Detecting entanglement harnessing Lindblad structure

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Keywords: detecting entanglement, positive maps, multipartite entanglement, Lindblad master equation

Abstract
The problem of entanglement detection is a long standing problem in quantum information theory. One of the primary procedures of detecting entanglement is to find the suitable positive but non-completely positive maps. Here we try to give a generic prescription to construct a positive map that can be useful for such scenarios. We study a class of positive maps arising from Lindblad structures. We show that two famous positive maps viz. transposition, reduction map and Choi map can be obtained as a special case of a class of positive maps having Lindblad structure. Generalizing the transposition map to a one parameter family we have used it to detect genuine multipartite entanglement. Finally being motivated by the negativity of entanglement, we have defined a similar measure for genuine multipartite entanglement.

1. Introduction

The core structure of quantum information processing is predominantly governed by presence of quantum entanglement [1] in a non-local system [2–5]. For the efficient implementation of many information processing tasks and different quantum algorithms, not only the presence of entanglement is necessary [6, 7] but also it is the key property that helps a quantum state to outperform its classical counterpart [8–10]. Hence for all practical purposes, detecting entanglement in a given quantum system is one of the key domain of research in the corresponding literature. There are different detection criteria for entanglement in bipartiite or multipartite systems [11] such as entanglement witnesses [12, 13]. Also to quantify the amount of the same in a given system, there are several well studied measures for bipartiite systems, such as negativity [14], concurrence [15, 16] and many more [17, 18]. While the field of detecting and measuring entanglement in bipartiite system is a well explored area in quantum information theory [19–23], moving to more parties and/or higher dimensions are still much uncharted territory in the literature. In most of the realisations of quantum algorithms the systems under consideration consists of more than two number of parties and hence motivates the community to investigate the properties of entanglement in these many-party circumstances.

The primary mathematical domain of entanglement detection problem is of positive but non completely positive maps. In some low dimensions, such as for $2 \times 2$ and $2 \times 3$ dimensional states, one of such maps namely transposition [12] gives us the necessary and sufficient condition for entanglement detection. However, for higher dimensions this fails to give such criterion because there exists entangled states that are also positive under partial transposition (PT) and in case of arbitrary dimensional systems, entanglement detection falls in the category of NP hard problems [24]. The entanglement of such states, which cannot be detected by transposition map are not distillable and hence they are called bound entangled states [25]. These states are shown to be useful when the entanglement is unlocked [26, 27]. The study of bound entanglement and its detection is a fascinating and difficult area of research in quantum information theory [28–31] and is still far from exhaustive. It is already clear that a transposition map or any positive map which is decomposable with respect to transposition, can not...
detect a PPT entangled state [32]. So when a positive but not completely positive map detects PPT entanglement, then it is definitely an indecomposable map; i.e. such a map cannot be written as \( CP_1 + T \circ CP_2 \), where \( CP_1 \) and \( CP_2 \) are two complete positive maps and \( T \) denotes transposition. In case of these higher dimensional PPT entangled states several detection protocols has been discovered over the years, such as computable cross norm or realignment criterion (CCNR criterion) [33, 34], range criterion [35, 36] and others [31, 37, 38]. In spite of all these, positive but non completely positive maps still play key role in entanglement detection. Here we try to put one step forward by trying to introduce a generic prescription to find such useful positive maps from a well defined physically realisable structure, borrowed from the theory of open quantum systems.

The theory of open quantum systems, tells us that the most general quantum dynamics are represented by completely positive trace preserving (CPTP) maps, the generators of which (if exists) can be associated with Lindblad type super-operators [39–48]. Let us consider a system is interacting with the environment governed by a given interaction Hamiltonian. By considering the fact that the initial state of the composite system is in a product form, with invoking stationary bath and Born-Markov approximation [39–42], one can obtain a master equation that in turn dictates the evolution of the system’s state with time. Interestingly, the Lindblad type master equation can also be derived for some cases, without even invoking stationary bath and Born-Markov approximation [43–46]. Hence the study of Lindblad type dynamics is an integral part of research in open quantum systems. We note that it is possible to construct a set of positive maps from a Lindblad operator by parameterizing the same [49]. These positive maps are in turn useful to detect both bipartite and multipartite entanglement. The main motivation of this paper is to give a generic umbrella structure based on the dynamics of a system interacting with an environment. We show that it is possible to obtain different famous positive but not CP maps from this structure for suitable choice of parameters. Alongside we find that the family of positive map arising from the given master equation can detect entanglement for broader range of the map parameter.

The main results of the paper are as follows. We first consider the simplest possible case of qubit systems. In this scenario, while detection of entanglement in bipartite quantum states is well studied in literature, genuine multi-partite entanglement detection is much uncharted territory in contrast to the bi-partite case. From the Lindblad structure we first construct an one parameter family of linear maps which acts on a qubit and helps detecting genuine multipartite entanglement (GME). We construct the corresponding witness operator and also give a quantitative measure for GME. The structure of positive maps on this qubit systems is presented in section 2. Next we move on to the problem of entanglement detection in higher dimension. Here also we construct suitable maps from the Lindblad structure. We check if the maps are decomposable or not, as it has direct connection with detection of PPT entanglement. Finally from the general structure we construct various well known positive maps which are useful in detecting entanglement in higher dimensional systems. For this purpose we restrict ourselves to qutrit systems and the processes are illustrated in section 3. We find transposition, reduction map and the indecomposible Choi map from the structure of the Lindbladian. Finally in section 4, we sum up the findings of the paper and shed some light on possible new avenue to study in the direction of the theory of entanglement detection.

2. Positive maps on \( \mathcal{M}_2 \)

Let \( \mathcal{M}_2 \) stands for the algebra of 2 by 2 complex matrices. Let us now define an one parameter family of linear maps \( \Lambda \) on \( \mathcal{M}_2 \) as, \( \Lambda : \mathcal{M}_2 \longrightarrow \mathcal{M}_2 \) such that

\[
\Lambda (X) = X + \gamma \left( \sigma_1 X \sigma_1 - \frac{1}{2} \sigma_1 \sigma_3 X - \frac{1}{2} X \sigma_3 \sigma_1 - \gamma \left( \sigma_2 X \sigma_2 - \frac{1}{2} \sigma_2 \sigma_3 X - \frac{1}{2} X \sigma_3 \sigma_2 + \gamma \left( \sigma_3 X \sigma_3 - \frac{1}{2} \sigma_3 \sigma_2 X - \frac{1}{2} X \sigma_2 \sigma_3 \right) \right) \right)
\]  

(1)

Here, \( X \in \mathcal{M}_2 \), \( \gamma \in \mathbb{R} \), and \( \sigma_i \)’s (for \( i = 1, 2, 3 \)) are the Pauli matrices.

The construction of the map is motivated from the structure of time independent (\( \gamma \) being independent of time) Lindblad form. Further simplification leads the linear map to the given form,

\[
\Lambda (X) = \begin{bmatrix}
\frac{\eta_1}{2} & 2 \eta_1 \gamma \\
2 \eta_2 \gamma & \eta_2
\end{bmatrix}
\]  

(2)

for any \( X = \begin{bmatrix}
\eta_1 & \eta_2 \\
\eta_2 & \eta_2
\end{bmatrix} \in \mathcal{M}_2 \) and \( \gamma \in \mathbb{R} \). Note that, for \( \gamma = \frac{1}{2} \) it reproduces the famous transposition map \( T \).

Thus this family of map can be regarded as an one parameter generalisation of transposition map arising from Lindblad like structure. Hence it is interesting to search the range of the parameter \( \gamma \) for which the map is positive and can potentially contribute to the literature of the problem of entanglement detection. Moreover we try to find whether it is completely positive (CP) for some \( \gamma \). Clearly for \( \gamma = \frac{1}{2} \) the map is positive but not completely positive.

Let us consider a unit vector \( \eta \) in two dimensional Hilbert space. Applying the map \( \Lambda \) on \( \eta \eta^* \), where \( \eta^* \) is the adjoint of \( \eta \), we find that the family of maps \( \Lambda \) is positive for \( -1/2 \leq \gamma \leq 1/2 \). Moreover, computing the Choi matrix of the corresponding family of maps, we find that the maps are not completely positive for any \( \gamma \in \mathbb{R} \).
Therefore we have an one parameter family of positive but not completely positive (PNCP) maps $\Lambda$ for $-1/2 \leq \gamma \leq 1/2$ which are certainly useful for detecting entangled states. Now it is well known that in $\mathcal{M}_2$, all positive maps are decomposible. In fact transposition map gives necessary and sufficient condition to detect all entangled states in $\mathcal{M}_2 \otimes \mathcal{M}_2$ and $\mathcal{M}_2 \otimes \mathcal{M}_2$ state space. Hence it is trivial to discuss bi-partite entanglement detection by a positive map acting on $\mathcal{M}_2$. Therefore, we now move on to the case of genuinely multi-partite entanglement detection by positive maps.

2.1. Multipartite entanglement

It is well known that the transposition map plays an instrumental role in entanglement detection. Seminal results by Stormer, Woronowich and Horodecki's have established that transposition is the only PNCP map acting on $\mathcal{M}_2$ which is responsible for entanglement detection in two qubit or qubit-qutrit systems [12]. In this paper we do not focus on entanglement detection on two qubit or qubit-qutrit systems. Rather we are interested in the effectiveness of the map in detecting multipartite entanglement. The domain of entanglement detection in this scenario becomes drastically non trivial as it involves the concept of genuine multipartite entanglement along with bi-separability as more than two parties are involved. In this paper, we concentrate on the simplest case in this genre where there are three qubits present. In order to do that, we need to construct a lifted map on $\mathcal{M}_2 \otimes \mathcal{M}_2 \otimes \mathcal{M}_2$ space and show that this new map can separate out genuinely multi-partite entangled states from all possible bi-separable states. For this purpose, we need to first prove the statement given in the following proposition.

**Proposition 1.** The minimum eigenvalue of the Choi state in $2 \otimes 2$ is the minimum eigenvalue for any state when subjected to $\Lambda$.

**Proof.** Let us now try to find the minimum eigenvalue of the output matrix corresponding to any pure input state. Without loss of generality we can consider an input pure state in its Schmidt form. A pure state can be of two types: either product or entangled. For a product input state the eigenvalues of the output matrix is non-negative always as the map is positive. So to get the minimum eigenvalue we need to consider entangled states as input. Let us consider $|\psi\rangle = c_1|00\rangle + c_2|11\rangle$, where $|c_1|^2 + |c_2|^2 = 1$. Now $\Lambda \otimes T(|\psi\rangle\langle\psi|)$ produces the output matrix as

$$
\begin{bmatrix}
|c_1|^2 & 0 & 0 & 0 \\
0 & 0 & 2c_2c_1^*\gamma & 0 \\
0 & 2c_2c_1^*\gamma & 0 & 0 \\
0 & 0 & 0 & |c_2|^2
\end{bmatrix}
$$

whose minimum eigenvalue is given by $\mathcal{E}_{\min} = -2|c_1|\langle c_1|\gamma$. Clearly, since $|c_1|^2 + |c_2|^2 = 1$, $|c_1||c_2| \leq \frac{1}{\sqrt{2}}$. Here the upper bound is achieved at $c_1 = c_2 = \frac{1}{\sqrt{2}}$. Therefore the minimum eigenvalue of the output matrix is $-\gamma$ and it is obtained for two qubit maximally entangled state as input. Clearly for $\gamma = \frac{1}{2}$ we are getting back the previous result of transposition map.

We now concentrate on detecting genuinely multipartite entanglement by using positive maps. Let us consider a positive linear map $\hat{\Lambda}$ corresponding to the positive map $\Lambda: \mathcal{M}_2 \rightarrow \mathcal{M}_2$ and the action of the map on a three-qubit system can be seen as,

$$
\hat{\Lambda}[\ast] = (\Lambda_A \otimes I_B \otimes I_C + I_A \otimes \Lambda_B \otimes I_C + I_A \otimes I_B \otimes \Lambda_C + c I \cdot \text{Tr}[\ast])
$$

where $A, B$ and $C$ are the labels of the parties and $I_X$ is the identity map on party $X \in \{A, B, C\}$ and $c$ is a constant. We can call this map as the lifting of the positive map $\Lambda$. It was shown in [50], that the similar lifting of the transposition map $T$ can detect three qubit $W$ states. Our proposed map given in equation (2) is a one parameter generalization of the Transposition map structure. In order to show its usefulness in detecting Genuine entanglement, we prove the following statement.

**Statement 1:** The minimum value of the constant $c$, for all bi-separable three qubit states $\rho_{BS}$, for which $\hat{\Lambda}(\rho_{BS}) \geq 0$, is given by $2\gamma$.

**Proof.** The proof of this statement can be sketched following the [48, 50, 51]. Let us consider that the three qubit bi-separable states $\rho_{BS}$ are of the form
Here $p_Z^X$ is of course the probability distribution, $\rho_Z^X$ denotes the qubit density matrix of the party $Z$ and subsequently $\rho_Z^{\bar{X}}$ corresponds to the complementary two qubit quantum state, which can be entangled or otherwise. The sum $\sum_Z$ runs over all possible bi-separations $Z|\bar{Z}$ for $Z \in \{A, B, C\}$. Therefore, we have

$$\tilde{A}[\rho_{BS}] = \sum_{Z \in \{A, B, C\}} (A_Z \otimes I_Z)(\rho_{BS}) + c \cdot I \cdot \text{Tr}[\rho_{BS}].$$

This gives the following condition on the minimum eigenvalue of the matrix $\tilde{A}[\rho_{BS}]$

$$\mathcal{E}_{\text{min}}[\tilde{A}[\rho_{BS}]] \geq \sum_{\lambda, I} \mathcal{E}_{\text{min}}((A_I \otimes I_I)[\rho_Z^I \otimes \rho_{\bar{Z}}^I]) + \sum_{\lambda, I} \mathcal{E}_{\text{min}}((A_I \otimes I_I)[\rho_Z^I \otimes \rho_{\bar{Z}}^I]) + \epsilon,$$

where $\mathcal{E}_{\text{min}}(X)$ means minimum eigenvalue of the matrix $X$. Here we have used proposition 1 to get the lower bound $-2\gamma$ of the second term in the right hand side of equation (5). It is important to mention that the first term on the right hand side will always be positive, because the positive map $A_I$ is acting on the qubit part of the bi-separable state. Based on this, it is evident that to have the output of the action of the given lifted map on any bi-separable state, the condition $c = 2\gamma$ is needed to be satisfied.

Armed with Statement 1, we now know that lifted map (3) with $c = 2\gamma$ gives positive outcome when acted upon any bi-separable state. Therefore, it can only give negative eigenvalue for genuinely entangled states and hence can be utilised to detect such quantum states. Investigating further, we have found that $\tilde{A}[\{|W\}\langle W|]) \neq 0$ for the values of $\gamma \in \left[-\frac{1}{2}, -\frac{\sqrt{5}}{4}\right] \cup \left(\frac{\sqrt{5}}{4}, \frac{1}{2}\right]$, where $\{|W\} = \frac{1}{\sqrt{3}}(|100\rangle + |010\rangle + |100\rangle)$ represents the three qubit W state. Clearly $\gamma = \frac{1}{2}$ corresponds to the the transposition map.

Now we consider the case when the W state is mixed with some white noise. Hence the state can be written as, $\tilde{W} = p|W\rangle\langle W| + (1 - p)|\tilde{W}\rangle\langle \tilde{W}|$. It was shown in [48, 50, 51] that for $\gamma = \frac{1}{2}$ genuine entanglement can be detected for all $p$ strictly greater than 0.73. Our finding is that the same is also true for $\gamma = -\frac{1}{2}$. Moreover $\gamma = \frac{1}{2}$ and $\gamma = -\frac{1}{2}$ are the best choice for the class of noisy W state as for $\gamma \in \left(-\frac{1}{2}, -\frac{\sqrt{5}}{4}\right]$ no genuine entanglement can be detected by $\tilde{A}$. Beyond the above mentioned region, $\gamma = \frac{1}{2}$ and $\gamma = -\frac{1}{2}$ are the only values of the map parameter for which the largest class of genuinely entangled noisy W states can be detected by this maps. In a similar way GHZ state can also be detected by using the map $\tilde{A}$ followed by a rotation by $\sigma_x$ operator. The result has been graphically represented in figure 1.

2.1.1. Construction of witness for genuine multipartite entanglement (GME)

Further, we can also construct a witness for detection of GME in the multipartite scenario from the above class of transposition maps. A witness $W$ is a Hermitian operator such that $\text{Tr}[W_{\text{sep}}] \geq 0$ for all separable and bi-separable states $\sigma_{\text{sep}}$ and $\text{Tr}[W_{\text{GME}}] \neq 0$ for at least one genuinely entangled state. We know that entanglement witnesses are directly measurable quantities and hence a very useful tool for detecting entanglement in experiments.

![Graph](笑意)
Proposition 2. \( W_{\text{GME}} = \tilde{\Lambda}(|W\rangle \langle W|) \) is a witness detecting GME.

**Proof.** Let us consider the map \( \tilde{\Lambda} \) introduced in equation (3) obtained by lifting the map \( \Lambda \) action of which has been shown in equation (1) and can be simplified as,

\[
\Lambda(\rho) = \rho + \sum_i \gamma_i \left( \sigma_i \rho \sigma_i - \frac{1}{2} (\sigma_i \rho + \rho \sigma_i) \right)
\]

Expanding as given in equation (6), it can be easily seen that, \( Tr[\Lambda(\rho)|W\rangle \langle W|] = Tr[\Lambda(|W\rangle \langle W|) \rho] \). Hence we have,

\[
Tr[\tilde{\Lambda}(\sigma_{2-\text{sep}})|W\rangle \langle W|] = Tr[\tilde{\Lambda}(|W\rangle \langle W|) \sigma_{2-\text{sep}}]
\]

We know from \([51]\) that,

\[
\tilde{\Lambda}(\sigma_{2-\text{sep}}) \geq 0
\]

\[
\Rightarrow Tr[\tilde{\Lambda}(\sigma_{2-\text{sep}})|W\rangle \langle W|] \geq 0
\]

Hence we get,

\[
Tr[\tilde{\Lambda}(|W\rangle \langle W|) \sigma_{2-\text{sep}}] \geq 0
\]

The minimum eigenvalue for the map \( \Lambda \) is \( \gamma \) i.e \( E_{\text{min}}(\Lambda) = \gamma \). Hence, the constant \( c \) in equation (3) is \( 2\gamma \). We can easily see that \( \tilde{\Lambda}(\sigma_{2-\text{sep}}) \geq 0 \) when \( c \) is atleast \( 2\gamma \) for the tripartite scenario. Also, we found that \( \tilde{\Lambda}(|W\rangle \langle W|) \neq 0 \) for the above range of \( \gamma \). Finally following equation (8), \( W_{\text{GME}} = \tilde{\Lambda}(|W\rangle \langle W|) \) acts as GME witness. □

**Remark.** Since the proof works for any general \( \tilde{\Lambda} \) constructed from the Lindbladian structure, all the obtained maps in this paper have corresponding witnesses that can be obtained in a similar way.

2.1.2. Negativity for GME

In bipartite systems a well known measure for entanglement is Negativity and it is defined in terms of the negative eigenvalue of the partially transposed density matrix. Here, we move one step forward by introducing a similar measure for genuine multipartite entanglement (GME). Let us consider the lifted Transposition map \( \tilde{T} \).

One should note that \( \tilde{T} \) is not trace preserving. Therefore let us consider the normalized map \( \tilde{T} = \frac{1}{N} \tilde{T} \), where \( N \) is the normalizing factor. It is defined as follows,

\[
N_{\text{GME}}(\rho) = \frac{\|\tilde{T}[\rho]\|_1 - 1}{K},
\]

where \( K \) is a factor of normalization and to be chosen suitably.

**Proposition 3.** \( N_{\text{GME}} \) introduced in equation (9) is a valid entanglement measure.

**Sketch of the proof.** Let us here discuss the sketch of the proof and one can find the details in the appendix. To prove the above proposition, we must show that,

1. \( N_{\text{GME}}(\rho_{2-\text{sep}}) = 0 \),
2. \( N_{\text{GME}} \) is convex,
3. \( N_{\text{GME}} \) is monotone under LOCC.

Here \( \rho_{2-\text{sep}} \) denotes any bi-separable state and LOCC stands for local operation and classical communication. The detailed proof based on the above mentioned points can be found in the [appendix].

As an example of the usefulness of the measure, we calculate it for the state \( |W\rangle \) mixed with a maximally mixed state by a varying noise parameter \( p \) given by

\[
\rho_W = p|W\rangle \langle W| + (1 - p) \frac{I}{8}
\]

Note that the measure takes a non-zero value from \( p = 0.9 \) to \( p = 1 \) as there does not exist any negative eigenvalues of the density matrix after the action of the map other than the mentioned region. We show the variation of the same in the figure.

In this context, it is important to mention that in a recent work \([52]\), a robust measure of tri-partite entanglement has been constructed. In connection to that ‘concurrence triangle’ based measure defined in \([52]\), we want to comment that though the triangle measure is indeed more robust that the GME negativity based...
measure defined by us, it is restricted to the case of tri-partition. Though we have given example from three qubit system, the lifted map can be directly generalised to higher parties [31] and hence the negativity of GME can be consequently extended to the such cases. Therefore, though the negativity measure has certain shortcomings, it has other advantages in terms of possible extension to higher number of parties.

3. Positive maps on $\mathcal{M}_3$

In this section, we consider positive maps in higher dimensions, arising from the Lindblad structure. Other than generalising our method, the reason behind considering higher dimensional maps is to explore the possibilities of both decomposable and indecomposable maps arising from the structure of Lindbladians. One of the maps we will look into, can be called a Choi-like map which turns out to be decomposable and not stronger than the famous transposition map. On the other hand we will also derive the indecomposable Choi map useful for detecting bound entanglement by the proposed method.

Let $\{ G_i \}$ be the set of Gell Mann matrices with $i = 1, 2, 3, \ldots, 8$ and $G_9 = I_3$ is the 3-dimensional identity matrix. In the similar fashion with the previous section the algebra of $3 \times 3$ complex matrices is denoted by $\mathcal{M}_3$. We define a linear map, $\Phi: \mathcal{M}_3 \rightarrow \mathcal{M}_3$ such that,

$$\Phi[\rho] = (I + L)[\rho], \quad \forall \rho \in \mathcal{M}_3$$

where $L: \mathcal{M}_3 \rightarrow \mathcal{M}_3$ is another linear map given by,

$$L[\rho] = \sum_i x_i (G_i \rho G_i - \frac{1}{2} (G_i \rho + \rho G_i)).$$

From this structure, we are now going to construct various well known positive maps in the following.

3.1. Transposition map

Note that for some specific values for the Lindblad coefficients, we get back the transposition map. By putting $\gamma_2 = \gamma_4 = \gamma_5 = \gamma_7 = 1/2, \gamma_3 = \gamma_6 = \gamma_8 = -1/2, \gamma_9 = 1/6$, we have the transposition map. Now if we generalize the map by considering $\gamma_2 = \gamma_4 = \gamma_5 = \gamma_7 = \alpha, \gamma_3 = \gamma_6 = \gamma_8 = - \alpha$, and $\gamma_9 = \alpha/3$, we find an one parameter family of maps $\Phi_\alpha$, which is positive for the range $0 \leq \alpha \leq 1/2$.

First let us write the action of the map explicitly. $\Phi_\alpha: \mathcal{M}_3 \rightarrow \mathcal{M}_3$ is given by,

$$\Phi_\alpha(X) = \begin{bmatrix} x_{11} & x_{12} - 2x_{12} \alpha + 2x_{33} \alpha & x_{13} - 2x_{13} \alpha + 2x_{33} \alpha \\ x_{12} + 2x_{12} \alpha - 2x_{33} \alpha & x_{22} & x_{23} - 2x_{23} \alpha + 2x_{33} \alpha \\ x_{13} + 2x_{13} \alpha - 2x_{33} \alpha & x_{23} + 2x_{33} \alpha - 2x_{32} \alpha & x_{33} \end{bmatrix}$$

for any

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \in \mathcal{M}_3$$

To check the positivity of the map, it suffices to check positivity for pure states. Hence, for a general pure state $\rho = \ket{\psi}\bra{\psi}$ where $\ket{\psi} = [\psi_1, \psi_2, \psi_3]^T$, the matrix $\Phi_\alpha[\rho]$ has to be a positive semi-definite. Thus, the determinants of all the principal minors of the matrix must be greater than or equal to zero [53]. We have the

![Figure 2. Variation of the measure of GME with the white noise parameter introduced in $\rho_{W}$. For $p = 1$, it gives back $W$ state where the GME takes the maximum value.](image-url)
principal minors as, $2\alpha (2\alpha - 1)(\psi_2 \bar{\psi}_3 - \psi_3 \bar{\psi}_2)^2$, $2\alpha (2\alpha - 1)(\psi_1 \bar{\psi}_3 - \psi_3 \bar{\psi}_1)^2$, $2\alpha (2\alpha - 1)(\psi_1 \bar{\psi}_2 - \psi_2 \bar{\psi}_1)^2$. For the map to be positive, we must have the above mentioned quantities to be greater than or equal to zero which is obtained for $0 \leq \alpha \leq 1/2$. Therefore, the map is positive for this range of $\alpha$.

Note that for the map to be able to detect entanglement we need the map to be positive but not completely positive. Hence we check at what range of $\alpha$, the map is not completely positive. We do this by checking if the corresponding Choi matrix is a positive semi definite matrix or not. The Choi matrix corresponding to the map is given by,

$\Psi_C = (I \otimes \Phi_{\alpha})[\rho_{in}]$ with, $\rho_{in} = |\psi_{in} \rangle \langle \psi_{in}|$ where, $|\psi_{in} \rangle = \frac{1}{\sqrt{d}} \sum_{i=0}^{d-1} |ii \rangle$.

The distinct eigenvalues for the Choi matrix for the map $\Phi_{\alpha}$ are $\{1 - \frac{4\alpha}{3}, \frac{2\alpha}{3}, \frac{2\alpha}{3}\}$. Hence, the map is not completely positive for $\alpha \geq 0$ and finally positive but not completely positive for $0 \leq \alpha \leq 1/2$.

Next We show that the above class of maps is decomposable and cannot detect any PPT entangled states. We note that the action of the map $\Phi_{\alpha}$ can be decomposed as the following:

$$\Phi_{\alpha}(X) = (1 - 2\alpha) \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} + 2\alpha \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

for any $X \in \mathcal{M}_3$. Hence, the map can be written as,

$$\Phi_{\alpha} = (1 - 2\alpha)I + 2\alpha T_0 I$$

where $I$ and $T$ are the identity map and transposition map respectively. Identity map being a completely positive, $\Phi_{\alpha}$ becomes decomposable map in the range $0 \leq \alpha \leq 1/2$ as it can be written as sum of a completely positive map and a completely co-positive map.

### 3.2. Decomposable Choi map or reduction map

Note that for $\gamma_2 = 1/4, \gamma_3 = 1/4, \gamma_4 = 1/4, \gamma_5 = 1/4, \gamma_6 = 1/4, \gamma_7 = 1/4, \gamma_8 = 1/4, \gamma_9 = 1/12$, we have one of the Choi maps i.e.

$$\Phi_{\alpha}^2[\rho] = \begin{bmatrix} \rho_{22} + \rho_{33} & -\frac{\rho_{12}}{2} & -\frac{\rho_{11}}{2} \\ -\frac{\rho_{21}}{2} & \rho_{11} + \rho_{33} & -\frac{\rho_{23}}{2} \\ -\frac{\rho_{31}}{2} & -\frac{\rho_{32}}{2} & \rho_{11} + \rho_{22} \end{bmatrix}$$

It is interesting to note that this Choi map is nothing but the well known reduction map [54], which is of the form

$$\mathcal{R}(\rho) = \frac{1}{d - 1}(Tr(\rho)I - \rho),$$

where $d$ stands for dimension of the state. Similarly, we parameterize it with $\alpha$ i.e $\gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \gamma_7 = \gamma_8 = \alpha$ and $\gamma_9 = \alpha/3$, where we find the map $\Phi_{\alpha}^2$ is positive for the range $0 \leq \alpha \leq 1/4$. On the other hand the map is not complete positive for $\alpha > 3/16$. Hence, the map is Positive but not completely positive for $3/16 < \alpha \leq 1/4$. From theorem 3.4 of the Generalized Choi maps in three-dimensional matrix algebra of [55], we get a necessary and sufficient condition for a map on qutrit system to be decomposable. And the Choi like map $\Phi_{\alpha}^2$, we obtain is proved to be decomposable family of maps. We cannot obtain the original Choi map from this structure for any values of the $\gamma_i$’s.

### 3.3. Indecomposable Choi map

Though we cannot obtain the well known indecomposable Choi map from (11), we can however reconstruct the original Choi map by taking a combination sum of Lindblad-type sums. Consider the map $\Phi_{\alpha}^C$ given below

$$\Phi_{\alpha}^C(\rho) = \left\{ \sum_{j=1}^{3} A_j \rho A_j^\dagger - \frac{i}{2} \{ A_j, A_j^\dagger \}, \rho \right\} + \left\{ \sum_{j=1}^{3} B_j \rho B_j^\dagger - \frac{i}{2} \{ B_j, B_j^\dagger \}, \rho \right\} + \left\{ \sum_{k=1}^{3} C_k \rho C_k^\dagger - \frac{i}{2} \{ C_k, C_k^\dagger \}, \rho \right\}$$

where $A_1 = |1 \rangle \langle 2|, A_2 = |2 \rangle \langle 3|, A_3 = |3 \rangle \langle 1|$ and $B_1 = |1 \rangle \langle 1|, B_2 = |2 \rangle \langle 2|, B_3 = |3 \rangle \langle 3|$ and $C_1 = |1 \rangle \langle 1| - |2 \rangle \langle 2|, C_2 = |2 \rangle \langle 2| - |3 \rangle \langle 3|, C_3 = |3 \rangle \langle 3| - |1 \rangle \langle 1|$ respectively. Here $|1 \rangle, |2 \rangle, |3 \rangle$ denotes the computational basis in qutrit system. We can see that the following map

$$\Phi_{\alpha}^C(\rho) = \left[ I + \frac{1}{2} \Phi_{\alpha}^C \right](\rho),$$
is nothing but the well known Choi map,

$$\Phi_{C}^{\beta}[\rho] = \begin{bmatrix} \rho_{11} + \rho_{22} & -\rho_{12} & -\rho_{13} \\ -\rho_{21} & \rho_{22} + \rho_{33} & -\rho_{23} \\ -\rho_{31} & -\rho_{32} & \rho_{33} + \rho_{11} \end{bmatrix}. \quad (16)$$

We would want to parameterize the above structure to find a range of Choi-like indecomposable maps. We can consider the following one parameter family of maps,

$$\Phi_{b}^{\beta}[\rho] = S_1 - \beta(S_2 - S_3) \quad (17)$$

where $\beta$ is the single parameter and at $\beta = 1$, we get back the Choi map $\Phi_{C}^{C}$.

$$S_1 = \sum_{i=1}^{3} A_i \rho A_i^\dagger - \frac{1}{2}\{A_i A_i^\dagger, \rho\}$$

$$S_2 = \sum_{j=1}^{3} B_j \rho B_j^\dagger - \frac{1}{2}\{B_j B_j^\dagger, \rho\}$$

$$S_3 = \sum_{k=1}^{3} C_k \rho C_k^\dagger - \frac{1}{2}\{C_k C_k^\dagger, \rho\}$$

Note that, the map $\Phi_{C}^{b,\beta}$ (which is the $\beta$ parameterized linear map corresponding to the positive map $\Phi_{C}^{C}$) is a positive map for the range $0 \leq \beta \leq 1$ and it is completely positive for $\beta \leq 3/4$.

3.3.1. A note on GME detection

A direct extension of three qubit GME detection can be done for qutrit positive maps discussed in this section. Moreover, it can also be directly extended to the case of $n$-partite qudit quantum states, though the calculation of scaling factor $c$ as discussed in Statement 1 becomes more and more involved as we increase the number of parties. Moreover, as we construct this detection protocol for different positive maps, Proposition 1 is needed to be verified, since it is imperative for separating out the bi-separable states by employing Statement 1. A detailed discussion on this aspect can be found in [51].

4. Conclusion

In the literature of quantum information theory there exists an involved connection between entanglement detection and positive but not completely positive map. Also as entanglement is a basic and vastly used quantum resource in different information processing tasks, detecting it in an unknown system is one of the most crucial jobs in any experimental scenario. In this work, we give a generalized prescription to construct such positive maps from a given structure. Here we have investigated the structure of Lindblad superoperators for the case of qubits and qutrits and constructed various well known positive maps from them. We have found that in different parameter regions, we can reconstruct the transposition map and different variants of Choi map from the structure of the Lindbladians. We have further investigated a process of GME detection by exploiting positive maps, for the case of three qubits scenario. We have also constructed a class of linear GME witness operators. Moreover, we have also been able to propose a negativity measure for GME, based on a modified map constructed out of transposition. This gives a non zero value only for GME states, in turn distinguishing the bi-separable states from them. As an example we find the quantification of the measure for the noisy $W$ state. Our work can be directly generalized to higher dimensions and hence gives rise to a novel method to both detect and measure bi-partite and multi-partite entanglement.

Acknowledgments

BB acknowledges Ritabrata Sengupta for various discussions on the theory of positive maps. SG acknowledges partial support from the Department of Science and Technology, Government of India through the QuEST grant (grant number DST/ICPS/QUST/Theme-3/2019/120).

Data availability statement

No new data were created or analysed in this study.
Appendix

Proof of proposition 3

Proof. As mentioned in the main text, to prove that $\mathcal{N}_{\text{GME}}$ is a valid entanglement measure, it is enough to show 1. faithfulness, 2. convexity and 3. monotonicity under LOCC of the given measure.

1. For any Hermitian operator $A$, the trace norm $\|A\|_1 = \text{Tr} \sqrt{A^\dagger A}$ is the sum of the absolute values of the eigenvalues of $A$. Here we have, $\|\hat{T}[\rho_{2-\text{sep}}]\|_1 = 1$. Hence, $\mathcal{N}_{\text{GME}}(\rho_{2-\text{sep}}) = 0$.

2. To prove the convexity, it is enough to show that probabilistic mixing can not increase the value of $\mathcal{N}_{\text{GME}}$, i.e. $\mathcal{N}_{\text{GME}}(\sum_i r_i \sigma_i) \leq \sum_i r_i \mathcal{N}_{\text{GME}}(\sigma_i)$ where $\{r_i\}$ is a probability distribution. We recall that, $\mathcal{N}_{\text{GME}} = \frac{\|\hat{T}[\rho]\|_1 - 1}{K}$. Therefore the proof follows from the fact that the trace norm obeys triangle inequality and it is homogeneous of degree one for positive factors.

3. To prove the monotonicity of $\mathcal{N}_{\text{GME}}$ under LOCC, we have to keep in mind that here three parties viz. Alice, Bob and Charlie are present. In each round of the protocol, some of the parties performs some measurement in his/her own lab and let the other parties know about the outcome of the measurement. Depending upon that classical outcome, the subsequent party chooses his/her measurement in their respective lab. Mathematically speaking, if one party starts with a round with some initial state $\sigma$ and after performing some measurement obtains a result $j$ with the probability $q_j$ which in turn produces the output state $\sigma_j$, then an entanglement monotone $\mathcal{N}$ be such that

$$\mathcal{N}(\sigma) \geq \sum_j q_j \mathcal{N}(\sigma_j)$$

(18)

In our case one should keep in mind that $\mathcal{N}_{\text{GME}}$ does not make any distinction among Alice, Bob and Charlie. Hence it is sufficient to consider only one local measurement by any one of them. Without loss of generality lets say Alice starts the protocol. If the initial state is $\rho$, then after the measurement of Alice the final unnormalised state becomes

$$\mathcal{M}_\tau(\rho) = (\mathcal{M}_i \otimes \mathcal{I}_B \otimes \mathcal{I}_C) \rho (\mathcal{M}_i^\dagger \otimes \mathcal{I}_B \otimes \mathcal{I}_C)$$

where

$$\sum_i M_i^\dagger M_i \leq \mathcal{I}_A.$$

We note that,

$$\mathcal{M}_\tau(\mathcal{T} \otimes \mathcal{I}_B \otimes \mathcal{I}_C(\rho)) = \mathcal{T} \otimes \mathcal{I}_B \otimes \mathcal{I}_C(\mathcal{M}_\tau(\rho))$$

(19)

Due to symmetry, Bob and Charlie can perform the similar operation. Moreover this invariance is true even if they do nothing. In this scenario one round of LOCC is considered whenever Alice, Bob and Charlie complete their operation once. Hence following Vidal’s approach [14] we can establish the monotonicity of $\mathcal{N}_{\text{GME}}$ under LOCC. Hence the claim.

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