Irradiated bilayer graphene

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Abstract
We describe the gated bilayer graphene system when it is subjected to intense terahertz frequency electromagnetic radiation. We examine the electron band structure and density of states via exact diagonalization methods within Floquet theory. We find that dynamical states are induced which lead to modification of the band structure. We first examine the situation where there is no external magnetic field. In the unbiased case, dynamical gaps appear in the spectrum which manifest as dips in the density of states. For finite inter-layer bias (where a static gap is present in the band structure of unirradiated bilayer graphene), dynamical states may be induced in the static gap. These states can show a high degree of valley polarization. When the system is placed in a strong magnetic field, the radiation induces coupling between the Landau levels which allows dynamical levels to exist. For strong fields, this means the Landau levels are smeared to form a near-continuum of states.

1. Introduction
Graphene [1] and its bilayer [2] have attracted much attention recently due to the novel fundamental physics that they display and huge scope that they have for device applications [3, 4]. In particular, electrons in monolayer graphene (whose low-energy quasi-particles mimic chiral Dirac fermions with linear dispersion) have displayed relativistic-like phenomena, including Klein tunnelling [5, 6] and the half-integer quantum Hall effect [7, 8]. Electrons in bilayer graphene show properties which are a hybrid between the monolayer and traditional two-dimensional semiconductor systems, since the low-energy quasi-particles are chiral, but the inter-layer coupling induces an effective mass and corresponding quadratic energy dispersion. A fully tunable gap can be opened at the charge neutrality point by application of a transverse electric field (either by gating [9] or by doping [2]), a feature which is unique to this system.

Monolayer graphene which is irradiated by monochromatic, high-intensity laser light has been studied theoretically for the bulk [10–15], in nanoribbons [16, 17] and in np junctions [18, 19]. Experimental investigations of few-layer graphene have also been carried out [20]. However, the transport properties of bilayer graphene which is irradiated by intense laser light have not been considered in much detail. Ryzhii et al proposed that a phototransistor could be implemented using bilayer graphene [21], Wright et al have found a large peak conductance in the terahertz and far-infrared frequency range for bilayer graphene nanoribbons [22], and we have previously suggested that valley-polarized electrons can be produced in gapped bilayer graphene [23].

In this paper, we give a comprehensive description of the band structure and density of states of irradiated bilayer graphene, both in zero external magnetic field and in a strong field. This is of fundamental physical importance, but also has application in the realm of devices and technology because of the growing consensus that graphene and its bilayer have vast potential in the fields of optoelectronics and photonics [4], and in the design of new electronic devices such as ambipolar transistors [24]. Also, the spin-like degrees of freedom in graphene (such as the lattice pseudospin and the valley) may allow for electronic implementations of the ideas of spintronics [25] which have been discussed in the literature [23, 26–30]. In all of these areas, a thorough understanding of the basic properties of irradiated bilayer graphene is an essential building block for the design and application of devices. In particular, this present work focuses on the generation of valley-polarized states which may be used as a filter for the generation of valley-polarized currents. This is an essential step in the realization of valleytronics devices.

We briefly outline the structure of our paper. In section 2 we describe the theoretical framework which we employ for both the zero-field and strong-field cases. Then we present and discuss the results of our calculations in section 3 before summarizing and placing our work in the context of valleytronic devices in section 4. Various important formulae and derivations are collected in the appendices.
2. Theoretical framework

The Hamiltonian of irradiated bilayer graphene is written as

$$\mathcal{H}(t) = H_0 + H_U + h(t)$$

where $H_0$ is the continuum limit of the tight binding Hamiltonian, $H_U$ describes the external electrostatic field due to a top gate or dopants and $h(t)$ is the time-dependent part which depends only on the irradiating field. We take the nearest-neighbour approximation of this $H_0$, but in principle any combination of hopping terms can be included by computing the eigenfunctions of the static Hamiltonian numerically. The static part of the Hamiltonian $H_0 + H_U$ determines single-particle wavefunctions $\psi_x$, which span the spatial part of the Hilbert space of solutions of $\mathcal{H}(t)$ and which we shall use as a basis for the time-dependent solutions $\Psi(t)$ of $\mathcal{H}(t)$. The energy spectrum and wavefunctions of the static Hamiltonian are presented in appendix A, both in zero external magnetic field and when a strong magnetic field quantizes the Hamiltonian. The static part of the Hamiltonian retain these three quantities as good quantum numbers which are called ‘Floquet states’. Substituting equation (3) into the time-dependent Schrödinger equation yields $\mathcal{H}(t)\Phi(t) = \varepsilon \Phi(t)$, an eigenvalue equation for the operator $\mathcal{F}(t) = \mathcal{H}(t) - i\hbar \frac{\partial}{\partial t}$ with $\mathcal{F}(t)\Psi(t) = 0$.

In order to solve the Schrödinger equation for $\mathcal{F}(t)$ and $\Phi(t)$, we consider an expanded Hilbert space $\mathcal{R} \otimes \mathcal{T}$ of square-integrable functions of space and functions of time with period $t_0$ (see [32] for a full description). In this space, the scalar product is defined as the regular spatial scalar product with the average over one period:

$$\langle \langle n|m \rangle \rangle = \frac{1}{t_0} \int_0^{t_0} \langle n|m \rangle \, dt$$

where $n$ and $m$ label arbitrary states in $\mathcal{R} \otimes \mathcal{T}$. Since the wavefunctions are periodic and the eigenfunctions of the static Hamiltonian $H_0 + H_U$ form a complete set for the spatial coordinates, we can write the wavefunctions as

$$\Phi_m(r, t) = \sum_{j, \mathcal{X}} e_{jX}^m \tilde{\Phi}_{jX}(r, t)$$

(4)

where $m$ labels the Floquet state and the Hilbert space $\mathcal{R} \otimes \mathcal{T}$ is spanned by the infinite set of functions:

$$\tilde{\Phi}_{jX}(r, t) = e^{i\Omega_j t} \psi_{X}(r)$$

such that $j \in \{ \ldots, -2, -1, 0, 1, 2, \ldots \}$, and $X$ labels the eigenstates of $H_0 + H_U$. The label $X$ contains all appropriate single-particle quantum numbers, but its exact composition depends on whether there is a magnetic field present in the system. The operator in equation (1) does not couple states with different momenta, different spins, or which are in different valleys. Therefore the Floquet states which result from diagonalization of the $t$-dependent Hamiltonian retain these three quantities as good quantum numbers. The Hamiltonian can be written as a matrix by computing the matrix elements of $\mathcal{F}(t)$ over these states. This yields an infinite-dimensional matrix which can be truncated for a sufficiently large number of terms in the Fourier expansion and numerically diagonalized to give quasienergies and wavefunctions to arbitrary precision. These matrix elements are discussed in appendix B.

We now introduce the two-time formalism which we use to compute the Green’s function and hence the density of states (DoS) in irradiated bilayer graphene. In this formalism, the time associated with the expanded Hilbert space (which was previously labelled $t$ but which we shall call $\zeta$ from now on) is separated from the evolution of the system such that the full time-dependent solution of the Schrödinger equation $\Psi(r, t)$ is [31]

$$\Psi(r, t) = \Psi(r, \zeta, t)|_{\zeta = 0}$$

(5)

The two-time wavefunction is then defined to be

$$\Psi(r, \zeta, t) = \exp \left(-\frac{i}{\hbar} \mathcal{F}(\zeta)(t - t') \right) \Psi(r, \zeta, t')$$

where $\Phi(r, t + t_0) = \Phi(r, t)$. The scalar constant $\varepsilon$ is called the quasienergy. This theorem is the temporal analogue of the Bloch theorem, so that the quasienergy is equivalent to the quasimomentum, and the time period is equivalent to the lattice constant of the reciprocal lattice. The periodic wavefunctions $\Phi$ are called ‘Floquet states’.
where $\mathcal{F}$ is the Floquet operator introduced earlier. Full time-dependent solutions are given by the limiting procedure in equation (5), but we shall generally be interested in the dynamics of the system on timescales much longer than $t_0$ so we shall instead take the time average with respect to the field.

In the Matsubara formalism, we utilize the grand canonical ensemble and define the associated energy scale $\kappa = \epsilon - \mu$, where $\mu$ is the chemical potential. The operator for this energy is $\mathcal{K} = \mathcal{F} - \mu \mathcal{N}$. The imaginary time $\tau = i\beta$ is defined and the evolution of the field operators associated with the Floquet states is given by (the coordinate dependence is implicit)

$$
\psi(\xi, \tau) = \sum_n e^{-\kappa_n \tau / \beta} \Phi_n(\xi) a_n,
$$

$$
\psi^\dagger(\xi, \tau) = \sum_n e^{\kappa_n \tau / \beta} \Phi^*_n(\xi) a_n^\dagger,
$$

where the index $n$ runs over all Floquet states. The Matsubara Green’s function is

$$
\mathcal{G}(\mathbf{r}, \mathbf{r}', \tau - \tau') = \frac{-1}{\hbar} \text{Tr}[e^{-\beta(\mathcal{K} - \Omega)}/\Omega_1 \mathcal{G}(\xi, \tau) \psi^\dagger(\xi', \tau')].
$$

The operator $\mathcal{T}_\tau$ is the $\tau$-ordering operator, $\Omega$ is the thermodynamic potential and serves as the normalizing factor for the thermodynamic average and $\beta = 1/(k_B T)$. The Fourier transform of this Green’s function is the quantity from which the density of states can be calculated. The Fourier transform is

$$
\mathcal{G}(\mathbf{p}, \omega) = \frac{1}{2} \int_0^{\beta} d\tau \int_0^{\beta} d\tau' e^{-\omega (\tau - \tau')} \int d^2 \mathbf{r} \int d^2 \mathbf{r}' e^{i \mathbf{p} \cdot \mathbf{r} - i \mathbf{p} \cdot \mathbf{r}'} \mathcal{G}(\mathbf{r}, \mathbf{r}', \tau, \tau')
$$

which includes an averaging procedure over the period of the fast oscillation associated with the radiation. The retarded Green’s function $\mathcal{G}^R$ can then be found by carrying out the analytic continuation $\text{Im} \omega \to \infty$ and the density of states can be extracted from this Green’s function in the standard way:

$$
\rho(\omega) = -\frac{1}{\pi} \text{Im} \sum \rho \text{Tr} \mathcal{G}^R(\mathbf{p}, \omega)
$$

where $\text{Tr}$ denotes the summation of the diagonal elements of $\mathcal{G}^R$, which is a $4 \times 4$ matrix in the sublattice space.

3. Results

3.1. Zero magnetic field

In the case of zero external magnetic field, the single-particle quantum states in the system are of plane wave nature and are characterized by a two-dimensional wavevector $\mathbf{k}$. In addition, there are two (real) spins $\sigma$, two inequivalent valleys $\xi$, and four bands within each valley (labelled by the conduction or valence band $\nu$ and the high-energy or split branch $b$). The wavefunctions associated with these states are given in appendix A.1. The energy spectrum is found by substituting the matrix elements of the radiation operator into the Hamiltonian and numerically diagonalizing the resulting matrix. The spectrum is shown in figure 1. Panel (a) shows the unirradiated low-energy bands for comparison with the other three plots. In zero magnetic field, the effective coupling parameter is $\nu e F / h \Omega_1$ which implies that the strongest coupling occurs for smaller frequencies. This is illustrated by panels (b) and (c) which show the effects of an irradiating field with $\Omega_2 = 2$ and $5$ THz, respectively, with a field strength of $F = 5$ kV cm$^{-1}$. The mixing between different Fourier harmonics is stronger for the lower frequency, but the dynamic gap opened by the mixing is larger for the higher frequency. The dynamic gaps only occur when states from opposite bands mix. States from the same band run parallel to each other (separated in energy by $h \Omega_4$ and therefore cannot cross. There is no gap at $k = 0$ because the distribution of the wavefunction across the four lattice sites forbids coupling for small momenta (see appendix A).

When a gap is introduced to the spectrum by doping one side of the graphene bilayer or by electrostatic gating, the effect of the radiation is markedly different. This is because the gap means that consecutive Fourier harmonics from opposite bands do not cross, and therefore the spectral weight is spread only between Floquet states which originate from the same band. Therefore no anticrossings appear, although the effect of a broadening of the band will be seen. This is illustrated in panel (d) of figure 1. However, as figure 2 shows, the coupling to right-handed circularly polarized light is stronger in the K valley than in the K’ valley. This is due to the redistribution of the wavefunction among the four lattice sites as a result of the inter-layer bias potential which generates the gap. If the direction of the potential or the orientation of
the irradiating field are changed, then the coupling becomes stronger in the K’ valley instead. If linearly polarized light is used, the response of electrons in the two valleys are identical since linearly polarized radiation can be represented as the sum of two circularly polarized components.

The Floquet wavefunctions which are calculated numerically can now be used in equations (4) and (8) to compute the Green’s function and hence the density of states. The Green’s function is

\[
G^R(p, \omega) = \frac{1}{\hbar} \sum_{n'} \frac{1}{\omega - \kappa_{np} / \hbar + i\delta} \sum_{x} |c_{np}^{x}|^2
\]

where \(n'\) labels the discrete set of Floquet states with wavevector \(p\) and \(x\) labels the unirradiated basis states with wavevector \(p\). The density of states can be calculated by substituting this expression into equation (9) and numerically evaluating the integral over momentum:

\[
\rho = \int_{0}^{\Lambda} dp \sum_{x} \frac{\delta}{(\hbar \omega - \kappa_{x})^2 + \delta^2} \sum_{x} |c_{0x}^{p}|^2
\]

where \(\Lambda\) is the momentum cutoff determined by requiring the summation over all states yields the correct electron density at half-filling. Plots of this function are shown in figure 3. In (a), the inter-layer bias potential is zero so that the graphene does not have a static gap. The density of states for unirradiated graphene is constant in this case, as predicted by straightforward analysis of the band structure. When the graphene is irradiated, dynamic gaps open at \(\Omega_{\lambda}/2\) and are clearly visible for stronger fields. The responses of electrons in the two valleys are identical. In (b), the same system parameters are used, except that now a small static gap is present \((U = 10 \text{ meV})\) so that the band edge occurs at roughly \(U/2 = 5 \text{ meV}\). In the unirradiated case (thin solid line), the static gap manifests as a region with near-zero DoS for small energies \((\hbar \omega < 5 \text{ meV})\). For \(\hbar \omega > 5 \text{ meV}\), the dynamic gaps are still present. However, for \(\hbar \omega < 5 \text{ meV}\), a finite DoS is present under strong radiation. This is due to the dynamical states that are induced by the radiation. In (c), the gap is wider \((U = 20 \text{ meV})\) and now the conduction and valence bands are separated to an extent where significant coupling between them is not permitted. However, states within the same band do still couple, causing the band edge to be significantly smeared and for electron density to be present in the static gap.

\[\text{Figure 2. Valley asymmetry when inter-layer bias potential is present. In these plots } F = 5 \text{ kV cm}^{-1}, \Omega_{\lambda} = 2 \text{ THz and } U = 30 \text{ meV.}\]

3.2. Quantizing magnetic field

When a strong magnetic field is present in the system, the motion of the electrons is quantized into Landau levels. Using the Landau gauge \(A_B = [0, B x, 0]\), the discrete single-particle quantum numbers are the same as in the zero-field case, but the momentum is continuous in the \(y\) direction and discrete in the \(x\) direction. This is due to the gauge field which breaks the translational symmetry. Therefore, the sum over the two-dimensional momenta may be split into two separate one-dimensional sums, one discrete (over \(k_x\)) and the other represented by an integral with periodic boundary conditions (over \(k_y\)).

Figure 4 shows the evolution of the Landau level spectrum with increasing intensity of linearly polarized incident radiation. At \(F = 0\), the standard bilayer Landau level spectrum is evident. For weak field, \((F < 1 \text{ kV cm}^{-1})\) the mixing of dynamical states is small and the original Landau level spectrum is recognizable. For strong coupling \((F > 3 \text{ kV cm}^{-1})\), the Landau level spectrum is replaced by a near-continuum of levels, each with rather small weight, and there are the beginnings of states visible in the gapped region. Notice, however, that the two \(\delta\) states barely change their intensity, showing that they are only very weakly coupled to the radiation due to their unique distribution of wavefunction weight between the four sublattice sites.

The DoS in this case is evaluated using the same steps as in the zero magnetic field situation. The analytical expression

\[\text{Figure 3. Density of states in bilayer graphene in zero magnetic field. (a) For unirradiated bilayer graphene, the valley degeneracy is intact and gaps open at intervals of } \Omega_{\lambda}/2 \approx 4.1 \text{ meV.}\]

\[\text{(b) } U = 10 \text{ meV. A significant electron density is induced in the spectral gap by the radiation. (c) } U = 20 \text{ meV. The spectral gap is too wide to allow significant coupling of electron and hole states.}\]

\[\text{Figure 4 shows the evolution of the Landau level spectrum with increasing intensity of linearly polarized incident radiation. At } F = 0, \text{ the standard bilayer Landau level spectrum is evident. For weak field, } F < 1 \text{ kV cm}^{-1} \text{ the mixing of dynamical states is small and the original Landau level spectrum is recognizable. For strong coupling } F > 3 \text{ kV cm}^{-1} \text{, the Landau level spectrum is replaced by a near-continuum of levels, each with rather small weight, and there are the beginnings of states visible in the gapped region. Notice, however, that the two } \delta \text{ states barely change their intensity, showing that they are only very weakly coupled to the radiation due to their unique distribution of wavefunction weight between the four sublattice sites.}\]

\[\text{The DoS in this case is evaluated using the same steps as in the zero magnetic field situation. The analytical expression}\]
polarized electrons [23]. If a current is incident on a region of other allows us to consider the possibility of generating valley-
where there is significant DoS in one valley but not in the
device, we discuss how the existence of parameter ranges
to demonstrate how this might be applicable in the design of
electromagnetic field is not the same in either valley. In order
We have shown that, under certain conditions, the coupling
4. Conclusions and summary
We have shown that, under certain conditions, the coupling
electrons depends critically on the wavefunction components.
the difference between the behaviour in the conduction and
The features associated with the dynamical gaps also become
the static gap, which exist only for small parameter ranges.
features caused by additional dynamical states appearing in
field strength is increased. This is in contrast to polarization
In panel (b) we demonstrate that polarization can approach
orientations of light induce polarization in opposite valleys.
In panel (a) we demonstrate that the left-handed and right-handed
potentials. This is important if highly polarized electron
irradiated bilayer graphene where the radiation and inter-layer
bias are tuned such that states only exist in one valley, then
incident current in the valley where there are no states will not
be able to traverse the irradiated region. This region can then
act as a filter for the electron valley, leading to the possibility
of switching devices and ‘valleytronic’ applications [23]. We
define the polarization to be
so that \( \mathcal{P} = +1 \) implies that all electrons are in the K valley
and \( \mathcal{P} = -1 \) means that all are in the K’ valley. Figure 6 shows
the polarization for several different scenarios. Firstly, in panel
(b) we demonstrate that polarization can approach unity for specific values of the radiation parameters and bias
potential. This is important if highly polarized electron currents are to be produced. Lastly, in panel (c) we show that the polarization induced near the dynamical gaps becomes
more pronounced and exists for a wider energy range when the
field strength is increased. This is in contrast to polarization
features caused by additional dynamical states appearing in
the static gap, which exist only for small parameter ranges.
The features associated with the dynamical gaps also become
stronger as the frequency \( \Omega_A \) is increased.

In the case where a strong magnetic field is present, valley
polarization may also be generated, but only as a result of the
shift in energy of the Landau levels created by the radiation.
inter-layer dimer bond and sites in the upper layer at the inter-layer bias potential which sets the energy of lattice valley and the basis π under specific conditions. Solid line has(b) Demonstration that polarization of almost unity can be achieved for right-handed (solid line) and left-handed (dashed line) radiation programme and the NSERC Discovery grant. This work was supported by the Canada Research Chairs Acknowledgments

We take the basis \{ψ_A, ψ_B, ψ_a, ψ_b\} with ξ = +1 in the K valley and the basis \{ψ_B, ψ_A, ψ_B, ψ_a\} with ξ = −1 in the K' valley so that the single-particle nearest-neighbour tight binding Hamiltonian for unbiased bilayer graphene can be written as [33]

\[
H_0 = \begin{pmatrix}
0 & 0 & 0 & ξ v_F π^\dagger \\
0 & 0 & ξ v_F π & 0 \\
0 & ξ v_F π^\dagger & 0 & γ_1 \\
ξ v_F π & 0 & γ_1 & 0 \\
\end{pmatrix}, \quad (A.1)
\]

where \( π = p_x + ip_y \) is the linear expansion of the transfer integral in the tight binding formalism, \( γ_1 \) is the energy of the inter-layer dimer bond and \( v_F \) is the Fermi velocity. Similarly, the inter-layer bias potential which sets the energy of lattice sites in the upper layer at \( U/2 \) and sites in the lower layer at \( −U/2 \) causes a gap of magnitude \( U \) to open at the K points, and is described in [34]

\[
H_U = \begin{pmatrix}
ξ U/2 & 0 & 0 & 0 \\
0 & −ξ U/2 & 0 & 0 \\
0 & 0 & −ξ U/2 & 0 \\
0 & 0 & 0 & ξ U/2 \\
\end{pmatrix}. \quad (A.2)
\]

A.1. Zero magnetic field

In the zero-field case, the \( π \) operators are constructed from the usual single-particle momentum operators \( π = −i\hbar \partial_x − h\partial_y \). The energy spectrum associated with the Hamiltonian \( H_0 + H_U \) is

\[
E_{k,n,b} = v \left( \frac{y_1^2}{2} + \frac{U^2}{4} + h^2 v_F^2 k^2 + b \sqrt{\frac{y_1^4}{4} + h^2 v_F^2 k^2 (U^2 + γ_1^2)} \right)^{1/2}.
\]

The quantum numbers \( v = \pm 1 \) and \( b = \pm 1 \) label the band and branch, respectively. Henceforth, we denote the band and valley indices by the label \( α = [ξ_α, ν_α, b_α] \) to shorten the notation. The wavefunctions are given by

\[
φ_{k,n,b} = C_{k,n,b} \left( \begin{array}{c}
ξ_α n_b + f_{k,n,b} \\
ξ_α ν_α + f_{k,n,b} \\
1 \\
ξ_α ν_α + f_{k,n,b} \gamma_1 \\
\end{array} \right) \frac{1}{\sqrt{A}},
\]

\[
f_{k,n,b} = \frac{h^2 v_F^2 k^2}{(ξ_α ν_1 + E_{k,n,b})^2} - 1,
\]

\[
C_{k,n,b} = \left( \frac{h^2 v_F^2 k^2}{(ξ_α ν_1 + E_{k,n,b})^2} + 1 \right)^{-1/2} \left[ \frac{(ξ_α ν_1 + E_{k,n,b})^2 γ_1}{γ_1^2} f_{k,n,b} \right]^{1/2},
\]

where \( Σ = U/2 \), \( k = |k| \) and \( θ_0 \) is the angle of the wavevector \( k \) and the \( x \) axis in the graphene plane.

A.2. Quantizing magnetic field

In this case, the \( π \) operators are constructed from the gauge-invariant momentum in a magnetic field found by making the minimal coupling substitution in the momentum operator so that \( p \to p + eA \). We label the Landau levels with a set of quantum numbers \( n = [n_α, ν_α, ξ_α, q_α] \), where \( n > 0 \) is the Landau level index, \( ν = +1(−1) \) in the conduction (valence) band, \( ξ \) is the valley and \( q \) is the \( z \) component of the wavevector which defines the guiding centre coordinate in the Landau gauge and we assume that all Landau levels are in the low-energy bands. Then, the Landau level energies \( E_a \) are found by solving a polynomial equation derived using the Landau level operators \( π J_n = −i\hbar \frac{ξ_α ν_α}{θ_0} J_{n−1} \) and \( π^\dagger J_n = i\hbar \frac{ξ_α ν_α}{θ_0} J_{n+1} \):

\[
[2(n_α + 1) − (ξ_α ν_α + E_a)]^2(2(n_α + 1) − (ξ_α ν_α + E_a)]^2) = γ_1^2 (ξ_α ν_α^2 + E_a^2) = 0 \quad (A.3)
\]

where the energies are measured in units of \( h v_F / θ_0 \) and \( Σ_α = ξ U/2 \). The wavefunction associated with each Landau level
is defined by the index of the Landau function in the third and fourth components. The \( \chi \) functions are only defined for \( n_a \geq 0 \) so components of the \( n_a = 0 \) wavefunctions which contain indices outside of this range have zero weight on those components. The wavefunctions are

\[
\varphi_a = \frac{d_a e^{i\delta_a}}{\sqrt{L \lambda_B \mu}} \left( \begin{array}{c} \Xi a_{n_a+1}(r_a) \\ g_a X_{n_a}(r_a) \\ 0 \\ 0 \end{array} \right) = \frac{d_a e^{i\delta_a}}{\sqrt{L \lambda_B \mu}} \left( \begin{array}{c} \Xi a_{n_a+1}(r_a) \\ g_a X_{n_a}(r_a) \\ 0 \\ 0 \end{array} \right),
\]

where \( d_a \) is defined by the index of the Landau function in the third component.

Since we discuss radiation incident at the perpendicular to the graphene plane, this angle is immaterial and we can substitute \( \theta = 0 \) without loss of generality. On the other hand, circularly polarized light is given by \( A_\pm(\tau) = A[\cos(\Omega_\tau), \pm \sin(\Omega_\tau)] \), where the positive or negative sign corresponds to right- or left-handed orientations of the polarization:

\[
h_\pm(\tau) = \frac{\xi \nu e F}{\Omega_A} \sigma_z \otimes \left( \begin{array}{cc} 0 & e^{i\Omega_\tau} \\ e^{-i\Omega_\tau} & 0 \end{array} \right).
\]

### Appendix B. Matrix elements of \( \mathcal{F}(\tau) \)

For convenience, we restate the definition of \( \mathcal{F}(\tau) = \mathcal{H}(\tau) - i\hbar \frac{\partial}{\partial \tau} \mathcal{J}(\tau) \). The matrix elements contain several terms. The terms contributing to the static Hamiltonian and the time derivative are

\[
\langle (j'X') | H_0 + H_U - i\hbar \frac{\partial}{\partial \tau} | jX \rangle = \langle E_X + j\hbar \Omega_\lambda \delta_{j,j'} \delta_X \rangle.
\]

The matrix elements of the term associated with the irradiating field \( h(\tau) \) depend on the specific nature of the field and the wavefunctions. For example, linearly polarized light can be described by the vector potential \( A_{\text{lin}}(\tau) = A \cos(\Omega_\tau) \cos \theta, \sin \theta \), where \( \theta \) is the angle of polarization in the plane of the graphene with respect to the x axis. This yields

\[
h_{\text{lin}}(\tau) = \frac{\xi \nu e F}{\Omega_A} \sigma_z \otimes \left( \begin{array}{cc} 0 & e^{i\Omega_\tau} \\ e^{-i\Omega_\tau} & 0 \end{array} \right)
\]

and

\[
\langle (j'X') | h_{\text{lin}}(\tau) | jX \rangle = \frac{\xi \nu e F}{\Omega_A} \sigma_z \otimes \left( \begin{array}{cc} 0 & e^{i\Omega_\tau} \\ e^{-i\Omega_\tau} & 0 \end{array} \right).
\]

### B.1. Zero magnetic field

In this case the matrix elements of linearly polarized light (taking \( \theta = 0 \)) are zero for states with unequal wavevectors, and as follows for states in the same valley and with the same spin, which also have identical wavevector:

\[
\langle (j'X') | h_{\text{lin}}(\tau) | jX \rangle = \frac{\xi \nu e F}{\Omega_A} \sigma_z \otimes \left( \begin{array}{cc} 0 & e^{i\Omega_\tau} \\ e^{-i\Omega_\tau} & 0 \end{array} \right).
\]

### B.2. Quantizing magnetic field

The matrix elements in this case are

\[
\langle (j'X') | h_{\text{lin}}(\tau) | jX \rangle = \frac{\xi \nu e F}{\Omega_A} \sigma_z \otimes \left( \begin{array}{cc} 0 & e^{i\Omega_\tau} \\ e^{-i\Omega_\tau} & 0 \end{array} \right).
\]

and

\[
\langle (j'X') | h_{\text{lin}}(\tau) | jX \rangle = \frac{\xi \nu e F}{\Omega_A} \sigma_z \otimes \left( \begin{array}{cc} 0 & e^{i\Omega_\tau} \\ e^{-i\Omega_\tau} & 0 \end{array} \right).
\]

and
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