Lower Critical Field $H_{c1}(T)$ and Pairing Symmetry Based on Eilenberger Theory

Takanobu Akiyama, Masanori Ichioka, and Kazushige Machida

Department of Physics, Okayama University, Okayama 700-8530, Japan

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Vortex physics plays an important role in the study of unconventional superconductors. In this short note, based on Eilenberger theory we study the temperature ($T$) dependence of lower critical field $H_{c1}(T)$ of vortex states in anisotropic superconductors.

Recent developments of experimental technique make it possible to observe $H_{c1}(T)$ exactly, and discuss it in the relation to the mechanism of unconventional superconductivity, such as in new iron-based superconductors$^{1,2}$ and URu$_2$Si$_2$. In traditional Ginzburg-Landau (GL) theory, $H_{c1}$ is given by$^4$

$$H_{c1} \propto \lambda(T)^{-2} (\ln \kappa + c_0) \tag{1}$$

with penetration depth $\lambda$, GL parameter $\kappa$, and a constant $c_0$. In the London theory $\lambda(T) = \lambda(T = 0) + \delta \lambda(T)$ behaves as $\delta \lambda(T) \propto \exp(-\Lambda/k_{B} T)$ at low $T$ in the s-wave pairing, reflecting superconducting gap $\Delta$. In d-wave pairing with line nodes, $\delta \lambda(T) \propto T$ at low $T$ in the clean limit. These indicate that $H_{c1}(T)$ depends on the pairing symmetry of anisotropic superconductors.

We note that GL theory is a phenomenological theory valid near the transition temperature $T_c$. Thus it is not clear whether the above-discussion on $H_{c1}(T)$ is quantitatively valid. Therefore, it is expected that $H_{c1}(T)$ is evaluated by Eilenberger theory, which is quantitatively reliable in vortex states even far from $T_c$. To study contributions by the pairing symmetry, we calculate $H_{c1}(T)$ for s-wave pairing with full gap and $d_{x^2-y^2}$-wave pairing with line nodes, as typical examples, by quantitative Eilenberger theory. The previous work for s-wave pairing was done in a single vortex.$^5$ Our calculation is performed in vortex lattice. We also study $H_{c1}(T)$ for chiral $p_{+}$-wave pairings, to see dependences on the chirality directions, i.e., parallel or anti-parallel to applied fields.

In this study, for simplicity, we use isotropic cylindrical Fermi surface $k = k_F (\cos \theta, \sin \theta)$ and magnetic fields are applied to the $c$ direction. The quasiclassical Green’s functions $g(\omega, k, r), f(\omega, k, r)$, and $f^\dagger(\omega, k, r)$ are calculated by the Eilenberger equation

$$\{\omega_n + v \cdot (\nabla + iA)\}f = \Delta \phi g,$$
$$\{\omega_n - v \cdot (\nabla - iA)\}f^\dagger = \Delta^* \phi^* g, \tag{2}$$

$g = (1 - ff^\dagger)_{1/2}$ in the vortex lattice state, with the selfconsistent conditions of pair potential

$$\Delta(r) = g_0 N_0 T \sum_{0 < \omega_n \leq \omega_{cut}} \left\langle \phi^* (k) \left( f + f^\dagger \right) \right\rangle_k \tag{3}$$

and the vector potential

$$\nabla \times (\nabla \times A) = -\frac{2T}{\kappa^2} \sum_{0 < \omega_n} \left\{ \nu m g \right\}_k \tag{4}$$

in Eilenberger unit,$^6,7$ with Matsubara frequency $\omega_n$, $(g_0 N_0)^{-1} = \ln T + 2T \sum_{0 < \omega_n \leq \omega_{cut}} \omega_n^{-1}$, where $v = k/k_F$ is the direction of Fermi velocity $v_F$, $r$ is the center-of-mass coordinate, and $\left\langle \cdots \right\rangle_k$ indicates the Fermi surface average. We use $\omega_{cut} = 20k_BT_c$. The internal field $B(r) = B + \nabla \times a(r)$ is related to the vector potential $A(r) = \frac{1}{2} B \times r + a(r)$ in the symmetric gauge, where $B = (0, 0, B)$ is a uniform flux density. The pairing function is defined as $\phi(k) = 1$ for s-wave pairing, $\phi(k) = \sqrt{2} \cos \theta$ for $d_{x^2-y^2}$-wave pairing. In the chiral p-wave pairing, we consider two-component order parameter $\Delta_+ (r) \phi_+ (k) + \Delta_- (r) \phi_- (k)$ instead of $\Delta (r) \phi(k)$, where $\phi_\pm (k) = e^{\pm i\theta}$.$^8$ In the $p_{+}$- (p$_-$-) wave pairing, $\Delta_+$ ($\Delta_-$) is main component with singular vortex, and $\Delta_-$ ($\Delta_+$) is passive component induced around vortices.

Our calculation is done for $\kappa = 2$ and triangular vortex lattice. We iterate calculations of eqs. (2)-(4) under given $B$, and obtain selfconsistent vortex solutions for spatial structures of $\Delta(r)$, $A(r)$, and quasiclassical Green’s functions, as done in previous works.$^7,8$ Using the solutions, we calculate the external magnetic field $H$ by

$$H = B + \left\langle (B(r) - B) \right\rangle_r / B \tag{5}$$

$$+ \frac{T}{\kappa^2 B} \sum_{0 < \omega_n} \left\{ \left\langle \frac{1}{2} \text{Re} \left\{ \frac{f^\dagger \Delta + f \Delta^*}{g + 1} \right\} \right\rangle_r \right\}_{k} \text{Re}(g - 1)$$

which is derived by Doria-Gubernatis-Rainer scaling$^7,9,10$ and $\left\langle \cdots \right\rangle_r$ indicates the spatial average. Magnetic fields are in unit of $B_0 = \phi_0/2\pi R_0^2$ with the flux quantum $\phi_0$ and $R_0 = h v_F / 2 \pi k_BT_c$.

Figure 1(a) present magnetization curves of $B$ as a function of $H$ at some $T$ for $d_{x^2-y^2}$-wave pairing. There, $H_{c1}$ is defined as onset of $B$. In Meissner states at $H < H_{c1}$, $B = 0$. In Fig. 1(b), we present $H_{c1}(T)$ as a function of $T$ for some pairing symmetries, and we replot them as $H_{c1}(T)/H_{c1}(T = 0.05T_c)$ in Fig. 1(c) to compare the $T$-dependence each other.

First, we discuss the differences between the s-wave and the d-wave pairings. $H_{c1}$ in d-wave pairing is smaller than that in s-wave pairing, because the condensation energy of d-wave pairing is weaker due to the line node contributions, compared to that in the full-gap s-wave pairing. $H_{c1}$ is related to the energy for creation of a vortex in Meissner states.$^4$ As for $T$-dependence, reflecting low energy excitations by line nodes, $H_{c1}(T)$ in d-wave pairing decreases rapidly at low $T$, compared with s-wave pairing.

To discuss quantitative validity of the relation in eq. (1), in Fig. 1(c) we also present $\lambda^2_{L_{\text{London}}}(T)$ given by

$$\lambda^2_{L_{\text{London}}} \propto T \sum_{\omega_n} \left\langle \frac{\Delta \phi^2 v^2}{(\omega_n^2 + |\Delta \phi|^2)^{3/2}} \right\rangle_k \tag{6}$$

in London theory, where $T$-dependence of $\Delta$ is deter-
an opposite effect to the both contributions inside and outside of vortex cores. Simple in $\lambda$ energy is weaker than estimate by Eilenberger theory appears smaller than $H_{c1}(T)$ with $\lambda^2(T)$. 

Next, we study $H_{c1}(T)$ in chiral $p$-wave superconductors. $H_{c1}(T)$ in $p_{-}$-wave pairing is smaller than that in $p_{+}$-wave pairing. This difference in quantitative estimate is consistent to previous results by phenomenological GL theory.\(^{13,14}\) In chiral $p$-wave superconductors, opposite chiral component is induced around vortices of main chiral component, and core energy becomes smaller by the induced component. Compared with $p_{+}$-wave pairing, the induced component is larger in $p_{-}$-wave pairing, and the core energy is smaller, making $H_{c1}$ smaller. If domains of $p_{+}$-wave pairing and $p_{-}$-wave pairing coexist at a zero-field, on increasing fields vortices penetrate at lower $H_{c1}$ only into the $p_{-}$-wave domain, where chirality is antiparallel to the applied field.\(^{14}\) As for the $T$-dependence, in Fig. 1(c) we see that normalized $H_{c1}$ both for $p_{+}$- and $p_{-}$-wave pairings have similar $T$-dependence to that in $s$-wave pairing. This is reasonable, because $p_{\pm}$-wave pairing with $|\phi_{\pm}| = 1$ has full gap, as in $s$-wave pairing.

In summary, we quantitatively estimated different $T$-dependences of $H_{c1}$ between $s$-wave and $d$-wave pairings by Eilenberger theory. The $T$-dependences of $H_{c1}(T)$ show quantitative deviation from $\lambda_{\text{London}}^2(T)$. We also studied differences of $H_{c1}(T)$ between $p_{+}$ and $p_{-}$-wave pairing in chiral $p$-wave superconductors. We expect that future experimental studies will confirm the relations of $H_{c1}(T)$ and the pairing symmetry in various anisotropic superconductors.

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Fig. 1. (Color online) (a) Magnetization curve of $B$ as a function of applied field $H$ at $T/T_c = 0.1, 0.3, 0.5$ and 0.7 for $d$-wave pairing. (b) $T$-dependence of $H_{c1}(T)$ for $s$, $d$, $p_{+}$ and $p_{-}$-wave pairings, estimated by Eilenberger theory. (c) $H_{c1}(T)$ in (b) is replotted as normalized $H_{c1}(T)/H_{c1}(T = 0.05T_c)$. We also present normalized $\lambda_{\text{London}}^{-2}(T)$ for $s$- and $d$-wave pairings.

determined by gap eq. (3) in uniform states. In the $s$-wave pairing, as shown in Fig. 1(c), normalized $H_{c1}(T)$ in Eilenberger theory appears smaller than $\lambda_{\text{London}}^{-2}(T)$, and shows decreases even at low $T$. This indicates that the vortex core energy still has $T$-dependence at low $T$, rather than saturation expected by $\lambda_{\text{London}}^{-2}(T)$. This may include the contribution of vortex core shrink on lowering $T$ by Kramer-Pesch effect.\(^{11}\) On the other hand, in the $d$-wave pairing, $H_{c1}(T)$ in Eilenberger theory is higher than $\lambda_{\text{London}}^{-2}(T)$. Thus, $T$-dependence of the core energy is weaker than estimate by $\lambda_{\text{London}}^{-2}(T)$. This is an opposite effect to the $s$-wave pairing case, and indicates that the estimate of core creation energy is not simple in $d$-wave pairing because we have to consider both contributions inside and outside of vortex cores. The latter is contributions by quasiparticles extending toward node-directions.\(^{12}\) These behaviors of $H_{c1}(T)$ is also confirmed for $\kappa = 6.9$. We expect that the relation in eq. (1) will be examined in experiments, comparing $H_{c1}(T)$ with $\lambda^{-2}(T)$.

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