Self-organized dynamics in local load sharing fiber bundle models

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We study the dynamics of a local load sharing fiber bundle model, where initially load is applied only at a single point. Since more and more fibers share the load after the initial fibers break, the breaking process will eventually stop, no matter how large the initial load is. One has to increase further the load to continue the breaking dynamics in the model. This leads the system to a self-organized state, where scale free behaviour of avalanche size distribution and other associated quantities are seen. We present a scaling argument to explain the value of the exponent. We also analyse the effect of having a lower cut-off in the threshold distribution of the fibers.

I. INTRODUCTION

The fiber bundle model, since its introduction [1], have been studied widely as a prototypical model of failure dynamics [2–5]. While being simple and often analytically tractable, it captures the attention of physicists observing dynamical critical phenomena [6–14]. This discrete element model involving disorder and non-linear dynamics (due to thresholds) also enables engineers to apply it to analyse the breaking properties of real materials (e.g., fiber reinforced composites [17]).

There are mainly two extreme versions of this model. Both the versions consider a bunch of fibers (Hooke springs) hanging from a rigid ceiling and a platform is connected to the ends of the fibers and a load hangs from that platform. The fibers have a given limit or load-carrying threshold (usually taken randomly from some distribution function), beyond which it breaks. Completely different behaviours are observed when the elastic property of the lower platform changes. The two extremes arise when the platform is either absolutely rigid or absolutely soft. The difference is of course, when a fiber breaks in the former case, due to rigidity of the lower platform the load is equally shared by all other remaining fibers. In the later case, however, due to local deformation of the platform, the load of the broken fiber is only to be carried by nearest surviving neighbours (stress concentration occurs around the failure). While in the global load sharing version the load at which the system completely fails scales with system size linearly, for the local load sharing case the increase is only sub-linear (in fact $N/\log(N)$; $N$ being system size [2]). The implication being, the critical load per fiber $\sigma_c$ at which the system fails, becomes finite for the global load sharing case and goes to zero in the large system size limit for the local load sharing case. Therefore observations like divergence of relaxation time, proper scale free distribution of avalanches are not seen here; unlike the global load sharing version, where these are analyzed in detail [2].

However, the situation can be quite different if one also makes the initial applied load localized (at an arbitrarily chosen central site) in a local load sharing fiber bundle model. Let this load be increased at a slow but constant rate. Initially no load is present on any fiber except for one at the central site. As the applied load increases beyond the threshold of this central fiber, it breaks and the load is distributed among nearest neighbours and so on. In general, whenever a fiber breaks, the load carried by that fiber is equally distributed among all the fibers that have at least one broken neighbour. In this way, the fibers which are newly exposed to the load, say, after an avalanche, has a relatively low load compared to the ones which are accumulating load shares from the earlier failures and are still surviving. As we shall see later, this helps in maintaining a compact structure of the cluster of the broken fibers. This rule of force redistribution is justified from the view that the newly exposed fibers are presumably further away from the point of loading and therefore has to carry a smaller fraction of the load.

Since the fibers in the perimeter, which are the only ones carrying the entire load, are increasing in number, the load per fiber decreases as more and more damages occur. This would stop the growth of the cluster. Note that in this model we assume that the process of fiber breaking and load redistribution happens at a much faster rate than the rate of external load increase. Therefore, the external load is not increased during an ongoing avalanche. However, as the load on the bundle increases at a constant rate, dynamics will continue again until some critical load per fiber is reached and so on. The failure of fibers in the process of avalanches is seen to have a scale free size distribution. In the following we shall analyse the scale free nature of this avalanche size distribution and other related quantities and will try to estimate the exponent values from a scaling argument.
FIG. 1: The compact structure of the patch of the broken fibers in a two dimensional square lattice \((200 \times 200)\) of local load sharing fiber bundle models. The breaking thresholds are distributed uniformly in the range \([0 : 1]\).

II. APPLICATION OF LOCAL LOAD: SELF-ORGANIZATION

As mentioned before, usually one does not find a finite critical load for local load sharing fiber bundle models. Since the load, no matter how small, is applied system wide, there will always be a large enough weak patch, which will be broken and due to concentration of the load on the boundary and the fact that more load is nucleated if the fracture progresses, keep the patch growing, leading to a system wide failure without any further input from outside. However, the situation will be very different when the initial load also is applied locally and not on all elements of the system. The algorithm is as follows: Initially we apply a load on one fiber, say the one at the middle. We continue to increase the load until this fiber breaks. When this fiber breaks, the load is distributed equally among its four neighbours. We then increase the load upto the point when one of those four fibers breaks. In general, whenever a fiber breaks, the load carried by that fiber is equally shared by all fibers that have at least one broken neighbour. Note that in this way of redistribution, the fibers that are newly exposed to the load have a rather low load per fiber compared to the ones which are gathering and withstanding the loads of broken fibers from a few steps earlier. This ensures that the compact structure of the patch of the broken fibers is maintained, since a rather weak but new fiber is more likely to survive than a stronger but old one, thereby eliminating the possibilities of fingering like effects from the patch boundary. Also note that, following the usual definition of avalanche, we do not increase the external load when a fiber breaks or more fibers break due to load redistribution. In other words, the process of load redistribution happens in a much faster time scale.

FIG. 2: The time evolution of the load per fiber value when the threshold is uniformly distributed between \([0 : 1]\). The solid line is the analytical estimate \((1/3)\) and the simulation result is close to that value. The inset shows the comparison of the simulation results and analytical estimates of the saturation load per fiber value, when the threshold is distributed uniformly between \([a : 1]\).

FIG. 3: The distributions of the load per fiber (red curves) and failure thresholds of the fibers (blue curves), both only along the boundary of the patch of broken fibers, for different values of the lower cut-offs in the otherwise uniform threshold distributions.

In all subsequent discussions, we will deal with a uniform distribution of for the thresholds of the fibers \([a : 1]\), particularly the special case when \(a = 0\).

A. Saturation of average load per fiber

First let us consider the case when \(a = 0\). Then we follow the algorithm for load increase as stated before. If
one measures the average load per fiber with time (one time step being one avalanche), one finds that after a transient it saturates to a given value (see Fig. 2). For this case, the value is close to 1/3. This is a signature of self-organization, where in spite of external drive (say, energy input) the average energy of the system remains constant. While in most systems, the necessary dissipation comes from the presence of open boundary [18, 19], in our case, there is no apparent dissipation but the same effect is produced from the fact that the system size (the perimeter of the broken patch) is not conserved but increases. Before going into the dynamics of the system in the self-organized state, let us first attempt to understand the stationary behaviour, i.e. the stationary probability density function of the load per fiber along the perimeter and the average value for that matter. In doing that we consider that dynamically all values of force (between [0 : 1]) are generated with equal probability. This may be justified as follows: As all values of load redistribution is likely (avalanche size is scale free) we do not have a special value for force. Once this is assumed, we have the probability of having a load on a fiber between \( x \) and \( x + dx \) is \( P(x)dx \). Now, the probability that the fiber has a threshold greater than \( x \) is \( 2(1 - x) \). So, the probability that we will find a fiber carrying a load between \( x \) and \( x + dx \) is \( 2(1 - x)P(x)dx \). With \( P(x) = 1 \) in this case (as \( a = 0 \)), we have the probability density function for the force distribution as \( 2(1 - x) \). This is very close to what we get numerically (Fig. 3). Of course, the average

\[
\int_0^1 2(1 - x)xdx = \frac{1}{6},
\]

which is again close to what we see numerically (Fig. 3). In the same way, one can calculate the probability density function for the load, which is \( 2x \) and is also seen numerically (Fig. 3).

Now we go to the case when \( a \) is finite, i.e., the threshold distribution function has a lower cut-off. Now, since all fibers have threshold higher than \( a \) and we have assumed before that force is uniformly distributed, we take the probability density function for force to be uniform between \([0 : a]\) and then decreasing linearly as before. From normalization condition, the height of the uniform part is \( \frac{1}{a + 1} \). If we now calculate the average force, it comes out to be

\[
\langle \sigma_c \rangle = \frac{a^2}{1 + a} + \frac{2}{1 - a^2} \left[ \frac{1}{6} + \frac{a^2}{2} + \frac{a^3}{3} \right].
\]

where the angular brackets denote spatial average. We compare the prediction of this calculation with the values obtained numerically in Fig. 3.

B. Avalanche size distribution

As indicated before, we define an avalanche as follows: We increase the load upto the point when the weakest fiber (the fiber having smallest difference between its threshold and the load it carries) breaks. The load is redistributed following the algorithm mentioned before.

There may be further breaking of fibers after this load redistribution and so on until the process stops. During this whole time external load is not increased any further. All the fibers that have broken between two successive stoppages of dynamics, constitute one avalanche. The size distribution of avalanches measured in this way are plotted in Fig. 4. This shows a power-law distribution with exponent value close to 3/2. Note that similar exponent value was obtained before when one measures avalanches only near the critical point in global load sharing models [20] and also in other versions of self-organized models with fiber regeneration [21]. This exponent value remains unchanged when a lower cut-off is present in the threshold distribution (shown in Fig. 4) and also for other types of threshold distribution (not shown here). We have also measured the duration of an avalanche. This is defined as the number of times the load is redistributed during one process of avalanche. This is also a power-law decay with exponent value close to 2 (see Fig. 4). One can also measure the distribution of the steps size of the load increase. Since one has to increase the load upto which the weakest fiber breaks, the size of load increase require to restart the dynamics \( \Delta W \) is not fixed. It turns out that its distribution also follows a power-law decay with exponent value close to 2 (see inset of Fig. 5).

Now to estimate the value of the avalanche size exponent, we first assume that the average load per fiber on the perimeter of the damaged region has a distribution which is Gaussian around its mean: \( P(\sigma) \sim e^{-(\sigma - \sigma_c)^2/2\sigma^2} \). Hence, on a dimensional analysis, mean-squared fluctuation \( \delta \sigma \sim (\sigma - \sigma_c)^2 \). Also the avalanche size \( \Delta \) scales as \( (\delta \sigma)^{-1} \) since it may be viewed as the number of broken fibers after a load increase of \( \delta \sigma \). This gives

\[
(\sigma - \sigma_c) \sim \Delta^{-1/2}.
\]
The probability of an avalanche being of the size between $\Delta$ and $\Delta + d\Delta$ is $D(\Delta)d\Delta$. Now, the deviation from the critical point scales with the cumulative of all avalanches up to that point; giving $(\sigma - \sigma_c) \sim \int_{\Delta}^{\infty} D(\Delta)d\Delta$.

If we take $D(\Delta) \sim \Delta^{-\gamma}$, then

$$(\sigma - \sigma_c) \sim \Delta^{1-\gamma}. \tag{3}$$

Comparing Eqs. (2) and (3) therefore we have $\gamma = \frac{3}{2}$. So the probability density function for the avalanche size becomes $D(\Delta) \sim \Delta^{-3/2}$, which fits well with simulation results (Fig. 4).

### III. SUMMARY AND CONCLUSION

Under the application of any non-zero system wide load, a local load sharing fiber bundle model will fail completely. Hence a relaxation time divergence or scale free avalanche size distribution etc. are not seen in this version of the model. Here we have taken a local load sharing fiber bundle model where the Hooke springs (with random breaking thresholds), are hanging from a rigid platform on the top and are arranged in a square lattice ($L \times L$). The load is hanging from a central site, say, at ($L/2, L/2$) of the lower platform, which is completely soft though inextensible. Failure dynamics and load redistribution are assumed to occur at a much faster rate. After the failure of the central fiber, the load will be shared by its four nearest neighboring sites, which initially had no load on them. Hence once any of them fails, its load will be shared equally by all the surviving fibers on the perimeter of the damaged region and so on. Because of the softness of the lower platform, the load of the failed fiber gets equally shared by all the fibers on the perimeter of the broken or damaged region. These load shares accumulate on all the fibers on the perimeter and grows with time since their respective appearance on the perimeter of the damaged central region. This may be interpreted as the fact that further the surviving fibers are from the central point, lower are the loads on them and closest fiber (also the longest surviving fiber on the perimeter) to the central site accumulates the maximum load share. Clearly, the size of the perimeter of the broken patch of fibers increases, and the load is shared by increasingly more number of fibers. This decreases the load per fiber value and the breaking dynamics temporarily stops until the external load is increased further and the perimeter of the damaged zone increase further to accommodate the extra load released during breaking. This leads to a self-organized dynamics of failure in the model. We have studied the breaking statistics in such a self-organized dynamical state.

The average load per fiber, no matter how far the central region is damaged by increasing the load, comes back to a particular average value. We have made an estimate of this average value for a system with uniform threshold distribution between $[0 : 1]$, by assuming a uniform distribution of the load per fiber values. It turns out, for $a = 0$, this average value of $\sigma_c$ is estimated to be 1/3, which is in close agreement with simulation results. However, for non-zero values of $a$, we find significant depart-ure from the analytical estimate (given by Eq. (1)). This self-organization of the average load per fiber value is one signature of the self-organization. While the usual ingredients for self organization are external drive and dissipation [18, 19], in our case the later is apparently absent. However a similar effect is coming from the fact that the effective system size, i.e. the perimeter length of the patch of the broken fibers, is not a constant but increases with the increase of load with time. This leads the system into a self-organized dynamical state, from which if one makes a departure (by increasing the load) the response of the system (size of the avalanche and its duration) can be of any size, leading to scale free (indicating perhaps self-organized critical) dynamics.

Note that one could not have obtained this self-organization in an one dimensional array of local load sharing fibers. This is because, the effect of absorption of extra load due to the increase of the damaged region size, will not take place there (always two fibers on the two sides of the damaged region will have to carry the load). This self-organization will therefore occur only for two and higher dimensions.

In conclusion, the self-organized dynamics of failure in this local load sharing fiber bundle model has interesting critical (scale free) properties. The avalanche size distribution follows a power-law with exponent value close to 3/2. One can estimate this value from a scaling argument. The duration of avalanche follows a power-law with exponent value close to 2 and the distribution of the
size of load increase between two avalanches also follows a power-law with exponent value 2. We have checked that these responses remain unchanged with other types of distribution of the breaking thresholds of the fibers.

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