SPECTRAL PROPERTIES OF A LIMIT-PERIODIC SCHRÖDINGER OPERATOR IN DIMENSION TWO.

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Abstract. We study Schrödinger operator $H = -\Delta + V(x)$ in dimension two, $V(x)$ being a limit-periodic potential. We prove that the spectrum of $H$ contains a semiaxis and there is a family of generalized eigenfunctions at every point of this semiaxis with the following properties. First, the eigenfunctions are close to plane waves $e^{i\langle\vec{k},x\rangle}$ at the high energy region. Second, the isoenergetic curves in the space of momenta $\vec{k}$ corresponding to these eigenfunctions have a form of slightly distorted circles with holes (Cantor type structure). Third, the spectrum corresponding to the eigenfunctions (the semiaxis) is absolutely continuous.

1. Introduction.

We study the operator

$$H = -\Delta + V(x)$$

(1.1)

in two dimensions, $V(x)$ being a limit-periodic potential:

$$V(x) = \sum_{r=1}^{\infty} V_r(x),$$

(1.2)

where $\{V_r\}_{r=1}^{\infty}$ is a family of periodic potentials with doubling periods and decreasing $L_\infty$-norms. Namely, $V_r$ has orthogonal periods $2^{r-1}\vec{d}_1$, $2^{r-1}\vec{d}_2$ and $\|V_r\|_\infty < C\exp(-2^{\eta r})$ for some $\eta > \eta_0 > 0$. Without loss of generality we assume that $\tilde{C} = 1$, $\vec{d}_1 = (d_1,0)$, $\vec{d}_2 = (0,d_2)$ and $\int_{Q_r} V_r(x)dx = 0$, $Q_r$ being the elementary cell of periods corresponding to $V_r(x)$. We assume that all $V_r(x)$ are trigonometric polynomials with the lengths growing at most linearly with period. Namely, there exists a positive number $R_0 < \infty$, such that each potential admits Fourier representation:

$$V_r(x) = \sum_{q \in \mathbb{Z}^2 \setminus \{0\}, \ 2^{-r+1}|q| < R_0} v_{r,q} \exp i\langle 2^{-r+1}\vec{q},x\rangle, \quad \vec{q} = 2\pi \left(\frac{q_1}{d_1}, \frac{q_2}{d_2}\right),$$

$\langle \cdot, \cdot \rangle$ being the canonical dot product in $\mathbb{R}^2$.

The one-dimensional analog of (1.1), (1.2) is already thoroughly investigated. It is proven in [1]–[7] that the spectrum of the operator $H_1 u = -u'' + Vu$ is generically a Cantor type set. It has a positive Lebesgue measure [1] [6]. The spectrum is absolutely continuous [1] [2], [5]–[9]. Generalized eigenfunctions can be represented...
in the form of $e^{ikx}u(x)$, $u(x)$ being limit-periodic [5, 6, 7]. The case of a complex-valued potential is studied in [10]. Integrated density of states is investigated in [11]-[14]. Spectral properties of Schrödinger operators in $l^2(Z)$ with limit-periodic potentials are recently investigated in [15]. It regards such potentials as generated by continuous sampling along the orbits of a minimal translation of a Cantor group. It is shown that the spectrum is a Cantor set of positive Lebesgue measure and purely singular continuous for a dense set of sampling functions, and it is a Cantor set of zero Lebesgue measure and purely singular continuous for a dense $G_δ$ set of sampling functions. Properties of eigenfunctions of discrete multidimensional limit-periodic Schrödinger operator are studied in [16]. As to the continuum multidimensional case, it is proven in [14] that the integrated density of states for (1.1) is the limit of densities of states for periodic operators.

We concentrate here on properties of the spectrum and eigenfunctions of (1.1), (1.2) in the high energy region. We prove the following results for the two-dimensional case.

1) The spectrum of the operator (1.1), (1.2) contains a semi-axis. A proof of an analogous result by different means can be found in the paper [17]. In [17], the authors consider the operator $H = (-δ)^l + V$, $8l > d+3$, $d \not\equiv 1(\mod 4)$, $d$ being the dimension of the space. This obviously includes our case $l = 1$, $d = 2$. However, there is an additional rather strong restriction on the potential $V(x)$ in [17], which we don’t have here: in [17] all the lattices of periods $Q_r$ of periodic potentials $V_r$ need to contain a nonzero vector $γ$ in common, i.e., $V(x)$ is periodic in a direction $γ$.

2) There are generalized eigenfunctions $Ψ_∞(k, x)$, corresponding to the semi-axis, which are close to plane waves: for every $k$ in an extensive subset $G_∞$ of $R^2$, there is a solution $Ψ_∞(k, x)$ of the equation $HΨ_∞ = λ_∞Ψ_∞$ which can be described by the formula:

$$Ψ_∞(k, x) = e^{ikx}(1 + u_∞(k, x)),$$  \hspace{1cm} (1.3)

$$∥u_∞∥ = |k|→∞ O(|k|^{-γ_1}), \hspace{0.5cm} γ_1 > 0,$$ \hspace{1cm} (1.4)

where $u_∞(k, x)$ is a limit-periodic function:

$$u_∞(k, x) = ∑_{r=1}^∞ u_r(k, x),$$ \hspace{1cm} (1.5)

$u_r(k, x)$ being periodic with periods $2^{r-1}d_1$, $2^{r-1}d_2$. The eigenvalue $λ_∞(k)$ corresponding to $Ψ_∞(k, x)$ is close to $|k|^2$:

$$λ_∞(k) = |k|→∞ |k|^2 + O(|k|^{-γ_2}), \hspace{0.5cm} γ_2 > 0.$$ \hspace{1cm} (1.6)

The “non-resonant” set $G_∞$ of the vectors $k$, for which (1.3) – (1.6) hold, is an extensive Cantor type set: $G_∞ = ∩_{n=1}^∞ G_n$, where $G_∞ = ∩_{n=1}^∞ G_n$ is a decreasing sequence of sets in $R^2$. Each $G_n$ has a finite number of holes in each bounded region. More and more holes appears when $n$ increases, however holes added
at each step are of smaller and smaller size. The set $\mathcal{G}_\infty$ satisfies the estimate:

$$\frac{|(\mathcal{G}_\infty \cap \mathcal{B}_R)|}{|\mathcal{B}_R|} = R \to \infty 1 + O(R^{-\gamma_3}), \quad \gamma_3 > 0,$$

(1.7)

where $\mathcal{B}_R$ is the disk of radius $R$ centered at the origin, $| \cdot |$ is the Lebesgue measure in $\mathbb{R}^2$.

(3) The set $\mathcal{D}_\infty(\lambda)$, defined as a level (isoenergetic) set for $\lambda_\infty(\vec{k})$,

$$\mathcal{D}_\infty(\lambda) = \{ \vec{k} \in \mathcal{G}_\infty : \lambda_\infty(\vec{k}) = \lambda \},$$

is proven to be a slightly distorted circle with the infinite number of holes. It can be described by the formula:

$$\mathcal{D}_\infty(\lambda) = \{ \vec{k} : \vec{k} = \kappa_\infty(\lambda, \vec{\nu})\vec{\nu}, \; \vec{\nu} \in \mathcal{B}_\infty(\lambda) \},$$

(1.8)

where $\mathcal{B}_\infty(\lambda)$ is a subset of the unit circle $S_1$. The set $\mathcal{B}_\infty(\lambda)$ has a Cantor type structure and an asymptotically full measure on $S_1$ as $\lambda \to \infty$:

$$L(\mathcal{B}_\infty(\lambda)) = \lambda \to \infty 2\pi + O\left(\lambda^{-\gamma_3/2}\right),$$

(1.9)

here and below $L(\cdot)$ is the length of a curve. The value $\kappa_\infty(\lambda, \vec{\nu})$ in (1.8) is the “radius” of $\mathcal{D}_\infty(\lambda)$ in a direction $\vec{\nu}$. The function $\kappa_\infty(\lambda, \vec{\nu}) - \lambda^{1/2}$ describes the deviation of $\mathcal{D}_\infty(\lambda)$ from the perfect circle of the radius $\lambda^{1/2}$. It is proven that the deviation is asymptotically small:

$$\kappa_\infty(\lambda, \vec{\nu}) = \lambda \to \infty \lambda^{1/2} + O\left(\lambda^{-\gamma_4}\right), \quad \gamma_4 > 0.$$

(1.10)

(4) Absolute continuity of the branch of the spectrum (the semiaxis) corresponding to $\Psi_\infty(\vec{k}, \vec{x})$ is proven.

To prove the results listed above we develop a modification of the Kolmogorov-Arnold-Moser (KAM) method. This paper is inspired by [18, 19, 20], where the method is used for periodic problems. In [18] KAM method is applied to classical Hamiltonian systems. In [19, 20] the technique developed in [18] is applied to semiclassical approximation for multidimensional periodic Schrödinger operators at high energies.

We consider a sequence of operators

$$H_0 = -\Delta, \quad H^{(n)} = H_0 + \sum_{r=1}^{M_n} V_r, \quad n \geq 1, \; M_n \to \infty \text{ as } n \to \infty.$$

Obviously, $\|H - H^{(n)}\| \to 0$ as $n \to \infty$, where $\| \cdot \|$ is the norm in the class of bounded operators. Clearly,

$$H^{(n)} = H^{(n-1)} + W_n, \quad W_n = \sum_{r=M_{n-1}+1}^{M_n} V_r.$$

(1.11)

We consider each operator $H^{(n)}$, $n \geq 1$, as a perturbation of the previous operator $H^{(n-1)}$. Every operator $H^{(n)}$ is periodic, however the periods go to infinity as $n \to$
We show that there is a $\lambda_\star$, $\lambda_\star = \lambda_\star(V)$, such that the semiaxis $[\lambda_\star, \infty)$ is contained in the spectra of all operators $H^{(n)}$. For every operator $H^{(n)}$ there is a set of eigenfunctions (corresponding to the semiaxis) being close to plane waves: for every $\vec{k}$ in an extensive subset $\mathcal{G}_n$ of $\mathbb{R}^2$, there is a solution $\Psi_n(\vec{k}, \vec{x})$ of the differential equation $H^{(n)}(\vec{k})\Psi_n = \lambda_n \Psi_n$, which can be described by the formula:

$$\Psi_n(\vec{k}, \vec{x}) = e^{i\langle \vec{k}, \vec{x} \rangle} \left(1 + \tilde{u}_n(\vec{k}, \vec{x})\right),$$

where $\tilde{u}_n(\vec{k}, \vec{x})$ has periods $2^{M_n-1}\vec{d}_1, 2^{M_n-1}\vec{d}_2$. The corresponding eigenvalue $\lambda^{(n)}(\vec{k})$ is close to $|\vec{k}|^2$:

$$\lambda^{(n)}(\vec{k}) = |\vec{k}|^2 + O\left(|\vec{k}|^{-\gamma_1}\right), \quad \gamma_1 > 0,$$

(1.12) where $\tilde{u}_n(\vec{k}, \vec{x})$ has periods $2^{M_n-1}\vec{d}_1, 2^{M_n-1}\vec{d}_2$. The non-resonant set $\mathcal{G}_n$ for which (1.13) holds, is proven to be extensive in $\mathbb{R}^2$:

$$\frac{|\mathcal{G}_n \cap B_R|}{|B_R|} = R \to \infty 1 + O(R^{-\gamma_3}).$$

(1.14) Estimates (1.12) – (1.14) are uniform in $n$. The set $\mathcal{D}_n(\lambda)$ is defined as the level (isoenergetic) set for non-resonant eigenvalue $\lambda^{(n)}(\vec{k})$:

$$\mathcal{D}_n(\lambda) = \left\{\vec{k} \in \mathcal{G}_n : \lambda^{(n)}(\vec{k}) = \lambda\right\}.$$  

This set is proven to be a slightly distorted circle with a finite number of holes (Fig. 1, 2). The set $\mathcal{D}_n(\lambda)$ can be described by the formula:

$$\mathcal{D}_n(\lambda) = \left\{\vec{k} : \vec{k} = \kappa_n(\lambda, \vec{\nu})\vec{\nu}, \quad \vec{\nu} \in \mathcal{B}_n(\lambda)\right\},$$

(1.15) where $\mathcal{B}_n(\lambda)$ is a subset of the unit circle $S_1$. The set $\mathcal{B}_n(\lambda)$ can be interpreted as the set of possible directions of propagation for almost plane waves $1.12$. It is shown that $\left\{\mathcal{B}_n(\lambda)\right\}_{n=1}^\infty$ is a decreasing sequence of sets, since on each step more and more
directions are excluded. Each $B_n(\lambda)$ has an asymptotically full measure on $S_1$ as $\lambda \to \infty$:

$$L(B_n(\lambda)) =_{\lambda \to \infty} 2\pi + O\left(\lambda^{-\gamma_3/2}\right),$$

the estimate being uniform in $n$. The set $B_n$ has only a finite number of holes, however their number is growing with $n$. More and more holes of a smaller and smaller size are added at each step. The value $\varkappa_n(\lambda, \vec{\nu}) - \lambda^{1/2}$ gives the deviation of $D_n(\lambda)$ from the perfect circle of the radius $\lambda^{1/2}$ in the direction $\vec{\nu}$. It is proven that the deviation is asymptotically small uniformly in $n$:

$$\varkappa_n(\lambda, \vec{\nu}) = \lambda^{1/2} + O\left(\lambda^{-\gamma_4}\right), \quad \frac{\partial \varkappa_n(\lambda, \vec{\nu})}{\partial \varphi} = O\left(\lambda^{-\gamma_5}\right), \quad \gamma_4, \gamma_5 > 0, \quad (1.17)$$

$\varphi$ being an angle variable $\vec{\nu} = (\cos \varphi, \sin \varphi)$.

On each step more and more points are excluded from the non-resonant sets $G_n$, thus $\{G_n\}_{n=1}^{\infty}$ is a decreasing sequence of sets. The set $G_\infty$ is defined as the limit set: $G_\infty = \cap_{n=1}^{\infty} G_n$. It has the infinite number of holes at each bounded region, but nevertheless satisfies the relation $(1.7)$. For every $\vec{k} \in G_\infty$ and every $n$, there is a generalized eigenfunction of $H^{(n)}$ of the type $(1.12)$. It is proven that the sequence of $\Psi_n(\vec{k}, \vec{x})$ has a limit in $L_\infty(\mathbb{R}^2)$ as $n \to \infty$, when $\vec{k} \in G_\infty$. The function $\Psi_\infty(\vec{k}, \vec{x}) = \lim_{n \to \infty} \Psi_n(\vec{k}, \vec{x})$ is a generalized eigenfunction of $H$. It can be written in the form $(1.3) - (1.5)$. Naturally, the corresponding eigenvalue $\lambda_\infty(\vec{k})$ is the limit of $\lambda^{(n)}(\vec{k})$ as $n \to \infty$.

We consider the limit $B_\infty(\lambda)$ of $B_n(\lambda)$:

$$B_\infty(\lambda) = \bigcap_{n=1}^{\infty} B_n(\lambda), \quad B_n \subset B_{n-1}.$$  

This set has a Cantor type structure on the unit circle. It is proven that $B_\infty(\lambda)$ has an asymptotically full measure on the unit circle (see $(1.9)$). We prove that the sequence $\varkappa_n(\lambda, \vec{\nu})$, $n = 1, 2, ..., \infty$, describing the isoenergetic curves $D_n$, quickly converges as $n \to \infty$. Hence, $D_\infty(\lambda)$ can be described as the limit of $D_n(\lambda)$ in the sense $(1.8)$, where $\varkappa_\infty(\lambda, \vec{\nu}) = \lim_{n \to \infty} \varkappa_n(\lambda, \vec{\nu})$ for every $\vec{\nu} \in B_\infty(\lambda)$. It is shown that the derivatives of the functions $\varkappa_n(\lambda, \vec{\nu})$ (with respect to the angle variable $\varphi$ on the unit circle) have a limit as $n \to \infty$ for every $\vec{\nu} \in B_\infty(\lambda)$. We denote this limit by $\frac{\partial \varkappa_\infty(\lambda, \vec{\nu})}{\partial \varphi}$. Using $(1.17)$ we prove that

$$\frac{\partial \varkappa_\infty(\lambda, \vec{\nu})}{\partial \varphi} = O\left(\lambda^{-\gamma_5}\right). \quad (1.18)$$

Thus, the limit curve $D_\infty(\lambda)$ has a tangent vector in spite of its Cantor type structure, the tangent vector being the limit of the corresponding tangent vectors for $D_n(\lambda)$ as $n \to \infty$. The curve $D_\infty(\lambda)$ looks as a slightly distorted circle with the infinite number of holes.

Absolute continuity of the branch of the spectrum $[\lambda_*(V), \infty)$, corresponding to the functions $\Psi_\infty(\vec{k}, \vec{x})$, $\vec{k} \in G_\infty$, follows from continuity properties of level curves $D_\infty(\lambda)$ with respect to $\lambda$, and from convergence of spectral projections corresponding
to \(\Psi_n(\vec{k}, \vec{x}), \vec{k} \in \mathcal{G}_\infty\), to spectral projections of \(H\) in the strong sense and uniformly in \(\lambda, \lambda > \lambda_*\).

The limit-periodic operator \(H = (-\Delta)^l + V, l \geq 6, d = 2\) is considered in \([24, 25]\). The results proved in \(24, 25\) are analogous to 1-4 on pages 2, 3. The main difficulty of the case \(l = 1\) comparing with \(l \geq 6\) is in starting the recurrent procedure. Step 1 here is really cumbersome comparing with the case \(l \geq 6\). This technical difficulty is related to the fact that perturbation theory for a periodic operator \((-\Delta)^l + V_{\text{per}}\) is much simpler for \(l > 1\) then for \(l = 1\), since Bloch eigenvalues are well-spaced for \(l > 1\) at high energies.

Further steps of approximation procedure are similar to those in \([24, 25]\) up to some technical modifications. The main technical difficulty to overcome is construction of non-resonance sets \(B_n(\lambda)\) for every sufficiently large \(\lambda, \lambda > \lambda_*(V)\), the last bound being uniform in \(n\). The set \(B_n(\lambda)\) is obtained by deleting a “resonant” part from \(B_{n-1}(\lambda)\). Definition of \(B_{n-1} \setminus B_n\), naturally, includes Bloch eigenvalues of \(H^{(n-1)}\). To describe \(B_{n-1} \setminus B_n\) one needs both non-resonant eigenvalues (1.13) and resonant eigenvalues. No suitable formulae are known for resonant eigenvalues. Absence of formulae causes difficulties in estimating the size of \(B_n \setminus B_{n-1}\). Let us describe shortly how we treat this problem in the second and higher steps of approximation. Indeed, on \(n\)-th step of approximation we start with the operator \(H^{(n-1)}\). It has elementary cell of periods \(Q_{n-1}\). The corresponding Bloch decomposition is denoted by \(H^{(n-1)}\), quasimomentum \(\ell^{(n-1)}\) belonging to the dual elementary cell \(K_{n-1}\).

We take \(t^{(n-1)}(\varphi)\) being equal to \(\tilde{x}_{n-1}(\varphi)\) modulo \(K_{n-1}\), where \(\tilde{x}_{n-1}(\varphi)\) describes \(\mathcal{X}_{n-1}(\varphi)\) (Fig. 1,2), \(\tilde{x}_{n-1}(\varphi) = x_{n-1}(\lambda, \vec{v})\vec{v}, \vec{v} = (\cos \varphi, \sin \varphi)\). When \(\tilde{x}_{n-1}(\varphi)\) is a point on \(\mathcal{X}_{n-1}(\lambda)\), vector \(\vec{v} = (\cos \varphi, \sin \varphi)\) belongs to \(B_{n-1}(\lambda)\). The operator \(H^{(n-1)}\) has a simple eigenvalue equal to \(\lambda\), formulas (1.12)–(1.17) with \(n = 1\) instead of \(n\) being valid. All other eigenvalues are separated from \(\lambda\) by the distance much greater than \(|W_{n-1}|\), see (1.11). It is convenient to denote such \(H^{(n-1)}\) by \(H^{(n-1)}(\tilde{x}_{n-1}(\varphi))\). Next, perturbation \(W_n\) of operator \(H^{(n-1)}\) has bigger periods than \(H^{(n-1)}\). We assign these bigger periods to \(H^{(n-1)}\). The corresponding Bloch decomposition we denote by \(\tilde{H}^{(n-1)}(t^{(n)}(\varphi))\), where \(t^{(n)}\) is quasimomentum in the dual elementary cell \(K_n\). According to Bloch’s theory, for any \(t^{(n)}\) the spectrum of \(\tilde{H}^{(n-1)}(t^{(n)})\) is the union of the spectra of \(H^{(n-1)}(t^{(n)} + \vec{b})\), where \(\vec{b}\) belongs to the lattice \(P^{(n)}\) generated by \(K_n\) in \(K_{n-1}\), see Fig.4, page 35. We take \(t^{(n)}(\varphi)\) being equal to \(\tilde{x}_{n-1}(\varphi)\) modulo \(K_n\). This means that \(t^{(n-1)}(\varphi)\) and \(t^{(n)}(\varphi)\) satisfy the relation \(t^{(n-1)}(\varphi) = t^{(n)}(\varphi) + \vec{b}_s\) for some \(\vec{b}_s \in P^{(n)}\) where \(t^{(n-1)}(\varphi)\) is \(\tilde{x}_{n-1}(\varphi)\) modulo \(K_{n-1}\). It is convenient to denote such operator \(\tilde{H}^{(n-1)}(t^{(n)}(\varphi))\) by \(\tilde{H}^{(n-1)}(\tilde{x}_{n-1}(\varphi))\). Obviously, \(\tilde{H}^{(n-1)}(\tilde{x}_{n-1}(\varphi))\) has an eigenvalue equal to \(\lambda\), since \(H^{(n-1)}(\tilde{x}_{n-1}(\varphi) + \vec{b}_s)\) does, all other eigenvalues of \(H^{(n-1)}(\tilde{x}_{n-1}(\varphi) + \vec{b}_s)\) being separated from \(\lambda\). We say that \(\varphi\) is resonant if \(H^{(n-1)}(\tilde{x}_{n-1}(\varphi) + \vec{b})\) has an eigenvalue close to \(\lambda\) for some \(\vec{b} \neq \vec{b}_s\). Note that it happens if and only if the operator

\[
I + S_n(\varphi) = \left( H^{(n-1)}(\tilde{x}_{n-1}(\varphi) + \vec{b}) - \lambda - \epsilon \right) \left( H_0(\tilde{x}_{n-1}(\varphi) + \vec{b}) + \lambda \right)^{-1} \tag{1.19}
\]
has an eigenvalue equal to zero for some small \( \epsilon \) and \( \vec{b} \neq \vec{b}_s \), the operator \( H_0 \) being the free operator with the same periods as \( H^{(n-1)} \). Assume for a moment we consider a polyharmonic operator \( H = (-\Delta)^l + V, \ l > 1 \). Then, \( S_n \) is in the trace class and the determinant of \( I + S_n \) exists. The resonant set \( B_{n-1} \setminus B_n \) can be described (in terms of \( \varphi \)) as the set of solutions of the equation \( \text{Det}(I + S_n(\varphi)) = 0 \). To obtain \( B_{n-1} \setminus B_n \) we take all values of \( \epsilon \) in a small interval and values of \( \vec{b} \) in a finite set, \( \vec{b} \neq 0 \). To estimate the size of \( B_{n-1} \setminus B_n \) we introduce a complex angle variable \( \varphi \), i.e., we extend our considerations to a complex neighborhood \( \Phi_0 \) of \([0, 2\pi] \). We show that the determinant (1.19) is an analytic function of \( \varphi \) in \( \Phi_0 \), and, by this, reduce the problem of estimating the size of \( B_{n-1} \setminus B_n \) to a problem in complex analysis. We use Rouche’s theorem to count zeros of the determinants and to investigate how far the zeros move when \( \epsilon \) changes or \( W_n \) is added to \( H^{(n-1)} \). It enables us to estimate the size of the zero set of the determinants, and, hence, the size of the non-resonance set \( \Phi_n \subset \Phi_0 \), which is defined as a non-zero set for the determinants. Proving that the non-resonance set \( \Phi_n \) is sufficiently large, we obtain estimates (1.16) for \( B_n \), the set \( B_n \) being defined by the real part of \( \Phi_n \).

To obtain \( \Phi_n \) we delete from \( \Phi_0 \) more and more holes of smaller and smaller radii at each step. Thus, the non-resonance set \( \Phi_n \subset \Phi_0 \) has a structure of Swiss Cheese (Fig. 4 [3], pages 11 [8]). Deleting resonance set from \( \Phi_0 \) at each step of the recurrent procedure we call a “Swiss Cheese Method”. The essential difference of our method from those applied in similar situations before (see e.g. [18]–[21]) is that we construct a non-resonance set not only in the whole space of a parameter \( \Phi \in \mathbb{R}^2 \) here), but also on all isoenergetic curves \( D_n(\lambda) \) in the space of parameter, corresponding to sufficiently large \( \lambda \). Estimating for the size of non-resonance sets on a curve require more subtle technical considerations (“Swiss Cheese construction”) than those sufficient for description of a non-resonant set in the whole space of the parameter.

When \( l = 1 \) (the present case) the determinant of \( I + S_n(\varphi) \) (see (1.19)) does not exist, since \( S_n(\varphi) \) is not from a trace class. We approximate \( S_n(\varphi) \) by finite dimensional operators \( S_n^{(N)} \) and consider the solutions of \( \text{Det}(I + S_n^{(N)}(\varphi)) = 0 \). The accumulation points of these solutions as \( N \to \infty \) we call the solutions of “Det” \( (I + S_n(\varphi)) \). Swiss Cheese method is applied here with such a modification.

The requirement for super exponential decay of \( \|V_r\| \) as \( r \to \infty \) is essential, since it is needed to ensure convergence of the recurrent procedure. At every step we use the upper bounds on \( \|V_r\| \) to prove perturbation formulae for Bloch eigenvalues and eigenfunctions when \( \lambda > \lambda_s(V) \), \( \lambda_s \) being the same for all steps. It is not important that potentials \( V_r \) have doubling periods, in the sense that the periods of the type \( q^{r-1} \vec{d}_1, q^{r-1} \vec{d}_2, q \in \mathbb{N} \), can be treated in the same way as the doubling.

The plan of the paper is the following. Section 2 (page 8) is devoted to the first step of the recurrent procedure. Sections 3 (page 34), 4 (page 50) and 5 (page 55) describe second, third and \( n \)-th steps of the recurrent procedure, respectively. Discussion of convergence of the procedure and proofs of the results 1–3, listed above, are in Section 6 (page 66). Absolute continuity is proved in Section 7 (page 69). Proofs of geometric lemmas and appendices are in Sections 8 (page 72) and 9 (page 85), respectively.
Further, we denote by $c, C$ absolute constants, by $C(\|V\|)$ a value depending only on $\|V\|$, etc.

2. The First Approximation

We fix $0 < s_1 < 10^{-4}$. Let $k > 0$ and be large enough so that $k^{s_1} > 500R_0, d_1, d_2, d_1^{-1}, d_2^{-1}$.

We define the first operator $H^{(1)}$ by

$$H^{(1)} := -\Delta + W_1, \quad W_1 := \sum_{r=1}^{M_1} V_r, \quad 2^{M_1} \approx k^{s_1}. \quad (2.1)$$

Obviously, operator $H^{(1)}$ has a periodic potential. We denote its periods by $(a_1, 0), (0, a_2)$, $a_1 = 2^{M_1-1}d_1, a_2 = 2^{M_1-1}d_2$. We employ Bloch theory (see e.g. [22]) for this operator, i.e., consider a family of operators $H^{(1)}(t), t \in K_1$, where $K_1$ is the elementary cell of the dual lattice, $K_1 = [0, 2\pi a_1^{-1}] \times [0, 2\pi a_2^{-1}]$. Vector $t$ is called quasi-momentum. Each operator $H^{(1)}(t)$ acts in $L_2(Q_1), Q_1 = [0, a_1] \times [0, a_2]$. The operator $H^{(1)}(t)$ is defined by (2.1) and quasi-periodic boundary conditions:

$$u(a_1, x_2) = \exp(it_1 a_1)u(0, x_2), \quad u(x_1, a_2) = \exp(it_2 a_2)u(x_1, 0),$$
$$u_{x_1}(a_1, x_2) = \exp(it_1 a_1)u_{x_1}(0, x_2), \quad u_{x_2}(x_1, a_2) = \exp(it_2 a_2)u_{x_2}(x_1, 0). \quad (2.2)$$

Each operator $H^{(1)}(t), t \in K_1$, has a discrete bounded below spectrum $\Lambda^{(1)}(t)$,

$$\Lambda^{(1)}(t) = \bigcup_{n=1}^{\infty} \{\lambda^{(1)}_n(t)\}, \quad \lambda^{(1)}_n(t) \to \infty \text{ as } n \to \infty.$$ 

The spectrum $\Lambda^{(1)}$ of the original operator $H^{(1)}$ is the union of the spectra of the operators $H^{(1)}(t)$ over all $t \in K_1$: $\Lambda^{(1)} = \cup_{t \in K_1} \Lambda^{(1)}(t)$. The functions $\lambda^{(1)}_n(t)$ are continuous, so $\Lambda^{(1)}$ has a band structure. Extending all the eigenfunctions of $H^{(1)}(t)$ for all $t \in K_1$ by the quasi-periodic boundary conditions to $\mathbb{R}^2$ yields a complete system of generalized eigenfunctions of $H^{(1)}$.

Let $H^{(1)}_0$ be the operator (2.1) corresponding to $W_1 = 0$. We consider that it has periods $a_1, a_2$ and that operators $H^{(1)}_0(t)$ are defined in $L_2(Q_1)$. The eigenfunctions of the operator $H^{(1)}_0(t)$ are plane waves satisfying (2.2). They are naturally indexed by points in $\mathbb{Z}^2$:

$$\Psi^0_j(t, x) = |Q_1|^{-1/2} \exp i\langle \tilde{p}_j(t), x \rangle, \quad |Q_1| = a_1 a_2,$$

where here and below $\tilde{p}_j(t) := (2\pi j_1 / a_1 + t_1, 2\pi j_2 / a_2 + t_2)$. The eigenvalue corresponding to $\Psi^0_j(t, x)$ is equal to $p^2_j(t)$, $p_j(t) := |\tilde{p}_j(t)|$.

Next, we introduce an isoenergetic surface $S_0(\lambda)$ of the free operator $H^{(1)}_0$. A point $t \in K_1$ belongs to $S_0(\lambda)$ if and only if $H^{(1)}_0(t)$ has an eigenvalue equal to $\lambda$, i.e., there exists $j \in \mathbb{Z}^2$ such that $p^2_j(t) = \lambda$. This surface can be obtained as follows: the circle of radius $k = \sqrt{\lambda}$ centered at the origin is divided into pieces by the dual lattice $\{\tilde{p}_q(0)\}_{q \in \mathbb{Z}^2}$, and then all pieces are translated in a parallel manner into the

\footnote{We write $a(k) \approx b(k)$ when the inequalities $\frac{1}{2}b(k) \leq a(k) \leq 2b(k)$ hold.}

\footnote{“Surface” is a traditional term. In our case, it is a curve.}
cell $K_1$ of the dual lattice. We also can get $S_0(\lambda)$ by drawing sufficiently many circles of radii $k$ centered at the dual lattice $\{\vec{p}_q(0)\}_{q \in \mathbb{Z}^2}$ and by looking at the figure in the cell $K_1$. As the result of either of these two procedures, we obtain a circle of radius $k$ “packed into the bag $K_1$,” as is shown in the Fig. 3. Note that each piece of $S_0(\lambda)$ can be described by an equation $p_j^2(t) = \lambda$ for a fixed $j$. If $t \in S_0(\lambda)$, then $j$ can be uniquely defined from the last equation, unless $t$ is a point of self-intersection of the isoenergetic surface. A point $t$ is a self-intersection of $S_0(\lambda)$ if and only if $p_q^2(t) = p_j^2(t) = \lambda$ for at least one pair of indices $q, j, q \neq j$.

Note that any vector $\vec{x}$ in $\mathbb{R}^2$ can be uniquely represented in the form $\vec{x} = \vec{p}_j(t)$, where $j \in \mathbb{Z}^2$ and $t \in K_1$. Let $K_1$ be the parallel shift into $K_1$:

$$K_1 : \mathbb{R}^2 \to K_1, \quad K_1 \vec{p}_j(t) = t.$$ 

Suppose $\Omega \subset \mathbb{R}^2$. In order to obtain $K_1 \Omega$, it is necessary to partition $\Omega$ by the lattice with nodes at the points $\vec{p}_q(0), q \in \mathbb{Z}^2$ and to shift all parts in a parallel manner into a single cell. It is obvious that $|K_1 \Omega| \leq |\Omega|$ for any $\Omega$. If $\Omega$ is a smooth curve, then $L(K_1 \Omega) \leq L(\Omega)$. For any pair of sets $\Omega_1$ and $\Omega_2$, $K_1 (\Omega_1 \cup \Omega_2) = K_1 \Omega_1 \cup K_1 \Omega_2$. Obviously, $K_1 S_k = S_0(\lambda)$ and $L(S_0(\lambda)) = L(S_k) = 2\pi k, k = \sqrt{\lambda}$, $S_k$ being the circle of radius $k$ centered at the origin.

The operator $H^{(1)}(t), t \in K_1$, has the following matrix representation in the basis of plane waves $\Psi_{0,j}(t,x), j \in \mathbb{Z}^2$:

$$H^{(1)}(t)_{mq} = p_m^2(t) \delta_{mq} + w_{m-q}, \quad m, q \in \mathbb{Z}^2.$$ 

Here and below, $\delta_{mq}$ is the Kronecker symbol, $w_{m-q}$ are Fourier coefficients of $W_1$:

$$w_q = \frac{1}{|Q_1|} \int_{Q_1} W_1(x) \exp -i \langle \vec{p}_q, x \rangle dx, \quad \overline{w}_q = w_{-q}.$$
Using assumptions on potentials $V_r$, we easily obtain:
\[ w_0 = 0, \quad |w_q| \leq \|W_1\|, \quad w_q = 0, \text{ when } p_q > R_0. \] (2.3)

The matrix $H^{(1)}(t)_{mq}$ describes an operator in $l_2(\mathbb{Z}^2)$ unitary equivalent to $H^{(1)}(t)$ in $L_2(Q_1)$. From now on, we denote the operator in $l_2(\mathbb{Z}^2)$ also by $H^{(1)}(t)$. Since the canonical basis in $l_2(\mathbb{Z}^2)$ does not depend on $t$ and all dependence on $t$ is in the matrix, the matrix $H^{(1)}(t)_{mq}$ and, hence, the operator $H^{(1)}(t) : l_2(\mathbb{Z}^2) \to l_2(\mathbb{Z}^2)$ can be analytically extended in $t$ from $K_1$ to $\mathbb{C}^2$. From now on, when we refer to $H^{(1)}(t)$ for $t \in \mathbb{C}^2$, we mean the operator in $l_2(\mathbb{Z}^2)$.

2.1. Perturbation theory. We formulate the main results of the perturbation theory considering $H^{(1)}(t)$ as a perturbed operator of the free operator $H^{(1)}_0$, i.e., we construct the perturbation series for eigenvalues and spectral projections for $t$ in a small neighborhood of a non-resonant set $\chi_1$.

Lemma 2.1 (Geometric Lemma). For arbitrarily small positive $\delta < s_1$ and $\beta : 4s_1 < 2\beta \leq 1 - 15s_1 - 8\delta$ and sufficiently large $\lambda$, $\lambda > \lambda_0(\beta, s_1, \delta)$, there exists a non-resonant set $\chi_1(\lambda, \beta, s_1, \delta) \subset S_0(\lambda)$ satisfying

(i) For any $t \in \chi_1(\lambda, \beta, s_1, \delta)$, there exists a unique $j \in \mathbb{Z}^2$ such that $p_j(t) = k$, $k := \sqrt{\lambda}$. The following inequalities hold:

\[ 4 \min_{0 < p_q < k^{s_1}} \min_{t, i + q \neq j} |p_{i+q}^2(t) - p_j^2(t)||p_{i+q}^2(t) - p_j^2(t)| > k^{2\beta}. \] (2.4)

\[ 2 \min_{0 < p_q < k^{s_1}} \min_{t} |p_{i+q}^2(t) - p_j^2(t)| > k^{1 - 3s_1 - \delta}. \] (2.5)

(ii) For any $t$ in the complex $(k^{2\beta - 2 - s_1 - 2\delta})$-neighborhood of the non-resonance set in $\mathbb{C}^2$, there exists a unique $j \in \mathbb{Z}^2$ such that

\[ |p_j^2(t) - k^2| < 3k^{2\beta - 1 - s_1 - 2\delta}. \] (2.6)

Estimates (2.4) and (2.5) hold.

(iii) The non-resonance set $\chi_1(\lambda, \beta, s_1, \delta)$ has an asymptotically full measure on $S_0(\lambda)$ in the sense of

\[ \lim_{\lambda \to \infty} \frac{L(S_0(\lambda) \setminus \chi_1(\lambda, \beta, s_1, \delta))}{L(S_0(\lambda))} = O(k^{-\delta}). \] (2.7)

Corollary 2.2. If $t$ belongs to the complex $(k^{2\beta - 2 - s_1 - 2\delta})$-neighborhood of the non-resonant set $\chi_1(\lambda, \beta, s_1, \delta)$ in $\mathbb{C}^2$, then for any $z \in \mathbb{C}$ lying on the circle

\[ C_1 = \{ z : |z - k^2| = k^{2\beta - 1 - s_1 - \delta} \} \] (2.8)

and for any $i$ and $q$ in $\mathbb{Z}^2$ with $0 < p_q < k^{s_1}$

\[ 2|p_i^2(t) - z| \geq k^{2\beta - 1 - s_1 - \delta} \] (2.9)

\[ 16|p_i^2(t) - z||p_{i+q}^2(t) - z| > k^{2\beta - 4s_1 - 2\delta} \] (2.10)
The set $\chi_1(\lambda, \beta, s_1, \delta)$ is defined below. The lemma and the corollary are proven in Section 8.1. Section 8.1.1 is introductory. Properties (i), (ii) and Corollary 2.2 are proven in Section 8.1.2 (Corollary 8.5). Section 8.1.3 is devoted to the proof of (iii). Here we just note the following. An analog of the lemma and corollary is proven earlier in [23]. However, in this work we need more detailed description of $\chi_1(\lambda, \beta, s_1, \delta)$ than that in [23]. Moreover, we need a complex analog of $\chi_1(\lambda, \beta, s_1, \delta)$.

We define $\chi_1(\lambda, \beta, s_1, \delta)$ by the formula:

$$\chi_1(\lambda, \beta, s_1, \delta) = \mathcal{K}_1 \mathcal{D}_0(\lambda)_{\text{nonres}},$$

(2.11)

where $\mathcal{D}_0(\lambda)_{\text{nonres}}$ is a subset of the sphere $S_k$,

$$\mathcal{D}_0(\lambda)_{\text{nonres}} = \{ k\vec{v}, \vec{v} = (\cos \varphi, \sin \varphi), \varphi \in \Theta_1 \},$$

(2.12)

the set $\Theta_1$ being defined as the real part of a set $\Phi_1 \subset \mathbb{C}$:

$$\Theta_1 = \Phi_1 \cap [0, 2\pi),$$

(2.13)

$$\Phi_1 = \Phi_0 \setminus \mathcal{O}^{(1)},$$

(2.14)

where $\Phi_0$ is the complex $k^{-\delta}$ neighborhood of $[0, 2\pi)$,

$$\Phi_0 = \{ \varphi \in \mathbb{C} : \Re \varphi \in [0, 2\pi), |\Im \varphi| < k^{-\delta} \},$$

(2.15)

$\mathcal{O}^{(1)}$ is the union of discs $\mathcal{O}^{\pm}_m$:

$$\mathcal{O}^{(1)} = \cup_{m \in \mathbb{Z}^2, 0 < p_m < 4k, \pm} \mathcal{O}^{\pm}_m.$$  

(2.16)

Discs $\mathcal{O}^{\pm}_m$ are centered at the points $\varphi^\pm_m$, which are solutions in $\mathbb{C}$ (two for each $m$) of the equations:

$$|\vec{k}(\varphi) + \vec{p}_m|^2 = k^2, \quad \vec{k}(\varphi) = k(\cos \varphi, \sin \varphi),$$

(2.17)

here and below $|\vec{x}|^2 = x_1^2 + x_2^2$ for any $\vec{x} = (x_1, x_2), x_1, x_2 \in \mathbb{C}$.

Let $E_j(t)$ be the spectral projection of the free operator corresponding to the eigenvalue $p_j(t)$, $(E_j)_{rm} = \delta_{jr}\delta_{jm}$. In the $(k^{23-2-s_1-2\delta})$-neighborhood of $\chi_1(\lambda, \beta, s_1, \delta)$,
we define functions $g_r^{(1)}(k,t)$ and operator-valued functions $G_r^{(1)}(k,t)$, $r = 1, 2, \ldots$ as follows.

\[
g_r^{(1)}(k,t) = \frac{(-1)^r}{2\pi i} \text{Tr} \oint_{C_1} \left( (H_0^{(1)}(t) - z)^{-1} W_1 \right)^r dz,
\]

where $G_r^{(1)}(k,t) = \frac{(-1)^{r+1}}{2\pi i} \oint_{C_1} \left( (H_0^{(1)}(t) - z)^{-1} W_1 \right)^r (H_0^{(1)}(t) - z)^{-1} dz.

To find $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$, it is necessary to compute the residues of a rational function of a simple structure, whose numerator does not depend on $z$, while the denominator is a product of factors of the type $(p_i^2(t) - z)$. For all $t$ in the non-resonance set the integrand has a single pole within $C_1$ at the point $z = k^2 = p_i^2(t)$.

By computing the residue at this point, we obtain explicit expressions for $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$. For example, $g_1^{(1)}(k,t) = 0$,

\[
g_2^{(1)}(k,t) = \sum_{q \in \mathbb{Z}^2, q \neq 0} |w_q|^2 (p_j^2(t) - p_j^2 + q(t))^{-1}
\]

\[
= - \sum_{q \in \mathbb{Z}^2, q \neq 0} \frac{|w_q|^2 p_q^2(0)}{(p_j^2(t) - p_j^2 + q(t))(p_j^2(t) - p_j^2 - q(t))},
\]

\[
G_1^{(1)}(k,t)_{rm} = \frac{w_{j-m}}{p_j^2(t) - p_m^2(t)} \delta_{rm} + \frac{w_{j-m}}{p_j^2(t) - p_j^2(t)} \delta_{mj}, \quad G_1^{(1)}(k,t)_{jj} = 0.
\]

For technical reasons, we introduce parameter $\alpha$ in front of the potential $W_1$, $H_0^{(1)} = (-\Delta)^\ell + \alpha W_1$, $0 \leq \alpha \leq 1$. We denote the operator $H_0^{(1)}$ with $\alpha = 1$ simply by $H^{(1)}$.

**Theorem 2.3.** Suppose $t$ belongs to the $(k^{2\beta-2-s_1-2\delta})$-neighborhood in $K_1$ of the non-resonant set $\chi_1(\lambda, \beta, s_1, \delta)$, $0 < \delta < s_1$, $8s_1 + 6\delta < 2\beta < 1 - 15s_1 - 8\delta$. Then for sufficiently large $k$, $k > k_0(\|W_1\|, s_1, \beta, \delta)$ and for all $\alpha$, $-1 \leq \alpha \leq 1$, there exists a single eigenvalue of the operator $H_0^{(1)}(t)$ in the interval $\varepsilon_1(k, \delta) := (k^2 - k^{2\beta-1-s_1-\delta}, k^2 + k^{2\beta-1-s_1-\delta})$. It is given by the series

\[
\lambda_j^{(1)}(\alpha, t) = p_j^2(t) + \sum_{r=2}^{\infty} \alpha^r g_r^{(1)}(k,t),
\]

converging absolutely, where the index $j$ is defined as in Lemma 2.1. The spectral projection corresponding to $\lambda_j^{(1)}(\alpha, t)$ is given by the series

\[
E_j^{(1)}(\alpha, t) = E_j + \sum_{r=1}^{\infty} \alpha^r G_r^{(1)}(k,t),
\]

which converges in the trace class $S_1$.

Moreover, $g_r^{(1)}(k,t)$ and $G_r^{(1)}(k,t)$ satisfy the estimates:

\[
|g_r^{(1)}(k,t)| < k^{2\beta-1-s_1-\delta} (c\|W_1\|^{k^{2\beta+4s_1+2\delta}})^r,
\]

\[
\|G_r^{(1)}(k,t)\|_1 < r(c k^{2\beta+4s_1+2\delta})^r,
\]
\[ \| \cdot \|_{1} \text{ being the norm in the trace class } S_{1}. \] In addition,
\[
\| g_{2}^{(1)}(k, t) \| < c \| W_{1} \|^{2} k^{-2+10s_{1}+4s}, \quad | g_{3}^{(1)}(k, t) | < c \| W_{1} \|^{3} k^{-2+10s_{1}+4s}, \quad (2.26)
\]
\[
g_{2}^{(1)}(k, t) > 0, \text{ when } \| W_{1} \| \neq 0, \quad (2.27)
\]
Operators \( G_{r}^{(1)}(k, t) \) are finite dimensional:
\[
G_{r}^{(1)}(k, t) = 0 \text{ when } |i - j| > r R_{0} \text{ or } |l - j| > r R_{0}. \quad (2.28)
\]
The series \( (2.22) \) and \( (2.23) \) converge uniformly with respect to \( \alpha \) in the complex disk \( |\alpha| \leq 1 \).

**Corollary 2.4.** The perturbed eigenvalue and its spectral projection satisfy
\[
|\lambda_{j}^{(1)}(\alpha, t) - p_{j}^{2}(t)| \leq \alpha^{2} C(\| W_{1} \|) k^{-1-2\beta+15s_{1}+11s}, \quad (2.29)
\]
\[
\| E_{j}^{(1)}(\alpha, t) - E_{j} \| \leq c |\alpha| \| W_{1} \| k^{-\beta+4s_{1}+3s}. \quad (2.30)
\]

**Proof.** The proof of the theorem is based on expanding the resolvent \( (H_{0}^{(1)}(t) - z)^{-1} \) in a perturbation series for \( z \) belonging to the contour \( C_{1} \) about the unperturbed eigenvalue \( p_{j}^{2}(t) \). It is completely analogous to the proofs of Theorems 2.1 and 3.1 in [23]. Indeed, it is obvious that
\[
(H_{0}^{(1)}(t) - z)^{-1} = (H_{0}^{(1)}(t) - z)^{-1/2}(I - \alpha A_{1})^{-1}(H_{0}^{(1)}(t) - z)^{-1/2}, \quad (2.31)
\]
\[
A_{1} = A_{1}(z, t) := -(H_{0}^{(1)}(t) - z)^{-1/2}W_{1}(H_{0}^{(1)}(t) - z)^{-1/2}. \quad (2.32)
\]
It follows from Corollary 2.2 estimates \( (2.3) \) and \( R_{0} < k^{s_{1}} \), that
\[
\| (H_{0}^{(1)}(t) - z)^{-1} \| < 2k^{-2\beta+1+s_{1}+\delta}, \quad z \in C_{1}, \quad (2.33)
\]
\[
\| A_{1} \| < 16\| W_{1} \| k^{-\beta+4s_{1}+2s}. \quad (2.34)
\]
Thus, \( \| A_{1} \| < 1 \) for sufficiently large \( k, k > k_{0}(\| W_{1} \|, \beta, s_{1}, \delta) \). Expanding \( (I - \alpha A_{1})^{-1} \) in powers of \( \alpha A_{1} \), we obtain
\[
(H_{0}^{(1)}(t) - z)^{-1} = (H_{0}^{(1)}(t) - z)^{-1/2}A_{1}^{(r)}(H_{0}^{(1)}(t) - z)^{-1/2}. \quad (2.35)
\]

Note that \( (H_{0}^{(1)}(t) - z)^{-1} \in S_{2} \). Taking into account estimates \( (2.33) \), we see that the series \( (2.35) \) converges in the class \( S_{1} \) uniformly with respect to \( \alpha \) in the whole complex disk \( |\alpha| \leq 1 \). For real \( \alpha \)-s we substitute the series into the following formula for a spectral projection
\[
E_{j}^{(1)}(\alpha, t) = -\frac{1}{2\pi i} \int_{C_{1}} (H_{0}^{(1)}(t) - z)^{-1} dz. \quad (2.36)
\]
Integrating termwise, we arrive at \( (2.23) \). Next, we prove estimate \( (2.25) \) for \( G_{r}^{(1)}(k, t) \). Indeed, it is easy to see that
\[
G_{r}^{(1)}(k, t) = -\frac{1}{2\pi i} \int_{C_{1}} (H_{0}^{(1)}(t) - z)^{-1/2}A_{1}(H_{0}^{(1)}(t) - z)^{-1/2} dz. \quad (2.37)
\]
We introduce the operator $A_0 = (I - E_j)A_1(I - E_j)$. It is obvious that $\|A_0\| \leq \|A_1\|$. In addition
\[ \oint_{C_1} (H_0^{(1)}(t) - z)^{-1/2} A_0'(H_0^{(1)}(t) - z)^{-1/2} dz = 0, \] (2.38)
since the integrand is holomorphic inside the circle. Thus,
\[ G_r^{(1)}(k, t) = -\frac{1}{2\pi i} \oint_{C_1} (H_0^{(1)}(t) - z)^{-1/2} (A_1^r - A_0^r)(H_0^{(1)}(t) - z)^{-1/2} dz. \] (2.39)
Since $A_1 - A_0 = E_j A_1(I - E_j) + (I - E_j) A_1 E_j$, $E_j A_1 E_j = 0$ and $E_j$ is a one-dimensional projection, we get
\[ \|A_1 - A_0\|_1 \leq \|A_1\|. \] (2.40)
Since
\[ \|A_1^r - A_0^r\|_1 \leq r \|A_1 - A_0\|_1 \|A_1\|^{r-1}, \] (2.41)
we obtain from relation (2.33) that
\[ \|A_1^r - A_0^r\|_1 \leq r \left( c \|W_1\| \|k^{-\beta + 4s_1 + 2\delta}\right)^r. \] (2.42)
Noting that the length of $C_1$ is equal to $2\pi k^{2\beta - 1 - s_1 - \delta}$, we obtain from formula (2.40):\
\[ \|G_r^{(1)}(k, t)\|_1 \leq k^{2\beta - 1 - s_1 - \delta} \|H_0^{(1)}(t) - z\|^{1/2} \|A_1^r - A_0^r\|_1 \] (2.43)
Using inequalities (2.33) and (2.42), we get (2.25). Convergence of the series in the complex disk $|\alpha| < 1$ easily follows. Note that $(A_1)_{it} = 0$ if $|i - l| > R_0$. Now it is not difficult to check (2.28), for details see [23].

We show now that operator $H_\alpha^{(1)}$, $-1 < \alpha < 1$, has a single eigenvalue ($n_\alpha = 1$) in the interval $\varepsilon(k, \delta)$. It is easy to see that $n_\alpha$ is determined from the formula $n_\alpha - n_0 = \text{Tr}(E_j^{(1)} - E_j)$. Considering formula (2.23), we obtain
\[ |n_\alpha - n_0| < \sum_{r=1}^{\infty} \|G_r^{(1)}(k, t)\|_1 = o(1). \]
Since $n_\alpha$ and $n_0$ are integers, and $n_0 = 1$ by the hypothesis of the theorem, the operator $H_\alpha^{(1)}(t)$ for all $\alpha$, $-1 \leq \alpha \leq 1$, has a single eigenvalue in $\varepsilon(k, \delta)$.

Further, we use the well-known formula:
\[ \frac{\partial \lambda^{(1)}(\alpha, t)}{\partial \alpha} = -\frac{1}{2\pi i} \text{Tr} \oint_{C_1} W_1(H_\alpha^{(1)}(t) - z)^{-1} dz \] (2.44)
Using formula (2.35) and considering that $\text{Tr}(W_1(H_0^{(1)}(t) - z)^{-1}) = 0$, we obtain:
\[ \frac{\partial \lambda^{(1)}(\alpha, t)}{\partial \alpha} = \sum_{r=2}^{\infty} r \alpha^{r-1} g_r^{(1)}(k, t). \] (2.45)
Integrating the last relation with respect to $\alpha$ and noting that $\lambda^{(1)}(0, t) = p_0^2(t)$, we get formula (2.22). To prove estimate (2.24) we note that $g_r^{(1)} = -(2\pi ir)^{-1} \text{Tr} \oint_{C_1} A_1^r dz$, and, therefore,
\[ |g_r^{(1)}(k, t)| \leq r^{-1} k^{2\beta - 1 - s_1 - \delta} \|A_1^r - A_0^r\|_1, \] (2.46)
Lemma 2.5. Coefficients $g_r^{(1)}(k, t)$, and $G_r^{(1)}(k, t)$ can be continued as holomorphic functions of two variables from the real $(k^{2\beta'-2-s_1-2\delta})$-neighborhood of the non-resonance set $\chi_1(k, \beta, s_1, \delta)$ to its complex $(k^{2\beta-2-s_1-2\delta})$-neighborhood. In this complex neighborhood the following estimates hold.

$$|T(m)g_r^{(1)}(k, t)| < m!k^{2\beta-1-s_1-\delta}(c||W_1||k^{-\beta+4s_1+2\delta})^r k^{m(2-2\beta+s_1+2\delta)},$$

(2.47)

$$|T(m)G_r^{(1)}(k, t)| < m!r(c||W_1||k^{-\beta+4s_1+2\delta})^r k^{m(2-2\beta+s_1+2\delta)}.$$  

(2.48)

The coefficients $g_r(k, t)$, $r < k^s R_0^{-1}$ can be continued to the complex $(k^{-3s_1-2\delta})$-neighborhood of $\chi_1(k, \beta, s_1, \delta)$. They obey the estimates:

$$|T(m)g_r^{(1)}(k, t)| < cm!||W_1||^2 k^{-2+10s_1+4\delta} k^{m(3s_1+2\delta)},$$

(2.49)

$$|T(m)g_r^{(1)}(k, t)| < cm!||W_1||^3 k^{-2+10s_1+4\delta} k^{m(3s_1+2\delta)}.$$  

(2.50)

$$|T(m)g_r^{(1)}(k, t)| < m!k^{2\beta-1-s_1-\delta}(c||W_1||k^{-\beta+4s_1+2\delta})^r k^{m(3s_1+2\delta)},$$

(2.51)

Proof. Coefficients $g_r^{(1)}(k, t)$, and $G_r^{(1)}(k, t)$ can be extended analytically to the $(k^{2\beta'-2-s_1-2\delta})$-neighborhood of $\chi_1(k, \beta, s_1, \delta)$, since estimates (2.47), (2.48) are stable in this neighborhood. The coefficients $g_r(k, t)$, $r < k^s R_0^{-1}$, can be extended to a bigger neighborhood, since only (2.5) is required to estimate them. The estimates for the derivatives are obtained by means of Cauchy integrals. For details, see Theorem 3.3 and Corollary 3.3 in [23].

Lemma 2.5 implies the following theorem.

Theorem 2.6. The series (2.22) and (2.23) can be continued as holomorphic functions of two variables from the real $(k^{2\beta'-2-s_1-2\delta})$-neighborhood of the non-resonance set $\chi_1(k, \beta, s_1, \delta)$ to its complex $(k^{2\beta-2-s_1-2\delta})$-neighborhood. The following estimates hold in the complex neighborhood:

$$|T(m)(\chi_r^{(1)}(\alpha, t) - p_r^{(1)}(t))| < m! C(W_1)\alpha^2 k^{-1-2\beta+15s_1+11\delta+|m|(2-2\beta+s_1+2\delta)},$$

(2.52)

$$\|T(m)(E_r^{(1)}(\alpha, t) - E_r^{(1)})\| < cm!||W_1||\alpha k^{-\beta+4s_1+3\delta+|m|(2-2\beta+s_1+2\delta)}.$$  

(2.53)
There are stronger estimates for $|m| = 1, 2$:

$$\left| \nabla \lambda_j^{(1)}(\alpha, t) - 2\vec{p}_j(t) \right| < C(W_1)\alpha^2 k^{-1-2\beta+18s_1+13\delta},$$  \hfill (2.54)

$$\left| T(m) \left( \lambda_j^{(1)}(\alpha, t) - \tilde{p}_j^2 \right) \right| < C(W_1)\alpha^2 k^{-1-2\beta+21s_1+15\delta}, \text{ if } |m| = 2. \hfill (2.55)$$

The next lemma is used in the second step of approximation, where the operator $H^{(1)}(t)$ plays a role of the initial (unperturbed) operator.

**Lemma 2.7.** For any $z$ on the circle $C_1$ given in (2.8) and $t$ in the $(k^{2\beta-2s_1-2\delta})$-neighborhood of $\chi_1(k, \delta)$ in $K_1$,

$$\| (H^{(1)}(t) - z)^{-1} \| \leq 2k^{-2\beta+1+s_1+\delta}. \hfill (2.56)$$

**Proof.** The estimate follows from (2.31), (2.33) and (2.34). \hfill \blacksquare

### 2.2. Non-resonant part of the Isoenergetic Set of $H^{(1)}$

In this subsection we choose $\beta$ to have the biggest possible value $\beta = \beta_0$, $2\beta_0 = 1 - 15s_1 - 9\delta$. Let $S_1(\lambda)$ be the isoenergetic surface of the perturbed operator $H^{(1)}_{\alpha}$, i.e.,

$$S_1(\lambda) = \{ t \in K_1 : \exists \kappa \in \mathbb{N} \text{ s.t. } \lambda^{(1)}_n(\alpha, t) = \lambda \}, \hfill (2.57)$$

where $\{\lambda^{(1)}_n(\alpha, t)\}_{n=1}^{\infty}$ is the complete set of eigenvalues of $H^{(1)}_{\alpha}(t)$. We construct a “non-resonance” subset $\chi_1^{*}(\lambda)$ of $S_1(\lambda)$, which corresponds to the non-resonance eigenvalues $\lambda_j^{(1)}(\alpha, t)$ given by the perturbation series. By Lemma 2.1 for every $t$ belonging to the non-resonant set $\chi_1(\lambda, \beta, s_1, \delta)$, there is a single $j \in \mathbb{Z}^2$ such that $p_j(t) = k$, $k = \sqrt{\lambda}$. This means that formula (2.11) establishes a one-to-one correspondence between $\chi_1(\lambda, \beta, s_1, \delta)$ and $D_0(\lambda)_{\text{nonres}}$. Let $\vec{z} \in D_0(\lambda)_{\text{nonres}}$ and $j \in \mathbb{Z}^2$, $t \in \chi_1(\lambda, \beta, s_1, \delta)$ are defined by the relation $\vec{z} = \tilde{p}_j(t)$.

According to Theorem 2.3 for sufficiently large $k$, there exists an eigenvalue of the operator $H^{(1)}_{\alpha}(t)$, $t = K_1 \vec{z}$, $0 \leq \alpha \leq 1$, given by (2.22). It is convenient here to denote $\lambda_j^{(1)}(\alpha, t)$ by $\lambda_j^{(1)}(\alpha, \vec{z})$: we can do this since there is a one-to-one correspondence between $\vec{z}$ and the pair $(t, j)$. We rewrite (2.22) in the form

$$\lambda^{(1)}(\alpha, \vec{z}) = \vec{x}^2 + f_1(\alpha, \vec{z}), \quad \vec{x} = |\vec{z}|, \quad f_1(\alpha, \vec{z}) = \sum_{r=2}^{\infty} \alpha^r g_r^{(1)}(\vec{z}), \hfill (2.58)$$

$g_r^{(1)}(\vec{z})$ being defined by (2.18) with $j$ and $t$ such that $\tilde{p}_j(t) = \vec{z}$. By Theorem 2.6 $f_1(\alpha, \vec{z})$ satisfies the following estimates when $2\beta = 1 - 15s_1 - 9\delta$:

$$|f_1(\alpha, \vec{z})| \leq \alpha^2 C(W_1)k^{-2+30s_1+20\delta}, \hfill (2.59)$$

$$|T(m)f_1(\alpha, \vec{z})| \leq \alpha^2 C(W_1)k^{-2+30s_1+20\delta+|m|(1+16s_1+11\delta)}. \hfill (2.60)$$

Estimates (2.54), (2.55) yield:

$$\nabla f_1(\alpha, \vec{z}) = O \left( k^{-2+33s_1+22\delta} \right), \hfill (2.61)$$

$^4$Usually the vector $\tilde{p}_j(t)$ is denoted by $\vec{k}$, the corresponding plane wave being $e^{i\vec{k}\cdot \vec{x}}$. We use the less common notation $\vec{z}$, since we already have other $k$'s in the text.
\[ \nabla \lambda^{(1)}(\alpha, \vec{z}) = 2\vec{z} + O \left( k^{-2+33s_1+22\delta} \right), \]  
\[ T(m)f_1(\alpha, \vec{z}) = O \left( k^{-2+36s_1+24\delta} \right), \quad T(m)\lambda^{(1)}(\alpha, \vec{z}) = O(1), \quad \text{if } |m| = 2. \]  

By Theorem 2.6, the series (2.58) converges in the \((k^{-1-16s_1-11\delta})\)-neighborhood of \(D_0(\lambda)_{\text{nonres}}\), the estimate (2.59)–(2.63) hold.

Let us recall that operator \(H^{(1)}(t)\) is defined in \(l^2\), i.e., just by the matrix depending on \(t\). We define \(H^{(1)}(\vec{z})\) by substituting \(\vec{z}\) instead of \(\vec{p}_j(t)\) into \(H^{(1)}(t)\). For any \(t \in \chi_1(k, \beta_0, s_1, \delta)\) matrices \(H^{(1)}(t)\) and \(H^{(1)}(\vec{z})\) are the same, up to the shift of indices by \(j\). Further we consider \(H^{(1)}(\vec{z})\) for \(\vec{z} = \kappa\vec{v}\), when \(\kappa\) is in the complex \(2k^{-2-16s_1-11\delta}\) neighborhood of \(\Phi_1\) and \(\kappa\) is in the complex \(2k^{-1-16s_1-11\delta}\) neighborhood of \(k\). Using Lemma 8.9 we easily obtain the estimates analogous to (2.33), (2.34), which provide the convergence of the series for the resolvent. Operator \(H^{(1)}(\vec{z})\) is not self-adjoint for complex \(\kappa\) and we do not try to make spectral analysis of it. However, we notice that the series (2.58) converges when \(\kappa\) is in the complex \(2k^{-2-16s_1-11\delta}\) neighborhood of \(\Phi_1\) and \(\kappa\) is in the complex \(2k^{-1-16s_1-11\delta}\) neighborhood of \(k\). Lemma 2.7 admits the following generalization.

**Lemma 2.8.** For any \(z\) on the circle \(C_1\) given in (2.8) and \(\vec{z} = \kappa\vec{v}\), \(\kappa\) being in the complex \(2k^{-2-16s_1-11\delta}\) neighborhood of \(\Phi_1\), \(\kappa\) being in the complex \(2k^{-1-16s_1-11\delta}\) neighborhood of \(k\),

\[ \| (H^{(1)}(\vec{z}) - z)^{-1} \| \leq 2k^{-2\delta+1+s_1+\delta}. \]  

Let \(B_1(\lambda)\) be a set of unit vectors corresponding to \(D_0(\lambda)_{\text{nonres}}\):

\[ B_1(\lambda) = \{ \vec{v} \in S_1 : k\vec{v} \in D_0(\lambda)_{\text{nonres}} \} = \{ \vec{v} \in S_1 : \vec{v} = (\cos \varphi, \sin \varphi), \ \varphi \in \Theta_1 \}. \]

It is easy to see that \(B_1(\lambda)\) is a unit circle with holes, centered at the origin. Since formulas (2.58)–(2.63) hold in the \((k^{-1-16s_1-11\delta})\)-neighborhood of \(D_0(\lambda)_{\text{nonres}}\), they hold for any \(\kappa\vec{v}\) such that \(\vec{v} \in B_1(\lambda), |\kappa - k| < k^{-1-16s_1-11\delta}\). We define \(D_1(\lambda)\) as the level set of the function \(\lambda^{(1)}(\alpha, \vec{z})\) in this neighborhood:

\[ D_1(\lambda) := \{ \vec{z} = \kappa\vec{v} : \vec{v} \in B_1(\lambda), |\kappa - k| < k^{-1-16s_1-11\delta}, \lambda^{(1)}(\alpha, \vec{z}) = \lambda \}. \]  

We prove in Lemma 2.10 that \(D_1(\lambda)\) is a distorted circle with holes, which is close to the circle of radius \(k\); see Fig. 1. First, we prove that the equation \(\lambda^{(1)}(\alpha, \vec{z}) = \lambda\) is solvable with respect to \(\kappa = |\vec{z}|\) for any \(\vec{v} = \frac{\vec{z}}{|\vec{z}|} \in B_1(\lambda)\).

**Lemma 2.9.** For every \(\vec{v} \in B_1(\lambda)\) and every \(0 \leq \alpha \leq 1\), and sufficiently large \(\lambda\), there is a unique \(\kappa = \kappa_1(\lambda, \vec{v})\) in the interval

\[ I_1 := [k - k^{-1-16s_1-11\delta}, k + k^{-1-16s_1-11\delta}], \quad k^2 = \lambda, \]

such that

\[ \lambda^{(1)}(\alpha, \kappa_1 \vec{v}) = \lambda. \]  

Furthermore, \(|\kappa_1 - k| \leq C(W_1)k^{-3+30s_1+20\delta}\).

**Proof.** Formula (2.62) yields

\[ \frac{\partial \lambda^{(1)}(\alpha, \kappa)}{\partial \kappa} = 2\kappa, \]  

(2.67)
when \(|\varpi - k| < k^{-1-16\delta - 11\delta}\). Now the lemma easily follows from (2.59). For details see Lemma 2.10 in [24].

Let us introduce new notations. Let \(\hat{\Phi}_1\) be the \((k^{-2-16\delta - 11\delta})\)-neighborhood of \(\Phi_1\). Note that (2.58)–(2.63) hold for \(\varphi\) in \(\hat{\Phi}_1\) and even its \((k^{-2-16\delta - 11\delta})\)-neighborhood when \(\varphi = \varpi \vec{v}\), \(|\varpi - k| < k^{-1-16\delta - 11\delta}\). Let
\[
\varphi_1(\varphi) := \varphi_1(\lambda, \vec{v}), \quad \vec{v} = (\cos \varphi, \sin \varphi), \quad \varphi_1(\varphi) = \varphi_1(\varphi) \vec{v}, \quad h_1(\varphi) = \varphi_1(\varphi) - k, \quad (2.68)
\]
\(\varphi_1(\lambda, \vec{v})\) being defined by Lemma 2.9.

**Lemma 2.10.** (1) For sufficiently large \(\lambda\), the set \(D_1(\lambda)\) is a distorted circle with holes; it can be described by the formula
\[
D_1(\lambda) = \{ \varpi \in \mathbb{R}^2 : \varphi = \varphi_1(\varphi) \vec{v}, \quad \vec{v} \in B_1(\lambda) \}, \quad (2.69)
\]
where \(\varphi_1(\varphi) = k + h_1(\varphi)\) and \(h_1(\varphi)\) obeys the inequalities
\[
|h_1| < C(W_1)k^{-3-30s_1+20\delta}, \quad \left| \frac{\partial h_1}{\partial \varphi} \right| < C(W_1)k^{-1+33s_1+22\delta}. \quad (2.70)
\]
(2) The total length of \(B_1(\lambda)\) satisfies the estimate
\[
L(B_1) = 2\pi(1 + O(k^{-\delta})). \quad (2.71)
\]
(3) The function \(h_1(\varphi)\) can be extended as a holomorphic function of \(\varphi\) from \(\Theta_2\) to \(\hat{\Phi}_1\). Estimates (2.70) hold in \(\hat{\Phi}_1\) too.
(4) The curve \(D_1(\lambda)\) has a length which is asymptotically close to that of the whole circle in the sense that
\[
L(D_1(\lambda)) \sim 2\pi k(1 + O(k^{-\delta})), \quad \lambda = k^2. \quad (2.72)
\]

**Proof.** Here are the main points of the proof, for details see Lemma 2.11 in [24].

1. Inequalities (2.70) easily follow from (2.58)–(2.60) and the definition of \(D_1(\lambda)\), Implicit function theorem being applied.
2. By definition, \(B_1(\lambda)\) is the set of directions corresponding to \(D_0(\lambda)_{nonres}\), the latter set being a subset of the sphere of radius \(k\). Formula (2.11) establishes a one-to-one correspondence between \(\chi_1(\lambda, \beta, s_1, \delta)\) and \(D_0(\lambda)_{nonres}\), their lengths being equal. Considering (2.7), we obtain \(L(D(\lambda)_{nonres}) = 2\pi k(1 + O(k^{-\delta}))\). Hence, (2.71) holds.
3. The series (2.58) converges for \(\varphi = \varpi \vec{v}\), when \(\varphi\) in \(\hat{\Phi}_1\) and even its \((k^{-2-16\delta - 11\delta})\)-neighborhood and \(\varpi \in \omega\), \(\omega\) being the complex \((k^{-1-16\delta - 11\delta})\)-neighborhood of \(k\), \(\omega = \{ \varpi \in \mathbb{C} : |\varpi - k| < k^{-1-16\delta - 11\delta}\}\). Function \(\lambda_1(\alpha, \varphi)\) is analytic in \(\varphi\) and \(\varpi\). The estimate (2.59) holds. Thus, \(|\lambda_1(\alpha, \varphi) - \varpi|^2 < C(W_1)k^{-2+30s_1+20\delta}\), and \(|\varpi - k|^2 \approx k^{-16\delta - 11\delta}\) when \(\varpi \in \partial \omega\). Applying Rouche’s theorem in \(\omega\), we obtain that the equations \(\lambda_1(\alpha, \varphi) = k^2\) and \(\varpi^2 = k^2\) have the same number of solutions \(\varpi\) in \(\omega\) for every \(\varphi \in \hat{\Phi}_1\). Obviously, \(\varpi^2 = k^2\) has just one solution. Therefore, \(\varphi_1(\lambda, \vec{v})\) is analytic in \(\varphi\) in \(\hat{\Phi}_1\) and estimates (2.70) hold.
4. Estimate (2.72) follows from (2.70) and (2.71).
Next, we define the non-resonance subset $\chi_1^*(\lambda)$ of the isoenergetic set $S_1(\lambda)$ as the parallel shift of $D_1(\lambda)$ into $K_1$:

$$\chi_1^*(\lambda) := \mathcal{K}_1 D_1(\lambda).$$

**(Lemma 2.11)** The set $\chi_1^*(\lambda)$ belongs to the $(k^{-3+30s_1+20\delta})$-neighborhood of $\chi_1(\lambda)$ in $K_1$. If $t \in \chi_1^*(\lambda)$, then the operator $H^{(1)}_\alpha(t)$ has a simple eigenvalue $\lambda^{(1)}(\alpha, t)$, $n \in \mathbb{N}$, equal to $\lambda$, no other eigenvalues being in the interval $\varepsilon_1(k, \delta)$, $\varepsilon_1(k, \delta) := (k^2 - k^{-16s_1 - 11\delta}, k^2 + k^{-16s_1 - 11\delta})$. This eigenvalue is given by the perturbation series (2.22), where $j$ is uniquely defined by $t$ from the relation $p^2_j(t) \in \varepsilon_1(k, \delta)$.

**Proof.** By Lemma (2.10) $D_1(\lambda)$ is in the $(k^{-3+30s_1+20\delta})$-neighborhood of $D_0(\lambda)$. Since $\chi_1(\lambda) = \mathcal{K}_1 D_0(\lambda)$ and $\chi_1^*(\lambda) = \mathcal{K}_1 D_1(\lambda)$, we immediately obtain that $\chi_1^*(\lambda)$ is in the $(ck^{-3+30s_1+20\delta})$-neighborhood of $\chi_1(\lambda)$. The size of this neighborhood is less than $k^{-1-16s_1-11\delta}$, here $-1-16s_1-11\delta = 2\beta_0 - 2 - s_1 - 2\delta$ for $2\beta_0 = 1 - 15s_1 - 9\delta$. Hence, Theorem 2.3 holds for any $t \in \chi_1^*(\lambda)$: there is a single eigenvalue of $H^{(1)}_\alpha(t)$ in the interval $\varepsilon_1(k, \delta)$. Since $\chi_1^*(\lambda) \subseteq S_1(\lambda)$, this eigenvalue is equal to $\lambda$. By the theorem, the eigenvalue is given by the series (2.22), $j$ being uniquely defined by $t$ from the relation $p^2_j(t) \in \varepsilon_1(k, \delta)$.

**(Lemma 2.12)** Formula (2.73) establishes a one-to-one correspondence between $\chi_1^*(\lambda)$ and $D_1(\lambda)$.

**Remark 1.** From the geometric point of view, this means that $\chi_1^*(\lambda)$ does not have self-intersections.

**Proof.** Suppose there is a pair $\mathcal{Z}_{1,1}, \mathcal{Z}_{1,2} \in D_1(\lambda)$, such that $\mathcal{K}_1 \mathcal{Z}_{1,1} = \mathcal{K}_1 \mathcal{Z}_{1,2} = t$, $t \in \chi_1^*(\lambda)$. By the definition (2.65) of $D_1(\lambda)$, we have $\lambda^{(1)}(\alpha, \mathcal{Z}_{1,1}) = \lambda^{(1)}(\alpha, \mathcal{Z}_{1,2}) = \lambda$, i.e., the eigenvalue $\lambda$ of $H^{(1)}_\alpha(t)$ is not simple. This contradicts the previous lemma.

### 2.3. Preparation for the Next Approximation.

In the next steps of approximations we will need estimates for the resolvent $\left( H^{(1)}(\mathcal{Z}_1(\varphi) + \vec{b})^{-1} - k^2 \right)^{-1}$, $\vec{b} \in K_1 \setminus \{0\}$, $\varphi \in \Phi_1$. Let $b_0$ be the distance of the point $\vec{b}$ to the nearest corner of $K_1$:

$$b_0 = \min_{m=(0,0),(0,1),(1,0),(1,1)} |\vec{b} - 2\pi m/\alpha|. \quad (2.74)$$

We will consider separately two cases: $b_0 \geq k^{-1-16s_1-12\delta}$ and $0 < b_0 < k^{-1-16s_1-12\delta}$. Our goal is to prove Lemmas 2.20 and 2.23, which give the estimates for the resolvent on a subset of $\Phi_1$ for the cases $b_0 \geq k^{-1-16s_1-12\delta}$ and $0 < b_0 < k^{-1-16s_1-12\delta}$, respectively. In this subsection we choose $\beta$ to be relatively small, $100s_1 < \beta < 1/12 - 28s_1 - 14\delta$. We denote such $\beta$ by $\beta_1$. 
2.3.1. The case \( b_0 \geq k^{-1-16s_1-12\delta} \). The goal is to construct a set \( O_s^{(1)}(\vec{b}) \) such that Lemma \( \lfloor 2.29 \rfloor \) holds. The general scheme of considerations is similar to that for the case \( H = (-\Delta)^l + V, \ l \geq 6 \), see \([24]\), Section 3.5.1. First, we describe shortly the scheme in \([24]\). Further, we explain what changes have to be made to adjust it for the present case. Indeed, in \([24]\) we constructed a set \( O(\vec{b}) \), which consists of small discs centered at the poles of the resolvent

\[
\left( H_0^{(1)}(\vec{k}(\varphi) + \vec{b}) - k^2 \right)^{-1}.
\]

However, the size of discs was chosen sufficiently large to ensure that

\[
\left\| \left( H_0^{(1)}(\vec{k}(\varphi) + \vec{b}) - k^2 \right)^{-1} \right\| < k^{-\mu}, \ \mu > 0, \ \text{when} \ \varphi \not\in O(\vec{b}).
\]

It was shown that the number of poles of \( (3.25) \) in \( \Phi_0 \) does not exceed \( c_0k^{2+2s_1} \). Using perturbation arguments we showed that the estimate analogous to \( (3.25) \) holds for the perturbed resolvent

\[
\left( H^{(1)}(\vec{\varphi}_1(\varphi) + \vec{b}) - k^2 \right)^{-1}
\]

in \( \Phi_1 \setminus O(\vec{b}) \):

\[
\left\| \left( H^{(1)}(\vec{\varphi}_1(\varphi) + \vec{b}) - k^2 \right)^{-1} \right\| < k^{-\mu}, \ \mu > 0.
\]

Next, suppose \( O_c(\vec{b}) \) is a connected component of \( O(\vec{b}) \), which is entirely in \( \Phi_1 \): \( O_c(\vec{b}) \subset \Phi_1 \). Then \( (3.26) \) is an analytic function in \( O_c(\vec{b}) \). Using determinants and Rouche’s theorem, we showed that \( (3.25) \) and \( (3.26) \) have the same number of poles inside each connected component of \( O(\vec{b}) \). Therefore, the total number of poles of \( (3.26) \) in such components does not exceed \( c_0k^{2+2s_1} \). Next, we replaced the discs in \( O(\vec{b}) \) by much smaller discs around the poles of \( (3.26) \). By doing this, we obtained a much smaller set \( O_s(\vec{b}), O_s(\vec{b}) \subset O(\vec{b}) \). Using analyticity of \( (3.26) \) inside \( O(\vec{b}) \) and the maximum principle, we arrived at the estimate:

\[
\left\| \left( H^{(1)}(\vec{\varphi}_1(\varphi) + \vec{b}) - k^2 \right)^{-1} \right\| < ck^{-\mu} \left( \frac{R_b}{r^{(1)}} \right)^J, \ J = c_0k^{2+2s_1},
\]

where \( R_b \) is the size of the biggest component of \( O(\vec{b}) \) and \( r^{(1)} \) is the radius of the discs constituting \( O_s(\vec{b}) \). Thus, we shrank the set \( O(\vec{b}) \), but still obtained a bound for the norm of the resolvent, which is essential for the next step of approximation in \([24]\). It may happen, however, that \( O_c(\vec{b}) \cap \Phi_1 \neq \emptyset \), but \( O_c(\vec{b}) \not\subset \Phi_1 \). In this case we showed that the radii of the discs in \( O_c(\vec{b}) \) were sufficiently small to make sure that the function \( \varphi_1(\varphi) \) can be analytically extended from \( \Phi_1 \) to the interior each \( O_c(\vec{b}) \) when \( O_c(\vec{b}) \cap \Phi_1 \neq \emptyset \) even if \( O_c(\vec{b}) \not\subset \Phi_1 \). For \( l \geq 6 \) this condition on radii does not contradict to \( (3.26) \).

In the present case \( l = 1 \) the plan is basically the same. However, we have essential technical complications. The set \( O(\vec{b}) \), constructed for \( l \geq 6 \), is now too small to provide convergence of perturbation series for \( (3.26) \) when \( \varphi \not\in O(\vec{b}) \). Therefore, we need to construct a bigger set \( O^{(1)}(\vec{b}) \), such that the series converges when \( \varphi \not\in O^{(1)}(\vec{b}) \).
It can be constructed by analogy with (2.16). Let $O_c^{(1)}(\tilde{b})$ be a connected component of $O^{(1)}(\tilde{b})$. If $O_c^{(1)}(\tilde{b}) \subset \Phi_1$, then considering as in [24], we obtain estimates similar to (2.79). A difficult case is $O_c^{(1)}(\tilde{b}) \cap \Phi_1 \not= \emptyset$, but $O_c^{(1)}(\tilde{b}) \not\subset \Phi_1$. On one hand, the function $\tilde{x}_1(\varphi)$, included in (2.16), is defined on the set $\Phi_1$, which has holes as small as $\varepsilon = k^{-2 - 12 s_1 - 9 \delta}$. These holes are essential in the construction of $\Phi_1$. Hence, the function $\tilde{x}_1(\varphi)$ can be analytically extended no further than $\varepsilon_\varepsilon$, $c < 1$, neighborhood of $\Phi_1$. Thus, to extend $\tilde{x}_1(\varphi)$ analytically into a connected component $O_c^{(1)}(\tilde{b})$ of a set $O^{(1)}(\tilde{b})$, such that $O_c^{(1)}(\tilde{b}) \not\subset \Phi_1$, we have to make sure that the size of $O_c^{(1)}(\tilde{b})$ is smaller than $\varepsilon$. On the other hand, deleting only such a small set from $\Phi_1$ cannot provide convergence of perturbations series for (2.77) in $\Phi_1 \setminus O^{(1)}(\tilde{b})$. The properties of the smallest set $O^{(1)}(\tilde{b})$, which we can construct to provide convergence of the perturbation series, is given in Lemma 2.16. Its connected component still can have a size up to $k^{-11/6 + \beta_1 + 12 s_1 + 4 \delta}$, at least this is the strongest estimate we can prove. Obviously, the size of such a component is still greater then the size $\varepsilon$ of the neighborhood of $\Phi_1$ where $x_1(\varphi)$ is holomorphic. Thus, we can not construct the set $O^{(1)}(\tilde{b})$ which fits into $\varepsilon$-neighborhood of $\Phi_1$, even though it is much smaller then (2.16). To overcome this difficulty we need additional considerations. We represent $\tilde{x}_1$ in the form $\tilde{x}_1(\varphi) = \tilde{x}_s(\varphi) + O \left( k^{-c k^{3/4}} \right)$, where $\tilde{x}_s(\varphi)$ is an analytic function in a set $\tilde{\Phi}_0$ larger then $\Phi_1$, $\tilde{\Phi}_0 \supset \Phi_1$, see Lemma 2.21. The smallest hole in $\tilde{\Phi}_0$ has the size $k^{-7/4 - 12 s_1 - 9 \delta}$. Since the size of each connected component of $O^{(1)}(\tilde{b})$ is smaller than $k^{-11/6 + \beta_1 + 12 s_1 + 4 \delta}$, $\tilde{x}_s(\varphi)$ is holomorphic in every $O_c^{(1)}(\tilde{b})$, such that $O_c^{(1)}(\tilde{b}) \cap \Phi_1 \not= \emptyset$. We prove an estimate for the norm of

$$
H^{(1)}(\tilde{x}_s(\varphi) + \tilde{b}) - k^2 \right)^{-1}
$$

outside $O_c^{(1)}(\tilde{b})$. Since, the set $O^{(1)}(\tilde{b})$ is rather small, the proof of an estimate for the norm is quite technical (Lemmas 2.16, 2.20, 2.23, 2.25). Further we construct a set $O_s^{(1)}(\tilde{b})$ taking small discs around the poles of (2.80). Using the analyticity of the resolvent and the maximum principle, we obtain (Lemma 2.25) an estimate of the type (2.79) for (2.80):

$$
\left\Vert \left( H^{(1)}(\tilde{x}_s(\varphi) + \tilde{b}) - k^2 \right)^{-1} \right\Vert < c k^{\mu} \left( \frac{R_b}{r(1)} \right)^J, \quad J = c_0 k^{2 + 2 s_1}, \quad \mu > 0.
$$

The size of each connected component of $O_s^{(1)}(\tilde{b})$ is much less than $k^{-2 - 12 s_1 - 9 \delta}$. Therefore, the function $x_1(\varphi)$ can be analytically extended into interior of every connected component of $O_s^{(1)}(\tilde{b})$ intersecting with $\Phi_1$. Using the estimate $\tilde{x}_1(\varphi) = \tilde{x}_s(\varphi) + O \left( k^{-c k^{3/4}} \right)$ in the $\varepsilon$ neighborhood of $\Phi_1$, we prove that (2.77) obeys an estimate similar to (2.81), see Lemma 2.29. This lemma is used in the second and step of approximation.

One more technical difficulty is related to the fact that the resolvent (2.75) is not from the trace class when $l = 1$, while it is from the trace class when $l > 1$. This means
that the determinant of the operator \( H^{(1)}(\vec{k}(\varphi) + \vec{b}) - k^2 \) \( H^{(1)}_0(\vec{k}(\varphi) + \vec{b}) - k^2 \)^{-1}, which we considered in [24] as a complex function of \( \varphi \) for \( l \geq 6 \), is not defined for the present case \( l = 1 \). To overcome this difficulty, we take a family of finite dimensional projections \( P_N, P_N \to s I \) as \( N \) goes to infinity. We consider a finite dimensional operator \( H^{(1)}_N = P_N H^{(1)} P_N \) and the determinant of \( \left( H^{(1)}_N(\vec{k}(\varphi) + \vec{b}) - k^2 \right) \left( H^{(1)}_{0,N}(\vec{k}(\varphi) + \vec{b}) - k^2 \right)^{-1} \).

We prove all necessary results for \( H^{(1)}_N \) and, then, the analogous results for \( H^{(1)} \) by sending \( N \) to infinity.

**Definition 2.13.** Let

\[
\Phi'_0 = \Phi_0 \setminus \left( \cup_{q \leq p_q < 4k^{-1}, \pm} O^+_{q} \right),
\]

Obviously, \( \Phi_1 \subset \Phi'_0 \subset \Phi_0 \). Note that the circles \( O^+_{q} \) deleted from \( \Phi_0 \) to obtain \( \Phi'_0 \) are relatively large. Their radius is larger than \( \frac{1}{4} k^{-4s_1-\delta} \), see (8.4). These circles are essentially bigger than other \( O^\pm_m \) constituting \( \Phi^{(1)} \). Properties of \( \Phi'_0 \) are stable in its \( k^{-4s_1-2\delta} \) neighborhood. We denote such neighborhood of \( \Phi'_0 \) by \( \Phi'_0 \).

**Definition 2.14.** The set \( \Phi^{(1)}(\vec{b}) \) is defined by the formula:

\[
\Phi^{(1)}(\vec{b}) = \cup_{0 < p_m < 4k, \pm} O^\pm_m(\vec{b}),
\]

where \( O^\pm_m(\vec{b}) \) are discs in complex plane centered at \( \varphi^\pm_m(\vec{b}) \), which are zeros of \( |\vec{k}(\varphi) + \vec{p}_m(\vec{b})|^2 - k^2 \). The radii of the discs are given by Definition 8.8.

**Definition 2.15.** The total size of \( \Phi^{(1)}(\vec{b}) \) is the sum of the sizes of its connected components.

**Lemma 2.16.** Let \( 100s_1 < \beta_1 < 1/12 - 28s_1 - 14\delta \).

1. If \( \varphi \in \Phi'_0 \setminus \Phi^{(1)}(\vec{b}) \) and \( m \) is such that \( |\vec{k}(\varphi) + \vec{p}_m(\vec{b})|^2 - k^2 | < k^{\beta_1} \), then

\[
\min_{0 < p_q < k^{-1}} |\vec{k}(\varphi) + \vec{p}_{m+q}(\vec{b})|^2 - k^2 | > k^{\beta_1},
\]

\[
|\vec{k}(\varphi) + \vec{p}_m(\vec{b})|^2 - k^2 | |\vec{k}(\varphi) + \vec{p}_{m+q_1}(\vec{b})|^2 - k^2 | |\vec{k}(\varphi) + \vec{p}_{m+q_2}(\vec{b})|^2 - k^2 | > \frac{1}{64} k^{2\beta_1}
\]

when \( 0 < p_{q_1}, p_{q_2} < k^{-1} \). This property is preserved in the \( k^{-4+2\beta_1-2s_1-\delta} \) neighborhood of \( \Phi'_0 \setminus \Phi^{(1)}(\vec{b}) \).

2. The size of each connected component \( \Phi^{(1)}_c(\vec{b}) \) of \( \Phi^{(1)}(\vec{b}) \) is less than \( k^{-\gamma} \), \( \gamma = 11/6 - \beta_1 - 12s_1 - 4\delta \). Each component contains no more than \( c_1 k^{2/3+s_1} \), \( c_1 = c_1(d_1, d_2) \) discs. The total size of \( \Phi^{(1)}(\vec{b}) \) does not exceed \( 2\pi c_1 k^{-5/6+\beta_1+12s_1+4\delta} \).

The set \( \Phi^{(1)}(\vec{b}) \) contains less than \( c_0 k^{2+2s_1} \) discs.

**Corollary 2.17.** For every \( \varphi \) in the \( k^{-4+2\beta_1-2s_1-\delta} \) neighborhood of \( \Phi'_0 \setminus \Phi^{(1)}(\vec{b}) \) and for every \( m \in \mathbb{Z}^2 \)

\[
|\vec{k}(\varphi) + \vec{p}_m(\vec{b})|^2 - k^2 | > k^{-2+2\beta_1-2s_1}.
\]
Proof of Corollary 2.17. Obviously, 
\[ |\tilde{k}(\varphi) + \tilde{p}_{m+q_1,2}(b)| - k^2 | \leq |\tilde{k}(\varphi) + \tilde{p}_m(b)| - k^2 | + 2 |\tilde{k}(\varphi) + \tilde{p}_m(b), \tilde{p}_{q_1,2}| + \tilde{p}^2_q. \]

If \[ |\tilde{k}(\varphi) + \tilde{p}_m(b)| - k^2 | < 1, \] then, clearly, \[ |\tilde{k}(\varphi) + \tilde{p}_{m+q_1,2}(b)| - k^2 | \leq ck^{1+s_1}. \] Now the corollary follows from (2.85) and the last estimate.

Corollary 2.18. If \( \varphi \) in the \( k^{-4+2\beta_1-2s_1-\delta} \) neighborhood of \( \hat{\Phi}_0 \setminus O^{(1)}(\tilde{b}) \), then the following estimates hold:
\[ \| (H^{(1)}_0 (\tilde{k}(\varphi) + \tilde{b})^{-1} - k^2 )^{-1} \| < ck^{2-2\beta_1+2s_1}. \] (2.87)
The resolvent has no more than \( c_1 k^{2/3+s_1} \) poles in each connected component of \( O^{(1)}(\tilde{b}) \). The total number of poles in \( \Phi_0 \) is less than \( c_0 k^{2+2s_1} \).

Corollary 2.19. If \( O^{(1)}(\tilde{b}) \cap \Phi'_0 \neq \emptyset \), then \( O^{(1)}(\tilde{b}) \subset \hat{\Phi}'_0 \).

This corollary follows from the statement that the size of \( O^{(1)}(\tilde{b}) \) does not exceed \( k^{-7} \) and the definition of \( \hat{\Phi}'_0 \) as the \( k^{-4s_1-2\delta} \) neighborhood of \( \Phi'_0 \).

The lemma is proven is Section 8.2. Note that the total size \( k^{-5/6+\beta_1+12s_1+4\delta} \) of the set \( O^{(1)}(\tilde{b}) \cap \hat{\Phi}'_0 \) is small comparing even with the smallest of the circle \( O_q, p_q < k^{s_1} \).

Lemma 2.20. If \( \varphi \in \hat{\Phi}'_0 \setminus O^{(1)}(\tilde{b}) \), then
\[ \| (H(\tilde{k}(\varphi) + \tilde{b}) - k^2 )^{-1} \| < c k^4. \] (2.88)
The only possible singularities of the resolvent \( (H(\tilde{k}(\varphi) + \tilde{b}) - k^2 )^{-1} \) in each connected \( O^{(1)}(\tilde{b}) \) component of \( O^{(1)}(\tilde{b}) \), such that \( O^{(1)}(\tilde{b}) \cap \Phi'_0 \neq \emptyset \), are poles. The number of poles (counting multiplicity) inside each component does not exceed \( c_1 k^{2/3+s_1} \). The total number of such poles is less than \( c_0 k^{2+2s_1} \).

Remark 2. In Lemma 2.20 we could prove convergence of perturbation series and the estimate for the resolvent analogous to (2.87) by somewhat longer considerations. However, (2.88) is good enough for our purposes. The main reason that convergence holds, in spite of weaker conditions on \( \varphi \) are formulas of the type (2.20).

Proof. Let \( \varphi \in \hat{\Phi}'_0 \setminus O^{(1)}(\tilde{b}) \). We define the set \( \Omega(\varphi) \subset Z^2 \) as follows:
\[ \Omega = \left\{ m \in Z^2 : \|\tilde{k}(\varphi) + \tilde{p}_m(\tilde{b})\| - k^2 | < k^{\beta_1} \right\}. \]

Let \( P_0 \) be the diagonal projection, corresponding to \( \Omega \): \( P_{0mm} = 1 \) if and only if \( m \in \Omega \). Let \( P_1 = I - P_0 \).

It follows from Lemma 2.16 that \( P_0W_1P_0 = 0 \). Indeed, suppose it is not so. Then, there is a pair \( m, m + q \in \Omega \) such that \( w_q \neq 0 \). Function \( W_1 \) is a trigonometric polynomial, see (2.3). Hence \( 0 < p_q < k^{s_1} \). This contradicts to (2.84).

Next, we consider Hilbert identity:
\[ (H^{(1)} - k^2)^{-1} = (H^{(1)}_0 - k^2)^{-1} - (H^{(1)}_0 - k^2)^{-1}W_1(H^{(1)}_0 - k^2)^{-1}, \]
abbreviations $H^{(1)} = H^{(1)}(\tilde{k}(\varphi) + \tilde{b})$, $H_0^{(1)} = H_0^{(1)}(\tilde{k}(\varphi) + \tilde{b})$ being used. Applying $P_1$ from two sides and solving for $P_1(H^{(1)} - k^2)^{-1}P_1$, we obtain:

$$P_1(H^{(1)} - k^2)^{-1}P_1 = (I + \mathcal{E})^{-1}P_1(H_0^{(1)} - k^2)^{-1}(I - W_1P_0(H^{(1)} - k^2)^{-1}P_1), \quad (2.89)$$

where $\mathcal{E} : P_1^2 \to P_1^2$, $\mathcal{E} = P_1(H_0^{(1)} - k^2)^{-1}W_1P_1$. By the definition of $P_1$,

$$\|P_1(H_0^{(1)} - k^2)^{-1}\| \leq k^{-\beta_1}, \quad (2.90)$$

$$\|\mathcal{E}\| \leq \|W_1\|k^{-\beta_1}. \quad (2.91)$$

Next, we apply $P_0$ from the left and $P_1$ from the right to the Hilbert identity. Considering that $P_0W_1P_0 = 0$ and $P_0(H_0^{(1)} - k^2)^{-1}P_1 = 0$, we obtain:

$$P_0(H^{(1)} - k^2)^{-1}P_1 = -P_0(H_0^{(1)} - k^2)^{-1}W_1P_1(H^{(1)} - k^2)^{-1}P_1. \quad (2.92)$$

Substituting $(2.89)$ into $(2.92)$ gives:

$$P_0(H^{(1)} - k^2)^{-1}P_1 = BP_0(H^{(1)} - k^2)^{-1}P_1 - C, \quad (2.93)$$

$$B = P_0QP_1(I + \mathcal{E})^{-1}P_1QP_0, \quad C = P_0QP_1(I + \mathcal{E})^{-1}P_1(H_0^{(1)} - k^2)^{-1}, \quad Q = (H_0^{(1)} - k^2)^{-1}W_1.$$

It follows from $(2.87)$ that

$$\|Q\| < C(W_1)k^{2-2\beta_1+2s_1}. \quad (2.94)$$

Using also $(2.90)$ and $(2.91)$ we get:

$$\|C\| < C(W_1)k^{2-3\beta_1+2s_1}. \quad (2.95)$$

Let us prove that

$$\|B\| < C(W_1)k^{2-2\beta_1+8s_1}. \quad (2.96)$$

Using $(I + \mathcal{E})^{-1} = I + \sum_{r=1}^{N}(\mathcal{E})^r + (\mathcal{E})^{N+1}(I + \mathcal{E})^{-1}$, $N = [40\beta_1^{-1}]$, we expand $B$ into sum of $N + 2$ operators:

$$B = \sum_{r=0}^{N+1} B_r, \quad B_0 = P_0QP_1QP_0, \quad B_r = (-1)^r P_0QP_1(QP_1)^rQP_0, \quad r = 1, ..., N,$$

$$B_{N+1} = (-1)^{N+1} P_0QP_1(QP_1)^{N+1}(I + \mathcal{E})^{-1}P_1QP_0.$$

Let us estimate $\|B_r\|$, $r = 0, ..., N$. First, we prove that $B_0$ is diagonal and

$$\|B_0\| < k^{-2\beta_1+6s_1}. \quad (2.97)$$

Indeed, suppose it is not diagonal. Then there is a pair $m, m + q \in \Omega$, $q \neq 0$, such that $B_{0m,m+q} = 0$. Considering that $W_1$ is a trigonometric polynomial, we obtain that $p_q < 2R_0 < k^{s_1}$. Now we easily arrive to contradiction with $(2.84)$, Lemma 2.16. Thus, $\|B_0\| = \sup_{m \in \Omega} |B_{0mm}|$,

$$B_{0mm} = \sum_{q \in \mathbb{Z}^2, 0 < p_q < R_0} \frac{|w_q|^2}{(|k(\varphi) + \tilde{p}_m(b)|^2 - k^2) (|k(\varphi) + \tilde{p}_{m+q}(b)|^2 - k^2)}.$$
Considering that $w_q = \bar{w}_{-q}$, we easily show that the right-hand side is equal to

$$\sum_{q \in \mathbb{Z}^2, 0 < p_q < R_0} w_q \frac{|\bar{k}(\varphi) + \bar{p}_m(\bar{b})|^2 - k^2 + p_q^2}{(|\bar{k}(\varphi) + \bar{p}_m(\bar{b})|^2 - k^2)(|\bar{k}(\varphi) + \bar{p}_{m+q}(\bar{b})|^2 - k^2)(|\bar{k}(\varphi) + \bar{p}_{m-q}(\bar{b})|^2 - k^2)}.$$ 

Using (2.81) and (2.85), we obtain $B_{0mm} = O(k^{-2\beta_1+6s_1})$. Therefore, (2.97) holds. Similar argument yields that $B_1$ is diagonal and $\|B_1\| = \sup_{m \in \Omega} |B_{1mm}|$.

$$B_{1mm} = \sum_{q_1, q_2 \in \mathbb{Z}^2, 0 < |p_{q_1, q_2}| < R_0, q_1 + q_2 \neq 0} \frac{-w_{q_1}w_{q_2}w_{-q_1-q_2}}{(|\bar{k}(\varphi) + \bar{p}_m(\bar{b})|^2 - k^2)(|\bar{k}(\varphi) + \bar{p}_{m+q_1}(\bar{b})|^2 - k^2)(|\bar{k}(\varphi) + \bar{p}_{m+q_2}(\bar{b})|^2 - k^2)}.$$ 

Applying (2.85) in Lemma 2.10, we obtain

$$\|B_1\| < \|W_1\|^3k^{-2\beta_1+8s_1}. \quad (2.98)$$

Similar estimates hold for all $B_r$, $r = 2, ..., N$:

$$\|B_r\| \leq (c\|W_1\|)^{r+2} k^{-(\beta_1-4s_1)(r+1)}. \quad (2.99)$$

To estimate $B_{N,1}$, we note that $\|P_1Q\| < \|W_1\|k^{-\beta_1}$. Considering that $\|P_0Q\| < ck^{-2\beta_1+2s_1}$ and $N$ is sufficiently large, we arrive to $\|B_{N,1}\| < ck^{-2\beta_1+8s_1}$. Hence, (2.96) is proven. Next, considering (2.93) and (2.95), (2.96), we obtain:

$$\|P_0(H^{(1)} - k^2)^{-1}P_1\| < ck^{2-3\beta_1+2s_1}. \quad (2.100)$$

Using the last estimate in (2.89), we get

$$\|P_1(H^{(1)} - k^2)^{-1}P_1\| < ck^{2-4\beta_1+2s_1}. \quad (2.101)$$

Taking into account that $P_1(H^{(1)} - k^2)^{-1}P_0$ corresponding to $\varphi$ is the adjoint of $P_0(H^{(1)} - k^2)^{-1}P_1$ corresponding to $\varphi$ and the set $\mathcal{O}_c(\bar{b})$ is symmetric with respect to real axis, we obtain:

$$\|P_1(H^{(1)} - k^2)^{-1}P_0\| < ck^{2-3\beta_1+2s_1}. \quad (2.102)$$

Applying $P_0$ to both parts of Hilbert equations, using $P_0W_1P_0 = 0$ and (2.87), (2.102), we obtain:

$$\|P_0(H^{(1)} - k^2)^{-1}P_0\| < ck^{4-5\beta_1+4s_1} < ck^4. \quad (2.103)$$

Combining (2.100) – (2.103), we obtain (2.88). Note that (2.88) holds on the boundary $\partial \mathcal{O}_c(\bar{b})$ of each connected component $\mathcal{O}_c(\bar{b})$, such that $\mathcal{O}_c(\bar{b}) \cap \Phi_0 \neq \emptyset$. It is true, since the size of $\mathcal{O}_c(\bar{b})$ is less than $ck^{-\gamma}$, i.e., much smaller than $k^{-4s_1-2\delta}$.

It remains to show that all singularities of $(H^{(1)} - k^2)^{-1}$ inside each connected component $\mathcal{O}_c(\bar{b})$, such that $\mathcal{O}_c(\bar{b}) \cap \Phi_0 \neq \emptyset$, are poles and the number of poles, counting multiplicity, does not exceed $ck^{2/3+s_1}$. We follow here the approach developed in [24] for $(-\Delta)^l + V$, $l > 6$. The plan in [24] is the following. We consider the
operator $I + A := (H^{(1)} - k^2)(H_0^{(1)} - k^2)^{-1}$ and, use a well-known relation (see e.g. [22]):

$$\left| \det(I + A) - 1 \right| \leq \| A \|_1 e^{\| A \|_1 + 2}, \quad A \in S_1. \tag{2.104}$$

on the boundary of $O_c^{(1)}(\vec{b})$. Using Rouche’s theorem, we arrive at the conclusion that $I + A$ has the same number of zeros and poles inside $O_c^{(1)}(\vec{b})$. Therefore, the resolvents $(H^{(1)} - k^2)^{-1}$ and $(H_0^{(1)} - k^2)^{-1}$ have the same number of poles. Using Corollary 2.18 we obtain that the number of poles of $(H_0^{(1)} - k^2)^{-1}$ does not exceed $c_1k^{2/3 + s_1}$. However, there is a technical obstacle on the way of this proof in the present case $l = 1$: the determinant of $(H^{(1)} - k^2)(H_0^{(1)} - k^2)^{-1}$ is not from $S_1$. To overcome this obstacle, we introduce a family of expanding diagonal finite dimensional projections $P_N : P_N \to I$. We consider a finite dimensional analog of $H^{(1)}$ with a multiplier $\alpha$ in front of $W_1$:

$$H_{\alpha, N}^{(1)} = H_0^{(1)} P_N + \alpha P_N W_1 P_N, \quad 0 \leq \alpha \leq 1. \tag{2.105}$$

First, we show that all $(H_{\alpha, N}^{(1)} - k^2)^{-1}, \quad 0 \leq \alpha \leq 1$, have the same number of poles inside each $O_c^{(1)}(\vec{b})$. Indeed, let $\alpha, \alpha_0 \in [0, 1]$. We introduce operator $A$ by the formula: $I + (\alpha - \alpha_0)A = (H_{\alpha, N}^{(1)} - k^2)(H_{\alpha_0, N}^{(1)} - k^2)^{-1}$. Obviously, $A = P_N W_1 P_N (H_{\alpha_0, N}^{(1)} - k^2)^{-1}$. Applying formula (2.104) to $I + (\alpha - \alpha_0)A$, we obtain:

$$\left| \det(H_{\alpha, N}^{(1)} - k^2)(H_{\alpha_0, N}^{(1)} - k^2)^{-1} - 1 \right| \leq |\alpha - \alpha_0| \| A \|_1 e^{\| A \|_1 + 2}. \tag{2.106}$$

Clearly, $\| A \| \leq \| W_1 \| (H_{\alpha_0, N}^{(1)} - k^2)^{-1}$, $A$ is an $N$ dimensional operator: $\| A \|_1 \leq \| A \| N$. Combining the last three inequalities, we get: $\| A \|_1 < ck^4 N$. Using this estimate in (2.106), we obtain:

$$\left| \det(H_{\alpha, N}^{(1)}(\varphi) - k^2)(H_{\alpha_0, N}^{(1)}(\varphi) - k^2)^{-1} - 1 \right| \leq c|\alpha - \alpha_0| Nk^4, \tag{2.107}$$

when $\varphi$ is on the boundary of $O_c^{(1)}(\vec{b})$. If $|\alpha - \alpha_0|$ is sufficiently small, then the right-hand side of (2.107) is less than 1. By Rouche’s theorem $\det(H_{\alpha, N}^{(1)} - k^2)(H_{\alpha_0, N}^{(1)} - k^2)^{-1}$ has the same number of poles and zeros inside $O_c^{(1)}(\vec{b})$. Therefore, $\det(H_{\alpha, N}^{(1)} - k^2)$ and $\det(H_{\alpha_0, N}^{(1)} - k^2)$ have the same number of zeros. Covering $[0, 1]$ by small intervals and using a finite step induction, we obtain that $\det(H_{\alpha, N}^{(1)} - k^2)$ has the same number of zeros as $\det(H_{0, N}^{(1)} - k^2)$ for any $\alpha \in [0, 1]$.

Obviously the zeros of $\det(H_{0, N}^{(1)} - k^2)$ in $O_c^{(1)}(\vec{b})$ stay the same after $N$ surpasses a certain number. They are solutions of the equations $|\vec{k}(\varphi) + \vec{p}_m|^2 = k^2$ in $O_c^{(1)}(\vec{b})$. Hence, the number of zeros of $\det(H_{1, N}^{(1)} - k^2)$ is the same for all sufficiently large $N$. It is equal to the number of discs in $O_c^{(1)}(\vec{b})$. We denote this number by $M$. By Lemma
We denote the zeros of \( \det(H_{1,N}^{(1)}-k^2) \) as \( \varphi_1^{(N)},...\varphi_M^{(N)} \), multiplicity being taken into account. Obviously, the function \( \prod_{n=1}^{M} (\varphi - \varphi_n^{(N)}) (H_{1,N}^{(1)}-k^2)^{-1} \) is holomorphic in \( O^{(1)}(\vec{b}) \). For a fixed \( n \) the sequence \( \{\varphi_n^{(N)}\}_{N=1}^{\infty} \) has an accumulation point. We choose a subsequence \( N_i \) such that each \( \{\varphi_n^{(N_i)}\}_{i=1}^{\infty} \) converges. With slight abuse of notations we drop the index \( i \) and consider that each \( \{\varphi_n^{(N)}\}_{N=1}^{\infty} \) has a limit \( \varphi_n \). Considering as in the proof of (2.88), we obtain:
\[
\left\| (H_{1,N}^{(1)}(\varphi) - k^2)^{-1} \right\| < ck^4, \quad \varphi \in \partial O^{(1)}(\vec{b})
\]
(2.108)

Therefore,
\[
\left\| \prod_{n=1}^{M} (\varphi - \varphi_n^{(N)}) (H_{1,N}^{(1)}(\varphi) - k^2)^{-1} \right\| < cr^M k^4,
\]
(2.109)

when \( \varphi \in \partial O^{(1)}(\vec{b}) \), \( r \) being the size of \( O^{(1)}(\vec{b}) \), \( r < k^{-11/6+\beta_1+12s_1+4\delta} \). By the maximum principle, the above estimate holds inside \( O^{(1)}(\vec{b}) \) too. Suppose \( |\varphi - \varphi_n| > 2\varepsilon \) for all \( n \) and some \( \varepsilon > 0 \). Then \( |\varphi - \varphi_{n,N}| > \varepsilon \) for all \( n \) and sufficiently large \( N \). Using (2.109), we obtain:
\[
\left\| (H_{1,N}^{(1)}(\varphi) - k^2)^{-1} \right\| < cr^M k^4\varepsilon^{-M}.
\]
(2.110)

Thus, all resolvents are bounded uniformly in \( N \), when \( \varphi \neq \varphi_n, n = 1,...M \). Now, it is easy to show now that \( (H_{1,N}^{(1)}(\varphi) - k^2)^{-1} \) tends to \( (H_{1}^{(1)}(\varphi) - k^2)^{-1} \) in the class of bounded operators when \( \varphi \neq \varphi_n, n = 1,...M \). Taking the limit in (2.109), we obtain:
\[
\left\| \prod_{n=1}^{M} (\varphi - \varphi_n) (H_{1}^{(1)}(\varphi) - k^2)^{-1} \right\| < cr^M k^4.
\]
(2.111)

This means that the only possible singularities of \( (H_{1}^{(1)}(\varphi) - k^2)^{-1} \) are poles at the points \( \varphi = \varphi_n, n = 1,...M \). The number of poles, counting multiplicity does not exceed \( M = c_1 k^{2/3+s_1} \).

Let us recall that function \( \varphi_1(\varphi) \), defined by Lemma 2.9 is holomorphic in \( \hat{\Phi}_1 \), see Lemma 2.10. Now we prove existence of a function \( \varphi_*(\varphi) \), which is a good approximation of \( \varphi_1(\varphi) \) in \( \hat{\Phi}_1 \) and defined on a larger set \( \hat{\Phi}_* \). Let
\[
\hat{\Phi}_* = \Phi_0 \setminus \hat{O}^{(1)}, \quad \hat{O}^{(1)} = \cup_{i \in Z^2, 0 < p_i < k^{3/4} \partial_1}.
\]

The size of the discs constituting \( \hat{O}^{(1)} \) is at least \( k^{-7/4-16s_1-9\delta} \). Let \( \Phi_* \) be the \( k^{-7/4-16s_1-10\delta} \)-neighborhood of \( \hat{\Phi}_* \). Since the size of the neighborhood is much smaller than the size of the discs, \( \hat{\Phi}_* \) and \( \hat{\Phi}_* \) are the sets of the same type. The following relations hold: \( \Phi_1 \subset \Phi_* \subset \Phi_0 \), \( \hat{\Phi}_1 \subset \hat{\Phi}_* \subset \hat{\Phi}_0 \subset \Phi_0 \).

**Lemma 2.21.** Let \( 100s_1 < \beta_1 < 1/12 - 28s_1 - 14\delta \). There is a function \( \varphi_*(\varphi) \) holomorphic in \( \Phi_* \), which satisfies the estimates analogous to (2.70) on this set:
\[
|\varphi_* - k| < C(W_1)k^{-3+30s_1+2\delta}, \quad \left| \frac{\partial \varphi_*}{\partial \varphi} \right| < C(W_1)k^{-1+33s_1+22\delta}.
\]
(2.112)
It satisfies the following estimate when \( \varphi \in \hat{\Phi}_1 \):
\[
|\kappa_1(\varphi) - \kappa_1(\varphi)| < ck^{-\frac{1}{4}}k^{3/4}
\]  
(2.113)

**Corollary 2.22.** Function \( \kappa_+ \) is holomorphic and obeys (2.114) inside each connected component \( O_c^{(1)}(b) \) of \( O^{(1)}(b) \), such that \( O_c^{(1)}(b) \cap \Phi_+ \neq \emptyset \).

**Proof.** Let us consider the function:
\[
\lambda_+^{(1)}(\alpha, \vec{x}) = \varphi^2 + \varphi_1(\alpha, \vec{x}), \quad \varphi = |\vec{x}|, \quad f_1(\alpha, \vec{x}) = \sum_{r=2}^{[k^{3/4}]+1} \alpha^r g_r^{(1)}(\vec{x}).
\]
(2.114)

Obviously \( f_1(\alpha, \vec{x}) \) is a finite sum for the series \( f_1(\alpha, \vec{x}) \), see (2.58). It satisfies the estimates analogous to (2.59), when \( \varphi \in \Phi_1 \)
\[
|f_1(\alpha, \vec{x})| \leq 2\alpha^2 k^{-2+30s_1-20\delta}, \quad |T(m)f_1(\alpha, \vec{x})| \leq 2\alpha^2 k^{-2+30s_1-20\delta+|m|(1+16s_1+11\delta)}
\]
(2.115)

By the construction of \( \Phi_+ \), the estimates (2.4) and (2.5) hold for all \( \varphi \in \Phi_+ \) and \( p_i \leq k^{3/4} \). Now it is easy to see that all coefficients \( g_r^{(1)}(\vec{x}) \), \( 2 \leq r \leq k^{3/4} \), are holomorphic functions in \( \Phi_+ \) and even to its \( 2k^{-7/4-16s_1-10\delta} \) neighborhood. The estimates (2.24), (2.26), (2.47), (2.49)–(2.51) hold when \( r \leq k^{3/4} \). Therefore, the finite sum can be analytically extended to \( 2k^{-7/4-16s_1-10\delta} \) neighborhood of \( \Phi_+ \), the estimates (2.115) being valid. Estimating the tail of the series, we get
\[
|f_1 - f_1| < k^{-\frac{1}{4}}k^{3/4}. \]

Solving the equation \( \lambda_+^{(1)}(\alpha, \vec{x}) = k^2 \) for \( \varphi \), we obtain that there is a function \( \kappa_+(\varphi) \) defined in the \( \hat{\Phi}_+ \), such that (2.113) holds. This function obviously obeys estimates (2.112).  

**Lemma 2.23.**

1. Let \( 100s_1 < \beta_1 < 1/12 - 28s_1 - 14\delta \). If \( \varphi \in \hat{\Phi}_+ \setminus O^{(1)}(b) \), and \( m \) is such that \( |\vec{z}_+(\varphi) + \vec{p}_m(b) - k^2| < \frac{1}{2}k^{\beta_1} \), then
\[
\min_{0 < p_1 < k_1} |\vec{z}_+(\varphi) + \vec{p}_{m+q}(b) - k^2| > \frac{1}{2}k^{\beta_1},
\]
(2.116)
\[
|\vec{z}_+(\varphi) + \vec{p}_m(b)^2 - k^2| |\vec{z}_+(\varphi) + \vec{p}_{m+q_1}(b)^2 - k^2| |\vec{z}_+(\varphi) + \vec{p}_{m+q_2}(b)^2 - k^2| > \frac{1}{128}k^{2\beta_1}
\]
(2.117)
when \( 0 < p_{1}, p_{2} < k_1 \). This property is preserved in the \( \frac{1}{2}k^{-4+2\beta_1-2s_1-\delta} \) neighborhood of \( \hat{\Phi}_+ \setminus O^{(1)}(b) \).

2. Each equation \( |\vec{z}_+(\varphi) + \vec{p}_m(b)|^2 = k^2 \) has the same number of solutions as the “unperturbed” equation \( |k(b) + \vec{p}_m(b)|^2 = k^2 \) inside every \( O_c^{(1)}(b) \), \( O_c^{(1)}(b) \cap \Phi_+ \neq \emptyset \).
Corollary 2.24. The number of points \( \varphi \) satisfying one of the equations \( |\mathcal{Z}_s(\varphi) + \overline{p}_m(\overline{b})|^2 = k^2 \), \( m \in \mathbb{Z}^2 \), in \( \mathcal{O}_c^{(1)}(\overline{b}) \), \( \mathcal{O}_c^{(1)}(\overline{b}) \cap \Phi_* \neq \emptyset \), does not exceed \( c_1k^{2/3+s_1} \). The total number of such points is less then \( c_0k^{2+2s_1} \).

The corollary follows from the above lemma and the last statement of Lemma 2.16.

Proof. Let \( \varphi \in \Phi_* \setminus \mathcal{O}_c^{(1)}(\overline{b}) \). Noticing that \( \Phi_* \setminus \mathcal{O}_c^{(1)}(\overline{b}) \subset \Phi_0' \setminus \mathcal{O}_c^{(1)}(\overline{b}) \), we obtain that \( \varphi \) satisfies the conditions of Lemma 2.16. Considering the first inequality in (2.112), we conclude:

\[
|\mathcal{Z}_s(\varphi) + \overline{p}_m(\overline{b})|^2 = O(k^{-2+30s_1+20\delta}) \quad \text{for any } m \in \mathbb{Z}^2.
\]

Taking into account that \( 30s_1 + 20\delta < 2\beta_1 - 2s_1 \) and (2.86), we get

\[
|\mathcal{Z}_s(\varphi) + \overline{p}_m(\overline{b})|^2 < \frac{1}{2}|\overline{k}(\varphi) + \overline{p}_m(\overline{b})|^2 - k^2 \quad \text{for any } m \in \mathbb{Z}^2.
\]

Therefore, \( 2|\mathcal{Z}_s(\varphi) + \overline{p}_m(\overline{b})|^2 - k^2 > |\overline{k}(\varphi) + \overline{p}_m(\overline{b})|^2 - k^2 \) for all \( m \in \mathbb{Z}^2 \). Now, the first statement of the lemma easily follows from the first statement of Lemma 2.16.

In particular, (2.119) holds on the boundary of each \( \mathcal{O}_c^{(1)}(\overline{b}) \), \( \mathcal{O}_c^{(1)}(\overline{b}) \cap \Phi_* \neq \emptyset \), see Corollary 2.22. By Rouche’s theorem, each equation \( |\mathcal{Z}_s(\varphi) + \overline{p}_m(\overline{b})|^2 = k^2 \) has the same number of solutions inside every \( \mathcal{O}_c^{(1)}(\overline{b}) \) as the “unperturbed” equation \( |\overline{k}(\varphi) + \overline{p}_m(\overline{b})|^2 = k^2 \).

\[\blacksquare\]

Lemma 2.25. Let \( 100s_1 < \beta_1 < 1/12 - 28s_1 - 14\delta \). If \( \varphi \in \Phi_* \setminus \mathcal{O}_c^{(1)}(\overline{b}) \), then

\[
\left\| (H(\mathcal{Z}_s(\varphi) + \overline{b}) - k^2)^{-1} \right\| < ck^4.
\]

This estimate is stable in the \( \frac{1}{2}k^{-4+2\beta_1-2s_1-\delta} \) neighborhood of \( \Phi_* \setminus \mathcal{O}_c^{(1)}(\overline{b}) \). The only possible singularities of the resolvent \( (H(\mathcal{Z}_s(\varphi) + \overline{b}) - k^2)^{-1} \) inside each \( \mathcal{O}_c^{(1)}(\overline{b}) \), \( \mathcal{O}_c^{(1)}(\overline{b}) \cap \Phi_* \neq \emptyset \), are poles. The number of poles, counting multiplicity, inside each such \( \mathcal{O}_c^{(1)}(\overline{b}) \) does not exceed \( c_1k^{2/3+s_1} \). The total number of such poles is less then \( c_0k^{2+2s_1} \).

Proof. The proof of The lemma is analogous to that of Lemma 2.20 up to replacement of Lemma 2.16 by Lemma 2.23. \[\blacksquare\]

Definition 2.26. Let us numerate all components \( \mathcal{O}_c^{(1)}(\overline{b}) \), \( \mathcal{O}_c^{(1)}(\overline{b}) \cap \Phi_* \neq \emptyset \), by index \( i, i = 1, \ldots, I \). We denote the poles of \( (H(\mathcal{Z}_s(\varphi) + \overline{b}) - k^2)^{-1} \) in a component \( \mathcal{O}_c^{(1)}(\overline{b}) \) by \( \varphi_{i,n_i}, n_i = 1, \ldots, M_i, M_i < c_1k^{2/3+s_1} \). Let us consider the discs \( \mathcal{O}_s(\overline{b})_{i,n_i} \) of the radius \( r^{(1)} = k^{-4-6s_1-3\delta} \) around these poles. This radius is much less then the size of discs constituting \( \mathcal{O}_c^{(1)}(\overline{b}) \). Let \( \mathcal{O}_s^{(1)}(\overline{b})_i \) be the union of all small discs corresponding to a component \( \mathcal{O}_c^{(1)}(\overline{b})_i \):

\[
\mathcal{O}_s^{(1)}(\overline{b})_i = \bigcup_{n_i=1}^{M_i} \mathcal{O}_s^{(1)}(\overline{b})_{i,n_i}.
\]

Obviously, \( \mathcal{O}_s^{(1)}(\overline{b})_i \subset \mathcal{O}_c^{(1)}(\overline{b})_i \), and, therefore, \( \mathcal{O}_s^{(1)}(\overline{b})_i \cap \mathcal{O}_s^{(1)}(\overline{b})_{i'} = \emptyset \) if \( i \neq i' \). Let

\[
\mathcal{O}_s^{(1)}(\overline{b}) = \bigcup_{i=1}^{I} \mathcal{O}_s^{(1)}(\overline{b})_i.
\]
Lemma 2.27. The size of each connected component of $O_{s_1}^{(1)}(\tilde{b})$ does not exceed $c_1k^{-3\delta -5s_1}$ and the total size of $O_{s_1}^{(1)}(\tilde{b})$ does not exceed $c_0k^{-2-4s_1-3\delta}$.

Proof. Since each disc has the radius $r^{(1)} = k^{-4-6s_1-3\delta}$ and $M_i < c_1k^{2/3+s_1}$, the size of a connected component of $O_{s_1}^{(1)}(\tilde{b})$ does not exceed $c_1k^{-3\delta -5s_1}$. The total number of discs in $O_{s_1}^{(1)}(\tilde{b})$ does not exceed $c_0k^{2+2s_1}$. Now the second statement of the lemma easily follows.

Further we consider only those connected components $O_{sc}^{(1)}(\tilde{b})$ of $O_{s_1}^{(1)}(\tilde{b})$ who have non-empty intersection with $\Phi_1$: $O_{sc}^{(1)}(\tilde{b}) \cap \Phi_1 \neq \emptyset$.

Lemma 2.28. Function $\varphi_1(\varphi)$ is analytic in every $O_{sc}^{(1)}(\tilde{b})$ such that $O_{sc}^{(1)}(\tilde{b}) \cap \Phi_1 \neq \emptyset$. The estimates (2.126) hold inside $O_{sc}^{(1)}(\tilde{b})$.

Proof. By Lemma 2.27, the length of each connected component of $O_{s_1}^{(1)}(\tilde{b})$ is less than $c_1k^{-3\delta -5s_1}$. Thus, each connected component of $O_{s_1}^{(1)}(\tilde{b})$, which has a non-empty intersection with $\Phi_1$, is, in fact, in $\tilde{\Phi}_1$. By Lemma 2.10 (3), $\varphi_1(\varphi)$ is analytic in in $\tilde{\Phi}_1$ and, therefore, in $O_{s_1}^{(1)}(\tilde{b})$.

Lemma 2.29. If $100s_1 < \beta_1 < 1/12 - 28s_1 - 14\delta$ and $\varphi \in \tilde{\Phi}_1 \setminus O_{s_1}^{(1)}(\tilde{b})$, then
\[
\left\| \left( H(\tilde{\varphi}_1(\varphi) + \tilde{b}) - k^2 \right)^{-1} \right\| < k^{J^{(1)}}, \quad J^{(1)} = 5c_1k^{2/3+s_1}. \tag{2.123}
\]

This estimate is stable in the $k^{-4-6s_1-4\delta}$ neighborhood of $\tilde{\Phi}_1 \setminus O_{s_1}^{(1)}(\tilde{b})$. The resolvent $(H(\tilde{\varphi}_1(\varphi) + \tilde{b}) - k^2)^{-1}$ is an analytic function of $\varphi$ in every component of $O_{s_1}^{(1)}(\tilde{b})$, whose intersection with $\Phi_1$ is not empty. The only singularities of the resolvent are poles. The number of poles in each connected component does not exceed $c_1k^{2/3+s_1}$. The total number of poles in all components of $O_{s_1}^{(1)}(\tilde{b})$, whose intersection with $\Phi_1$ is not empty, is less than $c_0k^{2+2s_1}$.

Proof. Suppose $\varphi \in O_{c}^{(1)}(\tilde{b})_i$. Using Lemma 2.25, we obtain
\[
\left\| \prod_{n_i=1}^{M_i} (\varphi - \varphi_i, n_i) (H(\tilde{\varphi}_s(\varphi) + \tilde{b}) - k^2)^{-1} \right\| < cr^c_k^{2/3} k^4, \tag{2.124}
\]
\(r\) being the maximal size of the components $O_{c}^{(1)}(\tilde{b})_i$, $r = k^{-\gamma}$. Let $\varphi \in O_{c}^{(1)}(\tilde{b})_i \setminus O_{s_1}^{(1)}(\tilde{b})$. Considering that $|\varphi - \varphi_i, n_i| > k^{-4-6s_1-3\delta}$, we obtain
\[
2 \left\| \left( H(\tilde{\varphi}_s(\varphi) + \tilde{b}) - k^2 \right)^{-1} \right\| < k^{5c_1k^{2/3+s_1}}. \tag{2.125}
\]
Combining the last estimate with (2.120), we obtain that
\[
2 \left\| \left( H(\tilde{\varphi}_s(\varphi) + \tilde{b}) - k^2 \right)^{-1} \right\| < k^{5c_1k^{2/3+s_1}} \text{ when } \varphi \in \tilde{\Phi}_1 \setminus O_{s_1}^{(1)}(\tilde{b}). \tag{2.126}
\]
Hilbert identity yields:
\[
(H^{(1)}(\tilde{\varphi}_1(\varphi) + \tilde{b}) - k^2)^{-1} - (H^{(1)}(\tilde{\varphi}_s(\varphi) + \tilde{b}) - k^2)^{-1} = (H^{(1)}(\tilde{\varphi}_1(\varphi) + \tilde{b}) - k^2)^{-1} \mathcal{E}, \tag{2.127}
\]
where $\mathcal{E}$ is a complex remainder term.
\[ \mathcal{E} = (H_0(\tilde{z}_s(\varphi) + \tilde{b}) - H_0(\tilde{z}_1(\varphi) + \tilde{b})) \left( H^{(1)}(\tilde{z}_s(\varphi) + \tilde{b}) - k^2 \right)^{-1}. \]

Let us show that \( \|\mathcal{E}\| < \frac{1}{2} \). Indeed, we write \( \mathcal{E} \) in the form \( \mathcal{E} = \mathcal{E}_1A \),

\[ \mathcal{E}_1 = \left( H_0(\tilde{z}_s(\varphi) + \tilde{b}) - H_0(\tilde{z}_1(\varphi) + \tilde{b}) \right) \left( H^{(1)}(\tilde{z}_s(\varphi) + \tilde{b}) + k^2 \right)^{-1}, \]

\[ A = \left( H_0(\tilde{z}_s(\varphi) + \tilde{b}) + k^2 \right) \left( H^{(1)}(\tilde{z}_s(\varphi) + \tilde{b}) - k^2 \right)^{-1}. \]

It is easy to show that \( \mathcal{E}_1 \) is a diagonal operator and \( \|\mathcal{E}_1\| \leq |\varkappa_2(\varphi) - \varkappa_1(\varphi)|. \) Using \( \eqref{2.123} \), we obtain:

\[ \|\mathcal{E}_1\| \leq c k^{-1 - (\beta_0 - 4\delta_1 - 4\delta)k^{3/4}}. \]

It easily follows from formula for \( A \) that \( \|A\| \leq 1 + 2k^2 \| (H^{(1)}(\tilde{z}_s(\varphi) + \tilde{b}) - k^2)^{-1} \|. \) Using \( \eqref{2.125} \), we get \( \|A\| \leq k^{6\delta_1 k^{2/3} + s_1} \). Multiplying the estimates for the norms \( \mathcal{E}_1 \) and \( \|A\| \), we get \( \|\mathcal{E}\| = o(1) \). Now \( \eqref{2.127} \) yields:

\[ \| (H^{(1)}(\tilde{z}_1(\varphi) + \tilde{b}) - k^2)^{-1} \| \leq 2 \| (H^{(1)}(\tilde{z}_s(\varphi) + \tilde{b}) - k^2)^{-1} \| < k^{5\delta_1 k^{2/3} + s_1}. \] \( \eqref{2.128} \)

Considering that the discs in \( O^{(1)}(\tilde{b}) \) have the radius \( k^{-4 - 6\delta_1 - 3\delta} \), we easily get that the estimate \( \eqref{2.123} \) is stable in the \( k^{-4 - 6\delta_1 - 4\delta} \) neighborhood of \( \tilde{\Phi}_s \cap O^{(1)}(\tilde{b}) \).

Next, we use determinants to estimate the number of poles of \( (H^{(1)}(\tilde{z}_1(\varphi) + \tilde{b}) - k^2)^{-1} \). We will follow the scheme established in Lemma 2.20. First, we note that \( \eqref{2.127} \) and \( \eqref{2.128} \) hold not only for \( \tilde{z}_1(\varphi) \), but also for all points in the segment between \( \tilde{z}_s(\varphi) \) and \( \tilde{z}_1(\varphi) \), i.e., for \( \tilde{z}_s(\varphi) + \alpha(\tilde{z}_1(\varphi) - \tilde{z}_s(\varphi)) \), \( \alpha \in [0, 1] \). Second, we consider the family of projectors \( P_N \) and the operator \( H^{(1)}_N = P_N H^{(1)} P_N \). We use analogs of \( \eqref{2.125} \), \( \eqref{2.127} \) for operators \( H_N \) and \( \alpha \in [0, 1] \). Using a multi-step procedure similar to that in Lemma 2.20, see \( \eqref{2.105} \)–\( \eqref{2.107} \), we obtain that \( (H^{(1)}_N(\tilde{z}_s(\varphi) + \tilde{b}) - k^2)^{-1} \) and \( (H^{(1)}_N(\tilde{z}_1(\varphi) + \tilde{b}) - k^2)^{-1} \) have the same number of poles inside each connected component of \( O^{(1)}(\tilde{b}) \), whose intersection with \( \Phi_1 \) is not empty. Considering exactly as in Lemma 2.20, see page 26, we show that all operators \( (H^{(1)}_N(\tilde{z}_s(\varphi) + \tilde{b}) - k^2)^{-1} \) with sufficiently large \( N \) have the same number of poles \( M, M < c_1 k^{2/3 + s_1} \), inside a component of of \( O^{(1)}(\tilde{b}) \), whose intersection with \( \Phi_1 \) is not empty. Hence, the same is true for \( (H^{(1)}_N(\tilde{z}_1(\varphi) + \tilde{b}) - k^2)^{-1} \). Considering further as in the proof of Lemma 2.20, we obtain that \( (H^{(1)}(\tilde{z}_1(\varphi) + \tilde{b}) - k^2)^{-1} \) has no more than \( c_1 k^{2/3} \) poles. \[ \Box \]

2.3.2. The set \( O^{(1)}(\tilde{b}) \) for small \( b_0 \). Everything we considered so far is valid for \( \tilde{b} \) obeying the inequality \( b_0 \geq k^{-1 - 16\delta_1 - 12\delta} \), here \( b_0 \) is the distance from \( \tilde{b} \) to the nearest vertex of \( K_1 \). However, in the next section and later, \( b_0 \) will be taken smaller, since the reciprocal lattice is getting finer with each step. To prepare for this, let us consider \( \tilde{b} \) being close to a vertex of \( K_1 \):

\[ 0 < b_0 < k^{-1 - 16\delta_1 - 12\delta}. \] \( \eqref{2.129} \)
We show that for such \( \vec{b} \) the resolvent \( (H^{(1)}(\vec{y}(\varphi)) - k^2)^{-1} \), \( \vec{y}(\varphi) = \vec{z}_1(\varphi) + \vec{b} \) has no more than two poles \( \varphi^\pm \) in \( \Phi_1 \). We surround these poles by two contours \( \gamma^{(1)} \pm \) and obtain estimate (2.131) for \( (H^{(1)}(\vec{y}(\varphi)) - k^2)^{-1} \) when \( \varphi \) is outside \( \gamma^{(1)} \pm \).

Suppose \( |\vec{b}| = b_0 \), i.e., the closest vertex of \( K_1 \) for \( \vec{b} \) is \((0, 0)\). The perturbation series (2.58) converge for both \( \lambda^{(1)}(\vec{z}_1(\varphi)) \) and \( \lambda^{(1)}(\vec{y}(\varphi)) \) when \( \varphi \in \Phi_1 \), and both functions are holomorphic in \( \Phi_1 \) (Lemma 2.10). Note that \( \lambda^{(1)}(\vec{z}_1(\varphi)) = k^2 \) for all \( \varphi \in \Phi_1 \). We base our further considerations on these perturbation series expansions. For \( \vec{b} \) being close to a vertex \( \vec{c} \) other than \((0, 0)\), we take \( \vec{y}(\varphi) = \vec{z}_1(\varphi) + \vec{b} - \vec{c} \).

We define \( \varphi_b \in [0, 2\pi) \) by the formula \( \vec{b} = b_0(\cos \varphi_b, \sin \varphi_b) \) when \( |\vec{b}| = b_0 \), and by the analogous formula \( \vec{b} - \vec{c} = b_0(\cos \varphi_b, \sin \varphi_b) \) when \( \vec{b} \) is close to a vertex \( \vec{c} \) other than \((0, 0)\).

**Lemma 2.30.** If \( \vec{b} \) satisfies (2.129) and \( |\epsilon_0| < b_0k^{-1-16s_1-12\delta} \), then the equation

\[
\lambda^{(1)}(\vec{y}(\varphi)) = k^2 + \epsilon_0
\]

(2.130)

has no more than two solutions, \( \varphi^\pm_{\epsilon_0} \), in \( \Phi_1 \). They satisfy the inequality

\[
|\varphi^\pm_{\epsilon_0} - (\varphi_b \pm \pi/2)| < \frac{1}{8}k^{-2-16s_1-11\delta}. \tag{2.131}
\]

**Proof.** Suppose \( W_1 = 0 \) and \( |\vec{b}| = b_0 \), i.e., the closest vertex of \( K_1 \) for \( \vec{b} \) is \((0, 0)\). Then the equation (2.130) has the form \( |k\vec{\nu} + \vec{b}|^2 = k^2 + \epsilon_0 \), \( \vec{\nu} = (\cos \varphi, \sin \varphi) \). It is easy to show that it has two solutions \( \varphi^\pm_{\epsilon_0} \) satisfying (2.131). Applying perturbative arguments and Rouché’s theorem, we prove the lemma for nonzero \( W_1 \). A detailed proof can be found in Appendix 1. In the case when \( \vec{b} \) is close to a vertex other than \((0, 0)\), the considerations are the same up to a parallel shift. ■

**Lemma 2.31.** Suppose \( \vec{b} \) satisfies (2.129), \( \varphi \in \Phi_1 \) and obeys the inequality analogous to (2.137): \( |\varphi - (\varphi_{\epsilon_0} \pm \pi/2)| < k^{-2-16s_1-11\delta} \). Then,

\[
\frac{\partial}{\partial \varphi} \lambda^{(1)}(\vec{y}(\varphi)) = k \to \infty \pm 2b_0k(1+o(1)). \tag{2.132}
\]

**Proof.** Let \( W_1 = 0 \) and \( |\vec{b}| = b_0 \). Then \( \lambda^{(1)}(\vec{y}(\varphi)) = |k\vec{\nu} + \vec{b}|^2 \) and

\[
\frac{\partial}{\partial \varphi}|k\vec{\nu} + \vec{b}|^2 = 2\langle k\vec{\nu} + \vec{b}, k\vec{\mu} \rangle, \quad \vec{\mu} = \frac{\partial \vec{\nu}}{\partial \varphi} = (-\sin \varphi, \cos \varphi).
\]

For \( \varphi \) close to \( \varphi_{\epsilon_0} \pm \pi/2 \), we have \( \langle \vec{b}, \vec{\mu} \rangle = \pm b_0(1+o(1)) \). Considering also that \( \langle \vec{\mu}, \vec{\nu} \rangle = 0 \), we obtain \( \frac{\partial}{\partial \varphi}|k\vec{\nu} + b|^2 = \pm 2b_0k(1+o(1)) \). Applying perturbative arguments, we get a similar formula for nonzero \( W_1 \). For a detailed proof see Appendix 2. In the case when \( \vec{b} \) is close to a vertex other than \((0, 0)\), the considerations are the same up to a parallel shift. ■

**Definition 2.32.** Let \( \Gamma^{(1)\pm}(\vec{b}) \) be two open disks centered at \( \varphi_{\epsilon_0}^\pm \in \Phi_1 \) with the radius \( r^{(1)} = k^{-4-6s_1-3\delta} \), \( \gamma^{(1)\pm}(\vec{b}) \) be their boundary circles and \( \mathcal{O}_s^{(1)}(\vec{b}) = \Gamma^{(1)+} \cup \Gamma^{(1)-} \).

\( ^5 \varphi_{\epsilon_0}^\pm \) is \( \varphi_{\epsilon_0}^\pm \) for \( \epsilon_0 = 0 \).
Lemma 2.33. For any \( \varphi \) in \( \Phi_1 \setminus O_s^{(1)}(\tilde{b}) \),
\[
|\lambda^{(1)}(\tilde{y}(\varphi)) - k^2| \geq b_0 k r^{(1)}. \tag{2.133}
\]
This estimate is stable in the \( k^{-4-6s_1-4\delta} \) neighborhood of \( \Phi_1 \setminus O_s^{(1)}(\tilde{b}) \).

Proof. Suppose (2.133) does not hold for some \( \varphi \) in \( \Phi_1 \setminus O_s^{(1)}(\tilde{b}) \). This means that \( \varphi \) satisfies equation (2.131) with some \( \varepsilon_0 : |\varepsilon_0| < b_0 k r^{(1)} \). By Lemma 2.30 \( \varphi \) obeys (2.131). Thus \( \varphi \) could be either \( \varphi_{\varepsilon_0^+} \) or \( \varphi_{\varepsilon_0^-} \). Without loss of generality, assume \( \varphi = \varphi_{\varepsilon_0^+} \). Obviously, the \( r^{(1)} \)-neighborhood of \( \varphi_{\varepsilon_0^+} \) also satisfies conditions of Lemma 2.31, since \( r^{(1)} \) is small considering with the size \( k^{-2-16s_1-11\delta} \) of the neighborhood \( \Phi_1 \). Using (2.132), we obtain that \( |\lambda^{(1)}(\tilde{y}(\varphi)) - k^2 - \varepsilon_0| = 2k b_0 r^{(1)}(1 + o(1)) \) on the boundary of this neighborhood. Applying Rouche's Theorem, we obtain that there is a point \( \varphi_{\varepsilon_0^+} \) in this neighborhood. It immediately follows that \( |\varphi - \varphi_{\varepsilon_0^+}| < r^{(1)} \), i.e., \( \varphi \in \Gamma^{(1)+} \subset O_s(\tilde{b}) \), which contradicts the assumption \( \varphi \in \Phi_1 \setminus O_s^{(1)}(\tilde{b}) \).

Lemma 2.34. Let \( 0 < b_0 < k^{-1-16s_1-12\delta} \). For any \( \varphi \) in \( \Phi_1 \setminus O_s^{(1)}(\tilde{b}) \) or its \( k^{-4-6s_1-4\delta} \) neighborhood
\[
\left\| \left( H^{(1)}(\tilde{y}(\varphi)) - k^2 \right)^{-1} \right\| < \frac{4k^{3+6s_1+4\delta}}{b_0}. \tag{2.134}
\]
The resolvent is an analytic function of \( \varphi \) in every component of \( O_s^{(1)}(\tilde{b}) \), whose intersection with \( \Phi_1 \) is not empty. The only singularities of the resolvent are poles. The number of poles in each connected component does not exceed two.

Proof. Suppose we have proven that
\[
\left\| \left( \lambda^{(1)}(\tilde{y}(\varphi)) - k^2 \right) \left( H^{(1)}(\tilde{y}(\varphi)) - k^2 \right)^{-1} \right\| < 2. \tag{2.135}
\]
for all \( \varphi \) in \( \Phi_1 \). Then, using (2.133) with \( r^{(1)} = k^{-4-6s_1-3\delta} \), we easily get (2.134).

Let us prove (2.135). By Lemma 2.10, \( \tilde{z}^{(1)}(\varphi) \) is a holomorphic function in \( \Phi_1 \) and estimates (2.70) hold. Therefore, \( \tilde{y}(\varphi) = \tilde{k}(\varphi) + O(k^{-1-16s_1-12\delta}) \). Using (2.64) with \( 2\beta = 1 - 15s_1 - 9\delta \), we obtain
\[
\left\| \left( H_{\alpha}^{(1)}(\tilde{y}(\varphi)) - z \right)^{-1} \right\| < k^{16s_1+10\delta}, \quad z \in C_1, \quad \alpha \in [0, 1], \tag{2.136}
\]
\( C_1 \) being given by (2.8). Using considerations similar to those in Lemma 2.20 (see (2.105) and further), we prove that \( \left( H_{\alpha}^{(1)}(\tilde{y}(\varphi)) - z \right)^{-1} \) has at most the same number of poles \( z \) inside \( C_1 \) as \( \left( H_{0}^{(1)}(\tilde{y}(\varphi)) - z \right)^{-1} \), i.e., at most one pole. The pole obviously exists and is located at the point \( z = \lambda^{(1)}(\tilde{y}(\varphi)) \). Hence, \( \left( \lambda^{(1)}(\tilde{y}(\varphi)) - z \right) \left( H^{(1)}(\tilde{y}(\varphi)) - z \right)^{-1} \) is a holomorphic function of \( z \) inside \( C_1 \) for a fixed \( \varphi \). From the definition of \( C_1 \) and estimate (2.20) it follows that \( |\lambda^{(1)}(\tilde{y}(\varphi)) - z| < 2k^{-16s_1-10\delta} \). Multiplying (2.136) and the last estimate, we get
\[
\left\| \left( \lambda^{(1)}(\tilde{y}(\varphi)) - z \right) \left( H^{(1)}(\tilde{y}(\varphi)) - z \right)^{-1} \right\| < 2, \quad z \in C_1.
\]
Using the maximum principle, we obtain the same estimate inside the circle and, therefore, for $z = k^2$. Thus, we have proved (2.134) for $\varphi \in \Phi_1 \setminus O^{(1)}_s(\bar{b})$. It is easy to see that estimate (2.134) is stable with respect to a perturbation of $\varphi$ of order $k^{-4 - 6s_1 - 4\delta}$. 

Suppose $O^{(1)}_s(\bar{b}) \cap \Phi_1 \neq \emptyset$. This means that $O^{(1)}_s(\bar{b}) \subset \Phi_1$. Considering (2.135), we see that the only poles of the resolvent are the zeros of the function $\lambda^{(1)}(\bar{y}(\varphi)) - k^2$. By Lemma 2.30, the number of zeros does not exceed two. 

3. The Second Approximation 

Let us start with establishing a lower bound for $k$. Let $\eta > 3 \cdot 10^4$. Since $\eta s_1 > 2 + 4s_1$, there is a number $k_* > e$ such that

$$C_*(1 + s_1)k^{2 + 4s_1} \ln k < k^{\eta s_1}, \quad C_* = 400(c_0 + c_1 + 1)^2, \quad c_0 = 32d_1d_2. \tag{3.1}$$

for any $k > k_*$. Assume also, that $k_*$ is sufficiently large to ensure validity of all estimates in the first step for any $k > k_*$. In particular we assume that all $o(1)$ in the first step satisfy the estimate $|o(1)| < 10^{-2}$ when $k > k_*$. 

3.1. Operator $H^{(2)}_\alpha$. Choosing $s_2 = 2s_1$, we define the second operator $H^{(2)}_\alpha$ by the formula:

$$H^{(2)}_\alpha = H^{(1)} + \alpha W_2, \quad (0 \leq \alpha \leq 1), \quad W_2 = \sum_{r=M_1 + 1}^{M_2} V_r, \tag{3.1}$$

where $H^{(1)}$ is defined by (2.41), $M_2$ is chosen in such a way that $2^{M_2} \approx k^{s_2}$. Obviously, the periods of $W_2$ are $2^{M_2}(d_1,0)$ and $2^{M_2}(0,d_2)$. We will write them in the form: $N_1(a_1,0)$ and $N_1(0,a_2)$, where $a_1, a_2$ are the periods of $W_1$ and $N_1 = 2^{M_2 - M_1}, \quad \frac{1}{4}k^{s_2-s_1} < N_1 < 4k^{s_2-s_1}$. Note that

$$\|W_2\|_\infty \leq \sum_{r=M_1 + 1}^{M_2} \|V_r\|_\infty \leq \sum_{r=M_1 + 1}^{M_2} \exp(-2^{\eta r}) < \exp(-k^{\eta s_1}). \tag{3.2}$$

3.1.1. Multiple Periods of $W_1(x)$. The operator $H^{(1)} = H_0 + W_1(x)$ has the periods $a_1, a_2$. The corresponding family of operators, $\{H^{(1)}(t)\}_{t \in K_1}$, acts in $L_2(Q_1)$, where $Q_1 = [0,a_1] \times [0,a_2]$ and $K_1 = [0,2\pi/a_1] \times [0,2\pi/a_2]$. Eigenvalues of $H^{(1)}(t)$ are denoted by $\lambda^{(1)}_n(t)$, $n \in \mathbb{N}$, and its spectrum by $\Lambda^{(1)}(t)$. Now let us consider the same $W_1(x)$ as a periodic function with the periods $N_1a_1, N_1a_2$. Obviously, the definition of the operator $H^{(1)}$ does not depend on the way how we define the periods of $W_1$. However, the family of operators $\{H^{(1)}(t)\}_{t \in K_1}$ does change, when we replace the periods $a_1, a_2$ by $N_1a_1, N_1a_2$. The family of operators $\{H^{(1)}(t)\}_{t \in K_1}$ has to be replaced by a family of operators $\{\tilde{H}^{(1)}(\tau)\}_{\tau \in K_2}$ acting in $L_2(Q_2)$, where $Q_2 = [0,N_1a_1] \times [0,N_1a_2]$ and $K_2 = [0,2\pi/N_1a_1] \times [0,2\pi/N_1a_2]$. We denote eigenvalues of $\tilde{H}^{(1)}(\tau)$ by $\tilde{\lambda}^{(1)}_n(\tau)$, $n \in \mathbb{N}$ and its spectrum by $\tilde{\Lambda}^{(1)}(\tau)$. The next lemma establishes a connection between spectra of operators $H^{(1)}(t)$ and $\tilde{H}^{(1)}(\tau)$. It easily follows from Bloch theory (see e.g. [22]).
Lemma 3.1. For any $\tau \in K_2$,

$$\tilde{\Lambda}^{(1)}(\tau) = \bigcup_{p \in P} \Lambda^{(1)}(t_p), \quad (3.3)$$

where

$$P = \{p = (p_1, p_2) \in \mathbb{Z}^2 : 0 \leq p_1 \leq N_1 - 1, \ 0 \leq p_2 \leq N_1 - 1\} \quad (3.4)$$

and $t_p = (t_{p,1}, t_{p,2}) = (\tau_1 + 2\pi p_1/N_1 a_1, \tau_2 + 2\pi p_2/N_1 a_2) \in K_1$, see Fig. 6.

We defined isoenergetic set $S_1(\lambda) \subset K_1$ of $H^{(1)}(t)$ by formula (2.57). Obviously, this definition is directly associated with the family of operators $H^{(1)}(t)$ and, therefore, with periods $a_1, a_2$, which we assigned to $W_1(x)$. Now, assuming that the periods are equal to $N_1 a_1, N_1 a_2$, we give an analogous definition of the isoenergetic set $\tilde{S}_1(\lambda)$ in $K_2$:

$$\tilde{S}_1(\lambda) := \{\tau \in K_2 : \exists n \in \mathbb{N} : \tilde{\lambda}_n^{(1)}(\tau) = \lambda\}. \quad (3.5)$$

By Lemma 3.1, $\tilde{S}_1(\lambda)$ can be expressed as follows:

$$\tilde{S}_1(\lambda) = \bigg\{\tau \in K_2 : \exists n \in \mathbb{N}, \ p \in P : \lambda_n^{(1)}(\tau + 2\pi p/N_1 a) = \lambda\bigg\}, \quad (3.6)$$

where

$$2\pi p/N_1 a = \left(\frac{2\pi p_1}{N_1 a_1}, \frac{2\pi p_2}{N_1 a_2}\right).$$

The relation between $S_1(\lambda)$ and $\tilde{S}_1(\lambda)$ can be easily understood from the geometric point of view as

$$\tilde{S}_1(\lambda) = K_2 S_1(\lambda), \quad (3.7)$$

where $K_2$ is the parallel shift into $K_2$, i.e.,

$$K_2 : \mathbb{R}^2 \to K_2, \ K_2(\tau + 2\pi m/N_1 a) = \tau, \ m \in \mathbb{Z}^2, \ \tau \in K_2. \quad (3.8)$$

Thus, $\tilde{S}_1(\lambda)$ is obtained from $S_1(\lambda)$ by cutting $S_1(\lambda)$ into pieces of the size $K_2$ and shifting them together in $K_2$.

Definition 3.2. We say that $\tau$ is a point of self-intersection of $\tilde{S}_1(\lambda)$, if there is a pair $m, \hat{m} \in \mathbb{N}, \ m \neq \hat{m}$ such that $\tilde{\lambda}_m^{(1)}(\tau) = \tilde{\lambda}_{\hat{m}}^{(1)}(\tau) = \lambda$. 

Figure 5. Relation between $\tau$ (×) and $t_p$ (·)
Remark 3.3. By Lemma 3.1, \( \tau \) is a point of self-intersection of \( \tilde{S}_1(\lambda) \), if there is a pair \( p, \hat{p} \in P \) and a pair \( n, \hat{n} \in N \) such that |\( p - \hat{p} \) + |n - \( \hat{n} \)| \( \neq 0 \) and \( \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1\alpha) = \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{n}/N_1\alpha) = \lambda \).

Now let us recall that the isoenergetic set \( S_1(\lambda) \) consists of two parts: \( S_1(\lambda) = \chi_1^*(\lambda) \cup (S_1(\lambda) \setminus \chi_1^*(\lambda)) \), where \( \chi_1^*(\lambda) \) is the first non-resonance set given by (2.73). Obviously \( K_2\chi_1^*(\lambda) \subset K_2S_1(\lambda) = \tilde{S}_1(\lambda) \) and can be described by the formula:

\[
K_2\chi_1^*(\lambda) = \{ \tau \in K_2 : \exists p \in P : \tau + 2\pi p/N_1\alpha \in \chi_1^*(\lambda) \}.
\]

Let us consider only those self-intersections of \( \tilde{S}_1(\lambda) \) which belong to \( K_2\chi_1^*(\lambda) \), i.e., we consider the points of intersection of \( K_2\chi_1^*(\lambda) \) both with itself and with \( \tilde{S}_1(\lambda) \setminus K_2\chi_1^*(\lambda) \).

Lemma 3.4. A self-intersection \( \tau \) of \( \tilde{S}_1(\lambda) \) belongs to \( K_2\chi_1^*(\lambda) \) if and only if there are a pair \( p, \hat{p} \in P \), \( \hat{p} \neq \hat{p} \) and a pair \( n, \hat{n} \in N \) such that \( \tau + 2\pi p/N_1\alpha \in \chi_1^*(\lambda) \) and \( \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1\alpha) = \lambda \), the eigenvalue \( \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1\alpha) \) being given by the series (2.22) with \( t = \tau + 2\pi p/N_1\alpha \) and \( j \) uniquely defined by \( t \) from the relation \( p_j^2(t) \in \varepsilon_1 \),

\[
\varepsilon_1 = (k^2 - 3k^{-16\delta_1 - 11\delta}, k^2 + 3k^{-16\delta_1 - 11\delta}).
\] (3.9)

Proof. Suppose \( \tau \) is a point of self-intersection of \( \tilde{S}_1(\lambda) \) belonging to \( K_2\chi_1^*(\lambda) \). Since, \( \tau \in K_2\chi_1^*(\lambda) \), there is a \( p \in P \) such that \( \tau + 2\pi p/N_1\alpha \in \chi_1^*(\lambda) \). By Lemma 2.11 there is a single eigenvalue \( \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1\alpha) = \lambda \) of \( H^{(1)}(\tau + 2\pi \hat{p}/N_1\alpha) \) in \( \varepsilon_1 \). It is given by the series (2.22) with \( t = \tau + 2\pi p/N_1\alpha \) and \( j \) uniquely defined by \( t \) from the relation \( p_j^2(t) \in \varepsilon_1 \). Uniqueness means: \( \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1\alpha) \neq \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{n}/N_1\alpha) \) when \( \hat{n} \neq \hat{n} \). Since \( \tau \) is a point of self-intersection of \( \tilde{S}_1(\lambda) \), \( \lambda_{\hat{n}}^{(1)}(\tau + 2\pi p/N_1\alpha) = \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1\alpha) = \lambda \) for some \( \hat{p} \neq p \). The converse part of the lemma is trivial. \( \blacksquare \)

To obtain a new non-resonance set \( \chi_2(\lambda) \) we remove from \( K_2\chi_1^*(\lambda) \) a neighborhood of its intersections (quasi-intersections) with the whole isoenergetic surface \( \tilde{S}_1(\lambda) \) given by (3.5)–(3.7). More precisely, we remove from \( K_2\chi_1^*(\lambda) \) the following set:

\[
\Omega_1(\lambda) = \{ \tau \in K_2\chi_1^*(\lambda) : \exists n, \hat{n} \in N, p, \hat{p} \in P, p \neq \hat{p} : \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1\alpha) = \lambda, \\
\tau + 2\pi p/N_1\alpha \in \chi_1^*(\lambda), \ |\lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{p}/N_1\alpha) - \lambda_{\hat{n}}^{(1)}(\tau + 2\pi \hat{n}/N_1\alpha)| \leq \varepsilon_1 \},
\] (3.10)

where \( ^6 \)

\[
\varepsilon_1 = e^{-\frac{1}{4} k^{6\delta_1}}.
\] (3.11)

We define \( \chi_2(\lambda) \) by the formula:

\[
\chi_2(\lambda) = K_2\chi_1^*(\lambda) \setminus \Omega_1(\lambda).
\] (3.12)
3.2. **Perturbation Formulae.** Before proving the main result, we formulate Geometric Lemma:

**Lemma 3.5 (Geometric Lemma).** If \( \lambda > k_2^2 \), there exists a non-resonance set \( \chi_2(\lambda, \delta) \subset K_2 \chi_1^* \) such that:

1. For any \( \tau \in \chi_2 \), the following conditions hold:
   
   (a) There exists a unique \( p \in P \) such that \( \tau + 2\pi p/N_1a \in \chi_1^* \).
   
   (b) The following relation holds:

   \[
   \lambda_j^{(1)}(\tau + 2\pi p/N_1a) = k^2,
   \]

   where \( \lambda_j^{(1)}(\tau + 2\pi p/N_1a) \) is given by the perturbation series (2.22) with \( \alpha = 1 \), \( j \) being uniquely defined by \( t = \tau + 2\pi p/N_1a \) and the relation \( p^2_j(t) \in \varepsilon_1 \).

   (c) The eigenvalue \( \lambda_j^{(1)}(\tau + 2\pi p/N_1a) \) is a simple eigenvalue of \( \tilde{H}_1(\tau) \) and its distance from all other eigenvalues \( \lambda_n^{(1)}(\tau + 2\pi \hat{p}/N_1a) \), \( \hat{n} \in \mathbb{N} \) of \( \tilde{H}_1(\tau) \) is greater than \( \epsilon_1 = e^{-\frac{1}{2}\sum_{i=1}^s} \):

   \[
   |\lambda_j^{(1)}(\tau + 2\pi p/N_1a) - \lambda_n^{(1)}(\tau + 2\pi \hat{p}/N_1a)| > \epsilon_1. \quad (3.13)
   \]

2. For any \( \tau \) in the real \( (\varepsilon_1 k^{-1-\delta}) \)-neighborhood of \( \chi_2 \), there exists a unique \( p \in P \) such that \( \tau + 2\pi p/N_1a \) is in the \( (\varepsilon_1 k^{-1-\delta}) \)-neighborhood of \( \chi_1^* \) and

   \[
   |\lambda_j^{(1)}(\tau + 2\pi p/N_1a) - k^2| < 2\varepsilon_1 k^{-\delta}, \quad (3.14)
   \]

   \( j \) being uniquely defined by \( t = \tau + 2\pi p/N_1a j \) and the relation \( p^2_j(t) \in \varepsilon_1 \). An estimate analogous to (3.13) holds:

   \[
   2|\lambda_j^{(1)}(\tau + 2\pi p/N_1a) - \lambda_n^{(1)}(\tau + 2\pi \hat{p}/N_1a)| > \epsilon_1. \quad (3.15)
   \]

3. The second non-resonance set \( \chi_2 \) has an asymptotically full measure in \( \chi_1^* \) in the following sense:

   \[
   \frac{L(K_2 \chi_1^* \setminus \chi_2)}{L(\chi_1^*)} < c_0 k^{-2-2s_1}. \quad (3.16)
   \]

**Corollary 3.6.** If \( \tau \) belongs to the complex \( (\varepsilon_1 k^{-1-\delta}) \)-neighborhood of the second non-resonance set \( \chi_2(\lambda, \delta) \), then for any \( z \in \mathbb{C} \) lying on the circle

   \[
   C_2 = \{ z : |z - k^2| = \varepsilon_1/2 \}, \quad (3.17)
   \]

   the following inequalities hold:

   \[
   \| (\tilde{H}_1(\tau) - z)^{-1} \| \leq \frac{4}{\varepsilon_1}, \quad (3.18)
   \]

   Corollary is proven in Appendix 3.

---

Footnote 7: From geometric point of view this means that \( \chi_2(\lambda) \) does not have self-intersections.
Proof. Let us consider
\[
\mathcal{O}^{(2)} = \bigcup_{p' \in P \setminus \{0\}} \mathcal{O}_s \left( \frac{2\pi p'}{N_1 a} \right),
\]
where \( \mathcal{O}_s(\cdot) \) is defined by (2.122) in Definition 2.26, \( b = \frac{2\pi p'}{N_1 a} \). Note, that the definition make sense, since \( \left| \frac{2\pi p'}{N_1 a} \right| > d_{\max}^{-1}k^{-s_2} > k^{-1-16s_1-12s} \). Let

\[
\Phi_3 = \Phi_1 \setminus \mathcal{O}^{(2)}, \quad \Theta_3 = \Phi_3 \cap [0, 2\pi) .
\]

By Lemma 2.29,
\[
\left\| \left( \mathcal{H}^{(1)} \left( \tilde{\nu}_1(\varphi) + \frac{2\pi p'}{N_1 a} \right) - k^2 \right)^{-1} \right\| < k^{J(1)}, \quad J(1) = c_1k^{2/3+s_1} .
\]

for all \( p' \in P \setminus \{0\} \) and \( \varphi \in \Phi_3 \). Estimate (3.1) yields \( k^{J(1)} < \epsilon_1^{-1} \). We consider \( D_{1,nonres} = D_{1,nonres} \subset D_1 \):

\[
D_{1,nonres} = \{ \tilde{\nu}_1(\varphi), \ \varphi \in \Theta_3 \} .
\]

We define \( \chi_2 \) by the formula:

\[
\chi_2 = K_2 D_{1,nonres} .
\]

By definition of \( K_2 \), for every \( \tau \) in \( \chi_2 \), there are \( p \in P \) and \( j \in Z^2 \) such that

\[
\tau + \frac{2\pi p}{N_1 a} = \tilde{\nu}_1(\varphi), \quad \tilde{\nu}_1(\varphi) \in D_{1,nonres} .
\]

Considering (3.21), the estimate \( k^{J(1)} < \epsilon_1^{-1} \) and the definition of \( D_{1,nonres} \), we obtain:

\[
\left\| \left( \mathcal{H}^{(1)} \left( \tau + \frac{2\pi \tilde{p}}{N_1 a} \right) - k^2 \right)^{-1} \right\| < \epsilon_1^{-1}, \quad \tilde{p} = p + p'.
\]

Note that the index \( j \) does not play a role, since it just produces the shift \( j \) of indices of the matrix elements of the resolvent. Considering that \( p' \) can be any but zero, we obtain that (3.25) holds for all \( \tilde{p} \in P \setminus \{p\} \). Taking into account that \( \lambda_j \left( \tau + \frac{2\pi p}{N_1 a} \right) = k^2 \) and inequality (3.25), we arrive at (3.13) for all \( \tilde{p} \neq p \). It remains to check (3.13) for \( p = \tilde{p} \). Let \( t = \tau + \frac{2\pi p}{N_1 a} \). By (3.24), \( t \in K_1 D_1 = \chi_1 \). By Theorem 2.3...
$\lambda_j \left( \tau + \frac{2\pi p}{N_1 a} \right)$ is a holomorphic function of $\tau$ in $(\epsilon_1 k^{-1-\delta})$-neighborhood of $\chi_2(\lambda, \delta)$ and

$$|\lambda_j^{(1)}(\tau + 2\pi p/N_1 a) - \lambda_j^{(1)}(\tau + 2\pi p/N_1 a)| > k^{2\beta - 2 - 2s_1} > \epsilon_1.$$  

Part 2 follows from stability of all estimates with respect to perturbation of $\tau$ smaller then $(\epsilon_1 k^{-1-\delta})$. Indeed, suppose $\tau$ is in the real $(\epsilon_1 k^{-1-\delta})$-neighborhood of $\chi_2(\lambda, \delta)$. Then, there is a $\tau_0 \in \chi_2(\lambda, \delta)$, such that $|\tau - \tau_0| < \epsilon_1 k^{-1-\delta}$. Let $p, j$ be defined by $\tau_0$ as in Part 1. Obviously, perturbation series for $\lambda_j^{(1)}(\tau + 2\pi p/N_1 a)$ converges and (3.14) holds. Using (3.13), we easily obtain (3.15). Therefore, $p, j$ are defined uniquely by (3.13).

Let us estimate the size of $O_2$. According to Lemma 2.27, the total size of each $O_s \left( \frac{2\pi p}{N_1 a} \right)$ is less then $c_0 k^{-2-4s_1}$. Considering that the number of $p$ does not exceed $4k^{2s_1}$, we obtain that the total size of $O_2$ is less then $c_0 k^{-2-2s_1}$. Using (2.70), we arrive at (3.16).

**Remark 3.7.** Note that every point $\frac{2\pi m}{N_1 a}$ $(m \in \mathbb{Z}^2)$ of a dual lattice corresponding to the larger periods $N_1 a_1, N_1 a_2$ can be uniquely represented in the form $\frac{2\pi m}{N_1 a} = \frac{2\pi j}{a} + \frac{2\pi p}{N_1 a}$, where $m = N_1 j + p$ and $\frac{2\pi j}{a}$ is a point of a dual lattice for periods $a_1, a_2$, while $p \in P$ is responsible for refining the lattice.

Let us consider a normalized eigenfunction $\psi_n(t, x)$ of $H(t)$ in $L_2(Q_1)$. We extend it quasiperiodically to $Q_2$, renormalize in $L_2(Q_2)$ and denote the new function by $\tilde{\psi}_n(\tau, x), \tau = K_2 t$. The Fourier representations of $\psi_n(t, x)$ in $L_2(Q_1)$ and $\tilde{\psi}_n(\tau, x)$ in $L_2(Q_2)$ are simply related. If we denote Fourier coefficients of $\psi_n(t, x)$ with respect to the basis of exponential functions $\frac{1}{|Q_1|^{1/2}} e^{i(t + \frac{2\pi j}{a} x)}$, $j \in \mathbb{Z}^2$, in $L_2(Q_1)$ by $C_{nj}$, then, the Fourier coefficients $\tilde{C}_{nm}$ of $\tilde{\psi}_n(\tau, x)$ with respect to the basis of exponential functions $\frac{1}{|Q_2|^{1/2}} e^{i(r \tau + \frac{2\pi m}{N_1 a} x)}$, $m \in \mathbb{Z}^2$, in $L_2(Q_2)$ are given by the formula:

$$\tilde{C}_{nm} = \begin{cases} C_{nj}, & \text{if } m = jN_1 + p; \\ 0, & \text{otherwise}, \end{cases}$$

$p$ being defined from the relation $t = \tau + \frac{2\pi p}{N_1 a}$, $p \in P$. Hence, matrices of the projections on $\psi_n(t, x)$ and $\tilde{\psi}_n(\tau, x)$ with respect to the above bases are simply related:

$$(\tilde{E}_n)_{jj} = \begin{cases} (E_n)_{mm}, & \text{if } m = jN_1 + p, \hat{m} = jN_1 + p; \\ 0, & \text{otherwise}, \end{cases}$$

$\tilde{E}_n$ and $E_n$ being projections in $L_2(Q_2)$ and $L_2(Q_1)$, respectively.

Let us denote by $\tilde{E}_j^{(1)}(\tau + \frac{2\pi p}{N_1 a})$ the spectral projection $E_j^{(1)}(\alpha, t)$ (see (2.23)) with $\alpha = 1$ and $t = \tau + \frac{2\pi p}{N_1 a}$, “extended” from $L_2(Q_1)$ to $L_2(Q_2)$.

By analogy with (2.18), (2.19), we define functions $g_\nu^{(2)}(k, \tau)$ and operator-valued functions $G^{(2)}(k, \tau), \nu = 1, 2, \cdots$, as follows:

$$g_\nu^{(2)}(k, \tau) = \left( \frac{-1}{2\pi i r} \right)^r \text{Tr} \int_{C_2} \left( \tilde{H}^{(1)}(\tau) - z \right)^{-1} W_2 \right)^r dz,$$  

(3.26)
\[ G_r^{(2)}(k, \tau) = \frac{(-1)^{r+1}}{2\pi i} \oint_{C_2} \left( (\tilde{H}^{(1)}(\tau) - z)^{-1}W_2 \right)^r (\tilde{H}^{(1)}(\tau) - z)^{-1} dz, \]  
\[ (3.27) \]

\( \tilde{H}^{(1)}(\tau) \) being defined at the beginning of Section 3.1.1. \( C_2 \) being given by (3.17). We consider the operators \( H_0^{(2)} = H^{(1)} + \alpha W_2 \) and the family \( H_\alpha^{(2)}(\tau), \tau \in K_2, \) acting in \( L_2(Q_2) \). By (3.2) and (3.11)

\[ \|W_2\| < \epsilon_1^4, \]  
\[ (3.28) \]

\( \|W_2\| \) being the norm of the operator here.

**Theorem 3.8.** Suppose \( \tau \) belongs to the \( (\epsilon_1 k^{-1-\delta}) \)-neighborhood in \( K_2 \) of the second non-resonance set \( \chi_2(\lambda, \delta), 0 < \delta < s_1, \epsilon_1 = e^{-\frac{1}{4}k^{s_1}} \). Then, for sufficiently large \( \lambda, \lambda > k_\ast^2 \) and for all \( 0 \leq \alpha \leq 1 \), there exists a unique eigenvalue of the operator \( H_\alpha^{(2)}(\tau) \) in the interval \( \varepsilon_2(k) := (k^2 - \epsilon_1/2, k^2 + \epsilon_1/2) \). It is given by the series:

\[ \lambda_j^{(2)}(\alpha, \tau) = \lambda_j^{(1)}(\tau + 2\pi p/N_1 a) + \sum_{r=1}^{\infty} \alpha^r g_r^{(2)}(k, \tau), \quad \tilde{j} = j + p/N_1, \]  
\[ (3.29) \]

converging absolutely in the disk \( |\alpha| \leq 1 \), where \( p \in P \) and \( j \in \mathbb{Z}^2 \) are described as in Geometric Lemma 3.3. The spectral projection corresponding to \( \lambda_j^{(2)}(\alpha, \tau) \) is given by the series:

\[ E_j^{(2)}(\alpha, \tau) = \tilde{E}_j^{(1)}(\tau + 2\pi p/N_1 a) + \sum_{r=1}^{\infty} \alpha^r G_r^{(2)}(k, \tau), \]  
\[ (3.30) \]

which converges in the trace class \( S_1 \) uniformly with respect to \( \alpha \) in the disk \( |\alpha| \leq 1 \).

The following estimates hold for coefficients \( g_r^{(2)}(k, \tau), G_r^{(2)}(k, \tau), \ r \geq 1 : \)

\[ |g_r^{(2)}(k, \tau)| < \frac{3\epsilon_1}{2} (4\epsilon_1^3)^r, \quad \| G_r^{(2)}(k, \tau) \|_1 < 6r(4\epsilon_1^3)^r. \]  
\[ (3.31) \]

**Corollary 3.9.** The following estimates hold for the perturbed eigenvalue and its spectral projection:

\[ |\lambda_j^{(2)}(\alpha, \tau) - \lambda_j^{(1)}(\tau + 2\pi p/N_1 a)| \leq 12\epsilon_1^4, \]  
\[ (3.32) \]

\[ \| E_j^{(2)}(\alpha, \tau) - \tilde{E}_j^{(1)}(\tau + 2\pi p/N_1 a) \|_1 \leq 48\epsilon_1^3. \]  
\[ (3.33) \]

**Remark 3.10.** The theorem states that \( \lambda_j^{(2)}(\alpha, \tau) \) is a single eigenvalue in the interval \( \varepsilon_2(k, \delta) \). This means that \( |\lambda_j^{(2)}(\alpha, \tau) - k_\ast^2| < \epsilon_1/2 \). Formula (3.32) provides a stronger estimate on the location of \( \lambda_j^{(2)}(\alpha, \tau) \).

**Proof.** The proof of the theorem is based on expanding the resolvent \( (H_\alpha^{(2)}(\tau) - z)^{-1} \) in a perturbation series for \( z \in C_2 \). Integrating the resolvent yields the formulae for an eigenvalue of \( H_\alpha^{(2)}(\tau) \) and its spectral projection. In fact, it is obvious that

\[ (H_\alpha^{(2)}(\tau) - z)^{-1} = (\tilde{H}^{(1)}(\tau) - z)^{-1}(I - \alpha A_2)^{-1}, \quad A_2 := -W_2(\tilde{H}^{(1)}(\tau) - z)^{-1}. \]  
\[ (3.34) \]
Suppose $z \in C_2$. Using Corollary 3.6 and estimate (3.28), we obtain:

$$\| (\tilde{H}^{(1)}(\tau) - z)^{-1} \| \leq \frac{4}{\epsilon_1}, \quad \| A_2 \| \leq \frac{4\| W_2 \|}{\epsilon_1} < 4\epsilon_1^3 < 1. \quad (3.35)$$

The last inequality makes it possible to expand $(I - \alpha A_2)^{-1}$ in the series in powers of $\alpha A_2$. Integrating the series for the resolvent and considering as in the proof of Theorem 2.3 we obtain formulae (3.29), (3.30). Estimates (3.31) follow from the estimates (3.35).

Next, we show that the series (3.29), (3.30) can be extended as holomorphic functions of $\tau$ in a complex neighborhood of $\chi_2$: they can be differentiated any number of times with respect to $\tau$ and retain their asymptotic character.

**Lemma 3.11.** The following estimates hold for the coefficients $g^{(2)}_x(k, \tau)$ and $G^{(2)}_x(k, \tau)$ in the complex $(\frac{1}{2}\epsilon_1 k^{-1-\delta})$-neighborhood of the non-resonance set $\chi_2$:

$$|T(m)g^{(2)}_x(k, \tau)| < m! \cdot 3 \cdot 2^{2r-1+|m|} \epsilon_1^{-3r+1-|m|} k^{|m|(1+\delta)}, \quad (3.36)$$

$$\|T(m)G^{(2)}_x(k, \tau)\| < m! \cdot 3r \cdot 2^{2r+1+|m|} \epsilon_1^{-3r-|m|} k^{|m|(1+\delta)}. \quad (3.37)$$

**Proof.** Since (3.18) is valid in the complex $(\epsilon_1 k^{-1-\delta})$-neighborhood of the second non-resonance set, it is not hard to see that the coefficients $g^{(2)}_x(k, \tau)$ and $G^{(2)}_x(k, \tau)$ can be continued from the real $(\epsilon_1 k^{-1-\delta})$-neighborhood of $\tau$ to the complex $(\epsilon_1 k^{-1-\delta})$-neighborhood as holomorphic functions of two variables and inequalities (3.31) are hereby preserved. Estimating, by means of the Cauchy integral formula, the value of the derivative with respect to $\tau$ in terms of the value of the function itself on the boundary of the $(\frac{1}{2}\epsilon_1 k^{-1-\delta})$-neighborhood of $\tau$ (formulas (3.31)), we obtain (3.36) and (3.37).

From this lemma the following theorem easily follows.

**Theorem 3.12.** Under the conditions of Theorem 3.8 the series (3.29), (3.30) can be extended as holomorphic functions of two variables from the real $(\epsilon_1 k^{-1-\delta})$-neighborhood of the non-resonance set $\chi_2$ to its complex $(\epsilon_1 k^{-1-\delta})$-neighborhood and the following estimates hold in the complex neighborhood:

$$|T(m)\left(\lambda^{(2)}_j(\alpha, \tau) - \lambda^{(1)}_j(\tau + 2\pi p/N_1 a)\right)| < \alpha C_m \epsilon_1^{1-|m|} k^{|m|(1+\delta)}, \quad (3.38)$$

$$\|T(m)\left(E^{(2)}_j(\alpha, \tau) - E^{(1)}_j(\tau + 2\pi p/N_1 a)\right)\| < \alpha C_m \epsilon_1^{3-|m|} k^{|m|(1+\delta)}, \quad (3.39)$$

Here and below $C_m = 48 m! 2^{|m|}$.

**Corollary 3.13.**

$$|\nabla \lambda^{(2)}_j(\alpha, \tau) - 2\overline{k}| < 2C(W_1)k^{-1-2\beta+15s_1+11\delta}, \quad \overline{k} = \overline{p}_j(\tau + 2\pi p/N_1 a) \quad (3.40)$$

$$|T(m)\lambda^{(2)}_j(\alpha, \tau)| < 2 + 2C(W_1)k^{-1-2\beta+21s_1+15\delta}, \text{ if } |m| = 2. \quad (3.41)$$
The next lemma will be used in the third step of approximation. The operator $H^{(2)}(\tau)$ is $H^{(2)}_n(\tau)$ with $\alpha = 1$. It will play a role of the initial (unperturbed) operator in the third step.

**Lemma 3.14.** For any $z$ on the circle $C_2$ and $\tau$ in the complex $(\epsilon_1 k^{-1-\delta})$-neighborhood of $\chi_2$,

$$\|(H^{(2)}(\tau) - z)^{-1}\| \leq \frac{8}{\epsilon_1}. \quad (3.42)$$

**Proof.** Considering the Hilbert relation

$$(H^{(2)}(\tau) - z)^{-1} = (\tilde{H}^{(1)}(\tau) - z)^{-1} + (\tilde{H}^{(1)}(\tau) - z)^{-1} (-W_2) (H^{(2)}(\tau) - z)^{-1},$$

and the estimate (3.18), together with the estimate (3.28), we obtain:

$$\|(H^{(2)}(\tau) - z)^{-1}\| \leq \frac{\|(\tilde{H}^{(1)}(\tau) - z)^{-1}\|}{1 - \|(\tilde{H}^{(1)}(\tau) - z)^{-1} W_2\|} \leq 2\|(\tilde{H}^{(1)}(\tau) - z)^{-1}\| \leq \frac{8}{\epsilon_1}. \quad (3.43)$$

**3.3. Non-resonant part of the isoenergetic set of $H^{(2)}$**. Let $S_2(\lambda)$ be an isoenergetic set of the operator $H^{(2)}_n$: $S_2(\lambda) = \{\tau \in K_2 : \exists n \in \mathbb{N} : \lambda^{(2)}_n(\alpha, \tau) = \lambda\}$, here $\{\lambda^{(2)}_n(\alpha, \tau)\}_{n=1}^{\infty}$ is the spectrum of $H^{(2)}_n(\tau)$. Now we construct a non-resonance subset $\chi^{(2)}_\lambda(\lambda)$ of $S_2(\lambda)$. It corresponds to non-resonance eigenvalues $\lambda^{(2)}_j(\tau)$ given by the perturbation series (3.29). Recall that $D_1(\lambda)_{nonres}$ and $\chi_2(\lambda)$ are defined by the formulae (3.22) and (3.23) respectively. Recall also that $\chi_2 \subset K_2 \chi_1^*(\lambda)$ (see the Geometric Lemma) and $\chi_1^*(\lambda) = K_1 D_1(\lambda)$, see (2.73). Hence, $\chi_2 \subset K_2 D_1(\lambda)$.

**Lemma 3.15.** The formula $K_2 D_1(\lambda)_{nonres} = \chi_2$ establishes one-to-one correspondence between $D_1(\lambda)_{nonres}$ and $\chi_2$.

**Proof.** Suppose there is a pair $\tilde{\varphi}_1, \tilde{\varphi}_{1*} \in D_1(\lambda)_{nonres}$ such that $K_2 \tilde{\varphi}_1 = K_2 \tilde{\varphi}_{1*} = \tau, \tau \in \chi_2$. We introduce also $t_1 = K_1 \tilde{\varphi}_1$ and $t_{1*} = K_1 \tilde{\varphi}_{1*}$. The definition (2.73) of $\chi^{(2)}_1(\lambda)$ implies that $t_1, t_{1*} \in \chi^{(2)}_1(\lambda)$, since $\tilde{\varphi}_1, \tilde{\varphi}_{1*} \in D_1(\lambda)_{nonres} \subset D_1(\lambda)$. Clearly, $K_2 t_1 = K_2 t_{1*} = \tau$ and, hence, $t_1 = \tau + 2\pi p_1/N_1 a, t_{1*} = \tau + 2\pi p_2/N_1 a$ for some $p_1, p_2 \in P$. Now, by Part 1a of Geometric Lemma 3.5 $p_1 = p_2$, and, therefore, $t_1 = t_{1*}$. Next, by Lemma 2.12, $\tilde{\varphi}_1 = \tilde{\varphi}_{1*}$.

We define $B_2(\lambda)$ as the set of directions corresponding to the set $\Theta_2$ given by (3.20):

$$B_2(\lambda) = \{\tilde{\varphi} \in B_1(\lambda) : \varphi \in \Theta_2\}.$$

Note that $B_2(\lambda)$ is a unit circle with holes, centered at the origin, and $B_2(\lambda) \subset B_1(\lambda)$.

Let $\tilde{\varphi} \in D_1(\lambda)_{nonres}$. By (3.23), $\tau = K_2 \tilde{\varphi} \in \chi_2(\lambda)$. According to Theorem 3.8 for sufficiently large $\lambda$, there exists an eigenvalue of the operator $H^{(2)}_n(\tau)$, given by (3.29). It is convenient here to denote $\lambda^{(2)}_j(\alpha, \tau)$ by $\lambda^{(2)}(\alpha, \tilde{\varphi})$. We can do this, since, by Lemma 3.15 there is one-to-one correspondence between $\tilde{\varphi} \in D_1(\lambda)_{nonres}$ and the pair $(\tau, j)$, $\tilde{\varphi} = 2\pi j/\alpha + \tau$. We rewrite (3.29) in the form:

$$\lambda^{(2)}(\alpha, \tilde{\varphi}) = \lambda^{(1)}(\tilde{\varphi}) + f_2(\alpha, \tilde{\varphi}), \quad f_2(\alpha, \tilde{\varphi}) = \sum_{r=1}^{\infty} \alpha^r g^{(2)}_r(\tilde{\varphi}), \quad (3.44)$$
here \( g_r^{(2)}(\vec{\zeta}) \) is given by (3.26). The function \( f_2(\alpha, \vec{\zeta}) \) satisfies the estimates:

\[
|f_2(\alpha, \vec{\zeta})| \leq 12\alpha\epsilon_4^1, \quad (3.45)
\]

\[
|\nabla f_2(\alpha, \vec{\zeta})| \leq 96\alpha\epsilon_3^1k^{1+\delta}. \quad (3.46)
\]

By Theorems 3.8 and 3.12, the formulas (3.44) – (3.46) hold even in the real \((\epsilon_1k^{-1-\delta})\)-neighborhood of \(D_1(\lambda)_{\text{nonres}}\), i.e., for any \( \vec{\zeta} = \vec{x}\vec{\nu} \) such that \( \vec{\nu} \in B_2(\lambda) \) and \( |\vec{x} - \vec{x}_1(\lambda, \vec{\nu})| < \epsilon_1k^{-1-\delta} \). We define \( D_2(\lambda) \) as a level set for \( \lambda^{(2)}(\alpha, \vec{\zeta}) \) in this neighborhood:

\[
D_2(\lambda) := \left\{ \vec{\zeta} = \vec{x}\vec{\nu} : \vec{\nu} \in B_2, \quad |\vec{x} - \vec{x}_1(\lambda, \vec{\nu})| < \epsilon_1k^{-1-\delta}, \quad \lambda^{(2)}(\alpha, \vec{\zeta}) = \lambda \right\}. \quad (3.47)
\]

Next two lemmas are to prove that \( D_2(\lambda) \) is a distorted circle with holes.

**Lemma 3.16.** For every \( \vec{\nu} \in B_2 \) and every \( \alpha, 0 \leq \alpha \leq 1 \), there is a unique \( \vec{x} = \vec{x}_2(\lambda, \vec{\nu}) \) in the interval \( I_2 := [\vec{x}_1(\lambda, \vec{\nu}) - \epsilon_1k^{-1-\delta}, \vec{x}_1(\lambda, \vec{\nu}) + \epsilon_1k^{-1-\delta}] \) such that

\[
\lambda^{(2)}(\alpha, \vec{x}_2\vec{\nu}) = \lambda. \quad (3.48)
\]

Furthermore,

\[
|\vec{x}_2(\lambda, \vec{\nu}) - \vec{x}_1(\lambda, \vec{\nu})| \leq 12\alpha\epsilon_4^1k^{-1}. \quad (3.49)
\]

The proof is based on (3.44), (3.45), (3.40) and completely analogous to that of Lemma 3.41 in [24], set \( l = 1 \).

Further, we use the notations:

\[
\vec{x}_2(\varphi) \equiv \vec{x}_2(\lambda, \vec{\nu}), \quad h_2(\varphi) = \vec{x}_2(\varphi) - \vec{x}_1(\varphi), \quad \vec{x}_2(\varphi) = \vec{x}_2(\varphi)\vec{\nu}. \quad (3.50)
\]

**Lemma 3.17.** Let \( 10^{-4} < s_1 < 10^{-3}, \eta > 10^4 \). Then the following statements hold for \( \lambda > k_2^4 \):

1. The set \( D_2(\lambda) \) is a distorted circle with holes: it can be described by the formula:

\[
D_2(\lambda) = \left\{ \vec{\zeta} \in \mathbb{R}^2 : \vec{\zeta} = \vec{x}_2(\varphi), \quad \varphi \in \Theta_2(\lambda) \right\}, \quad (3.51)
\]

where \( \vec{x}_2(\varphi) = \vec{x}_1(\varphi) + h_2(\varphi) \), \( \vec{x}_1(\varphi) \) is the “radius” of \( D_1(\lambda) \) and \( h_2(\varphi) \) satisfies the estimates

\[
|h_2| \leq 12\alpha\epsilon_4^1k^{-1}, \quad \frac{\partial h_2}{\partial \varphi} \leq 96\alpha\epsilon_3^1k^{1+\delta}. \quad (3.52)
\]

2. The total length of \( B_2(\lambda) \) satisfies the estimate:

\[
L(B_1 \setminus B_2) < c_0k^{-2-2s_1}. \quad (3.53)
\]

3. The function \( h_2(\varphi) \) can be extended as a holomorphic function of \( \varphi \) into the complex non-resonance set \( \Phi_2 \) and its \( k^{-4-6s_1-4\delta} \) neighborhood \( \Phi_2 \), estimates (3.52) being preserved.

4. The curve \( D_2(\lambda) \) has a length which is asymptotically close to that of \( D_1(\lambda) \) in the following sense:

\[
L(D_2(\lambda)) \underset{\lambda \to \infty}{=} L(D_1(\lambda)) \left( 1 + O(k^{-2-2s_1}) \right), \quad (3.54)
\]

where \( O(k^{-2-2s_1}) = (1 + o(1))c_0k^{-2-2s_1}, |o(1)| < 10^{-2} \) when \( k > k_* \).
Proof. The proof is completely analogous to that of Lemma 3.42 in [24], set $l = 1$. Here we give just a short version. Indeed, the first inequality in (3.52) is equivalent to (3.49). Differentiating the identity $\lambda^{(2)}(\tilde{z}_2(\varphi)) = \lambda^{(1)}(\tilde{z}_1(\varphi)) = k^2$ with respect to $\varphi$ and using (3.40), (3.41), (3.46), (3.49), we easily obtain the second estimate in (3.52). Estimate (3.53) valid, since the total size of $O(2)$ is less than $c_0 k^{-2 - 2s_1}$. To prove the analyticity of $h_2(\varphi)$ in $\Phi_2$ we check the convergence of the perturbation series for $\lambda^{(2)}(\tilde{z}_1(\varphi))$, $\varphi \in \Phi_2$. It is enough to show that

$$
\left\| \left( H^{(1)}(\tilde{z}_1(\varphi)) - z \right)^{-1} \right\| \leq \frac{4}{\varepsilon_1}, \quad z \in C_2.
$$

(3.55)

This inequality immediately follows from

$$
\left\| \left( H^{(1)}(\tilde{z}_1(\varphi)) + \frac{2\pi p}{N_1 a} - z \right)^{-1} \right\| \leq \frac{4}{\varepsilon_1}
$$

(3.56)

proven for all $p \in P$. Let $p = 0$. By Lemma 2.8

$$
\left\| (H^{(1)}(\tilde{z}_1(\varphi)) - z)^{-1} \right\| \leq 2k^{-2\beta + 1 + s_1 + \delta}, \quad \text{when} \quad z \in C_1, \quad 2\beta = 1 - 15s_1 - 9\delta.
$$

It is not difficult to show that the resolvent has a single pole inside $C_1$ at the point $z = k^2$. The circle $C_2$ has the same centrum and a smaller radius $\varepsilon_1/2$. Hence

$$
\left\| (H^{(1)}(\tilde{z}_1(\varphi)) - z)^{-1} \right\| \leq \frac{4}{\varepsilon_1}, \quad \text{when} \quad z \in C_2.
$$

(3.57)

Next we use (2.123) with $\tilde{b} = \frac{2\pi p}{N_1 a}$. The right-hand side of (2.123) is less that $\varepsilon_1^{-1}$, since $s_1 \eta > 1$. Hence, (3.56) holds for $p \neq 0$ too. Convergence of perturbation series follows from (3.55) and (3.28). Note that the smallest circle in $O(2)$ has the size $k^{-4 - 6s_1 - 3\delta}$. Since $k^{-4 - 6s_1 - 4\delta}$ is much smaller than the radius of circles, all estimates are stable in the $k^{-4 - 6s_1 - 4\delta}$ of $\Phi_2$. Now, using Rouche’s and Implicit function theorems we easily show the equation $\lambda^{(2)}(\tilde{z}_2) = k^2$ has a solution $\tilde{z} = \tilde{z}_2(\varphi)$ which is holomorphic in $\Phi_2$ and coincides with $\tilde{z}_1(\varphi) + h_2(\varphi)$ for real $\varphi$. Estimate (3.54) follows from (3.52) and (3.53).

Let us record a remark for the sequel. Convergence of the series for the resolvent $(H^{(2)}(\tilde{z}_1(\varphi)) - z)^{-1}$, $z \in C_2$, means that the resolvent has a single pole $z = \lambda^{(2)}(\tilde{z}_1(\varphi))\tilde{v}$ inside $C_2$. Similar result holds when we replace $\tilde{z}_1(\varphi)$ by $\tilde{z}_2(\varphi)$, since $\tilde{z}_1(\varphi)$ and $\tilde{z}_2(\varphi)$ are close: $|\tilde{z}_2(\varphi) - \tilde{z}_1(\varphi)| = o(\varepsilon_1)$. Considering that $\lambda^{(2)}(\tilde{z}_2(\varphi)) = \lambda$, we obtain that $(z - \lambda) (H^{(2)}(\tilde{z}_2(\varphi)) - z)^{-1}$ is holomorphic inside $C_2$ and the estimate following holds:

$$
\left\| (z - \lambda) (H^{(2)}(\tilde{z}_2(\varphi)) - z)^{-1} \right\| < 32.
$$

(3.58)

Now define the non-resonance set $\chi^*_2(\lambda)$ in $S_2(\lambda)$ by

$$
\chi^*_2(\lambda) := K_2 D_2(\lambda).
$$

(3.59)

Lemma 3.18. The set $\chi^*_2(\lambda)$ belongs to the $(12\alpha\varepsilon_1^4 k^{-1})$-neighborhood of $\chi_2(\lambda)$ in $K_2$. If $\tau \in \chi^*_2(\lambda)$, then the operator $H^{(2)}_\alpha(\tau)$ has a simple eigenvalue equal to $\lambda$. This
eigenvalue is given by the perturbation series \((3.24)\), where \(p \in P, j \in \mathbb{Z}^2\) are uniquely defined by \(\tau\) as it is described in Geometric Lemma \(3.5\) part 2.

**Proof.** By Lemma \(3.17\), \(D_2(\lambda)\) is in the \((12\alpha \epsilon^4 k^{-1})\) neighborhood of \(D_1(\lambda)_{\text{nonres}}\). Considering that \(\chi^*_2(\lambda) = K_2 D_2(\lambda)\) and \(\chi_2(\lambda) = K_2 D_1(\lambda)_{\text{nonres}}\) (see \((3.23)\)), we immediately obtain that \(\chi^*_2(\lambda)\) is in the \((12\alpha \epsilon^4 k^{-1})\)-neighborhood of \(\chi_2(\lambda)\). The size of this neighborhood is less than \(\epsilon_1 k^{-1-\delta}\), hence Theorem \(3.3\) holds in it, i.e., for any \(\tau \in \chi^*_2(\lambda)\) there is a single eigenvalue of \(H^{(2)}_\alpha(\tau)\) in the interval \(\varepsilon_2(k, \delta)\). Since \(\chi^*_2(\lambda) \subset \mathcal{S}_2(\lambda)\), this eigenvalue is equal to \(\lambda\). By the theorem, the eigenvalue is given by the series \((3.29)\), where \(p \in P, j \in \mathbb{Z}^2\) are uniquely defined by \(\tau\) as it is described in Geometric Lemma \(3.5\) part 2. ■

**Lemma 3.19.** Formula \((3.3)\) establishes one-to-one correspondence between \(\chi^*_2(\lambda)\) and \(D_2(\lambda)\).

**Remark 3.20.** From geometric point of view this means that \(\chi^*_2(\lambda)\) does not have self-intersections.

**Proof.** Suppose there is a pair \(\vec{z}, \vec{z}^* \in D_2(\lambda)\) such that \(K_2 \vec{z} = K_2 \vec{z}^* = \tau, \tau \in \chi^*_2(\lambda)\). By the definition \((3.4)\) of \(D_2(\lambda)\), we have \(\lambda^{(2)}(\alpha, \vec{z}) = \lambda^{(2)}(\alpha, \vec{z}^*) = \lambda\), i.e., the eigenvalue \(\lambda\) of \(H^{(2)}_\alpha(\tau)\) is not simple. This contradicts to the previous lemma. ■

### 3.4. Preparation for the Next Approximation

Let \(\vec{b}^{(2)} \in K_2\) and \(b_0^{(2)}\) be the distance of the point \(\vec{b}^{(2)}\) to the nearest corner of \(K_2\):

\[
b_0^{(2)} = \min_{m=(0,0),(0,1),(1,0),(1,1)} |\vec{b} - 2\pi m/N_1a|.
\]

We assume \(b_0^{(2)} = |\vec{b}^{(2)}|\). In the case when \(\vec{b}\) is closer to a vertex other than \((0,0)\), the considerations are the same up to a parallel shift. We consider two cases: \(b_0^{(2)} \geq \epsilon_1 k^{-1-2\delta}\) and \(0 < b_0^{(2)} < \epsilon_1 k^{-1-2\delta}\). Let

\[
\vec{y}^{(1)}(\varphi) = \vec{z}_1(\varphi) + \vec{b}^{(2)}.
\]

#### 3.4.1. The case \(b_0^{(2)} \geq \epsilon_1 k^{-1-2\delta}\)

**Definition 3.21.** We define the set \(O^{(2)}(\vec{b}^{(2)})\) by the formula:

\[
O^{(2)}(\vec{b}^{(2)}) = \bigcup_{p \in P} O^{(1)}_s(2\pi p/N_1a + \vec{b}^{(2)}).
\]

In the above formula we assume \(O^{(1)}_s \cap \hat{\varphi}_1 \neq \emptyset\).

**Lemma 3.22.** If \(\varphi \in \hat{\varphi}_1 \setminus O^{(2)}(\vec{b}^{(2)})\) or its \(k^{-4-6s_1-4\delta}\) neighborhood, then

\[
\left\| \left( \hat{F}^{(1)}(\vec{y}^{(1)}(\varphi)) - k^2 \right)^{-1} \right\| \leq \frac{k^{4+6s_1+5\delta}}{\epsilon_1}
\]

The resolvent is an analytic function of \(\varphi\) in every connected component \(O^{(2)}_c(\vec{b}^{(2)})\) of \(O^{(2)}(\vec{b}^{(2)})\), whose intersection with \(\hat{\varphi}_1\) is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside \(O^{(2)}_c(\vec{b}^{(2)})\) is less than \(c_0 k^{1+2s_2}\). The size of each connected
component is less than $k^{-3-4s_1-2\delta}$. The total number of poles (counting multiplicity) of the resolvent inside $O^2(\vec{b}(2))$ is less than $c_0k^{2+2s_2}$. The total size of $O_2(\vec{b}(2))$ is less then $c_0k^{-2-2s_1}$.

**Proof.** Combining Lemmas 2.29 and 2.34 for $\vec{b}(1) = 2\pi p/N_1a + \vec{b}(2)$, we obtain:

$$\left\| \left( H^{(1)}(\vec{z}_1(\varphi) + 2\pi p/N_1a + \vec{b}(2)) - k^2 \right)^{-1} \right\| \leq \max \left\{ k^{J(1)} \frac{4k^{3+6s_1+4\delta}}{b_0^{(2)}} \right\}, \quad (3.64)$$

$$J^{(1)} = 5c_1k^{2/3+s_1} < c_0k^{2+2s_1}, \ p \in \mathcal{P},$$

for any $\varphi \in \hat{\Phi}_1 \setminus O_2(\vec{b}(2))$. Using (3.61), we easily obtain $k^{J(1)} < \epsilon_1^{-1}$. Considering the condition $b_0^{(2)} \geq \epsilon_1k^{-1-\delta}$ and the definition of $\hat{H}^{(1)}$, we obtain (3.63) in $\hat{\Phi}_1 \setminus (O^2(\vec{b}(2)))$.

The estimate is stable with respect of the perturbation of order $k^{-4-6s_1-4\delta}$, since the size of the discs in $O_2(\vec{b}(2))$ is $k^{-4-6s_1-4\delta}$.

Now we prove the second part of the lemma. Let $\Delta_*$ be a rectangle in $\mathbb{C}$: $\Delta_* = \{ \varphi : |R\varphi - \varphi_*| \leq k^{-1}, |3\varphi| < k^{-\delta} \}$ for some $\varphi_* \in [0, 2\pi)$. It is shown in the proof of Lemma 8.10 (see the estimate for the number of points in $I_*$) that the total number of points $\vec{\varphi}_n^*(\vec{b})$ in $\Delta_*$ does not exceed $5c_0k^{1+2s_2}$ for any $\vec{b} \in K_1$. Therefore, the number of discs of $O^{(1)}(\vec{b})$ which intersect $\Delta_*$ does not exceed $5c_0k^{1+2s_2}$. Therefore, the number of discs of $O_s^{(1)}(\vec{b})$ which intersect $\Delta_*$ does not exceed $5c_0k^{1+2s_2}$ for any $\vec{b} \in K_1$. Hence, the number of discs of $O^2(\vec{b}(2))$ which intersect $\Delta_*$ does not exceed $5c_0k^{1+2s_2}$. Considering that the size of each disc is less then $k^{-4-6s_1-4\delta}$, we obtain that the total size of $\Delta_*$ is less than $k^{-3-4s_1-2\delta}$. It is obvious now that the size of each connected component of $O^2(\vec{b}(2))$ does not exceed $k^{-3-4s_1-2\delta}$. If $O^2(\vec{b}(2)) \cap \hat{\Phi}_1 \neq \emptyset$, then $O^2(\vec{b}(2)) \subset \hat{\Phi}_1$. Therefore, the resolvent is an analytic function in this $O^2(\vec{b}(2))$. By construction, the poles of the resolvent are in the centres of the discs in $O^2(\vec{b}(2))$. Therefore, the number of poles in each connected component does not exceed $6c_0k^{1+2s_2}$.

By Lemmas 2.29 and 2.34 the number of poles of the resolvent $\left( H^{(1)}(\vec{z}_1(\varphi) + 2\pi p/N_1a + \vec{b}(2)) - k^2 \right)^{-1}$ inside $O_s(2\pi p/N_1a + \vec{b}(2))$ is less than $c_0k^{2+2s_2}$ for any $p \in \mathcal{P}$. Considering that there are no poles in $\Phi_1$ outside $O_s(2\pi p/N_1a + \vec{b}(2))$, we obtain that the number of poles of the resolvent $\left( H^{(1)}(\vec{z}_1(\varphi) + 2\pi p/N_1a + \vec{b}(2)) - k^2 \right)^{-1}$ inside $O^2(\vec{b}(2))$ is less than $c_0k^{2+2s_2}$. Taking into account that the number of $p \in \mathcal{P}$ does not exceed $ck^{2(s_2-s_1)}$, $s_2 = 2s_1$, we obtain the estimate for the number of poles for the resolvent $\left( \hat{H}^{(1)}(\vec{y}(1)(\varphi)) - k^2 \right)^{-1}$.

We estimate the size of $O^2(\vec{b}(2))$ the same way as we estimated the size of $O(2)$. Indeed, according to Lemma 2.27 the total size of each $O_s(\frac{2\pi p}{N_1a} + \vec{b}(2))$ is less then $c_0k^{-2-4s_1}$. Considering that the number of $p$ does not exceed $4k^{2s_1}$, we obtain that the total size of $O_2(\vec{b}(2))$ is less then $c_0k^{-2-2s_1}$. 

\[ \square \]
Definition 3.23. We denote the poles of the resolvent \( \left( \hat{H}^{(1)}(\tilde{y}^{(1)}(\varphi)) - k^2 \right)^{-1} \) in \( O^{(2)}(\tilde{b}^{(2)}) \) as \( \varphi^{(2)}_n \), \( n = 1, \ldots, M^{(2)}, M^{(2)} < c_0 k^{2 + 2s_2} \). Let us consider the discs \( O^{(2)}(\tilde{b}^{(2)}) \) of the radius \( r^{(2)} = k^{-2 - 4s_2 - \delta} r^{(1)} = k^{-6 - 7s_2 - 4\delta} \) centered at these poles. Let
\[
O^{(2)}_n(\tilde{b}^{(2)}) = \bigcup_{n=1}^{M^{(2)}} O^{(2)}_n(\tilde{b}^{(2)}).
\]

Lemma 3.24. The total size of \( O^{(2)}_s(\tilde{b}^{(2)}) \) is less then \( c_0 k^{-2 - 4s_2 - \delta} r^{(1)} = c_0 k^{-4 - 5s_2 - 4\delta} \).

Proof. The lemma easily follows from the formula \( r^{(2)} = k^{-2 - 4s_2 - \delta} r^{(1)} = k^{-6 - 7s_2 - 4\delta} \) and the estimate \( M^{(2)} < c_0 k^{2 + 2s_2} \).

Lemma 3.25. If \( \varphi \in \hat{\Phi}_1 \setminus O^{(2)}_s(\tilde{b}^{(2)}) \), then
\[
\left\| \left( \hat{H}^{(1)}(\tilde{y}^{(1)}(\varphi)) - k^2 \right)^{-1} \right\| \leq \frac{1}{\epsilon_1^2}.
\]

The estimate is stable in the \( (r^{(2)}k^{-\delta}) \)-neighborhood of \( \Phi_1 \setminus O^{(2)}_s(\tilde{b}^{(2)}) \). The resolvent is an analytic function of \( \varphi \) in every component of \( O^{(2)}_s(\tilde{b}^{(2)}) \), whose intersection with \( \Phi_1 \) is not empty. The only singularities of the resolvent are poles. The number of poles (counting multiplicity) of the resolvent inside \( O^{(2)}_s(\tilde{b}^{(2)}) \) is less than \( ck^{2 + 2s_2} \).

Proof. By the definition of \( O^{(2)}_s(\tilde{b}^{(2)}) \), the number of poles (counting multiplicity) of the resolvent inside this set is less than \( ck^{2 + 2s_2} \). Considering as in Lemma 2.29 we obtain
\[
\left\| \left( \hat{H}^{(1)}(\tilde{y}^{(1)}(\varphi)) - k^2 \right)^{-1} \right\| \leq \nu^{-M^{(2)}} k^{4 + \delta},
\]
where \( \nu \) is the coefficient of contraction, when we reduce \( O^{(2)}(\tilde{b}^{(2)}) \) to \( O^{(2)}_s(\tilde{b}^{(2)}) \). Namely \( \nu \) is the ratio of \( r^{(2)} \) to the maximal size of \( O^{(2)}(\tilde{b}^{(2)}) \). By Lemma 3.22 and the definition of \( r^{(2)} \), \( \nu = k^{-3 - 5s_2 - 2\delta} \). It is not difficult to show that that \( \nu^{-M^{(2)}} k^{4 + \delta} < \epsilon_1^{-1} \), when \( k > k_s \), see (3.1). The estimate (3.66) easily follows.

Obviously the total size of \( O^{(2)}_s(\tilde{b}^{(2)}) \) is much smaller the smallest disc in \( O^{(2)} \). Therefore, the function \( \hat{\varphi}_2(\varphi) \) is holomorphic inside each connected component of \( O^{(2)}_s(\tilde{b}^{(2)}) \) which has a non-empty intersection with \( \Phi_2 \). Let
\[
\tilde{y}^{(2)}(\varphi) = \hat{\varphi}_2(\varphi) + \tilde{b}^{(2)}.
\]

Lemma 3.26. If \( \varphi \in \hat{\Phi}_2 \setminus O^{(2)}_s(\tilde{b}^{(2)}) \), then
\[
\left\| \left( \hat{H}^{(2)}(\tilde{y}^{(2)}(\varphi)) - k^2 \right)^{-1} \right\| \leq \frac{2}{\epsilon_1^2}.
\]

The estimate is stable in the \( (r^{(2)}k^{-\delta}) \)-neighborhood of \( \Phi_2 \setminus O^{(2)}_s(\tilde{b}^{(2)}) \). The resolvent is an analytic function of \( \varphi \) in every component of \( O^{(2)}_s(\tilde{b}^{(2)}) \), whose intersection with \( \Phi_2 \) is not empty. The only singularities of the resolvent are poles. The number of poles (counting multiplicity) of the resolvent inside \( O^{(2)}_s(\tilde{b}^{(2)}) \) is less than \( ck^{2 + 2s_2} \).
Proof. We use the Hilbert identity
\[
\left( H^{(2)}(\tilde{y}^{(2)}) - k^2 \right)^{-1} - \left( \tilde{H}^{(1)}(\tilde{y}^{(1)}) - k^2 \right)^{-1} = \mathcal{E} \left( H^{(2)}(\tilde{y}^{(2)}) - k^2 \right)^{-1},
\]
where \( y^{(1)}, y^{(2)} \) are given by (3.61), (3.68), \( \mathcal{E} = \mathcal{E}_1 + \mathcal{E}_2 \mathcal{E}_3 \),
\[
\mathcal{E}_1 = -\left( \tilde{H}^{(1)}(\tilde{y}^{(1)}) - k^2 \right)^{-1} W_2, \quad \mathcal{E}_2 = \left( \tilde{H}^{(1)}(\tilde{y}^{(1)}) - k^2 \right)^{-1} \left( \tilde{H}_0^{(1)}(\tilde{y}^{(1)}) + k^2 \right),
\]
\[
\mathcal{E}_3 = \left( \tilde{H}_0^{(1)}(\tilde{y}^{(1)}) + k^2 \right)^{-1} \left( \tilde{H}_0^{(1)}(\tilde{y}^{(1)}) - \tilde{H}_0^{(1)}(\tilde{y}^{(2)}) \right).
\]
Using the estimates (3.28) and (3.66), we see that \( \| \mathcal{E}_1 \| < \epsilon_1^2 \). Using (3.66), we get \( \| \mathcal{E}_2 \| < 1 + \| W_1 - 2k^2 \| \epsilon_1^{-2} < 3k^2 \epsilon_1^{-2} \). It is easy to show that \( \| \mathcal{E}_3 \| < k^{-1} \delta \). Considering (3.49), we get \( \| \mathcal{E}_3 \| < 12 \epsilon_1^2 k^{-2} \). Therefore, \( \| \mathcal{E} \| < 4 \epsilon_1^2 / 1 < 1/2 \) when \( k > k_* \). It is easy to see now that
\[
\left\| \left( H^{(2)}(\tilde{y}^{(2)}) - k^2 \right)^{-1} \right\| < 2 \left\| \left( \tilde{H}^{(1)}(\tilde{y}^{(1)}) - k^2 \right)^{-1} \right\| < 2 \epsilon_1^2.
\]
Thus, we have obtained (3.69). Introducing operators \( P_N \) and using the same technique as in the proof of Lemma 2.20 (see (2.105) and further) we prove that the number of poles of \( \left( H^{(2)}(\tilde{y}^{(2)}(\varphi)) - k^2 \right)^{-1} \) obeys the same estimate as the number of poles of \( \left( \tilde{H}^{(1)}(\tilde{y}^{(1)}(\varphi)) - k^2 \right)^{-1} \), i.e., it does not exceed \( c_0 k^{2 + 2 s_2} \).

3.4.2. The set \( \mathcal{O}^{(2)} b(\tilde{y}^{(2)}) \) for small \( \tilde{y}^{(2)} \). Everything we considered so far is valid if \( b_0^{(2)} > \epsilon_1 k^{-1 - 24} \). However, in the next section and later, \( b_0^{(2)} \) is taken smaller, since the reciprocal lattice is getting finer with each step. To prepare for this, let us consider \( \tilde{y}^{(2)} \) being close to a vertex of \( K_2 \): \( 0 < b_0^{(2)} \leq \epsilon_1 k^{-1 - 28} \). We show that for such \( \tilde{y}^{(2)} \), the resolvent \( \left( H^{(2)}(\tilde{y}^{(2)}(\varphi)) - k^2 \right)^{-1} \) has no more than two poles in \( \tilde{\Phi}_2 \). We surround these poles by two contours \( \gamma^{(2)} \) and prove an estimate for the norm of \( \left( H^{(2)}(\tilde{y}^{(2)}(\varphi)) - k^2 \right)^{-1} \) when \( \varphi \) is outside these contours.

Suppose \( |\tilde{y}^{(2)}| = b_0^{(2)} \), i.e., the closest vertex of \( K_2 \) for \( \tilde{y}^{(2)} \) is \( (0, 0) \). We consider the functions \( \lambda^{(1)}(\tilde{y}^{(1)}(\varphi)) \) and \( \lambda^{(2)}(\tilde{y}^{(2)}(\varphi)) \) defined by perturbation series (2.58) and (3.44) for \( \varphi \in \tilde{\Phi}_2 \). The convergence of these series can be easily justified. In fact, by Lemmas 2.10 and 3.17, \( \tilde{z}_2(\varphi) \) and \( \tilde{z}_2(\varphi) \) are holomorphic functions of \( \varphi \) in \( \tilde{\Phi}_2 \). The perturbation series (2.58) and (3.44) converge for \( \lambda^{(1)}(\tilde{z}_1(\varphi)) \) and \( \lambda^{(2)}(\tilde{z}_2(\varphi)) \), respectively. Since the estimates involved are stable with respect to a change of \( \tilde{z}_1, \tilde{z}_2 \) not exceeding \( \epsilon_1 k^{-1 - \delta} \), the perturbation series for \( \lambda^{(1)}(\tilde{y}^{(1)}(\varphi)) \) and \( \lambda^{(2)}(\tilde{y}^{(2)}(\varphi)) \) also converge, both functions being holomorphic in \( \tilde{\Phi}_2 \). We base our further considerations on these perturbation series expansions. For \( \tilde{y}^{(2)} \) being close to a vertex \( \tilde{e} \) other than \( (0, 0) \), we take \( \tilde{y}^{(2)}(\varphi) = \tilde{z}_2(\varphi) + \tilde{e} \).

From now on, we denote the solutions \( \varphi \) of the equation \( \lambda^{(1)}(\tilde{y}^{(1)}(\varphi)) = k^2 + \epsilon_0 \), introduced in Lemma 2.30, by \( \varphi^{(1)}_{\epsilon_0} \). We set \( r^{(2)} = r^{(1)} k^{-2 - 4 s_2 - \delta} \).
Lemma 3.27. If \(0 < b_0^{(2)} \leq \epsilon_1 k^{-1-2\delta}\) and \(|\epsilon_0| < b_0^{(2)} k^{1-\delta} r^{(2)}\), then the equation
\[
\lambda^{(2)} \left( \hat{y}^{(2)}(\varphi) \right) = k^2 + \epsilon_0 \tag{3.70}
\]
has no more than two solutions \(\varphi^{(2)\pm}_{\epsilon_0}\) in \(\hat{\Phi}_2\). For any \(\varphi^{(2)\pm}_{\epsilon_0} \in \hat{\Phi}_2\) there is \(\varphi^{(1)\pm}_{0} \in \hat{\Phi}_1\) such that
\[
\left| \varphi^{(2)\pm}_{\epsilon_0} - \varphi^{(1)\pm}_{0} \right| < r^{(2)}/4, \tag{3.71}
\]
here and below \(\varphi^{(1)\pm}_{0}\) is \(\varphi^{(1)\pm}_{\epsilon_0}\) for \(\epsilon_0 = 0\).

Proof. First, we expand \(\lambda^{(2)}(\hat{y}^{(2)}(\varphi)) - \lambda^{(1)}(\tilde{y}^{(1)}(\varphi))\) near the point \(\hat{b}^{(2)} = 0\) and consider that \(\lambda^{(2)}(\tilde{y}^{(2)}(\varphi)) = \lambda^{(1)}(\tilde{y}^{(1)}(\varphi)) = \lambda\). Then, using (3.46), (2.63), (3.49) and (3.41), we check that
\[
\left| \lambda^{(2)}(\hat{y}^{(2)}(\varphi)) - \lambda^{(1)}(\tilde{y}^{(1)}(\varphi)) \right| < b_0^{(2)} \epsilon_1 \tag{3.72}
\]
in \(\hat{\Phi}_2\) and even in its \(r^{(2)}\)-neighborhood, the neighborhood being a subset of \(\hat{\Phi}_1\). Suppose (3.70) holds for some \(\varphi \in \hat{\Phi}_2\). By (3.72), \(\lambda^{(1)}(\tilde{y}^{(1)}(\varphi)) = k^2 + \epsilon_0, \epsilon_0 < \epsilon_0 + b_0^{(2)} \epsilon_1 < 2b_0^{(2)} k^{1-\delta} r^{(2)}\) when \(k > k_+\). Hence, \(\varphi\) satisfies conditions of Lemmas 2.30 and 2.31. Surrounding \(\varphi\) by a circle \(C\) of the radius \(r^{(2)}/4\) and using (2.132), we see that \(\lambda^{(1)}(\tilde{y}^{(1)}(\varphi)) - k^2 - \epsilon_0 \approx \frac{1}{4} k b_0^{(2)} r^{(2)} \gg |\epsilon_0|\) on this circle. Applying Rouché’s theorem, we obtain that there is a solution of \(\lambda^{(1)}(\tilde{y}^{(1)}(\varphi)) = k^2\) inside this circle. Thus, any solution of (3.70) is in the circle of the radius \(r^{(2)}/4\) surrounding \(\varphi^{(1)\pm}_{0}\), the point \(\varphi^{(1)\pm}_{0}\) being in the \((r^{(2)}/4)\)-neighborhood of \(\hat{\Phi}_2\). It remains to check that (3.70) has no more than two solutions and no more than one in a vicinity of each \(\varphi^{(1)\pm}_{0}\). We construct the disk of the radius \(r^{(2)}/4\) centered at \(\varphi^{(1)\pm}_{0}\) and note that \(\lambda^{(1)}(\tilde{y}^{(1)}(\varphi)) - k^2 > \frac{1}{4} k b_0^{(2)} r^{(2)}\) outside the circle. Using (3.72) and Rouché’s Theorem, we obtain that there is only one solution of (3.70) in the disk.

Lemma 3.28. Suppose \(0 < b_0^{(2)} \leq \epsilon_1 k^{-1-2\delta}\) and \(\varphi \in \hat{\Phi}_2\) obeys the inequality analogous to (3.71):
\[
\left| \varphi - \varphi^{(1)\pm}_{0} \right| < r^{(2)}. \tag{3.71}
\]
Then,
\[
\frac{\partial}{\partial \varphi} \lambda^{(2)}(\hat{y}^{(2)}(\varphi)) = k \to \infty \pm 2b_0^{(2)} k(1 + o(1)), \tag{3.73}
\]
where \(|o(1)| < 10^{-2} + \epsilon_1\) when \(k > k_+\).

The proof is completely analogous to that of Lemma 2.31.

Definition 3.29. Let \(\Gamma^{(2)\pm}(\hat{b}^{(2)})\) be the open disks centered at \(\varphi^{(2)\pm}_{0}\) with radius \(r^{(2)}\); \(\gamma^{(2)\pm}(\hat{b}^{(2)})\) be their boundary circles and \(O^{(2)}_s(\hat{b}^{(2)}) = \Gamma^{(2)+} \cup \Gamma^{(2)-}\).

Lemma 3.30. For any \(\varphi\) in \(\hat{\Phi}_2 \setminus O^{(2)}_s(\hat{b}^{(2)})\),
\[
\left| \lambda^{(2)}(\hat{y}^{(2)}(\varphi)) - k^2 \right| \geq b_0^{(2)} k^{1-\delta} r^{(2)}. \tag{3.74}
\]

The proofs of this and the next lemma are analogous to those of Lemma 2.33 and 2.34.
Lemma 3.31. For any \( \varphi \in \Phi_2 \setminus \mathcal{O}_s^{(2)}(\tilde{\mathcal{u}}^{(2)}) \),
\[
\left\| \left( H^{(2)} \left( \tilde{\mathcal{u}}^{(2)}(\varphi) \right) - k^2 \right)^{-1} \right\| < \frac{16}{k_0^{(2)} r^{(2)} k^{1-\delta}},
\]
(3.75)

The estimate is stable in the \( (r^{(2)} k^{-\delta}) \)-neighborhood of \( \Phi_2 \setminus \mathcal{O}_s^{(2)}(\tilde{\mathcal{u}}^{(2)}) \). The resolvent is an analytic function of \( \varphi \) in every component of \( \mathcal{O}_s^{(2)}(\tilde{\mathcal{u}}^{(2)}) \), whose intersection with \( \Phi_2 \) is not empty. The only singularities of the resolvent are poles. The resolvent has at most two poles inside \( \mathcal{O}_s^{(2)}(\tilde{\mathcal{u}}^{(2)}) \).

4. The Third Approximation

Squaring both sides of (3.1), we easily obtain the relation which coincide with (3.1) up to the substitution of \( s_2 \) instead of \( s_1 \):
\[
C_s(1 + s_2) k^{2+4s_2} \ln k < k^{\eta s_2}, \quad C_s = 400(c_0 + 1)^2, \quad c_0 = 32d_1, \quad s_2 = 2s_1.
\]
(4.1)
for any \( k > k_+ \). We will use (4.1) in the next step.

4.1. Operator \( H^{(3)}_\alpha \). Choosing \( s_3 = 2s_2 \), we define the third operator \( H^{(3)}_\alpha \) by the formula:
\[
H^{(3)}_\alpha = H^{(2)} + \alpha W_3, \quad (0 \leq \alpha \leq 1), \quad W_3 = \sum_{r=M_2+1}^{M_3} V_r,
\]
where \( M_3 \) is chosen in such a way that \( 2^{M_3} \approx k^{s_3} \). Obviously, the periods of \( W_3 \) are \( 2^{M_3-1}(d_1,0) \) and \( 2^{M_2-1}(0,d_2) \). We write them in the form: \( N_2 N_1(a_1,0) \) and \( N_2 N_1(0,a_2) \), here \( N_2 = 2^{M_3-M_2}, \quad \frac{1}{4} k^{s_1-s_2} < N_2 < 4 k^{s_1-s_2} \). Note that
\[
\| W_3 \| \leq \sum_{r=M_2+1}^{M_3} \| V_r \| \leq \sum_{r=M_2+1}^{M_3} \exp(-2^r) < \exp(-k^{\eta s_2}).
\]

4.2. Multiple Periods of \( W_2(x) \). The operator, \( H^{(2)} = H_1 + W_2(x) \), has the periods \( N_1 a_1, N_1 a_2 \). The corresponding family of operators, \( \{ H^{(2)}(\tau) \}_{\tau \in K_2} \), acts in \( L_2(Q_2) \), where \( Q_2 = [0,N_1 a_1] \times [0,N_1 a_2] \) and \( K_2 = [0,2\pi/N_1 a_1] \times [0,2\pi/N_1 a_2] \). Since now on we denote quasimomentum \( t \) from the first step by \( t^{(1)} \), quasimomentum \( \tau \) from the second step by \( t^{(2)} \). Correspondingly, quasimomentum for \( H^{(3)}_\alpha \) we denote by \( t^{(3)} \). Eigenvalues of \( H^{(2)}(t^{(2)}) \) are denoted by \( \lambda^{(2)}_n(t^{(2)}) \), \( n \in \mathbb{N} \) and its spectrum by \( \Lambda^{(2)}(t^{(2)}) \).

Next, let us consider \( W_2(x) \) as a periodic function with the periods \( N_2 a_1, N_2 a_2 \). When changing the periods, the family of operators \( \{ H^{(2)}(t^{(2)}) \}_{t^{(2)} \in K_2} \) is replaced by the family of operators \( \{ \tilde{H}^{(2)}(t^{(3)}) \}_{t^{(3)} \in K_3} \), acting in \( L_2(Q_3) \), where \( Q_3 = [0,N_2 a_1] \times [0,N_2 a_2] \) and \( K_3 = [0,2\pi/N_2 a_1] \times [0,2\pi/N_2 a_2] \). We denote eigenvalues of \( \tilde{H}^{(2)}(t^{(3)}) \) by \( \tilde{\lambda}^{(2)}_n(t^{(3)}) \), \( n \in \mathbb{N} \), and its spectrum by \( \tilde{\Lambda}^{(2)}(t^{(3)}) \).
We denote now by $P^{(1)}$ the set $P$, introduced by (3.4), its elements being $p^{(1)}$. By Bloch theory (see e.g. [22]), for any $t^{(3)} \in K_3,$

\[
\tilde{\Lambda}^{(2)}(t^{(3)}) = \bigcup_{p^{(2)} \in P^{(2)}} \Lambda^{(2)} \left( t^{(3)} + 2\pi p^{(2)}/N_2N_1a \right), \tag{4.2}
\]

where $P^{(2)} = \{ p^{(2)} = (p_1^{(2)}, p_2^{(2)}) \in \mathbb{Z}^2 : 0 \leq p_1^{(2)} \leq N_2 - 1, 0 \leq p_2^{(2)} \leq N_2 - 1 \}.$

\[
2\pi p^{(2)}/N_2N_1a = \left( \frac{2\pi p_1^{(2)}}{N_2N_1a_1}, \frac{2\pi p_2^{(2)}}{N_2N_1a_2} \right).
\]

An isoenergetic set $\tilde{S}_2(\lambda_0) \subset K_3$ of the operator $\tilde{H}^{(2)}$ is defined by the formula:

\[
\tilde{S}_2(\lambda) = \left\{ t^{(3)} \in K_3 : \exists n \in \mathbb{N} : \tilde{\Lambda}_n^{(2)}(t^{(3)}) = \lambda \right\}
\]

\[
= \left\{ t^{(3)} \in K_3 : \exists n \in \mathbb{N}, p^{(2)} \in P^{(2)} : \lambda_n^{(2)} \left( t^{(3)} + 2\pi p^{(2)}/N_2N_1a \right) = \lambda \right\}.
\]

Obviously, $\tilde{S}_2 = K_3S_2$, where $K_3$ is the parallel shift into $K_3$, that is,

\[
K_3 : \mathbb{R}^2 \to K_3, \quad K_3 \left( t^{(3)} + 2\pi m/N_2N_1a \right) = t^{(3)}, \quad m \in \mathbb{Z}^2, \quad t^{(3)} \in K_3.
\]

We denote index $j$, introduced in Part 1 of Lemma 3.4 (Geometric Lemma for the First approximation), by $j^{(1)}$ and $j$, introduced in Part 1 of Lemma 3.5 (Geometric Lemma for the Second approximation), by $j^{(2)}, j^{(2)} = j^{(1)} + p^{(1)}/N_1$.

### 4.3. Perturbation Formulae.

**Lemma 4.1** (Geometric Lemma). For a sufficiently large $\lambda$, $\lambda > k_*^2$, there exists a non-resonance set $\chi_3(\lambda, \delta) \subset K_3\chi_2^*$ such that:

1. For any point $t^{(3)} \in \chi_3$, the following conditions hold:
   
   (a) There exists a unique $p^{(2)} \in P^{(2)}$ such that $t^{(3)} + 2\pi p^{(2)}/N_2N_1a \in \chi_2^*$.
   
   (b) The following relation holds:

\[
\lambda^{(2)}_{j^{(2)}} \left( t^{(3)} + 2\pi p^{(2)}/N_2N_1a \right) = \lambda,
\]

where $\lambda^{(2)}_{j^{(2)}}(t^{(3)} + 2\pi p^{(2)}/N_2N_1a)$ is given by the perturbation series (3.29) with $\alpha = 1$ and $j^{(2)} = j + p/N_1$, here $j$ and $p$ are defined by the point $\tau = t^{(3)} + 2\pi p^{(2)}/N_2N_1a$ as it is described in Part 1(b) of the Geometric Lemma for the previous step.

(c) The eigenvalue $\lambda^{(2)}_{j^{(2)}}(t^{(3)} + 2\pi p^{(2)}/N_2N_1a)$ is a simple eigenvalue of $\tilde{H}^{(2)}(t^{(3)})$ and its distance to all other eigenvalues $\lambda^{(2)}_n(t^{(3)} + 2\pi p^{(2)}/N_2N_1a)$ of $\tilde{H}^{(2)}(t^{(3)})$ is greater than $\epsilon_2 = e^{-4k_{xy}^2}$:

\[
\left| \lambda^{(2)}_{j^{(2)}} \left( t^{(3)} + 2\pi p^{(2)}/N_2N_1a \right) - \lambda^{(2)}_n \left( t^{(3)} + 2\pi p^{(2)}/N_2N_1a \right) \right| > \epsilon_2. \tag{4.3}
\]
(2) For any $t^{(3)}$ in the real $(\epsilon_k k^{-1-\delta})$-neighborhood of $\chi_3$, there exists a unique $p^{(2)}(\varphi) \in P^{(2)}$ such that $t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a$ is in the $(\epsilon_k k^{-1-\delta})$-neighborhood of $\chi_2$ and

$$
\left| \lambda_j^{(2)} \left( t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a \right) - k^2 \right| < \epsilon_k k^{-\delta}, \tag{4.4}
$$

$j^{(2)} = j + p/N_1$, here $j$ and $p$ are defined by the point $\tau = t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a$ as it is described in Part 2 of the Geometric Lemma for the previous step. An estimate analogous to (4.3) holds:

$$
\left| \lambda_j^{(2)} \left( t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a \right) - \lambda_j^{(1)} \left( t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a \right) \right| > (1 + o(1)) \epsilon_k. \tag{4.5}
$$

(3) The third nonresonance set $\chi_3$ has an asymptotically full measure on $\chi_2$ in the following sense:

$$
\frac{L(K_3 \chi_2 \setminus \chi_3)}{L(\chi_2)} < c_0 k^{-4 - 2s_1 - 2s_2}. \tag{4.6}
$$

**Proof.** The proof of the lemma is analogous to that for Geometric Lemma in the second step. Indeed, let us consider

$$
O^{(3)} = \bigcup_{p^{(2)} \in P^{(2)} \setminus \{0\}} O_s^{(2)} \left( \frac{2\pi p^{(2)}}{N_2 N_1 a} \right), \tag{4.7}
$$

where $O_s^{(2)} \left( \frac{2\pi p^{(2)}}{N_2 N_1 a} \right)$ is defined by (3.65) with $b^{(2)} = \frac{2\pi p^{(2)}}{N_2 N_1 a}$. Note, that the definition make sense, since $\left| \frac{2\pi p^{(2)}}{N_2 N_1 a} \right| > d_{\max}^{-1} k^{-s_1} > \epsilon_k k^{-1 - 2\delta}$. Let

$$
\Phi_3 = \Phi_2 \setminus O^{(3)}, \quad \Theta_3 = \Phi_3 \cap [0, 2\pi). \tag{4.8}
$$

By Lemma 3.26

$$
\left\| \left( H^{(2)} \left( \tilde{\varphi}_2(\varphi) + \frac{2\pi p^{(2)}}{N_2 N_1 a} \right) - k^2 \right) \right\|^{-1} < \frac{2}{\epsilon_k^2} \tag{4.9}
$$

for all $p^{(2)} \in P^{(2)} \setminus \{0\}$ and $\varphi \in \Phi_3$, here and below $\tilde{\Phi}_3$ is the $(r^{(2)} k^{-\delta})$-neighborhood of $\Phi_3$. We consider $D_{2,nonres} \subset D_2$:

$$
D_{2,nonres} = \{ \tilde{\varphi}_2(\varphi), \quad \varphi \in \Theta_3 \}. \tag{4.10}
$$

We define $\chi_3$ by the formula:

$$
\chi_3 = K_3 D_{2,nonres}. \tag{4.11}
$$

By the definition of $K_3$, for every $t^{(3)}$ in $\chi_3$, there are $p^{(1)}(\varphi) \in P^{(1)}$, $p^{(2)} \in P^{(2)}$ and $j \in Z^2$ such that

$$
t^{(3)} + \frac{2\pi p^{(2)}}{N_2 N_1 a} + \frac{2\pi p^{(1)}}{N_1 a} + \frac{2\pi j}{a} = \tilde{\varphi}_2(\varphi), \quad \tilde{\varphi}_2(\varphi) \in D_{2,nonres}. \tag{4.12}
$$

Considering the definition of $D_{2,nonres}$, we obtain:

$$
\left\| \left( H^{(2)} \left( t^{(3)} + \frac{2\pi p^{(2)}}{N_2 N_1 a} \right) - k^2 \right) \right\|^{-1} < \frac{2}{\epsilon_k^2}, \tag{4.13}
$$
\[ \hat{p}^{(2)} = p^{(2)} + \tilde{p}^{(2)}. \] Note that the indices \( \hat{j}, \ p^{(1)} \) do not play a role, since they just produce a shift of indices of the matrix elements of the resolvent. Considering that \( \hat{p}^{(2)} \) can be any but zero, we obtain that (4.13) holds for all \( \hat{p}^{(2)} \in P^{(2)} \setminus \{ p^{(2)} \} \). Taking into account that \( \lambda_j^{(2)}(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a) = k^2 \) and \( \epsilon_2 < \epsilon_1^2 \), we arrive at (4.5) for all \( \hat{p} \neq p \). It remains to check (4.13) for \( p = \hat{p} \). Let \( t^{(2)} = t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a \). By (4.12), \( t^{(2)} \in K_2 D_2 \). Using (3.59), we get \( t^{(2)} \in \chi_s^* \). By Theorem 3.8 \( \lambda_j^{(2)}(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a) \) is the only eigenvalue of \( H^{(2)}(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a) \) in the interval \( \epsilon_2 \). Hence,\[
abla_j^{(2)}(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a) - \lambda_n^{(2)}(t^{(3)} + 2\pi p^{(2)}/N_2 N_1 a) > \epsilon_1.
\]

Part 2 holds, since all estimates are stable with respect to the perturbation of \( t^{(3)} \) less then \( \epsilon_2 k^{-1-\delta} \).

Let us estimate the size of \( O^{(3)} \). According to Lemma 3.24 the total size of each \( O^{(2)} \) is less then \( c_0 k^{-4-s_2-4\delta} \). Considering that the number of \( p^{(2)} \) does not exceed \( k^{2(s_3-s_2)} = k^{2s_2} \), we arrive at (4.6).

**Corollary 4.2.** If \( t^{(3)} \) belongs to the complex \( (\epsilon_2 k^{-1-\delta}) \)-neighborhood of \( \chi_3(\lambda, \delta) \), then for any \( z \) lying on the circle \( C_3 = \{ z: |z-k^2| = \epsilon_2/2 \} \), the following inequality holds:

\[
\left\| \left( \hat{\cal H}^{(2)}(t^{(3)}) - z \right)^{-1} \right\| < \frac{4}{\epsilon_2}.
\]

**Remark 4.3.** Every point \( 2nq/N_2 N_1 a \) \( q \in \mathbb{Z}^2 \) of the dual lattice for periods \( N_2 N_1 a_1, N_2 N_1 a_2 \) can be uniquely represented in the form: \( 2nq/N_2 N_1 a = 2\pi m/N_1 a + 2\pi p^{(2)}/N_2 N_1 a \), where \( m \in \mathbb{Z}^2 \), \( p^{(2)} \in P^{(2)} \). Note that \( 2\pi m/N_1 a \) is a point of a dual lattice for periods \( N_1 a_1, N_1 a_2 \) and \( p^{(2)} \in P^{(2)} \) is responsible for refining the lattice. By Remark 3.7, \( \hat{\cal H}^{(2)}(t^{(3)}) \) also can be uniquely represented as \( 2nq/N_2 N_1 a = 2\pi j/a + 2\pi p^{(1)}/N_1 a + 2\pi p^{(2)}/N_2 N_1 a \), here \( j \in \mathbb{Z}^2 \), \( p^{(1)} \in P^{(1)} \), \( p^{(2)} \in P^{(2)} \).

Let us consider a normalized eigenfunction \( \tilde{\psi}_n(t^{(2)}, x) \) of \( H^{(2)}(t^{(2)}) \) in \( L_2(Q_2) \). We extend it quasiperiodically to \( L_2(Q_3) \), renormalize and denote the new function by \( \tilde{\psi}_n(t^{(3)}, x) \), \( t^{(3)} = K_3 t^{(2)} \). The Fourier representations of \( \psi_n(t^{(2)}, x) \) in \( L_2(Q_2) \) and \( \tilde{\psi}_n(t^{(3)}, x) \) in \( L_2(Q_3) \) are simply related. If we denote Fourier coefficients of \( \psi_n(t^{(2)}, x) \) with respect to the basis \( |Q_2|^{-1/2} e^{-i(2\pi m/N_1 a + \epsilon^{(2)}, x)}, m \in \mathbb{Z}^2 \), in \( L_2(Q_2) \) by \( C_{nm} \), then, obviously, the Fourier coefficients \( \tilde{C}_{nm} \) of \( \tilde{\psi}_n(t^{(3)}, x) \) with respect to the basis \( |Q_3|^{-1/2} e^{-i(2\pi q/N_2 N_1 a + \epsilon^{(3)}, x)}, q \in \mathbb{Z}^2 \), in \( L_2(Q_3) \) are given by the formula:

\[
\tilde{C}_{nm} = \begin{cases} C_{nm}, & \text{if } q = m N_2 + p^{(2)}; \\ 0, & \text{otherwise}, \end{cases}
\]

\( p^{(2)} \) being defined from the relation \( t^{(2)} = t^{(3)} + 2\pi p^{(2)}/N_1 N_2 a \). Correspondingly, matrices of the projections on \( \psi_n(\tau, x) \) and \( \tilde{\psi}_n(t^{(3)}, x) \) with respect to the above
bases are simply related:

\[
(\tilde{E}_n)_{\hat{q}q} = \begin{cases} 
(E_n)_{m\hat{m}}, & \text{if } q = mN + p(2), \hat{q} = \hat{m}N + p(2); \\
0, & \text{otherwise,}
\end{cases}
\]

\(\tilde{E}_n\) and \(E_n\) being projections in \(L_2(Q_3)\) and \(L_2(Q_2)\), respectively.

We define functions \(g_r^{(3)}(k, t^{(3)})\) and operator-valued functions \(G_r^{(3)}(k, t^{(3)}), r = 1, 2, \cdots\), as follows:

\[
g_r^{(3)}(k, t^{(3)}) = \frac{(-1)^r}{2\pi i} \text{Tr} \oint_{C_3} \left( (\tilde{H}^{(2)}(t^{(3)}) - z)^{-1} W_3 \right)^r dz,
\]

\[
G_r^{(3)}(k, t^{(3)}) = \frac{(-1)^{r+1}}{2\pi i} \oint_{C_3} \left( (\tilde{H}^{(2)}(t^{(3)}) - z)^{-1} W_3 \left( \tilde{H}^{(2)}(t^{(3)}) - z \right)^{-1} \right)^{(2)} dz.
\]

**Theorem 4.4.** Suppose \(t^{(3)}\) belongs to the \((\epsilon_2 k^{-1-\delta})\)-neighborhood in \(K_3\) of the third nonresonance set \(\chi_3(\lambda, \delta), 0 < \delta < s_1, \epsilon_2 = e^{-\frac{4}{3}k^{2/3}}\). Then, \(\lambda > k^2\) and for all \(\alpha, 0 \leq \alpha \leq 1\), there exists a unique eigenvalue of the operator \(H_3^{(3)}(t^{(3)})\) in the interval \(\tilde{\chi}_3(k) := (k^2 - \epsilon_2/2, k^2 + \epsilon_2/2)\). It is given by the series:

\[
\lambda_{j^{(3)}}^{(3)}(\alpha, t^{(3)}) = \lambda_{j^{(2)}}^{(2)} \left( t^{(3)} + 2\pi p^{(2)} / N_2 N_1 a \right) + \sum_{r=1}^{\infty} \alpha^r g_r^{(3)}(k, t^{(3)}),
\]

converging absolutely in the disk \(|\alpha| \leq 1\), where \(j^{(3)} := j^{(2)} + p^{(2)} / N_2 N_1, p^{(2)}, j^{(2)}\) being described in Geometric Lemma 4.7. The spectral projection, corresponding to \(\lambda_{j^{(3)}}^{(3)}(\alpha, t^{(3)})\), is given by the series:

\[
E_{j^{(3)}}^{(3)}(\alpha, t^{(3)}) = \tilde{E}_{j^{(2)}}^{(2)} \left( t^{(3)} + 2\pi p^{(2)} / N_2 N_1 a \right) + \sum_{r=1}^{\infty} \alpha^r G_r^{(3)}(k, t^{(3)}),
\]

which converges in the trace class \(S_1\) uniformly with respect to \(\alpha\) in the disk \(|\alpha| \leq 1\).

The following estimates hold for coefficients \(g_r^{(3)}(k, t^{(3)}), G_r^{(3)}(k, t^{(3)}):\)

\[
\left\| g_r^{(3)}(k, t^{(3)}) \right\| < \frac{3\epsilon_2}{2} \left( 4\epsilon_2 \right)^r, \quad \left\| G_r^{(3)}(k, t^{(3)}) \right\|_1 < 6\epsilon_2 \left( 4\epsilon_2 \right)^r.
\]

**Corollary 4.5.** The following estimates hold for the perturbed eigenvalue and its spectral projection:

\[
\left| \lambda_{j^{(3)}}^{(3)}(\alpha, t^{(3)}) - \lambda_{j^{(2)}}^{(2)} \left( t^{(3)} + 2\pi p^{(2)} / N_2 N_1 a \right) \right| \leq 12\alpha\epsilon_2^4,
\]

\[
\left\| E_{j^{(3)}}^{(3)}(\alpha, t^{(3)}) - \tilde{E}_{j^{(2)}}^{(2)} \left( t^{(3)} + 2\pi p^{(2)} / N_2 N_1 a \right) \right\|_1 \leq 48\alpha\epsilon_2^3.
\]

Proof of the theorem is analogous to that of the Theorem 3.8. The series (4.15), (4.16) can be extended as holomorphic functions of \(t^{(3)}\) in the complex \((\epsilon_2 k^{-1-\delta})\)-neighborhood of \(\chi_3\), they can be differentiated any number of times with respect to \(t^{(3)}\) and retain their asymptotic character. The results analogous to Lemma 3.11, Theorem 3.12, Corollary 3.13 and Lemma 3.14 hold.
4.4. **Nonresonant part of the isoenergetic set of \( H_α^{(3)} \).** This section is analogous to Section 3.3 for the second step. Indeed, let

\[ S \]

\[ H_α^{(3)} \]

\[ \text{the spectrum of} \]

\[ 4.4. \]

\[ \text{Nonresonant part of the isoenergetic set of} \]

\[ B \]

\[ D \]

\[ \text{ries} (4.15). \]

Recall that

\[ (4.8): \]

Recall also that

\[ \text{it corresponds to non-resonance eigenvalues} \]

\[ \lambda_3 \]

\[ \text{between} \]

\[ \text{formulations and proofs are analogous to those of Lemmas 3.16 and 3.17.} \]

**Lemma 4.6.** The formula \( K_3D_2(\lambda)_{\text{nonres}} = \chi_3 \) establishes one-to-one correspondence between \( D_2(\lambda)_{\text{nonres}} \) and \( \chi_3 \).

**Proof.** The proof is analogous to that of Lemma 3.16 up to the shift of indices by 1, i.e., \( \chi_2 \to \chi_3 \), \( \chi_2^2(\lambda) \to \chi_3^2(\lambda) \), \( \tau = t(2) \to t(3) \); we use formula (3.59) instead of (2.73), Part 1a of the Geometric Lemma for the third step instead of Part 1a of the Geometric Lemma for the second step, and Lemma 3.19 instead of 2.12.

We define \( B_3(\lambda) \) as the set of directions corresponding to \( \Theta_3 \), \( \Theta_3 \) being given by (4.8):

\[ B_3(\lambda) = \{ \vec{v} \in B_2(\lambda) : \varphi \in \Theta_3 \}. \]

Note that \( B_3(\lambda) \) is a unit circle with holes, centered at the origin, and \( B_3(\lambda) \subset B_2(\lambda) \subset B_1(\lambda) \). We define \( D_3(\lambda) \) as a level set for \( \lambda_3(\alpha, \vec{z}) \) in the \((\epsilon_2k^{-1-\delta})\)-neighborhood of \( D_2(\lambda)_{\text{nonres}} \):

\[ D_3(\lambda) := \{ \vec{z} = \varphi \vec{v} : \vec{v} \in B_3(\lambda), \quad |\varphi - \varphi_2(\lambda, \vec{v})| < \epsilon_2k^{-1-\delta}, \lambda_3(\alpha, \vec{z}) = \lambda \}. \]

Next two lemmas are to prove that \( D_3(\lambda) \) is a distorted circle with holes. Their formulations and proofs are analogous to those of Lemmas 3.16 and 3.17.

**Lemma 4.7.** For every \( \vec{v} \in B_3(\lambda) \) and every \( \alpha \), \( 0 \leq \alpha \leq 1 \), there is a unique \( \varphi = \varphi_3(\lambda, \vec{v}) \) in the interval \( I_3 := [\varphi_2(\lambda, \vec{v}) - \epsilon_2k^{-1-\delta}, \varphi_2(\lambda, \vec{v}) + \epsilon_2k^{-1-\delta}] \) such that

\[ \lambda_3(\alpha, \varphi \vec{v}) = \lambda. \] (4.20)

Furthermore,

\[ |\varphi_3(\lambda, \vec{v}) - \varphi_2(\lambda, \vec{v})| \leq 2\epsilon_2k^{-1}. \] (4.21)

Further we use the notations \( \varphi_3(\varphi) \equiv \varphi_3(\lambda, \vec{v}) \), \( h_3(\varphi) \equiv \varphi_3(\varphi) - \varphi_2(\varphi), \varphi_3(\varphi) = \varphi_3(\varphi) \).

**Lemma 4.8.** The following statements hold for \( \lambda > k_2^2 \):

1. The set \( D_3(\lambda) \) is a distorted circle with holes: it can be described by the formula:

\[ D_3(\lambda) = \{ \vec{z} \in \mathbb{R}^2 : \vec{z} = \varphi_3(\varphi), \quad \varphi \in \Theta_3(\lambda) \}. \] (4.22)
where \( \varphi_3(\varphi) = \varphi_2(\varphi) + h_3(\varphi) \), \( \varphi_2(\varphi) \) is the “radius” of \( D_2(\lambda) \) and \( h_3(\varphi) \) satisfies the estimates

\[
|h_3| < 12\alpha\epsilon_2^4k^{-1}, \quad \left| \frac{\partial h_3}{\partial \varphi} \right| \leq 96\alpha\epsilon_2^3k^1+\delta.
\]  

(4.23)

(2) The total length of \( B_3(\lambda) \) satisfies the estimate:

\[
L(B_2 \setminus B_3) < 4\pi k^{-1-2s_1-2s_2}.
\]  

(4.24)

(3) The function \( h_3(\varphi) \) can be extended as a holomorphic function of \( \varphi \) to the complex non-resonance set \( \Phi_3 \) and its \( (k^{-\delta_1(2)}) \)-neighborhood \( \Phi_3 \), estimates \( (4.23) \) being preserved.

(4) The curve \( D_3(\lambda) \) has a length which is asymptotically close to that of \( D_2(\lambda) \) in the following sense:

\[
L(D_3(\lambda)) = L(D_2(\lambda)) \left( 1 + O(k^{-1-2s_1-2s_2}) \right),
\]  

(4.25)

where \( O(k^{-1-2s_1-2s_2}) = (1 + o(1))c_0k^{-1-2s_1-2s_2}, \ |o(1)| < 10^{-2} \) when \( k > k_* \).

Proof. The proof is analogous to that of Lemma 3.17. Note only that, in Part 2, when proving convergence of the series for the resolvent \( (H^{(3)}(\bar{z}_2(\varphi)) - z)^{-1} \), we use the estimate

\[
\left\| \left( \tilde{H}^{(2)}(\bar{z}_2(\varphi)) - z \right)^{-1} \right\| < \frac{4}{\epsilon_2}, \quad z \in C_3,
\]  

(4.26)

analogous to (3.55), the operator \( \tilde{H}^{(2)} \) acting in \( L_2(Q_3) \). The estimate (4.26) follows from (4.19) and (4.42). As a side result of these considerations, we obtain an estimate analogous to (3.58) for the new resolvent and \( z \) being inside \( C_3 \).

We define the non-resonance set, \( \chi_3^*(\lambda) \) in \( S_3(\lambda) \) by the formula analogous to (3.59):

\[
\chi_3^*(\lambda) := K_3D_3(\lambda).
\]  

(4.27)

The following lemmas are analogous to Lemmas 3.18 and 3.19.

**Lemma 4.9.** The set \( \chi_3^*(\lambda) \) belongs to the \( (12\alpha\epsilon_2^3k^{-1}) \)-neighborhood of \( \chi_3(\lambda) \) in \( K_3 \). If \( t^{(3)} \in \chi_3^*(\lambda) \), then the operator \( H_\alpha^{(3)}(t^{(3)}) \) has a simple eigenvalue equal to \( \lambda \). This eigenvalue is given by the perturbation series (4.15).

**Lemma 4.10.** Formula (4.27) establishes one-to-one correspondence between \( \chi_3^*(\lambda) \) and \( D_3(\lambda) \).

**Remark 4.11.** From geometric point of view this means that \( \chi_3^*(\lambda) \) does not have self-intersections.

4.5. **Preparation for the Next Approximation.** Let \( \bar{b}^{(3)} \in K_3 \) and \( b_0^{(3)} \) be the distance of the point \( \bar{b}^{(3)} \) to the nearest corner of \( K_3 \):

\[
b_0^{(3)} = \min_{m = (0,0),(0,1),(1,0),(1,1)} |\bar{b}^{(3)} - 2\pi m/N_2N_1a|.
\]  

(4.28)
We assume $b_0^{(3)} = |\overline{\tilde{b}(3)}|$. In the case when $\overline{\tilde{b}(3)}$ is closer to a vertex other than $(0,0)$, the considerations are the same up to a parallel shift. We consider two cases: $b_0^{(3)} \geq \epsilon_2 k^{-1-2\delta}$ and $0 < b_0^{(3)} < \epsilon_2 k^{-1-2\delta}$. Let

$$\tilde{y}^{(2)}(\varphi) = \tilde{z}_2(\varphi) + \overline{\tilde{b}(3)}. \tag{4.29}$$

4.5.1. The case $b_0^{(3)} \geq \epsilon_2 k^{-1-2\delta}$.

**Definition 4.12.** We define the set $O^{(3)}(\overline{\tilde{b}(3)})$ by the formula:

$$O^{(3)}(\overline{\tilde{b}(3)}) = \bigcup_{p^{(2)} \in P^{(2)}} O_s^{(2)}(2\pi p^{(2)}/N_2 N_1 a + \overline{\tilde{b}(3)}), \tag{4.30}$$

set $O_s^{(2)}$ being defined by formula (3.65). We assume $O_s^{(2)} \cap \hat{\Phi}_2 \neq \emptyset$.

The set $O^{(3)}(\overline{\tilde{b}(3)})$ consists the disks with the radius $r^{(2)}(2+2\pi p^{(2)}/N_2 N_1 a + \overline{\tilde{b}(3)})$ centered at poles of the resolvent $\left(\tilde{H}_2(\tilde{y}^{(2)}(\varphi)) - k^2\right)^{-1}$.

**Lemma 4.13.** If $\varphi \in \hat{\Phi}_2 \setminus O^{(3)}(\overline{\tilde{b}(3)})$, then

$$\left\|\left(\tilde{H}^{(2)}(\tilde{y}^{(2)}(\varphi)) - k^2\right)^{-1}\right\| \leq \frac{17r^{(2)}k^{3\delta}}{\epsilon_2} \tag{4.31}$$

The estimate is stable in the $(r^{(2)}k^{-\delta})$-neighborhood of $\hat{\Phi}_2 \setminus O^{(3)}(\overline{\tilde{b}(3)})$. The resolvent is an analytic function of $\varphi$ in every connected component $O^{(3)}(\overline{\tilde{b}(3)})$ of $O^{(3)}(\overline{\tilde{b}(3)})$, whose intersection with $\Phi_2$ is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside $O^{(3)}(\overline{\tilde{b}(3)})$ is less than $ck^{2+2s_3}$. The total size of $O^{(3)}(\overline{\tilde{b}(3)})$ does not exceed $c_0 k^{-4-2s_1-2s_2}$.

**Proof.** The proof is analogous to that of Lemma 3.22. To obtain (4.31) we combine Lemmas 3.26 and 3.31 for $\tilde{y}^{(2)} = 2\pi p^{(2)}/N_2 N_1 a + \overline{\tilde{b}(3)}$, $p^{(2)} \in P^{(2)}$, and take into account that $\epsilon_2^2 > \epsilon_2$. By the same lemmas the number of poles of $\left(\tilde{H}^{(2)}(\tilde{z}_2(\varphi) + 2\pi p^{(2)}/N_2 N_1 a + \overline{\tilde{b}(3)}) - k^2\right)^{-1}$ inside $O^{(2)}(2\pi p^{(2)}/N_2 N_1 a + \overline{\tilde{b}(3)})$ is less than $ck^{2+2s_2}$. Considering that this resolvent does not have poles outside $O^{(2)}(2\pi p^{(2)}/N_2 N_1 a + \overline{\tilde{b}(3)})$, we obtain that the total number of poles of $\left(\tilde{H}^{(2)}(\tilde{z}_2(\varphi) + 2\pi p^{(2)}/N_2 N_1 a + \overline{\tilde{b}(3)}) - k^2\right)^{-1}$ inside $O^{(3)}(\overline{\tilde{b}(3)})$ is less than $c_0 k^{2+2s_2}$. Taking into account that the number of $p^{(2)} \in P^{(2)}$ does not exceed $ck^{2(s_3-s_2)}$, $s_3 = 2s_2$, we obtain the estimate for the number of poles for the resolvent $\left(\tilde{H}^{(2)}(\tilde{y}^{(2)}(\varphi)) - k^2\right)^{-1}$.

We estimate the size of $O^{(3)}(\overline{\tilde{b}(3)})$ the same way as we estimated the size of $O^{(3)}$. Indeed, according to Lemma 3.24 the total size of each $O^{(2)}(2\pi p^{(2)}/N_2 N_1 a)$ is less then $c_0 k^{-4-5s_2-4\delta}$. Considering that the number of $p^{(2)}$ does not exceed $k^{2(s_3-s_2)} = k^{2s_2}$, $s_2 = 2s_1$, we arrive at the estimate for $O^{(3)}(\overline{\tilde{b}(3)})$.
Definition 4.14. We denote the poles of the resolvent \( \left( \hat{H}^{(2)}(\hat{y}^{(2)}(\varphi)) - k^2 \right)^{-1} \) in \( \mathcal{O}^{(3)}(\tilde{b}^{(3)}) \) as \( \varphi_n^{(3)}, n = 1, ..., M^{(3)} \). Let us consider the circles \( \mathcal{O}^{(3)}_n(\tilde{b}^{(3)}) \) of the radius \( r^{(3)} = k^{-2 - 4s_3 - \delta r^{(2)}} \) around these poles. Let
\[
\mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) = \bigcup_{n=1}^{M^{(3)}} \mathcal{O}^{(3)}_n(\tilde{b}^{(3)}).
\] (4.32)

Lemma 4.15. The total size of \( \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \) is less then \( c_0 k^{-2s_3 - \delta r^{(2)}} \).

Proof. The lemma easily follows from the formula \( r^{(3)} = k^{-2 - 4s_3 - \delta r^{(2)}} \) and the estimate \( M^{(3)} < c_0 k^{2+2s_3} \).

Lemma 4.16. If \( \varphi \in \hat{\Phi}_2 \setminus \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \), then
\[
\left\| \left( \hat{H}^{(2)}(\hat{y}^{(2)}(\varphi)) - k^2 \right)^{-1} \right\| \leq \frac{1}{\epsilon_2^{(3)}}.
\] (4.33)

The estimate is stable in the \( (r^{(3)}k^{-\delta}) \)-neighborhood of \( \hat{\Phi}_2 \setminus \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \). The resolvent is an analytic function of \( \varphi \) in every connected component \( \mathcal{O}^{(3)}_{s_2}(\tilde{b}^{(3)}) \) of \( \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \), whose intersection with \( \hat{\Phi}_2 \) is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside \( \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \) is less than \( c_0 k^{2+2s_3} \).

Proof. By the definition of \( \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \), the number of poles (counting multiplicity) of the resolvent inside this set is less than \( c_3 k^{2+2s_3} \). Considering as in Lemma 2.29 we obtain
\[
\left\| \left( \hat{H}^{(2)}(\hat{y}^{(2)}(\varphi)) - k^2 \right)^{-1} \right\| \leq \nu^{-M^{(3)}} \frac{17r^{(2)}k^{3\delta}}{\epsilon_2},
\] (4.34)

where \( \nu \) is the coefficient of contraction, when we reduce \( \mathcal{O}^{(3)}(\tilde{b}^{(3)}) \) to \( \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \). Namely \( \nu \) is the ratio of \( r^{(3)} \) to the maximal size of \( \mathcal{O}^{(3)}(\tilde{b}^{(3)}) \). By Lemma 2.13 and the definition of \( r^{(3)}, \nu = k^{-4 - 4s_3 - 4s_2 - 2\delta} \). Considering that \( \nu^{-M^{(3)}}r^{(2)}k^{\delta} < \epsilon_2^{-1} \), we obtain (4.33).

By Lemma 4.15, the total size of \( \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \) is less then \( r^{(2)} \). Since the smallest circle in \( \mathcal{O}^{(3)} \) has the size \( r^{(2)} \), the function \( \hat{z}_3(\varphi) \) is holomorphic inside each connected component of \( \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \) which has non-empty intersection with \( \hat{\Phi}_3 \). Let
\[
\hat{g}^{(3)}(\varphi) = \hat{z}_3(\varphi) + \tilde{b}^{(3)}.
\] (4.35)

Lemma 4.17. If \( \varphi \in \hat{\Phi}_3 \setminus \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \), then
\[
\left\| \left( \hat{H}^{(3)}(\hat{g}^{(3)}(\varphi)) - k^2 \right)^{-1} \right\| \leq \frac{2}{\epsilon_2^{(3)}}.
\] (4.36)

The estimate is stable in the \( (r^{(3)}k^{-\delta}) \)-neighborhood of \( \hat{\Phi}_3 \setminus \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \). The resolvent is an analytic function of \( \varphi \) in every component of \( \mathcal{O}^{(3)}_s(\tilde{b}^{(3)}) \), whose intersection with \( \hat{\Phi}_3 \) is not empty. The only singularities of the resolvent are poles. The number of poles (counting multiplicity) of the resolvent inside \( \mathcal{O}^{(3)}(\tilde{b}^{(3)}) \) is less than \( c_0 k^{2+2s_3} \).
Proof. The proof is analogous to that of Lemma 3.26 up to the shift of indices by one. We use Lemma 4.16 \( \|W_3\|_\infty < \epsilon_4^2 \), and the first estimate in (4.23).

4.5.2. The set \( \mathcal{O}_s^{(3)}(\vec{b}^{(3)}) \) for small \( \vec{b}^{(3)} \). The considerations of this section are analogous to those of Section 3.4.2 up to the shift of indices by one. The following lemmas and the definitions are completely analogous to 3.27 – 3.31.

Lemma 4.18. If \( 0 < b_0^{(3)} \leq \epsilon_2 k^{-1-2\delta} \) and \( |\epsilon_0| < b_0^{(3)} k^{1-\delta} r^{(3)} \), then the equation

\[
\lambda^{(3)}(\vec{y}^{(3)}(\varphi)) = k^2 + \epsilon_0
\]

has no more than two solutions \( \varphi^{(3)}_{\epsilon_0} \) in \( \hat{\Phi}_3 \). For any \( \varphi^{(3)}_{\epsilon_0} \) there is \( \varphi^{(2)}_0 \in \hat{\Phi}_2 \) such that

\[
|\varphi^{(3)}_{\epsilon_0} - \varphi^{(2)}_0| < r^{(3)}/4,
\]

here and below \( \varphi^{(2)}_0 \) is \( \varphi^{(2)}_{\epsilon_0} \) for \( \epsilon_0 = 0 \).

Lemma 4.19. Suppose \( 0 < b_0^{(3)} \leq \epsilon_2 k^{-1-2\delta} \) and \( \varphi \in \hat{\Phi}_3 \) obeys the inequality analogous to (4.38):

\[
|\varphi - \varphi^{(2)}_0| < r^{(3)}.
\]

Then, \( \partial_{\varphi}\lambda^{(3)}(\vec{y}^{(3)}(\varphi)) = k \to \infty \pm 2b_0^{(3)} k (1 + o(1)) \), where \( |o(1)| < 10^{-2} + \epsilon_1 + \epsilon_2 \) when \( k > k_* \).

Definition 4.20. Let \( \Gamma^{(3)}(\vec{b}^{(3)}) \) be the open disks centered at \( \varphi^{(3)}_{\epsilon_0} \) with radius \( r^{(3)} \); \( \gamma^{(3)}_{\epsilon_0}(\vec{b}^{(3)}) \) be their boundary circles and \( \mathcal{O}_s^{(3)}(\vec{b}^{(3)}) = \Gamma^{(3)} + \Gamma^{(3)-} \).

Lemma 4.21. For any \( \varphi \in \hat{\Phi}_3 \setminus \mathcal{O}_s^{(3)}(\vec{b}^{(3)}) \), \( |\lambda^{(3)}(\vec{y}^{(3)}(\varphi)) - k^2| \geq b_0^{(3)} k^{1-\delta} r^{(3)} \).

Lemma 4.22. For any \( \varphi \in \hat{\Phi}_3 \setminus \mathcal{O}_s^{(3)}(\vec{b}^{(3)}) \),

\[
\left\| \left( H^{(3)}(\vec{y}^{(3)}(\varphi)) - k^2 \right)^{-1} \right\| < \frac{16}{b_0^{(3)} r^{(3)} k^{1-\delta}}.
\]

The estimate is stable in the \( (r^{(3)} k^{-\delta}) \)-neighborhood of \( \hat{\Phi}_3 \setminus \mathcal{O}_s^{(3)}(\vec{b}^{(3)}) \). The resolvent is an analytic function of \( \varphi \) in every component of \( \mathcal{O}_s^{(3)}(\vec{b}^{(3)}) \), whose intersection with \( \hat{\Phi}_3 \) is not empty. The only singularities of the resolvent are poles. The resolvent has at most two poles inside \( \mathcal{O}_s^{(3)}(\vec{b}^{(3)}) \).

5. The \( n \)-th Step of Approximation. Swiss Cheese Method.

5.1. Introduction. On the \( n \)-th step, \( n \geq 4 \), we choose \( s_n = 2s_{n-1} \) and define the operator \( H^{(n)}_\alpha \) by the formula:

\[
H^{(n)}_\alpha = H^{(n-1)} + \alpha W_n, \quad 0 \leq \alpha \leq 1, \quad W_n = \sum_{r=M_{n-1}+1}^{M_n} V_r,
\]

where \( M_n \) is chosen in such a way that \( 2^{M_n} \approx k^{s_n} \). Obviously, the periods of \( W_n \) are \( 2^{M_{n-1}}(d_1, 0) \) and \( 2^{M_{n-1}}(0, d_2) \). We write them in the form: \( N_{n-1}, \ldots, N_1(a_1, 0) \) and \( N_{n-1}, \ldots, N_1(0, a_2) \), here \( N_{n-1} = 2^{M_{n-1}-M_n-1} = k^{s_n-s_{n-1}} \). Note that \( N_{n-1} < 4k^{s_{n-1}} < N_{n-1} < 4k^{s_n-s_{n-1}} \).
that \( \|W_n\|_\infty \leq \sum_{r=M_n}^{M_n+1} \|V_r\|_\infty < \exp(-k^{\eta s_n}) \).

By analogy with the definition of \( \epsilon_1, \epsilon_2 \) we introduce the notation \( \epsilon_n = \exp(-k^{\eta s_n}) \).

The \( n \)-th step is analogous to the second step up to replacement the indices 3 by \( n, 3 \) by \( n - 1 \), the product \( N_2N_1 \) by \( N_{n-1} \cdot \cdots \cdot N_1 \), etc.

We note that \( k \), satisfying (3.1), obeys the analogous condition for with any \( s_n \) instead of \( s_1 \)
\[
C_s(1 + s_n)k^{2 + 4s_n} \ln k < k^{\eta s_n} \quad (5.1)
\]
with the same constant \( C_s \). The inequality (5.1) can be obtained from (3.1) by induction. This is an important fact: it means that the lower bound for \( k \) does not grow with \( n \), i.e., all steps hold uniformly in \( k \) for \( k > k_s \), \( k_s \) being introduced by (3.1). Further we assume \( k > k_s \).

The formulation of the geometric lemma for \( n \)-th step is the same as that for Step 2 up to a shift of indices, we skip it here for shortness. Note only that in the lemma we use the set \( \chi_{n-1}^s(\lambda) \) to define \( \chi_n(\lambda) \). In fact, we started with the definition of \( \chi_1(\lambda) \) and then use it to define \( \chi_1^s(\lambda) \) (Step 1). Considering \( \chi_1^s(\lambda) \), we constructed \( \chi_2^s(\lambda) \) (Lemma 3.5) and later used it to define \( \chi_3^s(\lambda) \) (Section 3.3). Using \( \chi_3^s(\lambda) \), we introduced \( \chi_3(\lambda) \) (Lemma 4.1) and then it to define \( \chi_3^s(\lambda) \) (Section 4.4). Thus, the process goes like \( \chi_1 \to \chi_1^s \to \chi_2 \to \chi_2^s \to \chi_3 \to \chi_3^s \). Here we start with the set \( \chi_{n-1}^s(\lambda) \) defined by (4.27) for \( n = 4 \) and by (5.13) for \( n > 4 \). The estimate (4.6) for \( n \)-th step takes the form:
\[
L\left(\frac{K_n\chi_{n-1}^s(\lambda_n)}{\chi_{n-1}^s(\lambda_n)}\right) < k^{-S_n}, \quad S_n = 2\sum_{i=1}^{n-1} (1 + s_i) \quad (5.2)
\]
The formulation of the main results (perturbation formulae) for \( n \)-th step is the same as for the second and third step: Theorems analogous up to the shift of indices to Theorem 3.8 [4.4], Lemma 3.11, Theorem 3.12, Corollary 3.13 and Lemma 3.14 hold.

5.2. Proof of Geometric Lemma. The proof of the lemma is analogous to that for Geometric Lemma in the second step. Let us consider
\[
\mathcal{O}^{(n)} = \bigcup_{\tilde{\mathcal{P}}^{(n-1)} \in \mathcal{P}^{(n-1)} \setminus \{0\}} \mathcal{O}^{(n-1)}_s \left( \frac{2\pi \tilde{p}^{(n-1)}}{N_{n-1} \cdots N_1} \right),
\quad (5.3)
\]
where \( \mathcal{O}_s^{(n-1)} \left( \frac{2\pi \tilde{p}^{(2)}}{N_{n-1} \cdots N_1} \right) \) is defined by Definition 4.14 with \( \tilde{p}^{(3)} = \frac{2\pi \tilde{p}^{(3)}}{N_3N_2} \) when \( n = 4 \). When \( n > 4 \) we use Definition 5.6 with \( n - 1 \) instead of \( n \) and take \( \tilde{p}^{(n-1)} = \frac{2\pi \tilde{p}^{(n-1)}}{N_{n-1} \cdots N_1} \). The radius \( r^{(n-1)} \) of \( \mathcal{O}_s^{(n-1)} \) is defined by the recurrent formula: \( r^{(n-1)} = k^{-2 + 4s_{n-1} - \delta r(n-2)} \). This means
\[
r^{(n-1)} = k^{-2n-4} \sum_{k=1}^{n-1} s_n - 2s_1 - 2\delta = k^{-2n-2} \sum_{k=1}^{n} s_n - 2\delta = k^{-S_{n+1}-2\delta}, \quad (5.4)
\]

\[8\text{Strictly speaking we assume that there is a subset } \chi_{n-1}^s(\lambda) \text{ of the isoenergetic surface } S_{n-1}(\lambda) \text{ of } H^{(n-1)} \text{ such that perturbation series of the type } (4.15), (4.16) \text{ converges for } t^{(n-1)} \in \chi_{n-1}^s(\lambda) \text{ and } \chi_{n-1}^s(\lambda) \text{ has properties described in Section 4.4 up to replacement of 3 by } n - 1. \text{ In particular, we assume that } \chi_{n-1}^s(\lambda) = K_{n-1}D_{n-1}(\lambda), \text{ where } D_{n-1}(\lambda) \text{ satisfies the analog of Lemma 4.8} \text{ and that the analogs of Lemmas 4.9 and 4.10 hold too. Here, } K_{n-1} \text{ is the parallel shift into } K_{n-1}. \text{ Further in this section we describe the next set } \chi_n^s(\lambda) \text{ which has analogous properties.} \]
$S_n$ being defined by (5.2). Note, that the definition make sense, since $|\frac{2\pi p^{(n-1)}}{N_{n-1}...N_1a}| > d_{\max}^{-1}k^{-s_n} > \epsilon_n^{-1}k^{-1-2\delta}$, when $k > k_*$, the estimate (5.1) has been used. The last inequality can be easily proved by induction. Let

$$\Phi_n = \Phi_{n-1} \setminus \Omega^{(n)}, \quad \Theta_n = \Phi_n \cap [0, 2\pi),$$

(5.5)

$\Phi_3, \Theta_3$ are given by (4.8). By Lemmas 4.17, 5.9

$$\left\| \left( H^{(n-1)} \left( \tilde{\varphi}_{n-1}(\varphi) + \frac{2\pi p^{(n-1)}}{N_{n-1}...N_1a} - k^2 \right) \right)^{-1} \right\| \leq \frac{2}{\epsilon_n^{-2}}$$

(5.6)

for all $p^{(n-1)} \in P^{(n-1)} \setminus \{0\}$ and $\varphi \in \Phi_n$, here and below $\Phi_n$ is $p^{(n-1)}k^{-\delta}$-neighborhood of $\Phi_n$. We consider $D_{n-1, \text{nonres}} \subset D_{n-1}$:

$$D_{n-1, \text{nonres}} = \{ \tilde{\varphi}_{n-1}(\varphi), \varphi \in \Theta_n \}. \quad (5.7)$$

We define $\chi_n$ by the formula:

$$\chi_n = K_n D_{n-1, \text{nonres}}. \quad (5.8)$$

By the definition of $K_n$, for every $t^{(n)}$ in $\chi_n$, there are $p^{(1)} \in P^{(1)}, ..., p^{(n-1)} \in P^{(n-1)}$ and $j \in \mathbb{Z}^2$ such that

$$t^{(n)} + \frac{2\pi p^{(n-1)}}{N_{n-1}...N_1a} + \ldots + \frac{2\pi p^{(1)}}{N_1a} + \frac{2\pi j}{a} = \tilde{\varphi}_{n-1}(\varphi), \quad \tilde{\varphi}_{n-1}(\varphi) \in D_{n-1, \text{nonres}}. \quad (5.9)$$

Considering the definition of $D_{n-1, \text{nonres}}$ and (5.6), we obtain:

$$\left\| \left( H^{(n-1)} \left( \tilde{\varphi}^{(n)} + \frac{2\pi p^{(n-1)}}{N_{n-1}...N_1a} - k^2 \right) \right)^{-1} \right\| \leq \frac{2}{\epsilon_n^{-2}}, \quad (5.10)$$

$\hat{p}^{(n-1)} = p^{(n-1)} + \hat{p}^{(n-1)}$. Note that the indices $j, p^{(1)}, ..., p^{(n-1)}$ do not play a role, since they just produce a shift of indices of the matrix elements of the resolvent. Considering that $\hat{p}^{(n-1)}$ can be any but zero, we obtain that (5.10) holds for all $\hat{p}^{(n-1)} \in P^{(n-1)} \setminus \{p^{(n-1)}\}$. Taking into account that $\lambda_j^{(n-1)}(t^{(n)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a) = k^2$ and $\epsilon_n^{-1} < \epsilon_n^{-2}$. We arrive at the analog of (4.5) for all $\hat{p}^{(n-1)} \neq p^{(n-1)}$. This also proves that $\hat{p}^{(n-1)}$ is uniquely defined by (5.9). It remains to check the analog of (4.5) for $p^{(n-1)} = \hat{p}^{(n-1)}$. Let $t^{(n-1)} := t^{(n)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a$. By (4.9), $t^{(n-1)} \in K_{n-1} D_{n-1}$. Using (4.27) for $n = 4$ and (5.15) with $n - 1$ instead of $n$ for $n > 4$, we get $t^{(n-1)} \in \chi_{n-1}$. By the analog of Theorem 4.4 for step $n - 1$, $\lambda_j^{(n-1)}(t^{(n)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a)$ is the only eigenvalue of $H^{(n-1)}(t^{(n)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a)$ in the interval $\epsilon_{n-1}$. Hence,

$$|\lambda_j^{(n-1)}(t^{(3)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a) - \lambda^{(n-1)}_m(t^{(n)} + 2\pi p^{(n-1)}/N_{n-1}...N_1a)| > \epsilon_{n-2}.$$  

Thus, the analog of (4.3) holds for all $p^{(n-1)} \in P^{(n-1)}$. Part 2 holds, since all estimates are stable with respect to the perturbation of $t^{(3)}$ less then $\epsilon_{n-1}k^{-1-\delta}$. Let us estimate the size of $\mathcal{O}_n$. According to Lemma 4.15, the total size of each $\mathcal{O}_s^{(n-1)}(2\pi p^{(n-1)}/N_{n-1}...N_1a)$ is less then $c_0k^{-2s_n-\delta}r^{(n-2)}$. Considering that
the number of $p^{(n-1)}$ does not exceed $k^{2s_n-s_{n-1}} = k^{2s_n-1}$, $s_n = 2s_n-1$, we obtain that the size of $O_n$ is less than $k^{-\delta}r^{(n-2)}$. Using the formula for $p^{(n-2)}$, we obtain that the total size of $O_n$ does not exceed $k^{-2(n-1)-2}\sum_{k=1}^{n-1}s_k$. Using this estimate we easily arrive at (5.2). It is easy to see that $S_n = 2(n - 1) + (2n - 2)s_1$ and $S_n \approx 2^n s_1 \approx s_n$ for large $n$. The lemma is proved.

5.3. Nonresonant part of the isoenergetic set of $H^{(n)}_\alpha$. Now we construct a nonresonance subset $\chi_n^*(\lambda)$ of the isoenergetic surface $S_n(\lambda)$ of $H^{(n)}_\alpha$ in $K_n$, $S_n(\lambda) \subset K_n$. It corresponds to nonresonance eigenvalues given by perturbation series. The sets $\chi_1^*(\lambda), \chi_2^*(\lambda), \chi_3^*(\lambda)$ are defined in the previous steps as well as the non-resonance sets $\chi_1(\lambda), \chi_2(\lambda), \chi_3(\lambda)$. Let us recall that we started with the definition of $\chi_1(\lambda)$ and then use it to define $\chi_1^*(\lambda)$. Considering $\chi_1^*(\lambda)$, we constructed $\chi_2(\lambda)$ (Step 2). Next, we defined $\chi_2^*(\lambda)$. Using $\chi_2^*(\lambda)$, we introduced $\chi_3(\lambda)$ and, then $\chi_3^*(\lambda)$. Thus, the process looks like $\chi_1 \rightarrow \chi_1^* \rightarrow \chi_2 \rightarrow \chi_2^* \rightarrow \chi_3 \rightarrow \chi_3^*$. The geometric lemma in this section gives us $\chi_1$ and every next $\chi_n$ if $\chi_n^*(\lambda)$ is defined. To ensure the recurrent procedure we show now how to define $\chi_n^*(\lambda)$ using $\chi_n(\lambda)$.

We define $B_n$ as the set of directions corresponding to $\Theta_n$:

$$B_n(\lambda) = \{ \vec{v} \in S_1 : \varphi \in \Theta_n \}.$$ 

Note that $B_n$ is a unit circle with holes centered at the origin and $B_n(\lambda) \subset B_n(\lambda)$. We define $D_n(\lambda)$ as a level set for $\lambda^{(n)}(\alpha, \vec{z})$ in the $(\epsilon_n k^{-1-\delta})$-neighborhood of $D_{n-1,\text{nonres}}(\lambda)$:

$$D_n(\lambda) = \left\{ \vec{z} = \lambda \vec{v} : \vec{v} \in B_n, |\lambda - \lambda_n-1(\lambda, \vec{v})| < \epsilon_n k^{-1-\delta}, \lambda^{(n)}(\alpha, \vec{z}) = \lambda \right\}.$$ 

Considering as in the previous step, we prove the analogs of Lemmas 4.7 and 4.8. For shortness, we provide here only the second lemma. By analogy with previous sections, we shorten notations here: $\lambda_n-1(\varphi) \equiv \lambda_n-1(\lambda, \vec{v}), \vec{z}_n-1(\varphi) \equiv \lambda_n-1(\lambda, \vec{v})\vec{v}$.

**Lemma 5.1.** For $\lambda > k^2$:

1. The set $D_n(\lambda)$ is a distorted circle with holes: it can be described by the formula:

$$D_n(\lambda) = \{ \vec{z} \in \mathbb{R}^2 : \vec{z} = \lambda \vec{v}, \varphi \in \Theta_n(\lambda) \},$$

where $\varphi(\varphi) = \lambda_n-1(\varphi) + h_n(\varphi)$, and $h_n(\varphi)$ satisfies the estimates

$$|h_n| < 12\alpha e_n k^{-1}, \quad \left| \frac{\partial h_n}{\partial \varphi} \right| \leq 4\alpha e_n^{3} k^{1+\delta}.$$ (5.12)

2. The total length of $B_n(\lambda)$ satisfies the estimate:

$$L(B_n-1 \setminus B_n) < 4\pi k^{-S_n}.$$ (5.13)

3. Function $\varphi_n(\varphi)$ can be extended as a holomorphic function of $\varphi$ to $\Phi_n$, estimates (5.12) being preserved.

4. The curve $D_n(\lambda)$ has a length which is asymptotically close to that of $D_{n-1}(\lambda)$ in the following sense:

$$L(D_n(\lambda)) \approx L(D_{n-1}(\lambda)) \left( 1 + O(k^{-S_n}) \right).$$ (5.14)
Now define the nonresonance set, $\chi^*_n(\lambda)$ in $S_n(\lambda)$ by the formula analogous to (4.27). Indeed,

$$\chi^*_n(\lambda) := K_nD_n(\lambda).$$

(5.15)

The following lemmas are analogous to Lemmas 4.9 and 4.10.

**Lemma 5.2.** The set $\chi^*_n(\lambda)$ belongs to the $(2\alpha \epsilon_{n-1}^4 k^{-1})$-neighborhood of $\chi_n(\lambda)$ in $K_n$. If $t^{(n)} \in \chi^*_n(\lambda)$, then the operator $H^{(n)}_\alpha(t^{(n)})$ has a simple eigenvalue equal to $\lambda$. This eigenvalue is given by the perturbation series analogous to (4.15).

**Lemma 5.3.** Formula (5.15) establishes one-to-one correspondence between $\chi^*_n(\lambda)$ and $D_n(\lambda)$.

### 5.4. Preparation for the Next Approximation

Let $\vec{b}^{(n)} \in K_n$ and $b_0^{(n)}$ be the distance of the point $\vec{b}^{(n)}$ to the nearest corner of $K_n$. We assume $b_0^{(n)} = |\vec{b}^{(n)}|$. We consider two cases: $b_0^{(n)} \geq \epsilon_{n-1} k^{-1-2\delta}$ and $0 < b_0^{(n)} < \epsilon_{n-1} k^{-1-2\delta}$. Let

$$y^{(n-1)}(\varphi) = \vec{z}_{n-1}(\varphi) + \vec{b}^{(n)}.$$

(5.16)

5.4.1. The case $b_0^{(n)} \geq \epsilon_{n-1} k^{-1-2\delta}$.

**Definition 5.4.** We define the set $O^{(n)}(\vec{b}^{(n)})$ by the formula:

$$O^{(n)}(\vec{b}^{(n)}) = \bigcup_{\vec{b}^{(n-1)} \in P^{(n-1)} \setminus O^{(n-1)}} \left( 2\pi p^{(n-1)}/N_{n-1}...N_1 a + \vec{b}^{(n)} \right),$$

(5.17)

set $O^{(n-1)}$ being defined by Definition 5.4 for $n = 4$ and by 5.6 for $n > 4$.

The set $O^{(n)}(\vec{b}^{(n)})$ consists the disks with the radius $r^{(n-1)} = r^{(n-2)} k^{2-4s_{n-1}-\delta}$ centered at poles of the resolvent $\left( \vec{H}_{n-1}(\vec{y}^{(n-1)}(\varphi)) - k^2 \right)^{-1}$.

**Lemma 5.5.** If $\varphi \in \Phi_{n-1} \setminus O^{(n)}(\vec{b}^{(n)})$, then

$$\left\| \left( \vec{H}^{(n-1)}(\vec{y}^{(n-1)}(\varphi)) - k^2 \right)^{-1} \right\| \leq \frac{17 r^{(n-1)} k^{3\delta}}{\epsilon_{n-1}}$$

(5.18)

The estimate is stable in the $(r^{(n-1)} k^{-\delta})$-neighborhood of $\Phi_{n-1} \setminus O^{(n)}(\vec{b}^{(n)})$. The resolvent is an analytic function of $\varphi$ in every connected component $O^{(n)}(\vec{b}^{(n)})$ of $O^{(n)}(\vec{b}^{(n)})$, whose intersection with $\Phi_{n-1}$ is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside $O^{(n)}(\vec{b}^{(n)})$ is less than $ck^{2+2s_n}$. The total size of $O^{(n)}(\vec{b}^{(n)})$ does not exceed $c_0 k^{-S_n}$.

**Proof.** The proof is analogous to that of Lemma 3.22. Let $n = 4$. To obtain (5.18) we combine Lemmas 4.17 and 4.22 for $\vec{y}^{(n-1)} = 2\pi p^{(n-1)}/N_{n-1}...N_1 a + \vec{b}^{(n)}$ taking into account $\epsilon_{n-2}^{\prime} > \epsilon_{n-1}$ and the estimate for $b_0^{(n)}$. If $n > 4$, then using the recurrent procedure, we apply Lemmas 5.9 and 5.14 with $n - 1$ instead of $n$. Moreover, the number of poles of $\left( \vec{H}^{(n-1)}(\vec{z}_{n-1}(\varphi)) + \vec{b}^{(n-1)} - k^2 \right)^{-1}$ inside $O^{(n)}(\vec{b}^{(n)})$ is less than $ck^{2+2s_n-1}$. Considering that the number of $p^{(n-1)} \in P^{(n-1)}$ does not exceed
\[ c k^{2(s_n-s_{n-1})}, s_n = 2s_{n-1}, \]

we obtain the estimate for the number of poles for the resolvent \( \left( \tilde{H}^{(n-1)}(\tilde{g}^{(n-1)}(\varphi)) - k^2 \right)^{-1} \). We estimate the size of \( O^{(n)}(\tilde{b}^{(n)}) \) the same way as we estimated the size of \( O^{(n)} \).

**Definition 5.6.** We denote the poles of the resolvent \( \left( \tilde{H}^{(n-1)}(\tilde{g}^{(n-1)}((\varphi)) - k^2 \right)^{-1} \)

in \( O^{(n)}(\tilde{b}^{(n)}) \) as \( \varphi_m^{(n)}, m = 1, \ldots, M^{(n)} M^{(n)} < c_0 k^{2+2s_n} \). Let us consider the circles \( O_m^{(n)}(\tilde{b}^{(n)}) \) of the radius \( r^{(n)} = k^{-2-4s_n} r^{(n-1)} \) around these poles. Let

\[
O^{(n)}_s(\tilde{b}^{(n)}) = \bigcup_{m=1}^{M^{(n)}} O_m^{(n)}(\tilde{b}^{(n)}).
\]

**Lemma 5.7.** The total size of \( O^{(n)}_s(\tilde{b}^{(n)}) \) is less then \( c_0 k^{-2s_n} r^{(n-1)} \).

**Proof.** The lemma easily follows from the formula \( r^{(n)} = k^{-2-4s_n} r^{(n-1)} \) and the estimate \( M^{(n)} < c_0 k^{2+2s_n} \).

**Lemma 5.8.** If \( \varphi \in \Phi_{n-1} \setminus O^{(n)}_s(\tilde{b}^{(n)}) \), then

\[
\left\| \left( \tilde{H}^{(n-1)}(\tilde{g}^{(n-1)}((\varphi)) - k^2 \right)^{-1} \right\| \leq \frac{1}{\epsilon^{2n_1}}
\]

The estimate is stable in the \((r^{(n)}k^{-\delta})\)-neighborhood of \( \Phi_{n-1} \setminus O^{(n)}_s(\tilde{b}^{(n)}) \). The resolvent is an analytic function of \( \varphi \) in every connected component \( O^{(n)}_s(\tilde{b}^{(n)}) \) of \( O^{(n)}_s(\tilde{b}^{(n)}) \), whose intersection with \( \Phi_{n-1} \) is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside \( O^{(n)}_s(\tilde{b}^{(n)}) \) is less than \( c k^{2+2s_n} \).

**Proof.** By the definition of \( O^{(n)}_s(\tilde{b}^{(n)}) \), the number of poles (counting multiplicity) of the resolvent inside this set is less than \( M^{(n)} = c_0 k^{2+2s_n} \). Considering as in Lemma 2.20 we obtain

\[
\left\| \left( \tilde{H}^{(n-1)}(\tilde{g}^{(n-1)}((\varphi)) - k^2 \right)^{-1} \right\| \leq \nu^{M^{(n)}(1717 r^{(n-1)} k^{3\delta})},
\]

where \( \nu \) is the coefficient of contraction, when we reduce \( O^{(n)}(\tilde{b}^{(n)}) \) to \( O^{(n)}(\tilde{b}^{(n)}) \). Namely \( \nu \) is the ratio of \( r^{(n)} \) to the maximal size of \( O^{(n)}(\tilde{b}^{(n)}) \). By Lemma 5.6 and formula (5.4), \( \nu = k^{-4-4s_n-4s_{n-1}} \). Considering that \( r^{(n-1)} k^\delta < 1 \) and using (5.1), we obtain \( \nu^{M^{(n)}(1717 r^{(n-1)} k^{3\delta})} < \epsilon^{-1} \) when \( k > k_\ast \). Thus, we obtain (5.20).

By Lemma 5.7 the total size of \( O^{(n)}_s(\tilde{b}^{(n)}) \) is less then \( r^{(n-1)} \). Therefore, the function \( \tilde{z}_n(\varphi) \) is holomorphic inside each connected component of \( O^{(n)}_s(\tilde{b}^{(n)}) \) which has non-empty intersection with \( \Phi_n \). Let

\[
\tilde{y}^{(n)}(\varphi) = \tilde{z}_n(\varphi) + \tilde{b}^{(n)}.
\]

**Lemma 5.9.** If \( \varphi \in \Phi_n \setminus O^{(n)}_s(\tilde{b}^{(n)}) \), then

\[
\left\| \left( H^{(n)}(\tilde{y}^{(n)}(\varphi)) - k^2 \right)^{-1} \right\| \leq \frac{2}{\epsilon^{2n_1}}
\]
Lemma 5.14. For any \( r \in O \) at most two poles inside \( \Phi \).

Lemma 5.13. For any \( \Gamma \),

Definition 5.12.

The estimate is stable in the \((r^{(n)}k^{-\delta})\)-neighborhood of \( \hat{\Phi}_n \setminus O^{(n)}_s(\tilde{b}^{(n)}) \). The resolvent is an analytic function of \( \varphi \) in every connected component \( O^{(n)}_s(\tilde{b}^{(n)}) \) of \( O^{(n)}(\tilde{b}^{(n)}) \), whose intersection with \( \Phi_n \) is not empty. The only singularities of the resolvent in such a component are poles. The number of poles (counting multiplicity) of the resolvent inside \( O^{(n)}(\tilde{b}^{(n)}) \) is less than \( c_0 k^{2+2s_n} \).

Proof. The proof is analogous to that of Lemma 3.26 up to the shift of indices by two. We use Lemma 5.8, \[ \|W_n\|_\infty < \epsilon_n^{4}, \] and the first estimate in (5.12). \( \square \)

5.4.2. The set \( O^{(n)}_s(\tilde{b}^{(n)}) \) for small \( \tilde{b}^{(n)} \). The considerations of this section are analogous to those of By analogy with the previous subsection, we choose \( r^{(n)} = k^{-2-2s_n-\delta r^{(n)}-1} \) here. The following Lemmas 5.10 and 5.11 are identical to 4.18 and 4.19 up to replacement of indices (3) by (n), (2) by \((n-1)\), etc. Next, Definition 5.12 is analogous to 4.20. Lemmas 5.13, 5.14 are analogous to 4.21, 4.22.

**Lemma 5.10.** If \( 0 < b_{0}^{(n)} \leq \epsilon_{n-1}^{-1-2s_n} \) and \( |\epsilon_0| < b_{0}^{(n)} k^{1-\delta r^{(n)}} \), then the equation

\[
\lambda^{(n)}\left(\gamma^{(n)}(\varphi)\right) = k^2 + \epsilon_0
\]

has no more than two solutions \( \varphi^{\pm}(n) \) in \( \hat{\Phi}_n \). For any \( \varphi_{\epsilon_0}^{\pm}(n) \) there is \( \varphi_{0}^{\pm}(n-1) \in \hat{\Phi}_{n-1} \) such that

\[
\left| \varphi_{\epsilon_0}^{\pm}(n) - \varphi_{0}^{\pm}(n-1) \right| < r^{(n)}/4,
\]

here and below \( \varphi_{0}^{\pm}(n-1) \) is \( \varphi_{\epsilon_0}^{\pm}(n-1) \) for \( \epsilon_0 = 0 \).

**Lemma 5.11.** Suppose \( 0 < b_{0}^{(n)} \leq \epsilon_{n-1}^{-1-2s_n} \) and \( \varphi \in \hat{\Phi}_n \) obeys the inequality analogous to (5.25):

\[
\left| 2k + 2b_{0}^{(n)} \left( 1 + o(1) \right) \right| < r^{(n)}.\]

Then, \( \frac{\partial}{\partial \varphi} \lambda^{(n)}(\gamma^{(n)}(\varphi)) \equiv_{k \to \infty} \pm 2b_{0}^{(n)} k \left( 1 + o(1) \right) \), where \( |o(1)| < 10^{-2} + \epsilon_1 + \ldots + \epsilon_{n-1} \) when \( k > k_s \).

**Definition 5.12.** Let \( \Gamma^{(n)} \) be the open disks centered at \( \varphi_{0}^{\pm}(n) \) with radius \( r^{(n)} \); \( \gamma^{(n)}(\tilde{b}^{(n)}) \) be their boundary circles and \( O^{(n)}_s(\tilde{b}^{(n)}/1) \) be their intersection with \( \Phi_n \).

**Lemma 5.13.** For any \( \varphi \in \hat{\Phi}_n \cup O^{(n)}_s(\tilde{b}^{(n)}) \),

\[
|\lambda^{(n)}(\gamma^{(n)}(\varphi)) - k^2| \geq b_{0}^{(n)} k^{1-\delta r^{(n)}}.
\]

**Lemma 5.14.** For any \( \varphi \in \hat{\Phi}_n \setminus O^{(n)}_s(\tilde{b}^{(n)}) \),

\[
\left\| \left( H^{(n)}(\gamma^{(n)}(\varphi)) - k^2 \right)^{-1} \right\| < \frac{16}{b_{0}^{(n)} r^{(n)} k^{1-\delta}},
\]

The estimate is stable in the \((r^{(n)}k^{-\delta})\)-neighborhood of \( \hat{\Phi}_n \setminus O^{(n)}_s(\tilde{b}^{(n)}) \). The resolvent is an analytic function of \( \varphi \) in every component of \( O^{(n)}_s(\tilde{b}^{(n)}) \), whose intersection with \( \Phi_n \) is not empty. The only singularities of the resolvent are poles. The resolvent has at most two poles inside \( O^{(n)}_s(\tilde{b}^{(n)}) \).
6. Limit-Isoenergetic Set and Eigenfunctions

6.1. Limit-Isoenergetic Set and Proof of the Bethe-Sommerfeld Conjecture. At every step \( n \) we constructed a subset \( B_n(\lambda) \) of the unit circle, and a function \( \varphi_n(\lambda, \vec{\nu}) \), \( \vec{\nu} \in B_n(\lambda) \), with the following properties. The sequence \( B_n(\lambda) \) is decreasing: \( B_n(\lambda) \subset B_{n-1}(\lambda) \). The set \( D_n(\lambda) \) of vectors \( \vec{x} = \varphi_n(\lambda, \vec{\nu}) \vec{\nu}, \vec{\nu} \in B_n(\lambda) \), is a slightly distorted circle with holes, see Fig1, Fig2, formula (1.15) and Lemmas 2.10, 3.17, 4.8, 5.1. For any \( \vec{z}_n(\lambda, \vec{\nu}) \in D_n(\lambda) \) there is a simple eigenvalue of \( H^{(n)}(\vec{z}_n) \) equal to \( \lambda \) and given by a perturbation series. \(^9\) Let \( B_\infty(\lambda) = \bigcap_{n=1}^\infty B_n(\lambda) \). Since \( B_{n+1} \subset B_n \) for every \( n \), \( B_\infty(\lambda) \) is a unit circle with the infinite number of holes, more and more holes of smaller and smaller size appearing at each step.

**Lemma 6.1.** The length of \( B_\infty(\lambda) \) satisfies estimate (1.1) with \( \gamma_3 = \delta \).

**Proof.** Using (2.72), (3.53), (4.21) and (5.13) and considering that \( S_n \approx 2^n s_1 \), we easily conclude that \( L(B_n) = \left(1 + O(k^{-\delta}) \right), k = \lambda^{1/2} \) uniformly in \( n \). Since \( B_n \) is a decreasing sequence of sets, (1.9) holds.

Let us consider \( \varphi_\infty(\lambda, \vec{\nu}) = \lim_{n \to \infty} \varphi_n(\lambda, \vec{\nu}), \vec{\nu} \in B_\infty(\lambda) \).

**Lemma 6.2.** The limit \( \varphi_\infty(\lambda, \vec{\nu}) \) exists for any \( \vec{\nu} \in B_\infty(\lambda) \) and the following estimates hold when \( n \geq 1 \):

\[
|\varphi_\infty(\lambda, \vec{\nu}) - \varphi_n(\lambda, \vec{\nu})| < 14\epsilon_n^4 k^{-1}, \quad \epsilon_n = \exp\left(-\frac{1}{4}k^{n/s_1}\right), \quad s_n = 2^{n-1}s_1. \tag{6.1}
\]

**Corollary 6.3.** For every \( \vec{\nu} \in B_\infty(\lambda) \) estimate (1.10) holds, where \( \gamma_4 = 3 - 30s_1 - 20\delta > 0 \).

**Proof.** The lemma easily follows from the estimates (3.52), (4.23) and (5.12). To obtain corollary we use (2.70).

Estimates (3.52), (4.23) and (5.12) justify convergence of the series \( \sum_{m=1}^{\infty} \frac{\partial h}{\partial \varphi} \), and hence, of the sequence \( \frac{\partial \varphi_\infty}{\partial \varphi} \). We denote the limit of this sequence by \( \frac{\partial \varphi_\infty}{\partial \varphi} \).

**Lemma 6.4.** The estimate (1.18) with \( \gamma_5 = 1 - 33s_1 - 22\delta > 0 \) holds for any \( \vec{\nu} \in B_\infty(\lambda) \).

**Proof.** The lemma easily follows from (2.70), (3.52), (4.23) and (5.12).

We define \( D_\infty(\lambda) \) by (1.8). Clearly, \( D_\infty(\lambda) \) is a slightly distorted circle of radius \( k \) with the infinite number of holes. We can assign a tangent vector \( \frac{\partial \varphi_\infty}{\partial \varphi} \vec{\nu} + \varphi_\infty \vec{\mu} \), \( \vec{\mu} = (-\sin \varphi, \cos \varphi) \) to the curve \( D_\infty(\lambda) \), this tangent vector being the limit of corresponding tangent vectors for curves \( D_n(\lambda) \) at points \( \vec{z}_n(\lambda, \vec{\nu}) \) as \( n \to \infty \).

**Remark 6.5.** We easily see from (6.1), that any \( \vec{z} \in D_\infty(\lambda) \) belongs to the \( (14\epsilon_n^4 k^{-1}) \)-neighborhood of \( D_\infty(\lambda) \). Applying perturbation formulae for \( n \)-th step, we easily obtain that there is an eigenvalue \( \lambda^{(n)}(\vec{z}) \) of \( H^{(n)}(\vec{z}) \) satisfying the estimate

\(^9\)The operator \( H^{(n)}(\vec{z}) \) is defined for every \( \vec{z} \in \mathbb{R}^2 \). The perturbation series is given by a formula analogous to (4.14), which coincides with (4.29) up to a shift of indices corresponding to the parallel shift of \( \vec{z} \) into \( K_n \).
\[ \lambda^{(n)}(\vec{z}) = \lambda + \delta_n, \quad \delta_n = O\left(\epsilon_n^4\right), \] the eigenvalue \( \lambda^{(n)}(\vec{z}) \) being given by a perturbation series of the type (3.44). Hence, for every \( \vec{z} \in \mathcal{D}_\infty(\lambda) \) there is a limit:
\[ \lim_{n \to \infty} \lambda^{(n)}(\vec{z}) = \lambda. \] (6.2)

**Theorem 6.6** (Bethe-Sommerfeld Conjecture). The spectrum of operator \( H \) contains a semi-axis.

**Proof.** By Remark 6.5, there is a point of the spectrum of \( H_n \) in the \( \delta_n \)-neighborhood of \( \lambda \) for every \( \lambda > k_s^2 \), \( k_s \) being introduced by (3.1). Since \( \|H_n - H\| < \epsilon_n^4 \), there is a point of the spectrum of \( H \) in the \( \delta_s^* \)-neighborhood of \( \lambda \), \( \delta_s^* = \delta_n + \epsilon_n^4 \). Since it is true for every \( n \) and the spectrum of \( H \) is closed, \( \lambda \) is in the spectrum of \( H \). \( \blacksquare \)

### 6.2. Generalized Eigenfunctions of \( H \)

A plane wave is usually denoted by \( e^{i(\vec{k} \cdot x)} \), \( \vec{k} \in \mathbb{R}^2 \). Here we use \( \vec{z} \) instead of \( \vec{k} \) to comply with our previous notations. We show that for every \( \vec{z} \) in a set
\[ \mathcal{G}_\infty = \bigcup_{\lambda \geq \lambda_c} \mathcal{D}_\infty(\lambda), \quad \lambda_c = k_s^2, \] (6.3)
there is a solution \( \Psi_\infty(\vec{z}, x) \) of the equation for eigenfunctions:
\[ -\Delta \Psi_\infty(\vec{z}, x) + V(\vec{z}) \Psi_\infty(\vec{z}, x) = \lambda_\infty(\vec{z}) \Psi_\infty(\vec{z}, x), \] (6.4)
which can be represented in the form
\[ \Psi_\infty(\vec{z}, x) = e^{i(\vec{z} \cdot x)} \left(1 + u_\infty(\vec{z}, x)\right), \quad \|u_\infty(\vec{z}, x)\|_{L_\infty(\mathbb{R}^2)} < c(V) |\vec{z}|^{-\gamma_1}, \] (6.5)
where \( u_\infty(\vec{z}, x) \) is a limit-periodic function, \( \gamma_1 = 1/2 - 15s_1 - 8\delta \); the eigenvalue \( \lambda_\infty(\vec{z}) \) satisfies the asymptotic formula:
\[ \lambda_\infty(\vec{z}) = |\vec{z}|^2 + O(|\vec{z}|^{-\gamma_2}), \quad \gamma_2 = 2 - 30s_1 - 20\delta. \] (6.6)

We also show that the set \( \mathcal{G}_\infty \) satisfies (3.7).

In fact, by (6.1), any \( \vec{z} \in \mathcal{D}_\infty(\lambda) \) belongs to the \( (\epsilon_n k_s^{-1} - \delta) \)-neighborhood of \( \mathcal{D}_n(\lambda) \). Applying (2.29), (2.30) with \( 2\beta = 2\beta_0 = 1 - 15s_1 - 9\delta \) and the perturbation formulae proved for next steps, we obtain the following inequalities:
\[ \|E^{(1)}(\vec{z}) - E^{(0)}(\vec{z})\|_1 < c \|W_1\| k_s^{-1/2 + 12s_1 + 8\delta}, \quad \|E^{(n+1)}(\vec{z}) - \tilde{E}^{(n)}(\vec{z})\|_1 < 48 \epsilon_n^{3n}, \quad n \geq 1, \] (6.7)
\[ |\lambda^{(1)}(\vec{z}) - |\vec{z}|^2| < C(W_1) k_s^{-2 + 30s_1 + 20\delta}, \quad |\lambda^{(n+1)}(\vec{z}) - \lambda^{(n)}(\vec{z})| < 12 \epsilon_n^{4n}, \quad n \geq 1, \] (6.8)
where \( E^{(n+1)}, \tilde{E}^{(n)} \) are one-dimensional spectral projectors in \( L_2(Q_{n+1}) \) corresponding to potentials \( W_{n+1} \) and \( W_n \), respectively; \( \lambda^{(n+1)}(\vec{z}) \) is the eigenvalue corresponding to \( E^{(n+1)}(\vec{z}) \), \( E^{(0)}(\vec{z}) \) corresponds to \( V = 0 \) and the periods \( a_1, a_2 \). This means that for properly chosen eigenfunctions \( \Psi_{n+1}(\vec{z}, x) \):
\[ \|\Psi_1 - \Psi_0\|_{L_2(Q_1)} < c \|W_1\| k_s^{-1/2 + 12s_1 + 8\delta} |Q_1|^{1/2}, \quad \Psi_0(x) = e^{i(\vec{z} \cdot x)}, \] (6.9)
\[ \|\Psi_{n+1} - \tilde{\Psi}_n\|_{L_2(Q_{n+1})} < 100 \epsilon_n^{3n} |Q_{n+1}|^{1/2}, \] (6.10)
where \( \tilde{\Psi}_n \) is \( \Psi_n \) extended quasi-periodically from \( Q_n \) to \( Q_{n+1} \). Eigenfunctions \( \Psi_n, n \geq 1 \), are chosen to obey two conditions. First, \( \|\Psi_n\|_{L_2(Q_n)} = |Q_n|^{1/2}; \) second

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\[ ^{10} \] The condition \( \|\Psi_n\|_{L_2(Q_n)} = |Q_n|^{1/2} \) implies \( \|\Psi_n\|_{L_2(Q_{n+1})} = |Q_{n+1}|^{1/2} \).
(Ψ_n, ˜Ψ_{n-1})_n > 0, here (·, ·)_n is an inner product in L_2(Q_n). These two conditions, obviously, provide a unique choice of each Ψ_n. Considering that Ψ_{n+1} and Ψ_n satisfy equations for eigenfunctions and taking into account (6.8), (6.10) we obtain:

\[ \|\Psi_{n+1} - \Psi_n\|_{L_2(Q_{n+1})} < 2k^2\epsilon_n^3|Q_{n+1}|^{1/2}, \ n \geq 1, \]

and, hence, \( \|\Psi_{n+1} - \Psi_n\|_{L_\infty(Q_{n+1})} < 6k^2\epsilon_n^3|Q_{n+1}|^{1/2} \). Since Ψ_{n+1} and Ψ_n obey the same quasiperiodic conditions, the same inequality holds in the whole space \( \mathbb{R}^2 \):

\[ \|\Psi_{n+1} - \Psi_n\|_{L_\infty(\mathbb{R}^2)} < 6k^2\epsilon_n^3|Q_{n+1}|^{1/2}, \ n \geq 1, \]  

(6.11)

where Ψ_{n+1}, Ψ_n are quasiperiodically extended to \( \mathbb{R}^2 \). Obviously, we have a Cauchy sequence in \( L_\infty(\mathbb{R}^2) \). Let Ψ_∞( ˜x, x) = lim_n→∞ Ψ_n( ˜x, x). This limit is defined pointwise uniformly in \( x \) and in \( W_{2,loc}^2(\mathbb{R}^2) \).

**Theorem 6.7.** For every sufficiently large \( \lambda, \lambda > \lambda_0(\sum_{r=1}^\infty \|V_r\|, \delta) \), and \( \tilde{x} \in D_\infty(\lambda) \) the sequence of functions \( \Psi_n(\tilde{x}, x) \) converges in \( L_\infty(\mathbb{R}^2) \) and \( W_{2,loc}^2(\mathbb{R}^2) \). The limit function \( \Psi_\infty(\tilde{x}, x) \), \( \Psi_\infty(\tilde{x}, x) = \lim_n \Psi_n(\tilde{x}, x) \), satisfies the equation

\[ -\Delta \Psi_\infty(\tilde{x}, x) + V(x)\Psi_\infty(\tilde{x}, x) = \lambda\Psi_\infty(\tilde{x}, x). \]  

(6.12)

It can be represented in the form

\[ \Psi_\infty(\tilde{x}, x) = e^{i(\tilde{x}, x)}(1 + u_\infty(\tilde{x}, x)), \]  

(6.13)

where \( u_\infty(\tilde{x}, x) \) is a limit-periodic function:

\[ u_\infty(\tilde{x}, x) = \sum_{n=1}^\infty \tilde{u}_n(\tilde{x}, x), \]  

(6.14)

\( \tilde{u}_n(\tilde{x}, x) \) being a periodic function with the periods \( 2^{M_n-1}d_1, 2^{M_n-1}d_2 \) with \( 2^{M_n} \approx k^{2n-1}s_1 \),

\[ \|\tilde{u}_1\|_{L_\infty(\mathbb{R}^2)} < c(V)k^{-\gamma_1}, \ \gamma_1 = 1/2 - 15s_1 - 8\delta, \]  

(6.15)

\[ \|\tilde{u}_n\|_{L_\infty(\mathbb{R}^2)} < 6k^2\epsilon_n^3|Q_n|^{1/2}, \ n \geq 2. \]  

(6.16)

**Corollary 6.8.** Function \( u_\infty(\tilde{x}, x) \) obeys the estimate (6.7).

**Proof.** Let us show that Ψ_∞ is a limit-periodic function. Obviously, \( \Psi_\infty = \Psi_0 + \sum_{n=0}^\infty (\Psi_{n+1} - \Psi_n) \), the series converging in \( L_\infty(\mathbb{R}^2) \) by (6.11). Introducing the notation \( \tilde{u}_{n+1} = e^{-i(\tilde{x}, x)}(\Psi_{n+1} - \Psi_n) \), we arrive at (6.13), (6.14). Note that \( \tilde{u}_n \) is periodic with the periods \( 2^{M_n-1}d_1, 2^{M_n-1}d_2 \). Estimate (6.16) follows from (6.11).

We check (6.15). Indeed, by (6.9), Fourier coefficients \( (\tilde{u}_1)_m, m \in \mathbb{Z}^2 \), satisfy the estimate \( |(\tilde{u}_1)_m| < c(V)k^{-1/2+12s_1+8\delta}|Q_1|^{1/2} < c(V)k^{-1/2+13s_1+8\delta} \). We will use this estimate for \( m : |m - j| < k^{s_1} \). Next, we obtain a stronger estimate for other \( m \)-s. Indeed, the inequality \( |E^{(1)}(x)|_m < (c(V)k^{-1/2-12s_1-7\delta})|m-j|R_0^{-1} \) follows from (2.25) and (2.28). Hence, similar estimates holds for Fourier coefficients of \( \Psi_1: |(\tilde{u}_1)_m| < (c(V)k^{-1/2-12s_1-7\delta})|m-j|R_0^{-1} \). Summarizing the inequalities, we obtain that (6.15) holds. It remains to prove (6.12). Indeed, \( \Psi_n(\tilde{x}, x), n \geq 1 \), satisfy equations for eigenfunctions: \( H_\lambda^0\Psi_n = \lambda^{(n)}(\tilde{x})\Psi_n \). Considering that \( \Psi_n(\tilde{x}, x) \) converges to \( \Psi(\tilde{x}, x) \) in \( W_{2,loc}^2(\mathbb{R}^2) \) and relation (6.2), we arrive at (6.12).
Theorem 6.9. Formulae (6.4), (6.5) and (6.6) hold for every $\vec{z} \in G_\infty$. The set $G_\infty$ is Lebesgue measurable and satisfies (1.7) with $\gamma_3 = \delta$.

Proof. By Theorem 6.7, (6.4), (6.5) hold, where $\lambda_\infty(\vec{z}) = \lambda$ for $\vec{z} \in D_\infty(\lambda)$. Using (1.10), which is proven in Corollary 6.3 with $\varphi_\infty = |\vec{z}|$, we easily obtain (6.6). It remains to prove (1.7). Let us consider a small region $U_n(\lambda_0)$ around an isoenergetic surface $D_n(\lambda_0)$, $\lambda_0 > k^2_\infty$. Namely, $U_n(\lambda_0) = \cup_{|\lambda - \lambda_0| < \rho_n} D_n(\lambda)$, $\rho_n = \epsilon_n^{-1} k^{-28}$, $k = \lambda^{1/2}$. Considering an estimate of the type (2.62) for $\lambda(\vec{z})$, which holds in the $(\epsilon_{n-1} k^{-1-28})$-neighborhood of $D_n(\lambda_0)$, we see that $U_n(\lambda_0)$ is an open set (a distorted ring with holes) and the length of the ring is of order $\epsilon_n^{-1} k^{-1-28}$. By Part 4 of Lemmas 2.10, 3.17, 4.8, 5.1, the length of $D_n(\lambda_0)$ is $2\pi k (1 + O(k^{-8}))$. Hence, $|U_n(\lambda_0)| = 2\pi k \rho_n (1 + O(k^{-8}))$. It easily follows from Lemma 5.1 that $U_{n+1} \subset U_n$. Definition of $D_\infty(\lambda_0)$ yield: $D_\infty(\lambda_0) = \cap_0^\infty U_n(\lambda_0)$. Hence, $G_\infty = \cap_0^\infty G_n$, where

$$G_n = \cup_{\lambda_0 > \lambda_n} \cup_{\lambda > \lambda_0 - \rho_n(\lambda)} D_n(\lambda).$$

(6.17)

Considering that $U_{n+1} \subset U_n$ for every $\lambda_0 \geq \lambda$, we obtain $G_{n+1} \subset G_n$. Hence, $|G_\infty \cap B| = \lim_{n \to \infty} |G_n \cap B|$. Summarizing volumes of the regions $U_n$, we easily conclude $|G_n \cap B| = |B| (1 + O(R^{-\delta}))$ uniformly in $n$. Thus, we have obtained (1.7) with $\gamma_3 = \delta$.

7. Proof of Absolute Continuity of the Spectrum

The proof is completely analogous to that for the case $l \geq 6$. Here we give only the list of lemmas and the main result. For details, see [24], [26].

7.1. Projections $E_n(G_n')$, $G_n' \subset G_n$. Let us consider the open sets $G_n$ given by (6.17). There is a family of Bloch eigenfunctions $\Psi_n(\vec{z}, x)$, $\vec{z} \in G_n$, of the operator $H^{(n)}$, which are described by the perturbation formulas (1.12). Let $G_n'$ be a Lebesgue measurable subset of $G_n$. We consider the spectral projection $E_n(G_n')$ of $H^{(n)}$, corresponding to functions $\Psi_n(\vec{z}, x)$, $\vec{z} \in G_n'$. By [26], $E_n(G_n') : L_2(\mathbb{R}^2) \to L_2(\mathbb{R}^2)$ can be presented by the formula:

$$E_n(G_n') F = \frac{1}{4\pi^2} \int_{G_n'} (F, \Psi_n(\vec{z})) \Psi_n(\vec{z}) d\vec{z}$$

(7.1)

for any $F \in C_0^\infty(\mathbb{R}^2)$, here and below $(\cdot, \cdot)$ is the canonical scalar product in $L_2(\mathbb{R}^2)$, i.e.,

$$(F, \Psi_n(\vec{z})) = \int_{\mathbb{R}^2} F(x) \overline{\Psi_n(\vec{z}, x)} dx.$$ 

The above formula can be rewritten in the form:

$$E_n(G_n') = S_n(G_n') T_n(G_n'),$$

(7.2)

$$T_n : C_0^\infty(\mathbb{R}^2) \to L_2(G_n')$$,

(7.3)
\( T_n \) being in \( L_\infty (G'_n) \), and,

\[
S_n f = \frac{1}{2\pi} \int_{G'_n} f(\vec{\xi})\Psi_n(\vec{\xi}, x) d\vec{\xi} \quad \text{for any } f \in L_\infty (G'_n).
\]  

(7.4)

By [26], \( \| T_n \|_{L_2(G'_n)} \leq \| F \|_{L_2(\mathbb{R}^2)} \) and \( \| S_n f \|_{L_2(G'_n)} \leq \| f \|_{L_2(G'_n)} \). Hence, \( T_n, S_n \) can be extended by continuity from \( C_0^\infty (\mathbb{R}^2) \), \( L_\infty (G'_n) \) to \( L_2(\mathbb{R}^2) \) and \( L_2(G'_n) \), respectively. Thus, the operator \( E_n (G'_n) \) is described by (7.2) in the whole space \( L_2(\mathbb{R}^2) \).

Obviously, for every \( \vec{\xi} \) in \( G_n \), there exists a pair \( (\lambda_n, \varphi) \) such that \( \lambda_n = \lambda(n)(\vec{\xi}) \) and that \( (\cos \varphi, \sin \varphi) = \frac{\vec{\xi}}{\| \vec{\xi} \|} \). Let us introduce new coordinates \( (\lambda_n, \varphi) \) in \( G_n \): \( \lambda_n = \lambda(n)(\vec{\xi}) \), \( (\cos \varphi, \sin \varphi) = \frac{\vec{\xi}}{\| \vec{\xi} \|} \).

**Lemma 7.1.** Every point \( \vec{\xi} \) in \( G_n \) is represented by a unique pair \( (\lambda_n, \varphi) \), \( \lambda_n > \lambda_s \), \( \varphi \in [0, 2\pi) \), such that

\[
\vec{\xi}(\lambda_n, \varphi) = \kappa_n(\lambda_n, \vec{v}) \vec{v}, \quad \vec{v} = (\cos \varphi, \sin \varphi),
\]

(7.5)

\( \kappa_n(\lambda_n, \vec{v}) \) being the “radius” of the isoenergetic curve \( D_n(\lambda_n) \) in the direction \( \vec{v} \). where \( \lambda_s = \kappa_s^2 \).

For any function \( f(\vec{\xi}) \) integrable on \( G_n \), we use the new coordinates and write

\[
\int_{G_n} f(\vec{\xi}) d\vec{\xi} = \int_{\mathbb{R}^2} \chi(G_n, \vec{\xi}) f(\vec{\xi}) d\vec{\xi} = \int_0^{2\pi} \int_{\lambda_n}^\infty \chi(G_n, \vec{\xi}(\lambda_n, \varphi)) f(\vec{\xi}(\lambda_n, \varphi)) \frac{\partial \lambda_n}{\partial \varphi} d\varphi d\lambda_n,
\]

where \( \chi(G_n, \vec{\xi}) \) is the characteristic function on \( G_n \), \( \vec{\xi}(\lambda_n, \varphi) \) is given by (7.5) and \( \frac{\partial \lambda_n}{\partial \varphi} = (\nabla \lambda(n)(\vec{\xi}), \vec{v}) \bigg|_{\vec{\xi} = \vec{\xi}_n(\lambda_n, \vec{v})} \). Let

\[
G_{n, \lambda} = \{ \vec{\xi} \in G_n : \lambda(n)(\vec{\xi}) < \lambda \}.
\]

(7.6)

This set is Lebesgue measurable, since \( G_n \) is open and \( \lambda(n)(\vec{\xi}) \) is continuous on \( G_n \).

**Lemma 7.2.** \( |G_{n, \lambda + \varepsilon} \setminus G_{n, \lambda}| \leq 2\pi \varepsilon \) when \( 0 \leq \varepsilon \leq 1 \).

By (7.1), \( E_n(G_{n, \lambda + \varepsilon}) - E_n(G_{n, \lambda}) = E_n(G_{n, \lambda + \varepsilon} \setminus G_{n, \lambda}) \). Let us obtain an estimate for this projection.

**Lemma 7.3.** For any \( F \in C_0^\infty (\mathbb{R}^2) \) and \( 0 \leq \varepsilon \leq 1 \),

\[
\| (E_n(G_{n, \lambda + \varepsilon}) - E_n(G_{n, \lambda})) F \|_{L_2(\mathbb{R}^2)}^2 \leq C(F)\varepsilon,
\]

(7.7)

where \( C(F) \) is uniform with respect to \( n \) and \( \lambda \).

**7.2. Sets \( G_\infty \) and \( G_{\infty, \lambda} \).** The sets \( G_\infty, G_n \) are given by (6.3), (6.17). As it was shown in the proof of Theorem 6.9, \( G_n+1 \subset G_n \), \( G_\infty = \bigcap_{n=1}^\infty G_n \). Therefore, the perturbation formulas for \( \lambda(n)(\vec{\xi}) \) and \( \Psi_n(\vec{\xi}) \) hold in \( G_\infty \) for all \( n \). Moreover, coordinates \( (\lambda_n, \varphi) \) can be used in \( G_\infty \) for every \( n \). Let

\[
G_{\infty, \lambda} = \{ \vec{\xi} \in G_\infty : \lambda_\infty(\vec{\xi}) < \lambda \}.
\]

(7.8)

The function \( \lambda_\infty(\vec{\xi}) \) is a Lebesgue measurable function, since it is a limit of the sequence of measurable functions. Hence, the set \( G_{\infty, \lambda} \) is measurable.
Lemma 7.4. The measure of the symmetric difference of two sets \( G_{\infty,\lambda} \) and \( G_{n,\lambda} \) converges to zero as \( n \to \infty \) uniformly in \( \lambda \) in every bounded interval:

\[
\lim_{n \to \infty} |G_{\infty,\lambda} \Delta G_{n,\lambda}| = 0.
\]

7.3. Spectral Projections \( E(G_{\infty,\lambda}) \). In this section, we show that spectral projections \( E_n(G_{\infty,\lambda}) \) have a strong limit \( E_\infty(G_{\infty,\lambda}) \) in \( L_2(\mathbb{R}^2) \) as \( n \) tends to infinity. The operator \( E_\infty(G_{\infty,\lambda}) \) is a spectral projection of \( H \). It can be represented in the form \( E_\infty(G_{\infty,\lambda}) = S_\infty T_\infty \), where \( S_\infty \) and \( T_\infty \) are strong limits of \( S_n(G_{\infty,\lambda}) \) and \( T_n(G_{\infty,\lambda}) \), respectively. For any \( F \in C_0^\infty(\mathbb{R}^2) \), we show:

\[
E_\infty(G_{\infty,\lambda}) F = \frac{1}{4\pi^2} \int_{G_{\infty,\lambda}} (F, \Psi_\infty(\tilde{z})) \Psi_\infty(\tilde{z}) d\tilde{z}, \quad (7.9)
\]

Then we consider the sequence of operators \( T_n(G_{\infty,\lambda}) \) which are given by (7.3) and act from \( L_2(\mathbb{R}^2) \) to \( L_2(G_{\infty,\lambda}) \). We prove that the sequence has a strong limit and describe its properties.

Lemma 7.5. The sequence \( T_n(G_{\infty,\lambda}) \) has a strong limit \( T_\infty(G_{\infty,\lambda}) \). The operator \( T_\infty(G_{\infty,\lambda}) \) satisfies \( \|T_\infty\| \leq 1 \) and can be described by the formula \( T_\infty F = \frac{1}{2\pi} (F, \Psi_\infty(\tilde{z})) \) for any \( F \in C_0^\infty(\mathbb{R}^2) \). The convergence of \( T_n(G_{\infty,\lambda}) F \) to \( T_\infty(G_{\infty,\lambda}) F \) is uniform in \( \lambda \) for every \( F \in L_2(\mathbb{R}^2) \).

We now consider the sequence of operators \( S_n(G_{\infty,\lambda}) \) which are given by (7.4) with \( G'_n = G_{\infty,\lambda} \):

\[
S_n(G_{\infty,\lambda}) : L_2(G_{\infty,\lambda}) \to L_2(\mathbb{R}^2).
\]

Lemma 7.6. The sequence of operators \( S_n(G_{\infty,\lambda}) \) has a strong limit \( S_\infty(G_{\infty,\lambda}) \). The operator \( S_\infty(G_{\infty,\lambda}) \) satisfies \( \|S_\infty\| \leq 1 \) and can be described by the formula

\[
(S_\infty f)(x) = \frac{1}{2\pi} \int_{G_{\infty,\lambda}} f(\tilde{z}) \Psi_\infty(\tilde{z}, x) d\tilde{z}, \quad (7.11)
\]

for any \( f \in L_\infty(G_{\infty,\lambda}) \). The convergence of \( S_n(G_{\infty,\lambda}) f \) to \( S_\infty(G_{\infty,\lambda}) f \) is uniform in \( \lambda \) for every \( f \in L_2(G_{\infty}) \).

Lemma 7.7. Spectral projections \( E_n(G_{\infty,\lambda}) \) have a strong limit \( E_\infty(G_{\infty,\lambda}) \) in \( L_2(\mathbb{R}^2) \), the convergence being uniform in \( \lambda \) for every element. The operator \( E_\infty(G_{\infty,\lambda}) \) is a projection. For any \( F \in C_0^\infty(\mathbb{R}^2) \) it is given by (7.9) and formula (7.10) holds.

Lemma 7.8. There is a strong limit \( E_\infty(G_{\infty}) \) of the projections \( E_\infty(G_{\infty,\lambda}) \) as \( \lambda \) goes to infinity.

Corollary 7.9. The operator \( E_\infty(G_{\infty}) \) is a projection.

Lemma 7.10. Projections \( E_\infty(G_{\infty,\lambda}), \lambda \in \mathbb{R} \), and \( E_\infty(G_{\infty}) \) reduce the operator \( H \).
Lemma 7.11. The family of projections $E_{\infty}(\mathcal{G}_{\infty}, \lambda)$ is the resolution of the identity of the operator $HE_{\infty}(\mathcal{G}_{\infty})$ acting in $E_{\infty}(\mathcal{G}_{\infty})L_2(\mathbb{R}^2)$.

7.4. Proof of Absolute Continuity. Now we show that the branch of spectrum (semi-axis) corresponding to $\mathcal{G}_{\infty}$ is absolutely continuous.

Theorem 7.12. For any $F \in C^\infty(\mathbb{R}^2)$ and $0 \leq \varepsilon \leq 1$,
$$|(E_{\infty}(\mathcal{G}_{\infty},\lambda+\varepsilon)F,F) - (E_{\infty}(\mathcal{G}_{\infty},\lambda)F,F)| \leq C_F \varepsilon. \quad (7.12)$$

Corollary 7.13. The spectrum of the operator $HE_{\infty}(\mathcal{G}_{\infty})$ is absolutely continuous.

Proof. By formula (7.3),
$$|(E_{\infty}(\mathcal{G}_{\infty},\lambda+\varepsilon)F,F) - (E_{\infty}(\mathcal{G}_{\infty},\lambda)F,F)| \leq C_F \mathcal{G}_{\infty,\lambda+\varepsilon} \mathcal{G}_{\infty,\lambda}. \quad (7.12)$$
Applying Lemmas 7.2 and 7.4 we immediately get (7.12).

8. Geometrical Lemmas.

8.1. Proof of Lemma 2.1 (Geometric Lemma, step I). Corollaries 8.5 and 8.7 obtained in Sections 8.1.2 and 8.1.3 together give Lemma 2.1 and its corollary.

8.1.1. Set $\chi_1(\lambda, \beta, s_1, \delta)$. The set $\chi_1(\lambda, \beta, s_1, \delta)$ is given by formulas (2.11)–(2.17). It remains to define radii $r_m$ of the discs $O_m^\pm$. First, we need more notations. Angle $\varphi_m$ is defined as a polar coordinate of $\vec{p}_m$:
$$\vec{p}_m := \vec{p}_m(0) = p_m(\cos \varphi_m, \sin \varphi_m), \quad m = (m_1, m_2) \in \mathbb{Z}^2.$$
In addition, let $\Pi_{m,q,\pm}$ be the distance in $\mathbb{C}$ between $|\vec{k}(\varphi) + \vec{p}_m|^2$ and $|\vec{k}(\varphi) + \vec{p}_m + \vec{q}|^2$ when $\varphi = \varphi_m^\pm$, i.e., when $\varphi$ solves $|\vec{k}(\varphi) + \vec{p}_m|^2 = k^2$. Obviously,
$$\Pi_{m,q,\pm} := |(|\vec{k}(\varphi_m^\pm) + \vec{p}_m, \vec{p}_q)| + |\vec{q}|^2. \quad (8.1)$$
We will be interested in values of $\Pi_{m,q,\pm}$ for $q: 0 < p_q < k^{s_1}$. Let
$$\Pi'_{m,q,\pm} := |(|\vec{k}(\varphi_m^\pm) + \vec{p}_m, \vec{p}_q)| + k^{2s_1}. \quad (8.2)$$
Obviously, $\Pi'_{m,q,\pm} = \Pi_{m,q,\pm} + O(k^{2s_1})$, $\Pi'_{m,q,\pm} \geq k^{2s_1}$. Next,
$$\Pi_m := \min_{0 < p_q < k^{s_1}} \Pi'_{m,q,\pm} = \min_{0 < p_q < k^{s_1}} |(|\vec{k}(\varphi_m^\pm) + \vec{p}_m, \vec{p}_q)| + k^{2s_1}. \quad (8.3)$$
The value $\Pi_m$ characterizes the smallest distance between $|k(\varphi) + \vec{p}_m|^2$ and $|\vec{k}(\varphi) + \vec{p}_m + \vec{q}|^2$ when $|k(\varphi) + \vec{p}_m|^2 = k^2$, $0 < p_q < k^{s_1}$. It coincides with this distance up to the value $O(k^{2s_1})$. Further we use that $\Pi_m \geq k^{2s_1}$.
Lemma 8.3. For every $m$ with $0 < p_m < 4k$, $r_m = o(1)$ as $k \to \infty$.

Proof. Let us consider the following four cases:

If $0 < p_m \leq 4k$, then $r_m < k^{3s_1-\delta} = o(1)$, since the periods $a_1$, $a_2$ of $H^{(1)}$ are of order $k^{s_1}$.

If $4k < p_m \leq k$, then $r_m < \frac{k^{2s_1-2}}{k^{s_1}+4^{1/2}} < 2k^{3s_1-6s_1} = o(1)$, since $\Pi_m \geq k^{2s_1}$.

If $k < p_m < 4k$ and $1 - \left| \frac{p_m^2}{4k^2} \right| \geq \frac{k^{2s_1-2}}{4k}$, then $r_m < \frac{k^{4s_1-2}}{k^{s_1}+4^{1/2}} < k^{3s_1-2s_1} = o(1)$.

If $1 - \left| \frac{p_m^2}{4k^2} \right| < \frac{k^{2s_1-2}}{4k}$, then of course $p_m > k$ and $r_m < \frac{8k^{3s_1-2}}{k^{3s_1-1}} = o(1)$.

8.1.2. Proof of the statements (i), (ii) in Lemma 2.1.

Lemma 8.4. Let $4s_1 < 2\beta < 1 - 15s_1 - 8\delta$, $\varphi$ is in $\Phi_0$ or its $(k^{3s_1-3s_1-2\delta})$-neighborhood, $m, q \in \mathbb{Z}^2$, $m, q, m + q \neq 0$, $0 < p_q < k^{s_1}$. Then

$$2\left| \tilde{k}(\varphi) + \tilde{p}_m \right|^2 - k^2 \left| \tilde{k}(\varphi) + \tilde{p}_m + \tilde{p}_q \right|^2 > k^{2\beta},$$
(8.5)

$$\left| \tilde{k}(\varphi) + \tilde{p}_m \right|^2 - k^2 > k^{1-3s_1-\delta}. $$
(8.6)

Corollary 8.5 (Statements (i), (ii)) in Lemma 2.1 Corollary 2.1. If $t \in \chi_1(\lambda, \beta, s_1, \delta)$, then there is a unique $j \in \mathbb{Z}^2$ such that $p_j(t) = k$ and (2.4), (2.5) hold. For any $t$ in the $(k^{2s_1-3s_1-2\delta})$-neighborhood of the non-resonance set in $\mathbb{C}^2$, there exists a unique $j \in \mathbb{Z}^2$ such that (2.6) and (2.4), (2.5), (2.9), (2.10) hold.

Proof of Corollary 8.5. If $t \in \chi_1(\lambda, \beta, s_1, \delta)$, then, by the definition of $\chi_1$, there is $\varphi \in \Theta_1$ such that $t = K_1 \tilde{k}(\varphi)$. Hence, there exists a $j$ such that $p_j(t) = k$. Let us show that $j$ is unique. Suppose there is $j' = j + m, m \neq 0$ such that $p_{j+m}(t) = k$. Then $\tilde{p}_{j+m}(t) = \tilde{k}(\varphi) + \tilde{p}_m$ and $|\tilde{k}(\varphi) + \tilde{p}_m| = k$. The last relation contradicts to (8.5). Therefore, $j' = j$.

Substituting $\tilde{k}(\varphi) = \tilde{p}_j(t)$ into (8.5), (8.6) and using the notation $j + m = i$, we obtain (2.4) and (2.5). Similar arguments work when $t$ is in the $(k^{2s_1-3s_1-2\delta})$-neighborhood of $\chi_1(\lambda, \beta, s_1, \delta)$ in $\mathbb{C}^2$.

The inequalities (2.9), (2.10) are obvious, when $p_i > 4k$. We assume now $p_i < 4k$. Let us prove (2.9). When $i = j$, (2.9) follows directly from (2.3) and (2.6).
Suppose $i \neq j$. Clearly, $|p_i^2(t) - p_0^2(t)| = O(k^{1+s_1})$. This together with (2.4) yields: 

$$|p_i^2(t) - p_j^2(t)| > ck^{2\beta - 1 - s_1}. $$

Considering (2.8), we obtain

$$2|p_i^2(t) - z| \geq |p_i^2(t) - p_j^2(t)|. $$

The estimate (2.9) easily follows. Let us prove (2.10). In the case of $i = j$, using (2.5) and the definition of $C_1$, we get

$$|p_j^2(t) - z| \geq |p_j^2(t) - p_j^2(t)| - k^{2\beta - 1 - s_1 - \delta} > \frac{1}{2}k^{1-3s_1-\delta}. $$

Using this together with the definition of $C_1$, we get (2.10) in the case $i = j$. In the case of $i \neq j$ (2.4) and (8.7) yield (2.10). It is easy to see that all the estimates are stable under a perturbation of $t$ of order $k^{2\beta - 2 - s_1 - 2\delta}$. Therefore, the estimate (2.10) can be extended to the complex $(k^{2\beta - 2 - s_1 - 2\delta})$-neighborhood of $\chi_1(k, \beta, s_1, \delta)$. □

**Proof of Lemma 8.7** If $p_m > 4k$ then (8.5) is obvious, since both factors are greater than $k^2$. Assume $0 < p_m < 4k$. Noting that

$$|\tilde{k}(\varphi) + \tilde{p}_m|^2 - k^2 = 2kp_m \cos(\varphi - \varphi_m) + p_m^2 $$

and recalling that $\varphi^\pm_m$ are the solutions of $|\tilde{k}(\varphi) + \tilde{p}_m|^2 = k^2$, we see:

$$\cos(\varphi^\pm_m - \varphi_m) = -\frac{p_m}{2k}, \quad |\sin(\varphi^\pm_m - \varphi_m)| = \sqrt{\frac{1 - p_m^2}{4k^2}}. $$

Now let $\varphi$ be on the boundary of $\mathcal{O}_m^+$ or $\mathcal{O}_m^-$. Expanding (8.9) around $\varphi^\pm_m$, we get:

$$|\tilde{k}(\varphi) + \tilde{p}_m|^2 - k^2 = 2kp_m \sin(\varphi^\pm_m - \varphi_m)r_m \left(1 + O(r_m^2)\right) - kp_m \cos(\varphi^\pm_m - \varphi_m)r_m \left(1 + O(r_m^2)\right) + o(1).$$

Next, we prove

$$|\tilde{k}(\varphi) + \tilde{p}_m|^2 - k^2 > k^{2\beta}/\Pi_m. $$

in three cases as in Definition 8.1(i). In special, we get (8.6) in the first case.

Case (i) : $r_m = \frac{k^{1-3s_1-\delta}}{p_m}$ when $0 < p_m \leq 4k^{s_1}$. The modulus of the first term in (8.11) is

$$2k^{1-3s_1-\delta}\sqrt{1 - \frac{p_m^2}{4k^2}(1 + o(1))} > \frac{3}{2}k^{1-3s_1-\delta} and that of the second term is

$$O(k^{-6s_1-2\delta}),$$

which is much smaller than the first term. Thus, we obtain (8.6).

Considering that $k^{1-3s_1-\delta} > \frac{2k^{2\beta}}{\Pi_m}$, we obtain (8.12).

Case (ii) : $r_m = \frac{k^{2\beta-1}}{p_m\Pi_m \sqrt{1 - \frac{p_m^2}{4k^2}}}$ when $4k^{s_1} < p_m < 4k$ and

$$\left|1 - \frac{p_m^2}{4k^2}\right| > \frac{k^{2\beta-2}}{\Pi_m}. $$

Substituting $r_m$ into (8.11), we get that the modulus of the first term is

$$\frac{k^{4\beta-2}}{\Pi_m^2} \left(1 + o(1)\right)$$

and that of the second term is

$$\frac{k^{4\beta-2}}{\Pi_m^2 \left|1 - \frac{p_m^2}{4k^2}\right|} \left(1 + o(1)\right).$$

Using the condition $\left|1 - \frac{p_m^2}{4k^2}\right| > \frac{k^{2\beta-2}}{\Pi_m}$, one can easily see that the former is at least twice greater than the latter. Thus, we get (8.12).
Case (iii) : $r_m = \frac{8k^\beta}{p_m \sqrt{\Pi_m}}$ when $1 - \frac{p_m^2}{4k^2} < \frac{k^{2\beta - 2}}{\Pi_m}$. This time the modulus of the second term is $\frac{2k^{2\beta}}{\Pi_m} (1 + o(1))$ and that of the first is smaller than $\frac{16k^{2\beta}}{\Pi_m} (1 + o(1))$. Therefore we again have (8.12).

Now we prove (8.5). If $0 < p_m, p_{m+q} \leq 4k^{s_1}$, then (8.5) easily follows from (8.6) proven in the Case (i) above for such $m, m + q$. Next, if both factors in the left hand side of (8.5) is greater than $k^\beta$, then (8.5) is obvious. Therefore, without loss of generality, we can assume

$$|\vec{k}(\varphi) + \vec{p}_m|^2 - k^2| \leq k^\beta, \quad p_m > 4k^{s_1}. \quad (8.13)$$

Suppose we have proved that

$$|\vec{k}(\varphi) + \vec{p}_m + q|^2 - k^2| > \frac{1}{2} \Pi_m \quad (8.14)$$

for every $m$ : $\Pi_m > k^\beta$ and $\varphi$ satisfying (8.13).\footnote{The only conditions required for (8.14) are $\Pi_m > k^\beta$ and (8.13). We will use (8.14) under this condition in Lemma 8.9.} Then, notice that $\Pi_m > k^\beta$ follows from (8.12). Considering (8.12) and (8.14) together, we obtain (8.5).

It remains to prove (8.14). First, we check that

$$||\vec{k}(\varphi) + \vec{p}_m|^2 - \vec{k}(\varphi) + \vec{p}_{m+q}|^2| \geq \frac{3}{2} \Pi_m + O(k^{2s_1}) \quad (8.15)$$

for every $m$ : $\Pi_m > k^\beta$ and $\varphi$ satisfying (8.13). Indeed, taking into account (8.11) and considering as in the proof of (8.12), we conclude that the set of $\varphi$ satisfying (8.13) is inside the circle of the radius $r'_m$ around $\varphi_m$, where

$$r'_m = \begin{cases} \frac{k^{\beta - 1}}{p_m \sqrt{1 - \frac{p_m^2}{4k^2}}} & \text{when } 1 - \frac{p_m^2}{4k^2} > k^{\beta - 2} \\ 10k^{\beta/2 - 1} & \text{otherwise.} \end{cases} \quad (8.16)$$

It is easy to show that for any $\varphi$ inside those circles:

$$||\vec{k}(\varphi) + \vec{p}_m|^2 - |\vec{k}(\varphi) + \vec{p}_{m+q}|^2| - |\vec{k}(\varphi_m)^\pm + \vec{p}_m|^2 - |\vec{k}(\varphi_m)^\pm + \vec{p}_{m+q}|^2| \leq 2k^\beta q_m' < \frac{1}{2} \Pi_m + o(1). \quad (8.17)$$

Considering the definition (8.3) of $\Pi_m$, we easily get:

$$|\vec{k}(\varphi_m)^\pm + \vec{p}_m|^2 - |\vec{k}(\varphi_m)^\pm + \vec{p}_{m+q}|^2| \geq 2\Pi_m + O(k^{2s_1}). \quad (8.18)$$

Combining (8.17) and (8.18), we arrive at (8.15). Now inequalities (8.13) and (8.15) yield (8.14). Thus, (8.5) is proven.

Note that all estimates are stable with respect the perturbation of $\varphi$ of order of $k^{2\beta - 3 - s_1 - 2\delta}$. Therefore, (8.5), (8.6) hold not only in $\Phi_1$, but also in its $(k^{2\beta - 3 - s_1 - 2\delta})-$neighborhood.
8.1.3. Proof of the statement (iii) in Lemma 2.1

**Lemma 8.6.**

\[
\sum_{m \in \mathbb{Z}^2} r_m = O(k^{-\delta}), \quad (8.19)
\]

\[
\sum_{m \in \mathbb{Z}^2, 0 < m < 4k^4} r_m = O(k^{-\gamma_0}), \quad (8.20)
\]

where

\[
\gamma_0 = \begin{cases} 
1 - 2\beta - 15s_1 - 7\delta, & \text{if } \beta \geq 1/6; \\
5/6 - \beta - 15s_1 - 7\delta, & \text{if } \beta < 1/6.
\end{cases}
\]

the sum \(\sum\) in (8.20) includes only such \(m\) that \(O_m \not\subset \bigcup_{0 < q < 4k^4} \pm O_q^\pm\).

**Corollary 8.7.** If \(2\beta < 1 - 15s_1 - 8\delta\), then the total length of \(O(1) \cap [0, 2\pi)\) does not exceed \(O(k^{-\delta})\). Estimate (2.7) holds.

**Proof.** We split \(m \in \mathbb{Z}^2\) with \(0 < m < 4k\) into five sets:

\[
I_1 := \{ m \in \mathbb{Z}^2 : 0 < m \leq 4k^4 \}, \\
I_2 := \{ m \in \mathbb{Z}^2 : 4k^4 < m \leq k^{1 - 4s_1 - 4\delta} \}, \\
I_3 := \{ m \in \mathbb{Z}^2 : m > k^{1 - 4s_1 - 4\delta}, 2k - m \geq k^{1 - 8s_1 - 4\delta} \}, \\
I_4 := \{ m \in \mathbb{Z}^2 : 2k - m < k^{1 - 8s_1 - 4\delta}, 1 - \frac{p_m^2}{4k} \geq \frac{k^{2\beta - 2}}{4\Pi_m} \}, \\
I_5 := \{ m \in \mathbb{Z}^2 : \left| 1 - \frac{p_m^2}{4k} \right| < \frac{k^{2\beta - 2}}{4\Pi_m} \}.
\]

and define \(\Sigma_j := \sum_{m \in I_j} r_m, j = 1, \ldots, 5\) and get an upper bound for each.

(i) By (8.19),

\[
\Sigma_1 = \sum_{m \in I_1} \frac{k^{-3s_1 - \delta}}{p_m}.
\]

Considering that the size of the lattice formed by \(\vec{p}_m\) is \(ck^{-s_1}\), we obtain:

\[
\sum_{m \in I_1} \frac{1}{p_m} \leq c k^{2s_1} \int_0^{2\pi} \int_0^{k^{s_1}} \frac{1}{r} \cdot r dr d\varphi = ck^{3s_1}.
\]

Substituting the last estimate into (8.21), we obtain (8.19).

(ii) Now we estimate \(\Sigma_2\). Let us check that

\[
\Pi_m \geq \frac{1}{2} k^{-1 + 3s_1 - 3\delta}, \text{ when } m \in I_2.
\]

Indeed, suppose \(\Pi_m < \frac{1}{2} k^{-1 + 3s_1 - 3\delta}\). Then for some \(q \in \mathbb{Z}\) with \(0 < q < k^{s_1}:

\[
\left| (\vec{k}(\varphi_m) + \vec{p}_m, \vec{p}_q)_* + k^{2s_1} \right| < \frac{1}{2} k^{-1 + 3s_1 - 3\delta}, \text{ where } \pm \text{ means } + \text{ or } -.
\]

Hence, \(\left| (\vec{k}(\varphi_m^\pm), \vec{p}_q)_* \right| < k^{-1 + 3s_1 - 3\delta}, \text{ since } \left| (\vec{p}_m, \vec{p}_q)_* \right| \leq k^{-1 + 4s_1 - 4\delta} \cdot k^{s_1} = k^{1 - 3s_1 - 4\delta} \text{ when } m \in I_2\). Therefore,

\[
\left| (\vec{k}(\varphi_m^\pm) + \vec{p}_q)_* - k^2 \right| = \left| 2(\vec{k}(\varphi_m^\pm), \vec{p}_q)_* + p_q^2 \right| \leq 2k^{1 - 3s_1 - 3\delta} + k^{2s_1} < 3k^{1 - 3s_1 - 3\delta} \text{, which}
\]

\[
\geq \frac{1}{2} k^{-1 + 3s_1 - 3\delta}.
\]
means \( \varphi_m^\pm \in \mathcal{O}_q^\pm \), see (8.4) in Lemma 8.4. Moreover, \( \varphi_m^\pm \) is relatively close to the centrum of \( \mathcal{O}_q^\pm \), its distance to the centrum of \( \mathcal{O}_q^\pm \) is less than \( r_m k^{-2\delta} \). Using (8.4), one can easily check that \( r_m << r_q k^{-2\delta} \) when \( m \in I_2 \). Thus \( \mathcal{O}_q^\pm \) is completely contained in \( \mathcal{O}_q^\pm \). We do not consider such an \( m \) in the sum (8.20). Thus, we can use \( \Pi_m \geq \frac{1}{2} k^{1-3s_1-3\delta} \). By (8.4)

\[
\Sigma_2 = \sum_{m \in I_2} \frac{k^{2\beta-1}}{p_m \Pi_m \sqrt{1 - \frac{p_m^2}{4k^2}}} \leq \frac{ck^{2\beta-1}}{k^{1-3s_1-3\delta}} \sum_{m \in I_2} \frac{1}{p_m}
\]

We estimate the sum on the r.h.s. by the integral \( k^{2s_1} \int_0^{2\pi} \int_{4k^{s_1}}^{k^{1-4s_1-4\delta}} \frac{1}{r} r \, dr \, d\varphi \). Computing the integral, we obtain:

\[
\Sigma_2 \leq ck^{2\beta-1+8s_1-\delta}.
\] (8.23)

(iii) Let \( m \in I_3 \). First we note that

\[
\sqrt{1 - \frac{p_m^2}{4k^2}} \geq \frac{1}{2} k^{-4s_1-2\delta}
\] (8.24)
since \( 2k - p_m \geq k^{1-8s_1-4\delta} \). For the moment we assume to have

\[
\sum_{m \in I_3} \frac{1}{\Pi_m} \leq ck^{1+7s_1+\delta}.
\] (8.25)

Then,

\[
\Sigma_3 = \sum_{m \in I_3} \frac{k^{2\beta-1}}{p_m \Pi_m \sqrt{1 - \frac{p_m^2}{4k^2}}} \leq \frac{ck^{2\beta-1}}{k^{1-4s_1-4\delta} \cdot k^{-4s_1-2\delta}} \sum_{m \in I_3} \frac{1}{\Pi_m}
\]

where we used the definition of \( I_3 \) and (8.24). Considering also (8.25), we obtain:

\[
\Sigma_3 \leq ck^{2\beta-1+15s_1+7\delta}.
\] (8.26)

Now we need to show (8.25). Obviously, \( \varphi_m^\pm \) are real, when \( m \in I_3 \). Therefore, all \( \langle \cdot, \cdot \rangle_m \) and \( \cdot \cdot \cdot \) in the following formulas are real too. We note that \(|\tilde{k}(\varphi_m^\pm)_m|^2 = k^2\), \(|\tilde{k}(\varphi_m^\pm) - \tilde{p}_m|^2 = k^2\) and \(|\langle \tilde{k}(\varphi_m^\pm) + \tilde{p}_m, \tilde{p}_m \rangle_m + k^{2s_1} = \Pi_m \) for some \( q_m \), \( 0 < p_m < k^{s_1} \) and \( \varphi_m^+ \) or/and \( \varphi_m^- \). Assume for definiteness that the last equality holds for \( \varphi_m^- \). Denoting \( \tilde{k}(\varphi_m^-) + \tilde{p}_m \) by \( \tilde{k}_m \) gives us:

\[
|\tilde{k}_m - \tilde{p}_m|^2 = k^2, \quad |\tilde{k}_m|^2 = k^2
\] (8.27)

\[
|\langle \tilde{k}_m, \tilde{p}_m \rangle_m| + k^{2s_1} = \Pi_m
\] (8.28)

Obviously, \( k^{2s_1} \leq \Pi_m \leq k^{1+s_1} \). Define \( \Omega_j \) as the set of \( \tilde{p}_m \) in \( I_3 \) satisfying the inequality:

\[
k^{2s_1+j\delta} < \Pi_m \leq k^{2s_1+(j+1)\delta}, \quad j = 0, \ldots, J, \quad J = \lfloor \frac{1-s_1}{\delta} \rfloor.
\]

Here \(|r| = \max \{ z \in \mathbb{Z} : z \leq r \} \). Obviously, \( \Omega_j \subset \cup_{0 \leq p < k^{s_1}} \Omega_{j,q} \), where \( \Omega_{j,q} \) consists of \( \tilde{p}_m \) in \( I_3 \) satisfying (8.27) and

\[
k^{2s_1+j\delta} < |\langle \tilde{k}_m, \tilde{p}_m \rangle_m| + k^{2s_1} \leq k^{2s_1+(j+1)\delta}.
\] (8.29)
Let \( \bar{p}_m \in \Omega_{j,q} \). By (8.27), (8.29), \( \bar{p}_m \) belongs to a circle of radius \( k \) centered at a point \( \bar{k}_m \), where \( \bar{k}_m \) belongs to a circle of the radius \( k \) centered at the origin and satisfies the inequality \( k^{2s_1+3\delta} < \| (\bar{k}_m, \bar{P}_q) \| + k^{2s_1} \leq k^{2s_1+(j+1)\delta} \), i.e., \( \bar{k}_m \) belongs to one of two arcs. Considering that \( p_q > k^{-s_1} \), we easily see that all such points \( \bar{p}_m \) belong to a couple of rings of the same radii \( k - k^{3s_1+(j+1)\delta}, k + k^{3s_1+(j+1)\delta} \) centered at two points \( \bar{k}^q \pm \) satisfying the conditions \( |\bar{k}^q| = k, \langle \bar{k}^q, \bar{p}_q \rangle = 0 \). Estimating the number of points \( \bar{p}_m \) at this region by its area, we obtain, that the number of points in \( \Omega_{j,q} \) does not exceed \( ck^{1+5s_1+(j+1)\delta} \). Therefore, the number of points in \( \Omega_j \) does not exceed \( ck^{1+9s_1+(j+1)\delta} \). Thus, we obtain the estimate

\[
\sum_{m \in I_3} \frac{1}{\Pi_m} \leq \sum_{j=0}^{J} \sum_{m \in \Omega_j} \frac{1}{\Pi_m} \leq \sum_{j=0}^{J} c k^{1+9s_1+(j+1)\delta} \cdot \frac{1}{k^{2s_1+3\delta}} \leq c k^{1+7s_1+\delta},
\]

which finishes this case.

(iv) Let \( m \in I_4 \). For the moment, we assume the following two estimates,\(^\tag{12}\)

\[
\Pi_m > \frac{1}{4} k^{1-9s_1-4\delta},
\]

\[
\omega := \sum_{m \in I_4} \frac{1}{|2k - p_m|} < \left\{ \begin{array}{ll}
k^{3/2+3s_1+\delta}, & \text{if } \beta \geq 1/6; \\
ck^{-5/3+3s_1+\delta}, & \text{if } \beta < 1/6.
\end{array} \right.
\]

Then,

\[
\Sigma_4 = \sum_{m \in I_4} \frac{k^{2\beta-1}}{p_m \Pi_m \sqrt{|1 - \frac{p_m^2}{4k^2}|}} \leq \frac{ck^{2\beta-1}}{k \cdot k^{1-9s_1-4\delta} \cdot k^{-1/2}} \sum_{m \in I_4} \frac{1}{|2k - p_m|} \leq \left\{ \begin{array}{ll}
ck^{-1+2\beta+12s_1+5\delta}, & \text{if } \beta \geq 1/6; \\
ck^{-5/6+\beta+12s_1+5\delta}, & \text{if } \beta < 1/6.
\end{array} \right.
\]

Now we show (8.30). We consider two cases: \( |2k - p_m| \leq k^{1-8s_1-4\delta} \) and \( 2k - p_m < -k^{1-8s_1-4\delta} \). We start with the former. Suppose, (8.30) is not true, i.e., \( \Pi_m \leq \frac{1}{4} k^{1-9s_1-4\delta} \). Then, there is a \( q \) such that \( 0 < p_q < k^{s_1} \) and

\[
|\langle k(\varphi_m^\pm) + \bar{p}_m, \bar{p}_q \rangle| \leq \frac{1}{4} k^{1-9s_1-4\delta}
\]

for \( \varphi_m^+ \) or/and \( \varphi_m^- \). Note that \( \varphi_m^\pm \) has a non-zero imaginary part when \( p_m > 2k \). Denoting \( k(\varphi_m^\pm) + \bar{p}_m \) by \( \hat{k}m \), \( \hat{k}m \in C^2 \), we get (8.27) for both \( \hat{k}m \). It follows from (8.27) that \( \hat{k}m = k(\varphi_m^\pm) \), we mean here that the pairs of vectors are the same. Therefore, there is at least one \( k(\varphi_m^\pm) \), such that \( |\langle k(\varphi_m^\pm) + \bar{p}_m, \bar{p}_q \rangle| \leq \frac{1}{4} k^{1-9s_1-4\delta} \). We denote it by \( \varphi_m^\pm \). The last inequality means that \( \varphi_m^\pm \) is in \( O_q, O_q = O_q^+ \cup O_q^- \), and even relatively close to the center of a disc \( O_q^\pm \): its distance to the center is less than \( r_q k^{2s_1} \), the radius \( r_q \) of \( O_q^\pm \) being given by \( r_q = k^{-3s_1-\delta} p_q^{-1} \), see (8.4). The distance between centers of \( O_q^+ \) and \( O_q^- \) is \( \pi + O(p_q k^{-1}) \). It is easy to see also

\(^{12}\)To prove (8.30) we use only the following conditions: \( 2k - p_m \leq -k^{1-8s_1-4\delta} \) or \( 2k - p_m \leq k^{1-8s_1-4\delta} \) and at least one of \( \varphi_m^\pm \) is in \( \Phi_0 = \Phi_0 \setminus \cup_{0 < r_q < k^{3s_1+2\delta} \cdot O_q^\pm} \) or its \( 2k^{-4s_1-2\delta} \)-neighborhood. Under these conditions, we use estimate (8.30) also in Lemmas 3.9 8.10.
that $\varphi_{\pm m} = \varphi_{m}^\pm + \pi$ modulo $2\pi$, we mean here that the pairs of angles are the same. Considering the last two relations, we obtain that $\varphi_{m}^+$ or $\varphi_{m}^-$ is also in $O_q$. Using $|2k - p_m| < k^{1-8s_1-4\delta}$, we get from (8.10): $|\varphi_{m}^+ - \varphi_{m}^-| < 2k^{-4s_1-2\delta}$. It is much less than $r_q$. Hence, both $\varphi_{m}^+$ and $\varphi_{m}^-$ are in either $O_q^+$ or $O_q^-$, relatively close to a center. It is easy to show that $r_m = o(k^{3-1}) = o(r_q)$. This means $O_m \subset O_q$. We don’t include include such $r_m$ in the sum (8.20). Thus, (8.30) is proven for the case $|2k - p_m| \leq k^{1-8s_1-4\delta}$. Let $2k - p_m < -k^{1-8s_1-4\delta}$. By (8.10),

$$|\Im \varphi_m| = |\Im \varphi_{m}^\pm| \approx \sqrt{\frac{p_m^2}{4k^2}} - 1 \geq \frac{1}{2}k^{-4s_1-2\delta}.$$ 

It is easy to see that from the definition of $\varphi_{m}^\pm$ that $k(\varphi_{m}^\pm) = \bar{k}(\varphi_{m}^-)$. Hence,

$$|k(\varphi_{m}^\pm) + \bar{p}_m, \bar{p}_q| = |k(\varphi_{m}^-), \bar{p}_q| = kp_q \cos(\varphi_{m} - \varphi_q) \geq \frac{1}{4} kp_q k^{-8s_1-4\delta} \geq \frac{1}{4} k^{1-9s_1-4\delta}.$$ 

Therefore, $\Pi_m > \frac{1}{4} k^{1-9s_1-4\delta}$. Thus, we proved (8.30).

To show (8.31), we split $I_4$ into three subsets $I_4 = I'_4 + I''_4 + I'''_4$: $I'_4 = \{ m \in I_4 : |2k - p_m| < k^{-1/3} \}$, $I''_4 = \{ m \in I_4 : k^{-1/3} < |2k - p_m| < k^{3/4} \}$, $I'''_4 = \{ m \in I_4 : |2k - p_m| \geq k^{3/4} \}$.

Correspondingly, $\omega = \omega' + \omega'' + \omega'''$. Let us estimate $\omega'$. We use a well known estimate (see e.g. (27)) for the number $N(k)$ of a rectangular lattice points in the circle of radius $k$: $N(k) = \pi v_- k^2 + O(k^{2/3})$, where $v$ is the area of the elementary cell of the lattice and the implicit constant depend on the periods. Considering that our lattice has a size of order $k^{-s_1}$, we rescale the last estimate, so it becomes: $N(k) = \pi v_d^{-1} k^{2+2s_1} + O(k^{2/3+2s_1/3})$, $v_d = 4\pi^2 d_1^{-1} d_2^{-1}$. This means that $I'_4$ contains less than $ck^{2/3+2s_1}$ points. Note also that $|2k - p_m| > \frac{1}{4} k^{2\beta-2-s_1-\delta}$ follows from $|1 - \frac{k^2}{4k^2}| \geq \frac{2\beta-2}{111}$ and $\Pi_m \leq k^{1+s_1}$. Hence,

$$\omega' < cv_d^{-1} k^{2/3+2s_1} \cdot k^{-\beta+1+s_1/2+\delta/2} = o(k^{-\beta+5/3+3s_1+\delta}).$$

Next, considering that the region $I''_4$ contains no more than $cv_d^{-1} k^{1+\delta+2s_1}$ of points and $|2k - p_m| \geq k^{-1/3}$, we obtain

$$\omega'' < cv_d^{-1} k^{1+\delta+2s_1} k^{1/6} < cv_d^{-1} k^{7/6+2s_1+\delta}.$$ 

Further, we estimate $\omega'''$ by an integral:

$$\omega''' < ck^{2s_1} v_d^{-1} \int_{0}^{2\pi} \int_{|2k-r|<2k} \frac{1}{\sqrt{|2k-r|}} r drd\varphi \leq cv_d^{-1} k^{\frac{3}{2}}.$$ 

Combining the last three estimates, we obtain (8.31).

(v) As in the previous case, we have $\Pi_m > \frac{1}{4} k^{1-9s_1-4\delta}$. The number of points in $I_5$ admits the estimate: $#(I_5) < cv_d^{-1} k^{3/2+2s_1}$. Therefore,

$$\Sigma_5 = \sum_{m \in I_5} \frac{4k^\beta}{p_m \sqrt{\Pi_m}} \leq \frac{8k^\beta}{k \cdot k^{(1-9s_1-4\delta)}/2} \sum_{m \in I_5} 1.$$
Hence, $\Sigma_2 < cv_d^{-1}k^{\beta-\frac{3}{8}+7s_1+2\delta}$. Adding (8.23), (8.26), (8.32), and the last estimate, we obtain (8.20).

**8.2. Proof of Geometric Lemma 2.16**

The set $O^{(1)}(\vec{b})$ is defined by formula (2.83), where $O^\pm_m(\vec{b})$ are open disks around $\varphi^\pm_m(\vec{b})$, the points $\varphi^\pm_m(\vec{b})$ being the two zeros in $\mathbb{C}$ of $|\vec{k}(\varphi)+\bar{p}_m(\vec{b})|^2 = k^2$. Radius $r_m(\vec{b})$ of $O^\pm_m(\vec{b})$ is defined in Section 8.2.2. Lemma 2.16 is the combination of Lemma 8.9 proven in Section 8.2.1 and Lemma 8.8 proven in Section 8.2.3.

**8.2.1. Definition of the set $O^{(1)}(\vec{b})$**

Let us recall that $r_m$ are the radii of the discs of the first non-resonant set $O^{(1)}$. They are defined by formula (8.4). The radii $r_m(\vec{b})$ of the discs in $O^{(1)}(\vec{b})$ are, roughly speaking, defined as follows: $r_m(\vec{b}) = k^{-2+2\beta_1+22s_1+15\delta}$, when $m = 0$; $r_m(\vec{b}) \approx r_m$, when $p_m(\vec{b}) \geq 3k^{s_1}, |2k - p_m(\vec{b})| > k^{1-8s_1-4\delta}, r_m(\vec{b}) < r_m$, when $|2k - p_m(\vec{b})| \leq k^{1-8s_1-4\delta}$ or $p_m(\vec{b}) < 3k^{s_1}$. We have to reduce the $r_m(\vec{b})$ for smaller $p_m(\vec{b})$ and for $p_m(\vec{b})$ close to $2k$ in order to ensure that each component of $O^{(1)}(\vec{b})$ is sufficiently small (by (8.4), $r_m$ becomes bigger, when $p_m$ is small or tends to $2k$).

**Definition 8.8.** The radius $r_m(\vec{b})$ of the two open disks $O^\pm_m(\vec{b})$ in $\mathbb{C}$ centered at $\varphi^\pm_m(\vec{b})$ are defined by

\[
\begin{cases}
  k^{-2+2\beta_1+22s_1+15\delta} & \text{if } m = 0, \\
  \frac{k^{-2+2\beta_1+3s_1+2\delta}}{p_m(\vec{b})} & \text{if } m \neq 0, p_m(\vec{b}) < k^{1-8s_1-4\delta}, \\
  \frac{k^{-1+2\beta_1}}{p_m(\vec{b})\Pi_m(\vec{b})\sqrt{1 - \frac{p^2_m(\vec{b})}{4k^2}}} & \text{if } p_m(\vec{b}) \geq k^{1-8s_1-4\delta}, |2k - p_m(\vec{b})| > k^{1-8s_1-4\delta}, \\
  \frac{k^{-4+2\beta_1+10s_1+4\delta}}{\sqrt{1 - \frac{p^2_m(\vec{b})}{4k^2}}} & \text{if } |2k - p_m(\vec{b})| \leq k^{1-8s_1-4\delta}, 1 - \frac{p^2_m(\vec{b})}{4k^2} \geq \frac{1}{4}k^{-4+2\beta_1+10s_1+4\delta}, \\
  4k^{-2+\beta_1+5s_1+2\delta} & \text{if } 1 - \frac{p^2_m(\vec{b})}{4k^2} < \frac{1}{4}k^{-4+2\beta_1+10s_1+4\delta},
\end{cases}
\]

where $\Pi_m(\vec{b})$ is defined by (8.3), $\bar{p}_m(\vec{b})$ being substituted instead of $\bar{p}_m$.

Considering as in Lemma 8.9 we easily show $r_m(\vec{b}) = o(1)$ as $k \to \infty$.

**8.2.2. Lemma 8.9.**

**Lemma 8.9.** Let $100s_1 < \beta_1 < 1/12 - 28s_1 - 14\delta$. Suppose $\varphi \in \Phi_0 \setminus O^{(1)}(\vec{b})$. Then for every $m \in \mathbb{Z}^2$ such that

\[
|\vec{k}(\varphi) + \bar{p}_m(\vec{b})|^2 - k^2 < k^{\beta_1}
\]

(8.34)
the following inequalities hold:
\begin{equation}
\min_{0 < \rho_i < k^{s_1}} \left| \langle \tilde{k}(\varphi) + \tilde{\rho}_m(\tilde{b}) \rangle \right|^2 - k^2 > k^{\beta_1} \tag{8.35}
\end{equation}
\begin{equation}
\left| \langle \tilde{k}(\varphi) + \tilde{\rho}_m(\tilde{b}) \rangle \right|^2 - k^2 < \left| \langle \tilde{k}(\varphi) + \tilde{\rho}_m + \rho_{m+q_i}(\tilde{b}) \rangle \right|^2 - k^2 \right| > k^{2\beta_1}, \tag{8.36}
\end{equation}
when \(0 < p_{q_1}(0), p_{q_2}(0) < k^{s_1}\). This property is preserved in the \(k^{-4+2\beta_1-2s_1-\delta}\) neighborhood of \(\tilde{\Phi}_0 \setminus O^{(1)}(\tilde{b})\).

\textbf{Proof.} First, we consider the case \(m = 0\). It is easy to see that \(\left| \langle \tilde{k}(\varphi) + \tilde{b} \rangle \right|^2 - k^2 \geq k^b_0 r_0\), when \(\varphi \notin O^\pm(\tilde{b})\). Using the estimate \(b_0 \geq k^{-1-16s_1-12\delta}\) and the formula for \(r_0\), we get
\begin{equation}
\left| \langle \tilde{k} + b \rangle \right|^2 - k^2 \geq k^{-2+2\beta_1+6s_1+3\delta}. \tag{8.37}
\end{equation}

Next,
\begin{equation}
\left| \langle \tilde{k}(\varphi) + \tilde{\rho}_q(\tilde{b}) \rangle \right|^2 - k^2 = \left| \langle \tilde{k}(\varphi) + \tilde{\rho}_q(\tilde{b}) \rangle \right|^2 - \langle \tilde{k} + b \rangle \right|^2 + O(k^{\beta_1}) = 2 \left| \langle \tilde{k}(\varphi), \tilde{\rho}_q(\tilde{b}) \rangle \right|^2 + O(k^{\beta_1}), \quad i = 1, 2.
\end{equation}

Considering that \(\varphi \in \tilde{\Phi}_0\), we obtain
\begin{equation}
2 \left| \langle \tilde{k}(\varphi), \tilde{\rho}_q(\tilde{b}) \rangle \right|^2 > k^{1-3s_1-\delta}. \tag{8.38}
\end{equation}

Hence,
\begin{equation}
\left| \langle \tilde{k}(\varphi) + \tilde{\rho}_q(\tilde{b}) \rangle \right|^2 - k^2 > \frac{1}{4} k^{1-3s_1-\delta}, \quad i = 1, 2. \tag{8.39}
\end{equation}

Thus, (8.35) holds for \(m = 0\). Multiplying (8.37) and (8.39) for \(i = 1, 2\), we arrive at (8.36) for \(m = 0\).

Let \(p_m(\tilde{b}) < 3k^{1-8s_1-4\delta}, m \neq 0\). It easily follows from the definition of \(O^\pm(\tilde{b})\),
\begin{equation}
\left| \langle \tilde{k}(\varphi) + \tilde{\rho}_m(\tilde{b}) \rangle \right|^2 - k^2 \geq k p_m(\tilde{b}) r_m \geq k^{-1+2\beta_1+3s_1+\delta}. \tag{8.40}
\end{equation}

Next,
\begin{equation}
\left| \langle \tilde{k}(\varphi) + \tilde{\rho}_{m+q_i}(\tilde{b}) \rangle \right|^2 - \left| \langle \tilde{k}(\varphi) + \tilde{\rho}_m(\tilde{b}) \rangle \right|^2 = 2 \left| \langle \tilde{k}(\varphi), \tilde{\rho}_q(\tilde{b}) \rangle \right|^2 + O(k^{1-7s_1-4\delta}), \quad i = 1, 2.
\end{equation}

Considering that \(\varphi \in \tilde{\Phi}_0\), we again obtain (8.35). Hence,
\begin{equation}
\left| \langle \tilde{k}(\varphi) + \tilde{\rho}_m+q_i(\tilde{b}) \rangle \right|^2 - \left| \langle \tilde{k}(\varphi) + \tilde{\rho}_m(\tilde{b}) \rangle \right|^2 > \frac{1}{2} k^{1-3s_1-\delta}. \tag{8.41}
\end{equation}

Taking into account (8.34), we obtain
\begin{equation}
\left| \langle \tilde{k}(\varphi) + \tilde{\rho}_{m+q_i}(\tilde{b}) \rangle \right|^2 - k^2 > \frac{1}{4} k^{1-3s_1-\delta}. \tag{8.41}
\end{equation}

Thus, (8.35) holds for the case \(p_m(\tilde{b}) < 3k^{1-8s_1-4\delta}\). Multiplying (8.40) and (8.41) for \(i = 1, 2\), we obtain (8.36).

If \(p_m(\tilde{b}) \geq k^{1-8s_1-4\delta}\), \(|2k - p_m(\tilde{b})| > k^{1-8s_1-4\delta}\), we use the same considerations as in Lemma 8.2 to show that
\begin{equation}
\left| \langle \tilde{k}(\varphi) + \tilde{\rho}_m(\tilde{b}) \rangle \right|^2 - k^2 \geq \left| \langle \tilde{k}(\varphi) + \tilde{\rho}_{m+q_i}(\tilde{b}) \rangle \right|^2 - k^2 \right| > k^{2\beta_1}, \quad i = 1, 2.
\end{equation}

The estimates (8.35), (8.36) easily follow.
Let \(|2k - p_m(\bar{b})| < k^{1-8s_1-4\delta}\). Using (8.11) and considering as in Lemma 8.4, we obtain that
\[
|\bar{k}(\varphi) + \bar{p}_m(\bar{b})|^2 - k^2 > k^{-2+2\beta_1+10s_1+4\delta}
\]  
(8.42)
when \(\varphi \notin O^{(1)}(\bar{b})\). Suppose at least one of \(\varphi^\pm_m(\bar{b})\) is in the \((k^{-4s_1-2\delta})\)-neighborhood of \(\Phi_0\). Then, \(\Pi_m(\bar{b}) > \frac{1}{4} k^{1-9s_1-4\delta}\), see formula (8.30) and the footnote there. Using the inequality (8.14), see the footnote there, we obtain:
\[
|\bar{k}(\varphi) + \bar{p}_m(\bar{b})|^2 - k^2 > \frac{1}{2} \Pi_m(\bar{b}), \quad i = 1, 2.
\]  
(8.43)
Estimates (8.35), (8.36) easily follow. To finish the proof it remains to show that at least one \(\varphi^\pm_m(\bar{b})\) is in the \(k^{-4s_1-2\delta}\) neighborhood of \(\Phi_0\). If both \(\varphi^\pm_m(\bar{b})\) are not in the \(k^{-4s_1-2\delta}\) neighborhood of \(\Phi_0\), then each is inside a disc \(O^\pm_q\), \(0 < p_q < k^{s_1}\) and further then \(2k^{-4s_1-2\delta}\) from its boundary. Therefore, the distance of \(\varphi\) to \(\varphi^\pm_m\) is greater than \(k^{-4s_1-2\delta}\). However, (8.34) and (8.16), yield \(|\varphi - \varphi^\pm_m| \leq 10 k^{\beta_1/2-1}\). This contradiction proves that this case is not possible. Since all estimates are stable with respect to the perturbation of \(\varphi\) of order \(k^{-4+2\beta_1-2s_1-\delta}\), the statement of the lemma holds in the \(k^{-4+2\beta_1-2s_1-\delta}\) neighborhood of \(\Phi_0 \setminus O^{(1)}(\bar{b})\).

8.2.3. Lemma 8.10

**Lemma 8.10.** Suppose \(100s_1 < \beta_1 < 1/12 - 28s_1 - 14\delta\) and \(O^{(1)}(\bar{b})\) is a connected component of \(O^{(1)}(\bar{b})\). Then, the size of \(O^{(1)}(\bar{b})\) does not exceed \(ck^{-\gamma}\), \(\gamma = 11/6 - \beta_1 - 12s_1 - 4\delta\). A component \(O^{(1)}(\bar{b})\) contains no more than \(c_1 k^{2/3+2s_1}\) discs. The total size of \(O^{(1)}(\bar{b})\) does not exceed \(2\pi c_1 k^{-5/6+\beta_1+12s_1+4\delta}\). The total number of discs in \(O^{(1)}(\bar{b})\) is less than \(ck^{2+2s_1}\).

**Proof.** Let \(\Delta_*\) be a rectangle in \(\mathbb{C}\): \(\Delta_* = \{\varphi : |\Re \varphi - \varphi_*| \leq k^{-1}, \ |\Im \varphi| < k^{-\delta}\}\) for some \(\varphi_* \in [0, 2\pi]\). Clearly, \(\Phi_0\) is the union of such rectangles. Let
\[
I_* = \{m \in \mathbb{Z}^2 : p_m(\bar{b}) < 4k, \ \varphi^\pm_m(\bar{b}) \in \Delta_*\},
\]
where \(\varphi^\pm_m(\bar{b}) \in \Delta_*\) means that either \(\varphi^+_m(\bar{b}) \in \Delta_*\) or \(\varphi^-_m(\bar{b}) \in \Delta_*\). We will show that the number of points in \(I_*\) does not exceed \(6c_0 k^{1+2s_1}\) and
\[
\Sigma_* := \sum_{m \in I_*} r_m(\bar{b}) < ck^{-\gamma}
\]  
(8.44)
This means that the total size of \(O^{(1)}(\bar{b}) \cap \Delta_*\) is less than \(k^{-\gamma}\). Considering that \(\gamma > 1\), we easily obtain that the size of each \(O^{(1)}(\bar{b})\) is less than \(ck^{-\gamma}\). Since \(\Phi_0\) consists of \(2\pi k\) rectangles \(\Delta_*\), the total size of \(O^{(1)}(\bar{b})\) does not exceed \(k^{-5/6+\beta_1+12s_1+4\delta}\). Let us prove (8.44). In accordance with the Definition 8.8 and by analogy with the proof of Lemma 8.6 we split \(I_*\) into six sets:
\[
I_{*0} := \{m = 0\}, \quad I_{*1} := \{m \in I_* \setminus \{0\} : p_m(\bar{b}) < k^{1-8s_1-4\delta}\},
\]
\[
I_{*2} := \{m \in I_* : p_m(\bar{b}) \geq k^{1-8s_1-4\delta}, \ 2k - p_m(\bar{b}) > k^{1-8s_1-4\delta}\},
\]
\[
I_{*3} := \{m \in I_* : p_m(\bar{b}) - 2k > k^{1-8s_1-4\delta}\},
\]
\[
I_{*4} := \{m \in I_* : p_m(\bar{b}) - 2k < k^{1-8s_1-4\delta}\},
\]
\[
I_{*5} := \{m \in I_* : p_m(\bar{b}) - 2k = k^{1-8s_1-4\delta}\}.
\]
Let us prove (8.48). Definition of 

\[ I_{s4} := \left\{ m \in I_s : |2k - p_m(\vec{b})| \leq k^{1-8s_1-\delta}, \left| 1 - \frac{p_m^2(\vec{b})}{4k^2} \right| \geq \frac{1}{4} |k - 4 + 2\beta_1 + 10s_1 + 4\delta| \right\}, \]

\[ I_{s5} := \left\{ m \in I_s : \left| 1 - \frac{p_m^2(\vec{b})}{4k^2} \right| < \frac{1}{4} |k - 4 + 2\beta_1 + 10s_1 + 4\delta| \right\}. \]

We define \( \Sigma_{s_j} := \sum_{m \in I_{s_j}} r_m(\vec{b}), \ j = 0, \ldots, 5, \) and get an upper bound for each sum. Formula

\[ \Sigma_{s0} = k^{-2+2\beta_1+22s_1+15\delta} \tag{8.45} \]

immediately follows from Definition 8.8. Obviously,

\[ \Sigma_{s1} = k^{-2+2\beta_1+3s_1+2\delta} \sum_{m \in I_{s1}} \frac{1}{p_m(\vec{b})}. \tag{8.46} \]

We estimate the sum in the right-hand side by an integral:

\[ \sum_{m \in I_{s1}} \frac{1}{p_m(\vec{b})} \leq k^{2s_1} + k^{2s_1} \int_{I'_{s1}} \frac{1}{r} r dr d\varphi < k^{3s_1}, \]

where \( I'_{s1} \) is a \( (ck^{-s_1}) \) neighborhood of \( I_{s1} \). Substituting the last estimate into (8.46), we obtain:

\[ \Sigma_{s1} < ck^{-2+2\beta_1+6s_1+2\delta}. \tag{8.47} \]

Let us estimate \( \Sigma_{s2} \). First we note that \( \sqrt{1 - \frac{p_m^2(\vec{b})}{4k^2}} > \frac{1}{4} k^{-4s_1-2\delta} \), since \( 2k - p_m > k^{1-8s_1-4\delta} \). For the moment we assume that

\[ \sum_{m \in I_{s2}} \frac{1}{\Pi_m(\vec{b})} \leq ck^{7s_1+3\delta}. \tag{8.48} \]

Then,

\[ \Sigma_{s2} = \sum_{m \in I_{s2}} \frac{k^{-1+2\beta_1}}{p_m \Pi_m(\vec{b})\sqrt{1 - \frac{p_m^2(\vec{b})}{4k^2}}} \leq \frac{ck^{-1+2\beta_1}}{k^{1-8s_1-4\delta} \cdot k^{-4s_1-2\delta}} \sum_{m \in I_{s3}} \frac{1}{\Pi_m(\vec{b})}. \]

Thus,

\[ \Sigma_{s2} < ck^{-2+2\beta_1+19s_1+9\delta}. \tag{8.49} \]

Let us prove (8.48). Definition of \( \varphi_m^\pm(\vec{b}) \) yields:

\[ |\vec{k} (\varphi_m^\pm(\vec{b}))| = k, \quad |\vec{k} (\varphi_m^\pm(\vec{b})) + \vec{p}_m(\vec{b})| = k. \tag{8.50} \]

Considering that \( |\varphi_m^\pm(\vec{b}) - \varphi_s| < k^{-1} \), we obtain: \( |\vec{k}(\varphi_s)| = k, \ |\vec{k}(\varphi_s) + \vec{p}_m(\vec{b})| = k + O(1), \ |O(1)| < 1 \). This means that \( \vec{p}_m(\vec{b}) \) belongs to the ring \( \mathcal{R} \) of radii \( k \pm 1 \) centered at \( \vec{k}(\varphi_s) \). We split \( \mathcal{R} \) into several components depending on the value of \( \Pi_m(\vec{b}) \). Indeed, let

\[ \Omega_{s_j} = \{ m \in \mathbb{Z}^2 : \vec{p}_m(\vec{b}) \in \mathcal{R}, \ k^{2s_1+j\delta} \leq \Pi_m(\vec{b}) < k^{2s_1+(j+1)\delta} \}, \]

\( j = 0, \ldots, J, \ J = \left\lfloor \frac{1-8s_1}{\delta} \right\rfloor \). Let us show that

\[ \sum_{m \in \Omega_{s_j}} \Pi_m^{-1}(\vec{b}) < 4k^{7s_1+2\delta}. \tag{8.51} \]
We estimate the number of points in $\Omega_s$. Indeed, by the definition of $\Pi_m$ for every $m \in \Omega_s$ there is a $\vec{p}_m$, such that $0 < p_{qm} < k^{s_1}$ and

$$< \vec{k} \left( \varphi_m^G(b) + \vec{p}_m(b), \vec{p}_m > \right) + k^{2s_1} < k^{2s_1+(j+1)\delta}.$$ 

Considering that $|\varphi_m^G(b) - \varphi_s| < k^{-1}$, we obtain that

$$< \vec{k}(\varphi_s) + \vec{p}_m(b), \vec{p}_q > < 2k^{2s_1+(j+1)\delta}$$

for a $q : 0 < p_q < k^{s_1}$, and $\vec{p}_m \in \mathcal{R}$.

The number of such points in $\mathcal{R}$, obviously, does not exceed $4k^{5s_1+(j+1)\delta}$ for a fixed $q$ and $4k^{5s_1+(j+1)\delta}$ for all $q$. Using now the estimate $\Pi_m \geq k^{2s_1+j\delta}$, we arrive at (8.51).

The estimate (8.48) easily follows.

Let us estimate $\Sigma_{s3}$. Using (8.30), we get $r_m < 2k^{-3+2\beta_1+13s_1+6\delta}$.

By (8.10), $\mathbb{R}\varphi_m^G + \pi = \varphi_m$ (mod $2\pi$) when $m \in I_{s3}$. Therefore, for any $m \in I_{s3}$, we have $|\varphi_m - \varphi_s| < k^{-1}$. Now it is obvious that the number of points in $I_{s3}$ does not exceed $ck$. The estimate

$$\Sigma_{s3} < ck^{-2+2\beta_1+13s_1+6\delta}$$

(8.52)

easily follows. Next, we estimate $\Sigma_{s4}$. Suppose we have checked that

$$\sum_{m \in I_{s4}} \frac{1}{\sqrt{|2k - p_m(b)|}} < ck^{5/3 - \beta_1 + 2s_1}.$$  

(8.53)

Then,

$$\Sigma_{s4} = \sum_{m \in I_{s4}} \frac{k^{-4+2\beta_1+10s_1+4\delta}}{\sqrt{1 - \frac{p_m^2}{4k^2}}} \leq ck^{-1/6 + \beta_1 + 12s_1 + 4\delta} \sum_{m \in I_{s4}} \frac{1}{\sqrt{|2k - p_m|}}.$$ 

Therefore,

$$\Sigma_{s4} < ck^{-11/6 + \beta_1 + 12s_1 + 4\delta}.$$  

(8.54)

To show (8.53), we note that $|2k - p_m| \geq \frac{1}{2}k^{-3+2\beta_1+10s_1+4\delta}$ follows from $1 - \frac{p_m^2}{4k^2} \geq \frac{1}{2}k^{-4+2\beta_1+10s_1+4\delta}$. We split $I_{s4}$ into three regions:

- $\omega_{s1} := \{m \in I_{s4}, k^\delta \leq |2k - p_m| \leq k^{1-8s_1-4\delta}\}$,
- $\omega_{s2} := \{m \in I_{s4}, k^{-1} \leq |2k - p_m| < k^\delta\}$,
- $\omega_{s3} := \{m \in I_{s4}, \frac{1}{2}k^{-3+2\beta_1+10s_1+4\delta} \leq |2k - p_m| < k^{-1}\}$.

The corresponding sums we denote as $\sigma_j$, $j = 1, 2, 3$. It is easy to estimate $\sigma_1$ by an integral:

$$\sigma_1 := \sum_{k^\delta \leq |2k - p_m| \leq k^{1-8s_1-4\delta}} \frac{2}{\sqrt{|2k - p_m|}} < ck \int_{k^\delta}^{k^{1-8s_1-4\delta}} \int_0^{2\pi} \frac{1}{\sqrt{t}} d\varphi dt \leq ck^{5/2 - 4s_1 - 2\delta}.$$ 

Now we estimate $\sigma_2$. It is easy to see that the number of points in $\omega_{s2}$ does not exceed $c_0k^{1+\delta + 2s_1}$. Using the estimate $2k - p_m > k^{-1}$, we obtain:

$$\sigma_2 < ck^{3/2 + 2s_1 + \delta}.$$
Let us estimate \( \sigma_3 \). Definition of \( \varphi_m^\pm(\bar{b}) \) yields \( |k\left( \varphi_m^\pm(\bar{b}) \right)_{\ast}^2 = k^2, |k\left( \varphi_m^\pm(\bar{b}) \right) + \bar{p}_m(\bar{b}) |^2 = k^2 \). By (8.10), \( |\Re\varphi_m^\pm| < k^{-1} \) when \( m \in \omega_3 \). Therefore, \( |\varphi_m^\pm - \varphi| < 2k^{-1} \). Hence, we obtain: \( |k(\varphi_\ast)| = k, |k(\varphi_\ast + \bar{p}_m(\bar{b}))| = k + O(1), |O(1)| < 1 \). This means that \( \bar{p}_m(\bar{b}) \) belongs to the ring \( \mathcal{R} \) of radii \( k \pm 1 \) centered at \( k(\varphi_\ast) \). We also introduce the ring \( \mathcal{R}_\ast = \{ \bar{k} : |\bar{k}| - 2k < k^{-1} \} \). Obviously \( \omega_{\ast 3} \subset \mathcal{R}_\ast \). Thus, \( \omega_{\ast 3} \subset \mathcal{R} \cap \mathcal{R}_\ast \). It is not difficult to show that the ring \( \mathcal{R} \) cuts from \( \mathcal{R}_\ast \) an “arc” of the length \( O(k^{1/2}) \). Using well-known results from the theory of lattices, see [27], we obtain that \( \mathcal{R} \cap \mathcal{R}_\ast \) contains no more than \( ck^{1/6+\varepsilon_1} \) points \( \bar{p}_m(\bar{b}) \) for any \( \bar{b} \). Considering the estimate \( |2k - p_m| \geq \frac{1}{2} |k - 3 + 2\beta_1 + 10s_1 + 4\delta | \), we arrive at the inequality:

\[
\sigma_3 < ck^{5/3 - \beta_1 - 4s_1 - 2\delta}.
\]

Adding the estimates for \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) and considering that \( \beta_1 < 1/6 - 10s_1 - 11\delta \), we obtain (8.55).

Let us estimate \( \Sigma_{\ast 5} \). The number of points in \( I_{\ast 5} \) can be estimated the same way as the number of points in \( \omega_{\ast 3} \), i.e., it is less than \( ck^{1/6+\varepsilon_1} \). Hence,

\[
\Sigma_{\ast 5} < ck^{-11/6 + \beta_1 + 6s_1 + 2\delta}.
\] (8.55)

Adding the estimates (8.45), (8.47), (8.52), (8.49), (8.54) and (8.55) and considering that \( \beta_1 < 1/6 - 10s_1 - 11\delta \), we obtain (8.44).

Let us show that the number of points in \( I_\ast \) does not exceed \( c_0 k^{1/2+\varepsilon_1} \). First, we count all real \( \varphi_m^\pm(\bar{b}) \). Obviously, \( |\varphi_\ast - \varphi_m^\pm(\bar{b})| < k^{-1} \). Using (8.50), we obtain \( |k(\varphi_\ast + \bar{p}_m(\bar{b}))| = k + O(1), |O(1)| < 1 \). The number of points \( \bar{p}_m(\bar{b}) \) in this ring does not exceed \( c_0 k^{1/2+\varepsilon_1} \), \( s_1 \) appearing since the size of our lattice is of order \( k^{-s_1} \). If \( \Im\varphi_m^\pm(\bar{b}) \neq 0 \), then, by (8.10), \( \varphi_m = \Re\varphi_m^\pm(\bar{b}) + \pi \) modulo \( 2\pi \). Hence, \( |\varphi_m - \pi - \varphi_\ast| < k^{-1} \). Considering that \( p_m(\bar{b}) < 4k \), we obtain that the number of such points does not exceed \( 4c_0 k^{1/2+\varepsilon_1} \). Hence, the total number of points in \( I_\ast \) does not exceed \( 6c_0 k^{1/2+\varepsilon_1} \).

It remains to show that a component \( \mathcal{O}_c^{(1)}(\bar{b}) \) contains no more than \( c_1 k^{2/3+\varepsilon_1} \) discs. Indeed, the length of \( \mathcal{O}_c^{(1)}(\bar{b}) \) is less than \( ck^{-\gamma} \). This means that all points \( \varphi_m^\pm(\bar{b}) \) belonging to \( \mathcal{O}_c^{(1)}(\bar{b}) \), are, in fact, in a square of the size \( ck^{-\gamma} \) centered at a point \( \varphi_\ast \in \Delta_\ast \). Let us consider all real \( \varphi_m^\pm(\bar{b}) \) in the component. Obviously \( |\Re\varphi_\ast - \varphi_m^\pm(\bar{b})| < ck^{-\gamma} \). Using (8.50), we obtain \( |k(\Re\varphi_\ast + \bar{p}_m(\bar{b}))| = k + O(k^{1-\gamma}) \). By (27), the number of points \( \bar{p}_m(\bar{b}) \) in this ring does not exceed \( c_1 k^{2/3+\varepsilon_1} \), \( s_1 \) appearing since the size of our lattice is of order \( k^{-s_1} \). If \( \Im\varphi_m^\pm(\bar{b}) \neq 0 \), then, by (8.10), \( \varphi_m = \Re\varphi_m^\pm(\bar{b}) + \pi \) modulo \( 2\pi \), and \( \cos\Im\varphi_m^\pm(\bar{b}) = p_m(\bar{b})/2k \). Hence, \( |\varphi_m - \pi - \Re\varphi_\ast| < ck^{-\gamma} \) and \( |p_m - x_\ast| < ck^{-\gamma}, x_\ast = k \cos\Im\varphi_\ast \). Obviously the number of points \( \bar{p}_m(\bar{b}) \) is such a region is \( O(1) \). Hence the total number of points \( \varphi_m^\pm(\bar{b}) \) in \( \mathcal{O}_c^{(1)}(\bar{b}) \) is less than \( c_1 k^{2/3+\varepsilon_1} \). Considering that the number of points \( \bar{p}_m(\bar{b}) \) satisfying the inequality \( p_m(\bar{b}) < 4k \) is less than \( ck^{2+2\varepsilon_1} \), we obtain that the total number of discs in \( \mathcal{O}_c^{(1)}(\bar{b}) \) is less than \( ck^{2+2\varepsilon_1} \).

9. Appendices.
Appendix 1. Proof of Lemma [2.30]. By Lemma [2.10] Part 2, the function \( z_1(\varphi) \) is holomorphic in \( \hat{\Phi}_1 \) and \( \lambda^1(\bar{z}_1(\varphi)) = k^2 \). Hence, the equation (2.130) is equivalent to 
\[
\lambda^1(\bar{g}^1(\varphi)) = \lambda^1(\bar{g}^1(\varphi) - \bar{b}) + \epsilon_0.
\]
We use perturbation formula (2.58): 
\[
|g^1(\varphi)|^2 + f_1(\bar{g}^1(\varphi)) = |\bar{g}^1(\varphi) - \bar{b}|^2 + f_1(\bar{g}^1(\varphi) - \bar{b}) + \epsilon_0.
\]
This equation can be rewritten as 
\[
2(\bar{g}^1(\varphi), \bar{b}) + \epsilon_0 = 0.
\]
Using the notation \( \bar{b} = b_0(\cos \varphi_b, \sin \varphi_b) \), dividing both sides of the equation (9.1) by \( 2b_0k \), and considering that \( \bar{g}^1(\varphi) = \bar{z}_1(\varphi) + \bar{b} = (k + h_1)\bar{\nu} + \bar{b} \), we obtain:
\[
\cos(\varphi - \varphi_b) - \epsilon_0 g_1(\varphi) + g_2(\varphi) = 0,
\]
where \( g_1(\varphi) = (2b_0k)^{-1} \) and 
\[
g_2(\varphi) = \frac{\langle \bar{h}_1(\varphi), \bar{b} \rangle}{b_0k} - \frac{b_0}{2k} + \left( f_1(\bar{g}^1(\varphi)) - f_1(\bar{g}^1(\varphi) - \bar{b}) \right) g_1(\varphi), \quad \bar{h}_1(\varphi) = h_1(\varphi)\bar{\nu}.
\]
Let us estimate \( g_2(\varphi) \). Using the inequality (2.70) for \( h_1 \), and considering that \( b_0 < k^{-1-16\delta} \), we easily obtain:
\[
\left| \frac{\langle \bar{h}_1(\varphi), \bar{b} \rangle}{b_0k} \right| \leq \frac{2h_1}{k} = O(k^{-4+30\delta}), \quad \frac{b_0}{2k} \leq \frac{1}{2}k^{-2-16\delta}.
\]
By (2.61), 
\[
|f_1(\bar{g}^1(\varphi)) - f_1(\bar{g}^1(\varphi) - \bar{b})| \leq \sup |\nabla f_1| |b_0| = O(k^{-2+33\delta}b_0),
\]
and therefore, 
\[
\left| \left( f_1(\bar{g}^1(\varphi)) - f_1(\bar{g}^1(\varphi) - \bar{b}) \right) g_1(\varphi) \right| = O(k^{-3+33\delta}b_0).\]
Thus, we have 
\[
g_2(\varphi) = O(k^{-2-16\delta}).\]
Using \( \epsilon_0 < b_0k^{-1-16\delta} \), we obtain \( \epsilon_0 g_1(\varphi) < k^{-2-16\delta} \).

Thus, 
\[
g_2(\varphi) - \epsilon_0 g_1(\varphi) = O(k^{-2-16\delta}).
\]
Suppose that \( \varphi_b \pm \frac{\pi}{2} \) is in the \( \left( \frac{4}{5}k^{-2-16\delta} \right) \)-neighborhood of \( \hat{\Phi}_1 \). We draw two circles \( C_\pm \) centered at \( \varphi_b \pm \frac{\pi}{2} \) with the radius \( \frac{1}{5}k^{-2-16\delta} \). They are both inside the complex \( 2k^{-2-16\delta} \)-neighborhood of \( \hat{\Phi}_1 \), the perturbation series converging and the estimate (9.3) holds. For any \( \varphi \) on \( C_\pm \), \( |\varphi - (\varphi_b \pm \pi/2)| = \frac{1}{5}k^{-2-16\delta} \) and, therefore, \( |\cos(\varphi - \varphi_b)| > \frac{1}{16}k^{-2-16\delta} > |g_2(\varphi) - \epsilon_0 g_1(\varphi)| \) for any \( \varphi \in C_\pm \). By Rouché’s Theorem, there is only one solution of the equation (9.2) inside each \( C_\pm \). If \( \varphi_b + \pi/2 \) is not in the \( \left( \frac{4}{5}k^{-2-16\delta} \right) \)-neighborhood of \( \hat{\Phi}_1 \), then \( |\cos(\varphi - \varphi_b)| > \frac{1}{16}k^{-2-16\delta} \) in \( \hat{\Phi}_1 \) and, hence, equation (9.2) has no solution. Thus, there are at most two solutions in \( \hat{\Phi}_1 \) and \( |\varphi_{\epsilon_0} - (\varphi_b \pm \pi/2)| < \frac{1}{5}k^{-2-16\delta} \).

Appendix 2. Proof of Lemma [2.31]. Using the perturbation formula (2.58), we obtain:
\[
\frac{\partial}{\partial \varphi} \lambda^1(\bar{g}^1(\varphi)) = \frac{\partial}{\partial \varphi} \left[ \lambda^1(\bar{g}^1(\varphi)) - k^2 \right] = \frac{\partial}{\partial \varphi} \left[ \lambda^1(\bar{g}^1(\varphi)) \right] = \left\langle \nabla_{\bar{g}} \lambda^1(\bar{g}^1(\varphi)) - \nabla_{\bar{g}} \lambda^1(\bar{g}^1(\varphi) - \bar{b}), \frac{\partial}{\partial \varphi} \bar{g}^1(\varphi) \right\rangle.*
\]
where $\vec{v} = (\cos \varphi, \sin \varphi)$ and $\vec{\mu} = \vec{v}' = (- \sin \varphi, \cos \varphi)$. Note that
\[
\nabla |\vec{y}^{(1)}(\varphi)|^2 - \nabla |\vec{y}^{(1)}(\varphi) - \vec{b}|^2 = \vec{y}^{(1)}(\varphi) - 2(\vec{y}^{(1)}(\varphi) - \vec{b}) = 2\vec{b}
\]
Substituting (9.5) into (9.4), we get
\[
T_1 = 2 \left( \vec{b}, (k + h_1)\vec{\mu} + h_1'\vec{v} \right)_* \\
T_2 = \left( \nabla f_1(\vec{y}^{(1)}(\varphi)) - \nabla f_1(\vec{y}^{(1)}(\varphi) - \vec{b}), (k + h_1)\vec{\mu} + h_1'\vec{v} \right)_*
\]
Considering that $\varphi$ is close to $\varphi_0 \pm \pi/2$, we readily obtain: $\langle \vec{b}, \vec{\mu} \rangle_* = o(b_0)$, $\langle \vec{b}, \vec{\mu} \rangle_* = \pm b_0(1 + o(1))$. Using also estimates (2.70) for $h_1$, we get $T_1 = \pm 2b_0k(1 + o(1))$. By (2.63), $\left| \nabla f_1(\vec{y}^{(1)}(\varphi)) - \nabla f_1(\vec{y}^{(1)}(\varphi) - \vec{b}) \right| = O(b_0k^{-2+36\delta_1+24\delta})$. Hence, $T_2 = o(b_0k)$.

Adding the estimates for $T_1, T_2$, we get (2.132).

**Appendix 3. Proof of Corollary 3.6**

Let $\tau_0 \in \chi_2$. Taking into account the relation $\lambda_j^{(1)}((\tau_0 + 2\pi p/N_1a) = k^2$ and the definition of $C_2$, we see that $|\lambda_j^{(1)}((\tau_0 + 2\pi p/N_1a) - z| = \epsilon_1/2$. Using (3.15) and the last equality, we easily obtain: $|\lambda_j^{(1)}((\tau_0 + 2\pi p/N_1a) - z| \geq \epsilon_1/2$ for $\lambda_j^{(1)}((\tau_0 + 2\pi p/N_1a) \neq \lambda_j^{(1)}((\tau_0 + 2\pi p/N_1a)$. Therefore, for any $z \in C_2$,
\[
\|(\tilde{H}^{(1)}(\tau_0) - z)^{-1}\| \leq 2/\epsilon_1, \quad (9.6)
\]
i.e., (3.18) is proved for $\tau_0 \in \chi_2$. Now we consider $\tau$ in the complex $(\epsilon_1 k^{-1-\delta})$—neighborhood of $\chi_2$. By Hilbert relation,
\[
(\tilde{H}^{(1)}(\tau) - z)^{-1} = (\tilde{H}^{(1)}(\tau_0) - z)^{-1} + T_1T_2(\tilde{H}^{(1)}(\tau) - z)^{-1},
\]
\[
T_1 = (\tilde{H}(\tau_0) - z)^{-1}(\tilde{H}_0(\tau_0) + k^2), \quad T_2 = (\tilde{H}_0(\tau_0) + k^2)^{-1}(\tilde{H}_0(\tau_0) - \tilde{H}_0(\tau)).
\]
Suppose we have checked that $\|T_1T_2\| < k^{-\delta}$. Then, using (9.6), we easily arrive at (3.18). The estimate $\|T_1\| < 4k^2/\epsilon_1$, easily follows from (9.6). The estimate $\|T_2\| < 2\epsilon_1 k^{-2-\delta}$ easily follows from $|\tau - \tau_0| < \epsilon_1 k^{-1-\delta}$. Thus, $\|T_1T_2\| < 8k^{-\delta}$ and, hence, (3.18) is proved.

**REFERENCES**

[1] J. Avron, B. Simon *Almost Periodic Schrödinger Operators I: Limit Periodic Potentials*. Commun. Math. Physics, 82 (1981), 101 – 120.
[2] V.A. Chulaevski *On perturbation of a Schrödinger Operator with Periodic Potential*. Russian Math. Surv., 36(5), (1981), 143 – 144.
[3] J. Moser *An Example of the Schrödinger Operator with Almost-Periodic Potentials and Nowhere Dense Spectrum*. Comment. Math. Helv., 56 (1981), 198 – 224.
[4] B. Simon *Almost Periodic Schrödinger Operators. A Review*. Advances in Applied Mathematics, 3 (1982), 463 – 490.
[5] L.A. Pastur, V.A. Tkachenko On the Spectral Theory of the One-Dimensional Schrödinger Operator with Limit-Periodic Potential. Dokl. Akad. Nauk SSSR, 279 (1984), 1050 – 1053; Engl. Transl.: Soviet Math. Dokl., 30 (1984), no. 3, 773 – 776

[6] L.A. Pastur, V.A. Tkachenko Spectral Theory of a Class of One-Dimensional Schrödinger Operators with Limit-Periodic Potentials. Trans. Moscow Math. Soc., 51 (1989), 115 – 166. in disordered, nonlinear (1986), 247–274. MR 87i:82083

[7] L. Pastur, A. Figotin Spectra of Random and Almost-Periodic Operators. Springer-Verlag, 1992, 583 pp.

[8] J. Avron, B. Simon Cantor Sets and Schrödinger Operators: Transient and Recurrent Spectrum. J. Func. Anal., 43 (1981), 1 – 31.

[9] S.A. Molchanov and V.A. Chulaevskii Structure of the Spectrum of Lacunary Limit-Periodic Schrödinger Operator. Func. Anal. Appl., 18 (1984), 91 – 92.

[10] L. Zelenko On a Generic Topological Structure of the Spectrum to One-Dimensional Schrödinger Operators with Complex Limit-Periodic Potentials. Integral Equations and Operator Theory, 50 (2004), 393 – 430.

[11] M.A. Shubin The Density of States for Selfadjoint Elliptic Operators with Almost Periodic Coefficients. Trudy sem. Petrovskii (Moscow University), 3 (1978), 243 – 275.

[12] M.A. Shubin Spectral Theory and Index of Elliptic Operators with Almost Periodic Coefficients. Russ. Math. Surveys, 34(2), (1979), 109 – 157.

[13] J. Avron, B. Simon Almost Periodic Schrödinger Operators II: The Integrated Density of States. Duke Math. J., 50 (1983), 1, 369 – 391.

[14] G.V. Rozenblum, M.A. Shubin, M.Z. Solomyak Spectral Theory of Differential Operators. Encyclopaedia of Mathematical Sciences, 64, 1994.

[15] D. Damanik, Zh. Gan Spectral properties of Limit-periodic Schrödinger operators. ArXiv: 0906.3337v1 [math.SP], 18 June, 2009, pp 1-12.

[16] Yu. P. Chuburin On the Multidimensional Discrete Schrödinger Equation with a Limit Periodic Potential. Theoretical and Mathematical Physics, 102 (1995), no. 1, 53 – 59.

[17] M.M. Skriganov, A.V. Sobolev On the spectrum of a limit-periodic Schrödinger operator Algebra i Analiz, 17 (2005), 5; English translation: St. Petersburg Math. J. 17 (2006), 815-833.

[18] G. Gallavotti, Perturbation Theory for Classical Hamiltonian Systems. Scaling and Self-Similarity in Progr. Phys., 7, edited by J. Froehlich, Birkhäuser, Basel, Switzerland, 1983, 359 – 424.

[19] L.E. Thomas, S. R. Wassel, Stability of Hamiltonian systems at high energy. J. Math. Phys. 33(10), (1992), 3367 – 3373.

[20] L.E. Thomas and S. R. Wassel, Semiclassical Approximation for Schrödinger Operators at High Energy, Lecture Notes in Physics, 403, edited by E. Balslev, Springer-Verlag, 1992, 194 – 210.

[21] J. Bourgain Quasiperiodic Solutions of Hamilton Perturbations of 2D Linear Schrödinger Equation. Ann. of Math. (2), 148 (1998), 2, 363–439.

[22] Reed, M., Simon, B. Methods of Modern Mathematical Physics., Vol IV, Academic Press, 3rd ed., New York – San Francisco – London (1987), 396 pp.

[23] Yu. Karpeshina Perturbation theory for the Schrödinger operator with a periodic potential. Lecture Notes in Mathematics, 1663, Springer-Verlag, 1997, 352 pp.

[24] Yu. Karpeshina and Y.-R. Lee, Spectral properties of polyharmonic operators with limit-periodic potential in dimension two, Journal d’Analyse Mathematique, 102 (2007), 225-310.

[25] Yu. Karpeshina and Y.-R. Lee, Absolutely Continuous Spectrum of a Polyharmonic Operator with a Limit Periodic Potential in Dimension Two, Communications in Partial Differential Equations, 33, 9, 2008, 1711-1728.

[26] I.M. Gel’fand Expansion in Eigenfunctions of an Equation with Periodic Coefficients. Dokl. Akad. Nauk SSSR, 73 (1950), 1117-1120 (in Russian).

[27] E.Krtzce Lattice Points. Series: Mathematics and its Applications, 33, 1989, 324 p.
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