On the universal ac optical background in graphene

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Abstract. Latest experiments have confirmed the theoretically expected universal value $\pi e^2/2h$ of the ac conductivity of graphene and have revealed departures of the quasi-particle dynamics from predictions for the Dirac fermions in idealized graphene. We present analytical expressions for the ac conductivity in graphene that allow one to study how it is affected by interactions, temperature, external magnetic field and the opening of a gap in the quasi-particle spectrum. We show that the ac conductivity of graphene does not necessarily give a metrologically accurate value of the von Klitzing constant $h/e^2$, because it is depleted by the electron–phonon interaction. In a weak magnetic field, the ac conductivity oscillates around the universal value and the Drude peak evolves into a peak at the cyclotron frequency.

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1. Introduction

The fascinating progress in the study of graphene [1, 2] in many ways is based on the usage of various experimental methods that provide new information on the intriguing electronic properties of this material. An impressive example is provided by the measurements of ac response which include reflectance, transmission and, extracted from them, the conductivity of monolayer graphene [3]–[5], bilayer graphene [6], epitaxial graphene [7] and graphite [8], as reported in 2008. These measurements have made possible a quantitative comparison of the value of the universal ac conductivity background extracted from the experimental data and the theoretical result [9, 10] predicted earlier which at zero temperature can be written in a particularly simple analytical form

\[
\text{Re } \sigma_{xx}(\Omega) = \frac{e^2}{h} |\mu| \frac{4\Gamma}{\Omega^2 + 4\Gamma^2} + \frac{\pi e^2}{2h} \theta (\Omega - 2|\mu|). \tag{1}
\]

Here \(\Omega (> 0)\) is the photon energy, \(\mu\) is the chemical potential tunable by the gate voltage and \(\Gamma\) is the impurity scattering rate which is set to zero in the second, interband term. It is remarkable that the interband conductivity is universal, independent of the band structure parameters. The dependence of the absorption edge on concentration in semiconductors is known as the Burstein–Moss effect. It allows one to measure directly the value of the chemical potential of the system. For finite temperature versions of equation (1) see, for example, [10]–[12].

The theoretical value \(\pi e^2/(2h)\) of the universal ac conductivity of monolayer graphene is in good agreement with experiments in the infrared range of wavelengths \(\lambda \simeq 1500–2500\, \text{nm}\) [5] and even in the visible range with \(\lambda \simeq 400–700\, \text{nm}\) [4]. The last result seems to be well beyond the validity of the Dirac approximation used in [10, 11], but as shown in [13], the corrections to the Dirac-cone approximation are a few per cent only.

Even earlier similar measurements were made in an applied magnetic field on epitaxial graphene [14, 15], monolayer graphene [16, 17], bilayer graphene [18] and highly oriented pyrolytic graphite [19]. The ac conductivity was extracted from the reflectance in this last paper only, while in [14, 16, 18] the relative transmission was considered. Knowledge of relative transmission allows one to investigate the magnetic field dependence of the energy levels and find a band velocity, although this as yet does not allow one to study the redistribution of the
optical spectral weight between different energy regions as done in [8]. On the theoretical side, the magneto-optical response of graphene has been studied in [20]–[23], and including the role of interactions and collective excitations, in [24, 25].

Besides the agreement with early theoretical predictions [9]–[12], [20, 21], several observations [5] indicate the relevance of many-body interactions to the electromagnetic response of graphene. Those include unusually large broadening around 2|μ| which cannot be explained by a thermal smearing of the curve described by (1). Also the conductivity below the threshold of 2|μ| does not vanish due to Pauli blocking as described by equation (1), but shows a significant residual conductivity as strong as 0.3πe^2/(2h) in a wide range of energies at finite doping.

An attempt to explain these results taking into account the effects of disorder and phonons (both optical and acoustic) was made in [26, 27]. Earlier, the influence of optical phonons on the ac conductivity was studied in [28] and recently corrections due to electron–electron interaction in undoped (μ = 0) graphene were discussed [29]–[32].

Thus one may conclude that the investigation of the ac response of graphene is already reaching the same maturity as, for example, for high-Tc superconductors (HTSC) (see [33], sections IV B and C, for a review), where the connection between conductivity and quasi-particle self-energy is exploited to extract information about the bosonic modes and their interaction with carriers. The case of graphene is, however, more complicated than that of HTSC, because one has to take into account the presence of interband transitions.

In this paper, we present some of our latest results on the ac conductivity accentuating the effect of various parameters such as temperature, impurity scattering rate, doping, magnetic field and of interactions such as the electron–phonon interaction on the value of the universal background.

2. Model and general expressions for conductivity

The frequency-dependent electrical conductivity tensor \( \sigma_{\alpha\beta}(\Omega) \) is calculated by using the Kubo formula

\[
\sigma_{\alpha\beta}(\Omega) = \frac{K_{\alpha\beta}(\Omega + i0)}{-i(\Omega + i0)}, \quad K_{\alpha\beta}(\Omega + i0) \equiv \frac{\langle \tau_{\alpha\beta} \rangle}{V} + \frac{\Pi^R_{\alpha\beta}(\Omega + i0)}{\hbar V},
\]

where the retarded current–current correlation function is given by

\[
\Pi^R_{\alpha\beta}(\Omega) = \int_{-\infty}^{\infty} dt \, e^{i\Omega t} \Pi^R_{\alpha\beta}(t), \quad \Pi^R_{\alpha\beta}(t) = -i\theta(t)\text{Tr}\left(\hat{\rho}[J_\alpha(t), J_\beta(0)]\right).
\]

Here, \( V \) is the volume (area) of the system, \( \hat{\rho} = \exp(-\beta H_0)/Z \) is the density matrix with the Hamiltonian \( H_0 \) in the grand canonical ensemble, \( \beta = 1/T \) is the inverse temperature, \( Z = \text{Tr}\exp(-\beta H_0) \) is the partition function, and \( J_\alpha \) are the total paramagnetic current operators. The diamagnetic or stress tensor \( \langle \tau_{\alpha\beta} \rangle \) in the Kubo formula (2) is a thermal average of the diamagnetic part that is nonzero only when a tight-binding model is considered. Accordingly, the restricted optical conductivity sum rule (see e.g. [34, 35] for a review) in graphene has the form [36]

\[
\frac{1}{\pi} \int_{-\infty}^{\infty} d\Omega \, \text{Re} \, \sigma_{xx}(\Omega) = \frac{\langle \tau_{xx} \rangle}{V} = -\frac{e^2a^2}{3h^2} \int_{BZ} \frac{d^2k}{(2\pi)^2} [n_F(\epsilon(k)) - n_F(-\epsilon(k))] \epsilon(k),
\]

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where $\epsilon(k)$ is the full band dispersion of graphene [37], $n_F(\omega) = [\exp((\omega - \mu)/T) + 1]^{-1}$ is the Fermi distribution function and the integration is done over the two-dimensional Brillouin zone (BZ). At large $\Omega \sim t$, where $t$ is the nearest neighbor hopping parameter, the expression for $\sigma_{xx}(\Omega)$ has to be derived from a tight-binding model [36]. As studied in [27], for such large energies it is necessary to include in the analysis a hopping parameter for second-nearest neighbors with a value of the order of $0.1t$.

Still, one may consider the partial sum rules in the low-energy regime [38] based on the linearized continuum approximation for the tight-binding Hamiltonian. A big advantage of this approximation is that it allows one to obtain many transparent analytical results that elucidate the physics of graphene. Even more such insight comes when the low-energy description of graphene is mapped onto 2+1-dimensional electrodynamics [39] (see [40] for a review). In many cases, the effective low-energy QED description provides a very good starting point for the investigation of various transport properties making it possible to profit from the analogies with particle physics and to use its powerful field theoretical methods. In this case, the low-energy quasi-particle excitations in graphene are described in terms of a four-component Dirac spinor $\Psi^{T} = (\psi_{K+As}, \psi_{K+Bs}, \psi_{K-As}, \psi_{K-As})$. The spinor (with a given spin index $s = \pm$) combines the Bloch states on two different sublattices (A and B) of the hexagonal graphene lattice and with momenta near the two inequivalent Dirac points ($K_+$ and $K_-$) of the two-dimensional Brillouin zone. The low-energy Lagrangian density for quasi-particles can be written in a relativistic-like form as

$$L = \sum_{\sigma=\pm} \bar{\Psi}_{\sigma}(t, r) \left[ i \gamma^0 (\hbar \partial_t - i \mu_\sigma) + i \hbar v_F \gamma^1 D_x + i \hbar v_F \gamma^2 D_y - \Delta \right] \Psi_{\sigma}(t, r),$$

(5)

where $\bar{\Psi}_{\sigma} = \Psi^\dagger_{\sigma} \gamma^0$ is the Dirac conjugated spinor, the Fermi velocity $v_F = 3ta_{cc}/(2\hbar) \approx c/300$ plays the role of the speed of light $c$. The orbital effect of a perpendicular magnetic field $B = \nabla \times A$ is included via the covariant derivative $D_\alpha = \partial_\alpha + (ie/\hbar c) A_\alpha$, where $\alpha = x, y$ and the Zeeman energy is included via splitting $\mu_\sigma = \mu - \sigma \mu_B B$ of the chemical potential $\mu$ with $\mu_B = e\hbar/(2mc)$ being the Bohr magneton. The $4 \times 4$ matrices $\gamma^\nu$ furnish a reducible representation of the Dirac algebra. Here, we use the following representation:

$$\gamma^0 = \tilde{\tau}_1 \otimes \tau_0, \quad \gamma^\nu = -i \tilde{\tau}_2 \otimes \tau_\alpha,$$

(6)

where the Pauli matrices $\tilde{\tau}_\alpha$ and $\tau_\alpha$ (as well as the $2 \times 2$ unit matrices $\tau_0$ and $\tau_0$) act on the valley ($K_+$, $K_-$) and the sublattice (A, B) indices, respectively. This representation is derived from the tight-binding model for graphene and to make the treatment more general we also include a mass (gap) term with $\Delta$ (see e.g. review [40]). Accordingly, the Kubo formula (2) can now be calculated using the electric current density operator $j_\alpha(t, r)$ that follows from the Lagrangian (5):

$$J_\alpha(t) = \int d^2r j_\alpha(t, r), \quad j_\alpha(t, r) = \frac{\delta L}{\delta (A_\alpha/c)} = -e v_F \sum_\sigma \bar{\Psi}_\sigma(t, r) \gamma^\alpha \Psi_\sigma(t, r).$$

(7)

All results in this paper are obtained by neglecting the vertex corrections. In this case, the calculation of the conductivity reduces to the evaluation of the bubble diagram. The corresponding polarization operator can be written in the following form:

$$\frac{\Pi_{\alpha\beta}(\Omega + i0)}{V} = -\frac{e^2 v_F^2}{2\pi^2} \int_{-\infty}^\infty d\omega n_F(\omega) \int \frac{d^2k}{(2\pi)^2} \text{tr} \left[ \gamma^\alpha (G^R(\omega, k) - G^A(\omega, k)) \gamma^\beta (G^A(\omega - \Omega, k) + G^R(\omega + \Omega, k)) \right],$$

(8)

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where the tr also includes the summation over spin index and $G^{R(A)}(\omega, \mathbf{k})$ are retarded (advanced) fermion Green’s functions (GF). The approximation of equation (8) does not correctly reproduce the Boltzmann dc conductivity because of the omission of vertex corrections. The current vertex corrections for electron–phonon scattering in graphene within the Kubo formalism for the dc conductivity were considered in [41]. For the Coulomb interaction and ac conductivity, the vertex corrections were considered in [32]. In the absence of a magnetic field, these GF are given by (we also set $\hbar = 1$, so that $\Omega$ can be regarded both as the frequency and energy of the photon)

$$G^{R(A)}(\omega, \mathbf{k}) = \frac{\gamma^0(\omega - \Sigma^{R(A)}(\omega)) - v_F k y + \Delta}{(\omega - \Sigma^{R(A)}(\omega))^2 - v_F^2 k^2 - \Delta^2}. \quad (9)$$

In (9) we included the retarded (advanced) electronic self-energy $\Sigma^{R(A)}(\omega)$ which represents the contribution due to impurities, phonons or electron–electron interactions and which in general also depends on the wave vector $\mathbf{k}$. However, because we study self-energy corrections due to optical phonons in what follows, we omit this dependence. We write all expressions in terms of the retarded self-energy $\Sigma^R(\omega)$ related to the advanced by a complex conjugation $\Sigma^A(\omega) = [\Sigma^R(\omega)]^*$, so that the superscript ‘R’ is dropped.

The calculation of the conductivity follows closely that described in appendix B of [21], hence, we present a final result for the case of a zero magnetic field:

$$\sigma_{xx}(\Omega) = \frac{e^2}{\pi^2 \Omega} \text{Re} \int_{-\infty}^{\infty} d\omega [n_F(\omega) - n_F(\omega + \Omega)] \left[ \frac{(\omega - \Sigma(\omega))(\omega' - \Sigma(\omega')) - \Delta^2}{[\Sigma(\omega') - \Sigma(\omega) - \Omega][2\omega + \Omega - \Sigma(\omega) - \Sigma(\omega')]} - \frac{(\omega - \Sigma(\omega))(\omega' - \Sigma^*(\omega')) - \Delta^2}{[\Sigma^*(\omega') - \Sigma^*(\omega) - \Omega][2\omega + \Omega - \Sigma^*(\omega) - \Sigma^*(\omega')]} \right] \ln[\Delta^2 - (\omega - \Sigma(\omega))^2] + (\omega \leftrightarrow \omega'), \quad (10)$$

where $\omega' = \omega + \Omega$. The expression (10) can be explicitly calculated as was done in [21] for $\text{Re}\Sigma(\omega) = 0$ and in [27] for $\Delta = 0$ including both $\text{Re}\Sigma(\omega)$ and $\text{Im}\Sigma(\omega)$. These final expressions are somewhat cumbersome as compared to the representation given in (10).

3. Ac conductivity in the absence of an external magnetic field and electron–phonon interaction

As mentioned in section 4, for the correct analysis of the ac conductivity with phonons, it is crucial to include both real and imaginary parts of the self-energy $\Sigma(\omega)$. However, to gain some first insight into equation (10) it is useful to consider first the case $\text{Re}\Sigma(\omega) = 0$. Considering the imaginary part of the self-energy, $\Gamma(\omega) = -\text{Im}\Sigma(\omega)$ to be small, one can obtain the following representation [10] for the conductivity

$$\text{Re} \sigma_{xx}(\Omega, T) = \frac{e^2}{\pi^2 \hbar} \int_{-\infty}^{\infty} d\omega \left[ n_F(\omega) - n_F(\omega') \right] \frac{\pi}{4\omega\omega'} \times \left[ \frac{2\Gamma(\omega)}{\Omega^2 + 4\Gamma^2(\omega)} - \frac{2\Gamma(\omega)}{(\omega + \omega')^2 + 4\Gamma^2(\omega)} \right] (|\omega| + |\omega'|)(\omega^2 + \omega'^2), \quad (11)$$
where the first term in square brackets of equation (11) describes the intraband transitions and the second term describes the interband transitions. In the clean limit, \( \Gamma \to 0 \), both terms in the square brackets reduce to delta functions and we obtain a simple analytical expression,

\[
\text{Re} \sigma_{xx}(\Omega) = \frac{4e^2}{h} \frac{2T\Gamma}{\Omega^2 + 4\Gamma^2} \ln \left( 2 \cosh \frac{\mu}{2T} \right) + \frac{\pi e^2}{2h} \left[ n_F \left( -\frac{\Omega}{2} \right) - n_F \left( \frac{\Omega}{2} \right) \right],
\]

where, as in equation (1), the impurity scattering rate is only kept in the intraband (Drude) term for a regularization. In the limit \( \Gamma \to 0 \), the real part of the intraband conductivity can be written as a Dirac delta function

\[
\text{Re} \sigma_{xx}^{\text{intra}}(\Omega) = \frac{4\pi e^2 T}{h} \ln \left[ 2 \cosh \left( \frac{\mu}{2T} \right) \right] \delta(\Omega).
\]

The analysis of experimental data [5, 7] created a need for the various generalizations of equation (1) to finite temperature, finite impurity scattering rate in the interband term and a nonzero gap \( \Delta \). Postponing the analysis of formulae with nonzero gap to the end of the section, we rewrite equation (12) in a slightly different form (see also [11, 12] and a review [42]):

\[
\sigma_{xx}(\Omega) = \sigma_{xx}^{\text{intra}}(\Omega) + \sigma_{xx}^{\text{inter}}(\Omega),
\]

\[
\text{Re} \sigma_{xx}^{\text{intra}}(\Omega) = \frac{e^2}{h} \frac{8T\Gamma}{\Omega^2 + 4\Gamma^2} \ln \left( 2 \cosh \frac{\mu}{2T} \right),
\]

\[
\text{Re} \sigma_{xx}^{\text{inter}}(\Omega) = \frac{\pi e^2}{2h} \frac{\sinh(\Omega/2T)}{\cosh(\mu/T) + \cosh(\Omega/2T)}.
\]

For a finite \( T \) and small \( \Omega \) the interband term in equation (14) is linear in \( \Omega \) with decreasing slope as \( \mu \) is increased. For \( \Omega \gg |\mu| \) and \( \Omega/T \gg 1 \) we recover the universal background value. Therefore, finite temperature effects do not deplete this background. These features are shown in figure 1(a), where we show numerical results for the interband conductivity at two temperatures \( T = 10 \) and 50 K for the neutrality point, \( \mu = 0 \), and for \( \mu = 40 \) meV. We notice the depletion below \( \pi e^2/2h \) of the low frequency region with increasing temperature and particularly with increasing value of the chemical potential. At large frequency \( \Omega \), the background remains unaltered. In all cases, however, the missing spectral weight is transferred to the intraband Drude term not shown in the figure. To establish this we show that a sum rule applies to the spectral weight under the real part of the conductivity. Integrating \( \text{Re} \sigma_{xx}(\Omega) \) up to \( \Lambda \) chosen to be \( \Lambda \gg |\mu| \gg T \), we find

\[
\int_0^\Lambda \text{Re} \sigma_{xx}(\Omega)d\Omega = W_{\text{intra}} + W_{\text{inter}} = \frac{\pi e^2}{2h} \Lambda,
\]

where the intraband and interband spectral weights are

\[
W_{\text{intra}} = \frac{2\pi e^2}{h} T \ln \left( 2 \cosh \frac{\mu}{2T} \right), \quad W_{\text{inter}} = \frac{\pi e^2}{h} T \ln \left( \frac{\cosh \Lambda/2T + \cosh(\mu/T)}{1 + \cosh(\mu/T)} \right).
\]

For \( \Lambda \gg |\mu| \gg T \), we arrive at the last equality in equation (15). This value is exactly equal to its zero temperature and zero \( \mu \) value, when \( \text{Re} \sigma_{xx}(\Omega) \) is simply equal to \( \pi e^2/2h \) for all \( \Omega \).
Figure 1. (a) The interband conductivity $\text{Re} \sigma_{xx}^{\text{inter}}(\Omega)$ in units $\pi e^2/2h$ (universal background): for $\mu = 0$ meV the long-dashed (red) line at $T = 10$ K, the solid (blue) line at $T = 50$ K, and for $\mu = 40$ meV the dash-dotted (black) line at $T = 10$ K, the short-dashed (green) line at $T = 50$ K. There are no interactions and the impurity scattering is set to zero. (b) The conductivity $\text{Re} \sigma_{xx}(\Omega)$ in units $\pi e^2/2h$ for $\mu = 40$ meV. The long-dashed (red) line at $T = 10$ K, the solid (blue) line at $T = 50$ K both are for $\Gamma = 2$ meV. The dash-dotted (black) line at $T = 10$ K and the short-dashed (green) line at $T = 50$ K are for $\Gamma = 10$ meV.

Our results show that the interband contribution to the dc conductivity is zero for any finite temperature. However, there is also an intraband Drude contribution which in the clean limit gives an infinite contribution to the conductivity (see equation (13)). This agrees with the physical expectation that $\sigma_{dc}$ should be infinite at any finite $T$, even at the neutrality point $\mu = 0$. There are always thermally excited electrons and holes present at finite $T$ and these do not scatter in the clean limit; hence the conductivity will be infinite and not zero as claimed by Peres and Stauber [43].

In figure 1(b), we show results for the case of increasing impurity scattering $\Gamma$. In both cases shown the chemical potential is taken to be 40 meV and two values of the temperature are used: $T = 10$ and 100 K. The long-dashed (red) curve and solid (blue) curve are for $\Gamma = 2$ meV, while the dash-dotted (black) curve and short-dashed (green) curve are for $\Gamma = 10$ meV. What is important to note is that for $\Omega > 200$ meV, all curves have risen to the universal background value, equal to one in our units. The width of the absorption edge at $\Omega = 2\mu = 80$ meV depends both on $T$ and $\Gamma$. For example, for the same temperature, larger $\Gamma$ corresponds to a wider edge.

3.1. Ac conductivity in the presence of a gap

Next we consider the possibility of the opening of a gap $\Delta$. The most general expressions for the complex conductivity which include a finite $\Delta$ were derived in [23] (equations (13) and (14)), then rederived in [7] and used to analyze conductivity data in epitaxial graphene (see also [44] for an alternative scenario that preserves the linear dispersion law of carriers in graphene).
In this case, the clean limit generalization of equation (14) is
\[
\text{Re } \sigma_{xx}^{\text{intra}}(\Omega) = \frac{2\pi e^2}{h} \delta(\Omega) \int_{-\infty}^{\infty} d\omega (-n'_F(\omega))(\omega^2 - \Delta^2)\theta(\omega^2 - \Delta^2),
\]
\[
\text{Re } \sigma_{xx}^{\text{inter}}(\Omega) = \frac{\pi e^2}{2h} \frac{\sinh(\Omega/2T)}{\cosh(\mu/T) + \cosh(\Omega/2T)} \frac{\Omega^2 + 4\Delta^2}{\Omega^2} \theta(\Omega^2 - 4\Delta^2).
\] (17)

In the \( T = 0 \) limit, the last expression reduces to a result given in [10]
\[
\sigma_{xx}(\Omega) = \frac{2\pi e^2}{h} \delta(\Omega) \frac{(\mu^2 - \Delta^2)\theta(\mu^2 - \Delta^2)}{\mu} + \frac{\pi e^2}{2h} \frac{\Omega^2 + 4\Delta^2}{\Omega^2} \theta(\Omega - 2\max(\vert\mu\vert, \Delta)).
\] (18)
The first term in equation (18) gives the modified intraband Drude contribution and the second is the modified interband background. It is clear that for \( \Omega \gg \vert\mu\vert, \Delta \) this background again reduces to its universal value.

4. Optical conductivity in the presence of electron–phonon interaction

One may notice that the expression for the GF (9) does not contain the chemical potential. This is because during the derivation of equation (8), the replacement \( \omega \rightarrow \omega - \mu \) moves \( \mu \) from the GF to the Fermi function \( n_F(\omega) \), so that in the GF (9) the value \( \omega = \mu \) corresponds to the Fermi level. In the usual many-body calculations, \( \mu \) is kept in the GF. This implies that when the corresponding self-energy is inserted in the GF (9), its dependence on energy has to be accordingly shifted \( \omega \rightarrow \omega - \mu \). For example, for the simplified single-parameter model proposed in [45] that captures the qualitative features of the e–ph interaction in graphene, the imaginary part of the self-energy reads as
\[
\text{Im } \Sigma(\omega, \mu) = -G[\omega - \text{sgn}(\omega - \mu)\omega_0] \theta[(\omega - \mu)^2 - \omega_0^2].
\] (19)

Here, \( \omega_0 \approx 0.2 \text{ eV} \) is the energy of the in-plane optical phonon mode and \( G \) is the dimensionless parameter determined by matching the model self-energy with the full \textit{ab initio} result [45]. The corresponding real part of the self-energy can be obtained from equation (19) using Kramers–Krönig relations or by directly calculating the self-energy for Dirac quasi-particles coupled to a single Einstein phonon [28, 46, 47]:
\[
\text{Re } \Sigma(\omega, \mu) = \begin{cases} 
\frac{G}{\pi} \left[ \omega_0 \ln \frac{(\omega + \omega_0)^2}{(\omega - \mu)^2 - \omega_0^2} - \omega \ln \frac{W^2(\omega - \mu + \omega_0)}{(\omega + \omega_0)^2(\omega - \mu - \omega_0)} \right], & \mu > 0, \\
\frac{G}{\pi} \left[ \omega_0 \ln \frac{(\omega - \mu)^2 - \omega_0^2}{(\omega - \omega_0)^2} - \omega \ln \frac{W^2(\omega - \mu - \omega_0)}{(\omega - \omega_0)^2(\omega - \mu + \omega_0)} \right], & \mu < 0.
\end{cases}
\] (20)

Here, \( W = \sqrt{\pi \sqrt{3} t} \approx 7 \text{ eV} \) is the energy cutoff which preserves the number of states, with \( t \) being the nearest neighbor hopping parameter. By definition, the carrier effective mass renormalization \( \lambda_{\text{eff}} \) due the electron–phonon interaction is given by
\[
\text{Re } \Sigma(\omega, \mu) = -\lambda_{\text{eff}}(\omega - \mu) + \text{Re } \Sigma^R(\omega = \mu, \mu).
\] (21)
where from (20) we find that
\[ \lambda_{\text{eff}} = \frac{2G}{\pi} \left( \ln \frac{W}{|\mu + \omega_0|} - 1 + \frac{|\mu|}{\omega_0} \right). \] (22)

The role of \( \lambda_{\text{eff}} \) will be clarified in what follows. Choosing the value of \( G = 0.1 \) as done in figure 2(a), for \( \mu = 0 \) one obtains \( \lambda_{\text{eff}} \approx 0.16 \). This is of the order of the theoretical estimate \[ 45 \] and smaller than the value found experimentally \[ 48, 49 \].

So far we have not distinguished the bare chemical potential \( \mu \) of the noninteracting system from the chemical potential \( \tilde{\mu} \) of the interacting system. The value of \( \mu \) is set by the doping \( \rho = \text{sgn}(\mu)\mu^2/\pi \hbar^2 v_F^2 \) (here \( \hbar \) is restored), which is controlled by the gate voltage. In the interacting system, the dispersion law of the quasi-particle excitations is determined by the pole of the GF (9). For \( \Delta = 0 \) and neglecting the imaginary part of the self-energy, one obtains
\[ \omega - \text{Re} \Sigma(\omega, \mu) = \pm v_F |k|, \] (23)
or counting the energy \( E \) of the excitations from the true Fermi surface of the interacting system, \( \omega \to E + \tilde{\mu} \), we obtain the following from equation (23):
\[ E + \tilde{\mu} - \text{Re} \Sigma(E + \tilde{\mu}, \tilde{\mu}) = \pm v_F |k|, \] (24)
where in the vicinity of the Fermi surface, \( \text{Re} \Sigma(E + \tilde{\mu}, \tilde{\mu}) = -\lambda_{\text{eff}} E + \text{Re} \Sigma(\tilde{\mu}, \tilde{\mu}) \), as follows from equation (21). Hence, we arrive at
\[ E(1 + \lambda_{\text{eff}}) + \tilde{\mu} - \text{Re} \Sigma(\tilde{\mu}, \tilde{\mu}) = \pm v_F |k|, \] (25)
which allows us to identify the quantity

$$\tilde{\mu} - \text{Re} \Sigma(\mu, \tilde{\mu}) = \mu$$

(26)

as the bare chemical potential of the noninteracting system. Due to the smallness of $G$, we can assume that the self-energy is also small and solve equation (26) by iterations:

$$\tilde{\mu} = \mu + \text{Re} \Sigma(\mu, \mu).$$

(27)

Thus, as discussed in [50] (see also references therein), in the presence of the electron–phonon interaction one should take into account the renormalization of the noninteracting chemical potential $\mu$ to its interacting value $\tilde{\mu}$. In the numerical results presented below, we fix the value of $\mu$ and then in accordance with equation (27) compute the value of $\tilde{\mu}$ which has to be used as the true chemical potential of the interacting system. As one can see from equation (25), for the massless carriers in graphene, the role of $\lambda_{\text{eff}}$ is to renormalize their velocity: $v_F \rightarrow v_F/(1 + \lambda_{\text{eff}})$ and the value of chemical potential $\mu \rightarrow \mu/(1 + \lambda_{\text{eff}}) \approx \tilde{\mu}$. Restoring the imaginary part of the self-energy, $\Gamma = -\text{Im} \Sigma$ in (25), we find that the quasi-particle width is also renormalized: $\Gamma \rightarrow \Gamma/(1 + \lambda_{\text{eff}})$. These properties allow us to use the previously obtained expressions with the corresponding renormalizations.

Figure 2(a) compares results for the ac conductivity $\text{Re} \sigma_{xx}(\Omega)$ as a function of $\Omega$ to 800 meV with and without phonons at two temperatures: $T = 10$ and 100 K. For $T = 10$ K the dash-dotted (black) curve was calculated using bare bands, a constant residual scattering rate $\Gamma = 0.5$ meV and a chemical potential $\mu = 190$ meV. This curve is to be compared with the long-dashed (red) curve which includes phonons with $G = 0.1$, $\omega_0 = 0.2$ eV and is at the same temperature. To regularize computation when the self-energy $\text{Im} \Sigma(\omega, \mu)$ given by (19) is zero, we use the same value of the residual scattering rate $\Gamma = 0.5$ meV. While the main rise in the conductivity to its universal background is at $2\mu = 380$ meV in the bare band case, the electron–phonon renormalization according to equation (27) has shifted this rise to $2\tilde{\mu}$, the renormalized chemical potential which falls at considerably lower energy. Also note that at higher energies, the bare conductivity rises to its universal value, while the renormalized one falls below, i.e. is slightly less than the one in the chosen units. Increasing the coupling $G$ reduces the background value even more, as can be seen in figure 2 of [28]. We find that for $G = 5$ (which is certainly larger than expected for graphene), the conductivity at $\Omega = 800$ meV has dropped to a value of 0.87. This reduction is expected on physical grounds. Phonons shift optical spectral weight to energies beyond the bare band cutoff on the scale of a few phonon energies. This spectral weight must come from lower energy to conserve the optical sum. We find that the depletion starts for $\Omega > |\mu|$ and shows no structure. There are no wiggles (excess conductivity associated with the Stokes and anti-Stokes lines) of the kind reported in [26, 27]. Our result agrees with an earlier observation made in [28] that these features disappear when the real part (20) of the self-energy is included. This is further confirmed in [51]. The reduction in universal conductivity found here is important, because it shows that the ac conductivity of graphene does not necessarily give a metrologically accurate value of the von Klitzing constant $h/e^2$, although it does give a good estimate. A second feature of the long-dashed (red) dressed curve as compared with the dash-dotted (black) curve for bare bands is the region starting at $\Omega = \omega_0$ and extending to around $2\tilde{\mu}$ (twice the dressed chemical potential). The bare case is totally flat and small in this region, but the dressed curve shows additional absorption starting with a clear ‘kink’ at $\Omega = \omega_0$ and increasing substantially as $\Omega$ grows toward $2\tilde{\mu}$. These are the Holstein boson assisted absorption processes which proceed with the creation of a hole–particle.
pair and in addition a phonon is present in the final state. These processes are well known in ordinary metals and correspond to the incoherent part of the electronic spectral density brought about by coupling to the phonons. At the same time, the coherent Drude like contribution centered at $\Omega = 0$ is reduced by a factor $1 + \lambda_{\text{eff}}$ and contains only a fraction $1/(1 + \lambda_{\text{eff}})$ of the optical spectral weight, while the remainder $\lambda_{\text{eff}}/(1 + \lambda_{\text{eff}})$ is to be found in the incoherent phonon assisted Holstein side band [52]. The short-dashed (green) and solid (blue) curves give results at $T = 100$ K. First note that temperature does not significantly change the value of Re $\sigma_{xx} (\Omega)$ at large $\Omega$ and so the ac background value is robust with respect to temperature changes. In the region of the main rise around the bare (for the short-dashed curve) and renormalized (for solid curve) chemical potential, however, temperature does provide smearing. For the bare band this smearing is more symmetric about twice the chemical potential than it is for the phonon renormalized case, where the signature of a phonon contribution remains as extra absorption in the region above the phonon energy $\omega_0$ and below $2\tilde{\mu}$. Comparison with the long-dashed (red) curve at $T = 10$ K with that at $T = 100$ K shows that the phonon structures are themselves smeared by temperature and are best seen at low temperature.

This is seen better in figure 2(b) where we show the phonon renormalized value of Re $\sigma_{xx} (\Omega)$ for four temperatures, namely, $T = 10, 50, 100$ and $200$ K. At $T = 200$ K, the short-dashed (green) curve becomes rather smooth and no sharp phonon structure remains, but smearing about $\Omega = 2\tilde{\mu}$ is nevertheless still quite asymmetric and in that sense retains information on phonon renormalization.

There is another way to observe phonon renormalization in Re $\sigma_{xx} (\Omega)$ curves that is worth commenting on. It was shown in [50] that at $T = 0$ for $|\mu| > \Omega$ (photon energy) and small residual scattering rate $\Gamma$, the low frequency conductivity is dominated by the intraband transitions and that Re $\sigma_{xx} (\Omega)$ takes on a Drude like profile with a single change over the bare band case, namely,

$$\text{Re } \sigma_{xx} (\Omega) = \frac{e^2}{h} |\mu| \frac{4\Gamma}{\Omega^2(1 + \lambda_{\text{eff}})^2 + 4\Gamma^2} \tag{28}$$

with the phonon renormalization $\lambda_{\text{eff}}$ as defined in equation (21) and in the lowest approximation given by equation (22). Defining the ratio of renormalized to unrenormalized conductivity as $R(\Omega, T)$, we obtain at zero temperature

$$R(\Omega, T = 0) = \frac{\Omega^2 + 4\Gamma^2}{\Omega^2(1 + \lambda_{\text{eff}})^2 + 4\Gamma^2}, \tag{29}$$

which starts at a value of 1 for $\Omega = 0$ and saturates to a value of $1/(1 + \lambda_{\text{eff}})^2$ at $\Omega \gg 2\Gamma$.

Numerical results for a value of $R(\Omega, T)$ are shown in figure 3 for four temperatures: $T = 10, 100, 200$ and $300$ K. All curves start at a value of 1 and are close to a saturation value of $\sim 0.72$ by $\Omega = 6$ meV with a small decrease in saturation value at higher temperature taken to mean that the effective phonon renormalization $\lambda_{\text{eff}}$ has increased slightly with temperature. This is a well-known effect in simple metals [52]. Of course, in an actual experiment, one does not have access to the bare band conductivity and the ratio $R(\Omega, T)$ cannot be formed. However, one can take ratios at different temperatures to verify that the electron–phonon renormalization is nearly a constant in temperature, as found in this work. What is more important is that figure 3 clearly shows that the intraband Drude type renormalizations of equation (28) are significant for the parameters used here and should be observable.

While we have digressed somewhat from our main theme to describe the more general effects of phonons on conductivity, our important result is that, in principle, the electron–phonon
interaction reduces the value of the universal background below its bare band value. Admittedly for the parameters characteristic of graphene, this effect is not expected to be large. We note that in [53], the conclusion was drawn that in general the effects of Fermi liquid corrections on the universal background are small.

5. Ac conductivity in an external magnetic field without phonons

As we already mentioned in the introduction, in papers [21]–[23], [36] we have derived and analyzed various expressions for the diagonal and off-diagonal magneto-optical conductivity of graphene. The derivation is also based on the analytical calculation of the bubble diagram (8), but instead of the GF (9) one has to substitute the GF describing Dirac fermions in an external magnetic field.

Particularly useful are equations (11) and (12) of [23]. The first of these equations is for the complex diagonal conductivity:

\[
\sigma_{xx}(\Omega) = e^2 v_F^2 |eB|(\Omega + 2i\Gamma) \sum_{n=0}^\infty \left\{ \left( 1 - \frac{\Delta^2}{M_n M_{n+1}} \right) \times \left[ \frac{n_F(M_n) - n_F(M_{n+1})}{(M_{n+1} - M_n)^2 - (\Omega + 2i\Gamma)^2} \right] + \frac{1}{M_{n+1} - M_n} \right\} + \left( 1 + \frac{\Delta^2}{M_n M_{n+1}} \right) \left[ \frac{n_F(-M_n) - n_F(-M_{n+1})}{(M_{n+1} + M_n)^2 - (\Omega + 2i\Gamma)^2} \right] + \frac{1}{M_{n+1} + M_n} \right\} \]  

\[
\times \left[ \frac{n_F(-M_n) - n_F(M_{n+1})}{(M_{n+1} + M_n)^2 - (\Omega + 2i\Gamma)^2} \right] + \frac{1}{M_{n+1} + M_n} \right\}.  
\]  

(30)
where the energies of the relativistic Landau levels are
\[ E_n = \pm M_n, \quad M_n = \sqrt{\Delta^2 + 2n v_f^2 |eB|/c}. \]  (31)

In equation (31) the Landau energy scale \( L(B) \) associated with the magnetic field expressed in units of temperature and energy reads (restoring \( \hbar \)) as
\[ L^2(B) = \frac{2eBv_f^2}{c} \rightarrow \frac{2eBhv_f^2}{c^2} = 1.7696 \times 10^{-7} K^2 v_f^2 (m s^{-1}) B(T), \]
\[ L^2(B) \approx 1314 \text{ meV}^2 B(T), \quad L(B) \approx 36.3 \text{ meV}\sqrt{B(T)}, \quad E_n \approx \pm \sqrt{nL(B)}, \]  (32)

where the field \( B \) is given in tesla and \( v_f \) is the Fermi velocity in graphene given in m s\(^{-1}\) and in the second line it is chosen to be \( v_f \approx 10^8 \text{ m s}^{-1} \) (in what follows we set \( \Delta = 0 \)). It is assumed in equation (30) that all Landau levels have the same width; for the case of Landau levels with different widths, see equation (9) of [23].

To study the limit of a weak magnetic field, we apply the Poisson summation formula and express the conductivity (30) as follows:
\[ \text{Re} \sigma_{xx}(\Omega) = \frac{e^2}{4\pi\hbar} \left[ \frac{1}{2} L^2 f(0) + \int_0^\infty dx f(x) + \sum_{k=1}^\infty \int_0^\infty dx f(x) \cos \left( \frac{2\pi kx}{L^2} \right) \right], \]  (33)

where \( f(x) = f^{\text{intra}}(x) + f^{\text{inter}}(x) \). The intraband and interband contributions are given by
\[ f^{\text{intra}}(x) = \frac{n_F(\sqrt{x}) - n_F(\sqrt{x + L^2}) + n_F(\sqrt{x} - L^2) - n_F(\sqrt{x})}{\sqrt{x + L^2} - \sqrt{x}} \times \left[ \frac{2\Gamma}{(\sqrt{x + L^2} - \sqrt{x} - \Omega)^2 + (2\Gamma)^2} + \frac{2\Gamma}{(\sqrt{x + L^2} - \sqrt{x} + \Omega)^2 + (2\Gamma)^2} \right], \]  (34)

and
\[ f^{\text{inter}}(x) = \frac{n_F(\sqrt{x}) - n_F(\sqrt{x + L^2}) + n_F(\sqrt{x} - L^2) - n_F(\sqrt{x})}{\sqrt{x + L^2} + \sqrt{x}} \times \left[ \frac{2\Gamma}{(\sqrt{x + L^2} + \sqrt{x} - \Omega)^2 + (2\Gamma)^2} + \frac{2\Gamma}{(\sqrt{x + L^2} + \sqrt{x} + \Omega)^2 + (2\Gamma)^2} \right]. \]  (35)

Although the representation (33)–(35) looks rather complicated, its limiting case provides a lot of insight into the behavior of the conductivity. For example, evaluation of equation (33) in the high frequency limit, \( \Omega \gg L(B), |\mu|, \Gamma \), gives
\[ \text{Re} \sigma_{xx}(\Omega) \approx \frac{e^2}{2\hbar} \left[ \frac{\pi + 4\Gamma}{\Omega} \sum_{k=1}^\infty \int_0^\infty du \left[ \frac{1}{(u + 1)^2 + (2\Gamma/\Omega)^2} + \frac{1}{(u - 1)^2 + (2\Gamma/\Omega)^2} \right] \right] \times \cos \left( \pi k \frac{u^2\Omega^2}{2L^2(B)} \right). \]  (36)

The first term of equation (36) corresponds to the universal ac background, while the second term describes an oscillatory behavior about that value. In figure 4(a) we illustrate this analytical result and present the ac conductivity \( \text{Re} \sigma_{xx}(\Omega) \) in units \( \pi e^2/2\hbar \) as a function of \( \Omega \) to 150 meV.
**Figure 4.** (a) The conductivity $\sigma_{xx}(\Omega)$ in units $\pi e^2/2h$ (universal background) for temperature $T = 10$ K, scattering rate $\Gamma = 2.5$ meV. Long-dashed (red) line, chemical potential $\mu = 30$ meV and magnetic field $B = 0.1$ T, dash-dotted (black) line $\mu = 30$ meV and $B = 0.2$ T, solid (blue) line $\mu = 30$ meV and $B = 0.3$ T, short-dashed (green) line $\mu = 20$ meV and $B = 0.2$ T. (b) The conductivity $\sigma_{xx}(\Omega)$ in units $\pi e^2/2h$ (universal background) for magnetic field $B = 0.1$ T, temperature $T = 10$ K and scattering rate $\Gamma = 1$ meV. Long-dashed (red) line, chemical potential $\mu = 50$ meV, dash-dotted (black) $\mu = 40$ meV, solid (blue) $\mu = 30$ meV, short-dashed (green) $\mu = 10$ meV. All thick lines are computed using the full expression (30) and the thin lines using the first line of equation (39).

computed using formula (30). It turns out that even for the smallest value of the field considered, $B = 0.1$ T, it is sufficient to include 100 terms in the sum. We observe indeed that for large $\Omega$, the conductivity oscillates around the universal value.

In the clean limit, $\Gamma \rightarrow 0$, and for fields $L(B) \gg |\mu|$ the main contribution to the integrals in equation (33) comes from the Lorentzians, because they reduce to delta functions, and thus we obtain

$$
\sigma_{xx}(\Omega) = \frac{e^2}{4h} \left[ n_F(-L) - n_F(L) \right] \left[ \frac{2\Gamma L}{\pi [(L - \Omega)^2 + (2\Gamma)^2]} + \frac{2\Gamma L}{\pi [(L + \Omega)^2 + (2\Gamma)^2]} \right] + \left[ n_F \left( \frac{L^2 - \Omega^2}{2\Omega} \right) - n_F \left( \frac{-L^2 - \Omega^2}{2\Omega} \right) - n_F \left( \frac{L^2 + \Omega^2}{2\Omega} \right) + n_F \left( \frac{-L^2 + \Omega^2}{2\Omega} \right) \right] \times \left[ \frac{\Omega^4 - L^4}{2\Omega^4} + \frac{2}{\pi \Omega} \sum_{k=1}^{\infty} \int_0^\infty dx \left[ \frac{2\Gamma}{(\sqrt{x + L^2} - \sqrt{x - \Omega})^2 + (2\Gamma)^2} \right. \cos \left( \frac{2\pi kx}{L^2} \right) \right] \right],
$$

(37)
where we kept $\Gamma$ finite in the places crucial for the regularization of the conductivity. This high field regime $L(B) \gg |\mu|$ is, in principle, well described by the original representation (30). For example, the first term of (37) corresponds to the absorption line at $\Omega = E_1 - E_0 = L(B)$. In the weak fields, $L(B) \lesssim |\mu|$, the intraband part has an additional sharp (at small temperatures) maximum which gives the following contribution:

$$
\int_0^\infty \! dx f(x) \simeq \int_0^\infty \! dx \left( -\frac{\partial n_F(\sqrt{x})}{\partial \sqrt{x}} + \frac{\partial n_F(-\sqrt{x})}{\partial \sqrt{x}} \right) \left[ \frac{2\Gamma}{(\frac{L^2}{2\sqrt{x}} - \Omega)^2 + (2\Gamma)^2} + \frac{2\Gamma}{(\frac{L^2}{2\sqrt{x}} + \Omega)^2 + (2\Gamma)^2} \right].
$$

(38)

Since the maximum of the integral is at the point $x = \mu^2$, one can replace $L^2/2\sqrt{x}$ by the cyclotron frequency $\omega_c = |eB|v_F^2/(c|\mu|)$;\(^4\) then the remaining integrals are evaluated exactly and we obtain

$$
\text{Re} \, \sigma_{xx}(\Omega) = \frac{2e^2}{h} T \ln \left( 2 \cosh \frac{\mu}{2T} \right) \left[ \frac{2\Gamma}{(\omega_c - \Omega)^2 + 4\Gamma^2} + \frac{2\Gamma}{(\omega_c + \Omega)^2 + 4\Gamma^2} \right]$$

$$
+ \frac{e^2 L}{2h} \left[ n_F(-L) - n_F(L) \right] \left[ \frac{2\Gamma}{(L - \Omega)^2 + 4\Gamma^2} + \frac{2\Gamma}{(L + \Omega)^2 + 4\Gamma^2} \right]$$

$$
+ \frac{\pi e^2}{4h} \left[ \tanh \frac{\Omega + 2\mu}{4T} + \tanh \frac{\Omega - 2\mu}{4T} \right], \quad E_1 = L(B) \lesssim |\mu|,
$$

(39)

where similarly to equations (1) and (14), the impurity scattering rate $\Gamma$ was set to zero in the interband term. We note that setting $B = 0$ in equation (39), we arrive at equations (13) and (14) from [23] mentioned in section 3.1. Equation (39) illustrates that the universal ac background is robust to the perturbations provided by a weak magnetic field. For nonzero, but small $B$ such that $E_1 = L(B) \lesssim |\mu|$ (or equivalently, for $2\hbar \omega_c \lesssim |\mu|$), the first line of (39) with the same coefficient $\ln[2 \cosh(\mu/2T)]$ as in the $B = 0$ expression (14), is much larger than the second line of (39). In this case, the spectral weight $W_{\text{intra}}$ given by equation (16) of the Drude peak at $\Omega = 0$ from equation (14) is transferred to the cyclotron peak at $\Omega = \omega_c$ which has exactly the same field independent spectral weight. Further, as the field increases, the dependence on this field spectral weight of the second line also grows and a second peak at $\Omega = E_1$ develops. As the field grows and becomes $E_1 = L(B) \gtrsim |\mu|$, one has to use the appropriate expression in this limit (37) which has only one low-energy peak at $\Omega = E_1$.

In figure 4(b) we show the evolution of the Drude peak to the cyclotron peak at $\Omega = \omega_c$ observed when $E_1 \lesssim |\mu|$. One can see that for $\mu = 50$ meV, the thick long-dashed (red) curve is indistinguishable from the thin long-dashed (red) curve computed by using the approximate expression based on the first line of equation (39). For $\mu = 40$ meV, thick and thin dash-dotted (black) lines are already distinguishable, but show that the agreement between the full expression (30) and the first line of equation (30) is very good. The agreement remains good for a lower value of $\mu = 30$ meV (for solid (blue) lines). However, when $\mu$ gets close to $E_1 = 11.5$ meV, as shown in the short-dashed (green) curve for $\mu = 12$ meV, the approximation based on (39) loses its validity. The position of the first peak on the thick short-dashed line is shifted from $\Omega = \omega_c$ to lower energy and a second peak at $\Omega = E_1$ develops.

\(^4\) Accordingly, we have $\hbar \omega_c (\text{meV}) = 657 \, \text{meV}^2 B(T)/\mu(\text{meV})$.

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**Supplementary Material**

**Supplementary Figure 1**: This figure provides additional insights into the theoretical framework discussed in the main text. It illustrates the impact of magnetic field on the spectral weight distribution, highlighting the transition from Drude to cyclotron peaks. The theoretical predictions are compared with experimental data, demonstrating the robustness of the model under varying magnetic field strengths.

**Supplementary Table 1**: This table outlines the parameters used in the theoretical calculations, including the magnetic field strength, temperature, and key energy values, which are crucial for understanding the spectral weight distribution.

**Supplementary Equation A**: This equation extends the theoretical framework to include corrections for finite temperature effects, providing a more comprehensive model for the observed spectral weight distribution.

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6. Discussion

We have studied how the universal ac conductivity background in graphene is affected by a finite chemical potential, finite temperature, elastic impurity scattering, application of a weak magnetic field, as well as the opening of a finite gap which changes the quasi-particles into massive relativistic-like fermions. While large changes in the ac conductivity are found to arise at low energies, when the photon energy is increased to a value much larger than any of the energy scales involved, the corrections to the universal conductivity coming from the above-mentioned sources are small and the background value remains close to $\pi e^2/(2h)$. On the other hand, the changes found when the electron–phonon interaction is accounted for can be more significant for large values of coupling to high energy phonons. These renormalizations increase the bandwidth on the magnitude of the phonon energy scale and thus transfer optical weight to states beyond the bare band cutoff and consequently reduce the background at intermediate energies. Of course, if the actual electron–phonon coupling in graphene is very small, then these effects will also be small. Nevertheless, it is important to note that the background value can be affected by various agencies and need not be exactly $\pi e^2/(2h)$. The electron–phonon interaction also renormalizes the value of the chemical potential, measurable as a threshold in the absorption.

We find that this interaction, while providing phonon assisted sidebands called Holstein processes in metal physics, does not introduce wiggles in the background value as confirmed in [51]. We demonstrated that in a weak external magnetic field, the universal value of the ac conductivity does not change while the conductivity itself oscillates around this value. Finally, we studied the evolution of the Drude peak in graphene with a magnetic field.

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