SEMICONtinuity of Eigenvalues under Intrinsic Flat Convergence

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Abstract. We use the theory of rectifiable metric spaces to define a normalized Dirichlet energy of Lipschitz functions defined on the support of integral currents. This energy is obtained by integration of the square of the norm of the tangential derivative, or equivalently of the approximate local dilatation, of the Lipschitz functions. We define min-max values based on the normalized energy and show that when integral current spaces converge in the intrinsic flat sense without loss of volume, the min-max values of the limit space are larger than or equal to the upper limit of the min-max values of the currents in the sequence. In particular, the infimum of the normalized energy is semicontinuous. On spaces that are infinitesimally Hilbertian, we can define a linear Laplace operator. We can show that semicontinuity under intrinsic flat convergence holds for eigenvalues below the essential spectrum, if the total volume of the spaces converges as well.

1. Introduction

Riemannian manifolds come with a natural definition of a Laplace operator, which is involved in describing heat flow or diffusion on the manifold. The properties of the Laplace operator, such as its eigenvalues and eigenfunctions, give some information on the geometry of the manifold, and vice versa (see for instance [7], [8] and [5]). In this context it is natural to ask whether the spectra of the Laplace operator on the manifolds converge when the manifolds do.

It is easy to see that the spectrum of the Laplace operator varies continuously under $C^2$-convergence of manifolds. As for weaker types of convergence, Fukaya has constructed examples demonstrating that under Gromov-Hausdorff convergence, the eigenvalues of the Laplace operator need not be continuous [14]. One way to retrieve continuity is to keep track of a measure on the space as well, that is, to consider Fukaya’s metric measure convergence rather than just Gromov-Hausdorff convergence. Fukaya showed that with uniform bounds on the sectional curvature and the diameter, the eigenvalues of the Laplace operator on manifolds are continuous under measured Gromov-Hausdorff convergence [14]. He also conjectured that the assumption of uniform bounds on the sectional curvature could be replaced by a lower bound on the Ricci curvature. He proved that without curvature bounds, the eigenvalues are still upper semicontinuous [14].

Cheeger and Colding proved Fukaya’s conjecture, in that they showed that with a uniform lower bound on the Ricci curvature, a Laplace operator can be defined on the limit space, and the eigenvalues and eigenfunctions are continuous [11]. Moreover, these limit spaces are almost everywhere Euclidean.

Rather than considering metric measure convergence, we will study what happens to the eigenvalues of the Laplace operator when the underlying manifolds converge in the intrinsic flat sense.

Ambrosio and Kirchheim showed how one can define currents on metric spaces [3], following an idea by De Giorgi [12]. Sormani and Wenger [21] applied their work to define an intrinsic flat distance between oriented Riemannian manifolds of finite volume (and more generally, between integral current spaces). The flat distance was originally introduced by Whitney for submanifolds of Euclidean space, and extended to integral currents by Federer and Fleming [13]. Similar to how the Gromov-Hausdorff distance relates to the Hausdorff distance, the intrinsic flat distance between two manifolds is determined as the infimum of the flat distance between isometric embeddings of the manifolds taken over all isometric embeddings into all possible common metric spaces.

There are important differences between metric-measure convergence and intrinsic flat convergence. The metric measure limit of compact metric measure spaces is always compact, while
this need not be the case under intrinsic flat convergence. Limits obtained under intrinsic flat convergence are always rectifiable, while Fukaya’s metric measure limits are only metric spaces with a doubling measure. Moreover, there are examples of sequences of spaces that do not have a Gromov-Hausdorff limit, but do have an intrinsic flat limit [21].

Sequences of manifolds that converge in the intrinsic flat sense occur naturally, since by Wenger’s compactness theorem for integral currents on metric spaces [23], a sequence of integral current spaces with bounded diameter, mass and mass of the boundary has a subsequence converging in the intrinsic flat sense. In other words, a sequence of oriented Riemannian manifolds with boundary, with uniform bounds on the volumes and the volumes of the boundary, will have a subsequence converging in the intrinsic flat sense to an integral current space.

The following theorem presents our first main result formulated for closed, oriented Riemannian manifolds, instead of more general integral currents.

**Theorem 1.1.** Let \( M_i (i = 1, 2, \ldots) \) and \( M \) be closed, oriented Riemannian manifolds such that as \( i \to \infty \), \( M_i \) converges to \( M \) in the intrinsic flat sense. Moreover, assume that \( \text{Vol}(M_i) \to \text{Vol}(M) \).

Then, for \( k = 1, 2, \ldots \),

\[
\limsup_{i \to \infty} \lambda_k(M_i) \leq \lambda_k(M),
\]

where \( \lambda_k(M) \) is the \( k \)-th eigenvalue of the Laplace operator on a manifold \( M \).

The precise definition of min-max values \( \lambda_k(M) \) when \( M \) is not a Riemannian manifold is given through a min-max variational problem involving a normalized energy in (4.3). We show semicontinuity of \( \lambda_k \) in Theorem 4.2 under flat convergence without loss of volume and in Theorem 5.3 we prove semicontinuity of \( \lambda_k \) under intrinsic flat convergence without loss of volume. In particular, it follows that the infimum of the normalized energy is semicontinuous. When the currents involved are supported on an infinitesimally Hilbertian rectifiable metric space, we can define an unbounded self-adjoint operator on their rectifiable sets. The min-max values of these operators correspond to the \( \lambda_k \) defined before, and consequently, all \( \lambda_k \) below the essential spectrum correspond to eigenvalues of the operators. However, in the general case it follows from the work by Kirchheim [16] and Ambrosio and Kirchheim [1] that rectifiable metric spaces are in a measure-theoretic sense locally Finsler. Even for Finsler spaces, there does not seem to be a canonical choice of a Laplace operator, see for instance the works by Bao and Lackey [4], Centore [9] and Barthelmé [6]. We will use this local Finsler structure to define (in Definition 5.6) a normalized Dirichlet energy \( E_{\Gamma}(f) \) of a function \( f \) on the support of an integral current \( T \). For Riemannian manifolds, this normalized energy corresponds to the usual Dirichlet energy. A similar definition of the normalized energy for instance appears in Gromov’s work [15] and the work by Ambrosio, Gigli and Savaré [2]. This definition also corresponds to the energy that is at the basis of the non-linear Laplacian as introduced by Shen [20], when restricted to Finsler manifolds.

Our definition differs from the one used by Gromov [15] and Ambrosio, Gigli and Savaré [2]. We will measure an approximate local dilatation only, rather than a full local dilatation as used in [15] and [2]. We will show that the approximate dilatation corresponds almost everywhere to the norm of the tangential derivative. Unlike the definitions by Gromov [15] and Shen [20], we do not use the Hausdorff measure on the space, but rather the mass measure as introduced by Ambrosio and Kirchheim [1], which is the natural measure on an integral current space.

The structure of this paper is as follows. In Section 2 we review preliminaries from the work by Kirchheim [16], Ambrosio and Kirchheim [3, 4], and Sormani and Wenger [21]. In Section 3 we introduce the normalized energy and show we can define it either using the norm of the tangential derivative or the approximate local dilatation. Subsequently, we introduce the min-max functionals \( \lambda_k \) in Section 3 and show their semicontinuity under flat convergence in a fixed space without loss of volume. We then show semicontinuity of the \( \lambda_k \) under intrinsic flat convergence in Section 5 and the semicontinuity of eigenvalues for infinitesimally Hilbertian spaces in Section 6. Finally, in Section 7 we show by examples that under conditions of Theorem 1.1 the eigenvalues may jump up, and that we cannot remove the condition on convergence of the volumes or else the eigenvalues may jump down in the limit.
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2. Background

In this section, we will summarize the properties of currents on metric spaces that we will use. Currents on metric spaces were introduced by Ambrosio and Kirchheim \[3\], following an idea by DeGiorgi \[12\].

2.1. Currents on metric spaces. We first review the main concepts from the paper by Ambrosio and Kirchheim on currents on metric spaces \[3\].

Let \( Z \) be a complete metric space. For a positive integer \( n \), let \( \mathcal{D}^n(Z) \) denote the set of all \((n+1)\)-tuples \( \omega = (f, \pi_1, \ldots, \pi_n) \), such that \( f \) is bounded and Lipschitz and the \( \pi_i \) are Lipschitz.

In a smooth setting, \( \omega \) would correspond to \( f d\pi_1 \wedge \cdots \wedge d\pi_n \).

The exterior differential \( d \) is an action that creates an \((n+2)\)-tuple out of an \((n+1)\)-tuple as follows

\[
d(f, \pi_1, \ldots, \pi_n) = (1, f, \pi_1, \ldots, \pi_n).
\]

If \( \phi \) is a Lipschitz map from \( Z \) to another metric space \( E \), we define the pullback \( \phi^* \omega \in \mathcal{D}^n(Z) \) of \( \omega \in \mathcal{D}^n(E) \) by

\[
\phi^* \omega = \phi^* (f, \pi_1, \ldots, \pi_n) = (f \circ \phi, \pi_1 \circ \phi, \ldots, \pi_n \circ \phi).
\]

An \( n \)-dimensional metric functional is a function \( T : \mathcal{D}^n(Z) \to \mathbb{R} \) that is subadditive and positively 1-homogeneous with respect to \( f \) and the \( \pi_i, i = 1, \ldots, n \).

The boundary \( \partial T \) of a metric functional \( T \) is defined by

\[
\partial T(\omega) = T(d\omega), \quad \omega \in \mathcal{D}^n(Z).
\]

and the push-forward of \( T \) under a map \( \phi \) is given by

\[
\phi_* T(\omega) = T(\phi^* \omega), \quad \omega \in \mathcal{D}^n(Z).
\]

A metric functional \( T \) is said to have finite mass if there exists a finite Borel measure \( \mu \) such that for every \((f, \pi_1, \ldots, \pi_n) \in \mathcal{D}^n(Z)\),

\[
|T(f, \pi_1, \ldots, \pi_n)| \leq \prod_{i=1}^n \text{Lip}(\pi_i) \int_Z |f|d\mu.
\]

The mass of \( T \), which we denote by \( \|T\| \), is defined as the minimal \( \mu \) satisfying \( (2.5) \).

Definition 2.1 (\[3\]). An \( n \)-current on a metric space \( Z \) is defined to be a metric functional \( T \) with the additional properties that

1. \( T \) is multilinear in \((f, \pi_1, \ldots, \pi_n)\),
2. \( \lim_{j \to \infty} T(f, \pi_1, \ldots, \pi_n) = T(f, \pi_1, \ldots, \pi_n) \) whenever \( \pi_j \to \pi_j \) pointwise in \( Z \) with \( \text{Lip}(\pi_j) \leq C \) for some constant \( C \).
3. \( T(f, \pi_1, \ldots, \pi_n) = 0 \) if for some \( i \in \{1, \ldots, n\} \) the function \( \pi_i \) is constant on a neighborhood of \( \{f \neq 0\} \).

We denote by \( \mathcal{M}_n(Z) \) the Banach space of \( n \)-dimensional currents on \( Z \) with finite mass, with norm \( \|T\| \).

A function \( g \in L^1(\mathbb{R}^n) \) induces a current \([g] \in \mathcal{M}_n(\mathbb{R}^n)\) by

\[
[g](f, \pi_1, \ldots, \pi_n) := \int_{\mathbb{R}^n} g f \det(\nabla \pi) dx.
\]
A normal current is a current \( T \in M_n(Z) \) such that \( \partial T \in M_{n-1}(Z) \). A current \( T \) is called rectifiable if \( ||T|| \) is concentrated on a countably \( H^n \)-rectifiable set, and vanishes on \( H^n \)-negligible Borel sets. We define

\[
\text{set}(T) := \left\{ x \in Z \mid \liminf_{r \downarrow 0} \frac{||T||{(B(x, r))}}{r^n} > 0 \right\}.
\]

For rectifiable currents, set(\( T \)) is a rectifiable set. It is said to be integer rectifiable if, in addition, for any \( \phi \in \text{Lip}(Z, \mathbb{R}^n) \) and any open \( O \subset Z \), \( \phi_B(T, O) = \langle [g] \rangle \) for some \( g \in L^1(\mathbb{R}^n, Z) \). Integral currents are integer rectifiable currents that are also normal. We denote the class of \( n \)-dimensional integral currents on \( Z \) by \( I_n(Z) \). The boundary operator \( \partial \) maps from \( I_{n+1}(Z) \) to \( I_n(Z) \).

2.2. The flat distance. Federer and Fleming [13] extended the concept of flat distance to Euclidean integral currents. In [22], Wenger explores the properties of the following analogous flat distance between integral currents on metric spaces.

**Definition 2.2** ([22]). The flat distance between \( T_1 \in I_n(Z) \) and \( T_2 \in I_n(Z) \) is given by

\[
d^F(T_1, T_2) = \inf \left\{ M(U) + M(V) \mid S - T = U + \partial V \right\}
\]

where the inf is taken over all \( U \in I_n(Z) \) and \( V \in I_{n+1}(Z) \), \( i = 1, 2 \).

2.3. The intrinsic flat distance. Subsequently, Sormani and Wenger introduced the concept of integral current spaces, and an intrinsic flat distance between them.

**Definition 2.3** ([21]). An \( n \)-dimensional integral current space \((X, d, T)\) is a triple of a separable metric space \( X \) with a metric \( d \) and an integer rectifiable current \( T \in I_n(X) \) on the completion \( \overline{X} \) of \( X \), with the additional condition that

\[
X = \left\{ x \in \overline{X} \mid \liminf_{r \downarrow 0} \frac{||T||{(B(x, r))}}{r^n} > 0 \right\}.
\]

An \( n \)-dimensional oriented manifold \( M \) induces an integral current space \([M] = (M, d_M, T)\), with \( d_M \) the geodesic distance on \( M \), by

\[
T(f, \pi_1, \ldots, \pi_n) = \int f \ d\pi_1 \wedge \cdots \wedge d\pi_n.
\]

Imitating Gromov’s definition of the Gromov-Hausdorff distance, Sormani and Wenger [21] introduced the intrinsic flat distance as follows.

**Definition 2.4** ([21]). The intrinsic flat distance between two integral current spaces \( M_i = (X_i, d_i, T_i) \), \( i = 1, 2 \), is defined by

\[
d^F(M_1, M_2) := \inf d^F((\phi_1)_#T_1, (\phi_2)_#T_2)
\]

where the infimum is over all complete metric spaces \( Z \) and all isometric embeddings \( \phi_i : X_i \rightarrow Z \).

As in Gromov’s definition, by an isometric embedding from a metric space \( X \) with metric \( d_X \) into a metric space \( Z \) with metric \( d_Z \) we mean a map \( I : X \rightarrow Z \) such that for every \( x, y \in X \),

\[
d_X(x, y) = d_Z(I(x), I(y)).
\]

We will also use the following result by Sormani and Wenger [21].

**Theorem 2.5** ([21]). Let \( M_i = (X_i, d_i, T_i) \) be a sequence of integral current spaces converging in the intrinsic flat sense to a limit integral current space \( M = (X, d, T) \). Then there exist a complete, separable metric space \( Z \), and isometric embeddings \( \phi_i : \overline{X_i} \rightarrow Z \), \( \phi : \overline{X} \rightarrow Z \) such that \( (\phi_i)_#T_i \rightarrow \phi_#T \) in the flat distance in \( Z \).

We finally include the following simple lemma for later use.

**Lemma 2.6.** Suppose \( Z \) is a complete metric space, and \( T_i \in M_n(Z) \), \( i = 1, 2, \ldots \) and \( T \in M_n(Z) \) such that \( T_i \rightarrow T \) weakly. Moreover, assume that \( M(T_i) \rightarrow M(T) \). Then \( ||T_i|| \rightarrow ||T|| \) weakly as measures.
We say that a finite Borel measure $\mu$ is a linear map $L$ on $\mathbb{R}^n$ such that
\begin{equation}
(2.19)
H = \omega
\end{equation}
for any $O \subset Z$ open. By assumption, $\|T_i\| (Z) \to \|T\| (Z)$. Since $Z$ is a complete metric space, it follows that $\|T_i\| \to \|T\|$ by for instance the portmanteau theorem (cf. [17]).

### 2.4. Rectifiable sets in metric and Banach spaces

In this section, we will review some important results on rectifiable sets on metric spaces, that were obtained mainly by Kirchheim [16] and Ambrosio and Kirchheim [1]. First of all, we review two concepts of differentiability, namely metric differentiability and $w^*$-differentiability.

**Definition 2.7 ([1] Definition 3.1).** Let $Z$ be a metric space. A function $g : \mathbb{R}^n \to Z$ is called metrically differentiable at a point $x \in \mathbb{R}^n$ if there is a seminorm $md_x g(.)$ on $\mathbb{R}^n$ such that
\begin{equation}
(2.14)
d(g(y), g(x)) - md_x g(y - x) = o(||y - x||),
\end{equation}
as $y \to x$. We call $md_x g$ the metric differential of $g$ at $x$.

**Definition 2.8 ([1] Definition 3.4).** Let $Y$ be a $w^*$-separable dual space, and let $g : \mathbb{R}^n \to Y$. We say that $g$ is $w^*$-differentiable at $x \in \mathbb{R}^n$ if there is a linear map $wd_x g : \mathbb{R}^n \to Y$ such that
\begin{equation}
(2.15)
w^* - \lim_{y \to x} \frac{g(y) - g(x) - wd_x g(y - x)}{||y - x||} = 0.
\end{equation}
The map $wd_x g$ is called the $w^*$-differential of $g$ at $x$.

Next, we recall the definition of the Jacobian of a linear map between two Banach spaces.

**Definition 2.9 ([1] Definition 4.1).** Let $V$ and $W$ be Banach spaces, with $\dim V = n$. Let $L$ be a linear map $L : V \to W$. Define the $n$-Jacobian of $L$ by
\begin{equation}
(2.16)
J_n(L) := \frac{\omega_n}{\mathcal{H}^n(\{x \in V \mid \|L(x)\| \leq 1\})}.
\end{equation}
Similarly, when $s$ is a seminorm on $\mathbb{R}^n$, define
\begin{equation}
(2.17)
J_n(s) := \frac{\omega_n}{\mathcal{H}^n(\{x \in \mathbb{R}^n \mid s(x) \leq 1\})}.
\end{equation}
Here, $\omega_n$ is the volume of the Euclidean unit ball in $n$ dimensions.

**Definition 2.10.** The upper and lower $n$-dimensional densities of a finite Borel measure $\mu$ at a point $x$ are defined respectively as
\begin{equation}
(2.18)
\Theta^*_n(\mu, x) := \limsup_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_n r^n}, \quad \Theta_*(\mu, x) := \liminf_{r \downarrow 0} \frac{\mu(B_r(x))}{\omega_n r^n}.
\end{equation}
When $\Theta^*_n(\mu, x) = \Theta_*(\mu, x)$ in a point $x$ we define the $n$-dimensional density of $\mu$ at $x$ by $\Theta_n(\mu, x) := \Theta^*_n(\mu, x)$.

**Definition 2.11.** A subset $S \subset Z$ is called countably $\mathcal{H}^n$-rectifiable if there exists a sequence of Lipschitz functions $g_j : A_j \subset \mathbb{R}^n \to Z$ such that
\begin{equation}
(2.19)
\mathcal{H}^n \left( S \setminus \bigcup_j g_j(A_j) \right) = 0.
\end{equation}
We say that a finite Borel measure $\mu$ is $n$-rectifiable if $\mu = \Theta \mathcal{H}^n, S$ for a countably $\mathcal{H}^n$-rectifiable set $S$ and a Borel function $\theta : S \to (0, \infty)$.

**Definition 2.12 ([1] Definition 5.5).** If $Y$ is a $w^*$-separable dual space, and $S \subset Y$ is countably $\mathcal{H}^n$-rectifiable, with functions $g_j$ as in Definition 2.11 the approximate tangent space to $S$ at a point $x$ is defined as
\begin{equation}
(2.20)
\text{Tan}(S, x) = \text{wd}_x g_i(\mathbb{R}^n),
\end{equation}
when for some $i \in \mathbb{N}$, $y = g_i^{-1}(x)$ and $g_i$ is metrically and $w^*$-differentiable at $y$, with $J_n(\text{wd}_x g_i) > 0$. It is shown in [1] that this is a good definition for almost every $x \in S$. 

Proof. Since $T_i \to T$ weakly, the mass is lower semicontinuous, in that
\begin{equation}
(2.13)
\text{lim inf}_{i \to \infty} \|T_i\| (O) \geq \|T\| (O)
\end{equation}
for any $O \subset Z$ open. By assumption, $\|T_i\| (Z) \to \|T\| (Z)$. Since $Z$ is a complete metric space, it follows that $\|T_i\| \to \|T\|$ by for instance the portmanteau theorem (cf. [17]).
In case \( X \) is an arbitrary separable metric space, the approximate tangent space can still be defined by using an isometric embedding \( j : X \to Y \) of \( X \) into a \( w^* \)-separable dual space \( Y \), and setting

\[
\Tan(S, x) = \Tan(j(S), j(x)).
\]

It is shown that this definition does not depend on the choice of \( j \) and \( Y \), in the sense that \( \Tan(S, x) \) is uniquely determined \( \mathcal{H}^d \)-a.e. up to linear isometries \([1, 5]\).

The next theorem by Ambrosio and Kirchheim \([1]\) shows the existence of tangential derivatives of Lipschitz functions on rectifiable sets in \( w^* \)-separable dual spaces. In the formulation of the theorem, the distance \( d_w \) metrizes the \( w^* \)-topology of a \( w^* \)-separable dual space \( \bar{Y} \), that is, for \( x, y \in \bar{Y} \),

\[
d_w(x, y) := \sum_{j=0}^{\infty} 2^{-j} |\langle x - y, g_j \rangle|,
\]

where \( \langle g_j \rangle_{j=1}^{\infty} \subset \bar{Y} \) is a countable dense set in the unit ball of \( \bar{Y} \).

**Theorem 2.13** ([1] Theorem 8.1). Let \( S \subset Y \) be a countably \( \mathcal{H}^n \)-rectifiable set of a \( w^* \)-separable dual space \( Y \), and let \( f \) be a Lipschitz function from \( Y \) into another \( w^* \)-separable dual space \( \bar{Y} \). Let \( \theta : S \to (0, \infty) \) be integrable with respect to \( \mathcal{H}^n \llcorner S \) and let \( \mu = \theta \mathcal{H}^n \llcorner S \) be the corresponding rectifiable measure.

Then for \( \mathcal{H}^n \)-almost every \( x \in S \), there exist a \( w^* \)-continuous and linear map \( L : Y \to \bar{Y} \), and a Borel set \( S^x \subset S \) such that \( \Theta^*_{\infty}(\mu, S^x, x) = 0 \) and

\[
\lim_{y \to S, y \to x} \frac{d_w(f(y), f(x) + L(y - x))}{|y - x|} = 0.
\]

The map \( L \) is uniquely determined on \( \Tan(S, x) \). We denote its restriction to \( \Tan(S, x) \) by

\[
d^S_x f : \Tan(S, x) \to \bar{Y}
\]

and call it the tangential differential. It is characterized by the property that for any Lipschitz map \( g : D \subset \mathbb{R}^n \to S \),

\[
wd_y(f \circ g) = d^S_{g(y)} f \circ wd_y g, \quad \text{for } \mathcal{L}^n \text{-a.e. } y \in D.
\]

Finally, we recall the area formula by Ambrosio and Kirchheim \([1]\).

**Theorem 2.14** (Area formula [1] Theorem 8.2). Let \( f : Z \to \tilde{Z} \) be a Lipschitz function and let \( S \subset Z \) be a countably \( \mathcal{H}^n \)-rectifiable set. Then, for any Borel function \( \theta : S \to [0, \infty] \),

\[
\int_S \theta(x) J_n(d^S f) d\mathcal{H}^n(x) = \int_{\tilde{Z}} \sum_{x \in S \cap \theta^{-1}(y)} \theta(x) d\mathcal{H}^n.(y).
\]

Moreover, for any Borel set \( A \) and any Borel function \( \theta : \tilde{Z} \to [0, \infty] \),

\[
\int_A \theta(g(x)) J_n(d^S f) d\mathcal{H}^n(x) = \int_{\tilde{Z}} \theta(y) J_n(A \cap f^{-1}(y)) d\mathcal{H}^n.(y).
\]

In the Theorem above, \( J_n(d^S f) \) is calculated after embedding \( Z \) and \( \tilde{Z} \) in \( w^* \)-separable metric spaces and calculating the tangential derivative of the appropriate lift of \( f \). We would also like to mention that in special cases it may be easier to apply the less general version of the area formula \([1]\) Theorem 5.2].

### 3. The normalized energy

In this section we define a normalized energy of functions on sets of integral currents on metric spaces. First, we introduce two quantities in Sections 3.1 and 3.2 respectively the norm of the tangential derivative and the approximate local dilatation. In Section 3.3, we prove that the two coincide. Subsequently, we integrate them in Section 3.4 to obtain the normalized energy.
3.1. The norm of the tangential derivative. Let $Y$ be a $w^*$-separable dual space and let $\mu = \theta\mathcal{H}^n \llcorner S$ be an $n$-rectifiable measure on $Y$. Let $f$ be a Lipschitz function on $S$. Since $S$ is rectifiable, $f$ is $\mu$-a.e. tangentially differentiable, see [1, Theorem 8.1]. If it exists, we denote the tangential derivative of $f$ to $S$ at $x$ by $d_x^S f$. Note that $d_x^S f$ is a linear functional on $\text{Tan}(S, x)$. We denote by $|d_x^S f|$ its dual norm.

If $X$ is an arbitrary separable metric space and $\mu = \theta\mathcal{H}^n \llcorner S$ an $n$-rectifiable measure on $X$, we first isometrically embed $X$ into a $w^*$-separable dual space. Consider two such embeddings: $j_1 : X \to Y_1$ and $j_2 : X \to Y_2$. By [1], for $\mu$-a.e. $x \in X$, the approximate tangent spaces $\text{Tan}(j_1(S), j_1(x))$ and $\text{Tan}(j_2(S), j_2(x))$ are isometric. Therefore, one can make sense of $\text{Tan}(S, x)$. The isometry between the approximate tangent spaces induces an isometry between the dual spaces. Consequently, the functionals $d_{j_1(x)}^{j_1(S)}(f \circ j_1^{-1})$ and $d_{j_2(x)}^{j_2(S)}(f \circ j_2^{-1})$ are linked through this isometry, and it makes sense to define

\begin{equation}
|d_x^S f| = |d_{j_1(x)}^{j_1(S)}(f \circ j_1^{-1})|,
\end{equation}

for $\mu$-a.e. $x \in X$.

3.2. Approximate local dilatation. We may also give a different definition that does not mention approximate tangent spaces.

**Definition 3.1.** Let $\mu = \theta\mathcal{H}^n \llcorner S$ be a rectifiable measure on a metric space $X$. Let $f$ be a Lipschitz function defined on $S$. Define the set $D_x(t)$ by

\begin{equation}
D_x(t) = \left\{ y \in X \mid \frac{|f(y) - f(x)|}{d(y, x)} > t \right\}.
\end{equation}

Then, we define the approximate local dilatation of $f$ at a point $x \in X$ by

\begin{equation}
ap\text{dil}_x f := \inf \{ t > 0 \mid \Theta^*_{\mathcal{L}}(\|T\|_1, D_x(t), x) = 0 \}.\end{equation}

We note that ap dil$(f)$ is a bounded, measurable function.

3.3. Norm tangential derivative equals approximate local dilatation. In this section, we will show that for almost every $x$, the norm of the tangential derivative and the approximate local dilatation actually coincide.

The following Lemma gives a nice parametrization of $S$.

**Lemma 3.2** (dividing a rectifiable set). Let $S$ be countably $\mathcal{H}^n$-rectifiable subset of a metric space $X$. Then there exist compact sets $K_i \subset \mathbb{R}^n \ (i = 1, 2, \ldots)$ and bi-Lipschitz maps $g_i : K_i \to S$ such that

- $g_i(K_i) \cap g_j(K_j) = \emptyset$ for $i \neq j$.
- The $g_i(K_i)$ cover $\mathcal{H}^n$-almost all of $S$, that is,

\begin{equation}
\mathcal{H}^n \left( S \setminus \bigcup_{i=1}^{\infty} g_i(K_i) \right) = 0.
\end{equation}

- The function $\theta \circ g_i$ is continuous on $K_i$.
- For every $x \in K_i$, $g_i$ is metrically differentiable in $x$, with $J_i(md_x g_i) > 0$, and $x \mapsto md_x g_i$ is continuous, and for some moduli of continuity $\omega_i$ and all $y, x \in K_i$,

\begin{equation}
|d(g_i(y), g_i(x)) - md_x g_i(y - x)| \leq \omega_i(d(g_i(y), g_i(x)))d(g_i(y), g_i(x)).
\end{equation}

Moreover, if $j : X \to Y$ is an isometric embedding into a $w^*$-separable dual space, the compact sets $K_i$ and the maps $g_i : K_i \to X$ can be chosen so that additionally,

- For every $i$, the map $j \circ g_i$ is $w^*$-differentiable in every $x \in K_i$, and the map $x \mapsto wd_x (j \circ g_i)(v)$ is $w^*$-continuous for any $v \in \mathbb{R}^n$.

**Proof.** Compact sets $K_i \subset \mathbb{R}^n$ and maps $g_i : K_i \to X$ that satisfy the first two items can be obtained as in [3, Lemma 4.1]. By Lusin’s theorem, we may replace each $K_i$ by a countable union of compact sets that cover $K_i$ up to a set of $\mathcal{H}^n$-measure zero such that on these new sets the restriction of $\theta \circ g_i$ is continuous. Next, the compact sets thus obtained can each be divided in
countably many other compact sets satisfying the other bullet points, except technically (3.5), by the work of Ambrosio and Kirchheim [1 Theorem 3.3 and Remark 3.6] and the area formula [1 Theorem 8.2].

Finally, equation (3.3) is very similar to the conclusion of [1 Theorem 3.3], and is proven in a similar way: by repeatedly applying Egorov’s theorem to the sequence of functions

\[ h_j(x) := \sup_{y \in K_i, y \neq x} \left( \frac{md_x g_i(y - x)}{d(g_i(y), g_i(x))} - 1 \right), \]

that converge to zero on \( K_i \), we find sets \( K_{ij} \) the union of which covers \( K_i \) almost everywhere, and such that for moduli of continuity \( \omega_{ij} \) and all \( y \in K_i \) and \( x \in K_{ij} \),

\[ |d(g_i(y), g_i(x)) - md_x g_i(y - x)| \leq \omega_{ij}(d(g_i(y), g_i(x)))d(g_i(y), g_i(x)). \]

The lemma follows after reindexing. \( \square \)

The main part of the following Theorem was proven by Kirchheim [16]. The formulation in [1] is slightly stronger.

**Theorem 3.3** ([1]). Let \( \mu = \theta \mathcal{H}^n \cdot S \) be a rectifiable measure on a metric space \( X \). Then

\[ \Theta_n(\mu, x) = \theta(x), \quad \text{for } \mathcal{H}^n \text{-a.e. } x \in S. \]

As a by-product, it follows with the area formula that for \( \mathcal{H}^n \)-almost every \( x \in S \), in the sense of density the neighborhood of \( x \) in \( S \) is approximately given by the image of one of the parametrization maps.

**Corollary 3.4.** Let \( \mu = \theta \mathcal{H}^n \cdot S \) be a rectifiable measure on a metric space \( X \), and let compact sets \( K_i \subset \mathbb{R}^n \) and maps \( g_i : K_i \to \mathbb{R}^n \) be given as in Lemma 3.5. Then, for \( \mathcal{H}^n \)-almost every \( x \in S \), there exists an \( i_x \in \mathbb{N} \) such that \( x \in g_{i_x}(K_{i_x}) \) and moreover,

\[ \Theta_n(\mu, x) = \Theta_n(\mu \circ g_{i_x}(K_{i_x}), x) = \theta(x). \]

We will now show how to calculate the density of a set closely related to \( D_x(t) \).

**Lemma 3.5.** Let \( X \) be a separable metric space, \( \mu = \theta \mathcal{H}^n \cdot S \) be a rectifiable measure on \( X \), and \( f \) be a Lipschitz function defined on \( S \). Define for \( x \in S \),

\[ D_x^>(t) := \left\{ x' \in X \mid f(x') - f(x) \left| \frac{d(x', x)}{d(x', x)} \right| > t \right\}. \]

Let \( j : X \to Y \) be an isometric embedding of \( X \) into a \( w^* \)-separable dual space \( Y \).

Then, for \( \mu \)-a.e. \( x \in X \) and all \( t \in \mathbb{R} \) such that \(|t| \neq |d^j_x f|\),

\[ \Theta_n(\mu, x, D_x^>(t), x) = \theta(x) \mathcal{H}^n(W_x^>(t) \cap B_1(0)), \]

where \( W_x^>(t) \subset \text{Tan}(j(S), j(x)) \) is the cone given by

\[ W_x^>(t) := \left\{ w \in \text{Tan}(j(S), j(x)) \mid \left\| d^{j(x)}(f \circ j^{-1})(w) \right\| > t \|w\| \right\}. \]

**Proof.** Let \( j : X \to Y \) be an isometric embedding of \( X \) into a \( w^* \)-separable dual space \( Y \). Define \( \bar{f} : j(S) \to \mathbb{R} \) by \( \bar{f} = f \circ j^{-1} \). Let us apply Lemma 3.5 to get \( K_i \subset \mathbb{R}^n \) and bi-Lipschitz maps \( g_i : K_i \to X \) with the properties mentioned in the Lemma. Let \( x \in X \). By Corollary 3.4 we may assume that there exists an \( i_x \in \mathbb{N} \) such that \( x \in g_{i_x}(K_{i_x}) \) and

\[ \Theta_n(\mu, x) = \Theta_n(\mu \circ g_{i_x}(K_{i_x}), x) = \theta(x). \]

Let \( y = j(x) \). Without loss of generality, we may assume that \( g_{i_x}^{-1}(x) = 0 \in \mathbb{R}^n \), and that \( K_{i_x} \) has Lebesgue density 1 at 0.

Let \( t \in \mathbb{R} \) be such that \(|t| \neq |d^j_x \bar{f}|\). Define the cone \( C^>(t) \subset \mathbb{R}^n \) as

\[ C^>(t) := w_d(a \circ g_{i_x})^{-1}(W_x^>(t)), \]

and let \( B \) be the induced unit ball on \( \mathbb{R}^n \),

\[ B := w_d(a \circ g_{i_x})^{-1}(B_1(0)). \]
We calculate
\[ \Theta_n(\mu, D_x(t), x) = \lim_{r \downarrow 0} \frac{\mu(B_r(x) \cap D_x(t))}{\omega_n r^n} \]
\[ = \lim_{r \downarrow 0} \frac{\mu(g_{i_s}(K_{i_s}) \cap D_x(t) \cap B_r(x))}{\omega_n r^n} \]
\[ = \lim_{r \downarrow 0} \frac{1}{\omega_n r^n} \int_{K_{i_s}} \theta(g(z)) \chi_{D_x(t) \cap B_r(x)}(g(z)) J_n(m d z g_{i_s}) dz \]
\[ = \lim_{r \downarrow 0} \frac{1}{\omega_n} \int_{K_{i_s} / r} \theta(g(rp)) \chi_{D_x(t) \cap B_r(x)}(g(rp)) J_n(m d r p g_{i_s}) dp. \quad (3.16) \]

Here, \( \chi_A \) denotes the characteristic function of a set \( A \). For \( p \in \mathbb{R}^n \) write \( w_p := w d (j \circ g_{i_s})(p) \).

Note that
\[ \lim_{r \downarrow 0} \frac{d(g(rp), x)}{r} = \|w_p\|, \]
and
\[ \lim_{r \downarrow 0} \frac{f(g(rp)) - f(x)}{d(g(rp), x)} = \frac{\partial_j \omega(S) f(w_p)}{\|w_p\|} > t. \quad (3.19) \]

Consequently,
\[ \lim_{r \downarrow 0} \chi_{D_x(t) \cap B_r(x)}(g(rp)) = \begin{cases} 1 & \text{if } \|w_p\| < 1 \text{ and } \|d_j^{(n)}(w_p)\| > t \|w_p\|, \\ 0 & \text{if } \|w_p\| > 1 \text{ or } \|d_j^{(n)}(w_p)\| < t \|w_p\|. \end{cases} \]

Although the limit may not exist for other values of \( p \), the set
\[ \{ p \in \mathbb{R}^n \mid \|w_p\| = 1 \} \cup \{ p \in \mathbb{R}^n \mid \|d_j^{(n)}(w_p)\| = t \|w_p\| \text{ and } \|w_p\| \leq 1 \} \]
has zero \( n \)-dimensional Lebesgue measure when \( t \neq \|d_j^{(n)}(\bar{f})\| \). Since \( \theta \circ g_{i_s} \) and \( z \mapsto m d z g_{i_s} \) are continuous and the set \( K_{i_s} \) has Lebesgue density one at \( 0 \), we find
\[ \Theta_n(\mu, D_x(t), x) = \lim_{r \downarrow 0} \frac{1}{\omega_n} \int_{K_{i_s} / r} \theta(g(rp)) \chi_{D_x(t) \cap B_r(x)}(g(rp)) J_n(m d r p g_{i_s}) dp \]
\[ = \lim_{r \downarrow 0} \frac{1}{\omega_n} \int_{C > (t) \cap B} \theta(x) J_n(m d_0 g_{i_s}) dp \]
\[ = \frac{\theta(x)}{\omega_n} J_n(m d_0 g_{i_s}) \mathcal{L}^n(C > (t) \cap B) \]
\[ = \theta(x) \mathcal{H}^n(W_x^>(t) \cup B_1(0)). \quad (3.21) \]

\[ \square \]

It follows that for almost every \( x \), the approximate local dilatation and the norm of the weak tangential derivative coincide:

**Theorem 3.6.** Let \( X \) be a separable metric space, and let \( \mu = \theta \mathcal{H}^n \downarrow S \) be a rectifiable measure on \( X \). Let \( f \) be a Lipschitz function defined on \( S \). Then, for \( \mu \text{-a.e. } x \in S \), \( |d_x^S f| = \text{ap dil}_x f \).

**Proof.** By Definition 3.1, we need to show that
\[ \text{ap dil}_x f = \inf \{ t > 0 \mid \Theta_n(\mu, D_x(t), x) = 0 \} = |d_x^S f|. \]

This equality immediately follows from Lemma 3.5, since by symmetry of the cone \( W \), for \( 0 < t < |d_x^S f| \), for \( \mu \text{-a.e. } x \in S \),
\[ \Theta_n(\mu, D_x(t), x) = 2 \Theta_n(\mu, D_x^>(t), x) = 2 \theta(x) \mathcal{H}^n(W_x^>(t) \cup B_1(0)) > 0, \]
while for \( t > |d_x^S f| \),
\[ \Theta_n(\mu, D_x(t), x) = 2 \Theta_n(\mu, D_x^>(t), x) = 2 \theta(x) \mathcal{H}^n(W_x^>(t) \cup B_1(0)) = 0. \]

\[ \square \]
3.4. The normalized energy.

**Definition 3.7.** Let $X$ be a metric space, and let $T \in \mathcal{I}_n(X)$. Denote $S = \text{set}(X)$ and let $f$ be a Lipschitz function defined on $S$. Then we define the renormalized energy of $f$ by

$$E_T(f) := \frac{\int_X |dS_x f|^2 \, d\|T\|}{\int_X |f|^2 \, d\|T\|},$$

where $\text{ap dil}(f)$ is determined with respect to the $n$-rectifiable measure $\|T\|$. The definition behaves well under isometric embeddings, as expressed by the following proposition.

**Proposition 3.8.** Suppose $X$ is a metric space and $T \in \mathcal{I}_n(X)$. Let $j : X \to Z$ be an isometric embedding of $X$ into another metric space $Z$. Let $f$ be a Lipschitz function on $Z$. Then

$$E_T(j\# f) = E_{j\# T}(f).$$

**Proof.** Since $j : X \to Z$ is an isometric embedding, for $\|T\|$-a.e. $x \in X$

$$|dS_x j\# f|^2 = j\# |d_j(S) f|^2$$

and

$$j\# \|T\| = \|j\# T\|.$$

From this, the statement follows immediately. □

**Corollary 3.9.** If $\|M\|$ is an integral current space induced by a Riemannian manifold $M$, the renormalized energy coincides with the usual Rayleigh quotient, that is, for every Lipschitz function $f$,

$$E_{\|M\|}(f) = \frac{\int_M |\nabla_M f|^2 \, d\text{Vol}_M}{\int_M |f|^2 \, d\text{Vol}_M}.$$

**Proof.** This follows immediately from Proposition 3.8 since $|\nabla_M f| = |df|$ for almost every $x \in M$. □

**Remark 3.10.** In some cases, as in Gromov’s work [15] and the work by Ambrosio, Gigli and Šaueré [2], the normalized energy is introduced with the local dilatation rather than with the approximate local dilatation. The local dilatation of a function $f : X \to \mathbb{R}$ in a point $x$ is defined as

$$\text{dil}_x f = \lim_{r \downarrow 0} \sup_{y \in B_r(x)} \frac{|f(y) - f(x)|}{d(y, x)}.$$

We note that these may have different values. Consider for instance the following example.

For a point $x \in \mathbb{R}^3$, and $r > 0$, let $D_r(x)$ denote the disk

$$D_r(x) = \{ (y_1, y_2, x_3) \in \mathbb{R}^3 \mid (y_1 - x_1)^2 + (y_2 - x_2)^2 < r^2 \}.$$

Pick a dense sequence of points $p_i \in B_1(0)$, and a sequence $r_i \downarrow 0$ as $i \to \infty$ such that the current $T$ associated to

$$D_1(0) \cup \bigcup_{i=1}^{\infty} D_r(p_i)$$

is normal. Now consider the function $f(x) = x_3$ restricted to supp$T$. Then, for almost every $x \in D_1(0)$, $\text{dil}_x f = 1$, while $\text{ap dil}_x f = 0$. 
4. Semicontinuity of min-max values

Let $X$ be a metric space, and let $T \in \mathcal{I}_u(X)$. By the space $L^2(\|T\|)$ we mean the usual Hilbert space of functions on $X$ with inner product

\begin{equation}
(f, g)_{L^2(\|T\|)} := \int_X f g d\|T\|.
\end{equation}

We define the function space $W^{1,2}(\|T\|)$ as the completion of the set of bounded Lipschitz functions on $S$ with respect to the norm $\|\cdot\|_{W^{1,2}}$ given by

\begin{equation}
\|f\|_{W^{1,2}} := \int_X f^2 d\|T\| + \int_X ap \text{dil}(f)^2 d\|T\|.
\end{equation}

We define

\begin{equation}
\mathcal{V}(\|T\|) = \left\{ f \in W^{1,2}(\|T\|) \mid \int_X f d\|T\| = 0 \right\},
\end{equation}

and the first eigenvalue of a current $T$, $\lambda_1$, as the smallest critical point of $\mathcal{E}_T$. That is,

\begin{equation}
\lambda_1 = \inf_{f \in \mathcal{V}(\|T\|)} \mathcal{E}_T(f).
\end{equation}

We also define higher order min-max values $\lambda_k$

\begin{equation}
\lambda_k(T) := \inf_{\{\phi_1, \ldots, \phi_k\} \subset \mathcal{V}(\|T\|)} \sup_{i=1, \ldots, k} \mathcal{E}_T(\phi_i).
\end{equation}

Recall that this corresponds to the Rayleigh quotient on Riemannian manifolds. More generally, in Section 6 we will show by the min-max principle that these have significance as eigenvalues only if the approximate tangent spaces are almost everywhere inner product spaces. Regardless, semicontinuity of these values holds in the general case as well. Before we prove this, we first state a helpful lemma.

**Lemma 4.1** (dividing a rectifiable set, the domain of Lipschitz functions). Let $Y$ be a $w^*$-separable dual space, let $T \in \mathcal{I}_u(Y)$ and let $f_1, \ldots, f_k \in \text{Lip}(S)$, where $S = \text{set}(T)$. Moreover, let $\epsilon > 0$.

Then there exist compact sets $K_i \subset \mathbb{R}^n$ and bi-Lipschitz maps $g_i : K_i \to Y$ such that the $g_i$ and $K_i$ have all the properties of Lemma 3.2 with respect to $\mu = \|T\|$, and additionally, for every $i \in \mathbb{N}$, $m = 1, \ldots, k$, the functions $x \mapsto |d_z^2 f_m(x)|$ restricted to $g_i(K_i)$ are continuous, and for moduli of continuity $\tilde{\omega}_i$, and all $y, z \in K_i$,

\begin{equation}
|f \circ g_i(y) - (f \circ g_i)(z) - d_z(f \circ g_i)(y - z)| \leq \tilde{\omega}_i(\|g_i(y) - g_i(z)\|) \|g_i(y) - g_i(z)\|,
\end{equation}

and therefore in particular, it can be assured that, with $c^2_m = \text{Lip}(f_m |_{g_i(K_i)})$,

\begin{equation}
(c^2_m)^2 - \epsilon \leq |d_z^2 f_m|^2 \leq (c^2_m)^2, \quad \text{for all } x \in g_i(K_i).
\end{equation}

**Proof.** After obtaining the $K_i$ and $g_i$ as in Lemma 3.2 we can apply Lusin’s Theorem repeatedly to find compact sets $K_{ij} \subset K_i$ that for fixed $i$ cover the sets $K_i$ up to a set of measure zero, that is

\begin{equation}
\mathcal{H}^n \left( K_i \setminus \bigcup_{j=1}^\infty K_{ij} \right) = 0,
\end{equation}

and such that $x \mapsto |d_z^2 f_m|$ is continuous on $g_i(K_{ij})$.

Note that on the other hand, for all $z \in K_i$,

\begin{equation}
\lim_{K_{ij}, z \to y} \frac{|f \circ g_i(y) - (f \circ g_i)(z)|}{\|g_i(y) - g_i(z)\|} \frac{d_z(f \circ g_i)(y - z)}{\|g_i(y) - g_i(z)\|} = 0,
\end{equation}

so that by Egorov’s Theorem, there are compact sets $K_{ij} \subset K_i$, $j = 1, 2, \ldots$, again covering $K_i$ up to measure zero, such that (4.9) holds for $\tilde{\omega}_i$ replaced by a modulus of continuity $\omega_{ij}$. After taking intersections and reindexing, we have shown the first part of the lemma.

Now let $\epsilon > 0$, let $y, z \in K_i$. From (4.9), we find

\begin{equation}
|f \circ g_i(y) - (f \circ g_i)(x)| \leq |d_z(f \circ g_i)(y - z)| + \tilde{\omega}_i(\|g_i(y) - g_i(z)\|)\|g_i(y) - g_i(z)\|.
\end{equation}
Since $d_z(f \circ g_i) = d_{g_i(z)}^S(f \circ \omega_z g_i)$ and $\|\omega_z g_i(y - z)\| = md_z g_i(y - z)$, we find also using (3.5),
\[
\begin{align*}
(f \circ g_i)(y) - (f \circ g_i)(z) &\leq [(d_{g_i(z)}^S f) ||\omega_z g_i(y - z)||]
\quad + \omega_i(||g_i(y) - g_i(z)||) ||g_i(y) - g_i(z)||
\quad + \omega_i(||g_i(y) - g_i(z)||) ||g_i(y) - g_i(z)||
\end{align*}
\]
(4.11)
Again replacing each set $K_i$ by a countable collection of compact subsets, with small enough
diameters, the union of which covers $K$, up to a set of measure zero, we ensure (4.13). \hfill \Box

**Theorem 4.2.** Suppose $Y$ is a $w^*$-separable dual space and $T_i \in I_\kappa(Y)$ converge in the flat
distance on $Y$ to a current $T \in I_\kappa(Y)$ such that $\mathbf{M}(T_i) \to \mathbf{M}(T)$ as $i \to \infty$. Then
\[
\limsup_{i \to \infty} \lambda_\kappa(T_i) \leq \lambda_\kappa(T).
\]
*Proof.* Let $\sigma > 0$ and choose bounded Lipschitz functions $f_1, \ldots, f_k$ such that they are $L^2(||T||)$-
orthonormal and for $j = 1, \ldots, k$
\[
\mathcal{E}_T(f_j) \leq \mathcal{E}_T(f_k) \leq \lambda_k(T) + \sigma.
\]
(4.13)
Let $\epsilon > 0$ and apply Lemma 4.1 to the current $T$ and the functions $f_j$, to obtain functions $g_\ell$
and sets $K_\ell$ ($\ell = 1, 2, \ldots$) as in the Lemma.
Select $N$ large enough such that
\[
(4.14) \quad \|T\| \left(\bigcup_{\ell=1}^N g_\ell(K_\ell)\right) < \epsilon.
\]
The sets $g_\ell(K_\ell)$ are compact and disjoint, so the minimal distance $\delta$ between the sets is positive,
\[
\delta := \min_{i \leq j} \min_{1 \leq \ell \leq N} \text{dist} \left(g_\ell(K_i), g_\ell(K_j)\right) > 0.
\]
(4.15)
Define the open sets $U_j \subset Y$ as the $\delta/10$ neighborhoods of $K_j$.

We claim that we can extend the functions $f_j$ to bounded Lipschitz functions $\hat{f}_j$ on the whole
of $Y$, so that $\text{Lip} (\hat{f}_j) \leq 2 \text{Lip} (f_j)$, $\sup |\hat{f}_j| \leq 2 \sup |f_j| :=: M_j$, and
\[
(4.16) \quad \text{Lip} \left(\hat{f}_j|_{U_j}\right) \leq \text{Lip} \left(f_j\right)_{g_\ell(K_i)} =: c_j^\ell.
\]
This can be done as follows. First, we define
\[
\hat{f}_j(x) := \inf_{a \in g_\ell(K_i)} f_j(a) + c_j^\ell \|a - x\|, \quad x \in U_\ell,
\]
(4.17)
and if necessary, we truncate $\hat{f}_j := (\hat{f}_j \wedge M_j) \vee (-M_j)$. Note that $\text{Lip}(\hat{f}_j) \leq c_j^\ell$. Subsequently, we consider the functions $\hat{f}_j : \cup U_\ell \to \mathbb{R}$, given by
\[
\hat{f}_j(x) = \hat{f}_j^\ell(x), \quad \text{if } x \in U_\ell.
\]
(4.18)
Note that the Lipschitz constant of $\hat{f}_j$ is less than $2 \text{Lip}(f_j)$. Indeed, if $x \in U_{\ell_1}$, $y \in U_{\ell_2}$, $\ell_1 \neq \ell_2$, there exist $x_0 \in g_\ell(K_{\ell_1})$ and $y_0 \in g_\ell(K_{\ell_2})$ such that $|x - x_0| < \delta/5$ and $|y - y_0| < \delta/5$ and therefore
\[
|\hat{f}_j(x) - \hat{f}_j(y)| \leq |\hat{f}_j(x) - \hat{f}_j(x_0)| + |\hat{f}_j(x_0) - \hat{f}_j(y_0)| + |\hat{f}_j(y_0) - \hat{f}_j(y)|
\quad \leq c_j^{\ell_1} |x - x_0| + \text{Lip}(f_j) |x_0 - y_0| + c_j^{\ell_2} |y_0 - y|
\quad \leq \text{Lip}(f_j) \left(\frac{1}{4} |x - y| + \frac{5}{4} |x - y| + \frac{1}{4} |x - y|\right)
\quad < 2 \text{Lip}(f_j) |x - y|,
\]
(4.19)
Consequently, we can extend the functions $\hat{f}_j$ to Lipschitz functions on the whole of $Y$ as claimed.
As explained in Lemma 2.7 the portmanteau theorem (cf. [17]), \( \|T_i\| \to \|T\| \) weakly as measures on \( Y \). This implies that

\[
\begin{align*}
(4.20a) \quad & \lim_{i \to \infty} \int_Y \hat{f}_{j_1} \hat{f}_{j_2} d\|T_i\| = \lim_{i \to \infty} \int_Y \hat{f}_{j_1} \hat{f}_{j_2} d\|T\| = \delta_{j_1,j_2}, \\
(4.20b) \quad & \lim_{i \to \infty} \int_Y \hat{f}_{j_1} d\|T_i\| = \lim_{i \to \infty} \int_Y \hat{f}_{j_1} d\|T\| = 0.
\end{align*}
\]

for all \( j_1, j_2 \in \{1, \ldots, k\} \).

We can restrict the functions \( \hat{f}_j \) to set(\( T_i \)), subtract the average,

\[
(4.21) \quad f^*_j := \hat{f}_j - \int_Y \hat{f}_j d\|T_i\|,
\]

and then apply Gram-Schmidt to obtain an \( L^2(\|T_i\|) \)-orthonormal system \( \psi^*_1, \ldots, \psi^*_k \). That is,

\[
\begin{align*}
(4.22) \quad & \psi^*_j := \frac{\hat{f}_j}{\|\hat{f}_j\|_{L^2(\|T_i\|)}}, \\
(4.23) \quad & \psi^*_{j+1} := f^*_{j+1} - (f^*_{j+1}, \psi^*_1)_{L^2(\|T_i\|)} \psi^*_1 - \cdots - (f^*_{j+1}, \psi^*_j)_{L^2(\|T_i\|)} \psi^*_j, \\
(4.24) \quad & \psi^*_{j+1} := \frac{\psi^*_{j+1}}{\|\psi^*_{j+1}\|_{L^2(\|T_i\|)}}, \quad j = 1, 2, \ldots, k - 1.
\end{align*}
\]

Note that in particular, \( \int_Y \psi^*_j d\|T_i\| = 0 \), so that \( \psi^*_j \in \mathcal{V}(\|T_i\|) \). As a result of the Gram-Schmidt algorithm, the functions \( \psi^*_j \) satisfy, for \( j_1 = 1, \ldots, k, \)

\[
(4.25) \quad \psi^*_j = \sum_{j_2=1}^k a^i_{j_1,j_2} \hat{f}_{j_2} + b^i_{j_1},
\]

where \( a^i_{j_1,j_2} \) and \( b^i_{j_1} \) are constants, \( j_1, j_2 \in \{1, \ldots, k\} \), which by (4.20) satisfy

\[
\begin{align*}
(4.26a) \quad & \lim_{i \to \infty} a^i_{j_1,j_2} = \delta_{j_1,j_2}, \\
(4.26b) \quad & \lim_{i \to \infty} b^i_{j_1} = 0.
\end{align*}
\]

Observe that this implies that

\[
(4.27) \quad \lim_{i \to \infty} \int_Y \left| d\psi^*_j \right|^2 = \lim_{i \to \infty} \left| d\hat{f}_j \right|^2 d\|T_i\| = 0.
\]

By (4.27), (4.14) and the fact that Lip(\( \hat{f}_j \)) < 2Lip(\( f_j \)),

\[
(4.28) \quad \limsup_{i \to \infty} \int_Y |d\psi^*_j|^2 d\|T_i\| \leq \limsup_{i \to \infty} \sum_{\ell=1}^N \int_{U_\ell} |d\psi^*_j|^2 d\|T_i\| + \limsup_{i \to \infty} \int_{(\bigcup_{\ell} U_\ell)^c} |d\psi^*_j|^2 d\|T_i\|
\]

\[
\leq \limsup_{i \to \infty} \sum_{\ell=1}^N \int_{U_\ell} |d\psi^*_j|^2 d\|T_i\| + 2\epsilon \sup_j \text{Lip}(f_j)^2.
\]
For the first term, we have by (4.27), (4.10), and the bound (4.7) from the application of Lemma 4.1

$$\limsup_{i \to \infty} \sum_{\ell=1}^{N} \int_{U_{\ell}} |d\psi_{j}^{i}|^{2} d\|T_{i}\| \leq \limsup_{i \to \infty} \sum_{\ell=1}^{N} \int_{U_{\ell}} |d\hat{f}_{j}^{i}|^{2} d\|T_{i}\|$$

(4.29)

$$\leq \limsup_{i \to \infty} \sum_{\ell=1}^{N} (\epsilon f_{j})^{2} d\|T_{i}\|(U_{\ell})$$

$$\leq \sum_{\ell=1}^{N} \int_{U_{\ell}} |df_{j}|^{2} d\|T\| + \epsilon M(T)$$

$$\leq \lambda_{k}(T) + \sigma + \epsilon M(T).$$

Since \(\{\psi_{j}\}_{j=1}^{k} \subset \mathcal{V}(\||T_{i}\||)\) are \(L^{2}(\||T_{i}\||)\)-orthonormal, we conclude from (4.28) and (4.29) that

$$\limsup_{i \to \infty} \lambda_{k}(T_{i}) \leq \lambda_{k}(T) + \sigma + \epsilon M(T) + 2\epsilon \sup_{j} \text{Lip}(f_{j}).$$

(4.30)

Because \(\sigma\) and \(\epsilon\) were arbitrary, and the \(f_{j}\) do not depend on \(\epsilon\), this implies the theorem. \(\square\)

5. **Semicontinuity for min-max values under intrinsic flat convergence**

In this section we define the infimum of the normalized energy \(\lambda_{1}\) and the other min-max values \(\lambda_{k}\) for integral current spaces, and show that they are semicontinuous under intrinsic flat convergence if the mass converges as well.

5.1. **Min-max values for integral current spaces.** We first define the infimum of the normalized energy \(\lambda_{1}\) and the min-max values \(\lambda_{k}\) following the definitions in (4.4) and (4.5).

**Definition 5.1.** Given a nonzero integral current space \(M = (X, d, T)\) we define \(\lambda_{1}(M)\) as the infimum of the normalized energy

$$\lambda_{1}(M) = \inf_{f \in \mathcal{V}(\||T||)} \mathcal{E}_{T}(f),$$

(5.1)

and the min-max values \(\lambda_{k}(M)\)

$$\lambda_{k}(M) := \inf_{\{\phi_{1}, \ldots, \phi_{k}\} \subset \mathcal{V}(\||T||)} \sup_{L^{2}(\||T||)\text{-orthonormal}} \mathcal{E}_{T}(\phi_{i}),$$

(5.2)

with \(\mathcal{E}_{T}\) as in Definition 3.7.

Observe that when \(M\) is induced by an oriented Riemannian manifold, \(\lambda_{k}(M)\) is its \(k\)th Neumann eigenvalue. In Section 3 we will see that also in case the currents are infinitesimally Hilbertian, the min-max values \(\lambda_{k}\) have interpretations as eigenvalues of a self-adjoint operator defined on functions on the currents.

The intrinsic flat distance between two integral current spaces is zero if and only if there exists a current preserving isometry between the two spaces. The following Lemma states that in that case, the min-max values of the two spaces coincide.

**Lemma 5.2.** Let \(M_{1} = (X_{1}, d_{1}, T_{1})\) and \(M_{2} = (X_{2}, d_{2}, T_{2})\) be integral current spaces and let \(\phi : X_{1} \to X_{2}\) be a current-preserving isometry. That is, besides being an isometry from \(X_{1}\) to \(X_{2}\), \(\phi\) also satisfies \(\phi_{*} T_{1} = T_{2}\). Then for all \(k = 1, 2, \ldots\), it holds that \(\lambda_{k}(M_{1}) = \lambda_{k}(M_{2})\).

**Proof.** This follows immediately from Proposition 3.8. \(\square\)
5.2. Semicontinuity for min-max values under intrinsic flat convergence. Theorem 4.2 immediately implies semicontinuity of the min-max values $\lambda_k$ under intrinsic flat convergence when the total mass is conserved as well.

**Theorem 5.3** (Upper-semicontinuity of min-max values). Let $(X_i, d_i, T_i)$, $(i = 1, 2, \ldots)$, be a sequence of integral current spaces converging in the intrinsic flat distance to a nonzero integral current space $(X, d, T)$ such that additionally, $M(T_i) \to M(T)$ as $i \to \infty$. Then one has semicontinuity of the min-max values

$$\limsup_{i \to \infty} \lambda_k(T_i) \leq \lambda_k(T).$$

**Proof.** Since the integral current spaces $(X_i, d_i, T_i)$ converge in the intrinsic flat sense to $(X, d, T)$, by Theorem 2.5 established by Sormani and Wenger, and the Kuratowski embedding, there exist a $w^*$-separable dual space $Y$ and isometric embeddings $\phi_i : X_i \to X$, $\phi : X \to Y$ such that $(\phi_i)_# T_i \to (\phi)_# T$ in the flat distance in $Y$. Then we may apply Theorem 4.2. \(\square\)

6. Infinitesimally Hilbertian integral currents

In this section we additionally assume that the currents involved are infinitesimally Hilbertian, that is, that (almost everywhere) the norm on the tangent spaces to their rectifiable set is induced by an inner product. This assumption is similar to the one made by Cheeger and Colding [11], and ensures that there is a quadratic form associated to the energy.

More precisely, for a metric space $X$ and an integral current $T \in I_0(X)$, we assume that for $\|T\|$-a.e. $x \in X$, $\Tan(set T, x)$ is an inner product space. Denote the inner product on the dual space to $\Tan(set T, x)$ by $g_x(\ldots)$.

**Theorem 6.1.** Let $X$ be a metric space and let $T$ be an integral rectifiable current on $X$. Then the quadratic form

$$Q_T(f, g) := \int_X g_x(d^S_x f, d^S_x g) \, d\|T\|,$$

defined on $W^{1,2}(\|T\|)$ is closed. Consequently, there is a unique associated (unbounded) nonnegative self-adjoint operator $\Delta_T$ on $L^2(\|T\|)$ satisfying $Q_T(\phi, \phi) = (\phi, \Delta_T \phi)$ for every $\phi \in \mathcal{D}(\Delta_T)$.

**Definition 6.2.** By the min-max values $\mu_k(A)$ of an unbounded nonnegative self-adjoint operator $A$ on a Hilbert space $H$ with domain $\mathcal{D}(A)$, we mean

$$\mu_k(A) := \inf_{\{\phi_1, \ldots, \phi_k\} \subset H, \text{orthonormal}} (\phi, A\phi)_H.$$

From standard functional analysis (cf. e.g. [11]), it follows that there are two options. It could be that $\mu_k \to \infty$ as $k \to \infty$. In that case, the spectrum of $\Delta_T$ completely consists of eigenvalues $\mu_k$. Alternatively, there is a $\mu \geq 0$, the bottom of the essential spectrum, and if $\mu_k < \mu$, it is the $k$th eigenvalue counting degenerate eigenvalues a number of times equal to their multiplicity. If $\mu_k = \mu$, then also $\mu_j = \mu$ for all $j > k$.

For $f \in W^{1,2}$, it holds that $\mathcal{E}_T(f) = Q_T(f, f)$, therefore the operator $\Delta_T$ defined in Theorem 6.1 satisfies

$$\mu_k(\Delta_T) = \inf_{\{\phi_1, \ldots, \phi_k\} \subset V} \sup_{i=1, \ldots, k} Q_T(\phi_i, \phi_i)$$

$$= \inf_{\{\phi_1, \ldots, \phi_k\} \subset V} \sup_{i=1, \ldots, k} \mathcal{E}_T(\phi_i)$$

$$= \lambda_k(T).$$

Hence, when we combine Theorem 6.1 and Theorem 5.3 we obtain semicontinuity of min-max values of the self-adjoint operators under intrinsic flat convergence without loss of volume.

**Theorem 6.3.** Let $M_i = (X_i, d_i, T_i)$, $(i = 1, 2, \ldots)$ be a sequence of integral current spaces converging in the intrinsic flat distance to a nonzero integral current space $M_\infty = (X_\infty, d_\infty, T_\infty)$ such that also $M(T_i) \to M(T_\infty)$ as $i \to \infty$. Moreover, assume that the norm on $\|T_i\|$-a.e.
approximate tangent space to set $(T_i)$ is Hilbert, $i = 1, 2, \ldots, \infty$. Then, for every $i$, we may define an unbounded operator $\Delta_{T_i}$ on $L^2(||T_i||)$ as in Theorem 6.3 and its min-max values satisfy
\begin{equation}
\limsup_{i \to \infty} \mu_k(\Delta_{T_i}) \leq \mu_k(\Delta_{T_\infty}).
\end{equation}

Theorem [11] in the introduction is a translation of Theorem 6.3 to the simpler setting of closed oriented Riemannian manifolds.

When we combine Theorem 5.1 together with Theorem 4.2 we obtain a version of a Theorem 1.1 in the introduction is a translation of Theorem 6.3 to the simpler setting of closed oriented Riemannian manifolds.

Let
\begin{equation}
\epsilon > 0
\end{equation}
and consider for $\epsilon > 0$ the smooth functions $h_\epsilon : [-2, \epsilon] \to \mathbb{R}$, $h_\epsilon \geq 0$, such that
\begin{equation}
h_\epsilon(x) = \begin{cases} 
\sqrt{1 - (x + 1)^2} & -2 \leq x \leq -\epsilon, \\
\epsilon & \epsilon \leq x \leq 2 - \epsilon, \\
\epsilon - (x - 2 + \epsilon)^2 & 2 - \epsilon \leq x \leq 2.
\end{cases}
\end{equation}

Moreover, assume that $h_\epsilon$ is decreasing on $(-1, 2)$. We construct manifolds $M_\epsilon$ by revolving the graphs $y = h_\epsilon(x)$ around the $x$-axis. Sormani has shown that in this case the manifolds $M_\epsilon$ converge in the intrinsic flat sense as $\epsilon \to 0$ to the unit sphere $S^1$ [21, Example A.4]. Also observe that $\text{Vol}(M_\epsilon) \to \text{Vol}(S^1)$. We choose test functions $f_\epsilon$ on $M_\epsilon$, that only depend on $x$, increasing in $x$, such that
\begin{equation}
f_\epsilon(x) := \begin{cases} 
c_\epsilon & x \leq -\epsilon, \\
\sin(\frac{\pi x}{2\epsilon}) & \epsilon \leq x \leq 2 - \epsilon.
\end{cases}
\end{equation}

where $c_\epsilon$ is chosen in such a way that $f_\epsilon$ has zero average on $M_\epsilon$. After calculating the Rayleigh-quotient for $f_\epsilon$, we observe that
\begin{equation}
\limsup_{\epsilon \to 0} \lambda_1(M_\epsilon) \leq \limsup_{\epsilon \to 0} \frac{\int_{M_\epsilon} |\nabla f_\epsilon|^2 dH^2}{\int_{M_\epsilon} |f_\epsilon|^2 dH^2} = \left(\frac{\pi}{4}\right)^2 < 2 = \lambda_1(S^1).
\end{equation}

This shows that the first eigenvalue can actually jump up in the limit.

### 7.2. Example: cancellation can make eigenvalues drop.

We conclude with an example that shows that we cannot remove the assumption of the convergence of the volumes from Theorem 5.1. Without this assumption, cancellation can occur, which can make the eigenvalues drop down in the limit.

For $x \in \mathbb{R}^3$ let $Q_{q,x}(x) \subset \mathbb{R}^3$ denote the following rectangular box:
\begin{equation}
Q_{q,x}(x) = [x_1 - q, x_1 + q] \times [x_2 - q, x_2 + q] \times [x_3 - r, x_3 + r].
\end{equation}

Let $e_1$ denote the unit vector $(1, 0, 0) \in \mathbb{R}^3$.

Consider the sequence $M_j = \partial W_j$ where $W_j$ is given as a set by
\begin{equation}
W_j = Q_{1,1}(2e_1) \cup Q_{1,1}(-2e_1) \cup Q_{1,1}(-2e_1).
\end{equation}

At first sight, one might be tempted to think that as $j \to \infty$ the $M_j$ converge in the intrinsic flat sense to the boundary of the two cubes $Q_{1,1}(2e_1)$ and $Q_{1,1}(-2e_1)$. However, the induced
embedding of $M_j$ into $\mathbb{R}^3$ is not \textit{isometric}. Yet, we can use the construction by Sormani in [21]
Example A.19], and create manifolds $\tilde{M}_j$ with a lot of tunnels from one side of the thin sheet to
the other side. To be more precise, let

$$ P_j := Q_{1,4^{-j}}(0) \setminus \bigcup_{k,\ell = -2^{j+1}}^{2^{j-1}} Q_{4^{-j},4^{-j}} \left( \frac{k}{2^{j}}, \frac{\ell}{2^j}, 0 \right) $$

and

$$ \tilde{W}_j = Q_{1,1}(2e_1) \cup P_j \cup Q_{1,1}(-2e_1). $$

and let $\tilde{M}_j = \partial \tilde{W}_j$ (to be interpreted as the boundary of the set in Euclidean space). We claim that

$$ \tilde{M}_j \to \partial (Q_{1,1}(2e_1) \cup Q_{1,1}(-2e_1)) =: \tilde{M}. $$

in the intrinsic flat sense, where $\tilde{M}$ is endowed with the metric $d_Y$, which is the length metric on
the space $Y$ given by

$$ Y = \partial Q_{1,1}(2e_1) \cup \partial Q_{0,1}(0) \cup \partial Q_{1,1}(-2e_1), $$

which is the Gromov-Hausdorff limit of $M_j$.

By the isometric product $A \times B$ of two metric spaces $(A,d_A)$ and $(B,d_B)$ we mean the Cartesian
product endowed with the metric

$$ d_{A \times B}((a_1,b_1),(a_2,b_2)) = \sqrt{(a_1-a_2)^2 + (b_1-b_2)^2}. $$

We consider the metric space

$$ Z_j := \left( \tilde{M}_j \times [0,\frac{1}{j}] \right) \cup \left( (\partial Q_{1,1}(2e_1) \cup Q_{1,4^{-j}}(0) \cup \partial Q_{1,1}(-2e_1)) \times \{0\} \right) \cup \left( Y \times [-\frac{1}{j},0] \right). $$

Note that for $j$ large enough, the embeddings

$$ \phi_j : M_j \to Z_j, \quad \phi_j(x) = (x,\frac{1}{j}), $$

$$ \psi_j : Y \to Z_j, \quad \psi_j(y) = (y,-\frac{1}{j}), $$

are isometric. Let $B_j \in I_\delta(Z_j)$ be the current

$$ B_j := [\tilde{M}_j \times [0,\frac{1}{j}]] + [\tilde{M} \times [-\frac{1}{j},0]] - [P_j \times \{0\}]. $$

Then

$$ (\phi_j)_{\#}[\tilde{M}_j] - (\psi_j)_{\#}[\tilde{M}] = \partial [B_j]. $$

Since

$$ M(\|B_j\|) \leq \frac{1}{j} \left( \text{Vol}(\tilde{M}_j) + \text{Vol}(\tilde{M}) \right) + \text{Vol}(Q_{1,4^{-j}}(0)) \to 0, $$

as $j \to \infty$, it follows that indeed $\tilde{M}_j$ converges to $\tilde{M}$ in the intrinsic flat sense.

Recall that for a smooth $n$-dimensional manifold $M$, the Cheeger’s constant $h(M)$ is defined as

$$ h(M) := \inf_{E} \frac{\mathcal{H}^{n-1}(E)}{\min(\mathcal{H}^n(A),\mathcal{H}^n(B))}, $$

where the infimum runs over all smooth $(n-1)$-dimensional submanifolds $E$ of $M$ that divide $M$
into two disjoint submanifolds $A$ and $B$ [10]. Cheeger’s inequality states that

$$ \lambda_1(M) \geq \frac{h(M)^2}{4}. $$

As we can take a minimizing function with average zero that is constant on each of the two
cubes, we have $\lambda_1(M) = 0$. However, Cheeger’s inequality implies that $\lambda_1(\tilde{M}_j)$ is uniformly
bounded away from zero. The idea is as follows. Let $E$ be a 1-dimensional submanifold of $\tilde{M}_j$,\nseparating $M_j$ into two disjoint submanifolds $A$ and $B$, such that $\mathcal{H}^2(A) \leq \mathcal{H}^2(B)$. Without loss
of generality we may assume that $A$ is connected. If $\text{diam}(E) \leq 1$, there are constants $c_1$ and $c_2$ (independent of $j$) such that

$$H^2(A) \leq c_1 \text{diam}(E)^2, \quad H^1(E) \geq c_2 \text{diam}(E),$$

so that

$$H^1(E) \geq \frac{c_2}{c_1} \frac{1}{\text{diam}(E)} \geq \frac{c_2}{c_1}.$$  

(7.19)

If $\text{diam}(E) > 1$, $H^1(E) > c_3 > 0$ and

$$H^1(E) \geq \frac{c_3}{2H^2(M_j)}.$$  

(7.20)

As the volume $H^2(M_j)$ is uniformly bounded, the uniform lower bound on $h(M_j)$ follows.

It is true that Cheeger’s inequality in this form applies to smooth manifolds, but we can for instance approximate the spaces in the flat distance in Euclidean space to conclude the bound still holds. Therefore, in the case of this example,

$$\limsup_{j \to \infty} \lambda_1(M_j) > \lambda_1(M).$$  

(7.22)

The example can be easily modified to obtain a connected limit space, by adding a thin tube connecting the one cube to the other. In that case $\lambda_1$ of the limit space can be made arbitrarily small by making the tube arbitrarily thin.

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