On the prescribed $Q$-curvature problem in Riemannian manifolds

Received: 18 November 2019 / Accepted: 26 March 2020 / Published online: 17 April 2020

Abstract. We prove the existence of metrics with prescribed $Q$-curvature under natural assumptions on the sign of the prescribing function and the background metric. In the dimension four case, we also obtain existence results for curvature forms requiring only restrictions on the Euler characteristic. Moreover, we derive a prescription result for open submanifolds which allow us to conclude that any smooth function on $\mathbb{R}^n$ can be realized as the $Q$-curvature of some Riemannian metric.

1. Introduction

The problem of prescribing Riemannian curvatures has attracted considerable attention in the last decades. Such a problem provides an interesting interplay between differential geometry and nonlinear partial differential equations, since it can rely on to solve a system of PDE on the fundamental tensor of the Riemannian metric.

In this paper we are interested in the problem of prescribing a well-known fourth order conformal invariant introduced by Tom Branson [3] called $Q$-curvature. In other words, we present conditions that allow us to find a solution $g$ for the four order differential equation

$$Q_g = f,$$

(1.1)

for a given smooth function $f$, where $Q_g$ denotes the $Q$-curvature of the metric $g$. We emphasize that, by an entirely different method such as the one considered by us, (1.1) may be solved using a metric $g$ that is pointwise conformal to a fixed metric $g_0$, say $g = e^\varphi g_0$, for some function $\varphi$. (For instances we refer the reader to [1,5,6,12,14,30] and references therein).

For surfaces the $Q$-curvature is the half of the scalar curvature while that for conformally flat manifolds of dimension four, its integral is a multiple of the Euler characteristic, that obviously refers to Gauss–Bonnet Theorem. The $Q$-curvature
also shares the same conformal behavior that the scalar curvature and satisfies analogous transformations laws under conformal rescaling of the metric (see [9]). It’s worth to point out that this scalar invariant has also been studied in theoretical physics with applications in quantum field theory and higher-derivative field theories (for more details, c.f. [28,32]).

Kazdan and Warner [22,24,26] completely solved the prescribed scalar curvature problem on closed manifolds under signal constraint. Considering the analogy between scalar curvature and $Q$-curvature, it is reasonable to ask whether similar results and/or techniques can be generalized to $Q$-curvature. Then, we prove the following result.

**Theorem 1.1.** Let $(M^n, g)$, $n \geq 3$, be a compact Riemannian manifold with constant $Q$-curvature. Any smooth function having the same sign as $Q_g$ somewhere is the $Q$-curvature of some metric, if $Q_g \neq 0$. Moreover, any smooth function that changes sign is the $Q$-curvature of some metric, if the $Q$-curvature is zero.

As already mentioned, in a four dimensional Riemannian manifold $(M^4, g)$ the $Q$-curvature satisfies a formula that reminds Gauss–Bonnet theorem:

$$\int_M \left( Q_g + \frac{|W_g|^2}{4} \right) dv = 8\pi^2 \chi(M),$$

(1.2)

where $W_g$ stands for the Weyl tensor of $g$. As $W_g$, the total $Q$-curvature, denoted by $\kappa_P = \int_M Q_g dv$, is invariant under conformal changes. Moreover, there is an analogous to the uniformization theorem in dimension four involving the $Q$-curvature proved by Djadli and Malchiodi in [13], which says that if $\kappa_P \neq 8k\pi^2$ for $k = 1, 2, \ldots$ and $\ker P_g = \{\text{const}\}$, where $P_g$ is the Paneitz-Operator, then $(M, g)$ admits a conformal metric with constant $Q$-curvature. Using this existence result, we can prove the following converse to the Gauss Bonnet Theorem for locally conformally flat manifold, which says that (1.2) imposes a sign constraint on $Q$ depending on $\chi(M)$, and conversely.

**Corollary 1.2.** Let $(M^4, g)$ be a compact locally conformally flat 4-manifold such that $\ker P_g = \{\text{const}\}$ and $\kappa_P \neq 8k\pi^2$ for $k = 1, 2, \ldots$. Then, a smooth function $f$ on $M$ is the $Q$-curvature of some metric on $M$ iff

(a) $f$ is positive somewhere, if $\chi(M) > 0$;
(b) $f$ changes sign or $f \equiv 0$, if $\chi(M) = 0$;
(c) $f$ is negative somewhere, if $\chi(M) < 0$.

Another interesting question that seems natural is to ask if a given function defined on a non-compact manifold is the curvature of some Riemannian metric. Related to this question, we proved the following result for open submanifolds of compact manifolds.

**Theorem 1.3.** Let $(M^n, g)$ be a non-compact Riemannian manifold, $n \geq 3$, diffeomorphic to an open submanifold of some compact manifold $(N^n, h)$ of constant $Q$-curvature $Q_0 \neq 0$. Any smooth function $f$ on $M^n$ can be realized as the $Q$-curvature of some Riemannian metric on $M$. 
In particular, we obtain:

**Corollary 1.4.** Any \( f \in C^\infty(\mathbb{R}^n), \ n \geq 3, \) is the \( Q \)-curvature of some Riemannian metric on \( \mathbb{R}^n \).

We should mention that is possible to rephrase the problem of prescribing curvature depending on the Euler characteristic in terms of curvature forms. Recall that the generalized Gauss–Bonnet theorem says that

\[
\frac{1}{16} \int_M \text{Pfaff} = 8\pi^2 \chi(M),
\]

where \( M \) is a 4-dimensional compact orientable Riemannian manifold boundaryless and \( \text{Pfaff} \) is the Pfaffian 4-form. We wonder to know if the conversely is true, that is, given any 4-form \( \Omega \) satisfying \( \int_M \Omega = 8\pi^2 \chi(M) \), there exists a metric \( g \) on \( M \) such that \( \Omega = \text{Pfaff} \)? We give an affirmative answer to this question that is analogue of the main result of Wallach-Warner [35]. It should be emphasized that this problem for higher dimensions was posed in [21], p. 3, and here we solve it in dimension four for a certain class of manifolds.

**Theorem 1.5.** Let \((M, g)\) be a compact, connected, orientable Riemannian 4-manifold such that \( \ker P_g = \{\text{const}\} \). Given any 4-form \( \Omega \) that satisfies

\[
\int_M \Omega = 8\pi^2 \chi(M),
\]

then there exists a metric pointwise conformal to \( g \) such that \( \Omega \) is a curvature 4-form.

The paper is organized as follows. In Sect. 2, we establish the fundamental concepts and we prove a local surjectivity result for the \( Q \)-curvature map in Sect. 2. In Sect. 4, we prove Theorems 1.1 and 1.3. Theorem 1.5 is proved in Sect. 5.

### 2. Preliminaries

Throughout this section, \( M^n \) will denote a compact connected Riemannian manifold without boundary, \( n \geq 3 \), \( S_2(M) \) the set of all smooth symmetric 2-tensors on \( M \), and \( \mathcal{M}^{4,p} \) the space of class \( W^{4,p} \) of the symmetric \((0, 2)\)-tensors. The \( Q \)-curvature of order four, denoted by \( Q_g \), is defined as

\[
Q_g = a_n \Delta_g R_g + b_n |Ric_g|^2_g + c_n R_g^2,
\]

where \( a_n = -\frac{1}{2(n-1)} \), \( b_n = -\frac{2}{(n-2)^2} \), \( c_n = \frac{n^2(n-4)+16(n-1)}{8(n-1)^2(n-2)^2} \), \( \Delta_g := g^{ij} \nabla_i \nabla_j \), \( R_g \) is the scalar curvature and \( |Ric_g| \) is the norm of the Ricci tensor.

Now consider the following nonlinear fourth order differential operator

\[
Q : \mathcal{M}^{4,p} \to L^p(M), \quad q \mapsto Q_g
\]
where \( \mathcal{M}^{4,p} \) denotes the open subset of \( S_2^{4,p} \) of the Riemannian metrics on \( M \). It is possible to show, using multiplicative properties of Sobolev spaces, see for example [31], the \( Q \)-curvature map is well defined and smooth for \( 2p > n \). In order to study the local surjectivity of the \( Q \)-curvature, we have to study the kernel of \( L^2 \)-formal adjoint for the linearization of \( Q \)-curvature.

Before stating results giving the linearization and \( L^2 \) formal adjoint of the map \( q \mapsto Q_g \), we first need a few definitions. The Lichnerowicz Laplacian acting on \( h \in S_2(M) \) is defined to be

\[
\Delta_L h_{jk} = \Delta h_{jk} + 2(Rm \cdot h)_{jk} - R_{ji}h_{ik}^j - R_{ki}h_{ij}^j,
\]

where \((Rm \cdot h)_{jk} := R_{ijkl}h^{kl}\).

Following notation in [29] we have

**Proposition 2.1.** (Lin-Yuan [29]) Given an infinitesimal variation \( h \), the linearization of the \( Q \)-curvature \( Q \) at \( g \), denoted by \( L_g \), in the direction of \( h \) is given by

\[
L_g h = a_n[-\Delta^2 (tr h) + \Delta \delta^2 h + \frac{1}{2} dR \cdot (d(tr h) + 2\delta h)
- \Delta(Ric \cdot h) - \nabla^2 R \cdot h] - b_n[Ric \cdot \Delta_L h + Ric \cdot \nabla^2 (tr h)
+ 2Ric \cdot \nabla(\delta h) + 2[Ric \times Ric] \cdot h + 2c_n(R(-\Delta(tr h))
+ \delta^2 h - Ric \cdot h],
\]

where \( \nabla^2 = \nabla_i \nabla_j, (\delta h)_{ij} := -(div v_g h)_i = -\nabla^i h_{ij} and (Ric \times Ric)_{ij} := R^l_i R_{lj} \).

The next theorem address the \( L^2 \) formal adjoint, denoted by \( L_g^* \).

**Proposition 2.2.** (Lin-Yuan [29]) The \( L^2 \) formal adjoint of \( L_g \) is given by

\[
L_g^* f = a_n[-g \Delta^2 f + \nabla^2 \Delta f - Ric \Delta f + \frac{1}{2} g \delta(f dR) + \nabla(f dR)
- f \nabla^2 R] - b_n[\Delta(f Ric) + 2 f Rm \cdot Ric + g \delta^2(f Ric)
+ 2\nabla \delta(f Ric)] - 2c_n(g \Delta(f R) - \nabla^2(f R) + f RRic).
\]

Recall that the principal symbol of a differential operator is an invariant that captures some very strong properties of the operator, as example, the ellipticity. In our case, it was observed in [29] that the principal symbol of \( L_g^* \) is

\[
\sigma_\xi(L_g^*) = -a_n \left(g|\xi|^2 - \xi \otimes \xi\right) |\xi|^2.
\]

Notice that \( L_g^* \) has an injective symbol. Indeed, taking the trace we obtain

\[
tr \sigma_\xi(L_g^*) = -a_n (n - 1) |\xi|^4.
\]

If \( \sigma_\xi(L_g^*) \) were zero, then \( tr \sigma_\xi(L_g^*) \) would be zero with \( \xi \neq 0 \). Therefore, \( L_g^* \) is overdetermined elliptic and, thus, \( L_g L_g^* \) is elliptic. This fact plays a fundamental
role in the proof of Theorem 3.1. We also point out that the the $L^2$-formal adjoint $L^*$ is also essential to derive a higher-order analogue of Ricci curvature on Riemannian manifolds, see [27]

Following Chang et al. [11] we define the notion of $Q$-singular space.

**Definition 2.3.** (Chang et al. [11]) A complete Riemannian manifold $(M, g)$ is said to be $Q$-singular, if $L^*_g$ possesses non-trivial kernel, that is,

$$\ker L^*_g \neq \{0\}.$$  

In this case, we say that $(M, g, f)$ is a $Q$-singular space, where $f \not\equiv 0$ is in the kernel of $L^*_g$.

Taking the trace of (2.3) one obtain

$$\text{tr}_g L^*_g f = \frac{1}{2} \left( P_g - \frac{n + 4}{2} Q_g \right) f,$$

which allows to prove that the condition of non-$Q$-singularity is satisfied for generic metrics.

**Theorem 2.4.** (Chang et al. [11]) Suppose that $(M^n, g)$ is a $Q$-singular space, then it has constant $Q$-curvature and

$$\frac{n + 4}{2} Q_g \in \text{Spec}(P_g).$$

**Remark 2.5.** It is possible to show that Theorem 2.4 implies that the set of non-$Q$-singular metrics on $M$ is open and dense in the $W^{4,p}$ topology for any $2p > n$.

3. **Local surjectivity**

We are now prepared to prove the following proposition about the local surjectivity of the $Q$-curvature map. However, before we do this, we will briefly discuss some useful facts.

Let $E, F$ be vector bundles over $M$, and let $D : W^{s,p}(E) \to W^{s-k,p}(F)$ be a $k$-th order differential operator, where $k \leq s \leq \infty$, $1 < p < \infty$. We notice that we can make use of a splitting lemma of Berger and Ebin [2]. Recall that if $D^* : W^{s+k,p}(F) \to W^{s,p}(E)$, its $L^2$ formal adjoint, has injective symbol, then

$$W^{s,p}(F) = \text{Im} D \oplus \ker D^*.$$  

A useful consequence is the following: If $D^*$ is injective and has injective symbol then we can conclude that $D$ is surjective. Inspired by the arguments in [23,25,26] (see also [16,17]), we prove the following result.

**Theorem 3.1.** Let $f \in L^p(M)$ and $2p > n$. Assume that $(M, g_1)$ is not $Q$-singular. Then there is an $\eta > 0$ such that if

$$\|f - Q_{g_1}\|_p < \eta,$$

then there is a $g \in \mathcal{M}^{4,k}$ such that $Q_g = f$. Furthermore, $g$ is smooth if $f$ is smooth.
Proof. The map \( g \mapsto Q_g \) is a quasilinear differential operator of fourth order which can be extend from \( \mathcal{M}^{4,p} \) into \( L^p \), for \( 2p > n \), using the Sobolev Embedding Theorem. Let \( F \) be a map from a sufficiently small neighborhood of zero \( U \subset W^{8,p} \) into \( L^p \) given by

\[
F(u) = Q_{g_1} + L_{g_1}^* u.
\]

We will use the implicit function theorem for Banach spaces in order to solve this eighth order quasilinear elliptic equation. First, it is straightforward to see that \( F'(0) \) is elliptic at \( u = 0 \) since

\[
F'(0)v = L_{g_1}^* v.
\]

Further, it is invertible since

\[
\ker L_{g_1} L_{g_1}^* = \ker L_{g_1}^* = 0.
\]

Indeed, note that in \( L^2 \),

\[
\langle L_{g_1} L_{g_1}^* v, v \rangle = \| L_{g_1}^* v \|^2 = 0.
\]

Recall that \( F(u) \) is a quasilinear map, then for a sequence \( \{u_j\} \) converging to \( u \) in \( W^{8,p} \), standard Sobolev embedding arguments imply that this convergence is uniform in \( C^6, \alpha(M) \), \( \alpha \in (0,1) \), function (observe that also the coefficients related to \( F \) as quasilinear map are in \( C^6, \alpha(M) \)). Using a bootstrap argument and elliptic regularity theory the result follows. \( \square \)

4. Prescribing curvature on compact and open manifolds

In this section we will prove results that establish conditions to prescribing the \( Q \)-curvature. First, we fix a non-\( Q \)-singular metric \( g_1 \) and set \( Q_{g_1} = Q_1 \). The idea is as follows. In order to solve

\[
Q_g = f,
\]

for a given smooth function \( f \), we have to apply Theorem 3.1 which holds only for functions \( f \) near \( Q_{g_1} \) in some appropriated sense, that is, \( \| Q_1 - f \| < \varepsilon \) in \( L^p \) norm. To overcome this difficult we make use of the existence of a diffeomorphism \( \varphi \) such that \( \| Q_1 - f \circ \varphi \|_p < \varepsilon \) (see Lemma 4.1). Hence, we can conclude that there exists a metric up to diffeomorphism that solves (4.1) for \( f \).

We will make use of the following Approximation Lemma.

Lemma 4.1. (Approximation Lemma [24]) Let \( M^n, n \geq 2 \), be a compact manifold and let \( f, r \in C^1(M) \cap L^p(M) \). If there exists a positive constant \( c \) such that the range of \( r \) is contained in the range of \( cf \), that is,

\[
r(x) \in [\inf cf, \sup cf]
\]

for almost all \( x \) on \( M \), then given any \( \varepsilon > 0 \) there is a diffeomorphism \( \varphi \) of \( M \) such that

\[
\| cf \circ \varphi - r \|_p < \varepsilon,
\]

and conversely.
Remark 4.2. The above result is trivially false for the uniform metric.

Now we can prove our first prescribing result.

Proposition 4.3. Let \((M^n, g_0)\), \(n \geq 3\), be a smooth compact Riemannian manifold with \(Q\)-curvature, \(Q\), and let \(f \in C^{j+\alpha}(M)\), for some \(j \in \mathbb{N}\) and \(\alpha \in (0, 1)\). Assume that there is a positive constant \(c\) such that

\[
Q(x) \in [\min cf, \max cf]
\]

for all \(x \in M\), then there is a metric \(g \in C^{j+\alpha+4}\) with \(Q_g = f\).

Proof. Assume that \((M, g_0)\) is non-\(Q\)-singular. By Lemma 4.1, there exists a diffeomorphism \(\varphi\) such that

\[
\|Q - cf \circ \varphi\|_p < \varepsilon
\]

for all \(\varepsilon > 0\) and \(2p > n\). Since \(\ker L_g^* \neq \{0\}\), it follows from Theorem 3.1 that there is a metric \(g_1\) with \(Q_{g_1} = cf \circ \varphi\). Since the \(Q\)-curvature is invariant by diffeomorphism and homogeneous of degree \(-2\), the metric having \(Q\)-curvature \(f\) is given by

\[
g = (\varphi^{-1})^*(\sqrt{cg_1}).
\]

Otherwise, if \(M\) is \(Q\)-singular (which implies that \(Q\) is constant) we may modify slightly \(g_0\) in order to obtain a metric \(\tilde{g}\) with non-constant \(Q\)-curvature still satisfying (4.2) and the result follows. \(\Box\)

As a consequence, we prove the following result that corresponds to Theorem 1.1.

Theorem 4.4. (Theorem 1.1) Let \((M^n, g_0)\) be a compact, Riemannian, \(n\)-manifold \((n \geq 3)\) with constant \(Q\)-curvature \(Q_0\). Any function \(f\) having the same sign as \(Q_0\) somewhere is the \(Q\)-curvature of some metric, if \(Q_0 \neq 0\), while if \(Q_0 = 0\), then any function \(f\) that changes sign is the \(Q\)-curvature of some metric.

Proof. The proof of this result is a consequence of Proposition 4.3. Indeed, note that (4.2) is satisfied by any function \(f\) having the same sign as \(Q_0\) at some point of \(M\), moreover if \(Q_0\) is identically zero, then (4.2) is satisfied if \(f\) changes sign on \(M\). Observe that instead \(f \in C^{j+\alpha}(M)\), for some \(j \in \mathbb{N}\) and \(\alpha \in (0, 1)\) we assume by elliptic regularity that \(f \in C^\infty(M)\). \(\Box\)

More recently, Lin and Yuan [29] have shown that non-\(Q\)-singular spaces are linearized stable, which turns to be very useful to finding solution in a given direction. Thus, it is possible to prescribe some kinds of \(Q\)-curvature problems. See also [8], for some related results.

The problem concerning the existence of metrics of constant \(Q\)-curvature in compact 4-manifolds was developed by Chang and Yang [10], Gursky [20] and Wei and Xu [36], and more recently, Djadli and Malchiodi [13] provided extensions of these works. In dimension four we have the following result whose assumptions are conformally invariant and generics (see also [33], for dimension higher than four).
Theorem 4.5. (Djadli and Malchiodi [13]) Suppose \( \ker P_g = \{ \text{const} \} \), and assume that the total \( Q \)-curvature

\[
\kappa_P \notin \{ 8\pi^2, 16\pi^2, 32\pi^2, \ldots \}.
\]

Then \((M, g)\) admits a conformal metric with constant \( Q \)-curvature.

We obtain the following corollary of Theorem 1.1 for compact locally conformally flat (or l.c.f) manifolds of dimension four, whose notion is characterized by the Weyl tensor.

Corollary 4.6. (Corollary 1.2) Let \((M^4, g)\) be a compact locally conformally flat 4-manifold such that \( \ker P_g = \{ \text{const} \} \) and \( \kappa_P \neq 8k\pi^2 \) for \( k = 1, 2, \ldots \). Then, a smooth function \( f \) on \( M \) is the \( Q \)-curvature of some metric on \( M \) if and only if

(a) \( \{ f > 0 \} \neq 0 \), if \( \chi(M) > 0 \);
(b) \( \{ f > 0 \} \neq 0 \) and \( \{ f < 0 \} \neq 0 \); or \( f \equiv 0 \), if \( \chi(M) = 0 \);
(c) \( \{ f > 0 \} \neq 0 \), if \( \chi(M) < 0 \).

Proof. For the necessity, notice that the signal of the Euler Characteristic implies directly in the signal of the \( Q \) curvature by (1.2). To prove the sufficiency, we observe that the existence of metrics with constant \( Q \)-curvature is given by Theorem 4.5, thus the result follows immediately from Theorem 1.1.

Remark 4.7. \( \kappa_P \) has an upper sharp inequality (see [20]) besides being multiple of the Euler characteristic on conformally flat structures.

Next we prescribe the \( Q \)-curvature of open submanifolds which reads as follows.

Theorem 4.8. (Theorem 1.3) Let \( M^n \) be a non-compact Riemannian manifold, \( n \geq 3 \), diffeomorphic to an open submanifold of some compact \( n \)-manifold \( N \) of constant \( Q \)-curvature \( Q_0 \neq 0 \). Then every \( f \in C^\infty(M) \) is the \( Q \)-curvature of some Riemannian metric on \( N \).

Remark 4.9. Since on surfaces the \( Q \)-curvature is essentially the Gaussian curvature, we conclude using Bonnet-Myers theorem and completeness that if the sign of \( f \) were positive, then we would have compactness. Hence, one cannot always hope to achieve a complete metric which has a given \( f \).

Proof of Theorem 4.8. Assume with no loss of generality that \( N \setminus M \) contains an open set and that \( M \) and \( N \) are connected. Now, extend \( Q \) to \( N \) by defining it to be identically equal to \( Q_0 \) on \( N \setminus M \). By Approximation Lemma 4.1, there exists a diffeomorphism \( \varphi \) on \( N \) such that

\[
\| f \circ \varphi - Q_0 \|_p < \varepsilon,
\]

where \( 2p > n \). Since \( \varphi^{-1}(M) \subset N \), by Theorem 3.1 there is a metric \( g_1 \) with

\[
Q_{g_1} = f \circ \varphi.
\]

Hence \( f \) is the curvature of the pulled-back metric \((\varphi^{-1})^*(g_1)\) on \( M \) and the result follows.
An immediate and interesting consequence of the above theorem is the following.

**Corollary 4.10.** Any \( Q \in C^\infty(\mathbb{R}^n), \ n \geq 3, \) is the \( Q \)-curvature of some Riemannian metric on \( \mathbb{R}^n. \)

Due to the complexity of the \( L^2 \)-formal adjoint of \( L_g \) (2.3), it does not seem easy to study the geometry of \( Q \)-singular manifolds. On the other hand, as we can see in Theorem 1.3 of [29], this is more treatable if we restrict to Einstein manifolds with non-negative scalar curvature, sometimes called of nonnegative Einstein \( Q \)-singular spaces, then we can obtain some interesting examples, as for example, Ricci-flat spaces and round spheres. Based in these comments and in the light of the previous result, we see that a necessary condition for a complete Einstein metric \( g \) is \( Q \)-curvature of some some function if the associated scalar curvature satisfies

\[
\lim_{r \to +\infty} \inf_{x \geq r} R_g(x) \leq 0,
\]

where \( R_g \) is the scalar curvature at \( x \in \mathbb{R}^n. \) Indeed, this follows from the Bonnet-Myer’s Theorem.

### 5. Prescribing 4-forms

Given a Riemannian manifold \((M^n, g)\) let us recall some basic facts concerning its Riemannian geometry. The first one deals with the classical decomposition of the Riemannian curvature tensor with respect to the Hilbert-Schmidt inner product

\[
\text{Riem}_g = \frac{R_g}{2n(n-1)} g \odot g + \frac{1}{n-2} \left( \text{Ric}_g - \frac{R_g}{n} g \right) \odot g + W_g, \quad (5.1)
\]

where \( \odot \) stands for the Kulkarni–Nomizu product of symmetric bilinear forms.

In 4 dimension, consider the following curvature 4-form

\[
\Omega_g = \left( Q_g + \frac{1}{4} |W_g|^2 \right) dv ol g. \quad (5.2)
\]

Observe its close relation to the Pfaffian, defined as

\[
Pfaff_g = \left( 16 Q_g + 4 W^{abcd} W_{abcd} - \frac{8}{3} \Delta_g R_g \right) dv ol g.
\]

Using the Gauss–Bonnet–Chern formula and (5.1) we have that

\[
32\pi^2 \chi(M) = \int_M \left( |\text{Riem}_g|^2 - 4 |\text{Ric}_g|^2 + R_g^2 \right) dv ol g
\]

\[
= \int_M (4 Q_g + |W_g|^2) dv ol g
\]

\[
= 4 \int_M \Omega_g.
\]
Next, we show that given a 4-form $\omega$ satisfying

$$\int_M \omega = 8\pi^2 \chi(M),$$

we find a metric $\tilde{g}$ that satisfies $\omega = \Omega_{\tilde{g}}$. This new metric is obtained pointwise conformal to $g$.

In order to proceed we need some preliminaries definitions. Recall that in four dimension, the Paneitz-operator is a 4-th order differential operator defined by

$$P_g u = \Delta_g^2 u + \text{div}_g \left( \frac{2}{3} R_g g - 2 \text{Ric}_g \right) du,$$

where $d$ is the differential (acting on functions). $P_g$ it is conformally invariant. Indeed, performing the conformal change of metric $\tilde{g} = e^{2\varphi} g$ we get that $P_{\tilde{g}} = e^{-4\varphi} P_g$. In this sense, the transformation law by conformal metric of the Paneitz operator represents an analogue of the Laplace-Beltrami operator. Moreover, it is well known that, as well as the $Q$-curvature, $P_g$ is natural, that is, $\phi^* P_g = P_{\phi^* g}$ for all smooth diffeomorphism $\phi : M \to M$, and self-adjoint with respect to the $L^2$-scalar product (see e.g. [15,18,34]).

Now we are in position to prescribe curvature 4-forms in dimension four.

**Theorem 5.1.** (Theorem 1.5) Let $(M, g)$ be a compact, connected, orientable Riemannian 4-manifold such that $\ker P_g = \{\text{const}\}$. Given any 4-form $\omega$ that satisfies

$$\int_M \omega = 8\pi^2 \chi(M),$$

then there exist a metric pointwise conformal to $g$ such that $\omega$ is a curvature 4-form.

**Proof.** The proof consists in seeking a metric $\tilde{g} = e^{2\varphi} g$, or more precisely a function $\varphi$, in order to realize a given $\omega$ as $\Omega_{\tilde{g}}$. First we recall that for pointwise conformal metrics one has

$$P_g \varphi + Q_g = Q_{\tilde{g}} e^{4\varphi}, \quad (5.3)$$

Thus

$$\Omega_{\tilde{g}} = \left( Q_{\tilde{g}} + \frac{1}{4} |W_{\tilde{g}}|^2 \right) dvol_{\tilde{g}} = \Omega_g + P_g \varphi \ dvol_g, \quad (5.4)$$

where we have used that

$$dvol_{\tilde{g}} = e^{-4\varphi} dvol_g \quad \text{and} \quad e^{4\varphi} \tilde{g}(W_{\tilde{g}}, W_{\tilde{g}}) = g(W_g, W_g).$$

Thus, solve the linear equation

$$\Omega_{\tilde{g}} - \Omega_g = P_g \varphi \ dvol_g, \quad (5.5)$$
for some \( \varphi \), is equivalent to realize the 4-form \( \omega \) as \( \Omega_{\tilde{g}} \). Moreover, observe that (5.5) can be rewritten as

\[
P_g \varphi = \ast (\Omega_{\tilde{g}} - \Omega_g), \tag{5.6}
\]

where \( \ast \) stands for the Hodge star operation with respect to \( g \).

Taking into account that \( \int_M (\Omega_{\tilde{g}} - \Omega_g) = 0 \) and that \( P_g \) is self-adjoint with \( \text{ker} \ P_g = \{\text{const}\} \), it follows from elliptic theory that (5.6) has a unique solution up to an additive constant. \( \square \)

Although, \( P_g \) and \( \kappa_P \) are conformal invariant objects on four manifolds, they provide some interesting geometric information. Indeed, if a manifold of non-negative Yamabe invariant \( Y(g) \) satisfies also \( \kappa_P \geq 0 \), then \( \text{ker} \ P_g \) consists only of the constant functions and \( P_g \) is a non-negative operator. Thus, instead of assuming \( P_g \) with trivial kernel, one may suppose \( Y(g) \geq 0 \) and \( \kappa_P \geq 0 \), so we have.

**Corollary 5.2.** Let \( (M, g) \) be a compact, connected, orientable Riemannian 4-manifold of non-negative Yamabe invariant \( Y(g) > 0 \) and \( \chi(M) > 0 \). Given any 4-form \( \omega \) that satisfies

\[
\int_M \omega = 8\pi^2 \chi(M),
\]

then there exist a metric pointwise conformal to \( g \) such that \( \omega \) is a curvature 4-form.

Graham et al. [19] have defined a family of conformally invariant operators \( P_{k,g} \) (in odd dimensions, \( k \) is any positive integer, while in dimension \( n \) even, \( k \) is a positive integer no more than \( \frac{n}{2} \)), whose leading term is \( \Delta \), that are high-order analogues to the Laplace-Beltrami operator and to the Paneitz operator for high dimensional compact manifolds. As the case treated here, these operators, the so-called of GJMS operators, have associated curvature invariants \( Q_{k,g} \). For more detail, see [7,19,33,34]. Furthermore, it was proved in [4] that given a closed locally conformally flat manifold \( (M, g) \) of even dimension \( n \), we have that

\[
C_n \int_M Q_{k,g} dV = \chi(M),
\]

where \( C_n = \frac{1}{((n-2)/2)! |S^{n-1}|} \) (\( |S^{n-1}| \) denotes the volume of the standard (n-1)-sphere of radius 1). Hence, the methods of Theorem 1.5 apply equally well, with minor modifications, in order to prescribe curvature n-forms of locally conformally flat manifolds using \( Q_{k,g} \) and \( P_{k,g} \).

**Acknowledgements** T.C would like to thank C. Arezzo for his kind interest in this work. We thank the referee for valuable comments and suggestions.
References

[1] Baird, P., Fardoun, A., Regbaoui, R.: Prescribed $Q$-curvature on manifolds of even dimension. J. Geom. Phys. 59(2), 221–233 (2009)
[2] Berger, M., Ebin, D.: Some decompositions of the space of symmetric tensors on a Riemannian manifold. J. Differ. Geom. 3, 379–392 (1969)
[3] Branson, T.P.: Differential operator canonically associated to a conformal structure. Math. Scand. 57, 293–345 (1985)
[4] Branson, T., Gilkey, P., Pohjanpelto, J.: Invariants of locally conformally flat manifolds. Trans. Am. Math. Soc. 347, 939–953 (1995)
[5] Brendle, S.: Global existence and convergence for a higher order flow in conformal geometry. Ann. Math. (2) 158, 323–343 (2003)
[6] Brendle, S.: Convergence of the $Q$-curvature flow on $S^4$. Adv. Math. 205, 1–32 (2006)
[7] Canzani, Y., Gover, R., Jakobson, D., Ponge, R.: Conformal invariants from nodal sets. I. Negative eigen values and curvature prescription. Int. Math. Res. Notice IMRN 9, 2356–2400 (2014)
[8] Case, J., Lin, Y., Yuan, W.: Conformally variational riemannian invariants. Trans. Am. Math. Soc. 371(11), 8217–8254 (2019)
[9] Chang, S.Y.A., Eastwood, M., Ørsted, B., Yang, P.: What is $Q$-curvature? Acta Appl. Math. 102(2–3), 119–125 (2008)
[10] Chang, S.Y.A., Yang, P.C.: Extremal metrics of zeta function determinants on 4-manifolds. Ann. Math. 142, 171–212 (1995)
[11] Chang, S.-Y.A., Gursky, M., Yang, P.: Remarks on a fourth order invariant in conformal geometry. Aspects Math. HKU. 353–372 (2019)
[12] Chtioui, H., Rigane, A.: On the prescribed $Q$-curvature problem on $S^n$. J. Funct. Anal. 261, 2999–3043 (2011)
[13] Djadli, Z., Malchiodi, A.: Existence of conformal metrics with constant $Q$-curvature. Ann. Math. 168(3), 813–858 (2008)
[14] Delanoé, P., Robert, F.: On the local Nirenberg problem for the $Q$-curvatures. Pacific J. Math. 231, 293–304 (2007)
[15] Fefferman, C., Graham, C.R.: $Q$-curvature and Poincaré metrics. Math. Res. Lett. 9, 139–151 (2002)
[16] Fischer, A., Marsden, J.: Linearization stability of nonlinear partial differential equations. In: Proceedings of a Symposium in Pure Mathematics, vol. 27. American Mathematical Society, Providence, pp. 219–263 (1975)
[17] Fisher, A.: Mardsen, Linearization stability of nonlinear partial differential equations. In: Proceedings of a Symposium in Pure Mathematics, vol. 27, Part 2. pp. 219–263 (1975)
[18] Graham, C.R., Zworski, M.: Scattering matrix in conformal geometry. Invent. Math. 152, 89–118 (2003)
[19] Graham, C.R., Jenne, R., Mason, L.J., Sparling, G.A.J.: Conformally invariant powers of the Laplacian I Existence. J. Lond. Math. Soc. (2) 46(3), 557–565 (1992)
[20] Gursky, M.: The principal eigenvalue of a conormally invariant differential operator, with an application to semilinear elliptic PDE. Commun. Math. Phys. 207, 131–147 (1999)
[21] Kazdan, J.: Prescribing the Curvature of a Riemannian Manifold. American Mathematics Society, New York (1984) (CBMS Regional Conference Series 57)
[22] Kazdan, J., Warner, F.: Curvature functions for compact 2-manifold. Ann. Math. 99, 14–47 (1974)
On the prescribed $Q$-curvature problem

[23] Kazdan, J., Warner, F.: Curvature functions for open 2-manifold. Ann. Math. 99, 203–219 (1974)

[24] Kazdan, J., Warner, F.: Existence and conformal deformation of metrics with prescribed Gaussian and scalar curvature. Ann. Math. (2) 101, 317–331 (1975)

[25] Kazdan, J., Warner, F.: Scalar Curvature and conformal deformation of Riemannian structure. J. Differ. Geom. 10, 113–134 (1975)

[26] Kazdan, J., Warner, F.: A direct approach to the determination of Gaussian and scalar curvature functions. Invent. Math. 28, 227–230 (1975)

[27] Lin, J., Yuan, W.: A symmetric 2-tensor canonically associated to $Q$-curvature and its applications. Pacific J. Math. 291, 425–438 (2017)

[28] Levy, T., Oz, Y.: Liouville conformal field theories in higher dimensions (2018). arXiv:1804.02283. [hep-th]

[29] Lin, Y.-J., Yuan, W.: Deformations of $Q$-curvature I. Calc. Var. Partial Differ. Equ. 55(4):Paper No. 101, 29 (2016)

[30] Malchiodi, A., Struwe, M.: $Q$-curvature flow on $S^4$. J. Differ. Geom. 73, 1–44 (2006)

[31] Maz’ya, V.G., Shaposhnikova, T.O.: Theory of Multipliers in Spaces of Differentiable Functions. Monographs and Studies in Mathematics, vol. 23. Pitman, Boston (1985)

[32] Nakayama, Y.: Canceling the Weyl anomaly from a position-dependent coupling. Phys. Rev. D 97(4), 045008 (2018). https://doi.org/10.1103/PhysRevD.97.045008. arXiv:1711.06413

[33] Ndiaye, C.B.: Constant $Q$-curvature metrics in arbitrary dimension. J. Funct. Anal. 251(1), 1–58 (2007)

[34] Robert, F.: Admissible $Q$-Curvatures Under Isometries for the Conformal GJMS Operators. Contemporary Mathematics, vol. 540, pp. 241–259. American Mathematics Society, Providence (2011)

[35] Wallach, N., Warner, F.: Curvature forms for 2-manifolds. Proc. Am. Math. Soc. 25, 712–713 (1970)

[36] Wei, J., Xu, X.: On conformal deformations of metrics on $S^n$. J. Funct. Anal. 157, 292–325 (1998)

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.