Adaptive synchronization of fractional-order memristor-based Chua’s system

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Adaptive synchronization of fractional-order memristor-based Chua’s circuit is discussed in this paper. Two adaptive synchronization schemes are presented for the fractional-order memristor-based Chua’s circuit based on the stability theory of fractional-order differential system, the drive–response concept and the adaptive control principle. Also the case of only one parameter known and the case of all the other parameters fully unknown are discussed. The results show that controller parameters can be adjusted to enhance the convergence rate of the error system. Finally, numerical simulations are drawn to demonstrate these results.

Keywords: fractional-order; memristor; Chua’s circuit; adaptive synchronization

1. Introduction

As Professor Chua pointed in 1971 (Chua, 1971), memristor is considered as the missing fourth passive circuit element. It has attracted many scientists to investigate the property of the memristor-based systems. The memristor-based systems (El-Sayed, Elsaid, Nour, & Elsonbaty, 2013; Wang, Wang, & Tan, 2011) display abundant dynamical behaviors such as bifurcation, chaos and so on, which are very complicated and have already been applied in other fields such as secure communication (Kinzel, Englert, & Kanter, 2010). Also researchers have discussed dynamical behavior and stability analysis of the fractional-order memristor-based Chua’s circuit in Petrás (2010a). This system exhibits chaos phenomena under some special conditions. As an important and interesting behavior, synchronization has extensively been investigated. Many researchers have studied the synchronization of chaotic systems by adopting many schemes such as feedback control (Wu, Wen, & Zeng, 2012), impulsive control (Zhong, Yu, & Yu, 2010), sliding mode control (Yau, 2004), backstepping method (Song, Shen, & Chang, 2011) and so on. System structure and parameters are usually known in most methods. However, in fact, some parameters are uncertain or unavailable. Adaptive control principle (Wen, Zeng, & Huang, 2012) can be used to estimate the uncertain parameters of the system.

Motivated by the above discussion, two adaptive synchronization schemes are proposed for the fractional-order memristor-based Chua’s circuit. In this paper two cases of only one parameter known and all the other parameters unknown are discussed. Also different controller functions and parameter update rules are proposed. The simulation results show that the parameters in the update rules can be chosen to enhance the convergence rate. This method can also be applied in the chaotic system without a memristor.

2. Fractional-order memristor-based systems

The fractional-order operator is the generalization of integer-order operator. There are three commonly used definition of the fractional-order differential operator: Grunwald–Letnikov, Riemann–Liouville and Caputo. These definitions can be found in Ding, Wang, and Ye (2012), Dong, Wang, and Gao (2013), Hu, Wang, Shen, and Gao (2013) and Kan, Wang, and Shu (2013). In this paper, we will study the dynamical behavior of fractional-order system with the Grunwald–Letnikov (GL) definition, which is defined as

\[ aD^q_t f(t) = \lim_{\Delta h \to 0} h^{-q} \sum_{i=0}^{[t/q]/h} (-1)^i \binom{q}{i} f(t - ih), \]

where \([\cdot]\) means the integer part. Therefore, in the rest of this paper, the notation \(D^q_t\) is chosen as GL fractional derivative operator \(aD^q_t\) for brevity’s sake. General numerical solution of the fractional differential equation

\[ D^q_t x(t) = f(x(t), t) \]

can be expressed as

\[ x(kh) = f(x(kh), kh)h^q - \sum_{i=0}^{k} c_i^{(q)} x((k - i)h), \]

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where parameters \( c_i^{(q)} (i = 0, 1, \ldots) \) are binomial coefficients, which is calculated with the following formula (Monje, Chen, Vinagre, Xue, & Feliu, 2010):

\[
c_i^{(q)} = 1, e_i^{(q)} = \left(1 - \frac{1 + q}{i} \right) e_{i-1}^{(q)}.
\]

For the memory term expressed in the above equation, we put \( v = 1 \) for all \( k \). In order to investigate the dynamical behavior, we will employ the following lemma.

**Lemma 1** (Zhao, Hu, & Liu, 2010) For fractional-order system \( D^q x = f(x) \), when fractional order \( q \) is 1, if there exists one positive matrix \( P \) such that function \( J = x^T P D^q x \leq 0 \) holds, the state \( x \) is asymptotically stable.

A new fractional-order memristor-based Chua’s circuit has been proposed in Petráš (2010a):

\[
\begin{align*}
D^q x &= \alpha (y - x + \xi x - W(\omega)x), \\
D^q y &= x - y + z, \\
D^q z &= -\beta y - \gamma z, \\
D^q \omega &= x,
\end{align*}
\]

(1)

where the piece-wise linear function \( W(\omega) \) is given as

\[
W(\omega) = \begin{cases} 
  a, & |\omega| > 1, \\
  b, & |\omega| < 1,
\end{cases}
\]

where \( a, b > 0 \). As pointed in Petráš (2010a), when parameters are taken as \( \alpha = 10, \beta = 13, \gamma = 0.1, \xi = 1.5, a = 0.3, b = 0.8 \) and fractional order \( q = 0.97 \), this system displays chaotic attractors in the 3D state space which is shown in Figure 1.

3. **Adaptive synchronization**

Let system (1) be the drive system. In this section, according to the property of parameter \( \alpha \), we discuss two cases. Suppose other parameters are unknown.

### 3.1 Parameter \( \alpha \) known

The response system is as follows:

\[
\begin{align*}
D^q \hat{x} &= \alpha (\hat{y} - \hat{x} + \xi \hat{x} - W(\hat{\omega})\hat{x}) + u_1, \\
D^q \hat{y} &= \hat{x} - \hat{y} + \hat{z} + u_2, \\
D^q \hat{z} &= -\hat{\beta} \hat{y} - \hat{\gamma} \hat{z} + u_3, \\
D^q \hat{\omega} &= \hat{x} + u_4,
\end{align*}
\]

(2)

where parameters \( \hat{\beta}, \hat{\xi} \) and \( \hat{\gamma} \) of the response system (2) are unknown and need to be estimated. The piece-wise linear function \( W(\hat{\omega}) \) is given by

\[
W(\hat{\omega}) = \begin{cases} 
  a, & |\hat{\omega}| > 1, \\
  b, & |\hat{\omega}| < 1.
\end{cases}
\]

It is easy to obtain the error system from Equations (1) and (2) as follows:

\[
\begin{align*}
D^q e_1 &= \alpha (y - x + \xi x - \hat{x}) - W(\omega)x + W(\omega)\hat{x}) - u_1, \\
D^q e_2 &= x - \hat{x} - y + \hat{y} + z - \hat{z} - u_2, \\
D^q e_3 &= -\beta y - \gamma z - \gamma \hat{z} - u_3, \\
D^q e_4 &= x - \hat{x} - u_4,
\end{align*}
\]

(3)

where \( e_1 = x - \hat{x}, e_2 = y - \hat{y} \) and \( e_3 = z - \hat{z}, e_4 = \omega - \hat{\omega} \).

The controller functions are designed as follows:

\[
\begin{align*}
u_1 &= \alpha [-(W(\omega) - W(\hat{\omega}))x + k_1 e_1], \\
u_2 &= k_2 e_2, \\
u_3 &= k_3 e_3, \\
u_4 &= k_4 e_4
\end{align*}
\]

(4)

where \( k_1, k_2, k_3, k_4 \) are adjustable parameters. Let \( e_\beta = \beta - \hat{\beta}, e_\gamma = \gamma - \hat{\gamma}, e_\xi = \xi - \hat{\xi} \). Then the update rules for three unknown parameters are designed as follows:

\[
\begin{align*}
D^q e_\beta &= \theta (e_\beta \hat{y} - k_5 e_\beta), \\
D^q e_\gamma &= \lambda (e_\gamma \hat{z} - k_6 e_\gamma), \\
D^q e_\xi &= \tau (-e_\xi \hat{x} - k_7 e_\xi),
\end{align*}
\]

(5)

where \( k_5, k_6, k_7 \) are control gains of parameters \( e_\beta, e_\gamma, e_\xi \), respectively. Thus, we have the following main result.

**Theorem 1** For error system (3), when parameters \( k_1, k_2, k_3, k_4 \geq 0 \) are properly chosen so that the following
where blocking matrices $\Sigma_1$ and $\Sigma_2$ are given by

$$
\Sigma_1 = \begin{pmatrix}
-1 + \alpha - a - k_1 & 1 & 0 & 0.5 \\
* & -1 - k_2 & 0.5(1 - \beta) & 0 \\
* & * & -\gamma - k_3 & 0 \\
* & * & * & -k_4
\end{pmatrix},
$$

$$
\Sigma_2 = \begin{pmatrix}
-k_5 & 0 & 0 \\
* & -k_6 & 0 \\
* & * & -k_7
\end{pmatrix},
$$

where $\bar{a} = \min(a, b)$, then the fractional-order memristor-based Chua’s systems (1) and (2) can be synchronized under the adaptive control of Equations (4) and (5).

Proof From Lemma 1, one can choose $J$ function as follows:

$$
J = \phi^T P D^e \phi,
$$

where $\phi = (e_1, e_2, e_3, e_4, e_\beta, e_\gamma, e_\epsilon)^T$, $P = \text{diag} \left\{ \frac{1}{\alpha}, 1, 1, 1, \frac{1}{\theta'}, \frac{1}{\lambda}, \frac{1}{\tau} \right\}$.

Then function $J$ along trajectories of Equation (3) is calculated as

$$
J = \frac{1}{\alpha} e_1 D^e e_1 + e_2 D^e e_2 + e_3 D^e e_3 + e_4 D^e e_4 + \frac{1}{\theta} e_\beta D^e e_\beta + \frac{1}{\lambda} e_\gamma D^e e_\gamma + \frac{1}{\tau} e_\epsilon D^e e_\epsilon
$$

$$
\begin{aligned}
&= e_1 \left( y - \hat{y} - x + \hat{x} + \chi x - \hat{\chi} x - W(\omega)x \\
&+ W(\hat{\omega})\hat{x} - \frac{1}{\alpha} u_1 \right) + e_2(x - \hat{x} - y + \hat{y} + z - \hat{z} - u_2) \\
&+ e_3(-\beta y + \hat{\beta} \hat{y} - \gamma z + \hat{\gamma} \hat{z} - u_3) + e_4(x - \hat{x} - u_4) \\
&+ \frac{1}{\theta} e_\beta D^e e_\beta + \frac{1}{\lambda} e_\gamma D^e e_\gamma + \frac{1}{\tau} e_\epsilon D^e e_\epsilon
\end{aligned}
$$

$$
= e_1 \left( e_2 - e_1 + \xi e_1 + e_\epsilon \hat{x} - (W(\omega) - W(\hat{\omega}))x \\
+ W(\hat{\omega})x + W(\hat{\omega})\hat{x} - W(\hat{\omega})x - \frac{1}{\alpha} u_1 \right) \\
+ e_2(e_1 - e_2 + e_3 - u_2) + e_3(-\beta y + \hat{\beta} \hat{y} - \gamma z + \hat{\gamma} \hat{z} - u_3) + e_4(x - \hat{x} - u_4) \\
+ \frac{1}{\theta} e_\beta D^e e_\beta + \frac{1}{\lambda} e_\gamma D^e e_\gamma + \frac{1}{\tau} e_\epsilon D^e e_\epsilon
$$

$$
= e_1 \left( e_2 - e_1 + \xi e_1 + e_\epsilon \hat{x} - (W(\omega) - W(\hat{\omega}))x \\
+ W(\hat{\omega})e_1 - \frac{1}{\alpha} u_1 \right) + e_2(e_1 - e_2 + e_3 - u_2) + e_3(-\beta e_2 - e_\beta \hat{y} - \gamma e_3 - e_\gamma \hat{z} - u_3) + e_4(e_1 - u_4) \\
+ \frac{1}{\theta} e_\beta D^e e_\beta + \frac{1}{\lambda} e_\gamma D^e e_\gamma + \frac{1}{\tau} e_\epsilon D^e e_\epsilon
$$

where $Q_1$ is given in Equation (6). From Lemma 1, $J$ function is negative, which implies that the state variables $e_1, e_2, e_3$ and $e_4$ are asymptotically stable and parameters $e_\beta, e_\gamma,$ and $e_\epsilon$ are estimated with controller functions (4) and update rules (5). Thus response system (2) is synchronized with drive system (1).

**3.2. Parameter $\alpha$ unknown**

When parameter $\alpha$ is unknown, the response system is given by

$$
D^e \dot{\hat{x}} = \hat{\alpha}(\hat{y} - \hat{x} + \hat{\chi} \hat{x} - W(\hat{\omega})\hat{x}) + u_1,
$$

$$
D^e \dot{\hat{y}} = \hat{x} - \hat{y} + \hat{z} + u_2,
$$

$$
D^e \dot{\hat{z}} = -\hat{\beta} \hat{y} - \gamma \hat{z} + u_3,
$$

$$
D^e \dot{\hat{\omega}} = \hat{x} + u_4,
$$

where parameters $\alpha, \hat{\beta}, \hat{\chi},$ and $\hat{\gamma}$ of response system (7) are unknown and need to be estimated. It is easy to obtain the error system from Equations (1) and (7) as follows:

$$
D^e e_1 = \alpha(y - x + \chi x - W(\omega)x) \\
- \hat{\alpha}(\hat{y} - \hat{x} + \hat{\chi} \hat{x} - W(\hat{\omega})\hat{x}) - u_1,
$$

$$
D^e e_2 = x - \hat{x} - y + \hat{y} + z - \hat{z} - u_2,
$$

$$
D^e e_3 = -\beta y + \hat{\beta} \hat{y} - \gamma z + \hat{\gamma} \hat{z} - u_3,
$$

$$
D^e e_4 = x - \hat{x} - u_4.
$$

The controller functions are designed as follows:

$$
u_1 = \alpha [x e_1(W(\omega) - W(\hat{\omega})) + k_1 e_1],
$$

$$
u_2 = k_2 e_2,
$$

$$
u_3 = k_3 e_3,
$$

$$
u_4 = k_4 e_4.$$
Let $e_\alpha = \alpha - \dot{\alpha}$. Then the update rules for unknown parameters are designed as follows:

\begin{align}
D^\alpha e_\alpha &= \delta(-e_1 \dot{\hat{y}} + e_1 \dot{\hat{x}} - \xi e_1 \dot{x} + (x + e_1)e_1 W(\dot{\hat{\alpha}}) - k_5 e_\alpha), \\
D^\alpha e_\beta &= \theta(e_1 \dot{\hat{y}} - k_5 e_\beta), \\
D^\alpha e_\gamma &= \lambda(e_1 \dot{\hat{y}} - k_5 e_\gamma), \\
D^\alpha e_\zeta &= \tau(-\dot{\hat{\alpha}} e_1 \dot{x} - k_\gamma e_\zeta),
\end{align}

where $k_5, k_6, k_7$ and $k_8$ are the adjustable parameters. From the similar proof of Theorem 1, we obtain the following result.

**Theorem 2** For error system (8), when parameters $k_1, k_2, k_3, k_4 \geq 0$ are properly chosen so that the following linear matrix inequality holds:

\begin{align}
Q_2 &= \begin{pmatrix} \Pi_1 & 0 \\ 0 & \Pi_2 \end{pmatrix} < 0,
\end{align}

where blocking matrices $\Pi_1$ and $\Pi_2$ are given by

\begin{align}
\Pi_1 &= 
\begin{pmatrix}
-\alpha + a \xi - a \tilde{a} - k_1 & 0.5(1 + \alpha) & 0 \\
* & -1 - k_2 & 0.5(1 - \beta) \\
* & * & -\gamma - k_3 \\
* & * & * & -k_4
\end{pmatrix}, \\
\Pi_2 &= 
\begin{pmatrix}
-k_5 & 0 & 0 & 0 \\
* & -k_6 & 0 & 0 \\
* & * & -k_7 & 0 \\
* & * & * & -k_8
\end{pmatrix},
\end{align}

where $\tilde{a} = \min\{a, b\}$, then the fractional-order memristor-based Chua’s systems (1) and (7) can be synchronized under the adaptive control of Equation (9) and (10).

**Proof** From Lemma 1, one can choose another $J$ function as follows:

\begin{align}
J &= \psi^T M D^\alpha \psi,
\end{align}

where $\psi = (e_1, e_2, e_3, e_4, e_\alpha, e_\beta, e_\gamma, e_\zeta)^T$,

\begin{align}
M &= \text{diag}\left\{1, 1, 1, 1, \frac{1}{\delta}, \frac{1}{\theta}, \frac{1}{\lambda}, \frac{1}{\tau}\right\}.
\end{align}

The similar proof can be referred to Theorem 1. Thus the response system (7) is synchronized with the drive system (1).

**4. Numerical example**

The numerical simulation is carried out by using the approximation on Grunwald–Letnikov method proposed in Petráš (2010b). And for the memory terms, a short memory principle is used.

**4.1. Simulation with known $\alpha$**

The initial states of the drive system and the response system are $[0.8, 0.05, 0.007, 0.6]$ and $[0.81, 0.051, 0.0071, 0.61]$, respectively. The parameters of the drive system are $\alpha = 10, \beta = 13, \gamma = 0.1, \xi = 1.5, a = 0.3, b = 0.8$ and $q = 0.97$. Choose $k_1 = 7.2, k_2 = 6, k_3 = 6.9$ and $k_4 = 2.5$, the eigenvalues of matrix $\Sigma_1$ are $-13.0831, -7.0534, -2.4484$ and $-0.9151$. Also we choose $k_5 = 1, k_6 = 1$ and $k_7 = 1$. This can guarantee that the matrix $Q_1$ is negative. In addition, the initial values of parameters $e_\beta, e_\gamma$ and $e_\zeta$ are $e_\beta(0) = 13.1, e_\gamma(0) = 0.11, e_\zeta(0) = 1.51$ and $\theta = 1, \lambda = 1, \tau = 1$, respectively. Then the adaptive error states $e_1, e_2, e_3$ and $e_4$ and unknown parameters curve of $\beta, \gamma$ and $\zeta$ are drawn in Figures 2 and 3, respectively.

**4.2. Simulation with unknown $\alpha$**

For unknown $\alpha$, from Theorem 2, the similar parameters are the same as that in Section 4.1. Moreover, parameters are
chosen as $k_1 = 11, k_2 = 8, k_3 = 8.9$ and $k_4 = 2.5$. Then the eigenvalues of matrix $\Pi_1$ are $-17.1433, -9.0208, -2.5098$ and $-0.8261$. Also we choose $k_5 = 1, k_6 = 1, k_7 = 1$ and $k_8 = 1$. This can guarantee the matrix $Q_2$ is negative. Take $\delta = 1, \theta = 1, \lambda = 1$ and $\tau = 1$. Then the synchronization errors $e_1, e_2, e_3$ and $e_4$ and unknown parameters of $\alpha, \beta, \gamma$ and $\zeta$ are drawn in Figures 4 and 5, respectively.

It is obvious that the convergence rate of synchronization errors and parameters $\hat{\alpha}, \hat{\beta}, \hat{\gamma}$ and $\hat{\zeta}$ are slower. In order to enhance the convergence rate, parameters $\delta, \theta, \lambda$ and $\tau$ are revised as $\delta = 5, \theta = 5, \lambda = 5$ and $\tau = 5$. The corresponding synchronization errors are drawn in Figure 6.

4.3. Discussion with $k_5 = 0, k_6 = 0, k_7 = 0$ for known $\alpha$

Figure 7 describes the synchronization errors with $k_5 = 0, k_6 = 0$ and $k_7 = 0$. It is obvious that the convergence rate of synchronization errors are slower than that of $k_5 = 1, k_6 = 1$ and $k_7 = 1$. Also in order to enhance the convergence rate, we choose $\theta = 15, \lambda = 15$ and $\tau = 15$.

5. Conclusions

With the intensive study on the fractional-order system and the memristor, the investigation on the fractional-order memristor-based system becomes the hot issue. The adaptive synchronization for the fractional-order memristor-based Chua’s circuit has been investigated with unknown parameters. Some controller schemes and update rules for unknown parameters have been proposed to guarantee the adaptive synchronization. Finally, numerical simulation has been carried out to show the efficiency of the proposed schemes.

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