Uniqueness of Nonextensive entropy under Rényi’s Recipe

Ambedkar Dukkipati‡, M Narasimha Murty and Shalabh Bhatnagar
Department of Computer Science and Automation, Indian Institute of Science, Bangalore-560012, India.
E-mail: ambedkar@csa.iisc.ernet.in, mnm@csa.iisc.ernet.in, shalabh@csa.iisc.ernet.in

Abstract. By replacing linear averaging in Shannon entropy with Kolmogorov-Nagumo average (KN-averages) or quasilinear mean and further imposing the additivity constraint, Rényi proposed the first formal generalization of Shannon entropy. Using this recipe of Rényi, one can prepare only two information measures: Shannon and Rényi entropy. Indeed, using this formalism Rényi characterized these additive entropies in terms of axioms of quasilinear mean. As additivity is a characteristic property of Shannon entropy, pseudo-additivity of the form \( x \oplus_q y = x + y + (1 - q)xy \) is a characteristic property of nonextensive (or Tsallis) entropy. One can apply Rényi’s recipe in the nonextensive case by replacing the linear averaging in Tsallis entropy with KN-averages and thereby imposing the constraint of pseudo-additivity. In this paper we show that nonextensive entropy is unique under the Rényi’s recipe, and thereby give a characterization.

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‡ Corresponding author
1. Introduction

In recent years, interest in generalized information measures has increased dramatically, after the introduction of nonextensive entropy in Physics in 1988 by Tsallis [1]. One can get this nonextensive entropy or Tsallis entropy by generalizing the information of single event in the definition of Shannon entropy, by replacing logarithm with so called $q$-logarithm, which is defined as $\ln_q x = \frac{x^q - 1}{q-1}$. Tsallis entropy does not satisfy the additivity property which is a characteristic property of Shannon entropy. Instead, it satisfies pseudo-additivity of the form $x \oplus_q y = x + y + (1 - q)xy$ and this definition of entropy (also known as nonextensive entropy) led to the field of nonextensive statistical mechanics in Physics. In this paper we use the term pseudo-addition to represent the binary operation $x \oplus_q y = x + y + (1 - q)xy$ for any $q \in \mathbb{R}$ and $q > 0$.

Tsallis entropy is considered as a useful measure in describing the thermostatistical properties of a certain class of physical systems that entail long-range interactions, long-term memories and multi-fractal structures. Tsallis entropy is also studied in information theory and Shannon-Khinchin axioms have been generalized to nonextensive case. While canonical distributions resulting from maximization of Shannon entropy are exponential in nature, in the Tsallis case, these result in power-law distributions. To a great extent, the success of Tsallis proposal is due to the ubiquity of power law distributions in nature.

Indeed, the starting point of the theory of generalized measures of information is due to Alfred Rényi [2, 3]. By using Kolmogorov-Nagumo averages (KN-average) Rényi introduced a generalized information measure, known as $\alpha$-entropy or Rényi entropy, the first formal well-known generalization of Shannon entropy. KN-average or quasilinear mean (we use these two terms interchangeably) is of the form $\langle x \rangle_{\psi} = \psi^{-1} (\sum_k p_k \psi(x_k))$, where $\psi$ is an arbitrary continuous and strictly monotone function. Replacing linear averaging in Shannon entropy with KN-averages and further imposing the additivity constraint – a characteristic property of underlying information associated with single event, which is logarithmic – leads to Rényi entropy. Using this recipe of Rényi, one can prepare only two information measures: Shannon and Rényi entropy. Using this formalism Rényi characterized these additive entropies in terms of axioms of KN-averages.

One can apply Rényi’s recipe in the nonextensive case by replacing the linear averaging in Tsallis entropy with KN-averages and thereby imposing the constraint of pseudo-additivity. A natural question arises: what are all the pseudo-additive information measures one can prepare with this recipe? We prove that only Tsallis entropy is possible in this case, which allows us to characterize Tsallis entropy based on axioms of KN-averages.

To understand these generalizations, the so called Hartley function [4] of a single stochastic event plays a fundamental role. We discuss Hartley function in $\S$ 2 along with a brief discussion on quasilinear mean and Rényi entropy. The main results of this paper, on uniqueness of Tsallis entropy under Rényi’s recipe and a result on characterization
of Tsallis entropy are presented in §3 and §4 respectively.

2. KN-averages and Information measures

2.1. Hartley Function and Shannon Entropy

Let $X$ be a discrete random variable (r.v) defined on some probability space, which takes only $n$ values, $n < \infty$. We denote the set of all such random variables by $\mathcal{X}$. Corresponding to the $n$-tuple $(x_1, \ldots, x_n)$ of values which $X$ takes, probability mass function (pmf) of $X$ is denoted by $p = (p_1, \ldots, p_n)$, where $p_k \geq 0$ for $k = 1, \ldots n$ and $\sum_{k=1}^{n} p_k = 1$. Expectation of r.v $X$ is denoted by $E_X$ or $\langle X \rangle$; in this paper we use both the notations, interchangeably.

Shannon entropy, a logarithmic measure of information on $X$ denoted by $S(X)$, reads [5]

$$S(X) = - \sum_{k=1}^{n} p_k \ln p_k,$$  

and measures the average lack of information that is inherent in $p$.

This motivation to quantify information in terms of logarithmic functions is due to Hartley [4], who first used a logarithmic function to define uncertainty associated with a finite set. This is known as Hartley information measure. The Hartley information measure of a finite set $A$ with $n$ elements is defined as $H(A) = \log_b n$. If the base of the logarithm is 2, then the uncertainty is measured in bits, and in the case of natural logarithm, the unit is nats. Throughout this paper we use only natural logarithm as a convention.

One can give a more general definition of Hartley information measure, which is a special case of Shannon entropy as follows. Define a function $H : \{x_1, \ldots, x_n\} \rightarrow \mathbb{R}$ of the values taken by r.v $X \in \mathcal{X}$ with corresponding p.m.f $p = (p_1, \ldots, p_n)$ as [6]

$$H(x_k) = \ln \frac{1}{p_k}, \quad \forall k = 1, \ldots n. \tag{2}$$

$H$ is also known as entropy of a single event and plays an important role in all classical measures of information. It can be interpreted either as a measure of how unexpected the event was, or as measure of the information yielded by the event. Hartley function satisfies: (i) $H$ is nonnegative: $H(x_k) \geq 0$ (ii) $H$ is additive: $H(x_i x_j) = H(x_i) + H(x_j)$ (iii) $H$ is normalized: $H(x_k) = 1$, whenever $p_k = \frac{1}{2}$ (in the case of logarithm with base 2, the same satisfied for $p_k = \frac{1}{2}$). These properties are both necessary and sufficient [6].

Now, Shannon entropy [1] can be written as expectation of Hartley function as

$$S(X) = \langle H \rangle = \sum_{k=1}^{n} p_k H_k,$$  

where $H_k = H(x_k), \forall k = 1, \ldots n$, with the understanding that $\langle H \rangle = \langle H(X) \rangle$.

The characteristic additive property of Shannon entropy

$$S(X \times Y) = S(X) + S(Y),$$  

where $H_k = H(x_k), \forall k = 1, \ldots n$, with the understanding that $\langle H \rangle = \langle H(X) \rangle$. 

The characteristic additive property of Shannon entropy
for two independent random variables $X$ and $Y$ now follows as a consequence of the additivity property of Hartley function.

There are two postulates involved in defining Shannon entropy as expectation of Hartley function. One is the additivity of information which is the characteristic property of Hartley function, and the other is that if different amounts of information occur with different probabilities, the total information will be the average of the individual informations weighted by the probabilities of their occurrences.

The basic idea behind Rényi’s generalization is any putative candidate for an entropy should be a mean and there by use a well known idea in mathematics that the linear mean, though most widely used, is not the only possible way of averaging, however, one can define the mean with respect to an arbitrary function. Here we briefly discuss generalized averages and its properties which are essential for the results we present in this paper.

2.2. Kolmogorov-Nagumo Averages or Quasilinear Mean

In the general theory of means, quasilinear mean of a random variable $X$ is defined as

$$E_\psi X = \langle X \rangle_\psi = \psi^{-1} \left( \sum_{k=1}^{n} p_k \psi(x_k) \right),$$

(5)

where $\psi$ is continuous and strictly monotonic (increasing or decreasing) in which case it has an inverse $\psi^{-1}$ which satisfies the same conditions. In the context of generalized means, $\psi$ is referred to as Kolmogorov-Nagumo function or KN-function. If, in particular, $\psi$ is linear, then (5) reduces to the expression of linear averaging,

$$EX = \langle X \rangle = \sum_{k=1}^{n} p_k x_k.$$

The following theorem qualifies quasilinear means.

**Theorem 2.1.** If $\psi$ is continuous and strictly monotone in $a \leq x \leq b$, $a \leq x_k \leq b$, $k = 1, \ldots, n$, $p_k > 0$ and $\sum_{k=1}^{n} p_k = 1$, then $\exists$ unique $x_0 \in (a, b)$ such that

$$\psi(x_0) = \sum_{k=1}^{n} p_k \psi(x_k)$$

and $x_0$ is greater than some and less than others of the $x_k$ unless all $x_k$ are zero.

§ Kolmogorov \([7]\) and Nagumo \([8]\) first characterized the quasilinear mean $\langle x \rangle_\psi$ for a vector $(x_1, \ldots, x_n)$ as $\langle x \rangle_\psi = \psi^{-1} \left( \sum_{k=1}^{n} \frac{1}{n} \psi(x_k) \right)$ where $\psi$ is a continuous and strictly monotone function. De Finetti \([9]\) extended their result to the case of simple (finite) probability distributions. The version of the quasilinear mean representation theorem referred to in §4 is due to Hardy, Littlewood and Pólya \([10]\), which followed closely the approach of de Finetti. Aczél \([11]\) proved a characterization of the quasilinear mean using functional equations. Ben-Tal \([12]\) showed that quasilinear means are ordinary arithmetic means under suitably defined addition and scalar multiplication operations. Norris \([13]\) did a survey of quasilinear means and its more restrictive forms in Statistics. More recent survey of generalized means can be found in \([14]\). Applications of quasilinear means can be found in economics (for example, \([15]\)) and decision theory (for example, \([16]\)). Recently Czachor and Naudts \([17]\) studied generalized thermostatistics based on quasilinear means.
Thus, the mean $\langle . \rangle_\psi$ is determined when the function $\psi$ is given. We may ask whether the converse is true: if $\langle X \rangle_{\psi_1} = \langle X \rangle_{\psi_2}$ for all $X \in \mathcal{X}$, is $\psi_1$ necessarily the same function as $\psi_2$? First we give the following definition.

**Definition 2.2.** Continuous and strictly monotone functions $\psi_1$ and $\psi_2$ are said to be KN-equivalent if $\langle X \rangle_{\psi_1} = \langle X \rangle_{\psi_2}$ for all $X \in \mathcal{X}$.

Note that when we compare two means, it is to be understood that the underlying probabilities are same. The following theorem characterizes KN-equivalent functions.

**Theorem 2.3.** In order that two continuous and strictly monotone functions $\psi_1$ and $\psi_2$ are KN-equivalent, it is necessary and sufficient that

$$\psi_1 = \alpha \psi_2 + \beta,$$

where $\alpha$ and $\beta$ are constants and $\alpha \neq 0$.

**Corollary 2.4.** Let $\psi$ be a KN-function then $\langle X \rangle_\psi = \langle X \rangle_{-\psi}$.

Hence, when ever required, without loss of generality, one can assume that $\psi$ is an increasing function. The following theorem characterizes additivity of quasilinear means.

**Theorem 2.5.** Let $\psi$ be a KN-function and $c$ be a real constant then $\langle X+c \rangle_\psi = \langle X \rangle_\psi + c$ i.e.,

$$\psi^{-1}\left(\sum_{k=1}^{n} p_k \psi(x_k + c)\right) = \psi^{-1}\left(\sum_{k=1}^{n} p_k \psi(x_k)\right) + c$$

if and only if $\psi$ is either linear or exponential.

Proof of Theorems 2.3, 2.4 and 2.5 can be found in the book on inequalities by Hardy, Littlewood, Pólya [10].

### 2.3. Rényi Entropy

In the definition of Shannon entropy [3], if the standard mean of Hartley function $H$ is replaced with the quasilinear mean [5], one can obtain a generalized measure of information of r.v $X$ with respect to a KN-function $\psi$ as

$$S_\psi(X) = \psi^{-1}\left(\sum_{k=1}^{n} p_k \psi\left(\frac{1}{p_k}\right) \right) = \psi^{-1}\left(\sum_{k=1}^{n} p_k \psi(H_k)\right),$$

where $\psi$ is a KN-function. We refer to (6) as quasilinear entropy with respect to the KN-function $\psi$. If we impose the constraint of additivity on $S_\psi$, then $\psi$ should satisfy [2]

$$\langle X+c \rangle_\psi = \langle X \rangle_\psi + c,$$

for any random variable $X \in \mathcal{X}$ and a constant $c$.

Rényi employed this formalism to define a one-parameter family of measures of information ($\alpha$-entropies) as follows:

$$S_\alpha(X) = \frac{1}{1-\alpha} \ln \left(\sum_{k=1}^{n} p_k^\alpha\right),$$

(8)
where the KN-function $\psi$ is chosen in (6) as $\psi(x) = e^{(1-\alpha)x}$ whose choice is motivated by Theorem 2.5. If we choose $\psi$ as a linear function in quasilinear entropy (6), what we get is Shannon entropy. Rényi entropy is a one-parameter generalization of Shannon entropy in the sense that the limit $\alpha \to 1$ in (8) retrieves Shannon entropy.

Despite its formal origin Rényi entropy proved important in a variety of practical applications in coding theory [6], statistical inference [18, 19], quantum mechanics [20], chaotic dynamics systems [21]. Thermodynamic properties of systems with multi-fractal structures have been studied by extending the notion of Gibbs-Shannon entropy into a more general framework - Rényi entropy [22].

3. Rényi’s Recipe and Tsallis Entropy

3.1. Tsallis Entropy

Due to an increasing interest in long-range correlated systems and non-equilibrium phenomena there has recently been much focus on the Tsallis (or nonextensive) entropy. Although, first introduced by Havrda and Charvat [23] in the context of cybernetics theory and later studied by Daróczy [24], it was Tsallis [1] who exploited its nonextensive features and placed it in a physical setting. Hence it is also known as Harvda-Charvat-Daróczy-Tsallis entropy. Throughout this paper we refer to this as Tsallis or nonextensive entropy. Tsallis entropy of a r.v $X \in \mathcal{X}$ with p.m.f $p = (p_1, \ldots, p_n)$ is defined as

$$S_q(X) = \frac{1 - \sum_{k=1}^{n} p_k^q}{q - 1},$$

where $q > 0$ is called the nonextensive index. Tsallis entropy too, like Rényi entropy, is a one-parameter generalization of Shannon entropy in the sense that $q \to 1$ in (9) retrieves Shannon entropy. Tsallis entropy is concave for all $q > 0$, but Rényi entropy is concave only for $0 < \alpha < 1$. The index $q$ characterizes the degree of nonextensivity reflected in the pseudo-additivity property

$$S_q(X \times Y) = S_q(X) \oplus_q S_q(Y) = S_q(X) + S_q(Y) + (1 - q)S_q(X)S_q(Y),$$

where $X, Y \in \mathcal{X}$ are two independent random variables.

3.2. Nongeneralizability of Tsallis Entropy

Though the derivation of Tsallis entropy, when it was proposed in 1988 [1] is slightly different, one can understand this generalization using $q$-logarithm function (see (12)), where one would first generalize logarithm in the Hartley information with $q$-logarithm and define $q$-Hartley function $\tilde{H} : \{x_1, \ldots, x_n\} \to \mathbb{R}$ of r.v $X$ as [25]

$$\tilde{H}_k = \tilde{H}(x_k) = \ln_q \frac{1}{p_k}, \quad k = 1, \ldots, n.$$  

The $q$-logarithm in (11) is defined as

$$\ln_q(x) = \frac{x^{1-q} - 1}{1-q}.$$  

which satisfies pseudo-additivity of the form \( \ln_q(xy) = \ln_q x \oplus_q \ln_q y \) and in the limit \( q \to 1 \), we have \( \ln_q x \to \ln x \). Now Tsallis entropy \((10)\) can be defined as the expectation of \(q\)-Hartley function \(\tilde{H}\) as

\[
S_q(X) = \left\langle \tilde{H} \right\rangle.
\]  

(13)

Note that the characteristic pseudo-additivity property of Tsallis entropy \((10)\) is a consequence of additivity property of Hartley function.

Before we present the main results of this paper, we briefly discuss the context of quasilinear means where there is a relation between Tsallis and Rényi entropy. The \(q\)-Hartley function can be written as

\[
\tilde{H}_k = \ln_q \frac{1}{p_k} = \phi_q(H_k),
\]

where

\[
\phi_q(x) = \frac{e^{(1-q)x} - 1}{1 - q} = \ln_q(e^x).
\]  

(14)

Note that \(\phi_q\) is KN-equivalent to \(e^{(1-q)x}\) (by Theorem 2.3), the KN-function used in Rényi entropy. Hence Tsallis entropy is related to Rényi entropies as

\[
S^T_q = \phi_q(S^R_q),
\]

(15)

where \(S^T_q\) and \(S^R_q\) denote the Tsallis and Rényi entropy respectively with a real number \(q\) as a parameter. Hence, Tsallis entropy and Rényi entropy are monotonic functions of each other and, as a result, both must be maximized by the same probability distribution.

Now a natural question that arises is whether one could generalize Tsallis entropy using Rényi’s recipe i.e., by replacing linear average in \((13)\) by KN-averages and impose the condition of pseudo-additivity. It is equivalent to determining the KN-function \(\psi\) for which so called \(q\)-quasilinear entropy defined as

\[
\tilde{S}_\psi(X) = \left\langle \tilde{H} \right\rangle_{\psi} = \psi^{-1} \left[ \sum_{k=1}^{n} p_k \psi \left( \tilde{H}_k \right) \right],
\]  

(16)

where \(\tilde{H}_k = \tilde{H}(x_k) \forall k = 1, \ldots, n\), satisfies the pseudo-additive property.

First, we present the following result which characterizes the pseudo-additivity of quasilinear means.

**Theorem 3.1.** Let \(X, Y \in \mathcal{X}\) be two independent random variables. Let \(\psi\) be any KN-function. Then

\[
\left\langle X \oplus_q Y \right\rangle_{\psi} = \left\langle X \right\rangle_{\psi} \oplus_q \left\langle Y \right\rangle_{\psi}
\]

(17)

if and only if \(\psi\) is linear.

**Proof.** Let \(p\) and \(r\) be the p.m.f.s of random variables \(X, Y \in \mathcal{X}\) respectively. The proof of sufficiency is simple which follows from

\[
\left\langle X \oplus_q Y \right\rangle_{\psi} = \sum_{i=1}^{n} \sum_{j=1}^{n} p_i r_j (x_i \oplus_q y_j),
\]
and by the definition of $\oplus_q$, we have

$$
\langle X \oplus_q Y \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} p_i r_j (x_i + y_j + (1 - q)x_i y_j)
$$

$$
= \sum_{i=1}^{n} p_i x_i + \sum_{j=1}^{n} r_j y_j + (1 - q) \sum_{i=1}^{n} p_i x_i \sum_{j=1}^{n} r_j y_j .
$$

To prove the converse, we need to determine all forms of $\psi$ which satisfy

$$
\psi^{-1}\left(\sum_{i=1}^{n} \sum_{j=1}^{n} p_i r_j \psi (x_i \oplus_q y_j)\right) = \psi^{-1}\left(\sum_{i=1}^{n} p_i \psi (x_i)\right) \oplus_q \psi^{-1}\left(\sum_{j=1}^{n} r_j \psi (y_j)\right) .
$$

Since (18) must hold for arbitrary p.m.fs $p, r$ and for arbitrary numbers $\{x_1, \ldots, x_n\}$ and $\{y_1, \ldots, y_n\}$, one can choose $y_j = c$ independently of $j$. Then (18) yields

$$
\psi^{-1}\left(\sum_{i=1}^{n} p_k \psi (x_i \oplus_q c)\right) = \psi^{-1}\left(\sum_{i=1}^{n} p_k \psi (x_i)\right) \oplus_q c .
$$

That is, $\psi$ should satisfy

$$
\langle X \oplus_q c \rangle_\psi = \langle X \rangle_\psi \oplus_q c ,
$$

for any $X \in \mathcal{X}$ and any constant $c$. This can be rearranged as

$$
\langle (1 + (1 - q)c)X + c \rangle_\psi = (1 + (1 - q)c)\langle X \rangle_\psi + c
$$

by using the definition of $\oplus_q$. Since $q$ is independent of other quantities, $\psi$ should satisfy an equation of the form

$$
\langle dX + c \rangle_\psi = d\langle X \rangle_\psi + c ,
$$

where $d \neq 0$ (by writing $d = (1 + (1 - q)c)$). Finally $\psi$ must satisfy

$$
\langle X + c \rangle_\psi = \langle X \rangle_\psi + c
$$

and

$$
\langle dX \rangle_\psi = d\langle X \rangle_\psi ,
$$

for any $X \in \mathcal{X}$ and any constants $d, c$. From Theorem 2.5, the condition (22) is satisfied only when $\psi$ is linear or exponential.

To complete the theorem we have to show that KN-averages do not satisfy condition (23) when $\psi$ is exponential. For a particular choice of $\psi(x) = e^{(1-\alpha)x}$, assume that

$$
\langle dX \rangle_\psi = d\langle X \rangle_\psi ,
$$

where

$$
\langle dX \rangle_{\psi_1} = \frac{1}{1 - \alpha} \ln \left(\sum_{k=1}^{n} p_k e^{(1-\alpha)d x_k}\right) ,
$$
and
\[ d\langle X \rangle_{\psi_1} = \frac{d}{1 - \alpha} \ln \left( \sum_{k=1}^{n} p_k e^{(1-\alpha)x_k} \right). \]

Now define a KN-function \( \psi' \) as \( \psi'(x) = e^{(1-\alpha)dx} \), for which
\[ \langle X \rangle_{\psi'} = \frac{1}{d(1 - \alpha)} \ln \left( \sum_{k=1}^{n} p_k e^{(1-\alpha)x_k} \right). \]

Condition (24) implies
\[ \langle X \rangle_{\psi} = \langle X \rangle_{\psi'}, \]
and by Theorem 2.3, \( \psi \) and \( \psi' \) are KN-equivalent which gives a contradiction.

One can observe that the above proof avoids solving functional equations as in the case of Theorem 2.3 (see [6]). Instead it makes use of basic results of KN-averages. The following corollary is the immediate consequence of Theorem 3.1.

**Corollary 3.2.** \( q \)-quasilinear entropy \( \tilde{S}_\psi \) (defined as in (16)) with respect to a KN-function \( \psi \) satisfies pseudo-additivity if and only if \( \tilde{S}_\psi \) is Tsallis entropy.

**Proof.** Let \( X, Y \in \mathcal{X} \) be two independent random variables and let \( p, r \) be their corresponding pmfs. By the pseudo-additivity constraint, \( \psi \) should satisfy
\[ \langle \tilde{H}_p^q \oplus q \tilde{H}_r^q \rangle_{\psi} = \langle \tilde{H}_p^q \rangle_{\psi} \oplus q \langle \tilde{H}_r^q \rangle_{\psi}. \]

From the property of \( q \)-logarithm that \( \ln_q xy = \ln_q x \oplus_q \ln_q y \), we need
\[ \psi^{-1} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} p_i r_j \psi \left( \ln_q \frac{1}{p_i r_j} \right) \right) = \psi^{-1} \left( \sum_{i=1}^{n} p_i \psi \left( \ln_q \frac{1}{p_i} \right) \right) \oplus_q \psi^{-1} \left( \sum_{j=1}^{n} r_j \psi \left( \ln_q \frac{1}{r_j} \right) \right). \]

Equivalently, we need
\[ \psi^{-1} \left( \sum_{i=1}^{n} \sum_{j=1}^{n} p_i r_j \psi \left( \tilde{H}_p^q \oplus_q \tilde{H}_r^q \right) \right) = \psi^{-1} \left( \sum_{i=1}^{n} p_i \psi \left( \tilde{H}_p^q \right) \right) \oplus_q \psi^{-1} \left( \sum_{j=1}^{n} r_j \psi \left( \tilde{H}_r^q \right) \right), \]

where \( \tilde{H}_p \) and \( \tilde{H}_r \) represent the \( q \)-Hartley functions corresponding to probability distributions \( p \) and \( r \) respectively. That is, \( \psi \) should satisfy
\[ \langle \tilde{H}_p^q \oplus q \tilde{H}_r^q \rangle_{\psi} = \langle \tilde{H}_p^q \rangle_{\psi} \oplus_q \langle \tilde{H}_r^q \rangle_{\psi}. \]

Also from Theorem 3.1, \( \psi \) is linear and hence \( \tilde{S}_\psi \) is Tsallis. \( \square \)
Corollary shows that using the Rényi’s recipe in the nonextensive case one can prepare only Tsallis entropy, while in the classical there are two possibilities.

4. A Characterization Theorem for Tsallis Entropy

The importance of Rényi’s formalism to generalize Shannon entropy is a characterization of Shannon entropy in terms of axiom of quasilinear means [2]. By the result, Theorem 3.1, that we presented in this paper, one can give a characterization of Tsallis entropy in terms of axioms of quasilinear means. For such a characterization one would assume that entropy is the expectation of a function of underlying r.v. In the classical case, the function is Hartley function, while in the nonextensive case it is $q$-Hartlay function.

Since characterization of quasilinear means is given in terms of cumulative distribution of a random variable, we use the following definitions and notation.

Let $F : \mathbb{R} \rightarrow \mathbb{R}$ denote the cumulative distribution function of random variable $X \in \mathcal{X}$. Corresponding to a KN-function $\psi : \mathbb{R} \rightarrow \mathbb{R}$, generalized mean of $F$ (or $X$) can be written as

$$E_\psi(F) = E_\psi(X) = \langle X \rangle_\psi = \psi^{-1}\left(\int \psi \, dF\right),$$

which is continuous analogue to (5) and it is axiomized by Kolmogorov, Nagumo and De Finetti (see [10, Theorem 215]) as follows.

**Theorem 4.1.** Let $\mathcal{F}_I$ be the set of all cumulative distribution functions defined on some interval $I$ of the real line $\mathbb{R}$. A functional $\kappa : \mathcal{F}_I \rightarrow \mathbb{R}$ satisfies the following axioms:

- **axiom 1:** $\kappa(\delta_x) = x$, where $\delta_x \in \mathcal{F}_I$ denotes the step function at $x$ (Consistency with certainty),

- **axiom 2:** $F, G \in \mathcal{F}_I$, if $F \leq G$ then $\kappa(F) \leq \kappa(G)$; the equality holds if and only if $F = G$ (Monotonicity) and,

- **axiom 3:** $F, G \in \mathcal{F}_I$, if $\kappa(F) = \kappa(G)$ then $\kappa(\beta F + (1 - \beta)H) = \kappa(\beta G + (1 - \beta)H)$, for any $H \in \mathcal{F}_I$ (Quasilinearity)

if and only if there is a continuous strictly monotone function $\psi$ such that

$$\kappa(F) = \psi^{-1}\left(\int \psi \, dF\right).$$

The modified axioms for quasilinear mean can be found in [26, 27, 14]. Now we give our characterization theorem for Tsallis entropy that is similar to the characterization of Shannon entropy given by Rényi [2].

**Theorem 4.2.** Let $X \in \mathcal{X}$ be a random variable. An information measure defined as a (generalized) mean $\kappa$ of $q$-Hartley function of $X$ is Tsallis entropy if and only if

(i) $\kappa$ satisfies axioms of quasilinear means given in Theorem 4.1 and,
(ii) If $X, Y \in X$ are two random variables which are independent, then

$$\kappa(X \oplus_q Y) = \kappa(X) \oplus_q \kappa(Y).$$

Theorem 4.2 is a direct consequence of Theorems 3.1 and 4.1. This characterization of Tsallis entropy only replaces the additivity constraint in the characterization of Shannon entropy given by Rényi in [2], with pseudo-additivity, which further does not make use of the postulate $\kappa(H) + \kappa(-H) = 0$. (This postulate is needed to distinguish Shannon entropy from Rényi entropy). This is possible because Tsallis entropy is unique by means of KN-averages and under pseudo-additivity.

5. Conclusions

Passing an information measure through Rényi formalism – procedure followed by Rényi to generalize Shannon entropy – allows one to study the possible generalizations and characterize information measure in the context in terms of axioms of quasilinear means. In this paper we studied this technique for nonextensive entropy and showed that Tsallis entropy is unique under Rényi’s recipe. Considering the attempts to study generalized thermostatistics based on KN-averages (for example [17]), the results presented in this paper further the relation between entropic measures and generalized averages.

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