Abstract Cesàro Spaces. Optimal Range

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Abstract. Abstract Cesàro spaces are investigated from the optimal domain and optimal range point of view. There is a big difference between the cases on $[0, \infty)$ and on $[0, 1]$, as we can see in Theorem 1. Moreover, we present an improvement of Hardy’s inequality on $[0, 1]$ which plays an important role in these considerations.

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1. Introduction and Basic Definitions

For a Banach ideal space $X$ on $I = [0, 1]$ or $I = [0, \infty)$ let us consider, as in [6], the abstract Cesàro space $CX$ on $I$ defined as $CX = \{ f \in L^0(I) : C|f| \in X \}$ with the norm given by

$$\|f\|_{CX} = \|C|f|\|_X,$$

where $C$ is the Cesàro operator

$$Cf(x) = \frac{1}{x} \int_0^x f(t) \, dt, \ x \in I.$$

One may look at these spaces, on one hand, as on generalization of the well-known Cesàro spaces $Ces_p[0, 1]$ and $Ces_p[0, \infty)$ which were investigated for example in [1]. On the other hand, $CX$ is the optimal domain of $C$ for $X$ since, just by definition, $C : CX \to X$ is bounded and $CX$ is the largest ideal space satisfying this relation. Consequently, the abstract Cesàro spaces may be considered also from the optimal domain point of view, as it was done in [3,9–11]. In this paper we discuss the Cesàro function spaces on $[0, \infty)$ and on $[0, 1]$ from the point of view of optimal domain and optimal range of the Cesàro operator $C$. Such concept was already considered for $X = L^p(\cdot)$ on $[0, 1]$ in [10,11] and for $X = L^p(\cdot)$ on $\mathbb{R}^n$ in [9], although the most interesting situation of $CX$ on $[0, 1]$ was omitted there. We develop and complete the

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discussion under some minimal assumptions. In this more interesting case of interval \([0, 1]\) a very important role is played by the improvement of Hardy inequality presented in Theorem 2.

We present some basic definitions to understand further description of results. By \(L^0 = L^0(I)\) we denote the space of Lebesgue measurable functions (in fact, respective equivalence classes with respect to equality almost everywhere) on \(I = [0, 1]\) or \(I = [0, \infty)\). A Banach space \(X \subseteq L^0\) is called a Banach ideal space on \(I\) if \(g \in X, f \in L^0(I), |f| \leq |g|\) a.e. on \(I\) implies \(f \in X\) and \(\|f\| \leq \|g\|\). We will also assume that \(\text{supp} X = I\), i.e. there exists \(f \in X\) with \(f(x) > 0\) for each \(x \in I\).

For a given Banach ideal space \(X\) on \(I\) and a function \(w \in L^0(I)\) such that \(w(x) > 0\) a.e. on \(I\), the weighted Banach ideal space \(X(w)\) is defined as \(X(w) = \{f \in L^0(I) : fw \in X\}\) with the norm \(\|f\|_{X(w)} = \|fw\|_X\).

In the whole paper only two concrete weights on \(I = [0, 1]\) will appear, namely \(v\) and \(1/v\) where
\[
v(x) = 1 - x.
\]

We will need also a non-increasing majorant \(\tilde{f}\) of a given function \(f\), which is just
\[
\tilde{f}(x) = \text{ess sup}_{t \in I, t \geq x} |f(t)|, \quad x \in I.
\]

Moreover, for a given Banach ideal space \(X\) on \(I\), we define a new Banach ideal space \(\tilde{X} = \tilde{X}(I)\) as \(\tilde{X} = \{f \in L^0(I) : \tilde{f} \in X\}\) with the norm given by
\[
\|f\|_{\tilde{X}} = \|\tilde{f}\|_X.
\]

By a symmetric function space on \(I\) with the Lebesgue measure \(m\) (symmetric space in short), we mean a Banach ideal space \((X, \| \cdot \|_X)\) with the additional property that for any two equimeasurable functions \(f \sim g, f, g \in L^0(I)\) (that is, they have the same distribution functions \(d_f = d_g\), where \(d_f(\lambda) = m(\{x \in I : |f(x)| > \lambda\})), \lambda \geq 0\) and \(f \in X\) we have \(g \in X\) and \(\|f\|_X = \|g\|_X\). In particular, \(\|f\|_X = \|f^*\|_X\), where \(f^*(t) = \inf\{\lambda > 0 : d_f(\lambda) < t\}, \quad t \geq 0\).

The dilation operators \(\sigma_a (a > 0)\) defined on \(L^0(I)\) by
\[
\sigma_a f(x) = f(x/a)\chi_I(x/a) = f(x/a)\chi_{[0, \min(1, a)]}(x), \quad x \in I,
\]
are bounded in any symmetric space \(X\) on \(I\) and \(\|\sigma_a\|_{X \to X} \leq \max(1, a)\) (see [2, p. 148] and [5, pp. 96–98]). They are also bounded in some Banach ideal spaces which are not necessarily symmetric spaces. Furthermore, recall that the Cesàro operator \(C\), the Copson operator \(C^*\) and the Hardy–Littlewood maximal operator \(M\) are defined, respectively, by
\[
Cf(x) = \frac{1}{x} \int_0^x f(t)dt, \quad x \in I, \quad C^*f(x) = \int_{I \cap [x, \infty)} \frac{f(t)}{t}dt, \quad x \in I,
\]
\[
Mf(x) = \sup_{a, b \in I, 0 \leq a \leq x \leq b} \frac{1}{b-a} \int_a^b |f(t)|dt, \quad x \in I.
\]
We refer the reader to [6], where basic facts about the spaces \( CX \) and \( \widetilde{X} \) were presented with more details. For more references on Banach ideal spaces and symmetric spaces we refer to [2,4,5,7,8].

2. Optimal Domain and Optimal Range

Let \( X \) and \( Y \) be two Banach ideal spaces on \( I \) and let \( T : X \to Y \) be a bounded linear or sublinear operator. A Banach ideal space \( Z \) on \( I \) is called the optimal domain of \( T \) for \( Y \) within the class of Banach ideal spaces on \( I \), if \( T : Z \to Y \) is bounded and for each Banach ideal space \( W \) on \( I \), \( T : W \to Y \) is bounded implies that \( W \subset Z \). The last implication may be formulated equivalently as: if \( Z \) and \( W \) are Banach ideal spaces on \( I \) and if \( Z \subset W \), then \( T : W \not\to Y \). Of course in such a case \( X \subset W \).

Similarly, we shall say that a Banach ideal space \( Z \) on \( I \) is the optimal range of \( T \) for \( X \) within the class of Banach ideal spaces on \( I \), if \( T : X \to Z \) is bounded and for each Banach ideal space \( W \) on \( I \), \( T : X \to W \) is bounded implies that \( Z \subset W \). Once again, the last condition may be replaced by: \( W \subset Z \) implies \( T : X \not\to W \). Such optimal range satisfies of course \( Z \subset Y \).

The following theorem describes the optimal domain and optimal range problem for Cesàro operator within the class of Banach ideal spaces on \( I \).

**Theorem 1.** Let \( X \) be a Banach ideal space on \( I \) such that the maximal operator \( M \) is bounded on \( X \).

(i) If \( I = [0, \infty) \), then \( C : CX \to \widetilde{X} \) is bounded. Moreover, the space \( CX \) is the optimal domain of \( C \) for \( X \) and for \( \widetilde{X} \) (also for \( CX \) if the dilation operator \( \sigma_a \) is bounded on \( X \) for some \( 0 < a < 1 \)). The space \( \widetilde{X} \) is the optimal range of \( C \) for \( CX \), \( X \) and \( \widetilde{X} \). In particular, \( CX = C\widetilde{X} \).

(ii) If \( I = [0, 1] \) and \( v \) is from (1.1), then \( C : CX \to X(1/v)(v) \) is bounded. The space \( CX \) is the optimal domain of \( C \) for \( X \) and also for \( X(1/v)(v) \). Moreover, if the maximal operator \( M \) is bounded on \( X' \), then the space \( X(1/v)(v) \) is the optimal range of \( C \) for \( CX \) and \( X(v) \) (cf. Diagram 2). In particular, \( CX = C[X(1/v)(v)] \).

(iii) If \( I = [0, 1] \) and the dilation operator \( \sigma_{1/2} \) is bounded on \( X \), then \( C : C\widetilde{X} \to \widetilde{X} \) is bounded. Moreover, the space \( C\widetilde{X} \) is the optimal domain of \( C \) for \( \widetilde{X} \) and the space \( \widetilde{X} \) is the optimal range of \( C \) for \( C\widetilde{X} \), \( X \) and \( \widetilde{X} \). One also has \( C\widetilde{X} = CX \cap L^1 \).

Before we prove the theorem, let us comment on the situation. Suppose that the corresponding assumptions in Theorem 1 are satisfied. Of course, boundedness of \( M \) on \( X \) implies also boundedness of \( C \) on \( X \), therefore the support of \( CX \) is for sure the same as support of \( X \) (cf. [6]). Let \( I = [0, \infty) \). Then the statement of (i) may be therefore pictured, putting the boundedness of \( C \) and respective embeddings, on Diagram 1.
Moreover, point (i) says that, in fact, $CX$ is the optimal domain of $C$ for $\tilde{X}$, since $CX = C\tilde{X}$. Even more can be said when the dilation operator $\sigma_a$ is bounded on $X$ for a certain $0 < a < 1$. Then $CX$ is the optimal domain of $C$ even for $CX$ since, by Lemma 6 in [6], it follows that $CCX = CX$. On the other hand, we will see that $\tilde{X}$ is the optimal range of $C$ for $\tilde{X}$, which by Diagram 1 means that also for $X$ and for $CX$.

Much more interesting and delicate is the case of interval $[0, 1]$. Suppose that $C : X \to X$ is bounded and all assumptions of (ii) and (iii) are satisfied. Then $C : CX \to X$ is bounded, where $CX$ is by definition the optimal domain of $C$ for $X$. The case (ii) says that the optimal range of $C$ for $CX$ is then $X(1/v)(v)$. It is however interesting that one may look at the situation also in another way. Let’s start once again with $C : X \to X$ and find first the optimal range. It appears to be just $\tilde{X}$ (cf. [10, Theorem 8.2], [11, Theorem 3.16] and [9, Theorem 4.1]) which is much smaller than $X(1/v)(v)$. If we now find optimal domain of $C$ for $\tilde{X}$ it is then just $CX \cap L^1 = C(\tilde{X})$. The diagram describing this dichotomy is now more complicated (see Diagram 2).

In general, there is no inclusion relation between $X(v)$ and $C\tilde{X}$. For example, if $X$ is a symmetric space on $I = [0, 1]$, we have for $f(x) := \frac{1}{1-x}$ that $f \in X(v)$ while $f \not\in C\tilde{X}$ because $Cf(x) \to \infty$ as $x \to 1^-$ and so $\tilde{C}f$ is not defined (or just $\infty$ everywhere). Therefore, $X(v) \not\subset C\tilde{X}$. This means also that $C$ does not act from $X(v)$ into $\tilde{X}$. On the other hand, let $X = L^2$ and put $f(x) = |\frac{1}{2} - x|^{-1/2}$. Then $f \not\in L^2$, but $Cf \in L^\infty$ and so $\tilde{C}f \in L^\infty \subset L^2$. This
gives $C \hat{X} \not\subset X(v)$. For general symmetric space $X$ on $I$ such that $C : X \to X$ is bounded, one could take $f \in L^1$ in such a way that $f - f \chi_{[1/2-\epsilon,1/2+\epsilon]} \in L^\infty$ for each $0 < \epsilon < 1/2$ but $f \not\in X$, to achieve the same effect.

**Proof of Theorem 1.** (ii). Let $0 \leq f \in CX$. Suppose first that $0 \leq y \leq t \leq 2y \leq 1$. Then

$$Cf(t) = \frac{1}{t} \int_0^t f(s)ds \geq \frac{1}{2y} \int_0^y f(s)ds = \frac{1}{2} Cf(y). \quad (2.1)$$

If now $0 \leq x \leq y$ and $y \leq \frac{1}{2}$, then applying (2.1) one gets

$$MCf(x) \geq \frac{1}{2y-x} \int_x^{2y} Cf(t)dt \geq \frac{1}{2y} \int_y^{2y} Cf(t)dt \geq \frac{1}{4} Cf(y) \geq \frac{1-y}{4(1-x)} Cf(y).$$

Suppose now that $\frac{1}{2} \leq y \leq t \leq 1$. Then, similarly as in (2.1),

$$Cf(t) = \frac{1}{t} \int_0^t f(s)ds \geq \int_0^y f(s)ds \geq \frac{1}{2} Cf(y). \quad (2.2)$$

In consequence, when $0 \leq x \leq y$ and $\frac{1}{2} \leq y \leq 1$, applying (2.2) we obtain

$$MCf(x) \geq \frac{1}{1-x} \int_x^1 Cf(t)dt \geq \frac{1}{1-x} \int_y^1 Cf(t)dt \geq \frac{1}{1-x} \frac{1-y}{2(1-x)} Cf(y).$$

Consequently,

$$MCf(x) \geq \frac{1}{4(1-x)} \text{ess sup}_{0 \leq x \leq y \leq 1} (1-y)Cf(y) = \frac{1}{4(1-x)} [vCf](x). \quad (2.3)$$

Since $M$ is bounded on $X$, by our assumption, it follows that

$$\|Cf\|_{\hat{X}(1/v)(v)} = \|[vCf]/v\|_X \leq 4 \|M\|_{X \to X} \|Cf\|_X = 4 \|M\|_{X \to X} \|f\|_{CX}.$$ 

This means that $C : CX \to \hat{X}(1/v)(v)$ is bounded and the first statement of (ii) is proved. It remains to show that the space $\hat{X}(1/v)(v)$ is the optimal range of $C$ for CX (in fact, even for $X(v)$). Suppose that there is a Banach ideal space $Z$ on $I$ such that

$$Z \not\subset Y \text{ but } C : CX \to Z \text{ is bounded.}$$

Let $0 \leq f \in Y \setminus Z$. Define

$$g(x) = \frac{1}{1-x} [vf](x), x \in I.$$ 

Then $f \leq g$ and $g \in \hat{X}(1/v)(v) \subset X$ because $\frac{1}{1-x} [vg](x) = \frac{1}{1-x} [vf](x)$. We have
\[C(g/v)(x) = \frac{1}{x} \int_0^x \frac{[vg](t)}{(1-t)^2} dt \geq \frac{[vf](x)}{x} \int_0^x \frac{1}{(1-t)^2} dt = \frac{[vf](x)}{x} \frac{x}{(1-x)} \geq f(x),\]

which means that \(C(g/v) \notin Z\). However, \(g \in X\) and so \(g/v \in X(v)\). Also, by Theorem 2 below, \(X(v) \subset CX\) and therefore \(g/v \in CX\) which means that \(C : CX \not\to Z\). Note that we have already shown \(C : X(v) \not\to Z\), which by inclusion \(X(v) \subset CX\) means that \(\widehat{X(1/v)(v)}\) is the optimal range also for \(X(v)\).

(iii). The argument is analogous to the one from statement (5.1) in [10]. However, we need to modify it because in [10] the maximal operator is defined on a larger interval than \([0, 1]\). Let \(0 \leq f \in CX \cap L^1[0, 1]\). We shall understand that \(f(x) = 0\) for \(x > 1\). Of course, inequality from (2.1) remains true in this case, since \(f \in CX\). Suppose that \(0 < x \leq 1\) and consider two cases. If \(y/2 \leq x\), then

\[M\sigma_{1/2}Cf(x) \geq \frac{2}{y} \int_{y/2}^y \sigma_{1/2}Cf(u) du.\]

If \(x \leq y/2\), then

\[M\sigma_{1/2}Cf(x) \geq \frac{1}{y-x} \int_x^y \sigma_{1/2}Cf(u) du \geq \frac{1}{y} \int_{y/2}^y \sigma_{1/2}Cf(u) du.\]

Altogether we get

\[M\sigma_{1/2}Cf(x) \geq \frac{1}{y} \int_{y/2}^y \sigma_{1/2}Cf(u) du = \frac{1}{2y} \int_y^{2y} Cf(t) dt \geq \frac{1}{4} Cf(y).\]

Therefore, similarly as before,

\[M\sigma_{1/2}Cf(x) \geq \frac{1}{4} \text{ ess sup}_{x \leq y} Cf(y) = \frac{1}{4} \widehat{Cf}(x),\]

which gives

\[\|f\|_{C\bar{X}} = \|\widehat{Cf}\|_X \leq 4 \|M\sigma_{1/2}Cf\|_X \leq 4 \|M\|_{X \to X} \|\sigma_{1/2}\|_{X \to X} \|Cf\|_X \leq 4 \|M\|_{X \to X} \|\sigma_{1/2}\|_{X \to X} \|f\|_{CX \cap L^1}.\]

On the other hand, if \(0 \leq f \in C\bar{X}\), then

\[\|f\|_{L^1} = \int_0^1 f(t) dt \frac{\|\chi_{[0,1]}\|_X}{\|\chi_{[0,1]}\|_X} = \frac{\|(f \in L^1(t) dt)\chi_{[0,1]}\|_X}{\|\chi_{[0,1]}\|_X} \leq \frac{\|\widehat{Cf}\|_X}{\|\chi_{[0,1]}\|_X}.\]

Thus also

\[\|f\|_{CX \cap L^1} \leq \max\{1, \frac{1}{\|\chi_{[0,1]}\|_X}\}\|\widehat{Cf}\|_X,\]

which means that \(C\bar{X} = CX \cap L^1\). For the sake of completeness we present the argument that \(\bar{X}\) is the optimal range of \(C\) for \(C\bar{X}\), although it works just like in [10, Theorem 8.2]. Let \(Z\) be a Banach ideal space on \(I\) and suppose that \(0 \leq f \in \bar{X} \setminus Z\). Then also \(\tilde{f} \in \bar{X} \setminus Z\) and \(C\tilde{f} \geq \tilde{f}\). However \(\tilde{f} \notin Z\), which means that \(C\tilde{f} \notin Z\) and \(C : C\bar{X} \not\to Z\).
(i) This case is easier and may be deduced directly from [9]. Since for $0 < y$ also $2y \in I$ it is enough to follow (2.1) and after that to get for $y \geq x \geq 0$

$$MCf(x) \geq \frac{1}{2y - x} \int_x^{2y} Cf(t)dt \geq \frac{1}{4} Cf(y).$$

Then

$$\|Cf\|_{\tilde{X}} = \|\tilde{C}f\|_X \leq 4\|MCf\|_X \leq 4\|M\|_{X \to X} \|Cf\|_X = 4\|M\|_{X \to X} \|f\|_{CX},$$

which means that $C : CX \to \tilde{X}$ is bounded and $CX = \tilde{CX}$. The optimal range of $C$ for $\tilde{X}$, $X$, $CX$ is once again $\tilde{X}$ and the proof is the same as in (iii) (see also [10, Theorem 8.2], [11, Theorem 3.16] and [9, Theorem 4.1]). □

3. Hardy Inequality

We present an improvement of the Hardy inequality which appears for spaces on $I = [0, 1]$.

**Theorem 2.** If $C$ is bounded on a Banach ideal space $X$ on $I = [0, 1]$ and the maximal operator $M$ is bounded on $X'$, then

$$C : X(v) \to X$$

is also bounded, where $v$ is from (1.1).

**Proof.** Let $0 \leq f \in X$. We have for $0 < x \leq \frac{1}{2}$

$$C(f/v)(x) = \frac{1}{x} \int_0^x \frac{f(s)}{1 - s} ds \leq \frac{2}{x} \int_0^x f(s) ds$$

and for $\frac{1}{2} < x \leq 1$

$$C(f/v)(x) = \frac{1}{x} \int_0^x \frac{f(s)}{1 - s} ds \leq \frac{2}{x} \int_0^x f(s) ds.$$

If we define the operator $T$ as $Tf(x) = \int_0^x \frac{f(s)}{1 - s} ds$, then

$$C(f/v) \leq 2(Cf + Tf).$$

Therefore, we need to show that $T$ is bounded on $X$. Consider an involution operator $\tau : f(x) \mapsto f(1 - x)$. Then

$$Tf(x) = \int_0^x \frac{f(s)}{1 - s} ds = \int_{1-x}^1 \frac{f(1-s)}{s} ds = \tau C^* \tau f(x). \quad (3.1)$$

Observe that the space

$$X^- = \{ f : \tau f \in X \}$$

with its natural norm $\|f\|_{X^-} = \|\tau f\|_X$ is also a Banach ideal space on $I$ and so $(X^-)^-$. Just by definition $\sigma : X \to X^-$, $\tau : X^- \to X$ are bounded and $\tau \sigma = id$. Thus $T$ is bounded on $X$ if and only if $C^*$ is bounded on $X^-$. We will prove the last equivalence. Notice that simply

$$Mf(1 - x) = \sup_{a \neq b, 0 \leq a \leq 1 - x \leq b \leq 1} \frac{1}{b - a} \int_a^b f(s) ds \quad (3.2)$$
\[
M\tau f = \tau Mf \text{ which means that for any Banach ideal space } Y, M \text{ is bounded on } Y \text{ if and only if } M \text{ is bounded on } Y^- \text{, which by our assumption gives that } M \text{ is bounded on } (X')^- \text{. Thus also } C \text{ is bounded on } (X')^- \text{ and by duality } C^* \text{ is bounded on } [(X')^-]' \text{. However, it is evident that for any Banach ideal space } Y \text{ there holds } (Y')^- = (Y^-)' \text{. Then } [(X')^-]' = (X'')^- = X^- \text{ and so } C^* \text{ is bounded on } X^-.
\]

\( \square \)

**Remark 1.** If \( X \) is a symmetric space, then evidently \( X = X^- \) and we get Lemma 10 from [6], whose proof was a generalization of the Astashkin–Maligranda result from [1]. Moreover, our Theorem 2 includes Theorem 9 in [6] for the weighted \( L^p(x^\alpha) \) spaces when \( 1 \leq p < \infty \) and \( -1/p < \alpha < 1 - 1/p \).

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