Approximation by modified Kantorovich–Stancu operators

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Abstract
In the present paper, we study a new kind of Kantorovich–Stancu type operators. For this modified form, we discuss a uniform convergence estimate. Some Voronovskaja-type theorems are given.

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1 Introduction
Let \( 0 \leq \alpha \leq \beta \) and \( m \in \mathbb{N} \). In [15], D.D. Stancu introduced the linear positive operators

\[ P_{m}^{(\alpha, \beta)} : C([0, 1]) \rightarrow C([0, 1]) \]

defined by

\[ P_{m}^{(\alpha, \beta)}(f; x) = \sum_{k=0}^{m} p_{m,k}(x) f\left( \frac{k + \alpha}{m + \beta} \right) , \tag{1.1} \]

where

\[ p_{m,k}(x) = \binom{m}{k} x^{k} (1-x)^{m-k} \]

are the fundamental Bernstein polynomials [3].

When \( \alpha = \beta = 0 \),

\[ P_{m}^{(0,0)}(f; x) = B_{m}(f; x) \]

is the classical Bernstein operator.

L.V. Kantorovich [8] introduced the linear positive operators

\[ K_{m} : L_{1}([0, 1]) \rightarrow C([0, 1]) \]

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defined for any nonnegative integer \( m \) by

\[
K_m(f; x) = (m + 1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k}{m+1}}^{\frac{k+1}{m+1}} f(s) \, ds.
\]

(1.2)

By combining (1.1) and (1.2), D. Bărbosu [2] introduced

\[
K_{\alpha, \beta}^m : L_1([0,1]) \to C([0,1])
\]

defined for any \( m \in N \) by

\[
K_{\alpha, \beta}^m(f; x) = (m + \beta + 1) \sum_{k=0}^{m} p_{m,k}(x) \int_{\frac{k+\alpha}{m+\beta+1}}^{\frac{k+\alpha+1}{m+\beta+1}} f(s) \, ds.
\]

(1.3)

\( K_{\alpha, \beta}^m \) are linear positive operators called Kantorovich–Stancu operators.

In recent years, Bernstein–Kantorovich–Stancu operators have been modified and studied by many mathematicians. For instance, in [4] Cai et al. defined a new type \( \lambda \)-Bernstein operators, and a Kantorovich variant of the modified Bernstein operators was introduced and studied in [7]. In the last three years, Mursaleen et al. investigated several approximation properties for a Kantorovich type generalization of \( q \)-Bernstein–Stancu operators in [14], applied \( (p,q) \)-calculus in approximation theory, and constructed the \( (p,q) \)-analogue of Bernstein operators [12], \( (p,q) \)-Bernstein–Kantorovich operators [13], and a Kantorovich variant of \( (p,q) \)-Szász–Mirakjan operators [11]. Also, in [1] Ansari and Karaisa introduced and studied Chlodowsky variant of \( (p,q) \)-Bernstein operators.

H. Khosravian-Arab, M. Delghan, and M.R. Eslahchi introduced in [9] the following operators:

\[
B_{m}^{M,1}(f; x) = \sum_{k=0}^{m} p_{m,k}^{M,1}(x) f\left(\frac{k}{m}\right),
\]

(1.4)

where

\[
p_{m,k}^{M,1}(x) = a(x; m)p_{m-1,k}(x) + a(1-x; m)p_{m-1,k-1}(x), \quad x \in [0,1]
\]

(1.5)

and

\[
a(x; m) = a_1(m)x + a_0(m), \quad m = 0, 1, \ldots
\]

(1.6)

Here, \( a_0(m) \) and \( a_1(m) \) are two unknown sequences which are determined in an appropriate way. Note that, for \( a_0(m) = 1 \) and \( a_1(m) = -1 \), (1.5) becomes the well-known identity for the fundamental Bernstein polynomials

\[
p_{m,k}(x) = (1-x)p_{m-1,k}(x) + xp_{m-1,k-1}(x), \quad 0 < k < m.
\]
From (1.5), the operators (1.4) become

\[
B_m^{(1)}(f;x) = a(x;m) \sum_{k=0}^{m} p_{m,k}(x) (k/m) + a(1-x;m) \sum_{k=0}^{m} p_{m-1,k-1}(x) (k/m)
\]

\[= a(x;m)p_{m-1}^{(1)}(f;x) + a(1-x;m)p_{m-1}^{(1)}(f;x).\]

We try to extend some results to the Kantorovich–Stancu operators considering the operators denoted by

\[
\mathcal{K}_m^{(a,p)}(f;x) = (m+\beta+1) \sum_{k=0}^{m} p_{m,k}^{(m+1)}(x) \int_{[0,1]} f(s) ds, \quad m \in \mathbb{N}, x \in [0,1].
\]

### 2 Auxiliary results

**Lemma 2.1** For \( p \in \mathbb{N}^* \), we have

(i) \( p_m^{(a,p)}((t-x)^p;x) = \sum_{i=0}^{p} \binom{p}{i} \frac{1}{(m+\beta)^{p-i}} B_m((\alpha - t\beta)^{p-i}(t-x)^i;x)\)

(ii) \( K_m^{(a,p)}((t-x)^p;x) = \frac{1}{p+1} \sum_{i=0}^{p+1} \binom{p+1}{i} \frac{1}{(m+\beta+1)^{p+1-i}} p_m^{(a,p+1)}((t-x)^{p+1-i};x).\)

**Proof** (i)

\[
p_m^{(a,p)}((t-x)^p;x) = \sum_{k=0}^{m} p_{m,k}(x) \left( \frac{k+\alpha}{m+\beta} - x \right)^p = \sum_{k=0}^{m} p_{m,k}(x) \left( \frac{k}{m} - x + \frac{m\alpha - k\beta}{m(m+\beta)} \right)^p
\]

\[= \sum_{k=0}^{m} p_{m,k}(x) \left( \sum_{i=0}^{p} \binom{p}{i} \left( \frac{k}{m} - x \right)^i \left( \frac{m\alpha - k\beta}{m(m+\beta)} \right)^{p-i} \right)
\]

\[= \sum_{k=0}^{m} p_{m,k}(x) \left( \sum_{i=0}^{p} \binom{p}{i} \frac{1}{(m+\beta)^{p-i}} \left( \alpha - \frac{k}{m} \beta \right)^{p-i} \left( \frac{k}{m} - x \right)^i \right)
\]

\[= \sum_{i=0}^{p} \binom{p}{i} \frac{1}{(m+\beta)^{p-i}} \sum_{k=0}^{m} p_{m,k}(x) \left( \alpha - \frac{k}{m} \beta \right)^{p-i} \left( \frac{k}{m} - x \right)^i
\]

\[= \sum_{i=0}^{p} \binom{p}{i} \frac{1}{(m+\beta)^{p-i}} B_m((\alpha - t\beta)^{p-i}(t-x)^i;x).
\]

(ii) We have that

\[
K_m^{(a,p)}((t-x)^p;x)
\]

\[= \frac{m+\beta+1}{p+1} \sum_{k=0}^{m} p_{m,k}(x) \left[ \left( \frac{k+\alpha}{m+\beta+1} - x + \frac{1}{m+\beta+1} \right)^{p+1} - \left( \frac{k+\alpha}{m+\beta+1} - x \right)^{p+1} \right]
\]

\[= \frac{m+\beta+1}{p+1} \sum_{k=0}^{m} p_{m,k}(x) \sum_{i=1}^{p+1} \binom{p+1}{i} \frac{1}{(m+\beta+1)^{p+1-i}} \left( \frac{k+\alpha}{m+\beta+1} - x \right)^{p+1-i}.
\]
\[
\sum_{i=1}^{p+1} \frac{1}{(m+\beta+1)^{i-1}} \left( \begin{array}{c} p+1 \\ i \end{array} \right) \frac{1}{p+1} \sum_{k=0}^{m} p_{m,k}(x) \left( \frac{k+\alpha}{m+\beta+1}-x \right)^{p+1-i} \left( \begin{array}{c} m+\beta+1 \\ i-1 \end{array} \right) \left( \frac{x}{(m+\beta+1)^{i-1}} \right) \]

\[
= \frac{1}{p+1} \sum_{i=1}^{p+1} \frac{1}{(m+\beta+1)^{i-1}} \left( \begin{array}{c} p+1 \\ i \end{array} \right) \frac{1}{p+1} p_{m,\alpha,\beta+1}((t-x)^{p+1-i};x). \]

Corollary 2.2 For any \( p \in \mathbb{N}^* \), there exists a constant \( C(p) \), independent of \( m \) and \( x \), such that

\[
|K_m^{(\alpha, \beta)}((t-x)^p; x)| \leq C(p) \left( \frac{x(1-x)}{m} \right)^{\frac{p}{2}} + O\left( \frac{1}{m^p} \right) \tag{2.1}
\]

for every \( x \in [0,1] \).

Proof First we have

\[
|P_m^{(\alpha, \beta)}((t-x)^p; x)| \leq \sum_{i=0}^{p} \left( \begin{array}{c} p \\ i \end{array} \right) \left( \frac{M}{m+\beta} \right)^{p-i} B_m(|t-x|^i; x), \tag{2.2}
\]

where \( M = \max\{|\alpha, \beta - \alpha|\} \) for \( x \in [0,1] \).

The following inequality

\[
B_m(|t-x|^i; x) \leq c(i) \sqrt{\left( \frac{X}{m} \right)^i}, \tag{2.3}
\]

where \( c(i) \) is a constant independent of \( m \), can be found in [16] for \( mX \geq 1 \), \( X = x(1-x) \) and in [5] for \( mX < 1 \).

Taking \( c(p) = \max_{i \leq p} c(i) \) in (2.3), by (2.2) it follows

\[
|P_m^{(\alpha, \beta)}((t-x)^p; x)| \leq c(p) \sum_{i=0}^{p} \left( \begin{array}{c} p \\ i \end{array} \right) \sqrt{\left( \frac{x(1-x)}{m} \right)^i} \left( \frac{M}{m+\beta} \right)^{p-i}
\]

\[
\leq c(p) \left( \sqrt{\left( \frac{x(1-x)}{m} \right)} + \left( \frac{M}{m+\beta} \right) \right)^p \]

\[
\leq 2^{p-1} c(p) \left( \sqrt{\left( \frac{x(1-x)}{m} \right)} + \left( \frac{M}{m+\beta} \right) \right)^p \]

\[
\leq C(p) \left( \frac{x(1-x)}{m} \right)^{\frac{p}{2}} + O\left( \frac{1}{m^p} \right). \tag{2.4}
\]

From (2.4) and Lemma 2.1, we obtain estimate (2.1).

The first four central moments for \( K_m^{(\alpha, \beta)} \) are as follows:

\[
K_m^{(\alpha, \beta)}(t-x; x) = -\frac{\beta + 1}{m + \beta + 1} x + \frac{2\alpha + 1}{2(m + \beta + 1)}.
\]
\[ K_m^{(\alpha, \beta)}((t-x)^2; x) = \frac{m - (2\alpha + 1)(\beta + 1)}{(m + \beta + 1)^2} x(1-x) \]
\[ + \frac{(\beta + 1)(\beta - 2\alpha)}{(m + \beta + 1)^2} x^2 + \frac{3\alpha^2 + 3\alpha + 1}{3(m + \beta + 1)^2} x(1-x) \]
\[ K_m^{(\alpha, \beta)}((t-x)^3; x) = -\frac{(3\beta + 5)m - (\beta + 1)^3}{(m + \beta + 1)^3} x^2(1-x) \]
\[ + \frac{(12\alpha + 10)m - 6(2\alpha + 1)(\beta + 1)^2 + 4(\beta + 1)^3}{4(m + \beta + 1)^3} x(1-x) \]
\[ - \frac{4(3\alpha^2 + 3\alpha + 1)(\beta + 1) - 6(2\alpha + 1)(\beta + 1)^2 + 4(\beta + 1)^3}{4(m + \beta + 1)^3} x \]
\[ + \frac{4\alpha^3 + 6\alpha^2 + 4\alpha + 1}{4(m + \beta + 1)^3} \]
\[ K_m^{(\alpha, \beta)}((t-x)^4; x) = \frac{3m^2 - 2(3 + 4(\beta + 1) + 3(\beta + 1)^2)m}{(m + \beta + 1)^4} x(1-x)^2 \]
\[ - \frac{[4(2\alpha + 1) + 2(6\alpha + 1)(\beta + 1) - 6(\beta + 1)^2]m - 2(2\alpha + 1)(\beta + 1)^3}{(m + \beta + 1)^4} x^2(1-x) \]
\[ + \frac{(6\alpha^2 + 10\alpha + 5)m + 2(2\alpha + 1)(\beta + 1)^3 - 2(3\alpha^2 + 3\alpha + 1)(\beta + 1)^2}{(m + \beta + 1)^4} x(1-x) \]
\[ - \frac{2(2\alpha + 1)(\beta + 1)^3 - 2(3\alpha^2 + 3\alpha + 1)(\beta + 1)^2 + 4\alpha^3 + 6\alpha^2 + 4\alpha + 1)(\beta + 1)}{(m + \beta + 1)^4} x \]
\[ + \frac{5\alpha^4 + 10\alpha^3 + 10\alpha^2 + 5\alpha + 1}{5(m + \beta + 1)^4}. \]

**Remark 2.3** Using the results obtained by Gavrea and Ivan ([5], Theorem 14, Theorem 15, Remark 16), it is straightforward to give the following estimates:

(i) For any \( p \geq 4 \) and \( x \in (0, 1) \), there exists a constant \( A(p) \) independent of \( m \) and \( x \) such that
\[
\frac{\|K_m^{(\alpha, \beta)}(|t-x|^{p+1}; x)\|}{\|K_m^{(\alpha, \beta)}(|t-x|^p; x)\|} \leq \frac{A(p)}{\sqrt{m}}, \quad m \geq 5. \tag{2.5}
\]

(ii) For any \( p \geq 1 \) and \( x \in (0, 1) \), there exists a positive constant \( B(p) \) independent of \( m \) and \( x \) such that
\[
\frac{\|K_m^{(\alpha, \beta)}(|t-x|^{p+1}; x)\|}{\|K_m^{(\alpha, \beta)}(|t-x|^p; x)\|} \geq \frac{B(p)}{\sqrt{m}}, \tag{2.6}
\]
where \( \| \cdot \| \) is the uniform norm on \([0,1]\).

(iii) From (i) and (ii) it follows
\[
\frac{\|K_m^{(\alpha, \beta)}(|t-x|^{p+1}; x)\|}{\|K_m^{(\alpha, \beta)}(|t-x|^p; x)\|} = O\left(\frac{1}{\sqrt{m}}\right). \tag{2.7}
\]

**Remark 2.4** From Mamedov’s theorem [10] it follows that:
If \( p \in N^* \) is even and \( f \in C^p([0,1]) \), for any \( x \in [0,1] \), we have that

\[
\lim_{m \to \infty} \frac{1}{K_m^{(\alpha,\beta)}((t-x)^p;x)} \left( K_m^{(\alpha,\beta)}(f;x) - f(x) - \sum_{i=1}^{p} K_m^{(\alpha,\beta)}((t-x)^i;x) \frac{f^{(i)}(x)}{i!} \right) = 0. \quad (2.8)
\]

### 3 Modified Kantorovich–Stancu operators

Now, we modify the Kantorovich–Stancu operator as follows:

\[
K_m^{(\alpha,\beta)}(f;x) = a(x;m)K_m^{(\alpha,\beta+1)}(f;x) + a(1-x;m)K_m^{(\alpha+1,\beta+1)}(f;x). \quad (3.1)
\]

**Lemma 3.1** The moments \( \overline{K}_m^{(\alpha,\beta)}(t^i;x) \), \( i = 0, 1, 2 \), are given by

\[
\overline{K}_m^{(\alpha,\beta)}(1;x) = 2a_0(m) + a_1(m),
\]

\[
\overline{K}_m^{(\alpha,\beta)}(t;x) = \frac{(2a_0(m) + a_1(m))m}{m + \beta + 1} x
+ \frac{4(\alpha + 1)a_0(m) + (2\alpha + 3)a_1(m) - 4(a_0(m) + a_1(m))x}{2(m + \beta + 1)},
\]

\[
\overline{K}_m^{(\alpha,\beta)}(t^2;x) = \frac{(2a_0(m) + a_1(m))m^2}{(m + \beta + 1)^2} x^2
+ \frac{[2((2\alpha + 3)a_0(m) + (\alpha + 2)a_1(m))x - (6a_0(m) + 5a_1(m))x^2]m}{(m + \beta + 1)^2}
+ \frac{12(a_0(m) + a_1(m))x^2 - 6(a_0(m) + a_1(m))(2\alpha + 3)x}{3(m + \beta + 1)^2}
+ \frac{2(3\alpha^2 + 6\alpha + 4)a_0(m) + (3\alpha^2 + 9\alpha + 7)a_1(m)}{3(m + \beta + 1)^2}.
\]

**Lemma 3.2** The central moments of the operators \( \overline{K}_m^{(\alpha,\beta)} \), \( \overline{K}_m^{(\alpha,\beta)}((t-x)^i;x) \), \( i = 1, 2, 3, 4 \), are given by

\[
\overline{K}_m^{(\alpha,\beta)}((t-x);x) = \frac{2(\beta + 2)a_0(m) + (\beta + 3)a_1(m)}{m + \beta + 1} x
+ \frac{4(\alpha + 1)a_0(m) + (2\alpha + 3)a_1(m)}{2(m + \beta + 1)},
\]

\[
\overline{K}_m^{(\alpha,\beta)}((t-x)^2;x) = \frac{(2a_0(m) + a_1(m))m}{(m + \beta + 1)^2} x(1-x) + O\left( \frac{1}{m^2} \right),
\]

\[
\overline{K}_m^{(\alpha,\beta)}((t-x)^3;x) = \frac{((3\beta + 8)(2a_0(m) + a_1(m)) + 3a_1(m))m}{(m + \beta + 1)^3} x^2(1-x)
+ \frac{((12\alpha + 10)(2a_0(m) + a_1(m)) + 12(a_0(m) + a_1(m))m}{4(m + \beta + 1)^3} x(1-x)
+ O\left( \frac{1}{m^3} \right),
\]

\[
\overline{K}_m^{(\alpha,\beta)}((t-x)^4;x) = \frac{3(2a_0(m) + a_1(m)m^2}{(m + \beta + 1)^4} x(1-x)^2 + O\left( \frac{1}{m^4} \right).
\]

We will study the uniform convergence of the sequence \( (\overline{K}_m^{(\alpha,\beta)} f)_{m \in N} \) for the case

\[
2a_0(m) + a_1(m) = 1. \quad (3.2)
\]
We observe that (3.2) implies $K_m^{(α, β)}(1; x) = 1$.

We are interested in the following cases:

Case 1:

$$a_0(m) \geq 0 \quad \text{and} \quad a_0(m) + a_1(m) \geq 0.$$  

(3.3)

Case 2:

$$a_1(m) < 0 \quad \text{or} \quad a_0(m) + a_1(m) < 0.$$  

(3.4)

Combining (3.2) and (3.3), we obtain $a_0(m) \in [0, 1]$ and $a_1(m) \in [-1, 1]$, which implies that the sequences $a_0(m)$ and $a_1(m)$ are bounded. The operators $K_m^{(α, β)}$ are bounded and positive.

Combining (3.2) and (3.4), we obtain that $a_0(m) + a_1(m) > 1$ if $a_1(m) < 0$ and $a_0(m) > 1$ if $a_0(m) + a_1(m) < 0$. In these cases, the operators $K_m^{(α, β)}$ are not positive.

Remark that, for $α = β = 0$ and $a_0(m) = \frac{3}{2}$, $a_1(m) = -2$, we obtain the modified operators introduced and studied in [7].

In order to prove the uniform convergence of the operators $K_m^{(α, β)}$, we give the Korovkin theorem:

**Theorem 3.3** ([9], Theorem 10) Let $0 < h \in C([a, b])$ be a function and suppose that $(L_n)_{n \geq 1}$ is a sequence of positive linear operators such that $\lim_{n \to \infty} L_n(e_i) = he_i$, $i = 0, 1, 2$, uniformly on $[a, b]$. Then, for a given function $f \in C([a, b])$, we have $\lim_{n \to \infty} L_n(f) = hf$ uniformly on $[a, b]$.

For the first case, we obtain the following result:

**Theorem 3.4** Given two sequences $a_0(m)$ and $a_1(m)$ that satisfy conditions (3.2) and (3.3), the sequence $(K_m^{(α, β)}, f)_{m \in N}$ converges to $f$, uniformly on $[0, 1]$, for any function $f \in C([0, 1])$.

**Proof** The operator $K_m^{(α, β)}f$ is a linear convex combination of positive operators $K_m^{(α, β + 1)}f$ and $K_m^{(α + 1, β + 1)}f$. Consequently, the result follows from Theorem 3.3. □

In the second case, we have the following:

**Theorem 3.5** For any function $f \in C([0, 1])$ and all bounded sequences $a_0(m), a_1(m)$ that satisfy conditions (3.2) and (3.4), the sequence $(K_m^{(α, β)}, f)_{m \in N}$ converges to $f$, uniformly on $[0, 1]$.

**Proof**

$$K_m^{(α, β)}(f; x) = a(x; m)K_m^{(α, β + 1)}(f; x) + a(1 - x; m)K_m^{(α + 1, β + 1)}(f; x)$$

$$= (a_1(m)x + a_0(m))K_m^{(α, β + 1)}(f; x)$$

$$+ (-a_1(m)x + a_0(m) + a_1(m))K_m^{(α + 1, β + 1)}(f; x)$$

$$= \left[ a_0(m)K_m^{(α, β + 1)}(f; x) + (a_0(m) - a_1(m)x)K_m^{(α + 1, β + 1)}(f; x) \right]$$

$$- \left[ -a_1(m)xK_m^{(α, β + 1)}(f; x) - a_1(m)K_m^{(α + 1, β + 1)}(f; x) \right].$$
Taking
\[ K_{m,1}^{(\alpha,\beta)}(f;x) = a_0(m)K_{m-1}^{(\alpha,\beta)}(f;x) + (a_0(m) - a_1(m)x)K_{m-1}^{(\alpha,\beta+1)}(f;x) \]
and
\[ K_{m,2}^{(\alpha,\beta)}(f;x) = -a_1(m)xK_{m-1}^{(\alpha,\beta)}(f;x) - a_1(m)K_{m-1}^{(\alpha,\beta+1)}(f;x), \]
we have
\[ K_m^{(\alpha,\beta)}(f;x) = K_{m,1}^{(\alpha,\beta)}(f;x) - K_{m,2}^{(\alpha,\beta)}(f;x). \tag{3.5} \]

Using the remarks for case 2, it follows that the operators \( K_{m,1}^{(\alpha,\beta)} \) and \( K_{m,2}^{(\alpha,\beta)} \) are positive. According to Theorems 3.3 and 3.4, we obtain that
\[ \lim_{m \to \infty} (K_m^{(\alpha,\beta)} f)(x) = \lim_{m \to \infty} (K_{m,1}^{(\alpha,\beta)} f)(x) - \lim_{m \to \infty} (K_{m,2}^{(\alpha,\beta)} f)(x) = (2l_0 - l_1)f(x) + l_1(1 + x)f(x) = (2l_0 + l_1)f(x) = f(x), \]
where \( l_i = \lim_{m \to \infty} a_i(m), \) \( i = 0, 1. \)

The following theorems are Voronovskaja-type results for the operators \( K_m^{(\alpha,\beta)} \).

**Theorem 3.6** Let \( a_0(m), a_1(m) \) be two convergent sequences that verify conditions (3.2) and (3.3) and \( l_i = \lim_{m \to \infty} a_i(m), \) \( i = 0, 1. \) If \( f \in C^3([0,1]) \), then
\[ \lim_{m \to \infty} m(K_m^{(\alpha,\beta)}(f;x) - f(x)) = \left( \alpha + 1 - (\beta + 2)x \right) \left( \frac{1 - 2x}{2} \right) f'(x) + \frac{1}{2} x(1-x)f''(x) \tag{3.6} \]
uniformly on \([0,1].\)

**Proof** Applying Taylor’s formula to the operators \( K_m^{(\alpha,\beta)} \), we have
\[ K_m^{(\alpha,\beta)}(f;x) = f(x) + \frac{1}{1!}K_m^{(\alpha,\beta)}(t-x)f'(x) + \frac{1}{2!}K_m^{(\alpha,\beta)}(t-x)^2f''(x) + K_m^{(\alpha,\beta)}(\rho(t;x)(t-x)^2;x), \tag{3.7} \]
where \( \rho \in C([0,1]) \) and \( \lim_{t \to x} \rho(t;x) = 0. \)

It is sufficient to prove that \( \lim_{m \to \infty} mK_m^{(\alpha,\beta)}(\rho(t;x)(t-x)^2;x) = 0 \) uniformly on \([0,1].\)

Using the Cauchy–Schwarz theorem, we obtain that
\[ K_m^{(\alpha,\beta)}(\rho(t;x)(t-x)^2;x) \leq \sqrt{K_m^{(\alpha,\beta)}((\rho(t;x))^2;x)} \cdot \sqrt{K_m^{(\alpha,\beta)}((t-x)^4;x)}. \]
Since $\rho(x,x) = 0, \rho^2(\cdot;x) \in C([0,1])$, by Theorem 3.4, we have
\[
\lim_{m \to \infty} \mathcal{K}_m^{(\alpha,\beta)} ((\rho(t;x))^2;x) = 0,
\]
and by Lemma 3.2, we get
\[
\lim_{m \to \infty} m^{2} \mathcal{K}_m^{(\alpha,\beta)} ((t-x)^4;x) = 3(x(1-x))^2
\]
uniformly on $[0,1]$. Hence, we obtain the above limit.

Finally, Lemma 3.2 gives us (3.6).

\[ \square \]

**Theorem 3.7** Let $a_0(m), a_1(m)$ be two bounded convergent sequences that verify conditions (3.2) and (3.4) and $l_i = \lim_{m \to \infty} a_i(m), i = 0,1$. If $f \in C^2([0,1])$, then
\[
\lim_{m \to \infty} m(\mathcal{K}_m^{(\alpha,\beta)}(f;x) - f(x)) = \left( (\alpha + 1 - (\beta + 2)x) + \frac{(1-2x)a_0}{2} \right)f'(x) + \frac{1}{2}x(1-x)f''(x)
\]
uniformly on $[0,1]$.

**Proof** From (3.5), we have
\[
\mathcal{K}_m^{(\alpha,\beta)}(f;x) = \mathcal{K}_m^{(\alpha,\beta)}(f;x) - \mathcal{K}_m^{(\alpha,\beta)}(f;x),
\]
where
\[
\mathcal{K}_m^{(\alpha,\beta)}(f;x) = a_0(m) \mathcal{K}_m^{(\alpha,\beta)}(f;x) + (a_0(m) - a_1(m)) \mathcal{K}_m^{(\alpha+1,\beta+1)}(f;x)
\]
and
\[
\mathcal{K}_m^{(\alpha,\beta)}(f;x) = -a_1(m) x \mathcal{K}_m^{(\alpha,\beta)}(f;x) - a_1(m) K_m^{(\alpha+1,\beta+1)}(f;x).
\]

Applying Theorem 3.6 to the operators $\mathcal{K}_m^{(\alpha,\beta)}$ and $\mathcal{K}_m^{(\alpha,\beta)}$, we obtain
\[
\lim_{m \to \infty} m(\mathcal{K}_m^{(\alpha,\beta)}(f;x) - f(x)) = \left( -(2l_0 - l_1)x(\beta + 2)x + \frac{4(\alpha + 1)x + (2\alpha + 3)x}{2} \right)f'(x) + \frac{1}{2}(2l_0 - l_1)x(1-x)f''(x)
\]
and
\[
\lim_{m \to \infty} m(\mathcal{K}_m^{(\alpha,\beta)}(f;x) - f(x)) = \left( l_1(x + 1)(\beta + 2)x - \frac{(2\alpha + 1)x + (2\alpha + 3)x}{2} \right)f'(x) - \frac{1}{2}l_1(x + 1)x(1-x)f''(x)
\]
uniformly on $[0,1]$.

Combining these two results, the proof is finished. \[ \square \]
In what follows, we will denote by \( \omega(f; \cdot) \) the first order modulus of continuity of the function \( f \)
\[
\omega(f; \delta) = \sup \left\{ \left| f(x') - f(x'') \right| \middle| x', x'' \in I, \left| x' - x'' \right| \leq \delta \right\}, \quad \text{where } I = [0,1], f: I \to \mathbb{R}.
\]

**Theorem 3.8** Let \( a_0(m), a_1(m) \) be two bounded sequences that verify (3.2). If \( f(x) \) is bounded for \( x \in [0,1] \), then
\[
\| K^{(\alpha, \beta)}_m f - f \| \leq \frac{3}{2} \left( 3|a_1(m)| + 1 \right) \omega \left( f; \frac{1}{\sqrt{m + \beta + 1}} \right),
\]
where \( \| \cdot \| \) is the uniform norm on \([0,1]\).

**Proof** By (3.1), we have that
\[
\left| K^{(\alpha, \beta)}_m (f;x) - f(x) \right| \leq \left| a(x;m) \right| \left| K^{(\alpha, \beta+1)}_{m-1} (f;x) - f(x) \right|
+ \left| a(1-x;m) \right| \left| K^{(\alpha+1, \beta+1)}_{m-1} (f;x) - f(x) \right|.
\]

We need an upper bound for \( a(x;m) \) and \( a(1-x;m) \). Note that this is the same upper bound for both. From (3.2), it follows that
\[
\left| a(x;m) \right| = \left| a_1(m)x + a_0(m) \right| \leq \left| a_1(m) \right| + \left| a_0(m) \right|
= \left| a_1(m) \right| + \left| 1 - a_1(m) \right| \leq 3 \left| a_1(m) \right| + 1.
\]

and (3.9) becomes
\[
\left| K^{(\alpha, \beta)}_m (f;x) - f(x) \right| \leq \frac{3}{2} \left| a_1(m) \right| + 1 \left| K^{(\alpha, \beta+1)}_{m-1} (f;x) - f(x) \right|
+ \left| K^{(\alpha+1, \beta+1)}_{m-1} (f;x) - f(x) \right|.
\]

By ([2], Theorem 2.6), we have
\[
\left| K^{(\alpha, \beta+1)}_{m-1} (f;x) - f(x) \right| \leq 2 \omega \left( f; \sqrt{\delta^{(\alpha, \beta+1)}_{m-1,1}} \right)
\]
and
\[
\left| K^{(\alpha+1, \beta+1)}_{m-1} (f;x) - f(x) \right| \leq 2 \omega \left( f; \sqrt{\delta^{(\alpha+1, \beta+1)}_{m-1,1}} \right),
\]
where
\[
\delta^{(\alpha, \beta+1)}_{m-1,1} = \frac{(\beta + 2)^2}{(m + \beta + 1)^2} + \frac{(2\alpha + 1)(m - 1)^2}{(m + \beta)(m + \beta + 1)^2} + \frac{m - 1}{4(m + \beta + 1)^2}
+ \frac{3\alpha^2(3m + \beta - 2)}{3(m + \beta)(m + \beta + 1)^2}.
\]
\[\delta_{m-1,1}^{(\alpha+1, \beta+1)} = \frac{(\beta + 2)^2}{(m + \beta + 1)^2} + \frac{(2\alpha + 3)(m - 1)^2}{(m + \beta)(m + \beta + 1)^2} + \frac{m - 1}{4(m + \beta + 1)^2} \]
\[+ \frac{3(\alpha + 1)^2(3m + \beta - 2) + (m + \beta)(1 - 3m - 3\beta)}{3(m + \beta)(m + \beta + 1)^2}.\]

So,
\[|K_m^{(\alpha, \beta)}(f; x) - f(x)| \leq (3|a_1(m)| + 1)\left(\omega\left(f; \sqrt{\delta_{m-1,1}^{(\alpha, \beta+1)}}\right) + \omega\left(f; \sqrt{\delta_{m-1,1}^{(\alpha+1, \beta+1)}}\right)\right). \tag{3.11}\]

By using the properties of the first order modulus of continuity together with the above forms of \(\delta_{m-1,1}^{(\alpha, \beta+1)}\) and \(\delta_{m-1,1}^{(\alpha+1, \beta+1)}\) in (3.11), we obtain (3.8).

Assume that \(\beta = 2\alpha\), \(K_m^{(\alpha, \beta)}(1; x) = 1\) and \(K_m^{(\alpha, \beta)}(t; x) = x\).
Consequently, we get
\[2a_0(m) + a_1(m) = 1 \quad \text{and} \quad a_0(m) + a_1(m) = -\frac{2\alpha + 1}{2},\]
which implies that
\[a_0(m) = \frac{2\alpha + 3}{2}, \quad a_1(m) = -2(\alpha + 1)\]
and from (3.4), it follows that the operators \(K_m^{(\alpha, \beta)}\) are not positive.

Now, we can formulate a new quantitative Voronovskaja-type result:

**Theorem 3.9** For \(g \in C^2([0, 1]), x \in [0, 1]\) fixed, we have the following estimate:
\[\left|K_m^{(\alpha, \beta)}(g; x) - g(x) - \frac{1}{2}K_m^{(\alpha, \beta)}\left((t - x)^2; x\right)g''(x)\right| \leq C \frac{1}{m} \omega\left(g''; \frac{1}{\sqrt{m + \beta + 1}}\right), \tag{3.12}\]
where \(C\) is a positive constant independent of \(m\) and \(x\).

**Proof** Under the above assumptions, by applying Taylor’s formula to the operators \(K_m^{(\alpha, \beta)}\), we have
\[K_m^{(\alpha, \beta)}(g; x) = g(x) + \frac{1}{2}K_m^{(\alpha, \beta)}\left((t - x)^2; x\right)g''(x) + K_m^{(\alpha, \beta)}(r(t; x); x), \tag{3.13}\]
where
\[r(t; x) = \int_x^t (t - u)\left[g''(t) - g''(u)\right]du.\]
From the mean value theorem, it follows that there exists \(\xi \in (\min(x, t), \max(x, t))\) such that
\[r(t; x) = \left[g''(x) - g''(\xi)\right] \int_x^t (t - u) du = \left[g''(x) - g''(\xi)\right]\frac{(t - x)^2}{2}.\]
So,

\[ |r(t; x)| \leq \omega(g''; |t - x|) \frac{(t - x)^2}{2} \]
\[ \leq (1 + \sqrt{m + \beta + 1}|t - x|) \omega\left( g''; \frac{1}{\sqrt{m + \beta + 1}} \right) \frac{(t - x)^2}{2}. \]  
(3.14)

When \( x \in [0, 1] \), an upper bound for \( a(x; m) \) and \( a(1 - x; m) \) is

\[ |a(x; m)| = |a_1(m)x + a_0(m)| = \left| -2(\alpha + 1)x + \frac{2\alpha + 3}{2} \right| \leq \frac{2\alpha + 3}{2}. \]  
(3.15)

From (3.15) and (3.1), we get

\[ |K_{m-1}^{(\alpha, \beta)}(r(t; x); x)| \leq \frac{2\alpha + 3}{2} \left| K_{m-1}^{(\alpha, \beta)}(r(t; x); x) + K_{m-1}^{(\alpha+1, \beta+1)}(r(t; x); x) \right|. \]  
(3.16)

Using (3.14), it follows that

\[ |K_{m-1}^{(\alpha+1, \beta+1)}(r(t; x); x)| \leq \frac{1}{2} \left| K_{m-1}^{(\alpha, \beta)} \left( (t - x)^2; x \right) \right| + \sqrt{m + \beta + 1} K_{m-1}^{(\alpha, \beta+1)} \left( |t - x|((t - x)^2); x \right). \]  
(3.17)

Applying Corollary 2.2, it follows that there exists a constant \( C' \) independent of \( m \) and \( x \) such that (3.17) becomes

\[ |K_{m-1}^{(\alpha, \beta+1)}(r(t; x); x)| \leq C' \frac{1}{m-1} \omega\left( g''; \frac{1}{\sqrt{m + \beta + 1}} \right). \]  
(3.18)

Thus,

\[ |K_{m}^{(\alpha, \beta)}(r(t; x); x)| \leq C \frac{1}{m} \omega\left( g''; \frac{1}{\sqrt{m + \beta + 1}} \right), \]  
(3.19)

and the proof is completed.  \( \square \)

**Corollary 3.10** For \( g \in C^2([0, 1]), \ x \in [0, 1] \) fixed, we have

\[ \lim_{m \to \infty} m(K_{m}^{(\alpha, \beta)} g; x) = \frac{1}{2} x(1 - x)g''(x). \]  
(3.20)

**Proof** By Theorem 3.9 and Lemma 3.2(ii), we obtain (3.20).  \( \square \)

**Corollary 3.11** For \( g \in C^2([0, 1]), \) the following estimate holds:

\[ \|K_{m}^{(\alpha, \beta)} g - g\| \leq \frac{C}{m} \|g''\|, \]  
(3.21)

where \( \| \cdot \| \) is the uniform norm on \([0, 1]\).
Proof Since $\omega(g''; \delta) \leq 2\|g''\|$, by Lemma 3.2(ii) and Theorem 3.9, we obtain (3.21). □

We can reformulate Theorem 3.9 in terms of second order moduli of continuity.

**Theorem 3.12** Assuming $\beta = 2\alpha$, for $a_0(m) = \frac{2\alpha + 3}{2}$, $a_1(m) = -2(\alpha + 1)$, and $g \in C([0,1])$, we have

$$\|K_m^{(\alpha, \beta)} g - g\| \leq C \omega_2\left(g, \frac{1}{\sqrt{m}}\right).$$

(3.22)

Proof The operators $K_m^{(\alpha, \beta)}$ are bounded, and by (3.1), we have

$$\|K_m^{(\alpha, \beta)} g\| \leq (|a_0(m)| + |a_1(m)|)\|g\|.$$

It is well known that the second order modulus of continuity is equivalent to the $K$-functional

$$K_2(g, t^2) = \inf_{h \in C^2([0,1])} \{\|g - h\| + t^2 \|h''\|\}.$$

From Gonska ([6], Corollary 2.7),

$$K_2(g, t^2) \leq \frac{7}{2} \omega_2(g, t), \quad t \geq 0, g \in C([0,1]).$$

Combining the above inequalities and taking the infimum over all $h \in C^2([0,1])$ in the following inequality

$$\|K_m^{(\alpha, \beta)} g - g\| \leq \|K_m^{(\alpha, \beta)} (g - h) - (g - h)\| + \|K_m^{(\alpha, \beta)} h - h\|$$

$$\leq C_1 \|g - h\| + \frac{C_2}{m} \|g''\| \leq C_3 \left\{\|g - h\| + \frac{1}{m} \|g''\|\right\}$$

leads to the desired result. □

4 Conclusions
In this paper, we introduce and study a modified form of the Kantorovich–Stancu operators.

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