Parabola approximation by a hyperbola when modeling steering mechanism characteristics of tracked vehicles

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Abstract. Steering control quality improvement and movement safety ensuring of tracked vehicles, primarily high-speed ones, is an urgent problem. Characteristic feature of most tracked and transport vehicles transmissions is an almost linear relationship between values of calculated (maximum) turning radii in gears and movement speed, which leads to underutilization of traction and dynamic capabilities of machines. It would be more advantageous to provide a parabolic view for such a dependence, or at least its non-linear character. Two-stream transmission and rotation mechanisms proposed today with a nonlinear characteristic make it possible to realize only a hyperbolic dependence between these quantities, which makes it urgent to search for constructing methods of the best parabola approximation by a hyperbola. Article shows constructing principle of a hyperbolic swing mechanism diagram for transmission with central gearbox and parallel branch drive through three-link differential mechanism, provides an example scheme for it and proposes a technique for constructing best uniform parabola approximation by a hyperbola on a given interval, based on classical problem of uniform approximation of a function by a polynomial. Construction results can be used to determine main parameters of two-line hyperbolic swing mechanisms for transport tracked vehicles with central gearboxes.

1. Introduction
The curvilinear movement of tracked vehicles is accompanied by shear and shift of the deformable soil, including in the transverse direction. When working on arable land, this effect can be neglected in most cases. When driving on virgin soil, the lateral deformation of the soil causes no less damage to the ecological system than the processes associated with the formation of a rut. Lateral deformations increase with an increase in the speed of movement, therefore, transport tracked vehicles, in a first approximation, pose an even greater danger to ecosystems (especially with a weak surface layer, see, for example, works [1,2], in comparison with tracked tractors. However, a particular danger is posed by skidding when the chassis is sliding in the lateral direction. Wheeled transport and traction machines often find themselves in this situation, especially in connection with the tendency to an increase in the specific power of the power plant of tractors. The skid situation is also undesirable from the point of view of traffic safety.
Reducing the risk of skidding is achieved, on the one hand, by improving traffic control systems [3], on the other hand, by developing mechanisms responsible for controlling the distribution of traction force along the sides. For a tracked vehicle, such units are traditionally called swing mechanisms, for wheeled vehicles - power distribution mechanisms.

In the works [4,5,6,7,8,9] it is shown that for tracked and wheeled platforms the most promising way to ensure control of the distribution of traction forces is the development of two-flow mechanisms in the transmission. However, in most of the completed designs and promising developments, there remains an unrealized reserve for optimizing the characteristics of mechanisms, analyzed, for example, in the article [10].

From the point of view of the theory of movement of tracked vehicles [9,10,11], most of the used steering mechanisms and power distribution mechanisms are united by the almost linear nature of the dependence of the calculated (largest) turning radii in gears on the vehicle speed when turning, which leads to an underutilization of traction and dynamic capabilities transport tracked vehicle. It would be more advantageous to provide a parabolic form for such a dependence, or at least its nonlinear character.

Taking as a basis a two-line differential swing mechanism for a machine with a central gearbox and introducing a differential series into it (see, for example, [10, 11], figure 1), it seems possible to ensure the hyperbolic nature of such a dependence. The kinematics of “hyperbolic” rotation mechanisms does not allow otherwise. However, it seems possible to approach the characteristic of the “ideal” (i.e., “parabolic”) rotation mechanism by using a parabola, which best approximates the hyperbola, in determining the parameters of the hyperbolic mechanism. The construction of such an approximation is the goal of this work.

![Diagram of the differential drive of the swing mechanism branch and the view of the plan of angular velocities of such a differential: 0 – power supply from the engine; α – drive of the parallel branch of the swing mechanism; x – power diversion to summing rows; k – kinematic parameter of the planetary mechanism.](image)

**Figure 1.** Diagram of the differential drive of the swing mechanism branch and the view of the plan of angular velocities of such a differential: 0 – power supply from the engine; α – drive of the parallel branch of the swing mechanism; x – power diversion to summing rows; k – kinematic parameter of the planetary mechanism.

2. **Materials and methods**

The object of the research is the transmission and turning mechanisms of transport-traction and transport tracked and wheeled vehicles using the principle of side turning. When simulating the operation of these mechanisms, the methods of theoretical mechanics were used, adapted for use in the theory of motion of tracked and wheeled vehicles (kinematic and power analysis of planetary mechanisms, methods of synthesis of planetary mechanisms, etc.). The results presented in this article are based on the approximation methods used in modern mathematical analysis (Chebyshev's alternance theorem, the problem of the best uniform approximation of a function continuous on an interval by a polynomial, theorems on the existence and uniqueness of a polynomial of the best uniform approximation). At various stages of work, other theoretical methods were used (synthesis, abstraction, generalization, deduction, analogy, computer modeling) and empirical methods (description, comparison).
3. Theoretical research results

The parabola, given by the expression \( P(x) = x^2 \) on the interval \( x \in [a, b] \), is proposed to be approximated uniformly by a hyperbola in the best way.

The hyperbola is specified using three independent parameters (1), on which natural constraints will be imposed (2):

\[
f(x) = \frac{b}{(a-x)} - \tilde{a},
\]

\( \tilde{a} > 1, \quad \tilde{b} > 0, \quad \tilde{d} > 0. \) \hspace{1cm} (2)

Then the problem can be formulated as follows: among functions of the form (1), find one for which the margin of error of the uniform approximation on a given interval is minimal:

\[
\Delta(f) = \max_{[a,b]} |P(x) - f(x)|.
\] \hspace{1cm} (3)

Equivalent formulation: for a function \( g(t) = 1/t \) on an interval \( t \in [\alpha, \beta] \) (hereinafter referred to as an auxiliary interval, a segment), find the polynomial of the best uniform second order approximation \( Q(t) = At^2 - Bt + C \):

\[
\gamma \left( \frac{1}{A} \right) = \gamma \left( \frac{1}{B} \right) = \gamma \left( \frac{1}{C} \right) = 0, \quad i = 1,2
\] \hspace{1cm} (4)

The problem of uniform approximation of a function by a polynomial is the classical problem of the best uniform approximation of a function that is continuous on a segment by a polynomial. The theorems on the existence and uniqueness of the polynomial of best uniform approximation and Chebyshev's theorem on alternance \[12\] are extended to this problem. From the last theorem applied to the problem under consideration, it follows that

\[
Q(\alpha) - g(\alpha) = E, \quad Q(t_1) - g(t_1) = -E, \quad Q(t_2) - g(t_2) = E, \quad Q(\beta) - g(\beta) = -E,
\]

\[ E = \gamma \max_{[\alpha, \beta]} |Q(x) - g(x)|, \quad \gamma = 1 \text{ or } \gamma = -1, \quad \alpha < t_1 < t_2 < \beta. \] \hspace{1cm} (5)

Points \( t_1 \) and \( t_2 \) are called alternance points. These points are the coordinates of the extremum of the function \( Q(x) - g(x) \), and for them the equality is realized:

\[
Q'(t_i) - g'(t_i) = 0, \quad i = 1,2
\] \hspace{1cm} (6)

To search for the optimal parameters of the approximation, we compose a system of equations: in the conditions (5,6) we substitute the dependences adopted for the parabola and hyperbola. The resulting system of equations is nonlinear with respect to the parameters of interest to us:

\[
\frac{1}{\alpha} - A\alpha^2 + B\alpha - C = E, \quad \frac{1}{t_1} - At_1^2 + Bt_1 - C = -E, \quad \frac{1}{t_2} - At_2^2 + Bt_2 - C = E,
\]

\[
\frac{1}{\beta} - A\beta^2 + B\beta - C = -E, \quad -\frac{1}{t_1} - 2At_1 + B = 0, \quad -\frac{1}{t_2} - 2At_2 + B = 0.
\] \hspace{1cm} (7)

The solution of the system is the parameters of the parabola, expressed through the boundaries of the auxiliary interval:
Using dependencies (4), it is now possible to determine the optimal parameters of the hyperbola for approximating the parabola through the boundaries of the auxiliary interval:

\[
A = \frac{4}{\alpha \beta (\sqrt{\beta} + \sqrt{\alpha})}, \quad B = \frac{4 (\alpha + \beta + \sqrt{\alpha \beta})}{\alpha \beta (\sqrt{\beta} + \sqrt{\alpha})}, \quad C = \frac{\beta \sqrt{\beta} + 5 \beta \sqrt{\alpha} + 5 \alpha \sqrt{\beta} + \alpha \sqrt{\alpha}}{2 \alpha \beta (\sqrt{\beta} + \sqrt{\alpha})},
\]

\[
t_1 = \frac{\sqrt{\alpha (\sqrt{\beta} + \sqrt{\alpha})}}{2}, \quad t_2 = \frac{\sqrt{\beta (\sqrt{\beta} + \sqrt{\alpha})}}{2}, \quad E = \frac{(\sqrt{\beta - \sqrt{\alpha}})^3}{2 \alpha \beta (\sqrt{\beta} + \sqrt{\alpha})} = \frac{(\beta - 2 \sqrt{\alpha \beta} + \alpha)^2}{2 \alpha \beta (\beta - \alpha)}.
\]

For the initially specified interval, after transition from \([\alpha, \beta]\) to \([a, b]\), we obtain:

\[
(\alpha + \beta)^2 = 5b^2 + 6ab + 5a^2, \quad \beta - \alpha = b - a, \quad a + b = \sqrt{\alpha \beta},
\]

\[
\bar{b} = \frac{(a + b)^2 \left(\sqrt{5b^2 + 6ab + 5a^2} + 2a + 2b\right)}{4}, \quad \bar{a} = \frac{1}{2} \left(\sqrt{5b^2 + 6ab + 5a^2} + a + b\right),
\]

\[
\bar{d} = \frac{1}{8} \left[7(a + b)^2 - \left(\sqrt{5b^2 + 6ab + 5a^2} - a - b\right)^2\right].
\]

The relationship between the boundaries of the intervals is established by dependencies:

\[
a = \frac{1}{2} \left(\alpha - \beta + \sqrt{\alpha \beta}\right), \quad b = \frac{1}{2} \left(\beta - \alpha + \sqrt{\alpha \beta}\right),
\]

\[
\alpha = \frac{1}{2} \left(-b + a + \sqrt{5b^2 + 6ab + 5a^2}\right), \quad \beta = \frac{1}{2} \left(b - a + \sqrt{5b^2 + 6ab + 5a^2}\right).
\]

For simplicity of notation, we use designations:

\[
a + b = v, \quad 5b^2 + 6ab + 5a^2 = 4u^2, \quad b - a = 2\sqrt{u^2 - v^2}, \quad u > 0
\]

\[
\bar{b} = \frac{v^2 (u + v)}{2}, \quad \bar{a} = \frac{1}{2} (2u + v), \quad \bar{d} = \frac{1}{4} (3v^2 - 2u^2 + 2uv).
\]

The expression \(v/2 = (a + b)/2\) can be interpreted as the middle of the original interval, and \(2\sqrt{u^2 - v^2} = b - a\) as the length of this interval.

Using the introduced notation, the hyperbola is described by the expression:

\[
f(x) = \frac{\bar{b}}{\bar{a} - x} - \bar{d} = \frac{v^2 (u + v)}{v + 2(u - x)} - \frac{1}{4} (3v^2 - 2u^2 + 2uv).
\]

The approximation margin of error is determined by the expressions:
4. Discussion

Thus, the parabola \( P(x) = x^2 \) on the interval \( x \in [a, b] \) can be uniformly approximated by the hyperbola:

\[
a + b = v, \quad 5b^2 + 6ab + 5a^2 = 4u^2, \quad u > 0,
\]

\[
f(x) = \frac{\sqrt{v^2(u+v)}}{a-x} - \frac{d}{2} = \frac{v}{2u+v-2x} - \frac{1}{4} (3v^2 - 2u^2 + 2uv),
\]

\[
(3v^2 + 2uv - 2u^2) = \frac{1}{3} (3v + u - u\sqrt{7})(3v + u + u\sqrt{7}) = \frac{1}{3} ([3v + u^2 - 7u^2]),
\]

\[
(3v^2 - 2u^2 + 2uv) = v(v+2u) - 2(u^2 - v^2).
\]

The minimum approximation margin of error \( \Delta f \) is achieved at four points of the interval, two of which are boundary points, and the rest are given by the expressions:

\[
x_1 = \frac{5a + 3b - \sqrt{5b^2 + 6ab + 5a^2}}{4} = \frac{2v - u - \sqrt{u^2 - v^2}}{2},
\]

\[
x_2 = \frac{5b + 3a - \sqrt{5b^2 + 6ab + 5a^2}}{4} = \frac{2v - u + \sqrt{u^2 - v^2}}{2}.
\]

On the segment \([0, 1]\), (i.e., at \( a = 0, \quad b = 1 \)) we obtain the expressions:

\[
a + b = v = 1, \quad 5b^2 + 6ab + 5a^2 = 4u^2 = 5, \quad 2u = \sqrt{5},
\]

\[
f(x) = \frac{1}{2} \frac{\sqrt{5} + 2}{\sqrt{5 + 1 - 2x}} - \frac{1}{8} (1 + 2\sqrt{5}), \quad \Delta f = \frac{1}{4} (\sqrt{5} - 2),
\]

\[
x_1 = \frac{2v - u - \sqrt{u^2 - v^2}}{2} = \frac{3}{4} - \frac{\sqrt{5}}{4}, \quad x_2 = \frac{2v - u + \sqrt{u^2 - v^2}}{2} = \frac{5}{4} - \frac{\sqrt{5}}{4}.
\]

If the parabola is specified with a coefficient \( \tilde{P}(x) = kx^2 \), this option can be reduced to the problem discussed above by scaling the interval (segment) to \( \sqrt{k} \), that is \( \tilde{a} = a/\sqrt{k}, \quad \tilde{b} = b/\sqrt{k} \).

It should be noted that the proposed hyperbola does not intersect the parabola at the point with coordinates \((0, 0)\). This point corresponds to zero turning radius (turning the machine around the center of gravity). Transmissions of some tracked vehicles with a differential swing mechanism allow this mode of operation (if the power of the power plant is sufficient). At the same time, significant slipping and skidding of the tracks can develop (up to the breakdown of the soil and the impossibility of turning), but the threat of skidding is not of interest, except in the case of turning on a slope. Therefore, such a feature of the proposed approximation can be neglected in most cases.

5. Conclusion

It seems promising to use the final expressions describing the hyperbola that best approximates the parabola in the computational determination of the parameters of the hyperbolic steering mechanism of the tracked vehicle.
The use of “hyperbolic” steering mechanisms will increase the average speed of movement of transport and traction-transport vehicles (in particular, those used in agricultural and forestry complexes), reduce the destructive effect of the propeller on the ground, increase the safety of tractors and reduce the load on the operator (driver).

As applied to wheeled vehicles, “hyperbolic” power distribution mechanisms will improve the environmental characteristics of the propulsion unit and open up additional opportunities in the future for the complete automation of control of agricultural tractors.

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