PIONIUM LIFETIME CORRECTIONS

H. SAZDJIAN

Groupe de Physique Théorique, Institut de Physique Nucléaire,
Université Paris XI, F-91406 Orsay Cedex, France

ABSTRACT

Pionium lifetime corrections are evaluated in the frameworks of constrained Bethe-Salpeter equation and chiral perturbation theory. Corrections of order $O(\alpha)$ are calculated with respect to the conventional lowest-order formula, in which the strong interaction amplitude has been calculated to two-loop order with charged pion masses. The total correction is found to be of the order of $(6.1 \pm 2.0)\%$.

1. Introduction

Our aim is to evaluate the sizable corrections to the nonrelativistic formula of pionium lifetime:

$$\frac{1}{\tau_0} = \Gamma_0 = \frac{16\pi}{9} \sqrt{\frac{2\Delta m_\pi (a_0^2 - a_2^2)^2}{m_{\pi^+}^2}} |\psi_{+-}(0)|^2, \quad \Delta m_\pi = m_{\pi^+} - m_{\pi^0}. \quad (1)$$

Here, $a_I^0$ is the strong interaction (dimensionless) $S$-wave $\pi\pi$ scattering length in the isospin $I$ channel and $\psi_{+-}(0)$ is the ground state wave function of the pionium at the origin (in $x$-space). The strong interaction scattering lengths $a_0^2$ and $a_2^2$ have been evaluated in the literature in the framework of chiral perturbation theory ($\chi$PT) to two-loop order of the chiral effective lagrangian.

Formula (1) has been obtained by treating the strong interaction scattering amplitude at threshold as a first-order perturbation with respect to the Coulomb potential between the charged pions. We shall evaluate in the following the corrections of order $O(\alpha)$ ($\alpha$ being the fine structure constant) to formula (1). Most of the details of the calculations can be found in Ref. 5.

2. Bound state equation

To deal with the bound state problem, we use the constraint theory approach to the Bethe-Salpeter equation, which consists in reducing the latter, by means of a constraint applied to the external relative energy, to a three-dimensional equation. The constraint that is chosen is the following: $C(P, p) \equiv (p_1^2 - p_2^2) - (m_1^2 - m_2^2) \approx 0$, where $p_1$ and $p_2$ are the external particle momenta and $m_1$ and $m_2$ their physical masses. One then establishes a three-dimensional eigenvalue equation that takes the form

$$\tilde{g}_0^{-1}\Psi = -\tilde{V}\Psi, \quad (2)$$
where $\bar{g}_0^{-1}$ is the wave equation operator defined for two-spinless particle systems by $\bar{g}_0 = -1/(p_1^2 - m_1^2 + ie)_{C} = -1/(p_2^2 - m_2^2 + ie)_{C}$, the index $C$ denoting the use of the constraint. $\tilde{V}$ is the potential, related to the renormalized off-mass shell scattering amplitude $T$ by a Lippmann-Schwinger type equation:

$$
\tilde{V} = \tilde{T} + \tilde{V} \bar{g}_0 \tilde{T}, \quad \tilde{T} = \frac{i}{2\sqrt{s}} T|_{C},
$$

where $s = P^2 = (p_1 + p_2)^2$. The amplitude $\tilde{T}$ contains the usual Feynman diagrams, where the external particles are submitted to the constraint $C$. The second term in the right-hand side of the first of Eqs. (3) generates an iteration series, the diagrams of which are called “constraint diagrams”, where the integrations relative to the factor $\bar{g}_0$ are three-dimensional, because of the presence of constraint $C$. The constraint diagrams cancel, in the $s$-channel, the singularities of the two-particle reducible diagrams of $\tilde{T}$. As long as perturbation theory is concerned, Eq. (2) is equivalent in content to the exact Bethe-Salpeter equation, with, however, a different arrangement of the perturbation series. The interest of Eq. (2) is that it generates a three-dimensional systematic perturbation theory with covariant propagators.

We use for our calculations the chiral effective lagrangian in the $SU(2) \times SU(2)$ case. We shall be working with the pseudoscalar densities defined as $P^a = i\gamma_5 \tau^a q$ ($a=1,2,3$), where $q$ are the quark fields and $\tau^a$ the Pauli matrices. The off-mass shell scattering amplitude is defined as the amputated connected four-point Green’s function of the pseudoscalar densities, multiplied by the corresponding wave function renormalization factors $2B_0F_0 Z_a^{-1/2}(p_a)$, where $F_0$ is the pion decay constant $F_\pi$ in the chiral limit ($F_\pi = 92.4$ MeV) and $B_0$ the quark condensate parameter. With the above definitions, the amplitude $T$ is renormalization group invariant. Furthermore, Eqs. (3) imply that the constraint propagators $\bar{g}_{0,ab}$ are kinematic operators with physical masses and are not concerned with the renormalization factors.

For the analysis of the decay of the pionium into two neutral pions, it is convenient to use a coupled-channel formalism. Defining wave functions, potentials and propagators in matrix form with the indices $\{+-\}$ for the charged pion sector and $\{00\}$ for the neutral pion sector, Eq. (2) is transformed into two coupled equations. Eliminating the wave function $\Psi_{00}$ in terms of $\Psi_{+-}$ and neglecting unimportant unitarity correction factors, one ends up with a single wave equation for $\Psi_{+-}$:

$$
- \bar{g}_{0,+-}^{-1} \Psi_{+-} = \left[ V_{\text{Coul.}} + \nabla_{+-} + \frac{1}{2} V_{+-,00} \bar{g}_{0,00} V_{00,+-} \right] \Psi_{+-},
$$

(4)

$\nabla_{+-}$ represents the potential relative to the transition in the charged sector and from which the Coulomb potential $V_{\text{Coul.}} = -m_+ \alpha/r$ has been separated. $V_{+-,00}$ represents the potential relative to the transition from the charged to the neutral sector, etc.. In the constraint propagators $\bar{g}_{0,+-}$ and $\bar{g}_{0,00}$ it is the physical masses of the charged and neutral pions that appear, respectively. The nonrelativistic Coulomb potential will be considered as the zeroth-order potential and the remaining potentials in Eq. (4) will be treated as perturbations.

At the pionium energy, the operator $\bar{g}_{0,00}$ in Eq. (4) lies in the scattering region of the $\pi^0\pi^0$ sector and therefore contributes with imaginary values in $x$-space, thus inducing an imaginary part to the pionium energy.
3. Separation of quark-photon interaction terms

In first-order of perturbation theory, it is the third term of the right-hand side of Eq. (1) that contributes to the imaginary part of the energy and hence to the decay width. The simplest approximation at this stage consists in taking for the potential $V_{00,+-} (= V_{+-,00})$ the strong interaction amplitude $T_{00,+-}^{str}$ at threshold (deviations from the pionium energy to the charged pion threshold yielding corrections of order $\alpha^2$), calculated to two-loop order. Defining $P_0 = P_0 + i\Gamma/2$, one recovers, at leading order in $\Delta m_\pi$, the nonrelativistic formula (1). Corrections of order $\alpha$ to $\Gamma_0$ arise from the presence of additional terms of electromagnetic origin in $\nabla_{+-,+-}$ and $V_{00,+-}$, as well as from second-order perturbation theory contributions of the potentials. It is natural at this stage to isolate from the rest the electromagnetic interactions coming from quark-photon interactions. The reason for this separation is that electromagnetic interactions with the explicit presence of the photon field lead, in bound state problems, or near threshold, to infra-red singularities for this separation is that electromagnetic interactions with the explicit presence of the rest the electromagnetic interactions coming from quark-photon interactions. The reason

The quark-photon interaction is represented in the chiral effective lagrangian at order $e^2 p^0 \hat{Q}$ by the term $e^2 C \langle QUQU^\dagger \rangle$, where $U$ is the chiral field, $Q$ the quark charge matrix and $C$ an unknown constant. Using for the field $U$ the representation $U = \sigma + i\pi \cdot \tau / F_0$, $\sigma = \sqrt{1 - \pi^2 / F_0^2}$, where $\tau$ are the Pauli matrices and $\pi$ the pion fields, one finds:

$$e^2 C \langle QUQU^\dagger \rangle = -\frac{2e^2 C}{F_0^2} \pi^+ \pi^-.$$  

(5)

This term induces for the charged pions the mass shift $(\Delta m_\pi^2)_{q\gamma} \equiv (m_{\pi^+}^2 - m_{\pi^0}^2)_{q\gamma} = 2e^2 C / F_0^2$, which is nonvanishing in the chiral limit and is numerically close to the physical pion mass shift. It has no effect on the scattering amplitudein lowest order, but acts essentially through insertions (together with counterterms of higher order) in the pion loop propagators, where the charged pion masses are replaced by their (almost) physical masses.

We then split the scattering amplitude $\mathcal{M} (=-iT)$ into two parts:

$$\mathcal{M} = \mathcal{M}^{str,+q\gamma} + \mathcal{M}^{em},$$  

(6)

where $\mathcal{M}^{em}$ contains all electromagnetic interaction terms generated by explicit photon fields of the chiral effective lagrangian, while $\mathcal{M}^{str,+q\gamma}$ contains the strong interaction terms together with the quark-photon interaction terms.

Let us now neglect the amplitude $\mathcal{M}^{em}$. It can be shown that the potential $V_{00,+-}$, calculated from Eq. (3), where one takes also into account the contributions of the constraint diagrams, is equal, to order $\alpha$, to the real part of the on-mass shell scattering amplitude $T_{00,+-}^{str,+q\gamma}$ taken at threshold:

$$V_{00,+-}^{str,+q\gamma} = \text{Re} \tilde{T}_{00,+-}^{str,+q\gamma} (p_1^2 = p_2^2 = m_{\pi^+}^2, p_3^2 = p_4^2 = m_{\pi^0}^2, s = 4m_{\pi^+}^2, t = u = -\Delta m_\pi^2),$$  

(7)

where $\Delta m_\pi^2 = m_{\pi^+}^2 - m_{\pi^0}^2$. (An error in Ref. [3] concerning the mass-shell condition of the neutral pions has been corrected.)
4. Second-order perturbation theory

At this order, it is the interference of the last two potential terms of Eq. (4) that contributes to the imaginary part of the energy.

The contribution of the discrete states of the pionium spectrum is found to be:

\[(\Delta \Gamma)_{\text{discr.}} = (\alpha/3)(2a_0^0 + a_2^0)\Gamma_0.\]

The contribution of the continuous spectrum is ultra-violet divergent, with a linear and a logarithmic divergences. In order to avoid these divergences, it is necessary to associate with the previous terms contributions coming from the constraint diagrams present in Eq. (4). The constraint diagram with one pion loop has a linear divergence that cancels the previous one. The three constraint diagrams with one pion loop and one photon exchanged between the two internal pions have logarithmic divergences the sum of which cancels the previous logarithmic divergence. The sum of the contributions of the continuous spectrum and of the constraint diagrams is then finite and one finds:

\[(\Delta \Gamma)_{\text{cont.}} + (\Delta \Gamma)_{\text{constr.}} = 1.1\alpha(2a_0^0 + a_2^0)\Gamma_0.\]

The total contribution of the discrete and continuous states, together with the constraint diagrams, is:

\[(\Delta \Gamma)_{\text{str.}} = 1.5\alpha(2a_0^0 + a_2^0)\Gamma_0 = 0.004\Gamma_0.\] (8)

5. Vacuum polarization

Vacuum polarization in positronium and hydrogen-like atoms provides corrections of order \(\alpha^3\). In the pionium problem, vacuum polarization, because of the presence of electron loops and of the large ratio of the reduced mass of the two-pion system to the electron mass, yields corrections of order \(\alpha\). This point was emphasized by Labelle and Buckley. Using the nonrelativistic QED effective field theory formalism, they have found: \((\Delta \Gamma)_{\text{vac. pol.}} = 0.43\alpha\Gamma_0.\)

In the present formalism, vacuum polarization contributes through the potential \(V^{++}_{\text{pol.}}\) with a local term. It contributes to the decay width through the interference term between \(V^{++}\) and the last term of Eq. (4) at the second order of perturbation theory. The contribution of the discrete states is found to be: \((\Delta \Gamma)_{\text{discr.}} = 0.03\alpha\Gamma_0,\) while that of the continuous states, which is finite, is \((\Delta \Gamma)_{\text{cont.}} = 0.38\alpha\Gamma_0.\) The total contribution is:

\[(\Delta \Gamma)_{\text{vac. pol.}} = 0.41\alpha\Gamma_0 = 0.003\Gamma_0,\] (9)

in agreement with the previous estimate.

6. Electromagnetic radiative corrections

These corrections arise from pion-photon interaction terms contained in \(\mathcal{M}^{\text{em.}}\) [Eq. (4)]. The chiral effective lagrangian for the \(SU(3)\times SU(3)\) case including electromagnetism to order \(e^2p^2\) was presented by Urech. It was presented for the \(SU(2)\times SU(2)\) case by Meissner, Müller and Steininger and by Knecht and Urech. We use for our subsequent calculations the \(SU(2)\times SU(2)\) lagrangian with the notations of Ref. 14.

The pion-photon radiative corrections to order \(e^2p^2\) are represented by the pion self-energy corrections and the vertex correction of the lowest order scattering amplitude.
of the process \( \pi^+ \pi^- \to \pi^0 \pi^0 \). The sum of these contributions to the off-mass shell scattering amplitude \( \mathcal{M}_{00,++} \) at the pionium energy has a spurious \( O(\alpha^0) \) term. The latter is cancelled, however, by the contribution of the corresponding constraint diagram of Eq. (3) (with one photon exchanged between the charged pions). The corresponding total contribution is:

\[
\left[ \mathcal{M}_{00,++}^{\text{em.}} + \mathcal{M}_{00,++}^{\text{em.}(\text{constr.})} \right]_{\pi \gamma} = \left( \frac{s - m_{\pi^0}^2}{F_0^2} \right) \left[ -\frac{\alpha}{2\pi} - \frac{3\alpha}{4\pi} \left( \ln \left( \frac{m_{\pi^0}^2}{\mu^2} \right) + 1 \right) - 4e^2k_r^r \right.
\]
\[
\left. - \frac{2e^2}{9}(k_2^r + 10k_{10}^r) \right] + 2e^2 \frac{m_{\pi^0}^2}{F_0^2} \left[ 2k_3^r + 3(k_7^r + 2k_8^r) \right],
\]

where the \( k_r^r \)’s are the renormalized low energy constants appearing in the counterterm lagrangian and \( \mu \) is the renormalization mass, chosen in the following for the numerical estimates equal to the \( \rho \)-meson mass (770 MeV). We have also incorporated in the mass term of the tree level amplitude the mass shift of the neutral pion. For the numerical estimate of the corrections, we have adopted the values of the \( k_r^r \)’s obtained by Baur and Urech\(^{16} \) (in the Feynman gauge) using a resonance model for the saturation of sum rules and assigned to them conservative 100\% uncertainties. The numerical values that we use (after conversion to the \( SU(2) \times SU(2) \) case) are, in units of \( 10^{-3} \): \( k_2^r + 10k_{10}^r = 122.2 \), \( k_3^r = 6.4 \), \( k_7^r + 2k_8^r = 0.8 \). One finds for the modification of the decay width:

\[
(\Delta \Gamma)_{\pi \gamma} = (-0.0015 \pm 0.0075)\Gamma_0, \quad (11)
\]

where the uncertainty comes from the \( k_r^r \)’s.

7. Electromagnetic mass shift corrections

These corrections are those contained in the amplitude \( \text{Re} \mathcal{M}_{00,++}^{\text{str.}+q\gamma} \) [Eqs. (6)-(7)]. They are induced by the term (3) and by its counterterms of the effective lagrangian and the effect of which is the generation of mass shifts in the scattering amplitude. The corrections are evaluated with respect to the strong interaction scattering amplitude, in which all pions have the same mass, chosen to be the charged pion mass.\(^2,3,4 \)

In the strong interaction case, the pion mass is essentially determined in terms of the quark mass parameter \( \hat{m} \)\(^2 \) and the quark condensate parameter \( B_0 \). In lowest order, one has \( m_{\pi^0}^2 = 2\hat{m}B_0 \). To one-loop order, one has:

\[
(m_{\pi^0}^2)^{\text{str.}} = 2\hat{m}B_0 + \frac{(2\hat{m}B_0)^2}{F_0^2} \left( 2l_3^r + \frac{1}{32\pi^2} \ln \left( \frac{m_{\pi^0}^2}{\mu^2} \right) \right) \equiv 2\hat{m}B, \quad (12)
\]

where \( l_3^r \) is one of the low-energy constants of the effective lagrangian.

According to the result (5), the neutral pion mass remains unchanged when the \( O(e^2p^0) \) term is introduced in the strong interaction lagrangian. Therefore, to order \( p^4 \) of the strong interaction lagrangian and to order \( e^2p^0 \) of the electromagnetic interaction the mass appearing in the left-hand side of Eq. (12) can be identified with the neutral pion mass. Higher-order effects in the masses play only a corrective role and modify slightly this equality, which can be represented in the approximate form as \( m_{\pi^0}^2 = 2\hat{m}B + O(e^2p^2) + O(p^6) \approx 2\hat{m}B \).
Let us now rewrite the amplitude $\text{Re}\mathcal{M}_{00,+,-}^{\text{str.}+q\gamma}$ by specifying in detail its kinematic and parametric conditions, as emerging from Eqs. (7) and (12):

$$\text{Re}\mathcal{M}_{00,+,-}^{\text{str.}+q\gamma} = \text{Re}\mathcal{M}_{00,+,-}^{\text{str.}+q\gamma}(s = 4m_{\pi^+}^2, p_1^2 = p_2^2 = m_{\pi^+}^2, p_3^2 = p_4^2 = m_{\pi^0}^2, 2\hat{m}B \simeq m_{\pi^0}^2).$$  \hfill (13)

On the other hand, the strong interaction amplitude is defined in the following way:

$$\text{Re}\mathcal{M}_{00,+,-}^{\text{str.}} = \text{Re}\mathcal{M}_{00,+,-}^{\text{str.}}(s = 4m_{\pi^+}^2, p_1^2 = p_2^2 = m_{\pi^+}^2, p_3^2 = p_4^2 = m_{\pi^+}^2, 2\hat{m}B = m_{\pi^+}^2).$$  \hfill (14)

We have to calculate the difference between the two previous amplitudes. The latter is the result of three effects: (i) The dynamical effect of the quark-photon interaction term (1), which acts through insertions in internal pion loop propagators. (ii) Shift of the neutral pion momenta squared, $p_3^2$ and $p_4^2$, from their value $m_{\pi^0}^2$, of the strong interaction case to their final value $m_{\pi^0}^2$. (iii) Shift of the mass parameter $2\hat{m}B$ from $m_{\pi^+}^2$ (strong interaction case) to $m_{\pi^0}^2$. As long as one considers linear effects in the pion mass shift, these three effects can be calculated separately. We shall stick to this approximation in the following. One then obtains:

$$\left(\Delta\text{Re}\mathcal{M}_{00,+,-}\right)_{m.\,\text{shift}} = \text{Re}\mathcal{M}_{00,+,-}^{\text{str.}+q\gamma} - \text{Re}\mathcal{M}_{00,+,-}^{\text{str.}} = \frac{\Delta m_{\pi}^2}{F_0^4} - \frac{m_{\pi^0}^2\Delta m_{\pi}^2}{16\pi^2 F_0^4} \left[ \frac{22}{3} \ln \left( \frac{m_{\pi^0}^2}{\mu^2} \right) - \frac{23}{6} \right]$$

$$+ \frac{m_{\pi^0}^2\Delta m_{\pi}^2}{F_0^4} \left[ 8l_1^r - 4l_3^r \right] + 2e^2\frac{m_{\pi^0}^2}{F_0^2} \left[ (k_2^r - 2k_4^r) + (k_7^r - 2k_8^r) \right].$$  \hfill (15)

The $l^r$'s and $k^r$'s are the renormalized low energy constants of the strong interaction and electromagnetic interaction lagrangians, respectively. The logarithms that appear in the strong interaction amplitude, associated with the $l^r$'s, are calculated with the charged pion mass. The masses that appear in the corrective terms are taken equal to the neutral pion mass. The $\Delta m_{\pi}^2$ factor of the lowest-order term represents the physical pion mass squared shift, for the $O(e^2p^2)$ terms of the neutral pion mass, coming from pion-photon and quark-photon interactions, have been included in the tree level amplitude. We neglect, in our numerical evaluations, isospin breaking effects coming from the quark masses, which are small. We use, as in Sec. 6., the numerical values of the $k^r$'s obtained in Ref. 18 (in the Feynman gauge). These are, in units of $10^{-3}$: $k_2^r = 5.4, k_4^r = 6.2, k_7^r - 2k_8^r = -2.3, k_9^r + k_{11}^r = -0.3$. The constants $l^r$ are obtained from Refs. 20; they are, in units of $10^{-3}$: $l_1^r = -5.4 \pm 3.9, l_3^r = 0.8 \pm 3.8, l_4^r = 5.6 \pm 5.7$. We also take $F_0 = 88$ MeV.

The correction for the decay width is:

$$\frac{(\Delta\Gamma)_{m.\,\text{shift}}}{\Gamma_0} = 0.049 \pm 0.004 \pm 0.001,$$  \hfill (16)

where the first uncertainty comes from the $l^r$'s and the second one from the $k^r$'s. The contribution to the above number of the tree level correction, represented by the first term of the right-hand side of Eq. (17) and coming from the effect of type (iii), is 0.037. Therefore, the $O(e^2p^2)$ terms contribute with an amount of 30% with respect to the tree level effect.
Knecht and Urech have provided, from a direct evaluation of the generating functional to one loop, the expression of the on-mass shell $\pi^+\pi^- \rightarrow \pi^0\pi^0$ scattering amplitude in the presence of electromagnetism. A finite result at threshold is obtained after subtraction of the infra-red singularity of the photon. The sum of our evaluations (10) and (15) agrees with the result presented by the above authors, to linear effects in $\Delta m_\pi$.

8. $O(e^2p^4)$ effects

$O(e^2p^4)$ effects come from diagrams having a pion loop with one photon exchanged or having two pion loops with a mass shift insertion. Most of these effects are expected to be small. However, enhancements may appear due to the presence of infra-red singularities. The latter generally arise from kinematic regions where the loop momentum $k$ is small. Two cases are distinguished. In the first case, the component $k_0$ is of the order of $k^2/m_\pi$; in the second case, $k_0$ is of the order of $|k|$. The singularities of the first case, which are the strongest ones, are generally cancelled by those of the corresponding constraint diagrams. The singularities of the second case are generally cancelled either by those of the crossed diagrams (if present) or by those of the self-energy diagrams. A typical example of such cancellations was met in the vertex correction evaluation [Sec. 6], where the sum of the contributions of the vertex diagram and of the corresponding constraint and self-energy diagrams was found to be free of singularities.

By considering groups of diagrams of the above kinds, one arrives at the conclusion that the sum of all $O(e^2p^4)$ diagrams is free of infra-red singularities (at the pionium energy). However, in our treatment of the second-order perturbation theory problem [Sec. 4], contributions of the constraint diagrams of the diagram with a photon exchanged between the two loop-pions were already taken into account to cancel the ultra-violet divergence of the continuous spectrum of the intermediate states. Therefore, the infra-red contribution of the corresponding four-dimensional diagram should also be isolated. This diagram has an infra-red singularity of the first kind mentioned above and has been evaluated in the literature [5]. Its contribution to the pionium decay width, leaving aside its ultra-violet divergent part, is:

$$\langle \Delta \Gamma \rangle_{O(e^2p^4)} = -\frac{\alpha}{3}(2a_0^0 + a_0^2)(2\ln \alpha + 3\ln 2 + 21\zeta(3)/(2\pi^2))\Gamma_0 = 0.006\Gamma_0.$$  (17)

The remaining groups of diagrams, being infra-red regular, will have orders of magnitude fixed by $\chi$PT. Let us consider the $O(e^2p^2)$ radiative correction of Sec. 5. The corresponding relative correction to the decay width, without the counterterms $k^r$, is of order $\alpha$ [Eq. (11)]. Pion loops have typical relative order of magnitude of 5% (compared to 1). The number of the $O(e^2p^4)$ groups of diagram being of the order of 25 and assuming their signs are distributed at random, one arrives at an estimate of 0.2% for the corresponding relative contribution to the decay with. A similar contribution, 0.2%, should also be expected from the low energy constants of the counterterm lagrangian. Mass shift corrections due to quark-photon interactions have produced a 30% effect at the $O(e^2p^2)$ level relative to the tree level, with a 1% global effect [Eq. (16) and comment following it]. Assuming that the corresponding $O(e^2p^4)$ effect (two-loop diagrams) is of

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relative order of 40% with respect to the $O(e^2p^2)$ effect (taking into account the increase in the number of diagrams), one arrives at an estimate of 0.4% for its contribution. The sum of the previous contributions is then 0.8%. The total contribution of the $O(e^2p^4)$ diagrams is thus:

$$(\Delta \Gamma)_{O(e^2p^4)} = (0.006 \pm 0.008)\Gamma_0.$$  (18)

9. Summary and concluding remarks

The corrections to the nonrelativistic formula of the pionium decay width can be represented in the following form:

$$\Gamma = \Gamma_0 \sqrt{\left(1 - \frac{\Delta m_\pi}{2m_\pi}\right)\left(1 + \frac{\Delta \Gamma}{\Gamma_0}\right)},$$  (19)

where $\Delta \Gamma$ is composed of the following contributions:

$$\Delta \Gamma = (\Delta \Gamma)_{str.} + (\Delta \Gamma)_{vac. pol.} + (\Delta \Gamma)_{\pi\gamma} + (\Delta \Gamma)_{m. shift} + (\Delta \Gamma)_{O(e^2p^4)}.$$  (20)

One finds for $\Delta \Gamma/\Gamma_0$:

$$\frac{\Delta \Gamma}{\Gamma_0} = 0.061 \pm 0.004 \pm 0.008 \pm 0.008,$$  (21)

the first uncertainty coming from the low energy constants $l_i$, the second one from the $k_i$'s and the third one from the $O(e^2p^4)$ diagrams. $\Gamma_0$ [Eq. (1)] is calculated with the strong interaction scattering lengths evaluated up to two-loop order ($a_0 - a_2 = 0.258$). One finds $\tau_0 = 3.19 \times 10^{-15}$ s. With the corrective terms, the lifetime becomes: $\tau = (3.02 \pm 0.07) \times 10^{-15}$ s.

The different contributions are summarized in Table 1.

| Effect          | $\Delta \Gamma/\Gamma_0$ (%) |
|-----------------|-------------------------------|
| strong          | 0.4                           |
| vac. pol.       | 0.3                           |
| $\pi\gamma$ rad. corr. | $-0.1 \pm 0.7$ |
| mass shift      | $4.9 \pm 0.4 \pm 0.1$         |
| $O(e^2p^4)$      | $0.6 \pm 0.8$                |
| total           | $6.1 \pm 2.0$                |

Table 1: The various corrections to the decay width.

Evaluations similar to those of the present work were done by Ivanov et al. solving the Bethe-Salpeter equation in the Coulomb gauge and using, for the mass shift corrections, the result provided in Ref. [15]. The results found by Ivanov et al. and ours are qualitatively and quantitatively compatible among themselves. In particular, the sum of their evaluations corresponding to strong and one Coulomb exchange corrections ($-0.22 + 1.55$) should be compared with the sum of our evaluations of strong and $O(e^2p^4)$ effects ($0.4 + 0.6$). The latter effect, not evaluated in Ref. [8], has reintroduced an infra-red logarithm present in Ref. [19]. The sum of the quantities corresponding to mass shift and
electromagnetic radiative corrections of Ref. 19 \((2.99 + 1.73)\) should be compared with the sum of our evaluations of mass shift and pion-photon radiative corrections \((4.9 − 0.1)\).

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