The spatial Functional Renormalization Group
and Hadamard states
on cosmological spacetimes

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Abstract

A spatial variant of the Functional Renormalization Group (FRG) is introduced on (Lorentzian signature) globally hyperbolic spacetimes. Through its perturbative expansion it is argued that such a FRG must inevitably be state dependent and that it should be based on a Hadamard state. A concrete implementation is presented for scalar quantum fields on flat Friedmann-Lemaître spacetimes. The universal ultraviolet behavior of Hadamard states allows the flow to be matched to the one-loop renormalized flow (where strict removal of the ultraviolet cutoff requires a tower of potentials, one for each power of the Ricci scalar). The state-dependent infrared behavior of the flow is investigated for States of Low Energy, which are Hadamard states deemed to be viable vacua for a pre-inflationary period. A simple time-dependent infrared fixed point equation (resembling that in Minkowski space) arises for any scale factor, with analytically computable corrections coding the non-perturbative ramifications of the Hadamard property in the infrared.

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1. Introduction

The Functional Renormalization Group (FRG) is a work-horse for non-perturbative Quantum Field Theory (QFT) and has found applications in areas as diverse as solid-state physics, QCD, and quantum gravity; see [1, 2, 3, 4] for recent accounts. It is primarily used in Euclidean signature and in combination with heat kernel methodology. For certain applications, however, this standard framework does not convincingly capture the physics situation one seeks to model, the one in focus here being QFT on cosmological backgrounds. Cosmological spacetimes in general do not admit a satisfactory notion of Wick rotation and the formal application of the (pseudo-)heat kernel expansion to non-elliptic operators is mathematically dubious. More importantly, many cosmological signatures refer to the infrared dynamics (super-Hubble wavelengths) of the quantum fields and these are not convincingly captured by a resummation of the structurally unique heat kernel expansion. In fact, vacuum-like states in QFTs on curved backgrounds are inherently non-unique and this non-uniqueness should manifest itself on the level of the FRG. For the widely studied de Sitter background (where the Bunch-Davies vacuum provides a unique group invariant vacuum) the issue does not arise [5, 6, 7], and of course neither does it in Minkowski space [8, 9], but for generic Friedmann-Lemaître backgrounds it will.

The main goal of the present note is to develop a spatial variant of the FRG on globally hyperbolic spacetimes where the issue of state dependence can successfully be addressed. A schematic summary is presented in Figure 1. By a spatial FRG we mean one where the mode modulation affects only the eigenvalues of the spatial Laplacian, while the temporal dynamics is left unaffected. In this way Lorentzian signature can be maintained, while implementing a mode suppression where nearly homogeneous configurations rather than near-null configurations correspond to small spectral values. In addition, in a perturbative solution of the FRG at each order the Green’s function of a well-defined hyperbolic wave equation enters, a feature that would be spoiled by a temporal mode modulation. The Cauchy data then fix a Green’s function but the admissible data themselves are insufficiently constrained by physics requirements so that the ambiguity is best attributed to the Green’s function.

It is through the non-uniqueness of such Green’s functions that one can make contact to the selection criteria independently developed in the framework of perturbative QFT in curved backgrounds. Recall that for perturbatively defined QFTs on globally hyperbolic spacetimes the free state on which perturbation theory is based must be a Hadamard state. This is because by-and-large the Hadamard property is necessary and sufficient for the existence of Wick powers of arbitrary order and hence for the perturbative series to be termwise well-defined, see e.g. [14, 15]. The Hadamard property entails that the associated Green’s function has a universal ultraviolet behavior, the same for all Hadamard states. In contrast, the infrared behavior of the Green’s function will be specific for the Hadamard state under consideration and whatever construction principle has been used to obtain it. In the perturbative expansion of the FRG a similar Green’s function will occur, just with its spatial mode content modulated. By the very nature of a FRG modulator it should
however leave an ultraviolet asymptotic expansion of the Green’s function unaffected. This leads to the one-to-one correspondence depicted in Figure 1, as far the FRG’s perturbative expansion is concerned.

Figure 1: Correspondence between state choice in perturbative QFT and choice of non-perturbative spatial FRG.

Next, we consider the Effective Potential Approximation (EPA) of the spatial FRG. Apart from the now scale dependent potential the same Green’s function will occur and the correspondence carries over to the non-perturbative aspects covered by the EPA. Beyond the EPA it will of course become increasingly difficult to construct a Green’s function for the (modified) Hessian of the effective action, but computationally this inversion is also difficult in the covariant Euclidean case. Further, at present no systematic non-perturbative formulation of QFTs on non-static backgrounds and the associated vacuum-like states exists. Our proposal is to use the right hand side of Figure 1 to explore the left hand side, i.e. to use the spatial FRG to define and explore Lorentzian signature QFTs on non-static backgrounds beyond perturbation theory.

Importantly, the infrared aspects of the spatial FRG flow will always be state dependent. This is manifest in the perturbative expansion and on the level of the EPA. We expect it to be true generally whenever the full spatial FRG can be rendered well-defined. A systematic proposal to do so based on a spatial hopping expansion [10, 11] will be presented elsewhere. The state dependence of the spatial EPA flow will be studied in Section 4.
As with other FRGs, the spatial FRG requires the specification of a boundary condition in the ultraviolet. The boundary functional is normally identified with the renormalized action at arbitrarily high renormalization scale. Assuming that the theory is renormalizable the latter coincides with the bare action and thus is known. In the case at hand, the renormalization of a scalar field theory with a quartic self-interaction in curved background is well understood [33, 34, 35, 36]. At the one-loop level the familiar “tracelog” of the action’s Hessian needs to be evaluated, which can efficiently be done via heat kernel (for Euclidean signature) or pseudo-heat kernel (for Lorentzian signature) techniques [33]. We shall find that even at one-loop order such a Wilsonian renormalization of a scalar field theory (with strict removal of the UV cutoff) demands the inclusion of an infinite tower of Ricci scalar terms $R(g)^n$, each coming with its own potential

$$U(\phi) \mapsto U(\phi, g) = \sum_{n \geq 0} nU(\phi)R(g)^n.$$ (1.1)

Here $0U(\phi) = U(\phi)$ is the original potential and the usual non-minimal coupling would invoke only a quadratic $1U(\phi) = \xi \phi^2$. As soon as $U(\phi)$ contains a sextic term, an infinite tower of additional interactions is required to absorb the additional divergences generated. This holds for any non-trivial background metric, though we shall focus on Friedmann-Lemaître backgrounds. The scalar field theory with the generalized potential (1.1) will be shown to be one-loop renormalizable in Section 3 and the beta functions for the relevant and marginal couplings will be computed. This will be done by re-integrating the one-loop spatial FRG flow itself, which ensures that it can consistently be used to set boundary conditions for the spatial FRG in the ultraviolet, see Figure 1.

With this framework in place, we elaborate the EPA for the spatial FRG in Friedmann-Lemaître backgrounds in Section 4. Here we focus on the so-called States of Low Energy (SLE). These are Hadamard states that arise by minimizing a suitable time averaged energy functional [17]. The function $f$ used for the time averaging is chosen to have support in the cosmological epoch in focus, e.g. a pre-inflationary period of non-accelerated expansion. Importantly, the SLE admit an infrared expansion [19] where the dependence on the potential is rendered explicit. This allows one to extract analytically the dependence of the EPA’s right hand side on the dimensionless version $V_k(\varphi_0, t)$ of the generalized potential (1.1). Here $\varphi_0$ is the constant dimensionless background and, as the Ricci scalar may appear in a temporally nonlocal form in the infrared, we merely indicate the time dependence. In addition, the flow equation carries an explicit time dependence. We analyze both by projecting onto a complete, dimensionless, orthonormal system of polynomials $p_l[R](t)$ built from the Ricci scalar $R$. In particular, one has the expansion $V_k(\varphi_0, t) = \sum_{l \geq 0} lV_k(\varphi_0)p_l[R](t)$. The scale dependent ‘infrared’ potentials $V_k(\varphi_0)$, $l \geq 0$, then obey the following small $k$ flow equation (with suitable normalizations and in $1+d$ dimensions)

$$k \frac{d}{dk} V_k(\varphi_0) + (1+d)V_k(\varphi_0) - \frac{d-1}{2}\varphi_0 V'_k(\varphi_0) = \int_0^\infty \frac{d\varphi \varphi^{d-1}}{\sqrt{\varphi^2 a^2}} \frac{\rho(\varphi)}{\sqrt{oV_k(\varphi_0) + o\rho(\varphi)}} + O(k^2),$$

$$\rho(\varphi) := \int dt N(t) a(t)^d p_l(t) \left[ r(\varphi^2/a^2) - (\varphi^2/a^2) r'(\varphi^2/a^2) \right], \quad l \geq 0.$$ (1.2)
Here $r: \mathbb{R}_+ \rightarrow \mathbb{R}_+, \ r(0) = 1$, is a mode modulator of rapid decay, $a(t)$ is the cosmological scale factor, $N(t)$ is the lapse function, the integration is over a dimensionless spatial momentum modulus $\varrho = |p|/k$, and $\dot{}$ on $\mathcal{V}_k$ denotes a derivative with respect to $\varphi_0$. The $O(k^2)$ and higher order corrections are analytically computable. For $k \rightarrow 0$ the flow (1.2) has a well-defined fixed point equation where all infrared fixed point potentials $\mathcal{V}_* (\varphi_0)$, $l \geq 1$, are determined by $0 \mathcal{V}_* (\varphi_0)$ (which is proportional to the temporal average of $\mathcal{V}_* (\varphi_0, t)$). Moreover the $l = 0$ fixed point equation resembles that in Minkowski space.

By construction, the independently computed one-loop flow is consistent with the Hadamard property of the SLE and can be used to set boundary conditions for the dimensionless ‘ultraviolet’ potentials $^a \mathcal{V}_k (\varphi_0)$ for large $k$. Solving the flow numerically into the infrared fixed point regime governed by (1.2) will allow one to explore the non-perturbative ramifications of the Hadamard property for a generic self-interacting scalar field in the very early universe.

The paper is organized as follows. In Section 2 we introduce the spatial FRG on a generic globally hyperbolic background. Via its perturbative expansion we establish the one-to-one correspondence between a choice of Hadamard state and a choice of spatial FRG, see Fig 1. In Section 3 we specialize to spatially flat Friedmann-Lemaître backgrounds with generic scale factor $a$. The SLE are introduced as the Hadamard states of choice and analytically controllable expansions for the ultraviolet and infrared regimes are prepared. The ultraviolet asymptotics is used in Section 3 to extract the divergent part of the spatially regularized one-loop effective action. For a beyond quartic potential the successful absorption of these terms mandates the generalized potential (1.1). In Section 4 the spatial EPA flow equation associated with the SLE is introduced, leading to (1.2) with explicitly computable subleading terms. Appendix A contains computational details for Section 3. Appendix B presents the “instantaneous limit” of the SLE based flow equation, which resembles the spatial EPA flow in Minkowski space. In Appendix C the $O(k^2)$ corrections in (1.2) are computed.
2. Spatial FRG on globally hyperbolic backgrounds

Here we derive the spatial FRG on generic globally hyperbolic manifolds and discuss some of its basic properties. It admits a perturbative expansion in powers of $\hbar$ in which the correspondence to the choice of a modified Hadamard state is transparent.

2.1 Basics

For comparison's sake we initially consider foliated background manifolds with both signatures and line element

$$ds^2 = \epsilon_g N^2 dt^2 + g_{ij}(N^i dt + dx^i)(N^j dt + dx^j).$$

Here $N, N^i, g_{ij}, i, j = 1, \ldots, d$ are the lapse, shift, spatial metric as usual and $\epsilon_g = \pm 1$ is the signature parameter, and we take $d \geq 3$ throughout. On such a foliated background we consider the action of a self-interacting scalar field non-minimally coupled to it

$$S[\tilde{\chi};g] = \epsilon_g \int d^p y \sqrt{\epsilon_g g} \left\{ \frac{1}{2} g^{\mu\nu} \partial_\mu \tilde{\chi} \partial_\nu \tilde{\chi} + U(\tilde{\chi}, R) \right\}$$

$$= \int_{t_i}^{t_f} dt \int_{\Sigma} dx \left\{ \frac{1}{2n} e_0(\tilde{\chi})^2 + \frac{\epsilon_g}{2} n_g g^{ij} \partial_i \tilde{\chi} \partial_j \tilde{\chi} + \epsilon_g n_g U(\tilde{\chi}, R) \right\}. \tag{2.1}$$

Here, the metric $g_{\mu\nu}$ is treated as a background field and $R(g)$ is its Ricci scalar. In the second line $e_0 = \partial_t - \mathcal{L}_N$ is the derivation transversal to the leaves of the foliation, $n = N/\sqrt{g}$ is the lapse anti-density. Further, $R(g)$ is to be read as its $1 + d$ decomposition $R(g) = R(g) + 2 \epsilon_g N^{-1} e_0(K) - \epsilon_g (K^2 + K^i_k K^j_j) - 2N^{-1} \nabla^2 N$, where $K_{ij} = -e_0(g_{ij})/(2N)$ is the extrinsic curvature. Finally, $U(\tilde{\chi}, R)$ is a generalized potential of the form

$$U(\tilde{\chi}, R) = \sum_{n \geq 0} n^n U(\tilde{\chi}) R(g)^n, \quad n^n U(\tilde{\chi}) = \sum_{j \geq 0} n^{u_{2j}} \frac{\tilde{\chi}^{2j}}{(2j)!}, \tag{2.2}$$

with real coupling constants $n^{u_{2j}}, n, j \geq 0$. The potentials $n^n U(\tilde{\chi})$ have mass dimension $d+1-2n$, the field has $(d-1)/2$, giving $n^n u_{2j}$, a mass dimension $d+1-2n-(d-1)j$. Despite the proliferation of couplings only a small number have non-negative mass dimension, 6 for $d = 3$ and $[(d+1)/2] + 3$ for $d \geq 4$.

The derivation of the FRG with a covariant modulator is by now standard [1]. In order to set conventions, trace the occurrence of the signature parameter, and highlight the step where a purely spatial modulator enters, we quickly run though the basic steps. In the schematic functional integrals both metric signatures are treated in parallel with conventions

$$\text{Euclidean:} \quad \epsilon_g = 1, \quad \sqrt{\epsilon_g} = 1, \quad \text{Lorentzian:} \quad \epsilon_g = -1, \quad \sqrt{\epsilon_g} = i. \tag{2.3}$$

We stress the difference between a geometrically meaningful “Wick flip” ($\epsilon_g \mapsto -\epsilon_g$) and a problematic “Wick rotation” ($t \mapsto -it$) that complexifies the line element. A meaningful
notion of a Wick rotation on a generic globally hyperbolic manifold would have to meet a number of desiderata, see e.g. [13]. Any known notion fails to meet one or more of these criteria. In contrast, the Wick flip is a weaker notion that simply maps a given Lorentzian manifold onto a Riemannian one, with no claims to analyticity or bijectivity.

Adopting the background field formalism we write \( \tilde{\chi} = \varphi + \chi \), where \( \varphi \) is an arbitrary background field, and add a modulator term

\[
S_k[\chi, \varphi; g] = S[\varphi + \chi; g] + \Delta S_k[\chi; g], \quad \Delta S_k[\chi; g] = \frac{\epsilon_g}{2} \chi \cdot R_k(g) \cdot \chi.
\]

The “…” shorthand represents an integration with respect to the standard (pseudo) Riemannian volume density. The regulator kernel \( R_k \) is normally taken to suppress the eigenvalues of the full Laplacian \( \nabla^2 \) in Euclidean signature (possibly shifted by a mass-like term) below a scale \( k^2 \). In the Riemannian setting, the ellipticity of \( \nabla^2 \) in principle allows for a well-defined spectral resolution with a clear discrimination between modes associated with large and small eigenvalues. Hence, modulating the mode content according to the eigenvalues of \( \nabla^2 \) leads to a well defined covariant modulation procedure which for \( k \to \infty \) suppresses almost all of the modes. On the other hand, for Lorentzian signature \( \nabla^2 \) is hyperbolic and the modes associated with vanishing spectral values are associated with field propagation along lightcones. A mode modulation \( R_k \) of the same form would leave such null degrees of freedom (equivalent to a \( d \)-dimensional field theory) unaffected. The associated functional integral would still be intractable, rendering a naive transcription of the Euclidean procedure untenable. While in Minkowski space the simple form of the wave front sets allows for practically usable Lorentzian signature modulators \( R_k \) [8, 9], such an adaptation seems both mathematically problematic and physically obscure for generic pseudo-Riemannian manifolds. A local Wick rotation might suffice for the ultraviolet aspects captured by a (pseudo-)heat kernel expansion [26], but for the infrared aspects needed to recover the unmodulated QFT results one encounters an impasse.

The route pursued here is to fix a foliation and to replace the covariant modulation with a merely spatial one through \( R_k(t, x; t', x') = N(t, x)^{-1/2}N(t', x')^{-1/2}\delta(t - t')R_k(t, x, x') \), while leaving the temporal dynamics unaffected. Moreover, the equal time kernel \( R_k \) is taken to suppress modes associated with eigenvalues of the spatial part of the Laplacian \( \nabla^2_s \) (see 2.11) at time \( t \) below a scale \( k^2 \). The modes unaffected in the limit \( k \to \infty \) are then equivalent to those of a 1-dimensional field theory, i.e. quantum mechanics. Since quantum mechanics typically does not require renormalization in itself this should lead to a legitimate mode modulation that mirrors correctly the scale dependence of the full functional integral and its divergences. This extends the use of a spatial regulator employed in [5, 6, 7] for the de Sitter background to generic foliated manifolds.

Having introduced the mode modulation, the background field generating functional \( W_k[J; \varphi] \) and effective average action \( \Gamma_k[\varphi; \varphi] \) are defined as usual. For notational simplicity we momentarily suppress the parametric dependence on the metric. One then has

\[
e^{\frac{1}{\sqrt{\epsilon_g}}W_k[J; \varphi]} = \int D\chi e^{-\frac{1}{\sqrt{\epsilon_g}}(S[\varphi + \chi] + \Delta S_k[\chi]) + \frac{1}{\sqrt{\epsilon_g}}J \cdot \chi},
\]
\[ \Gamma_k[\phi; \varphi] := J_k[\phi; \varphi] \cdot \phi - W_k[J=J_k[\phi; \varphi]] - \Delta S_k[\phi], \quad \frac{\delta W_k}{\delta J}[J=J_k[\phi; \varphi]] = \phi. \] (2.5)

In general the regulator \( \mathcal{R}_k \) may depend on the background field \( \varphi \), in which case the action \( S[\varphi + \zeta; g] + \Delta S_k[\chi; \varphi; g] \) depends the fluctuation field and the background field separately, i.e. not just through their sum. On the level of the effective action this means that the splitting symmetry \( \Gamma_k[\phi + \zeta; \varphi - \zeta] = \Gamma_k[\phi; \varphi] \) is violated. For the present purposes it suffices to take \( \mathcal{R}_k \) independent of the background field \( \varphi \), in which case splitting symmetry is maintained, i.e. \( \Gamma_k[\phi; \varphi] = \Gamma_k[\phi + \varphi] \) (by slight abuse of notation).

Throughout we use weighted functional derivatives adapted to the \( \cdot \) convolutions. A covariant source coupling \( J \cdot \chi = \int dy \sqrt{g(y)} J(y) \chi(y) = \int dt dx N \sqrt{g} J(t, x) \chi(t, x) \) suggests to define

\[ \lim_{\epsilon \to 0} \frac{1}{\epsilon} (F[J + \epsilon H] - F[J]) = \int dt dx (N \sqrt{g})(t, x) H(t, x) \frac{\delta F[J]}{\delta J(t, x)} =: H \cdot \frac{\delta F}{\delta J}, \] (2.6)

where the measure contribution is taken out of the functional derivative. Whenever \( d \) is essential we write \( dt dx \) for \( dt^d x \), the \( d \)-dimensional flat Euclidean measure. Then \( \delta F/\delta J(t, x) \) transforms as a temporal scalar and has length dimension \( \delta F/\delta J = -(d+1) - [J] = -(d-1)/2 \), for any dimensionless covariant functional \( F[J] \). The covariant delta function is normalized by \( \int dt dx (N \sqrt{g})(t, x) J(t, x) \delta(t, x, t', x') = J(t', x') \).

In this setting we quickly run through the familiar steps of deriving the FRG. Differentiating \( W_k[J; \varphi] \) with respect to \( k \) yields the Polchinski equation

\[ k \partial_k W_k[J; \varphi] = -\frac{\epsilon_g \sqrt{\epsilon_g \hbar}}{2} \text{Tr} \left\{ k \partial_k \mathcal{R}_k \left[ \frac{\delta^2 W_k}{\delta J \delta J} + \sqrt{\epsilon_g \hbar} \frac{\delta^2 W_k}{\delta J \delta \phi} \right] \right\}, \] (2.7)

where we note that the trace over spacetime indices is with respect to the usual volume density. Implementing the modified Legendre transformation in (2.5) one has

\[ k \partial_k \Gamma_k[\phi; \varphi] = \frac{\epsilon_g \sqrt{\epsilon_g \hbar}}{2} \text{Tr} \left\{ k \partial_k \mathcal{R}_k \frac{\delta^2 W_k}{\delta J \delta J} \right|_{J=J_k[\phi; \varphi]} \right\} \],

\[ \frac{\delta^2 W_k}{\delta J \delta J}[J=J_k[\phi; \varphi]]^{-1} = \left[ \frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi} + \epsilon_g \mathcal{R}_k \right]. \] (2.8)

For our purposes a background independent \( \mathcal{R}_k \) suffices and we can set the mean field \( \phi \) to zero, writing \( \Gamma_k[\varphi] = \Gamma_k[0; \varphi] \). Further, splitting symmetry entails \( \frac{\delta^2 \Gamma_k}{\delta \phi \delta \phi} = \frac{\delta^2 \Gamma_k}{\delta \varphi \delta \varphi} \), yielding the Wetterich equation in the schematic form

\[ k \partial_k \Gamma_k[\varphi] = \frac{\sqrt{\epsilon_g \hbar}}{2} \text{Tr} \left\{ k \partial_k \mathcal{R}_k \cdot G_k[\varphi] \right\}, \quad \left[ \frac{\delta^2 \Gamma_k}{\delta \varphi \delta \varphi} + \epsilon_g \mathcal{R}_k \right] \cdot G_k[\varphi] = \epsilon_g \mathbb{I}. \] (2.9)

The last relation highlights the key difference between both signatures. For the sake of argument, consider an \( \hbar \) expansion of \( \Gamma_k \) starting with \( S[\varphi] \) and a corresponding \( \hbar \) expansion of \( G_k[\varphi] \). This gives a recursion relation for the coefficients of \( G_k[\varphi] \) detailed in the next
For clarity’s sake we summarize the key relations of the spatial FRG in the geodesic completeness of the spatial metric
\[ N^{-\Delta N} = -\nabla^2 + \mathcal{U}'(\varphi, R) + \mathcal{R}_k \]. For Euclidean signature this will typically be a positive (elliptic) second order operator, which has a unique inverse on general grounds. In contrast, for Lorentzian signature \( S^{(2)}(\varphi) \) is a wave operator (hyperbolic) and although a unique Green’s function is determined by the Cauchy data [16] the data themselves are insufficiently constrained by physically reasonable requirements (like being induced by a vacuum-like two-point function). The resulting ambiguity can be attributed to the Green’s function and is the mathematical origin of the state selection problem in perturbative QFT on curved backgrounds [14, 15]. We stress that analogous choices enters the FRG’s \( \hbar \) expansion via the need to invert \( S^{(2)} - \mathcal{R}_k \).

As described above, the proposal here is to fix a foliation and to use a purely spatial modulator. For \( \epsilon_g = -1 \) a foliated geometry \( ds^2 = -N^2dt^2 + g_{ij}(N^iN^j + dx^i dx^j)(N^j dt + dx^j) \) will be assumed to be associated with a globally hyperbolic (connected, time oriented, Lorentzian) manifold \( M \). A useful characterization (see e.g. [16], Thm. 1.3.10) of a globally hyperbolic manifold is: \( \bar{M} \) is isometric to \( \mathbb{R} \times \Sigma \) with metric \( -\bar{N}^2dt^2 + \bar{g}_{ij}(\bar{t}, \bar{x}) d\bar{x}^i d\bar{x}^j \), where \( \bar{N} \) is real and the second term is a Riemannian metric on \( \Sigma \) that depends smoothly on \( \bar{t} \). Further, each \( \{ \bar{t} \} \times \Sigma \) is a smooth Cauchy surface in \( M \). In physicist’s terminology this means in particular that the ‘shift zero’ gauge \( \bar{N} \equiv 0 \) is always attainable and that a preferred foliation of \( \bar{M} \) in terms of Cauchy surfaces exists. The residual covariance group is then the direct product of purely temporal reparameterizations and the group \( \text{Diff}(\Sigma) \) of time independent spatial diffeomorphisms. Occasionally we shall restrict the group isometric to \( \text{Diff}(\Sigma) \times \) the direct product of purely temporal reparameterizations and the group \( \text{Diff}(\Sigma) \) of time independent spatial diffeomorphisms. Occasionally we shall restrict the foliation time to an interval \([t_i, t_f]\); the allowed temporal reparameterizations must the preserve the end points and we write \( \text{Diff}[t_i, t_f] \) for the resulting group. From now on we shall always work in the ‘shift zero’ gauge and drop the overbars from the notation, i.e. take \( ds^2 = -N^2dt^2 + g_{ij} dx^i dx^j \) as the metric. After a conformal rescaling to obtain \( -\bar{dt}^2 + N^{-2}g_{ij} dx^i dx^j \) the global hyperbolicity of the spacetime is closely related to the geodesic completeness of the spatial metric \( N^{-2}g_{ij}dx^i dx^j \) [23].

For clarity’s sake we summarize the key relations of the spatial FRG in the \( N^i = 0 \) foliation:

**Modulator term:**

\[
\Delta S_k[\chi] = -\frac{1}{2} \int dt dx dx' \sqrt{\bar{N}} g(t, x) \sqrt{\bar{N}} g(t, x') \chi(t, x) \chi(t, x') R_k(t, x, x', \chi(t, x')). \tag{2.10}
\]

Here \( R_k(t, x, x') \) is a bi-scalar under spatial diffeomorphisms and a scalar under temporal reparameterizations, so as to render (2.7) invariant. As noted before, we assume it to be independent of the background field and to depend on the spatial metric \( g_{ij}(t, x) \) only through the spectral values of the spatial part \( \nabla^2_s \) of the Laplacian. Explicitly,

\[
\nabla^2 = -\sqrt{\bar{g}^{-1}} N^{-1} \partial_t \left( \sqrt{\bar{g}} N^{-1} \partial_t \right) + \nabla^2_s,
\]

\[
\nabla^2_s := N^{-1} \sqrt{\bar{g}^{-1}} \partial_i \left( N \sqrt{\bar{g}^{ij}} \partial_j \right). \tag{2.11}
\]

For closed spatial sections \( \{ t \} \times \Sigma \) the operator \( -\nabla^2_s \) is symmetric and positive definite with respect to the inner product \( (f_1, f_2)_s := \int dt dx N \sqrt{\bar{g}} f_1^* f_2 \). In order to render the
temporal part symmetric the domain has to be suitably restricted, either by imposing fall-off conditions for \( t \in \mathbb{R} \) or by imposing Sturm-Liouville boundary conditions for \( t \in [t_i, t_f] \).

Flow equation:

\[
\partial_k \Gamma_k[\varphi] = \frac{i\hbar}{2} \int dt dx dx' \sqrt{N_g(t, x)} \sqrt{N_g(t, x')} \partial_k R_k(t, x, x') G_k[\varphi](t, x'; t, x), \tag{2.12}
\]

where we assume the temporal coincidence limit of the Green’s function \( G_k \) to be well-defined and non-zero. Then \( G_k(t, x; t, x') \) is a temporal scalar and a spatial bi-scalar which carries a functional dependence on \( \varphi(t, x) \). For generic \( \varphi(t, x) \) it is not spatially translation invariant, so no overcounting of the volume occurs.

Green’s function relation:

\[
-\int dt' dx' N(t', x') \sqrt{g(t', x')} \delta^2(\Gamma_k[\varphi] + \Delta S_k[\varphi]) \left( \frac{\delta^2}{\delta \varphi(t, x) \delta \varphi(t', x')} G_k[\varphi](t', x'; t'', x'') \right) = \delta(t, x; t'', x''), \tag{2.13}
\]

where \( \delta(t, x; t'', x'') \) is the covariantly normalized \( \delta \)-function.

### 2.2 Perturbative expansion and correspondence to Hadamard states

The spatial FRG is defined by the coupled system (2.12), (2.13). An analogous coupling would hold for any other Lorentzian signature FRG, but the spatially modulated version is more amenable to analysis. To start, a perturbative analysis turns out to be instructive. We consider Ansätze for \( \Gamma_k \) and \( G_k \) of the form

\[
\Gamma_k[\varphi] = S[\varphi] + \sum_{n \geq 1} h^n \Gamma_{k,n}[\varphi], \quad G_k[\varphi] = G_{k,0}[\varphi] + \sum_{n \geq 1} h^n G_{k,n}[\varphi], \tag{2.14}
\]

where \( S[\varphi] \) is the renormalized action. Inserted into (2.12), (2.13) this gives rise to a closed recursive system for the coefficients. We first present the condensed version arising from the \( \epsilon_g = -1 \) version of (2.9):

\[
\begin{align*}
[-S^{(2)}(\varphi) + R_k] \cdot G_{k,0}[\varphi] &= 1, \\
[-S^{(2)}(\varphi) + R_k] \cdot G_{k,n}[\varphi] &= \sum_{l=1}^{n} \Gamma_{k,l}^{(2)}(\varphi) \cdot G_{k,n-l}[\varphi], \quad n \geq 1, \tag{2.15}
\end{align*}
\]

\[
k \partial_k \Gamma_{k,n}[\varphi] = \frac{i}{2} \text{Tr} \left\{ k \partial_k R_k \cdot G_{k,n-1}[\varphi] \right\}, \quad n \geq 1.
\]

In principle, this determines iteratively the pairs \( (G_{k,n}, \Gamma_{k,n+1}) \), \( n \geq 0 \), viz

\[
G_{k,0} \rightarrow \Gamma_{k,1} \rightarrow G_{k,1} \rightarrow \Gamma_{k,2} \rightarrow \ldots . \tag{2.16}
\]
The only - though major - complication is to address the ambiguities that arise at each step and how they accumulate. The flow equation is integrated between a reference scale $\mu$ and an ultraviolet cutoff $\Lambda$ leading to

$$
\Gamma_{\mu,n}[\varphi] = \Gamma_{\Lambda,n}[\varphi] - \frac{i}{2} \int_{\mu}^{\Lambda} dk \text{Tr} \{ \partial_k R_k \cdot G_{k,n-1}[\varphi] \}.
$$

We seek to interpret $\Gamma_{\mu,n}$ as the $n$-th order renormalized effective action at scale $\mu$. Field and coupling renormalizations are introduced with the goal of strictly removing the UV cutoff $\Lambda$, although this strict removal may or may not be possible depending on the self-interaction under consideration. The successful removal of the UV cutoff by tractable coupling and field renormalizations amounts to establishing the perturbative renormalizability of the QFT. In Section 3 we shall detail this process at one-loop order for a generic (beyond quartic) scalar potential on Friedmann-Lemaître backgrounds, with the result that the Ricci couplings (1.1) need to be turned on to achieve strict renormalizability.

Here we focus on the, in the context of the FRG, novel ambiguities arising from the inversion of $S^{(2)}(\varphi) - R_k$, which drives the other part of the recursion. We write $S^{(2)}(\varphi) = -D \delta$ with

$$
D := -\nabla^2 + \sum_{n \geq 0} n U^{(2)}(\varphi) R(g)^n.
$$

In the second line the decomposition (2.11) was inserted. Despite the complicated form of the potential $U''(\varphi, R)$ it can in the context of (2.15) be viewed as a given (smooth) function on $M$ transforming as a scalar. The operator $-\nabla^2 s + U''$ can be rendered selfadjoint subject to suitable positivity conditions on $U''$ [23].

For the Green’s function of such wave operators powerful general results are available, including the unique solubility of the Cauchy problem and the existence of asymptotic expansions of the Hadamard form [16]. Unfortunately, the defining relation for $G_{k,0}$ is not quite of this form. In terms of $D$ and the spatial kernel $R_k$ it reads

$$
D_{t,x} G_{k,0}[\varphi](t, x, t', x') + \int dx'' (N \sqrt{g})(t, x''; t', x') G_{k,0}[\varphi](t, x''; t', x') = \delta(t, x; t', x').
$$

Although considerably more complicated than the standard case, locality in $t$ is preserved in that a spatial integro-differential operator acts at fixed $t$ on $G_{k,0}[\varphi](t, x; t', x')$. No ready-made theory seems to be available for this situation. To proceed, we note that for $k \to 0$ the relation (2.19) reduces to the standard one, $D_{t,x} G_{0,0}[\varphi](t, x, t', x') = \delta(t, x; t', x')$, and hence is tractable. For the moment we assume that a solution has been chosen according to some criterion and for simplicity we take it to be a Feynman Green’s function. The relation (2.19) can then be converted into an integral equation

$$
G_{k,0}[\varphi](t, x; t', x') + \int dt'' dx'' (N \sqrt{g})(t'', x'') R_k(t, x; t'', x'') G_{k,0}(t'', x''; t', x')
$$
\[ = G_{0,0} [\varphi](t, x; t', x') , \]
\[ R_k (t, x; t'', x'') = \int dx_1 (N \sqrt{g})(t'', x_1) G_{0,0} [\varphi](t, x; t'', x_1) R_k (t'', x_1, x'') . \]  
(2.20)

In order to obtain a Feynman-type Green’s function this needs to be augmented by suitable boundary conditions for the (first derivative of the) coincidence limit. Both the inhomogeneity and the kernel \( R_k \) depend on the choice of \( G_{0,0} \). Schematically (2.20) is of the form \([1 + R_k] \cdot G_{k,0} = G_{0,0}\) and for ‘small’ \( R_k \) should have a unique solution. We postpone an analysis of the conditions under which this holds true and note the important conclusion:

\[\text{Whenever (2.20) has for given } R_k \text{ and a choice of Feynman Green’s function } G_{0,0} \text{ a unique solution it defines a one-to-one correspondence between Hadamard states and the perturbative expansion of the spatial FRG.}\]

This rests on the fact that physically viable Feynman Green’s function solutions of the basic wave equation \( D_{t,x} G_{0,0} [\varphi](t, x, t', x') = \delta(t, x; t', x') \) are in one-to-one correspondence to Hadamard states. For completeness’ sake we briefly outline the (known) rationale underlying the last assertion. A Feynman Green’s function can always be written as \( 2G_{0,0} = G^c + i\omega^s \), where \( G^c \) is the unique real valued causal Green’s function and \( \omega^s \) is the symmetrized Wightman function. On any globally hyperbolic manifold the unique existence of \( G^c \) is ensured even globally, see [16] Section 3.4. The symmetrized Wightman function is said to be locally of Hadamard form if it can be written as

\[ \omega^s(t, x; t', x') = N_d [H_\epsilon(t, x; t', x') + W(t, x; t', x')] , \quad N_d = \frac{\Gamma(d-1)}{2(2\pi)^{(d+1)/2}} , \]  
(2.21)

where \( H_\epsilon \) is the “Hadamard parametrix” and \( W \) is a smooth symmetric function. The Hadamard parametrix formalizes the notion of ‘Minkowski-like’ short distance singularities, which occur for small \( \sigma_\epsilon(t, x; t', x') = \sigma(t, x; t', x') + i\epsilon(t-t') + O(\epsilon^2) \), where \( \sigma(t, x; t', x') \) is the Syng function (one-half of the geodesic distance squared between points with coordinates \( (t, x) \) and \( (t', x') \)). The normalization \( N_d \) is chosen such that leading singularity of \( H_\epsilon \) is \( U(t, x', t', x') \sigma_\epsilon^{-\frac{d-1}{2}} \), with \( U(t, x; t, x) = 1 \). For \( d \) even this is the only singularity, for \( d \) odd there is an additional \( V(t, x; t', x') \mu^2 \sigma_\epsilon(t; t', x') \) term, for some mass parameter \( \mu \). Here \( U \) and \( V \) are smooth symmetric functions uniquely associated with the differential operator \( D \). They can be expanded in terms of the “Hadamard coefficients”, which are recursively computable and related to those of the formal pseudo-heat kernel [27]. On the other hand, \( W \) in (2.21) is not uniquely associated with \( D_{t,x} \) and is co-determined by the quantum state underlying the Wightman function. A quantum state for which (2.21) holds is called a “local Hadamard state”. There is also a global characterization of a “Hadamard state” in terms of the “wave front set” of the differential operator. Nevertheless, the inherent ambiguity signaled in (2.21) through the underdetermination of \( W \) remains. In the global characterization a (Feynman) Green’s function is uniquely determined by the Cauchy data on a spacelike hypersurface [16]. Alas, on a general globally hyperbolic manifold there is no
physically preferred way of choosing these data such that a preferred $W$ is associated with them. The converse implication however works: given a Hadamard state there are entailed Cauchy data as well as a unique $W$ associated with it. By a physically viable Feynman Green’s function we mean one of the form $2G_{0,0} = G^c + i\omega^s$, where $\omega^s$ is associated with a Hadamard state. As such a choice of Feynman Green’s function is in one-to-one correspondence to the choice of a Hadamard state. Via the above italicized statement this lifts to a one-to-one correspondence between Hadamard states and the spatial FRG.

In practice, it has proven difficult to construct Hadamard states explicitly. Since the italicized statement conceptually rests on such a construction we shall not try to establish a general result here. Among non-static globally hyperbolic manifolds Friedmann-Lemaître backgrounds are of prime interest. For them a fairly explicit construction of Hadamard states is possible \cite{17} and concomitant results \cite{19} allow one to analyze (2.19), (2.20) in detail.

2.3 FRG vs Hadamard correspondence for Friedmann-Lemaître backgrounds

We consider spatially flat Friedmann-Lemaître line elements of the form

$$ds^2 = -N(t)dt^2 + a(t)^2\delta_{ij}dx^idx^j = g^{\text{FL}}_{\mu\nu}dy^\mu dy^\nu,$$  \hspace{1cm} (2.22)

initially without imposing any field equations. It is globally hyperbolic because the flat spatial metric is geodesically complete, see \cite{16}, Lemma A.5.14. Here the shift $N_i$ has been set to zero and we continue to write $N$ for the merely $t$ dependent lapse. The form of the line element (2.22) is preserved under $\text{Diff}[t_i, t_f] \times \text{ISO}(d)$ transformations, where the rotation group acts as global $\text{Diff}(\Sigma)$ transformations connected to the identity. Under the temporal reparameterizations $a(t)$ and $\tilde{\chi}(t, x)$ transform as scalars, while $N(t)$ and $n(t) = N(t)/a(t)^d$ are temporal densities, $n'(t') = n(t)/|\partial t'/\partial t|$, etc.. The scalar field action from (2.1) specialized to $g_{ij} = a(t)^2\delta_{ij}$, $n = n(t)$, and $N^i = 0$ reads

$$S[\tilde{\chi}] = \int_{t_i}^{t_f} dt \int d^d x \left\{ \frac{1}{2n(t)}(\partial_t \tilde{\chi})^2 - \frac{1}{2}n(t)a(t)^{2d-2}\partial_i \tilde{\chi}\delta^{ij}\partial_j \tilde{\chi} - n(t)a(t)^{2d}U(\tilde{\chi}, R) \right\} + \text{boundary terms},$$

(2.23)

where $n(t)$, $a(t)$ specify the geometry (2.22) and

$$R(g^{\text{FL}}) = 2d\frac{(N^{-1}\partial_t)^2 a}{a} + d(d-1)\frac{(N^{-1}\partial_t a)^2}{a^2} : R(t).$$

(2.24)

As indicated, we normally use a finite temporal interval but do not keep track of boundary terms. In the basic action, for example, the term proportional to $R(g^{\text{FL}})$ could be rewritten – modulo a Gibbons-Hawking boundary term – to obtain a kinetic term for the scale factor $(N^{-1}\partial_t a)^2$. Since we shall not consider equations of motion for $a$ and $N$, subtraction of
such boundary terms is inessential for the present purpose. Later on the effective action will similarly be evaluated only modulo temporal boundary terms.

In a next step we transition to the background field formalism, using the $\epsilon_g = -1$ versions of (2.4), (2.5), (2.9). The main simplifying feature of (2.22) as a globally hyperbolic background is that the flat spatial sections allow for a spatial Fourier transform of all relevant quantities. Our conventions are that of $\mathbb{R}^d$, $\chi(t, x) = \int \frac{d^dp}{(2\pi)^d} e^{ipx} \chi(t, p)$, $\chi(t, p) = \int d^d x e^{-ipx} \chi(t, x)$, with a `$\cdot$' omitted on $\chi(t, p)$. The associated delta distribution is $\delta(x-x')$, so that the previous covariant delta distribution factorizes according to

$$
\delta(t, x; x', t') = \delta(t, t') \delta(x-x'), \quad \delta(t, t') := (Na^d)^{-1} \delta(t-t').
$$

(2.25)

Since $R_k(t, x, x')$ in Section 2.2 was normalized by $R_k(t, x; t', x') = N(t)^{-1} \delta(t-t') R_k(t, x, x')$ additional powers of $a^{-d}$ may occur in the specialization of the general formulas.

For example, the modulator term in the action reads

$$
\Delta S_k[\chi] = -\frac{1}{2} \int dt N(t) a(t)^d \int \frac{d^dp}{(2\pi)^d} R_k(t, p) |\chi(t, p)|^2,
$$

(2.26)

with $Na^d$ instead of $Na^{2d}$ occurring. The Fourier representation of the equal time kernel is defined by

$$
R_k(t, x, x') = a(t)^{-d} \int \frac{d^dp}{(2\pi)^d} e^{ip(x-x')} R_k(t, p),
$$

$$
R_k(t, p) = k^2 r \left( \frac{p^2}{k^2 a(t)^2} \right), \quad r \in \mathcal{S}(\mathbb{R}_+), \quad r(0) = 1,
$$

(2.27)

where the factor of $a^{-d}$ is required to arrive at (2.26) from (2.10). Similar modulators have been used in [5, 6, 7]. As indicated, the $t, p$ dependence is stipulated to enter through the comoving momentum square $p^2/a(t)^2$ only. Further, $\mathcal{S}(\mathbb{R}_+)$ is the class of radial, real valued Schwartz functions on $\mathbb{R}_+$, decaying together with all its derivatives faster than any power. The $r(0) = 1$ normalization could be relaxed to any nonzero positive constant while the smooth fast decay will be important later on; note that it rules out the widely used ‘hockey stick’ $r(u) = (1-u)\theta(1-u)$. An admissible example is the exponential modulator, $r(u) = \alpha u/(e^{\alpha u} - 1)$, $\alpha > 0$.

Since the background field $\varphi$ is a function of $t$ only the Green’s function likewise admits a Fourier realization

$$
G_k[\varphi](t, x; t, x') = \int \frac{d^dp}{(2\pi)^d} e^{ip(x-x')} G_k[\varphi](t, t', p),
$$

(2.28)

and is defined by the relation

$$
- \int dt' N(t') a(t')^d \Gamma_k^{(2)}[\varphi](t, t', p) G_k[\varphi](t', t'', p) + R_k(t, p) G_k[\varphi](t, t'') = \delta(t, t').
$$

(2.29)
Here we used $(\Delta S_k)^{(2)}(t, x, t', x') = -R_k(t, x, x')\delta(t, t')$ in the conventions (2.18), (2.25) and wrote $\Gamma_k^{(2)}[\varphi](t, t', p)$ for the Fourier transform of $\Gamma_k^{(2)}[\varphi](t, x; t', x')$.

Due to the spatial homogeneity the volume in $\Gamma_k$ and its FRG is overcounted and will be factored out later on. With this understanding the spatial FRG (2.12) assumes the form

$$\partial_k \Gamma_k[\varphi] = \frac{i\hbar}{2} \int dt d^d x \ N(t) a(t)^d \int \frac{d^dp}{(2\pi)^d} \partial_k R_k(t, p) G_k[\varphi](t, t, p).$$

(2.30)

The system (2.30), (2.29) is the central object in the subsequent sections and will be referred to as the Friedmann-Lemaître spatial FRG, FL-sFRG. A similar spatial FRG has been considered on a de Sitter background in [5, 6, 7]. However, these constructions utilize the maximal symmetry of de Sitter spacetime to select the Bunch-Davies vacuum as the unique de Sitter invariant Hadamard state, and hence the issue of state dependence of the flow equation does not arise. In cosmology one of the prime applications of the FL-sFRG is to the exploration of a pre-inflationary, kinetic energy dominated era [29, 30].

The perturbative expansion proceeds as before via Ansätze (2.14) transferred to Fourier space. The recursion reads

\[
[D_{t,p} + R_k(t, p)]G_{k,0}[\varphi](t, t'; p) = \delta(t, t'),
\]

\[
[D_{t,p} + R_k(t, p)]G_{k,n}[\varphi](t, t'; p) = \sum_{l=1}^{n} \int dt'' N(t'') a(t'')^d \Gamma_{k,l}^{(2)}(t, t''; p) G_{k,n-l}[\varphi](t'', t'; p).
\]

\[
\partial_k \Gamma_{k,n}[\varphi] = \frac{i}{2} \int dt d^d x N(t) a(t)^d \int \frac{d^dp}{(2\pi)^d} \partial_k R(t, p) G_{k,n-1}[\varphi](t, t, p).
\]

(2.31)

Here we used $S^{(2)}[\varphi](t, t'; p) = -D_{t,p} \delta(t, t')$, for the Fourier kernel of the leading order. The differential operator reads

\[
D_{t,p} = a^{-2d}(a^d N^{-1} \partial_t)^2 + a^{-2} p^2 + W(t),
\]

\[
W(t) := \mathcal{U}'(\varphi(t), R(t)) = \sum_{n \geq 0}^n U^{(2)}(\varphi(t)) R(t)^n.
\]

(2.32)

Importantly, the Green’s function relations are pointwise in $p$ and $G_{k,0}$ is the Green’s function of an ordinary, second order, differential operator with a parametric dependence on $p^2$ and $k^2$. For $p^2 \ll k^2$ the $R_k$ term will by (2.27) just give rise to an additive term in the potential, $W \rightarrow W + O(k^2)$. For $p^2 \gg k^2$ the influence of the $R_k$ term will be negligible due to the fast decay. The construction of a Green’s function $G_{k,0}$ for $D_{t,p} + R_k(t, p)$ should therefore be a ‘small variation’ of the construction of a Green’s function $G_{0,0}$ of $D_{t,p}$. In technical terms, the relation between a given $G_{0,0}$ and the $G_{k,0}$ sought reads

\[
G_{k,0}[\varphi](t, t'; p) + \int dt''(Na^d)(t'')G_{0,0}[\varphi](t, t''; p)R_k(t'', p)G_{k,0}[\varphi](t'', t'; p) = G_{0,0}[\varphi](t, t'; p),
\]

(2.33)
where \( [a^{-2d}(a^d N^{-1} \partial_t)^2 + a^{-2} p^2 + W] G_{0,0} [\varphi](t, t'; p) = \delta(t, t') \). Again, this has to be augmented by conditions on the (first derivative's) temporal coincidence limit to characterize a Feynman-type Green’s function. We postpone the analysis of (2.33) and first describe how to obtain a physically viable \( G \) in Fourier space be realized as

\[
G_{k,0}[\varphi](t, t'; p) = i \theta(t - t') T_k(t, p) T_k(t, p)^* + i \theta(t' - t) T_k(t, p)^* T_k(t', p),
\]

(2.34)

where \( T_k(t, p) \) is a Wronskian normalized solution of the homogeneous wave equation

\[
\left( a^d N^{-1} \partial_t + \omega_k(t, p) + p^2 \omega_2(t) \right) T_k(t, p) = 0,
\]

\[
\omega_2(t)^2 = a(t)^{2d-2}, \quad \omega_k(t, p)^2 = a(t)^{2d} \left[ W(t) + k^2 r \left( \frac{p^2}{k^2 a^2} \right) \right],
\]

\[
a^d N^{-1} \partial_t T_k(t, p) T_k(t, p)^* - a^d N^{-1} \partial_t T_k(t, p)^* T_k(t, p) = -i.
\]

(2.35)

Here \( \lim_{\omega \to 0} \omega_k(t, p)^2 = a(t)^{2d} W(t) = \omega_0(t)^2 \) recovers the standard case with solution \( T_0(t, p) \); the subscript “2” in \( \omega_2(t) \) signals the coefficient of \( p^2 \) (not \( k = 2 \) in \( \omega_k(t, p)^2 \), of course). Note that \( G_{k,0}[\varphi](t, t'; p) = i |T_k(t, p)|^2 \), \( a^d N^{-1} \partial_t G_{k,0}[\varphi](t, t'; p)|_{t=t} = 1/2 \), so that (2.34) is fully determined by \( |T_k(t, p)| \). In fact \( T_k(t, p) \) itself is - up to constant phase - determined by its modulus

\[
T_k(t, p) = |T_k(t, p)| \exp \left\{ - \frac{i}{2} \int_{t_0}^{t} dt' \frac{1}{a(t')^d |T_k(t, p)|^2} \right\}.
\]

(2.36)

Inserting (2.36) into (2.34) produces the decomposition of the Feynman Green’s function into the state-independent causal Green’s function and the state-dependent symmetrized Wightman function

\[
G_{k,0}[\varphi](t, t'; p) = \frac{1}{2} \Delta_0(t, t'; p) + \frac{i}{2} \omega_k(t, t'; p),
\]

(2.37)

both of which are fully determined by the modulus of \( T_k \).

Remarkably, the modulus square satisfies itself a nonlinear second order differential equation

\[
2|T_k|^2 (a^d N^{-1} \partial_t)^2 |T_k|^2 - \left( a^d N^{-1} \partial_t |T_k|^2 \right)^2 + 4 \left[ \omega_2(t)^2 p^2 + \omega_k(t, p)^2 \right] |T_k|^4 = 1,
\]

(2.38)

as can be verified from (2.35). This is a variant of the Gelfand-Dickey equation satisfied by the diagonal of the resolvent kernel of a Schrödinger operator. Indeed, \( G_{k,0}(t, t; p) = \)
\[i|T_k(t, p)|^2\] is heuristically the diagonal of the inverse of the differential operator in (2.35). Due to the \(a^dN^{-1}\) factors the differential operator in (2.35) is not directly a Schrödinger operator and there is no immediate resolvent parameter. After a suitable transformation and without the \(r\) term the coefficient of \(p^2\) is \(N^2/a^2\), which is in general not constant and \(p^2\) does not quite play the role of a resolvent parameter for the Schrödinger operator. One can, however, develop a generalized resolvent expansion which allows the coefficient of the large parameter – here \(p^2\) – to be non-constant. Moreover, the structure of the expansion characterizes a Hadamard state:

**Result** [19]: Let \(T_0(t, p)\) be a solution of \([(a^dN^{-1}\partial_t)^2 + \omega_0(t)^2+p^2\omega_2(t)^2]T_0(t, p) = 0\) and let \(\omega_0^*(t, t'; p)\) be the symmetrized Wightman function (2.37) built from it. Then, its inverse Fourier transform is associated with a Hadamard state if and only if \(|T_0(t, p)|^2\) admits an asymptotic expansion of the form

\[-iG_{0,0}[\bar{\varphi}](t, t; p) = |T_0(t, p)|^2 \approx \frac{1}{2\omega_2(t)p}\left\{1 + \sum_{n \geq 1} (-)^n \bar{G}_n(t)p^{-2n}\right\}.\] (2.39)

Here \(\bar{G}_0 = 1\) the coefficients \(\bar{G}_n(t), n \geq 1\), are fixed uniquely by the recursion relation

\[
\bar{G}_n = \sum_{j,l \geq 0, j+l=n-1} \left\{ \frac{1}{4} \frac{G_j}{\omega_2} (a^dN^{-1}\partial_t) (\bar{G}_l) - \frac{1}{8} (a^dN^{-1}\partial_t) (G_j) (a^dN^{-1}\partial_t) (\bar{G}_l) \right\} \\
+ \frac{1}{2} \frac{\omega_0^2}{\omega_2} G_j \bar{G}_l \right\} - \frac{1}{2} \sum_{j,l \geq 1, j+l=n} \bar{G}_j \bar{G}_l. \] (2.40)

Here the \(\sim\) sign indicates that the difference between the left hand side and the \(N^{th}\) partial sum of the right hand side is \(O(p^{-2(N+1)})\), without control over the \(N\) and \(t\) dependent coefficient. The implied asymptotic expansion of \(1/|T_0(t, p)|^2\) and hence of the phase is understood.

**Remarks:**

(i) The recursion follows by inserting an Ansatz of the form (2.39) into (2.38). It expresses \(\bar{G}_n\) in terms of \(\bar{G}_{n-1}, \ldots, \bar{G}_1\), and involves only differentiations. It follows that \(\bar{G}_n\) equals \(\omega_2^{-4n}\) times a differential polynomial in \(\omega_0^2, \omega_2^2\), with \(a^dN^{-1}\partial_t\) as basic derivative. Moreover, one can show that the \(\bar{G}_n, n \geq 2\), can be expressed as differential polynomials in only \(\bar{G}_1\), using \(\omega_2^{-1}a^dN^{-1}\partial_t\) as basic derivative. For (2.35) one has \(\omega_2^2 = a^{2d-2}, \omega_0^2 = a^{2d}W\), where positivity of \(W\) is not needed.

(ii) Writing \(v := \omega_0^2, w := \omega_2^2\) and denoting \(a^dN^{-1}\partial_t\) differentiations momentarily by a “\(\frac{\partial}{\partial t}\)” one finds:

\[
\bar{G}_1 = \frac{v}{2w} + \frac{5}{32} \frac{w^2}{w^3} - \frac{1}{8} \frac{w''}{w^2},
\]

\[
\bar{G}_2 = \frac{3}{8w^2} \left( \frac{v^2}{3} + \frac{v^2}{w'} - \frac{5}{16w^3} \left( vw'' + v'w' - \frac{7}{4} \frac{w^2}{w} \right) \right).
\]

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The recursion (2.40) is easily programmed in Mathematica and produces the \( \bar{G}_n \) to reasonably high orders. 

(iii) The familiar adiabatic iteration [33] is primarily organized according to the number of time-derivatives of \( a(t) \), but is provenly asymptotic only for large \( p \) [18]. In this form it enters the original proof [17] of the Hadamard property of SLE. The above result eliminates the reorganization of the adiabatic iteration, and characterizes directly the coefficients of \( p^{-2n-1} \).

(iv) Being merely asymptotic, the expansion (2.39) does not fully determine \( G_{0,0} \). Rather, an independent construction principle for a specific solution \( T_0(t,p) = T_0^{\text{Had}}(t,p) \), of Hadamard type, valid for all \( p \), needs to be known and implemented. While the large \( p \) behavior of \( T_0^{\text{Had}}(t,p) \) will always be governed by (2.39) the small \( p \) behavior will reflect the nature of the construction principle used to obtain it.

Occasionally, one will want to relate (2.35) and the results based on it to a standard Schrödinger operator and its resolvent expansion. This can be done by expanding \((a^d N^{-1} \partial_t)^2 = a^{2d} N^{-2} [\partial_t^2 - \partial_t \ln(N a^{-d}) \partial_t] \) and then removing the first order term by redefining \( T_k(t,p) = (N(t) a(t)^{-d})^{1/2} \chi_k(t,p) \). Doing so, (2.35) transcribes into

\[
\left[ \partial_t^2 + N^2 k^2 \left( \frac{p^2}{a^2 k^2} + r \left( \frac{p^2}{a^2 k^2} \right) \right) \right] + N^2 W + \frac{1}{2}S(t) \chi_k(t,p) = 0 ,
\]

\[\partial_t \chi_k(t,p) \chi_k(t,p)^* - \partial_t \chi_k^*(t,p) \chi_k(t,p) = -i . \quad (2.42)\]

where \( S(t) = -2(N a^{-d})^{1/2} \partial_t^2 (N^{-1} a^d)^{1/2} \) is the induced Schwarzian. The associated Gelfand-Dickey equation is

\[
2|\chi_k|^2 \partial_t^2 |\chi_k|^2 - (\partial_t |\chi_k|^2)^2 + \left[ 4N^2 k^2 \left( \frac{p^2}{a^2 k^2} + r \left( \frac{p^2}{a^2 k^2} \right) \right) \right] + 4N^2 W + 2S \right]|\chi_k|^4 = 1 . \quad (2.43)
\]

Without the \( r \) term the coefficient of \( p^2 \) in (2.42) is \( N^2/a^2 \), which is in general not constant and \( p^2 \) does not quite play the role of a resolvent parameter for the Schrödinger operator. This can be remedied in several ways: One can gauge fix temporal reparameterization invariance by choosing conformal time, \( N(t) = a(t) \). Without the \( r \) term then \( p^2 \) literally plays the role of a resolvent parameter for the Schrödinger operator with potential \( a^2 W + S/2 \). Alternatively, one can introduce a resolvent parameter for the original Hessian in (2.18), \( D \mapsto D + z \). Then \( U'' \) is everywhere replaced with \( U'' + z \), and in (2.42) it occurs with pre-factor \( N^2 \). This can now be rendered constant by choosing cosmological time, \( N(t) \equiv 1 \), which results in a Schrödinger operator with potential \( W + S/2 \). With the above result in place we can return to the issue of how to lift a Feynman Green’s function \( G_{0,0} \) to its scale dependent counterpart \( \bar{G}_{k,0} \). Instead of analyzing (2.33) directly...
we return to (2.35), (2.38) and note

\[-iG_{k,0}[\varphi](t, t; p) = |T_k(t, p)|^2 \propto \frac{1}{2\omega_2(t)p}\left\{ 1 + \sum_{n \geq 1} (-)^n \bar{G}_n(t) \frac{p^{-2n}}{\omega_0^2 - \omega^2_k} \right\}. \tag{2.44}\]

This holds for \( k \) bounded away from zero because the \( r \) term in \( \omega_k(t, p)^2 \) decays faster than any power in \( p \), and thus does not interfere with the derivation of the recursion relation obtained by comparing powers of \( 1/p^2 \). The only difference is that in (2.40) \( \omega_0^2 \) will be replaced with \( \omega_0^2 \), which results in (2.44). For given \( r \) one obtains a one-parameter deformation of the original Hadamard state. Note that no modified notion of a Hadamard state is needed, the asymptotic expansion (2.39) simply allows for the deformation.

In Section 3.1 we shall keep the dimensionless ratio \( \phi = p/k \) fixed and interpret the differential operator in (2.35) as \((a^dN^{-1}\partial_t)^2 + a^{2d}W + k^2a^{2d-2}[\varphi^2 + a^2r(\varphi^2/a^2)]\). The general result (2.39), (2.40) then applies with \( p \mapsto k \), \( \omega_0^2 \mapsto a^{2d}W \), \( \omega_2^2 \mapsto a^{2d-2}[\varphi^2 + a^2r(\varphi^2/a^2)] \). The resulting series in \( k^{-2n-1} \) can be seen as a re-organization of the one in (2.44). In particular, it can still be viewed as a deformation of the underlying Hadamard state.

A similar deformation exists in the infrared regime. While this can be seen at the level of the integral equation (2.33) for the Green’s function, it is simpler to consider the lift from \( T_0(t, p) \) to \( T_k(t, p) \) via an expansion in convolution powers of \( a(t)^{2dk^2r(p^2/(k^2a(t)^2))} \).

**Result:** Let \( T_0^{\text{Had}}(t, p) \) be a given Wronskian normalized solution of Hadamard type of \([(a^dN^{-1}\partial_t)^2 + a^{2d}W + a^{2d-2}p^2]T_0(t, p) = 0 \), and let \( G_0^{\text{Had}}(t, t'; p) \) be the associated Feynman Green’s function, formed according to (2.34). Then, there exists a solution of the wave equation in (2.35) given by

\[
\tilde{T}_k(t, p) = T_0^{\text{Had}}(t, p) + \sum_{n \geq 1} (-)^n \left( \left( C_0^{\text{Had}} \cdot a^{2dk^2r}(p^2/(a^2k^2)) \right)^{-n} \cdot T_0^{\text{Had}} \right)(t, p), \tag{2.45}
\]

where \( \cdot \) denotes a temporal integration over \([t_i, t_f]\) and \( \cdot \) its \( n \)-fold nested iteration. Moreover, the series converges uniformly on \([t_i, t_f]\) for \( k < k_c(p) \). The solution \( \tilde{T}_k(t, p) \) can be re-normalized to satisfy the Wronskian condition in (2.35) as well.

We only sketch the proof, which parallels the one in Prop. 3.1 of [19]. On an interval \([t_i, t_f]\) the differential equation in (2.35) can be recast as an integral equation

\[
\tilde{T}_k(t, p) = -k^2 \int_{t_i}^{t_f} dt' G_0^{\text{Had}}(t, t'; p)a(t')^{2d}r(p^2/(k^2a(t')^2)) \tilde{T}_k(t', p), \tag{2.46}
\]

where the given Hadamard Green’s function is used to invert the the \( k = 0 \) part of the differential operator. The three terms comprising the integral kernel are all bounded in \([t_i, t_f]\). For \( G_0^{\text{Had}} \) this holds because it is built in analogy to (2.34) from \( T_0^{\text{Had}} \); for \( a^{2d} \) it
holds trivially; for the $r$-term it holds by assumption on $r$ in (2.27). This implies that the integral operator (depending parametrically on $k$ and $p$)

$$
f \mapsto -k^2 \int_{t_i}^{t_f} dt' G^\text{Had}_0(t, t'; p)a(t')^2r(p^2/(k^2a(t')^2))f(t'),
$$

(2.47)
is well-defined as a self-map on the Banach space $\mathcal{C}([t_i, t_f]), \|\cdot\|_{\sup}$ of continuous functions on $[t_i, t_f]$ equipped with the sup norm. Moreover, for sufficiently small $k < k_s(p)$ it is a contraction. As such, it possesses a fixed point $\bar{T}_k(t, p)$ which solves (2.46) exactly, and which may be reached from $T^\text{Had}_0$ by the series (2.45). A similar argument applies to the time derivative $a^dN^{-1}\partial_t \bar{T}_k(t, p)$, for which the termwise time derivative of the series (2.45) converges uniformly in $[t_i, t_f]$ for $k < k_s(p)$; c.f. the proof of Prop. 3.1 in [19].

Taken together, this implies the constancy of the Wronskian $a^dN^{-1}\partial_t \bar{T}_k(t, p)\bar{T}_k(t, p)^* - a^dN^{-1}\partial_t T_k(t, p)^* T_k(t, p) := -iwr_k(p)$, and of the induced series expansion within the radius of convergence $k < k_s(p)$. By construction, $wr_k(p)$ is real with $wr_0(p) = 1$. Re-normalizing the solution $T_k(t, p) := wr_k(p)^{-1/2} T_k(t, p)$ produces for $k < k_s(p)$ an exact solution of (2.35), which is a deformation of the given $T^\text{Had}_0(t, p)$. This shows the result.

In summary, a given Hadamard type Feynman Green’s function $G^\text{Had}_{0,0}[\varphi](t, t', p)$ has a unique deformation to a Feynman Green’s function $G_{k,0}[\varphi](t, t', p)$, both in the ultraviolet (large $p$ at fixed $k$, or large $k$ at fixed $\varphi = p/k$) and in the infrared (small $k$ at fixed $p$, in particular for small $p$). The deformation is also of Hadamard type but modified, see (2.44). Both Green’s functions are related by (2.33) and since $G^\text{Had}_{0,0}[\varphi](t, t', p)$ is defined for all $p$, we expect that the deformation is unique also in the crossover region.

### 2.4 States of Low Energy and their infrared behavior

States of Low Energy (SLE) are Hadamard states for a minimally coupled scalar field on a generic Friedmann-Lemaître background. The name originates from the fact that a time averaged energy functional is minimized as part of the construction. The energy functional derives from the time-time component of the energy momentum tensor, which changes if the coupling to gravity is non-minimal. The original construction [17] starts from an arbitrary (exact) fiducial solution $S_0(t, p)$ of the homogeneous $k = 0$ wave equation (2.34) and associates to it a special solution $T^\text{SLE}_0[S_0](t, p)$ of the same wave equation. The construction proceeds by minimization of the time averaged energy functional and fixes the $p$-dependence of $T^\text{SLE}_0[S_0](t, p)$ completely, for all $0 < p = \sqrt{p_1^2 + \ldots + p_d^2} < \infty$. For large $p$ it is such that the expansion (2.39) holds. The associated Feynman Green’s function (2.28), (2.34) can then be shown to have Hadamard property in the global, wave front sense. Finally, it can be shown that although a choice of $S_0$ is made in the construction of the SLE $T^\text{SLE}_0[S_0]$, it is actually independent of this choice (up to a constant phase), i.e. any other $\bar{S}_0$ related to $S_0$ by a Bogoliubov transformation gives the same SLE, see (2.52) below.
While the direct link to the energy-momentum tensor and the minimization procedure are physically appealing, the mathematics of the Hadamard property rests more on the time averaging and the invariance under Bogoliubov transformations of $S_0$. For the standard non-minimal coupling, $\xi R\chi^2$, the positivity properties of the energy functional are obscured for generic values of $\xi$, and a complete proof of the Hadamard property is currently not available. For our Ricci tower potential (2.2) even the definition of a viable energy-momentum tensor would require a lengthy detour and the analysis of the minimizers may have to remain incomplete. Instead, we use here the minimizing functionals associated with the minimally coupled scalar and the proven Hadamard property of the associated SLE. Whenever the distinction needs to be highlighted we shall refer to them as the minimal SLE. This route sacrifices a direct link to the energy-momentum tensor and its renormalizability would have to be explored separately. In fact, one of the advantages of the FRG formalism is that for most applications the construction of composite operators is not needed.

For a minimally coupled scalar field the dispersion relation is $\omega_p(t)^2 = a(t)^2 [p^2/a^2 + 0U''(\varphi(t))]$, see (2.32). For later use we generalize this to

$$\omega_p(t)^2 = \omega_0(t)^2 + p^2 w_{2,p}(t), \quad p \mapsto w_{2,p}(\cdot) \text{ bounded and smooth. \quad (2.48)}$$

Specifically, $w_{2,p}(t) - w_{2,\infty}(t)$ is assumed to be of rapid decay in $p$ uniformly in $t \in [t_i, t_f]$, with some $w_{2,\infty}(t) =: \omega_2(t)^2 > 0$. Similarly, $\lim_{p \to 0} w_{2,p}(t)^2 = w_{2,0}(t)$, for some nonzero $w_{2,0}(\cdot)$. However, $\omega_p(t)^2 \geq 0$, $t \in [t_i, t_f]$, is assumed throughout. This may be viewed as the dispersion of an IR modified QFT, where the rapid decay in the ultraviolet leaves the Hadamard property unaffected. The $k$-modified dispersion $\omega_p(t)^2 = p^2 \omega_2(t)^2 + \omega_k(t,p)^2$ from (2.35) is a prime example, another one will arise in Section 4.3. For the dispersion (2.48) we consider Wronskian normalized fiducial solutions solving the modified wave equation $[\{a^dN^{-1}\partial_t\}^2 + \omega_p(t)^2]S(t,p) = 0$. To any some-such solution the associated minimal, IR modified SLE solution is defined by

$$T^{\text{SLE}}[S](t,p) = \lambda_p[S]S(t,p) + \mu_p[S]S(t,p)\ast. \quad (2.49)$$

Suppressing the subscripts momentarily, the coefficients are given by

$$\lambda[S] = -e^{-i\text{Arg}D[S]} \sqrt{\frac{E[S]}{2\sqrt{E[S]^2 - |D[S]|^2}}} + \frac{1}{2},$$

$$\mu[S] = \sqrt{\frac{E[S]}{2\sqrt{E[S]^2 - |D[S]|^2}}} - \frac{1}{2}. \quad (2.50)$$

Here $E[S]$ and $D[S]$ are functionals that arise by expanding the time averaged free Hamilton operator in terms of creation and annihilation operators. The averaging is done with a positive smooth function $f(\cdot)$ of compact support in $[t_i, t_f]$. Explicitly,

$$E[S] = \frac{1}{2} \int dt \, n(t)f(t) \left\{ n^{-1}\partial_t S|^2 + \omega_p(t)^2 |S(t)|^2 \right\},$$
\[ \mathcal{D}[S] = \frac{1}{2} \int dt \; n(t) f(t) \left\{ (n^{-1} \partial_t S)^2 + \omega_p(t)^2 S(t)^2 \right\}. \]  

(2.51)

In (2.50) a choice of sign has been made that renders \( \mu[\cdot] \) real. Consistently, we require \( \mathcal{E}[S] > |\mathcal{D}[S]| \), which is satisfied on account of \( \omega_p(t)^2 \geq 0 \), \( t \in [t_i, t_f] \).

An obvious concern about the construction is the dependence on the fiducial solutions. Recall that the general Wronskian normalized solution of a homogeneous second order differential equation can always be written as the Bogoliubov transform of a fixed solution and its complex conjugate. Hence, one can probe the extent to which \( T^{\text{SLE}}[S](t, p) \) depends on the choice of fiducial solutions by subjecting them to a Bogoliubov transformation. Remarkably, \( T^{\text{SLE}}[S](t, p) \) turns out to be invariant up to a time independent phase [19].

Such a phase drops out in the Green’s function (2.37) and one has in particular

\[ \omega^s[aS + bS^*](t, t', p) = \omega^s[S](t, t', p), \quad |a|^2 - |b|^2 = 1. \]  

(2.52)

Viewed as a quantum state underlying the Green’s function, the IR-modified SLE therefore is independent of the choice of \( S \).

The key property of the original SLE is that the inverse Fourier transform of the two-point function \( T_0^{\text{SLE}}[S_0](t, p)T_0^{\text{SLE}}[S_0](t', p) \) has the Hadamard property [17]. One can show that this remains true for the IR modified SLE modes \( T^{\text{SLE}}[S](t, p) \) defined above. The proofs in [17] are given for a constant potential (mass squared), the extension to the dispersion \( \omega_p(t)^2 = a(t)^2 + \varphi(t) \) is straightforward [19]. For the IR modified dispersion (2.48) the non-quadratic \( p \) dependence needs to be taken into account. A major step in the proof of the Hadamard property is to show that \( \lambda[S] - 1 \) and \( \mu[S] \) carry a \( p \) dependence that falls off for large \( p \) faster than any power. Since in (2.48) we take the difference \( \omega_p(t)^2 - p^2 \omega_2(t)^2 - \omega_0(t)^2 \) to decay itself faster than any power in \( p \) (uniformly in \( t \in [t_i, t_f] \)), the additional \( p \) dependence will not interfere with these estimates.

In accordance with the general result the IR modified SLE admit a large \( p \) asymptotic expansion of the form (2.44) entering (2.36). It is significant that the dependence on the window function \( f(\cdot) \) drops out to all orders of the expansion. This is one manifestation of the universality of a Hadamard state’s ultraviolet behavior. In contrast, the coefficients of the small \( p \) expansion depend manifestly on the window function. For later use we describe this expansion to low orders explicitly, allowing for a dispersion of the form (2.48).

The invariance of an SLE under Bogoliubov transformations can be rendered manifest by expressing the solution \( T_0^{\text{SLE}}(t, p) \) solely in terms of the commutator function \( \Delta_p(t, t') = i[S(t, p)S(t', p)^* - S(t, p)^*S(t', p)] \) [19]. For the modulus square one has

\[ |T^{\text{SLE}}(t, p)|^2 = \frac{J_p(t)}{2 \mathcal{E}_p^{\text{SLE}}}, \]  

(2.53)

with

\[ J_p(t) = \frac{1}{2} \int dt' \; n(t') f(t') \left( (n(t')^{-1} \partial_t \Delta_p(t', t))^2 + \omega_p(t')^2 \Delta_p(t', t)^2 \right), \]
Remarkably, the leading term is where \( \Delta = \Delta_p(t', t) \) is the commutator function \( \Delta \) has mass dimension \(-1\) so has \( J_p(t) \), while \( E^{\text{SLE}} \) is dimensionless. Moreover, the quantities (2.53), (2.54) are known \cite{19} to admit for \( \omega_p(t) = \omega_0(t)^2 + p^2 \omega_2(t)^2 \) a convergent expansion in powers of \( p^2 \). In particular,

\[
J_p(t) = \sum_{n \geq 0} J_n(t) p^{2n}, \quad (E^{p})^2 = \sum_{n \geq 0} \varepsilon_n^2 p^{2n},
\]

where \( J_n(t) \) has mass dimension \(-(2n+1)\) and \( \varepsilon_n \) has mass dimension \(-n\). Note that the commutator function \( \Delta_p(t, t') = \sum_{n \geq 0} \Delta_n(t, t') p^{2n} \) needs to be expanded as well. We extend the expansions (2.55) to the generalized dispersion (2.48) simply by substituting \( w_{2,p}(t) \) for \( \omega_2(t)^2 \) and write

\[
J_{n,p}(t) := J_n(t)\left|_{\omega_2^2 \rightarrow w_{2,p}} \right., \quad \varepsilon_{n,p} := \varepsilon_n^2 \left|_{\omega_2^2 \rightarrow w_{2,p}} \right., \quad \Delta_{n,p}(t, t') := \Delta_n(t, t')\left|_{\omega_2^2 \rightarrow w_{2,p}} \right.. \tag{2.56}
\]

The induced \( p \) dependence is of bounded variation. Although it could in principle be expanded as well for small \( p \), it turns out to be advantageous not to do so.

For nonzero \( \omega_0 \) one has \( \varepsilon_0 > 0 \) and the expansions (2.55) starts at \( O(p^0) \). For \( \omega_0 \equiv 0 \) one has \( \varepsilon_0 = 0 \) and the expansion for the generalized dispersion (2.48) reads

\[
E^{\text{SLE}} = \varepsilon_{1,p} p + \frac{\varepsilon_{2,p}^2}{2\varepsilon_{1,p}} p^3 + O(p^5),
\]

\[
|T^{\text{SLE}}(t, p)|^2 = \frac{J_0}{2\varepsilon_{1,p} p} - \frac{2J_{1,p}(t)\varepsilon_{1,p}^2 - J_0\varepsilon_{2,p}^2}{4\varepsilon_{1,p}^3} p + O(p^3). \tag{2.57}
\]

Remarkably, the leading term is \( O(1/p) \), indicating a Minkowski-like infrared behavior for all scale factors, – a characteristic property of ‘massless’ SLE. This second case will be more relevant later on, so we present explicit expressions for the coefficients:

\[
J_0 = \frac{1}{2} \int dt \ n(t) f(t),
\]

\[
J_{1,p}(t) = \frac{1}{2} \int dt' n(t') f(t') \left[ 2n(t')^{-1} \partial_{t'} \Delta_{1,p}(t', t) + \Delta_0(t, t')^2 w_{2,p}(t') \right], \tag{2.58}
\]

where \( \Delta_0, \Delta_1 \) are the terms in the expansion of the \( \omega_0 \equiv 0 \) commutator function

\[
\Delta_0(t, t') = \int_t^{t'} ds n(s), \quad \Delta_{1,p}(t, t') = \int_t^{t'} ds n(s)w_{2,p}(s)\Delta_0(t, s)\Delta_0(t', s). \tag{2.59}
\]

\(^1\)This corrects a factor 1/2 in the second term of \( J_1 \) and an overall sign in \( \Delta_1 \) in Eqs (102), (103) of \cite{19}. 

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Finally,

\[ \varepsilon_{1,p}^2 = \frac{1}{2} J_0 \int dt n(t) f(t) w_{2,p}(t), \]

\[ \varepsilon_{2,p}^2 = \frac{1}{8} \int dt dt' n(t) n(t') f(t) f(t') \left[ \left( n(t)^{-1} n(t')^{-1} \partial_t \partial_{t'} \Delta_{1,p}(t, t') \right)^2 \right] \]

\[ - 4w_{2,p}(t') n(t)^{-1} \partial_t \Delta_{1,p}(t', t) + w_{2,p}(t) w_{2,p}(t') \Delta_0(t, t')^2 \]. \quad (2.60)

Note that for given \( n \) and window function \( f \) these are simple functionals of \( w_{2,p} \). In the special case \( n \equiv 1 \) and constant \( w_{2,p} = w_2 \) the expressions simplify drastically: \( \Delta_0(t, t') = t - t' \), \( \Delta_1(t, t') = -w_2(t - t')^3/6 \), giving \( J_1(t) \equiv 0 \), \( \epsilon_1 = w_2 J_0 \), \( \varepsilon_2^2 = 0 \). Expressions similar to (2.58), (2.60) exist for the case with nonzero \( \omega_0 \), but \( \Delta_0, \Delta_1 \) can then in general not be determined explicitly.
3. Wilsonian one-loop renormalization

On general grounds all Hadamard states share the same ultraviolet behavior. For the Friedmann-Lemaitre backgrounds considered this manifests itself in the universality of the asymptotic expansion (2.39). Here we use this non-covariant expansion to extract the divergent parts of the one-loop effective action by re-integrating the one-loop flow equation, \( \partial_k \Gamma_{k,1} = ... \), in (2.31). For simplicity we focus on \( d = 3 \) and use conformal time. Remarkably, the result is again covariant, i.e. the \( a \) dependence only enters through the measure and the Ricci scalar. This holds for any potential, with or without the Ricci tower (1.1).

Next we aim at the absorption of the divergent parts through coupling and field renormalizations. Traditional one-loop renormalization includes only power counting renormalizable terms in the action, evaluates the divergent part of the Hessian’s ‘tracelog’ using dimensional regularization, and then removes the divergent parts using minimal subtraction, see [33, 36, 34, 35] in the present context. In a Wilsonian framework the action is meant to contain all interaction monomials compatible with a stipulated symmetry requirement, here \( \varphi \mapsto -\varphi \) even terms in the scalar field. Regularization invokes a momentum-like ultraviolet (UV) cutoff \( \Lambda \) and a running scale \( \mu \). After evaluation of the divergent part of the effective action \( L_1^{\text{div}} \) one seeks coupling and field redefinitions entering the bare action \( L_\Lambda \) such that the UV cutoff \( \Lambda \) can strictly be removed

\[
L = \lim_{\Lambda \to \infty} \left[ L_\Lambda + \hbar L_1^{\text{div}} \right].
\]  

(3.1)

Conceptually, the renormalized Lagrangian \( L \) ought to be identified with the scale dependent Wilsonian one at some scale \( \mu \). The strict removal of the UV cutoff amounts to the successful identification of the unstable manifold of a UV fixed point of the \( \mu \)-flow. As such it is an integral part of the generalized notion of renormalizability aimed at. For a counterterm renormalization to be consistent therewith non-minimal subtraction terms need to be included to ensure that the bare quantities (associated with scale \( \Lambda \)) and the renormalized quantities (associated with scale \( \mu \)) coincide when \( \Lambda = \mu \). Through the finite terms this innocuous “matching principle” [12] affects the parts of the beta functions driven by powerlike divergences. We shall aim at this “strict Wilsonian one-loop renormalizability”.

For a scalar QFT on Friedmann-Lemaitre backgrounds we find that any beyond-quartic, non-conformal self-interaction requires the inclusion of an infinite tower of potentials (2.2) in order to meet (3.1). Despite the proliferation of couplings only 6 turn out to be relevant or marginal (in \( d = 3 \)) and we compute the associated beta functions.

3.1 Ultraviolet Divergences of Hadamard states

From (2.31) one sees that the one-loop effective action is determined by the diagonal \( G_{k,0}[\varphi](t,t,p) \) of the tree level Green’s function \( G_{k,0} \). We transition to conformal time gauge \( N(t) = a(t) \), and write \( \eta = t \) for the so-defined time variable. The one-loop flow
equation from (2.31) then reads
\[ k \partial _k \Gamma _{k,1}[\varphi ] = \frac{i}{2} \int d\eta d^dx \: a(\eta )^{d+1} \int \frac{d^dp}{(2\pi)^d} k \partial _k R_k(\eta , p) G_{k,0}[\varphi ](\eta , \eta , p) . \] (3.2)

Here, the Green’s function satisfies the first equation in (2.31) with \( D_{t,x} = a^{-2d} (a^{-d-1} \partial _t) \) + \( a^{-2} p^2 + W, \delta (\eta , \eta ') = a^{-d-1} \delta (\eta - \eta '), \) and \( W(\eta ) := \sum _{n \geq 0} a^nu''(\varphi (\eta )) R(\eta )^n . \) As in (2.17) one can integrate (3.2) between a reference scale \( \mu \) and an ultraviolet scale \( \Lambda \) to obtain
\[ \Gamma _{\mu ,1}[\varphi ] = \Gamma _{\Lambda ,1}[\varphi ] - \frac{i}{2} \int d\eta a(\eta )^{d+1} d^dx \int \frac{d^dp}{(2\pi)^d} \int _\mu ^\Lambda dk \partial _k R_k(\eta , p) G_{k,0}[\varphi ](\eta , \eta , p) . \] (3.3)

Note that for a \( k \)-independent \( D \) one can interpret \( \partial _k R_k G_{k,0} \) as \( \partial _k \ln G_{k,0} \), so that the \( k \) integral leads to the expected ‘trace-log’ structure. One loop renormalizability amounts to the existence of coupling and field renormalizations such that the \( \Lambda \rightarrow \infty \) limit exists. We shall take up this task in Section 3.2. A prerequisite is the isolation of the parts of the \( k \) integral in (3.3) that diverge as \( \Lambda \rightarrow \infty , \)

It is convenient to transition to a genuine Schrödinger operator problem as in (2.42), (2.43). Defining \( G_{k,0}[\varphi ](\eta , \eta '; \varphi ) := [a(\eta )a(\eta ';)]^{d+2} G_{k,0}[\varphi ](\eta , \eta '; k \varphi ), \varphi = p/k, \) one has
\[ \{ \partial _\eta ^2 + k^2 \omega _2(\eta , \varphi )^2 + \omega _0(\eta )^2 \} G_{k,0}[\varphi ](\eta , \eta '; \varphi ) = \delta (\eta - \eta '), \]
\[ \omega _0(\eta , \varphi )^2 := \varphi ^2 + a(\eta )^2 \left( \frac{\varphi ^2}{a(\eta )^2} \right), \quad \omega _0(\eta )^2 := a(\eta )^2 W(\eta ) - \frac{d-1}{4d} R(\eta ) . \] (3.4)

Here we used \( S(\eta ) = -\frac{d-1}{2d} a(\eta )^2 R(\eta ) \), obtained by rewriting the definition from (2.42) via (2.24). Only the diagonal \( G_{k,0}[\varphi ](\eta , \eta ; \varphi ) = i \chi _k(\eta , k \varphi )^2 \) enters the flow equation. Moreover, the diagonal is governed by the Gelfand-Dickey equation (2.43) with \( t = \eta , N = a, \) and \( S = -\frac{d-1}{2d} a^2 R . \) Its solution \( G_{k,0}(\eta , \varphi ) := G_{k,0}[\varphi ](\eta , \eta ; \varphi ) \) enters the flow equation
\[ \partial _k \Gamma _{k,1}[\varphi ] = -\frac{2i k^{d+1}}{(4\pi)^{d/2} \Gamma (d/2)} \int d\eta a(\eta )^{d+1} d^dx \int _0 ^\infty d\varphi \varphi ^{d-1} \left[ r(\varphi ^2 / a^2) - (\varphi ^2 / a^2)r'(\varphi ^2 / a^2) \right] G_{k,0}(\eta , \varphi ) . \] (3.5)

Since \( G_{k,0}(\eta , \varphi ) = i \chi _k(\eta , k \varphi )^2 \) and \( \chi _k(\eta , p) \) solves \( \partial _\eta ^2 + \omega _0(\eta )^2 + k^2 \omega _2(\eta , \varphi )^2 \chi _k(\eta , m) = 0, \) the standard adiabatic iteration [33] applies. The resulting partial sums are, however, only provenly asymptotic to an exact solution for large \( k \) [18] (this is the same argument as in Remark (iii) following the result (2.39), (2.40) with \( k \) playing the role of \( p \); see also the comment after (2.44)). Accordingly, it is advantageous to apply the generalized resolvent expansion (2.39), (2.40) to directly extract the coefficients of \( k^{-2n-1} \). The latter expansion applies with \( \partial _\eta \) as the basic derivative, the indicated \( \omega _0, \omega _2, \) and \( k^2 \) as the generalized resolvent parameter. Note that for \( r \equiv 0 \) the standard resolvent expansion would apply, just as it does for the \( N = a \) form of (2.43) without the \( r \) term. With the \( r \) term the generalized resolvent expansion is indispensable, however. It reads
\[ G_{k,0}(\eta , \varphi ) \approx \frac{i}{2 \omega _2(\eta , \varphi )k} \left\{ 1 + \sum _{n \geq 1} (-)^n \tilde{G}_n(\eta , \varphi ) k^{-2n} \right\} , \] (3.6)
where the coefficients are uniquely defined by

\[ \bar{G}_n = \sum_{j,l \geq 0,j+l=n-1} \left\{ \frac{1}{4} \frac{\bar{G}_j \partial^2_{\eta} (\bar{G}_l)}{\omega_2 (\omega_2) - \frac{1}{8} \partial_{\eta} (\frac{\bar{G}_j}{\omega_2}) \partial_{\eta} (\frac{\bar{G}_l}{\omega_2}) + \frac{1}{2} \frac{\omega_0^2}{\omega_2} \bar{G}_j \bar{G}_l} \right\} - \frac{1}{2} \sum_{j,l \geq 1,j+l=n} \bar{G}_j \bar{G}_l, \tag{3.7} \]

with \( \bar{G}_0 = 1 \) and \( \omega_0, \omega_2 \) from (3.4). The explicit expressions (2.41) carry over with ' now simply \( \partial_{\eta} \). Ultimately, it is the uniqueness ensured by the explicit recursion (3.7) that underlies the universality of the FL-sFRG’s large \( k \) behavior. Note also that (3.7) is dimension independent; it is only upon insertion of (3.6) into (3.5) that the divergence structure depends on \( d \).

For definiteness we now specialize to \( d = 3 \). Then only the \( n = 0,1,2 \) terms in (3.6) give rise to divergences. Evaluating the integral \(- \int_\mu^{\Lambda} dk \partial_k \Gamma_1^{\text{div}}[\varphi] \) yields the UV-divergent contribution to the one-loop effective action

\[ \Gamma_1^{\text{div}}[\varphi] = \frac{1}{(4\pi)^2} \int d\eta dx a(\eta)^4 \left\{ q_0 (\Lambda^4 - \mu^4) + g_1(\eta)(\Lambda^2 - \mu^2) + g_2(\eta) \ln(\Lambda/\mu) \right\}. \tag{3.8} \]

Here we keep the terms from the lower integration boundary, see remark (i) below. For the coefficients it is convenient to redefine \( \varphi = a(\eta) \varphi \) and \( G_j(\eta, \varphi) := \bar{G}_j(\eta, \varphi) |_{\varphi=a(\eta)\varphi}^2 \). Then

\[ q_0 = \int_0^\infty d\varrho \varrho^2 \frac{[r(\varrho^2) - \varrho^2 r'(\varrho^2)]}{[\varrho^2 + r(\varrho^2)]^{1/2}}, \]

\[ g_1(\eta) = -2 \int_0^\infty d\varrho \varrho^2 \frac{[r(\varrho^2) - \varrho^2 r'(\varrho^2)]}{[\varrho^2 + r(\varrho^2)]^{1/2}} G_1(\eta, \varphi), \]

\[ g_2(\eta) = 4 \int_0^\infty d\varrho \varrho^2 \frac{[r(\varrho^2) - \varrho^2 r'(\varrho^2)]}{[\varrho^2 + r(\varrho^2)]^{1/2}} G_2(\eta, \varphi). \tag{3.9} \]

Inserting \( G_1, G_2 \) from (2.41) one finds

\[ g_1 = \tilde{\eta} R - q_1(W - R/6) \]

\[ g_2 = \frac{1}{2} (W - R/6)^2 - \frac{1}{6} \nabla^2 (W - R/6) + B_1 \frac{a^{(4)}}{a^5} + B_2 \frac{a^{(1) a^{(3)}}}{a^6} + B_3 \frac{a^{(2)} a^{(2)}}{a^6} + B_4 \frac{a^{(1)2} a^{(2)}}{a^7} + B_5 \frac{a^{(1)4}}{a^8} + \left( B_6 \frac{R}{6} + B_7 \frac{a^{(1)2}}{a^4} \right) (W - R/6) + B_8 \frac{a^{(1)}}{a^3} \partial_\eta (W - R/6). \tag{3.10} \]

Here \( \tilde{\eta}, q_1, B_1, \ldots, B_8 \) are real constants that depend on the regulator \( r \) via the \( \varphi \)-integrals in (3.9). The \( \eta \)-dependence is carried by \( W, R, a \), where we wrote \( a^{(n)} := \partial_\eta^n a \) and \( \nabla^2 = -a^{-2} \partial_\eta (a^2 \partial_\eta) \), see (2.18).

\footnote{It is to be understood that the redefinition \( \varphi = a(\eta) \varphi \) is performed after all the time derivatives in (3.7) have been taken.}
As may be expected from our use of a purely spatial regulator, the structure of the logarithmic divergence from the second and third lines of \( g_2 \) above has a non-covariant form. Remarkably, it follows from the properties of the regulator function (2.27) that \( g_2 \) can be “covariantized” and expressed solely in terms of

\[
\begin{align*}
R &= \frac{6a^{(2)}}{a^3}, \quad \nabla^2 R = 0 \frac{a^{(4)}}{a^5} + 24 \frac{a^{(1)}a^{(3)}}{a^6} + 18 \frac{a^{(2)}^2}{a^6} - 30 \frac{a^{(1)}^2a^{(2)}}{a^7}, \\
-R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 &= -12 \frac{a^{(1)}^4}{a^8} + 12 \frac{a^{(1)}^2a^{(2)}}{a^7} = \frac{4}{a^4}\partial_\eta \left( \frac{a^{(1)}}{a^3} \right),
\end{align*}
\]

for generic regulator functions \( r(x) \). This is the content of the Lemma A in Appendix A.

Specifically, it follows from the explicit expressions (3.11) for the curvature invariants together with Lemma A(a), (b) that

\[
\begin{align*}
B_1 \frac{a^{(4)}}{a^5} + B_2 \frac{a^{(1)}a^{(3)}}{a^6} + B_3 \frac{a^{(2)}^2}{a^6} + B_4 \frac{a^{(1)}^2a^{(2)}}{a^7} + B_5 \frac{a^{(1)}^4}{a^8} &= b_1 \nabla^2 R + b_2 \left[ -R_{\mu\nu}R^{\mu\nu} + \frac{1}{3}R^2 \right].
\end{align*}
\]

Moreover, Lemma A(c) entails that the final line of \( g_2(\eta) \) in (3.10) may be expressed in terms of a single coefficient \( b_3 := B_8 \),

\[
\left( B_6 \frac{R}{6} + B_7 \frac{a^{(1)}^2}{a^4} \right)(W - R/6) + B_8 \frac{a^{(1)}}{a^3} \partial_\eta(W - R/6) = b_3 a^{-4} \partial_\eta [aa^{(1)}(W - R/6)].
\]  

(3.13)

Explicit expressions for \( \bar{q}, q_1 \) and \( b_1, b_2, b_3 \) are relegated to Appendix A. All but the first term of \( g_2(\eta) \) are thus total derivatives. Upon \( \int d\eta d^3x a(\eta)^4 \) integration in (3.8) these evaluate to boundary terms, which we omit in accordance with (2.23). In summary, up to boundary terms, the UV-divergent correction to the one-loop effective action is \( \hbar \Gamma^{\text{div}}[\varphi] = \int d\eta d^4x a^4L_1^{\text{div}}, \) with

\[
L_1^{\text{div}} = \frac{\hbar}{(4\pi)^2} \left\{ q_0(\Lambda^4 - \mu^4) + [\bar{q}R - q_1(W - R/6)](\Lambda^2 - \mu^2) + \frac{1}{2}(W - R/6)^2 \ln(\Lambda/\mu) \right\},
\]

(3.14)

and \( W(\eta) = \sum_{n \geq 0}^n U^n(\varphi(\eta))R(\eta)^n. \)

Remarks:

(i) In (3.14) we kept the finite terms from the lower integration boundary. This ensures \( L_1^{\text{div}}|_{\mu = \Lambda} = 0 \) and the bare action \( L_\Lambda \) can be identified with the renormalized one \( L_\mu \) at the UV scale \( \mu = \Lambda \). The non-minimal subtraction implemented in the next subsection preserves this structure and after strict removal of the UV cutoff (3.1) allows one to literally identify the renormalized quantities with those at scale \( \mu \).

(ii) Although the heat kernel is not defined for non-elliptic Hessians, the heat kernel coefficients remain well-defined. Often a ‘pseudo-heat kernel’ is used as generating functional for the coefficients, obtained by formally replacing the heat kernel time \( s \) with
\( s = i \tilde{s} \), treating \( \tilde{s} \) as real; see e.g. [33]. The divergent part of the effective action can then be computed along the usual lines for any pseudo-Riemannian background metric \( g_{\mu\nu}(y)dy^\mu dy^\nu \). In the notation of Section 2.1 one has \( S^{(2)}(\varphi) = \epsilon_0 D \), with \( D \) from (2.18), and we normalize \( L_1^{\text{div}} \) by \( \sqrt{\epsilon_0} h_0^1 \left( \text{Tr} \ln D \right)_{\mu \Lambda} \propto \int d^D y \sqrt{g(y)} L_1^{\text{div}} \). Here the regularized ‘tracelog’ enters, which for momentum type cutoffs \( \mu \leq \Lambda \) can be defined in several equivalent ways [12, 37]. In \( D = 4 \) only the diagonal heat kernel coefficients \( E_0, E_2, E_4 \) enter. The result can be expressed in terms of the Ricci scalar \( R(g) \) and the square of the Weyl tensor \( K \) of (3.14). The \( \mu \)-dependent terms are kept with same rationale as in (i). For a conformally flat metric like \( g^{\text{FL}} \), the \( C^2 \) term vanishes and (3.15) has the same structure as (3.14). The coefficient of the \( \ln \Lambda / \mu \) term is universal as expected, while the regulator dependent constants \( q_0^{\text{cov}}, q_1^{\text{cov}} \) vs \( q_0 \) and \( q_1^{\text{cov}} / 2 \) vs \( q_1 \) will in general differ. Notably, (3.14) contains an extra term \( \dot{q} R \) which is induced by the \( a(t) \) dependence of the spatial modulator in (2.27). It will lead to an additional renormalization of Newton’s constant later on.

(iii) One can check that the \( \dot{q} R \) term in (3.14) is indeed induced by the \( a \) dependence of the spatial modulator and not by the use of the generalized resolvent expansion (2.39), (2.40) by pursuing a hybrid approach. In the situation at hand the pseudo-heat kernel \( K_\delta(t, x; t', x') \) can also be spatially Fourier transformed and its Fourier kernel \( K_\delta(t; t') \) then satisfies the defining relations

\[
\left( i \frac{\partial}{\partial \tilde{s}} - D_{t,p} \right) K_\delta(t, t'; p) = 0, \quad \lim_{\tilde{s} \to 0} K_\delta(t, t'; p) = a(t)^{-d} \delta(t, t'),
\]

with \( D_{t,p} \) from (2.31). In cosmological time gauge \( N(t) = 1 \) one can use the Gelfand-Dickey coefficients \( G_{\alpha} \) to compute the coefficients \( E_n^{\text{GD}} \) induced by an asymptotic expansion of \( K_\delta(t; t; p) \). With matched normalizations one finds \( E_0^{\text{GD}}(t) = 1 \) and

\[
E_2^{\text{GD}}(t) = -W(t) + \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2},
\]
\[
E_4^{\text{GD}}(t) = \frac{1}{2} W(t)^2 + \frac{1}{6} \dot{W} - W \left( \frac{\dot{a}}{a} + \frac{\dot{a}^2}{a^2} \right) + \frac{1}{2} \dot{W} \frac{\dot{a}}{a} + \frac{1}{2} \frac{\ddot{a}}{a} \frac{\dot{a}}{a} + \frac{29}{15} \frac{\dot{a}^2}{a^3} + \frac{3}{10} \frac{\dot{a}^2}{a^2} - \frac{3}{5} \frac{\dot{a}^2}{a^2} - \frac{3}{5} \frac{a^{(3)}}{a^2}.
\]

Discarding boundary terms in the integrals the expressions (3.17) lead to the same divergence structure as (3.15).

29
3.2 Counterterm renormalization and one-loop beta functions

Next, we introduce the general structure of the counterterm Lagrangian. Throughout we denote bare quantities (diverging as $\Lambda \to \infty$) by a subscript $\Lambda$ while renormalized quantities carry no subscript but tacitly refer to a finite renormalization scale $\mu$. Both are as usual related by a renormalization constant where the subscript $\Lambda$ is omitted. Writing

$$-L_\Lambda = -\frac{1}{2} \tilde{\chi}_\Lambda \nabla^2 \tilde{\chi}_\Lambda + \sum_{n \geq 0} n U_\Lambda(\tilde{\chi}_\Lambda) R^n,$$

(3.18)

the bare field $\tilde{\chi}_\Lambda$ is related to the renormalized one $\tilde{\chi}$ by $\tilde{\chi}_\Lambda = Z^{1/2} \tilde{\chi}$. In the background field formalism the wave function renormalization constant of the fluctuation field $Z_\chi$ and that of the background field $Z_\varphi$ will coincide as long as the splitting symmetry $\chi \to \chi + \zeta, \varphi \to \varphi - \zeta$, is preserved. Since we assume the latter to be the case (see the discussion after (2.5)) we can identify $Z_\chi = Z_\varphi = Z$, set the mean fluctuation field $\phi$ to zero, and renormalize the background field according to $\varphi_\Lambda = Z^{1/2}\varphi$. Using the parameterization (2.2) the bare potentials $n U_\Lambda(\varphi_\Lambda)$ then expand according to

$$n U_\Lambda(\varphi_\Lambda) = \sum_{j \geq 0} \frac{1}{(2j)!} n U_{2j,\Lambda} \varphi_\Lambda^{2j} = \sum_{j \geq 0} \frac{1}{(2j)!} n Z_{2j} n U_{2j} \varphi^{2j},$$

(3.19)

where all the renormalization constants are $1 + O(\hbar)$, and hence $L_\Lambda = L + \hbar L_{ct}$ defines the counter term action.

We now examine the general structure of the counterterm Lagrangian. It is clear from (3.14) that the one-loop divergence structure does not necessitate a non-trivial wave function renormalization, i.e. $Z = 1 + O(\hbar^2)$. Thus the bare and renormalized couplings in (3.19) are related by

$$n U_{2n,\Lambda} = Z^{n} n U_{2j} \varphi^{2j}, \quad n \geq 0, \ j \geq 0,$$

(3.20)

and we choose appropriate Ansätze for the renormalization constants:

$$0 Z_0 = 1 + \hbar \frac{1}{(4\pi)^2} \left\{ 0 z_{0,0} + 0 z_{0,1} \ln \Lambda / \mu + 0 z_{0,2} (\Lambda / \mu)^2 + 0 z_{0,4} (\Lambda / \mu)^4 \right\},$$

$$0 Z_{2j} = 1 + \hbar \frac{1}{(4\pi)^2} \left\{ 0 z_{2j,0} + 0 z_{2j,1} \ln \Lambda / \mu + 0 z_{2j,2} (\Lambda / \mu)^2 \right\}, \quad j \geq 1,$$

$$n Z_{2j} = 1 + \hbar \frac{1}{(4\pi)^2} \left\{ n z_{2j,0} + n z_{2j,1} \ln \Lambda / \mu + n z_{2j,2} (\Lambda / \mu)^2 \right\}, \quad n \geq 1, \ j \geq 0.$$

(3.21)

In all cases $O(\hbar^2)$ corrections are implicit. The Ansätze (3.21) contain non-divergent terms, which are however related to the coefficients of the powerlike terms by the “matching principle” of [12]: We seek to identify the renormalized couplings with the bare ones at scale $\mu$. This constrains the coefficients of the non-logarithmic terms as follows

$$0 z_{0,0} + 0 z_{0,2} + 0 z_{0,4} = 0, \quad 0 z_{2j,0} + 0 z_{2j,2} = 0, \quad j \geq 1.$$
\[ n z_{2j,0} + n z_{2j,2} = 0, \quad n \geq 1, \quad j \geq 0. \quad (3.22) \]

The counterterm action thus reads
\[
L_\Lambda = L - \frac{\hbar}{(4\pi)^2} \left\{ \mu^{-4} 0 u_0 0 z_{0,4} \Lambda^4 + F_2 \Lambda^2 + F_4 \ln(\Lambda/\mu) + O(\Lambda^0) \right\}.
\]
\[
F_2 = \mu^{-2} \sum_{n \geq 0} \sum_{j \geq 0} \frac{1}{(2j)!} n z_{2j,2} n u_{2j} \phi^{2j} R^n,
\]
\[
F_4 = \sum_{n \geq 0} \sum_{j \geq 0} \frac{1}{(2j)!} n z_{2j,1} n u_{2j} \phi^{2j} R^n. \quad (3.23)
\]

Then, the condition \( L_\Lambda + L_1^{\text{div}} \) is finite as \( \Lambda \to \infty \) amounts to three conditions: \( 0 u_0 0 z_{0,4} \not\equiv \mu^4 q_0 \), which fixes \( 0 z_{0,4} \), and
\[
F_2 \equiv \bar{q} R - q_1 (W - R/6), \quad F_4 \equiv \frac{1}{2} (W - R/6)^2. \quad (3.24)
\]

The occurrence of \( W - R/6 \) in (3.24) makes it convenient to define couplings \( \bar{u}_2 := u_2 - \frac{1}{6} \),
\[ n \bar{u}_{2j} := n u_{2j}, \quad j \neq 1, \]
and associated potentials
\[
\bar{U}(\varphi) := U(\varphi) - \frac{1}{12} \varphi^2, \quad n \bar{U}(\varphi) := n U(\varphi), \quad n \neq 1. \quad (3.25)
\]

Re-expressing (3.24) in terms of these yields
\[
2 F_4 \equiv 0 \bar{U}''(\varphi)^2 + 2 0 \bar{U}''(\varphi) \bar{U}''(\varphi) R + \sum_{n \geq 2} \left[ \sum_{n_1 + n_2 = n} n_1 \bar{U}''(\varphi) n_2 \bar{U}''(\varphi) \right] R^n. \quad (3.26)
\]

We proceed by computing the renormalization constants \( n Z_{2j} \) by matching powers of \( \phi^{2j} R^n \) in (3.26). From the \( F_2 \)-relation one obtains
\[
0 z_{2j,2} 0 u_{2j} = -\mu^2 q_1 0 \bar{u}_{2j+2}, \quad j \geq 0, \\
1 z_{0,2} 1 u_0 = \mu^2 (\bar{q} - q_1 1 \bar{u}_2), \\
1 z_{2j,2} 1 u_{2j} = -\mu^2 q_1 1 \bar{u}_{2j+2}, \quad j \geq 1, \\
n z_{2j,2} n u_{2j} = -\mu^2 q_1 n \bar{u}_{2j+2}, \quad n \geq 2, \quad j \geq 0, \quad (3.27)
\]
while the logarithmic divergence yields
\[
\frac{1}{2} \sum_{n_1 + n_2 = n} \sum_{j_1 + j_2 = j} \frac{(2j)!}{(2j_1)!(2j_2)!} n_1 \bar{u}_{2j_1+2} n_2 \bar{u}_{2j_2+2}. \quad (3.28)
\]
These solutions can now be inserted into the Ansätze (3.21) and produce expressions for the \( ^nZ \), \( n, j \geq 0 \), relating the bare and the renormalized couplings via (3.20). Instead of presenting the \( ^nZ \) we insert them into (3.19) and note the resulting relations between the bare and the renormalized potentials

\[
\begin{align*}
0U_\Lambda(\varphi_\Lambda) &= 0U(\varphi) + \frac{\hbar}{(4\pi)^2} \left\{ (\Lambda^4 - \mu^4)q_0 - (\Lambda^2 - \mu^2)q_1 \ 0U''(\varphi) + \ln(\Lambda/\mu) \frac{1}{2} 0U''(\varphi)^2 \right\}, \\
1U_\Lambda(\varphi_\Lambda) &= 1U(\varphi) + \frac{\hbar}{(4\pi)^2} \left\{ (\Lambda^2 - \mu^2)[\tilde{q} - q_1 \ 1U''(\varphi)] + \ln(\Lambda/\mu) \ 0U''(\varphi) \ 1U''(\varphi) \right\}, \\
nU_\Lambda(\varphi_\Lambda) &= nU(\varphi) + \frac{\hbar}{(4\pi)^2} \left\{ - (\Lambda^2 - \mu^2)q_1 \ nU''(\varphi) \\
&\quad + \frac{1}{2} \ln(\Lambda/\mu) \sum_{n_1+n_2=n, n_1, n_2 \geq 0} n_1U''(\varphi) n_2U''(\varphi) \right\}, \quad n \geq 2. 
\end{align*}
\]

Of course, \( O(\hbar^2) \) corrections are implicit in all relations.

Remarks:

(i) The renormalization of \( 0U \) is the same as in flat space and is independent of the non-minimal potentials. In contrast, the renormalization of each \( nU, n \geq 1 \), depends on \( lU, l = 0, \ldots, n-1 \).

(ii) This upward cascade in the order of the Ricci scalar halts only in two special cases. One is the standard case of a \( \varphi^4 \)-theory, where one may set \( ^nU_j = 0 \) for all \( n \geq 1, j \geq 0 \) except for \( \xi = 1u_2 \), which corresponds to non-minimal coupling \( \frac{\lambda}{6} \varphi^2 R \) [33]. The other is the “conformal sector”, where

\[
1U(\varphi) = \frac{1}{12} \varphi^2, \quad nU(\varphi) \equiv 0, \quad n \geq 2, \quad (3.30)
\]

is consistent for generic even scalar potential \( 0U(\varphi) \).

(iii) For a generic scalar potential \( 0U(\varphi) \) and non-conformal coupling \( 1u_2 \neq 1/6 \), the need to include Ricci-couplings of all orders in the action (2.23), (3.18), follows from the structure of the logarithmic divergence. Consider, for example, a sextic potential \( 0U(\varphi) = \frac{1}{2} m^2 \varphi^2 + \frac{1}{4} \lambda \varphi^4 + \frac{1}{6} g \varphi^6 \) for non-conformal coupling. Then the logarithmic divergence in (3.14) contains a term proportional to \( (1u_2 - 1/6) g \varphi^4 R \), whose absorption requires \( 1u_4 \neq 0 \) in the action (3.18), (3.19). This in turn leads to the logarithmic divergence proportional to \( (1u_2 - 1/6) 1u_4 \varphi^2 R^2 \), which necessitates \( 2u_2 \neq 0 \), and so on.

(iv) In an attempt to avoid the infinite tower of Ricci couplings, one might try to remove some of the \( R^n \) divergences in (3.14) by a non-linear field renormalization

\[
\varphi_\Lambda = \varphi + \frac{\hbar}{(4\pi)^2} \ln(\Lambda/\mu) \ \zeta(\varphi) R + O(\hbar^2), \quad \zeta(\varphi) = \sum_{n \geq 0} n \zeta(\varphi) R^n. \quad (3.31)
\]
However, since

\[ S_\Lambda[\varphi_\Lambda] = S_\Lambda[\varphi] + \frac{\hbar}{(4\pi)^2} \ln(\Lambda/\mu) \int d\eta d^3x a^4 \frac{\delta S}{\delta \varphi} [\varphi] \zeta(\varphi) + O(\hbar^2), \] (3.32)

such a field renormalization does not produce any counterterms for on-shell background fields with \( \delta S_\Lambda/\delta \varphi = 0 \). When keeping the background off-shell, a non-trivial \( \zeta(\varphi) \) clashes with a standard kinetic term and would require further modifications.

The flow equations for the couplings are derived by differentiating the defining relations: bare coupling = (renormalization constant) \( \times \) (renormalized coupling) with respect to the scale \( \mu \). For the original (usually dimensionful) couplings, the response is always \( O(\hbar) \) so that only the explicit \( \mu \)-dependence needs to be taken into account. One can view the renormalized potentials \( nU(\varphi) = \sum_{j \geq 0} n_{u_j} \varphi^j/(2j)! \) as generating functionals for the couplings and differentiate the relations (3.29) with respect to \( \mu \). This gives

\[ \mu \partial_\mu n_{U_0} = \frac{\hbar}{(4\pi)^2} \left\{ 4q_0 n_{U_0} - 2q_1 n_{U_0}'' + \frac{1}{2} n_{U_0}'' n_{U_0}'' \right\}, \]

\[ \mu \partial_\mu n_{U_1} = \frac{\hbar}{(4\pi)^2} \left\{ 2q_1 n_{U_1}'' + 0 n_{U_1}'' n_{U_1}'' \right\}, \]

\[ \mu \partial_\mu n_{U_n} = \frac{\hbar}{(4\pi)^2} \left\{ -2q_1 n_{U_n}'' + 1 \sum_{n_1+n_2=n} n_{1U_1''}(\varphi) n_{2U_1''}(\varphi) \right\}, \quad n \geq 2. \] (3.33)

By expansion, the induced flows of all couplings \( n_{u_{2j}}, n, j \geq 0 \), can be obtained.

Next, we transition to dimensionless counterparts of the dimensionful couplings and rewrite the flow equations in terms of them. In preparation, we recall that the couplings \( n_{u_{2j}} \) have mass dimension \( 4 - 2(n+j) \). Only 6 have non-negative mass dimension and for orientation we note their standard notations

\[ 0_{u_0} = \frac{\Lambda_{\text{cosm}}}{\kappa}, \quad 0_{u_2} = m^2, \quad 0_{u_4} = \lambda, \]

\[ 1_{u_0} = -\frac{1}{\kappa}, \quad 1_{u_2} = \xi, \quad 2_{u_0} = \omega_0. \] (3.34)

Generally, we denote the dimensionless counterparts of the couplings \( n_{u_{2j}} \) by

\[ n_{v_{2j}} := \mu^{-4+2(n+j)} n_{u_{2j}}, \quad n, j \geq 0, \] (3.35)

and \( \varepsilon := 0_{v_0} = \mu^{-4} 0_{u_0} \) for the dimensionless vacuum energy. The redefinitions (3.35) ensure that the dimensionless coupling-flow equations obtained from (3.33) are all autonomous, i.e. carry no explicit \( \mu \)-dependence. The resulting flow equations for the dimensionless couplings of \( 0_{U} \) read:

\[ \mu \frac{d}{d\mu} \varepsilon = -4\varepsilon + \frac{\hbar}{(4\pi)^2} \left\{ 4q_0 - 2q_1 0_{v_2} + \frac{1}{2} 0_{v_2}^2 \right\}, \]
\[ \mu \frac{d}{d\mu} v_{2j} = (2j - 4) v_{2j} \]  
\[ + \frac{\hbar}{(4\pi)^2} \left\{ -2q_1 v_{2j+2} + \frac{1}{2} \sum_{1+2j = j, j_1, j_2 \geq 0} \frac{(2j)!}{(2j_1)!(2j_2)!} v_{2j_1+2} v_{2j_2+2} \right\}, \quad j \geq 1, \]  
(3.36)

These equations are precisely those obtained on a flat spacetime, and are not closed as the one for \( v_{2j} \) also invokes \( v_{2j+2} \), \( j \geq 0 \). In order to solve them a truncation is required that sets all \( v_{2j} \) to zero for all \( j \geq j_0 \), for some \( j_0 \in \mathbb{N} \). Truncation at order \( j_0 = 4 \) yields the fixed point
\[ \varepsilon^* = \frac{\hbar}{(4\pi)^2} g_0, \quad v_2^* = v_4^* = v_6^* = 0, \]  
(3.37)
consistent with the existence of only a Gaussian UV fixed point for a scalar theory in four spacetime dimensions. It is noteworthy, however, that \( \varepsilon \) has a mass independent positive fixed point that depends only mildly on the choice of the regulator. This is also a feature of the effective potential approximation of the FRG, but it cannot be seen in dimensional regularization.

For the non-minimal couplings we focus on the three which are power counting relevant or marginal, see (3.34). The equation for \( v_0 \) transcribes into a flow equation for the dimensionless Newton constant \( g_N = \mu^2 \kappa \) via \( v_0 = -1/(2g_N) \). For the other two we retain the general definitions. The resulting flow equations are
\[ \mu \frac{d}{d\mu} g_N = 2g_N + \frac{\hbar}{(4\pi)^2} \left\{ 4\ddot{q} g_N^2 + (0 v_2 - 2q_1) (1 v_2 - 1/6) g_N^2 \right\}, \]  
\[ \mu \frac{d}{d\mu} v_2 = \frac{\hbar}{(4\pi)^2} \left\{ (1 v_2 - 1/6) v_4 + 0 v_2 v_4 - 2q_1 v_4 \right\}, \]  
\[ \mu \frac{d}{d\mu} v_0 = \frac{\hbar}{(4\pi)^2} \left\{ \frac{1}{2} (1 v_2 - 1/6)^2 + 0 v_2 v_2 - 2q_1 v_2^2 \right\}. \]  
(3.38)

The \( g_N \) flow equation admits a nontrivial fixed point that is best interpreted as one for the \( 1/g_N \) flow,
\[ 1/g_N^* = -\frac{\hbar}{2(4\pi)^2} \left\{ 4\ddot{q} + (0 v_2^* - 2q_1) (1 v_2^* - 1/6) \right\}, \]  
(3.39)
where typically \( v_2^* = 0 \) by (3.37). We note that compared to the covariant formulation, the beta function (3.38) and the fixed point (3.39) feature an additional contribution proportional to \( \ddot{q} \) arising from the time dependence of the spatial regulator (3.2). Since \( \ddot{q} > 0 \) this contribution (as is typical of matter) tends to drive the Newton coupling to negative values. However, in a full quantum gravity plus matter computation one expects that the quantum gravity contribution will turn \( g_N^* \) positive again. Furthermore, it is noteworthy that the \( \ddot{q} \) term does not vanish for conformal coupling to matter.

This concludes our discussion of the one-loop renormalization flow. As noted in the introduction it will be used to set boundary conditions for the FL-sFRG.
4. Spatial cosmological EPA flow

Generally, the effective potential approximation (EPA) truncates the nonlocal effective action to a local functional of the same form as the basic action but with a scale dependent potential Ansatz. This concept carries over to the FL-sFRG (2.30), (2.29) straightforwardly. Since the (coefficient of the) kinetic term is taken to be \( k \)-independent the left hand side of (2.30) reduces to 

\[
\int dt d^d x \, N a^d \partial_t \mathcal{U}_k, \quad \text{where} \quad \mathcal{U}_k(\varphi, R) \quad \text{is the scale dependent effective potential to be determined.}
\]

As noted after (2.31) one has for the original action \( S[\varphi](t, t'; p) = -\mathcal{D}_p \delta(t, t') \). The \( k \)-dependent modification leads to 

\[
a^{-2d}(a^d N^{-1} \partial_t)^2 + a^{-2} p^2 + \mathcal{U}''_k \quad \text{as the differential operator entering the specialization of (2.29). Since we assume} \quad \varphi \quad \text{to be a function of} \quad t \quad \text{only the spatial volume in (2.30) is overcounted on both sides and can be dropped.}
\]

The temporal average can likewise be omitted. This is because the EPA flow equation is meant to be valid for arbitrary compact time intervals \( [t_i, t_f] \), which enforces pointwise equality of the (continuous) integrands. Moreover, one typically seeks to study the flow of the potential rather than its spatio-temporal average; see Section 3.2. The resulting spatial EPA (FL-sEPA) flow equation is

\[
\partial_t \mathcal{U}_k = -\frac{i \hbar}{2} \int \frac{d^d p}{(2\pi)^d} \, \partial_k R_k(t, p) G_k[\varphi](t, t; p),
\]

\[
\left\{ a^{-2d}(a^d N^{-1} \partial_t)^2 + a^{-2} p^2 + R_k(t, p) + \mathcal{U}''_k \right\} G_k[\varphi](t, t'; p) = \delta(t, t'). \quad (4.1)
\]

Here, \( \mathcal{U}_k \) depends on \( \varphi(t) \) and the right hand side also carries an explicit time dependence through the background, which will be further discussed in Section 4.1. As indicated, we continue to write \( G_k \) for the Green’s function entering, no confusion with the solution of the untruncated (2.29) should arise. Not incidentally (4.1) coincides with the one-loop equation in (2.31) upon substitution of the scale dependent potential on the right hand side \( G_k[\varphi](t, t'; p) = G_{k,0}[\varphi](t, t'; p)|_{\mathcal{U}'' \rightarrow \mathcal{U}'_k} \). This will become relevant later on when examining large \( k \) regime of the flow equation. In that case it is \( G_{k,0} \)’s large \( k \) behavior that needs to be known as a functional of \( \mathcal{U}'_k \), and the limiting case of the flow equation arises upon substitution of the unknown \( \mathcal{U}'_k \). A similar interplay holds in the dimensionless formulation introduced below for the small \( k \) regime.

Since only the diagonal of the Green’s function enters the flow equation it is advantageous to replace its defining relation by the associated Gelfand-Dickey equation for \( G_k = G_k[\varphi](t, t; p) \)

\[
2G_k(a^d N^{-1} \partial_t)^2 G_k - (a^d N^{-1} \partial_t G_k)^2 + 4a^{2d} \left[ k^2 \left( \frac{p^2}{a^2 k^2} + r \left( \frac{p^2}{a^2 k^2} \right) \right) + \mathcal{U}''_k \right] G_k^2 = -1. \quad (4.2)
\]

Again, the solution is related to that in (2.38) by \( i |T_k(t, p)|^2 |_{\mathcal{W} \rightarrow \mathcal{U}''_k} = G_k[\varphi](t, t; p) \). Of course \( G_k[\varphi] \) depends on \( \varphi \) only through \( \mathcal{U}'_k(\varphi) \) but the explicit \( \tilde{U}'_k(\varphi) \) dependence is difficult to extract. For generic scale factor \( a \) this will only be possible via series expansions, either for large or for small \( k \). Below we shall develop such series expansions, highlighting
the structural difference between the universal large $k$ expansion and the inevitably state dependent small $k$ expansion.

4.1 Explicit time-dependence and dimensionless formulation

A noteworthy feature of the flow equation (4.1) is its explicit time dependence not carried by $\varphi(t)$. In the ultraviolet the explicit time dependence of the Green's function $G_{k,0}[\varphi](t,t;p)$ will turn out to be solely carried by a local function of $R(t)$; in particular $U_k$ will for large $k$ be of the form

$$U_k(\varphi,R) = \sum_{n \geq 0} n U_k(\varphi) R(t)^n. \quad (4.3)$$

For small $k$, on the other hand, $U_k$ is expected to carry a nonlocal time dependence, so in general we write $U_k(t,\varphi)$ for the generalized effective potential sought. In a Euclidean setting a nonlocal flow equation would be expected from heat kernel resummations; see e.g. [37] and the references therein. In the present context, the infrared behavior is more drastically altered and lies outside the realm of any heat-kernel methodology.

To proceed, we shall expand the unknown time dependence in a suitable basis of orthonormal polynomials. In order to retain contact to (4.3) we use polynomials in $R(t)$ rather than polynomials in $t$. To this end, we assume the powers of $R(t)$ to be linearly independent. This is warranted, as for monotonically increasing $a(t)$ the Ricci scalar $R(t)$ given by (2.24) is typically strictly monotonically decreasing in $t$ on bounded intervals, like $[t_i,t_f]$ in the present context. Next, we choose a nonnegative weight function $w$ with mass dimension $+1$, smooth with compact support in $[t_i,t_f]$. In particular, this renders the measure $dtw(t)$ dimensionless. In terms of it we define

$$\sigma_i = \int dtw(t) R(t)^i, \quad i \geq 0, \quad D_l = \det(\sigma_{i+j})_{0 \leq i,j \leq l}, \quad l \geq 0,$$

and $p_0 = \sigma_0^{-1/2}$, while for $l \geq 1$,

$$p_l(t) = \frac{1}{\sqrt{D_l D_{l-1}}} \det \begin{bmatrix} \sigma_0 & \sigma_1 & \ldots & \sigma_{l-1} & 1 \\ \sigma_1 & \sigma_2 & \ldots & \sigma_l & R(t) \\ \vdots & \vdots & & \vdots & \vdots \\ \sigma_{l-1} & \sigma_l & \ldots & \sigma_{2l-2} & R(t)^{l-1} \\ \sigma_l & \sigma_{l+1} & \ldots & \sigma_{2l-1} & R(t)^l \end{bmatrix}. \quad (4.5)$$

Each $p_l(t)$ is a polynomial of degree $l$ in $R(t)$ with coefficients whose squares are rational functions of the averages of the Ricci powers. Hence $p_l$ is both a (nonlocal) functional of $R(\cdot)$ and a (local) function of $R(t)$. Whenever needed we indicate this by writing $p_l[R](t)$ and $\sigma_l[R]$. Collectively the $p_l, l \geq 0$, form a system of orthonormal polynomials

$$\int dtw(t) p_l(t)p_{l'}(t) = \delta_{l,l'}. \quad (4.6)$$
They are also invariant under constant rescalings of $R$
\[
p_l[\lambda R](t) = p_l[R](t), \quad \lambda > 0, \quad l \geq 0,
\]
as can be seen by noting that all diagonals in the determinants have constant degrees. The proof of (4.6) specializes Theorem 2.1 of [25] to powers of the Ricci scalar. Further, the set of these polynomials is closed in $[t_i, t_f]$ and a square integrable function $h \in L^2(wdt)$ can be expanded such that Parseval’s identity holds
\[
h(t) = \sum_{l \geq 0} h_l p_l(t), \quad h_l = \int dt w(t) p_l(t) h(t), \quad \sum_{l \geq 0} |h_l|^2 = \int dt w(t) |h(t)|^2.
\]
Convergence holds in the $L^2$ sense, see Theorem 3.1.5 of [25]. This implies almost everywhere pointwise convergence of a subsequence of the partial sums. Later on, the functions $h$ considered will be at least continuous, so that pointwise convergence everywhere is ensured. Finally, since the measure $wdt$ is dimensionless and so are by (4.7) the $p_l$, the coefficients $h_l$ will have the same dimension as $h$.

In the context of the FL-sFRG (4.1), we apply this expansion to parameterize the nonlocal time dependence of the generalized potential $U_k$ expected to arise in the infrared part of the flow. This yields an alternative set of potentials $\mathcal{U}_k(\varphi), \, l \geq 0$, which in turn are expanded in powers of the field,
\[
U_k(\varphi, t) = \sum_{l \geq 0} \mathcal{U}_k(\varphi) p_l(t), \quad \mathcal{U}_k(\varphi) = \sum_{j \geq 0} l u_{2j,k} \frac{\varphi^{2j}}{(2j)!}.
\]
In this parameterization $U_k$ depends on the Ricci scalar also nonlocally through the $p_l[R](t)$. As in (4.5) we do not display this dependence and just indicate the time dependence.

As seen in Section 3, in the ultraviolet the potentials $^nU(\varphi), \, n \geq 0$, entering (4.3) are preferred and for large $k$ both sets are related by
\[
\mathcal{U}_k(\varphi) = \sum_{n \geq l} ^n U_k(\varphi) \int dt w(t) p_l(R(t)) R(t)^n,
\]
and similarly for the couplings. Here we used that $p_l$ is orthogonal to any monomial of degree less than $l$. Generally, the $\mathcal{U}_k(\varphi)$ may be viewed as potentials whose coefficients are determined nonlocally in time.

We now first implement the transition to dimensionless variables for the ultraviolet potentials in (4.3). In order to render $R(t)$ dimensionless, while preserving the autonomous nature of the coupling flow in the ultraviolet, we introduce a mass scale $\kappa(k)$ that interpolates monotonically between $k$ in the ultraviolet and some inverse background length scale in the infrared:
\[
\kappa, \kappa': \mathbb{R}_+ \to \mathbb{R}_+, \quad \kappa(k) - k \in \mathcal{S}(\mathbb{R}_+), \quad \lim_{k \to 0} \kappa(k) < \infty,
\]

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with $S(\mathbb{R}_+)$ defined in the paragraph following (2.27). In particular, $\lim_{k \to 0} [k \partial_k \ln \kappa(k)] = 0$. The property $\kappa(k) \sim k$ for large $k$ will ensure in Section 4.2 the autonomy of the dimensionless coupling flow, mirroring (3.35) ff. In the infrared, this cannot expected to be true, and $\kappa(0)$ is needed to set an external, finite, infrared mass scale. As an external scale, it is natural to relate $\kappa(0)$ to a background scale, e.g. the inverse Hubble radius $1/d_H(t_*)$ at some reference time $t_*$. Using $k$ and $\kappa(k)$ as reference scales, dimensionless couplings and fields are introduced by

$$n v_{2j,k} := c_0^{-j} k(2n_k)^{(d-1)/2} (d+1) n u_{2j,k}, \quad n, j \geq 0; \quad \varphi_0 = c_0^{-1/2} k^{-d+1} \varphi. \quad (4.12)$$

where $c_0$ is a fudge factor inserted for later convenience. This entails

$$n U_k(\varphi) = c_0 k^{d+1} (\kappa(k) - 2n) n V_k(c_0^{-1/2} k^{-d+1} \varphi), \quad n V_k(\varphi_0) := \sum_{j \geq 0} n v_{2j,k} \frac{\varphi_0^{2j}}{(2j)!}, \quad (4.13)$$

Note that $\kappa(k)$ only enters the non-minimal sectors, $n \geq 1$. In terms of $r_k(t) := R(t)/\kappa(k)^2$ this gives

$$U_k(\varphi, R) = c_0 k^{d+1} V_k(\varphi_0, r_k), \quad \frac{\partial^2 U_k}{\partial^2 \varphi} = k^2 \frac{\partial^2 V_k}{\partial^2 \varphi}, \quad (4.14)$$

where $V_k(\varphi_0, r_k) = \sum_{n \geq 0} n V_k(\varphi_0) r_k^n$ and we also write $V_k'' = \partial^2 V_k/\partial \varphi_0^2$. Inevitably (4.12) imprints on $V_k$ an additional $k$ dependence beyond that of the couplings $n v_{2j,k}$, namely that carried by $r_k$. The property $\kappa(k) = k \in S(\mathbb{R}_+)$ ensures that this parameterization agrees with the standard one for large $k$ and then leads to autonomous flow equations for all $n v_{2j,k}$, $n, j \geq 0$.

In the infrared we aim at a dimensionless version of (4.9). As noted earlier, the orthogonal polynomial basis $p_l$ is dimensionless, so in $\sum_{l \geq 0} U_k(\varphi)p_l(t)$ the dimension enters only through the potentials $U_k(\varphi)$. Clearly, these have the same dimension as the generalized potential itself. One may thus transition to a dimensionless formulation by introducing

$$i v_{2j,k} = c_0^{-j} k(2n_k)^{(d-1)/2} (d+1) i u_{2j,k}, \quad n, j \geq 0; \quad \varphi_0 = c_0^{-1/2} k^{-d+1} \varphi,$$

$$i U_k(\varphi) = c_0 k^{d+1} i V_k(c_0^{-1/2} k^{-d+1} \varphi), \quad i V_k(\varphi_0) = \sum_{j \geq 0} i v_{2j,k} \frac{\varphi_0^{2j}}{(2j)!}. \quad (4.15)$$

This entails

$$U_k(\varphi, t) = c_0 k^{d+1} V_k(\varphi_0, t), \quad V_k(\varphi_0, t) = \sum_{l \geq 0} i V_k(\varphi_0)p_l(t), \quad \frac{\partial^2 U_k}{\partial^2 \varphi} = k^2 \frac{\partial^2 V_k}{\partial^2 \varphi_0}. \quad (4.16)$$

Since the parameterization (4.9), (4.16) is valid for all $k$, not just in the infrared, we discuss the connection between the $i V_k(\varphi_0)$ and $n V_k(\varphi_0)$ in a cross-over regime. The link follows
by expressing the polynomials $p_l$ as

$$p_l(R(t)) = \sum_{n=0}^{l} \Xi^n_l(\sigma_1[R], \ldots, \sigma_{2l}[R]) R(t)^n,$$  \hspace{1cm} (4.17)

where in the coefficients $\Xi^n_l$ we indicate the dependence of the $\sigma_i[R]$ of (4.4) on $R$. It follows from the scaling invariance (4.7) that

$$\Xi^n_l(\sigma_1[\lambda R], \ldots, \sigma_{2l}[\lambda R]) = \lambda^{-n} \Xi^n_l(\sigma_1[R], \ldots, \sigma_{2l}[R]).$$  \hspace{1cm} (4.18)

We may thus write

$$V_k(\varphi_0, t) = \sum_{l \geq 0} i \mathcal{V}_k(\varphi_0) p_l(t) = \sum_{n \geq 0} \left( \sum_{l \geq n} \Xi^n_l(\sigma_1[r_k], \ldots, \sigma_{2l}[r_k]) i \mathcal{V}_k(\varphi_0) \right) r_k(t)^n,$$  \hspace{1cm} (4.19)

where we used $R(t) = \kappa(k)^2 r_k(t)$ and (4.18) to absorb the resulting $\kappa(k)^{2n}$ factor into $\Xi^n_l$. Comparing to the parameterization $V_k = \sum_{n \geq 0}^{\sim} \mathcal{V}_k(\varphi_0) r_k(t)^n$ identifies

$$^{\sim} \mathcal{V}_k(\varphi_0) = \kappa(k)^{2n} \sum_{l \geq n} \Xi^n_l(\sigma_1[r_0], \ldots, \sigma_{2l}[r_0]) i \mathcal{V}_k(\varphi_0).$$  \hspace{1cm} (4.20)

This last relation highlights that the distinction between ‘explicit’ and ‘implicit’ $k$ dependence will become blurred in a cross-over regime. While the $^{\sim} \mathcal{V}_k(\varphi_0)$ are designed to obey a system of autonomous (not explicitly $k$-dependent) flow equations for large $k$ (see Section 4.2), this will in general not be the case towards the infrared. As illustrated in Appendix B, this feature is rooted in the temporal nonlocality argued after (4.3) to be a generic feature of the infrared dynamics. Hence, the autonomy of the small $k$ flow equation cannot be used as a criterion to adjust the transition formulas to a dimensionless formulation. Instead, the flow equation itself will govern the total $k$-dependence of the couplings $\nu_{2j,k}$ defined by (4.15). As a mnemonic we shall write $kd/dk$ for the derivative terms.

With the above preparations at hand, the original form (4.1) of the FL-sEPA flow equation can be converted into a dimensionless form useful to explore the infrared (IR) regime. The Green’s function relation in (4.1) is expressed in terms of a dimensionless momentum $\varphi = p/k$ and assumes the form

$$\{ a^{-2d}(a^d N^{-1} \partial_t)^2 + k^2[\varphi^2/a^2 + r(\varphi^2/a^2) + \mathcal{V}_k(\varphi)] \} G_k[\varphi_0](t, t'; \varphi) = \delta(t, t').$$  \hspace{1cm} (4.21)

We regard the solution of (4.21) as a functional of $\varphi_0$ entering via $\mathcal{V}_k(\varphi_0, t)$, expanded as in (4.16). After projection onto the $p_l(t)$ basis one has for $\mathcal{V}_k = \mathcal{V}_k(\varphi_0)$

$$k \frac{d}{dk} \mathcal{V}_k + (1+d) \mathcal{V}_k = \frac{d-1}{2} \frac{\partial}{\partial \varphi_0} \mathcal{V}_k = \frac{-i2\hbar}{c_0(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty d\varphi \varphi^{d-1} \times \int dt \, w(t) \mathcal{V}_k(t) \left[ r(\varphi^2/a(t)^2) - (\varphi^2/a(t)^2)r'(\varphi^2/a(t)^2) \right] k G_k[\varphi_0](t, t; \varphi).$$  \hspace{1cm} (4.22)
In Section 4.3 we shall use (4.22) to examine the IR fixed point regime, $k \to 0$, of the FL-sEPA flow. Since this is inevitably state dependent we will choose specifically a State of Low Energy from Section 2.4 as the underlying Hadamard state.

### 4.2 Universal large $k$ flow and UV fixed point

The large $k$ regime is best studied in the dimensionful formulation (4.1). As in Section 3 we now focus on $d = 3$ and adopt the conformal time gauge $N = a$, renaming $t$ into $\eta$. Following the same steps as in (3.4) – (3.5) one finds

$$\partial_k U_k = - \frac{i \hbar k q}{(4\pi)^2 a^2} \int_0^{\infty} d\phi \varphi^2 \left[ r(\varphi^2/a^2) - (\varphi^2/a^2)r'(\varphi^2/a^2) \right]G_{k,0}(\eta, \varphi) \big|_{W=\Phi k'}. \quad (4.23)$$

where $\varphi = p/k$ and $-iG_{k,0}(\eta, \varphi) = -iG_{k,0}[\varphi(\eta, \eta; p)]|_{p=k\varphi}$ is a solution of (2.43) with $N = a$. For large $k$ the right hand side of the flow equation (4.23) can be expanded using (3.6). This gives

$$\partial_k U_k = \frac{\hbar}{(4\pi)^2} \left\{ 4k^3 q_0 + 2k g_1(\eta) |_{k} + \frac{g_2(\eta)}{k} + \sum_{n \geq 3} \frac{2-n}{2} \frac{g_n(\eta)}{k^{2n-3}} \right\}. \quad (4.24)$$

Here $q_0, g_1(\eta), g_2(\eta)$ are as in (3.9) and we extend the definition to

$$g_n(\eta) := (-)^n \frac{2}{2-n} \int_0^{\infty} d\phi \varphi^2 \left[ r(\varphi^2/a^2) - (\varphi^2/a^2)r'(\varphi^2/a^2) \right] G_n(\eta, \varphi), \quad n \geq 3. \quad (4.25)$$

For all $n \geq 1$ we use $g_n(\eta)|_k := g_n(\eta)|_{W=\Phi k'}$ as a shorthand. As seen in Section 3.1 the $g_1, g_2$ evaluate up boundary terms to

$$g_1(\eta)|_k = \bar{q}R - q_1 (U_k'' - R/6)$$

$$= -q_1 \ 0^0 \bar{U}_k''(\varphi) + \left[ \bar{q} - q_1 \ 1^1 \bar{U}_k''(\varphi) \right] R - q_1 \sum_{n \geq 2} n \ 0^n \bar{U}_k''(\varphi) R^n,$$

$$g_2(\eta)|_k \simeq \frac{1}{2} (U_k'' - R/6)^2 = \frac{1}{2} \ 0^0 \bar{U}_k''(\varphi)^2 + \ 0^0 \bar{U}_k''(\varphi) \ 1^1 \bar{U}_k''(\varphi) R$$

$$+ \sum_{n \geq 2} \left[ \frac{1}{2} \sum_{n_1 + n_2 = n} n_1 \ 0^n \bar{U}_k''(\varphi) \ n_2 \ 0^n \bar{U}_k''(\varphi) \right] R^n. \quad (4.26)$$

Here the $n \ 0^n \bar{U}_k''(\varphi)$ are defined as in (3.25) but refer to $k$-dependent couplings $n \bar{a}_{2j,k}$, $n, j \geq 0$. Comparing powers of $R$ one finds

$$\partial_k \ 0^0 \bar{U}_k = \frac{\hbar}{(4\pi)^2} \left\{ 4k^3 q_0 - 2kq_1 \ 0^0 \bar{U}_k'' + \frac{1}{2k} (0^0 \bar{U}_k'')^2 + O(1/k^3) \right\},$$

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\begin{align}
\partial_k U_k &= \frac{\hbar}{(4\pi)^2} \left\{ 2k (\tilde{q} - q_k) U_k^n + \frac{1}{k} U_k^n U_k^n + O(1/k^3) \right\} \quad \text{(4.27)} \\
\partial_k U_k &= \frac{\hbar}{(4\pi)^2} \left\{ -2k q_k U_k^n + \frac{1}{2k} \sum_{n_1,n_2} n_1 U_k^n(\phi) n_2 U_k^n(\phi) + O(1/k^3) \right\}, \quad n \geq 2.
\end{align}

One may check that the transition formulas (4.14) lead to an autonomous system of flow equations for the dimensionless potentials \( n V_k(\varphi_0), \; n \geq 0 \). The subleading terms are determined by \( G_n, n \geq 3 \), in the recursion (3.7). We omit their analysis for now and relate the displayed terms in (4.27) to the perturbative analysis of Section 3. Comparison with (3.33) exhibits an exact match to the one-loop flows of all potentials. Hence, the coupling flow determined from the EPA (4.1), (4.23) coincides – for large enough \( k \) – with the one-loop flow of the renormalized couplings after strict removal of the UV cutoff:

\[ n u_{2,j,k} = n u_{2,j}(\mu), \quad n, j \geq 0, \quad \text{for} \quad \mu = k \text{ large} \quad (4.28) \]

Underlying this seemingly trivial match are some structural elements [12] that are easily overlooked:

- The use of a non-minimal subtraction scheme, see (3.21), which allows one to treat \( \mu \) and \( \Lambda \) as instances of \( k \) and the \( k = \mu \) quantities as the renormalized ones.

- The inclusion of all interaction monomials, here (2.2), needed for strict removal of the UV cutoff. Only then can one make \( k < \Lambda \) arbitrarily large and identify the unstable manifold so as to justify the interpretation of the couplings as the renormalized ones (as opposed to merely running Wilsonian parameters).

- Use of the same regulator to set up perturbation theory and the EPA.

The previous comments are not specific to the FL-sFRG. With Euclidean signature the reintegration of the covariant \( O(h) \) FRG, \( 2\partial_k \Gamma_{k,1} = \text{Tr} [\partial_k R_k (S^{(2)}(\varphi) + R_k)^{-1}] \), is often used in combination with the heat kernel expansion to obtain the full renormalized one-loop effective action at \( k = 0 \). Since the heat kernel expansion (even in variants where nonlocal terms are included) is an asymptotic expansion tailored to the ultraviolet, the integration down to \( k = 0 \) is conceptually unwarranted. Replacing it with an integration from \( \mu \) to \( \Lambda \) for some Wilsonian scale \( \mu \), it can be seen to produce termwise (precisely) the same results for the divergent parts as the direct evaluation of \( L_\Lambda + L_1^{\text{div}} \), provided a regularization is used that facilitates such a comparison. The next step, of attempting to strictly remove the UV cut-off \( \Lambda \), is often omitted. Doing so somewhat trivializes the notion of one-loop renormalizability, fails to identify the unstable manifold, and may lead to inconsistent truncations.

In Section 3 of [37] the removal of the UV cutoff is discussed for a scalar field theory in flat space. Unfortunately, in their Eq. (49) several typos crept in which obscure the fact that the UV beta functions (\( \Lambda \partial / \partial \Lambda \) response of the bare couplings) must in this framework...
coincide with the IR beta functions ($k\partial/\partial k$ response of the parameters in the EPA ansatz).

In Section 7 of [37] gravity coupled to a scalar is studied. In their Eq. (180) it is noted that several terms are generated not present in the original action, but no attempt is made to absorb them through coupling or field redefinitions.

The EPA flow for scalar QFT on a generic Riemannian manifold has also been considered in [31, 32, 39, 40], primarily using the heat kernel methodology to compute the right hand side of the flow equation. In [40] a potential for the nonminimal coupling proportional to $R$ was included (our $U(\varphi)R$ term). Taking into account $n\dot{U}_k^n = nU_k^n - (1/6)\delta_{n,1}$, the structure of the $n = 0, 1$ flow equations (4.27) parallels those in Eqs (26),(27) of [40] and the modulator independent coefficients agree exactly. The restriction to linear order in $R$ is seen as truncation, so neither does the criterion (3.1) enter nor the ensued Ricci tower.

4.3 Infrared regime for SLE based EPA flow

We return to the dimensionless form (4.22) of the EPA flow and seek to analyze its structure for small $k$. On the right-hand side the dependence of the Green’s function on $V_k''$ needs to be known. In the case of a de Sitter background [5, 6, 7], as mentioned after (2.30), the issue of state dependence of the flow equation does not arise due to the maximal symmetry of the spacetime. The right-hand side of (4.1) can be then be determined in closed form for the ‘hockey stick’ FRG modulator, and (assuming that $V_k''$ remains bounded as $k \to 0$) the well known infrared enhancement of the Bunch-Davies Green’s function leads to a “dimensional reduction” in the small $k$ form of the dimensionless EPA flow equation [7]. For a generic Friedmann-Lemaître background, however, the small $k$ behavior of the FL-sEPA flow will normally be very different.

In analyzing the small $k$ form of the EPA flow (4.1), we seek to keep the scale factor $a(t)$ and the modulator function (2.27) generic, and may allow for a time dependent $\varphi_0(t)$. Then even the modulator independent basic wave equation can not be solved in terms of known special functions. As argued before, the inevitable state dependence of the small $k$ flow is a major difference to the large $k$ regime. Rather than restricting attention to some drastic specializations and truncations we aim at developing an analytically controllable series expansion of the Green’s function for small $k$. Inevitably, this requires a choice of underlying Hadamard state, and we shall focus on the States of Low Energy (SLE) described in Section 2.4. With the assumption that $V_k''$ remains bounded as $k \to 0$, an analytically controllable small $k$ expansion turns out to be feasible.

The role of $p$ as an expansion parameter in (2.57) is played by $k$ after transition to dimensionless variables. Using (4.16) the dimensionless form of the basic wave equation reads

$$\{(a^dN^{-1}\partial_t)^2 + k^2a^{2d}[\varphi^2/a^2 + r(\varphi^2/a^2) + V_k'']\}S_k(t, \varphi) = 0.$$  \hspace{1cm} (4.29)

An arbitrary Wronskian normalized solution of (4.29) is used as input for the construction of the SLE solution $T_{k}^{\text{SLE}}[S_k](t, k\varphi)$, see (2.49). Comparing with (2.48) and the wave
equation based on it, one is lead to the identifications $\omega_0(t) \to 0$ and

$$ p \to k, \quad w_{2,p}(t) \to a(t)^{2d} \left[ V''_k + \varphi^2/a^2 + r(\varphi^2/a^2) \right] =: w_2(V'', \varphi), \tag{4.30} $$

We relegate the justification of this substitution rule to Appendix C and note here its important consequence: the coincidence limit $kG_k[\varphi_0](t; \varphi) = ik|T^\text{SLE}_k(t, k\varphi)|^2$ of the intractable (dimensionless) Green’s function admits a convergent series expansion in powers of $k^2$. The coefficients can be computed analytically from (2.55) – (2.60) in combination with (4.30). We thus define

$$ \varepsilon_{n,\varphi}[V''_k] := \varepsilon_n|_{\omega_0^2 = w_2(V''_k, \varphi)} \quad J_{n,\varphi}[V''_k](t) := J_n(t)|_{\omega_0^2 = w_2(V''_k, \varphi)}, \quad n \geq 0, $$

$$ \bar{a}_{\varphi}[V''_k] := J_{0,\varphi} \varepsilon_{1,\varphi} = \left( \frac{\int dt n(t) a(t)^{2d} f(t)}{\int dt n(t) a(t)^{2d} f(t)} \right)^{1/2}. \tag{4.31} $$

The trivial $J_0 = J_{0,\varphi} = \frac{1}{2} \int dt n(t) f(t)$ could be normalized to some value, but is kept for dimensionality reasons. Then, $\bar{a}_{\varphi}$ is dimensionless while $J_{n,\varphi}$ and $\varepsilon_{n,\varphi}$ have mass dimension $-(2n+1)$ and $-n$, respectively. In this notation

$$ |T^\text{SLE}_k(t, k\varphi)|^2 = \frac{\bar{a}_{\varphi}[V''_k]}{2k} \left\{ 1 + \left( \frac{J_{1,\varphi}[V''_k](t)}{J_0} - \frac{\bar{a}_{\varphi}[V''_k] \varepsilon_{2,\varphi}[V''_k]}{2J_0^2} \right) k^2 + O(k^4) \right\}. \tag{4.32} $$

This has to be inserted into the projected FL-sEPA equation (4.22). From now on we choose the weight in the orthonormal basis proportional to the window function of the SLE, specifically

$$ w(t) = \kappa(0) n(t) a(t)^{2d} f(t), \tag{4.33} $$

where $n(t) = N(t) a(t)^{-d}$ and the mass scale $\kappa(0)$ simply accounts for dimensions, see the paragraph preceding (4.4). Further, we take $\varphi$ to be time independent. Due to the orthonormality (4.6), in the denominator of $\bar{a}_{\varphi}[V''_k]$ then only the $l = 0$ term in the decomposition $V''_k(\varphi_0, t) = \sum_{l \geq 0} V''_k(\varphi_0) p_l(t)$ contributes:

$$ \bar{a}_{\varphi}[V''_k] = \left( \frac{2J_0(0) p_0}{\int \varphi^2_0 + \omega \rho_0(\varphi)} \right)^{1/2}, $$

$$ \omega \rho(\varphi) := \int dt w(t) p_0(t) \left[ \varphi^2/a(t)^2 + r(\varphi^2/a(t)^2) \right]. \tag{4.34} $$

In turn, this suggests to define for a function $h(t, \varphi)$

$$ lQ_m(h|V) := \frac{h}{c_0(4\pi)^{d/2} \Gamma(d/2)} \int_0^{\infty} d\varphi \varphi^{d-1} \left( \frac{2J_0(0)p_0}{V + \omega \rho(\varphi)} \right)^{1/2+m}. \tag{4.35} $$
The flow equation (4.36) is non-autonomous, i.e.

\[ k \frac{d}{dk} \mathcal{V}_k(\varphi_0) + (d+1) \mathcal{V}_k(\varphi_0) - \frac{d-1}{2} \varphi_0 \frac{\partial}{\partial \varphi_0} \mathcal{V}_k(\varphi_0) \]

\[ = iQ_0(1|0) \mathcal{V}_k'' - \frac{k^2}{2J_0} iQ_1(1|0) \mathcal{V}_k'' \sum_{l_1,l_2 \geq 0} i_l \mathcal{V}_k''(\varphi_0) i_{l_2} \mathcal{V}_k''(\varphi_0) E_{l_1,l_2} \]  \hspace{1cm} (4.36)

\[ + \frac{k^2}{J_0} \sum_{l \geq 0} i_l \mathcal{V}_k''(\varphi_0) \left( iQ_0(j|0) \mathcal{V}_k'' - \frac{1}{2J_0} iQ_1(E|0) \mathcal{V}_k'' \right) \]

\[ + \frac{k^2}{J_0} \left( iQ_0(j|0) \mathcal{V}_k'' - \frac{1}{2J_0} iQ_1(E|0) \mathcal{V}_k'' \right) + O(k^4). \]

All terms on the right hand side of (4.22) can be expressed in terms of these averages. We arrive at the SLE based FL-sEPA flow equation:

The functions \( E_{l,p}, E_{l}(\varphi), E(\varphi) \) and \( j_l(t), j(t, \varphi) \) are collected in Appendix C. Equation (4.36) is the main result of this section.

Remarks:

(i) Both sides of (4.36) depend on the choice of window function \( f \). The support of \( f \) will be chosen in accordance with the cosmological period where one seeks to inject information about the primordial vacuum state. Since general relativity demands a pre-inflationary period, locating \( f \) immediately after a quantum gravity epoch is a natural choice.

(ii) The scale factor \( a \) can be chosen to model a realistic expansion history. For example, one can have a pre-inflationary period of non-accelerated expansion followed by a de Sitter like acceleration, etc. This avoids the technically clumsy matching of piecewise powerlike \( a \)'s.

(iii) The flow equation (4.36) is non-autonomous, i.e. \( k \) occurs explicitly in the dimensionless formulation. The \( O(k^4) \) and higher order corrections refer to (2.55), (2.56) and can be computed systematically. This feature differs from the UV flow (4.27) where the same substitution (4.12), (4.13) leads to an autonomous system of flow equations for the dimensionless potentials \( ^nV_k(\varphi_0), n \geq 0 \). In Appendix B we analyze the instantaneous limit of (4.36) where again an autonomous system of flow equations for the \( ^nV_k(\varphi_0) \) arises. The non-autonomous character of the (4.36) therefore originates in the temporal averaging (and the ensued temporal nonlocality) needed to render the underlying state (of low energy) mathematically and physically well-defined [28, 17].

(iv) The small \( k \) expansion (4.32) entails that the flow equation specializes correctly to the spatial EPA flow in Minkowski space. Since \( n \equiv 1 = a \) one has \( \Delta_p(t, t') = p^{-1} \sin p(t - t') \). Then (2.54) reduces to \( J_p(t) = J_0, \mathcal{E}_p^{\text{SLE}} = p \varepsilon_1 \) and only the leading term on the right hand side of (4.32) remains. For a constant \( \varphi_0 \) the window function drops out in \( a_p[V_k''] \) and the corresponding flow equation reduces to (B.5), as required.
Importantly, (4.36) has a well-defined IR fixed point equation

\[(d+1)\mathcal{V}_* - \frac{d-1}{2}\frac{\partial}{\partial \varphi_0} \mathcal{V}_* = iQ_0(1|0\mathcal{V}''_*) , \tag{4.37}\]

which resembles that in Minkowski space but refers to a curvature dependent generalized potential, \(\mathcal{V}_* = \mathcal{V}_*(\varphi_0, t) = \sum_{l \geq 0} i\mathcal{V}_*(\varphi_0)p_l(t)\). To proceed, we insert power series expansions

\[i\mathcal{V}_*(\varphi_0) = \sum_{j \geq 0} i^{v_{,2j}}(\varphi_0) = \sum_{j \geq 1} i^{v_{,2j+2}}(\varphi_0) , \tag{4.38}\]

and expand \(iQ_0(1|0\mathcal{V}''_*)\) in powers of the field. This gives

\[(d+1)i^{v_{,0}} = iQ_0(1|0v_{,2}) , \tag{4.39}\]

\[|d+1 - j(d-1)|i^{v_{,2j}} = \sum_{m=0}^{j} \left( -\frac{1}{2} \right) \frac{i^{Q_m}(1|0v_{,2})}{(2J_0p_0)^m} \times \sum_{j_1 + \ldots + j_m = j, j_1 \ldots j_m \geq 1} \left( 2j_1, \ldots, 2j_m \right) 0^{v_{,2j_1+2}} \ldots 0^{v_{,2j_1+2}} .\]

The \(l = 0\) instance of the second equation is a coupled system for the \(0v_{,2j}, j \geq 0\). Assuming that a solution has been found all other \(i^{v_{,2j}}, l \geq 1, j \neq 1\) are determined by them. The same holds for \(d > 3\) for \(j = 1\), while for \(d = 3\) and \(j = 1\) the coefficient of \(i^{v_{,4}}\) vanishes and the \(0v_{,2j}\) are further constrained. With the possible exception of \(i^{v_{,4}}\) the couplings associated with the temporally averaged potential \(0\mathcal{V}_*(\varphi_0)\) therefore determine all other \(i\mathcal{V}_*(\varphi_0), l \geq 0\). Although time dependent, the fixed point solution \(\mathcal{V}_*(\varphi, t)\) is therefore essentially determined by its temporal average.

The derivation of the \(O(k^2)\) correction terms in (4.36) is relegated to Appendix C. The further analysis of (4.36) will begin with a linearization around the fixed point solutions(s) and then use a shooting technique to numerically relate the infrared couplings \(i^{v_{,2j}}, l \geq 0\), to the ultraviolet ones \(n^{v_{,2j}}, n \geq 0\), using the respective flow equations. For length reasons a detailed investigation will have to be relegated to a separate paper. To the best of our knowledge the flow (4.36) is the first result that allows one to explore the consequences of the Hadamard property for the non-perturbative infrared dynamics of quantum fields on a Friedmann-Lemaître background with a realistic expansion history.
5. Conclusions

The main take away from the results obtained here is the inevitable state dependence of a Lorentzian signature FRG on a generic foliated background. At present, no selection criteria for physically viable vacuum like states are known beyond perturbation theory. The advocated correspondence between vacuum-like states and a spatial FRG may in fact help to develop such criteria. The resulting interplay between state selection, Wilsonian perturbation theory, and a non-perturbative spatial FRG flow has been summarized in Figure 1. Details were elaborated for Friedmann-Lemaître backgrounds with States of Low Energy as the Hadamard states of choice. In particular, systematic expansions for the infrared and ultraviolet regime of the associated effective potential flow equations have been presented. For length reasons, a detailed quantitative investigation of the ensued flows has to be relegated to a separate study.

There are many threads of research to be taken up from here. Most immediate is the non-perturbative flow of the scalar power spectrum. A case study in [19] suggests that the States of Low Energy are viable candidates for pre-inflationary vacua. In particular, the low multipole moments in the CMB spectrum are suppressed in a natural way. The approximately scale invariant power spectrum arises in a cross over region, which in a spatial FRG setting has to be explored numerically. On a de Sitter background Stochastic Inflation provides an alternative framework to capture the non-perturbative inflaton dynamics and has some interface with the FRG flow [5, 6, 7]. Since general relativity demands a pre-inflationary (kinetic energy dominated) period, the generalization of this interplay to Friedmann-Lemaître backgrounds other than de Sitter is an important desideratum. A third thread is the extension to other types of cosmological backgrounds. Since the framework of Figure 1 requires some degree of computational control over the Hadamard states in question, the choices are presently limited. The States of Low Energy can however be extended to spatially inhomogeneous generalized FRW geometries [20]. Another option are temporally averaged Sorkin-Johnston states [21, 22]. Eventually, one would like to go beyond the Effective Potential Approximation, for which the spatial hopping expansion [10, 11] provides a starting point.

A spatial FRG should also be the appropriate tool to investigate Quantum Gravity at strong coupling. This takes foliated geometries as basic as in [38], though with a different dynamics that underlies an associated Anti-Newtonian expansion [11], antipodal to perturbation theory.

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A: Regulator dependence of $\Gamma_{k,1}^{\text{div}}$ coefficients

In this appendix we present a proof of the Lemma entering (3.12), (3.13) together with explicit formulas for the regulator dependent coefficients $\hat{q}, q_1, b_1, b_2, b_3$. We begin by formulating

**Lemma A:** Let $r(x)$ be a generic FRG regulator function as defined in (2.27).

(a) The regulator dependent coefficients $B_1, B_2, B_3$ always satisfy the ratio

$$B_1 : B_2 : B_3 = -1 : 4 : 3,$$

and hence can be expressed in terms of a single regulator dependent constant $b_1$,

$$B_1 = -6b_1, \quad B_2 = 24b_1, \quad B_3 = 18b_1.$$

(b) $B_4 + 36b_1 + B_5 = 0$.

(c) $B_6 = B_7 = B_8$.

Before proceeding to the proof, we note the explicit expressions for $B_1, \ldots, B_8$ (obtained with Mathematica):

$$B_1 = -\int_0^\infty d\rho \rho^2 \left[ r(\rho^2) - \rho^2 r^{(1)}(\rho^2) \right]^2 \frac{4}{\left[ \rho^2 + r(\rho^2) \right]^{7/2}},$$

$$B_2 = -\int_0^\infty d\rho \rho^2 \left[ r(\rho^2) - \rho^2 r^{(1)}(\rho^2) \right]^2 \left\{ \rho^4 r^{(2)}(\rho^2) (\rho^2 + r(\rho^2)) \right\} - 4 \left( r(\rho^2) - \rho^2 r^{(1)}(\rho^2) \right) \left( 9r(\rho^2) - \rho^2 (2 + 7r^{(1)}(\rho^2)) \right),$$

$$4B_3 = 3B_2,$$

$$B_4 = \frac{3}{8} \int_0^\infty d\rho \rho^2 \left[ r(\rho^2) - \rho^2 r^{(1)}(\rho^2) \right]^2 \left\{ \rho^2 (\rho^2 + r(\rho^2)) \right\} - 35r(\rho^2)^3 + \rho^4 r(\rho^2)^2 \left[ 42 + 147r^{(1)}(\rho^2) + 108\rho^2 r^{(2)}(\rho^2) + 16\rho^4 r^{(3)}(\rho^2) \right]
+ \rho^6 \left[ 42r^{(1)}(\rho^2)^2 + 77r^{(1)}(\rho^2)^3 + 24\rho^2 r^{(2)}(\rho^2)^2 - 84\rho^2 r^{(1)}(\rho^2)^2 r^{(2)}(\rho^2) + 16\rho^4 r^{(3)}(\rho^2) \right]
+ \rho^4 r(\rho^2)^2 \left[ 132\rho^2 r^{(2)}(\rho^2)^2 - 21r^{(1)}(\rho^2)^2 (4 + 9r^{(1)}(\rho^2)^2) + 4\rho^2 r^{(2)}(\rho^2)^2 + 32\rho^2 r^{(3)}(\rho^2)^2 \right],$$

$$B_5 = -\int_0^\infty d\rho \rho^2 \left[ r(\rho^2) - \rho^2 r^{(1)}(\rho^2) \right] \left\{ \rho^2 (\rho^2 + r(\rho^2)) \right\},$$

$$B_6 = B_7 = B_8.$$
Proof of Lemma A.

(b) The ratio

\[-24r^{(1)}(\theta^2)^3 - 1155r^{(1)}(\theta^2)^4 + 240r^{(2)}(\theta^2) + 84r^{(1)}(\theta^2)^2(22r^{(2)}(\theta^2) - 1) - 112r^{(1)}(\theta^2)(3r^{(2)}(\theta^2) + 4r^{(3)}(\theta^2))
\]

\[+ 16\theta^4(4\theta^2 + 20r^{(3)}(\theta^2) - 21r^{(2)}(\theta^2)^2)\]

\[+ 4\theta^6r^{(2)}(\theta^2)^3 + 264\theta^2r^{(2)}(\theta^2) + 21r^{(1)}(\theta^2)^2(31 + 22\theta^2r^{(2)}(\theta^2))
\]

\[-14r^{(1)}(\theta^2)(16\theta^4r^{(3)}(\theta^2) + 78\theta^2r^{(2)}(\theta^2) - 3)
\]

\[+ 8\theta^2(6\theta^4(\theta^2) + 44r^{(3)}(\theta^2) - 21r^{(2)}(\theta^2))\]

\[+ 2\theta^2r^{(2)}(\theta^2)^2\]

Indeed, it follows by direct computation that

\[\rho^{m(1)}(\theta^2) + \rho^{m(2)}(\theta^2) + \rho^{m(3)}(\theta^2) + \rho^{m(4)}(\theta^2)\]

\[+ 4\rho^2r^{(2)}(\theta^2)^3\]

\[+ 4\rho^6r^{(2)}(\theta^2)^3 + 264\rho^2r^{(2)}(\theta^2) + 21r^{(1)}(\theta^2)^2(31 + 22\rho^2r^{(2)}(\theta^2))
\]

\[-14r^{(1)}(\theta^2)(16\rho^4r^{(3)}(\theta^2) + 78\rho^2r^{(2)}(\theta^2) - 3)
\]

\[+ 8\rho^2(6\rho^4(\theta^2) + 44r^{(3)}(\theta^2) - 21r^{(2)}(\theta^2))\]

\[+ 2\rho^2r^{(2)}(\theta^2)^2\]

\[-2121r^{(1)}(\theta^2)^2 + 6(270\rho^2r^{(2)}(\theta^2) - 7)
\]

\[-14r^{(1)}(\theta^2)(16\rho^4r^{(3)}(\theta^2) + 144\rho^2r^{(2)}(\theta^2) + 87)
\]

\[+ 8\rho^2(12\rho^2r^{(4)}(\theta^2) + 116r^{(3)}(\theta^2) - 21r^{(2)}(\theta^2)^2)\}

\[B_6 = \int_0^\infty d\theta \rho^2 r^{(1)}(\theta^2) + \rho^2 r^{(2)}(\theta^2) + \rho^2 r^{(3)}(\theta^2) + \rho^2 r^{(4)}(\theta^2)
\]

\[B_7 = \int_0^\infty d\theta \rho^2 r^{(1)}(\theta^2) + \rho^2 r^{(2)}(\theta^2) + \rho^2 r^{(3)}(\theta^2) + \rho^2 r^{(4)}(\theta^2)
\]

\[B_8 = B_6.
\]

Note that we use the notation \(r^{(n)}(x) := \frac{d^n}{dx^n} r(x), n \geq 0, \) throughout.

**Proof of Lemma A.**

(a) The ratio \(B_2 : B_3 = 4 : 3\) is clear from (A.3), so it remains to show

\[4B_1 + B_2 = 0.
\]  

(A.4)

Indeed, it follows by direct computation that

\[4B_1 + B_2 = \frac{1}{2} \int_0^\infty d\theta \frac{\partial}{\partial \theta} \left\{ \frac{\rho^2 r^{(1)}(\theta^2)^2 - 2\rho^5r^{(1)}(\theta^2)^2 + \rho^7 r^{(1)}(\theta^2)^2}{[\theta^2 + r(\theta^2)]^{7/2}} \right\}
\]

\[= \lim_{\theta \to \infty} \frac{\rho^2 r^{(1)}(\theta^2)^2 - 2\rho^5r^{(1)}(\theta^2)^2 + \rho^7 r^{(1)}(\theta^2)^2}{2[\theta^2 + r(\theta^2)]^{7/2}}.
\]  

(A.5)

By (2.27) \(r(\theta^2)\) and its derivatives approach zero as \(\theta \to \infty\) faster than any polynomial in \(\theta\). Thus (A.5) vanishes, i.e. \(4B_1 + B_2 = 0\), establishing Lemma A(a).
We end this appendix with a summary of the expressions for the regulator dependent coefficients in (3.12), (3.13).

\[ \bar{q} = \frac{1}{12} \int_0^\infty d\varrho \varrho^2 \left[ \frac{r(\varrho^2) - \varrho^2 r^{(1)}(\varrho^2)}{[\varrho^2 + r(\varrho^2)]^{5/2}} \right], \quad q_1 = \int_0^\infty d\varrho \varrho^2 \frac{r(\varrho^2) - \varrho^2 r^{(1)}(\varrho^2)}{[\varrho^2 + r(\varrho^2)]^{3/2}}, \]

\[ b_1 = \frac{3}{2} \int_0^\infty d\varrho \varrho^2 \left[ \frac{r(\varrho^2) - \varrho^2 r^{(1)}(\varrho^2)}{[\varrho^2 + r(\varrho^2)]^{7/2}} \right], \quad b_2 = -\frac{B_5}{12}, \]

\[ b_3 = \frac{1}{2} \int_0^\infty d\varrho \varrho^2 \frac{r(\varrho^2) - \varrho^2 r^{(1)}(\varrho^2)}{[\varrho^2 + r(\varrho^2)]^{7/2}} \left\{ -3r(\varrho^2) + \varrho^2 (2 + 5r^{(1)}(\varrho^2)) \right\}. \]
B: Spatial EPA in Minkowski space and SLE instantaneous limit

The analysis of the spatial EPA (4.1) is normally hampered by the difficulty of gaining analytical control over the Green’s function $G_k[\varphi](t, t'; p)$. There are two instructive exceptions which we present in this appendix: Minkowski space and the “instantaneous limit” of the version of (4.1) based on a State of Low Energy (SLE), see Section 2.4.

In Minkowski space $a = N = 1$, the generalized potential reduces to the standard one $U_k(\varphi, R) = 0 U_k(\varphi) =: U_k(\varphi)$, and we may take the field $\varphi$ to be constant. In line with the general discussion we choose the Feynman type solution of the Green’s function relation in (4.1), viz.

$$G_k[\varphi](t, t'; p) = i e^{-i|t-t'| \sqrt{p^2 + R_k(p) + U_k''(\varphi)}}.$$  \hfill (B.1)

This corresponds to (2.34) with a Wronskian normalized positive frequency solution of the homogeneous wave equation. The spatial EPA flow equation then reads

$$\partial_k U_k = \frac{\hbar}{4} \int d^d p \frac{\partial_k R_k(p)}{(2\pi)^d} \sqrt{p^2 + R_k(p) + U_k''}.$$  \hfill (B.2)

We note that no contour deformation or other indirect reference to a Euclidean time or momentum regime enters in arriving at (B.1), (B.2). Other Minkowski space EPA flow equations can be found in [8, 9]. After transition to radial momenta $|p|$ and $\varphi = |p|/k$ one has

$$\partial_k U_k = \hbar \frac{k^d}{(4\pi)^{d/2} \Gamma(d/2)} \int_0^\infty d\varphi \varphi^{d-1} \frac{r(\varphi^2) - \varphi^2 r'(\varphi^2)}{\sqrt{\varphi^2 + r(\varphi^2) + U_k''/k^2}}.$$  \hfill (B.3)

using $\text{vol}(S^{d-1}) = 2\pi^{d/2} / \Gamma(d/2)$ for the volume of the $S^{d-1}$ sphere. The transition to a dimensionless formulation is effected by

$$U_k(\varphi) = c_0 k^{1+d} V_k(\varphi_0) \bigg|_{\varphi_0 = c_0^{-1/2} \varphi_0^1 \varphi^1},$$  \hfill (B.4)

where $c_0 > 0$ a fudge factor to be adjusted later on. In terms of $V_k(\varphi_0)$ the spatial EPA flow equation reads

$$k \partial_k V_k(\varphi_0) + (1+d) V_k(\varphi_0) - \frac{d-1}{2} \varphi_0 V_k'(\varphi_0) = \hbar q_0(V_k''(\varphi_0)).$$  \hfill (B.5)

For re-use later on we defined here

$$q_\nu(v) := \frac{1}{c_0 (4\pi)^{d/2} \Gamma(d/2)} \int_0^\infty d\varphi \varphi^{d-1} \frac{r(\varphi^2) - \varphi^2 r'(\varphi^2)}{(\varphi^2 + r(\varphi^2) + v)^{1/2+\nu} },$$  \hfill (B.6)
with $\nu \in \mathbb{R}$, $v \geq 0$. Following the general discussion in Section 4 we use (B.3) for the large $k$ expansion and (B.5) to study the fixed point regime.

For the large $k$ expansion the dimensionful potential should be treated as $k$ independent, with $U'' \rightarrow U''_k$ re-substituted after expansion. This gives

$$\partial_k U_k = \hbar c_0 \sum_{n \geq 0} \binom{-1/2}{n} q_n(0) \left(\frac{U''_k}{k^2}\right)^n.$$  \hfill (B.7)

The coefficients are just the binomial ones for which we note $\binom{-1/2}{n} = (-)^n \frac{(2n-1)!!}{2^n n!}$. Comparing with the general large $k$ expansion (4.24), (4.25) one can read off

$$G_n = \frac{(2n-1)!!}{2^n(\varphi^2 + r(\varphi^2))^n} \frac{(U'')^n}{n!}, \quad n \geq 0.$$  \hfill (B.8)

For constant $\varphi$ the $(U'')^n/n!$ are in fact the conventionally normalized heat kernel coefficients of the Schrödinger operator $\partial_t^2 + U''(\varphi)$. For $d=3$ the $n = 0, 1, 2$ terms match those for $U_k$ in (4.27), using $4\pi^2 c_0 q_0(0) = q_0$, $4\pi^2 c_0 q_1(0) = q_1$, $4\pi^2 c_0 q_2(0) = 1/3$.

The fixed point regime of the flow is explored via the dimensionless flow equation (B.5). Its right hand side can be expressed as $q_0(V'')$. Typically, $V''$ has a field independent part $v_2$ which one will split off when comparing powers of $\varphi^2_0$ later on. The relevant expansion then is

$$q_0(v_2 + \Delta V'') = \sum_{n \geq 0} \binom{-1/2}{n} q_n(v_2)(\Delta V'')^n,$$  \hfill (B.9)

where $\Delta V'' = \sum_{j \geq 1} (\varphi^2_0/(2j)!!) v_{2j+2}$. Expanding around a non-zero $v_2$ gives a better approximation to the exact $q_0(V'')$ than the $q_n(0)$’s entering (B.7). Note that neither (B.7) nor (B.9) explore the large field regime of the nonlinear flow (B.3) or (B.5), which would amount to an expansion in inverse powers of $U''_k$ or $V'$. This reflects the choice of a potential Ansatz in positive (even) powers of the fields.

Comparing in (B.5) powers of $\varphi^2_0$ results in a ‘upward’ coupled system of flow equations for the dimensionless couplings. Setting $\hbar = 1$ the first few read

$$k \partial_k v_{k,0} = -(d + 1)v_{k,0} + q_0(v_{k,2}),$$

$$k \partial_k v_{k,2} = -2v_{k,2} - \frac{1}{2} q_1(v_{k,2})v_{k,4},$$

$$k \partial_k v_{k,4} = (d - 3)v_{k,4} - \frac{1}{2} q_1(v_{k,2})v_{k,6} + \frac{9}{4} q_2(v_{k,2})v_{k,4}^4.$$  \hfill (B.10)

The further analysis now proceeds along the familiar lines: Truncating the system (B.1) at some order $N$ (i.e. $v_{k,2j} \equiv 0$, $j \geq N$) results in a closed coupled system of ordinary differential equations which can be integrated numerically. The fixed point equations are coupled algebraic equations for which one finds many solutions. In $d = 2$ only two of
them have a stability matrix with a single negative eigenvalue, however. These are the Gaussian fixed point and the Minkowski space counterpart of the Wilson-Fisher fixed point. For $d \geq 3$ only the Gaussian fixed point has a stability matrix with a single negative eigenvalue. Linearizing the flow around the chosen fixed point sets the target for the numerical shooting technique. In the ultraviolet boundary conditions are set by the expansion of (B.7). The quantitative results so obtained do not differ significantly from those obtained with Euclidean signature and the standard EPA flow, see e.g. [1] and references therein. This validates (B.2) as a viable Minkowski space EPA flow equation.

A flow equation of the form (B.2) also arises in the instantaneous limit of the SLE based version of (4.1). Recall that the equal time limit of the Green's function $G_k[\varphi](t, t; p)$ entering (4.1) is then given by $i|T^\text{SLE}[S_k](t, p)|^2$, where the fiducial solution $S_k(t, p)$ is an arbitrary solution of (2.35). The modulus square can be expressed as in (2.53) and involves a window function $f(t)$ of compact support in $[t_i, t_f]$. Although this quantity is amenable to analytical control, it is fairly complicated. A remarkable simplification occurs when the window function $f$ of the SLE is increasingly centered around an instant $t_0 \in (t_i, t_f)$, giving $f(t)^2 \rightarrow n(t)^{-1} \delta(t-t_0)$ in a formal limit. If the underlying second order operator is of Schrödinger type (i.e. without first order term) the limit of the modulus square of the SLE solution equals $1/(2\sqrt{\text{full potential}})$, see [19]. In the situation at hand this gives

$$|T^\text{SLE}[S_k](t, k\varphi)|^2 \rightarrow \frac{1}{2k a(t_0)^{d} \sqrt{\varphi^2/a(t_0)^2 + r(\varphi^2/a(t_0)^2)} + U^\prime \prime(\varphi, R(t_0))/k^2},$$

(B.11)

with $\varphi = p/k$. Let us stress that the instantaneous limit ruins the Hadamard property and is in general neither mathematically nor physically viable [28]. Another caveat is that the operations “taking the instantaneous limit” and “selecting a time function through a choice of $n$” do not commute because the (non-instantaneous) Schwarzian terms arising from the elimination of the first order term are missed.

It is nevertheless instructive to see that the limit produces a flow equation of the form (B.3) which is formally applicable to generic Friedmann-Lemaître spacetimes merely by replacing $\varphi$ in the integrand by $\varphi/a(t_0)$ and $U_k$ by $U_k = \sum_{n \geq 0} n U_k(\varphi) R(t_0)^n$. The explicit time dependence of the SLE EPA flow disappears and the projection onto the orthonormal basis (4.4) is both unnecessary and tautological. Generally, if in (4.8) the weight $w(t)$ approaches $\delta(t-t_0)$, the coefficients $h_0$ approach $p_0(t_0) h(t_0)$, and on account of the completeness relation the series resums to $h(t) = \varphi(t_0) \delta(t-t_0)$. In the expansion (4.32) one finds from the definitions (2.58), (2.59) that $J_{1,\varphi}[\nabla_{p}^{\prime\prime}](t_0) \rightarrow 0$ and $\varepsilon_{2,\varphi}^{2}[\nabla_{k}^{\prime\prime} \rightarrow 0$ in the instantaneous limit. The right hand side (4.32) therefore reduces to (B.11).

In order to compare with the results of Section 3 we fix conformal time and consider the SLE solution $\chi_{k}^\text{SLE}(\eta, p)$ associated with the (homogeneous) wave equation (3.4). The shifted potential is $W(\eta) - \frac{d-1}{d} R(\eta)$, and we write $\nabla^{\prime\prime}(\varphi(\eta), r_{k}(\eta))$ for its dimensionless counterpart, in parallel to (3.25). In both cases, as in (B.11), we omit the subscript $k$ referring to the dynamical $k$ dependence. The instantaneous limit of the SLE solution in
conformal time and \( d = 3 \) then reads
\[
|x_k^{\text{SLE}}(\eta_0, k\varphi)|^2 \rightarrow \frac{1}{2k\alpha(\eta_0)\sqrt{V''(\varphi_0(\eta_0), \tau_k(\eta_0)) + \varphi^2/a(\eta)^2 + r(\varphi^2/a(\eta_0)^2)}},
\]
with \( \eta_0 \in (\eta_1, \eta_f) \). Restoring the subscript and abbreviating momentarily \( V_k(\eta_0) := V_k(\varphi_0(\eta_0), \tau_k(\eta_0)) \), etc., this results in
\[
k\partial_k V_k(\eta_0) + (1 + d) V_k(\eta_0) - k\partial_k \ln \kappa(k)^2 \tau_k(\eta_0) \frac{\partial V_k(\eta_0)}{\partial \tau_k(\eta_0)} - \frac{d - 1}{2} \varphi_0(\eta_0) \frac{\partial V_k(\eta_0)}{\partial \varphi_0(\eta_0)}
= \frac{h}{c_0(4\pi)^{d/2}\Gamma(d/2)} \int_0^\infty dq q^{d-1} \frac{r(\varphi) - \varphi^2 r'(\varphi)}{\sqrt{V_k''(\eta_0) + \varphi^2 + r(\varphi)}}.
\]

The right hand side coincides with that of the spatial EPA flow equation in Minkowski space (B.5) up to the replacement of \( V_k''(\varphi_0) = 0V_k''(\varphi_0) \) by the generalized potential \( V_k''(\eta_0) \) depending also on the Ricci scalar \( \tau_k(\eta_0) = R(\eta_0)/\kappa(k)^2 \). Upon expansion of (B.13) in powers of \( \tau_k(\eta_0) \) one obtains (B.5) at lowest order and (with \( h = 1 \))
\[
k\partial_k^1 V_k(\eta_0) + (1 + d - k\partial_k \ln \kappa(k)^2)^1 V_k(\eta_0) - \frac{d - 1}{2} \varphi_0(\eta_0) \frac{\partial V_k(\eta_0)}{\partial \varphi_0(\eta_0)}
= -\frac{1}{2} q_v(0) V_k''(\eta_0) \right)^1 V_k''(\eta_0),
\]
\[
k\partial_k^2 V_k(\eta_0) + (1 + d - 2k\partial_k \ln \kappa(k)^2)^2 V_k(\eta_0) - \frac{d - 1}{2} \varphi_0(\eta_0) \frac{\partial V_k(\eta_0)}{\partial \varphi_0(\eta_0)}
= -\frac{1}{2} q_v(0) V_k''(\eta_0) \right)^2 V_k''(\eta_0) + \frac{3}{8} q_v(0) V_k''(\eta_0) \right)^2 V_k''(\eta_0))^2,
\]
at the next two orders, with \( q_v(v) \) from (B.6). Here \( V_k''(\varphi_0) = \frac{V_k(\varphi_0) - \frac{d-1}{4d} \delta_{n,1}}{4\pi} \).

It is instructive to compare the UV flow (4.27) with the instantaneous limit flow (B.14). The flow equations (B.13), (B.14) are not limited to large \( k \) but an expansion in powers of the potential can be compared with the dimensionless form of (4.27). Since \( \kappa(k) \sim k \) for large \( k \) the latter is obtained via the transition relations \( \kappa(U_k(\varphi) = c_0 k^{d+1-2n} V_k(\varphi_0) \). For the comparison of the right hand sides we note \( q_v(v) = q_v(0) - \frac{1}{2}(2\nu + 1)q_{\nu+1}(0) + \frac{1}{8}(2\nu + 1)(2\nu + 3)q_{\nu+2}(0) + O(\nu^3) \), in the notation (B.6). Using this in the \( d = 3 \) version of (B.14) together with \( 1/[(4\pi)^{d/2}\Gamma(d/2)] \) one finds for \( n = 0, 2 \) an exact match, while for \( n = 1 \) the \( q_v \) term in (4.27) is missed. Since the latter arises from differentiating the time dependence in the modulator, it is unsurprising that this term is not seen in the instantaneous limit.
C: Subleading small $k^2$ terms in SLE based EPA flow

Here we justify the substitution recipe (4.30) and use it to compute the explicit form of the $O(k^2)$ corrections displayed in (4.36).

As seen in Appendix B, in the instantaneous limit $|T_k^{\text{SLE}}(t_0,k\varphi)|^2$ is a shifted inverse square root in the (generalized) potential $U'_k$ and it is plain that the substitutions (4.14) or (4.16) transitioning to the dimensionless formulation do not affect the dependence on the re-interpreted (generalized) potential. In the non-instantaneous case, however, the dimensionful formulation (4.1) does not allow one to isolate the $U'_k$ dependence of the Green’s function explicitly, not even in a series expansion in powers of $k^2$. This is because in an expansion like (C.3) below the lowest order Green’s function of $(n^{-1}\partial t)^2 + \omega_0(t)^2$ needed to solve the recursion cannot be obtained explicitly for generic $\omega_0(t)^2 = a(t)^2W(t)$. In contrast, the iterative solution of (4.21) for the power series expansion of the dimensionless Green’s function $G_k[\varphi_0](t,t';\varphi)$ is in itself straightforward, as the only a Green’s function for the trivial $(n^{-1}\partial t)^2$ is needed. For the SLE this $\omega_0 \equiv 0$ expansion can be implemented and leads to the explicit result (4.32). The gap to the dimensionful expansion is bridged by the following

Lemma C: The SLE Green’s function $kG_k[\varphi](t,t;p)$ in (4.1) with dimensionful generalized potential $U''(\varphi,t)$ expanded in powers of $k^2$ has upon substitution of $\varphi = \varphi[\varphi_0]$, $p = k\varphi$, $U''(\varphi,t) = k^2V'(\varphi_0,t)$, and re-expansion in powers of $k^2$ (with $V''(\varphi_0,t)$ held fixed) the same coefficients as the directly expanded dimensionless $kG_k[\varphi_0](t,t;\varphi)$ leading to (4.32).

Remarks:

(i) A result of this form does not hold for the large $k$ expansion (2.44). One way of seeing this is by noting that for $\omega_0 \equiv 0$ the coefficients $\bar{C}_n$ vanish for constant $w_2$. Since constant $w_2$ occurs in Minkowski space (for constant field) this clashes with the direct expansion in the dimensionful formulation, where the coefficients (B.8) arise. The origin of the clash is that upon substitution $U''(\varphi,R) \mapsto k^2V''(\varphi_0,t_0)$ the $\bar{G}_n$ are $O(k^{2n})$, which invalidates the organizing principle in (B.8).

(ii) A simple toy model may illustrate the point of the argument. Consider the following three series (where $\partial^2$, $W$, $V$, $r$ are suggestive notations for positive numbers)

$$\frac{1}{\partial^2 + W + k^2r} = \sum_{n\geq 0} (-)^n \frac{r^n k^{2n}}{\partial^2 + W)^{n+1}},$$

$$\frac{1}{\partial^2 + W)^{n+1}} \bigg|_{W \rightarrow k^2V} = \sum_{j \geq 0} \left( \frac{-n-1}{j} \right) \frac{V^j k^{2j}}{(\partial^2)^{n+j}},$$

$$\frac{1}{\partial^2 + k^2 (r+V)} = \sum_{n \geq 0} (-)^n \frac{(V+r)^{n k^{2n}}}{(\partial^2)^{n+1}}. \quad (C.1)$$

Inserting the first two series into each other (re-)produces the third, on account of $\sum_{j=0}^{n} (-)^{n-j} \left( \frac{-n-1+j}{j} \right) V^j r^{n-j} = (-)^n (V+r)^n$. 

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Proof of Lemma C. We apply an abstracted version of (C.1) to the fiducial solutions entering the SLE. To this end, we search for a fiducial solution \( S_k(t, \varphi) \) of the homogeneous wave equation (2.35) for fixed \( \varphi = p/k \) in the form of a power series ansatz

\[
S_k(t, \varphi) = S_0(t) + \sum_{n \geq 1} S_n(t, \varphi) k^{2n}, \quad \varphi = p/k \text{ fixed.} \tag{C.2}
\]

Inserted into (2.35) one obtains the recursive system

\[
[(a^d N^{-1} \partial_t)^2 + a^{2d} W] S_0(t) = 0, \tag{C.3}
\]

\[
[(a^d N^{-1} \partial_t)^2 + a^{2d} W] S_n(t, \varphi) = -a^{2d-2} \left( \varphi^2 + a^2 r(\varphi^2/a^2) \right) S_{n-1}(t, \varphi), \quad n \geq 1,
\]

with \( W = U'' \) as in (2.32). The coefficients \( S_n \) are in addition slightly constrained by the expansion of the Wronskian condition. Since an SLE does not depend on the choice of fiducial solution we may begin with a fiducial solution of the dimensionful wave equation whose initial data are \( k, \varphi \) independent. The recursion relations (C.3) can then be solved unambiguously in terms of the (universal) retarded Green’s functions for \((n^{-1} \partial_t)^2 + a^{2d} U''\), \( n = Na^{-d} \). The resulting series has finite radius of convergence and thus yields an exact solutions for \( k < k_* \). Since its initial data are \( k \) independent one can appeal to a standard result on ordinary differential equations (see e.g. [24], Chapter 5.2) to conclude that the dependence of the exact solution on \( U''\)’s overall scale is smooth. The same holds for the Green’s function of \((n^{-1} \partial_t)^2 + a^{2d} U''\). Replacing now \( U'' \) by \( k^2 \nu'' \) the basic Green’s function needed for the iteration as well as the exact solution can be re-expanded in powers of \( k^2 \). The re-expanded series is a solution of the \( \omega_0^2 = a^{2d} W \equiv 0 \) recursion relations uniquely characterized by the initial data induced by re-expanding the exact solution, and the trivial \( \omega_0^2 \equiv 0 \) retarded Green’s function. It also has finite radius of convergence \( k < k_{*0} \), see [19]. Within the (nonempty) intersection of the intervals of convergence the two-step expanded solution and the solution of the \( \omega_0^2 \equiv 0 \) recursion will coincide. This justifies the substitution recipe (4.30) for the fiducial solutions.

Either of these expansions can now be inserted into the SLE solution \( T_k^{\text{SLE}}[S_k](t, p) \), viewed as a functional of \( S_k \). The functionals also carry an explicit \( k, \varphi \) dependence entering via the \( \mathcal{E}, \mathcal{D} \) from (2.51). Since analyticity in \( k^2 \) is maintained in the substitution process the explicit \( k \) dependence allows for a power series expansion in \( k^2 \). The same holds for the \( k^2 \) dependence carried by the two distinct series realizations for the fiducial solution. The respective expansions are of the form (2.55) (with \( k \) playing the role \( p \)) but the \( \varepsilon_0 = 0 \) version must by the previous step be the re-expansion of the \( \varepsilon_0 > 0 \) version. This justifies the substitution recipe (4.30) for \( |T_k^{\text{SLE}}[S_k](t, p)|^2 \), constructed from the above specific fiducial solution \( S_k(t, p) \). Since \( |T_k^{\text{SLE}}[S_k](t, p)|^2 \) does not depend on the choice of \( S_k \) the result follows. □

Application of Lemma C yields (4.32) so that the small \( k \) regime of the SLE based FL-sFRG flow equation (4.22) is governed by the coefficients \( J_{1, \varphi}[V''](t) \) and \( \varepsilon_{2, \varphi}^2[V''] \), where
$\mathcal{V}'(t)$ is short for $\mathcal{V}'(\varphi_0, t)$. Using the averages (4.35) the right hand side of (4.22) becomes

$$\text{RHS of (4.36)} = iQ_0(1|\phi_k') + \frac{k^2}{\hbar} iQ_0(\mathcal{J}_{1.,|}\phi_k') - \frac{k^2}{2\hbar^2} iQ_1(\varepsilon_2, [\mathcal{V}'], \phi_k') \,. \quad (C.4)$$

In view of the linearity of the $i\mathcal{Q}_m(h|\mathcal{V})$ in the first argument we prepare expansions for $\mathcal{J}_{1.,|}(t)$ and $\varepsilon^2_{2,|}$. To this end the expansion of the commutator function $\Delta_{1,|}$ and its derivatives is needed. In parallel to (4.31) we write

$$n(t)^{-1} \partial_t \Delta_{1,|}[\mathcal{V}'](t', t) = \int\limits_{t'}^t ds n(s) a(s)^2 [\mathcal{V}''(s) + \tau(\varphi/a(s))] \Delta_0(s, t') \,,$$

$$\left[n(t)^{-1} n(t')^{-1} \partial_t \partial_r \Delta_{1,|}[\mathcal{V}'](t', t)\right]^2 = \int\limits_{t'}^t ds'ds n(s)n(s') a(s)^2 a(s')^{2d}$$

$$\times \left[\mathcal{V}''(s) + \tau(\varphi/a(s))\right] \left[\mathcal{V}''(s') + \tau(\varphi/a(s'))\right] \,,$$

with the abbreviation $\tau(\varphi/a(t)) := \varphi^2/a(t)^2 + r(\varphi^2/a(t)^2)$. Using (2.58), (2.60), (4.30), these enter

$$\mathcal{J}_{1,|}[\mathcal{V}'](t) = \frac{1}{2} \int dt' n(t') f(t') \left\{2n(t')^{-1} \partial_t \Delta_{1,|}[\mathcal{V}'](t', t) + \Delta_0(t, t') a(t')^{2d} \left[\mathcal{V}''(t') + \tau(\varphi/a(t'))\right]\right\} , \quad (C.6)$$

and

$$\varepsilon^2_{2,|}[\mathcal{V}'] = \frac{1}{8} \int dtdt' n(t)n(t') f(t)f(t') \left\{(n(t)^{-1}n(t')^{-1} \partial_t \partial_r \Delta_{1,|}[\mathcal{V}'](t', t))^2ight.$$

$$-4a(t')^{2d} [\mathcal{V}''(t') + \tau(\varphi/a(t'))] n(t)^{-1} \partial_t \Delta_{1,|}[\mathcal{V}'](t', t)$$

$$+ a(t)^{2d} a(t')^{2d} \Delta_0(t, t')^2 [\mathcal{V}''(t') + \tau(\varphi/a(t'))] [\mathcal{V}''(t') + \tau(\varphi/a(t'))] \right\} . \quad (C.7)$$

Next, we expand $\mathcal{V}''(t) = \mathcal{V}''(\varphi_0, t) = \sum_{l \geq 0} i\mathcal{V}''(\varphi_0) p_l(t)$ and insert into (C.6). This gives

$$\mathcal{J}_{1,|}[\mathcal{V}'](t) = \sum_{l \geq 0} i\mathcal{V}''(\varphi_0) \mathcal{J}_l(t) + \mathcal{J}(t, \varphi) , \quad (C.8)$$

$$\mathcal{J}_l(t) = \int dt' n(t') f(t') \left\{\int\limits_{t'}^t ds n(s) a(s)^2 p_l(s) \Delta_0(s, t') + \frac{1}{2} a(t')^{2d} \Delta_0(t, t')^2 p_l(t')\right\} ,$$

$$\mathcal{J}(t, \varphi) = \int dt' n(t') f(t') \left\{\int\limits_{t'}^t ds n(s) a(s)^2 \tau(\varphi/a(s)) \Delta_0(s, t') + \frac{1}{2} a(t')^{2d} \Delta_0(t, t')^2 \tau(\varphi/a(t'))\right\} .$$

Finally,

$$\mathcal{E}^2_{2,|}[\mathcal{V}'] = \sum_{l, m \geq 0} i\mathcal{V}''(\varphi_0) \mathcal{V}''(\varphi_0) E_{l, m} + \sum_{l \geq 0} i\mathcal{V}''(\varphi_0) E_l(\varphi) + E(\varphi) , \quad (C.9)$$

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\begin{align}
E_{l,l'} &= \frac{1}{8} \int dt dt' n(t) n(t') f(t') \left\{ \int_{t'}^{t} ds n(s) a(s)^2 \varphi_l(s) \int_{s'}^{s} ds' n(s') a(s')^2 \varphi_{l'}(s') \\
& \quad - 4a(t')^2 \varphi_{l'}(t') \int_{s'}^{s} ds a(s)^2 \varphi_{l}(s) \Delta_0(s, s') + a(t)^2 a(t')^2 \Delta_0(t, t')^2 \varphi_l(t) \varphi_{l'}(t') \right\}, \\
E_{l}(\varphi) &= \frac{1}{4} \int dt dt' n(t) n(t') f(t') \left\{ \int_{t'}^{t} ds n(s) a(s)^2 \varphi_l(s) \int_{s'}^{s} ds' n(s') a(s')^2 \varphi(t) a(s') \Delta_0(s, t') \\
& \quad - 2a(t')^2 \varphi_l(t') \int_{s'}^{s} ds a(s)^2 \varphi(t) a(s) \Delta_0(s, t') \\
& \quad - 2a(t')^2 \varphi_l(t') a(t') \int_{s'}^{s} ds n(s) a(s)^2 \varphi_l(s) \Delta_0(s, t') \\
& \quad + a(t)^2 a(t')^2 \Delta_0(t, t')^2 \varphi_l(t) \varphi(t) \right\}, \\
E(\varphi) &= \frac{1}{8} \int dt dt' n(t) n(t') f(t') \left\{ \left[ \int_{t'}^{t} ds n(s) a(s)^2 \varphi(a(s)) \right]^2 \\
& \quad - 4a(t')^2 \varphi(a(t')) \int_{s'}^{s} ds n(s) a(s)^2 \varphi(a(s)) \Delta_0(s, t') \\
& \quad + a(t)^2 a(t')^2 \Delta_0(t, t')^2 \varphi(a(t)) \varphi(a(t')) \right\}. 
\end{align}

These expansions are now inserted into (C.4) and yield the explicit form of (4.36)'s right hand side.
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