Hopping in the Glass Configuration Space: Subaging and Generalized Scaling Laws

Bernd Rinn,1 Philipp Maass,1,2 and Jean-Philippe Bouchaud2
1 Fachbereich Physik, Universität Konstanz, 78457 Konstanz, Germany and
2 Service de Physique de l’Etat Condensé, CEA Saclay, 91191 Gif sur Yvette Cedex, France

(Dated: April 17, 2001)

Aging dynamics in glassy systems is investigated by considering the hopping motion in a rugged energy landscape whose deep minima are characterized by an exponential density of states \( \rho(E) = T^{-1} \exp(E/T_0) \), \(-\infty < E \leq 0\). In particular we explore the behavior of a generic two-time correlation function \( \Pi(t_w + t, t_w) \) below the glass transition temperature \( T_0 \) when both the observation time \( t \) and the waiting time \( t_w \) become large. We show the occurrence of ordinary scaling behavior, \( \Pi(t_w + t, t_w) \sim F_1(t/t^*_w) \), where \( \mu_1 = 1 \) (normal aging) or \( \mu_1 < 1 \) (subaging), and the possible simultaneous occurrence of generalized scaling behavior, \( F^*_w \sim [1 - \Pi(t_w + t, t_w)] \sim F_2(t/t^*_w) \) with \( \mu_2 < \mu_1 \) (subaging). Which situation occurs depends on the form of the effective rate differences between the low lying states. Employing a “Partial Equilibrium Concept”, the exponents \( \mu_{1,2} \) and the asymptotic form of the scaling functions are obtained both by simple scaling arguments and by analytical calculations. The predicted scaling properties compare well with Monte-Carlo simulations in dimensions \( d = 1 \) – 1000 and it is argued that a mean-field type treatment of the hopping motion fails to describe the aging dynamics in any dimension. Implications for more general situations involving different forms of transition rates and the occurrence of many scaling regimes in the \( t-t_w \) plane are pointed out.

PACS numbers: 02.50.-r, 75.10.Nr, 05.20.-y

I. INTRODUCTION

The history of glass formation strongly affects the relaxation dynamics of glassy materials [1, 2]. This dynamics is found to become slower with the “age” of the system, that means with the time \( t_w \) expired since the material was brought into the glassy state. Such aging phenomena have been identified in many systems and various dynamical probes. Prominent examples are shear stress relaxations in structural glasses [3], thermoremanent magnetizations or a.c. susceptibility in spin glasses [4, 5]. Similar effects have been observed on the di-electric constant of dipolar glasses [6, 7], of structural glasses [8, 9], and on the structure factor of Lennard–Jones systems [10]. More recent experiments in colloidal gels [11], or other ‘soft glassy materials’ have been reported [12, 13, 14, 15, 16], and even electronic relaxations in Anderson insulators were found to exhibit aging effects [17]. Aging is also expected for pinned systems (pinned domain walls [18], pinned vortex lines [19], polymer melts [20] and granular materials (see e.g. [21, 22]). From a theoretical point of view, several pictures have been proposed [23]. The simplest one is based on domain coarsening ideas [24], and is probably well suited to describe aging in, say, disordered ferromagnets where a well defined order wants to establish across the system. However, in spin glasses and even more evidently in glasses, the idea of some long range order which progressively invades the system is far from trivial. Mean field models for spin glasses, which are formally equivalent to the Mode Coupling Theory of glasses [25], do indeed lead to aging phenomena below the glass transition. In this case, aging is of geometric origin [26]. As time grows, the system progressively exhausts the possibilities of lowering its energy, and finds itself around saddle points from which it is more and more difficult to escape. Thermal activation is irrelevant in these models. Although this picture might be justified for supercooled liquids above the Mode-Coupling temperature [27], it certainly breaks down at lower temperature, where activated events become dominant. In this regime, one expects that a coarse-grained dynamical model of thermally activated hops between metastable states is a proper description of the dynamics. In fact, recent MD simulations support this view [27, 28, 29]. Landscape models have been widely discussed in the past [30, 31], but their relevance for aging phenomena was recognized later [32, 33, 34, 35]. These later developments were recently extended to treat rheological phenomena [36].

The ‘trap’ models studied up to now lead to correlation or response functions that depend on the ratio \( t/t_w \) of the observation time \( t \) to the waiting time \( t_w \) (full aging), or, for long-range correlated energy landscapes, on the ratio \( \log t/\log t_w \) [37]. However, many experimental systems reveal subaging behavior, that is, the relevant variable is \( t/t^*_w \) with \( \mu < 1 \). Furthermore, it is possible that there exist, for given waiting time \( t_w \), various scaling regimes in time \( t \), which are governed by different relaxation times \( \propto t^*_w s, s = 1, 2, \ldots \). The occurrence of different scaling regions has recently been conjectured on the basis of analytical results for mean-field spin glass models [23, 38]. So far, however, it was not possible to illustrate intu-
II. HOPPING IN A RANDOM ENERGY LANDSCAPE

A. Model

Slow dynamics in glasses is often attributed to a thermally activated motion of a point (henceforth denoted as “particle”) that jumps among metastable states in a rugged configuration space (see Fig. 1). In a coarse grained description, only transitions between the deepest (free) energy minima \( E_i \) of a complex configuration space. Some of the results discussed in this paper already appeared in a Letter [43].

The absence of an equilibrium state for \( \theta < 1 \) is reflected in the fact that the mean trapping time \( \langle \tau \rangle \) becomes infinite. For \( \alpha > 0 \) it is convenient to operate with the \( \tau \) defined in (6) as well, although these \( \tau \) no longer have the meaning of a trapping time. The hopping rate (8) can then be written as

\[
E_i \left( \frac{\tau_1}{\tau_1 - \alpha} \right)
\]

As first glance the model defined here seems to be similar to the model considered earlier in [35]. However, there is an important difference. In the model studied in [35] the disorder is of “annealed” type, i.e. the energy of each site is drawn anew from \( \psi(E) \) after each jump. This can be viewed as a mean–field treatment and simplifies the problem to a great extent. As will be shown in sec. [14], in the annealed case one can map any parameters \( (\theta', \alpha > 0) \) to parameters \( (\theta, \alpha = 0) \), that means

\[
0 \leq \alpha < 1.
\]
the α parameter is irrelevant. In the present model by contrast the energy disorder is “quenched” and such a mapping is not possible. We will show that this leads to the occurrence of much richer aging dynamics, including subaging behavior and generalized scaling forms.

B. Aging function

For studying aging properties we consider a quench from θ = ∞ to θ < 1 at time 0, then wait for some time \( t_w \gg 1 \) and ask for the behavior of correlation functions followed during an observation time \( t \) after the waiting period. Since the physical observables are functions of the coordinates of the configuration space, they will essentially de-correlate after the system has undergone a single transition from one deep minimum to another one. Hence, a “generic correlation function” in the model is to consider the probability that the particle has not jumped between \( t_w \) and \( t_w + t \). We denote this probability, after performing the disorder average, by \( \Pi(t + t_w, t_w) \).

Let us note that \( \Pi(t_w + t, t_w) \) can also be regarded as the spin-spin-correlation function of the generalized Sherrington–Kirkpatrick model with \( p \)-spin interactions in the limit \( p \to \infty \), which maps onto the random energy model \([12]\). It can also be viewed as an incoherent intermediate scattering function at large wave numbers in diffusion processes \([12]\).

A typical result for \( \Pi(t + t_w, t_w) \) from continuous–time Monte Carlo simulations is shown in Fig. 2 for parameters \((d, \theta, \alpha) = (10, 1/4, 3/8)\) and different \( t_w \) (for details regarding the simulation see appendix \([A]\)). Indeed, we can identify a pronounced aging phenomenon: The decay of \( \Pi(t_w + t, t_w) \) with time \( t \) becomes slower and slower with increasing waiting time \( t_w \). The linear behavior of the curves at long times in the double-logarithmic plot indicates a common asymptotic power–law decay, and suggests that the various curves may be collapsed onto a common master after rescaling the observation time \( t \) by a proper function of the waiting time \( t_w \). The understanding of such scaling properties of \( \Pi(t + t_w, t_w) \) when both \( t \) and \( t_w \) become large \( (t, t_w \gg 1) \) will be the central issue in the following.

III. ANNEALED SITUATION

In the annealed case the model corresponds to a continuous–time random walk (CTRW) irrespective of the value of \( \alpha \), since all times \( \tau_j \) are drawn anew after each transition. The whole process can be characterized by a distribution \( \rho_{\text{eff}}(\tau_{\text{eff}}) \) of effective trapping times \( \tau_{\text{eff}} \) (residence times between two transitions).

In order to calculate \( \rho_{\text{eff}}(\tau_{\text{eff}}) \), let us consider the particle to be at site \( i = 0 \). Then

\[
\tau_{\text{eff}} = \frac{1}{\sum_{j=1}^{2d} \tau_{\text{eff}}^\alpha} = \frac{\tau_0^{1-\alpha}}{\sum_{j=1}^{2d} \tau_j^\alpha},
\]

and the distribution is given by

\[
\rho_{\text{eff}}(\tau_{\text{eff}}) = \left[ \prod_{i=0}^{2d} \int d\tau_i \rho(\tau_i) \right] \delta \left( \tau_{\text{eff}} - \frac{\tau_0^{1-\alpha}}{\sum_{j=1}^{2d} \tau_j^\alpha} \right) \times \delta \left( \tau_0 - \tau_{\text{eff}}^{1/(1-\alpha)} \sum_{j=1}^{2d} \tau_j^\alpha \right)^{1/(1-\alpha)}.
\]

The δ function contributes if \( \tau_{\text{eff}} \sum_{j=1}^{2d} \tau_j^\alpha \geq 1 \), i.e. there is no restriction for \( \tau_{\text{eff}} \geq 1/2d \). Hence, for \( \tau_{\text{eff}} \geq 1/2d \):

\[
\rho_{\text{eff}}(\tau_{\text{eff}}) = C_{\text{eff}} \theta' \tau_{\text{eff}}^{1-\theta'} \quad \theta' = \frac{\theta}{(1-\alpha)},
\]

where \( C_{\text{eff}} = \langle \sum_{j=1}^{2d} \tau_j^p \rangle^{-\theta'} \), and \( \langle \ldots \rangle \) denotes an average over \( 2d \) uncorrelated random numbers \( \tau_j \) distributed according to \( \rho_0 \). The CTRW theory applies with a rescaled temperature \( \theta' \), and the aging properties of this model have been worked out in \([8]\).
distribution. The dynamical phase transition in the annealed model is defined by the diverging mean waiting time of the distribution \( p_{\text{eff}}(\tau_{\text{eff}}) \) that occurs for \( \theta < 1 \). The absence of aging for \( 0 < 1 - \theta < \alpha \), that is when \( \theta > 1 \) but \( 0 < \theta < 1 \), points to the fact that the annealed model may not provide a valid mean field description of the quenched model in any dimension \( d \).

In Fig. 3 the aging function \( \Pi(t_w + t, t_w) \) is shown for the annealed model as resulting from simulations for two different parameter sets, \((d, \theta, \alpha) = (10, 1/3, 0)\) and \((d, \theta, \alpha) = (10, 1/4, 1/4)\) that both refer to the same \( \theta' = 1/3 \). As expected, the aging functions for the two parameter sets are indistinguishable. Moreover, as predicted in \([22]\), \( \Pi(t_w + t, t_w) \) scales with \( t/t_w \), \( \Pi(t_w + t, t_w) = F(t/t_w) \), where \( F(u) \sim 1-u^{-1-x} \) for \( u \ll 1 \) (see Fig. 3b) and \( F(u) \sim u^{-x} \) (see Fig. 3a).

In the following section we present simple arguments why and how \( \Pi(t_w + t, t_w) \) is expected to scale. In particular we propose the presence of two characteristic relaxation times \( t_i(t_w), i = 1, 2 \), that grow as power laws with \( t_w \).

\[
\Pi(t_w + t, t_w) \equiv \left( \sum_{j=1}^{S(t_w)} \tau_j \exp \left( -t \tau_j^{\alpha-1} \sum_{n_j} \tau_j^{\alpha} \right) \right)^{\frac{1}{1+\gamma}}.
\]

Here \( \langle \ldots \rangle \) denotes an average over \((2d-1)S(t_w)\) uncorrelated random numbers \( \tau_j \) that are distributed according to \( F(u) \). We will see that \( \Pi(t_w + t, t_w) \) and \( \Pi(t_w + t, t_w) \) exhibit the same scaling properties, which in view of the generic character of \( \Pi(t_w + t, t_w) \) (cf. Sec. IV B) are of primary interest for us here. In quantitative terms both functions turn out to be different. Since \( \Pi(t_w + t, t_w) \) depends on \( t_w \) only through \( S = S(t_w) \), we will, in the following, also denote this function by \( \Pi(t, S) \).

As shown in appendix \( \[3\] \) the number of distinct visited sites \( S(t_w) \) grows with increasing waiting time \( t_w \) as

\[
S(t_w) \sim t_w^\gamma,
\]

where

\[
\gamma = \begin{cases} 
\frac{d}{d+(2-d)\theta} & \text{for } d < 2 \\
\frac{1}{\theta} & \text{for } d > 2 
\end{cases}
\]

In \( d = 2 \) there are logarithmic corrections, \( S(t_w) \sim [t_w/\log t_w]^{\theta/\gamma} \).

### V. SIMPLE SCALING ARGUMENTS

In this section we present simple arguments, why and how \( \Pi(t_w + t, t_w) \) is expected to scale. In particular we propose the presence of two characteristic relaxation times \( t_i(t_w), i = 1, 2 \), that grow as power laws with \( t_w \).

\[
t_i(t_w) \sim t_w^{\mu_i},
\]

where \( \mu_i \) are exponents depending on \( \theta \) and \( \alpha \).

The first characteristic relaxation time \( t_1(t_w) \) can be obtained as follows. After the waiting time \( t_w \), the particle has visited \( S = S(t_w) \) distinct sites and the occupation probability of these sites is dominated by the one for the site with lowest energy \( E_{\min} \) (in the glassy phase \( \theta < 1 \) the extreme values are dominant). The typical value of \( E_{\min} \) can be estimated from

\[
E_{\min}(t_w) \sim -\ln S(t_w) \sim -\gamma \ln(t_w),
\]

or, in terms of the variable \( \tau_{\max} \) corresponding to \( E_{\min} \),

\[
\tau_{\max}(t_w) = \exp(-E_{\min}/\theta) \sim S(t_w)^{1/\theta} \sim t_w^{\mu_1/\theta}.
\]
that it should occur with a probability 1 of length rare events depicted in Fig. 4b, which turn out to be im-
1
function of Λ
state remains finite when

FIG. 3: Aging functions (a) Π(t + t_w, t_w) and its complement (b) 1 − Π(t + t_w, t_w) from simulations of the annealed model with
parameters (d, θ, α) = (10, 1/3, 0) (lines) and (d, θ, α) = (10, 1/4, 1/4) (points). The waiting times range from 10^3 to 10^12. The
scaling is so good that the various curves cannot be distinguished on the scale of the line thickness.

FIG. 4: Jumps out of a valley with energy E_min after time t_w: (a) a situation corresponding to the typical situation, where
the energies of the neighboring valleys have values close to zero; (b) a rare event where at least one of the neighboring
valleys has an energy of order E_min.

escape time t_1 from the site with energy E_min is the inverse hopping rate given in eq. (1), t_1(t_w) ∼ τ_{max}^{-1−α} ∼ S(1−α)/θ ∼ \tau_{\text{max}}^{\gamma(1−α)/\theta}, i.e.

\mu_1 = \frac{\gamma(1−\alpha)}{\theta} \quad (18)

Hence, for both t and t_w becoming large (t, t_w \gg 1) and \Lambda_1 = t/t_w^{\mu_1} being fixed we expect \tilde{\Pi}(t, S) to become a function of \Lambda_1 only,

\tilde{\Pi}(t, S) \sim F_1(\Lambda_1), \quad \Lambda_1 = t/t_w^{\mu_1}. \quad (19)

The second relaxation time t_2(t_w) is associated with rare events depicted in Fig. 4b, which turn out to be important at small times. The deepest state on an interval of length S has a typical value determined by the fact that it should occur with a probability 1/S. To obtain the scaling of the second deepest state with S we note that for a distribution exhibiting no peculiar long–time tails, the gap between the deepest and second deepest state remains finite when S → ∞. Accordingly, with

probability of order 1/S ∼ τ_{\text{max}}^{-\theta} one of the neighboring sites can also have an energy comparable to E_min. In this case the hopping rate specified in eq. (8) yields

\mu_2 = \frac{\gamma(1−2\alpha)}{\theta} \quad (20)

Clearly this is a relevant time growing with t_w only for \alpha < 1/2. Moreover, there is a further condition on the relevance of t_2. To see this, we may consider the formal short–time expansion of \tilde{\Pi}(t, S) from eq. (13). Using the fact that a sum over random \tau’s raised to some power with an exponent larger than \theta scales as the maximal term appearing in the sum (Lévy statistics), we can estimate

\tilde{\Pi}(t, S) = \left\langle \sum_{m=0}^{\infty} \frac{(-t)^m}{m!} \sum_{j=1}^{S} b_j \tau_j^{-(1−\alpha)m} \left( \sum_{n_j} \tau_{n_j}^\alpha \right)^m \right\rangle

\approx 1 + \sum_{m=1}^{\infty} \frac{(-t)^m}{m! (\tau_{\text{max}}^{1−\alpha})^m} \left[ 1 + \tau_{\text{max}}^{-\theta} \alpha^m \right]. \quad (21)

Here the first term in the rectangular brackets corresponds to the typical situation, while the second term corresponds to the rare events and is weighted by a factor \tau_{\text{max}}^{-\theta} (we have neglected constant prefactors). Comparing these two terms, we see that, if \theta < \alpha, the latter term dominates for all m. In this case we thus expect the scaling behavior of the form

S \left[ 1 − \tilde{\Pi}(t, S) \right] \sim F_2(\Lambda_2), \quad \Lambda_2 = t/t_2(t_w) \quad (22)

for times t, t_w \gg 1 and \theta < \alpha < 1/2.

For \alpha < \theta to the contrary, the first term in the rectangular brackets dominates for small 1 ≤ m ≤ \theta/\alpha, and
be at a site with energy in an interval \((E_1, E_2 + dE_1)\) is 
\[
\psi(E_1) \exp(-E_1/\theta) dE_1 \propto \tau_1^{-1-\theta} \tau_1 d\tau_1 \quad \text{("equilibrated initial site")}
\]
and the probability for the energy of a neighboring site to be in an interval \((E_2, E_2 + dE_2)\) is 
\[
\psi(E_2) dE_2 \propto \tau_2^{-1-\theta} d\tau_2 \quad \text{("random target site")}
\]
Hence the probability \(\psi(t) dt\) for the particle to leave the site reached after \(t_w\) in a time interval \((t, t + dt)\) can be estimated by 
\[
\psi(t, \tau_{\max}) \propto \int_{t_1}^{\tau_{\max}} d\tau_1 \tau_1^{-\theta} \int_{t_2}^{\tau_{\max}} d\tau_2 \tau_2^{-1-\theta} \delta \left( t - \frac{\tau_1^{1-\theta}}{\tau_2^{1-\theta}} \right)
\]
\[
= \phi(t, \tau_{\max}) \quad \text{(25)}
\]
Normalizing \(\psi(t)\) on its support \(\tau_{\max}^{-\alpha} \leq t \leq \tau_{\max}^{1-\alpha}\) we obtain 
\[
\psi(t, \tau_{\max}) = \frac{\phi(t, \tau_{\max})}{\tau_{\max}^{1-\alpha}} \quad \text{(26)}
\]
Since \(1 - \tilde{\Pi}(t_w + t, t_w)\) is the probability of the particle to leave the site reached after \(t_w\) within the time interval \([0, t)\), we expect 
\[
1 - \tilde{\Pi}(t_w + t, t_w) \sim \int_{0}^{t} dt' \psi(t', \tau_{\max}(t_w)) \quad \text{(27)}
\]
for \(t, t_w \gg 1\) and \(t/t_w^{\mu_2} \ll 1\). Performing the integrals in eq. (25) gives 
\[
\phi(t, \tau_{\max}) = \frac{1 - \alpha}{\alpha - \theta} \tau_{\max}^{-\theta/\alpha} \left[ \left( \frac{t}{\tau_{\max}^{1-\alpha}} \right)^{\frac{\theta}{1-\alpha}} - 1 \right] \quad \text{(28)}
\]
Since \(t/\tau_{\max}^{1-\alpha} \sim \Lambda_1\), we have, in the limit \(\Lambda_1 \to 0\), to distinguish between \(\alpha > \theta\) and \(\alpha < \theta\), yielding 
\[
\phi(t, \tau_{\max}) \sim \begin{cases} 
\frac{1 - \alpha}{\alpha - \theta} t^{-1+\theta/\alpha} \tau_{\max}^{1-\theta/\alpha}, & \alpha > \theta \\
\frac{1 - \alpha}{\theta - \alpha} t^{-1+\theta/\alpha}, & \alpha < \theta 
\end{cases} \quad \text{(29)}
\]
Inserting this into eqs. (26,27) and using (17) then gives 
\[
1 - F_1(\Lambda_1) \sim \Lambda_1^\theta, \quad \Lambda_1 \to 0 \quad \text{(30a)}
\]
with 
\[
\varepsilon \equiv \begin{cases} 
\varepsilon > \equiv \frac{\theta}{\alpha}, & \alpha > \theta \\
\varepsilon < \equiv \frac{1 - \theta}{1 - \alpha}, & \alpha < \theta 
\end{cases} \quad \text{(30b)}
\]
As discussed above, by matching this small \(\Lambda_1\) behavior of \(F_1(\Lambda_1)\) to the large \(\Lambda_2\) behavior of \(F_2(\Lambda_2)\) we furthermore obtain for \(\alpha > \theta\) 
\[
F_2(\Lambda_2) \sim \Lambda_2^{\theta/\alpha}, \quad \Lambda_2 \to \infty \quad \text{(31)}
\]

VI. TEST OF THE PEC
FIG. 5: First scaling function for a parameter set with $\alpha > \theta$: (a) $\Pi(t_w + t, t_w)$ and (b) $1 - \Pi(t_w + t, t_w)$ as a function of $t/t_w^{\mu_2}$ for $d = 1$, 10, 100 and 1000 by means of a continuous–time Monte–Carlo algorithm (see appendix A for details of the simulation procedure). Averages were taken over $10^5$ (for $d = 1, 10, 100$) and $10^4$ (for $d = 1000$) energy landscapes.

Figures 3 and 4 show, respectively, $\Pi(t + t_w, t_w)$ and $1 - \Pi(t + t_w, t_w)$ as a function of the scaled variable $\Lambda_1 = t/t_w^{\mu_1}$ (cf. eqs. (30a,b) for $(\theta, \alpha) = (1/4, 3/8)$, i.e. a case where $\alpha > \theta$. As expected from the PEC and the scaling arguments outlined in the previous section, the data collapse onto master curves $F_1(\Lambda_1)$ for all dimensions $d$. In particular we find $F_1(\Lambda_1) \sim \Lambda_1^{\epsilon_\gamma}$ in agreement with eqs. (30a,b) and (24), respectively.

Correspondingly scaled data for $(\theta, \alpha) = (1/3, 1/4)$, i.e. a case where $\alpha < \theta$, are shown in fig. 5. Again there is a good data collapse and the scaling functions $F_1(\Lambda_1)$ have the expected asymptotic behavior for small and large $\Lambda_1$. In particular we now find $F_1(\Lambda_1) \sim \Lambda_1^{\epsilon_\gamma}$ with $\epsilon_\gamma$ from eq. (30a).

Moreover, for $\alpha > \theta$, PEC and the scaling arguments predict the presence of a second time scale $t_2 = t_w^{\mu_2}$ [cf. eq. (20)] and an associated generalized scaling $t_w^{\mu_2} [1 - \Pi(t + t_w, t_w)] \sim F_2(t/t_w^{\mu_2})$ [cf. eqs. (22, 24)]. The occurrence of this generalized scaling is verified in fig. 6 for $d = 1, 10$ and 100. The master curves scale $F_2(\Lambda_2) \sim \Lambda_2^{\epsilon_\alpha}$ for $\Lambda_2 \to \infty$ as predicted by eq. (31). The critical reader may note that the simulated data do not collapse at large $\Lambda_2$, the deviations from scaling setting in at larger $\Lambda_2$ for larger $t_w$. The reason for this is that the limits $t_w \to \infty$ and $\Lambda_2 = t/t_w^{\mu_2} \to \infty$ must not be commuted. One first has to take the limit $t_w \to \infty$ to obtain the scaling function $F_2(\Lambda_2)$ and then has to consider the asymptotic behavior for large $\Lambda_2$.

The behavior for $\Lambda_2 \to 0$ is not so clear. For $d > 1$, the exact treatment of the PEC formula (13) yields $F_2(\Lambda_2) \sim \Lambda_2^{\epsilon_\alpha}$ also (with a smaller prefactor in the scaling law), while for $d = 1$, $F_2(\Lambda_2) \sim \Lambda_2$ seems to be correct. In fact, for $d = 1$ the data in fig. 6 are in fair agreement with a linear behavior. For $d = 100$ the data also show some linear dependence for intermediate $\Lambda_2$ values (see the dashed lines in fig. 6), and in fact, such intermediate regime is predicted to occur based on the PEC formula (although it should be less pronounced, see the discussion in sec. VII F). For very small $\Lambda_2$, however, we expect $F_2(\Lambda_2) \sim \Lambda_2^{\epsilon_\alpha}$, although this asymptotic regime can not be identified in fig. 6. At best we can say that there is a change in curvature at very small $\Lambda_2$. The true asymptotics, however, could not be obtained within reasonable CPU time.

Accounting for the existence of two different time scales is very important to properly rescale the numerical results. Had one assumed a single time scale $t_w^{\mu_2}$ one would have obtained an approximate data collapse with an effective value of $\mu$ intermediate between $\mu_1$ and $\mu_2$. This remark might be of importance for analyzing experimental data: the assumption of a single time scale

A. Scaling properties

In order to test the PEC, we performed simulations of the model in $d = 1, 10, 100$ and 1000 by means of a continuous–time Monte–Carlo algorithm (see appendix A for details of the simulation procedure). Averages were taken over $10^5$ (for $d = 1, 10, 100$) and $10^4$ (for $d = 1000$) energy landscapes.

Figures 5a and 5b show, respectively, $\Pi(t + t_w, t_w)$ and $1 - \Pi(t + t_w, t_w)$ as a function of the scaled variable $\Lambda_1 = t/t_w^{\mu_1}$ (cf. eqs. (30a,b) for $(\theta, \alpha) = (1/4, 3/8)$, i.e. a case where $\alpha > \theta$. As expected from the PEC and the scaling arguments outlined in the previous section, the data collapse onto master curves $F_1(\Lambda_1)$ for all dimensions $d$. In particular we find $F_1(\Lambda_1) \sim \Lambda_1^{\epsilon_\gamma}$ in agreement with eqs. (30a,b) and (24), respectively.

Correspondingly scaled data for $(\theta, \alpha) = (1/3, 1/4)$, i.e. a case where $\alpha < \theta$, are shown in fig. 6. Again there is a good data collapse and the scaling functions $F_1(\Lambda_1)$ have the expected asymptotic behavior for small and large $\Lambda_1$. In particular we now find $F_1(\Lambda_1) \sim \Lambda_1^{\epsilon_\gamma}$ with $\epsilon_\gamma$ from eq. (30a).

Moreover, for $\alpha > \theta$, PEC and the scaling arguments predict the presence of a second time scale $t_2 = t_w^{\mu_2}$ [cf. eq. (20)] and an associated generalized scaling $t_w^{\mu_2} [1 - \Pi(t + t_w, t_w)] \sim F_2(t/t_w^{\mu_2})$ [cf. eqs. (22, 24)]. The occurrence of this generalized scaling is verified in fig. 6 for $d = 1, 10$ and 100. The master curves scale $F_2(\Lambda_2) \sim \Lambda_2^{\epsilon_\alpha}$ for $\Lambda_2 \to \infty$ as predicted by eq. (31). The critical reader may note that the simulated data do not collapse at large $\Lambda_2$, the deviations from scaling setting in at larger $\Lambda_2$ for larger $t_w$. The reason for this is that the limits $t_w \to \infty$ and $\Lambda_2 = t/t_w^{\mu_2} \to \infty$ must not be commuted. One first has to take the limit $t_w \to \infty$ to obtain the scaling function $F_2(\Lambda_2)$ and then has to consider the asymptotic behavior for large $\Lambda_2$.

The behavior for $\Lambda_2 \to 0$ is not so clear. For $d > 1$, the exact treatment of the PEC formula (13) yields $F_2(\Lambda_2) \sim \Lambda_2^{\epsilon_\alpha}$ also (with a smaller prefactor in the scaling law), while for $d = 1$, $F_2(\Lambda_2) \sim \Lambda_2$ seems to be correct. In fact, for $d = 1$ the data in fig. 6 are in fair agreement with a linear behavior. For $d = 100$ the data also show some linear dependence for intermediate $\Lambda_2$ values (see the dashed lines in fig. 6), and in fact, such intermediate regime is predicted to occur based on the PEC formula (although it should be less pronounced, see the discussion in sec. VII F). For very small $\Lambda_2$, however, we expect $F_2(\Lambda_2) \sim \Lambda_2^{\epsilon_\alpha}$, although this asymptotic regime can not be identified in fig. 6. At best we can say that there is a change in curvature at very small $\Lambda_2$. The true asymptotics, however, could not be obtained within reasonable CPU time.

Accounting for the existence of two different time scales is very important to properly rescale the numerical results. Had one assumed a single time scale $t_w^{\mu_2}$ one would have obtained an approximate data collapse with an effective value of $\mu$ intermediate between $\mu_1$ and $\mu_2$. This remark might be of importance for analyzing experimental data: the assumption of a single time scale
could lead to a systematic underestimation of the true, asymptotic value of \( \mu \) (see the discussion in [14]).

**B. Full comparison with simulations**

While the predictions of PEC regarding the scaling properties and the asymptotic behavior of the scaling functions are fulfilled, \( \Pi(t + t_w, t_w) \) and its PEC equivalent \( \bar{\Pi}(t + t_w, t_w) \) are different in quantitative terms. To see this, we have computed \( \bar{\Pi}(t + t_w, t_w) \) numerically according to eq. (13) and compared it with \( \Pi(t + t_w, t_w) \) obtained from the Monte–Carlo simulations. As shown in fig. 8, the scaling functions associated with \( \Pi(t + t_w, t_w) \) and \( \bar{\Pi}(t + t_w, t_w) \) differ by a factor in the scaling variable and the precise form in a transient region between the asymptotic regimes for small and large arguments.

In summary we can conclude that all predictions of the PEC concerning the scaling properties can be corroborated by the simulations. The PEC thus turns out to be a powerful tool to uncover the mechanisms of aging in quenched random energy landscapes.

**VII. EXACT EVALUATION OF THE PARTIAL EQUILIBRIUM FORMULA**

In this part we show how the scaling arguments presented in sec. VI can be validated by an exact evaluation of the PEC formula (13) in the limit of large \( S \) (large \( t_w \)). The reader who is not interested in these more mathematical derivations, may skip this section and proceed with the summary in sec. VIII.

When we replace the denominator of eq. (13) by \( \int_0^\infty d\lambda \exp(-\lambda \sum_{k=1}^S \tau_k) \) and notice that all but three random \( \tau_k \) are uncorrelated with the \( \tau_j, \tau_{n_j} \) appearing in the numerator of (13), we find

\[
\bar{\Pi}(t, S) = S \int_0^\infty d\lambda \exp(-\lambda) \langle e^{-\lambda \tau} \rangle^{S-3} g(t; \lambda),
\]

where \( g(t; \lambda) \) is defined by

\[
g(t; \lambda) = \bigg\langle \tau \exp \left( -t \sum_{j=1}^{2d} \tau_j^\alpha - \lambda \sum_{j=1}^{2d} (\tau_j + \tau_1 + \tau_2) \right) \bigg\rangle
\]

\[
= \int_1^\infty \frac{\theta d\tau}{\tau^\theta} e^{-\lambda \tau} \int_1^\infty \frac{\theta d\tau_1}{\tau_1^\theta} e^{-\lambda \tau_1} \int_1^\infty \frac{\theta d\tau_2}{\tau_2^\theta} e^{-\lambda \tau_2} \times \exp \left( -t \frac{\tau_1^\alpha + \tau_2^\alpha}{\tau_1^\alpha + \tau_2^\alpha} \right) \bigg[ f \left( \frac{t}{\tau_1} \right) \bigg]^{2(d-1)}
\]

and

\[
f(z) = \int_1^\infty \frac{\theta d\tau}{\tau^{1+\theta} e^{-\lambda \tau}} = \frac{\theta}{\alpha} z^{\theta/\alpha} \Gamma \left( 1 - \frac{\theta}{\alpha}, \frac{z}{\alpha} \right)
\]

Here \( \Gamma(a, z) = \int_0^z dt e^{-t^a} \) denotes the incomplete Gamma function. Note that 0 \( \leq f(z) \leq 1 \) and that for \( z \to 0 \)

\[
f(z) = 1 - \Gamma \left( 1 - \frac{\theta}{\alpha}, \frac{z}{\alpha} \right) - \frac{\theta}{\theta - \alpha} z + O(z^2),
\]

where \( \Gamma(a) = \Gamma(a, 0) \) is the Gamma function. For \( z \to \infty \) \( |argz| < 3\pi/2 \)

\[
f(z) \sim \frac{\theta}{\alpha} e^{-\frac{z}{\alpha}}.
\]
In the limit \( S \to \infty \), the asymptotic form of eq. (B2) is [see eq. (C3)]

\[
\tilde{\Pi}(t, S) \approx S \int_0^\infty d\lambda e^{-\lambda S} g(t; \lambda),
\]
where

\[
\tilde{S} \equiv \kappa S, \quad \kappa \equiv \Gamma(1 - \theta).
\]

A. First scaling regime: \( F_1(\Lambda_1) \)

By substituting \( u = \tilde{S}^{1/\theta} \lambda \) in eq. (33) and \( v = \tau/\tilde{S}^{1/\theta} \) in eq. (33) we obtain

\[
\tilde{\Pi}(t, S) \sim \frac{\theta}{\kappa} \int_0^\infty du e^{-u^\theta} \int_{\tilde{S}/u}^\infty \frac{dv}{v^{\theta}} e^{-u v} 
\times \int_1^\infty \theta d\tau_1 e^{-u \tau_1/\tilde{S}^{1/\theta}} \int_1^\infty \theta d\tau_2 e^{-u \tau_2/\tilde{S}^{1/\theta}}
\times \exp \left( -t \tilde{S}^{1/\theta} \tau_1^{\alpha} + \tau_2^{\alpha} \right) \left[ f \left( \frac{t \tilde{S}^{1-\alpha}}{v^{1-\alpha}} \right) \right]^{2(d-1)}.
\]

When keeping

\[
\Lambda_1 \equiv t \tilde{S}^{-(1-\alpha)/\theta}
\]
fixed, (33) has a well defined limit for \( \tilde{S} \to \infty \). Before taking this limit, however, let us note at this point that if we would have considered all \( \tau_n \) in eq. (13) to belong to the Brownian path, we had obtained

\[
\tilde{\Pi}(t, S) \sim \frac{\theta}{\kappa} \int_0^\infty du e^{-u^\theta} \int_{\tilde{S}/u}^\infty \frac{dv}{v^{\theta}} e^{-u v} 
\times \prod_{i=1}^{2d} \left[ \int_0^\infty \theta d\tau_i \exp \left( -\frac{u \tau_i}{\tilde{S}^{1/\theta}} - \Lambda_1 \frac{\tau_i^{\alpha}}{v^{1-\alpha}} \right) \right]
\]

instead of (37). Now, \( v^{-\theta} \exp(-u^\theta - uv) \prod_{i=1}^{2d} \theta \tau_i^{-1-\theta} \) is an integrable majorant for the integrand in both eqs. (37,39). Hence, by Lebesgue’s theorem we obtain from both equations the same scaling function

\[
F_1(\Lambda_1) \equiv \frac{\theta}{\kappa} \int_0^\infty du e^{-u^\theta} \int_0^\infty \frac{dv}{v^{\theta}} e^{-u v} \left[ f \left( \frac{\Lambda_1}{v^{1-\alpha}} \right) \right]^{2d}.
\]

We can conclude that it makes no difference here whether we consider the path of distinct visited sites to have a one-dimensional or a compact topology or anything in between. As required by normalization, \( F_1(0) = 1 \).

B. Limit \( \Lambda_1 \to \infty \)

When transforming to variables \( \xi \equiv \Lambda_1^{1/(1-\alpha)} u \) and \( \zeta \equiv \Lambda_1^{1/(1-\alpha)} v \) in eq. (40), we find

\[
F_1(\Lambda_1) = \Lambda_1^{-\theta/(1-\alpha)} \frac{\theta}{\kappa} \int_0^\infty d\xi e^{-\Lambda_1^{\theta/(1-\alpha)} \xi^\theta} 
\times \int_0^\infty d\zeta \zeta^{-\theta} e^{-\zeta} f^{2d}(\zeta^{-(1-\alpha)}) .
\]

Since the integrand in the limit \( \Lambda_1 \to \infty \) is an integrable majorant for all \( \Lambda_1 \), we can take the \( \Lambda_1 \to \infty \) limit under the integral to obtain

\[
F_1(\Lambda_1) \sim c_\infty \Lambda_1^{-\theta/(1-\alpha)} \quad \text{for} \quad \Lambda_1 \to \infty ,
\]
where

\[
c_\infty \equiv \frac{\theta}{\kappa} \int_0^\infty d\zeta \zeta^{-\theta} f^{2d}(\zeta^{-(1-\alpha)}) .
\]

Note that this integral is well defined because of the asymptotic behavior of \( f(.) \) given in eqs. (34b,c).

C. Limit \( \Lambda_1 \to 0 \)

Since \( F_1(\Lambda_1) \to 1 \) for \( \Lambda_1 \to 0 \), it is convenient to consider

\[
1 - F_1(\Lambda_1) = \frac{\theta}{\kappa} \int_0^\infty du e^{-u^\theta} \int_0^\infty \frac{dv}{v^{\theta}} e^{-uv} 
\times \left[ 1 - f^{2d} \left( \frac{\Lambda_1}{v^{1-\alpha}} \right) \right].
\]
For $\theta < \alpha$ it follows from eqs. (34a-c) $1 - f(x)^{2d} = 2dGamma(1 - \theta/\alpha) x^{\theta/\alpha} \varphi_<(x)$, where $\varphi_<(x)$ is a bounded function, $\varphi_<(x) < M_<$ for $0 \leq x < \infty$, with $\varphi_<(x) \to 1$ for $x \to 0$. Hence,

$$1 - F_1(\Lambda_1) = 2d Gamma\left(1 - \frac{\theta}{\alpha}\right) \frac{\theta}{\kappa} \frac{\Lambda_1^{\theta/\alpha}}{\Gamma(1 - \theta/\alpha)} \int_0^\infty du e^{-u^\theta} \times \int_0^\infty \frac{dv}{v^{\theta/\alpha}} e^{-uv} \varphi_<(\frac{\Lambda_1}{v^{1/\alpha}})$$

(43)

Since $M_<^{1/\theta} \exp(-uv - u^\theta)$ is an integrable majorant of the integrand, we find

$$1 - F_1(\Lambda_1) \sim c_< \frac{\Lambda_1^{\theta/\alpha}}{\Lambda_1} \quad \Lambda_1 \to 0 \quad \theta < \alpha \quad (44a)$$

where

$$c_< = \frac{2d \theta \Gamma(1 - \frac{\theta}{\alpha})}{\Gamma(1 - \theta)} \int_0^\infty du e^{-u^\theta} \int_0^\infty \frac{dv}{v^{\theta/\alpha}} e^{-uv} = \frac{2d \theta \Gamma(1 - \frac{\theta}{\alpha})^2 \Gamma(1/\alpha)}{\Gamma(1 - \theta).}$$

(44b)

For $\theta > \alpha$ the $v$ integral in eq. (43) would become divergent when taking the $\Lambda_1 \to 0$ limit in $\varphi_<(\Lambda_1/v^{1-\alpha})$. We thus transform the $v$ variable in eq. (42), $w = \Lambda_1/v^{1-\alpha}$, to obtain

$$1 - F_1(\Lambda_1) = \frac{\theta \Lambda_1^{1-\theta}}{(1-\alpha) \kappa} \int_0^\infty du e^{-u^\theta} \int_0^\infty \frac{dv}{w^{1+\frac{\theta}{1-\alpha}}} \times \left[1 - f^{2d}(w)\right] \exp\left(\frac{u \Lambda_1^{1-\theta}}{w^{1+\frac{\theta}{1-\alpha}}}\right)$$

(45)

From eqs. (34a-c) it follows for $\theta > \alpha$ that $1 - f^{2d}(w) = w \varphi_>(w)$, where $\varphi_>(w)$ is a bounded function, $\varphi_>(w) < M_>$ for $0 \leq w < \infty$, with $\varphi_>(w) \to (\theta - \alpha)/2d\theta$ for $w \to 0$. Hence, $M_> \exp(-w^\theta w^{1-\theta})^{(1-\alpha)}$ is an integrable majorant of the integrand in (43) and we can take the $\Lambda_1 \to 0$ limit under the integral, yielding

$$1 - F_1(\Lambda_1) \sim c_> \frac{\Lambda_1^{1-\theta}}{\Lambda_1} \quad \Lambda_1 \to 0 \quad \theta > \alpha \quad (46a)$$

where

$$c_> = \frac{\Gamma(\frac{\theta}{\alpha})}{(1-\alpha) \Gamma(1 - \theta)} \int_0^\infty \frac{dv}{w^{1+\frac{\theta}{1-\alpha}}} \left[1 - f^{2d}(w)\right].$$

(46b)

### D. Second scaling regime: $F_2(\Lambda_2)$

The second scaling regime is more difficult to extract from eqs. (34a-c). We start by taking advantage of the normalization $\Pi(t=0,S) \sim 1$ in order to write

$$1 - \Pi(t,S) \sim \frac{\theta}{\kappa} \int_0^\infty du e^{-u^\theta} \int_{\Delta - 1/\theta}^\infty \frac{dv}{v^{\theta/\alpha}} e^{-uv} \left(\prod_{j=1}^{2d} \int_1^\infty \frac{d\tau_j}{\tau_j^{1+\theta}}\right) \exp\left(-\frac{u}{\tau_j} \sum_{k=1}^n \tau_k\right) \times \left[1 - f^{2d}(w)\right] \exp\left(\frac{u \Lambda_1^{1-\theta}}{w^{1+\frac{\theta}{1-\alpha}}}\right).$$

(47)

This corresponds to eqs. (47b, 33), if the number of neighboring sites belonging to the Brownian path is $n$, $2 \leq n \leq 2d$ (see the discussion in sec. [34]).

After the transformation $v \to w = \Lambda_1 \sum_{j=1}^{2d} \tau_j^{\alpha} / w^{1-\alpha}$
this can be rewritten as
\[
1 - \overline{\Pi}(t, S) \sim \frac{\theta \Lambda_1}{(1 - \alpha) \kappa} \int_0^\infty du \, e^{-u^\alpha} u^{-(\alpha - \theta)}
\]
\[
\times \left( \prod_{j=1}^{2d} \int_0^\infty \frac{d\tau_j}{\tau_j^{1+s/\theta}} \right) \exp \left( -\frac{u}{S \theta^{1/\theta}} \sum_{k=1}^n \tau_k \right) \left( \sum_{j=1}^{2d} e^{\alpha \tau_j} \right)^{\frac{1+\theta}{1-\theta}}
\]
\[
\times \left[ \Lambda_1 u^{1-\alpha} \sum_{j=1}^{2d} \tau_j^\alpha \right]^{\frac{1+\theta}{1-\theta}} \int_0^{\tau_{2d+1}} \frac{\tau_j^\alpha d \tau_j}{(1 - e^{-w}) w^{1+\frac{1+\theta}{1-\theta}}} \exp \left( -\frac{\Lambda_1 u^{1-\alpha} \sum_{j=1}^{2d} \tau_j^\alpha}{w^{1+\frac{1+\theta}{1-\theta}}} \right).
\]

We now decompose this expression into the sum of two parts, corresponding to a decomposition \( \sum_{j=1}^n \tau_j^\alpha + \sum_{j=n+1}^{2d} \tau_j^\alpha \) in the second line. We can use the symmetry with respect to \( \{\tau_j\} \) in order to replace \( \sum_{j=1}^n \tau_j^\alpha \) by \( n \tau_1^\alpha \) in the first part and \( \sum_{j=n+1}^{2d} \tau_j^\alpha \) by \( (2d - n) \tau_{2d}^\alpha \) in the second part. After the transformation \( \zeta = \tilde{S}^{-1/\theta} \tau_1 \) in the first part, and the transformation \( \zeta = \tilde{S}^{-1/\theta} \tau_{2d} \) in the second part one can, for \( \theta < \alpha \), again use Lebesgue's theorem to perform the limit \( S \to \infty \) for fixed
\[
\Lambda_2 = \tilde{S}^{1/\theta} \Lambda_1 = \frac{t}{\tilde{S}^{(1-2\alpha)/\theta}}, \quad \theta < \alpha < \frac{1}{2}.
\]
This yields
\[
\tilde{S}[1 - \Pi(t, S)] \sim F_2(\Lambda_2),
\]
\[
F_2(\Lambda_2) = n F_2^{(1)}(\Lambda_2) + (2d - n) F_2^{(2)}(\Lambda_2),
\]
where
\[
F_2^{(1)}(\Lambda_2) = \frac{\theta^2 \Lambda_2}{(1 - \alpha) \kappa} \int_0^\infty du \, e^{-u^\alpha} u^{-(\alpha - \theta)}
\]
\[
\times \int_0^\infty \frac{d\zeta e^{-u^{\alpha - \theta}}}{\zeta^{1-(\alpha - \theta)}} h(\Lambda_2 u^{1-\alpha} \zeta^\alpha),
\]
\[
h(x) \equiv x^{1-\alpha} \int_0^\infty \frac{dw (1 - e^{-w})}{w^{1+\frac{1+\theta}{1-\theta}}} e^{-x^{(1/\alpha)}/(1-\alpha)},
\]
and
\[
F_2^{(2)}(\Lambda_2) = \frac{\theta^2 \Lambda_2}{(1 - \alpha) \kappa} \int_0^\infty du \, e^{-u^\alpha} u^{-(\alpha - \theta)} \int_0^\infty \frac{d\zeta h(\Lambda_2 u^{1-\alpha} \zeta^\alpha)}{\zeta^{1-(\alpha - \theta)}}
\]
\[
= \frac{\theta^2 \Lambda_2 \theta/\alpha}{\alpha(1 - \alpha) \kappa} \int_0^\infty du \, e^{-u^\alpha} u^{-(\alpha - \theta)} \int_0^\infty \frac{d\zeta h(v)}{\zeta^{1-(\alpha - \theta)}}
\]
\[
= \frac{\Gamma(1 - \frac{\theta}{2})^2 \Gamma(\frac{1}{\alpha})}{\Gamma(1 - \theta)} \Lambda_2 \theta/\alpha.
\]
The restriction to \( \alpha < 1/2 \) in eq. (52) follows from the fact that in order for the scaling regime to be relevant, \( t_2 \sim \tilde{S}^{(1-2\alpha)/\theta} \sim \tau_2^{(1-2\alpha)/\theta} \) should increase with increasing \( t_w \). The function \( h(x) \) has the asymptotic behavior
\[
h(x) \sim \begin{cases} (1 - \alpha) \Gamma(\alpha - \theta), & x \to 0^+ \\ (1 - \alpha) \Gamma(1 - \theta) x^{-1}, & x \to \infty. \end{cases}
\]
(53)

The importance of the condition \( \theta < \alpha \) now becomes clear, since for the \( \zeta \) integrand to be integrable in eqs. (50, 52) for \( \zeta \to 0, \alpha - \theta \) has to be positive.

E. Limit \( \Lambda_2 \to \infty \)

After the transformation \( \zeta \to v = \Lambda_2 u^{1-\alpha} \zeta^\alpha \), eq. (50) gives
\[
F_2^{(1)}(\Lambda_2) = \frac{\theta^2 \Lambda_2^{\theta/\alpha}}{\alpha(1 - \alpha) \kappa} \int_0^\infty du \, e^{-u^\alpha} u^{-(\alpha - \theta)}
\]
\[
\times \int_0^\infty \frac{dv}{v^{\theta/\alpha}} e^{-u^{1/\alpha} \Lambda_2^{\theta/\alpha} u^{-(\alpha - 1)/\alpha} \Gamma(1 - \theta)} h(v).
\]
(54)

The limit \( \Lambda_2 \to \infty \) can then be taken under the integral, yielding [cf. eq. (52)]
\[
F_2^{(1)}(\Lambda_2) \sim F_2^{(2)}(\Lambda_2).
\]
(55)

Using eqs. (50, 52) we finally obtain
\[
F_2(\Lambda_2) \sim c_2^{(\infty)} \Lambda_2^{\theta/\alpha}, \quad c_2^{(\infty)} = \frac{2d \Gamma(1 - \frac{\theta}{2})^2 \Gamma(\frac{1}{\alpha})}{\Gamma(1 - \theta)}. \quad (56)
\]
As required by matching, \( c_2^{(\infty)} = c_\infty \) [cf. eq. (44b)].

F. Limit \( \Lambda_2 \to 0 \)

Since \( h(x) \) is monotonically decreasing with \( x \) for \( x > 0 \), we can replace \( h(\Lambda_2 u^{1-\alpha} \zeta^\alpha) \) by \( h(0) = (1 - \alpha) \Gamma(\alpha - \theta) \) in eq. (55). Hence,
\[
F_2^{(1)}(\Lambda_2) \sim c_2 \Lambda_2,
\]
(57)

where
\[
c_2 = \frac{\theta^2 \Gamma(\alpha - \theta)}{\kappa} \int_0^\infty du \, e^{-u^\alpha} u^{-(\alpha - \theta)} \int_0^\infty \frac{d\zeta e^{-u^{\alpha - \theta}}}{\zeta^{1-(\alpha - \theta)}}
\]
\[
= \frac{\theta}{1 - \alpha} \frac{\Gamma(\alpha - \theta) \Gamma\left(\frac{1-2(\alpha-\theta)}{\theta}\right)}{\Gamma(1-\theta)}.
\]
(58)

This means that for small \( \Lambda_2 \), \( F_2(\Lambda_2) \) should be dominated by \( F_2^{(2)}(\Lambda_2) \), except for \( d > 1 \) (where \( F_2(\Lambda_2) = 2F_2^{(1)}(\Lambda_2) \)). For a given (mean) value of \( n \) we thus find
\[
F_2(\Lambda_2) \sim c_2^{(0)} \Lambda_2^{\theta/\alpha}, \quad c_2^{(0)} \equiv \frac{1 - \frac{n}{2d}}{c_2^{(\infty)}}.
\]
(59)

This \( \Lambda_2 \to 0 \) limit of the generalized scaling form depends on the number of neighbors being considered to belong
to the Brownian path (see the discussion in sec. [13]). According to eq. (57), there can occur an intermediate regime, where \( F_2(\Lambda_2) \) depends linearly on \( \Lambda_2 \). This intermediate regime seems to be more pronounced for the “true dynamics” (see Fig. 6) than for the dynamics predicted by the PEC formula (13).

VIII. SUMMARY AND CONCLUSIONS

We have studied aging within the framework of a simple hopping model mimicking a system that performs thermally activated transitions between the deep free energy minima of its configuration space. Based on general arguments from the statistical theory of extremes we have chosen the free energy density of states to exhibit an exponential tail. In order to effectively quantify the influence of the initial and target site on the energy barrier to be surmounted during a transition, we introduced a parameter \( \alpha, 0 \leq \alpha < 1 \), in the hopping rates that turned out to strongly influence the aging properties. We have found that generically, subaging occurs in these models, an effect related to the multiple visits of deep traps. We have also found that different time scales, corresponding to different scaling regions, appear in these models.

These aging properties can be understood from a partial equilibrium concept (PEC) that, despite not being an exact quantitative description, provides a powerful tool to study the scaling properties of the aging dynamics. Based on the PEC we first motivated the occurrence of subaging behavior and generalized scaling forms in terms of simple scaling arguments. We then presented a detailed analysis of the PEC formula (13) and calculated the aging functions following from (13) and their asymptotics exactly. With respect to the scaling properties the predictions could be confirmed by Monte-Carlo simulations in \( d = 1, 10, 100 \) and 1000 dimensions. The fact that even for \( d = 1000 \) the “quenched model” has aging properties different from the “annealed model” studied earlier in [15] is rather surprising, since the number of distinct visited sites \( S \) in \( d > 2 \) scales as the number of all transitions \( N \) between minima for large \( N \), \( S \sim N \). From this one tends to conclude that the system effectively explores a new minimum in each transition, which would correspond to the annealed situation. However, in the quenched situation one can imagine that there is always some local equilibrium established at the site with minimal energy reached after time \( t_w \), and this local equilibration effect slows down the diffusion in configuration space on all time scales, i.e. instead of \( S \sim N \sim t_w^\theta \) with \( \theta = \min(1, \theta/(1 - \alpha)) \) in the annealed situation we have \( S \sim N \sim t_w^\theta \) in the quenched situation for \( d > 2, 0 < \theta < 1 \).

The existence of a local equilibrium around the “dominant” site with minimal energy after time \( t_w \) does not imply that there must be a true equilibrium on all visited sites as it is assumed in the PEC formula (13). In fact, by studying the disorder averaged participation ratios \( Y_q(t) \equiv \langle \sum_j P_j(t)q \rangle \), where \( P_j(t) \) is the probability for the system to be at minimum \( j \) at time \( t \), we find that the PEC never becomes exact in the limit \( t \to \infty \), not even in \( d = 1 \) where each trap is visited an infinite number of times. This behavior offers the possibility to define an effective temperature in the non-equilibrium aging regime, which enters a modified fluctuation-dissipation theorem, similar as it was found for mean-field spin glass models [28]. The role of an effective temperature in the landscape model considered here will be discussed elsewhere [50] (for recent progress in our understanding of this problem coming from MD simulations, see [51]).

With respect to the applicability of the analysis outlined above the question arises, whether the characteristics of the aging dynamics can be worked out also for general hopping rates not exhibiting the specific form given in eq. (9). For the PEC to be applicable, the system should have the tendency to approach equilibrium (that truly exists only for \( \theta > 1 \)), so that one may require the jump rates to obey detailed balance. It is then indeed straightforward, by using the simple arguments presented in sec. [13] to extract all characteristic time scales \( t_j(t_w) \) and to predict the scaling properties. A necessary ingredient for this procedure to work correctly, however, is the robustness of the scaling relation \( S(t_w) \sim t_w^\gamma [\text{cf. eqs. (14a,b)}] \). Preliminary results indicate that for “physical choices” of the jump rates (meaning that the dependence of the energies of the initial and target site on the saddle point energy is reasonable), eqs. (14a,b) always hold true.

Moreover, it is possible also to consider some random distribution of the form of the jump rates (as it is expected to occur when the dynamics in configuration space is mapped onto a jump process by means of some quantitative analysis), and to work out the aging features of such more realistic models. It turns out then, that in principle infinitely many aging regimes can exist in the two-time plane \( t, t_w \geq 0 \). A thorough discussion of these issues, however, is beyond the scope of the present work.

ACKNOWLEDGMENTS

We should like to thank W. Dieterich and E. Bertin for stimulating discussions. P.M. gratefully acknowledges financial support from the Deutsche Forschungsgemeinschaft by a Heisenberg fellowship (Ma 1636/2-1).

APPENDIX A: TECHNICAL DETAILS OF THE MONTE-CARLO SIMULATIONS

We use the standard continuous-time Monte Carlo algorithm as discussed in detail e.g. in [19] to simulate the stochastic process defined in sec. [1A]. A special problem arises for large dimensions \( d \gg 1 \), where it is not possible to save the energies within a hypercube of even small
linear dimension. To resolve this problem, we use hash-maps, as, for example, the hash_map template provided by the Standard Template Library of ISO-C++.

The hash-function should be computable quickly and at the same time the sites \( x \) being encountered must be mapped to different hash-values as often as possible. For dimensions \( d \geq 10 \) we found

\[
f(x) = \sum_{n=1}^{2d} n x_n \quad \text{(A1)}
\]

to do a good job. For large \( d \) the RAM consumption of the computer program is the limiting factor when trying to access longer times \( t_w \). While simulations for \( d \leq 10 \) can easily be performed on workstations, for \( d \geq 100 \) computers with 4GB of RAM and more are necessary.

**APPENDIX B: CONNECTION BETWEEN THE “NUMBER OF DISTINCT VISITED SITES” AND THE WAITING TIME**

For \( \alpha = 0 \), i.e. the trap model, we use a scaling argument discussed in [17] to derive the behavior of \( S(t_w) \) for large \( t_w \). Then we show by a finite size scaling argument that in \( d = 1 \) the behavior of \( S(t_w) \) for \( \alpha = 0 \) is not expected to change for \( 0 < \alpha < 1 \). Furthermore, we give general arguments for the invariance of eq. (1) with respect to \( \alpha \) for all \( d \). Finally we confirm eqs. (14a,b) by Monte-Carlo simulations.

1. Trap model (\( \alpha = 0 \))

After \( N \gg 1 \) transitions of the system, the typical elapsed time \( t_w(N) \) is

\[
t_w(N) \simeq N \sum_{\mu} g_\mu(N) \tau_\mu \Delta \tau_\mu \sim N \int_1^{\tau_{\max}(S(N))} d\tau \rho(\tau) \sim N [\tau_{\max}(S(N))]^{(1-\theta)} , \quad \text{(B1)}
\]

where \( g_\mu(N) \) is the typical number of \( \tau_\mu \) falling in some interval \( \tau_\mu - \Delta \tau_\mu / 2 \leq \tau_\mu \leq \tau_\mu + \Delta \tau_\mu / 2 \), \( S(N) \) is the typical number of distinct visited sites after \( N \) jumps and \( \tau_{\max}(S) \) is the typical maximal \( \tau \) obtained after the system encountered \( S \) distinct sites.

Since \( \tau_{\max}(S) \sim S^{1/\theta} \) and (see e.g. [23])

\[
S(N) \sim \begin{cases} N^{d/2} , & 1 \leq d < 2 \\ N / \ln N , & d = 2 \\ N , & d > 2 \end{cases} , \quad \text{(B2)}
\]

we find

\[
t_w(S) \sim \begin{cases} S^{d+(2-d)\theta}/d\theta , & 1 \leq d < 2 \\ S^{1/\theta} \ln S , & d = 2 \\ S^{1/\theta} , & d > 2 \end{cases} . \quad \text{(B3)}
\]

This yields eqs. (14a,b) for \( \alpha = 0 \).

We note that in the annealed model eq. (22) remains valid, while \( \tau_{\max} \sim N^{1/\theta} \), leading to \( S(t_w) \sim t_w^{d/2} / 2 \) for \( 1 \leq d < 2 \), \( S(t_w) \sim t_w^{d/\ln t_w} \ln t_w \) for \( d = 2 \), and \( S(t_w) \sim t_w^{\theta} \) for \( d > 2 \). Due to our discussion in sec. (11) one can replace \( \theta \) by \( \theta' = \min(1, \theta/(1-\alpha)) \) in these formulae to obtain the behavior for \( 0 \leq \alpha < 1 \).

2. One dimension (\( 0 \leq \alpha < 1 \))

Let us consider a finite chain with \( L \) sites and site energies distributed according to eq. (1). Then the mean square displacement \( \langle \Delta x^2(t) \rangle \) of a particle performing a random walk on this chain with the hopping rates given in (3) is expected to scale as

\[
\langle \Delta x^2(t) \rangle \sim \begin{cases} t^\eta & \text{for } \langle \Delta x^2(t) \rangle \ll L^2 \\ D(L) t & \text{for } \langle \Delta x^2(t) \rangle \gg L^2 \end{cases} \quad \text{(B4)}
\]

for large \( L \). The diffusion coefficient \( D(L) \) can be written as

\[
D(L) = \frac{L^2}{\sum_{j=1}^{L} (p_{j}^{(eq)} w_{j+1})^{-\gamma}} \quad \text{(B5)}
\]

with \( p_{j}^{(eq)} = \exp(-\beta E_j) / \sum_{k=1}^{L} \exp(-\beta E_k) = \tau_j / \sum_{k} \tau_k \). The denominator then reads, using (3), \( \sum_{j=1}^{L} \tau_j \sum_{j=1}^{L} (\tau_j \tau_{j+1})^{-\alpha} \), since \( (\tau_j \tau_{j+1})^{-\alpha} \) exists (for \( \alpha > -\theta \)), the second sum gives a contribution \( \propto L \) for large \( L \), while the first sum has no finite average and is dominated by the maximum \( \tau_{\max} \sim L^{1/\theta} \). Thus we find

\[
D(L) \sim \frac{L^2}{L^{1-\theta}} \sim L^{1-1/\theta} . \quad \text{(B6)}
\]

At the crossover time \( t_x \), where \( \langle \Delta x^2(t_x) \rangle \) changes its behavior in eq. (3), we obtain from continuity

\[
t_x^\eta \sim D(L) t_x \sim L^2 . \quad \text{(B7)}
\]

This implies \( t_x \sim L^{2/\eta} \) and \( t_x \sim L^{1+1/\theta} \) yielding

\[
\eta = \frac{2 \theta}{1 + \theta} . \quad \text{(B8)}
\]

Since \( S(t_w) \sim \langle \Delta x^2(t) \rangle^{1/2} \), we finally obtain for \( 0 \leq \alpha < 1 \)

\[
S(t_w) \sim t_w^{\theta/2} \sim t_w^{\theta/(1+\theta)} \quad \text{(B9)}
\]

in agreement with eqs. (14a,b).

3. General arguments

The very physical difference between the trap model \( (\alpha = 0) \) and models with weighted rates \( (\alpha > 0) \) is the occurrence of forward–backward jump correlations. When
the system jumps from a site with low energy to a site with energy close to zero (such energies are most likely), it has high tendency to jump back for $\alpha > 0$. More generally, when the system enters a region of connected low energy sites, it will, before escaping this region, perform more and more jumps between the low energy sites the larger the value of $\alpha$ is. Once it left the region, it again has high tendency to jump back to it.

One may regard a cluster of sites with deep energies and the surrounding shell of sites with higher energy as a “supertrap”. On a coarse–grained level with respect to time, the particle performs “superhops” between these supertraps. Then the essential difference between the $\alpha = 0$ and the $\alpha > 0$ situation disappears, since there are no increased backward jump correlations between the superhops. One thus suspects that $\alpha$ only rescales the time $t_w$ of the $S(t_w)$ relation but does not change its exponent. Indeed this is what we have shown more explicitly in $d = 1$ in the previous subsection, and there is no reason why the argument should fail in higher dimensions.

It is worth to note that the same arguments also apply to $\Pi(t + t_w, t_w)$, if one generalizes it to a quantity $\Pi_n(t + t_w, t_w)$ which is defined as the averaged probability that the system after a waiting time $t_w$ does not leave a region of radius $n$ in configuration space. Clearly, $\Pi(t + t_w, t_w) = \Pi_0(t + t_w, t_w)$, but for larger $n$ only superhops should lead to a decrease in $\Pi_n(t + t_w, t_w)$. In fact we found that for $n \geq 1$ $\Pi_n(t + t_w, t_w)$ shows normal aging, i.e. $\Pi_n(t + t_w, t_w) \sim F_1^{(n)}(t/t_w)$ for $\alpha > 0$.

4. Monte–Carlo results

![FIG. 9: Number of distinct visited sites $S(t_w)$ for $(\theta, \alpha) = (1/4, 3/8)$ and different dimensions $d$.](image)

Figs. [3] and [4] show $S(t_w)$ from Monte Carlo simulations in $d = 1$, 10, 100, and 1000 for parameter sets

\[(\theta, \alpha) = (1/4, 3/8)\] and \[(1/6, 1/4)\], respectively. In all cases $S(t_w)$ shows the behavior predicted by eqs. (14a,b).

APPENDIX C: $\tilde{\Pi}(t, S)$ IN THE LIMIT $S \to \infty$

To derive the large $S$ limit of eq. (22),

$$\tilde{\Pi}(t, S) = S \int_0^\infty d\lambda \, \langle e^{-\lambda \tau} \rangle^{S-3} g(t; \lambda), \quad (C1)$$

we need to consider the small $\lambda$ limit of

$$\langle e^{-\lambda \tau} \rangle = \int_1^\infty \frac{d\theta}{\tau^{1+\theta}} \, e^{-\lambda \tau} = \theta^{\lambda \theta} \Gamma(-\theta, \lambda)$$

$$= 1 - \Gamma(1-\theta) \lambda^{\theta} + \frac{\theta}{1-\theta} \lambda + O(\lambda^2)$$

$$= e^{-\varphi(\lambda)} \lambda^{\theta}$$

where $\varphi(\lambda)$ is a continuous function with $\varphi(\lambda) \to \Gamma(1-\theta)$ for $\lambda \to 0$. Furthermore, $\langle e^{-\lambda \tau} \rangle$ has an upper bound $e^{-a \lambda^\theta}$ for $\lambda \geq 0$,

$$\langle e^{-\lambda \tau} \rangle < e^{-a \lambda^\theta} \quad (C2)$$

with some constant $a$, $0 < a \leq 1$, being independent of $\lambda$. To proof this for $\lambda \geq 1$ we compare

$$e^{-a \lambda^\theta} = \int_1^\infty d\tau \left[ a\lambda^\theta \tau^{-1+\theta} e^{-a(\lambda \tau)^\theta} \right]$$

with

$$\langle e^{-\lambda \tau} \rangle = \int_1^\infty d\tau \left[ \theta \tau^{-1-\theta} e^{-\lambda \tau} \right].$$

![FIG. 10: Number of distinct visited sites $S(t_w)$ for $(\theta, \alpha) = (1/6, 1/4)$ and different dimensions $d$.](image)
For $0 < a < 1$, we note that for small argument is strictly monotonously decreasing with $a$ this remains valid also for $0 < a < \theta$. To proof property (C2) for $\lambda < 1$ we write asymptotically

$$e^{-\varphi(\epsilon)} \epsilon^\theta + \mathcal{O}(\epsilon) = 1 - \Gamma(1 - \theta) \epsilon^\theta$$

$$< 1 - \alpha \epsilon^\theta = e^{-a \epsilon^\theta} + \mathcal{O}(\epsilon^{2\theta}) ,$$

since $\Gamma(1 - \theta) > 1$ for $0 < a < \theta$, we can deduce that $\langle \epsilon^{-\lambda \tau} \rangle$ is strictly monotonously decreasing it is $\langle \epsilon^{-\lambda} \rangle < 1$. When choosing $a_0 \equiv -\ln \langle \epsilon^{-\lambda} \rangle > 0$ it holds that

$$\langle \epsilon^{-\lambda} \rangle < e^{-a_0 \lambda} \quad \text{for} \quad 0 < \lambda < 1$$

and the proof of eq. (C2) is complete.

With the transformation $\lambda \to u \equiv S^{1/\lambda}$ eq. (C1) gives

$$\tilde{\Pi}(t, S) = \int_0^\infty du \exp \left( -\varphi \left( \frac{u^\theta}{S} \right) \frac{S - 3}{S} u^\theta \right) \times S^{1-1/\theta} g(t; \frac{u}{S^{1/\theta}}) .$$

When using eq. (C2) we can estimate

$$\exp \left( -\varphi \left( \frac{u^\theta}{S} \right) \frac{S - 3}{S} u^\theta \right) < \exp \left( -a \frac{S - 3}{S} u^\theta \right)$$

for $S > 6$ and from eq. (B3),

$$S^{1-1/\theta} g(t; S^{-1/\theta} u) \leq S^{1-1/\theta} \int_1^\infty \frac{\theta du}{\tau^\theta} e^{-S^{-1/\theta} u \tau} \leq S^{1-1/\theta} \theta \Gamma(1, -S^{-1/\theta} u)$$

$$\leq S^{1-1/\theta} \theta \Gamma(1 - \theta) \mu^{-1} .$$

These estimation will allow us to use LEBESGUE’s theorem when considering the limit $S \to \infty$ in eqs. (C3). Thus we can write asymptotically

$$\tilde{\Pi}(t, S) \sim S^{1-1/\theta} \int_0^\infty du e^{-u^\theta} g(t; S^{-1/\theta} u)$$

$$\sim \int_0^\infty d\lambda e^{-\lambda^\theta \epsilon S} g(t; \lambda) ,$$

where the shortcut $\kappa = \Gamma(1 - \theta)$ [cf. eq. (B6)] has been used.

REFERENCES

[1] Glasses and Amorphous Materials, Materials Science and Technology, Vol. 9, edited by J. Zarzycki (VCH, Weinheim, 1991).

[2] G. Parisi, e-print cond/mat 9910375.

[3] L. C. E. Struik, “Physical Aging in Amorphous Polymers and Other Materials” (Elsevier, Houston, 1978).

[4] E. Vincent, J. Hammann, M. Ocio, J. P. Bouchaud, and L. Cugliandolo, in Complex Behaviour of Glassy Systems, M. Rubi Editor, Lecture Notes in Physics (Springer Verlag, Berlin, 1997), Vol.492, pp.184-219, and refs. therein.

[5] P. Nordblad, in: Spin Glasses and Random Fields, P. Young Editor, World Scientific (1998).

[6] F. Alberici, P. Doussineau, and A. Levelut, J. Phys I (France) 7, 329 (1997); F. Alberici, P. Doussineau, and A. Levelut, Europhys. Lett. 39, 329 (1997).

[7] J. P. Bouchaud, P. Doussineau, T. de Lacerda-Aros, and A. Levelut, cond-mat/0011190 and refs. therein.

[8] R. L. Leheny and S. R. Nagel, Phys. Rev. B 57, 5154 (1998).

[9] L. Bellon, S. Ciliberto, and C. Laroche, Europhys. Lett. 51 551 (2000).

[10] W. Kob, J. L. Barrat, Eur. Phys. J. B13, 319 (2000).

[11] L. Cipelli, S. Mansley, R. C. Ball, D. A. Weitz, Phys. Rev. Lett. 84, 2275 (2000).

[12] C. Derec, A. Ajdari, G. Ducouret, F. Lequeux, C.R.A.S. Paris, 1 1115 (2000).

[13] M. Cloitre, R. Borrega, L. Leibler, submitted to Phys. Rev. Lett.

[14] A. Knaebel, M. Bellour, J.-P. Munch, V. Viasnoff, F. Lequeux, and J. L. Harden, Europhys. Lett. 52, 73 (2000).

[15] B. Abou, D. Bonn, and J. Meunier, preprint cond-mat/0101327.

[16] L. Bellon, S. Ciliberto, and C. Laroche, Europhys. Lett. 53, 511 (2001).

[17] A. Vaknin, Z. Ovadyahu, and M. Pollak, Phys. Rev. Lett. 5, 3402 (2001).

[18] E. Vincent, V. Dupuis, M. Alba, J. Hammann, and J. P. Bouchaud, Europhys. Lett. 50, 674 (2000).

[19] M. Nicodemi, H. J. Jensen, preprint cond-mat/0103070.

[20] E. Pitard, E. Shaknovitch, preprint cond-mat/9910431.

[21] A. Barrat, V. Loreto, Europhys. Lett. 53, 297 (2001).

[22] D. Head, Phys. Rev. E 62, 2439 (2000).

[23] J. P. Bouchaud, L. Cugliandolo, J. Kurchan, and M. Mézard, Out of Equilibrium dynamics in spin-glasses and other glassy systems, in ’Spin-glasses and Random Fields’, A. P. Young Editor, (World Scientific, Singapore, 1998), and refs. therein.

[24] A. J. Bray, Adv. Phys. 43, 357 (1994).

[25] J. Kurchan and L. Laloux, J. Phys. A29, 1929 (1996).

[26] A. Cavagna, Europhys. Lett. 53, 490 (2001).

[27] C. Donati, F. Sciortino, and P. Tartaglia, Phys. Rev. Lett. 85, 1464 (2000).

[28] L. Angelani, R. Di Leonardo, G. Ruocco, A. Scala, and F. Sciortino, Phys. Rev. Lett. 85, 5356 (2000).
[29] K. Broderix, K. K. Bhattacharya, A. Cavagna, A. Zippelius, and I. Giardina, Phys. Rev. Lett. 85, 5360 (2000).
[30] M. Goldstein, J. Chem. Phys. 51, 3728 (1969).
[31] C. A. Angell, Science 267, 1924 (1995).
[32] J. P. Bouchaud, J. de Physique I 20, 1705 (1992).
[33] M. Feigel’man and V. Vinokur, J. Physique I (France) 49, 1731 (1988).
[34] J. P. Bouchaud and D. S. Dean, J. Physique I (France) 5, 265 (1995).
[35] C. Monthus and J. P. Bouchaud, J. Phys. A 29, 3847 (1996).
[36] P. Sollich, F. Lequeux, P. Hebraud, and M. Cates, Phys. Rev. Lett. 70, 2020 (1997); P. Sollich, Phys. Rev. E 58, 738 (1998); S. Fielding, P. Sollich, and M. Cates, cond-mat/9907101.
[37] D. S. Fisher, P. Le Doussal, C. Monthus, Phys. Rev. E 59, 4795 (1999).
[38] L. Cugliandolo and J. Kurchan, J. Phys. A 27, 5749 (1994).
[39] L. Berthier, J.-L. Barrat, and J. Kurchan, Phys. Rev. E 63, 016105 (2001).
[40] P. Embrecht, C. Klüppelberg, and Th. Mikosch, Modelling Extremal Events (Springer, Berlin, 1997)
[41] J. P. Bouchaud and M. Mézard, J. Phys. A 30, 7997 (1997).
[42] J. C. Schön and P. Sibani, Europhys. Lett., 49, 196 (2000).
[43] B. Rinn, P. Maass, and J. P. Bouchaud, Phys. Rev. Lett. 84, 5403 (2000).
[44] J. P. Bouchaud, Aging in glassy systems: new experiments, simple models and open questions, in ‘Soft and Fragile Matter’, edited by M. E. Cates and M. Evans, IOP (2000).
[45] B. Derrida, Phys. Rev. B 24, 2613 (1981).
[46] A. Compte and J. P. Bouchaud, J. Phys. A 31, 6113 (1998).
[47] H. Harder, S. Havlin, and A. Bunde, Phys. Rev. B 36, 3874 (1987).
[48] L. Cugliandolo, J. Kurchan, and L. Peliti, Phys. Rev. E 55, 3898 (1997), and refs. therein.
[49] K. Binder and D. W. Heermann, Monte Carlo Simulations in Statistical Physics, Springer Series in Solid State Science, Vol. 80, 2nd ed. (Springer, Berlin, 1992).
[50] E. Bertin, J. P. Bouchaud, and P. Maass, to be published.
[51] F. Sciortino and P. Tartaglia, Phys. Rev. Lett. 86, 107 (2001).
[52] B. D. Hughes, Random walks in random environments, Vol. I (Clarendon Press, Oxford, 1995).
[53] R. Kutner, D. Knödler, P. Pendzig, R. Przenioslo, and W. Dieterich, in Diffusion Processes: Experiment, Theory, Simulations, (Lecture Notes in Physics 438), edited by A. Pekalski (Springer Verlag, Heidelberg), p. 197.