), no. 4, 535–549.

[KS06] Masaki Kashiwara and Pierre Schapira, *Categories and sheaves*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 332, Springer-Verlag, Berlin, 2006.

[Lak] Rosanna Laking, *Infinite-dimensional representations of algebras*, Modern trends in algebra and representation theory, Cambridge University Press, to appear.

[Lip02] Joseph Lipman, *Lectures on local cohomology and duality*, Local cohomology and its applications (Guanajuato, 1999), Lecture Notes in Pure and Appl. Math., vol. 226, Dekker, New York, 2002, pp. 39–89.

[LP19] Inna Los and Gena Puninski, *The Ziegler spectrum of the D-infinity plane singularity*, Colloq. Math. 157 (2019), no. 1, 35–63.

[LW12] Graham J. Leuschke and Roger Wiegand, *Cohen-Macaulay representations*, Mathematical Surveys and Monographs, vol. 181, American Mathematical Society, Providence, RI, 2012.

[Mat89] Hideyuki Matsumura, *Commutative ring theory*, Cambridge University Press, Cambridge, 1989, translated from the Japanese by M. Reid, second ed.

[MS11] Daniel Murfet and Shokrollah Salarian, *Generic modules over Artin algebras*, Adv. Math. 226 (2011), no. 2, 1096–1133.

[Nak] Tsutomu Nakamura, *Large Cohen–Macaulay modules and Ziegler spectra*, in preparation.

[Nee01] Amnon Neeman, *Triangulated categories*, Annals of Mathematics Studies, vol. 148, Princeton University Press, Princeton, NJ, 2001.

[Nee08] Amnon Neeman, *The homotopy category of flat modules, and Grothendieck duality*, Invent. Math. 174 (2008), no. 2, 255–308.

[NT20] Tsutomu Nakamura and Peder Thompson, *Minimal semi-flat-cotorsion modules and cosupport*, J. Algebra 562 (2020), 587–620.

[Pé16] Marco A. Pérez, *Introduction to Abelian model structures and Gorenstein homological dimensions*, Memoirs of the American Mathematical Society, vol. 246, American Mathematical Society, Providence, RI, 2016.

[PR22] Chrysostomos Psaroudakis and Wolfgang Rump, *Exact categories, big Cohen-Macaulay modules and finite representation type*, J. Pure Appl. Algebra 226 (2022), no. 4, 106891, 30 pages.

[Pre09] Mike Prest, *Purity, spectra and localisation*, Encyclopedia of Mathematics and its Applications, vol. 121, Cambridge University Press, Cambridge, 2009.

[Pun18] Gena Puninski, *The Ziegler spectrum and Ringel’s quilt of the A-infinity plane curve singularity*, Algebr. Represent. Theory 21 (2018), no. 2, 419–446.

[Rin00] Claus Michael Ringel, *Infinite length modules. Some examples as introduction*, Infinite length modules (Bielefeld, 1998), Trends Math., Birkhäuser, Basel, 2000, pp. 1–73.

[Rin11] Claus Michael Ringel, *The minimal representation-infinite algebras which are special biserial*, Representations of algebras and related topics, EMS Ser. Congr. Rep., Eur. Math. Soc., Zürich, 2011, pp. 501–560.
[RT74] Claus Michael Ringel and Hiroyuki Tachikawa, QF – 3 rings, J. Reine Angew. Math. 272 (1974), 49–72.

[Sim05] Daniel Simson, On Corner type endo-wild algebras, J. Pure Appl. Algebra 202 (2005), no. 1-3, 118–132.

[SS20] Manuel Saorín and Jan Šťovíček, t-Structures with Grothendieck hearts via functor categories, arXiv:2003.01401

[War69] R. B. Warfield, Jr., A Krull-Schmidt theorem for infinite sums of modules, Proc. Amer. Math. Soc. 22 (1969), no. 2, 460–465.

[Yos90] Yuji Yoshino, Cohen-Macaulay modules over Cohen-Macaulay rings, London Mathematical Society Lecture Note Series, vol. 146, Cambridge University Press, Cambridge, 1990.

[Zie84] Martin Ziegler, Model theory of modules, Ann. Pure Appl. Logic 26 (1984), no. 2, 149–213.