Efficiently Generating Independent Replicates Directly from the Posterior Distribution for a Large Class of Bayesian Generalized Linear Mixed Effects Models

Jonathan R. Bradley

Abstract

Markov chain Monte Carlo (MCMC) allows one to generate dependent replicates from a posterior distribution for effectively any Bayesian hierarchical model. However, MCMC can produce a significant computational burden. This motivates us to consider finding closed-form expressions of the posterior distribution that are computationally straightforward to obtain independent replicates from directly. We focus on a broad class of Bayesian generalized linear mixed-effects models (GLMM) that allows one to jointly model data of different types. First, we derive a new class of distributions we refer to as the generalized conjugate multivariate (GCM) distribution. The GCM distribution’s theoretical development is similar to that of the CM distribution with two main differences; namely, (1) the GCM allows for multiple data types and prior distributional assumptions, and (2) the GCM explicitly accounts for hyperparameters through marginalization. The development of GCM is needed for to obtain independent replicates directly from the exact posterior distribution, which have an efficient projection/regression form. Hence, we refer to our method as Exact Posterior Regression (EPR). Illustrative examples are provided including simulation studies and an analysis of multivariate spatial data from the U.S. Census Bureau’s American Community Survey (ACS).

Keywords: Bayesian hierarchical model; Big data; Gibbs sampler; Log-Linear Models; Markov chain Monte Carlo; Non-Gaussian; Nonlinear.

1(to whom correspondence should be addressed) Department of Statistics, Florida State University, 117 N. Woodward Ave., Tallahassee, FL 32306-4330, jrbradley@fsu.edu
1 Introduction

MCMC has become an invaluable tool in statistics and is covered in standard textbooks (Robert and Casella, 2004). MCMC is an all-purpose strategy that allows one to obtain dependent samples from a generic posterior distribution. There are several theoretical considerations that one needs to consider when implementing MCMC to obtain samples from the posterior distribution including, ergodicity, irreducibility, and positive recurrence of the MCMC. In addition to theoretical considerations, practical implementation issues arise, including, a potential for high computational costs, assessing convergence (Gelman and Rubin, 1992; Cowles and Carlin, 1996), tuning the MCMC (Roberts and Rosenthal, 2009), and computing the effective sample size of the Markov chain (Vats et al., 2019), among other considerations. One of the current state-of-the-art techniques in MCMC is Hamiltonian Monte Carlo (HMC, Neal, 2011). HMC is a Metropolis–Hastings algorithm, where Hamiltonian dynamic evolution is used to propose a new value. In general, HMC leads to “fast mixing” (i.e., converges relatively quickly to the posterior distribution) because it provides a sample from the joint posterior distribution of all processes and parameters, and moreover, has been optimized efficiently using the software Stan (Carpenter et al., 2017).

Of course, if one can obtain independent replicates directly from the posterior distribution MCMC is not needed. In this article, we revisit the problem of generating independent replicates directly from the posterior distribution for a broad class of generalized linear mixed effects models (GLMM, McCulloch et al., 2008). Much of the current literature does not consider solving this problem, since obtaining independent replicates directly from the exact posterior distribution for Bayesian GLMMs is a difficult problem, and MCMC can easily be adapted to many settings. We consider Bayesian GLMMs for Gaussian distributed data, binomial distributed data, and Poisson distributed data. Our approach uses the conjugate multivariate (CM) distribution (Bradley et al., 2020) as the class of priors for fixed and random effects (e.g., see Chen and Ibrahim, 2003, for the first formal use of conjugate priors in generalized linear regression). The samples from our proposed model are independently drawn, and hence avoid issues with convergence, tuning, and positive autocorrelations in a MCMC. Moreover, our exact replicates have an interpretable projection formulation. This regression-type projection can be computed efficiently using known block matrix inversion formulas (Lu and Shiou, 2002). Thus, we refer to our method as Exact Posterior Regression (EPR), which is the primary contribution of this article.

The conjugate prior distribution is often restricted to the data type. For example, for binomial, negative binomial, Bernoulli, and multinomial distributed data, the fixed and random effects are conjugate with the multivariate logit-beta distribution (Gao and Bradley, 2019; Bradley et al., 2019), which is the special case of the conjugate multivariate (CM) distribution. Similarly, Poisson and Weibull distributed data are conjugate with the multivariate log-gamma distribution (Bradley et al., 2018; Hu and Bradley, 2018; Xu et al., 2019; H.-C. Yang et al., 2019; Parker et al., 2020, 2021), another special case of the CM distribution. Finally, mixed effects models for Gaussian distributed data regularly make use of Gaussian priors for fixed and random effects (Gelman et al., 2013), which is also a type of CM distribution. Thus, our second major contribution is that we develop an extension of the CM distribution, which allows one to model fixed and random effects in a GLMM using any class of CM distributions. Additionally, the CM distribution does not allow one
to explicitly account for hyperparameters. We call this new distribution the generalized CM (GCM) distribution, which allows for standard latent Gaussian process model specifications of GLMMs (e.g., see [Gelfand and Schliep, 2016] for a recent discussion). Furthermore, we develop conditional distributions for GCM distributed random vectors.

A key step in our formulation is the incorporation of what we call “bypass location parameters,” which are simply additive location parameters introduced into a generalized mixed effects model. Traditionally, these parameters are set equal to zero. When these location parameters are not set equal to zero and instead given an improper prior then the implied posterior distribution for fixed and random effects is of the form of a GCM, which we can directly sample from (bypassing the need for MCMC). However, we show that posterior replicates from this GCM can overfit the latent process. Thus, we suggest marginalizing across these bypass location parameters and estimating them to be zero.

We emphasize the high potential impact of the contributions of EPR and GCM, since much of the literature places a high consistent emphasis on using MCMC strategies to obtain asymptotically exact correlated samples from the posterior distribution. For example, at the time of writing this manuscript the following papers use MCMC in a GLMM setting: [Collins et al., 2021], [Covernton et al., 2021], [Gañán-Cardenas et al., 2021], [Liu et al., 2021], [Jobst and Rösch, 2021], and [Sapri et al., 2021], among others. All of these analyses can easily be adapted to be implemented using EPR, which completely avoids MCMC.

EPR allows one to efficiently jointly analyze several types of correlated data. In particular, we consider jointly modeling four “types of data,” namely, conditionally Gaussian, Poisson, binomial, and Student-t distributed data. Joint analysis of multiple response-type datasets is often accomplished using Bayesian GLMMs with Gaussian priors (e.g., see [Christensen and Amemiya, 2002], [Schliep and Hoeting, 2013], [Wu et al., 2015], [Clarke et al., 2017], [Todd et al., 2018], among several others) and make use of computationally expensive MCMC techniques. Recently, [Bradley, 2022] introduced an approach that transforms the data and makes use of the CM distribution, but models transformed data and does not model the data directly. Consequently, our approach, which allows for joint modeling of multiple data types directly, offers a contribution to this growing literature.

To summarize, the contributions of this article can be classified into two groups:

1. The first group of contributions of this article develops the GCM distribution. This includes expressions for the GCM distribution and the conditional GCM distribution. The key literature on conjugate modeling began with [Diaconis and Ylvisaker, 1979]’s seminal paper which developed univariate conjugate models for the exponential family. Then [Chen and Ibrahim, 2003] developed [Diaconis and Ylvisaker, 1979]’s work in the context of fixed effects models and [Bradley et al., 2020] developed [Diaconis and Ylvisaker, 1979]’s work in the context of mixed effects models. However, all of these papers require one to match the form of the prior distribution with that of the likelihood. The use of the GCM allows one to consider any class of CM distributions for models for fixed and random effects. Moreover, this literature often does not emphasize hyperparameters; however, our development explicitly addresses hyperparameters through marginalization. It should be noted that the theoretical development of the GCM is similar to that of the CM distribution ([Bradley et al., 2020]). However,
the GCM has an enormous practical advantage over the CM by allowing one to use a richer class of prior distributions and avoids MCMC updates of hyperparameters. For example, when using the CM for a Poisson GLMM, one uses multivariate log-gamma priors for fixed and random effects and updates shape/rate parameters in an MCMC. When using the GCM one can also use Gaussian priors (or possibly different priors from Diaconis and Ylvisaker (1979)) and avoid sampling hyperparameters in an MCMC.

2. The second group of contributions of this article allows one to use the GCM in a Bayesian GLMM context to produce what we call exact posterior regression (EPR). Much of the Bayesian literature is shifting its focus on avoiding MCMC through the use of approximate Bayesian methods (e.g., see Wainwright and Jordan, 2008; Rue et al., 2009) or through direct sampling of the posterior distributions in special cases (Zhang et al., 2021). EPR adds to this growing literature by allowing one to independently sample from the posterior from a broad class of GLMMs. By “broad” we mean that many existing GLMMs can be written in terms of our formulation, and we allow for multiple data types (e.g., Gaussian, binomial, Poisson, etc.). We show that the posterior distribution for fixed and random effects are GCM. Furthermore, we use matrix algebra techniques to aid in the computation of EPR (see Theorems 4.3 and 4.4).

The remainder of the article proceeds as follows. In Section 2 we review the CM distribution and in Section 3 give an expression of the GCM and conditional GCM distribution. Then, in Section 4 we develop EPR, and we describe how to efficiently sample independent replicates directly from the marginal posterior of the fixed effects, and random effects. Illustrations are provided in Section 5, which includes several simulations/comparisons and a multiple response-type ACS dataset. The main goal of our illustrations is to compare to several common existing strategies for Bayesian GLMMs. Proofs are given in the Appendix, and a discussion is given in Section 6.

2 Review: The Conjugate Multivariate Distribution

In this section, we give the reader a review of the CM distribution from Bradley et al. (2020). Suppose the observed data is distributed according to the natural exponential family (Diaconis and Ylvisaker, 1979; Lehmann and Casella, 1998). That is, suppose that the probability density function/probability mass function (pdf/pmf) of the observed datum \( Z_k \) is given by,

\[
f(Z_k|Y_k, b_k) = \exp \left\{ Z_k Y_k - b_k \psi_k(Y_k) + c_k(Z_k) \right\}; \quad Z_k \in \mathcal{Z}_k, Y_k \in \mathcal{Y}_k,
\]

where \( f \) denotes a generic pdf/pmf, \( \mathcal{Z}_k \) is the support of \( Z_k \), \( \mathcal{Y}_k \) is the support of the unknown parameter \( Y_k \), \( b_k \) is a possibly known real-value, both \( \psi_k(\cdot) \) and \( c_k(\cdot) \) are known real-valued functions, and \( k = 1, \ldots, K \) is used to index the specific member of the exponential family (e.g., Gaussian, Poisson, binomial, etc.). The function \( b_1 \psi_1(Y_k) \) is often called the log partition function (Lehmann and Casella, 1998). We focus on Gaussian responses, which sets \( \psi_1(Z) = Z^2 \), \( b_1 = 1/2\sigma^2 \), \( \mathcal{Z}_1 = \mathbb{R} \), and \( \mathcal{Y}_1 = \mathbb{R} \) with \( \sigma^2 > 0 \); Poisson responses, which sets \( \psi_2(Z) = \exp(Z) \), \( b_2 = 1 \),
\[ \mathcal{Z}_2 = \{0, 1, 2, \ldots\}, \text{ and } \mathcal{Z}_3 = \mathbb{R}; \text{ and binomial responses, which sets } \psi_3(Z) = \log\{1 + \exp(Z)\}, \]

\[ b_3 = m, \mathcal{Z}_3 = \{0, 1, \ldots, m\}, \text{ and } \mathcal{Y}_3 = \mathbb{R} \text{ with } m \text{ a strictly positive integer.} \]

It follows from [Diagonis and Ylvisaker (1979)] that the conjugate prior distribution for \( Y_k \) (when it exists) is given by,

\[ f(Y_k|\alpha_k, \kappa_k) = \mathcal{N}_k(\alpha_k, \kappa_k) \exp \left\{ \alpha_k Y_k - \kappa_k \psi_k(Y_k) \right\}; Y_k \in \mathcal{Y}_k, \frac{\alpha_k}{\kappa_k} \in \mathcal{Z}_k, \kappa_k > 0, k = 1, \ldots, K, \quad (2) \]

where \( \mathcal{N}_k(\alpha_k, \kappa_k) \) is a normalizing constant. Let \( \text{DY}(\alpha_k, \kappa_k; \psi_k) \) denote a shorthand for the pdf in (2). Here “DY” stands for “Diaconis-Ylvisaker.” Of course, there are several special cases of the DY distribution other than the Gaussian \( (k = 1) \), log-gamma \( (k = 2) \), and logit-beta \( (k = 3) \), distributions, several of which do not correspond to a member of the exponential family. For example, \( \alpha_4 = 0, \psi_4(Z) = \log \left\{ 1 + \frac{Z^2}{V} \right\} \), and \( \kappa_4 = (\nu + 1)/2 \) with \( \nu > 0 \), results in a Student- \( t \) distribution with degrees of freedom \( \nu \), which is not a member of the exponential family of distributions.

By conjugate we mean that the posterior distribution is from the same family of distributions as the prior distribution. In the case of (1) and (2), we obtain conjugacy as

\[ Y_k|Z_k, \alpha_k, \kappa_k \sim \text{DY}(\alpha_k + Z_k, \kappa_k + b_k; \psi_k). \quad (3) \]

[Bradley et al. (2020)] derived a multivariate version of \( \text{DY}(\alpha_k, \kappa_k; \psi_k) \). Specifically, define the \( n_k \)-dimensional random vector \( y_k \) using the following transformation:

\[ y_k = \mu_k + V_kw_k; k = 1, \ldots, K, \quad (4) \]

where \( \mu_k \) is an \( n_k \)-dimensional real-valued vector called the “location vector,” \( V_k \) is an \( n_k \times n_k \) real-valued invertible “covariance parameter matrix,” the elements of the \( n_k \)-dimensional random vector \( w_k \) are mutually independent, and the \( i \)-th element of \( w_k \) is \( \text{DY}(\alpha_k, \kappa_k; \psi_k) \) with \( \alpha_{k,i}/\kappa_{k,i} \in \mathcal{Z}_k \) and shape/scale (depending on \( k \)) \( \kappa_{k,i} > 0 \), respectively. Straightforward change-of-variables of the transformation in (4) yields the following expression for the pdf of \( y_k \):

\[ f(y_k|\mu_k, V_k, \alpha_k, \kappa_k) = \det(V_k^{-1}) \left\{ \prod_{i=1}^n \mathcal{N}_k(\kappa_{k,i}, \alpha_{k,i}) \right\} \exp \left[ \alpha_k^t V_k^{-1}(y_k - \mu_k) - \kappa_k^t \psi_k \{ V_k^{-1}(y_k - \mu_k) \} \right] , \quad (5) \]

where the \( j \)-th element of \( \psi_k \{ V_k^{-1}(y_k - \mu_k) \} \) contains \( \psi_k \) evaluated at the \( j \)-th element of the \( n_k \)-dimensional vector \( V_k^{-1}(y_k - \mu_k) \), “det” denotes the determinant function, \( \alpha_k \equiv (\alpha_{k,1}, \ldots, \alpha_{k,n_k})^t \), and \( \kappa_k \equiv (\kappa_{k,1}, \ldots, \kappa_{k,n_k})^t \). The density in (5) is referred to as the CM distribution, and we use the shorthand CM(\( \alpha_k, \kappa_k, \mu_k, V_k, \psi_k \)).
3 The Generalized Conjugate Multivariate Distribution

We now propose the generalized conjugate multivariate (GCM) distribution. This development is similar to the development of the CM distribution from [Bradley et al. (2020)]. The difference between the GCM and CM is that the GCM drops the assumption of identical classes of DY random variables among the elements of $\mathbf{w}_k$ in (4), and marginalizes across a generic $d$-dimensional real-valued parameter vector $\mathbf{\theta}$. The GCM is needed for our main contribution of EPR in Section 4.

The GCM is defined by the transformation,

$$ y = \mathbf{\mu}_M + \mathbf{V}_M \mathbf{D}(\mathbf{\theta}) \mathbf{w}_M, \quad (6) $$

where the $N = \sum_{k=1}^K n_k$-dimensional random vector $\mathbf{y} = (y_1, \ldots, y_K)'$, $N$-dimensional random vector $\mathbf{w}_M = (w_1', \ldots, w_K')'$ with $(k,i)$-th element $w_{k,i} \sim \text{DY}(\alpha_{k,i}, \kappa_{k,i}; \psi_k)$, the subscript “M” stands for “Multi-type,” the $N \times N$ real-valued matrix $\mathbf{V}_M$ is an invertible covariance parameter matrix, and $\mathbf{\mu}_M$ is an unknown $N$-dimensional real-valued location parameter vector. Let $\mathbf{D} : \Omega \rightarrow \mathbb{R}^N \times \mathbb{R}^N$ be a known $N \times N$ matrix valued function, such that $\mathbf{D}(\mathbf{\theta})^{-1}$ exists for every $d$-dimensional $\mathbf{\theta} \in \Omega$ for a generic real-valued set $\Omega$. Let $\mathbf{\theta}$ be distributed according to the proper density $\pi(\mathbf{\theta})$, where $\mathbf{\theta}$ is independent of $\mathbf{\mu}_M$, $\mathbf{\alpha}_M$, $\mathbf{\kappa}_M$, and $\mathbf{V}_M$. Sampling from the marginal distribution $f(\mathbf{y}|\mathbf{\mu}_M, \mathbf{V}_M, \mathbf{\alpha}_M, \mathbf{\kappa}_M)$ (marginalizing across $\mathbf{\theta}$) is straightforward; namely, first sample $\mathbf{\theta}$ from $\pi(\mathbf{\theta})$ and then compute the transformation in (6) to produce a sample from $f(\mathbf{y}|\mathbf{\mu}_M, \mathbf{V}_M, \mathbf{\alpha}_M, \mathbf{\kappa}_M)$.

The pdf $f(\mathbf{y}|\mathbf{\mu}_M, \mathbf{V}_M, \mathbf{\alpha}_M, \mathbf{\kappa}_M)$ is stated in Theorem 3.1.

**Theorem 3.1.** Let $\mathbf{y}$ be defined as in (6). Then the pdf for $\mathbf{y}$ is given by,

$$ f(\mathbf{y}|\mathbf{\mu}_M, \mathbf{V}_M, \mathbf{\alpha}_M, \mathbf{\kappa}_M) = \int_{\Omega} \pi(\mathbf{\theta}) \left\{ \prod_{k=1}^K \prod_{i=1}^{n_k} N(k_i; \alpha_{k,i}, \kappa_{k,i}) \right\} \exp \left[ \mathbf{\alpha}_M^t \mathbf{D}(\mathbf{\theta})^{-1} \mathbf{V}_M^{-1} (\mathbf{y} - \mathbf{\mu}_M) - \mathbf{\kappa}_M^t \mathbf{\psi}_M \left\{ \mathbf{D}(\mathbf{\theta})^{-1} \mathbf{V}_M^{-1} (\mathbf{y} - \mathbf{\mu}_M) \right\} \right] d\mathbf{\theta}, \quad (7) $$

where $\mathbf{y} \in \mathcal{Y}$, $\mathcal{Y} = \{ \mathbf{y} : \mathbf{y} = \mathbf{\mu}_M + \mathbf{V}_M \mathbf{D}(\mathbf{\theta}) \mathbf{c}, \mathbf{c} = (c_{k,i}), c_{k,i} \in \mathcal{Y}_k, \mathbf{\theta} \in \Omega, i = 1, \ldots, n_k, k = 1, \ldots, K \}$, $\alpha_{k,i}/\kappa_{k,i} > 0$, $\mathbf{\psi}_M \left\{ \mathbf{V}_M (\mathbf{y} - \mathbf{\mu}_M) \right\} = (\psi_1 \{ J_1 \mathbf{V}_M (\mathbf{y} - \mathbf{\mu}_M) \}', \ldots, \psi_K \{ J_K \mathbf{V}_M (\mathbf{y} - \mathbf{\mu}_M) \}')'$, the $n_k \times N$ matrix $J_k = \left( \mathbf{0}_{n_k \times 
_k}, \mathbf{I}_{n_k}, \mathbf{0}_{n_k \times \n_k} \right)$, $\mathbf{0}_{n \times m}$ is an $n \times m$ matrix of zeros, $\mathbf{I}_{n_k}$ is an $n_k \times n_k$ identity matrix, the $N$-dimensional vector $\mathbf{\alpha}_M = (\mathbf{\alpha}_1', \ldots, \mathbf{\alpha}_K')'$, and the $N$-dimensional vector $\mathbf{\kappa}_M = (\mathbf{\kappa}_1', \ldots, \mathbf{\kappa}_K')'$.

**Proof:** See Appendix A.

We use the shorthand $\text{GCM}(\mathbf{\alpha}_M, \mathbf{\kappa}_M, \mathbf{\mu}_M, \mathbf{V}_M, \pi, \mathbf{D}; \mathbf{\psi}_M)$ for the density in (7). Our class of GLMMs will imply a GCM for the posterior predictive distribution for fixed effects, random effects, and bypass location parameters. This form arises since the data model in a Bayesian hierarchical model has a similar form to that of a possibly non-Gaussian CM distribution, and when stacked over certain CM priors (e.g., Gaussian priors), we obtain the form of a GCM. The GCM
allows one the flexibility to specify any CM prior (e.g., Gaussian, multivariate log-gamma, multivariate logit-beta, etc.) for fixed and random effects. It also allows you to specify the data model to be DY, which is not necessarily part of the exponential family. For example, one might consider $Z_{k,i} Y_{k,i}$ to be distributed as Student-$t$, which is not part of the exponential family, but is part of the DY family. Moreover, integrating across $\theta$ will allow one to marginalize across hyperparameters for posterior inference on fixed and random effects.

We now provide the expression for the conditional GCM in Theorem 3.2

**Theorem 3.2.** Let $y = (y^{(1)}, y^{(2)})' \sim \text{GCM}(\alpha_M, \kappa_M, \mu_M, V_M, \pi, D; \psi_M)$, where $y^{(1)}$ is $r$-dimensional and $y^{(2)}$ is $(N - r)$-dimensional. Also, let $V_M^{-1} = (H, Q)$, where $H$ is a $N \times r$ and $Q$ is $N \times (N - r)$. Then, it follows

$$f(y^{(1)}|y^{(2)}, \mu_M, V_M, \alpha_M, \kappa_M) \propto \int_{\Omega} \pi(\theta) \exp \left[ \alpha_M^T D(\theta)^{-1} H y^{(1)} - \alpha_M^T \mu_M^* - \kappa_M^T \psi_M \left( D(\theta)^{-1} H y^{(1)} - \mu_M^* \right) \right] d\theta,$$

where $\mu_M^* = D(\theta)^{-1} V_M^{-1} \mu_M - D(\theta)^{-1} Q y^{(2)}$.

**Proof:** See Appendix A.

We use the shorthand cGCM($\alpha_M, \kappa_M, \mu_M^*, H, \pi, D; \psi_M$) for the conditional GCM in Theorem 3.2. It is not known how to simulate directly from a cGCM.

### 4 Exact Posterior Regression

In this section, we outline how to sample from the posterior distribution of fixed and random effects from a general class of GLMMs. We define EPR in Section 4.1, address computational issues in Sections 4.2, and outline implementation in Section 4.3.

#### 4.1 Generalized Linear Mixed Effects Models

Suppose we observe $K$ different types of data from the exponential family, and we observe multiple observations of each data type. Let the total number of observations of type $k$ be denoted with $n_k$. Denote the $n_k$-dimensional data vectors with $z_k = (Z_{k,1}, \ldots, Z_{k,n_k})'$. Each of which depend on a latent unobserved vector $y_k = (Y_{k,1}, \ldots, Y_{k,n_k})'$. Then $Z_{k,i}|Y_{k,i}$ is distributed independently according to (1) for $k = 1, 2, 3, 4$. When $k = 1$ we assume $Z_{1,i}|Y_{1,i}, \sigma_i^2$ is Gaussian distributed with mean $Y_{1,i}$ and variance $\sigma_i^2 > 0$. Additionally, we assume $Z_{2,i}|Y_{2,i}$ is Poisson distributed with mean $\exp(Y_{2,i})$, $Z_{3,i}|Y_{3,i}, m_i$ is binomial distributed with sample size $m_i$ and probability of success $\exp(Y_{3,i})/(1 + \exp(Y_{3,i}))$, the $n_3$-dimensional vector $m = (m_1, \ldots, m_{n_3})'$, and consider a shifted Student-$t$ specification $Z_{4,i}|Y_{4,i} \sim \text{CM}(0, \psi^{-1} V_{4,i}, 1)$ with $\nu > 0$. Binomial distributed data allows for Bernoulli distributed data as a special case (i.e., $m_i = 1$), and multinomial distributed data when
using a stick-breaking representation of the multinomial (e.g., see [Bradley et al., 2019] for stick-breaking in the context of CM prior distributions). It is straightforward to extend our setup to a process model setting. We give these details in Appendix B.

Consider the following linear model assumption for the \( n_k \)-dimensional random vector \( y_k \) (Mc-Cullagh and Nelder, 1989):

\[
y_k = X_k \beta + G_k \eta + (\xi_k - \mu_k); \quad k = 1, \ldots, K,
\]

where we consider \( K = 4 \), \( X_k \) is a \( n_k \times p \) matrix of known covariates, and \( \beta \) is an unknown \( p \)-dimensional vector of regression coefficients. Let \( \beta \) have a CM prior with location vector \( \mu_\beta \), shape vector \( \alpha_\beta \), scale vector \( \kappa_\beta \), and \( p \times p \) covariance parameter matrix \( D_\beta(\theta) : \Omega \to \mathbb{R}^p \times \mathbb{R}^p \). Let \( G_k \) be a \( n_k \times r \) matrix of coefficients for the \( r \)-dimensional random effects \( \eta \). In this article, \( G_k \) will be set equal to a known pre-specified matrix of basis functions (e.g., splines [Wahba, 1990], wavelets [Novikov et al., 2005], Moran’s I basis functions [Hughes and Haran, 2013], etc.), the \( n_k \times n_k \) identity matrix \( I_{n_k} \), or an incidence matrix. We assume \( \eta \) is CM with location vector \( \mu_\eta \), shape vector \( \alpha_\eta \), scale vector \( \kappa_\eta \), and \( r \times r \) covariance parameter matrix \( D_\eta(\theta) : \Omega \to \mathbb{R}^r \times \mathbb{R}^r \). Let \( \beta \) be a generic \( d \)-dimensional parameter vector with prior distribution \( \pi(\theta) \).

Stack the \( n_k \)-dimensional random vectors \( \xi_k \) into a \( N \)-dimensional vector \( \xi = (\xi_1', \ldots, \xi_K')' \). Let the prior on \( \xi \) be proportional to a cGCM(\( \alpha_\xi, \kappa_\xi, \mu_\xi^\ast, H_\xi, \pi_\xi, D_\xi; \psi_\xi \)), where the \( 2N \)-dimensional location parameter \( \mu_\xi^\ast = (\mu_D' - \beta'X' - \eta'G', \mu_\xi')' \), \( \mu_\xi \) is a \( N \)-dimensional real-vector, the \( N \)-dimensional vector \( \mu_D = (\mu_1', \ldots, \mu_K')' \), the \( N \times p \) matrix \( X = (X_1', \ldots, X_K')' \), and \( 2N \times N \) matrix \( G = (X_1', \ldots, X_K')' \). The shape parameter for the prior on \( \xi \) is set to \( \alpha_\xi = (\alpha_{\xi 1, n_1}, \alpha_{\xi 1, n_2}, \alpha_{\xi 1, n_3}, \alpha_{\xi 1, N+n_4})' \) and the scale/shape parameter is set to \( \kappa_\xi = (0_{1,n_1}, 0_{1,n_2}, 2\alpha_{\xi 1, n_3}, 0_{1,n_4}, \frac{1}{2}1_{1,N})' \), where \( 1_{r,N} \) is a \( r \times N \) matrix of ones. Let \( \pi_\xi(\{\sigma_\xi^2, \alpha_\xi \}) \) be a point mass on pre-specified “small” values for \( \sigma_\xi^2 > 0 \) and \( \alpha_\xi > 0 \), and let \( D_\xi \equiv I_{2N} \). The unit-log partition function \( \psi_\xi(h) \) is,

\[
\psi_\xi(h) = (\psi_1(h_1), \ldots, \psi_1(h_{n_1}), \ldots, \psi_K(h_{n_K-n_{K-1}}), \ldots, \psi_K(h_N), \psi_1(h_1'), \ldots, \psi_1(h_{n_1}'))',
\]

for any \( h = (h_1, \ldots, h_{n_K}, h_1', \ldots, h_{n_1}')' \in \mathbb{R}^{2N} \). It is straightforward to verify that when \( \alpha_\xi = 0 \) we have that cGCM(\( \alpha_\xi, \kappa_\xi, \mu_\xi^\ast, H_\xi, \pi_\xi, D_\xi; \psi_\xi \)) is proportional to a Gaussian distribution with mean zero and covariance \( \sigma_\xi^2 I_N \). This choice of cGCM with \( \alpha_\xi > 0 \) will ensure that the implied posterior distribution has parameters that do not lie on the boundary of the parameter space.

Stack the location parameters into a \((2N + p + r)\)-dimensional vector \( \mu = (\mu_1', \ldots, \mu'_K, \mu_\beta', \mu_\eta', \mu_\xi')' \). To obtain a posterior distribution that is GCM, \( \mu \) needs to have a particular form. Specifically, let \( \mu = -D(\theta)^{-1}Qq \), where \( Q \) are the \((2N + p + r) \times N \) eigenvectors of the orthogonal
complement of the \((2N + p + r) \times (N + p + r)\) matrix,

\[
H = \begin{pmatrix}
I_N & X & G \\
0_{p,N} & I_p & 0_{p,r} \\
0_{r,N} & 0_{r,p} & I_r \\
I_N & 0_{N,p} & 0_{N,r}
\end{pmatrix},
\]

so that \(QQ' = I_{2N+p+r} - H(H' H)^{-1} H'\) and \(H'Q = 0_{N+p+r,N}\), where recall that idempotent matrices have eigenvalues equal to zero or one. Let \(D(\theta)^{-1} = \text{blkdiag}(I_N, D_\beta(\theta)^{-1}, D_\eta(\theta)^{-1}, \frac{1}{\sigma^2} I_N)\), where “blkdiag” be the block diagonal operator. We refer to \(q\) as the “bypass location parameters,” which are assumed unknown. Several GLMMs in the literature set \(q = 0_{N,1}\). However, if one instead assumes an improper prior on \(q\) then the posterior distribution of \(\zeta = (\xi', \beta', \eta')'\) and \(q\) is GCM, as seen in Theorem 4.1. In Appendix C, we derive the posterior distribution when these location parameters are set equal zero, which can not be simulated from directly; hence the name bypass location parameters.

**Theorem 4.1.** Suppose \(Z_{k,i} | Y_{k,i}, \theta_D\) is independently distributed according to either (1) or (2) with \(i = 1, \ldots, n_k\), and \(k = 1, \ldots, 4\). Let \(\theta_D \subset \theta\) be all data model parameters (e.g., \(\theta_D\) contains \(\{\sigma^2\}\) and \(v\)). Let \(j\) index the choice of GCM prior for \(\beta | \alpha, \kappa, \mu, \sigma, D(\theta)\), and let \(b\) index choice of GCM prior for \(\eta | \alpha, \kappa, \mu, \sigma, D(\theta)\) and let \(\xi\) have density proportional to \(c_{\text{GCM}}(\alpha_{\xi}, \kappa_{\xi}, \mu_{\xi}, \sigma_{\xi}, D_{\xi}, \Psi_{\xi})\). Let \(y_k\) be defined as in (8). \(\mu = -D(\theta)^{-1} Q q, D(\theta)^{-1} = \text{blkdiag}(I_N, D_\beta(\theta)^{-1}, D_\eta(\theta)^{-1}, \frac{1}{\sigma^2} I_N)\), assume the improper prior \(f(q) = 1\), and let \(\theta\) have the proper prior \(\pi(\theta)\). Then

\[
(\zeta', q')_{\mid z} \sim \text{GCM}(\alpha_M, \kappa_M, \mu_M, V_M, \pi, D; \Psi_M),
\]

where \(V_M^{-1} = (H, Q)\) is defined by (9), and the \((2N + p + r)\)-dimensional location and shape/scale parameter vectors,

\[
\begin{align*}
\mu_M &= \begin{pmatrix} (H'H)^{-1} H' \\ Q' \end{pmatrix} \\
\alpha_M &= (1_{N + p + r, 1} D'_{\sigma}, 1_{N + p + r, 1})' \\
\kappa_M &= \begin{pmatrix} 1_{N + p + r, 1} D'_{\sigma}, 1_{N + p + r, 1} m' + 2 \alpha_{\xi} \sigma_{\xi}, \frac{1}{2} \sigma_{\xi} \sigma_{\xi} \\
\end{pmatrix}
\end{align*}
\]
where \( D_\sigma = \text{diag} \left( \frac{1}{\sigma_i^2} : i = 1, \ldots, n_1 \right) \) and the \((2N + p + r)\)-dimensional unit-log partition function \\
\[ \psi_M(H) = (\psi_1(h_1), \ldots, \psi_1(h_{n_1}), \psi_2(h_{n_1+1}), \ldots, \psi_2(h_{n_1+n_2}), \ldots, \psi_K(h_N), \psi_{j,1}(h_{j,1}), \ldots, \psi_{j,p}(h_{j,p}), \]
\[ \psi_{b,1}(h_{b,1}), \ldots, \psi_{b,r}(h_{b,r}), \psi_1(h_1), \ldots, \psi_1(h_N) \)' \]
for \((2N + p + r)\)-dimensional real-valued vector \( H = (h_1, \ldots, h_N, h_{j,1}, \ldots, h_{j,p}, h_{b,1}, \ldots, h_{b,r}, h_1, \ldots, h_N)' \).

Proof: See Appendix A.

It is straightforward to adjust Theorem 4.1 when not all four data types are observed. For example, suppose we only observe Poisson distributed data and binomial distributed data with CM priors for \( \beta \) and \( \eta \). Then the posterior distribution is GCM with \( \mu_M = 0_{2n_2+2n_3+p+r}, \alpha_M = (z_2' + \alpha_z^1 1_{1,n_2}, z_3' + \alpha_z^2 1_{1,n_3}, \alpha_\beta^1 \eta_1, 0_{1,n_2+n_3})' \), \( \kappa_M = (1_{1,n_2}, m' + 2\alpha_\xi^2 1_{1,n_3}, \kappa_\beta^1, \kappa_\eta^1, \frac{1}{2} 1_{1,n_2+n_3})' \), \\
\[ \psi_M(H) = (\psi_2(h_1), \ldots, \psi_2(h_{n_2}), \psi_3(h_{n_2+1}), \ldots, \psi_3(h_{n_2+n_3}), \psi_j(h_{j,1}), \ldots, \psi_j(h_{j,p}), \]
\[ \psi_b(h_{b,1}), \ldots, \psi_b(h_{b,r}), \psi_1(h_1), \ldots, \psi_1(h_{n_2+n_3}) \)' \\
with \( H = (h_1, \ldots, h_{n_2+n_3}, h_{j,1}, \ldots, h_{j,p}, h_{b,1}, h_{b,r}, h_1, \ldots, h_{n_2+n_3})' \in \mathbb{R}^{2n_2+2n_3+p+r} \), the first block row of \( H \) is replaced with \((I_{n_2+n_3}, (X_2', X_3')', (G_2', G_3')') \), and \( Q \) are the eigenvectors of the orthogonal complement of this adjusted expression of \( H \).

In Theorem 4.1 the presence of \( \alpha_z^2 > 0 \) makes \( \alpha_M \) and \( \kappa_M \) strictly positive when elements of the Poisson and binomial data vectors \( z_2 \) and \( z_3 \) are zero. Hence, the presence of a fine-scale term allows one to avoid the boundaries of the parameter space, leading to a well-defined GCM. Theorem 4.1 allows one to obtain replicates directly from the posterior distribution \( f(\zeta, q | z) \) using a familiar projection expression, as seen below in Theorem 4.2.

Theorem 4.2. Denote a replicate of \( \zeta, q, \) and \( y \) using \( f(\zeta, q | z) \) from Theorem 4.1 with \( \zeta_{rep}, q_{rep}, \) and \( y_{rep} \). Then \\
\[ \zeta_{rep} = (H'H)^{-1}H'w \]  
(10) \\
\[ q_{rep} = Q'w, \]  
(11) \\
\[ y_{rep} = (I_N, 0_{N,N+p+r})H\zeta_{rep} + (I_N, 0_{N,N+p+r})Qq_{rep} = (I_N, 0_{N,N+p+r})w \]  
(12)
where the \((2N + p + r)\)-dimensional random vector \( w \) is \( \text{GCM}(\alpha_M, \kappa_M, \mu_M, I_{2N+p+r}, \pi, D; \psi_M) \), where \( \alpha_M, \kappa_M, \mu_M, \pi, D, \) and \( \psi_M \) are the same as defined in Theorem 4.1.

Proof: See Appendix A.

The vector \((2N + p + r)\)-dimensional vector \( w \equiv D(\theta)w_M = (y_{rep}', w_\beta', w_\eta', w_\xi')' \), where \( y_{rep} \) is easy to generate since it consists of independent DY random variables, \( w_\beta \sim f(\beta | \alpha_\beta, \kappa_\beta, \mu_\beta = 0_{p,1}) \)
is $p$-dimensional, $\mathbf{w}_\eta \sim f(\mathbf{\eta} | \mathbf{\alpha}_\beta, \mathbf{\kappa}_\beta, \mathbf{\mu}_\eta = \mathbf{0}_{r,1})$ is $r$-dimensional, and $\mathbf{w}_\xi$ is $N$-dimensional consisting of independent Gaussian random variables with mean zero and variance $\sigma^2_\xi$. Note that $\mathbf{w}_\beta$ and $\mathbf{w}_\eta$ are simply samples from the respective marginal prior distributions for $\mathbf{\beta}$ and $\mathbf{\eta}$ after marginalizing across $\mathbf{\theta}$ and with location vector zero. Thus, it is straightforward to compute $\mathbf{w}$ when it is straightforward to sample from the marginal prior distributions for $\mathbf{\beta}$ and $\mathbf{\eta}$. To do this one can, for example, sample from the joint distribution of $\mathbf{\beta}$ and $\mathbf{\theta}$, where first one samples $\mathbf{\theta}$ from $\pi$ then samples from $f(\mathbf{\beta} | \mathbf{\alpha}_\beta, \mathbf{\kappa}_\beta, \mathbf{\mu}_\beta = \mathbf{0}_{p,1}, \mathbf{D}_\beta(\mathbf{\theta}))$. The projection $(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\mathbf{w}$ can be computed on the order of $N + p^3 + r^3$ operations with storage on the order of $N(p + r) + p^2 + r^2$, when $\mathbf{G}$ is dense. When $\mathbf{G}$ is identity with $r = N$, $(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\mathbf{w}$ can be computed on the order of $p^3$ operations with storage on the order of $Np + p^2$. For the details on computing $(\mathbf{H}'\mathbf{H})^{-1}\mathbf{H}'\mathbf{w}$ see Section 4.2.

Remark 1: Theorem 4.2 provides the motivation for including the location parameter $\mathbf{q}$. Namely, this location parameter leads to easy-to-compute direct simulations from the posterior distribution. However, the incorporation of $\mathbf{q}$ leads to a model that is clearly overparameterized. Thus, a simple solution is to perform inference on $\mathbf{\zeta}$ using exact replicates from (10), which generates values from the marginal distribution $f(\mathbf{\zeta} | \mathbf{z})$. Then use the estimator of $\mathbf{q} = \mathbf{0}_{N,1}$. This is the general strategy used in the CM literature (Bradley et al., 2020) implemented using a block Gibbs sampler. Let $\hat{\mathbf{y}}$ represent the profile of $\mathbf{y} = (y_1, \ldots, y_K)'$ using the plug-in estimator $\mathbf{q} = \mathbf{0}_{N,1}$, so that $\hat{\mathbf{y}}_{\text{rep}} = (\mathbf{I}_N, \mathbf{0}_{N,N+p+r})\mathbf{H}\mathbf{\zeta}_{\text{rep}} = \mathbf{X}\mathbf{\beta}_{\text{rep}} + \mathbf{G}\mathbf{\eta}_{\text{rep}} + \mathbf{\xi}_{\text{rep}}$, where $\mathbf{\zeta}_{\text{rep}} = (\mathbf{\zeta}'_{\text{rep}}, \mathbf{\beta}'_{\text{rep}}, \mathbf{\eta}'_{\text{rep}})'$. Moreover, one might similarly use $\hat{\mathbf{y}}_{\text{rep}} = \mathbf{X}\mathbf{\beta}_{\text{rep}} + \mathbf{G}\mathbf{\eta}_{\text{rep}}$ for inference on $\mathbf{y}$, which would implicitly estimate both $\mathbf{q}$ and $\mathbf{\xi}$ to be zero after marginalizing them from the posterior distribution.

Remark 2: The random vector $\mathbf{y}_{\text{rep}}$ has a very important interpretation. If one assumes $Z_{ik} | Y_{ik}$ is distributed according to the exponential family in (11), and $Y_{ik}$ is independently distributed according to the DY distribution in (2) then we have that the implied posterior distribution for $\{Y_{ik}\}$ in (3) is equal in distribution to $\mathbf{y}_{\text{rep}}$ in Theorem 4.2. Thus, $\mathbf{y}_{\text{rep}}$ represents a replicate from the posterior distribution from a saturated model. Recall in the goodness-of-fit literature that saturated models define a separate parameter for each datum and is meant to overfit the data, and then, measures of deviance from the saturated model are used to select more parsimonious models (e.g., see Bradley, 2022 for a recent paper). This provides additional motivation for using the marginal distribution $f(\mathbf{\zeta} | \mathbf{z})$ and $\hat{\mathbf{y}}_{\text{rep}}$ (or $\tilde{\mathbf{y}}_{\text{rep}}$) to perform inference on $\mathbf{y}$, which implies the use of the estimator of $\mathbf{q} = \mathbf{0}_{N,1}$ (and $\mathbf{\xi} = \mathbf{0}_{N,1}$).

4.2 Computational Considerations

For large $N$ the EPR formulation may not look practically feasible. However, standard block matrix inversion techniques can be used to reduce the order of operations to inverses of $r \times r$ matrices, $p \times p$ matrices, and $N \times N$ diagonal matrices (Lu and Shiou, 2002).
Theorem 4.3. The following expression holds,

\[
(H'H)^{-1} = \begin{pmatrix}
A^{-1} + A^{-1}B(D - B'A^{-1}B)^{-1}B'A^{-1} & -A^{-1}B(D - B'A^{-1}B)^{-1} \\
-(D - B'A^{-1}B)^{-1}B'A^{-1} & (D - B'A^{-1}B)^{-1}
\end{pmatrix},
\]

(13)

where \( A = 2I_N \), the \( N \times (p + r) \) matrix \( B = (X, G) \), the \( (p + r) \times (p + r) \) matrix

\[
D = \begin{pmatrix}
X'X + I_p & X'G \\
G'X & G'G + I_r
\end{pmatrix},
\]

(14)

the \( (p + r) \times (p + r) \) matrix

\[
(D - B'A^{-1}B)^{-1} = \begin{pmatrix}
A^{*-1} + A^{*-1}B(D^* - C^*A^{*-1}B^*)^{-1}C^*A^{*-1} & -A^{*-1}B(D^* - C^*A^{*-1}B^*)^{-1} \\
-(D^* - C^*A^{*-1}B^*)^{-1}C^*A^{*-1} & (D^* - C^*A^{*-1}B^*)^{-1}
\end{pmatrix},
\]

the \( p \times p \) matrix \( A^* = \frac{1}{2}X'X + I_p \), the \( p \times r \) matrix \( B^* = \frac{1}{2}X'G \), the \( r \times p \) matrix \( C^* = \frac{1}{2}G'X \), and the \( r \times r \) matrix \( D^* = \frac{1}{2}G'G + I_r \).

Proof: See Appendix A.

Theorem 4.3 allows us to reduce the inverse of the \( (N + p + r) \times (N + p + r) \) matrix \( H'H \) to the inverse of the \( p \times p \) matrix \( A^* \), and the \( r \times r \) matrix \( (D^* - C^*A^{*-1}B^*)^{-1} \). When \( p \) and \( r \) are both “small,” these inverses are computationally efficient.

Simulation from the posterior using EPR does not necessarily require first computing a matrix of the form \( (H'H)^{-1} \), storing this matrix, and then computing a \( (N + p + r) \)-dimensional vector of the form \( (H'H)^{-1}H'w \). In fact this order of operations may require impractical storage, since the \( (N + p + r) \times (N + p + r) \) matrix \( (H'H)^{-1} \) may be high-dimensional. To avoid these issues one can instead compute/store the \( (N + p + r) \)-dimensional vector of the form \( (H'H)^{-1}H'w \) that avoids storage of high-dimensional matrices.

Theorem 4.4. Let \( w = (w_\beta', w_\eta', w_q', w_e', w_\beta, w_\eta, w_q)' \), \( w_e \in \mathbb{R}^N \), \( w_\beta \in \mathbb{R}^p \), \( w_\eta \in \mathbb{R}^r \), and \( w_q \in \mathbb{R}^q \). Then the following expression holds,

\[
(H'H)^{-1}H'w = \begin{pmatrix}
(F - KL^{-1}K')^{-1}(R - KL^{-1}P) \\
-L^{-1}K'(F - KL^{-1}K')^{-1}(R - KL^{-1}P) + L^{-1}P
\end{pmatrix},
\]

(15)

where the \( (N + p) \)-dimensional vector \( R = (w_\beta' + w_q', w_e' + w_\beta)' \), the \( r \)-dimensional vector \( P = G'w_e + w_\eta \), the \( r \times (N + p) \) matrix \( K' = (G', G'X) \), the \( r \times r \) matrix \( L = G'G + I_r \), the \( (N + p) \times
\[(N + p) \text{ matrix } F = \begin{pmatrix} 2I_N & X \\ X' & X'X + I_p \end{pmatrix}, \text{ and the } (N + p) \times (N + p) \text{ matrix} \]
\[F - KL^{-1}K' = \begin{pmatrix} F_1 & B_{12} \\ B'_{12} & F_2 \end{pmatrix}. \]

The \((N + p) \times (N + p) \) matrix,
\[(F - KL^{-1}K')^{-1} = \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}, \]

where the \(N \times N \) matrix \(F_1 = 2I_N - GL^{-1}G'\), the \(N \times p \) matrix \(B_{12} = X - GL^{-1}G'X\), the \(p \times p \) matrix \(F_2 = X'X + I_p - X'GL^{-1}G'X\), the \(N \times N \) matrix \(F_{11} = F_1^{-1} + F_1^{-1}B_{12}(F_2 - B'_{12}F_1^{-1}B_{12})^{-1}B'_{12}F_1^{-1}, \)
the \(N \times p \) matrix \(F_{12} = F_1^{-1}B_{12}(F_2 - B'_{12}F_1^{-1}B_{12})^{-1}, \)
the \(p \times N \) matrix \(F_{21} = (F_2 - B'_{12}F_1^{-1}B_{12})^{-1}B'_{12}F_1^{-1}, \)
the \(p \times p \) matrix \(F_{22} = (F_2 - B'_{12}F_1^{-1}B_{12})^{-1}, \) and the \(N \times N \) matrix \(F_1^{-1} = \frac{1}{2}I_N + \frac{1}{4}G(L - \frac{1}{2}G'G)^{-1}G'. \)

Proof: See the Appendix.

Careful examination of the order of operations show that Theorem (4.4) allows one to compute the vector \((H'H)^{-1}H'w\) by storing/computing the \(N \times p \) matrix \(X\), the \(N \times r \) matrix \(G\) (when \(r = N\) we set \(G = I_N\)), the \(r \times r \) matrix \(L^{-1}\), the \(r \times r \) matrix \((L - \frac{1}{2}G'G)^{-1}\), the \(p \times p \) matrix \(F_2\), and the \(p \times p \) matrix \(F_{22}\). These computations are straightforward when \(r\) and \(p\) are “small” or when \(p\) is small and \(G\) is diagonal.

### 4.3 Implementation of EPR

The following gives step-by-step instructions on obtaining efficient independent replicates directly from the posterior distribution of fixed effects and random effects using Theorem (4.1), which we refer to as EPR. We consider Gaussian data with unknown non-constant variance, Poisson data, binomial data, and Gaussian priors on \(\beta\) and \(\eta\). Minor adjustments to these steps are needed for other settings (e.g., jointly modeling just two data types, etc.).

1. Store/compute the \(N \times p \) matrix \(X\), the \(N \times r \) matrix \(G\), the \(r \times r \) matrix \(L^{-1}\), the \(r \times r \) matrix \((L - \frac{1}{2}G'G)^{-1}\), the \(p \times p \) matrix \(F_2\), and the \(p \times p \) matrix \(F_{22}\).

2. Simulate \(w\) according to Theorem (4.2).

3. Use the matrices computed in Step 1 and \(w\) in Step 2 to compute \(\zeta_{rep}, y_{rep}, \hat{y}_{rep}\), and \(\tilde{y}_{rep}\) according to Theorem (4.4).

4. Repeat Steps 2–3 \(B\) times.
Figure 1: In the first row we plot a single replicated dataset. The red line in the first row represents the true mean. The second row contains a plot of \( y_{true,k} \) (red) and the posterior mean of \( \tilde{y}_k \) (blue). In the second row, the gray lines represent the posterior mean of \( \hat{y}_k \), which is left out when \( k = 1 \) for visualization purposes.

Repeated matrix operations that one might see in a Gibbs sampler are avoided, since matrix inversions are only required a single time in Step 1. Additionally, \( B \) does not have to be as large as what one requires for an MCMC, since one does not require a burn-in period, thinning, or have concerns of mixing and positive autocorrelations in the MCMC.

5 Illustrations

We provide several illustrations to: (1) investigate which specification of EPR is generally preferable (Section 5.1), and (2) compare to existing choices in the literature (Sections 5.2 and 5.3). We evaluate both estimation of \( \beta \) and prediction of \( y \). All R code is provided on the Github page, https://github.com/JonathanBradley28/CM.

5.1 Simulations: EPR Specification

The goal of Section 5.1 is to provide guidance on which specification of EPR to use in practice. In particular, to perform inference on \( y \), one can use posterior summaries of \( y, \hat{y}, \) and \( \tilde{y} \). Thus, we investigate both the estimation performance of \( \beta \) and the predictive performance of \( y \) using posterior summaries of \( y, \hat{y}, \) and \( \tilde{y} \).
Let $K = 3$ and $n_1 = n_2 = n_3 = 5000$. We specify a $5000 \times 3$ matrix $M$ to have a column of ones, the second column consist of an equally spaced sequence from zero to one, and the third column consisting of the element-wise square of the second column. Set $X = \text{blkdiag}(M, M, M)$. This sets up a simple quadratic expression for the mean, where there are different intercepts, linear effects, and quadratic effects per response type. Let the $5000 \times 100$ matrix $G$ consists of exponential radial basis functions, where the $(i, j)$-th element is given by $\exp\{-\text{abs}(i - j)\}$, where “abs” is the absolute value operator. Let the $15000 \times 100$ matrix $G = \{I_{15000} - X(X'X)^{-1}X'\} (M'_G, M'_G, M'_G)'$, where the orthogonal complement of $X$ is introduced to avoid confounding. The index $i$ can be interpreted as a time index. We arbitrarily fix $\beta_{true} = (9, -3, 3, -0.2, -1, 2, 2.6, -0.5, 2, 2.6, -0.5, 2)'$ and $\eta_{true}$ is generated from a mean zero normal with variance equal to that implied by $X \beta_{true}$. This implies that the true value of $y$ in this simulation is $y_{true} = (y'_{true,1}, y'_{true,2}, y'_{true,3})' = X \beta_{true} + G_{true} \eta_{true}$. Normal data are generated with constant variance $\sigma^2_{true} = 1$. For binomial data we set the $i$-th element of $m$ to be a sample from a Poisson distribution with mean 20. Figure 1 contains a plot of simulated replicate according to this design.

To evaluate the performance of EPR we consider the coverage of 95% joint credible intervals over independently replicated datasets, the root mean squared prediction error (RMSPE), and the root mean squared error (RMSE), where

$$\text{RMSPE} \{E(\tilde{y}_k|z)\} = \left[\frac{1}{5000} \{y_{true,k} - E(\tilde{y}_k|z)\}' \{y_{true,k} - E(\tilde{y}_k|z)\}\right]^{1/2} ; k = 1, 2, 3,$$

$$\text{RMSE}_\beta = \left[\frac{1}{9} \{\beta_{true} - E(\beta|z)\}' \{\beta_{true} - E(\beta|z)\}\right]^{1/2},$$

$\tilde{y}_k = X_k \beta + G_k \eta$, and we consider RMSPE $\{E(\tilde{y}_k|z)\}$, RMSPE $\{E(\tilde{y}_k|z)\}$, and RMSPE $\{E(Y_k|z)\}$. Small values of RMSPE$_k$ and RMSE$_\beta$ are preferable. Coverage at least as big as 95% is preferable, where recall that 95% credible intervals are interpreted differently from confidence intervals so that
there is no guarantee that this holds. The credible intervals are jointly 95%, and are of the form $E(\beta_i | z) \pm C \text{var}(\beta_i | z)^{1/2}$, where $C$ is chosen so that at least 95% of the replicates of $\beta$ are all contained within these intervals. Since we have independent replicates the effective sample size for our method is $B$ as there are no autocorrelations induced by a Markov chain. Moreover, there is no need for any additional computational overhead/subjectivity of examining trace plots and computing MCMC diagnostics.

We implement EPR according to the steps outlined in Section 4.3 with Student-t prior specifications for $\beta$ and $\eta$ with shape and scale parameter set equal to 2. In Figure 1, we present a plot of $y_{true,k}$ and $E(\tilde{y}_k | z)$ for one simulated replicate, along with $E(\hat{y}_k | z)$. In general, the predictions are fairly close to the truth in this example, and there is considerably more noise in the estimates based on $\hat{y}_k$ instead of $\tilde{y}_k$. In Figure 2, we plot $E(\hat{\beta}_i - \beta_{true,i} | z)$, which are close to zero. Moreover, the credible intervals for $\hat{\beta}_i - \beta_{true,i}$ all contain zero, indicating reasonable estimates.

In Table 1 we present average (over 100 independent replicates of $z$) RMSE$_k$. In general, $E(\tilde{y}_k | z)$ produced smaller RMSPE than $E(\hat{y}_k | z)$ followed by $E(y_k | z)$ for this simulation. The coverage rate of $\beta_{true,i}$ and $y_{true,i}$ (using credible intervals for either $\tilde{y}_k$, $\hat{y}_k$, or $y_k$) over the 100 simulations were all one.

### 5.2 Comparison to Standard Approaches in Bayesian GLMMs

The goal of this section is to compare EPR to standard and current approaches in Bayesian GLMMs. In particular, we compare EPR to,

- HMC for logistic regression and Poisson regression implemented using the R package \texttt{rstan} (Neal, 2011; Carpenter et al., 2017),
- INLA for logistic regression and Poisson regression implemented using the R package \texttt{inla} (Rue et al., 2009; R-INLA Development Team, 2022),
- the data augmentation strategy from Albert and Chib (1993) for logistic regression implemented using R code,
- and the data augmentation strategy from Polson et al. (2013) for logistic regression using R code.
We simulate Poisson distributed data and Bernoulli distributed data as follows,

\[
Z_{i,2} \sim \text{Poisson}\left\{ \exp\left(0.5 x_{i,2} + 0.1 g_{i,2}\right)\right\}
\]

\[
Z_{i,3} \sim \text{Bernoulli}\left\{ \frac{\exp\left(-5 x_{i,3} + 0.5 g_{i,3}\right)}{1 + \exp\left(-5 x_{i,3} + 0.5 g_{i,3}\right)} \right\},
\]

for \( i = 1, \ldots, 100, \ x_{i,2} \equiv x_{i,3} \equiv 1, \) and \( \{g_{i,k}\} \) generated from a uniform distribution over the interval 1 to 20. HMC is implemented with 3 chains of size 1000 and a burn-in of 500, and the data augmentation strategies are implemented using a chain of size 3000 and a burn-in of 1000. EPR is implemented with \( B = 500 \) and predictions are done with \( \tilde{y}_k \) on the inverse-link scale. Each method is computed using a single data type (\( K = 1 \)), which is done to compare methods in more standard logistic regression and Poisson regression settings. Figure 3 presents plots of the true means and the estimates from each of the competing methods for one simulated replicate. In general, all methods perform reasonably well in terms of estimation for the simulated example in Figure 3.

To assess the performance over multiple replicates, we use the central processing unit (CPU) time (seconds) and the mean squared error (MSE),

\[
\text{MSE}_\lambda = \frac{1}{100} \sum_{i=1}^{100} \left( \hat{\lambda}_i - \exp\left(0.5 x_{i,2} + 0.1 g_{i,2}\right) \right)^2
\]

\[
\text{MSE}_p = \frac{1}{100} \sum_{i=1}^{100} \left\{ \hat{p}_i - \frac{\exp\left(-5 x_{i,3} + 0.5 g_{i,3}\right)}{1 + \exp\left(-5 x_{i,3} + 0.5 g_{i,3}\right)} \right\}^2,
\]

where \( \hat{\lambda}_i \) is the posterior median of \( \exp\left(0.5 x_{i,2} + 0.1 g_{i,2}\right) \), and \( \hat{p}_i \) is the posterior median of \( \exp\left(-5 x_{i,3} + 0.5 g_{i,3}\right)/\left(1 + \exp\left(-5 x_{i,3} + 0.5 g_{i,3}\right)\right) \). In Table 2, we provide the average MSE plus or minus two standard deviations over 50 independent replicates by method and data type. In general, we see that EPR performs similarly to each competing method in both logistic regression and Poisson regression in terms of average MSE.

Table 2 shows that we outperform each method in terms of CPU time (the confidence interval for EPR is below that of the competing methods). The CPU times for MCMC based methods do not include examining trace plots to assess convergence. Thus, not only do we have an exact method that performs similar to standard approximate approaches, this exact method is faster. However, it is important to emphasize that the data augmentation methods are computed using R code written by the author, and have not been optimized by a development team as done with \texttt{inla} and \texttt{rstan}.

It is common for \( r > 1 \) and \( n_k \gg 100 \). Thus, we consider setting \( n_k = 10,000 \) and \( r = 30 \) such
Figure 3: Circles represent the true mean and proportion according to our simulation model. The left (right) panel contains results for Poisson (logistic) regression. The overlapping lines are the posterior means (for logistic regression) and posterior medians (for Poisson regression) by method. The legend indicates the method “AC” for the method from Albert and Chib (1993), EPR, INLA, “PG” for the method from Polson et al. (2013), and HMC. The lines are difficult to distinguish since all methods perform similarly in these standard settings.

That,

\[
Z_{i,2} \sim \text{Poisson} \left\{ \exp \left( 0.5 x_{i,2} + 0.005 \sum_{w=1}^{30} g_{i,2}^{(w)} \right) \right\}
\]

\[
Z_{i,3} \sim \text{Bernoulli} \left\{ \frac{\exp \left( -5 x_{i,3} + 0.025 \sum_{w=1}^{30} g_{i,3}^{(w)} \right)}{1 + \exp \left( -5 x_{i,3} + 0.025 \sum_{w=1}^{30} g_{i,3}^{(w)} \right)} \right\},
\]

for \( i = 1, \ldots, 10000 \), \( x_{i,2} \equiv x_{i,3} \equiv 1 \), and \( \{g_{i,k}^{(w)}\} \) generated from a uniform distribution over the interval 1 to 20. We implement EPR with \( B = 50 \) independent replicates from the posterior distribution, with again \( K = 1 \) for both settings. EPR and INLA are practical to implement for datasets of size 10,000 multiple times. Table 3 contains the MSE and CPU times for this simulation study. Here, the approximate solution from INLA leads to considerably higher MSE than the exact EPR method, the average MSE and confidence intervals are lower for EPR. The CPU time is comparable. Both methods take seconds, where Poisson regression is favorable for INLA and logistic regression is favorable for EPR.

5.3 A Comparison to the Normal Approximation to the Poisson Distribution

The normal approximation to the Poisson (or binomial distribution) is a standard choice in GLMM settings (e.g., see Lesch and D. R. Jeske [2009] for a review and other choices). We compare EPR (with \( K = 2 \)) to a Bayesian Gaussian linear mixed model using a dataset from the American Com-
| Method | Regression Type        | Average MSE | CI MSE            | Average CPU | CI CPU     |
|--------|------------------------|-------------|-------------------|-------------|------------|
| AC     | Logistic Regression    | 0.0029      | (0.0022, 0.0036)  | 1.7         | (1.68, 1.72)|
| EPR    | Logistic Regression    | 0.0022      | (0.0016, 0.0028)  | 0.28        | (0.19, 0.37)|
| INLA   | Logistic Regression    | 0.0029      | (0.0022, 0.0036)  | 0.87        | (0.83, 0.91)|
| PG     | Logistic Regression    | 0.003       | (0.0023, 0.0037)  | 5.33        | (5.28, 5.38)|
| HMC    | Logistic Regression    | 0.0036      | (0.0027, 0.0045)  | 3.59        | (1.58, 5.6)|
| EPR    | Poisson Regression     | 0.15        | (0.12, 0.18)      | 0.31        | (0.23, 0.39)|
| INLA   | Poisson Regression     | 0.12        | (0.09, 0.15)      | 0.94        | (0.89, 0.99)|
| HMC    | Poisson Regression     | 0.12        | (0.09, 0.15)      | 2.43        | (0.24, 4.62)|

Table 2: Fifty independent replicates data vectors are drawn according to (16), and several methods are applied to each replicated data vector. The method column indicates “AC” for the method from Albert and Chib (1993), EPR, INLA, “PG” for the method from Polson et al. (2013), and HMC. The regression type indicates logistic regression versus Poisson regression. The average MSE is across 50 independent replicates. The column “CI MSE” provides the average MSE plus or minus two standard deviations computed across 50 independent replicates. The average CPU is across 50 independent replicates. The column “CI CPU” provides the average CPU plus or minus two standard deviations computed across 50 independent replicates.

The Community Survey (ACS) consisting of mean income estimates and count-valued poverty estimates. The ACS provides demographic statistics for the U.S. over 1-year and 5-year time periods, and over several geographies. For EPR we assume the population of those in poverty is Poisson distributed, and mean income is Gaussian distributed with known variances set equal to the direct survey variances. Poverty incidences over counties in Florida are large enough that it would be reasonable to use the normal approximation to the Poisson distribution. We compare EPR to the Bayesian linear mixed model using the Gaussian approximation implemented via a Gibbs sampler using R code. That is, we compare EPR to the bivariate Gaussian model that replaces the Poisson(λ) assumption with $\text{Normal}(\hat{\lambda}, \hat{\lambda})$, where $\hat{\lambda}$ is the direct survey variances from ACS.

We make use of the Student-\(t\) distribution for $\boldsymbol{\beta}$ with $N = 134$ and $n_1 = n_2 = 67$. A log transformation for mean income is used for symmetry, and the direct survey variances are adjusted using the delta method. Both responses appear to have a quadratic relationship with their respective direct survey variances from ACS, and consequently, we use the direct survey variance as covariates. We also define different intercepts for the different responses. Consider $G = I_N$ and a type of multivariate conditional autoregressive model (CAR; Besag, 1974, 1986; Besag et al., 1991) specification for $\eta = (\eta'_1, \eta'_2)'$, where both $\eta_1$ and $\eta_2$ are $(N/2)$-dimensional. That is, let $\eta_2 = \gamma \eta_1 + \epsilon_\eta$, where $\gamma$ is Student-\(t\) with 3 degrees of freedom and $\epsilon_\eta$ is Gaussian with mean zero and covariance matrix $\sigma_\eta^2 \sigma_\gamma^{-1} I_{N/2}$ with $\sigma_\eta^2 > 0$ for $j = 1, 2$. Then assume $\eta_2$ is Gaussian with mean zero and co-
smaller and the correlations are smaller. To worse out-of-sample performance than that of EPR, since the relative-cross-validation errors are also provides metrics for the normal approximation to the Poisson (NAP), which in general, leads that leave-one-out predictions are roughly within 4% of the hold-out observation indicating reasonable predictions, and in general the Pearson’s correlation coefficient suggests a strong relationship between the observations and the leave-one out predictions (roughly 90 to 95%). Table 4 gives a table of both metrics. The relative cross-validation errors suggest that leave-one-out predictions are roughly within 4% of the hold-out observation indicating reasonable predictions, and in general the Pearson’s correlation coefficient suggests a strong relationship between the observations and the leave-one out predictions (roughly 90 to 95%). Table 4 also provides metrics for the normal approximation to the Poisson (NAP), which in general, leads to worse out-of-sample performance than that of EPR, since the relative-cross-validation errors are smaller and the correlations are smaller.

| Method | Regression Type | Average MSE | CI MSE | Average CPU | CI CPU |
|--------|-----------------|-------------|--------|-------------|--------|
| EPR    | Logistic        | 0.0015      | (0.0014, 0.0016) | 9.76 | (9.6, 9.92) |
| INLA   | Logistic        | 0.0594      | (0.05936, 0.05944) | 10.55 | (10.42, 10.68) |
| EPR    | Poisson         | 0.0124      | (0.009, 0.0158) | 10.25 | (9.57, 10.93) |
| INLA   | Poisson         | 0.0447      | (0.0411, 0.0483) | 4.24 | (4.03, 4.45) |

Table 3: Fifty independent replicates data vectors are drawn according to [17], and both EPR and INLA are applied to each replicated data vector. The average MSE is across 50 independent replicates. The column “CI MSE” provides the average MSE plus or minus two standard deviations computed across 50 independent replicates. The average CPU is across 50 independent replicates. The column “CI CPU” provides the average CPU plus or minus two standard deviations computed across 50 independent replicates.

The variance matrix \( \sigma^2_{\eta,1}(D_a - \rho A)^{-1} \), where \((D_a - \rho A)\) is the precision matrix from a CAR model, \(A\) is a binary adjacency matrix, and the diagonal matrix \(D_a\) has \(i\)-th diagonal element set equal to the \(i\)-th row total of \(A\). The model between \(\eta_1\) and \(\eta_2\) is a type of multivariate CAR model (e.g., see Cressie and Wikle [2011]). Let \(\sigma^2_{\eta,j}\) be distributed according to an inverse gamma distribution with shape 3 and rate equal to 2, and \(\rho\) be distributed from a uniform distribution between \(\rho_L\) and \(\rho_U\), where \(\rho_L\) and \(\rho_U\) are the smallest and largest values so that \(D_a - \rho A\) is positive definite (Banerjee et al. [2015]). In this case \(D_{\eta}(\theta) = \text{chol}\{\text{cov}(\eta|\rho, \sigma^2_{\eta,1}, \sigma^2_{\eta,2})\}\), “chol” is the Cholesky operator, \(D_\beta = \text{diag}(\sigma^2_{\beta,1}, \ldots, \sigma^2_{\beta,p})\), “diag” is a diagonal matrix, \(\sigma^2_{\beta,i}\) is inverse gamma with shape 1.5 and rate \(1/2\), \(\theta = \{\rho, \sigma^2_{\eta,1}, \sigma^2_{\eta,2}, \sigma^2_{\beta,1}, \ldots, \sigma^2_{\beta,p}\}\), and \(\pi(\theta) = \pi(\rho)\pi(\sigma^2_{\eta,1})\pi(\sigma^2_{\eta,2})\pi(\sigma^2_{\beta,1})\ldots\pi(\sigma^2_{\beta,p})\) is as defined above. We fit EPR according to Section [4.3] with \(B = 500\) independent replicates from the posterior distribution.

Plots of the predicted mean and standard deviation of \(\tilde{y}_k\) versus the log-data using EPR are provided in Appendix D. In general, we see predictions that reflect the pattern of the data with spatial smoothing. The leave-one-out cross validation error ([Wahba, 1990]) is used to assess the predictive performance. Specifically, an observation is left out, and the model is used to predict this value. We compute the relative cross-validation error as \(CV_1 = \frac{\text{median}_{i \in \{1, \ldots, n_1\}} \left\{ \left\{ \frac{\text{abs}(Z_{1,i} - E_{-i}[\tilde{Y}_{1,i}])}{\text{abs}(Z_{1,i})} \right\} \right\} }{10} \), and the Pearson’s correlation between the observations and the predictions based on leaving one observation out. Table 4 gives a table of both metrics. The relative cross-validation errors suggest that leave-one-out predictions are roughly within 4% of the hold-out observation indicating reasonable predictions, and in general the Pearson’s correlation coefficient suggests a strong relationship between the observations and the leave-one out predictions (roughly 90 to 95%). Table 4 also provides metrics for the normal approximation to the Poisson (NAP), which in general, leads to worse out-of-sample performance than that of EPR, since the relative-cross-validation errors are smaller and the correlations are smaller.
| Method | k | CV$_k$ | Corr |
|--------|---|--------|------|
| EPR    | 1 | 0.036  | 0.91 |
| NAP    | 1 | 0.064  | 0.89 |
| EPR    | 2 | 0.041  | 0.968|
| NAP    | 2 | 0.045  | 0.899|

Table 4: Let “NAP” represent the normal approximation to the Poisson. Let CV$_2$ be the relative cross-validation error for poverty computed on the log scale. Let “Corr” represent Pearson’s correlation between $\{Z_{1,i}\}$ and predictions, and $\{log(Z_{2,i})\}$ and predictions, respectively.

6 Discussion

This paper describes how to efficiently sample independent replicates directly from the posterior distribution for multiple-response type data modeled using a broad class of generalized linear mixed effects models. This development required the introduction of the GCM distribution and the conditional GCM distribution. The use of the GCM allows one to consider any class of CM’s for their prior distributions on fixed and random effects. Our development explicitly addresses hyperparameters through marginalization. We make use of the GCM in a GLMM context to produce what we call exact posterior regression, which represents an efficiently generated independent sample from the posterior distribution. We show that the posterior distribution for fixed and random effects in this GLMM are GCM, which we can directly sample from. Furthermore, we use matrix algebra techniques to aid in the computation of EPR.

The results in this paper solve an important problem for Bayesian analysis that is regularly overlooked (i.e., obtaining efficient independent replicates directly from the posterior distribution in Bayesian GLMMs). Our solution also allows one to sample independent replicates directly from the posterior predictive distribution when using point mass specification of $\pi$ (i.e., point mass on an estimate), which avoids MCMC in empirical Bayesian settings as well. Specifically, Theorems 4.1 – 4.3 can be used with a plug-in estimator of $\vec{\theta}$. However, plug-in estimators have unchecked sampling variability (provided that the plug-in estimator is a non-constant function of the data), and the development of the GCM provides a straightforward solution that accounts for all sources of variability.

While we feel that the results in this manuscript represent a significant advancement in Bayesian modeling of GLMMs, it is important to state that MCMC will always be a standard tool. This is because the GLMM specification we consider does not represent the wide variety of GLMMs used in the literature (e.g., allowing for mixture components, inference on hyperparameters, data models outside the class of distributions we consider, etc.). However, we hope the theory developed in this article leads to further theoretical developments that allows one to sample independent replicates from the posterior distribution in other settings.
Appendix A : Proofs

Proof Theorem 3.1
We first derive \( f(y|\mu_M, \alpha_M, \kappa_M, V_M, \theta) \). Upon multiplying independent DY random variables we see that the distribution of \( w_M \) is,

\[
    f(w_M|\alpha_M, \kappa_M) = \prod_{k=1}^{K} \prod_{i=1}^{n_k} N_k(\kappa_{k,i}, \alpha_{k,i}) \exp \{ \alpha'_M w_M - \kappa'_M \psi_M(w_M) \}.
\]

The inverse transform is \( w_M = D(\theta)^{-1} V_M^{-1} (y - \mu_M) \), and the corresponding Jacobian is \( \det \{ D(\theta)^{-1} \} \det(V_M^{-1}) \).

By standard change-of-variables (e.g., see Casella and Berger [2002]), we have that,

\[
    \begin{align*}
    f(y|\mu_M, V_M, \alpha_M, \kappa_M, \theta) &= \det \{ D(\theta)^{-1} \} \det(V_M^{-1}) \prod_{k=1}^{K} \prod_{i=1}^{n_k} N_k(\kappa_{k,i}, \alpha_{k,i}) \exp \left[ \alpha'_M D(\theta)^{-1} V_M^{-1} (y - \mu_M) - \kappa'_M \psi_M \{ D(\theta)^{-1} V_M^{-1} (y - \mu_M) \} \right] . \tag{18}
    \end{align*}
\]

From our independence assumption,

\[
    f(y|\mu_M, V_M, \alpha_M, \kappa_M) = \int_{\Omega} f(y|\mu_M, V_M, \alpha_M, \kappa_M, \theta) d\theta
    \]

which upon substituting (18) completes the result.

Proof Theorem 3.2
From Theorem 3.1, the conditional distribution is given by

\[
    f(y^{(1)}, \theta|y^{(2)}, \mu_M, V_M, \alpha_M, \kappa_M) \propto f(y|\mu_M, V_M, \alpha_M, \kappa_M, \theta) f(\theta),
\]

\[
    \propto \frac{f(\theta)}{\det \{ D(\theta) \}} \exp \left[ \alpha'_M D(\theta)^{-1} (H Q) \left\{ \begin{pmatrix} y^{(1)} \\ y^{(2)} \end{pmatrix} - \mu_M \right\} \right.
    \]

\[
    \left. - \kappa'_M \psi_M \left\{ D(\theta)^{-1} (H Q) \begin{pmatrix} y^{(1)} \\ y^{(2)} \end{pmatrix} - D(\theta)^{-1} V_M^{-1} \mu_M \right\} \right],
\]

\[
    = \frac{f(\theta)}{\det \{ D(\theta) \}} \exp \left\{ \alpha'_M D(\theta)^{-1} H y^{(1)} - \alpha'_M \mu_M^* - \kappa'_M \psi_M \left( D(\theta)^{-1} H y^{(1)} - \mu_M^* \right) \right\}.
\]

Integrating across \( \theta \) completes the result.
Proof Theorem 4.1

Our strategy is to show that $f(\zeta, q|z) \propto \int_\Omega \pi(\theta)f(\zeta, q, z|\theta)d\theta$ is the GCM stated in Theorem 4.1. The four data models can be written as:

\[
f(z_1|\xi_1, \beta, \eta, \mu_1) \propto \exp \left[ z'_1 D'_\sigma \left\{ (I_{n_1}, X_1, G_1) \begin{pmatrix} \xi_1 \\ \beta \\ \eta \end{pmatrix} - \mu_1 \right\} - \frac{1}{2} 1_{1,n_1} D'_\sigma \psi_1 \left\{ (I_{n_1}, X_1, G_1) \begin{pmatrix} \beta \\ \eta \end{pmatrix} - \mu_1 \right\} \right],
\]

\[
f(z_2|\xi_2, \beta, \eta, \mu_2) \propto \exp \left[ z'_2 \left\{ (I_{n_2}, X_2, G_2) \begin{pmatrix} \xi_2 \\ \beta \\ \eta \end{pmatrix} - \mu_2 \right\} - 1_{1,n_2} \psi_2 \left\{ (I_{n_2}, X_2, G_2) \begin{pmatrix} \beta \\ \eta \end{pmatrix} - \mu_2 \right\} \right],
\]

\[
f(z_3|\xi_3, \beta, \eta, \mu_3) \propto \exp \left[ z'_3 \left\{ (I_{n_3}, X_3, G_3) \begin{pmatrix} \xi_3 \\ \beta \\ \eta \end{pmatrix} - \mu_3 \right\} - m' \psi_3 \left\{ (I_{n_3}, X_3, G_3) \begin{pmatrix} \beta \\ \eta \end{pmatrix} - \mu_3 \right\} \right],
\]

\[
f(z_4|\xi_4, \beta, \eta, \nu, \mu_4) \propto \exp \left[ - \frac{v+1}{2} 1_{1,n_4} \psi_4 \left\{ (I_{n_4}, X_4, G_4) \begin{pmatrix} \xi_4 \\ \beta \\ \eta \end{pmatrix} - \mu_4 - z_4 \right\} \right].
\]

Upon stacking the four data models we obtain:

\[
f(z|\xi, \beta, \eta, \mu_1, \ldots, \mu_K) \propto \exp \left[ a' \left\{ (I_N, X, G) \begin{pmatrix} \xi \\ \beta \\ \eta \end{pmatrix} - \mu_D \right\} - b' \Psi_D \left\{ (I_N, X, G) \begin{pmatrix} \xi \\ \beta \\ \eta \end{pmatrix} - \mu_D^* \right\} \right],
\]

where $a = (z'_1 D'_\sigma, z'_2, z'_3, 0_{1,n_4})'$, $b = (\frac{1}{2} 1_{1,n_1} D'_\sigma, 1_{1,n_2}, m', \frac{v+1}{2} 1_{1,n_4})'$, $\mu_D^* = \mu' = (\mu'_1, \ldots, \mu'_4 + z'_4)'$, and $\Psi_D(h_D) = (\psi_1(h_1), \ldots, \psi_1(h_{n_1}), \psi_2(h_{n_1}+1), \ldots, \psi_2(h_{n_1+n_2}), \ldots, \psi_K(h_N))'$ for $N$-dimensional real-valued vector $h_D = (h_1, \ldots, h_N)'$. Equivalently

\[
f(z|\xi, \beta, \eta, \{\sigma_i\}, \mu_D) \propto \exp \left[ a' \left\{ (I_N, X, G) \{ \xi - \mu_D \} \right\} - b' \Psi_D \left\{ (I_N, X, G) \{ \xi - \mu_D^* \} \right\} \right],
\]

where recall $\xi = (\xi'_1, \beta'_1, \eta')'$. One can re-order or drop blocks that define $a$, $b$, $\mu_D^*$ to allow for fewer data types (e.g., Poisson, binomial, and Student-$t$ distributed data only). The density
\( f(\xi, q, \theta | z, \alpha, \beta, \kappa, \kappa') \) is proportional to the product

\[
f(z | \xi, \beta, \eta, \mu_D) f(\xi | \beta, \eta, \mu_2, \mu_3, \mu_3') \]

Now,

\[
f(\xi | \beta, \eta, \mu_2, \mu_3, \mu_3') \propto \exp \left[ \alpha' \begin{pmatrix} I_N & X & G \\ \frac{1}{\sigma_\xi^2} I_N & 0_{N,p} & 0_{N,r} \end{pmatrix} \begin{pmatrix} \xi \\ \mu_D^* \end{pmatrix} - \begin{pmatrix} \mu_D^* \\ \mu_3 \end{pmatrix} \right]
\]

with \( \sigma_\xi^2 \) and \( \alpha_\xi \) known,

\[
\alpha_\xi = (0_{1,N_1}, \alpha_\xi 1_{1,2}, \alpha_\xi 1_{1,3}, 0_{1,4}, 0_{1,1,N}),
\]

\[
\kappa_\xi = (0_{1,N_1}, 0_{1,1,2}, 2\alpha_\xi 1_{1,3}, 0_{1,4}, \frac{1}{2} 1_{1,1}),
\]

\[
\psi_D,\xi(h_D,\xi) = (\psi_1(h_1), ..., \psi_1(h_{N_1}), \psi_2(h_{N_1}+1), ..., \psi_2(h_{N_1}+N_2), ..., \psi_N(h_N), \psi_1(h_{1}), ..., \psi_1(h_{N})),
\]

for 2N-dimensional real-valued vector \( h_D,\xi = (h_1, ..., h_N, h_{11}, ..., h_{N1})' \). Notice that \( \mu_1 \) and \( \mu_4 - z_4 \) are contained within the expression of \( \mu_D^* \), however these terms are multiplied by zero through \( a_\xi \) and \( b_\xi \). We write \( f(\xi | \beta, \eta, \mu_2, \mu_3, \mu_3') \) in this way so that the following expression is more immediate,

\[
f(z | \xi, \beta, \eta, \mu_D) f(\xi | \beta, \eta, \mu_2, \mu_3, \mu_3') \propto \exp \left[ a'_Z \begin{pmatrix} I_N & X & G \\ \frac{1}{\sigma_\xi^2} I_N & 0_{N,p} & 0_{N,r} \end{pmatrix} \begin{pmatrix} \xi \\ \mu_D^* \end{pmatrix} - \begin{pmatrix} \mu_D^* \\ \mu_3 \end{pmatrix} \right]
\]

where

\[
a_Z = (z_1 D'_\sigma, z_2 + \alpha_\xi 1_{1,2}, z_3 + \alpha_\xi 1_{1,3}, 0_{1,4}, 0_{1,1,N})',
\]

\[
b_Z = (\frac{1}{2} 1_{1,1} D'_\sigma, 1_{1,1}, m' + 2\alpha_\xi 1_{1,3}, \frac{v + 1}{2} 1_{1,4}, \frac{1}{2} 1_{1,1,N})'.
\]

Notice that the implied shape.scale parameters \( a_Z \) and \( b_Z \) are not on the boundary of the parameter
space (when zero counts are present), which is a motivation for including \( \xi \) in the GLMM. Multiplying by \( \pi(\theta)f(\beta|\alpha, \kappa, \mu, D_\beta(\theta)) \) \( f(\eta|\alpha, \kappa, \mu, D_\eta(\theta)) \) and stacking vector and matrices leads to

\[
f(\zeta, q, \theta|z) \propto \pi(\theta) \exp \left[ \alpha'_{M} \left\{ \begin{pmatrix} I_N & X & G \\ 0_{p,N} & D_\beta(\theta)^{-1} & 0_{p,r} \\ 0_{r,N} & 0_{r,p} & D_\eta(\theta)^{-1} \end{pmatrix} - \begin{pmatrix} \mu^*_D \\ \mu_\beta \\ \mu_\eta \\ \mu_\xi \end{pmatrix} \right\} + \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \\ \theta_4 \end{pmatrix} \right] - \kappa'_{M} \psi_{M} \exp \left[ \begin{pmatrix} I_N & X & G \\ 0_{p,N} & D_\beta(\theta)^{-1} & 0_{p,r} \\ 0_{r,N} & 0_{r,p} & D_\eta(\theta)^{-1} \end{pmatrix} - \begin{pmatrix} \mu^*_D \\ \mu_\beta \\ \mu_\eta \\ \mu_\xi \end{pmatrix} \right] \right] d\theta \]

Substituting \( \mu = -D(\theta)^{-1}Qq \) and integrating with respect to \( \theta \) leads to

\[
f(\zeta, q|z) \propto \int_{\Omega} \frac{\pi(\theta)}{\det \{D(\theta)\}} \exp \left[ \alpha'_{M} \left\{ D(\theta)^{-1}(H, Q) \begin{pmatrix} \zeta \\ q \end{pmatrix} - D(\theta)^{-1}(H, Q)\mu_{M} \right\} - \kappa'_{M} \psi_{M} \left\{ D(\theta)^{-1}(H, Q) \begin{pmatrix} \zeta \\ q \end{pmatrix} - D(\theta)^{-1}(H, Q)\mu_{M} \right\} \right] d\theta \]

\[\propto \text{GCM}(\alpha_M, \kappa_M, \mu_M, \lambda_M, \pi, D; \psi_M),\]

which completes the result.

**Proof Theorem 4.2**

Equations (10) and (11) from the main text follows from (6) from the main text, Theorem 4.1, and that

\[(H, Q)^{-1}D(\theta) = (H(H'H)^{-1}, Q)'D(\theta).\]

Let \( w_M \) consist of independent DY random variables with respective shape and scale parameters in \( \alpha_M \) and \( \kappa_M \). Let

\[
w = D(\theta) \begin{pmatrix} w_M \\ z_4 \\ 0_{N+p+r,1} \end{pmatrix} = D(\theta)w_M + \begin{pmatrix} w_M \\ z_4 \\ 0_{N+p+r,1} \end{pmatrix},\]

24
where recall $D(\theta)$ is a block diagonal matrix with first $N \times N$ block diagonal equaling the identity matrix. It follows from Theorem 3.1 that $w$ has the stated GCM distribution in Theorem 4.2. Then

$$
\begin{pmatrix}
\zeta_{rep} \\
q_{rep}
\end{pmatrix} = \begin{pmatrix}
(H'H)^{-1}H' \\
Q'
\end{pmatrix} w =
\begin{pmatrix}
0_{n_1+n_2+n_3,1} \\
z_4 \\
0_{N+p+r,1}
\end{pmatrix} + \begin{pmatrix}
(H'H)^{-1}H'
\end{pmatrix} D(\theta)w_M.
$$

From Theorem 3.1 it follows that $(\zeta'_{rep}, q'_{rep})'$ is the GCM stated in Theorem 4.2. Equation (12) in the main text follows from the fact that $y = (I_N, 0_{N,N+p+r})H\zeta + (I_N, 0_{N,N+p+r})Qq$ so that

$$
y_{rep} = (I_N, 0_{N,N+p+r})H\zeta_{rep} + (I_N, 0_{N,N+p+r})Qq_{rep}
= (I_N, 0_{N,N+p+r})H(H'H)^{-1}H'w + (I_N, 0_{N,N+p+r})QQ'w
= (I_N, 0_{N,N+p+r})w = (I_N, 0_{N,N+p+r})w.
$$

**Proof Theorem 4.3**

From \cite{Lu and Shiou 2002} the inverse of a $2 \times 2$ block matrix is,

$$
\begin{pmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22}
\end{pmatrix}^{-1} = \begin{pmatrix}
A_{11}^{-1} + A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & -A_{11}^{-1}A_{12}(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \\
-(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}A_{21}A_{11}^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1}
\end{pmatrix},
$$

for generic real-valued $M \times M$ matrix $A_{11}$, $M \times (p+r)$ matrix $A_{12}$, $(p+r) \times M$ matrix $A_{21}$, and $(p+r) \times (p+r)$ matrix $A_{22}$. Equation (13) of the main text follows from applying this known inverse of $2 \times 2$ block matrices \cite{Lu and Shiou 2002} to,

$$
(H'H) = \begin{pmatrix}
2I_N & X & G \\
X' & X'X + I_p & X'G \\
G' & G'X & G'G + I_r
\end{pmatrix}.
$$

Similarly, Equation (14) from the main text follows from applying the same inverse identity to

$$
(D - B'A^{-1}B) = \begin{pmatrix}
A^* & B^* \\
C^* & D^*
\end{pmatrix}.
$$
Proof Theorem 4.4

(Lu and Shiou, 2002) gave two identities for the inverse of a $2 \times 2$ block matrix,

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} A_{11}^{-1} + A_{11}^{-1} A_{12} (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} A_{21} A_{11}^{-1} & -A_{11}^{-1} A_{12} (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} \\ -(A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} A_{21} A_{11}^{-1} & (A_{22} - A_{21} A_{11}^{-1} A_{12})^{-1} \end{pmatrix},$$

and

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} & -(A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} A_{12} A_{22}^{-1} \\ -A_{22}^{-1} A_{21} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} & A_{22}^{-1} + A_{22}^{-1} A_{21} (A_{11} - A_{12} A_{22}^{-1} A_{21})^{-1} A_{12} A_{22}^{-1} \end{pmatrix},$$

for generic real-valued $M \times M$ matrix $A_{11}$, $M \times (p + r)$ matrix $A_{12}$, $(p + r) \times M$ matrix $A_{21}$, and $(p + r) \times (p + r)$ matrix $A_{22}$. Apply this second identity to,

$$(H'H) = \begin{pmatrix} 2I_N & X & G \\ X' & X'X + I_p & X'G \\ G' & G'X & G'G + I_r \end{pmatrix},$$

to produce

$$(H'H)^{-1} = \begin{pmatrix} (F - KL^{-1}K')^{-1} & -(F - KL^{-1}K')^{-1}KL^{-1} \\ L^{-1}K'(F - KL^{-1}K')^{-1} & L^{-1} + L^{-1}K'(F - KL^{-1}K')^{-1}KL^{-1} \end{pmatrix}.$$ 

Since, we have that $H'w = (R', P')'$, Equation (15) from the main text follows immediately. Apply the Sherman-Morrison Woodbury identity (Cressie and Johannesson, 2008) to obtain $F_1^{-1}$.

Appendix B: Process Modeling

The mixed effects model specification in Section 4.1 may be deceptively simple; however, we emphasize that several modern statistical models use EPR including process models (e.g., spatial and spatio-temporal statistical models). See Hodges (2013) for an thorough treatment of how spatial and temporal statistical models can be written as a richly parameterized mixed effects model. Although, of course, process models are different from mixed effects models, implementation of additive process models are similar to that of mixed effects models for a given collection of location/times. For example, consider locations $s \in D$, where $D$ is a generic spatial domain (e.g., a lattice or subset of $\mathbb{R}^d$). We introduce process into our notation functionally so that, for example, $Z_{1,i}$ is written as $Z_k(s_i)$, where $s_{k,1}, \ldots, s_{n_k,k} \in D$. Consider the following multivariate spatial statistical model,

$$Y_k(s) = x_k(s)' \beta + \eta(s) + (\xi(s) - \mu(s)); s \in D,$$
Theorem 4.2 shows that posterior inference on parameters Appendix C: Enforcing Sparsity Before Marginalizing Bypass Location Parameters

Theorem 4.2 shows that posterior inference on \( \xi \) can be interpreted as a type of regression when using \( f(\xi | z) \) for inference, which marginalizes across \( q \). Another choice is to enforce sparsity on the location parameter (i.e., setting \( q \) equal to zero) before marginalizing it out instead of after, which we do in this article. That is, one might instead use the informative point mass prior of the location parameter (i.e., setting \( q = 0 \)) for inference \cite{McCulloch et al. 2008}.

Result: Suppose \( Z_{k,i} | Y_{k,i} \) is independently distributed according to either (1) or (2) of the main text with \( i = 1, \ldots, n_k \), and \( k = 1, \ldots, K \). Let \( y_k \) be defined as in Theorem 4.1 of the main text with the added assumption that \( f(q) = I(q = 0_{N,1}) \). Then \( \xi | z \) is cGCM(\( \alpha_M, \kappa_M, \mu_q, H, \pi, D, \psi_M \)), with \( \mu_q = (0_{1, n_1 + n_2 + n_3, z_4', 0_{1, N + p + r}})' \), and \( \alpha_M, \kappa_M, H, D, \) and \( \psi_M \) defined in Theorem 4.1.
\textit{Proof:} We have

\[
\begin{align*}
  f(\zeta, \theta | z, \alpha, \alpha, \kappa, & \kappa, q = 0_{N,1}) \\
  \propto \frac{\pi(\theta)}{\det\{D(\theta)\}} \exp \left[ \alpha' D(\theta)^{-1} (H Q) \begin{pmatrix} \zeta \\ 0_{N,1} \end{pmatrix} \\
  - \kappa' \psi_M \begin{pmatrix} D(\theta)^{-1} (H Q) \begin{pmatrix} \zeta \\ 0_{N,1} \end{pmatrix} - \begin{pmatrix} 0_{n_1+n_2+n_3,1} \\ z_4 \\ 0_{N+p+r,1} \end{pmatrix} \end{pmatrix} \right] ,
\end{align*}
\]

and when integrating across \( \theta \) we obtain the cGCM in the statement above.

The result above shows that \( f(\zeta | z, q = 0_{N,1}) \) is a conditional GCM, however, it is currently unknown how to simulate from the conditional GCM in many settings.
Appendix D: Additional Figure for Section 5.3

Figure 4: We plot the log ACS estimates (as a reference) along with a plot of the posterior mean and standard deviations computed from 500 independent replicates of EPR. The first row contains values for $k = 1$, and the second row contains values when $k = 2$. The third row provides the respective posterior standard deviations.
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