Stability and Control of Fluid Queueing Systems with Stochastically Switching Capacities

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Abstract

We consider a piecewise-deterministic queueing (PDQ) model to study traffic queues due to stochastic capacity fluctuations in transportation facilities. The saturation rate (capacity) of the PDQ model switches between a finite set of values (modes) according to a Markov chain. The inflow to the PDQ is controlled by a state-feedback policy. The main results of this article are stability conditions of PDQs, i.e. conditions under which the distribution of the queue length converges to a unique invariant probability measure. On one hand, a necessary condition for stability is that the average inflow does not exceed the average saturation rate. On the other hand, based on the Foster-Lyapunov criteria, we derive a sufficient condition that requires a bilinear matrix inequality to admit positive solutions and the invariant probability measure to be unique. We also study the rate of convergence for stable PDQs. Furthermore, for PDQs with two modes, a necessary and sufficient condition for stability is available. In addition, we present examples for the stability analysis of feedback control policies for single PDQs as well as a network of two PDQ links in parallel.

Index Terms

Traffic control, queueing systems, stability analysis, stochastic switching systems.

I. INTRODUCTION

Capacity fluctuations in transportation systems often cause congestion delays to travelers and efficiency losses to system operators [1], [2], [3]. Typical examples include incidents on freeways [4] and weather-related capacity drops at airports [5]. In many cases, capacity fluctuations are

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frequent and hard to predict deterministically [4], [5], [6]. Consequently, traffic management strategies that assume fixed (nominal) link capacities may lead to inefficient decisions when capacity fluctuations are not negligible (in terms of their intensity and frequency). In this article, we study a simple fluid queueing model that accounts for stochastically varying capacity of transportation systems, and investigate its stability under a class of feedback control policies.

We contribute to the growing literature on the analysis and control of transportation facilities (links or nodes) and networks with uncertain link/node capacities [5], [7], [8], [9]. Current approaches either assume a static (but uncertain) capacity model, or consider time-varying capacities. In the former class of models, the actual capacity is unknown to the system operator; however, its value is assumed to lie in a known set [9], [10], or is realized according to a given probability distribution [11], [12]. Such capacity models can be useful for evaluating the networks’ resilience against worst-case disturbances, such as cascading failures [9] or natural disasters [12]. However, in situations where capacity is inherently dynamic, the system operator can achieve better performance efficiency using control policies that account for the dynamic nature of capacity rather than using control policies designed for the worst-case capacity. The latter class of models, i.e. time-varying capacity models, are thus relevant in such situations. For example, Ziliaskopoulos [13] utilizes the knowledge of time-varying link capacities over a time horizon of interest for solving a dynamic traffic assignment problem in road networks. Similar deterministic time-varying capacity models have been proposed for control of air transportation systems [2], [14]. However, the work presented in this article is motivated by the practical situations where capacity can be naturally modeled as a stochastic process [5], [15], [16], [17] as opposed to a deterministic one.

Since we focus on the behavior of aggregate traffic flows under fluctuating capacities, fluid queueing models are better suited to our objectives than the conventional queueing models (e.g. $M/M/1$) [5], [18]. Fluid queueing systems with stochastically switching saturation rates have been studied previously; see [7], [21], [22], [23], [24] for notable contributions. This line of work focuses on the analysis of the stationary distribution of queue length under uncontrolled (or open-loop) inflow. Some results are available on feedback-controlled fluid queueing systems with

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1It is well known that the stability of conventional queueing models is closely related to the stability of their fluid counterparts [19], [20].
stochastic capacities. For example, Peterson et al. [5] considered a Markov/semi-Markov capacity model and presented a recursive approach to derive the expected queue length under controlled inflows. In a related work, Yu and Cassandras [8] developed an unbiased gradient estimator for throughput and queue length of feedback-controlled stochastic fluid queueing systems with respect to the buffer storage capacity. The authors of [5], [8] focused on finite-horizon stochastic optimization problems for feedback-controlled systems. However, our focus is on the stability of these systems. Specifically, we relate the finiteness and convergence (in terms of its distribution) of the queue length to the stability of stochastic fluid queueing systems under a class of feedback control policies.

In Section II, we formally introduce our fluid queueing model with stochastically switching saturation rate (capacity). This model is called the piecewise-deterministic queueing (PDQ) model, since the saturation rate switches between a finite set of values, or modes, according to a Markov chain, while the evolution of queue length between mode switches is deterministic. Our PDQ model is a special case of the (more general) piecewise-deterministic Markov processes (PDMP, see [25]). The PDQ model can be calibrated using publicly available traffic data [6]. Furthermore, the capacity and/or the queue length can be measured using modern traffic sensing technologies [5], [26], [27], [28], [29]. Thus, the problem of implementing capacity-aware control policies become relevant from a practical viewpoint. In this article, we use the knowledge of the frequency and intensity of capacity fluctuation to design feedback control policies that guarantee the convergence of traffic queue in a link. Given the widespread applicability of fluid queueing models [18], [20], [30], [31], [32], we hope that our analysis can be useful to study the performance of transportation systems with stochastic capacity fluctuation.

The main results of this article (Theorems 1–3) provide stability conditions for feedback-controlled PDQs. We derive these results under fairly mild assumptions: (i) the mode transition process is ergodic, (ii) the feedback control policy is bounded, continuous, and non-increasing in the queue length, and (iii) the PDQ dynamics in various modes do not have an identical attracting state. Following [33], [34], we say that a PDQ is stable if the joint distribution of its states (the mode and the queue length) converges to a unique invariant probability measure, which is essentially a distribution invariant with respect to the system’s stochastic dynamics.

Our work builds on known results on stability analysis of continuous-time Markov processes [34], [35], [36] and properties of PDMPs [25], [33]. A necessary condition for a Markov process
to be stable is that the process is non-evanescent, i.e. that the state is finite almost surely. A
sufficient condition is that the process admits a Lyapunov function satisfying the Foster-Lyapunov
drift condition \([34]\) and has a unique invariant probability measure. We build on the above results
and derive stability conditions that can be directly applied to feedback controlled PDQs. We show
that, in general, there is a gap between the necessary condition and the sufficient condition;
however, under particular assumptions, these conditions are equivalent.

In Section III, we derive a necessary condition (Theorem 1) and a sufficient condition (Theo-
rem 2) for the stability of feedback-controlled PDQs. The necessary condition is that the time-
average inflow does not exceed the time-average saturation rate. The sufficient condition requires
the infima of the control policy in various modes to verify a bilinear matrix inequality (BMI),
which essentially imposes a bound on the controlled inflow. The BMI results from the application
of the Foster-Lyapunov drift condition to our setting with a suitably chosen Lyapunov function.
The sufficient condition also restricts the form of the control policy to ensure the uniqueness of
the invariant probability measure. Importantly, our sufficient condition also leads to an estimate
of an exponential rate of convergence towards the invariant probability measure. We use these
results to study the stability of several practically relevant control policies, including mode-
responsive and linear feedback control policies.

In Section IV, we analyze PDQs with two modes, or bimodal PDQs (BPDQs). These simpler
systems are of practical interest, since they can model traffic links that stochastically switch
between a nominal mode and a disrupted mode (with reduced or zero capacity). In addition,
our analysis of BPDQs provides a clear intuition and some useful insights. The main result of
this section (Theorem 3) is that a controlled BPDQ is stable if and only if a weighted average
of the infima of the control policy in various modes is less than the average saturation rate. In
addition, for stable BPDQs, we are able to derive bounds on the steady-state queue length and
the exponential rate of convergence to the invariant probability measure, in terms of the model
parameters and characteristics of the control policy.

In Section V, we illustrate some applications of our results to traffic routing under stochastic
capacity fluctuation. We consider the stability of typical traffic routing policies over a network of
two links in parallel. We apply our results to analyze the stability of static and mode-responsive
routing policies. Under particular assumptions, we also compute optimal routing policies that
minimize the expected travel time. Finally, we study the stability and performance of the well-
known logit routing model [9], [37], which, for our purposes, can be viewed as a queue-responsive routing policy.

II. PIECEWISE-DETERMINISTIC QUEUEING MODEL

In this section, we formally define the PDQ model, and introduce the assumptions that we use in our analysis.

\[ f(t) \xrightarrow{\text{buffer}} q(t) \xrightarrow{\text{buffer}} r(t) \]

Fig. 1. An illustration of the PDQ model.

Consider the system in Figure 1. Traffic arrives at the system at the inflow rate \( f(t) \) at time \( t \geq 0 \). We assume that \( f(t) \) can take values in an admissible set \( \mathcal{F} \subseteq \mathbb{R}_{\geq 0} \). Traffic is temporarily stored in a queueing buffer and discharged downstream. We denote the queue length by \( q(t) \). Let \( u(t) \) denote the saturation rate, i.e. the maximum rate at which traffic can be released. If \( q(t) = 0 \) and \( f(t) \leq u(t) \), the discharge rate \( r(t) \), i.e. the rate at which traffic departs from the system, is given by \( r(t) = f(t) \); otherwise \( r(t) = u(t) \). Figure 2 shows a typical evolution of queue length \( q(t) \) with time. We assume an infinite buffer size; i.e. \( q(t) \) can take value in the set \( Q := \mathbb{R}_{\geq 0} \). This assumption is made to account for all the traffic arriving at the system and not just the traffic that is ultimately discharged by the system.

Fig. 2. A typical sample path of \( q(t) \) (bottom) for a PDQ with a constant inflow rate \( f(t) \) and a piecewise-constant saturation rate \( u(t) \) (top). The queue length grows when \( f(t) > u(t) \), and reduces when \( f(t) < u(t) \).
The saturation rate of the PDQ model stochastically switches between a finite set of values. Specifically, let $I$ be the set of modes of the PDQ and $m = |I|$. We denote the mode of the PDQ at time $t$ by $i(t)$. Each mode $i \in I$ is associated with a fixed saturation rate, denoted by $u_i$, which is distinct for each mode; i.e., for all $i, j \in I$, $u_i \neq u_j$ if $i \neq j$. The evolution of mode $i(t)$ is governed by a finite-state Markov process with state space $I$ and constant transition rates $\{\lambda_{ij}; i, j \in I\}$. We assume that $\lambda_{ii} = 0$ for all $i \in I$\footnote{This is without loss of generality since self-transitions do not change the saturation rate; thus, including them will not affect the PDQ dynamics.}

Let

$$\nu_i := \sum_{j \in I} \lambda_{ij},$$

which is the rate at which the system leaves mode $i$. Given an initial mode $i_0 \in I$ at $t = t_0 = 0$, let $\{t_k; k = 1, 2, \ldots\}$ be the epochs at which the mode transitions occur. Let $i_{k-1}$ be the mode during $[t_{k-1}, t_k)$ and $s_k = t_k - t_{k-1}$; see Figure 2 Then, $s_k$ follows an exponential distribution with the cumulative distribution function (CDF) \footnote{This is without loss of generality since self-transitions do not change the saturation rate; thus, including them will not affect the PDQ dynamics.}:

$$F_{s_k}(s) = 1 - \exp(-\nu_{i_{k-1}}s), \quad k = 1, 2 \ldots$$

We can write the transition rates in the $m \times m$ matrix:

$$\Lambda := \begin{bmatrix}
-\nu_1 & \lambda_{12} & \cdots & \lambda_{1m} \\
\lambda_{21} & -\nu_2 & \cdots & \lambda_{2m} \\
\vdots & \vdots & \ddots & \vdots \\
\lambda_{m1} & \lambda_{m2} & \cdots & -\nu_m
\end{bmatrix}.$$  

Our first assumption is concerned with the mode transition process:

**Assumption 1.** The finite-state Markov process \{i(t); t \geq 0\} is ergodic.

This assumption ensures that the process \{i(t); t \geq 0\} converges towards a unique steady-state distribution, i.e. a row vector $p = [p_1, \ldots, p_m]$ satisfying the following \footnote{This is without loss of generality since self-transitions do not change the saturation rate; thus, including them will not affect the PDQ dynamics.} [38, Theorem 7.2.7]:

$$p\Lambda = 0, \quad pe = 1, \quad p \geq 0, \quad (4)$$

where $e$ is the $m$-dimensional vector of 1’s.

In addition, we define the effective capacity of a PDQ as

$$\bar{u} = pu, \quad (5)$$
where

\[ u := [u_1, u_2, \ldots, u_m]^T. \]  \tag{6} \]

Under Assumption 1 the limiting time-average saturation rate is given by \( \overline{u} \).

The discharge rate \( r \) of the PDQ can be written as a function of the mode \( i \), the queue length \( q \), and the inflow \( f \):

\[ r(i, q, f) := \begin{cases} f, & q = 0, f \leq u_i, \\ u_i, & \text{o.w.} \end{cases} \]  \tag{7} \]

Then, the evolution of the hybrid state \( (i(t), q(t)) \) of the PDQ is specified by the matrix \( \Lambda \) and a vector field \( D : \mathcal{I} \times \mathcal{Q} \times \mathcal{F} \rightarrow \mathbb{R} \) such that

\[ \begin{align} i(0) &= i_0, \quad q(0) = q_0, \quad (i_0, q_0) \in \mathcal{I} \times \mathcal{Q}, \\ \Pr\{i(t + \Delta t) = j|i(t) = i\} &= \lambda_{ij} \Delta t + o(\Delta t), \\ \frac{dq(t)}{dt} &= D\left( i(t), q(t), f(t) \right) := f(t) - r\left( i(t), q(t), f(t) \right). \end{align} \]  \tag{8} \]

We can now formally define PDQs as follows:

**Definition 1.** A piecewise-deterministic queueing system is a tuple \( \langle \mathcal{I}, \mathcal{Q}, \mathcal{F}, \Lambda, D \rangle \), where

- \( \mathcal{I} \) is the finite discrete state space,
- \( \mathcal{Q} = \mathbb{R}_{\geq 0} \) is the continuous state space,
- \( \mathcal{F} \subseteq \mathbb{R}_{\geq 0} \) is the set of admissible inflow values,
- \( \Lambda \) is the matrix that governs the evolution of the discrete state, and
- \( D : \mathcal{I} \times \mathcal{Q} \times \mathcal{F} \rightarrow \mathbb{R} \) is the vector field that governs the evolution of the continuous state.

We allow the inflow \( f \) to be specified by a control policy \( \phi : \mathcal{I} \times \mathcal{Q} \rightarrow \mathcal{F} \), i.e. \( f(t) = \phi(i(t), q(t)) \). Under such a control policy, the PDQ becomes a controlled Markov process \(39\). For a given \( \phi \), the queue length evolves according to the vector field \( D(i, q, \phi(i, q)) = \phi(i, q) - r(i, q, \phi(i, q)) \).

For notational simplicity, we will henceforth use \( r(i, q, \phi) \) to denote \( r(i, q, \phi(i, q)) \) and \( D(i, q, \phi) \) to denote \( D(i, q, \phi(i, q)) \).

Note that a control policy need not be responsive to both \( i \) and \( q \). If \( \phi \) only depends on the mode, we call it a mode-responsive control policy and denote it for short as \( \phi(i) \). Similarly, if
φ only depends on the queue length, we call it a *queue-responsive* control policy and denote it as φ(q).

Our second assumption restricts the class of control policies that we consider in this article:

**Assumption 2.** The control policy φ : I × Q → F is bounded, non-increasing, and continuous in the second argument.

Assumption 2 is motivated by the control policies deployed in transportation systems which restrict the inflows for large queue lengths; see e.g. [9], [40], [41].

Under Assumption 2 one can check that, for any initial condition (i₀, q₀) ∈ I × Q, the integral curve induced by the vector field D(i₀, q, φ) is unique and continuous. Furthermore, q(t) is not reset after mode transitions. Thus, the controlled process {((i(t), q(t)); t ≥ 0} is a right continuous with left limits (RCLL, or càdlàg) PDMP [25].

Assumptions 1 and 2 enable us to focus on the stability issues arising from the interplay between the stochastic saturation rate (supply) and the controlled inflow (demand). Under these assumptions, we can exclude stability concerns arising from non-ergodicity of the mode transitions and lack of regularity of the control policy.

Next, we follow [33], [34], [36] and introduce some standard definitions for our subsequent analysis.

The *transition kernel* of a PDQ at time t ≥ 0 is a map Pᵣ from I × Q to the set of probability measures on I × Q. Essentially, for a measurable set A ⊆ I × Q, Pᵣ(i₀, q₀; A) is the probability of (i(t), q(t)) ∈ A given the initial condition (i₀, q₀) ∈ I × Q. We also consider Pᵣ as an operator acting on probability measures µ on I × Q via

$$\mu Pᵣ(A) = \int_{I \times Q} Pᵣ(i, q; A)d\mu. \quad (11)$$

An *invariant probability measure* of a PDQ with control policy φ is a probability measure µ_φ on I × Q such that

$$\mu Pᵣ = \mu_φ, \quad \forall t ≥ 0.$$  

³We use the subscript φ to emphasize the dependence of µ_φ on φ.
Since the process \( \{i(t), q(t); t \geq 0\} \) is RCLL, following [25, Theorem 5.5], the infinitesimal generator \( \mathcal{L} \) of a PDQ with control policy \( \phi \in \Phi \) is given by

\[
\mathcal{L} g(i, q) = D(i, q, \phi) \frac{\partial}{\partial q} g(i, q) + \sum_{j \in I} \lambda_{ij} \left( g(j, q) - g(i, q) \right),
\]

\((i, q) \in I \times Q, \quad (12)\)

where \( g \) is any function on \( I \times Q \) smooth in the continuous argument.

A controlled PDQ is non-evanescent if, for any \((i_0, q_0) \in I \times Q\),

\[
\Pr \left\{ \lim_{t \to \infty} q(t) = \infty \mid i(0) = i_0, q(0) = q_0 \right\} = 0;
\]

\((13)\)

i.e., the PDQ is non-evanescent if the queue length is finite almost surely (a.s.).

A controlled PDQ is stable if, for any initial condition \((i_0, q_0) \in I \times Q\), the joint distribution of \( i \) and \( q \) converges to a unique invariant probability measure \( \mu_{\phi} \), i.e.

\[
\lim_{t \to \infty} \| P_t(i_0, q_0; \cdot) - \mu_{\phi}(\cdot) \|_{TV} = 0, \quad \forall (i_0, q_0) \in I \times Q,
\]

\((14)\)

where \( \| \cdot \|_{TV} \) is the total variation distance[^1]. A PDQ is said to be unstable if \((14)\) does not hold.

In addition, a controlled PDQ is exponentially stable if it is stable and there exist constants \( B > 0 \) and \( c > 0 \), and a function \( W : I \times Q \to [1, \infty) \) going to infinity as \( q \to \infty \), such that, for any \((i_0, q_0) \in I \times Q\),

\[
\| P_t(i_0, q_0; \cdot) - \mu_{\phi}(\cdot) \|_{TV} \leq BW(i_0, q_0) \exp(-ct), \quad \forall t \geq 0.
\]

**Remark 1.** As pointed out by Meyn and Tweedie [34], stability necessarily implies non-evanescence. In other words, if \( \Pr \{ \lim_{t \to \infty} q(t) = \infty \mid i(0) = i_0, q(0) = q_0 \} > 0 \) for some \((i_0, q_0) \in I \times Q\), then the PDQ is unstable; such a case may happen when the inflow rate is too large in comparison to the effective capacity.

Before proceeding further, let us consider the PDQ dynamics under a special class of control policies with the following property:

\[
\exists \hat{q} \in Q, \forall i \in I, \phi(i, \hat{q}) = u_i.
\]

[^1]: The total variation distance between probability measures \( \mu_1, \mu_2 \) on \( I \times Q \) is defined as \( \| \mu_1 - \mu_2 \|_{TV} = \sup_{A \subseteq I \times Q} \left\{ \int_A |d\mu_1 - d\mu_2| \right\} \).
Then, two cases can arise:

If \( \hat{q} \) is not unique, then the PDQ admits multiple invariant probability measures. Consider, for example, a PDQ with a mode-responsive control policy \( \phi(i, q) = u_i \) for all \((i, q) \in \mathcal{I} \times \mathcal{Q}\). Under this control policy, one can see from (7) and (10) that \( q(t) = q(0) = q_0 \) for all \( q_0 \in \mathcal{Q} \) and for all \( t \geq 0 \). Thus, every probability measure taking the form \( \mu(i, dq) = p_i \delta_{q_0} \) is invariant, where \( \delta_{q_0} \) is the Dirac delta function centered at \( q_0 \).

If \( \hat{q} \) is unique, noting Assumptions 1 and 2, one can show that, for any initial condition \((i_0, q_0) \in \mathcal{I} \times \mathcal{Q}\),

\[
\lim_{t \to \infty} q(t) = \hat{q}, \quad a.s.
\]

That is, \( \hat{q} \) is the unique attracting state of the PDQ. In other words, the (marginal) distribution of the queue length at steady-state is degenerate (i.e. supported over a single point \( \hat{q} \)). This case is of limited practical interest, since it entails ensuring that the PDQ dynamics in various modes have an exactly identical attracting state.\(^5\)

Based on the above arguments, we do not consider the control policies satisfying (15). Specifically, in addition to Assumption 2, we impose the following assumption on the control policies:

**Assumption 3.** For every \( q \in \mathcal{Q} \), there exists \( i \in \mathcal{I} \) such that \( \phi(i, q) \neq u_i \).

We use \( \Phi \) to denote the set of control policies satisfying Assumptions 2 and 3.

Finally, we emphasize that all our subsequent analysis are based on Assumptions 1–3.

### III. Stability of Feedback-Controlled PDQs

In this section, we study the stability of PDQs with feedback control policies. The main results are Theorems 1 and 2, which are a necessary condition and a sufficient condition for the stability of controlled PDQs, respectively. Their proofs are provided in Sections III-A and III-B. Some applications of these results are described in Section III-C.

The necessary condition requires that the time-average inflow does not exceed the effective capacity:

\(^5\)In fact, this case (unique \( \hat{q} \)) does not arise for mode-responsive or queue-responsive control policies.
Theorem 1. If a PDQ $(I, Q, F, \Lambda, D)$ with a control policy $\phi \in \Phi$ is stable, then there exists $\bar{\phi} > 0$ such that, for any $(i(0), q(0)) = (i_0, q_0) \in I \times Q$,
\[
\lim_{t \to \infty} \frac{1}{t} \int_0^t \phi(i(\tau), q(\tau)) d\tau \overset{a.s.}{=} \bar{\phi} \leq \bar{u},
\]
where $\bar{u}$ is the effective capacity of the PDQ, given by (5).

Theorem 1 provides a way of identifying unstable control policies. Specifically, given a control policy $\phi \in \Phi$, if $\bar{\phi} > \bar{u}$, then the PDQ is unstable. To apply this theorem, one needs to compute $\bar{\phi}$. If $\phi$ is mode-responsive, one has $\bar{\phi} = \sum_{i \in I} p_i \phi(i)$. However, if $\phi$ also depends on the queue length, then $\bar{\phi}$ is, in general, not easy to compute. In this case, one can check a (weaker) necessary condition as follows. By Assumption 2, for any $i \in I$, $\inf_{q} \phi(i, q)$ necessarily exists. Let
\[
\phi_{\inf} = \left[ \inf_{q} \phi(1, q), \inf_{q} \phi(2, q), \ldots, \inf_{q} \phi(m, q) \right]^T.
\]
(17)
Then, Theorem 1 leads to the following result:

Corollary 1. If a PDQ $(I, Q, F, \Lambda, D)$ with a control policy $\phi \in \Phi$ is stable, then
\[
\bar{\phi}_{\inf} := p\phi_{\inf} \leq \bar{u},
\]
(18)
where $p$ is the solution to (4).

Note that (18) is easier to check than (16), since the former only involves the infima.

Remark 2. In Section III-C we provide an example to show that the upper bound in (16) is attainable in some situations.

Our next result provides a sufficient condition for the stability of feedback-controlled PDQs:

Theorem 2. A PDQ $(I, Q, F, \Lambda, D)$ with a control policy $\phi \in \Phi$ is stable if there exist $b > 0$ and $a = [a_1, \ldots, a_m]^T > 0$ such that
\[
\left( \text{diag}(\phi_{\inf} - u) \right) b + \Lambda \right) a \leq -e,
\]
(19)
where $\phi_{\inf}$ is defined in (17), $u$ is defined in (6), and $e$ is the $m$-dimensional vector of 1’s. Furthermore, under (19), there exists a positive constant $c = \min_i 1/(2a_i)$ such that, for some $B > 0$,
\[
\|P_i(i_0, q_0; \cdot) - \mu_\phi(\cdot)\|_{TV} \leq B \left( a_{i_0} \exp(bq_0) + 1 \right) \exp(-ct),
\]
∀(i_0, q_0) ∈ \mathcal{I} \times \mathcal{Q}, \forall t \geq 0, \quad (20)

where $\mu_{\phi}$ is the unique invariant probability measure.

Theorem 2 essentially imposes upper bounds on the infima of the controlled inflow in each mode. This result is based on the Foster-Lyapunov drift condition \cite{34}. In fact, the positive constants $b$ and $a$ are parameters of a Lyapunov function for the controlled PDQ (as defined in Section III-B). To check this condition, one needs to determine whether the bilinear matrix inequality (BMI) (19) admits a strictly positive solution for the scalar $b$ and the vector $a$. This can be done using the known computational methods to solve BMIs (see e.g. \cite{42}.) However, if the control policy has a special structure, solutions for $b$ and $a$ can be constructed in a more straightforward manner; see Section III-C for an example.

Remark 3. The BMI (19) only involves the infima of the control policy. However, the constant $c$ in (20) depends on the specific form of $\phi$; see Section III-B.

In general, the necessary condition (Theorem 1) and the sufficient condition (Theorem 2) here are not equivalent. We provide an example in Section III-C to illustrate this point. However, in Section IV, we present a sufficient and necessary condition for the stability of PDQs with two modes.

A. Proof of Theorem 1 and Corollary 1

Suppose that the PDQ $\langle \mathcal{I}, \mathcal{Q}, \mathcal{F}, \Lambda, D \rangle$ with the control policy $\phi \in \Phi$ is stable.

From (10), we obtain that $q(t) = \int_{\tau=0}^{t} (\phi(\tau) - r(\tau)) d\tau + q(0)$ for $t \geq 0$. We know that $\lim_{t \to \infty} q(0)/t = 0$ for all $q(0) = q_0 \in \mathcal{Q}$. Thus, we have

$$0 = \lim_{t \to \infty} \frac{1}{t} \left( \int_{\tau=0}^{t} (\phi(\tau) - r(\tau)) d\tau + q(0) - q(t) \right)$$

$$= \lim_{t \to \infty} \frac{1}{t} \left( \int_{\tau=0}^{t} (\phi(\tau) - r(\tau)) d\tau - q(t) \right). \quad (21)$$

By (7), $r(\tau) \leq u(\tau)$ for all $\tau \geq 0$. Therefore, (21) yields

$$\lim_{t \to \infty} \frac{1}{t} \left( \int_{\tau=0}^{t} (\phi(\tau) - u(\tau)) d\tau - q(t) \right) \leq 0. \quad (22)$$
Now we show the existence of the limit on the left-hand side of (22). First, note that, if the PDQ is stable and thus non-evanescent (recall Remark 1), then

$$\lim_{t \to \infty} q(t)/t = 0, \quad a.s.$$ \hfill (23)

Second, for every $i \in \mathcal{I}$, let $S_i(t)$ be the amount of time that the PDQ is in mode $i$ up to time $t$, i.e.

$$S_i(t) = \int_{\tau=0}^{t} 1_{i(\tau) = i} d\tau.$$  

Then, by [38, Theorem 7.2.6], we have

$$\lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^{t} u(\tau) d\tau = \lim_{t \to \infty} \frac{1}{t} \sum_{i \in \mathcal{I}} S_i(t) u_i = \sum_{i \in \mathcal{I}} p_i u_i,$$ \hfill (24)

Third, the ergodic theorem of Markov processes ([43, p. 169], see also [35]) implies that

$$\lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^{t} \phi(i(\tau), q(\tau)) d\tau = \int_{\mathcal{I} \times \mathcal{Q}} \phi(i, q) d\mu_\phi, \quad a.s. \hfill (25)$$

The existence of the integral on the right-hand side of (25) is guaranteed by Assumption 2. Hence, we can let

$$\bar{\phi} = \int_{\mathcal{I} \times \mathcal{Q}} \phi(i, q) d\mu_\phi.$$ \hfill (26)

Combining (23)–(25), we can conclude that the limit on the left-hand side of (22) exists. Finally, we can write

$$\bar{\phi} - \bar{u} = \lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^{t} \phi(\tau) d\tau - \lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^{t} u(\tau) d\tau \quad a.s. \hfill (22)$$

$$\leq 0,$$

which implies (16). Thus, we have proved Theorem 1.

Then, Corollary 1 can be obtained in a straightforward manner:

$$\bar{\phi}_{\inf} \sum_{i \in \mathcal{I}} p_i \inf_w \phi(i, w) = \int_{\mathcal{I} \times \mathcal{Q}} \inf_w \phi(i, w) d\mu_\phi.$$
\[
\lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^{t} \inf_{w} \phi(i(\tau), w) \, d\tau \quad \text{a.s.}
\]

\[
\leq \lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^{t} \phi(i(\tau), q(\tau)) \, d\tau \quad \text{a.s.}
\]

\[
\leq \frac{\|u\|}{\bar{u}}.
\]

**B. Proof of Theorem 2**

The proof of Theorem 2 is based the Foster-Lyapunov criteria ([34, Theorem 6.1], see also [36, Theorem 2.11]), which we recall here for the sake of completeness. The criteria require the following two conditions:

(A) There exist a norm-like function \( V : \mathcal{I} \times \mathcal{Q} \to \mathbb{R}_{\geq 0} \) (called the Lyapunov function) and constants \( c > 0 \) and \( d < \infty \) such that

\[
L V(i, q) \leq -c V(i, q) + d, \quad \forall (i, q) \in \mathcal{I} \times \mathcal{Q}. \tag{27}
\]

(B) For every \( C > 0 \) and for any two initial conditions \( (i_1, q_1), (i_2, q_2) \in \{(i, q) \in \mathcal{I} \times \mathcal{Q} | V(i, q) \leq C\} \), there exist \( \delta > 0 \) and \( T > 0 \) such that

\[
\|P_T(i_1, q_1; \cdot) - P_T(i_2, q_2; \cdot)\|_{TV} \leq 1 - \delta. \tag{28}
\]

Condition (A) is often referred to as the drift condition, which essentially ensures the existence of invariant probability measures [34, Theorem 4.5]. In addition, condition (B) is required for the uniqueness of the invariant probability measure [44].

For PDQs (and more generally, PDMPs), consider the following condition introduced by Benaïm et al [33]:

(B') There exists \( q^* \in \mathcal{Q} \) such that, for any \( \epsilon > 0 \) and any initial condition \( (i_0, q_0) \in \mathcal{I} \times \mathcal{Q} \), there exists \( T > 0 \) such that

\[
\Pr\left\{ q(T) \in \left( (q^* - \epsilon, q^* + \epsilon) \cap \mathcal{Q} \right) \mid i(0) = i_0, q(0) = q_0 \right\} > 0, \tag{29}
\]

The argument in [33, Theorem 4.6] can be adapted to conclude the uniqueness of the invariant probability measure under condition (B').

\[\text{According to [34], a function } V : \mathcal{I} \times \mathcal{Q} \to \mathbb{R}_{\geq 0} \text{ is norm-like if } \lim_{q \to \infty} V(i, q) = \infty \text{ for all } i \in \mathcal{I}.\]
The Foster-Lyapunov criteria state that, under conditions (A) and (B) (or (B’)), the controlled PDQ admits a unique invariant probability measure $\mu_\phi$; furthermore, there exists $B > 0$ such that, for all $(i_0, q_0) \in \mathcal{I} \times \mathcal{Q}$ and all $t \geq 0$,

$$\|P_t(i_0, q_0; \cdot) - \mu_\phi(\cdot)\|_{\text{TV}} \leq B(V(i_0, q_0) + 1) \exp(-ct).$$

In this article, we consider the Lyapunov function

$$V(i, q) = a_i \exp(bq), \quad (i, q) \in \mathcal{I} \times \mathcal{Q},$$

where $a_1, \ldots, a_m$, and $b$ are positive constants.

We are now ready to prove Theorem 2:

**Proof of Theorem 2.** Suppose that the hypotheses in Theorem 2 hold. From these conditions, we derive conditions (A) and (B’) in two separate steps.

**Condition (A):**

Let

$$\eta = \min_{i \in \mathcal{I}} 1/(2a_i b).$$

By the definition of infima, for each $i \in \mathcal{I}$, there (necessarily) exists a sufficiently large $L_i > 0$ such that $\phi(i, L_i) - \inf_q \phi(i, q) \leq \eta$. Let $L = \max_i L_i$. From Assumption 2, we note that $L$ satisfies

$$\phi(i, L) - \inf_q \phi(i, q) \leq \eta, \quad \forall i \in \mathcal{I}.$$  

One can interpret $L$ as a quantity that characterizes how fast $\phi(i, q)$ approaches $\inf_w \phi(i, w)$ as $q$ increases.

We claim that the constants

$$c := \min_{i \in \mathcal{I}} \{1/(2a_i)\},$$

$$d := \max_{(i, q) \in \mathcal{I} \times [0, L]} |\mathcal{L}V(i, q) + cV(i, q)|,$$

verify the drift condition (27). Let us prove this claim.

Plugging the Lyapunov function as defined in (30) into the expression of the infinitesimal generator $\mathcal{L}$, we have

$$\mathcal{L}V(i, q)$$
\[ (\phi(i, q) - r(i, q, \phi)) a_i b + \sum_{j \in I} \lambda_{ij} (a_j - a_i) \] \exp(bq). \quad (34)

To check the drift condition (27), we need to consider two cases:

**Case A.1:** \( q > L \). Since \( \phi(i, q) \leq \phi(i, L) \) (recall that \( \phi \) is by assumption non-increasing in \( q \)) and \( r(i, q, \phi) = u_i \) (see (7)), we have, for all \( i \in I \),

\[ \left( \phi(i, q) - r(i, q, \phi) \right) a_i b + \sum_{j \in I} \lambda_{ij} (a_j - a_i) \leq \left( \phi(i, L) - u_i \right) a_i b + \sum_{j \in I} \lambda_{ij} (a_j - a_i) \tag{32} \]

\[ \leq \left( \inf_w \phi(i, w) + \eta - u_i \right) a_i b + \sum_{j \in I} \lambda_{ij} (a_j - a_i) \tag{19} \]

\[ \leq -1 + a_i b \eta \leq -\frac{1}{2}. \tag{31} \]

Then, we have

\[ \mathcal{L}V(i, q) \leq -\frac{1}{2} \exp(bq) \leq -ca_i \exp(bq) \leq -cV. \tag{30} \]

Hence, (27) holds for all \( i \in I \) and \( q > L \), for any \( d \geq 0 \).

**Case A.2:** \( q \leq L \). We can verify in a straightforward manner that, with \( c \) and \( d \) given by (33), \( \mathcal{L}V \leq -cV + d \) for all \( i \in I \) and all \( q \leq L \).

Thus, the drift condition (27) holds for all \( (i, q) \in I \times Q \).

**Condition (B’):**

To derive (B’), we first choose a \( q^* \), and then verify (29) for the chosen \( q^* \).

For each \( i \in I \), let \( E_i \) denote the set of solutions to the following equation in \( q \):

\[ \phi(i, q) = r(i, q, \phi). \]

Because of Assumption [2] and the definition of \( r \) in (7), \( E_i \) may be a singleton, an interval, or an empty set.

For a given \( i \in I \), by (7), \( E_i \neq \emptyset \) if \( \inf_q \phi(i, q) < u_i \). In fact, (19) ensures the existence of at least one \( i \in I \) such that \( \inf_q \phi(i, q) < u_i \). To see this, assume by contradiction that \( \inf_q \phi(i, q) \geq u_i \) for all \( i \in I \). Then, letting \( j = \arg \min_{i \in I} a_i \), we have

\[ \left( \inf_q \phi(j, q) - u_j \right) b_a + \sum_{j \in I} \lambda_{ij} (a_i - a_j) \geq 0, \]
which contradicts (19).

Let \( I_1 \) be the set of modes \( i \) such that \( \mathcal{E}_i \) is non-empty. For \( i \in I_1 \), we denote \( \mathcal{E}_i = [l_i^1, l_i^2] \) (which may reduce to a singleton if \( l_i^1 = l_i^2 \)). Then, we choose \((i^*, q^*)\) as follows:

\[
i^* = \arg \min_{i \in I_1} l_i^2, \quad q^* = \min_{i \in I_1} l_i^2.
\]

Now, we need to verify (29), i.e. that the interval \((q^* - \epsilon, q^* + \epsilon) \cap Q\) can be attained from any initial condition with positive probability. We need to consider two cases.

**Case B.1:** \( q^* = 0 \). Consider an initial condition \( i(0) = i_0, q(0) = q_0 \). By Assumptions 1 and 2.1, there exists a sufficiently large time \( X > 0 \) and a sufficiently large queue length \( \hat{L} > 0 \), such that

\[
\Pr \left\{ i(X) = i^*, q(X) \leq \hat{L} \mid i(0) = i_0, q(0) = q_0 \right\} = P_1
\]

for some \( P_1 > 0 \). Note that \( \phi(i^*, q) < \phi(i^*, 0) = \phi(i^*, q^*) = r(i^*, q^*, \phi) \leq u_{i^*} \) for \( q > q^* = 0 \).

Thus, we have \( u_{i^*} - \phi(i^*, \epsilon) > 0 \) for any \( \epsilon > 0 \). Therefore, we can define

\[
Y = \frac{2\hat{L}}{u_{i^*} - \phi(i^*, \epsilon)}, \quad T = X + Y.
\]

Recall that the inter-transition times follow (2). Thus, we have

\[
P_2 = \Pr \left\{ i(T) = i^*, q(T) \in [0, \epsilon] \mid i(X) = i^*, q(X) \leq \hat{L} \right\}
\]

\[
\geq \exp(-\nu_{i^*} Y).
\]

Finally, we can write, for any \( \epsilon > 0 \),

\[
\Pr \left\{ q(T) \in [0, \epsilon] \mid i(0) = i_0, q(0) = q_0 \right\}
\]

\[
\geq \Pr \left\{ i(T) = i^*, q(T) \in [0, \epsilon] \mid i(0) = i_0, q(0) = q_0 \right\}
\]

\[
\geq \Pr \left\{ i(T) = i^*, q(T) \in [0, \epsilon] \mid i(X) = i^*, q(X) \leq \hat{L} \right\}
\]

\[
\times \Pr \left\{ i(X) = i^*, q(X) \leq \hat{L} \mid i(0) = i_0, q(0) = q_0 \right\}
\]

\[
\geq P_2 P_1 > 0.
\]

**Case B.2:** \( q^* > 0 \). For this case, we need to consider various initial conditions.

If the initial queue length \( q_0 \geq q^* \), then the proof is analogous to case B.1.

If the initial queue length \( q_0 < q^* \), we show that the set \((q^* - \epsilon, q^*] \cap Q\) can be attained with positive probability. Without loss of generality, we only need to consider \( \epsilon \in (0, q^*] \). By
Assumption 3, there exists $j \in \mathcal{I}$ such that $\phi(j, q) > u_j$ for all $q < q^*$. Therefore, there exists $0 < X < \infty$ and $0 < \hat{l} < q^*$ such that

$$\Pr\{i(X) = j, q(X) \leq q^* - \hat{l} \mid i(0) = i_0, q(0) = q_0\} = P_1$$

for some $P_1 > 0$. Then, since $\phi(j, q) \geq \phi(j, q^* - \epsilon) > u_j$ for all $q \leq q^* - \epsilon$, there exists $Y \geq 0$ and $T = X + Y$ such that

$$\Pr\{i(T) = j, q(T) \in (q^* - \epsilon, q^*) \mid i(X) = j, q(X) \leq q^* - \hat{l}\} = P_2$$

for some $P_2 > 0$. Hence, we have

$$\Pr\{q(T) \in (q^* - \epsilon, q^*) \mid i(0) = i_0, q(0) = q_0\} \geq \Pr\{i(T) = j, q(T) \in (q^* - \epsilon, q^*) \mid i(0) = i_0, q(0) = q_0\} \geq P_1 P_2 > 0.$$

Finally, note that we have verified conditions (A) and (B’) for the controlled PDQ. Thus, we conclude from the Foster-Lyapunov criteria that the controlled PDQ is exponentially stable.

**Remark 4.** Although (20) involves total variation distance, our proofs do not directly involve this notion. Instead, we are able to base our proofs on checking the Foster-Lyapunov criteria and conclude (20), following Meyn and Tweedie [34].

C. Stability analysis of two typical classes of control policies

Next, we apply Theorems 1 and 2 to two controlled PDQs; one with a mode responsive control policy, and the other with a linear feedback control policy.

1) Mode-responsive control policy: Consider a four-mode PDQ with the following parameters:

$$\mathcal{I} = \{1, 2, 3, 4\}, \ u = [1, 0.8, 0.6, 0.4]^T,$$

$$\Lambda = \begin{bmatrix}
-2 & 1 & 1 & 0 \\
1 & -2 & 1 & 0 \\
0 & 1 & -2 & 1 \\
0 & 1 & 1 & -2
\end{bmatrix}.$$
Let us define a mode-responsive control policy as follows:

\[
\phi(i) = \begin{cases} 
  f_1, & i = 1, \\
  f_2, & i = 2, 3, 4.
\end{cases}
\]

This control policy sends \( f_1 \) amount of flow to the PDQ when \( u(t) = 1 \), and sends \( f_2 \) amount of flow when \( u(t) < 1 \). For ease of presentation, we assume \( \mathcal{F} = [0, 1] \) and consider \([f_1, f_2]^T \in [0, 1]^2\). We are concerned with the set of \([f_1, f_2]^T\) such that the PDQ is stable.

First, we solve (4) and obtain

\[
p = [0.25, 0.25, 0.25, 0.25].
\]

Then, we have

\[
\bar{\phi} = 0.25 f_1 + 0.75 f_2, \quad a.s.
\]

The effective capacity is given by

\[
\bar{\pi} = pu = 0.7.
\]

Hence, by Theorem 1 a necessary condition for the PDQ to be stable is

\[
0.25 f_1 + 0.75 f_2 \leq 0.7. \quad (35)
\]

The contrapositive is that, if \( 0.25 f_1 + 0.75 f_2 > 0.7 \), then the PDQ is unstable. This set of \([f_1, f_2]^T\) is depicted as the “Unstable” region in Figure 3.

![Fig. 3. Stable, unstable, and ambiguous regions of the set of control parameters \([f_1, f_2]^T\).](image-url)
Second, to apply Theorem 2, we plug various values of \([f_1, f_2]^T\) satisfying (35) and search for feasible \(b\) and \(a\) verifying (19). Using YALMIP [46], we find that positive solutions are available over the white and light gray regions in Figure 3. Thus, the PDQ is stable over these regions.

In fact, from (7) and (10), if \(f_1 < 1\) and \(f_2 < 0.4\), then, for any initial condition, the queue vanishes in finite time, and \(q = 0\) is the unique steady state. For this set of \([f_1, f_2]^T\), which is depicted as the “Stable without queue” region in Figure 3, we can conclude stability even without applying Theorem 2. For the “Stable with queue” region, a non-zero queue is possible for any \(t \geq 0\).

Finally, note that there is a gap, labeled as “Ambiguous” in Figure 3, for which our results do not provide a conclusive answer for stability.

2) Linear feedback control policy: Consider a PDQ with modes \(I = \{1, 2, \ldots, m\}\) and a transition matrix \(\Lambda\) satisfying Assumption 1. In addition, we assume that the saturation rates satisfy

\[u_1 > u_2 > \cdots > u_m \geq 0.\]

If the PDQ is subject to a constant inflow \(f > \bar{u}\), then, by Theorem 1, the system is unstable.

Now suppose that the inflow is controlled by a control policy as follows

\[\phi(q) = \max\{0, f - kq\},\]

(36)

where \(k > 0\) is a parameter. That is, the controlled inflow decreases linearly with the queue length until the controlled inflow vanishes. This control policy is motivated by practically deployed freeway ramp metering policies [47].

Next, we use Theorem 2 to show that this controlled PDQ is exponentially stable for all \(k > 0\). To apply Theorem 2, we need to check (19). By Assumption 1, there exists \(j \neq m\) such that \(\lambda_{mj} > 0\). Select an arbitrary \(b > 0\) and let

\[a_i = \max\left\{\frac{bu_j + \nu_j}{bu_j \lambda_{mj}} + \frac{1}{bu_j}, \frac{1}{bu_{m-1}}\right\}, \quad i \neq j,
\]

\[a_j = a_i - \frac{1}{\lambda_{mj}},\]

where \(\nu_j\) is as defined in (1). Then, noting that \(\inf_q \phi(q) = 0\), one can show that (19) holds with the above \(a_i\) and \(b\). By Theorem 2, we conclude that the controlled PDQ is exponentially stable.
for all \( k > 0 \). Furthermore, we have the following upper bound on the convergence:

\[
\| P_t(i_0, q_0) - \mu_\phi \| \leq B \left( a_{i_0} \exp(bq_0) + 1 \right) \exp \left( -\frac{t}{2a_j} \right)
\]

for some \( B > 0 \).

Finally, as mentioned in Remark 2, we can also use this example to show that the upper bound given by Theorem 1 can be attained. To see this, note that, for every mode \( i \), there exists a unique \( \tilde{q}_i = \max\{0, (f - u_i)/k\} \).

such that

\[
D(i, \tilde{q}_i, \phi) = \phi(\tilde{q}_i) - r(i, \tilde{q}_i, \phi) = 0.
\]

Let \( q_{\text{min}} = \min_i \tilde{q}_i \) and \( q_{\text{max}} = \max_i \tilde{q}_i \). Since \( \phi \) is decreasing in \( q \), it is not hard to see that

\[
D(i, q, \phi) > 0, \quad \forall (i, q) \in \mathcal{I} \times [0, q_{\text{min}}),
\]

\[
D(i, q, \phi) < 0, \quad \forall (i, q) \in \mathcal{I} \times (q_{\text{max}}, \infty).
\]

Therefore, after sufficiently long time, \( q(t) \) enters the compact set \([q_{\text{min}}, q_{\text{max}}]\) and stays therein.

In fact, the interval \([q_{\text{min}}, q_{\text{max}}]\) is the accessible set of the PDQ, which supports the invariant probability measure \( \mu_\phi \); for details regarding the relation between invariant probability measures and accessible sets, see [33]. If \( q_{\text{min}} > 0 \) (i.e. if \( f > u_1 \)), the accessible set does not contain \( q = 0 \). Since the controlled PDQ is stable, we have \( \lim_{t \to \infty} q(t)/t = 0 \) a.s. Therefore, since

\[
q(t) = \int_0^t (\phi(\tau) - r(\tau))d\tau,
\]

the time-average inflow \( \overline{\phi} \) is given by

\[
\overline{\phi} = \lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^t \phi(\tau)d\tau = \lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^t r(\tau)d\tau + q(t)
\]

\[
= \lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^t r(\tau)d\tau = \int_{\mathcal{I} \times [q_{\text{min}}, q_{\text{max}}]} r(i, q, \phi)d\mu_\phi, \quad \text{a.s.}
\]

where the last equality results from the ergodic theorem of Markov processes. One can see from (7) that \( r(i, q, \phi) = u_i \) for \( q \in [q_{\text{min}}, q_{\text{max}}] \). Thus,

\[
\overline{\phi} = \int_{\mathcal{I} \times [q_{\text{min}}, q_{\text{max}}]} r(i, q, \phi)d\mu_\phi = \sum_{i \in \mathcal{I}} u_i p_i = \overline{\pi}.
\]

Therefore, the upper bound in (16) is tight for this example.
IV. STABILITY OF FEEDBACK-CONTROLLED BIMODAL PDQS

We now focus on the stability of bimodal PDQs (i.e. $\mathcal{I} = \{1, 2\}$) under control policies satisfying Assumption 2.

Without loss of generality, we assume that $u_1 > u_2$. One can view mode 1 as the nominal mode and mode 2 as the disrupted mode with reduced capacity. The transition rate $\lambda_{12}$ (respectively $\lambda_{21}$) can be viewed as the occurrence (respectively clearance) rate of capacity disruption. We represent a BPDQ by the tuple $\langle\{1, 2\}, Q, \mathcal{F}, \Lambda, D\rangle$. Under Assumption 1 the steady-state probabilities of the process $\{i(t); t \geq 0\}$ can be obtained from (4):

$$p_1 = \frac{\lambda_{21}}{\lambda_{12} + \lambda_{21}}, \quad p_2 = \frac{\lambda_{12}}{\lambda_{12} + \lambda_{21}}.$$  \hspace{1cm} (37)

The effective capacity is obtained as

$$\overline{u} = p_1 u_1 + p_2 u_2.$$  

The main result of this section is a necessary and sufficient condition for the stability of feedback-controlled BPDQs:

**Theorem 3.** A BPDQ $\langle\{1, 2\}, Q, \mathcal{F}, \Lambda, D\rangle$ with a control policy $\phi \in \Phi$ is stable if and only if

$$\phi_{\inf} := p_1 \inf_q \phi(1, q) + p_2 \inf_q \phi(2, q) < \overline{u}. \hspace{1cm} (38)$$

Furthermore, under (38), the BPDQ admits a unique invariant probability measure $\mu_{\phi}$, and there exist positive constants $a_1$, $a_2$, $b$, and $c$ such that, for some $B > 0$,

$$\| P_t(i_0, q_0; \cdot) - \mu_\phi \|_{TV} \leq B \left( a_{i_0} \exp(bq_0) + 1 \right) \exp(-ct), \quad \forall (i_0, q_0) \in \mathcal{I} \times Q, \forall t \geq 0. \hspace{1cm} (39)$$

Theorem 3 is stronger than Theorems 1 and 2 in the following respects. First, Theorem 3 is a necessary and sufficient condition, while Theorems 1 and 2 provide a necessary condition and a sufficient condition, respectively. Second, (38) is easier to check than (16) and (19), since it only involves the steady-state probabilities of the mode transition process and the infima of the control policy in various modes. Third, the derivation of Theorem 3 leads to explicit expressions for the constants $a_1$, $a_2$, $b$, and $c$ in (39); see Section IV-B.
We prove Theorem 3 in Sections IV-A and IV-B and investigate how the characteristics of the control policy (in addition to $\varphi_{\text{inf}}$) affect the average queue length and the rate of convergence in Section IV-C.

A. Proof of necessity in Theorem 3

First, let us define the following quantities

\begin{align}
D_{\text{min}} &= \min \left\{ \inf_{q} \phi(1, q) - u_1, \inf_{q} \phi(2, q) - u_2 \right\}, \quad (40a) \\
D_{\text{max}} &= \max \left\{ \inf_{q} \phi(1, q) - u_1, \inf_{q} \phi(2, q) - u_2 \right\}, \quad (40b) \\
i_{\text{min}} &= \begin{cases} 1, & \text{if } D_{\text{min}} = \inf_{q} \phi(1, q) - u_1, \\ 2, & \text{o.w.} \end{cases} \quad (40c) \\
i_{\text{max}} &= \begin{cases} 2, & \text{if } D_{\text{min}} = \inf_{q} \phi(1, q) - u_1, \\ 1, & \text{o.w.} \end{cases} \quad (40d) \\
\lambda_{\text{min}} &= \begin{cases} \lambda_{12}, & \text{if } D_{\text{min}} = \inf_{q} \phi(1, q) - u_1, \\ \lambda_{21}, & \text{o.w.} \end{cases} \quad (40e) \\
\lambda_{\text{max}} &= \begin{cases} \lambda_{21}, & \text{if } D_{\text{min}} = \inf_{q} \phi(1, q) - u_1, \\ \lambda_{12}, & \text{o.w.} \end{cases} \quad (40f)
\end{align}

Note that $D_{\text{min}} \leq D_{\text{max}}$.

To prove the necessity of (38), we show that, if the BPDQ is stable, then we can obtain this condition.

Recall that a stable PDQ has to be non-evanescent; i.e. the queue length has to be finite a.s. Hence, we can adapt the proof of Theorem 1 to show that

\begin{align}
0 &= \lim_{t \to \infty} \frac{q(t)}{t} \quad \text{a.s.} \\
&= \lim_{t \to \infty} \frac{1}{t} \left( q_0 + \int_{\tau=0}^{t} (\phi(\tau) - r(\tau)) d\tau \right) \\
&\geq \lim_{t \to \infty} \frac{1}{t} \int_{\tau=0}^{t} \left( \inf_{q} \phi(i(\tau), q) - u(\tau) \right) d\tau \\
&= \int_{\{1,2\} \times \mathcal{Q}} \left( \inf_{q} \phi(i, q) \right) d\mu_{\phi} - \sum_{i=1}^{2} u_i p_i, \quad \text{a.s.}
\end{align}
Therefore, we obtain that \( \bar{\phi}_{\text{inf}} \leq \bar{u} \).

To show the necessity of (38), we still need to show that the BPDQ is not stable if \( \bar{\phi}_{\text{inf}} = \bar{u} \).

Now, we need to consider two cases:

**Case 1:** There exists an \( i \in \{1, 2\} \) such that \( \phi(i, q) > \inf_q \phi(i, q) \) for all \( q \in Q \).

By the ergodic theorem of Markov processes [43, p. 169], we have

\[
\bar{\phi} = \int_{\{1,2\} \times Q} \phi(i, q) d\mu_\phi > \int_{\{1,2\} \times Q} \inf_w \phi(i, w) d\mu_\phi
\]

\[
= p_1 \inf_w \phi(1, w) + p_2 \inf_w \phi(2, w) \quad \text{(17)}
\]

which implies that \( \bar{\phi} > \bar{u} \). By Theorem [1] this contradicts that the BPDQ is stable.

**Case 2:** There exists \( q \in Q \) such that \( \phi(i, q) = \inf_{l'} \phi(i, l') \) for \( i = 1 \) and \( i = 2 \).

In this case, we show that, when \( \bar{\phi}_{\text{inf}} = \bar{u} \), the invariant probability measure \( \mu_\phi \) is either non-
unique (subcase 2.1) or non-existent (subcase 2.2). Recall (40) for definitions of \( D_{\text{min}}, D_{\text{max}},
\)
\( i_{\text{min}}, \) and \( i_{\text{max}} \). Since \( p_{i_{\text{min}}} D_{\text{min}} + p_{i_{\text{max}}} D_{\text{max}} = \bar{\phi}_{\text{inf}} - \bar{u} = 0 \), we only need to consider two
subcases:

**Subcase 2.1:** \( D_{\text{max}} = D_{\text{min}} = 0 \).

In this subcase, we show that the BPDQ admits multiple invariant probability measures.

Define

\[
l_i = \min \left\{ q : \phi(i, q) = \inf_{l'} \phi(i, l') \right\}, \quad i = 1, 2,
\]

\[
l = \max_i l_i. \quad \text{(41b)}
\]

Thus, for any initial condition \( q(0) = q_0 \in [l, \infty) \), we can obtain from (10) that \( q(t) = q_0 \) for
all \( t \geq 0 \). Therefore, any probability measure \( \mu \) on \( \{1, 2\} \times Q \) given by

\[
\mu(i, dq) = p_i \delta_{q_0},
\]

for some \( q_0 \in [l, \infty) \), where \( \delta_{q_0} \) is the Dirac delta function centered at \( q_0 \), satisfies \( \mu P_t = \mu \)
for all \( t \). Hence, the invariant probability measures are not unique and thus the BPDQ does not
converge in the sense of (14).

**Subcase 2.2:** \( D_{\text{min}} < 0, \) \( D_{\text{max}} > 0 \); see Figure [4].
In this subcase, we show that the PDQ admits no invariant probability measure.

First, let \( l \) be as defined in (41). Then, the invariant probability measure \( \mu \), if exists, has to satisfy
\[
\mu(\{1, 2\} \times (0, l)) = 0
\]
(see Figure 4). To see this, note that, if \( \mu(\{1, 2\} \times (0, l)) > 0 \), then
\[
\phi(x) = \int_{\{1, 2\} \times (0, l)} \phi(i, q) d\mu + \int_{\{1, 2\} \times (l, \infty)} \phi(i, q) d\mu
\]
\[
\geq \int_{\{1, 2\} \times (0, l)} \inf_{w} \phi(i, w) d\mu + \int_{\{1, 2\} \times (l, \infty)} \phi(i, q) d\mu
\]
which implies \( \overline{\phi} > \pi \) and contradicts the stability of the BPDQ, according to Theorem 1.

Second, we show that every measure \( \mu \) such that \( \mu(\{1, 2\} \times (0, l)) = 0 \) is not invariant with respect to the transition kernel \( P_t \). For \( \mu \) supported over \( \{1, 2\} \times [l, \infty) \), there exists \( M > 0 \) such that \( \mu(\{1, 2\} \times [l, M]) > 0 \). Since \( D_{\min} = \phi(i_{\min}, l) - u_{i_{\min}} < 0 \) and \( \phi \) is continuous (recall Assumption 2), there exists an interval \( (l - \epsilon, l) \) for some \( \epsilon > 0 \) such that \( \phi(i_{\min}, q) - u_{i_{\min}} < 0 \) for all \( q \in (l - \epsilon, l) \); see Figure 4. Then, there exists a finite \( T > 0 \) such that, for any \( (i_0, q_0) \in \{1, 2\} \times [l, M] \),
\[
\Pr\{i(T) = i_{\min}, q(T) \in (l - \epsilon, l)|i(0) = i_0, q(0) = q_0\} > 0.
\]
Recalling (11), we have \( \mu_P T(\{i_{\min}\} \times (l - \epsilon, l)) > 0 \). However, we have argued that \( \mu(\{i_{\min}\} \times [0, l]) = 0 \). Therefore, \( \mu \neq \mu_P T \) and thus \( \mu \) is not invariant with respect to \( P_t \).

Finally, since cases 1 and 2 are exhaustive, we conclude that the BPDQ is unstable if \( \overline{\phi}_{\min} = \pi \).
Hence, if the BPDQ is stable, then we have \( \overline{\phi}_{\min} < \pi \).
B. Proof of sufficiency in Theorem 3

To prove the sufficiency of (38), we construct constants \( a_{i_{\text{min}}} \), \( a_{i_{\text{max}}} \), and \( b \) satisfying the BMI (19). The same Lyapunov function as in (30) is considered here.

Recall the definition of \( D_{\text{min}}, D_{\text{max}}, \lambda_{\text{min}}, \) and \( \lambda_{\text{max}} \) from (40). Condition (38) states that \( \phi_{\text{inf}} < \bar{\pi}, \) i.e.

\[
\phi_{\text{inf}} - \bar{\pi} = p_{i_{\text{min}}} D_{\text{min}} + p_{i_{\text{max}}} D_{\text{max}} < 0 \tag{42}
\]

Since \( D_{\text{min}} \leq D_{\text{max}} \), (42) implies that \( D_{\text{min}} < 0 \). Thus, we only need to consider two cases:

Case 1: \( D_{\text{min}} < 0, D_{\text{max}} \leq 0 \). In this case, we can select an arbitrary \( a_{i_{\text{min}}} > \max_i \{1/\lambda_i\} \) and let

\[
a_{i_{\text{max}}} = 2a_{i_{\text{min}}}, \quad b = \frac{\lambda_{\text{min}} a_{i_{\text{min}}} + 1}{-D_{\text{min}} a_{i_{\text{min}}}}. \tag{43}
\]

It is not hard to see that \( a_{i_{\text{min}}}, a_{i_{\text{max}}}, \) and \( b \) are positive. One can check that the above constants satisfy the BMI (19).

Case 2: \( D_{\text{min}} < 0, D_{\text{max}} > 0 \). In this case, we let

\[
b = \frac{(\lambda_{12} + \lambda_{21})(\bar{\pi} - \phi_{\text{inf}})}{-2D_{\text{min}} D_{\text{max}}}, \tag{44a}
\]

\[
a_{i_{\text{min}}} = \frac{-D_{\text{max}} b + \lambda_{12} + \lambda_{21}}{\det[\text{diag}(\phi_{\text{inf}} - u)b + \Lambda]}, \tag{44b}
\]

\[
a_{i_{\text{max}}} = \frac{-D_{\text{min}} b + \lambda_{12} + \lambda_{21}}{\det[\text{diag}(\phi_{\text{inf}} - u)b + \Lambda]}, \tag{44c}
\]

where \( \phi_{\text{inf}} \) and \( \phi_{\text{inf}} \) are as defined in (18) and (17), respectively.

Now, we show that these constants are positive. First, since \( D_{\text{min}} < 0 \) and \( D_{\text{max}} > 0 \), and since \( \bar{\pi} > \phi_{\text{inf}}, b \) is positive. Second, to see that \( a_{i_{\text{min}}} > 0 \), note that

\[
\text{diag}(\phi_{\text{inf}} - u)b + \Lambda
\]

\[
= \begin{bmatrix}
\lambda_{12} & \lambda_{12} \\
\lambda_{21} & \lambda_{21}
\end{bmatrix} + \begin{bmatrix}
b(\inf_q \phi(1,q) - u_1) - \lambda_{12} & \lambda_{12} \\
\lambda_{21} & b(\inf_q \phi(2,q) - u_2) - \lambda_{21}
\end{bmatrix},
\]

and

\[
\det[\text{diag}(\phi_{\text{inf}} - u)b + \Lambda]
\]

\[
= b^2 \left( \inf_q \phi(1,q) - u_1 \right) \left( \inf_q \phi(2,q) - u_2 \right)
\]
Again, since $D_{\text{min}} < 0$ and $D_{\text{max}} > 0$, one can check that the $b$ given in (44a) ensures that 
\[ \det[\text{diag}(\phi_{\text{inf}} - u)b + \Lambda] > 0. \]
In addition, note that
\[
\begin{align*}
    b &= \frac{(\lambda_{12} + \lambda_{21})(\overline{u} - \bar{\phi}_{\text{inf}})}{-2D_{\text{min}}D_{\text{max}}} \\
    &= \frac{\lambda_{12} + \lambda_{21}}{D_{\text{max}}} \left( \frac{-p_{\text{min}}D_{\text{min}} - p_{\text{max}}D_{\text{max}}}{-2D_{\text{min}}} \right) \\
    &< \frac{\lambda_{12} + \lambda_{21}}{2D_{\text{max}}} < \frac{\lambda_{12} + \lambda_{21}}{D_{\text{max}}},
\end{align*}
\]
which, along with $D_{\text{max}} > 0$, implies $a_{i_{\text{min}}} > 0$. Finally, since $D_{\text{min}} < 0$, $a_{i_{\text{max}}}$ is positive.

Then, one can see from Cramer’s rule that the above constants satisfy
\[ [\text{diag}(\phi_{\text{inf}} - u)b + \Lambda]a = -e, \]
and thus satisfy the BMI (19) as well; details are omitted for the sake of conciseness.

Thus, we can conclude by Theorem 2 that the controlled BPDQ is exponentially stable.

In fact, with the expressions for $a_1$, $a_2$, and $b$ in (43) or (44) for the respective cases, we can also follow the proof of Theorem 2 and construct the constants $c$ and $d$ in the drift condition (27):
\[
\begin{align*}
c &= 1/(2a_{i_{\text{max}}}), \quad (45a) \\
d &= \left( a_{i_{\text{max}}}b\phi_{\text{max}} + \lambda_{\text{min}}(a_{i_{\text{max}}} - a_{i_{\text{min}}}) \right)e^{\delta L}, \quad (45b)
\end{align*}
\]
where
\[
\phi_{\text{max}} := \max_{i,q} \phi(i, q) = \max_{i} \phi(i, 0), \quad (46)
\]
and $L$ is a sufficiently large constant such that
\[ \phi(i, q) - \inf_{w} \phi(i, w) \leq \eta, \quad \forall i \in \{1, 2\}, \forall q \geq L, \quad (47) \]
where $\eta := 1/(2a_{i_{\text{max}}}b)$. These results can be used to further investigate the relation between the control policy and the long-time behavior of the controlled BPDQ, which is the topic of the next subsection.

C. Effects of other characteristics of $\phi$

Theorem 3 shows how the quantity $\bar{\phi}_{\text{inf}}$, i.e. the “average” infimum, affects the stability of the BPDQ. Now, for stable BPDQs, we study the effect of other characteristics of the control policy $\phi$, including $\phi_{\text{max}}$ (defined in (46)), $L$ (as in (47)), and $(\inf_q \phi(1, q) - \inf_q \phi(2, q))$; see Figure 5 for illustration of these quantities.

Fig. 5. Characteristics of a control policy $\phi(i, q)$.

1) Effect of $\phi_{\text{max}}$: The maximum inflow $\phi_{\text{max}}$ affects on the average queue length. To see this, note that the average queue length $\bar{q}$ satisfies

$$\bar{q} := \int_{\{1,2\}\times\mathcal{Q}} q d\mu_\phi \leq \int_{\{1,2\}\times\mathcal{Q}} \frac{1}{b} e^{bq} d\mu_\phi$$

$$= \int_{\{1,2\}\times\mathcal{Q}} \frac{a_{\text{min}}}{a_{\text{max}}} b e^{bq} d\mu_\phi \leq \frac{1}{a_{\text{min}} b} \int_{\{1,2\}\times\mathcal{Q}} a_i e^{bq} d\mu_\phi$$

(30)  $\frac{1}{a_{\text{min}} b} \int_{\{1,2\}\times\mathcal{Q}} V(i, q) d\mu_\phi$,

where $a$ and $b$ are given by (43) or (44). By [48] Theorem 4.1, we have

$$\int_{\{1,2\}\times\mathcal{Q}} V(i, q) d\mu_\phi \leq d/c.$$

Recalling (45b) for the expression of $d$, we have

$$\bar{q} \leq \frac{d}{a_{\text{min}} b c} = \frac{a_{\text{max}} b \phi_{\text{max}} + \lambda_{\text{min}} (a_{\text{max}} - a_{\text{min}})}{a_{\text{min}} b c} e^L$$

(48)

Therefore, the upper bound for $\bar{q}$ increases with $\phi_{\text{max}}$. This is intuitive: a large inflow leads to a long queue.
2) Effect of $L$: Note that $L$ characterizes how fast the controlled inflow $\phi$ decreases as $q$ grows. We can easily see that the upper bound on queue length provided by (48) increases with $L$. This is also intuitive: a small $L$ implies that $\phi$ decreases fast as $q$ increases, which results in a small queue.

3) Effect of $\inf_q \phi(1, q) - \inf_q \phi(2, q)$: This difference affects the steady-state queue length as well as the exponential coefficient $c$ in (39). For ease of presentation, we illustrate this by considering the mode-responsive control policy as follows:

$$\phi(i) = \begin{cases} f_1, & i = 1, \\ f_2, & i = 2, \end{cases}$$

(49)

where $f_1, f_2 \in F$. Thus, we have $\inf_q \phi(1, q) = f_1$ and $\inf_q \phi(2, q) = f_2$, and $\phi_{\inf} = \phi = p_1 f_1 + p_2 f_2$. By Theorem 3, the BPDQ is stable if and only if $\phi_{\inf} < u$.

For this control policy, we can analytically compute the invariant probability measure $\mu_{\phi m}$ and thus the average queue length $\overline{q}$ if the system is stable:

$$\overline{q} = \begin{cases} 0, & D_{\max} \leq 0, \\ \lambda_{\min} \frac{D_{\max} (D_{\max} - D_{\min})}{\pi - \phi_{\inf}}, & D_{\max} > 0, \end{cases}$$

(50)

where $D_{\min}, D_{\max}, \lambda_{\min}$, and $\lambda_{\max}$ are given by (40). We refer to [23] for the details of the derivation. The numerical example below shows how the difference between $f_1$ and $f_2$ affects the queue length and the rate of convergence:

**Example 1.** Consider a BPDQ with $u_1 = 1$, $u_2 = 0.5$, $\lambda_{12} = \lambda_{21} = 1$, and the control policy (49). One can easily see that $p_1 = p_2 = 0.5$ and $\pi = 0.75$. We fix the average inflow as follows:

$$\phi_{\inf} = 0.5 f_1 + 0.5 f_2 = 0.6.$$  

By Theorem 3, since $\phi_{\inf} < \pi$, the BPDQ is exponentially stable. In addition, we can compute $a_{i_{\min}}, a_{i_{\max}}, b, c,$ and $d$ using (43)–(45) and the average queue length $\overline{q}$ using (50).

Figure 6(a) shows that $\overline{q}$ decreases with $f_1 - f_2$. Intuitively, since $u_1 > u_2$ (by assumption), the system can discharge the queue faster in mode 1; thus, a large $f_1$ means efficient utilization of the larger saturation rate $u_1$. In addition, when $f_1 - f_2 \geq 0.2$, i.e. when $f_2 \leq 0.5 = u_2$, the average queue length vanishes, which corresponds to case 1 in Section IV-B when $f_1 - f_2 < 0.2$, i.e. when $f_2 > 0.5 = u_2$, the average queue length is non-zero, which corresponds to case 2 in Section IV-B.
Fig. 6. Impact of $f_1 - f_2$ on queue length and rate of convergence.

Figure 6(b) shows that the coefficient $c$ increases with $f_1 - f_2$. This is also intuitive. As we have argued above, a large $f_1$ implies efficient utilization of $u_1$, and thus leads to faster convergence. Once again, the curve to the left (respectively right) of $f_1 - f_2 = 0.2$ corresponds to case 1 (respectively case 2) in Section IV-B.

V. IMPLICATIONS FOR ROUTING OVER PARALLEL LINKS

In this section, we try to demonstrate how our results can provide insights for traffic flow routing under stochastic capacity fluctuation.

Fig. 7. A network of two parallel links, with capacity fluctuation on link 1.

For the sake of illustration, consider a network of two parallel links as shown in Figure 7. Link 1 is modeled by a BPDQ with modes $I = \{1, 2\}$ and a saturation rate $u_1$ switching between $u_1^1 = 1$ and $u_1^2 = 0.5$, with symmetric transition rates $\lambda_{12} = \lambda_{21} = 1$. Thus, the steady-state probabilities are $p_1 = p_2 = 0.5$, and the effective capacity of link 1 is $\bar{u}_1 = 0.75$. Link 2 has a constant saturation rate $u_2 = 0.45$. In addition, the nominal travel times of link 1 and link 2 are $v_1 = 1$ and $v_2 = 2$, respectively. The source node $s$ receives a unit inflow $f_s = 1$. The
queue lengths and discharge rates of the links are denoted as \( q_1(t) \) and \( q_2(t) \), and \( r_1(t) \) and \( r_2(t) \), respectively.

The control variable here is the flow sent to link 1; the rest of the incoming flow is sent to link 2. Suppose that the flow sent to link 1 is specified by a control policy, or a routing policy, \( \phi \), which may be static, mode-responsive, or queue-responsive. This example is motivated by the situation where a system operator needs to determine how traffic should be split between a faster route with a fluctuating capacity and a slower route with deterministic capacity.

We now apply our results to study the stability of the above-mentioned network. In addition, we attempt to find routing policies that minimize expected total travel times, i.e. the sum of nominal travel time and queueing times on both links.

A. Static routing

Suppose that the system operator sends a fixed (time-invariant) flow \( \phi_s = f \in [0, 1] \) to link 1; the remaining \( (f_s - f) \) amount of traffic is sent to link 2. If the network is stable, the objective function \( J_s \) is defined as the expected total travel time for the network:

\[
J_s(f) = v_1 f + \bar{q}_1(f) + v_2 (f_s - f) + \bar{q}_2(f_s - f),
\]

where \( \bar{q}_1 \) and \( \bar{q}_2 \) can be computed using (50).

We first determine the set of stable values for the inflow to link 1. Theorem 3 requires \( \phi_{\text{inf}} = f < u_1 = 0.75 \) for the stability of link 1. In addition, it is easy to see that, if \( 1 - f > 0.45 \) (i.e. \( f < 0.55 \)), then \( \lim_{t \to \infty} q_2(t) = \infty \); if \( 1 - f = 0.45 \) (i.e. \( f = 0.55 \)), then \( q_2(t) = q_2(0) \) for all \( t \geq 0 \) (not convergent); otherwise \( \lim_{t \to \infty} q_2(t) = 0 \). Hence, the set of stable inflow values is \((0.55, 0.75)\). Note that \( \bar{q}_2(f) = 0 \) for \( f \in (0.55, 0.75) \). Thus, we have

\[
J_s(f) = \bar{q}_1(f) + v_1 f + v_2 (f_s - f) = \frac{1}{8} \left( \frac{f - 0.5}{0.75 - f} \right) - f + 2.
\]

Note that \( J_s \) is convex in \( f \) over \((0.55, 0.75)\). Therefore, setting \( dJ_s/df = 0 \), we obtain the optimal solution \( f^* = 0.573 \) and the optimal objective value \( J_s^* = 1.479 \); see Figure 8.
B. Mode-responsive routing

Suppose that the system operator is able to divert additional flow from link 1 to link 2 when the former is experiencing reduced capacity (mode 2). Consider the mode-responsive routing policy given by

\[ \phi_m(i) = \begin{cases} 
    f_1, & i = 1, \\
    f_2, & i = 2,
\end{cases} \]

where \( i \) is the mode of link 1, and \( \phi_m \) is the traffic sent to link 1 (the remaining \( (f_s - \phi_m) \) is sent to link 2). The decision vector is \([f_1, f_2]^T \in [0, 1]^2\).

Theorem 3 can be separately applied to both link 1 and link 2. The application to link 1 is straightforward. Since link 2 is subject to an inflow that switches between \((1 - f_1)\) and \((1 - f_2)\) according to the Markov process \(\{i(t); t \geq 0\}\), this link can also be viewed as a BPDQ. Thus, by Theorem 3 for link 1 to be stable, we need

\[ \overline{\phi}_{\text{inf}} = 0.5f_1 + 0.5f_2 < 0.75; \]

for link 2 to be stable, we need

\[ 0.5(f_s - f_1) + 0.5(f_s - f_2) < 0.45. \]

In addition, let us impose the practically motivated constraint:

\[ |f_1 - f_2| \leq 0.3, \quad (51) \]

i.e. the diverted traffic cannot exceed 30% of the total inflow. Under the above constraints, the feasible set can be represented by the unshaded region in Figure 9.
The objective function (i.e. total expected travel time) can be expressed as

\[ J_m(f_1, f_2) = \bar{q}_1(f_1, f_2) + v_1 p_1 f_1 + v_1 p_2 f_2 \]

\[ + \bar{q}_2(f_s - f_1, f_s - f_2) + v_2 p_1 (f_s - f_1) + v_2 p_2 (f_s - f_2) \]

\[ = \frac{1}{4} (f_2 - 0.5) (f_2 - f_1 + 0.5) + \frac{1}{4} (0.65 - f_2) (f_1 - f_2) \]

\[ - 0.5 f_1 - 0.5 f_2 + 2, \]

where the expected queue lengths are computed using (50). One can show that \( J_m \) is convex in \( f_1 \) and \( f_2 \) over the feasible set. Therefore, applying standard optimality conditions [49], we obtain the optimal solution \([f_1^*, f_2^*] = [0.844, 0.544]\), as shown in Figure 9. The optimal value is \( J_m^* = 1.282 \). Compared to the optimal value \( J_s^* \) obtained via the static routing policy \( \phi_s \), mode-responsive routing improves the optimal value by 13%.

C. Queue-responsive routing

Now, suppose that the traffic is routed according to the classical logit model, which is often used for traffic assignment over networks [50]. That is, the flow into link 1 is given by

\[ \phi_l(q_1, q_2) = \frac{\exp(-\beta(q_1/u_1 + v_1))}{\exp(-\beta(q_1/u_1 + v_1)) + \exp(-\beta(q_2/u_2 + v_2))}; \]

the remaining \( (f_s - \phi_l) \) amount of traffic is sent to link 2. This routing policy assigns more traffic to the link with a smaller travel time, i.e. the sum of nominal time and queueing time. The coefficient \( \beta \) can be viewed as the sensitivity of route choice decision with respect to the
travel time on individual links. Next, we show that, for all $\beta > 0$, this routing policy is stable for the network in Figure 7. Furthermore, we study via simulation the relation between $\beta$ and the expected total travel time $J_l$ defined as follows:

$$J_l = \int_{\{1,2\} \times \mathbb{R}_+^2} \left(q_1 + v_1\phi(i, q_1, q_2) + q_2 + v_2(f_s - \phi(i, q_1, q_2))\right) d\mu_\phi.$$  

Theorem 2 does not directly apply here, since $\phi_l$ depends on both $q_1$ and $q_2$. However, we show below that Theorem 2 can be extended to this setting.

Consider the Lyapunov function for the network:

$$V(i, q_1, q_2) = a_i \exp \left(b(q_1 + q_2)\right), \quad i \in \mathcal{I}, q_1, q_2 \geq 0;$$

the parameters $a_1, a_2$, and $b$ are defined as follows:

$$a_1 = \frac{-(f_s - U_1)b + \lambda_{12} + \lambda_{21}}{\det[\text{diag}(F - U)b + \Lambda]},$$

$$a_2 = \frac{-(f_2 - U_2)b + \lambda_{12} + \lambda_{21}}{\det[\text{diag}(F - U)b + \Lambda]},$$

$$b = \frac{(\lambda_{12} + \lambda_{21})(\overline{U} - f_s)}{2(U_1 - f_s)(f_s - U_2)},$$

where

$$U_1 = \frac{4}{5}u_1^1 + u_2, \quad U_2 = u_2^2 + u_2, \quad \overline{U} = p_1U_1 + p_2U_2,$$

$$F = [f_s, f_s], \quad U = [U_1, U_2], \quad \Lambda = \begin{bmatrix} -\lambda_{12} & \lambda_{12} \\ \lambda_{21} & -\lambda_{21} \end{bmatrix}.$$  

One can check that $a_1, a_2, b > 0$. We can show that, for $i = 1, 2$ and for all $q_1, q_2 \geq 0$, there exist constants

$$c = 1/a_2 > 0,$$

$$d = \max \left\{ \left(a_1bf_s + \lambda_{12}(a_2 - a_1) + ca_1\right) \exp \left(b(L_1 + L_2)\right), \quad (a_2bf_s + ca_2) \exp \left(b(L_1 + L_2)\right) \right\},$$

where

$$L_1 = \left| \frac{\pi_1}{\beta} \left( \log \frac{u_2}{1 - u_2} + \beta(v_2 - v_1) \right) \right|.$$
\[
L_2 = \left| \frac{u_2}{\beta} \left( \log 5 - \beta (v_2 - v_1) \right) \right|
\]
such that the drift condition (A) in the Foster-Lyapunov criteria holds, i.e.
\[
\mathcal{L}V = a_i \left( \left( \phi_l - r_1 \right) + \left( f_s - \phi_l - r_2 \right) \right) b
\]
\[+ \sum_{j=1}^{2} \lambda_{ij} (a_j - a_i) \exp \left( b(q_1 + q_2) \right) \]
\[
\leq -ca_1 \exp \left( b(q_1 + q_2) \right) + d = -cV + d
\]
for all \((i, [q_1, q_2]^T) \in \{1, 2\} \times \mathbb{R}^2_{\geq 0}\).

In addition, one can easily show that both \(q_1\) and \(q_2\) will vanish in finite time if link 1 stays in mode 1 for sufficiently long time, which implies that the set \(\{1, 2\} \times \{[0, 0]^T\}\) can be attained with positive probability for any initial condition. This ensures condition (B') in the Foster-Lyapunov criteria.

Thus, the stability of the network with the logit routing policy follows from the Foster-Lyapunov criteria.

![Graph](image)

**Fig. 10.** The objective value \(J_l\) for the logit routing policy with respect to the parameter \(\beta\). The values of \(J_s^*\) and \(J_m^*\) are also indicated for comparison.

For the logit routing policy, the expected total travel time \(J_l\) is not easy to express analytically. We simulate the network with various \(\beta\) values and obtain the corresponding objective values. Figure [10] shows the result. The simulated optimal solution is \(\beta^* = 0.63\), and the optimal value is \(J_l^* = 1.473\), which is close to the optimal value \(J_s^*\) resulting from static routing policy.
VI. Concluding Remarks

In this article, we studied the stability and control of the traffic queue in a simple fluid queueing model (i.e. the PDQ model) under stochastic capacity fluctuations. The PDQ model has a saturation rate that randomly switches between a finite set of values (modes). We derived a necessary condition (Theorem 1) for stability, which states that the average inflow cannot exceed the average saturation rate. We also derived a sufficient condition (Theorem 2) based on properties of PDMPs and the Foster-Lyapunov criteria along with an argument for the uniqueness of the invariant probability measure. For bimodal PDQs, we obtained a stronger result, i.e. a sufficient and necessary condition for stability (Theorem 3), and analyzed the impact of control policy on the average queue length and the rate of convergence. The stability conditions in Theorems 1–3 can be analytically or numerically verified. We also applied our results to study the stability of typical control policies for individual links and routing policies for a network of two PDQ links in parallel.

As a final remark, note that our analysis of traffic routing is based on the assumption of infinite buffer size. In practice, finite buffer size may have a significant impact on the behavior of queueing systems, especially when the inflow leads to a large expected queue length and the interaction between multiple queues is strong. The analysis and control of such queueing systems are part of our ongoing work [51].

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References

[1] J. Kwon, M. Mauch, and P. Varaiya, “Components of congestion: Delay from incidents, special events, lane closures, weather, potential ramp metering gain, and excess demand,” Transportation Research Record: Journal of the Transportation Research Board, vol. 1959, no. 1, pp. 84–91, 2006.
[2] J. Le Ny and H. Balakrishnan, “Feedback control of the National Airspace System,” Journal of Guidance, Control, and Dynamics, vol. 34, no. 3, pp. 832–846, 2011.
[3] D. Schrank, B. Eisele, and T. Lomax, “TTI’s 2012 urban mobility report,” *Proceedings of the 2012 annual urban mobility report. Texas A&M Transportation Institute, Texas, USA*, 2012.

[4] A. Skabardonis, K. F. Petty, R. L. Bertini, P. P. Varaiya, H. Noeimi, and D. Rydzewski, “I-880 field experiment: Analysis of incident data,” *Transportation Research Record: Journal of the Transportation Research Board*, vol. 1603, no. 1, pp. 72–79, 1997.

[5] M. D. Peterson, D. J. Bertsimas, and A. R. Odoni, “Models and algorithms for transient queueing congestion at airports,” *Management Science*, vol. 41, no. 8, pp. 1279–1295, 1995.

[6] L. Jin and S. Amin, “Calibration of a macroscopic traffic flow model with stochastic saturation rates,” in *Transportation Research Board 96th Annual Meeting*, to appear.

[7] D. Anick, D. Mitra, and M. M. Sondhi, “Stochastic theory of a data-handling system with multiple sources,” *The Bell System Technical Journal*, vol. 61, no. 8, pp. 1871–1894, 1982.

[8] H. Yu and C. G. Cassandras, “Perturbation analysis of feedback-controlled stochastic flow systems,” *IEEE Transactions on Automatic Control*, vol. 49, no. 8, pp. 1317–1332, 2004.

[9] G. Como, K. Savla, D. Acemoglu, M. A. Dahleh, and E. Frazzoli, “Robust distributed routing in dynamical networks Part I: Locally responsive policies and weak resilience,” *Automatic Control, IEEE Transactions on*, vol. 58, no. 2, pp. 317–332, 2013.

[10] G. Como, K. Savla, D. Acemoglu, M. A. Dahleh, and E. Frazzoli, “Robust distributed routing in dynamical networks Part II: Strong resilience, equilibrium selection and cascaded failures,” *Automatic Control, IEEE Transactions on*, vol. 58, no. 2, pp. 333–348, 2013.

[11] G. D. Glockner and G. L. Nemhauser, “A dynamic network flow problem with uncertain arc capacities: Formulation and problem structure,” *Operations Research*, vol. 48, no. 2, pp. 233–242, 2000.

[12] G. Barbarosoglu and Y. Arda, “A two-stage stochastic programming framework for transportation planning in disaster response,” *Journal of the Operational Research Society*, vol. 55, no. 1, pp. 43–53, 2004.

[13] A. K. Ziliaskopoulos, “A linear programming model for the single destination system optimum dynamic traffic assignment problem,” *Transportation Science*, vol. 34, no. 1, pp. 37–49, 2000.

[14] D. Bertsimas and S. S. Patterson, “The traffic flow management rerouting problem in air traffic control: A dynamic network flow approach,” *Transportation Science*, vol. 34, no. 3, pp. 239–255, 2000.

[15] C. G. Cassandras, Y. Wardi, B. Melamed, G. Sun, and C. G. Panayiotou, “Perturbation analysis for online control and optimization of stochastic fluid models,” *IEEE Transactions on Automatic Control*, vol. 47, no. 8, pp. 1234–1248, 2002.

[16] M. Baykal-Gürsoy, W. Xiao, and K. Ozbay, “Modeling traffic flow interrupted by incidents,” *European Journal of Operational Research*, vol. 195, no. 1, pp. 127–138, 2009.

[17] L. Jin and S. Amin, “A piecewise-deterministic Markov model of freeway accidents,” in *Decision and Control (CDC), 2014 IEEE 53rd Annual Conference on*, IEEE, 2014.

[18] G. F. Newell, *Applications of Queueing Theory*, vol. 4. Springer Science & Business Media, 2013.

[19] J. G. Dai, “On positive Harris recurrence of multiclass queueing networks: a unified approach via fluid limit models,” *The Annals of Applied Probability*, pp. 49–77, 1995.

[20] D. Bertsimas, D. Gamarnik, and J. N. Tsitsiklis, “Stability conditions for multiclass fluid queueing networks,” *IEEE Transactions on Automatic Control*, vol. 41, no. 11, pp. 1618–1631, 1996.

[21] H. Chen and D. D. Yao, “A fluid model for systems with random disruptions,” *Operations Research*, vol. 40, no. 3-supplement-2, pp. S239–S247, 1992.
[22] O. Kella and W. Whitt, “A storage model with a two-state random environment,” Operations Research, vol. 40, no. 3-supplement-2, pp. S257–S262, 1992.
[23] V. G. Kulkarni, “Fluid models for single buffer systems,” Frontiers in queueing: Models and applications in science and engineering, vol. 321, p. 338, 1997.
[24] W. Whitt, Stochastic-process limits: an introduction to stochastic-process limits and their application to queues. Springer Science & Business Media, 2002.
[25] M. H. A. Davis, “Piecewise-deterministic Markov processes: A general class of non-diffusion stochastic models,” Journal of the Royal Statistical Society. Series B. Methodological, vol. 46, no. 3, pp. 353–388, 1984.
[26] A. Khattak, X. Wang, and H. Zhang, “Incident management integration tool: dynamically predicting incident durations, secondary incident occurrence and incident delays,” IET Intelligent Transport Systems, vol. 6, no. 2, pp. 204–214, 2012.
[27] W. Zhang, M. Kamgarpour, D. Sun, and C. J. Tomlin, “A hierarchical flight planning framework for air traffic management,” Proceedings of the IEEE, vol. 100, no. 1, pp. 179–194, 2012.
[28] A. A. Kurzhanskiy, “Online traffic simulation service for highway incident management,” Tech. Rep. SHRP 2 L-15(C), Relteq Systems, Inc., Albany, CA, February 2013.
[29] J. Thai and A. M. Bayen, “State estimation for polyhedral hybrid systems and applications to the godunov scheme for highway traffic estimation,” Automatic Control, IEEE Transactions on, vol. 60, no. 2, pp. 311–326, 2015.
[30] C. F. Daganzo, “Queue spillovers in transportation networks with a route choice,” Transportation Science, vol. 32, no. 1, pp. 3–11, 1998.
[31] M. Ball, C. Barnhart, M. Dresner, M. Hansen, K. Neels, A. Odoni, E. Peterson, L. Sherry, A. A. Trani, and B. Zou, “Total delay impact study: A comprehensive assessment of the costs and impacts of flight delay in the United States,” 2010.
[32] A. Muralidharan, R. Pedarsani, and P. Varaiya, “Analysis of fixed-time control,” Transportation Research Part B: Methodological, vol. 73, pp. 81–90, 2015.
[33] M. Ben-Akiva, M. Cyna, and A. de Palma, “Dynamic model of peak period congestion,” Transportation Research Part B: Methodological, vol. 18, no. 4, pp. 339–355, 1984.
[34] R. G. Gallager, Stochastic Processes: Theory for Applications. Cambridge University Press, 2013.
[35] A. Abate, M. Prandini, J. Lygeros, and S. Sastry, “Probabilistic reachability and safety for controlled discrete time stochastic hybrid systems,” Automatica, vol. 44, no. 11, pp. 2724–2734, 2008.
[36] M. Papageorgiou, H. Hadj-Salem, and J.-M. Blosseville, “ALINEA: A local feedback control law for on-ramp metering,” Transportation Research Record, no. 1320, 1991.
[37] S. Coogan and M. Arcak, “A compartmental model for traffic networks and its dynamical behavior,” IEEE Transactions on Automatic Control, vol. 60, no. 10, pp. 2698–2703, 2015.
[42] J. G. VanAntwerp and R. D. Braatz, “A tutorial on linear and bilinear matrix inequalities,” *Journal of process control*, vol. 10, no. 4, pp. 363–385, 2000.

[43] J. Azema, M. Kaplan-Duflo, and D. Revuz, “Mesure invariante sur les classes récurrentes des processus de Markov,” *Zeitschrift für Wahrscheinlichkeitstheorie und Verwandte Gebiete*, vol. 8, no. 3, pp. 157–181, 1967.

[44] J. G. Dai and S. P. Meyn, “Stability and convergence of moments for multiclass queueing networks via fluid limit models,” *IEEE Transactions on Automatic Control*, vol. 40, no. 11, pp. 1889–1904, 1995.

[45] M. Benaim, S. Le Borgne, F. Malrieu, P.-A. Zitt, *et al.*, “On the stability of planar randomly switched systems,” *The Annals of Applied Probability*, vol. 24, no. 1, pp. 292–311, 2014.

[46] J. Löfberg, “YALMIP: A toolbox for modeling and optimization in MATLAB,” in *Computer Aided Control Systems Design, 2004 IEEE International Symposium on*, pp. 284–289, IEEE, 2004.

[47] M. Papageorgiou, J.-M. Blosseville, and H. Haj-Salem, “Modelling and real-time control of traffic flow on the southern part of boulevard périphérique in paris: Part ii: Coordinated on-ramp metering,” *Transportation Research Part A: General*, vol. 24, no. 5, pp. 361–370, 1990.

[48] M. Hairer, “Convergence of markov processes,” *lecture notes*, 2010.

[49] S. Boyd and L. Vandenberghe, *Convex optimization*. Cambridge university press, 2009.

[50] M. E. Ben-Akiva and S. R. Lerman, *Discrete choice analysis: Theory and application to travel demand*, vol. 9. MIT press, 1985.

[51] L. Jin and S. Amin, “Analysis of a stochastic switched model of freeway traffic incidents,” *arXiv preprint arXiv:1601.00204*, 2016.