Research Article

An Analytical Solution for the Vibration and Far-Field Sound Radiation Analysis of Finite, Semisubmerged Cylindrical Shells

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Vibration response and far-field sound radiation of a semisubmerged, finite cylindrical shell with low-frequency excitation are studied. The solution to this problem can be divided into two steps. The first step is to apply the wave propagation approach to determine the vibration response of the cylindrical shell. In the cylindrical coordinate system, the Flügge shell equations and Laplace equation are used to describe the cylindrical shell and surrounding fluid so that the vibration responses of the shell can be addressed analytically. The fluid free surface effect is taken into account by applying the sine series to force the velocity potential on the free surface to be zero. Furthermore, compared with the FEM (the finite element method), the present method is not only reliable but also effective. In the second step, the far-field sound radiation is solved by the Fourier transform technique and the stationary phase method in accordance with the vibration responses of the shell from the previous step. The boundary element method is applied to validate the reliability of the acoustical radiation calculation. The circumferential directivity of far-field sound pressure is discussed, and it is found that the maximum value of the sound pressure always appears directly under the structure when the driving frequencies are relatively low. Besides, in consideration of simplicity and less computation effort, the present method can be used for the rapid prediction of the vibration and far-field sound pressure of a semisubmerged cylindrical shell with low-frequency excitation.

1. Introduction

Vibroacoustical characteristics of cylindrical shells are the subject studied extensively due to their widespread use in many industrial applications, such as underwater vehicles, aircrafts, and pipelines. There are many researchers [1, 2] who have studied the shell and plate vibration problems. And with respect to the coupling effect of acoustics and structure, previous works [3–11] focused mainly on shells that were in complete contact with a fluid. So, the influence of a free surface on the vibration and sound radiation of the cylindrical shell tended to be neglected. Obviously, as far as the structure located close to the water surface or afloat on the water surface is concerned, it is unreasonable to consider the fluid region around the cylindrical shell unbounded. Cylindrical shells partially coupled with the fluid (the shell centerline is parallel to the free surface) are frequently encountered in ocean engineering, such as ships or submarines floating on the sea.

The coupling dynamic behavior, in particular the free vibration behavior, has attracted a lot of attention from researchers. In order to study the vibration of a finite cylindrical shell in a partially liquid-filled state, Amabili [10] applied appropriate boundary conditions on the free surface. Afterward, using the above method, Amabili [11] further studied the vibration of a cylindrical shell in a partially immersed state. For the purpose of solving the problem about the vibration of a partially liquid-filled and submerged finite cylindrical shell, Ergin [12] combined the boundary integral method and the image method to study it for the first time. Lakis et al. [13, 14] made the element division in the circumferential direction of the finite cylindrical shell and found that the energy method is very suitable for the analysis of the vibration and sloshing of the finite cylindrical
shell in a partially liquid-filled state. Li et al. [15] adopted the wave propagation method and Fourier transform technique to study the vibroacoustic characteristics of a semi-submerged infinite cylindrical shell. Guo et al. [16] investigated the free and forced vibration of a semi-submerged finite cylindrical shell by applying the Rayleigh–Ritz method.

The sound radiation from structures partially coupled with fluid has received growing attention. Wu et al. [17, 18] studied the vibroacoustic characteristics of a fluid-filled cylindrical shell semisubmerged in a coaxial flow with the Fourier series expansion method. Furthermore, the analytical solution for the vibration of the cylindrical shell can be obtained by using this coordinate system. Besides, at the point \((R, x_0, \varphi_0)\) of the cylindrical shell, there is a harmonic point force \(f_0\), whose amplitude and driving frequency are separately \(F_0\) and \(f_0\).

The motion state of the cylindrical shell can be represented by the Flügge shell equations [24]:

\[
\begin{bmatrix}
L u \\
L v \\
L w \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} + \begin{bmatrix}
0 \\
0 \\
0 \\
\end{bmatrix} f_r, \quad f_p = \begin{bmatrix}
f_0 + f \end{bmatrix}.
\]

The meaning of each character in the formula is as follows: \(U\) represents the axial displacement of the shell, \(V\) represents the tangential displacement, \(W\) represents the radial displacement, and \([L]\) denotes the classical Flügge differential operator for the thin shell theory:

\[
L u = R^2 \frac{\partial^2}{\partial x^2} + \frac{1 - \mu}{2} (K + 1) \frac{\partial^2}{\partial \varphi^2} - \frac{\rho R^2 (1 - \mu^2)}{E} \frac{\partial^2}{\partial t^2},
\]

\[
L v = L_{22} = L_{23} = L_{33} = L_{12} = L_{21} = L_{31} = L_{32} = \ldots
\]

\[
L_{11} = R^2 \frac{\partial^2}{\partial x^2} + \frac{1 - \mu}{2} (K + 1) \frac{\partial^2}{\partial \varphi^2} - \frac{\rho R^2 (1 - \mu^2)}{E} \frac{\partial^2}{\partial t^2}.
\]

\[
L_{22} = R \frac{\partial^2}{\partial x^2} + R \frac{\partial^2}{\partial \varphi^2} + \frac{\rho R^2 (1 - \mu^2)}{E} \frac{\partial^2}{\partial t^2},
\]

\[
\psi^4 = R^4 \left( \frac{\partial^4}{\partial x^4} + 2 R^2 \frac{\partial^4}{\partial x^2 \partial \varphi^2} + \frac{\partial^4}{\partial \varphi^4} \right),
\]

\[
K = \frac{h^2}{12 R^2},
\]

\[
L_{12} = L_{21} = L_{31} = L_{32} = \ldots
\]

\[
L_{11} = R \frac{\partial^2}{\partial x^2} + R \frac{\partial^2}{\partial \varphi^2} + \frac{\rho R^2 (1 - \mu^2)}{E} \frac{\partial^2}{\partial t^2}.
\]

2. Theoretical Analysis of the Shell Vibration

2.1. The Motion Equations of the Cylindrical Shell. There is a simply supported finite thin circular cylindrical shell in Figure 1, whose length, thickness, radius, Young’s modulus, Poisson’s ratio, and density are, respectively, \(L, h, R, E, \mu, \) and \(\rho\). Meanwhile, this cylindrical shell model submerges in such a fluid, whose density and sound velocity are, respectively, \(\rho_f\) and \(c_f\).

The hypothesis of this paper is a semi-infinite fluid domain. That is to say, there is a free surface that divides the whole space into two parts: the upper part is a vacuum and the lower part is the fluid domain. Since the cylindrical coordinate system \((r, x, \varphi)\) is very convenient to solve and calculate in this paper, this coordinate system is adopted, whose origin is point O. Furthermore, the analytical solution for the vibration of the cylindrical shell can be obtained by using this coordinate system. Besides, at the point \((R, x_0, \varphi_0)\) of the cylindrical shell, there is a harmonic point force \(f_0\), whose amplitude and driving frequency are separately \(F_0\) and \(f_0\).
The meaning of \( f_p \) is the fluid pressure on the cylindrical shell, which can be defined as
\[
f_p = \begin{cases} 
p, & 0 \leq \phi \leq \pi, \\
0, & \text{others}, 
\end{cases}
\] (3)
where \( p \) is the fluid pressure on the outer surface of the cylindrical shell.

Because the motion of the cylindrical shell is a kind of harmonic vibration, the axial displacement, the tangential displacement, the radial displacement, and the fluid pressure can be separately written as follows:
\[
\begin{align*}
\mathbf{u} &= \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} U_{mn} \cos(k_m x) \exp(i \phi) \cos(\omega t), \\
\mathbf{v} &= \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} V_{mn} \sin(k_m x) \exp(i \phi) \cos(\omega t), \\
\mathbf{w} &= \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} W_{mn} \sin(k_m x) \exp(i \phi) \cos(\omega t), \\
\mathbf{f}_r &= \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} F_{mn} \sin(k_m x) \exp(i \phi) \cos(\omega t), \\
\mathbf{f}_p &= \sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} f_{mn} \sin(k_m x) \exp(i \phi) \cos(\omega t).
\end{align*}
\] (4)

The meaning of each character in the aforementioned formulas is as follows: \( U_{mn} \) represents the axial displacement amplitude, \( V_{mn} \) represents the tangential displacement amplitude, and \( W_{mn} \) represents the radial displacement amplitude. \( F_{mn} \) refers to the amplitudes of harmonic point force \( f_r \), and \( f_{mn} \) refers to the amplitudes of the fluid pressure \( f_p \) on a cylindrical shell. As for \( m \) and \( n \), they separately represent the expansion coefficients in the axial and the circumferential directions. Besides, the circular frequency is expressed with \( \omega \), and the axial wavenumber is expressed with \( k_m \), and \( k_m = m \pi / L \) [25] and \( i = \sqrt{-1} \).

Substituting equation (4) into equation (1), the following equations can be further gained after the orthogonalization:
\[
\begin{bmatrix} U_{mn} \\ V_{mn} \\ W_{mn} \end{bmatrix} = \begin{bmatrix} 1 - \mu^2 \end{bmatrix} \frac{R^2}{Eh} \begin{bmatrix} 0 \\ 0 \\ F_{mn} - f_{mn} \end{bmatrix}.
\] (5)

Matrix \([T]\) contains multiple elements, whose expressions are as follows:
\[
\begin{align*}
\zeta &= k_m R, \\
T_{11} &= \Omega^2 - \zeta^2 - n^2(1 + K)(1 - \mu)/2, \\
T_{12} &= i\zeta n(1 + \mu)/2, \\
T_{13} &= \mu \zeta^3 + \mu K \zeta^3 - K(1 - \mu) n^2/2, \\
T_{21} &= -T_{12}, \\
T_{22} &= \Omega^2 - \zeta^2 - n^2(1 + 3K)(1 - \mu)/2, \\
T_{23} &= i n + i Kn \zeta^3 - (3 - \mu)/2, \\
T_{31} &= -T_{13}, \\
T_{32} &= T_{23}, \\
T_{33} &= 1 + K + K \zeta^4 + 2Kn \zeta^2 + Kn^4 - 2Kn^2 - \Omega^2.
\end{align*}
\] (6)

From the perspective of dimension, \( \Omega \) belongs to a nondimensional frequency, which is only related to the ring frequency of the cylindrical shell, and \( \Omega = \omega \sqrt{\rho R^2 (1 - \mu^2)/E} \).

It is not difficult to obtain the vibration equation about radial displacement by solving equation (5):
\[
W_{mn} = \frac{(1 - \mu^2) R^2}{Eh \zeta_m} (F_{mn} - f_{mn}),
\] (7)
where \( I_{mn} = |T|/(T_{11} T_{22} - T_{12} T_{21}) \).

The expression for \( f_r \) is as follows:
\[
f_r(x, \phi) = F_0 \delta(x - x_0) \delta(\phi - \phi_0).
\] (8)

The meaning of each character in the aforementioned formula is as follows: \( F_0 \) is the maximum value of the harmonic point force \( f_r \), \( \delta(\cdot) \) is the delta function, and \( (x_0, \phi_0) \) is the position of \( f_r \).

Substituting equation (8) into equation (4), \( F_{mn} \) can be further gained after the orthogonalization:
\[ F_{mn} = F_0 \sin(k_m x_0) \exp(-im\omega_0) \frac{1}{L \pi} \]  

2.2. The Motion Equations of the Fluid. Combined with the incompressibility of the fluid and velocity potential function \( \phi(r, x, \varphi, t) \), a function meeting the fluid Laplace equation, the characteristics of the fluid field can be reflected. 

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \phi}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \varphi^2} + \frac{\partial^2 \phi}{\partial x^2} = 0. \]  

(10)

Consider that the liquid at infinity is still; that is, 

\[ \frac{\partial \phi}{\partial r} \bigg|_{r=\infty} = 0. \]  

(11)

At the free surface, both the gravity effect and the surface tension of the fluid could be neglected [26]. Therefore, the fluid boundary condition at the free surface can be expressed as 

\[ \phi(r, x, \varphi, t)\bigg|_{\text{outer}} = 0, \quad \phi(r, x, \varphi, t)\bigg|_{\varphi=0} = 0. \]  

(12)

The separation of variables method can be employed to solve the Laplace equation [10], and the velocity potential function which meets all the conditions could be expressed as 

\[ \phi(r, x, \varphi, t) = \sum_{m=1}^{\infty} \sum_{a=1}^{\infty} \phi_{ma} \sin(k_m x) \sin(a \varphi) K_a(k_m r) \sin(\omega t), \]  

(13)

where \( \phi_{ma} \) denotes the amplitude of the velocity potential function and \( K_a(\cdot) \) is the \( a \)-th order modified Bessel function of the second kind. Besides, sound pressure should be equal to 0 for an arbitrary point on the free surface. Thus, the velocity potential would be equivalent to zero on the free surface. That is why we choose the sine series.

2.3. The Velocity Continuity Condition. Since the radial velocity on the outer surface of the cylindrical shell should be equal to that of the fluid, 

\[ \frac{\partial \phi}{\partial r}\bigg|_{r=R} = -\frac{\partial \omega}{\partial t}\bigg|_{r=R}. \]  

(14)

By conducting orthogonal processing and integrating about equation (14), one can get 

\[ \int_0^\pi \left( \frac{\partial \phi}{\partial r}\bigg|_{r=R} \right) \sin(a \varphi) d\varphi = \int_0^\pi \left(-\frac{\partial \omega}{\partial t}\bigg|_{r=R} \right) \sin(a \varphi) d\varphi, \]  

(15)

where \( \sin(a \varphi) \) is a weight function, and equation (15) is a typical Galerkin integral equation.

Furthermore, the amplitude of the velocity potential function can be deduced from equation (15):

\[ \phi_{ma} = \frac{2\omega}{\pi k_m K_a(k_m R)} \sum_{b=\infty}^{\infty} W_{mb} \lambda_{ba}, \]  

(16)

where 

\[ \lambda_{ba} = \int_0^\pi \sin(a \varphi) \exp(ib\varphi) d\varphi. \]  

(17)

The load \( p \) which the fluid imposed on the cylindrical shell can be calculated by Bernoulli’s equation:

\[ p = F \frac{\partial \phi}{\partial r}\bigg|_{r=R}. \]  

(18)

By substituting equations (13) and (17) into equation (18), the fluid pressure \( p \) is expressed as

\[ p = \frac{2p}{\pi} \int_{-\pi}^{\pi} f_p \exp(-im\varphi) \sin(k_m x) d\varphi dx \]  

(20)

By applying orthogonal processing about equation (3), one can get 

\[ \int_{-\pi}^{\pi} f_p \exp(-im\varphi) \sin(k_m x) d\varphi dx = \int_{-\pi}^{\pi} p \exp(-im\varphi) \sin(k_m x) d\varphi dx. \]  

(21)

Thus, the amplitude of the load on the shell’s surface imposed by the fluid pressure can be obtained after solving equation (20).

\[ f_{mn} = \frac{p}{\pi} \sum_{b=\infty}^{\infty} \sum_{a=1}^{\infty} W_{mb} \lambda_{ma} K_a(k_m R) \lambda_{ba}, \]  

(21)

where \( \lambda_{ma} = \int_0^\pi \sin(a \varphi) \exp(-im\varphi) d\varphi. \)

By substituting equations (9) and (21) into (5), the coupling equation of the system can be expressed as

\[ I_{na} W_{mn} + \frac{\Omega^2 R}{2 \pi h} \sum_{b=\infty}^{\infty} \sum_{a=1}^{\infty} W_{mb} \lambda_{ma} \frac{K_a(\zeta_m)}{\zeta_m K_a(\zeta_m) \lambda_{ba}} \left( 1 - \mu^2 \right) R^2 F_0 \sin(k_m x_0) \exp(-im\omega_0) = \frac{EhL}{\pi}, \]  

(22)
where $\zeta_m = k_m R$.

Since there exist infinite series in equation (22), a truncation process is required to make the solution possible.

\[
\begin{bmatrix}
I_{m,-N} + G_{m,-N,-N} & G_{m,-N,1-N} & \ldots & G_{m,-N,N-1} \\
G_{m,1-N,-N} & I_{m,1-N} + G_{m,1-N,1-N} & \ldots & G_{m,1-N,N-1} \\
\vdots & \vdots & \ddots & \vdots \\
G_{m,N-1,-N} & G_{m,N-1,1-N} & \ldots & I_{m,N-1} + G_{m,N-1,N-1} \\
G_{m,N,-N} & G_{m,N,1-N} & \ldots & G_{m,N,N-1} & I_{m,N} + G_{m,N,N} \\
\end{bmatrix} \begin{bmatrix}
W_{m,-N} \\
W_{m,-N-1} \\
\vdots \\
W_{m,N-1} \\
W_{m,N} \\
\end{bmatrix} = \begin{bmatrix}
\exp(iN\varphi_0) \\
\exp[i(1-N)\varphi_0] \\
\exp(-iN\varphi_0) \\
\end{bmatrix}
\]

where $G_{mn\alpha} = W_{mn}\Omega^2 R^2/h \sum_{n=1}^{K} \lambda_{m,n} (K_m,\zeta_m) (K_m,\zeta_m) \lambda_{n\alpha}$.

For the free vibration analysis, there is no exciting force, and the above problem changes into the solution of eigenvalues.

### 3. Theoretical Analysis of the Far-Field Sound Radiation

In this section, the far-field sound radiation of the semi-submerged finite shell will be derived. In Figure 2, point P is a far-field observation point in the acoustic field. In the figure, the length of line segment OP is defined as $R_0$, the angle formed by rotating counterclockwise from the vector OP to the positive half of the x-axis is defined as $\theta$, and the vector OP is written as $\vec{R}_0$. It is obvious that the coordinate system $(R_0, \theta, \phi)$ is a typical spherical coordinate system.

In the x-direction, the fluid is free and unbounded, which would bring about such a problem that the domain of the sound field is incompatible with that of the shell’s surface velocity. In order to avoid this problem, we have processed the cylindrical shell properly, that is, setting up a semi-infinite rigid column barrier [26] at each end of the shell, respectively. Therefore, the radial displacement $w(r, \phi)$ of the shell is rewritten as

\[
w(r, \phi) = \left\{ \begin{array}{ll}
\sum_{m=1}^{\infty} \sum_{n=-\infty}^{\infty} W_{mn} \sin(k_n r) \exp(in\phi), & 0 \leq r \leq L, \\
0, & \text{else}.
\end{array} \right.
\] (24)

When dealing with the acoustic radiation problem, the fluid is considered as an acoustic medium, and it is compressible. Thus, the sound pressure $P(r, x, \phi)$ in the fluid field should satisfy the Helmholtz equation:

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial P}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 P}{\partial \phi^2} + \frac{\partial^2 P}{\partial x^2} + k_p^2 P = 0,
\] (25)

The truncated numbers of $a$, $m$, and $n$ are $K$, $M$, and $N$, where $N$ represents the truncated numbers of $N$ and $B$, and $n = b$.

Then, equation (22) can be rewritten as

\[
\begin{bmatrix}
I_{m,-N} + G_{m,-N,-N} & G_{m,-N,1-N} & \ldots & G_{m,-N,N-1} \\
G_{m,1-N,-N} & I_{m,1-N} + G_{m,1-N,1-N} & \ldots & G_{m,1-N,N-1} \\
\vdots & \vdots & \ddots & \vdots \\
G_{m,N-1,-N} & G_{m,N-1,1-N} & \ldots & I_{m,N-1} + G_{m,N-1,N-1} \\
G_{m,N,-N} & G_{m,N,1-N} & \ldots & G_{m,N,N-1} & I_{m,N} + G_{m,N,N} \\
\end{bmatrix} \begin{bmatrix}
W_{m,-N} \\
W_{m,-N-1} \\
\vdots \\
W_{m,N-1} \\
W_{m,N} \\
\end{bmatrix} = \begin{bmatrix}
\exp(iN\varphi_0) \\
\exp[i(1-N)\varphi_0] \\
\exp(-iN\varphi_0) \\
\end{bmatrix}
\]

where $k_f = \omega/c_f$ is the compression wavenumber, and $c_f$ is the sound speed in the fluid.

Since there exist infinite series in equation (22), a truncation process is required to make the solution possible.

\[
\begin{bmatrix}
I_{m,-N} + G_{m,-N,-N} & G_{m,-N,1-N} & \ldots & G_{m,-N,N-1} \\
G_{m,1-N,-N} & I_{m,1-N} + G_{m,1-N,1-N} & \ldots & G_{m,1-N,N-1} \\
\vdots & \vdots & \ddots & \vdots \\
G_{m,N-1,-N} & G_{m,N-1,1-N} & \ldots & I_{m,N-1} + G_{m,N-1,N-1} \\
G_{m,N,-N} & G_{m,N,1-N} & \ldots & G_{m,N,N-1} & I_{m,N} + G_{m,N,N} \\
\end{bmatrix} \begin{bmatrix}
W_{m,-N} \\
W_{m,-N-1} \\
\vdots \\
W_{m,N-1} \\
W_{m,N} \\
\end{bmatrix} = \begin{bmatrix}
\exp(iN\varphi_0) \\
\exp[i(1-N)\varphi_0] \\
\exp(-iN\varphi_0) \\
\end{bmatrix}
\]

Therefore, the sound pressure expression, which satisfies the sound pressure release boundary condition on the free surface should be met:

\[
P(r, x, \phi)|_{\varphi=\pi} = 0, \quad P(r, x, \phi)|_{\varphi=0} = 0.
\] (26)

In order to solve the sound-structural coupling model, the Fourier transform and Fourier inverse transform are introduced as follows:

\[
\left\{ \begin{array}{l}
\tilde{f}(k_x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(x) \exp(-ik_x x) dx, \\
\end{array} \right.
\] (28a)

\[
\left\{ \begin{array}{l}
f(x) = \int_{-\infty}^{\infty} \tilde{f}(k_x) \exp(ik_x x) dx. \\
\end{array} \right.
\] (28b)

The meaning of each character in the aforementioned formulas is as follows: $\tilde{f}(k_x)$ refers to the function of $f(x)$, $k_x$ refers to the wavenumber $i = \sqrt{-1}$, and they are all in the wavenumber domain.

By substituting equation (27) into equation (25) and conducting the Fourier transform technique, equation (25) is further changed to

\[
\frac{\partial^2 \tilde{p}_a(r, k_x)}{\partial r^2} + \frac{1}{r} \frac{\partial \tilde{p}_a(r, k_x)}{\partial r} + \left( k_f^2 - k_x^2 - \frac{a^2}{r^2} \right) \tilde{p}_a(r, k_x) = 0.
\] (29)

Then, the expression of $p_a(r, x)$ can be acquired by applying the separation of variables method.
\[ \bar{p}_a(r, k_x) = \bar{A}_a(k_x) H_1^{(1)}(k, r), \quad (30) \]

where \( \bar{A}_a(k_x) \) is the sound pressure amplitude in the wavenumber domain, \( H_1^{(1)}(\cdot) \) is the first-order Hankel function of the first kind, and \( k_x = \sqrt{k_x^2 - k_z^2} \) is the radial wavenumber.

Afterward, the sound pressure in the wavenumber domain can be expressed as

\[ \bar{P}(r, k_x, \varphi) = \sum_{n=1}^{\infty} \bar{A}_a(k_x) H_1^{(1)}(k, r) \sin(a\varphi). \quad (31) \]

Similarly, combined with the Fourier transform technique, the radial displacement can be converted into the wavenumber domain:

\[ \bar{w}(k_x, \varphi) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} W_{mn} k_m [1 - (-1)^m \exp(-i k_x L)] \exp(i n \varphi). \quad (32) \]

Accordingly, the expression of the radial displacement is as follows:

\[ \bar{w}(k_x, \varphi) = \sum_{n=-\infty}^{\infty} \bar{w}_n(k_x) \exp(i n \varphi), \quad (33) \]

\[ P(r, x, \varphi) = \int_{-\infty}^{\infty} \sum_{a=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2 \rho_f \omega^2}{\pi k_x H_1^{(1)}(k, R)} \bar{w}_n(k_x) \lambda_{ma} H_1^{(1)}(k, r) \sin(a \varphi) \exp(i k_x x) dk_x. \quad (37) \]

where \( x = k_x r \), and when \( r \) goes to infinity, \( x \) also goes to infinity.

Substituting equation (38) into equation (37), the sound pressure can be further written as

\[ P(r, x, \varphi) = \int_{-\infty}^{\infty} \sum_{a=1}^{\infty} \sum_{n=-\infty}^{\infty} \frac{2 \rho_f \omega^2}{\pi k_x H_1^{(1)}(k, R)} \bar{w}_n(k_x) \lambda_{ma} \frac{2}{\pi k_x r} \exp \left[ i \left( k_x r - \frac{a \pi}{2} - \frac{\pi}{4} \right) \right] \sin(a \varphi) \exp(i k_x x) dk_x. \quad (39) \]
Converting equation (39) into the expression under the spherical coordinate system \((R_0, r, x)\), where the conversion

\[
P(R_0, \theta, \phi) = \sum_{d=1}^{\infty} \sum_{m=-\infty}^{\infty} 2\rho f \omega^2 \lambda_m \sin(\alpha \phi) \exp \left\{ -\left( \frac{4\pi}{2} + \frac{\pi}{4} \right) \right\} \frac{2}{\pi^2 R_0 \sin \theta} \int_{-\infty}^{\infty} \hat{w}_n(k_x) \exp \left\{ i(k_x R_0 \sin \theta + k_x R_0 \cos \theta) \right\} \frac{\sqrt{k_x^2 H^{(1)}_n(k_x R_0)}}{n! H_n^{(1)}(k_x R)} \, dk_x.
\]  

(40)

The far-field sound pressure is examined in this study, and in the acoustic field [27] and electromagnetic field [28], it is often necessary to calculate the infinite integral, which could lead to a large amount of calculation and inconvenience. In order to solve this problem well, a common approach is to introduce the stationary phase method, which can transform the infinite integral problem into a linear operation one. Therefore, the solution of equation (40) is relatively easier.

Two related functions are given here to explain the method—the stationary phase method:

\[
\Phi(k_x) = \frac{\hat{w}_n(k_x)}{k_x^2 H_n^{(1)}(k_x R)},
\]  

(41a)

\[
\Psi(R_0, \theta, k_x) = \exp \left\{ i(k_x R_0 \sin \theta + k_x R_0 \cos \theta) \right\}.
\]  

(41b)

According to the aforementioned formulas, the infinite integral in equation (40) is further changed into

\[
I = \int_{-\infty}^{\infty} \Phi(k_x) \exp \left\{ i\Psi(R_0, \theta, k_x) \right\} dk_x.
\]  

(42)

Then, the stationary phase method [27] is used to process equation (40), and the result is

\[
I \approx \sqrt{2\pi} \Phi(k_x) \exp \left\{ i\Psi(R_0, \theta, k_x) - \pi/4 \right\} \left| \frac{\partial^2 \Psi(R_0, \theta, k_x)}{\partial k_x^2} \right|^{1/2} \bigg|\bigg|_{k_x = k_x^*}.
\]  

(43)

where \(k_x^*\) refers to the stationary phase point.

Actually, \(k_x^*\) is the solution of equation (43):

\[
\frac{\partial \Psi(R_0, \theta, k_x)}{\partial k_x} = R_0 \left( -\frac{k_x \sin \theta}{\sqrt{k_f^2 - k_x^2}} + \cos \theta \right) = 0.
\]  

(44)

Therefore, the stationary phase point \(k_x^* = k_f \cos \theta\), and equation (43) is further changed into

\[
I = \sqrt{2\pi} \hat{w}_n(k_f \cos \theta) \exp \left\{ -\pi i/4 \right\} \exp \left\{ ik_f R_0 \right\} H_n^{(1)}(k_f R \sin \theta) k_f \sqrt{R_0 \sin \theta}
\]  

(45)

Thus, the expression of the far-field sound pressure in the spherical coordinate system is obtained as

\[
P(R_0, \theta, \phi) = \sum_{d=1}^{\infty} \sum_{m=-\infty}^{\infty} 2\rho f \omega^2 \lambda_m \sin(\alpha \phi) \exp \left\{ (-\pi/2)i\sin(\alpha \phi) \cdot \lambda_m \hat{w}_n(k_f \cos \theta) \right\} \frac{\exp \left\{ -(\alpha n/2)i \sin(\alpha \phi) \cdot \lambda_m \hat{w}_n(k_f \cos \theta) \right\}}{H_n^{(1)}(k_f R \sin \theta)}
\]  

(46)

where \(W_m k_m / k_m^2 - \frac{1}{1 - (-1)^m} \exp(-ik_f \cos \theta L)\).

Should be mentioned that when the stationary phase point \(k_x = k_f \cos \theta\) is close to the axial wavenumber \(k_m\), the expression of \(\hat{w}_n(k_f \cos \theta)\) is also convergent; the reason is as follows:

\[
\lim_{k_f \cos \theta \to k_m} \frac{k_m \left[ 1 - (-1)^m \exp(-ik_f \cos \theta L) \right]}{k_m^2 - (k_f \cos \theta)^2} = \frac{iL}{2}
\]  

(47)

To be clear, the calculation formula of the sound pressure level is \(SPL = 20 \log_{10}(|P|/p_0)\), and the value of \(P_0\) is \(10^{-6}\) Pa.

4. Numerical Examples of the Shell Vibration

The relevant calculation parameters used in this paper are as follows: for the cylindrical shell, its elastic modulus \(E\) is 206 GPa, Poisson’s ratio \(\mu\) is 0.3, density \(\rho\) is 7850 kg/m³, length \(L\) is 1.284 m, radius \(R\) is 0.18 m, and thickness \(h\) is 0.003 m; for the harmonic point force, its amplitude \(F_0\) is 1 N; for the fluid, its density \(\rho_f\) is 1025 kg/m³, and sound velocity \(c_f\) is 1500 m/s. In this paper, the complex elastic modulus \(E_c\) is used instead of the elastic modulus \(E\) for calculation. The relationship between them is as follows: \(E_c = E (1 + i\eta)\), where \(\eta\) is the damping coefficient of the structure, and its value is 0.01.

4.1. The Convergence Analysis of the Radial Velocity. In Section 2, in order to calculate the infinite series in equation (23), we introduced the fixed-phase method, which is an approximate algorithm, so it is necessary to prove its convergence. The main proof process is as follows: if \(M, K, N\) are equal, then the quadratic velocity \(V_m\) of the cylindrical shell can be further calculated, and its expression is \(V_m = (1/2s) \int_s^2 |v_s|^2 \, ds\), where the meaning of \(s\) is the surface area of the cylindrical shell. The calculation results of \(V_m\) are
shown in Figure 3. The quadratic velocity level is defined as
\[ QVL = 10 \log_{10}(V_m/V_0) \text{ (unit: dB)} \], where \( V_0 = 10^{-12} \text{ m}^2/\text{s}^2 \).

It is not hard to see in the figure that when \( N \) is close to
12, \( V_m \) begins to converge, so it is reasonable to take \( N = 14 \)
in this paper.

4.2. The Verification of the Present Method for Vibration Analysis. In this section, we will continue to demonstrate the reliability of the present method. In finite element software MSC Patran/Nastran, a cylindrical shell model with the external fluid is established (see Figure 4), and the model is simplified. For example, in MSC Patran/Nastran, the cylindrical shell is regarded as the shell element, and the virtual added mass matrix [29] in the virtual mass method is used to simulate the influence of the fluid on the free vibration and forced vibration of the cylindrical shell. In addition, the model is divided into 40 equal parts in both axial and tangential directions, with a total of 1600 elements, all of which are quadrilateral.

4.2.1. Free Vibration. In order to illustrate the calculation accuracy of the present method, the natural frequencies and mode shapes obtained by the present method are compared with those obtained by the FEM, respectively (see Table 1 and Figure 5), where \( f_a \) and \( f_b \) represent the natural frequencies obtained by the present method and the FEM, respectively. The relative error of the natural frequencies between both methods is defined as
\[ \text{Err1} = \frac{|f_a - f_b|}{f_b} \times 100\% \]

It is easy to see that the method is reliable and accurate from the comparison results of Figure 5 and Table 1. For the natural frequencies of the first ten orders, the relative errors are less than 1%, where mode number \((i,j)\) indicates the axial mode number and circumferential mode number in Table 1, respectively.

In order to further demonstrate the accuracy and reliability of the present method, the natural frequencies calculated with the present method are compared with those in [11]. The material parameters in [11] are listed as follows: \( E = 206 \text{ GPa}, \mu = 0.3, \rho = 7680 \text{ kg/m}^3 \), and the dimensions of the shell are \( L = 0.664 \text{ m}, R = 0.175 \text{ m} \), and thickness \( h = 0.001 \text{ m} \). The density of the fluid is \( \rho_f = 1000 \text{ kg/m}^3 \). The relative error of the natural frequencies obtained by the two methods is defined as
\[ \text{Err2} = \frac{|f_{pre} - f_{ref}|}{f_{ref}} \times 100\% \]

where \( f_{pre} \) refers to the natural frequencies obtained by the present method and \( f_{ref} \) represents the natural frequencies in [11].

In Table 2, the differences between results of the present method and [11] are relatively small (less than 3%), proving that the present method for the free vibration is accurate and reliable.

4.2.2. Forced Vibration. By using the same model parameters as those in Table 1, the point excitation is exerted at \((R, L/2, \pi/2)\), and then the dynamic responses at \((R, L/4, \pi/2)\) and \((R, L/4, -\pi/2)\) can be obtained. They are evaluated by the radial displacement level, whose calculation formula is
\[ RDL = 20 \log_{10}(|w_0|/|w_0|) \cdot (\text{dB}) \], where the value of \( w_0 \) is \( 10^{-12} \text{ m} \).

The comparison between the finite element simulation results about \( RDL \) and the theoretical calculation ones is shown in Figure 6.

As shown in Figure 6, it is obvious that this method is reliable, and its calculation results are in good agreement with the finite element ones. More importantly, the computational efficiency of the present method is much higher than that of the finite element method. The former is about 20 times faster than the latter.

It is noteworthy that this method can solve the problem of low frequency very well. When the frequency of simple harmonic point force is relatively low (more than 500 Hz), it is not so suitable. The reason for this phenomenon lies in the following: in software Nastran, the fluid is regarded as virtual mass and infinite in the axial direction of the cylindrical shell; however, when the problem is solved analytically, the fluid is limited in the axial direction of the cylindrical shell; besides, the calculation accuracy of the analytical method depends on the frequency of the harmonic point force.

5. Numerical Analysis of the Far-Field Sound Radiation

5.1. The Verification of the Accuracy of the Stationary Phase Method. It can be seen from the first section that the solution content of this paper is mainly vibration response and acoustic radiation. Since the vibration response has been calculated in the previous sections, the task of this section is to solve the sound radiation and compare this result with that of the half-space boundary element method (BEM) [26].

The excitation point is located at \((R, L/2, \pi/2)\), and its maximum value \( F_0 \) is 1 N. When \( R_0 = 1000 \text{ m}, \theta = \pi/4 \), and \( \varphi = 0-\pi \), the sound pressure level calculated by the present method and that calculated by the half-space boundary element method are calculated, respectively, and the differences between them are compared (see Figure 7).

As shown in the figure, the results gained by the two methods are very consistent, and the calculation process of the present method (the stationary phase method) is actually a linear calculation process, so this method is not only reliable but also efficient. Under the same conditions, the computational speed of the present method is 40 times faster than that of the half-space BEM.

Besides, since the driven frequencies are relatively low, the directivity of far-field sound pressure in the circumferential direction is monotonous. Moreover, it can be seen that the maximum value of the sound pressure in the circumferential direction is at the angle of the excitation position \((\varphi_0 = \pi/2)\) in Figure 7.

5.2. Analysis of the Circumferential Directivity of the Far-Field Sound Radiation. In order to learn more about the relationship between the circumferential directivity and the excitation position, we discussed the circumferential directivity at different excitation position angles \( \varphi_0 \). In particular, \( \varphi_0 \) is set to be \( \pi/2, \pi/4, 0, -\pi/4, \) and \(-\pi/2\); when \( R_0 = 1000 \text{ m}, \theta = \pi/3 \), and \( \varphi = 0-\pi \), the circumferential
Figure 3: The convergence curves of the quadratic velocity with different exciting frequencies.

Figure 4: The finite element model of the cylindrical shell in software MSC Patran/Nastran.

Table 1: Comparison of the natural frequencies of the first ten orders (Hz).

| Order Mode number | 1, 2 | 1, 2 | 1, 3 | 1, 3 | 1, 4 | 1, 4 | 1, 5 | 1, 5 | 1, 6 | 1, 6 |
|-------------------|------|------|------|------|------|------|------|------|------|------|
| $f_a$             | 111.39 | 113.32 | 160.06 | 165.44 | 239.34 | 242.54 | 265.10 | 267.71 | 320.79 | 320.90 |
| $f_b$             | 111.53 | 113.41 | 160.48 | 164.85 | 240.68 | 243.85 | 266.97 | 269.36 | 321.88 | 322.43 |
| Err1 (%)          | 0.13  | 0.08  | 0.26  | 0.36  | 0.56  | 0.54  | 0.70  | 0.61  | 0.34  | 0.47  |

Figure 5: The circumferential mode shapes of the first four orders.
The directivity of the sound pressure is studied at a frequency of 350 Hz. It is found in Figure 8 that the maximum value of the sound pressure on the circumferential direction is located at the bottom, and it is independent of the excitation position. The physical explanation for the directivity is given below.

The sound pressure mentioned in this paper is that at infinity, and the distance from any point of the cylindrical shell to the sound pressure point studied is the same, so the cylindrical shell can be replaced by a point source.

According to the image method [30, 31], another equidistant (the distance between the point source and free surface) and opposite phase source is applied to force the sound pressure on the free surface to be zero. In fact, the above physical model is called a dipole model.

Furthermore, it is not difficult to deduce the sound pressure of the dipole model [27]:

\[ P = \frac{A}{r} \exp \left( -ik_1r \right) \left[ -2i \sin(k_1D \sin \varphi \sin \theta) \right], \quad (48) \]

where the meaning of \( A \) is the maximum value of the sound pressure of the point source, \( r \) is the distance between the point source and the sound pressure point studied, and \( D \) represents the distance between the source and the image source \( (D/R < 2) \).

In fact, the exact value of \( D \) is unknown, but it does not affect our analysis of sound pressure directivity. This is

| Order number | 1 | 2 | 3 | 4 |
|--------------|---|---|---|---|
| Mode number  | 1, 4 | 1, 4 | 1, 5 | 1, 5 |
| Present method | 98.87 | 99.94 | 128.12 | 131.83 |
| Amabili [11] | 96.30 | 97.64 | 124.63 | 129.12 |
| Err2 (%)     | 2.67 | 2.36 | 2.80 | 2.10 |

Figure 6: The comparison of the radial displacement levels with the present method and with the numerical simulation: (a) measure point located at \((R, L/4, -\pi/2)\); (b) measure point located at \((R, L/4, 0)\).

Table 2: Comparison of the first four natural frequencies (Hz).
because the driving frequency is relatively low, and the product of \( k_f \) and \( D \) is less than \( \pi/2 \) \((k_f D < \pi/2)\); thus, when \( \sin \varphi \sin \theta \) takes a maximum value, the sound pressure also does.

Therefore, this is a good explanation for why the extreme value of sound pressure always appears at \( \varphi = \pi/2 \) and why the circumferential directivity diagram has good symmetry, although the velocity distribution is asymmetrical.

Besides, from equation (48), the sound pressure will also reach a maximum value obviously when \( \theta = \pi/2 \). For the purpose of illustrating the viewpoint, we calculated the sound pressure at different angles \( \theta \) when the excitation position angle \( \varphi_0 \) is equal to \( \pi/4 \). In particular, \( \vartheta \) is set to be \( \pi/8 \), \( \pi/4 \), and \( \pi/2 \); when \( R_0 = 1000 \text{ m} \) and \( \varphi = 0~\pi \), the circumferential directivity of the sound pressure is studied at the same driven frequency (350 Hz). Figure 9 presents the circumferential directivity of the sound pressure with different angles \( \theta \). From Figure 9, it is seen that the sound pressures increase with the increase of the angle \( \theta \). This further proves that the dipole model is suitable to explain the law of directivity.

Figure 7: Comparison of the sound pressure levels between the stationary phase method and boundary element method: (a) \( f = 100 \text{ Hz} \); (b) \( f = 200 \text{ Hz} \); (c) \( f = 300 \text{ Hz} \); (d) \( f = 400 \text{ Hz} \).
An approximate analytical method is proposed to study the vibration and far-field acoustic radiation of a semisubmerged, finite cylindrical shell with low-frequency excitation. In order to improve the efficiency of the solving process, the solution to this problem can be divided into two parts. After comparing the results of the present method with those of the FEM and BEM, this method is very reliable and can be used to calculate far-field sound pressure with high efficiency.

Besides, the results show the following sound pressure characteristics. The circumferential directivity diagram has good symmetry, although the velocity distribution is asymmetrical. Furthermore, the circumferential directivity of far-field sound pressure is independent of the excitation position, and the maximum value of the sound pressure always appears at the angle \( \varphi \) equal to \( \pi/2 \) or the angle \( \theta \) equal to \( \pi/2 \) (directly under the structure). In addition, the sound pressures increase with the increase of the angle \( \theta \).

### Data Availability

All the data included in this study are available from the corresponding author upon request.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

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