Bayesian Nonparametrics for Directional Statistics

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Abstract

A density basis of the trigonometric polynomials, suitable for mixture modelling, is introduced. Statistical and geometric properties are derived, suggesting it as a circular analogue to the Bernstein polynomial densities. Nonparametric priors are constructed and strong posterior consistency is obtained for a wide class of densities. We conclude by comparing posterior mean estimates to other circular density estimation methods, also based on trigonometric polynomials, previously suggested in the literature.

1 Introduction

There is increasing interest in the statistical analysis of non-euclidean data, such as data lying on a circle, on a sphere or on a more complex manifold or metric space. Applications range from the analysis of seasonal and angular measurements to the statistics of shapes and configurations (Jammalamadaka and SenGupta, 2001; Bhattacharya and Bhattacharya, 2012). In bioinformatics, for instance, an important problem is that of using the chemical composition of a protein to predict the conformational angles of its backbone (Al-Lazikani et al., 2001). Bayesian nonparametric methods, accounting for the wrapping of angular data, have been successfully applied in this context (Lennox et al., 2009, 2010).

Directional statistics deals in particular with univariate angular data and provides basic building blocks for more complex models. Among the most commonly used model for the probability density function of a circular random variable is the Von-Mises density defined by

\[ u \mapsto \exp(\kappa \cos(u - \mu))/(2\pi I_0(\kappa)), \]

where \( \mu \) is the circular mean, \( \kappa > 0 \) is a shape parameter and \( I_0 \) is the modified Bessel function of the first kind and order 0. This function is nonnegative, \( 2\pi \)-periodic and integrates to one on the interval \([0, 2\pi]\). It can be regarded a circular analogue to normal distribution (Jammalamadaka and SenGupta, 2001) (see also Coeurjolly and Le Bihan (2012) for a comparison with the geodesic normal distribution). Mixtures of Von-Mises densities and other

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log-trigonometric densities are also frequently used (Kent, 1983). Another natural approach
is to model circular densities using trigonometric polynomials

\[ u \mapsto \frac{1}{2\pi} + \sum_{k=1}^{n} (a_k \cos(ku) + b_k \sin(ku)). \] (1.1)

These densities have tractable normalizing constants, but the coefficients \(a_k\) and \(b_k\) must be
constrained as to ensure nonnegativity (Fejér, 1916; Fernández-Durán, 2004).

For a review of common circular distributions, see Mardia and Jupp (2000); Jammalamadaka and SenGupta (2001). Notable Bayesian approaches to directional statistics problems include Ghosh and Ramamoorthi (2003); McVinish and Mengersen (2008); Ravindran and Ghosh (2011); Hernandez-Stumpfhauser et al. (2017).

In this paper, we introduce a basis of the trigonometric polynomials consisting only of
probability density functions. Properties shown in section 2 suggest it as a circular analogue
to the Bernstein polynomial densities and we argue that it is particularly well suited to
mixture modelling. In section 3, we use this basis to devise nonparametric priors on the space
of bounded circular densities. We compare their posterior mean estimates to other density
estimation methods based on the usual trigonometric representation (1.1) in section 4.

An important aspect of nonparametric prior specification is the posterior consistency
property, which entails almost sure convergence (in an appropriate topology) of the posterior
mean estimate. In section 3.2, we thus develop a general prior specification framework that
immediately provides consistency of a class of sieve priors. Particular instances of this frame-
work appeared previously in the literature. For instance, Petrone and Wasserman (2002)
obtained consistency of the Bernstein-Dirichlet prior on the set of continuous densities on the
interval \([0,1]\). More recently Xing and Ranneby (2009) (see also Walker (2004); Lijoi et al.
(2005)) have obtained a simple condition for models of this kind ensuring consistency on the
Kullback-Leibler support of the prior. As an application, they quickly revisit the problem of
Petrone and Wasserman (2002) but without discussing what contains the Kullback-Leibler
support. Our main contribution here is the proof that the Kullback-Leibler support of the
priors specified in our framework contain every bounded density.

## 2 De la Vallée Poussin mixtures for circular densities

### 2.1 The basis

We propose the basis \(\mathcal{B}_n\) for \(2\pi\)-periodic densities of circular random variables given by

\[ C_{j,n}(u) = \frac{2^{2n}}{2\pi \binom{2n}{n}} \left(1 + \cos\left(u - \frac{2\pi j}{2n+1}\right)\right)^n, \quad u \in \mathbb{R}, \quad j = 0, \ldots, 2n, \] (2.1)

with \(\int_0^{2\pi} C_{j,n}(x)dx = 1\).
The rescalings $C_{j,n}^* = (2\pi/(2n+1))C_{j,n}$, $j = 0, \ldots, 2n$, were considered in Röth et al. (2009) in the context of Computer Aided Geometric Design (CAGD). It was shown therein to actually form a basis for the vector space of trigonometric polynomials (of order at most $n \geq 1$) given by

$$\mathcal{V}_n = \text{span}\{1, \cos u, \sin u, \ldots, \cos nu, \sin nu\}.$$ 

One important property of these rescalings to the CAGD community is that the resulting basis forms a partition of unity, meaning that $\sum_{j=0}^{2n} C_{j,n}^*(u) = 1$, for all $u \in \mathbb{R}$. The function $\omega_n = 2\pi C_{0,n}$ is the so-called De la Vallée Poussin kernel which has been studied by Pólya and Schoenberg (1958).

We argue here that $\mathcal{B}_n$ provides an interesting model for densities of circular random variables, representing an angle or located on the circumference of a circle. Here is a formal definition of the angular domain on which we work.

Circular random variables take their values on a circle $S^1$, which we identify to the real line modulo $2\pi$. We therefore write $S^1 = \mathbb{R}$ (mod $2\pi$), so that $S^1$ consists of equivalence classes $\{x + 2\pi k : k \in \mathbb{Z}\}$ and is represented by any half-open interval of length $2\pi$. In the following, we do not distinguish equivalence classes from their representatives. We endow $S^1$ with the angular distance $d$ defined as $d_{S^1}(u, v) = \min_{k \in \mathbb{Z}} |u - v + 2\pi k|$. By the embedding $\theta \mapsto e^{i\theta}$ of $S^1$ as the unit circle of the complex plane $\mathbb{C}$, the angular distance $d_{S^1}$ becomes the arc length distance. For instance, an interval $[a, b) \subset S^1$, $b - a < 2\pi$, can be viewed as an arc of length $b - a$ on the unit circle.

The following result gives elementary properties of the distributions corresponding to the densities in $\mathcal{B}_n$.

**Theorem 2.1.** The random variables on $S^1$ given by $U_j = U + \frac{2\pi j}{2n+1}$, $j = 0, \ldots, 2n$, where $U = (1 - 2V)\cos^{-1}(1 - 2W)$, with $V$ and $W$ independently distributed, $V \sim \text{Ber}(1/2)$ and $W \sim \text{Beta}(1/2, 1/2 + n)$, have (2.1) as densities. Furthermore, by letting $Z_j = e^{iU_j}$ be the
corresponding random variable on the unit circle of \( \mathbb{C} \), we have

\[
E(Z^p_j) = \begin{cases} 
\frac{2^n}{(\frac{n}{2})^n} e^{\frac{2\pi np}{2n+1}}, & \text{if } p \in \{-n, \ldots, n\}, \\
0 & \text{if } p \in \mathbb{Z} \setminus \{-n, \ldots, n\}.
\end{cases}
\] (2.2)

**Proof.** The first part is a straightforward application of the change of variables formula. For the integer moments, we have the equality

\[
E(Z^p_j) = e^{\frac{2\pi np}{2n+1}} E(Z^0_j).
\]

Using the identity

\[
C_{0,n}(u) = \frac{2^{2n}}{2\pi (\frac{2n}{n})^n} \cos^{2n}(u/2), \quad u \in [0,2\pi),
\] (2.3)

and letting \( S \sim \mathcal{U}(S^1) \), we find

\[
E(Z^0_j) = \frac{1}{(\frac{2n}{n})^n} \sum_{k=0}^{2n} \binom{2n}{k} E(e^{-i(n-k-p)S}) = \begin{cases} 
\frac{2^n}{(\frac{n}{2})^n}, & \text{if } p \in \{-n, \ldots, n\}, \\
0 & \text{if } p \in \mathbb{Z} \setminus \{-n, \ldots, n\}.
\end{cases}
\]

\( \square \)

The above integer moments (2.2) are also known as the Fourier coefficients in Feller (1971, p. 631) and as trigonometric moments in the directional statistics jargon, see for instance Mardia and Jupp (2000), Jammalamadaka and SenGupta (2001) and recently Coeurjolly and Le Bihan (2012). From the result for \( p = 1 \), we get that the mean direction of the \( j \)th component is \( e^{\frac{2\pi ip}{2n+1}} \) with the so-called circular variance equal to \( 1/(n+1) \).

### 2.2 The circular density model

Let \( \Delta_{2n} \) be the \( 2n \)-dimensional simplex \( \Delta_{2n} = \{(c_0, \ldots, c_{2n}) \in [0,1]^{2n+1} : c_0 + \cdots + c_{2n} = 1\} \). Our model consists in mixtures of the form

\[
C_n(u; c_0, \ldots, c_{2n}) = \sum_{j=0}^{2n} c_j C_{j,n}(u), \quad u \in \mathbb{R},
\] (2.4)

with \( (c_0, \ldots, c_{2n}) \in \Delta_{2n} \), and \( n \geq 0 \). Let \( \mathcal{C}_n, n \geq 0, \) represent the set of mixtures obtained this way; our model is therefore

\[
\mathcal{C} = \bigcup_{n \geq 0} \mathcal{C}_n.
\] (2.5)

We now give a characterization of the model in terms of trigonometric polynomials. We use the following degree elevation lemma, which is a reformulation of Róth et al. (2009, Theorem 6).

**Lemma 2.2 (Degree elevation formula).** Each \( C_{j,n} \in \mathcal{B}_n \) given by (2.1) can be expressed as

\[
C_{j,n}(u) = \sum_{\ell=0}^{2(n+r)} d^{\ell,r}_{j,n} C_{\ell,n+r}(u),
\] (2.6)
with

$$d^{n,r}_{j,\ell} = \frac{1}{2(n+r) + 1} \left\{ 1 + \frac{2^{(n+r)}}{(2n)} \sum_{k=0}^{n-1} \binom{2n}{k} \cos \left( \frac{2(n-k)\pi\ell}{2(n+r) + 1} - \frac{2(n-k)\pi j}{2n+1} \right) \right\}, \quad (2.7)$$

for $\ell \in \{0, 1, \ldots, 2(n+r)\}$, and $r \geq 0$.

To give the characterization, let $\mathcal{P}_n \subset \mathcal{Y}_n$ be the subset of trigonometric polynomial densities (of order at most $C$ since $k$ is negative here. However, by the degree elevation lemma we have

$$D_{\mathcal{P}_n} \subset \mathcal{P}_n$$

for every $C$. Some of the $c_j$'s may be negative here. However, by the degree elevation lemma we have

$$C_n(u) = \sum_{\ell=0}^{2(n+r)} \left\{ \sum_{j=0}^{2n} c^n_{j,\ell} d^{n,r}_{j,\ell} \right\} C_{\mathcal{P}_n}(u),$$

with $d^{n,r}_{j,\ell}$ given by (2.7). The resulting coefficients $c^{n+r}_{\ell,\ell} = \sum_{j=0}^{2n} c^n_{j,\ell} d^{n,r}_{j,\ell}$ also have the property

$$\sum_{\ell=0}^{2(n+r)} c^{n+r}_{\ell,\ell} = 1,$$

and so it remains to show that there is some $r \geq 0$ such that $c^{n+r}_{\ell,\ell} \geq 0$, for every $\ell = 0, \ldots, 2(n+r)$. To see this, use (2.3) and the binomial identity to write

$$C_n \left( \frac{2\pi \ell}{2(n+r) + 1} \right) = \frac{1}{2\pi} \left\{ 1 + \frac{2^{(n+r)}}{(2n)} \sum_{k=0}^{n-1} \binom{2n}{k} \sum_{j=0}^{2n} c^n_j \cos \left( \frac{2(n-k)\pi\ell}{2(n+r) + 1} - \frac{2(n-k)\pi j}{2n+1} \right) \right\} \quad (2.8)$$

After some manipulations, and using the fact that $k \mapsto \binom{2(n+r)}{k}$ is increasing on $\{0, \ldots, n-1\}$, we find

$$\left| \frac{2(n+r) + 1}{2\pi} c^{n+r}_{\ell,\ell} - C_n \left( \frac{2\pi \ell}{2(n+r) + 1} \right) \right| \leq \alpha_1(n) \left( \sum_{k=0}^{n-1} \binom{2n}{k} \left| \binom{2(n+r)}{k} \right| - 1 \right) \leq \alpha_2(n) \left( \frac{2(n+r)}{(2n+r)} - 1 \right),$$

where $\alpha_1(n), \alpha_2(n) > 0$. A final calculation shows that

$$\frac{2(n+r)}{(2n+r)} - 1 = \frac{(2n+r)(2n+r-1) \cdots (n+r+1)}{(n+r)(n+r-1) \cdots (r+1)} - 1 \leq (1 + n/r)^n - 1.$$

Since $C_n \in \mathcal{P}_n^+$ is positive by assumption, this shows that for large enough $r$, we have $c^{n+r}_{\ell,\ell} > 0$, for every $\ell = 0, \ldots, 2(n+r)$, and therefore $C_n \in \mathcal{C}$.  

As mentioned in the introduction, a criticism made by (Ferreira, Juárez, and Steel, 2008, p.298) concerning the nonnegative trigonometric polynomials proposed by Fernández-Durán (2004) and Fernández-Durán (2007) is that “approximating a function (using nonnegative
trigonometric polynomials) often results in a wiggly approximation, unlikely to be useful in most real applications”.

In the following, we define the notion of cyclic variations to formalize “wiggliness” and show that it can be controlled using our basis.

One way of quantifying “wiggliness” was discussed by Pólya and Schoenberg (1958) via the cyclic variations. For a finite sequence \(x = (x_1, \ldots, x_m)\), \(m \geq 2\), denote by \(v(x)\) the number of sign changes (from positive to negative or vice versa) in the terms of the sequence. Denote by \(\hat{v}(x) = v(x_i, x_{i+1}, \ldots, x_m, x_1, x_2, \ldots, x_{i-1}, x_i)\), \(x_i \neq 0\), the cyclic variation of the sequence, with \(\hat{v}(x) = 0\) if \(x = 0\). This is well defined because \(\hat{v}\) does not depend on the particular index \(i\) such that \(x_i \neq 0\). Notice that the value of \(\hat{v}\) is always an even number not exceeding \(m\). The sequence \(x\) is said to be periodically unimodal if \(\hat{v}(\Delta x) = 2\), where \(\Delta x = (x_2 - x_1, \ldots, x_m - x_{m-1}, x_1 - x_m)\). For a function \(f : \mathbb{S}^1 \to \mathbb{R}\), we make use of the notation

\[
\hat{v}(f) = \sup\{\hat{v}(f(x_i)_m) : 0 \leq x_1 < x_2 < \cdots < x_m < 2\pi, m \geq 2\}
\]

and \(Z(f) = \#\{x \in [0,2\pi) : f(x) = 0\}\). Similarly to the discrete case, such a function \(f\) is said to be periodically unimodal, also called periodically monotone by Pólya and Schoenberg (1958), if \(\hat{v}(f') = 2\), provided \(f'\) exists (a more general definition without the differentiability assumption is given in the latter paper but is not needed in our case).

We have the following results.

**Theorem 2.4.** For \(C_n = \sum_{j=0}^{2n} c_j C_{j,n} \in \mathcal{C}_n\), let \(c = (c_0, \ldots, c_{2n}) \in \Delta_{2n}\). We have

(i)

\[
\hat{v}(C_n - \alpha) \leq Z(C_n - \alpha) \leq \hat{v}\left(\frac{2n + 1}{2\pi} c - \alpha\right), \quad \text{for all } \alpha \geq 0.
\]

(ii) A bound for the total variation of \(C_n\) is given by

\[
\text{TV}(C_n) := \int_0^{2\pi} |C_n'(u)| \, du \leq \frac{2n + 1}{2\pi} \sum_{j=0}^{2n} |c_{j+1} - c_j| \leq (2n + 1)/\pi,
\]

where \(c_{2n+1} = c_0\).

(iii) If \(c = (c_0, \ldots, c_{2n})\) is periodically unimodal, then \(C_n\) is also periodically unimodal.

**Proof.** The proof of (i) follows by Pólya and Schoenberg (1958, Lemma 3) by noticing that

\[
C_n(u) - \alpha = \sum_{j=0}^{2n} \left\{ \frac{c_j}{2\pi} - \frac{\alpha}{2n + 1} \right\} \omega_n \left( u - \frac{2\pi j}{2n + 1} \right), \quad u \in \mathbb{S}^1,
\]

with \(\omega_n = 2\pi C_{0,n}\) the De la Vallée Poussin kernel. Their result says (in this case) that

\[
Z(C_n - \alpha) \leq \hat{v}\left(\frac{c_j}{2\pi} - \alpha/(2n + 1)\right)_{j=0}^{2n},
\]

which implies (i).
To show (ii), let \( P_n : S^1 \to \mathbb{R} \) be the continuous and \( 2\pi \)-periodic, piecewise linear interpolation of the points \((2\pi j/(2n+1), (2n+1)c_j/2\pi) \in S^1 \times \mathbb{R}, j \in \{0, \ldots, 2n\} \). For definiteness,

\[
P_n(u) = \sum_{j=0}^{2n} c_j L_j(u), \quad u \in S^1,
\]

where \( L_j(u) = 0 \vee 2n+1/(2\pi) \alpha 2 \)} \Vert_1 \leq 2, which leads to the assertion \( \text{TV}(C_n) \leq (2n + 1)/\pi \).

For (iii), we assume \( \hat{\nu}(\Delta c) = 2 \) and we want to show that \( \hat{\nu}(C'_n) = 2 \). First, if \( \hat{\nu}(C'_n) = 0 \) then \( C'_n \) is either nonnegative or nonpositive. By continuity of \( C'_n \), we have \( 0 = C_n(2\pi) - C_n(0) = \int_{0}^{2\pi} C'_n(u) \, du \), which implies \( C'_n(u) = 0 \), for all \( u \in [0,2\pi] \), and this gives \( c_i = 1/(2n+1), i = 0, \ldots, 2n \). Thus, \( \hat{\nu}(C'_n) = 2k \), for some \( 1 \leq k \leq n \). The unit circle \( S^1 \) can therefore be partitioned into \( 2k \) open arcs \( A_1, \ldots, A_{2k} \) with \((-1)^jC_n\) being nondecreasing on \( A_j, j = 1, \ldots, 2k \) and with (anticlockwise) end points \( a_1, \ldots, a_{2k} \) (listed in anticlockwise order) being interlaced local minima \( \{a_1, a_3, \ldots, a_{2k-1}\} \) and maxima \( \{a_2, \ldots, a_{2k}\} \) of \( C_n \). Assume \( k > 1 \) and without loss of generality \( a_2 \leq a_4 \). Let \( m = \max\{a_1, a_3\} \). By the monotonicity of \( C_n \) on each arc, each of which being a connected set (relatively to the topology induced by the angular distance \( d \)), the intermediate value theorem gives \( Z(C_n - \alpha) > 2 \) for all \( \alpha \in (m, a_2) \).

By the same argument, using the fact that \( \hat{\nu}(\Delta c) = 2 \), we obtain

\[
\hat{\nu}\left(\frac{2n+1}{2\pi}c - \alpha\right) = \begin{cases} 
2, & \text{if } \alpha \in (\min(c), \max(c)), \\
0, & \text{otherwise},
\end{cases}
\]

contradicting (i), and this implies \( k = 1 \).

\[\square\]

3 Prior specification

We introduce a nonparametric prior on bounded circular densities based on De la Vallée Poussin basis densities (2.1) and the model (2.5). Strong posterior consistency at all bounded density is obtained from a general result of section 3.2. In section 3.3, we relate and apply our approach to Dirichlet process mixtures.
3.1 Circular density prior

Our prior $\Pi$ on the space $\mathbb{F} = \mathbb{F}(S^1)$ of bounded circular densities, parametrized by a Dirichlet process $\mathcal{D}$ and a distribution $\rho$ on $\{1, 2, 3, \ldots\}$, is induced by the random density

$$2N \sum_{j=0}^{2N} \mathcal{D}(R_{j,N})C_{j,N}, \quad N \sim \rho,$$

where $R_{j,n} = \left[ \frac{\pi(2j-1)}{2n+1}, \frac{\pi(2j+1)}{2n+1} \right) \subset S^1$. If $\mathcal{D}$ has a base probability measure $G$ and a concentration parameter $M > 0$, then

$$\Pi(B) = \sum_{n \geq 0} \rho(n)\Pi_n(B \cap \mathcal{C}_n), \quad B \in \mathcal{D},$$

where $\Pi_n = \Pi_{\Delta_{2n}} \circ l_{n}^{-1}$, $\Pi_{\Delta_{2n}}$ is the Dirichlet distribution of parameters $MG(R_{j,n}), \ j = 0, 1, \ldots 2n$, and where $l_{n} : \Delta_{2n} \ni (c_0, \ldots, c_{2n}) \mapsto \sum_{j=0}^{2n} c_j C_{j,n} \in \mathcal{C}_n$.

Strong posterior consistency is obtained using theorem 3.3 of section 3.2. The theorem requires the conditional distributions $\Pi_n$ to have full support on $\mathcal{C}_n$, that $0 < \rho(n) < ce^{-Cn}$ for some $c, C > 0$, and that proper approximation properties of the sieves $\mathcal{C}_n$ are assessed by a sequence $T_n : L^1(\mathbb{M}) \to L^1(\mathbb{M})$ of linear operators, mapping densities to densities, such that $T_n(\mathbb{F}) = \mathcal{C}_n \subset \mathbb{F}$. Here we let $T_n$ be defined by

$$T_n f = \sum_{j=0}^{2n} \int_{R_{j,n}} f(u)C_{j,n}.$$  

The only condition of the theorem that is not readily verified is given in the following lemma.

**Lemma 3.1.** For every continuous function $f$ on $S^1$, $\|T_n f - f\|_\infty \to 0$.

**Proof.** We use lemma A.3, in the appendix (a result is similar to that of Lorentz (1986, theorem 1.2.1)), which gives three sufficient conditions (i) – (iii) for uniform convergence. We denote $d_{3}\left(u, R_{j,n}\right) = \inf_{v \in R_{j,n}} d(u, v)$, and $\text{diam}(R_{j,n}) = \sup_{u, v \in R_{j,n}} d_{3}(u, v)$. Here (i) is immediate by $\text{diam}(R_{j,n}) = 2\pi/(2n + 1)$, $j = 0, \ldots, 2n$, and (iii) follows from the partition of unity property of $\frac{2\pi}{2n+1}C_{j,n}$. Assumption (ii) follows since $C_{0,n}$ is unimodal with mode at 0, and $d_{3}(u, R_{j,n}) \geq \delta > 0$ implies

$$C_{j,n}(u) = C_{0,n}\left( d_{3}\left( u, \frac{2\pi j}{2n+1} \right) \right) \leq C_{0,n}\left( d_{3}(u, R_{j,n}) \right) \leq C_{0,n}(\delta),$$

therefore $\sum_{j: d_{3}(u, R_{j,n}) \geq \delta} \frac{2\pi}{2n+1}C_{j,n}(u) \leq 2\pi C_{0,n}(\delta) \to 0, \ n \to \infty$, uniformly over $u \in S^1$. \hfill $\Box$

The prior may be interpreted similarly as the Bernstein-Dirichlet prior of Petrone (1999). Conditionally on a fixed $n$, the random histogram $H_n = \frac{2n+1}{2\pi} \sum_{j=0}^{2n} c_{j,n} \mathbb{I}_{R_{j,n}}$ is immediately understood through the Dirichlet distribution on $(c_{0,n}, \ldots, c_{2n,n})$. Since $\sum_{j=0}^{2n} c_{j,n}C_{j,n} = T_n H_n$, the following proposition together with lemma 3.1 shows that the finite mixture (3.1) may be seen as a smooth, variation diminishing approximation to $H_n$.
Proposition 3.2 (Variation diminishing property). For every density \( f \) on \( S^1 \), continuous on \( R_{j,n}, \) \( j = 0, \ldots, 2n \), we have \( \hat{v}(T_n f - \alpha) \leq \hat{v}(f - \alpha) \) for all \( \alpha > 0 \).

Proof. This is a straightforward consequence of theorem 2.4 (i). Indeed, by continuity of \( f \), the mean value theorem says that \( P_f(R_{j,n}) = \frac{2\pi}{2n+1} f(u_j) \), for some \( u_j \in R_{j,n}, \) \( j = 0, \ldots, 2n \). It follows that

\[
\hat{v}(T_n f - \alpha) \leq \hat{v}((P_f(R_{0,n}), \ldots, P_f(R_{2n,n})) - \alpha) \leq \hat{v}(f - \alpha), \quad \alpha > 0.
\]

\[ \square \]

3.2 Strong posterior consistency

We show the strong posterior consistency of a general class of priors for bounded density spaces on compact metric spaces. These include sieve priors such as (3.2), as well as a class of Dirichlet process location mixtures (see §3.3). In contrast with Bhattacharya and Dunson (2012), who also obtained general strong consistency result, we consider a prior specification framework, with a different applicability, that does not require continuity and positivity assumptions on the true density from which observations are made.

Here, strong consistency on \( F \) means that if \( X_1, \ldots, X_n \) are independent random variables and identically distributed according to the probability distribution \( P_{f_0} \) with density \( f_0 \in F \), denoted \( (X_i)_{i \geq 1} \sim P_{f_0}^{(\infty)} \), then for all \( \varepsilon > 0 \),

\[
\Pi \left( \left\{ f \in F : \int |f - f_0| < \varepsilon \left( (X_i)_{i=1}^{n} \right) \right\} \right) \rightarrow 1, \quad P_{f_0}^{(\infty)} \text{-a.s.} \quad (3.4)
\]

The general framework is the following. Suppose \( F \) is the space of all bounded densities with respect to some finite measure \( \mathcal{M} \) on a compact metric space \( (M, d) \). Let \( T_n : L^1(M) \rightarrow L^1(M), \) \( n \in \mathbb{N} \), be a sequence of linear operators mapping densities to densities. Consider a model having the form \( \mathcal{C} = \cup_{n \geq 0} \mathcal{C}_n, \) with \( \mathcal{C}_n := T_n(F) \subset F \). Let \( \mathfrak{B} \) be the Borel \( \sigma \)-algebra of \( F \) for the \( L^1 \) metric and let \( \mathfrak{B}_n \) be the restriction of \( \mathfrak{B} \) to \( \mathcal{C}_n, \) \( n \geq 0 \). A prior \( \Pi \) on \( F \) can be specified through priors \( \Pi_n \) on \( (\mathcal{C}_n, \mathfrak{B}_n) \) and a distribution \( \rho \) on \( n \in \{0, 1, 2, \ldots\} \) as

\[
\Pi(B) = \sum_{n \geq 0} \rho(n)\Pi_n(B \cap \mathcal{C}_n), \quad B \in \mathfrak{B}. \quad (3.5)
\]

In theorem 3.3 below, we give simple conditions on \( \Pi_n, T_n \) and \( \rho, \) in this framework, ensuring strong posterior consistency on all of \( F \). The proof is given in the appendix.

Theorem 3.3. Let \( F, \Pi_n, \Pi \) and \( T_n \) be as above. Suppose that \( T_n(F) \subset F \) of finite dimensions bounded by an increasing sequence \( d_n \in \mathbb{N} \), and also that \( ||T_n f - f||_\infty \rightarrow 0, \) \( n \rightarrow \infty, \) for every continuous function \( f \) on \( M \). If \( 0 < \rho(n) < ce^{-Cd_n} \), for some \( c > 0, \) \( C > 0 \) and if \( \Pi_n \) has support \( T_n(F) \), then the posterior distribution of \( \Pi \) is strongly consistent on \( F \).
Remark 3.4. The result still holds when the space $F$ is constrained such as being some convex subset of bounded densities containing at least one density that is bounded away from zero or a star-shaped subset around such a density (e.g. $F$ may be a set of bounded unimodal densities or a set of continuous multivariate copula densities). The precise conditions required on $F$ are stated at the beginning of the appendix.

3.3 Relationship with Dirichlet process mixtures

Here we consider Dirichlet process location mixtures (DPM) on $F$ induced by the random density

$$f = \int_{\mathcal{M}} f(\cdot | \mu, n) \mathcal{D}(d\mu), \quad (3.6)$$

where $\{f(\cdot | \mu, n) | \mu \in \mathcal{M}\} \subset F$ are families of densities, $\mathcal{D}$ is a Dirichlet process and $n$ follows some distribution $\rho$ on $\{1, 2, 3, \ldots\}$. Our circular density prior (3.1) can be seen to take the form (3.6) by letting $f(u | \mu, n) = \sum_{j=0}^{2n} I_{R_j,n}(\mu)C_{j,n}(u)$. This point of view is especially useful in view of the slice sampler of Walker (2007); Kalli et al. (2011) which is tailored to the Dirichlet process mixture.

Furthermore, theorem 3.3 may be applied to a class of such DPMs. The idea is the following. In order to describe properties of (3.6), consider the linear operators $T_n$, $n \in \mathbb{N}$, which maps a probability measure $P$ on $\mathcal{M}$ to the density

$$T_n P = \int_{\mathcal{M}} f(\cdot | \mu, n) P(d\mu). \quad (3.7)$$

If $P$ has some continuous density $p$, then it is natural to require that $\|T_n P - p\|_{\infty} \xrightarrow{n \to \infty} 0$ (see e.g. (Bhattacharya and Dunson, 2012, assumption A2)). If also the image under $T_n$ of all absolutely continuous probability measures is a finite dimensional space, then theorem 3.3 can be applied to ensure strong posterior consistency.

For instance, we can let $f(u | \mu, n) = C_{0,n}(u-\mu)$ (3.8) to obtain a Dirichlet process mixture over a continuous range of locations. The associated operator $T_n$ defined by (3.7), when seen as acting on probability densities, is the De la Vallée Poussin mean of Pólya and Schoenberg (1958). Now for any density $f$ on $\mathbb{S}^1$, $T_nf$ is a trigonometric polynomial of degree $n$ (Pólya and Schoenberg, 1958). Hence the dimension of $T_n(F)$ is bounded above by $2n + 1$. Following general theory about integral operators (DeVore and Lorentz, 1993), it is straightforward to verify that $\|T_n f - f\|_{\infty} \to 0$ uniformly for all continuous $f$. Theorem 3.3 is therefore readily applied to obtain strong posterior consistency.

In section 4, this prior is compared to our circular density prior. Both yield very similar posterior mean estimates.
4 Comparison of density estimates

We compare density estimates based on the De la Vallée Poussin basis and the nonnegative trigonometric sums of Fernández-Durán (2004). Focus is on the expected Kullback-Leibler and $L^1$ losses in the estimation of target densities exhibiting a range of smoothness, skewness and multimodal characteristics.

This section is organized as follows. Section 4.1 introduces the nonnegative trigonometric sums of Fernández-Durán (2004). Section 4.2 defines five density estimators and the method we use for their comparison. The results are presented in section 4.3 while computational implementation is summarized in section 4.4.

4.1 Nonnegative trigonometric sums

Trigonometric polynomials that are probability density functions on the circle can be parameterized by the surface of a complex sphere (Fernández-Durán, 2004). A circular distribution of the corresponding family takes the form

$$f(u; c_0, \ldots, c_M) = \left\| \sum_{k=0}^{M} c_k e^{iku} \right\|^2,$$

where the coefficients $c_k$ are complex numbers such that $\sum_{k=0}^{M} \|c_k\|^2 = \frac{1}{2\pi}$.

The parameterization (4.1) is exploited in Fernández-Durán (2004, 2007); Fernández-Durán and Gregorio-Domínguez (2010); Fernández-Durán and Gregorio-Domínguez (2014a,b) to model distributions of circular random variables. Circular density estimates from i.i.d. samples are obtained therein by maximum likelihood. Goodness of fit for different degrees $M$ of the trigonometric polynomials is assessed using Akaike’s information criterion (AIC) and the Bayesian information criterion (BIC). Recently, a uniform prior on the coefficients $c_k$, with respect to hyperspherical surface measure, has also been considered by Fernández-Durán and Gregorio-Domínguez (2016b) for the Bayesian analysis of circular distributions.

4.2 Methods

The following five estimates of circular densities, denoted $pd$, $pc$, $nAIC$, $nBIC$ and $fdbayes$, are to be compared.

$pd$: The posterior mean estimate based on the De la Vallée Poussin prior (3.1). This prior is parameterized by a Dirichlet process $\mathcal{D}$ and a probability distribution $\rho$ on $\mathbb{N}$. We chose $\mathcal{D}$ to be centered on the circular uniform distribution with concentration parameter $\alpha = 1$, and we let $\rho(n) \propto e^{-n/5}$.

$pc$: The posterior mean estimate based on the Dirichlet process location mixture (3.8). This prior is also parameterized by a Dirichlet process and a distribution $\rho$ on $\mathbb{N}$. We use the same hyperparameters as above.
\( nAIC \): The maximum likelihood estimate of (4.1) where the dimension \( M \) is chosen as to minimize Akaike’s information criterion.

\( nBIC \): The maximum likelihood estimate of (4.1) where the dimension \( M \) is chosen as to minimize the Bayesian information criterion.

\( fdbayes \): The posterior mean estimate based on a uniform hyperspherical distributions on the coefficients \( c_k \) of (4.1) and a uniform prior on \( \{0, 1, 2, \ldots, 5\} \) for the dimension \( M \). This prior on \( M \), uniform on a range \( \{0, 1, \ldots, m\} \) of values, is suggested in Fernández-Durán and Gregorio-Domínguez (2016b). The value of \( m = 5 \), also suggested therein, was chosen as to provide the best performance of this estimator in the comparison of section 4.3.

We assess the quality of a density estimate \( f \) using the Kullback-Leibler loss defined by
\[
\int_{S} \log \left( \frac{f_0(u)}{f(u)} \right) f_0(u) du,
\]
where \( f_0 \) is the target density (Kullback and Leibler, 1951), as well as the \( L^1 \) loss defined by
\[
\int_{S} |f_0(u) - f(u)| du.
\]
This Kullback-Leibler loss is appropriate in the context of discrimination between density estimates (Hall, 1987), while the \( L^1 \) loss is relevant in view of Theorem 3.3. Results obtained using the \( L^2 \) and Hellinger losses were highly similar to those using the \( L^1 \) loss and we omit their presentation.

4.2.1 Target densities

We consider the following two families of target densities to be estimated.

1. The skewed Von Mises family parameterized by \( \alpha \in [0, 1] \) and with densities
\[
v_\alpha(u) \propto (1 + \alpha \sin(u + 1)) \exp(3\alpha \cos(u - \pi)).
\]
2. The chaotic family parameterized by \( \alpha \in [0, 2\pi) \) and with densities
\[
w_\alpha(u) \propto \exp(\sin(\cos(2u) + \sin(3u) + \alpha)).
\]

The first family was obtained by applying the skewing technique of Abe and Pewsey (2011) to Von Mises circular densities. Figure 2 illustrates the two families.

4.3 Results

We estimated the mean Kullback-Leibler loss in 200 repetitions of the estimation of our target densities, for a range of parameter values, using independent samples of sizes 30 and 100. The results are shown in Figure 3 and Figure 4. Confidence intervals at the 95\% level, obtained by bootstrap, are illustrated by vertical bars.

We note that \( pc \) and \( pd \) yield very similar estimates and perform almost exactly the same. The \( pc \) and \( pd \) estimates perform globally well and have the lowest losses in a majority of cases. The \( nAIC \) estimates are highly penalized under the Kullback-Leibler loss, due to their tendency to approach zero at some angular values, and less so under the \( L^1 \) loss.
Figure 2: The skewed Von Mises family of densities (left panel) and the chaotic family of densities (right panel).

4.4 Implementation summary

The $nAIC$ and $nBIC$ density estimates are obtained using the CircNNTSR R package (Fernández-Durán and Gregorio-Domínguez, 2016a). Precisely, we ran the function `nntsmanifoldnewtonestimation` twice from random starting points provided by `nntsrandominitial` and for each degree $M$ of the trigonometric polynomials ranging in $\{0, 1, \ldots, 7\}$. Density estimates with the best AIC and BIC scores were retrieved.

Posterior means corresponding to the $pc$ and $pd$ estimates are approximated using the slice sampler described in Kalli et al. (2011) (see also Yanyun Zhao (2013)). The implementation is straightforward. We ran 80 thousand iterations of the algorithm, of which 20 thousand were treated as burn-in, and sub-sampled down to 20 thousand iterations in order to calculate the posterior mean. Each iteration consisted in the update of every variable in the slice sampler following their full conditional distribution. The distribution of the model dimension $n$ was truncated to the range $\{1, 2, 3, \ldots, 60\}$.

Posterior means for the $fdbus$ estimates are approximated using an adaptation of the reversible jump MCMC algorithm of Fernández-Durán and Gregorio-Domínguez (2016b). We ran a million iterations of the algorithm, treating 100 thousand as burn-in, and sub-sampled down to 20 thousand observations in order to calculate the posterior mean. This number of iterations, exceeding what we observed to be necessary for convergence, was used to ensure convergence across the 7200 different datasets. Each iteration consisted of either a same-dimensional move, or the proposal of a trans-dimensional move.

5 Discussion

We introduced the density basis $C_{j,n}$, $j \in \{0, 1, \ldots, 2n\}$, of the trigonometric polynomials. It is well suited to mixture modelling in the sense that different characteristics of the mixture density $f = \sum_{j=0}^{2n} c_{j,n} C_{j,n}$ can be easily related to the vector $c = (c_{0,n}, c_{1,n}, \ldots, c_{2n,n})$ of
Figure 3: Mean Kullback-Leibler losses for the skewed Von Mises family \( \{v_\alpha\} \) of target densities and different values of the parameter \( \alpha \).
Figure 4: Mean Kullback-Leibler losses for the chaotic family \( \{ w_\alpha \} \) of target densities and different values of the parameter \( \alpha \).
coefficients. For instance, Theorem 2.4 shows that \( f \) is constant if and only if \( c \) is constant; that it is periodically unimodal if \( c \) is periodically unimodal; and that the range of \( f \) is contained between \( \frac{2n+1}{2\pi} \min \{c_{j,n}\}_{j=0}^{2n} \) and \( \frac{2n+1}{2\pi} \max \{c_{j,n}\}_{j=0}^{2n} \). From the cyclic symmetry of the basis, it also follows that \( f \) is symmetric about 0 if the vector \((c_{n+1,n}, \ldots, c_{2n,n}, c_{0,n}, c_{1,n}, \ldots, c_{n,n})\) is symmetric about its center coefficient \( c_{0,n} \). As yet another example, consider the problem of modelling a bivariate angular copula density \( g : S^1 \times S^1 \to [0, \infty) \). Using the De la Vallée Poussin basis, we may let \( g(u, v) = \sum_{i,j=0}^{2n} c_{i,j} C_{i,n}(u) C_{j,n}(v) \). The fact that \( g \) has constant marginal densities follows if the row sums and column sums of the matrix of coefficients \([c_{i,j}]_{i,j}\) are constant. On the interval \([0, 1]\), similar properties of the Bernstein polynomial densities have been exploited for copula modelling and shape constrained regression (Guillotte and Perron, 2012; Chang et al., 2007). The De la Vallée Poussin basis may thus be used to adapt the procedures developed in the unit interval case to the topology of the circle.

Strong posterior consistency of our sieve priors was obtained through the approximation properties of operators based on the De la Vallée Poussin basis. This approximation-theoretic relationship was exploited to ensure that the Kullback-Leibler support of the priors contained every bounded density. A possible refinement of Theorem 3.3 would exploit convergence rates of the operator, as to either weaken our assumption on the prior on the dimension or to obtain posterior contraction rates.
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Appendix A  Proof of theorem 3.3

Proof. Let \( F \) be any space of bounded densities such that for all \( f \in F \), there exists \( h \in F \) with \( \inf_x h(x) > 0 \) and \( \{(1 - \alpha)f + \alpha h : 0 < \alpha < 1\} \subset F \) (the assumption is used only at the end of the proof in Claim 3). We also recall the hypothesis \( \mathcal{C}_n := T_n(F) \subset F \).

Some notations. Let \( \| \cdot \|_\infty \) denote the supremum norm, let \( \| \cdot \|_1 \) denote the \( L^1 \)-norm, and write \( B_1(f_0, \varepsilon) = \{ f \in F : \| f - f_0 \|_1 < \varepsilon \}, \varepsilon > 0 \), for an \( L^1 \)-ball. For a subset \( A \subset F \) and \( \delta > 0 \), let \( N(A, \delta) \) be the minimum number of \( L^1 \)-balls of radius \( \delta \) and centered in \( F \) needed to cover \( A \). Let \( \text{KL}(f_0, f) = \int_{\{f_0 > 0\}} f_0 \log f_0/f \, d\mu \) be the Kullback-Leibler divergence between the densities \( f_0 \) and \( f \), and denote \( B_{\text{KL}}(f_0, \varepsilon) := \{ f \in F : \text{KL}(f_0, f) < \varepsilon \} \). The Kullback-Leibler support of \( \Pi \) is the set of all densities \( f_0 \) such that \( \Pi(B_{\text{KL}}(f_0, \varepsilon)) > 0 \), for all \( \varepsilon > 0 \). Note that the \( \mathcal{B} \)-measurability of \( B_{\text{KL}}(f_0, \varepsilon) \) is shown in Barron, Schervish, and Wasserman (1999, lemma 11).

-A result of Xing and Ranneby (2009). Strong consistency on the Kullback-Leibler support of \( \Pi \) is ensured as a particular case of Xing and Ranneby (2009, theorem 2) (see also Walker (2004); Lijoi et al. (2005)) which we state here in the following lemma (their result is stated in terms of the Hellinger distance which is topologically equivalent to the \( L^1 \)-distance). The fact that \( M \) is a compact metric space satisfies the conditions on \( M \) and \( F \) stated therein. Therefore, once we show that the lemma applies, all we need is to compute the Kullback-Leibler support.

Lemma A.1. Let \( \mathcal{F}_n \subset F, n \geq 0 \), be such that \( \Pi( \cup_n \mathcal{F}_n ) = 1 \). If for any \( 0 < \delta < 1 \) there exists \( \alpha = \alpha(\delta) \) such that \( 1/2 \leq \alpha < 1 \) and

\[
\sum_{n=0}^{\infty} N(\mathcal{F}_n, \delta)^{1-\alpha} \Pi(\mathcal{F}_n)^{\alpha} < \infty, \tag{A.1}
\]

then \( \Pi \) is strongly consistent at all density \( f_0 \) of its Kullback-Leibler support.

-The lemma A.1 applies. Denote \( \overline{\mathcal{C}}_n \) the \( L^1 \)-closure of \( \mathcal{C}_n = T_n(F) \) in \( F \) and let \( 0 < \delta < 1 \). We apply lemma A.1 with the disjoint \( \mathcal{B} \)-measurable sets \( \mathcal{F}_n = \overline{\mathcal{C}}_n \cap \bigcup_{0 \leq k < n} \overline{\mathcal{E}}_k \), so that \( \Pi(\cup_n \mathcal{F}_n ) = \Pi(\cup_n \overline{\mathcal{C}}_n) = 1 \) and \( \Pi(\mathcal{F}_n ) = \sum_{k \geq 0} \rho(k) \Pi_k(\mathcal{F}_n \cap \mathcal{C}_k) \leq \sum_{k \geq 0} \rho(k) \). Let \( d_k \) be the strictly increasing sequence bounding \( \dim(\mathcal{F}_k) \) and such that \( \rho(k) < C e^{-Cd_k} \), so that we find \( \sum_{k \geq n} \rho(k) < C \sum_{k \geq n} e^{-Cd_k} \leq C \sum_{k \geq d_n} e^{-Ck} \propto e^{-Cd_n} \). Moreover, from Lorentz (1966, lemma 1), \( \mathcal{F}_n \) being of dimension at most \( d_n \) and contained in an \( L^1 \)-ball of radius 2, we have...
Lemma A.2. We use the following result from Ghosal, Ghosh, and Ramamoorthi (1999, lemma 5.1).

\[ \rho \]

We can find \( \parallel \cdot \parallel_\infty \) monotone and since \( C_1 \) is finite and equivalent to \( \parallel \cdot \parallel \) on \( C_N \) we can find \( 0 < \varepsilon < \delta \) such that \( B(\rho_{\varepsilon}(f_1, \varepsilon) \cap C_k) > \rho(N)B_{KL}(B_1(T_N f_1, \delta) \cap C_N) > 0 \),

so that for all \( f \in F^+ \),

\[ \text{KL}(f_1, f) \leq \|f_1/f\|_{\infty} f_1 - f_1 \leq \|f_1/f\|_{\infty} (f_1 - T_N f_1) + \|T_N f_1 - f_1\|_1. \]  

(A.2)

Now put \( 0 < \inf_{x \in M} f_1(x) =: m \leq M := \sup_{x \in M} f_1(x) \). By the first claim, there exists \( N \geq 0 \) such that \( \|T_N f_1 - f_1\|_1 < \frac{m}{2}\varepsilon \), for all \( n \geq N \). Furthermore, since \( f \to T_N f \) is monotone and since \( \|T_N m - m\|_\infty \to 0 \), we can assume \( N \) is large enough so that we also have \( \inf_{x \in M} T_N f_1(x) \geq \inf_{x \in M} T_N m(x) \geq m/2 \). Since \( C_N = T_N(F) \subset F \) and is finite dimensional, \( \|T_N m - m\|_\infty \to 0 \), we can find \( 0 < \delta < \frac{m}{2}\varepsilon \) such that \( B_1(T_N f_1, \delta) \cap C_N \subset B_{KL}(T_N f_1, m/4) \cap C_N \). Now for any \( f \in B_1(T_N f_1, \delta) \cap C_N \), the quantity \( \|f_1/f\|_{\infty} \leq 4M/m \), so that by plugging \( N \) in (A.2) we get \( \text{KL}(f_1, f) < \varepsilon \).

Claim 3: \( F \setminus F^+ \subset \text{KL}(I) \).

Let \( f_0 \in F \setminus F^+ \) and let \( 0 < \varepsilon < 6 \). By assumption there is an \( h \in F^+ \) such that \( \{(1-\alpha)f_0 + \alpha h : 0 < \alpha < 1\} \subset F \). Now take \( f_1 = \frac{f_0 + \gamma h}{1 + \gamma} \in F^+ \), with \( \gamma = \varepsilon/6 \), so \( f_0 < (1+\gamma)f_1 \).

We use the following result from Ghosal, Ghosh, and Ramamoorthi (1999, lemma 5.1).

Lemma A.2. If \( f_0 \) and \( f_1 \) are densities with \( f_0 \leq C f_1 \), for some \( C \geq 1 \), then for any density \( f \),

\[ \text{KL}(f_0, f) \leq (C + 1) \log C + C \left[ \text{KL}(f_1, f) + \sqrt{\text{KL}(f_1, f)} \right]. \]

Here \( (2 + \gamma) \log(1 + \gamma) < \varepsilon/2 \). By the second claim and the above lemma, there exists \( \delta > 0 \) and \( N \geq 0 \) such that for \( f \in B_1(T_N f_1, \delta) \cap C_N \), we have \( \text{KL}(f_0, f) < \varepsilon \).

Lemma A.3. Let \( \mu \) be a finite measure on the compact metric space \((M, d)\). For each \( n \geq 0 \), \( d_n \geq 0 \), let \( \{\phi_{i,n}\}_{i=0}^d \) be a set of densities (with respect to \( \mu \)) and let \( \{R_{i,n}\}_{i=0}^d \) be a partition of \( M \). Let \( T_nf = \sum_{i=0}^d \left( \int_{R_{i,n}} f \, d\mu \right) \phi_{i,n} \), \( f \in L^1(M) \). If the three following conditions hold:
(i) \( \max_i \text{diam}(R_{i,n}) \to 0, \) as \( n \to \infty, \) where \( \text{diam}(R_{i,n}) = \sup \{ d(x,y) : x, y \in R_{i,n} \} \),

(ii) for all \( \delta > 0, \) \( \sum_{i : d(x,R_{i,n}) \geq \delta} \mu(R_{i,n}) \phi_{i,n}(x) \to 0, \) uniformly in \( x \in \mathbb{M}, \) where \( d(x,R_{i,n}) := \inf \{ d(x,y) : y \in R_{i,n} \} \),

(iii) \( \sum_{i=0}^{d_n} \mu(R_{i,n}) \phi_{i,n} = 1, \) so that \( T_n c = c, \) for all \( c \in \mathbb{R}, \)

then we have \( \| T_n f - f \|_{\infty} \to 0 \) for every continuous density \( f. \)

Proof. Let \( f \) be a (uniformly) continuous density on \( \mathbb{M} \) and let \( \varepsilon > 0. \) From (iii) we have \( |T_n f(x) - f(x)| \leq \sum_{i=0}^{d_n} \int_{R_{i,n}} |f(y) - f(x)| \mu(dy) \phi_{i,n}(x). \) Take \( \varepsilon > 0, \) there exists \( \delta > 0, \) such that \( |f(y) - f(x)| < \varepsilon/2, \) for all \( y \in B_d(x,\delta). \) Using (i), let \( N \geq 0 \) be chosen so that \( \max_i \text{diam}(R_{i,n}) < \delta/2, \) for all \( n \geq N. \) Notice that for \( n \geq N, \) we have \( \mathbb{M} = B_d(x,\delta) \cup \{ i : d(x,R_{i,n}) \geq \delta/2 \} R_{i,n}; \) this follows from the fact that \( d(x,y) \leq d(x,S) + \text{diam}(S), \) for all \( y \in S \subset \mathbb{M}. \) Therefore,

\[
|T_n f(x) - f(x)| \leq \varepsilon + \delta/2 \int_{R_{i,n} \subseteq B_d(x,\delta)} \mu(dy) \phi_{i,n}(x) + 2 \int_{i \in \{ i : d(x,R_{i,n}) \geq \delta/2 \}} \mu(dy) \phi_{i,n}(x),
\]

this follows from (iii) and (ii) provided \( N \) is further chosen large enough. \( \square \)

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