DIAGRAMS OF DIVIDE LINKS

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Abstract. Recently N. A’Campo suggested a construction of a link from a generic immersion of a curve into a 2-disk. It is tightly related to the singularity theory. In this paper, we give a simple procedure to draw a diagram of the link from a picture of the curve.

There are several fascinating relations of plane immersed curves and links. One of them which goes through Legendrian links led Arnold [5] to the discovery of three simple invariants $J^+$, $J^-$, $St$ of such a curve. N. A’Campo in [2] suggested another construction of a link from a generic immersion of a curve into a 2-disk. It is tightly related to the singularity theory. If $f_\mathbb{R} : (\mathbb{R}^2, 0) \rightarrow (\mathbb{R}, 0)$ is a germ of an analytic function whose complexification $f_\mathbb{C}$ has an isolated critical point, then one can define a link $L$ of the singularity $f_\mathbb{C}$ as an intersection of the zero-level variety of $f_\mathbb{C}$ with a small 3-sphere in $\mathbb{C}^2$ centered at the critical point. Such links are sometimes called algebraic links. On the other hand, in singularity theory it is useful [1, 9] (see also [6]) to consider a small perturbation $D$ of the real plane singular curve \{ $f_\mathbb{R}$ = 0 \} which is a generic immersed curve with the maximal possible number of double points. A’Campo [2] restored the link $L$ directly from the curve $D$.

In this paper we give a simple procedure to draw a diagram of the link $L$ from a picture of $D$ (Theorem 2.2). It is essentially a particular case of the results of [8]. An advantage of our approach is that we obtain a link diagram directly from a divide picture without deforming it into a so-called ordered Morse signed divide as in [8]. A similar method to draw diagrams was found by M. Hirasawa in [10]. Our diagrams are obviously symmetrical in a sense that the rotation of $\mathbb{R}^3$ by 180° around the $x$-axis reverses the orientation of $L$.

1. Divides and their links

Definition 1.1 ([2, 3]). A divide $D$ is the image of a generic immersion of a finite number of copies of the unit interval $I = [0,1]$ in the unit disk $B \subset \mathbb{R}^2$ such that $\partial I$ is embedded in $\partial B$ and double points are the only singularities allowed.

We consider divides up to isotopy of the disk $B$. The isotopy does not assume to be identical on the boundary $\partial B$. 

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Example 1.2. The curve $x^3 + y^4 = 0$ has a singularity of type $E_6$ at the origin [6]. A small perturbation of it is a divide which looks as follows.

![Diagram of a divide]

Definition 1.3 ([2, 3]). Let $x$ be the horizontal coordinate on the disk $B$ and $y$ be the vertical coordinate. A divide link $L_D$ is a link in the 3-sphere $S^3 = \{(x, y, u, v) \in \mathbb{R}^4 | x^2 + y^2 + u^2 + v^2 = 1\}$ such that $(x, y)$ is a point on $D$ while $u$ and $v$ are the coordinates of a tangent vector to $D$ at the point $(x, y)$.

Therefore each interior point of $D$ has two corresponding points on $L_D$, and a boundary point of $D$ gives a single point on $L_D$.

The link $L_D$ has a natural orientation. Indeed, choose any orientation of every branch of $D$. Let $(u, v)$ be the tangent vector to $D$ at $(x, y)$ pointing to the direction of the chosen orientation of $D$. Then the orientation of $L_D$ is given by the vector $(\dot{x}, \dot{y}, \dot{u}, \dot{v})$. It is easy to see that this orientation of $L_D$ does not depend on the choice of orientations of branches of $D$.

The number of components of $L_D$ equals to the number of branches of the divide $D$ which is the number of the copies of the unit interval $I$ in Definition 1.1. In particular, if $D$ consists of only one branch as in Example 1.2, then $L_D$ will be a knot.

Remark 1.4. Topological type of a divide link does not change under a regular transversal isotopy of the disk $B$. Therefore it does not depend on the choice of coordinates in Definition 1.3. Also it does not change under a moving of a piece of the curve $D$ through a triple point [8]. In particular, the following two divides have the same knot type as the one in Example 1.2:

![Diagram of two divides]

Remark 1.5. In [2] A’Campo proved that all algebraic links are divide links. In [3] he showed that the links $L_D$ corresponding to a connected divide $D$ are fibered, and computed their monodromy in terms of the combinatorics of divide $D$. Moreover, he proved that the unknotted number of a one-branch divide knot $L_D$ equals the number of double points of $D$. Not all fibered links have the form $L_D$. Figure eight knot $4_1$ is not a divide knot. It is not clear how large is the class of divide links in the class of all fibered links.

Remark 1.6. It has been known for a long time [7] that algebraic knots are classified by the Alexander polynomial. N. A’Campo ([4]) found two different divide knots with the same Alexander polynomial.
2. Diagrams of divide links

**Definition 2.1.** Let us call a divide *generic* if its points with vertical tangents differ from the double points and the boundary points and their $x$-coordinates are pairwise different. The divide in Example 1.2 is generic. Any divide can be made generic by a small deformation.

**Theorem 2.2.** For a generic divide $D$ a link diagram of $\mathcal{L}_D$ can be drawn in the following way:

1. Consider a horizontal line below the disk $B$, say the line $\{y = -1.5\}$. Let $s$ be the symmetry with respect to the line. We are going to draw the link diagram of $\mathcal{L}_D$ by modifications of the union $D \cup s(D)$.
2. Replace each double point of the union by a crossing of the type $\times\times$.
3. Connect each boundary point $p$ of $D$ with $s(p)$ by a vertical string.
4. Replace a small piece of our curve near each point $p$ with a vertical tangent by two vertical strings connecting the upper and lower parts of the picture as in the examples below. The strings make a positive half-twist at the line of the reflection $s$. Note that at the upper part of the picture the right string goes below all intersected intervals of $D$ while the left string goes above the intervals. Correspondingly at the lower part of the picture the right string goes above the intervals of $s(D)$ and the left string goes below the intervals. Examples 2.3 and 2.4 demonstrate this.

**Example 2.3.** For the divide of Example 1.2 the theorem gives the diagram shown in Figure 1.
Example 2.4. We borrowed this divide from [3]. The corresponding knot is $10_{145}$.

Remark 2.5. As it was noted in [8] the theorem is also valid in the situation when we allow closed immersed components in the definition of a divide.

2.1. Proof of the Theorem 2.2. The theorem follows from the results [8] which give a representative braid for an ordered Morse signed divide (OMS). An OMS is a divide $D$ such that the $x$-coordinate as a function on $D$ has only two critical values: $a$ as the minimum critical value and $b$ as the maximum critical value; and the $x$-coordinate of each double point is between $a$ and $b$. Besides this a sign $+$ or $-$ is attached to each double point of $D$. See the details in [8]. An OMS is not a generic divide in our sense. Also in [8] there is an algorithm transforming an arbitrary divide to OMS (and attaching signs to double points). We use such a particular transformation pulling down a narrow tail near each critical point of the function $x|_D$ and then move a minimum (resp. maximum) left (resp. right) to the level $a$ (resp. $b$). For the divide of Example 2.4 this transformation gives the following OMS:

Then applying Proposition 4.2 from [8] we get the following representative braid:

The closure of this braid gives a knot diagram isotopic to ours. It is clear that the same arguments work for any generic divide as well.
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