

VARIATION OF HODGE STRUCTURES FOR NON-KĀHLER MANIFOLDS

WEI XIA

Abstract. In this note, we discuss variation of Hodge structures on non-Kähler manifolds. In particular, given a holomorphic family of ∂̅-manifolds the period map is shown to be holomorphic and transversal.

1. Introduction

The theory of variation of Hodge structures (VHS for short) is a extensively studied subject in complex geometry and mathematical physics [GGK13, CMSP17, KP16, CK99]. The purpose of this note is to extend some parts of Griffiths’ classical theory of VHS [Gri68] to the non-Kähler setting. There are two motivations to do this, the first one is the works of Popovici and Anthes-Cattaneo-Rollenske-Tomassini [Pop19, ACRT18] where they showed local Torelli theorem holds for Calabi-Yau ∂̅-manifolds and ∂̅-complex symplectic manifolds, respectively; the second one is that we want to understand the proof of Voisin on a density criterion of complex projective manifolds [Voi05].

Given a holomorphic family of ∂̅-manifolds π : (X, X) → (B, 0) and 0 ≤ p ≤ k ≤ n =: dimC X, the period map is defined as follows

\[ \Phi_{p,k} : B \longrightarrow \text{Grass}(f_{p,k} H^k(X, \mathbb{C})), \quad t \mapsto F^p H^k(X_t), \]

where \( f_{p,k} := \dim F^p H^k(X) \) and

\[ F^p H^k(X_t) := \text{im} \left( \oplus_{r \geq p} H^{r,k-r}_{BC}(X) \longrightarrow H^k(X, \mathbb{C}) \right). \]

Theorem 1.1 (=Theorem 4.4). Let π : (X, X) → (B, 0) holomorphic family of ∂̅-manifolds and assume X is equipped with a fixed Hermitian metric.

(i) The period map \( \Phi_{p,k} \) as defined in (1.1) is holomorphic;

(ii) Griffiths transversality: the tangent map

\[ d\Phi_{p,k}^p : T_0 B \longrightarrow \text{Hom}(F^p H^k(X), H^k(X, \mathbb{C})/F^p H^k(X)) \]

has values in \( \text{Hom}(F^p H^k(X), F^{p-1} H^k(X)/F^p H^k(X)) \).

A main ingredient of this theorem is the fact that the exponential operator preserves Hodge structures, see Theorem 4.2.

\[ \text{Date: May 20, 2022.} \]

This work was supported by the National Natural Science Foundation of China No. 11901590 and Scientific Research Foundation of Chongqing University of Technology.
The exponential operator

$$e^{i\phi(t)} := \sum_{k=0}^{\infty} \frac{i^{k}}{k!}$$

which first appear in the work of Todorov (see Remark 2.4.9. in [Tod89, pp. 339]) turns out to be very useful for the studying of deformations of complex manifolds, see e.g. [Cle05, LSY09, LRY15, RZ18, Xia21b, LS20]. Let $\pi : (X, X) \rightarrow (B, 0)$ be a complex analytic family in the sense of Kodaira-Spencer, that is, $\pi$ is a holomorphic submersion which is surjective and proper over the connected complex manifold $B$. Assume $B$ is a small polydisc (centered in $0$) with coordinates $t = (t_1, t_2, \cdots)$, then by Ehresmann’s theorem we have the following commutative diagram

$$
\begin{array}{ccc}
X \times B & \xrightarrow{F} & X \\
\downarrow \pi & & \downarrow \pi \\
B & & B,
\end{array}
$$

where $F$ is a diffeomorphism. For any $t \in B$ set $f_t := F_{|X \times \{t\}}$, then $f_t$ is a diffeomorphism from $X$ to $X_t$. We use the isomorphism

$$f_t^{*} : A^{*}(X_t) \rightarrow A^{*}(X)$$

to identify forms/cohomology on $X_t$ with forms/cohomology on $X$. Let $z^{1}, \cdots, z^{n}$ and $w^{1}, \cdots, w^{n}$ be holomorphic coordinates on $X$ and $X_t$ respectively, the Beltrami differential can be defined by

$$(2.1) \quad \phi(t) := \left( \frac{\partial w^{\gamma}}{\partial z^{\alpha}} \right)^{-1} \frac{\partial w^{\gamma}}{\partial z^{\beta}} d\bar{z}^{\beta} \otimes \frac{\partial}{\partial z^{\alpha}} \in A^{0,1}(X, T^{1,0}) ,$$

where by abuse of notations, we write $w^{i} = f_t^{*}w^{i} = w^{i} \circ f_t$ for each $i = 1, 2, \cdots, n$. From

$$(2.2) \quad dw^{\beta} = \frac{\partial w^{\beta}}{\partial z^{\alpha}} dz^{\alpha} + \frac{\partial w^{\beta}}{\partial \bar{z}^{\alpha}} d\bar{z}^{\alpha} = \frac{\partial w^{\beta}}{\partial z^{\alpha}} (1 + i\phi(t)) dz^{\alpha},$$

and

$$e^{i\phi(t)} (dz^{i_1} \wedge \cdots \wedge dz^{i_p}) = (dz^{i_1} + i\phi(t) dz^{i_1}) \wedge \cdots \wedge (dz^{i_p} + i\phi(t)) dz^{i_p}),$$

We see that

$$(2.3) \quad e^{i\phi(t)} : A^{p,0}(X) \rightarrow A^{p,0}(X_t).$$

We will use the following formula (c.f. [Man04, LRY15, Xia19]):

$$(2.4) \quad d_{\phi(t)} := e^{-i\phi(t)} d e^{i\phi(t)} = \partial + \bar{\partial} \phi(t) = \partial + \bar{\partial} - L^{1,0}_{\phi(t)} ,$$

where $L^{1,0}_{\phi(t)} := i\phi(t) \partial - \partial i\phi(t)$ is the Lie derivative. A remarkable property about the exponential operator (as observed in [FM06, FM09, WZ20]) is that while it does not preserve all the types of $(p, q)$-forms, it does map elements in a filtration on $X$ to elements in the corresponding filtration on $X_t$:

$$(2.5) \quad e^{i\phi(t)} : F^{p}A^{k}(X) = \oplus_{p+\lambda \leq n} A^{\lambda, k-\lambda}(X) \rightarrow F^{p}A^{k}(X_t),$$
where \( n = \text{dim}_\mathbb{C} X \). In fact, let \( \varphi \in A^{\lambda,k-\lambda}(X) \) with \( p \leq \lambda \leq k \), locally we write \( \varphi = \varphi_I dz^I \wedge d\bar{z}^I \) where \( dz^I = dz^{i_1} \wedge \cdots \wedge dz^{i_p} \) and \( d\bar{z}^I = d\bar{z}^{j_1} \wedge \cdots \wedge d\bar{z}^{j_k} \), then
\[
e^{i\phi(t)}\varphi = \varphi_I(e^{i\phi(t)} dz^I) \wedge d\bar{z}^I
= \varphi_I(e^{i\phi(t)} dz^I) \wedge (\frac{\partial \bar{z}^{j_1}}{\partial w^l} dw^l + \frac{\partial \bar{z}^{j_1}}{\partial \bar{w}^l} d\bar{w}^l) \wedge \cdots \wedge (\frac{\partial \bar{z}^{j_k-\lambda}}{\partial w^l} dw^l + \frac{\partial \bar{z}^{j_k-\lambda}}{\partial \bar{w}^l} d\bar{w}^l),
\]
which is clearly an element in \( F^p A^k(X_t) \).

3. The canonical Bott-Chern deformation of \((p,q)\)-forms

The Bott-Chern cohomology of \( X \) is defined by
\[
H^{\bullet\bullet}_{BC}(X) := \frac{\ker d \cap A^{\bullet\bullet}(X)}{\text{im } \partial \cap A^{\bullet\bullet}(X)}.
\]

3.1. Hodge decomposition for Bott-Chern cohomology. Let \( X \) be a compact complex manifold equipped with a fixed Hermitian metric. The Bott-Chern Laplacian operator \( \Box_{BC} : A^{\bullet\bullet}(X) \to A^{\bullet\bullet}(X) \) is defined as
\[
\Box_{BC} := (\Box)(\partial\bar{\partial})^* + (\partial\bar{\partial})^*(\partial\bar{\partial}) + (\bar{\partial}^*\partial)(\partial\bar{\partial}) + (\bar{\partial}^*\partial)^*(\partial\bar{\partial}) + \bar{\partial}^*\partial + \partial^*\partial,
\]
Define the Bott-Chern harmonic space by
\[
\mathcal{H}_{BC} := \ker \Box_{BC} = \ker \partial \cap \ker \partial \cap \ker (\partial\bar{\partial})^*,
\]
then the following orthogonal direct sum decomposition holds:
\[
A^{\bullet\bullet}(X) = \ker \Box_{BC} \oplus \text{im } \partial\bar{\partial} \oplus (\text{im } \partial^* + \text{im } \bar{\partial}^*),
\]
which is equivalent to the existence of the Green operator \( G_{BC} \) such that
\[
1 = \mathcal{H}_{BC} + \Box_{BC} G_{BC},
\]
where \( \mathcal{H}_{BC} \) denotes the orthogonal projection operator to the Bott-Chern harmonic space.

3.2. Canonical Bott-Chern deformations. Recall the following

Definition 3.1. [Xia21b] Let \( \pi : (X, X) \to (B, 0) \) be a small deformation of a compact complex manifold \( X \) over a small polydisc \( B \) centered in 0 such that for each \( t \in B \) the complex structure on \( X_t \) is represented by Beltrami differential \( \phi(t) \). Given \( y \in \ker d \cap A^{p,q}(X_t) \) and \( T \subseteq B \), which is a complex subspace (i.e. analytic subset) of \( B \) containing 0, a Bott-Chern deformation of \( y \) (w.r.t. \( \pi : (X, X) \to (B, 0) \)) on \( T \) is a family of \((p,q)\)-forms \( \sigma(t) \in A^{p,q}(X_t) \) such that
1. \( \sigma(t) \) is holomorphic in \( t \in T \) and \( \sigma(0) = y \);
2. \( d_{\phi(t)}\sigma(t) = 0, \forall t \in T \).

We will always assume \( X \) has been equipped with a fixed Hermitian metric. A Bott-Chern deformation \( \sigma(t) \) of \( y \) on \( T \) is called canonical if
\[
\sigma(t) = \sigma_0 - G_{BC}(\bar{\partial}^*\partial^* + \bar{\partial}^*)\partial_i\phi(t)\sigma(t), \quad \forall t \in T.
\]
It can be shown that for any given \( \sigma_0 \in \mathcal{H}_{BC}^{p,q}(X) \), its canonical Bott-Chern deformation exists on an analytic subset of \( B \) and is uniquely given by a convergent power series as follows,

\[
\sigma(t) = \sum_k \sigma_k \in A^{p,q}(X) \quad \text{and} \quad \sigma_k = -G_{BC}(\bar{\partial}^* \partial^* + \bar{\partial}^*) \sum_{i+j=k} \partial_i \phi_j \sigma_i,
\]

where \( \phi(t) = \sum_j \phi_j \) and each \( \phi_j \), \( \sigma_k \) is a homogeneous polynomial of degree \( k \) with coefficients in \( A^{0,1}(X, T^{1,0}) \), \( A^{p,q}(X) \) respectively. Note that up to contraction to a smaller polydisc if possible we may assume the power series \( \sigma(t) \) converges on \( B \). It may happen that for some \( t \in B \), \( d_{\phi(t)} \sigma(t) \neq 0 \) or \( \sigma(t) \) is \( d_{\phi(t)} \)-exact. These are exactly the two facts which make \( \dim H_{BC\phi(t)}^{p,q}(X) \) different from \( \dim H_{BC}^{p,q}(X) \), where

\[
H_{BC\phi(t)}^{p,q}(X) := \frac{\ker d_{\phi(t)} \cap A^{p,q}(X)}{\im \partial \partial_{\phi(t)} \cap A^{p,q}(X)}
\]

is the deformed Bott-Chern cohomology and

\[
\partial_{\phi(t)} : A^{p,q}(X) \longrightarrow A^{p,q+1}(X),
\]

is the deformed Dolbeault operator defined by \( \partial_{\phi(t)} := \bar{\partial} - \bar{\partial}_{\phi(t)} \) (c.f. [Xia22, Xia21a])

In fact, for any \( t \in B \) and a vector subspace \( V \subseteq \mathcal{H}_{BC}^{p,q}(X) \), we set

\[
V_t := \{ \sigma_0 \in V \mid \sigma(t) \in \ker d_{\phi(t)} \}, \quad \text{where } \sigma(t) \text{ is the canonical deformation of } \sigma_0.
\]

The following mapping

\[
\tilde{f}_t : V_t \longrightarrow \ker (\partial \bar{\partial}^*) \cap \ker d_{\phi(t)} \cap A^{p,q}(X),
\]

\[
\sigma_0 \mapsto \sigma(t) = \sum_k \sigma_k, \quad \text{where } \sigma_k = -G_{BC}(\bar{\partial}^* \partial^* + \bar{\partial}^*) \sum_{i+j=k} \partial_i \phi_j \sigma_i, \quad \forall k \neq 0,
\]

is an isomorphism and

\[
\tilde{f}_t : V_t \longrightarrow \frac{\ker (\partial \bar{\partial}^*) \cap \ker d_{\phi(t)} \cap A^{p,q}(X)}{\ker (\partial \bar{\partial}^*) \cap \im \partial \partial_{\phi(t)} \cap A^{p,q}(X)} \cong H_{BC\phi(t)}^{p,q}(X),
\]

\[
\sigma_0 \mapsto [\tilde{f}_t \sigma_0].
\]

is surjective with kernel equal to \( \ker (\partial \bar{\partial}^*) \cap \im \partial \partial_{\phi(t)} \cap A^{p,q}(X) \). The following result give a precise description about the relation between the behavior of canonical Bott-Chern deformations and the changing of \( \dim H_{BC\phi(t)}^{p,q}(X) \) (see Theorem 4.13 and Proposition 4.3 in [Xia21b]):

**Theorem 3.2.** Let \( \pi : (X, X) \to (B, 0) \) be a small deformation of a compact Hermitian manifold \( X \) such that for each \( t \in B \) the complex structure on \( X_t \) is represented by a Beltrami differential \( \phi(t) \). For each \( (p, q) \in \mathbb{N} \times \mathbb{N} \), set

\[
u^{p,q}_t := \dim H_{BC}^{p,q}(X) - \dim \ker d_{\phi(t)} \cap \ker (\partial \bar{\partial})^* \cap A^{p,q}(X) \geq 0,
\]

and

\[
u^{p,q}_t := \dim H_{BC}^{p,q}(X) - \dim \ker \partial \partial_{\phi(t)} \cap (\mathcal{H}_{BC}^{p,q}(X) + \im (\partial \bar{\partial}^*)) \cap A^{p,q}(X) \geq 0,
\]

for each \( t \in B \). Then there exists a homotopy of the form

\[
\tilde{f}_t : V_t \longrightarrow \ker (\partial \bar{\partial}^*) \cap \im \partial \partial_{\phi(t)} \cap A^{p,q}(X)
\]

such that

\[
\tilde{f}_t : V_t \longrightarrow \ker (\partial \bar{\partial}^*) \cap \im \partial \partial_{\phi(t)} \cap A^{p,q}(X) \cong H_{BC\phi(t)}^{p,q}(X),
\]

\[
\sigma_0 \mapsto [\tilde{f}_t \sigma_0].
\]
then we have
\[ \dim \ker(\overline{\partial}^*) \cap \im \bar{\partial} \phi(t) \cap A^{p,q}(X) = \dim u_t^{p-1,q-1}, \]
and
\[ (3.4) \quad \dim H_{BC}^{p,q}(X) = \dim H_{BC \phi(t)}^{p,q}(X) + u_t^{p,q} + u_t^{p-1,q-1}. \]

3.3. Bott-Chern deformations on \( \partial \bar{\partial} \)-manifold. Let \( X \) be a compact complex manifold satisfying the \( \partial \bar{\partial} \)-lemma, see [DGMS75, Pop19, AT13] for more information. Such manifolds will be called a \( \partial \bar{\partial} \)-manifold.

**Proposition 3.3.** Let \( \pi : (X, X) \to (B, 0) \) be a deformation of a \( \partial \bar{\partial} \)-manifold \( X \) (equipped with a fixed Hermitian metric) over a small polydisc \( B \) such that for each \( t \in B \) the complex structure on \( X_t \) is represented by a Beltrami differential \( \phi(t) \). Then for any \( \sigma_0 \in H_{BC}^{p,q}(X) \) its canonical Bott-Chern deformation \( \sigma(t) = \sum_k \sigma_k \) exists on \( B \) and if \( \sigma(t) \in \im \bar{\partial} \phi(t) \) then \( \sigma_0 = 0 \).

**Proof.** First, since \( X \) satisfies the \( \partial \bar{\partial} \)-lemma, we have that \( X_t \) satisfies the \( \partial \bar{\partial} \)-\phi-lemma [Wu06, AT13] and also the \( \partial \bar{\partial} \phi(t) \)-lemma holds for each \( t \in B \) (see [Xia21b, Coro. 5.5]) which implies that
\[ \dim H_{BC \phi(t)}^{p,q}(X_t) = \dim H_{\phi(t)}^{p,q}(X_t) = \dim H_{BC}^{p,q}(X_t), \quad \forall t \in B, \]
where in the last equality we have used the deformation invariance of Hodge numbers for \( \partial \bar{\partial} \)-manifolds. Now by Theorem 3.2 we have
\[ u_t^{p,q} = \dim H_{BC(t)}^{p,q}(X) - \dim \ker d_{\phi(t)} \cap \ker(\overline{\partial}^*) \cap A^{p,q}(X) = 0, \]
and
\[ u_t^{p-1,q-1} = \dim \ker(\overline{\partial}^*) \cap \im \bar{\partial} \phi(t) \cap A^{p,q}(X) = 0. \]
The former equality says that for \( V = H_{BC}^{p,q}(X) \), the mapping
\[ \tilde{f}_t : H_{BC}^{p,q}(X) \to \ker(\overline{\partial}^*) \cap \ker d_{\phi(t)} \cap A^{p,q}(X) \]
defined as above is an isomorphism which shows that for any \( \sigma_0 \in H_{BC}^{p,q}(X) \) its canonical Bott-Chern deformation \( \sigma(t) = \sum_k \sigma_k \) is \( d_{\phi(t)} \)-closed for any \( t \in B \). The latter equality implies that if \( \sigma(t) \in \im \bar{\partial} \phi(t) \) then \( \sigma_0 = 0 \) in view of the mapping \( f_t \). \( \square \)

4. Deformations of Hodge filtration

On a \( \partial \bar{\partial} \)-manifold \( X \), there is a natural Hodge decomposition given by\(^1\)
\[ (4.1) \quad H^k(X, \mathbb{C}) = \oplus_{p+q=k} H_{BC}^{p,q}(X), \quad \text{with} \quad H_{BC}^{p,q}(X) := \frac{\ker d \cap A^{p,q}(X)}{\im \partial \bar{\partial} \cap A^{p,q}(X)}. \]

\(^1\)We could also use the pure type de Rham cohomology \( H_{d}^{p,q}(X) := \frac{\ker d \cap A^{p,q}(X)}{\im d \cap A^{p,q}(X)} \) in this decomposition. Here we prefer to adopt the Bott-Chern cohomology because later we need to apply the deformation theory of Bott-Chern classes established in [Xia21b]. Moreover, since \( \partial \bar{\partial} \)-lemma holds on \( X \) the natural map \( H_{BC}^{p,q}(X) \to H^k(X, \mathbb{C}) \) is injective, so we can regard \( H_{BC}^{p,q}(X) \) as a subspace of \( H^k(X, \mathbb{C}) \).
Together with $H^k(X, \mathbb{Z})/\text{Tor}$, where $\text{Tor}$ denotes the torsion subgroup of $H^k(X, \mathbb{Z})$, (4.1) define an integral Hodge structure of weight $k$. Equivalently, this Hodge structure can be determined by the Hodge filtration

$$H^k(X, \mathbb{C}) = F^0 H^k(X) \supseteq \cdots \supseteq F^k H^k(X) = H^{k,0}(X),$$

where

$$F^p H^k(X_t) := \text{im} \left( \oplus_{r \geq p} H^{r,k-r}_{BC}(X) \rightarrow H^k(X, \mathbb{C}) \right).$$

**Lemma 4.1.** Let $X$ be a $\partial\bar{\partial}$-manifold. If $\sigma \in F^p A^k(X) \cap \ker d$, then there exists $x \in A^{k-1}(X)$ and $\beta^{r,k-r} \in A^{r,k-r}(X) \cap \ker d$ for each $r \geq p$ such that

$$\sigma = dx + \sum_{r \geq p} \beta^{r,k-r}.$$

**Proof.** Since $X$ is a $\partial\bar{\partial}$-manifold, we have $[\sigma] \in H^k(X, \mathbb{C}) = \oplus_{p+q=k} H^{p,q}_{BC}(X)$ which implies that there exists $x \in A^{k-1}(X)$ and $\beta^{r,k-r} \in A^{r,k-r}(X) \cap \ker d$ for each $r \geq 0$ such that

$$\sigma = dx + \sum_{r \geq 0} \beta^{r,k-r}.$$

Now because $\sigma \in F^p A^k(X)$, we have

$$\sigma = \sum_{r \geq p} \partial x^{r-1,k-r} + \sum_{r \geq p} \bar{\partial} x^{r,k-r-1} + \sum_{r \geq p} \beta^{r,k-r},$$

where $x = \sum_{r \geq 0} x^{r,k-r}$ with $x^{r,k-r} \in A^{r,k-r}(X)$. Rearranging this identity, we get

$$\sigma = d \sum_{r \geq p + 1} x^{r-1,k-r} + \partial x^{p-1,k-p} + \sum_{r \geq p} \beta^{r,k-r},$$

which shows that $\partial x^{p-1,k-p} \in A^{p,k-p}(X) \cap \ker d$. Hence $\partial x^{p-1,k-p} = \partial \bar{\partial} y^{p-1,k-p-1}$ for some $y^{p-1,k-p-1} \in A^{p-1,k-p-1}(X)$ by the $\partial\bar{\partial}$-lemma. The conclusion follows. \qed

**Theorem 4.2.** Let $\pi : (X, X_t) \rightarrow (B, 0)$ be a deformation of a $\partial\bar{\partial}$-manifold $X$ (equipped with a fixed Hermitian metric) over a small polydisc $B$ such that for each $t \in B$ the complex structure on $X_t$ is represented by a Beltrami differential $\phi(t)$. For any $t \in B$ and $p \leq k$, the exponential operator induces the following isomorphism of vector spaces (still denoted by $e^{i\phi(t)}$)

$$e^{i\phi(t)} : H^k(X, \mathbb{C}) \rightarrow H^k(X_t, \mathbb{C}) : [\sigma_0] \mapsto [e^{i\phi(t)}\sigma(t)],$$

so that

$$e^{i\phi(t)} F^p H^k(X) = F^p H^k(X_t),$$

where $\sigma(t)$ is the canonical Bott-Chern deformation of $\sigma_0$.

**Proof.** First of all, note that (4.3) is the composition of the following two mappings

$$H^k(X, \mathbb{C}) \rightarrow H^k_{d\phi(t)}(X) : [\sigma_0] \mapsto [\sigma(t)],$$

where $\sigma(t)$ is the canonical Bott-Chern deformation of $\sigma_0$ and

$$H^k_{d\phi(t)}(X) \rightarrow H^k(X_t, \mathbb{C}) : \sigma \mapsto [e^{i\phi(t)}\sigma],$$

and $\sigma(t)$ is the canonical Bott-Chern deformation of $\sigma_0$. \qed
where \( H^k_{\partial\bar{\partial}}(X) := \ker d_{\partial\bar{\partial}} \cap A^k(X) \). Since \( X \) is a \( \partial\bar{\partial} \)-manifold, it follows from Proposition 3.3 that (4.4) is well-defined. To show (4.3) is an isomorphism we only need to show (4.4) is injective because (4.5) is clearly an isomorphism and \( \dim H^k(X, \mathbb{C}) = \dim \mathcal{H}^r_{BC}(X) \). Let \( \sigma_0 \in \mathcal{H}^r_{BC}(X) \subset H^k(X, \mathbb{C}) \) be a Bott-Chern harmonic form, if \( \sigma_0 \neq 0 \), then by the \( \partial\bar{\partial}_\phi \)-lemma [Xia21b, Coro. 5.5] we know that \( \sigma(t) \in \im \partial\bar{\partial}_\phi \). Thus we must have \( \sigma_0 = 0 \) by Proposition 3.3. This shows that (4.3) is an isomorphism.

It is left to show the image of the subspace \( F^p H^k(X) \subseteq H^k(X, \mathbb{C}) \) under \( e^{\phi(t)} \) is exactly \( F^p H^k(X_t) \subseteq H^k(X_t, \mathbb{C}) \). Indeed, let \( \sigma_0 \in \mathcal{H}^r_{BC}(X) \) be a Bott-Chern harmonic form with \( r \geq p \), then it follows from (2.5) that \( e^{\phi(t)} \sigma(t) \in F^p A^k(X_t) \cap \ker d \). Since \( X_t \) is also a \( \partial\bar{\partial} \)-manifold, we have that \( [e^{\phi(t)} \sigma(t)] \in F^p H^k(X_t) \) by Lemma 4.1. This shows \( e^{\phi(t)} F^p H^k(X) \subseteq F^p H^k(X_t) \). On the other hand, since \( H^*_{BC}(X) \) is deformation invariant for \( \partial\bar{\partial} \)-manifolds, we have

\[
\dim F^p H^k(X) = \sum_{r \geq p} \dim H_{BC}^{r,k-r}(X) = \sum_{r \geq p} \dim H_{BC}^{r,k-r}(X_t) = \dim F^p H^k(X_t).
\]

The proof is complete. \( \square \)

**Remark 4.3.** Note that the mapping (4.3) may depend on the choices of the Hermitian metric on \( X \).

In the context of Theorem 4.2, the **period map** can be defined as follows

\[(4.6) \quad \phi^{p,k} : B \to \text{Grass}(F^p H^k(X, \mathbb{C})), \quad t \mapsto F^p H^k(X_t),\]

where \( F^p H^k(X) := \dim F^p H^k(X) \). The period map \( \phi^{p,k} \) has the following properties (c.f. [Voï02, Chp. 10] for case of Kähler manifolds),

**Theorem 4.4.** Let \( X \) be a \( \partial\bar{\partial} \)-manifold equipped with a fixed Hermitian metric.

(i) The period map \( \phi^{p,k} \) is holomorphic;

(ii) Griffiths transversality: the tangent map

\[d\phi^{p,k}_0 : T_0 B \to \text{Hom}(F^p H^k(X), H^k(X, \mathbb{C})/F^p H^k(X))\]

has values in \( \text{Hom}(F^p H^k(X), F^{p-1} H^k(X)/F^p H^k(X)) \);

(iii) The following diagram is commutative:

\[
\begin{array}{ccc}
T_0 B & \xrightarrow{\kappa} & H^1(X, T^1_X) \\
\downarrow{d\phi^{p,k}_0} & & \downarrow{\mathcal{H}_{\bar{\partial}}} \\
\text{Hom} \left( \mathcal{H}^{r,k-r}_{BC}(X), H^{r-1,k-r+1}_{BC}(X) \oplus H^{r,k-r}_{BC}(X) \right) & \xrightarrow{\iota} & \mathcal{H}^{0,1}_{\bar{\partial}}(X, T^1_X),
\end{array}
\]

where \( \kappa \) is the Kodaira-Spencer map, \( \mathcal{H}_{\bar{\partial}} \) is the \( \bar{\partial} \)-harmonic projection map and \( \iota \) is defined as follows: for any \( \varphi \in \mathcal{H}^{0,1}_{\bar{\partial}}(X, T^1_X) \),

\[
\iota(\varphi) : H^{r,k-r}_{BC}(X) \to H^{r-1,k-r+1}_{BC}(X) \oplus H^{r,k-r}_{BC}(X) \\
x \mapsto (i \varphi x - \bar{\partial}(\bar{\partial}\bar{\partial})^{}* G_{BC} \partial i \varphi x) - \mathcal{H}^{r,k-r}_{BC} \partial(\bar{\partial})^{}* G_{BC} \partial i \varphi x.
\]
Proof. These statements can be deduced from Theorem 4.2. In fact, for any $t \in B$ we now have
\[
\Phi^{p,k}(t) = F^pH^k(X_t) = \mathbb{C}\{e^{i\phi(t)}\sigma^l(t)\}_{l=1}^{p,k},
\]
where $\{\sigma^l\}_{l=1}^{p,k}$ is a basis of $\bigoplus_{r \geq p} \mathcal{H}_BC^{r-k-r}(X)$ such that for each $q \geq p,$
\[
\mathbb{C}\{|\sigma^l_0| 1 + \sum_{r=p}^{q-1} \dim \mathcal{H}_BC^{r-k-r}(X) \leq l \leq \sum_{r=p}^{q} \dim \mathcal{H}_BC^{r-k-r}(X)\} = \mathcal{H}_BC^{a,k-q}(X)
\]
and $\sigma^l(t)$ is the canonical deformation of $\sigma^l_0.$ This shows $\Phi^{p,k}$ is holomorphic because both $\phi(t)$ and each $\sigma^l(t)$ are holomorphic. This is (i).

Next, let $\sigma(t)$ be the canonical deformation of some $\sigma_0 \in \mathcal{H}_BC^{r-k-r}(X)$ with $r \geq p.$ Without loss of generality assume dim $B = 1$ and we can write $\phi(t) = \sum_{i} \phi_j t^j,$ $\sigma(t) = \sum_{k \geq 0} \sigma_k t^k,$ where $\sigma_k = -G_{BC}(\tilde{\sigma}^* \partial \sigma^{*} + \tilde{\sigma}^*) \sum_{i+j=k} \partial i_{\phi_j} \sigma_i.$ So we have
\[
\frac{\partial}{\partial t} \big|_{t=0} \sigma(t) = \sigma_1 = -G_{BC}(\tilde{\sigma}^* \partial \sigma^{*} + \tilde{\sigma}^*) \partial i_{\phi_1} \sigma_0.
\]
Now if we differentiate $e^{i\phi(t)}\sigma(t)$ with respect to $t$ at $t = 0,$ we get
\[
(4.7) \quad \frac{\partial}{\partial t} \big|_{t=0} [e^{i\phi(t)}\sigma(t)] = [i_{\phi_1} \sigma_0 - G_{BC}(\tilde{\sigma}^* \partial \sigma^{*} + \tilde{\sigma}^*) \partial i_{\phi_1} \sigma_0].
\]
Notice that since $(i_{\phi_1} \sigma_0 - G_{BC}(\tilde{\sigma}^* \partial \sigma^{*} + \tilde{\sigma}^*) \partial i_{\phi_1} \sigma_0) \in \ker d,$ it follows that for $y := G_{BC}(\tilde{\sigma}^* \partial \sigma^{*} + \tilde{\sigma}^*) \partial i_{\phi_1} \sigma_0 \in A^{r-k-r}(X),$ we have
\[
\partial y = 0 = \partial i_{\phi_1} \sigma_0 \quad \text{and} \quad \partial i_{\phi_1} \sigma_0 = \partial y,
\]
which by the $\partial \bar{\partial}$-lemma implies that there exists $u \in A^{r-1,k-r}(X), \tau \in A^{r-1,k-r+1}(X) \cap \ker \partial$ and $\mu \in A^{r,k-r}(X) \cap \ker \partial$ such that
\[
i_{\phi_1} \sigma_0 = \bar{\partial} u + \tau \quad \text{and} \quad y = -\partial u + \mu.
\]
From which we may observe that $\tau \in A^{r-1,k-r+1}(X) \cap \ker d$ and $\mu \in A^{r,k-r}(X) \cap \ker d.$ Hence, we have the following equality in $H^k(X, \mathbb{C}):$
\[
[i_{\phi_1} \sigma_0 - y] = [\tau] - [\mu] \in F^{p-1}H^k(X).
\]
This is (ii).

For (iii), since
\[
[\tau] = [i_{\phi_1} \sigma_0 - \bar{\partial} u] \in H^{r-1,k-r+1}_BC(X),
\]
and
\[
[\mu] = [H^{r,k-r}_BC(y + \partial u)] = [H^{r,k-r}_BC \partial u] \in H^{r,k-r}_BC(X),
\]
by using the Hodge decomposition $H^k(X, \mathbb{C}) = \bigoplus_{p+q=k} H^{p,q}_BC(X),$ we have the following equality in $H^k(X, \mathbb{C}):$
\[
[i_{\phi_1} \sigma_0 - y] = [\tau] - [\mu] = [i_{\phi_1} \sigma_0 - \bar{\partial} u] - [H^{r,k-r}_BC \partial u] \in H^{r-1,k-r+1}_BC(X) \oplus H^{r,k-r}_BC(X).
\]
Since $u \in A^{r-1,k-r}(X)$ is a solution of $\partial \bar{\partial} u = \partial i_{\phi_1} \sigma_0,$ we may set $u = (\partial \bar{\partial})^* G_{BC} \partial i_{\phi_1} \sigma_0.$
Note that \( \{ e^{i\psi(t)} \sigma^i(t) \}_{i=1}^{p,h} \) is a holomorphic frame of the Hodge bundle \( F^p := \bigcup_{t \in B} F^p H^k(X_t) \) (c.f. [LS18]).

**References**

[ACRT18] B. Anthes, A. Cattaneo, S. Rollenske, and A. Tomassini. \( \partial \bar{\partial} \)-complex symplectic and Calabi-Yau manifolds: Albanese map, deformations and period maps. *Ann. Global Anal. Geom.*, 54(3):377–398, 2018.

[AT13] D. Angella and A. Tomassini. On the \( \partial \bar{\partial} \)-lemma and Bott-Chern cohomology. *Invent. Math.*, 192(1):71–81, 2013.

[CK99] David A. Cox and Sheldon Katz. *Mirror symmetry and algebraic geometry*, volume 68 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 1999.

[Cle05] H. Clemens. Geometry of formal Kuranishi theory. *Adv. Math.*, 198(1):311–365, 2005.

[CMSP17] J. Carlson, S. Müller-Stach, and C. Peters. *Period mappings and period domains*, volume 168 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2017. Second edition.

[DGMS75] P. Deligne, P. Griffiths, J. Morgan, and D. Sullivan. Real homotopy theory of Kähler manifolds. *Invent. Math.*, 29(3):245–274, 1975.

[FM06] D. Fiorenza and M. Manetti. L-infinity algebras, cartan homotopies and period maps. arXiv:math/0605297, 2006.

[FM09] D. Fiorenza and M. Manetti. A period map for generalized deformations. *J. Noncommut. Geom.*, 3(4):579–597, 2009.

[GGK13] M. Green, P. Griffiths, and M. Kerr. *Hodge theory, complex geometry, and representation theory*, volume 118 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2013.

[Gri68] P. Griffiths. Periods of integrals on algebraic manifolds. ii: Local study of the period mapping. *Amer. J. Math.*, 90(54):805–865, 1968.

[KP16] M. Kerr and G. Pearlstein, editors. *Recent advances in Hodge theory*, volume 427 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, Cambridge, 2016.

[LRY15] K. Liu, S. Rao, and X. Yang. Quasi-isometry and deformations of Calabi-Yau manifolds. *Invent. Math.*, 199(2):423–453, 2015.

[LS18] K. Liu and Y. Shen. Moduli spaces as ball quotients I, local theory. arXiv:1810.10892, 2018.

[LS20] K. Liu and Y. Shen. Canonical sections of Hodge bundles on moduli spaces. *J. Iranian Math. Soc.*, 1(1):97–115, 2020.

[LSY09] K. Liu, X. Sun, and S.-T. Yau. Recent development on the geometry of the Teichmüller and moduli spaces of Riemann surfaces. In *Geometry of Riemann surfaces and their moduli spaces*, volume XIV of *Surveys in differential geometry*, pages 221–259. 2009.

[Man04] M. Manetti. Lectures on deformations of complex manifolds (deformations from differential graded viewpoint). *Rend. Mat. Appl. (7)*, 24(1):1–183, 2004.

[Pop19] D. Popovici. Holomorphic deformations of balanced Calabi-Yau \( \partial \bar{\partial} \)-manifolds. *Ann. Inst. Fourier (Grenoble)*, 69(2):673–728, 2019.

[RZ18] S. Rao and Q. Zhao. Several special complex structures and their deformation properties. *J. Geom. Anal.*, 28(4):2984–3047, 2018.

[Tod89] A. N. Todorov. The Weil-Petersson geometry of the moduli space of SU(\( \geq 3 \)) (Calabi-Yau) manifolds i. *Comm. Math. Phys.*, 126(2):325–346, 1989.

[Voi02] C. Voisin. *Hodge theory and complex algebraic geometry. I*, volume 76 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2002. Translated from the French original by Leila Schneps.
[Voi05] C. Voisin. Hodge theory and the topology of compact Kähler and complex projective manifolds. In *Lecture Notes for the Seattle AMS Summer Institute*, 2005.

[Wu06] C.-C. Wu. *On the geometry of superstrings with torsion*. ProQuest LLC, Ann Arbor, MI, 2006. Thesis (Ph.D.)–Harvard University.

[WZ20] D. Wei and S. Zhu. Note on invariance of Hodge numbers for complex manifolds. preprint, 2020.

[Xia19] W. Xia. Derivations on almost complex manifolds. *Proc. Amer. Math. Soc.*, 147:559–566, 2019. Errata in arXiv:1809.07443v3.

[Xia21a] Wei Xia. Deformations of Dolbeault cohomology classes for Lie algebra with complex structures. *Ann. Global Anal. Geom.*, 60(3):709–734, 2021.

[Xia21b] Wei Xia. On the deformed Bott-Chern cohomology. *J. Geom. Phys.*, 166:Paper No. 104250, 19, 2021.

[Xia22] Wei Xia. Deformations of Dolbeault cohomology classes. *Math. Z.*, 300:2931–2973, 2022.

Wei Xia, Mathematical Science Research Center, Chongqing University of Technology, Chongqing, P.R.China, 400054.

Email address: xiawei@cqut.edu.cn, xiaweiwei3@126.com