COMPLEXITY OF SHIFT SPACES ON SEMIGROUPS

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Abstract. Let \( G = \langle S | R_A \rangle \) be a semigroup with generating set \( S \) and equivalences \( R_A \) among \( S \) determined by a matrix \( A \). This paper investigates the complexity of \( G \)-shift spaces by yielding the topological entropies. After revealing the existence of topological entropy of \( G \)-shift of finite type (\( G \)-SFT), the calculation of topological entropy of \( G \)-SFT is equivalent to solving a system of nonlinear recurrence equations. The complete characterization of topological entropies of \( G \)-SFTs on two symbols is addressed, which extends [Ban and Chang, arXiv:1803.03082] in which \( G \) is a free semigroup.

1. Introduction

Let \( \mathcal{A} \) be a finite alphabet and \( G \) be a group. A configuration is a function \( f : G \to \mathcal{A} \) and a pattern is a function from a finite subset of \( G \) to \( \mathcal{A} \). A subset \( X \subseteq \mathcal{A}^G \) is called a shift space if \( X = X_F \) which is a set of configurations which avoid patterns from some set \( \mathcal{F} \) of patterns. If \( \mathcal{F} \) is finite we call such a shift space shift of finite type (SFT). Let \( n \in \mathbb{N} \) and denote by \( E_n \) the set of elements in \( G \) whose length are less than or equal to \( n \). We define \( \Gamma_n(X) \) the set of all possible patterns of \( X \) in \( E_n \) and set \( \gamma_n = |\Gamma_n| \), i.e., the number of \( \Gamma_n \). The topological entropy (entropy for short) of \( X \) is defined as

\[
(1) \quad h(X) = \limsup_{n \to \infty} \frac{\ln \gamma_n}{|E_n|}.
\]

It is known that the value \((1)\) measures the exponential growth rate of the number of admissible patterns. From the dynamics viewpoint, it is a measure of the randomness or complexity of a given physical or dynamical system (cf. \([25, 11, 17, 22]\)). From the information theory viewpoint, it measures how much information can be stored in the set of allowed sequences.

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The value $h(X)$ (we write it simply $h$) exists for $G = \mathbb{Z}^1$ under the classical subadditive argument, and its entropy formula and algebraic characterization is given by D. Lind [20, 21]; that is, the nonzero entropies of $\mathbb{Z}^1$-SFTs are exactly the non-negative rational multiples of the logarithm of Perron numbers. Generally, if $G$ is amenable ($\mathbb{Z}^d$ is amenable), $h$ exists due to the fact that $G$ satisfies the Følner condition [16, 19, 12]. The characterization of the entropies for $G = \mathbb{Z}^d$, $d \geq 2$ is given by Hochman-Meyerovitch [18], namely, the entropies of such $G$-SFTs are the set of right recursively enumerable numbers. In [10], the authors found the recursive formula of $h$ in $\mathbb{Z}^2$-SFTs, some explicit value of $h$ can be computed therein. Finally, the quantity $H$ can also be used to characterize the chaotic spatial behavior for $\mathbb{Z}^2$ lattice dynamical systems (cf. [15, 13, 14, 9, 8]).

For $G = FS_d$, the free semigroup with generators $S = \{s_1, \ldots, s_d\}$ (it is not amenable), the existence of the limit (1) is due to the recent result of Petersen-Salama [23]. Its entropy formula for $d = 2$ and $|A| = 2$ is presented by Ban-Chang [6]. For $G = F_2$, the free group with 2 generators, Piantadosi [24] finds an approximation of $h \approx 0.909155$ for the $F_2$-golden mean shift, i.e., the forbidden set $F \subset A \times S \times A$ is defined by $F = \{(2, s_i, 2) : i = 1, 2\}$ for $|A| = 2$. Either the existence of the limit of (1) or to find the exact value of $h$ for a $F_d$-SFT is an open problem. The specification properties or the chaotic behavior of $FS_d$-SFTs can be found in [4, 3].

A semigroup is a set $G = \langle S \mid R \rangle$ together with a binary operation which is closed and associative, where $R$ is a set of equivalences which describe the relations among $S$. A monoid is a semigroup with an identity element $e$. Suppose $A \in \{0, 1\}^{d \times d}$ is a binary matrix. A semigroup/monoid $G$ of the form $G = \langle S \mid R_A \rangle$ means that $s_i s_j = s_i$ if and only if $A(i, j) = 0$. We note that $FS_d = \langle S \mid R_A \rangle$, where $S = \{s_1, \ldots, s_d\}$ and $A = E_d$, the $d \times d$ matrix with all entries being 1's. The aim of this paper is to find the entropy formula of $G$-SFTs. Although the discussion works for general cases, we focus on the case where $d = |A| = 2$ and $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ for the clarity and compactness of the idea.

In Section 2 we demonstrate that $\gamma_n$ solves some nonlinear recurrence equation with respect to the lattice $G$ and the rules $T = (T_1, T_2)$ (Theorem 2.2). Various types of recurrence equations, namely, the zero entropy type (type Z, Proposition

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1. A **Perron number** is a real algebraic integer greater than 1 and greater than the modulus of its algebraic conjugates.
2. The infimum of a monotonic recursive sequence of rational numbers
3. This means that the consecutive 22 is a forbidden pattern along the generators $s_1$ and $s_2$. 
In this section, we give the notations and some known results of $G$-SFTs. Let $\mathcal{A}$ be a finite set and $t \in \mathcal{A}^G$, for $g \in G$, $t_g = t(g)$ denotes the label attached to the vertex $g$ of the right Cayley graph $G$. The full shift $\mathcal{A}^G$ collects all configurations, and the shift map $\sigma : G \times \mathcal{A}^G \rightarrow \mathcal{A}^G$ is defined as $(\sigma gt_g') = t_g t_g'$ for $g, g' \in G$. For $n \geq 0$, let $E_n = \{g \in G : |g| \leq n\}$ denote the initial $n$-subgraph of the Cayley graph. An $n$-block is a function $\tau : E_n \rightarrow \mathcal{A}$. A configuration $t$ accepts an $n$-block $\tau$ if there exists $g \in G$ such that $t_{gg'} = \tau_{g'}$ for all $g' \in E_n$; otherwise, we call $\tau$ a forbidden block of $t$ (or $t$ avoids $\tau$). A $G$-shift space is a set $X \subseteq \mathcal{A}^G$ of all configurations which avoid a set of forbidden blocks. For $n \in \mathbb{N}$ and $g \in G$, let $\Gamma_n[g](X)$ be the set of $n$-blocks of $X$ rooted at $g$, i.e., the support of each block of $\Gamma_n[g](X)$ is $gE_n$. Let $\gamma_n[g] = \left|\Gamma_n[g](X)\right|$, the cardinality of $\Gamma_n[g](X)$, and denote $\gamma_n(X) = \gamma_n[c](X)$. The topological degree of $X$ is defined as

$$\deg(X) = \limsup_{n \rightarrow \infty} \frac{\ln \gamma_n(X)}{n}. \tag{2}$$

In $[2]$, the authors show that the limit of (2) exists for $G$-SFTs, where $G = \langle S | R_A \rangle$ and $A = \left(\begin{array}{cc}1 & 1 \\ 1 & 0 \end{array}\right)$. The relation of $\deg(X)$ and $h$ is described as follows. For a $G$-SFT, $\gamma_n$ behaves approximately like $\lambda_1 \lambda_2^n$ for some $\kappa, \lambda_1, \lambda_2 \in \mathbb{R}$ while $\deg(X) = \ln \kappa$. The formulation $\lambda_1 \lambda_2^n$ reveals that we could use $\lambda_2$ colors (in average) to fill up the elements of $E_n$ in $G$. The $\deg(X)$ represents the logarithm of the degree $\kappa$ (in average) of $G$ in the viewpoint of the graph theory. For examples, let $\mathcal{A} = \{1, 2\}$ and $X$ be a $\mathbb{Z}^1$-full shift over $\mathcal{A}$, then $\gamma_n = 2^n$ while $|E_n| = n$ and $\deg(X) = 0$. If $X$ is a $FS_3$-full shift with $\mathcal{A}$, then $\gamma_n = 2^{3n}$ while $|E_n| = 3^n$ and $\deg(X) = \ln 3$. Thus the formulation $\lambda_1 \lambda_2^n$ can also be symbolized as $\gamma_n \approx |\mathcal{A}|^{|E_n|}$. It can be easily checked that if $\deg(X) = \ln \rho_A$, where $\rho_A$ denotes the the spectral radius of $G$ with respect to $S$ is the directed graph whose vertex set is $G$ and its set of arcs is given by $E = \{(g, gs) : g \in G, s \in S\}$.
the matrix $A$, then $h = \ln \lambda_2$. However, it is not always the case, we prove that
\[
\{ \kappa(X_F) : X_F \text{ is a } FS_d\text{-SFT} \} = \{ \xi : \xi \in \mathcal{P}, \ p \geq 1 \}
\] for $FS_d$-SFTs, where \(\mathcal{P}\) is the set of Perron numbers. Suppose $G = \langle S|R_A \rangle$ has at least one right free generator\(^5\). Ban-Chang-Huang provide the necessary and sufficient conditions for $\kappa = \rho_A$. Roughly speaking, the more information of the tuple $(\kappa, \lambda_1, \lambda_2) \in \mathbb{R}^3$ we know, the more explicit value of $\gamma_n$ we obtain. A natural question arises: how to compute the value $\lambda_2$? If $G = FS_2$, i.e., $A = E_2$, the authors give the complete characterization of the entropies for $|A| = 2$\(^1\). This study is to extend the previous work to $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

2.1. The existence of the entropy. From now on, we assume that $G = \langle S|R_A \rangle$, where $S = \{s_1, s_2\}$, $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and $A = \{1, 2\}$. Theorem 2.1 below shows that the limit (1) exists for $G$-shifts. The proof is based on the concept of the proof for $FS_2$-shifts\(^2\). Since the proof for $G$-shifts is not straightforward, we include the proof for the convenience of the reader.

**Theorem 2.1.** Let $X$ be a $G$-shift. The entropy \(\underline{1}\) exists.

**Proof.** Let $L_n = \{g \in G : |g| = n\}$ and $l_n = |L_n|$ for $n \geq 0$. Set
\[
\underline{h} = \liminf_{n \to \infty} \frac{\ln \gamma_n}{|E_n|} = \liminf_{n \to \infty} \frac{\ln \gamma_n}{l_0 + l_1 + \cdots + l_n}.
\]
Since $l_0 = 1$, for $\varepsilon > 0$, there exists a large $m \in \mathbb{N}$ such that
\[
\frac{\ln \gamma_m}{l_1 + \cdots + l_m} < \underline{h} + \varepsilon.
\]
Write $n = pm + q$ where $0 \leq q \leq m - 1$, then we have
\[
\gamma_{pm+q} = \gamma_{(p-1)m+q} \cdot \gamma_m \leq \gamma_{(p-1)m+q} \cdot \gamma_{(p-2)m+q} \gamma_{(p-1)m+q} \leq \cdots \leq \gamma_q \gamma_m \gamma_{(p-2)m+q} \gamma_{(p-1)m+q}.
\]
On the other hand, we have
\[
\begin{pmatrix} l_{n+1} \\ l_n \end{pmatrix} = A \begin{pmatrix} l_n \\ l_{n-1} \end{pmatrix}.
\]

\(^5\)s\(_i\) is a right (resp. left) free generator if and only if $A(i, j) = 1$ (resp. $A(j, i) = 1$) for $1 \leq j \leq d$.\(^1\)
It can be easily checked that \( l_{m+n} = l_m l_{n+1} + l_{m-1} l_n \). Thus we have \( l_{m+n} > l_m l_{n+1} > l_m l_n \) and

\[
\begin{align*}
& l_{q+1} + l_{q+2} + \cdots + l_{q+pm} > l_q (l_1 + \cdots + l_m) + l_{q+m} (l_1 + \cdots + l_m) \\
& \quad + \cdots + l_{q+(p-1)m} (l_1 + \cdots + l_m) \\
& = (l_1 + \cdots + l_m) (l_q + l_{q+m} + \cdots + l_{q+(p-1)m}).
\end{align*}
\]

Therefore, for \( n \) large enough we have

\[
\begin{align*}
\mathcal{H} & \leq \frac{\ln \gamma_n}{|E_n|} + \frac{\ln \gamma_{pm+q}}{|E_{pm+q}|} \\
& \leq \frac{\ln \gamma_q}{|E_{pm+q}|} + \frac{l_q + l_{q+m} + \cdots + l_{q+(p-1)m}}{l_1 + l_2 + \cdots + l_{qm}} \ln \gamma_m \\
& \leq \frac{\ln \gamma_q}{|E_{pm+q}|} + \frac{l_q + l_{q+m} + \cdots + l_{q+(p-1)m}}{l_{q+1} + \cdots + l_{q+pm}} \ln \gamma_m \\
& \leq \frac{\ln \gamma_q}{|E_{pm+q}|} + \frac{\ln \gamma_m}{l_1 + \cdots + l_m} < \mathcal{H} + 2\varepsilon.
\end{align*}
\]

Thus we conclude that \( \mathcal{H} \) exists and equals \( \mathcal{H} \). This completes the proof. \( \square \)

2.2. Nonlinear recurrence equations. Let \( \mathcal{F} \subseteq \mathcal{A} \times S \times \mathcal{A} \) be a set of forbidden blocks and \( X = \mathcal{X}_\mathcal{F} \) be the corresponding G-SFT. The associated adjacency matrices \( T_1 = (t_{ij}^1)_{i,j=1}^2 \) and \( T_2 = (t_{ij}^2)_{i,j=1}^2 \) of \( \mathcal{F} \) are defined as follows. For \( i = 1, 2 \), \( T_1(a, b) = 0 \) if \((a, s_i, b) \in \mathcal{F} \) and \( T_1(a, b) = 1 \), otherwise. Let \( T = (T_1, T_2) \), the G-vertex shift \( X_T \) is defined.

\[
X_T = \{ t \in \mathcal{A}^G : T_i(t_g, t_{gs_i}) = 1 \ \forall g, gs_i \in G \}.
\]

It is obvious that \( X_T \) is equal to \( X \) and thus \( h(X) = h(X_T) \). Throughout the paper, we assume that \( T_i \) has no zero rows for \( i = 1, 2 \).

Fix \( i \in \mathcal{A} \), we set \( \Gamma_{i,n}^{[g]} \) the set which consists of all \( n \)-blocks \( \tau \) in \( \Gamma_{i,n}^{[g]} \) such that \( \tau_g = i \) and \( \gamma_{i,n}^{[g]} = |\Gamma_{i,n}^{[g]}| \), the cardinality of \( \Gamma_{i,n}^{[g]} \). For \( g \in G \), we define \( \mathbf{F}_g = \{ g' \in G : gg' \in G \) and \( |gg'| = |g| + |g'| \}. \) If \( g = g_1 g_2 \cdots g_n \in G \), it is easily seen that \( \mathbf{F}_g = G \) if \( g_n = s_1 \) and \( \mathbf{F}_g = \mathbf{F}_{s_2} \) if \( g_n = s_2 \), note that \( \mathbf{F}_{s_2} \neq G \). Thus

\[
\begin{align*}
\gamma_{i,n}^{[s_1]} & = \sum_{j_1, j_2=1}^1 t_{ij_1}^{1} t_{ij_2}^{1} \gamma_{j_1, n-1}^{[s_1]} \gamma_{j_2, n-1}^{[s_2]} = \sum_{j_1, j_2=1}^1 t_{ij_1}^{1} t_{ij_2}^{1} \gamma_{j_1, n-1}^{[s_1]} \gamma_{j_2, n-1}^{[s_2]}, \\
\gamma_{i,n}^{[s_2]} & = \sum_{j=1}^2 t_{ij}^{1} \gamma_{j, n-1}^{[s_2]} = \sum_{j=1}^2 t_{ij}^{1} \gamma_{j, n-1}^{[s_1]}, 1 \leq i \leq 2 \) and \( n \geq 3 \).
\end{align*}
\]

The first equality in (4) means that the \( \Gamma_{i,n}^{[s_1]} \) is generated by \( \Gamma_{j_1, n-1}^{[s_1]} \) and \( \Gamma_{j_2, n-1}^{[s_2]} \) according to the rule of \( t_{ij_1}^{1} \) (resp. \( t_{ij_2}^{1} \)) of \( T_1 \) (resp. \( T_2 \)). Since \( \Gamma_{j_1, n-1}^{[s_1]} \) and \( \Gamma_{j_2, n-1}^{[s_2]} \)
are in different branches, the summation \(\sum_{j_1,j_2=1}^{2} t_{j_1} t_{j_2} \gamma_{i,n-1,j_1} \gamma_{j_2,n-1}^{[s_1,s_2]}\) demonstrates the cardinality of \(\Gamma_{i,n}^{[s_1]}\). The second equality in (4) comes from the fact that \(F_{s_1,s_1} = F_{s_1}\) and \(F_{s_1,s_2} = F_{s_2}\). Similar argument derives (5), the only difference is that we do not have the item \(\gamma_{j,n-1}^{[s_2,s_2]}\) in the summation since \(A(2,2) = 0\), i.e., \(s_2 s_2 = s_2\).

Since the equations (4) and (5) are the nonlinear recurrence equations which describe the numbers \(\gamma_{i,n}^{[s_1]}\) and \(\gamma_{i,n}^{[s_2]}\) for \(i = 1, 2\), we continue to write \(\{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^{2}\) to represent nonlinear recurrence equations (4) and (5) by abuse of notation.

The nonlinear recurrence equation \(\{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^{2}\) can be described in an efficient way. Let \(\mathcal{M}_m\) be the collection of \(m \times m\) binary matrices. Let \(v = (v_i)_{i=1}^{n}\) and \(w = (w_i)_{i=1}^{n}\) be two vectors over \(\mathbb{R}\). Denote by \(\otimes\) the dyadic product of \(v\) and \(w\); that is,

\[v \otimes w = (v_1 w_1, v_1 w_2, \ldots, v_1 w_n, \ldots, v_n w_1, v_n w_2, \ldots, v_n w_n).\]

Let \(M \in \mathcal{M}_m\) and \(M^{(i)}\) denote the \(i\)th row of \(M\). Define

\[
\alpha_i = T_1^{(i)} \otimes T_2^{(i)} \quad \text{for } i = 1, 2
\]

and

\[
\Theta^{[s_j]}_n = \left\{ \begin{array}{ll}
(\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}) \otimes (\gamma_{1,n}^{[s_1]}, \gamma_{2,n}^{[s_2]}), & \text{if } j = 1; \\
(\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}) \otimes (1, 1), & \text{if } j = 2.
\end{array} \right.
\]

Let “\(\cdot\)” be the usual inner product, we define

\[
\tilde{\gamma}_{i,n}^{[s_j]} = \alpha_i \cdot \Theta^{[s_j]}_n \quad \text{for } 1 \leq i, j \leq 2.
\]

**Theorem 2.2.** Let \(T = (T_1, T_2) \in \mathcal{M}_2 \times \mathcal{M}_2\), the formula (4) and (5) can be reformulated as follows. For \(i = 1, 2\)

1. \(\gamma_{i,n}^{[s_1]} = \tilde{\gamma}_{i,n}^{[s_1]}\)
2. \(\gamma_{i,n}^{[s_2]}\) is derived from \(\tilde{\gamma}_{i,n}^{[s_2]}\) by letting all coefficients of the items \(\gamma_{1,n-1}^{[s_1]}\) and \(\gamma_{2,n-1}^{[s_2]}\) in \(\tilde{\gamma}_{i,n}^{[s_2]}\) to be 1.

**Proof.** It follows from (4)
Since $\gamma_{j_1,n-1}^{[s_1]} = \gamma_{j_1,n-1}^{[s_1]}$ and $\gamma_{j_1,n-1}^{[s_1,s_2]} = \gamma_{j_1,n-1}^{[s_2]}$, we conclude that

$$\gamma_{i,n}^{[s_1]} = \left[(t_{i1}^1 t_{i2}^1) \otimes (t_{i1}^2 t_{i2}^2)\right] \cdot \left[(\gamma_{1,n-1}^{[s_1]}, \gamma_{2,n-1}^{[s_1]}) \otimes (\gamma_{1,n-1}^{[s_2]}, \gamma_{2,n-1}^{[s_2]})\right]$$

$$= \alpha_i \cdot \Theta_{n-1}^{[s_1]} = \gamma_{i,n}^{[s_1]}.$$  

Thus $\gamma_{i,n}^{[s_1]} = \gamma_{i,n}^{[s_1]}$. On the other hand, it follows from (8) we have

$$\gamma_{i,n}^{[s_2]} = \alpha_i \cdot \Theta_{n-1}^{[s_2]}$$

$$= (t_{i1}^1 t_{i2}^1 t_{i1}^2 t_{i2}^2 t_{i1}^2 t_{i2}^1 t_{i1}^2 t_{i2}^2) \cdot (\gamma_{1,n-1}^{[s_1]}, \gamma_{2,n-1}^{[s_1]} \otimes (1,1))$$

$$= (t_{i1}^1 t_{i2}^1 t_{i1}^2 t_{i2}^2 t_{i1}^2 t_{i2}^1 t_{i1}^2 t_{i2}^2) \cdot (\gamma_{1,n-1}^{[s_1]}, \gamma_{1,n-1}^{[s_1]}, \gamma_{2,n-1}^{[s_1]}, \gamma_{2,n-1}^{[s_1]})$$

$$= t_{i1} t_{i2} \gamma_{1,n-1}^{[s_1]} + t_{i1} t_{i2} \gamma_{1,n-1}^{[s_1]} + t_{i2} t_{i1} \gamma_{2,n-1}^{[s_1]} + t_{i2} t_{i1} \gamma_{2,n-1}^{[s_1]}$$

(9)

Suppose that there is no restriction on the node $s_2$ in $G$, the same reasoning as $\gamma_{i,n}^{[s_1]}$ applied to $\gamma_{i,n}^{[s_2]}$ implies

$$\gamma_{i,n}^{[s_2]} = t_{i1} t_{i2} \gamma_{1,n-1}^{[s_1]} + t_{i1} t_{i2} \gamma_{1,n-1}^{[s_1]} + t_{i1} t_{i2} \gamma_{2,n-1}^{[s_1]} + t_{i1} t_{i2} \gamma_{2,n-1}^{[s_1]}$$

(10)

Since $s_2 s_2 = s_2$, the formula (9) is derived from (10) by letting $\gamma_{i,n-1}^{[s_2]} = 1$ for $i = 1, 2$. However, compare to the formula (9), $\gamma_{i,n}^{[s_2]} = \sum_{j=1}^2 t_{ij} \gamma_{j,n-1}^{[s_1]}$, the formula (10) counts $\gamma_{i,n}^{[s_2]}$ repeatedly, e.g., if $t_{i1}^1 t_{i2}^2 = t_{i1}^2 t_{i2}^1 = 1$, then (10) counts the item $\gamma_{i,n}^{[s_1]}$ twice. Thus $\gamma_{i,n}^{[s_2]}$ can be derived from $\gamma_{i,n}^{[s_1]}$ by letting all coefficients of the terms $\gamma_{1,n-1}^{[s_1]}$ and $\gamma_{2,n-1}^{[s_1]}$ in $\gamma_{i,n}^{[s_1]}$ to be 1. This completes the proof. \(\square\)

It is worth noting that the intrinsic meaning of (8) is that the effect of $T$ (rules) comes from the factor $\alpha_i$ (since $\alpha_i = \alpha_i(T)$) and the effect of $G$ (lattice) comes from the factor $\Theta_{n-1}^{[s_1]}$.

Example 2.3. Let $T = (T_1, T_2)$, where $T_i = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ for $i = 1, 2$. Then we have $\alpha_1 = (1, 1, 1, 1)$ and $\alpha_2 = (1, 0, 0, 0)$. Apply Theorem 2.2, we have $\gamma_{1,n}^{[s_1]} = \sum_{j=1}^2 \gamma_{j,n-1}^{[s_1]} \gamma_{j,n-1}^{[s_1]}$ and $\gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_2]}$. On the other hand, it follows from $\gamma_{1,n}^{[s_2]} = 2 \gamma_{1,n-1}^{[s_1]} + 2 \gamma_{2,n-1}^{[s_1]}$ and $\gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}$ that we have $\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]}$ and $\gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}$.

Since $T = (T_1, T_2) \in M_2 \times M_2$, there are only finite possibilities for $T_i^{(1)}$ and $T_i^{(2)}$, namely, (1, 1), (1, 0) and (0, 1). Hence we have only finite choices of $\alpha_i$ (recall
Remark 2.4. (1) Given \((\alpha, \alpha') = (v_k, v_l)\), the corresponding recurrence equation is
\[\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{1,n-1}^{[s_2]} \gamma_{2,n-1}^{[s_2]},\]
where the existence of the limit is due to Theorem 2.1.

(2) Note that \(F_g = G\) if \(g_n = s_1\). The entropy \(\gamma_n\) can also be represented as
\[h = \lim_{n \to \infty} \frac{\ln \gamma_n}{|E_n|} = \lim_{n \to \infty} \frac{\ln (\gamma_{1,n}^{[s_1]} + \gamma_{2,n}^{[s_1]})}{|E_n|},\]
where the existence of the limit is due to Theorem 2.1.

2.3. Equivalence of the recurrence equations. Given two nonlinear recurrence equations \(F_{kl}\) and \(F_{pq}\), we say that \(F_{kl}\) is equivalent to \(F_{pq}\) (write \(F_{kl} \simeq F_{pq}\)) if \(F_{kl}\) is equal to \(F_{pq}\) by interchanging items \(\gamma_{1,n}^{[s_1]}\) with \(\gamma_{2,n}^{[s_1]}\) and \(\gamma_{1,n}^{[s_2]}\) with \(\gamma_{2,n}^{[s_2]}\). It follows from (12) that the entropies of two G-SFTs are equal if their corresponding nonlinear recurrence equations are equivalent.

Example 2.5. \(F_{48} \simeq F_{75}\).

Proof. It follows from Theorem 2.2 we obtain
\[F_{48} = \left\{ \begin{array}{l}
gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]} + \gamma_{1,n-1}^{[s_1]} \gamma_{2,n-1}^{[s_2]}, \\
gamma_{2,n}^{[s_1]} = \gamma_{2,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]}, \\
gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
gamma_{2,n}^{[s_2]} = \gamma_{2,n-1}^{[s_1]}, \\
gamma_{1,1}^{[s_1]} = 2, \gamma_{1,1}^{[s_1]} = 1, \gamma_{1,1}^{[s_2]} = 1, \gamma_{2,1}^{[s_2]} = 1. \end{array} \right.\]
If we interchange $\gamma^{[s_1]}_{1,n}$ (resp. $\gamma^{[s_2]}_{1,n}$) with $\gamma^{[s_2]}_{2,n}$ (resp. $\gamma^{[s_2]}_{2,n}$) we have

$$
\gamma^{[s_1]}_{1,n} = \gamma^{[s_1]}_{2,n-1}\gamma^{[s_2]}_{2,n-1} + \gamma^{[s_1]}_{2,n-1}\gamma^{[s_2]}_{1,n-1},
$$
$$
\gamma^{[s_2]}_{2,n} = \gamma^{[s_2]}_{1,n-1},
$$
$$
\gamma^{[s_2]}_{2,n} = \gamma^{[s_1]}_{1,n-1},
$$
$$
\gamma^{[s_1]}_{1,1} = 1, \gamma^{[s_1]}_{2,1} = 2, \gamma^{[s_2]}_{1,1} = 1, \gamma^{[s_2]}_{2,1} = 1.
$$

One can check that it is indeed $F_{75}$. This completes the proof. \[\square\]

3. Formula and estimate of entropy

In what follows, $\lambda$ stands for the spectral radius of $A$, i.e., $\lambda = \frac{1+\sqrt{5}}{2}$ and $\bar{\lambda}$ is its conjugate. We provide various types of nonlinear recurrence equations in which the formula (or estimate) of $h$ are presented in this section. By abuse of notation we also denote by $|v| = \sum_{i=1}^{n} |v^{(i)}|$ the norm of $v \in \mathbb{R}^n$ and $v^{(i)}$ the $i$th coordinate of $v$. It can be easily checked that $|E_{n}| = \left(\sum_{i=0}^{n} A^i \mathbf{1}\right)^{(1)}$, where $\mathbf{1} = (1,1)'$ and $v'$ denotes the transpose of $v$.

3.1. Zero entropy type. Proposition 3.1 below indicates that $h_{kl} = 0$ if $F_{kl}$ satisfies $|v_k| = |v_l| = 1$, e.g., $k,l = 6,7,8,9$. We call such $F_{kl}$ zero entropy type (write type Z).

**Proposition 3.1.** Let $T = (T_1, T_2) \in \mathcal{M}_2 \times \mathcal{M}_2$, and $\alpha_i = \alpha_i(T)$ be defined as above for $i = 1,2$. If $|\alpha_1| = |\alpha_2| = 1$, then $h = 0$.

**Proof.** For simplicity we only prove the case $F_{67}$, the other cases can be treated similarly. Indeed, $F_{67}$ is of the following form.

$$
\left\{
\begin{aligned}
\gamma^{[s_1]}_{1,n} &= \gamma^{[s_1]}_{1,n-1}\gamma^{[s_2]}_{1,n-1}, \\
\gamma^{[s_2]}_{2,n} &= \gamma^{[s_2]}_{1,n-1}\gamma^{[s_2]}_{2,n-1}, \\
\gamma^{[s_1]}_{1,1} &= \gamma^{[s_1]}_{2,1} = \gamma^{[s_2]}_{1,1} = \gamma^{[s_2]}_{2,1} = 1 \\
\end{aligned}
\right.
$$

Note that $\gamma^{[s_1]}_{i,1} = 1$, and if we assume that $\gamma^{[s_2]}_{i,k-1} = 1$ for $1 \leq i,j \leq 2$, (13) infers that $\gamma^{[s_2]}_{i,k} = 1$. Thus $\gamma^{[s_1]}_{i,n} = 1$ for all $1 \leq n$ and $1 \leq i,j \leq 2$ by induction. This shows that $h_{67} = 0$. This completes the proof. \[\square\]

3.2. Equal growth type. Let $T = (T_1, T_2)$ and $F = \{\gamma^{[s_1]}_{i,n}, \gamma^{[s_2]}_{i,n}\}_{i=1}^{2}$ be its nonlinear recurrence equations, we say that $F$ is of the equal growth type ($F \in E$) if $|\alpha_1| = |\alpha_2|$. Denote by $k_{i,j}$ the number of different items of $\gamma^{[s_2]}_{i,n}$ for $1 \leq i,j \leq 2$. If
\[|\alpha_1| = |\alpha_2|, \text{ it can be checked that } k_{1,1} = k_{2,1} = \alpha, \text{ but } k_{1,2} \text{ may not equal to } k_{2,2} \text{ in general.} \]

**Theorem 3.2.** Let \( T = (T_1, T_2) \) and the corresponding \( \alpha_1, \alpha_2 \) satisfy \( |\alpha_1| = |\alpha_2| = \alpha \in \mathbb{N} \). If \( k_{1,2} = k_{2,2} = \beta \), then
\[
h = \left( 1 - \frac{\ln \beta}{\lambda^2} \right) \ln \alpha.
\]
Furthermore, if \( k_{1,2} = k_{2,2} = \alpha \), then \( h = \frac{\ln \alpha}{\alpha} \).

**Proof.** 1 First we claim that \( \gamma_{1,n} = \gamma_{2,n} \) for \( 1 \leq j \leq 2 \) and we prove it by induction. Note that \( \gamma_{1,0} = \gamma_{2,0} = 1 \) and assume that \( \gamma_{1,n} = \gamma_{2,n} \) for \( 1 \leq j \leq 2 \). Theorem 2.2 is applied to show that
\[
\gamma_{i,n+1} = \gamma_{i,n+1} = \alpha_i \cdot \Theta_{n} = \alpha_i \cdot [(\gamma_{i,n} \cdot \gamma_{2,n}) \otimes (\gamma_{1,n} \cdot \gamma_{2,n})],
\]
and \( \gamma_{i,n+1} \) is constructed by letting all the coefficients of
\[
\gamma_{i,n+1} = \alpha_i \cdot \Theta_{n} = \alpha_i \cdot [(\gamma_{i,n} \cdot \gamma_{2,n}) \otimes (1, 1)]
\]
to be 1 (Theorem 2.2). Since \( k_{1,2} = k_{2,2} \), we conclude that \( \gamma_{1,n+1} = \gamma_{2,n+1} \).
Combining (14) with \( \gamma_{1,n+1} = \gamma_{2,n+1} \) we can assert that \( \gamma_{1,n+1} = \gamma_{2,n+1} \), this proves the claim.

2 Since \( |\alpha_1| = |\alpha_2| \) and \( \gamma_{1,n} = \gamma_{2,n} \) for \( 1 \leq j \leq 2 \). We have \( \gamma_{i,n} = |\alpha_i| \gamma_{1,n-1} \gamma_{i,n-1} = \alpha_i \gamma_{1,n-1} \gamma_{i,n-1} \) and \( \gamma_{i,n} = 2 \gamma_{1,n-1} \). Thus, the nonlinear equation \( F = \{ \gamma_{1,n}, \gamma_{i,n}, \gamma_{i,n+1} \} \)
can be reduced to the simplified form
\[
\begin{cases}
\gamma_{1,n} = \alpha \gamma_{1,n-1}, \\
\gamma_{i,n} = \beta \gamma_{1,n-1}.
\end{cases}
\]
Let \( w_n = (\ln \gamma_{1,n}^{(1)}, \ln \gamma_{1,n}^{(2)}) \). We have \( w_n = A w_{n-1} + b \), where \( b = (\ln \alpha, \ln \beta) \).
Iterate \( w_n \) we have \( w_n = A^{n-1} w_{1} + \sum_{i=0}^{n-2} A^i b \). Observe that \( w_{1} = b \), thus
\[
w_n = \sum_{i=0}^{n-1} A^i b = \ln \alpha \left( \sum_{i=0}^{n-1} A^i b \right),
\]
where \( b = (1, \ln \beta, \ln \alpha) \). Combining (12) with the fact that \( \gamma_{1,n} = \gamma_{2,n} \) we assert that
\[
h = \lim_{n \to \infty} \frac{\ln \sum_{i=1}^{n} \gamma_{i,n}^{(1)}}{|E_n|} = \lim_{n \to \infty} \frac{\ln \gamma_{1,n}^{(1)}}{|E_n|} = \lim_{n \to \infty} \frac{w_n^{(1)}}{|E_n|} = \lim_{n \to \infty} \left( \frac{(\sum_{i=0}^{n-1} A^i b)^{\gamma_{1,n}^{(1)}}}{|E_n|} \right).
\]
(17)
Substituting \( |E_n| = (\sum_{i=0}^{n} A^i 1) (1) \) into (17) yields
\[
h = (\ln \alpha) \lim_{n \to \infty} \left( \frac{\sum_{i=0}^{n-1} A^i \hat{b}}{\sum_{i=0}^{n-1} A^i 1 } \right) (1).
\]

Set \( A = PDP^{-1}, P = (p_{ij}), P^{-1} = (q_{ij}), \) and \( D = \text{diag}(\lambda, \bar{\lambda}), \) we have
\[
\left( \sum_{i=0}^{n-1} A^i \hat{b} \right) (1) = \left[ P \left( \sum_{i=0}^{n-1} D^i \right) P^{-1} \hat{b} \right] (1)
\]
\[
= \left[ P \left( \sum_{i=0}^{n-1} D^i \right) (q_{11} + q_{12} \ln \beta \frac{\ln \alpha}{\ln \alpha}, q_{21} + q_{22} \ln \beta \frac{\ln \alpha}{\ln \alpha} ) \right] (1)
\]
\[
= \sum_{i=0}^{n-1} \left[ \lambda^i p_{11} \left( q_{11} + q_{12} \ln \beta \frac{\ln \alpha}{\ln \alpha} \right) + \bar{\lambda}^i p_{12} \left( q_{21} + q_{22} \ln \beta \frac{\ln \alpha}{\ln \alpha} \right) \right]
\]
\[
= p_{11} \left( q_{11} + q_{12} \ln \beta \frac{\ln \alpha}{\ln \alpha} \right) \frac{\lambda^n - 1}{\lambda - 1} + p_{12} \left( q_{21} + q_{22} \ln \beta \frac{\ln \alpha}{\ln \alpha} \right) \frac{\bar{\lambda}^n - 1}{\bar{\lambda} - 1}.
\]

It follows the same computation we have
\[
\left( \sum_{i=0}^{n-1} A^i 1 \right) (1) = p_{11} \left( q_{11} + q_{12} \right) \frac{\lambda^n - 1}{\lambda - 1} + p_{12} \left( q_{21} + q_{22} \right) \frac{\bar{\lambda}^n - 1}{\bar{\lambda} - 1}.
\]

Thus
\[
\lim_{n \to \infty} \left( \frac{\sum_{i=0}^{n-1} A^i \hat{b}}{\sum_{i=0}^{n-1} A^i 1 } \right) (1) = \frac{q_{11} + q_{12} \ln \beta \frac{\ln \alpha}{\ln \alpha}}{q_{11} + q_{12} \lambda}.
\]

Direct computation shows that
\[
P = \left( \begin{array}{cc} \frac{1 + \sqrt{5}}{2} & \frac{1 - \sqrt{5}}{2} \\ \frac{1 - \sqrt{5}}{2} & \frac{1 + \sqrt{5}}{2} \end{array} \right) \quad \text{and} \quad P^{-1} = \frac{1}{\sqrt{5}} \left( \begin{array}{cc} 1 & \frac{-1 + \sqrt{5}}{2} \\ -1 & \frac{1 + \sqrt{5}}{2} \end{array} \right).
\]

Thus \( \lim_{n \to \infty} \left( \frac{\sum_{i=0}^{n-1} A^i \hat{b}}{\sum_{i=0}^{n-1} A^i 1 } \right) (1) = \frac{1 - \frac{\ln \beta}{\ln \alpha}}{\lambda^2}. \) If \( \alpha = \beta, \) we have \( \frac{1 - \frac{\ln \beta}{\ln \alpha}}{\lambda^2} = \frac{1}{\lambda}. \) This completes the proof.

\[\square\]

**Example 3.3.**

1. \( h_{42} = \frac{1 - \frac{\ln \beta}{\ln \alpha}}{\lambda^2} \ln 4 = \frac{1}{2} \ln 4 = \ln 2. \)

2. \( h_{23} = \frac{\ln \beta}{\ln \alpha} \ln 2 = \frac{1}{\lambda} \ln 2 \approx 0.42839. \) Since \( \deg(X) = \ln \lambda \) (cf. [2]) we have \( \gamma_n \approx \left( 2^\frac{1}{\lambda} \right)^n = 2^{\lambda^{-1}}, \) e.g., \( \gamma_9 \approx (2)^{\lambda^{-1}} \approx 1.3868 \times 10^{14}. \)

3. \( h_{45} = \frac{1}{\lambda^2} \ln 2 \approx 0.26476. \)

3.3. **Dominating type.** Let \( \gamma_{i,n}^{[s]} = \sum_{j=l}^{n} f_{l,n-1}^{ij} \) denotes the \( l \)th item of \( \gamma_{i,n}^{[s]} \). We say that \( \gamma_{i,n}^{[s]} \) has a **dominate item** if there exists integer \( 1 \leq r \leq n_{ij} \) such that \( f_{r,n}^{ij} \geq f_{l,n}^{ij} \) for all \( r \neq l \) and \( n \geq 1 \). We say \( F = \{ \gamma_{i,n}^{[s]} \} \) is of the **dominating type** \( (F \in \mathbf{D}) \) if each \( \gamma_{i,n}^{[s]} \) has a dominate item for all \( 1 \leq i, j \leq 2 \). If
$F \in \mathbf{D}$, we assume that $f_{1,n-1}^{ij}$ is the corresponding dominate item for all $1 \leq i, j \leq 2$. Thus,

$$\gamma_{1,n}^{[s]} = \sum_{l=1}^{n} f_{1,n-1}^{ij} = f_{1,n-1}^{ij}(1 + \sum_{l=2}^{n} \frac{f_{1,n-1}^{ij}}{f_{1,n-1}^{ij}})$$

and $1 \leq 1 + \sum_{l=2}^{n} \frac{f_{1,n-1}^{ij}}{f_{1,n-1}^{ij}} \leq 4$, where the number 4 comes from the extreme case

where $n_{ij} = 4$ and $f_{1,n-1}^{ij} \leq 1$ for $l = 2, \ldots, n_{ij}$. Let $w_n = (\ln \gamma_{1,n}^{[s]}, \ln \gamma_{2,n}^{[s]}, \ln \gamma_{1,n}^{[x]}, \ln \gamma_{2,n}^{[x]})'$, it follows immediately from (14) and (15) that

$$w_n = K w_{n-1} + b_{n-1},$$

for some $K \in \mathcal{M}_4$ and

$$b_{n-1} = \begin{pmatrix} \ln(1 + \frac{\sum_{l=1}^{n} f_{1,n-1}^{ij}}{f_{1,n-1}^{ij}}) \\ \ln(1 + \frac{\sum_{l=2}^{n} f_{1,n-1}^{ij}}{f_{1,n-1}^{ij}}) \\ \ln(1 + \frac{\sum_{l=2}^{n} f_{1,n-1}^{ij}}{f_{1,n-1}^{ij}}) \\ \ln(1 + \frac{\sum_{l=2}^{n} f_{1,n-1}^{ij}}{f_{1,n-1}^{ij}}) \end{pmatrix}.$$  

Ban-Chang [2] prove that if the symbol $\gamma_{i,n}^{[x]}$ is essential for $1 \leq i, j \leq 2$, then $\rho_K = \lambda$, where $\rho_B$ is the spectral radius of the matrix $B$. Let $v, w \in \mathbb{R}^n$ we say $v \geq w$ if $v_i \geq w_i$ for $1 \leq i \leq n$.

**Proposition 3.4.** Suppose $\alpha_1 > \alpha_2$ (or $\alpha_2 > \alpha_1$), then $F \in \mathbf{D}$.

**Proof.** We only prove the case where $\alpha_1 > \alpha_2$ and the other case is similar. The proof is divided into small cases.

1. $\alpha_1 = \nu_1$. In this case, there are eight possibilities of $\alpha_2$, namely, $\alpha_2 = \nu_i$ for $i = 2, \ldots, 9$. If $\alpha_2 = \nu_2$, the nonlinear recurrence equation $F_{12} = \{\gamma_{i,n}^{[s]}\}_{i=1}^{2}$ is as follows

$$\begin{cases} 
\gamma_{1,n}^{[s]} = \gamma_{1,n-1}^{[s]} \gamma_{1,n}^{[s]} + \gamma_{1,n-1}^{[s]} \gamma_{1,n}^{[s]} + \gamma_{1,n-1}^{[s]} \gamma_{1,n-1}^{[s]} + \gamma_{1,n}^{[s]} \gamma_{1,n}^{[s]} + \gamma_{1,n}^{[s]} \gamma_{1,n}^{[s]}, \\
\gamma_{2,n}^{[s]} = \gamma_{1,n-1}^{[s]} \gamma_{1,n}^{[s]} + \gamma_{1,n}^{[s]} \gamma_{1,n}^{[s]}, \\
\gamma_{1,n}^{[x]} = \gamma_{1,n-1}^{[x]} \gamma_{1,n}^{[x]} + \gamma_{1,n}^{[x]} \gamma_{1,n}^{[x]}, \\
\gamma_{2,n}^{[x]} = \gamma_{1,n-1}^{[x]} \gamma_{1,n}^{[x]} + \gamma_{1,n}^{[x]} \gamma_{1,n}^{[x]}, \\
\gamma_{1,n}^{[x]} = 4, \gamma_{2,n}^{[x]} = 2, \gamma_{1,n}^{[x]} = 1, \gamma_{2,n}^{[x]} = 1.
\end{cases}$$

Since $\gamma_{1,n}^{[s]} \geq \gamma_{2,n}^{[s]}$ and $\gamma_{1,n}^{[x]} \geq \gamma_{2,n}^{[x]}$ we deduce that $\gamma_{1,n-1}^{[s]} \gamma_{1,n}^{[s]}$ (resp. $\gamma_{1,n}^{[s]}$) is the dominate item for $\gamma_{1,n}^{[s]}$ and $\gamma_{1,n}^{[x]}$ (resp. $\gamma_{1,n}^{[x]}$ and $\gamma_{2,n}^{[x]}$). Thus $F_{12} \in \mathbf{D}$.

---

6We call the symbol $\gamma_{i,n}^{[x]}$ *essential* if there exists $n \in \mathbb{N}$ such that $\gamma_{i,n}^{[x]} > 1$, and *inessential* otherwise. In [3], the authors find a finite checkable conditions to characterize whether $\gamma_{i,n}^{[x]}$ is essential or inessential.
\[ F_{13}, F_{14}, F_{15} \in D \] can be treated in the same manner. For \( \alpha_2 = v_6, v_7, v_8, v_9 \), let \( \alpha_2 = v_6, F_{16} \) is of the following form
\[
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]} + \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]} + \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]}, \\
\gamma_{2,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n}^{[s_1]} + \gamma_{1,n-1}^{[s_2]}, \\
\gamma_{2,n}^{[s_2]} = \gamma_{1,n}^{[s_1]} + \gamma_{1,n-1}^{[s_2]}, \\
\gamma_{1,1}^{[s_1]} = 2, \gamma_{2,1}^{[s_1]} = 1, \gamma_{1,1}^{[s_2]} = 2, \gamma_{2,1}^{[s_2]} = 1.
\]

Since \( \gamma_{2,n}^{[s_1]} \) (resp. \( \gamma_{2,n}^{[s_2]} \)) have only one item, it is the dominate item. It follows from the fact that \( \gamma_{1,n}^{[s_1]} \geq \gamma_{2,n}^{[s_1]} \) and \( \gamma_{1,n}^{[s_2]} \geq \gamma_{2,n}^{[s_2]} \), it is concluded that \( \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]} \) (resp. \( \gamma_{1,n-1}^{[s_1]} \)) is still the dominate item of \( \gamma_{1,n}^{[s_1]} \) (resp. \( \gamma_{1,n}^{[s_2]} \)). Thus, \( F \in D \).

\( F_{17}, F_{18}, F_{19} \in D \) can be treated similarly.

2. \( \alpha_1 = v_2, v_3, v_4, v_5 \). Assume \( \alpha_1 = v_2 \), since \( \alpha_1 > \alpha_2 \), it suffices to check \( \alpha_2 = v_6 \) and \( v_3 \). For \( \alpha_1 = v_2 \), under the same argument as above we conclude that \( \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]} \) (resp. \( \gamma_{1,n-1}^{[s_1]} \)) is the dominate item for \( \gamma_{1,n}^{[s_1]} \) (resp. \( \gamma_{2,n}^{[s_2]} \)), thus \( F_{26} \in D \). The same reasoning applies to other cases. This completes the proof. \( \square \)

The entropy can be computed for \( F \in D \). Let us denote by \( \mathcal{E} \) (resp. \( \mathcal{I} \)) the set of all essential (resp. inessential) symbols of \( \{ \gamma_i^{[s_j]} \}_{i,j=1} \), we note that \( \{ \gamma_i^{[s_j]} \}_{i,j=1} = \mathcal{E} \cup \mathcal{I} \). The computation methods of \( h \) is divided into two subcases, namely, \( \mathcal{I} = \emptyset \) and \( \mathcal{I} \neq \emptyset \). First, we take \( F_{16} \) as an example to illustrate how to compute \( h \) for this type, and \( \mathcal{I} = \emptyset \) in this case.

**Theorem 3.5.** Let \( X \) be a G-SFT in which the corresponding nonlinear recurrence equation is \( F_{16} \); that is, \( T_1 = T_2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \). Let \( b_n \) be constructed as \((20)\),
\[
K = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} \lambda & \lambda & 0 & 0 \\ \lambda & \lambda & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{pmatrix}
\]
be such that \( QDQ^{-1} = K \) and \( D = \text{diag}(\lambda, \bar{\lambda}, 0, 0) \). Then
\[
b_{16} = \frac{(\lambda - 1)A_\infty}{\bar{\lambda}^2} \approx 0.23607A_\infty,
\]
where
\[
A_\infty = \lim_{n \to \infty} \left( \hat{w}_1^{(1)} + \lambda^{-1}\hat{b}_{1}^{(1)} + \cdots + \lambda^{-n+1}\hat{b}_{n}^{(1)} \right)
\]
and \( \hat{b}_n = Q^{-1}b_n \). Moreover, the limit \((22)\) exists.
\textbf{Proof.} Note that

\[
F_{16} = \left\{ \begin{array}{l}
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_2]}, \\
\gamma_{2,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_1]}.
\end{array} \right.
\]

It can be easily checked that $\gamma_{1,i}^{[s_2]} \geq 4$ for $1 \leq i, j \leq 2$. Thus, there is no inessential symbol and $\mathcal{I} = \emptyset$. Since $\gamma_{1,i}^{[s_1]} \geq \gamma_{2,n}^{[s_2]}$ and $\gamma_{1,i}^{[s_1]} \geq \gamma_{2,n}^{[s_2]}$ for $1 \leq i, j \leq 2$, $\gamma_{1,i}^{[s_1]}$ is the dominate item of $\gamma_{1,i}^{[s_1]}$ (resp. $\gamma_{1,i}^{[s_1]}$, $F \in \mathcal{D}$. The above argument indicates that $w_n = Kw_{n-1} + b_{n-1}$, and $K$ is indeed (21). Along the identical line of the proof in Theorem 3.2 we have $w_n = K^{n-1}w_1 + \sum_{i=0}^{n-2} K^i b_i$,

\[
h = \lim_{n \to \infty} \frac{\ln \sum_{i=1}^{n} \gamma_{1,1}^{[s_1]}}{|E_n|} = \lim_{n \to \infty} \frac{\ln \gamma_{1,1}^{[s_1]}}{|E_n|} = \lim_{n \to \infty} \frac{w_n^{(1)}}{|E_n|},
\]

and

\[
w_n^{(1)} = \left( K^{n-1}w_1 + K^{n-2}b_1 + \cdots + b_n \right)^{(1)}
\]

\[
= \left( QD^{n-1}Q^{-1}w_1 + QD^{n-2}Q^{-1}b_1 + \cdots + QQ^{-1}b_n \right)^{(1)}.
\]

Combining (24) with direct computation yields

\[
w_n^{(1)} = \left( QD^{n-1}\hat{w}_1 + QD^{n-2}\hat{b}_1 + \cdots + \hat{Q}\hat{b}_n \right)^{(1)}
\]

\[= \lambda^{n-1}Q_{11} \left( \hat{w}_1^{(1)} + \lambda^{-1}\hat{b}_1^{(1)} + \cdots + \lambda^{-n+1}\hat{b}_n^{(1)} \right) + O(\lambda^n)
\]

\[= \lambda^n \left( \hat{w}_1^{(1)} + \lambda^{-1}\hat{b}_1^{(1)} + \cdots + \lambda^{-n+1}\hat{b}_n^{(1)} \right) + O(\lambda^n).
\]

Combining (23) we obtain that

\[
h = \lim_{n \to \infty} \frac{\lambda^n \left( \hat{w}_1^{(1)} + \lambda^{-1}\hat{b}_1^{(1)} + \cdots + \lambda^{-n+1}\hat{b}_n^{(1)} \right)}{\left( \sum_{i=0}^{n} A^i \right)^{(1)}}
\]

Let $A_\infty = \lim_{n \to \infty} \left( \hat{w}_1^{(1)} + \lambda^{-1}\hat{b}_1^{(1)} + \cdots + \lambda^{-n+1}\hat{b}_n^{(1)} \right)$, such limit exists due to the fact that $\hat{b}_n^{(1)}$ is bounded for all $n$. Combining (25) with (38) yields

\[
h = \lim_{n \to \infty} \frac{\lambda^n A_\infty}{\left( \sum_{i=0}^{n} A^i \right)^{(1)}}
\]

\[= A_\infty \lim_{n \to \infty} \frac{\lambda^n}{\left[ p_{11} (q_{11} + q_{12}) + p_{12} (q_{21} + q_{22}) - \Delta \right]^{n-1} \lambda^n}
\]

\[= \frac{1}{\lambda - 1} p_{11} (q_{11} + q_{12}) \frac{\lambda^n}{\lambda - 1} + p_{12} (q_{21} + q_{22}) \frac{\lambda^n}{\lambda - 1}
\]

\[= \frac{1}{\lambda^2} (\lambda - 1) A_\infty
\]

\[= 0.23607 A_\infty \approx 0.5011681177.
\]
This completes the proof. \(\square\)

Next, we use \(F_{39}\) to illustrate the computation of \(h\) for the case where \(\mathcal{I} \neq \emptyset\).

**Proposition 3.6.** \(h_{39} = 0\).

**Proof.** Note that

\[
\begin{aligned}
\gamma_{1,n} &= \gamma_{1,n-1} \gamma_{2,n-1} + \gamma_{2,n-1} \gamma_{2,n-1}, \\
\gamma_{2,n} &= \gamma_{2,n} - \gamma_{2,n}, \\
\gamma_{1,n} &= \gamma_{1,n} - \gamma_{2,n}, \\
\gamma_{2,n} &= \gamma_{2,n}, \\
\gamma_{1,1} &= 2, \gamma_{2,1} = 1, \gamma_{1,1} = 2, \gamma_{2,1} = 1.
\end{aligned}
\]

It can be checked that \(\gamma_{2,n} = 1\) for all \(n \in \mathbb{N}\), and \(F_{39} \in D\). Thus, \(\mathcal{I} = \{\gamma_{2,1}, \gamma_{2,1}\} \neq \emptyset\). Under the same argument in the beginning of this section, we construct

\[
K = \begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 1 & 0 & 1 \\
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0
\end{pmatrix}.
\]

Since \(w_{n}^{(2)} = w_{n}^{(4)} = 0\), the formula (27) can be reduced to the following form.

\[
\tilde{w}_{n} = \tilde{K} \tilde{w}_{n-1} + \tilde{b}_{n-1},
\]

where \(\tilde{w}_{n} = (w_{n}^{(1)}, w_{n}^{(2)}, w_{n}^{(3)})\), \(\tilde{b}_{n} = (\ln(1 + \frac{\gamma_{1,n}}{\gamma_{1,n-1}}), \ln(1 + \frac{\gamma_{1,n}}{\gamma_{1,n-1}}))\) and \(\tilde{K} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\) which is derived by deleting the 2nd and 4th columns and rows of \(K\). Note that if \(\rho_{\tilde{K}} > 1\), the method used in Theorem 3.5 still works. The induction formula (27) in fact provides us the formula of \(w_{n}^{(1)}\) and \(w_{n}^{(3)}\). Indeed, since \(\gamma_{2,n} = 1\), \(w_{n}^{(1)} = \ln \gamma_{1,n} = \ln(\gamma_{1,n} + \gamma_{2,n-1}) = \ln \gamma_{1,n} = w_{n}^{(3)}\), the recurrence equation (27) is reduced to the simple form

\[
\begin{aligned}
w_{n}^{(1)} &= w_{n-1}^{(1)} + \ln(1 + \frac{\gamma_{1,n}}{\gamma_{1,n-1}}) \\
&= w_{1}^{(1)} + \ln(1 + \frac{\gamma_{1,1}}{\gamma_{1,1}}) + \cdots + \ln(1 + \frac{\gamma_{1,n-1}}{\gamma_{1,n-1}}) \\
&= \ln \gamma_{1,1} + \ln(1 + \frac{\gamma_{1,1}}{\gamma_{1,1}}) + \cdots + \ln(1 + \frac{\gamma_{1,n-1}}{\gamma_{1,n-1}}).
\end{aligned}
\]
The equality (28) actually demonstrates the inductive formula $w_n^{(1)} = \ln \gamma_{1,n}^{[s_1]}$. Indeed,

$$\ln \gamma_{1,1}^{[s_1]} + \ln (1 + \frac{\gamma_{2,1}^{[s_1]}}{\gamma_{1,1}^{[s_1]}}) + \cdots + \ln (1 + \frac{\gamma_{2,n-1}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}})$$

$$= \ln \gamma_{1,1}^{[s_1]} \left(1 + \frac{\gamma_{2,1}^{[s_1]}}{\gamma_{1,1}^{[s_1]}}\right) \cdots \left(1 + \frac{\gamma_{2,n-1}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}}\right)$$

$$= \ln \gamma_{1,1}^{[s_1]} \left(\frac{\gamma_{1,2}^{[s_1]}}{\gamma_{1,1}^{[s_1]}}\right) \cdots \left(\frac{\gamma_{1,n}^{[s_1]}}{\gamma_{1,n-1}^{[s_1]}}\right)$$

$$= \ln \gamma_{1,n}^{[s_1]}.$$

The second equality comes from the recurrence formula (20). For the computation of $h$, we know that $\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} + 1$. Thus $\gamma_{1,n}^{[s_1]} = n$ and $h = 0$.

The same reasoning applies to the cases $h_{62} = h_{64} = h_{72} = h_{58} = h_{59} = 0$.

**Corollary 3.7.** $h_{46} = h_{47} = h_{44} = h_{45} = \frac{1}{46} \ln 2$.

**Proof.** It follows from Theorem 3.2 we have $h_{44} = h_{45}$. It suffices to prove that $h_{46} = h_{44}$, the case where $h_{47} = h_{44}$ is similar. Since

$$F_{46} = \begin{cases} 
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_2]}, \\
\gamma_{2,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_2]}, \\
\gamma_{1,n}^{[s_1]} = 2, \gamma_{2,n}^{[s_2]} = 1, \gamma_{1,n}^{[s_2]} = 1, \gamma_{2,1}^{[s_2]} = 1.
\end{cases}$$

Thus, we have $\gamma_{1,n}^{[s_2]} = \gamma_{2,n}^{[s_2]}$, and it follows from the fact that $F_{46} \in D$, we reduce $F_{46}$ to (note $h_{46} = \lim_{n \to \infty} \frac{\ln \gamma_{1,n}^{[s_1]} + \gamma_{1,n}^{[s_2]}}{|E_n|}$) $\lim_{n \to \infty} \frac{\ln \gamma_{1,n}^{[s_1]}}{|E_n|}$)

$$\begin{cases} 
\gamma_{1,n}^{[s_1]} = 2\gamma_{1,n-1}^{[s_1]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_2]}, \\
\gamma_{1,n}^{[s_1]} = 2, \gamma_{1,n}^{[s_2]} = 1.
\end{cases}$$

The same argument as in the proof of Theorem 3.2 infers that $h_{46} = h_{44} = \frac{1}{11} \ln 2$. This completes the proof. □
3.4. Oscillating type. We call an $F = \{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^2$ the oscillating type ($F \in \mathbb{O}$) if there exist two sequences $\{m_1^n\}, \{m_2^n\}$ of $\mathbb{N}$ with $\{m_1^n\} \cap \{m_2^n\} = \emptyset$ and $\{m_1^n\} \cup \{m_2^n\} = \mathbb{N}$ such that $\gamma_{1,n}^{[s_1]} \geq \gamma_{2,n}^{[s_1]}$ for $n \in \{m_1^n\}$ and $\gamma_{1,n}^{[s_1]} < \gamma_{2,n}^{[s_1]}$ if $n \in \{m_2^n\}$. We say $F \in \mathbb{O}_2$ if the two sequences are odd and even numbers. For $F \in \mathbb{O}_2$, $h$ can be computed along the same line of Theorem 3.5. The steps are listed as follows.

1. Expand $F = \{\gamma_{i,n}^{[s_1]}, \gamma_{i,n}^{[s_2]}\}_{i=1}^2$ to $(n-2)$-order, say $F^{(2)}$; that is, expand each item of $\gamma_{i,n}^{[s_1]}$ to next level according to the rule of $F$.
2. Since $\gamma_{1,n}^{[s_1]} \geq \gamma_{2,n}^{[s_1]}$ for $n$ being even and $\gamma_{1,n}^{[s_1]} < \gamma_{2,n}^{[s_1]}$ for $n$ being odd, one assures that $F^{(2)} \in \mathbb{D}$.
3. Construct $w_n = K w_{n-2} + b_{n-2}$ as in the case of dominating type, and note that $K \in \mathcal{M}_{4 \times 4}$ with $\rho_K = \lambda^2$ (cf. [7]).
4. Iteration $w_n$ and compute the growth rate of $\lim_{n \to \infty} \frac{w_n}{\log_2 n}$. Since the limit $h$ exists (Theorem 2.1), we have $h = \lim_{n \to \infty} \frac{w_n}{\log_2 n}$.

The following Proposition characterizes whether $F \in \mathbb{O}_2$.

**Proposition 3.8.** $F_{36}, F_{56}, F_{92}, F_{94} \in \mathbb{O}_2$.

**Proof.** Note that $F_{36} \simeq F_{92}$ and $F_{56} \simeq F_{94}$. Thus we only need to prove $F_{36}$ and $F_{56}$. Since the proofs of $F_{36}$ and $F_{56}$ are identical, it suffices to prove the case of $F_{36}$. $F_{36}$ is of the following form

$$F_{36} = \begin{cases} 
\gamma_{1,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} \gamma_{2,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]} \gamma_{2,n-1}^{[s_2]}, \\
\gamma_{2,n}^{[s_1]} = \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]}, \\
\gamma_{1,n}^{[s_2]} = \gamma_{1,n-1}^{[s_2]} \gamma_{2,n-1}^{[s_1]}, \\
\gamma_{2,n}^{[s_2]} = \gamma_{1,n-1}^{[s_2]}, \\
\gamma_{1,1}^{[s_1]} = 2, \gamma_{1,1}^{[s_2]} = 1, \gamma_{2,1}^{[s_1]} = 2, \gamma_{2,1}^{[s_2]} = 1.
\end{cases}$$

Let $\tau_n = \frac{\gamma_{1,n}^{[s_1]}}{\gamma_{2,n}^{[s_2]}}$ and $\chi_n = \frac{\gamma_{1,n}^{[s_2]}}{\gamma_{2,n}^{[s_2]}}$, we have

$$\tau_n = \frac{\gamma_{1,n-1}^{[s_1]} \gamma_{2,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]} \gamma_{2,n-1}^{[s_2]}}{\gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]} - \gamma_{1,n-1}^{[s_2]} \gamma_{2,n-1}^{[s_1]}} = \frac{1}{\chi_n - 1} + \frac{1}{\tau_{n-1} \chi_{n-1}}.$$

$$\chi_n = \frac{\gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_2]}}{\gamma_{1,n-1}^{[s_1]}} = 1 + \frac{1}{\tau_{n-1}}.$$

The direct computation shows that $(\tau_1, \chi_1) = (\frac{1}{2}, \frac{1}{2})$ and $(\tau_2, \chi_2) = (2, 3)$. Note that if $\tau_n \leq \frac{1}{2}$ and $\chi_n \leq \frac{3}{2}$, then

$$\tau_{n+1} = \frac{1}{\chi_{n+1}} + \frac{1}{\tau_n \chi_{n+1}} \geq \frac{2}{3} + \frac{4}{3} = 2,$$

$$\chi_{n+1} = 1 + \frac{1}{\tau_{n+1} \chi_n} \geq 1 + 2 = 3.$$
Theorem 4.1. Let $G$ be an $S$-SFT with $G = \langle S | R_A \rangle$, where $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Then $F$ is either one of the following four types.

1. $F$ is of the zero entropy type.
2. $F$ is of the equal growth type.
3. $F$ is of the dominating type.
4. $F$ is of the oscillating type.

Proof. Without loss of generality, we assume that $|\alpha_1| \geq |\alpha_2|$. The proof is divided into two subcases.

1. $|\alpha_1| = 4$. In this case, under the same proof of Theorem 3.4, we see that $F \in D$. If $|\alpha_2| = 1$ or $2$, then $F \in O_2$. Thus we only need to discuss the cases of $F_{36}, F_{56}, F_{38}, F_{57}, F_{72}, F_{92}, F_{94}$ and $F_{84}$. Since $F_{36} \simeq F_{92}, F_{56} \simeq F_{94}, F_{38} \simeq F_{72}$ and $F_{57} \simeq F_{84}$. It suffices to check $F_{36}, F_{56}, F_{38}$ and $F_{57}$. Proposition 3.8 indicates that $F_{36}, F_{56} \in O_2$. Thus we only need to discuss the cases of $F_{38}$ and $F_{57}$. Actually, we prove $F_{38} \in O \setminus O_2$ ($F_{57}$ is similar).

F_{38} is of the form

$$
\begin{align*}
\gamma_{1,1}^{[s_1]} &= \gamma_{1,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_2]} \gamma_{2,n-1}^{[s_1]} + \gamma_{2,n-1}^{[s_2]}, \\
\gamma_{2,n}^{[s_1]} &= \gamma_{2,n-1}^{[s_1]} + \gamma_{1,n-1}^{[s_1]} \gamma_{1,n-1}^{[s_2]}, \\
\gamma_{1,n-1}^{[s_2]} &= \gamma_{1,n-1}^{[s_2]} + \gamma_{2,n-1}^{[s_1]}, \\
\gamma_{2,n-1}^{[s_2]} &= \gamma_{2,n-1}^{[s_2]}, \\
\gamma_{1,1} &= 2, \gamma_{2,1} = 1, \gamma_{1,1}^{[s_2]} = 2, \gamma_{2,1}^{[s_2]} = 1.
\end{align*}
$$
Let $\tau_n = \frac{\gamma_{s_1}^{[s_2]_n}}{\gamma_{s_2}^{[s_1]_n}}$ and $\chi_n = \frac{\gamma_{s_2}^{[s_1]_n}}{\gamma_{s_1}^{[s_2]_n}}$, then

$$\tau_n = \frac{\tau_{n-1} + 1}{\chi_{n-1}} = \frac{\chi_n}{\tau_{n-1}} + 1$$

Direct examination shows that $(\tau_1, \chi_1) = (2, 2), (\tau_2, \chi_2) = (\frac{5}{3}, 3), (\tau_3, \chi_3) = (\frac{5}{2}, 3), (\tau_4, \chi_4) = (\frac{11}{5}, \frac{11}{6}), (\tau_5, \chi_5) = (\frac{52}{29}, \frac{26}{15}), \ldots$, thus $F_{38} \notin O_2$.

3 $|\alpha_1| = 1$. Proposition 3.1 indicates that all these cases belong to type $Z$. This completes the proof. □

The following table indicates all types of $F_{ij}$ for $1 \leq i, j \leq 9$.

| $\alpha_2 \setminus \alpha_1$ | $v_1$ | $v_2$ | $v_3$ | $v_4$ | $v_5$ | $v_6$ | $v_7$ | $v_8$ | $v_9$ |
|-----------------------------|-------|-------|-------|-------|-------|-------|-------|-------|-------|
| $v_1$                       | E     | D     | D     | D     | D     | D     | D     | D     | D     |
| $v_2$                       | D     | E     | E     | D     | D     | D     | O     | D     | O     |
| $v_3$                       | D     | E     | E     | D     | D     | D     | D     | D     | D     |
| $v_4$                       | D     | D     | D     | E     | E     | D     | D     | D     | D     |
| $v_5$                       | D     | D     | D     | O     | O     | Z     | Z     | Z     | Z     |
| $v_6$                       | D     | D     | O     | D     | O     | Z     | Z     | Z     | Z     |
| $v_7$                       | D     | D     | D     | O     | Z     | Z     | Z     | Z     | Z     |
| $v_8$                       | D     | D     | O     | D     | Z     | Z     | Z     | Z     | Z     |
| $v_9$                       | D     | D     | D     | D     | Z     | Z     | Z     | Z     | Z     |

4.1. **Numerical results.** The numerical result of $h$ is presented. We give some explanations as follows.

1. We note that for each $F_{kl}$, there exists a unique $F_{pq}$ such that $F_{kl} \simeq F_{pq}$.

which gives $h_{kl} = h_{pq}$, e.g., $h_{48} = h_{75}, h_{14} = h_{51}$ etc.

2. $h_{kl} = 0$ for $k, l \in \{6, 7, 8, 9\}$ (Proposition 3.1).

3. $h_{11} = \ln 2$ and $h_{kl} = \frac{\ln 2}{\lambda}$ if $k, l \in \{2, 3\}$ and $h_{kl} = \frac{\ln 2}{\lambda^2}$ if $k, l \in \{4, 5\}$ (Theorem 3.2).

4. $h_{44} = h_{45} = h_{46} = h_{47}$ (Corollary 3.7).

5. $h_{39} = h_{62} = h_{64} = h_{58} = h_{59} = 0$ (Proposition 3.6).
We list the results of this paper as follows.

(1) The existence of the entropy \( (1) \) for a \( G \)-shift is illustrated in Theorem 2.1.
(2) The nonlinear recurrence equation which describes the growth behavior of the admissible patterns \( \gamma_n(X) \) in a \( G \)-SFT (or \( G \)-vertex shift) is established in Section 2.
(3) The \( Z, E, D \) and \( O (O_2) \) types of nonlinear recurrence equations are introduced. The algorithms of the entropy computations for these types are also presented (cf. Section 3).
(4) The characterization of the nonlinear recurrence equations of \( G \)-SFTs with two symbols is presented (Theorem 4.1).

We emphasize that the computation method of \( h \) can be easily extended to the case of more symbols. However, the general entropy formula for arbitrary \( G \)-SFTs is far from being solved. We list some problems in the further study.

**Problem 5.1.** Can we give the characterization for \( G \)-SFTs over symbol set \( \mathcal{A} \) with \( |\mathcal{A}| > 2 \)?

**Problem 5.2.** Let \( H = \langle S | R_B \rangle \) with \( S = \{s_1, \ldots, s_d\} \) and \( B \) be an arbitrary \( d \)-dimensional \( \{0,1\} \)-matrix, can we develop the entropy theory for \( H \)-SFTs?
Problem 5.3. Can we extend the methods of $G$-SFTs to $F_d$-SFTs? More precisely, can we establish the entropy formula for $F_d$-SFTs?

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