NONLINEAR COMMUTATORS FOR THE FRACTIONAL \( p \)-LAPLACIAN AND APPLICATIONS

ARMIN SCHIKORRA

Abstract. We prove a nonlocal, nonlinear commutator estimate concerning the transfer of derivatives onto testfunctions. For the fractional \( p \)-Laplace operator it implies that solutions to certain degenerate nonlocal equations are higher differentiable. Also, weak fractional \( p \)-harmonic functions which a priori are less regular than variational solutions are in fact classical. As an application we show that sequences of uniformly bounded \( \frac{1}{s} \)-harmonic maps converge strongly outside at most finitely many points.

1. Introduction

The fractional \( p \)-Laplacian of order \( s \in (0,1) \) on a domain \( \Omega \subset \mathbb{R}^n \), \((-\Delta)^s_{\Omega}u \) is a distribution given by

\[
(-\Delta)^s_{\Omega}u[\varphi] := \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) \left( \varphi(x) - \varphi(y) \right)}{|x-y|^{n+sp}} \, dx \, dy
\]

for \( \varphi \in C^\infty_c(\Omega) \). It appears as the first variation of the \( \dot{W}^{s,p} \)-Sobolev norm

\[
[u]_{W^{s,p}(\Omega)}^p := \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^p}{|x-y|^{n+sp}} \, dx \, dy.
\]

In this sense it is related to the classical \( p \)-Laplacian

\[
\Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u)
\]

which appears as first variation of the \( \dot{W}^{1,p} \)-Sobolev norm \( \|\nabla u\|_p^p \).

If \( p = 2 \) the fractional \( p \)-Laplacian on \( \mathbb{R}^n \) becomes the usual fractional Laplace operator

\[
(-\Delta)^s f = \mathcal{F}^{-1}(c |\xi|^{2s} \mathcal{F} f),
\]

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where $\mathcal{F}$ and $\mathcal{F}^{-1}$ denote the Fourier transform and its inverse, respectively. As a distribution

$$(-\Delta)^s f[\varphi] = \int_{\mathbb{R}^n} (-\Delta)^s f \varphi.$$  

For an overview on the fractional Laplacian and fractional Sobolev spaces we refer to, e.g., [11, 4].

Due to the degeneracy for $p \neq 2$, regularity theory for equations involving the $p$-Laplacian is quite delicate, for example $p$-harmonic functions may not be $C^2$. The fractional $p$-Laplacian has recently received quite some interest, for example we refer to [2, 9, 10, 21, 18, 16, 13, 17, 23].

Higher regularity is one interesting and very challenging question where only very partial results are known, e.g. in [2] they obtain for $s \approx 1$ estimates in $C^{1,\alpha}$.

Our first result is a nonlinear commutator estimate for the fractional $p$-Laplacian. It measures how and at what price one can “transfer” derivatives to the testfunction. In the linear case $p = 2$ this is just integration by parts: Let $c$ be the constant depending on $s$ and $\varepsilon$ so that $(-\Delta)^s + \varepsilon = c(-\Delta)^{\varepsilon} \circ (-\Delta)^s$. Then for any testfunction $\varphi$,

$$(-\Delta)^{s+\varepsilon} u[\varphi] = c(-\Delta)^s u[(-\Delta)^\varepsilon \varphi]$$

In the nonlinear case $p \neq 2$ (we shall restrict our attention for technical simplicity to $p \geq 2$) this is not true anymore. Instead we have

**Theorem 1.1.** Let $s \in (0, 1)$, $p \in [2, \infty)$ and $\varepsilon \in [0, 1 - s)$. Take $B \subset \mathbb{R}^n$ a ball or all of $\mathbb{R}^n$. Let $u \in W^{s,p}(B)$ and $\varphi \in C^\infty_c(B)$. For a certain constant $c$ depending on $s, \varepsilon, p$, denote the nonlinear commutator

$$R(u, \varphi) := (-\Delta)^{s+\varepsilon} u[\varphi] - c(-\Delta)^s u[(-\Delta)^\varepsilon \varphi].$$

Then we have the estimate

$$|R(u, \varphi)| \leq C \left[ u_{W^{s+\varepsilon,p}(B)} \right] \left[ \varphi_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \right].$$

The fact that the $\varepsilon$ appears in the estimate of $R(u, \varphi)$ is the main point in Theorem 1.1. It relies on a logarithmic potential estimate:

**Lemma 1.2.** Let for $\alpha, \beta \in (0, n)$,

$$k(x, y, z) = \left( |x - z|^{\alpha - n} \log \frac{|x - z|}{|x - y|} - |y - z|^{\alpha - n} \log \frac{|y - z|}{|x - y|} \right).$$
Let $\gamma \in (0,1)$, $p \in (1,\infty)$ and assume that $s := \gamma + \beta - \alpha \in (0,1)$. We consider the following semi-norm expression for $\varphi \in C_c^\infty(\mathbb{R}^n)$

$$A(\varphi) := \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} k(x, y, z) (-\Delta)^{\beta/2} \varphi(z) \, dz \right)^p \frac{dx \, dy}{|x - y|^{n+\gamma p}} \right)^{\frac{1}{p}}.$$

We have

$$A(\varphi) \leq C[\varphi]_{W^{s,p}(\mathbb{R}^n)}.$$

The additional factor $\varepsilon$ in Theorem 1.1 facilitates estimates “close to the differential order $s$”. More precisely

**Theorem 1.3.** Let $s \in (0,1)$, $p \in [2,\infty)$, and a domain $\Omega \subset \mathbb{R}^n$, and $u \in W^{s,p}(\Omega)$ be a solution to $(-\Delta)^s u = f$, i.e.

$$\int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{n+sp}} \, dx \, dy = f[\varphi]$$

for all $\varphi \in C_c^\infty(\Omega)$. Then there is an $\varepsilon_0 > 0$ only depending on $s$, $p$, and $\Omega$, so that for $\varepsilon \in (0,\varepsilon_0)$ the following holds: If $f \in (W^{s-\varepsilon(p-1),p}(\Omega))^*$ then $u \in W^{s+\varepsilon,p}_{loc}(\Omega)$.

More precisely, we have for any $\Omega_1 \Subset \Omega$ a constant $C = C(\Omega_1, \Omega, s, p)$ so that

$$[u]_{W^{s+\varepsilon,p}_{loc}(\Omega_1)} \leq C \|f\|_{(W^{s-\varepsilon(p-1),p}(\Omega))^*} + C[u]_{W^{s,p}(\Omega)}.$$

Also, by Sobolev imbedding, the higher differentiability $W^{s+\varepsilon,p}_{loc}$ implies higher integrability i.e. $W^{s,p}_{loc}$-estimates.

In the regime $p = 2$, a higher differentiability result similar to Theorem 1.3 was proven by Kuusi, Mingione, and Sire [18]. It seems also possible to extend their approach to the case $p > 2$. Their argument is based on a generalization of Gehring’s Lemma and it is also valid for nonlinear versions, see [16]. Our method is similarly robust. Indeed one can show

**Theorem 1.4.** Let $s \in (0,1)$, $p \in [2,\infty)$, and a domain $\Omega \subset \mathbb{R}^n$. Let $\phi : \mathbb{R} \to \mathbb{R}$ and $K(x,y)$ be a measurable kernel so that for some $C > 1$,

$$|\phi(t)| \leq C|t|^{p-1}, \quad \phi(t)t \geq |t|^p \quad \forall t \in \mathbb{R},$$

and

$$C^{-1}|x - y|^{-n-sp} \leq K(x,y) \leq C|x - y|^{-n-sp}.$$
We consider for \( u \in W^{s,p}(\Omega) \), the distribution \( \mathcal{L}_{\phi,K,\Omega}(u) \)

\[
\mathcal{L}_{\phi,K,\Omega}(u)[\varphi] := \int_{\Omega} \int_{\Omega} K(x,y) \phi(u(x) - u(y)) \ (\varphi(x) - \varphi(y)) \ dx \ dy
\]

Then the conclusions of Theorem 1.3 still hold if the fractional \( p \)-Laplace \((-\Delta)^s_{p,\Omega}\) is replaced with \( \mathcal{L}_{\phi,K,\Omega} \).

Since the arguments for Theorem 1.4 follow closely the proof of Theorem 1.3, we leave this as an exercise to the interested reader.

There is also a reminiscent result to Theorem 1.3 the usual \( p \)-Laplace: A nonlinear potential estimate due to Iwaniec [14]. It implies that for \( u \) with \( \text{supp}\ u \subset \Omega \) there are maps \( v, R \), so that

\[
|\nabla u|^{q} \nabla u = \nabla v + R,
\]

with \( ||\nabla v||^{\frac{q}{q+\epsilon},\Omega} \lesssim ||\nabla u||^{1+\epsilon}_{q,\Omega} \) for all \( q \) and

\[
||R||^{\frac{q}{q+\epsilon},\Omega} \lesssim \epsilon ||\nabla u||^{1+\epsilon}_{p+\epsilon,\Omega}.
\]

In this situation, the additional \( \epsilon \) in the last estimate allows for estimates “close to the integrability order \( p \)”. Indeed

\[
||\nabla u||^{p+\epsilon}_{p+\epsilon,\Omega} = \int_{\Omega} |\nabla u|^{p-2} \nabla u \nabla v + \int_{\Omega} |\nabla u|^{p-2} \nabla u R,
\]

and thus,

\[
||\nabla u||^{p+\epsilon}_{p+\epsilon,\Omega} \lesssim |\Delta_{p} u[v]| + \epsilon ||\nabla u||^{p-1}_{p+\epsilon,\Omega} ||\nabla u||^{1+\epsilon}_{p+\epsilon,\Omega}.
\]

In particular, if \( \epsilon \) is small enough and \( \Delta_{p} u \) is in \((W^{1,\frac{p+\epsilon}{p+\epsilon}}(\Omega))^{*}\), then \( u \in W^{1,p+\epsilon}(\Omega) \).

The commutator estimate in Theorem 1.1 also allows to estimate very weak solutions - i.e. solutions whose initial regularity assumptions are below the variationally natural regularity:

In the local regime, the distributional \( p \)-Laplacian \( \Delta_{p} u[\varphi] \) is well defined for \( \varphi \in C_{c}^{\infty}(\Omega) \) whenever \( u \in W^{1,p-1}_{\text{loc}}(\Omega) \). The variationally natural regularity assumption is however \( W^{1,p}_{\text{loc}} \), since \( \Delta_{p} \) appears as first variation of \( ||\nabla u||^{p}_{p,\Omega} \). For the \( p \)-Laplacian, Iwaniec and Sbordone [15] showed that some weak \( p \)-harmonic functions are in fact classical variational solutions:

**Theorem 1.5** (Iwaniec-Sbordone). For any \( p \in (1,\infty) \), \( \Omega \subset \mathbb{R}^{n} \), there are exponents \( 1 < r_1 < p < r_2 < \infty \) so that every (weakly) \( p \)-harmonic map,

\[
\Delta_{p} u = 0,
\]
satisfying \( u \in W^{1,r_1}_{\text{loc}}(\Omega) \) indeed belongs to \( W^{1,r_2}_{\text{loc}}(\Omega) \).

Again, while the \( p \)-Laplace improves its solution’s integrability, the fractional \( p \)-Laplace improves its solution’s differentiability. The distributional fractional \( p \)-Laplace \((-\Delta)^s_{p,\Omega} u[\varphi]\) is well defined for \( \varphi \in C_c^\infty(\Omega) \) whenever \( u \in W^{q,p-1}(\Omega) \) for any \( q > 0 \) with \( q \geq \left( \frac{2p-1}{p-1} \right)_+ \). We have

**Theorem 1.6.** For any \( s \in (0,1) \) \( p \in (2,\infty) \), \( \Omega \subset \mathbb{R}^n \), there are exponents \( 1 < r_1 < p < r_2 < \infty \) and \( t_1 < s < t_2 \) so that every (weakly) \( s\)-\( p \)-harmonic map,

\[
(-\Delta)^s_{p,\Omega} u = 0,
\]
satisfying \( u \in W^{t_1,r_1}(\Omega) \) indeed belongs to \( W^{t_2,r_2}_{\text{loc}}(\Omega) \).

The arguments for Theorem 1.6 are quite similar to the ones in Theorem 1.3, and we shall skip them.

Let us state an important application of Theorem 1.3: It is concerning degenerate fractional harmonic maps into spheres \( S^N \subset \mathbb{R}^{N+1} \): In [21] we proved that for \( s \in (0,1) \) critical points of the energy

\[
\mathcal{E}_s(u) := \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^2}{|x-y|^{n+\frac{s}{2}}} \, dx \, dy, \quad u : \Omega \subset \mathbb{R}^n \to S^N
\]

are Hölder continuous. Indeed, together with Theorem 1.3 the estimates in [21] imply a sharper result

**Theorem 1.7** (\( \varepsilon \)-regularity for fractional harmonic maps). For any open set \( \Omega \subset \mathbb{R}^n \) there is a \( \delta > 0 \) so that for any \( \Lambda > 0 \) there exists \( \varepsilon > 0 \) and the following holds: Let \( u \in W^{s,\frac{2}{s}}(\Omega,S^N) \) with

\[
[u]_{W^{s,\frac{2}{s}}(\Omega)} \leq \Lambda
\]

be a critical point of \( \mathcal{E}_s(u) \), i.e.

\[
\frac{d}{dt} \bigg|_{t=0} \mathcal{E}_s \left( \frac{u + t\varphi}{|u + t\varphi|} \right) = 0 \quad \forall \varphi \in C_c^\infty(\Omega,\mathbb{R}^N).
\]

If on a ball \( 2B \subset \Omega \) we have

\[
[u]_{W^{s,\frac{2}{s}}(2B)} \leq \varepsilon,
\]

then on the ball \( B \) (the ball concentric to \( 2B \) with half the radius),

\[
[u]_{W^{s+\delta,\frac{2}{s+\delta}}(B)} \leq C_{\Lambda,B}.
\]
This kind of $\varepsilon$-regularity estimate is crucial for compactness and bubble analysis for fractional harmonic maps. Da Lio obtained quantization results [6] for $n = 1$ and $s = \frac{1}{2}$. With the help of Theorem 1.7 one can extend her compactness estimates to all $s \in (0, 1)$, $n \in \mathbb{N}$. More precisely, we have the following result extending the first part of [6, Theorem 1.1].

**Theorem 1.8.** Let $u_k \in \dot{W}^{s, \frac{n}{2}}(\mathbb{R}^n, S^{N-1})$ be a sequence of $(s, \frac{n}{2})$-harmonic maps in the sense of (1.2) such that

$$[u_k]_{W^{s, \frac{n}{2}}(\mathbb{R}^n, S^{N-1})} \leq C.$$

Then there is $u_\infty \in \dot{W}^{s, \frac{n}{2}}(\mathbb{R}^n, S^{N-1})$ and a possibly empty set \{\(\alpha_1, \ldots, \alpha_l\)\} such that up to a subsequence we have strong convergence away from \{\(\alpha_1, \ldots, \alpha_l\)\}, that is

$$u_k \xrightarrow{k \to \infty} u_\infty \quad \text{in} \quad W^{s, \frac{n}{2}}_{loc}(\mathbb{R}^n \setminus \{\alpha_1, \ldots, \alpha_l\}).$$

A more precise analysis of compactness and the formation of bubbles will be part of a future work.

2. **Outline and Notation**

In Section 3 we will prove the commutator estimate, Theorem 1.1. Roughly speaking, we compute the kernel $\kappa_\varepsilon(x, y, z)$ of the commutator and show that its derivative in $\varepsilon$ (which gives a logarithmic potential) induces a bounded operator. The latter estimate is contained in Lemma 1.2 which we shall prove via Littlewood-Paley theory in Section 4.

Based on Theorem 1.1 we will then proceed in Section 5 with the proof of Theorem 1.3. Finally, the consequences of this analysis, i.e. higher differentiability result for $p$-fractional harmonic maps is sketched in Section 6, and the proof of Theorem 1.8 in Session 7. In the appendix we record a few necessary tools used throughout the proofs.

We try to keep the notation as simple as possible. For a ball $B$, $\lambda B$ denotes the concentric ball with $\lambda$-times the radius. With

$$(u)_B := |B|^{-1} \int_B u$$

we denote the mean value.
The dual norm of the $p$-Laplacian is denoted as
\[ \|(-\Delta)^s_{p,\Omega} u\|_{(W^{1,p}(\Omega))^*}, \equiv \sup_{\varphi} \|(-\Delta)^s_{p,\Omega} u[\varphi]\] where the supremum is taken over $\varphi \in C_c^\infty(\Omega)$ with $[\varphi]_{W^{1,p}(\mathbb{R}^n)} \leq 1$.

We already defined the fractional Laplacian $(-\Delta)^{\frac{s}{2}}$. Its inverse $I^s$ is the Riesz potential, which for some constant $c \in \mathbb{R}$ can be written as
\[ I^s g(x) = c \int_{\mathbb{R}^n} |x - z|^{s-n} g(z) \, dz. \] (3.1)

In the estimates, the constants can change from line to line. Whenever we deem the constant unimportant to the argument, we will drop it, writing $A \lesssim B$ if $A \leq C \cdot B$ for some constant $C > 0$. Similarly will use $A \approx B$ whenever $A$ and $B$ are comparable.

3. The commutator estimate: Proof of Theorem 1.1

Proof. Recall that for $t \in (0, n)$ there is a constant $c \in \mathbb{R}$ so that for any $\varphi \in C_c^\infty(\mathbb{R}^n)$,
\[ c \int_{\mathbb{R}^n} |x - z|^{t-n} (-\Delta)^{\frac{t}{2}} \varphi(z) \, dz = I^t (-\Delta)^{\frac{t}{2}} \varphi(x) = \varphi(x). \] (3.1)

We write
\[ (-\Delta)^{s+\varepsilon}_{p,B} u[\varphi] = \int \int \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) \langle \varphi(x) - \varphi(y) \rangle}{|x - y|^{n+sp}} \, dx \, dy \]
\[ = \int \int \int \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) \langle |x - z|^{t+\varepsilon p-n} - |x - z|^{t+\varepsilon p-n} \rangle}{|x - y|^{n+sp}} \, dx \, dy \, (-\Delta)^{\frac{t+\varepsilon p}{2}} \varphi(z) \, dz \]
\[ + \int \int \int \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) \kappa_\varepsilon(x, y, z)}{|x - y|^{n+sp}} \, dx \, dy \, (-\Delta)^{\frac{t+\varepsilon p}{2}} \varphi(z) \, dz \]

with
\[ \kappa_\varepsilon(x, y, z) := \left( \frac{|x - z|^{t+\varepsilon p-n} - |y - z|^{t+\varepsilon p-n}}{|x - y|^{\varepsilon p}} \right) - (|x - z|^{t-n} - |x - y|^{t-n}). \]

Using again (3.1) this reads as
\[ R(u, \varphi) := (-\Delta)^{s+\varepsilon}_{p,B} u[\varphi] - c(-\Delta)^s_{p,B} u[(-\Delta)^{\frac{s}{2}} \varphi] \]
\[ \int_{\mathbb{R}^n} \int_{B} \int_{B} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))\kappa_{\varepsilon}(x, y, z)}{|x - y|^{n+sp}} \, dx \, dy \left( -\Delta \right)^{\frac{t+sp}{2}} \varphi(z) \, dz. \]

Since \( \kappa_0(x, y, z) = 0 \) for almost all \( x, y, z \in \mathbb{R}^n \),

\[ \kappa_{\varepsilon}(x, y, z) = \int_{0}^{\varepsilon} \frac{d}{d\delta} \kappa_{\delta}(x, y, z) \, d\delta. \]

We thus set

\[ k_{\delta}(x, y, z) := |x - y|^{\delta p} \frac{d}{d\delta} \kappa_{\delta}(x, y, z) \]

and arrive at \( R(u, \varphi) \) being equal to

\[ \int_{0}^{\varepsilon} \int_{B} \int_{B} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))}{|x - y|^{(s+\varepsilon)(p-1)}} \left( \int_{\mathbb{R}^n} \kappa_{\delta}(x, y, z) \left( -\Delta \right)^{\frac{t+sp}{2}} \varphi(z) \, dz \right) \frac{1}{|x - y|^{s+\varepsilon - (\varepsilon - \delta)p}} \, dx \, dy \, d\delta. \]

With Hölder inequality we get the upper bound for \( |R(u, \varphi)| \)

\[ \varepsilon [u]_{W^{s+\varepsilon, p}(B)}^{p-1} \sup_{\delta \in (0, \varepsilon)} \left( \int_{B} \int_{B} \left( \int_{\mathbb{R}^n} \kappa_{\delta}(x, y, z) \left( -\Delta \right)^{\frac{t+sp}{2}} \varphi(z) \, dz \right)^{\frac{p}{2}} \frac{1}{|x - y|^{s+\varepsilon - (\varepsilon - \delta)p}} \right)^{\frac{1}{p}}. \]

This falls into the realm of Lemma 1.2 for

\[ \alpha := t + \delta p, \quad \beta := t + \varepsilon p, \quad \gamma := s + \varepsilon - (\varepsilon - \delta)p, \quad \gamma + \beta - \alpha = s + \varepsilon. \]

This concludes the proof. \( \square \)

4. **Logarithmic potential estimate: Proof of Lemma 1.2**

For the proof of Lemma 1.2 we will use the Littlewood-Paley decomposition: We refer to the Triebel monographs, e.g. [22] and [12] for a complete picture of this tool. We will only need few properties:

For a tempered distribution \( f \) we define \( f_j \) to be the Littlewood-Paley projections \( f_j := P_j f \), where

\[ P_j f(x) := \int_{\mathbb{R}^n} 2^{jn} p(2^j(x - z)) f(z) \, dz. \]

Here, \( p \) is a Schwartz function, and it can be chosen in a way such that

\[ \sum_{j \in \mathbb{Z}} f_j = f \quad \text{for all } f \in S'. \]
For any $j \in \mathbb{Z}$ we have the estimate for Riesz potentials and derivatives (cf. (2.1))

\[(4.2) \quad \|I^s(-\Delta)^{\frac{t}{2}} f_j\|_p \lesssim \sum_{i=j-1}^{j+1} 2^{j(t-s)} \|f_i\|_p\]

The homogeneous semi-norm for the Triebel space $\dot{F}^s_{p,p} = \dot{B}^s_{p,p}$ is

\[(4.3) \quad \|f\|_{\dot{F}^s_{p,p}} := \left( \sum_{j \in \mathbb{Z}} 2^{jsp} \|f_j\|_p^p \right)^{\frac{1}{p}}.\]

Crucially to us, the Triebel spaces are equivalent to Sobolev spaces: For $s \in (0,1)$ we have the identification

\[(4.4) \quad \|f\|_{\dot{F}^s_{p,p}} \approx [f]_{W^{s,p}(\mathbb{R}^n)}.\]

**Proof of Lemma 1.2.** For $k \in \mathbb{Z}$, we use the annular cutoff function

\[\chi_{|y| \approx 2^{-k}} := \chi_{B_{2^{-k}}(0) \setminus B_{2^{-k-1}}(0)}(y).\]

With this and (4.1), setting

\[T \varphi(x,y) := \int_{\mathbb{R}^n} k(x,y,z) (-\Delta)^{\frac{t}{2}} \varphi(z) \, dz,\]

we decompose

\[A(\varphi)^p \lesssim \sum_{k \in \mathbb{Z}, j \in \mathbb{Z}} I_{j,k},\]

where

\[I_{j,k} := \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{|x-y| \approx 2^{-k}} |T \varphi(x,y)|^{p-1} |T \varphi_j(x,y)| \frac{dx \, dy}{|x-y|^{n+\gamma p}}.\]

Set

\[a_k := \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{|x-y| \approx 2^{-k}} |T \varphi(x,y)|^p \frac{dx \, dy}{|x-y|^{n+\gamma p}} \right)^{\frac{1}{p}},\]

and

\[b_j := 2^{j(\gamma + \beta - \alpha)} \|\varphi_j\|_p.\]

Note that with (4.3) and (4.4)

\[(4.5) \quad \left( \sum_{k \in \mathbb{Z}} a_k^p \right)^{\frac{1}{p}} \approx A(\varphi) \quad \text{and} \quad \left( \sum_{j \in \mathbb{Z}} b_j^p \right)^{\frac{1}{p}} \approx \|\varphi\|_{\dot{F}^s_{p,p}} \approx [\varphi]_{W^{s,p}(\mathbb{R}^n)}.\]
Then with Hölder inequality,
\[
I_{j,k} \lesssim a_k^{p-1} \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \chi_{|x-y|\approx 2^{-k}} |T\varphi_j(x,y)| \, dx \, dy \right)^{\frac{1}{p}}
\]
\[
\quad =: a_k^{p-1} \tilde{I}_{j,k}.
\]

Now we have to possibilities of estimating \( \tilde{I}_{j,k} \):

**Firstly**, for any small \( \sigma \in (0, \alpha) \) we can employ the estimate
\[
|\log \frac{|x-z|}{|x-y|}\| \lesssim \frac{|x-y|^{\sigma}}{|x-z|^{\sigma}} + \frac{|x-z|^{\sigma}}{|x-y|^{\sigma}},
\]
and have an estimate with Riesz potentials \((2.1)\)
\[
\int_{\mathbb{R}^n} |x-z|^{\alpha-n} \log \frac{|x-z|}{|x-y|} |(-\Delta)^{\frac{\sigma}{2}} \varphi_j(z)| \, dz
\]
\[
\lesssim |x-y|^{-\sigma} I^{\alpha+\sigma} |(-\Delta)^{\frac{\sigma}{2}} \varphi_j|(x) + |x-y|^{\sigma} I^{\alpha-\sigma} |(-\Delta)^{\frac{\sigma}{2}} \varphi_j|(x).
\]
Having in mind \((1.2)\) we obtain the estimate
\[
\tilde{I}_{j,k} \lesssim 2^{k(\frac{\alpha+n}{p})} 2^{k\sigma} 2^{-k^{\frac{\sigma}{2}}} |I^{\alpha+\sigma} |(-\Delta)^{\frac{\sigma}{2}} \varphi_j||_p + 2^{k(\frac{\alpha+n}{p})} 2^{-k\sigma} 2^{-k^{\frac{\sigma}{2}}} |I^{\alpha-\sigma} |(-\Delta)^{\frac{\sigma}{2}} \varphi_j||_p
\]
\[
\lesssim 2^{(k-j)(\gamma+\sigma)} (b_{j-1} + b_j + b_{j+1}) + 2^{(k-j)(\gamma-\sigma)} (b_{j-1} + b_j + b_{j+1}).
\]
This is our first estimate:
\[
(4.6) \quad \tilde{I}_{j,k} \lesssim 2^{(k-j)(\gamma-\sigma)} (2^{2\sigma(k-j)} + 1) (b_{j-1} + b_j + b_{j+1}).
\]

**Secondly**, by a substitution we can write
\[
T\varphi_j(x,y) = \int_{\mathbb{R}^n} |z|^{\alpha-n} \log \frac{|z|}{|x-y|} \left( (-\Delta)^{\frac{\sigma}{2}} \varphi_j(z + x) - (-\Delta)^{\frac{\sigma}{2}} \varphi_j(z + y) \right) \, dz.
\]
We use now \( |f(x) - f(y)| \lesssim |x-y| (|\mathcal{M}| \nabla f|(x) + |\mathcal{M}| \nabla f|(y)) \), where \( \mathcal{M} \) is the Hardy-Littlewood maximal function. Then, again for any \( \sigma > 0 \),
\[
|T\varphi_j(x,y)|
\]
\[
\lesssim |x-y| |z|^{\alpha-n} \left| \log \frac{|z|}{|x-y|} \right| |\mathcal{M}| (-\Delta)^{\frac{\sigma}{2}} \nabla \varphi_j|(z + x) \, dz
\]
\[
+ |x-y| |z|^{\alpha-n} \left| \log \frac{|z|}{|x-y|} \right| |\mathcal{M}| (-\Delta)^{\frac{\sigma}{2}} \nabla \varphi_j|(z + y) \, dz
\]
\[
\lesssim |x-y|^{1-\sigma} I^{\alpha+\sigma} |(-\Delta)^{\frac{\sigma}{2}} \varphi_j|(x)
\]
\[
+ |x-y|^{1-\sigma} I^{\alpha+\sigma} |(-\Delta)^{\frac{\sigma}{2}} \varphi_j|(y)
\]
\[
+ |x-y|^{1+\sigma} I^{\alpha-\sigma} |(-\Delta)^{\frac{\sigma}{2}} \varphi_j|(x)
\]
\[
+ |x-y|^{1+\sigma} I^{\alpha-\sigma} |(-\Delta)^{\frac{\sigma}{2}} \varphi_j|(y)
\]
Consequently, our second estimate is
\[ \tilde{I}_{k,j} \lesssim 2^{k(\gamma-1+\sigma)} \| I_{\alpha+\sigma} \mathcal{M} (-\Delta)^{\frac{\sigma}{2}} \nabla \phi_j \|_{L^p} + 2^{k(\gamma-1-\sigma)} \| I_{\alpha-\sigma} \mathcal{M} (-\Delta)^{\frac{\sigma}{2}} \nabla \phi_j \|_{L^p} \]
\[ \lesssim 2^{k(\gamma-1+\sigma)} 2^{j(-\alpha+\sigma+\beta+1)} \| \phi_j \|_{L^p} + 2^{k(\gamma-1-\sigma)} 2^{j(-\alpha+\sigma+\beta+1)} \| \phi_j \|_{L^p}. \]
Together with (4.6) we thus have
\[ \tilde{I}_{k,j} \lesssim \min \{ 2^{(k-j)(\gamma-\sigma)} (2^{2\sigma(k-j)+1}), 2^{(j-k)(1-\gamma-\sigma)} (1+2^{(j-k)(2\sigma)}) \} (b_{j-1} + b_j + b_{j+1}). \]
In particular, since \( \gamma \in (0, 1) \) pick any \( 0 < \sigma < \min \{ \gamma, 1-\gamma \} \) – which, as we shall see in a moment, makes the following sums convergent:
\[
A(\varphi)^p \lesssim \sum_{j \in \mathbb{Z}} \sum_{k=j+1}^{\infty} 2^{(j-k)(1-\gamma-\sigma)} (b_{j-1} + b_j + b_{j+1}) a_j^{p-1}
+ \sum_{j \in \mathbb{Z}} \sum_{k=-\infty}^{j-1} 2^{(k-j)(\gamma-\sigma)} (b_{j-1} + b_j + b_{j+1}) a_j^{p-1}
+ \sum_{j \in \mathbb{Z}} (b_{j-1} + b_j + b_{j+1}) a_j^{p-1}
=: I + II + III.
\]
With Hölder inequality and (4.5),
\[ III \lesssim (\sum_{j \in \mathbb{Z}} b_j^p)^{\frac{1}{p}} (\sum_{j \in \mathbb{Z}} a_j^p)^{\frac{1}{p-1}} = A(\varphi)^{p-1} [\varphi]_{W^{\gamma,p}(\mathbb{R}^n)}. \]
As for \( I \), for any \( \varepsilon > 0 \),
\[
I = \sum_{j \in \mathbb{Z}} \sum_{k=j}^{\infty} 2^{(j-k)(1-\gamma-\sigma)} b_j a_k^{p-1}
\lesssim \sum_{j \in \mathbb{Z}} \sum_{k=j}^{\infty} 2^{(j-k)(1-\gamma-\sigma)} (\varepsilon b_j^p + \varepsilon^{-p} a_k^p)
= C_{1-\gamma-\sigma} \varepsilon^p \sum_{j \in \mathbb{Z}} b_j^p + \varepsilon^{-p} \sum_{j \in \mathbb{Z}} \sum_{k=j}^{\infty} 2^{(j-k)(1-\gamma-\sigma)} a_k^p
= C_{1-\gamma-\sigma} \varepsilon^p \sum_{j \in \mathbb{Z}} b_j^p + \varepsilon^{-p} \sum_{k \in \mathbb{Z}} \sum_{j=-\infty}^{k} 2^{(j-k)(1-\gamma-\sigma)} a_k^p
= C_{1-\gamma-\sigma} \varepsilon^p \sum_{j \in \mathbb{Z}} b_j^p + \varepsilon^{-p} C_{1-\gamma-\sigma} \sum_{k \in \mathbb{Z}} a_k^p
\approx \varepsilon^p [\varphi]_{W^{\gamma,p}(\mathbb{R}^n)}^p + \varepsilon^{-p} C_{1-\gamma-\sigma} A(\varphi)^p.
The same works for $II$:

$$II = \sum_{j \in \mathbb{Z}} \sum_{k = -\infty}^{j-1} 2^{(k-j)(\gamma-\sigma)} b_j a_k^{p-1} \lesssim \varepsilon^p [\varphi]_{W^{s,p}(\mathbb{R}^n)}^p + \varepsilon^{-p'} C_{1-\gamma-\sigma} A(\varphi)^p$$

Together,

$$I + II \lesssim \varepsilon^p [\varphi]_{W^{s,p}(\mathbb{R}^n)}^p + \varepsilon^{-p'} C_{1-\gamma-\sigma} A(\varphi)^p,$$

which holds for any $\varepsilon > 0$. Pick

$$\varepsilon := \left[ \varphi \right]_{W^{s,p}(\mathbb{R}^n)}^{-\frac{1}{p'}} A(\varphi)^{\frac{1}{p'}}.$$

Then

$$A(\varphi)^p \leq I + II + III \lesssim A(\varphi)^p \left[ \varphi \right]_{W^{s,p}(\mathbb{R}^n)}^{p-1}.$$

We conclude dividing both sides by $A(\varphi)^{p-1}$.

---

5. Higher Differentiability: Proof of Theorem 1.3

In view of Lemma 4.1 we can assume w.l.o.g. that $\Omega$ is a bounded open set, and that the support of $u$ is strictly contained in some open set $\Omega_1 \subset \Omega$. Then Theorem 1.3 follows from

**Lemma 5.1.** Let $\Omega_1 \subset \Omega$ two open, bounded sets, $s \in (0,1)$, $p \in [2, \infty)$. Then there exists an $\varepsilon_0 > 0$ so that for any $\varepsilon \in (0, \varepsilon_0)$,

$$[u]_{W^{s+\varepsilon,p}(\Omega)}^{p-1} \lesssim [u]_{W^{s,p}(\Omega)}^{p-1} + \|(-\Delta)_{p,\Omega} u\|_{W^{s-(p-1),p}(\Omega)}^s.$$

**Proof.** We can find finitely many balls $(B_k)_{k=1}^K \subset \Omega$ so that $\bigcup_{k=1}^K B_k \supset \Omega_1$. We denote with $10B_k$ the concentric balls with ten times the radius, and may assume $\bigcup_{k=1}^N 10B_k \subset \Omega$.

Denote

$$\Gamma_s := [u]_{W^{s,p}(\Omega)}^{p}, \quad \Gamma_{s+\varepsilon} := [u]_{W^{s+\varepsilon,p}(\Omega)}^{p}.$$

We then have

$$\Gamma_{s+\varepsilon} \lesssim \sum_{k=1}^K [u]_{W^{s+\varepsilon,p}(2B_k)}^{p} + \sum_{k=1}^K \int_{\Omega \setminus 2B_k} \int_{B_k} \frac{|u(x) - u(y)|^p}{|x-y|^{n+(s+\varepsilon)p}} \, dx \, dy.$$

As for the second term, because of the disjoint support of the integrals we find

$$\int_{\Omega \setminus 2B_k} \int_{B_k} \frac{|u(x) - u(y)|^p}{|x-y|^{n+(s+\varepsilon)p}} \, dx \, dy \lesssim (\text{diam } B_k)^{-ep} \Gamma_s.$$
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That is
$$\Gamma_{s+\varepsilon} \lesssim \sum_{k=1}^{K} [u]_{W^{s+\varepsilon,p}(2B_k)}^p + \Gamma_s.$$ 

With Lemma A.2 and Poincaré inequality, Proposition A.3, for any $\delta > 0$,
$$\Gamma_{s+\varepsilon} \lesssim \delta^p \Gamma_{s+\varepsilon} + C_\delta \Gamma_s + \sum_{k=1}^{K} \delta^{-p'} \left( \sup_{\varphi} (-\Delta)^{s+\varepsilon}_{p,8B_k} u[\varphi] \right)^{\frac{p}{p-1}}$$
where the supremum is over all $\varphi \in C_c(4B_k)$ and $[\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1$. Here we also used that $\bigcup_{k=1}^{K} 8B_k$ covers no more than $\Omega$. Choosing $\delta$ sufficiently small, we can estimate $\Gamma_{s+\varepsilon}$ by
$$\Gamma_s + \sum_{k=1}^{K} \left( \sup \{|(-\Delta)^{s+\varepsilon}_{p,8B_k} u[\varphi]| : \varphi \in C_c(4B_k), [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1 \} \right)^{\frac{p}{p-1}}.$$ 

With Theorem 1.1 this can be estimated by
$$\Gamma_s + \varepsilon^{\frac{p}{p-1}} \Gamma_{s+\varepsilon} + \sum_{k=1}^{K} \left( \sup \{|(-\Delta)^{s+\varepsilon}_{p,8B_k} u[(-\Delta)^{\frac{p}{2}}\varphi]| : \varphi \in C_c(4B_k), [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1 \} \right)^{\frac{p}{p-1}}.$$ 

If $\varepsilon \in [0, \varepsilon_0)$ for $\varepsilon_0$ small enough, we can again absorb $\Gamma_{s+\varepsilon}$. The estimate for $\Gamma_{s+\varepsilon}$ becomes
$$\Gamma_s + \sum_{k=1}^{K} \left( \sup \{|(-\Delta)^{s+\varepsilon}_{p,8B_k} u[(-\Delta)^{\frac{p}{2}}\varphi]| : \varphi \in C_c(4B_k), [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq 1 \} \right)^{\frac{p}{p-1}}.$$ 

Next, we need to transform $(-\Delta)^{\frac{p}{2}}\varphi$ into a feasible testfunction, and denoting the usual cutoff function with $\eta_{6B_k} \in C_c(6B_k)$, $\eta_{6B_k} \equiv 1$ in $5B_k$

$$(-\Delta)^{\frac{p}{2}}\varphi =: \psi + (1 - \eta_{6B_k})(-\Delta)^{\frac{p}{2}}\varphi.$$ 

Then $\psi \in C_c(6B_k)$
$$[\psi]_{W^{s-\varepsilon(p-1),p}(\Omega)} \lesssim C_k [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)} \leq C_k.$$ 

Moreover, the disjoint support of $(1 - \eta_{6B_k})$ and $\varphi$ implies (see, e.g., Lemma A.1)
$$[(1 - \eta_{6B_k})(-\Delta)^{\frac{p}{2}}\varphi]_{\text{Lip}} \leq C_k [\varphi]_{W^{s+\varepsilon,p}(\mathbb{R}^n)}.$$ 

Consequently,
$$|(-\Delta)^{s+\varepsilon}_{p,8B_k} u[(-\Delta)^{\frac{p}{2}}\varphi - \psi]| \lesssim [u]_{W^{s,p}(\Omega)}^{p-1}.$$
Hence, our estimate for $\Gamma_{s+\varepsilon}$ now looks like

$$\Gamma_s + \sum_{k=1}^K \left( \sup \left\{ \left| (-\Delta)^s_{p,8B_k} u[\psi] \right| : \psi \in C_c^\infty(6B_k), [\psi]_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^n)} \leq 1 \right\} \right)^{\frac{p}{p-1}}.$$

Finally, we need to transform the support of $(-\Delta)^s_{p,8B_k}$ from $8B_k$ to $\Omega$. Since $\text{supp} \psi \subset 6B_k$, the disjoint support of the integrals gives

$$\left| (-\Delta)^s_{p,8B_k} u[\psi] - (-\Delta)^s_{p,\Omega} u[\psi] \right| \leq C_k [u]_{W^{s-\varepsilon(p-1),p}(\Omega)} [\psi]_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^n)}.$$

This implies the final estimate of $\Gamma_{s+\varepsilon}$ by

$$\Gamma_s + \left( \sup \left\{ \left| (-\Delta)^s_{p,\Omega} u[\psi] \right| : \psi \in C_c^\infty(\Omega), [\psi]_{W^{s-\varepsilon(p-1),p}(\mathbb{R}^n)} \leq 1 \right\} \right)^{\frac{p}{p-1}}.$$

\[ \square \]

6. Differentiability of $p$-harmonic maps: Proof of Theorem 1.7

For $B \subset \mathbb{R}^n$, $t \in (0,1)$, we set

$$T_{t,B} u(z) = \int_B \int_B \frac{|u(x) - u(y)|^{n-2}(u(x) - u(y)) (|x-z|^{t-n} - |y-z|^{t-n})}{|x-y|^{n+\frac{n}{p}}} \, dx \, dy.$$

$T_{t,B} u$ was introduced in [21] because of the following relation

\begin{equation}
(6.1) \quad c \int_{\mathbb{R}^n} T_{t,B} u(z) \varphi(z) \, dz = \int_B \int_B \frac{|u(x) - u(y)|^{n-2}(u(x) - u(y)) (I^t \varphi(x) - I^t \varphi(y))}{|x-y|^{n+\frac{n}{p}}} \, dx \, dy.
\end{equation}

From [21] in particular (3.1), Lemma 3.3, 3.4, 3.5] we have the following

**Theorem 6.1.** Let $u$ satisfy \(1.1\) and \(1.2\) in an open set $\Omega$. Assume that on the Ball $2B$ for a small enough $\varepsilon > 0$ (depending on $\Lambda$) \(1.3\) holds. Then there is $t_0 < s$, $\sigma > 0$, so that for some $\gamma_2 > \gamma_1 \gg 1$ for any ball $B_{\gamma_2 \rho} \subset B$

\begin{equation}
(6.2) \quad [u]_{W^{s,\frac{n}{p}}(B_{\rho})} \lesssim C_{\Lambda} \rho^{\sigma},
\end{equation}

and

\begin{equation}
(6.3) \quad \|T_{t_0, B_{\gamma_1 \rho}} u\|_{n-t_0, B_{\rho}} \leq C_{\Lambda} \rho^{\sigma}.
\end{equation}
Estimate (6.3) looks almost as if $T_{t_0,B}$ belongs locally to a Morrey space. But the domain dependence on $B_{\gamma \rho}$ bars us from exploiting this. The following proposition removes the domain dependence.

**Proposition 6.2.** Under the assumptions of Theorem 6.1 there exists $\gamma > 1$, $\sigma > 0$ so that

$$\|T_{t_0,B}u\|_{n-t_0,B_{\rho}} \leq C_{B,\Lambda} \rho^\sigma$$

for any ball so that $B_{\gamma \rho} \subset B$.

**Proof.** Set $\kappa_1 \geq \kappa_2 \geq \kappa_3 \geq 1$ to be chosen later. Take $\gamma := 2\gamma_1$ with $\gamma_1$ from (6.3). We will always assume $\rho < 1$.

For some $\phi \in C_c^\infty(B_{\rho})$, $\|\phi\|_{n-t_0} \leq 1$ we have

$$\left\|T_{t_0,B}u\right\|_{n-t_0,B_{\rho}} \approx \int_{B} \int_{B} |u(x) - u(y)|^{\frac{1}{s}} \left|u(x) - u(y)\right| \left(I_{t_0} \phi(x) - I_{t_0} \phi(y)\right) dx\ dy.$$  

We will now use several cutoffs to slice $\phi$ into the right form. This kind of arguments and the consequent (tedious) estimates have been used several times in work related to fractional harmonic maps, cf. e.g. [8, 7, 5, 3, 21, 19, 20], and we will not repeat them in detail. We will also assume that $\kappa_1 > \kappa_2 > \kappa_3$. If they are equal, to keep the “disjoint support estimates” working one needs to use cutoff functions on twice, four times etc. of the Balls.

For a cutoff function $\eta_{B_{\rho_2}} \in C_c^\infty(B_{2\rho_2})$, $\eta_{B_{\rho_2}} \equiv 1$ on $B_{\rho_2}$, we have

$$I_{t_0} \phi := \psi + (1 - \eta_{B_{\rho_2}}) I_{t_0} \phi.$$  

Note that $\psi \in C_c^\infty(B_{2\rho_2})$ and

$$\|(-\Delta)^{\frac{t_0}{2}} \psi\|_{n-t_0} \approx \|\phi\|_{n-t_0}.$$  

The disjoint support of $(1 - \eta)$ and $\phi$ ensures (see [3, Lemma A.1])

$$\|I_{t_0} \phi - \psi\|_{W^{s,n}(\mathbb{R}^n)} \approx \rho^{(\kappa_1 - \kappa_2)(n-t_0)} \|\phi\|_{n-t_0}.$$  

We furthermore decompose

$$(-\Delta)^{\frac{t_0}{2}} \psi =: \phi + (1 - \eta_{B_{\rho_3}})(-\Delta)^{\frac{t_0}{2}} \psi.$$  

1This is true if $\frac{n}{n-t_0} \geq 2$, since then $\|f\|_{W^{s,n}} \leq \|(-\Delta)^{\frac{t_0}{2}} \psi\|_{n-t_0}$. If $\frac{n}{n-t_0} < 2$ one has to adapt the estimate, but the results remains true.
Then \( \phi \in \mathcal{C}_c^{\infty}(B_{2\rho'^3}) \) and
\[
\|\phi\|_{\frac{n}{2}} \lesssim \|\varphi\|_{\frac{n}{2}},
\]
(6.6)
\[
\|
\nabla (\psi - I_0 \phi)\|_\infty \lesssim \rho^{-\kappa_3 + (\kappa_2 - \kappa_3)n} \|\varphi\|_{\frac{n}{2}}.
\]
Again with (6.1), we then have
\[
\|T_{0,B}u\|_{\frac{n}{2\rho},B} \lesssim |I| + |II| + |III| + |IV|
\]
where
\[
I := \int T_{0,B_{\rho'}} u \phi
\]
\[
II := \int_{B_{\rho'}} \int_{B_{\rho'}} |u(x) - u(y)|^{\frac{n}{2} - 2} (u(x) - u(y)) \left( (\psi - I_0 \phi)(x) - (\psi - I_0 \phi)(y) \right) dx dy
\]
\[
III := \int_{B \setminus B_{\rho'}} \int_{B_{\rho'}} |u(x) - u(y)|^{\frac{n}{2} - 2} (u(x) - u(y)) \left( \psi(x) - \psi(y) \right) dx dy
\]
and
\[
IV := \int_B \int_B |u(x) - u(y)|^{\frac{n}{2} - 2} (u(x) - u(y)) \left( (I_0 \varphi - \psi)(x) - (I_0 \varphi - \psi)(y) \right) dx dy
\]
With (6.6), \( \text{supp } \phi \subset B_{2\rho'^3} \subset B_{2\rho} \), and (6.3),
\[
|I| \lesssim \rho^\rho.
\]
With (6.2), (6.7) (for \( \rho \) small enough),
\[
|II| \lesssim \|u\|_{W^{s,\frac{n}{s}}(B_{\rho'})} [\psi - I_0 \phi]_{W^{s,\frac{n}{s}}(B_{\rho'})} \lesssim \rho^{s(\frac{n}{s} - 1)} \rho^{-\kappa_3 - 1} \rho^{(\kappa_2 - \kappa_3)n}.
\]
With the disjoint support of the integrals, Hölder inequality \( (\frac{n}{2}, \frac{n}{s}) \), and (6.4),
\[
|III| \lesssim \|u\|_{W^{s,\frac{n}{s}}(B)}^{p-1} \rho^{t_0 - s} \rho^{\kappa_2(s - t_0)} \|\psi\|_{W^{s,\frac{n}{s}}(B)} \lesssim \rho^{(\kappa_2 - 1)(s - t_0)}.
\]
Lastly, with (6.5)
\[
|IV| \lesssim \|u\|_{W^{s,\frac{n}{s}}(B)}^{p-1} \|I_0 \varphi - \psi\|_{W^{s,\frac{n}{s}}(B)} \lesssim \rho^{(\kappa_1 - \kappa_2)(n - t_0)}.
\]
If we choose \( \kappa_1 = \kappa_2 = \kappa_3 = 1 \), we obtain
\[
\|T_{0,B}u\|_{\frac{n}{2\rho},B} \lesssim 1,
\]
whenever \( B_{2\rho'} \subset B \). In particular
\[
\|T_{0,B}u\|_{\frac{n}{2\rho},B} \lesssim 1.
\]
On the other hand, we may take
\[
\kappa_1 > \kappa_2 > \kappa_3 = 1.
\]
Then we have shown that
\[ \| T_{t_0,B}u \|_{\frac{n}{n-t_0},B_{\rho^{\frac{s}{t_0}}}} \lesssim \rho^{\tilde{\sigma}}, \]
which holds whenever \( B_{\gamma \rho} \subset B \). Equivalently, for an even smaller \( \tilde{\sigma} \),
\[ \| T_{t_0,B}u \|_{\frac{n}{n-t_0},B_{\rho}} \lesssim \rho^{\tilde{\sigma}}, \]
which holds whenever \( B_{\gamma \rho} \subset B \). With (6.8) this estimate also holds whenever \( B_{2 \gamma \rho} \subset B \), with a constant depending on the radius of \( B \). □

In [21] it is shown that for \( t_1 > t_0 \), \( T_{t_1,B}u = I^{t_1-t_0} T_{t_0,B}u \). Since according to Proposition 6.2 \( T_{t_0,B}u \) belongs to a Morrey space, we can apply Adams estimates on Riesz potential acting on Morrey spaces [1, Theorem 3.1 and Corollary after Proposition 3.4] and obtain an increased integrability estimate for \( T_{t_1,B}u \).

**Proposition 6.3.** Under the assumptions of Theorem 6.1 there are \( \gamma > 1 \), \( t_0 < t_1 < s \), and \( p_1 > \frac{n}{n-t_1} \) so that
\[ \| T_{t_1,B}u \|_{p_1,B_{\rho}} \leq C_{\Lambda} \rho^{\sigma} \]
for any ball so that \( B_{\gamma \rho} \subset B \).

Now we exploit (6.1): For any \( \varphi \in C_c^\infty(\mathbb{R}^n) \)
\[ (-\Delta)^{\frac{s}{n},B} u[\varphi] = \int_{\mathbb{R}^n} T_{t_1,B}u (-\Delta)^{\frac{t_1}{2}} \varphi. \]
Let \( \varphi \in C_c^\infty(B_{\gamma \rho}^\circ) \) for \( B_{\gamma \rho} \subset B \). With the usual cutoff-function \( \eta \in C_c^\infty(B_\rho) \), \( \eta \equiv 1 \) on \( B_{\frac{1}{2}\rho} \)
\[ \|(-\Delta)^{\frac{s}{n},B} u[\varphi]\| \lesssim \|T_{t_1,B}u\|_{p_1,B_{\rho}}\|(-\Delta)^{\frac{t_1}{2}} \varphi\|_{p_1,B_{\rho}} + \|T_{t_1,B}u\|_{\frac{n}{n-t_1},B_{\rho}}\|(-\Delta)^{\frac{t_1}{2}} \varphi\|_{\frac{n}{p_1}, R^n \setminus B_{\frac{1}{2}\rho}}. \]

By the Sobolev inequality for Gagliardo-Norms [21 Theorem 1.6], and the disjoint support [3 Lemma A.1], this implies
\[ \|(-\Delta)^{\frac{s}{n},B} u[\varphi]\| \lesssim C_{\Lambda}[\varphi] \| W^{s+t_1-\frac{n}{p_1},\frac{n}{p_1}}(\mathbb{R}^n) \].

Since \( p_1 > \frac{n}{n-t_1} \), we have \( s+t_1 - \frac{n}{p_1} < s \), and the claim of Theorem 1.7 follows from Theorem 1.3 by a covering argument. □
7. Compactness for $\frac{n}{s}$-harmonic maps: Proof of Theorem 1.8

From the arguments in [6, Proof of Lemma 2.3.] one has the following:

**Proposition 7.1.** For $s \in (0,1)$, $p \in (1,\infty)$ let $(u_k)_{k=1}^{\infty} \in W^{s,p}(\mathbb{R}^n, S^{N-1})$, $\Lambda := \sup_{k \in \mathbb{N}} [u_k]_{W^{s,p}(\mathbb{R}^n)} < \infty$ and $\varepsilon_0 > 0$ given. Then up to a subsequence there is $u_\infty \in W^{s,p}(\mathbb{R}^n, S^{N-1})$ and a finite set of points $J = \{a_1, \ldots, a_l\}$ such that

$$u_k \rightharpoonup u_\infty \text{ in } W^{s,p}(\mathbb{R}^n, S^{N-1}) \text{ as } k \to \infty,$$

and for all $x \notin J$ there is $r = r_x > 0$ so that

$$\limsup_{k \to \infty} [u_k]_{W^{s,p}(B_r(x))} < \varepsilon_0.$$

This, Theorem 1.7 and the compactness of the embedding $W^{s+p,\frac{n}{s}}(B_r(x)) \hookrightarrow W^{s,\frac{n}{s}}(B_r(x))$ immediately implies that

$$u_k \xrightarrow{k \to \infty} u_\infty \text{ in } W^{s,\frac{n}{s}}_\text{loc}(\mathbb{R}^n \setminus J).$$

\[\square.\]

**Appendix A. Useful Tools**

The following Lemma is used to restrict the fractional $p$-Laplacian to smaller sets.

**Lemma A.1** (Localization Lemma). Let $\Omega_1 \Subset \Omega_2 \Subset \Omega_3 \Subset \Omega \subset \mathbb{R}^n$ be open sets so that $\text{dist}(\Omega_1, \Omega_2^c), \text{dist}(\Omega_2, \Omega_3^c), \text{dist}(\Omega_3, \Omega^c) > 0$. Let $s \in (0,1)$, $p \in [2,\infty)$.

For any $u \in W^{s,p}(\Omega)$ there exists $\tilde{u} \in W^{s,p}(\mathbb{R}^n)$ so that

1. $\tilde{u} - u \equiv \text{const } \text{in } \Omega_1$
2. $\text{supp } \tilde{u} \subset \Omega_2$
3. $[\tilde{u}]_{W^{s,p}(\mathbb{R}^n)} \lesssim [u]_{W^{s,p}(\Omega)}$
4. For any $t \in (2s-1, s)$,

$$\|(-\Delta)^{s}_p_{\Omega_3} \tilde{u}\|_{(W^{t,p}(\Omega))^*} \lesssim \|(-\Delta)^{s}_p_{\Omega} u\|_{(W^{t,p}(\Omega))^*} + [u]_{W^{s,p}(\Omega)}^{p-1}.$$

The constants are uniform in $u$ and depend only on $s, t, p$ and the sets $\Omega_1, \Omega_2, \Omega_3, \text{ and } \Omega$. 
Proof. Let $\Omega_1 \Subset \Omega$, let $\eta \equiv \eta_{\Omega_1} \in C^\infty_c(\Omega_2)$, $\eta_{\Omega_1} \equiv 1$ on $\Omega_1$. We set

$$\tilde{u} := \eta_{\Omega_1}(u - (u)_{\Omega_1}).$$

Clearly $\tilde{u}$ satisfies property (1) and (2). We have property (3), too:

$$[\tilde{u}]_{W^{s,p}(\mathbb{R}^n)} \lesssim [u]_{W^{s,p}(\Omega)}.$$  

We write

$$\tilde{u}(x) - \tilde{u}(y) = \underbrace{\eta(x)(u(x) - u(y))}_{a(x,y)} + \underbrace{(\eta(x) - \eta(y))(u(y) - (u)_{\Omega_1})}_{b(x,y)}.$$ 

Setting

$$T(a) := |a|^{p-2}a,$$

observe that

$$|T(a + b) - T(a)| \lesssim |b| \left( |a|^{p-2} + |b|^{p-2} \right).$$

Also note that

$$T(a(x,y)) = \eta^{p-1}(x)|u(x) - u(y)|^{p-2}(u(x) - u(y)).$$

We thus have for any $\varphi \in C^\infty_c(\Omega_3)$,

$$(-\Delta)^s_{p,\Omega} \tilde{u}[\varphi] = \int_{\Omega} \int_{\Omega} \frac{|\tilde{u}(x) - \tilde{u}(y)|^{p-2}(\tilde{u}(x) - \tilde{u}(y)) (\varphi(x) - \varphi(y))}{|x - y|^{n+sp}} \, dx \, dy$$

$$= \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) (\eta^{p-1}(x)\varphi(x) - \eta^{p-1}(y)\varphi(y))}{|x - y|^{n+sp}} \, dx \, dy$$

$$+ \int_{\Omega} \int_{\Omega} \frac{(T(a + b) - T(a)) (\varphi(x) - \varphi(y))}{|x - y|^{n+sp}} \, dx \, dy$$

$$= \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) (\eta^{p-1}(x)\varphi(x) - \eta^{p-1}(y)\varphi(y))}{|x - y|^{n+sp}} \, dx \, dy$$

$$- \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) (\eta^{p-1}(x) - \eta^{p-1}(y))\varphi(y)}{|x - y|^{n+sp}} \, dx \, dy$$

$$+ \int_{\Omega} \int_{\Omega} \frac{(T(a + b) - T(a)) (\varphi(x) - \varphi(y))}{|x - y|^{n+sp}} \, dx \, dy$$

$$= (-\Delta)^s_{p,\Omega} u[\eta^{p-1}\varphi]$$

$$- \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) (\eta^{p-1}(x) - \eta^{p-1}(y))\varphi(y)}{|x - y|^{n+sp}} \, dx \, dy$$

$$+ \int_{\Omega} \int_{\Omega} \frac{(T(a + b) - T(a)) (\varphi(x) - \varphi(y))}{|x - y|^{n+sp}} \, dx \, dy.$$
So we have that
\[ |(-\Delta)^s_{\Omega} \tilde{u}[\varphi]| \]
\[ \lesssim \|(-\Delta)^s_{\Omega} u\|_{(W^{s,p}_0(\Omega))^*} \cdot [\eta^{p-1} \varphi]_{W^{s,p}(\Omega)} \]
\[ + \int_{\Omega} \int_{\Omega} \frac{|u(x) - u(y)|^{p-1} |\eta^{p-1}(x) - \eta^{p-1}(y)| |\varphi(y)|}{|x-y|^{n+sp}} \, dx \, dy \]
\[ + \int_{\Omega} \int_{\Omega} \frac{|\eta(x) - \eta(y)| |u(y) - (u)_{\Omega}\|^2 |u(x) - u(y)|^{p-2} |\varphi(x) - \varphi(y)|}{|x-y|^{n+sp}} \, dx \, dy \]
\[ + \int_{\Omega} \int_{\Omega} \frac{|\eta(x) - \eta(y)|^{p-1} |u(y) - (u)_{\Omega}\|^p |\varphi(x) - \varphi(y)|}{|x-y|^{n+sp}} \, dx \, dy. \]

That is for any \( t < s \)
\[ |(-\Delta)^s_{\Omega} \tilde{u}[\varphi]| \]
\[ \lesssim \|(-\Delta)^s_{\Omega} u\|_{(W^{s,p}_0(\Omega))^*} \cdot [\eta^{p-1} \varphi]_{W^{s,p}(\Omega)} \]
\[ + [u]_{W^{s,p}(\Omega)}^{p-1} \left( \int_{\Omega} \int_{\Omega} \frac{|\eta^{p-1}(x) - \eta^{p-1}(y)|^p |\varphi(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \right)^{\frac{1}{2^p}} \]
\[ + [\varphi]_{W^{s,p}(\Omega)} [u]_{W^{s,p}(\Omega)}^{p-2} \left( \int_{\Omega} \int_{\Omega} \frac{|\eta(x) - \eta(y)|^p |u(y) - (u)_{\Omega}\|^p}{|x-y|^{n+(2s-t)p}} \, dx \, dy \right)^{\frac{1}{2^p}} \]
\[ + [\varphi]_{W^{s,p}(\Omega)} \left( \int_{\Omega} \int_{\Omega} \frac{|\eta(x) - \eta(y)|^p |u(y) - (u)_{\Omega}\|^p}{|x-y|^{n+(2s-t)p}} \, dx \, dy \right)^{\frac{p-1}{2^p}}. \]

Since \( \eta \) is bounded and Lipschitz, \( \text{supp} \, \eta \subset \Omega_2 \), and \( \varphi \in C^\infty_c(\Omega_3) \) we have that
\[ [\eta^{p-1} \varphi]_{W^{s,p}(\Omega)} \lesssim [\varphi]_{W^{s,p}(\mathbb{R}^n)}. \]

Also, choosing some bounded \( \Omega_4 \Subset \Omega \) so that \( \Omega_3 \Subset \Omega_4 \),
\[ \int_{\Omega} \int_{\Omega} \frac{|\eta^{p-1}(x) - \eta^{p-1}(y)|^p |\varphi(y)|^p}{|x-y|^{n+sp}} \, dx \, dy \]
\[ \lesssim \int_{\Omega_3} \int_{\Omega_4} |x-y|^{(1-s)p-n} \, dx \, |\varphi(y)|^p \, dy \]
\[ + \int_{\Omega_3} \int_{\mathbb{R}^n \setminus \Omega_4} |x-y|^{-n-sp} \, dx \, |\varphi(y)|^p \, dy \]
\[ \lesssim \|\varphi\|_p^p \lesssim [\varphi]_{W^{s,p}(\mathbb{R}^n)}. \]
Finally, using Lipschitz continuity of \( \eta \) and that \( 2s - 1 < t < s \)
\[
\int_{\Omega} \int_{\Omega_2} \frac{\left| \eta(x) - \eta(y) \right|^p |u(y) - (u)_{\Omega_1}|^p}{|x - y|^{n+(2s-t)p}} \, dx \, dy
\]
\[
\lambda \int_{\Omega_3} |u(y) - (u)_{\Omega_1}|^p \int_{\Omega_2} |x - y|^{-n+(t+1-2s)p} \, dx \, dy
\]
\[
+ \int_{\Omega \setminus \Omega_3} |u(y) - (u)_{\Omega_1}|^p \int_{\Omega_2} \frac{1}{|x - y|^{n+sp}} \, dx \, dy
\]
\[
\lambda \int_{\Omega_1} \int_{\Omega_3} |u(y) - u(z)|^p \, dy \, dz
\]
\[
+ \int_{\Omega_1} \int_{\Omega_1 \setminus \Omega_3} |u(y) - u(z)|^p \int_{\Omega_2} \frac{1}{|x - y|^{n+sp}} \, dx \, dy \, dz
\]
Note that for \( x, z \in \Omega_2 \) and \( y \in \Omega_3 \) we have that \( |x - y| \approx |y - z| \), and since \( \Omega_1, \Omega_2, \Omega_3 \) are bounded we then have
\[
\int_{\Omega} \int_{\Omega_2} \frac{\left| \eta(x) - \eta(y) \right|^p |u(y) - (u)_{\Omega_1}|^p}{|x - y|^{n+(2s-t)p}} \, dx \, dy \lesssim [u]_{W^{s,p}(\Omega)}
\]
Thus we have shown that for any \( \varphi \in C_c^\infty(\Omega_3) \),
\[
|(-\Delta)^s_{p,\Omega} \tilde{u}[\varphi]| \lesssim \left( \|(-\Delta)^s_{p,\Omega} u\|_{W^{1,p}(\Omega)} + [u]_{W^{s,p}(\Omega)}^{p-1} \right) [\varphi]_{W^{1,p}(\mathbb{R}^n)}.
\]
Since moreover, \( \text{supp } \tilde{u} \subset \Omega_2 \), for any \( \varphi \in C_c^\infty(\Omega_3) \),
\[
|(-\Delta)^s_{p,\Omega} \tilde{u}[\varphi]| \lesssim |(-\Delta)^s_{p,\Omega} \tilde{u}[\varphi]| + [u]_{W^{s,p}(\Omega)}^{p-1} [\varphi]_{W^{1,p}(\mathbb{R}^n)},
\]
we get the claim. \(\square\)

The next Lemma estimates the \( W^{s,p} \)-norm in terms of the fractional \( p \)-Laplacian.

**Lemma A.2.** Let \( B \subset \mathbb{R}^n \) be a ball and \( 4B \) the concentric ball with four times the radius. Then for any \( \delta > 0 \), \( [u]_{W^{s,p}(4B)} \) can be estimated by
\[
\delta^p [u]_{W^{s,p}(4B)}^p
\]
\[
+ \frac{C}{\delta^p} \left( \sup_{\varphi} \int_{4B} \int_{4B} \frac{|u(x) - u(y)|^p - 2(u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{n+sp}} \, dx \, dy \right)^{\frac{p}{p-1}}
\]
\[
+ \frac{C}{\delta^p} \text{diam}(B)^{-sp} \int_{4B} |u(x) - (u)_B|^p \, dx
\]
where the supremum is over all \( \varphi \in C_c^\infty(2B) \) and \( [\varphi]_{W^{s,p}(\mathbb{R}^n)} \leq 1 \).
Proof. Let $\eta \in C_c^\infty (2B)$, $\eta \equiv 1$ in $B$ be the usual cutoff function in $2B$.

$\psi (x) := \eta (x)(u(x) - (u)_B)$, and $\varphi (x) := \eta^2 (x)(u(x) - (u)_B)$.

Then,

$\psi(A.1) \lesssim [\psi]_{W^{s,p} (\mathbb{R}^n)} + [\varphi]_{W^{s,p} (\mathbb{R}^n)} \lesssim [u]_{W^{s,p} (2B)}$.

We have

$$[u]^p_{W^{s,p} (B)} \leq \int \int_{4B} \frac{|u(x) - u(y)|^{p-2}(\psi(x) - \psi(y)) (\psi(x) - \psi(y))}{|x - y|^{n+sp}} dx \ dy$$

Now we observe

$$(\psi(x) - \psi(y))^2 = (\psi(x) - \psi(y))(\eta(x) - \eta(y))(u(x) - (u)_B) + \psi(x)(\eta(y) - \eta(x))(u(x) - u(y)) + (\varphi(x) - \varphi(y))(u(x) - u(y)).$$

That is,

$$[u]^p_{W^{s,p} (B)} \lesssim I + II + III,$$

with

$I := \int \int_{4B} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+sp}} dx \ dy,$

$II := \int \int_{4B} \frac{|u(x) - u(y)|^{p-2}\eta(x) - \eta(y)| |\psi(x) - \psi(y)|}{|x - y|^{n+sp}} |u(x) - (u)_B| dx \ dy,$

$III := \int \int_{4B} \frac{|u(x) - u(y)|^{p-1}\eta(x) - \eta(y)|}{|x - y|^{n+sp}} |\psi(x)| dx \ dy.$

With (A.1),

$I \leq [u]_{W^{s,p} (4B)} \sup_{[\varphi]_{W^{s,p} (\mathbb{R}^n)} \leq 1} \int \int_{4B} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y))(\varphi(x) - \varphi(y))}{|x - y|^{n+sp}} dx \ dy.$

As for $II$,

$II \lesssim \|\nabla \eta\|_\infty \int \int_{4B} \frac{|u(x) - u(y)|^{p-2}\psi(x) - \psi(y)| |u(x) - (u)_B|}{|x - y|^{n+sp-1}} dx \ dy.$
For any $t_2 > 0$ so that $t_2 = 1 - s$, we have with Hölder’s inequality

$$II \lesssim \|\nabla \eta\|_{\infty} \int_{4B} \int_{4B} \frac{|u(x) - u(y)|^{p-2}|\psi(x) - \psi(y)| |u(x) - (u)_B|}{|x - y|^{n+sp-2-t_2}} \, dx \, dy$$

$$\lesssim \text{diam}(B)^{-1} [u]_{W^{s,p}(4B)}^{p-2} [\psi]_{W^{s,p}(4B)} \left( \int_{4B} \int_{4B} \frac{|u(x) - (u)_B|^p}{|x - y|^{n-t_2}} \, dx \, dy \right)^{\frac{1}{p}}.$$ 

Since $t_2 > 0$,

$$\int_{4B} \int_{4B} \frac{|u(x) - (u)_B|^p}{|x - y|^{n-t_2}} \, dx \, dy \lesssim (\text{diam}(B))^{t_2p} \int_{4B} |u(x) - (u)_B|^p \, dx$$

So using again (A.1), we arrive at

$$II \lesssim \text{diam}(B)^{-s} [u]_{W^{s,p}(4B)}^{p-1} \left( \int_{4B} |u(x) - (u)_B|^p \, dx \right)^{\frac{1}{p}}.$$ 

$III$ can be estimated the same way as $II$, and we have the following estimate for $[u]_{W^{s,p}(\lambda B)}^{p}$

$$[u]_{W^{s,p}(\lambda B)}^{p} \sup_{\varphi} \int_{4B} \int_{4B} \frac{|u(x) - u(y)|^{p-2}(u(x) - u(y)) (\varphi(x) - \varphi(y))}{|x - y|^{n+sp}} \, dx \, dy$$

$$+ [u]_{W^{s,p}(4B)}^{p-1} \text{diam}(B)^{-s} \left( \int_{4B} |u(x) - (u)_B|^p \, dx \right)^{\frac{1}{p}}$$

We conclude with Young’s inequality. 

The next Proposition follows immediately from Jensen’s inequality and the definition of $[u]_{W^{s,p}(\lambda B)}^{p}$.

**Proposition A.3** (A Poincaré type inequality). Let $B$ be a ball and for $\lambda \geq 1$ let $\lambda B$ be the concentric ball with $\lambda$ times the radius. Then for any $t \in (0, 1)$, $p \in (1, \infty)$,

$$\int_{\lambda B} |u(x) - (u)_B|^p \, dx \lesssim \lambda^{n+tp} \text{diam}(B)^{tp} [u]_{W^{s,p}(\lambda B)}^{p}.$$
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Armin Schikorra, Universität Basel, armin.schikorra@unibas.ch