Notes on relative normalizations of ruled surfaces in the three-dimensional Euclidean space

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Abstract

This paper deals with relative normalizations of skew ruled surfaces in the Euclidean space \( \mathbb{E}^3 \). In section 2 we investigate some new formulae concerning the Pick invariant, the relative mean curvature and the curvature of the relative metric of a relatively normalized ruled surface \( \Phi \) and in section 3 we introduce some special normalizations of it. All ruled surfaces and their corresponding normalizations that make \( \Phi \) an improper or a proper relative sphere are determined in section 4. In the last section we study ruled surfaces, which are centrally normalized, i.e., their relative normals at each point lie on the corresponding central plane. Especially we study various properties of the Tchebychev vector field. We conclude the paper by the study of the central image of \( \Phi \).

Key Words: Ruled surfaces, relative normalizations, proper or improper relative sphere, Tchebychev vector field, Pick invariant

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1 Preliminaries

To set the stage for our work we present briefly some elementary facts regarding the relative Differential Geometry of surfaces and the Differential Geometry of ruled surfaces in the Euclidean space \( \mathbb{E}^3 \); we shall follow the notations and definitions of [4] and [6].

In the three-dimensional Euclidean space \( \mathbb{E}^3 \) let \( \Phi \) be a ruled \( C^r \)-surface of nonvanishing Gaussian curvature, \( r \geq 3 \), defined by an injective \( C^r \)-immersion \( \overline{\varphi} = \overline{\varphi}(u,v) \) on a region \( U := I \times \mathbb{R} \) \((I \subset \mathbb{R} \) open interval) of \( \mathbb{R}^2 \). Let \( \langle \cdot, \cdot \rangle \) denote the standard scalar product in \( \mathbb{E}^3 \). We introduce the so-called standard parameters \( u \in I, v \in \mathbb{R} \) of \( \Phi \), such that

\[
\overline{\varphi}(u,v) = \overline{s}(u) + v \overline{e}(u), \tag{1.1}
\]

with

\[
|\overline{e}| = |\overline{e}'| = 1, \quad \langle \overline{s}', \overline{e}' \rangle = 0, \tag{1.2}
\]

where the differentiation with respect to \( u \) is denoted by a prime. Here \( \Gamma : \overline{s} = \overline{s}(u) \) is the striction curve of \( \Phi \) and the parameter \( u \) is the arc length along the spherical curve \( \overline{e} = \overline{e}(u) \).

Let

\[
\delta(u) := \langle \overline{s}', \overline{e}, \overline{e}' \rangle
\]
be the distribution parameter,
\[ \kappa(u) := (\bar{\epsilon}, \bar{\epsilon}', \bar{\epsilon}'') \]
be the conical curvature and
\[ \sigma(u) := \varsigma(\bar{\tau}, \bar{\sigma}'), \quad \text{where} \quad -\frac{\pi}{2} < \sigma \leq \frac{\pi}{2}, \quad \text{sign} \sigma = \text{sign} \delta \]
be the striction of \( \Phi \). We consider yet the central normal vector \( \tau(u) := \bar{\epsilon}' \) and the central tangent vector \( \tau(u) := \tau \times \pi \). The moving frame \( D := \{\bar{\epsilon}, \pi, \tau\} \) of \( \Phi \) fulfills the equations [4, p. 280]
\[
\begin{align*}
\bar{\epsilon}' &= \pi, & \pi' &= -\bar{\epsilon} + \kappa \tau, & \tau' &= -\kappa \pi.
\end{align*}
\]
(1.3)
Then, we have
\[
\bar{\tau}' = \delta \lambda \bar{\tau} + \delta \tau, \quad \text{where} \quad \lambda(u) := \cot \sigma.
\]
(1.4)
We denote partial derivatives of a function (or a vector-valued function) \( f \) in the coordinates \( u^1 := u, u^2 := v \) by \( f /i \), \( f /ij \) etc. Then from (1.1) and (1.4) we obtain
\[
\begin{align*}
\bar{\tau}/1 &= \delta \lambda \bar{\tau} + v \pi + \delta \tau, & \bar{\tau}/2 &= \bar{\tau},
\end{align*}
\]
(1.5)
and thus the unit normal vector \( \xi(u, v) \) to \( \Phi \) is expressed by
\[
\xi = \delta \bar{\pi} - v \bar{\tau}, \quad \text{where} \quad w := \sqrt{\delta^2 + v^2}.
\]
(1.6)
Let \( I = g_{ij} du^i du^j \) be the first and \( II = h_{ij} du^i du^j \) be the second fundamental form of \( \Phi \), where
\[
\begin{align*}
g_{11} &= w^2 + \delta^2 \lambda^2, & g_{12} = \delta \lambda, & g_{22} = 1, \\
h_{11} &= -\frac{\kappa w^2 + \delta' v - \delta^2 \lambda}{w}, & h_{12} = \delta \frac{w}{w}, & h_{22} = 0.
\end{align*}
\]
(1.7)
(1.8)
The Gaussian curvature \( \tilde{K}(u, v) \) and the mean curvature \( \tilde{H}_I(u, v) \) of \( \Phi \) are respectively given by [4]
\[
\tilde{K} = \frac{\delta^2}{w^4}, \quad \tilde{H}_I = -\frac{\kappa w^2 + \delta' v + \delta^2 \lambda}{2w^3}.
\]
(1.9)
A \( C^s \)-relative normalization of \( \Phi \) is a \( C^s \)-mapping \( \bar{\gamma} = \bar{\gamma}(u, v), 1 \leq s < r \), defined on \( U \), such that
\[
\text{rank}\{\bar{\tau}/1, \bar{\tau}/2, \bar{\gamma}\} = 3, \quad \text{rank}\{\bar{\tau}/1, \bar{\tau}/2, \bar{\gamma}/i\} = 2, \quad i = 1, 2, \, \forall \,(u, v) \in U.
\]
(1.10)
The pair \( (\Phi, \bar{\gamma}) \) is called a relatively normalized ruled surface and the line issuing from a point \( P \in \Phi \) in the direction \( \bar{\gamma} \) is called the relative normal of \( \Phi \) at \( P \). When we move the vectors \( \bar{\gamma} \) to the origin, the endpoints of them describe the relative image of \( \Phi \).

Let \( q(u, v) := \langle \bar{\xi}, \bar{\gamma} \rangle \), denote the support function of the relative normalization \( \bar{\gamma} \) (see [3]). As follows from (1.10) \( q \) never vanishes on \( U \). Conversely, when a support function \( q \) is given, the relative normalization \( \bar{\gamma} \) of the ruled surface \( \Phi \) is uniquely determined and can be expressed in terms of the moving frame \( D \) as follows [7, p.179]:
\[
\bar{\gamma} = y_1 \bar{\epsilon} + y_2 \pi + y_3 \tau,
\]
(1.11)
where
\[ y_1 = -w \frac{\delta q/1 + q/2(\kappa w^2 + \delta' v)}{\delta^2 w}, \quad y_2 = \frac{\delta^2 q - w^2 v q/2}{\delta w}, \quad y_3 = -v q + w^2 q/2. \] (1.12)

One can easily verify the following relations:
\[ y_1 + y_{2/1} - \kappa y_3 = v \frac{\delta}{\delta}(y_{3/1} + \kappa y_2), \quad y_{2/2} = \frac{v}{\delta} y_{3/2}. \] (1.13)

For the coefficients \( G_{ij} \) of the relative metric \( G \) of \((\Phi, \bar{\gamma})\), which is undefined, the following applies
\[ G_{ij} = q^{-1} h_{ij}. \] (1.14)

Then, on account of (1.8), the coefficients of the inverse relative metric tensor are computed by
\[ G^{(11)} = 0, \quad G^{(12)} = \frac{w q}{\delta}, \quad G^{(22)} = w q \frac{\kappa w^2 + \delta' v - \delta^2 \lambda}{\delta^2}. \] (1.15)

For a function (or a vector-valued function) \( f \) we denote by \( \nabla^G f \) the first Beltrami differential operator and by \( \nabla^G_i f \) the covariant derivative, both with respect to the relative metric. The coefficients \( A_{ijk}(u, v) \) of the Darboux tensor are given by
\[ A_{ijk} := \frac{1}{q} \langle \xi, \nabla^G_k \nabla^G_j x_i \rangle. \] (1.16)

Then, by using the relative metric tensor \( G_{ij} \) for “raising and lowering the indices”, the Pick invariant \( J(u, v) \) of \((\Phi, \bar{\gamma})\) is defined by
\[ J := \frac{1}{2} A_{ijk} A^{ijk}. \] (1.17)

In [7] it was shown, that the coefficients of the Tchebychev vector \( \bar{T}(u, v) \) of \((\Phi, \bar{\gamma})\), which is defined by
\[ \bar{T} := T^m \bar{x}_m, \quad \text{where} \quad T^m := \frac{1}{2} A^{lm}, \] are given by
\[ T^1 = \frac{w^2 q/2 + v q}{\delta w}, \quad T^2 = \frac{2 \delta w^2 q/1 + \delta' q(\delta^2 - v^2)}{2 \delta^2 w} + T^1(\kappa w^2 + \delta' v - \delta^2 \lambda). \] (1.18)

\( \bar{T} \) can be expressed in terms of the moving frame \( D \) as follows [7]
\[ \bar{T} = w q \frac{(2 \kappa v + \delta') + 2 \delta q/1 + 2 q/2(\kappa w^2 + \delta' v)}{2 \delta^2 \delta w} \bar{v} + v q + w^2 q/2 \frac{v \bar{n} + \delta \bar{z}}{\delta w}. \]

The relative shape operator has the coefficients \( B^j_i(u, v) \) defined by
\[ \bar{y}_i := -B^j_i \bar{x}_j. \] (1.19)

Then, the relative curvature \( K(u, v) \) and the relative mean curvature \( H(u, v) \) are defined by
\[ K := \det \left( B^j_i \right), \quad H := \frac{B^1_1 + B^2_2}{2}. \] (1.20)

We mention finally for later use, that among the surfaces \( \Phi \subset E^3 \) with negative Gaussian curvature the ruled surfaces are characterized by the relation [8]
\[ 3H - J - 3S = 0, \] (1.21)
where \( S(u, v) \) is the scalar curvature of the relative metric \( G \), which is defined formally as the curvature of the pseudo-Riemannian manifold \((\Phi, G)\).
2 Some formulae for $J, K, H$ and $S$

In this section we express the relative magnitudes $J, K, H$ and $S$ of the relatively normalized ruled surface $\Phi$ in terms of the fundamental invariants $\delta, \kappa$ and $\lambda$ of $\Phi$ and the support function $q$. Firstly we compute the Pick invariant $J$. We notice that by virtue of the symmetry of the Darboux tensor (1.16) we have

$$J = \frac{3}{2} (A_{112} A^{112} + A_{122} A^{122}) + \frac{1}{2} (A_{111} A^{111} + A_{222} A^{222}) .$$

(2.1)

By using the well known equation [3, p. 196]

$$A_{ijk} = \frac{1}{q} \langle \xi, \pi_{ijk} \rangle - \frac{1}{2} (G_{ij/k} + G_{jk/i} + G_{ki/j}) ,$$

and the relations (1.3), (1.5), (1.11), (1.12) and (1.19) we get by straightforward calculations

$$A^{111} = A_{222} = 0 ,$$

$$A_{112} = \frac{-1}{2q^2 w} \{ (w^2 q/2 + q v) \left[ \kappa v^2 + \delta' v + \delta^2 (\kappa - \lambda) \right] + \delta' w^2 q - 2\delta (w^2 q/1 + \delta \delta' q) \} ,$$

$$A^{112} = \frac{q}{\delta^2} \left( w^2 q/2 + q v \right) ,$$

$$A_{122} = \frac{1}{w^3 q^2} \left( w^2 q/2 + q v \right) ,$$

$$A^{122} = \frac{q}{\delta^2} \{ 3 (w^2 q/2 + q v) \left[ \kappa v^2 + \delta' v + \delta^2 (\kappa - \lambda) \right] - \delta' w^2 q + 2\delta (w^2 q/1 + \delta \delta' q) \} .$$

Inserting these relations in (2.1) we obtain

$$J = \frac{3}{2\delta^2 w^3 q} \left\{ w^2 \left[ \kappa q v + 2\delta q/1 + q/2 \left( \kappa w^2 + \delta' v - \delta^2 \lambda \right) \right] - \delta^2 q \left( \lambda v - \delta' \right) \right\} .$$

(2.2)

Next, we wish now to compute the relative curvature and the relative mean curvature. To this end we find on account of (1.3), (1.5), (1.11), (1.12) and (1.19) firstly the coefficients $B_i^j$ of the relative shape operator:

$$B_1^1 = -\frac{1}{\delta^2 w^3} \left[ \delta^2 \delta' q v + \kappa w^2 \left( \delta^2 q - \delta' w^2 q/2 \right) - \delta w^2 \left( \delta \delta' q/2 + v q/1 + w^2 q/12 \right) \right] ,$$

$$B_2^1 = \frac{1}{\delta^2 w^3} \left\{ \delta^4 w^2 q - \delta^2 v w^4 q/2 - \delta^2 w^2 \left( \delta^2 + 2v^2 \right) q/2 \left( \kappa w^2 + \delta' v \right) + \delta q/1 \right\} + \delta^2 q/2 \left( 2\delta' \kappa + \kappa' w^2 + \delta^2 \nu + \delta q/1 + q/12 \left( \kappa w^2 + \delta' v \right) + \delta q/11 \right) ,$$

$$B_1^2 = \frac{1}{\delta w^3} \left( 2q/2 v w^2 + \delta^2 q + v^4 q/22 \right) ,$$

$$B_2^2 = \frac{1}{\delta^2 w^3} \left\{ - \delta^2 \lambda \left( 2q/2 v w^2 + \delta^2 q + w^4 q/22 \right) + v w^2 \left[ q/2 \left( \kappa w^2 + \delta' v \right) + \delta q/1 \right] + w^4 \left( 2\kappa v + \delta' \right) + q/22 \left( \kappa w^2 + \delta' v \right) + \delta q/12 \right\} .$$
Substituting the above relations in (1.20) we get

$$K = \frac{-1}{\delta^4 w_0} \left\{ \lambda \delta^2 [-2v^3 q_{/2}^2 - \delta^2 (q + 2vq_{/2}) - w^4 q_{/22}] + v w^2 [q_{/2} (\kappa w^2 + \delta' v) + \delta q_{/1}] + w^4 [q_{/2} (2\kappa v + \delta') + q_{/22} (\kappa w^2 + \delta' v) + \delta q_{/12}] \right\} \delta^2 (\delta q v + \delta^2 q_{/2}) + \kappa w^2 (\delta^2 q - v w^2 q_{/2}) - \delta w^2 (2\delta \delta' q_{/2} + \delta q_{/1} + \delta \lambda (\delta^2 \delta' (q v + w^2 q_{/2}) + \delta q_{/1}) - 2\delta' w^4 [q_{/2} (\kappa w^2 + \delta' v) + \delta q_{/1}] + \delta \lambda (\delta^2 \delta' (q v + w^2 q_{/2}) + \delta q_{/1}) + \delta q_{/12}) \right\} \right\} \\
= \frac{1}{\delta^2 w^3 q} \left\{ -\delta^2 q (\kappa w^2 + \delta' v + \delta^2 \lambda) + 2w^2 q_{/2} \left[ (2\kappa v + \delta') w^2 - \delta^2 \lambda v \right] + w^4 q_{/22} (\kappa w^2 + \delta' v - \delta^2 \lambda) + 2\delta v w^2 q_{/1} + 2\delta w^4 q_{/12} \right\}.
$$

Inserting (2.1) and (2.4) in (1.21) we infer the scalar curvature of the relative metric $G$

$$S = \frac{1}{\delta^2 w^2 q} q^2 \left[ -\kappa w^4 + \delta^2 (\lambda v^2 - 2\delta' v - \delta^2 \lambda) \right] + w^4 q_{/2} (2\kappa v + \delta') + w^4 (\kappa w^2 + \delta' v - \delta^2 \lambda) (q_{/22} - q_{/2}^2) - 2\delta w^4 q_{/1} + 2\delta w^4 q_{/12} \right\}.
$$

The divergence $\text{div}^G \overline{T}$ of $\overline{T}$ with respect to the relative metric $G$ of $\Phi$ is given by [9]

$$\text{div}^G \overline{T} = \frac{|G^{1/2} T^i|}{|G|^{1/2}} / i, \quad \text{where} \quad G := \text{det} (G_{ij}) = -\frac{\delta^2}{w^2 q^2}.
$$

By taking (1.17), (2.5) and (2.6) into consideration it turns out that

$$\text{div}^G \overline{T} - 2S = \frac{2\kappa w q}{\delta^2}.
$$

### 3 Special relative normalizations

In [7] I. Kaffas and S. Stamatakis have studied the so called asymptotic relative normalizations of a given ruled surface, that is relative normalizations such that the relative normal at each point $P$ of $\Phi$ lies on the corresponding asymptotic plane $\{P, \overline{\tau}, \overline{\pi}\}$ of $\Phi$. Following this idea we consider relative normalizations such that the relative normal at each point $P$ lies a) on the corresponding central plane $\{P, \overline{\tau}, \overline{\tau}\}$, or b) on the corresponding polar plane $\{P, \overline{\pi}, \overline{\tau}\}$.

The first case occurs iff $y_2 = 0$, or, because of (1.12b), iff the support function of $\overline{\tau}$ is of the form

$$q = \frac{g v}{w},
$$

(3.1)
where \( g = g(u) \) is an arbitrary nonvanishing \( C^3 \)-function. We call the corresponding relative normalization central. Obviously in this case it is

\[
\eta = \frac{g'v + \delta \kappa g}{\delta} \varphi - g \zeta,
\]

(3.2)

cf. (1.11), (1.12). The second case occurs iff \( y_1 = 0 \), or, because of (1.12a), iff the support function of \( \eta \) is of the form

\[
q = f(V), \quad \text{where} \quad V = \arctan \frac{v}{\delta} - \int \kappa \, du
\]

and \( f \) is an arbitrary nonvanishing \( C^2 \)-function of \( V \). We call the arising relative normalization polar. We find

\[
\eta = \frac{\delta q - v \dot{q}}{w} \varphi - \frac{v q + \delta \dot{q}}{w} \zeta,
\]

where the dot denotes the derivative in \( V \).

Finally, let the relative image be as well as \( \Phi \) a ruled surface whose generators are parallel to those of \( \Phi \). Then \( y_{2/2} = y_{3/2} = 0 \), from which, by means of (1.12), we obtain

\[
2v w^2 q_{/2} + \delta^2 q + q_{/22} w^4 = 0.
\]

Consequently

\[
q = \frac{f + g v}{w},
\]

(3.3)

where \( f \) and \( g \) are arbitrary \( C^3 \)-functions of \( u \), such that \( q \neq 0 \). In this case we have

\[
\eta = \bar{\eta} - \frac{\delta g' - \kappa f}{\delta^2} \varphi \bar{\varphi},
\]

(3.4)

where

\[
\bar{\eta}(u) = - \left[ \left( \frac{f}{\delta} \right)' + \kappa g \right] \bar{\varphi} + \frac{f}{\delta} \bar{\bar{\varphi}} - g \bar{\zeta}.
\]

From (3.4) it follows that the relative image of \( \Phi \) is a curve or a ruled surface whose generators are parallel to those of \( \Phi \) iff the function \( \delta g' - \kappa f \) vanishes everywhere or nowhere in \( I \), respectively. We call in the sequel such a normalization right. We recognise immediately that both asymptotic and central normalizations belong to the right ones. In section 5 of this paper we investigate the central normalizations leaving the study of the polar and the right ones for a subsequent paper.

4 \( \Phi \) is an improper or a proper relative sphere

In this section we investigate all ruled surfaces \( \Phi \) and the corresponding support functions \( q(u, v) \) so that \( \Phi \) is an improper or a proper relative sphere.

It is easily verified from (1.3), (1.11) and (1.13) that \( \Phi \) is an improper relative sphere, i.e., by definition [2], its relative image degenerates into a point \( (\overline{\eta}/i = 0, i = 1, 2) \), iff the following relations hold true

\[
y_{1/1} - y_2 = \kappa y_2 + y_{3/1} = y_{1/2} = y_{3/2} = 0.
\]

(4.1)
It is however obvious that \( K = H = 0 \). By means of \( y_{3/2} = 0 \) and (1.12c) we derive that the support function has the form (3.3), i.e., the normalization is right. We distinguish now two cases:

**Case I.** \( \Phi \) is conoidal \((\kappa = 0)\). From (1.12), (3.3) and (4.1) we find

\[
f = \delta (c_1 \cos u + c_2 \sin u), \quad g = c_3, \quad c_1, c_2, c_3 \in \mathbb{R}, \quad c_1^2 + c_2^2 + c_3^2 \neq 0.
\]  

(4.2)

The corresponding relative normalization of \( \Phi \) then results

\[
\overline{y} = (c_1 \sin u - c_2 \cos u) \overline{\pi} + (c_1 \cos u + c_2 \sin u) \overline{n} + \overline{c},
\]  

(4.3)

where \( \overline{c} = -c_3 \overline{\pi} \) is a constant vector. One can easily verify that the converse is valid as well, i.e., that the relative normalization (4.3) is constant. From (1.21), (2.2), (3.3) and (4.2) we find

\[
J = -3S = 3c_3 \frac{\delta (2c_2 \cos u - 2c_1 \sin u - c_3 \lambda) + \delta' (c_1 \cos u + c_2 \sin u)}{2 \delta [c_3 v + \delta (c_1 \cos u + c_2 \sin u)]}.
\]

**Case II.** \( \Phi \) is non-conoidal. From the relations (1.12) and (4.1) we take

\[
f = \frac{\delta g'}{\kappa},
\]  

(4.4)

while the function \( g \) fulfils the equation

\[
\left( \frac{g'}{\kappa} \right)' + \frac{g'}{\kappa} + (\kappa g)' = 0.
\]  

(4.5)

In this case we find

\[
\overline{y} = - \left[ \kappa g + \left( \frac{g'}{\kappa} \right)' \right] \overline{\pi} + \frac{g'}{\kappa} \overline{n} - g \overline{\pi}.
\]  

(4.6)

The inverse is valid as well: The relative normalization (4.6), under the assumption (4.5), is constant. From (1.21), (2.2), (3.3) and (4.4) we obtain

\[
J = -3S = 3g \frac{\kappa^2 g \left( \kappa v^2 + \delta^2 \kappa - \delta^2 \lambda \right) + \delta \left[ g' \left( 2\kappa v + \delta' \right) - 2\delta \kappa' + 2\kappa g'' \right]}{2\delta^2 \kappa \left( \kappa g v + \delta g' \right)}.
\]

So we arrive at

**Proposition 4.1.** A relatively normalized ruled surface \( \Phi \subset \mathbb{E}^3 \) is an improper relative sphere iff the relative normalization is right and one of the following properties holds:

(a) \( \Phi \) is conoidal and \( f \) and \( g \) are the functions (4.2).

(b) \( \Phi \) is non-conoidal, the function \( g \) fulfils (4.5) and \( f \) is the function (4.4).

Let now \( \Phi \) be a proper relative sphere, i.e., by definition [3], its relative normals pass through a fixed point. It is obvious, that this is valid iff there exists a constant \( c \in \mathbb{R}^n \) and a constant vector \( \overline{\pi} \), such that

\[
\overline{x} = c \overline{y} + \overline{\pi}.
\]  

(4.7)

Taking into consideration (1.19) and (4.7), we observe that

\[
B^j_i = - \frac{\delta^j}{c}, \quad \forall \ i, j = 1, 2.
\]
Consequently, by (1.20), it is
\[ H = -\frac{1}{c}, \quad K = \frac{1}{c^2}. \]
Furthermore, taking partial derivatives of (4.7) on account of (1.3), (1.5), (1.11) and (1.13) we obtain
\[ \delta \lambda = c \left( y_1 - y_2 \right), \quad \delta = c \left( \kappa y_2 + y_3 \right), \quad 1 = c y_1, \quad 0 = c y_3. \] (4.8)
Because of (4.8d) the support function is again of the form (3.3), i.e., the normalization is right. We distinguish two cases

**Case I.** $\Phi$ is conoidal. From (1.12) and (4.8c) we take
\[ g = \frac{c_3 - \int \delta du}{c}, \quad c_3 \in \mathbb{R}. \] (4.9)
Then from (4.8a) we have
\[ \left( \frac{f}{\delta} \right)^{\prime\prime} + \frac{f}{\delta} + \frac{\delta \lambda}{c} = 0, \]
and (4.8b) becomes an identity. For the relative normalization holds (3.4), where the functions $f$ and $g$ are given by (4.10) and (4.9), respectively. Conversely, let the relative normalization (3.4) be given, where $f$ and $g$ are the functions (4.10) and (4.9), respectively. By using (1.3) and (1.4) we infer
\[ \overline{s}' = (c \overline{y} - v\overline{e})'. \] (4.11)
It follows
\[ \overline{s} = c \overline{y} - v\overline{e} + \overline{\pi}, \]
where $\overline{\pi}$ is a constant vector. Thus, (4.7) is valid and therefore $\Phi$ is a proper relative sphere, whose striction curve $\Gamma$ is parametrized by
\[ \overline{s} = -\left[ \cos u \left( c_2 - \int \delta \lambda \cos u du \right) + \sin u \left( c_1 + \int \delta \lambda \sin u du \right) \right] \overline{e} - \left( \int \delta du - c_3 \right) \overline{e} \]
\[ + \left[ \cos u \left( c_1 + \int \delta \lambda \sin u du \right) + \sin u \left( c_2 - \int \delta \lambda \cos u du \right) \right] \overline{\pi} + \overline{\pi}. \]
Finally, from (2.2) and (3.3) we find
\[ J = 3g \frac{2g'v - \delta^2 \lambda g - \delta' f + 2\delta' f'}{2g^2 (g v + f)}. \]

**Case II.** $\Phi$ is non conoidal. From (1.12) and (4.8c) we obtain
\[ f = \frac{\delta (\delta + c g')}{c \kappa}. \] (4.12)
By using (4.8a) we take
\[ \left[ \frac{\delta (\delta + c g')}{c \kappa} \right]'' + \frac{\delta (\delta + c g')}{c \kappa} + c (\kappa g)' + \delta \lambda = 0, \] (4.13)
and (4.8b) becomes an identity. The relative normalization results
\[ \mathcal{g} = \left\{ \frac{v}{c} - \kappa g - \left[ \frac{\delta (\delta + cg')}{c \kappa} \right]' \right\} \overline{v} + \frac{\delta (\delta + cg')}{c \kappa} \overline{w} - g \overline{z}. \]  
(4.14)

Conversely, let the relative normalization (4.14) be given, where the function \( g \) satisfies (4.13). If we proceed as in case I, we easily verify that (4.11) holds true and therefore \( \Phi \) is a proper relative sphere, whose striction curve \( \Gamma \) is parametrized by
\[ s = \left\{ -c \kappa g - \left[ \frac{\delta (\delta + cg')}{\delta} \right]' \right\} \overline{v} + \frac{\delta (\delta + cg')}{\kappa} \overline{w} - c g \overline{z} + \overline{a}. \]

where \( \overline{a} \) is a constant vector. Finally, for the Pick invariant we have
\[ J = 3g \frac{[\kappa v^2 + \delta^2 (\kappa - \lambda)] - \delta f + 2\delta (g'v + f')}{2\delta^2 (gv + f)}. \]

Thus the following has been shown

**Proposition 4.2.** A relatively normalized ruled surface \( \Phi \subset \mathbb{E}^3 \) is a proper relative sphere iff
the relative normalization is right and one of the following properties holds:

(a) \( \Phi \) is conoidal and \( f \) and \( g \) are the functions (4.10) and (4.9), respectively.

(b) \( \Phi \) is non conoidal, the function \( g \) fulfils (4.13) and \( f \) is the function (4.12).

We wish to conclude this section by determining the relative normalizations, which are constantly linked to the moving frame \( \mathcal{D} \), i.e., \( y_{ij} = 0 \) for \( i = 1, 2, 3 \) and \( j = 1, 2 \). An elementary treatment of the last system of equations yields: \( \Phi \) is a conoidal relatively minimal ruled surface and the support function is of the form
\[ q = \frac{c_1 v + c_2 \delta}{w}, \quad c_1, c_2 \in \mathbb{R}, c_1^2 + c_2^2 \neq 0. \]

Whenever \( c_2 = 0 \) (\( \Phi, \overline{y} \)) is an improper relative sphere (\( \overline{y}' = \overline{u} \)), otherwise the relative image of \( \Phi \) degenerates into a piece of circle of radius 1 : \( |c_2| \) which is parametrized by
\[ \overline{y} = c_2 \overline{w} - c_1 \overline{z}. \]

### 5 Central normalizations

Let \( \overline{y} \) be a central normalization of a given ruled surface \( \Phi \). From (2.2) - (2.5) and (3.1) we obtain
\[ H = \frac{g'}{\delta}, \quad K = \frac{g'^2}{\delta^2}, \quad S = -g \left[ \kappa v^2 + \delta^2 (\kappa - \lambda) \right], \quad J = 3 \left[ \frac{\kappa g v^2 + 2\delta g' v + \delta^2 g (\kappa - \lambda)}{2\delta^2 v} \right]. \]  
(5.1)

It is obvious that \( \Phi \) is a relative minimal surface (or of vanishing relative curvature) iff \( g = c \in \mathbb{R}^* \). Furthermore, the scalar curvature of the relative metric \( G \) vanishes identically iff
\[ \kappa v^2 + \delta^2 (\kappa - \lambda) = 0. \]

After successive differentiations of this last equation relative to \( v \), we deduce, that \( \kappa = \kappa - \lambda = 0. \) Thus, the scalar curvature of the relative metric vanishes identically iff \( \Phi \) is a right conoid.
(κ = λ = 0). In the same way one may see that the Pick invariant vanishes identically iff Φ is a relative minimal right conoid.

We notice that all points of Φ are relative umbilics (H^2 - K ≡ 0). Hence, for the relative principal curvatures k_1 and k_2 holds k_1 = k_2 = H. If g = c ∈ R*, then, because of (3.2), the central normalization of Φ reads \( \vec{y} = -g \vec{d} \), i.e., the central image degenerates into a curve parallel to the Darboux vector of Φ [1]. If g is not constant the parametrization of the unique relative focal surface of Φ which by definition is given by

\[
\vec{x}^* = \vec{s} + v \vec{e} + \frac{1}{H} \vec{y}
\]

becomes

\[
\vec{x}^* = \vec{s} - \frac{\delta g}{g'} \vec{d},
\]

i.e., the focal surface degenerates into a curve \( \Gamma^* \) and all relative normals along each generator form a pencil of straight lines whose centers lie on the curve \( \Gamma^* \). Let \( P(u_0) \) be a point of the striction curve \( \Gamma \) of Φ and \( R(u_0) \) the corresponding point on the focal curve \( \Gamma^* \). We consider all central normalizations of Φ. Therefore, the locus of the points \( R(u_0) \) is a straight line parallel to the vector \( \vec{d}(u_0) \). Thus, we obtain a ruled surface \( \Phi^* \), whose generators are parallel to the vectors \( \vec{d}(u) \), a parametrization of which reads

\[
\vec{x}^* = \vec{s} + v^* \vec{d}.
\]

A parametrization of its striction curve is

\[
\vec{s}^* = \vec{s} - \frac{\delta \lambda}{\kappa'} \vec{d}.
\]

One can easily verify that \( \Phi^* \) is developable.

By using (3.2) and (4.1) we may infer:

- A centrally normalized ruled surface \( \Phi \) is an improper relative sphere iff \( \Phi \) is a relative minimal surface of constant conical curvature. Then, the central image of \( \Phi \) degenerates into a curve parallel to the Darboux vector.

- A centrally normalized ruled surface \( \Phi \) is a proper relative sphere iff

\[
g = \frac{1}{c} \left( c_1 - \int \delta \, du \right), c, c_1 ∈ \mathbb{R} \quad \text{and} \quad \delta (\kappa - \lambda) + \kappa' \left( \int \delta \, du - c c_1 \right) = 0.
\]

We focus now on the field \( \vec{T}(u, v) \) of the Tchebychev vectors of \((\Phi, \vec{y})\). By using (1.18) we find

\[
\vec{T} = \frac{2 \kappa g v^2 + (\kappa' g + 2 \delta g') v + 2 \kappa \delta^2 g}{2 \delta^2} \vec{e} + \frac{g}{\delta} (\delta v + \delta \vec{s}).
\]  

(5.2)

By taking (2.7) and (5.1c) into consideration it turns out that the divergence \( \text{div}^G \vec{T} \) of \( \vec{T} \) with respect to the relative metric \( G \) of \( \Phi \) is given by

\[
\text{div}^G \vec{T} = \frac{g}{\delta^2 v} \left[ \kappa v^2 - \delta^2 (\kappa - \lambda) \right].
\]

Consequently we have

\[
\text{div}^G \vec{T} = 0
\]
\[ \kappa v^2 - \delta^2(\kappa - \lambda) = 0. \]

After successive differentiations of this last equation relative to \( v \), we deduce that \( \kappa = \kappa - \lambda = 0 \). So we have: **The vector field \( \overrightarrow{T} \) is incompressible iff \( \Phi \) is a right conoid.**

The vectors \( \overrightarrow{T} \) are orthogonal to the generators of \( \Phi \) iff

\[ \langle \overrightarrow{\tau}, \overrightarrow{T} \rangle = 0. \]

On account of (5.2) we have

\[ 2k\kappa g^2 + (\delta' g + 2\delta' g') v + 2\delta^2 \kappa g = 0. \]

Treating analogously this equation we conclude: **The tangent vector to a curve \( \Lambda : v = v(u) \) of \( \Phi \) is**

\[ \overrightarrow{x}' = (\delta \lambda + v') \overrightarrow{\tau} + v \overrightarrow{n} + \delta \overrightarrow{x}. \] **(5.3)**

We consider now the following families of curves on \( \Phi \): a) the curved asymptotic lines, b) the curves of constant striction distance (\( u \)-curves) and c) the \( \tilde{K} \)-curves, i.e., the curves along which the Gaussian curvature is constant [5]. The corresponding differential equations of these families of curves are

\[ \kappa v^2 + \delta' v + \delta^2 (\kappa - \lambda) - 2\delta v' = 0, \] **(5.4)**

\[ v' = 0, \] **(5.5)**

\[ 2\delta vv' + \delta' (\delta^2 - v^2) = 0. \] **(5.6)**

From (5.2) and (5.3) it follows: \( \overrightarrow{x}' \) and \( \overrightarrow{T} \) are parallel or orthogonal iff

\[ 2k\kappa g^2 + (\delta' g + 2\delta' g') v + 2\delta^2 \kappa g - 2\delta g (\delta \lambda + v') = 0 \] **(5.7)**

or

\[ (\delta \lambda + v') \left[ 2k\kappa g^2 + (\delta' g + 2\delta' g') v + 2\delta^2 \kappa g \right] + 2\delta gw^2 = 0, \] **(5.8)**

respectively. From (5.4) and (5.7) we infer, that **\( \overrightarrow{T} \) is tangential to the curved asymptotic lines iff**

\[ \kappa g^2 + 2\delta g' v + \delta^2 g (\kappa - \lambda) = 0, \]

that is iff \( \kappa = \lambda = 0 \) and \( g = \text{const.} \) and therefore **iff \( \Phi \) is a relative minimal right conoid.**

Analogous reasoning in the case of the families of curves b) and c) leads to the following results:

- **\( \overrightarrow{T} \) is tangential to the \( u \)-curves of \( \Phi \) iff \( \Phi \) is right conoid and \( g = c|\delta|^{-1/2}, c \in \mathbb{R}^* \).**
- **\( \overrightarrow{T} \) is orthogonal to the \( u \)-curves of \( \Phi \) iff the striction curve of \( \Phi \) is an Euclidean line of curvature and \( g = c|\delta|^{-1/2}, c \in \mathbb{R}^* \).**
- **\( \overrightarrow{T} \) is tangential to the \( \tilde{K} \)-curves of \( \Phi \) iff \( \Phi \) is a relative minimal right helicoid surface \( (\kappa = \lambda = 0, \delta = \text{const.}) \).**
- **\( \overrightarrow{T} \) is orthogonal to the \( \tilde{K} \)-curves of \( \Phi \) iff \( \Phi \) is a relative minimal Edlinger surface \( (\kappa \lambda + 1 = 0, \delta = \text{const.}, \text{see}[1], [5]) \).**
Next we assume that the vector field $\mathbf{T}$ is tangential to one family of Euclidean lines of curvature. Their differential equation which initially reads

$$g_{12}h_{11} - g_{11}h_{12} + (g_{22}h_{11} - g_{11}h_{22})v' + (g_{22}h_{12} - g_{12}h_{22})v'^2 = 0,$$

becomes, on account of (1.7) and (1.8),

$$\delta \left[ w^2 (\kappa \lambda + 1) + \delta' \lambda v \right] + (\kappa w^2 + \delta' v - \delta^2 \lambda) v' - \delta v'^2 = 0,$$

from which, by virtue of (5.7), we infer

$$2\kappa g \left( \delta' g - 2\delta g' \right) v^3 + \left[ 4\delta^2 g^2 (\kappa \lambda + 1) + \delta^2 g^2 - 4\delta^2 g'^2 \right] v^2$$

$$+ 2\delta^2 g \left[ \delta' g (\kappa + \lambda) - 2\delta g' (\kappa - \lambda) \right] v + 4\delta^4 g^2 (\kappa \lambda + 1) = 0.$$

This last equation holds true iff $\kappa \lambda + 1 = 0, \delta = c_1 \in \mathbb{R}^* \text{ and } g = c_2 \in \mathbb{R}^*$. Hence $\Phi$ is a relative minimal Edlinger surface. Because of (1.4) and

$$\mathbf{T}_{v=0} = g (\kappa \overline{e} + \overline{e})$$

we obtain $\langle \mathbf{T}_{v=0}, \mathbf{s}' \rangle = 0$. So, we have:

**Proposition 5.1.** $\mathbf{T}$ is tangential to the one family of the Euclidean lines of curvature of $\Phi$ iff $\Phi$ is a relative minimal Edlinger surface. Moreover the family of the Euclidean lines of curvature under consideration consists of the lines of curvature which are orthogonal to the striction curve.

In the rest of this section we assume that $\Phi$ is a non relatively minimal ruled surface. The central image $\Psi_1$ of $\Phi$ is also a ruled surface, whose generators are parallel to those of $\Phi$. Then, from (3.2) by direct computation, we find that the parametrization of its striction curve is

$$\Gamma_1: \overline{s}_1 = -g \overline{d}.$$

We set $\overline{y} = \overline{y}_1$ and we rewrite the parametrization of $\Psi_1$ as

$$\Psi_1: \overline{y}_1 = \overline{s}_1 + v_1 \overline{e}, \quad v_1 := -\frac{g' v}{\delta}.$$

It is obvious that $\Psi_1$ is parametrized like in (1.1) and (1.2). We use $\mathcal{D}$ as moving frame of $\Psi_1$. By direct computation we find the fundamental invariants of $\Psi_1$:

$$\kappa_1 = \kappa, \quad \delta_1 = -g', \quad \lambda_1 = \frac{(\kappa g')'}{g'}.$$

Thus the Darboux vectors of $\Phi$ and $\Psi_1$ are parallel. Furthermore it is $w_1 = |H|w$. Then, by (1.9), the Gaussian curvature $\tilde{K}_1$ of $\Psi_1$ is seen to be

$$\tilde{K}_1 = \frac{\overline{K}}{\overline{K}}.$$

From the above we list the following results, which can be checked easily:

- $\Phi$ and its central image $\Psi_1$ are congruent ($\delta = \delta_1, \kappa = \kappa_1, \lambda = \lambda_1$) iff $\Phi$ is a proper relative sphere.
- $\Psi_1$ is orthoid ($\lambda_1 = 0$) iff $\kappa g = c \in \mathbb{R}$.  

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• The striction curve $\Gamma_1$ of $\Psi_1$ is an asymptotic line of it ($\kappa_1 = \lambda_1$) iff $\Phi$ is of constant conical curvature.

• The striction curve $\Gamma_1$ of $\Psi_1$ is an Euclidean line of curvature of it ($1 + \kappa_1 \lambda_1 = 0$) iff

\[
g = \frac{c}{\sqrt{1 + \kappa^2}}, c \in \mathbb{R}^*.
\]

• $\Psi_1$ is an Edlinger surface iff

\[
g = c_1 + c_2u, c_1 \in \mathbb{R}, c_2 \in \mathbb{R}^* \quad \text{and} \quad 1 + \kappa^2 = \frac{c^2}{(c_1 + c_2u)^2}, c \in \mathbb{R}^*.
\]

We wish to conclude this paper by answering the following question: Is there a ruled surface $\Psi^*$ whose a central normalization is the given ruled surface $\Phi$? We suppose that such a ruled surface exists and let it be parametrized like in (1.1) and (1.2). We consider a central normalization of $\Psi^*$ via a support function of the form (3.1). Denoting by $\delta^*$, $\kappa^*$ and $\lambda^*$ its fundamental invariants we have:

\[
\kappa = \kappa^*, \quad \delta = -g^*, \quad \lambda = \frac{(\kappa g^*)'}{g^*},
\]

cf. (5.9), thus

\[
g^* = c_1 - \int \delta \, du \quad (c_1 \in \mathbb{R}), \quad \delta (\kappa - \lambda) - \kappa' \left( c_1 - \int \delta \, du \right) = 0. \quad (5.10)
\]

Conversely, we assume that a constant $c_1 \in \mathbb{R}$ exists, such that the fundamental invariants of $\Phi$ fulfil (5.10b). We consider an arbitrary skew ruled surface $\Psi^*$, whose generators are parallel to those of $\Phi$ and we normalize it centrally via a support function of the form (3.1), where $g^*$ is a function of the form (5.10a). By taking (5.9) and (5.10b) into account, we deduce that the fundamental invariants of the arising central image of $\Psi^*$ are $\delta$, $\kappa$ and $\lambda$. Hence, the central image of $\Psi^*$ and $\Phi$ are congruent. Summing up we have:

**Proposition 5.2.** A necessary and sufficient condition for the existence of a ruled surface $\Psi^*$, whose central normalization is the given ruled surface $\Phi$ is that there is a constant $c_1 \in \mathbb{R}$, such that the fundamental invariants of $\Phi$ fulfil (5.10b).

An example of a ruled surface, whose fundamental invariants fulfil (5.10b) is the right helicoid.

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