Noncommutative K3 Surfaces

Hoi Kim
Topology and Geometry Research Center, Kyungpook National University,
Taegu 702-701, Korea

and

Chang-Yeong Lee
Department of Physics, Sejong University, Seoul 143-747, Korea

ABSTRACT

We consider deformations of a toroidal orbifold $T^4/\mathbb{Z}_2$ and an orbifold of quartic in $\mathbb{C}P^3$. In the $T^4/\mathbb{Z}_2$ case, we construct a family of noncommutative K3 surfaces obtained via both complex and noncommutative deformations. We do this following the line of algebraic deformation done by Berenstein and Leigh for the Calabi-Yau threefold. We obtain 18 as the dimension of the moduli space both in the noncommutative deformation as well as in the complex deformation, matching the expectation from classical consideration. In the quartic case, we construct a $4 \times 4$ matrix representation of noncommutative K3 surface in terms of quartic variables in $\mathbb{C}P^3$ with a fourth root of unity. In this case, the fractionation of branes occurs at codimension two singularities due to the presence of discrete torsion.

1hikim@gauss.knu.ac.kr
2cylee@sejong.ac.kr
I. Introduction

Starting from mid eighties, there have been some works on noncommutative manifolds, notably on noncommutative torus \([1, 2]\). Yang-Mills solutions on noncommutative two torus were described as isomorphic to the commutative underlying torus by Connes and Rieffel \([1]\) and the case for higher dimensional tori has been dealt with Rieffel \([2]\).

After the work of Connes, Douglas, and Schwarz \([3]\), noncommutative space has been a focus of recent interest among high energy physicists in relation with string/M theory. The connections between string theory and noncommutative geometry \([4]\) and Yang-Mills theory on noncommutative space have been studied by many people \([3]\). However, much of these works have been related with noncommutative tori, the most well-known noncommutative manifold, or noncommutative \(\mathbb{R}^n\). Relevant to compactification, noncommutative tori in particular have been the main focus \([3, 7, 8]\). Among higher dimensional tori, Hoffman and Verlinde \([7]\) first described the moduli space of noncommutative four torus in the case of the projective flat connections from the viewpoint of physics, and more general solutions were obtained in Ref. \([8]\).

So far, physically more interesting noncommutative version of orbifolds or Calabi-Yau(CY) manifolds have been seldom addressed. Until recently, only a few cases of \(\mathbb{Z}_n\) type orbifolds of noncommutative tori have been studied \([3, 10, 11]\). Orbifolds of four tori by a discrete group were studied by Konechny and Schwarz \([11]\). They determined the K group of them. In Ref. \([11]\), projective modules on these \(T^4_\theta\) were explicitly constructed following the methods of Rieffel \([2]\), then the dual structure of \(\mathbb{Z}_2\) orbifolds of them was considered.

Recently, Berenstein, Jejjala, and Leigh \([12]\) initiated an algebraic geometry approach to noncommutative moduli space. Then, Berenstein and Leigh \([13]\) discussed noncommutative CY threefold from the viewpoint of algebraic geometry. They considered two examples: a toroidal orbifold \(T^6/\mathbb{Z}_2 \times \mathbb{Z}_2\) and an orbifold of the quintic in \(\mathbb{CP}^4\), each with discrete torsion \([14, 15, 16, 17, 18]\). There, they explained the fractionation of branes at singularities from noncommutative geometric viewpoint, when discrete torsion is present. This case is different from the fractionation of branes due to the resolution of singularities in the orbifolds without discrete torsion. When discrete torsion is present, singularities cannot be resolved.
by blow-up process [15]. Orbifold singularities without discrete torsion can be resolved by blow-ups. In Ref. [19] the orbifold singularities in a K3 surface were studied and it was shown that the K3 moduli space is related with two form $B$-field. The sixteen fixed points of $T^4/Z_2$ can be blown-up to give a smooth K3 surface. And among its 22-dimensional moduli space where two-form $B$-field lives, 16 components of $B$ come from the twisted sector due to orbifolding. Then Douglas and company [20, 21] showed that this orbifold resolution gives rise to fractional branes.

Considering a D-brane world volume theory in the presence of discrete torsion, Douglas [16] found that the resolution of singularities agrees with what Vafa and Witten [15] previously proposed. In this process, he found a new type of fractional branes bound to the singularities. Berenstein and Leigh [13] successfully described this type of fractionation of branes from noncommutative geometric viewpoint, in response to the issue raised by Douglas [16]. Douglas pointed out that there had been no satisfactory understanding of discrete torsion from geometric viewpoint and suggested noncommutative geometric approach as a possible wayout.

In Ref. [13], Berenstein and Leigh first considered the $T^6/Z_2 \times Z_2$ case and recovered a large slice of the moduli space of complex structures of the CY threefold from the deformation of the noncommutative resolution of the orbifolds via central extension of the local algebra of holomorphic functions. Then, they considered the orbifolds of the quintic Calabi-Yau threefolds and constructed an explicit representation of a family of the noncommutative algebra.

Here, we apply this algebraic approach to K3 surfaces in the cases of the orbifolds $T^4/Z_2$ and the orbifolds of the quartics in $\mathbb{CP}^3$. In the first example, we did in two steps. As a preliminary step, we follow Berenstein and Leigh [13] obtaining a similar result. We deformed $E_1 \times E_2$ yielding the two dimensional complex deformation of Kummer surface whose moduli space is of dimension 2. As a main step, we construct a family of noncommutative K3 surfaces by algebraically deforming $T^4/Z_2$ in both complex and noncommutative ways at the same time. Our construction shows that the dimensions of moduli space for both the complex structures and the noncommutative deformation are the same, 18. And this is the dimension of the moduli space of the complex structures of K3 surfaces constructed with
two elliptic curves. In the commutative K3 case, the moduli space for the K3 space itself has been known already (see for instance [22]), and even the moduli space for the bundles on K3 surfaces has been studied [23]. In the first example of CY threefold case [13], the three holomorphic coordinates $y_i$ anticommute with each other to be compatible with $Z_2$ discrete torsion. Thus, in this case the central extension of the local algebra can be understood as possible deformations of $su(2)$ in the noncommutative direction for the underlying 6 torus, but the deformations of the CY threefold were done via the newly obtained center from the central extension by which most of the moduli space of complex deformations was obtained. However, in our case we obtained the same dimension of the moduli spaces both in the complex deformation and in the noncommutative deformation.

In the case of the orbifold of the quartic in $\mathbb{C}P^3$, we obtain the result similar to the quintic threefold case: We see the fractionation of branes at codimension two singularities instead of codimension three singularities. And this case corresponds to a generalization of $su(2)$ deformation.

Our main methods are noncommutative algebraic geometry after Refs. [13] and [12]. In Refs. [13, 12], the algebra of holomorphic functions on a non-commutative algebraic space was used instead of (pre) $C^*$ algebra of (smooth) continuous functions. Then the center of the algebra is the commutative part inside the noncommutative algebra. They correspond to two geometries: a commutative geometry on which closed strings propagate, and a noncommutative version for open strings. The center of the algebra describes locally a commutative Calabi-Yau manifold with possible singularities. Here, our approach is a little different from theirs. We deform the invariant polynomials of the K3 surface itself, such that its center is $\mathbb{P}^1 \times \mathbb{P}^1$ not the classical K3 surface itself, unlike the Berenstein and Leigh’s case [13, 12].

In section II, we construct a family of noncommutative orbifolds of $T^4/Z_2$, and obtain the moduli space of noncommutative K3 (Kummer) surfaces. In section III, we deal with the orbifold of quartic surfaces in $\mathbb{C}P^3$. Here, we construct a four dimensional representation of a family of noncommutative algebra. In section IV, we conclude with discussion.
II. Orbifold of the torus

In this section, we consider the orbifold of $T^4/\mathbb{Z}_2$. Here we will use the compact orbifold as a target space for D-branes and consider $T^4$ as the product of two elliptic curves, each given in the Weierstrass form

$$y_i^2 = x_i(x_i - 1)(x_i - a_i)$$

(1)

with a point added at infinity for $i = 1, 2$. The $\mathbb{Z}_2$ will act by $y_i \to \pm y_i$ and $x_i \to x_i$ so that the holomorphic function $y_1 y_2$ is invariant under the orbifold action. This satisfies the CY condition on the quotient space. The four fixed points of the orbifold at each torus are located at $y_i = 0$ and at the point of infinity. By the following change of variables, the point at infinity is brought to a finite point:

$$y_i \to y_i' = \frac{y_i}{x_i},$$

$$x_i \to x_i' = \frac{1}{x_i}.$$ (2)

It is known that $H^2(\mathbb{Z}_m \times \mathbb{Z}_n, U(1)) = \mathbb{Z}_d$, where $d$ is the greatest common divisor of $m$ and $n$ \cite{24,18}. So, in this case there is no discrete torsion. In the CY threefold case, $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ has discrete torsion, and the presence of discrete torsion requires that $y_i$ ($i = 1, 2, 3$) for three elliptic curves be anti-commuting variables, where these variables can be represented with $su(2)$ generators.

In the $T^4/\mathbb{Z}_2$ case, we consider the quotient space by the invariant polynomials. They are $x_1, x_2$ and $y_1 y_2 = t$ with the constraints $t^2 = f_1(x_1) f_2(x_2)$ coming from $y_1^2 = f_1(x_1)$ and $y_2^2 = f_2(x_2)$. This is the singular Kummer surface doubly covering $\mathbb{P}^1 \times \mathbb{P}^1$ with 4 parallel lines meeting 4 parallel lines once as branch locus. These are just the locus of zeroes of $f_1(x_1) f_2(x_2)$, where $f_i$ is considered to be of degree 4 in $\mathbb{P}^1$ instead of $\mathbb{C}$ including the infinity.

In our approach for noncommutative K3 surface, we do in two steps. As a preliminary step, following the line of Berenstein and Leigh \cite{13}, we deform the covering space such that the center of the deformed algebra corresponds to the commutative K3 surface. In this process, the complex structure of the center is also deformed as a consequence of the
covering space deformation. As a main step, we deform the K3 (Kummer) surface itself in the noncommutative direction.

In the first step, we consider two variables for the classical variable $t = y_1 y_2$, namely $t_1$ for $y_1 y_2$ and $t_2$ for $y_2 y_1$. In the classical (commutative) case, both $y_1 y_2$ and $y_2 y_1$ are invariant under the $Z_2$ action, and they are the same. Now, we consider a variation of four torus with $y_1 y_2 + y_2 y_1 = 0$. Then the invariant polynomials under $Z_2$ action are generated by $x_1, x_2, t_1 = y_1 y_2 = -y_2 y_1 = -t_2$ and they satisfy $t_1^2 = t_2^2 = -f_1 f_2$. Now, we deform it into

$$y_1 y_2 + y_2 y_1 = P_0(x_1, x_2) \quad (3)$$

where $P_0$ is a polynomial of degree two for each variable. Then the subalgebra generated by invariant polynomials $(x_1, x_2, t_1, t_2 = -t_1 + P_0)$ is not the center but is a commuting subalgebra of the deformed algebra. They satisfy the condition $t_i^2 - P_0 t_i + f_1 f_2 = 0$, which can be rewritten as

$$(t_i - \frac{P_0}{2})^2 = \frac{P_0^2}{4} - f_1 f_2, \quad \text{for } i = 1, 2. \quad (4)$$

Since this is a commuting subalgebra, it has a geometrical meaning. It is a K3 surface doubly covering $P^1 \times P^1$ with branch locus $\frac{P_0^2}{4} - f_1 f_2 = 0$. Note that this is a polynomial of degree four for each $x_i, \quad i = 1, 2$. The noncommutative deformation of $E_1 \times E_2$ corresponding to the condition (3) induces a complex deformation of K3 surface corresponding to the condition (1). Now, we count the parameters of $P_0(x_1, x_2)$. It has 9 parameters since it is of degree two in each variable. But we must subtract one corresponding to multiplication by a constant and six coming from $PGL(2, \mathbb{C})$ action on both variables $x_1, x_2$. So, the number of the remaining degrees of freedom is two. Actually, both the dimension of the Kähler metric spaces in $E_1 \times E_2$ and the deformation dimension of the Kummer surface coming from $E_1 \times E_2$ are 2.

As a main step, we now consider the noncommutative deformation of K3 surface itself. From the previous consideration of commutative K3 surface, we may consider the case with four variables, $x_1, x_2, t_1, t_2$. This is simply because that there is no a priori reason that $t_1$ and $t_2$ be the same in the noncommutative case though they are the same in the classical case. Thus we begin with four independent variables for the noncommutative $T^4/\mathbb{Z}_2$. However, we do not know yet how to proceed to the noncommutative case in which all the variables are...
non-commuting each other. So, we consider a simpler case in which \( x_1, x_2 \) are commuting variables and belong to the center and only \( t_1, t_2 \) are non-commuting variables.

To proceed in that direction, we first consider the cases of the independent complex deformation for each \( t_i (i = 1, 2) \). Let us consider the case with \( t_1 \) but not with \( t_2 \) in which we deform the relation \( t_1^2 = f_1(x_1)f_2(x_2) \) to any polynomial \( h_1(x_1, x_2) \) of degree 4 in each variable. Then, for a generic \( h_1 \) it determines a smooth K3 surface doubly covering \( \mathbb{P}^1 \times \mathbb{P}^1 \). If we count the number of deformation parameters naively, we have 25 parameters since \( h_1 \) is a polynomial of degree 4 in each variable. However, we must ignore 1 parameter since multiplication by a constant defines an isomorphic K3 surface with exactly the same branch locus. We should also consider another symmetry giving isomorphic surfaces. These are \( PGL(2, \mathbb{C}) \) acting both \( \mathbb{P}^1 \) independently. Thus we have to subtract 6 parameters, and we are left with 18 parameters. This is what we expected since the moduli space of K3 surfaces doubly covering \( \mathbb{P}^1 \times \mathbb{P}^1 \) is of dimension 18. This is the classical(complex) deformation of the original singular Kummer surface.

We can consider in the same manner with \( t_2 \), by deforming \( t_2^2 = f_1(x_1)f_2(x_2) \) to any polynomial \( h_2(x_1, x_2) \) of degree 4 in each variable. Thus, this case also represents another complex deformation of the Kummer surface.

When we have both \( t_1 \) and \( t_2 \), and \( h_1(x_1, x_2) \neq h_2(x_1, x_2) \), the following two cases are seemingly possible.

\[
[t_1, t_2] = c(Z.A) \quad (5)
\]

Here, \( c(Z.A) \) is a function in the center of the local algebra, and “−” denotes commutator and “+” denotes anticommutator. However, our requirement that \( t_i^2 (i = 1, 2) \) belong to the center, namely \( [t_1, t_2] = 0 \), allows only the anticommutator case.

Since only two variables \( x_1, x_2 \) are in the center of the deformed algebra, the right hand side of (5) should be a polynomial and free of poles in each patch. Thus, (5) can be written in the following form

\[
t_1t_2 + t_2t_1 = P(x_1, x_2) \quad (6)
\]
where $P$ is a polynomial. The change of variables (2) on $x_i$, now changes into

$$t_j \rightarrow t'_j = \frac{t_j}{x_i^2}, \quad \text{for } j = 1 \text{ and } 2,$$

$$x_i \rightarrow x'_i = \frac{1}{x_i}, \quad \text{for } i = 1 \text{ or } 2,$$

and thus $P$ transforms as

$$P(x_1, x_2) \rightarrow x_1^4 P(1/x_1, x_2),$$

under the transform of $x_1$. Here, $P$ should be of degree at most four in $x_1$, since it has to transform into a polynomial. In a similar manner, it is easy to see that it is of degree at most four in $x_2$.

Now, we have a noncommutative K3 surface defined by $x_1, x_2, t_1, t_2$ whose noncommutativity is characterized by the relation (6), $t_1 t_2 + t_2 t_1 = P(x_1, x_2)$.

This is a deformation in another direction, a noncommutative deformation. Here, we also have 25 parameters by naive counting. But, due to the same reason as in the classical (complex) deformation case described before, we have only 18 parameters.

In order to understand the above obtained moduli space of noncommutative deformations, we first need to understand the algebraic and complex structures in the commutative case. The aspect in the commutative case can be understood by looking into $(p, q)$ forms preserved under the involution. In the present case, $(p, q) = (0, 0), (1, 1), (2, 2), (2, 0), (0, 2)$ are preserved under the involution. The dimension of preserved $(p, q)$ forms is one except the case of $(1, 1)$. The dimension of preserved $(1, 1)$ forms is four as in the four torus. The rank of the Picard group which is the intersection of $(1, 1)$ forms with the integral second cohomology class of the quotient space is at least two since we constructed the four torus with two elliptic curves. If we resolve singularities of the K3 surface, we have a two dimensional family of K3 surfaces whose Picard rank is at least 18, since we have additional 16 coming from 16 singular points. This corresponds to the so-called A model [15]. Since there is no torsion we can deform the singularities, and get a family of smooth K3 surfaces of dimension 18 described as above. Each member of this family has the Picard group containing those of the quotient spaces. This corresponds to the so-called B model [15]. Note that a K3 surface in the family we are considering is not generic since we started at a torus which is a product.
of two elliptic curves, not a generic torus. Vafa and Witten [15] considered the classical deformation of a Calabi-Yau threefold doubly covering \( \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \). Berenstein and Leigh [13] considered the anticommutation of the variables \( y_i \), while \( x_i \) are still commuting. Then they considered the center generated by four variables \( x_1, x_2, x_3, w = y_1y_2y_3 \) with the constraints \( w^2 = f_1(x_1)f_2(x_2)f_3(x_3) \) corresponding to Calabi-Yau threefolds. They deformed the 6 torus with anticommuting \( y_i \) and commuting \( x_i \) as \( y_iy_j + y_jy_i = P_{ij}(x_i, x_j) \) giving rise to the change of the center into another center generated by \( x_1, x_2, x_3, w' = y_1y_2y_3 + P_{13}y_2 - P_{23}y_1 - P_{12}y_3 \) with the constraints \( w'^2 = f_1f_2f_3 + h(x_1, x_2, x_3) \), where the function \( h \) is determined by \( P_{12}, P_{13}, P_{23} \). Notice that here \( y_i \) are not variables for Calabi-Yau threefold since they are not invariant under the action. So, the deformation of anticommuting variables affects the deformation of \( T^6 \) and the central Calabi-Yau threefold at the same time.

In our present case, \( t_1 \) for \( y_1y_2 \) and \( t_2 \) for \( y_2y_3 \) are all invariants of the K3 surface. In the preliminary step, we deformed \( E_1 \times E_2 \) noncommutatively giving rise to the two dimensional classical deformation of Kummer surfaces corresponding to the invariant polynomials which is not the center but is a commuting subalgebra. This is in line with Ref. [13] in the sense that noncommutative deformation of covering space yields complex deformation of the classical space which corresponds to the center of the deformed algebra. In the main step, we deformed the classical (commutative) space itself rather than the covering space. As a result the center in this case does not correspond to the classical space, and the deformation comprises both noncommutative deformation with anticommuting \( t_1, t_2 \) and complex deformation arising from the constraints on \( t_i^2, \ i = 1, 2 \). The dimensions of the moduli spaces of these deformations are 18 for both the noncommutative and complex deformation cases.

Our result in this section that the moduli space of the complex structures of the commutative K3 surfaces and the moduli space of the noncommutative K3 surfaces have the same dimension 18 seemingly suggests that there may be another kind of mirror symmetry in these deformations. However, since our deformation in the noncommutative direction is not as general as it can be, we leave this as an open question.
III. Orbifold of the quartic

In this section we consider a noncommutative version of an orbifold of the quartic in $\mathbb{CP}^3$. The complex structure moduli space of the quartic has a discrete symmetry group of $\mathbb{Z}_4 \times \mathbb{Z}_4$. The quartic is described by

$$\mathcal{P}(z_j) = z_1^4 + z_2^4 + z_3^4 + z_4^4 + \lambda z_1 z_2 z_3 z_4 = 0$$

where the $z_j$ $(j = 1, .., 4)$ are homogeneous coordinates on $\mathbb{CP}^3$. The $\mathbb{Z}_4 \times \mathbb{Z}_4$ action is generated by phases acting on the $z_j$ as $z_j \rightarrow w^{\bar{a}_j} z_j$, with $w^4 = 1$, and the vectors

$$\bar{a}_1 = (1, -1, 0, 0)$$
$$\bar{a}_2 = (1, 0, -1, 0)$$

consistent with the CY condition $\sum_{j=1}^4 a_j = 0 \mod 4$. Once we choose the action such that $z_4$ is invariant, then we can consider a coordinate patch where $z_4 = 1$. Here, the discrete torsion is given by $[24, 16]$

$$H^2(\mathbb{Z}_4^2, U(1)) = \mathbb{Z}_4$$

so we need a phase to determine the geometry.

Within the coordinate patch, $z_i^4$ and $z_1 z_2 z_3$ are invariant. We assume the following commutation relations among $z_j$

$$z_1 z_2 = \alpha z_2 z_1$$
$$z_1 z_3 = \beta z_3 z_1$$
$$z_2 z_3 = \gamma z_3 z_2,$$

and require that the above invariant quantities remain in the center of the noncommutative quotient space. From this requirement, one can easily see that

$$\alpha^4 = 1, \quad \beta = \alpha^{-1}, \quad \gamma = \alpha.$$  

Now, we consider a four-dimensional representation of this with the following matrices

$$P = \text{diag}(1, \alpha, \alpha^2, \alpha^3), \quad Q = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

10
where $\alpha$ is a fourth root of unity. In terms of $P$, $Q$ matrices, we put $z_1 = b_1 P$, $z_2 = b_2 Q$, and $z_3 = b_3 P^m Q^n$ where $b_j$ are arbitrary complex numbers. The integers $m, n$ can be determined from the requirement that $z_1 z_2 z_3 z_4$ belongs to the center of the algebra, and are given by $m = n = 3$. By the use of the commutation relation $PQ = \alpha QP$ and of $P^4 = Q^4 = 1$, the defining relation of the quartic (9) now becomes

$$b_1^4 + b_2^4 + \alpha^2 b_3^4 + \alpha \lambda b_1 b_2 b_3 = -1.$$  \hspace{1cm} (15)

Notice that only when $\alpha = 1$, (15) satisfies the same classical relation (9). When the phase $\alpha$ becomes $-1$ which is a fourth root of unity, the $z_j$’s become to anticommute, (12). In this case the algebra may be regarded as a deformation of $su(2)$. Indeed we have

$$z_i z_j = g_{ijk} z_k^{-1},$$  \hspace{1cm} (16)

where $g_{ijk}$ are totally antisymmetric scalars. Note that in $su(2)$, we have $z_i z_j = i f_{ijk} z_k = i f_{ijk} z_k^{-1}$, where we considered $su(2)$ as an algebra, not as a Lie algebra. This is in line with $T^6 \mathbb{Z}_2 \times \mathbb{Z}_2$ representation of the CY threefold with discrete torsion in Ref. [13] where algebras generated by three anticommuting variables were considered as deformations of $su(2)$.

If we think of the trace of the representation matrices as functions on the $b_i$ space with the defining equation above, then the representation becomes reducible only when two out of the three $z_i$ act by zero, which shows that there are fractional branes for the codimension two singularities. In the case of the quintic threefold, the representation becomes not reducible on the codimension two singularities. There the representation becomes reducible on the codimension three singularities where three out of the four $z_i$ act by zero. One may consider the coordinate transformations to other patches. It is not difficult to show that one can still do the standard coordinate changes of the quartic. One needs to be only careful with orderings of variables. In every coordinate patch the algebra is of the same type as (12) as in the quintic case.
IV. Discussion

In the first part of this paper, we studied the deformations of orbifolds of four tori which are products of two elliptic curves.

As a preliminary step, we followed Berenstein and Leigh [13] obtaining a similar result. We deformed $E_1 \times E_2$ yielding the two dimensional complex deformation of the Kummer surface. This agrees with the fact that the moduli space of the Kummer surface coming from the product of two elliptic curves is of dimension 2.

Then we deformed the singular K3 (Kummer) surface in both noncommutative and complex directions. We described the noncommutative K3 surfaces with the two non-commuting variables coming from one of the classically invariant variables. The squares of each of these two non-commuting variables belong to the center and represent complex deformations of the Kummer surfaces. Allowed deformations of the commutation relation for these two variables represent the noncommutative deformations. This result is different from the noncommutative deformation based on the $\mathbb{Z}_2$ orbifold of noncommutative four torus $T_\theta^4$ which has been carried out previously [11]. Here, we obtained 18 as the dimension of the moduli space of our noncommutative deformations. In the commutative case, the complete family of commutative K3 surfaces is of 20 dimension inside which algebraic K3 surfaces are of 19 dimension. However, the family coming from the $\mathbb{Z}_2$ quotient of $T^2 \times T^2$ is of 18 dimension. This shows that the dimension of the moduli space of our noncommutative deformations is the same as the one we get from the classical consideration of complex deformations.

Notice that what we have done here is a little different from the work of Berenstein and Leigh [13]. In their work they deformed the algebra of the covering space, in which the center of the deformed algebra corresponds to the classical commutative space of its undeformed algebra. On the other hand, we deformed the K3 surface itself rather than deforming the algebra of the covering space. As a consequence, its center is $\mathbb{P}^1 \times \mathbb{P}^1$ not the commutative K3 surface.

In the second part of the paper, we dealt with the orbifolds of the quartic K3 surfaces. In this case, we have nonvanishing discrete torsion. This turned out to be a deformation of $su(2)$ as in the $T^6/\mathbb{Z}_2 \times \mathbb{Z}_2$ case of Ref. [13]. We could find fractional branes on the
codimension two singularities rather than on the codimension three singularities as in the case of the quintic threefolds \[13\]. However, the reasoning was the same as in the quintic case.

We described almost all the complete family of K3 surfaces, but this does not fit to the most general algebraic K3 surfaces of 19 dimensional family. One should also find the suitable representation of the noncommutative algebra we used, so that one can find the dual noncommutative K3 and the symmetry group. Finally, from our result of the first part, one may speculate a kind of mirror symmetry between complex deformations and noncommutative deformations. Here, we will leave it as an open issue.

Acknowledgments

This work was supported by KOSEF Interdisciplinary Research Grant No. 2000-2-11200-001-5. We would like to thank Igor Dolgachev for useful discussions.

References

[1] A. Connes and M. Rieffel, Contemp. Math. 62 (1987) 237.

[2] M. Rieffel, Can. J. Math. Vol. XL (1988) 257.

[3] A. Connes, M.R. Douglas, and A. Schwarz, JHEP 9802 (1998) 003, hep-th/9711162.

[4] A. Connes, Noncommutative geometry (Academic Press, New York, 1994).

[5] See, for instance, N. Seiberg and E. Witten, JHEP 9909 (1999) 032, hep-th/9908142
and references therein for the development in this direction.

[6] D. Brace, B. Morariu, and B. Zumino, Nucl. Phys. B 545 (1999) 192, hep-th/9810099;
P.-M. Ho, Y.-Y. Wu, and Y.-S. Wu, Phys. Rev. D 58 (1998) 026006, hep-th/9712201.

[7] C. Hofman and E. Verlinde, Nucl. Phys. B 547 (1999) 157, hep-th/9810219.

[8] E. Kim, H. Kim, N. Kim, B.-H. Lee, C.-Y. Lee, and H. S. Yang, Phys. Rev. D 62 (2000) 046001, hep-th/9912272.
[9] A. Konechny and A. Schwarz, Nucl. Phys. B 591 (2000) 667, hep-th/9912185.
[10] A. Konechny and A. Schwarz, JHEP 0009 (2000) 005, hep-th/0005174.
[11] E. Kim, H. Kim, and C.-Y. Lee, J. Math. Phys. 42 (2001) 2677, hep-th/0005203.
[12] D. Berenstein, V. Jejjala, and R. Leigh, Nucl. Phys. B 589 (2000) 196, hep-th/0005087; Phys. Lett. B 493 (2000) 162, hep-th/0006168.
[13] D. Berenstein and R. G. Leigh, Phys. Lett. B 499 (2001) 207, hep-th/0009209.
[14] C. Vafa, Nucl. Phys. B 273 (1986) 592.
[15] C. Vafa and E. Witten, J. Geom. Phys. 15 (1995) 189, hep-th/9409188.
[16] M. R. Douglas, “D-branes and discrete torsion”, hep-th/9807233.
[17] M. R. Douglas and B. Fiol, “D-branes and discrete torsion II”, hep-th/9903031.
[18] J. Gomis, JHEP 0005 (2000) 006, hep-th/0001200.
[19] P. S. Aspinwall, Phys. Lett. B 357 (1995) 141, hep-th/9507012; “Resolution of orbifold singularities in string theory”, hep-th/9403123.
[20] M. Douglas, JHEP 9707 (1997) 004, hep-th/9612126.
[21] D. E. Diaconescu, M. Douglas, and J. Gomis, JHEP 9802 (1998) 013, hep-th/9712230.
[22] P. S. Aspinwall, “K3 surfaces and string duality”, TASI-96 lecture notes, hep-th/9611137.
[23] S. Mukai, “On the moduli space of bundles on K3 surfaces, I” in Vector Bundles on Algebraic Varieties, edited by M. Atiah et al. (Oxford Univ. Press, Oxford, 1985).
[24] G. Karpilovsky, Projective representations of finite groups (M. Dekker, New York, 1985).