A FLUX-BASED HDG METHOD

ISSEI OIKAWA

ABSTRACT. In this paper, we present a flux-based formulation of the hybridizable discontinuous Galerkin (HDG) method for steady-state diffusion problems and propose a new method derived by letting a stabilization parameter tend to infinity. Assuming an inf-sup condition, we prove its well-posedness and error estimates of optimal order. We show that the inf-sup condition is satisfied by some triangular elements. Numerical results are also provided to support our theoretical results.

1. INTRODUCTION

We consider the hybridizable discontinuous Galerkin (HDG) method for the steady-state diffusion problem with Dirichlet boundary condition

\begin{align}
\text{(1a)} & \quad q + \nabla u = 0 \quad \text{in } \Omega, \\
\text{(1b)} & \quad \nabla \cdot q = f \quad \text{in } \Omega, \\
\text{(1c)} & \quad u = 0 \quad \text{on } \partial \Omega,
\end{align}

where $\Omega \subset \mathbb{R}^d$ ($d = 2, 3$) is a bounded convex polygonal or polyhedral domain and $f$ is a given function. In the original HDG method [3], a numerical trace $\hat{u}_h$ is introduced as an unknown variable to approximate the trace of $u$ on element boundaries, which corresponds to a Dirichlet boundary condition, and a numerical flux $\hat{q}_h$ is properly defined. The other variables $u_h$ and $q_h$ approximating $u$ and $q$, respectively, can be eliminated in element-by-element fashion and we obtain a globally-coupled system of equations only in terms of $\hat{u}_h$, which is called static condensation.

In [2], a flux-based formulation is presented, in which the trace of $q$ on element boundaries instead of $\hat{u}_h$ is hybridized, in other words, $\hat{q}_h$ is an unknown variable and $\hat{u}_h$ is defined in terms of $\hat{q}_h$ and the other variables. The flux-based method is a rewrite of the original HDG method and provides the same solution, however, the local problem has a Neumann boundary condition, so that the static condensation is different from that of the original method. We note that the local solvability of the flux-based method is not obvious, which will be verified in Section 2.4.

In this paper, we propose a new flux-based method derived by passing the stabilization parameter to infinity. In our method, $\hat{q}_h$ is unknown and the numerical trace is defined by $\hat{u}_h = u_h$. Since our method has saddle point structure, its well-posedness depends on whether the inf-sup condition we define in Section 2.3 is satisfied. The inf-sup condition is

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fulfilled if we use triangular meshes and the polynomials of degree \( k \) and \( k+1 \) for \( \hat{q}_h \) and \( u_h \), respectively, with a non-negative integer \( k \). In addition, the proposed method using such approximation spaces achieves the optimal convergence rates in \( u_h \) and \( q_h \), like the HDG method with the so-called Lehrenfeld–Schöberl stabilization proposed in [5] and analyzed in [6, 7]. Although, in the Lehrenfeld–Schöberl stabilization, the \( L^2 \)-orthogonal projection onto the approximation space of \( \hat{u}_h \) is inserted in front of \( u_h \), such a projection is not used in our method because it is naturally incorporated through the transmission condition in a flux-based formulation.

The rest of the paper is organized as follows. In Section 2, we introduce notation and present the flux-based formulation, and a new method is derived from it. We verify the local solvability of the methods. In Section 3, we establish a priori estimate and error estimates of optimal order for our method, assuming that an inf-sup condition holds. In Section 4, we prove that the inf-sup condition is satisfied if the polynomial degrees for \( \hat{q}_h \) and \( u_h \) are \( k \) and \( k+1 \), respectively, and triangular meshes are used. In Section 5, numerical results are presented to validate our theoretical results.

2. A FLUX-BASED HDG FORMULATION

2.1. Notation. To begin with, we introduce notation to present the HDG method via flux hybridization. Let \( \{T_h\}_h \) be a family of meshes satisfying the quasi-uniform condition, where \( h \) stands for the mesh size. Let \( \mathcal{E}_h \) denote the set of all edges or faces of elements in \( T_h \). Let \( L^2(\mathcal{E}_h) \) denote the \( L^2 \)-space on \( \bigcup_{e \in \mathcal{E}_h} e \) and \( P_m(T_h) \) and \( P_m(\mathcal{E}_h) \) denote the spaces of element-wise and edge-wise polynomials of degree \( m \), respectively.

We use the usual symbols of Sobolev spaces [1], such as \( H^m(D) \), \( H^m(D)^d \), \( \| \cdot \|_{m,D} := \| \cdot \|_{H^m(D)} \), and \( | \cdot |_{m,D} := | \cdot |_{H^m(D)} \) for a domain \( D \) and an integer \( m \). We may omit the subscripts when \( D = \Omega \) or \( m = 0 \), such as \( \| \cdot \| = \| \cdot \|_{m,\Omega} \), \( \| \cdot \| = \| \cdot \|_{0,\Omega} \), and \( | \cdot | = | \cdot |_{m,\Omega} \).

The piecewise Sobolev space of order \( m \) is denoted by \( H^m(T_h) \). The inner products are defined as

\[
(q, v)_K = \int_K q \cdot v \, dx, \quad (q, v)_{T_h} = \sum_{K \in T_h} (q, v)_K,
\]

\[
(u, w)_K = \int_K u w \, dx, \quad (u, w)_{T_h} = \sum_{K \in T_h} (u, w)_K,
\]

\[
\langle u, w \rangle_{\partial K} = \int_{\partial K} u w \, ds, \quad \langle u, w \rangle_{\partial T_h} = \sum_{K \in T_h} \langle u, w \rangle_{\partial K},
\]

\[
\langle \mu, \lambda \rangle_{\mathcal{E}_h} = \sum_{e \in \mathcal{E}_h} \int_e \mu \lambda \, ds, \quad (u, w) = \int_{\Omega} u w \, dx.
\]

We define the induced norms from these inner products by

\[
\| v \|_{T_h} = (v, v)_{T_h}^{1/2}, \quad \| w \|_{T_h} = (w, w)_{T_h}^{1/2}, \quad \| w \|_{\partial T_h} = (w, w)_{\partial T_h}^{1/2}.
\]
Throughout the paper, we use the symbol $C$ to denote a generic constant independent of the mesh size $h$ and $n$ to stand for the unit (outer) normal vector to an edge $e \in \mathcal{E}_h$ or $\partial K$ for $K \in \mathcal{T}_h$.

2.2. Finite element spaces. Let $k$ be a non-negative integer. We define the local approximation spaces on $K \in \mathcal{T}_h$ as

$$V(K) = P_k(K)^d, \quad W(K) = P_{k+1}(K),$$

where $P_m(K)$ stands for the space of polynomials of degree $m$. We introduce an approximation space for $q|_e \cdot n$ on $e \in \mathcal{E}_h$,

$$N(e) = P_k(e),$$

and assume that $(I - n \otimes n)r = 0$ for $r \in N(e)$. The global finite element spaces are defined by

$$V_h := \{ v \in L^2(\Omega)^d : v|_K \in V(K) \forall K \in \mathcal{T}_h \},$$

$$W_h := \{ w \in L^2(\Omega) : w|_K \in W(K) \forall K \in \mathcal{T}_h \},$$

$$N_h := \{ r \in L^2(\mathcal{E}_h)^d : r|_e \in N(e) \forall e \in \mathcal{E}_h \}.$$

Let $P_V, P_W$, and $P_N$ denote the $L^2$-projections onto $V_h, W_h$, and $N_h$, respectively. The following approximation properties hold for $1 \leq s \leq k + 1$: If $u \in H^{k+2}(\Omega)$, then

(2a) $\| q - P_V q \| \leq Ch^s|q|_s,$

(2b) $\| q \cdot n - (P_V q) \cdot n\|_{\partial \mathcal{T}_h} \leq Ch^{s-1/2}|q|_s,$

(2c) $\| u - P_W u \| \leq Ch^s|u|_s,$

(2d) $\| u - P_W u \|_{\partial \mathcal{T}_h} \leq Ch^{s+1/2}|u|_{s+1},$

(2e) $\| (q - P_N q) \cdot n\|_{\partial \mathcal{T}_h} \leq Ch^{s+1/2}|q|_{s+1}.$

2.3. A flux-based HDG method. The solution of the original HDG method, $(q_h, u_h, \hat{u}_h) \in V_h \times W_h \times M_h$, is defined by

(3a) $\langle q_h, v \rangle_{\mathcal{T}_h} - (u_h, \nabla \cdot v)_{\mathcal{T}_h} + \langle \hat{u}_h, v \cdot n \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall v \in V_h,$

(3b) $-\langle q_h, \nabla w \rangle_{\mathcal{T}_h} + \langle \hat{q}_h \cdot n, w \rangle_{\partial \mathcal{T}_h} = \langle f, w \rangle \quad \forall w \in W_h,$

(3c) $\langle \hat{q}_h \cdot n, \mu \rangle_{\partial \mathcal{T}_h} = 0 \quad \forall \mu \in M_h,$

(3d) $\hat{q}_h \cdot n := q_h \cdot n + \tau(u_h - \hat{u}_h)$ on $\partial K \quad \forall K \in \mathcal{T}_h,$

where $M_h$ is an approximation space for the trace $u|_{\mathcal{E}_h}$ and $\tau$ is a stabilization parameter. Another formulation via flux hybridization is also stated in [2], which reads as follows:
Find \((q_h, u_h, \hat{q}_h) \in V_h \times W_h \times N_h\) such that

\[
(4a) \quad \langle q_h, v \rangle_{T_h} - (u_h, \nabla \cdot v)_{T_h} + \langle \hat{u}_h, v \cdot n \rangle_{\partial T_h} = 0 \quad \forall v \in V_h,
\]

\[
(4b) \quad - (q_h, \nabla w)_{T_h} + \langle \hat{q}_h \cdot n, w \rangle_{\partial T_h} = (f, w) \quad \forall w \in W_h,
\]

\[
(4c) \quad \langle \hat{u}_h, r \cdot n \rangle_{\partial T_h} = 0 \quad \forall r \in N_h,
\]

\[
(4d) \quad \hat{u}_h := u_h + \tau^{-1} (q_h - \hat{q}_h) \cdot n \text{ on } \partial K \quad \forall K \in T_h.
\]

This method is a rewrite of the original HDG method and its solution coincides with that of the original method. We can verify that by expressing the hybrid variables in terms of \(u_h\) and \(q_h\). Let \(K^+ + K^-\) be adjacent elements sharing an internal edge \(e \in \mathcal{E}_h\) and let \(n^+\) and \(n^-\) denote the outer unit normal vectors to \(\partial K^+\) and \(\partial K^-,\) respectively. For a function \(w\), let \(w^+\) and \(w^-\) stand for the trace of \((w|_{K^+})|_e\) and \((w|_{K^-})|_e\), respectively. In both methods, \(\hat{u}_h\) and \(\hat{q}_h\) are single valued on element boundaries from the transmission conditions (3c) and (4c). From (3d) or (4d), it follows that

\[
\hat{q}_h \cdot n^+ = q_h^+ \cdot n^+ + \tau (u_h^+ - \hat{u}_h),
\]

\[
\hat{q}_h \cdot n^- = q_h^- \cdot n^- + \tau (u_h^- - \hat{u}_h).
\]

Solving these equations, we have

\[
\hat{u}_h = \frac{1}{2} (u_h^+ + u_h^-) + \frac{1}{2\tau} (q_h^+ \cdot n^+ + q_h^- \cdot n^-),
\]

\*

\[
\hat{q}_h \cdot n = \frac{1}{2} (q_h^+ + q_h^-) \cdot n + \frac{\tau}{2} (u_h^+ n^+ + u_h^- n^-) \cdot n.
\]

Therefore, we see that the equations (3a)-(3b) and (4a)-(4b) give the same solution \(u_h\) and \(q_h\). However, note that the procedures of the static condensation are different and the local solvability of the flux-based method is not obvious, which we will prove later.

We consider the limiting case of \(\tau \to +\infty\) in (4). In this case, (4d) is naturally interpreted as \(\hat{u}_h = u_h\), which leads to the following scheme: Find \((q_h, u_h, \hat{q}_h) \in V_h \times W_h \times N_h\) such that

\[
(5a) \quad \langle q_h, v \rangle_{T_h} - (u_h, \nabla \cdot v)_{T_h} + \langle u_h, v \cdot n \rangle_{\partial T_h} = 0 \quad \forall v \in V_h,
\]

\[
(5b) \quad - (q_h, \nabla w)_{T_h} + \langle \hat{q}_h \cdot n, w \rangle_{\partial T_h} = (f, w) \quad \forall w \in W_h,
\]

\[
(5c) \quad \langle u_h, r \cdot n \rangle_{\partial T_h} = 0 \quad \forall r \in N_h.
\]

We remark that the above method is not always well-posed. Assume that \(f \equiv 0\). By taking \(v = q_h\), \(w = u_h\), and \(r = \hat{q}_h\) in (5), we have \(q_h = 0\). From (5a) with \(q_h = 0\) and (5c), it follows that \(u_h = 0\). However, \(\hat{q}_h\) still remains unknown, which depends on if the following equation implies \(\hat{q}_h \cdot n = 0\):

\[
(6) \quad \langle \hat{q}_h \cdot n, w \rangle_{\partial T_h} = 0 \quad \forall w \in W_h.
\]

Indeed, when \(N_h\) and \(W_h\) are piecewise constant spaces, it is easy to see that there exists \(\hat{q}_h \in N_h\) satisfying (6) and \(\hat{q}_h \cdot n \neq 0\). For this reason, we need the following inf-sup
condition for the well-posedness: There exists a constant \( C \) independent of \( h \) such that, for all \( r \in N_h \),
\[
\|h^{1/2}r \cdot n\|_{\partial T_h} \leq C \sum_{K \in T_h} \sup_{w \in W(K)} \frac{\langle r \cdot n, w \rangle_{\partial K}}{\|\nabla w\|_{L^2(K)} + \|h^{-1/2}P_Nw\|_{L^2(\partial K)}},
\]
where \( P_Nw := P_N(w|_{\partial K}) \cdot n \). In order to derive a priori estimates, we also use the transposed version of the inf-sup condition: There exists a constant \( C \) such that
\[
\|h^{-1/2}P_Nw\|_{\partial T_h} \leq C \sum_{K \in T_h} \sup_{r \in N(\partial K)} \frac{\langle r \cdot n, w \rangle_{\partial K}}{\|h^{1/2}r \cdot n\|_{L^2(\partial K)}} \quad \forall w \in W_h.
\]

2.4. Local solvability. We here verify the local solvability of the flux-based methods (4) and (5).

We first consider the local problem of (4). Let \( \tilde{q}_{\partial K} \) denote the restriction of \( \tilde{q}_h \) to \( \partial K \) and let \([w] \) denote the jump of a function \( w \). We define \( \|\mu\|_{\mathcal{E}_h} = \langle \mu, \mu \rangle_{\mathcal{E}_h}^{1/2} \) and \( \|r\|_{\mathcal{E}_h} = \langle r, r \rangle_{\mathcal{E}_h}^{1/2} \). We introduce the mean-zero subspace of \( W(K) \),
\[
W_0(K) := \{ w \in W(K) : (w, 1)_K = 0 \}.
\]
The local problem reads: Find \((q_K, u_{K0}) \in V(K) \times W_0(K)\) such that
\[
\begin{align}
(q_K, v)_K + \langle \tau^{-1}q_K \cdot n, v \cdot n \rangle_{\partial K} + (\nabla u_{K0}, v)_K &= \langle \tau^{-1}\tilde{q}_{\partial K} \cdot n, v \cdot n \rangle_{\partial K} \quad \forall v \in V(K), \\
-(q_K, \nabla u_{K0})_K &= (f, u_{K0})_K - \langle \tilde{q}_{\partial K} \cdot n, w_0 \rangle_{\partial K} \quad \forall u_{K0} \in W_0(K).
\end{align}
\]
To verify the well-posedness of the local problem, we let \( f \equiv 0 \) and \( \tilde{q}_{\partial K} \cdot n = 0 \). Taking \( v = q_K \) in (9a) and \( u_{K0} \equiv u_{K0} \) in (9b), we have \( q_K = 0 \). Since we can \( v = \nabla u_{K0} \), we get \( \nabla u_{K0} = 0 \), which implies \( u_{K0} = 0 \). Therefore, \( q_K \) and \( u_{K0} \) are uniquely determined if \( \tilde{q}_{\partial K} \) is given.

However, the piecewise constant part of \( u_h \), denoted by \( \bar{u}_h \), remains unknown. Eliminating \( q_K \) and \( u_{K0} \) in element-by-element fashion by static condensation, we obtain the global equations for \((\bar{q}_h, \bar{u}_h) \in N_h \times P_0(T_h)\)
\[
\begin{align}
\langle \bar{q}_h - \tau^{-1}\tilde{q}_{h \partial} \cdot n, r \cdot n \rangle_{\partial T_h} &= -\langle u_{h0} - \tau^{-1}q_h \cdot n, r \cdot n \rangle_{\partial T_h} =: F_1(r) \quad \forall r \in N_h, \\
\langle \bar{q}_h \cdot n, \bar{w} \rangle_{\partial T_h} &= (f, \bar{w}) \quad \forall \bar{w} \in P_0(T_h),
\end{align}
\]
where \( u_{h0} := u_h - \bar{u}_h \) and we note that \( u_{h0} \) has been determined by (9). We will show that the global problem is well posed. To this end, we first prove the following inf-sup condition.

**Theorem 2.1.** There exists a positive constant \( C \) such that
\[
C\|\bar{w}\|_{\mathcal{E}_h} \leq \sup_{r \in N_h} \frac{\langle r \cdot n, \bar{w} \rangle_{\partial T_h}}{\|r \cdot n\|_{\partial T_h}} \quad \forall \bar{w} \in P_0(\mathcal{E}_h).
\]

**Proof.** For \( \bar{w} \in P_0(T_h) \) with \( \bar{w} \neq 0 \), we define \( r = [\bar{w}] \). Then, it follows that
\[
\langle r \cdot n, \bar{w} \rangle_{\partial T_h} = \langle r, [\bar{w}] \rangle_{\mathcal{E}_h} = \|\bar{w}\|_{\mathcal{E}_h}^2 = \|r\|_{\mathcal{E}_h} \|\bar{w}\|_{\mathcal{E}_h}.
\]
From this and (11), it follows that
\[
\| [ \mathbf{w} ] \|_{\mathcal{E}_h} \leq \frac{ ( \mathbf{r} \cdot \mathbf{n}, \mathbf{w} )_{\partial \Omega_h} }{ \| \mathbf{r} \|_{\mathcal{E}_h} } \leq \sqrt{2} \frac{ ( \mathbf{r} \cdot \mathbf{n}, \mathbf{w} )_{\partial \Omega_h} }{ \| \mathbf{r} \cdot \mathbf{n} \|_{\partial \Omega_h} },
\]
which completes the proof.

We now prove an a priori estimate for the global problem (10) using the inf-sup condition in Theorem 2.1, which ensures that the problem admits a unique solution.

**Theorem 2.2.** There exists a positive constant $C$ independent of $h$ such that
\[
\| \tau^{-1/2} \hat{q}_h \cdot \mathbf{n} \|_{\partial \Omega_h} + \| \lbrack \mathbf{q}_h \rbrack \|_{\mathcal{E}_h} \leq C \left( \| u_{h0} \|_{\partial \Omega_h} + \| q_h \cdot \mathbf{n} \|_{\partial \Omega_h} + \| f \| \right).
\]

**Proof.** Taking $\mathbf{r} = \hat{q}_h$ in (10a) and $\mathbf{w} = \overline{u}_h$ in (10b), we have
\[
(\overline{u}_h, \hat{q}_h \cdot \mathbf{n})_{\partial \Omega_h} - \| \tau^{-1/2} \hat{q}_h \cdot \mathbf{n} \|^2_{\partial \Omega_h} = F_1(\hat{q}_h),
\]
\[
(\hat{q}_h \cdot \mathbf{n}, \overline{u}_h)_{\partial \Omega_h} = (f, \overline{u}_h).
\]
It then follows that
\[
\| \tau^{-1/2} \hat{q}_h \cdot \mathbf{n} \|^2_{\partial \Omega_h} = (f, \overline{u}_h) - F_1(\hat{q}_h) \leq \| f \| \| \overline{u}_h \| + \| F_1 \| \| \hat{q}_h \cdot \mathbf{n} \|_{\partial \Omega_h},
\]
where
\[
\| F_1 \| := \sup_{\mathbf{r} \in \mathcal{N}_h} \frac{ F(\mathbf{r}) }{ \| \mathbf{r} \cdot \mathbf{n} \|_{\partial \Omega_h} }.
\]
Using Young’s inequality, we deduce
\[
\| \tau^{-1/2} \hat{q}_h \cdot \mathbf{n} \|^2_{\partial \Omega_h} \leq C \left( \| F_1 \|^2 + \epsilon^2 \| f \|^2 \right) + \epsilon^2 \| \overline{u}_h \|^2_{\partial \Omega_h}
\]
for any $\epsilon > 0$. By Theorem 2.1, (10a) and (11), we have
\[
\| [ \mathbf{\overline{u}_h} ] \|_{\mathcal{E}_h} \leq C \sup_{\mathbf{r} \in \mathcal{N}_h} \frac{ ( \mathbf{r} \cdot \mathbf{n}, \overline{u}_h )_{\partial \Omega_h} }{ \| \mathbf{r} \cdot \mathbf{n} \|_{\partial \Omega_h} } \leq C \sup_{\mathbf{r} \in \mathcal{N}_h} \frac{ F_1(\mathbf{r}) + \langle \tau^{-1} \hat{q}_h \cdot \mathbf{n}, \mathbf{r} \cdot \mathbf{n} \rangle_{\partial \Omega_h} }{ \| \mathbf{r} \cdot \mathbf{n} \|_{\partial \Omega_h} } \leq C \left( \| F_1 \| + \| \tau^{-1} \hat{q}_h \cdot \mathbf{n} \|_{\partial \Omega_h} \right) \leq C \left( \| F_1 \| + \epsilon^2 \| f \| + \epsilon \| \overline{u}_h \|_{\partial \Omega_h} \right).
\]
Since both $\| \cdot \|_{\mathcal{E}_h}$ and $\| \cdot \|_{\partial \Omega_h}$ are norms on $P_0(\Omega_h)$, they are equivalent to each other and $\| \overline{u}_h \|_{\partial \Omega_h}$ is bounded by $C\| \lbrack \mathbf{q}_h \rbrack \|_{\mathcal{E}_h}$. Choosing $\epsilon$ sufficiently small, we get
\[
\| [ \mathbf{\overline{u}_h} ] \|_{\mathcal{E}_h} \leq C \left( \| F_1 \| + \| f \| \right).
\]
From this and (11), it follows that
\[
\| \tau^{-1/2} \hat{q}_h \cdot \mathbf{n} \|_{\partial \Omega_h} \leq C \left( \| F_1 \| + \| f \| \right).
\]
Since we can bound as $\| F_1 \| \leq C \left( \| u_{h0} \|_{\partial \Omega_h} + \| q_h \cdot \mathbf{n} \|_{\partial \Omega_h} \right)$, the proof is complete.
Next, we show that the local problem of the proposed method is well-posed. Let us define $R_0(\partial K) = \{w|_{\partial K} n : w \in P_0(K)\}$, whose dimension is one. The local problem of (5) is as follows: Find $(q_K, u_{K0}, \overline{u}_K) \in V(K) \times W_0(K) \times P_0(K)$ such that

\begin{align}
(12a) \quad & (q_K, v)_K + (\nabla u_{K0}, v)_K = 0 \quad & \forall v \in V(K), \\
(12b) \quad & -(q_K, \nabla w_0)_K = (f, w_0)_K - (\hat{q}_{\partial K} \cdot n, w_0)_{\partial K} \quad & \forall w_0 \in W_0(K), \\
(12c) \quad & (\overline{u}_K + u_{K0}, \mathbf{r}_0 \cdot n)_{\partial K} = 0 \quad & \forall \mathbf{r}_0 \in R_0(\partial K).
\end{align}

The well-posedness is verified by setting all terms on the right-hand side to zero and a straightforward computation.

3. Error Analysis

3.1. A priori estimate. We consider the proposed method in general form

\begin{align}
(13a) \quad & (q_h, v)_{\mathcal{T}_h} - (u_h, \nabla \cdot v)_{\mathcal{T}_h} + (u_h, v \cdot n)_{\partial \mathcal{T}_h} = F_1(v) \quad & \forall v \in V_h, \\
(13b) \quad & -(q_h, \nabla w)_{\mathcal{T}_h} + (\hat{q}_h \cdot n, w)_{\partial \mathcal{T}_h} = F_2(w) \quad & \forall w \in W_h, \\
(13c) \quad & (u_h, \mathbf{r} \cdot n)_{\partial \mathcal{T}_h} = F_3(\mathbf{r}) \quad & \forall \mathbf{r} \in N_h,
\end{align}

where $F_1 : V_h \to \mathbb{R}$, $F_2 : W_h \to \mathbb{R}$, and $F_3 : N_h \to \mathbb{R}$ are linear functionals and their norms are defined by

\begin{align*}
\|F_1\| &= \sup_{v \in V_h} \frac{F_1(v)}{\|v\|}, \\
\|F_2\| &= \sum_{K \in \mathcal{T}_h} \sup_{w \in W(K)} \|\nabla w\|_{L^2(K)} + \|h^{-1/2} P_N w\|_{L^2(\partial K)}, \\
\|F_3\| &= \sup_{\mathbf{r} \in N_h} \|h^{1/2} \mathbf{r} \cdot n\|_{\partial \mathcal{T}_h}.
\end{align*}

We first establish a priori estimate for the problem.

**Theorem 3.1.** Let $(q_h, u_h, \hat{q}_h) \in V_h \times W_h \times N_h$ be a solution of (13). Then there exists a constant $C$ such that

\[
\|q_h\| + \|h^{1/2} \hat{q}_h \cdot n\|_{\partial \mathcal{T}_h} + \|\nabla u_h\|_{\mathcal{T}_h} + \|h^{-1/2} P_N u_h\|_{\partial \mathcal{T}_h} \leq C \left(\|F_1\| + \|F_2\| + \|F_3\|\right).
\]

**Proof.** From the inf-sup condition (7) and (13b), it follows that

\[
\|h^{1/2} \hat{q}_h \cdot n\|_{\partial \mathcal{T}_h} \leq C \sum_{K \in \mathcal{T}_h} \sup_{w \in W(K)} \frac{\langle \hat{q}_h \cdot n, w \rangle_{\partial K}}{\|\nabla w\|_{L^2(K)} + \|h^{-1/2} P_N w\|_{L^2(\partial K)}}
\]

\[
= C \sum_{K \in \mathcal{T}_h} \sup_{w \in W(K)} \frac{\langle \hat{q}_h \cdot n, w \rangle_{\partial K}}{\|\nabla w\|_{L^2(K)} + \|h^{-1/2} P_N w\|_{L^2(\partial K)}}
\]

\[
\leq C \left(\|F_2\| + \|q_h\|\right).
\]

Integrating by parts in (13a), we have

\[
(q_h, v)_{\mathcal{T}_h} + (\nabla u_h, v)_{\mathcal{T}_h} = F_1(v).
\]
Substituting $v = \nabla u_h$ in the above equation, we get
\begin{equation}
\|\nabla u_h\|_{\mathcal{T}_h} \leq C (\|q_h\| + \|F_1\|). 
\end{equation}
Taking $v = q_h$ in (15), $r = \hat{q}_h$ in (13c), and $w = u_h$ in (13b), we have
\begin{align*}
\|q_h\|^2 &= F_1(q_h) + F_2(u_h) - F_3(\hat{q}_h) \\
&\leq \|F_1\|\|q_h\| + C\|F_2\| \sum_{K \in \mathcal{T}_h} \left( \|\nabla u_h\|_{L^2(K)} + \|h^{-1/2} P_N u_h\|_{\partial K} \right) + \|F_3\|\|h^{1/2} \hat{q}_h \cdot n\|_{\partial \mathcal{T}_h}.
\end{align*}

By (8), we have
\begin{align*}
\|h^{-1/2} P_N u_h\|_{\partial \mathcal{T}_h} &\leq C \sum_{K \in \mathcal{T}_h} \sup_{r \in N_h} \frac{\langle r \cdot n, P_N u_h \rangle_{\partial \mathcal{T}_h}}{\|h^{1/2} r \cdot n\|} \\
&= C \sum_{K \in \mathcal{T}_h} \sup_{r \in N_h} \frac{\langle r \cdot n, u_h \rangle_{\partial \mathcal{T}_h}}{\|h^{1/2} r \cdot n\|} \\
&\leq C \sum_{K \in \mathcal{T}_h} \sup_{r \in N_h} \frac{F_3(r)}{\|h^{1/2} r \cdot n\|} \\
&\leq C \|F_3\|.
\end{align*}

Combining this with (16), we get
\begin{align*}
\sum_{K \in \mathcal{T}_h} \left( \|\nabla u_h\|_{L^2(K)} + \|h^{-1/2} P_N u_h\|_{\partial K} \right) &\leq C (\|q_h\| + \|F_3\|).
\end{align*}
Thus we estimate as
\begin{align*}
\|q_h\|^2 &\leq \|F_1\|\|q_h\| + C\|F_2\| (\|q_h\| + \|F_1\|) + C\|F_3\| (\|F_2\| + \|q_h\|).
\end{align*}
Using Young’s inequality, we obtain
\begin{align*}
\|q_h\|^2 &\leq C (\|F_1\|^2 + \|F_2\|^2 + \|F_3\|^2).
\end{align*}
From this and (14), it follows that
\begin{align*}
\|h^{1/2} \hat{q}_h \cdot n\|_{\partial \mathcal{T}_h} &\leq C (\|F_2\| + \|q_h\|) \leq C (\|F_1\| + \|F_2\| + \|F_3\|),
\end{align*}
which completes the proof. \hfill \square

If $f \equiv 0$, it follows from Theorem 3.1 that $q_h = 0$, $u_h$ is constant on each element, $\hat{q}_h \cdot n = 0$, and $P_N u_h = 0$ on element boundaries. Thus, we have verified the existence and uniqueness of our method.

3.2. Optimal convergence of $q_h$. The projections of errors are defined as
\begin{align*}
e_q = P_q q - q_h, \quad e_u = P_w u - u_h, \quad e_{\hat{q}} \cdot n = P_N q \cdot n - \hat{q}_h \cdot n.
\end{align*}

**Theorem 3.2.** If $u \in H^{k+2}(\Omega)$, then we have
\begin{align*}
\|e_q\| + \|h^{1/2} e_{\hat{q}} \cdot n\|_{\partial \mathcal{T}_h} &\leq C h^{k+1} |u|_{k+2}.
\end{align*}
Proof. The problem (1) is rewritten into
\[(q, v)_{T_h} - (u, \nabla \cdot v)_{T_h} + \langle u, v \cdot n \rangle_{\partial T_h} = 0 \quad \forall v \in V_h, \]
\[- (q, \nabla w)_{T_h} + \langle q \cdot n, w \rangle_{\partial T_h} = (f, w) \quad \forall w \in W_h, \]
\[\langle u, r \cdot n \rangle_{\partial T_h} = 0 \quad \forall r \in N_h. \]
By the property of the $L^2$-projections, the above equations become
\[(17a) \quad (P_V q, v)_{T_h} - (P_W u, \nabla \cdot v)_{T_h} + \langle P_W u, v \cdot n \rangle_{\partial T_h} = G_1(v) \quad \forall v \in V_h, \]
\[(17b) \quad - (P_V q, \nabla w)_{T_h} + \langle P_N q \cdot n, w \rangle_{\partial T_h} = (f, w) + G_2(w) \quad \forall w \in W_h, \]
\[(17c) \quad \langle P_W u, r \cdot n \rangle_{\partial T_h} = G_3(r) \quad \forall r \in N_h, \]
where we have integrated by parts in the first equation and
\[G_1(v) := -\langle u - P_W u, v \cdot n \rangle_{\partial T_h}, \]
\[G_2(w) := -\langle q \cdot n - P_N q \cdot n, w \rangle_{\partial T_h}, \]
\[G_3(r) := -\langle u - P_W u, r \cdot n \rangle_{\partial T_h}. \]
The norms of $G_1$ and $G_3$ are bounded as
\[\|G_1\| = \sup_{v \in V_h} \frac{G_1(v)}{\|v\|} \leq \|u - P_W u\|_{\partial T_h} \cdot Ch^{-1/2} \leq Ch^{k+1/2} |u|_{k+2}, \]
\[\|G_3\| = \sup_{r \in N_h} \frac{G_3(r)}{h^{1/2} r \cdot n} \leq \|u - P_W u\|_{\partial T_h} \cdot h^{-1/2} \leq Ch^{k+1/2} |u|_{k+2}. \]
Using [6, Lemma 3], we can estimate
\[|G_2(w)| \leq \|\langle q - P_N q \rangle \cdot n, w \rangle_{\partial T_h} \|\leq \|q - P_N q \cdot n\|_{\partial T_h} \cdot Ch^{1/2} \|\nabla w\|_{T_h}. \]
The norm of $G_2$ is bounded as
\[\|G_2\| = \sum_{K \in T_h} \sup_{w \in W(K)} \frac{G_2(w)}{\|\nabla w\|_{L^2(K)} + \|h^{-1/2} P_N w\|_{L^2(\partial K)}} \leq Ch^{k+1} |q|_{k+1}. \]
Subtracting (5) from (17), we obtain the error equations
\[(18a) \quad (e_q, v)_{T_h} - (e_u, \nabla \cdot v)_{T_h} + \langle e_u, v \cdot n \rangle_{\partial T_h} = G_1(v) \quad \forall v \in V_h, \]
\[(18b) \quad - (e_q, \nabla w)_{T_h} + \langle e_q \cdot n, w \rangle_{\partial T_h} = G_2(w) \quad \forall w \in W_h, \]
\[(18c) \quad \langle e_u, r \cdot n \rangle_{\partial T_h} = G_3(r) \quad \forall r \in N_h. \]
Applying Theorem 3.1 to the error equations leads to
\[\|e_q\| + \|h^{1/2} e_q \cdot n\|_{\partial T_h} \leq C (\|G_1\| + \|G_2\| + \|G_3\|) \leq Ch^{k+1} |u|_{k+2}. \]
\[
\]
From Theorem 3.1, it also follows that
\[(19) \quad \|\nabla e_u\|_{T_h} + \|h^{-1/2} P_N e_u\|_{\partial T_h} \leq Ch^{k+1} |u|_{k+2}. \]
3.3. \textbf{L}^2\textbf{-error estimate of} \( u_h \). We consider the following adjoint problem: Find \((\theta, \xi) \in H^1(\Omega) \times (H^2(\Omega) \cap H^1_0(\Omega))\) such that
\[
\begin{align*}
\theta + \nabla \xi &= 0 \quad \text{in } \Omega, \\
\nabla \cdot \theta &= e_u \quad \text{in } \Omega, \\
\xi &= 0 \quad \text{on } \partial \Omega.
\end{align*}
\]

It is well known that the elliptic regularity holds:
\[
\|\theta\|_1 + \|\xi\|_2 \leq C\|e_u\|.
\]

We provide an \( L^2 \)-error estimate of \( u_h \) by the Aubin–Nitsche technique.

**Theorem 3.3.** If \( k \geq 1 \) and \( u \in H^{k+2}(\Omega) \), then there exists a constant \( C \) such that
\[
\|e_u\| \leq C h^{k+2}|u|_{k+2}.
\]

**Proof.** Since (17) holds for the adjoint problem, we have
\[
\begin{align*}
(21a) & \quad (P \theta, v)_{T_h} - (P W \xi, \nabla \cdot v)_{T_h} + (P W \xi, v \cdot n)_{\partial T_h} = G'_1(v) \quad \forall v \in V_h, \\
(21b) & \quad -(P \theta, \nabla w)_{T_h} + (P N \theta \cdot n, w)_{\partial T_h} = (e_u, w) + G'_2(w) \quad \forall w \in W_h, \\
(21c) & \quad (P W \xi, r \cdot n)_{\partial T_h} = G'_3(r) \quad \forall r \in N_h,
\end{align*}
\]

where
\[
\begin{align*}
G'_1(v) &= -\langle \xi - P W \xi, v \cdot n \rangle_{\partial T_h}, \\
G'_2(w) &= -\langle (\theta - P N \theta) \cdot n, w \rangle_{\partial T_h}, \\
G'_3(r) &= -\langle \xi - P W \xi, r \cdot n \rangle_{\partial T_h}.
\end{align*}
\]

Taking \( v = -e_q \) in (21a), \( w = e_u \) in (21b), and \( r = e_q \) in (21c), we have
\[
(22) \quad -(P \theta, e_q)_{T_h} - (\nabla P W \xi, e_q)_{T_h} - (P \theta, \nabla e_u)_{T_h} + (P N \theta \cdot n, e_u)_{\partial T_h} + (P W \xi, e_q \cdot n)_{\partial T_h}
\]
\[
= \|e_u\|^2 - G'_1(e_q) + G'_2(e_u) + G'_3(e_q).
\]

Choosing \( v = -P \theta \) in (18a), \( w = P W \xi \) in (18b), and \( r = P N \theta \) in (18c), we have
\[
(23) \quad -(e_q, P \theta)_{T_h} - (\nabla e_u, P \theta)_{T_h} - (e_q, \nabla P W \xi)_{T_h} + (e_q \cdot n, P W \xi)_{\partial T_h} + (e_u, P N \theta \cdot n)_{\partial T_h}
\]
\[
= -G'_1(P \theta) + G'_2(P W \xi) + G'_3(P N \theta).
\]

Subtracting (23) from (22) yields
\[
\|e_u\|^2 = -G'_1(P \theta) + G'_2(P W \xi) + G'_3(P N \theta) - G'_1(e_q) + G'_2(e_u) + G'_3(e_q).
\]

We will bound the terms on the right-hand side. The first and third terms are bounded as
\[
\begin{align*}
- G'_1(P \theta) + G'_3(P N \theta) &= \langle u - P_W u, (P \theta - P N \theta) \cdot n \rangle_{\partial T_h} \\
&\leq \|u - P_W u\|_{\partial T_h} (\|P \theta - \theta\|_{\partial T_h} + \|P N \theta \cdot n\|_{\partial T_h}) \\
&\leq Ch^{k+3/2}|u|_{k+2} Ch^{1/2} |\theta|_1 \\
&= Ch^{k+2}|u|_{k+2} |\theta|_1.
\end{align*}
\]
Let $P_1$ denote the $L^2$-projection from $L^2(\Omega)$ onto $P_1(\mathcal{T}_h)$. Note that $\langle (q - P_N q) \cdot n, P_1 \xi \rangle_{\partial \mathcal{T}_h} = 0$ since we assume $k \geq 1$. We have

$$
|G_2(P_W \xi)| = |\langle (q - P_N q) \cdot n, P_W \xi - P_1 \xi \rangle_{\partial \mathcal{T}_h}| \\
\leq \| (q - P_N q) \cdot n \|_{\partial \mathcal{T}_h} (\| P_W \xi - \xi \|_{\partial \mathcal{T}_h} + \| \xi - P_1 \xi \|_{\partial \mathcal{T}_h}) \\
\leq C h^{k+1/2} |q|_{k+1} \cdot C h^{3/2} |\xi|_2 \\
= C h^{k+2} |q|_{k+1} |\xi|_2.
$$

\[ (24) \]

The rest terms are bounded as follows:

$$
|G_1'(e_q)| \leq C h |\xi|_2 \| e_q \|, \\
|G_2'(e_u)| \leq C h |\theta|_1 \| \nabla e_u \|_{\mathcal{T}_h} \quad \text{(by [6, Lemma 3])} \\
\leq C h |\theta|_1 \left( \| e_q \| + h^{k+1} |u|_{k+2} \right), \quad \text{(by (19))} \\
|G_3'(e_q)| \leq C h |\xi|_2 \| h^{1/2} e_q \cdot n \|_{\partial \mathcal{T}_h}.
$$

Thus we deduce

$$
\| e_u \| \leq C \left( h \| e_q \| + h h^{1/2} e_q \cdot n \|_{\partial \mathcal{T}_h} + h^{k+2} |u|_{k+2} \right) \leq C h^{k+2} |u|_{k+2}.
$$

\[ \square \]

**Remark 3.4.** Theorem 3.3 also holds for $k = 0$. When $k = 0$, we can use the Crouzeix–Raviart interpolation $I_{CR}$ instead of $P_W$. Then, the right-hand sides of (18) and (21) are changed as $G_1(v) = - (\nabla (u - I_{CR} \xi), v)_{\mathcal{T}_h}$, $G_3(r) = 0$, $G_1'(v) = - (\nabla (\xi - I_{CR} \xi), v)_{\mathcal{T}_h}$, $G_3'(r) = 0$. It is clear that they are bounded by the Schwarz inequality and the interpolation error estimate, and we do not need to use the $L^2$-projection in (24) since $G_2(I_{CR} \xi) = \langle (q - P_N q) \cdot n, I_{CR} \xi \rangle_{\partial \mathcal{T}_h} = 0$.

4. **Proof of the inf-sup condition for triangular elements**

We show that the inf-sup conditions (7) and (8) are satisfied for the triangular $P_h$-element in the two-dimensional case.

Let $T_1$ and $T_2$ be the reference triangles whose vertices are $\{(0, 0), (0, 1), (1, 1)\}$ and $\{(1, 0), (0, 0), (0, 1)\}$, respectively. Let $\{e_1, e_2, e_3\}$ and $\{e_3, e_4, e_5\}$ denote the edges of $T_1$ and $T_2$, respectively, see Figure 1. For $1 \leq i \leq 5$, we define $F_i$ by the linear transforms from $e_i$ to $[-1, 1]$ such that

$$
F_1((0, 0)) = -1, \quad F_1((1, 0)) = 1, \\
F_2((0, 0)) = -1, \quad F_2((1, 1)) = 1, \\
F_3((0, 0)) = -1, \quad F_3((0, 0)) = 1, \\
F_4((0, 0)) = -1, \quad F_4((0, 1)) = 1, \\
F_5((0, 1)) = -1, \quad F_5((1, 1)) = 1.
$$
Let $\chi^{(i)}$ denote the characteristic function of $e_i$ and let $\varphi_p$ denote the Legendre polynomial of degree $p$ on $[-1, 1]$. We define the normalized Legendre polynomial on $e_i$ by

$$\varphi_p^{(i)} = \frac{(\varphi_p \circ F_i)\chi^{(i)}}{\left(\int_{e_i} (\varphi_p \circ F_i)^2 ds\right)^{1/2}}$$

for $0 \leq p \leq k$. It is clear that \{${\varphi_0^{(i)}, \varphi_1^{(i)}, \ldots, \varphi_k^{(i)}$} is a basis of $N(e_i) = P_k(e_i)$ and satisfies the orthogonality

$$\langle \varphi_p^{(i)}, \varphi_q^{(j)} \rangle_{\partial T_1 \cup \partial T_2} = \delta_{ij} \delta_{pq} \quad \text{for } 1 \leq i, j, p, q \leq 5.$$

Note that $\varphi_p^{(i)} = (-1)^p$ at the starting point of $e_i$ and $\varphi_p^{(i)} = 1$ at the end point of $e_i$.

**Lemma 4.1.** Let $r \in \bigoplus_{1 \leq i \leq 5} N(e_i)$. It holds that $r \cdot n = 0$ on $e_i$ ($1 \leq i \leq 5$) if and only if

$$\langle r \cdot n, w \rangle_{\partial T_1 \cup \partial T_2} = 0 \quad \forall w \in W(T_1) \oplus W(T_2).$$

**Proof.** We prove that only when $k$ is even, i.e., $k = 2k'$, since the proof when $k$ is odd is similar. We show that $r \cdot n = 0$ follows from (25). We can write $r \cdot n$ as

$$r \cdot n = \sum_{i=1}^{5} \sum_{p=0}^{2k'} a_p^{(i)} \varphi_p^{(i)}, \quad a_p^{(i)} \in \mathbb{R}.$$ 

First, we show that $a_{2q}^{(i)} = 0$ for $1 \leq q \leq k'$ and $1 \leq i \leq 5$. Let us define

$$w_1 = \varphi_1^{(2)} - \varphi_2^{2k'+1} + \varphi_2^{2k'+1}$$

for $1 \leq q \leq k'$, see also Figure 2. Since $w_1$ is continuous at the vertices, there exists $Ew_1 \in W(T_1) \oplus W(T_2)$ such that $Ew_1 = w_1$ on element boundaries. For simplicity,
we use the same symbol $w_1$ to denote $Ew_1$. Choosing $w = w_1$ in (25) and noting that
\[ \langle r \cdot n, \varphi^{(i)}_{2k'} \rangle_{\partial T_1 \cup \partial T_2} = 0 \] for any $i$, we have
\[ \langle r \cdot n, w_1 \rangle_{\partial T_1 \cup \partial T_2} = \sum_{p=0}^{2k'} \langle a_p^{(1)}, \varphi^{(1)}_p, \varphi^{(1)}_{2q} \rangle_{e_1} = a^{(1)}_{2q} = 0. \]

Taking $w = \varphi^{(1)}_{2k'+1} + \varphi^{(2)}_{2q} - \varphi^{(3)}_{2k'+1}$ and $w = -\varphi^{(1)}_{2k'+1} + \varphi^{(2)}_{2k'+1} + \varphi^{(3)}_{2q}$, we get $a^{(2)}_{2q} = a^{(3)}_{2q} = 0$ for $1 \leq q \leq k'$. Similarly, it follows that $a^{(4)}_{2q} = a^{(5)}_{2q} = 0$ for $1 \leq q \leq k'$.

**Figure 2.** Diagrams of the test functions $w_1$, $w_2$, and $w_3$, where the subscripts are discarded and the number at each vertex indicates the value of the function at the vertex.

Next, we show that $a^{(i)}_1 = a^{(i)}_3 = \cdots = a^{(i)}_{2k'-1}$ for $1 \leq i \leq 5$. In (25), choosing
\[ w = w_2 := \varphi^{(1)}_{2q-1} - \varphi^{(2)}_{2q-1} + \varphi^{(3)}_{2k'}, \]

in view of $a^{(i)}_{2k'} = 0$ for $1 \leq i \leq 5$, we have
\[ \langle r \cdot n, w_2 \rangle_{\partial T_1 \cup \partial T_2} = a^{(1)}_{2q-1} - a^{(2)}_{2q-1} = 0 \quad (1 \leq q \leq k'). \]

Choosing $w = \varphi^{(1)}_{2k'+1} + \varphi^{(2)}_{2q-1} - \varphi^{(3)}_{2q-1}$ in (25) yields $a^{(2)}_{2q-1} = a^{(3)}_{2q-1}$ for $1 \leq q \leq k'$. Similarly, we deduce that $a^{(3)}_{2q-1} = a^{(4)}_{2q-1} = a^{(5)}_{2q-1}$ for $1 \leq q \leq k'$. Hence, we define $a_{2q-1} := a^{(1)}_{2q-1} = \cdots = a^{(5)}_{2q-1}$ for $1 \leq q \leq k'$ to omit the subscripts.

Finally, we prove that $a_{2q-1} = 0$ for $1 \leq q \leq k'$. We take the following $w_3$ as a test function:
\[ w_3 := \varphi^{(1)}_{2q-1} + \varphi^{(2)}_{2k'} - \varphi^{(3)}_{2k'+1} - \varphi^{(3)}_{2k'} + \varphi^{(5)}_{2q-1}. \]

Since $w_3$ is single valued on $e_3$, we see that
\[ \langle r \cdot n, w_3 \rangle_{e_3} = \langle r \cdot n, -\varphi^{(3)}_{2k'+1} \rangle_{\partial T_1 \cup \partial T_2} = 0. \]
Noting that $a_{2k}^{(i)} = 0$ for $1 \leq i \leq 5$, we get
\[ \langle r \cdot n, \varphi_{2k}^{(2)} \rangle_{\partial T_1 \cup \partial T_2} = \langle r \cdot n, -\varphi_{2k}^{(4)} \rangle_{\partial T_1 \cup \partial T_2} = 0 \]
and
\[ \langle r \cdot n, w_3 \rangle_{\partial T_1 \cup \partial T_2} = \langle r \cdot n, \varphi_{2q-1}^{(1)} + \varphi_{2q-1}^{(5)} \rangle_{\partial T_1 \cup \partial T_2} = 2a_{2q-1} = 0. \]
Thus, we conclude that all coefficients equal zero. \hfill \square

Lemma 4.2. Let $K_1$ and $K_2 \in \mathcal{T}_h$ be two adjacent triangles. There exists a constant $C$ independent of $h$ such that, for $r \in \bigoplus_{e \in \partial K_1 \cup \partial K_2} N(e)$,
\[ \sum_{i=1}^{2} \| h^{1/2} r \cdot n \|_{L^2(\partial K_i)} \leq C \sum_{i=1,2} \sup_{w \in W(K_i)} \frac{\langle r \cdot n, w \rangle_{\partial K_i}}{\| \nabla w \|_{L^2(K_i)} + \| h^{-1/2} P_N \nabla w \|_{L^2(\partial K_i)}}. \]

Proof. Let $T_1$ and $T_2$ be the reference triangles and let $e_1, e_2, \ldots, e_5$ be the edges of $T_1$ and $T_2$. By Lemma 4.1, we see that
\[ \| r \cdot n \|_{\partial T_1 \cup \partial T_2} := \sum_{i=1,2} \sup_{w \in W(T_i)} \frac{\langle r \cdot n, w \rangle_{\partial T_i}}{\| \nabla w \|_{L^2(T_i)} + \| P_N w \|_{\partial T_i}} \]
is a norm on $\bigoplus_{1 \leq i \leq 5} N(e_i)$. Since any two norms on a finite-dimensional space are equivalent, there exists a constant $C$ such that
\[ \| r \cdot n \|_{\partial T_1 \cup \partial T_2} \leq C \| r \cdot n \|_{\partial T_1 \cup \partial T_2} \quad \forall r \in \bigoplus_{1 \leq i \leq 5} N(e_i). \]
By considering the Piola transforms from $T_i$ to $K_i (i = 1, 2)$ and the scaling argument, we obtain the assertion. \hfill \square

The inf-sup condition (7) immediately follows from Lemma 4.2. Similarly, we can prove the transposed inf-sup condition (8) from the following lemma.

Lemma 4.3. Let $K$ be an element of $\mathcal{T}_h$ and $w \in W(K)$. Then, $P_N w = 0$ on $\partial K$ if and only if
\[ \langle r \cdot n, w \rangle_{\partial K} = 0 \quad \forall r \in N(\partial K). \]

Proof. Since $P_N w \in N(\partial K)$, we can choose $r \cdot n = P_N w$. We then have
\[ \langle r \cdot n, w \rangle_{\partial K} = \langle r \cdot n, P_N w \rangle_{\partial K} = \| P_N w \|_{\partial K}^2 = 0, \]
which implies $P_N w|_{\partial K} = 0$. \hfill \square

The proof of the transposed inf-sup condition is the same as in Lemma 4.1, so we omit it here.

Remark 4.4. We have proved that the inf-sup condition (7) holds for the triangular elements where $N_h$ and $W_h$ are polynomials of degree $k$ and $k + 1$, respectively. However, it is still open whether there exists a pair of $N_h$ and $W_h$ satisfying the inf-sup condition (7) in the three- or higher-dimensional cases.
5. Numerical results

In this section, we examine the convergence property of the proposed method (5) by numerical experiments. As a test problem, we consider the Poisson equation with homogeneous Dirichlet boundary condition

\[-\Delta u = 2\pi^2 \sin(\pi x) \sin(\pi y) \quad \text{in} \; \Omega,\]
\[u = 0 \quad \text{on} \; \partial \Omega,\]

where \( \Omega = (0, 1)^2 \) and the exact solution is given by \( u(x, y) = \sin(\pi x) \sin(\pi y) \). We use the unstructured triangulations whose mesh sizes are approximately \( 0.4 \times 2^{-i} \) (1 \(< i \leq 4 \)) and compute the solution of (5), varying the polynomial degree \( k \) from 0 to 3. All numerical computations are carried out by FreeFEM [4]. The \( L^2 \)-errors of \( q_h \) and \( u_h \) are displayed in Table 5. We observe that the orders of convergence in \( q_h \) and \( u_h \) are \( k + 1 \) and \( k + 2 \), respectively, which are of optimal order and fully agrees with Theorems 3.2 and 3.3.

| \( k \) | \( h \)   | \( \| q - q_h \| \) | Order | \( \| u - u_h \| \) | Order |
|-------|--------|----------------|-------|----------------|-------|
| 0.1901| 1.660E-01 | 1.20           | 2.404E-03 | 2.35           |
| 0.0509| 8.355E-02 | 0.98           | 6.150E-04 | 1.94           |
| 0.0262| 4.120E-02 | 1.07           | 1.461E-04 | 2.17           |
| 0.1901| 1.660E-01 | —              | 2.404E-03 | 2.35           |
| 0.0509| 8.355E-02 | 0.98           | 6.150E-04 | 1.94           |
| 0.0262| 4.120E-02 | 1.07           | 1.461E-04 | 2.17           |
| 0.1901| 1.660E-01 | —              | 2.404E-03 | 2.35           |
| 0.0509| 8.355E-02 | 0.98           | 6.150E-04 | 1.94           |
| 0.0262| 4.120E-02 | 1.07           | 1.461E-04 | 2.17           |

Table 1. Convergence history for the method (5)

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Department of Mathematics, Institute of Pure and Applied Sciences, University of Tsukuba, 1-1-1 Tennodai, Tsukuba, Ibaraki 305-8571, Japan

Email address: ioikawa00@gmail.com