Construciton of analytic functions, which determine bounded Toeplitz operators on $H^1$ and $H^\infty$

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Abstract

For $f \in H^\infty$ we denote by $T_f$ the Toeplitz operator on $H^p$, defined by

$$T_fh = \int_T T(\zeta)h(\zeta) \frac{dm(\zeta)}{1 - \zeta z}, \quad h \in H^p.$$ 

In this paper we prove some sufficient conditions for the sequences of numbers $\alpha = (\alpha_n)_{n \geq 0}$ in which the functions

$$f * \alpha \overset{def}{=} \sum_{n \geq 1} f(n) \alpha_n z^n$$

determine bounded Toeplitz operators $T_{f*\alpha}$ on $H^1$ and $H^\infty$ for all $f \in H^\infty$.

1 Introduction

Let $A$ be the class of all functions analytic in the unit disk $\mathbb{D} = \{ \zeta : \ |\zeta| < 1 \}$, $m(\zeta)$ - normalized Lebesgue measure on the circle $\mathbb{T} = \{ \zeta : \ |\zeta| = 1 \}$. Let $H^p$ $(0 < p \leq \infty)$ is the space of all functions analytic in $\mathbb{D}$ and satisfying

$$\|f\|^p_{H^p} = \sup_{0 < r < 1} \int_\mathbb{T} |f(r\zeta)|^p dm(\zeta) < \infty, \quad 0 < p < \infty,$$
\[ \|f\|_{H^\infty} = \sup_{z \in \mathbb{D}} |f(z)| < \infty, \quad p = \infty. \]

Let \( M \) is the space of all finite, complex Borel measures on \( \mathbb{T} \) with the usual variation norm.

For \( \mu \in M \), the analytic function on \( \mathbb{D} \)

\[ K_\mu(z) = \int_\mathbb{T} \frac{1}{1 - \zeta z} \, d\mu(\zeta) \]

is called the Cauchy transforms of \( \mu \) and the set of functions

\[ K = \{ f \in A : f = K_\mu, \ \mu \in M \} \]

is called the space of Cauchy transforms.

For \( d\mu(\zeta) = \varphi(\zeta) \, dm(\zeta), \ 1 \leq p \leq \infty \), we denote \( K_\mu(z) = K_\varphi(z) \) and

\[ K^p = \{ f \in A : f = K_\varphi, \ \varphi \in L^p \}, \ \ 1 \leq p \leq \infty. \]

By the theorem of M. Riez \( K^p = H^p \) for \( 1 < p < \infty \), however \( H^1 \subsetneq K^1 \), \( H^\infty \subsetneq K^\infty \).

We note that \( K^\infty = BMOA \) (the space of analytic functions of bounded mean oscillation )[1].

For \( f \in H^\infty \) we denote by \( T_f \) the Toeplitz operator on \( H^p \), defined by

\[ T_f h = K_{f^*}(z) = \int_\mathbb{T} \frac{f(\zeta)h(\zeta)}{1 - \zeta z} \, dm(\zeta), \quad h \in H^p. \]

By the theorem of M. Riez for \( 1 < p < \infty \) the operator \( T_f \) is bounded on \( H^p \) for all \( f \in H^\infty \). But if \( p = 1 \) and \( p = \infty \) not every function \( f \in H^\infty \) gives rise to bounded Toeplitz operator \( T_f \) on \( H^1 \) and \( H^\infty \).

There is also an interesting connection between multipliers of the spaces \( K \) and \( K^p \), \( p = 1, \infty \) and the Toeplitz operators.

Let \( \mathcal{M} \) and \( \mathcal{M}^p \) be the class to all multipliers of the spaces \( K \) and \( K^p \):

\[ \mathcal{M} = \{ f \in A : \ f g \in K, \ \forall g \in K \}, \]

\[ \mathcal{M}^p = \{ f \in A : \ f g \in K^p, \ \forall g \in K^p \}. \]

Since \( K^p = H^p \) for \( 1 < p < \infty \), then \( \mathcal{M}^p = H^\infty \) for \( 1 < p < \infty \).

However

\[ \mathcal{M} = \mathcal{M}^1 \subsetneq H^\infty, \quad \mathcal{M}^\infty \subsetneq H^\infty \]

and

\[ \mathcal{M} = \mathcal{M}^1 = \{ f \in H^\infty : \ \|T_f\|_{H^\infty} < \infty \} [3], \]
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\[ \mathcal{M}^\infty = \{ f \in H^\infty : \|T_f\|_{H^1} < \infty \} \] [2].

Let’s note, that more information, bibliography and review of results for the spaces \( K \) and \( \mathcal{M} \) contains the new monograph [5].

Since \( \mathcal{M} = \mathcal{M}^1 \subsetneq H^\infty \), \( \mathcal{M}^\infty \subsetneq H^\infty \) i.e. not all function \( f \in H^\infty \) give rise to bounded Toeplitz operators on \( H^1 \) and \( H^\infty \), then naturally arises the following task:

To describe these sequences of numbers \( \alpha = (\alpha_n)_{n \geq 0} \), for which the functions

\[ f \ast \alpha \overset{\text{def}}{=} \sum_{n \geq 1} \hat{f}(n) \alpha_n z^n, \quad z \in \mathbb{D} \]

give rise to bounded Toeplitz operators \( T_{f \ast \alpha} \) on \( H^1 \) and \( H^\infty \) for all \( f \in H^\infty \).

In this paper we prove some sufficient conditions for the sequences \( \alpha = (\alpha_n)_{n \geq 0} \) in which Toeplitz operator \( T_{f \ast \alpha} \) is bounded on \( H^1 \) and \( H^\infty \) for all \( f \in H^\infty \).

Further we will use the following important theorem:

**Theorem of Smirnov.**

Let \( 0 < p < q \), \( f \in H^p \) and has \( L^q \) boundary values (\( f \in L^q(\mathbb{T}) \)). Then \( f \in H^q \).

We include also its proof for convenience of the reader.

**Proof.** Since \( f \in H^p \), then \( f = Bg \), where \( B \) is a Blaschke product, \( g \in H^p \) and \( g \neq 0 \) in \( \mathbb{D} \).

The function \( g^p \in H^1 \) and applying the formula of Poisson to the function \( g^p \) we have

\[ g^p(z) = \int_{\mathbb{T}} g^p(\zeta) P_z(\zeta) \, dm(\zeta), \quad P_z(\zeta) = \frac{1-|z|^2}{|\zeta-z|^2}, \quad \zeta \in \mathbb{T}, \quad z \in D. \]

From this formula, taking into account that

\[ |f(z)| \leq |g(z)| \quad \text{in} \ \mathbb{D}, \quad |f(\zeta)| = |g(\zeta)| \quad \text{for almost every} \ \zeta \in \mathbb{T}, \]

follows

\[ |f(z)|^p \leq \int_{\mathbb{T}} |f(\zeta)|^p P_z(\zeta) \, dm(\zeta). \]

If \( q = \infty \), then \( f \in L^\infty(\mathbb{T}) \) and \( \|f\|_{H^\infty} \leq \|f\|_{L^\infty(\mathbb{T})} < \infty. \)

If \( q < \infty \), then applying the Holder’s inequality we have

\[ |f(z)|^p \leq \int_{\mathbb{T}} |f(\zeta)|^p (P_z(\zeta))^{p/q} (P_z(\zeta))^{1-p/q} \, dm(\zeta) \leq \]

\[ \left( \frac{1}{p/q} \int_{T} |f(\zeta)|^q P_{\zeta}(\zeta) dm(\zeta) \right)^{p/q} \left( \int_{T} P_{\zeta}(\zeta) dm(\zeta) \right)^{1-p/q} = \]

\[ = \left( \frac{1}{p/q} \int_{T} |f(\zeta)|^q P_{\zeta}(\zeta) dm(\zeta) \right)^{p/q} \Rightarrow \]

\[ |f(z)|^q \leq \int_{T} |f(\zeta)|^q P_{\zeta}(\zeta) dm(\zeta). \]

Integrating on the circle \(|z| = r, 0 < r < 1\) we obtain

\[ \int_{T} |f(r\eta)|^q dm(\eta) \leq \int_{T} \int_{T} |f(\zeta)|^q \frac{1-r^2}{|\zeta - r\eta|^2} dm(\zeta) dm(\eta) \leq \| f \|_{L^q(\mathbb{T})} < \infty. \]

Consequently \( f \in H^q. \)

2 Main results

Let \( \mathfrak{N} \) is the class of all functions \( f \in H^\infty \) for which

\[ \Lambda(f) \overset{\text{def}}{=} \text{ess sup} \int_{\mathbb{T}} \frac{|f(\zeta) - f(\eta)|}{|\zeta - \eta|} dm(\zeta) < \infty. \]

For \( f \in \mathfrak{N} \) we denote \( \| f \|_{\mathfrak{N}} \overset{\text{def}}{=} \| f \|_{H^\infty} + \Lambda(f). \)

**Theorem 1.** If \( f \in \mathfrak{N} \), then Toeplitz operator \( T_f \) is bounded on \( H^p \) \( (p = 1, \infty) \) and

\[ \| T_f \|_{H^p} \leq \| f \|_{\mathfrak{N}}. \]

**Proof.** The case \( p = \infty \) is proved in [3,4] and is generalized in [6] for the multipliers of the integrals of Cauchy-Stieltjes type in domains with closed Jordan curve.

We shall prove the case \( p = 1 \).

Let \( f \in \mathfrak{N}, h \in H^1 \). Let \( E \) be a subset with total measure \( (m(E) = 1) \) lying on \( \mathbb{T} \) so that

\[ \| f \|_{H^\infty} = \sup_{\eta \in E} |f(\eta)|. \]

Then

\[ \| T_f h \|_{H^1} = \sup_{0 < r < 1} \int_{T} \left| \int_{T} \frac{\overline{f}(\zeta)h(\zeta)}{\zeta - r\eta} \zeta dm(\zeta) \right| dm(\eta) = \]
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\[
= \sup_{0 < r < 1} \int_T \left| \int_T \overline{f(\zeta) - f(r\eta)} \frac{h(\zeta) \zeta dm(\zeta)}{\zeta - r\eta} + \overline{f(r\eta)} \int_T \frac{1}{\zeta - r\eta} h(\zeta) \zeta dm(\zeta) \right| dm(\eta) \leq \\
\leq \sup_{0 < r < 1} \left\{ \int_T \int_T \left| \frac{f(\zeta) - f(r\eta)}{\zeta - r\eta} \right| \left| h(\zeta) \right| dm(\zeta) dm(\eta) + \int_T \left| \overline{f(r\eta)h(r\eta)} \right| dm(\eta) \right\} \leq \\
\leq \sup_{0 < r < 1} \sup_{\zeta \in E} \left( \int_T \frac{f(\zeta) - f(z)}{\zeta - z} \left| dm(\eta) + \left\| f \right\|_{H^\infty} \right) \right\| h \right\|_{H^1}.
\]

We denote for \( \zeta \in E \)

\[
F_\zeta(z) = \frac{f(\zeta) - f(z)}{\zeta - z}, \quad z \in \mathbb{D}.
\]

Then

\[
\left\| T_fh \right\|_{H^1} \leq \sup_{\zeta \in E} \left( \left\| F_\zeta \right\|_{H^1} + \left\| f \right\|_{H^\infty} \right) \left\| h \right\|_{H^1}.
\]

To end the proof is necessary to show

\[
f \in \mathfrak{M} \Rightarrow \sup_{\zeta \in E} \left\| F_\zeta \right\|_{H^1} < \infty.
\]

Since

\[
\frac{1}{\zeta - z} \in H^p \quad (0 < p < 1)
\]

and \( f \in H^\infty \), then \( F_\zeta(z) \in H^p \quad (0 < p < 1) \).

Furthermore

\[
f \in \mathfrak{M} \Rightarrow \sup_{\zeta \in E} \left\| F_\zeta \right\|_{L^1(T)} \leq \Lambda(f) < \infty
\]

and according to the Theorem of Smirnov

\[
F_\zeta(z) \in H^1, \quad \left\| F_\zeta \right\|_{H^1} = \left\| F_\zeta \right\|_{L^1(T)} \leq \Lambda(f) < \infty.
\]

Consequently

\[
\left\| T_f \right\|_{H^1} \leq \sup_{\zeta \in E} \left( \left\| F_\zeta \right\|_{H^1} + \left\| f \right\|_{H^\infty} \right) \leq \Lambda(f) + \left\| f \right\|_{H^\infty} = \left\| f \right\|_{\mathfrak{M}} < \infty. \square
\]
**Remark.** We note that from the Theorem of Stegenga [2] characterizing a class of bounded Toeplitz operators on $H^1$ does not follow Theorem 1 for $p = 1$.

**Lemma 1.**[3] If $p_n$ is a polynomial of degree $n$, then
\[
\|p_n\|_\mathfrak{M} \leq 3 \|p_n\|_{H^\infty} \log(n + 2).
\]

**Definition.** A sequence $\alpha = (\alpha_n)_{n \geq 0}$ of positive numbers is called concave if
\[
\alpha_{n+2} - \alpha_{n+1} \geq \alpha_{n+1} - \alpha_n \Leftrightarrow \alpha_n - 2\alpha_{n+1} + \alpha_{n+2} \geq 0.
\]

**Theorem 2.** Let $\alpha = (\alpha_n)_{n \geq 0}$ be a monotone decreasing, concave sequence of positive numbers and
\[
\|\alpha\| \overset{\text{def}}{=} \sum_{n \geq 0} \frac{\alpha_n}{n + 1} < \infty.
\]

Then $f * \alpha \in \mathfrak{M}$, Toeplitz operator $T_{f * \alpha}$ is bounded on $H^1$ and $H^\infty$ for all $f \in H^\infty$ and
\[
\|T_{f * \alpha}\|_{H^p} \leq \|f * \alpha\|_\mathfrak{M} \leq 12 \|f\|_{H^\infty} \|\alpha\|, \quad p = 1, \infty.
\]

**Proof.** Using Abel’s formula two times we obtain
\[
\sum_{n \geq 0} \frac{\alpha_n}{n + 1} = \sum_{n \geq 0} (\alpha_n - \alpha_{n+1}) \sum_{k=0}^n \frac{1}{k + 1} \geq \sum_{n \geq 0} (\alpha_n - \alpha_{n+1}) \log(n + 2) = \sum_{n \geq 0} (\alpha_n - 2\alpha_{n+1} + \alpha_{n+2}) \sum_{k=0}^n \log(k + 2).
\]

Since
\[
\sum_{k=0}^n \log(k + 2) \geq \sum_{k=\lfloor n/2 \rfloor}^n \log(k + 2) \geq (n/2 + 1) \log([n/2] + 2) \geq \frac{1}{4} (n + 1) \log(n + 2),
\]

then
\[
4 \sum_{n \geq 0} \frac{\alpha_n}{n + 1} \geq \sum_{n \geq 0} (\alpha_n - 2\alpha_{n+1} + \alpha_{n+2})(n + 1) \log(n + 2).
\]

Further let $f \in H^\infty$ and
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\[ S_n(f) = \sum_{k=0}^{n} \hat{f}(k) z^k; \quad \sigma_n(f) = \frac{1}{n+1} \sum_{k=0}^{n} S_k(f). \]

Applying the Abel’s formula we obtain

\[ f * \alpha = \sum_{n \geq 0} \hat{f}(n) \alpha_n z^n = \sum_{n \geq 0} (\alpha_n - \alpha_{n+1}) S_n(f) = \sum_{n \geq 0} (\alpha_n - 2\alpha_{n+1} + \alpha_{n+2})(n+1) \sigma_n(f). \]

Since by Lemma 1.

\[ \|\sigma_n(f)\|_{\mathcal{R}} \leq 3 \|\sigma_n(f)\|_{H^\infty} \log(n+2) \leq 3 \|f\|_{H^\infty} \log(n+2), \]

then

\[ \|f * \alpha\|_{\mathcal{R}} \leq \sum_{n \geq 0} (\alpha_n - 2\alpha_{n+1} + \alpha_{n+2})(n+1) \|\sigma_n(f)\|_{\mathcal{R}} \leq \]

\[ \leq 3 \|f\|_{H^\infty} \sum_{n \geq 0} (\alpha_n - 2\alpha_{n+1} + \alpha_{n+2})(n+1) \log(n+2) \leq \]

\[ \leq 12 \|f\|_{H^\infty} \sum_{n \geq 0} \frac{\alpha_n}{n+1} = 12 \|f\|_{H^\infty} \|\alpha\| < \infty. \square \]

The following proposition follows at once from Theorem 2.

**Theorem 3.** Let \( \alpha \) denote one of the sequences \((\varepsilon > 0)\):

\[ \left( \frac{1}{(n+1)^\varepsilon} \right)_{n \geq 0}; \]

\[ \left( \frac{1}{\log^{1+\varepsilon}(n+2)} \right)_{n \geq 0}; \]

\[ \left( \frac{1}{\log(n+2) \log^{1+\varepsilon}(n+3)} \right)_{n \geq 0}, \ldots. \]

Then \( f * \alpha \in \mathfrak{N} \), Toeplitz operator \( T_{f * \alpha} \) is bounded on \( H^1 \) and \( H^\infty \) for all \( f \in H^\infty \).
Remark. Theorem 3 was proved by another method in [3] (Theorem 7.) for the bounded Toeplitz operators $T_{f,\alpha}$ on $H^\infty$.

**Theorem 4.** Let the sequence $\alpha = (\alpha_n)_{n \geq 0}$ satisfy the conditions of Theorem 3. If the sequence $a = (a_n)_{n \geq 0} \in \ell^2$, then there exists a function $f \in \mathcal{N}$, satisfying

$$|\hat{f}(n)| \geq \alpha_n |a_n|, \quad \|f\|_{\mathcal{N}} \leq c_0 \|\alpha\| \|a\|_{\ell^2},$$

where $c_0$ is an absolute constant.

**Proof.** By the Theorem of Kislyakov [7] if $a = (a_n)_{n \geq 0} \in \ell^2$, then there exists a function $f \in H^\infty$, satisfying

$$|\hat{g}(n)| \geq |a_n|, \quad \|g\|_{H^\infty} \leq B \|a\|_{\ell^2},$$

where $B$ is an absolute constant. By Theorem 2.3 $f = g * \alpha \in \mathcal{N}$ and

$$\|f\|_{\mathcal{N}} \leq 12 \|\alpha\| \|g\|_{H^\infty} \leq 12B \|\alpha\| \|a\|_{\ell^2}. \Box$$

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