ALGEBRAIC CONNECTIONS ON PARALLEL UNIVERSES*)

R. Coquereaux

MPCE, Macquarie University, New South Wales, Australia

R. Häußling, F. Scheck

Institut für Physik
Johannes Gutenberg-Universität
D-55099 Mainz (Germany)

Abstract

For any manifold $M$, we introduce a $\mathbb{Z}$-graded differential algebra $\Xi$, which, in particular, is a bi-module over the associative algebra $C(M \cup M)$. We then introduce the corresponding covariant differentials and show how this construction can be interpreted in terms of Yang-Mills and Higgs fields. This is a particular example of noncommutative geometry. It differs from the prescription of Connes in the following way: The definition of $\Xi$ does not rely on a given Dirac-Yukawa operator acting on a space of spinors.

*) Work supported in part by PROCOPE project Mainz University and CPT Marseille-Luminy CPT 93/PE 2947

1) On leave of absence from Centre de Physique Théorique, CNRS Luminy, Case 907, F-13288 Marseille
1. Introduction

In physics, fundamental interactions are described in terms of covariant differentials, or connections. These covariant differentials appear because we want physics to be independent of the choice that we make, at every point of space-time $M$, of a particular “frame”. This “frame” can be a set of four independent vectors belonging to the tangent space at $x \in M$ (we then describe gravity) but it can also be a set of three independent complex vectors, smoothly dependent on $x \in M$ (we then describe chromodynamics). For electroweak interactions the situation is similar.

In all cases, the freedom of choosing different frames ("external" or "internal") at different space-time points leads to gauge fields (connection one-forms) appearing in the covariant derivatives and to different kinds of “forces”, described by the corresponding curvatures (assuming that our “external” and “internal” space-time is curved!). In mathematical language, and at the classical level, the fundamental interactions of physics are described by connections in appropriate principal (or vector) bundles.

There is however an important aspect of physics that is not described by the above mathematical structure: There are scales (or masses) in physics. In all classical treatments, gauge fields and matter fields have a nice geometrical interpretation but mass-related phenomena do not. In particle physics, the Higgs mechanism is taken as ultimately responsible for mass generation, but the Higgs field along with its self-coupling did not appear as very appealing – from the aesthetical point of view – when it was proposed more then twenty years ago.

Recently, several constructions of the fundamental interactions using “noncommutative geometry” were proposed [1, 2, 3, 4, 5, 6]. All these constructions share the following common features:

- The concepts of covariant derivative and connection are extended from the realm of manifolds (or commutative algebras) to arbitrary associative (not necessarily commutative) algebras. At the same time, the usual algebra of differential forms – that plays a key role in the usual construction – is replaced by some more general differential algebra.

- The Higgs field appears as part of the generalized connection, and its coupling to the Yang-Mills fields and the Higgs potential itself appear naturally in the expression of the generalized Yang-Mills functional (the trace of the squared generalized curvature).

- Mass generation for fermions, via Yukawa couplings to the Higgs, is now incorporated in a generalized Dirac operator that we shall call the Dirac-Yukawa operator.

The different constructions that were proposed are described in the literature and we do not review them here. The purpose of the present paper is to explain how the specific construction that was presented in [3] and continued in [4] fits into the general framework.
of noncommutative geometry. Indeed, [3] was written on purpose in a way that involves only elementary mathematical tools in order for the construction to be understood without having to rely on a number of rather nontrivial mathematical concepts. However, the mathematically oriented reader might wish to know if the construction presented in [3] is indeed a particular case of a “noncommutative connection” or if it is something different; he will want to obtain answers to the following questions:

• What is the underlying associative algebra?
• What is the underlying differential algebra?
• What is the module?

We shall answer these questions here.

The present paper is not conceived as a sequel to [3] (or [4]). It is self-contained, to a large extent, and, therefore, it is of independent interest. Furthermore, for any manifold, we shall construct a new \( \mathbb{Z} \)-graded differential algebra by using matrix-valued differential forms (cf. sec. 3). This algebra is neither commutative nor even graded-commutative (and therefore does not coincide with the usual algebra of differential forms); it underlies, albeit not explicitly, the construction made in [3] and is very similar to the algebra \( \Omega_D(A) \) introduced by Connes in [2].

2. Connections in noncommutative geometry

In order to define covariant differentials in noncommutative geometry, one needs three ingredients [7]:

1. an algebra \( A \), not necessarily commutative;
2. a \( \mathbb{Z} \)-graded differential algebra \( (\Xi = \oplus_{p=0}^{\infty} \Xi^p, d) \) such that \( \Xi^0 = A \). Notice that \( \Xi \) is, therefore, an \( A \)-bimodule.
3. a right-module \( S \) over \( A \). (This could actually be a left-module but our choice makes notations easier.)

In conventional gauge theories, i.e. in the framework of commutative geometry, the ingredients are as follows:

1. \( A \) is the (commutative) algebra \( C(M) \) of real or complex valued functions on a smooth manifold \( M \), with \( M \) representing, for example, space-time,
2. \( \Xi \) is the algebra of differential forms \( \Lambda(M) \) with its wedge product, and
3. \( S \) is the space of sections of a vector bundle (for instance the space of vector fields, scalar fields, tensor fields, spinor fields, quark fields).
The covariant differential $\nabla$ maps $S$ into $S \otimes_A \Xi^1$, and more generally $S \otimes_A \Xi^p$ to $S \otimes_A \Xi^{p+1}$. For instance, it maps vector fields to one-form valued vector fields. $\nabla$ should be such that

$$
\nabla(vf) = (\nabla v)f + v \otimes df, \quad v \in S, f \in A
$$

and more generally

$$
\nabla(v \otimes \lambda) = (\nabla v)\lambda + (-1)^p v \otimes d\lambda, \quad v \in S, \lambda \in \Xi^p.
$$

In usual differential geometry we can define not only covariant differentials but also covariant derivatives by evaluating the covariant differential $\nabla v$ along a vector field $e_\mu$: $\nabla_{e_\mu} v$. Indeed, in the usual case, the space of one-forms is dual to the space of vector fields and vector fields themselves are derivations on the commutative algebra $C^\infty(M)$. However, an arbitrary algebra may not have derivations at all (for instance the complex numbers); of course one could argue that “interesting” algebras have non trivial derivations (for instance the inner derivations). The problem is that, in any case, derivations on the algebra $A$, even if they exist, will not form a module over $A$ (unless we are in the commutative or graded-commutative case). Finally there is no reason why these derivations should be related to one-forms (elements of $\Xi^1$). To summarize: we have covariant differentials but no covariant derivatives (unless we introduce a space dual (over $A$) to $\Xi^1$ and evaluate $\nabla v$ on its elements.)

It should be stressed that, for a given algebra $A$ (commutative or not), every choice of a $\mathbb{Z}$-graded differential algebra $\Xi$ defines a differential calculus. Many nice properties of covariant differentials can be generalized to the noncommutative case \[4\]. In particular the square $\nabla^2$ of the covariant differential is an $A$-linear object: the curvature. Let us suppose that $S$ can be written as $S = pA^n$ where $p$ is a projector ($p^2 = p$). This is a mild assumption since all the modules obtained as spaces of sections of vector bundles are of this form (projective finite modules). The case $p = Identity$ means that the module (the vector bundle) is trivial. A covariant differential can be written as

$$
\nabla = pd + A
$$

where $A \in \Xi^1 \otimes_A EndS$ and should be such that $pAp = A$. Then, if $X \in S$, $\nabla X = pdX + AX$, which, when written in components, reads $(\nabla X)^i = p_k^i(dX)^k + A_k^i X^k$. It is easy to check that $\nabla$ is indeed a connection, viz.

$$
\nabla(Xf) = pd(Xf) + AXf
= p(dX)f + AXf + pXdf
= \nabla(X)f + Xdf
$$

where we used the fact that $pX = X$. Then we compute

$$
\mathcal{F} = \nabla^2
$$
\[ \nabla^2(X) = pd(pdX + AX) + A(pdX + AX) \]
\[ = pd(pdX + AX) + A(dX + AX), \text{ since } Ap = A \]
\[ = p(dpX + pd^2X + (dA)X - A(dX) + A(dX)) + A^2X \]
\[ = p(dpX + (dA)X) + A^2X \]
\[ = [pdpdp + pdA + A^2]X \]

where we have used the properties \( dX = d(pX) = (dp)X + pdX \) and \( p^2(dp)p = pd(p^2) - pdp = 0 \) implying \( pdpX = pdpdpX + p(dp)pX = pdpdpX \). The conclusion is that the curvature is

\[ \mathcal{F} = pdpdp + pdA + A^2 \] (7)

and is linear.

Notice that the associative algebra \( A \) could be \( \mathbb{Z}_2 \)-graded; the whole formalism of noncommutative geometry extends to the \( \mathbb{Z}_2 \)-graded case and this is discussed in [8].

3. A simple example of generalized connections

We now construct a very simple example that fits into the general framework but is slightly more general than the situation of pure Yang-Mills theory.

For the associative algebra \( A \) we take \( A = C(M) \oplus C(M) \) and we represent elements of \( A \) as \( 2 \times 2 \) diagonal matrices with elements \( f(x) \) und \( g(x) \), two numerical functions over the \( n \)-dimensional space (or space-time \( M \)). Notice that \( A \) is still commutative but does not coincide with the space of functions over a connected manifold: Indeed \( A \) is the space of functions over the non-connected disjoint union \( M \cup M \). Intuitively we can think of it as two parallel universes where left and right “movers” live (or where fermionic particles and antiparticles live [9]).

For \( S \) we take \( S = A \) itself, represented as the column vectors with two elements \( f(x) \) and \( g(x) \). An alternative choice, motivated by physical considerations, would be to take a space of vectors with two components \( \Psi_L \) and \( \Psi_R \), the first being a left-handed spinor field and the second being a right-handed spinor field. (Of course, here we have to assume that \( M \) is even dimensional and admits a spin structure.)

To fully set the stage we have to specify the \( \mathbb{Z} \)-graded differential algebra \( \Xi \). This is where our construction differs from [1]. Let us just define \( \Xi \) as a vector space. (To simplify the description we assume here that \( M \) is 4-dimensional, but generalizing the construction for any dimension \( n \), \( n \) even or odd, is obvious.)

- \( \Xi^0 \) is the space of \( 2 \times 2 \) matrices with zero-forms (functions) on the diagonal.
- \( \Xi^1 \) is the space of \( 2 \times 2 \) matrices with one-forms on the diagonal and zero-forms on the antidiagonal.
- \( \Xi^2 \) is the space of \( 2 \times 2 \) matrices with zero-forms and two-forms on the diagonal and one-forms on the antidiagonal.
\[ \Xi^3 \text{ is the space of } 2 \times 2 \text{ matrices with one-forms and three-forms on the diagonal and zero-forms and two-forms on the antidiagonal.} \]

\[ \Xi^4 \text{ is the space of } 2 \times 2 \text{ matrices with zero-forms and two-forms and four-forms on the diagonal and one-forms and three-forms on the antidiagonal.} \]

\[ \Xi^5 \text{ is the space of } 2 \times 2 \text{ matrices with one-forms and three-forms on the diagonal and zero-forms and two-forms and four-forms on the antidiagonal.} \]

Note that this sequence of spaces does not stop at \( \Xi^4 \), even though \( \dim M = 4 \). For the remainder, we take \( \Xi^6 \cong \Xi^8 \cong \Xi^{10} \cong \ldots \) as isomorphic copies of \( \Xi^4 \) and \( \Xi^7 \cong \Xi^9 \cong \Xi^{11} \cong \ldots \) as isomorphic copies of \( \Xi^5 \).

We then define \( \Xi \) itself as the direct sum \( \Xi = \bigoplus_{p=0}^{\infty} \Xi^p \).

It is to be noted that we define \( \Xi \) as a direct sum of vector spaces. This implies, in particular, that \( \Xi^p \cap \Xi^q = \emptyset \) whenever \( p \neq q \), and, for instance, that \( \Xi^2 \) is not included in \( \Xi^4 \)!

This direct sum construction is exactly analogous to writing \( \mathbb{R}^2 = \mathbb{R} \oplus \mathbb{R} \) for the two dimensional plane. Clearly, the first copy of \( \mathbb{R} \) is isomorphic to the second one but the \( \partial_x \)- and \( \partial_y \)-axes are different. It is probably not necessary to elaborate more on such a elementary remark but the fact itself is essential.

For a smooth manifold \( M \) of dimension \( n \) we call \( \Lambda(M) \) the algebra of differential forms and define \( \Xi^2^p \) as the space of \( 2 \times 2 \) matrices such that diagonal elements belong to \( \bigoplus_{j=0,(\text{even})}^{2p} \Lambda^j(M) \) and such that antidiagonal elements belong to \( \bigoplus_{j=1, (\text{odd})}^{2p-1} \Lambda^j(M) \); similarly we define \( \Xi^2^{p+1} \) as the space of \( 2 \times 2 \) matrices such that diagonal elements belong to \( \bigoplus_{j=1, (\text{odd})}^{2p+1} \Lambda^j(M) \) and such that antidiagonal elements belong to \( \bigoplus_{j=0, (\text{even})}^{2p} \Lambda^j(M) \).

As a vector space we have in all cases

\[ \Xi = \bigoplus_{q=0}^{\infty} \Xi^q \quad (8) \]

with

\[ \Xi^{2p} = \left[ \bigoplus_{j=0, (\text{even})}^{2p} \Lambda^j(M) \right] \oplus \left[ \bigoplus_{j=0, (\text{even})}^{2p} \Lambda^j(M) \right] \oplus \left[ \bigoplus_{j=1, (\text{odd})}^{2p-1} \Lambda^j(M) \right] \oplus \left[ \bigoplus_{j=1, (\text{odd})}^{2p-1} \Lambda^j(M) \right] \quad (9) \]

and a similar expression for \( \Xi^{2p+1} \). (Notice that, \( M \) being finite dimensional, we always get a periodicity modulo 2 for \( \Xi^q \) when \( q \) is big enough.)

To make \( \Xi \) an algebra, we define the following associative, graded product \( \odot \):

\[ (M \otimes f) \odot (N \otimes g) = (-1)^{\partial N \partial f} MN \otimes f \wedge g \quad (10) \]

where \( \partial f \) denotes the \( \mathbb{Z}_2 \)-grading of the form \( f \) (even or odd) and \( \partial N \) denotes the \( \mathbb{Z}_2 \)-grading of the \( 2 \times 2 \) matrix \( N \) (diagonal or antidiagonal). For the case of \( 2 \times 2 \) matrices
this reads
\[
\begin{pmatrix}
  A & C \\
  D & B
\end{pmatrix} \odot \begin{pmatrix}
  A' & C' \\
  D' & B'
\end{pmatrix} = \begin{pmatrix}
  A \wedge A' + (-1)^{\partial C} C \wedge D' \\
  D \wedge A' + (-1)^{\partial B} B \wedge D'
\end{pmatrix}
\begin{pmatrix}
  C \wedge B' + (-1)^{\partial A} A \wedge C' \\
  B \wedge B' + (-1)^{\partial D} D \wedge C'
\end{pmatrix}
\tag{11}
\]

It should be understood that the product of an element of $\Xi^p$ and an element of $\Xi^q$ belongs to $\Xi^{p+q}$. It is easy to check that the product $\odot$ makes $\Xi$ a $\mathbb{Z}$-graded algebra.

Finally we want to define a differential $d$ on $\Xi$. (Actually it will be a graded differential.) With
\[
X = \begin{pmatrix}
  A & C \\
  D & B
\end{pmatrix} \in \Xi^p
\tag{12}
\]
we define
\[
\delta_1 X = \begin{pmatrix}
  dA & -dC \\
  -dD & dB
\end{pmatrix} \in \Xi^{p+1}
\tag{13}
\]
\[
\delta_2 X = i \begin{pmatrix}
  C + D & -(A - B) \\
  (A - B) & C + D
\end{pmatrix} \in \Xi^{p+1},
\tag{14}
\]
with $d$ denoting the usual exterior derivative.

It is easy to check that $\delta_1^2 = \delta_2^2 = \delta_1 \delta_2 + \delta_2 \delta_1 = 0$. We therefore define
\[
dX = \delta_1 X + \delta_2 X \in \Xi^{p+1}
\tag{15}
\]
and verify that $d^2 X = 0$. Moreover,
\[
\text{for } X \in \Xi^{2p}, Y \in \Xi^r \text{ we have } d(X \odot Y) = dX \odot Y + X \odot dY,
\]
but for $X \in \Xi^{2p+1}, Y \in \Xi^r$ we have $d(X \odot Y) = dX \odot Y - X \odot dY$.

Therefore $\Xi$ is a $\mathbb{Z}$-graded and $\mathbb{Z}_2$-graded differential algebra. This completes the list of ingredients needed to construct an example of gauge theory in noncommutative geometry.

As explained in section 2, an arbitrary element of $\Xi^1$ defines a connection, moreover the module $\mathcal{S}$, in our case, is trivial, so that the curvature is simply:
\[
\mathcal{F} = dA + A \odot A \in \Xi^2
\tag{16}
\]
Writing
\[
A = \begin{pmatrix}
  L & i\phi \\
  i\phi & R
\end{pmatrix} \in \Xi^1
\tag{17}
\]
we obtain
\[
\mathcal{F} = \begin{pmatrix}
  \mathcal{F}_{11} & \mathcal{F}_{12} \\
  \mathcal{F}_{21} & \mathcal{F}_{22}
\end{pmatrix}
\]
with
\[
\mathcal{F}_{11} = dL - [(\phi + \phi) + \phi\phi]
\]
\[
\mathcal{F}_{12} = -i[d\phi + L\phi - \phi R + (L - R)]
\]
\[
\mathcal{F}_{21} = -i[d\phi - \phi L + R\phi - (L - R)]
\]
\[
\mathcal{F}_{22} = dR - [(\phi + \phi) + \phi\phi]
\tag{18}
\]
We now suppose that the underlying manifold is Riemannian (or pseudo-Riemannian); therefore we have a metric \( g = (g_{\mu\nu}) \), i.e. a scalar product on the tangent spaces. Assuming that it is not degenerate, we can extend this scalar product to one-forms and more generally to the whole algebra of differential forms. (The extension is not unique in the sense that we could introduce arbitrary positive constants, or rescaling factors, in the expression of \( g \) for \( p \)-forms and \( q \)-forms.)

Writing
\[
L = |\mathcal{F}_{11}|^2 + |\mathcal{F}_{12}|^2 + |\mathcal{F}_{21}|^2 + |\mathcal{F}_{22}|^2,
\]

we find (8), up to an overall rescaling,

\[
L = -\frac{1}{4}(F^L_{\mu\nu})^2 - \frac{1}{4}(F^R_{\mu\nu})^2 + 2D\phi D\phi + 2(\phi + \overline{\phi} + \phi\overline{\phi})^2
\]

with
\[
D\phi = \nabla\phi + (L - R)
\]
\[
= d\phi + L\phi - \phi R + (L - R)
\]

and \( F^L = dL \) and likewise for \( R \). It is then convenient to introduce new fields \( \gamma \) and \( Z \)

\[
\gamma = \frac{1}{\sqrt{2}}(L + R)
\]
\[
Z = \frac{1}{\sqrt{2}}(L - R).
\]

The above formalism describes therefore a \( U(1) \times U(1) \) theory with symmetry breaking. (The photon \( \gamma \) stays massless but the \( Z \) acquires a mass.) Notice that the potential

\[
V(\phi) = (\phi + \overline{\phi} + \phi\overline{\phi})^2
\]

is already shifted to a point of absolute minima (see the discussion in [3, 10]).

At this point it is useful to point out the fact that the algebra \( \Omega_D(A) \) introduced in [2] and the above algebra \( \Xi \) are very similar. The former is obtained from the universal differential algebra \( \Omega(A) \) by dividing out a differential ideal \( J = J_0 + \delta J_0 \) where \( J_0 \) is the kernel of the map

\[
\Omega(A) \ni a_0 \delta a_1 \ldots \delta a_p \mapsto a_0[D, a_1] \ldots [D, a_p] \in \mathcal{B}(H),
\]

\( D \) being the Dirac-Yukawa operator and \( \mathcal{B}(H) \) being the space of bounded operators on the Hilbert space of spinors \( H \). Actually \( J \) is not naturally graded and one has to grade it by taking its intersection with the vector subspaces of \( \Omega(A) \) associated with its \( \mathbb{Z} \)-grading. This division leads to the consequence, in Connes’ approach, that the kinetic term \( D\phi D\phi \) is proportional to \( \text{tr}(MM^\dagger) \) and that the Higgs potential is proportional to \( \text{tr}[(MM^\dagger)_\perp]^2 \), where

\[
(MM^\dagger)_\perp = (MM^\dagger) - \frac{1}{n}n\text{tr}MM^\dagger,
\]

8
where $M$ is a $n \times n$ fermionic mass matrix. Hence, this potential vanishes whenever $MM^\dagger$ is proportional to the unit matrix.

The spaces $\Omega^0_D(A)$, $\Omega^1_D(A)$ and $\Omega^2_D(A)$ for $A = C(M) \oplus C(M)$ have been computed in [2] and it happens that they are respectively equal to $\Xi^0$, $\Xi^1$ and $\Xi^2$. This comparison applies when $MM^\dagger \neq \mathbb{1}$, i.e., in Connes’ model, when the fermion masses are not degenerate. Note that in our construction no mass matrix appears.

To avoid possible confusions, we should mention the fact that the notation $\Omega^*_D(A)$ itself was introduced already in [11] to denote another differential algebra (namely, the homomorphic image of the universal differential envelope of $A$ into the algebra of $A$-valued multilinear forms on the space of derivations of $A$.)

4. Conclusions

Connes’ differential algebra $\Omega^*_D(A)$ depends explicitly on the fermionic masses entering the Dirac-Yukawa operator $D$. Apart from this difference, we have seen that the spaces that are relevant for physics, i.e. the spaces of degree 0, 1, and 2, are the same in our construction. In particular, the expression of $\mathcal{L}$ (the bosonic Lagrangian) is essentially the same in both cases. The only difference is the following: In our case, using $\Xi$, the coefficients in front of the individual terms of the Lagrangian are arbitrary. They stem from different normalizations of the scalar product in the space of $p$-forms. In the case of [2] (using $\Omega^*_D(A)$), these coefficients are computed in terms of masses of fermions.

It is not clear how quantization could be made to respect, or modify in a predictable way, the mass relations obtained using $\Omega^*_D(A)$. If one disregards these mass relations, the two approaches become completely equivalent, as far as physics and the standard model of electroweak interactions are concerned. While the approach in [3, 4] is more elementary and easily amenable to a physical interpretation, the approach [1, 2] has a bigger character of generality.

Also we would like to emphasize the following point: The above formulation does not use explicitly the $\mathbb{Z}$-grading of the algebra $\Xi$; everything can be done using only $2 \times 2$ matrices, i.e. representing the whole algebra $\Xi$ by $2 \times 2$ matrices. In this process, the $\mathbb{Z}$-grading is lost, only the $\mathbb{Z}_2$-grading remains (it is not a representation of $\mathbb{Z}$-graded algebra). This is what was done in [3] because it is simpler. The drawback was that the underlying $\mathbb{Z}$-graded algebra (which had to be there in order to put things in a more conventional mathematical framework) was not obvious. The main purpose of the present paper was to answer that question.

In the present paper we presented the example of $2 \times 2$ matrices because this makes the underlying structure clear. More generally, we could use $2N \times 2N$ matrices and this would lead to a $U(N)_L \times U(N)_R$ gauge theory. The structure group can subsequently be reduced to a Lie subgroup by imposing further constraints on the (algebraic) connection: trace conditions, projections or even symmetry conditions.
References

[1] A. Connes, J. Lott, Nucl. Phys. B 18 B (1990) 29-47

[2] A. Connes, Les Houches Lectures (1992)

[3] R. Coquereaux, G. Esposito-Farèse, G. Vaillant, Nucl. Phys. B 353 (1991) 689-706

[4] R. Coquereaux, G. Esposito-Farèse, F. Scheck, Int. J. Mod. Phys. A7 (1992) 6555-6593

[5] E. Chamsedine, G. Felder, J. Fröhlich, Phys. Lett. B 296 (1992) 109-116

[6] M. Dubois-Violette, R. Kerner, J. Madore, Class. Quant. Grav. 6 (1989) 1709-1724

[7] A. Connes, Publ. Math. IHES 62 (1985)

[8] D. Kastler, Cyclic cohomology within the differential envelope, Coll. Travaux en Cours, Hermann, Paris, 1985

[9] G. Chardin, Nucl. Phys. A558 (1993) 477-496

[10] R. Coquereaux, R. Häußling, N.A. Papadopoulos, F. Scheck, Int. J. Mod. Phys. A7 (1992) 2809-2824

[11] M. Dubois-Violette, C. R. Acad. Sci. Paris 307, Ser.1 (1988) 403-408