Evidence of Strong Correlation between Instanton and QCD-monopole on SU(2) Lattice

H. Suganuma\textsuperscript{a}, A. Tanaka\textsuperscript{a}, S. Sasaki\textsuperscript{a} and O. Miyamura\textsuperscript{b}

\textsuperscript{a}Research Center for Nuclear Physics (RCNP), Osaka University, Mihogaoka 10-1, Ibaraki 567, Japan
\textsuperscript{b}Department of Physics, Hiroshima University, Kagamiyama 1-3, Higashi-Hiroshima 739, Japan

The correlation between instantons and QCD-monopoles is studied both in the lattice gauge theory and in the continuum theory. An analytical study in the Polyakov-like gauge, where $A_4(x)$ is diagonalized, shows that the QCD-monopole trajectory penetrates the center of each instanton, and becomes complicated in the multi-instanton system. Using the SU(2) lattice with $16^4$, the instanton number is measured in the singular (monopole-dominating) and regular (photon-dominating) parts, respectively. The monopole dominance for the topological charge is found both in the maximally abelian gauge and in the Polyakov gauge.

1. Introduction

QCD is reduced to an abelian gauge theory with magnetic monopoles (QCD-monopoles) by the abelian gauge fixing through the diagonalization of a gauge dependent variable $X(x)$. The QCD-monopole appears from the hedgehog configuration on $X(x)$ corresponding to the non-trivial homotopy group $\pi_2(SU(N_c)/U(1)) = Z_{N_c-1}^\infty$, and its condensation plays an essential role to the nonperturbative QCD. The instanton is also an important topological object relating to $U_A(1)$ anomaly, and appears in the Euclidean 4-space corresponding to $\pi_3(SU(N_c)) = Z_\infty$. We study the correlation between instantons and QCD-monopoles both in the lattice theory \cite{3} and in the analytical framework \cite{4}.

Recent lattice studies \cite{2} indicate the abelian dominance for the nonperturbative quantities in the maximally abelian (MA) gauge and/or in the Polyakov gauge. If the system is completely described only by the abelian field, the instanton would lose the topological basis for its existence, and therefore it seems unable to survive in the abelian manifold. However, even in the abelian gauge, nonabelian components remain relatively large around the QCD-monopoles, which are nothing but the topological defects, so that instantons are expected to survive only around the QCD-monopole trajectories in the abelian-dominant system. The close relation between instantons and QCD-monopoles are thus suggested from the topological consideration.

2. Analytic Calculation

First, we demonstrate a close relation between instantons and QCD-monopoles within the continuum theory. \cite{4}. Using an ambiguity on $X(x)$ in the abelian gauge fixing, we choose $X(x) = A_4(x)$ to this end. This abelian gauge diagonalizing $A_4(x)$ will be called as the Polyakov-like gauge, where the Polyakov loop $P(x)$ is also diagonal. Since $A_4(x)$ takes a hedgehog configuration around each instanton, the QCD-monopole trajectory should pass through the center of instantons inevitably in the Polyakov-like gauge. We show this relation in the SU(2) gauge theory below.

Using the 't Hooft symbol $\bar{\eta}^{a\mu\nu}$, the multi-instanton solution is written as

$$A^\mu(x) = i\bar{\eta}^{a\mu\nu}\frac{\tau^a}{2}\partial^\nu \ln \left(1 + \sum_k \frac{a_k^2}{|x - x_k|^2}\right),$$

(1)

where $x_k^\mu = (x_k, t_k)$ and $a_k$ denote the center coordinate and the size of $k$-th instanton, respectively. Near the center of $k$-th instanton, $A_4(x)$ takes a hedgehog configuration,

$$A_4(x) \simeq \frac{\tau^a(x - x_k)^a}{|x - x_k|^2}.$$  

(2)
In the Polyakov-like gauge, $A_4(x)$ is diagonalized by a singular gauge transformation, which leads to the QCD-monopole trajectory on $A_4(x) = 0$: $x \simeq x$.

Thus, the center of each instanton is penetrated by a QCD-monopole trajectory with the temporal direction in the Polyakov-like gauge [1]. In other words, instantons only live along the QCD-monopole trajectories.

Here, we refer the magnetic charge of the QCD-monopole. In general, the abelian gauge fixing consists of two sequential procedures. One is the diagonalization of $X(x)$: $X(x) \to X_d(x)$. The other is the ordering of the diagonal elements of $X_d(x)$, e.g., $X_1^d \geq X_2^d \geq ... \geq X_d^N$. The gauge group $SU(N_c)$ is reduced to $U(1)^{N_c-1} \times P_{N_c}$ by the diagonalization of $X(x)$, and is reduced to $U(1)^{N_c-1}$ by the ordering condition on $X_d(x)$. The magnetic charge of the QCD-monopole is closely related to the ordering condition in the diagonalization in the abelian gauge fixing. For instance, in the $SU(2)$ case, the hedgehog configuration as $X(x) \sim (x \cdot \tau)$ and the anti-hedgehog one as $X(x) \sim -(x \cdot \tau)$ provide a QCD-monopole with an opposite magnetic charge respectively, because they are connected by the additional gauge transformation $\Omega = \exp\{i \pi (\tau^1 \cos \phi + \tau^2 \sin \phi)\}$, which interchanges the diagonal elements of $X_d(x)$ and leads a minus sign in the $U(1)_3$ gauge field.

For the single-instanton system, the QCD-monopole trajectory $x^h \equiv (x, t)$ is simply given by $x = x_1$ ($-\infty < t < \infty$) at the classical level.

For the two-instanton system, two instanton centers can be located on the $zt$-plane without loss of generality, so that one can set $x_1 = y_1 = x_2 = y_2 = 0$. Owing to the symmetry of the system, QCD-monopoles only appear on the $zt$-plane, and hence one has only to examine $A_4(x)$ on the $zt$-plane by setting $x = y = 0$. In this case, $A_4(x)$ in Eq. (3) is already diagonalized on the $zt$-plane: $A_4(z) = A_3^2(z,t)\tau^3$. Therefore, the QCD-monopole trajectory $x^h = (x, y, z, t)$ is simply given by $A_3^2(z,t) = 0$ and $x = y = 0$. However, the QCD-monopole trajectories are rather complicated even at the classical level in the two-instanton system. According to the parameters $x_k, a_k$ ($k = 1, 2$), the QCD-monopole trajectory has a loop or a folded structure as shown in Fig. 1 (a) or (b), respectively. Here, the QCD-monopole trajectories originating from instantons are very unstable against a small fluctuation relating to the location or the size of instantons.

The QCD-monopole trajectory tends to be highly complicated and unstable in the multi-instanton system even at the classical level, and the topology of the trajectory is often changed due to a small fluctuation of instantons. In addition, the quantum fluctuation would make it more complicated and more unstable, which leads to appearance of a long complicated trajectory as a result. Thus, instantons may contribute to promote monopole condensation, which is signaled by a long complicated monopole loop in the lattice QCD simulation [2].

We also study the thermal instanton system in the Polyakov-like gauge. At high temperature, QCD-monopole trajectories tend to be reduced to simple straight lines penetrating instantons in the temporal direction, which may corresponds to the deconfinement phase transition through the vanishing of QCD-monopole condensation.
For the thermal two-instanton system, the topology of the QCD-monopole trajectory is drastically changed at $T_c \simeq 0.6t^{-1}$, where $d$ is the distance between the two instantons. If one adopts $d \sim 1$fm as a typical mean distance between instantons, such a topological change occurs at $T_c \sim 120$MeV.

3. Instanton and Monopole on Lattice

We study the correlation between instantons and QCD-monopoles in the maximally abelian (MA) gauge and in the Polyakov gauge using the SU(2) lattice with 16$^4$ and $\beta = 2.4$. All measurements are done every 500 sweeps after a thermalization of 1000 sweeps using the heat-bath algorithm. After generating the gauge configurations, we examine the monopole dominance for the topological charge using the following procedure.

1) The abelian gauge fixing is done by diagonalizing $R(s) = \sum_\mu U_\mu(s) r^\mu U_\mu^{-1}(s)$ in the MA gauge, or the Polyakov loop $P(s)$ in the Polyakov gauge.

2) The SU(2) link variable $U_\mu(s)$ is factorized as $U_\mu(s) = M_\mu(s) u_\mu(s)$ with the ‘off-diagonal’ factor $M_\mu(s) \equiv \exp \{ i \tau^1 C^\mu_1(s) + i \tau^2 C^\mu_2(s) \}$ and the abelian link variable $u_\mu(s) = \exp \{ i \tau_3 \theta_\mu(s) \}$.

3) The abelian field strength $\theta_\mu(s) \equiv \partial_\mu \theta_\nu - \partial_\nu \theta_\mu$ is decomposed as $\theta_\mu(s) = \bar{\theta}_\mu(s) + 2\pi M_\mu(s)$ with $-\pi < \bar{\theta}_\mu(s) < \pi$ and $M_\mu(s) \in Z$. Here, $\bar{\theta}_\mu(s)$ and $2\pi M_\mu(s)$ correspond to the regular photon and the Dirac string, respectively.

4) The U(1) gauge field $\theta_\mu(s)$ is decomposed as $\theta_\mu(s) = \theta_\mu^{Ph}(s) + \theta_\mu^{Ds}(s)$ with a regular part $\theta_\mu^{Ph}(s)$ and a singular part $\theta_\mu^{Ds}(s)$, which are obtained from $\bar{\theta}_\mu(s)$ and $2\pi M_\mu(s)$, respectively, using the lattice Coulomb propagator in the Landau gauge. The singular part carries almost the same amount of magnetic current as the original U(1) field, whereas it scarcely carries the electric current. The situation is just the opposite in the regular part. For this reason, we regard the singular part as ‘monopole-dominating’, and the regular part as ‘photon-dominating’.

5) The corresponding SU(2) variables are reconstructed from $\theta_\mu^{Ph}(s)$ and $\theta_\mu^{Ds}(s)$ by multiplying the off-diagonal factor $M_\mu(s)$: $U_\mu^{Ph}(s) = M_\mu(s) \exp \{ i \tau_3 \theta_\mu^{Ph}(s) \}$ and $U_\mu^{Ds}(s) = M_\mu(s) \exp \{ i \tau_3 \theta_\mu^{Ds}(s) \}$.

6) The topological charge $Q = \int \frac{d^4x}{16\pi^2} \text{tr}(G_{\mu\nu} \tilde{G}_{\mu\nu})$, the integral of the absolute value of the topological density $I_Q = \int \frac{d^4x}{16\pi^2} \text{tr}(G_{\mu\nu} \tilde{G}_{\mu\nu})$, and the action $S$ are calculated by using $U_\mu(s)$, $U_\mu^{Ph}(s)$ and $U_\mu^{Ds}(s)$. Then, three sets of quantities are obtained, $\{ Q(SU(2)), I_Q(SU(2)), S(SU(2)) \}$ for the full SU(2) variable, $\{ Q(Ph), I_Q(Ph), S(Ph) \}$ for the regular part, and $\{ Q(Ds), I_Q(Ds), S(Ds) \}$ for the singular part. Here, $I_Q$ has been introduced to get information on the instanton and anti-instanton pair.

7) The correlations among these quantities are examined using the Cabibbo-Marinari cooling method.

Figure 2. Correlations between (a) $Q(Ds)$ and $Q(SU(2))$ at 80 cooling sweeps, (b) $Q(Ph)$ and $Q(SU(2))$ at 10 cooling sweeps.
We prepare 40 samples for the MA gauge and the Polyakov gauge, respectively. These simulations have been performed on the Intel Paragon XP/S(56node) at the Institute for Numerical Simulations and Applied Mathematics of Hiroshima University. Since quite similar results have been obtained in the MA gauge [3] and in the Polyakov gauge, only latter case is shown.

Fig.2 shows the correlation among $Q_{\text{SU(2)}}$, $Q_{\text{Ds}}$ and $Q_{\text{Ph}}$ after some cooling sweeps in the Polyakov gauge. A strong correlation is found between $Q_{\text{SU(2)}}$ and $Q_{\text{Ds}}$, which is defined in singular (monopole-dominating) part. Such a strong correlation remains even at 80 cooling sweeps. On the other hand, $Q_{\text{Ph}}$ quickly vanishes only by several cooling sweeps, and no correlation is seen between $Q_{\text{Ph}}$ and $Q_{\text{SU(2)}}$.

We show in Fig.3 the cooling curves for $Q$, $I_Q$ and $S$ in a typical example with $Q_{\text{SU(2)}} \neq 0$ in the Polyakov gauge. Similar to the full SU(2) case, $Q_{\text{Ds}}$, $I_Q_{\text{Ds}}$ and $S_{\text{Ds}}$ in the singular (monopole-dominating) part tends to remain finite during the cooling process. On the other hand, $Q_{\text{Ph}}$, $I_Q_{\text{Ph}}$ and $S_{\text{Ph}}$ in the regular part quickly vanish by only less than 10 cooling sweeps. Therefore, instantons seem unable to live in the regular (photon-dominating) part, but only survive in the singular (monopole-dominating) part in the abelian gauges.

The cooling curves for $Q$, $I_Q$ and $S$ are examined in the case with $Q_{\text{SU(2)}} = 0$. Similar to the full SU(2) case, $I_Q_{\text{Ds}}$ and $S_{\text{Ds}}$ decrease slowly and remain finite even at 70 cooling sweeps, which means the existence of the instanton and anti-instanton pair in the singular part. On the other hand, $I_Q_{\text{Ph}}$ and $S_{\text{Ph}}$ quickly vanish, which indicates the absence of such a topological pair excitation in the regular part.

In conclusion, the monopole dominance for the topological charge is found both in the MA gauge and in the Polyakov gauge. In particular, instantons would survive only in the singular (monopole-dominating) part in the abelian gauges, which agrees with the result in our previous analytical study. The monopole dominance for the $U_A(1)$ anomaly is also expected.

REFERENCES

1. G. ’t Hooft, Nucl. Phys. B190 (1981) 455.
2. A.S. Kronfeld, G. Schierholz and U.-J. Wiese, Nucl. Phys. B293 (1987) 461.
3. O. Miyamura and S. Origuchi, Color Confinement and Hadrons, (World Scientific, 1995).
4. H. Suganuma, H. Ichie, S. Sasaki and H. Toki, Color Confinement and Hadrons, (World Scientific, 1995).
5. T. DeGrand and D. Toussaint, Phys. Rev. D22 (1980) 2478.