STRUCTURE OF MULTICORRELATION SEQUENCES WITH INTEGER PART POLYNOMIAL ITERATES ALONG PRIMES

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ABSTRACT. Let $T$ be a measure preserving $\mathbb{Z}^d$-action on the probability space $(X,B,\mu)$, $q_1,\ldots,q_m: \mathbb{R} \to \mathbb{R}^d$ vector polynomials, and $f_0,\ldots,f_m \in L^\infty(X)$. For any $\epsilon > 0$ and multicorrelation sequences of the form $\alpha(n) = \int_X f_0 \cdot T^{q_1(n)} f_1 \cdots T^{q_m(n)} f_m \, d\mu$ we show that there exists a nilsequence $\psi$ for which
\[
\lim_{N \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \psi(n)| \leq \epsilon
\]
and
\[
\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} |\alpha(p) - \psi(p)| \leq \epsilon.
\]
This result simultaneously generalizes previous results of Frantzikinakis \cite{2} and the authors \cite{11,13}.

1. Introduction and main result

Since Furstenberg’s ergodic theoretic proof of Szemerédi’s theorem \cite{5}, there has been much interest in understanding the structure of multicorrelation sequences, i.e., sequences of the form
\[
\alpha(n) = \int_X f_0 \cdot T^{q_1} f_1 \cdots T^{q_m} f_k \, d\mu, \quad n \in \mathbb{N},
\]
where $(X,B,\mu,T)$ is a measure preserving system and $f_0,\ldots,f_k \in L^\infty(X)$. The first to provide deeper insight into the algebraic structure of such sequences were Bergelson, Host, and Kra, who showed in \cite{11} that if the system $(X,\mu,T)$ is ergodic then for any multicorrelation sequence $\alpha$ as in \cite{11} there exists a uniform limit of $k$-step nilsequences $\phi$ such that
\[
\lim_{N \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \phi(n)| = 0.
\]

Here, a $k$-step nilsequence is a sequence of the form $\psi(n) = F(g^n x)$, $n \in \mathbb{N}$, where $F$ is a continuous function on a $k$-step nilmanifold $X = G/\Gamma$ for $g \in G$, $x \in X$. A uniform limit of $k$-step nilsequences is a sequence $\phi$ such that for every $\epsilon > 0$ there exists a $k$-step nilsequence $\psi$ with $\sup_{n \in \mathbb{N}} |\phi(n) - \psi(n)| \leq \epsilon$.

Later, Leibman extended the result of Bergelson, Host and Kra to polynomial iterates in \cite{11}, and removed the ergodicity assumption in \cite{13}. Another extension was obtained by the second author in \cite{12}, and independently by Tao and Teräväinen in \cite{17}, answering a question raised in \cite{3}. There, it was shown that in addition to \cite{2} one also has
\[
\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} |\alpha(p) - \phi(p)| = 0,
\]
where $\mathbb{P}$ denotes the set of prime numbers, $[1,N] := \{1,\ldots,N\}$, and $\pi(N) := |\mathbb{P} \cap [1,N]|$.

The proofs of all the aforementioned results depend crucially on the structure theory ofHost and Kra, who established in \cite{11} that the building blocks of the factors that control

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\textsuperscript{1}A $k$-step nilmanifold is a homogeneous space $X = G/\Gamma$, where $G$ is a $k$-step nilpotent Lie group and $\Gamma$ is a discrete and co-compact subgroup of $G$. 

multiple ergodic averages are nilsystems. Since the analogous factors for \(Z^\ell\)-actions are unknown, extending the results above from \(Z\)-actions to \(Z^\ell\)-actions proved to be a challenge. Nevertheless, in [2] Frantzikinakis concocted a different approach and gave a description of the structure of multicorrelation sequences of \(Z^\ell\)-actions, which we now explain.

Henceforth, let \(\ell \in \mathbb{N}\) and let \(T\) be a measure preserving \(Z^\ell\)-action on a probability space \((X, \mathcal{B}, \mu)\). The system \((X, \mathcal{B}, \mu, T)\) gives rise to a more general class of multicorrelation sequences,

\[
\alpha(n) = \int_X f_0 \cdot T^{q_1(n)} f_1 \cdots T^{q_m(n)} f_m \, d\mu, \quad n \in \mathbb{N},
\]  

where \(q_1, \ldots, q_m: \mathbb{Z} \to \mathbb{Z}^\ell\) are integer-valued vector polynomials and \(f_0, \ldots, f_m \in L^\infty(X)\). Note that (1) corresponds to the special case of (4) when \(\ell = 1\) and \(q_1(n) = in\). Frantzikinakis showed in [2] that for every \(\alpha\) as in (1) and every \(\epsilon > 0\) there exists a \(k\)-step nilsequence \(\psi\) such that

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\alpha(n) - \psi(n)| \leq \epsilon,
\]  

where \(k\) only depends on \(\ell, m,\) and the maximal degree among the polynomials \(q_1, \ldots, q_m\). Moreover, in the special case where each polynomial iterate is linear, it was proved in [2] that one can take \(k = m\). It is still an open question whether in (5) one can replace \(\epsilon\) with a uniform limit of such sequences (see Question 2 in Section [3]).

For \(x \in \mathbb{R}\) we denote by \(|x|\) the largest integer which is smaller or equal to \(x\), while for \(x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell\) we let \(|x| := (|x_1|, \ldots, |x_\ell|)\). In [11], the first author extended Frantzikinakis’ results to all multicorrelation sequences of the form

\[
\alpha(n) = \int_X f_0 \cdot T^{q_1(n)} f_1 \cdots T^{q_m(n)} f_m \, d\mu, \quad n \in \mathbb{N},
\]  

where \(q_1, \ldots, q_m: \mathbb{R} \to \mathbb{R}^\ell\) are real-valued vector polynomials.

More recently, the last three authors showed that the conclusion of Frantzikinakis’ result holds also along the primes:

**Theorem 1 ([13 Theorems A and B]).** For every \(\ell, m, d \in \mathbb{N}\) there exists \(k \in \mathbb{N}\) with the following property. For any polynomials \(q_1, \ldots, q_m: \mathbb{Z} \to \mathbb{Z}^\ell\) with degree at most \(d\), measure preserving \(Z^\ell\)-action \(T\) on a probability space \((X, \mathcal{B}, \mu)\), functions \(f_0, f_1, \ldots, f_m \in L^\infty(X)\), \(\varepsilon > 0\), \(r \in \mathbb{N}\) and \(s \in \mathbb{Z}\), letting \(\alpha\) be as in (4), there exists a \(k\)-step nilsequence \(\psi\) satisfying (5) and

\[
\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} |\alpha(rp + s) - \psi(rp + s)| \leq \varepsilon.
\]  

In the special case \(d = 1\) one can choose \(k = m\).

Our main theorem simultaneously generalizes the main results from [11] and [13].

**Theorem A.** For every \(\ell, m, d \in \mathbb{N}\) there exists \(k \in \mathbb{N}\) with the following property. For any polynomials \(q_1, \ldots, q_m: \mathbb{R} \to \mathbb{R}^\ell\) with degree at most \(d\), any measure preserving \(Z^\ell\)-action \(T\) on a probability space \((X, \mathcal{B}, \mu)\), functions \(f_0, f_1, \ldots, f_m \in L^\infty(X)\), \(\varepsilon > 0\), \(r \in \mathbb{N}\) and \(s \in \mathbb{Z}\), letting \(\alpha\) be as in (4), there exists a \(k\)-step nilsequence \(\psi\) satisfying (5) and (7). In the special case \(d = 1\) one can choose \(k = m\).

The proof of Theorem A presented in the next section, follows closely the strategy implemented in [11], but uses Theorem [11] instead of Walsh’s theorem [13] as a blackbox.
Remark 2. Both Theorems 1 and A are equivalent to seemingly stronger versions involving commuting actions. We say that two actions $T_1$ and $T_2$ of a group $G$ commute if for every $g, h \in G$ we have $T_1^g \circ T_2^h = T_2^h \circ T_1^g$. When $G$ is an abelian group, a collection of $m$ commuting $G$-actions $T_1, \ldots, T_m$ can be identified with a single $G^m$-action $T$ via $T(g_1, \ldots, g_m) = T_1^{g_1 \ldots} \cdot \cdots \cdot T_m^{g_m}$. Using this observation, and the identification $(\mathbb{Z}^\ell)^m = \mathbb{Z}^{\ell m}$, one sees that, given commuting measure preserving $\mathbb{Z}^\ell$-actions $T_1, \ldots, T_m$ in a probability space $(X, \mu, T)$, Theorem 1 holds when (4) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{q_1(n)} f_1 \cdots T_m^{q_m(n)} f_m \, d\mu,$$

and Theorem A holds when (4) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{[q_1(n)]} f_1 \cdots T_m^{[q_m(n)]} f_m \, d\mu.$$ 

Remark 3. Let $[x]$ and $\lfloor x \rfloor$ denote the smallest integer $\geq x$ and the closest integer to $x$, respectively. Using the relations $\lfloor x \rfloor = -\lfloor -x \rfloor$ and $[x] = \lfloor x+1/2 \rfloor$, we see that Theorem A remains true if (4) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{[q_1(n)]} f_1 \cdots T_m^{[q_m(n)]} f_m \, d\mu,$$

where $[x]_i = ([x_1]_i, \ldots, [x_\ell]_i, \ell)$ and $\lfloor \cdot \rfloor, [\cdot], \lfloor \cdot \rfloor$ are any of $\lfloor \cdot \rfloor$, $[\cdot]$, or $\lfloor \cdot \rfloor$.

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2. Proof of main result

We start by proving a theorem concerning flows, which stands halfway in between Theorems 1 and A. The idea behind this result is that for a real polynomial $q(x) = a_d x^d + \ldots + a_1 x + a_0 \in \mathbb{R}[x]$ and a measure presenting flow $(S^t)_{t \in \mathbb{R}}$ we can write $S^{q(n)} = (S^{a_d})^n \cdots (S^{a_0})^1$, an expression which can be handled by Theorem 1.

Theorem 4. For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \ldots, q_m : \mathbb{R} \to \mathbb{R}^\ell$ of degree at most $d$, commuting measure preserving $\mathbb{R}^\ell$-actions $S_1, \ldots, S_m$ on a probability space $(X, \mathcal{B}, \mu)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$, and $s \in \mathbb{Z}$, letting

$$\alpha(n) = \int_X f_0 \cdot S_1^{q_1(n)} f_1 \cdots S_m^{q_m(n)} f_m \, d\mu,$$

there exists a $k$-step nilsequence $\psi$ satisfying (1) and (7). In the special case $d = 1$ one can choose $k = m$.

Proof. For each $i \in [1, m]$, let $q_i = (q_{i,1}, \ldots, q_{i,\ell})$ for some $q_{i,j} \in \mathbb{R}[x]$. Next, for each $j \in [1, \ell]$, write $q_{i,j}(x) = \sum_{h=0}^{d} a_{i,j,h} x^h$, where the $a_{i,j,h}$’s are real numbers. Also, for each $j \in [1, \ell]$, let $e_j$ be the $j$-th vector of the canonical basis of $\mathbb{R}^\ell$ and let $T_{i,j,h}$ be the measure preserving transformation defined by $T_{i,j,h} = S_i^{a_{i,j,h} e_j}$. Next, let $T_{i,h}$ be the composition $T_{i,h} = T_{i,1,h} \cdot \cdots \cdot T_{i,\ell,h}$, let $T_i$ be the $\mathbb{Z}^{d+1}$-action defined by $T_i^{(n_0, \ldots, n_d)} = T_{i,0}^{n_0} \cdots T_{i,d}^{n_d}$, and let $q : \mathbb{Z} \to \mathbb{Z}^{d+1}$ be the polynomial $q(n) = (1, n, \ldots, n^d)$.

With this setup, for each $i \in [1, m]$ and $n \in \mathbb{N}$, we have

$$S_i^{q_i(n)} = \prod_{j=1}^{\ell} S_i^{q_{i,j}(n) e_j} = \prod_{j=1}^{\ell} \prod_{h=0}^{d} T_{i,j,h}^{n_h} = \prod_{h=0}^{d} T_{i,h}^{n_h} = T_i^{q_i(n)}.$$
Since the $\mathbb{R}^\ell$-actions $S_1, \ldots, S_m$ commute, so do the $\mathbb{Z}^{d+1}$-actions $T_1, \ldots, T_m$. This implies that the multicorrelation sequence $\alpha$ can be represented by an expression of the form (9).

The conclusion now follows directly from Theorem 1 and Remark 2. □

Next we need a result concerning the distribution of real polynomials.

**Lemma 5.** Let $q \in \mathbb{R}[x]$ be a non-constant real polynomial, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$. Then, denoting by $\{\cdot\}$ the fractional part, we have

$$\lim_{\delta \to 0^+} \lim_{N \to \infty} \frac{1}{N-M} \left| \left\{ n \in [M,N) : \{q(n)\} \in [1-\delta,1) \right\} \right| = 0,$$

and

$$\lim_{\delta \to 0^+} \lim_{N \to \infty} \frac{1}{\pi(N)} \left| \left\{ p \in \mathbb{P} \cap [1,N] : \{q(rp+s)\} \in [1-\delta,1) \right\} \right| = 0.$$

**Proof.** Let

$$A(\delta) = \lim_{N \to \infty} \frac{1}{N-M} \left| \left\{ n \in [M,N) : \{q(n)\} \in [1-\delta,1) \right\} \right|,$$

and

$$B(\delta) = \lim_{N \to \infty} \frac{1}{\pi(N)} \left| \left\{ p \in \mathbb{P} \cap [1,N] : \{q(rp+s)\} \in [1-\delta,1) \right\} \right|.$$

If $q-q(0)$ has an irrational coefficient, then by Weyl’s Uniform Distribution Theorem [19] and Rhin’s Theorem [16] we have $A(\delta) = B(\delta) = \delta$ which approach 0 as $\delta \to 0^+$.

Assume otherwise that $q \in \mathbb{R}[x]$ satisfies $q-q(0) \in \mathbb{Q}[x]$, say $q(x) = q(0) + b^{-1} \sum_{j=1}^\ell a_j x^j$ where $b \in \mathbb{N}$, $a_j \in \mathbb{Z}$ for $1 \leq j \leq \ell$, and $q(0) \in \mathbb{R}$. It follows that for all $n \in \mathbb{N}$,

$$q(n) - q(0) \mod 1 \in \left\{ 0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b-1}{b} \right\}.$$

In particular, the fractional part $\{q(n)\}$ takes only finitely many values. Therefore, if $\delta$ is small enough, for every $n \in \mathbb{N}$ we have $\{q(n)\} \not\in [1-\delta,1)$ and hence $A(\delta) = B(\delta) = 0$, which implies the desired conclusion. □

For the proof of Theorem A we adapt arguments from [10, 11], i.e., we use a multidimensional suspension flow to approximate $\alpha$ by a multicorrelation sequence of the form (9). The arising error consists of terms of the form $1_{\{n \in \mathbb{N} : \{q(n)\} \not\in [1-\delta,1)\}}$ that can be controlled by Lemma 5.

**Proof of Theorem A.** Given $\ell, m, d \in \mathbb{N}$, let $k$ be as guaranteed by Theorem 1. Let $q_1, \ldots, q_m$, $T_1, \ldots, T_m$, $\epsilon > 0$, $r \in \mathbb{N}$, $s \in \mathbb{Z}$ and $\alpha$ be as in the statement. By multiplying each function by a constant if needed, we can assume without loss of generality that $\|f_i\|_{L^\infty} \leq 1$ for each $i \in [1, m]$.

We start by considering a multidimensional suspension flow with a constant 1 ceiling function. More precisely, let $Y := X \times [0,1)^{m \times \ell}$ and $\nu = \mu \otimes \lambda$, where $\lambda$ denotes the Lebesgue measure on $[0,1)^{m \times \ell}$. For each $i \in [1, m]$ define the measure preserving $\mathbb{R}^\ell$-action $S_i$ on $(Y, \nu)$ as follows: for any $i \in [1, m]$ and $(x; b_1, \ldots, b_m) \in Y = X \times ([0,1)^{\ell})^m$, let

$$S_i^j(x; b_1, \ldots, b_m) := (T^{[b_i+t]} x; b_1, \ldots, b_{i-1}, b_i+t, b_{i+1}, \ldots, b_m),$$

where $\{u\} := u - \lfloor u \rfloor$ for any $u \in \mathbb{R}^\ell$. Observe that the actions $S_1, \ldots, S_m$ commute.

Let $\pi : Y \to X$ be the natural projection and $\delta > 0$ a small parameter to be determined later. For each $i \in [1, m]$ let $f_i \in L^\infty(Y)$ be the composition $f_i := f_i \circ \pi$, and $f_0 := 1_{X \times [0,\delta)^{m \times \ell}} \cdot f_0 \circ \pi$. Define

$$\tilde{\alpha}(n) = \int_Y f_0 \cdot S_1^{q_1(n)} f_1 \cdots S_m^{q_m(n)} f_m \, d\nu.$$
By Theorem 4 there exists a $k$-step nilsequence $\hat{\psi}$ such that
\[
\lim_{N \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\tilde{\alpha}(n) - \tilde{\psi}(n)| \leq \delta^{\ell m} \epsilon / 2,
\] (10)
and
\[
\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} |\tilde{\alpha}(rp+s) - \tilde{\psi}(rp+s)| \leq \delta^{\ell m} \epsilon / 2.
\] (11)

On the other hand,
\[
\tilde{\alpha}(n) = \int_{[0,\delta^{\ell m}]} \int_X f_0(x) f_1 \left( T^{[q_i(n)+b_1]} x \right) \cdots f_m \left( T^{[q_m(n)+b_m]} x \right) d\mu(x) d\lambda(b_1, \ldots, b_m),
\]
which implies
\[
\alpha(n) - \frac{\tilde{\alpha}(n)}{\delta^{\ell m}} = \frac{1}{\delta^{\ell m}} \int_{[0,\delta^{\ell m}]} \int_X f_0(x) \left( \prod_{i=1}^{m} f_i \left( T^{[q_i(n)]} x \right) - \prod_{i=1}^{m} f_i \left( T^{[q_i(n)+b_i]} x \right) \right) d\mu d\lambda.
\] (12)

In particular, it follows from (12) that $|\alpha(n) - \delta^{-\ell m} \tilde{\alpha}(n)| \leq 2$ for all $n \in \mathbb{N}$. If $b_i \in [0,\delta^{\ell}]$ and $\{q_i(n)\} \in [0,1-\delta^{\ell}]$ then $[q_i(n)+b_i] = [q_i(n)]$. Therefore (12) also implies that $\alpha(n) = \delta^{-\ell m} \tilde{\alpha}(n)$ whenever
\[
n \notin \left\{ n \in \mathbb{N} : \{q_i(n)\} \in [1-\delta,1]^{\ell} \text{ for some } i \in [1,m] \right\}.
\]

In view of Lemma 6 by choosing $\delta$ small enough, we have
\[
\lim_{N \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \delta^{-\ell m} \tilde{\alpha}(n)| < \frac{\epsilon}{2}
\] (13)
and
\[
\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} |\alpha(rp+s) - \delta^{-\ell m} \tilde{\alpha}(rp+s)| < \frac{\epsilon}{2}.
\] (14)

Letting $\psi = \delta^{-m\ell} \hat{\psi}$ and combining (10) with (13) and (11) with (14) we obtain the desired conclusion.

\textbf{Remark 6.} As it was already mentioned in Section 1 it is an open problem whether one can improve upon the approximation in Frantzikinakis’ main result in \cite{2} and Theorem \ref{thm:main} and take $\epsilon = 0$ in \cite{3} and \cite{7} (see Question 2 below). However, as the following example shows, in the case of Theorem \ref{thm:main} it is not possible to improve upon the approximation in that manner.

\textbf{Example 7.} Take $X = T := \mathbb{R}/\mathbb{Z}$, $T(x) = x + 1/\sqrt{2}$, $q(n) = \sqrt{2} n$, $f_0(x) = e(x)$ and $f_1(x) = e(-x)$, where $e(x) := e^{2\pi i x}$. Then we have
\[
\alpha(n) = \int f_0 \cdot T^{[q(n)]} f_1 d\mu = \int e(x) e \left( -x - \frac{1}{\sqrt{2}} [\sqrt{2} n] \right) dx
\]
\[
= e \left( -\frac{1}{\sqrt{2}} [\sqrt{2} n] \right) = e \left( \frac{1}{\sqrt{2}} ([\sqrt{2} n]) \right).
\]
In particular, we can write $\alpha(n)$ as $F(T^n x_0)$ with $x_0 = 0 \in T$ and $F(x) = e(\{x\}/\sqrt{2})$ for $x \in T$. Assume for the sake of a contradiction that there exists a uniform limit of nilsequences $\phi$ for which
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\alpha(n) - \phi(n)| = 0.
\] (15)
By [9, Lemma 18], \( \phi \) can be written as \( \phi(n) = G(S^n y_0) \) for all \( n \in \mathbb{N} \), where \( G \) is a continuous function on an inverse limit of nilsystems \( (Y, S) \) and \( y_0 \in Y \).

We claim that \( \alpha(n) = \phi(n) \) for all \( n \in \mathbb{N} \). If not, then there exists \( \delta > 0 \) and \( n_0 \in \mathbb{N} \) such that
\[
|\alpha(n_0) - \phi(n_0)| = |F(T^{n_0} x_0) - G(S^{n_0} y_0)| \geq \delta.
\]
Since the system \( (X \times Y, T \times S) \) is the product of two distal systems, is a distal system itself. This implies that the point \( (T^{n_0} x_0, S^{n_0} y_0) \) is uniformly recurrent, i.e., the sequence \( (T^{n_0} x_0, S^{n_0} y_0) \) visits any neighborhood of \( (T^{n_0} x_0, S^{n_0} y_0) \) in a syndetic set. This fact together with (16) and the fact that both the real and imaginary parts of \( F \) are almost everywhere continuous and semicontinuous imply that the set
\[
\{n \in \mathbb{N} : |F(T^n x_0) - G(S^n y_0)| \geq \delta/2\}
\]
is syndetic, which contradicts (15). Hence \( \alpha(n) = \phi(n) \) for all \( n \in \mathbb{N} \). However, by [6, Proposition 4.2.5], the sequence \( \alpha \) is not a distal sequence; in particular, it is not a uniform limit of nilsequences, contradicting our assumption.

3. Open questions

We close this article with three open questions. Theorem \ref{approximation} provides an approximation result of multicorrelation sequences along an integer polynomial of degree one, evaluated at primes. We can ask whether a similar result is true along other classes of sequences.

Question 1. Let \( q \in \mathbb{R}[x] \) be a non-constant real polynomial, \( c > 0 \), and \( p_n \) denote the \( n \)-th prime. Suppose \( r_n = q(n), q(p_n), |n^c| \) or \( |p_n^c| \) for \( n \in \mathbb{N} \). Is it true that for any \( \alpha \) as in (6) and \( \epsilon > 0 \), there exists a nilsequence \( \psi \) satisfying
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\alpha(r_n) - \psi(r_n)| \leq \epsilon?
\]

Variants of the following question have appeared several times in the literature, e.g., [2, Remark after Theorem 1.1], [3, Problem 20], [4, Problem 1], and [8, Page 398].

Question 2. Let \( \alpha \) be as in (6). Does there exist a uniform limit of nilsequences \( \phi \) such that
\[
\lim_{N-M \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \phi(n)| = 0?
\]

As mentioned in Example [7] the answer to Question 2 is negative when \( \alpha \) is a multicorrelation sequence as in [6]. Nevertheless, it makes sense to ask for the following modification of it.

Question 3. Let \( \alpha \) be as in (6). Does there exist a uniform limit of Riemann integrable nilsequences \( \phi \) satisfying
\[
\lim_{N-M \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \phi(n)| = 0?
\]

Here we say that \( \phi \) is a uniform limit of Riemann integrable nilsequences if for every \( \epsilon > 0 \) there exists a nilmanifold \( X = G/\Gamma \), a point \( x \in X \), \( g \in G \) and a Riemann integrable function \( F : X \to \mathbb{C} \) such that \( \sup_{n \in \mathbb{N}} |\phi(n) - F(g^n x)| < \epsilon \).

\textsuperscript{2} A function \( F \) is Riemann integrable on a nilmanifold if its points of discontinuity is a null set with respect to the Haar measure.
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