Graviton one-loop effective action and inflationary dynamics

Tomas Janssen*, Shun-Pei Miao† and Tomislav Prokopec‡

Institute for Theoretical Physics, University of Utrecht Leuvenlaan 4, Postbus 80.195, 3508 TD Utrecht, The Netherlands

We consider the one-loop effective action due to gravitons in a FLRW background with constant $\epsilon = -\dot{H}/H^2$. By expanding around $\epsilon = 0$ (corresponding to an expansion around de Sitter space), we can study how the deviation from de Sitter space effects the quantum corrected Friedmann equations. We find that, at zeroth order in $\epsilon$, one-loop effects induce only a finite shift in the coupling constants. At linear order in $\epsilon$ there is however a divergent contribution to the equations of motion. This contribution leads to a nontrivial term in the renormalized equations that depends logarithmically on $H$ and thus cannot be absorbed in local counterterms. We find that deviations due to this term are unobservably small. Our study shows that quantum effects in quasi de Sitter space can be fundamentally different then in de Sitter space, albeit in the case under consideration the effect is unobservably small.

I. INTRODUCTION

Because of the potential relevance for inflationary cosmology, the quantum behavior of gravitons on a (locally) de Sitter background has been a widely studied subject over the past years [1, 2, 3] [4, 5, 6] [7, 8, 9] [10, 11, 12] [13, 14, 15, 16] [17, 18, 19] [20, 21, 22, 23, 24].

One line of research deals with the back-reaction of gravitational waves on the background spacetime [13, 14, 17, 18]. However of more interest for the present work is the one loop back-reaction by virtual gravitons on a de Sitter background which has been calculated by several authors using different techniques [10, 23, 26]. Since it is not clear whether in these works exactly the same quantity is calculated and the renormalization schemes differ, the

* T.M.Janssen@uu.nl
† S.P.Miao@phys.uu.nl
‡ T.Prokopec@uu.nl
numerical coefficients differ. However the main result is qualitatively the same: one loop graviton contributions to the expectation value of the energy momentum tensor result in a finite, time independent shift of the effective cosmological constant. Since the contribution can always be absorbed in a counterterm \[25\], the exact numerical coefficient has no real physical meaning.

The goal of this paper is to go beyond the works mentioned above and calculate the one loop effective action induced by gravitons in a more general background space-time using dimensional regularization. The geometry we consider is a Friedmann-Lemaître-Robertson-Walker (FLRW) geometry with Hubble parameter \( H = \dot{a}/a \) and the additional constraint that

\[
\epsilon \equiv - \frac{\dot{H}}{H^2}
\]

is a constant. Standard matter, radiation or dark energy dominated universes all satisfy this constraint (recall that in matter era \( \epsilon = 3/2 \), while in radiation era \( \epsilon = 2 \)) and de Sitter space is the special limit when \( \epsilon \to 0 \) \[27\]. One immediate problem with working in such a space-time, instead of in de Sitter space, is that, for consistency of the Einstein equations, the addition of matter fields is unavoidable. Whereas in de Sitter space, the only relevant metric fluctuations are the tensor modes (gravitational waves), in a more general setting also the scalar modes, due to the mixing of gravitational and matter degrees of freedom, have to be taken into account \[28\] \[29\] \[21\] \[22\] \[23\] \[24\]. This full treatment is considerably more complicated and is presented elsewhere \[30\]. For now we will only focus on the tensor modes, and do not consider the mixing of degrees of freedom.

The main motivation for this work is to show explicitly that new effects can occur when one considers loop effects in a more general background then de Sitter space. We find new effects first of all in quasi de Sitter space, where due to the presence of an ultraviolet divergence one generates small, but physical corrections to the quantum Friedmann equations that cannot be subtracted by local counterterms.

In section II we briefly review our background geometry. In section III we generalize the work of Ref. \[20\] and construct the massless minimally coupled scalar and graviton propagator in any \( \epsilon=\text{constant} \) space. In section V we calculate the one-loop effective action contribution to the quantum corrected Friedmann equations and renormalize the theory and in section VI we study the associated dynamics in quasi de Sitter spaces. We conclude in section VIII. We work in units: \( \hbar = 1 = c \).
II. GEOMETRY

The geometry we work in is the Friedmann-Lemaître-Robertson-Walker geometry in conformal coordinates

\[ g_{\mu\nu} = a^2 \eta_{\mu\nu} \quad ; \quad \eta_{\mu\nu} = \text{diag}(-1,1,1,1), \quad (2) \]

with the additional constraint,

\[ \epsilon \equiv -\frac{\dot{H}}{H^2} = \text{constant} \quad ; \quad H \equiv \frac{\dot{a}}{a}. \quad (3) \]

Here a dot indicates a derivative with respect to cosmological time \( t \), related to the conformal time \( \eta \) by \( dt = a d\eta \). The FLRW geometry obeys the Friedmann equations

\[ \frac{3H^2}{\kappa} - \frac{1}{2} \rho = 0 \quad ; \quad -2\frac{\dot{H}}{\kappa} - \frac{1}{2}(\rho + p) = 0, \quad \kappa = 16\pi G_N, \quad (4) \]

with \( G_N \) being the Newton constant, \( \rho \) and \( p \) are the energy density and pressure of the cosmological fluid. If one writes

\[ p = w\rho, \quad (5) \]

one immediately finds that (3) implies that \( w \) is constant. One can solve (4) for \( a \) to find

\[ a(\eta) = \left( (\epsilon - 1) H_0 \eta \right)^{-1/(1-\epsilon)} \quad ; \quad \epsilon = \frac{3}{2}(1 + w) \]

\[ H = H_0 \left( (\epsilon - 1) H_0 \eta \right)^{\epsilon/(1-\epsilon)} = H_0 a^{-\epsilon} \quad ; \quad a\eta = \frac{1}{1 - \epsilon} \frac{1}{H}. \quad (6) \]

Notice that if \( \epsilon < 1 \), \( \eta \) is negative and if \( \epsilon > 1 \), \( \eta \) is positive. \( H_0 \) is chosen such that the \( \epsilon \to 0 \) expansion of \( H \) corresponds to the one given in [20]. An important geometrical quantity is

\[ y \equiv y_{++} = \frac{\Delta x^2_{++}(x; \tilde{x})}{\eta \tilde{\eta}} = \frac{1}{\eta \tilde{\eta}}(-(|\eta - \tilde{\eta}| - i\epsilon)^2 + ||\tilde{x} - \bar{x}||^2). \quad (7) \]

Here the infinitesimal \( \epsilon > 0 \) refers to the Feynman (time-ordered) pole prescription. In de Sitter space \( y \) is related to the geodesic distance \( l \) as \( y = 4 \sin^2(\frac{1}{2}Hl) \). If \( y < 0 \), points \( \tilde{x} \) are timelike related to \( x \), and if \( y > 0 \), they are spacelike related. We define the antipodal point \( \bar{x} \) of \( x \) by the map \( \eta \to -\eta \). Notice that, since in our coordinates \( \eta \) is either positive or negative, this point is not covered by our coordinates. If \( y = 4 \), \( \tilde{x} \) lies on the lightcone of an unobservable image charge at the antipodal point \( \bar{x} \), see figure 1.
FIG. 1: The causal structure in the conformal coordinates \([2]\). The plot assumes \(\epsilon < 1\), so the coordinates (of an expanding universe) cover only the region \(\eta < 0\). The wavy line at \(\eta = 0\) indicates future infinity. The lightcone of the point \(x\) is given by \(y = 0\). If \(y = 4\), the point \(\tilde{x}\) lies on the light cone of an unobservable image charge at the antipodal point \(\bar{x}\).

The curvature tensors are given by

\[
R^{\alpha\mu\beta\nu} = \left(\frac{a''}{a} - 2\left(\frac{a'}{a}\right)^2\right) \left(\delta^\alpha_\nu \delta^\mu_\beta - \delta^\alpha_\nu \delta^\mu_\beta \eta_{\mu\beta} - \delta^\alpha_\beta \delta^\nu_\mu \eta_{\mu\beta} + \delta^\alpha_\beta \delta^\nu_\mu \eta_{\mu\beta}\right) \\
- \left(\frac{a'}{a}\right)^2 \left(\delta^\alpha_\beta \eta_{\mu\beta} - \delta^\alpha_\beta \eta_{\mu\beta}\right)
\]

\[
R_{\mu\nu} = \left(\frac{a''}{a} - 2\left(\frac{a'}{a}\right)^2\right) (\eta_{\mu\nu} - (D - 2) \delta^0_{\mu\nu}) + \left(\frac{a'}{a}\right)^2 (D - 1) \eta_{\mu\nu}
\]

\[
R = \left(\frac{a''}{a^3} - 2\left(\frac{a'}{a^2}\right)^2\right) 2(D - 1) + \left(\frac{a'}{a^2}\right)^2 D(D - 1),
\]

where \(D\) denotes the number of space-time dimensions and \(a' = da/d\eta\).

III. SCALAR PROPAGATOR

The construction of the graviton propagator in the geometry under consideration is very similar to the construction in quasi de Sitter space, as given in \([20]\). Therefore we will only
give the main steps here.

Since the graviton propagator can be expressed in terms of massless scalar propagators, we first consider the following Klein-Gordon equation for a massless scalar in $D$ dimensions

$$\sqrt{-g} (\Box - \xi R) \Delta(x; \tilde{x}) = \sqrt{-g} \left[ \Box - \xi (D - 1)(D - 2\epsilon) H^2 \right] \Delta(x; \tilde{x}) = \delta^D(x - \tilde{x}),$$  \hspace{1cm} (9)

with some constant $\xi$, where

$$\Box \equiv \frac{1}{\sqrt{-g}} \partial_\mu \sqrt{-g} g^{\mu\nu} \partial_\nu = \frac{1}{a^2} \left( \partial^2 - (D - 2) \frac{a'}{a} \partial_0 \right), \quad (\partial^2 = \eta^{\mu\nu} \partial_\mu \partial_\nu) \hspace{1cm} (10)$$

denotes the scalar D’Alembertian.

We now make the following Ansatz for the propagator

$$\Delta(x; \tilde{x}) = (aa')^{(1-D/2)} \Xi(y).$$  \hspace{1cm} (11)

After the rescaling (11), the nonsingular part of the Klein-Gordon equation reads

$$a^D (aa')^{(1-D/2)} (1 - \epsilon)^2 H^2 \left[ y(4 - y) \left( \frac{d}{dy} \right)^2 - (D(y - 2)) \frac{d}{dy} 

- (1 - \epsilon)^{-2} \left[ (D - 1)(D - 2\epsilon) \xi - \frac{1}{2} (D - 1)(D - 2) \epsilon + \frac{D}{4} (D - 2) \epsilon^2 \right] \right] \Xi(y) = 0. \hspace{1cm} (12)$$

This hypergeometric equation has a general solution

$$\Xi(y) = A_2 F_1 \left( \frac{D - 1}{2} + \nu_D, \frac{D - 1}{2} - \nu_D; \frac{D}{2}; \frac{y}{4} \right) + B_2 F_1 \left( \frac{D - 1}{2} + \nu_D, \frac{D - 1}{2} - \nu_D; \frac{D}{2}; 1 - \frac{y}{4} \right),$$ \hspace{1cm} (13)

where

$$\nu_D^2 = \left( \frac{D - 1}{2} \right)^2 - (1 - \epsilon)^{-2} \left[ (D - 1)(D - 2\epsilon) \xi - \frac{1}{2} (D - 1)(D - 2) \epsilon + \frac{D}{4} (D - 2) \epsilon^2 \right].$$ \hspace{1cm} (14)

The constants $A$ and $B$ are fixed by the singularity conditions. The vanishing of the antipodal singularity (at $y = 4$), which leads to $\alpha$-vacua [33, 34, 35, 36, 37], fixes $A = 0$. The constant $B$ is fixed by requiring that the Hadamard singularity at $y = 0$ sources the $\delta$-function correctly. Notice however that there are values for $\nu_D$ where this is not possible. In particular if $\nu_D$ is half integer, larger or equal than 1/2, the hypergeometric equation is no longer a valid solution. For these particular cases one finds that one cannot source the $\delta$ function correctly and remove the $\alpha$-vacua. However, our solution is valid arbitrary close to these points. A
well known example of such behavior is the massless minimally coupled (MMC) scalar field in de Sitter space \[38, 39\].

The most singular term sourcing the $\delta$-function is

$$i\Xi(y)_{\text{sing}} = B \left( \frac{y}{4} \right)^{1-D/2} \frac{\Gamma(D/2)\Gamma(D/2-1)}{\Gamma(D/2+\nu_D)\Gamma(D/2-\nu_D)}. \quad (15)$$

using

$$\partial^2 \frac{1}{\Delta x_{++}^{D-2}} = \frac{4\pi^{D/2}}{\Gamma(D/2-1)}i\delta^D(x-x) \quad (16)$$

and (6), we find

$$B = \frac{\Gamma(D-1)/2 + \nu_D)\Gamma(D-1)/2 - \nu_D)}{\Gamma(D/2)(1-|\epsilon|H_0)^{D-2}}. \quad (17)$$

It is important to notice that, due to the rescaling (11), $B$ is indeed – as required – a constant, constituting a nontrivial consistency check of our Ansatz (11). The MMC scalar propagator for a general, constant $\epsilon$ reads

$$i\Delta(x; \bar{x}) = (a\bar{a})^{-\epsilon(D/2-1)}\frac{1-\epsilon\partial^2 \hat{g}}{(4\pi)^{D/2}} \frac{\Gamma(D-1)/2 + \nu_D)\Gamma(D-1)/2 - \nu_D)}{\Gamma(D/2)(1-|\epsilon|H_0)^{D-2}}$$

$$\times {}_2F_1\left( \frac{D-1}{2} + \nu_D, \frac{D-1}{2} - \nu_D; \frac{D}{2}; 1 - \frac{y}{4} \right). \quad (18)$$

Equation (18) is the generalization of the scalar Chernikov-Tagirov propagator \[40\] to FLRW spaces with general, but constant, equation of state parameter $w$, and thus from Eq. (6) a constant $\epsilon$.

**IV. GRAVITON PROPAGATOR**

Next we consider the graviton propagator. The only difference from the analysis as given in \[20\] is in the coefficients $\nu_{D,n}$. We consider the following action for gravity plus an arbitrary scalar field $\hat{\phi} = \hat{\phi}(x)$.

$$S = \frac{1}{\kappa} \int d^D x \sqrt{-\hat{g}} \left( \hat{R} - (D-2)\Lambda \right) + \int d^D x \sqrt{-\hat{g}} \left( -\frac{1}{2} \partial_{\alpha}\hat{\phi} \partial_{\beta}\hat{\phi} \hat{g}^{\alpha\beta} - V(\hat{\phi}) \right)$$

$$\equiv S_{EH} + S_M \quad (19)$$

and we write the metric tensor $\hat{g}$ as a background contribution $g$ plus a perturbation $h$

$$\hat{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}. \quad (20)$$
For the scalar fluid $\dot{\phi} \rightarrow \phi = \phi(t)$, the background energy density and pressure are
\begin{align}
\rho_M &= -\frac{1}{2} (\partial \phi)^2 + V(\phi) = \frac{1}{2} \dot{\phi}^2 + V(\phi) \\
p_M &= -\frac{1}{2} (\partial \phi)^2 - V(\phi) = \frac{1}{2} \dot{\phi}^2 - V(\phi),
\end{align}
where the background scalar field $\phi = \phi(\eta)$ is a function of (conformal) time only. The corresponding tree level Friedmann equations are
\begin{align}
H^2 &= \frac{1}{D-1} \Lambda - \frac{\kappa}{(D-1)(D-2)} \rho_M = 0 \\
\dot{H} + \frac{D-1}{2} H^2 - \frac{1}{2} \Lambda + \frac{\kappa}{2(D-2)} p_M = 0 \\
\phi'' + (D-2) a H \phi' + a^2 \frac{\partial V}{\partial \phi}(\phi) &= 0,
\end{align}
(21)

Expanding the action up to quadratic order in $h_{\mu \nu}$ and using the background equations of motion for $\phi$ gives \[11\[20\]
\begin{align}
S_\chi &= \int d^D x \chi_{\alpha \beta} \eta^{\alpha \mu} \eta^{\beta \nu} \left[ (\partial^2 + \frac{D-2}{2} \dot{H} a^2 + \frac{D}{4} (D-2) H^2 a^2) \left( \frac{1}{4} \delta_\mu^\rho \delta_\nu^\sigma - \frac{1}{8} \eta_{\mu \nu} \eta^{\rho \sigma} \right) \\
&- \frac{D-2}{2} \left( H^2 + \dot{H} a^2 \right) \delta_\mu^\alpha \delta_\nu^\beta \right] \chi_{\rho \sigma} + S_{\text{os}},
\end{align}
(23)

where we defined,
\begin{align}
h_{\mu \nu} &= \sqrt{\kappa} a^2 \psi_{\mu \nu} = \sqrt{\kappa} a^{3-D/2} \chi_{\mu \nu}, \quad \tilde{\chi}_{\mu \nu} = \chi_{\mu \nu} - \frac{1}{2} g_{\mu \nu} \chi, \quad \chi = g_{\mu \nu} \chi_{\mu \nu}, \quad (\psi_{\mu \nu} = a^{1-D/2} \chi_{\mu \nu}),
\end{align}
(24)

we added a gauge fixing term
\begin{align}
-\frac{1}{2} a^{D+4} \left[ \nabla_\mu (a^{1-D/2} \tilde{\chi}^{\alpha \mu}) \right] \nabla_\nu (a^{1-D/2} \tilde{\chi}^{\nu \alpha})
\end{align}
(25)
and where $S_{\text{os}}$ denotes the second-order terms that vanish on-shell (cf. Eqs. \(22\)),
\begin{align}
S_{\text{os}} &= -\int d^D x a^2 \chi_{\alpha \beta} \left( \frac{1}{4} \eta^{\alpha \rho} \eta^{\beta \sigma} - \frac{1}{8} \eta^{\alpha \beta} \eta^{\rho \sigma} \right) 2(D-2) \left( \dot{H} + \frac{D-1}{2} H^2 - \frac{\Lambda}{2} + \frac{\kappa p_M}{2(D-2)} \right) \chi_{\rho \sigma} \\
&+ \int d^D x a^2 \chi_{\alpha \beta} \left( -\delta_0^{(\alpha \beta)} (\rho \delta_0^\rho + \frac{1}{2} \delta_0^{(\alpha \beta)} \eta^{\rho \sigma}) \right) (D-2) \dot{H} + \frac{\kappa}{2} (\rho_M + p_M) \chi_{\rho \sigma}.
\end{align}
(26)

Even though these terms do not contribute to the graviton propagator, they do contribute to the one loop effective action and hence we must keep them.

From the action (23) we find the graviton propagator for $\chi_{\mu \nu}$
\begin{align}
a^{1-D/2} \tilde{a}^{1-D/2} \Gamma_{[\rho \sigma \Delta]}^{\alpha \beta} = (T_0)_{\rho \sigma}^{\alpha \beta} \Delta_0 + (T_1)_{\rho \sigma}^{\alpha \beta} \Delta_1 + (T_0)_{\rho \sigma}^{\alpha \beta} \Delta_2,
\end{align}
(27)
where
\[\begin{align*}
(T_0)_{\rho\sigma}^{\alpha\beta} &= 2\delta_0^\rho \delta_0^\sigma - \frac{2}{D-3} \bar{\eta}_{\rho\sigma} \bar{\eta}^{\alpha\beta}, \\
(T_1)_{\rho\sigma}^{\alpha\beta} &= 4\delta_0^\rho \delta_0^\sigma \\
(T_2)_{\rho\sigma}^{\alpha\beta} &= \frac{2}{(D-2)(D-3)}(\eta_{\rho\sigma} + (D-2)\delta_\rho^\alpha \delta_\sigma^\beta)(\eta^{\alpha\beta} + (D-2)\delta_0^\alpha \delta_0^\beta)
\end{align*}\] (28)
denote the relevant graviton tensor structures and
\[\bar{\eta}_{\mu\nu} = \eta_{\mu\nu} + \delta_\mu^0 \delta_\nu^0\] (29)
is the spatial part of the metric tensor. The prefactor \(a^{1-D/2}\bar{a}^{1-D/2}\) in Eq. (27) comes from the fact that the \(\chi_{\mu\nu}\) field differs by a rescaling, \(\chi_{\mu\nu} = a^{-1+D/2}\psi_{\mu\nu}\), from the \(\psi_{\mu\nu}\) field for which the propagator is calculated in [20]. The scalar propagators \(\Delta_n\) for the pseudograviton \(\psi_{\mu\nu}\) obey
\[\sqrt{-g}\left(\Box - n(D - n - 1)(1 - \epsilon)H^2\right)i\Delta_n = i\delta^D(x - x'),\quad (n = 0, 1, 2),\] (30)
and thus they are given by (18) with
\[\nu_{D,n}^2 = \frac{(D - 1)^2}{4} + \frac{\frac{1}{2}(D - 1)(D - 2)\epsilon - n(1 - \epsilon)(D - n - 1) - \frac{D}{4}(D - 2)\epsilon^2}{1 - \epsilon^2}.\] (31)
The ghost action associated with our gauge fixing is given by
\[S_{\text{ghost}} = \int d^Dx \tilde{\eta}_{\alpha\beta} \bar{U}^\beta \left[(\partial^2 + \frac{D - 2}{2} \hat{H}a^2 + \frac{D}{4}(D - 2)H^2a^2)\delta_\alpha^\mu - (D - 2)a^2(H^2 - \hat{H})\delta_\mu^0 \delta_0^\alpha\right] U^\mu,\] (32)
where \(U\) is the ghost field, related to the ghost field of Ref. [20] by a scale transformation \(U = a^{D/2-1}V\). The ghost propagator is found to be
\[a^{1-D/2}\bar{a}^{1-D/2}l_{[\alpha}\hat{\Delta}^\rho]|(x; \bar{x}) = i\delta_0^\rho \hat{\Delta}_0(x; \bar{x}) + i\delta_0^0 \delta_0^\rho \hat{\Delta}_1(x; \bar{x}),\] (33)
where the \(\hat{\Delta}_n\) propagators satisfy
\[\sqrt{-g}\left(\Box - n(D - n - 1)(1 + \epsilon)H^2\right)i\hat{\Delta}_n = i\delta^D(x - x'),\quad (n = 0, 1).\] (34)
Therefore the corresponding propagators are given by (18) with
\[\hat{\nu}_{D,n}^2 = \frac{(D - 1)^2}{4} + \frac{\frac{1}{2}(D - 1)(D - 2)\epsilon - n(1 + \epsilon)(D - n - 1) - \frac{D}{4}(D - 2)\epsilon^2}{1 - \epsilon^2}.\] (35)
Note that both the graviton and ghost propagators given above differ by a scaling \(a^{1-D/2}\bar{a}^{1-D/2}\) from the ones given in Ref. [20].
V. ONE-LOOP EFFECTIVE ACTION

The one-loop effective action is defined as

$$\Gamma = -i \langle \text{out}, 0 | 0, \text{in} \rangle. \quad (36)$$

While in flat space and in the absence of external sources such a vacuum-to-vacuum transition can be normalized to unity, this is not possible in general curved space-times. Since our lagrangian is quadratic in the graviton and ghost fields, we can integrate them out to get the one loop effective action

$$\exp[i\Gamma] = \int D h_{\mu\nu} D W D \bar{W} \exp \left[ i \left( S_{EH} + S_M + S_X + S_{os} + S_{ghost} \right) \right]$$

$$= \int D \chi_{\mu\nu} D U D \bar{U} \exp \left[ i \left( S_{EH} + S_M + S_X + S_{os} + S_{ghost} \right) \right]$$

$$= \exp \left[ i \left( S_{EH} + S_M \right) \right] \frac{\det(F^\alpha_\mu)}{\sqrt{\det(D^{\rho\sigma}_{\mu\nu} + \delta D^{\rho\sigma}_{\mu\nu})}}, \quad (37)$$

where the step from the first to the second line can be made by noticing that the Jacobian of the transformation contributes as a D-dimensional delta function evaluated at zero, \(\delta^D(0)\). In dimensional regularization such a term does not contribute. In (37) \(D\) and \(F\) are the kinetic operators from the rescaled graviton (23), including the off-shell contribution (26), and the ghost (32), respectively, given by

$$D^{\rho\sigma}_{\mu\nu} = \left( \partial^2 + \frac{1}{4} (D - 2)(D - 2\epsilon) H^2 a^2 \right) \left( \frac{1}{2} \delta^{(\rho}_\mu \delta^{(\sigma)}_{\nu) - \frac{1}{4} \eta^{(\rho}_\mu \eta^{(\sigma)}_{\nu) - (D - 2)(1 - \epsilon) H^2 a^2 \delta^{(\rho}_\mu \delta^{(\sigma)}_{\nu) - \frac{1}{4} \eta^{(\rho}_\mu \eta^{(\sigma)}_{\nu) - (D - 2)(1 - \epsilon) H^2 a^2 \delta^{(\rho}_\mu \delta^{(\sigma)}_{\nu) \eta^{\mu\nu}} \right) - (D - 2)(1 - \epsilon) H^2 a^2 \delta^{(\rho}_\mu \delta^{(\sigma)}_{\nu) \eta^{\mu\nu}} \right)$$

$$\delta D^{\rho\sigma}_{\mu\nu} = -a^2 \left( \frac{1}{2} \delta^{(\rho}_\mu \delta^{(\sigma)}_{\nu) - \frac{1}{4} \eta^{(\rho}_\mu \eta^{(\sigma)}_{\nu) \right) \left( (D - 2) \dot{H} + \frac{D - 1}{2} H^2 - \frac{\Lambda}{2} + \frac{\kappa p}{2(D - 2)} \right)$$

$$+ a^2 \left( 2 \delta^{(\rho}_0 \delta^{(\sigma)}_\mu \delta^{0}_\nu) + \delta^{(\rho}_0 \delta^{(\sigma)}_\mu \eta^{\mu\nu} \right) \right)$$

$$F^\alpha_\mu = \left( \partial^2 + \frac{1}{4} (D - 2)(D - 2\epsilon) H^2 a^2 \right) \delta^\alpha_\mu - (D - 2)(1 + \epsilon) H^2 a^2 \delta^0_\mu \delta^0_\nu \right). \quad (38)$$

From Eq. (37) we obtain

$$\Gamma = S_{EH} + S_M + \frac{i}{2} \text{Tr} \ln[D^{\rho\sigma}_{\mu\nu} + \delta D^{\rho\sigma}_{\mu\nu}] - i \text{Tr} \ln[F^\alpha_\mu]$$

$$\equiv S_{EH} + S_M + \Gamma_{1L}. \quad (39)$$

While in principle one could – at least formally – evaluate the effective action, the object one is eventually interested in is the effective Friedmann equation, i.e. the equations of motion of the metric. Moreover in the present case there is the technical complication that
we need to work under the constraint that $\epsilon$ is constant. As long as $\dot{\epsilon}$ remains small, there is no problem with imposing such a constraint in the equations of motion. On the other hand, imposing such a constraint in the action might change the dynamics substantially.

By taking the functional derivative with respect to the scale factor $a = a(\eta)$, we obtain the Einstein trace equation, that is the $-(00) + 3(ii)$ component of the Einstein equation. Since in a FLRW universe there are only two independent equations, the second equation can be obtained by imposing the Bianchi identity. Thus our first equation of motion is given by

$$\frac{\delta \Gamma}{\delta a(l)} = \frac{\delta (S_{\text{HE}} + S_{M})}{\delta a(l)} + \frac{\delta \Gamma_{1L}}{\delta a(l)} = V a^3 \left[ \frac{24}{\kappa} \left( H^2 - \frac{1}{3} \Lambda + \frac{1}{2} \dot{H} \right) + 3p_{M} - \rho_{M} \right] + \frac{\delta \Gamma_{1L}}{\delta a(l)}, \quad (40)$$

where $V = \int d^{D-1}x$ denotes the volume of space and $p_{M}$ and $\rho_{M}$ are the pressure and energy density associated to the matter action $S_{M}$ and they are defined by, $T_{\mu \nu} = (2/\sqrt{-g}) \delta S_{M}/\delta g^{\mu \nu} = -g_{\mu \nu}p_{M} - a^{2} \delta_{\mu}^{0} \delta_{\nu}^{0}(\rho_{M} + p_{M})$. We first focus on the graviton contribution to $\delta \Gamma_{1L}/\delta a(l)$. We first write the effective graviton action in terms of $a$:

$$\Gamma_g[a] = \frac{i}{2} \text{Tr} \ln \left\{ \left[ \eta_{\mu(\rho} \eta_{\nu)(\sigma)} \right] (D - 2) \left( \frac{a''}{a} - \frac{a'^{2}}{a^{2}} \right) + \left[ \frac{1}{2} \eta_{\mu(\rho} \eta_{\sigma)\nu} - \frac{1}{4} \eta_{\mu\nu} \eta_{\rho\sigma} \right] \left( \partial^{2} + \frac{1}{2} (D - 2) \frac{1}{2} (D - 4) \frac{a'^{2}}{a^{2}} + \frac{a''}{a} \right) + \delta D_{\rho\sigma\mu\nu} \right\}. \quad (41)$$

Now, since the term within the logarithm is just the kinetic operator, upon variation we will generate the inverse of this object. This inverse is of course the propagator. Taking the trace implies here both tracing over the indices, and evaluation at coincidence [41]. After taking the functional derivative we get,

$$\frac{1}{V} \frac{\delta \Gamma_g[a]}{\delta a(l)} = \frac{1}{V} \frac{\delta \Gamma_g'[a]}{\delta a(l)} + \frac{1}{V} \frac{\delta \Gamma_g''[a]}{\delta a(l)}, \quad (42)$$

where the latter term originates from variation of $\delta D_{\rho\sigma\mu\nu}$ in Eq. (41). We have

$$\frac{1}{V} \frac{\delta \Gamma_g'[a]}{\delta a(l)} = (D - 2) \left\{ \left( -\delta_{0}^{0} \delta_{\rho}^{(\sigma)} \right) \frac{1}{2a} \frac{d^{2}}{dl^{2}} + \left( \frac{1}{2} \delta_{\mu}^{(\rho} \delta_{\nu)^{\sigma}} - \frac{1}{4} \eta_{\mu\nu} \eta^{\rho\sigma} \right) \right\} \times \left[ \frac{D - 2}{4} \left( \frac{a'^{2}}{a^{2}} - \frac{a''}{a} \frac{d}{dl} \right) + \frac{1}{4a} \frac{d^{2}}{dl^{2}} \right] \eta^{\mu(\alpha} \eta^{\beta)\nu} \Gamma_{\alpha\beta} x, x \right\}.
and
\[
\frac{1}{V} \frac{\delta \Gamma''_{g[a]}}{\delta \alpha(l)} = (D-2) \left\{ \left( \frac{1}{2} \delta^{(\rho \delta \sigma)}_{\mu \nu} - \frac{1}{4} \eta_{\mu \nu} \eta^{\rho \sigma} \right) \right. \\
\times \left[ (D-3) \left( \frac{a''}{a^2} - \frac{a''}{a^3} + \frac{a' d}{a^2 \, dl} \right) + a \Lambda + a \frac{\kappa}{D-2} V(\phi) - \frac{1}{a \, dl^2} \right] \\
+ \left( \delta^0_{(\mu \nu)} \delta^0_{\delta \sigma} + \frac{1}{2} \delta^0_{(\mu \nu)} \eta^{\rho \sigma} \right) \left[ 2 \left( \frac{a''}{a^2} - \frac{a''}{a^3} + \frac{a' d}{a^2 \, dl} \right) + \frac{1}{a \, dl^2} \right] \right\} \eta^{(\alpha \eta \beta \nu)} \left[ \alpha \beta \delta \sigma \right] (x; x),
\]
where \( \alpha \beta \delta \sigma \left[ x; x \right] \) denotes the graviton propagator (27) evaluated at coincidence. Eq. (28) implies the following contractions,
\[
\begin{align*}
\eta^{\mu (\alpha \eta \beta \nu)} \left( \frac{1}{2} \delta^{(\rho \delta \sigma)}_{\mu \nu} - \frac{1}{4} \eta_{\mu \nu} \eta^{\rho \sigma} \right) (T_0)_{\alpha \beta \rho \sigma} &= \frac{1}{2} D(D-1), \\
\eta^{\mu (\alpha \eta \beta \nu)} \left( \frac{1}{2} \delta^{(\rho \delta \sigma)}_{\mu \nu} - \frac{1}{4} \eta_{\mu \nu} \eta^{\rho \sigma} \right) (T_1)_{\alpha \beta \rho \sigma} &= D-1, \\
\eta^{\mu (\alpha \eta \beta \nu)} \left( \frac{1}{2} \delta^{(\rho \delta \sigma)}_{\mu \nu} - \frac{1}{4} \eta_{\mu \nu} \eta^{\rho \sigma} \right) (T_2)_{\alpha \beta \rho \sigma} &= 1, \\
\eta^{\mu (\alpha \eta \beta \nu)} \left( \delta^0_{(\mu \nu)} \delta^0_{\delta \sigma} \right) (T_1)_{\alpha \beta \rho \sigma} &= D-1, \\
\eta^{\mu (\alpha \eta \beta \nu)} \left( \delta^0_{(\mu \nu)} \delta^0_{\delta \sigma} \right) (T_2)_{\alpha \beta \rho \sigma} &= \frac{2(D-3)}{D-2}, \\
\eta^{\mu (\alpha \eta \beta \nu)} \left( \delta^0_{(\mu \nu)} \delta^0_{\delta \sigma} \right) (T_2)_{\alpha \beta \rho \sigma} &= \frac{4}{D-2},
\end{align*}
\]
where \( (T_i)_{\alpha \beta \rho \sigma} \equiv (T_i)_{\alpha \beta \gamma \delta} \eta_{\gamma \rho} \eta_{\delta \sigma} \) \((i = 0, 1, 2)\). Other contractions vanish. After substituting (45) into (44) one obtains
\[
\begin{align*}
\frac{1}{V} \frac{\delta \Gamma''_{g[a]}}{\delta \alpha(l)} &= \frac{1}{8} D(D-1)(D-2) \left[ (D-2) \left( \frac{a''}{a^2} - \frac{a''}{a^3} + \frac{a' d}{a^2 \, dl} \right) + \frac{1}{a \, dl^2} \right] \eta \Delta_0(x; x)a^{D-2} \\
&+ \frac{1}{4} (D-1)(D-2) \left[ (D-2) \left( \frac{a''}{a^2} - \frac{a''}{a^3} + \frac{a' d}{a^2 \, dl} \right) - \frac{1}{a \, dl^2} \right] \eta \Delta_1(x; x)a^{D-2} \\
&+ \left[ \frac{1}{4} (D-2)^2 \left( \frac{a''}{a^2} - \frac{a''}{a^3} + \frac{a' d}{a^2 \, dl} \right) - \frac{1}{4} (3D-10) \frac{1}{a \, dl^2} \right] \eta \Delta_2(x; x)a^{D-2} \\
&= (D-2) \left\{ \frac{D(D-1)}{2} \left[ (D-3) \left( \frac{a''}{a^2} - \frac{a''}{a^3} + \frac{a' d}{a^2 \, dl} \right) + a \Lambda + a \frac{\kappa V(\phi)}{D-2} - \frac{1}{a \, dl^2} \right] \\
&\times \eta \Delta_0(x; x)a^{D-2} \\
&+ (D-1) \left[ (D-1) \left( \frac{a''}{a^2} - \frac{a''}{a^3} + \frac{a' d}{a^2 \, dl} \right) + a \Lambda + a \frac{\kappa V(\phi)}{D-2} \right] \eta \Delta_1(x; x)a^{D-2} \\
&+ (D+1) \left( \frac{a''}{a^2} - \frac{a''}{a^3} + \frac{a' d}{a^2 \, dl} \right) + a \Lambda + a \frac{\kappa V(\phi)}{D-2} + \frac{1}{a \, dl^2} \right\} \eta \Delta_2(x; x)a^{D-2} \right\}.
\end{align*}
\]
This can be further simplified by making use of the on-shell relation [22]

\[ \Lambda + \frac{\kappa V(\phi)}{D-2} = (D-1)H^2 + \dot{H}. \]  

(48)

For the ghost field we follow exactly the same procedure to obtain

\[
\frac{1}{V} \frac{\delta \Gamma_{gh}[a]}{\delta a(l)} = (D-2) \left\{ \frac{1}{8} (D-1)(D-4) \left[ (D-2) \left( \frac{a'^2}{a^3} - \frac{a''}{a^2} - \frac{a'}{a} \frac{d}{dl} \right) + \frac{1}{a} \frac{d^2}{dl^2} \right] i \delta_0(x;x)a^{D-2}
+ \left[ \frac{1}{2} (D-10) \left( \frac{a'^2}{a^3} - \frac{a''}{a^2} - \frac{a'}{a} \frac{d}{dl} \right) - \frac{3}{2} \frac{1}{a} \frac{d^2}{dl^2} \right] i \delta_1(x;x)a^{D-2} \right\} .
\]  

(49)

Upon combining (46) and (49) we obtain,

\[
\frac{1}{V} \frac{\delta \Gamma'_{gh}[a]}{\delta a(l)} = (D-2) \left\{ \frac{1}{8} (D-1)(D-4) \left[ (D-2) \left( \frac{a'^2}{a^3} - \frac{a''}{a^2} - \frac{a'}{a} \frac{d}{dl} \right) + \frac{1}{a} \frac{d^2}{dl^2} \right] i \delta_0(x;x)a^{D-2}
+ \frac{1}{4} (D-1) \left[ (D-2) \left( \frac{a'^2}{a^3} - \frac{a''}{a^2} - \frac{a'}{a} \frac{d}{dl} \right) - \frac{1}{a} \frac{d^2}{dl^2} \right] i \delta_1(x;x)a^{D-2}
+ \left[ \frac{1}{2} (D-10) \left( \frac{a'^2}{a^3} - \frac{a''}{a^2} - \frac{a'}{a} \frac{d}{dl} \right) - \frac{3}{2} \frac{1}{a} \frac{d^2}{dl^2} \right] i \delta_1(x;x)a^{D-2}
+ \frac{1}{4} (D-2) \left( \frac{a'^2}{a^3} - \frac{a''}{a^2} - \frac{a'}{a} \frac{d}{dl} \right) - \frac{1}{4} (3D-10) \frac{1}{a} \frac{d^2}{dl^2} \right] i \delta_2(x;x)a^{D-2} \right\} ,
\]  

(50)

where we made use of \( i \Delta_0 = i \hat{\Delta}_0 \).

The next step is to evaluate the propagator at coincidence. From this point on we need to constrain our calculation to the case where \( \epsilon \) is constant. In this case the coincidence limit of the propagators are given by

\[ i \Delta_n(x;x) = |1 - \epsilon|^{D-2} H^{D-2} \frac{\Gamma(1 - \frac{D}{2}) \Gamma(\frac{D-1}{2} + \nu_{D,n}) \Gamma(\frac{D-1}{2} - \nu_{D,n})}{(4\pi)^{\frac{D}{2}} \Gamma(\frac{1}{2} + \nu_{D,n}) \Gamma(\frac{1}{2} - \nu_{D,n})} \]  

(51)

\[ \frac{d}{d\eta} a^{D-2} \Delta_n(x;x) = Ha(D-2)(1-\epsilon) a^{D-2} \Delta_n(x;x) \]  

(52)

\[ \frac{d^2}{d\eta^2} a^{D-2} \Delta_n(x;x) = H^2 a^2 (D-1)(D-2) (1-\epsilon)^2 a^{D-2} \Delta_n(x;x) , \]  

(53)

where the \( \nu \) parameters for the ghost and the graviton are given by \([35]\) and \([31]\), respectively. Notice that the propagator is first evaluated at coincidence and only then hit by the derivative. Using these results in Eqs. (50) and (47) and Eq. (48) we obtain

\[
\frac{1}{V} \frac{\delta \Gamma'_{gh}[a]}{\delta a(l)} = (D-1)(D-2)(1-\epsilon) H^2 a^{D-1} \left\{ \frac{1}{8} (D-1)(D-2)(D-4) \epsilon i \Delta_0(x;x)
- \frac{1}{4} (D-1)(D-2)(2-\epsilon) i \Delta_1(x;x)
- \frac{1}{2} [2(D+2) - 3(D-2) \epsilon] i \Delta_1(x;x)
- \frac{1}{4} [4(D-3) - (3D-10) \epsilon] i \Delta_2(x;x) \right\} .
\]  

(54)
and
\[
\frac{1}{V} \frac{\delta \Gamma''[a]}{\delta a(l)} = (D - 2)H^2 a^{D-1} \left\{ \frac{D(D-1)(D-2)}{2} \epsilon [D - (D-1)\epsilon] t \Delta_0(x; x) + (D-1) \left[ D(D-1) - (D^2 - 2D + 2)\epsilon \right] t \Delta_1(x; x) + 2D(D-1) - (3D^2 - 6D + 4)\epsilon + (D-1)(D-2) \epsilon^2 \right\} t \Delta_2(x; x) \right\}.
\] (55)

We substitute (51) in (54) and (55), add the two contributions and expand around \( D = 4 \) to obtain the nonrenormalized one loop effective action [46]
\[
\frac{1}{H^4 a^{D-1} V} \frac{\delta \Gamma_{g+gh}[a]}{\delta a(l)} = -\frac{\epsilon(198 - 241\epsilon + 63\epsilon^2) \mu^{D-4}}{4\pi^2} \frac{1}{D-4}
+ \frac{1}{16\pi^2} \left\{ (84 - 1810\epsilon + 2307\epsilon^2 - 791\epsilon^3 + 54\epsilon^4) - 2\epsilon(198 - 241\epsilon + 63\epsilon^2) \left[ \ln \left( \frac{(1 - \epsilon)^2}{4\pi} \right) + 2 \ln \left( \frac{H}{\mu} \right) + \gamma_E \right] - 8\epsilon(36 - 40\epsilon + 9\epsilon^2) \left[ \psi \left( \frac{1}{1 - \epsilon} \right) + \psi \left( 1 - \frac{1}{1 - \epsilon} \right) \right] - 54\epsilon(1 - \epsilon)(2 - \epsilon) \left[ \psi \left( \frac{1}{2} + \sqrt{1 - 14\epsilon + \epsilon^2} \right) + \psi \left( \frac{1}{2} - \sqrt{1 - 14\epsilon + \epsilon^2} \right) \right] \right\}
+ O(D - 4),
\] (56)

where \( \psi(z) = (d/dz)\Gamma(z) \) denotes the digamma function, and we made use of
\[
H^D = \mu^{D-4} H^4 \left( 1 + (D - 4) \ln \left( \frac{H}{\mu} \right) \right) + O((D-4)^2),
\] (57)

where \( \mu \) is an arbitrary renormalization scale.

Since \( \epsilon = -\dot{H}/H^2 \) is in general a dynamical quantity, in order to renormalize the theory properly, one needs to subtract all divergent terms in Eq. (56) containing powers of \( \epsilon \). In order to do this we shall make use of the following counter lagrangian,
\[
\mathcal{L}_c = \sqrt{-g} \left( a_0 R^2 + a_1 \kappa g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) R + a_2 \kappa g^{\mu\nu} (\partial_\mu \phi)(\partial_\nu \phi) \frac{\partial^2 V(\phi)}{\partial^2 \phi} + a_3 [R^2 - 3 R_{\mu\nu} R^{\mu\nu}] \right),
\] (58)

where the last term denotes the Gauss-Bonnet term in FLRW spaces. This can be related to the standard form of the Gauss-Bonnet term by noticing that, since FLRW spaces are conformally flat and thus have a vanishing Weyl tensor, \( R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} \) can be expressed as a linear combination of \( R^2 \) and \( R_{\mu\nu} R^{\mu\nu} \),
\[
R_{\mu\nu\rho\sigma} R^{\mu\nu\rho\sigma} = -\frac{2}{(D - 1)(D - 2)} (R^2 - 2(D - 1) R_{\mu\nu} R^{\mu\nu}) \right\}.
\] (59)
In order to fully renormalize the effective action \( \Gamma_{g+gh} \) \(^{(56)}\) non-geometric counterterms are required. These terms appear as a consequence of including the terms in the effective action \( S_{os} \) \(^{(26)}\) that vanish on shell. The counter lagrangian \(^{(58)}\) is not unique. Indeed we could have chosen different counter terms \(^{(30)}\). Since, based on the available information, there is no unique way to fix the counterterms, the form \(^{(58)}\) of the counter lagrangian suffices for the purpose of this work.

Varying the individual terms in the counter lagrangian \(^{(58)}\) results in:

\[
\frac{1}{V} \delta a \int d^D x \sqrt{-g} R^2 = a^{D-1} H^4 \left( -432 \epsilon (1 - \epsilon)(2 - \epsilon) \right.
\]
\[+ 36 \left( 4 - 34 \epsilon + 35 \epsilon^2 - 8 \epsilon^3 \right) (D-4) + \mathcal{O}\left((D-4)^2, \dot{\epsilon}\right)\]

\[
\frac{1}{V} \delta a \int d^D x \kappa \sqrt{-g} g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) R = a^{D-1} H^4 \left( 144 \epsilon^2 (1 - \epsilon) \right.
\]
\[- 24 \epsilon(2 - 8 \epsilon + 5 \epsilon^2)(D-4) \left. + \mathcal{O}\left((D-4)^2, \dot{\epsilon}\right)\right)\]

\[
\frac{1}{V} \delta a \int d^D x \kappa \sqrt{-g} g^{\mu\nu}(\partial_\mu \phi)(\partial_\nu \phi) \frac{\partial^2 V}{\partial \phi^2}(\phi) = a^{D-1} H^4 \left( -16 \epsilon^2 (3 - \epsilon) \right.
\]
\[- 16 \epsilon^2 (4 - \epsilon)(D-4) \left. + \mathcal{O}\left((D-4)^2, \dot{\epsilon}\right)\right)\]

\[
\frac{1}{V} \delta a \int d^D x \sqrt{-g} \left[ R^2 - 3 R_{\mu\nu} R^{\mu\nu} \right] = a^{D-1} H^4 \left( 36 (1 - \epsilon)^3 \right)(D-4) + \mathcal{O}\left((D-4)^2, \dot{\epsilon}\right), \quad \text{(60)}
\]

where we used \(^{(8)}\) and in the last step we again used the background equations of motion \(^{(22)}\) and the following on-shell identities,

\[\sqrt{\kappa} \dot{\phi} = \sqrt{2(D - 2)} e \alpha H; \quad \frac{\partial^2 V}{\partial \phi^2}(\phi) = 2(D - 1 - \epsilon) \epsilon H^2 + \mathcal{O}(\dot{\epsilon}). \quad \text{(61)}\]

The divergent part of Eq. \(^{(56)}\) cancels when the coefficients \(a_i\) \((i = 0, 1, 2)\) in the counter lagrangian \(^{(58)}\) are

\[a_0 = -\frac{11}{192 \pi^2} \frac{\mu^{D-4}}{D-4} + a_0^f, \quad a_1 = \frac{13}{288 \pi^2} \frac{\mu^{D-4}}{D-4} + a_1^f, \quad a_2 = -\frac{5}{32 \pi^2} \frac{\mu^{D-4}}{D-4} + a_2^f, \quad \text{(62)}\]

where the \(a_i^f\) \((i = 0, 1, 2)\) indicates a possible finite part of \(a_i\). \(a_3\) remains a free (infinite) parameter.
The renormalized effective action $\Gamma_{1L,\text{ren}}$ is then obtained from

$$\frac{1}{H^4 a^{D-1} V} \frac{\delta \Gamma_{1L,\text{ren}}[a]}{\delta a(l)} = \frac{1}{16\pi^2} \left\{ \left( \beta_0 + \beta_1 \epsilon + \beta_2 \epsilon^2 + \beta_3 \epsilon^3 + \beta_4 \epsilon^4 \right) \right.$$

$$- 2\epsilon(198 - 241\epsilon + 63\epsilon^2) \left[ \ln \left((1 - \epsilon)^2\right) + 2 \ln \left(\frac{H}{H_0}\right) \right]$$

$$- 8\epsilon(36 - 40\epsilon + 9\epsilon^2) \left[ \psi\left(\frac{1}{1 - \epsilon}\right) + \psi\left(1 - \frac{1}{1 - \epsilon}\right) \right]$$

$$- 54\epsilon(1 - \epsilon)(2 - \epsilon) \left[ \psi\left(\frac{1}{2} + \frac{\sqrt{1 - 14\epsilon + \epsilon^2}}{2(1 - \epsilon)}\right) + \psi\left(\frac{1}{2} - \frac{\sqrt{1 - 14\epsilon + \epsilon^2}}{2(1 - \epsilon)}\right) \right] \right\}$$

$$+ \mathcal{O}(D - 4),$$

(63)

where the coefficients of the terms multiplying $\epsilon^i/(16\pi^2)$ ($i = 0, 1, 2, 3, 4$) are given by

$$\beta_0 = -48 + 576\pi^2(D - 4)a_3$$

$$\beta_1 = -\frac{2168}{3} - 1728\pi^2 \left[ 8a_0^f + (D - 4)a_3 \right] + 396 \left[ \ln(4\pi) - \gamma_E + 2 \ln \left(\mu/H_0\right) \right]$$

$$\beta_2 = \frac{4352}{3} + 192\pi^2 \left[ 108a_0^f + 12a_1^f - 4a_2^f + 9(D - 4)a_3 \right] - 482 \left[ \ln(4\pi) - \gamma_E + 2 \ln \left(\mu/H_0\right) \right]$$

$$\beta_3 = -\frac{1961}{3} + 64\pi^2 \left[ -108a_0^f - 36a_1^f + 4a_2^f - 9(D - 4)a_3 \right] + 126 \left[ \ln(4\pi) - \gamma_E + 2 \ln \left(\mu/H_0\right) \right]$$

$$\beta_4 = 54,$$

(64)

where $H_0$ is the expansion rate at which the $\ln(H/H_0)$ term in Eq. (63) vanishes. The formula (63) is one of the central results of our work. Even though $\beta_4$ in Eq. (64) seems to be fully specified by the one loop calculation, this is in fact not the case. Indeed, one can show that, upon adding to the counter lagrangian the counter term $L'_c = a_4\sqrt{-g}R(\partial^2 V/\partial\phi^2)$, $\beta_4$ will become a function of $a_4^f$, and thus unspecified. A similar statement holds for $\beta_0$: in the absence of the Gauss-Bonnet counterterm $\beta_0$ has a definite value ($\beta_0 = -48$). Since currently there are no physical measurements that specify the value of the Gauss-Bonnet counterterm, we conclude that $a_3$ – and hence also $\beta_0$ – is unspecified by the one loop calculation (see also Refs. [30, 44]).

Other terms in Eq. (63), in particular the logarithm and polygamma functions, cannot be altered by local counterterms, and hence these terms constitute the physical graviton one loop contributions. According to the analysis of Ref. [30], when mode mixing is taken account of, the poles of the ghost propagators coincide with those of the graviton, such that in the full analysis the digamma functions in the last line of Eq. (63) are absent (the same holds for the non-renormalized result (56)).
VI. DYNAMICS IN QUASI DE SITTER SPACES

Equations (40) and (63), together with the Bianchi identity, give the quantum modified Friedmann equations. However, because of the complexity of (63), we will expand the correction in the limit of small $\epsilon$ (quasi de Sitter space) \[47\].

When expanded in powers of $\epsilon$ Eq. (63) gives,
\[
\frac{1}{a^3 V} \frac{\delta \Gamma_{1L,\text{ren}}[a]}{\delta a(\eta)} = \frac{H^4}{16\pi^2} \left\{ \left[ \beta_0 - 252 \right] + \left[ \beta_1 + 374 - 792 \ln \left( \frac{H}{H_0} \right) + 792 \gamma_E \right] \epsilon + O(\epsilon^2, \dot{\epsilon}) \right\}. \tag{65}
\]

Inserting (65) into Eq. (40) we obtain the following approximate Friedmann trace equation,
\[
H^2 - \frac{\Lambda}{3} + \frac{1}{2} \dot{H} + \frac{\beta_0 - 252}{24\pi} G_N H^4 + \frac{33}{\pi} \ln \left( \frac{H}{H_0} \right) - \gamma_E - \frac{\beta_1 + 374}{792} G_N H^2 \dot{H} + O(\epsilon^2, \dot{\epsilon}) = \frac{2\pi G_N}{3} (\rho_M - 3p_M). \tag{66}
\]

The quantum correction to the trace of the Einstein equation is the correction to the expectation value of the trace of the (quantum) Einstein tensor,
\[
\delta G \equiv \langle \Omega | \delta \hat{G} | \Omega \rangle = - \frac{D - 2}{2} \langle \Omega | \delta \hat{R} | \Omega \rangle. \tag{67}
\]

From the symmetry of the background FLRW space we know that $\delta G_{\mu\nu}$ contains two independent components: the first is the trace, and the second can be inferred from the corresponding Bianchi identity for $\delta G_{\mu\nu}$, which is a consequence of the Bianchi identity for the background space Einstein tensor and of the covariant conservation of the matter stress energy tensor. Equivalently, one can view $\delta G_{\mu\nu}$ the 'stress energy' tensor $(T_{\mu\nu})_Q$ corresponding to the quantum corrections to (66); then the symmetries of the FLRW determine its form to be,
\[
(T^\mu_\nu)_Q = \text{diag}(\rho_Q, -p_Q, -p_Q, -p_Q). \tag{68}
\]

The covariant conservation of (68) implies the following perfect fluid-like conservation law,
\[
\frac{d}{dt}(a^4 \rho_Q) = a^4 H (\rho_Q - 3p_Q). \tag{69}
\]

To solve for $\rho_Q$, we use the following Ansatz:
\[
\rho_Q = \lambda H^4 + \nu H^2 \dot{H} + (\sigma H^4 + \tau H^2 \dot{H}) \ln \left( \frac{H}{H_0} \right) + O(\epsilon^2, \dot{\epsilon}) \tag{70}
\]

which implies for the fluid equation (69) that
\[
\frac{d}{dt}(a^4 \rho_Q) = a^4 \left[ 4\lambda H^5 + (4v + 4\lambda + \sigma) H^3 \dot{H} + (4\sigma H^5 + (4\tau + 4\sigma)H^3 \dot{H}) \ln \left( \frac{H}{H_0} \right) \right]. \tag{71}
\]
We read off $(\rho_Q - 3p_Q)$ from (66) and find that

$$\lambda = -\frac{1}{64\pi^2}(\beta_0 - 252), \quad v = \frac{1}{64\pi^2}(\beta_0 + \beta_1 + 122 + 792\gamma_E), \quad \sigma = 0, \quad \tau = -\frac{99}{8\pi^2},$$

and thus

$$\rho_Q = \frac{1}{64\pi^2}\left[-(\beta_0 - 252)H^4 + \left(\beta_0 + \beta_1 + 122 + 792\gamma_E\right)H^2\dot{H} - 792H^2\dot{H} \ln\left(\frac{H}{H_0}\right)\right] + \mathcal{O}(\epsilon^2, \dot{\epsilon})$$

$$p_Q = \frac{1}{64\pi^2}\left[(\beta_0 - 252)H^4 + \left(\frac{1}{3}\beta_0 - \beta_1 - 458 - 792\gamma_E\right)H^2\dot{H} + 792H^2\dot{H} \ln\left(\frac{H}{H_0}\right)\right] + \mathcal{O}(\epsilon^2, \dot{\epsilon})$$

$$\rho_Q + p_Q = \frac{1}{48\pi^2}(\beta_0 - 252)H^2\dot{H} + \mathcal{O}(\epsilon^2, \dot{\epsilon}).$$

The quantum corrected Friedmann equations become (cf. Eqs. (22)):

$$3H^2 - \Lambda + \frac{\beta_0 - 252}{8\pi}G_NH^4 - \beta_0 + \beta_1 + 122 + 792\gamma_E H^2\dot{H} + \frac{99}{\pi}G_NH^2\dot{H} \ln\left(\frac{H}{H_0}\right)$$

$$+ \mathcal{O}(\epsilon^2, \dot{\epsilon}) = 8\pi G_N\rho_M$$

$$- 2\dot{H} - \frac{\beta_0 - 252}{6\pi}G_NH^2\dot{H} + \mathcal{O}(\epsilon^2, \dot{\epsilon}) = 8\pi G_N(p_M + p_M).$$

We shall assume that the matter contribution obeys an equation of state $p_M = w\rho_M$, with $w$ constant. In this case we can combine the two equations as

$$3H^2 + \frac{2\dot{H}}{1 + w} - \Lambda + AG_NH^4 + \left[B + \frac{99}{\pi} \ln\left(\frac{H}{H_0}\right)\right]G_NH^2\dot{H} + \mathcal{O}(\epsilon^2, \dot{\epsilon}) = 0,$$

where we defined

$$A \equiv \frac{\beta_0 - 252}{8\pi}$$

$$B \equiv \frac{1}{8\pi}\left[\frac{4}{3(1 + w)}(\beta_0 - 252) - (\beta_0 + \beta_1 + 122 + 792\gamma_E)\right].$$

Notice that, since we are free to choose $\beta_0$ and $\beta_1$ by a suitable choice of the coefficients $a_0^f$ and $a_3$ in the counterterms (see Eq. (64)) (the Gauss-Bonnet terms must be also included). Indeed, choosing $\beta_0 = 252$ and $\beta_1 = -374 - 792\gamma_E$ results in $A = 0 = B$. In fact, $A$ and $B$ are not completely independent for a general value of $w$ since from (76) it follows that

$$B = 4A/[3(1 + w)] + \text{const.}$$

One can integrate Eq. (74). The result can be expressed in terms of the roots of the quartic equation,

$$\frac{A}{3}G_NH^4 + H^2 - \frac{\Lambda}{3} = 0.$$
In the case when $A < 0$ all four roots $\pm H_\pm$ are real,

$$H_\pm^2 = \frac{3}{2AG_N} \left[-1 \pm \sqrt{1 + \frac{4A}{9G_N\Lambda}}\right].$$  \hspace{1cm} (78)

The positive root $H_+ > 0$ corresponds to the one-loop corrected de Sitter attractor. From equation (75) it follows that $H_+$ is approached exponentially fast. More precisely, the late time limit can be approximated by the form,

$$H = H_+ \left[1 + 2 \exp \left(-\Omega t + \delta_Q\right)\right],$$ \hspace{1cm} (79)

where to order $G_N\Lambda$ and at late times $\Omega t \gg 1$,

$$\Omega \simeq 3(1 + w)\left\{1 + \frac{G_N\Lambda}{3} \left[\frac{A}{2} - \frac{1 + w}{2} \left(B + \frac{99}{2\pi} \ln \left(\frac{\Lambda}{3\bar{H}_0^2}\right)\right)\right]\right\} \sqrt{\frac{\Lambda}{3}} \hspace{1cm} (80)$$

and $\delta_Q/\Omega$ represents an order $G_N\Lambda$ shift in time, which is unphysical since it can be absorbed in the definition of time. This means that quantum effects during quasi de Sitter phase induce an order $G_N\Lambda$ shift of the late time de Sitter attractor (which can be read off from $H_+$ in Eq. (78)). At late times this de Sitter attractor is approached exponentially fast, with the characteristic time scale given by $\Omega^{-1}$, which equals the classical time scale plus an order $G_N\Lambda$ correction, as expected. In addition, there is an order $G_N\Lambda$ shift $\delta_Q$, which implies a time delay of $\delta_Q/\Omega$. Note that $\delta_Q$ can be both positive and negative, depending on the sign and magnitude of $A$ and $B$ defined in Eq. (76). (The sign of $\delta_Q$ depends also on $\bar{H}_0$, but a change in $\bar{H}_0$ can always be absorbed in a change in $B$.) This agrees with figure 2 where we show $H$ as a function of time both when $\delta_Q$ is positive (left panel) and when it is negative (right panel) (in the plots we have chosen $A = 0$ and $B = 0$). A positive (negative) correction $\delta_Q$ implies a greater (smaller) expansion rate $H$, and therefore a universe that has expanded more (less) before entering the late time de Sitter phase.

At early times the quantum solution deviates more and more the classical solution, which approaches the Big Bang singularity at $t = 0$. Formally, the quantum one loop solution is not singular, and at large and ‘negative’ times (any negative time can be of course transformed to a positive time by an appropriate time shift) the solution approaches the quantum attractor $[42, 43, 44] \; H \rightarrow H_- \; \text{defined in Eq. (78)}$. At this point the expansion rate becomes of the order the Planck scale, implying large higher loop corrections, such that this behavior cannot be trusted.
FIG. 2: Numerical solution to (75) for $H$ as a function of time. The red, dashed curve represents the classical behavior and the blue, solid curve includes our one-loop corrections. The late time de Sitter limit is clearly obtained. The quantum corrections lower the effective cosmological constant.

In all plots we choose $\Lambda = 3$. This implies that we are effectively plotting the dimensionless variables: $h = \sqrt{3/\Lambda}H; \tau = \sqrt{\Lambda/3}t$ and $g = (\Lambda/3)G_N$ ($\Lambda = 3$; $w = 1/3$; $G_N = 0.001$; $A = 0$; $B = 0$; $H(t = 0) = 10$; $\bar{H}_0 = 0.1$ (left panel) $\bar{H}_0 = 10$ (right panel)).

VII. DISCUSSION

Before specializing the discussion to the two cases discussed above, we make some general remarks on the validity of our results. First of all, the correction we calculate is only valid when $\epsilon$ is strictly constant. A nonconstant $\epsilon$ would induce corrections to the propagators as calculated in section III and unfortunately it is not yet known how to calculate these. This of course does not prevent one from using these propagators to calculate quantum corrections. One can then reasonably assume that, as long as in the final answer the change in $\epsilon$ is sufficiently small, the error one is making is small and thus the results can be trusted. From figure 3 it is clear however, that there are regimes where $\epsilon$ is far from constant and one should be careful to trust our results there.

A second general concern is the issue of gauge invariance (invariance under infinitesimal coordinate transformations). When both a gravitational field and a matter field are present, the fluctuations in those fields are coupled and do not transform independently. Therefore one cannot self consistently quantize the gravitational fluctuations, without quantizing the matter fluctuations. The types of structures (e.g. the poles in the digamma functions), how-
FIG. 3: $\epsilon$ as a function of time. The red, dashed curve represents the classical behavior and the blue, solid curve includes our one-loop corrections. The strong dependence of the behavior at early times on $\bar{H}_0$ is clearly visible. The parameters are: $\Lambda = 3$; $w = 1/3$; $G_N = 0.001$; $A = 0$; $B = 0$; $H(t = 0) = 10$; $\bar{H}_0 = 0.1$ (left panel) $\bar{H}_0 = 10$ (right panel), see also figure 2.

ever, are generic since they are naturally generated by the coincident limit of any propagator of the form (18), and they do not disappear when matter fluctuations are included [30].

Because the poles of the digamma functions $\epsilon_p = 1/2, 2/3, 3/4, \ldots, 4/3, 3/2$ that yield a divergent one loop effective potential (56) are sufficiently distant from the quasi-de Sitter limit $\epsilon \to 0$ considered here, the results of our dynamical analysis can be trusted as long as $\epsilon \ll 1/2$. When this condition is satisfied, quantum effects do not change the fact that at late times the Universe asymptotes a de Sitter attractor, albeit with a modified expansion rate given by $H_+$ in Eq. (78). The leading order quantum effect at late times has a contribution proportional to $H^4$ to the effective energy momentum tensor. A contribution of this form has also been found in earlier studies of graviton one-loop effects in de Sitter space [10][25][26]. The exact contribution is unknown because of the ambiguity in the counterterms. Depending on the choice of counterterms, the contribution could slightly increase or decrease the effective late time cosmological constant. Although in our more general treatment, divergencies appear in the effective action, leading to the logarithmic correction to (75), these corrections have no significant effect at late times.

At early times the contribution of quantum effects becomes more significant. However in this regime we have lost predictability, since the results strongly depend on the unknown part of the counterterms and the renormalization scale $\mu$ ($\bar{H}_0$). Moreover, the assumption
that $\epsilon < 1/2$ and nearly constant appears to be violated.

VIII. CONCLUSION AND OUTLOOK

In this paper we calculated the quantum corrected Friedmann equations due to the one loop vacuum bubble from gravitons in a FLRW universe with constant $\epsilon \equiv -\dot{H}/H^2$. The result has a divergence that contains terms proportional to $\epsilon H^4$, $\epsilon^2 H^4$ and $\epsilon^3 H^4$, which can be renormalized using local counterterms, which include both geometric and scalar field counterterms. This is consistent with the result that in de Sitter space ($\epsilon = 0$) one loop effects lead to a finite constant shift of the cosmological term $\propto H^4$. We study the dynamics in the quasi de Sitter limit and find that they are not much different from the dynamics in true de Sitter space. Indeed, the quantum effects induce a shift in the effective, late time, cosmological constant $\sim (A/36)G_N H^4$, where $A$ is an unknown parameter, that can be expressed in terms of the Gauss-Bonnet counterterm with an $\mathcal{O}(1/(D - 4))$ coefficient.

Although our results are correct within the approximations used, the results described above should not be taken too literally. The propagators we used (and hence the singularity structure we find) are strictly speaking only valid when $\epsilon = \text{constant}$. Our analysis is correct as long as any time variation in $\epsilon$ is small enough, which is indeed the case sufficiently close to de Sitter space. Indeed, our late time solution does have $\dot{\epsilon} \to 0$. Therefore we have good reasons to believe that our solution approximates well the solution of the full theory, at least at late times and sufficiently close to de Sitter space.

Another issue is that we choose our propagator such to describe a physically meaningful vacuum state. However, due to the evolution and mixing of modes, close to the de Sitter attractor the Universe will not be in a vacuum state, but in some excited state (that can be described by mode mixing in momentum space), which might influence our results. We postpone a study of this question for future publication.

The next issue is the question of gauge invariance. Since there is both matter and gravity in our model, one should self-consistently take both fluctuations in matter and gravitons into account. This issue is complicated due to the mixing of the degrees of freedom, and hence it is addressed in a separate publication [30]. Taking this mixing into account changes our results quantitatively, but since the singularity structure is inherent in the propagators (and those do not change), the logarithmic terms do not cancel, such that qualitative features of
the analysis presented here remain unchanged.

Finally, an important question is what are the dynamics near the poles of the digamma functions $\epsilon_p = 1/2, 2/3, 3/4, ..., 4/3, 3/2$ where the one loop effective potential diverges. The corresponding dynamical analysis is performed in Ref. [30]. Here we just note that the Universe typically gets stuck near the poles, such that each pole acts as a late time attractor. Probably the most important attractors are the two highest poles $\epsilon_p = 3/2$ and $\epsilon_p = 4/3$ (the latter is also the value of $\epsilon$ in matter era). The latter pole is the late time attractor of a universe filled mostly with radiation and a cosmological term [30], which represents a realistic composition of the Universe immediately after the Big Bang.

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[45] The analysis of Ref. [30] shows that, when the mode graviton-matter mixing is taken account of, the ghost propagators \( \hat{\Delta}_n(x; \tilde{x}) \) (\( n = 0, 1 \)) become identical to the graviton propagators \( \Delta_n(x; \tilde{x}) \) (\( n = 0, 1 \)).

[46] Had we not included the contribution (55) from the off shell terms (26) in the effective action, instead of Eq. (56) we would get:

\[
\frac{1}{H^4a^{D-1}V} \frac{\delta \Gamma'[g+gh][a]}{\delta a(l)} = -\frac{21 \mu^{D-4} \epsilon(1 - \epsilon)(2 - \epsilon)}{4\pi^2(D - 4)} + \frac{1 - \epsilon}{16\pi^2} \left\{ -12 + 22\epsilon - 35\epsilon^2 + 30\epsilon^3 \right\} + \epsilon(2 - \epsilon) \left[ -7 \ln \left( \frac{1 - \epsilon}{4\pi} \right) - 14 \ln \left( \frac{H}{\mu} \right) - 7\gamma_E + 2\psi \left( \frac{1}{1 - \epsilon} \right) + 2\psi \left( -\frac{\epsilon}{1 - \epsilon} \right) \right. \\
\left. - 9 \left( \psi \left( \frac{1}{2} + \frac{\sqrt{1 - 14\epsilon + \epsilon^2}}{2(1 - \epsilon)} \right) + \psi \left( \frac{1}{2} - \frac{\sqrt{1 - 14\epsilon + \epsilon^2}}{2(1 - \epsilon)} \right) \right) \right\} + O(D - 4). 
\]

Note that the structure of the divergent term in this expression is simpler such that – unlike the divergence in Eq. (56) – it can be renormalized by making use of the \( R^2 \) counterterm alone.

[47] The question of the dynamics around those \( \epsilon \) for which the one-loop potential (63) exhibits poles is addressed in the companion paper (30).