Exact determination of the volume of an inclusion in a body having constant shear modulus

Andrew E Thaler and Graeme W Milton

Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA
E-mail: andythaler05@gmail.com and milton@math.utah.edu

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Abstract
We derive an exact formula for the volume fraction of an inclusion in a body when the inclusion and the body are linearly elastic materials with the same shear modulus. Our formula depends on an appropriate measurement of the displacement and traction around the boundary of the body. In particular, the boundary conditions around the boundary of the body must be such that they mimic the body being placed in an infinite medium with an appropriate displacement applied at infinity.

Keywords: volume fraction, Dirichlet-to-Neumann map, size estimation

1. Introduction

A fundamental and interesting problem in the study of materials is the estimation of the volume fraction occupied by an inclusion $D$ in a body $\Omega$. Although the volume fraction could be determined by weighing the body, the densities of the materials may be close or unknown or weighing the body may be impractical. Because of this, many methods have been developed which utilize measurements of certain fields around $\partial \Omega$ to derive bounds on the volume fraction $|D|/|\Omega|$, where $|U|$ is the Lebesgue measure of the set $U$ [1–23]. In this paper, we show that under certain circumstances the volume fraction $|D|/|\Omega|$ can be computed exactly from a single appropriate boundary measurement around $\partial \Omega$. We note that many of the results in the literature (and our results in this paper) can also be applied when $\Omega$ contains a two-phase composite with microstructure much smaller than the dimensions of $\Omega$.

We consider an inclusion $D$ in a body $\Omega$ (or a two-phase composite inside $\Omega$), where $\Omega$ is a subset of $\mathbb{R}^d$ ($d = 2$ or 3). We assume that the inclusion and body are filled with linearly elastic materials with the same shear modulus $\mu$ and Lamé moduli $\lambda_1$ and $\lambda_2$, respectively. Our
goal is to determine the volume fraction occupied by the inclusion, namely $|D|/|\Omega|$, in terms of a measurement of the displacement and traction around $\partial\Omega$. The boundary conditions around $\partial\Omega$ are taken to be such that they mimic the body $\Omega$ being placed in an infinite medium with a suitable field at infinity. The starting point for our result is based on an exact relation due to Hill [24], which we now describe.

One of the most important problems in the study of composite materials is the determination of effective moduli given information about the local moduli—see, e.g., the work by Hashin [25] and the book by Milton [26] (chapters 1 and 2 in particular). In general, it is extremely difficult (if not impossible) to determine effective parameters exactly, even if the microgeometry of the composite is known and relatively uncomplicated. However, many useful approximation techniques and bounds on effective properties of composites have been derived in the literature—see the book by Milton [26] and the references therein for a vast collection of such results. Surprisingly, there are several circumstances in which exact links between effective moduli (or exact formulas for the moduli themselves) can be derived regardless of the complexity of the microstructure; such links are known as exact relations.

Exact relations exist for a variety of problems including elasticity and coupled problems such as thermoelasticity, thermoelectricity, piezoelectricity, thermo–piezo–electricity, and others—see the review articles by Dvorak and Benveniste [27] and Milton [28], the work by Grabovsky et al [29], the works by Hegg [30, 31], and the references therein for summaries of numerous previous and current results on exact relations.

Perhaps even more surprising than the existence of exact relations is the existence of a general mathematical theory of exact relations, developed by Grabovsky, Milton, and Sage [29, 32–35], that allows us to determine all of the above mentioned exact relations and many more. For example, Hegg applied this general theory to the study of fiber–reinforced elastic composites [30, 31].

Rather than study the general theory, we focus on a specific exact relation derived by Hill [24, 36]. In particular, Hill considered a two-phase composite material consisting of two homogeneous and isotropic phases with the same shear modulus $\mu$ but different Lamé moduli $\lambda_1$ and $\lambda_2$. Hill proved that such a composite is macroscopically elastically isotropic with shear modulus $\mu$ and effective Lamé modulus $\lambda_\star$; he also derived an exact formula for $\lambda_\star$ that holds regardless of the complexity of the microgeometry. His derivation of this formula [24, 36] provides the starting point of our work in this paper.

We begin by assuming that the body $\Omega$ is embedded in an infinite medium with Lamé modulus $\lambda_E$ and shear modulus $\mu$ (we take $\lambda_E = \lambda_2$ for simplicity) and that a displacement $u = Vg$ is applied at infinity. Using a method similar to Hill’s derivation of $\lambda_\star$, we derive a formula for $|D|/|\Omega|$ in terms of a measurement of the displacement around $\partial\Omega$, the (known) parameters $\lambda_1$, $\lambda_2$, and $\mu$, and the (known) function $g$. In order to make the situation more practical, we derive a certain nonlocal boundary condition that can be applied to $\partial\Omega$ that forces the body to behave as if it actually were embedded in an infinite medium with Lamé modulus $\lambda_2$, shear modulus $\mu$, and an applied displacement $u = Vg$ at infinity. This nonlocal boundary condition couples the measurements of the traction and displacement around $\partial\Omega$.

Nonlocal boundary conditions which mimic infinite media similar to the one mentioned above are common tools used in the numerical solution of PDEs (and ODEs) in infinite domains—see, e.g., the review article by Givoli [37] and references therein for examples specific to scattering problems, the work by Han and Wu on the Laplace equation [38] and elasticity equations [39], the work by Lee et al on the elasticity equations [40], and references therein.

As discussed above, our formula for the volume fraction $|D|/|\Omega|$ holds as long as the body $\Omega$ is embedded in an infinite medium with an applied displacement $u = Vg$ at infinity.
In section 4 we derive a nonlocal boundary condition such that if this boundary condition is applied to $\partial \Omega$ the solution inside $\Omega$ will be equal to the restriction to $\Omega$ of the solution to the infinite problem. Our boundary condition depends on the function $g$ and on the exterior Dirichlet–Neumann (DtN) map on $\partial \Omega$ (which, when the body $\Omega$ is absent, maps the displacement on $\partial \Omega$ to the traction on $\partial \Omega$ when no fields are applied at infinity). Thus it is closely related to the boundary condition of Han and Wu [38, 39] and Bonnaillie–Noël et al [41]—see section 4 for complete details.

The rest of this paper is organized as follows. In section 2 we briefly review the linear elasticity equations and relevant results from homogenization theory. Next, in section 3 we derive a formula that gives the exact volume fraction of an inclusion in a body when the inclusion and the body have the same shear modulus $\mu$ and the body is embedded in an infinite medium with shear modulus $\mu$. We discuss the nonlocal boundary condition relevant to our problem in section 4 so we can focus on a (more realistic) finite domain. Finally, in section 5 we present the expression of the nonlocal boundary condition in the particular case when $\Omega$ is a disk in $\mathbb{R}^2$—this expression was first derived by Han and Wu [38, 39]. A complete derivation of our nonlocal boundary condition is given in work by one of the authors of this paper [23].

2. Elasticity

Let $d = 2$ or 3 be the dimension under consideration; then we define $\text{Sym}(\mathbb{R}^d)$ to be the set of all symmetric linear mappings from $\mathbb{R}^d$ to itself, i.e.,

$$\text{Sym}(\mathbb{R}^d) \equiv \left\{ B \in \mathbb{R}^d \otimes \mathbb{R}^d \mid B = B^T \right\}.$$ 

Similarly, the set $\text{Sym}(\text{Sym}(\mathbb{R}^d))$ is defined as the set of symmetric linear mappings from $\text{Sym}(\mathbb{R}^d)$ to itself. If $A \in \text{Sym}(\text{Sym}(\mathbb{R}^d))$ and $B \in \text{Sym}(\mathbb{R}^d)$, then $A \cdot B \in \text{Sym}(\mathbb{R}^d)$ with elements

$$(A \cdot B)_{ij} = A_{ijkl} B_{kl}, \quad (1)$$

where here and throughout the paper “:” represents double contraction of indices and we use the Einstein summation convention that repeated indices are summed from 1 to $d$.

Consider a linearly elastic body which occupies an open, bounded set $\Omega \subset \mathbb{R}^d$ with Lipschitz boundary. Let $u(x)$, $\epsilon(x)$, and $\sigma(x)$ denote the displacement, linearized strain tensor, and Cauchy stress tensor, respectively, at the point $x = (x_1, \ldots, x_d) \in \Omega$. Then $u \in \mathbb{R}^d$ while $\epsilon$ and $\sigma$ belong to $\text{Sym}(\mathbb{R}^d)$. By Hooke’s law, the stress and strain tensor are related through the linear constitutive relation

$$\sigma(x) = C(x) : \epsilon(x), \quad (2)$$

where $C \in \text{Sym}(\text{Sym}(\mathbb{R}^d))$ is the elasticity (or stiffness) tensor. We assume $C$ is elliptic for all $x \in \Omega$, i.e., there are positive constants $a$ and $b$ such that

$$B : [C(x) : B'] \leq a \|B\| \|B\| \quad \text{and} \quad B : [C(x) : B] \geq b \|B\|^2$$

for all $B, B' \in \text{Sym}(\mathbb{R}^d)$ and where $\|B\|^2 = \frac{1}{2}(B : B)$. If there are no body forces present, then at equilibrium the elasticity equations are

$$\nabla \cdot \sigma(x) = 0, \quad \epsilon(x) = \frac{1}{2} \left[ \nabla u(x) + \nabla u(x)^T \right], \quad \text{and} \quad \epsilon(x) = C(x) : \sigma(x); \quad (3)$$

see, e.g., chapter 2 of the book by Milton [26].
If the composite is locally isotropic, the local elasticity tensor takes the form

\[ C(x) = \lambda(x)I \otimes I + 2\mu(x)I, \]

where \( \lambda \) is the Lamé modulus, \( \mu \) is the shear modulus, \( I \in \text{Sym}(\mathbb{R}^d) \) is the second-order identity tensor with elements \( I_{ij} = \delta_{ij} \) (where \( \delta_{ij} \) is the Kronecker delta which is 1 if \( i = j \) and 0 otherwise), and \( I \in \text{Sym(Sym}(\mathbb{R}^d)) \) is the fourth-order identity tensor which maps an element in \( \text{Sym}(\mathbb{R}^4) \) to itself under contraction [26]. In this case, Hooke’s law (2) reduces to

\[ Su(x) \equiv \sigma(x) = \lambda(x) \text{Tr}[\epsilon(x)]I + 2\mu(x)\epsilon(x) \quad (4) \]

\[ = \lambda(x)[V \cdot u(x)]I + \mu(x)[Vu(x) + Vu(x)^T]. \quad (5) \]

where \( S: \mathbb{R}^d \to \text{Sym}(\mathbb{R}^d) \) is the linear stress operator that maps the displacement \( u \) to the stress \( \sigma \) (note that \( S \) itself depends on \( x \) through \( \lambda(x) \) and \( \mu(x) \)).

The bulk modulus, Young’s modulus, and the Poisson ratio are related to \( \lambda \) and \( \mu \) by

\[ \kappa = \lambda + \frac{2\mu}{d}, \quad E = \frac{2\mu(d\lambda + 2\mu)}{(d-1)\lambda + 2\mu}, \quad \nu = \frac{\lambda}{(d-1)\lambda + 2\mu}, \]

respectively. Throughout this paper we assume that the elasticity tensor \( C(x) \) is positive definite, i.e.,

\[ \mu(x) > 0 \quad \text{and} \quad d\lambda(x) + 2\mu(x) > 0. \quad (6) \]

3. Exact volume fraction

In this section we derive a formula that gives the exact volume fraction occupied by an inclusion in a body, where our formula depends on a boundary measurement of the displacement. Let \( D \) and \( \Omega \) be open, bounded sets in \( \mathbb{R}^d \) with Lipschitz boundary such that \( \Omega \subset D \). We note that \( D \) may intersect \( \partial\Omega \). Suppose \( \mathbb{R}^d \) is filled with a linearly elastic, locally isotropic material with constant shear modulus \( \mu \) and Lamé modulus \( \lambda \). Since the material is locally isotropic, the elasticity tensor is

\[ C(x) = \lambda(x)I \otimes I + 2\muI. \quad (8) \]

We can write \( Su(x) \) from (5) as

\[ Su(x) = \begin{cases} 
S_1u(x) \equiv \lambda_1[V \cdot u(x)]I + \mu[Vu(x) + Vu(x)^T] & \text{for } x \in D, \\
S_2u(x) \equiv \lambda_2[V \cdot u(x)]I + \mu[Vu(x) + Vu(x)^T] & \text{for } x \in \mathbb{R}^d \setminus D.
\end{cases} \quad (9) \]

Let the function \( f = Vg \) be given and satisfy \( L_2f = L_2Vg = 0 \) for all \( x \in \mathbb{R}^d \), where \( L_ju = -(\lambda_j + \mu)V(V \cdot u) - \mu Au \) (for \( j = 1, 2 \)) is the Lamé operator. To avoid possible technical complications we assume that \( g \) is at least three times continuously differentiable in \( \mathbb{R}^d \). We will see that \( f \) represents the “displacement at infinity”; perhaps the simplest example of such a function is \( f(x) = x \), in which case \( g = \frac{1}{\tau}(x \cdot x) + \text{constant}. \)
According to the elasticity equations in (3), the displacement \( u \) satisfies
\[
\begin{cases}
\mathcal{L}_1 u = 0 & \text{in } D, \\
\mathcal{L}_2 u = 0 & \text{in } \mathbb{R}^d \setminus D, \\
u, \sigma \cdot n_D = (Su) \cdot n_D & \text{continuous across } \partial D, \\
u - f = \mathcal{O}(|x|^{-d}) & \text{as } |x| \to \infty,
\end{cases}
\] (10)
where \( n_D \) is the outward unit normal vector to \( \partial D \), and \( \sigma = Su \) is the stress tensor associated with \( u \).

As shown in chapters 9 and 10 of the book by Ammari and Kang [42], there exists a unique solution \( u \) to (10) if \( D \) is a Lipschitz domain, where if \( d = 3 \) then \( u - f \in H^1(\mathbb{R}^3) \), while if \( d = 2 \) then \( \mathcal{I}(u - f) \in L^2(\mathbb{R}^2) \) and
\[
\int_{\mathbb{R}^2} \frac{|u(x) - f(x)|^2}{\sqrt{1 + |x|^2}} \, dx < \infty,
\]
which defines the space with a weighted norm in which \( u - f \) belongs (H Kang, personal communication); also see the equation after (2.13) in the work by Ammari et al [43].

Following Hill’s work [24, 36], we assume there is a continuously differentiable potential \( \phi \) such that \( u = V\phi \). In particular, we assume \( \phi \) and \( V\phi \) are continuous across \( \partial D \) (by (10), \( \phi = V\phi \) must be continuous across \( \partial D \)). Also, note that the matrix \( V V\phi \) is symmetric in each phase. We only assume that \( \phi \) and \( V\phi \) are continuous across \( \partial D \) (indeed, as shown by Hill [44], \( \partial^2\phi/\partial x_i \partial x_j \) is discontinuous across \( \partial D \)). Then from (3) we have
\[
\epsilon = \frac{1}{2} \left( V u + u^T \right) = \frac{1}{2} \left[ V V\phi + (V V\phi)^T \right] = V V\phi.
\] (11)
(Strictly speaking, (11) holds in \( \mathbb{R}^d \setminus \partial D \), since \( V V\phi \) is discontinuous across \( \partial D \); however, (11) also holds in a weak sense, e.g., in the sense of distributions.) From (11) we have \( Tr (\epsilon) = Tr (VV\phi) = \Delta\phi \), where \( \Delta = V \cdot V = \partial^2/\partial x_i \partial x_i \) is the Laplacian. Then (4) and (11) imply
\[
\sigma(x) = C(x); \; \epsilon(x) = \lambda(x) \Delta \phi I + 2\mu V V\phi.
\] (12)
Finally, for \( j = 1 \) and \( j = 2 \) we have
\[
\mathcal{L}_j u = -(\lambda_j + \mu) V (V \cdot V\phi) - \mu \Delta (V\phi)
\]
\[
= -(\lambda_j + \mu) V (\Delta\phi) - \mu V (\Delta\phi)
\]
\[
= -(\lambda_j + 2\mu) V (\Delta\phi).
\] (13)
By assumption, we have
\[
0 = \mathcal{L}_2 f = -(\lambda_2 + 2\mu) V (\Delta g)
\] (14)
for all \( x \in \mathbb{R}^d \), so \( V (\Delta g) = 0 \) for all \( x \in \mathbb{R}^d \). Then we must have \( \Delta g = C_g \neq 0 \) for all \( x \in \mathbb{R}^d \), where \( C_g \) is a constant. (The constant \( C_g \) is known since \( g \) is known; we will see later why we must take \( C_g \neq 0 \).) Thus the function \( g \) must be chosen so that
\[
g = \frac{C_g}{2} x \cdot x + g_h,
\] (15)
where \( g_h \) is harmonic in \( \mathbb{R}^d \); this implies that \( g \) is infinitely differentiable in \( \mathbb{R}^d \) [45] and not just thrice continuously differentiable. (Although the requirement that \( g \) is thrice continuously
differentiable is not strictly necessary for \( g \) to be of the form given in (15), we place this smoothness requirement on \( g \) to ensure that (14) and the remaining analysis in this paragraph hold in a strong sense.)

Recalling that \( \mathbf{u} = V\phi \) and \( \mathbf{f} = Vg \), we see that (13) implies that (10) becomes

\[
\begin{cases}
V(D\phi) = 0 & \text{in } D \text{ and } \mathbb{R}^d \setminus D, \\
V\phi, \quad \sigma \cdot \mathbf{n}_D = (S V\phi) \cdot \mathbf{n}_D & \text{continuous across } \partial D, \\
V\phi - Vg = \mathcal{O}(|x|^{1-d}) & \text{as } |x| \to \infty,
\end{cases}
\]

where \( \sigma = SV\phi \) is given in (12).

### 3.1. Behavior of \( \Delta \phi \)

In this section we study the behavior of \( \Delta \phi \). Recall that we assume \( \phi \) to be at least continuously differentiable in \( \mathbb{R}^d \); in particular, this implies that \( \phi \) and \( \mathbf{u} = V\phi \) are continuous across \( \partial D \) (see (16)). If \( \mathbf{f} \) is smooth the solution \( \mathbf{u} \) to (10) is smooth in \( \mathbb{R}^d \setminus D \) and in \( D \) [42, equation (10.2)] (although it is only continuous across \( \partial D \)).

Since \( \phi \) is smooth in \( D \) and \( \mathbb{R}^d \setminus D \), (16) implies that \( \Delta \phi \) is constant in each phase, i.e.,

\[
\Delta \phi = \begin{cases}
C_1 & \text{in } D, \\
C_2 & \text{in } \mathbb{R}^d \setminus D.
\end{cases}
\]

We will now derive relationships between the constants \( C_1 \), \( C_2 \), and \( C_g \). We begin by recalling some useful results regarding the jump in values of a function across a boundary.

**Definition 3.1.** Let \( U \subset \mathbb{R}^d \) be a bounded, open set with Lipschitz boundary. Suppose \( h_+ \) is a function defined and continuous in a neighborhood \( U^+ \) of \( \partial U \) that is outside \( U \) and that \( h_+ \) can be made to be continuous up to \( \partial U \); also assume \( h_- \) is a function that is defined and continuous in a neighborhood \( U^- \) of \( \partial U \) that is inside \( U \) and that \( h_- \) can be made to be continuous up to \( \partial U \). Define

\[
h(x) \equiv \begin{cases}
h_+(x) & \text{for } x \in U^+ \setminus \partial U, \\
h_-(x) & \text{for } x \in U^-.
\end{cases}
\]

Then the jump in \( h \) across \( \partial U \), denoted by \( [h]_{\partial U} \), is defined for each \( x \in \partial U \) by

\[
[h]_{\partial U}(x) \equiv \lim_{y \to x^+} h_+(y) - \lim_{y \to x^-} h_-(y),
\]

where \( \lim_{y \to x^+} h_+(y) = \lim_{y \to x^-} h_-(y) \) denotes the limit from outside \( U \) (i.e., in \( U^+ \)) while \( \lim_{y \to x^+} h_-(y) = \lim_{y \to x^-} h_+(y) \) denotes the limit from inside \( U \) (i.e., in \( U^- \)).

The following lemma is due to Hill [24, 36]. We include the details of the proof here for completeness.

**Lemma 3.2.** Suppose \( \mathbf{u} = V\phi \) solves (10) with \( \mathbf{f} = Vg \) where \( g = \frac{C}{2} \mathbf{x} \cdot \mathbf{x} + g_h \), \( C \neq 0 \) is an arbitrary constant, and \( g_h \) is an arbitrary harmonic function in \( \mathbb{R}^d \) (so \( \phi \) satisfies (16)). Then
Δϕ = V \cdot u = \begin{cases} C_1 = \left( \frac{\lambda_2 + 2\mu}{\lambda_1 + 2\mu} \right) C_g & \text{in } D, \\ C_2 = C_g & \text{in } \mathbb{R}^d \setminus \partial D. \end{cases} \tag{19}

Proof. First, since \( u - f = V\phi - Vg \to 0 \) as \( |x| \to \infty \) (by (16)), by (17) we have \( V \cdot (u - f) = \Delta \phi - \Delta g = C_2 - C_g \to 0 \) as \( |x| \to \infty \) as well. This can only happen if \( C_2 = C_g \).

Recall that we assume \( \phi \) is continuous across \( \partial D \); also, \( V\phi = u \) must be continuous across \( \partial D \) by (16). Results by Hill [44] then imply that the second derivatives of \( \phi \) undergo a jump

\[
[\nabla \nabla \phi]_{\partial D}(x) = \tau(x)n_D(x)n_D(x)^T
\]

across \( \partial D \) at the point \( x \in \partial D \) where \( \tau \) is a function to be determined. To determine the function \( \tau \), we note that (17) and (18) imply

\[
[\Delta \phi]_{\partial D}(x) = C_2 - C_1
\]

for all \( x \in \partial D \). Then from (20) and (21) we have

\[
C_2 - C_1 = [\Delta \phi]_{\partial D}(x) = [\text{Tr} (\nabla \nabla \phi)]_{\partial D}(x) = \text{Tr} \left( [\nabla \nabla \phi]_{\partial D}(x) \right) = \tau(x)
\]

for all \( x \in \partial D \).

We now use the requirement from (16) that the traction \( \sigma \cdot n_D = (SV\phi) \cdot n_D \) must be continuous across \( \partial D \). From (12) and (17) we have

\[
\sigma = \begin{cases} \lambda_1 C_1 I + 2\mu \nabla \nabla \phi(x) & \text{if } x \in D, \\ \lambda_2 C_2 I + 2\mu \nabla \nabla \phi(x) & \text{if } x \in \Omega \setminus \partial D. \end{cases} \tag{23}
\]

Then (23) implies that the jump in traction across \( \partial D \) at \( x \in \partial D \) is

\[
[\sigma \cdot n_D]_{\partial D}(x) = \left( \lambda_2 C_2 I + 2\mu \nabla \nabla \phi|_{\partial D} \right) \cdot n_D(x) - \left( \lambda_1 C_1 I + 2\mu \nabla \nabla \phi|_{\partial D} \right) \cdot n_D(x)
\]

\[
= \left( \lambda_2 C_2 - \lambda_1 C_1 \right)n_D(x) + 2\mu \left( [\nabla \nabla \phi]_{\partial D}(x) \right) \cdot n_D(x).
\]

By (20) and (22)

\[
[\nabla \nabla \phi]_{\partial D}(x) = \tau(x)n_D(x)n_D(x)^T = (C_2 - C_1)n_D(x)n_D(x)^T,
\]

so

\[
\left( [\nabla \nabla \phi]_{\partial D}(x) \right) \cdot n_D(x) = (C_2 - C_1)n_D(x).
\]

Since \( [\sigma \cdot n_D]_{\partial D}(x) \) must be \( 0 \) for each \( x \in \partial D \) (due to the continuity of the traction across \( \partial D \)), (24) and (25) imply, for each \( x \in \partial D \), that

\[
0 = (\lambda_2 C_2 - \lambda_1 C_1 + 2\mu (C_2 - C_1))n_D(x).
\]

Thus

\[
C_1 = \left( \frac{\lambda_2 + 2\mu}{\lambda_1 + 2\mu} \right) C_2 = \left( \frac{\lambda_2 + 2\mu}{\lambda_1 + 2\mu} \right) C_g.
\]

□
3.2. Main Result

Recalling that \( u \) solves (10), the divergence theorem and (19) imply

\[
\int_{\Omega} \nabla \cdot u \, d\Omega = \int_{\partial \Omega} \nabla \cdot u \, ds + \int_{\Omega} \nabla \cdot u \, dx = (C_1 - C_2)|D| + C_2 |\Omega|.
\]

Therefore the volume fraction of the inclusion is given by the formula

\[
\frac{|D|}{|\Omega|} = \frac{1}{C_1 - C_2} \left( \frac{1}{|\Omega|} \int_{\partial \Omega} u \cdot n_{\Omega} \, dS - C_2 \right),
\]

(26)

where \( C_1 \) and \( C_2 \) are related to \( C_g \) by (19). Since we are assuming we have complete knowledge of \( u \) around \( \partial \Omega \) from our measurement, and since \( \Delta = C_g \) is given, (26) allows us to exactly determine \(|D|/|\Omega|\). Note also that we must take \( g \neq C_g \). If \( g = C_g \), then (19) implies that \( C_1 = C_2 = 0 \), which makes the formula in (26) undefined. We have thus proved the following theorem.

**Theorem 3.3.** Let \( D \) and \( \Omega \) be open, bounded sets in \( \mathbb{R}^d \) \((d = 2\) or \(3\)) with Lipschitz boundary such that \( \Omega \subset D \). Suppose \( \mathbb{R}^d \) is filled with a material described by the local elasticity tensor given by (8) and (7). Also suppose \( f = Vg \) is given (where \( g \) is of the form given in (15)) and \( L_2 f = -(\lambda_1 + \mu) \nabla (\nabla \cdot f) - \mu \Delta f = 0 \) \((\Leftrightarrow \Delta g = C_g \neq 0)\) for all \( x \in \mathbb{R}^d \). Assume that \( u \cdot n_{\Omega} \) is known around \( \partial \Omega \). Then the volume fraction of the inclusion \( D \) is given by (26).

4. Finite medium

Consider again the linear elasticity problem from section 3, namely that of an inclusion \( D \) in a body \( \Omega \) which in turn is embedded in an infinite medium \( \mathbb{R}^d \setminus \overline{\Omega} \). The isotropic and homogeneous materials in \( D \) and \( \mathbb{R}^d \setminus \overline{D} \) have Lamé moduli \( \lambda_1 \) and \( \lambda_2 \), respectively; we also assume that both materials have the same shear modulus \( \mu \). If a displacement \( f = Vg \) is applied at infinity, then the displacement \( u = V\phi \) satisfies (10) \((\phi \) satisfies (16)). Recall that we require \( L_2 f = 0 \) in \( \mathbb{R}^d \), which implies \( \Delta g = C_g \) in \( \mathbb{R}^d \). Since \( g \) and \( f = Vg \) are smooth in \( \mathbb{R}^d \), \( g \), \( f \), and \( \Delta f \) are continuous up to \( \partial D \) from outside \( D \); in other words, the limits \( g|_{\partial D}^+ \), \( f|_{\partial D}^+ \), and \( \Delta f|_{\partial D}^+ \) exist and are finite at each point of \( \partial D \), where \( h|_{\partial D}^+ \) and \( h|_{\partial D}^- \) denote the restriction of the function \( h \) to \( \partial D \) from outside and inside \( D \), respectively.

We now derive a boundary condition \( P \) so that the solution \( u' \in H^1(\Omega) \) to

\[
\begin{aligned}
L_1 u' &= 0 \quad \text{for } x \in D, \\
L_2 u' &= 0 \quad \text{for } x \in \Omega \setminus D, \\
u' \cdot \sigma' \cdot n_D &= (Su') \cdot n_D \quad \text{continuous across } \partial D, \\
P(u_0', t_0', f_0', F_0) &= 0 \quad \text{on } \partial D
\end{aligned}
\]

(27)

is equal to the solution \( u \) to (10) restricted to \( \overline{\Omega} \), i.e., \( u' = u|_{\overline{\Omega}} \); we have defined

\[
u_0' \equiv u'|_{\partial D^+}, \quad t_0' \equiv (\sigma'|_{\partial D^+}) \cdot n_{\partial D^+}, \quad f_0' \equiv f|_{\partial D^+}, \quad \text{and} \quad F_0 \equiv \left[ \left( S_2 f \right)|_{\partial D^+} \right] \cdot n_{\partial D^+}.
\]

(28)

This allows us to apply our formula (26) to the problem (27), which is posed on the finite domain \( \Omega \). For details on a related problem (including proofs of the well-posedness of problems similar to (27)), see the papers of Han and Wu [38, 39].
To derive the boundary condition $P$, we begin by considering the following exterior problem

$$
\begin{aligned}
\mathcal{L}_E \tilde{v}_E &= 0 \quad \text{for } x \in \mathbb{R}^d \setminus \mathcal{D}, \\
\tilde{v}_E &= \tilde{v} \quad \text{on } \partial \Omega, \\
\tilde{v}_E &= \mathcal{O}(|x|^{-d}) \quad \text{as } |x| \to \infty,
\end{aligned}
$$

(29)

where $\mathcal{L}_E \tilde{v}_E = -(\lambda_E + \mu) \nabla (\nabla \cdot \tilde{v}_E) - \mu \Delta \tilde{v}_E$, $\lambda_E = \lambda_2$, and $\tilde{v} \in H^{1/2}(\partial \Omega)$ is a given displacement on $\partial \Omega$. Note that (29) has a unique solution $\tilde{v}_E$ for each $\tilde{v} \in H^{1/2}(\partial \Omega)$ where if $d = 3$, then $\tilde{v}_E \in H^1(\mathbb{R}^d)$ while if $d = 2$, then $\nabla \tilde{v}_E \in L^2(\mathbb{R}^2)$ and

$$\int_{\mathbb{R}^d} \frac{|\tilde{v}_E|^2}{\sqrt{1 + |x|^2}} \, dx < \infty.$$

Ultimately we wish to find the normal stress distribution $(\tilde{\sigma}_E|_{\partial \Omega^+}) \cdot n_{\partial \Omega}$ around $\partial \Omega$ given $\tilde{v}$—this mapping from the displacement on the boundary to the traction on the boundary is defined as the exterior DtN map.

**Definition 4.1.** The exterior DtN map $\Lambda_E$ is defined by

$$\Lambda_E(\tilde{v}_E|_{\partial \Omega^+}) = \Lambda_E(\tilde{v}) \equiv (\tilde{\sigma}_E|_{\partial \Omega^+}) \cdot n_{\partial \Omega},$$

(30)

where $\tilde{v}_E$ solves (29) and $\tilde{\sigma}_E$ and $S \tilde{v}_E$ are given by (5) (with $\lambda(x) = \lambda_E$).

### 4.1. Equivalent boundary value problems

We now return to the problem (10), which has a unique solution $u$. We introduce exterior fields

$$u_E(x) \equiv u \bigg|_{\mathbb{R}^d \setminus \Omega}, \quad \text{and} \quad \sigma_E(x) \equiv \sigma \bigg|_{\mathbb{R}^d \setminus \Omega};$$

(31)

we also introduce interior fields

$$u_I(x) \equiv u \bigg|_{\mathcal{D}}, \quad \text{and} \quad \sigma_I(x) \equiv \sigma \bigg|_{\mathcal{D}}.$$ 

(32)

Recall that $\lambda_E = \lambda_2$.

**Lemma 4.2.** Define $\tilde{u}_E \equiv u_E - f$ where $u_E$ is defined in (31) and $f = Vg$ satisfies $\mathcal{L}_E f = 0$ in $\mathbb{R}^d$. Then $\tilde{v}_E = \tilde{u}_E$ solves (29) with $\tilde{v} = \tilde{u} \equiv (u_I|_{\partial \Omega^+}) - f_0$, where $f_0 = f|_{\partial \Omega^+}$ is defined in (28).

**Proof.** First, since $\mathcal{L}_E u_E = 0$ in $\mathbb{R}^d \setminus \mathcal{D}$ (by (10)) and $\mathcal{L}_E f = 0$ in $\mathbb{R}^d \setminus \mathcal{D}$, we have

$$\mathcal{L}_E \tilde{u}_E = \mathcal{L}_E (u_E - f) = \mathcal{L}_E u_E - \mathcal{L}_E f = 0$$

in $\mathbb{R}^d \setminus \mathcal{D}$ as well. Second, $\tilde{u} \equiv \tilde{u}_E|_{\partial \Omega^+} = (u_E|_{\partial \Omega^+}) - f_0$. Since $\lambda_E = \lambda_2$, $u$ must be continuous across $\partial \Omega$, i.e., $u_I|_{\partial \Omega^+} = u_I|_{\partial \Omega^-}$. Hence $\tilde{u} = (u_I|_{\partial \Omega^+}) - f_0$. Finally, $\tilde{u}_E = u_E - f \to 0$ as $|x| \to \infty$ by (10). Thus $\tilde{u}_E$ solves (29) with $\tilde{u} = (u_I|_{\partial \Omega^+}) - f_0$. \qed
Theorem 4.3. Suppose \( \mathbf{u} \) solves (10) with \( f = Vg \) and \( g = (C_g/2) \mathbf{x} \cdot \mathbf{x} + \mathbf{g}_0 \) where \( C_g \neq 0 \) is an arbitrary constant and \( \Delta g_0 = 0 \) in \( \mathbb{R}^d \). Define \( \mathbf{u}_E, \sigma_E \) and \( \mathbf{u}_I, \sigma_I \) as in (31) and (32), respectively. Finally, define \( \mathbf{u}_E = \mathbf{u}_E - f \). Then \( \mathbf{u}_I \) satisfies

\[
\begin{align*}
L_1 \mathbf{u}_I &= 0 & \text{in } D,
L_2 \mathbf{u}_I &= 0 & \text{in } \Omega \setminus \overline{D},
\mathbf{u}_I, \sigma_I \cdot \mathbf{n}_D &= (S \mathbf{u}_I) \cdot \mathbf{n}_D & \text{continuous across } \partial D, \\
P(\mathbf{u}_I|_{\partial D^+}, (\sigma_I|_{\partial D^+}) \cdot \mathbf{n}_\Omega, f_0, F_0) &= 0 & \text{on } \partial \Omega,
\end{align*}
\]

where

\[
P(\mathbf{u}_I|_{\partial D^+}, (\sigma_I|_{\partial D^+}) \cdot \mathbf{n}_\Omega, f_0, F_0) \equiv (\sigma_I|_{\partial D^+}) \cdot \mathbf{n}_\Omega - A_E \left( (\mathbf{u}_I|_{\partial D^+}) - f_0 \right) - F_0.
\]

and \( f_0 \) and \( F_0 \) are defined in (28).

Proof. By definition (see (10) and (32)), \( \mathbf{u}_I \) satisfies the differential equations and continuity conditions in (33). By lemma 4.2, \( \mathbf{u}_E = \mathbf{u}_E - f \) solves (29) with \( \tilde{\mathbf{u}} = (\mathbf{u}_I|_{\partial D^+}) - f|_{\partial D^+} \). By (30), then, we have

\[
(\tilde{\mathbf{u}}_E|_{\partial D^+}) \cdot \mathbf{n}_\Omega = A_E \left( \tilde{\mathbf{u}}_E \right) = A_E \left( (\mathbf{u}_I|_{\partial D^+}) - f_0 \right).
\]

Since \( S_E \) is linear, we have

\[
(\tilde{\mathbf{u}}_E|_{\partial D^+}) \cdot \mathbf{n}_\Omega = \left( (S_E \tilde{\mathbf{u}}_E)|_{\partial D^+} \right) \cdot \mathbf{n}_\Omega = \left( (S_E \mathbf{u}_E)|_{\partial D^+} \right) \cdot \mathbf{n}_\Omega - F_0.
\]

Then (35) and (36) imply

\[
\left( (S_E \mathbf{u}_E)|_{\partial D^+} \right) \cdot \mathbf{n}_\Omega = A_E \left( (\mathbf{u}_I|_{\partial D^+}) - f_0 \right) + F_0.
\]

Since \( \lambda_E = \lambda_2 \), the traction across \( \partial \Omega \) must be continuous, i.e.,

\[
\left( (S_E \mathbf{u}_E)|_{\partial D^+} \right) \cdot \mathbf{n}_\Omega = (\sigma_I|_{\partial D^+}) \cdot \mathbf{n}_\Omega = (\sigma_I|_{\partial D^+}) \cdot \mathbf{n}_\Omega.
\]

Inserting this into (37) gives

\[
A_E \left( (\mathbf{u}_I|_{\partial D^+}) - f_0 \right) + F_0 = (\sigma_I|_{\partial D^+}) \cdot \mathbf{n}_\Omega.
\]

We define \( P(\mathbf{u}_I|_{\partial D^+}, (\sigma_I|_{\partial D^+}) \cdot \mathbf{n}_\Omega, f_0, F_0) \) as in (34). Then, due to (38), the interior part of the solution \( \mathbf{u}_I \), namely \( \mathbf{u}_I \), satisfies (33).

We can thus identify the solution \( \mathbf{u}' \) of (27) with \( \mathbf{u}_I \) which solves (33), i.e., \( \mathbf{u}' = \mathbf{u}_I = \mathbf{u}|_{\partial D} \). In other words, the solution to (27) in the finite domain \( \Omega \) will be exactly the same as if \( \Omega \) were placed in an infinite medium with Lamé parameters \( \lambda = \lambda_2 \) and \( \mu \) and a displacement \( Vg \) were applied at infinity. Therefore, if we apply the boundary condition

\[
P(\mathbf{u}_0', t_0, f_0, F_0) = t_0 - A_E (\mathbf{u}_0' - f_0) - F_0 = 0
\]

on \( \partial \Omega \), where \( \mathbf{u}_0', t_0, f_0, \) and \( F_0 \) are defined in (28), we can use the measurement of \( \mathbf{u}' \cdot \mathbf{n}_\Omega \) around \( \partial \Omega \) (i.e. \( \mathbf{u}_0' \cdot \mathbf{n}_\Omega \)) along with (26) (with \( \mathbf{u} \) replaced by \( \mathbf{u}' \)) to find the volume fraction occupied by \( D \).
Remark 4.4. Since the geometry inside the body $\Omega$ is unknown, we cannot write $\sigma' \cdot \mathbf{n}_\Omega$ in terms of $u'$ (since the interior DtN map is unknown). Practically, we would typically apply a displacement $u'_0$ around $\partial\Omega$ with a known $f$ and measure the resulting traction $t'_0$ around $\partial\Omega$. The question of whether or not one can find $u'_0$ and $t'_0$ satisfying (39) is tied to the question of whether or not the infinite problem (10) always has a solution since $u'_0$ and $t'_0$ are the displacement and traction around $\partial B_R$ in that problem. The fact that (10) has a unique solution for Lipschitz domains $D$ was established by Ammari and Kang [42, chapter 9].

5. Two dimensional example

The results presented here were first derived in a slightly different form by Han and Wu [38, 39]. We consider the case when $d = 2$ and $\Omega$ is a disk of radius $R$ centered at the origin, denoted $B_R$. In this geometry, it is possible to determine $\Lambda_E$ exactly by first solving (29) for the displacement $\tilde{u}_E$ in terms of 

$$
\tilde{u}_E |_{\partial B_R} = \left( \tilde{\sigma}_E \cdot \frac{x}{R} \right) |_{\partial B_R} = \left[ S_E (\tilde{u}_E) \cdot \frac{x}{R} \right] |_{\partial B_R}.
$$

We state the main results here; the complete calculations are given in work by one of the authors [23]. For more general regions, $\Lambda_E$ may have to be computed numerically.

5.1. Exterior DtN map

We denote the polar components of $\tilde{u}_E$ by $\tilde{u}_{E, r}$ and $\tilde{u}_{E, \theta}$. It is convenient to write

$$
\tilde{u}_E (r, \theta) = \tilde{u}_{E, r} (r, \theta) + i \tilde{u}_{E, \theta} (r, \theta),
$$

where $i = \sqrt{-1}$; see the books by Muskhelishvili [46] and England [47] for more details.

We begin by expanding $\tilde{u} (\theta) = \tilde{u}_r (\theta) + i \tilde{u}_\theta (\theta)$ in a Fourier series, namely

$$
\tilde{u}_r (\theta) + i \tilde{u}_\theta (\theta) = \sum_{n = -\infty}^{\infty} \tilde{u}_n e^{i n \theta},
$$

where

$$
\tilde{u}_n = \frac{1}{2\pi} \int_{0}^{2\pi} \left[ \tilde{u}_r (\theta') + i \tilde{u}_\theta (\theta') \right] e^{-i n \theta'} d\theta'.
$$

Then it can be shown that [23]

$$
\tilde{u}_{E, r} (r, \theta) + i \tilde{u}_{E, \theta} (r, \theta) = \tilde{u}_r R r^{-1} + \sum_{n = 1}^{\infty} \tilde{u}_n R^{n-1} r^{-(n+1)} e^{-i n \theta}
$$

$$
+ \sum_{n=1}^{\infty} \left[ \tilde{u}_n R^{n+1} r^{-(n+1)} \left( \frac{n-1}{\rho_{E}} \right) \tilde{u}_n R^{n-1} r^{-(n+1)} \left( r^2 - R^2 \right) \right] e^{i n \theta}
$$

for $r \geq R$ and where $\rho_{E} \equiv (\lambda_{E} + 3\mu)/(\lambda + \mu)$.

Next we recall from (30) that $\Lambda_E (\tilde{u}_E) = (\bar{\sigma}_E |_{\partial B_R}) \cdot \mathbf{n}_{B_R}$. In polar coordinates, the components of the traction around the boundary of the disk of radius $r \geq R$ are $\tilde{\sigma}_{E, r} (r, \theta) + i \tilde{\sigma}_{E, \theta} (r, \theta)$ (where $\tilde{\sigma}_{E, r}$ is the radial component of the traction and $\tilde{\sigma}_{E, \theta}$ is the angular component of the traction). In particular, the traction around $\partial B_R$ is given by
\[ \mathcal{E}_{E,r}(R^+, \theta) + i\mathcal{E}_{E,\theta}(R^+, \theta) = \Lambda_E(\mathbf{u}) = \sum_{n=-\infty}^{\infty} \mathcal{E}_n e^{i\theta n}, \]  

(42)

where \( \mathcal{E}_{E,r}(R^+, \theta) + i\mathcal{E}_{E,\theta}(R^+, \theta) = (\mathcal{E}_{E,r} + i\mathcal{E}_{E,\theta})|_{\partial B^2}, \)

\[
\mathcal{E}_n = \frac{-2\mu}{\rho}(n + 1)\tilde{u}_n \quad (n \geq 0), \\
\mathcal{E}_{-n} = \frac{-2\mu}{\rho\mu}(n - 1)\tilde{u}_{-n} \quad (n \geq 1),
\]

(43)

and the coefficients \( \tilde{u}_n \) are defined in (40).

5.2. Nonlocal boundary condition

In this section, we derive an expression for the boundary condition \( P(u_0, t_0, f_0, F_0) = 0, \) where \( P \) is defined in (39). We begin by expanding \( f_0 \) in a Fourier series around \( \partial B^2 \); we have

\[
\left. \left( f_r + i\mathcal{E}_r \right) \right|_{\partial B^2} = f_r(R^+, \theta) + i\mathcal{E}_r(R^+, \theta) = f_{0,r}(\theta) + i\mathcal{E}_{0,r}(\theta) = \sum_{n=-\infty}^{\infty} f_{0,n} e^{in\theta},
\]

(44)

where

\[ f_{0,n} = \frac{1}{2\pi} \int_{0}^{2\pi} \left( f_{0,r}(\theta') + i\mathcal{E}_{0,r}(\theta') \right) e^{-in\theta'} d\theta'. \]

Next we define

\[ F \equiv S_E f = S_E V g = \lambda_E \Delta g I + 2\mu V V g, \]

where the last equality holds by (9). Recall from (28) that \( F_0 = (F|_{\partial B^2}) \cdot n_{\partial B^2}. \) In complex notation, the normal components of \( F \) around the boundary of a disk of radius \( r > 0 \) are given by \( F_r(r, \theta) + iF_{\theta}(r, \theta), \) where \( F_r \) is the radial component and \( F_{\theta} \) is the angular component. We can expand \( (F_r + iF_{\theta})|_{\partial B^2} \) in a Fourier series as

\[
\left. \left( F_r + iF_{\theta} \right) \right|_{\partial B^2} = F_r(R^+, \theta) + iF_{\theta}(R^+, \theta) = F_{0,r}(\theta) + iF_{0,\theta}(\theta) = \sum_{n=-\infty}^{\infty} F_{0,n} e^{in\theta},
\]

(45)

where

\[ F_{0,n} = \frac{1}{2\pi} \int_{0}^{2\pi} \left( F_{0,r}(\theta') + iF_{0,\theta}(\theta') \right) e^{-in\theta'} d\theta'. \]

Next we expand \( g \) in a Fourier series around the disk of radius \( r > 0 \) as

\[ g(r, \theta) = \sum_{n=-\infty}^{\infty} g_n(r) e^{in\theta}, \quad \text{where} \quad g_n(r) = \frac{1}{2\pi} \int_{0}^{2\pi} g(r, \theta) e^{-in\theta'} d\theta'. \]
We can write the Fourier coefficients $F_{0,n}$ in terms of the coefficients $g_n$ as

$$F_{0,n} = (\lambda E + 2\mu) \frac{\partial^2 g_n(r)}{\partial r^2} \bigg|_{r=R^+}$$

$$+ \lambda E \left[ \frac{1}{R} \frac{\partial g_n(r)}{\partial r} \bigg|_{r=R^+} - \frac{n^2}{R^2} g_n(R^+) \right]$$

$$+ 2\mu \left[ -\frac{n}{R} \frac{\partial g_n(r)}{\partial r} \bigg|_{r=R^+} + \frac{n}{R^2} g_n(R^+) \right].$$

(46)

Returning to (34), recall that $(u'|_{\partial B_R^+}) - \mathbf{f}_0|_{\partial B_R^+} = \mathbf{u}'_0 - \mathbf{f}_0 = \bar{u}$, where $\mathbf{u}'$ solves (27). Thus if we write

$$\left( u'_r + iu'_\theta \right)|_{\partial B_R^+} = u'_r(R^+, \theta) + iu'_\theta(R^+, \theta)$$

$$= u'_r(\theta) + iu'_\theta(\theta) = \sum_{n=-\infty}^{\infty} u'_{0,n} e^{in\theta},$$

where

$$u'_{0,n} = \frac{1}{2\pi} \int_0^{2\pi} \left[ u'_r(\theta') + iu'_\theta(\theta') \right] e^{in\theta'} d\theta'.$$

then $u'_{0,n} - f_{0,n} = \bar{a}_n$—see (40). The components of the traction $(\sigma'|_{\partial B_R^+}) \cdot \mathbf{n}_{B_R} = t'_0$ can be written in polar coordinates as $t_{0,r}(\theta) + i t_{0,\theta}(\theta)$. This can be expanded in a Fourier series as well, namely

$$\left( \sigma'_r + i\sigma'_\theta \right)|_{\partial B_R^+} = \sigma'_r(R^+, \theta) + i\sigma'_\theta(R^+, \theta)$$

$$= t_{0,r}(\theta) + i t_{0,\theta}(\theta) = \sum_{n=-\infty}^{\infty} F_{0,n} e^{in\theta},$$

where

$$t_{0,n} = \frac{1}{2\pi} \int_0^{2\pi} \left[ t_{0,r}(\theta') + i t_{0,\theta}(\theta') \right] e^{-in\theta'} d\theta'.$$

Recalling the Fourier expansions of $f_0$ given in (44), $F_0$ given in (45) (and (46)), and $\Lambda_E(\bar{u}) = \Lambda_E(\mathbf{u}'_0 - \mathbf{f}_0)$ given in (42)–(43), the condition $P(u'_0, t'_0, f_0, F_0) = 0$ is equivalent to

$$t'_0 - F_0 - \Lambda_E(u'_0 - f_0) = 0$$

$$\Leftrightarrow \sum_{n=-1}^{\infty} \left[ t_{0,n} - F_{0,n} + \frac{2\mu}{R}(n+1)(u'_{0,n} - f_{0,n}) \right] e^{in\theta}$$

$$+ \sum_{n=2}^{\infty} \left[ t'_{0,n} - f_{0,n} + \frac{2\mu}{R\rho_E}(n-1)(u'_{n} - f_{0,n}) \right] e^{-in\theta} = 0.$$  

(47)

Therefore we have the following relationships between the Fourier coefficients of the polar components of the displacement, traction, and applied stress around $\partial B_R^+$.
\[
\begin{aligned}
& l_{0,n} - F_{0,n} + \frac{2\nu}{R} (n + 1) \left( u_{n}^{r} - f_{0,n} \right) = 0 & (n \geq -1), \\
& l_{0,-n} - F_{0,-n} + \frac{2\nu}{R_{E}} (n - 1) \left( u_{-n}^{r} - f_{0,-n} \right) = 0 & (n \geq 2).
\end{aligned}
\]
(48)

**Remark 5.1.** Recall from (27) that \( t_{0}^{r} \) is the traction around \( \partial BR \) due to the applied displacement \( u_{0} \). In practice, one could consider applying a displacement \( u_{0}^{r} \) around \( \partial \Omega \) with a known \( f \) and then measuring \( t_{0}^{r} \) around \( \partial BR \). The applied displacement \( u_{0}^{r} \) and measured traction \( t_{0}^{r} \) have to be such that (47) (and, hence, (48)) holds.

### 5.3. Previous results

Previously, Han and Wu also derived an expression for the exterior DtN map \( \Lambda_{E}(\vec{u}) \) [38, 39]. They found the solution \( \vec{u}_{E} \) to (29) by a method slightly different from the one we used; they then computed the Cartesian components of the traction \( \vec{t}_{E} \cdot n_{E} \) around \( \partial BR \). In particular, if we denote the Cartesian components of \( \vec{u}_{E} \) by \( \vec{u}_{E}^{u} \) and \( \vec{v}_{E}^{v} \), the Cartesian components of the traction \( \vec{t}_{E} \cdot n_{E} \) by \( \vec{X} \) and \( \vec{Y} \), then \( \Lambda_{E}(\vec{u}) = \Lambda_{E}(\vec{u} + i\vec{v}) = \vec{X} + i\vec{Y} \). In particular they showed ([39] equations (29) and (4.1))

\[
\begin{align*}
\vec{X} &= \frac{2 + 2\eta}{1 + 2\eta} \sum_{n=1}^{\infty} \int_{0}^{2\pi} \frac{d^{2}\vec{u}(\theta)}{d\theta^{2}} \cos n(\theta - \theta') \, d\theta' \\
&\quad - \frac{2\eta}{1 + 2\eta} \sum_{n=1}^{\infty} \int_{0}^{2\pi} \frac{d^{2}\vec{v}(\theta)}{d\theta^{2}} \sin n(\theta - \theta') \, d\theta', \\
\vec{Y} &= \frac{2 + 2\eta}{1 + 2\eta} \sum_{n=1}^{\infty} \int_{0}^{2\pi} \frac{d^{2}\vec{u}(\theta)}{d\theta^{2}} \cos n(\theta - \theta') \, d\theta' \\
&\quad + \frac{2\eta}{1 + 2\eta} \sum_{n=1}^{\infty} \int_{0}^{2\pi} \frac{d^{2}\vec{v}(\theta)}{d\theta^{2}} \sin n(\theta - \theta') \, d\theta',
\end{align*}
\]
(49)

where \( \vec{u}(\theta') = \vec{u}_{E}(R^{+}, \theta'), \vec{v}(\theta') = \vec{v}_{E}(R^{+}, \theta') \), and \( \eta = \mu/\lambda_{E} + \mu \). Also see the books by Muskhelishvili [46, section 83] and England [47, section 4.2] for solutions to problems related to (29) based on potential formulations. A proof that our formulas (42)–(43) agree with (49) as long as \( \vec{u} \) is smooth enough is given in the work by Thaler [23].

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