Evaluating the solution accuracy of the heat equation by a posteriori error estimation

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Abstract. A produce based on a guaranteed posterior error estimates is proposed to estimate the accuracy of solution of the time-dependent heat equation. The time-dependent heat equation is discretized by the backward Euler scheme in time and conforming finite elements in space. The error between the exact solution and the numeric solution in space is bounded by a guaranteed posterior error estimates which is calculated at every time step. Numerical results demonstrate that the produce can be used to evaluate the accuracy of numerical solutions.

1. Introduction

Finite element methods have been used to simulate the physical phenomena which is described by the partial differential equations in many engineering applications and scientific predictions[1-2]. Finite element methods which are used to solve the partial differential equations in some finite-dimensional space can only obtain an approximate solution. Thus, how large of the error between the exact and the approximate solution and where the error localized in space are two important questions.

The theory of a posteriori error estimation for finite element approximation is studied to answer the questions. There is a well-developed literature on a posteriori error estimation [3-10]. Several types of posteriori error estimators have been developed, such as the residual estimator for the error measured in the energy norm, the equilibrated residual estimator and recovery-based error estimator, etc. In the majority of the cases, however, these estimates are not guaranteed, meaning that they are not fully computable because some coefficients involved in these estimates are not given. This means that the associated estimation can only be used as local refinement indicators. It cannot give an exact error bound which can be used to estimate the error between the numeric solution and the real solution. To remedy this inconvenience, a posteriori error estimate which is guaranteed and fully computable is introduced in Ref. [11, 12]. The guaranteed estimator proposed in Ref. [11, 12] do not involve unknown coefficients. So it can be directly used in practice to estimate the error between the numeric solution and the real solution. In this paper, we give the method which use the guaranteed estimator to estimate the accuracy of the solution of the unsteady heat equation which solved by the finite element method.

The rest of this paper is organized as follows. We review some preliminaries in Section 2. In Section 3, we introduce the guaranteed estimator which is used to estimate the error between the numeric solution and the real solution. In Section 4, experimental results are presented to show the effectiveness of the method.
2. Preliminaries
This section briefly describes the model problem and the finite element method to solve the model problem.

2.1. The Problem model
We consider the heat equation

\[ \partial_t u - \mu \Delta u = f \quad \text{in } Q := \Omega \times (0,T), \]  
\[ \mu \nabla u \cdot n = g \quad \text{in } \partial \Omega \times (0,T), \]  
\[ u(\cdot,0) = u_0 \quad \text{in } \Omega. \]  

where \( \Omega \subset \mathbb{R}^2 \) is a polyhedral domain. \( T \) stands for the finite simulation time, \( \mu \) for a positive coefficient, \( f \) for the source term, and \( u_0 \) for the initial temperature. We assume that \( f \in L^2(Q) \) and \( u_0 \in L^2(\Omega) \). Equations (1a)-(1c) describe the unsteady heat flow.

2.2. Finite element method
We give firstly the variational formulation of problem (1a)-(1c). It can be checked that problem (1a)-(1c) admits the variational formulation:

Find \( u \) in \( H^1(0,T;H^1(\Omega)) \cap C^0(0,T;L^2(\Omega)) \), such that

\[ u(\cdot,0) = u_0 \quad \text{in } \Omega, \]  

And that, for \( t \in (0,T) \) and for all \( v \in H^1(\Omega) \),

\[ (\partial_t u, v) + (\mu \nabla u, \nabla v) = (f, v) + g, v \geq \Omega \]  

We divide the interval \([0,T]\) into subintervals \([t_{n-1},t_n]\), \(1 \leq n \leq N\), such that \( 0 = t_0 < \cdots < t_N = T \). We denote by \( \tau_n \) the length \( t_n - t_{n-1} \). Let’s suppose that \( f \) belongs to \( C^0(0,T;H^{-1}(\Omega)) \) and we denote by \( f^\tau \) the function \( f(\cdot,t_n) \). Using the backward Euler scheme to the variational formulation (2)-(3), we get the semi-discrete problem:

Find \( u^n \in L^2(\Omega) \times \{H^1(\Omega)\}^N \), \( 0 \leq n \leq N \), such that

\[ u^0 = u_0 \quad \text{in } \Omega, \]  

and that, for \( t \in (0,T) \) and for all \( v \in H^1(\Omega) \),

\[ (u^n / \tau_n, v) + (\mu \nabla u^n, \nabla v) = (u^{n-1} / \tau_n, v) + (f^n, v) + g^n, v \geq \Omega \]  

Now, we describe the space discretization of problem (4)-(5). For each \( n, 0 \leq n \leq N \), Let \( \{Z_h^n\} \) be a conforming, shape-regular family of affine meshes of \( \Omega \) consisting of simplices. An element in \( Z_h^n \) is denoted by \( K \), \( h_K \) stands for its diameter, and \( n_K \) for its unit vector of outward normal. Let the interior and boundary faces are collected in the sets \( F_h^i \) and \( F_h^b \) respectively, and faces set \( F_h = F_h^i \cup F_h^b \).

We introduce finite-dimensional subspaces \( V_h^n \)
\[ V_h^n = \{ v_h \in H^1(\Omega); \forall K \in Z_h^n, v_{h_K} \in P_1(K) \} \]  
\hspace{1cm} (6) 

Where \( P_1(K) \) denotes the space of restrictions to \( K \) of affine functions in \( \mathbb{R}^2 \). Let \( \Pi_h \) denote a projection operator from \( \tilde{L}^2(\Omega) \) onto \( V_h^0 \). The fully discrete problem constructed from problem (4)-(5) by finite element method is:

Find \( u_h^n \in \Pi_{w,t}^N V_h^n, 0 \leq n \leq N \), such that

\[ u_h^0 = \Pi_h u_0 \quad \text{in} \ \Omega, \]  
\hspace{1cm} (7)

and that, for all \( n, 1 \leq n \leq N \), and for all \( v_h \in V_h^n \),

\[ (u_h^n / \tau_n, v_h) + (\mu \nabla u_h^n, \nabla v_h) = (u_h^{n-1} / \tau_n, v_h) + (f^n, v_h) + <g^n, v >_{\partial \Omega} \]  
\hspace{1cm} (8)

3. A posterior error estimates

A guaranteed posterior error estimate for the Laplace equation is proposed in [12]. Based the guaranteed posterior error estimates, we give the posterior error estimates of the problem (1a)-(1c) at every time step. Let \( u \) be the weak solution of problem (1a)-(1c) and \( u_h \) is solution of (7)-(8), then \( u_h^n \in V_h^n, 1 \leq n \leq N \) is a approximation of \( u \). Let \( \sigma_h^n \) denote the equilibrated flux which is constructed from \( u_h^n \) and satisfies

\[ \sigma_h^n \in H(div, \Omega) \]  
\hspace{1cm} (9a)

\[ (\nabla \cdot \sigma_h^n, 1)_K = (f^n - u_h^n - u_h^{n-1}) / \tau_n, 1)_K \]  
\hspace{1cm} (9b)

\[ \sigma_h^n \cdot n_F |_{F} \in L^2(F) \quad \forall F \in F_h \cap \partial \Omega, \]  
\hspace{1cm} (9c)

\[ <\sigma_h^n \cdot n_F >_F = = -<g^n, 1 >_F \quad \forall F \in F_h \cap \partial \Omega, \]  
\hspace{1cm} (9d)

where \( H(div, \Omega) = \{ v: v \in [L^2(\Omega)]^2, \nabla \cdot v \in L^2(\Omega) \} \). Several methods which can be used to determinate the equilibrated flux \( \sigma_h^n \) are given in [12]. Defining the residual estimator \( \eta_{RK}^n \), the diffusive flux estimator \( \eta_{DF,K}^n \), the edge residual estimator \( \eta_{FR,K}^n \) as

\[ \eta_{RK}^n := h_K \frac{\mu}{\tau_n} \left\| f^n - \frac{u_h^n - u_h^{n-1}}{\tau_n} - \nabla \cdot \sigma_h^n \right\|_K, \]  
\hspace{1cm} (10a)

\[ \eta_{DF,K}^n := \mu^{-1} \left\| \mu \nabla u_h^n + \sigma_h^n \right\|_K, \]  
\hspace{1cm} (10b)

\[ \eta_{FR,K}^n := C_{t,K,F} h_F^{1/2} \left\| \sigma_h^n \cdot n_\Omega + g^n \right\|_{H^1}, \]  
\hspace{1cm} (10c)

where \( n_\Omega \) is unit vector of outward normal of \( \Omega \), \( C_{t,K,F} \) can be referenced in [12].

Let \( h_h^n \) denote the numeric solution of problem (1a)-(1c) and define the error \( e^n = u - u_h^n \) expressed in energy norm
In every time step, the error \( ||e^n|| \) satisfies

\[
||e^n|| = \sqrt{\sum_{k \in Z^a} (\eta_{R_K}^n + \eta_{DF,K}^n + \sum_{P \in \Omega} \eta_{FR,P}^n)^2} = \eta_n, \tag{11}
\]

where \( \eta_n \) is the posteriori error estimates of the problem (1a)-(1c) at the \( n \)th time step.

\[4. \textbf{Numerical experiments}\]

We consider problem (1a)-(1c) with \( \Omega \) in Fig.1, where \( L = 0.1 \text{m} \) and \( H = 0.05 \text{m} \). The coefficient \( \mu \) equals to \( k / \rho c_p \), where \( k = 48 \text{W/(m⋅K)} \), \( \rho = 7850 \text{kg/m}^3 \), \( c_p = 461 \text{J/(kg⋅K)} \). The initial condition and boundary conditions are \( \mu_0 = T_0 = 300 \text{K} \), \( \mu_{|x=0} = T_1 = 400 \text{K} \), \( \mu_{|x=1} = T_2 = 300 \text{K} \), \( \nabla \mu \cdot n_\Omega |_{y=0} = 0 \), \( \nabla \mu \cdot n_\Omega |_{y=0.05} = 0 \). The source term \( f \) equals to zero.

The exact solution of above problem is

\[
u(x,y,t) = T_1 - T_s \frac{T_1-T_s}{L} x + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} (T_0-T_1 + (T_x-T_0)(-1)^n) \exp\left(\frac{n^2 \pi^2 t}{L^2} \cdot \mu \cdot t\right) \sin\left(\frac{n \pi x}{L}\right) \tag{13}\]

The numeric solutions which marked by dash line and the exact solutions which marked by solid line are given in Fig. 2. The two solutions are given in time step equal to \( 30\Delta t, 60\Delta t, 90\Delta t, 120\Delta t \), where \( \Delta t = 5e^{-4}/2\mu \). We can see that the numeric solutions are coincident with the exact solutions very well.

![Figure 1. \( \Omega \) of problem (1a)-(1c).](image)

The real error \( \eta_n^* \) expressed by energy norm and the posteriori error \( \eta_n \) of the first time step to the 30th time step are given in Fig. 3. The three subfigures in Fig.3 are the error \( \eta_n^* \) and the posteriori error \( \eta_n \) calculated by finite element method using 366, 1344 and 5376 elements. In Fig.3, we can see that, with the element number increasing, the posteriori error \( \eta_n \) is close to the error \( \eta_n^* \) and the two errors
all became small. The posteriori error $\eta_n$ is always a upper bound of the error $\eta^*_n$. So the real error $\eta^*_n$ can be estimated by the posteriori error $\eta_n$ which is used to evaluate the accuracy of the numeric solutions.

![Figure 2. The exact solution and the numeric solution.](image)

![Figure 3. The error $\eta^*_n$ and the posteriori error $\eta_n$ calculated by finite element method using 366,1344 and 5376 elements.](image)

5. Conclusions
In this paper, we use a guaranteed estimator to estimate the accuracy of the solution of the unsteady heat equation which solved by the finite element method. The time-dependent heat equation is discretized by the backward Euler scheme in time and conforming finite elements in space. The error between the exact solution and the numeric solution in space is bounded by the guaranteed posteriori error estimation which is calculated at every time step. Numerical example shows that our method can
be used to evaluate the accuracy of numerical solutions. The posteriori error is always an upper bound of the real error. With the element number increasing, the posteriori error is close to the real error and the two errors all became small.

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