Formation of the Hayward black hole from a collapsing shell

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(Dated: December 11, 2019)

Abstract

We consider a collapsing shell of matter to form the Hayward black hole and investigate semiclassically quantum radiation from the shell. Using the Israel’s formulation, we obtain the mass relation between the collapsing shell and the Hayward black hole. By using the functional Schrödinger formulation for the massless quantum radiation, the evolution of a vacuum state for a scalar field is shown to be unitary. We find that the number of quanta at a low frequency decreases for a large length parameter characterizing the Hayward black hole. Moreover, in the limit of low frequency, the Hawking temperature can be read off from the occupation number of excited states when the shell approaches its own horizon.

Keywords: Hayward black hole, regular black hole, collapsing shell, domain wall, Hawking temperature

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I. INTRODUCTION

Since physical singularities of black holes have been interpreted as a breakdown of general relativity, there has been much attention to singularity-free black holes, the so-called regular black holes [1–6]. Among these black-hole solutions, in particular, the metric of the Hayward black hole reduces to the geometry of Schwarzschild black hole for a large radius, and to the metric of the de Sitter spacetime for a small radius so that the curvature becomes nonsingular at the center [4]. In the Hayward black hole, various aspects have been also studied in Refs. [7–12].

On the other hand, the functional Schrödinger picture was formulated in order for studying quantum radiation from a collapsing shell [13, 14]. The process of the collapsing shell to form the Schwarzschild black hole was constructed, so massless quantum radiation as seen by an observer at asymptotic infinity turned out to be non-thermal. Subsequently, there have been several works employing this method such as studies on radiation as seen by an infalling observer [15], collapsing shells to form the BTZ black string [16], Hawking radiation from the Reissner-Nordström domain wall [17], and a massive quantum radiation [18]. In addition, thanks to an analytic solution of the functional Schrödinger equation in Ref. [19], the density matrix of the quantum radiation was also calculated explicitly, which shows that the process of the radiation is unitary during the evolution [20, 21]. This fact was confirmed in the other singular black holes such as the anti-de Sitter Schwarzschild black hole and the Reissner-Nordström black hole [22, 23]. Now, one might wonder how this collapsing process including the functional Schrodinger method works in the Hayward black hole as a regular black hole.

In this paper, we will study the process of a collapsing shell to form the Hayward black hole and investigate the quantum radiation in the context of the functional Schrödinger equation. In Sec. II, we will obtain a mass relation between the collapsing shell and the Hayward black hole by using the Israel formulation [24]. Next, in Sec. III, a massless quantum radiation from the shell will be investigated in the context of the functional Schrödinger equation. The analytic solution of the functional Schrödinger equation will be obtained in the Hayward black hole along the line of Refs. [13, 19], and eventually the process is shown to be unitary. In Sec. IV, we will calculate the occupation number of excited states and discuss the spectrum of the occupation number. The occupation number at a low frequency will be shown to
decrease for a larger value of $\ell$ being a length parameter in the Hayward black hole. Finally, conclusion and discussion will be given in Sec. V.

II. CLASSICAL THEORY FOR A COLLAPSING SHELL TO FORM THE HAYWARD BLACK HOLE

The spacetime will be divided into the interior and the exterior regions of an infinitely thin shell with a radius $R(t)$ [13–18, 20–23]. In the exterior region of the shell, i.e., $r > R(t)$, the spacetime is described by the Hayward metric [4]

$$ (ds_{+})^2 = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2d\Omega^2 \quad (1) $$

with $f(r) = 1 - 2Mr^2/(r^3 + 2\ell^2M)$, where $M$ is the mass of the Hayward black hole and $\ell$ is the length parameter related to cosmological constant as $\Lambda = 3/\ell^2$. The metric (1) reduces to the Schwarzschild metric for $\ell = 0$, and it becomes Minkowski spacetime for $M = 0$. Note that we will consider the case of $M > M^*$ in order to study a non-extremal black hole, where the critical mass is given as $M^* = 3\sqrt{3}\ell/4$ [4]. Next, in the interior region of the shell, the spacetime is assumed to be Minkowski spacetime described by the metric

$$ (ds_{-})^2 = -dT^2 + dr^2 + r^2d\Omega^2, \quad (2) $$

which is valid for $r < R(t)$. On the shell $r = R(t)$, the proper distance is required to be continuous as $(ds_{+})^2 |_{r=R(t)} = (ds_{-})^2 |_{r=R(t)}$, so a relation between the exterior and interior times is obtained as

$$ \frac{dT}{dt} \bigg|_{r=R(t)} = \sqrt{f(R) + \left(1 - \frac{1}{f(R)}\right)\hat{R}^2}, \quad (3) $$

where $\hat{R} = dR/dt$.

In the Israel formulation [24], a combined metric can be obtained as $g_{\mu\nu} = \Theta(r - R(t))\tilde{g}_{\mu\nu}^+ + \Theta(-r + R(t))\tilde{g}_{\mu\nu}^-$, where $\Theta(x)$ is the unit step function defined as $\Theta(x) = 1(x > 0)$ and $\Theta(x) = 0(x < 0)$ with the metrics being $\tilde{g}_{\mu\nu}^+$ and $\tilde{g}_{\mu\nu}^-$ in the exterior and the interior regions, respectively. Then, a junction condition requires that $g_{\mu\nu}$ be continuous on the shell, so the Einstein tensor can be calculated as

$$ G_{\mu\nu} = \Theta(r - R(t))\tilde{G}_{\mu\nu}^+ + \Theta(-r + R(t))\tilde{G}_{\mu\nu}^- + \delta(r - R(t))\tilde{G}_{\mu\nu}^0, \quad (4) $$
where $\mathbf{G}^\pm_{\mu\nu}$ and $\mathbf{G}^0_{\mu\nu}$ are Einstein tensors calculated in the exterior and interior regions, and on the shell. Explicitly, $\mathbf{G}^0_{\mu\nu} = \frac{\kappa_{\mu\nu} + \kappa^{\gamma}_{\mu\nu}}{2}$, where $\kappa_{\mu\nu} = \kappa^{\gamma}_{\mu\nu}$ is a unit normal vector to the shell.

From the Einstein tensor (4), the corresponding energy-momentum tensor can also be calculated as

$$
T_{\mu\nu} = \Theta(r - R(t)) \tilde{T}^+_{\mu\nu} + \Theta(-r + R(t)) \tilde{T}^-_{\mu\nu} + \delta(r - R(t)) \tilde{T}^0_{\mu\nu}.
$$

(5)

Projecting $\tilde{G}^0_{\mu\nu} = 8\pi \tilde{T}^0_{\mu\nu}$ to the shell by using the projection operator $h_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu$, we obtain the so-called Israel relation

$$
[K h_{\mu\nu}] - [K_{\mu\nu}] = 8\pi S_{\mu\nu},
$$

(6)

where $K_{\mu\nu}$ is the extrinsic curvature of the shell and $[...]$ denotes $[A] = \lim_{r \to R^+} A(r) - \lim_{r \to R^-} A(r)$. Now, the source on the shell is assumed to be described by a perfect fluid; thus, $S_{\mu\nu} = h^{\gamma}_{\mu} \tilde{T}^0_{\gamma\nu} = (\sigma - \tau) u^\mu u^\nu - \tau h_{\mu\nu}$, where $\sigma$ is an energy density and $\tau$ is a surface tension [24]. In particular, a domain wall satisfying the relation $\sigma = \tau$ can be chosen as a source, so the source on the shell reduces to $S_{\mu\nu} = -\sigma h_{\mu\nu}$. Note that the energy density turns out to be constant thanks to the conservation equation on the shell $h_{\mu\gamma} D_{\nu} S_{\gamma\nu} = 0$, where $D_{\mu} = h_{\nu}^{\mu} \nabla_{\nu}$ [25, 26].

Plugging $S_{\mu\nu}$ into Eq. (6), we finally obtain the mass relation

$$
\frac{f(R)}{\sqrt{f(R) - \frac{R^2}{f(R)}}} - \frac{1}{\sqrt{1 - \frac{R^2}{B^2}}} = -4\pi \sigma R
$$

(7)

with $B^2 = (dT/dt)^2|_{r=R(t)}$. For a static limit, i.e., $\dot{R} = 0$, the mass in Eq. (1) can be expressed in terms of the mass $M_0 = 4\pi R^2 \sigma$ on the shell in such a way that

$$
M = \frac{M_0 - \frac{M_0^2}{2R}}{1 - \frac{2f}{B^2} \left( M_0 - \frac{M_0^2}{2R} \right)},
$$

(8)

which can be reduced to the case of the Schwarzschild black hole as $\ell \to 0$ [13]. From Eq. (7) with Eq. (3), a first-order differential equation can be obtained as

$$
\frac{dR}{dt} = -\frac{f(R)}{\sqrt{1 + f(R) \left[ 1 - \frac{R^2}{4M_0^2} (M_0^2 + 1 - f(R))^2 \right]^{-1}}.
$$

(9)

When the shell starts to collapse at $R = R_0$ that is actually infinite, or when the collapsing shell starts to form the black hole as $R(t) \to R_H$, the leading order of terms in Eq. (9) are
coincident, so Eq. (9) reduces to a simple form as [13–18, 20–23]

\[
\frac{dR}{dt} \approx -f(R).
\] (10)

Solving this equation in this incipient limit, we get the radius \( R(t) = R_H + (R_0 - R_H)e^{-f(R_H)t} \) for \( 0 \leq t \leq t_f \), while the initial and the final radii are \( R_0 = R(t_0) \) for \( t < 0 \) and \( R_f = R(t_f) \) for \( t > t_f \), respectively.

III. QUANTUM RADIATION FROM THE COLLAPSING SHELL

If an observer at asymptotic infinity sees the formation of the black hole, one can ask what radiation that characterizes gravitational collapse might be observed. Hence, we consider a quantum scalar field on the background of the collapsing shell and derive a quantized theory of the scalar field on the Hayward black hole.

The total action for a single scalar field is assumed to consist of both the exterior and the interior actions such as

\[
S_\Phi = S_\Phi^+ + S_\Phi^-, \tag{11}
\]

\[
S_\Phi^\pm = \int d^4 x^\pm \sqrt{-g^\pm} \left( -\frac{1}{2} g^\pm_{\mu \nu} \partial^\pm_\mu \Phi \partial^\pm_\nu \Phi \right), \tag{12}
\]

where \( g^\pm_{\mu \nu} \) and \( g^\pm_{\mu \nu} \) are the Hayward metric (1) and the Minkowski metric (2), and \( x^\mu_\pm \) are the exterior and the interior coordinates, respectively. By using the spherical symmetries of the metrics, the scalar field can be decomposed into real spherical harmonics as

\[
\Phi(x_\pm^0, r, \theta, \phi) = \sum_{j, m_j} \Phi_j(x_\pm^0, r) Y_{j, m_j}(\theta, \phi) \tag{13}
\]

with \( j = 0, 1, 2, \ldots \) and \( m_j = -j, -j + 1, \ldots, j \). Then, the actions (12) can be written as

\[
S_\Phi^+ = \sum_j 4\pi(2j + 1) \int dt \int_R^\infty dr \left( \frac{r^2}{2f(r)} \left( \frac{\partial \Phi_j}{\partial t} \right)^2 - \frac{r^2 f(r)}{2} \left( \frac{\partial \Phi_j}{\partial r} \right)^2 - \frac{j(j + 1)}{2} \Phi_j^2 \right), \tag{14}
\]

\[
S_\Phi^- = \sum_j 4\pi(2j + 1) \int dT \int_0^R dr \left( \frac{r^2}{2} \left( \frac{\partial \Phi_j}{\partial T} \right)^2 - \frac{r^2 B}{2} \left( \frac{\partial \Phi_j}{\partial r} \right)^2 - \frac{j(j + 1)}{2} \Phi_j^2 \right). \tag{15}
\]

Especially, the interior action (15) is written as

\[
S_\Phi^- = \sum_j 4\pi(2j + 1) \int dt \int_0^R dr \left( \frac{r^2}{2B} \left( \frac{\partial \Phi_j}{\partial T} \right)^2 - \frac{r^2 B}{2} \left( \frac{\partial \Phi_j}{\partial r} \right)^2 - \frac{j(j + 1)}{2} \Phi_j^2 \right), \tag{16}
\]
where we have used the relation \( B = (dT/dt)_{r=R(t)} \) in Eq. (3).

We decompose \( \Phi_j \) into a complete set of real basis functions denoted by \( \{d_{k,j}\} \) [13]

\[
\Phi_j = \sum_k a_{k,j}(t)d_{k,j}(r),
\]

where \( a_{k,j}(t) \) is a mode amplitude. By using Eq. (17), in the incipient limit \( R \to R_H \), the action (12) is obtained as

\[
S_\Psi \approx \sum_{k,k',j} (2j+1) \int dt \left( \frac{1}{2f(R)} \dot{a}_{k,j} \dot{\tilde{A}}_{k,k',j} a_{k',j} - \frac{1}{2} a_{k,j} (\tilde{B}_{k,k',j} + \tilde{C}_{k,k',j}) a_{k',j} \right)
\]

(18)
after spatial integrations, where the coefficients are defined by

\[
\tilde{A}_{k,k',j} = 4\pi \int_0^{R_H} drr^2 d_{k,j} d_{k',j},
\]

\[
\tilde{B}_{k,k',j} = 4\pi \int_{R_H}^{\infty} drr^2 f(r)(dd_{k,j}/dr)(dd_{k',j}/dr),
\]

and \( \tilde{C}_{k,k',j} = 4\pi j(j+1) \int_{R_H}^{\infty} drr^2 d_{k,j} d_{k',j} \).

Since the coefficients \( \tilde{A}_{k,k',j}, \tilde{B}_{k,k',j} \) and \( \tilde{C}_{k,k',j} \) are Hermitian operators, they can be simultaneously diagonalized by a set of eigenvectors \( \{b_{k,j}\} \) [13, 27],

\[
S_\Psi = \sum_{k,j} (2j+1) \int dt \left( \frac{1}{2f(R)} \alpha_{k,j} \dot{b}_{k,j}^2 - \frac{1}{2} (\beta_{k,j} + \gamma_{k,j}) b_{k,j}^2 \right),
\]

(19)
where \( \alpha_{k,j}, \beta_{k,j} \) and \( \gamma_{k,j} \) are eigenvalues of \( \tilde{A}_{k,k',j}, \tilde{B}_{k,k',j} \) and \( \tilde{C}_{k,k',j} \), respectively.

We are now in a position to impose the quantization rule as \( [b_{k,j}, \Pi_{k',j}] = i\delta_{k,k'}\delta_{j,j'} \)

represented by \( \Pi_{k,j} \to -i\partial/\partial b_{k,j} \), where \( \Pi_{k,j} = \partial L_\Psi / \partial \dot{b}_{k,j} = (2j+1)\alpha_{k,j} \dot{b}_{k,j} / f(R) \). Hence, the Hamiltonian for the quantum radiation can be obtained as

\[
H_\Psi = \sum_{k,j} \left( -\frac{f(R)}{2(2j+1)\alpha_{k,j}} \frac{\partial^2}{\partial b_{k,j}^2} + \frac{1}{2} (\beta_{k,j} + \gamma_{k,j}) b_{k,j}^2 \right).
\]

(20)
From the functional Schrödinger equation \( H_\Psi \Psi = i\partial \Psi / \partial t \) [13], the Schrödinger equation of the wave function \( \psi \) for one eigenvector \( b \in \{b_{k,j}\} \) can be written as

\[
\left\{ -\frac{f(R)}{2\alpha} \frac{\partial^2}{\partial b^2} + \frac{1}{2} \alpha \omega_0^2 b^2 \right\} \psi(t,b) = \frac{i}{\partial t} \psi(t,b),
\]

(21)
where \( \omega_0 = \sqrt{(\beta + \gamma)/\alpha} \) with \( \alpha \in \{(2j+1)\alpha_{k,j}\}, \beta \in \{\beta_{k,j}\}, \) and \( \gamma \in \{\gamma_{k,j}\} \). For convenience, a new time parameter is defined as

\[
\eta = \int_0^t dt' f(R(t'))
\]

(22)
so that Eq. (21) becomes the Schrödinger equation for a harmonic oscillator with the mass \( \alpha \) and the time-dependent frequency \( \omega(\eta) \) as

\[
\left\{ -\frac{1}{2\alpha} \frac{\partial^2}{\partial b^2} + \frac{1}{2} \alpha \omega^2(\eta) b^2 \right\} \psi(\eta,b) = \frac{i}{\partial \eta} \psi(\eta,b),
\]

(23)
where $\omega(\eta) = \omega_0/\sqrt{f(R)}$. For $\eta < 0$ and $\eta > \eta_f$, the frequency $\omega(\eta)$ must be constant, so the eigenstates are given by those of simple harmonic oscillators,

$$\phi_n(b) = \left(\frac{\alpha \bar{\omega}}{\pi}\right)^{\frac{1}{4}} \frac{1}{\sqrt{2^n n!}} H_n(\sqrt{\alpha \bar{\omega} b}) e^{-\frac{1}{2} \alpha \bar{\omega} b^2},$$  

(24)

where $H_n(x)$ are Hermite polynomials and $\bar{\omega}$ is $\omega_0$ for $\eta < 0$ or $\omega_f = \omega_0/\sqrt{f(R)}$ for $\eta > \eta_f$.

In addition, in case of $0 \leq \eta \leq \eta_f$, the solutions for the time-dependent harmonic oscillator have been shown to be coherent state, which are the unitary transform of eigenstates [28], and thus, a vacuum state is given by a coherent state with the lowest energy as

$$\psi(\eta, b) = e^{i\theta(\eta)} \left(\frac{\alpha}{\pi \zeta^2}\right)^{\frac{1}{4}} e^{i\left(\frac{1}{2} \frac{d\zeta}{d\eta} + \frac{x}{\zeta}\right)\frac{\zeta^2}{2}},$$

(25)

where $\theta(\eta) = (-1/2) \int_0^\eta d\eta' \zeta^{-2}(\eta')$. Note that $\zeta(\eta)$ is a real solution satisfying

$$\frac{d^2 \zeta}{d\eta^2} + \frac{\omega_0^2}{f(R)} \zeta = \frac{1}{\zeta^3}$$

(26)

with initial conditions: $\zeta(0) = 1/\sqrt{\omega_0}$ and $[d\zeta/d\eta](0) = 0$. In the incipient limit, the metric function is approximately written as $f(R) \approx 1 - H\eta$, where $H = [df(R)/dR]|_{R=R_H} = (R^2_H - 3\ell^2)/R^2_H$, so that the real solution $\zeta(\eta)$ in Eq. (26) can be obtained as [19]

$$\zeta(\eta) = \frac{1}{\sqrt{\omega_0}} \sqrt{\xi(\tilde{\eta}) + \chi^2(\tilde{\eta})},$$

(27)

where

$$\xi(\tilde{\eta}) = \frac{\pi y}{2} (Y_0(\tilde{x})J_1(y) - J_0(\tilde{x})Y_1(y)),$$

(28)

$$\chi(\tilde{\eta}) = \frac{\pi y}{2} (Y_1(\tilde{x})J_1(y) - J_1(\tilde{x})Y_1(y)),$$

(29)

and $J_n$ and $Y_n$ are Bessel functions of the first kind and the second kind, respectively with $\tilde{\eta} = H\eta$, $x = 2\omega_0/H$, and $y = (2\omega_0/H)\sqrt{1 - \tilde{\eta}}$. In the incipient limit, a basis is taken as $\{\phi_n(b)|\bar{\omega}=\omega_f\}$, and; thus, the probability amplitude in the $n$-th eigenstate in the vacuum state can be calculated as

$$c_n(\eta) = \int_{-\infty}^{\infty} db \phi_n(b) \psi(\eta, b)$$

(30)

$$= \begin{cases} e^{i\theta(\eta)\frac{(n-1)!}{\sqrt{n!}} \frac{1}{\sqrt{\omega_0 \zeta^2}} \sqrt{(-1)^n \frac{\omega_0}{2} (1 - \frac{2}{P})}} & (n = 0, 2, 4, \ldots) \\ 0 & (n = 1, 3, 5, \ldots) \end{cases}$$

(31)
where $P = 1 - (i/\omega_f\zeta)(d\zeta/d\eta) + 1/(\omega_f\zeta^2)$.

In connection with the probability amplitude (31), the density matrix in case of $0 \leq \eta \leq \eta_f$ takes the form of $\rho(\eta, b, b') = \sum_{n,m} c_n(\eta)c_m(\eta)\phi_n(b)\phi_m(b')$ in the incipient limit. Then the trace of the density matrix is calculated as

$$\text{Tr} \rho = \sum_{n=0}^{\infty} \int db \int db' \phi_n(b)\rho(\eta, b, b')\phi_n(b') = \sum_{n=0}^{\infty} |c_n(\eta)|^2 = \sqrt{\frac{H}{\zeta^2\omega_f |P|}} \frac{1}{\sqrt{1 - |1 - \frac{2}{P}|^2}} \quad (32)$$

which can be simplified as

$$\text{Tr} \rho = \frac{|\omega_f\zeta^2|}{\sqrt{\omega_f\zeta^2 \left( \text{Re}[\omega_f\zeta^2] - \text{Im}[\omega_f\zeta^2] \text{Re}[\zeta \frac{d\zeta}{d\eta}] + \text{Re}[\omega_f\zeta] \text{Im}[\zeta \frac{d\zeta}{d\eta}] \right)}} = 1 \quad (33)$$

because $\zeta(\eta)$ is real regardless of $H$. Therefore, the process of radiation turns out to be unitary during the evolution in the Hayward black hole, which is similar to the evolution process of singular black holes [20–23]. This fact can also be understood from the conservation law of a probability current. Using Eq. (21) in the incipient limit, we define the probability and the probability current as $P_{\text{pro}} = |\psi(t, b)|^2$ and $J(t, b) = (f(R)/(2\alpha)) (\psi^* \partial_b \psi - \psi \partial_b \psi^*)$, respectively, where they naturally satisfy the continuity equation $\partial P_{\text{pro}}/\partial t + \partial J(t, b)/\partial b = 0$. It leads to $\text{Tr} \rho = 1$.

**IV. OCCUPATION NUMBER AND HAWKING TEMPERATURE**

If we consider detectors at asymptotic infinity designed to register particles for the quantum radiation, the number of quanta is obtained by evaluating the occupation number of the excited states [13]

$$N(\eta, \omega) = \sum_{n=0}^{\infty} n|c_n(\eta)|^2, \quad (34)$$

where $c_n(\eta)$ is the probability amplitude in the $n$-th excited state (31). From the spectrum of the occupation number (34), the radiation of the collapsing shell to form a singular black hole at early times turned out to be non-thermal [13–15, 17, 19–22]; however, the spectrum interestingly resembles the thermal Hawking distribution when the shell approaches its own horizon.

Likewise, the spectrum of the occupation number in the Hayward black hole can also be
calculated from Eqs. (31) and (34) as

\[
N(\eta_f, \omega_f) = \frac{1}{4} \omega_f \zeta^2 \left[ \left( 1 - \frac{1}{\omega_f \zeta^2} \right)^2 + \frac{1}{\omega_f^2 \zeta^2} \left( \frac{d\zeta}{d\eta} \right)^2 \right]_{\omega_0 = \sqrt{1 - H \eta_f \omega_f}}. \tag{35}
\]

By using Eq. (22), a frequency for the original time \( t \) can be defined as \( \Omega(t) = (d\eta/dt) \omega(\eta) = f(R) \omega(\eta) \), so the occupation number (35) can be written in terms of \( t_f \) and \( \Omega_f \) as [19]

\[
N(t_f, \Omega_f) = \frac{1}{4} e^{\frac{1}{2} H t_f} (\xi^2 + \chi^2) \left[ \left( 1 - \frac{1}{e^{\frac{1}{2} H t_f} (\xi^2 + \chi^2)} \right)^2 + \left( \frac{H e^{-H t_f} \xi \dot{\xi} + \chi \dot{\chi}}{\Omega_f \xi^2 + \chi^2} \right)^2 \right], \tag{36}
\]

where \( \Omega_f \approx e^{-H t_f} \omega_f \) with \( \dot{\xi} = d\xi/d\tilde{\eta} \) and \( \dot{\chi} = d\chi/d\tilde{\eta} \).

On the other hand, in order to study the limiting case we rewrite the occupation number (36) in terms of \( x \) and \( y \) for convenience as

\[
N(x, y) = \frac{x}{4y} (\xi^2 + \chi^2) \left( 1 - \frac{1}{x/y} (\xi^2 + \chi^2) \right)^2 + \frac{1}{y^4} \left\{ x(x h_1 + \chi h_2) \right\}^2, \tag{37}
\]

where

\[
h_1(x, y) = e^{-H t_f} \dot{\xi} = -\frac{\pi y^2}{4} \left\{ Y_0(x) J_0(y) - J_0(x) Y_0(y) \right\}, \tag{38}
\]

\[
h_2(x, y) = e^{-H t_f} \dot{\chi} = -\frac{\pi y^2}{4} \left\{ Y_1(x) J_0(y) - J_1(x) Y_0(y) \right\}. \tag{39}
\]

FIG. 1. The occupation number (36) is plotted in Fig. (a) in case of a finite time. Fig. (b) also shows the spectrum of Eq. (42) which is the occupation number with the infinite time. Note that \( \Omega_f R_H \) is a dimensionless quantity, and we can set \( R_H = 1 \) for simplicity. In Figs. (a) and (b), the black and the grey curves are for the Hayward black hole with \( \ell = 0.2 \) and \( \ell = 0.3 \), respectively, and the dashed curves are for the Schwarzschild black hole where \( \ell = 0 \).
In the incipient limit, \( x \) and \( y \) are approximately 
\( x \approx \frac{2\Omega f e^{H t_f} / H}{2\Omega f / H} \) and 
\( y \approx \frac{2\Omega f / H}{2\Omega f / H} \), respectively. Using the asymptotic forms of Bessel functions in the limit \( t_f \to \infty \), we get

\[
\frac{x}{y}(\xi^2 + \chi^2) \to \frac{\pi y^2}{2} \{J_1^2(y) + Y_1^2(y)\},
\]

\[
x(\xi h_1 + \chi h_2) \to -\frac{\pi y^3}{4}\{J_0(y)J_1(y) + Y_0(y)Y_1(y)\}.
\]

From Eqs. (40) and (41), the occupation number (37) for \( t_f \to \infty \) is obtained as

\[
N_\infty(\Omega_f) = \frac{\pi}{8} \left[ \frac{y}{J_1^2(y) + Y_1^2(y)} \left( J_1^2(y) + Y_1^2(y) - \frac{2}{\pi y} \right)^2 + \{J_0(y)J_1(y) + Y_0(y)Y_1(y)\}^2 \right].
\]

The spectrum of the occupation number in terms of \( \Omega_f \) for a finite time and the infinite time are shown in Fig. 1(a) and Fig. 1(b). The radiation at early times in Fig. 1(a) turns out to be non-thermal, however, the spectrum in Fig. 1(b) is very close to the thermal Hawking distribution if the shell approaches its own horizon. In particular, in Fig. 1(a), we can find that the number of quanta at a low frequency decreases when \( \ell \) is getting larger. Extremely, if \( \ell \to \ell_{\text{max}} = 4M/(3\sqrt{3}) \), the occupation number (36) eventually vanishes because 
\( H = (R_H^2 - 3\ell^2)/R_H^3 \) approaches zero with \( R_H = 1 \). This extremal limit looks similar to the fact that the Hawking radiation vanishes for extremal black holes.

Now, we can read off a temperature in a low frequency region. In the low frequency \( \Omega \ll 1 \), Eq. (42) is approximated as

\[
N_\infty(\Omega_f) \approx \frac{H}{4\pi\Omega_f}
\]

and the Planckian distribution can also be written as

\[
N_{\text{Planck}}(\Omega) = \frac{1}{e^\Omega - 1} \approx \frac{T}{\Omega}
\]

Comparing Eq. (43) and Eq. (44), we can obtain the temperature

\[
T = \frac{H}{4\pi}
\]

which is the same as the Hawking temperature of the Hayward Black hole.

V. CONCLUSION

We studied the collapsing shell to form the Hayward black hole and investigated the quantum radiation. Using the Israel formulation, we obtained the mass relation between
the energy density of the shell and the mass of the Hayward black hole. The main difference from the Schwarzschild black hole is that the energy density in the exterior region of the shell as well as the energy density on the shell contributes to the source of the Hayward black hole. Next, we investigated the quantum radiation from the shell by employing the functional Schrödinger equation which takes the form of the harmonic oscillator with time-dependent frequency. From the vacuum state defined by the coherent states, the density matrix, the probability current, and the occupation number were exactly calculated. By using these quantities, we showed that the process of the radiation is unitary during the collapsing. The spectrum of the radiation does not coincide with the thermal Hawking radiation at early times; however, it is very close to the thermal Hawking distribution when the shell approaches the horizon. For the infinite time, the temperature of the radiation could be estimated in the limit of the low frequency and it turned out to be the Hawking temperature of the Hayward black hole. In addition, the number of quanta at a low frequency decreases for a larger value of \( \ell \). In the extremal limit of \( \ell \to \ell_{\text{max}} \), the occupation number (36) eventually vanishes, which is reminiscent of the extremal limit of black holes where the Hawking radiation vanishes.

Finally, we comment on the incipient limit \( R \to R_H \) employed in our paper. In the incipient limit, we would get the simplified analytic results such as Eqs. (18) and (27), so we could easily show that the whole process of the collapsing must be unitary because \( \text{Tr} \rho = 1 \) regardless of \( \eta \) in Eq. (33). However, the incipient limit corresponds to the moment when the shell approaches the horizon, that is, almost the final stage of the collapsing. If the incipient limit is released, then the vacuum state (25) will be modified accordingly. Although we expect that this modification of the vacuum state will not affect the final results in our paper, but it deserves further study.

**ACKNOWLEDGMENTS**

We would like to thank Myungseok Eune and Yongwan Gim for exciting discussions. This work was supported by the National Research Foundation of Korea (NRF) grant funded by
the Korea government (MSIP) (2017R1A2B2006159).

[1] J. Bardeen, *Proceedings of GR5, Tiflis, USSR* (1968) 174.

[2] A. Borde, *Regular black holes and topology change*, Phys. Rev. D55 (1997) 7615–7617, [gr-qc/9612057].

[3] E. Ayon-Beato and A. Garcia, *Nonsingular charged black hole solution for nonlinear source*, Gen. Rel. Grav. 31 (1999) 629–633, [gr-qc/9911084].

[4] S. A. Hayward, *Formation and evaporation of regular black holes*, Phys. Rev. Lett. 96 (2006) 031103, [gr-qc/0506126].

[5] C. Bambi and L. Modesto, *Rotating regular black holes*, Phys. Lett. B721 (2013) 329–334, [1302.6075].

[6] L. Balart and E. C. Vagenas, *Regular black holes with a nonlinear electrodynamics source*, Phys. Rev. D90 (2014) 124045, [1408.0306].

[7] M. Halilsoy, A. Ovgun and S. H. Mazharimousavi, *Thin-shell wormholes from the regular Hayward black hole*, Eur. Phys. J. C74 (2014) 2796, [1312.6665].

[8] G. Abbas and U. Sabiullah, *Geodesic Study of Regular Hayward Black Hole*, Astrophys. Space Sci. 352 (2014) 769–774, [1406.0840].

[9] M. Amir and S. G. Ghosh, *Rotating Haywards regular black hole as particle accelerator*, JHEP 07 (2015) 015, [1503.08553].

[10] B. Pourhassan, M. Faizal and U. Debnath, *Effects of Thermal Fluctuations on the Thermodynamics of Modified Hayward Black Hole*, Eur. Phys. J. C76 (2016) 145, [1603.01457].

[11] T. Chiba and M. Kimura, *A note on geodesics in the Hayward metric*, PTEP 2017 (2017) 043E01, [1701.04910].

[12] S. H. Mehdipour and M. H. Ahmadi, *Black Hole Remnants in Hayward Solutions and Noncommutative Effects*, Nucl. Phys. B926 (2018) 49–69, [1604.08584].

[13] T. Vachaspati, D. Stojkovic and L. M. Krauss, *Observation of incipient black holes and the information loss problem*, Phys. Rev. D76 (2007) 024005, [gr-qc/0609024].

[14] T. Vachaspati and D. Stojkovic, *Quantum radiation from quantum gravitational collapse*, Phys. Lett. B663 (2008) 107–110, [gr-qc/0701096].
[15] E. Greenwood and D. Stojkovic, *Hawking radiation as seen by an infalling observer*, JHEP 09 (2009) 058, [0806.0628].

[16] E. Greenwood, E. Halstead and P. Hao, *Classical and Quantum Equations of Motion for a BTZ Black String in AdS Space*, JHEP 02 (2010) 044, [0912.1860].

[17] E. Greenwood, *Hawking Radiation from a Reisner-Nordstrom Domain Wall*, JCAP 1001 (2010) 002, [0910.0024].

[18] E. Greenwood, D. I. Podolsky and G. D. Starkman, *Pre-Hawking Radiation from a Collapsing Shell*, JCAP 1111 (2011) 024, [1011.2219].

[19] M. Kolopanis and T. Vachaspati, *Quantum Excitations in Time-Dependent Backgrounds*, Phys. Rev. D87 (2013) 085041, [1302.1449].

[20] A. Saini and D. Stojkovic, *Radiation from a collapsing object is manifestly unitary*, Phys. Rev. Lett. 114 (2015) 111301, [1503.01487].

[21] A. Saini and D. Stojkovic, *Hawking-like radiation and the density matrix for an infalling observer during gravitational collapse*, Phys. Rev. D94 (2016) 064028, [1609.06584].

[22] A. Saini and D. Stojkovic, *Gravitational collapse and Hawking-like radiation of a shell in AdS spacetime*, Phys. Rev. D97 (2018) 025020, [1711.08182].

[23] A. Das and N. Banerjee, *Unitarity in ReissnerNordstrom background: striding away from information loss*, Eur. Phys. J. C79 (2019) 475, [1902.03378].

[24] W. Israel, *Singular hypersurfaces and thin shells in general relativity*, Nuovo Cim. B44S10 (1966) 1.

[25] J. Ipser and P. Sikivie, *The Gravitationally Repulsive Domain Wall*, Phys. Rev. D30 (1984) 712.

[26] C. A. Lopez, *Dynamics of Charged Bubbles in General Relativity and Models of Particles*, Phys. Rev. D38 (1988) 3662–3666.

[27] H. Goldstein, *Classical Mechanics*. Addison-Wesley, 1980.

[28] C. M. A. Dantas, I. A. Pedrosa and B. Baseia, *Harmonic oscillator with time-dependent mass and frequency and a perturbative potential*, Phys. Rev. A45 (1992) 1320–1324.