Weak coupling limit for the ground state energy of the 2D Fermi polaron

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Abstract

We analyze the ground state energy for \( N \) fermions in a two-dimensional box interacting with an impurity particle via two-body point interactions. We allow for mass ratios \( M > 1.225 \) between the impurity mass and the mass of a fermion and consider arbitrarily large box sizes while keeping the Fermi energy fixed. Our main result shows that the ground state energy in the limit of weak coupling is given by the polaron energy. The polaron energy is an energy estimate based on trial states up to first order in particle-hole expansion, which was proposed by Chevy in the physics literature. For the proof we apply a Birman–Schwinger principle that was recently obtained by Griesemer and Linden. One main new ingredient is a suitable localization of the polaron energy.

1 Introduction and main result

It is a universal challenge in quantum theory to understand the physics of few particles immersed into a complex environment in terms of properties of quasi-particles. A famous example of a quasi-particle is the Fröhlich polaron developed in a series of influential works by Landau, Pekar and Fröhlich [Lan33, LP48, Pek54, Frö54]. They suggested to describe the motion of an electron through a polarizable crystal in terms of a polaron, that is, a quasi-particle composed of an electron dressed by a local deformation of the crystal. This picture leads to a drastic simplification as the complex many-body problem is replaced by a self-consistent non-linear one-body model, which is much more accessible to computations. While the Fröhlich polaron is certainly the most prominent example for a polaron model, the concept of quasi-particles and polarons has turned out very useful far beyond its original application in the theory of electrons moving through crystals. For instance the experimental realization of impurities immersed into ultracold atomic gases during the last two decades has triggered the invention and analysis of many new models such as the Fermi polaron [Che06],
the Bose polaron [GD15] and the angulon [SL15]. In the present work we are interested
in the two-dimensional Fermi polaron which is a popular model in theoretical physics to
describe strongly population imbalanced Fermi gases at low temperature confined to the two-
dimensional plane. In case of extreme imbalance there is only a single particle interacting with
a gas of non-interacting fermions via a two-body short range interaction.

We consider $N$ identical fermions and an additional distinguished particle, called impurity
particle, in a two-dimensional box $\Omega = [-L/2, L/2]^2$ with periodic boundary conditions.
The underlying Hilbert space is $L^2(\Omega) \otimes \mathcal{H}_N$ where $\mathcal{H}_N = \bigwedge^N L^2(\Omega)$ denotes the space of
anti-symmetric $N$-particle wave functions. For a short-range potential, the Pauli principle
suppresses the interaction among the fermions which is therefore neglected. The Hamiltonian
of the system is formally described by

$$-\frac{1}{M} \Delta_y - \sum_{i=1}^N \Delta_{x_i} - g \sum_{i=1}^N \delta(x_i - y),$$  \hspace{1cm} (1.1)$$

where $y$ represents the coordinate of the impurity, $\Delta$ is the Laplace operator and $M$ denotes
the ratio between the mass of the impurity particle and the mass of a fermion. The interaction
is given by a Dirac-delta-potential $\delta(x)$ with coupling strength $g > 0$. This model is known as
the 2D Fermi polaron and has been analyzed to a great extent in the physics literature, see
e.g. [Che06, CG08, PS08, CM09, PDZ09, BM10, Par11, SEPD12, PL13]. The Fermi polaron
is of interest, among other reasons, because of the occurrence of a pairing mechanism some-
what analogous to the famous BCS–BEC crossover. In two space dimensions, one expects a
transition of the ground state as a function of the coupling strength. While for weak coupling,
the impurity particle is expected to be surrounded by a cloud of particle-hole excitations,
in the strong coupling regime it is predicted that the impurity is closely bound by a single
fermion forming a molecular state.

Here we provide a rigorous analysis of the ground state energy in the limit of weak coupling
by which we confirm its asymptotic form conjectured in the physics literature. From the
mathematical point of view, our work is a continuation of recent articles by Griesemer and
Linden [Lin17, GL18, GL19] in which they provide a definition of the self-adjoint Hamiltonian
$H$ associated with the formal expression (1.1), derive a Birman-Schwinger type principle for
this Hamiltonian and prove stability of the Fermi polaron at zero density. The Birman–
Schwinger priciple characterizes the low energy spectrum by means of an operator $\phi(\lambda)$ with
spectral parameter $\lambda$. Compared to $H$ the operator $\phi(\lambda)$ is given more explicitly and thus
provides a suitable tool for the analysis of the low energies, in particular for upper and lower
bounds for the ground state energy $\inf \sigma(H)$. Two such upper bounds, called polaron and
molecule energy, respectively, were discussed in [GL19]. Motivated by the derivation of these
upper bounds, we shall provide a matching lower bound for $\inf \sigma(H)$ in the limit of weak
1.1 The model

A possible approach to define the Fermi polaron is to start with a regularized version of the point interaction and then remove the regularization in a suitable sense [GL19]. Since this lays the foundation for our work, we provide a short summary.

For reasons of convenience we describe the fermions in the formalism of second quantization. This means that we think of \( \mathcal{H}_N \) as the \( N \)-particle sector of \( \mathcal{F} = \bigoplus_{n=0}^{\infty} L^2(\Omega) \), the fermionic Fock space over \( L^2(\Omega) \). We denote the vacuum state with zero particles by \( |0\rangle = (1, 0, 0, ...) \) and define creation and annihilation operators \( a_k^*, a_k : \mathcal{F} \to \mathcal{F} \) of plane waves \( \varphi_k(x) = L^{-1}e^{ikx}, k \in (2\pi/L)\mathbb{Z}^2 \),

\[
(a_k \Psi)^{(n)} = \sqrt{n + 1} \int_{\Omega} dx_{n+1} \overline{\varphi_k(x_{n+1})} \Psi^{(n+1)}(x_1, ..., x_{n+1}),
\]

\[
(a_k^* \Psi)^{(n)} = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} (-1)^j \varphi_k(x_j) \Psi^{(n-1)}(x_1, ..., x_{j-1}, x_{j+1}, ..., x_n)
\]

for \( \Psi = (\Psi^{(n)})_{n \geq 0} \in \mathcal{F} \). The creation and annihilation operators satisfy the usual canonical anti-commutation relations (CAR),

\[
a_k a_l^* + a_l^* a_k = \delta_{kl}, \quad a_k a_l + a_l a_k = 0
\]

for all pairs \( k, l \in (2\pi/L)\mathbb{Z}^2 \).

For any number \( E_B < 0 \) we introduce the inverse coupling constant

\[
g_n^{-1} = \sum_{k^2 \leq n} \frac{1}{(1 + \frac{1}{M})k^2 - E_B}
\]

and define the sequence of regularized Hamiltonians \( (H_n)_{n \in \mathbb{N}} \), acting on \( L^2(\Omega) \otimes \mathcal{H}_N \), by

\[
H_n = -\frac{1}{M} \Delta_y + \sum_{k} k^2 a_k^* a_k - g_n \sum_{k^2, l^2 \leq n} e^{i(k-l)y} a_l^* a_k.
\]

If not stated otherwise, sums run over the two-dimensional momentum lattice \((2\pi/L)\mathbb{Z}^2\) with possible restrictions indicated, e.g., as \( k^2 \leq n \).

The following statement proves the existence of the self-adjoint Hamiltonian describing the 2D Fermi polaron.

**Proposition 1.1.** (see [GL19, Theorem 6]) For given \( L > 0, N \geq 1, M > 0 \) and \( E_B < 0 \) there exists a self-adjoint Hamiltonian \( H : D(H) \subseteq L^2(\Omega) \otimes \mathcal{H}_N \to L^2(\Omega) \otimes \mathcal{H}_N \) such that
$H_n \to H$ in strong resolvent sense as $n \to \infty$. $H$ is bounded from below.

Remarks.

1.1. From Proposition 5.1 [GL19] we know that the spectrum $\sigma(H)$ is purely discrete.

1.2. The choice of $g_n$ ensures the following renormalization condition (note that $g_n$ has a logarithmic divergence as $n \to \infty$): For $N = 1$ the Hamiltonian $H$ has exactly one negative eigenvalue which coincides with $E_B < 0$. Hence the number $E_B$ corresponds to the binding energy of the 1 + 1-particle model and can be used as a suitable coupling parameter of the point interaction.

The goal of this work is to derive an asymptotic formula for the ground state energy

$$\min \sigma(H)$$

in the limit of weak coupling $E_B \nearrow 0$ or in the limit of large density $NL^{-2} \to \infty$ (the two limits turn out to be closely connected). Instead of working with the particle number $N$ as a free parameter, it is more convenient to fix a chemical potential $\mu > 0$ and then choose the number of fermions by $N = N(\mu)$ with

$$N(\mu) = \left| \{ k \in \kappa \mathbb{Z}^2 : k^2 \leq \mu \} \right|, \quad \kappa = \frac{2\pi}{L}. \quad (1.7)$$

Since the number of fermions now coincides with the number of eigenvalues of $-\Delta$ that are less or equal than $\mu$, counting multiplicities, the parameter $\mu$ plays the role of the Fermi energy. We write the ground state energy of $H$ as a function of $\mu$ and $E_B$ as $E(\mu, E_B) = \min \sigma(H)$ and introduce the energy of $N(\mu)$ non-interacting fermions inside the box $\Omega$,

$$E_0(\mu) = \sum_{k^2 \leq \mu} k^2. \quad (1.8)$$

Our main result, Theorem 1.2, shows that the energy difference $E(\mu, E_B) - E_0(\mu)$ is given at leading order by the polaron energy $e_p(\mu, E_B)$, that is

$$\frac{E(\mu, E_B) - E_0(\mu)}{e_p(\mu, E_B)} = 1 + o(1) \quad \text{as} \quad \frac{\mu}{|E_B|} \to \infty. \quad (1.9)$$

The polaron energy $e_p(\mu, E_B) < 0$ is the lowest solution to the polaron equation

$$e_p(\mu, E_B) = -\frac{1}{L^2} \sum_{k^2 \leq \mu} \frac{1}{G(k, -k^2 - e_p(\mu, E_B))}. \quad (1.10)$$

\footnote{The energies $E(\mu, E_B)$ and $E_0(\mu)$ depend of course also on $L$ but we omit this in our notation.}
where $G(q, \tau)$ is defined for $\tau > -\mu$ and $q \in \kappa \mathbb{Z}^2$ by

$$G(q, \tau) = \frac{1}{L^2} \sum_k \left( \frac{1}{(1 + \frac{1}{M})k^2 - E_B} - \frac{\chi(\mu, \infty)(k^2)}{\frac{1}{M}(q - k)^2 + k^2 + \tau} \right).$$

(1.11)

Here $\chi(\mu, \infty)(s)$ denotes the characteristic function $\chi(\mu, \infty)(s) = 1$ for $s > \mu$ and $\chi(\mu, \infty)(s) = 0$ otherwise. That (1.10) admits a lowest negative solution was shown in [GL19, Proposition 7.1]. Let us mention that our main result (1.9) holds in particular in the thermodynamic limit, i.e. after taking the limit $L \to \infty$.

The polaron equation (1.10) was proposed in [Che06] based on a formal variational calculation with trial states $w_P \in L^2(\Omega) \otimes \mathcal{H}_N(\mu)$ of the form

$$w_P = \alpha_0 \varphi_0 \otimes |FS_\mu\rangle + \sum_{k^2 \leq \mu} \sum_{l^2 > \mu} \alpha_{k,l} \varphi_{k-l} \otimes a_l^* a_k |FS_\mu\rangle$$

(1.12)

where $\alpha_0, \alpha_{k,l} \in \mathbb{C}$, $\varphi_k(y) = L^{-1} e^{iky}$ and

$$|FS_\mu\rangle = \prod_{k^2 \leq \mu} a_k^* |0\rangle$$

(1.13)

denotes the ground state of the kinetic operator $\sum_k k^2 a_k^* a_k \upharpoonright \mathcal{H}_N(\mu)$ (called the Fermi sea). A rigorous proof of the upper bound $E(\mu, E_B) \leq E_0(\mu) + e_P(\mu, E_B)$ was given in [GL19] utilizing a generalized Birman–Schwinger principle for the Hamiltonian $H$ (see Section 2).

In the physics literature the polaron energy is considered to be a good approximation in the weak coupling limit $E_B \nearrow 0$ as well as in the large density limit $\mu \to \infty$ [Che06, CG08, Par11]. In the regime of strong coupling $E_B \to -\infty$, it is expected that the ground state undergoes a transition to states in which the impurity is tightly bound by a single fermion. This behavior is represented by the so-called molecule or dimer ansatz [CM09, PDZ09, Par11]. In contrast to the latter, the polaron state (1.12) is interpreted as an impurity that is surrounded by weak density fluctuations in the Fermi sea. The two classes of trial states were investigated extensively in the physics literature leading to indications for the anticipated difference between the shape of the ground state in the weak and strong coupling limits (see, e.g., the literature quoted in the previous section). For this reason the Fermi polaron is also discussed in the context of the BCS–BEC crossover. Most results in the physics literature, however, are based on variational estimates using suitable classes of trial states. We remark that this can only justify upper bounds for the ground state energy, whereas here we provide a corresponding lower bound.

The Fermi polaron has been studied also in three dimensions. The problem of defining a semi bounded self-adjoint Hamiltonian in this case was solved in [Min11, CDFMT12, MS17]. Contrary to the 2D model, it is known that the Hamiltonian is semi-bounded in three dimen-
sions only if \( M \geq M_* \) for some critical mass ratio \( M_* > 0 \). Rigorous results concerning the ground state energy mostly addressed the question of stability and the existence of a lower bound that is uniform in the particle number \( N \). In [MS17] it was shown that at zero density there is such a uniform lower bound under the condition that \( M > 0.36 \). In a more recent work, Moser and Seiringer generalized their findings to the positive density setup by proving that the energy shift caused by the impurity particle depends only on the average density and the interaction strength but not on the size of the system [MS19]. The question whether the polaron energy describes the correct asymptotic form of the ground state energy similar to (1.9) is still open for the three-dimensional model.

Quantum models with \( N + 1 \) particles interacting via two-body point interactions have been studied in the mathematical literature from various points of views. Besides the works already quoted, we refer to [DFT94, DR04, AGHH05, Min11, CDFMT12, MO18] and references therein.

1.2 Main result

We are now ready to state our main result which provides an asymptotic estimate for the ground state energy \( \min \sigma(H) \) of the 2D Fermi polaron.

**Theorem 1.2.** Set \( M > 1.225 \) and for \( L > 0 \) and \( \mu > 0 \), fix the number of particles \( N(\mu) \) by (1.7). Moreover, let the Hamiltonian \( H \) be the limit operator of \( (H_n)_{n \in \mathbb{N}} \) as stated in Proposition 1.1. Then the ground state energy \( E(\mu, E_B) = \min \sigma(H) \) and the lowest solution \( e_P(\mu, E_B) < 0 \) of the polaron equation (1.10) satisfy the following property. There exist constants \( c_0, C > 0 \) (possibly depending on \( M \)) such that

\[
\left| E(\mu, E_B) - E_0(\mu) - e_P(\mu, E_B) \right| \leq C \frac{|e_P(\mu, E_B)|}{\log(\mu/|E_B|)}
\]

for all \( L > 0, \mu > 0 \) and \( E_B < 0 \) with \( L^2|E_B| \geq 1 \) and \( \mu/|E_B| \geq c_0 \).

**Remarks.**

1.3. In Lemma 3.3 we show that \( e_P(\mu, E_B) = O(\mu/\log(\mu/|E_B|)) \) as \( \mu/|E_B| \to \infty \).

1.4. The condition \( L^2|E_B| \geq 1 \) characterizes the range of parameters in which the two-body binding energy \( E_B \) is at least of the order of the minimal kinetic excitation energy which equals \((2\pi/L)^2\). In this sense our analysis is beyond the perturbative regime.

1.5. Since the constant on the right side of (1.14) does not depend on \( L > 0 \), we can directly infer a statement about the ground state energy in the thermodynamic limit,

\[
\limsup_{L \to \infty} \left| \frac{E(\mu, E_B) - E_0(\mu)}{e_P(\mu, E_B)} - 1 \right| \leq \frac{C}{\log(\mu/|E_B|)}
\]

(1.15)
for all $\mu/|E_B| \geq c_0$.

1.6. The condition $M > 1.225$ is related to the problem of stability (of second kind), that is, to find a uniform lower bound for the ground state energy in the thermodynamic limit $L \to \infty$. While it is known that $H$ is bounded from below for all $M > 0$ [DFT94, GL19], it is unclear whether a uniform bound exists when $M \leq 1.225$. This is an unsolved problem also in the case of zero density, see [GL18].

1.7. The upper bound in (1.14) was proven in [GL19]. For the convenience of the reader, we give a brief sketch of the argument in Section 2.2. The novel contribution of the present work is the derivation of the lower bound.

1.8. A similar result was obtained in [LM19] for the case of an infinitely heavy impurity, formally corresponding to $M = \infty$. In this case the $N$ fermions interact with an external delta potential which simplifies the analysis significantly.

The rest of the article is organized as follows. In the next section we introduce the Birman–Schwinger operator $\phi(\lambda)$ associated to the Hamiltonian $H$ and state the corresponding Birman–Schwinger principle. Upper and lower bounds for the ground state energy follow from suitable bounds for $\phi(\lambda)$. In Section 2.2 we recall how to obtain the upper bound in (1.14). Sections 3–6 are about the matching lower bound. They account for the main part of this work. In Section 3 we derive a localization of the polaron energy inside a suitable subspace of the Hilbert space. In the two subsequent sections we provide lower bounds for the Birman–Schwinger operator on the localization subspace and its orthogonal complement. On the localization subspace, we obtain a perturbed polaron equation whose solution we compare to the polaron energy, see Section 4. In Section 5 we analyze the Birman–Schwinger operator on the orthogonal complement of the localization subspace. The lower bound on this subspace can be understood as a proof of stability of the Fermi polaron at positive density which generalizes analogous findings for the zero density model [GL19]. In Section 5.1 we combine the obtained results to conclude the proof of Theorem 1.2. The last section contains the proof of a technical lemma that is used several times throughout the article.

2 Preliminaries and upper bound

In this section we discuss the Birman–Schwinger principle for the Hamiltonian $H$ which provides a suitable tool for the analysis of upper and lower bounds for $E(\mu, E_B) = \min \sigma(H)$.

2.1 The Birman–Schwinger operator $\phi(\lambda)$

Our starting point for the proof of Theorem 1.2 is a Birman–Schwinger type principle for the operator $H$. This is the second result from [GL19] which is important for our analysis. For
the precise statement, let us introduce the resolvent set \( \rho(H_0) \subset \mathbb{C} \) of the non-interacting Hamiltonian

\[
H_0 = \left( -\frac{1}{M} \Delta_y + T \right) \upharpoonright L^2(\Omega) \otimes \mathcal{H}_{N(\mu)},
\]

with \( T = \sum_k k^2 a_k^* a_k \) the kinetic energy operator on the fermionic Fock space.

**Proposition 2.1.** (See [GL19, Sections 5 and 6]) There exists a family of operators \( \phi(\lambda) \), \( \lambda \in \rho(H_0) \), acting on \( L^2(\Omega) \otimes \mathcal{H}_{N(\mu)} \) with \( \lambda \)-independent domain \( D \), such that for all real-valued \( \lambda \), \( \phi(\lambda) \) is essentially self-adjoint and its closure (denoted again by \( \phi(\lambda) \)) satisfies

\[
\inf \sigma(\phi(\lambda)) \leq 0 \quad \Leftrightarrow \quad E(\mu, E_B) \leq \lambda,
\]

with equality on one side implying equality on both sides. Moreover the \( \phi(\lambda) \) form an analytic family of type (A) and for \( \lambda \in \mathbb{R} \cup (\mathbb{C} \setminus \mathbb{R}) \subset \rho(H_0) \) they are given explicitly by

\[
\phi(\lambda) = F(i\nabla_y, T - \lambda) + \frac{1}{L^2} \sum_{k,l} a_k^* e^{iky} \frac{1}{\frac{1}{M} \Delta_y + T + k^2 + l^2 - \lambda} e^{-ily} a_l
\]

where

\[
F(q, \tau) = \frac{1}{L^2} \sum_k \left( \frac{1}{mk^2 - E_B} - \frac{1}{\frac{1}{M} q^2 + k^2 + \tau} \right), \quad m = \frac{M + 1}{M}.
\]

**Remarks.**

2.1 Note that while \( H \) is defined on the Hilbert space \( L^2(\Omega) \otimes \mathcal{H}_{N(\mu)} \), the Birman–Schwinger operator \( \phi(\lambda) \) acts on \( L^2(\Omega) \otimes \mathcal{H}_{N(\mu)} \). We also remark that the domain \( D \) is given by the set of all finite linear combinations of states of the form \( \varphi_q \otimes \varphi_{k_1} \wedge \ldots \wedge \varphi_{k_{N(\mu)} - 1} \) with \( q, k_1, \ldots, k_{N(\mu)} - 1 \in \kappa \mathbb{Z}^2 \) and \( \varphi_k \) the normalized plane waves in \( L^2(\Omega) \).

2.2. The operator defined in (2.3) coincides with the Birman–Schwinger operator \( \phi(z) \) from [GL19, Lemma 6.3] up to a multiplicative factor \( L^{-2} \). Apart from renaming \( z \) into \( \lambda \), we write the impurity degree of freedom in first quantization whereas in [GL19], all degrees of freedom are expressed in second quantization. Proposition 2.1 is a direct consequence of the statements from [GL19, Section 5 and Lemma 6.3].

For explicit computations it is useful to invert the normal order of creation and annihilation operators in (2.3) when \( k^2, l^2 \leq \mu \). With \( G(q, \tau) \) defined in (1.11), this leads for \( \lambda \in \mathbb{R} \cup (\mathbb{C} \setminus \mathbb{R}) \) to

\[
\phi(\lambda) = G(i\nabla_y, T - \lambda) - \frac{1}{L^2} \sum_{k,l^2 \leq \mu} a_k^* e^{iky} \frac{1}{\frac{1}{M} \Delta_y + T - \lambda} e^{-ily} a_l^*
\]
understood as an operator on $L^2(\Omega) \otimes \mathcal{H}_{N(\mu)-1}$. Through analytic continuation the above identity extends to $\lambda < E_0(\mu)$. This explicit expression of $\phi(\lambda)$ will be the main object to be analyzed.

To arrive at (2.5) we made use of the CAR and the pull-through formula, which for suitable functions $f : \kappa \mathbb{Z}^2 \times \mathbb{R} \to \mathbb{C}$ reads

$$a_k f(P, T) = f(P + k, T + k^2) a_k, \quad a_k^* f(P, T) = f(P - k, T - k^2) a_k^*.$$  \hspace{1cm} (2.6)

Here $P = \sum_k k a_k^* a_k$ denotes the momentum operator of the fermions.

### 2.2 Upper bound

We show how to use Proposition 2.1 to obtain an upper bound for $E(\mu, E_B)$. This resembles the analysis performed in the first part of Section 7 [GL19]. For an upper bound, it is sufficient to find a trial state $w$ and a suitable $\lambda$ that satisfy $\langle w, \phi(\lambda) w \rangle \leq 0$. As such we choose $\lambda = E_0(\mu) + e_P(\mu, E_B)$ and the wave function

$$w = \sum_{k^2 \leq \mu} \frac{1}{G(k, -k^2 - e_P(\mu, E_B))} \varphi_k \otimes a_k |FS_{\mu}\rangle.$$  \hspace{1cm} (2.7)

With the aid of (2.5), a straightforward computation leads to

$$\langle w, \phi(E_0(\mu) + e_P(\mu, E_B)) w \rangle = \sum_{k^2 \leq \mu} \frac{1}{G(k, -k^2 - e_P(\mu, E_B))} \left[ 1 + \frac{1}{L^2} \sum_{k^2 \leq \mu} \frac{1}{G(k, -k^2 - e_P(\mu, E_B))} \cdot \frac{1}{e_P(\mu, E_B)} \right],$$  \hspace{1cm} (2.8)

which is identically zero because of (1.10). By Proposition 2.1 this implies the upper bound

$$E(\mu, E_B) \leq E_0(\mu) + e_P(\mu, E_B).$$  \hspace{1cm} (2.9)

### 2.3 Momentum decomposition of $\phi(\lambda)$

For the analysis of the lower bound it is convenient to make use of the translational invariance of the model, in particular, that $\phi(\lambda)$ commutes with the total momentum operator $P_{tot} = \sum_k k a_k^* a_k$. 

\[ -\left( \frac{1}{L^2} \sum_{k^2 \leq \mu, l^2 > \mu} e^{iky} a_k a_l^* - \frac{1}{\lambda^2} \Delta_y + T + l^2 - \lambda e^{-ily} + \text{h.c.} \right) \]

\[ + \frac{1}{L^2} \sum_{k^2, l^2 > \mu} a_k^* e^{ily} \frac{1}{\lambda^2} \Delta_y + T + k^2 + l^2 - \lambda e^{-ily} a_k \]
\[ -i \nabla_y + P_t \text{ with } P_t = \sum_k k a_k^* a_k \uparrow \mathcal{H}_{N(\mu) - 1}. \] This guarantees a total momentum decomposition of \( \phi(\lambda) \), meaning that there is a unitary map

\[ V : L^2(\Omega) \otimes \mathcal{H}_{N(\mu) - 1} \to \bigoplus_{p \in \kappa \mathbb{Z}^2} \mathcal{H}_{N(\mu) - 1} \] (2.10)

that diagonalizes \( P_{\text{tot}} \) by eliminating the \( y \) coordinate in favor of the total momentum \( p \in \kappa \mathbb{Z}^2 \). This unitary is given by \( (Vw)_p = \langle \phi_p | \otimes 1 \rangle \mathcal{H}_{N(\mu) - 1} \) where \( \langle \phi_p | \otimes 1 \rangle \mathcal{H}_{N(\mu) - 1} \) shall indicate to take the scalar product in the coordinate \( y \) with the plane wave \( \phi_p \in L^2(\Omega) \). To see that the parameter \( p \) describes the total momentum, use \( (VP_{\text{tot}}w)_p = p(Vw)_p \) to verify

\[ \langle w, P_{\text{tot}}w \rangle = \sum_{p \in \kappa \mathbb{Z}^2} p \langle (Vw)_p, (Vw)_p \rangle. \] (2.11)

The map \( V \) is called Lee–Low–Pines transformation [LLP53] and its inverse is given by \( V^*(w_p) = e^{-iP_y} (\phi_p \otimes w_p) \).

From this definition it is not difficult to check that \( \phi(\lambda) \) in (2.5) transforms into \( V\phi(\lambda)V^* = \sum_{p \in \kappa \mathbb{Z}^2} \phi_p(\lambda) \) where

\[ \phi_p(\lambda) = G(p - P_t, T - \lambda) - H_p(\lambda) - X_p(\lambda) + P_p(\lambda) \] (2.12)

is defined as an operator on \( \mathcal{H}_{N(\mu) - 1} \) with

\[ H_p(\lambda) = a(\eta) \frac{1}{M(p - P)^2 + T - \lambda} a^*(\eta), \] (2.13)

\[ X_p(\lambda) = a(\eta) A^*_p(\lambda) + A_p(\lambda) a^*(\eta), \] (2.14)

\[ P_p(\lambda) = \frac{1}{L^2} \sum_{k^2, l^2 > \mu} a_k^* \frac{1}{M(p - P - k - l)^2 + T + k^2 + l^2 - \lambda} a_k, \] (2.15)

and

\[ a(\eta) = \frac{1}{L} \sum_{k^2 \leq \mu} a_k, \quad A_p(\lambda) = \frac{1}{L} \sum_{k^2 > \mu} \frac{1}{M(p - P - k)^2 + T + k^2 - \lambda} a_k. \] (2.16)

Note that the first summand in \( \phi_p(\lambda) \) defines an unbounded operator whereas the three other terms can be shown to be bounded operators. The domain of essential self-adjointness of \( \phi_p(\lambda) \) is the dense subspace consisting of all finite linear combinations of states of the form \( \varphi_{k_1} \wedge \ldots \wedge \varphi_{k_{N(\mu) - 1}} \) with \( k_1, \ldots, k_{N(\mu) - 1} \in \kappa \mathbb{Z}^2 \).
3 Localization of the polaron energy

By proposition 2.1 the lower bound \( E(\mu, E_B) \geq \lambda \) is equivalent to \( \phi(\lambda) \geq 0 \). The next four sections are therefore devoted to the analysis of the condition \( \phi_p(\lambda) \geq 0 \) for the operator (2.12) with \( p \in \kappa \mathbb{Z}^2 \). In view of the upper bound (2.2) it is sufficient to consider \( \lambda \leq E_0(\mu) + e_p(\mu, E_B) \) from now on.

To prepare our first main statement we need to introduce a suitable orthogonal projector in the Hilbert space \( \mathcal{H}_{N(\mu) - 1} \). For its definition let us give names to the subsets of the momentum lattice \( \kappa \mathbb{Z}^2 \) that correspond to hole and particle momenta w.r.t. the Fermi sea,

\[
\Lambda_h = \{ k \in \kappa \mathbb{Z}^2 : k^2 \leq \mu \}, \quad \Lambda_p = \{ k \in \kappa \mathbb{Z}^2 : k^2 > \mu \}.
\]

Moreover for \( \varepsilon > 0 \) we set

\[
\Lambda_{p,\varepsilon}^{\leq} = \left\{ k \in \Lambda_p : \mu < k^2 \leq \left( 1 + \frac{1}{\varepsilon \log \tilde{\mu}} \right) \mu \right\}
\]

(3.2)

and define the orthogonal projectors \( \Pi_\varepsilon \) and \( \Pi_\varepsilon^\perp = I - \Pi_\varepsilon \) through

\[
\text{Ran}(\Pi_\varepsilon) = \overline{\text{lin}} \{ a_{l_1}^* \ldots a_{l_{m-1}}^* a_{k_1} \ldots a_{k_m} | \text{FS}_\mu : m \geq 1, k_1, \ldots, k_m \in \Lambda_h, l_1, \ldots, l_{m-1} \in \Lambda_{p,\varepsilon}^{\leq} \}.
\]

(3.3)

Here \( \overline{\text{lin}} V \) stands for the closure in \( \mathcal{H}_{N(\mu) - 1} \) of the linear hull of the subset \( V \subseteq \mathcal{H}_{N(\mu) - 1} \). For a better understanding of \( \text{Ran}(\Pi_\varepsilon) \) and its orthogonal complement, let us recall that the set of all anti-symmetric products of \( N(\mu) - 1 \) plane waves,

\[
D = \{ a_{l_1}^* \ldots a_{l_{m-1}}^* a_{k_1} \ldots a_{k_m} | \text{FS}_\mu : m \geq 1, k_1, \ldots, k_m \in \Lambda_h, l_1, \ldots, l_{m-1} \in \Lambda_p \},
\]

(3.4)

is a total set of the Hilbert space \( \mathcal{H}_{N(\mu) - 1} \), i.e. \( \overline{\text{lin}} D = \mathcal{H}_{N(\mu) - 1} \). A comparison with (3.3) shows that \( \Pi_\varepsilon \) projects on all states in \( \mathcal{H}_{N(\mu) - 1} \) that have particle modes occupied solely in the momentum lattice region \( \Lambda_{p,\varepsilon}^{\leq} \) (this includes all states with zero particle modes occupied), whereas the range of \( \Pi_\varepsilon^\perp \) consists of states that have at least one mode occupied in \( \Lambda_{p,\varepsilon}^{>\perp} = \Lambda_p \setminus \Lambda_{p,\varepsilon}^{\leq} \).

In the next proposition we provide a lower bound for \( \phi_p(\lambda) \) in terms of two operators that act only on \( \text{Ran}(\Pi_\varepsilon) \) and \( \text{Ran}(\Pi_\varepsilon^\perp) \), respectively. The physical meaning of the two subspaces is the following: On \( \text{Ran}(\Pi_\varepsilon^\perp) \) it is not clear how to obtain a suitable \( L \)-independent bound for the operator \( P_p(\lambda) \) which is one of the main obstacles in the analysis. On this subspace we estimate the negative part of \( P_p(\lambda) \) in terms of \( G(p - P_f, T - \lambda) \). This is closely connected to the problem of obtaining a lower bound of \( H \) uniformly in the system size \( L \to \infty \). Such a bound, though necessary for the proof of Theorem 1.2, is however not much related to the asymptotic form of \( E(\mu, E_B) - E_0(\mu) \). The latter will be determined on \( \text{Ran}(\Pi_\varepsilon) \) on which the
Lemma 3.2. The corresponding integral which can be evaluated explicitly.

$q \sim \tau$ as $\mu/|E_B|$ is easily estimated with a suitable uniform bound. Hence on this subspace all of $G(p - P_I, T - \lambda)$ is available (and needed) for the analysis of the correct energy asymptotics. This explains the motivation behind the following decomposition of $\phi_p(\lambda)$. Since the operator $G(p - P_I, T - \lambda)$ is needed on both subspaces separately, it is an important step in our argument.

**Proposition 3.1.** There are constants $c_0, \varepsilon_0 > 0$ such that for all $p \in \kappa \mathbb{Z}^2$, $L^2|E_B| \geq 1$, $\mu/|E_B| \geq c_0$ and $\varepsilon \in (0, \varepsilon_0)$, it holds that $\phi_p(\lambda) \geq \Phi_p(\lambda, \varepsilon) + \Psi_p(\lambda, \varepsilon)$, with

\[
\Phi_p(\lambda, \varepsilon) = \Pi_\varepsilon \left(G(p - P_I, T - \lambda) - H_p(\lambda) - \varepsilon^{-1}\right) \Pi_\varepsilon, \tag{3.5}
\]

\[
\Psi_p(\lambda, \varepsilon) = \Pi_\varepsilon \left(G(p - P_I, T - \lambda) + P_p(\lambda) - K(\varepsilon, \tilde{\mu})\right) \Pi_\varepsilon, \tag{3.6}
\]

and $K(\varepsilon, \tilde{\mu}) = \varepsilon^{-1/2}(\varepsilon^{-1/2} + \sqrt{\log \tilde{\mu}} + \varepsilon \log \tilde{\mu})$. We use the notation $\tilde{\mu} = \mu/|E_B|$.

**Remark 3.1.** From the discussion above it is clear that the $\varepsilon$- and $\tilde{\mu}$-dependent errors in (3.5) and (3.6) have physically different meanings. Eventually only the error in (3.5) enters the constant on the right side of (1.14). For that reason, we do not optimize the error terms as $\tilde{\mu} \to \infty$, and always consider $\varepsilon$ sufficiently small but fixed w.r.t. $\tilde{\mu}$ and $L > 0$.

Before we come to the proof of the proposition, we state two helpful results about the asymptotics of $G(q, \tau)$ and $e_p(\mu, E_B)$.

### 3.1 Asymptotics of $G(q, \tau)$ and $e_p(\mu, E_B)$

As the following bound will be used several times, we note that it follows easily with the aid of Lemma A.1,

\[
\left| \frac{1}{L^2} \sum_{a_\mu \leq k^2 < b_\mu} 1 - \frac{(b - a)\mu}{4\pi} \right| \leq \frac{2}{\pi L} \left( \sqrt{a\mu} + \sqrt{b\mu} \right) + \frac{6}{L^2} \tag{3.7}
\]

for any $b > a \geq 0$.

The first lemma of this section tells us the error for replacing the sum in $G(q, \tau)$ by the corresponding integral which can be evaluated explicitly.

**Lemma 3.2.** There are constants $c_0, C > 0$ such that

\[
\left| G(q, \tau) - \frac{1}{4\pi m} \log \left( \frac{q^2 + m\mu + \tau}{|E_B|} \right) \right| \leq C \left( 1 + \frac{\mu}{(\mu + \tau) \log(\mu/|E_B|)} \right)^3 \tag{3.8}
\]

for all $q \in \mathbb{R}^2$, $\tau > -\mu$, $L^2|E_B| \geq 1$ and $\mu/|E_B| \geq c_0$. Recall $m = 1 + 1/M$. 

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The proof of the lemma is postponed to Section 6. Utilizing this lemma, we can derive an asymptotic formula for $e_P(\mu, E_B)$ as $\mu/|E_B| \to \infty$. The precise statement is

**Lemma 3.3.** There are constants $c_0, C > 0$ such that the polaron energy satisfies

$$
|e_P(\mu, E_B) + (1 + \frac{1}{M}) \frac{\mu}{\log(\mu/|E_B|)}| \leq C \frac{\mu}{(\log(\mu/|E_B|))^2}
$$

(3.9)

for all $L^2|E_B| \geq 1$ and $\mu/|E_B| \geq c_0$.

**Proof.** Let us set $z_P = |e_P(\mu, E_B)|$ and $\tilde{\mu} = \mu/|E_B|$. To prove suitable upper and lower bounds for $z_P$ we first show $z_P \leq \mu$ for all $\mu/|E_B| \geq c_0$ given that the constant $c_0$ is chosen large enough.

Consider the set of parameters for which $z_P$ exceeds the value $\mu$,

$$
M_{c_0} = \{(L, \mu, E_B) : z_P > \mu \geq c_0|E_B| \text{ and } L^2|E_B| \geq 1\} \subseteq \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}_-.
$$

(3.10)

By monotonicity of $G(q, \tau)$ in the $\tau$ variable, we have $G(k, -k^2 + z_p) \geq G(k, 0)$ for all $k^2 \leq \mu$ and $(L, \mu, E_B) \in M_{c_0}$. By Lemma 3.2 this implies

$$
G(k, 0) \geq \frac{1}{4\pi m} \log(mc_0) - C \left(1 + \frac{1}{\log c_0}\right)^3 \geq \frac{1}{8\pi m} \log(mc_0).
$$

(3.11)

Inserting this into the polaron equation (1.10) and employing (3.7) leads to

$$
z_P \leq \frac{2m\mu}{\log(mc_0)} \left(1 + \frac{8}{\sqrt{c_0}} + \frac{24\pi}{c_0}\right),
$$

(3.12)

which implies $z_P \leq \mu$ for $c_0$ large enough. Hence $M_{c_0}$ is empty and we can assume $z_P \leq \mu$.

Utilizing again monotonicity of $G(q, \tau)$, we get $G(k, -k^2 + z_P) \leq G(k, \mu)$. By (3.8) we have for all $k^2 \leq \mu$,

$$
G(k, \mu) \leq \frac{1}{4\pi m} \log \left(\frac{\mu + m\mu + \mu}{|E_B|}\right) + C_1 \leq \frac{1}{4\pi m} \log \tilde{\mu} + C_2
$$

(3.13)

for two constants $C_1, C_2 > 0$. Using (1.10) together with (3.7), we obtain the lower bound

$$
z_P \geq \frac{\mu}{m^{-1} \log \tilde{\mu} + C_2} \left(1 - \frac{C_3}{\sqrt{\mu}}\right) \geq \frac{m\mu}{\log \tilde{\mu}} - C/(\log \tilde{\mu})^2.
$$

(3.14)
With the lower bound (3.14) we can estimate for $0 \leq k^2 \leq \mu$,

$$G(k, -k^2 + z_p) \geq \frac{1}{4\pi m} \log(m\tilde{\mu} - \tilde{\mu}) - C_1 \left(1 + \frac{\mu}{z_p \log \tilde{\mu}}\right)^3 \geq \frac{1}{4\pi m} \log \tilde{\mu} - C_2.$$ (3.15)

Similarly as above, using the polaron equation and (3.7), one finds

$$z_p \leq \frac{m\mu}{\log \tilde{\mu}} \left(1 - C_2 / \log \tilde{\mu}\right) \left(1 + \frac{C_3}{\sqrt{\mu}}\right),$$ (3.16)

which gives the desired upper bound.

**Remark 3.2.** For $\lambda \leq E_0(\mu) + e_p(\mu, E_B)$ it follows from $T \upharpoonright \mathcal{H}_{N(\mu)-1} \geq E_0(\mu) - \mu$ that there are constants $c_0, C > 0$ such that

$$\pm \left(G(p - P_t, T - \lambda) - \frac{1}{4\pi m} \log \left(\frac{(p - P_t)^2 + m\mu + T - \lambda}{|E_B|}\right)\right) \leq C$$ (3.17)

as operator inequalities on $\mathcal{H}_{N(\mu)-1}$ for all $L^2|E_B| \geq 1$ and $\mu/|E_B| \geq c_0$. A useful implication of this bound is

$$G(p - P_t, T - \lambda) \upharpoonright \mathcal{H}_{N(\mu)-1} \geq \frac{1}{4\pi m} \log(\mu/|E_B|) - C.$$ (3.18)

### 3.2 Proof of Proposition 3.1

Since $G(p - P_t, T - \lambda)$ and $H_p(\lambda)$ both commute with the projector $\Pi_\varepsilon$, we have

$$\phi_p(\lambda) = \Pi_\varepsilon \phi_p(\lambda) \Pi_\varepsilon + \Pi_\varepsilon^\perp \phi_p(\lambda) \Pi_\varepsilon^\perp + \left(\Pi_\varepsilon \left( - X_p(\lambda) + P_p(\lambda)\right) \Pi_\varepsilon + \text{h.c.} \right).$$ (3.19)

The statement of Proposition 3.1 is a consequence of the following estimates. Note that for notational convenience we estimate the constant $C$ from above by $\varepsilon^{-1/2}$.

**Lemma 3.4.** There are constants $c_0, \varepsilon_0, C > 0$ such that for all $p \in \kappa\mathbb{Z}^2$, $L^2|E_B| \geq 1$, $\tilde{\mu} = \mu/|E_B| \geq c_0$ and $\varepsilon \in (0, \varepsilon_0)$,

$$\Pi_\varepsilon P_p(\lambda) \Pi_\varepsilon \geq - \frac{C}{\sqrt{\varepsilon}} \Pi_\varepsilon,$$ (3.20)

$$\Pi_\varepsilon^\perp H_p(\lambda) \Pi_\varepsilon^\perp \leq C\varepsilon \log \tilde{\mu} \Pi_\varepsilon^\perp,$$ (3.21)

$$\Pi_\varepsilon^\perp X_p(\lambda) \Pi_\varepsilon^\perp \leq C\sqrt{\log \tilde{\mu}} \Pi_\varepsilon^\perp.$$ (3.22)
and

$$\Pi_\varepsilon X_\beta(\lambda) \Pi_\varepsilon^\perp + \text{h.c.} \leq C\varepsilon^{1/2} \log \tilde{\mu} \Pi_\varepsilon^\perp + C\varepsilon^{-1/2} \Pi_\varepsilon,$$

(3.23)

$$\Pi_\varepsilon P_\beta(\lambda) \Pi_\varepsilon^\perp + \text{h.c.} \geq -\frac{C}{\sqrt{\varepsilon}}.$$

(3.24)

Proof. Line (3.20). Using \(a_k \Pi_\varepsilon = 0\) for all \(k \in \Lambda_p \setminus \Lambda_{p,\varepsilon}^\leq\) together with the pull-through formula (2.6), we obtain

$$\Pi_\varepsilon P_\beta(\lambda) \Pi_\varepsilon = \Pi_\varepsilon \left( \frac{1}{L^2} \sum_{k,l \in \Lambda_{p,e}^\leq} a_k^* a_k \frac{1}{M(p-P_l - l)^2 + T + l^2 - \lambda} \right) \Pi_\varepsilon.$$  

(3.25)

By means of the commutation relations \(a_k a_l^* + a_l^* a_k = \delta_{kl}\) and

$$\frac{1}{L^2} \sum_{l \in \Lambda_{p,\varepsilon}^\leq} \frac{1}{M(p-P_l - l)^2 + T + l^2 - \lambda} \geq 0$$

(3.26)

on \(\mathcal{H}_{N(\mu)}\), we further get

$$\Pi_\varepsilon \left( \frac{1}{L^2} \sum_{k,l \in \Lambda_{p,e}^\leq} a_k^* a_k \frac{1}{M(p-P_l - l)^2 + T + l^2 - \lambda} a_l^* \right) \Pi_\varepsilon \geq 0$$

(3.27)

For \(f(\mu, E_B) > 0\) we proceed by estimating the right side from below in terms of the operator

$$-\frac{f(\mu, E_B)}{2} \left( \frac{1}{L^2} \sum_{k,l \in \Lambda_{p,e}^\leq} a_k a_l^* \right) - \frac{1}{2f(\mu, E_B)} \left( \frac{1}{L^2} \sum_{k,l \in \Lambda_{p,e}^\leq} a_k \frac{1}{M(p-P_l)^2 + T - \lambda} a_l^* \right)$$

(3.28)

acting on \(\Pi_\varepsilon \mathcal{H}_{N(\mu)}\). In the first summand, we use the CAR together with (3.7) for \(a = \mu, b = \mu(1 + \frac{1}{\varepsilon \log \tilde{\mu}})\), and further employ \(L^2|E_B| \geq 1\) and \(\tilde{\mu} \geq c_0\). This gives

$$\frac{1}{L^2} \sum_{k,l \in \Lambda_{p,e}^\leq} a_k a_l^* \leq \frac{1}{L^2} \sum_{k \in \Lambda_{p,e}^\leq} 1 \leq \frac{C\mu}{\varepsilon \log \tilde{\mu}}.$$  

(3.29)

In the second summand in (3.28), we use \(T - \lambda > 0\) on \(\mathcal{H}_{N(\mu)}\) in order to neglect the positive operator \(\frac{1}{M}(p-P_l)^2\) in the denominator. Then we use again the pull-through formula and the CAR to obtain

$$\frac{1}{L^2} \sum_{k,l \in \Lambda_{p,e}^\leq} a_k \frac{1}{M(p-P_l)^2 + T - \lambda} a_l^* \leq \frac{1}{L^2} \sum_{k,l \in \Lambda_{p,e}^\leq} a_k \frac{1}{(T - \lambda)^2} a_l^*$$

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\[ \leq \frac{1}{L^2} \sum_{k \in \Lambda_{\tilde{w},\varepsilon}} \frac{1}{(T + k^2 - \lambda)^2}. \] (3.30)

Note that in the last step, we applied the inequality
\[ \frac{1}{L^2} \sum_{k,l \in \Lambda_{\tilde{w},\varepsilon}} a_l^* \frac{1}{(T + k^2 + l^2 - \lambda)} a_k \geq 0, \] (3.31)
which is verified by writing
\[ \frac{1}{(T + k^2 + l^2 - \lambda)^2} = \int_0^\infty \exp \left( -t(T - \lambda + k^2 + l^2) \right) \, dt \] (3.32)
and estimating
\[ \exp \left( -t(T - \lambda + k^2 + l^2) \right) \geq \exp \left( -t4l^4 \right) \exp \left( -t2(T - \lambda) \right) \exp \left( -t4k^4 \right). \] (3.33)

With \( T \geq E_0(\mu) - \mu \) on \( \mathscr{H}_N(\mu)_1 \) and \( E_0(\mu) - \lambda \geq |e_P(\mu, E_B)| \) we next get
\[ (3.30) \leq \frac{1}{L^2} \sum_{k^2 > \mu} \frac{1}{(k^2 - \mu + |e_P(\mu, E_B)|)^2}. \] (3.34)

By Lemma A.1 and the estimate
\[ \int_0^\infty \frac{\, dt}{\sqrt{t} - \mu + |e_P(\mu, E_B)|} \leq \int_0^\infty \frac{\, dt}{\sqrt{(t - \sqrt{\mu})^2 + |e_P(\mu, E_B)|}^3} = \frac{\pi}{4|e_P(\mu, E_B)|^{3/2}}, \] (3.35)
one finds the upper bound
\[ (3.30) \leq \frac{1}{4\pi|e_P(\mu, E_B)|} + \frac{1}{L|e_P(\mu, E_B)|^{3/2}} + \left( \frac{4\sqrt{\mu}}{L} + \frac{6}{L^2} \right) \frac{1}{|e_P(\mu, E_B)|^2} \leq C \log \tilde{\mu}. \] (3.36)

Hence,
\[ \Pi_\varepsilon \tilde{P}(\lambda) \Pi_\varepsilon \geq -C \left( \frac{f(\mu, E_B)\mu}{\varepsilon \log \mu} + \frac{\log \tilde{\mu}}{f(\mu, E_B)\mu} \right) \Pi_\varepsilon \geq -\frac{C}{\sqrt{\varepsilon}} \Pi_\varepsilon, \] (3.37)
if we choose \( f(\mu, E_B) = (\sqrt{\varepsilon} \log \tilde{\mu})/\mu \).

**Line (3.21).** Since states of the form \( \tilde{w} = a^*(\eta)w \in \mathscr{H}_N(\mu) \) with \( w \in \text{Ran}(\Pi_\varepsilon^+) \) have at least
one momentum mode occupied in $\Lambda_{p,\varepsilon}^{\geq}$, it follows that

$$\Pi_{\varepsilon}^\perp H_p(\lambda) \Pi_{\varepsilon}^\perp \leq \Pi_{\varepsilon}^\perp \left( \frac{a(\eta)a^*(\eta)}{|e_p(\mu, E_B)| + \mu/\log \tilde{\mu}} \right) \Pi_{\varepsilon}^\perp.$$  \hspace{1cm} (3.38)

The remaining expression is estimated using

$$a(\eta)a^*(\eta) \leq \frac{1}{L^2} \sum_{k^2 \leq \mu} 1 \leq C\mu$$  \hspace{1cm} (3.39)

which follows from the CAR in combination with (3.7).

**Lines (3.22) and (3.23).** It is straightforward to verify that for any two orthogonal projectors $Q, \tilde{Q}$ acting on $\mathcal{H}_{N(\mu)-1}$ and for any $f(\mu, E_B), g(\mu, E_B) > 0$,

$$Q X_p(\lambda) \tilde{Q} + \text{h.c.} \leq Q \left( f(\mu, E_B) A_p(\lambda) A_p^*(\lambda) + \frac{a(\eta)a^*(\eta)}{g(\mu, E_B)} \right) Q + \tilde{Q} \left( g(\mu, E_B) A_p(\lambda) A_p^*(\lambda) + \frac{a(\eta)a^*(\eta)}{f(\mu, E_B)} \right) \tilde{Q}.$$  \hspace{1cm} (3.40)

Similar as in the analysis of (3.25), one further shows

$$A_p(\lambda) A_p^*(\lambda) \upharpoonright \mathcal{H}_{N(\mu)-1} \leq C\frac{\log \tilde{\mu}}{\mu}.$$  \hspace{1cm} (3.41)

Together with (3.39) and (3.40) this leads to

$$Q X_p(\lambda) \tilde{Q} + \text{h.c.} \leq C \left( f(\mu, E_B) \log \tilde{\mu} + \frac{\mu}{g(\mu, E_B)} \right) Q + C \left( g(\mu, E_B) \log \tilde{\mu} + \frac{\mu}{f(\mu, E_B)} \right) \tilde{Q}.$$  \hspace{1cm} (3.42)

For $f(\mu, E_B) = g(\mu, E_B) = \mu/\sqrt{\log \tilde{\mu}}$, this shows (3.22), whereas the inequality in (3.23) follows from $f(\mu, E_B) = \sqrt{\varepsilon} \mu$ and $g(\mu, E_B) = \mu/\sqrt{\varepsilon \log \tilde{\mu}}$. (In the latter case we set $Q = \Pi_{\varepsilon}^\perp$ and $\tilde{Q} = \Pi_{\varepsilon}$.)

**Line (3.24).** Using $a_l \Pi_{\varepsilon} = 0$ for $l \in \Lambda_p \setminus \Lambda_{p,\varepsilon}^{\leq}$ and the pull-through formula together with the CAR, we find

$$\Pi_{\varepsilon} P_p(\lambda) \Pi_{\varepsilon}^\perp = \Pi_{\varepsilon} \left( -\frac{1}{L^2} \sum_{l \in \Lambda_{p,\varepsilon}} a_k \frac{1}{M(p - P_l)^2 + T - \lambda} a_l^* \right) \Pi_{\varepsilon}^\perp,$$  \hspace{1cm} (3.43)
where we made use of
\[
\Pi_\varepsilon \frac{1}{M} (p - P_l - l)^2 + T + l^2 - \lambda \Pi_\varepsilon \Pi_\varepsilon^\perp = 1.
\]

From here the proof works the same way as for (3.28) (with the difference that we end up with an identity on the right side). We obtain
\[
\Pi_\varepsilon P_p(\lambda) \Pi_\varepsilon^\perp + \text{h.c.} \geq -\frac{C}{\sqrt{\varepsilon}},
\]
which completes the proof of the lemma and thus also the proof of Lemma 3.1.

4 Analysis of $\Phi_p(\lambda, \varepsilon)$: perturbed polaron equation

In this section we show that the condition $\Phi_p(\lambda, \varepsilon) \geq 0$ leads to a perturbed polaron equation for $\lambda$ and then provide a suitable estimate for the solution of this equation.

In order to obtain the presumably optimal asymptotics of the error in (1.14), we introduce another orthogonal projector $\Pi_{\varepsilon,1}$ with $\text{Ran}(\Pi_{\varepsilon,1}) \subseteq \text{Ran}(\Pi_\varepsilon)$ defined as the closed subspace of all states containing exactly one unoccupied momentum mode (a hole) in the lattice region $\Lambda_{h,\varepsilon} \subseteq \Lambda_h$.

More precisely we set for $n \geq 0$, $\text{Ran}(\Pi_{\varepsilon,n}) = \text{lin} V_{\varepsilon,n}$ with $V_{\varepsilon,n} \subset H^N(\mu) - 1$ the subset
\[
V_{\varepsilon,n} = \left\{ w = a_{l_1}^* \ldots a_{l_{m-1}}^* a_{k_1} \ldots a_{k_m} | FS_\mu | m \geq 1, k_1, \ldots, k_m \in \Lambda_h, l_1, \ldots, l_{m-1} \in \Lambda_{h,\varepsilon}^\leq, \right. \\
\left. \quad \sum_{k \in \Lambda_{h,\varepsilon}^\leq} a_k a_k^* w = nw \right\}. \tag{4.2}
\]

Clearly $\text{Ran}(\Pi_\varepsilon) = \bigoplus_{n \geq 0} \text{Ran}(\Pi_{\varepsilon,n})$. (Note that the operator $\sum_{k \in \Lambda_{h,\varepsilon}^\leq} a_k a_k^*$ counts the number of holes in $\Lambda_{h,\varepsilon}^\leq$.)

The next lemma provides a more accurate localization of the polaron energy inside the subspace $\text{Ran}(\Pi_{\varepsilon,1})$.

**Lemma 4.1.** There are constants $c_0, \varepsilon_0 > 0$ such that for all $p \in \kappa \mathbb{Z}^2$, $L^2 |E_B| \geq 1$, $\mu/|E_B| \geq c_0$ and $\varepsilon \in (0, \varepsilon_0)$, we have
\[
\Phi_p(\lambda, \varepsilon) \geq \Pi_{\varepsilon,1} (\text{pol}(\lambda) - \varepsilon^{-3}) \Pi_{\varepsilon,1} \tag{4.3}
\]
where $\text{pol}(\lambda) = G(0, T - \lambda) - a(\eta)(T - \lambda)^{-1} a^*(\eta)$. 

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Proof. We write $\Pi_{\varepsilon} = \Pi_{\varepsilon,0} + \Pi_{\varepsilon,1} + \Pi_{\varepsilon,2+}$ with $\Pi_{\varepsilon,2+} = \sum_{n \geq 2} \Pi_{\varepsilon,n}$. Below we prove the inequality

$$\Phi_p(\lambda, \varepsilon) \geq \Pi_{\varepsilon,1}(G(p - P_t, T - \lambda) - H_p(\lambda) - C\varepsilon^{-2})\Pi_{\varepsilon,1}$$

$$+ (\Pi_{\varepsilon,0} + \Pi_{\varepsilon,2+}) (G(p - P_t, T - \lambda) - C\varepsilon \log \tilde{\mu})(\Pi_{\varepsilon,0} + \Pi_{\varepsilon,2+})$$

(4.4)

from which the statement of the lemma follows by

$$G(p - P_t, T - \lambda) - H_p(\lambda) \geq G(0, T - \lambda) - a(\eta)(T - \lambda)^{-1}a^*(\eta) - C$$

(4.5)

together with inequality (3.18). By choosing $\varepsilon_0$ small enough the second line in (4.4) is positive for all $\tilde{\mu} \geq c_0$. The bound in (4.5) is a direct consequence of (3.17) and the fact that $T - \lambda \geq |e_P(\mu, E_B)|$ on $\mathcal{H}_N(\mu)$.

The derivation of (4.4) occupies the remainder of this proof. To this end, note

$$\Pi_{\varepsilon} G(p - P_t, T - \lambda) \Pi_{\varepsilon} = \Pi_{\varepsilon,1} G(p - P_t, T - \lambda) \Pi_{\varepsilon,1}$$

$$+ (\Pi_{\varepsilon,0} + \Pi_{\varepsilon,2+}) G(p - P_t, T - \lambda)(\Pi_{\varepsilon,0} + \Pi_{\varepsilon,2+}).$$

(4.6)

• Introducing $\Lambda_{h,\varepsilon}^> = \Lambda_h \setminus \Lambda_{h,\varepsilon}^\leq$ and $a(\eta_{\varepsilon}^>) = \sum_{k \in \Lambda_{h,\varepsilon}^>} a_k$, we can start with

$$\Pi_{\varepsilon,0} H_p(\lambda) \Pi_{\varepsilon,0} \leq \Pi_{\varepsilon,0} \left( \frac{a(\eta_{\varepsilon}^>)a^*(\eta_{\varepsilon}^>)}{|e_P(\mu, E_B)|} \right) \Pi_{\varepsilon,0} \leq C\varepsilon^{-1} \Pi_{\varepsilon,0}$$

(4.7)

which follows from

$$\left( \frac{1}{M}(p - P_t)^2 + T - \lambda \right)|\mathcal{H}_N(\mu)| \geq |e_P(\mu, E_B)| \geq \frac{C\mu}{\log \tilde{\mu}},$$

(4.8)

$a^*(\eta)\Pi_0(\varepsilon) = a^*(\eta_{\varepsilon}^>)\Pi_0(\varepsilon)$, and

$$a(\eta_{\varepsilon}^>)a^*(\eta_{\varepsilon}^>) \leq \frac{1}{L^2} \sum_{k \in \Lambda_{h,\varepsilon}^>} 1 \leq \frac{C\mu}{\varepsilon \log \tilde{\mu}}.$$

(4.9)

The latter is obtained via (3.7).

• Next we consider

$$\Pi_{\varepsilon,0} H_p(\lambda) \Pi_{\varepsilon,1} + \text{h.c.}$$

$$= \Pi_{\varepsilon,0} a(\eta_{\varepsilon}^>) \frac{1}{M}(p - P_t)^2 + T - \lambda a^*(\eta) \Pi_{\varepsilon,1} + \text{h.c.}$$
\[
\frac{(\varepsilon \log \tilde{\mu})^2}{\mu} \Pi_{\varepsilon,0} a(\eta_{\varepsilon}^>) a^*(\eta_{\varepsilon}^>) \Pi_{\varepsilon,0} + \frac{\mu}{(\varepsilon \log \tilde{\mu})^2} \Pi_{\varepsilon,1} a(\eta) \frac{1}{(T - \lambda)^2} a^*(\eta) \Pi_{\varepsilon,1}
\]

\[
\leq C(\varepsilon \log \tilde{\mu} \Pi_{\varepsilon,0} + \varepsilon^{-2} \Pi_{\varepsilon,1}),
\]

where we made use of (3.39), (4.8) and (4.9).

- The contribution

\[
\Pi_{\varepsilon,0} H_{\rho}(\lambda) \Pi_{\varepsilon,2+} + \text{h.c.} = 0
\]

vanishes identically since

\[
a(\eta) \frac{1}{\mathcal{M}(p - P_t)^2 + T - \lambda} a^*(\eta) \Pi_{\varepsilon,2+} w \in \text{Ran}(\Pi_{\varepsilon,1}) \oplus \text{Ran}(\Pi_{\varepsilon,2+})
\]

for any \( w \in \mathscr{H}_{\mathcal{N}(\mu) - 1} \) and \( \Pi_{\varepsilon,0} \Pi_{\varepsilon,1} = \Pi_{\varepsilon,0} \Pi_{\varepsilon,2+} = 0 \). (The operator \( a^*(\eta) \) can reduce the number of unoccupied modes at most by one.)

- We proceed with

\[
\Pi_{\varepsilon,1} H_{\rho}(\lambda) \Pi_{\varepsilon,2+} + \text{h.c.} = \Pi_{\varepsilon,1} a(\eta_{\varepsilon}^>) \frac{1}{\mathcal{M}(p - P_t)^2 + T - \lambda} a^*(\eta) \Pi_{\varepsilon,2+} + \text{h.c.}
\]

which holds because of

\[
(a(\eta) - a(\eta_{\varepsilon}^>)) \frac{1}{\mathcal{M}(p - P_t)^2 + T - \lambda} a^*(\eta) \Pi_{\varepsilon,2+} w \in \text{Ran}(\Pi_{\varepsilon,2+})
\]

and \( \Pi_{\varepsilon,1} \Pi_{\varepsilon,2+} = 0 \). (Note that \( a(\eta) - a(\eta_{\varepsilon}^>) \) adds an unoccupied mode in \( \Lambda_{\tilde{h},\varepsilon}^< \).) We estimate the r.h.s. of (4.13) from above by

\[
\frac{\varepsilon \log \tilde{\mu}}{\mu} \Pi_{\varepsilon,1} a(\eta_{\varepsilon}^>) a^*(\eta_{\varepsilon}^>) \Pi_{\varepsilon,1} + \frac{\mu}{\varepsilon \log \tilde{\mu}} \Pi_{\varepsilon,2+} a(\eta) \frac{1}{\mathcal{M}(p - P_t)^2 + T - \lambda} a^*(\eta) \Pi_{\varepsilon,2+}
\]

\[
\leq C(\Pi_{\varepsilon,1} + \varepsilon \log \tilde{\mu} \Pi_{\varepsilon,2+}),
\]

where we used another time that states of the form \( \psi = a^*(\eta) \Pi_{\varepsilon,2+} w \in \mathscr{H}_{\mathcal{N}(\mu)} \) are either zero or have at least one unoccupied mode in \( \Lambda_{\tilde{h},\varepsilon}^< \). The latter implies

\[
\langle \psi, (T - \lambda)^{-s} \psi \rangle \leq \langle \psi, \psi \rangle \left( \frac{\varepsilon \log \tilde{\mu}}{\mu^s} \right)^s (s > 0).
\]
In the bound for \( \Pi_{\varepsilon,2+} H_p(\lambda) \Pi_{\varepsilon,2+} \), we use (4.16) with \( s = 1 \) to get
\[
\Pi_{\varepsilon,2+} H_p(\lambda) \Pi_{\varepsilon,2+} \leq \frac{C\varepsilon \log \tilde{\mu}}{\mu} \Pi_{\varepsilon,2+} a(\eta)a^*(\eta) \Pi_{\varepsilon,2+} \leq C\varepsilon \log \tilde{\mu} \Pi_{\varepsilon,2+}
\] (4.17)
by means of (3.39).

So far we have shown
\[
\Pi_{\varepsilon} H_p(\lambda) \Pi_{\varepsilon} - \Pi_{\varepsilon,1} H_p(\lambda) \Pi_{\varepsilon,1} \geq C\varepsilon^{-2} \Pi_{\varepsilon,1} + C\varepsilon \log \tilde{\mu} (\Pi_{\varepsilon,0} + \Pi_{\varepsilon,2+}).
\] (4.18)

For the bounds involving \( X_p(\lambda) \), we recall (3.42).

With \( f(\mu, E_B) = g(\mu, E_B) = \mu/\sqrt{\varepsilon \log \tilde{\mu}} \), we obtain
\[
(\Pi_{\varepsilon,0} + \Pi_{\varepsilon,2+}) X_p(\lambda) (\Pi_{\varepsilon,0} + \Pi_{\varepsilon,2+}) \leq C\sqrt{\log \tilde{\mu}} (\Pi_{\varepsilon,0} + \Pi_{\varepsilon,2+}).
\] (4.19)

Choosing \( f(\mu, E_B) = \mu/(\varepsilon \log \tilde{\mu}) \) and \( g(\mu, E_B) = \varepsilon \mu \) leads to
\[
\Pi_{\varepsilon,1} X(\lambda) (\Pi_{\varepsilon,0} + \Pi_{\varepsilon,2+}) + h.c. \leq C\varepsilon^{-1} \Pi_{\varepsilon,1} + C\varepsilon \log \tilde{\mu} (\Pi_{\varepsilon,0} + \Pi_{\varepsilon,2+}).
\] (4.20)

For the term with \( \Pi_{\varepsilon,1} \) on both sides, the estimate in (3.42) is not good enough (for obtaining an error of order one w.r.t. \( \tilde{\mu} \)). A possible improvement, however, is readily obtained from
\[
\Pi_{\varepsilon,1} X_p(\lambda) \Pi_{\varepsilon,1} = \Pi_{\varepsilon,1} (A_p(\lambda)a^*(\eta^\gep) + h.c.) \Pi_{\varepsilon,1}
\] (4.21)
which is true since \( A_p(\lambda)(a^*(\eta) - a^*(\eta^\gep))\Pi_{\varepsilon,1}w \in \text{Ran}(\Pi_{\varepsilon,0}) \) and \( \Pi_{\varepsilon,1}\Pi_{\varepsilon,0} = 0 \). Following now the same steps that led to (3.42) and using in addition (4.9), we obtain
\[
\Pi_{\varepsilon,1} X_p(\lambda) \Pi_{\varepsilon,1} \leq \Pi_{\varepsilon,1} \left( \frac{f(\mu, E_B) A_p(\lambda)A^*_p(\lambda)}{g(\mu, E_B)\varepsilon \log \tilde{\mu}} \right) \Pi_{\varepsilon,1}
\]
\[
\leq C \left( \frac{f(\mu, E_B) \log \tilde{\mu}}{\mu} + \frac{\mu}{g(\mu, E_B)\varepsilon \log \tilde{\mu}} \right) \Pi_{\varepsilon,1}.
\] (4.22)

With \( f(\mu, E_B) = \mu/(\sqrt{\varepsilon \log \tilde{\mu}}) \) and \( g(\mu, E_B) = \sqrt{\varepsilon \mu/\log \tilde{\mu}} \), this provides \( \Pi_{\varepsilon,1} X_p(\lambda) \Pi_{\varepsilon,1} \leq C\varepsilon^{-1/2} \Pi_{\varepsilon,1} \).

Bringing the above bounds together proves (4.4).

The goal of the next lemma is to analyze the condition \( \text{pol}(\lambda) - r \geq 0 \) for a given number \( r \geq 0 \). To see for which \( \lambda \) such a bound may hold, we use the fact that this operator is given by an expression of the form \( K - V^*V \) with \( K = G - r \) and \( V = (T - \lambda)^{-1/2}a^*(\eta) \). If \( K \) is
self-adjoint and $K \geq c$ for some number $c > 0$, it follows easily that
\[
K - V^*V = (K - V^*V)K^{-1}(K - V^*V) + V^*(1 - VK^{-1}V)V \geq V^*(1 - VK^{-1}V)V. \tag{4.23}
\]

This is a key argument in the proof of the following proposition.

**Proposition 4.2.** For any fixed $r > 0$ there exists a constant $c_0 > 0$ such that for all $L^2|E_B| \geq 1$ and $\mu/|E_B| \geq c_0$ the following implication holds: $\text{pol}(\lambda) \geq r$ if $\lambda$ satisfies
\[
E_0(\mu) - \lambda - \frac{1}{L^2} \sum_{k^2 \leq \mu} G(0, E_0(\mu) - \lambda - k^2) - r = 0. \tag{4.24}
\]

**Remark 4.1.** We call (4.24) the perturbed polaron equation.

**Proof.** Since $T \geq E_0(\mu) - \mu$ on $\mathcal{H}_{N(\mu)-1}$ and $\lambda \leq E_0(\mu) + e_P(\mu, E_B)$, it follows by (3.18) that $G(0, T - \lambda)$ exceeds the value of $r$ for $\tilde{\mu}$ large enough. For such $\tilde{\mu}$ we can use (4.23) to find
\[
\text{pol}(\lambda) - r \geq a(\eta) \frac{1}{T - \lambda} \mathcal{F}(T - \lambda, r) \frac{1}{T - \lambda} a^*(\eta) \upharpoonright \mathcal{H}_{N(\mu)-1} \tag{4.25}
\]
where
\[
\mathcal{F}(T - \lambda, r) = \left( T - \lambda - a^*(\eta) \frac{1}{G(0, T - \lambda) - r} a(\eta) \right) \upharpoonright \mathcal{H}_{N(\mu)}. \tag{4.26}
\]

From here it follows similarly as in the proof of [LM19, Lemma 4.2] that $\mathcal{F}(T - \lambda, r) \geq 0$ if $\lambda$ satisfies the inequality
\[
E_0(\mu) - \lambda - \frac{1}{L^2} \sum_{k^2} G(0, E_0(\mu) - k^2 - \lambda) - r \geq 0. \tag{4.27}
\]

For convenience of the reader we provide the proof of the last statement in Appendix B. □

Next we prove the existence of a unique solution to the perturbed polaron equation (4.24) in the interval $(-\infty, E_0(\mu) + e_P(\mu, E_B)]$ and provide a suitable estimate for the difference of this solution and $E_0(\mu) + e_P(\mu, E_B)$.

**Lemma 4.3.** For any fixed $r > 0$ there exists a constant $c_0 > 0$ such that for all $L^2|E_B| \geq 1$ and $\mu/|E_B| \geq c_0$, the perturbed polaron equation (4.24) admits a unique solution in the interval $(-\infty, E_0(\mu) + e_P(\mu, E_B)]$. We call this solution $\lambda(\mu, E_B)$.\(^2\) Moreover there exists a constant

\(^2\)The omission of the $r$ dependence of $\lambda(\mu, E_B)$ is justified by (4.28).
$C > 0$ such that

$$E_0(\mu) + e_P(\mu, E_B) - \lambda(\mu, E_B) \leq C(1 + r) \frac{|e_P(\mu, E_B)|}{\log(\mu/|E_B|)} \quad (4.28)$$

for all $L^2|E_B| \geq 1$ and $\mu/|E_B| \geq c_0$.

Proof. To prove the existence of a solution we write (4.24) as $E_0(\mu) - \lambda = f(\lambda)$ with

$$f(\lambda) = \frac{1}{L^2} \sum_{k^2 \leq \mu} G(0, E_0(\mu) - \lambda - k^2) - r$$

(4.29)

a continuous monotonically increasing function $f(\lambda) : (-\infty, E_0(\mu) + e_P(\mu, E_B)] \to \mathbb{R}$. By definition of $G(q, \tau)$ we have $f(\lambda) \to 0$ as $\lambda \to -\infty$. Next consider $f(\Lambda(\mu, E_B))$ with $\Lambda(\mu, E_B) = E_0(\mu) + e_P(\mu, E_B)$. With the help of (1.10),

$$f(\Lambda(\mu, E_B)) = -e_P(\mu, E_B) + \frac{1}{L^2} \sum_{k^2 \leq \mu} r + G(k, -e_P(\mu, E_B) - k^2) - G(0, -e_P(\mu, E_B) - k^2)$$

$$+ \frac{1}{L^2} \sum_{k^2 \leq \mu} \left( G(0, -e_P(\mu, E_B) - k^2) - r \right) G(k, -e_P(\mu, E_B) - k^2), \quad (4.30)$$

and by way of Lemma 3.2, $G(k, -e_P(\mu, E_B) - k^2) - G(0, -e_P(\mu, E_B) - k^2) \geq -2C$, we see that the second line in (4.30) is bounded from below by a constant times $-\mu/(\log \tilde{\mu})^2$. Hence for all $\tilde{\mu}$ large enough, we infer $f(\Lambda(\mu, E_B)) \geq \frac{1}{2}|e_P(\mu, E_B)| > 0$. These observations imply that there is a unique $\lambda(\mu, E_B) \in (-\infty, \Lambda(\mu, E_B))$ such that $f(\lambda(\mu, E_B)) = E_0(\mu) - \lambda(\mu, E_B)$.

The difference between $\Lambda(\mu, E_B)$ and $\lambda(\mu, E_B)$ is estimated by

$$\Lambda(\mu, E_B) - \lambda(\mu, E_B)$$

$$= \frac{1}{L^2} \sum_{k^2 \leq \mu} \left( r + G(k, -e_P(\mu, E_B) - k^2) - G(0, E_0(\mu) - k^2 - \lambda(\mu, E_B)) \right)$$

$$\leq C(1 + r) \left( \mu + \frac{1}{L^2} \sum_{k^2 \leq \mu} \left( G(k^2, -e_P(\mu, E_B) - k^2) - G(0, E_0(\mu) - k^2 - \lambda(\mu, E_B)) \right) \right), \quad (4.31)$$

where we used Lemma 3.2 to estimate the denominator from below by a constant times $(\log \tilde{\mu})^2$. In the remainder we show

$$\frac{1}{L^2} \sum_{k^2 \leq \mu} \left( G(k, -e_P(\mu, E_B) - k^2) - G(0, E_0(\mu) - k^2 - \lambda(\mu, E_B)) \right) \leq C \mu \quad (4.32)$$

which proves that the left side of (4.28) is bounded from above by $C(1 + r)\mu/(\log(\mu/|E_B|))^2$. 

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To verify (4.32) we use \( \lambda(\mu, E_B) \leq E_0(\mu) + e_P(\mu, E_B) \) and again Lemma 3.2 to estimate the expression inside the brackets from above by

\[
\frac{1}{4\pi m} \log \left( \frac{k^2 - e_P(\mu, E_B) + m\mu - k^2}{e_P(\mu, E_B) + m\mu - k^2} \right) + 2C. \tag{4.33}
\]

With \( 0 \leq k^2 \leq \mu \), \( 0 \leq -e_P(\mu, E_B) \leq \mu \) and \( |e_P(\mu, E_B)| = O(\mu/\log \tilde{\mu}) \), one further verifies that the logarithm is bounded from above by \( \log((2M^2 + 4M + 1)/(M + 1)) \leq C \).

Let us summarize the result of this section.

**Corollary 4.4.** For any fixed \( \varepsilon > 0 \) there exist constants \( c_0, C > 0 \) such that \( \lambda(\mu, E_B) \leq E_0(\mu) + e_P(\mu, E_B) \), the unique solution of the perturbed polaron equation (4.24) with \( r = \varepsilon^{-3} \), satisfies the following two properties: \( \Phi_p(\lambda(\mu, E_B)) \geq 0 \) and

\[
\lambda(\mu, E_B) - E_0(\mu) - e_P(\mu, E_B) \geq -C(1 + \varepsilon^{-3}) \frac{|e_P(\mu, E_B)|}{\log(\mu/|E_B|)} \tag{4.34}
\]

for all \( p \in \kappa \mathbb{Z}^2 \), \( L^2|E_B| \geq 1 \) and \( \mu/|E_B| \geq c_0 \).

In the next section we show that \( \Psi_p(\lambda, \varepsilon) \geq 0 \) for all \( \lambda \leq E_0(\mu) + e_P(\mu, E_B) \) provided that \( M > 1.225 \) and \( \varepsilon \) is sufficiently small.

## 5 Analysis of \( \Psi_p(\lambda, \varepsilon) \): stability condition

On the subspace \( \text{Ran}(\Pi_\varepsilon^+) \) it is not clear how to obtain a suitable \( L \)-independent bound for the operator \( P_p(\lambda) \). A possible solution to this difficulty is to estimate its negative part in terms of \( G(p - P_t, T - \lambda) \). Such a bound was derived in [Lin17, GL18] in the context of the 2D Fermi polaron at zero density (there the model is defined on \( \mathbb{R}^2 \) instead of the box \( \Omega \) and the kinetic energy \( E_0(\mu = 0) \) is zero). The strategy of our proof follows the one developed there, but several new obstacles need to be dealt with in the present case. The new obstacles are due to \( \mu > 0 \) and the fact that we have to work with momentum sums instead of integrals.

We write \( P_p(\lambda) = P_p(\lambda) - \tilde{P}_p(\lambda, \varepsilon) + \tilde{P}_p(\lambda, \varepsilon) \) where

\[
\tilde{P}_p(\lambda, \varepsilon) = \frac{1}{L^2} \sum_{k^2, l^2 > \mu/\varepsilon} a_k^* \frac{1}{M(p - P_t - k - l)^2 + T + k^2 + l^2 - \lambda} a_k. \tag{5.1}
\]

The operator \( P_p(\lambda) - \tilde{P}_p(\lambda, \varepsilon) \) is the easy part and can be estimated by the following lemma.

**Lemma 5.1.** There are constants \( c_0, \varepsilon_0, C > 0 \) such that

\[
P_p(\lambda) - \tilde{P}_p(\lambda, \varepsilon) \geq -C \sqrt{\varepsilon^{-1} \log(\mu/|E_B|)} \tag{5.2}
\]
on $\mathcal{H}_{N(\mu)-1}$ for all $p \in \kappa \mathbb{Z}^2$, $L^2|E_B| \geq 1$, $\mu/|E_B| \geq c_0$ and $\varepsilon \in (0,\varepsilon_0)$.

Proof. Write

$$P_p(\lambda) - \tilde{P}_p(\lambda,\varepsilon) = \frac{1}{L^2} \sum_{\mu < k^2, p \leq \mu/\varepsilon} a_k^* \frac{1}{M(p - P_1 - k - l)^2 + T + k^2 + l^2 - \lambda} a_k \tag{5.3}$$

$$+ \frac{1}{L^2} \sum_{\mu < k^2 \leq \mu/\varepsilon \atop l^2 > \mu/\varepsilon} a_k^* \frac{1}{M(p - P_1 - k - l)^2 + T + k^2 + l^2 - \lambda} a_k + \text{h.c.} \tag{5.4}$$

We proceed as in the proof of (3.20) to obtain

$$(5.3) \geq -C \left( f(\mu, E_B) \frac{\mu}{\varepsilon} + \frac{1}{f(\mu, E_B)|\mu, E_B|} \right). \tag{5.5}$$

Choosing $f(\mu, E_B) = \sqrt{\varepsilon^{-1} \log \mu/\mu}$ we get the desired bound for this line. The proof for the second line works in complete analogy.

To a large extent this section is about the derivation of a lower bound for $\tilde{P}_p(\lambda,\varepsilon)$. For that purpose we need to introduce some further notation. For $\varepsilon$ small enough, let the function $\beta(\cdot, \varepsilon) : [0, 1] \to (0, 1]$ be given by

$$\beta(u, \varepsilon) = \min \left\{ 1, \frac{(M + 1 - u)(M + 2)(1 - (1 + \frac{M+1-u}{M(M+2)})\sqrt{\varepsilon})}{(M + 1 - u)(1 - 2\sqrt{\varepsilon}) + M(M + 2)(1 - \sqrt{\varepsilon})} \right\} \tag{5.6}$$

and set

$$\alpha(M, \varepsilon) = \frac{1}{2} \left( \frac{1}{M(1 - \sqrt{\varepsilon}) + 1} + \int_0^1 \frac{1}{\beta(u, \varepsilon)(M(1 - \sqrt{\varepsilon}) + 1 - u)} du \right). \tag{5.7}$$

Proposition 5.2. There are constants $c_0, \varepsilon_0, C > 0$ such that

$$\tilde{P}_p(\lambda, \varepsilon) \geq -\frac{1}{1 - \varepsilon} \left( \frac{\alpha(M, \varepsilon)}{4\pi} \log \left( 1 + \frac{T - \lambda + 2\mu}{\mu} \right) + \frac{C}{\sqrt{\mu/|E_B|}} \right) \tag{5.8}$$

on $\mathcal{H}_{N(\mu)-1}$ for all $p \in \kappa \mathbb{Z}^2$, $L^2|E_B| \geq 1$, $\mu/|E_B| \geq c_0$ and $\varepsilon \in (0,\varepsilon_0)$.

To prove Proposition 5.2 we combine the next two lemmas.

Lemma 5.3. Let $T_{>\mu/\varepsilon} = \sum_{k^2 > \mu/\varepsilon} k^2 a_k^* a_k$. For any $\varepsilon \in (0, 1)$, it holds that

$$\frac{T_{>\mu/\varepsilon}}{T - \lambda + 2\mu} \uparrow \mathcal{H}_{N(\mu)-1} \leq \frac{1}{1 - \varepsilon}. \tag{5.9}$$
Lemma 5.4. There are constant $c_0, \varepsilon_0, C > 0$ such that

\[ \tilde{P}_p(\lambda, \varepsilon) \geq -\frac{T_{>\mu/\varepsilon}}{T - \lambda + 2\mu} \left( \frac{\alpha(M, \varepsilon)}{4\pi} \log \left( 1 + \frac{T - \lambda + 2\mu}{\mu} \right) + \frac{C}{\sqrt{\mu/|E_B|}} \right) \]  

(5.10)
on $\mathcal{H}_{N(\mu)-1}$ for all $p \in \kappa \mathbb{Z}^2$, $L^2|E_B| \geq 1$, $\mu/|E_B| \geq c_0$ and $\varepsilon \in (0, \varepsilon_0)$.

Proof of Proposition 5.2. Since $T - \lambda \upharpoonright \mathcal{H}_{N(\mu)-1} \geq -\mu$, the operator

\[ \log \left( 1 + \frac{T - \lambda + 2\mu}{\mu} \right) \upharpoonright \mathcal{H}_{N(\mu)-1} \geq \log(2) \]  

(5.11)
is positive. Since $T$ and $T_{>\mu/\varepsilon}$ commute, we can use Lemma 5.3 to prove Proposition 5.2 with the aid of (5.10).

Proof of Lemma 5.3. Using again $T - \lambda \upharpoonright \mathcal{H}_{N(\mu)-1} \geq -\mu$ in combination with $0 \leq T_{>\mu/\varepsilon} \leq T$, the operator

\[ \frac{T_{>\mu/\varepsilon}}{T - \lambda + 2\mu} \leq 1 + \frac{\lambda - 2\mu}{T - \lambda + 2\mu} \leq 1 + \frac{E_0(\mu) - 2\mu}{\mu} \]  

(5.12)
is bounded when restricted to $\mathcal{H}_{N(\mu)-1}$. Hence it is sufficient to show (5.9) on the dense subspace $\text{lin} D \subseteq \mathcal{H}_{N(\mu)-1}$ given by all finite linear combinations of anti-symmetric products of plane waves, see (3.4). Since the states in $\text{lin} D$ are linear combinations of simultaneous eigenstates of $T \upharpoonright \mathcal{H}_{N(\mu)-1}$ and $T_{>\mu/\varepsilon} \upharpoonright \mathcal{H}_{N(\mu)-1}$, we can restrict the argument further to the set $D$ itself. This becomes particularly useful when writing $D = \bigcup_{n \geq 0} W_n(\varepsilon)$ with

\[ W_n(\varepsilon) = \left\{ w \in D : \left( \sum_{k^2 > \mu/\varepsilon} a_k^* a_k \right) w = nw \right\} \]  

(5.13)
the set of anti-symmetric products of plane waves with exactly $n$ momentum modes occupied in $\{ k \in \kappa \mathbb{Z}^2 : k^2 > \mu/\varepsilon \}$.

Since $T_{>\mu/\varepsilon} w = 0$ for $w \in W_0(\varepsilon)$, we consider $w \in W_n(\varepsilon)$, $n \geq 1$, with $\|w\|^2 = 1$. We call $\beta(w)$ the eigenvalue of $T$ and $\gamma(w)$ the eigenvalue of $T_{>\mu/\varepsilon}$. It follows that

\[ \gamma(w) = \langle w, T_{>\mu/\varepsilon} w \rangle > n\mu\varepsilon^{-1} \]  

(5.14)
as well as

\[ \beta(w) - \gamma(w) = \langle w, (T - T_{>\mu/\varepsilon}) w \rangle \geq E_0(\mu) - (n + 1) \mu. \]  

(5.15)
To derive the last inequality we denote the eigenvalues of $-\Delta$ by $\lambda_i(-\Delta)$ ($i \geq 1$, numbered
with increasing order and counting multiplicities) and use

$$\langle w, (T - T_{\geq \mu/\varepsilon})w \rangle = \langle w, (\sum_{k^2 \leq \mu/\varepsilon} k^2 a_k^* a_k)w \rangle \geq E_0(\mu) - \sum_{i=N(\mu) - n}^N \lambda_i(-\Delta). \quad (5.16)$$

From $\lambda_i(-\Delta) \leq \mu$ for $i \leq N(\mu)$, we obtain (5.15). The latter together with $\lambda \leq E_0(\mu)$ implies

$$\beta(w) - \lambda \geq \gamma(w) - (n + 1)\mu, \quad (5.17)$$

and combining this with (5.14), we get

$$\langle w, \frac{r_{\geq \mu/\varepsilon}}{T - \lambda + 2\mu}w \rangle = \frac{\gamma(w)}{\beta(w) - \lambda + 2\mu} \leq \frac{\gamma(w)}{\gamma(w) - (n - 1)\mu} \leq \frac{n\mu\varepsilon^{-1}}{n\mu\varepsilon^{-1} - (n - 1)\mu}. \quad (5.18)$$

Since the expression on the right does not exceed the value $\frac{1}{1-\varepsilon}$, we have proven the statement.

**Proof of Proposition 5.2.** We start by introducing the abbreviations

$$\hat{k} = k + \frac{1}{M + 2} (p - P_t), \quad \hat{l} = l + \frac{1}{M + 2} (p - P_t) \quad (5.19)$$

by which one writes the denominator in the expression defining $\tilde{P}(\lambda, \varepsilon)$ as

$$m(\hat{k}^2 + \hat{l}^2) + \frac{2}{M} \hat{k} \cdot \hat{l} + \frac{1}{M + 2} (p - P_t)^2 + T - \lambda. \quad (5.20)$$

For $w \in \mathcal{H}_{N(\mu) - 1}$ we define $\tilde{w} \in L^2(\kappa\mathbb{Z}^2; \mathcal{H}_{N(\mu) - 2})$ by $\tilde{w}(k) = a_k w$. Moreover we define the unitary operator $U \in \mathcal{L}(L^2(\kappa\mathbb{Z}^2; \mathcal{H}_{N(\mu) - 2}))$ by

$$(U\varphi)(k; k_1, ..., k_{N(\mu) - 2}) = \varphi(k + \frac{1}{M + 2} (p - \sum_{i=1}^{N(\mu) - 2} k_i); k_1, ..., k_{N(\mu) - 2}), \quad (5.21)$$

where we use the notation $(U\varphi)(k; k_1, ..., k_{N(\mu) - 2})$ for the Fourier space representation of $(U\varphi)(k) \in \mathcal{H}_{N(\mu) - 2}$. With these definitions at hand, it is not difficult to compute

$$\langle w, \tilde{P}_p(\lambda, \varepsilon)w \rangle = \frac{1}{L^2} \sum_{k,l} \langle (\chi_{\mu/\varepsilon} \tilde{w})(k), U\sigma(k, l)U^*(\chi_{\mu/\varepsilon} \tilde{w})(l) \rangle \quad (5.22)$$

\[Note that we omit the p-dependence of the unitary operator U.\]
where \(\chi_{(\mu/\varepsilon, \infty)}\) stands for the characteristic function \(k \mapsto \chi_{(\mu/\varepsilon, \infty)}(k^2)\) and where

\[
\sigma(k, l) = \frac{1}{L^2} \frac{m(k^2 + l^2)}{2M} k \cdot l + \frac{1}{M+2} (p - P_l)^2 + T - \lambda. \tag{5.23}
\]

Denoting the scalar product on \(L^2(\kappa \mathbb{Z}^2; \mathcal{H}_{\mathcal{N}(\mu)} - 2)\) by \(\langle \cdot, \cdot \rangle\), (5.22) is rewritten as

\[
\langle w, \tilde{P}_p(\lambda, \varepsilon)w \rangle = \langle \chi_{(\mu/\varepsilon, \infty)} \tilde{w}, U \sigma U^* \chi_{(\mu/\varepsilon, \infty)} \tilde{w} \rangle \tag{5.24}
\]

where \(\sigma\) is the operator on \(L^2(\kappa \mathbb{Z}^2; \mathcal{H}_{\mathcal{N}(\mu)} - 2)\) with operator-valued kernel \(\sigma(k, l)\). Next, we show that the negative part of \(\sigma\) has the kernel \(\sigma^-(k, l) = \frac{1}{2} (\sigma(-k, l) - \sigma(k, l))\). To this end, consider the reflection operator \(R\) defined by \((R \tilde{w})(k) = \tilde{w}(-k)\) for any \(\tilde{w} \in L^2(\kappa \mathbb{Z}^2; \mathcal{H}_{\mathcal{N}(\mu)} - 2)\). It is straightforward to verify \(R \sigma = \sigma R\). Moreover, \(R \sigma\) is a positive operator, which can be seen as follows. The integral kernel of \(R \sigma\) is given by \((R \sigma)(k, l) = \sigma(-k, l)\) and has the integral representation

\[
\sigma(-k, l) = \frac{1}{L^2} \int_0^\infty e^{-tk^2} \left( e^{-t(k-l)^2/M} - e^{-t(\frac{1}{M+2}(p - P_l)^2 + T - \lambda)} \right) \, e^{-tl} \, dt. \tag{5.25}
\]

Then use the following identity for \(\psi \in L^2(\Omega)\) and its Fourier transform \(\hat{\psi} \in \ell^2(\kappa \mathbb{Z}^2)\),

\[
\frac{1}{L^2} \sum_{k,l} \hat{\tilde{\psi}}(k) e^{-t(k-l)^2/M} \hat{\psi}(l) = \int_{\Omega} |\psi(x)|^2 \sum_k e^{ikx} e^{-tk^2/M} \, dx. \tag{5.26}
\]

This together with Poisson’s summation formula (see e.g. [Gra14, Section 3.2]) and the fact that the Fourier transform of a Gaussian is a positive function implies that \(R \sigma\) is a positive operator. Consequently, we have \(R \sigma = |\sigma|\) since \(R \sigma\) is positive and \(\sigma^2 = (R \sigma)(R \sigma)\). The positive and negative parts of \(\sigma\) are thus given by \(\sigma^\pm = \pm(\sigma \pm R \sigma)/2\) and the corresponding kernels by \(\sigma^\pm(k, l) = \pm(\sigma(k, l) \pm \sigma(-k, l))/2\).

We proceed by writing the kernel of the negative part as \(\sigma^-(k, l) = \frac{1}{2} \int_{-1}^{1} \frac{d}{du} \sigma(-uk, l) \, du\), and hence

\[
\sigma^-(k, l) = \frac{Mk \cdot l}{L^2} \int_{-1}^{1} \frac{1}{[(M+1)(k^2 + l^2) - 2uk \cdot l + B]^2} \, du, \tag{5.27}
\]

where \(B = \frac{M}{M+2}(p - P_l)^2 + M(T - \lambda)\). Using this in combination with (5.24), we get

\[
\tilde{P}_p(\lambda, \varepsilon) \geq -\frac{M}{L^2} \sum_{k^2 + l^2 > \mu/\varepsilon} a_k^* \left( \int_{-1}^{1} \frac{\hat{k} \cdot \hat{l}}{[(M+1)(\hat{k}^2 + \hat{l}^2) - 2\hat{u} \hat{k} \cdot \hat{l} + B]^2} \, du \right) a_l. \tag{5.28}
\]
To the expression on the right we apply the following inequality which is a version of the Schur test and is easily proven by applying the Cauchy-Schwarz inequality two times,

\[
\sum_{k^2, l^2 > \mu/\varepsilon} a_k^* J(k, l) a_l \leq \sum_{k^2 > \mu/\varepsilon} k^2 a_k^* \left( \sum_{l^2 > \mu/\varepsilon} \frac{|J(k, l)|}{l^2} \right) \]

(5.29)

for any family of bounded operators \((J(k, l))_{k, l \in \mathbb{Z}^2}\) on \(\mathcal{F}\) satisfying \(J(k, l)^* = J(k, l)\). This provides

\[
\tilde{P}_p(\lambda, \varepsilon) \geq -M \sum_{k^2 > \mu/\varepsilon} k^2 a_k^* \left\{ \frac{1}{l^2} \sum_{l^2 > \mu/\varepsilon} \int_{-1}^{1} \frac{|\hat{k} \cdot \hat{l}|}{l^2[(M + 1)(\hat{k}^2 + \hat{l}^2) - 2u\hat{k} \cdot \hat{l} + B]}\right\} a_k
\]

(5.30)

as an operator inequality on \(\mathcal{H}_{N(\mu) - 1}\). Our next goal is to find a suitable function \(g\) such that for \(k^2 > \mu/\varepsilon\), we have \(f(k^2, p - P_1, T) \leq g(T + k^2)\) on \(\mathcal{H}_{N(\mu) - 2}\). For such a function we have

\[
\tilde{P}_p(\lambda, \varepsilon) \geq -M \sum_{k^2 > \mu/\varepsilon} k^2 a_k^* g(T + k^2) a_k \geq -MT_{\mu/\varepsilon} g(T)
\]

(5.31)

since \(g(T + k^2) a_k = a_k g(T)\) and \(T_{\mu/\varepsilon} = \sum_{k^2 > \mu/\varepsilon} k^2 a_k^* a_k\).

To find a suitable function \(g\), it is helpful to check that the expression inside the square brackets in the denominator in (5.30) is positive for \(\varepsilon\) small enough. To see this we use

\[
\hat{k}^2 \geq \sqrt{\varepsilon} k^2 - \frac{\sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}} \frac{(p - P_1)^2}{(M + 2)^2},
\]

(5.32)

and similarly for \(\hat{l}^2\), together with \(k^2, l^2 > \mu/\varepsilon\) and \(T - \lambda \geq -2\mu\) on \(\mathcal{H}_{N(\mu) - 2}\) to find

\[
(M + 1)(\hat{k}^2 + \hat{l}^2) - 2u\hat{k} \cdot \hat{l} + B
\]

\[
\geq 2M(\varepsilon^{-1/2} - 1)\mu + \left(1 - \frac{\sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}} \frac{1}{M + 2}\right)\frac{M}{M + 2}(p - P_1)^2.
\]

(5.33)

Next we use that \(-2u\hat{k} \cdot \hat{l} \geq 0\) either for \(u \in [-1, 0]\) or for \(u \in [0, 1]\). This makes the quotient in the definition of \(f\) larger and also independent of \(u\) on the respective interval. On the other interval, we employ \(0 \geq -2u\hat{k} \cdot \hat{l} \geq |u|(|\hat{k}^2 + \hat{l}^2)|\). In both cases this leads to

\[
\int_{-1}^{1} \frac{|\hat{k} \cdot \hat{l}|}{l^2[(M + 1)(\hat{k}^2 + \hat{l}^2) - 2u\hat{k} \cdot \hat{l} + B]}du \leq \frac{|\hat{k} \cdot \hat{l}|}{l^2[(M + 1)(\hat{k}^2 + \hat{l}^2) + B]}
\]

(5.34a)
\[ \int_0^1 \frac{[\hat{k} \cdot \hat{l}]}{l^2[(M + 1 - u)(\hat{k}^2 + \hat{l}^2) + B]^2} du. \quad (5.34b) \]

In the denominators we proceed with the bound

\[ (M + 1 - u)(\hat{k}^2 + \hat{l}^2) + B \geq 2|\hat{k} \cdot \hat{l}|(M(1 - \sqrt{\varepsilon}) + 1 - u). \quad (5.35) \]

The latter is verified by

\[ (M + 1 - u)(\hat{k}^2 + \hat{l}^2) + \frac{(p - P_t)^2 M}{(M + 1)(M + 2)} (M + 1 - u - \frac{2 \mu M}{\hat{k}^2 + \hat{l}^2 + \frac{(p - P_t)^2}{(M + 1)(M + 2)}}) \quad (5.36) \]

on \( \mathcal{H}_{N(\mu) - 2} \) in combination with

\[ \hat{k}^2 + \hat{l}^2 + \frac{(p - P_t)^2 M}{(M + 1)(M + 2)} \geq \frac{2 \mu}{\sqrt{\varepsilon}} \quad (5.37) \]

which, in turn, follows from (5.32) and \( k^2 + l^2 \geq 2\mu/\varepsilon \). Putting the different steps together, one obtains

\[ f(k^2, p - P_t, T) \leq \tilde{f}(k^2, p - P_t, T, 0) + \int_0^1 \tilde{f}(k^2, p - P_t, T, u) du \quad (5.38) \]

with

\[ \tilde{f}(k^2, p - P_t, T, u) = \frac{1}{L^2} \sum_{l \geq \mu/\varepsilon} \frac{1}{2l^2(M(1 - \sqrt{\varepsilon}) + 1 - u)[(M + 1 - u)(\hat{k}^2 + \hat{l}^2) + B]} \quad (5.39) \]

In the expression inside the square brackets we estimate \( \hat{k}^2 \) and \( \hat{l}^2 \) by (5.32) to get the lower bound

\[ [... \geq (M + 1 - u) \left( \sqrt{\varepsilon} l^2 + \sqrt{\delta} k^2 - \frac{2 \mu M}{(M + 1 - u)} \right) + M(T - \lambda + 2 \mu) \]

\[ + \left( M(M + 2) - \frac{\sqrt{\varepsilon}}{1 - \sqrt{\varepsilon}}(M + 1 - u) - \frac{\sqrt{\delta}}{1 - \sqrt{\delta}}(M + 1 - u) \right) \frac{(p - P_t)^2}{(M + 2)^2}. \quad (5.40) \]

Requiring that the second line vanishes implies

\[ \sqrt{\delta} = \frac{M(M + 2)(1 - \sqrt{\varepsilon}) - \sqrt{\varepsilon}(M + 1 - u)}{M(M + 2)(1 - \sqrt{\varepsilon}) + (M + 1 - u)(1 - 2\sqrt{\varepsilon})}. \quad (5.41) \]
Hence we can bound the expression in square brackets by

\[ (M + 1 - u)(\hat{k}^2 + \hat{l}^2) + B \geq \frac{\sqrt{\varepsilon}}{2} l^2 (M + 1 - u) + M\beta(u, \varepsilon)(T + k^2 - \lambda + 2\mu) \]  \hfill (5.42)

with

\[ \beta(u, \varepsilon) = \min\{1, \sqrt{\delta}(M + 1 - u)/M\}. \]  \hfill (5.43)

Note that for \( \varepsilon \) small enough \( \beta(0, \varepsilon) = 1. \) Applying this to (5.39), we obtain

\[ \tilde{f}(k^2, p - P_f, T, u) \leq \frac{1}{2} \sum_{l^2 > \mu/\varepsilon} \frac{1}{l^2 (1 + Y l^2)} \]  \hfill (5.44)

Here we sum a non-negative and monotonically decreasing function of \( l^2 \) so that we can apply Lemma A.1. To follow the next steps with more ease, let us write

\[ (5.44) = \frac{1}{X} \left( \frac{1}{L^2} \sum_{l^2 > \mu/\varepsilon} \frac{1}{l^2 (1 + Y l^2)} \right) \]  \hfill (5.45)

with (all understood as operator on \( \mathcal{H}_{N(\mu)-2} \))

\[ X = 2(M(1 - \sqrt{\varepsilon}) + 1 - u)Z, \quad Y = \frac{\sqrt{\varepsilon}(M + 1 - u)}{2Z}, \]  \hfill (5.46)

and \( Z = M\beta(u, \varepsilon)(T + k^2 - \lambda + 2\mu). \) Since for \( b > 0 \)

\[ \int_{\sqrt{\mu/\varepsilon}}^{\infty} \frac{1}{s(1 + bs^2)} \, ds = \frac{1}{2} \log \left( 1 + \frac{\varepsilon}{\mu b} \right), \quad \int_{\sqrt{\mu/\varepsilon}}^{\infty} \frac{1}{s^2(1 + bs^2)} \, ds \leq \frac{1}{\sqrt{\mu/\varepsilon}} + \frac{\pi \sqrt{b}}{2}, \]  \hfill (5.47)

we obtain the bound

\[ \frac{1}{L^2} \sum_{l^2 > \mu/\varepsilon} \frac{1}{l^2 (1 + Y l^2)} \leq \frac{1}{4\pi} \log \left( 1 + \frac{\varepsilon}{\mu Y} \right) \]

\[ + \frac{2}{\pi L} \left( \frac{1}{\sqrt{\mu/\varepsilon}} + \frac{\pi}{2} \sqrt{Y} \right) + \left( \frac{4\sqrt{\mu/\varepsilon}}{\pi L} + \frac{6}{L^2} \right) \frac{1}{\varepsilon(1 + \mu Y)}. \]  \hfill (5.48)

Using \( T - \lambda \geq -2\mu \) on \( \mathcal{H}_{N(\mu)-2} \), \( k^2 \geq \mu/\varepsilon \) and \( L^2|E_B| \geq 1 \), the second line is easily seen to
be bounded by a constant times $\tilde{\mu}^{-1/2}$. In the first line, we estimate
\[
\log \left( 1 + \frac{2\sqrt{\varepsilon}M\beta(u, \varepsilon)(T + k^2 - \lambda + 2\mu)}{(M + 1 - u)\mu} \right) \leq \log \left( 1 + \frac{T + k^2 - \lambda + 2\mu}{\mu} \right)
\]
by choosing $\varepsilon$ sufficiently small. This together with (5.45) leads to
\[
\tilde{f}(k^2, p - P_t, T, u) \leq \frac{1}{T + k^2 - \lambda + 2\mu} \left( \log \left( 1 + \frac{T + k^2 - \lambda + 2\mu}{\mu} \right) + \frac{C}{\sqrt{\mu}} \right). \tag{5.50}
\]
Recalling definition (5.7) for $\alpha(M, \varepsilon)$, we set
\[
g(T) = \frac{1}{T - \lambda + 2\mu} \left( \frac{\alpha(M, \varepsilon)}{4\pi M} \log \left( 1 + \frac{T - \lambda + 2\mu}{\mu} \right) + \frac{C}{\sqrt{\mu}} \right) \tag{5.51}
\]
for some suitable constant $C$. In view of (5.38) and (5.50), it follows that $f(k^2, p - P_t, T) \leq g(T + k^2)$ as desired. With the aid of (5.31) this leads to
\[
\tilde{P}_p(\lambda, \varepsilon) \geq -\frac{T_{>\mu/\varepsilon}}{T - \lambda + 2\mu} \left( \frac{\alpha(M, \varepsilon)}{4\pi} \log \left( 1 + \frac{T - \lambda - 2\mu}{\mu} \right) + \frac{C}{\sqrt{\mu}} \right) \tag{5.52}
\]
for some constant $C > 0$ and thus the proof of the lemma is complete.

The next statement is the main result of this section. Let us mention that the condition $M > 1.225$ enters as a technical assumption and is not expected to be optimal.

**Corollary 5.5.** Let $M > 1.225$ and $\lambda \leq E_0(\mu) + c_p(\mu, E_B)$. There exist constants $c_0, \varepsilon_0 > 0$ such that $\Psi_p(\lambda, \varepsilon) \geq 0$ for all $p \in \kappa \mathbb{Z}^2$, $L^2|E_B| \geq 1$, $\mu/|E_B| \geq c_0$ and $\varepsilon \in (0, \varepsilon_0)$.

**Proof.** Recalling the definition of $\Psi_p(\lambda, \varepsilon)$ in (3.6), we write
\[
\Psi_p(\lambda, \varepsilon) = \Pi^\perp(\varepsilon)(\Psi_{p,1}(\lambda, \varepsilon) + \Psi_{p,2}(\lambda, \varepsilon))\Pi^\perp(\varepsilon) \tag{5.53}
\]
with
\[
\Psi_{p,1}(\lambda, \varepsilon) = \varepsilon^{1/3} G(p - P_t, T - \lambda) + P_p(\lambda) - \tilde{P}_p(\lambda, \varepsilon) - K(\varepsilon, \tilde{\mu}) - d, \tag{5.54}
\]
\[
\Psi_{p,2}(\lambda, \varepsilon) = (1 - \varepsilon^{1/3}) G(p - P_t, T - \lambda) + \tilde{P}_p(\lambda, \varepsilon) + d, \tag{5.55}
\]
where $K(\varepsilon, \tilde{\mu}) = \varepsilon^{-1} + \varepsilon^{-1/2} \sqrt{\log \mu + \varepsilon^{1/2} \log \tilde{\mu}}$ and $d > 0$ is a constant that we choose large enough but fixed w.r.t. all parameters.
By means of inequality (3.18) and Lemma 5.1, we estimate
\[
\Psi_{p,1}(\lambda, \varepsilon) \geq \frac{\varepsilon^{1/3}}{4\pi m} \log \mu - C(\varepsilon^{1/3} + d + \sqrt{\varepsilon^{-1} \log \mu + K(\varepsilon, \mu)}) \tag{5.56}
\]
for some (\varepsilon-independent) \( C > 0 \). With \( \varepsilon_0 > 0 \) small enough, the right side is non-negative for all \( \mu \) large enough.

In line (5.55) we apply (3.17) to get
\[
(1 - \varepsilon^{1/3})G(p - P_t, T - \lambda) + \frac{d}{2} \geq \frac{1 - \varepsilon^{1/3}}{4\pi m} \log \left( \frac{T - \lambda + m\mu}{|E_B|} \right) \tag{5.57}
\]
on \( \mathcal{H}_{N(\mu)} - 1 \). Since \( m = 1 + \frac{1}{M}, \mu/|E_B| \geq c_0 \geq 2M \) as well as \((T - \lambda + \mu) \uparrow \mathcal{H}_{N(\mu)} - 1 \geq 0\), we can estimate the logarithm further by
\[
\log \left( \frac{T - \lambda + m\mu}{|E_B|} \right) \geq \log \left( \frac{T - \lambda + \mu + c_0\mu/M}{\mu} \right) \geq \log \left( \frac{T - \lambda + 3\mu}{\mu} \right). \tag{5.58}
\]
Proposition 5.2 gives a bound for the second term in (5.55),
\[
\tilde{P}(\lambda, \varepsilon) + \frac{d}{2} \geq -\frac{1}{1 - \varepsilon} \frac{\alpha(M, \varepsilon)}{4\pi} \log \left( 1 + \frac{T - \lambda + 2\mu}{\mu} \right). \tag{5.59}
\]
Adding both estimates together leads to
\[
\Psi_{p,2}(\lambda, \varepsilon) \geq \left( \frac{1 - \varepsilon^{1/3}}{4\pi m} - \frac{1}{1 - \varepsilon} \frac{\alpha(M, \varepsilon)}{4\pi} \right) \log \left( 1 + \frac{T - \lambda + 2\mu}{\mu} \right) \tag{5.60}
\]
on \( \mathcal{H}_{N(\mu)} - 1 \).

The condition \( \Psi_p(\lambda, \varepsilon) \geq 0 \) is thus satisfied if
\[
(1 - \varepsilon^{1/3}) \frac{M}{M + 1} - \frac{1}{1 - \varepsilon} \alpha(M, \varepsilon) \geq 0. \tag{5.61}
\]
This is similar to the stability condition at zero density that was derived in [GL18]. There it was shown that the Fermi polaron defined on \( \mathbb{R}^2 \) is stable if
\[
\frac{M}{M + 1} - \alpha(M, 0) \geq 0 \tag{5.62}
\]
which was proven to hold for all \( M > 1.225 \) [GL18, Theorem 1]. Since \( \alpha(M, \varepsilon) \) depends continuously on \( \varepsilon \), we can conclude that (5.61) holds for any given \( M > 1.225 \) if we choose \( \varepsilon \) sufficiently small. This completes the proof of the corollary.
5.1 Proof of Theorem 1.2

The lower bound in (1.14) is a direct consequence of the Birman–Schwinger principle (2.2) together with Corollaries 4.4 and 5.5. As the upper bound was already discussed in Section 2.1, we have completed the proof of Theorem 1.2.

6 Proof of Lemma 3.2

As a first step we replace $G(q, \tau)$ by

$$\tilde{G}(q, \tau) = \frac{1}{L^2} \sum_k \left( \frac{1}{mk^2 - E_B} - \frac{\xi_\mu(k^2)}{M(q - k)^2 + k^2 + \tau} \right), \quad (6.1)$$

where

$$\xi_\mu(s) = \begin{cases} 0 & (s \leq \mu) \\ \frac{1}{2} \cos \left( \frac{\pi(s-\mu) \log \tilde{\nu}}{\mu} \right) + \frac{1}{2} & (\mu \leq s \leq \mu + \mu/ \log \tilde{\nu}) \\ 1 & (s \geq \mu + \mu/ \log \tilde{\nu}). \end{cases} \quad (6.2)$$

Compared to $G(q, \tau)$ we have replaced the characteristic function $\chi_{(\mu, \infty)}(k^2)$ with a smoother cutoff described by $\xi_\mu(k^2)$. The error for this can be controlled by a crude estimate like

$$|G(q, \tau) - \tilde{G}(q, \tau)| \leq \frac{1}{L^2} \sum_{k^2 \geq \mu} \frac{\chi_{(\mu, \mu + \mu/ \log \tilde{\nu})}(k^2)}{M(q - k)^2 + k^2 + \tau} \leq C \frac{\mu}{(\mu + \tau) \log \tilde{\nu}}, \quad (6.3)$$

which is easily justified by means of (3.7). Next we write $\tilde{G}(q, \tau) = L^{-2} \sum_k g(k)$ with

$$g(k) = \frac{1}{mk^2 + |E_B|} - \frac{\xi_\mu(k^2)}{M(q - k)^2 + k^2 + \tau}, \quad (6.4)$$

and apply Poisson’s summation formula (see e.g. [Gra14, Section 3.2]) to find

$$\tilde{G}(q, \tau) - \frac{1}{4\pi^2} \int_{\mathbb{R}^2} g(k) d^2k = \frac{1}{4\pi^2} \left( \frac{2\pi}{L} \right)^2 \sum_{k \in \mathbb{Z}^2} g(k) - (2\pi)\hat{g}(0) = \frac{1}{2\pi} \sum_{z \in L\mathbb{Z}^2 \setminus \{0\}} \hat{g}(z). \quad (6.5)$$

To compute $\hat{g}(0)$ we replace $\xi_\mu(s)$ again with $\chi_{(\mu, \infty)}(k^2)$ and estimate the difference

$$\left| (2\pi)\hat{g}(0) - \int_{\mathbb{R}^2} \left( \frac{1}{mk^2 + |E_B|} - \frac{\chi_{(\mu, \infty)}(k^2)}{M(q - k)^2 + k^2 + \tau} \right) d^2k \right| \leq C \frac{\mu}{(\mu + \tau) \log \tilde{\nu}}. \quad (6.6)$$
The integral can be evaluated explicitly,
\[
\int_{\mathbb{R}^2} \left( \frac{1}{mk^2 + |E_B|} - \frac{\chi(\mu, \infty)(k^2)}{M(q - k)^2 + k^2 + \tau} \right) d^2k \\
= \frac{\pi}{m} \log \left( \frac{1}{M+1s + \tau + m\mu} \right) + \frac{m}{m} \log \left( 1 - \frac{F(q^2, \tau)}{2} \right)
\] (6.7)

where
\[
F(s, \tau) = \frac{1}{M+1s + \tau + m\mu} \left( 1 - \sqrt{1 - \frac{4s \mu}{(1/Ms + \tau + m\mu)^2}} \right).
\] (6.8)

The last expression is not larger than \(1 + 1/M\) such that for \(M > 1\), we have
\[
\left| \log \left( 1 - \frac{F(q^2, \tau)}{2} \right) \right| \leq C.
\] (6.9)

It follows that
\[
\left| (2\pi\hat{g}(0) - \frac{\pi}{m} \log \left( \frac{1}{M+1s + \tau + m\mu} \right) \right| \leq C \left( 1 + \frac{\mu}{(\mu + \tau) \log \mu} \right).
\] (6.10)

Next we need to estimate the right side in (6.5). To this end, write \(g(k) = g_1(k) + g_2(k)\) with
\[
g_1(k) = \frac{1}{mk^2 + |E_B|}, \quad g_2(k) = \frac{\xi(\mu)(k^2)}{M(q - k)^2 + k^2 + \tau},
\] (6.11)

and use rotational symmetry, i.e. \(\hat{g}_i(z) = \hat{g}_i(|z|e_u), \ i = 1, 2\), where \(e_u\) denotes the first unit vector in the \((k_u, k_v)\) plane. We can then use integration by parts to compute the Fourier transform for \(z \neq 0\),
\[
\hat{g}_1(z) = m^{-1} \int_{-\infty}^{\infty} dk_u e^{ik_u |z|} \int_{-\infty}^{\infty} dk_v \frac{1}{k^2 + |E_B|/m} \\
= \frac{1}{m(|z|)^3} \int_{-\infty}^{\infty} dk_u \left( \frac{\partial^3}{\partial k_u^3} e^{ik_u |z|} \right) \int_{-\infty}^{\infty} dk_v \frac{1}{k^2 + |E_B|/m} \\
= \frac{1}{im(|z|)^3} \int_{-\infty}^{\infty} dk_u e^{ik_u |z|} \int_{-\infty}^{\infty} dk_v \frac{18k_u(k^2 + |E_B|/m) - 24k^3}{(k^2 + |E_B|/m)^4}.
\] (6.12)
Of the last expression we estimate the absolute value to get
\[
|\tilde{g}_1(z)| \leq \frac{1}{m|z|^3} \int d^2k \left( \frac{18|k|}{(k^2 + |E_B|/m)^3} + \frac{24|k|^3}{(k^2 + |E_B|/m)^4} \right) \leq \frac{C}{|z|^3|E_B|^{3/2}}. \tag{6.13}
\]

The bound for $|\tilde{g}_2(z)|$ works similarly but is slightly more cumbersome. We start again with
\[
\tilde{g}_2(z) = \frac{1}{im|z|^3|E_B|^{3/2}} \int_{-\infty}^{\infty} dk_u e^{ik_u |z|} \int_{-\infty}^{\infty} dk_v \frac{\partial^3}{\partial k^3_u} \left( \frac{\xi_\mu(k^2)|E_B|^{3/2}}{\frac{1}{M}(q - k)^2 + k^2 + \tau} \right) \tag{6.14}
\]
for which we need to compute the different derivatives. A straightforward computation shows
\[
\left| \frac{\partial^n}{\partial k^n_u} \xi_\mu(k^2) \right| \leq C \frac{(\log \tilde{\mu})^n}{\mu^{n/2}} \chi_{(\mu,\mu/\log \tilde{\mu})}(k^2), \quad n \in \{1, 2, 3\}. \tag{6.15}
\]

Abbreviating the denominator as $D(k) = \frac{1}{M}(q - k)^2 + k^2 + \tau$ it is not difficult to verify
\[
\chi_{(\mu,\mu/\log \tilde{\mu})}(k^2) \left| \frac{\partial}{\partial k_u} \frac{1}{D(k)} \right| \leq C \left( \frac{1}{\mu^{1/2}(k^2 + \tau)} + \frac{\mu^{1/2}}{(k^2 + \tau)^2} \right), \tag{6.16}
\]
\[
\chi_{(\mu,\mu/\log \tilde{\mu})}(k^2) \left| \frac{\partial^2}{\partial k^2_u} \frac{1}{D(k)} \right| \leq C \left( \frac{1}{(k^2 + \tau)^2} + \frac{\mu}{(k^2 + \tau)^3} \right), \tag{6.17}
\]
\[
\chi_{(\mu,\mu/\log \tilde{\mu})}(k^2) \left| \frac{\partial^3}{\partial k^3_u} \frac{1}{D(k)} \right| \leq C \left( \frac{1}{\mu^{1/2}(k^2 + \tau)^2} + \frac{\mu^{1/2}}{(k^2 + \tau)^3} + \frac{\mu^{3/2}}{(k^2 + \tau)^4} \right). \tag{6.18}
\]

To illustrate this for the first line, we compute
\[
\left| \frac{\partial}{\partial k_u} \frac{1}{D(k)} \right| = \left| \frac{1}{D(k)^2} \left( \frac{2M}{k_u - q_u} + 2k_u \right) \right| \leq C \left( \frac{|k - q|}{D(k)^2} + \frac{|k|}{D(k)^2} \right) \tag{6.19}
\]
and use $D(k) \geq \frac{1}{M}(k - q)^2$ and $k^2 \leq 2\mu$ in combination with a balanced Cauchy-Schwarz estimate. This leads to
\[
\left| \frac{\partial}{\partial k_u} \frac{1}{D(k)} \right| \leq C \left( \frac{1}{D(k)^{3/2}} + \frac{\sqrt{\mu}}{D(k)^2} \right) \leq C \left( \frac{1}{\sqrt{\mu} D(k)} + \frac{\sqrt{\mu}}{D(k)^2} \right) \tag{6.20}
\]
from which the bound in (6.16) follows by $D(k) \geq k^2 + \tau$. The other two lines are obtained in close analogy.

Summing up the different combinations we obtain
\[
\left| \frac{\partial^3}{\partial k^3_u} \left( \frac{\xi_\mu(k^2)|E_B|^{3/2}}{\frac{1}{M}(p - k)^2 + k^2 + \tau} \right) \right|
\]
\[ \leq \frac{C(\log \tilde{\mu})^3}{\tilde{\mu}^{3/2}} \left( \chi(\mu, \mu+\mu/\log \tilde{\mu})(k^2) \frac{1}{k^2 + \tau} + \chi(\mu, \infty)(k^2) \sum_{j=2}^{4} \frac{\mu^{j-1}}{(k^2 + \tau)^j} \right) \]  

(6.21)

by which we can estimate the integral

\[ \int_{\mathbb{R}^2} \left| \frac{\partial^3}{\partial k^3} \left( \frac{\xi_{\mu}(k^2)|E_B|^{3/2}}{1 + \sqrt{(q-k)^2 + \gamma^2 + \tau^2}} \right) \right| \, d^2k \leq C \left( 1 + \frac{\mu}{(\mu + \tau) \log \tilde{\mu}} \right)^3. \]  

(6.22)

Together with (6.13) and (6.14), this gives

\[ |\hat{g}(z)| \leq \frac{C}{|z|^3 |E_B|^{3/2}} \left( 1 + \frac{\mu}{(\mu + \tau) \log \tilde{\mu}} \right)^3. \]  

(6.23)

The remaining series can be bounded as

\[ \sum_{z \in \mathbb{L}^2, z \neq 0} |z|^{-3} |E_B|^{-3/2} = (L|E_B|^{1/2})^{-3} \sum_{z \in \mathbb{Z}^2, z \neq 0} |z|^{-3} \leq C \]  

(6.24)

because of \( L^2|E_B| \geq 1 \) and

\[ \sum_{z \in \mathbb{Z}^2, z \neq 0} \frac{1}{|z|^3} = \sum_{n,m \geq 1} \frac{4}{(n^2 + m^2)^{3/2}} + \sum_{n \geq 1} \frac{4}{n^3} \leq \int_{1}^{\infty} \left( \frac{8\pi}{s^2} + \frac{4}{s^3} \right) \, ds \leq C \]  

(6.25)

by the integral test of convergence.

We conclude that the absolute value of the right side in (6.5) is bounded from above by

\[ \frac{1}{2\pi} \sum_{z \in \mathbb{L}^2, z \neq 0} |\hat{g}(z)| \leq C \left( 1 + \frac{\mu}{(\mu + \tau) \log \tilde{\mu}} \right)^3. \]  

(6.26)

Hence the proof of the lemma is complete.

### A Replacing sums by integrals

For a short proof of the following lemma, see [LM19, Appendix B].

**Lemma A.1.** (a) Let \( f : [0, \infty) \to [0, \infty) \) be monotonically decreasing. Then,

\[ \left| \frac{1}{L^2} \sum_{k} f(k^2) - \frac{1}{2\pi} \int_{0}^{\infty} f(t^2) t \, dt \right| \leq \frac{2}{\pi L} \int_{0}^{\infty} f(t^2) \, dt + \frac{3f(0)}{L^2}. \]  

(A.1)
\[(b) \text{Let } m \geq 0 \text{ and } f : [m, \infty) \to [0, \infty) \text{ be monotonically decreasing. Then,}
\[
\left| \frac{1}{L^2} \sum_{k^2 \geq m} f(k^2) - \frac{1}{2\pi} \int_{\sqrt{m}}^{\infty} f(t^2) t \, dt \right| \leq \frac{2}{\pi L} \int_{\sqrt{m}}^{\infty} f(t^2) dt + \left( \frac{4\sqrt{m}}{\pi L} + \frac{6}{L^2} \right) f(m).
\] (A.2)

\section*{B \ Completing the proof of Lemma 4.2}

It remains to analyze the condition \(\mathcal{F}(T - \lambda, r) \geq 0\). For that we approximate \(\mathcal{F}(T - \lambda, r)\) by \(\mathcal{F}^{(n)}(T - \lambda, r)\) where the operator \(\mathcal{F}^{(n)}(T - \lambda, r)\) arises by replacing the function \(G(0, \tau)\) in (4.26) by
\[
G^{(n)}(0, \tau) = \frac{1}{L^2} \sum_{k^2 \leq n} \left( \frac{1}{k^2 - E_B} - \frac{\chi_{(\mu, \infty)}(k^2)}{k^2 + \tau} \right).
\] (B.1)

Note that \(G^{(n)}(0, \tau) \to G(0, \tau)\) as \(n \to \infty\) for every \(\tau > -\mu\). Thus, \(G^{(n)}(0, T - \lambda) \psi \to G\mu(0, T - \lambda) \psi\) as \(n \to \infty\) for every \(\psi \in D\) (recall that \(D \subset \mathcal{H}_{N(\mu)}\) is the set of all anti-symmetric product states, see (3.4)). Since \(D\) forms a total set of eigenstates of \(G(0, T - \lambda)\) on \(\mathcal{H}_{N(\mu)}\), its linear hull \(\text{lin}D \subseteq \mathcal{H}_{N(\mu)}\) is a domain of essential self-adjointness for this operator. Furthermore, \(G^{(n)}(0, T - \lambda) \geq G^{(n)}(0, -\mu - \epsilon_p(\mu, E_B))\) and thus, by the convergence of \(G^{(n)}(0, \tau)\) and the fact that \(G(0, -\mu - \epsilon_p(\mu)) \geq C \log \tilde{\mu}\), it follows that there is a \(c > 0\) such that \(G^{(n)}(0, T - \lambda) - r > c\) for \(n\) large enough. Hence as \(n \to \infty\), \((G^{(n)}(0, T - \lambda) - r)^{-1} \to (G(0, T - \lambda) - r)^{-1}\) and \(\mathcal{F}^{(n)}(T - \lambda, r) \to \mathcal{F}(T - \lambda, r)\) strongly.

Using the pull-through formula (2.6), we can write
\[
\mathcal{F}^{(n)}(T - \lambda, r) = T - \lambda - \frac{1}{L^2} \sum_{k^2 \leq \mu} (G^{(n)}(0, T - k^2 - \lambda) - r)^{-1}
\]
\[
+ \frac{1}{L^2} \sum_{k^2, l^2 \leq \mu} a_k(G^{(n)}(0, T - k^2 - l^2 - \lambda) - r)^{-1} a_l^* \]
(B.2)
on \(\mathcal{H}_{N(\mu)}\). Assuming that the last term in (B.2), which we call \(\mathcal{P}^{(n)}(T - \lambda, r)\) in the following, is a positive operator on \(\mathcal{H}_{N(\mu)}\), we obtain
\[
\mathcal{F}^{(n)}(T - \lambda, r) \geq E_0(\mu) - \lambda - \frac{1}{L^2} \sum_{k^2 \leq \mu} (G^{(n)}(0, E_0(\mu) - k^2 - \lambda) - r)^{-1},
\] (B.3)
since \(T \geq E_0(\mu)\) on \(\mathcal{H}_{N(\mu)}\). In view of (4.25), this completes the proof of Lemma 4.2.
It remains to show \( P^{(n)}(T - \lambda, r) \geq 0 \). For \( \psi \in \mathcal{H}_N(\mu) \),

\[
L^2 \langle \psi, P^{(n)}(T - \lambda, r) \psi \rangle = \int_0^\infty \sum_{k^2, l^2 \leq \mu} \langle \psi, a_k \exp(-t [G^{(n)}(0, T - k^2 - l^2 - \lambda) - r) a^*_k \psi \rangle dt
\]
\[
= \int_0^\infty \exp(-t[L^{-2} \sum_{p^2 \leq n} \frac{1}{p^2 - E_B} - r]) I^{(n)}(t) dt
\]

(B.4)

with

\[
I^{(n)}(t) = \sum_{k^2, l^2 \leq \mu} \langle \psi, a_k \prod_{\mu < q^2 \leq n} \exp(t(q^2 + T - k^2 - l^2 - \lambda)^{-1}) a^*_l \psi \rangle.
\]

(B.5)

We show that \( I^{(n)}(t) \geq 0 \) for all \( t \in [0, \infty) \) and \( n \in \mathbb{N} \). Note that the product in the definition of \( I^{(n)}(t) \) has only finitely many factors, because \( A_n = \{ q \in \mathbb{Z}^2 \mid \mu < q^2 \leq n \} \) is a finite set. We consider the exponential series and obtain

\[
I^{(n)}(t) = \sum_{m: A_n \rightarrow \mathbb{N}_0} \sum_{k^2, l^2 \leq \mu} \langle \psi, a_k \prod_{q \in A_n} \left( \sum_{m=0}^\infty \frac{t^m}{m!} \frac{1}{(q^2 + T - k^2 - l^2 - \lambda)^m} \right) a^*_l \psi \rangle.
\]

(B.6)

By the absolute convergence of the exponential series, we can rearrange the product of series to get

\[
I^{(n)}(t) = \sum_{m: A_n \rightarrow \mathbb{N}_0} \sum_{k^2, l^2 \leq \mu} \langle \psi, a_k \prod_{q \in A_n} \left( \frac{1}{m(q)!} \frac{1}{(q^2 + T - k^2 - l^2 - \lambda)^{m(q)}} \right) a^*_l \psi \rangle,
\]

(B.7)

where we sum over all \( \mathbb{N}_0 \)-valued functions \( m \) on the finite set \( A_n \). Note that the factor in parentheses indexed by \( q \) is equal to 1 if \( m(q) = 0 \). For all factors with \( m(q) \neq 0 \), we use the identity

\[
\frac{1}{a^\tau} = \frac{1}{c_\tau} \int_0^\infty ds \ e^{-s^1/\tau} \quad \text{with} \quad c_\tau = \int_0^\infty ds \ e^{-s^1/\tau}
\]

(B.8)

for \( a, \tau > 0 \) to rewrite each of the summands in the \( m \)-sum as

\[
\sum_{k^2, l^2 \leq \mu} \prod_{q \in A_n \atop m(q) \neq 0} \left( \frac{1}{m(q)!} \frac{1}{c_{m(q)}} \right) \int_0^\infty ds_q \langle \psi, a_k \prod_{p \in A_n \atop m(p) \neq 0} e^{-(p^2 + T - k^2 - l^2 - \lambda)s_p^{1/m(p)}} a^*_l \psi \rangle
\]
\[
= \prod_{q \in A_n, m(q) \neq 0} \left( \frac{t^{m(q)}}{m(q)! c_{m(q)}} \int_0^{\infty} ds_q \right) \left\| \sum_{k^2 \leq \mu} \prod_{p \in A_n, m(p) \neq 0} e^{-\frac{1}{2}(p^2 + T - 2k^2 - \lambda)s_p^{1/m(p)}} a_k^* \psi \right\|^2.
\] (B.9)

This yields \( I^{(n)}(t) \geq 0 \) and thus \( \langle \psi, \mathcal{P}^{(n)}(T - \lambda, r) \psi \rangle \geq 0 \) for all \( n \in \mathbb{N} \).

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