Stokes Matrices and Monodromy of the Quantum Cohomology of Projective Spaces

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Abstract

In this paper we compute Stokes matrices and monodromy for the quantum cohomology of projective spaces. This problem can be formulated in a "classical" framework, as the problem of computation of Stokes matrices and monodromy of (systems of) differential equations with regular and irregular singularities. We prove that the Stokes' matrix of the quantum cohomology coincides with the Gram matrix in the theory of derived categories of coherent sheaves. We also study the monodromy group of the quantum cohomology and we show that it is related to hyperbolic triangular groups.

1 Introduction

In this paper we compute Stokes' matrices and monodromy group for the Frobenius manifold given by the quantum cohomology of the projective space \( \mathbb{CP}^d \). Our main motivation is to study the links between quantum cohomology and the theory of coherent sheaves.

Stokes matrices first appeared in the theory of WDVV equations of associativity in the paper [7] by B. Dubrovin. WDVV equations were formulated in a geometrical setting: the theory of Frobenius manifolds. From then on, the notion of Frobenius manifold has been largely studied in many papers, of which we cite [9], [10], [11].

The Stokes' matrix is a part of the monodromy data for a semisimple Frobenius manifold. Monodromy data can serve as natural moduli of semisimple Frobenius manifolds. More precisely, any local chart of the atlas covering the manifold is reconstructed from the monodromy data. The glueing of the local charts is described by the action of the braid group on the data, particularly, on the central connection matrix and on the Stokes' matrix [9] [10].

One well-known example of Frobenius manifolds is the quantum cohomology of smooth projective varieties [20] [19] [2] [21]. It was conjectured [11] that the Stokes matrix for the quantum cohomology of a good Fano variety \( X \) is equal to the Gram matrix of the bilinear form \( \chi(E,F) := \sum_k (-1)^k \dim \text{Ext}^k(E,F) \) computed on a full collection of exceptional objects in the derived category \( \text{Der}^b(\text{Coh}(X)) \) of coherent sheaves on \( X \). More precisely, let \( \text{Der}^b(\text{Coh}(X)) \) be the derived category of coherent sheaves on a smooth projective variety \( X \) of dimension \( d \). An object \( E \) of \( \text{Der}^b(\text{Coh}(X)) \) is called exceptional if \( \text{Ext}^i(E,E) = 0 \) for \( 0 < i < d \), \( \text{Ext}^0(E,E) = \mathbb{C} \) and \( \text{Ext}^d(E,E) \) is of the smallest dimension (if \( X \) is a projective space, then \( \text{Ext}^d(E,E) = 0 \)). A collection \( \{E_1,...,E_s\} \) of exceptional objects is an exceptional collection if for any \( 1 \leq m < n \leq s \) we have \( \text{Ext}^i(E_n,E_m) = 0 \) for any \( i \geq 0 \), \( \text{Ext}^i(E_m,E_n) = 0 \) for any \( i \geq 0 \) except possibly for one value of \( i \). A full exceptional collection is an exceptional collection which generates \( \text{Der}^b(\text{Coh}(X)) \) as a triangulated category. This theory is developed in [23] [24] [1]. We say that a Fano variety is good if it has a full exceptional collection.

It is known that \( X = \mathbb{CP}^d \) is good, the collection of sheaves on \( \mathbb{CP}^d \) \( \{O(n)\}_{n \in \mathbb{Z}} \) is exceptional, and \( \{E_1,E_2,...,E_{d+1}\} := \{O,O(1),...,O(d)\} \) is a full exceptional collection [3], [16]. In this case, \( s_{ij} = \chi(O(i-1),O(j-1)) \), \( i,j = 1,2,...,d+1 \) has the “canonical form”:

\[
s_{ij} = \begin{cases} 
  d + j - i & i < j \\
  j - i & i > j 
\end{cases}
\]

\[s_{ii} = 1, \quad s_{ij} = 0 \quad i > j\]
The inverse to this matrix has entries $a_{ij}$

$$a_{ij} = (-1)^{j-i} \left( \frac{d+1}{j-i} \right) \quad i < j$$

$$a_{ii} = 1, \quad a_{ij} = 0 \quad i > j$$

This matrix is equivalent to the one above with respect to the action of the braid group. We will also call it “canonical”.

The mentioned conjecture claims that the Stokes matrix of the quantum cohomology of $\mathbf{CP}^d$ is equal to the above Gram matrix (modulo the action of the braid group: remarkably, this action on the Stokes matrix for the Frobenius manifold coincides with the natural action of the braid group on the collections of exceptional objects \cite{23, 24}).

This conjecture has its origin in the paper by Cecotti and Vafa \cite{8}, where another Stokes matrix introduced in \cite{8} for the $tt^*$ equations was found in the case of the $\mathbf{CP}^2$ topological sigma model. It was suggested, on physical arguments, that the entries of the Stokes’ matrix $S = \begin{pmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{pmatrix}$ are integers.

They must satisfy a Diophantine equation $x^2 + y^2 + z^2 - xyz = 0$ whose integer solutions $(x, y, z)$ are all equivalent to $(3,3,3)$ modulo the action of the braid group. The authors of \cite{8} also suggested that their matrix must coincide with the Stokes matrix defined in the theory of WDVV equations of associativity, that is, in the geometrical theory of Frobenius manifolds for 2D topological field theories \cite{8, 10}.

Later, in \cite{25}, the links between $N = 2$ supersymmetric field theories and the theory of derived categories were further investigated and the coincidence of $\chi(E_i, E_j)$ with the Stokes matrix of $tt^*$ for $\mathbf{CP}^d$ was conjectured.

The conjecture may probably be derived from more general conjectures by Kontsevich in the framework of categorical mirror symmetry. To my knowledge, the subject was discussed in \cite{18} (I thank B. Dubrovin for this reference).

The main result of this paper is the proof (Theorem 2, 2’) that the conjecture about coincidence of the Stokes matrix for quantum cohomology of $\mathbf{CP}^d$ and the Gram matrix $\chi(E_i, E_j)$ of a full exceptional collection in $\text{Der}^b(\text{Coh}(\mathbf{CP}^d))$ is true. In this way, we generalize to any $d$ the result obtained in \cite{10} for $d = 2$.

We remark that it has not yet been proved that the Stokes’ matrix for $tt^*$ equations and the Stokes’ matrix for the corresponding Frobenius manifold coincide. This point deserves further investigation.

We also study the structure of the monodromy group of the quantum cohomology of $\mathbf{CP}^d$. The notion of monodromy group of a Frobenius manifold was introduced in \cite{3}. We prove (Theorem 3) that for $d = 3$ the group is isomorphic to the subgroup of orientation preserving transformations in the hyperbolic triangular group $[2, 4, \infty]$. In \cite{10} it was proved that for $d = 2$ the monodromy group is isomorphic to the direct product of the subgroup of orientation preserving transformations in $[2, 3, \infty]$ and the cyclic group of order 2, $C_2 = \{\pm\}$. Our numerical calculations also suggest that for any $d$ even the monodromy group may be isomorphic to the orientation preserving transformations in $[2, d + 1, \infty]$, and for any $d$ odd to the direct product of the orientation preserving transformations in $[2, d + 1, \infty]$ by $C_2$.

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2 The system corresponding to $\mathbf{CP}^{k-1}$

We introduce here the linear system of differential equations whose Stokes matrices are the Stokes matrices for the quantum cohomology of $\mathbf{CP}^{k-1}$ (we use the more convenient choice $k = d + 1$).

In the quantum cohomology of $\mathbf{CP}^{k-1}$ we choose flat coordinate $t^1, \ldots, t^k$ for the symmetric non degenerate bilinear form $<,>$:

$$\eta_{\alpha\beta} = <\partial_{\alpha}, \partial_{\beta}> = \delta_{\alpha+\beta, k+1} \quad \text{where} \quad \partial_{\alpha} = \frac{\partial}{\partial t^\alpha}$$
Let \( \eta \) be the matrix \((\eta_{\alpha \beta})\). In flat coordinates the Euler vector field is

\[
\eta = \sum_{\alpha \neq 2} (1 - q_{\alpha}) t^\alpha \frac{\partial}{\partial t^\alpha} + k \frac{\partial}{\partial t^2}
\]

\( q_1 = 0, q_2 = 1, q_3 = 2, \ldots, q_k = k - 1 \)

the multiplication is

\[
\partial_\alpha \cdot \partial_\beta = c_{\alpha \beta}(t) \partial_\gamma \quad \text{where} \quad c_{\alpha \beta}(t) := \eta_{\alpha \beta} c_{\alpha \beta}(t) = \partial_\alpha \partial_\beta \partial_\gamma F(t)
\]

\[
F(t) = \frac{1}{6} \sum_{\alpha + \beta + \gamma = k+2} t^\alpha t^\beta t^\gamma + \sum_{l=1}^{\infty} \Phi_l(t, \ldots, t^k) e^{lt^2}
\]

\[
\Phi_l(t, \ldots, t^k) = \sum_{n=2}^{\infty} \sum_{\alpha_1 + \ldots + \alpha_n = (l+1)(k-1)+l-n} I(l; \alpha_1, \ldots, \alpha_n) \frac{t^{\alpha_1} \ldots t^{\alpha_n}}{n!}, \quad I(1, k, k) = 1
\]

and the unity vector field is \( e = \frac{\partial}{\partial t} \). Finally, let

\[
\mu = \text{diag}(\mu_1, \ldots, \mu_k) = \text{diag}(\frac{k-1}{2}, \ldots, \frac{k-3}{2}, \ldots, \frac{k-1}{2}), \quad \mu_\alpha = q_\alpha - \frac{d}{2}
\]

Consider the system of differential equations determining deformed flat coordinates (see (1) (2)):

\[
\begin{align*}
\partial_z \xi &= (U(t) + \frac{1}{z} \mu) \xi \\
\partial_\alpha \xi &= z C_\alpha(t) \xi
\end{align*}
\]

where \( z \in \mathbb{C}, \partial_z := \frac{\partial}{\partial z} \) and \( \xi \) is a column vector of components

\[
\xi^\alpha = \eta^{\alpha \beta} \partial_\beta \tilde{t}(t, z)
\]

Here \((\eta^{\alpha \beta}) = (\eta_{\mu \nu})^{-1}\) and \(\tilde{t}(t, z)\) is one of the \( k \) (deformed) flat coordinates. \( U(t) \) is the matrix of multiplication by the Euler vector field \( E(t) \), and \( C_\alpha(t) \) is the matrix of entries \((C_\alpha)^\beta := c_{\alpha \beta}\). The monodromy data of the system (2) are, by definition, the monodromy data of the quantum cohomology of \( \mathbb{C}P^{k-1} \) in the local chart containing \( t \). Let us compute \( C_2(t) \) and \( U(t) \) at the semisimple point \((0, t^2, 0, \ldots, 0)\):

\[
E \cdot \partial_\beta = E \cdot c_{\gamma \beta} \partial_\alpha = E^2 c_{2 \beta} \partial_\alpha = k e_{2 \beta} \partial_\alpha = U^\alpha \partial_\alpha
\]

Moreover \( c_{2 \alpha}(0, t^2, 0, \ldots, 0) = \partial_\alpha \partial_\beta \partial F(0, t^2, \ldots, 0) \). This immediately yields

\[
C_2(0, t^2, 0, \ldots, 0) = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
1 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 1 \\
1 & 0 & \cdots & 1 & 0
\end{pmatrix}, \quad U(0, t^2, 0, \ldots, 0) = \begin{pmatrix}
k & 0 & \cdots & 0 & ke^{t^2} \\
0 & k & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & k & 0
\end{pmatrix}
\]

Let \( y_\alpha := \eta_{\alpha \beta} \xi^\beta \equiv \partial_\alpha \tilde{t} \). It satisfies

\[
\begin{align*}
\partial_z y &= (\tilde{t}(t^2) - \frac{1}{z} \mu) y \\
\partial_2 y &= z \tilde{C}_2(t^2) y
\end{align*}
\]

where

\[
\tilde{C}_2(t^2) := \eta \cdot C_2(0, t^2, \ldots, 0) \eta^{-1} = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 1 \\
1 & 0 & \cdots & 1 & 0
\end{pmatrix}
\]
\[ \hat{U}(t^2) := \eta \mathcal{U}(0, t^2, \ldots, 0) \eta^{-1} = \begin{pmatrix} k & 0 & k \\ 0 & k & 0 \\ \vdots & \ddots & \ddots \\ k e t^2 & \cdots & 0 & k \end{pmatrix} \]

**Lemma 1:** Let \( y(t^2, z) = (y_1(t^2, z), \ldots, y_k(t^2, z))^T \) be a column vector solution of the above system (4). With the following substitution

\[ y_\alpha(t^2, z) = \frac{1}{k^{\alpha-1}} z^{k^{\alpha-1}-\alpha+1} (z \partial_z)^{(\alpha-1)} \varphi(t^2, z) \equiv z^{k^{\alpha-1}-\alpha+1} \partial_z^{\alpha-1} \varphi(t^2, z), \quad \alpha = 1, 2, \ldots, k \]

the above system is equivalent to the equations

\[ (z \partial_z)^k \varphi = (kz)^k e^{t_z} \varphi \]

\[ \partial_z^k \varphi = z^{k e t_z} \varphi \]

The proof is a simple calculation we leave to the reader.

The substitution of the lemma implies \( \partial_z \varphi = \frac{1}{k} z \partial_z \varphi \). Then

\[ e^{\frac{z}{k} t_z} \frac{\partial}{\partial \frac{z}{k} t_z} \varphi(t^2, z) = z \frac{\partial}{\partial z} \varphi(t^2, z) \]

which implies (with abuse of notation)

\[ \varphi(t^2, z) \equiv \varphi(ze^{\frac{t_z}{k}}) \]

Namely, \( \varphi \) (at \((0, t^2, \ldots, 0)\)) depends on one argument \( w = ze^{\frac{t_z}{k}} \) and satisfies the *generalized hypergeometric equation*

\[ \left( w \frac{d}{dw} \right)^k \varphi(w) = (k w)^k \varphi(w) \]

The equation is equivalent to the system

\[ \frac{dY}{dw} = \left[ \hat{U} + \hat{\mu} \frac{w}{\mu} \right] Y \]

where

\[ Y_n(w) = \frac{1}{k^{n-1}} w^{\frac{k^{n-1}}{2} - n+1} (w \partial_w)^{(n-1)} \varphi(w) \quad n = 1, 2, \ldots, k \]

\[ \hat{U} := \hat{U}(0) = \begin{pmatrix} 0 & k & 0 & k & \cdots & 0 & k \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots \\ k & \cdots & 0 & k & \cdots & \ddots & \ddots \\ k & \cdots & \cdots & \ddots & \ddots & \cdots & \ddots \end{pmatrix} \]

\[ \hat{\mu} := -\mu = \text{diag} \left( \frac{k-1}{2}, \frac{k-3}{2}, \frac{k-5}{2}, \ldots, -\frac{k-3}{2}, -\frac{k-1}{2} \right) \]

The system \( 4 \) may also be interpreted as the system \( 2 \) with \( t^2 = 0 \). We will return later (section 4) on the connection between its monodromy data and the monodromy data of the system \( 2 \).

Let us study system \( 4 \). We change notation and choose the more familiar letter \( z \) instead of \( w \). So, the system \( 2 \) is re-written as

\[ \frac{dY}{dz} = \left[ \hat{U} + \hat{\mu} \frac{z}{w} \right] Y \]
and (10) (8) become

\[ Y_n(z) = \frac{1}{k^{n-1}} z^{\frac{k-1}{2} - n + 1} (z \partial_z)^{(n-1)} \varphi(z) \quad n = 1, 2, ..., k \] (12)

\[ \left( z \frac{d}{dz} \right)^k \varphi(z) = (kz)^k \varphi(z) \] (13)

The point \( z = 0 \) is a fuchsian singularity, and \( z = \infty \) is a singularity of the second kind. (11) has a fundamental matrix solution \( Y_0(z) \) whose behaviour at \( z = 0 \) is

\[ Y_0(z) = (I + O(z)) z^R \quad R = \begin{bmatrix} 0 & k & 0 & k & 0 & k & \cdots & k \ 0 & k & \cdots & k & 0 & k & \cdots & k \ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \ddots & \vdots \ 0 & k & \cdots & k & 0 \end{bmatrix} \]

and the monodromy for a counter-clockwise loop around the origin is \( e^{2\pi i (\hat{\mu} + R)} \).

The characteristic polynomial of the matrix \( \hat{U} \) is

\[ 0 = \det(\hat{U} - \mu) = (-\mu)^k + (-1)^{k+1} k^k \]

It has \( k \) distinct eigenvalues \( \mu_n = k e^{2\pi i (n-1)/k}, n = 1, ..., k \). The equations for the eigenvector \( x_n \) corresponding to \( \mu_n \), namely

\[ \hat{U} x_n = \mu_n x_n \]

written for the components \( x_1, ..., x_k \) of the column vector \( x_n \) are

\[ x_{l+1} = e^{\frac{2\pi i (n-1)}{k}} x_l \quad l = 1, 2, ..., k - 1 \]

\[ x_1 = e^{\frac{2\pi i (n-1)}{k}} x^n \]

With the choice \( x_1 = e^{\frac{2\pi i (n-1)}{k}} \) we get

\[ x_n = \left( e^{\frac{2\pi i (n-1)}{k}}, e^{\frac{3\pi i (n-1)}{k}}, e^{\frac{4\pi i (n-1)}{k}}, ..., e^{\frac{(k-1)\pi i (n-1)}{k}} \right)^T \]

(\( T \) stands for transpose). The matrix

\[ X = \frac{1}{\sqrt{k}} \begin{bmatrix} |x_1| & |x_2| & ... & |x_k| \end{bmatrix} = \frac{1}{\sqrt{k}} \begin{bmatrix} x_1^n & x_2^n & \cdots & x_k^n \end{bmatrix} \quad x_j^n = e^{(2j-1)i\pi \frac{k-1}{k}} \]

puts \( \hat{U} \) in diagonal form:

\[ U = X^{-1} \hat{U} X = \text{diag}(u_1, u_2, ..., u_n, ..., u_k) \quad u_n = k e^{2\pi i (n-1)/k} \]

We stress that \( u_i \neq u_j \) for \( i \neq j \). The system (11) is transformed by the gauge \( X \) in an equivalent form

\[ \frac{d\tilde{Y}}{dz} = \left[ U + \frac{V}{z} \right] \tilde{Y} \] (14)

\[ \tilde{Y} = X^{-1} Y, \quad U = X^{-1} \hat{U} X, \quad V = X^{-1} \hat{\mu} X \]

Observe that

\[ \eta \hat{\mu} + \hat{\mu} \eta = 0 \quad XX^T = \eta^{-1} \]

This implies that \( V \) is skew-symmetric

\[ V + V^T = 0 \]

With the gauge \( X \), \( Y_0(z) \) transforms into

\[ \tilde{Y}_0(z) = (X^{-1} + O(z)) z^R, \quad z \to 0 \]
3 Asymptotic Behaviour and Stokes’ Phenomenon

Our aim is to explicitly compute a Stokes’ matrix for the above system (14), or for the system (11). The system (14) has formal solution

\[ \tilde{Y}_F = \left[I + \frac{F_1}{z} + \frac{F_2}{z^2} + \ldots\right] e^{zU} \]

where \( F_j \)'s are \( k \times k \) matrices. It is a well known result that fundamental matrix solutions exist which have \( \tilde{Y}_F \) as asymptotic expansion for \( z \to \infty \) in some “admissible” sectors of the complex plane of angular width greater than \( \pi \). In order to find such sectors we need the so called Stokes’ rays, defined by

\[ \Re \left( (u_r - u_s)z \right) = 0, \quad r \neq s \quad \Rightarrow \quad \arg z = -\arg(u_r - u_s) + \frac{\pi}{2} + m\pi, \quad m = 0, 1 \]

There exists a unique solution of the system asymptotic to \( \tilde{Y}_F \) in a sector greater than \( \pi \) and bounded by the first two Stokes’ rays we meet extending over \( \pi \) the angular width of the sector. The general theory of Stokes’ phenomenon is found in the classical paper by W. Balser, W.B. Jurkat, D.A. Lutz \[1\]. Stokes matrices are also a natural monodromy datum in the theory of isomonodromy deformation \[12\].

A possible choice for the labelling of the rays is the following: we call \( R_{rs} \) the Stokes’ ray

\[ R_{rs} = \{ z = -i\rho(\bar{u}_r - \bar{u}_s), \quad \rho > 0 \} \quad r \neq s \]

**Lemma 2:** For \( r < s \) the Stokes’ rays of the system (14) are

\[ R_{rs} = \left\{ z = \rho \exp \left(i \left[ \frac{2\pi}{k} - \frac{\pi}{k} (r + s) \right] \right), \quad \rho > 0 \right\} \\
R_{sr} = -R_{rs} \]

**Proof:** Just compute

\[ -i(\bar{u}_r - \bar{u}_s) = -i(e^{-i\frac{2\pi}{k}(r-1)} - e^{-i\frac{2\pi}{k}(s-1)}) = 2 \sin \left( \frac{\pi}{k} (s - r) \right) e^{i(\frac{2\pi}{k} - \frac{\pi}{k}(r+s))} \]

Then we note that \( \sin \left( \frac{\pi}{k} (s - r) \right) \) is positive because \( 0 < s - r \leq k - 1 \).

**Remark 1:** \( R_{rs} = R_{pq} \) for \( r + s = p + q \). \( R_{12} \) is at \( \arg z = -\frac{\pi}{k} \), \( R_{13} \) is at \( \arg z = -\frac{2\pi}{k} \), and so on. For \( r + s = k + 2 \) the corresponding \( R_{rs} \)'s are at \( \arg z = -\pi \) and the \( R_{sr} \)'s are at \( \arg z = 0 \). \( R_{k-1,k} \) is at the angle \( -2\pi + \frac{2\pi}{k} \) or, equivalently, at \( \frac{2\pi}{k} \). See figure 1.

We choose two “admissible” overlapping sectors in a canonical way. Let \( l \) be an “admissible” line through the origin, namely a line not containing Stokes’ rays. For our purposes we take

\[ l = \{ z \mid z = \rho e^{i\epsilon}, \quad \rho \in \mathbb{R}, \quad 0 < \epsilon < \frac{\pi}{k} \} \]

\( l \) has a natural orientation inherited from \( \mathbb{R} \). We call \( \Pi_R \) and \( \Pi_L \) the half planes to the right/left of \( l \) w.r.t its orientation.

\[ \Pi_R = \{ -\pi + \epsilon < \arg z < \epsilon \} \quad \Pi_L = \{ \epsilon < \arg z < \pi + \epsilon \} \]

We then define two different “admissible” sectors \( S_L, S_R \) which contain \( l \)

\[ S_L = \{ z \in \mathbb{C} \mid 0 < \arg z < \pi + \frac{\pi}{k} \} \supset \Pi_L \]
\[ S_R = \{ z \in \mathbb{C} \mid -\pi < \arg z < \frac{\pi}{k} \} \supset \Pi_R \]
We call the corresponding solutions $\hat{Y}_L(z)$ and $\hat{Y}_R(z)$.

**Definition:** The *Stokes’ matrix* of the system (14) with respect to the admissible line $l$ is the connection matrix $S$ such that

$$\hat{Y}_L(z) = \hat{Y}_R(z)S \quad 0 < \text{arg} \ z < \frac{\pi}{k}$$

On the opposite overlapping region one can prove (as a consequence of the skew-symmetry of $V$, see [9]) that

$$\hat{Y}_L(z) = \hat{Y}_R(z e^{-2\pi i}) S^T \quad \pi < \text{arg} \ z < \pi + \frac{\pi}{k}$$

In [1] $S$ is called a “Stokes’ multiplier”. The terminology in this field changes from one author to the other...

**Definition:** We call *central connection matrix* the connection matrix $C$ such that

$$\hat{Y}_0(z) = \hat{Y}_R(z)C \quad z \in \Pi_R$$

It is clear that the system (11) has solutions $Y_0(z) = X\hat{Y}_0(z)$, and $Y_L(z) = X\hat{Y}_L(z)$, $Y_R(z) = X\hat{Y}_R(z)$ asymptotic to $X\hat{Y}_F(z)$ as $z \to \infty$ in $S_L$ and $S_R$ respectively, which are connected by the same $S$ and $C$.

In order to compute the entries of $S$ explicitly, we use the reduction of (11) to the generalized hypergeometric equation (13). If $\varphi^{(1)}(z), \ldots, \varphi^{(k)}(z)$ is a basis of $k$ linearly independent solutions of (13), then the matrix $Y(z)$ of entries $(n, j)$ defined by

$$Y_{n,j}(z) := \frac{1}{k^{n-1}} z^{\frac{k-1}{2} - n + 1} (z\partial_z)^{(n-1)} \varphi^{(j)}(z)$$

is a fundamental matrix for (11).

**Lemma 3:** The generalized hypergeometric equation (13) has two bases of linearly independent solutions $\varphi^{(1)}_{L/R}(z), \ldots, \varphi^{(k)}_{L/R}(z)$ having asymptotic behaviours

$$\varphi^{(n)}_{L/R} = \frac{1}{\sqrt{k}} e^{i\frac{\pi}{k}(n-1)} \exp \left[ k e^{i\frac{2\pi}{k}(n-1)} z \right] \left( 1 + O \left( \frac{1}{z} \right) \right), \quad z \to \infty$$

in $S_L$ and $S_R$ respectively. Let $\Phi(z)$ denote the row vector $[\varphi^{(1)}(z), \ldots, \varphi^{(k)}(z)]$. The fundamental matrices $Y_L(z)$, $Y_R(z)$ of (14) are expressed through formula (14) in terms of $\Phi_L(z)$ and $\Phi_R(z)$ and

$$Y_L(z) = Y_R(z)S \quad 0 < \text{arg} \ z < \frac{\pi}{k}$$
if and only if
\[ \Phi_L(z) = \Phi_R(z)S \quad 0 < \arg z < \frac{\pi}{k} \]

**Proof:** Simply observe that for a fundamental solution in \( S_L \) or \( S_R \) (we omit subscripts \( L, R \))

\[ Y(z) = \begin{bmatrix} z^{\frac{k-1}{2}} \varphi^{(1)} & \ldots & z^{\frac{k-1}{2}} \varphi^{(k)} \end{bmatrix} = XYZ \]

which is asymptotic, for \( z \to \infty \), to

\[
\sim \begin{bmatrix}
1 & e^{i\frac{k-1}{2}z} & e^{i\frac{3k-1}{2}z} & \ldots & e^{i\frac{k-1}{2}z} \\
0 & 1 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \ldots & 1
\end{bmatrix} \begin{bmatrix}
\exp(kz) \\
\exp(ke^{\frac{k+1}{2}z}) \\
\vdots \\
\exp(ke^{\frac{2n(k-1)}{k}})
\end{bmatrix}
\]

Now, the first row of \( Y(z) \) is \( z^{\frac{k-1}{2}} \Phi(z) \)

\( \square \)

The Stokes' matrix \( S \) has entries

\[ s_{ii} = 1 \]

\[ s_{ij} = 0 \quad \text{if } R_{ij} \subset \Pi_R \]

This follows from the fact that on the overlapping region \( 0 < \arg z < \frac{\pi}{k} \) there are no Stokes' rays and

\[ e^{zU} S e^{-zU} \sim I, \quad z \to \infty, \quad \text{then } e^{z(u_i - u_j)} s_{ij} \to \delta_{ij} \]

Moreover, \( \Re(z(u_i - u_j)) > 0 \) to the left of the ray \( R_{ij} \), while \( \Re(z(u_i - u_j)) < 0 \) to the right (the natural orientation on \( R_{ij} \), from \( z = 0 \) to \( \infty \) is understood). This implies

\[ |e^{zu_i}| > |e^{zu_j}| \quad \text{and } e^{z(u_i - u_j)} \to \infty \quad \text{as } z \to \infty \]

on the left, while on the right

\[ |e^{zu_i}| < |e^{zu_j}| \quad \text{and } e^{z(u_i - u_j)} \to 0 \quad \text{as } z \to \infty \]

With this observations in mind, we prove the following

**Lemma 4 :** \( S \) has a column whose entries are all zero but one. More precisely:

For \( k \) even

\[ s_{i, \frac{k}{2} + 1} = 0 \quad \forall i \neq \frac{k}{2} + 1, \quad s_{\frac{k}{2} + 1, \frac{k}{2} + 1} = 1 \]

For \( k \) odd

\[ s_{i, \frac{k+1}{2}} = 0 \quad \forall i \neq \frac{k+1}{2}, \quad s_{\frac{k+1}{2}, \frac{k+1}{2}} = 1 \]

**Proof:** Let us determine \( n \) such that \( s_{in} = 0 \) for any \( i \neq n \) and \( s_{mn} = 1 \). We need to find all rays in \( \Pi_R \). We start with \( R_{rs} \) with \( r < s \). We know that for \( r + s = k + 2 \) the ray is the negative real half-line (at angle \( -\pi \)). Then \( R_{rs} \subset \Pi_R \) for \( r + s \leq k + 1 \) (\( r < s \)). Then, in \( \Pi_R \) we have

\[
\begin{align*}
R_{12} & \quad R_{13} & \quad \ldots & \quad R_{1k} \\
R_{23} & \quad R_{24} & \quad \ldots & \quad R_{2,k-1} \\
R_{34} & \quad R_{35} & \quad \ldots & \quad R_{3,k-2} \\
& \quad \vdots & \quad \vdots & \quad \vdots \\
& \quad R_{ab} \\
\end{align*}
\]

where \( R_{ab} = R_{\frac{a}{2} + 1} \) for \( k \) even, and \( R_{\frac{a+1}{2} + 1} \) for \( k \) odd. In \( \Pi_R \) we have also \( R_{rs} \) with \( r + s = k + 2 \) and \( r > s \). For fixed \( n \) we require \( R_{in} \subset \Pi_R \) for any \( i \). Namely,

\[
\forall i < n \quad i + n \leq k + 1, \quad \forall i > n \quad i + n \geq k + 2
\]
This yields \( n = \frac{k}{2} + 1 \) for \( k \) even, \( n = \frac{k+1}{2} \) for \( k \) odd.

Let \( n(k) \) be \( \frac{k}{2} + 1 \), or \( \frac{k+1}{2} \). Lemma 4 implies that the \( n(k) \) th columns of \( Y_L \) and \( Y_R \) coincide. In particular, their asymptotic representation holds for \(-\pi < \arg z < \pi + \frac{\pi}{k}\). Actually, this domain can be further enlarged, up to
\[
\begin{align*}
-\frac{\pi}{k} < \arg z < \pi + \frac{\pi}{k} & \quad \text{k even} \\
-\pi < \arg z < \frac{2\pi}{k} & \quad \text{k odd}
\end{align*}
\]

To see this recall that \(|e^{z+1}| < |e^{zu}|\) on the right of \( R_{ij} \), and conversely on the left. Then it is easy to see that for \( k \) even \( |\exp(z u_{\frac{k}{2}+1})|\) dominates all exponentials in the sector \(-\frac{\pi}{k} - \pi < \arg z < \frac{\pi}{k} - \pi\), while for \( k \) odd \( |\exp(z u_{\frac{k+1}{2}+1})|\) dominates all exponentials in the sector \( \pi < \arg z < \pi + \frac{2\pi}{k}\).

The first entry of the \( n(k) \) th column is \( \varphi_L^{(n(k))}(z) = \varphi_R^{(n(k))}(z) \) times \( z^{\frac{k-1}{2}} \). Then \( \varphi^{(n(k))} \) has the established asymptotic behaviour on the enlarged domains above.

We now introduce an integral representation for a solution \( \varphi(z) \) of the generalized hypergeometric equation which will allow us to compute the entries of \( S \).

**Lemma 5:** The function
\[
g^{(n)}(z) = \frac{1}{(2\pi)^{\frac{c+1}{2}}} e^{i\frac{\pi}{k}(n-1)} \int_{-c-i\infty}^{-c+i\infty} ds \, \Gamma^k(-s) \, e^{-i\pi ks} \, e^{i2(n-1)\pi s} \, z^s
\]
defined for \( \frac{\pi}{k} - 2(n - 1)\pi < \arg z < \frac{\pi}{2} - 2(n - 1)\pi \), \( z \neq 0 \) and for any positive number \( c > 0 \), is a solution of the generalized hypergeometric equation (17) (the path of integration is a vertical line through \(-c\)). It has asymptotic behaviour
\[
g^{(n)}(z) \sim \frac{1}{\sqrt{k}} \frac{e^{i\frac{\pi}{k}(n-1)}}{z^{\frac{k-1}{2}}} \exp\left(ke^{i\frac{\pi}{2}(n-1)}z\right) \quad z \to \infty
\]

In particular, for \( n(k) = \frac{k}{2} + 1 \) (\( k \) even), or \( n(k) = \frac{k+1}{2} \) (\( k \) odd), the analytic continuation of \( g^{(n(k))}(z) \) has the above asymptotic behaviour in the domains
\[
-\pi - \frac{\pi}{k} < \arg z < \pi + \frac{\pi}{k} \quad \text{k even}
\]
\[
-\pi < \arg z < \frac{2\pi}{k} \quad \text{k odd}
\]
and coincides with the solution \( \varphi_L^{(n(k))} = \varphi_R^{(n(k))} \) appearing in the first rows of the fundamental matrices \( Y_L \) and \( Y_R \) of the system (17).

The following identity holds
\[
\sum_{m=0}^{k} (-1)^{m-k} \binom{k}{m} g^{(n)}(z \, e^{i\frac{\pi}{k}m}) = 0 \quad (16)
\]

We’ll sketch the proof in Appendix 1.

**Remark 2:** Observe that for basic solutions of the hypergeometric equation \( \Phi_{L/R} = \{\varphi^{(1)}_{L/R}, ..., \varphi^{(k)}_{L/R}\} \),
\[
\varphi^{(n)}(z) \sim \frac{1}{\sqrt{k}} \frac{e^{i\frac{\pi}{k}(n-1)}}{z^{\frac{k-1}{2}}} \exp(ke^{i\frac{\pi}{2}(n-1)}z)
\]
on some sector, and
\[
\varphi^{(n)}(ze^{\frac{2\pi}{k}i}) \sim (-1) \frac{1}{\sqrt{k}} \frac{e^{i\frac{\pi}{k}(n+1-1)}}{z^{\frac{k-1}{2}}} \exp(ke^{i\frac{\pi}{2}(n+1-1)}z)
\]
like \(-\varphi^{(n+1)}\), on the sector rotated by \(-\frac{2\pi}{k}\). Note however that \( \varphi^{(k)}(ze^{\frac{2\pi}{k}i}) \sim (\sqrt{k} \, ze^{\frac{k-1}{2}})^{-1} \, e^{kz} \), like \( \varphi^{(1)}(z) \).

Also, note that \( g^{(n)}(ze^{\frac{2\pi}{k}i}) = -g^{(n+1)}(z) \) (we mean analytic continuations), with asymptotic behaviour on rotated domain.
4 Monodromy Data of the Quantum Cohomology of $\mathbb{CP}^{k-1}$

Let us return to the system (2):

$$\partial_z y = \left( \hat{U}(t^2) + \frac{1}{z^\mu} \right) y \quad (17)$$

In this section we use the original notation $w = z e^{\frac{2\pi i}{k}}$. The system has a fundamental matrix

$$y_0(t^2, z) = (I + A_1(t^2)z + A_2(t^2)z^2 + \ldots) z^\mu z^R, \quad z \to 0$$

where $R$ is the same of system (11). The series appearing in the solutions converges near $z = 0$. The matrix $\hat{U}(t^2)$ has eigenvalues and eigenvectors

$$u_n(t^2) = e^{i\frac{2\pi n(n-1)}{k}} e^{\frac{t^2}{2} \frac{k}{2}} \equiv u_n e^{\frac{t^2}{2} \frac{k}{2}} \quad n = 1, \ldots, k$$

$$x_n(t^2) \text{ of entries } x^j_n(t^2) = e^{i(2j-1)(n-1)\frac{2\pi}{k}} e^{\frac{2j^2+n^2}{k}} e^{z^2} \equiv x^j_n e^{\frac{2j^2+n^2}{k}}$$

Let $X(t^2) = (x^j_n(t^2))$. With the gauge $y = X(t^2) \hat{y}(t^2, z)$ we obtain the equivalent system

$$\partial_z \hat{y} = \left[ U(t^2) + \frac{V(t^2)}{z} \right] \hat{y} \quad (18)$$

$$U(t^2) = X^{-1}(t^2) \hat{U}(t^2) X(t^2) = \text{diag}(u_1(t^2), \ldots, u_k(t^2))$$

$$V(t^2) = X^{-1}(t^2) \hat{\mu} X(t^2))$$

$$V(t^2)^T + V(t^2) = 0$$

The skew symmetry of $V$ follows from $\eta \hat{\mu} + \hat{\mu} \eta = 0$ and from the choice of the normalization of the eigenvectors, such that $X(t^2)^T X(t^2) = \eta^{-1}$.

Let us fix an initial point $t_0 = (0, t_0^2, 0, \ldots, 0)$. The system (18) has fundamental matrices $y_R(t_0^2, z)$, $y_L(t_0^2, z)$, which are asymptotic to the formal solution

$$\hat{y}_F(t_0^2, z) = \left[ I + \frac{F_1(t_0^2)}{z} + \frac{F_2(t_0^2)}{z^2} + \ldots \right] e^{z \ U(t_0^2)}$$

in the sectors

$$S_L(t_0) = \{ z \in \mathbb{C} \ | \ 0 < \arg \left[ z \exp \left( \frac{t_0^2}{k} \right) \right] < \pi + \frac{\pi}{k} \}$$

$$S_R(t_0) = \{ z \in \mathbb{C} \ | \ -\pi < \arg \left[ z \exp \left( \frac{t_0^2}{k} \right) \right] < \frac{\pi}{k} \}$$

and

$$\hat{y}_L(t_0^2, z) = y_R(t_0^2, z) S \quad 0 < \arg \left[ z \exp \left( \frac{t_0^2}{k} \right) \right] < \frac{\pi}{k}$$

with respect to the admissible line

$$l_{t_0} := \{ z \ | \ z = \rho \exp \left( i \epsilon - \frac{\Im m t_0^2}{k} \right), \ \rho > 0 \}$$

The Stokes’ matrix is precisely the matrix $S$ of system (11) with respect to the admissible line $l_{t_0}$. Also the central connection matrix defined by

$$y_0(t_0^2, z) = y_R(t_0^2, z) C \quad -\pi < \arg \left[ ze^{\frac{t_0^2}{k}} \right] < \frac{\pi}{k}$$

is the same of the system (11).

Definition: $C$ and $S$, together with $\hat{\mu}$, $R$, and $e = \frac{\partial}{\partial R}$ are the monodromy data of the quantum cohomology of $\mathbb{CP}^{k-1}$ in a local chart containing $t_0$. 

Recall that we fixed a point \( t_0 = (0, t_0^2, ..., 0) \). When we consider a point \( t \) away from \( t_0 \), the system \([17]\) acquires the general form

\[
\partial z y = \left[ \hat{U}(t) + \hat{\mu} \right] y
\]  

(19)

where \( \hat{U}(t^1, ..., t^k) = \eta \ U(t) \ \eta^{-1} \) and \( y^{(j)}_{\alpha}(t^1, ..., t^k; z) = \partial_{\alpha} \tilde{t}^{(j)}(t, z) \).

The admissible line \( l_{t_0} \) must be considered fixed once and for all. Instead, the Stokes’ rays change. This is because they are functions of the eigenvalues \( u_1(t), ..., u_k(t) \) of the matrix \( \hat{U}(t^1, ..., t^k) \). For example, if just \( t^2 \) varies, while \( t^1 = t^3 = ... = t^k = 0 \), the system \([17]\) has Stokes’ rays

\[
R_{rs}(t^2) = \{ z \mid = \rho \ \exp \left( \frac{2\pi i}{k} - i \frac{\pi}{k} (r + s) - i \frac{3m}{k} t^2 \right), \ \rho > 0 \}
\]

The dependence of the coefficients of the system \([19]\) on \( t \) is isomonodromic \([9]\) \([10]\). Then \( \mu \) and \( R \) are the same for any \( t \). \( S \) and \( C \) do not change if we move in a sufficiently small neighbourhood of \( t_0 \). Problems arise when some Stokes’ rays cross \( l_{t_0} \). \( S \) and \( C \) must be modified by an action of the braid group. We will return to this point later.

5 Computation of \( S \)

To compute \( S \), we factorize it in “Stokes’ factors”. Our fundamental matrix \( Y_L \) has the required asymptotic form on the sector between \( R_{k2} \) (arg \( z = 0 \)) and \( R_{1k} \) (arg \( z = \pi + \frac{\pi}{k} \)). \( Y_R \) has the same behaviour between \( R_{2k} \) (arg \( z = -\pi \)) and \( R_{k1} \) (arg \( z = \frac{2\pi}{k} \)).

Of course, we can consider fundamental matrices with the same asymptotic behaviour on other sectors of angular width less than \( \pi + \frac{\pi}{k} \) and bounded by two Stokes’ rays. We introduce the following notation: consider a fundamental matrix of \([11]\) having the required asymptotic behaviour on such a sector. If we go all over the sector clockwise we meet Stokes rays belonging to the sector at each displacement of \( \frac{\pi}{k} \). Let \( R_{ij} \) be the last ray we meet before reaching the boundary (the boundary is still a Stokes ray not belonging to the sector). Then we will call the fundamental matrix \( Y_{ij} \). For example, \( Y_L = Y_{k1} \) and \( Y_R = Y_{1k} \). See figure 2.

Sometimes, a different labelling is used in the literature. The rays must be enumerated as in figure 3. The numeration refers to the line \( l \) : the rays are labelled in counter-clockwise order starting from the first one in \( \Pi_R \) (which will be \( R_0 \); then \( R_0, R_1, ..., R_{k-1} \) are in \( \Pi_R \), and \( R_k, ..., R_{2k-1} \) are in \( \Pi_L \)). For our particular choice of \( l \), \( R_0 \equiv R_{1k} \) (at arg \( z = -\pi + \frac{\pi}{k} \)); \( R_1 \equiv R_{1,k-1} \) follows counter-clockwise... Then we proceed until we reach \( R_{k-1} \equiv R_{k2} \) before crossing \( l \), and so on. The fundamental matrices
are labelled as we prescribed above, namely $Y_j$ if its sector contains $R_j$ as the last ray met going all over the sector clockwise before the boundary. The sector itself is denoted by $S_j$. See figure 3.

We define Stokes’ factors to be the connection matrices $K_j$ such that

$$Y_{j+1}(z) = Y_j(z)K_j$$

on the overlapping region of width $\pi$. We warn the reader that also the Stokes’ factors will be labelled with both conventions above, according to our convenience (for example $K_0 \equiv K_{1k}$).

As a consequence of the above definitions, we can factorize $S$ as follows

$$Y_L = Y_{k1} = Y_{k2}K_{k2} = Y_{k3}K_{k3}K_{k2} = ...$$

$$... = Y_{1k}K_{1k}K_{1,k-1}K_{k,k-1}K_{k,k-2}...K_{k3}K_{k2} \equiv Y_RS$$

Then

$$S = K_{1k}K_{1,k-1}K_{k,k-1}K_{k,k-2}...K_{k3}K_{k2}$$

(20)

We observe that, being the first row of $Y(z)$ equat to $z^{\frac{k-1}{2}}\Phi(z)$, the following holds:

$$\Phi_{j+1}(z) = \Phi_j K_j$$

**Remark 3:** The Stokes’ factors of the system (11) and of the gauge-equivalent system (14) are the same. From the skew-symmetry of $V$ it follows that $K_{ji} = (K_{ij}^{-1})^T$.

Before computing the Stokes factors explicitly, we show that just two of them are enough to compute all the others. Let

$$F(z) := \left( \frac{1}{\sqrt{k}} \frac{1}{z^{\frac{k-1}{2}}} \exp(kz), \frac{1}{\sqrt{k}} \frac{e^{i\pi}}{z^{\frac{k-1}{2}}} \exp(k e^{2\pi i} z), ..., \frac{1}{\sqrt{k}} \frac{e^{i\pi(k-1)}}{z^{\frac{k-1}{2}}} \exp(k e^{2\pi i(k-1)} z) \right)$$

be the row vector whose entries are the first terms of the asymptotic expansions of an actual solution $\Phi(z)$ of the generalized hypergeometric equation. By a straightforward computation we see that

$$F(z e^{2\pi i}) = F(z)T_F$$

$$T_F = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ -1 & -1 & \ddots \\ \vdots & \ddots & \ddots & \ddots \\ -1 & 0 & \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$
We use now the convention of enumeration of Stokes’ rays $R_0, R_1, ...$ starting from $l$ (see above). Let \( \Phi_m(z) \) be an actual solution of the hypergeometric equation having asymptotic behaviour $F(z)$ on $S_m$. 

\[
\Phi_m(z) \sim F(z), \quad z \to \infty, \quad z \in S_m
\]

Then

\[
\Phi_{m+2}(ze^{\pm i\pi}) \sim F(z)T_F, \quad z \in S_m
\]

Namely,

\[
\Phi_{m+2}(ze^{\pm i\pi})T_F^{-1} \sim F(z), \quad z \in S_m
\]

then, by unicity of actual solutions having asymptotic behaviour $F(z)$ in a sector wider then $\pi$, we have

\[
\Phi_{m+2}(ze^{\pm i\pi}) = \Phi_m(z)T_F, \quad z \in S_m
\]

**Lemma 6:** For any $p \in \mathbb{Z}$

\[
K_{m+2p} = T_F^{-p}K_mT_F^p
\]

**Proof:** For $z \in S_m \cap S_{m+1}$

\[
\Phi_{m+1}(z) = \Phi_m(z)K_m = \Phi_{m+2}(ze^{\pm i\pi})T_F^{-1}K_m = \Phi_{m+3}(ze^{\pm i\pi})K_m^{-1}T_F^{-1}K_m = \Phi_{m+1}(z)T_F^{-1}K_{m+2}^{-1}T_F^{-1}K_m
\]

Then $K_{m+2} = T_F^{-1}K_mT_F$. By induction we prove the lemma. 

\[\Box\]

From the lemma it follows that just two Stokes’ factors are enough to compute all the others. We are ready to give a concise formula for $S$:

**Theorem 1:** Let $l$ be an admissible line (i.e. not containing Stokes’ rays), and let us enumerate the rays in counter-clockwise order starting from the first one in $\Pi_R$ (which will be $R_0$, then $R_0$, $R_1$, ..., $R_{k-1}$ are in $\Pi_R$, and $R_k$, ..., $R_{2k-1}$ are in $\Pi_L$). Then the Stokes’ matrix for $[1.1], [1.4], [1.3]$ and $k > 3$ is

\[
S = \begin{cases} 
(0 K_1 T_f^{-1})^k T_F^k & \text{k even} \\
(0 K_1 T_f^{-1})^{k-1} K_0 T_{f-1} \equiv T_F^{-1} (T_F^{-1}K_{k-2}K_{k-1})^{\frac{k}{2}}, & \text{k odd}
\end{cases}
\]

**Proof:**

\[
S = K_0K_1K_2...K_{k-1}, K_{2p} = T_F^{-p}K_0T_F^p \text{ and } K_{2p+1} = T_F^{-p}K_1T_F^p. \text{ Then }
\]

\[
S = K_0K_1(T_F^{-1}K_0T_F)(T_F^{-1}K_1T_F)K_4K_5...K_{k-1}
\]

\[
= (K_0K_1T_F^{-1})K_0K_1T_FK_4K_5...K_{k-1}
\]

\[
= (K_0K_1T_F^{-1})K_0K_1T_F(T_F^{-2}K_0T_F^2)(T_F^{-2}K_1T_F^2)K_6...K_{k-1}
\]

\[
= (K_0K_1T_F^{-1})(K_0K_1T_F^{-1})K_0K_1T_F^2K_6...K_{k-1}
\]

Now observe that $K_{k-1} = T_F^{-\left(\frac{k-1}{2}\right)}K_1T_F^{-1}$ for $k$ even, while $K_{k-1} = T_F^{-\left(\frac{k-1}{2}\right)}K_0T_F^{\frac{k+1}{2}}$ for $k$ odd. Then, for $k$ even

\[
S = (K_0K_1T_F^{-1})^{\frac{k}{2}-2} K_0K_1T_F^{\frac{k}{2}-2} K_{k-2}K_{k-1}
\]

\[
= (K_0K_1T_F^{-1})^{\frac{k}{2}-2} K_0K_1T_F^{\frac{k}{2}-2} T_F^{-\left(\frac{k}{2}-1\right)}K_0T_F^{\frac{k}{2}-1} = \left(K_0K_1T_F^{-1}\right)^{\frac{k}{2}} T_F^\frac{k}{2}
\]

For $k$ odd:

\[
S = (K_0K_1T_F^{-1})^{\frac{k+1}{2}} K_0K_1T_F^{\frac{k+1}{2}} K_{k-1}
\]

\[
= (K_0K_1T_F^{-1})^{\frac{k+1}{2}} K_0K_1T_F^{\frac{k+1}{2}} T_F^{-\left(\frac{k+1}{2}\right)}K_0T_F^{\frac{k+1}{2}} = \left(K_0K_1T_F^{-1}\right)^{\frac{k+1}{2}} K_0T_F^{\frac{k+1}{2}}
\]
If instead we write the Stokes’ factors in term of $K_{k-2}$ and $K_{k-1}$ we obtain the other two formulas in the same way.

\[ S = T_F K_{32} (T_F^{-1} K_{12} K_{32}) = K_{13} K_{12} K_{32} \]

**Remark 4:** For our particular choice of $l$, $K_0 \equiv K_{1k}$, $K_1 \equiv K_{1,k-1}$, $K_{k-2} \equiv K_{k3}$ and $K_{k-1} \equiv K_{k2}$. For $k = 3$

\[ S = T_F K_{32} (T_F^{-1} K_{12} K_{32}) = K_{13} K_{12} K_{32} \]

It is now worth deriving some properties of the monodromy of $Y(z)$ (for (11)) and $\Phi(z)$ (for (13)), which will be useful later. Consider $\Phi_m(z)$ with asymptotic behaviour $F(z)$ on $S_m$. Then

\[ \Phi_m(z) = \Phi_{m-2}(z) K_{m-2} K_{m-1} \equiv \Phi_m(z e^{\pm \pi i}) T_F^{-1} K_{m-2} K_{m-1} \]

On the other hand

\[ \Phi_m(z) = \Phi_{m+2}(z) K_{m+1} K_{m} \equiv \Phi_m(z e^{-\pm \pi i}) T_F K_{m+1} K_{m} \]

This proves the following

**Lemma 7:** The basic solution $\Phi_m(z)$ of the generalized hypergeometric equation (13) with asymptotic behaviour $F(z)$ on $S_m$, satisfies the identity

\[ \Phi_m(z e^{\pm \pi i}) = \Phi_m(z) T_m \]

where

\[ T_m := K_{m-1}^{-1} K_{m-2}^{-1} T_F = T_F K_{m+1}^{-1} K_{m}^{-1} \]

**Corollary 1:** The monodromy (at $z = 0$) of $\Phi_m(z)$ is

\[ \Phi_m(z e^{2\pi i}) = \Phi_m(z) (T_m)^k \]

Now, for our particular choice of the line $l$ and for $m = k$, $\Phi_m(z) = \Phi_L(z)$. For the solution $Y_L(z)$ of (11), the relations $Y_R(z) = Y_L(z) S^{-1}$ ($0 < \arg z < \frac{\pi}{2}$), $Y_L(z) = Y_R(z e^{-2\pi i}) S^T$ ($\pi < \arg z < \frac{\pi}{2}$) immediately imply

\[ Y_L(z e^{2\pi i}) = Y_L(z) S^{-1} S^T \]

Recall that the $(n,j)$-th entry of $Y(z)$ is $Y_{n,j}(z) \equiv Y^{(j)}(z) = \frac{1}{\varphi(j)} z^{\frac{1}{2} - n} (z \partial_z)^{(n-1)} \varphi(j)(z)$, and observe that $(z e^{2\pi i})^{\frac{1}{2} - n} = (-1)^{k-1} z^{\frac{k}{2} - n}$. Then, from Corollary 1 we get the following:

**Corollary 2:** Let $T$ be the $k$-monodromy matrix of $\Phi_L$ (namely, $T \equiv T_k$ for our choiche of $l$). Then

\[ T^k = (-1)^{k-1} S^{-1} S^T \]

Our formula (21) allows us to easily compute $S$. The recipe is simply to take $K_{k3}$, $K_{k2}$ (which we are going to compute explicitly) and substitute them into

\[ S = T_F^\frac{k}{2} (T_F^{-1} K_{k3} K_{k2})^\frac{1}{2} = T_F^\frac{k}{2} T^{-\frac{k}{2}} \] (22)

for $k$ even, or into

\[ S = T_F^\frac{k+1}{2} K_{k2} (T_F^{-1} K_{k3} K_{k2})^\frac{k+1}{2} = T_F^\frac{k+1}{2} K_{k2} T^{-\frac{k+1}{2}} \] (23)

for $k$ odd.

**Computation of Stokes’ factors:** We need to distinguish between $k$ odd and even. In the following $g(z)$ will mean $g^{(n(k))}(z)$ ($n(k) = \frac{k}{2} + 1$ or $\frac{k+1}{2}$ for $k$ even or odd respectively).

For $k$ odd:

\[ g(z) = \varphi_L^{(\frac{k+1}{2})}(z) \equiv \varphi_R^{(\frac{k+1}{2})}(z) \]
with asymptotic behaviour on
\[ -\pi < \arg z < \pi + \frac{2\pi}{k} \]
If we iterate the map \( z \mapsto ze^{\frac{2\pi i}{k}} \) for \( m = 1, 2, \ldots, \frac{k-1}{k} \) times, the domain of \( g(ze^{\frac{2\pi i}{k}}) \) for each \( m \) covers \( S_R \). When we reach \( m = \frac{k-1}{k} \) a new iteration (i.e. a new rotation of the domain of \(-\frac{2\pi}{k}\)) will live the sector \(-\frac{\pi}{k} < \arg z < \frac{\pi}{k}\) of \( S_R \) uncovered. The same, if we do \( z \mapsto ze^{-\frac{2\pi i}{k}} \) the sector \(-\pi < \arg z < -\pi + \frac{2\pi}{k}\) of \( S_R \) remains uncovered. Then, by remark 2:
\[
\Phi_{1k}(z) = \Phi_R(z) = \left( (-1)^{\frac{k-1}{2}} g(ze^{i\frac{k-1}{2}}, \ldots, \frac{k-3}{2} \text{ unknown terms} \ldots), \right.
\]
\[
g(z), -g(ze^{\frac{2\pi i}{k}}), \ldots, (-1)^{\frac{k-1}{2}} g(ze^{-i\frac{k-1}{2}}) \right)
\]
In the same way we see that
\[
\Phi_{k1}(z) = \Phi_L(z) = \left( (-1)^{\frac{k-1}{2}} g(ze^{-i\frac{k-1}{2}}), \ldots, (-1)^{\frac{k-1}{2}} g(ze^{i\frac{k-1}{2}}) \right)
\]
and similar expressions for \( \Phi_{1,k-1}, \Phi_{k,k-1}, \Phi_{k,k-2}, \ldots, \Phi_{k3,\Phi_{k2}} \). The unknown terms are computed using the identity (14) and simple considerations on the dominant exponentials \(|e^{g(z)}|\) on the sectors which remain uncovered in the iterations of \( z \mapsto ze^{\pm i\frac{\pi}{k}} \).

**Example:** A simple example will clarify this procedure. Let \( k = 7 \); then \( g = \varphi^{(4)} \),
\[
\Phi_{17} = \Phi_R = (-g(ze^{\frac{2\pi i}{7}}), \varphi^1, \varphi^2, g(z), -g(ze^{\frac{4\pi i}{7}}), g(ze^{\frac{2\pi i}{7}}), -g(ze^{\frac{6\pi i}{7}})).
\]
We look for \( \varphi^{(3)}_R \). If we take \((-1)g(ze^{-\frac{2\pi i}{7}}) \) we miss to cover \(-\pi < \arg z < -\pi + \frac{2\pi}{7}\) in \( S_R \). On \(-\pi < \arg z < -\pi + \frac{2\pi}{7}\) between two nearby Stokes’ rays we have the relations \(|e^{2\pi i R_4}| > |e^{2\pi i 5}| > |e^{2\pi i 3}| \) (later on we will simply write \( 4 > 3 \)). Then
\[
\varphi^{(3)}_R(z) = (-1)g(ze^{-\frac{2\pi i}{7}}) + c_4 \varphi^{(4)}_R(z) + c_5 \varphi^{(5)}_R(z)
\]
To find \( c_4, c_5 \) we need another representation for \( \varphi^{(3)}_R \). We consider \((-1)g(ze^{\frac{2\pi i}{7}}) \), has the correct asymptotic behaviour, but on a domain which leaves uncovered \(-\frac{2\pi}{7} < \arg z < -\frac{2\pi}{7} \). The relations are: on \( 0 < \arg z < \frac{2\pi}{7}, 1 > 7 > 2 > 6 > 3 \); on \(-\frac{2\pi}{7} < \arg z < 0, 1 > 2 > 7 ; on \(-\frac{2\pi}{7} < \arg z < -\frac{2\pi}{7}, 2 > 1 > 3 \); on \(-\frac{2\pi}{7} < \arg z < -\frac{2\pi}{7}, 2 > 3 \). Then
\[
\varphi^{(3)}_R(z) = (-1)g(ze^{\frac{2\pi i}{7}}) + d_1 \varphi^{(1)}_R(z) + d_2 \varphi^{(2)}_R(z) + d_6 \varphi^{(6)}_R(z) + d_7 \varphi^{(7)}_R(z)
\]
In the same way one finds
\[
\varphi^{(2)}_R(z) = \begin{cases} g(ze^{-\frac{2\pi i}{7}}) + a_3 \varphi^{(3)}_R(z) + a_4 \varphi^{(4)}_R(z) + a_5 \varphi^{(5)}_R(z) + a_6 \varphi^{(6)}_R(z) \\ g(ze^{\frac{4\pi i}{7}}) + b_1 \varphi^{(1)}_R(z) + b_2 \varphi^{(2)}_R(z) \end{cases}
\]
\( \varphi^{(i)} \)'s are known for \( i = 1, 4, 5, 6, 7 \). Using the identity (14) we compute \( a, b, c, d \). We get
\[
\varphi^{(2)}_R(z) = g(ze^{-\frac{2\pi i}{7}}) - \left( \left( \frac{7}{1} \right) g(ze^{\frac{2\pi i}{7}}) + \left( \frac{7}{2} \right) g(z) - \left( \frac{7}{3} \right) g(ze^{\frac{4\pi i}{7}}) + \left( \frac{7}{4} \right) g(ze^{\frac{6\pi i}{7}}) \right)
\]
\[
\varphi^{(3)}_R(z) = -g(ze^{-\frac{2\pi i}{7}}) + \left( \left( \frac{7}{1} \right) g(z) - \left( \frac{7}{2} \right) g(ze^{\frac{2\pi i}{7}}) \right)
\]
A similar computation gives \( \Phi_{71} = \Phi_L \).
\[
\Phi_{71} = \begin{bmatrix} -g(ze^{-\frac{2\pi i}{7}}) \\
-g(ze^{\frac{4\pi i}{7}}) \\
g(z) \\
-g(ze^{\frac{6\pi i}{7}}) + a \left( \frac{7}{6} \right) g(z) \\
g(ze^{\frac{4\pi i}{7}}) - b \left( \frac{7}{6} \right) g(ze^{\frac{2\pi i}{7}}) + c \left( \frac{7}{5} \right) g(z) - d \left( \frac{7}{4} \right) g(ze^{\frac{4\pi i}{7}}) \\
-g(ze^{\frac{6\pi i}{7}}) + e \left( \frac{7}{6} \right) g(ze^{\frac{2\pi i}{7}}) + f \left( \frac{7}{5} \right) g(z) - g \left( \frac{7}{4} \right) g(ze^{\frac{4\pi i}{7}}) + h \left( \frac{7}{2} \right) g(ze^{\frac{6\pi i}{7}}) \end{bmatrix}
\]

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and

$$\Phi_{72} = \begin{bmatrix}
-g(ze^{-\frac{7\pi i}{7}}) & g(ze^{-\frac{4\pi i}{7}}) & -g(ze^{-\frac{2\pi i}{7}}) & g(z) & -g(ze^\frac{2\pi i}{7}) \\
-g(ze^{-\frac{6\pi i}{7}}) & g(ze^{-\frac{3\pi i}{7}}) & -g(ze^{-\frac{1\pi i}{7}}) & g(z) & -g(ze^\frac{1\pi i}{7}) \\
g(ze^{-\frac{5\pi i}{7}}) - \left( \frac{7}{6} \right) g(ze^{-\frac{2\pi i}{7}}) + \left( \frac{7}{5} \right) g(z) & g(ze^{-\frac{2\pi i}{7}}) - \left( \frac{7}{6} \right) g(ze^{-\frac{1\pi i}{7}}) + \left( \frac{7}{5} \right) g(z) & -g(ze^\frac{1\pi i}{7}) + g(ze^\frac{2\pi i}{7})
\end{bmatrix}$$

Notice that in each of the last three entries of $\Phi_{72}$ there is a term missing w.r.t the corresponding entries of $\Phi_{71}$. This immediately implies

$$K_{72} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & \left( \frac{k}{2} \right)
0 & 1 & 0 & 0 & 0 & \left( \frac{k}{4} \right) & \left( \frac{k}{8} \right)
0 & 0 & 1 & 0 & 0 & \left( \frac{k}{8} \right) & \left( \frac{k}{16} \right)
0 & 0 & 0 & 1 & 0 & \left( \frac{k}{16} \right) & \left( \frac{k}{32} \right)
0 & 0 & 0 & 0 & 1 & \left( \frac{k}{32} \right) & \left( \frac{k}{64} \right)
0 & 0 & 0 & 0 & 0 & 1 & \left( \frac{k}{64} \right)
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

The next step is the computation of $\Phi_{73}$ and $K_{73}$, through $\Phi_{72} = \Phi_{73}K_{73}$. It is done in the same way...

The above procedure is extended to the general case. In Appendix 2 we give, for example, the general expressions of $\Phi_R$ and $\Phi_L$. The factors of interest are:

**k odd:**

$$(K_{k2})_{j, k-j+2} = \left( \frac{k}{2(j-1)} \right)$$ for $j = 2, \ldots, \frac{k+1}{2}$. $(K_{k2})_{j,j} = 1$ for $j = 1, \ldots, k$. All the other entries are zero.

$$(K_{k3})_{2,1} = - \left( \frac{k}{1} \right); \ (K_{k3})_{j, k-j+3} = \left( \frac{k}{2j-3} \right)$$ for $j = 3, \ldots, \frac{k+1}{2}$. $(K_{k3})_{j,j} = 1$ for $j = 1, \ldots, k$. All the other entries are zero. Namely:

$$K_{k2} = \begin{bmatrix}
1 & 1 & \left( \frac{k}{4} \right) & \left( \frac{k}{8} \right)
1 & 1 & \left( \frac{k}{8} \right) & \left( \frac{k}{16} \right)
1 & \left( \frac{k}{8} \right) & \left( \frac{k}{16} \right) & \left( \frac{k}{32} \right)
1 & \left( \frac{k}{16} \right) & \left( \frac{k}{32} \right) & \left( \frac{k}{64} \right)
1 & \left( \frac{k}{32} \right) & \left( \frac{k}{64} \right)
1 & \left( \frac{k}{64} \right)
1 & \left( \frac{k}{64} \right)
1 & \left( \frac{k}{64} \right)
\end{bmatrix}$$

$$K_{k3} = \begin{bmatrix}
1 & 0 & \left( \frac{k}{k-2} \right) & \left( \frac{k}{k-4} \right)
1 & 0 & \left( \frac{k}{k-4} \right) & \left( \frac{k}{k-8} \right)
1 & 0 & \left( \frac{k}{k-8} \right) & \left( \frac{k}{k-16} \right)
1 & 0 & \left( \frac{k}{k-16} \right) & \left( \frac{k}{k-32} \right)
1 & 0 & \left( \frac{k}{k-32} \right)
1 & 0
1 & 0
1 & 0
1 & 0
\end{bmatrix}$$

**k even:**
\((K_{k2})_{j, k-j+2} = \binom{k}{2(j-1)}\) for \(j = 2, \ldots, \frac{k}{2}\). \((K_{k2})_{j,j} = 1\) for \(j = 1, \ldots, k\). All the other entries are zero.

\((K_{k3})_{2,1} = -\binom{k}{1}\); \((K_{k3})_{j, k-j+3} = \binom{k}{2j-3}\), for \(j = 3, \ldots, \frac{k}{2}+1\). \((K_{k3})_{j,j} = 1\) for \(j = 1, \ldots, k\). All the other entries are zero. Namely

\[
K_{k2} = \begin{pmatrix}
1 & & & & & & 0 \\
1 & 1 & & & & & \\
1 & 0 & (k) & & & & \\
1 & (k-2) & (k-2) & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & &
\end{pmatrix}
\]

\[
K_{k3} = \begin{pmatrix}
1 & & & & & & 0 \\
1 & 1 & & & & & \\
1 & 0 & (k) & & & & \\
1 & (k-4) & (k-2) & & & & \\
1 & (k-4) & (k-2) & & & & \\
& & & & & & \\
& & & & & & \\
& & & & & &
\end{pmatrix}
\]

6 Reduction of \(S\) to “Canonical” Form

Some examples of computations of \(S\) are in Appendix 2. The reader may observe that \(S\) is not in a nice upper triangular form (see also Lemma 4) and quite strange numbers (complicated combinations of sum and products of binomial coefficients) appear.

Some natural operations are allowed on the Stokes matrices of a Frobenius manifold:

a) Permutations. Let us consider the system

\[
\frac{d\tilde{Y}}{dz} = \left[ U + \frac{1}{z} V \right] \tilde{Y}
\]

where \(U = \text{diag}(u_1, u_2, \ldots, u_k)\). Let \(\sigma: (1, 2, \ldots, k) \mapsto (\sigma(1), \sigma(2), \ldots, \sigma(k))\) be a permutation. It is represented by an invertible matrix \(P\) such that

\[
P U P^{-1} = \text{diag}(u_{\sigma(1)}, u_{\sigma(2)}, \ldots, u_{\sigma(k)})
\]

The system

\[
\frac{dY}{dz} = \left[ PUP^{-1} + \frac{1}{z} PVP^{-1} \right] Y
\]

has solutions

\[
Y_{L/R}(z) := P \tilde{Y}_{L/R}(z) P^{-1} \sim (I + O(\frac{1}{z})) e^{zPUP^{-1}} \tilde{Y}_{L}(z) = Y_{R}(z) PSP^{-1}
\]

\[
Y_{0}(z) := P\tilde{Y}_{0}(z) = (PX^{-1} + O(z))z^{\mu}z^{R}, \quad Y_{0}(z) = Y_{R}(z) PC
\]
that
relations:
We first recall that the braid group is generated by \( u \) given chart, consisting in the permutation \( \sigma \) u

This abstract group is realized as the fundamental group of \((C,0)\). Then also \( S \) and \( C \) change when we go from one local chart to another. Consequently, Stokes’ rays move. Let us start from the point \( u \). Then we must change “Left” and “Right” solutions of (19). Then also \( S \) and \( C \) change. This is the reason why the monodromy data \( S \) and \( C \) change when we go from one local chart to another.

The motions of the points \( u_1(t) \) change when we go from one local chart to another. Actually, a braid \( \beta_{i,i+1} \) can be represented as an “elementary” deformation consisting of a permutation of \( u_1 \), \( u_{i+1} \) moving counter-clockwise (clockwise or counter-clockwise is a matter of convention).

Suppose \( u_1 \), ..., \( u_k \) are already in lexicographical order w.r.t. \( l \), so that \( S \) is upper triangular (recall that this configuration can be reached by a suitable permutation \( P \)). The effect on \( S \) of the deformation of \( u_1 \), \( u_{i+1} \) representing \( \beta_{i,i+1} \) is the following:

\[
S \mapsto S_{\beta_{i,i+1}} := A_{\beta_{i,i+1}}(S) S A_{\beta_{i,i+1}}(S)
\]

where

\[
(A_{\beta_{i,i+1}}(S))_{nn} = 1 \quad n = 1, \ldots, k \quad n \neq i, i+1
\]

\[
(A_{\beta_{i,i+1}}(S))_{i+1,i+1} = -s_{i,i+1}
\]

\[
(A_{\beta_{i,i+1}}(S))_{i,i+1} = (A_{\beta_{i,i+1}}(S))_{i+1,i} = 1
\]

and all the other entries are zero.

For the inverse braid \( \beta_{i,i+1}^{-1} \) (\( u_1 \) and \( u_{i+1} \) move clockwise) the representation is

\[
(A_{\beta_{i,i+1}^{-1}}(S))_{nn} = 1 \quad n = 1, \ldots, k \quad n \neq i, i+1
\]

\[
(A_{\beta_{i,i+1}^{-1}}(S))_{i,i} = -s_{i,i+1}
\]

\[
(A_{\beta_{i,i+1}^{-1}}(S))_{i,i+1} = (A_{\beta_{i,i+1}^{-1}}(S))_{i+1,i} = 1
\]

and all the other entries are zero.

For a generic braid \( \beta \), which is a product of \( N \) elementary braids (for some \( N \) \( \beta = \beta_{j_1,j_1+1} \cdots \beta_{j_N,j_N+1} \)) the action is

\[
S \mapsto S^\beta = A^\beta(S) S [A^\beta(S)]^T
\]

where

\[
A^\beta(S) = A_{\beta_{j_1,j_1+1}}(S) A_{\beta_{j_1,j_1+1}}(S) \cdots A_{\beta_{j_N,j_N+1}}(S) A_{\beta_{j_N,j_N+1}}(S)
\]

We remark that \( S^\beta \) is still upper triangular.
In figure 4 we have drawn some lines $L_j = \{ \lambda = u_j + \rho e^{i (\frac{\pi}{2} - \epsilon)} \mid \rho > 0 \}$ ($0 < \epsilon < \frac{\pi}{2}$ is the angle of $l$), which help us to visualize the topological effect of the braids action (they are the branch cuts for the fuchsian system which will be introduced in section 8). We are going to prove that the braid whose effect is to set the deformed points in cyclic order and the cuts in the configuration of figure 4 (namely, the last two cuts remain unchanged, the others are alternatively “inverted”), brings $S$ in a canonical form:

$$s_{i,i+1} = {k \atop i}, \quad s_{i,i+2} = {k \atop i+1}, \quad s_{i,i+n} = {k \atop i+n}, \quad \forall i = 1, \ldots, k-n+1;$$

$$s_{i,k} = -{k \atop k-i}.$$  

Namely

$$S^\beta = \begin{pmatrix}
1 & (k \atop 1) & (k \atop 2) & (k \atop 3) & (k \atop 4) & \cdots & -{k \atop k-1} \\
1 & (k \atop 1) & (k \atop 2) & (k \atop 3) & \cdots & -{k \atop k-2} \\
1 & (k \atop 1) & (k \atop 2) & \cdots & -* \\
1 & (k \atop 1) & \cdots & -* \\
& \ddots & \vdots & \ddots & \vdots \\
& & & & & 1
\end{pmatrix}$$  

Note that the last column is negative. Its sign is inverted by $S \mapsto ISI$, where $I := \text{diag}(1,1,\ldots,-1)$.

**Lemma 8:** Let the points $u_j$ ($j = 1, \ldots, k$) be in lexicographical order w.r.t the admissible line $l$. Then
Figure 5: Effect of the sequences of braids which bring $S$ to the canonical form (the figure refers to $k$ even)
the braid

\[ \beta := (\beta_{k-5,k-4} \beta_{k-6,k-5} \ldots \beta_{12}) (\beta_{k-6,k-5} \beta_{k-7,k-6} \ldots \beta_{23}) (\beta_{k-7,k-6} \ldots \beta_{34}) \ldots \beta_{k-2,\frac{k}{2}-1} (\beta_{k-3,k-2} \beta_{k-4,k-3} \ldots \beta_{12}) \]

for \( k \) even, or

\[ \beta := (\beta_{k-5,k-4} \beta_{k-6,k-5} \ldots \beta_{12}) (\beta_{k-6,k-5} \beta_{k-7,k-6} \ldots \beta_{23}) (\beta_{k-7,k-6} \ldots \beta_{34}) \ldots (\beta_{\frac{k-1}{2},\frac{k-1}{2}} \beta_{\frac{k+1}{2},\frac{k+1}{2}}) (\beta_{k-3,k-2} \beta_{k-4,k-3} \ldots \beta_{12}) \]

for \( k \) odd, brings the points in cyclic counter-clockwise order, \( u_1 \) being the first point in \( \Pi_L \) (figure 4, right side, or figure 5).

Note that we have collected the braids in \( \frac{k}{2} - 1 \) \( (k \) even), or \( \frac{k-2}{2} \) \( (k \) odd) sequences \((\ldots)\).

**Proof:** Let \( k \) be even. The first braid \( \beta_{k-5,k-4} \) interchanges \( u_{k-4} \) and \( u_{k-5} \). The second braid interchanges \( u_{k-5} \) and \( u_{k-6} \). One easily sees that the effect of the first sequence of braids \((\beta_{k-5,k-4} \beta_{k-6,k-5} \ldots \beta_{12})\) is to bring \( u_1 \) in the (old) position of \( u_{k-4} \), \( u_{k-4} \) in the position of \( u_{k-5} \), \( u_{k-5} \) in the position of \( u_{k-6} \), \( \ldots \), \( u_4 \) in the position of \( u_3 \) and \( u_2 \) in the position of \( u_1 \) (figure 5). \( u_k \), \( u_{k-1} \), \( u_{k-2} \), \( u_{k-3} \) are not moved.

The second sequence of braids \((\beta_{k-6,k-5} \beta_{k-7,k-6} \ldots \beta_{23})\) acts in a similar way, bringing \( u_2 \) in \( u_{k-5} \), \( u_{k-5} \) in \( u_{k-6} \), \( \ldots \), \( u_3 \) in \( u_2 \), \( u_k \), \( u_{k-1} \), \( u_{k-2} \), \( u_{k-3} \), \( u_{k-4} \) are not moved.

We go on in this way. After the action of

\[
(\beta_{k-5,k-4} \beta_{k-6,k-5} \ldots \beta_{12}) (\beta_{k-6,k-5} \beta_{k-7,k-6} \ldots \beta_{23}) (\beta_{k-7,k-6} \ldots \beta_{34}) \ldots \beta_{\frac{k-1}{2},\frac{k}{2}}
\]

the points are as in figure 5: \( u_k \) is on the positive real axis, \( u_{k-2} \) is the first point met in counter-clockwise order, \( u_1 \) is the second, \( u_2 \) is the third; the points are in cyclic order up to \( u_{k-3} \); finally, \( u_{k-1} \) is the last point before reaching again the positive real axis from below.

Then, \((\beta_{k-3,k-2} \beta_{k-4,k-3} \ldots \beta_{12})\) brings \( u_1 \) in \( u_{k-2} \), \( u_{k-2} \) in \( u_{k-3} \), \( u_{k-3} \) in \( u_{k-4} \), and so on. The cyclic order is reached.

For \( k \) odd the proof is similar. \( \square \)

A careful consideration of the effect of the braid \( \beta \) on the lines \( L_j \) (which we leave as an exercise for the reader) shows that they are alternatively inverted as in figure 4. To reconstruct uniquely this configuration we just need to know the oriented line \( l \), namely, its angle \( \epsilon \) w.r.t the positive real axis. The points \( u_{k-1} \), \( u_k \) and the lines \( L_{k-1} \) and \( L_k \) are unchanged (angle \( \frac{\pi}{2} - \epsilon \)). The line at \( u_1 \) starts in the opposite direction, it goes around \( u_2 \), \( \ldots \), \( u_{k-2} \) without intersecting other cuts, and then goes to \( \infty \) with the original asymptotic direction \( \frac{\pi}{2} - \epsilon \). Moving in the direction opposite to that of \( l \) we meet \( u_{k-2} \). Its line has the original direction \( \frac{\pi}{2} - \epsilon \). Then we meet \( u_2 \), and the corresponding line starts with opposite direction, goes around \( u_3 \), \( \ldots \), \( u_{k-3} \) and then goes to \( \infty \) with asymptotic direction \( \frac{\pi}{2} - \epsilon \). And so on.

Now we find the matrix representation for the braid \( \beta \).

**Proposition 1:** The braid \( \beta \) of Lemma 8 has the following matrix representation:

\[
A^\beta(S) =
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & (\frac{k}{2}) & 0 & (\frac{k}{2}) & 0 & 0 & 0 & 0 & 0 \\
(\frac{k}{2}) & 0 & (\frac{k}{2}) & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & (\frac{k}{2}) & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & (\frac{k}{2}) & 0 & (\frac{k}{2}) & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & (\frac{k}{2}) & 0 & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0
\end{pmatrix}
\]
for \( k \) even.

\[
A^\beta(S) = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
1 & 0 & (k) & 0 & (k) & 0 & 0 & 0 & 0 & 0 \\
(k) & 0 & (k) & 0 & (k) & 0 & 0 & 0 & 0 & 0 \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
& & & & & & & & & \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

for \( k \) odd.

The "\(*\)" means \( \binom{k}{j} \), and \( j \) increases by one when we move downwards row by row.

**Proof:** We need some steps.

1) For an upper triangular Stokes’ matrix \( S \) (with entries \( s_{ij} \)) the entries of the matrix \( A^\beta \) which are different from zero are the following:

\( k \) odd:

\[
A^\beta_{k-1,k-1} = A^\beta_{k,k} = 1
\]

For \( j \) even, \( 2 + 2i \leq j \leq k - 2 \), and \( i = 1, 2, ..., \frac{k}{2} - 1 \)

\[
A^\beta_{\frac{k}{2} - i, 2i} = 1
\]

\[
A^\beta_{\frac{k}{2} - i, j} = -s_{2i,j} + \sum_{n=1}^{\frac{k}{2} - i} (-1)^{n+1}F_{n,i,j}
\]

\[
F_{n,i,j} = \sum_{\alpha_1=1}^{\frac{k}{2} - i - n} \left( \sum_{\alpha_2=1}^{\frac{k}{2} - i - \alpha_1 - n + 1} \left( \sum_{\alpha_3=1}^{\frac{k}{2} - i - \alpha_1 - \alpha_2 - n + 2} \left( \sum_{\alpha_4=1}^{\frac{k}{2} - i - \sum_{r=1}^{p-1} \alpha_r - n + p - 1} \left( \sum_{\alpha_5=1}^{\frac{k}{2} - i - \sum_{r=1}^{n-1} \alpha_r - 1} f(i,j,\alpha) \right) \right) \right) \right)
\]

\[
f(i,j,\alpha) = s_{2i,2i+2\alpha_1} s_{2i+2\alpha_1,2i+2\alpha_1+2\alpha_2} s_{2i+2\alpha_1+2\alpha_2,2i+2\alpha_1+2\alpha_2+2\alpha_3} \cdots s_{2i+2\alpha_1+\cdots+2\alpha_{n-1},j}
\]

For \( j \) even, \( 2i + 2 \leq j \leq k - 2 \) and \( i = 0, 1, ..., \frac{k}{2} - 2 \)

\[
A^\beta_{\frac{k}{2} + i, 2i+1} = 1
\]

\[
A^\beta_{\frac{k}{2} + i, j} = -s_{2i+1,j} + \sum_{n=1}^{\frac{k}{2} - i - n} (-1)^{n+1}G_{n,i,j}
\]

\[
G_{n,i,j} = \sum_{\alpha_1=1}^{\frac{k}{2} - i - n} \left( \sum_{\alpha_2=1}^{\frac{k}{2} - i - \alpha_1 - n + 1} \left( \sum_{\alpha_3=1}^{\frac{k}{2} - i - \alpha_1 - \alpha_2 - n + 2} \left( \sum_{\alpha_4=1}^{\frac{k}{2} - i - \sum_{r=1}^{p-1} \alpha_r - n + p - 1} \left( \sum_{\alpha_5=1}^{\frac{k}{2} - i - \sum_{r=1}^{n-1} \alpha_r - 1} g(i,j,\alpha) \right) \right) \right) \right)
\]

\[
g(i,j,\alpha) = s_{2i+1,2i+2\alpha_1} s_{2i+2\alpha_1,2i+2\alpha_1+2\alpha_2} s_{2i+2\alpha_1+2\alpha_2,2i+2\alpha_1+2\alpha_2+2\alpha_3} \cdots s_{2i+2\alpha_1+\cdots+2\alpha_{n-1},j}
\]
For odd:\n
\[ A^3(S) = \begin{pmatrix}
  \mathcal{H} \\
  \mathcal{V}
\end{pmatrix}
\]

More explicitly,\n
\[ A^{k-1,k-1} = A_{k,k} = 1 \]

For odd, \(2i + 1 \leq j \leq k - 2, \ i = 1, \ldots, \frac{k-3}{2} \):\n
\[ A_{\frac{k-1}{2}i+1,j} = (-1)^{i+1} \sum_{n=1}^{i+1} (-1)^n H(n, i, j) \]

\[ H(n, i, j) = \sum_{\alpha=1}^{i+1} \left( \sum_{\gamma=1}^{i+1-n} \alpha - n + p - 1 \right) \]

\[ v(i, j, \alpha) = s_{2i+1+2\alpha} s_{2i+1+2\alpha+1} s_{2i+1+2\alpha+2} + \ldots + 2\alpha_n, j \]

For odd, \(2i + 3 \leq j \leq k - 2, \ i = 0, 1, \ldots, \frac{k-3}{2} \):\n
\[ A_{\frac{k-1}{2}i+1,j} = (-1)^{i+1} \sum_{n=0}^{i+1} (-1)^n V(n, i, j) \]

\[ V(n, i, j) = \sum_{\alpha=1}^{i+1} \left( \sum_{\gamma=1}^{i+1-n} \alpha - n + p - 1 \right) \]

\[ v(i, j, \alpha) = s_{2i+1+2\alpha} s_{2i+1+2\alpha+1} s_{2i+1+2\alpha+2} + \ldots + 2\alpha_n, j \]

More explicitly:\n
\[ A^3(S) = \begin{pmatrix}
  \mathcal{A} \\
  \mathcal{B}
\end{pmatrix}
\]
In order to prove the above expressions, we have to find each matrix $A^{\beta_{i,i+1}}$ corresponding to the elementary braid $\beta_{i,i+1}$ appearing in $\beta$. This means computing its entry $(i+1, i+1)$. This is a rather complicated problem, since the entry $(i+1, i+1)$ of a given $A^{\beta_{i,i+1}}$ is minus the entry $(i, i+1)$ of the Stokes’ matrix resulting from the action of the elementary braids acting before $\beta_{i,i+1}$, which in general is a sum of products of the elements $s_{ij}$ of the initial Stokes matrix $S$.

First we recall that $S \mapsto A^{\beta_{i,i+1}} S A^{\beta_{i,i+1}}$ has the following effect on the entries of $S$:

\[
\begin{align*}
s_{n,i} &\mapsto s_{n,i+1}, \\
s_{i,i+1} &\mapsto s_{i,i} - s_{i,i+1}s_{n,i+1}, \\
s_{i,i+1} &\mapsto -s_{i,i+1}
\end{align*}
\]

while all the other entries of $S$ remain unchanged.

We start from $A^{\beta_{5-6,k-4}}$, whose non trivial entry is simply $-s_{k-5,k-4}$. Its action on $S$ brings $s_{k-6,k-5}$ to $s_{k-6,k-4}$ (the reader may compute all the elements of $S^{\beta_{5-6,k-4}}$).

Then, the entry of $A^{\beta_{5-6,k-5}}$ is $-s_{k-6,k-4}$. Proceeding in this way, the reader may check that for the first sequence of braids $(\beta_{5-6,k-4}\beta_{5-6,k-5} ... \beta_{12})$ the entries $(i, i+1)$ of the matrices are:

$-s_{i,k-4}$ for $A^{\beta_{i,i+1}}$

Now, observe that

\[
\begin{pmatrix}
1 & ... & 1 \\
... & 0 & 1 \\
1 & x_1 & 1 \\
... & 1 & 1
\end{pmatrix}
\begin{pmatrix}
1 & ... & 1 \\
... & 0 & 1 \\
1 & x_2 & 1 \\
... & 1 & 1
\end{pmatrix}
= 
\begin{pmatrix}
1 & ... & 1 \\
0 & 0 & 1 \\
1 & 0 & x_1 \\
0 & 1 & x_2 \\
\end{pmatrix}
\]

and recall that $A^{\beta_1\beta_2} = A^{\beta_2} A^{\beta_1}$. This implies for the first sequence of braids:

\[
\begin{pmatrix}
0 & 1 \\
1 & 0 & * \\
1 & 0 & * \\
\vdots & \vdots & \vdots \\
0 & * & 1 \\
1 & * & 1 \\
\end{pmatrix}
\]

the entries $*$ are exactly those of the $A_{i,i+1}$’s, namely $-s_{1,k-4}, -s_{2,k-4}, ... , -s_{k-5,k-4}$ from the top to the bottom of the $(k-4)^{th}$ column.

For the second sequence of braids $(\beta_{5-7,k-6} ... \beta_{34})$, the entries are

$-s_{i-1,k-6}$ for $A^{\beta_{i,i+1}}$

and, as above:

\[
m_2 := A^{\beta_{5-7,k-6} \beta_{5-7,k-6} ... \beta_{23}} = 
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & * \\
1 & 0 & * \\
\vdots & \vdots & \vdots \\
0 & * & 1 \\
1 & 1 & 1 \\
\end{pmatrix}
\]

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and the entries $s$ are those of the $A_{i,i+1}$'s (namely, $-s_{1,k-6}$, ..., $-s_{k-7,k-6}$ from the top to the bottom).

For the third sequence $(\beta_{k-6,k-5} \ldots \beta_{23})$, they are $-s_{i-2,k-8}$ for $A_{i,i+1}$

and:

$$m_3 := A^{\beta_{k-7,k-6} \ldots \beta_{44}} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & 1 \\
0 & 1 & \ddots \\
& \ddots & \ddots & \ddots & 1 \\
& & & 0 & 1 \\
& & & 1 & 1 \\
& & & 1 & 1 \\
& & & 1 & 1 \\
& & & 1 & 1
\end{pmatrix}$$

And so on. We reach the last but one “sequence”, namely $\beta_{k-2,k-1}$ for $k$ even, or $(\beta_{k-3,k-2} \beta_{k-1,k})$ for $k$ odd. The entries are $-s_{12}$, or $-s_{23}, -s_{13}$ respectively. Then

$$m_{\frac{k-2}{2}} := A^{\beta_{k-3,k-2} \beta_{k-2,1}} = \begin{pmatrix}
1 & \ddots & & & & \\
& 1 & 0 & 1 & & \\
& & 1 & -s_{12} & 1 & \\
& & & \ddots & \ddots & \ddots & \ddots \\
& & & & 1 & 0 & 1 \\
& & & & 1 & -s_{13} & 1 \\
& & & & & 1 & -s_{23} \\
& & & & & & 1 & 1
\end{pmatrix}$$

or

$$m_{\frac{k-5}{2}} := A^{\beta_{k-3,k-2} \beta_{k-5,k-3}} = \begin{pmatrix}
1 & \ddots & & & & & \\
& \ddots & 1 & 0 & 1 & & \\
& & 1 & 0 & -s_{13} & 1 & \\
& & & \ddots & \ddots & \ddots & \ddots \\
& & & & 1 & 0 & -s_{13} \\
& & & & 1 & -s_{23} & 1 \\
& & & & & 1 & -s_{23} \\
& & & & & & 1 & 1
\end{pmatrix}$$

The entries for the last braid are more complicated, because the entries on the first upper sub-diagonal of the Stokes' matrix have been shuffled by the preceding braids. We give the result ($A_{i,i+1}$ stands for $A_{\beta_{i,i+1}}$)

$$A_{k-3,k-2} := -s_{k-3,k-2}$$
$$A_{k-4,k-3} := -s_{k-5,k-2} + s_{k-5,k-4} s_{k-4,k-2}$$
$$A_{k-5,k-4} := -s_{k-7,k-2} + (s_{k-7,k-4} s_{k-4,k-2} + s_{k-7,k-6} s_{k-6,k-2}) - s_{k-7,k-6} s_{k-6,k-4} s_{k-4,k-2}$$

$$A_{\frac{k-5}{2},1} := -s_{1,k-2} + (s_{12} s_{2,k-2} + s_{14} s_{4,k-2} + \ldots + s_{1,k-4} s_{k-4,k-2}) + \ldots + (-1)^{k-1} s_{12} s_{24} s_{48} \ldots s_{k-4,k-2}$$

$$A_{\frac{k-3}{2},1} := -s_{2,k-2} + (s_{24} s_{4,k-2} + s_{26} s_{6,k-2} + \ldots + s_{2,k-4} s_{k-4,k-2}) + \ldots + (-1)^{k-2} s_{24} s_{48} \ldots s_{k-4,k-2}$$

$$A_{34} := -s_{k-8,k-2} + (s_{k-8,k-4} s_{k-4,k-2} + s_{k-8,k-6} s_{k-6,k-2}) - s_{k-8,k-6} s_{k-6,k-4} s_{k-4,k-2}$$
$$A_{23} := -s_{k-6,k-2} + s_{k-6,k-4} s_{k-4,k-2}$$
$$A_{12} := -s_{k-4,k-2}$$

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For \( k \) odd
\[
A_{k-3,k-2} = -s_{k-3,k-2}
\]
\[
A_{k-4,k-3} = -s_{k-5,k-2} + s_{k-5,k-4}s_{k-4,k-2}
\]
\[
A_{k-5,k-4} = -s_{k-7,k-2} + \left( s_{k-7,k-4}s_{k-4,k-2} + s_{k-7,k-6}s_{k-6,k-2} \right) - s_{k-7,k-6}s_{k-6,k-4}s_{k-4,k-2}
\]
\[
\vdots
\]
\[
A_{\frac{k-1}{2}, \frac{k+1}{2}} = -s_{2,k-2} + \left( s_{23}s_{3,k-2} + s_{25}s_{5,k-2} + \ldots + s_{2,k-4}s_{k-2} \right) + \ldots + \left( -1 \right)^{\frac{k-3}{2}}s_{23}s_{3}s_{5}s_{7}s_{9}\ldots s_{k-4,k-2}
\]
\[
A_{\frac{k-3}{2}, \frac{k-1}{2}} = -s_{1,k-2} + \left( s_{13}s_{3,k-2} + s_{15}s_{5,k-2} + \ldots + s_{1,k-4}s_{k-2} \right) + \ldots + \left( -1 \right)^{\frac{k-3}{2}}s_{13}s_{3}s_{5}s_{7}s_{9}\ldots s_{k-4,k-2}
\]
\[
\vdots
\]
\[
A_{34} = -s_{k-8,k-2} + \left( s_{k-8,k-4}s_{k-4,k-2} + s_{k-8,k-6}s_{k-6,k-2} \right) - s_{k-8,k-6}s_{k-6,k-4}s_{k-4,k-2}
\]
\[
A_{23} = -s_{k-6,k-2} + s_{k-6,k-4}s_{k-4,k-2}
\]
\[
A_{12} = -s_{k-4,k-2}
\]

and:
\[
A^\beta_{k-3,k-2} \beta_{k-4,k-3} \ldots \beta_{12} = \begin{pmatrix}
0 & 1 \\
1 & 0 & * \\
& 1 & 0 & * \\
& & \ddots & \ddots \\
& & & 0 & * \\
& & & & 1 & * \\
& & & & & 1 & 0 \\
\end{pmatrix}
\]

which we call \( m_{\frac{k-1}{2}} \) for \( k \) even, \( m_{\frac{k-3}{2}} \) for \( k \) odd.

Then, for \( k \) even
\[
A^\beta = m_{\frac{k-1}{2}}m_{\frac{k-3}{2}} \ldots m_3m_2m_1
\]

and for \( k \) odd
\[
A^\beta = m_{\frac{k-1}{2}}m_{\frac{k-3}{2}} \ldots m_3m_2m_1
\]

Doing a careful computation, we obtain \((25)\) and \((26)\).

2) The second step consists of expressing \( A^\beta \) in terms of the entries of the Stokes’ factors of \( S \), which are simply binomial coefficients. First we prove the following

**Lemma 9**: Given an upper triangular \( k \times k \) matrix \( S \), with entries \( s_{ii} = 1 \), we can uniquely determine numbers \( a_{ij} \) such that, for \( k, i, j \) all even or all odd:

\[
s_{ij} = a_{ij} + \left( a_{i+2,i+2} + a_{i+4,i+4} + \ldots + a_{i+2j-2j, i+2j-2j} \right) + \left( a_{i,i+2} + a_{i,i+4} + \ldots + a_{i,j-2} + a_{j-2, i} \right) + \left( a_{i+2,i+4} + a_{i+4,i+6} + \ldots + a_{j-2,j, i} \right)
\]

If \( k \) is even, but \( i \) is odd, just replace in the formula \( i + 2 \) with \( i + 1 \), \( i + 4 \) with \( i + 3 \), etc. If \( k \) is even, but \( j \) is odd, just replace \( j - 2 \) with \( j - 1 \), \( j - 4 \) with \( j - 3 \), etc.

If \( k \) is odd, but \( i \) is even, or \( j \) is even, just do the same replacements as above. More explicitly:

\[
S = \begin{pmatrix}
1 & a_{12} & a_{13} + a_{12}a_{23} & a_{14} + a_{12}a_{24} & a_{15} + (a_{12}a_{25} + a_{14}a_{45}) & a_{16} + (a_{12}a_{26} + a_{14}a_{46}) & \ldots \\
a_{23} & a_{24} & a_{25} + a_{24}a_{45} & a_{26} + a_{24}a_{46} & a_{27} + (a_{24}a_{47} + a_{26}a_{67}) & \ldots \\
a_{34} & a_{35} + a_{34}a_{45} & a_{36} + a_{34}a_{46} & a_{37} + (a_{34}a_{47} + a_{36}a_{67}) & a_{38} + (a_{36}a_{68} + a_{37}a_{78}) & \ldots \\
1 & a_{45} & a_{46} & a_{47} + a_{46}a_{67} & a_{48} + (a_{46}a_{68} + a_{47}a_{78}) & \ldots \\
1 & a_{56} & a_{57} + a_{56}a_{67} & a_{58} + (a_{56}a_{68} + a_{57}a_{78}) & a_{59} + (a_{58}a_{89} + a_{57}a_{79}) & \ldots \\
1 & a_{67} & a_{68} + a_{67}a_{78} & a_{69} + (a_{67}a_{79} + a_{68}a_{89}) & a_{70} + (a_{68}a_{810} + a_{67}a_{710}) & \ldots \\
1 & a_{78} & a_{79} + a_{78}a_{89} & a_{80} + (a_{78}a_{810} + a_{79}a_{910}) & a_{81} + (a_{80}a_{1011} + a_{81}a_{911}) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
Proof: Just substitute the factorization of \( k \) for \( k \).

The sum of the differences between the indices of the factors \( a \) in \( s_{ij} \) is equal to the difference of the indices of \( s_{ij} \), namely \( j - i \).

If from this it follows that the terms non-linear in the \( a \)'s occur on sub-diagonals which lie above all the sub-diagonals containing the factors of the non-linear terms.

Then, the system \( F_{ij}(a) = s_{ij} \) is uniquely solvable, starting from the first sub-diagonal and successively determining all the \( a_{rs} \) going up diagonal by diagonal.

Corollary 3: With the above factorization, the matrix \( A^\beta \) becomes:

\[
A^\beta(S) = \begin{pmatrix}
1 & a_{12} & a_{13} & a_{14} + a_{13} a_{34} & a_{15} + a_{13} a_{35} & a_{16} + (a_{13} a_{36} + a_{15} a_{56}) + a_{13} a_{35} a_{56} & a_{17} + (a_{13} a_{37} + a_{15} a_{57}) + a_{13} a_{35} a_{57} & \cdots \\
1 & a_{23} & a_{24} + a_{23} a_{34} & a_{25} + a_{23} a_{35} & a_{26} + (a_{23} a_{36} + a_{25} a_{56}) + a_{23} a_{35} a_{56} & a_{27} + (a_{23} a_{37} + a_{25} a_{57}) + a_{23} a_{35} a_{57} & \cdots \\
1 & a_{34} & a_{35} & a_{36} + a_{35} a_{56} & a_{37} + a_{35} a_{57} & \cdots \\
1 & a_{45} & a_{46} + a_{45} a_{56} & a_{47} + a_{45} a_{57} & \cdots \\
1 & a_{56} & & a_{57} & \cdots \\
1 & & & a_{67} & \cdots \\
& & & & & \cdots \\
& & & & & \cdots
\end{pmatrix}
\]

for \( k \) even.

for \( k \) odd.

Proof: We have to solve a non-linear system \( F_{ij}(a) = s_{ij} \).

The sum of the differences between the indices of the factors \( a \) in \( s_{ij} \) is equal to the difference of the indices of \( s_{ij} \), namely \( j - i \).

\( \square \)

In section 5 we computed the Stokes’ factors for \( S \). If we sum all the factors appearing in formula
we get a matrix of the form:

\[
M := K_{1k} + K_{1,k-1} + K_{k,k-1} + \ldots + K_{k3} + K_{k2} = \begin{pmatrix}
  k & * & 0 & 0 \\
  * & k & 0 & * \\
  * & * & k & * \\
  * & * & \ddots & \ddots \\
  * & * & \ddots & \ddots \\
  * & 0 & k & * \\
  0 & 0 & \cdots & k
\end{pmatrix}
\]

The * are the binomial coefficient appearing in the factors. If we know \( M \), we can determine all the entries of the single Stokes' factors, because if the entry \((i,j)\) is not zero for one factor, then it is zero for all the other factors. Now, we rename the entries of the factors according to the following rule:

\[
M := \begin{pmatrix}
  k & a_{k-2,k} & a_{k-4,k-2} & k & 0 & a_{k-2,k-1} \\
a_{k-4,k} & a_{k-4,k-2} & k & \ddots & \ddots & \ddots \\
  \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
a_{4k} & a_{4,k-2} & \ldots & a_{46} & k & 0 & 0 & 0 & a_{45} & a_{47} & \ldots & a_{4,k-1} \\
a_{2k} & a_{2,k-2} & \ldots & a_{26} & a_{24} & k & 0 & a_{23} & a_{25} & a_{27} & \ldots & a_{2,k-1} \\
a_{1k} & a_{1,k-2} & \ldots & a_{16} & a_{14} & a_{12} & k & a_{13} & a_{15} & a_{17} & \ldots & a_{1,k-1} \\
a_{3k} & a_{3,k-2} & \ldots & a_{36} & a_{34} & 0 & 0 & k & a_{35} & a_{37} & \ldots & a_{3,k-1} \\
a_{5k} & a_{5,k-2} & \ldots & a_{56} & 0 & 0 & 0 & k & a_{57} & \ldots & a_{5,k-1} \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \\
a_{k-3,k} & a_{k-3,k-2} & \ldots & a_{k-45} & 0 & 0 & 0 & k & a_{k-46} & \ldots & a_{k-3,k-1} \\
a_{k-1,k} & a_{k-3,k-2} & \ldots & a_{k-45} & 0 & 0 & 0 & k & a_{k-46} & \ldots & a_{k-3,k-1} \\
  0 & 0 & \cdots & k & a_{k-3,k-1} & \ldots & k & a_{k-3,k-1} & \ldots & \ldots & \ldots & k
\end{pmatrix}
\]

for \( k \) even. The strange labelling is simply the one such that

\[
PMP^{-1} = \begin{pmatrix}
  k & a_{12} & a_{13} & a_{14} & \ldots \\
ak & a_{23} & a_{24} & \ldots \\
  \ddots & \ddots & \ddots \\
  k & a_{34} & \ldots \\
  k & \ldots & \ldots & \ldots & \ldots & k
\end{pmatrix}
\]

where the matrix of permutation is

\[
P = \begin{pmatrix}
  1 & 0 & & & & & & & & & \\
  1 & 0 & 0 & & & & & & & & \\
  0 & 0 & 0 & 1 & 0 & & & & & & \\
  1 & 0 & 0 & 0 & 0 & & & & & & \\
  0 & 0 & 0 & 0 & 0 & 1 & 0 & & & & \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  1 & & & & & & & & & & \\
  0 & & & & & & & 0 & 1 & & \\
  & & & 1 & 0 & & & & & 0 & 0
\end{pmatrix}
\]
for $k$ even (the 1 on the first row is on the $\frac{k}{2} + 1$-th column); and
\[
P = \begin{pmatrix}
0 & 1 \\
0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 1 \\
0 & 0 \\
1 & 0
\end{pmatrix}
\]
for $k$ odd (the 1 on the first row is on the $\frac{k+1}{2}$-th column).

With this choice of the labelling, the product $S_{\text{upper}} := P(K_{1k}K_{1,k-1}K_{k,k-1} \ldots K_{k3}K_{k2})P^{-1}$ is precisely factorized as in lemma 9.

Then we can write the entries of $A^\beta$ (formulae (27) (28)) from the entries of the Stokes factors (which are binomial coefficient).

The final result is precisely the claim of the proposition.

We are ready to prove the main result:

**Theorem 2:** Consider the Stokes matrix $S = T_F^k T^{-\frac{k-1}{2}}$ ($k$ even) or $S = T_F^{k-1}K_{k2}T^{-\frac{k-1}{2}}$ ($k$ odd) for the quantum cohomology of $\mathbb{CP}^{k-1}$ and set it in the upper triangular form $S_{\text{upper}} = P S P^{-1}$ by the permutation $P$. Then, there exists a braid $\beta$ (Lemma 8), represented by a matrix $A^\beta$ (Proposition), which sets $S_{\text{upper}}$ in the form (24). The last column is negative, but conjugation by $I = \text{diag}(1,1,\ldots,1,-1)$ makes it positive. We reach the canonical form:

\[
s_{ij} = \binom{k}{j-i}, \quad i < j
\]

Another conjugation by $\text{diag}(-1,1,-1,1,-1,\ldots)$ brings the matrix in the equivalent canonical form

\[
s_{ij} = (-1)^{j-i}\binom{k}{j-i}, \quad i < j
\]

Finally, by the action of the braid group, the last matrix can be put in the form

\[
s_{ij} = \binom{k-1 + j - i}{j-i}, \quad i < j
\]

In all the above matrices

\[
s_{ii} = 1, \quad s_{ij} = 0 \quad i > j
\]

**Proof:** First, we want to explain which is the braid which brings the upper triangular matrix with entries $s_{ij} = (-1)^{j-i}\binom{k}{j-i}$ in the matrix $s_{ij} = \binom{k-1 + j - i}{j-i}$. We make use of the following known result [25]:

Consider the upper triangular Stokes’ matrix $S$, the braid $\beta = \beta_1 \beta_2 \beta_3 \ldots (\beta_{n-1} \beta_n \beta_{n-2} \ldots)$ and the permutation

\[
P = \begin{pmatrix}
\vdots & \vdots & \vdots & \vdots & \vdots & 1 \\
1 & \ddots & \ddots & \ddots & \ddots & \ddots \\
& \ddots & \ddots & \ddots & \ddots & \ddots \\
& & \ddots & \ddots & \ddots & \ddots \\
& & & \ddots & \ddots & \ddots \\
& & & & \ddots & \ddots
\end{pmatrix}
\]

Then, the relation

\[
[S^{-1}]^\beta = PS^TP
\]
holds.
Observe that for the matrix $S$, whose upper triangular part has entries $s_{ij} = \left(\begin{array}{cc} k & 1 + j - i \\ & j - i \end{array}\right)$, we have $PS^TP \equiv S$. Moreover, $S^{-1}$ is upper triangular with entries $s_{ij} = (-1)^{j-i} \left(\begin{array}{c} k \\ & j - i \end{array}\right)$. This proves that $S$ and $S^{-1}$ are equivalent w.r.t the action of the braid group.

Let us now prove the theorem staring from $k$ even. We have to prove that $A^3 \cdot P \cdot T^k \cdot T^{-k} \cdot P^{-1} [A^3]^T$ is in “canonical form” (24). The proof “reduces” to the computation of products of matrices explicitly given. We do the products in an shrewd way. First we rewrite

$$S^3 = A^3 \cdot (P \cdot T^k \cdot P^{-1})^k \cdot (P \cdot T^{-1} \cdot P^{-1})^k \cdot [A^3]^T$$

and we compute:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & \ldots & 0 \\ -1 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & -1 \end{pmatrix}$$

(in $P^{-1}$ the 1 on the first column is on the $\frac{k}{2} + 1$-th row)

Thus

$$\begin{pmatrix} 1 & \ldots & \ldots & \ldots \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \ldots & 1 & \ldots \end{pmatrix}$$

Then, using the expression of $A^3$ from the proposition:

$$F := A^3 \cdot (P \cdot T^k \cdot P^{-1})^k = \begin{pmatrix} 0 & \ldots & \ldots & \ldots \\ \ldots & \ddots & \ddots & \vdots \\ \ldots & \ddots & \ddots & \ldots \\ 0 & \ldots & \ldots & 1 \end{pmatrix}$$
Then, multiplying \( F^{-1} K_{k3} K_{k2} \) of section 5 we compute

\[
T^{-1} = T_{F}^{-1} K_{k3} K_{k2} = \begin{pmatrix}
0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
K_{k3} K_{k2} =
\begin{pmatrix}
0 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0
\end{pmatrix}
\]

\[
(\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}^{-1})^{-1} = \begin{pmatrix}
-\binom{k}{1} & \binom{k}{1} & \binom{k}{1} \\
-\binom{k}{1} & -\binom{k}{1} & -\binom{k}{1} \\
-\binom{k}{1} & -\binom{k}{1} & -\binom{k}{1} \\
-\binom{k}{1} & -\binom{k}{1} & -\binom{k}{1}
\end{pmatrix}
\]

\[
P T^{-1} P^{-1} = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

After this we computed \( F [P T^{-1} P^{-1}], F [P T^{-1} P^{-1}] [P T^{-1} P^{-1}], F [P T^{-1} P^{-1}]^3, \ldots, F [P T^{-1} P^{-1}]^\frac{1}{2} \). We omit the intermediate computations and we give the final result:

\[
F_1 := F (P T^{-1} P^{-1})^\frac{1}{2} = \begin{pmatrix}
\binom{k}{1} & \binom{k}{1} & \binom{k}{1} & \binom{k}{1} & \binom{k}{1} \\
\binom{k}{1} & \binom{k}{1} & \binom{k}{1} & \binom{k}{1} & \binom{k}{1} \\
\binom{k}{1} & \binom{k}{1} & 0 & 0 & 0 \\
\binom{k}{1} & \binom{k}{1} & 0 & 0 & 0 \\
\binom{k}{1} & \binom{k}{1} & 0 & 0 & 0 \\
\binom{k}{1} & \binom{k}{1} & 0 & 0 & 0 \\
\binom{k}{1} & \binom{k}{1} & 0 & 0 & 0 \\
\binom{k}{1} & \binom{k}{1} & 0 & 0 & 0
\end{pmatrix}
\]

Now, multiplying \( F_1 [A^3]^T \), we obtain precisely the “canonical form” \([24] \).
For $k$ odd, we did a similar computation. We omit the detail, but we indicate the order of multiplications which yielded the most simple expressions to multiply step by step. Our aim is to compute $A^\beta P T_{F^{-1}} K_{k_2} (T_{F^{-1}} K_{k_3} K_{k_2}) T_{F^{-1}} P^{-1} [A^\beta]^T$. First, we computed $PT_F P^{-1}$, then $(PT_F P^{-1}) T_{F^{-1}}$, then $F := A^\beta (PT_F P^{-1}) T_{F^{-1}}$. After this, we computed $PK_{k_2} P^{-1}$, and $F_1 := F PK_{k_2} P^{-1}$. Finally, we calculated $m := P T_{F^{-1}} K_{k_3} K_{k_2} P^{-1}$ and $F_1 m, F_1 m m, ..., F_2 := F_1 m T_{F^{-1}}$. The matrix $F_2 [A^\beta]^T$ proved to be in “canonical form”.

\[\square\]

7 Canonical form of $S^{-1}$

The matrix $S^{-1}$ such that $Y_R(z) = Y_L(z)$ $S^{-1}$ can be put in the same canonical form of $S$, as a consequence of the relation (2). The only remarks we want to add concern the braid which brings $S^{-1}$ to the canonical form, because it arranges the lines $L_i$ in a “beautiful” shape.

Lemma 8': Let the points $u_j$ be in lexicographical order w.r.t the admissible line $l$. Let us denote $\sigma_{i,i+1} := \beta_{i+1}^{-1}$. Then, the following braid arranges the points in cyclic clockwise order, $u_1$ being the first point in $\Pi_L$ for $k$ even, or the last in $\Pi_R$ for $k$ odd (w.r.t the clockwise order) (see figure 6):

$\beta' := \left[(\sigma_{34} \sigma_{56} \sigma_{78} \ldots \sigma_{k-3,k-2} \sigma_{k-1,k}) (\sigma_{45} \sigma_{67} \sigma_{89} \ldots \sigma_{k-4,k-3} \sigma_{k-2,k-1}) \ldots (\sigma_{\frac{k}{2}+1,\frac{k}{2}+2} \sigma_{\frac{k}{2}+2,\frac{k}{2}+3} \sigma_{\frac{k}{2}+1,\frac{k}{2}+2}) \right]$ for $k$ even, and

$\beta' := \beta_{12} \left[(\sigma_{45} \sigma_{67} \sigma_{89} \ldots \sigma_{k-3,k-2} \sigma_{k-1,k}) (\sigma_{56} \sigma_{78} \ldots \sigma_{k-4,k-3} \sigma_{k-2,k-1}) \ldots (\sigma_{\frac{k}{2}+3,\frac{k}{2}+4} \sigma_{\frac{k}{2}+2,\frac{k}{2}+3} \sigma_{\frac{k}{2}+3,\frac{k}{2}+4}) \right]$ for $k$ odd.

A careful consideration of the topological effect of the braid on the lines $L_j$ shows that they are arranged as in figure 6. To reconstruct the configuration it is enough to know the admissible line $l$ (at angle $\epsilon$ w.r.t. the positive real axis). In fact, $u_1$ is the first point in $\Pi_L$ (in clockwise order) for $k$ even, or the last in $\Pi_R$ for $k$ odd. The lines come out of the points in centrifugal directions. They go to infinity, without intersections (so preserving their lexicographical order w.r.t $l$) with the original asymptotic direction $\frac{\pi}{2} - \epsilon$.

Proposition 1' The matrix representing $\beta'$ is

\[
A^\beta' = \begin{pmatrix}
1 & 1 \\
-\left(\frac{k}{k-3}\right) & 1 \\
-\left(\frac{k}{k-4}\right) & 0 & -\left(\frac{k}{k-5}\right) & 1 \\
-\left(\frac{k}{k-5}\right) & 0 & -\left(\frac{k}{k-4}\right) & 1 \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
& & & \\
\end{pmatrix}
\]
Figure 6: Lines $L_j$ (Branch cuts) after the braid which brings $S^{-1}$ to canonical form

for $k$ even, and

$$A^{\beta'} = 
\begin{pmatrix}
0 & 1 & \cdots & 1 \\
1 & -\left(\frac{k}{2} - 1\right) & \cdots & -\left(\frac{k}{2} - 1\right) \\
\vdots & \ddots & \ddots & \ddots \\
1 & -\left(\frac{k}{2} - 1\right) & \cdots & 1
\end{pmatrix}
$$

for $k$ odd.

The proposition is proved as in the previous section. Finally, the analogous of Theorem 2 holds:

**Theorem 2'** Consider the Stokes matrix $S^{-1} = T_F^{-\frac{k}{2}}T_F^{-\frac{k}{2}}$ (k even) or $S = T_F^{-\frac{k}{2}}K_{k/2}^{-1}T_F^{-\frac{k}{2}}$ (k odd) for the quantum cohomology of $\text{CP}^{k-1}$ and set it in the upper triangular form $S_{\text{upper}}^{-1} = P S^{-1} P^{-1}$ by the permutation $P$. Then, there exists a braid $\beta'$ (Lemma $8'$), represented by a matrix $A^{\beta'}$ (Proposition $1'$), which sets $S_{\text{upper}}^{-1}$ in the “canonical form”. The entries $s_{ij}$ have minus sign for $j > \frac{k}{2} + 1$, $i \leq \frac{k}{2} + 1$ and $k$ even, or $j > \frac{k + 3}{2}$, $i \leq \frac{k + 3}{2}$ and $k$ odd. A suitable conjugation by $\mathcal{L} = \text{diag}(1, 1, \ldots, -1, \ldots, -1)$ sets all signs positive and the final matrix has entries

$$s_{ij} = \begin{cases} 
\frac{k}{j - i} & \text{for } i < j, \\
0 & \text{for } i = j, \\
1 & \text{for } i > j
\end{cases}$$
Proof: The proof is similar to the one of theorem 2.
Examples are found in Appendix 2.

8 Relation between Irregular and Fuchsian systems

Let us consider the fuchsian system

\[(U - \lambda) \frac{d\phi}{d\lambda} = \left(\frac{1}{2} + V\right) \phi\]

which can also be written

\[\frac{d\phi}{d\lambda} = \sum_{j=1}^{k} \frac{A_j}{\lambda - u_j} \phi\]

\[A_j = -E_j \left(\frac{1}{2} + V\right), \quad (E_j)_{jj} = 1, \quad \text{otherwise} \quad (E_j)_{nm} = 0\]

Around the point \(u_j\) a fundamental matrix has the form

\[\begin{bmatrix} B_0 + O(\lambda - u_j) \end{bmatrix} (\lambda - u_j)^M\]

where \(M = \text{diag}(\frac{-1}{2}, 0, ..., 0)\) and the columns of \(B_0\) are the eigenvectors of \(A_j\); in particular, the first column is \((0, ..., 0, 1, 0, ..., 0)^T\), and 1 occurs at the \(j^{th}\) position. Then, the system has \(k\) independent vector solutions, of which \(k - 1\) are regular near \(u_j\) and the last is

\[\phi^{(j)}(\lambda) = \frac{1}{\sqrt{\lambda - u_j}} \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} + O\left(\sqrt{\lambda - u_j}\right) \quad \lambda \to u_j\]

where 1 occurs at the \(j^{th}\) row. For any \(u_j\) we can construct such a basis of solutions. The branch of \(\sqrt{\lambda - u_j}\) is chosen as follows: let us consider an angle \(\eta\) with a range of 2\(\pi\), for example \(-\frac{\pi}{2} \leq \eta < \frac{\pi}{2}\), such that \(\eta \neq \arg(u_i - u_j), \forall i \neq j\). Then consider the cuts \(L_j = \{\lambda = u_j + \rho e^{i\eta}, \quad -\pi < \rho < 0\}\). Actually, the cuts have two sides, \(L_j^+ = \{\lambda = u_j + \rho e^{i\eta}, \quad \rho > 0\}\) and \(L_j^- = \{\lambda = u_j + \rho e^{i(\eta + 2\pi)}, \quad \rho > 0\}\). The branch is determined by the choice \(\log(\lambda - u_j) = \log|\lambda - u_j| + i\eta\) on \(L_j^+\) and \(\log(\lambda - u_j) = \log|\lambda - u_j| + i(\eta + 2\pi)\) on \(L_j^-\). On \(\mathbb{C} \setminus \bigcup_j L_j\), \(\sqrt{\lambda - u_1}, ..., \sqrt{\lambda - u_k}\) are single valued.

For any two (column) vector solutions \(\phi(\lambda), \psi(\lambda)\) we define the symmetric bilinear form:

\[(\phi, \psi) := \phi(\lambda)^T (\lambda - U) \psi(\lambda)\]

which is independent of \(\lambda\) and \(u_1, ..., u_k\). Let \(G\) be the matrix whose entries are \(G_{ij} = (\phi^{(i)}, \phi^{(j)})\). In particular, \(G_{ii} = 1\). Then, it can be proved (see [2] and also [10]) that near \(u_j\)

\[\phi^{(i)}(\lambda) = G_{ij} \phi^{(j)}(\lambda) + r_{ij}(\lambda)\]

where \(r_{ij}(\lambda)\) is regular near \(u_j\). For a counter-clockwise loop around \(u_j\) the monodromy of \(\phi^{(i)}\) is

\[\phi^{(i)} \to R_{ij} \phi^{(i)} := \phi^{(i)} - 2 \frac{(\phi^{(i)}, \phi^{(j)})}{(\phi^{(i)}, \phi^{(j)})} \phi^{(j)} \equiv \phi^{(i)} - 2G_{ij} \phi^{(j)}\]

Then, the monodromy group of \((\mathbb{R})\) acts on \(\phi^{(1)}, ..., \phi^{(k)}\) as a reflection group whose Gram matrix is \(2G\). In particular, \(\phi^{(1)}, ..., \phi^{(k)}\) are linearly independent (and then a basis) if and only if \(\det G \neq 0\).

Now consider an oriented line \(l\) of argument \(\theta = \frac{\pi}{2} - \eta\), and for any \(j\) define the following vector

\[\tilde{Y}^{(j)} = -\frac{\sqrt{z}}{2\sqrt{\pi}} \int_{\gamma_j} d\lambda \, \phi^{(j)}(\lambda) \, e^{\lambda z}\]

(31)
which is a Laplace transform of \( \phi^{(j)} \). The path \( \gamma_j \) comes from infinity near \( L_j^+ \), encircles \( u_j \) and returns to infinity along \( L_j^- \). We can define \( \Pi_L = \{ \theta < \arg z < \theta + \pi \} \) and \( \Pi_R = \{ \theta - \pi < \arg z < \theta \} \). \( \lambda = \infty \) is a regular singularity for (33), then the integrals exist for \( z \in \Pi_L \), and the non-singular matrix
\[
\hat{Y}(z) := \left[ \hat{Y}^{(1)}|...|\hat{Y}^{(k)} \right]
\]
has the asymptotic behaviour
\[
\hat{Y}(z) \sim \left( I + O \left( \frac{1}{z} \right) \right) e^{zU} \quad z \to \infty, \quad z \in \Pi_L
\]
and satisfies the system (14). Then it is a fundamental matrix \( \hat{Y}_L \). Note that \( l \) is admissible, since it does not contain Stokes’ rays.

It is a fundamental result [10] that the Stokes’ matrix of (14) satisfies
\[
S + S^T = 2G
\]

9 Monodromy Group of the Quantum Cohomology of \( \mathbb{C}P^{k-1} \)

A system like (33) comes about in the theory of Frobenius manifolds (replace \( U \mapsto U(t) \), \( V \mapsto V(t) \)). It determines flat coordinates \( x^1(t, \lambda), ..., x^k(t, \lambda) \) for a linear pencil of metrics \(( , ) - \lambda < , > \) is the intersection form \([3] [13]\). We write a gauge equivalent form (gauge \( X(t^2) \)) at the semisimple point \((0, t^2, 0, ..., 0)\)
\[
(\hat{U}(t^2) - \lambda) \frac{d\psi}{d\lambda} = \left( \frac{1}{2} + \hat{\mu} \right) \psi
\]
A fundamental matrix \( \psi(t, \lambda) \) has entries \( \psi^{(j)}(t, \lambda) = \partial_\alpha x^{(j)}(t, \lambda) \). Moreover, by (31)
\[
\partial_\alpha \hat{V}(t, z) = -\frac{\sqrt{z}}{2\sqrt{\pi}} \int d\lambda \partial_\alpha x^1(t, \lambda) e^{\lambda z}
\]

The Monodromy group of the Frobenius manifold \( M \) is the group of the transformations which
\((x^1(t, \lambda), ..., x^k(t, \lambda))\) undergo when \( t \) moves in \( M \backslash \Sigma_\lambda \), where \( \Sigma_\lambda = \{ t \in M \mid \det [( , ) - \lambda < , >] = 0 \} \) is the discriminant of the linear pencil.

Due to formula (33), for \( \mathbb{C}P^{k-1} \) this group is generated by the monodromy of the solutions of (32) when \( \lambda \) describes loops around \( u_1(t), ..., u_k(t) \) (see [3] [13]). To these loops, we must add the effect of the displacement \( t^2 \mapsto t^2 + 2\pi i \). In fact, in this case
\[
[\varphi^{(1)}(ze^{i\pi}), ..., \varphi^{(k)}(ze^{i\pi})] \mapsto [\varphi^{(1)}(ze^{i\pi}), ..., \varphi^{(k)}(ze^{i\pi})] T
\]
and the same holds for \( \tilde{t}(z, (0, t^2, ..., 0)) \).

Then, the monodromy group of the quantum cohomology of \( \mathbb{C}P^{k-1} \) is generated by the transformations \( R_1, R_2, ..., R_k, T \) introduced in the preceding sections.

We are going to study the structure of the monodromy group of \( \mathbb{C}P^{k-1} \) for any \( k \geq 3 \). Recall that the matrix \( S \) for (14) is not upper triangular, because in \( U \) the order of \( u_1, ..., u_k \) is not lexicographical w.r.t. the line \( l \). Then, Coxeter identity is \(-S^{-1}S^T = \text{product of the } R_j \)'s in the order referred to \( l \). For example, for \( k = 3 \), \( S^{-1}S^T = -R_3R_3R_1 \), since the lexicographical ordering would be \( u_2, u_3, u_1 \). From the identity \( S^{-1}S^T = (-1)^{k-1}T^k \) it follows a first general relation in the group
\[
T^k = (-1)^{k-1} \text{ product of } R_j \text{’s in suitable order}
\]

Two cases must now be distinguished.

\( k \) odd: As a general result [3], \( \det G = 0 \) if and only if \( V + \frac{1}{2} \) has an integer eigenvalue. The eigenvalues of \( V \) are \( \frac{-k-1}{2}, \frac{-k-3}{2}, ..., \frac{k-1}{2} \). Then, for \( k \) odd, \( \det G \neq 0 \), and \( \phi^{(1)}, ..., \phi^{(k)} \) are a basis. The matrices \( R_j \) are
\[
R_j = \begin{pmatrix}
1 & & & \\
& \ddots & & \\
& & \ddots & \\
-2G_{j1} & -2G_{j2} & ... & -1 & ... & -2G_{jk} \\
& & & 1
\end{pmatrix}
\]
where $S^T + S = 2G$.

In concrete examples, we have “empirically” found other relations like

$$R_2 = p_1(T, R_1)$$
$$R_2 = p_2(T, R_1)$$
$$\vdots$$
$$R_k = p_k(T, R_1)$$

where $p_j(T, R_1)$ means a product of the elements $T$ and $R_1$. We have also found the relation

$$(TR_1)^k = -I$$

We investigated the following cases:

**CP²** ($k = 3$)

$$R_2 = TR_3T^{-1}, ~ R_3 = T(R_1R_2R_1)T^{-1}$$
$$(TR_1)^3 = -I$$
$$T^3 = -R_2R_3R_1$$

**CP⁴** ($k = 5$)

$$\begin{cases}
R_2 = TR_3T^{-1}, & R_3 = TR_2T^{-1} \\
R_4 = T(R_2R_3R_2)T^{-1}, & R_5 = T^{-1}(R_2R_1R_2)T
\end{cases}$$
$$(TR_1)^5 = -I$$
$$T^5 = -R_3R_4R_2R_3R_1$$

**CP⁶** ($k = 7$)

$$\begin{cases}
R_2 = TR_3T^{-1}, & R_3 = TR_2T^{-1}, & R_4 = TR_3T^{-1} \\
R_5 = T(R_3R_4R_3)T^{-1}, & R_6 = T(TR_1)^3R_2[T(TR_1)^3]^{-1} \\
R_7 = T^{-1}(R_3R_2R_3)T^2
\end{cases}$$
$$(TR_1)^7 = -I$$
$$T^7 = -R_4R_5R_6R_3R_2R_7R_1$$

Note that one relation, for example that for $R_6$, can be derived from the others, and that just $R_1, T, -I$ are enough to generate the monodromy group in each of the examples. They satisfy (in the examples) the relations:

$$R_1^2 = (-I T R_1)^k = (-I)^2 = I$$
$$R_1(-I) ((-I)R_1)^{-1} = I, \quad T(-I) ((-I)T)^{-1} = I$$

The last two relations mean simply the commutativity of $-I$ with $R_1$ and $T$. The relations are not only satisfied, but also “fulfilled” (namely, $(-I T R_1)^n \neq I$ for $n < k$). Now call

$$X := R_1, \quad Y := -ITR_1, \quad Z = -I$$

These elements generate the monodromy group of **CP⁶⁻¹** with at least the relations

$$X^2 = Y^k = Z^2 = 1$$
$$(ZX)(XZ)^{-1} = 1, \quad (ZY)(YZ)^{-1} = 1$$

Note that $Z$ generates the cyclic group $C_2$ of order 2.

If there were no other relations (which we did not find “empirically”), we would conclude that the monodromy group of the quantum cohomology of **CP⁶** (in the examples) is isomorphic to the direct product

$$< X, Y \mid X^2 = Y^k = 1 > \times C_2$$

where $< X, Y \mid X^2 = Y^k = 1 >$ means the group generated by $X, Y$ with relations $X^2 = Y^k = 1$. 

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$k$ even: Now $\det G = 0$, since $V + \frac{1}{2}$ has integer eigenvalues. $G$ has rank $k - 1$ and the eigenspace of its eigenvalue 0 has dimension 1. Let $(z^1, ..., z^k)^T$ be an eigenvector of eigenvalue 1. The vector $v := \sum_{j=1}^k z^j \phi^{(j)}$ is zero, because

$$(v, \phi^{(i)}) = \sum_{j=1}^k z^j G_{ji} = 0 \ \forall i$$

then

$$z^1 \phi^{(1)} + z^{(2)} \phi^{(2)} + ... + z^k \phi^{(k)} = 0$$

and $k - 1$ of the $\phi^{(j)}$’s are linearly independent. The fuchsian system (31) has a regular (vector) solution $\phi_0(\lambda) = \sum_{n=0}^d \phi_n \lambda^n$, where $\phi_n$ are constant (column) vectors, and $\phi_d$ is the eigenvector of $V + \frac{1}{2}$ relative to the largest integer eigenvalue less or equal to zero; this eigenvalue is precisely $-d$ (see [2]). In our case, $d = 0$ and $\phi_0(\lambda) = \phi_0$, a constant vector. $\phi^{(1)}, \phi^{(2)}, ..., \phi^{(k-1)}, \phi_0$ is then a possible choice for a basis of solutions.

Observe that in the gauge equivalent form $\psi = X \phi$, $\psi_0$ is the eigenvector of $\frac{1}{2} + \hat{\mu}$ with eigenvalue zero. Then

$$\psi_0 = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix} \equiv \frac{\partial x}{\partial x}$$

where all the entries are zero but the one at position $\frac{1}{2} + 1$. $x$ is the flat coordinate for $(\ , \ ) - \lambda < , >$ corresponding to $\psi_0$. Then, we can choose the following flat coordinates:

$$x^1(\lambda, t), \ x^2(\lambda, t), \ ..., \ x^{k-1}(\lambda, t), \ t^{\frac{k}{2}+1}$$

The monodromy group then acts on a $k - 1$ dimensional space.

Let us determine the reduction of $R_1, R_2, ..., R_k, T$ to the $k - 1$ dimensional space. The entries of $T$ on the vectors $\phi^{(j)}$ are: $T \phi^{(i)} = \sum_{j=1}^k T_{ji} \phi^{(j)}$, $i = 1, ..., k$. On the new basis $\phi^{(i)}, ..., \phi^{(k-1)}, \phi_0$ the matrices are rewritten

$$R_j \phi^{(i)} = \phi^{(i)} - 2G_{ij} \phi^{(j)} \ \ i = 1, ..., k - 1 \ \ j \neq k$$

$$R_j \phi_0 = \phi_0 \ \ \ \ \ \ j \neq k$$

$$R_k \phi^{(i)} = \phi^{(i)} - 2G_{ik} \left( -\frac{1}{z^k} \sum_{j=1}^{k-1} z^j \phi^{(j)} \right) \ \ i \neq k$$

$$T \phi^{(i)} = \sum_{j=1}^{k-1} T_{ji} \phi^{(j)} + T_{ki} \left( -\frac{1}{z^k} \sum_{j=1}^{k-1} z^j \phi^{(j)} \right)$$

Then the matrices assume a reduced form

$$\begin{pmatrix} * & * & 0 \\ * & * & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

We studied two examples; besides Coxeter identity $T^k =$ product of $R_j$’s, we found relations similar to the case $k$ odd:

$$R_2 = p_1(T, R_1)$$

$$R_2 = p_2(T, R_1)$$

$$\vdots$$

$$R_k = p_k(T, R_1)$$

and

$$(TR_1)^k = I$$

Namely:
$$\mathbb{CP}^4 \ (k = 4)$$

\[
\begin{align*}
R_2 &= TR_1 T^{-1}, \quad R_3 = TR_2 T^{-1} \\
R_4 &= T^{-1} (R_2 R_1 R_2) \ T \\
\end{align*}
\]

\[
(T R_1)^4 = I \\
T^4 = R_3 R_2 R_4 R_1
\]

$$\mathbb{CP}^5 \ (k = 6)$$

\[
\begin{align*}
R_2 &= TR_1 T^{-1}, \quad R_3 = TR_2 T^{-1} \\
R_4 &= TR_3 T^{-1}, \quad R_6 = T (R_2 R_3 R_4 R_2) \ T^{-1} \\
R_5 &= T^{-1} (R_2 R_1 R_2) \ T \\
\end{align*}
\]

\[
(T R_1)^6 = I \\
T^6 = R_4 R_3 R_5 R_2 R_6 R_1
\]

The same remarks of $k$ odd hold here. Call

\[
X := R_1, \quad Y := R_1 T
\]

then, if there were no other hidden relations, the monodromy group of the quantum cohomology of $\mathbb{CP}^k$ (in the examples) would be isomorphic to

\[
< X, Y, \ | \ X^2 = Y^k = 1 >
\]

Note that $< X, Y, \ | \ X^2 = Y^k = 1 >$ is (isomorphic to) the subgroup of orientation preserving transformations of the hyperbolic triangular group $[2, k, \infty]$.

**Lemma 10:** The subgroup of the orientation preserving transformations of the hyperbolic triangular group $[2, k, \infty]$ is isomorphic to the subgroup of $\text{PSL}(2, \mathbb{R})$ generated by

\[
\begin{align*}
\tau \mapsto -\frac{1}{\tau} \\
\tau \mapsto \frac{1}{2 \cos \frac{\pi}{k} - \tau}
\end{align*}
\]

$\tau \in H := \{ z \in \mathbb{C} \mid \Im z > 0 \}$

**Proof:** Consider three integers $m_1, m_2, m_3$ such that

\[
\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} < 1
\]

In the Bolyai-Lobatchewsky plane $H$, the triangular group $[m_1, m_3, m_3]$ of hyperbolic reflections in the sides of hyperbolic triangles of angles $\frac{\pi}{m_1}, \frac{\pi}{m_2}, \frac{\pi}{m_3}$ is generated by three reflections $r_1, r_2, r_3$ satisfying the relations

\[
r_1^2 = r_2^2 = r_3^2 = (r_2 r_3)^{m_1} = (r_3 r_1)^{m_2} = (r_2 r_1)^{m_3} = 1
\]

and the subgroup of orientation preserving transformation is generated by $X = r_2 r_3, Y = r_3 r_1$. Then

\[
X^{m_1} = Y^{m_2} = (XY)^{m_3} = 1
\]

For $m_1 = 2, m_2 = k, m_3 = \infty$, a fundamental triangular region is $\{ 0 < \Re z < \cos \frac{\pi}{k} \} \cap \{|z| > 1\}$. Then

\[
r_1(\tau) = -\bar{\tau}, \quad r_2(\tau) = \frac{1}{\tau}, \quad r_3(\tau) = 2 \cos \frac{\pi}{k} - \bar{\tau}
\]

The bar means complex conjugation. Then

\[
X(\tau) = -\frac{1}{\tau}, \quad Y(\tau) = \frac{1}{2 \cos \frac{\pi}{k} - \tau}
\]

\[\square\]

**Remark:** The orientation preserving transformations of $[2, 3, \infty]$ are the modular group $\text{PSL}(2, \mathbb{Z})$. 

38
Theorem 3: The monodromy group of the quantum cohomology of $\mathbb{CP}^2$ is isomorphic to

$$\langle X, Y, \mid X^2 = Y^3 = 1 \rangle \times C_2 \cong PSL(2, \mathbb{Z}) \times C_2$$

(34)

The monodromy group of the quantum cohomology of $\mathbb{CP}^3$ is isomorphic to

$$\langle X, Y, \mid X^2 = Y^4 = 1 \rangle \cong \text{orient. preserv. transf. of } [2, 4, \infty]$$

(35)

The theorem for the case of $\mathbb{CP}^2$ is already proved in [10].

Proof: a) $\mathbb{CP}^2$:

$$R_1 = \begin{pmatrix} -1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 0 & 1 \\ -1 & 3 & 3 \\ 0 & -1 & 0 \end{pmatrix}$$

and $X = R_1, Y = -ITR_1$ and $Z = -I$ satisfy the relations of (34). They act on the column vector $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. The quadratic form $q(x, y, z) = x^T G x$ is $R_1$ and $T$-invariant. Then $T, R_1$ act on two dimensional invariant subspaces $q(x, y, z) = \text{constant}$. On each of these subspaces we introduce new coordinates $\chi \in \mathbb{R}$ and $\varphi \in [0, 2\pi)$. Let $\tau = e^{i\chi} e^{i\varphi}$ and

$$x = \frac{a}{2}(\tau - \bar{\tau}) \quad y = \frac{a}{2}(\tau - \bar{\tau}) \quad z = \frac{a}{2}(\tau - \bar{\tau})$$

$a \in \mathbb{R}, a \neq 0$. Note that $q(x, y, z) = a^2 > 0$. Then, it is easily verified that

$$x \left( \frac{1}{\tau} \right) = -X x(\tau)$$

$$x \left( \frac{1}{1 - \tau} \right) = Y x(\tau)$$

$$x(\tau, -a) = Z x(\tau, a)$$

This implies the 1 to 1 correspondence between the generators of the modular group and $X$ and $Y$.

b) Case of $\mathbb{CP}^3$.

$$R_1 = \begin{pmatrix} -1 & 4 & -10 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T = \begin{pmatrix} 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 3 \\ 0 & -1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The matrices are already written on $\phi^{(1)}, \phi^{(2)}, \phi^{(3)}, \phi_0$. Recall that the monodromy acts only on $x^1, x^2, x^3$, because the last flat coordinate is $t^3$. This action is given by the following three dimensional matrices, acting on a three dimensional space of vectors $x = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$$r_1 = \begin{pmatrix} -1 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad t := \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 3 \\ 0 & -1 & 3 \end{pmatrix}$$

We redefine $X = r_1$ and $Y = tr_1$, which satisfy the relations (35). We proceed as above, defining

$$x = a(\tau \bar{\tau} - \frac{1}{\sqrt{2}}(\tau + \bar{\tau}) + \frac{1}{3}) \frac{i}{\tau - \bar{\tau}}$$
\[ y = a \left( \frac{2}{3} \tau \bar{\tau} - \frac{2\sqrt{2}}{3} (\tau + \bar{\tau}) + \frac{2}{3} \frac{i}{\tau - \bar{\tau}} \right) \]
\[ z = a \left( \frac{1}{3} \tau \bar{\tau} - \frac{\sqrt{2}}{6} (\tau + \bar{\tau}) + \frac{1}{3} \frac{i}{\tau - \bar{\tau}} \right) \]

\( a \neq 0 \). Note that \( x^T g x = (8/9)a^2 \), where \( g \) is the \( 3 \times 3 \) reduction of \( G \). It is easily verified that

\[ x \left( \frac{1}{\tau} \right) = -X x(\tau) \]
\[ x \left( \frac{1}{\sqrt{2} - \tau} \right) = -Y x(\tau) \]

which proves the theorem.

\[ \square \]

APPENDIX 1: Proof of Lemma 5

Let us consider the function

\[ g(z) = C \int_{-c-i\infty}^{-c+i\infty} ds \; \Gamma^k(-s) e^{i\pi f s} z^{ks} \]

where \( C \) and \( f \) are constants to be determined later, \( c > 0 \). The path of integration is a vertical line through \(-c\).

a) Domain of definition. We use Stirling formula

\[ \Gamma(-s) = e^s e^{-(s+\frac{1}{2})\log(-s)} \sqrt{2\pi(1+O(1/s))} \quad s \to \infty, \quad |\arg(-s)| < \pi \]

where \( \log(-s) = \log(s) + i\pi \).

The integrand is

\[ \Gamma^k(-s) e^{i\pi f s} z^{ks} \sim e^{ks(1+\log z)} e^{-k(s+\frac{1}{2})\log s} e^{-ik\pi(s+\frac{1}{2})+i\pi sf} \quad s \to \infty \]

Now let \( s = -c + i\eta \), \( ds = id\eta \). The integral in \( \eta \) is on the real axis. The dominant part in the integrand is

\[ e^{\eta[k \arg(z-c)+k\pi-k\arg z-\pi f]} \]

The condition of (uniform) convergence of the integral is

\[ -\pi/2 - f \pi/k < \arg z < \pi/2 - f \pi/k \]

b) \( g \) solves \([K]\). In fact (for simplicity \( C = 1 \) here):

\[ (z\partial_z)^k g(z) = k^k \int_{-c-i\infty}^{-c+i\infty} ds \; \Gamma^k(-s) e^{i\pi f s} z^{ks} = (-1)^k k^k \int_{-c-i\infty}^{-c+1+i\infty} ds \; \Gamma^k(1-s) e^{i\pi f s} z^{ks} \]

where we have used the identity \( s\Gamma(-s) = -\Gamma(1-s) \). Now let \( t = s-1 \):

\[ (z\partial_z)^k g(z) = (-1)^k k e^{i\pi f} (zk)^k \int_{-c-1-i\infty}^{-c-1+i\infty} dt \; \Gamma^k(-t) e^{i\pi f t} z^{kt} \]

\[ = (-1)^k k e^{i\pi f} (zk)^k g(z) \]

Now we impose \((-1)^k e^{i\pi f} = 1\), namely

\[ f = k + 2m \quad m \in \mathbb{Z} \]

Point b) is proved.
c) Asymptotic behaviour of $g(z)$. We use Laplace method for analytic functions (the so called “steepest descent method”). Put $C = 1$. By Stirling’s formula

$$g(z) = (2\pi)^k e^{i\pi k} \int_{-\epsilon-i\infty}^{-\epsilon+i\infty} ds \ e^{\phi(s)}$$

$$\phi(s) = s \ [k(1+\log z) + i\pi f - i\pi k] - k \left( s + \frac{1}{2} \right) \log s + O\left( \frac{1}{s} \right)$$

The stationary point of $\phi(s)$ is

$$s_0 = z e^{-i\pi(k-f)} - \frac{1}{2} + O\left( \frac{1}{z} \right) \quad z \to \infty$$

In the hypothesis $z \to \infty$ a straightforward computation gives

$$\phi(s_0) \sim k z e^{-i\pi(k-f)} - \frac{k}{2} \log z + i\frac{\pi}{2} (k-f) \quad \frac{d^2 \phi(s_0)}{ds^2} \sim -\frac{k}{z} e^{i\pi(k-f)}$$

We are ready to apply the steepest descent method. We deform the path of integration in such a way that it passes through $s_0$. Let us call it $\gamma$.

$$g(z) = (2\pi)^k e^{-i\pi k} e^{\phi(s_0)} \int_{\gamma} ds \ e^{\phi(s)-\phi(s_0)}$$

$$\sim (2\pi)^k e^{-i\pi k} e^{\phi(s_0)} \int_{\gamma} ds \ e^{\frac{1}{2} \frac{d^2 \phi(s_0)}{ds^2} (s-s_0)^2}$$

Let us divide $\gamma$ in two paths: $\gamma_1$ from $s_0$ to $+\infty$ and $\gamma_2$ from $-\infty$ to $s_0$. The integrals becomes the sum of two integrals. In $\int_{\gamma_1}$ we change variable. Let $\tau > 0$ and

$$-\tau^2 = \frac{1}{2} \frac{d^2 \phi(s_0)}{ds^2} (s-s_0)^2 = -\frac{k}{2z} \left[ e^{i\pi(k-f)} (s-s_0)^2 \right]$$

Then

$$\int_{\gamma_1} e^{\frac{1}{2} \frac{d^2 \phi(s_0)}{ds^2} (s-s_0)^2} = \frac{1}{2} \sqrt{\frac{\pi}{k}} (2\pi)^k e^{-i\pi(k-f)-i\pi} + O(e^{-\alpha |z|}) \quad \alpha > 0$$

Note that $\int_{\gamma_1} = \int_{\gamma_2}$. Now, recalling that $f = k + 2m$, we conclude that

$$g(z) \sim (2\pi)^e e^{-i(m+1)\pi} e^{-i\pi k} \frac{1}{\sqrt{k}} \frac{e^{i\pi m}}{z^{\frac{1}{2}+n}} \exp(ke^{i\pi m}z) \quad z \to \infty$$

If we choose

$$m = n - 1 + kl \quad l \in \mathbb{Z}$$

and $C = \left[ (2\pi)^e e^{i\pi(l-k-nkl)} \right]^{-1}$ we have

$$g(z) \sim \frac{1}{\sqrt{k}} \frac{e^{i\pi (n-1)}}{z^{\frac{1}{2}+n}} \exp(ke^{i\pi (n-1)}z)$$

Then $g$ is a solution $\varphi^{(n)}$ on the domain

$$-\frac{3\pi}{2} - 2(n-1)\frac{\pi}{k} - 2\pi l < \arg z < -\frac{\pi}{2} - 2(n-1)\frac{\pi}{k} - 2\pi l$$

d) We now determine the domain where the analytic extension of $g(z)$ still has the above asymptotic behaviour. From Lemma 4 we derived that the first entries of the $n(k)$th columns of $Y_L$ and $Y_R$ are equal and coincide with $\varphi^{(n(k))}_L(z) \equiv \varphi^{(n(k))}_R(z)$ times $z^{\frac{k-1}{2}}$, and $\varphi^{(n(k))}$ has the established asymptotic behaviour on the enlarged domains

$$-\frac{\pi}{k} - \pi < \arg z < \pi + \frac{\pi}{k}$$
\[-\pi < \arg z < \pi + \frac{2\pi}{k}\]

If in \(g(z)\) we chose \(n = n(k)\) and \(l = -1\) the integral representation holds for \(-\frac{\pi}{2} < \arg z < \frac{\pi}{2}\) for \(k\) even, and for \(-\frac{\pi}{2} < \arg z < \frac{\pi}{2} + \frac{\pi}{k}\) for \(k\) odd. Then its analytic continuation is precisely \(\varphi^{n(k)}(z) \equiv \varphi_R^{n(k)}(z)\).

\((e)\) We prove the identity \(\left[6\right]\). Observe that \((z\partial_z)^k \varphi = (kz)^k \varphi\) is invariant for \(z \mapsto ze^{\frac{2\pi i}{k}}\). A generic solution can be represented near \(z = 0\) as

\[
\varphi(z) = \sum_{m=0}^{\infty} \frac{z^m}{(m!)^k} \left[ a_m^{(1)} + a_m^{(2)} \log z + \ldots + a_m^{(k)} \log^{k-1} z \right]
\]

It follows that the operator \((A\varphi)(z) = \varphi(ze^{\frac{2\pi i}{k}})\) has eigenvalues \(1\), because on the basis obtained with \((a_0^{(1)} = 1, a_0^{(2)} = \ldots = a_0^{(k)} = 0), (a_0^{(1)} = 0, a_0^{(2)} = 1, a_0^{(3)} = \ldots = a_0^{(k)} = 0), \ldots, (a_0^{(1)} = \ldots = a_0^{(k-1)} = 0, a_0^{(k)} = 1)\) it is represented by a lower triangular matrix having 1’s on the diagonal. Then \((A - 1)^k g(z) = 0\) and

\[(A - 1)^k g(z) = 0\]

is precisely our identity. Lemma 4 is proved.

\[\square\]

**APPENDIX 2:**

First we give \(\Phi_R\) and \(\Phi_L\). \(k\) odd:

\[
\begin{bmatrix}
(-1)^{k-1} \left( \frac{k-1}{2} \right) g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})}) - \left( \frac{k}{1} \right) g(ze^{\frac{2\pi i}{k}(\frac{k-3}{2})}) + \ldots + \left( \frac{k}{k-1} \right) g(ze^{\frac{2\pi i}{k}(\frac{k-3}{2})}) \\
\vdots \\
(-1)^{k-1} g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})})
\end{bmatrix}
\]

\[
\phi_R(z) = 
\begin{bmatrix}
(-1)^{k-1} g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})}) - \frac{1}{2} g(z) + \frac{k-1}{2} g(ze^{\frac{2\pi i}{k}(\frac{k-3}{2})}) \\
\vdots \\
(-1)^{k-1} g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})})
\end{bmatrix}
\]

\[
\begin{bmatrix}
(-1)^{k-1} \left( \frac{k-1}{2} \right) g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})}) - \left( \frac{k}{1} \right) g(ze^{\frac{2\pi i}{k}(\frac{k-3}{2})}) + \ldots + \left( \frac{k}{k-1} \right) g(ze^{\frac{2\pi i}{k}(\frac{k-3}{2})}) \\
\vdots \\
(-1)^{k-1} g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})})
\end{bmatrix}
\]

\[
\phi_L(z) = 
\begin{bmatrix}
(-1)^{k-1} g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})}) - \frac{1}{2} g(z) + \frac{k-1}{2} g(ze^{\frac{2\pi i}{k}(\frac{k-3}{2})}) \\
\vdots \\
(-1)^{k-1} g(ze^{\frac{2\pi i}{k}(\frac{k-1}{2})})
\end{bmatrix}
\]
\( k \) even:

\[
\Phi_{HI}(z)^T = \begin{bmatrix}
\vdots \\
g(z(xe^{-i\pi}) - \left( \frac{k}{k-1} \right) g(z(xe^{-i\pi} + \frac{2\pi i}{k})) + \cdots + \left( \frac{k}{k-1} \right) g(z(xe^{-i\pi} - \frac{2\pi i}{k})) \\
\vdots
\end{bmatrix}
\]

\[
\Phi_{A}(z)^T = \begin{bmatrix}
\vdots \\
g(z(xe^{-i\pi})) \\
\vdots
\end{bmatrix}
\]

We give all the matrices of interest up to \( k = 10 \). \( S_{\text{upper}} \) is \( PSP^{-1} \). \( A \) stands for \( A^\beta, A' \) for \( A^\beta' \).

**CP**

\( S^3 = A S_{\text{upper}} A^T, S^3' = A' S_{\text{upper}}^{-1} [A']^T. \sigma_{i,i+1} := \beta_{i,i+1}^{-1} \)

\[
K_{12} = \begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \quad K_{13} := \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-3 & 0 & 1
\end{bmatrix} \quad K_{32} := \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 3 \\
0 & 0 & 1
\end{bmatrix}
\]

\[
T := \begin{bmatrix}
0 & 0 & 1 \\
-1 & 3 & 3 \\
0 & -1 & 0
\end{bmatrix}
\]

\[
S := \begin{bmatrix}
1 & 0 & 0 \\
-3 & 1 & 3 \\
-3 & 0 & 1
\end{bmatrix} \quad PSP^{-1} = \begin{bmatrix}
1 & 3 & -3 \\
0 & 1 & -3 \\
0 & 0 & 1
\end{bmatrix}
\]

**CP**

\[
K_{42} := \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 6 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix} \quad K_{43} := \begin{bmatrix}
1 & 0 & 0 & 0 \\
-4 & 1 & 0 & 0 \\
0 & 0 & 1 & 4 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
T := \begin{bmatrix}
0 & 0 & 0 & 1 \\
-1 & 0 & 6 & 4 \\
0 & -1 & 4 & 0 \\
0 & 0 & -1 & 0
\end{bmatrix} \quad T_1 := \begin{bmatrix}
0 & 0 & 1 & 0 \\
-1 & 0 & 3 & 0 \\
0 & -1 & 3 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

\[
S := \begin{bmatrix}
1 & 0 & 0 & 0 \\
-4 & 1 & 0 & 6 \\
10 & -4 & 1 & -20 \\
-4 & 0 & 0 & 1
\end{bmatrix}
\]

\[
P := \begin{bmatrix}
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0
\end{bmatrix} \quad S_{\text{upper}} = \begin{bmatrix}
1 & -4 & -20 & 10 \\
0 & 1 & 6 & -4 \\
0 & 0 & 1 & -4 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]
\[ A := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 4 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A' := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -4 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad S^3 = S'^{3'} := \begin{bmatrix} 1 & 4 & 6 & -4 \\ 0 & 1 & 4 & -6 \\ 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \]

where \( \beta = \beta_{12} \)

**CP⁴**

\[
K_{52} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 10 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad K_{53} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
T := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 10 & 5 \\ 0 & -1 & 5 & 10 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \end{bmatrix}, \quad S := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 10 \\ 15 & -5 & 1 & 5 & -40 \\ -40 & -10 & 0 & 1 & -95 \\ -5 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
P := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad S_{upper} := \begin{bmatrix} 1 & 5 & -5 & -40 & 15 \\ 0 & 1 & -10 & -95 & 40 \\ 0 & 0 & 1 & 10 & -5 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
A := \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 5 & 0 & 0 \\ 0 & 1 & 10 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad A' := \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 5 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -5 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}
\]

\[
S'^3 = S'^{3'} := \begin{bmatrix} 1 & 5 & 10 & 10 & -5 \\ 0 & 1 & 5 & 10 & -10 \\ 0 & 0 & 1 & 5 & -10 \\ 0 & 0 & 0 & 1 & -5 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

where \( \beta = \beta_{23}, \beta' = \beta_{12} s_{45} \).

**CP⁵**

\[
K_{62} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 15 & 0 \\ 0 & 0 & 1 & 10 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad K_{63} := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
\]

\[
T := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 15 \\ 0 & -1 & 0 & 15 & 20 \\ 0 & 0 & -1 & 6 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix}, \quad S := \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -6 & 1 & 0 & 0 & 0 \\ 21 & -6 & 1 & 0 & 15 \\ -56 & 21 & -6 & 1 & -84 \\ 210 & -56 & -20 & 0 & 0 \\ -6 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
T_1 := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 & 5 \\ 0 & -1 & 0 & 15 & 25 \\ 0 & 0 & -1 & 6 & 1 \\ 0 & 0 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}
\]

\[
P := \begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}
\]
\[
\beta_{CP} = \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 6 & 0 & 0 \\
1 & 6 & 0 & 15 & 0 & 0 \\
0 & 0 & 1 & 20 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
\beta'_{CP} = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -20 & 1 & 0 & 0 \\
0 & 0 & -15 & 0 & -6 & 1 \\
0 & 0 & 6 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]
\[
S^\beta := \begin{bmatrix}
1 & 6 & 15 & 20 & 15 & -6 \\
0 & 1 & 6 & 15 & 20 & -15 \\
0 & 0 & 0 & 1 & 6 & -15 \\
0 & 0 & 0 & 0 & 1 & -6 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
S'^\beta := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

where \( \beta = \beta_{12}(\beta_{34}\beta_{23}\beta_{12}), \beta' = [(\sigma_{34}\sigma_{56})\sigma_{45}]\sigma_{56}. \)

\( \text{CP}^6 \)

\[
A := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
A' := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & -20 & 1 & 0 & 0 \\
0 & 0 & -15 & 0 & -6 & 1 \\
0 & 0 & 6 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0
\end{bmatrix}
\]
\[
K_{72} := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 35 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
K_{73} := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-7 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
T := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 21 \\
0 & -1 & 0 & 0 & 0 & 21 \\
0 & 0 & -1 & 0 & 0 & 21 \\
0 & 0 & 0 & -1 & 0 & 21 \\
0 & 0 & 0 & 0 & -1 & 21
\end{bmatrix}
\]
\[
S := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
-7 & 1 & 0 & 0 & 0 & 0 \\
-7 & 1 & 0 & 0 & 0 & 0 \\
-7 & 1 & 0 & 0 & 0 & 0 \\
-7 & 1 & 0 & 0 & 0 & 0 \\
-7 & 1 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
P := \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
\[
A := \begin{bmatrix}
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 7 & 0 \\
1 & 0 & 7 & 0 & 21 & 0 \\
0 & 1 & 21 & 0 & 35 & 0 \\
0 & 0 & 0 & 1 & 35 & 0 \\
0 & 0 & 0 & 0 & 1 & 1
\end{bmatrix}
\]
\[
A' := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 7 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & -35 & 1 & 0 \\
0 & 0 & 0 & -21 & 0 & -7 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
S^\beta := \begin{bmatrix}
1 & 7 & 21 & 35 & 21 & -7 \\
0 & 1 & 7 & 21 & 35 & -21 \\
0 & 0 & 1 & 7 & 21 & 35 \\
0 & 0 & 0 & 1 & 7 & -21 \\
0 & 0 & 0 & 0 & 1 & -7 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
S'^\beta := \begin{bmatrix}
1 & 7 & 21 & 35 & 21 & -7 \\
0 & 1 & 7 & 21 & 35 & -21 \\
0 & 0 & 1 & 7 & 21 & 35 \\
0 & 0 & 0 & 1 & 7 & -21 \\
0 & 0 & 0 & 0 & 1 & 7 \\
0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

where \( \beta = (\beta_{23}\beta_{12})(\beta_{45}\beta_{34}\beta_{23}\beta_{12}), \beta' = \beta_{12}[(\sigma_{45}\sigma_{56})\sigma_{56}]\sigma_{57}. \)

\( \text{CP}^7 \)

\[
K_{82} := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 28 \\
0 & 0 & 1 & 0 & 0 & 0 & 70 \\
0 & 0 & 0 & 1 & 0 & 0 & 28 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
\[
K_{83} := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-8 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 56 \\
0 & 0 & 0 & 1 & 0 & 0 & 56 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
where $\beta = (\beta_{34}\beta_{23}\beta_{12})\beta_{23}(\beta_{36}\beta_{45}\beta_{34}\beta_{23}\beta_{12})$, $\beta' = [(\sigma_{44}\sigma_{55}\sigma_{78})(\sigma_{45}\sigma_{67})\sigma_{56}][(\sigma_{67}\sigma_{78})\sigma_{67}].$

\[C_{08}
\]

\[K_{02} := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 36 \\
0 & 0 & 1 & 0 & 0 & 0 & 126 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 84 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]

\[K_{03} := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 126 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 84 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
\]
where $\beta = (\beta_{15}\beta_{54}\beta_{23}\beta_{12})(\beta_{34}\beta_{23})(\beta_{67}\beta_{56}\beta_{45}\beta_{34}\beta_{23}\beta_{12})$, and $\beta' = \beta_{12}[\sigma_{15}\sigma_{67}\sigma_{89}](\sigma_{56}\sigma_{78})\sigma_{67}][\sigma_{78}\sigma_{89}].$

$\mathbf{Cp}^g$
\[ S := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
55 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-220 & 55 & -10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
715 & -220 & 55 & -10 & 1 & 0 & 0 & 0 & 0 & 0 \\
-2002 & 715 & -220 & 55 & -10 & 1 & 0 & 0 & 0 & 0 \\
17160 & -4752 & 990 & -120 & 0 & 0 & 0 & 0 & 0 & 0 \\
-11880 & 2310 & -252 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1155 & -120 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ K_{10,2} := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \]

\[ K_{10,3} := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-10 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix} \]

\[ T := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 10 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 210 & 120 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 210 & 252 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 45 & 120 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 10 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
\end{bmatrix} \]

\[ P := \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix} \]

\[ S_{upper} := \begin{bmatrix}
1 & -10 & -440 & 55 & 10395 & -220 & -34320 & 715 & 15015 & -2002 \\
0 & 1 & 45 & -10 & -1980 & 55 & 9240 & -220 & -5148 & 715 \\
0 & 0 & 1 & -120 & -25190 & 990 & 177705 & -4752 & -120120 & 17160 \\
0 & 0 & 0 & 1 & 210 & -10 & -1848 & 55 & 1485 & -220 \\
0 & 0 & 0 & 0 & 1 & -252 & -52910 & 2310 & 73755 & -11880 \\
0 & 0 & 0 & 0 & 0 & 1 & 210 & -10 & -330 & 55 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & -120 & -5390 & 1155 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 45 & -10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -10 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \\
\end{bmatrix} \]
\[
A = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 10 & 0 & 0 \\
0 & 1 & 0 & 10 & 0 & 45 & 0 & 0 \\
1 & 10 & 0 & 45 & 0 & 120 & 0 & 210 & 0 & 0 \\
0 & 0 & 1 & 120 & 0 & 210 & 0 & 252 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 252 & 0 & 210 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 120 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
A' := \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -120 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -210 & 0 & -252 & 1 & 0 & 0 & 0 \\
0 & 0 & -252 & 0 & -210 & 0 & -120 & 1 & 0 \\
0 & 0 & -210 & 0 & -120 & 0 & -45 & 0 & -10 \\
0 & 0 & 120 & 0 & 45 & 0 & 10 & 0 & 1 \\
0 & 0 & 45 & 0 & 10 & 0 & 1 & 0 & 0 \\
0 & 0 & 10 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}
\]

\[
S^\beta := \begin{bmatrix}
1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & 45 & -10 \\
0 & 1 & 10 & 45 & 120 & 210 & 252 & 210 & 120 & -45 \\
0 & 0 & 1 & 10 & 45 & 120 & 210 & 252 & 210 & -120 \\
0 & 0 & 0 & 0 & 1 & 10 & 45 & 120 & 210 & 252 \\
0 & 0 & 0 & 0 & 0 & 1 & 10 & 45 & 120 & 210 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & -45 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
\end{bmatrix}
\]

\[
S'^\beta := \begin{bmatrix}
1 & 10 & 45 & 120 & 210 & 252 & -210 & -120 & -45 & -10 \\
0 & 1 & 10 & 45 & 120 & 210 & -252 & -210 & -120 & -45 \\
0 & 0 & 1 & 10 & 45 & 120 & -210 & -252 & -210 & -120 \\
0 & 0 & 0 & 1 & 10 & 45 & -120 & -210 & -252 & -210 \\
0 & 0 & 0 & 0 & 1 & 10 & -45 & -120 & -210 & -252 \\
0 & 0 & 0 & 0 & 0 & 1 & -10 & -45 & -120 & -210 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & 45 & 120 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 10 & 45 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

where \( \beta = (\beta_{56}, \beta_{45}, \beta_{34}, \beta_{23}, \beta_{12}) (\beta_{45}, \beta_{34}, \beta_{23}) (\beta_{23}, \beta_{12}, \beta_{56}, \beta_{45}, \beta_{34}, \beta_{23}, \beta_{12}) \),
and \( \beta' = [(\sigma_{45}, \sigma_{56}, \sigma_{78}, \sigma_{9,10}) (\sigma_{45}, \sigma_{67}, \sigma_{89}) (\sigma_{56}, \sigma_{78})(\sigma_{67})][(\sigma_{78}, \sigma_{89}, \sigma_{9,10})(\sigma_{78}, \sigma_{89})\sigma_{78}] \).

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