A Fractal Space-filling Complex Network

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Abstract

We study in this work the properties of the $Q_{mf}$ network which is constructed from an
anisotropic partition of the square, the multifractal tiling. This tiling is build using a single
parameter $\rho$, in the limit of $\rho \to 1$ the tiling degenerates into the square lattice that is asso-
ciated with a regular network. The $Q_{mf}$ network is a space-filling network with the following
characteristics: it shows a power-law distribution of connectivity for $k > 7$ and it has an high
clustering coefficient when compared with a random network associated. In addition the $Q_{mf}$
network satisfy the relation $N \propto \ell^{d_f}$ where $\ell$ is a typical length of the network (the average
minimal distance) and $N$ the network size. We call $d_f$ the fractal dimension of the network. In
the limit case of $\rho \to 1$ we have $d_f \to 2$.

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I - INTRODUCTION

The last years have seen an increasing interest in network studies in physics [1, 2]. Despite graph theory have been a research topic of mathematics and science of computation, the physicists have driven their attention to networks that show distribution of connectivity, $P(k)$, following a power law and that present small world effect: $\ell \propto \ln N$, for $\ell$ a typical network distance and $N$ the size of the network. A recent article [3] points the difficulty to put together these two aspects in a common broad scale-free framework, it means, a fractal paradigm. In fact, a real fractal should have $N \propto \ell^{d_f}$ for $d_f$ the fractal dimension, and not a logarithm dependence between $N$ and $\ell$. In this work we present a network that have the following characteristics: it is fractal, $N \propto \ell^{d_f}$, it is scale-free $P(k) \propto k^{-\gamma}$ for a large range of $k$, and it is complex in the sense of having large clustering coefficient. The scaling relation $N \propto \ell^d$ is trivially fulfilled for regular lattices where $d$ is an integer, the topologic dimension of the space. The complex fractal network we explore in this paper has a non integer $d_f$.

Recently networks embedded in metric spaces have been investigated in the literature [4] because of the large applications of networks that effectively occupy a volume in 3-dimensions. In addition, motivated by microprocessors design space-filling networks [5] have been studied. We cite these new trends in networks because the distinguished network we analyze in this paper is embedded in a metric space and furthermore it is space filling. In fact our network has a geometrical inspiration, it comes from a partition of the square. Indeed this network is originated from a singular tiling that has the additional property of being a multifractal partition of the square [6].

We follow the previous literature [6, 7, 8, 9, 10] and call this object the multifractal tiling, $Q_{mf}$. The $Q_{mf}$ tiling was developed in the context of modeling transport and percolation in heterogeneous porous media. In fact, a broad set of irregular and heterogeneous systems are model in the literature using a multifractal approach. We cite systems in geology [11, 12], atmospheric science [13] and chemistry [14]. Oil reservoirs are complex anisotropic structures whose treatment have been challenged science and technique because of its non trivial geometry. Inspired in the description of oil reservoirs it was developed [6] the $Q_{mf}$ tiling. In reference [7] an exhaustive study of the percolation threshold of $Q_{mf}$ was performed, in [8] a random version of $Q_{mf}$ was created, in [9] some of its percolation critical exponents were found, and in [10] a numerical study of its coordination number is done.

In this work we explore the network properties of the $Q_{mf}$, it means its topology, the study of the connections (neighborhood) among the cells of the tiling. The paper is organized as follows. In section 2 we describe in some detail the process of construction of the $Q_{mf}$ tiling and show its more important properties. In section 3 we show the main results concerning the properties of the network: the distribution of connectivity, the scaling of the the average minimal distance between two sites and the clustering
coefficient. Finally in section 4 we present our conclusions, discuss the main implications of our results and compare the properties of the $Q_{mf}$ network with other networks in the literature.

II - THE MULTIFRACTAL OBJECT

The multifractal tiling is a peculiar partition of the square. It is interesting to think about the $Q_{mf}$ tiling in contrast to the square lattice. The square lattice can be constructed using the following algorithm: take a square and cut it symmetrically with horizontal and vertical lines. This procedure produces four square cells. Repeat this procedure $n$ times inside each new block and you have finally a square lattice with $2^n$ cells. The $Q_{mf}$ object is generated in a similar way as the square lattice above described, but instead of using a symmetric partition we perform horizontal and vertical sections following a given ratio.

In Fig. 1 we exemplify the five initial steps of the construction of the multifractal for the parameter $\rho = 1/3$, or $(s, r) = (1, 3)$. In (a) we show, $n = 0$, the initial square that by convenience we assume of size $L = 1$. In (b), $n = 1$, a vertical cut is performed and two rectangles are formed. We call this a $(s, r) = (1, 3)$ object because the square is divided in 4 parts such that 1 part stays at one side and 3 parts at the other side. In (c), $n = 2$, two horizontal lines are drawn using the same section rate as before. At this level the initial square generates four rectangular blocks. Using as the area unit a square of size $\epsilon = 1/(s + r)$, the largest block has area $r^2$, there are two blocks of areas $rs$ and the smallest block has area $s^2$. In (d) and (e), $n = 3$ and $n = 4$, respectively, the same procedure is repeated inside the initial four blocks. In reference [15] it is explored the group of eight possibilities of cutting a square lattice with a given ratio. In this work, as in other papers about $Q_{mf}$, it is followed the recipe of Fig. 1.

We remark that the number of blocks at step $n$ is $2^n$. These blocks do not have all the same area, we call the subsets of blocks of same area by a $k$-set. It is easy to check that the block area distribution follows a binomial distribution and the number of $k$-sets
FIG. 2: A $Q_{mf}$ multifractal tiling and a $Q_{mf}$ network, for $\rho = 1/2$ and $n = 6$. In (a) the original tiling and in (b) the corresponding network.

is $k = n + 1$. This fact implies that the $Q_{mf}$ has the remarkable property: in the limit of $n \to \infty$ the area of its forming blocks follows a multifractal distribution $\mathcal{D}$ [6]. The spectrum of fractal distributions comes from a box counting reasoning:

$$D_X = \lim_{\epsilon \to 0} \frac{\log N(X)}{\log (1/\epsilon)}$$

for $N(X)$ the number of unitary cells of size length $\epsilon$ that cover the set of blocks of a given area $X$. Once the initial square is partitioned $n$ times, the size of the unitary cell is $\epsilon = 1/(s + r)^n$. For each $k$-set the total area of blocks (using $\epsilon$ area units) is done by:

$$N_k = C_n^k s^k r^{(n-k)},$$

where $C_n^k$ is the binomial coefficient that express the number of elements $k$-type, and $s^k r^{(n-k)}$ is the area of each element of this set. We put together all these elements to have the fractal dimension of each $k$-set:

$$D_k = \lim_{\epsilon \to 0} \frac{\log N_k}{\log (1/\epsilon)} = \lim_{n \to \infty} \frac{\log (C_n^k s^k r^{(n-k)})}{\log (s + r)^n}. \tag{3}$$

This distribution show a concave shape with a maximum at $k = \rho n$. The case $r = s = 1$ is degenerated. In this situation the subsets of the lattice are composed uniquely by square cells of the same area. Therefore the tiling is formed by a single subset of dimension 2.
In Fig. 2 (a) it is shown an example of $Q_{mf}$ construction for $(s, r) = (1, 2)$ and $n = 6$. In Fig. 2 (b) we build a network corresponding to this tiling. The nodes of the network are the $2^n$ blocks of the $Q_{mf}$ and the vertices are established according to a neighborhood criterion. These last figures offer a glimpse of the metric heterogeneity and the topology of the multifractal. In the next section we explore in detail the network properties of this class of objects.

III - RESULTS

We start the analysis of the properties of the $Q_{mf}$ network discussing its distribution of connectivity, $P(k)$. In Fig. 3 we show the cumulative sum of $P(k)$ versus $k$ for several values of $\rho$ as indicated in the figure. The option for the cumulative sum instead of $P(k)$ itself is due to the strong fluctuation of the data. Fig. 3 confirms the results of a previous work [10]. For low $k$, typically $k < 7$, the curve suggests an exponential behavior and above this threshold the network depicts a scale-free behavior. The values of the exponents $\gamma$ of the power-law $P(k) \propto k^{-\gamma}$ are indicated in Table I for several $\rho$, trivially the values of $\gamma$ are estimated from the slopes of the curve of the cumulative sum decreased by one. The exponent $\gamma$ goes to an asymptotic limit for large $N$ [10]. We observe that in the limit of $\rho \to 1$ the $Q_{mf}$ tiling gets more symmetric and at $\rho = 1$ the $Q_{mf}$ degenerates into the square lattice. For the regular square lattice $P(k)$ is a delta of Dirac centered at $k = 4$ and the cumulative sum a step function. Fig. 3 corroborates this idea, the skewed curves in the $\rho \to 1$ limit anticipate the phase transition at $\rho = 1$.

We explore the distance characterization of the $Q_{mf}$ network in Fig. 4 where we display the behavior of the average shortest distance for all couple of distinct vertices of the network, $\ell$, versus network size, $N$. The simulation is performed for some values
FIG. 4: In (a) it is displayed the average distance $\ell$ as a function of $N$ for several values of $\rho$. We find $2 < d_f < 4$, the full set of $d_f$ is shown in Table I. Two limit cases are interesting. The limit $\rho \to 1$, which corresponds to the square lattice, has $d_f \to 2$ as it is expected in a bidimensional space. The opposit limit $\rho \to 0$, which is associated with very anisotropic structures, shows large $d_f$. We cannot affirm that 4 is an asymptotic threshold, further numerical investigation should test this hypothesis. We remark that the $Q_{mf}$ network does not follow a small world relationship $\ell \propto \ln N$ that is common to most of power-law and random like networks.

An analysis of the clustering coefficient, $C$, versus network size, $N$, is shown in Fig. 5. The general view of this figure points to a stable behavior of $C$ in the limit of large $N$. The dispersion of $C$ among $\rho$ is not large, the numerics show $C = 0.37 \pm 0.01$. Smaller values of $\rho$, however, show a significant larger $C$. The discussion about $C$ is intriguing once we compare the numerical values of $C$ with the clustering coefficient of a random network associated to the $Q_{mf}$ network. An associated random network is defined as a network with the same $N$ and $\langle k \rangle$ of the original network (we do not compare our results with a random network with a same $P(k)$ because such random network would alterate the space filling characteristics that we are interested in). For a random network $C = \langle k \rangle / N$, in the case of our network: $\langle k \rangle$ is a constant number smaller than 6 and $N$ a number that can grow without limit. As a consequence the associated random
network has $C \to 0$ in the limit $N \to \infty$. Therefore the $Q_{mf}$ network has a $C$ that is infinitely larger than the clustering coefficient of the associated random network. Because of the high $C$ and the power-law behavior of distribution of connectivity we call the $Q_{mf}$ network a complex network.

In Table I it is shown some parameters related to the $Q_{mf}$ network for several values of $\rho$. Most of these data was already commented in the text. We focus now on the average connectivity $<k>$ of the network. For all $\rho$ studied we have $<k> \sim 5.43$ which characterizes a sparse network. This is not surprising, since there is a result in topology that shows that for two dimensions the average coordination number of a tiling cannot exceed 6. The average connectivity confirms Fig. 3 where we can see that the majority of vertices are situated in the range: $4 \leq k \leq 6$. Otherwise we note that most of interesting results concerning the distribution of connectivity, in special the power-law behavior, are satisfied only in the range $k > 7$. Indeed, the $Q_{mf}$ network, as most of complex networks, also have hubs that determine the distinguished characteristics of the network.
TABLE I: The average quantities: clustering coefficient, \( C \), minimal distance, \( \ell \), and connectivity, \( < k > \); the slope \( \gamma \) of the distribution of connectivity and the fractal dimension \( d_f \). The data corresponding to the average parameters are estimated for \( N = 2^{16} \).

| \( \rho \) | (3,4) | (2,3) | (1,2) | (1,3) | (1,4) | (1,8) |
|---|---|---|---|---|---|---|
| \( C \) | 0.3603 | 0.3610 | 0.3703 | 0.3735 | 0.3769 | 0.3777 |
| \( \ell \) | 122.02 | 115.26 | 96.43 | 75.12 | 57.12 | 41.70 |
| \( < k > \) | 5.4357 | 5.4357 | 5.4338 | 5.4275 | 5.4357 | 5.4357 |
| \( \gamma \) | 16.6 | 11.1 | 6.4 | 4.2 | 3.4 | 2.9 |
| \( d_f \) | 2.09 | 2.14 | 2.25 | 2.52 | 2.94 | 3.79 |

IV - CONCLUSION

In this work we explore some properties of a space filling network that come from a multifractal partition of the square lattice, the \( Q_{mf} \) network. An analysis of the distribution of the connectivity, \( P(k) \), assures that the \( Q_{mf} \) network shows a power-law tail that is more accentuated as increases the anisotropy of the underlying \( Q_{mf} \) tiling. Roughly the power-law tail of \( P(k) \) starts at \( k \sim 7 \). We remark that there is no regular lattice in 2 dimensions with \( k > 6 \) and typical Voronoi lattices have an exponential small number of vertices in this range. In addition the \( Q_{mf} \) network has a clustering coefficient that approaches a constant, \( C = 0.37 \pm 0.01 \), that does not depend on \( N \). This fact is in contrast to random networks that (for a constant \( < k > \)) have \( C \propto N^{-1} \). Because the value of \( C \) is much larger than the value of \( C \) of the associated random network we call the \( Q_{mf} \) network a complex network.

The most interesting aspect of the \( Q_{mf} \) network concerns its fractal behavior. For the average minimal path \( \ell \) we observe that \( N \propto \ell^{d_f} \) for the fractal dimension, \( d_f \). The simulations show that \( 2 < d_f < 4 \). The lower limit correspond to the case \( \rho = 1 \) where the multifractal tiling degenerates into the square lattice. In this situation the slope \( \gamma \) (from the power law \( P(k) \propto k^{-\gamma} \)) increases dramatically, this situation corresponds to the \( P(k) \) of the square lattice that is of the form of a Delta of Dirac. The opposite limit, \( \rho = 0 \), corresponding to very anisotropic tilings, presents comparatively small values of \( \gamma \).

We point that, diversly from [3], the \( Q_{mf} \) network shows an actual fractal behavior \( N \propto \ell^{d_f} \) that is obtained without any renormalization artefact. In the reference [3] an ingenious procedure is used to calculate two fractal dimensions \( d_B \) and \( d_f \). The first dimension depends on a suitable embedding in a metric space and a box counting methodology. The second is based on network distance and a mass (number of vertices) inside a given
radios. In our case, we have calculated $d_f$ in the standard way, the box-counting, however, depends on the methodology we use to make the embedding of the $Q_{mf}$ object. The simplest embedding is the $Q_{mf}$ lattice itself that is a 2-dimensional object and as a result $d_B = 2$. Note that the multifractal property of $Q_{mf}$ appears when we consider the subsets of blocks of same area, if we disregard the area set a bidimensional tiling assumes the trivial topologic dimension, $d_f = 2$.

The $Q_{mf}$ tiling is indeed a remarkable mathematical object, from a metric perspective it is a multifractal: it is formed by a denumerable quantity of sets of different areas each one with a given fractal dimension. In a topologic perspective the connections among the vertices (the cells of the tiling) form a fractal network. We remark that regular networks (generated from lattices for instance) and the Bethe tree satisfy the criterium $N \propto \ell^d$, but these structures are regular. For the best knowledgement of the authors the $Q_{mf}$ network is the only case of a true fractal, scale-free and with high clustering coefficient.

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