Higher-Loop Corrections to the Infrared Evolution of a Gauge Theory with Fermions

Thomas A. Ryttov and Robert Shrock
C. N. Yang Institute for Theoretical Physics
State University of New York
Stony Brook, NY 11794

We consider a vectorial, asymptotically free gauge theory and analyze the effect of higher-loop corrections to the beta function on the evolution of the theory from the ultraviolet to the infrared. We study the case in which the theory contains $N_f$ copies of a fermion transforming according to the fundamental representation and several higher-dimensional representations of the gauge group. We also calculate higher-loop values of the anomalous dimension of the mass, $\gamma_m$ of $\bar{\psi}\psi$ at the infrared zero of the beta function. We find that for a given theory, the values of $\gamma_m$ calculated to three- and four-loop order, and evaluated at the infrared zero computed to the same order, tend to be somewhat smaller than the value calculated to two-loop order. The results are compared with recent lattice simulations.

I. INTRODUCTION

In this paper we investigate how higher-loop corrections to the beta function affect the evolution of a vectorial, asymptotically free gauge theory (in $(3+1)$ dimensions, at zero temperature) from the ultraviolet to the infrared. We assume that the theory contains a certain number, $N_f$, of massless Dirac fermions $\psi$ transforming according to a representation $R$ of the gauge group. We consider cases where $R$ is the fundamental, adjoint, and rank-2 symmetric or antisymmetric tensor representation. We also study the effect of higher-loop corrections to the anomalous dimension $\gamma_m$ of the fermion mass. This work yields more complete information on the nature of the evolution of the theory from the ultraviolet to the infrared, in particular, on the determination of the infrared zero of the beta function and the scaling behavior of the $\bar{\psi}\psi$ operator in the vicinity of this zero. We will give a number of results for a general gauge group $G$ but will focus on the case $G = SU(N)$.

We denote the running gauge coupling of the theory as $g(\mu)$, with $\alpha(\mu) = g(\mu)^2/(4\pi)$, where $\mu$ is the Euclidean energy/momentum scale (which will often be suppressed in the notation). The property that the SU($N$) gauge interaction is asymptotically free means that $\lim_{\mu \to \infty} \alpha(\mu) = 0$, and, since the beta function is negative for small $\alpha$, it follows that, as the energy/momentum scale $\mu$ decreases from large values, $\alpha$ increases. As $\mu$ decreases and the theory evolves into the infrared, two different types of behavior may occur, depending on the fermion content. In a theory with a sufficiently small number of fermions in small enough representations $R$, as $\mu$ decreases through a scale $\Lambda$, the coupling $\alpha$ exceeds a critical value $\alpha_{R\text{,cr}}$, depending on $R$, for the formation of bilinear fermion condensates, and these condensates are produced. This may or may not be associated with an infrared (IR) zero of the two-loop beta function at a value $\alpha = \alpha_{1R}$; if the two-loop ($2\ell$) beta function does have an IR zero, $\alpha_{1R,2\ell}$, then this type of behavior requires that $\alpha_{1R,2\ell} > \alpha_{R\text{,cr}}$ [3], [4]. As $\mu$ decreases toward $\Lambda$ and $\alpha$ increases toward the IR zero of the beta function, the increase of $\alpha$ as a function of decreasing $\mu$ is reduced. This gives rise to an $\alpha$ that is of order unity, but varies slowly as a function of $\mu$. This behavior is interestingly different from the behavior of the gauge coupling in quantum chromodynamics (QCD). As the condensates form, the fermions gain dynamical masses of order $\Lambda$, so that in the low-energy effective theory applicable at scales $\mu < \Lambda$, they are integrated out, and the further evolution of the theory into the infrared is controlled by the $N_f = 0$ beta function.

Alternatively, if the theory has a sufficiently large number $N_f$ of fermions and/or if these fermions are in a large enough representation $R$ (as bounded above by the requirement of asymptotic freedom), then the IR zero of the beta function occurs at a value smaller than $\alpha_{R\text{,cr}}$, so that as the scale $\mu$ decreases from large values, the theory evolves into the infrared without ever spontaneously breaking chirally symmetry. In this latter case, the IR zero of the beta function is an exact IR fixed point (IRFP), approached from below as $\mu \to 0$. More complicated behavior occurs in theories containing fermions in several different representations [3,4]; here we restrict to the case of fermions in a single representation. For a given asymptotically free theory that features an IR fixed point at $\alpha = \alpha_{1R}$, the value of this IRFP decreases as a function of $N_f$. There is thus a critical value of $N_f$, denoted $N_f\text{,cr}$, depending on $R$, at which $\alpha_{1R}$ decreases below $\alpha_{R\text{,cr}}$. This value serves as the boundary, as a function of $N_f$, between the interval of nonzero $1 \leq N_f < N_{f\text{,cr}}$, where the theory evolves into the infrared in a manner that involves fermion condensate formation and associated spontaneous chiral symmetry breaking (SxSB), and the interval $N_{f\text{,cr}} < N_f < N_{f\text{,max}}$, where the theory evolves into the infrared without this condensate formation and chiral symmetry breaking, with $N_{f\text{,max}}$ denoting the maximal value of $N_f$ consistent with the requirement of asymptotic freedom.

The anomalous dimension $\gamma_m$ contains important information about the scaling behavior of the operator $\bar{\psi}\psi$ for which $m$ is the coefficient, as probed on different momentum scales. In a theory with an $\alpha_{1R} \sim O(1)$, it follows that $\gamma_m$ may also be $O(1)$, which can produce significant enhancement of dynamically generated fermion
masses due to the renormalization-group factor
\[ \eta = \exp \left[ \int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} \gamma_m(\alpha(\mu)) \right]. \quad (1.1) \]

In a phase where no dynamical mass is generated, \( \gamma_m \) simply describes the scaling behavior of the \( \bar{\psi}\psi \) operator.

There are several motivations for the study of higher-order corrections to this evolution of an asymptotically free theory into the infrared. First, the critical coupling, \( \alpha_{R,cr} \), is generically of order unity, and hence there is a need to have a quantitative assessment of the importance of higher-loop corrections to the evolution of the theory. Second, besides being of fundamental field-theoretic interest, a knowledge of this evolution plays an important role in modern technicolor (TC) models with dynamical electroweak symmetry breaking, in which the slow running of the coupling associated with an approximate infrared zero of the beta function provide necessary enhancement of quark and lepton masses \([1, 2]\) (referred to as the hierarchy problem). Third, there can be an enhancement of the mass of particles due to the renormalization-group factor \( \eta \), which can be compared with these lattice studies. The present work is to provide higher-order calculations that can be compared with these lattice studies.

II. GENERAL THEORETICAL FRAMEWORK

A. Beta Function

The beta function of the theory is denoted \( \beta = dg/dt \), where \( dt = d\ln \mu \). In terms of the variable
\[ a = \frac{\alpha}{4\pi}, \quad (2.1) \]

the beta function can be written equivalently as \( \beta_a = da/dt \), expressed as a series
\[ \frac{da}{dt} = -2a \sum_{\ell=1}^{\infty} b_\ell a^\ell = -2a \sum_{\ell=1}^{\infty} b_\ell \alpha^\ell, \quad (2.2) \]

where \( \ell \) denotes the number of loops involved in the calculation of \( b_\ell \) and \( b_\ell = b_\ell/(4\pi)^\ell \). Although this series and series for other quantities in quantum field theories do not have finite radii of convergence but are only asymptotic, experience shows that in situations where the effective expansion parameter (here, \( \alpha/\pi \) times appropriate group invariants) is not too large, the first few terms can provide both qualitative and quantitative insight into the physics. The first two coefficients in the expansion \( (2.2) \), which are scheme-independent, are \([25]\)
\[ b_1 = \frac{1}{3}(11C_A - 4T_f N_f) \quad (2.3) \]

and \([26]\)
\[ b_2 = \frac{1}{3}[34C_A^2 - 4(5C_A + 3C_f)T_f N_f]. \quad (2.4) \]

Here \( C_f \equiv C_2(R) \) is the quadratic Casimir invariant for the representation \( R \) to which the \( N_f \) fermions belong, \( C_A \equiv C_2(G) \) is the quadratic Casimir invariant for the adjoint representation, and \( T_f \equiv T(R) \) is the trace invariant for the fermion representation \( R \). Higher-order coefficients, which are scheme-dependent \([27]\), have been calculated up to four-loop order \([28, 29]\). Some further details are given in Appendix I.

B. Anomalous Dimension of the \( \bar{\psi}\psi \) Operator

The anomalous dimension \( \gamma_m \) for the fermion bilinear \( \bar{\psi}\psi \) describes the scaling properties of this operator and can be expressed as a series in \( a \) or equivalently, \( \alpha \):
\[ \gamma_m = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} \tilde{c}_\ell \alpha^\ell \quad (2.5) \]

where \( \tilde{c}_\ell = c_\ell/(4\pi)^\ell \) is the \( \ell \)-loop series coefficient. Via Eq. \((1.1)\), the anomalous dimension \( \gamma_m \) governs the running of a dynamically generated fermion mass. The coefficients \( c_\ell \) have been calculated to four-loop order \([30]\).

The first two are
\[ c_1 = 6C_f \quad (2.6) \]

and
\[ c_2 = 2C_f \left[ \frac{3}{2} C_f + \frac{97}{6} C_A - \frac{10}{3} T_f N_f \right]. \quad (2.7) \]

For reference, the coefficient \( c_3 \) is listed in Appendix I. Since as \( N_f \) approaches \( N_{f,\text{max}} \) from below, \( b_1 \to 0 \) with nonzero \( b_2 \) and hence \( \alpha_{1R} \to 0 \), and since the perturbative calculations expresses \( \gamma_m \) in a power series in \( \alpha \), it follows that as \( \gamma_m \to 0 \) as \( N_f \) approaches \( N_{f,\text{max}} \) from below. We note that a conjectured beta function that directly relates \( \beta \) to \( \gamma \) has been proposed \([31]\).

III. PROPERTIES OF BETA FUNCTION COEFFICIENTS AND APPLICATION TO FUNDAMENTAL REPRESENTATION

In this section we discuss some general properties of the beta function coefficients as functions of \( N_f \), and
give particular results for the case of fermions in the fundamental representation. In later sections we consider fermions in two-index representations.

A. \( b_1 \)

Since we restrict our considerations to an asymptotically free theory, we require that, with our sign conventions, \( b_1 > 0 \). This, in turn, implies that

\[
N_f < N_{f,\text{max}} ,
\]

(3.1)

where

\[
N_{f,\text{max}} = \frac{11C_A}{4T_f} .
\]

(3.2)

Thus, for fermions in the fundamental representation, \( N_{f,\text{max,fund}} = (11/2)N \).

B. \( b_2 \) and Condition for Infrared Zero of \( \beta \)

We next proceed to characterize the behavior of the higher-loop coefficients of the beta function, \( b_\ell \) with \( \ell = 2, 3, 4 \), and the resultant zero(s) of the beta function, in terms of their dependence on \( N_f \). The two-loop results are well-known and are included here so that the discussion will be self-contained. Since only the first two coefficients of the beta function are scheme-independent, it follows that, to the extent that one is in a momentum regime where one can reliably use the perturbative beta function, the zeros obtained from these first two coefficients should be sufficient to characterize the physics at least qualitatively. When one includes higher-loop contributions to the beta function, one expects shifts of zeros, and there are, indeed, generically substantial shifts if zeros of the two-loop beta function occur at \( \alpha \sim O(1) \). However, if inclusion of three- and/or higher-loop contributions to \( \beta \) leads to a qualitative change in behavior, relative to the behavior obtained from the two-loop \( \beta \) function, then the results cannot be considered fully reliable, since they are scheme-dependent. For example, for a given gauge group \( G \) and fermion content, if the two-loop beta function does not have an infrared zero but the three-loop beta function does, one could not conclude reliably that this is a physical prediction of the theory. Moreover, it should be noted that even if there is no zero of the two-loop beta function away from the origin, i.e., a perturbative IRFP, the beta function may exhibit a nonperturbative slowing of the running associated with the fact that at energy scales below the confinement scale, the physics is not accurately described in terms of the Lagrangian degrees of freedom (fermions and gluons) \[32]-[34].

Another general comment is that the expression of the beta function in Eq. (2.2) is semiperturbative and does not incorporate certain nonperturbative properties of the physics, such as instantons, whose contributions involve essential zeros of the form \( \exp(-\kappa \pi/\alpha) \), where \( \kappa \) is a numerical constant. These instanton effects are absent to any order of the perturbative expansion in Eq. (2.2) but play an important role in the theory. For example, they break the global U(1)_A symmetry \[35] and also enhance spontaneous chiral symmetry breaking \[37]-[39].

Estimates of the effects of instantons on the running of \( \alpha \) in quantum chromodynamics have found that they increase this running, i.e., they make \( \beta \) more negative in the region of small to moderate \( \alpha \) values \[30]. If one were to model the effect of instantons crudely via a modification of \( \beta \) such as

\[
\beta_\alpha = \frac{d\alpha}{dt} = -2\alpha^2 \sum_{\ell=1}^{\infty} b_\ell \alpha^{\ell-1} + \lambda \exp\left(-\frac{\kappa \pi}{\alpha}\right) ,
\]

(3.3)

then, since \( \lambda > 0 \), this would have the effect of increasing the value of the smallest (nonzero, positive) IR zero \( \alpha_{IR} \) of \( \beta \). For a given minimal value of \( \alpha_{cr,R} \) for condensate formation and spontaneous chiral symmetry breaking, since at least at the perturbative level \( \alpha_{IR} \) is a decreasing function of \( N_f \), it would follow that incorporating instanton effects would increase the value of \( N_{f,cr} \), i.e., would increase the interval in \( N_f \) where there is S\chi SB. Furthermore, since instantons enhance chiral symmetry breaking, they would tend to reduce the value of \( \alpha_{cr,R} \), which also has the same effect of increasing \( N_{f,cr} \). We shall comment below on how, although the semiperturbative one-gluon exchange approximation to the Dyson-Schwinger (DS) equation does not directly include effects of confinement or instantons, it may nevertheless yield an approximately correct value of \( N_{f,cr} \) because of another approximation involved that has the opposite effect on the estimate.

If one knows the beta function calculated to a maximal loop order \( \ell_{max} \), then the equation for the zeros of the beta function, aside from the zero at \( a = 0 \), is

\[
\sum_{\ell=1}^{\ell_{max}} b_\ell \alpha^{\ell-1} = b_1 \left[ 1 + \sum_{\ell=2}^{\ell_{max}} \left( \frac{b_\ell}{b_1} \right) \alpha^{\ell-1} \right] = 0 .
\]

(3.4)

As is clear from Eq. (3.4), the zeros of \( \beta \) away from the origin depend only on the \( \ell_{max} - 1 \) ratios \( b_{\ell}/b_1 \) for \( 2 \leq \ell \leq \ell_{max} \).

The coefficients \( b_1 \) and \( b_2 \) are linear functions of \( N_f \), while \( b_3 \) and \( b_4 \) are, respectively, quadratic and cubic functions of \( N_f \). With our sign convention in which an overall minus sign is extracted in Eq. (2.2), each of these coefficients is positive for \( N_f = 0 \). The coefficients \( b_1 \) and \( b_2 \) are both monotonically and linearly decreasing function of \( N_f \). As \( N_f \) increases sufficiently, \( b_2 \) thus reverses sign, from positive to negative, vanishing at \( N_f = N_{f,02z} \), where

\[
N_{f,02z} = \frac{17C_A^2}{2T_f(5C_A + 3C_f)} .
\]

(3.5)
(The subscript \( b\ell z \) stands for the condition that \( b\ell \) is zero). Since
\[
N_{f,\text{max}} - N_{f,\text{b2z}} = \frac{3CA(11CF + 7CA)}{4TF(3CF + 5CA)} > 0 ,
\]
i.e., \( N_{f,\text{max}} > N_{f,\text{b2z}} \), it follows that there is always a nonvacuous interval in the variable \( N_f \) where the theory is asymptotically free and the two-loop \((2\ell)\) beta function has an infrared zero, namely
\[
N_{f,\text{b2z}} < N_f < N_{f,\text{max}} .
\]
This zero occurs at
\[
\alpha_{\text{IR},2\ell} = -\frac{4\pi b_1}{b_2}
\]
and is physical for \( b_2 < 0 \). Explicitly, for the fundamental representation,
\[
N_{f,b2z,\text{fund}} = \frac{34N^3}{13N^2 - 3}
\]
and
\[
\alpha_{\text{IR},2\ell,\text{fund}} = -\frac{4\pi(11N - 2N_f)}{-34N^2 + N_f(13N - 3N^2)}.
\]
The sizes of \( \ell\)-loop contributions are determined by \((\alpha/\pi)^\ell\) multiplied by corresponding powers of various group invariants. Illustrative values of \( \alpha_{\text{IR},2\ell,\text{fund}} \) are given in Table \( \text{[IV]} \) for \( N = 2, 3, 4 \) and the subset of the interval \([3\ell]\) for which \( \alpha_{\text{IR},2\ell,\text{fund}} \) is not so large as to render the two-loop perturbative calculation obviously unreliable. Here and below, when \( \alpha \) and \( \gamma \) values are listed without an explicit \( R \), it is understood that they refer to the fundamental representation. Examples of cases that we do not include in the table because the two-loop result cannot be considered reliable include the following (with formal values of \( \alpha_{\text{IR},2\ell,\text{fund}} \) listed): \( N = 2, N_f = 5 \), where \( \alpha_{\text{IR},2\ell,\text{fund}} = 11.4 \); \( N = 3, N_f = 9 \), where \( \alpha_{\text{IR},2\ell,\text{fund}} = 5.2 \); and \( N = 4, N_f = 11, 12 \), where \( \alpha_{\text{IR},2\ell,\text{fund}} = 14, 3.5 \).

For reference, the estimate in Eq. \((3.10)\) of \( \alpha_{\text{cr}} \) from the analysis of the Dyson-Schwinger equation for the fermion propagator, in the one-gluon exchange approximation, takes the form in Eq. \((3.4)\) for a fermion in the fundamental representation. This has the respective values 1.4, 0.79, and 0.56 for \( N = 2, 3, 4 \), respectively (where we quote the results to two significant figures but do not mean to imply that they have such a high degree of accuracy). Setting \( \alpha_{\text{cr}} = \alpha_{\text{IR},2\ell} \) yields the resultant estimates of \( N_{f,\text{cr}} \), which, rounded to the nearest integers, are 8, 12, and 16 for these values of \( N \). We denote these as \( \beta DS \) estimates since they combine a calculation of \( \alpha_{\text{IR}} \) from the perturbative two-loop \( \beta \) function with the (one-gluon exchange approximation to the) Dyson-Schwinger equation.

As \( N_f \) approaches its maximum value, \( N_{f,\text{max}} \), allowed by the constraint that the theory be asymptotically free, \( b_2 \) reaches its most negative value, namely \( b_2 = -C_A(7C_A + 11CF) \). Clearly, for \( N_f \) values such that \( b_2 \) is only negative by a small amount and \( \alpha_{\text{IR},2\ell} \) is large, the perturbative calculation is not reliable. As \( N_f \) increases further in the range \([3\ell]\) and \( \alpha_{\text{IR},2\ell} \) decreases, the calculation becomes more reliable. In Table \( \text{[IV]} \) we list the numerical values of \( N_{f,\text{b2z}} \) for some illustrative values of \( N \). At the two-loop level, depending on whether \( \alpha_{\text{IR},2\ell} \) is smaller or larger than a critical value for fermion condensation, this is an exact or approximate infrared fixed point (IRFP) of the renormalization group for the gauge coupling. The existence of such an IRFP is of fundamental importance in determining how the theory evolves from the ultraviolet to the infrared \([10]\). In particular, as mentioned above, this determines whether, as the scale \( \mu \) decreases sufficiently to a scale \( \Lambda \) (depending on the group \( G \) and the fermion content), \( \alpha \) grows to a large enough size to produce fermion condensates or, on the contrary, the coupling never gets this large and the theory evolves into the infrared in a chirally symmetric manner, without ever producing such fermion condensates. Note that in the former case, the fermions involved in the condensates get dynamical masses of order \( \Lambda \) and are integrated out of the effective low-energy field theory applicable for scales \( \mu < \Lambda \), so that the further evolution into the infrared is governed by a different beta function.

It is useful to observe how rapidly the numbers \( N_{f,\text{b2z}} \) approach their large-\( N \) values. The number \( N_{f,b2z,\text{fund}} \) has the large-\( N \) expansion
\[
N_{f,b2z,\text{fund}} = N \left[ \frac{34}{13} + \frac{102}{(13N)^2} + \frac{306}{(13)^3N^4} + O\left(\frac{1}{N^6}\right)\right] = N \left[ 2.615 + \frac{0.60355}{N^2} + \frac{0.1393}{N^4} + O\left(\frac{1}{N^6}\right)\right] .
\]
As is evident from Table \( \text{[IV]} \) the values of \( N_{f,b2z,\text{fund}} \) approach the leading asymptotic form for moderate \( N \), as a result of the fact that the subleading term in Eq. \((3.11)\) is suppressed by \( 1/N^2 \).

It is of interest to consider the 't Hooft large-\( N \) limit, where
\[
N \to \infty \quad \text{with } \alpha N \text{ fixed}.
\]
In a theory with fermions in the fundamental representation, in order for them to have a non-negligible effect in this
limit, one considers the simultaneous Veneziano limit

$$N_f \rightarrow \infty \quad \text{with} \quad r \equiv \frac{N_f}{N} \text{ fixed}. \quad (3.13)$$

In the combined limit of Eqs. (3.12) and (3.13), the range of \( r \) satisfying the requirement of asymptotic freedom and the condition that \( b_2 < 0 \) so that the two-loop beta function has an IR zero is \([41]\)

$$\frac{34}{13} < r < \frac{11}{2}, \quad \text{i.e.,} \quad 2.615 < r < 5.5. \quad (3.14)$$

### C. Coefficient \( b_3 \) and Three-Loop Behavior of the Beta Function

The three-loop beta function coefficient \( b_3 \) is a quadratic function of \( N_f \) with positive coefficients of its \( N_f^0 \) and \( N_f^2 \) terms and a negative coefficient of its \( N_f \) term. Hence, regarded as a function of the formal real variable \( N_f \), it is positive for large negative and positive \( N_f \), and positive at \( N_f = 0 \). The derivative of \( b_3 \) with respect to \( N_f \) is

$$\frac{db_3}{dN_f} = T_f \left[ -\frac{1415}{27} C_A^2 - \frac{205}{9} C_A C_f + 2C_f^2 + T_f N_f \left( \frac{88}{9} C_f + \frac{316}{27} C_A \right) \right]. \quad (3.15)$$

For the fermions representations \( R \) that we consider here, for small values of \( N_f \), this derivative \( db_3/dN_f \) is negative, so that in this region of \( N_f \), \( b_3 \) decreases from its positive value at \( N_f = 0 \) as \( N_f \) increases. Because \( b_3 \) is a quadratic polynomial in \( N_f \), the condition that it vanishes gives two formal solutions for \( N_f \), namely

$$N_{f,b3z,\pm} = \left( \frac{1415C_A^2 + 615C_A C_f - 54C_f^2 \pm 3\sqrt{F_{Rb3}}}{4T_f(79C_A + 66C_f)} \right), \quad j = 1, 2, \quad (3.16)$$

and

$$F_{Rb3} = 122157C_A^4 + 109578C_A^3 C_f + 25045C_A^2 C_f^2 - 7380C_A C_f^3 + 324C_f^4. \quad (3.17)$$

Given that \( F_{Rb3} \geq 0 \), as is the case here, so that the values \( N_{f,b3z,j} \) are real, it follows that \( b_3 \) is positive in the intervals \( N_f < N_{f,b3z,-} \) and \( N_f > N_{f,b3z,+} \) and negative in the interval \( N_{f,b3z,-} < N_f < N_{f,b3z,+} \). The value \( N_{f,b3z,+} \) and the neighborhood of \( N_f \) values in the vicinity of \( N_{f,b3z,+} \) are not of interest here because they are larger than the maximal value \( N_{f,max} \) allowed by the requirement of asymptotic freedom,

$$N_{f,b3z,+} > N_{f,max}. \quad (3.18)$$

Thus, \( b_3 \) only changes sign once for \( N_f \) in the asymptotically free interval \( 0 \leq N_f < N_{f,max} \). As \( N_f \) approaches \( N_{f,max} \) from below, \( b_3 \) decreases to a negative value given by

$$(b_3)_{N_f=N_{f,max}} = -\frac{C_A}{24} \left[ 1127C_A^2 + 44C_f(14C_A - 3C_f) \right]. \quad (3.19)$$

For fermions in the fundamental representation, this is

$$(b_3)_{N_f=N_{f,max,fund}} = \frac{701}{12} N_c^3 + \frac{121}{12} N_c + \frac{11}{8N_c}. \quad (3.20)$$

As is clear from Table III for this case

$$N_{f,b3z,1} < N_{f,b3z}. \quad (3.21)$$

We noted above that any physically reliable zero of the beta function must be present already at the level of the two-loop beta function, since this is the maximal scheme-independent part of this function. Hence, in analyzing such a zero for the case under consideration where the fermions transform according to the fundamental representation of SU(\( N \)), we only consider the interval \([3.7]\). Combining this fact with our results (3.21) and (3.18), it follows that \( b_3 \) is negative throughout all of the interval \([3.7]\) of interest here. For this fundamental-representation case, the \( N_{f,b3z,j} \) with \( j = 1, 2 \) have the large-\( N \) expansions

$$N_{f,b3z,1} = N \left[ 1.911 + \frac{0.324}{N^2} + \frac{0.0684}{N^4} + O \left( \frac{1}{N^6} \right) \right], \quad (3.22)$$

and

$$N_{f,b3z,2} = N \left[ 13.348 + \frac{1.667}{N^2} + \frac{0.3978}{N^4} + O \left( \frac{1}{N^6} \right) \right]. \quad (3.23)$$

Here, \( N_{f,max} = 5.5N \).

At three-loop order, the equation for the zeros of the beta function, aside from the zero at \( a = 0 \), is \( b_1 + b_2 a + \)
\( b_3 a^2 = 0 \). Formally, this equation has two solutions for \( a \) and hence for \( \alpha \), namely

\[
\alpha_{\beta,3\ell,\pm} = \frac{2\pi}{b_3} \left[ -b_2 \pm \sqrt{b_2^2 - 4b_1b_3} \right].  \tag{3.24}
\]

Since \( b_2 \) must be negative in order for the beta function to have a scheme-independent infrared zero, and since for fermions in the fundamental representation we have shown that \( b_3 < 0 \) in the relevant interval \( \mathbb{R}^- \), we can rewrite this equivalently as

\[
\alpha_{\beta,3\ell,\pm} = \frac{2\pi}{|b_3|} \left[ -|b_2| \mp \sqrt{b_2^2 + 4b_1|b_3|} \right].  \tag{3.25}
\]

In order for a given solution to be physical, it must be real and positive. As is evident from Eq. (3.25), the solution corresponding to the the + sign in Eq. (3.24) (i.e., the \( - \) sign in Eq. (3.25)) is negative and hence unphysical. Thus, there is a unique physical solution for the IR zero of the beta function to three-loop order, namely

\[
\alpha_{IR,3\ell} = \alpha_{\beta,3\ell,\pm} < 0.  \tag{3.26}
\]

Illustrative values for this IR zero of the beta function at three-loop order are listed in Table IV.

For an arbitrary fermion representation for which \( \beta \) has a two-loop IR zero, we observe that the value of this zero decreases when one calculates it to three-loop order, i.e.,

\[
\alpha_{IR,3\ell} < \alpha_{IR,2\ell}.  \tag{3.27}
\]

This can be proved as follows. We have

\[
\alpha_{IR,2\ell} - \alpha_{IR,3\ell} = \frac{2\pi}{|b_2b_3|} \left[ 2b_1|b_3| + b_2^2 - |b_2|\sqrt{b_2^2 + 4b_1|b_3|} \right].  \tag{3.28}
\]

The expression in square brackets is positive if and only if

\[
(2b_1|b_3| + b_2^2 - b_2^2(4b_1|b_3|)) > 0.  \tag{3.29}
\]

But the difference in (3.29) is equal to the positive-definite quantity \( b_1^2b_3^2 \), which proves the inequality (3.27). This inequality is evident in Table IV.

D. Coefficient \( b_4 \) and Four-Loop Behavior of \( \beta \)

The four-loop beta-function coefficient, \( b_4 \), was calculated in Ref. [29]. We next analyze its behavior as a function of \( N_f \) and the result four-loop IR zero of the beta function. The coefficient \( b_4 \) is a cubic polynomial in \( N_f \) which has positive coefficients of its \( N_f^0, \) and \( N_f^3 \) terms. Hence, regarded as a function of the formal real variable \( N_f, b_4 \) is negative for large negative \( N_f, \) positive for \( N_f = 0, \) and also positive for large positive \( N_f. \) For fermions in the fundamental representation, the derivative at \( N_f = 0 \) is

\[
\frac{db_4}{dN_f} \big|_{N_f=0} = -\left(\frac{485513}{1944} + \frac{20}{9} \zeta(3)\right)N^3 + \left(\frac{58583}{1944} - \frac{548}{9} \zeta(3)\right)N - \frac{2341}{216} + \frac{44}{9} \zeta(3)N^{-1} - \frac{23}{8} N^{-3} \tag{3.30}
\]

where \( \zeta(z) \) is the Riemann zeta function,

\[
\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3.31}
\]

and \( \zeta(3) = 1.20205690.... \) This derivative is negative for all \( N. \) (In the complex \( N \) plane, it has six zeros at three complex-conjugate pairs of \( N \) values.) It follows that, again as a function of the formal real variable \( N_f, b_4 \) has a local maximum at a negative value of \( N_f \) and then decreases through positive values as \( N_f \) increases toward 0 and passes through 0 into the interval of physical values.

The detailed behavior of \( b_4 \) in the physical asymptotically free interval \( 0 \leq N_f \leq N_{f,max} \) depends on \( N. \) In particular, one may determine the value of \( N_f \) where \( b_4 \) has a minimum and whether \( b_4 \) has any zeros for positive \( N_f. \) For \( SU(2), b_4 \) decreases to a minimum positive value as \( N_f \) ascends through the approximate value \( N_f = 5.8, \) and then increases monotonically for larger \( N_f, \) so that it is positive-definite for all non-negative \( N_f, \) in particular, the asymptotically free region \( 0 \leq N_f < 11. \) For \( SU(3), b_4 \) is also positive-definite for all non-negative \( N_f, \) reaching a local minimum as \( N_f \) ascends through a value of approximately 8.2 and then increasing monotonically for larger \( N_f. \) However, for \( SU(4), b_4 \) is positive for \( 0 \leq N_f \leq 9.51, \) negative for the interval \( 9.51 \leq N_f \leq 11.83, \) and positive again for \( N_f > 11.83, \) with zeros at \( N_f \approx 9.51 \) and \( N_f \approx 11.83. \) We list these zeros of \( b_4 \) as a function of \( N_f \) in Table II (Again, we recall that the physical values of \( N_f \) are, of course, restricted to non-negative integers.) Thus, this reversal of sign occurs in the interval of interest here, \( 0 \leq N_f < 22, \) where the \( SU(4) \) theory is asymptotically free. For \( SU(5), b_4 \) behaves in a manner qualitatively sim-
ilar to the SU(4) case; it is positive for $0 \leq N_f \leq 11.18$, negative in the interval $11.18 \leq N_f \leq 15.18$, and positive for larger values of $N_f$, vanishing at $N_f \simeq 11.18$ and $N_f \simeq 15.18$. Thus, again, $b_1$ reverses sign in the region $0 \leq N_f < 22.5$ where the SU(5) theory is asymptotically free. Thus, in contrast with $b_2$ and $b_3$, which are negative throughout the interval of $N_f$ of interest (and $b_1$, which is positive), $b_4$ can, for $N \geq 4$, vanish and reverse sign in this interval.

At the four-loop level, the equation for the zeros of the beta function, aside from $a = 0$, is the cubic equation

$$b_1 + b_2a + b_3a^2 + b_4a^3 = 0.$$  (3.32)

This equation has three solutions for $a$ and hence for $\alpha$, which will be denoted $\alpha_{3,4,5,6}$, $j = 1, 2, 3$. Since the coefficients $b_j$ are real, there are two generic possibilities for these three roots, namely that they are all real, or that one is real and the other two form a complex-conjugate pair. The properties of the roots are further restricted by the asymptotic freedom condition that $b_1 > 0$, the existence of a two-loop IR zero, which requires that $b_2 < 0$, and the fact that, as we have shown, for the relevant range [3.7] of $N_f$, where these conditions are met, $b_3 < 0$. As is evident in Table IV we find that for the values of $N$ and $N_f$ that we consider, the roots of Eq. (3.32) are real. For all of the values of $N$ and $N_f$ where there is a reliable two-loop value for an IR zero of the beta function (i.e., where it does not occur at such a large value of $\alpha$ as to render the perturbative calculation untrustworthy), one of these roots is negative and hence not physical, one of them, namely the minimal positive one, is the physical IR zero, which we will denote $a_{1R,4L} = \alpha_{1R,4L}/(4\pi)$, and there is a third root at a larger positive value. This third root, denoted $a_{4L,u} = \alpha_{4L,u}/(4\pi)$, is not relevant for our analysis, since the initial value of $\alpha$ at a high energy scale $\mu$ is assumed to be close to zero, so that as the scale $\mu$ decreases, $\alpha$ increases and approaches the (positive) zero of the beta function closest to the origin, namely $\alpha_{1R,4L}$.

It is straightforward to display the analytic expressions for the root $a_{1R,4L}$, but we shall not need this for our analysis. We list numerical values for $\alpha_{1R,4L}$ for various values of $N$ and $N_f$ in Table IV. For completeness, we note the specific sets $(N, N_f)$ where $\alpha_{1R,2L}$ is so large that we consider the analysis via the perturbative beta function unreliable: these are $(N, N_f) = (2, 6)$, (3, 9), (4, 11), and (4, 12).

### E. Estimates of Zeros of the Four-Loop Beta Function via Padé Approximants

For the beta function, or more conveniently, the reduced function with the prefactor removed, $\sum_{j=0}^{\ell-1} b_j \beta^{j-1}$, it is useful to calculate and analyze Padé approximants, since these provide closed-form expressions that, by construction, agree with the series to the maximal order to which it is calculated. The expansion for $\beta_l$ to $\ell = 4$ loop order can be used in two ways. First, one can simply solve the cubic equation $\beta_3 = b_1 + b_2a + b_3a^2 + b_4a^3 = 0$ and obtain the three roots, one of which is the root of interest, giving the IR zero. Secondly, one can calculate Padé approximants, e.g., the [2,1] and [1,2] approximants, and determine their zeros. The [1,2] Padé approximant has a single zero at

$$a_{\beta_3,4L,[2,1]} = \frac{\alpha_{1R,4L,[1,2]}}{4\pi} = \frac{b_1(b_1b_3 - b_2^2)}{b_2^3 - 2b_2b_3b_4 + b_4^2}.$$  (3.33)

Taking into account the fact that $b_2$ and $b_3$ are negative in the relevant interval [3.7], this can be rewritten as

$$a_{\beta_3,4L,[1,2]} = \frac{\alpha_{1R,4L,[1,2]}}{4\pi} = \frac{b_1(b_1b_3 + b_2^2)}{|b_2|^3 + 2b_1|b_2||b_3| - b_2^2|b_4|}.$$  (3.34)

The two zeros from the [2,1] approximant are

$$a_{\beta_3,4L,[2,1]} = \frac{b_2b_3 - b_1b_4 \pm \left(\left(|b_2||b_3| - b_1b_4\right)^2 + 4b_1b_3(b_3^2 - b_2b_4)^2\right)^{1/2}}{2(b_2b_4 - b_2^2)}.$$  (3.35)

Taking account of the fact that $b_2$ and $b_3$ are negative in the relevant interval [3.7], this can be rewritten as

$$a_{\beta_3,4L,[2,1]} = \frac{\alpha_{1R,4L,[1,2]}}{4\pi} = \frac{b_1(b_1b_3 + b_2^2)}{2(|b_2||b_3| - b_1b_4)}.$$  (3.36)

The expression in Eq. (3.36) with the $-$ sign in front of the square root is negative and unphysical, while the expression with the $+$ sign in front of the square root yields the estimate of the IR fixed point, as $\alpha_{1R,4L,[2,1]} = 4\pi a_{\beta_3,4L,[2,1]}$. As is evident from Eqs. (3.33) and (3.35), the zeros of the [1,2] and [2,1] Padé approximants incorporate information on $\beta$ up to four loops. One readily verifies that in the limit $b_4 \to 0$, the zero of the [1,2] Padé reduces to the two-loop result $a = -b_1/b_2$, and the two zeros of the [2,1] Padé reduce to those obtained from the
three-loop beta function, \( \alpha_{IR} \). We list the values of \( \alpha_{IR} \) obtained from the zeros of the \([1,2]\) and \([2,1]\) Padé approximants to the four-loop beta function for the case of fermions in the fundamental representation in Table IV.

From our calculations of \( \alpha_{IR} \) at the three- and four-loop level for SU(\( N \)) with fermions in the fundamental representation, we can make several remarks. Although \( n \)-loop calculations of the beta function for \( n \geq 3 \) loops are scheme-dependent, the results obtained with the present \( \overline{MS} \) scheme provide a quantitative measure of the accuracy of the scheme-independent two-loop result. For a given \( N_f \), as \( N_f \) increases above the minimal value \( N_{f,b_{22}} \), where the IR zero first appears, and as the resultant \( \alpha_{IR,2\ell} \) decreases to values \( \lesssim 1 \), the difference between \( \alpha_{IR,2\ell} \) and the higher-loop values \( \alpha_{IR,n\ell} \) for \( n = 3, 4 \) decrease. As is evident from Table IV, the value of \( \alpha_{IR,n\ell} \) generically decreases as one goes from \( n = 2 \) to \( n = 3 \) loops and then increases by a smaller amount as one goes from \( n = 3 \) to \( n = 4 \) loops, so that \( \alpha_{IR,4\ell} \) is smaller than \( \alpha_{IR,2\ell} \). In the same region of \( N_f \) values such that \( \alpha_{IR,2\ell} \) is reasonably small, the values obtained via the \([1,2]\) and \([2,1]\) Padé approximants to the four-loop beta function are close to those obtained from the zeros of this beta function itself.

IV. EVALUATION OF THE ANOMALOUS DIMENSION \( \gamma_m \) AT THE INFRARED ZERO OF \( \beta \)

In this section we evaluate the anomalous dimension of \( \gamma \equiv \gamma_m \), calculated to the \( n \)-loop order in perturbation theory, at the (approximate or exact) IR zero of the beta function to this order, \( \gamma_{n\ell}(\alpha_{IR,n\ell}) \). We denote these as \( \gamma_{n\ell}(\alpha_{IR,n\ell}) \). We focus here on general results and their application to the case of fermions in the fundamental representation, and discuss higher-dimensional representations in subsequent sections. In general, this anomalous dimension must be positive to avoid unphysical singularities in fermion correlation functions.

A running fermion mass, \( \Sigma(k) \), that is dynamically generated at a scale \( \Lambda \), decays with Euclidean momentum \( k > \Lambda \) like

\[
\Sigma(k) \sim \Lambda \left( \frac{\Lambda}{k} \right)^{2-\gamma_m}
\]

up to logs. Since for \( k > \Lambda \), the running coupling \( \alpha \) is smaller than the critical value \( \alpha_{R,cr} \) and there is no spontaneous chiral symmetry breaking, it follows that \( \Sigma(k) \) must decrease toward zero as \( k/\Lambda \to \infty \). In turn, this implies that \( \gamma_m < 2 \). Hence, a physical value of \( \gamma_m \) must lie in the range

\[
0 < \gamma_m < 2.
\]

For values of \( N_f \) such that the theory evolves into the infrared in a chirally symmetric manner, so that the IR zero of the beta function is exact, the same upper bound follows from a related unitarity consideration [43].

Using the two-loop result for \( \gamma \) and evaluating it at the two-loop value of the IR zero of the beta function, we have

\[
\gamma_{2\ell}(\alpha_{IR,2\ell}) = \frac{C_f(11CA - 4T_fN_f)(455C_A^2 + 99CAf + (180f - 248CA)T_fN_f + 80T_f^2N_f^2)}{12(-17C_A^2 + 10CAf + 6C_f)T_fN_f)^2}
\]

(4.3)

For the fundamental representation, this is

\[
\gamma_{2\ell}(\alpha_{IR,2\ell}) = \frac{(N_f^2 - 1)(11N_f - 2N_f)(1009N_f^3 - 99N_f - (158N_f^2 + 90N_f + 40N_f^2))}{12(-34N_f^2 + (13N_f^2 - 3)N_f)^2}
\]

(4.4)

We list numerical values of \( \gamma(\alpha_{IR,2\ell}) \) in Table VI for the illustrative values \( N = 2, 3, 4 \) and, for each \( N \), a set of \( N_f \) values in the range \( [7,7] \). For sufficiently small \( N_f > N_{\ell,22} \) in each \( N \) case, \( \alpha_{IR,2\ell} \) is so large that the formal value of \( \gamma_{2\ell}(\alpha_{IR,2\ell}) \) is larger than 2 and hence unphysical; we enclose these values in parentheses to indicate that they are unphysical artifacts of a perturbative calculation at an excessively large value of \( \alpha \).

In the large-\( N \), large-\( N_f \) limit of Eqs. (3.12) and (3.13) with \( r = N_f/N \), Eq. (4.4) reduces to

\[
\gamma_{2\ell}(\alpha_{IR,2\ell}) = \frac{(11 - 2r)(1009 - 158r + 40r^2)}{12(-34 + 13r)^2} + O\left( \frac{1}{N^2} \right).
\]

(4.5)

For \( r = 4 \) corresponding to the asymptotic value of \( N_{f,cr, \text{fund}} \) in Eq. (4.8), \( \gamma_{2\ell}(\alpha_{IR,2\ell}) = 113/144 \approx 0.785 \), which is the same as the large-\( N \) limit of Eq. (4.7).

One may evaluate \( \gamma_{2\ell}(\alpha_{IR,2\ell}) \) at \( N_f \) equal to the value \( N_{f,cr, \text{fund}} \) predicted by the one-gluon exchange (ladder) approximation to the Dyson-Schwinger equation for the fermion propagator, given in Eq. (4.8). This is somewhat
formal, since these values of \(N_{f,cr,fund}\) are not, in general, integers and hence not actually physical; for example, \(N_{f,cr,fund} = 7.86, 11.91, 15.94\) for \(N = 2, 3, 4\). This procedure yields the result

\[
\gamma_{2\ell}(\alpha_{IR,2\ell}; N_{f,cr,fund}) = \frac{565N^4 - 706N^2 + 225}{144(N^2 - 1)(5N^2 - 3)}. \tag{4.6}
\]

For the illustrative cases \(N = 2, 3, 4\), this anomalous dimension takes the values 0.88, 0.82, and 0.80, respectively. As \(N \to \infty\), Eq. (4.6) has the expansion

\[
\gamma_{2\ell}(\alpha_{cr,fund}) = \frac{113}{144} + \frac{11}{40N^2} + O\left(\frac{1}{N^4}\right). \tag{4.7}
\]

Since the estimate (9.8) is close to \(4N\) even for the smallest value, \(N = 2\), and asymptotically approaches \(4N\) as \(N \to \infty\), it is worthwhile to compare the above values of \(\gamma\), viz., 0.88, 0.82, and 0.80 for \(N = 2, 3, 4\), with \(\gamma_{2\ell}(\alpha_{IR,2\ell})\) evaluated at the nearest physical, integer values of \(N_f\), namely \(N_f = 8, 12, 16\) for \(N = 2, 3, 4\). This procedure yields \(\gamma_{2\ell}(\alpha_{IR,2\ell}) = 0.75, 0.77, 0.78\), as recorded in Table VI. To within the strong-coupling theoretical uncertainties of these calculations, these values are mutually consistent.

A closely related approach is to evaluate the two-loop expression for \(\gamma_m\) at \(\alpha = \alpha_{cr,R}\), where \(\alpha_{cr,R}\) is the estimate of the critical coupling for fermion condensation obtained from the one-gluon exchange approximation to the Dyson-Schwinger equation, given in Eq. (9.3). If one were to use the first term in Eq. (2.5) by itself, one would obtain, to this order, \(\gamma_m = c_1(\alpha_{cr,R}/\pi) = 1/2\), for an arbitrary fermion representation \(R\). The second term, \(c_2(\alpha_{cr,R}/\pi)^2\), does depend on \(R\). Combining these, i.e., evaluating \(c_1(\alpha/\pi) + c_2(\alpha/\pi)^2\) at \(\alpha = \alpha_{cr,R}\), we find, for a general fermion representation \(R\),

\[
\gamma_{2\ell}(\alpha_{cr,R}) = \frac{97CA + 225C_f - 20TfN_f}{432C_f}. \tag{4.8}
\]

For consistency, one would then substitute \(N_f\) equal to the estimate \(N_{f,cr,R}\) from the same Dyson-Schwinger analysis, given in Eq. (9.7). For a general representation this yields

\[
\gamma_{2\ell}(\alpha_{cr,R}; N_f = N_{f,cr,R}) = \frac{21C_A^2 + 128C_A C_f + 225C_f^2}{144C_f(C_A + 3C_f)}. \tag{4.9}
\]

For the fundamental representation, this reduces to the same result as was obtained in Eq. (4.6). We have also evaluated the three-loop result for \(\gamma\) at the three-loop value of the IR zero of the beta function, which we denote as \(\gamma_{a}(\alpha_{IR,3\ell})\), and the four-loop result for \(\gamma\) at the four-loop value of the IRFP, which we denote as \(\gamma_{a}(\alpha_{IR,4\ell})\). We list the resultant values in Table VI.

From our calculations of \(\gamma_m\) for the case of fermions in the fundamental representation, we can make several observations. Although computations of \(\alpha_{IR,4\ell}\) and \(\gamma_{nl}(\alpha_{IR,4\ell})\) are scheme-dependent for \(n \geq 3\) loops, they provide a useful measure of the accuracy of the lowest-order results. As was the case with the position of \(\alpha_{IR,n\ell}\) itself, we find that, for a given \(N\) and for \(N_f\) reasonably well above \(N_{f,4\ell}\), so that the perturbative calculation of \(\alpha_{IR,n\ell}\) is not too large, the value of \(\gamma_{nl}(\alpha_{IR,n\ell})\) generically decreases as one goes from \(n = 2\) to \(n = 3\) loops. Some of this decrease can be ascribed to the decrease in \(\alpha_{IR,n\ell}\) going from \((n = 2)\)-loop to \((n = 3)\)-loop order. At the four-loop level, \(\gamma_{4\ell}(\alpha_{IR,4\ell})\) tends to be smaller than \(\gamma_{3\ell}(\alpha_{IR,3\ell})\) for values of \(N_f\) from \(N_{f,4\ell}\) to values of \(N_f\) slightly above the middle of the range (9.7), while for values of \(N_f\) in the upper end of this range, \(\gamma_{4\ell}(\alpha_{IR,4\ell})\) is slightly larger than \(\gamma_{3\ell}(\alpha_{IR,3\ell})\). In general, for the values of \(N_f\) where \(\alpha_{IR}\) is sufficiently small that the calculation may be trustworthy, the value of the anomalous dimension evaluated at the IR zero of the beta function (both calculated to \(n\)-loop order) \(\gamma_{nl}(\alpha_{IR,n\ell})\), is somewhat smaller than unity.

Several recent high-statistics lattice simulations have been carried out on an SU(3) gauge theory with a varying number \(N_f\) of fermions in the fundamental representation in the range \(6 \leq N_f \leq 12\) [11, 13, 8, 24]. This work has yielded evidence for a regime of slowly running gauge couplings for \(N_f \lesssim 12\), consistent with the presence of an IR zero of the beta function, in agreement with the earlier continuum estimates in Ref. [1]. Ref. [11] also found a considerable enhancement of \(\langle \bar{\psi}\psi \rangle/f^3\) in the SU(3) theory with \(N_f = 6\). Further lattice simulations and analysis of data should yield values of \(\gamma_m\) that can be compared with our higher-loop calculations in this paper. A preliminary study of the SU(2) theory with \(N_f = 6\) fermions has also been reported [21].

V. ADJOINT REPRESENTATION

In this section we analyze the SU(\(N\)) theory with \(N_f\) copies of a Dirac fermion, or equivalently, \(2N_f\) copies of a Majorana fermion, in the adjoint representation. For this case, the general expression for the maximal value of \(N_f\) allowed by the requirement of asymptotic freedom, Eq. (8.2), reduces to

\[
N_{f,max,adj} = \frac{11}{4}, \tag{5.1}
\]

i.e., restricting \(N_f\) to the integers, \(N_{f,max} = 2\). The general expression in Eq. (3.3) for the value of \(N_f\) at which \(b_2\) changes sign from positive to negative with increasing \(N_f\) reduces to

\[
N_{f,b_2z,adj} = \frac{17}{16} = 1.0625. \tag{5.2}
\]

Hence there is only one (integer) value of \(N_f\), namely \(N_f = 2\) Dirac fermions (equivalently, \(N_f = 4\) Majorana fermions), for which the theory is asymptotically free and has an IR zero of the two-loop beta function. This zero occurs at

\[
\alpha_{IR,2\ell,adj} = \frac{2\pi}{5N} \simeq \frac{1.257}{N} \quad \text{for} \quad N_f = 2. \tag{5.3}
\]
Specializing the general formula for the critical coupling \( \alpha_{cr,R} \) from the one-gluon exchange approximation to the Dyson-Schwinger equation, Eq. (3.3) (see Appendix II) for the present case where \( R \) is the adjoint representation, one obtains \( \alpha_{cr,adj} = \pi/(3N) \). Formally setting \( \alpha_{IR,2\ell,adj} = \alpha_{adj,cr} \) yields the corresponding estimate for the critical number \( N_{f,cr} = 83/40 = 2.075 \). This may be rounded off to the nearest integer, giving \( N_{f,cr} \) for the adjoint representation. In view of the theoretical uncertainty in such an estimate, due to the strong-coupling nature of the physics involved, an \( SU(N) \) gauge theory with \( N_f = 2 \) adjoint fermions could be either slightly inside the chirally broken, confined side of \( N_{f,cr} \) or slightly on the other side, where the theory is chirally symmetric and the evolution into the infrared is governed by an exact conformal IR fixed point.

For the present case of \( N_f = 2 \) fermions in the adjoint representation of \( SU(N) \), the coefficients of the beta function are \( b_1 = N, \ b_2 = -10N^2, \ b_3 = -(101/2)N^3 \), and

\[
b_4 = \left[ \frac{5977}{54} N^4 + \frac{64}{27} N^3 - \frac{448}{27} N^2 + \frac{128}{9} N \right] + \frac{176}{9} \frac{\zeta(3)}{N^2} \left[ \frac{124}{9} N^4 - \frac{208}{9} N^3 + \frac{3232}{9} N^2 - \frac{416}{3} N - \frac{128}{3} + \frac{128}{N^2} \right]. \tag{5.4}
\]

At the three-loop level, the beta function has two zeros away from the origin, one of which is negative and unphysical, and the other of which is the physical IR zero,

\[
\alpha_{IR,3\ell,adj} = \frac{4\pi(-10 + \sqrt{302})}{101N} \simeq \frac{0.9180}{N}. \tag{5.5}
\]

At the four-loop level, the beta function has three zeros away from the origin, one of which is negative and unphysical, one of which is the four-loop IR zero, \( \alpha_{IR,4\ell,adj} \), and the third of which, denoted \( \alpha_{adj,u} \), occurs at a larger positive value of \( \alpha \) and is not relevant to our study, since it is not reached by evolution of the coupling starting at small \( \alpha \) for large \( \mu \). We list the numerical values of these zeros in Table VII.

The coefficients \( \tilde{c}_\ell \) in Eq. (2.5) for \( \gamma \) for this case are \( \tilde{c}_1 = 3N/(2\pi), \tilde{c}_2 = (11N^3)/(8\pi^2), \) and \( \tilde{c}_3 = -N^3/(2\pi^3), \) with \( \tilde{c}_4 \) given by

\[
\pi^4\tilde{c}_4 = \left[ -\frac{1817}{512} N^4 + \frac{1}{96} N^3 - \frac{5}{192} N^2 - \frac{7}{96} N + \frac{1}{32} \frac{1}{4N} - \frac{3}{16N^3} \right] + \zeta(3) \left[ \frac{155}{128} N^4 - \frac{5}{64} N^3 + \frac{25}{128} N^2 + \frac{35}{64} N - \frac{15}{64} - \frac{15}{8N} - \frac{45}{32N^3} \right]. \tag{5.6}
\]

(The term in \( \tilde{c}_3 \) proportional to \( \zeta(3) \) and the terms in \( \tilde{c}_4 \) proportional to \( \zeta(4) \) and \( \zeta(5) \) vanish for the adjoint representation for arbitrary \( N_f \).)

Evaluating the two-loop expression in Eq. (4.3) for \( \gamma_m \) at the IR zero of the beta function, also calculated at the two-loop level, \( \alpha_{IR,2\ell,adj} \), we obtain

\[
\gamma_{2\ell,adj}(\alpha_{IR,2\ell,adj}) = \frac{(11 - 4N_f)(277 - 34N_f + 40N_f^2)}{6(-17 + 16N_f)^2}, \tag{5.7}
\]

so that for the \( N_f = 2 \) case of interest here,

\[
\gamma_{2\ell,adj}(\alpha_{IR,2\ell,adj}) = \frac{41}{50} = 0.820 \quad \text{for} \quad N_f = 2. \tag{5.8}
\]

It is also of interest to evaluate the two-loop \( \gamma_m \) at the value of \( \alpha_{cr} \) from the one-gluon exchange (ladder) approximation to the Dyson-Schwinger equation. Specializing the general result in Eq. (4.3) to the adjoint representation with \( N_f = 2 \) yields

\[
\gamma_{2\ell}(\alpha_{cr,adj}) = \frac{47}{72} \simeq 0.653. \tag{5.9}
\]

Evaluating the three-loop result for \( \gamma_m \) at the IR zero of the beta function calculated at the three-loop level, \( \alpha_{IR,3\ell,adj} \), for the \( N_f = 2 \) case of interest, we obtain

\[
\gamma_{3\ell,adj}(\alpha_{IR,3\ell,adj}) = 0.543 \quad \text{for} \quad N_f = 2. \tag{5.10}
\]

which is again independent of \( N \). At the four-loop level, the value of \( \gamma_{4\ell,adj}(\alpha_{IR,4\ell,adj}) \) does depend slightly on \( N \). We list the values of these anomalous dimensions in Table VIII. The most recent simulations of a lattice gauge theory with \( SU(2) \) gauge group and \( N_f = 2 \) fermions in the adjoint representation report \( \gamma_m = 0.49 \pm 0.13 \) [23]. This is in agreement with the calculations of \( \gamma_m \) here at the three- and four-loop level, to within the uncertainties of the respective calculations.

VI. SYMMETRIC AND ANTISYMMETRIC RANK-2 TENSOR REPRESENTATIONS

In this section we consider the \( SU(N) \) theory with \( N_f \) fermions in the symmetric or antisymmetric rank-2 representation, denoted \( S_2 \) and \( A_2 \). Since a number of formulas are similar for these two cases, we will often give these in a unified way for both cases, denoted \( T_2 \) (for rank-2 tensor representation), with \( \pm \) signs distinguishing them. For \( S_2 \), our analysis applies for any \( N_f \), while for \( A_2 \), we restrict to \( N \geq 4 \), since the \( A_2 \) representation is the singlet for \( SU(2) \) and is equivalent to the conjugate fundamental representation for \( SU(3) \). Note that for \( SU(4) \), the \( A_2 \) representation is self-conjugate. Also, since for \( SU(2) \) the \( S_2 \) representation is the same as the adjoint representation, which has already been analyzed, we only consider the illustrative values \( N = 3, 4 \).
For the two T2 cases, the general expression for the maximal value of \(N_f\) allowed by the requirement of asymptotic freedom, Eq. (3.2), reduces to

\[
N_{f,max,T2} = \frac{11N}{2(N \pm 2)},
\]

(6.1)

where the \(\pm\) refers to S2 and A2, respectively. As \(N\) increases from 2 to \(\infty\), \(N_{f,max,S2}\) increases monotonically from 2.75 to 11/2 = 4.5, and as \(N\) increases from 3 to \(\infty\), \(N_{f,max,A2}\) decreases monotonically from 16.5 to the same limit, 4.5. The physical values of \(N_{f,max}\) in both cases are the greatest integral parts of these rational numbers.

For these representations, the general expression in Eq. (3.5) for the value of \(N_f\) at which the beta function coefficient \(b_2\) changes sign from positive to negative with increasing \(N_f\) takes the form

\[
N_{f,b2z,T2} = \frac{17N^2}{(N \pm 2)(8N + 3 - 6N^{-1})}.
\]

(6.2)

As a consequence of the general inequality (3.6), it follows that \(N_{f,b2z,S2} < N_{f,max,S2}\) and \(N_{f,b2z,A2} < N_{f,max,A2}\). For \(N = 2\), the S2 representation is just the adjoint representation, so we only consider the illustrative values \(N = 3, 4\). The respective intervals \(N_{f,b2z,S2} < N_f < N_{f,max,S2}\) for which the SU(\(N\)) gauge theory is asymptotically free and has an IR zero of \(\alpha\) are 1.06 < \(N_f < 2.75\) for \(N = 3\) and 1.22 < \(N_f < 3.30\) for \(N = 4\). These ranges imply that the only physical integral values of \(N_f\) satisfying these conditions are \(N_f = 2, 3\) for both SU(3) and SU(4).

For large \(N\), \(N_{f,b2z,T2}\) has the series expansion

\[
N_{f,b2z,T2} = \frac{17}{2N} + \frac{323}{29N^2} + \frac{6137}{29N^3} + \frac{103547}{212N^4} + \mathcal{O}(\frac{1}{N^4}).
\]

(6.3)

We list values of \(\gamma_{2\ell,T2}(\alpha_{IR,2\ell,T2})\) for \(N = 2, 3, 4\) in Table XI and values of \(\gamma_{2\ell,A2}(\alpha_{IR,2\ell,A2})\) for \(N = 4\) in Table XII.

It is also of interest to evaluate the two-loop expression for \(\gamma\) at the estimated \(\alpha = \alpha_{cr,T2}\). This yields

\[
\gamma_{2\ell,T2}(\alpha_{cr,T2}) = \frac{322N^2 + 225N - 450 - 10N(N \pm 2)N_f}{432(N \pm 2)(N \mp 1)}.
\]

(6.6)

We list these values in Tables XI and XII.

As \(N\) increases from 2 to \(\infty\), \(N_{f,b2z,S2}\) increases monotonically from 17/16 = 1.0625 to 17/8 = 2.125, and as \(N\) increases from 3 to \(\infty\), \(N_{f,b2z,A2}\) decreases monotonically from 8.05 to the same limit, 2.125. This limit is twice the (\(N\)-independent) value of \(N_{f,b2z,adj} = 17/16\) for the adjoint representation. Thus, for large \(N\), the range (3.7) where the SU(\(N\)) theory with \(N_f\) fermions in the S2 or A2 representation is asymptotically the same for both, namely, 17/8 < \(N_f < 11/2\); restricting \(N_f\) to physical, integer values, this range consists of the three values \(N_f = 3, 4, 5\).

For our further discussion we assume that \(N_f\) is in the range \(N_{f,b2z,T2} < N_f < N_{f,max,T2}\) where the theory is asymptotically free and the two-loop beta function has an IR zero, for the respective cases S2 and A2. This zero occurs at the value

\[
\alpha_{IR,2\ell,T2} = \frac{2\pi(11N - 2N_f(N \pm 2))}{-17N^2 + N_f(8N^2 + 19N + 12N^{-1})}.
\]

(6.4)

Evaluating Eq. (6.4) from the ladder approximation to the Dyson-Schwinger equation for the fermion propagator for S2 or A2 representations yields the estimate (9.10) given in Appendix II. The resultant BDS estimates for \(N_{f,cr}\) in the case of the S2 representation and \(N_f = 2, 3, 4\) are \(N_{f,cr,S2} = 2.1, 2.5, 2.8\), respectively. For the A2 representation with \(N = 4\), one has \(N_{f,cr,A2} = 8.1\).

The two-loop expression for the anomalous dimension, evaluated at \(\alpha = \alpha_{IR,2\ell,T2}\), is

\[
\gamma_{2\ell,T2}(\alpha_{IR,2\ell,T2}) = \frac{374N^4 + 578N^3 - 931N^2 + 900N + 900}{144(N \mp 1)(4N^2 + 3N - 6)}.
\]

(6.7)
This has the large-$N$ expansion
\[
\gamma_{2, T^2}(\alpha_{1 R, 2, T^2})|_{N_f = N_{f, cr, T^2}} = \frac{187}{288} + \frac{17}{128 N} + O \left( \frac{1}{N^2} \right).
\]
(6.8)

The leading term has the value $187/288 \simeq 0.649$.

From a lattice study of SU(3) gauge theory with $N_f = 2$ fermions in the $S_2$ (sextet) representation, Ref. 18 found that this theory is characterized by slow running behavior consistent with an (exact or approximate) IR fixed point, and further reported that $\gamma_m < 0.6$ where it was measured. For SU(3), the estimate of $\alpha_{cr, S_2}$ in Eq. 18 gives $\alpha_{cr, S_2} = \pi/10 = 0.31$. Our results for the IR zero of $\beta$ and the value of $\gamma_m$ at this zero for $N = 3$ and $N_f = 2$ are listed in Tables IX. and XI. We find that $\alpha_{1 R, ref, S_2}$ is approximately 0.84 at $n = 2$ loop level, decreases somewhat to 0.50 at three-loop level, and then increases slightly to 0.52 at four-loop level. The two-loop result for $\gamma_m$ is unphysically large, while the three- and four-loop values of $\gamma_m$ at the corresponding three- and four-loop IR zeros of $\beta$ are about 1.3 and 1.4. These are somewhat larger than the values reported in Ref. 18, although in assessing this comparison, one must take account of the significant strong-coupling uncertainties in our calculation stemming from the fact that $\alpha_{1 R, S_2} \sim O(1)$. Our evaluation of the two-loop expression for $\gamma_m$ at the ladder-Dyson-Schwinger estimate of $\alpha_{cr, S_2}$, is 0.65.

VII. EFFECTS OF NONZERO FERMION MASSES

The global chiral symmetry that is operative if the fermions are massless, and the way that it is broken by fermion condensates, is well-known, and we do not review it here. However, it is worthwhile to comment on the situation in which some fermion masses are nonzero. In this paper we have studied the evolution of an asymptotically free vectorial SU(N) gauge theory from high scales to the infrared taking account of higher-loop corrections to the beta function and the anomalous dimension $\gamma_m$ for fermions in the fundamental, adjoint, and rank-2 symmetric and antisymmetric representations $S_2$ and $A_2$. We have compared our results with lower-order calculations. We have shown that, for fixed $N$ and $N_f$, in the range for which the two-loop beta function has an IR zero, the value of this zero decreases as one goes from the two-loop to the three-loop calculations, and we have determined this decrease quantitatively. Going further, we have shown that there is a smaller fractional increase in the value of this IR zero when calculated to four-loop accuracy, with the final four-loop result still smaller than the two-loop value. We have analyzed instanton effects and have demonstrated that they tend to increase the value of the IR zero of the beta function somewhat. A major part of our work has been the evaluation of the anomalous dimension $\gamma_m$ of $\bar{\psi}\psi$ at the IR zero of the beta function at the $\ell = 2, 3, 4$ loop levels. This zero is approximate or exact, depending on whether for a given $N$, the value of $N_f$ is below or above the critical value $N_{f,cr}$ below which there is spontaneous chiral symmetry breaking associated with the formation of a fermion condensate. We have found that this $\gamma_m$ at the (approximate or exact) IR zero of the beta function decreases as one goes from two-loop to three-loop order, and that the four-loop values also tend to be somewhat less than those at the two-loop level. The values that we have calculated for $\gamma_m$ at the IR zero of the beta function tend to be somewhat smaller than unity. We have compared our higher-loop calculations with results from recent lattice simulations and have found general agreement. We believe that the
higher-loop calculations reported here should provide a useful reference for comparison with ongoing and future lattice measurements.

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### IX. APPENDIX I

For the reader’s convenience, we list the three-loop beta function coefficient, in the $\overline{MS}$ scheme [28],

$$b_3 = \frac{2857}{54} C_A^3 + +T_f N_f \left[ 2 C_f^2 - \frac{205}{9} C_A C_f - \frac{1415}{27} C_A^2 \right] + (T_f N_f)^2 \left[ \frac{44}{9} C_f + \frac{158}{27} C_A \right].$$

(9.1)

The four-loop coefficient is given in Ref. [29] and is a cubic polynomial in $N_f$. We note that the coefficients of the $N_f^0$ (which is independent of the fermion representation) is positive, and the coefficient of the $N_f^3$ term is positive for an arbitrary fermion representation.

Our normalizations for the quadratic Casimir and trace invariants of a Lie group are standard. The quadratic Casimir invariant $C_2(R)$ for the representation $R$ is given by $\sum_{a=1}^{\text{dim}(R)} \sum_{j=1}^{\text{dim}(R)} [D_R(T_a)]_{ij} [D_R(T_a)]_{jk} = C_2(R) \delta_{ik}$, where $a, b$ are group indices, $o(G)$ is the order of the group, $T_a$ are the generators of the associated Lie algebra, and $D_R(T_a)$ is the matrix form of the $T_a$ in the representation $R$. The trace invariant $T(R)$ is defined by $\sum_{i,j=1}^{\text{dim}(R)} [D_R(T_a)]_{ij} [D_R(T_b)]_{ji} = T(R) \delta_{ab}$.

From the calculations of the coefficients of the perturbative expansion of the anomalous dimension $\gamma_m$ in the $\overline{MS}$ scheme to four-loop order in Ref. [30], we record the three-loop coefficient

$$c_3 = 2 C_f \left[ \frac{129}{2} C_f^2 - \frac{129}{4} C_f C_A + \frac{11413}{108} C_A^2 + C_f T_f N_f (-46 + 48 \zeta(3)) - C_A T_f N_f \left( \frac{556}{27} + 48 \zeta(3) \right) - \frac{140}{27} T_f^2 N_f^2 \right].$$

(9.2)

We have used the four-loop coefficient $c_4$ from Ref. [30] for our calculations, but it is too lengthy to reproduce here.

### A. Appendix II: Beta-Dyson-Schwinger Estimate of $N_{f,cr}$

In this appendix we collect some useful known formulas from the one-gluon exchange (also called ladder) approximation to the Dyson-Schwinger equation for the fermion propagator $[1, 2, 49]$. With this approximation, a solution of the Dyson-Schwinger in the case of an initial zero input mass of a fermion in the representation $R$ of the gauge group, yields dynamically generated mass if the coupling $\alpha(\mu)$ exceeds a critical value $\alpha_{cr,R}$ given by $[1, 2, 49]$

$$\alpha_{cr,R} = \frac{\pi}{3 C_f}.$$  
(9.3)

For the representations of interest here, this has the form

$$\alpha_{cr,\text{fund}} = \frac{2 \pi N}{3( N^2 - 1)}.$$  
(9.4)

$$\alpha_{cr,\text{adj}} = \frac{\pi}{3 N}.$$  
(9.5)

$$\alpha_{cr, T2} = \frac{\pi N}{3(N \pm 2)(N \mp 1)}.$$  
(9.6)

where, as in the text, the $\pm$ signs apply for the S2 and A2 representations, respectively.

For the illustrative values $N = 2, 3, 4$, this has the following values: (i) 1.40, 0.79, 0.56 for the fundamental representation; (ii) Setting this equal to the two-loop expression for the IR zero of $\beta$ then yields an estimate for $N_{f,cr}$ to this order, namely

$$N_{f, cr} = \frac{C_A(66 C_f + 17 C_A)}{10 T_f(C_A + 3 C_f)}.$$  
(9.7)

For the various representations of interest here, this has the explicit forms

$$N_{f, cr, \text{fund}} = \frac{2 N (50 N^2 - 33)}{5 (5 N^2 - 3)}.$$  
(9.8)

$$N_{f, cr, \text{adj}} = \frac{83}{40} = 2.075.$$  
(9.9)

and

$$N_{f, cr, T2} = \frac{N (83 N^2 + 66 N - 132)}{5 (N \pm 2)(4 N^2 \mp 3 N - 6)}.$$  
(9.10)

where, as in the text, T2 refers to S2 or A2 and $\pm$ corresponds to S2 and A2, respectively. In the same ladder
One finds \( \gamma_m = 1 \) at \( \alpha = \alpha_{cr,R} \) (which also holds for the DS analysis at a UV-stable fixed point \[48\]). For the gauge group SU\((N)\) with the illustrative values of \( N \) used for the tables, namely \( N = 2, 3, 4, N_{f,cr,fund} \), it is equal to 7.9, 11.9, and 15.9, respectively, with the large-\( N \) form \( N_f \sim 4N \). For \( S_2 \), the symmetric rank-2 tensor representation, for \( N = 3 \), the result is the same as for the fundamental representation, for \( N = 4 \), one has \( N_{f,cr,A2} \simeq 8.1 \), and as \( N \to \infty \), \( N_{f,cr,A2} \) decreases toward the limit 11/2.

One understands that, \textit{a priori}, there could be significant uncertainty in these estimates because of the strong-coupling nature of the physics involved and the one-gluon approximation used for the solution of the Dyson-Schwinger equation. Moreover, the DS equation analysis is semi-perturbative in the sense that it contains polynomial dependence on \( \alpha \), and it neglects nonperturbative effects associated with confinement and instantons. However, corrections to the one-gluon exchange approximation have been analyzed and found not to be too large \[2\]. Recent lattice simulations for SU(3) are in broad agreement, to within the uncertainties, with the above prediction of \( N_{f,cr} \sim 12 \) \[11–15, 24\]. Some of the success of the \( \beta DS \) prediction for \( N_{f,cr} \) may arise from the fact that two major physical effects that it ignores would shift \( N_{f,cr} \) in opposite directions and hence tend to cancel each other out \[24\]. The DS equation is an integral equation, and the standard analysis of this equation involves an integration over Euclidean loop momentum \( k \) from \( k = 0 \) to \( k = \infty \). If the theory confines, then the lower bound for the Euclidean loop momentum should actually not be \( k = 0 \), but instead \( k = k_{min} \sim r_c^{-1} \) where \( r_c \) is the spatial confinement scale \[24\]. The use of \( k = 0 \) thus overestimates the tendency toward \( S_\chi SB \). Instantons enhance \( S_\chi SB \) \[36–39\], and the neglect of instanton effects leads to an underestimate of the tendency toward \( S_\chi SB \). Since these two neglected aspects of the physics - confinement and instantons - produce errors that are of opposite sign as regards the tendency for \( S_\chi SB \), it is plausible that these errors tend to cancel out, so that the \( \beta DS \) method may be reasonably accurate in predicting \( N_{f,cr} \).

\section{APPENDIX III - PADÉ RESULTS}

In this appendix we collect some relevant results on Padé approximants. Given a Taylor (or asymptotic) series expansion around \( z = 0 \) for the function \( f(z) \),

\[ f(z) = \sum_{n=0}^{n_{max}} f_n z^n \]

one can construct a set of \([p, q]\) Padé approximants, namely rational functions comprised of a numerator polynomial of degree \( p \) and a denominator polynomial of degree \( q \) such that \( p + q = n_{max} - 1 \), of the form \( (\sum_{j=0}^{p} p_j z^j) / (\sum_{k=0}^{q} q_k z^k) \). Without loss of generality, one can divide numerator and denominator by \( q_0 \), so that, after redefinition of the coefficients, one has

\[ [p, q] f(z) = \frac{\sum_{j=0}^{p} p_j z^j}{1 + \sum_{k=1}^{q} q_k z^k}. \]

The \( p + q + 1 \) coefficients \( p_j \) with \( 0 \leq j \leq p \) and \( q_k \) with \( 1 \leq k \leq q \) are uniquely determined in terms of the \( f_n \) coefficients with \( 0 \leq n \leq n_{max} \) by expanding the \([p, q]\) Padé approximant in a Taylor series around \( z = 0 \) and solving the set of \( n_{max} \) linear equations.

\[ 1 \] T. Appelquist, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. Lett. \textbf{57}, 957 (1986); T. Appelquist and L. C. R. Wijewardhana, Phys. Rev. D \textbf{35}, 774 (1987); Phys. Rev. D \textbf{36}, 568 (1987); T. Appelquist, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. Lett. \textbf{77}, 1214 (1996).

\[ 2 \] T. Appelquist, K. Lane, and U. Mahanta, Phys. Rev. Lett. \textbf{61}, 1553 (1988).

\[ 3 \] T. A. Ryttov and R. Shrock, Phys. Rev. D \textbf{81}, 116003 (2010); erratum, \textit{ibid.} D \textbf{82}, 059903 (2010).

\[ 4 \] C. T. Hill and E. H. Simmons, Phys. Rep. \textbf{381}, 235 (2003).

\[ 5 \] \textit{Workshop on Dynamical Electroweak Symmetry Breaking},Southern Denmark Univ. 2008 [http://hep.sdu.dk/dewsb].

\[ 6 \] F. Sannino, Acta Phys. Polon. B \textbf{40}, 3533 (2009).

\[ 7 \] T. Appelquist and F. Sannino, Phys. Rev. D \textbf{59}, 067702 (1999); M. Harada, M. Kurachi, and K. Yamawaki, Phys. Rev. \textbf{70}, 033009 (2004); M. Kurachi and R. Shrock, Phys. Rev. D \textbf{74}, 056003 (2006).

\[ 8 \] T. Appelquist et al., \texttt{arXiv:1009.5967}.

\[ 9 \] K. Lane and E. Eichten, Phys. Lett. B \textbf{222}, 274 (1989); D. Hong, S. Hsu, and F. Sannino, Phys. Lett. B \textbf{597} (2004) 89; F. Sannino and K. Tuominen, Phys. Rev. D \textbf{71}, 051901(R) (2005); N. D. Christensen and R. Shrock, Phys. Lett. B \textbf{632}, 92 (2006); D. D. Dietrich and F. Sannino, Phys. Rev. D \textbf{75}, 085018 (2007); R. Foadi, M. T. Frandsen, T. A. Ryttov, F. Sannino, Phys. Rev. D \textbf{76}, 055005 (2007); T. A. Ryttov and F. Sannino, Phys. Rev. D \textbf{76}, 105004 (2007).

\[ 10 \] T. Appelquist and J. Terning, Phys. Rev. D \textbf{50}, 2116 (1994); T. Appelquist and R. Shrock, Phys. Lett. B \textbf{548}, 204 (2002); T. Appelquist and R. Shrock, Phys. Rev. Lett. \textbf{90}, 201801 (2003); T. Appelquist, M. Piai, and R. Shrock, Phys. Rev. D \textbf{69}, 015002 (2004); N. C. Christensen and R. Shrock, Phys. Rev. D \textbf{74}, 015004 (2006); T. A. Ryttov and R. Shrock, Phys. Rev. D \textbf{81}, 115013 (2010).

\[ 11 \] T. Appelquist, G. Fleming, and E. Neil, Phys. Rev. Lett.
[10] A. J. Hietanen, K. Rummukainen, and K. Tuominen, Phys. Rev. D 79, 076010 (2009); T. Appelquist et al., Phys. Rev. Lett. 104, 071601 (2010).

[11] A. Deuzeman, M. P. Lombardo, and E. Pallante, Phys. Lett. B 670, 41 (2008); arXiv:0810.1719.

[12] X.-Y. Jin and R. Mawhinney, PoS Lattice-2008:059.

[13] T. DeGrand, Y. Shamir, and B. Svetitsky, Phys. Rev. D 81, 243003 (2010).

[14] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, Phys. Lett. B 681, 353 (2009); arXiv:0911.4263.

[15] S. Catterall, J. Giedt, F. Sannino, and J. Schneible, JHEP 0811, 009 (2008); arXiv:0910.3387.

[16] T. DeGrand, Y. Shamir, and B. Svetitsky, Phys. Rev. D 79, 034501 (2009).

[17] T. DeGrand, Y. Shamir, and B. Svetitsky, Phys. Rev. D 82, 054503 (2010).

[18] Z. Fodor, K. Holland, J. Kuti, D. Nogradi, and C. Schroeder, Phys. Lett. B 681, 353 (2009); JHEP 2009, 103 (2009).

[19] A. J. Hietanen, K. Rummukainen, and K. Tuominen, Phys. Rev. D 80, 094504 (2009).

[20] L. Del Debbio, B. Lucini, A. Patella, C. Pica, and A. Rago, Phys. Rev. D 80, 074507 (2009); F. Bursa, L. Del Debbio, L. Keegan, C. Pica, and T. Pickup, Phys. Rev. D 81, 014505 (2010).

[21] J. B. Kogut and D. K. Sinclair, arXiv:1002.2988.

[22] S. Catterall, L. Del Debbio, J. Giedt, and L. Keegan, arXiv:1010.5099.

[23] G. Fleming, talk at the Int. Conf. on High Energy Physics ICHEP 2010, Paris, http://www.ichep2010.fr.

[24] D. J. Gross and F. Wilczek, Phys. Rev. Lett. 30, 1343 (1973); H. D. Politzer, Phys. Rev. Lett. 30, 1346 (1973); G. ’t Hooft, unpublished.

[25] W. E. Caswell, Phys. Rev. Lett. 33, 244 (1974); D. R. T. Jones, Nucl. Phys. B 75, 531 (1974).

[26] D. J. Gross and F. Wilczek, Phys. Rev. D 8, 3633 (1973); D. J. Gross, in R. Balian and J. Zinn-Justin, eds. Methods in Field Theory, Les Houches 1975 (North Holland, Amsterdam, 1976).

[27] O. V. Tarasov, A. A. Vladimirov, and A. Yu. Zharkov, Phys. Lett. B 93, 429 (1980); S. A. Larin and J. A. M. Vermaseren, Phys. Lett. B 303, 334 (1993).

[28] T. van Ritbergen, J. A. M. Vermaseren, and S. A. Larin, Phys. Lett. B 400, 379 (1997).

[29] J. A. M. Vermaseren, S. A. Larin, and T. van Ritbergen, Phys. Lett. B 405, 327 (1997).

[30] T. A. Ryttov and F. Sannino, Phys. Rev. D 78, 065001 (2008).

[31] S. J. Brodsky and G. F. de Téramond, in M. Harada, M. Tanabashi, and K. Yamawaki, eds., The Origin of Mass and Strong Coupling Gauge Theories, SCGT2006 (World Scientific, Singapore, 2008), p. 31.

[32] M. Creutz, unpublished.

[33] S. Brodsky and R. Shrock, Phys. Lett. B 666, 95 (2008).

[34] G. ’t Hooft, Phys. Rev. Lett. 37, 8 (1976); Phys. Rev. D 14, 3432 (1976); erratum ibid. D 18, 2199 (1978).

[35] C. G. Callan, R. F. Dashen, and D. J. Gross, Phys. Rev. D 17, 2717 (1978); Phys. Rev. D 20, 3279 (1979).

[36] D. Caldi, Phys. Rev. Lett. 39, 121 (1977).

[37] T. Appelquist and S. Selipsky, Phys. Lett. B 400, 364 (1997).

[38] E. Shuryak and M. Velkovsky, Phys. Lett. B 437, 398 (1998).

[39] T. Banks and A. Zaks, Nucl. Phys. B 296, 189 (1982).

[40] Here and elsewhere, when expressions are given for $N_f$ that formally evaluate to non-integral real values, it is understood implicitly that one infers an appropriate integral value of $N_f$ from them, either the greatest integral part or the nearest integer, depending on the context.

[41] Parenthetically, we observe that if (i) the exact beta function of a theory were to have a zero at a (nonzero, positive) value $\alpha_1$ with $\beta/\alpha > 0$ at $\alpha_1$, and (ii) another zero at a larger value, $\alpha_2$ with $\beta/\alpha < 0$ at $\alpha_2$ with (iii) $\beta > 0$ for $\alpha_1 < \alpha < \alpha_2$, and if (iv) the initial condition in the deep ultraviolet is that as $\mu \to \infty$, $\alpha(\mu)$ approaches $\alpha_2$ from below, then as the scale $\mu$ decreases, $\alpha$ would decrease from the UV fixed point $\alpha_2$ and approach the IR fixed point $\alpha_1$ from above as $\mu \to 0$. This type of behavior is not relevant to our theory with the initial condition on $\alpha$ that we assume in the ultraviolet.

[42] G. Mack, Commun. Math. Phys. 55, 1 (1977); M. Flato and C. Fronsdal, Lett. Math. Phys. 8, 159 (1984); V.K. Dobrev and V.B. Petkova, Phys. Lett. B 162, 127 (1985).

[43] Here a “hard” fermion mass is a mass that would remain in the hypothetical limit in which one turned off the SU(N) interactions. If these masses are dynamically generated, then they are actually soft on the higher scale where they arise, as discussed in N. D. Christensen and R. Shrock, Phys. Rev. Lett. 94, 214501 (2005).

[44] T. Appelquist, M. Soldate, T. Takeuchi, and L. C. R. Wijewardhana, in G. Domokos and S. Kovesis-Domokos, eds., Proc. Johns Hopkins Workshop on Current Problems in Particle Theory (World Scientific, Singapore, 1988); K. I. Kondo, H. Mino, and K. Yamawaki, Phys. Rev. D 39, 2430 (1989).

[45] M. Kurachi, R. Shrock, and K. Yamawaki, Phys. Rev. D 76, 035003 (2007); H. S. Fukano and F. Sannino, Phys. Rev. D 82, 035021 (2010).

[46] A. Martin and K. Lane, Phys. Rev. D 71, 015011 (2005). F. Braam, M. Flosdorf, R. S. Chivukula, S. Di Chiara, and E. H. Simmons, Phys. Rev. D 77, 055005 (2008); T. A. Ryttov and R. Shrock, Phys. Rev. D 82, 055012 (2010).

[47] B. Holdom, Phys. Rev. Lett. 60, 1233 (1988); K. Yamawaki, M. Bando, and K. Matumoto, Phys. Rev. Lett. 56, 1335 (1986); V. Miransky and K. Yamawaki, Phys. Rev. D 55, 5051 (1997); Phys. Rev. D 56, E 3768 (1997).

[48] K. Lane, Phys. Rev. D 10, 2605 (1974).

[49] C. Pica and F. Sannino, arXiv:1011.3832.
TABLE I: Values of the ℓ-loop beta function coefficients $b_\ell$ defined in Eq. (2.2) in the SU($N$) gauge theory with $N_f$ fermions transforming according to the fundamental representation, as functions of $N$ and $N_f$, for the range where the theory is asymptotically free.

| $N$ | $N_f$ | $b_1$  | $b_2$  | $b_3$  | $b_4$  |
|-----|-------|--------|--------|--------|--------|
| 2   | 0     | 0.584  | 0.287  | 0.213  | 0.268  |
| 2   | 1     | 0.5305 | 0.235  | 0.154  | 0.191  |
| 2   | 2     | 0.477  | 0.184  | 0.099  | 0.127  |
| 2   | 3     | 0.424  | 0.132  | 0.047  | 0.078  |
| 2   | 4     | 0.371  | 0.080  | -0.0003 | 0.044  |
| 2   | 5     | 0.318  | 0.0285 | -0.044 | 0.024  |
| 2   | 6     | 0.265  | -0.023 | -0.084 | 0.020  |
| 2   | 7     | 0.212  | -0.075 | -0.120 | 0.030  |
| 2   | 8     | 0.159  | -0.127 | -0.152 | 0.057  |
| 2   | 9     | 0.106  | -0.178 | -0.180 | 0.099  |
| 2   | 10    | 0.053  | -0.230 | -0.205 | 0.156  |
| 3   | 0     | 0.875  | 0.646  | 0.720  | 1.173  |
| 3   | 1     | 0.822  | 0.566  | 0.582  | 0.910  |
| 3   | 2     | 0.769  | 0.485  | 0.450  | 0.681  |
| 3   | 3     | 0.716  | 0.405  | 0.324  | 0.485  |
| 3   | 4     | 0.663  | 0.325  | 0.205  | 0.322  |
| 3   | 5     | 0.610  | 0.245  | 0.091  | 0.194  |
| 3   | 6     | 0.557  | 0.165  | -0.016 | 0.099  |
| 3   | 7     | 0.504  | 0.084  | -0.118 | 0.039  |
| 3   | 8     | 0.451  | 0.004  | -0.213 | 0.015  |
| 3   | 9     | 0.398  | -0.076 | -0.303 | 0.025  |
| 3   | 10    | 0.345  | -0.156 | -0.386 | 0.072  |
| 3   | 11    | 0.292  | -0.236 | -0.463 | 0.154  |
| 3   | 12    | 0.239  | -0.317 | -0.534 | 0.273  |
| 3   | 13    | 0.186  | -0.397 | -0.599 | 0.429  |
| 3   | 14    | 0.133  | -0.477 | -0.658 | 0.622  |
| 3   | 15    | 0.080  | -0.557 | -0.711 | 0.852  |
| 3   | 16    | 0.0265 | -0.637 | -0.758 | 1.121  |
| 4   | 0     | 1.17   | 1.15   | 1.71   | 3.50   |
| 4   | 1     | 1.11   | 1.04   | 1.46   | 2.88   |
| 4   | 2     | 1.06   | 0.932  | 1.22   | 2.31   |
| 4   | 3     | 1.01   | 0.824  | 0.986  | 1.80   |
| 4   | 4     | 0.955  | 0.716  | 0.762  | 1.36   |
| 4   | 5     | 0.902  | 0.607  | 0.546  | 0.972  |
| 4   | 6     | 0.849  | 0.499  | 0.339  | 0.647  |
| 4   | 7     | 0.796  | 0.391  | 0.140  | 0.385  |
| 4   | 8     | 0.743  | 0.283  | -0.051 | 0.184  |
| 4   | 9     | 0.690  | 0.175  | -0.234 | 0.046  |
| 4   | 10    | 0.637  | 0.066  | -0.409 | -0.029 |
| 4   | 11    | 0.584  | -0.042 | -0.575 | -0.040 |
| 4   | 12    | 0.531  | -0.150 | -0.733 | 0.013  |
| 4   | 13    | 0.477  | -0.258 | -0.883 | 0.131  |
| 4   | 14    | 0.424  | -0.366 | -1.025 | 0.314  |
| 4   | 15    | 0.371  | -0.474 | -1.16  | 0.562  |
| 4   | 16    | 0.318  | -0.583 | -1.28  | 0.877  |
| 4   | 17    | 0.265  | -0.691 | -1.40  | 1.26   |
| 4   | 18    | 0.212  | -0.799 | -1.51  | 1.71   |
| 4   | 19    | 0.159  | -0.907 | -1.61  | 2.22   |
| 4   | 20    | 0.106  | -1.015 | -1.70  | 2.81   |
| 4   | 21    | 0.053  | -1.124 | -1.79  | 3.46   |
TABLE II: Values of $N_{f,b2z}$, $N_{f,b3z,\pm}$, and $N_{f,b4z,j}$, $i = 2,3$, for SU($N_f$) with $N_f$ fermions in the fundamental representation. We only list physical, i.e., real, non-negative values. Thus, since $N_{f,b4z,1} < 0$, it is not included.

| $N$ | $N_{f,max}$ | $N_{f,b2z}$ | $(N_{f,b3z,\pm}, N_{f,b3z,\mp})$ | $(N_{f,b4z,2}, N_{f,b4z,3})$ |
|-----|-------------|-------------|---------------------------------|---------------------------------|
| 2   | 11          | 5.55        | (3.99, 27.6)                    | none                            |
| 3   | 16.5        | 8.05        | (5.84, 40.6)                    | none                            |
| 4   | 22          | 10.61       | (7.73, 53.8)                    | (9.51, 11.83)                   |

TABLE III: Estimates of $\alpha_{cr,R}$ from the one-gluon exchange approximation to the Dyson-Schwinger equation for the fermion propagator. Values are listed for SU($N_f$) with $2 \leq N \leq 6$ and the representations $R = (i)$ fundamental (fund), (ii) adjoint (adj), (iii) symmetric rank-2 tensor (S2), and (iv) antisymmetric rank-2 (A2).

| $N$ | $\alpha_{cr, fund}$ | $\alpha_{cr, adj}$ | $\alpha_{cr, S2}$ | $\alpha_{cr, A2}$ |
|-----|----------------------|---------------------|-------------------|-------------------|
| 2   | 1.40                 | 0.52                | 0.52              | –                 |
| 3   | 0.79                 | 0.35                | 0.31              | 0.79              |
| 4   | 0.56                 | 0.26                | 0.23              | 0.42              |
| 5   | 0.44                 | 0.21                | 0.19              | 0.29              |
| 6   | 0.36                 | 0.17                | 0.16              | 0.22              |
TABLE IV: Values of the (approximate or exact) IR zeros in $\alpha$ of the $\text{SU}(N)$ beta function with $N_f$ fermions in the fundamental representation, for $N = 2, 3, 4$, calculated at $n$-loop order, and denoted as $\alpha_{IR,n\ell}$. For each $N$, we only give results for the integral $N_f$ values in the range (3.7) where the theory is asymptotically free and the two-loop beta function has an infrared zero. For the four-loop beta function, the cubic equation (3.32) has three zeros, one of which is negative, one of which is $\alpha_{IR,4\ell}$, and the third of which is positive but farther from the origin. We include the latter, denoted as $\alpha_{4\ell,u}$. We also list zeros from the [1,2] and [2,1] Padé approximants to the four-loop beta function.

| $N$ | $N_f$ | $\alpha_{IR,2\ell}$ | $\alpha_{IR,3\ell}$ | $\alpha_{IR,4\ell}$ | $\alpha_{IR,4\ell}[1,2]$ | $\alpha_{IR,4\ell}[2,1]$ | $\alpha_{4\ell,u}$ |
|-----|-------|---------------------|---------------------|---------------------|--------------------------|--------------------------|------------------|
| 2   | 7     | 2.83                | 1.05                | 1.21                | 2.30                     | 1.16                     | 4.12             |
| 2   | 8     | 1.26                | 0.688               | 0.760               | 0.952                    | 0.741                    | 3.11             |
| 2   | 9     | 0.595               | 0.418               | 0.444               | 0.475                    | 0.438                    | 2.395            |
| 2   | 10    | 0.231               | 0.196               | 0.200               | 0.202                    | 0.200                    | 1.97             |
| 3   | 10    | 2.21                | 0.764               | 0.815               | 1.47                     | 0.807                    | 5.62             |
| 3   | 11    | 1.23                | 0.578               | 0.626               | 0.871                    | 0.616                    | 3.29             |
| 3   | 12    | 0.754               | 0.435               | 0.470               | 0.561                    | 0.462                    | 2.295            |
| 3   | 13    | 0.468               | 0.317               | 0.337               | 0.367                    | 0.333                    | 1.78             |
| 3   | 14    | 0.278               | 0.215               | 0.224               | 0.231                    | 0.222                    | 1.48             |
| 3   | 15    | 0.143               | 0.123               | 0.126               | 0.127                    | 0.125                    | 1.29             |
| 3   | 16    | 0.0416              | 0.0397              | 0.0398              | 0.0398                   | 0.0398                   | 1.15             |
| 4   | 13    | 1.85                | 0.604               | 0.628               | 1.14                     | 0.625                    | 6.94             |
| 4   | 14    | 1.16                | 0.489               | 0.521               | 0.776                    | 0.516                    | 3.49             |
| 4   | 15    | 0.783               | 0.397               | 0.428               | 0.556                    | 0.422                    | 2.30             |
| 4   | 16    | 0.546               | 0.320               | 0.345               | 0.407                    | 0.340                    | 1.73             |
| 4   | 17    | 0.384               | 0.254               | 0.271               | 0.298                    | 0.267                    | 1.40             |
| 4   | 18    | 0.266               | 0.194               | 0.205               | 0.215                    | 0.203                    | 1.19             |
| 4   | 19    | 0.175               | 0.140               | 0.145               | 0.149                    | 0.145                    | 1.05             |
| 4   | 20    | 0.105               | 0.091               | 0.092               | 0.0930                   | 0.0921                   | 0.947            |
| 4   | 21    | 0.0472              | 0.044               | 0.044               | 0.0444                   | 0.0443                   | 0.870            |
TABLE V: Values of the $\ell$-loop coefficients $\bar{c}_\ell$ in the series expansion $leg\ 3$ for the anomalous dimension $\gamma_\ell$, as functions of $N$ and $N_f$, for the range $leg\ 3$, where the theory is asymptotically free.

| $N$ | $N_f$ | $\bar{c}_1$ | $\bar{c}_2$ | $\bar{c}_3$ | $\bar{c}_4$ |
|-----|-------|-------------|-------------|-------------|-------------|
| 2   | 0     | 0.358       | 0.318       | 0.310       | 0.329       |
| 2   | 1     | 0.358       | 0.302       | 0.254       | 0.234       |
| 2   | 2     | 0.358       | 0.286       | 0.195       | 0.143       |
| 2   | 3     | 0.358       | 0.270       | 0.134       | 0.0577      |
| 2   | 4     | 0.358       | 0.254       | 0.0712      | $-0.0218$   |
| 2   | 5     | 0.358       | 0.239       | 0.00656     | $-0.0952$   |
| 2   | 6     | 0.358       | 0.223       | $-0.0601$   | $-0.162$    |
| 2   | 7     | 0.358       | 0.207       | $-0.129$    | $-0.222$    |
| 2   | 8     | 0.358       | 0.191       | $-0.199$    | $-0.274$    |
| 2   | 9     | 0.358       | 0.175       | $-0.272$    | $-0.319$    |
| 2   | 10    | 0.358       | 0.1595      | $-0.346$    | $-0.355$    |
| 3   | 0     | 0.637       | 0.853       | 1.26        | 2.03        |
| 3   | 1     | 0.637       | 0.825       | 1.11        | 1.64        |
| 3   | 2     | 0.637       | 0.796       | 0.957       | 1.27        |
| 3   | 3     | 0.637       | 0.768       | 0.801       | 0.909       |
| 3   | 4     | 0.637       | 0.740       | 0.642       | 0.561       |
| 3   | 5     | 0.637       | 0.712       | 0.479       | 0.227       |
| 3   | 6     | 0.637       | 0.684       | 0.312       | $-0.0926$   |
| 3   | 7     | 0.637       | 0.656       | 0.142       | $-0.396$    |
| 3   | 8     | 0.637       | 0.628       | $-0.0313$   | $-0.683$    |
| 3   | 9     | 0.637       | 0.599       | $-0.208$    | $-0.953$    |
| 3   | 10    | 0.637       | 0.571       | $-0.389$    | $-1.21$     |
| 3   | 11    | 0.637       | 0.543       | $-0.573$    | $-1.44$     |
| 3   | 12    | 0.637       | 0.515       | $-0.760$    | $-1.65$     |
| 3   | 13    | 0.637       | 0.487       | $-0.951$    | $-1.85$     |
| 3   | 14    | 0.637       | 0.459       | $-1.145$    | $-2.02$     |
| 3   | 15    | 0.637       | 0.431       | $-1.34$     | $-2.18$     |
| 3   | 16    | 0.637       | 0.402       | $-1.54$     | $-2.31$     |
| 4   | 0     | 0.895       | 1.60        | 3.17        | 6.86        |
| 4   | 1     | 0.895       | 1.56        | 2.89        | 5.88        |
| 4   | 2     | 0.895       | 1.52        | 2.61        | 4.93        |
| 4   | 3     | 0.895       | 1.48        | 2.32        | 4.00        |
| 4   | 4     | 0.895       | 1.44        | 2.03        | 3.09        |
| 4   | 5     | 0.895       | 1.40        | 1.73        | 2.21        |
| 4   | 6     | 0.895       | 1.36        | 1.43        | 1.36        |
| 4   | 7     | 0.895       | 1.33        | 1.12        | 0.526       |
| 4   | 8     | 0.895       | 1.29        | 0.808       | $-0.275$    |
| 4   | 9     | 0.895       | 1.25        | 0.492       | $-1.05$     |
| 4   | 10    | 0.895       | 1.21        | 0.170       | $-1.79$     |
| 4   | 11    | 0.895       | 1.17        | $-0.157$    | $-2.50$     |
| 4   | 12    | 0.895       | 1.13        | $-0.488$    | $-3.18$     |
| 4   | 13    | 0.895       | 1.09        | $-0.825$    | $-3.83$     |
| 4   | 14    | 0.895       | 1.05        | $-1.17$     | $-4.45$     |
| 4   | 15    | 0.895       | 1.01        | $-1.51$     | $-5.025$    |
| 4   | 16    | 0.895       | 0.969       | $-1.86$     | $-5.57$     |
| 4   | 17    | 0.895       | 0.930       | $-2.22$     | $-6.08$     |
| 4   | 18    | 0.895       | 0.890       | $-2.58$     | $-6.54$     |
| 4   | 19    | 0.895       | 0.850       | $-2.95$     | $-6.97$     |
| 4   | 20    | 0.895       | 0.811       | $-3.32$     | $-7.37$     |
| 4   | 21    | 0.895       | 0.771       | $-3.69$     | $-7.71$     |
TABLE VI: Values of the anomalous dimension in the SU(N) theory with $N_f$ fermions in the fundamental representation, $\gamma_m$, calculated to the $n$-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, $\alpha_{IR,n}$, for $\ell = 2, 3, 4$. We denote these as $\gamma_n(\alpha_{IR,n})$. For sufficiently small $N_f > N_{f,bz}$ in each $N$ case, $\alpha_{IR,2}$ is so large that the formal value of $\gamma_2(\alpha_{IR,2})$ is larger than 2 and hence unphysical; we indicate this by placing these values in parentheses.

| $N$ | $N_f$ | $\gamma_2(\alpha_{IR,2})$ | $\gamma_3(\alpha_{IR,3})$ | $\gamma_4(\alpha_{IR,4})$ |
|-----|-------|--------------------------|--------------------------|--------------------------|
| 2   | 7     | 2.67                     | 0.457                    | 0.0325                   |
| 2   | 8     | 0.752                    | 0.272                    | 0.204                    |
| 2   | 9     | 0.275                    | 0.161                    | 0.157                    |
| 2   | 10    | 0.0910                   | 0.0738                   | 0.0748                   |
| 3   | 10    | 4.19                     | 0.647                    | 0.156                    |
| 3   | 11    | 1.61                     | 0.439                    | 0.250                    |
| 3   | 12    | 0.773                    | 0.312                    | 0.253                    |
| 3   | 13    | 0.404                    | 0.220                    | 0.210                    |
| 3   | 14    | 0.212                    | 0.146                    | 0.147                    |
| 3   | 15    | 0.0997                   | 0.0826                   | 0.0836                   |
| 3   | 16    | 0.0272                   | 0.0258                   | 0.0259                   |
| 4   | 13    | 5.38                     | 0.755                    | 0.192                    |
| 4   | 14    | 2.45                     | 0.552                    | 0.259                    |
| 4   | 15    | 1.32                     | 0.420                    | 0.281                    |
| 4   | 16    | 0.778                    | 0.325                    | 0.269                    |
| 4   | 17    | 0.481                    | 0.251                    | 0.234                    |
| 4   | 18    | 0.301                    | 0.189                    | 0.187                    |
| 4   | 19    | 0.183                    | 0.134                    | 0.136                    |
| 4   | 20    | 0.102                    | 0.0854                   | 0.0865                   |
| 4   | 21    | 0.0440                   | 0.0407                   | 0.0409                   |

TABLE VII: Values of the (approximate or exact) IR zeros in $\alpha$ of the SU(N) beta function with $N_f = 2$ fermions in the adjoint representation, for $N = 2, 3, 4$, calculated at $n$-loop order, and denoted as $\alpha_{IR,n}$, adj. For the four-loop beta function, the cubic equation (3.32) has three zeros, one of which is negative, one of which is $\alpha_{IR,4}$, and the third of which is positive but farther from the origin. We include the latter, denoted as $\alpha_{4,u}$, adj. We also list zeros from the [1,2] and [2,1] Padé approximants to the four-loop beta function.

| $N$ | $\alpha_{IR,2}$, adj | $\alpha_{IR,3}$, adj | $\alpha_{IR,4}$, adj | $\alpha_{IR,4}$, [1,2], adj | $\alpha_{IR,4}$, [2,1], adj | $\alpha_{4,u}$, adj |
|-----|----------------------|----------------------|----------------------|-----------------------------|-----------------------------|---------------------|
| 2   | 0.628                | 0.459                | 0.493                | 0.522                       | 0.484                       | 2.235               |
| 3   | 0.419                | 0.306                | 0.323                | 0.341                       | 0.320                       | 1.80                |
| 4   | 0.314                | 0.2295               | 0.241                | 0.253                       | 0.239                       | 1.47                |
TABLE VIII: Values of the anomalous dimension $\gamma_m$ in an SU($N$) gauge theory with $N_f = 2$ (Dirac) fermions in the adjoint representation, calculated to the $n$-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, for $n = 2, 3, 4$. We denote these as $\gamma_{n,\text{adj}}(\alpha_{IR,n,\text{adj}})$. We also list the value of $\gamma_{2,\text{adj}}$ evaluated at $\alpha$ equal to the BDS estimate, Eq. 9.3, for $\alpha_{cr,\text{adj}}$.

| $N$ | $\gamma_{2,\text{adj}}(\alpha_{IR,2,\text{adj}})$ | $\gamma_{3,\text{adj}}(\alpha_{IR,3,\text{adj}})$ | $\gamma_{4,\text{adj}}(\alpha_{IR,4,\text{adj}})$ | $\gamma_{2,\text{adj}}(\alpha_{cr,\text{adj}})$ |
|-----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 2   | 0.820                           | 0.543                           | 0.571                           | 0.653                           |
| 3   | 0.820                           | 0.543                           | 0.561                           | 0.653                           |
| 4   | 0.820                           | 0.543                           | 0.557                           | 0.653                           |

TABLE IX: Values of the (approximate or exact) IR zero in $\alpha$ of the SU($N$) beta function with $N_f = 2$ fermions in the symmetric rank-2 (i.e., S2) representation, for $N = 3, 4$, calculated at $n$-loop order, and denoted as $\alpha_{IR,n,\text{S}2}$.

| $N_f$ | $\alpha_{1R,2\text{f,S}2}$ | $\alpha_{1R,3\text{f,S}2}$ | $\alpha_{1R,4\text{f,S}2}$ |
|-------|----------------------------|----------------------------|----------------------------|
| 3     | 2.44                       | 0.500                      | 0.522                      |
| 3     | 0.085                      | 0.079                      | 0.080                      |
| 4     | 0.967                      | 0.485                      | 0.485                      |
| 4     | 0.152                      | 0.129                      | 0.132                      |

TABLE X: Values of the (approximate or exact) IR zero in $\alpha$ of the SU(4) beta function with $N_f$ fermions in the antisymmetric rank-2 (i.e., A2) representation, for the range $5 \leq N_f \leq 10$ where the theory is asymptotically free and has an IR zero of the beta function, calculated at $n$-loop order, and denoted as $\alpha_{1R,n,\text{A}2}$.

| $N_f$ | $\alpha_{1R,2\text{f,A}2}$ | $\alpha_{1R,3\text{f,A}2}$ | $\alpha_{1R,4\text{f,A}2}$ |
|-------|----------------------------|----------------------------|----------------------------|
| 4     | 0.664                      | 0.723                      | 0.723                      |
| 4     | 0.437                      | 0.482                      | 0.482                      |
| 4     | 0.287                      | 0.312                      | 0.312                      |
| 4     | 0.174                      | 0.183                      | 0.183                      |
| 4     | 0.080                      | 0.082                      | 0.082                      |

TABLE XI: Values of $\gamma_m$ in an SU($N$) gauge theory with $N_f$ fermions in the symmetric rank-2 tensor representation S2, calculated to the $n$-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, for $n = 2, 3, 4$. We denote these as $\gamma_{n,\text{S}2}(\alpha_{1R,n,\text{S}2})$. We also list $\gamma_{2,\text{S}2}$ evaluated at $\alpha$ equal to the estimate Eq. 9.3, for $\alpha_{cr,\text{S}2}$.

| $N_f$ | $\gamma_{2,\text{S}2}(\alpha_{1R,2\text{f,S}2})$ | $\gamma_{3,\text{S}2}(\alpha_{1R,3\text{f,S}2})$ | $\gamma_{4,\text{S}2}(\alpha_{1R,4\text{f,S}2})$ | $\gamma_{2,\text{S}2}(\alpha_{cr,\text{S}2})$ |
|-------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 3     | 2.44                            | 1.28                            | 1.38                            | 0.653                           |
| 3     | 0.144                           | 0.133                           | 0.134                           | 0.619                           |
| 4     | 4.82                            | 2.08                            | 2.27                            | 0.659                           |
| 4     | 0.381                           | 0.313                           | 0.319                           | 0.629                           |
TABLE XII: Values of $\gamma_m$ in an SU($N$) gauge theory with $N_f$ fermions in the antisymmetric rank-2 tensor representation $A_2$, calculated to the $n$-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, for $N = 4$ and $n = 2, 3, 4$. We denote these as $\gamma_{n,f,A_2}(\alpha_{IR,n,A_2})$. We also list $\gamma_{2\ell,A_2}$ evaluated at $\alpha$ equal to the estimate Eq. (9.3) for $\alpha_{cr,A_2}$.

| $N$ | $N_f$ | $\gamma_{2\ell,A_2}(\alpha_{IR,2\ell,A_2})$ | $\gamma_{3\ell,A_2}(\alpha_{IR,3\ell,A_2})$ | $\gamma_{4\ell,A_2}(\alpha_{IR,4\ell,A_2})$ | $\gamma_{2\ell,A_2}(\alpha_{cr,A_2})$ |
|-----|-------|----------------------------------------|----------------------------------------|----------------------------------------|----------------------------------------|
| 4   | 6     | (9.78)                                 | 1.38                                   | 0.647                                  | 0.769                                  |
| 4   | 7     | (2.19)                                 | 0.695                                  | 0.505                                  | 0.750                                  |
| 4   | 8     | 0.802                                  | 0.402                                  | 0.375                                  | 0.732                                  |
| 4   | 9     | 0.331                                  | 0.228                                  | 0.231                                  | 0.713                                  |
| 4   | 10    | 0.117                                  | 0.101                                  | 0.102                                  | 0.695                                  |
Higher-Loop Corrections to the Infrared Evolution of a Gauge Theory with Fermions

Thomas A. Ryttov and Robert Shrock
C. N. Yang Institute for Theoretical Physics
State University of New York
Stony Brook, NY 11794

We consider a vectorial, asymptotically free gauge theory and analyze the effect of higher-loop corrections to the beta function on the evolution of the theory from the ultraviolet to the infrared. We study the case in which the theory contains $N_f$ copies of a fermion transforming according to a representation $R$ of the gauge group. We also calculate higher-loop values of the anomalous dimension of the mass, $\gamma_m$ of $\bar{\psi}\psi$ at the infrared zero of the beta function. We find that for a given theory, the values of $\gamma_m$, calculated to three- and four-loop order, and evaluated at the infrared zero computed to the same order, tend to be somewhat smaller than the value calculated to two-loop order. The results are compared with recent lattice simulations.

I. INTRODUCTION

In this paper we investigate how higher-loop corrections to the beta function affect the evolution of a vectorial, asymptotically free gauge theory (in $(3 + 1)$ dimensions, at zero temperature) from the ultraviolet to the infrared. We assume that the theory contains a certain number, $N_f$, of massless Dirac fermions $\psi$ transforming according to a representation $R$ of the gauge group. We consider cases where $R$ is the fundamental, adjoint, and rank-2 symmetric or antisymmetric tensor representation. We also study the effect of higher-loop corrections to the anomalous dimension $\gamma_m$ of the fermion mass. This work yields more complete information on the nature of the evolution of the theory from the ultraviolet to the infrared, in particular, on the determination of the infrared zero of the beta function and the scaling behavior of the $\bar{\psi}\psi$ operator in the vicinity of this zero. We will give a number of results for a general gauge group $G$ but will focus on the case $G = SU(N)$.

We denote the running gauge coupling of the theory as $g(\mu)$, with $\alpha(\mu) = g(\mu)^2/(4\pi)$, where $\mu$ is the Euclidean energy/momentum scale (which will often be suppressed in the notation). The property that the $SU(N)$ gauge interaction is asymptotically free means that $\lim_{\mu \to \infty} \alpha(\mu) = 0$, and, since the beta function is negative for small $\alpha$, it follows that, as the energy/momentum scale $\mu$ decreases from large values, $\alpha$ increases. As $\mu$ decreases and the theory evolves into the infrared, two different types of behavior may occur, depending on the fermion content. In a theory with a sufficiently small number of fermions in small enough representations $R$, as $\mu$ decreases through a scale $\Lambda$, the coupling $\alpha$ exceeds a critical value $\alpha_{R, cr}$ depending on $R$, for the formation of bilinear fermion condensates, and these condensates are produced. This may or may not be associated with an infrared (IR) zero of the two-loop beta function at a value $\alpha = \alpha_{IR}$; if the two-loop ($2\ell$) beta function does have an IR zero, $\alpha_{IR, 2\ell}$, then this type of behavior requires that $\alpha_{IR, 2\ell} > \alpha_{R, cr}$ [3], [4]. As $\mu$ decreases toward $\Lambda$ and $\alpha$ increases toward the IR zero of the beta function, the increase of $\alpha$ as a function of decreasing $\mu$ is reduced. This gives rise to an $\alpha$ that is of order unity, but varies slowly as a function of $\mu$. This behavior is interestingly different from the behavior of the gauge coupling in quantum chromodynamics (QCD). As the condensates form, the fermions gain dynamical masses of order $\Lambda$, so that in the low-energy effective theory applicable at scales $\mu < \Lambda$, they are integrated out, and the further evolution of the theory into the infrared is controlled by the $N_f = 0$ beta function.

Alternatively, if the theory has a sufficiently large number $N_f$ of fermions and/or if these fermions are in a large enough representation $R$ (as bounded above by the requirement of asymptotic freedom), then the IR zero of the beta function occurs at a value smaller than $\alpha_{R, cr}$, so that as the scale $\mu$ decreases from large values, the theory evolves into the infrared without ever spontaneously breaking chiral symmetry. In this latter case, the IR zero of the beta function is an exact IR fixed point (IRFP), approached from below as $\mu \to 0$. More complicated behavior occurs in theories containing fermions in several different representations [5]; here we restrict to the case of fermions in a single representation. For a given asymptotically free theory that features an IR fixed point at $\alpha = \alpha_{IR}$, the value of this IRFP decreases as a function of $N_f$. There is thus a critical value of $N_f$, denoted $N_f,cr$, depending on $R$, at which $\alpha_{IR}$ decreases below $\alpha_{R, cr}$. This value serves as the boundary, as a function of $N_f$, between the interval of nonzero $1 \leq N_f < N_{f,cr}$ where the theory evolves into the infrared in a manner that involves fermion condensate formation and associated spontaneous chiral symmetry breaking ($\chi^2 B$), and the interval $N_{f,cr} < N_f < N_{f,max}$, where the theory evolves into the infrared without this condensate formation and chiral symmetry breaking, with $N_{f,max}$ denoting the maximal value of $N_f$ consistent with the requirement of asymptotic freedom.

The anomalous dimension $\gamma_m$ contains important information about the scaling behavior of the operator $\bar{\psi}\psi$ for which $m$ is the coefficient, as probed on different momentum scales. In a theory with an $\alpha_{IR} \sim O(1)$, it follows that $\gamma_m$ may also be $O(1)$, which can produce significant enhancement of dynamically generated fermion
masses due to the renormalization-group factor
\[ \eta = \exp \left[ \int_{\mu_1}^{\mu_2} \frac{d\mu}{\mu} \gamma_m(\alpha(\mu)) \right]. \tag{1.1} \]

In a phase where no dynamical mass is generated, \( \gamma_m \) simply describes the scaling behavior of the \( \bar{\psi}\psi \) operator.

There are several motivations for the study of higher-order corrections to this evolution of an asymptotically free theory into the infrared. First, the critical coupling, \( \alpha_{R,cr} \), is generically of order unity, and hence there is a need to have a quantitative assessment of the importance of higher-loop corrections to the evolution of the theory. Second, besides being of fundamental field-theoretic interest, a knowledge of this evolution plays an important role in modern technicolor (TC) models with dynamical electroweak symmetry breaking, in which the slow running of the coupling associated with an approximate infrared zero of the beta function provides necessary enhancement of quark and lepton masses \([1, 2]\) (recent reviews include \([4]-[6]\)), and can reduce technicolor to precision electroweak quantities \([7, 8]\). In addition to the fundamental representation, fermions have finite radii of convergence but are only asymptotic, higher-dimensional representations of chiral gauge groups have long played a valuable role in studies of extended technicolor (ETC) models that were relatively ultraviolet-complete and explicitly worked out the details of the sequential breaking of the ETC chiral gauge symmetries down to the TC group \([10]\). Recently, there has been a considerable amount of effort devoted to lattice studies of gauge coupling evolution and condensate formation in vectorial SU\((N)\) gauge theories as a function of \( N_f \), for fermions in both the fundamental representation \([8, 11]-[15]\) and higher representations \([16]-[23]\) (a recent review is \([24]\)). Thus, another important motivation for the present work is to provide higher-order calculations that can be compared with these lattice studies.

II. GENERAL THEORETICAL FRAMEWORK

A. Beta Function

The beta function of the theory is denoted \( \beta = dg/dt \), where \( dt = d\ln \mu \). In terms of the variable
\[ a \equiv \frac{g^2}{16\pi^2} = \frac{\alpha}{4\pi}, \tag{2.1} \]
the beta function can be written equivalently as \( \beta_a \equiv da/dt \), expressed as a series
\[ \frac{da}{dt} = -2a \sum_{\ell=1}^{\infty} b_\ell a^\ell = -2a \sum_{\ell=1}^{\infty} b_\ell \alpha^\ell, \tag{2.2} \]
where \( \ell \) denotes the number of loops involved in the calculation of \( b_\ell \) and \( b_\ell = b_\ell/(4\pi)^\ell \). Although this series and series for other quantities in quantum field theories do not have finite radii of convergence but are only asymptotic, experience shows that in situations where the effective expansion parameter (here, \( (\alpha/\pi) \) times appropriate group invariants) is not too large, the first few terms can provide both qualitative and quantitative insight into the physics. The first two coefficients in the expansion \((2.2)\), which are scheme-independent, are \([25]\)
\[ b_1 = \frac{1}{3}(11C_A - 4T_f N_f) \tag{2.3} \]
and \([26]\)
\[ b_2 = \frac{1}{3}[34C_A^2 - 4(5C_A + 3C_f)T_f N_f]. \tag{2.4} \]

Here \( C_f = C_2(R) \) is the quadratic Casimir invariant for the representation \( R \) to which the \( N_f \) fermions belong, \( C_A = C_2(G) \) is the quadratic Casimir invariant for the adjoint representation, and \( T_f = T(R) \) is the trace invariant for the fermion representation \( R \). Higher-order coefficients, which are scheme-dependent \([27]\), have been calculated up to four-loop order \([28, 29]\). Some further details are given in Appendix I. Values of \( b_\ell \) for \( 1 \leq N \leq 4 \) and relevant ranges of \( N_f \) are given in Table \( I \).

B. Anomalous Dimension of the \( \bar{\psi}\psi \) Operator

The anomalous dimension \( \gamma_m \) for the fermion bilinear \( \bar{\psi}\psi \) describes the scaling properties of this operator and can be expressed as a series in \( a \) or equivalently, \( \alpha \),
\[ \gamma_m = \sum_{\ell=1}^{\infty} c_\ell a^\ell = \sum_{\ell=1}^{\infty} c_\ell \alpha^\ell \tag{2.5} \]
where \( c_\ell = c_\ell/(4\pi)^\ell \) is the \( \ell \)-loop series coefficient. Via Eq. \((1.1)\), the anomalous dimension \( \gamma_m \) governs the running of a dynamically generated fermion mass. The coefficients \( c_\ell \) have been calculated to four-loop order \([30]\). The first two are
\[ c_1 = 6C_f \tag{2.6} \]
and
\[ c_2 = 2C_f \left[ \frac{3}{2} C_f + \frac{97}{6} C_A - \frac{10}{3} T_f N_f \right]. \tag{2.7} \]

For reference, the coefficient \( c_3 \) is listed in Appendix I. Since as \( N_f \) approaches \( N_f^{\text{max}} \) from below, \( b_1 \to 0 \) with nonzero \( b_2 \) and hence \( \alpha_{R,cr} \to 0 \), and since the perturbative calculation expresses \( \gamma_m \) in a power series in \( \alpha \), it follows that as \( \gamma_m \to 0 \) as \( N_f \) approaches \( N_f^{\text{max}} \) from below. We note that a conjectured beta function that directly relates \( \beta \) to \( \gamma \) has been proposed \([31]\).
III. PROPERTIES OF BETA FUNCTION COEFFICIENTS AND APPLICATION TO FUNDAMENTAL REPRESENTATION

In this section we discuss some general properties of the beta function coefficients as functions of \( N_f \), and give particular results for the case of fermions in the fundamental representation. In later sections we consider fermions in two-index representations.

A. \( b_1 \)

Since we restrict our considerations to an asymptotically free theory, we require that, with our sign conventions, \( b_1 > 0 \). This, in turn, implies that

\[
N_f < N_{f,\text{max}} ,
\]

where

\[
N_{f,\text{max}} = \frac{11CA}{4T_f}.
\]

Thus, for fermions in the fundamental representation, \( N_{f,\text{max, fund}} = (11/2)N \).

B. \( b_2 \) and Condition for Infrared Zero of \( \beta \)

We next proceed to characterize the behavior of the higher-loop coefficients of the beta function, \( b_\ell \) with \( \ell = 2, 3, 4 \), and the resultant zero(s) of the beta function, in terms of their dependence on \( N_f \). The two-loop results are well-known and are included here so that the discussion will be self-contained. Since only the first two coefficients of the beta function are scheme-independent, it follows that, to the extent that one is in a momentum regime where one can reliably use the perturbative beta function, the zeros obtained from these first two coefficients should be sufficient to characterize the physics at least qualitatively. When one includes higher-loop contributions to the beta function, one expects shifts of zeros, and there are, indeed, generically substantial shifts if zeros of the two-loop beta function occur at \( \alpha \sim O(1) \). However, if inclusion of three- and/or higher-loop contributions to \( \beta \) leads to a qualitative change in behavior, relative to the behavior obtained from the two-loop \( \beta \) function, then the results cannot be considered fully reliable, since they are scheme-dependent. For example, for a given gauge group \( G \) and fermion content, if the two-loop beta function does not have an infrared zero but the three-loop beta function does, one could not conclude reliably that this is a physical prediction of the theory. Moreover, it should be noted that even if there is no zero of the two-loop beta function away from the origin, i.e., a perturbative IRFP, the beta function may exhibit a nonperturbative slowing of the running associated with the fact that at energy scales below the confinement scale, the physics is not accurately described in terms of the Lagrangian degrees of freedom (fermions and gluons) \([32]-[34]\).

Another general comment is that the expression of the beta function in Eq. (2.2) is semiperturbative and does not incorporate certain nonperturbative properties of the physics, such as instantons, whose contributions involve essential zeros of the form \( \exp(-\pi\kappa/\alpha) \), where \( \kappa \) is a numerical constant. These instanton effects are absent to any order of the perturbative expansion in Eq. (2.2), but play an important role in the theory. For example, they break the global \( U(1) \) symmetry \([35]\) and also enhance spontaneous chiral symmetry breaking \([36]-[39]\). Estimates of the effects of instantons on the running of \( \alpha \) in quantum chromodynamics have found that they increase this running, i.e., they make \( \beta \) more negative in the region of small to moderate \( \alpha \) values \([37]\). If one were to model the effect of instantons crudely via a modification of \( \beta \) such as

\[
\beta_\alpha = \frac{d\alpha}{dt} = -2\alpha^2 \left[ \sum_{\ell=1}^{\infty} b_\ell \alpha^{\ell-1} + \lambda \exp \left( -\frac{\pi\kappa}{\alpha} \right) \right] ,
\]

then, since \( \lambda > 0 \), this would have the effect of increasing the value of the smallest (nonzero, positive) IR zero \( \alpha_{IR} \) of \( \beta \). For a given minimal value of \( \alpha_{cr,R} \) for condensate formation and spontaneous chiral symmetry breaking, since at least at the perturbative level \( \alpha_{IR} \) is a decreasing function of \( N_f \), it would follow that incorporating instanton effects would increase the value of \( N_{f,cr} \), i.e., would increase the interval in \( N_f \) where there is \( S\chi SB \). Furthermore, since instantons enhance chiral symmetry breaking, they would tend to reduce the value of \( \alpha_{cr,R} \), which also has the same effect of increasing \( N_{f,cr} \). We shall comment below on how, although the semiperturbative one-gluon exchange approximation to the Dyson-Schwinger (DS) equation does not directly include effects of confinement or instantons, it may nevertheless yield an approximately correct value of \( N_{f,cr} \) because of another approximation involved that has the opposite effect on the estimate.

If one knows the beta function calculated to a maximal loop order \( \ell_{\text{max}} \), then the equation for the zeros of the beta function, aside from the zero at \( a = 0 \), is

\[
\sum_{\ell=1}^{\ell_{\text{max}}} b_\ell a^{\ell-1} = b_1 \left[ 1 + \sum_{\ell=2}^{\ell_{\text{max}}} \left( \frac{b_\ell}{b_1} \right) a^{\ell-1} \right] = 0 .
\]

As is clear from Eq. (3.4), the zeros of \( \beta \) away from the origin depend only on the \( \ell_{\text{max}} / 1 \) ratios \( b/\ell \) for \( 2 \leq \ell \leq \ell_{\text{max}} \). The coefficients \( b_1 \) and \( b_2 \) are linear functions of \( N_f \), while \( b_3 \) and \( b_4 \) are, respectively, quadratic and cubic functions of \( N_f \). With our sign convention in which an overall minus sign is extracted in Eq. (2.2), each of these coefficients is positive for \( N_f = 0 \). The coefficients \( b_1 \) and \( b_2 \) are both monotonically and linearly decreasing functions of \( N_f \). As \( N_f \) increases sufficiently, \( b_2 \) thus reverses
sign, from positive to negative, vanishing at \( N_f = N_{f, b_2z} \), where

\[
N_{f, b_2z} = \frac{17C_A^2}{2T_f(5C_A + 3C_f)}.
\]

(The subscript \( b_2z \) stands for the condition that \( b_2 \) is zero). Since

\[
N_{f, \text{max}} - N_{f, b_2z} = \frac{3C_A(11C_f + 7C_A)}{4T_f(3C_f + 5C_A)} > 0,
\]

i.e., \( N_{f, \text{max}} > N_{f, b_2z} \), it follows that there is always a nonvacuous interval in the variable \( N_f \) where the theory is asymptotically free and the two-loop \( (2\ell) \) beta function has an infrared zero, namely

\[
N_{f, b_2z} < N_f < N_{f, \text{max}}.
\]

This zero occurs at

\[
\alpha_{\text{IR}, 2\ell} = -\frac{4\pi b_1}{b_2}
\]

and is physical for \( b_2 < 0 \). Explicitly, for the fundamental representation,

\[
N_{f, b_2z, \text{fund}} = \frac{34N^3}{13N^2 - 3}
\]

and

\[
\alpha_{\text{IR}, 2\ell, \text{fund}} = \frac{4\pi(11N - 2N_f)}{-34N^2 + N_f(13N - 3N^{-1})}.
\]

Illustrative values of \( N_{f, b_2z, \text{fund}} \) are given in Table II. The sizes of \( \ell \)-loop contributions are determined by \( (\alpha/\pi)^\ell \) multiplied by corresponding powers of various group invariants. Illustrative values of \( \alpha_{\text{IR}, 2\ell, \text{fund}} \) are given in Table III for \( N = 2, 3, 4 \) and the subset of the interval \([5, 7]\) for which \( \alpha_{\text{IR}, 2\ell, \text{fund}} \) is not so large as to render the two-loop perturbative calculation obviously unreliable. Here and below, when \( \alpha \) and \( \gamma \) values are listed without an explicit \( R \), it is understood that they refer to the fundamental representation. Examples of cases that we do not include in the table because the two-loop result cannot be considered reliable include the following (with formal values of \( \alpha_{\text{IR}, 2\ell, \text{fund}} \) listed): \( N = 2, N_f = 5 \), where \( \alpha_{\text{IR}, 2\ell, \text{fund}} = 11.4 \); \( N = 3, N_f = 9 \), where \( \alpha_{\text{IR}, 2\ell, \text{fund}} = 5.2 \); and \( N = 4, N_f = 11, 12 \), where \( \alpha_{\text{IR}, 2\ell, \text{fund}} = 14, 3.5 \).

For reference, the estimate in Eq. (9.3) of \( \alpha_{\text{cr}} \) from the analysis of the Dyson-Schwinger equation for the fermion propagator, in the one-gluon exchange approximation, takes the form in Eq. (9.3) for a fermion in the fundamental representation. As listed in Table IV this has the respective values 1.4, 0.79, and 0.56 for \( N = 2, 3, 4 \), respectively (where we quote the results to two significant figures but do not mean to imply that they have such a high degree of accuracy). Setting \( \alpha_{\text{cr}} = \alpha_{\text{IR}, 2\ell} \) yields the resultant estimates of \( N_{f, \text{cr}} \), which, rounded to the nearest integers, are 8, 12, and 16 for these values of \( N \). We denote these as \( \beta_{\text{DS}} \) estimates since they combine a calculation of \( \alpha_{\text{IR}} \) from the perturbative two-loop beta function with the (one-gluon exchange approximation to the) Dyson-Schwinger equation.

As \( N_f \) approaches its maximum value, \( N_{f, \text{max}} \), allowed by the constraint that the theory be asymptotically free, \( b_2 \) reaches its most negative value, namely \( b_2 = -C_A(7C_A + 11C_f) \). Clearly, for \( N_f \) values such that \( b_2 \) is only negative by a small amount and \( \alpha_{\text{IR}, 2\ell} \) is large, the perturbative calculation is not reliable. As \( N_f \) increases further in the range \([3, 7]\) and \( \alpha_{\text{IR}, 2\ell} \) decreases, the calculation becomes more reliable. In Table III we list the numerical values of \( N_{f, b_2z} \) for some illustrative values of \( N \). At the two-loop level, depending on whether \( \alpha_{\text{IR}, 2\ell} \) is smaller or larger than a critical value for fermion condensation, this is an exact or approximate infrared fixed point (IRFP) of the renormalization group for the gauge coupling. The existence of such an IRFP is of fundamental importance in determining how the theory evolves from the ultraviolet to the infrared \([10]\). In particular, as mentioned above, this determines whether, as the scale \( \mu \) decreases sufficiently to a scale \( \Lambda \) (depending on the group \( G \) and the fermion content), \( \alpha \) grows to a large enough size to produce fermion condensates or, on the contrary, the coupling never gets this large and the theory evolves into the infrared in a chirally symmetric manner, without ever producing such fermion condensates. Note that in the former case, the fermions involved in the condensates get dynamical masses of order \( \Lambda \) and are integrated out of the effective low-energy field theory applicable for scales \( \mu < \Lambda \), so that the further evolution into the infrared is governed by a different beta function.

It is useful to observe how rapidly the numbers \( N_{f, b_2z} \) approach their large-\( N \) values. The number \( N_{f, b_2z, \text{fund}} \) has the large-\( N \) expansion

\[
N_{f, b_2z, \text{fund}} = N \left[ \frac{34}{13} + \frac{102}{(13N)^2} + \frac{306}{(13)^3N^4} + O\left( \frac{1}{N^6} \right) \right] = N \left[ 2.615 + \frac{0.60355}{N^2} + \frac{0.1393}{N^4} + O\left( \frac{1}{N^6} \right) \right].
\]

As is evident from Table II the values of \( N_{f, b_2z, \text{fund}} \) approach the leading asymptotic form for moderate \( N \), as a result of the fact that the subleading term in Eq. (3.11) is suppressed by \( 1/N^2 \).
It is of interest to consider the 't Hooft large-$N$ limit, where
\[ N \to \infty \quad \text{with } \alpha N \text{ fixed.} \tag{3.12} \]
In a theory with fermions in the fundamental representation, in order for them to have a non-negligible effect in this limit, one considers the simultaneous Veneziano limit
\[ N_f \to \infty \quad \text{with } r = \frac{N_f}{N} \text{ fixed.} \tag{3.13} \]
In the combined limit of Eqs. (3.12) and (3.13), the range of $r$ satisfying the requirement of asymptotic freedom and the condition that $b_2 < 0$ so that the two-loop beta function has an IR zero is
\[ \frac{34}{13} < r < \frac{11}{2}, \; \text{i.e.,} \; \quad 2.615 < r < 5.5. \tag{3.14} \]

C. Coefficient $b_3$ and Three-Loop Behavior of the Beta Function

The three-loop beta function coefficient $b_3$ is a quadratic function of $N_f$ with positive coefficients of its $N_f^0$ and $N_f^2$ terms and a negative coefficient of its $N_f$ term. Hence, regarded as a function of the formal real variable $N_f$, it is positive for large negative and positive $N_f$, and positive at $N_f = 0$. The derivative of $b_3$ with respect to $N_f$ is
\[ \frac{db_3}{dN_f} = T_f \left[ -\frac{1415}{27}C_A^2 - \frac{205}{9}C_AC_f + 2C_f^2 + T_fN_f \left( \frac{88}{9}C_f + \frac{316}{27}C_A \right) \right]. \tag{3.15} \]
For the fermion representations $R$ that we consider here, for small values of $N_f$, this derivative $db_3/dN_f$ is negative, so that in this region of $N_f$, $b_3$ decreases from its positive value at $N_f = 0$ as $N_f$ increases. Because $b_3$ is a quadratic polynomial in $N_f$, the condition that it vanishes gives two formal solutions for $N_f$, namely
\[ N_{f,b3,z,\pm} = \frac{(1415C_A^2 + 615C_AC_f - 54C_f^2 \pm 3\sqrt{F_{b33}})}{4T_f(79C_A + 66C_f)}, \tag{3.16} \]
and
\[ F_{b33} = 122157C_A^4 + 109578C_A^3C_f + 25045C_A^2C_f^2 - 7380C_AC_f^3 + 324C_f^4. \tag{3.17} \]

Given that $F_{b33} > 0$, as is the case here, so that the values $N_{f,b3,j}$ are real, it follows that $b_3$ is positive in the intervals $N_f < N_{f,b3,-}$ and $N_f > N_{f,b3,+}$ and negative in the interval $N_{f,b3,-} < N_f < N_{f,b3,+}$. The value $N_{f,b3,+}$ and the neighborhood of $N_f$ values in the vicinity of $N_{f,b3,+}$ are not of interest here because they are larger than the maximal value $N_{f,max}$ allowed by the requirement of asymptotic freedom,
\[ N_{f,b3,+} > N_{f,max}. \tag{3.18} \]
Thus, $b_3$ only changes sign once for $N_f$ in the asymptotically free interval $0 \leq N_f < N_{f,max}$. As $N_f$ approaches $N_{f,max}$ from below, $b_3$ decreases to a negative value given by
\[ (b_3)_{N_f=N_{f,max}} = -\frac{C_A}{24} \left[ 1127C_A^2 + 44C_f(14C_A - 3C_f) \right]. \tag{3.19} \]
For fermions in the fundamental representation, this is
\[ (b_3)_{N_f=N_{f,max,fund}} = -\frac{701}{12}N_c^3 + \frac{121}{12}N_c + \frac{11}{8N_c}. \tag{3.20} \]

As is clear from Table III for this case
\[ N_{f,b3,-} < N_{f,b3,+}. \tag{3.21} \]
We noted above that any physically reliable zero of the beta function must be present already at the level of the two-loop beta function, since this is the maximal scheme-independent part of this function. Hence, in analyzing such a zero for the case under consideration where the fermions transform according to the fundamental representation of SU($N$), we only consider the interval $[3.7]$. Combining this fact with our results (3.21) and (3.18), it follows that $b_3$ is negative throughout all of the interval $[3.7]$ of interest here. For this fundamental-representation case, the $N_{f,b3,j}$ with $j = 1, 2$ have the large-$N$ expansions
\[ N_{f,b3,1} = N \left[ 1.911 + \frac{0.3244}{N^2} + \frac{0.06844}{N^4} + O \left( \frac{1}{N^6} \right) \right]. \tag{3.22} \]
and
\[ N_{f,b_3,2} = N \left[ 13.348 + \frac{1.667}{N^2} + \frac{0.3978}{N^4} + O\left(\frac{1}{N^6}\right) \right]. \]

(3.23)

Here, \( N_{f,max} = 5.5N \).

At three-loop order, the equation for the zeros of the beta function, aside from the zero at \( a = 0 \), is \( b_1 + b_2a + b_3a^2 = 0 \). Formally, this equation has two solutions for \( a \) and hence for \( \alpha \), namely
\[ \alpha_{\beta z,3\ell,\pm} = \frac{2\pi}{b_3} \left[ -b_2 \pm \sqrt{b_2^2 - 4b_1b_3} \right]. \]

(3.24)

Since \( b_2 \) must be negative in order for the beta function to have a scheme-independent infrared zero, and since for fermions in the fundamental representation we have found that \( b_3 < 0 \) in the relevant interval \([3.7]\), we can rewrite this equivalently as
\[ \alpha_{\beta z,3\ell,\pm} = \frac{2\pi}{|b_3|} \left[ -|b_2| \mp \sqrt{b_2^2 + 4|b_1b_3|} \right]. \]

(3.25)

In order for a given solution to be physical, it must be real and positive. As is evident from Eq. (3.22), the solution corresponding to the the + sign in Eq. (3.24) (i.e., the – sign in Eq. \([3.22] \)) is negative and hence unphysical. Thus, there is a unique physical solution for the IR zero of the beta function to three-loop order, namely
\[ \alpha_{IR,3\ell} = \alpha_{\beta z,3\ell,-}. \]

(3.26)

Illustrative values for this IR zero of the beta function at three-loop order are listed in Table [III].

For an arbitrary fermion representation for which \( \beta \) has a two-loop IR zero, we observe that the value of this zero decreases when one calculates it to three-loop order, i.e.,
\[ \alpha_{IR,3\ell} < \alpha_{IR,2\ell}. \]

(3.27)

This can be proved as follows. We have
\[ \alpha_{IR,2\ell} - \alpha_{IR,3\ell} = \frac{2\pi}{|b_2b_3|} \left[ 2b_1|b_3| + b_2^2 - |b_2|^2 + |b_2|b_1b_3 \right]. \]

(3.28)

The expression in square brackets is positive if and only if
\[ (2b_1|b_3| + b_2^2 - b_2^2(b_2^2 + 4|b_1b_3|) > 0. \]

(3.29)

But the difference in \([3.29] \) is equal to the positive-definite quantity \( b_2b_3^2 \), which proves the inequality \([3.27] \). This inequality is evident in Table [III].

D. Coefficient \( b_4 \) and Four-Loop Behavior of \( \beta \)

The four-loop beta-function coefficient, \( b_4 \), was calculated in Ref. [29]. We next analyze its behavior as a function of \( N_f \) and the result four-loop IR zero of the beta function. The coefficient \( b_4 \) is a cubic polynomial in \( N_f \) which has positive coefficients of its \( N_f^0 \) and \( N_f^3 \) terms. Hence, regarded as a function of the formal real variable \( N_f \), \( b_4 \) is negative for large negative \( N_f \), positive for \( N_f = 0 \), and also positive for large positive \( N_f \). For fermions in the fundamental representation, the derivative at \( N_f = 0 \) is

\[ \left( \frac{db_4}{dN_f} \right)_{N_f=0} = \left( \frac{485513}{1944} + \frac{20}{9} \zeta(3) \right) N^3 + \left( \frac{58583}{1944} - \frac{548}{9} \zeta(3) \right) N^4 + \left( \frac{23}{8} - \frac{2341}{216} + \frac{44}{9} \zeta(3) \right) N^{-3}. \]

(3.30)

where \( \zeta(z) \) is the Riemann zeta function,
\[ \zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}. \]

(3.31)

and \( \zeta(3) = 1.20205690... \). This derivative is negative for all \( N \). (In the complex \( N \) plane, it has six zeros at three complex-conjugate pairs of \( N \) values.) It follows that, again as a function of the formal real variable \( N_f \), \( b_4 \) has a local maximum at a negative value of \( N_f \) and then decreases through positive values as \( N_f \) increases toward 0 and passes through 0 into the interval of physical values.

The detailed behavior of \( b_4 \) in the physical asymptotically free interval \( 0 \leq N_f \leq N_{f,max} \) depends on \( N \). In particular, one may determine the value of \( N_f \) where \( b_4 \) has a minimum and whether \( b_4 \) has any zeros for positive \( N_f \). For SU(2), \( b_4 \) decreases to a minimum positive value as \( N_f \) ascends through the approximate value \( N_f = 5.8 \), and then increases monotonically for larger \( N_f \), so that it is positive-definite for all non-negative \( N_f \). In particular, the asymptotically free region \( 0 \leq N_f < 11 \). For SU(3), \( b_4 \) is also positive-definite for all non-negative \( N_f \), reaching a local minimum as \( N_f \) ascends through a value of approximately 8.2 and then increasing monotonically for larger \( N_f \). However, for SU(4), \( b_4 \) is positive for \( 0 \leq N_f \leq 9.51 \), negative for the interval of values 9.51 < \( N_f \leq 11.83 \), and positive again for
It is straightforward to display the analytic expressions for the root \( \alpha_{IR,4\ell} \), but we shall not need this for our analysis. We list numerical values for \( \alpha_{IR,4\ell} \) for various values of \( N \) and \( N_f \) in Table III. For completeness, we note the specific sets \((N, N_f)\) where \( \alpha_{IR,2\ell} \) is so large that we consider the analysis via the perturbative beta function unreliable: these are \((N, N_f) = (2, 6), (3, 9), (4, 11), \) and \((4, 12)\).

E. Estimates of Zeros of the Four-Loop Beta Function via Padé Approximants

For the beta function, or more conveniently, the reduced function with the prefactor removed, \( \sum_{j=0}^{\ell-1} b_j \alpha^j \), it is useful to calculate and analyze Padé approximants, since these provide closed-form expressions that, by construction, agree with the series to the maximal order to which it is calculated. The expansion for \( \beta_\alpha \) to \( \ell = 4 \) loop order can be used in two ways. First, one can simply solve the cubic equation \( \beta_\alpha = b_1 + b_2 \alpha + b_3 \alpha^2 + b_4 \alpha^3 = 0 \) and obtain the three roots, one of which is the root of interest, giving the IR zero. Secondly, one can calculate Padé approximants, e.g., the \([2,1]\) and \([1,2]\) approximants, and determine their zeros. The \([1,2]\) Padé approximant has a single zero at

\[
\alpha_{\beta,4\ell, [1,2]} = \frac{\alpha_{IR,4\ell, [1,2]}}{4\pi} = \frac{b_1 (b_3 - b_3^2)}{b_2^2 - 2b_1 b_2 b_3 + b_2^2 b_4} \quad (3.33)
\]

Taking into account the fact that \( b_2 \) and \( b_3 \) are negative in the relevant interval \( \ell \), this can be rewritten as

\[
a_{\beta,4\ell, [1,2]} = \frac{\alpha_{IR,4\ell, [1,2]}}{4\pi} = \frac{b_1 (b_3 + b_3^2)}{|b_2^3 + 2b_1 b_2 |b_3| - b_2^2 b_4|} \quad (3.34)
\]

The two zeros from the \([2,1]\) approximant are

\[
a_{\beta,4\ell, [2,1], \pm} = \frac{b_2 b_3 - b_1 b_4 \pm \left( (b_2 b_3 - b_1 b_4)^2 - 4b_1 b_3 (b_3^2 - b_2 b_4) \right)^{1/2}}{2(b_2 b_4 - b_2^2)} \quad (3.35)
\]

Taking account of the fact that \( b_2 \) and \( b_3 \) are negative in the relevant interval \( \ell \), this can be rewritten as

\[
\frac{b_2 b_3 - b_1 b_4 \pm \left( (b_2 b_3^2 - b_1 b_4)^2 + 4b_1 b_3 (b_3^2 + b_2^2 b_4) \right)^{1/2}}{2(b_2 b_4 - b_2^2)} \quad (3.36)
\]
4παβz,4ℓ, [2,1]. As is evident from Eqs. (3.33) and (3.35), the zeros of the [1,2] and [2,1] Padé approximants incorporate information on β up to four loops. One readily verifies that in the limit b1 → 0, the zero of the [1,2] Padé reduces to the two-loop result a = −b1/b2, and the two zeros of the [2,1] Padé reduce to those obtained from the three-loop beta function, (3.26). We list the values of αIR obtained from the zeros of the [1,2] and [2,1] Padé approximants to the four-loop beta function for the case of fermions in the fundamental representation in Table III.

From our calculations of αIR at the three- and four-loop level for SU(N) with fermions in the fundamental representation, we can make several remarks. Although n-loop calculations of the beta function for n ≥ 3 loops are scheme-dependent, the results obtained with the present $\bar{M}$S scheme provide a quantitative measure of the accuracy of the scheme-independent two-loop result. For a given N, as Nf increases above the minimal value $N_{f,\bar{MS}}$, where the IR zero first appears, and as the resultant αIR,2ℓ decreases to values ≈ 1, the difference between αIR,2ℓ and the higher-loop values αIR,nℓ for n = 3, 4 decrease. As is evident from Table III, the value of αIR,nℓ generically decreases as one goes from n = 2 to n = 3 loops and then increases by a smaller amount as one goes from n = 3 to n = 4 loops, so that αIR,4ℓ is smaller than αIR,2ℓ. In the same region of Nf values such that αIR,2ℓ is reasonably small, the values obtained via the [1,2] and [2,1] Padé approximants to the four-loop beta function are close to those obtained from the zeros of this beta function itself.

IV. EVALUATION OF THE ANOMALOUS DIMENSION γm AT THE INFRARED ZERO OF β

In this section we evaluate the anomalous dimension of γ ≡ γm, calculated to the n-loop order in perturbation theory, at the (approximate or exact) IR zero of the beta function to this order, αIR,nℓ, for n = 2, 3, 4. We denote these as γnℓ(αIR,nℓ). We focus here on general results and their application to the case of fermions in the fundamental representation, and discuss higher-dimensional representations in subsequent sections. In general, this anomalous dimension must be positive to avoid unphysical singularities in fermion correlation functions. The coefficients $\tilde{c}_ℓ$ that enter in Eq. (2.5) used in this calculation are listed in Table V.

A running fermion mass, Σ(k), that is dynamically generated at a scale Λ, decays with Euclidean momentum $k > \Lambda$ like

$$\Sigma(k) \sim \Lambda \left( \frac{\Lambda}{k} \right)^{2-\gamma_m}$$

up to logs. Since for $k > \Lambda$, the running coupling α is smaller than the critical value $\alpha_{R,cr}$ and there is no spontaneous chiral symmetry breaking, it follows that $\Sigma(k)$ must decrease toward zero as $k/\Lambda \to \infty$. In turn, this implies that $\gamma_m < 2$. Hence, a physical value of $\gamma_m$ must lie in the range

$$0 < \gamma_m < 2.$$  (4.2)

For values of Nf such that the theory evolves into the infrared in a chirally symmetric manner, so that the IR zero of the beta function is exact, the same upper bound follows from a related unitarity consideration [43].

Using the two-loop result for γ and evaluating it at the two-loop value of the IR zero of the beta function, we have

$$\gamma_{2\ell}(\alpha_{IR,2\ell}) = \frac{C_f(11C_A - 4T_f N_f)(455C_A^2 + 99C_A C_f + (180C_f - 248C_A)T_f N_f + 80T_f^2 N_f^2)}{12(-17C_A^2 + (10C_A + 6C_f)T_f N_f)^2}$$  (4.3)

For the fundamental representation, this is

$$\gamma_{2\ell}(\alpha_{IR,2\ell}) = \frac{(N^2 - 1)(11N - 2N_f)(100N^3 - 99N - (158N^2 + 90)N_f + 40N^2 N_f^2)}{12(-34N^3 + (13N^2 - 3)N_f)^2}$$  (4.4)

We list numerical values of $\gamma(\alpha_{IR,2\ell})$ in Table VI for the illustrative values $N = 2$, 3, 4 and, for each N, a set of Nf values in the range $[2,\bar{MS}]$. For sufficiently small Nf > $N_{f,\bar{MS}}$ in each N case, αIR,2ℓ is so large that the formal value of $\gamma_{2\ell}(\alpha_{IR,2\ell})$ is larger than 2 and hence unphysical; we enclose these values in parentheses to indicate that

they are unphysical artifacts of a perturbative calculation at an excessively large value of α.

In the large-N, large-Nf limit of Eqs. (5.12) and (5.13),
with \( r = N_f/N \), Eq. (4.4) reduces to

\[
\gamma_{2\ell}(\alpha_{IR,2\ell}) = \frac{(11 - 2r)(1009 - 158r + 40r^2)}{12(-34 + 13r)^2} + O\left(\frac{1}{N^2}\right).
\]  

(4.5)

For \( r = 4 \) corresponding to the asymptotic value of \( N_{f,cr,fund} \) in Eq. (4.4), \( \gamma_{2\ell}(\alpha_{IR,2\ell}) = 113/144 \approx 0.7856 \), which is the same as the large-\( N \) limit of Eq. (4.7).

One may evaluate \( \gamma_{2\ell}(\alpha_{IR,2\ell}) \) at \( N_f \) equal to the value of \( N_{f,cr,fund} \) predicted by the one-gluon exchange (ladder) approximation to the Dyson-Schwinger equation for the fermion propagator, given in Eq. (3.3). This is somewhat formal, since these values of \( N_{f,cr,fund} \) are not, in general, integers and hence not actually physical; for example, \( N_{f,cr,fund} = 7.86, 11.91, 15.94 \) for \( N = 2, 3, 4 \). This procedure yields the result

\[
\gamma_{2\ell}(\alpha_{IR,2\ell}; N_{f,cr,fund}) = \frac{565N^4 - 706N^2 + 225}{144(N^2 - 1)(5N^2 - 3)} + O\left(\frac{1}{N^2}\right).
\]  

(4.6)

For the illustrative cases \( N = 2, 3, 4 \), this anomalous dimension takes the values 0.88, 0.82, and 0.80, respectively. As \( N \to \infty \), Eq. (4.6) has the expansion

\[
\gamma_{2\ell}(\alpha_{cr,fund}) = \frac{113}{144} + \frac{11}{40N^2} + O\left(\frac{1}{N^4}\right).
\]  

(4.7)

Since the estimate (9.3) is close to 4\( N \) even for the smallest value, \( N = 2 \), and asymptotically approaches 4\( N \) as \( N \to \infty \), it is worthwhile to compare the above values of \( \gamma \), viz., 0.88, 0.82, and 0.80 for \( N = 2, 3, 4 \), with \( \gamma_{2\ell}(\alpha_{IR,2\ell}) \) evaluated at the nearest possible, integer values of \( N_f \), namely \( N_f = 8, 12, 16 \) for \( N = 2, 3, 4 \). This procedure yields \( \gamma_{2\ell}(\alpha_{IR,2\ell}) = 0.75, 0.77, 0.78 \), as recorded in Table IV. To within the strong-coupling theoretical uncertainties of these calculations, these values are mutually consistent.

A closely related approach is to evaluate the two-loop expression for \( \gamma_m \) at \( \alpha = \alpha_{cr,R} \), where \( \alpha_{cr,R} \) is the estimate of the critical coupling for fermion condensation obtained from the one-gluon exchange approximation to the Dyson-Schwinger equation, and then substitute \( N_f = N_{f,cr} \) from the \( \beta \)DS analysis (see Appendix II). This yields the result

\[
\gamma_{2\ell}(\alpha_{cr,R}; N_f = N_{f,cr}) = \frac{21C_A^2 + 128CA_{Cf} + 225C_f^2}{144C_f(C_A + 3C_f)}.
\]  

(4.8)

For the fundamental representation, this reduces to the same result as was obtained in Eq. (4.6).

We have evaluated the three-loop result for \( \gamma \) at the three-loop value of the IR zero of the beta function, which we denote as \( \gamma_3(\alpha_{IR,3\ell}) \), and the four-loop result for \( \gamma \) at the four-loop value of the IRFP, which we denote as \( \gamma_4(\alpha_{IR,4\ell}) \). We list the resultant values in Table VI. From our calculations of \( \gamma_m \) for the case of fermions in the fundamental representation, we can make several observations. Although computations of \( \alpha_{IR,n\ell} \) and \( \gamma_{n\ell}(\alpha_{IR,n\ell}) \) are scheme-dependent for \( n \geq 3 \) loops, they provide a useful measure of the accuracy of the lowest-order results. As was the case with the position of \( \alpha_{IR,n\ell} \) itself, we find that, for a given \( N \) and for \( N_f \) reasonably well above \( N_{f,b2\ell} \), so that the perturbative calculation of \( \alpha_{IR,n\ell} \) is not too large, the value of \( \gamma_{n\ell}(\alpha_{IR,n\ell}) \) generically decreases as one goes from \( n = 2 \) to \( n = 3 \) loops. Some of this decrease can be ascribed to the decrease in \( \alpha_{IR,n\ell} \) going from \((n-2)\)-loop to \((n-3)\)-loop order. At the four-loop level, \( \gamma_4(\alpha_{IR,4\ell}) \) tends to be smaller than \( \gamma_3(\alpha_{IR,3\ell}) \) for values of \( N_f \) from \( N_{f,b2\ell} \) to values of \( N_f \) slightly above the middle of the range (5.7), while for values of \( N_f \) in the upper end of this range, \( \gamma_4(\alpha_{IR,4\ell}) \) is slightly larger than \( \gamma_3(\alpha_{IR,3\ell}) \). In general, for the values of \( N_f \) where \( \alpha_{IR} \) is sufficiently small that the calculation may be trustworthy, the value of the anomalous dimension evaluated at the IR zero of the beta function (both calculated to \( n \)-loop order) \( \gamma_{n\ell}(\alpha_{IR,n\ell}) \), is somewhat smaller than unity.

Several recent high-statistics lattice simulations have been carried out on an SU(3) gauge theory with a varying number \( N_f \) of fermions in the fundamental representation in the range \( 6 \leq N_f \leq 12 \) [11]-[15], [6], [24]. This work has yielded evidence for a regime of slowly running gauge couplings for \( N_f \leq 12 \), consistent with the presence of an IR zero of the beta function, in agreement with the earlier continuum estimates in Ref. [1]. Ref. [11] also found a considerable enhancement of \( \langle \bar{\psi}\psi \rangle/f_P^2 \) in the SU(3) theory with \( N_f = 6 \). Further lattice simulations and analysis of data should yield values of \( \gamma_m \) that can be compared with our higher-loop calculations in this paper. A preliminary study of the SU(2) theory with \( N_f = 6 \) fermions has also been reported [21].

V. ADJOINT REPRESENTATION

In this section we analyze the SU(\( N \)) theory with \( N_f \) copies of a Dirac fermion, or equivalently, \( 2N_f \) copies of a Majorana fermion, in the adjoint representation. For this case, the general expression for the maximal value of \( N_f \) allowed by the requirement of asymptotic freedom, Eq. (3.2), reduces to

\[
N_{f,max,adj} = \frac{11}{4},
\]  

(5.1)

i.e., restricting \( N_f \) to the integers, \( N_{f,max} = 2 \). The general expression in Eq. (3.5) for the value of \( N_f \) at which \( b_2 \) changes sign from positive to negative with increasing \( N_f \) reduces to

\[
N_{f,b2,adj} = \frac{17}{16} = 1.0625.
\]  

(5.2)

Hence there is only one (integer) value of \( N_f \), namely \( N_f = 2 \) Dirac fermions (equivalently, \( N_f = 4 \) Majorana fermions), for which the theory is asymptotically free and has an IR zero of the two-loop beta function. This zero occurs at

\[
\alpha_{IR,2\ell,adj} = \frac{2\pi}{5N} \approx \frac{1.257}{N} \quad \text{for} \quad N_f = 2.
\]  

(5.3)
Specializing the general formula for the critical coupling \( \alpha_{cr,R} \) from the one-gluon exchange approximation to the Dyson-Schwinger equation, Eq. (5.3) (see Appendix II) for the present case where \( R \) is the adjoint representation, one obtains \( \alpha_{cr,adj} = \pi/(3N) \). Formally setting \( \alpha_{IR,2\ell,adj} = \alpha_{adj,cr} \) yields the corresponding estimate for the critical number \( N_{f,cr} = 83/40 = 2.075 \). This may be rounded off to the nearest integer, giving \( N_{f,cr} \) for the adjoint representation. In view of the theoretical uncertainty in such an estimate, due to the strong-coupling nature of the physics involved, an SU(\( N \)) gauge theory with \( N_f = 2 \) adjoint fermions could be either slightly inside the chirally broken, confined side of \( N_{f,cr} \) or slightly on the other side, where the theory is chirally symmetric and the evolution into the infrared is governed by an exact conformal IR fixed point.

For the present case of \( N_f = 2 \) fermions in the adjoint representation of SU(\( N \)), the coefficients of the beta function are \( b_1 = N, ~ b_2 = -10N^2, ~ b_3 = -(101/2)N^3, \) and

\[
b_4 = N^2 \left[ \frac{1843}{18} N^2 - 312 \right] - 4\zeta(3)N^2(N^2 + 72) . \tag{5.4}
\]

At the four-loop level, the beta function has three zeros away from the origin, one of which is the four-loop IR zero, denoted \( \alpha_{IR,4\ell,adj} \). For \( N = 2 \), the others form an unphysical complex-conjugate pair, while for the other representations for arbitrary \( N \) that we consider, the others consist of a negative one and a another, denoted \( \alpha_{R,u} \), which is not relevant to our study, since it is not reached by evolution of the coupling starting at small \( \alpha \) for large \( \mu \). We list the numerical values of these zeros in Table VI.

The coefficients \( \tilde{c}_\ell \) in Eq. (2.5) for \( \gamma \) for this case are \( \bar{c}_1 = 3N/(2\pi) \), \( \bar{c}_2 = (11N^2)/(8\pi^4) \), and \( \bar{c}_3 = -3N^3/(2\pi^3) \), with \( \bar{c}_4 \) given by

\[
\pi^4 \bar{c}_4 = \frac{N^2}{8} \left[ 9 - \frac{5395}{192} N^2 \right] + \frac{15}{16} \zeta(3)N^2(N^2 - 9). \tag{5.5}
\]

(The term in \( \bar{c}_3 \) proportional to \( \zeta(3) \) and the terms in \( \bar{c}_4 \) proportional to \( \zeta(4) \) and \( \zeta(5) \) vanish for the adjoint representation for arbitrary \( N_f \).)

Evaluating the two-loop expression in Eq. (4.3) for \( \gamma_m \) at the IR zero of the beta function, also calculated at the two-loop level, \( \alpha_{IR,2\ell,adj} \), we obtain

\[
\gamma_{2\ell,adj}(\alpha_{IR,2\ell,adj}) = \frac{(11 - 4N_f)(277 - 34N_f + 40N_f^2)}{6(-17 + 16N_f)^2}, \tag{5.6}
\]

so that for the \( N_f = 2 \) case of interest here,

\[
\gamma_{2\ell,adj}(\alpha_{IR,2\ell,adj}) = \frac{41}{50} = 0.820 \quad \text{for } N_f = 2 . \tag{5.7}
\]

It is also of interest to evaluate the two-loop \( \gamma_m \) at the value of \( \alpha_{cr} \) from the one-gluon exchange (ladder) approximation to the Dyson-Schwinger equation. With \( N_f = 2 \), this yields

\[
\gamma_{2\ell}(\alpha_{cr,adj}) = \frac{47}{72} \approx 0.653 . \tag{5.8}
\]

Evaluating the three-loop result for \( \gamma_m \) at the IR zero of the beta function calculated at the three-loop level, \( \alpha_{IR,3\ell,adj} \), for the \( N_f = 2 \) case of interest, we obtain

\[
\gamma_{3\ell,adj}(\alpha_{IR,3\ell,adj}) = 0.543 \quad \text{for } N_f = 2 . \tag{5.9}
\]

which is again independent of \( N \). At the four-loop level, the value of \( \gamma_{4\ell,adj}(\alpha_{IR,4\ell,adj}) \) does depend slightly on \( N \). We list the values of these anomalous dimensions in Table VII. The most recent simulations of a lattice gauge theory with SU(2) gauge group and \( N_f = 2 \) fermions in the adjoint representation report \( \gamma_m = 0.49 \pm 0.13 \) [23]. This is in agreement with the calculations of \( \gamma_m \) here at the three- and four-loop level, to within the uncertainties of the respective calculations.

VI. SYMMETRIC AND ANTISYMMETRIC RANK-2 TENSOR REPRESENTATIONS

In this section we consider the SU(\( N \)) theory with \( N_f \) fermions in the symmetric or antisymmetric rank-2 representation, denoted S2 and A2. Since a number of formulas are similar for these two cases, we will often give these in a unified way for both cases, denoted T2 (for rank-2 tensor representation), with \( \pm \) signs distinguishing them. For S2, our analysis applies for any \( N \), while for A2, we restrict to \( N \geq 4 \), since the A2 representation is the singlet for SU(2) and is equivalent to the conjugate fundamental representation for SU(3). Note that for SU(4), the A2 representation is self-conjugate. Also, since for SU(2) the S2 representation is the same as the adjoint representation, which has already been analyzed, we only consider the illustrative values \( N = 3, 4 \).

For the two T2 cases, the general expression for the maximal value of \( N_f \) allowed by the requirement of asymptotic freedom, Eq. (5.2), reduces to

\[
N_{f,max,T2} = \frac{11N}{2(N \pm 2)} , \tag{6.1}
\]

where the \( \pm \) refers to S2 and A2, respectively. As \( N \) increases from 2 to \( \infty \), \( N_{f,max,S2} \) increases monotonically from 2.75 to 11/2 = 4.5, and as \( N \) increases from 3 to \( \infty \), \( N_{f,max,A2} \) decreases monotonically from 16.5 to the same limit, 4.5. The physical values of \( N_{f,max} \) in both cases are the greatest integral parts of these rational numbers.

For these representations, the general expression in Eq. (3.3a) for the value of \( N_f \) at which the beta function coefficient \( b_2 \) changes sign from positive to negative with increasing \( N_f \) takes the form

\[
N_{f,b2,T2} = \frac{17N^2}{(N \pm 2)(8N \pm 3 - 6N)^{-1}} . \tag{6.2}
\]

As a consequence of the general inequality (3.9), it follows that \( N_{f,b2,S2} < N_{f,max,S2} \) and \( N_{f,b2,A2} < N_{f,max,A2} \). For \( N = 2 \), the S2 representation is just the adjoint representation, so we only consider the illustrative values
\[ N = 3, 4. \] The respective intervals \( N_{f,b2z,S2} < N_f < N_{f,\max,S2} \) for which the SU(\( N \)) gauge theory is asymptotically free and has an IR zero of \( \beta \) are 1.06 < \( N_f < 2.75 \) for \( N = 3 \) and 1.22 < \( N_f < 3.30 \) for \( N = 4 \). These ranges imply that the only physical integral values of \( N_f \) satisfying these conditions are \( N_f = 2, 3 \) for both SU(3) and SU(4).

For large \( N \), \( N_{f,b2z,T2} \) has the series expansion

\[
N_{f,b2z,T2} = \frac{17}{2^2} + \frac{323}{2^4 N} + \frac{6137}{2^6 N^2} + \frac{103547}{2^8 N^3} + O\left(\frac{1}{N^4}\right) \tag{6.3}
\]

As \( N \) increases from 2 to \( \infty \), \( N_{f,b2z,S2} \) decreases monotonically from 17/16 = 1.0625 to 17/8 = 2.125, and as \( N \) increases from 3 to \( \infty \), \( N_{f,b2z,A2} \) decreases monotonically from 8.05 to the same limit, 2.125. This limit is twice the (\( N \)-independent) value of \( N_{f,b2z,adj} = 17/16 \) for the adjoint representation. Thus, for large \( N \), the range \( (6.4) \) where the SU(\( N \)) theory with \( N_f \) fermions in the S2 or A2 representation is asymptotically the same for both, namely, 17/8 < \( N_f < 11/2 \); restricting \( N_f \) to physical, integer values, this range consists of the three values \( N_f = 3, 4, 5 \).

For our further discussion we assume that \( N_f \) is in the range \( N_{f,b2z,T2} < N_f < N_{f,\max,T2} \) where the theory is asymptotically free and the two-loop beta function has an IR zero, for the respective cases S2 and A2. This zero occurs at the value

\[
\alpha_{IR,\ell,T2} = \frac{2\pi(11N - 2N_f(N \mp 2))}{-17N^2 + N_f(8N^2 \mp 19N \mp 12N^{-1})}. \tag{6.4}
\]

We have calculated \( \alpha_{IR,nf,S2} \) and \( \alpha_{IR,nf,A2} \) up to \( n = 4 \) loops and list the results in Table XI and Table XII. The resultant \( \beta \) DS estimates for \( N_{f,c,r} \) in the case of the S2 representation and \( N = 2, 3, 4 \) are \( N_{f,c,r,S2} = 2.1, 2.5, 2.8 \), respectively. For the A2 representation with \( N = 4 \), one has \( N_{f,c,r,A2} = 8.1 \).

The two-loop expression for the anomalous dimension, evaluated at \( \alpha = \alpha_{IR,\ell,T2} \), is

\[
\gamma_{2\ell,T2}(\alpha_{IR,\ell,T2}) = \frac{(N \mp 2)(N \mp 1)}{12}\left[11N - 2(N \mp 2)N_f\right]\left[N(554N^2 + 99N - 198) + (-34N^3 \pm 22N^2 \mp 360)N_f + 20N(N \mp 2)^2N_f^2\right].
\]

(6.5)

We list values of \( \gamma_{2\ell,S2}(\alpha_{IR,\ell,S2}) \) for \( N = 2, 3, 4 \) in Table XI and values of \( \gamma_{2\ell,A2}(\alpha_{IR,\ell,A2}) \) for \( N = 4 \) in Table XII with \( \ell = 2, 3 \).

It is also of interest to evaluate the two-loop expression for \( \gamma \) at the estimated \( \alpha = \alpha_{c,r,T2} \). This yields

\[
\gamma_{2\ell,T2}(\alpha_{c,r,T2}) = \frac{322N^2 \pm 225N - 450 - 10N(N \mp 2)N_f}{432(N \pm 2)(N \mp 1)} . \tag{6.6}
\]

We list these values in Tables XII and XIII.

Evaluating the two-loop anomalous dimensions at the two-loop IR zero of the beta function, \( \gamma_{2\ell,T2}(\alpha_{IR,\ell,T2}) \), for \( N_f \) equal to the respective \( \beta \) DS-estimated critical values, we obtain (again with T2 and the \( \pm \) signs referring respectively to S2 and A2)

\[
\gamma_{2\ell,T2}(\alpha_{IR,\ell,T2})|_{N_f = N_{c,r,T2}} = \frac{374N^4 \pm 578N^3 - 931N^2 \mp 900N + 900}{144(N \mp 2)(N \mp 1)(4N^2 \mp 3N - 6)} . \tag{6.7}
\]

This has the large-\( N \) expansion

\[
\gamma_{2\ell,T2}(\alpha_{IR,\ell,T2})|_{N_f = N_{c,r,T2}} = \frac{187}{288} \pm \frac{17}{128N} + O\left(\frac{1}{N^2}\right) . \tag{6.8}
\]

The leading term has the value 187/288 \( \approx 0.649 \).

From a lattice study of SU(3) gauge theory with \( N_f = 2 \) fermions in the S2 (sextet) representation, Ref. 18 found that this theory is characterized by slow running behavior consistent with an (exact or approximate) IR fixed point, and further reported that \( \gamma_m < 0.6 \) where it was measured. For SU(3), the estimate of \( \alpha_{c,r,S2} \) in Eq. (6.3) gives \( \alpha_{c,r,S2} = \pi/10 = 0.31 \). Our results for the IR zero of \( \beta \) and the value of \( \gamma_m \) at this zero for \( N = 3 \) and \( N_f = 2 \) are listed in Tables XII and XIII. We find that \( \alpha_{IR,nf,S2} \) is approximately 0.84 at \( n = 2 \) loop level and decreases somewhat to 0.50 at three-loop level. The two-loop result for \( \gamma_m \) is unphysically large, while the three-loop value of \( \gamma_m \) at the corresponding three-loop IR zeros of \( \beta \) is about 1.3. These are somewhat larger than the values reported in Ref. 18, although in assessing this comparison, one must take account of the significant strong-coupling uncertainties in our calculation stemming from the fact that \( \alpha_{IR,S2} \sim O(1) \). Our evaluation of the two-loop expression for \( \gamma_m \) at the ladder-Dyson-Schwinger estimate of \( \alpha_{c,r,S2} \), is 0.65.
VII. EFFECTS OF NONZERO FERMION MASSES

The global chiral symmetry that is operative if the fermions are massless, and the way that it is broken by fermion condensates, is well-known, and we do not review it here. However, it is worthwhile to comment on the situation in which some fermion masses are nonzero. In this paper we generally assume that the fermions have zero intrinsic masses in the Lagrangian describing the high-scale physics, and the only masses that they acquire arise dynamically if they are involved in condensates that form as the gauge interaction becomes sufficiently strongly coupled in the infrared. This is a well-motivated assumption if the vectorial gauge theory arises as a low-energy effective field theory from an ultraviolet completion which is a chiral gauge theory. In turn, this is natural if the latter theory becomes strongly coupled, since it can then form fermion condensates that self-break it down to the vectorial subgroup symmetry. However, one may also choose to focus on the vectorial gauge theory as an ultraviolet-complete theory in itself. In a vectorial gauge theory, an intrinsic (bare) mass term for a fermion $\psi$, $\mathcal{L}_m = -m\bar{\psi}\psi$, is allowed by the gauge invariance. Hence, one may consider a more general situation in which the fermions may have such intrinsic (hard) masses in the high-scale Lagrangian $[44]$. In this case, as the reference scale $\mu$ decreases below the value of the hard mass of some fermion $m_f$, the beta function changes from one that includes this to one that excludes this fermion. If the hard fermion masses are small compared with the scale $\Lambda$ in the situation where the theory confines and breaks chiral symmetry spontaneously, then these hard masses have only a small effect. However, if some of the hard fermion masses are sufficiently large, then as $\mu$ decreases below their scale and the corresponding fermions are integrated out of the low-energy theory below this scale, this can significantly change the infrared properties of the resultant theory.

In applications of slowly running gauge theories to technicolor theories, at the scale $\Lambda_T C$ where the SU($N_{TC}$) gauge coupling grows to O(1) and is influenced by the presence of an approximate IR zero of the TC beta function, there can also be non-negligible effects due to four-fermion operators arising from the higher-lying extended technicolor dynamics $[4, 47, 10, 47, 48]$, and these can affect the scaling properties of $\bar{\psi}\psi$. Similar comments apply for topcolor-assisted technicolor $[4, 47]$. 

VIII. CONCLUSIONS

In this paper we have studied the evolution of an asymptotically free vectorial SU($N$) gauge theory from high scales to the infrared taking account of higher-loop corrections to the beta function and the anomalous dimension $\gamma_m$ for fermions in the fundamental, adjoint, and rank-2 symmetric and antisymmetric representations $S_2$ and $A_2$. We have compared our results with lower-order calculations. We have shown that, for fixed $N$ and $N_f$, in the range for which the two-loop beta function has an IR zero, the value of this zero decreases as one goes from the two-loop to the three-loop calculations, and we have determined this decrease quantitatively. Going further, we have shown that there is a smaller fractional increase in the value of this IR zero when calculated to four-loop accuracy, with the final four-loop result still smaller than the two-loop value. We have analyzed instanton effects and have demonstrated that they tend to increase the value of the IR zero of the beta function somewhat. A major part of our work has been the evaluation of the anomalous dimension $\gamma_m$ of $\bar{\psi}\psi$ at the IR zero of the beta function at the $\ell = 2, 3, 4$ loop levels. This zero is approximate or exact, depending on whether for a given $N$, the value of $N_f$ is below or above the critical value $N_{f,cr}$ below which there is spontaneous chiral symmetry breaking associated with the formation of a fermion condensate. We have found that this $\gamma_m$ at the (approximate or exact) IR zero of the beta function decreases as one goes from two-loop to three-loop order, and that the four-loop values also tend to be somewhat less than those at the two-loop level. The values that we have calculated for $\gamma_m$ at the IR zero of the beta function tend to be somewhat smaller than unity. We have compared our higher-loop calculations with results from recent lattice simulations and have found general agreement. We believe that the higher-loop calculations reported here should provide a useful reference for comparison with ongoing and future lattice measurements.

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IX. APPENDIX I

For the reader’s convenience, we list the three-loop beta function coefficient, in the $\overline{MS}$ scheme $[28]$, 

\[ \alpha_3 = \frac{\pi^2}{12} \frac{\ln \frac{\Lambda^2}{\mu^2}}{\beta_0} \]
\[ b_3 = \frac{2857}{54} C_A^3 + T_f N_f \left[ 2C_f^2 - \frac{205}{9} C_A C_f - \frac{1415}{27} C_A^2 \right] + (T_f N_f)^2 \left[ \frac{44}{9} C_f + \frac{158}{27} C_A \right]. \]  

(9.1)

The four-loop coefficient is given in Ref. [28] and is a cubic polynomial in \(N_f\). We note that the coefficients of the \(N_f^0\) (which is independent of the fermion representation) is positive, and the coefficient of the \(N_f^2\) term is positive for an arbitrary fermion representation.

Our normalizations for the quadratic Casimir and trace invariants of a Lie group are standard. The quadratic Casimir invariant \(C^2(R)\) for the representation \(R\) is given by \(\sum_{a=1}^{a(G)} \sum_{b=1}^{\dim(G)} [D_R(T_a)]_{ij} [D_R(T_b)]_{jk} = C^2(R) \delta_{ik}\), where \(a, b\) are group indices, \(a(G)\) is the order of the group, \(T_a\) are the generators of the associated Lie algebra, and \(D_R(T_a)\) is the matrix form of the \(T_a\) in the representation \(R\). The trace invariant \(T(R)\) is defined by \(\sum_{i,j=1}^{\dim(R)} [D_R(T_a)]_{ij} [D_R(T_b)]_{ji} = T(R) \delta_{ab}\).

From the calculations of the coefficients of the perturbative expansion of the anomalous dimension \(\gamma_m\) in the \(\overline{MS}\) scheme to four-loop order in Ref. [30], we record the three-loop coefficient

\[ c_3 = 2C_f \left[ \frac{129}{2} C_f^2 - \frac{129}{4} C_f C_A + \frac{11413}{108} C_A^3 + C_f T_f N_f (-46 + 48\zeta(3)) - C_A T_f N_f \left( \frac{556}{27} + 48\zeta(3) \right) \right]. \]  

(9.2)

We have used the four-loop coefficient \(c_4\) from Ref. [30] for our calculations, but it is too lengthy to reproduce here.

### A. Appendix II: Beta-Dyson-Schwinger Estimate of \(N_{f,cr}\)

In this appendix we briefly review the \(\beta\) DS estimate of \(N_{f,cr}\). In the one-gluon exchange (also called ladder approximation to the Dyson-Schwinger equation for the fermion propagator with an initially massless fermion in the representation \(R\) of the gauge group, one finds a solution with a dynamically generated, nonzero fermion mass if the coupling \(\alpha(\mu)\) exceeds a critical value \(\alpha_{cr,R}\) given by \([1, 2, 49]\).

\[ \alpha_{cr,R} = \frac{\pi}{3C_f}. \]  

(9.3)

Setting this equal to the two-loop expression for the IR zero of \(\beta\) then yields an estimate for \(N_{f,cr}\) to this order, namely

\[ N_{f,cr} = \frac{C_A (66 C_f + 17 C_A)}{10 T_f (C_A + 3 C_f)}. \]  

(9.4)

We call this the \(\beta DS\) estimate of \(N_{f,cr}\) since it combines a calculation of \(\alpha_{IR}\) from the \(\beta\) function with the estimate of \(\alpha_{cr,R}\) from the ladder approximation to the Dyson-Schwinger equation for the fermion propagator. In the same ladder approximation, one finds \(\gamma_m = 1\) at \(\alpha = \alpha_{cr,R}[1]\) (which also holds for the DS analysis at a UV-stable fixed point \([48]\)). For the gauge group SU(\(N\)) with the illustrative values of \(N\) used for the tables, namely \(N = 2, 3, 4, N_{f,cr,fund}\) is equal to 7.9, 11.9, and 15.9, respectively, with the large-\(N\) form \(N_f \sim N^2\). For \(S_2\), the symmetric rank-2 tensor representation, \(N = 2, 3, 4, N_{f,cr,S_2}\) is equal to 2.075, 2.5, and 2.9, increasing toward the limit 11/2 = 5.5 in the large-\(N\) limit. In the case of \(A_2\), the antisymmetric rank-2 tensor representation, for \(N = 3\), the result is the same as for the fundamental representation, while for \(N = 4\), one has \(N_{f,cr,A_2} \sim 8.1\), and as \(N \to \infty\), \(N_{f,cr,A_2}\) decreases toward the limit 11/2.

One understands that, \textit{a priori}, there could be significant uncertainty in these estimates because of the strong-coupling nature of the physics involved and the one-gluon approximation used for the solution of the Dyson-Schwinger equation. Moreover, the DS equation analysis is semi-perturbative in the sense that it contains polynomial dependence on \(\alpha\), and it neglects nonperturbative effects associated with confinement and instantons. However, corrections to the one-gluon exchange approximation have been analyzed and found not to be too large \([2]\). Recent lattice simulations for SU(3) are in broad agreement, to within the uncertainties, with the above prediction of \(N_{f,cr} \sim 12\) \([11, 15, 24]\). Some of the success of the \(\beta DS\) prediction for \(N_{f,cr}\) may arise from the fact that two major physical effects that it ignores, namely confinement and instantons, would shift \(N_{f,cr}\) in opposite directions and hence tend to cancel each other out \([24]\).

### X. APPENDIX III - PADÉ RESULTS

In this appendix we collect some relevant results on Padé approximants. Given a Taylor (or asymptotic) series expansion around \(z = 0\) for the function \(f(z)\),

\[ f(z) = \sum_{n=0}^{n_{max}} f_n z^n \]  

(10.1)

one can construct a set of \([p, q]\) Padé approximants, namely rational functions comprised of a numerator poly-
nominal of degree $p$ and a denominator polynomial of degree $q$, such that $p + q = n_{\text{max}} - 1$, of the form $(\sum_{j=0}^{p} P_j z^j)/(\sum_{k=0}^{q} q_k z^k)$. Without loss of generality, one can divide numerator and denominator by $q_0$, so that, after redefinition of the coefficients, one has

$$[p, q]_f(z) = \frac{\sum_{j=0}^{p} P_j z^j}{1 + \sum_{k=1}^{q} q_k z^k}. \quad (10.2)$$

The $p + 1$ coefficients $p_j$ with $0 \leq j \leq p$ and $q_k$ with $1 \leq k \leq q$ are uniquely determined in terms of the $f_n$ coefficients with $0 \leq n \leq n_{\text{max}}$ by expanding the $[p, q]$ Padé approximant in a Taylor series around $z = 0$ and solving the set of $n_{\text{max}}$ linear equations.

[1] T. Appelquist, D. Karabali, and L. C. R. Wijewardhana, Phys. Rev. Lett. 57, 957 (1986); T. Appelquist and L. C. R. Wijewardhana, Phys. Rev. D 35, 774 (1987); Phys. Rev. D 36, 568 (1987); T. Appelquist, J. Terning, and L. C. R. Wijewardhana, Phys. Rev. Lett. 77, 1214 (1996).
[2] T. Appelquist, K. Lane, and U. Mahanta, Phys. Rev. Lett. 61, 1553 (1988).
[3] T. A. Ryttov and R. Shrock, Phys. Rev. D 81, 116003 (2010); erratum, ibid. D 82, 059903 (2010).
[4] C. T. Hill and E. H. Simmons, Phys. Rep. 381, 235 (2003).
[5] Workshop on Dynamical Electroweak Symmetry Breaking, Southern Denmark Univ. 2008 (http://hep.sdu.dk/dewsb).
D 17, 2717 (1978); Phys. Rev. D 20, 3279 (1979).
[38] T. Appelquist and S. Selipsky, Phys. Lett. B 400, 364 (1997).
[39] E. Shuryak and M. Velkovsky, Phys. Lett. B 437, 398 (1998).
[40] T. Banks and A. Zaks, Nucl. Phys. B 196, 189 (1982).
[41] Here and elsewhere, when expressions are given for $N_f$ that formally evaluate to non-integral real values, it is understood implicitly that one infers an appropriate integral value of $N_f$ from them, either the greatest integral part or the nearest integer, depending on the context.
[42] Parenthetically, we observe that if (i) the exact beta function of a theory were to have a zero at a (nonzero, positive) value $\alpha_1$ with $d\beta/d\alpha > 0$ at $\alpha_1$, and (ii) another zero at a larger value, $\alpha_2$ with $d\beta/d\alpha < 0$ at $\alpha_2$ with (iii) $\beta > 0$ for $\alpha_1 < \alpha < \alpha_2$, and if (iv) the initial condition in the deep ultraviolet is that as $\mu \to \infty$, $\alpha(\mu)$ approaches $\alpha_2$ from below, then as the scale $\mu$ decreases, $\alpha$ would decrease from the UV fixed point $\alpha_1$ and approach the IR fixed point $\alpha_1$ from above as $\mu \to 0$. This type of behavior is not relevant to our theory with the initial condition on $\alpha$ that we assume in the ultraviolet.
[43] G. Mack, Commun. Math. Phys. 55, 1 (1977); M. Flato and C. Fronsdal, Lett. Math. Phys. 8, 159 (1984); V.K. Dobrev and V.B. Petkova, Phys. Lett. B 162, 127 (1985).
[44] Here a “hard” fermion mass is a mass that would remain in the hypothetical limit in which one turned off the SU($N$) interactions. If these masses are dynamically generated, then they are actually soft on the higher scale where they arise, as discussed in N. D. Christensen and R. Shrock, Phys. Rev. Lett. 94, 241801 (2005).
[45] T. Appelquist, M. Soldate, T. Takeuchi, and L. C. R. Wijewardhana, in G. Domokos and S. Kovesi-Domokos, eds., Proc. Johns Hopkins Workshop on Current Problems in Particle Theory (World Scientific, Singapore, 1988); K. I. Kondo, H. Mino, and K. Yamawaki, Phys. Rev. D 39, 2430 (1989).
[46] M. Kurachi, R. Shrock, and K. Yamawaki, Phys. Rev. D 76, 035003 (2007); H. S. Fukano and F. Sannino, Phys. Rev. D 82, 035021 (2010).
[47] A. Martin and K. Lane, Phys. Rev. D 71, 015011 (2005). F. Braam, M. Flossdorf, R. S. Chivukula, S. Di Chiara, and E. H. Simmons, Phys. Rev. D 77, 055005 (2008); T. A. Ryttov and R. Shrock, Phys. Rev. D 82, 055012 (2010).
[48] B. Holdom, Phys. Rev. Lett. 60, 1233 (1988); K. Yamawaki, M. Bando, and K. Matumoto, Phys. Rev. Lett. 56, 1335 (1986); V. Miransky and K. Yamawaki, Phys. Rev. D 55, 5051 (1997); Phys. Rev. D 56, 3768(E) (1997).
[49] K. Lane, Phys. Rev. D 10, 2605 (1974).
[50] C. Pica and F. Sannino, arXiv:1011.3832.
TABLE I: Values of the $\ell$-loop beta function coefficients $\beta_{\ell}$ defined in Eq. (26) in the SU($N$) gauge theory with $N_f$ fermions transforming according to the fundamental representation, as functions of $N$ and $N_f$, for the range where the theory is asymptotically free.

| $N$ | $N_f$ | $b_1$   | $b_2$   | $b_3$   | $b_4$   |
|-----|-------|---------|---------|---------|---------|
| 2   | 0     | 0.584   | 0.287   | 0.213   | 0.268   |
| 2   | 1     | 0.5305  | 0.235   | 0.154   | 0.191   |
| 2   | 2     | 0.477   | 0.184   | 0.099   | 0.127   |
| 2   | 3     | 0.424   | 0.132   | 0.047   | 0.078   |
| 2   | 4     | 0.371   | 0.080   | -0.0003 | 0.044   |
| 2   | 5     | 0.318   | 0.0285  | -0.044  | 0.024   |
| 2   | 6     | 0.265   | -0.023  | -0.084  | 0.020   |
| 2   | 7     | 0.212   | -0.075  | -0.120  | 0.030   |
| 2   | 8     | 0.159   | -0.127  | -0.152  | 0.057   |
| 2   | 9     | 0.106   | -0.178  | -0.180  | 0.099   |
| 2   | 10    | 0.053   | -0.230  | -0.205  | 0.156   |
| 3   | 0     | 0.875   | 0.646   | 0.720   | 1.173   |
| 3   | 1     | 0.822   | 0.566   | 0.582   | 0.910   |
| 3   | 2     | 0.769   | 0.485   | 0.450   | 0.681   |
| 3   | 3     | 0.716   | 0.405   | 0.324   | 0.485   |
| 3   | 4     | 0.663   | 0.325   | 0.205   | 0.322   |
| 3   | 5     | 0.610   | 0.245   | 0.091   | 0.194   |
| 3   | 6     | 0.557   | 0.165   | -0.016  | 0.099   |
| 3   | 7     | 0.504   | 0.084   | -0.118  | 0.039   |
| 3   | 8     | 0.451   | 0.004   | -0.213  | 0.015   |
| 3   | 9     | 0.398   | -0.076  | -0.303  | 0.025   |
| 3   | 10    | 0.345   | -0.156  | -0.386  | 0.072   |
| 3   | 11    | 0.292   | -0.236  | -0.463  | 0.154   |
| 3   | 12    | 0.239   | -0.317  | -0.534  | 0.273   |
| 3   | 13    | 0.186   | -0.397  | -0.599  | 0.429   |
| 3   | 14    | 0.133   | -0.477  | -0.658  | 0.622   |
| 3   | 15    | 0.080   | -0.557  | -0.711  | 0.852   |
| 3   | 16    | 0.0265  | -0.637  | -0.758  | 1.121   |
| 4   | 0     | 1.17    | 1.15    | 1.71    | 3.50    |
| 4   | 1     | 1.11    | 1.04    | 1.46    | 2.88    |
| 4   | 2     | 1.06    | 0.932   | 1.22    | 2.31    |
| 4   | 3     | 1.01    | 0.824   | 0.986   | 1.80    |
| 4   | 4     | 0.955   | 0.716   | 0.762   | 1.36    |
| 4   | 5     | 0.902   | 0.607   | 0.546   | 0.972   |
| 4   | 6     | 0.849   | 0.499   | 0.339   | 0.647   |
| 4   | 7     | 0.796   | 0.391   | 0.140   | 0.385   |
| 4   | 8     | 0.743   | 0.283   | -0.051  | 0.184   |
| 4   | 9     | 0.690   | 0.175   | -0.234  | 0.046   |
| 4   | 10    | 0.637   | 0.066   | -0.409  | -0.029  |
| 4   | 11    | 0.584   | -0.042  | -0.575  | -0.040  |
| 4   | 12    | 0.531   | -0.150  | -0.733  | 0.013   |
| 4   | 13    | 0.477   | -0.258  | -0.883  | 0.131   |
| 4   | 14    | 0.424   | -0.366  | -1.025  | 0.314   |
| 4   | 15    | 0.371   | -0.474  | -1.16   | 0.562   |
| 4   | 16    | 0.318   | -0.583  | -1.28   | 0.877   |
| 4   | 17    | 0.265   | -0.691  | -1.40   | 1.26    |
| 4   | 18    | 0.212   | -0.799  | -1.51   | 1.71    |
| 4   | 19    | 0.159   | -0.907  | -1.61   | 2.22    |
| 4   | 20    | 0.106   | -1.015  | -1.70   | 2.81    |
| 4   | 21    | 0.053   | -1.124  | -1.79   | 3.46    |
TABLE II: Values of $N_{f,b2z}$, $N_{f,b3z}$, and $N_{f,b4z}$, $i = 2, 3$, for SU($N$) with $N_f$ fermions in the fundamental representation. We only list physical, i.e., real, non-negative values. Thus, since $N_{f,b4,z} < 0$, it is not included.

| $N$ | $N_{f,max}$ | $N_{f,b2z}$ | ($N_{f,b3z}$, $N_{f,b3z}$) | ($N_{f,b4z}$, $N_{f,b4z}$) |
|-----|-------------|-------------|---------------------------|---------------------------|
| 2   | 11          | 5.55        | (3.99, 27.6)              | none                      |
| 3   | 16.5        | 8.05        | (5.84, 40.6)              | none                      |
| 4   | 22          | 10.61       | (7.73, 53.8)              | (9.51,11.83)              |

TABLE III: Values of the (approximate or exact) IR zeros in $\alpha$ of the SU($N$) beta function with $N_f$ fermions in the fundamental representation, for $N = 2, 3, 4$, calculated at $n$-loop order, and denoted as $\alpha_{IR,n}$. For each $N$, we only give results for the integral $N_f$ values in the range $[3,7]$ where the theory is asymptotically free and the two-loop beta function has an infrared zero. For the four-loop beta function, the cubic equation has three zeros, one of which is negative, one of which is $\alpha_{IR,4L}$, and the third of which is positive but farther from the origin. We include the latter, denoted as $\alpha_{4L,u}$. We also list zeros from the $[1,2]$ and $[2,1]$ Padé approximants to the four-loop beta function.

| $N$ | $N_f$ | $\alpha_{IR,2L}$ | $\alpha_{IR,3L}$ | $\alpha_{IR,4L}$ | $\alpha_{IR,4L}[1,2]$ | $\alpha_{IR,4L}[2,1]$ | $\alpha_{4L,u}$ |
|-----|-------|-------------------|-------------------|-------------------|----------------------|----------------------|---------------|
| 2   | 7     | 2.83              | 1.05              | 1.21              | 2.30                 | 1.16                 | 4.12          |
| 2   | 8     | 1.26              | 0.688             | 0.760             | 0.952                | 0.741                | 3.11          |
| 2   | 9     | 0.595             | 0.418             | 0.444             | 0.475                | 0.438                | 2.395         |
| 2   | 10    | 0.231             | 0.196             | 0.200             | 0.202                | 0.200                | 1.97          |
| 3   | 10    | 2.21              | 0.764             | 0.815             | 1.47                 | 0.807                | 5.62          |
| 3   | 11    | 1.23              | 0.578             | 0.626             | 0.871                | 0.616                | 3.29          |
| 3   | 12    | 0.754             | 0.435             | 0.470             | 0.561                | 0.462                | 2.295         |
| 3   | 13    | 0.468             | 0.317             | 0.337             | 0.367                | 0.333                | 1.78          |
| 3   | 14    | 0.278             | 0.215             | 0.224             | 0.231                | 0.222                | 1.48          |
| 3   | 15    | 0.143             | 0.123             | 0.126             | 0.127                | 0.125                | 1.29          |
| 3   | 16    | 0.0416            | 0.0397            | 0.0398            | 0.0398               | 0.0398               | 1.15          |
| 4   | 13    | 1.85              | 0.604             | 0.628             | 1.14                 | 0.625                | 6.94          |
| 4   | 14    | 1.16              | 0.489             | 0.521             | 0.776                | 0.516                | 3.49          |
| 4   | 15    | 0.783             | 0.307             | 0.428             | 0.556                | 0.422                | 2.30          |
| 4   | 16    | 0.546             | 0.320             | 0.345             | 0.407                | 0.340                | 1.73          |
| 4   | 17    | 0.384             | 0.254             | 0.271             | 0.298                | 0.267                | 1.40          |
| 4   | 18    | 0.266             | 0.194             | 0.205             | 0.215                | 0.203                | 1.19          |
| 4   | 19    | 0.175             | 0.140             | 0.145             | 0.149                | 0.145                | 1.05          |
| 4   | 20    | 0.105             | 0.091             | 0.092             | 0.093                | 0.0921               | 0.947         |
| 4   | 21    | 0.0472            | 0.044             | 0.0444            | 0.0444               | 0.0443               | 0.870         |
TABLE IV: Estimates of $\alpha_{cr,R}$ from the one-gluon exchange approximation to the Dyson-Schwinger equation for the fermion propagator. Values are listed for SU($N$) with $2 \leq N \leq 6$ and the representations $R = \text{(i) fundamental (fund), (ii) adjoint (adj), (iii) symmetric rank-2 tensor (S2), and (iv) antisymmetric rank-2 (A2).}$

| $N$ | $\alpha_{cr,\text{fund}}$ | $\alpha_{cr,\text{adj}}$ | $\alpha_{cr,S2}$ | $\alpha_{cr,A2}$ |
|-----|-----------------|-----------------|-----------------|-----------------|
| 2   | 1.40            | 0.52            | 0.52            | –               |
| 3   | 0.79            | 0.35            | 0.31            | 0.79            |
| 4   | 0.56            | 0.26            | 0.23            | 0.42            |
| 5   | 0.44            | 0.21            | 0.19            | 0.29            |
| 6   | 0.36            | 0.17            | 0.16            | 0.22            |
| $N$ | $N_f$ | $c_1$  | $c_2$  | $c_3$  | $c_4$  |
|-----|-------|-------|-------|-------|-------|
| 2   | 0     | 0.358 | 0.318 | 0.310 | 0.329 |
| 2   | 1     | 0.358 | 0.302 | 0.254 | 0.234 |
| 2   | 2     | 0.358 | 0.286 | 0.195 | 0.143 |
| 2   | 3     | 0.358 | 0.270 | 0.134 | 0.0577 |
| 2   | 4     | 0.358 | 0.254 | 0.0712 | -0.0218 |
| 2   | 5     | 0.358 | 0.239 | 0.00656 | -0.0952 |
| 2   | 6     | 0.358 | 0.223 | -0.0601 | -0.162 |
| 2   | 7     | 0.358 | 0.207 | -0.129 | -0.222 |
| 2   | 8     | 0.358 | 0.191 | -0.199 | -0.274 |
| 2   | 9     | 0.358 | 0.175 | -0.272 | -0.319 |
| 2   | 10    | 0.358 | 0.1595 | -0.346 | -0.355 |
| 3   | 0     | 0.637 | 0.853 | 1.26 | 2.03 |
| 3   | 1     | 0.637 | 0.825 | 1.11 | 1.64 |
| 3   | 2     | 0.637 | 0.796 | 0.957 | 1.27 |
| 3   | 3     | 0.637 | 0.768 | 0.801 | 0.909 |
| 3   | 4     | 0.637 | 0.740 | 0.642 | 0.561 |
| 3   | 5     | 0.637 | 0.712 | 0.479 | 0.227 |
| 3   | 6     | 0.637 | 0.684 | 0.312 | -0.0926 |
| 3   | 7     | 0.637 | 0.656 | 0.142 | -0.396 |
| 3   | 8     | 0.637 | 0.628 | -0.0313 | -0.683 |
| 3   | 9     | 0.637 | 0.599 | -0.208 | -0.953 |
| 3   | 10    | 0.637 | 0.571 | -0.389 | -1.21 |
| 3   | 11    | 0.637 | 0.543 | -0.573 | -1.44 |
| 3   | 12    | 0.637 | 0.515 | -0.760 | -1.65 |
| 3   | 13    | 0.637 | 0.487 | -0.951 | -1.85 |
| 3   | 14    | 0.637 | 0.459 | -1.145 | -2.02 |
| 3   | 15    | 0.637 | 0.431 | -1.34 | -2.18 |
| 3   | 16    | 0.637 | 0.402 | -1.54 | -2.31 |
| 4   | 0     | 0.895 | 1.60 | 3.17 | 6.86 |
| 4   | 1     | 0.895 | 1.56 | 2.89 | 5.88 |
| 4   | 2     | 0.895 | 1.52 | 2.61 | 4.93 |
| 4   | 3     | 0.895 | 1.48 | 2.32 | 4.00 |
| 4   | 4     | 0.895 | 1.44 | 2.03 | 3.09 |
| 4   | 5     | 0.895 | 1.40 | 1.73 | 2.21 |
| 4   | 6     | 0.895 | 1.36 | 1.43 | 1.36 |
| 4   | 7     | 0.895 | 1.33 | 1.12 | 0.526 |
| 4   | 8     | 0.895 | 1.29 | 0.808 | -0.275 |
| 4   | 9     | 0.895 | 1.25 | 0.492 | -1.05 |
| 4   | 10    | 0.895 | 1.21 | 0.170 | -1.79 |
| 4   | 11    | 0.895 | 1.17 | -0.157 | -2.50 |
| 4   | 12    | 0.895 | 1.13 | -0.488 | -3.18 |
| 4   | 13    | 0.895 | 1.09 | -0.825 | -3.83 |
| 4   | 14    | 0.895 | 1.05 | -1.17 | -4.45 |
| 4   | 15    | 0.895 | 1.01 | -1.51 | -5.025 |
| 4   | 16    | 0.895 | 0.969 | -1.86 | -5.57 |
| 4   | 17    | 0.895 | 0.930 | -2.22 | -6.08 |
| 4   | 18    | 0.895 | 0.890 | -2.58 | -6.54 |
| 4   | 19    | 0.895 | 0.850 | -2.95 | -6.97 |
| 4   | 20    | 0.895 | 0.811 | -3.32 | -7.37 |
| 4   | 21    | 0.895 | 0.771 | -3.69 | -7.71 |
TABLE VI: Values of the anomalous dimension in the SU($N$) theory with $N_f$ fermions in the fundamental representation, $\gamma_m$, calculated to the $n$-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, $\alpha_{IR,n\ell}$, for $\ell = 2, 3, 4$. We denote these as $\gamma_{m}(\alpha_{IR,n\ell})$. For sufficiently small $N_f > N_{f,b}$ in each $N$ case, $\alpha_{IR,2\ell}$ is so large that the formal value of $\gamma_{2\ell}(\alpha_{IR,2\ell})$ is larger than 2 and hence unphysical; we indicate this by placing these values in parentheses.

| $N$ | $N_f$ | $\gamma_{2\ell}(\alpha_{IR,2\ell})$ | $\gamma_{3\ell}(\alpha_{IR,3\ell})$ | $\gamma_{4\ell}(\alpha_{IR,4\ell})$ |
|-----|-------|----------------------------------|----------------------------------|----------------------------------|
| 2   | 7     | (2.67)                           | 0.457                            | 0.0325                           |
| 2   | 8     | 0.752                            | 0.272                            | 0.204                            |
| 2   | 9     | 0.275                            | 0.161                            | 0.157                            |
| 2   | 10    | 0.0910                           | 0.0738                           | 0.0748                           |
| 3   | 10    | (4.19)                           | 0.647                            | 0.156                            |
| 3   | 11    | 1.61                             | 0.439                            | 0.250                            |
| 3   | 12    | 0.773                            | 0.312                            | 0.253                            |
| 3   | 13    | 0.404                            | 0.220                            | 0.210                            |
| 3   | 14    | 0.212                            | 0.146                            | 0.147                            |
| 3   | 15    | 0.0997                           | 0.0826                           | 0.0836                           |
| 3   | 16    | 0.0272                           | 0.0258                           | 0.0259                           |
| 4   | 13    | (5.38)                           | 0.755                            | 0.192                            |
| 4   | 14    | (2.45)                           | 0.552                            | 0.259                            |
| 4   | 15    | 1.32                             | 0.420                            | 0.281                            |
| 4   | 16    | 0.778                            | 0.325                            | 0.269                            |
| 4   | 17    | 0.481                            | 0.251                            | 0.234                            |
| 4   | 18    | 0.301                            | 0.189                            | 0.187                            |
| 4   | 19    | 0.183                            | 0.134                            | 0.136                            |
| 4   | 20    | 0.102                            | 0.0854                           | 0.0865                           |
| 4   | 21    | 0.0440                           | 0.0407                           | 0.0409                           |

TABLE VII: Values of the (approximate or exact) IR zeros in $\alpha$ of the SU($N$) beta function with $N_f = 2$ fermions in the adjoint representation, for $N = 2, 3, 4$, calculated at $n$-loop order, and denoted as $\alpha_{IR,n\ell,adj}$. For the four-loop beta function, the cubic equation (3.32) has three zeros, one of which is $\alpha_{IR,4\ell,adj}$. Depending on $N$, there may be another real zero, denoted $\alpha_{4\ell,u,adj}$. We also list zeros from the [1,2] and [2,1] Padé approximants to the four-loop beta function.

| $N$ | $\alpha_{IR,2\ell,adj}$ | $\alpha_{IR,3\ell,adj}$ | $\alpha_{IR,4\ell,adj}$ | $\alpha_{IR,4\ell,[1,2],adj}$ | $\alpha_{IR,4\ell,[2,1],adj}$ |
|-----|--------------------------|--------------------------|--------------------------|-------------------------------|-------------------------------|
| 2   | 0.628                    | 0.459                    | 0.450                    | 0.455                         |                               |
| 3   | 0.419                    | 0.306                    | 0.308                    | 0.317                         | 0.308                         |
| 4   | 0.314                    | 0.2295                   | 0.234                    | 0.242                         | 0.233                         | 3.29                         |
TABLE VIII: Values of the anomalous dimension $\gamma_m$ in an SU($N$) gauge theory with $N_f = 2$ (Dirac) fermions in the adjoint representation, calculated to the $n$-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, for $n = 2, 3, 4$. We denote these as $\gamma_{n\ell,\text{adj}}(\alpha_{IR,n\ell,\text{adj}})$. We also list the value of $\gamma_{2\ell,\text{adj}}$ evaluated at $\alpha$ equal to the $\beta_{DS}$ estimate, Eq. (9.3), for $\alpha_{cr,\text{adj}}$.

| $N$ | $\gamma_{2\ell,\text{adj}}(\alpha_{IR,2\ell,\text{adj}})$ | $\gamma_{3\ell,\text{adj}}(\alpha_{IR,3\ell,\text{adj}})$ | $\gamma_{4\ell,\text{adj}}(\alpha_{IR,4\ell,\text{adj}})$ | $\gamma_{2\ell,\text{adj}}(\alpha_{cr,\text{adj}})$ |
|-----|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 2   | 0.820                           | 0.543                           | 0.500                           | 0.653                           |
| 3   | 0.820                           | 0.543                           | 0.523                           | 0.653                           |
| 4   | 0.820                           | 0.543                           | 0.532                           | 0.653                           |

TABLE IX: Values of the (approximate or exact) IR zero in $\alpha$ of the SU($N$) beta function with $N_f = 2$ fermions in the symmetric rank-2 (i.e., S2) representation, for $N = 3, 4$, calculated at $n$-loop order, and denoted as $\alpha_{IR,n\ell,S2}$.

| $N$ | $N_f$ | $\alpha_{IR,2\ell,S2}$ | $\alpha_{IR,3\ell,S2}$ | $\alpha_{IR,4\ell,S2}$ |
|-----|-------|------------------------|------------------------|------------------------|
| 3   | 2     | 0.842                  | 0.500                  | 0.470                  |
| 3   | 3     | 0.085                  | 0.079                  | 0.079                  |
| 4   | 2     | 0.967                  | 0.485                  | 0.440                  |
| 4   | 3     | 0.152                  | 0.129                  | 0.131                  |

TABLE X: Values of the (approximate or exact) IR zero in $\alpha$ of the SU(4) beta function with $N_f$ fermions in the antisymmetric rank-2 (i.e., A2) representation, for the range $5 \leq N_f \leq 10$ where the theory is asymptotically free and has an IR zero of the beta function, calculated at $n$-loop order, and denoted as $\alpha_{IR,n\ell,A2}$.

| $N$ | $N_f$ | $\alpha_{IR,2\ell,A2}$ | $\alpha_{IR,3\ell,A2}$ | $\alpha_{IR,4\ell,A2}$ |
|-----|-------|------------------------|------------------------|------------------------|
| 4   | 6     | 2.17                   | 0.664                  | 0.770                  |
| 4   | 7     | 0.890                  | 0.437                  | 0.502                  |
| 4   | 8     | 0.449                  | 0.287                  | 0.319                  |
| 4   | 9     | 0.225                  | 0.174                  | 0.184                  |
| 4   | 10    | 0.090                  | 0.080                  | 0.082                  |

TABLE XI: Values of $\gamma_m$ in an SU($N$) gauge theory with $N_f$ fermions in the symmetric rank-2 tensor representation S2, calculated to the $n$-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, for $n = 2, 3, 4$. We denote these as $\gamma_{n\ell,S2}(\alpha_{IR,n\ell,S2})$. We also list $\gamma_{2\ell,S2}$ evaluated at $\alpha$ equal to the $\beta_{DS}$ estimate, Eq. (9.3), for $\alpha_{cr,S2}$.

| $N$ | $N_f$ | $\gamma_{2\ell,S2}(\alpha_{IR,2\ell,S2})$ | $\gamma_{3\ell,S2}(\alpha_{IR,3\ell,S2})$ | $\gamma_{4\ell,S2}(\alpha_{IR,4\ell,S2})$ | $\gamma_{2\ell,S2}(\alpha_{cr,S2})$ |
|-----|-------|---------------------------------|---------------------------------|---------------------------------|---------------------------------|
| 3   | 2     | (2.44)                          | 1.28                            | 1.12                            | 0.653                           |
| 3   | 3     | 0.144                           | 0.133                           | 0.133                           | 0.619                           |
| 4   | 2     | (4.82)                          | (2.08)                          | 1.79                            | 0.659                           |
| 4   | 3     | 0.381                           | 0.313                           | 0.315                           | 0.629                           |
TABLE XII: Values of $\gamma_m$ in an SU($N$) gauge theory with $N_f$ fermions in the antisymmetric rank-2 tensor representation $A_2$, calculated to the $n$-loop order in perturbation theory and evaluated at the IR zero of the beta function calculated to this order, for $N = 4$ and $n = 2, 3, 4$. We denote these as $\gamma_{nf,A2}(\alpha_{IR,nf,A2})$. We also list $\gamma_{2\ell,A2}$ evaluated at $\alpha$ equal to the estimate Eq. (9.3) for $\alpha_{cr,A2}$.

| $N$ | $N_f$ | $\gamma_{2\ell,A2}(\alpha_{IR,2\ell,A2})$ | $\gamma_{3\ell,A2}(\alpha_{IR,3\ell,A2})$ | $\gamma_{4\ell,A2}(\alpha_{IR,4\ell,A2})$ | $\gamma_{2\ell,A2}(\alpha_{cr,A2})$ |
|-----|-------|--------------------------------|--------------------------------|--------------------------------|-------------------------------|
| 4   | 6     | (9.78)                        | 1.38                          | 0.293                          | 0.769                         |
| 4   | 7     | (2.19)                        | 0.695                         | 0.435                          | 0.750                         |
| 4   | 8     | 0.802                         | 0.402                         | 0.368                          | 0.732                         |
| 4   | 9     | 0.331                         | 0.228                         | 0.232                          | 0.713                         |
| 4   | 10    | 0.117                         | 0.101                         | 0.103                          | 0.695                         |