Time-varying coupling strengths, nuclear forces and unification

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July 2002

Abstract

We investigate the dependence of the nucleon-nucleon force in the deuteron system on the values of coupling strengths at high energy, which will in general depend on the geometry of extra dimensions. The stability of deuterium at all times after nucleosynthesis sets a bound on the time variation of the ratio of the QCD confinement scale to light quark masses. We find the relation between this ratio, which is exponentially sensitive to high-energy couplings, and fundamental parameters, in various classes of unified theory. Model-dependent effects in the Higgs and fermion mass sector may dominate even over the strong dependence of the QCD scale Λ. The binding energy of the deuteron also has an important effect on nucleosynthesis: we estimate the resulting bounds on variation of couplings.

1 Introduction

In many models of particle physics, the universe is assumed to have more than 4 dimensions. The extra dimensions are either compactified to such a small size that we cannot (currently) probe them experimentally [1] or possess metrics with

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a nontrivial dependence on the transverse directions such that we can only detect the gravitational influence of our familiar 4 dimensions [3]. Cosmological solutions of the field equations of these theories often involve time evolution of the higher dimensions; the value of the gauge couplings in the low energy limit of these theories is invariably a function of the size or shape of the higher dimensions. This can be extremely problematic as variation in the gauge couplings over cosmological time scales may destroy the successful predictions of primordial nucleosynthesis [19]. Recent claims of a time-variation of the fine structure constant [10] also motivate the study of theories with dynamically-determined (and thus potentially time-varying) couplings.

In relation to nucleosynthesis, it is only clear what the effect of changing one coupling constant at a time has upon the light element abundance. It is possible that degeneracies in the effect on nucleosynthesis may arise when more than one gauge coupling changes at once: an overall change in all couplings might be acceptable at a level much greater than that permitted for any one on its own. In this situation it is not possible accurately to constrain models such as [4] where the gauge couplings oscillate with a fractional change of the order of 10% in the matter dominated era. Such a large fractional change would not be acceptable in, for example, the electromagnetic fine structure constant $\alpha$ at nucleosynthesis, if this were the only time-dependent coupling. This paper is an attempt to provide an additional constraint, which is independent of nucleosynthesis (but which may affect calculations of nucleosynthesis), which suffers as little as possible from the problem of relating nuclear forces to underlying theory, and which is sensitive to a well-defined combination of couplings.

Deuterium is only produced during nucleosynthesis, as it is too weakly bound to survive in the regions of stars where nuclear processes take place. The fact that deuterium is still observed today means that variations in the gauge coupling strengths or other fundamental parameters are non-trivially constrained by the requirement that the deuteron be stable at all times after nucleosynthesis. The fact that the deuteron is so weakly bound also makes it more sensitive to variations in the internuclear force. The strong running of $\alpha_3$ at low energies means that a change in the coupling strength at high energy is manifested in a change in the strong coupling scale $\Lambda_{QCD}$ (henceforth denoted by $\Lambda$), by the usual dimensional transmutation arguments. Changes in $\Lambda$ in turn lead to changes in the internuclear force.

As recently pointed out by Langacker et al. [13], one also expects changes in quark masses and in the Higgs v.e.v. $v$ if gauge couplings are unified at some scale. Any viable unified theory should accommodate (if not predict) a mechanism for electroweak symmetry-breaking, which may well depend sensitively on SUSY-breaking masses, and a mechanism of generating small Yukawa couplings, all of which may have a dependence on the unified coupling. Moreover, $\Lambda$ is sensitive to all coloured particle masses through threshold effects of RG running. These effects introduce a large measure of model-dependence, since the correct
theories of SUSY-breaking, electroweak symmetry-breaking and fermion masses are unknown. One may choose for simplicity to set to zero unknown effects in the electroweak, SUSY and Yukawa sectors [12], but this runs the risk of neglecting terms which are of equal size or larger than the terms kept in the analysis.

One might also consider “less unified” models, with more than one dynamically-determined fundamental coupling. The heterotic string dilaton $S$ and volume moduli $T$ provide a basic example, where gauge couplings and renormalisable Yukawa couplings (for canonically-normalised fields) have a universal dependence on $S$, but may be differently affected by changing $T$. The greater the number of independent quantities considered as time-dependent, the less predictive the theory becomes and the less meaningful are any constraints. Here we restrict ourselves to estimating the dependence of low-energy quantities in a somewhat idealised framework with a single dynamical unified coupling, corresponding to the v.e.v. of a dilaton-like field.

In the first part of this paper we calculate the deuteron binding energy by considering meson exchange forces, expressing the relevant parameters as a functions of $\Lambda$ and the light quark masses $m_u$, $m_d$. The result is rather simple: we find that the deuteron is stable as long as the ratio $m_q/\Lambda$ is greater than a certain value, where $m_q = m_u + m_d$. We perform similar calculations for the dineutron and diproton systems in the same isospin multiplet and investigate the criterion for their stability. Then we relate $\Lambda$ and the quark masses to the QCD coupling strength and running masses at high energy using renormalisation group (RG) evolution. We take two cases, supersymmetric models with RG running up to the GUT scale (similar results will be obtained in the case of power-law unification in large extra dimensions), and nonsupersymmetric low-scale models with RG running up to a scale of a few TeV. The main result of this section is the exponential dependence of $\Lambda$ on the perturbative strong coupling $\alpha_3$ at high energy. We also find how the Higgs v.e.v. and SUSY-breaking masses influence the low-energy parameters.

Finally we consider how the bounds deduced from the two-nucleon system apply to various types of high-energy model, in which $\alpha_3$ and the quark masses depend on model parameters (in particular the sizes of extra dimensions) which may be time-dependent. Thus, bounds on the possible cosmological evolution of such models since nucleosynthesis can be obtained. The constraints from the dinucleon system will in general apply to a different combination of theory parameters from those arising from nucleosynthesis — taking the two together bounds the variation of fundamental parameters in two directions. (In addition, there are many other observational bounds applying at various much later epochs, discussed for example in [12], including some from direct laboratory measurement.) We also point out for the first time and estimate the effect of changing the deuteron binding energy on the process of helium formation at nucleosynthesis, which may give rise to stronger bounds.
1.1 Relation to recent work

The relation between a time-varying fine structure constant and other observables in particle physics was also treated in [11, 12, 13]. In [11] the resulting variation in the vacuum energy $V_0$ was estimated from general principles of QFT and found to be enormously larger than the cosmological bounds; hence, the authors concluded that a large number of implausibly accurate fine-tunings would be necessary for a time variation of the size that has been claimed to be consistent with field theory and cosmology. Such an argument is rather weak since it assumes, crucially, that the cosmological constant problem is “solved” at the present time by cancellations of different field theory contributions, all of which are many orders of magnitude larger than the measured value of $V_0$. This can hardly be a solid starting-point from which to set theoretical limits. The alternative considered in [11] was a self-tuning mechanism which protects the four-dimensional spacetime were we live by dynamically “absorbing” the vacuum energy into the curvature of one or more extra dimensions. Such a mechanism probably disallows inflation, but it is by no means clear that it also rules out spontaneous symmetry-breaking as claimed (the argument being that anything that “cancels” the vacuum energy also removes the source for a scalar to roll to the minimum of its potential). The existence of a dynamical time-scale for the self-tuning, the role of thermal effects in creating an effective potential, and the possibility that the extra-dimensional model may break four-dimensional Lorentz invariance are possible ways out of the $D = 4$ field theory argument that $V(\phi)$ must vanish at all times (see [14] for related discussions).

We prefer to take the majority point of view on the cosmological constant, i.e. that it is our greatest theoretical area of ignorance and that no credible way to explain its smallness currently exists, therefore very little can be deduced from it, and certainly nothing related to quantum effects in the theory (which are the main difficulty in accommodating varying alpha). If we want to retain the semiclassical picture of matter coupled to gravity, the only sensible interpretation is a very light scalar, which interestingly would have about the same mass as quintessence; due to the coupling to electromagnetism, such a scalar would mediate composition-dependent forces which might have experimental signatures [15]. But the vanishingly small scalar mass (even if experimentally confirmed) would remain a mystery, in the absence of an underlying theory which would explain why spacetime curvature was apparently so insensitive to a cosmologically-evolving field theory.

Calmet and Fritsche [12] calculate some of the consequences for low-energy physics of changing $\alpha$, within a GUT-like theory which constrains the SM gauge couplings to be equal at a particular energy scale (or at least to satisfy some fixed relation). They consider exclusively the effects on the strong interactions, with the unstated assumption that the mechanisms of electroweak symmetry-breaking, supersymmetry-breaking and fermion mass generation are held constant despite
varying the unified coupling, hence that the quark and lepton masses, as well as \( W \) and \( Z \) masses, remain unchanged. Our calculation of the dependence of \( \Lambda \) and \( M_N \) on \( \alpha_3(\mu > m_t) \) is essentially identical to theirs, except for including the full dependence on thresholds. As noted above, such a study can only be a first step since the electroweak mass scales may also depend sensitively on the unified coupling (for example in models of hidden sector SUSY-breaking, radiative electroweak symmetry-breaking and anomalous U(1) flavour models).

Unfortunately, since we do not know the correct theories of electroweak and supersymmetry-breaking, let alone that of fermion masses, correlations between the time-variation of different low-energy observables (such as \( \alpha \) and the proton magnetic moment \( \mu_p \) or the mass ratio \( m_p/m_e \)) are necessarily model-dependent, unless one can find a combination insensitive to, say, the electroweak sector. We take the converse approach and quote in a hopefully less model-dependent way the bounds on high-energy parameters deriving from the low-energy system that we are studying. In particular one cannot claim yet (as in \([12]\)) that the current data on the time variation of \( \alpha \) and other quantities are inconsistent with unification; but given an observational bound one can derive bounds on time-varying fundamental parameters, in whatever model one is interested in.

With precise measurements of at least two different quantities at any particular epoch, of which at least one shows a nonzero variation, one can rule out classes of unified theory that predict the wrong relation between (variations in the) different quantities. Time-varying couplings in principle are a new way of doing phenomenology, which allow us to test relations between the derivatives of different quantities rather than their static values. This paper presents an additional bound which applies at all times after nucleosynthesis, thus for models in which time variation was more rapid in the early Universe (such as \([4]\)) it is likely quite restrictive.

Langacker et al. \([13]\) have a somewhat similar approach to the “high-energy” aspect of the problem, parameterising the effects of variations in the unified coupling strength on the electroweak and Yukawa sectors by unknown, model-dependent constants of proportionality which depend on the particular model used. By looking at other precision astronomical measurements besides \([10]\), they find that these constants must satisfy a rather precise relation, for the model to be consistent with observation.

\section{Nuclear forces and the stability of di-nucleons}

\subsection{Chiral symmetry breaking and the pion}

Previous calculations taking into account the effect of the time variation of the strong interaction on low energy nuclear phenomena were performed in the chiral limit, \( i.e. \) the limit of massless quarks and pions, where the only energy scale of
the system is $\Lambda$. The approach of [30] was to assume that certain dimensionful static quantities (for instance vector meson masses and nuclear binding energies) were directly proportional to $\Lambda$; in this way it is possible to place constraints on the variation of $\alpha_3$ over time.

Explicit breaking of chiral symmetry in QCD is achieved by the addition of non-zero quark masses to the Lagrangian. The spontaneous breaking of chiral symmetry occurs when the quark condensate $\langle 0 | \bar{q} q | 0 \rangle$ develops a non-zero value and dynamically lines up with the quark masses in the internal SU($n_f$) flavour space [23]. The pion is the Goldstone boson associated with this spontaneous symmetry breaking: it is not quite massless, since the chiral symmetry is explicitly broken by the quark masses. The pion is however much less massive than any other strongly-interacting particle, which means that the internuclear pion force is relatively long range and important in the analysis of the loosely bound deuteron and unbound di-neutron and di-proton.

Models of nuclear structure utilising just the exchange of the $\omega$, $\sigma$ and $\rho$ mesons [33] are quite successful. This is partly because the isospin dependence of the pion interaction is such that the average effect across an isospin symmetric nucleus consisting of many protons and neutrons cancels. However, no such cancellation occurs in two-nucleon systems, where the internuclear pion force is finite and well-defined. The large separation of the proton and neutron in the deuteron (and in the putative di-neutron and di-proton systems, if coupling strengths were to change so that they were weakly bound), relative to the typical range of internuclear forces, increases the importance of the long range pion force in the binding of the system. Consequently, the pion force is the dominant contribution to the binding of two-nucleon systems, and the contributions of $\omega$, $\sigma$ and $\rho$ meson exchange to the binding energy can be taken to be of secondary importance.

The pion mass is given by the Gell-Mann-Oakes-Renner relation [24]

$$m_\pi^2 f_\pi^2 = (m_u + m_d) \langle 0 | \bar{q} q | 0 \rangle$$  \hspace{1cm} (1)

The non-zero pion mass leads to a finite divergence of the total axial current, which compensates for the partially conserved axial current of the weak interaction Hamiltonian. This compensation leads to the Goldberger-Treiman relation for the pion-nucleon coupling $g_\pi$ [25]

$$g_\pi = \frac{2 M_N g_A}{f_\pi}$$  \hspace{1cm} (2)

where $M_N$ is the nucleon mass. Although this is an approximate relation, one would expect it to remain valid as the parameters $f_\pi$ and $M_N$ change, so long as the variation is much less than that necessary to restore the chiral symmetry of the vacuum at zero temperature. (If this were not the case, the problems created for nuclear physics in the early Universe would be much greater than those we discuss here, indeed such a scenario would be immediately ruled out).
Since the nucleon mass $M_N$ originates mainly from confinement of the quark colour charges (rather than current quark masses), $M_N$ simply scales in direct proportion to $\Lambda$. Combining this with Eqs. (1) and (2) we find the relation

$$\frac{g_\pi^2 m_\pi}{M_N} = \frac{g_\pi^3 (m_u + m_d)^{1/2} \langle 0|q\bar{q}|0\rangle^{1/2}}{2g_A M_N^2} \propto \frac{g_\pi^3 (m_u + m_d)^{1/2} \langle 0|q\bar{q}|0\rangle^{1/2}}{g_A \Lambda^2} (3)$$

Note that the product $m_q \langle 0|q\bar{q}|0\rangle$ is invariant under change of renormalisation scale in QCD (see e.g. [26]) so thus far we avoid the problem of having scale-dependent quantities in a context where mass scales may be time-dependent (see section 4.1).

To proceed further, we need the relationship between $\langle 0|q\bar{q}|0\rangle$ and $\Lambda$, obtained by looking at the effective potential for the chiral symmetry-breaking, which shows that the energy scale associated with chiral symmetry restoration varies as the value of the condensate:

$$\langle 0|q\bar{q}|0\rangle^{1/2} \sim \Lambda, \quad (4)$$

a result also predicted by sum rule calculations of the nucleon mass [28]. As it stands this is not a well-defined relation, since the LHS is renormalisation-scale dependent. To remedy this, we use the formalism of “invariant quark masses” described e.g. in [27]: to one loop the equation for the running quark mass $\tilde{m}_i(\mu)$ is solved by

$$\tilde{m}_i(\mu) = \hat{m}_i(-\beta_1 \alpha_3(\mu)/\pi)^{-\gamma_1/\beta_1} (5)$$

where $\hat{m}_i$ is a RG invariant quantity, $\beta_1$ is the one-loop beta-function coefficient and $\gamma_1$ the one-loop anomalous dimension of quark masses, such that for three flavours $-\gamma_1/\beta_1 = -2/(-9/2) = 4/9$. Thus for scales below $m_c$ we have $\tilde{m}_i(\mu) = \hat{m}_i \alpha_3(\mu)^{4/9}$, up to a constant universal factor which will drop out of fractional changes $\Delta(\ln \tilde{m}) \equiv \Delta \hat{m}/\hat{m}$. Then the bilinear order parameter must have the one-loop behaviour

$$\langle 0|q\bar{q}|0\rangle(\mu) = (\text{RG invariant}) \times \alpha_3(\mu)^{-4/9} \propto \Lambda^3 \alpha_3(\mu)^{-4/9} \quad (6)$$

which gives the correct dependence on both the RG scale and (a possibly time-varying) $\Lambda$. We can now write

$$\frac{g_\pi^2 m_\pi}{M_N} = \text{const.} \times g_A^2 \left( \frac{\hat{m}_u + \hat{m}_d}{\Lambda} \right)^{1/2} (7)$$

in which all RG dependence cancels neatly \footnote{Such a redefinition can be performed to any desired loop order.}. We will see in the following sections that the value of the ratio $(\hat{m}_q/\Lambda)^{1/2}$, which we will denote as $c$,

$$c^2 \equiv \frac{\hat{m}_q}{\Lambda} \quad (8)$$
is the parameter controlling the effect of varying coupling strengths on nuclear physics phenomena dominated by pion exchange. Later we shall see how \( c \) can be related to coupling strengths at high energy in some examples of unified models.

Many different estimates of the dependence on nuclear binding energies on the one dimensionless parameter \( c^2 \) exist; despite the fact that the deuteron is the simplest nuclear system, a level of understanding sufficient to estimate \( B_d \) reliably from first principles is lacking. A systematic approach based on expanding in the light quark masses has been proposed [16], but even in this framework the lack of control of four-nucleon operators introduces uncertainties in the estimation of \( B_d \) (highlighted in [17], which appeared after the first version of this paper). Since we do not claim to be doing precision calculations of \( B_d \), and would be satisfied with an estimate of its dependence on \( c^2 \) which was inaccurate by a factor of a few (remember that we are placing cosmological bounds on \( \delta \alpha \)), this is not a major concern. However, Beane and Savage claimed in [17] that the dependence on \( c^2 \) could be mostly or entirely erased for some values of coefficients of these four-nucleon operators, drawing the conclusion that bounds on varying couplings from nuclear physics were considerably weakened. Without further analysis we cannot tell to what extent this particular choice of coefficients is fine-tuned, but in the absence of an underlying reason it appears very unlikely that the ultraviolet effects (from the point of view of the chiral expansion) giving rise to the quark mass-dependence of four-nucleon operators would conspire with the low-energy effects of pion exchange in such a way. Since all previous estimates gave a rather steep dependence of \( B_d \) on \( c^2 \), such a conspiracy would be a very unlucky (or lucky?) coincidence.

### 2.2 The di-neutron

In this section we attempt to calculate how large a change in the parameter \( c \) would allow two neutrons to form a stable bound state. The total spin of the di-neutron ground state would be zero and the neutrons share the same orientation in isospin space so we use the potential for the \( S = 0, I = 1 \) state [22]

\[
V(r)_{(S=0,I=1)} = -\frac{f^2}{4\pi} \frac{e^{-m_\pi r}}{r} \tag{9}
\]

where the dimensionless coupling \( f^2 \) is given by

\[
\frac{f^2}{4\pi} = \frac{g^2}{4\pi} \left( \frac{m_\pi}{2M_N} \right)^2 = 7.95 \times 10^{-2} \tag{10}
\]

where the numerical value is derived from present-day measurements. We assume a trial wavefunction of the form

\[
\psi(r) = e^{-1/m_\pi r} e^{-bm_\pi r} \tag{11}
\]
In this equation $\psi(r) \equiv r\Psi(r)$, where $\Psi$ is the radial part of the wavefunction, such that $\psi^2(r)dr$ is the probability of finding the nucleon separation to be between $r$ and $r + dr$. At large $r$ where the Yukawa potential is negligible, the Schrödinger equation for the relative motion of the two nucleons gives

$$\frac{1}{M_N} \frac{d^2\psi(r)}{dr^2} = -E_B\psi(r)$$

(12)

where $E_B$ is the (negative) binding energy of the state and the wavefunction (11) takes the form

$$\psi(r) = e^{-\sqrt{-E_B}M_Nr}, \quad b \equiv \frac{\sqrt{-E_B}M_N}{m_\pi}.$$

(13)

The trial wavefunction (11) is a suitable choice since at small $r$ it is independent of the binding energy whereas at high $r$ it is completely determined by the value $E_B$. By applying the Hamiltonian to the wavefunction (11) we find the energy of the system [32]

$$E = \frac{\int_0^\infty H(r)\psi(r)dr}{\int_0^\infty |\psi(r)|^2dr}$$

$$= -\frac{\sqrt{b}f^2m_\pi K_0(2\sqrt{2} + 4\sqrt{b})}{4\pi K_1(4\sqrt{b})} + \frac{b^{1/2}m_\pi^2 K_2(4\sqrt{b})}{2M_NK_1(4\sqrt{b})}$$

(14)
where the $K_n$ are modified Bessel functions of the second kind. We define a dimensionless parameter $v$ which is directly proportional to $c$ defined above

$$v = \frac{f^2}{4\pi} \frac{2M_N}{m_\pi} \propto c. \quad \text{(15)}$$

Thus, we can express the scaled ground state energy, in units of $bm_\pi^2/2M_NK_1(4\sqrt{b})$, in the simplified form

$$E_{\text{scaled}} = bK_0 \left(2\sqrt{2} + 4b\right) - vK_1(4). \quad \text{(16)}$$

The dependence of $E_{\text{scaled}}$ on $b$ and $v$ is illustrated in Fig. 1. In order for a bound state to exist, we require $v \geq 2.8$, or

$$\frac{v}{2} = \frac{f^2}{4\pi} \frac{M_N}{m_\pi} = \frac{g^2_\pi}{4\pi} \frac{m_\pi}{M_N} > 1.4 \quad \text{(17)}$$

We denote by $v_0$ the currently-observed value of $\frac{f^2}{4\pi} \frac{2M_N}{m_\pi}$, numerically equal to 1.08. Then the di-neutron stability criterion is

$$E_B(nn) < 0 \Rightarrow \frac{v}{v_0} = \frac{c}{c_0} \geq 2.6 \quad \text{(18)}$$

thus if $c \equiv \sqrt{\hat{m}_q/\Lambda}$ increases by a factor of 2.6 the di-neutron will become bound.

### 2.3 The di-proton

The fact that nuclear forces are independent of electromagnetic charge is illustrated by the result that the difference between the binding energies of $H^3$ and $He^3$ is well explained by the energy of Coulomb repulsion between the two protons in $He^3$ [31]. One would expect the ground state of the di-proton to have the same nuclear quantum numbers as the ground state of the di-neutron. We can therefore see from the previous section that reducing the Coulomb repulsion to zero will not be enough to bind the di-proton: only a large variation in the strong force (more precisely, in the ratio $\hat{m}_q/\Lambda$) would achieve this.

One would expect the size of a marginally bound di-neutron or di-proton to be of the same order as the effective range of the potential responsible for the binding. For a Yukawa potential like the one in Eq. (9) that we are using, the effective range is of order of the pion mass

$$r_{\text{eff}} \sim \frac{1}{m_\pi} \quad \text{(19)}$$

and the Coulomb repulsion energy at that range is approximately 1 MeV. Thus the parameters necessary to obtain a bound di-neutron with a binding energy of
1 MeV will be similar to those required to bind the di-proton at threshold. We assume here that the effect of varying $\alpha$ on the electromagnetic repulsion will be dwarfed by the corresponding change in $\hat{m}_q/\Lambda$. This assumption is shown to be justified consistent in the next section. Equation (12) and Fig. 1 show that this corresponds to the condition on the parameter $v$ of

$$E_B(pp) < 0 \Rightarrow \frac{v}{v_0} = \frac{c}{c_0} \geq 3.2$$

so if $c$ increases by a factor of 3.2 the di-proton will become bound.

There also exists a hypothetical proton-neutron pure $L = 0$ state which one would expect to be stabilised for similar values of $c$ as the di-neutron.

### 2.4 The deuteron

The deuteron has total isospin $I$ is zero, so the spins of the two nucleons must be parallel by the Pauli exclusion principle. The $L = 0$ wavefunction is unbound in the same way as the di-neutron and the di-proton, however the parallel nucleon spins mean that the $L = 2$ wavefunction is bound by the tensor force and the ground state forms an admixture of these two wavefunctions. The exact form of the wavefunctions is given by the solution of a coupled pair of 2nd. order differential equations [22]. To solve these equations explicitly for different values of the parameters $f$, $m_\pi$ and $M_N$ is highly nontrivial, so we will resort to a simple square well model of the kind used in [37].

The analysis is then simply the bound state condition for a finite square well potential of depth $V_{sq}$ and width $a$, found (assuming $|E_B| \ll |V_{sq}|$, which is the case in the situations considered) by matching a sinusoidal solution for the wavefunction inside the well to an exponentially decaying mode outside. We assume $a = m_\pi^{-1}$ then the binding energy $E_B$ is then given by the solution of

$$\cot \left[ \frac{m_\pi^{-1}}{\sqrt{M_N(E_B - V_{sq})}} \right] = -\sqrt{\frac{E_B}{V_{sq} - E_B}}.$$  

(21)

Then, using the observed values of the pion mass and the binding energy $E_B = -2.226$ MeV we find that the depth of the well is $V_{sq} = -66.15$ MeV. Since the prefactor in the internuclear pion potential $f^2/4\pi$ is dimensionless we need to see how the quantities in this simple model scale with the parameter $c$ that we have been using to parameterise the variation in the underlying gauge couplings. The corresponding dimensionful ‘depth’ $V_y$ of a Yukawa potential is related to equation (9) by

$$V(r) = V_y \frac{e^{-m_\pi r}}{m_\pi r} = \frac{f^2}{4\pi} \frac{e^{-m_\pi r}}{r}$$

$$\frac{V_y}{m_\pi} = \frac{f^2}{4\pi}$$

(22)
so we can expect $V_{sq}$, the depth of our square well, to be directly proportional to $f^2m_\pi$. Making the substitution $\beta = E_B/V_{sq}$ equation (21) becomes

$$\cot \left[ \sqrt{\frac{M_N V_{sq}}{m_\pi^2}} (\beta - 1) \right] = -\sqrt{\frac{\beta}{1 - \beta}}$$  \hspace{1cm} (23)$$

so the relevant combination of parameters is $M_N V_{sq}/m_\pi^2$

$$\frac{M_N V_{sq}}{m_\pi^2} \propto f^2 \frac{M_N}{m_\pi} \propto c. \hspace{1cm} (24)$$

Using Eq. (23) we can see that the value of $c$ where the deuteron becomes unstable is

$$\frac{c_{\text{unstable}}}{c_o} = \frac{\pi^2}{4} \frac{m_\pi^2}{M_N |V_{sq}|} = 0.77 \hspace{1cm} (25)$$

so a 13% reduction in $c$ will give rise to an unbound deuteron.

3 Running of gauge couplings and unification

3.1 GUT-like models

The LEP precision measurements of gauge couplings suggest that the three standard model gauge groups become unified at some high energy $M_U \approx 2 \times 10^{16}$ GeV \cite{2}. Thus any variation in the unified gauge coupling $\alpha_U(M_U)$ leads to calculable changes in the low energy values of the gauge couplings. The dependence of low energy couplings on $\alpha_U$ is very similar in the case of large extra dimensions \cite{6} or non-factorisable geometries \cite{5} where the unification scale is much lower. This is because extra contributions to the renormalisation group equations from the massive Kaluza-Klein modes of the theories change the running in such a way that the effective energy scale and coupling strength of unification stays the same from the point of view of the low energy 4D theory (see, however, \cite{51}). In the following analysis we start with the simplest scenario of a “grand desert” between the weak and GUT scales with $N = 1$ supersymmetry. However, no matter what the matter content or other details, results of a similar order to those presented here seem unavoidable for any theory with gauge unification.

The running of the gauge couplings $\alpha_i, i = 1, 2, 3$, with the renormalisation scale $\mu$ is given at one loop by the expression

$$\frac{d\alpha_i}{dt} = -\frac{b_i \alpha_i^2}{2\pi}, \quad t \equiv \ln(\mu/\mu_0) \hspace{1cm} (26)$$

where $\mu_0$ is an arbitrary, constant reference scale and the $b_i$ depend upon the gauge groups and matter representations transforming under each group. In
SU(5) and SO(10) SUSY GUT’s and a large class of string models the gauge couplings are unified at some high energy scale $M_U$:

$$\alpha_3(M_U) = \alpha_2(M_U) = (5/3)\alpha_1(M_U) = \alpha_U(M_U)$$  \hspace{1cm} (27)

where the normalisation factor $5/3$ derives from the fact that the U(1) hypercharge gauge group must be embedded consistently in the unified group or string model. Other normalisations are possible, but are often inconsistent with unification [6]. At energies above the superpartner masses, the $b_3$ and $b_2$ of Eq. (26) are given by [7]

$$b_3 = 9 - 2n_G, \quad b_2 = 6 - 2n_G - \frac{n_H}{2}$$  \hspace{1cm} (28)

where $n_H$ is the number of Higgs doublets and $n_G$ is the number of fermion generations. Integration of Eq. (26) above the scale of supersymmetry-breaking yields

$$\frac{M_U}{M} = \exp \left[ -\frac{4\pi}{6 + n_H} \left( \frac{1}{\alpha_2(M)} - \frac{1}{\alpha_3(M)} \right) \right].$$  \hspace{1cm} (29)

Using the standard relation $\alpha_1 \cos^2 \theta_W = \alpha_2 \sin^2 \theta_W = \alpha$ and substituting the observed values of $\sin^2 \theta_W$, $\alpha_3(M_Z)$ and $\alpha(M_Z)$ in this equation, we find a unification energy $M_U \approx 10^{16}$ GeV and a unified coupling of $\alpha_U^{-1} \approx 25$.

In order to find the effect on the strong interactions of changing $\alpha_U$ we will use the $b_3$ of Eq. (28) to run $\alpha_3$ from $M_U$ down to low energy for different values of $\alpha_U$. Then we assume that all the superpartners of the standard model particles have the same mass $\tilde{m}$, which defines the scale below which supersymmetry is broken. At energies less than $\tilde{m}$ the running of $\alpha_3$ proceeds as in normal QCD, i.e.

$$\frac{1}{\alpha_3(M_{\text{high}})} - \frac{1}{\alpha_3(M_{\text{low}})} = -\frac{33 - 2n_f}{6\pi} \ln \left( \frac{M_{\text{low}}}{M_{\text{high}}} \right)$$  \hspace{1cm} (30)

where $n_f$ is the number of quark flavours of mass $m_q < m_{\text{high}}$. We use the observed values of the top, bottom and charm quark masses, defined at the self-consistent renormalisation scale $m_q(m_q)$, in this part of the running.

If one neglects quark masses, the Lagrangian of QCD contains no dimensionful parameters: however, on quantisation, loop corrections give rise to the renormalisation group invariant strong interaction scale $\Lambda$:

$$\Lambda = M \exp \left( \frac{-6\pi}{(33 - 2n_f)\alpha_3(M)} \right)$$  \hspace{1cm} (31)

where $M < \tilde{m}$. This scale sets the characteristic energy of the particles of the low energy effective theory. Even in the presence of massive quarks, this quantity
remains important in understanding low energy phenomena, as we saw in the calculation of internuclear forces.

For constant $\alpha_U$, the running of $\alpha_3$ below the superpartner thresholds and the value of $\Lambda/M_U$ depend strongly on the common superpartner mass $\tilde{m}$. But if $\tilde{m}/M_U$ is held fixed and $\alpha_U$ varied, the fractional change in $\Lambda/M_U$ for a given fractional change in $\alpha_U$ has little dependence on the value of $\tilde{m}$. For any constant value of $\tilde{m}/M_U$ corresponding to superpartner masses between 100 GeV and 2 TeV, we find the same ratio of fractional changes $(M_U/\Lambda)\Delta(\Lambda/M_U)/(\Delta\alpha_U/\alpha_U)$, to well within the accuracy of our results. Later, we give the explicit formula for $\Lambda$ in the case $\tilde{m}/M_U$ is also changing.

In order to obtain $\alpha_3$ at low energies and $\Lambda/M_U$ as a function of $\alpha_U(M_U)$ we include the masses of the top, bottom and charm quarks [8] and change the running accordingly, with results illustrated in Fig. 2. To first order in the fractional change $\Delta\alpha_U/\alpha_U$, we should be able verify the results for $\Delta(\ln \Lambda)$ in [12, 13]. Later, we will consider the effect of varying the threshold masses, but for the moment we consider them as fixed. Using the result quoted later as Eq. [33] we find

$$\Delta(\ln \Lambda/M_U) = \frac{6\pi}{27} \alpha_U^{-1} \Delta(\ln \alpha_U) \approx 17\Delta(\ln \alpha_U)$$

(32)

consistent with the result of both papers. (Note that any determination of the absolute value of $\Lambda/M_U$ will depend significantly on the renormalisation prescription, so without a detailed treatment, including also higher loop effects, our estimate of the prefactor is likely rather imprecise.)
3.2 Non-GUT (brane world-type) models

One can perform a similar analysis in the case of theories where the gauge couplings at the fundamental scale do not satisfy a GUT relation, for example intersecting brane models (see e.g. [42, 43]) where different gauge groups propagate along different world-volumes ([44]). In this case we just find the relation between the SU(3) coupling just below the fundamental scale $M_*$ and $\Lambda$, by the same method. For a high-scale model with softly-broken SUSY the calculation is identical, replacing $\alpha_U$ by $\alpha_3(\mu \approx M_*)$ (where $M_*$ is the scale below which $d = 4$ effective theory applies for gauge interactions). In the case of a low fundamental scale of order 1-10 TeV and no supersymmetry we simply run $\alpha_3$ from $M_* \sim$ TeV down through the heavy quark thresholds, with results as illustrated in Fig. 3. The main lesson from this exercise is the sensitivity of $\Lambda/M_*$ to the strong coupling $\alpha_3$ at high energies $M \gg M_N$: the slope of both graphs around $\Delta = 0$ is 10 to 20. In our conclusions, for simplicity and to avoid making highly model-dependent statements, we will pretend that the analysis finishes at this point, and express the results in terms of a fractional change in $\alpha_U$ under the (in general unrealistic) assumption of constant quark and superpartner masses.

4 Thresholds and RG running

In both cases (GUT-like and non-GUT) the running of $\alpha_3$ is complicated by the dependence on heavy quark and coloured superpartner (gaugino and squark)
thresholds. In the case with superpartners the one-loop result with tree-level matching at thresholds is

\[ \Lambda = M e^{-6\pi/27\alpha_3(M)} \left( \frac{m_c m_b m_t}{M^3} \right)^2 \]  

where \( m_\lambda \) is the gaugino mass, \( \overline{m_q} \) is the geometric average squark mass and \( M \) is an arbitrary scale above the superpartner masses; for the SM we find simply

\[ \Lambda = M' e^{-6\pi/27\alpha_3(M')} \left( \frac{m_c m_b m_t}{M'^3} \right)^2 \]  

where \( M' > m_t \). The quark and superpartner masses are the “decoupling masses” \( m_q(\mu = m_q) \) etc. in a convenient renormalisation scheme.

The possibility of varying threshold masses complicates the simple one-loop formula

\[
\frac{1}{\alpha(\mu)} \frac{\Delta \alpha(\mu)}{\alpha(\mu)} = \frac{1}{\alpha(\mu')} \frac{\Delta \alpha(\mu')}{\alpha(\mu')}
\]

derived in [12] for a fixed ratio \( \mu/\mu' \) if there is, say, a quark with \( \mu < m_{q_i} < \mu' \); we find

\[
\frac{1}{\alpha(\mu)} \frac{\Delta \alpha(\mu)}{\alpha(\mu)} = \frac{1}{\alpha(\mu')} \frac{\Delta \alpha(\mu')}{\alpha(\mu')} + \frac{b^{>m_q} - b^{<m_q}}{2\pi} \Delta \ln \frac{m_{q_i}(m_{q_i})}{\mu\mu'}
\]

where \( b^{>m_q} - b^{<m_q} \) simply denotes the contribution of \( q_i \) to the beta-function coefficient above its threshold.

One expects higher loop effects to change these relations, generically by the introduction of a power-law dependence of \( \Lambda \) on \( \alpha_3(M)^{(l)} \) (etc.!)  

4.1 RG scale ambiguities

What is meant, in the context of a time-varying theory, by quark masses “at \( M_Z \)”?

Since in all likelihood the Higgs v.e.v. and the SU(2) coupling will also be time-dependent in a unified theory, such an expression becomes ambiguous through the change in \( M_Z \). It is commonly noted that dimensionful quantities cannot meaningfully be said to be time-dependent, since one has no way to guarantee that units of measurement do not also change; the presence of RG scale dependence threatens to introduce dimensionful quantities through the back door unless we take care in our notation.

One convenient approach is to define a reference scale which can always be set to a constant without loss of generality; in discussing unified theories the obvious choice is \( M_U \) (or we may choose the Planck scale, or the string scale, or the “quantum gravity” scale \( M_* \) according to the variety of theory we are considering). Then we would write in all explicitness

\[ m_q(\mu/M_U = M_Z/M_U = f(\alpha_U)) \]
remembering that $M_Z$ will depend in the unified coupling which itself is time-dependent; alternatively one could set a fixed ratio $\mu/M_U$ equal to the current value $(M_Z/M_U)_0 \approx 4 \times 10^{-15}$ to avoid the complication of calculating $M_Z/M_U$ as a function of $\alpha_U$. Once we realise that the RG scale can thus be converted to a dimensionless parameter, the ambiguity disappears.

Now we tackle the case of time-varying decoupling masses for heavy quarks in more detail; let us take the bottom quark for definiteness. The running mass is given at one loop by

$$m_b(\mu) = \hat{m}_b(\alpha_3(\mu))^{\gamma_1/b_1}$$

where the bottom mass runs as

$$\frac{d}{dt} m_b = -m_b \left( \frac{\gamma_1}{2\pi} \alpha_3 + \cdots \right)$$

and $\gamma_1 = 4$ in QCD (note that $b_1 = 11 - 2n_f/3$ in our notation). The decoupling mass $m^d_b$ satisfies

$$\alpha_3^{-1}(M) + \frac{b_1}{2\pi} \ln \frac{m^d_b}{M} = \left( \frac{m^d_b}{M} \right)^{-b_1/\gamma_1}$$

where $M$ is a constant reference mass above $m_b$ but below $m_t$ and we have substituted the one-loop solution for $\alpha_3$ on the LHS. Differentiating and rearranging to isolate $\Delta \ln(m^d_b/M)$, we find

$$\left( 1 + \frac{\gamma_1}{2\pi} \alpha_3(m^d_b) \right) \Delta \ln(m^d_b/M) = \Delta \ln(\hat{m}_b/M) + \frac{\gamma_1 \alpha_3(m^d_b)}{b_1 \alpha_3(M)} \Delta \ln \alpha_3(M). \quad (35)$$

As remarked before, the variation in $\hat{m}_b/M$ is given just by the change in the Higgs v.e.v. and in the Yukawa coupling $y_b$ at high scale (since we are at the moment only considering the QCD contribution to the running of quark masses, which will be the dominant effect at low energies).

Note that the second term in brackets on the LHS of Eq. (35) is formally of higher order than the first, therefore we may consistently discard it for the heavy quarks for which $\alpha_3$ is perturbative. The second term on the RHS is also a factor of $\alpha_3$ down compared to the analogous expression for the variation of the QCD scale $\Lambda$ (see Eq. 38), but we cannot tell without knowing the dependence of $\hat{m}_b$ on $\alpha_U$ which term on the RHS may dominate. If the second term on the RHS does dominate, then the changing strength of QCD is the main cause of $m^d_b/M$ varying, in which case quark threshold effects will be a rather small correction to the change in $\Lambda$ and can be neglected. If the first term dominates, the main effect is through the variation of quark masses at high energy, in which case we can take $\Delta \ln(m^d_b/M) \simeq \Delta \ln(\hat{m}_b/M)$; this is the relevant formula if one expects quark mass thresholds to play a significant role.

We also note, on the subject of fermion masses, that since the fine structure constant is defined at the scale $m_e$, a variation in $m_e/M_U$ can induce a change in
\(\alpha\) even if \(\alpha_U\) is unchanged; this was pointed out by Wetterich \[54\] in a preprint appearing after the first version of this paper. However due to the very slow running of \(\alpha(\mu)\) one would require a rather large change in \(m_e/M_U\), or in any other charged particle threshold, to reproduce the observed variation in alpha. This restriction, taken together with the bound on variation of \(M_p/m_e\) \[55\], imply that a sizable contribution to varying alpha from this kind of threshold effect is unlikely \[54\].

### 4.2 Dependence of \(c\) on high-energy parameters

Now we attempt to express \(c^2 \equiv \left(\hat{m}_q/\Lambda\right)\) as a function of some more fundamental parameters. Substituting for \(\hat{m}_q \equiv \hat{y}_q \sqrt{2}\), where \(\hat{y}\) is the average light quark Yukawa coupling, modified by the appropriate function of \(\alpha_3(\mu)\), and using the expressions derived above for \(\Lambda\) we find

\[
c^2 = \text{const.} \times \frac{(\hat{y}_u + \hat{y}_d)v(M')^{6/27}e^{6\pi/27\alpha_3(M')}}{M'(y_c(m_c)y_b(m_b)y_t(m_t)v^3)^{2/27}} \tag{36}
\]

in the SM and

\[
c^2 = \text{const.} \times \frac{(\hat{y}_u s_\beta + \hat{y}_d c_\beta)vM^{2/3}e^{6\pi/27\alpha_3(M)}}{M(y_c(y_c)y_b(y_b)y_t(y_t)s_\beta^2 c_\beta v^3 m_\lambda^4 \alpha_3^2)^{2/27}} \tag{37}
\]

in the MSSM, where \(s_\beta(c_\beta) = \sin(\cos)\beta\). From now on we assume that the change in \(\tan \beta\) is small and lump the light quark Yukawas together as \(\hat{y}_q\). Since the running of the up, down and charm Yukawas as they appear in this expression (\(\alpha_3\)-corrected for the light quarks) over the range up to \(M_U \sim 10^{16}\) GeV is small, we set them equal to their value at unification \(y_q(M)\). In fact, to first approximation we neglect the nonlinear running of the \(b\) and \(t\) Yukawas also: in a careful calculation one would use the semi-analytic solution for the top Yukawa and include the effects of the top feeding into the other masses, but the effects that we are neglecting by effectively taking \(y_t(\mu) = y_t(M)\) for \(\mu > m_t\) are likely to be sub-leading compared to, say, a large fractional change in the Higgs v.e.v. or in SUSY-breaking masses.

Thus we reach the approximate expressions (which still likely contain the leading dependence on \(\alpha_U\))

\[
\Delta(\ln c^2) = -\frac{2\pi}{9\alpha_3(M_U)}\Delta(\ln \alpha_3(M_U)) + \frac{7}{9}\Delta(\ln v/M_U) + \Delta(\ln y_qU)
- \frac{2}{27}\Delta(\ln y_cU + \ln y_bU + \ln y_tU) - \frac{2}{9}\Delta(\ln m_\lambda/M_U + \ln \overline{m}_q/M_U) \tag{38}
\]

for MSSM unification and

\[
\Delta(\ln c^2) = -\frac{2\pi}{9\alpha_3(M_*)}\Delta(\ln \alpha_3(M_*)) + \frac{7}{9}\Delta(\ln v/M_*) + \Delta(\ln y_q*)
- \frac{2}{27}\Delta(\ln y_c* + \ln y_b* + \ln y_t*) \tag{39}
\]
for the SM running in low-scale models.

5 Finding bounds on fundamental models

If we assume that the supersymmetric model we are referring to is associated with the low energy effective theory of some heterotic superstring or M-theory compactification the parameters $\alpha_U$ and $M_U$ are related to the size of the six dimensional manifold upon which the theory is compactified.\(^3\) In this situation $\alpha_U \propto V^{-1} \propto (M_U)^6$ where $V$ is the volume of the compactification manifold [9]. Because of this we might expect that as one changes the size of the extra dimensions, both the GUT unification energy scale and coupling might vary together. However the effect of the changing energy scale has a negligible effect on the results because of both the 6th power in the relation between the energy scale and the coupling, and the slow running of the gauge couplings at high energies.

The supersymmetry-breaking masses entering into the expression (37) depend on the mechanism of SUSY-breaking, which can either take place through non-perturbative effects in a hidden sector (as is appropriate in models with high fundamental scale) or perturbatively by a Scherk-Schwarz mechanism in models with large (TeV\(^{-1}\)-size) extra dimensions [49, 52, 53].\(^4\) Taking the scenario of gaugino condensation in a hidden sector and string moduli mediation as a benchmark, the functional dependence on the unified coupling is

$$\tilde{m} \propto m_{3/2} \propto \Lambda_H^3 M_P^{-2}$$

(40)

where $M_P$ is the reduced Planck mass $1/\sqrt{8\pi G_N}$ and the hidden sector confinement scale is $\Lambda_H^3 \sim M_U^3 e^{-6\pi/b_H \alpha_H(M_U)}$. Thus the fractional change expected is related as

$$\Delta(\ln \tilde{m}/M_U) \simeq 2\Delta \log(M_U/M_P) - \frac{6\pi}{b_H \alpha_U} \Delta(\ln \alpha_U)$$

(41)

where the hidden sector gauge coupling is taken also to unify at $M_U$. Using the relation $M_U \sim M_s \propto g_U M_P$ from heterotic string theory, and imposing the correct magnitude of $e^{-6\pi/b_H \alpha_H(M_U)} \sim \tilde{m}/M_P \simeq 10^{-15}$ we find

$$\Delta(\ln \tilde{m}/M_U) \simeq (2 + 34.5)\Delta(\ln \alpha_U)$$

(42)

where the factor $34.5 \simeq 15 \ln 10$ is universal to models of SUSY-breaking by strong coupling effects in a hidden sector mediated by non-renormalisable operators (as also found in [13]).

---

\(^3\)Although the low energy limit of M-theory is 11 dimensional, the gauge degrees of freedom, at least in the strong coupling limit of the heterotic theory, are confined to a 10 dimensional brane. The appropriate volume to consider is then that of the 6d Calabi-Yau manifold, as in heterotic string theory.

\(^4\)If the fundamental scale is intermediate ($10^{11}$–$10^{13}$ GeV) then SUSY-breaking may occur directly at tree level in a non-supersymmetric hidden sector and be gravitationally-mediated, a possibility recently realized in certain D-brane models [46].
The quark Yukawa couplings gain a universal factor of \((S + \bar{S})^{-1/2} \sim g_U\) from the normalisation of the superpotential in going from SUGRA to softly-broken global SUSY, thus in particular \(y_{UU}\) will vary proportional to \(\alpha^{1/2}_U\) which will be of importance for electroweak breaking. The light quark Yukawas are extremely model-dependent, but estimates can be made in some classes of unified models. The most common mechanism employed to generate small Yukawa couplings is the Froggatt-Nielsen picture in which effective couplings arise from non-renormalisable operators when scalars charged under a U(1) group get v.e.v.’s \(\langle X \rangle\). In fact this picture finds a natural embedding in the heterotic string where the group is now “pseudo-anomalous” and is broken near the string scale due to a nonzero Fayet-Iliopoulos term at one loop destabilising the symmetric \(\langle X = 0 \rangle\) vacuum. The fact that this is a one-loop effect tells us the dependence of the v.e.v. on the unified coupling as \(\langle X \rangle/M_P \propto g_U\) (see e.g. [50]), thus the small Yukawas are generated schematically as

\[
y_{iU} \sim \langle X \rangle^{Q_i} \propto g^{Q_i}_U
\]  

(43)

where the quark \(q_i\) has a U(1) charge \(Q_i\) of opposite sign to \(Q_X\). Thus with knowledge of the charges, which also follow phenomenologically if we know the constant of proportionality of \(\langle X \rangle/M_P = k g_U\) and demand the correct values of \(y_{UU}\), one easily finds the variation in \(y_{UU}\) as a function of \(\Delta(\ln \alpha_U)\).

Finally, as is well known the standard scenario of radiative electroweak symmetry-breaking is sensitive to the soft masses \(\tilde{m}\) and the top Yukawa \(y_{UU}\) (for a review, see [47]). While the full RG equations for the top Yukawa and the relevant soft masses are complicated, the leading dependence on the input quantities was already considered in [48] in which the following estimate was made for the sensitivity of the \(W\) mass to \(y_t\)

\[
M_W(\mu < Q_0) \simeq \frac{g_2(\mu) v^2}{4} \propto y_t m_t \ln(\mu^2/Q_0^2)

\rightarrow \Delta(\ln M_W(\mu)/M_U) \approx \Delta(\ln m_t/M_U) + \Delta(\ln y_t) \frac{\ln M_U/\mu}{\ln Q_0/\mu}
\]  

(44)

where \(Q_0\) is the scale at which the up-type Higgs mass-squared crosses to negative values, given by \(M_U e^{-k/\mu^2}\); where \(k\) is a ratio of SUSY-breaking masses at the scale \(M_U\) and we choose a fixed renormalisation scale \(\mu \simeq M_W|_0\) since (analogously to the case of heavy quark thresholds) the running of \(g_2(\mu)\) is slow. Then we impose \(M_U/\mu \simeq 10^{14}\): one finds from a more careful analysis [49] that \(Q_0/\mu\) tends to be of order 10, in which case the enhancement factor \(\ln(M_U/\mu)/\ln(Q_0/\mu)\) is of order 15 (although this also depends strongly on the pattern of Higgs mass parameters at unification).

Hence in addition to the strong SUSY-breaking mass effect previously discussed, there is a milder dependence of \(M_W\) or \(v\) through \(y_t \propto \alpha_U^{1/2}\) and the overall effect on the Higgs v.e.v. may well be the largest contribution to any
low-energy physics that depends on quark masses, even relative to the exponential dependence of $\Lambda$ on the high-energy coupling strength in QCD. The reason for this is subtle: while the beta-function coefficients giving the running of the hidden sector gauge coupling are assumed constant, thus the hidden sector scale $\Lambda_3^H$ “feels” the whole of the hierarchy between $M_U$ and $M_W$, the presence of thresholds at low energy ($\ll M_U$) in QCD means that $\Lambda_{\text{QCD}}$ feels mainly the running at low energies ($\ll \tilde{m}$) thus the sensitivity to varying the coupling at energies above the thresholds is less. If there were also thresholds for charged states in the hidden sector at masses $\Lambda_H \lesssim M_H \ll M_U$, the leading dependence of SUSY-breaking on the unified coupling would be smaller, however such a case is atypical [56] for the vector-like matter representations usually considered.

However, since SUSY-breaking and electroweak breaking are highly model-dependent, we will not go through the resulting bounds in detail. Other types of model are likely to give completely different results, for example Scherk-Schwarz SUSY-breaking, where the electroweak and SUSY-breaking mass scales vary inversely with the radius of an extra dimension, as do the 4-d gauge couplings (more precisely, $\alpha_i \propto R^{-1}$) and so the fractional variation in $v$ would likely be of the same order as that in $\alpha_U$ (rather than maybe 50 times larger).

6 Discussion and Constraints

In drawing our conclusions about the bounds on the variation of $\alpha_U$, we will simply for the sake of an easy comparison set to zero all variation in quark masses and other thresholds, from no matter what source, leaving the effect of $\alpha_3$ on $\Lambda$ as the only varying quantity. This is not intended as a realistic treatment of unified theories but only as an illustration of one effect, which can be systematically extended to whatever unified theory by going back to the bounds on $c \equiv (\tilde{m}_q \Lambda)^{1/2}$ and substituting for the expressions we derived in Sections 4 and 5. With this caveat we proceed to discuss the bounds from the di-nucleon systems.

6.1 The deuteron

A 69% increase in $c^{-2} = \Lambda/(m_u + m_d)$ will destabilise the deuteron. On the assumption of fixed quark masses, Fig. 2 tells us that this corresponds to a 3% increase in the gauge coupling at the scale of unification for a conventional SUSY GUT. This creates problems for cosmological models where the gauge couplings vary significantly over cosmological time scales [1], since no such variation can have occurred at any time since nucleosynthesis.

For a model with a TeV scale GUT, $\alpha_3$ cannot have increased more than 8% at any time since nucleosynthesis, since this would also lead to the deuteron becoming unbound.

Recent observations suggest a negative variation in the electromagnetic fine
structure constant at high redshift \[10\], which would, if one assumes high scale
gauge unification and consequently a corresponding negative change in \(\alpha_3\),
increase the binding energy of the deuteron. Unless the field responsible for the
value of the gauge coupling is frozen by the Hubble expansion \[15\] one might
expect this variation to be much larger in the early universe (see e.g. \[38\]). This
might result in the deuteron having been more resilient to photo-dissociation by
the decay products of massive relic particles \[39\]. This possibility would be ben-
eficial to models where the decay of gravitinos can cause problems for the light
element abundances of nucleosynthesis.

On the other hand, such an increase in the binding energy would allow helium
to form at higher temperatures, leading to an increase in the primordial helium
abundance. This effect, neglected in \[19\] and \[13\], would be superimposed on
other effects due to changing couplings, \textit{i.e.} a change in freezeout temperature
and neutron lifetime. One can make a rough estimate of the sensitivity of the
helium abundance to the deuteron binding energy in the following way. The
ratio of number densities of the neutron to the proton \(n/p\) at the time of helium
formation determines the primordial abundance of helium, since 99.99% of the
neutrons go on to form helium. This is related to the initial ratio of neutron to
proton number density at the time of weak interaction freezeout \(n_0/p_0\), via the
decay of neutrons into protons, by

\[
\frac{n}{p} = \frac{e^{-t_{He}/\tau}}{p_0/n_0 + 1 - e^{-t_{He}/\tau}}
\]

where \(t_{He}\) is the time at which helium production takes place and \(\tau\) is the neutron
lifetime (currently measured to be 887 s). In this formula, weak interactions are
taken to freeze out at \(t = 0\). We have also assumed that all helium production
occurs instantaneously at \(T_{He}\). Although still a subject of some controversy, the
orthodox view of light element abundances determines the initial mass fraction
to an accuracy of \(\pm 5\%\) is the ratio \(n/p\) \[41\]. Translating this into a constraint
upon the variation of \(T_{He}\) we obtain

\[
-0.20 \leq \frac{\Delta T_{He}}{T_{He}} \leq +0.13
\]

It is reasonable to suggest that \(T_{He}\) is proportional to the binding energy of the
deuteron, since only below a certain temperature set by that binding energy can
helium formation proceed unimpeded by the photo-dissociation of the deuteron.
This is because the reactions which form helium all rely upon deuterium as an
intermediate building block :-

\[
\begin{align*}
\text{H}^2 & (\text{H}^2, n) \text{He}^3 (\text{H}^2, p) \text{He}^4 \\
\text{H}^2 & (\text{H}^2, p) \text{H}^3 (\text{H}^2, n) \text{He}^4 \\
\text{H}^2 & (\text{H}^2, \gamma) \text{He}^4 
\end{align*}
\]

(47)
Using Eq. (23) we can find out how the parameter $\beta$ changes for different values of the parameter $c$. In order to translate this into the binding energy $E_B = \beta V_{sq}$ it is necessary to take into account the scaling of the depth of the square well $V_{sq} \propto m_n/c^2 \propto m_n^2 \Lambda^{-\frac{1}{2}}$ so the result is not in terms of the dimensionless parameter $c$. Therefore we take the case where the variation in the quark Yukawa couplings is not significant and simply consider the variation in $\Lambda$. The result is

$$-0.04 \leq \frac{\Delta \Lambda}{\Lambda} \leq +0.04$$

(48)

which relates into constraints on the gauge couplings at 1 TeV in non-SUSY models or $10^{16}$ GeV for GUT models of

$$-0.005 \leq \frac{\Delta \alpha_3}{\alpha_3} \bigg|_{1 \text{ TeV, no SUSY}} \leq 0.005$$

$$-0.0023 \leq \frac{\Delta \alpha_3}{\alpha_3} \bigg|_{10^{16} \text{ GeV, SUSY}} \leq 0.0023$$

(49)

which is obviously very restrictive. In order to investigate this further it would be necessary to obtain a more detailed model for the deuteron binding energy and the various binding energies, cross sections and decay rates in nucleosynthesis. This effect then has to be added to the effect of varying neutron-proton mass difference and the effect of gauge coupling variation on weak coupling [19]. However the point remains that in general a variation in the gauge coupling at nucleosynthesis would have a very large effect upon nuclear binding energies, and consequently on the primordial abundance of light elements. This would create problems for models such as [45] where a change in $\alpha$ of order 1% at nucleosynthesis is considered.

If one decreases $\Lambda$, the binding energy of the deuteron increases, therefore helium is produced earlier when there are more neutrons and the helium abundance goes up. At the same time, a decrease in $\Lambda$ decreases the neutron-proton mass difference, which increases the number of neutrons and therefore the helium abundance, so one would expect that two effects should work together, implying a yet more restrictive bound.

### 6.2 Implications of a stable di-neutron or di-proton

A negative change in the gauge coupling at high redshift would decrease the Coulomb repulsion and increase the nuclear forces in the di-proton system. The di-proton and di-neutron systems only become stable if $\Lambda$ decreases to around one tenth of its present day value. This corresponds to a 10% decrease in the unified gauge coupling for a high scale model and a 25% decrease for a TeV GUT. The stability of the di-proton would have a catastrophic effect upon the lifetime of stars as it would provide a rapid channel for hydrogen fusion [40]. However
the constraints on the gauge coupling variation in the matter dominated epoch are several orders of magnitude too small for this to occur. The presence of the di-proton would also be disastrous for nucleosynthesis and might eliminate all the hydrogen in the universe. However, the large negative change in the gauge couplings required indicates that this is not a particularly strong constraint.

It is not immediately obvious how dangerous the stability of the di-neutron would be to nucleosynthesis, since the neutrons would probably still end up in Helium atoms. Still, we can safely say that the variation in the gauge coupling required for the di-proton and di-neutron to become stable is much larger than the typical orders of magnitudes being considered at the moment.

7 Conclusions

We have shown in this paper that a 3% increase in the QCD coupling constant $\alpha_3$ at the GUT scale would result in the deuteron becoming unbound. The deuteron binding depends only on nuclear forces, so this conclusion cannot be escaped by considering the variation of more than one gauge coupling at once. Only negative variations at the level of 10% could bind the di-proton and the di-neutron.

We have developed formalisms which enable one to calculate the variation in low energy parameters as a function of the variation of gauge and Yukawa couplings in the underlying theory: in many models one expects the variation in the electroweak and SUSY-breaking sectors to be the dominant effect at low energies, but the model-dependent nature of such effects means that no firm conclusions can yet be reached.

We have also considered the effect of variations in the binding energy of the deuteron on the time at which helium formation occurs, and consequently on the helium abundance. This effect is complementary to the other effects on nucleosynthesis due to variation of gauge couplings, but on its own it constrains variation in $\alpha_3$ at nucleosynthesis to within 0.25%.

Acknowledgements

We would like to thank David Bailin, John Barrow, Ed Copeland, Marty Einhorn, Gordy Kane, Bernard Pagel and Lian-Tao Wang for valuable conversations.

References

[1] Polchinski, J., String Theory, Cambridge University Press, (1998).
[2] Amaldi, U., de Boer, W. and Furstenau, H., Phys. Lett. B 260, 447 (1991).
[3] Randall, L. and Sundrum, R., Phys. Rev. Lett. 83, 4690 (1999).
[4] Brax, P. and Davis, A.C., JHEP 0105, 7 (2001).

[5] Pomarol, A., Phys. Rev. Lett. 85, 4004 (2000).

[6] Dienes, K. R., Dudas, E. and Gherghetta, T., Nucl. Phys. B 537, 47 (1998).

[7] Bailin, D. and Love, A., Supersymmetric Gauge Field Theory and String Theory, Institute of Physics Publishing (1994).

[8] Groom, D. E. et al., Europhys. J. C 15, 1 (2000).

[9] Witten, E., Nucl. Phys. B 471, 135 (1996).

[10] Webb, J. K. et al., Phys. Rev. Lett. 87, 091301 (2001), astro-ph/0012539.

[11] T. Banks, M. Dine and M. R. Douglas, “Time-varying alpha and particle physics,” hep-ph/0112059.

[12] X. Calmet and H. Fritzsch, “The Cosmological Evolution of the Nucleon Mass and the Electroweak Coupling Constants,” hep-ph/0112110.

[13] P. Langacker, G. Segre and M. J. Strassler, “Implications of Gauge Unification for Time Variation of the Fine Structure Constant,” hep-ph/0112233.

[14] C. Csaki, J. Erlich and C. Grojean, Nucl. Phys. B 604, 312 (2001) arXiv:hep-th/0012143.

[15] G. R. Dvali and M. Zaldarriaga, Phys. Rev. Lett. 88, 091303 (2002) arXiv:hep-ph/0108217.

[16] S. R. Beane, P. F. Bedaque, M. J. Savage and U. van Kolck, Nucl. Phys. A 700, 377 (2002) arXiv:nucl-th/0104030.

[17] S. R. Beane and M. J. Savage, arXiv:hep-ph/0206113.

[18] K. R. Dienes, Nucl. Phys. B 611 (2001) 146 hep-ph/0104274.

[19] Campbell, B. A. and Olive, K. A., Phys. Lett. B 345, 429 (1995).

[20] Dixit, V. V. and Sher, M., Phys. Rev. D 37, 1097 (1998).

[21] Davies, P. C. W., J. Phys. A 5, 1296 (1972).

[22] Ericson, T. and Weise, W., Pions and Nuclei, Oxford University Press (1988).

[23] Weinberg, S., The Quantum Theory of Fields (vol. 2), Cambridge University Press (1996).

[24] Gell-Mann, M., Oakes, R. J. and Renner, B., Phys. Rev. 175, 2195 (1968).
[25] Goldberger, M. L. and Treiman, S. B., Phys. Rev. **110**, 1178 (1958).

[26] M. Jamin, J. A. Oller and A. Pich, “Light quark masses from scalar sum rules,” [hep-ph/0110194](https://arxiv.org/abs/hep-ph/0110194).

[27] S. Narison, Nucl. Phys. Proc. Suppl. **86** (2000) 242 [hep-ph/9911454](https://arxiv.org/abs/hep-ph/9911454).

[28] Reinders, L. H., Rubinstein, H. and Yazaki, S., Phys. Rept. **127**, 1 (1985); Ioffe, B. L., Nucl. Phys. **B188**, 317 (1981).

[29] Stevenson, P. M., Ann. Phys. **132**, 383 (1981).

[30] Sisterna, P. and Vucetich, H., Phys. Rev. **D41**, 1034 (1990).

[31] Blatt, J. M. and Weisskopf, V. F., Theoretical Nuclear Physics, Wiley (1962).

[32] This method for calculating the stability is the work of David M. Wood, who developed it to investigate the stability of central Yukawa potentials.

[33] Serot, B. D. and Walecka, J. D., Int. J. Mod. Phys. **E 6**, 515 (1997).

[34] Hamber, H. and Parisi, G., Phys. Rev. Lett. **47**, 1792 (1981).

[35] Karsch, F., in Strong and Electroweak Matter '98, World Scientific (1999) [hep-lat/9903031](https://arxiv.org/abs/hep-lat/9903031).

[36] Kogut, J. B., Wyld, H.W., Karsch, F. and Sinclair, D.K., Phys. Lett. **B 188**, 353 (1987).

[37] Barrow, J. D., Phys. Rev. **D87**, 1805 (1987).

[38] Barrow, J. D., Sandvik, H. B. and Magueijo, J., [astro-ph/0109414](https://arxiv.org/abs/astro-ph/0109414).

[39] Ellis, J. R., Nanopoulos, D. V. and Sarkar, S., Nucl. Phys. **B 259**, 175 (1985).

[40] Dyson, F., Scientific American 225 (September issue), 25 (1971)

[41] Pagel, B., Phys. Rept. **333**, 433, (2000), Steigman, G., [astro-ph/0009506](https://arxiv.org/abs/astro-ph/0009506).

[42] M. Cvetic, G. Shiu and A. M. Uranga, Phys. Rev. Lett. **87** (2001) 201801 [hep-th/0107143](https://arxiv.org/abs/hep-th/0107143).

[43] G. Aldazabal, S. Franco, L. E. Ibáñez, R. Rabadán and A. M. Uranga, JHEP **0102** (2001) 047 [hep-ph/0011132](https://arxiv.org/abs/hep-ph/0011132).

[44] R. Blumenhagen, B. Kors, D. Lust and T. Ott, [arXiv:hep-th/0112015](https://arxiv.org/abs/hep-th/0112015).

[45] Avelino, P. P. *et al.*, Phys. Rev. **D64**, 103505, 2001.
[46] C. P. Burgess, L. E. Ibáñez and F. Quevedo, Phys. Lett. B 447 (1999) 257 [hep-ph/9810535].

[47] L. E. Ibáñez and G. G. Ross, “Electroweak breaking in supersymmetric models,” in Perspectives on Higgs physics, World Scientific (1993) [hep-ph/9204201].

[48] G. G. Ross and R. G. Roberts, Nucl. Phys. B 377 (1992) 571.

[49] I. Antoniadis, Phys. Lett. B 246 (1990) 377.

[50] J. Giedt, Annals Phys. 289, 251 (2001) [hep-th/0009104].

[51] D. M. Ghilencea and G. G. Ross, Nucl. Phys. B 606, 101 (2001) [hep-ph/0102306].

[52] I. Antoniadis, K. Benakli and M. Quiros, Nucl. Phys. B 583 (2000) 35 [hep-ph/0004091].

[53] A. Delgado, A. Pomarol and M. Quiros, Phys. Rev. D 60 (1999) 095008 [hep-ph/9812489].

[54] C. Wetterich, arXiv:hep-ph/0203266.

[55] A. V. Ivanchik, E. Rodriguez, P. Petitjean and D. A. Varshalovich, Astron. Lett. 28, 423 (2002) arXiv:astro-ph/0112323.

[56] D. Lust and T. R. Taylor, Phys. Lett. B 253 (1991) 335; B. de Carlos, J. A. Casas and C. Muñoz, Phys. Lett. B 263 (1991) 248.