Permutation equivalent maximal irreducible Goppa codes

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Abstract

We consider the problem of finding the number of permutation non-equivalent classical irreducible maximal Goppa codes having fixed parameters $q, n$ and $r$ from a group theory point of view.

Keywords: Goppa codes, Linear codes, Permutation groups

1 Introduction

The study of classical Goppa codes is important since they are a very large class of codes, near to random codes. They are easy to generate and possess an interesting algebraic structure. For these reasons they are used in McEliece’s public key cryptosystem [16]. This cryptosystem is based on the difficulty to find a generator matrix of a Goppa code when a “scrambled” of it is known.

In this paper we consider the problem of finding an upper bound for the number of permutation non-equivalent irreducible maximal Goppa codes. This question was considered by several authors (see for example [6], [2], [3], [7], [9]). In Section 3 we briefly recall these approaches. In particular, we describe the action of a group $FG$ isomorphic to $AGL(1,q^n)$ on the $q^n$ columns of a suitable parity-check matrix $H_\alpha$. This induces on maximal irreducible Goppa codes the same action which arises from [18]. This action does not describe exactly the orbits of Goppa codes, since in some cases the
number of permutation non-equivalent Goppa codes is less than the number of orbits of $FG$. The group $FG$ acts faithfully on columns of $H_{\alpha}$, so that it may be seen as a subgroup of the symmetric group $S_{q^n}$. It seems interesting to study if there is a proper subgroup of $S_{q^n}$ containing $FG$, acting on the set $\Omega$ of classical irreducible maximal Goppa codes of fixed parameters, and giving on $\Omega$ exactly the orbits of permutation equivalent codes. In order to consider this problem, we analyze the subgroups of $S_{q^n}$ containing $FG$ and in Section 4 we find that there exists exactly one maximal subgroup $M$, isomorphic to $AGL(nm,p)$ of $S_{q^n}(A_{q^n})$ containing $FG$ ($q = p^m$). This suggests that one could consider the action of this $M$ on codes to reach the right bound.

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2 Preliminaries

In this section we fix some notation and we recall some basic concepts about linear codes and in particular about Goppa codes. Our main references are [11] for coding theory and [5] for group theory.

We denote by $F_q$ the finite field with $q$ elements, where $q = p^m$ is a power of a prime $p$; let $N$, $k$, $n$ and $r$ be natural numbers, $k \leq N$. We consider two extensions of $F_q$, of degree $n$ and $nr$, $F_{q^n}$ and $F_{q^{nr}}$ respectively; $F_{q^n}[x]$ denotes the polynomial ring over $F_{q^n}$ and $\varepsilon$ is a primitive element of $F_{q^n}$, $F_{q^n} = \langle \varepsilon \rangle$. We refer to the vector space of dimension $N$ over $F_q$ as to $(F_q)^N$.

In the following if $H$ is an $(N - k) \times N$ matrix with entries in $F_q$ and rank equal to $N - k$, the set $C$ of all vectors $c \in (F_q)^N$ such that $Hc^T = 0$ is an $(N, k)$ linear code over $F_q$, of length $N$ and dimension $k$, i.e. a subspace of $(F_q)^N$ of dimension $k$. The elements of $C$ are called codewords and matrix $H$ is a parity-check matrix of $C$. Any $k \times N$ matrix $G$ whose rows form a vector basis of $C$ is called a generator matrix of $C$.

**Definition 2.1.** Let $E/K$ be a field extension. A linear code $C$ is called a subfield subcode if $C$ is obtained as the restriction to $K^n$ of a linear subspace $L$ of $E^n$.

By abuse of notation we call parity-check matrix also a matrix $H$ with entries in $E$ such that $Hc^T = 0$ for all $c \in C$. According to this assumption, $H_1$ and $H_2$ may be parity-check matrices for the same code even if their entries are in different extension fields or they have different ranks.
Definition 2.2 ([11]). Let $C_1$ and $C_2$ be two linear codes over $\mathbb{F}_q$ of length $N$, let $G_1$ be a generator matrix of $C_1$. Codes $C_1$ and $C_2$ are permutation equivalent provided there is a permutation $\sigma \in S_N$ of coordinates which sends $C_1$ in $C_2$. Thus $C_1$ and $C_2$ are permutation equivalent provided there is a permutation matrix $P$ such that $G_1P$ is a generator matrix for $C_2$. They are monomially equivalent provided there is a monomial matrix $M$ so that $G_1M$ is a generator matrix for $C_2$ and equivalent provided there is a monomial matrix $M$ and an automorphism $\gamma$ of the field $\mathbb{F}_q$ so that $C_2 = C_1M\gamma$.

If code $C_2$ is permutation equivalent to $C_1$ with parity-check matrix $H_1$, we can obtain a parity-check matrix $H_2$ for $C_2$ by permuting columns of $H_1$ (and vice versa).

Definition 2.3. Let $g(x) = \sum g_i x^i \in \mathbb{F}_q^n[x]$ and let $L = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}$ denote a subset of elements of $\mathbb{F}_q^n$ which are not roots of $g(x)$. Then the Goppa code $G(L, g)$ is defined as the set of all vectors $c = (c_1, c_2, \ldots, c_N)$ with components in $\mathbb{F}_q$ which satisfy the condition:

$$\sum_{i=0}^{N} \frac{c_i}{x - \varepsilon_i} \equiv 0 \mod g(x).$$

(1)

Usually, but now always, the set $L = \{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}$ is taken to be the set of all elements in $\mathbb{F}_q^n$ which are not roots of the Goppa polynomial $g(x)$. In this case the Goppa code is said maximal. If the degree of $g(x)$ is $r$, then the Goppa code is called a Goppa code of degree $r$. It is easy to see ([17]) that a parity-check matrix for $G(L, g)$ is given by

$$H = \begin{pmatrix}
\frac{1}{g(\varepsilon_1)} & \frac{1}{g(\varepsilon_2)} & \cdots & \frac{1}{g(\varepsilon_N)} \\
\frac{\varepsilon_1^{-1}}{g(\varepsilon_1)} & \frac{\varepsilon_2^{-1}}{g(\varepsilon_2)} & \cdots & \frac{\varepsilon_N^{-1}}{g(\varepsilon_N)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{\varepsilon_1^{-r}}{g(\varepsilon_1)} & \frac{\varepsilon_2^{-r}}{g(\varepsilon_2)} & \cdots & \frac{\varepsilon_N^{-r}}{g(\varepsilon_N)}
\end{pmatrix}.$$ 

Note that the code $C = \ker H$ is a subspace of $(\mathbb{F}_q^n)^N$ and the Goppa code $G(L, g)$ is its subfield subcode on $\mathbb{F}_q$.

Definition 2.4. A Goppa code $G(L, g)$ is called irreducible if $g(x)$ is irreducible over $\mathbb{F}_q^n$.

In the following by Goppa code we mean maximal irreducible classical Goppa code of degree $r$, so that $N = q^n$. By Definition 2.3, a vector $c =$
\((c_1, c_2, \ldots, c_{q^n}) \in (\mathbb{F}_q)^{q^n}\) is a codeword of \(G(L, g)\) if and only if it satisfies (1). If \(\alpha\) is any root of \(g(x)\), \(\alpha \in \mathbb{F}_{q^n}\), then \(g(x) = \prod_{i=0}^{r-1}(x - \alpha^{q^ni})\) and (1) is equivalent to the \(r\) equations

\[
\sum_{i=1}^{q^n} \frac{c_i}{\alpha^{q^nj} - \varepsilon_i} = 0, \quad 0 \leq j \leq r - 1.
\]  

Hence \(G(L, g)\) is completely described by any root \(\alpha\) of \(g(x)\) and we may denote this code by \(C(\alpha)\). From (2) we easily get a parity-check matrix \(H_\alpha \in Mat_{1 \times q^n}(\mathbb{F}_{q^n})\) for \(C(\alpha)\) (see [6]):

\[
H_\alpha = \left( \frac{1}{\alpha - \varepsilon_1}, \frac{1}{\alpha - \varepsilon_2}, \ldots, \frac{1}{\alpha - \varepsilon_{q^n}} \right). \tag{3}
\]

It is important to stress that by using parity-check matrix \(H_\alpha\) to define \(C(\alpha)\) we implicitly fix an order in \(L\). So, we set

\[
L = \{\varepsilon, \varepsilon^2, \ldots, \varepsilon^{q^n-1}, \varepsilon^{-\infty}\},
\]

where \(\varepsilon^{-\infty} = 0, \varepsilon_i = \varepsilon^i\) and the matrix \(H_\alpha\) is

\[
H_\alpha = \left( \frac{1}{\alpha - \varepsilon_1}, \frac{1}{\alpha - \varepsilon_2}, \ldots, \frac{1}{\alpha - 1}, \frac{1}{\alpha} \right).
\]

We observe that the Goppa code \(C(\alpha)\) is the subfield subcode of codes having as parity-check matrices both \(H\) and \(H_\alpha\). Moreover, there exist matrices having structure different from \(H\) and \(H_\alpha\), which are parity-check matrices for \(C\).

We denote by \(\Omega = \Omega(q, n, r)\) the set of Goppa codes, with fixed parameters \(q, n, r\).

In the following an action on set \(S\) is considered, where \(S = S(n, r)\) is composed of all elements in \(\mathbb{F}_{q^n}\) of degree \(r\) over \(\mathbb{F}_{q^n}\).

### 3 Three actions on \(\Omega\)

In this section we briefly present semiaffine actions introduced in [1] and in [8]. These actions have degrees \(\frac{|S|}{r}\) and \(|S|\) respectively. Moreover we consider an action of the group \(A\Gamma L(1, q^n)\) on entries of parity-check matrix of type \(H_\alpha\). This time the degree is \(q^n\).

In [1], the author works directly on polynomials by studying automorphism groups of several classes of codes.
If \( \psi \in AGL(1, q^n) \), \( \psi(z) = az + b \), \( a, b \in \mathbb{F}_{q^n} \), \( a \neq 0 \), he defines

\[
g^\psi(x) = \sum_{i=0}^{r} g_i(ax + b)^i.
\]

The map \( \psi \) acts also on set \( L = \mathbb{F}_{q^n} \), \( \mathbb{F}_{q^n}^* = \langle \varepsilon \rangle \), by

\[
L^\psi = (\varepsilon^{\psi^{-1}}, \ldots, (\varepsilon^{q^n-1})^{\psi^{-1}}, (\varepsilon^{-\infty})^{\psi^{-1}}).
\]

The code \( G(L, g)^\psi = G(L^\psi, g^\psi) \) is said the conjugate of code \( G(L, g) \) by \( \psi \).

**Proposition 3.1.** [1] The Goppa codes are invariant by conjugation under the affine group \( AGL(1, q^n) \), i.e. \( G(L, g)^\psi = G(L, g) \) for all \( \psi \) such that \( \psi(z) = az + b \), \( a, b \in \mathbb{F}_{q^n} \), \( a \neq 0 \).

We get

**Corollary 3.2.** Goppa codes \( G(L, g)^\psi \) is equivalent to Goppa code \( G(L, g) \).

**Proof.** We known that \( G(L, g)^\psi = G(L^\psi, g^\psi) \) and from Proposition 3.1 \( G(L, g)^\psi = G(L, g) \). From Definition 2.2 it follows that \( G(L, g)^\psi \) is equivalent to \( G(L^\psi, g^\psi) = G(L, g) \) so \( G(L, g) \) is equivalent to \( G(L, g^\psi) \).

More generally, if \( \psi \in AGL(1, q^n) \), \( \psi(z) = az^{q^t} + b \), with \( a, b \in \mathbb{F}_{q^n} \), \( a \neq 0 \) and \( t \in \{0, \ldots, n-1\} \), we define

\[
g^\psi(x) = \sum_{i=0}^{r} g_i(ax^{q^t} + b)^i \tag{4}
\]

Equation 4 suggests to consider an action \( \sigma \) on \( \mathbb{P} \subseteq \mathbb{F}_{q^n}[x] \), where \( \mathbb{P} \) is the set of irreducible polynomials of degree \( r \). For \( g \in \mathbb{P} \), \( g^{\sigma(\psi)} \) is the unique polynomial \( f \) of degree \( r \) such that \( g(\alpha) = 0 \) if and only if \( f(\beta) = 0 \) for \( \beta = (\frac{a-b}{a})^{q^{nr-t}} \) (note \( g^{\sigma(\psi)} \in \mathbb{P} \)).

Indeed, if \( g(x) = \sum_{i=0}^{r} g_i x^i \), there exist \( \tilde{g}_i, \forall i = 1, \ldots, r, \tilde{a}, \tilde{b} \) such that \( \tilde{g}_i^{q^t} = g_i, \forall i = 1, \ldots, r, \tilde{a}^{q^t} = a, \tilde{b}^{q^t} = b \) so that

\[
\sum_{i=0}^{r} \tilde{g}_i^{q^t}(ax^{q^t} + \tilde{b})^i = \left( \sum_{i=0}^{r} \tilde{g}_i(ax + \tilde{b})^i \right)^{q^t} = f(x)^{q^t}.
\]

It is immediate to recognize that \( g(\alpha) = 0 \), for \( \alpha \in \mathbb{S} \), if and only if \( f(\beta) = 0 \), with \( \beta = (\frac{a-b}{a})^{q^{nr-t}} \in \mathbb{S} \).

With similar arguments used for Proposition 3.1 one gets
Proposition 3.3. The Goppa codes are invariant by conjugation under the semiaffine group \( \Gamma L(1, q^n) \), i.e. \( G(L, g) = G(L, f) \), where \( f = g^\psi \in \mathbb{P} \).

Corollary 3.4. Goppa codes \( G(L, g) \) is equivalent to Goppa code \( G(L, f) \).

In [18] the same action on \( \Omega \) is obtained considering an action on \( S \) of an "affine" group \( T = AGL(1, q^n) \langle \sigma \rangle \), where \( \sigma \) is defined as \( \sigma : x \to x^q \); the group \( \langle \sigma \rangle \) has order \( nr \). The main result is the following:

Theorem 3.5. [18] If \( \alpha, \beta \in S \) are related as it follows

\[
\beta = \zeta \alpha^q + \xi
\]

for some \( \zeta, \xi \in \mathbb{F}_{q^n}, \zeta \neq 0, i = 1 \ldots nr \), then \( C(\alpha) \) is equivalent to \( C(\beta) \).

Orbits over \( S \) give orbits on \( \Omega \).

Fact 3.6. The above actions on \( S \) and on \( \mathbb{P} \) create the same orbits on \( \Omega \).

Proof. Let \( \alpha \in S \) be a root of \( g(x) \in \mathbb{P} \). Let \( \beta = \zeta \alpha^q + \xi \in S \). There exists an irreducible polynomial \( g_1 \in \mathbb{F}_{q^n}[x] \), such that \( g_1(\beta) = 0 \). From Proposition 3.3 we get that the orbit \( \alpha^{\mathcal{F}G} = \{ t(\beta), t \in T \} \) induces on \( \Omega \) the same orbit than \( g^T = \{ t(g), t \in T \} \). \( \square \)

The work in [1] is mainly directed to the study of automorphism group of a given code; [18] is deeply interested in counting the number of non-equivalent Goppa codes.

In [18] the exact number of orbits on \( S \) is given. Unfortunately, several examples are exhibited where the number of orbits \( T \) on \( S \) is greater than the number of non-equivalent Goppa codes.

We introduce an action on columns of \( H_\alpha = \left( \frac{1}{\alpha - \varepsilon}, \ldots, \frac{1}{\alpha - 1}, \frac{1}{\alpha} \right) \), which induces the same orbits on \( \Omega \) than \( T \). We state the results and give a sketch of the proofs. For more details see [9].

Let us consider the subgroup \( \mathcal{F}G \simeq AGL(1, q^n) \leq S_{q^n} \) in its natural action on points of \( \mathbb{F}_{q^n} \). If \( \psi \in \mathcal{F}G \), then \( \psi(x) = ax^q + b \), where \( a, b \in \mathbb{F}_{q^n}, a \neq 0 \) and \( i = 1, \ldots, n \). Since each entry (column) of \( H_\alpha \) is uniquely determined by an element of \( \mathbb{F}_{q^n} \), \( AGL(1, q^n) \) realizes a permutation of \( H_\alpha \) entries given by:

\[
\left( \frac{1}{\alpha - \varepsilon} \right)^\psi = \frac{1}{\alpha - \varepsilon^\psi}.
\]

Writing \( \mathcal{F}G \) we mean \( F = AGL(1, q^n) \) and \( G \) the automorphism group of \( \mathbb{F}_{q^n} \) over \( \mathbb{F}_q \).
The matrices $H_\beta$ and $\zeta H_\beta^q$ are parity-check matrices for the same Goppa code $C(\beta)$. We characterize the permutations mapping $H_\alpha$ into $\zeta H_\beta^q$ in the following proposition.

**Proposition 3.7.** Let $H_\alpha$ and $H_\beta$ be parity-check matrices for Goppa codes $C(\alpha)$ and $C(\beta)$. If there exists a permutation $\rho \in S_n$, such that $\rho(H_\alpha) = \zeta(H_\beta)^q$, for some $\zeta, \beta \in \mathbb{F}_{q^n}, \zeta \neq 0$ and $j = 1, \ldots, nr$, then $\rho \in FG$.

**Proof.** We consider

$$\rho(H_\alpha) = H'_\alpha = \left( \frac{1}{\alpha - \varepsilon^{t_1}}, \ldots, \frac{1}{\alpha - \varepsilon^{t_n}}, \frac{1}{\alpha - \varepsilon^{-1}} \right),$$

where $i_j = \rho(j)$ and matrix $\zeta(H_\beta)^q$:

$$\zeta(H_\beta)^q = \left( \frac{\zeta}{\beta^{q^j} - \varepsilon^q}, \ldots, \frac{\zeta}{\beta^{q^j} - \varepsilon^{-1}} \right).$$

Suppose $\zeta = 1$ and $j = 1$ so that $\forall t \in \{1, 2, \ldots, q^n\}$ we have $\frac{1}{\beta^{q^j} - \varepsilon^q} = \frac{1}{\alpha - \varepsilon^t}$ and then $\alpha - \beta^q = \varepsilon^t - \varepsilon^q$.

If $\alpha - \beta^q = 0$, $\rho$ is the permutation induced by the Frobenius map $\sigma$, since $\varepsilon^t = \varepsilon^q$; it follows that

$$\rho(t) = \begin{cases} tq & \text{if } t = 1, 2, \ldots, q^n - 1 \\ -\infty & \text{if } t = -\infty \end{cases}$$

and $\rho = \sigma$.

If $\alpha - \beta^q \neq 0$, as above $\alpha - \beta^q \in \mathbb{F}_{q^n}$ so that $\alpha - \beta^q = \varepsilon^k$ for some $k \in \{1, \ldots, q^n - 1\}$ and then permutation $\rho \in FG$; explicitly it acts as:

$$\rho(t) = \begin{cases} i_t = t + f_k(\varepsilon) & \text{if } t = 1, 2, \ldots, q^n - 1 \\ i_t = k & \text{if } t = -\infty \end{cases}$$

where $i_t$ is such that $\varepsilon^t = \varepsilon^{tq} + \varepsilon^k$, and $f_k(\varepsilon)$ is a function depending on representation of $\mathbb{F}_{q^n}$.

If $\zeta \in \mathbb{F}_{q^n}, \zeta \neq 1$ and $j = 1$, then $\zeta = \varepsilon^l$ for some $l \in \{1, \ldots, q^n - 2\}$. With same arguments used in the previous step, we get

$$\zeta\alpha - \zeta \varepsilon^{hl} = \beta^q - \varepsilon^{tq} \Rightarrow \zeta \varepsilon^{hl} = \zeta\alpha - \beta^q + \varepsilon^{tq} \Rightarrow \varepsilon^{hl} = \alpha - \varepsilon^{-\beta^q} + \varepsilon^{tq-l}.$$ 

Again $\alpha - \varepsilon^{-\beta^q} \in \mathbb{F}_{q^n}$; then there is $h \in \{1, \ldots, q^n\}$ so that $\varepsilon^{hl} = \varepsilon^h + \varepsilon^{tq-l}$, and

$$\rho(t) = \begin{cases} i_t = t - l + f_h(\varepsilon) & \text{if } t = 1, 2, \ldots, q^n - 1 \\ i_t = h & \text{if } t = -\infty \end{cases}$$
where $\varepsilon^{it} = \varepsilon^{tq^l} + \varepsilon^h$ and $f_h(\varepsilon)$ depending on the representation of $\mathbb{F}_{q^n}$.

Concluding $\rho = \tau_k \mu \zeta^{-1} \sigma$; here $\tau_k$ is the translation defined by $\tau_k : x \rightarrow x + \varepsilon^k$, $\mu \zeta$ is the map $\mu \zeta : x \rightarrow \zeta x$ and $\sigma$ is the Frobenius map; this proves $\rho \in FG$.

Finally, if $j \neq 1$ we have: $\zeta \beta^{qj} - \varepsilon^{tqj} = 1$, $\zeta \alpha - \varepsilon^{it} = \beta^{qj} - \varepsilon^{tqj}$. As $\zeta = \varepsilon^l$ for some $l \in \{1, \ldots, q^n - 2\}$, we gain:

$$\varepsilon^{it} = \alpha - \varepsilon^{-l} \beta^{qj} + \varepsilon^{tqj - l}.$$ 

So there is $v \in \{1, \ldots, q^n\}$ such that $\alpha - \varepsilon^l \beta^{qj} = \varepsilon^v$ and $\varepsilon^{it} = \varepsilon^v + \varepsilon^{tqj - l}$.

Permutation $\rho$ is:

$$\rho(t) = \begin{cases} i_t = tq^l - l + f_v(\varepsilon) & \text{if } t = 1, 2, \ldots, q^n - 1 \\ i_t = v & \text{if } t = -\infty \end{cases}$$

where $\varepsilon^{it} = \varepsilon^{tq^l - l} + \varepsilon^v$ and $f_v(\varepsilon)$ depends on the representation of $\mathbb{F}_{q^n}$.

Concluding $\rho = \tau_v \mu \zeta^{-1} \sigma^j$. Clearly in all cases $\rho \in FG$.

\[\Box\]

**Corollary 3.8.** Let $H_\alpha$ and $H_\beta$ be parity-check matrices for Goppa codes $C(\alpha)$ and $C(\beta)$.

If there exists a permutation $\rho \in S_{q^n}$, such that $\rho(H_\alpha) = \zeta H_\beta$, for same $\zeta, \beta \in \mathbb{F}_{q^n}, \zeta \neq 0$, then $\rho \in F$.

### 4 Maximal subgroups

The action of $A \Gamma L(1,q^n)$ does not reach the exact number of non-equivalent maximal Goppa codes. So we look for maximal subgroups of $S_{q^n}$ containing a fixed $A \Gamma L(1,q^n) = FG$.

**Theorem 4.1** ([5]). A maximal subgroup of $S_{q^n}$ is one of the following:

1. intransitive, $S_k \times S_l$, $k + l = q^n$;
2. transitive imprimitive: the wreath product $S_k W r S_l$ in the standard action, $kl = q^n$;
3. primitive non-basic, the wreath product $S_k W r S_l$ in the product action, $k^l = q^n, k \neq 2$;
4. affine $AGL(d,p), p^d = q^n$;
5. diagonal, $T^k.(Out(T) \times S_k)$, $T$ non abelian simple, $|T|^{k-1} = q^n$; here $Out(T)$ denotes, as usual, the factor group $\frac{Aut(T)}{T}$.

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6. almost simple, that is an automorphism group $G$ of a finite non abelian simple group $S$, $S \leq G \leq \text{Aut}(S)$.

A maximal subgroup of the alternating group is the intersection of one of these groups with the alternating group.

**Remark 4.2.** We explicitly observe that for $p$ even, $d \geq 3$, the group $AGL(d, p)$ is actually contained in the alternating group $A_d$. It is sufficient to realize that, in this case, the translations are product of $2^{d-1}$ cycles of length 2, as well as the transvections are product of $2^{d-2}$ cycles of length 2. As the transvections generate the general linear group $GL(p, 2)$, $AGL(p, 2)$ is contained in $A_d$.

**Proposition 4.3.** $FG$ is contained in $A_q^n$ if and only if $q$ is even.

**Proof.** The thesis follows from the following result.

**Claim 4.4.** [14] Let $X$ be a primitive permutation group of degree $n$. Then $X$ contains an abelian regular subgroup $G$ if and only if either

i) $X \leq AGL(d, p)$, where $p$ is a prime, $d \geq 1$ and $n = p^d$; or

ii) $X = (\tilde{T}_1 \times \ldots \times \tilde{T}_l) \cdot O \cdot P$, $G = G_1 \times \ldots \times G_l$ where $n = m^l$, $l \geq 1$, $G_i < \tilde{T}_i$, with $|G_i| = m$, $\tilde{T}_1 \cong \ldots \cong \tilde{T}_l$, $O \leq \text{Out}(\tilde{T}_1) \times \ldots \times \text{Out}(\tilde{T}_l)$, $P$ is a transitive permutation group of degree $l$ and one of the following holds:

(a) $(\tilde{T}_i, G_i) = (PSL(2, 11), Z_{11}), (M_{11}, Z_{11}), (M_{12}, Z_2 \times Z_3), (M_{23}, Z_{23})$ ($M_i$ are the Mathieu groups);

(b) $\tilde{T}_i = PGL(d, q)$ e $G_i = Z_{q^{d-1}}$ is a Singer group;

(c) $\tilde{T}_i = PGL(2, 8)$ and $G_i = Z_9 \not\cong PSL(2, 8)$;

(d) $\tilde{T}_i = S_m$ or $A_m$ and $G_i$ is an abelian group of order $m$.

Take $X = FG$. $FG$ contains the subgroup $A$ of translations, $A = \{ \tau_\varepsilon : x \to x + \varepsilon \}$, so that $FG$ is contained in $N_{S_q^n}(A) = AGL(n m, p)$. By the above remark, if $p = 2$, the group $FG$ is contained in $A_q^n$. If $p$ is odd, then the element $\mu_\varepsilon : x \to \varepsilon x$ belongs to $FG$ and it is odd, as its order is $q^n - 1$ (recall that an element of order $q^n - 1$ is said a Singer cycle); this proves that $FG$ (and $AGL(n m, p)$) is not a subgroup of $A_q^n$.

**Theorem 4.5.** Let $G = A_q^n$ if $q = 2$ and $G = S_q^n$ for $q$ odd. If $M$ is a maximal subgroup of $G$ containing $FG$, then $M$ is isomorphic to the affine group $AGL(n m, p)$. Moreover, there is exactly one maximal subgroup containing $FG$.

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Proof. As $FG$ is a primitive 2-transitive group of $G$, a maximal subgroup $M$ of $S_{q^n}$ containing $FG$, is an almost simple group or it is isomorphic to the affine group $AGL(nm, p)$ \([5]\). In the proof of Proposition 4.3 we have seen that $FG$ is contained in $AGL(mn, p)$. We prove that it is not contained in an almost simple group. By contradiction, let $M$ be an automorphism group of a simple non abelian group $S$, $S \leq M \leq Aut(S)$. If $M$ contains $FG$, the stabilizer of a point $\omega$ in $\mathbb{F}_{q^n}$ has index $q^n = p^{nm}$. As $S$ is normal in $M$, $S$ is transitive on $\mathbb{F}_{q^n}$, so that we are reduced to consider subgroups of prime power index in $S$. These were described by Guralnick and for the reader’s sake we write the main result of \([10]\).

Claim 4.6 \([10]\). Let $G$ be a nonabelian simple group with $H < G$ and $[G : H] = p^d = q^n$, $p$ prime. One of the following holds.

1. $G = A_{q^n}$ and $H \cong A_{q^n-1}$;
2. $G = PSL(s, t)$ and $H$ is the stabilizer of a line or hyperplane. Then $[G : H] = t^{s-1} = q^n$ (Note $s$ must be prime);
3. $G = PSL(2, 11)$ and $H \cong A_5$;
4. $G = M_{23}$ and $H \cong M_{22}$ or $G = M_{11}$ and $H \cong M_{10}$;
5. $G = PSU(4, 2) \cong PSp(4, 3)$, $H$ is the parabolic subgroup of index 27.

Cases 3, 4, 5, are easily ruled out, as $p^{mn}$ is neither a prime number, nor 27. Similarly, case 1 is ruled out when $p$ is odd, as, in this case, the element $\mu_\varepsilon$ is odd. If $p = 2$, then $FG$ is actually contained in $M \simeq A_{q^n}$. So, we are left with Case 2. Here, we use Claim 4.4 $X$ satisfies condition ii), with

$$X = S = PSL(s, t), \quad [S : H] = \frac{s^t - 1}{t - 1} = p^{nm}, \quad l = 1.$$ 

and it is easy to see that it is not the case.

Now, we prove that there is exactly one subgroup isomorphic to $AGL(nm, p)$ containing $FG$.

Let $q$ be odd: in $S_{q^n}$ there is exactly one conjugacy class of maximal subgroups of this type (see for example \([13]\)). So, let $FG \leq M \simeq AGL(nm, p)$, where the normal subgroup of the translation of $M$ is exactly the translation group $A$ of $FG$ \([12]\). The element $\mu_\varepsilon$ generates a Singer subgroup; it is well known that the Singer cycles are conjugated in $M$; from the knowledge of the overgroups of a Singer cycle \([13], [4]\), one easily proves that also the normalizers of Singer cycles contained in $M$ are conjugate in $M$. It follows
that if $FG^g$, $g \in S_{q^n}$ is contained in $M$, there exists an element $m \in M$, such that $FG^g = FG^m$. So, if $s$ denotes the number of the subgroups of $M$ containing $FG$, we get:

$$[S_{q^n} : NS_{q^n}(FG)] = \frac{[S_{q^n} : M] \cdot [M : NM(FG)]}{s};$$

now, from [14] one gets $NS_{q^n}(FG) \leq M$, so that $s = 1$.

Now, suppose $q$ is even. $FG \leq A_{q^n}$ and in $A_{q^n}$ the conjugacy class of $S_{q^n}$ subgroups which are isomorphic to $AGL(nm,p)$. In $A_{q^n}$ $AGL(nm,p)$ splits into two classes so that also the class of Singer cycles splits into two different classes. Same argument used for the odd case leads to the result. □

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