Curve Classes on Calabi–Yau Hypersurfaces in Toric Varieties

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Abstract

We prove the Integral Hodge Conjecture for curve classes on smooth varieties of dimension at least three with nef anticanonical divisor constructed either as an ample hypersurface in a smooth toric Fano variety or as a double cover of a smooth toric Fano variety. In fact, using results of Casagrande and the toric MMP, we prove that in each case, $H_2(X, \mathbb{Z})$ is generated by classes of rational curves.

1 Introduction

On a smooth complex projective variety of dimension $n$, there is a Hodge decomposition of $H^k(X, \mathbb{C})$ into subspaces $H^{p,q}(X, \mathbb{C})$, with $p + q = k$. The integral Hodge classes $H^{k,k}(X, \mathbb{Z})$ are the classes in $H^{2k}(X, \mathbb{Z})$ which map to $H^{k,k}(X, \mathbb{C})$ under the natural map

$$H^{2k}(X, \mathbb{Z}) \rightarrow H^{2k}(X, \mathbb{C}),$$

and the class of any algebraic subvariety is an integral Hodge class. The Integral Hodge Conjecture asks whether the classes of algebraic subvarieties generate the integral Hodge Classes as a group.

A basic result in this direction is the Lefschetz (1,1)-theorem. This theorem states that the Integral Hodge Conjecture holds for codimension 1 classes. By the Hard Lefschetz theorem, this also implies that Hodge Conjecture holds for degree $2n - 2$ classes, i.e., classes of algebraic curves generate $H^{n-1,n-1}(X, \mathbb{Q})$ as a vector space. However the Integral Hodge Conjecture might still fail for $H^{n-1,n-1}(X, \mathbb{Z})$.

Much work has been done on constructing counterexamples to the Integral Hodge Conjecture for degree $2n - 2$ classes, which we will also call the Integral Hodge Conjecture for curves. There are two ways in which the Integral Hodge conjecture can fail and there are counterexamples illustrating both. The first way is through torsion classes in $H^{k,k}(X, \mathbb{Z})$. Any torsion class is an integral Hodge class, and one can find counterexamples to the Integral Hodge Conjecture for curves by finding a torsion class in $H^{k,k}(X, \mathbb{Z})$ which is not algebraic. In fact, the first counterexample to the Integral Hodge conjecture was of this form. In [AH62], Atiyah and Hirzebruch construct a projective variety with a degree 4 torsion class that is nonalgebraic.

The Integral Hodge Conjecture for curves can even fail modulo torsion. In [BCC92], Kollár constructs counterexamples on projective hypersurfaces in $\mathbb{P}^4$ of high degree, on which there is a nontorsion, nonalgebraic class in $H^{n-1,n-1}(X, \mathbb{Z})$. 

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On the other hand, by imposing restrictions on the geometry of the variety $X$, many positive results in the direction of the Integral Hodge Conjecture have also been found. In [Voi06] Voisin proves that for a complex projective threefold $X$ that is either uniruled or satisfies $K_X = 0$ and $H^2(X, O_X) = 0$ the Integral Hodge Conjecture for curves holds. In [Tot19] Totaro shows more generally that it holds for all threefolds of Kodaira dimension 0 with $H^0(X, O(K_X)) \neq 0$. In [BO18] Benoist and Ottem construct a threefold $X$ such that $2K_X = 0$, and $X$ does not satisfy the Integral Hodge Conjecture, which shows that there is an important difference between assuming Kodaira dimension 0 and assuming that the canonical divisor is trivial. In [Voi06], Voisin also raises the question of whether the Integral Hodge Conjecture for curves holds for rationally connected varieties.

One reason for the interest in the Integral Hodge Conjecture for curves is to construct stable birational invariants of smooth projective varieties. Voisin introduced the group $Z^{2n-2} = H^{n-1,n-1}(X, \mathbb{Z})/H^{n-1,n-1}(X, \mathbb{Z})_{alg}$, which is a stable birational invariant, and is the trivial group for rational varieties. There are also other cases where the Integral Hodge Conjecture for varieties with trivial canonical divisor can give give answers to other geometric questions. For instance in [Voi17] Voisin relates the question of stable rationality of a cubic threefold to the question of whether a particular class in the intermediate Jacobian, an abelian variety of dimension 5, is algebraic.

In this paper, we will prove that the Integral Hodge Conjecture for curves holds on Calabi–Yau varieties resulting from two common constructions. Both constructions begin with a smooth toric Fano variety and produce a Calabi–Yau, or more generally a variety $X$ with nef anticanonical divisor $-K_X$, of any dimension $n \geq 3$. The main result is:

**Theorem 1.1.** Let $X \subset Y$ be a smooth ample hypersurface in a smooth toric variety $Y$, with $\dim X$ at least 3. Assume furthermore that $X$ has nef anticanonical divisor $-K_X$. Then the Integral Hodge Conjecture for curves holds for $X$. In fact $H_2(X, \mathbb{Z})$ is generated by classes of rational curves.

Starting from a toric Fano variety, we can also construct a Calabi–Yau variety as a double cover ramified over an ample divisor. We also prove that the Integral Hodge Conjecture for curves holds for Calabi–Yau varieties of dimension at least three constructed in this way:

**Theorem 1.2.** Let $X$ be a Calabi–Yau variety of dimension at least three that is a double cover of a smooth toric Fano variety. Then the Integral Hodge Conjecture for curves holds for $X$. In fact $H_2(X, \mathbb{Z})$ is generated by classes of rational curves.

These constructions are a source of the majority of known examples of Calabi–Yau varieties. Even in the case of threefolds, there are a large number (473,800,776) of
families of Calabi-Yau varieties constructed as anticanonical hypersurfaces in toric Fano varieties. These manifolds originally received a lot of attention due to their relation with Mirror Symmetry, as in [Bat94].

The topology of \(X\) is well-known in both cases, thanks to the Lefschetz hyperplane theorem in the case of Theorem 1.1 and \([LS89]\) in the case of Theorem 1.2. In fact, since \(H^{2,0}(X, \mathbb{C}) = 0\), the Hodge classes on \(X\) will be precisely \(H^2(X, \mathbb{Z})\), and this is isomorphic to \(H^2(Y, \mathbb{Z})\), where \(Y\) is the toric variety we begin with.

The main challenge in proving the Integral Hodge Conjecture for curves is finding algebraic representatives of generators of the group \(H^2(X, \mathbb{Z})\). In [Cas03], Casagrande proves that for the ambient toric variety \(Y\), the group \(H^2(Y, \mathbb{Z})\) is generated by the classes of so-called contractible curves, which are algebraic. So we will prove that \(H^2(X, \mathbb{Z})\) also contains algebraic representatives of classes of contractible curves. The proof of Theorem 1.1 is inspired by an argument given by Kollár in an appendix to [Bor91], where he proves that for an anticanonical hypersurface \(X\) in a Fano variety \(Y\), the cones of effective curves \(\overline{NE}(X)\) and \(\overline{NE}(Y)\) coincide.

The structure of the paper is as follows: In Section 2 we will recall the main definitions and results used in this paper, in particular the results from [Cas03]. In Section 3 we will give a proof of Theorem 1.1. Finally in Section 4 we prove Theorem 1.2.

Acknowledgements

I would like to thank my advisor John Christian Ottem for suggesting this question, and for his guidance in writing this paper and patience throughout the writing process.

2 Preliminaries

Let \(X\) be a smooth projective variety over \(\mathbb{C}\) of dimension \(n\). The integral Hodge classes \(H^{k-1,k-1}(X, \mathbb{Z})\) are the classes in \(H^{2k}(X, \mathbb{Z})\) that map to the subspace \(H^{k,k}(X, \mathbb{C})\) of the Hodge decomposition of \(H^{2k}(X, \mathbb{C})\) under the natural map

\[
H^{2k}(X, \mathbb{Z}) \rightarrow H^{2k}(X, \mathbb{C}).
\]

The Integral Hodge Conjecture asks if any integral Hodge class is a linear combination of classes of algebraic varieties, in other words \(H^{2k}(X, \mathbb{Z})_{\text{alg}} = H^{k-1,k-1}(X, \mathbb{Z})\).

We will focus on the Integral Hodge Conjecture for curves, which is the statement that \(H^{n-1,n-1}(X, \mathbb{Z})\) is generated by the classes of algebraic curves contained in \(X\). Recall that on a smooth toric, or more generally rational variety \(Y\), \(H^i(Y, \mathcal{O}_Y) = 0\) for \(i > 0\), and \(H_2(Y, \mathbb{Z})\) is torsion free. As a consequence,

\[
H^{n-1,n-1}(Y, \mathbb{Z}) = H_2(Y, \mathbb{Z}).
\]
By the Lefschetz Hyperplane theorem, the same is true for an ample hypersurface $X \subset Y$ of dimension at least 3.

The varieties we study in this paper are constructed starting from toric varieties. For a general introduction to toric varieties, one can see, e.g., [CLS11]. We will recall some facts about toric varieties that we will use throughout.

A toric variety $Y$ corresponds to a fan $\Sigma$ in a real vector space $\mathbb{R}^n$, and many geometric properties of $Y$ are encoded by combinatorial properties of $\Sigma$. On a smooth toric variety $Y$ defined by a fan $\Sigma$, the group generated by curve classes up to numerical equivalence is isomorphic to the relations between primitive generators $x_i$ of the rays of $\Sigma$. The relations corresponding to torus invariant curves are called wall relations. Furthermore, the intersection number $D_i \cdot C$ between a torus-invariant divisor $D_i$, corresponding to the ray spanned by $x_i$, and a curve $C$ corresponding to a wall relation $a_1x_1 + \cdots + a_{n+1}x_{n+1} = 0$ is the coefficient $a_i$ (thus 0 if the generator of the ray does not occur in the wall relation). In terms of exact sequences we have the following:

**Proposition 2.1** ([CLS11, Proposition 6.4.1]). Let $\Sigma$ be a simplicial fan in $\mathbb{R}^n$ with convex support of full dimension. Then there are dual exact sequences:

$$0 \to M_\mathbb{R} \xrightarrow{\alpha} \mathbb{R}^{\Sigma(1)} \xrightarrow{\beta} \text{Pic}(Y_\Sigma) \otimes \mathbb{R} \to 0$$

$$0 \to N_1(Y_\Sigma) \xrightarrow{\beta^*} \mathbb{R}^{\Sigma(1)} \xrightarrow{\alpha^*} M_\mathbb{R} \to 0$$

where

$$\alpha^*(e_\rho) = u_\rho$$

$$\beta^*([C]) = (D_\rho \cdot C)_{\rho \in \Sigma(1)}$$

Where $\Sigma(1)$ are the rays of $\Sigma$, $M_\mathbb{R}$ is the dual space to $N_\mathbb{R}$, $e_\rho$ the standard basis vectors of $\mathbb{R}^{\Sigma(1)}$, $u_\rho$ the primitive generator of the ray $\rho \in \Sigma(1)$ and $C \subset Y_\Sigma$ a complete irreducible curve.

We also recall the definition of the Neron-Severi space $N_1(Y)$, the space of 1-cycles modulo numerical equivalence:

**Definition 2.2.** Let $Y$ be a smooth variety. We define the vector space

$$N_1(Y) := \{1\text{-cycles in } Y\}/ \equiv_{\text{num}} \otimes \mathbb{R}$$

In the case where $Y$ is a smooth toric variety, $N_1(Y)$ is isomorphic to $H_2(Y, \mathbb{Z}) \otimes \mathbb{R}$ and $H_2(Y, \mathbb{Z})$ embeds into $N_1(Y)$. Furthermore on a smooth toric variety, $H_2(Y, \mathbb{Z})$ is generated by the classes of torus-invariant curves.

Since the anticanonical divisor $-K_Y$ is the sum of all the torus-invariant divisors, the intersection number $-K_Y \cdot C$ is the sum of all coefficients in the wall relation.
2.1 Contractible Classes of a Toric Variety

Because $H_2(X, \mathbb{Z})$ embeds into $N_1(X, \mathbb{R})$, we can use tools from the Minimal Model Program to study the question of the Integral Hodge Conjecture, in particular the results in [Cas03].

**Definition 2.3** ([Cas03, Definition 2.3]). A primitive curve class $\gamma \in H_2(Y, \mathbb{Z})$, where $Y$ is a toric variety, is called contractible if there exists an equivariant toric morphism $\pi: Y \to Y'$ with connected fibers such that for every irreducible curve $C \subset X$,

$$\pi(C) = \{pt\} \iff [C] \in \mathbb{Q}_{\geq 0} \gamma$$

Recall that a curve class $\gamma \in H_2(Y, \mathbb{Z})$ is primitive if it is not a positive integer multiple of any other class.

The structure of a contraction of a contractible class is described by the following result:

**Proposition 2.4** ([Cas03, Corollary 2.3]). Let $Y$ be a smooth complete toric variety of dimension $n$, $\gamma \in \text{NE}(Y)$ a contractible class and $\pi: Y \to Z$ the associated contraction.

Suppose that $\gamma$ is numerically effective, so its wall relation is:

$$x_1 + \cdots + x_h = 0.$$  

Then $Z$ is smooth of dimension $n - h + 1$ and $\pi: Y \to Z$ is a $\mathbb{P}^{h-1}$-bundle.

Suppose $\gamma$ is not numerically effective, so its wall relation is:

$$x_1 + \cdots + x_h - a_1 y_1 - \cdots - a_k y_k, = 0 \quad k > 0$$

Then $\pi$ is birational, with exceptional loci $E \subset Y$, $B \subset Z$, $\dim E = n - k$, $\dim B = n - h - k + 1$ and $\pi|_E: E \to B$ is a $\mathbb{P}^{h-1}$-bundle.

By $\mathbb{P}^{h-1}$-bundle we mean a bundle that is locally trivial in the Zariski topology. In particular, there is a vector bundle $\mathcal{E}$ on $B$ such that $E = \mathbb{P}(\mathcal{E})$.

**Remark 2.5.** If $\gamma$ is numerically effective, i.e., $\gamma \cdot D \geq 0$ for all divisors $D$, then from how intersection numbers can be computed from the wall relation corresponding to $\gamma$, the wall relation can not have any negative coefficients. The positive coefficients in a wall relation corresponding to a contractible curve are all equal to 1. It must therefore have the form described in the theorem. This happens precisely when curves of class $\gamma$ move to cover the entire toric variety.

In contrast to contractions of extremal rays, if $\pi: Y \to Y'$ is a contraction of a contractible class, the target variety $Y'$ is not necessarily projective. In fact, $Y'$ is projective if and only if the contraction is a contraction of an extremal ray. However, no difficulties arise from this in this paper, thanks to the following:
Remark 2.6. By looking at the standard toric affine charts, one can see that an irreducible torus-invariant subset of a smooth projective toric variety will also be a smooth projective toric variety. It therefore follows from Proposition 2.4 that if $\pi|_E : E \to B$ is the restriction of a contraction to its exceptional locus, $\pi|_E$ will be a map between smooth projective toric varieties.

The reason we wish to consider contractible classes, as opposed to only extremal ones is the following result by Casagrande, which says that these rays generate $H_2(Y,\mathbb{Z})$.

**Theorem 2.7** ([Cas03, Theorem 4.1]). Let $Y$ be a smooth projective toric variety. Then for every $\eta \in H_2(Y,\mathbb{Z}) \cap \overline{NE}(Y)$ there is a decomposition:

$$\eta = m_1\gamma_1 + \cdots + m_r\gamma_r$$

with $\gamma_i$ contractible and $m_i \in \mathbb{Z}_{>0}$ for all $i = 1,\ldots,r$.

The following immediate corollary will be what we need to prove the Integral Hodge Conjecture for curves.

**Corollary 2.8.** Let $Y$ be a smooth projective toric variety. Then the $H_2(Y,\mathbb{Z})$ is spanned over $\mathbb{Z}$ by contractible curves.

**Proof.** Since $Y$ is projective, the cone of effective curves $\overline{NE}(Y)$ has nonempty interior. Use this to find $v \in \overline{NE}(Y) \cap H^{n-1,n-1}(Y,\mathbb{Z})$ and a $\mathbb{Z}$-basis $e_1,\ldots,e_k$ of $H_2(Y,\mathbb{Z})$ such that $v + e_i$ is in the interior of $\overline{NE}(Y)$ for all $i$. By Theorem 2.7 we can write

$$v = m_1^v\gamma_1^v + \cdots + m_r^v\gamma_r^v$$

$$v + e_i = m_1^v\gamma_1^v + \cdots + m_r^v\gamma_r^v + m_j^i\gamma_j^i$$

with $\gamma^v,\gamma^i$ contractible classes, and $m_j^i, m_j^i \in \mathbb{Z}_{>0}$. So for each $i$ we can write

$$e_i = m_1^i\gamma_1^i + \cdots + m_r^i\gamma_r^i - m_1^v\gamma_1^i + \cdots + m_r^v\gamma_r^v,$$

so a $\mathbb{Z}$-basis for the $H_2(Y,\mathbb{Z})$ is contained in the integral span of contractible curves. $\blacksquare$

### 2.2 Ample Vector Bundles

As the contractions of contractible classes on smooth toric varieties define projective bundles, we will need to study projective bundles in more detail. Because of the ampleness hypothesis in Theorem 1.1 ample vector bundles in particular will be important. In this section we will collect some known results about ample projective bundles that will be useful later. Throughout the paper, we will use the convention that a projective bundle $\mathbb{P}(\mathcal{E})$ parametrizes one-dimensional quotients $\mathcal{E}$. 
Definition 2.9. Let $\mathcal{E}$ be a vector bundle on a variety $X$. Then $\mathcal{E}$ is ample if the line bundle $\mathcal{O}_\mathbb{P}(\mathcal{E})(1)$ is ample on the projective bundle $\mathbb{P}(\mathcal{E})$.

We will need some positivity properties for the Chern classes of ample vector bundles, which we include here for reference.

If $\mathcal{E}$ is a vector bundle, write $c(\mathcal{E})$ for the Chern polynomial of the vector bundle. For any partition of $n$, i.e., any decreasing sequence $\lambda$ of non-negative integers

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_e \geq 0$$

with $\sum \lambda_i = n$, write $\Delta_\lambda = \Delta_{\lambda_1 \lambda_2 \cdots \lambda_e}(c(\mathcal{E}))$ for the Schur polynomial

$$\Delta_{\lambda_1 \lambda_2 \cdots \lambda_e}(c(\mathcal{E})) = \begin{vmatrix}
  c_{\lambda_1}(\mathcal{E}) & c_{\lambda_1+1}(\mathcal{E}) & \cdots & c_{\lambda_1+n-1}(\mathcal{E}) \\
  c_{\lambda_2-1}(\mathcal{E}) & c_{\lambda_2}(\mathcal{E}) & \cdots & c_{\lambda_2+n-2}(\mathcal{E}) \\
  \vdots & \vdots & \ddots & \vdots \\
  c_{\lambda_n-n+1}(\mathcal{E}) & c_{\lambda_n-n+2}(\mathcal{E}) & \cdots & c_{\lambda_e}(\mathcal{E})
\end{vmatrix}$$

These polynomials form an additive basis of the space of polynomials in the Chern classes of $\mathcal{E}$ of weighted degree $n$.

We say a that a polynomial $P$ of weighted degree $n$ is *numerically positive* if

$$\int_X P(c_1(\mathcal{E}), c_2(\mathcal{E}), \ldots, c_e(\mathcal{E}))$$

is strictly positive for every projective variety $X$ of dimension $n$ and any ample vector bundle $\mathcal{E}$.

Using Schur polynomials we can see if a polynomial in the Chern classes of an ample vector bundle is numerically positive, by [FL83, Theorem 1].

Theorem 2.10 ([FL83, Theorem 1]). Let $P$ be a polynomial in the Chern classes of a vector bundle $\mathcal{E}$. Write

$$P = \sum a_\lambda \Delta_\lambda(c(\mathcal{E}))$$

then $P$ is numerically positive for ample vector bundles if and only if $P$ is non-zero and $a_\lambda \geq 0$ for all $\lambda$.

In particular [Theorem 2.10] implies that the Chern classes of any ample vector bundle are numerically positive, a result first proven in [BG71].

Schur polynomials also let us compute the pushforward of certain Chern classes on a Grassmannian bundle $\pi: Gr(d, \mathcal{E}) \to B$. However, for this it is more convenient to work with Schur polynomials in the Segre classes $s_i(\mathcal{E})$ of a vector bundle $\mathcal{E}$. We will write these $\Delta_\lambda(s(\mathcal{E}))$. There is the following relation between Schur polynomials in the Chern- and Segre classes of a vector bundle:
Lemma 2.11 ([Ful98, Lemma 14.5.1]). Let $\lambda, \mu$ be conjugate partitions. Then
\[ \Delta_\lambda(c(\mathcal{E})) = \Delta_\mu(s(\mathcal{E}^*)) \]
where the conjugate partition $\mu$ of a partition $\lambda$ of $n$ is defined as
\[ \mu_j = \# \{ \lambda_i \mid \lambda_i \geq j \} \]
and $\#$ denotes the cardinality of the set.

In [JLP81] we can find a formula describing the pushforwards of Schur polynomials in the Segre classes of $\mathcal{S}^*$, where $\mathcal{S}$ the universal subbundle on the relative Grassmannian and $\mathcal{S}^*$ its dual. For us, the useful part will be [JLP81, Corollary 1], which we adapt to our notation:

Proposition 2.12 ([JLP81, Corollary 1]). Let $\mathcal{E}$ be a vector bundle on a variety $X$. Let $r$ be the rank of $\mathcal{E}$, and let $\pi: Gr(d, \mathcal{E}) \to X$ be the relative Grassmannian of $d$-dimensional subspaces in the fibers of $\mathcal{E}$. Let
\[ 0 \to \mathcal{S} \to \pi^*(\mathcal{E}^*) \to \mathcal{Q} \to 0 \]
be the tautological exact sequence. Then for any integers $\lambda_1, \ldots, \lambda_d$:
\[ \pi_* (\Delta_{\lambda_1, \ldots, \lambda_d}(s(\mathcal{S}^*))) = \Delta_{\mu_1, \ldots, \mu_d}(s(\mathcal{E}^*)) \]
where $\mu_i = \lambda_i - r + d$.

3 Hypersurfaces in toric Fano varieties

In this section, we will prove Theorem 1.1. For an ample hypersurface $X$ in a smooth projective Fano variety $Y$ the topological side of the Integral Hodge Conjecture for curves is well-known. The Lefschetz hyperplane theorem gives an isomorphism $i_*: H_2(X, \mathbb{Z}) \to H_2(Y, \mathbb{Z})$. Furthermore, we know from Corollary 2.8 that $H_2(Y, \mathbb{Z})$ is generated by the classes of contractible curves. To prove the Integral Hodge Conjecture for curves for $X \subset Y$, our strategy is to prove that the image of $i_*: H_2(X, \mathbb{Z}) \to H_2(Y, \mathbb{Z})$ contains all contractible classes of $H_2(Y, \mathbb{Z})$. The Integral Hodge conjecture for curves on $X$ then follows from Corollary 2.8.

Note that some bound on the positivity of the divisor $X$ is necessary. Consider a hypersurface $X \subset \mathbb{P}^n$ of very high degree. Then $X$ contains no lines, so the image $i_*(H_2(X, \mathbb{Z}))$ will not contain a primitive generator of $H_2(\mathbb{P}^n, \mathbb{Z})$. In fact, Kollár showed in [BCC92] that for certain such hypersurfaces, the Integral Hodge Conjecture for curves fails. In our situation, this bound is that as a variety $X$ should have nef anticanonical divisor.

We begin by studying the exceptional locus of a contraction $\pi: Y \to Z$ of a contractible class.
**Proposition 3.1.** Let $Y$ be a toric variety of dimension $n$ and let $\pi: Y \rightarrow Z$ be the contraction of a contractible class $[C]$. Then restricted to the exceptional locus $\pi|_E: E \rightarrow Z$, $\pi|_E$ gives $E$ the structure of a projective bundle $\mathbb{P}(\mathcal{E})$ over a smooth projective toric variety $B$. Let

$$x_1 + \cdots + x_h - a_1y_1 - \cdots - a_ky_k = 0$$

be the wall relation corresponding to $C$. Then the rank of $\mathcal{E}$ is $h$, the dimension of $E$ is $n - k$, the dimension of $B$ is $n - h - k + 1$.

**Proof.** Combine Proposition 2.4 and Remark 2.6. ■

Since $X \subset Y$ is an ample divisor, it restricts on $E \subset Y$ to an ample divisor. The assumption that $X$ has nef anticanonical divisor, relating the degree of $X$ on a fiber of $E \rightarrow B$ with $\dim E$ and $\dim B$. For contractions of extremal rays in not necessarily toric varieties, there is an analogous result in [Wiś91, Theorem 1.1].

**Proposition 3.2.** Let $Y$ be a smooth Fano toric variety of dimension $n \geq 4$ and let $\pi: Y \rightarrow Z$ be a contraction of a contractible class $[C]$ with exceptional locus $E$. Let $X$ be a smooth divisor of $Y$ such that $(X + K_Y) \cdot [C] \leq 0$. Then the positive-dimensional fibers $F$ of $\pi$ are projective spaces of the same dimension, and

$$\dim F + \dim E \geq \dim Y + \deg X_F - 1$$

(1)

**Proof.** Let

$$x_1 + \cdots + x_h - a_1y_1 - \cdots - a_ky_k = 0$$

be the wall relation corresponding $C$, where the $x_i$ and $y_j$ are the generators of rays in the fan of $Y$. The $a_i$ are positive integers and $h + k \leq n + 1$. The degree $\deg X_F$ is the intersection number $X \cdot C$. Furthermore, by assumption this number is less that $-K_Y \cdot C$, which from toric intersection theory, we know is equal to $h - \sum a_i$.

Inserting this, and the numbers in Proposition 3.1 into (1) we see that (1) follows from the inequality:

$$h - 1 + n - k \geq n + h - \sum a_i - 1$$

which holds, since the $a_i$ are positive integers. ■

Knowing this, we make the following simple observation, which will be useful later:

**Corollary 3.3.** With assumptions and notation as in Proposition 3.2

$$\deg X_F \leq \dim F + 1,$$

where $X_F$ is the restriction of $X$ to $F$. 9
Proof. Using that \( \dim Y \geq \dim E \), we see that (1) implies that
\[
\dim F \geq \deg X_F - 1.
\]

Finally, we will use the following well known result about projective hypersurfaces, which can be found in, e.g., [EH16, Chapter 6].

**Proposition 3.4.** Let \( X \subset \mathbb{P}^k \) be a smooth projective hypersurface of degree \( d \). If \( d \leq 2k - 3 \), then \( X \) contains lines.

If \( \pi : Y \to Z \) is the contraction of a contractible class, then to find a contractible curve contained in \( X \), we must find a line in a positive-dimensional fiber of \( \pi \) that is contained in \( X \). In the case where the fibers of \( \pi \) have dimension at most one, we will apply the following lemma by Kollár from an appendix in [Bor91]:

**Lemma 3.5.** Let \( V \) be a normal projective variety.

(i) Let \( g : U \to V \) be a \( \mathbb{P}^1 \)-bundle. Let \( Y \subset U \) be a subvariety such that \( g_{|Y} : Y \to V \) is finite of degree 1. If \( Y \) is ample, then \( \dim V \leq 1 \).

(ii) Let \( g : U \to V \) be a conic bundle. Let \( Y \) be a subvariety such that \( g_{|Y} : Y \to V \) is finite of degree \( \leq 2 \). If \( Y \) is ample, then \( \dim V \leq 2 \).

For smooth toric varieties we can consider only projective bundles, and we recall a simplified version of the proof given in [Bor91] here, as it contains many ideas similar to the ones we will use when the fibers have higher dimension.

**Proof.** (1): Let \( g : \mathbb{P}(\mathcal{E}) \to V \) be the projective bundle, where \( \mathcal{E} \) is a vector bundle of rank 2. Assume that \( g_{|Y} \) is finite. Then \( g_{*}(\mathcal{O}(Y)) = \mathcal{E} \otimes \mathcal{L} \) for some line bundle \( \mathcal{L} \), and since \( Y \) is ample, so is \( \mathcal{E} \otimes \mathcal{L} \). Since \( Y \) does not contain any fibers, \( c_2(\mathcal{E} \otimes \mathcal{L}) \) must be zero, since this is the class of points in \( V \) such that \( Y \) contains the fiber over this point. However, since \( \mathcal{E} \otimes \mathcal{L} \) is an ample vector bundle, this is only possible if \( \dim V \leq 1 \) by Theorem 2.10.

(2): If \( g_{|Y} \) has degree 1 the statement follows from (1). If \( g_{|Y} \) has degree 2 let \( g : \mathbb{P}(\mathcal{E}) \to V \) be the projective bundle, where \( \mathcal{E} \) is a vector bundle of rank 2. Assume that \( g_{|Y} \) is finite. Then \( g_{*}(\mathcal{O}(Y)) = \text{Sym}^2(\mathcal{E}) \otimes \mathcal{L} \) for some line bundle \( \mathcal{L} \), and since \( Y \) is ample, so is \( \text{Sym}^2(\mathcal{E}) \otimes \mathcal{L} \). Since \( Y \) does not contain any fibers of \( g \), \( c_3(\text{Sym}^2(\mathcal{E}) \otimes \mathcal{L}) \) must be zero. By Theorem 2.10 this is only possible if \( \dim V \leq 2 \).
Lemma 3.6. Let $X \subset \mathbb{P}(\mathcal{E})$ be a smooth ample hypersurface in a projective bundle $\mathbb{P}(\mathcal{E}) \rightarrow B$, where $\mathcal{E}$ has rank $h \geq 2$. Assume further that $X$ has degree at most $h$ on any fiber, and that
\[d \leq \dim B + 2h - 5.\]
Then $X$ contains a line in a fiber of $p: \mathbb{P}(\mathcal{E}) \rightarrow B$.

Proof. First observe that if $h = 2$ the statement follows from Lemma 3.5, so from now on we will assume that $h \geq 3$.

We will first handle three easy cases:

- If $X$ contains a fiber of $p$, the conclusion is obviously true. So we may assume that the map $X \rightarrow B$ gives a flat family of degree $d$ hypersurfaces of $\mathbb{P}^{h-1}$.
- If $X$ has degree 1 on the fibers, the conclusion also clearly holds.
- If $h \geq 5$, then for any fiber $F \simeq \mathbb{P}^{h-1}$, we can use Corollary 3.3 to see that $X_F$ is a hypersurface in projective space of degree
\[d \leq h \leq 2(h - 1) - 3.\]
which contains lines by Proposition 3.4.

There are now five remaining cases:

(i) $h = 3, d = 2$
(ii) $h = 3, d = 3$
(iii) $h = 4, d = 2$.
(iv) $h = 4, d = 3$.
(v) $h = 4, d = 4$.

Cases (iii) and (iv) are easy to handle. In these cases any fiber $X_F$ will be a quadric or cubic surface, and these contain lines.

In cases (i), (ii), (v), the generic fibers do not contain lines. However, if $X$ is ample and $\dim B$ sufficiently large, some fiber will contain at least one line. We will prove this by computing the fundamental class of the Fano scheme of $X$, and use ampleness to argue this class is nonzero.

Let $\pi: \text{Gr}(2, \mathcal{E}^*) \rightarrow B$ be the relative Grassmannian variety of lines in the fibers of $p: \mathbb{P}(\mathcal{E}) \rightarrow B$, and let
\[0 \rightarrow \mathcal{S} \rightarrow \pi^* (\mathcal{E}^*) \rightarrow \mathcal{Q} \rightarrow 0\]
be the tautological exact sequence. Then $X$ gives a section of $\text{Sym}^d(S^*) \otimes p^*\mathcal{L}$ whose vanishing defines the Fano scheme, where $\mathcal{L}$ is a line bundle on $B$.

More precisely, since we have assumed that $X \to B$ is a flat family, we have an isomorphism of line bundles

$$\mathcal{O}_{\mathbb{P}(\mathcal{E})}(X) \simeq \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d) \otimes p^*\mathcal{L}.$$ 

Pushing this line bundle forward by $p: \mathbb{P}(\mathcal{E}) \to B$ yields the vector bundle $\text{Sym}^d(\mathcal{E}) \otimes \mathcal{L}$, which is ample since $X$ is ample. We wish to show that the top Chern class $c_{d+1}(\text{Sym}^d(S^*) \otimes \mathcal{L})$ is positive. Since this is the fundamental class of the Fano scheme, this will prove that $X$ contains a line in a fiber of the projective bundle. In fact, we will show that $\pi_*(c_{d+1}(\text{Sym}^d(S^*) \otimes \mathcal{L}))$ is positive, so there are fibers of $\pi|^X : X \to B$ that contain lines.

To simplify the computations, consider the $\mathbb{Q}$-vector bundle $\mathcal{E}' = \mathcal{E} \otimes 1/\sqrt{d}\mathcal{L}$ (See, e.g., [Laz04, Chapter 6.2] for some background material on $\mathbb{Q}$-vector bundles.) Then $\text{Sym}^d(\mathcal{E}') = \text{Sym}^d(\mathcal{E}) \otimes \mathcal{L}$ and since this bundle is ample, so is $\mathcal{E}'$. So without loss of generality we may assume that $\mathbb{P}(\mathcal{E}) \to B$ is the projectivization of an ample vector bundle and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(X) = \mathcal{O}_{\mathbb{P}(\mathcal{E})}(d)$. Using the exact sequence

$$0 \to \mathcal{S} \to \pi^*\mathcal{E}^* \to \mathcal{Q} \to 0$$

we can compute the pushforwards to $B$ of $c_{d+1}(\text{Sym}^d(S^*))$. This is the class of the points $p$ in $B$ such that the fiber $X_p$ over the point contains a line. In the cases 1, 2 and 5 we get:

(i) $d = 2$, $\dim B \geq 1$, $\pi_*(c_3(\text{Sym}^2(S^*))) = 4c_1(\mathcal{E})$.

(ii) $d = 4$, $\dim B \geq 1$, $\pi_*(c_5(\text{Sym}^4(S^*))) = 320c_1(\mathcal{E})$.

(iii) $d = 3$, $\dim B \geq 2$, $\pi_*(c_4(\text{Sym}^3(S^*))) = 18c_1(\mathcal{E})^2 + 27c_2(\mathcal{E})$.

Since $\mathcal{E}$ is an ample vector bundle, these are all effective classes. Where the details of the computations are done in Section 3.1.

Remark 3.7. From these computations we also get the well-known results that in a generic pencil of quartic surfaces, 320 members will contain a line, and in a pencil of conics there will be three singular elements, i.e., six lines. See, e.g., [EH16] for different ways of computing these numbers.

To get a pencil of conics, we can start with a trivial bundle $V = \mathcal{O}^\oplus 3$ of rank 3 on $\mathbb{P}^1$. Then divisor in $\mathbb{P}(V) \simeq \mathbb{P}^1 \times \mathbb{P}^2$ corresponding to the line bundle $\mathcal{O}_{\mathbb{P}(V)}(2) \otimes \pi^*(\mathcal{O}_{\mathbb{P}^1}(1))$, is a pencil of conics by the projection map to $\mathbb{P}^1$. We want a $\mathbb{Q}$-vector
bundle $\mathcal{E}$ such that this divisor corresponds to $O_E(2)$, so the appropriate choice is $O(E)\otimes 3$, where $E$ is a generator of Pic$(\mathbb{P}^1)$. Then

$$c_1(\mathcal{E}) = \frac{3}{2}H$$

so $4c_1(\mathcal{E})$ has degree 6, hence a general pencil of conics contains 6 lines, i.e., there are three singular fibers.

Similarly, for a pencil of quartic surfaces, the appropriate choice of $\mathbb{Q}$-vector bundle is $\mathcal{E} \simeq O_{\mathbb{P}^1}(H)\otimes 4$, so $c_1(\mathcal{E})$ has degree 1. Hence a general pencil of quartics has 320 elements that contain a line.

**Remark 3.8.** The hypothesis that the hypersurface $X$ is ample is essential to Lemma 3.6 as one can see by taking the trivial family of conics.

Using Lemma 3.6 we can prove the following result about when a hypersurface in a toric variety contains a contractible curve.

**Proposition 3.9.** Let $Y$ be a smooth projective toric variety with $\dim Y \geq 4$, and let $\pi: Y \to Z$ be the contraction of a contractible class $\gamma \in H_2(Y, \mathbb{Z})$. If $X$ is an ample hypersurface of $Y$ such that $(X + K_Y) \cdot \gamma \leq 0$, then $X$ contains a curve $C'$ such that $[C'] = \gamma$.

**Proof.** Let $\pi: Y \to Z$ be the contraction of a contractible class $\gamma$. We need to show that $X$ contains a curve of class $\gamma$, i.e., any line in any fiber $F$ of $\pi$. From Proposition 3.1 we know that the exceptional locus $E$ is a $\mathbb{P}(E)$-bundle over $B$, and $X_E$ satisfies the conditions of Lemma 3.6 by Corollary 3.3 and Proposition 3.2, so we may conclude. \[\Box\]

With this proposition, it is easy to prove Theorem 1.1, which we recall here:

**Theorem 3.10.** Let $X \subset Y$ be a smooth ample hypersurface in a smooth toric variety $Y$, with $\dim X$ at least 3. Assume furthermore that $X$ has nef anticanonical divisor $-K_X$. Then the Integral Hodge Conjecture for curves holds for $X$, in fact $H_2(X, \mathbb{Z})$ is generated by classes of rational curves.

**Proof.** Since by assumption $X$ has nef anticanonical divisor, the hypotheses of Proposition 3.9 are satisfied for all contractible curves in $Y$. Since the contractible curve classes span $H_2(X, \mathbb{Z})$, we can conclude that the Integral Hodge Conjecture holds for $X$. Since all contractible classes are rational, we also get the second statement. \[\Box\]

**Remark 3.11.** The hypothesis that $-K_X$ is nef is essential for all contractible classes to have representatives on $X$. Consider a divisor $X$ of bidegree $(d,3)$ in $\mathbb{P}^1 \times \mathbb{P}^3$ for $d \gg 0$. Then the projection to $\mathbb{P}^3$ will be finite, so $X$ will not contain a representative of the contractible class contracted by this projection. However, by Tsen’s theorem there is a section of the map $X \to \mathbb{P}^1$. This section, together with the class contracted by $X \to \mathbb{P}^1$ generate $H_2(X, \mathbb{Z})$, so the Integral Hodge Conjecture still holds on $X$. 

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3.1 Chern Class Computations

3.1.1 Lines in a Family of Plane Conics

It is a standard computation to find the total Chern class of a symmetric power as a function of the Chern classes of the bundle we start with. In the case of the second symmetric power of a rank 2 bundle we get:

**Lemma 3.12.** Let $F$ be a rank 2 vector bundle. The Chern polynomial of $\text{Sym}^2(F)$ is

$$1 + 3c_1(F) + 2(c_1(F))^2 + 4c_2(F) + 4c_1(F)c_2(F)$$

Using this, we can compute the Chern class $c_3(\text{Sym}^2(S^*))$ as used in the proof of **Lemma 3.6**. It is straightforward to compute that:

$$c_3(\text{Sym}^2(S^*)) = 4c_1(S^*)c_2(S^*) = -4c_1(S)c_2(S) = -4\Delta_{21}(c(S)) = -4\Delta_{21}(s(S^*))$$

Applying **Proposition 2.12** we find that:

$$\pi_*(c_3(\text{Sym}^2(S^*))) = -4\Delta_1(s(\mathcal{E})) = -4s_1(\mathcal{E}) = 4c_1(\mathcal{E})$$

3.1.2 Lines in a Family of Plane Cubics

The computation for the case of a family of plane cubics is very similar. Again we begin with a standard computation.

**Lemma 3.13.** Let $F$ be a rank 2 vector bundle. The Chern polynomial of $\text{Sym}^3(F)$ is

$$c(\text{Sym}^3(F)) = 1 + 6c_1(F) + 11(c_1(F))^2 + 10c_2(F) + 30c_1(F)c_2(F) + 6c_1(F)^3 + 18c_1(F)^2c_2(F) + 9c_2(F)^2$$

A simple computation gives $c_4(\text{Sym}^3(S^*))$ in terms of Schur polynomials

$$c_4(\text{Sym}^3(S^*)) = 18c_1(S^*)^2c_2(S^*) + 9c_2(S^*)^2 = 18c_1(S)^2c_2(S) + 9c_2(S)^2$$

We can therefore use **Proposition 2.12** to find that:

$$\pi_*(c_4(\text{Sym}^3(S^*))) = 18\Delta_2(s(\mathcal{E}')) + 27\Delta_{11}(s(\mathcal{E}')) = 18s_2(\mathcal{E}') + 27s_2(\mathcal{E}')^2 = 18c_2(\mathcal{E}') + 27c_1(\mathcal{E}')^2$$

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3.1.3 Lines in a Family of Quartic Surfaces

Again we start by writing the Chern classes of a symmetric power in terms of the Chern classes of the vector bundle.

**Lemma 3.14.** Let \( \mathcal{F} \) be a rank 2 vector bundle. Then

\[
c(S^4(\mathcal{F})) = 1 + 10c_1(\mathcal{F}) + 35c_1(\mathcal{F})^2 + 20c_2(\mathcal{F}) + 50c_1(\mathcal{F})^3 + 120c_1(\mathcal{F})c_2(\mathcal{F}) + 24c_1(\mathcal{F})^4 + 208c_1(\mathcal{F})^2c_2(\mathcal{F}) + 64c_2(\mathcal{F})^2 + 96c_1(\mathcal{F})c_2(\mathcal{F})^2 + 128c_1(\mathcal{F})^3c_2(\mathcal{F}) + 24c_1(\mathcal{F})^4c_2(\mathcal{F}) + 208c_1(\mathcal{F})^2c_2(\mathcal{F})^2 + 64c_2(\mathcal{F})^3 + 32c_1(\mathcal{F})^5 + 128c_1(\mathcal{F})c_2(\mathcal{F})^3 + 256c_1(\mathcal{F})^3c_2(\mathcal{F})^2 + 512c_1(\mathcal{F})^4c_2(\mathcal{F})^2 + 256c_1(\mathcal{F})^2c_2(\mathcal{F})^3 + 128c_2(\mathcal{F})^4 + 256c_1(\mathcal{F})^5c_2(\mathcal{F}) + 256c_1(\mathcal{F})^3c_2(\mathcal{F})^3 + 128c_1(\mathcal{F})^4c_2(\mathcal{F})^3 + 256c_1(\mathcal{F})^2c_2(\mathcal{F})^4 + 64c_2(\mathcal{F})^5 + 64c_1(\mathcal{F})^6 + 256c_1(\mathcal{F})c_2(\mathcal{F})^5 + 512c_1(\mathcal{F})^3c_2(\mathcal{F})^4 + 256c_1(\mathcal{F})^4c_2(\mathcal{F})^4 + 512c_1(\mathcal{F})^2c_2(\mathcal{F})^5 + 256c_2(\mathcal{F})^6.
\]

We can now write \( c_5(S^4(\mathcal{F})) \) in terms of Schur polynomials.

\[
c_5(S^4(\mathcal{F})) = 96c_1(\mathcal{F})^3c_2(\mathcal{F}) + 128c_1(\mathcal{F})c_2(\mathcal{F})^2 + 64c_2(\mathcal{F})^3 + 32c_1(\mathcal{F})^5 + 128c_1(\mathcal{F})c_2(\mathcal{F})^3 + 256c_1(\mathcal{F})^3c_2(\mathcal{F})^2 + 512c_1(\mathcal{F})^4c_2(\mathcal{F})^2 + 256c_1(\mathcal{F})^2c_2(\mathcal{F})^3 + 128c_2(\mathcal{F})^4 + 256c_1(\mathcal{F})^5c_2(\mathcal{F}) + 256c_1(\mathcal{F})^3c_2(\mathcal{F})^3 + 128c_1(\mathcal{F})^4c_2(\mathcal{F})^3 + 256c_1(\mathcal{F})^2c_2(\mathcal{F})^4 + 64c_2(\mathcal{F})^5 + 64c_1(\mathcal{F})^6 + 256c_1(\mathcal{F})c_2(\mathcal{F})^5 + 512c_1(\mathcal{F})^3c_2(\mathcal{F})^4 + 256c_1(\mathcal{F})^4c_2(\mathcal{F})^4 + 512c_1(\mathcal{F})^2c_2(\mathcal{F})^5 + 256c_2(\mathcal{F})^6.
\]

We can apply Proposition 2.12 to get:

\[
\pi_*(c_5(S^4(\mathcal{F}))) = -96\Delta_1(-1)s(\mathcal{F}) - 320\Delta_1(s(\mathcal{F}))
\]

where we have used that \( \Delta_1(-1)s(\mathcal{F}) = 0 \) and \( \Delta_1(s(\mathcal{F})) = 0 \).

4 Double Covers of Toric Fano varieties

In this section we prove Theorem 1.2.

In the following let \( Y \) be a smooth toric Fano variety of dimension at least 3, and let \( \pi: X \to Y \) be a double cover ramified over a hypersurface \( R \), which we will assume is ample and non-singular. The canonical divisor of \( X \) is given by:

\[
K_X = \pi^*K_Y + \frac{1}{2}\pi^*R,
\]

so if \( 2K_Y - R \) is nef, \( X \) will have nef anticanonical divisor. Of particular interest is the case \( R = -2K_Y \), in which case \( X \) is Calabi–Yau. In general, for the construction of the double cover to work, \( R \) must be linearly equivalent to \( 2D \) for some divisor \( D \). The simplest example of these Calabi–Yau varieties is a double cover of \( \mathbb{P}^3 \) ramified over a smooth hypersurface of degree 8.

The integral homology of \( X \) is described by the following result:
Proposition 4.1 ([LS89, Proposition 1.11]). Let \( \pi : X \to Y \) be a cyclic covering of connected projective \( n \)-folds ramified over an ample divisor, then

\[
\pi_* : H_q(X, \mathbb{Z}) \to H_q(Y, \mathbb{Z})
\]

is an isomorphism for \( q \leq k - 1 \).

Because of this isomorphism, we can use the following strategy to prove the Integral Hodge Conjecture for curves for \( X \). For any contractible class \( \gamma \in H_2(Y, \mathbb{Z}) \), we will find an algebraic curve \( C \subset X \) such that \( \pi_*([C]) = \gamma \). The Integral Hodge Conjecture for curves will then follow from Corollary 2.8. There are two techniques we will use to prove the existence of such curves, depending on if the branch divisor \( R \) contains a curve of class \( \gamma \) or not.

Lemma 4.2. Let \( \pi : X \to Y \) be a double cover of a smooth toric variety \( Y \) branched along \( R \subset Y \), and let \( \gamma \) be a class in \( H_2(Y, \mathbb{Z}) \). If there is a curve \( C \subset Y \) with class \( \gamma \) and \( C \subset R \), then there is a curve \( C' \subset X \) such that \( \pi_*[C'] = \gamma \).

Proof. We can prove this by finding a curve \( C' \) in \( X \) such that \( \pi|_{C'} : C' \to C \) is one-to-one. But \( \text{Supp}(\pi^{-1}(C)) \) is exactly such a curve, since \( \pi \) is an isomorphism between the supports of the ramification locus and the branch locus. \( \blacksquare \)

When the branch divisor contains no such curves, we will use the following lemma to find curves with the correct homology class in the cover:

Lemma 4.3. Let \( \pi : X \to Y \) be a double cover branched along \( R \subset Y \), and let \( f : \mathbb{P}^1 \to Y \) be a rational curve. If we can write \( f^*(R) = 2D \) for some divisor \( D \) on \( \mathbb{P}^1 \), then \( \pi^{-1}(f(\mathbb{P}^1)) \) has two irreducible components, each isomorphic to \( f(\mathbb{P}^1) \) via \( \pi \).

Proof. We can think of the double cover being defined as a hypersurface in the total space of \( \mathcal{O}_Y(R) \) with equation \( z^2 = \pi^*g \), where \( g \) is a section of \( \mathcal{O}_Y(R) \) and \( z \) is the zero section. If \( f^*(R) = 2D \), then restricting the equation to \( \mathbb{P}^1 \) yields a double cover defined by \( z^2 = (\pi*h)^2 \), with \( h \) a section of \( \mathcal{O}_{\mathbb{P}^1}(D) \). Such a cover must consist of two components, each of which is isomorphic to \( \mathbb{P}^1 \) via the covering map. \( \blacksquare \)

So we wish to prove that for any smooth toric variety \( Y \), any contractible class \( \gamma \in H_2(Y, \mathbb{Z}) \) and any ample divisor \( R \), divisible by 2, such that \(- (K_X + \frac{1}{2}R)\) is effective, there is a morphism \( f : \mathbb{P}^1 \to Y \) with \([f(\mathbb{P}^1)] = \gamma \) such that \( f^*R \) is \( 2D \) for some divisor \( D \). To do this, we will estimate the dimensions of such morphisms.

The following lemma tells us that requiring \( f^*R = 2D \) is a codimension \( d = \frac{1}{2} \deg f^*(R) \) condition.

Lemma 4.4. The divisors of the form \( 2D \) have codimension \( d = \frac{1}{2} \deg f^*(R) \) in the projective space \( \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(f^*(R)))) \) of dimension \( 2d \).
Proof. This follows from a simple dimension count, since \( \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(k))) \) has dimension \( k \).

To find a curve \( C' \in X \) such that \( \pi_*(\{C\}) = \gamma \) for any given contractible class \( \gamma \in H_2(Y, \mathbb{Z}) \) we will use the structure of contractions on smooth toric varieties described in [Proposition 3.1]. We first need to find a sufficiently large family of curves whose class in \( H_2(Y, \mathbb{Z}) \) is contractible.

**Lemma 4.5.** Let \( Y \) be a smooth toric Fano variety and \( C \subset X \) be a curve such that \( [C] \) is a contractible class in \( H_2(Y, \mathbb{Z}) \). Then there is a family of curves with class \( [C] \) of dimension at least \( -K_Y \cdot [C] \geq \frac{1}{2} R \cdot C \).

Proof. Let the corresponding wall relation be

\[
x_1 + \cdots + x_h - a_1 y_1 - \cdots - a_k y_k = 0
\]

The exceptional locus of the contraction of \([C]\) is a \( \mathbb{P}^{h-1} \) bundle over a variety \( B \) of dimension \( n - h - k + 1 \). The curves of class \([C]\) are the lines in the fibers of the \( \mathbb{P}^{h-1} \) bundle, which are parametrized by the relative Grassmannian \( Gr(2, h) \rightarrow B \), which has dimension

\[
2(h - 2) + \dim B = 2h - 4 + n - h - k + 1 = h - k + (n - 3) \geq h - k
\]

where the final inequality uses the assumption that \( \dim Y \geq 3 \). Furthermore, we know that \( h - k \geq -K_Y \cdot [C] \geq \frac{1}{2} R \cdot C \). 

Given a contractible class \([C] \in H_2(Y, \mathbb{Z})\), we wish to find a curve \( C' \subset X \) such that \( \pi_*([C]) = \gamma \) for any given contractible class \( \gamma \in H_2(Y, \mathbb{Z}) \). Using the previous lemma we can find a \( (\frac{1}{2}(R \cdot C) + 3) \)-dimensional family \( M \) of morphisms \( f: \mathbb{P}^1 \rightarrow Y \) such that \( f(\mathbb{P}^1) \) has class \([C]\). This gives a rational map \( \phi: M \dashrightarrow \mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(R \cdot C)) \), defined whenever the image of \( \phi \) is not contained in \( R \). If \( \phi \) is not defined at some point, then by [Lemma 4.2] we find the desired curve \( C' \). We may therefore assume that \( \phi \) is defined everywhere. If the image of \( \phi \) indeed has codimension at most \( \frac{1}{2} R \cdot [C] \), then \( R \) restricts to a divisor of the form \( 2D \) on some curve of class \([C]\), by [Lemma 4.3].

We will use an incidence correspondence argument to prove that the image of \( \phi \) has codimension at most \( \frac{1}{2} R \cdot [C] \). We first fix some notation. Let \( d := \frac{1}{2} R \cdot [C] \), \( \mathcal{P} := \mathbb{P}(H^0(\mathcal{O}_{\mathbb{P}^1}(d))) \), and write \( \mathcal{K} \) for \( \mathbb{P}(H^0(R)) \) and \( \kappa \) for \( \dim \mathcal{K} \). Consider the incidence correspondence:

\[
Z = \{ (f, p, R') \subset M \times \mathcal{P} \times \mathcal{K} \mid f^* R' = p \}.
\]

Since specifying \( f \) and \( R' \) determines \( p \) uniquely, we see that \( Z \) has dimension \( \dim M + \dim \mathcal{K} = d + 3 + \kappa \).

Now we study the projection \( Z \rightarrow \mathcal{P} \).

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Lemma 4.6. With notation as above, the projection $Z \to \mathcal{P}$ is surjective.

Proof. To prove this it suffices to find a line $L$ in a fiber of $\pi : \mathbb{P}(\mathcal{E}) \to B$ such that the map

$$H^0(\mathcal{O}_{\mathcal{P}(\mathcal{E})}(R)) \to H^0(\mathcal{O}_L(2d))$$

is surjective. We will do this in two steps. First we will prove that for a torus-invariant fiber $F$, the map

$$H^0(\mathcal{O}_{\mathcal{P}(\mathcal{E})}(R)) \to H^0(\mathcal{O}_F(2d))$$

is surjective, and then prove that the map $H^0(\mathcal{O}_F(2d) \to H^0(\mathcal{O}_L(2d))$ is surjective.

First fix a torus-invariant fiber $F$, with ideal sheaf $\mathcal{I}_F$. From the exact sequence

$$0 \to \mathcal{I}_F(R) \to \mathcal{O}_{\mathcal{P}(\mathcal{E})}(R) \to \mathcal{O}_F(R) \to 0$$

we see that we can prove that the map (3) is surjective by proving that $H^1(\mathcal{I}_F(R), \mathbb{P}(\mathcal{E})) = 0$. Let $p : W \to \mathbb{P}(\mathcal{E})$ be the blow-up of $F$, with exceptional divisor $E$. Then proving $H^1(\mathcal{I}_F(R)) = 0$ is equivalent to proving that $H^1(p^*R - E, W) = 0$. Since $F$ is a torus-invariant fiber, we can prove the vanishing of this cohomology group by proving that $p^*R - E$ is a nef divisor on $W$. As a blow-up of a toric variety in a torus-invariant subset, $W$ will also be a toric variety, and the higher cohomology groups of nef line bundles vanish on toric varieties ([CLS11, Theorem 9.2.3]). The cone of effective curves on $W$ is spanned by the following three kinds of curves. Curves contained in the exceptional divisor $E$, the strict transform of curves $C$ in $\mathbb{P}(\mathcal{E})$ such that $\pi(\mathcal{E}) \subset B$ is smooth in a neighborhood of $\pi(F)$ and the strict transform of curves in $\mathbb{P}(\mathcal{E})$ disjoint from $F$. Since $R$ is ample, it is clear that $p^*R - E$ intersects the last kind of curve positively. If $C$ is a curve contained in the exceptional divisor $E$, then $(p^*R - E) \cdot C = -E \cdot C$. We can think of $E$ as $\mathbb{P}(\mathcal{O}_F^{\oplus \dim B})$, on which $-E$ is the divisor corresponding to the line bundle $\mathcal{O}_F(\mathcal{O}_F^{\oplus \dim B})(1)$, so $C$ intersects $p^*R - E$ positively. Finally, if $C$ is the strict transform of a curve $C'$ passing through $F$, then $(p^*R - E) \cdot C = C' \cdot R = \mult_{\pi(F)}(\pi(C'))$. Since by assumption $\pi(C')$ is smooth at $\pi(F)$, and $R$ is ample, so $C' \cdot R \geq 1$, and therefore the intersection number $(p^*R - E) \cdot C$ is non-negative.

That $H^0(\mathcal{O}_F(2d)) \to H^0(\mathcal{O}_L(2d))$ is surjective, follows since linear subspaces of projective space are projectively normal. ■

Using the previous lemma, we can use a dimension count to see that for a generic choice of ramification divisor $R$, the map from $M$ to $\mathcal{P}$ defined by $f \to f^*(R)$ has image of dimension at least $d$. By the dimension count of $Z$, the generic fiber of $Z \to \mathcal{P}$ must have dimension $\kappa + 3$. Assume for contradiction that for a generic choice of $D \in |R|$ the induced map $\phi$ has image of dimension less than $d$. Then the
generic fiber of this map must have dimension at least 4. But then a general fiber of the map $Z \to \mathcal{P}$ has dimension at least $\kappa + 4$, a contradiction. With this in place, we can prove the following:

**Theorem 4.7.** Let $X$ be a smooth double cover of $Y$, a smooth toric Fano variety, branched over $R$ such that $X$ has nef anticanonical divisor. Assume further that $R$ is general in its linear system $|R|$. Then the integral Hodge conjecture for curves holds for $X$.

**Proof.** From Corollary 2.8 and Proposition 4.1 we know that $H_2(X, \mathbb{Z})$ is generated by contractible classes $\gamma \in H_2(X, \mathbb{Z})$ such that $\pi_*(\gamma)$ is the class of a contractible curve in $Y$. Let $\gamma$ be a contractible class. By Lemma 4.2 if the branch divisor contains a curve of class $\gamma$ we are done. Otherwise by Lemma 4.4 and Lemma 4.3 for a general choice of $R$, and any contractible class $\gamma \in H_2(X, \mathbb{Z})$ there is a curve $\ell$ with $[\ell] = \pi_*(\gamma)$ such that the inverse image in $X$ splits, and therefore the class $\gamma$ in $X$ is also algebraic. ■

**Remark 4.8.** Using the same technique, one can also prove the result corresponding to Theorem 1.2 for higher degree cyclic covers. To modify the proof of Theorem 4.7, let $R$ be the ramification divisor, which must be divisible by $k$. The condition that $X$ has nef anticanonical divisor translates to the condition that $-K_Y - \frac{k-1}{k}R$ is nef. Now, for each contractible curve $C$ with contraction $\pi : Y \to Y'$, we need to find a curve $L$ contracted by $\pi$ such that $R|_L = kD$ for some divisor $D$ on $L$. This is a codimension $\frac{k-1}{k}R \cdot C$ condition on the space of lines in the fibers contracted by $\pi$. However, in the dimension estimate (2), we know that $h - k \geq -K_Y \cdot C \geq \frac{k-1}{k}R \cdot C$. The remainder of the proof follows precisely the remainder of the proof of Theorem 1.2.

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