Dark energy with logarithmic cosmological fluid

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Abstract

We propose a dark energy model with a logarithmic cosmological fluid which can result in a very small current value of the dark energy density and avoid the coincidence problem without much fine-tuning. We construct a couple of dynamical models that could realize this dark energy at very low energy in terms of four scalar fields quintessence and discuss the current acceleration of the Universe. Numerical values can be made to be consistent with the accelerating Universe with adjustment of the two parameters of the theory. The potential can be given only in terms of the scale factor, but the explicit form at very low energy can be obtained in terms of the scalar field to yield of the form $V(\phi) = \exp(-2\phi)(\frac{4}{3}\phi + B)$. Some discussions and the physical implications of this approach are given.

1 Introduction

One of the most intriguing discovery of modern cosmology is the acceleration of the Universe [1, 2] and it is widely believed that dark energy of repulsive nature is causing the current acceleration. Many candidates of the dark energy have been proposed [3, 4, 5], among which the cosmological constant is the most accepted one. Along with yet another unidentified constituent of the Universe called dark matter, they compose standard cosmological model, Λ – CDM [6]. It is remarkable that the observable Universe can be well...
addressed with the $\Lambda$–CDM model. Still, extreme fine-tuning of the cosmological constant \[7\] has been unsatisfactory feature of the model. That is, the current cosmological constant must have an unnaturally small value compared to Planck scale.

An alternative proposal to explain the dark energy is the quintessence model \[8, 9\] in which a scalar field is added as an indispensable component of the universe. In this approach, the smallness of the cosmological constant is achieved by a dynamical decay of the scalar field energy density. It has the very attractive features of tracking behavior and attractor solutions \[1\] such that galaxy formation is not affected too much by the quintessence field and the dark energy becomes dominant only at the late stage of the Universe causing the current acceleration. However, in order to achieve the late time dark energy dominance, thus providing a possible solution of the coincidence problem \[3\], the theory has to be fine-tuned to a certain extent such that the energy density today must be very close to the critical density \[10\].

In this paper, we propose a dark energy model which can alleviate the fine-tuning problem substantially. Suppose a decaying cosmological term \[11\] according to $\frac{1}{a}\left(t\right)^2$, where $a(t)$ is the scale factor of the Universe \[12, 13\]. We regard it as the size of the Universe. This term decreases with the expansion of the Universe, and the current value is $\Lambda \sim 10^{-122}M_p^2$, with $a_0 \sim 10^{42}\text{Gev}^{-1}$, where we assumed that its value at the Planck scale is of the order $M_p^2$. Note that energy density of the cosmological constant from this value is very close too the critical density $\rho_{cr,0} \simeq 10^{-122}M_p^4$, and it has the potential of explaining the coincidence problem without fine-tuning. Also, some theoretical background was given for such a decay \[12, 13\]. On the other hand, the conservation of the Einstein tensor prevents the cosmological term to be varying in pure gravity. If matter contents are included, the varying cosmological constant term disrupts the matter continuity equations, which changes the predictions of the standard cosmology in the matter-dominated epoch \[12, 13\]. However, if continuity equation is being enforced, $1/a^2$ term behaves exactly like the curvature constant term and it alone cannot yield an accelerating universe.

It turns out that by adding another cosmological term which varies according to $\ln a/a^2$ to the original $1/a^2$, accelerating Universe can be realized. To see this in detail, let us first check the energy dominance of each epoch. We start from $\rho_{cr} \simeq 10^{-122}M_p^4$ and the matter energy density $\rho_m = \frac{\gamma}{(a/a_0)^3}\rho_{cr}$, $\gamma \simeq 0.27$ \[6\]. Let us assume that the energy density of the cosmological fluid composed of those two terms was approximately of the order of $\sim M_p^4$ when the inflation started\[4\]. At the end of the inflation when the scale factor becomes $10^3$ cm with the number of $e$-foldings given by $N$, the energy density would have decreased in

\[1\] The cosmological terms considered here decay very fast and cannot be responsible for the inflation itself. The inflaton should come from some other source.
magnitude by a factor of $\sim Ne^{-2N}$ and becomes of the order of $\sim Ne^{-2N}M_p^4$. We choose $N \sim 81$ which is bigger than the minimum number of e-foldings \[\text{10}\] so that the energy density becomes of the order of $\sim 10^{-72}M_p^4$ and therefore we propose the following dark energy density

$$\rho_D = \left[ \frac{c_* \ln(a/a_{\text{inf}})}{(a/a_{\text{inf}})^2} + \frac{d_*}{(a/a_{\text{inf}})^2} \right] \times 10^{-72}M_p^4, \tag{1.1}$$

where $c_*$ and $d_*$ are constants\[2\] and $a_{\text{inf}} \sim 10^3\text{cm}$. Note that the number $10^{-72}$ is a dynamical consequence of the inflation and it turns out that no extreme fine-tuning of $c_*$ and $d_*$ are necessary to describe the current accelerating Universe. Comparing with $\rho_{D,0} = \delta\rho_{cr}$, $\delta \simeq 0.73 \text{[6]}$, we have a relation

$$c_*(25 \ln 10) + d_* \sim \delta \tag{1.2}$$

Just after the inflation ended at the energy scale $\sim 10^{13}\text{Gev}$, the energy density is of the order of $10^{-24}M_p^4 >> \rho_D$, and $\rho_t \sim 10^{-24}/(a/a_{r,i})^4M_p^4$ with $a_{r,i} \sim a_{\text{inf}} \sim 10^3\text{cm}$. The normal expansion takes over and the Universe expands by a factor of $10^{21}$ until the radiation-matter equality around $a \sim 10^{24}\text{cm}$. When this occurs, $\rho_{r,f} \simeq \rho_{m,i} \simeq 10^{-108}M_p^4$, and the matter dominance takes over since the radiation energy density decays faster than the matter energy density. In the meantime, the magnitude of $\rho_D$ keeps on decreasing according to (1.1), and becomes of the order of $\sim 10^{-114}M_p^4$. Therefore, the dark energy is completely subdominant during this period.

Then, the matter-dominated epoch began around $a \sim 10^{-3}a_0$, $a_0 \sim 10^{28}\text{cm}$. Since, the matter energy density decays faster than dark energy, there exist a scale where $\rho_m \simeq \rho_D$ given by

$$\frac{a_{eq}}{a_0} \left[ c_* \ln \left( \frac{a_{eq}}{a_0} \right) + \delta \right] \simeq \gamma. \tag{1.3}$$

So far, there is only one restriction on the numerical values of the parameters $c_*$ and $d_*$ of (1.2), and a wide range of their values are allowed to fit into the current observation. One can impose one more condition by demanding that the Universe has begun its acceleration very recently. The acceleration equation with our $\rho_D$ and $p_D$ is given by

$$\frac{\ddot{a}}{a} = \frac{1}{6} \left[ \frac{c_*}{(a/a_0)^2} - \frac{\gamma}{(a/a_0)^3} \right] \rho_{cr}. \tag{1.4}$$

\[2\text{We also assume that the pressure is given by } p_D = -\frac{1}{3} \left[ \frac{c_* \ln(a/a_{\text{inf}})}{(a/a_{\text{inf}})^2} + \frac{c_* + d_*}{(a/a_{\text{inf}})^2} \right] \times 10^{-72}M_p^4. \rho_D \text{ and } p_D \text{ satisfy the continuity equation, } \dot{\rho}_D = -3H(\rho_D + p_D). \]
In the next sections, we will show that this equation can be realized within a couple of quintessence models at very low energy. From the above equation, we see that the acceleration began around \( a_{acc} \sim \gamma/c_\ast a_0 \). For example, if we choose \( c_\ast = 0.54 \), \( a_{acc} \sim 1/2a_0 \) and \( a_{eq} \sim 0.60a_0 \). This would determine \( d_\ast \sim -56 \). Therefore, the transition to dark energy dominance occurred very recently. With these values, the dark energy stays negative during most of the time until \( a_{-+} \sim 0.25a_0 \) and becomes positive at late stage of the matter-dominated era. After passing this point a maximum, \( a_{max} \) is reached, and eventually, it begin to be dominant around \( a_{eq} \). It seems that a priori there is no reason for it to stay positive always as long as the total energy density \( \rho = \rho_D + \rho_r \) or \( \rho = \rho_D + \rho_m \) remains positive. In addition, absolute value of the energy density is very small when it stays negative compared to the radiation or matter energy density. Therefore, it should not disturb the radiation- or matter-dominated evolution to alter the course of it. Note that the increasing behavior of dark energy, although limited until \( a_{max} \) in our case, also appears in the phantom model [14]. It is interesting to check that \( a_{-+}, a_{max}, a_{acc}, a_{eq} \) all happen very recently without too much fine-tuning of the parameters \( c_\ast \) and \( d_\ast \).

In Sec. 2, we construct a quintessence model with four scalar fields which can produce the dark energy behavior of (1.1) at very low energy. An explicit form of the potential in terms of the scalar field in the scalar-dominated region is given. In sec. 3, we consider a generalized quintessence model where the explicit construction can be extended to matter-dominated epoch. In Sec. 4, a critical analysis of the generalized quintessence model is performed. Sec. 5 contains conclusion and discussions.

2 Quintessence with four scalar fields

Let us consider an action of the form \((8\pi G = 1)\)

\[
S_1 = \int d^4x \sqrt{-g} \left[ \frac{R}{2} - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - V(\phi) - \alpha X \right] + S_m, 
\]

with

\[
X \equiv g^{\mu\nu} \delta_{ab} \partial_\mu \sigma^a \partial_\nu \sigma^b, \tag{2.5}
\]

where \( \sigma^a \) is an 3 component scalar field and \( \alpha \) is a parameter which we assume to be constant. \( S_m \) ia a matter action with \( p_m = 0 \). The spacetime metric tensor is given by

\[
ds^2 = -dt^2 + a(t)^2 (dx^2 + dy^2 + dz^2), \tag{2.6}
\]

\(^3\) It can be checked that \( a_{-+} \sim e^{-\delta/c_\ast} a_0 \) independently of the detailed numerical values for \( \rho_{cr} \) and \( a_{inf} \).
where $a(t)$ is the scale factor. With ansatz for the scalar field $\sigma^a$ of the form $\sigma^a = x^a$ [15], the evolution and the continuity equation for matter are given as follows;

\[
3H^2 = \frac{1}{2} \dot{\phi}^2 + \frac{3\alpha}{a^2} + V + \rho_m, \quad (2.7)
\]
\[
-2\dot{H} = \dot{\phi}^2 + \frac{2\alpha}{a^2} + \rho_m, \quad (2.8)
\]
\[
0 = \dot{\rho}_m + 3H\rho_m, \quad (2.9)
\]

where $H \equiv \dot{a}/a$ is a Hubble parameter, $\rho_m$ is the matter energy density. The scalar field satisfies

\[
\ddot{\phi} + 3H\dot{\phi} + V'(\phi) = 0,
\]

where the dot and prime denote partial differentiations with respect to $t$ and $\phi$, respectively.

Taking linear combinations of (2.7) and (2.8), one gets the acceleration equation

\[
\frac{\ddot{a}}{a} = \frac{1}{3} \left( V - \dot{\phi}^2 \right) - \frac{1}{6} \rho_m. \quad (2.10)
\]

In the quintessence model, the kinetic term and matter density decay very fast in the above equation (2.10), and sole potential dominance at late time is reached which causes the acceleration. We open the possibility that the late time acceleration comes from the first two terms in the above equation, and anticipating (2.10) to describe the current acceleration at very low energy, we require the following relation;

\[
V - \dot{\phi}^2 = \frac{A}{(a/a_0)^2} \rho_{cr}, \quad (2.11)
\]

where $A$ is a parameter which we assume to be the positive. With this, the expansion changes from the deceleration for the small value of the scale factor corresponding to the matter-dominated epoch to an acceleration for the large value corresponding to the scalar-dominated epoch including the kinetic energy density. We will omit $a_0$ and $\rho_{cr}$ in what follows unless confusion arises. Using the relation (2.11), the evolution equations (2.7) and (2.8) can be rewritten as follows:

\[
3H^2 = \frac{3}{2} V - \frac{A}{2a^2} + \frac{3\alpha}{a^2} + \rho_m, \quad (2.12)
\]
\[
-2\dot{H} = V - \frac{A}{a^2} + \frac{2\alpha}{a^2} + \rho_m. \quad (2.13)
\]

Differentiating equation (2.12) with respect to time and comparing with (2.13), we obtain the following first-order differential equation for the potential

\[
3aV'(a) + 6V(a) - \frac{4A}{a^2} = 0, \quad (2.14)
\]
which yields

\[ V = \frac{4A \ln a}{3a^2} + \frac{B}{a^2}, \quad (2.15) \]

which is precisely of the form (1.1). From equation (2.12), we define (with \( \alpha = -A/3 \), see eq. (2.18) and below)

\[ \rho_\phi = \frac{2A \ln a}{a^2} + \frac{3(B - A)}{2a^2}. \quad (2.16) \]

Comparing with Eq. (1.1), we obtain

\[ A = c_*/2, \quad 3B = 3c_*/2 + 2\delta. \]

Note that it is difficult to express the scalar field \( \phi \) in a closed form from the equation (2.11),

\[ \phi = \int \frac{dt}{\sqrt{4A \ln a + B - A/a^2}}, \quad (2.17) \]

so the potential \( V \) cannot be expressed in terms of the \( \phi \) explicitly. Also, positivity of the square root in the above expression restricts the applicable range of \( a \) which turns out to be \( a \geq a_{-+} \), which is the same as the positivity of the \( \rho_\phi \) of (2.16). Basically, this comes from the imposition of the acceleration condition (2.11), and this condition restricts the dynamically permitted region to \( a \geq a_{-+} \). A closed form of the potential is viable, if we neglect the matter density in the evolution equations, that is, in the scalar-dominated epoch. In this case, the ratio of \( H^2 \) and \( \dot{\phi}^2 \) is given by

\[ \frac{\dot{\phi}^2}{3H^2} = \frac{V - \frac{A}{a^2}}{\frac{3}{2}V - \frac{A}{2a^2} + \frac{3\delta}{a^2}}, \quad (2.18) \]

and we can adjust the parameter \( \alpha = -\frac{1}{2}A \). Then, we have

\[ \phi(a) = \sqrt{2} \ln a + C, \quad (2.19) \]

with an integration constant \( D \). This gives the following expression for the potential

\[ V(\phi) = e^{-\sqrt{2}\phi} \left( \frac{2\sqrt{2}A}{3} \phi + B \right). \quad (2.20) \]

where the term involving the constant \( C \) are absorbed into \( A \) and \( B \).

3 Generalized quintessence model

In this section, we consider a generalized quintessence model given by

\[ S_2 = \int d^4x \sqrt{-g} \left[ \frac{R}{2} - \frac{1}{2} \omega(\phi) \partial_{\mu} \phi \partial^{\mu} \phi - V(\phi) - \alpha X \right] + S_m, \quad (3.21) \]
where we introduced a scalar function $\omega(\phi)$ which will be arranged for our purpose. The last term in the gravity sector is the triplet of scalar fields as before, and we choose the same ansatz of the form $\sigma^a = a^a$, which solves the equation of motion. The potential $V(\phi)$ takes the following form

$$V(\phi) = e^{-2\phi} \left( \frac{4A}{3} \phi + B \right).$$

(3.22)

It is essentially of the form given by (2.20) with a redefinition of the scalar field. We assume the matter to be a cold dark matter with $w_m = 0$ as before and satisfies the continuity equation. The evolution equations are given by

$$3H^2 = \frac{1}{2} \dot{\omega}^2 + \frac{3\alpha}{a^2} + V + \rho_m,$$

(3.23)

$$-2\dot{H} = \omega \ddot{\phi} + \frac{2\alpha}{a^2} + \rho_m,$$

(3.24)

along with the continuity equation (2.9) and the scalar field satisfies

$$\omega \dddot{\phi} + \frac{1}{2} \omega' \dot{\phi}^2 + 3H \dot{\phi} + V' = 0.$$

(3.25)

Taking linear combinations of (3.23) and (3.24), one gets the acceleration equation

$$\frac{\ddot{a}}{a} = \frac{1}{3} \left( V - \omega \dot{\phi}^2 \right) - \rho_m.$$

(3.26)

We choose the same acceleration condition as before

$$V - \omega \dot{\phi}^2 = \frac{A}{a^2},$$

(3.27)

with a positive $A$. Note that Eqs. (3.23) and (3.24) imply (3.25) and recall $\rho_m = \frac{\gamma}{(a/a_0)^3} \rho_{cr}$. Therefore, we have two independent dynamical equations and one constraint for the three unknown functions to be determined, $a(t)$ or $H$, $\omega(\phi)$, and $\phi(t)$. It turns out that we can solve the equations with the ansatz

$$\phi(a) = \ln a,$$

(3.28)

which reproduces the same forms of the scale factor dependent potential (2.15) and energy density (2.16). Inserting this ansatz into (3.27), we obtain the following $\omega$ in terms of $\phi$

$$\omega(\phi) = \frac{8A\phi + 6(B - A)}{4A\phi + 3(B - A) + 2\gamma e^{-\phi}}.$$

(3.29)

Note that for the choice of $\alpha = -A/3$, $\omega(\phi) > 0$ for positive value of the energy density $\rho_\phi$ defined with the first three terms of eq. (3.23) as before. One can show that for $\alpha < -A/3$, $\omega(\phi)$ is always greater than zero for $a \gtrsim a_{-+}$. The above analysis shows that it is possible to construct an explicit form of the potential beyond the scalar-dominated region if we introduce a generalized quintessence model with an adjustable kinetic function $\omega(\phi)$. 

7
4 Critical Analysis

In this section, we perform the critical analysis of the evolution equations of the previous section and present some numerical result. For convenience, we choose $\alpha = -A$. For other choices, the qualitative feature of the stability does not change as long as $\alpha < -A/3$. Let us introduce the following dimensionless quantities

$$x \equiv \frac{\omega \dot{\phi}^2}{6H^2}, \quad y \equiv \frac{\tilde{V}}{3H^2},$$

with $\tilde{V} = V + 3\alpha/a^2$. Then the equations (3.23)-(3.25) can be written in the following form

$$\frac{dx}{dN} = -3x^2 + \frac{5}{3}x - \frac{1}{3}y, \quad (4.31)$$

$$\frac{dy}{dN} = \frac{1}{3}y - 3xy + 4\frac{3}{3}x, \quad (4.32)$$

where $N \equiv \ln a$ and together with a constraint equation

$$\frac{\rho_m}{3H^2} = 1 - x - y. \quad (4.33)$$

The critical points of the above system are easily obtained by setting the right-hand sides of the above equations (4.31) and (4.32) to zero. The only physically meaningful critical points $(x_c, y_c)$ of the system are

(A) : $(x_c, y_c) = (0, 0)$,

(B) : $(x_c, y_c) = \left(\frac{1}{3}, \frac{2}{3}\right).$

Point (A) corresponds to the matter-dominated point, whereas point (B) is a scalar-dominates one.

To gain some insight into the property of the critical points, we write the variables near the critical points $(x_c, y_c)$ in the form

$$x = x_c + \delta x \quad (4.34)$$

$$y = y_c + \delta y \quad (4.35)$$

where $\delta x$ and $\delta y$ are perturbations around the critical points. From (4.31) and (4.32), we obtain the linearized equations

$$\frac{d\delta x}{dN} = \left(\frac{5}{3} - 6x_c\right)\delta x - \frac{1}{3}\delta y, \quad (4.36)$$

$$\frac{d\delta y}{dN} = \left(\frac{4}{3} - 3y_c\right)\delta x + \left(\frac{1}{3} - 3x_c\right)\delta y, \quad (4.37)$$

8
which can be written by using a matrix $M$:

\[
\frac{d}{dN} \begin{pmatrix} \delta x \\ \delta y \end{pmatrix} = M \begin{pmatrix} \delta x \\ \delta y \end{pmatrix}, \quad M = \begin{pmatrix} \frac{5}{3} - 6x_c & -\frac{1}{3} \\ \frac{4}{3} - 3y_c & \frac{1}{3} - 3x_c \end{pmatrix}. \tag{4.38}
\]

One can study the stability of critical points against perturbations by evaluating the eigenvalues of the matrix $M$. For class (A) corresponding to the matter dominated epoch, one has $\lambda_1 = 1$ and $\lambda_2 = 1$, which means that it is an unstable point. For class (B) corresponding to the scalar field dominated epoch, we obtain $\lambda_1 = -1$ and $\lambda_2 = 0$. The appearance of zero eigenvalue means that the linear perturbation, which leads to the matrix (4.38), is not adequate and we must consider the higher order perturbations to determine whether the considered critical point is stable or not.

First note that we have

\[
2x - y > 0, \tag{4.39}
\]

for $\alpha = -A$. Therefore, it is useful to change from the $(x, y)$ to new variables $(X, Y)$ defined as

\[
X = x + 2y, \quad Y = 2x - y, \tag{4.40}
\]

where the $X, Y$ are positively defined, and then (4.31) and (4.32) can be rewritten as

\[
\frac{dX}{dN} = \frac{5}{3}Y - \frac{1}{5}X(3X + 6Y - 5), \tag{4.41}
\]

\[
\frac{dY}{dN} = -\frac{1}{5}Y(3X + 6Y - 5), \tag{4.42}
\]

and corresponding critical points in terms of the new variables are:

(A) : $(X_c, Y_c) = (0, 0),$

(B) : $(X_c, Y_c) = \left(\frac{5}{3}, 0 \right).$

In the higher order perturbations, we write the variables near the critical points $(X_c, Y_c)$ in the form

\[
X = X_c + \delta X^{(1)} + \delta X^{(2)} + \delta X^{(3)} + \cdots, \quad Y = Y_c + \delta Y^{(1)} + \delta Y^{(2)} + \delta Y^{(3)} + \cdots, \tag{4.43}
\]

where $\delta X^{(n)}$ and $\delta Y^{(n)}$ are $n$-th order perturbations of the variables near the critical points. For class (B), the perturbative equations of each order are given by

\[
\frac{d}{dN} \delta X^{(1)} = -\delta X^{(1)} - \frac{1}{3} \delta Y^{(1)}
\]
\[ \frac{d}{dN} \delta Y^{(1)} = 0 \]
\[ \frac{d}{dN} \delta X^{(2)} = -\frac{3}{5} \delta X^{(1)} (\delta X^{(1)} + 2 \delta Y^{(1)}) - \delta X^{(2)} - \frac{1}{3} \delta Y^{(2)} \]
\[ \frac{d}{dN} \delta Y^{(2)} = -\frac{3}{5} \delta Y^{(1)} (\delta X^{(1)} + 2 \delta Y^{(1)}) \]
\[ \frac{d}{dN} \delta X^{(3)} = -\frac{6}{5} \delta X^{(1)} \delta X^{(2)} - \frac{6}{5} (\delta X^{(1)} \delta Y^{(2)} + \delta X^{(2)} \delta Y^{(1)}) - \delta X^{(3)} - \frac{1}{3} \delta Y^{(3)} \]
\[ \frac{d}{dN} \delta Y^{(3)} = -\frac{12}{5} \delta Y^{(1)} \delta Y^{(2)} - \frac{3}{5} (\delta X^{(1)} \delta Y^{(2)} + \delta X^{(2)} \delta Y^{(1)}) \]

Note that the right hand side of the second equation in the first order perturbation equation is zero which reflects the fact that one of two eigenvalues is zero and we must focus on the next order equations.

The solutions of the above linear differential equations are given by
\[ X(N) = \frac{5}{3} + \delta A_1 + \delta B_1, \] (4.44)
\[ Y(N) = 0 + \delta A_2 + \delta B_2, \] (4.45)

with
\[ \delta A_1 = -\frac{1}{3} \delta Y_0 + \frac{1}{3} \delta Y_0^2 N - \frac{1}{3} \delta Y_0^3 N^2 + O(\epsilon^4) \] (4.46)
\[ \delta A_2 = \delta Y_0 - \delta Y_0^2 N + \delta Y_0^3 N^2 + O(\epsilon^4) \] (4.47)
\[ \delta B_1 = \delta \alpha_0 e^{-N} \left[ 1 + \frac{3}{5} e^{-N} \delta \alpha_0 - \left( \frac{1}{5} + N \right) \delta Y_0 + \frac{9}{25} e^{-2N} \delta \alpha_0^2 - \frac{3}{25} e^{-N} (1 + 10N) \delta \alpha_0 \delta Y_0 \right. \\
\left. + N \left( 1 + \frac{2}{5} \right) \delta Y_0^2 \right] + O(\epsilon^4), \] (4.48)
\[ \delta B_2 = \frac{3}{5} \delta Y_0 \delta \alpha_0 e^{-N} \left( 1 + \frac{3}{5} e^{-N} \delta \alpha_0 - 2N \delta Y_0 \right) + O(\epsilon^4), \] (4.49)

where \( \delta Y_0 \) and \( \delta \alpha_0 \) are the initial values at \( N = 0 \) that satisfy the \( \delta Y_0 = \delta Y^{(1)}(0) \), \( \delta \alpha_0 = \delta X^{(1)}(0) + 4 \delta Y^{(1)}(0) \), and \( \epsilon \) is the infinitesimal order parameter for the perturbation. If the higher order terms are included and when \( N \) becomes very large, \( \delta A_{1,2} \) can be expressed in a closed form with
\[ \delta A_1 \rightarrow -\frac{\delta Y_0}{3(1 + \delta Y_0 N)}, \quad \delta B_1 \rightarrow 0, \] (4.50)
\[ \delta A_2 \rightarrow \frac{\delta Y_0}{1 + \delta Y_0 N}, \quad \delta B_2 \rightarrow 0. \] (4.51)
Since the variable $Y$ is a positively defined value, perturbation around the zero point must be also positive. In this case, $\delta A_1$ and $\delta A_2$ smoothly go to zero when $N$ goes infinity. Therefore, critical point (B) is stable. Numerical result is given in Fig. 1 which demonstrates the stability of the scalar-dominated critical point.

![Flow Diagram](image)

Figure 1: The flow diagram of the system in terms of $X, Y$ for different initial values. We have drawn the physically allowed region with $\delta Y > 0$. The left red dot is the matter-dominated point (A), whereas the right red dot is the scalar-dominated point (B).

## 5 Conclusion

We showed that the coincidence problem can be avoided with logarithmic cosmological fluid of the form (1.1), and this can be realized dynamically as a couple of quintessence models at very low energy. Among the four scalar fields scalar fields, one of the scalar fields play a major role as in the standard quintessence model causing the current acceleration, whereas the triplet of scalar fields is not essential in dynamical evolution but it can provide the necessary energy density such that the potential is completely integrable as an exact expression in the dark-energy dominated era. In the generalized quintessence model, the construction was extended to matter-dominated epoch, and critical analysis indicates that the scalar-domination is dynamically stable. An analytic expression of the potential in
terms of the scalar field is unavailable in each case, but effective field theories could be considered separately at each stage of the evolution of the Universe. We also have checked that in such a scheme, the current value of extremely small dark energy density can be obtained without much fine-tuning; The constants $c_*$ and $d_*$ are only of the order $10^{-1}$ and $10^2$, and the small value is attributed to its decaying according to essentially as $1/a^2$.

Comparing with the quintessence model, the constraint (2.11) brings a crucial difference as far as energy dominance is concerned. When the scalar field begins to roll down the potential, the initial potential energy is converted into kinetic energy which dominates the energy of the scalar field. But the kinetic energy decreases rapidly and the potential energy dominance takes over around $1/2 < z < 1$, which is responsible for the dark energy domination. In our case, at the very low energy eq. (2.11) maintains throughout the evolution, and it implies that kinetic energy does not decay fast, but it remains comparable with potential energy $\sim V/2$. This would result in a relatively small absolute value for the equation of state $\omega_\phi$, but detailed comparison with the observational data needs to be done. We comment that this constraint is a phenomenological input to conform to the current acceleration, and a more theoretical basis is required.

Another comment is the feature that the dark energy density remains negative during most of the time until $a_{-\epsilon}$. Even though this behavior does not destroy the accelerating Universe, there seems to be no special meaning to $a \sim 0.25a_0$ in the evolution. It needs to be addressed how the matter-dominated evolution is effected by the small negative energy, if any. One could get rid of the negative energy simply by modifying the dark energy density into $\rho_D=-\text{RHS of (1.1)}$ for $a > a_{-\epsilon}$, but it seems rather ad-hoc. If this scale could somehow be raised to electro-weak breaking scale, this could endow more flexibility to the theory. Adding $(\ln a)^2/a^2$ term to the dark energy density (1.1) is another possible avenue to deal with the negative energy. It is likely that this will also extend the applicable range of the quintessence model similarly constructed, and might also improve the equation of state previously mentioned.

We conclude with an intriguing property of (1.1). One can check that that the status current accelerating Universe is rather insensitive to the numerical values of $a_{inf} \sim 10^3 cm$ and $\rho_{ct} \sim 10^{-122} M_p^4$ chosen. If these values are chosen differently, these changes can be always reabsorbed into $c_*$ and $d_*$. Suppose we change $\rho_{ct}$ into $\alpha^{-1} \rho_{ct}$ with some constant $\alpha$. Then, eqs. (1.1) and (1.4) retain the same form with $c_* \rightarrow \alpha c_*$, and $d_* \rightarrow \alpha d_*$. Likewise, if $a_{inf} \rightarrow \beta^{-1} a_{inf}$, then, $c_* \rightarrow \beta^2 c_*$, and $d_* \rightarrow \beta^2(c_* \ln \beta + d_*)$. Since $\alpha$ and $\beta$ can be at most of the order 1, $c_*$ and $d_*$ can change in maximum by an order of 2, and fine-tuning problem does not arise. And these changes do not alter the energy dominances.
of radiation epoch and matter epoch. This especially means that $a_{acc}$ can be made in fact independent on $a_{inf}$ without much fine-tuning, suggesting that the current status of the Universe is not effected too much by the early universe. Similarly, the scaling of the scale factor itself can also be absorbed into new definition of $c_*$ and $d_*$, which seems to suggest a kind of scaling behavior of the dark energy proposed. It would be interesting if some theoretical foundation could be given for it.

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