CONTINUITY PROPERTIES OF THE SEMI-GROUP AND ITS INTEGRAL KERNEL IN NON-RELATIVISTIC QED

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ABSTRACT. Employing recent results on stochastic differential equations associated with the standard model of non-relativistic quantum electrodynamics by B. Güneysu, J.S. Møller, and the present author, we study the continuity of the corresponding semi-group between weighted vector-valued $L^p$-spaces, continuity properties of elements in the range of the semi-group, and the pointwise continuity of an operator-valued semi-group kernel. We further discuss the continuous dependence of the semi-group and its integral kernel on model parameters. All these results are obtained for Kato decomposable electrostatic potentials and the actual assumptions on the model are general enough to cover the Nelson model as well. As a corollary we obtain some new pointwise exponential decay and continuity results on elements of low-energetic spectral subspaces of atoms or molecules that also take spin into account. In a simpler situation where spin is neglected we explain how to verify the joint continuity of positive ground state eigenvectors with respect to spatial coordinates and model parameters. There are no smallness assumptions imposed on any model parameter.

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1. INTRODUCTION

In this article we extend some well-known results on Schrödinger semi-groups to the case where the quantum mechanical matter particles modeled by the non-relativistic Schrödinger operator are coupled to relativistic quantized radiation fields. The
prime example for such a situation is the standard model of non-relativistic quantum electrodynamics (QED), describing the interaction of a fixed number of non-relativistic electrons with the second quantized photon field; here a quantized vector potential is introduced in the Hamiltonian via minimal coupling. Another example is Nelson’s model for the interaction of nucleons with a quantized, linearly coupled meson field.

In these cases the corresponding Feynman-Kac (FK) formula for the semi-group is given by a vector-valued expectation, where the target Hilbert space, call it $\mathcal{H}$, is given by the tensor product of a finite-dimensional space accounting for spin degrees of freedom (if any) with the infinite-dimensional state space of the quantized radiation field (bosonic Fock space). In comparison to the usual Schrödinger operator, the FK integrand involves an additional process, attaining values in the set of bounded operators on $\mathcal{H}$, which is defined by means of certain $\mathcal{H}$-valued semi-martingales solving a stochastic differential equation (SDE) associated with the model. While FK representations have been available in non-relativistic QED for quite some time [17, 22, 25], the latter SDE has been derived and investigated only recently in our earlier work together with B. Güneysu and J.S. Möller [12]. (Our article [12] also provides a new version of the FK formula with spin, which will be employed in our examples; compare Rem. 2.18 below.)

The additional operator-valued process in the FK integrand is – roughly speaking – a perturbation of the semi-group associated with the radiation field energy operator, and it therefore has a regularizing effect on the position coordinates of the bosons that constitute the quantized radiation field. Taking these new regularizing effects into account, in addition to the familiar regularity properties of Schrödinger semi-groups, poses a new mathematical problem. To discuss these effects we employ our SDE and derive some Burkholder-Davis-Gundy (BDG) type estimates involving unbounded weight functions of the radiation field energy. As a result we obtain, for instance, the following canonical regularity result: To start with it turns out that – as usual – elements, $\Psi$, in the range of the semi-group at strictly positive times are represented by continuous functions of the position of the matter particles (electrons, nucleons), in our case attaining values in $\mathcal{H}$. For a fixed position, $x \in \mathbb{R}^\nu$, of the matter particles we therefore obtain a well-defined element of $\Psi(x) \in \mathcal{H}$. By means of the BDK type estimates we can show that the $n$-boson functions constituting $\Psi(x)$ belong to Sobolev spaces of some order $\alpha \geq 1$, provided that the coefficient functions in the SDE that couple the matter and radiation degrees of freedom belong to a Sobolev space (with respect to the bosonic position variable) of the same order. If the coefficient functions depend continuously on $x$ as elements of that Sobolev space, then so does $\Psi(x)$. If the coefficient functions are continuous functions of $x$ as elements of Sobolev spaces of arbitrary high orders, then the Sobolev embedding theorem implies that elements in the range of the semi-group are given by sequences (indexed by the boson number) of complex-valued functions that are jointly continuous in the position variables of the matter particles and the bosons, together with all derivatives with respect to any boson positions. All this holds true for Kato decomposable electrostatic potentials.

To show that the regularity of $\Psi(x)$ is not worse than the one of the coefficients, sufficiently good bounds on multiple commutators of creation and annihilation operators with functions of the radiation field energy are required in the derivation of
the BDG type estimates. Obtaining such commutator bounds with sufficiently good right hand sides turns out to be non-trivial. Hence we study them systematically in the appendix, which might also be useful elsewhere.

Of course, the analysis of the usual Schrödinger semi-groups by means of FK formulas is by now a well-known subject, which has been extensively studied by many authors in the past decades. One standard reference, most of the time treating Kato decomposable electrostatic potentials and neglecting magnetic fields, is [33]; see, e.g., also Ch. I.1 of [36] for a short introduction. In the presence of singular, classical magnetic fields, various continuity properties of the semi-group and its integral kernel are studied in [3]. Parts of the analysis in [3] are pushed forward to a matrix-valued case in [11]. The present article is mainly motivated by the work quoted so far: for more information on the literature on Schrödinger semi-groups we refer to the remarks and reference lists in [3, 33, 36].

In the following we shall describe some more results obtained in this article and explain its organization.

In Sect. 2 we fix our notation and standing assumptions and survey some earlier results. In Sect. 3 we verify that our Feynman-Kac operators define a self-adjoint semi-group between $\mathcal{H}$-valued $L^p$-spaces; we also verify the Chapman-Kolmogorov equations for an operator-valued integral kernel of the semi-group. The BDG type estimates mentioned above are established in Sect. 4 employing the commutator bounds of App. A. The results of Sect. 4 will also play an important role in a forthcoming analysis of a Bismut-Elworthy-Li type formula in non-relativistic QED [26].

After that we start analyzing continuity properties of the semi-group: Sect. 5 is devoted to weighted operator norm bounds on the semi-group between $\mathcal{H}$-valued $L^p$-spaces. At the same time, we shall study the continuous dependence of the semi-group on the electrostatic potential, which is always assumed to be Kato decomposable. The continuous dependence in weighted $L^p$-to-$L^q$-norms of the semi-group on the choice of the coupling functions, that determine the interaction between the matter particles and the radiation field, is discussed in Sect. 6. After that we study the strong continuity with respect to the time parameter of the semi-group in Sect. 7. In Sect. 8 we show that, at strictly positive times, the semi-group maps bounded sets in $L^p$ into equicontinuous sets of functions from $\mathbb{R}^\nu$ to $\mathcal{H}$. The results of Sects. 5–8 will imply the regularity results discussed above; cf. Ex. 8.3. With our crucial BDG type estimates at hand, we may proceed along the lines of the usual Schrödinger semi-group theory in the proofs in these sections.

As an application we mention a simple argument proving the pointwise exponential decay of the partial Fock-space norms of elements of low-lying spectral subspaces of atomic or molecular Hamiltonians in non-relativistic QED; see Ex. 8.3. This complements earlier pointwise decay results [16, 25] in several aspects: it applies to several electrons, it takes spin into account, the bound on the exponential decay rate is the natural one given in terms of the ionization energy, and it is not restricted to eigenvectors. In fact, corresponding $L^2$-exponential localization results already exist [3] and all we have to do is to apply our weighted operator norm bounds to go from $L^2$ to $L^\infty$.

The results of Sects. 5–8 will all be necessary in Sect. 9 to prove the (joint) continuity on $(0, \infty) \times \mathbb{R}^\nu \times \mathbb{R}^\nu$ of the operator-valued integral kernel of the semi-group with respect to some weighted operator norm. The fact that we study the
operator norm} continuity of the kernel also causes some new technical problems that do not show up in the usual Schrödinger semi-group theory. We shall also discuss the dependence on model parameters of the semi-group kernel.

The continuity results on the semi-group kernel will be complemented in Sect. \[ \] There it is shown that, if no spin degrees of freedom are present, the integral kernel (at arbitrary fixed \((t,x,y), t > 0\)) is positivity improving with respect to a suitable positive cone in the Fock space. The well-known positivity improvement by the corresponding semi-group \[ \] is an immediate corollary of this result. The relatively simple proof given in this section is based on a novel factorization of the FK integrand found in \[ \] and traditional ideas associated with Perron-Frobenius type arguments in mathematical quantum field theory; see, e.g., \[ \] .

Sect. \[ \] also provides a link to the final Sect. \[ \] where, again in the absence of any spin degrees of freedom, the joint continuity of positive ground state eigenvectors with respect to spatial coordinates and model parameters is discussed. In fact, what remains to prove in this section is only the continuous dependence on the model parameters in the Hilbert space norm. Since the ground state eigenvalues are typically embedded in the continuous spectrum this is, however, still a non-trivial task. We shall show that a certain compactness argument used to prove the existence of ground states in \[ \] can be adapted in such a way that is reveals their continuous dependence on model parameters. In fact, we shall present a simplified version of the compactness argument that works for a larger class of coupling functions and does not require the photon derivative bound used in \[ \]. This last section is rather sketchy at some points; we shall only work out the new observations that we would like to communicate.

General notation. \( \mathcal{D}(T) \) (resp. \( \mathcal{Q}(T) \)) denotes the domain (resp. form domain) of a suitable linear operator \( T \). If \( \mathcal{X} \) and \( \mathcal{Y} \) are normed vector spaces, then \( \mathcal{B}(\mathcal{X},\mathcal{Y}) \) denotes the set of bounded operators from \( \mathcal{X} \) to \( \mathcal{Y} \) and \( \mathcal{B}(\mathcal{X}) := \mathcal{B}(\mathcal{X},\mathcal{X}) \). If \( \mathbf{v} = (v_1,\ldots,v_k) \) is a vector of elements \( v_j \) of a fixed normed vector space, then we abbreviate \( \|\mathbf{v}\|^2 := \|v_1\|^2 + \cdots + \|v_k\|^2 \). If \( \eta \) is a measure on the \( \sigma \)-algebra \( \mathcal{C} \), then \( \eta^{\otimes n} \) is the corresponding \( n \)-fold product measure on the \( n \)-fold product \( \sigma \)-algebra \( \mathcal{C}^{\otimes n} \). We set \( a \wedge b := \min\{a,b\} \) and \( a \vee b := \max\{a,b\} \), for \( a,b \in \mathbb{R} \). If not specified otherwise, then the symbols \( c_{a,b}^\ldots \) denote positive constants that depend only on the quantities \( a,b,\ldots \) (if any) and whose values might change from one equation array to another.

2. Definitions, Assumptions, and Earlier Results

This preliminary section is split into six subsections. In the first one, we recall the definition of the bosonic Fock space, which is the state space of the quantized radiation field, as well as the definitions of certain operators acting in it. In Subsect. 2.2 we introduce the full Hilbert space and vector-valued \( L^p \)-spaces. The Hamiltonian determining the models covered by our results is introduced, for vanishing electrostatic potentials to start with, in Subsect. 2.3. In particular, a standing hypothesis on the terms in the Hamiltonian that couple the matter particles to the radiation field is introduced in that subsection. In Subsect. 2.4 we fix our notation for probabilistic objects and recall some results of \[ \] on SDE’s associated with the models under consideration. In this article we shall only treat Kato decomposable electrostatic potentials. Their definition and some of their well-known properties are recalled in Subsect. 2.5. Finally, in Subsect. 2.6 the main objects of our present
study, certain FK operators, are defined and a FK formula of [12] is recalled in the special case of Kato decomposable potentials. In this article, the FK formula of Thm. 2.17 below will actually only be used in Ex. 8.3 and in Sect. 11 where applications to non-relativistic QED are discussed. (The definition of the standard model of non-relativistic QED is recalled in Ex. 2.12 and Ex. 2.11.) All other results are statements on the FK operators introduced in Def. 2.10 that rely on the analysis of our SDE in [12], but not on the FK formula.

2.1. Bosonic Fock space. Let \((M, \mathfrak{A}, \mu)\) be a \(\sigma\)-finite measure space. To ensure separability of the corresponding \(L^2\)-space,

\[
\mathcal{H} := L^2(M, \mathfrak{A}, \mu),
\]

we assume that \(\mathfrak{A}\) is generated by some countable semi-ring \(\mathcal{S}\) such that \(\mu|_\mathcal{S}\) is \(\sigma\)-finite. The corresponding bosonic Fock space is denoted by

\[
\mathcal{F} := C \oplus \bigoplus_{n=1}^{\infty} \mathcal{F}^{(n)} \ni \psi = (\psi^{(0)}, \psi^{(1)}, \ldots, \psi^{(n)}, \ldots).
\]

Hence, \(\mathcal{F}^{(1)} := \mathcal{H}\) and \(\mathcal{F}^{(n)} \subset L^2(M^n; \mathfrak{A}^{\otimes n}, \mu^{\otimes n})\) is the closed subspace of all functions \(\psi^{(n)}\) which are symmetric under permutations of their arguments,

\[
\psi^{(n)}(k_1, \ldots, k_n) = \psi^{(n)}(k_{\pi(1)}, \ldots, k_{\pi(n)}), \quad \mu^{\otimes n}\text{-a.e.,}
\]

for every permutation \(\pi\) of \(\{1, \ldots, n\}\). The symbols \(a^\dagger(f)\) and \(a(f)\) denote the usual creation and annihilation operators of a boson \(f \in \mathcal{H}\). We recall the convenient notation

\[
(a(k)\psi)^{(n)}(k_1, \ldots, k_n) = (n + 1)^{1/2}\psi^{(n+1)}(k_1, \ldots, k_n, k), \quad \mu^{\otimes n+1}\text{-a.e.,}
\]

for every \(n \in \mathbb{N}\) and \(\psi \in \mathcal{F}\), and \((a(k)\psi)^{(0)} := \psi^{(1)}(k)\). Then

\[
(a(f)\psi)^{(n)} := \int f(k) (a(k)\psi)^{(n)} d\mu(k), \quad n \in \mathbb{N}_0, \, \psi \in \mathcal{D}(a(f)),
\]

and \(a^\dagger(f) := a(f)^*\), where \(\mathcal{D}(a(f))\) is the set of all \(\psi \in \mathcal{F}\) for which the right hand side of (2.3) defines an element of \(\mathcal{F}\). Let \(\mathcal{F}_{\text{fin}}\) denote the subspace of all elements \((\psi^{(n)})_{n=0}^{\infty} \in \mathcal{F}\) such that \(\psi^{(n)} \neq 0\) holds for only finitely many \(n\). Then we have the following canonical commutation relations on (e.g.) \(\mathcal{F}_{\text{fin}}\),

\[
[a(f), a(g)] = [a^\dagger(f), a^\dagger(g)] = 0, \quad [a(f), a^\dagger(g)] = \langle f | g \rangle \mathbb{1}, \quad f, g \in \mathcal{H}.
\]

The field operators given by

\[
\varphi(f) := a^\dagger(f) + a(f), \quad f \in \mathcal{H},
\]

are essentially self-adjoint, and we denote their unique self-adjoint extensions again by \(\varphi(f)\). Given a vector of boson wave functions, \(f = (f_1, \ldots, f_n)\), we set \(\varphi(f) := (\varphi(f_1), \ldots, \varphi(f_n))\) and we shall employ an analogous convention for the creation and annihilation operators.

If \(\mathcal{K}\) is a real-valued measurable function on \(M\) and \(n \in \mathbb{N}\), then \(d\Gamma^{(n)}(\mathcal{K})\) denotes the maximal operator of multiplication with the function \((k_1, \ldots, k_n) \mapsto \sum_{\ell=1}^n \mathcal{K}(k_\ell)\) in \(\mathcal{F}^{(n)}\). We also set \(d\Gamma^{(0)}(\mathcal{K}) := 0\) and recall that the differential second quantization of \(\mathcal{K}\) is the self-adjoint operator in \(\mathcal{F}\) given by the direct sum
bounded, if and only if $e^{L}$ for norms on scalar

2.3. $x$-by $x$ing

In our mathematical analysis of semi-groups we shall, however, most of the time

$$\text{(2.11)}$$

$$x \leftrightarrow$$ with a quantized bosonic field is then given by

$$\text{(2.10)}$$

the fiber Hilbert space. The full Hilbert space for non-relativistic quantum mechanical

particles with a total number of $L$ “spin” degrees of freedom and interacting

with a quantized bosonic field is then given by

$$\text{(2.11)}$$

In our mathematical analysis of semi-groups we shall, however, most of the time

work in the more general spaces $L^{p}(R^{'}, \mathcal{H} ; e^{F} dx)$, $p \in [1, \infty]$, where the density

e^{F} with respect to the Borel-Lebesgue measure $dx$ is given by some globally Lipschitz continuous function $F : R^{'} \rightarrow R$. The norm on $L^{p}(R^{'}, \mathcal{H} ; e^{F} dx)$ is denoted by $\| \cdot \|_{p}$, for $p \in [1, \infty]$, so that $\| \cdot \|_{2} = \| \cdot \|_{L^{2}}$. The operator norm on $B(L^{p}(R^{'}, \mathcal{H} ; L^{q}(R^{'}, \mathcal{H})))$ is denoted by $\| \cdot \|_{p,q}$. We shall use the same symbols for norms on scalar $L^{p}$-spaces as well, which should not cause any confusion.

Of course, a linear operator $T : L^{p}(R^{'}, \mathcal{H} ; e^{F} dx) \rightarrow L^{q}(R^{'}, \mathcal{H} ; e^{F} dx)$ is bounded, if and only if $e^{F T e^{-F}} \in B(L^{p}(R^{'}, \mathcal{H} ; L^{q}(R^{'}, \mathcal{H})))$, and in the affirmative case

$$\|T\|_{B(L^{p}(R^{'}, \mathcal{H} ; e^{F} dx), L^{q}(R^{'}, \mathcal{H} ; e^{F} dx))} = \|e^{F T e^{-F}}\|_{p,q}.$$  

2.3. Interaction terms and free Hamiltonian. The intercation of the non-relativistic quantum mechanical particles with the radiation field is determined by $S \in \mathbb{N}$ Hermitian matrices $\sigma_{1}, \ldots, \sigma_{S} \in B(C^{L})$ and two vectors of boson wave functions, $G_{x} = (G_{1,x}, \ldots, G_{S,x}) \in \mathfrak{h}^{S}$ and $F_{x} = (F_{1,x}, \ldots, F_{S,x}) \in \mathfrak{h}^{S}$, parametrized by $x \in R^{'}$. Most of the time we regard the matrices as operators on $\mathcal{H}$ by identifying $\sigma_{j} \equiv \sigma_{j} \otimes 1$. We shall write $\sigma \cdot v = \sigma_{1} v_{1} + \cdots + \sigma_{S} v_{S}$, if $v = (v_{1}, \ldots, v_{S})$ is any vector of numbers, functions, or suitable operators. To state our precise standing assumptions on $G$ and $F$, we first recall that a conjugation $C$ on a Hilbert space is an anti-linear isometry with $C^{2} = 1$.

Hypothesis 2.1. $\omega : M \rightarrow (0, \infty)$ is $\mu$-a.e. strictly positive. The map $R^{'} \times M \ni (x, k) \mapsto (G_{x}, F_{x}) \in C^{+S}$ is measurable, $x \mapsto G_{x}$ belongs to $C^{2}(R^{'}, \mathfrak{h}^{S})$, and $x \mapsto F_{x} \in \mathfrak{h}^{S}$ is globally Lipschitz continuous on $R^{'}$. All components of $G_{x}$, $F_{x}$, and $\partial_{x} G_{x}$ belong to

$$\mathfrak{f} := L^{2}(M, \mathfrak{A}, (\omega^{-1} + \omega^{2}) \mu).$$
and the map
\[ \mathbb{R}^\nu \ni x \mapsto (G_x, \partial_x G_x, \ldots, \partial_{x^S} G_x, F_x) \in \ell^{\nu+1+S} \]
is bounded and continuous. Furthermore, there is a conjugation \( C : \mathfrak{h} \to \mathfrak{h} \), such that, for all \( t > 0, x \in \mathbb{R}^\nu, \ell = \{1, \ldots, \nu\} \), and \( j \in \{1, \ldots, S\} \),
\[ [C, e^{-t\mathfrak{h}}] = 0, \quad CG_{\ell, x} = G_{\ell, x}, \quad C F_{j, x} = F_{j, x}. \]

The previous hypothesis will be tacitly assumed throughout the whole paper. Explicitly, we shall only mention certain additional assumptions on \( G \) and \( F \), which are occasionally used to derive some of our results. To this end it is convenient to introduce a coefficient vector \( c \) by setting
\[ q_x := \text{div}_x G_x, \quad c_x := (G_x, q_x, \sigma \cdot F_x) \in \ell^{\nu+1+L^2}, \quad x \in \mathbb{R}^\nu. \]
Furthermore, we abbreviate
\[ \mathcal{U}(x) := \|M(x)\|_{\mathcal{B}(\mathcal{L})}^2, \quad \text{where} \quad M(x) := (\|\omega^{-1/2}(\sigma \cdot F_x)_{ij}\|)_{i,j=1}^L. \]
The interaction terms appearing in the free Hamiltonian defined below are now given by
\[ \varphi(G) := \int_{\mathbb{R}^\nu} \|\mathcal{L} \otimes \varphi(G_x)\,dx, \quad \sigma \cdot \varphi(F) := \sum_{j=1}^S \int_{\mathbb{R}^\nu} \sigma_j \otimes \varphi(F_{j, x})\,dx, \]
\[ \varphi(q) := \int_{\mathbb{R}^\nu} \|\mathcal{L} \otimes \varphi(q_x)\,dx. \]

These direct integrals of self-adjoint operators are well-defined, since \( \mathbb{R}^\nu \ni x \mapsto e^{i\varphi(c, x)} \psi \in \mathcal{H} \) is measurable, for all \( \psi \in \mathcal{H} \), if \( c_j \) denotes any component of \( c \).

**Example 2.2.** To cover the standard model of non-relativistic QED for \( N \in \mathbb{N} \) electrons we choose \( \nu = 3N \) and write \( x = (x_1, \ldots, x_N) \in (\mathbb{R}^3)^N \) instead of \( x \).
Moreover, we choose \( \mathcal{M} = \mathbb{R}^3 \times \{0, 1\} \), equipped with the product of the Lebesgue and counting measures, and set \( \omega(k, \lambda) := |k|, (k, \lambda) \in \mathbb{R}^3 \times \{0, 1\} \). The coupling function \( G \) is given by
\[ G_{2N}(k, \lambda) := (G_{2N}^x(k, \lambda), \ldots, G_{2N}^y(k, \lambda)) \in (\mathbb{C}^3)^N, \]
and
\[ G_{2N}^x(k, \lambda) := (2\pi)^{-3/2}(2\omega(k, \lambda))^{-1/2} \chi(k)e^{-ik \cdot x} \varphi(k, \lambda), \quad x \in \mathbb{R}^3, \]
where the ultra-violet cut-off function \( \chi : \mathbb{R}^3 \to [0, \infty) \) is always assumed to be even, \( \chi(k) = \chi(-k), k \in \mathbb{R}^3, \) with \( \omega^2 \chi \in L^2(\mathbb{R}^3 \times \{0, 1\}) \). In the above formulas we chose the Coulomb gauge, i.e., \( q_{\mathbb{R}^3} = 0 \), and by applying a suitable unitary transformation, if necessary, we may assume that the polarization vectors are given as
\[ \varphi(k, 0) = |e \times k|^{-1} e \times k, \quad \varphi(k, 1) = |k|^{-1} k \times \varphi(k, 0), \quad \text{a.e. } k, \]
for some \( e \in \mathbb{R}^3 \) with \( |e| = 1 \). Several more explicit choices for \( \chi \) are common in the literature. Often a sharp ultra-violet cutoff is chosen, in which case \( \chi \) is proportional to the characteristic function of some ball about the origin in \( \mathbb{R}^3 \). Sometimes it is, however, favorable for technical reasons if \( \chi \) is some Schwartz function, e.g., a Gaussian.
In many papers devoted to the mathematical analysis of non-relativistic QED, spin is neglected. In this situation we set \( L = 1 \) and \( F = 0 \). To include spin, we choose \( L = 2^N \), so that \( C^L = (C^2)^\otimes N \), \( S = 3N \), and

\[
\sigma_{3\ell+j} := \mathbb{1}_{C^2}^{\otimes j} \otimes \sigma_j \otimes \mathbb{1}_{C^2}^{\otimes N-\ell-1}, \quad \ell = 0, \ldots, N-1, \ j = 1, 2, 3,
\]

with the \( 2 \times 2 \) Pauli spin-matrices \( \sigma_1, \sigma_2, \) and \( \sigma_3 \), as well as

\[
F^{\chi,N}_x(k, \lambda) := (F^{\chi}_x(k, \lambda), \ldots, F^{\chi}_x(k, \lambda)) \in (C^3)^N, \quad F^\chi_x(k, \lambda) := -\frac{i}{2} k \times G^\chi_x(k, \lambda).
\]

A suitable conjugation is given by \((Cf)(k, \lambda) := (-1)^{1+\lambda}f(-k, \lambda), \) a.e. \( k \in \mathbb{R}^3 \) and \( \lambda \in \{0, 1\} \). Obviously, if \( \chi \) is rapidly decaying (resp. a Gaussian), then the corresponding coefficient vector \( c^{\chi,N}_\mathbb{A} = (G^\chi_\mathbb{A}, 0, F^\chi,N_\mathbb{A}) \) satisfies

\[
\sup_{\mathbb{A}} \| \omega^\alpha c^{\chi,N}_\mathbb{A} \|_b < \infty \quad \text{(resp. } \sup_{\mathbb{A}} \| e^{\delta x} c^{\chi,N}_\mathbb{A} \|_b < \infty), \]

for all \( \alpha > 0 \) (resp. \( \delta > 0 \)). When relevant, additional conditions of this type will be imposed on the coefficient vector \( c \) in the statements of our results.

We are now in a position to introduce the Hamiltonian determining the free particle-radiation field system, i.e., we first consider the case of vanishing electrostatic potentials. For short, we will denote operators of the type \( \mathbb{1}_{C^L} \otimes A \) and constant direct integrals of the type \( \int_{\mathbb{R}^\nu} \mathbb{1}_{C^L} \otimes A dx \) simply by \( A \) in what follows. This should not cause any confusion if the underlying Hilbert space is specified.

**Definition 2.3.** We define the free Hamiltonian acting in the Hilbert space \( \mathcal{H} \) by

\[
H^0 := \frac{1}{2}(-i \nabla_x - \varphi(G))^2 - \sigma \cdot \varphi(F) + d\Gamma(\omega), \quad \mathcal{D}(H^0) := \mathcal{D}(\Delta) \cap \mathcal{D}(d\Gamma(\omega)).
\]

For every fixed \( x \in \mathbb{R}^\nu \), we further define an operator acting in \( \mathcal{H} \) by

\[
\hat{H}(x) := \frac{1}{2} \varphi(G_x)^2 - \frac{i}{2} \varphi(q_x) - \sigma \cdot \varphi(F_x) + d\Gamma(\omega), \quad \mathcal{D}(\hat{H}(x)) := \mathcal{D}(d\Gamma(\omega)).
\]

The \( \mathcal{H} \)-valued second order Sobolev space \( \mathcal{D}(\Delta) \) and the nabla operators acting on \( \mathcal{H} \)-valued Sobolev functions appearing in Def. 2.3 are defined by means of a \( \mathcal{H} \)-valued Fourier transform. These objects are introduced in complete analogy to the scalar case upon replacing the Lebesgue integral by a Bochner-Lebesgue integral in the formula for the Fourier transform of an element of \( L^1 \cap L^2 \).

It is known \[14\] \[18\] \[20\] that \( H^0 \) is well-defined and self-adjoint. In view of \( (2.8), H^0 \) is bounded from below. It is essentially well-known and not difficult to prove (see, e.g., \[12\] App. A) that \( \hat{H}(x) \) is closed, and self-adjoint if \( q_x = 0 \). On account of \( (2.0), (2.7), (2.9) \), and Hyp. 2.1 there is a universal constant \( c > 0 \) such that

\[
\sup_{x \in \mathbb{R}^\nu} \| \hat{H}(x)(1 + d\Gamma(\omega))^{-1} \| \leq c(1 + \sup_{x \in \mathbb{R}^\nu} \| c_x \|_b^2) < \infty.
\]

**2.4. Associated SDE’s.** Before we continue with our discussion of the Hamiltonians in the presence of exterior potentials we shall introduce some probabilistic objects in the present subsection. In particular, we shall recall some results of \[12\] on a certain SDE involving \( \hat{H}(x) \). For basic definitions and results from stochastic analysis we refer to \[4\] \[13\] \[28\] \[29\].

Let \( I \) be a closed interval with \( \inf I = 0 \) and \( \mathbb{B} = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbb{P}) \) some stochastic basis (i.e. a filtered probability space) satisfying the usual assumptions (i.e. it is complete and right continuous). We shall also consider the time-shifted basis,

\[
I_\tau := \{ t \geq 0 : \tau + t \in I \}, \quad \mathbb{B}_\tau := (\Omega, \mathcal{F}, (\mathcal{F}_{\tau+t})_{t \in I_\tau}, \mathbb{P}), \quad \tau \in [0, \sup I).
\]
If $\mathcal{X}$ is a separable Hilbert space, then $S_{T^*}(\mathcal{X})$ denotes the space of continuous $\mathcal{X}$-valued semi-martingales with respect to $\mathcal{B}_r$ and we set $S_{T^*}(\mathcal{X}) := S_{T^*}(\mathbb{R}^\nu)$. The bold letter $\mathbf{B} \in S_{T^*}(\mathbb{R}^\nu)$ denotes an arbitrary $\nu$-dimensional $\mathbb{B}$-Brownian motion with covariance matrix $1$. If $\tau \in [0, \sup I]$, then $\mathbf{X}^q := \mathbf{X}^q_{\tau} \in S_{T^*}(\mathbb{R}^\nu)$ is always a solution of the Itô equation

$$
(2.18) \quad \mathbf{X}_t = q + \mathbf{B}_{\tau+t} + \int_0^t \beta(\tau+s, \mathbf{X}_s) \, ds, \quad t \in [0, \sup I^\tau),
$$

for some $\mathbf{F}_x$-measurable $q : \Omega \to \mathbb{R}^\nu$. We set $\mathbf{X}^q := 0 \mathbf{X}^q$. In fact, we shall only need to consider two different choices of the time-dependent vector field $\beta : [0, \sup I] \times \mathbb{R}^\nu \to \mathbb{R}^\nu$ in the present article. Once and for all we thus agree upon the following conventions:

- either $I = [0, t]$ with $t > 0$ and $\beta(s, x) = \frac{y-x}{s-t}$ with $y \in \mathbb{R}^\nu$, so that $\mathbf{X}^q$ with $\tau \in [0, t)$ is a Brownian bridge from some $\mathbf{F}_x$-measurable $q$ to $y$ in time $t - \tau$. In this case we denote $\mathbf{X}^q$ also by $\mathbf{X}^{b_1, q, y}$ and $\mathbf{X}^b$ by $\mathbf{X}^{b_1, q, y}$.

- or $I = [0, \infty)$ and $\beta = 0$, so that $\mathbf{X}^q = \mathbf{X}^b := q + \mathbf{B}_{\tau+t}$. For every $s \in I$, we denote the process obtained by reversing $\mathbf{X}^q$ at time $s$ by $(\mathbf{X}^q)^t_{\tau-s} : \mathbf{X}^q_{\tau-s}, \tau \in [0, s]$. The associated filtration $(\mathbf{F}_x, (\mathbf{X}^q)^t_{\tau-s})_{\tau \in [0, s]}$ is the standard extension of the filtration $(\mathbf{F}_x^T)_{\tau \in [0, s]}$ given by $\mathbf{F}_x^T := \sigma(\mathbf{X}^q_{\tau-s}, \mathbf{B}_u - \mathbf{B}_s : s - \tau \leq r \leq u \leq s)$; see $[15, 30]$. Replacing $(\mathbf{F}_x, (\mathbf{F}_x^T)_{\tau \in [0, s]}, \mathbb{P})$ by $[\mathbf{F}, \mathbf{F}^T, \mathbb{P}, \mathbb{F}^T)$ we obtain a new stochastic basis, denoted by $[\mathbf{F}, \mathbf{F}^T, \mathbb{P}, \mathbb{F}^T)$, again satisfying the usual assumptions.

**Example 2.4.**

1. Let $\Omega_W := C([0, \infty), \mathbb{R}^\nu)$ denote the Wiener space and $\mathbf{F}_x^W$ the completion of the Borel $\sigma$-algebra associated with the standard Polish topology on $\Omega_W$ with respect to the Wiener measure $\mathbb{P}_W$ on $\Omega_W$. Let $\mathbf{F}_x^W$ be the standard extension of the filtration $(\mathbf{F}_x)_{\tau \in [0, s]}$ generated by the evaluation maps $\text{pr}_x(\gamma) := \gamma(t), \gamma \in \Omega_W, t \geq 0$. Then $(\Omega, \mathbf{F}_x^W, (\mathbf{F}_x^W)_{t \leq t}, \mathbb{P}, \mathcal{F}) := (\Omega_W, \mathbf{F}_x^W, (\mathbf{F}_x^W)_{t \leq t}, \mathbb{P}_W, \mathcal{F})$ is the standard example of a Brownian motion.

2. Let $t > 0$ and $x \in \mathbb{R}^\nu$. Then there exists a Brownian motion $\mathbf{B}$ with respect to the stochastic basis $[\mathbf{F}, \mathbf{F}^T, \mathbb{P}, \mathcal{F}]$ such that, up to indistinguishability, $\mathbf{R} \mathbf{X}^x$ is the unique solution with initial condition $q = \mathbf{X}^x_{\tau}$ of the stochastic differential equation

$$
(2.19) \quad \mathbf{X}_{\tau} = q + \int_0^\tau \mathbf{X}_{\tau-s} \, du + \mathbf{B}_{\tau}, \quad \tau \in [0, t), \quad \mathbf{X}_t = x.
$$

These assertions follow from $[15, 30]$. We see that $\mathbf{R} \mathbf{B}^x$ is a Brownian bridge back to $x$, defined by means of a new stochastic basis and Brownian motion. We will denote the solution of $(2.19)$ corresponding to a bounded $[\mathbf{F}, \mathbf{F}^T, \mathbb{P}, \mathcal{F}]$-measurable initial condition $q \in \mathbb{R}^\nu$ by the symbol $\mathbf{b}^{t, q, x}$.

3. Let $t > s > 0$ and $x, y \in \mathbb{R}^\nu$. Then there exists a Brownian motion $\mathbf{B}$ with respect to the stochastic basis $[\mathbf{F}, \mathbf{b}^{t, x, y}]$ such that, up to indistinguishability, $\mathbf{R} \mathbf{b}^{t, x, y}$ is the unique solution with initial condition $q = \mathbf{b}^{t, x, y}$ of the stochastic differential equation

$$
(2.20) \quad \mathbf{b}_{\tau} = q + \int_0^\tau \mathbf{X}_{\tau-s} \, du + \mathbf{B}_{\tau}, \quad \tau \in [0, s), \quad \mathbf{b}_s = x.
$$

These assertions follow again from $[15, 30]$. We denote the solution of $(2.20)$ with an arbitrary $[\mathbf{F}, \mathbf{b}^{t, x, y}]$-measurable initial condition $q : \Omega \to \mathbb{R}^\nu$ by $\mathbf{b}^{t, q, x}$. Then
Remark 2.5. Let $(\Sigma, \mathcal{F})$ be a measurable space and $T$ be a closed operator acting in the separable Hilbert space $\mathcal{H}$. Consider its domain $\mathcal{D}(T)$ as a Hilbert space equipped with the graph norm of $\mathcal{D}'(\omega)$, whence it might make sense to recall the following:

In the following three theorems we quote some of the main results of [12]. In their statements we shall always consider $\mathcal{D}(d\Gamma'(\omega))$ as a Hilbert space equipped with the graph norm of $\mathcal{D}'(\omega)$, hence it might make sense to recall the following:

**Theorem 2.6** ([12]). Let $V : \mathbb{R}^r \to \mathbb{R}$ be bounded and continuous. Then, for all $\tau \in [0, \sup I)$ and $\mathcal{F}_T$-measurable $q : \Omega \to \mathbb{R}^r$, there exist operators $\mathcal{W}_t^V[\tau X^q](\gamma) \in \mathcal{B}(\mathcal{H})$, $t \in I^\gamma$, $\gamma \in \Omega$, such that the following holds:

1. For every $t \in I^\gamma$, the operator-valued map $\mathcal{W}_t^V[\tau X^q] : \Omega \to \mathcal{B}(\mathcal{H})$ is $\mathcal{F}_{t+\cdot}$-measurable, and $\mathcal{F}$-almost separably valued.
2. Using the notation (2.13), we have the following bound for all $t \in I^\gamma$,

\[
\ln \|\mathcal{W}_t^V[\tau X^q]\| \leq \int_0^t (\mathcal{H}(\tau X^q) - V(\tau X^q))ds \quad \text{on } \Omega.
\]

3. If $\eta : \Omega \to \mathcal{D}(d\Gamma'(\omega))$ is $\mathcal{F}_T$-measurable, then $\mathcal{W}_t^V[\tau X^q] \eta \in \mathcal{S}_{T^\gamma}(\mathcal{H})$ and, up to indistinguishability, $\mathcal{W}_t^V[\tau X^q] \eta$ is the unique element of $\mathcal{S}_{T^\gamma}(\mathcal{H})$ whose paths belong $\mathcal{F}$-a.e. to $C(I^\gamma, \mathcal{D}(d\Gamma'(\omega)))$ and which $\mathcal{F}$-a.e. solves

\[
Y_s = \eta + i \int_0^s \varphi(G_s X^q) Y_s d\tau X^q - \int_0^s (\bar{\mathcal{H}}(\tau X^q) + V(\tau X^q)) Y_s ds
\]
on $[0, \sup I^\gamma)$.

4. For all $0 \leq \tau \leq t \in I$, $x \in \mathbb{R}^r$, $\psi \in \mathcal{H}$, and $\gamma \in \Omega$, set

\[
\Lambda_{t, \tau}(x, \psi, \gamma) := (\tau X_{t-\cdot}^q, \mathcal{W}_{t-\cdot}^V[\tau X^q] \psi)(\gamma) \in \mathbb{R}^r \times \mathcal{H}.
\]

Then $(\Lambda_{t, \tau})_{0 \leq \tau \leq t \in I}$ is a stochastic flow for the system of SDE comprised of (2.18) and (2.22) and in particular we have, for all $\tau \in I$ and $\mathcal{F}_T$-measurable $(q, \eta) : \Omega \to \mathbb{R}^r \times \mathcal{H}$,

\[
\Lambda_{t, \tau+\cdot}(q(\gamma), \eta(\gamma), \gamma) = (\tau X^q, \mathcal{W}_t^V[\tau X^q] \psi)(\gamma) \quad \text{on } I^\gamma, \text{ for } \mathcal{F}$-a.e. $\gamma.
\]

5. Let $\mathcal{H}$ be a separable Hilbert space. Set

\[
(P_{t, \tau} f)(x, \psi) := \int_\Omega f(\Lambda_{t, \tau}(x, \psi, \gamma)) d\mathcal{F}(\gamma), \quad x \in \mathbb{R}^r, \psi \in \mathcal{H},
\]

for all $0 \leq \tau \leq t \in I$ and every bounded Borel measurable function $f : \mathbb{R}^r \times \mathcal{H} \to \mathcal{H}$. Then the transition operators $(P_{t, \tau})_{0 \leq \tau \leq t \in I}$ satisfy the following Markov property: If $0 \leq \sigma \leq \tau \leq t \in I$, if $f$ is a bounded real-valued Borel function on $\mathbb{R}^r \times \mathcal{H}$ or if $f : \mathbb{R}^r \times \mathcal{H} \to \mathcal{H}$ is bounded and continuous, and if $(q, \eta) : \Omega \to \mathbb{R}^r \times \mathcal{H}$ is $\mathcal{F}_T$-measurable, then it follows that, for $\mathcal{F}$-a.e. $\gamma$,

\[
(\mathbb{E}^{\mathcal{F}_T}[f(\Lambda_{\sigma, \tau}(q, \eta))]) = (P_{t, \sigma} f)(\Lambda_{\sigma, \tau}(q(\gamma), \eta(\gamma), \gamma)).
\]

Here $\mathbb{E}^{\mathcal{F}_T}$ denotes conditional expectation with respect to $\mathcal{F}$ given $\mathcal{F}_T$ and $f(\Lambda_{\sigma, \tau}(q, \eta))$ is the random variable given by $\Omega \ni \gamma \mapsto f(\Lambda_{\sigma, \tau}(q(\gamma), \eta(\gamma), \gamma))$. 

$(\mathbb{E}^{\mathcal{F}_T}, \mathcal{F}^{\mathcal{F}_T} \cup \mathcal{F}_T, B^{\mathcal{F}_T} \cup B_T)$ is another valid choice for a basis and a driving process to which Thm. 2.6 below applies.
\textbf{Theorem 2.7 \cite{12}.} Assume that $V$ is continuous and bounded. Let $(\Lambda_t^W)_{t \in I}$ denote the stochastic flow defined in Thm. 2.6(3) for the special choices $\mathcal{B} = (\Omega, \mathcal{F}, \mathbb{P})_t \in I$ and $\mathcal{B} = \mathbb{P};$ see Ex. 2.4(1). Then $(\Lambda_t^W)_{t \in I}$ is a strong solution of (2.18) \& (2.22) in the sense that, for any stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \in I}, \mathbb{P})$ and Brownian motion $\mathcal{B}$ as explained in the beginning of this subsection, and for any $\mathcal{F}_y$-measurable $(q, \eta) : \Omega \to \mathbb{R}^n \times \mathcal{H},$ the (up to indistinguishability) unique solution of (2.18) \& (2.22) with $t = 0$ is given, for $\mathbb{P}$-a.e. $\gamma \in \Omega,$ by
\begin{equation}
(\mathcal{X}_t^\eta, \mathcal{W}_t^\nu [\mathcal{X}^\eta](\gamma)) = \Lambda_t^\nu (q(\gamma), \eta(\gamma), \mathcal{B}_t(\gamma)), \quad t \in I.
\end{equation}

\textbf{Example 2.8.} (1) For all $t > s > 0,$ $y, z \in \mathbb{R}^n,$ and $\psi \in \mathcal{A},$ the random variables $\mathcal{W}_s^\nu [b^{t-s; x, y}] \psi$ and $\mathcal{W}_s^\nu [b^{t-s; y}] \psi$ have the same distribution. In fact, both processes $(b^{t-s; x, y}, \mathcal{W}_s^\nu [b^{t-s; y}] \psi)$ and $(b^{t-s; y}, \mathcal{W}_s^\nu [b^{t-s; y}] \psi)$ solve the same system of equations (2.18) \& (2.22). The only difference is that the second one is constructed means that the time shifted basis with filtration $(\mathcal{F}_s + t)_{t \in [0, t]}$ and the time shifted Brownian motion $(\mathcal{B}_s + t)_{t \in [0, t]}.$ Since $(\mathcal{B}_r)_{r \in [0, t]}$ and $(\mathcal{B}_s + t)_{t \in [0, t]}$ have the same distribution and since the initial conditions $q = \eta$ and $\psi$ are constant, the claim follows from Thm. 2.7.

(2) For all $s > 0,$ $x, y, z \in \mathbb{R}^n,$ and $\psi \in \mathcal{A},$ the random variables $\mathcal{W}_s^\nu [b^{t-s; z, x}] \psi,$ $\mathcal{W}_s^\nu [b^{t-s; z}] \psi,$ and $\mathcal{W}_s^\nu [b^{t-s; x}] \psi$ have the same distribution, where we use the notation introduced in Ex. 2.4(2) \& (3). In fact, $b^{t-s; z, x}$, $b^{t-s; z}$ and $b^{t-s; x}$ satisfy formally the same SDE; only the underlying bases and the driving Brownian motions are different. Hence, the claim follows again from Thm. 2.7.

\textbf{Theorem 2.9 \cite{12}.} Let $V : \mathbb{R}^n \to \mathbb{R}$ be bounded and continuous, $x \in \mathbb{R}^n,$ $s \in I,$ and let $\mathcal{W}_s^\nu [\mathcal{A}_s \mathcal{X}^\eta]$ be given by Thm. 2.6 applied with $\mathcal{A}_s \mathcal{B}$ as underlying stochastic basis. Then the following identity holds $\mathbb{P}$-a.s.,
\[ \mathcal{W}_s^\nu [\mathcal{X}^\eta]^* = \mathcal{W}_s^\nu [\mathcal{A}_s \mathcal{X}^\eta]. \]

If the Brownian bridge is chosen as driving process, then we also know the following:

\textbf{Proposition 2.10 \cite{12}.} Let $V : \mathbb{R}^n \to \mathbb{R}$ be bounded and continuous and $t > 0.$ Then, for all $x, y \in \mathbb{R}^n,$ we can choose versions of $\mathcal{W}_s^\nu [b^{t-s; x, y}]$ such that $0, t] \times \mathbb{R}^n \times \Omega \ni (s, x, y, \gamma) \mapsto \mathcal{W}_s^\nu [b^{t-s; x, y}] (\gamma) \in \mathcal{A}_s \mathcal{H}$ is measurable with a separable image and $(s, x, y) \mapsto \mathcal{W}_s^\nu [b^{t-s; x, y}] (\gamma)$ is continuous from $(0, t] \times \mathbb{R}^n$ into $\mathcal{B}_s \mathcal{H}$ for all $\gamma \in \Omega.$

\subsection{2.5. Kato-decomposable potentials.}
For $x \neq 0$ and $r > 0,$ we set
\[ g_r(x) := \frac{1}{|x|}, \quad \text{if} \quad |x| < r, \quad \text{if} \quad |x| > r, \]
and we recall that a real-valued measurable function $V : \mathbb{R}^n \to \mathbb{R}$ is in the Kato-class, in symbols $V \in K(\mathbb{R}^n),$ iff
\[ \lim_{r \downarrow 0} \sup_{x \in \mathbb{R}^n} \int_{|y| < r} g_r(x - y) |V(y)| \, dy = 0. \]

It is said to be in the local Kato-class, in symbols $V \in K_{\text{loc}}(\mathbb{R}^n),$ iff $1_K V \in K(\mathbb{R}^n),$ for every compact $K \subset \mathbb{R}^n.$ It is called Kato-decomposable, in symbols $V \in K_{\text{deg}}(\mathbb{R}^n),$ if there exist $V_+ \in K_{\text{loc}}(\mathbb{R}^n)$ and $V_- \in K_{\text{loc}}(\mathbb{R}^n)$ with $V_+ \geq 0$ and $V = V_+ - V_-.$

We refer to \cite{11, 33, 35} for detailed treatments of Kato decomposable potentials and examples; for instance, $L^p_{\text{deg}}(\mathbb{R}^n) \subset K(\mathbb{R}^n)$ and $L^p_{\text{deg}, \text{loc}}(\mathbb{R}^n) \subset K_{\text{loc}}(\mathbb{R}^n),$ if $p > n/2.$

In view of the subsequent discussion it makes sense to recall the following:
Remark 2.11. For all \( V \in \mathcal{K}_{\text{loc}}(\mathbb{R}^\nu) \) and \( x \in \mathbb{R}^\nu \), we \( \mathbb{P} \)-a.s. have \( |V| (B^x_t) \in L^1_{\text{loc}}([0, \infty)) \); see, e.g., [36 §1.2]. Likewise, if \( V \in \mathcal{K}_{\text{loc}}(\mathbb{R}^\nu) \), \( x, y \in \mathbb{R}^\nu \), and \( t > 0 \), then we \( \mathbb{P} \)-a.s. have \( |V| (b^{x,y}_t) \in L^1([0, t]) \); see, e.g., the proof of Lem. C.8 in [3].

Of particular importance for us are the standard facts collected in the following lemma, where the Euclidean heat kernel is denoted by

\[
p_t(x, y) := (2\pi t)^{-\nu/2} e^{-(x-y)^2/4t}, \quad t > 0, \ x, y \in \mathbb{R}^\nu.
\]

**Lemma 2.12.** Let \( V \in \mathcal{K}_\pm(\mathbb{R}^\nu) \) and \( p > 0 \). Then the following holds:

1. There exists \( c > 0 \), depending only on \( pV_- \), such that for all \( t > 0 \) and \( x, y \in \mathbb{R}^\nu \),
   \[
   \sup_{z \in \mathbb{R}^\nu} \mathbb{E}[e^{\int_0^t V(B^z_s)ds}] \leq 2 e^{ct},
   \]
   (2.27)

2. For every compact \( K \subset \mathbb{R}^\nu \),
   \[
   \limsup_{s \downarrow 0} \sup_{y \in K} \mathbb{E}[|e^{\int_0^s V(B^y_u)du} - 1|^p] = 0.
   \]
   (2.29)

In the case \( V \in \mathcal{K}(\mathbb{R}^\nu) \), the compact set \( K \) in (2.29) can be replaced by \( \mathbb{R}^\nu \).

**Proof.** (1): See, e.g., [36 §1.1.2]. (2) is proved, e.g., in Lem. C.3 and C.5 of [3]. \( \square \)

Let \( V \in \mathcal{K}_\pm(\mathbb{R}^\nu) \), \( F : \mathbb{R}^\nu \to \mathbb{R} \) be globally Lipschitz continuous, and \( f \in L^p(\mathbb{R}^\nu; e^F dx) \) with \( p \in [1, \infty) \). Then the expectations in

\[
(S^V_t f)(x) := \mathbb{E}[e^{-\int_0^t V(B^x_s)ds}], \quad x \in \mathbb{R}^\nu, \ t \geq 0,
\]
are well-defined and, for all \( x \in \mathbb{R}^\nu \) and \( t > 0 \), we further have the relation

\[
(S^V_t f)(x) = \int_{\mathbb{R}^\nu} S^y_t(x, y) f(y) dy,
\]
with

\[
S^y_t(x, y) := p_t(x, y) \mathbb{E}[e^{-\int_0^t V(b^{x,y}_s)ds}].
\]

Here the map \( (0, \infty) \times \mathbb{R}^{2\nu} \ni (t, x, y) \mapsto S^y_t(x, y) \) is continuous and can be dominated by means of (2.28); see, e.g., [36 §1.1.3]. Using (2.27), one can in fact verify that \( S^y_t \) with \( t > 0 \) maps \( L^p(\mathbb{R}^\nu; e^F dx) \) continuously into every \( L^q(\mathbb{R}^\nu; e^F dx) \) with \( q \in [p, \infty) \). Moreover,

\[
\sup_{\text{Lip}(F) \leq \alpha} \sup_{\tau_1 \leq t \leq \tau_2} \| e^{\int_0^t S^y_s V} e^{-F} \|_{p, q} < \infty, \quad 0 < \tau_1 \leq \tau_2 < \infty, \quad 1 \leq p \leq q \leq \infty,
\]

where \( \| \cdot \|_{p, q} \) is the norm on \( \mathcal{B}(L^p(\mathbb{R}^\nu), L^q(\mathbb{R}^\nu)) \) and the first supremum is taken over all Lipschitz continuous \( F : \mathbb{R}^\nu \to \mathbb{R} \) with Lipschitz constant \( \leq \alpha \in (0, \infty) \).

In the case \( p = q \) the choice \( \tau_1 = 0 \) is allowed for in (2.28) as well; compare, e.g., Lem. C.1 in [3] or the proof of Thm. 5.2(1) below.

We shall also make use of the following approximation results:

**Lemma 2.13.** (1) For every \( V \in \mathcal{K}_\pm(\mathbb{R}^\nu) \), there exist \( V_n \in C^\infty_0(\mathbb{R}^\nu, \mathbb{R}) \), \( n \in \mathbb{N} \), such that, for every compact \( K \subset \mathbb{R}^\nu \),

\[
\lim_{n \to \infty} \sup_{x \in \mathbb{R}^\nu} \int_K g_1(x-y)|V - V_n|(y) dy = 0,
\]

(2.34)
and, for some $A \geq 1$ and every $r \in (0, 1]$,
\begin{equation}
\sup_n \sup_{x \in \mathbb{R}^r} \int_{\mathbb{R}^r} g_r(x - y)V_n - (y)dy \leq A \sup_{x \in \mathbb{R}^r} \int_{\mathbb{R}^r} g_r(x - y)V_n^-(y)dy.
\end{equation}

(2) Let $V_n \in K_\pm(\mathbb{R}^\nu)$, $n \in \mathbb{N}$. Then the bound
\begin{equation}
\sup_n \sup_{x \in \mathbb{R}^r} \int_{\mathbb{R}^r} g_r(x - y)V_n^-(y)dy < \infty,
\end{equation}
imples
\begin{equation}
\forall \tau, p > 0 : \sup_n \sup_{t \in [0, \tau]} \sup_{x \in \mathbb{R}^\nu} \mathbb{E}[e^{-p \int_0^t V_n(B^x_s)ds}] < \infty.
\end{equation}

(3) Let $V_n \in K(\mathbb{R}^\nu)$, $n \in \mathbb{N}$. Then the analytic condition
\begin{equation}
\limsup_{n \to \infty} \sup_{r \in K, n \in \mathbb{N}, x \in \mathbb{R}^r} \int_0^t \mathbb{E}[V_n(B^x_s)]ds = 0.
\end{equation}

(4) Let $V \in K_\pm(\mathbb{R}^\nu)$. Then, for every sequence $V_n \in K_\pm(\mathbb{R}^\nu)$, $n \in \mathbb{N}$, satisfying (2.34) and (2.35) and every compact $K \subset \mathbb{R}^\nu$,
\begin{equation}
\forall \tau, p > 0 : \lim_{n \to \infty} \sup_{t \in [0, \tau]} \sup_{x \in K} \mathbb{E}[e^{-p \int_0^t V(B^x_s)ds} - e^{-p \int_0^t V_n(B^x_s)ds}] = 0.
\end{equation}

If in addition $V - V_n \in K(\mathbb{R}^\nu)$, $n \in \mathbb{N}$, and (2.34) holds with $K$ replaced by $\mathbb{R}^\nu$, then the compact set $K$ in (2.40) can be replaced by $\mathbb{R}^\nu$.

Proof. Detailed proofs of (1) (resp. (4)) can be found, e.g., in App. A (resp. Lem. C.4 and Lem. C.6) of [3]. In fact, Lem.C.4 of [3] proves a slightly weaker version of the last statement of (4), where the condition $V - V_n \in K(\mathbb{R}^\nu)$, $n \in \mathbb{N}$, is replaced by $V \in K_\pm(\mathbb{R}^\nu)$ and $V_n \in K(\mathbb{R}^\nu)$, $n \in \mathbb{N}$. Combining this weaker statement with (2.27), we see, however, that the full assertion follows from Hölder's inequality. Part (2) follows from a standard application of Hölder's inequality and Khasminskii's lemma; compare the proof of Lem. C.1 in [3]. \hfill \square

Lemma 2.14. Let $V \in K(\mathbb{R}^\nu)$. Then $V$ is infinitesimally form-bounded with respect to the Laplacian. More precisely,
\begin{equation}
\int_{\mathbb{R}^\nu} |V(x)||f(x)|^2dx \leq \frac{c_\gamma(V)}{2} \int_{\mathbb{R}^\nu} |\nabla f(x)|^2dx + \gamma c_\gamma(V)\|f\|^2,
\end{equation}
for all $\gamma > 0$ and $f \in H^1(\mathbb{R}^\nu)$, where
\begin{equation}
c_\gamma(V) := \sup_{x \in \mathbb{R}^\nu} \int_0^\infty e^{-t\gamma} \mathbb{E}[|V(B^x_t)|]dt,
\end{equation}
which, for all $\tau > 0$ and $\gamma \geq 1$, can be estimated as follows,
\begin{equation}
c_\gamma(V) \leq \sup_{x \in \mathbb{R}^\nu} \int_0^\tau \mathbb{E}[|V(B^x_t)|]dt + Ae^{-\gamma/2}\ln \left( \sup_{x \in \mathbb{R}^\nu} \mathbb{E}[e^{\int_0^\tau |V(B^x_t)|ds}] \right),
\end{equation}
with some universal constant $A > 0$. For all $\varepsilon > 0$ and every sequence $V_n \in K_\pm(\mathbb{R}^\nu)$, $n \in \mathbb{N}$, satisfying (2.36), we find some $\gamma \geq 1$ such that $c_\gamma(V_n) \leq \varepsilon$, $n \in \mathbb{N}$.
Proof: The bound (2.41) is, e.g., identical to [5] Eqn. (58) on p. 92. To derive the bound (2.42), where $A := \sum_{i=1}^{\infty} e^{-(\ell-1)/2}$, follows easily from

$$e^{f_0} \mathbb{E}[|V(B^*_{\tau})|] \leq \mathbb{E}[e^{f_0} |V(B^*_{\tau})|] \leq \| S^{|V|}_{\ell} \|_{\infty, \infty} \leq \| S^{|V|}_{1} \|_{\infty, \infty} = \left( \mathbb{E}[e^{f_0} |V(B^*_{\tau})|] \right)^{\ell},$$

where we applied Jensen’s inequality in the first step. In view of (2.37) and (2.39), we may first choose $\tau > 0$ small enough and then $\gamma > 1$ large enough (depending on $\tau$) to make $\sup_n c_{\gamma}(V_n)$ small as we please. \hfill \Box

Example 2.15. (1) Let us consider the standard model of non-relativistic QED (Ex. 2.2) for $N \in \mathbb{N}$ electrons in the electrostatic potential of $K \in \mathbb{N}$ nuclei with atomic numbers $Z = (Z_1, \ldots, Z_K) \in [0, \infty)^K$ whose positions are given by the components of $R = (R_1, \ldots, R_K) \in \mathbb{R}^{3K}$. Then the total Hamiltonian reads

$$(2.43) \quad H_{Z, R}^{N, \epsilon} := \sum_{\ell=1}^{N} \left( \frac{1}{\ell} | -1 \nabla_{x_\ell} - \varphi(G_{x_\ell}) |^2 - \sigma(\ell) \cdot \varphi(F_{x_\ell}) \right) + d\Gamma(\omega) + V_{Z, R}^{N, \epsilon},$$

where $\sigma(\ell) := (\sigma_{3\ell-2}, \sigma_{3\ell-1}, \sigma_{3\ell})$ and where we dropped the cumbersome direct integral signs of (2.15) in the notation. Here the Coulomb interaction potential is

$$(2.44) \quad V_{Z, R}^{N, \epsilon}(x_1, \ldots, x_N) := -\sum_{i=1}^{N} \sum_{\ell=1}^{K} \frac{e^2 Z_{\ell}}{|x_i - R_{\ell}|} + \sum_{1 \leq i < j \leq N} \frac{e^2}{|x_i - x_j|},$$

where $e > 0$ is the elementary charge. It is well-known that $V_{Z, R}^{N, \epsilon}$ belongs to $\mathcal{K}(\mathbb{R}^{3N})$ [1 p. 216/7]. It is also well-known that the Hamiltonian in (2.43) is well-defined and self-adjoint on $D(\Delta) \cap D(d\Gamma(\omega))$ [14] [13] [20]; it is a special case of the Hamiltonians constructed for general Kato-decomposable potentials in Thm. 2.17 below. According to the Pauli principle, the physically relevant Hamiltonian is actually not $H_{Z, R}^{N, \epsilon}$, but rather its restriction to the reducing subspace of those functions in $\mathcal{H}$ which are anti-symmetric under simultaneous permutations of their $N$ electronic position-spin variables, i.e., to

$$\mathcal{H}^{N}_{\text{phys}} := \left( \bigwedge_{j=1}^{N} L^2(\mathbb{R}^3, \mathbb{C}^2) \right) \otimes \mathcal{F},$$

if it is considered as a subspace of $\mathcal{H}$ in the canonical way. Since all results of this paper proven for $H_{Z, R}^{N, \epsilon}$ contain analogous results for $H_{Z, R}^{N, \epsilon} \upharpoonright_{\mathcal{H}^{N}_{\text{phys}}}$ as an obvious special case, we shall most of the time not comment on the Pauli principle anymore.

(2) Let $\epsilon_n > 0$, $n \in \mathbb{N}$, such that $\epsilon_n \to \epsilon$, $n \to \infty$, for some $\epsilon > 0$. Furthermore, let $Z_n \in [0, \infty)^K$, $n \in \mathbb{N}$, be a converging sequence of $K$ atomic numbers with limit $Z \in [0, \infty)^K$. Abbreviate $V := V_{Z, R}^{N, \epsilon}$ and $V_n := V_{Z_n, R_n}^{N, \epsilon_n}$, $n \in \mathbb{N}$. Then, quite obviously, $V_n$ converges to $V$ in the sense that (2.31) and (2.35) are satisfied.

Next, we consider $R_n \in \mathbb{R}^{2K}$, $n \in \mathbb{N}$, be a converging sequence of nuclear positions with limit $R \in \mathbb{R}^{2K}$ and set $V_n' := V_{Z_n, R_n}^{N, \epsilon_n}$, $n \in \mathbb{N}$. Let us explain why (2.31) and (2.35) are satisfied by $V$ and $V_n'$, $n \in \mathbb{N}$. The validity of (2.35) is again obvious, because neither its right nor its left hand side change when the potentials is replaced by their translates. To argue that (2.31) holds true as well, it suffices to
treat a single term of the form $|y_i - R_{x,n}|^{-1}$. Without loss of generality we may further assume that $R_{x,n} \to 0$. Writing

$$\frac{1}{|y_i - R_{x,n}|} - \frac{1}{|y_i|} = \left( \frac{1}{|y_i - R_{x,n}|^{1/2}} - \frac{1}{|y_i|^{1/2}} \right) \left( \frac{1}{|y_i - R_{x,n}|^{1/2}} + \frac{1}{|y_i|^{1/2}} \right),$$

using that

$$\left| \frac{1}{|y_i - R_{x,n}|^{1/2}} - \frac{1}{|y_i|^{1/2}} \right| \leq \frac{|R_{x,n}|^{1/2}}{|y_i - R_{x,n}|^{3/2} |y_i|^{1/2}} \leq \frac{|R_{x,n}|^{1/2}}{2 \left( |y_i - R_{x,n}|^{1/2} + |y_i|^{1/2} \right)}.$$

and applying Young’s inequality $(\frac{1}{3} + \frac{2}{3} = 1)$ twice, we see that

$$\left| \frac{1}{|y_i - R_{x,n}|} - \frac{1}{|y_i|} \right| \leq |R_{x,n}|^{1/2} \left( \frac{1}{|y_i - R_{x,n}|^{1/2}} + \frac{1}{|y_i|^{1/2}} \right).$$

Now, the validity of (2.34) for $V$ and $V'_n$ follows from the fact that $R^{3N} \ni x \mapsto |x_i|^{-\alpha}$ is in $K_{\text{loc}}(R^{3N})$, for every $\alpha < 2$ [11 p. 216/7].

2.6. FK operators. Let $V \in K_{\pm}(R^\nu)$ and let $X^x = (X_t^x)_{t \in I}$ be a Brownian motion or a Brownian bridge starting at $x \in R^\nu$ as explained in Subsect. 2.3. Then, according to Rem. 2.11, it $\mathbb{P}$-a.s. makes sense to define

$$(2.45) \quad \omega_t^V[X^x] := e^{-\int_0^t V(x_s^x)ds} \omega_t^{0}[X^x], \quad t \in I,$$

where $\omega_t^{0}[X^x]$ is given by Thm. 2.6.

**Definition 2.16 (Feynman-Kac operators).** Assume that $V \in K_{\pm}(R^\nu)$. If $\Psi \in L^p(R^\nu, \mathcal{H}; e^{-a|x|^2}dx)$, for some $p \in [1, \infty]$ and $a \geq 0$, then we define

$$(T_t^V \Psi)(x) := E[\omega_t^V[B^x]\Psi(B_t^x)], \quad x \in R^\nu, \quad t \geq 0.$$

Furthermore, we set

$$T_t^V(x, y) := p_t(x, y) E[\omega_t^V[b^{|x-y}|]], \quad x, y \in R^\nu, \quad t > 0.$$

Invoking some of the results collected in Subsect. 2.3, we may in fact ensure the existence of the expectations appearing in Def. 2.16 (see Sect. 3) For instance, the vector $(T_t^V \Psi)(x)$ does not depend on the representative of $\Psi \in L^p(R^\nu, \mathcal{H}; e^{-a|x|^2}dx)$ because $\mathbb{P}\{B_t^x \in N\} = 0$, for every Borel set $N \subset R^\nu$ of Lebesgue measure zero. The operator-valued expectation defining $T_t^V(x, y)$ exists due to Thm. 2.6(1). The set of equivalence classes of functions $\Psi(\cdot) : R^\nu \to \mathcal{H}$, for which we find $a \geq 0$ and $p \in [1, \infty]$ such that $\Psi \in L^p(R^\nu, \mathcal{H}; e^{-a|x|^2}dx)$, is a vector space and we consider $T_t^V$ as a linear map on that vector space with values in the measurable functions from $R^\nu$ to $\mathcal{H}$. Many of our results deal with suitable restrictions of $T_t^V$, which are often again denoted by the same symbol; it should always be clear from the context what is meant.

We close this survey section by stating the FK formula and clarifying the definition of the full Hamiltonian including a Kato decomposable potential $V$ in the next theorem. It follows immediately from the results of [12 §11], because any $V_- \in K(R^\nu)$ is infinitesimally $-\Delta$-form bounded and $V_+ \in K_{\text{loc}}(R^\nu)$ entails
Hence, the assertion follows from (2.30) and the remarks below it.

Eventually verify the Chapman-Kolmogoroff equations for the kernels from which we deduce the semi-group property of the operators.

The kernels obtained in [22], where the spin degrees of freedom are accounted for by a Poisson procedure which is avoided in (2.46) by putting the spin degrees of freedom in the jump process. The FK formula found in [22] involves an additional regularization result of [12]. For one spin-1/2 electron, a different version of (2.46) has been first in [17] by intricate applications of Trotter's theorem, there exists a unique semi-bounded self-adjoint operator \( H^V \) with \( Q(H^V) = Q(H^0) \cap Q(V_+) \) representing the closure of the semi-bounded closable quadratic form

\[
\mathcal{D}(H^0) \cap Q(V_+) \ni \Psi \mapsto \langle \Psi | H^0 \Psi \rangle + \int_{\mathbb{R}^n} V_+(x) \| \Psi(x) \|^2 \, dx.
\]

Of course, if \( A \) is a self-adjoint operator in some Hilbert space which is semi-bounded below, then \((e^{-tA})_{t \geq 0}\) denotes its semi-group defined by means of the spectral calculus; we also recall the notation (2.13).

**Theorem 2.17** ([12]). Let \( V \in \mathcal{K}_+(\mathbb{R}^n) \). Then \( \Psi \in Q(H^V) \) implies \( \| \Psi(\cdot) \| \in Q(-\Delta) \cap Q(V_+) \) with

\[
\langle \| \Psi(\cdot) \| \|(-\frac{1}{2}\Delta + V_+) \| \Psi(\cdot) \| \rangle \leq \langle \Psi \| (H^V + \| \mathcal{U} \|_\infty) \Psi \rangle.
\]

In particular, \( V_- \) is infinitesimally \( H^V \)-form bounded. Hence, by the KLMN theorem, there exists a unique semi-bounded self-adjoint operator \( H^V \) with \( Q(H^V) = Q(H^0) \cap Q(V_+) \) representing the closure of the semi-bounded closable quadratic form

\[
\mathcal{D}(H^0) \cap Q(V_+) \ni \Psi \mapsto \langle \Psi | H^0 \Psi \rangle + \int_{\mathbb{R}^n} V(x) \| \Psi(x) \|^2 \, dx.
\]

The semi-group of \( H^V \) is represented by the following Feynman-Kac formula,

\[
(e^{-tH^V} \Psi)(x) = (T^V_t \Psi)(x),
\]

for all \( t > 0 \), \( \Psi \in \mathcal{H} = L^2(\mathbb{R}^n, \mathcal{H}) \), and a.e. \( x \in \mathbb{R}^n \).

**Remark 2.18.** In the scalar case \( (F = 0) \) (2.46) has been proved (under slightly more restrictive conditions on \( G \)) first in [17] by intricate applications of Trotter’s product formula. That the FK integrand found in [17] actually solves a SDE is a result of [12]. For one spin-1/2 electron, a different version of (2.46) has been obtained in [22], where the spin degrees of freedom are accounted for by a Poisson jump process. The FK formula found in [22] involves an additional regularization procedure which is avoided in (2.46) by putting the spin degrees of freedom in the target space.

3. **Chapman-Kolmogoroff Equation and Semi-group Properties**

In this section we establish the usual basic identities relating the operators \( T^V_t \) and the kernels \( \hat{T}^V_t(x,y) \), for Kato decomposable potentials. In particular, we shall eventually verify the Chapman-Kolmogoroff equations for the kernels from which we deduce the semi-group property of the operators.

Recall the definition of \( \mathcal{U} \) in (2.13) as well as (2.20) and (2.21).

**Lemma 3.1.** Let \( V \in \mathcal{K}_+(\mathbb{R}^n) \), \( t > 0 \), \( 1 \leq p \leq q \leq \infty \), and \( F : \mathbb{R}^n \to \mathbb{R} \) be globally Lipschitz continuous. Then the formula for \( T^V_t \) in Def. 2.10 yields a well-defined element of \( \mathcal{B}(L^p(\mathbb{R}^n, \mathcal{H}; e^{\rho F} \, dx), L^q(\mathbb{R}^n, \mathcal{H}; e^{\rho F} \, dx)) \) satisfying

\[
\| e^{F} T^V_t e^{-F} \|_{p,q} \leq e^{\|\mathcal{U}\|_{\infty} t} \| e^F S^V_t e^{-F} \|_{p,q},
\]

Proof. \( \Psi \in L^p(\mathbb{R}^n, \mathcal{H}; e^{\rho F} \, dx) \) says that \( \| e^{F} \Psi(\cdot) \| \in L^p(\mathbb{R}^n; \, dx) \) and Thm. 2.6(1) implies

\[
\| \mathbb{E} [w^t \Psi(B^*_t)] \| \leq e^{\|\mathcal{U}\|_{\infty} t} \mathbb{E} [e^{-\int_0^t V(B^*_s) \, ds} \| \Psi(B^*_t) \|], \quad x \in \mathbb{R}^n.
\]

Hence, the assertion follows from (2.30) and the remarks below it. \( \Box \)
The previous lemma will be complemented later on in Thm. 5.2 by considering additional unbounded weight operators acting in $\mathcal{H}$.

Before we turn to the discussion of the integral kernel we first verify that a familiar transformation rule for the Brownian bridge applies to the processes given by Thm. 2.9 as well. To prepare ourselves for the approximation arguments in the proof of the next lemma it might make sense to recall that, for all $t > 0$, $x, y \in \mathbb{R}^\nu$, and every Borel set $N \subset \mathbb{R}^\nu$ of Lebesgue measure zero,

$$\int_0^t \mathbb{P}\{B^x_s \in N\} \, ds = \int_0^t \mathbb{P}\{b^y_s \mid x \in N\} \, ds.$$  \hfill (3.2)

**Lemma 3.2.** Let $t > s > 0$, $x, y \in \mathbb{R}^\nu$, $V \in K_+(\mathbb{R}^\nu)$, and let $A : \mathbb{R}^\nu \to \mathcal{B}(\mathbb{H})$ be measurable and bounded with a separable image. Then

$$\int_{\mathbb{R}^\nu} p_t(x, z) A(z) \mathbb{E}[\nu_{t}^{V}[(b^{t,x}, \cdot)^*]](z) \, dz = \mathbb{E}[A(B^x_t) \nu_{t}^{V}[B^x_t]],$$ \hfill (3.3)

$$p_t(x, y) \mathbb{E}[A(b^t_s, y) \nu_s^{V}[b^{t,x}, y]] = \mathbb{E}[p_{t-s}(B^x_s, y) A|(B^x_s) \nu_s^{V}[B^x_s]].$$ \hfill (3.4)

**Proof. Step 1.** First, we prove both (3.3) and (3.4) under the additional assumption that $V : \mathbb{R}^\nu \to \mathbb{R}$ be bounded and continuous.

As to Eqn. (3.3): First, we observe that the left hand side of (3.3) is well-defined thanks to Prop. 2.10 and since the $\mathcal{B}(\mathbb{H})$-valued Bochner-Lebesgue integral commutes with the adjoint.

We have seen in Ex. 2.4(2) that the reversed process $(B^x_t)_{t \in [0, t]}$ is a Brownian bridge from $B^x_t$ to $x$ in time $t$. Hence $\nu_{t}^{V}[B^x_t] = \nu_{t}^{V}[b^{t,x}, x]$, $\mathbb{P}$-a.s., by Thm. 2.9 where we use the notation introduced in Ex. 2.4(2). By the Markov property stated in Thm. 2.6(4) we further have, for every $\mathcal{F} (B^x_t)_{t \in [0, t]}$-measurable bounded $\eta : \Omega \to \mathcal{H}$,

$$\mathbb{E}[\mathcal{F}\nu_t[B^x_t] \eta] = \mathbb{E}[\mathcal{F} \eta], \quad \mathbb{P}\text{-a.s.,}$$

with $\mathcal{F}(z) := \mathbb{E}[\nu_{t}^{V}[b^{t,x}, \cdot]]$, $z \in \mathbb{R}^\nu$. (Here we apply the Markov property with $\mathbb{E}[\mathcal{F}[b^t_s]]$ as underlying stochastic basis, $X^t = b^{t,x}$, and with $f(\chi) := \chi(||\phi||)\phi$, $z \in \mathbb{R}^\nu$, $\phi \in \mathcal{H}$, where $\chi \in C_0(\mathbb{R})$ equals 1 on $[0, R]$ where $R > 0$ is chosen much larger than $e^{(\lambda_0 + ||V||_{\infty})t}||\eta||_{\infty}$.) As we may choose $\eta := A(B^x_t) \psi$ with an arbitrary $\psi \in \mathcal{H}$, this implies

$$\mathbb{E}[\nu_{t}^{V}[B^x_t] * A(B^x_t) \psi] = \mathbb{E}[\mathcal{F} \psi] = \mathbb{E}[\mathcal{F}[A(B^x_t) \psi]],$$

This identity is the adjoint of (3.3) since $B^x_t$ is $p_t(x, \cdot)$-distributed.

As to Eqn. (3.4): According to Ex. 2.4(3) and Thm. 2.9

$$\nu_{s}^{V}[b^{t,x}, y]^* = \nu_{s}^{V}[b^{s,y}}[b^{s,x}, x]], \quad \mathbb{P}\text{-a.s.,}$$

where we employ the notation introduced in Ex. 2.4(3). Furthermore, the Markov property of Thm. 2.6(4) yields, for all $\mathcal{F}[b^{t,x}, y]_{s \in [0, t]}$-measurable bounded $\eta : \Omega \to \mathcal{H}$,

$$\mathbb{E}[\mathcal{F}[b^{t,x}, y]_{s \in [0, t]} \eta] = \mathbb{E}[\mathcal{F}[b^{s,y}, x]_{s \in [0, t]} \eta], \quad \mathbb{P}\text{-a.s.,}$$

with $\mathcal{F}(z) := \mathbb{E}[\nu_{s}^{V}[b^{t,x}, \cdot]]$, $z \in \mathbb{R}^\nu$. Since we may choose $\eta := A(b^{s,y}, x) \psi$ with $\psi \in \mathcal{H}$ and the distribution of $b^{s,y}$ is given by $\mathbb{P} \ni z \mapsto p_t(x, z)p_{t-s}(z, y)/p_t(x, y),$
it follows altogether that
\[
\begin{align*}
p_t(x,y)E\left[|W_s^V[\tilde{b}^{u,z}:y]A(b^z_s,y)^*\right] &=\int_{E^v} p_s(x,z)E\left[|W_s^V[\tilde{b}^{u,z}:z]|\right] p_{t-s}(z,y)A(z)^*dz \\
&=E\left[|W_s^V[B^z]|p_{t-s}(B^z_s,y)A(B^z_s)^*\right].
\end{align*}
\]
Here we applied (3.3) in the second step, with $t$ replaced by $s$ and $A(z)$ replaced by $p_{t-s}(z,y)A(z)$, and taking into account that $|W_s^V[\tilde{b}^{u,z}:z]|$ and $|W_s^V[\tilde{b}^{u,z}:z]|$ have the same distribution by Ex. 2.3(2).

**Step 2.** Next, we extend (3.4) to all Kato decomposable potentials. If $V$ is measurable and bounded, then, by convolution with standard mollifiers, we find a uniformly bounded sequence of potentials $V_n \in C(\mathbb{R}^v)$ such that $V_n \to V$ a.e. and, hence, $\int_0^\infty V_n(b^{u,y}_t)dr \to \int_0^\infty V(b^{u,y}_t)dr$ and $\int_0^\infty V_n(b^z_s)dr \to \int_0^\infty V(b^z_s)dr$, $\mathbb{F}$-a.s., by (3.2). In view of (2.21), we may thus extend (3.4) to bounded $V$ by dominated convergence. If $V \in \mathcal{C}_+(\mathbb{R}^v)$ is arbitrary, then we plug $V_m := 1_{|V| \leq m}V$, $m \in \mathbb{N}$, into (3.4) and employ the dominated convergence theorem together with (2.21), (2.27), and (2.28) to pass to the limit $m \to \infty$.

Since the second identity in (3.2) holds true with $b^{u,y,x}$ replaced by $b^{u,z,x}$ as well, it should now be clear how to extend (3.3) to all $V \in \mathcal{C}_+(\mathbb{R}^v)$.

**Proposition 3.3.** Let $V \in \mathcal{K}_+(\mathbb{R}^v)$ and $t > 0$. Then, for all $x, y \in \mathbb{R}^v$, $T_t^V(x,y)$ is a well-defined element of $\mathcal{A}(\mathcal{H})$ with
\[
(3.5) \quad T_t^V(x,y)^* = T_t^V(y,x) \quad \text{and} \quad \|T_t^V(x,y)\| \leq e^{\|V\|t}S^V_t(x,y).
\]
Furthermore, the map $\mathbb{R}^{2v} \ni (x,y) \mapsto T_t^V(x,y) \in \mathcal{A}(\mathcal{H})$ is measurable with a separable image and
\[
(3.6) \quad (T_t^V\Psi)(x) = \int_{\mathbb{R}^v} T_t^V(x,y)\Psi(y)dy, \quad x \in \mathbb{R}^v,
\]
for all $\Psi \in L^p(\mathbb{R}^v,\mathcal{H},e^{a|x|}dx)$ with $p \in [1,\infty]$ and $a \geq 0$.

**Proof.** The existence of the integral defining the kernel $T_t^V(x,y)$ and the inequality in (3.5) follow immediately from (2.28) and Thm. 2.6(1)&(2). The assertions on its measurability and image are consequences of Prop. 2.10.

For bounded and continuous $V$, the identity in (3.5) has been shown in [12, Lem. 10.6]. (It follows from Thms. 2.7 and 2.9 and the fact that $b^{u,y,x}_{t-\tau}$ is a Brownian bridge from $x$ to $y$ in time $t$ with respect to some new stochastic basis and driving Brownian motion. The latter fact can be proved by adapting arguments of [30]; see [12, Lem. 10.3 & App. D].) It is, however, straightforward to extend the identity in (3.5) to general Kato decomposable potentials by the approximation scheme applied in Step 2 of the proof of Lem. 3.2.

Likewise, (3.6) follows from Ex. 10.4 in [12], provided that $V$ is bounded and continuous and $\Psi \in L^\infty(\mathbb{R}^v,\mathcal{H})$. If $\Psi \in L^p(\mathbb{R}^v,\mathcal{H},e^{-a|x|}dx)$ with $p \in [1,\infty]$, then we plug the functions $\Psi_n \in L^p(\mathbb{R}^v,\mathcal{H}) \cap L^\infty(\mathbb{R}^v,\mathcal{H})$ given by $\Psi_n(z) := 1_{\|\Psi(z)\| \leq n}\Psi(z)$ into (3.6) and pass to the limit $n \to \infty$ by means of the dominated convergence theorem, on the right hand side using $\|T_t^V(x,\cdot)\| \in L^p(\mathbb{R}^v,e^{a|x|}dx)$ and (3.2), on the left hand side employing $e^{-\int_0^t V(b^z_s)ds}\|\Psi(b^z_t)|\|$ as a $\mathbb{F}$-integrable majorant. After that we carry through the approximation scheme of Step 2 of the proof of Lem. 3.2 to generalize the result to Kato decomposable potentials. □
Before we state the next proposition we notice that, in view of (2.28) and (3.5), for \(V \in \mathcal{K}_+(\mathbb{R}^v)\) and all fixed \(t > 0\), \(y \in \mathbb{R}^v\), and \(\psi \in \mathcal{H}\), the function \(\mathbb{R}^v \ni x \mapsto T^V_t(x, y)\psi\) belongs to every \(L^p(\mathbb{R}^v, \mathcal{H})\) with \(p \in [1, \infty)\).

**Proposition 3.4.** Let \(V \in \mathcal{K}_+(\mathbb{R}^v)\), \(t > s > 0\), and \(x, y \in \mathbb{R}^v\). Then
\[
T^V_t(x, y) = \left( T^V_s(T^V_{t-s}(-, y)\psi) \right)(x), \quad \psi \in \mathcal{H},
\]
and the following Chapman-Kolmogoroff equation is valid,
\[
T^V_t(x, y) = \int_{\mathbb{R}^v} T^V_s(x, z)T^V_{t-s}(z, y)\,dz.
\]

**Proof.** Let \(V\) be bounded and continuous to start with. Let \((\mathcal{P}^y_{t,s})_{0 \leq t \leq s}\) denote the transition operators associated with the choices \(X^q = b^{t-s}y\) in Thm. 2.6 and define \(f : \mathbb{R}^v \times \mathcal{H} \to \mathcal{H}\) by \(f(z, \psi) := \chi(\|\psi\|)\psi\), where \(\chi \in C_0(\mathbb{R})\) is again equal to 1 on \([0, R]\) with \(R > 0\) much larger than \(e^{\|X\|_\infty + \|V\|_\infty}t\). Then we observe that, for all \(z \in \mathbb{R}^v\) and \(\psi \in \mathcal{H}\),
\[
(\mathcal{P}^y_{t,s}f)(z, \psi) = \mathbb{E}[W_{t-s}^V[b^{t-s}z; y]\psi] = A(z)\psi,
\]
with \(A(z) := \mathbb{E}[W_{t-s}^V[b^{t-s}z; y]]\), because \(W_{t-s}^V[b^{t-s}z; y]\psi\) and \(W_{t-s}^V[b^{t-s}z; y]\psi\) have the same distribution by Ex. 2.8(2). In this notation,
\[
(3.10) \quad T^V_{t-s}(y, z) = p_{t-s}(y, z)A(z).
\]
Therefore, we obtain, for all \(\psi \in \mathcal{H}\),
\[
T^V_t(x, y)\psi = p_t(x, y)(\mathcal{P}^y_{0,t}f)(x, \psi) = p_t(x, y)(\mathcal{P}^y_{0,s}p_{s,t}f)(x, \psi)
\]
\[
= p_t(x, y)\mathbb{E}[(\mathcal{P}^y_{0,s}f)(b^{t-s}z; y)W_{s}^V[b^{s}x; y]\psi]
\]
\[
= p_t(x, y)\mathbb{E}[A(b^{s}x; y)W_{s}^V[b^{s}x; y]\psi]
\]
\[
= \mathbb{E}[p_{t-s}(y, B^s_x)A(B^s_x)W_{s}^V[B^s_x]\psi]
\]
\[
= [T^V_{t-s}(y, B^s_x)W_{s}^V[B^s_x]\psi],
\]
where we applied (3.9) in the fourth step, Lem. 3.2 in the penultimate step, and (3.10) in the last one. In view of (3.2) this computation implies, for every \(\phi \in \mathcal{H}\),
\[
T^V_t(x, y)\phi = \mathbb{E}[W_{t-s}^V[B^s_x; y]T^V_{t-s}(B^s_x; y)\phi] = T^V_s(T^V_{t-s}(-, y)\phi)(x),
\]
which is (3.7) with a bounded and continuous \(V\). Applying (3.6) we also obtain (3.8) for bounded and continuous \(V\). In the proof of (3.5) we have, however, explained how to approximate \(T^V_t(x, y), t > 0\), with a bounded and measurable (resp. Kato decomposable) \(V\) in operator norm by kernels with bounded and continuous (resp. bounded and measurable) potentials. Since all kernels can be majorized by means of (2.28) and the bound in (3.5), we may thus employ the dominated convergence theorem to extend (3.8) to Kato decomposable \(V\). Together with (3.10) this will also prove (3.7) for Kato decomposable \(V\). \(\square\)

In what follows, we shall sometimes use the symbol \(T^{V:p,q,F}\) to denote the restriction of \(T^V_t\) to \(L^p(\mathbb{R}^v, \mathcal{H}; e^{pF}\,dx)\) considered (by means of Lem. 3.1) as an element of \(\mathcal{B}(L^p(\mathbb{R}^v, \mathcal{H}; e^{pF}\,dx), L^q(\mathbb{R}^v, \mathcal{H}; e^{qF}\,dx))\). Thanks to (2.2) we already know that \((T^{V:(2,2)}_t)_{t \geq 0}\) is a strongly continuous semi-group of bounded self-adjoint operators in the Hilbert space \(\mathcal{H} = L^2(\mathbb{R}^v, \mathcal{H})\). In fact, the following more general statement holds true:
Corollary 3.5. Let $V \in K_{\pm}(\mathbb{R}^r)$ and $F : \mathbb{R}^r \to \mathbb{R}$ be globally Lipschitz continuous. Then $(V_t^p)^{t \geq 0}$ defines self-adjoint semi-groups between the spaces $L^p(\mathbb{R}^r, \mathcal{H}; e^{tF} \, dx)$ in the sense that

$$T_{s+t}^p \equiv T_s^p \circ T_t^p,$$

for all $s, t > 0$ and $1 \leq p \leq q \leq r \leq \infty$, where $p'$ is the exponent conjugate to $p$ and $q'$ the one conjugate to $q$.

Proof: The second relation in (3.11) follows immediately from Prop. 3.3. The first asserted relation, expressing the semi-group property, is an easy consequence of (3.0), the Chapman-Kolmogoroff equation, and Fubini’s theorem. The latter is applicable by virtue of (3.5) which shows that the right hand side of

$$e^{-a|x|} \int_{\mathbb{R}^r} \left\| T_s^V(x, z) T_t^V(z, y) \Psi(y) \right\| d(y, z)$$

is finite, for all $\Psi \in L^p(\mathbb{R}^r, \mathcal{H}; e^{-a|x|} \, dx)$ with $p \in [1, \infty]$. $\square$

4. WEIGHTED ESTIMATES ON THE STOCHASTIC FLOW

In this section we study how the operator-valued process $\mathcal{W}^V[B^q]$ introduced in Thm. 2.6 behaves when it is multiplied or conjugated with certain weight functions, i.e., with certain unbounded multiplication operators acting in the Fock space which are defined and discussed in Subsect. 4.4. The weighted Burkholder-Davis-Gundy (BDG) type inequalities derived Subsect. 4.2 are extensions of similar ones in [12, §7] in the sense that we consider more general weights and an inhomogeneous SDE generalizing (2.22). In certain respects the situation here is, however, also simpler than in [12], where we had to study the convergence of a time-ordered integral series defining $\mathcal{W}^V[B^q]$ in addition. Moreover, we refrain from considering unbounded drift vectors $\beta$ in this section to shorten some arguments. A suitable version of the stochastic Gronwall lemma [32] obtained in Subsect. 4.2 turns out to be convenient in the derivation of our BDG type bounds.

In Subsect. 4.5 we shall further address continuity properties of $\mathcal{W}^V[B^q]$ under changes of the initial condition $q$ and the coefficient vector $c$, employing an elementary algebraic lemma that we prove first in Subsect. 4.2.

The introduction of the inhomogeneity $R$ in the SDE (4.14) will actually become relevant only in the discussion of differentiability properties of the stochastic flow in our companion paper [26], where the results of this section will serve as a crucial ingredient. The introduction of the inhomogeneity $\rho$ in (4.14) will already turn out to be useful in Ex. 4.17.

4.1. Definition and discussion of the weight functions. First, we introduce the weight functions we are interested in: Let $t_0 > 0$ and set

$$\forall t \in [0, t_0] : \quad \tau_{\alpha}(t) := \begin{cases} t/2\alpha, & \alpha > 0, \\ (t - t_0)/2\alpha, & \alpha < 0. \end{cases}$$

Let $\omega, \varkappa : \mathcal{M} \to [0, \infty)$ be measurable functions such that $\omega \leq \omega$ and define

$$v_{\alpha, t} := \tau_{\alpha}(t) \omega + \varkappa, \quad v_{\alpha, \varepsilon, t} := v_{\alpha, t}(1 + \varepsilon v_{\alpha, t})^{-1},$$

$$\Theta_{\alpha, t} := (1 + t\tau_{\alpha}(t) + d\Gamma(v_{\alpha, \varepsilon, t}))^{\alpha} (1 + \varepsilon (1 + t\tau_{\alpha}(t) + d\Gamma(v_{\alpha, \varepsilon, t})))^{-\alpha},$$
for all \( t \in [0, t_0] \), \( \varepsilon \in [0, 1] \), and \( \alpha \in \mathbb{R} \setminus \{0\} \), where \( \omega = 0 \), if \( \varepsilon = 0 \), and \( \omega = 1 \), if \( \varepsilon = 0 \). If one is only interested in the question whether domains of higher powers of the radiation field energy or the number operator stay invariant under the action of our semi-groups, then it suffices to choose \( \omega = 0 \) and \( \omega = \omega = 1 \), respectively. In order to show that \( \mathcal{W}_{t_0}^{\alpha}[B^q] \) improves the localization in the photon momentum spaces, we choose \( \omega = \omega = 1 \) for some \( r > 0 \); see, e.g., Ex. (8.31). Moreover, we choose \( \alpha > 0 \) to control weights to the left of \( \mathcal{W}_{t_0}^{\alpha}[B^q] \), and \( \alpha < 0 \) to control weights to the right of \( \mathcal{W}_{t_0}^{\alpha}[B^q] \); cf. Rem. 4.11 below. In technical proofs the regularization parameter \( \varepsilon \) will be chosen strictly positive, thus rendering the operators bounded, and eventually be send to zero again. We shall also be interested in the exponential weights given by

\[
\forall \varepsilon \in [0, 1], \; t \in [0, t_0] : \quad \Xi_{\varepsilon, t}^{(\delta)} := \begin{cases} e^{\delta \Theta_{\varepsilon, t}^{(\varepsilon)}}, & \delta \in (0, 1], \\ e^{\delta \Theta_{\varepsilon, 0-t}^{(\varepsilon)}}, & \delta \in [-1, 0]. \end{cases}
\]

**Remark 4.1.** For \( t \in [0, t_0] \), \( \varepsilon \in (0, 1] \), \( |\alpha| > 0 \), and \( 0 < |\delta| \leq 1 \), the following holds:

(1) \( \Theta_{\varepsilon, t_0-t}^{(\alpha)} = \Theta_{\varepsilon, t}^{(-\alpha)} \), \( \Xi_{\varepsilon, t_0-t}^{(\delta)} = \Xi_{\varepsilon, t}^{(-\delta)} \).

(2) If we consider the \( d\Gamma(\cdot) \)'s as multiplication operators, then the following pointwise bounds hold,

(4.1) \[ \Theta_{\varepsilon, t}^{(\alpha)} \frac{d}{dt} \Theta_{\varepsilon, t}^{(\alpha)} \leq \alpha \tau_{\omega}^\varepsilon(t)(1 + d\Gamma(\omega)) \leq \frac{1}{2} \left( d\Gamma(\omega) + \frac{1}{2} \right), \]

(4.2) \[ \Xi_{\varepsilon, t}^{(\delta)} \frac{d}{dt} \Xi_{\varepsilon, t}^{(\delta)} \leq \frac{1}{2} \left( d\Gamma(\omega) + \frac{1}{2} \right). \]

**Notation 4.2.** In what follows, \( \Theta_{s} \) denotes any of the weights \( \Theta_{\varepsilon, s}^{(\alpha)} \) or \( \Xi_{\varepsilon, s}^{(\delta)} \) with non-zero \( \varepsilon \in (0, 1] \) as described in the previous paragraphs.

Using that the operator norm of \( \Theta_{s} \) is bounded uniformly in \( s \) by some \( \varepsilon \)-dependent constant and employing the dominated convergence theorem together with (4.1) and (4.2), it is straightforward to verify that, for every \( \psi \in \mathcal{D}(d\Gamma(\omega)) \), the map \( s \mapsto \Theta_{s}\psi \) belongs to \( C^1([0, t_0], \mathcal{H}) \) and its derivative is given by \( \tilde{\Theta}_{s}\psi \), where \( \tilde{\Theta}_{s} \) denotes the maximal operator of multiplication with the function obtained by taking pointwise derivatives of the weight functions at \( s \in [0, t_0] \).

Since the \( \mathcal{B}(\mathcal{H}) \)-valued maps \( s \mapsto \Theta_{s} \) and \( s \mapsto \tilde{\Theta}_{s} \) are only strongly continuous, it might make sense to verify the following:

**Lemma 4.3.** Let \( Y : [0, t_0] \times \Omega \to \mathcal{H} \) be a semi-martingale whose paths belong to \( C([0, t_0], \mathcal{D}(d\Gamma(\omega))) \) and which can \( \mathcal{F} \)-a.s. be written as

(4.3) \[ Y_t = \eta + \int_0^t A_{0,s} ds + \int_0^t A_s dB_s, \quad t \in [0, t_0], \]
for some $\mathcal{F}_0$-measurable $\eta : \Omega \to \mathcal{D}(d\Gamma(\omega))$ and an adapted $\mathcal{B}(\mathbb{R}^{1+\nu}, \mathcal{H})$-valued process $(A_n, A)$ with continuous paths. Then $(\Theta_t Y_t)_{t \in [0, t_0]}$ is a $\mathcal{H}$-valued semi-martingale and it $P$-a.s. satisfies

$$
\Theta_t Y_t = \Theta_0 \eta + \int_0^t \Theta_s A_s ds + \int_0^t \Theta_s A_{0,s} ds + \int_0^t \Theta_s A_{s} d\mathcal{B}_s, \quad t \in [0, t_0],
$$

$$
\|\Theta_t Y_t\|^2 = \|\Theta_0 \eta\|^2 + \int_0^t 2 \text{Re}(\Theta_s Y_s | \Theta_s A_{0,s} + \Theta_s Y_s) ds
$$

$$
+ \int_0^t \|\Theta_s A_{s}\|^2 ds, \quad t \in [0, t_0].
$$

Proof. Let $0 = \tau_0 < \tau_1 < \ldots < \tau_n = t_0$, $n \in \mathbb{N}$, be partitions of $[0, t_0]$ with $\max |\tau_{n+1} - \tau_n| \to 0$, as $n \to \infty$. On $\Omega$ and for all $t \in [0, t_0]$ and $n \in \mathbb{N}$, we then have

$$
\Theta_t Y_t - \Theta_0 Y_0 = \sum_{\ell=0}^{n-1} (\Theta_{\tau_{\ell+1} \wedge t} Y_{\tau_{\ell+1} \wedge t} - \Theta_{\tau_\ell \wedge t} Y_{\tau_\ell \wedge t})
$$

$$
= \int_0^t \sum_{\ell=0}^{n-1} 1(\tau_{\ell}, \tau_{\ell+1}) (s) \Theta_s Y_{\tau_{\ell+1} \wedge t} ds + \sum_{\ell=0}^{n-1} \Theta_{\tau_\ell} (Y_{\tau_{\ell+1} \wedge t} - Y_{\tau_\ell \wedge t}).
$$

Employing (4.1) and (4.2) and using max$_{s \in [0, t_0]} \| (1 + d\Gamma(\omega)) \Theta_s Y_s \|$ pathwise as dominating (constant) function, we see that the Bochner-Lebesgue integrals in the previous identity converge to $\int_0^t \Theta_s Y_s ds$, pathwise and for every $t \in [0, t_0]$, as $n$ goes to infinity. Furthermore, we $P$-a.s. have

$$
\sum_{\ell=0}^{n-1} \Theta_{\tau_\ell} (Y_{\tau_{\ell+1} \wedge t} - Y_{\tau_\ell \wedge t}) = \int_0^t \sum_{\ell=0}^{n-1} 1(\tau_{\ell}, \tau_{\ell+1}) (s) \Theta_s dY_s
$$

$$
= \int_0^t \sum_{\ell=0}^{n-1} 1(\tau_{\ell}, \tau_{\ell+1}) (s) \Theta_s A_{0,s} ds + \int_0^t \sum_{\ell=0}^{n-1} 1(\tau_{\ell}, \tau_{\ell+1}) (s) \Theta_s A_{s} d\mathcal{B}_s,
$$

for all $t \in [0, t_0]$ and $n \in \mathbb{N}$. Taking the remarks preceding the statement into account, we see that $\Theta(A_n, A)$ is again an adapted, continuous $\mathcal{B}(\mathbb{R}^{1+\nu}, \mathcal{H})$-valued process, whose paths are bounded on $[0, t_0]$. Using $R_t := \max_{0 \leq s \leq t} \| \Theta_s (A_{0,s}, A_s) \|$, $t \in [0, t_0]$, as a predictable dominating process, we may invoke the dominated convergence theorem for stochastic integrals [23 Thm. 24.2] to conclude that, along a subsequence, the expression in the last line of (4.6) converges $P$-a.s. uniformly on $[0, t_0]$ to the sum of the last two integrals in (4.4).

Recall that a conjugation $C$ was introduced in Hyp. 4.1. To infer (4.5) from (4.4), we observe that $\mathcal{H} = \mathcal{F}_C^L + i \mathcal{F}_C^L$, with the completely real subspace $\mathcal{F}_C := \{ \psi \in \mathcal{F} : \Gamma(-C) \psi = \psi \}$, and that every $\psi \in \mathcal{H}$ has a unique decomposition $\psi = \psi_1 + i \psi_2$ with $\psi_1, \psi_2 \in \mathcal{F}_C^L$ and $\|\psi\|^2 = \|\psi_1\|^2 + \|\psi_2\|^2$. Hence, we can interpret the squared norm on $\mathcal{H}$ as a function on the real Hilbert space $\mathcal{F}_C^L$ and read (4.4) as a $P$-equality between $\mathcal{F}_C^L$-valued processes. Then (4.5) follows from (4.4) and the Itô formula of [6 Thm. 4.32].
Notation 4.4. The operator $\vartheta$ is equal to 1, if $\Theta_s = \Theta_{\varepsilon,s}^{(\alpha)}$, and equal to $1 + d\Gamma(\omega)$, if $\Theta_s = \Xi_{\varepsilon,s}^{(\delta)}$. Using this convention we set

$$T_{1,s} := \frac{1}{2} \vartheta^{-1/2} \Theta_s^{-1} \left[ (\sqrt{\Theta_s^2 + \varphi(G_{B_s^2})}) \right] \Theta_s^{-1} \vartheta^{-1/2},$$

$$T_{2,s} := \frac{1}{2} \Theta_s^{-1} \vartheta^{-1/2} + \frac{1}{2} \Theta_s^{-1} \left[ \Theta_s, \varphi(F_{B_s^2}) \right] \Theta_s^{-1} \vartheta^{-1/2},$$

$$T_{s}^\pm := 2 \left[ \Theta_s^{1/2}, \varphi(G_{B_s^2}) \right] \Theta_s^{-1} \vartheta^{-1/2}.$$ 

The operators in (4.7)–(4.9) are well-defined a priori on $\mathcal{D}(d\Gamma(\omega))$. In fact, under suitable additional conditions on the coefficient vector $c$, they extend to bounded operators on $\mathcal{H}$ whose norms are bounded on $\Omega$ uniformly in $s$ and $\varepsilon$. For we have the following result, whose proof is deferred to the appendix, where a systematic commutator estimates is presented:

Lemma 4.5. (1) If $|\alpha| \geq 1/2$ and the operators in (4.7)–(4.9) are defined by means of $\Theta_s = \Theta_{\varepsilon,s}^{(\alpha)}$ and $\vartheta = 1$, then we find some $c_0 > 0$ such that

$$\|T_{1,s}\|, \|T_{2,s}\| \leq c_0 \| (\vartheta + x) \| (1 + \vartheta + x) |\alpha|^{-1/2} G_{B_s^2}^2, \tag{4.10}$$

$$\|T_{2,s}\| \leq c_0 \| (\vartheta + x) \| (1 + \vartheta + x) |\alpha|^{-1/2} (q, \sigma \cdot F)_{B_s^2}, \tag{4.11}$$

for all $s \in [0,t_0]$ and $\varepsilon \in (0,1]$.

(2) If the operators in (4.7)–(4.9) are defined by means of $\Theta_s = \Xi_{\varepsilon,s}^{(\delta)}$ with $\vartheta = 1 + d\Gamma(\omega)$, then we find some $c > 0$ such that

$$\|T_{1,s}\|, \|T_{2,s}\| \leq c \| (\vartheta + x) \| (1 + \vartheta + x) |\delta|^{-1/2} G_{B_s^2}^2, \tag{4.12}$$

$$\|T_{2,s}\| \leq c |\delta|^{-1/2} \| (\vartheta + x) \| (1 + \vartheta + x) |\delta|^{-1/2} (q, \sigma \cdot F)_{B_s^2}, \text{ on } \Omega, \text{ for all } s \in [0,t_0] \text{ and } 0 < \varepsilon \leq |\delta|. \tag{4.13}$$

Proof. The proof of (1) can be found in Ex. A.7, the one of (2) in Ex. A.8. \hfill \square

Notation 4.6. For any vector $v$ with components in $\mathfrak{h}$, we write

$$\|v\| := \begin{cases} c_0 \| (\vartheta + x) \| (1 + \vartheta + x) |\alpha|^{-1/2} v_{\mathfrak{h}}, & \text{if } \Theta_s = \Theta_{\varepsilon,s}^{(\alpha)}, \\ c |\delta|^{-1/2} \| (\vartheta + x) \| (1 + \vartheta + x) |\delta|^{-1/2} v_{\mathfrak{h}}, & \text{if } \Theta_s = \Xi_{\varepsilon,s}^{(\delta)}, \end{cases}$$

where $c_0$ and $c$ are the constants in (4.10) and (4.11), respectively. If $v$ depends on $x \in \mathbb{R}^\nu$, then we abbreviate $\|v\|_{x,\infty} := \sup_{x \in \mathbb{R}^\nu} \|v_x\|_{\mathfrak{h}}$.

We will use the following simple remark without further notice:

Remark 4.7. Let $\mathbb{R}^\nu \ni x \mapsto v_x \in \mathfrak{h}$ be continuous and $\kappa : \mathcal{M} \to [0,\infty)$ be measurable. Then the numerical function $\mathbb{R}^\nu \ni x \mapsto \int_{\mathcal{M}} \kappa |v_x|^2 d\mu$ is measurable as well. In fact, $\int_{\mathcal{M}} \kappa |v_x|^2 d\mu = \sup_{n \in \mathbb{N}} \int_{\mathcal{M}} (\kappa \wedge n) |v_x|^2 d\mu$, for all $x \in \mathbb{R}^\nu$, by the monotone convergence theorem.

4.2. A remark on the stochastic Gronwall lemma. In our derivation of the BDK type estimates we shall apply a variant of the stochastic Gronwall lemma [32] that we state first. The proof of the following lemma is a simple modification of a proof appearing in [32].
Lemma 4.8. Let $t_0 > 0$. Consider real processes $M$, $Z$, $R$, and $b$ on $[0,t_0]$, where $M$ is a continuous local martingale starting at 0, $Z$ and $R$ are non-negative and adapted with continuous paths, and $b$ is non-negative and progressively measurable. Assume that, $\mathbb{P}$-a.s.,

$$Z_t \leq Z_0 + \int_0^t b_s Z_s ds + M_t + \int_0^t Z_s^\delta R_s ds, \quad t \in [0,t_0],$$

for some $\delta \in (0,1)$. Pick some $\gamma \in (0,1)$ and set $B_t := \int_0^t b_s ds$. Then the a priori bound

$$(4.13) \quad \mathbb{E}\left[ \sup_{s \leq t} e^{-\gamma B_s} Z_s^\gamma \right] < \infty,$$

implies the following inequality, for all $t \in [0,t_0]$,

$$\mathbb{E}\left[ \sup_{s \leq t} e^{-\gamma B_s} Z_s^\gamma \right] \leq c_{\gamma,\delta} \mathbb{E}[Z_0^\gamma] + c_{\gamma,\delta} \mathbb{E}\left[ \left( \int_0^t e^{-(1-\delta)B_s} R_s ds \right)^{\frac{\gamma}{\delta}} \right].$$

Here the constant is given by $c_{\gamma,\delta} = 2^{(\gamma+\delta-1)\gamma} \{ (4 \wedge \frac{1}{\gamma}) \frac{\gamma}{\sin(\pi \gamma)} + 1 \}.$

Proof. Let $H_s := \int_0^s Z_r^\delta R_r ds$ and $L_s := \int_0^s e^{-B_r} dM_r$. Then a pathwise application of Gronwall’s lemma and a partial integration $\mathbb{P}$-a.s. yields

$$e^{-B_t} Z_t \leq e^{-B_t} (Z_0 + M_t + H_t) + \int_0^t (Z_0 + M_s + H_s) b_s e^{-B_s} ds$$

$$= Z_0 + M_t + \int_0^t e^{-B_s} Z_s^\delta R_s ds, \quad t \in [0,t_0].$$

In particular, we $\mathbb{P}$-a.s. have

$$\sup_{s \leq t} e^{-B_s} Z_s \leq Z_0 + \sup_{s \leq t} L_s + \int_0^t e^{-B_s} Z_s^\delta R_s ds,$$

$$- \inf_{s \leq t} L_s = \sup_{s \leq t} (-L_s) \leq Z_0 + \int_0^t e^{-B_s} Z_s^\delta R_s ds,$$

for all $t \in [0,t_0]$. Now the key step consists in applying the inequality

$$\mathbb{E}\left[ (\sup_{s \leq t} L_s)^\gamma \right] \leq (c_{\gamma,0} - 1) \mathbb{E}\left[ (-\inf_{s \leq t} L_s)^\gamma \right], \quad t \in [0,t_0].$$

This is a special case of an inequality for continuous local martingales starting at zero due to Burkholder with an improved constant obtained in [32 Prop. 1]. In combination with $(x+y+z)^\gamma \leq x^\gamma + y^\gamma + z^\gamma$, $x, y, z \geq 0$, the above estimates entail

$$\mathbb{E}\left[ \sup_{s \leq t} e^{-\gamma B_s} Z_s^\gamma \right] \leq c_{\gamma,0} \mathbb{E}\left[ \left( Z_0 + \int_0^t e^{-B_s} Z_s^\delta R_s ds \right)^\gamma \right]$$

$$\leq c_{\gamma,0} \mathbb{E}\left[ \left( \sup_{s \leq t} e^{-\gamma B_s} Z_s^\gamma \right)^\delta \left( Z_0^1 - \delta + \int_0^t e^{-(1-\delta)B_s} R_s ds \right)^\gamma \right]$$

$$\leq c_{\gamma,0} \mathbb{E}\left[ \left( \sup_{s \leq t} e^{-\gamma B_s} Z_s^\gamma \right)^\delta \mathbb{E}\left[ \left( Z_0^1 - \delta + \int_0^t e^{-(1-\delta)B_s} R_s ds \right)^{\frac{\gamma}{\delta}} \right] \right]^{1-\delta}.$$

Now the a priori bound (4.13) permits to get

$$\mathbb{E}\left[ \sup_{s \leq t} e^{-\gamma B_s} Z_s^\gamma \right] \leq c_{\gamma,0} \mathbb{E}\left[ \left( Z_0^1 - \delta + \int_0^t e^{-(1-\delta)B_s} R_s ds \right)^{\frac{\gamma}{\delta}} \right], \quad t \in [0,t_0].$$
Finally, the desired bound follows from $(x + y)^r \leq 2^{(r-1)\nu_0}(x^r + y^r)$, $x, y, r \geq 0$. □

4.3. Weighted BDG type estimates. In the following lemma and its proof we will employ the conventions of Notation 4.12, 4.3 and 4.6. The reader should keep in mind that the weights $\Theta_\rho$ depend on $\varepsilon$ and notice that the constants in (4.17) and (4.18) are $\varepsilon$-independent.

**Lemma 4.9.** Let $V \in \mathcal{K}_+(\mathbb{R}^r)$ and assume that $\|c_x\|_0 < \infty$, for every $x \in \mathbb{R}^r$. Let $(\rho_t, R_t)_{t \in [0,t_0]}$ be an adapted $\mathcal{B}(\mathbb{R}^{1+\nu}, \mathcal{H})$-valued process with continuous paths. Let $p \in \mathbb{N}$ and assume that $(\psi_t)_{t \in [0,t_0]}$ is a $\mathcal{H}$-valued semi-martingale whose paths belong $\mathbb{P}$-a.s. to $C([0,t_0], \mathcal{D}(d\Gamma(\omega)))$ and which $\mathbb{P}$-a.s. satisfies

$$
\psi_t = \eta - \int_0^t \left( \tilde{H}(B^\eta_t) + V(B^\eta_t) \right) \psi_s ds + \int_0^t i\varphi(G_{B^\eta_t}) \psi_s dB_s
$$

(4.14)

where $(q, \eta) : \Omega \to \mathbb{R}^r \times \mathcal{D}(d\Gamma(\omega))$ is $\bar{\mathcal{G}}_0$-measurable with $\|\eta\|_{\mathcal{G}} \in L^p(\mathbb{P})$. Let $\mu > 0$, set

$$
f_{p,\mu}(s) := \begin{cases} 2p\|G_{B^\eta_s}\|^2_0 + \mu \|(q, \sigma \cdot F)_{B^\eta_t}\|_0, & \text{if } \vartheta = 1 + d\Gamma(\omega), \\ 0, & \text{if } \vartheta = 1,
\end{cases}
$$

(4.15)

and assume that $f_{p,\mu}(s) \leq 1/8$, for all $s \in [0,t_0]$. Abbreviate

$$
R_p(s) := \|(1 + d\Gamma(\omega))^{-1/2} \Theta_\rho \rho_s\|^2 + (p + \|G_{B^\eta_s}\|^2_0)\|\Theta_\rho R_s\|^2,
$$

(4.16)

and assume that

$$
\mathbb{E}\left[ \left( \int_0^{t_0} R_p(s) ds \right)^{p/2} \right] < \infty.
$$

Then the following bounds hold for all $t \in [0,t_0]$,

$$
\mathbb{E}\left[ \sup_{s \leq t} \left\{ e^{-p\int_0^s b_{p,\mu}(r) dr} \|\Theta_\rho \psi_s\|^p \right\} \right]
$$

(4.17) \leq (7t^7)^p \mathbb{E}\left[ \|\Theta_\rho \psi_0\|^p \right] + (28e^p t^{7/2})^p \mathbb{E}\left[ \left( \int_0^t e^{-2\int_0^s b_{p,\mu}(r) dr} R_p(s) ds \right)^{p/2} \right],
$$

$$
\mathbb{E}\left[ \left( \int_0^t e^{-2\int_0^s b_{p,\mu}(r) dr} \|d\Gamma(\omega)^{1/2} \Theta_\rho \psi_s\|^2 ds \right)^{p/2} \right]
$$

\leq c_p e^{2pt} \mathbb{E}\left[ \|\Theta_\rho \psi_0\|^p \right] + c_p e^{2pt} \mathbb{E}\left[ \left( \int_0^t e^{-2\int_0^s b_{p,\mu}(r) dr} \|G_{B^\eta_s}\|^2_0 \|\Theta_\rho \psi_s\|^2 ds \right)^{p/2} \right]
$$

(4.18)

$$
+ c_p e^{2pt} \mathbb{E}\left[ \left( \int_0^t e^{-2\int_0^s b_{p,\mu}(r) dr} R_p(s) ds \right)^{p/2} \right].
$$

Here $c_p > 0$ depends only on $p$.

**Proof.** Step 1. If $f$ is one of the components of $c = (G, q, F)$, then $x \mapsto \varphi(f_x)$, $x \mapsto \varphi(G_x)$, and, hence, $x \mapsto \tilde{H}(x)$ are well-defined continuous maps from $\mathbb{R}^r$ into $\mathcal{B}(d\Gamma(\omega), \mathcal{H})$ as a consequence of (2.0), (2.7), Hyp (2.1) and Ex (A.6). This shows that Lem. 4.3 applies to the semi-martingale $\psi$. It is also known that $\varphi(f_x)$ maps $\mathcal{D}(d\Gamma(\omega))$ continuously into $\mathcal{D}(d\Gamma(\omega)^{1/2}) \subseteq \mathcal{D}(\varphi(f_x))$, which altogether shows that all algebraic manipulations in the following steps are justified.
Step 2. Combining Lem. [1.3] with (1.14) we \( \mathbb{P} \)-a.s. find, for all \( t \in [0, t_0] \),
\[
\| \Theta_t \psi_t \|^2 = \| \Theta_0 \psi_0 \|^2 - \int_0^t 2 \text{Re} ( \psi_s \| \Theta_s^2 \frac{1}{2} \varphi (G_{B_s^2})^2 \psi_s ) \, ds \\
- \int_0^t 2 \text{Re} ( \psi_s \| \Theta_s^2 (d \Gamma (\omega) - \sigma \cdot \varphi (F_{B_s^2}) - \frac{\dot{\varphi}}{\varphi} + V (B_s^2)) \psi_s ) \, ds \\
+ \int_0^t 2 \text{Re} ( \psi_s \| \Theta_s^2 i \varphi (G_{B_s^2}) \psi_s ) \, ds + \int_0^t \| \Theta_s \varphi (G_{B_s^2}) \psi_s \|^2 \, ds \\
+ \int_0^t 2 \text{Re} ( \Theta_s \psi_s ) (\Theta_s \rho_s ) \, ds + \int_0^t 2 \text{Re} ( \Theta_s \psi_s ) (\Theta_s R_s ) \, dB_s \\
+ \int_0^t 2 \text{Re} ( \Theta_s R_s ) (\Theta_s \varphi (G_{B_s^2}) \psi_s ) \, ds + \int_0^t \| \Theta_s R_s \|^2 \, ds.
\]

Now we commute \( \Theta_s^2 \) with one of the factors \( \varphi (G_{B_s^2}) \) in the integral in the first line, take the cancellation with the second integral in the third line into account, and observe that
\[
\text{Re} ( \psi_s \| \Theta_s^2 \varphi (G_{B_s^2}) \varphi (G_{B_s^2}) \psi ) = \langle \varphi (G_{B_s^2}) \Theta_s T_1 (s) \varphi (G_{B_s^2}) \psi, \Theta_s \psi \rangle ,
\]
\[
2 \text{Re} ( \psi_s \| \Theta_s \sigma \cdot \varphi (F_{B_s^2}) \psi ) = \langle \psi_s \| [\Theta_s, \Theta_s \sigma \cdot \varphi (F_{B_s^2})] \psi \rangle .
\]

Moreover, we commute one factor \( \Theta_s \) with \( \varphi (G_{B_s^2}) \) in the first \( dB \)-integral and use that \( \text{Re} ( \psi_s \| \Theta_s i \varphi (G_{B_s^2}) \Theta_s \psi_s ) = 0 \). The same observation can be used in the second line with \( \frac{1}{2} \varphi (q_{B_s^2}) \) in place of \( i \varphi (G_{B_s^2}) \). In this way we \( \mathbb{P} \)-a.s. arrive at
\[
(4.19) \quad \| \Theta_t \psi_t \|^2 = \| \Theta_0 \psi_0 \|^2 + \int_0^t \mathcal{J} (s) \, ds + \int_0^t J (s) \, dB_s ,
\]
for all \( t \in [0, t_0] \), with
\[
\mathcal{J} (s) := -2 (\Theta_s \psi_s (d \Gamma (\omega) - \sigma \cdot \varphi (F_{B_s^2}) - \frac{\dot{\varphi}}{\varphi} + V (B_s^2)) \Theta_s \psi_s ) \\
- \langle \varphi (G_{B_s^2}) \Theta_s T_1 (s) \varphi (G_{B_s^2}) \psi, \Theta_s \psi \rangle + 2 \text{Re} ( \Theta_s \psi_s ) |T_2 (s) \varphi (G_{B_s^2}) \psi_s ) + 2 \text{Re} ( \Theta_s \psi_s ) (\Theta_s \rho_s ) + \| \Theta_s R_s \|^2 \\
+ 2 \text{Re} ( \Theta_s R_s ) (\Theta_s \varphi (G_{B_s^2}) \Theta_s \psi_s ) + \text{Re} ( \Theta_s R_s |T_s \varphi (G_{B_s^2}) \Theta_s \psi_s ) ,
\]
\[
(4.20) \quad J (s) := \text{Re} ( \Theta_s \psi_s ) |T_s \varphi (G_{B_s^2}) \Theta_s \psi_s ) + 2 \Theta_s R_s .
\]

Next, we apply the bounds
\[
(4.21) \quad \sigma \cdot \varphi (F_{B_s^2}) + \frac{\dot{\varphi}}{\varphi} \leq (1/4 + 1/2) d \Gamma (\omega) + 4 \tilde{\Omega} (B_s^2) + 1/2 ,
\]
\[
(4.22) \quad \| \varphi (G_{B_s^2}) (1 + d \Gamma (\omega))^{-1/2} \| \leq 2^{1/2} \| G_{B_s^2} \| t .
\]

The first one of them follows from (2.8) (with \( \alpha = \omega / 4 \)) and (4.1) (resp. (4.2), and the second one is implied by (2.6) and (2.7). We thus obtain
\[
\mathcal{J} (s) \leq - \left( \frac{1}{2} - 2 \mu' \right) \| d \Gamma (\omega) \|^{1/2} \| \Theta_s \psi_s \|^2 \\
+ \left( \| T_3 (s) \| + \| T_2 (s) \| + \frac{\| T_3 (s) \|}{\mu} \right) \| \varphi (G_{B_s^2}) \Theta_s \psi_s \|^2 \\
+ \left( 8 \tilde{\Omega} (B_s^2) + 1 - 2 \tilde{\Omega} (B_s^2) + \frac{\| T_3 (s) \|}{\mu} + 2 \mu' \right) \| \Theta_s \psi_s \|^2 \\
+ \frac{1}{\mu} \| (1 + d \Gamma (\omega))^{-1/2} \| \Theta_s \rho_s \|^2 + \left( \frac{5}{4} + \frac{2 \| G_{B_s^2} \|^2}{\mu'} \right) \| \Theta_s R_s \|^2 ,
\]
\[
(4.23) \quad \text{Re} ( \psi_s \| \Theta_s^2 \varphi (G_{B_s^2})^2 \psi_s ) \\
- \int_0^t 2 \text{Re} ( \psi_s \| \Theta_s^2 (d \Gamma (\omega) - \sigma \cdot \varphi (F_{B_s^2}) - \frac{\dot{\varphi}}{\varphi} + V (B_s^2)) \psi_s ) \, ds \\
+ \int_0^t 2 \text{Re} ( \psi_s \| \Theta_s^2 i \varphi (G_{B_s^2}) \psi_s ) \, ds + \int_0^t \| \Theta_s \varphi (G_{B_s^2}) \psi_s \|^2 \, ds \\
+ \int_0^t 2 \text{Re} ( \Theta_s \psi_s ) (\Theta_s \rho_s ) \, ds + \int_0^t 2 \text{Re} ( \Theta_s \psi_s ) (\Theta_s R_s ) \, dB_s \\
+ \int_0^t 2 \text{Re} ( \Theta_s R_s ) (\Theta_s \varphi (G_{B_s^2}) \psi_s ) \, ds + \int_0^t \| \Theta_s R_s \|^2 \, ds.
\]
for all \( s \in [0, t_0] \) and \( \mu, \mu' > 0 \). We further have
\[
\frac{1}{2} J(s)^2 \leq 2 \| \Theta_s \psi_s \|^2 + 2 \| T_s \| \| \Theta_s \psi_s \|^2 + 4 \| \Theta_s \psi_s \|^2 \| \Theta_s \psi_s \|^2.
\]

Step 3. Let \( p \geq 2 \). In view of (4.19), an application of Itô’s formula \( \mathbb{P} \)-a.s. yields
\[
\| \Theta_t \psi_t \|^2 = \| \Theta_0 \psi_0 \|^2 + 2 \int_0^t \| \Theta_s \psi_s \|^2 \| \Theta_s \psi_s \|^2 ds + \int_0^t \| \Theta_s \psi_s \|^2 \mathcal{J}(s) ds
\]
(4.25)
for all \( t \in [0, t_0] \). Using (4.20) with \( \mu' = 1/8 \), (4.21), and Lem. 4.10 together with (4.12), we \( \mathbb{P} \)-a.s. obtain
\[
\| \Theta_t \psi_t \|^2 = \| \Theta_0 \psi_0 \|^2 + 2 \int_0^t \| \Theta_s \psi_s \|^2 \| \Theta_s \psi_s \|^2 ds + \int_0^t \| \Theta_s \psi_s \|^2 \mathcal{J}(s) ds
\]
\[
\leq \| \Theta_0 \psi_0 \|^2 + 2 \int_0^t (b_{p, \mu}(s) + 1) \| \Theta_s \psi_s \|^2 ds + \int_0^t \| \Theta_s \psi_s \|^2 \mathcal{J}(s) ds
\]
\[
+ 8p \int_0^t \| \Theta_s \psi_s \|^2 \| \Theta_s \rho_s \|^2 ds
\]
(4.26)
for all \( t \in [0, t_0] \) and \( \mu > 0 \). Taking (4.19) and (4.21) into account, we see that (4.20) is actually valid for \( p = 1 \) as well.

Step 4. Applying (4.23)–(4.25) with \( \Theta_s := \vartheta := 1 \) and \( \mu' = 1/8 \), we \( \mathbb{P} \)-a.s. obtain
\[
\| \psi_t \|^2 + \frac{p}{4} \int_0^t \| \psi_s \|^2 \| \Theta_s \psi_s \|^2 ds \leq \| \eta \|^2 + 16p \int_0^t \| \psi_s \|^2 \| \Theta_s \psi_s \|^2 ds
\]
\[
+ 2p \int_0^t (4\Omega(B^q_s) + 1 + V^-(B^q_s)) \| \psi_s \|^2 ds
\]
(4.27)
for all \( t \in [0, t_0] \) and \( p \in \{1\} \cup [2, \infty) \). Together with Gronwall’s lemma and an integration by parts (4.27) \( \mathbb{P} \)-a.s. yields
\[
\sup_{s \leq t} \{ e^{-2pB_s} \| \psi_s \|^p \} \leq \| \eta \|^p + 16p \int_0^t \{ e^{-2p(2)B_s} \| \psi_s \|^2 \} \| \psi_s \|^2 \| \Theta_s \psi_s \|^2 ds
\]
\[
\leq \sup_{s \leq t} \{ e^{-2pB_s} \| \psi_s \|^p - 1/p \} 2^{1-1/p} \| \eta \|^p + 16p \int_0^t \| \psi_s \|^2 \| \Theta_s \psi_s \|^2 ds
\]
with \( B_s := \int_0^s (4\Omega(B^q_s) + 1 + V^-(B^q_s)) ds \). Hence, we \( \mathbb{P} \)-a.s. have
\[
\sup_{s \leq t} \{ e^{-pB_s} \| \psi_s \|^p \} \leq 2^{1/p} \| \eta \|^p + 4p \int_0^t \| \psi_s \|^2 \| \Theta_s \psi_s \|^2 ds
\]
(4.28)
for all \( s \leq t \). By the assumptions on \( \eta, \rho, \) and \( R \), and since \( \Theta_s \) is uniformly bounded by some \( \varepsilon \)-dependent constant, (4.28) implies that the expectation on the left hand side of (4.14) is finite for all \( t \in [0, t_0] \); this a priori information is needed in the next step.

Step 5. By the remarks in the first step, \( \{ \| \Theta_s \psi_s \|^{-2} \mathcal{J}(s) \}_{s \in [0, t_0]} \) is an adapted continuous \( \mathbb{P} \)-valued process, whence its integral with respect to the Brownian motion in (4.20) is a continuous local martingale starting from zero. Thanks to this and the remarks in Step 4, we may apply Lem. 4.8 with \( \gamma = 1/2, \delta = 1 - 1/p, \)
Step 6. Finally, we verify \((4.18)\). We abbreviate \(B_{p,t} := t + \int_0^t b_{p,\mu}(s)ds\). Then \((4.19)\) and Itô’s formula \(\mathbb{P}\)-a.s. imply

\[
e^{-2B_{p,t}}||\Theta_t \psi_t||^2 = ||\Theta_0 \psi||^2 + \int_0^t e^{-2B_{p,t}} \mathcal{J}(s)ds - 2 \int_0^t (b_{p,\mu}(s) + 1)e^{-2B_{p,t}}||\Theta_s \psi_s||^2 ds + \int_0^t e^{-2B_{p,t}} \mathcal{J}(s)dB_s,
\]

for all \(t \in [0,t_0]\). Next, we employ \((4.28)\) with \(\mu' := 1/8\) and taking into account that \(||T_{1,s}|| + ||T_{2,s}|| \leq f_{1,p}(s) \leq 1/8\) in the case \(\vartheta = 1 + d\Gamma(\omega)\), by Lem. 4.5 and our assumption on \(f_{p,\mu}\). In all cases this \(\mathbb{P}\)-a.s. yields

\[
e^{-2B_{p,t}}||\Theta_t \psi_t||^2 + \frac{1}{4} \int_0^t e^{-2B_{p,t}}d\Gamma(\omega)^{1/2}||\Theta_s \psi_s||^2 ds
\]

\[(4.29)\]

\[
\leq ||\Theta_0 \psi||^2 + 16 \int_0^t e^{-2B_{p,t}} \mathcal{R}_1(s)ds + \int_0^t e^{-2B_{p,t}} \mathcal{J}(s)dB_s,
\]

for all \(t \in [0,t_0]\). Let us consider the previous bound in the case \(\Theta_s = 1\) for the moment. Then \(\mathcal{J} = 0\) and, hence, the assumptions on \(\eta = \psi_0\), \(\rho\), and \(\mathcal{R}\) entail \(\int_0^t e^{-2B_{p,t}}d\Gamma(\omega)^{1/2}||\Theta_s \psi_s||^2 ds \in L^{p/2}(\mathbb{P})\). Since \(||\Theta_s||\) is uniformly bounded by some \(\varepsilon\)-dependent constant this ensures a priori that the left hand side of \((4.18)\) is finite.

Returning to general \(\Theta_s\), we raise \((4.29)\) to the power \(p/2\) and employ the bound \((a_1 + \cdots + a_n)^q \leq n^{(q-1)/q}(a_1^q + \cdots + a_n^q)\) afterwards, which leads to

\[
\mathbb{E}\left[\left(\int_0^t e^{-2B_{p,t}}d\Gamma(\omega)^{1/2}||\Theta_s \psi_s||^2 ds\right)^{p/2}\right] \leq 4^{p}\mathbb{E}[||\Theta_0 \psi||^p]
\]

\[+ 4^{2p}\mathbb{E}\left[\left(\int_0^t e^{-2B_{p,t}} \mathcal{R}_1(s)ds\right)^{p/2}\right] + 4^{p}\mathbb{E}\left[\left(\int_0^t e^{-2B_{p,t}} \mathcal{J}(s)dB_s\right)^{p/2}\right],
\]

\[(4.30)\]

for all \(t \in [0,t_0]\). Furthermore, there exists \(c_p > 0\) such that the following special case of the Burkholder-Davis-Gundy inequality holds for every \(t \in [0,t_0]\),

\[
\mathbb{E}\left[\sup_{r \leq t}\left(\int_r^t e^{-2B_{p,t}} \mathcal{J}(s)dB_s\right)^{p/2}\right] \leq c_p \mathbb{E}\left[\left(\int_0^t e^{-2B_{p,t}} \mathcal{J}(s)^2 ds\right)^{p/4}\right];
\]

\[(4.31)\]

see, e.g., [6] Thm. 4.36]. In particular, the last term in \((4.30)\) can be estimated as

\[
\mathbb{E}\left[\left(\int_0^t e^{-2B_{p,t}} \mathcal{J}(s)dB_s\right)^{p/2}\right] \leq c_p \mathbb{E}\left[\sup_{s \leq t} e^{-pB_{p,t}} \left(||\Theta_s \psi_s||^p\right)^{1/2}\right]
\]

\[\times \mathbb{E}\left[\left(\int_0^t e^{-2B_{p,t}} \left(||T^+_s\vartheta^{1/2}\Theta_s \psi_s + 2\Theta_s R_s||^2 ds\right)^{p/2}\right)^{1/2}\right].
\]
Using that, by Lem. [1.3] and our assumption on $f_{p,s}$, $\vartheta = 1 + d\Gamma(\omega)$ implies $\|T_t^+\| \leq 1$, we readily infer from the previous inequality that, in all cases and for all $t \in [0, t_0]$, 

$$
\mathbb{E} \left[ \left\| \int_0^t e^{-sB_p,\omega} J(s) dB_s \right\|^{p/2} \right] \leq \frac{1}{2^{p-1}} \mathbb{E} \left[ \left( \int_0^t e^{-sB_p,\omega} \|d\Gamma(\omega)^{1/2} \Theta_s \|_{\mathcal{S}}^2 ds \right)^{p/2} \right] + c_p' \mathbb{E} \left[ \|e^{-P(B_p,\omega)} \|_{\mathcal{S}}^p \left( \int_0^t e^{-sB_p,\omega} \|\Theta_s \|^2 ds \right)^{p/2} \right] + c_p' \mathbb{E} \left[ \left( \int_0^t e^{-sB_p,\omega} R(s) ds \right)^{p/2} \right],
$$

(4.32)

for some $c_p' > 0$ depending only on $p$. Inserting (4.32) into (4.30), solving the resulting inequality for the left hand side of (4.30) (which is finite as we observed above), and applying (4.17) afterwards, we obtain (4.18). $\square$

In the rest of this subsection we apply Lem. [4.9] to the process given by Thm. [2.7] extending the bounds (4.17) and (4.18) to unbounded weight functions ($\varepsilon = 0$) at the same time. The reader might want to recall the definition of the weights $\Theta_{\varepsilon,s}$ and $\Xi_{\varepsilon,s}$ in the beginning of Subsect. [4.1] before reading the next lemmas.

**Lemma 4.10.** (1) Let $t_0 > 0$, $|\alpha| \geq 1/2$, and assume that the coefficient vector satisfies $\|c\|_{\ell^\infty} := \sup_{x \in \mathbb{R}^p} \|c_x\|_\infty < \infty$ with $\| \cdot \|_\infty$ given by the first line in (4.12). Then, $\mathbb{P}$-a.s., $\mathcal{W}_t^q[B_t^q]$ maps $D(\Theta_{0,0})$ into $D(\Theta_{0,0})$, for all $t \in [0, t_0]$, and

$$
\mathbb{E} \left[ \sup_{s \leq t} \|\Theta_{s,x} \|^2 \|B_t^q \|^2 \right] \leq (7e^{\varepsilon(p \|e\|_{\ell^\infty}^2 + 4\|B\|_{\ell^\infty}^2 + 2)}t)^{p/2} \mathbb{E} \left[ \|\Theta_{0,0} \|^p \right],
$$

(4.33)

for all $p \in \mathbb{N}$, $t \in [0, t_0]$, and $\mathfrak{H}_0$-measurable $(q, \eta) : \Omega \to \mathbb{R}^\nu \times D(\Theta_{0,0})$ with $\|\Theta_{0,0} \|^2 \|B_t^q \|^2 \in L^p(\mathbb{P})$.

(2) Let $t_0 > 0$, $0 < |\delta| \leq 1$, $p \in \mathbb{N}$, and assume that $\|\langle q,F \rangle\|_{\ell^\infty} < \infty$ and $p\|G\|_{\ell^\infty} \leq 1/32$, where $\| \cdot \|_\infty$ is given by the second line in (4.12). Then, $\mathbb{P}$-a.s., $\mathcal{W}_t^q[B_t^q]$ maps $D(\Xi_{\varepsilon,0})$ into $D(\Xi_{\varepsilon,0})$, for all $t \in [0, t_0]$, and

$$
\mathbb{E} \left[ \sup_{s \leq t} \|\Xi_{\varepsilon,s} \|^2 \right] \leq (7e^{\varepsilon(p \|e\|_{\ell^\infty}^2 + 4\|B\|_{\ell^\infty}^2 + 2)}t)^{p/2} \mathbb{E} \left[ \|\Xi_{\varepsilon,0} \|^p \right],
$$

(4.34)

for all $t \in [0, t_0]$ and $\mathfrak{H}_0$-measurable $(q, \eta) : \Omega \to \mathbb{R}^\nu \times D(\Xi_{\varepsilon,0})$ with $\|\Xi_{\varepsilon,0} \| \in L^p(\mathbb{P})$.

**Remark 4.11.** Before we prove the lemma, let us clarify the purpose of the inverse weights in the polynomial case. In the situation of Lem. [4.10] assume in addition that $\alpha \leq -1/2$. By Rem. [4.1] we have $\Theta_{0,t_0}^- = \Theta_{0,t_0}^{-1}$ and $\Theta_{0,t_0}^{-1} = \Theta_{0,t_0}^-$. Substituting $\eta = \Theta_{0,t_0}^{-1}$ in (4.33) we thus arrive at

$$
\mathbb{E} \left[ \|\Theta_{0,t_0}^{-1} \|_{\mathcal{S}}^2 \|B_t^q \|_{\mathcal{S}}^2 \right] \leq (7e^{\varepsilon(p \|e\|_{\ell^\infty}^2 + 4\|B\|_{\ell^\infty}^2 + 2)}t)^{p/2} \mathbb{E} \left[ \|\eta \|^p \right],
$$

(4.35)

for all $p \in \mathbb{N}$ and $\mathfrak{H}_0$-measurable $(q, \zeta) : \Omega \to \mathbb{R}^\nu \times D(\Theta_{0,t_0}^{-1})$ with $\|\zeta \| \in L^p(\mathbb{P})$. That is, the purpose of the inverse weights is to obtain bounds where an unbounded weight stands to the right of $\mathcal{W}_t^q[B_t^q]$.

**Proof.** By virtue of Thm. [2.7] we know that Lem. [4.9] applies to the semi-martingale $(\mathcal{W}_t^q[B_t^q] \eta_{t_0})_{t \geq t_0}$, if we set $V$, $\rho$, and $R$ equal to zero in the statement of the lemma. Below we put $\eta_m := 1_{\{t \in [-m, m]\}} \eta$ and $\eta_{n,m} := (1 + n^{-1}d\Gamma(\omega))^{-1/2} \eta_m$, for $n, m \in \mathbb{N}$, so that each $\eta_{n,m}$ is a $\mathfrak{H}_0$-measurable $D(d\Gamma(\omega))$-valued initial condition.
(1): For all $\varepsilon \in (0, 1]$ and $n, m \in \mathbb{N}$, the bound
\begin{equation}
\left(4.36\right) \quad \mathbb{E} \left[ \sup_{s \leq t} \| \Theta_{\varepsilon,s} \mathcal{W}_s \mathbb{B}_q \| \eta_{m,n} \|^{2p} \right] \leq \left( 7 e^{p} \| \mathcal{W}_s \|_{\infty} + 4 \| \mathcal{E}_s \|_{\infty} + 2 \right)^{p} \mathbb{E} \left[ \| \Theta_{0,0} \eta_{m,n} \|^{2p} \right]
\end{equation}
follows easily from (4.16) and (4.17) with $\mu = 1$. Thanks to (2.21), we further know that
\begin{equation}
\left(4.37\right) \quad \sup_{s \leq t} \| \Theta_{\varepsilon,s} \mathcal{W}_s \mathbb{B}_q \| \eta_{m,n} \| \leq c_{\alpha,e} \varepsilon^{\| \mathcal{W}_s \|_{\infty} + 1} \| \eta_{m} \| \in L^{p}(\mathcal{F}), \quad \varepsilon \in (0, 1], \; m, n \in \mathbb{N},
\end{equation}
and the dominated convergence theorem implies the bounds
\begin{equation}
\left(4.38\right) \quad \mathbb{E} \left[ \sup_{s \leq t} \| \Theta_{\varepsilon,s} \mathcal{W}_s \mathbb{B}_q \| \eta_{m} \|^{2p} \right] \leq \left( 7 e^{p} \| \mathcal{W}_s \|_{\infty} + 4 \| \mathcal{E}_s \|_{\infty} + 2 \right)^{p} \mathbb{E} \left[ \| \Theta_{0,0} \eta_{m} \|^{2p} \right],
\end{equation}
for all $t \in [0, t_0]$, $\varepsilon \in (0, 1]$, and $m \in \mathbb{N}$.

Consider the case $\alpha > 0$. Then $\| \Theta_{\varepsilon,0} \eta_{m} \| \leq \| \Theta_{0,0} \eta \| \in L^{p}(\mathcal{F})$. Employing the monotone convergence theorem we may thus pass to the limits $m \to \infty$ and $\varepsilon \downarrow 0$ on the left hand side of (4.38), which proves (4.38) in this case.

In the case $\alpha < 0$, where $\Theta_{\varepsilon,s} \leq 2|\alpha|$ we apply the dominated convergence theorem first, to pass to the limit $\varepsilon \downarrow 0$ on both sides of (4.38). After that we estimate $\| \Theta_{\varepsilon,s} \eta_{m} \| \leq \| \Theta_{0,0} \eta \| \in L^{p}(\mathcal{F})$ on the right hand side and, finally, we let $m$ go to infinity on the left hand side using the monotone convergence theorem.

(2) can be proved analogously to (1), distinguishing the cases $\delta > 0$ and $\delta < 0$. The only difference is that we choose $\mu$ in (4.10) such that $\mu \| (q, F) \|_{\infty, \infty} = 1/16$, for non-zero $(q, F)$, when we apply (4.17). For then the condition $I_{p,\mu} \leq 1/8$ in Lem. 4.9 is satisfied, since we are assuming that $2p \| G \|_{\infty, \infty} \leq 1/16$.

**Lemma 4.12.** (1) Let $|\alpha| \geq 1/2$, $t_0 > 0$, $p \in \mathbb{N}$, and let $c$, $q$, and $\eta$ fulfill the conditions in Lem. 4.10(1). Then, for $dt \otimes \mathcal{F}$-a.e. $(t, \gamma) \in [0, t_0] \times \Omega$, $\mathcal{W}_t \mathbb{B}_q (\gamma)$ maps $\mathcal{D}(\Theta_{0,0}^{(\alpha)})$ into $\mathcal{D}(d\Gamma(\gamma)^{1/2} \Theta_{0,0}^{(\alpha)})$, and
\begin{equation}
\left(4.39\right) \quad \mathbb{E} \left[ \left( \int_{0}^{t} \| d\Gamma(\omega)^{1/2} \Theta_{0,0}^{(\alpha)} \mathcal{W}_s \mathbb{B}_q \| \eta \|^{2p} ds \right)^{p/2} \right] \leq c_p e^{p} \varepsilon^{p} \mathbb{E} \left[ \| \Theta_{0,0} \eta \|^{2p} \right], \quad t \in [0, t_0],
\end{equation}
for some universal constant $c > 0$ and some $c_p > 0$ depending only on $p$.

(2) Let $0 < |\delta| \leq 1$, $t_0 > 0$, $p \in \mathbb{N}$, and let $c$, $q$, and $\eta$ fulfill the conditions in Lem. 4.10(2). Then, for $dt \otimes \mathcal{F}$-a.e. $(t, \gamma) \in [0, t_0] \times \Omega$, $\mathcal{W}_t \mathbb{B}_q (\gamma)$ maps $\mathcal{D}(\Xi_{0,0})$ into $\mathcal{D}(d\Gamma(\omega)^{1/2} \Xi_{0,0})$, and (4.38) holds true with $\Theta_{0,s}^{(\alpha)}$ replaced by $\Xi_{0,s}^{(\delta)}$.

**Proof.** (1): Let $\eta_{n,m}$ and $\eta_{m}$ be defined as in the proof of Lem. 4.10. Then we can apply (4.16) with $\mathcal{W}_s = \mathcal{W}_t^{0} \mathbb{B}_q |_{n,m}$, $(\rho, R) = 0$, and (4.39) afterwards. In this way we easily arrive at
\begin{equation}
\left(4.40\right) \quad \mathbb{E} \left[ \left( \int_{0}^{t} \| d\Gamma(\omega)^{1/2} \Theta_{\varepsilon,s} \mathcal{W}_s \mathbb{B}_q \| \eta_{n,m} \|^{2p} ds \right)^{p/2} \right] \leq c_p e^{p} \varepsilon^{p} \mathbb{E} \left[ \| \Theta_{\varepsilon,0} \eta_{n,m} \|^{2p} \right],
\end{equation}
for all $t \in [0, t_0]$, $\varepsilon \in (0, 1]$, and $m \in \mathbb{N}$.

(2) will be treated analogously.
for all \( t \in [0, t_0] \), \( m, n, \in \mathbb{N} \), and \( \varepsilon \in (0, 1) \). To remove the regularization parameter \( n \), we start by considering the expressions

\[
\mathbb{E}\left[ \left( \int_0^t \|d\Gamma(\omega)^{1/2}(1 + \ell^{-1}d\Gamma(\omega))^{-1/2}\Theta^{(\alpha)}_{\varepsilon, \nu} \mathcal{B}^\nu \eta_{m, n}\|^2 ds \right)^{\gamma/2} \right] \leq c_{p, t} \mathbb{E}\left[ \|\Theta^{(\alpha)}_{\varepsilon, \nu} \eta_m\|^p \right],
\]

(4.41)

where \( c_{p, t} \) denotes the constant on the right hand side of (4.40). Because of the bounds

\[
\|d\Gamma(\omega)^{1/2}(1 + \ell^{-1}d\Gamma(\omega))^{-1/2}\Theta^{(\alpha)}_{\varepsilon, \nu} \mathcal{B}^\nu \eta_{m}\| \leq c_{\alpha, \varepsilon, \ell} e^{\|B\|_{L^p(\mathbb{F})}} t_0 \|\eta_m\| \in L^p(\mathbb{F}),
\]

that hold pointwise on \([0, t_0] \times \Omega\), for \( \varepsilon \in (0, 1) \) and \( m, n, \in \mathbb{N} \), we may apply the dominated convergence theorem to obtain (4.41) with \( \eta_{n, m} \) replaced by \( \eta_m \) on its left hand side. After that we can pass to the limits \( \varepsilon \downarrow 0 \) and \( m \to \infty \) as in the proof of Lem. 4.11 distinguishing the cases \( \alpha > 0 \) and \( \alpha < 0 \). This yields (4.41) with \( \eta_{n, m} \) and \( \eta_m \) replaced by \( \eta \) and for \( \varepsilon = 0 \). Finally, we let \( \ell \) go to infinity with the help of the monotone convergence theorem.

The proof of (2) is analogous. \( \square \)

4.4. An elementary algebraic identity. Before studying the behavior of \( \mathcal{B}^\nu \mathcal{B}^\nu \) under changes of \( \nu \) and perturbations of the coefficient vector, we derive the following algebraic lemma which is needed in the proof of Lem. 4.11 below.

**Lemma 4.13.** Let \( \mathcal{K} \) be a Hilbert space, \( \nu = (\nu_1, \ldots, \nu_\nu) \) and \( \tilde{\nu} = (\tilde{\nu}_1, \ldots, \tilde{\nu}_\nu) \) be tuples of self-adjoint operators acting in \( \mathcal{K} \), and \( \Theta = \Theta^* \in \mathcal{B}(\mathcal{K}) \) be such that \( \Theta D(w) \subset D(w) \) and \( \Theta D(w^2) \subset D(w^2) \), if the operator \( w \) is any of the components of \( \nu \) or \( \tilde{\nu} \). Suppose that the vectors \( \eta \) and \( \tilde{\eta} \) are elements of the domain \( \mathcal{C} := \bigcap_{\nu=1}^\nu D(\nu^2) \cap D(\nu \nu) \cap D(\tilde{\nu} \nu) \cap D(\tilde{\nu} \tilde{\nu}) \). Then the following identity holds,

\[
A := -\text{Re}\langle \eta - \tilde{\eta} | \Theta^2 \nu^2 \eta - \Theta^2 \tilde{\nu}^2 \tilde{\eta} \rangle + \|\Theta \nu \eta - \Theta \tilde{\nu} \tilde{\eta}\|^2
\]

\[
= \|\Theta (\nu - \tilde{\nu}) \eta\|^2 + \frac{1}{2} \left\langle \eta - \tilde{\eta}, [\nu, \Theta] \eta - \tilde{\eta} \right\rangle
\]

\[
- 2\text{Re}\left\{ \langle \Theta | [\nu, \Theta] \rangle + [\nu, \Theta] \eta \right\} (\nu \eta) - \tilde{\nu} \tilde{\eta} + \text{Re}\langle \nu - \tilde{\nu} | \Theta (\eta - \tilde{\eta}) \right\} (\nu - \tilde{\nu}) \eta
\]

\[
- \text{Re}\langle \Theta (\eta - \tilde{\eta}) | \Theta (\nu - \tilde{\nu}) \right\} (\nu - \tilde{\nu}) \eta
\]

(4.42)

**Proof.** First we write \( \Theta^2 \nu^2 = \nu \Theta^2 \nu + [\Theta^2, \nu] \nu \) and analogously for \( \tilde{\nu} \) and take the cancelations between the two terms in the first line of (4.42) into account to get

\[
A = \text{Re}\langle \eta - \tilde{\eta} | \nu \Theta^2 (\nu - \tilde{\nu}) \eta \rangle - \text{Re}\langle \eta - \tilde{\eta} | \Theta^2 (\nu - \tilde{\nu}) \theta \rangle
\]

(4.43)

In both terms appearing on the right hand side of the first line in (4.43) and in the right entry of scalar product in the second line of (4.43) we now write \( \eta = (\eta - \tilde{\eta}) + \tilde{\eta} \) and apply distributive laws to get

\[
A = \text{Re}\langle \eta - \tilde{\eta} | \nu \Theta^2 (\nu - \tilde{\nu}) \eta \rangle - \text{Re}\langle \eta - \tilde{\eta} | (\nu - \tilde{\nu}) \Theta^2 \tilde{\nu} \rangle
\]

\[
+ \text{Re}\langle \eta - \tilde{\eta} | \nu \Theta^2 (\nu - \tilde{\nu}) \rangle - \text{Re}\langle \eta - \tilde{\eta} | (\nu - \tilde{\nu}) \Theta^2 \tilde{\nu} \rangle
\]

(4.44)

\[
- \text{Re}\langle \eta - \tilde{\eta} | \Theta^2, \nu | (\eta - \tilde{\eta}) \rangle
\]
Using \( \text{Re}(\phi|\psi) = \text{Re}(\psi|\phi) \), we see that the term in the second line of (4.44) is equal to the first term in the second line of (4.42). Computing the real part we further see that the term in the third line of (4.44) is equal to the second term in the second line of (4.42). (Here we use \( \Theta^2 D(v_j^2) \subset D(v_j^2) \).) The sum of the first and fourth line of (4.44) is equal to \( \text{Re} \langle \eta - \overline{\eta}|B \eta \rangle \), where \( B \) is defined on the domain \( C \) by
\[
B := v \Theta^2(v - \tilde{v}) - (v - \tilde{v}) \Theta^2 \tilde{v}
\]
(4.45)
\[
- [\Theta^2, v](v - \tilde{v}) - [\Theta^2, v - \tilde{v}] \tilde{v}.
\]
Here we added and subtracted \( [\Theta^2, v] \tilde{v} \) in the second line. In the first line of (4.45) we now commute one factor \( \Theta \) to the left, and in the second line we use the Leibnitz rule \( [\Theta, a] = \Theta[a] + [\Theta, a] \Theta \). This results in
\[
B = \Theta \{ v \Theta(v - \tilde{v}) - (v - \tilde{v}) \Theta \tilde{v} - [\Theta, v](v - \tilde{v}) - [\Theta, v - \tilde{v}] \tilde{v} \}
\]
\[
+ [v, \Theta] \Theta(v - \tilde{v}) - [v - \tilde{v}, \Theta] \Theta \tilde{v}
\]
\[
- [\Theta, v] \Theta(v - \tilde{v}) - [\Theta, v - \tilde{v}] \Theta \tilde{v}
\]
\[
= \Theta \{ 2[v, \Theta] + [\Theta(v - \tilde{v})](v - \tilde{v}) - [\Theta^2, v] \tilde{v} \}
\]
\[
+ 2[v, \Theta] \Theta(v - \tilde{v}) \quad \text{on } C.
\]
Since \( \Theta[v, \Theta] + [v, \Theta] \Theta \) is skew symmetric on \( D(v_j) \), for every \( j \in \{1, \ldots, \nu\} \), this yields the desired identity. \( \square \)

4.5. Continuity of the stochastic flow in weighted spaces. In this subsection we prove the announced Lem. 4.16 on the behavior of the stochastic flow under perturbations of the initial condition \( q \) and the coefficient vector. We shall again employ Notation 4.14 and 4.15. The following abbreviations will be useful as well:

**Notation 4.14.** Given two coefficient vectors \( c = (G, q, \sigma \cdot F) \) and \( \tilde{c} = (\tilde{G}, \tilde{q}, \tilde{\sigma} \cdot \tilde{F}) \) satisfying Hyp. 2.1 with the same \( \omega \) and the same conjugation \( C \) and two \( \mathfrak{g}_0 \)-measurable initial conditions \( q, \tilde{q} : \Omega \to \mathbb{R}^\nu \), we set
\[
D_s^\pm := [\varphi(GB^\delta - \tilde{G}B^\delta), \Theta_s^\pm] \Theta_s^\pm \varphi^{-\frac{1}{2}},
\]
(4.46)
\[
D_s := [\frac{1}{2} \varphi(qB_\delta^2 - \tilde{q}B_\delta^2) + \sigma \cdot \varphi(FB_\delta^2 - \tilde{F}B_\delta^2), \Theta_s] \Theta_s^{-\frac{1}{2}},
\]
(4.47)
for all \( s \in [0, t_0] \). For any vector \( v \) with components in \( \mathfrak{h} \) we further write
\[
\| v \|_s := \| v \|_t + \| v \|_\infty.
\]
Finally, we abbreviate
\[
\theta := 1 + d\Gamma(\omega).
\]
(4.48)
Again the operators in (4.46) and (4.47) are well-defined a priori on \( D(d\Gamma(\omega)) \) and have bounded closures under suitable extra conditions on \( c \). For we have the following analogue of Lem. 4.5.

**Lemma 4.15.** Let \( s \in [0, t_0] \) and \( \varepsilon \in (0, 1] \). Assume in addition that \( \varepsilon \leq |\delta| \), if the operators in (4.46) and (4.47) are defined by means of the exponential weights \( \Theta_s = \mathfrak{g}_s(\delta) \) and \( \vartheta = 1 + d\Gamma(\omega) \). Then the following bounds hold on \( \Omega \),
\[
\| D_s^\pm \| \leq \| GB^\delta - \tilde{G}B^\delta \|_\infty, \quad \| D_s \| \leq \| (q, \sigma \cdot F)B_\delta^2 - (\tilde{q}, \tilde{\sigma} \cdot \tilde{F})B_\delta^2 \|_\infty.
\]
(4.49)
Proof. Ex. A.7 and Ex. A.9 immediately imply the asserted bounds, if we replace \( GB^\delta \) by \( G\tilde{B}^\delta \), etc., in the corresponding arguments. \( \square \)
Lemma 4.16. Let $p \in \mathbb{N}$ and $V, \tilde{V} : \mathbb{R}^\nu \to \mathbb{R}$ be bounded and continuous. Let $c = (G, q, \sigma \cdot F)$ and $\tilde{c} = (\tilde{G}, \tilde{q}, \sigma \cdot \tilde{F})$ be two coefficient vectors satisfying Hyp. \[2.7\] with the same $\omega$ and the same conjugation $C$ such that $\|c_x\|_0, \|\tilde{c}_x\|_0 < \infty$, for every $x \in \mathbb{R}^\nu$. Let $(\tilde{\rho}_t, \tilde{R}_t)_{t \in [0, t_0]}$ and $(\tilde{\rho}_t, \tilde{R}_t)_{t \in [0, t_0]}$ be adapted $\mathcal{B}(\mathbb{R}^{1+\nu}, \mathcal{H})$-valued processes with continuous paths such that $\int_0^{t_0} \|(\tilde{\theta}^{-1/2} \tilde{\rho}_s, \tilde{R}_s)^2\|ds$ and $\int_0^{t_0} \|(\tilde{\theta}^{-1/2} \tilde{\rho}_s, \tilde{R}_s)^2\|ds$ belong to $L^{\nu/2}(\mathbb{P})$. Define a family of closed operators

$$
\hat{H}(x) := \frac{1}{2} \varphi(G_x)^2 - \frac{1}{2} \varphi(\tilde{G}_x)^2 - \sigma \cdot \varphi(\tilde{F}_x) + d\Gamma(\omega), \quad x \in \mathbb{R}^\nu,
$$
on the constant domain $D(d\Gamma(\omega))$. Assume in addition that $f_{2p}(s) \leq 1/8$, for all $s \in [0, t_0]$, where $f_{2p}$ is defined by \[1.13\]. Let $(q, \tilde{q}, \eta) : \Omega \to \mathbb{R}^\nu \times \mathbb{R}^\nu \times D(d\Gamma(\omega))$ be $\mathbb{H}_0$-measurable and $(\psi_t)_{t \in [0, t_0]}$ and $(\tilde{\psi}_t)_{t \in [0, t_0]}$ be $\mathcal{H}$-valued semi-martingales whose paths belong $\mathbb{P}$-a.s. to $C([0, t_0], D(d\Gamma(\omega)))$ and which $\mathbb{P}$-a.s. satisfy

$$
\begin{align*}
\psi_t &= \eta - \int_0^t (\hat{H}(B^q_s) + V(B^q_s)) \psi_s ds + \int_0^t i\varphi(G_{B^q_s}) \psi_s dB_s \\
&+ \int_0^t \rho_s ds + \int_0^t R_s dB_s,
\end{align*}
$$

$$
\begin{align*}
\tilde{\psi}_t &= \eta - \int_0^t (\hat{H}(\tilde{B}^q_s) + \tilde{V}(\tilde{B}^q_s)) \tilde{\psi}_s ds + \int_0^t i\varphi(\tilde{G}_{\tilde{B}^q_s}) \tilde{\psi}_s dB_s \\
&+ \int_0^t \tilde{\rho}_s ds + \int_0^t \tilde{R}_s dB_s,
\end{align*}
$$

for all $t \in [0, t_0]$. Abbreviate

$$
\begin{align*}
\phi_s &= \Theta_s(\psi_s - \tilde{\psi}_s), \\
d_p(s) &= [V(B^q_s) - \tilde{V}(\tilde{B}^q_s)] + \|q, \sigma \cdot F\|_{B^q_s}^2 - (\tilde{q}, \sigma \cdot \tilde{F})_{\tilde{B}^q_s}^2, \\
Q_p(s) &= d_p(s)^2 \|\theta^{1/2} \Theta_s \psi_s\|^2 \\
&+ \|\theta^{-1/2} \Theta_s (\rho_s - \tilde{\rho}_s)\|^2 + (p + \|G_{B^q_s}\|^2) \|\Theta_s (R_s - \tilde{R}_s)\|^2,
\end{align*}
$$

(4.50)

for all $s \in [0, t_0]$. Then the following bounds hold, for all $t \in [0, t_0]$,

$$
\begin{align*}
\mathbb{E} \left[ \sup_{s \leq t} e^{-p \int_0^s b_{2p, \mu}(r) dr} \|\phi_s\|^p \right] \\
\leq (c e^{p^{1/2}})^p \mathbb{E} \left[ \left( \int_0^t e^{-2 \int_0^t b_{2p, \mu}(r) dr} Q_p(s) ds \right)^{\nu/2} \right],
\end{align*}
$$

(4.51)

$$
\begin{align*}
\mathbb{E} \left[ \left( \int_0^t d\Gamma(\omega)^{1/2} \phi_s^2 ds \right)^{\nu/2} \right] &\leq c_p e^{p t} \mathbb{E} \left[ \left( \int_0^t e^{-2 \int_0^t b_{2p, \mu}(r) dr} Q_p(s) ds \right)^{\nu/2} \right] \\
&+ c_p e^{p t} \mathbb{E} \left[ \left( \int_0^t e^{-2 \int_0^t b_{2p, \mu}(r) dr} \|G_{B^q_s}\|^2 \|\phi_s\|^2 ds \right)^{\nu/2} \right].
\end{align*}
$$

(4.52)

Here $c > 0$ is some universal constant and $c_p > 0$ depends only on $p$. The process $b_{2p, \mu}$ is defined by \[4.10\].

Proof. Step 1. Exactly as in the first step of the proof of Lem. \[4.3\] we see that Lem. \[3.3\] applies to both $(\psi_t)_{t \in [0, t_0]}$ and $(\tilde{\psi}_t)_{t \in [0, t_0]}$. As a consequence, $(\phi_t)_{t \in [0, t_0]}$
can \( F \)-a.s. be written as
\[
\phi_t = -\int_0^t \Theta_s(\theta G_B) - \Phi_s)\psi_s ds - \int_0^t \Theta_s(\theta G_B) - \Phi_s)\psi_s ds \\
- \int_0^t (d\Gamma(\omega) - \frac{\partial}{\partial s})\phi_s ds + \int_0^t \Theta_s(\rho_s - \rho_s) ds \\
+ \int_0^t \Theta_s(i\varphi(G_B^*)\psi_s - i\varphi(G_B^*)\psi_s) dB_s + \int_0^t \Theta_s(R_s - \bar{R}_s) dB_s,
\]
for all \( t \in [0, t_0] \), with
\[
\Phi_s := \frac{1}{2}\varphi(q_B) + \sigma \cdot \varphi(F_B) - V(B^*_s), \\
\bar{\Phi}_s := \frac{1}{2}\varphi(\bar{q}_B) + \sigma \cdot \varphi(\bar{F}_B) - V(B^*_s).
\]
Now Itô’s formula \( F \)-a.s. implies
\[
\|\phi_t\|^2 = \sum_{\ell=1}^7 \int_0^t \mathcal{I}_\ell(s) ds + \int_0^t I(s) dB_s, \quad t \in [0, t_0],
\]
where
\[
\mathcal{I}_1(s) := -\text{Re}\langle \phi_s | \Theta_s(\varphi(G_B) - \phi(\bar{G}_B^*)\bar{\psi}_s) \rangle, \\
\mathcal{I}_2(s) := -2\text{Re}\langle \phi_s | (d\Gamma(\omega) - \frac{\partial}{\partial s})\phi_s \rangle, \\
\mathcal{I}_3(s) := 2\text{Re}\langle \phi_s | \Theta_s(\Phi_s\bar{\psi}_s - \bar{\Phi}_s\bar{\psi}_s) \rangle, \\
\mathcal{I}_4(s) := \|\Theta_s(\varphi(G_B) - \phi(\bar{G}_B^*)\bar{\psi}_s)\|^2, \\
\mathcal{I}_5(s) := 2\text{Re}\langle \phi_s | \Theta_s(\rho_s - \rho_s) \rangle, \\
\mathcal{I}_6(s) := 2\text{Re}\langle \Theta_s(R_s - \bar{R}_s) | i\Theta_s(\varphi(G_B) - \phi(\bar{G}_B^*)\bar{\psi}_s) \rangle, \\
\mathcal{I}_7(s) := \|\Theta_s(R_s - \bar{R}_s)\|^2, \\
I(s) := 2\text{Re}\langle \phi_s | \Theta_s(\varphi(G_B) - \phi(\bar{G}_B^*)\bar{\psi}_s) + \Theta_s(R_s - \bar{R}_s) \rangle.
\]
Furthermore, the following identity (4.54) is a direct consequence of the algebraic Lem. 4.13 the commutation relation
\[
[\varphi(G_B^*), \varphi(\bar{G}_B^*)] = 2\text{Im}(G_B^* \bar{G}_B^*) = 0, \quad \text{on } \mathcal{D}(d\Gamma(\omega)),
\]
and the definitions (4.17), (4.39), and (4.40),
\[
\mathcal{I}_1(s) + \mathcal{I}_2(s) = \langle \theta^{1/2} \phi_s | T_{\theta}^{1/2} \phi_s \rangle - 2\text{Re} \langle T_{\theta}^{1/2} - T_{\theta}^{1/2} \phi_s | \zeta_s \rangle \\
+ \text{Re} \langle \varphi(G_B^* - \bar{G}_B^*) \phi_s | \zeta_s \rangle - \text{Re} \langle D_\theta^{1/2} \phi_s | \zeta_s \rangle + \| \zeta_s \|^2.
\]
Here we abbreviate
\[
\zeta_s := \Theta_s \varphi(G_B^* - \bar{G}_B^*)\bar{\psi}_s = (\varphi(G_B^* - \bar{G}_B^*) - D_\theta^{1/2} \theta^{1/2}) \Theta_s \bar{\psi}_s.
\]
Using the shorthand (4.48), we further have
\[
\| \zeta_s \| \leq c \| G_B^* - \bar{G}_B^* \|_\theta \| \theta^{1/2} \Theta_s \bar{\psi}_s \|,
\]
for some universal constant \( c > 0 \), where we used (4.49) and
\[
\| \varphi(G_B^* - \bar{G}_B^*) \phi' \| \leq 2^{1/2} \| G_B^* - \bar{G}_B^* \|_\theta \| \theta^{1/2} \phi' \|, \quad \phi' \in \mathcal{D}(\theta^{1/2}).
\]
We readily infer from these remarks that
\[
|\mathcal{F}_1(s) + \mathcal{F}_3(s)| \leq \|\phi_s\|^2/8 + \|d\Gamma(\omega)^{1/2}\phi_s\|^2/8 + \|T_{1,s}\||\theta^{1/2}\phi_s|^2
\]
with some universal constant $c' > 0$. Since $\text{Re}\langle\phi_s|\Phi_2\rangle = 0$, we further have
\[
\mathcal{F}_3(s) = -2\text{Re}\langle\phi_s|T_{2,s}\theta^{1/2}\phi_s\rangle + 2\text{Re}\langle\phi_s|(\sigma \cdot \varphi(F\Phi_2) - V(BF)\phi_s) + 2\text{Re}\langle\phi_s|\xi_s\rangle
\]
with
\[
\xi_s := \Theta_s(\Phi_s - \bar{\Phi}_s)\psi_s = (\Phi_s - \bar{\Phi}_s)\Theta_s\psi_s - D_s\theta^{1/2}\Theta_s\psi_s,
\]
so that
\[
\|\xi_s\| \leq c''d_1(s)||\theta^{1/2}\Theta_s\psi_s||.
\]
Hence, using also (4.21), it is easy to see that
\[
\mathcal{F}_2(s) + \mathcal{F}_3(s) \leq (8\lambda(B\Phi_2) - 2V(B\Phi_2) + \frac{1}{\mu}\|T_{2,s}\| + \frac{2}{\mu})\|\phi_s\|^2 - \|d\Gamma(\omega)^{1/2}\phi_s\|^2 / 2
\]
for all $\mu > 0$. Moreover, we have, again for every $\mu' > 0$,
\[
|\mathcal{F}_5(s)| \leq \mu'\|\theta^{1/2}\phi_s\|^2 + \mu\|\theta^{-1/2}\Theta_s(\rho_s - \bar{\rho}_s)\|^2,
\]
\[
|\mathcal{F}_6(s)| \leq 2\mu'\|\theta^{1/2}\phi_s\|^2 + (1 + \frac{1}{\mu'})\|\varphi(GB\Phi_2)\theta^{-1/2}\|^2 + \frac{1}{\mu'}\|T_s\|^2)\mathcal{F}_{7}(s) + ||\xi_s||^2.
\]
Summarizing the above bounds on $\mathcal{F}_1, \ldots, \mathcal{F}_7$, we obtain
\[
\sum_{\ell=1}^{7} \mathcal{F}_\ell(s) \leq -\left(\frac{\lambda}{\lambda} - 3\mu'\right)\|d\Gamma(\omega)^{1/2}\phi_s\|^2 + (\|T_{1,s}\| + \mu\|T_{2,s}\|)||\theta^{1/2}\phi_s||^2
\]
\[
+ (8\lambda(B\Phi_2) - 2V(B\Phi_2) + \frac{1}{\mu}\|T_{2,s}\| + 3\mu' + \frac{33}{\mu})\|\phi_s\|^2
\]
\[
+ cd_1(s)^2||\theta^{1/2}\Theta_s\psi_s||^2
\]
\[
+ \frac{1}{\mu'}||\theta^{-1/2}\Theta_s(\rho_s - \bar{\rho}_s)\|^2 + c'(1 + \frac{1}{\mu'})\|GB\Phi_2\|^2)\mathcal{F}_{7}(s),
\]
for all $s \in [0, t_0]$ and $\mu, \mu' > 0$. Using $\text{Re}\langle\phi_s|\varphi(GB\Phi_2)\rangle = 0$, we further have
\[
I(s) = -2\text{Re}\langle\phi_s|T_{s}\theta^{1/2}\phi_s\rangle + 2\text{Re}\langle\phi_s|\xi_s\rangle + 2\text{Re}\langle\phi_s|\Theta_s(\rho_s - \bar{\rho}_s)\rangle,
\]
which together with (4.55) permits to get
\[
\frac{1}{2}I(s)^2 \leq \|\phi_s\|^2(4\|T_s\|^2||\theta^{1/2}\phi_s||^2 + c'd_1(s)^2||\theta^{1/2}\Theta_s\psi_s||^2 + c''\mathcal{F}_{7}(s)).
\]

Step 2. Let $p \geq 2$. Then, in view of (4.53), an application of Itô’s formula P-a.s. yields
\[
\|\phi_s\|^{2p} = p\sum_{\ell=1}^{7}\int_0^t \|\phi_s\|^{2p-2}\mathcal{F}_\ell(s)ds + p\int_0^t \|\phi_s\|^{2p-2}I(s)dB_s
\]
\[
+ \frac{p(p-1)}{2}\int_0^t \|\phi_s\|^{2p-4}I(s)^2ds, \quad t \in [0, t_0].
\]
Applying the bound (4.58) with \( \mu' := 1/24 \) and (4.59), we may \( \mathbb{P} \)-a.s. deduce that
\[
\|\phi_t\|^{2p} \leq \|\phi_0\|^{2p} - \int_0^t \left( \frac{1}{4} - f_{2p,\mu}(s) \right) \|\phi_0\|^{2p-2} \|d\Gamma(\omega)\|^{(2p-2)} ds
\]
\[
\leq 2p \int_0^t (b_{2p,\mu}(s) + 1) \|\phi_0\|^{2p} ds + cp \int_0^t \|\phi_0\|^{2p-2} Q_p(s) ds + p \int_0^t \|\phi_0\|^{2p-2} I(s) dB_s,
\]
for all \( t \in [0, t_0] \). The previous bound holds actually true for \( p = 1 \) as well; see (4.53) and (4.58). Applying Lem. 4.18 with \( \gamma = 1/2 \) and \( \delta = 1 - 1/p \) (exactly as in the proof of Lem. 4.19), we arrive at (4.51). In fact, we know that the a priori bound (4.13) is fulfilled by the right hand side of (4.51) since the arguments of Step 4 in the proof of Lem. 4.19 apply to both \( \left( \psi_t \right)_{t \in [0, t_0]} \) and \( \left( \tilde{\psi}_t \right)_{t \in [0, t_0]} \).

Finally, (4.52) is derived from (4.58) (with \( \mu' := 1/24 \)) exactly in the same fashion as (4.18) was derived from (4.23) in Step 6 of the proof of Lem. 4.19. \( \square \)

**Example 4.17.** Let us apply Lem. 4.16 with with constant \( \eta = \phi \in \mathcal{D}(d\Gamma(\omega)) \) and \( \bar{c} = 0, V = \bar{V} = 0, \rho = 0, R = \bar{R} = 0, \) and \( \tilde{\rho} = d\Gamma(\omega)\phi \). Then, up to indistinguishability, we simply have \( \psi_t = \phi, t \geq 0 \). Choosing the \( s \)-independent weight \( \Theta_s := \Theta_s^{1/2} := (1 + \varepsilon (1 + d\Gamma(\omega_s)(1 + d\Gamma(\omega_s))^{-1/2}) \), with \( \omega_s := \omega(1 + \varepsilon)\omega^{-1} \). In this case we have \( |\alpha| = 1/2 \), so that the norm \( \| \cdot \|_c \) is dominated by \( \| \cdot \|_t \). We again write \( \theta = \theta_0 \) and \( \|c\|_{t,\infty} := \sup_{x \in \mathbb{R}^p} \|c_x\|_t \). Employing (4.51) and the dominated convergence theorem, we then find a universal constant \( c > 0 \) and \( c_p > 0 \), depending only on \( p \in \mathbb{N} \), such that
\[
\mathbb{E} \left[ \sup_{s \leq t} \left( \theta^{-1/2} \|\psi_{0s}(B_\theta) - 1\| \right)^p \right] \leq \mathbb{E} \left[ \sup_{s \leq t} \left( \theta^{-1/2} \|\psi_{0s}(B_\theta) - 1\| \right)^p \right]
\]
\[
\leq c_p(\|c\|_{t,\infty} \vee \|c\|_{t,\infty}^2)^p t^{p/2} (\|\psi_{0s}(B_\theta) - 1\|)^2 \|d\Gamma(\omega)\| \quad \text{by any arbitrary element of } \mathcal{F}, \text{ we arrive at}
\]
\[
\mathbb{E} \left[ \sup_{s \leq t} \left( 1 + d\Gamma(\omega_s) \right)^{-1/2} \|\psi_{0s}(B_\theta) - 1\| \psi \right]^p \]
\[
\leq c'_p(\|c\|_{t,\infty} \vee \|c\|_{t,\infty}^2)^p t^{p/2} (\|\psi_{0s}(B_\theta) - 1\|)^p \psi^p,
\]
for all \( t \geq 0, \psi \in \mathcal{F} \), and \( p \in \mathbb{N} \), where \( c'_p > 0 \) depends only on \( p \).

Finally, we apply Lem. 4.16 to two solution processes given by Thm. 2.4 extending (4.51) to unbounded weights at the same time. We could also extend (4.52), of course, but refrain from doing so here, as (4.52) will only become important in our companion paper [20].

**Lemma 4.18.** (1) Let \( t_0 > 0, |\alpha| \geq 1/2, \) and let \( c \) and \( \bar{c} \) be two coefficient vectors satisfying Hyp. 2.1 with the same \( \omega \) and \( C \) such that \( \|c\|_\infty < \infty \) and \( \|\bar{c}\|_\infty < \infty \). Let \( V, \bar{V} : \mathbb{R}^\nu \to \mathbb{R} \) be bounded and continuous. Finally, let \( \Psi(q, \hat{q}, \eta) : \Omega \to \mathbb{R}^{2 \nu} \times \mathcal{D}(\theta_0(q_0, \eta)) \) be \( \mathfrak{g}_0 \)-measurable with \( \|\Psi(q_0, \eta)\|_p \in L^p(\mathbb{P}) \), and let \( \Psi[B_0(q_0)] \) be the solution process given by Thm. 2.4 applied to \( \tilde{c}, \bar{V}, \) and \( \tilde{q} \). Then there exist a universal
constant $c > 0$ and for every $p \in \mathbb{N}$, some $c_p > 0$, depending only on $p$, such that with
\[ c_{p,t} := c_p e^{c(p)(t^\alpha \|w\|_\infty^2 + \|\xi\|_\infty + \|V\|_\infty + p\|\xi\|_\infty^2 + \|\tilde{\xi}\|_\infty + \|\tilde{V}\|_\infty + 1)p t} \]
the following bound holds true,
\[ E\left[ \sup_{s \leq t} \left\| \Theta^{(\alpha)}_{\epsilon,s} \left( \Psi^V_s \left[ B^q \right] - \tilde{\Psi}^V_s \left[ B^q \right] \right) \right\|^p \right] \]
(4.62) \[ \leq c_{p,t} E\left[ \sup_{s \leq t} d_p(s)^{2p} \right]^{1/2} (1 + t^{\alpha/2} \|\tilde{G}\|_{p,\infty}^p) E\left[ \left\| \Theta^{(\alpha)}_{\epsilon,0} \eta \right\|^{2p} \right]^{1/2}. \]
Here $d_p(s)$ is defined in (4.50).

(2) The assertion in (1) holds true with $\alpha$ replaced by $0 < |\delta| \leq 1$ and $\Theta^{(\alpha)}_{\epsilon,s}$ replaced by $\Theta^{(\delta)}_{\epsilon,s}$ everywhere.

Proof. (1): Let $\eta_{n,m}$ be defined as in the proof of Lem. 4.10. By virtue of (4.51) we then obtain
\[ e^{-c(p)(t^\alpha \|w\|_\infty^2 + \|\xi\|_\infty + \|V\|_\infty + 1)p t} E\left[ \sup_{s \leq t} \left\| \Theta^{(\alpha)}_{\epsilon,s} \left( \Psi^V_s \left[ B^q \right] - \tilde{\Psi}^V_s \left[ B^q \right] \right) \eta_{n,m} \right\|^p \right] \]
\[ \leq c_p E\left[ \left( \int_0^t d_p(s)^2 \left\| \Theta^{(\alpha)}_{\epsilon,s} \tilde{\Psi}^V_s \left[ B^q \right] \right\|^2 \right)^{n/2} ds \right] \]
\[ \leq c_p E\left[ \sup_{s \leq t} d_p(s)^{2} \right]^{1/2} E\left[ \left( \int_0^t \left\| \Theta^{(\alpha)}_{\epsilon,s} \tilde{\Psi}^V_s \left[ B^q \right] \right\|^2 ds \right)^{n/2} \right] \]
(4.63) \[ \leq c_{p,t} E\left[ \sup_{s \leq t} d_p(s)^{2p} \right]^{1/2} (1 + t^{\alpha/2} \|\tilde{G}\|_{p,\infty}^p) E\left[ \left\| \Theta^{(\alpha)}_{\epsilon,0} \eta_{n,m} \right\|^{2p} \right]^{1/2}. \]
Here we also applied (4.30) and (4.40) in the last step. Next, we let $n$ go to infinity on the left hand side of (4.63) using the dominated convergence theorem with majorizing functions analogous to the ones in (4.37). After that the regularization parameters $m$ and $\epsilon$ are removed in the same way as in the end of the proof of Lem. 4.10 (1), distinguishing the cases $\alpha > 0$ and $\alpha < 0$.

The proof of (2) is again completely analogous. \qed

5. Weighted $L^p$-Estimates and Continuity in the Potential

The next theorem complements the $L^p$-estimates of Lem. 3.1 by including unbounded multiplication operators in Fock space. The convergence result in the next theorem will imply strong convergence of the semi-group when its Kato decomposable potential is approximated in the sense of Lem. 2.13 see Cor. 5.4 below. It is also used to prove Thm. 8.1 later on. To shorten statements we introduce the following convention:

Notation 5.1. In the following table we introduce weight functions and corresponding norms used to state our main theorems in this and the subsequent sections. These theorems hold for the weights in the first column provided that the coefficient vector $c$ fulfills the corresponding hypothesis in the third column in addition to our standing hypothesis Hyp. 2.1. In the third column of the table we use the notation introduced in the second one. We always assume that $\alpha \geq 1/2, \delta \in (0,1], t_* \geq 2$, and that $\varpi$ and $\varkappa$ are non-negative measurable functions on $\mathcal{M}$ with $\varpi \leq \varkappa$. The constant $c_\alpha$ is the one appearing in Lem. 3.1(1), $c$ is the one appearing in Lem. 3.3(2). Recall that $\sup_{x} \|c_x\|_{t} < \infty$ is part of Hyp. 2.1.
Furthermore, we abbreviate (2) \( (5.2) \):

\[
\|v\|_* := \|v\|_1 + \|v\|_* \quad \text{with} \quad \|v\|_* := \\

1. \( (1 + d\Gamma(x)/2\alpha)^n \)
   \[
   c^n \|x^{1/2}(1 + x)^{n-1/2}v\|_b \quad \sup \|x^n c_x\|_b < \infty \\
   \\
   2. \( (1 + (t + td\Gamma(x))v)/(2\alpha)^n \)
   \[
   c^n \|w^{1/2}(1 + w)^{n-1/2}v\|_b \quad \sup \|w^n c_x\|_b < \infty \\
   \\
   3. \( e^{\delta t\Gamma(x)} \)
   \[
   c^{\delta t^{1/2}}(\|x^2 x\|^{1/2})^n e^{\delta x^2}v\|_b \quad \sup \|x_0^* (F, q)\|_* < \infty \\
   \sup \|G_x\|_* < 1/9 \\
   \\
   4. \( e^{\delta (t+t_x)(1+d\Gamma(pi))}/2 \)
   \[
   c\delta^{1/2}(1/2) e^{\delta x^2}v\|_b \quad \sup \|x_0^* (F, q)\|_* < \infty \\
   \sup \|G_x\|_* < 1/9 \\
   \]

Table 1: Weight functions with corresponding norm \( \|\cdot\|_* \), and additional hypothesis.

**Theorem 5.2.** Let \( V \in K_\pm(\mathbb{R}^n) \), let \( F: \mathbb{R}^n \to \mathbb{R} \) be globally Lipschitz continuous, i.e., \( |F(x) - F(y)| \leq a|x - y|, x, y \in \mathbb{R}^n \), for some \( a \geq 0 \), and let \( 1 \leq p \leq q \leq \infty \). Let \( (\bar{\Upsilon}_t)_{t \geq 0} \) and \( \|\cdot\|_* \) be given by one of the lines in Table 1 and assume that \( c \) fulfills Hyp(\( \Upsilon \)) given by the same line. Then the following holds:

1. For all \( t > 0 \), \( T_t^V \) maps \( \bar{\Upsilon}_0^{-1} L^p(\mathbb{R}^n, \mathcal{H}; e^{\nu t}) \) into the domain of \( \bar{\Upsilon}_t \), considered as a densely defined operator in \( L^q(\mathbb{R}^n, \mathcal{H}; e^{\nu t}) \), and

\[
\|e^F \bar{\Upsilon}_t T_t^V \bar{\Upsilon}_0^{-1} e^{-F} \|_{p,q} \leq c_{p,q} \|x|^{(1+|x|^2)} e^{\nu(x^2 + a^2)t} \sup \mathbb{E} \left[ e^{8 \int_0^t V^*(B_s^2)ds} \right]^{1/4}.
\]

Here the constant \( c_{p,q} > 0 \) depends only on \( p, q, \) and \( \nu \).

2. If \( V_n \in K_\pm(\mathbb{R}^n) \), \( n \in \mathbb{N} \), satisfy \( (2.3) \) and \( (2.3') \), then

\[
\lim_{n \to \infty} \sup_{t \in [\tau_1, \tau_2]} \|1_{K} e^F \bar{\Upsilon}_t (T_t^{V_n} - T_t^V) \bar{\Upsilon}_0^{-1} e^{-F} \|_{p,q} = 0,
\]

\[
\lim_{n \to \infty} \sup_{t \in [\tau_1, \tau_2]} \|e^F \bar{\Upsilon}_t (T_t^{V_n} - T_t^V) \bar{\Upsilon}_0^{-1} e^{-F} 1_K \|_{p,q} = 0,
\]

for all compact \( K \subset \mathbb{R}^n \) and \( \tau_2 > \tau_1 > 0 \). If \( p = q \), then the choice \( \tau_1 = 0 \) is allowed for in \( (2.2) \) and \( (2.3) \) as well. If \( V, V_n \in K(\mathbb{R}^n) \), for all \( n \in \mathbb{N} \), and \( (2.3') \) holds with \( K \) replaced by \( \mathbb{R}^n \), then \( K \) can be replaced by \( \mathbb{R}^n \) in \( (5.2) \) and \( (5.3) \), too. In all cases, the convergences \( (5.2) \) and \( (5.3) \) are uniform in all Lipschitz continuous \( F \) with Lipschitz constant \( \leq a \) and all coefficient vectors \( c \) satisfying Hyp(\( \Upsilon \)) and \( \|c\|_* \leq A \), for some given \( A \in (0, \infty) \).

**Proof.** In the subsequent five steps of this proof we fix \( t > 0 \), that plays the role of \( t_0 \) in Lem. 4.10. The weight function \( (\bar{\Upsilon}_s)_{s \in [0,t]} \) will either be \( (\bar{\Upsilon}_s)_{s \in [0,t]} \) or \( (\bar{\Upsilon}_{t-s}^{-1})_{s \in [0,t]} \). We have to consider these two choices because of the duality arguments used below. We shall make use of Rem. 4.11 without further notice. Moreover, we set \( \bar{\Upsilon}_s := \bar{\Upsilon}_{t-s}^{-1}, s \in [0,t] \). One subtlety appears when \( \Upsilon \) is given by Line 4 of Table 1: If \( t > t_* \), then \( \bar{\Upsilon} \) (resp. \( \bar{\Upsilon} \)) is chosen to be equal to \( (e^{\delta t_s s(1+d\Gamma(pi))/2})_{s \in [0,t]} \) with \( \delta_s := \delta t_s/t \).

**Step 1.** Let \( p \in [2, \infty] \). Pick \( \tau \in [0,t] \) and some measurable \( \Psi: \mathbb{R}^n \to D(\bar{\Upsilon}_{t-s}) \) with \( \Psi \in L^p(\mathbb{R}^n, \mathcal{H}) \) and \( \bar{\Upsilon}_{t-s}^{-1} \Psi \in L^p(\mathbb{R}^n, \mathcal{H}) \). For all \( x \in \mathbb{R}^n \) and \( \phi \in D(\bar{\Upsilon}_t) \), we
may then write
\[
\langle \hat{\Upsilon}_t \phi | (T^V_t e^{-F \hat{\Upsilon}_{t-1}} \Psi)(x) \rangle
\]
\[
= \mathbb{E}[e^{-\int_0^t \nu(V^\ast_s)ds} \langle \hat{\Upsilon}_t \nu^0_s[B^\ast] \hat{\Upsilon}_0^{-1} \phi \rangle \Psi(B^\ast_x)e^{-F(B^\ast_x)}],
\]
where we took Lem. 4.10 into account. An analogous formula holds true with \(T^V_t\) replaced by \((T^V_s - T^V_t)\) on the left hand side and \(e^{-\int_0^t \nu(V^\ast_s)ds} - e^{-\int_0^t \nu(B^\ast_s)ds}\) on the right hand side. Define \(q_1 = 1, (1, \infty)\) by \(q_1 = 1 - \frac{1}{p - 1} - 4^{-1} - 8^{-1}\), so that \(q_1 \leq 8\), and let us agree that, for \(p = \infty\), the symbol \(\mathbb{E}[\|\Psi(B^\ast_x)||^p]^{1/p}\) should be read as \(\|\Psi\|_\infty\) in what follows. Then the above remarks permit to get
\[
\sup_{s \in (\{1\})} e^{F(x)} | \langle \hat{\Upsilon}_s \phi | (T^V_s - T^V_t) e^{-F \hat{\Upsilon}_{s-1}} \Psi)(x) \rangle |
\]
\[
\leq \sup_{s \in \mathbb{R}^n} \mathbb{E}[e^{-8 \int_0^s \nu(V^\ast_s)ds}]^{1/8}
\]
(5.4) \[\cdot \mathbb{E}[e^{q_1 |B\ast|^{1/q_1}} \sup_{y \in \mathbb{R}^n} \sup_{T^V_s \in (\{1\})} \mathbb{E}[\|\hat{\Upsilon}_s \nu^{0}_s[B^\ast] \hat{\Upsilon}_0^{-1} \phi \|^{4}]^{1/4} \mathbb{E}[\|\Psi(B^\ast_x)||^p]^{1/p},\]
and, for all \(K \in \mathbb{R}^n, x \in K,\) and \(\tau_2 \geq t,\)
\[
\sup_{s \in (\{1\})} e^{F(x)} | \langle \hat{\Upsilon}_s \phi | (T^V_s - T^V_t) e^{-F \hat{\Upsilon}_{s-1}} \Psi)(x) \rangle |
\]
\[
\leq \sup_{s \in \mathbb{R}^n} \mathbb{E}[e^{-\int_0^t \nu(V^\ast_s)ds} - e^{-\int_0^t \nu(B^\ast_s)ds}]^{1/8}
\]
(5.5) \[\cdot \mathbb{E}[e^{q_1 |B\ast|^{1/q_1}} \sup_{y \in \mathbb{R}^n} \sup_{T^V_s \in (\{1\})} \mathbb{E}[\|\hat{\Upsilon}_s \nu^{0}_s[B^\ast] \hat{\Upsilon}_0^{-1} \phi \|^{4}]^{1/4} \mathbb{E}[\|\Psi(B^\ast_x)||^p]^{1/p},\]
By virtue of Lem. 4.10 (see also Rem. 4.11) and Hyp(\(\Upsilon\)),
\[
\sup_{y \in \mathbb{R}^n} \mathbb{E}[\|\hat{\Upsilon}_s \nu^{0}_s[B^\ast] \hat{\Upsilon}_0^{-1} \phi \|^{4}]^{1/4} \leq 7 e^{4(1 + \|\nu\|^2_{L^\infty})^\tau} \|\phi\|, \quad \phi \in \mathcal{H}, \tau \in [0, t].
\]
Furthermore,
\[
\mathbb{E}[e^{q_1 |B\ast|}] = \int_{\mathbb{R}^n} p_1(y, 0) e^{q_1 |y|^{1/2}} dy \leq 2^{q_1/2} e^{q_1 a^2}, \quad q \geq 0.
\]
In the case \(p = q = \infty\), the term appearing in the last lines of (5.4) and (5.5) is thus bounded by \(7 \cdot 2^{q_1/2} e^{8(1 + \|\nu\|^2_{L^\infty})^\tau} \|\Psi\|_\infty\). In the case \(2 \leq p < q < \infty\), we integrate the \(q\)-th power of (5.4) and (5.5) with respect to \(x\) and use that
\[
\int_{\mathbb{R}^n} \mathbb{E}[\|\Psi(B^\ast_x)||^p]^{1/p} dx = \|\tau^{\Delta/2} \|\Psi(\cdot)||^p\|_{\tilde{p}/p}
\]
(5.7) \[
\leq c_{\nu,p,q}^{\tau} e^{q_1 (1 - \nu/2) p} \|\Psi(\cdot)||^p_{\tilde{p}/p} = c_{\nu,p,q}^{\tau} e^{q_1 (1 - \nu/2) p} \|\Psi||^p_{\tilde{p}/p}, \tau > 0,
\]
with a constant, \(c_{\nu,p,q}^{\tau} > 0\), depending \(\nu, p,\) and \(q\). The case \(2 \leq p < q = \infty\) is dealt with analogously, using
\[
\sup_{x \in \mathbb{R}^n} \mathbb{E}[\|\Psi(B^\ast_x)||^p] = \sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} p_\tau(x, y) \Psi(y) dy \|\Psi||^p_{\tilde{p}}, \tau > 0.
\]
We may thus conclude that \((T^V_x e^{-F} \hat{T}^{-1}_{t-\tau} \Psi)(x) \in \mathcal{D}(\hat{T})\), for every \(x \in \mathbb{R}^\nu\), and obtain the following bounds, for all \(2 \leq p \leq q \leq \infty\),

\[
(5.9) \quad \left\| e^{F} \hat{T} T^V_x \hat{T}^{-1}_{t-\tau} e^{-F} \right\|_{p,q} \leq c''_{p,q} \frac{e^{8(1+\|\xi\|_2^2 + \alpha^2)\tau}}{\tau^{(1/p - 1/q)/2}} \sup_{x \in \mathbb{R}^\nu} E \left[ e^{-8 \int_0^\tau V(B^*_x) \, ds} \right]^{1/8},
\]

\[
\left\| 1_{R} e^{F} \hat{T} (T^{V_n}_\tau - T^V_x) \hat{T}^{-1}_{t-\tau} e^{-F} \right\|_{p,q} \leq c''_{p,q} \frac{e^{8(1+\|\xi\|_2^2 + \alpha^2)\tau}}{\tau^{(1/p - 1/q)/2}} \sup_{x \in \mathbb{R}^\nu} E \left[ e^{-8 \int_0^\tau V(B^*_x) \, ds} \right]^{1/8}.
\]

\[
(5.10)
\]

\[
\]

**Step 2.** Again, let \(2 \leq p \leq q \leq \infty\) and \(\tau \in (0,t]\). By their definition, we may replace \(\hat{T}\) by \(\hat{T}\) in (5.9) and (5.10). Then, in the analogue of (5.9), the expression \(e^{F} \hat{T} T^V_x \hat{T}^{-1}_{t-\tau} e^{-F}\) is a densely defined operator in \(L^p(\mathbb{R}^\nu, \mathcal{H})\) with domain

\[
\{ \Psi \in \mathcal{L}^p(\mathbb{R}^\nu, \mathcal{H}) : \Psi \in \mathcal{D}(\hat{T}^{-1}_{t-\tau}) \text{ a.e., } \hat{T}^{-1}_{t-\tau} \Psi \in \mathcal{L}^p(\mathbb{R}^\nu, \mathcal{H}) \};
\]

recall that \(T^V\) was well-defined on \(e^{-F} L^p(\mathbb{R}^\nu, \mathcal{H})\). Hence, (5.9) can be rephrased by saying that \(\hat{T} T^V_x \hat{T}^{-1}_{t-\tau}\) is a densely defined bounded operator between the weighted spaces \(L^p(\mathbb{R}^\nu, \mathcal{H}; e^{pF} \, dx)\) and \(L^q(\mathbb{R}^\nu, \mathcal{H}; e^{qF} \, dx)\) with domain

\[
\mathcal{X} := \{ \Psi \in \mathcal{L}^p(\mathbb{R}^\nu, \mathcal{H}; e^{pF} \, dx) : \Psi \in \mathcal{D}(\hat{T}^{-1}_{t-\tau}) \text{ a.e., } \hat{T}^{-1}_{t-\tau} \Psi \in \mathcal{L}^q(\mathbb{R}^\nu, \mathcal{H}; e^{qF} \, dx) \},
\]

whose norm is bounded by the right hand side of (5.9); recall the notation introduced for certain restrictions of \(T^V_x\) in the paragraph preceding Cor. 3.3. For all \(\Phi \in \mathcal{X}\) and \(\Phi \in \mathcal{L}^q(\mathbb{R}^\nu, \mathcal{H}; e^{-qF} \, dx)\) such that \(\Phi \in \mathcal{D}(\hat{T}_\tau)\) a.e. and \(T_\tau \Phi \in \mathcal{L}^p(\mathbb{R}^\nu, \mathcal{H}; e^{-pF} \, dx)\), we further infer from Cor. 3.3 and Step 1 that

\[
\left| \int_{\mathbb{R}^\nu} \left< (T^{V:(q',p')} - T^{V}_\tau) \Phi(x), \hat{T}^{-1}_{t-\tau} \Psi(x) \right> \, dx \right|
\]

\[
= \left| \int_{\mathbb{R}^\nu} \left< \Phi(x), (\hat{T} T^{V:(p,q)} \hat{T}^{-1}_{t-\tau} \Psi)(x) \right> \, dx \right|
\]

\[
\leq \left\| e^{F} \hat{T} T^V_x \hat{T}^{-1}_{t-\tau} e^{-F} \right\|_{p,q} \left\| \Phi \right\|_{L^p(\mathbb{R}^\nu, \mathcal{H}; e^{-qF} \, dx)} \left\| \Psi \right\|_{L^q(\mathbb{R}^\nu, \mathcal{H}; e^{pF} \, dx)}.
\]

Observing also that \(\hat{T}_\tau = \hat{T}^{-1}_{t-\tau}\), considered as a densely defined operator in \(L^p(\mathbb{R}^\nu, \mathcal{H}; e^{-pF} \, dx)\), is the adjoint of itself, considered as a densely defined operator in \(L^p(\mathbb{R}^\nu, \mathcal{H}; e^{pF} \, dx)\), we conclude that \(T^{V:(q',p')} - T^V_x \Phi\) is in the domain of \(\hat{T}_\tau\), acting in \(L^p(\mathbb{R}^\nu, \mathcal{H}; e^{-pF} \, dx)\), and that

\[
\left\| e^{F} \hat{T} T^V_x T^V_0 \psi_{q',p'} \right\| \leq c''_{p,q} \frac{e^{8(1+\|\xi\|_2^2 + \alpha^2)\tau}}{\tau^{(1/p - 1/q)/2}} \sup_{x \in \mathbb{R}^\nu} E \left[ e^{-8 \int_0^\tau V(B^*_x) \, ds} \right]^{1/8},
\]

where \(1/p - 1/q = 1/q' - 1/p'\). Altogether this proves (5.1) in the case \(1 \leq p \leq q \leq 2\).

**Step 3.** Let \(1 \leq p < 2 < q \leq \infty\). Then the semi-group property and the mapping properties of \(T^V\) established in Steps 1 and 2 imply

\[
\left\| e^{F} \hat{T} T^V_x \hat{T}^{-1}_{t-\tau} e^{-F} \right\|_{p,q} \leq \left\| e^{F} \hat{T} T^V_x \hat{T}^{-1}_{t-\tau} e^{-F} \right\|_{2,q} \left\| e^{F} \hat{T} T^V_x \hat{T}^{-1}_{t-\tau} e^{-F} \right\|_{p,2},
\]

Since Steps 1 and 2 also yield bounds on the first and second factor on the right hand side, respectively, this completes the proof of Part (1).
Step 4. Let \( p \in [1, 2) \) and \( q \in [2, \infty) \). Analogously as in Step 3 we then obtain

\[
\|1 K e^{F} \Upsilon_t(T_{t/V}^{\nu} - T_{t/V}^{\nu}) \Upsilon_0^{-1} e^{-F}\|_{p,q} \\
\leq \|1 K e^{F} \Upsilon_t(T_{t/V}^{\nu} - T_{t/V}^{\nu}) \Upsilon_0^{-1} e^{-F}\|_{2,q} \|e^{F} \Upsilon_{t/2} T_{t/2}^{\nu} \Upsilon_0^{-1} e^{-F}\|_{p,2},
\]

(5.11)

where \( D(x) := \text{dist}(x, K), x \in \mathbb{R}^\nu \), is globally Lipschitz continuous. According to the first two steps, the term in the second line of (5.11) can be estimated using (5.9) and (5.10). The duality arguments of Step 2 and the trivial bound \( \|e^{-D}\|_{2,2} \leq 1 \) further imply

\[
\|e^{-D} \Upsilon_{t/2} (T_{t/V}^{\nu} - T_{t/V}^{\nu}) \Upsilon_0^{-1} e^{-F}\|_{2,p'} \leq \|1 K e^{D} e^{-D} \Upsilon_t (T_{t/V}^{\nu} - T_{t/V}^{\nu}) \Upsilon_0^{-1} e^{-D}\|_{2,p'} \\
+ e^{-R} \|e^{D} T_{t/V}^{\nu} - T_{t/V}^{\nu} \Upsilon_0^{-1} e^{-D}\|_{2,p'},
\]

(5.12)

where \( K_R := D^{-1}([0, R]), R \geq 1 \), so that \( \|e^{-D} 1_{K_R}\|_{p',p'} \leq e^{-R} \). By (2.37) and (5.9), the norm in the second line of (5.12) is bounded uniformly in \( n \in \mathbb{N} \), all \( F \) with Lipschitz constant \( \leq a \), and all coefficient vectors \( c \) with \( \|c\|_{*, \infty} \leq A \). Therefore, the whole term in the second line can be made arbitrarily small by choosing \( R \) large. Moreover, (5.10) applies to the norm in the first line of the right hand side of (5.12) after we have fixed some large value of \( R \). Finally, we note that the first norm in the last line of (5.11) is bounded uniformly in \( n \in \mathbb{N} \) by (2.37) and (5.9) as well.

Step 5. We have already proved (5.2) in all cases except for \( 1 \leq p \leq q < 2 \). By duality, we have also proved (5.3) in all cases except for \( 2 < p \leq q \leq \infty \).

We will now treat (5.3) in these missing cases. So, let \( 2 < p \leq q \leq \infty \). Define \( K_R \) and \( D \) as in Step 4. On account of the already proven cases of (5.2) and (5.3), it is enough to show that

\[
\sup_{n \in \mathbb{N}} \sup_{t \in [\tau_1, \tau_2]} \|e^{-D} 1_{K_R} e^{D} \Upsilon_t (T_{t/V}^{\nu} - T_{t/V}^{\nu}) \Upsilon_0^{-1} e^{-D} 1_{K}\|_{p,q} \xrightarrow{R \to \infty} 0,
\]

where the convergence should in addition be uniform in \( F \) and \( c \) as in the last assertion of Part (2). This is, however, obvious from \( \|e^{-D} 1_{K_R}\| \leq e^{-R} \), (2.37), and (5.1). Again by duality, these arguments can also be used to cover (5.2) in the remaining cases \( 1 \leq p \leq q < 2 \). \( \square \)

Corollary 5.3. Let \( V \in \mathcal{K}_+(\mathbb{R}^\nu) \). Let \( (\Upsilon_t)_{t \geq 0} \) and \( \| \cdot \|_\ast \) be given by one of the lines in Table 1 and assume that \( c \) fulfills Hyp(\( \Upsilon \)) given by the same line. Then there exist a universal constant \( c > 0 \) such that \( \text{Ran}(T_{t/V}^V(x, y)) \subset D(\Upsilon_{t/V}) \) with

\[
\|\Upsilon_{t/V} T_{t/V}^V(x, y)\| \leq c_p t^{-r/2} e^{c \|e\|^2} \|x - y\|^2 / c t \sup_{z \in \mathbb{R}^\nu} \mathbb{E}[e^{8 \int_0^1 V_\ast(B_s^z)ds}]^{1/8},
\]

for all \( t > 0 \) and \( x, y \in \mathbb{R}^\nu \).
Proof. Set $F_a(z) := a|z - x|$, $z \in \mathbb{R}^r$, for some $a > 0$. By virtue of Prop. 3.3 we can then write, for any $s \in (0, t)$,

$$
\|Y_s T_t^V (x, y)\| = \sup_{\psi = 1} \|(e^{-F_a} Y_s T_t^V (T_{t-s}^V , y))\| (x) \\
\leq \|e^{-F_a} Y_s T_t^V e F_a\|_{1, \infty} \int_{\mathbb{R}^r} e^{-F_a(z)} \|T_{t-s}(z, y)\| \, dz \\
\leq c_{p, s} \frac{1}{2} e^{|y|^2 \|z\|_{\infty}^2 + 2} \sup_{z \in \mathbb{R}^r} E \left[ e^{-2 s} \int_{\mathbb{R}^r} e^{-F_a(z)} \|T_{t-s}(z, y)\| \, dz \right],
$$

where we used (5.1) in the last estimate. Furthermore, (3.3) yields

$$
\int_{\mathbb{R}^r} e^{-F_a(z)} \|T_{t-s}(z, y)\| \, dz \\
= \int_{\mathbb{R}^r} e^{-a|y - z| + \|y\|_{\infty} (t-s)} p_{t-s}(y, z) \mathbb{E} \left[ e^{-f_{1-r} V(b_r^{i-r} y)} \right] \, dz \\
\leq e^{|y|_{\infty} \frac{1}{2}} \left( \int_{\mathbb{R}^r} e^{-2 a|z|} \, p_{t-s}(y, z) \mathbb{E} \left[ e^{-f_{1-r} V(b_r^{i-r} y)} \right] \, dz \right)^{\frac{1}{2}} \\
\leq e^{|y|_{\infty} \frac{1}{2}} \left( e^{|y|_{\infty} \frac{1}{2}} \int_{\mathbb{R}^r} e^{-2 a|z|} \, \mathbb{E} \left[ e^{-f_{1-r} V(b_r^{i-r} y)} \right] \, dz \right)^{\frac{1}{2}}.
$$

On account of $S_{t-s}^{2V}(z, y) = S_{t-s}^{2V}(y, z)$ and (2.31) we further observe that

$$
\int_{\mathbb{R}^r} S_{t-s}^{2V}(z, y) \, dz = (S_{t-s}^{2V}(1)) (y) \leq \mathbb{E} \left[ e^{8 f_{1-r} V(b_r^{i-r} y)} \right]^{\frac{1}{4}}.
$$

Choosing $a := |x - y|/2(t-s)$, we finally have

$$
2a|z| + \frac{|x - y|}{2(t-s)} \geq \frac{|x| + |y|}{2} + \frac{(|z| - |x - y|)^2}{2(t-s)} = \frac{|z|^2}{2(t-s)} + \frac{|x - y|^2}{2(t-s)},
$$

Inserting this bound and choosing $s := t/8$ we arrive at the asserted estimate. □

Corollary 5.4. Let $V \in \mathcal{K}_+ (\mathbb{R}^r)$, let $V_n \in \mathcal{K}(\mathbb{R}^r)$, $n \in \mathbb{N}$, satisfy (2.31) and (2.35), and let $F : \mathbb{R}^r \to \mathbb{R}$ be globally Lipschitz continuous. Let $(Y_t)_{t \geq 0}$ and $\| \cdot \|_*$ be given by one of the lines in Table 1 and assume that $c$ fulfills Hyp(Y) given by the same line. Furthermore, let $p \in [1, \infty)$ be finite and $q \in [p, \infty]$. Then

$$
\lim_{n \to \infty} \sup_{t \in [\tau_1, \tau_2]} \left\| e^T Y_t (T_{t-s}^V - T_0^V) Y_0^{-1} e^{-F} \Psi \right\| = 0,
$$

for all $\tau_2 > \tau_1 > 0$ and $\Psi \in L^p(\mathbb{R}^r)$. In the case $q = p$, the value $\tau_1 = 0$ is allowed for in (5.13) as well. The convergence (5.13) is in fact uniform in all Lipschitz continuous $F$ with Lipschitz constant $\leq a$ and all coefficient vectors $c$ satisfying Hyp(Y) and $\|c\|_{*, \infty} \leq A$, for some given $a, A \in (0, \infty)$. 

Proof. Set $B_R := \{ x \in \mathbb{R}^r : |x| \leq R \}$, $R > 0$. Then $\|1_{B_R} \Psi \|_p \to 0$, $R \to \infty$, and the claim follows easily from (5.1) and (5.3) with $K = B_R$. □

6. Continuity in the coupling functions

In the following theorem we complement the discussion of the previous section by considering the behavior of the semi-group under perturbations of the coefficient vector $c$. Thanks to its last assertion, the following theorem can be combined with Thm. 5.2.2, where the convergences are uniform in coefficient vectors as in the statement of Thm. 6.1.

Theorem 6.1. Let $V \in K_\pm(\mathbb{R}^\nu)$, $F : \mathbb{R}^\nu \to \mathbb{R}$ be globally Lipschitz continuous with Lipschitz constant $a \geq 0$, and $1 \leq p \leq q \leq \infty$. Let $(\tilde{Y}_t)_{t \geq 0}$ and $\| \cdot \|$ be given by one of the lines in Table 1. Assume that $c = (G, q, F)$ and $c_n = (G_n, q_n, F_n)$, $n \in \mathbb{N}$, are coefficient vectors satisfying Hyp. 2.7 with the same $\omega$ and the same conjugation $C$ and Hyp. (T). Assume further that $\|c_n\|, \|c_n \cdot \| \leq A$, for some $A \in [1, \infty)$, all $x \in \mathbb{R}^\nu$, and all $n \in \mathbb{N}$, and that
\begin{equation}
\lim_{n \to \infty} \sup_{x \in K} \|c_n x - c x\| = 0,
\end{equation}
for all compact $K \subset \mathbb{R}^\nu$. Let $(T^V_{t^n})_{t \geq 0}$ denote the semi-group defined by means of $c_n$. Then
\begin{align}
(6.1) & \quad \lim_{n \to \infty} \sup_{t \in [\tau_1, \tau_2]} \left\| I^e_T Y_t (T^V_t - T^V_{t^n}) \right\|_{p, q} = 0, \\
(6.2) & \quad \lim_{n \to \infty} \sup_{t \in [\tau_1, \tau_2]} \left\| I^e_T Y_t (\tilde{Y}_t - \tilde{Y}_t^n) \right\|_{p, q} = 0, \\
(6.3) & \quad \lim_{n \to \infty} \sup_{t \in [\tau_1, \tau_2]} \left\| I^e_T Y_t (\tilde{Y}_t - \tilde{Y}_t^n) \right\|_{p, q} = 0,
\end{align}
for all $1 \leq p \leq q \leq \infty$, $0 < \tau_1 \leq \tau_2$, and compact $K \subset \mathbb{R}^\nu$. If $K$ can be replaced by $\mathbb{R}^\nu$ in (6.1), then it may also be replaced by $\mathbb{R}^\nu$ in (6.2) and (6.3). If $p = q$, then the choice $\tau_1 = 0$ is allowed for in (6.2) and (6.3) as well. In all cases, the convergences in (6.2) and (6.3) are uniform in all globally Lipschitz continuous $F : \mathbb{R}^\nu \to \mathbb{R}$ with Lipschitz constant $\leq a$ and all $V \in K_\pm(\mathbb{R}^\nu)$ satisfying sup$_{t \in [\tau_1, \tau_2]} \sup_{x \in \mathbb{R}^\nu} E[\|e^{-\int_0^t V(B_s^\nu)ds}\|] \leq D$, for some fixed $a$, $D \in (0, \infty)$.

Proof. We shall only explain a substitute of Step 1 of the proof of Thm. 5.2 as the remaining steps of this proof are completely analogous to the proof of (5.2) and (5.3). To this end, we let $(\nu^0_t(B^\nu))_{t \geq 0}$, $x \in \mathbb{R}^\nu$, denote the solution processes of Thm. 2.6 defined by means of $c_n$. Again, we fix $t > 0$ and define $(\tilde{Y}_s)_{s \in [0, t]}$ and $\tilde{Y}_s := \tilde{Y}_s^{-1}$, $s \in [0, t]$, in the same way as in the beginning of the proof of Thm. 5.2.

Assume first that $p \geq 2$. Define $q_1 > 0$ by $q_1^{-1} = 1 - p^{-1} - q^{-1} - 1^{-1}$. For all $x \in K \subset \mathbb{R}^\nu$ and $\Psi \in L^p(\mathbb{R}^\nu, \Omega)$ with $\Psi(z) \in D(\tilde{Y}_t^{-1})$, a.e. $z$, and $\tilde{Y}_t^{-1} \Psi \in L^p(\mathbb{R}^\nu, \Omega)$, we then obtain
\begin{align}
\|e^{F \tilde{Y}_t (T^V_t - T^V_{t^n})} \tilde{Y}_t^{-1} e^{-F} \Psi(x)\| & \leq \sup_{\phi \in D(\tilde{Y}_t^{-1})} E[|e^{F(x) - e^{F(B^\nu_t^s) - \int_0^t V(B^\nu_s^\nu)ds}}||\tilde{Y}_t (\nu^{0,n}_t[B^\nu] - \nu^{0,n}_t[B^\nu])\tilde{Y}_t^{-1} \phi||\Psi(B^\nu_t)||]| \\
& \leq \sup_{\phi \in D(\tilde{Y}_t^{-1})} E[\|\tilde{Y}_t (\nu^{0,n}_t[B^\nu] - \nu^{0,n}_t[B^\nu])\tilde{Y}_t^{-1} \phi||\Psi(B^\nu_t)||]|^{1/4} \\
& \cdot E[e^{q_1|B_t^\nu|}]^{1/4} \sup_{\Psi \in \mathbb{R}^\nu} E[|e^{-\int_0^t V(B^\nu_s^\nu)ds}||\Psi(B^\nu_t)||]^{|p|/4},
\end{align}
with $E[\|\Psi(B^\nu_t)||]^{|p|/4} := ||\Psi||_\infty$ in the case $p = \infty$. Applying Lem. 4.18 with $\tilde{c} := c_n$, we further obtain
\begin{align}
E[\|\tilde{Y}_t (\nu^{0,n}_t[B^\nu] - \nu^{0,n}_t[B^\nu])\tilde{Y}_t^{-1} \phi||^4] & \leq c A^4 e^{A t} E[\sup_{s \leq t} ||c_n[B^\nu] - c[B^\nu]||^4]^{1/2}||\phi||^4,
\end{align}
for all $y \in \mathbb{R}^\nu$, $\phi \in D(\tilde{Y}_t^{-1})$, and some universal constant $c > 0$. Since $\{B^\nu_t(\gamma) : \gamma \in \Omega \}$ is compact, for all $\gamma \in \Omega$ and all compact $K \subset \mathbb{R}^\nu$, the
Furthermore, let
\[ \lim_{n \to \infty} \sup_{y \in K} \mathbb{E} \left[ \sup_{s \leq \tau} \| c_n B^y - c B^y \|_s^4 \right] = 0, \quad \tau > 0. \]

Therefore, the term in the third line of (6.4) converges to zero as \( n \to \infty \), locally uniformly in \( \tau \geq 0 \) and for all compact \( K \). If \( K \) can be replaced by \( \mathbb{R}^\nu \) in (6.1), then it can be replaced by \( \mathbb{R}^\nu \) in (6.6) as well, which then shows that the term in the third line of (6.4) with \( K = \mathbb{R}^\nu \) tends to zero as \( n \to \infty \). We may now proceed along the lines of the proof of (5.2) and (5.3) to conclude. □

7. Strong continuity in the time parameter

Next, we consider the strong continuity with respect to \( t \geq 0 \) of our semi-groups in the \( L^p \)-spaces with a finite \( p \). The somewhat technical statement of Part (1) of the next theorem, which implies its second part, will be needed in the proof of Thm. 9.4 on the operator norm-continuity of the integral kernel. It is perhaps needless to recall that a set, \( M \), of functions from \( \mathbb{R}^\nu \) to \( \mathcal{H} \) is called uniformly equicontinuous on a subset \( Q \subset \mathbb{R}^\nu \), if

\[ \lim_{r \downarrow 0} \sup_{x \in Q} \sup_{|y - x| < r} \| \Psi(x) - \Psi(y) \| = 0. \]

In Part (2) of Thm. 7.1 we again use the notation for restrictions of \( T_t^V \) introduced in the paragraph preceding Cor. 4.5.

**Theorem 7.1.** Let \( V \in \mathcal{K}_\pm (\mathbb{R}^\nu) \) and \( p \in [1, \infty) \). Then the following holds:

1. Let \( \mathcal{M} \subset L^p(\mathbb{R}^\nu, \mathcal{H}) \cap L^\infty(\mathbb{R}^\nu, \mathcal{H}) \) be such that
   a. \( \mathcal{M} \) is uniformly equicontinuous on every compact subset of \( \mathbb{R}^\nu \);
   b. \( \sup_{\Psi \in \mathcal{M}} \| \Psi \|_\infty < \infty \) and, for every \( \Psi \in \mathcal{M} \), there exists \( \alpha_\Psi \in \mathbb{R}^\nu \) such that \( \sup_{\Psi \in \mathcal{M}} \| \Psi - \Psi_R \|_p \to 0 \), \( R \to \infty \), where \( \Psi_R(x) := 1_{|x - \alpha_\Psi| < R} \Psi(x) \); if \( V \notin K(\mathbb{R}^\nu) \), then we assume that the points \( \alpha_\Psi \), \( \Psi \in \mathcal{M} \), are contained in a compact set;
   c. for every \( \Psi \in \mathcal{M} \), we have \( \Psi(x) \in \mathcal{D}(d\Gamma(\omega)^{1/2}) \), a.e. \( x \in \mathbb{R}^\nu \), and \( \sup_{\Psi \in \mathcal{M}} \| f_\Psi \|_p < \infty \), where \( f_\Psi(x) := \| (1 + d\Gamma(\omega))^{1/2} \Psi(x) \| \).

Furthermore, let \( F : \mathbb{R}^\nu \to \mathbb{R} \) be globally Lipschitz continuous and set \( F_\Psi(x) := F(x - \alpha_\Psi), x \in \mathbb{R}^\nu \). Then

\[ \lim_{t \downarrow 0} \sup_{\Psi \in \mathcal{M}} \| e^{-F_\Psi (T_t^V - \mathbb{1})} e^{F_\Psi} \Psi \|_p = 0. \]

2. The semi-group \( (T_t^{V_{(p,p)}}, 0)_{t \geq 0} \) is strongly continuous.

**Proof.** (1): Let \( \chi \in C(\mathbb{R}, [0,1]) \) satisfy \( \chi_0 = 1 \) on \( (-\infty, 0) \) and \( \chi = 0 \) on \([1, \infty) \) and set \( \Psi_R(x) := \chi(|x - \alpha_\Psi| - R) \Psi(x), x \in \mathbb{R}^\nu, R \geq 1 \). On account of

\[ \sup_{\Psi \in \mathcal{M}} \sup_{0 \leq t \leq 1} \| e^{-F_\Psi} T_t^V e^{F_\Psi} \|_{p,p} < \infty, \]

Condition (b), and \( \| \Psi - \Psi_R \|_p \leq \| \Psi - \Psi_R \|_p \) it suffices to show that

\[ \lim_{t \downarrow 0} \sup_{\Psi \in \mathcal{M}} \| e^{-F_\Psi (T_t^V - \mathbb{1})} e^{F_\Psi} \Psi_R \|_p = 0, \]

\[ \Psi \in \mathcal{M}, \| \Psi \|_\infty < \infty. \]
for every \( R \geq 2 \). To do so we estimate, using \( \| \psi(t) \|_{L^p} \leq e^{\| \psi \|_{\infty} t} \),
\[
\| e^{-F_\psi(T^V_t - 1)} e^{F_\psi} \Psi_R \|_p^p = \int_{\mathbb{R}^p} \mathbb{E} \left[ \psi_0^t(\mathbb{B}^\psi)^* \left( e^{- \int_0^t V(\mathbb{B}^\psi) ds - F_\psi(x) + F_\psi(\mathbb{B}^\psi)} \Psi_R(\mathbb{B}^\psi) - \Psi_R(x) \right) + (\psi_0^t(\mathbb{B}^\psi)^* - 1) \Psi_R(x) \right] d\mathbf{x}
\] 
\[ \leq 2^{p-1} e^{\| \psi \|_{\infty} t} (I_{1,1}(\Psi, R, t) + I_{1,2}(\Psi, R, t)) + 2^{p-1} I_2(\Psi, t), \]
for all \( \Psi \in \mathcal{M} \) and \( R \geq 2 \), where
\[
I_{1,1}(\Psi, R, t) := \int_{|x - a| < 2R} \mathbb{E} \left[ \| e^{- \int_0^t V(\mathbb{B}^\psi) ds - F_\psi(x) + F_\psi(\mathbb{B}^\psi)} \Psi_R(\mathbb{B}^\psi) - \Psi_R(x) \|_p^p \right] d\mathbf{x},
\]
\[
I_{1,2}(\Psi, R, t) := \int_{|x - a| \geq 2R} \mathbb{E} \left[ \| e^{- \int_0^t V(\mathbb{B}^\psi) ds} \Psi_R'(\mathbb{B}^\psi) \|_p^p \right] d\mathbf{x},
\]
\[
I_2(\Psi, t) := \int_{\mathbb{R}^p} \mathbb{E} \left[ (\psi_0^0(\mathbb{B}^\psi)^* - 1) \Psi(x) \right] d\mathbf{x}.
\]
The first integral can be estimated as
\[
\sup_{\Psi \in \mathcal{M}} I_{1,1}(\Psi, R, t) \leq 3^{p-1} C R^{2p} \sup_{\Psi \in \mathcal{M}} \sup_{|x - a| < 2R} \mathbb{E} \left[ \| e^{F_\psi(\mathbb{B}^\psi)|F_\psi(x) - 1|} e^{- \int_0^t V(\mathbb{B}^\psi) ds} \right] \| \Psi \|_p^p
\] 
\[ + 3^{p-1} e_a R^{2p} \sup_{\Psi \in \mathcal{M}} \sup_{|x - a| \leq 2R} \mathbb{E} \left[ \| e^{- \int_0^t V(\mathbb{B}^\psi) ds - 1} \|_p \right] \| \Psi \|_p^p \]
(7.1) 
\[ + 3^{p-1} \sup_{\Psi \in \mathcal{M}} \int_{|x - a| < 2R} \int_{\mathbb{R}^p} p_t(x, y) \| \Psi_R'(y) - \Psi_R(x) \|_p^p d\mathbf{y} d\mathbf{x}. \]
Here the term in the first line of the right hand side goes to zero, as \( t \downarrow 0 \), due to (2.8) (with \( p \) replaced by \( 2p \)) and
\[
\mathbb{E} \left[ e^{F_\psi(\mathbb{B}^\psi)|F_\psi(x) - 1|} \right] \leq \mathbb{E} \left[ (\| a \|_{\mathcal{B}^\psi}^2)^{2p} \right] = \int_{\mathbb{R}^p} p_1(y, 0) (at^{-1/2})^2 |y|^{2p} d\mathbf{y},
\]
where \( a \) denotes a Lipschitz constant for \( F \). The term in the second line of the right hand side vanishes in the limit \( t \downarrow 0 \) by virtue of Lem. 2.12 and Condition (b). The term in the last line of (7.1) goes to zero as a consequence of Condition (a).
Furthermore,
\[
I_{1,2}(\Psi, R, t) \leq \| \Psi \|_{\infty} \sup_{x \in \mathbb{R}^p} \mathbb{E} \left[ e^{- \int_0^t V(\mathbb{B}^{\psi}_s) ds} \right]^{1/2} \| \psi(\mathbb{B}^{\psi}_s) \|_{\infty}^{1/2}
\] 
\[ \cdot \int_{|x| > 2R} \int_{|y| < R+1} p_t(x, y) d\mathbf{y} d\mathbf{x}, \]
so that \( \sup_{\Psi \in \mathcal{M}} I_{1,2}(\Psi, R, t) \to 0, t \downarrow 0 \), by Condition (b), (2.27), and (5.7). Here we also used that \( 2R - (R + 1) \geq 1 \), for \( R \geq 2 \). Finally,
\[
\| \Psi_0^t(\mathbb{B}^{\psi}) - 1 \|_p = \sup_{\| \psi \|_{\infty} = 1} \| (1 + d\Gamma(\omega))^{-1/2} \mathbb{E} (\psi_0^t(\mathbb{B}^{\psi}) - 1) \|^p_p f_\psi(x),
\]
where the supremum runs over normalized elements of \( \mathcal{H} \), which together with (4.01) leads to the bound \( I_2(\Psi, t) \leq O(t^{R/2}) \| f_\psi \|^p_p \). Invoking Condition (c) we see that \( \sup_{\Psi \in \mathcal{M}} I_2(\Psi, t) \to 0, t \downarrow 0 \).
(2) By the semi-group property it suffices to prove the strong continuity at zero only. Since $C_0(\mathbb{R}^\nu, \mathcal{H})$ is dense in $L^p(\mathbb{R}^\nu, \mathcal{H})$ with $p \in [1, \infty)$, $\|T^V_t\|_{p,p}$ is bounded uniformly in $t \in [0,1]$, and $\Psi_\varepsilon := (1 + \varepsilon d\Gamma(\omega))^{-1/2}\Psi$ \(\varepsilon \downarrow 0\), in $L^p(\mathbb{R}^\nu, \mathcal{H})$, it is even sufficient to show that $T^V_t \Psi_\varepsilon \rightarrow \Psi_\varepsilon$, $t \downarrow 0$, in $L^p(\mathbb{R}^\nu, \mathcal{H})$, for every $\Psi \in C_0(\mathbb{R}^\nu, \mathcal{H})$ and $\varepsilon > 0$. This follows, however, immediately from (2) upon choosing $\mathcal{M} = \{\Psi_\varepsilon\}$. □

8. Equicontinuity in the image of the semi-group

Our next theorem, implying that $T^V_t$ with $t > 0$ maps bounded sets in $L^p$ into equicontinuous ones, will be needed to prove the joint continuity of the integral kernel in Thm. 9.1 as well. In the succeeding corollary we combine the next theorem with Thms. 5.2 and 6.1.

Theorem 8.1. Let $V \in \mathcal{K}_\pm(\mathbb{R}^\nu)$, $0 < \tau_1 \leq \tau_2 < \infty$, $p \in [1, \infty)$, and let $(\Upsilon_t)_{t \geq 0}$ and $\| \cdot \|_*$ be given by either Line 2 or Line 4 of Table 1. Assume that $\mathbb{R}^\nu \ni x \mapsto c_x$ is bounded and uniformly continuous with respect to $\| \cdot \|_*$, then the following set of $\mathcal{H}$-valued functions,

\[ \{ \Upsilon_{1/2} T^V_{1/\tau} \Psi : \Psi \in L^p(\mathbb{R}^\nu, \mathcal{H}), \|\Psi\|_p \leq 1, t \in [\tau_1, \tau_2] \}, \]

is uniformly equicontinuous on every compact subset of $\mathbb{R}^\nu$. If $V \in \mathcal{K}(\mathbb{R}^\nu)$ and if $\mathbb{R}^\nu \ni x \mapsto c_x$ is bounded and uniformly continuous with respect to $\| \cdot \|_*$, then the set in (8.1) is uniformly equicontinuous on $\mathbb{R}^\nu$.

Proof. Step 1. First, we consider the case $\Upsilon = 1$. To this end we shall adapt an argument by Carmona [4]; see also Sect. 4 in [3]. On account of the semi-group property and (5.1) it suffices to treat the case $p = \infty$. On account of (5.1) and the well-known properties of the semi-group of the free Laplacian we know that, for fixed $0 < \tau < \tau_1$, the set

\[ \{ e^{\tau \Delta/2} T_{1-\tau}^V \Psi : \Psi \in L^\infty(\mathbb{R}^\nu, \mathcal{H}), \|\Psi\|_\infty \leq 1, t \in [\tau_1, \tau_2] \} \]

is uniformly equicontinuous on $\mathbb{R}^\nu$. Let $K \subset \mathbb{R}^\nu$ be compact, if $V \in \mathcal{K}_\pm(\mathbb{R}^\nu) \setminus \mathcal{K}(\mathbb{R}^\nu)$, and $K = \mathbb{R}^\nu$, if $V \in \mathcal{K}(\mathbb{R}^\nu)$. Then it suffices to show that

\[ \limsup_{\tau \downarrow 0} \sup_{t \in [\tau_1, \tau_2]} \sup_{x \in K} (\|D_{\tau, x} \Psi(x)\|) = 0, \]

where

\[ D_{\tau, x} := e^{\tau \Delta/2} T_{1-\tau}^V - T^V_\tau = (e^{\tau \Delta/2} - T^V_\tau) T^V_\tau. \]

Again by (5.1) we know that $\sup_{0 \leq \tau \leq \tau_1/2} \sup_{t \in [\tau_1, \tau_2]} \|e^{\theta \tau/2} T_{1-\tau}^V \|_{\infty, \infty} < \infty$, where $\theta := 1 + \text{d}\Gamma(\omega)$, and it remains to show that

\[ \limsup_{\tau \downarrow 0} \sup_{x \in K} (\|E[(\tau/2 - \theta) / e^{-\theta} \Psi(B^x_\tau)]\|) = 0, \]

for every compact $K \subset \mathbb{R}^\nu$. The convergence (8.2) follows, however, from

\[ \|E[e^{\theta \tau/2} (1 - e^{-f_\tau^0 V(B^x_\tau))} e^{-\theta/2} \Psi(B^x_\tau)]\| \leq e^{\|\Psi\|_\infty \tau} E[(1 - e^{-f_\tau^0 V(B^x_\tau))}) \|\Psi\|_\infty \]

and
in combination with (2.29) and from
\[
\left\| \mathbb{E} \left[ (\mathbb{W}_x^\nu [B^x]^\ast - 1) \theta^{-1/2} \Psi(B^x) \right] \right\| = \sup_{\|\phi\|=1} \left| \mathbb{E} \left[ \langle \theta^{-1/2} (\mathbb{W}_x^\nu [B^x] - 1) \phi \rangle \Psi(B^x) \right] \right|
\leq \sup_{\|\phi\|=1} \mathbb{E} \left[ \|\theta^{-1/2} (\mathbb{W}_x^\nu [B^x] - 1) \phi\| \right] \|\Psi\|_{\infty}
\]
in combination with (4.61). In the case \( \Upsilon_t = 1 \), these remarks already prove the theorem.

**Step 2.** Next, we consider arbitrary \( \Upsilon \) as in the statement but restrict our attention to \( V \in C_0(\mathbb{R}^n) \). Let \( \mathcal{E} \) be any uniformly bounded set of functions from \( \mathbb{R}^n \) to \( \mathcal{H} \) which is uniformly equicontinuous on \( \mathbb{R}^n \). Then, by the semi-group property, by Step 1, and by the obvious inclusion \( C_0(\mathbb{R}^n) \subset \mathcal{K}(\mathbb{R}^n) \), it suffices to show that the set \( \{ \Upsilon_t \} \) is uniformly equicontinuous as well.

To do so we fix \( t \in [\tau_1/2, \tau_2/2] \) and set \( \Upsilon_s := \Upsilon_{t-s}, s \in [0, t] \). Then we observe that, for all \( x, y \in \mathbb{R}^n \) and \( \Psi \in \mathcal{E} \),
\[
\left\| \Upsilon_t (T_t^\nu \Psi(x) - T_t^\nu \Psi(y)) \right\| \leq \left\| \Upsilon_t \left( \left( \mathbb{W}_x^\nu [B^x]^\ast - \mathbb{W}_y^\nu [B^y]^\ast \right) \Psi(B^x) \right) \right\|
+ \left\| \Upsilon_t \left( \left( \mathbb{W}_x^\nu [B^x]^\ast - \mathbb{W}_y^\nu [B^y]^\ast \right) \Psi(B^x) \right) \right\|
\leq \sup_{\phi \in \mathcal{D}(\mathcal{E})} \mathbb{E} \left[ \|\mathbb{W}_x^\nu [B^x]^\ast - \mathbb{W}_y^\nu [B^y]^\ast\| \Psi(B^x) \right] \sup_{x, y \in \mathbb{R}^n} \|\Psi(x) - \Psi(y)\|
(8.3)
\]
Next, we employ Lem. 4.18 with \( \hat{c} := c, V = \hat{V}, q = x, \) and \( \hat{q} = y \). This yields
\[
\sup_{s \in \mathcal{D}(\mathcal{E})} \mathbb{E} \left[ \left\| \mathbb{W}_x^\nu [B^x]^\ast - \mathbb{W}_x^{\nu, s} [B^y]^\ast \right\| \mathbb{W}_x^\nu [B^x]^\ast \Psi(x) \right] \leq c A^2 \pi^{1/2} \mathbb{E} \left[ \sup_{s \in [0, t]} f(s, x, y)^2 \right]^{1/2}
\]
for all \( x, y \in \mathbb{R}^n \), with a universal constant \( c > 0 \) and random variables
\[
f(s, x, y) := |V(B^x) - V(B^y)| + \|c_{B^x} - c_{B^y}\|_{\ast}
\]
Since \( \{B^x(\gamma) : s \in [0, t], \text{dist}(x, K) \leq 1 \} \subset \mathbb{R}^n \) is compact, for all \( \gamma \in \Omega \) and all compact \( K \subset \mathbb{R}^n \), \( V \) is uniformly continuous, and since \( x \mapsto c_x \) is uniformly continuous on every compact subset of \( \mathbb{R}^n \) with respect to the norm \( \| \cdot \|_{\ast} \), the dominated convergence theorem implies
\[
\lim_{r \to 0} \sup_{s, y \in \mathcal{D}(\mathcal{E})} \mathbb{E} \left[ \sup_{s \leq t} f(s, x, y)^2 \right] = 0, \quad \tau > 0.
(8.4)
\]
If \( x \mapsto c_x \) is uniformly continuous on \( \mathbb{R}^n \) with respect to \( \| \cdot \|_{\ast} \), then \( K \) can be replaced by \( \mathbb{R}^n \) in (8.4). Taking also (8.3), the uniform boundedness of \( \mathcal{E} \), and the uniform equicontinuity of \( \mathcal{E} \) on \( \mathbb{R}^n \), into account, we conclude that
\[
\lim_{r \to 0} \sup_{t \in [\tau_1/2, \tau_2/2]} \sup_{x \in \mathbb{R}^n, \|x - \gamma\| \leq \tau} \sup_{\Phi \in \mathcal{E}} \mathbb{E} \left[ \mathbb{E} \left[ \left\| \mathbb{W}_x^\nu [B^x] - \mathbb{W}_x^{\nu, s} [B^y] \Psi(x) \right\| \right] \right] = 0.
\]

**Step 3.** Now let \( V \in C_0(\mathbb{R}^n) \) be arbitrary and \( \Upsilon_{t/2} \) as in the statement. Let \( p \in [1, \infty] \). We pick \( V_n \in C_0(\mathbb{R}^n), n \in \mathbb{N} \), approximating \( V \) in the sense made precise in Lem. 2.13 (1). For each \( n \in \mathbb{N} \), we then know from Step 2 that the set \( \{ \Upsilon_{t/2} \} \) is uniformly equicontinuous on
\( \mathbb{R}^n \). On account of (8.2) (with \( q = \infty \)) and the boundedness of \( \mathcal{E} \) we further have
\[
(8.5) \quad \lim_{n \to \infty} \sup_{t \in [0, \tau_n]} \sup_{\| \mathbf{y} \| \leq 1} \| \mathbf{Y}_{t/2}(T^V_{t^\infty} \mathbf{y} - \Theta_t T^V_{t^\infty} \mathbf{y})(\mathbf{x}) \| = 0,
\]
for every compact \( K \subset \mathbb{R}^n \). In the case \( V \in \mathcal{K}(\mathbb{R}^n) \) we may even choose \( K = \mathbb{R}^n \) in (8.3) according to Thm. 5.2. Altogether this proves the theorem in full generality.

**Corollary 8.2.** Let \( t > 0 \), \( p \in [1, \infty] \), and let \( V, V_n \in \mathcal{K}_\pm(\mathbb{R}^n) \), \( n \in \mathbb{N} \), satisfy (2.34) and (2.35). Let \( (\mathbf{Y}_t)_{t \geq 0} \) and \( \| \cdot \| \) be given by either Line 2 or Line 4 of Table 1. Suppose that \( c, c_n, n \in \mathbb{N} \), are coefficient vectors satisfying Hyp. 2.1 and \( \mathbb{R}^n \) with the same \( \omega \) and \( C \) such that the maps \( \mathbb{R}^n \ni \mathbf{x} \mapsto \mathbf{c}_n \mathbf{x} \) and \( \mathbb{R}^n \ni \mathbf{x} \mapsto c_n \mathbf{x} \), \( n \in \mathbb{N} \), are continuous and uniformly bounded with respect to the norm \( \| \cdot \| \). Assume that the \( c_n \) converge to \( c \) in the sense that (6.1) holds, for all compact \( K \subset \mathbb{R}^n \). Finally, let \( \{ \mathbf{Y}_n \}_{n \in \mathbb{N}} \) be a converging sequence in \( L^p(\mathbb{R}^n, \mathcal{A}) \) with limit \( \mathbf{Y} \), \( \{ \mathbf{x}_n \}_{n \in \mathbb{N}} \) be a converging sequence in \( \mathbb{R}^n \) with limit \( \mathbf{x} \), and denote the semi-group defined by means of \( V_n \) and \( c_n \) by \( (T^{V_n,c}_t)_{t \geq 0} \). Then
\[
(8.6) \quad \lim_{n \to \infty} \mathbf{Y}_{t/2}(T^{V_n,c}_t \mathbf{y}_n)(\mathbf{x}_n) = \mathbf{Y}_{t/2}(T^V_t \Psi)(\mathbf{x}).
\]

**Proof.** Combining Thms. 5.2 and 6.1 reveals that
\[
\sup_{n \in \mathbb{N}} \| \mathbf{Y}_{t/2} T^{V_n,c}_t \|_{p, \infty} < \infty, \quad \sup_{y \in K} \| \mathbf{Y}_{t/2}(T^{V_n,c}_t \mathbf{y} - \mathbf{Y}_{t/2}(T^V_t \mathbf{y}))(\mathbf{x}) \| \to 0 \quad \text{as} \quad n \to \infty,
\]
for every compact \( K \subset \mathbb{R}^n \). We conclude by choosing \( K \) such that it contains the image of the sequence \( \{ \mathbf{x}_n \}_{n \in \mathbb{N}} \), writing
\[
(T^{V_n,c}_t \mathbf{y}_n)(\mathbf{x}_n) - (T^V_t \mathbf{y})(\mathbf{x}) = (T^V_t \mathbf{y})(\mathbf{x}_n) - (T^V_t \mathbf{y})(\mathbf{x}) + ((T^{V_n,c}_t - T^V_t) \mathbf{y})(\mathbf{x}_n) + (T^{V_n,c}_t(\mathbf{y} - \mathbf{y}_n))(\mathbf{x}_n),
\]
and employing the continuity of \( \mathbf{Y}_{t/2} T^V_t \mathbf{y} \) guaranteed by Thm. 8.1.

The preceding corollary will be applied in a more specific situation in Sect. 11.

Let us consider the standard model of non-relativistic QED for \( N \) spin one-half electrons to illustrate Thms. 5.2, 6.1 and 8.1.

**Example 8.3.** Assume that \( G = G^{X,N} \) and \( F = F^{X,N} \) are as in Ex. 2.2 and that \( V = V^{X,N} \) is the many-body Coulomb potential defined in Ex. 2.12 with atomic numbers \( Z \in [0, \infty)^N \). (Of course, the component \( Z_a \) of \( Z \) being zero means that the \( a \)-th nucleus is absent.) Let \( E^{X,N} \) denote the infimum of the spectrum of the physical Hamiltonian \( H^{X,N} \) and let
\[
\Sigma^{X,N} := \min_{M=1,\ldots,N} \{ E^{X,M} + E^{X,N-M} \}
\]
denote the ionization threshold. Physically, this is the minimal energy required for removing at least one electron from the confining potential of the molecule modeled by the first term in (2.14). Let \( \{ P_s \}_{s \in \mathbb{R}} \) denote the spectral family of \( H^{X,N} \). Then it is known that the range of \( P_s \) is contained in \( \mathcal{D}(e^{a|x|}) \), provided that \( s \) and \( a \) satisfy
\[
(8.7) \quad s + a^2/2 < \Sigma^{X,N}.
\]
(1) Assume that $a$ and $s$ satisfy (8.7). Then $e^{a|\cdot|} P_s e^{2H_2^{N,0}} P_s \in \mathcal{B}(\mathcal{H})$ and the relation $P_s = P_s^2$ entails

\[(8.8)\quad P_s = (e^{-2H_2^{N,0}} e^{-a|\cdot|}) e^{a|\cdot|} P_s e^{2H_2^{N,0}} P_s.\]

Thanks to Thm. 8.1 we know that every $\Psi_s \in \text{Ran}(P_s)$ with $s \in \mathbb{R}$ admits a unique uniformly continuous representative that we shall consider in the following. We shall also assume that $\|\Psi_s\| = 1$.

Assume that that the cut-off function $\chi$ appearing in Ex. 2.2 satisfies $\omega e^{\delta_0 \omega} \chi \in \mathfrak{h}$, for some $\delta_0 > 0$. For sufficiently small $\delta > 0$, we then infer the following pointwise bound from the above remarks and (5.1).

\[\|e^{\delta d\Gamma(\omega)} \Psi_s(\mathbf{x})\|_{\mathcal{A}} \leq e^{-\alpha|\mathbf{x}|} \|e^{\delta d\Gamma(\omega)+a|\cdot|} e^{-2H_2^{N,0}} e^{-a|\cdot|}\|_{L^2,\infty} \|e^{a|\cdot|} \chi_s e^{2H_2^{N,0}} P_s\|_{L^2,2},\]

for all $\mathbf{x} \in \mathbb{R}^{3N}$. Here $\delta > 0$ depends only on the coefficient vector, in this case $\delta_0$, $\chi$, and the number of electrons $N$; see (2.16). For every $r > 0$, put $\omega_r := \omega_1(\omega \geq r)$ and consider the weight given by Line 4 of Table 1 with $t_* := 2$ and $\omega := \omega_r$. Then can we fix $r$ large enough such that the corresponding condition Hyp(Υ) in Line 4 of the table is fulfilled. As above we now obtain

\[\|e^{\delta d\Gamma(\omega_r)} \Psi_s(\mathbf{x})\|_{\mathcal{A}} \leq c_{r,a,s,N,\chi,Z,R,e^{-\alpha|\mathbf{x}|}} \|e^{\delta_0 \omega r} e^{-a|\mathbf{x}|}\|, \quad \mathbf{x} \in \mathbb{R}^{3N}.\]

Let $\Psi_s = (\Psi_s^{(n)})_{n=0}^\infty$ the representation of $\Psi_s$ as a sequence indexed by the boson number according to the isomorphism $L^2(\mathbb{R}^{3N}, \mathcal{H}) = \bigoplus_{n=0}^\infty L^2(\mathbb{R}^{3N}, \mathbb{C}^{2^n} \otimes \mathcal{F}^{(n)})$. Then the previous bound implies

\[\|e^{\delta d\Gamma(n)}(\omega) \Psi_s^{(n)}(\mathbf{x})\|_{L^{2N} \otimes \mathcal{F}^{(n)}} \leq c_{r,a,s,N,\chi,Z,R,e^{-\alpha|\mathbf{x}|}} \|e^{\delta_0 nr} e^{-a|\mathbf{x}|}\|, \quad \mathbf{x} \in \mathbb{R}^{3N}, n \in \mathbb{N}.\]

This shows that at least any fixed $n$-boson function $\Psi_s^{(n)}$ has an exponential decay with respect to the photon momentum variables in the $L^2$-sense with a rate no less than $\delta_0$.

Likewise, if we suppose instead that $\omega^{\alpha+1/2} \chi \in \mathfrak{h}$, for some $\alpha \geq 1$, then

\[\| (1 + d\Gamma(\omega))^{\alpha} \Psi_s(\mathbf{x})\|_{\mathcal{A}} \leq c_{\alpha,a,s,N,\chi,Z,R,e^{-\alpha|\mathbf{x}|}} \| (1 + d\Gamma(\omega))^{\alpha} \Psi_s(\mathbf{x})\| \leq c_{\alpha,s,N,\chi,Z,R} \| \Psi_s \|.\]

For general elements in spectral subspaces that are not eigenvectors, these pointwise bounds are new. Moreover, in the case of the Coulomb potential, the relation (8.7) gives a more explicit and probably better bound on the decay rate $a$ as compared to earlier pointwise decay estimates on ground state eigenvectors [16, 21, 25]. We should also mention at this point that ground state eigenvectors in non-relativistic QED are in the domain of all powers of the number operator [21]. In order to show this one has to exploit the corresponding eigenvalue equation.

(2) Let $s \in \mathbb{R}$ be arbitrary, $\Psi_s \in \text{Ran}(P_s)$, and assume that $\omega^{\alpha+1/2} \chi \in \mathfrak{h}$, for some $\alpha \geq 1$. Then (8.8) with $a = 0$ and Thm. 5.2 imply, for all $n \in \mathbb{N}$ and $\mathbf{x} \in \mathbb{R}^{3N}$,

\[\|(1 + d\Gamma(n)(\omega))^{\alpha} \Psi_s^{(n)}(\mathbf{x})\|_{L^{2N} \otimes \mathcal{F}^{(n)}} \leq \|(1 + d\Gamma(\omega))^{\alpha} \Psi_s(\mathbf{x})\| \leq c_{\alpha,s,N,\chi,Z,R} \| \Psi_s \|.\]

Notice that, since $\omega(k) = |k|$, the previous bound can be read as an estimate on the norm of $\Psi_s^{(n)}(\mathbf{x})$ in the Sobolev space $H^\alpha(\mathbb{R}^{3n}, \mathbb{C}^{2^{N+n}})$. (Here $C^{2^{N+n}}$ accounts for the spin degrees of freedom of the $N$ electrons and the polarizations of the $n$ photons.) Under the present assumption on $\chi$ we further know that the maps $\mathbb{R}^{3N} \ni \mathbf{x} \mapsto \omega^\gamma(G^{N}_{Z,e}, F^{N}_{Z,e}) \in \mathfrak{h}$ are bounded and continuous, for all $\gamma \in [-1/2,\alpha]$. Hence, Thm. 8.1 implies continuity of the maps $\mathbb{R}^{3N} \ni \mathbf{x} \mapsto \Psi_s^{(n)}(\mathbf{x}) \in H^\alpha(\mathbb{R}^{3n}, \mathbb{C}^{2^{N+n}})$.
For fixed $\mathbf{x}$, let $\left(\Psi_{s}^{(n)}(\mathbf{x})\right)^{\wedge}(\mathbf{y})$ denote the Fourier transform of $\Psi_{s}^{(n)}(\mathbf{x})$ evaluated at $\mathbf{y} \in \mathbb{R}^{3n}$. If $\omega^{\alpha} \chi \in \mathcal{B}$, for all $\alpha \geq 1$, then we may employ the Sobolev embedding theorem to argue that every partial derivative $\partial_{\mathbf{y}}^{\beta}(\Psi_{s}^{(n)}(\mathbf{x}))^{\wedge}(\mathbf{y})$ with $\beta \in \mathbb{N}_{0}^{3n}$ is bounded and jointly continuous in $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{3(N+n)}$.

9. Continuity of the integral kernel

Next, we turn our attention to the operator-valued integral kernel $T_{t}^{V}(\mathbf{x}, \mathbf{y})$. The first aim of this section is to show that it is jointly continuous in the variables $t > 0$, $\mathbf{x}$, and $\mathbf{y}$: see the subsequent Thm. 9.1 which is obtained by extending the arguments applied to Schrödinger semi-groups in, e.g., [36]. In fact, the results of our Sects. 5, 7, and 8 are needed in its proof. At this point we should also mention the standard reference for Schrödinger semi-groups [33], the continuity of the semi-group kernel is investigated in the absence of magnetic fields by methods somewhat different from [36].

Proceeding as in [33], it is actually possible to combine the next theorem with the Feynman-Kac formula (2.46) and (3.6) and to argue that resolvents $s$ of the operator $H^{V}$ continuous in their arguments

are fulfilled, and let $K$ theorem to argue that every partial derivative $\partial_{\mathbf{y}}^{\beta}(\Psi_{s}^{(n)}(\mathbf{x}))^{\wedge}(\mathbf{y})$ with $\beta \in \mathbb{N}_{0}^{3n}$ is bounded and jointly continuous in $(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^{3(N+n)}$.

The next theorem can also be combined with Thm. 9.2 further below to prove joint continuity of the semi-group kernel in $t > 0$, $\mathbf{x}$, $\mathbf{y}$, and additional model parameters.

**Theorem 9.1.** Let $V \in \mathcal{K}_{\pm}(\mathbb{R}^{n})$, $\tau_{2} > \tau_{1} > 0$, and let $(\Upsilon_{t})_{t \geq 0}$ and $\| \cdot \|_{*}$ be given by either Line 2 or Line 4 of Table 1. Assume that the map $\mathbb{R}^{n} \ni \mathbf{x} \mapsto c_{\mathbf{x}}$ is bounded and continuous with respect to the norm $\| \cdot \|_{*}$. Then the map

$$
(9.1) \quad [\tau_{1}, \tau_{2}] \times \mathbb{R}^{n} \times \mathbb{R}^{n} \ni (t, \mathbf{x}, \mathbf{y}) \mapsto \Upsilon_{t/\tau_{2}} T_{t}^{V}(\mathbf{x}, \mathbf{y}) \in \mathcal{B}(\mathcal{H})
$$

is continuous with respect to the norm topology on $\mathcal{B}(\mathcal{H})$. If $V \in \mathcal{K}(\mathbb{R}^{n})$ and $\mathbb{R}^{n} \ni \mathbf{x} \mapsto c_{\mathbf{x}}$ is bounded and uniformly continuous with respect to $\| \cdot \|_{*}$, then (9.1) is uniformly continuous.

**Proof.** We split the proof into three steps. In Steps 1, 2 and 3 we verify uniform continuity in $\mathbf{x}$, $\mathbf{y}$, and $t$, respectively, as the other two parameters are varying in suitable sets. To this end we pick $\tau_{2} > \tau_{1} > 0$ and set $s := \tau_{1}/2$.

**Step 1.** As a consequence of the inequality in (2.7a) the $L^{1}(\mathbb{R}^{n}, \mathcal{H})$-norm of the functions $T_{s}^{V}(\cdot, \mathbf{y})\psi$ with $\mathbf{y} \in \mathbb{R}^{n}$ and $\psi \in \mathcal{H}$, $\| \psi \| = 1$, is uniformly bounded. In view of this fact and Thm. 8.1, the set of $\mathcal{H}$-valued functions

$$
\{ \Upsilon_{t/\tau_{2}} T_{t-s}^{V}(\cdot, \mathbf{y})\psi : \psi \in \mathcal{H}, \| \psi \| = 1, \mathbf{y} \in \mathbb{R}^{n}, t \in [\tau_{1}, \tau_{2}] \}
$$

is uniformly equicontinuous on every compact subset of $\mathbb{R}^{n}$, and uniformly equicontinuous on the whole $\mathbb{R}^{n}$ provided that $V \in \mathcal{K}(\mathbb{R}^{n})$ and $\mathbb{R}^{n} \ni \mathbf{x} \mapsto c_{\mathbf{x}}$ is bounded and uniformly continuous with respect to $\| \cdot \|_{*}$. Let $K = \mathbb{R}^{n}$, if the latter conditions are fulfilled, and let $K \subset \mathbb{R}^{n}$ be compact, if not. Combining the above observation
with Prop. 3.4 we then deduce that the expression
\[
\sup_{x,y \in K} \sup_{|x-y| < r} \sup_{t \in [\tau_1, \tau_2]} \| \mathcal{Y}_{\tau/4}(T^V_t(x, y) - T^V_t(x, y)) \|
\]
\[
= \sup_{x,y \in K} \sup_{|x-y| < r} \sup_{t \in [\tau_1, \tau_2]} \| \mathcal{Y}_{\tau/4}T^V_{t-s}(T^V_s(\cdot, y))\psi((x) - \mathcal{Y}_{\tau/4}T^V_{t-s}(T^V_s(\cdot, y))\psi(x) \|)
\]
converges to zero in the limit \( r \downarrow 0 \).

**Step 2.** Likewise, the \( L^1(\mathbb{R}^\nu, \mathcal{F}) \)-norm of the functions \( T^V_s(\cdot, x)\Theta_{\tau/4}\psi \) with \( x \in \mathbb{R}^\nu \) and \( \psi \in \mathcal{D}(\Theta_{\tau/4}), \| \psi \| = 1 \), is uniformly bounded; in fact, setting \( F(z) := |z - x|, z \in \mathbb{R}^\nu \), we infer from (3.5) and (3.7) that

\[
\int_{\mathbb{R}^\nu} \| T^V_s(z, x)\Theta_{\tau/4}\psi \| dz \leq \int_{\mathbb{R}^\nu} \sup_{\| \phi \| = 1} \| \mathcal{Y}_{\tau/4}T^V_s(x, z)\phi \| dz
\]
\[
= \int_{\mathbb{R}^\nu} \sup_{\| \phi \| = 1} \| (\mathcal{Y}_{\tau/4}e^{-F T^V_s z}e^F)(e^{-F T^V_s z}((z) \phi)(x)) \| dz
\]
\[
\leq \| \mathcal{Y}_{\tau/4}e^{-F T^V_s z}e^F \|_{1, \infty} \int_{\mathbb{R}^\nu} \sup_{\| \phi \| = 1} \| e^{-F \hat{z}}(\mathcal{Y}_{\tau/4} T^V_s(x, z)\phi) \| d\hat{z} dz
\]
\[
\leq c_{s,V,c} \int_{\mathbb{R}^\nu} e^{-|\hat{z}|^2/4s} d\hat{z} \int_{\mathbb{R}^\nu} e^{-|z|^2/4s} dz,
\]
for all \( x \in \mathbb{R}^\nu \) and normalized \( \psi \in \mathcal{D}(\mathcal{Y}_{\tau/4}) \). Employing Thm. 8.1 once more, we see that the set of \( \mathcal{F} \)-valued functions

\[
\{ T^V_{t-s}(T^V_s(\cdot, x)\Theta_{\tau/4}\psi) : \psi \in \mathcal{D}(\mathcal{Y}_{\tau/4}), \| \psi \| = 1, x \in \mathbb{R}^\nu, t \in [\tau_1, \tau_2] \}
\]
is uniformly equicontinuous on every compact subset of \( \mathbb{R}^\nu \), and uniformly equicontinuous on all of \( \mathbb{R}^\nu \), if the additional conditions in the last assertion of the statement are fulfilled. Let \( K \subset \mathbb{R}^\nu \) be given as in Step 1. Using

\[
\| \mathcal{Y}_{\tau/4}(T^V_t(x, y) - T^V_t(x, \hat{y})) \|
\]
\[
\leq \sup_{\| \phi \| = 1} \sup_{v \in \mathcal{D}(\mathcal{Y}_{\tau/4})} \| \psi \| \| (T^V_t(y, x) - T^V_t(\hat{y}, x))\mathcal{Y}_{\tau/4}\psi \|
\]
and invoking Prop. 3.4 once more we conclude that the expression

\[
\sup_{x,y \in K} \sup_{|x-y| < r} \sup_{t \in [\tau_1, \tau_2]} \| \mathcal{Y}_{\tau/4}(T^V_t(x, y) - T^V_t(x, \hat{y})) \|
\]
\[
= \sup_{x,y \in K} \sup_{|x-y| < r} \sup_{t \in [\tau_1, \tau_2]} \| T^V_{t-s}(T^V_s(\cdot, x)\mathcal{Y}_{\tau/4}\psi)(y) - T^V_{t-s}(T^V_s(\cdot, x)\mathcal{Y}_{\tau/4}\psi)(\hat{y}) \|
\]
goes to zero in the limit \( r \downarrow 0 \).

**Step 3.** For all \( t \geq \tau_1 \) and \( \psi \in \mathcal{H} \), Prop. 3.4 further implies

\[
T^V_t(x, y)\psi = \int_{\mathbb{R}^\nu} T^V_{t-s}(T^V_s(x, z)\mathcal{Y}_{\tau/4}\psi)(y) dz = \int_{\mathbb{R}^\nu} T^V_{t-s}(T^V_s(\cdot, z)\mathcal{Y}_{\tau/4}\psi((z)))(x) dz,
\]
with \( \Psi_{s,t,y} := T^{V}_{t,s}(T^{V}_{s,t}(\cdot, y)) \), which yields, again with \( F_{x}(z) := |z - x| \),
\[
\| T^{V}_{t,s}(x, y) - T^{V}_{t,s}(x, y) \| \leq \sup_{x \in \mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \left( e^{-F_{x}(\bar{z})} T^{V}_{t,s}(\bar{z}, \bar{z}) (\Psi_{s,t,y}(z) - \Psi_{s,t,y}(\bar{z})) \right) d\bar{z} \],
\[
\leq \sup_{x \in \mathbb{R}^{n}} \int_{\mathbb{R}^{n}} e^{-(\bar{z} - z)^{2}/4s} d\bar{z} \leq c_{s} e^{-F_{x}(\bar{z})} T^{V}_{t,s}(\bar{z}, \bar{z}) d\bar{z} \leq c_{s},
\]
for all \( t, s \geq t_{1} \). Employing the bound in (3.5) and (5.1), we find some \( c_{s} > 0 \) such that
\[
\| T^{V}_{t,s}(x, y) - T^{V}_{t,s}(x, y) \| \leq c_{s} \sup_{\|\psi\|=1} \| e^{-F_{x}} (T^{V}_{t,s} - T^{V}_{t,s}) \Phi_{s,y,\psi} \|_{1},
\]
(9.2)
with \( \Phi_{s,y,\psi} := T^{V}_{t,s}(\cdot, y, \psi) \). Next, we observe that (3.5) implies that
\[
\sup_{y \in \mathbb{R}^{n}} \sup_{\|\psi\|=1} \| \Phi_{s,y,\psi} \|_{1} \leq \sup_{y \in \mathbb{R}^{n}} \int_{\mathbb{R}^{n}} \| T^{V}_{t,s}(z, y) \| dz \leq c_{s} \int_{\mathbb{R}^{n}} e^{-|z|^{2}/4s} dz < \infty.
\]
As above, let \( K \) be equal to \( \mathbb{R}^{n} \), if \( V \in K(\mathbb{R}^{n}) \) and \( x \mapsto c_{x} \) is bounded and uniformly continuous with respect to \( \| \cdot \| \), and let \( K \) be a compact subset of \( \mathbb{R}^{n} \) otherwise.

Defining
\[
\mathcal{M} := \{ e^{-F_{x}} T^{V}_{t,s}, \Phi_{s,y,\psi} : \psi \in \mathcal{H}, \|\psi\|=1, x \in K, y \in \mathbb{R}^{n}, t \in [t_{1}, t_{2}] \}
\]
we then infer that \( \mathcal{M} \subset L^{1}(\mathbb{R}^{n}, \mathcal{H}) \cap L^{\infty}(\mathbb{R}^{n}, \mathcal{H}) \) by Lem. 3.1 and
(3.4)
and the additional conditions in the last assertion of the statement are fulfilled, then \( \mathcal{M} \) is uniformly equicontinuous on the whole \( \mathbb{R}^{n} \).

(b) \( \sup_{\Psi \in \mathcal{M}} \| \Psi \|_{\infty} < \infty \) by (3.4). Moreover, setting \( a_{\Psi} := x \), if \( \Psi \in \mathcal{M} \) is given as \( \Psi := e^{-F_{x}} T^{V}_{t,s} \Phi_{s,y,\psi} \), and abbreviating \( \Psi_{R}(z) := 1_{|z - a_{\Psi}| < R} \Psi(z) \), we verify that
\[
\| \Psi_{R} - \Psi_{R} \|_{1} \leq e^{-Rm}.
\]
(c) If \( f_{\Psi} \) is defined as in Condition (c) of Thm. 7.1, then (5.1) implies
\[
\sup_{\Psi \in \mathcal{M}} \| f_{\Psi} \|_{1} \leq \sup_{\Psi \in \mathcal{M}} \sup_{\tau \in [t_{1}, t_{2}]} \| f_{\Psi_{\tau}} \|_{1} \sup_{\|\psi\|=1} \| \Phi_{s,y,\psi} \|_{1} \leq \sup_{\tau \in [t_{1}, t_{2}]} \| (1 + d\Gamma(\omega))^{1/2} T^{V}_{t,s} \Phi_{s,y,\psi} \|_{1} \sup_{\|\psi\|=1} \| \Phi_{s,y,\psi} \|_{1} < \infty.
\]
We may hence apply Thm. 7.1(1) to deduce that
\[
\lim_{t \downarrow t_{0}, \tau \in [t_{1}, t_{2}]} \sup_{x \in K} \sup_{y \in \mathbb{R}^{n}} \sup_{\|\psi\|=1} \| e^{-F_{x}} (T^{V}_{t,s} - \| T^{V}_{t,s} \| - 1) \Phi_{s,y,\psi} \|_{1} \leq \lim_{t \downarrow t_{0}} \sup_{\Psi \in \mathcal{M}} \| e^{-F_{x}} (\Psi_{\tau} - \| T^{V}_{t,s} \| - 1) e^{F_{x}} \Psi \|_{1} = 0,
\]
where \( F_\psi := F_{a_\psi} \). Combining this with (9.2) we arrive at

\[
\lim_{\tau \downarrow 0} \sup_{t \in [\tau_1, \tau_2]} \sup_{x, y \in K} \| \mathcal{Y}_{\tau/s}(T^{V, n}_t(x, y) - T^V_t(x, y)) \| = 0,
\]

and we conclude.

\[\square\]

**Theorem 9.2.** Let \( \tau_2 > \tau_1 > 0 \) and \( V, V_n \in \mathcal{K}_\pm(\mathbb{R}^v) \), \( n \in \mathbb{N} \), such that (2.34) and (2.35) are satisfied. Let \( c, c_n, n \in \mathbb{N} \) be coefficient vectors satisfying Hyp. (2.7) with the same \( \omega \) and \( C \) and assume that \( \| c_x \|_1, \| c_{n, x} \|_1 \leq A \), for all \( x \in \mathbb{R}^v \), \( n \in \mathbb{N} \), and some \( A \in (0, \infty) \). Assume further that (6.1) holds true, for all compact \( K \subset \mathbb{R}^v \).

Then

\[
(9.5) \quad \lim_{n \to \infty} \sup_{t \in [\tau_1, \tau_2]} \sup_{x, y \in K} \| T_{\tau_1/s}(T^{V, n}_t(x, y) - T^V_t(x, y)) \| = 0,
\]

for all compact \( K \subset \mathbb{R}^v \).

**Proof.** We pick some \( s \in [\tau_1/4, \tau_1/2] \) and write

\[
T^{V, n}_t(x, y) - T^V_t(x, y) = ((T^{V, n}_{t-s} - T^V_{t-s})(T^{V, n}_s(\cdot, y)\psi))(x)
+ (T^V_{t-s}(T^{V, n}_s(\cdot, y)\psi - T^V_s(\cdot, y)\psi))(x),
\]

for a given \( \psi \in \mathcal{H} \). Next, we observe that

\[
\sup_{t \in [\tau_1, \tau_2]} \sup_{x, y \in K} \| \mathcal{Y}_{\tau_1/s}(T^{V, n}_{t-s} - T^V_{t-s})(T^{V, n}_s(\cdot, y)\psi) \| \xrightarrow{n \to \infty} 0,
\]

because, on the one hand, \( \{ T^{V, n}_s(\cdot, y)\psi : n \in \mathbb{N}, y \in K, \| \psi \| = 1 \} \) is a bounded subset of \( L^2(\mathbb{R}^v, \mathcal{H}) \) in view of Cor. 5.1 and (2.35), and, on the other hand, \( 1_K \mathcal{Y}_{\tau_1/s}(T^{V, n}_{t-s} - T^V_{t-s}) \) converges to zero in \( \mathcal{B}(L^2(\mathbb{R}^v, \mathcal{H}), L^\infty(\mathbb{R}^v, \mathcal{H})) \), uniformly in \( t \in [\tau_1, \tau_2] \), according to Thms. 5.2 and 6.1. Since \( \mathcal{Y}_{\tau_1/s}T^V_{t-s} \) is bounded from \( L^1(\mathbb{R}^v, \mathcal{H}) \) to \( L^\infty(\mathbb{R}^v, \mathcal{H}) \), uniformly in \( t \in [\tau_1, \tau_2] \), it thus remains to show that

\[
(9.6) \quad \sup_{\| \psi \| = 1} \sup_{y \in K} \int_{\mathbb{R}^v} \| T^{V, n}_s(z, y)\psi - T^V_s(z, y)\psi \| \, dz \xrightarrow{n \to \infty} 0.
\]

To this end we write

\[
T^{V, n}_s(z, y)\psi - T^V_s(z, y)\psi = (T^{V, n}_s(z, y) - T^V_s(z, y))\psi
+ (T^V_s(z, y) - T^V_s(z, y))\psi,
\]

and estimate, using (2.21),

\[
(9.8) \quad \| (T^{V, n}_s(z, y) - T^V_s(z, y))\psi \|
\leq p_s(z, y)\mathbb{E} \left[ \| \mathcal{W}^{V, n}_{s} [b^{s, y, z}]\psi - \mathcal{W}^{V, n}_{s} [b^{s, y, z}]\psi \| \right]
\leq p_s(z, y)\mathbb{E} \left[ e^{-\int_0^s \mathcal{A} s^n(\mathcal{A}^{s, y, z}) \, dz} - e^{-\int_0^s \mathcal{A} s^n(\mathcal{A}^{s, y, z}) \, dz} \right] \| \mathcal{A}_n \|_{\infty} \| \psi \|.
\]

Here, the terms \( \| \mathcal{A}_n \|_{\infty} \), which are defined by (2.13) with \( F \) replaced by \( F_n \), are bounded uniformly in \( n \in \mathbb{N} \) by our assumptions on \( c_n \). The relation between the
laws of $b^{x,y,z}$ and $B^y$ further yields
\[ \int_{\mathbb{R}^n} p_s(z,y) \mathbb{E} \left[ e^{-\int_0^s V_\sigma(b^{x,y,z}) \, dr} - e^{-\int_0^s V(b^{x,y,z}) \, dr} \right] \, dz \]
\[ = \mathbb{E} \left[ \int_{\mathbb{R}^n} p_{s-\sigma}(B^y, z) \, dz \left( e^{\int_0^s V_\sigma(b^{x,y}) \, dr} - e^{\int_0^s V(b^{x,y}) \, dr} \right) \right] \]
\[ = \mathbb{E} \left[ e^{\int_0^s V_\sigma(b^{x,y}) \, dr} - e^{\int_0^s V(b^{x,y}) \, dr} \right] \quad (9.9) \]
for all $\sigma \in (0, s)$, and, by dominated convergence, the equality between the right and left hand sides of (9.9) extends to $\sigma = s$. With the help of the dominated convergence theorem and (3.4) we further obtain
\[ (T^{V,n}_s(z,y) - T^{V}_s(z,y)) \psi = \lim_{\sigma \uparrow s} \mathbb{E} \left[ \psi_{\sigma}^{V,n}[(b^{x,y,z})] \psi - \psi_{\sigma}^{V,n}[(b^{x,y,z})] \psi \right] \]
\[ = \lim_{\sigma \uparrow s} \mathbb{E} \left[ p_{s-\sigma}(B^y, z) (\psi_{\sigma}^{V,n}[B^y] - \psi_{\sigma}^{V}[B^y]) \psi \right]. \]
From the latter relation and Fatou’s lemma we infer that
\[ \int_{\mathbb{R}^n} \| (T^{V,n}_s(z,y) - T^{V}_s(z,y)) \psi \| \, dz \]
\[ \leq \liminf_{\sigma \uparrow s} \int_{\mathbb{R}^n} \mathbb{E} \left[ p_{s-\sigma}(B^y, z) \| (\psi_{\sigma}^{V,n}[B^y] - \psi_{\sigma}^{V}[B^y]) \| \right] \, dz \]
\[ = \mathbb{E} \left[ \| \psi_{\sigma}^{V,n}[B^y] \psi - \psi_{\sigma}^{V}[B^y] \psi \| \right] \]
\[ \leq \mathbb{E} \left[ e^{\int_0^s V(b^{x,y}) \, dr} \right]^{1/2} \mathbb{E} \left[ \| \psi_{\sigma}^{V,n}[B^y] \psi - \psi_{\sigma}^{V}[B^y] \psi \|^2 \right]^{1/2}, \quad (9.10) \]
where we also applied Fubini’s theorem and the dominated convergence theorem in the second step. The convergence (9.6) is now implied by (2.28), (2.40), (6.5), (6.6), (9.7), (9.8), the extension of (9.9) to $\sigma = s$, and (9.10).

10. Positivity improvement by the semi-group kernel in the scalar case

In this section we complement the discussion of the semi-group kernel by showing that, in the scalar case $L = 1$ with either $F = 0$ or $G = 0$, $T^V_t(x,y)$ is positivity improving with respect to a suitable positive cone in the Fock space, for all $t > 0$ and $x,y \in \mathbb{R}^n$. As an immediate corollary we re-obtain a theorem, due to Hiroshima [19], stating that the semi-group in non-relativistic QED is positivity improving, if spin is neglected and the Pauli principle is discarded. We will apply this result in Sect. 11 to discuss the continuous dependence on model parameters of strictly positive ground state eigenvectors of scalar Hamiltonians.

As usual, the proof of the aforementioned results is an application of Perron-Frobenius type theorems [7, 31]. Let us point out, however, that our proof of Thm. 10.1 is based on a novel factorization of the Feynman-Kac integrand found in [12], from which the positivity improvement by the integrand can be read off more easily. In fact, the factorization used in Hiroshima’s proof involved unbounded operators that lead to additional technical difficulties.

Let us introduce the notation $\varpi(g) := \varphi(ig)$ and $W(g) := e^{-i\varpi(g)}$, $g \in \mathfrak{h}$, for the conjugate field and the associated Weyl operator, respectively. As usual, $\Omega = (1,0,0,\ldots) \in \mathcal{F}$ will denote the vacuum vector. There will be no danger of confusing it with the underlying probability space that is denoted by the same symbol.
A suitable self-dual convex cone in the Fock space $\mathcal{F}$ is now given by $\mathcal{P} := \mathcal{F}$, the closure of the set
\[ \mathcal{P} := \{ F(\varpi(g))|\Omega : F \in \mathcal{S}(\mathbb{R}^n), F \geq 0, g \in \mathfrak{h}_C^n, n \in \mathbb{N} \}. \]
Here $\mathcal{S}(\mathbb{R}^n)$ is the set of Schwartz test functions on $\mathbb{R}^n$ and, for every $F \in \mathcal{S}(\mathbb{R}^n)$, the bounded operator $F(\varpi(g)) := F(\varpi(g_1), \ldots, \varpi(g_n))$ is defined by the spectral calculus for finitely many commuting self-adjoint operators. For later reference, we observe that the latter operator is equal to the Bochner-Lebesgue integral
\[ F(\varpi(g)) = (2\pi)^{-n} \int_{\mathbb{R}^n} \hat{F}(-\xi)W(\xi \cdot g)\,d\xi. \]
In fact, there exists a probability space $(\mathcal{Q}, \Omega, \eta)$ and a unitary map $\mathcal{U} : \mathcal{F} \to L^2(\mathcal{Q}, \Omega, \eta)$, turning each conjugate field operator $\varpi(g)$, $g \in \mathfrak{h}_C$, into a multiplication operator with a Gaussian random variable, such that $\mathcal{P}$ is the pre-image under $\mathcal{U}$ of all non-negative elements of $L^2(\mathcal{Q}, \Omega, \eta)$. Furthermore, $\mathcal{U}\Omega = 1$, and in particular $\Omega$ is strictly positive with respect to $\mathcal{P}$. See, e.g., [23, 34] for the construction of $\eta$ and a detailed discussion of related matters.

Notice that $\psi \in \mathcal{F}$ belongs to the completely real subspace $\mathcal{F}_C := \{ \phi \in \mathcal{F} : \Gamma(-C)\phi = \phi \}$, if and only if there exist $\psi_+, \psi_- \in \mathcal{P}$ with $\psi_+ \perp \psi_-$ and $\psi = \psi_+ - \psi_-$. In fact, $F \in \mathcal{S}(\mathbb{R}^n)$ is real-valued, if and only if $\hat{F}(-\xi) = \hat{F}(\xi)$, and we further have $\Gamma(-C)W(g)\Omega = W(-g)\Omega$, $g \in \mathfrak{h}_C$. Thus, (10.1) implies $F(\varpi(g))\Omega \in \mathcal{F}_C$, for all real-valued $F \in \mathcal{S}(\mathbb{R}^n)$ and $g \in \mathfrak{h}_C^n$.

**Theorem 10.1.** Consider the scalar case $L = 1$ with $F = 0$. Let $t > 0$ and $x, y \in \mathbb{R}^n$. Then $T_t^V(x, y)$ maps $\mathcal{F}_C$ into itself and it is positivity improving with respect to $\mathcal{P}$.

**Proof.** According to [12] Rem. F.2(1)] we have the factorization
\[ \mathcal{U}_t^V[b^V|x,y] = e^{-tV}A_{[s]}[U_t^+e^{-tdf(\omega)}/\beta]A_{[s]}[-U_t^-]^*; \]
\[ A_s[f] := \sum_{n=0}^{\infty} \frac{i^n}{n!} e^{if} (f)^n e^{-sdf(\omega)}, \quad f \in \mathfrak{h}_C, s > 0, \]
where $(u_s^V)_{s \in [0,t]}$ is a real-valued semi-martingale and $(U_s^\pm)_{s \in [0,t]}$ are $\mathfrak{h}_C$-valued semi-martingales, whose dependence on $(t, x, y)$ has been dropped in the notation. The sum defining $A_s[f] \in \mathcal{B}(\mathcal{F})$ converges absolutely in the operator norm [12 Lem. F.1]. Standard arguments [34] now show that $A_s[f]^*$ is positivity preserving. To see this, let $g \in \mathfrak{h}_C$ and $W(g) = e^{-i\pi(g)}$ be the corresponding Weyl operator. Since $W(g)\Omega$ is an analytic vector for $a(f)$, we easily find
\[ A_s[f]^*W(g)\Omega = e^{-sdf(\omega)} \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} a(f)^nW(g)\Omega \]
\[ = e^{-\|g\|^2/2+i\|e^{-i\pi(g)}\|^2/2-i(f|g)}W(g), \quad f, g \in \mathfrak{h}_C. \]
Let $g_1, \ldots, g_n \in \mathfrak{h}_C$ and $F \in \mathcal{S}(\mathbb{R}^n)$. Since the integration in [10.1] commutes with the bounded operator $A_s[f]^*$, we may employ (10.3) to get
\[ A_s[f]^*F(\varpi(g))\Omega = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\|1-e^{-i\pi(g)}\xi g\|^2/2-i(f|g)\langle \xi \rangle} \hat{F}(-\xi)W(\xi \cdot g)\,d\xi. \]
Another use of [10.1] yields $A_s[f]^*F(\varpi(g))\Omega = \hat{F}(\varpi(g))\Omega$, where $\hat{F}$ is the inverse Fourier transform of the Schwartz function $\xi \mapsto e^{-\|1-e^{-i\pi}\xi g\|^2/2-i(f|g)\langle \xi \rangle}$. 

\[ A_s[f]^*F(\varpi(g))\Omega = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-\|1-e^{-i\pi\xi g\|^2/2-i(f|g)\langle \xi \rangle} \hat{F}(-\xi). \]

\[ (10.3) \]
\[ (10.1) \]
If $F$ is non-negative, then $\tilde{F}$ is non-negative as well. We have thus shown that, for all $f \in \mathfrak{h}_C$ and $s > 0$, $A_s[f]^*$ maps $\mathcal{P}$ into itself. Since it is bounded, it also maps $\mathcal{P}$ into itself, and this holds in particular for $e^{-s d\Gamma(\omega)} = A_s[0]^*$. It also follows that $A_s[f] = A_s[f]^{**}$ is positivity preserving. Since $\omega > 0$, $\mu$-a.e., we further know that $1$ is a non-degenerate eigenvalue of $e^{-s d\Gamma(\omega)}$, $s > 0$, with strictly positive eigenvector $\Omega$. Therefore, the Perron-Frobenius type theorem [31] Thm.XIII.44(a)$\Rightarrow$(e)] (applied to $\mathcal{H} e^{-s d\Gamma(\omega)} \mathcal{H}^*$) ensures that $e^{-s d\Gamma(\omega)}$ with $s > 0$ is actually positivity improving. As a consequence, $\mathcal{W}^\dagger[b^{i;x,y}]$ is positivity improving, pointwise on the underlying probability space, as a composition of positivity preserving operators one of which improves positivity. It is now clear that $T^\dagger_t(x, y) = \mu_i(x, y) \mathbb{E}[\mathcal{W}^\dagger[b^{i;x,y}]]$ is positivity improving, too.

Corollary 10.2. Consider the scalar case $L = 1$ with $F = 0$. Let $t > 0$. Then $e^{-t H^\dagger}$ is positivity improving with respect to the self-dual convex cone in $L^2(\mathbb{R}^\nu, \mathcal{H})$ given by

$$
\int_{\mathbb{R}^\nu} \mathcal{P} \, dx := \{ \Psi \in L^2(\mathbb{R}^\nu, \mathcal{H}) : \Psi(x) \in \mathcal{P}, \text{a.e. } x \}.
$$

In particular, if $\inf \sigma(H^\dagger)$ is an eigenvalue, then it is non-degenerate and the corresponding eigenvector can be chosen strictly positive.

Proof. The first assertion is obvious from Thm. 10.1. The last statement follows from Faris’ Perron-Frobenius theorem [7, Cor. 1.2].

If $G$ vanishes, instead of $F$, then one can use the cone $\mathcal{P}_t := \overline{\mathcal{P}_t} = \Gamma(i)\mathcal{P}$ with $\mathcal{P}_t := \{ F(\varphi(g))\Omega : F \in \mathcal{S}(\mathbb{R}^n), F \geq 0, g \in \mathfrak{h}_C^n, n \in \mathbb{N} \} = \Gamma(i)\mathcal{P}.$

Here the conjugate fields $\varphi(g)$ have been replaced by the fields $\varphi(g)$ in the definition of $\mathcal{P}_t$, and we used the relations $\Gamma(i)W(g)\Gamma(-i) = W(-ig)$ and (10.1) to get the second equality in (10.5); also recall $\varphi(-ig) = \varphi(g)$. Hence, $\mathcal{P}_t$ is obtained upon replacing $\mathfrak{h}_C$ by the completely real subspace $i\mathfrak{h}_C$ of $\mathfrak{h}$ associated with the conjugation $-C$. Likewise, $\mathcal{F}_{-C} := \{ \psi_+ - \psi_- : \psi_+, \psi_- \in \mathcal{P}_t \}$ is the completely real subspace of $\mathcal{F}$ associated with the conjugation $\Gamma(C)$.

Theorem 10.3. Consider the scalar case $L = 1$ with $G = 0$. Let $t > 0$ and $x, y \in \mathbb{R}^\nu$. Then $T^\dagger_t(x, y)$ maps $\mathcal{F}_{-C}$ into itself and it is positivity improving with respect to $\mathcal{P}_t$. In particular, $e^{-t H^\dagger}$ is positivity improving with respect to $\int_{\mathbb{R}^\nu} \mathcal{P}_t \, dx$ (defined as in (10.3)).

Proof. The only difference to the proofs of Thm. 10.1 and Cor. 10.2 is that the semi-martingales $(U_t^s)_{s \in [0,t]}$ appearing in the formula (10.2) are now $i\mathfrak{h}_C$-valued. The proof of Thm. 10.1 thus shows that the transformed kernel $\Gamma(i)T^\dagger_t(x, y)\Gamma(-i)$ is positivity improving with respect to $\mathcal{P}$. 

11. Continuity properties of ground states in the scalar case

In this section we discuss the joint continuity of ground state eigenvectors in non-relativistic QED with respect to position coordinates and model parameters. Thanks to Cor. 8.2 all what is left to do is to prove $L^2$-continuity with respect to model parameters of the ground state eigenvectors. This is, however, still a non-trivial task as the ground state eigenvalue typically is embedded in the continuous
spectrum so that the standard methods of analytic perturbation theory are not available.

Besides providing an example for an application of Cor. 8.2, the aim of the present section, thus, is to demonstrate that a certain compactness argument usually applied to prove the existence of ground states \[10\] can also be employed to prove continuous dependence on model parameters of the ground state eigenvectors. Furthermore, we shall present a simplified version of the compactness argument of \[10\] that allows to work with more general assumptions on the coefficient vector \(c\) and thus reduces the amount of technicalities. In particular, we do not have to specify a choice of the polarization vectors and we do not need a certain photon derivative bound \[10\]. The idea is, roughly, to apply the standard characterization of relatively compact sets in \(L^2(\mathbb{R}^d)\), instead of the Rellich-Kondrachov theorem used in \[10\], in combination with the usual formula for \(a(k)\) applied to a ground state eigenvector.

To implement these ideas we shall consider a simplified situation where only one electron is present (no Pauli principle) and spin is neglected. The main reason for this is that we require at least a constant, finite degeneracy of the ground state eigenvalues to infer the continuity from the compactness result. (If the geometric multiplicity of the ground state eigenvalue is bigger than one, then one could still try to prove norm continuity of the ground state eigenprojections.) Besides the case of one electron with spin neglected, where the methods discussed in Sect. 10 ensure non-degeneracy of ground state eigenvalues and where we may talk about distinguished positive eigenvectors, we are not aware of any non-perturbative method in non-relativistic QED that controls the degeneracy in a more general situation.

Since a detailed workout of all construction steps in \[9, 10\] required here would be too space consuming and too repetitive at the same time, we shall concentrate on those things that are changed and otherwise be rather sketchy in this section.

In the whole section we shall consider the following situation:

**Hypothesis 11.1.** (a) The one-boson space is given by \(\mathfrak{h} = L^2(\mathbb{R}^3 \times \{0, 1\})\), the \(L^2\)-space associated with the product of the Lebesgue measure on \(\mathbb{R}^3\) and the counting measure on \(\{0, 1\}\). We consider the scalar case where \(L = 1\) and set \(\nu = 3\), thus \(\mathcal{H} = L^2(\mathbb{R}^3, \mathcal{F})\). We choose \(\omega(k, \lambda) := |k|, k \in \mathbb{R}^3, \lambda \in \{0, 1\}\), and \(c = (G, q, 0), c_n = (G_n, q_n, 0), n \in \mathbb{N}\), are coefficient vectors fulfilling Hyp. 2.1 together with \(\omega\) and some fixed conjugation \(C\) on \(\mathfrak{h}\) that commutes with \(\omega\). We assume that

\[
(11.1) \quad \lim_{n \to \infty} \sup_{x \in K} \|c_{n,x} - c_x\|_2 = 0,
\]

for all compact \(K \subset \mathbb{R}^3\). We further set \(\tilde{G}_x := G_x - G_0\) and \(\tilde{G}_{n,x} := G_{n,x} - G_{n,0}\), \(n \in \mathbb{N}\), and assume that the maps \(\mathbb{R}^3 \ni x \mapsto (\omega^{-1}q_x, \omega^{-1}\tilde{G}_x) \in \mathfrak{h}^4\) and \(\mathbb{R}^3 \ni x \mapsto (\omega^{-1}q_n,x, \omega^{-1}\tilde{G}_{n,x}) \in \mathfrak{h}^4\) are well-defined, bounded, and continuous with

\[
\lim_{n \to \infty} \sup_{x \in K} \|(\omega^{-1}q_x, \omega^{-1}\tilde{G}_x) - (\omega^{-1}q_{n,x}, \omega^{-1}\tilde{G}_{n,x})\|_\mathfrak{h} = 0,
\]

for all compact \(K \subset \mathbb{R}^3\).

(b) \(V, V_n \in \mathcal{K}, n \in \mathbb{N}\), are such that \(V_n - V \in \mathcal{K}(\mathbb{R}^3)\), for all \(n \in \mathbb{N}\), and (2.34) and (2.35) hold true.

(c) We denote by \(H\) the Hamiltonian defined by means of \((\omega, c, V)\) (according to Thm. 2.17). For every \(n \in \mathbb{N}\), the Hamiltonian defined by means of \((\omega, c_n, V_n)\) is
denoted by $H_n$. We further define
\[ E := \inf \sigma(H), \quad E_n := \inf \sigma(H_n), \quad \Sigma := \lim_{R \to \infty} \Sigma_R, \quad \Sigma_n := \lim_{R \to \infty} \Sigma_{n,R}, \]
with
\[ \Sigma_R := \inf \{ \langle \Phi | H \Phi \rangle \mid \Phi \in \mathcal{Q}(H) \subset L^2(\mathbb{R}^3, \mathcal{F}), \| \Phi \| = 1, \supp(\Phi) \subset \{|x| \geq R\} \}, \]
for all $R \geq 1$, and analogous definitions of $\Sigma_{n,R}$. We assume that the following binding condition is satisfied,
\[ (11.2) \quad \Sigma > E. \]

**Example 11.2.** (1) In the situation of Ex. 2.2 with $N = 1$, let $\{\chi_n\}_{n \in \mathbb{N}}$ be a sequence of even, non-negative functions on $\mathbb{R}^3$ converging to $\chi$ in $L^2(\mathbb{R}^3, (\omega^{-1} + \omega)dk)$. Then the coupling functions $G^\chi, G^{\chi_n}, n \in \mathbb{N}$, fulfill Hyp. (11.1a) (with $q = q_n = 0$).

Notice that in this example the functions $\omega^{-1}G^{\chi_n}_x$ do not belong to $l^3$, if $\chi_n$ is constant in a neighborhood of 0, which is the reason why the function $G_x$ is introduced [2]. The pre-factor $\omega^{-1}$ appears upon estimating the resolvents in the key relation (11.15) below as $\|R(k)\| \lesssim \omega(k)^{-1}$.

(2) Let us again consider $G^\chi$ as in Ex. 2.2. Assume that $V(x) \to 0$, $|x| \to \infty$, and that $e_V := \inf \sigma(-\frac{1}{2} \Delta + V)$ is a strictly negative discrete eigenvalue of the Schrödinger operator $-\frac{1}{2} \Delta + V$. (Here the dot on the + indicates that the Schrödinger operator is defined via quadratic forms.) Then it follows from the arguments in [10] that $\Sigma - E \geq -e_V$ and in particular that the binding condition (11.2) is fulfilled.

There are also certain short range potentials $V$ where $e_V$ is equal to the lower end of the essential spectrum of the Schrödinger operator and nevertheless (11.2) is satisfied; see [23] for a non-perturbative discussion of this enhanced binding effect and for references to earlier perturbative studies.

(3) The binding condition (11.2) is clearly fulfilled if $V(x) \to \infty$, $|x| \to \infty$. In this case the ionization threshold $\Sigma$ is infinite.

Straightforward variational arguments employing (2.6) - (2.8) and Lem. 2.14 reveal that (in $(-\infty, \infty)$)
\[ (11.3) \quad \lim_{n \to \infty} E_n = E, \quad \lim_{n \to \infty} \Sigma_n = \Sigma, \quad \lim_{n \to \infty} \Sigma_{n,R} = \Sigma_R, \quad R \geq 1. \]

Although the existence part of the following proposition is not formally covered by the existing literature in its full generality, we shall treat it as well-known, too.

In this proposition and henceforth, strict positivity of a vector in $\mathcal{H}$ is understood with respect to the cone $\int_{\mathbb{R}^3} \mathcal{P} dx$ introduced in Cor. 10.2.

**Proposition 11.3.** In the situation of Hyp. (11.1) $E$ is a non-degenerate eigenvalue of $H$ and there exists a unique normalized, strictly positive, corresponding eigenvector of $H$ that we denote by $\Psi$. Furthermore, there exists $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$, $E_n$ is a non-degenerate eigenvalue of $H_n$ to which there corresponds a unique normalized, strictly positive eigenvector $\Psi_n$.

**Proof.** The non-degeneracy and strict positivity follow immediately from Cor. 10.2 as soon as we have pegged $E$ and $E_n$, $n \geq n_0$, as eigenvalues. Here the number $n_0$ is identical to the one found in Prop. (11.6) below, which ensures that the binding condition $\Sigma_n > E_n$ holds, for $n \geq n_0$, as well. To prove that $E$ and $E_n$, $n \geq n_0$, are eigenvalues, one would then first replace $\omega$ by $\omega_n(k) := (k^2 + n^{-2})^{1/2}$,
\[ \hat{n} \in \mathbb{N}, \] and show that the Hamiltonians, thus modified, have an eigenvalue at the bottom of their spectrum. This is by now fairly standard and works under the assumptions of Hyp. 11.1. Let \( \Psi^{(\hat{n})} \) and \( \Psi_n^{(\hat{n})}, n \geq n_0, \) be corresponding normalized eigenvectors. Then, in the next step, one removes the artificial photon mass \( \hat{n}^{-1} \) by a compactness argument implying that \( \{\Psi^{(\hat{n})}\}_{\hat{n} \in \mathbb{N}} \) contain norm convergent subsequences. The arguments of [10] do not apply here directly, because we work with less specific assumptions on the coefficient vectors. To remedy this one can, however, adapt the compactness argument presented in this section: just consider constant sequences of potentials \( V \) and coefficient vectors \( c, \) and the non-constant sequence \( \omega_n, \hat{n} \in \mathbb{N}, \) instead.

**Theorem 11.4.** In the setting of Hyp. 11.1 and using the notation of Prop. 11.3 we may conclude that the following holds true:

1. \( \Psi_n \to \Psi, n \to \infty, \) in \( L^2(\mathbb{R}^3, \mathcal{F}). \)
2. Let \( \{x_n\}_{n \in \mathbb{N}} \) be a converging sequence in \( \mathbb{R}^3 \) with limit \( x \) and let us consider the unique continuous representatives of the ground state eigenvectors. Then \( \Psi_n(x_n) \to \Psi(x), n \to \infty, \) in \( \mathcal{F}. \) If \( t > 0 \) and \( (\Upsilon_t)_{t \geq 0} \) are given by either Line 2 or Line 4 of Table 1, and if we additionally assume that the maps \( \mathbb{R}^\nu \ni x \mapsto c \) and \( \mathbb{R}^\nu \ni x \mapsto c_n \), \( n \in \mathbb{N}, \) are continuous and uniformly bounded with respect to the norm \( \| \cdot \|_*, \) then \( \Upsilon_t \Psi_n(x_n) \to \Upsilon_t \Psi(x), n \to \infty, \) in \( \mathcal{F}. \)

**Proof.** (1): Let \( \{\Psi_{n_j}\}_{j \in \mathbb{N}} \) be any subsequence of \( \{\Psi_n\}_{n \geq n_0}. \) By Prop. 11.3 below this subsequence contains another subsequence, say \( \{\Psi_{j_k}\}_{k \in \mathbb{N}}, \) such that \( \Psi_{j_k} \to \Psi_\infty, j \to \infty, \) for some normalized \( \Psi_\infty \in \mathcal{H}. \) Since each \( \Psi_{j_k} \) is strictly positive, \( \Psi_\infty \) is non-negative. Furthermore, Thm. 5.2, Thm. 6.1 and Hyp. 11.1 imply that \( H_n \) converges to \( H \) in the strong resolvent sense; recall the proof of Cor. 5.4 and that strong convergence of the semi-group implies strong resolvent convergence. Taking also (11.3) into account, we deduce that

\[
\Psi_\infty = \lim_{j \to \infty} \Psi_{j_k} = \lim_{j \to \infty} (H_{j_k} - E_{j_k} + 1)^{-1} \Psi_{j_k} = (H - E + 1)^{-1} \Psi_\infty,
\]

whence \( \Psi_\infty \in \mathcal{D}(H) \) and \( H \Psi_\infty = E \Psi_\infty. \) Since \( \Psi_\infty \) is normalized and non-negative, we must have that \( \Psi_\infty = \Psi. \) Hence, every subsequence of \( \{\Psi_n\}_{n \geq n_0} \) contains a subsequence that converges to \( \Psi. \) This is, however, only possible, if \( \Psi_n \to \Psi, n \to \infty, \) in \( \mathcal{H}. \)

(2): Set \( \tau := 2t. \) By (11.3) and Part (1) the vectors \( \Psi_n' := e^{\tau E} \Psi_n, n \geq n_0, \) converge in \( L^2(\mathbb{R}^3, \mathcal{F}) \) to \( \Psi' := e^{\tau E} \Psi, \) as \( n \) goes to infinity. Furthermore, \( \Psi = e^{-\tau H} \Psi' = T_\tau \Psi' \) and similarly \( \Psi_n = T_{\tau_n} \Psi_n', n \geq n_0, \) where we use some notation of Cor. 5.2. The latter now implies all statements under (2).

**Proposition 11.5.** Every subsequence of the eigenvector sequence \( \{\Psi_n\}_{n \geq n_0} \) found in Prop. 11.3 contains another subsequence that converges in \( \mathcal{H}. \)

The previous proposition is proved at the end of this section. The remainder of this section serves as a preparation for that proof.

The compactness argument carried through below (see in particular Lem. 11.11) requires some information on the uniform spatial localization of the ground state eigenvectors \( \Psi_n. \) This is provided by the next proposition which is a corollary of a result in [9].

In what follows, the symbol \( F(\hat{x}) \) will be a convenient notation for the multiplication operator associated with a measurable function \( \mathbb{R}^3 \ni x \mapsto F(x). \)
Proposition 11.6. In the situation of Hyp. (11.4) pick $a, b > 0$ and $\delta \geq 0$ with $E + \delta + b + a^2/2 < \Sigma$. Then we find some $n_0 \in \mathbb{N}$ such that

$$\Sigma_n \geq E_n + \delta + b + a^2/2, \quad n \geq n_0.$$  

We further find $c, c', c'' > 0$ such that, for every $n \geq n_0$ and every normalized element $\Phi_n$ of the range of the spectral projection $1_{(-\infty, E_n + \delta)}(H_n)$, the following bounds hold true,

$$\|e^{a|x|}\Phi_n\| \leq c,$$  

$$\int_{\mathbb{R}^3} e^{2a|x|} \left\|(-i\nabla - \varphi(G_{n,x}))\Phi_n(x)\right\|^2 \mathrm{d}x \leq c',$$  

$$\int_{\mathbb{R}^3} e^{2a|x|} \left\|\nabla \Phi_n(x)\right\|^2 \mathrm{d}x \leq c''.$$  

Furthermore, if $\sup\{\|\omega^n c_{n,x}\| : n \in \mathbb{N}, x \in \mathbb{R}^3\} < \infty$, for some $\alpha \geq 1$, then there exists $c''' > 0$ such that, for every $n \geq n_0$, the unique continuous representative of $\Psi_n$ satisfies

$$\|(1 + d\Gamma(\omega))^{\alpha}\Phi_n(x)\| \leq c''' e^{-a|x|}, \quad x \in \mathbb{R}^\nu.$$  

If $\sup\{\|\epsilon^{\delta\omega} c_{n,x}\| : n \in \mathbb{N}, x \in \mathbb{R}^3\} < \infty$, for some $\delta_0 > 0$, then there exists $c'''' > 0$ and $\delta \in (0, \delta_0)$ such that, for all $n \geq n_0$,

$$\|\epsilon^{\delta\Gamma(\omega)}\Phi_n(x)\| \leq c'''' e^{-a|x|}, \quad x \in \mathbb{R}^\nu.$$  

Remark 11.7. Again the pointwise bounds (11.8) and (11.9) are new for general elements in spectral subspaces that are not eigenvectors. If $V(x)$ goes to infinity, as $|x| \to \infty$, which clearly entails $\Sigma = \infty$, then one can actually observe a much faster, possibly super-exponential decay of ground state eigenfunctions [10, 21, 25]. The bounds of Thm. (11.6) are, however, more than sufficient for the present purpose.

Proof of Prop. (11.4). Combining the estimates in the proof of [9, Thm. 1] and analyzing the integral involving the almost analytic function appearing there, it is easy to see that there exists a universal constant $C > 0$ such that

$$\|e^{a|x|}\Phi_n\| \leq Ce^{2aR} (\Sigma_{n,R} - E_n)(\delta + b) \max\{b^{-1}, b^{-3}\}, \quad n \geq n_0,$$

provided that we can choose $R \geq 1$ and $n_0 \in \mathbb{N}$ so large that $16/R^2 \leq b$ and $\Sigma_{n,R} \geq E_n + \delta + 3b/4 + a^2/2$, for all $n \geq n_0$. That this is actually possible can be seen as follows: First, we fix some $R \geq 4/b^{1/2}$ such that $\Sigma_R \geq E + \delta + 7b/8 + a^2/2$. After that we avail ourselves of (11.3) and pick $n_0$ so large that $|\Sigma_{n,R} - \Sigma_R| \leq b/16$ and $|E_n - E| \leq b/16$, for all $n \geq n_0$. We conclude that (11.10) implies a bound on its left hand side that is uniform in $n \geq n_0$. This proves (11.5). Next, we set $F(x) := a|x|, \quad x \in \mathbb{R}^\nu$, and pick $n_0$ so large that $|\Sigma_{n,R} - \Sigma_R| \leq b/16$ and $|E_n - E| \leq b/16$, for all $n \geq n_0$. We conclude that (11.10) implies a bound on its left hand side that is uniform in $n \geq n_0$. This proves (11.5). Next, we set $F(x) := a|x|, \quad x \in \mathbb{R}^\nu$, and pick $n_0$ so large that $|\Sigma_{n,R} - \Sigma_R| \leq b/16$ and $|E_n - E| \leq b/16$, for all $n \geq n_0$. We conclude that (11.10) implies a bound on its left hand side that is uniform in $n \geq n_0$. This proves (11.5). Next, we set $F(x) := a|x|, \quad x \in \mathbb{R}^\nu$, and pick $n_0$ so large that $|\Sigma_{n,R} - \Sigma_R| \leq b/16$ and $|E_n - E| \leq b/16$, for all $n \geq n_0$. We conclude that (11.10) implies a bound on its left hand side that is uniform in $n \geq n_0$. This proves (11.5). Next, we set $F(x) := a|x|, \quad x \in \mathbb{R}^\nu$, and pick $n_0$ so large that $|\Sigma_{n,R} - \Sigma_R| \leq b/16$ and $|E_n - E| \leq b/16$, for all $n \geq n_0$. We conclude that (11.10) implies a bound on its left hand side that is uniform in $n \geq n_0$. This proves (11.5). Next, we set $F(x) := a|x|, \quad x \in \mathbb{R}^\nu$, and pick $n_0$ so large that $|\Sigma_{n,R} - \Sigma_R| \leq b/16$ and $|E_n - E| \leq b/16$, for all $n \geq n_0$. We conclude that (11.10) implies a bound on its left hand side that is uniform in $n \geq n_0$. This proves (11.5).
Let \( \chi \in C^\infty(\mathbb{R}^d) \) be bounded with bounded first order partial derivatives. Then it is easy to see that \( \chi \mathcal{Q}(H) \subset \mathcal{Q}(H) \subset \mathcal{D}(-i\partial_x - \varphi(G_n,j)), \ j \in \{1,2,3\} \), and we obtain
\[
(11.11) \quad \| \chi(-i\nabla - \varphi(G_n)) \Phi_n \| \leq \| (-i\nabla - \varphi(G_n)) \chi \Phi_n \| + \| (\nabla \chi) \Phi_n \|, \quad n \in \mathbb{N}.
\]
Denoting the quadratic form associated with \( H_n \) by \( q_n \), we further have, for some \( \beta > 0 \) and all \( n \geq n_0 \),
\[
\| (-i\nabla - \varphi(G_n)) \chi \Phi_n \|^2 \leq 4q_n[\chi \Phi_n] + 2\beta \| \chi \Phi_n \|^2
\]
\[
= 4\text{Re} q_n[\chi^2 \Phi_n, \Phi_n] + 4\| (\nabla \chi) \Phi_n \|^2 + 2\beta \| \chi \Phi_n \|^2.
\]
(11.12) \[
\frac{1}{2} \| e^F (-i\nabla - \varphi(G_n)) \Phi_n \|^2 \leq (4\Sigma + 2\beta + 5a^2)c^2.
\]
where we used a well-known identity for the form \( q_n \) in the second step; see, e.g., [9, p. 324]. We now infer from (11.11) and (11.12) that, if \( \chi \) is of the form \( \chi = e^\varphi \) with a bounded smooth \( \varphi \) properties of the coefficient vectors \( \alpha \) \( \chi \), we drop it here. We just remark that the technical details will be facilitated by the key identity in the compactness argument. It allows to exploit the convergence \( \alpha \).

In the situation described by Hyp. 11.1, let \( \varphi \) be the ground state eigenvector found in Prop. 11.3. Then the following identity holds, for a.e. \( \varphi \in (\mathbb{R}^3 \setminus \{0\}) \times \{0,1\} \),
\[
(a\Psi_n)(k) = B_n(k)G_{n,0}(k) \cdot \hat{x}\Psi_n + R_n(k)\frac{i}{2}q_n,\hat{x}(k)\Psi_n
\]
\[
+ R_n(k)\tilde{G}_{n,\hat{x}}(k) \cdot (-i\nabla - \varphi(G_n))\Psi_n.
\]
(11.13)

**Proof.** The proof follows a standard procedure (see, e.g., [2, 10, 21, 35]), whence we drop it here. We just remark that the technical details will be facilitated by exploiting that \( H_n \) is essentially self-adjoint on the complex linear span of the functions \( f\phi \), where \( f \) is in the domain of the Schrödinger operator \(-\frac{1}{2}\Delta + V\) and \( \phi \) is, e.g., an analytic vector for \( df(\omega) \); in the situation of Hyp. 11.1 the latter result does not yet follow from the existing literature, but it will be shown in the forthcoming note [27].

**Corollary 11.9.** In the situation of Prop. 11.8, \( \sup_{n \in \mathbb{N}} \| d\Gamma(1)^{1/2}\Psi_n \| < \infty. \)
Proof. As usual, this follows easily from Hyp. \([1.13]\) and the fact that \(\Phi \in O(d\Gamma(1))\), if and only if the right hand side of the formula \(\|d\Gamma(1)^{1/2}\Phi\|^2 = \sum_{\lambda=0}^{1} \int_{\mathbb{R}^3} a(k,\lambda)\Phi^2 d\mathbf{k}\) is finite, the latter being valid in the affirmative case. \(\square\)

**Corollary 11.10.** In the situation of Prop. \([1.13]\) let \(\bar{x} \in C^\infty(\mathbb{R}, \mathbb{R})\) be such that \(0 \leq \bar{x} \leq 1\), \(\bar{x} = 0\) on \((-\infty, 1]\) and \(\bar{x} = 1\) on \([2, \infty)\). Set \(\chi_\delta(k) := \bar{x}(|k|/\delta)\), for all \(k \in \mathbb{R}^3\) and \(\delta \in (0, 1]\). For all \(0 \leq r_0 < r_1 \leq \infty\), \(\delta \in (0, 1]\), and \(h \in \mathbb{R}^3\), we further define

\[
F(r_0, r_1) := \sup_{n \in \mathbb{N}} \sum_{\lambda=0}^{1} \int_{\mathbb{R}^3} 1_{\{r_0 \leq |k| \leq r_1\}} \|a\psi_n(k,\lambda)\|^2 d\mathbf{k},
\]

\[
\Delta_\delta(h) := \sup_{n \in \mathbb{N}} \sum_{\lambda=0}^{1} \int_{\mathbb{R}^3} (\chi_{\delta a}\psi_n(k,\lambda) - (\chi_{\delta a}\psi_n)(k + h,\lambda))^2 d\mathbf{k}.
\]

Then \(F(r_0, \infty) \to 0\), as \(r_0 \to \infty\), \(F(0, r_1) \to 0\), as \(r_1 \downarrow 0\), and \(\Delta_\delta(h) \to 0\), as \(h \to 0\), for every \(\delta \in (0, 1]\).

**Proof.** To start with we observe that \(\|B_n(k)\| \leq 1\), \(\|R_n(k)\| \leq 1/\omega(k)\), as well as \(\|\hat{K}_R(\hat{x})\psi_n\| \leq 1/R, R \geq 1\), and that, by \([1.13]\) and \([1.14]\), the norms of \(\langle \hat{x} \rangle^2 \psi_n\) and \(\langle \hat{x}(\hat{x}^2 - \varphi(G_n))\rangle \psi_n\) are bounded uniformly in \(n\).

Let \(\varepsilon > 0\) and define

\[
\Phi_n^R(k) := B_n(k)G_n(x)\hat{x}\psi_n + R_n(k)\hat{x}g_n(k)1_{K_R}(\hat{x})\psi_n
\]

\[
(1.14)
\]

By virtue of Hyp. \([1.1.1]\) and the preceding remarks, we may fix \(R \geq 1\) such that \(\|a\psi_n - \Phi_n^R\|_{L^2(\mathbb{R}^3 \times \{0, 1\}, \omega^{-1} \phi)} < \varepsilon\), for all \(n \in \mathbb{N}\). Moreover, it follows from \((1.1.4)\) that

\[
\sup_{n \in \mathbb{N}} \sum_{\lambda=0}^{1} \int_{\mathbb{R}^3} 1_{\{r_0 \leq |k| \leq r_1\}} \|\Phi_n^R(k,\lambda)\|^2 d\mathbf{k}
\]

\[
\leq c^2 \sup_{n \in \mathbb{N}} \sup_{\mathbf{x} \in K_R} \sum_{\lambda=0}^{1} \|g_n k\|_{h}^2 =: F_R(r_0, r_1),
\]

for some \((n, r_0, r_1)\)-independent \(c > 0\), where \(g_{n,x} : = (G_n, \omega^{-1}g_n, \omega^{-1}\hat{G}_n)\). By Hyp. \([1.1.1]\) the sequence of functions \(\{g_n\}_n\) converges in \(C(K_R, h^*)\). Therefore, if we turn \(\mathbb{N} := \mathbb{N} \cup \{\infty\}\) into a compact topological space by demanding that the map \(n \mapsto n^{-1}\) is a homeomorphism onto \(\{0\} \cup \{n^{-1} : n \in \mathbb{N}\} \subset \mathbb{R}\), then the map \(\mathbb{N} \times K_R \ni (n, x) \mapsto g_{n,x} \in h^*\) has a jointly continuous extension to \(\mathbb{N} \times K_R\) in particular its image is relatively compact in \(h^*\). We can thus find finitely many functions in \(h^\gamma\) with compact supports in \((\mathbb{R}^3 \setminus \{0\}) \times \{0, 1\}\), such that the set \(\{g_{n,x} : n \in \mathbb{N}, x \in K_R\}\) is contained in the union of the open balls of radius \(\varepsilon\) about these compactly supported functions. If \(r_0\) is large enough (resp. \(r_1\) small enough), then the supports of all these finitely many compactly supported functions are disjoint from \(\{\varepsilon \leq \omega\} \) (resp. \(\{\omega \leq \varepsilon\})\). Then it follows that \(\limsup_{r_0 \to \infty} F_R(r_0, \infty)^{1/2} \leq \varepsilon\), thus \(\limsup_{r_0 \to \infty} F(r_0, \infty)^{1/2} \leq (1 + c)\varepsilon\), and similarly \(\limsup_{r_1 \to 0} F(0, r_1)^{1/2} \leq (1 + c)\varepsilon\).

We may further verify that the operator-valued functions \(\mathbb{R}^d \setminus \{0\} \ni k \mapsto \chi_\delta(k)B_n(k,\lambda)\) and \(\mathbb{R}^d \setminus \{0\} \ni k \mapsto \chi_\delta(k)R_n(k,\lambda)\) with \(\lambda \in \{0, 1\}\) are continuously differentiable with \(\|\nabla_k \{\chi_\delta(k)B_n(k,\lambda)\}\| \leq c/\delta^2\) and \(\|\nabla_k \{\chi_\delta(k)R_n(k,\lambda)\}\| \leq c/\delta\).}
\[c'/\delta^2, \text{ for some universal constant } c' > 0 \text{ and all } \delta \in (0, 1]. \text{ Likewise, } |\nabla_k (\chi_\delta / \omega)| \leq c'/\delta^2, \delta \in (0, 1]. \text{ Combining these remarks with the observations made in the first paragraph of this proof, we easily find } c'', c''' > 0 \text{ such that, for all } \delta \in (0, 1] \text{ and } h \in \mathbb{R}^d,
\]

\[
\sup_{n \in \mathbb{N}} \sum_{\lambda = 0, 1} \int_{\mathbb{R}^d} \| (\chi_\delta \Phi_n^R(k, \lambda) - (\chi_\delta \Phi_n^R)(k + h, \lambda)) \|^2 \, dk \\
\leq \frac{c''}{\delta^2} |h|^2 \sup_{n \in \mathbb{N}} \sum_{x \in K_R} \int_{\mathbb{R}^d} |\omega g_{n,x}|^2 (k + h, \lambda) \, dk \\
+ c''' \sup_{n \in \mathbb{N}} \sum_{x \in K_R} \int_{\mathbb{R}^d} |(\omega g_{n,x})(k + h, \lambda) - (\omega g_{n,x})(k, \lambda)|^2 \, dk.
\]

The supremum in the first line of the right hand side of (11.15) is obviously \(h\)-independent; it is finite, because \(\{g_n\}_n\) converges in \(C(K_R, h^2)\). The supremum in the second line of (11.15) goes to 0 as \(h \to 0\) since, by the above arguments, the set \(\{\omega g_{n,x} : n \in \mathbb{N}, x \in K_R\}\) is relatively compact in \(h^2\) and in particular the shift operation applied to its elements is norm-continuous uniformly in \((n, x) \in \mathbb{N} \times K_R\). Altogether this implies \(\lim_{h \to 0} \sup_n \Delta_\delta(h)^{1/2} \leq 2\varepsilon\) and we conclude recalling that \(\varepsilon > 0\) was arbitrary.

Employing the canonical isomorphism \(\mathcal{H} = \bigoplus_{m=0}^{\infty} L^2(\mathbb{R}^3, \mathcal{F}^{(m)})\), we represent the ground state eigenvectors found in Prop. 11.13 as sequences \(\Psi_n = (\Psi_n^{(m)})_{m \in \mathbb{N}_0}\) with \(\Psi_n^{(m)} \in \mathcal{F}^{(m)}\) in the next lemma.

**Lemma 11.11.** Let \(m \in \mathbb{N}, \lambda = (\lambda_1, \ldots, \lambda_m) \in \{0, 1\}^m, \text{ and } \delta \in (0, 1]. \) Put

\[
\Psi_{\delta, n}^{(m, \lambda)}(x, k_1, \ldots, k_m) := \left( \prod_{j=1}^{m} \chi_\delta(k_j) \right) \Psi_n^{(m)}(x, k_1, \lambda_1, \ldots, k_m, \lambda_m),
\]

where \(\chi_\delta\) is defined in the statement of Cor. 11.10. Then the set

\[
\mathcal{K}_{\delta}^{(m, \lambda)} := \{ \Psi_{\delta, n}^{(m, \lambda)} : n \in \mathbb{N} \}
\]

is relatively compact in \(L^2(\mathbb{R}^{3(m+1)})\).

**Proof.** For a start, it is clear that \(\mathcal{K}_{\delta}^{(m, \lambda)}\) is contained in the closed unit ball about the origin in \(L^2(\mathbb{R}^{3(m+1)})\). Furthermore, writing \(k_{[m]} := (k_1, \ldots, k_m)\), etc., we observe the following pointwise bounds between functions defined on \(\mathbb{R}^{3(m+1)}\),

\[
1_{\{|x, k_{[m]}| \geq R\}} \leq 1_{\{|x| \geq R/(m+1)\}} + \sum_{j=1}^{m} 1_{\{|k_j| \geq R/(m+1)\}}, \quad R > 0.
\]
We shall show that \( \parallel \) weakly converging subsequence, say \( \parallel \chi \), these remarks together and applying Cor. 11.10, we deduce that 
\[
\delta \in O. MATTE (K)
\]
Since the expression tending to zero in the last line is \( n \)-independent, we see that \( K_\delta^{(m, \Delta)} \) is uniformly integrable.

Next, let \( S(y, h_{(m)}) \) be the unitary operator that shifts the variables \( (x, k_{(m)}) \) by the vector \( (y, h_{(m)}) \in \mathbb{R}^{(m+1)} \) and let \( S^{(1)}_{h_j} \) be the unitary operator shifting only the variable \( k_1 \) by \( h_j \). By a telescopic sum argument and the permutation symmetry of \( \Psi_{\delta,n}^{(m)} \) in its last \( m \) variables, we then obtain
\[
\| \Psi_{\delta,n}^{(m, \Delta)} - S(y, h_{(m)}) \Psi_{\delta,n}^{(m, \Delta)} \| \leq \| \Psi_n - S(y, h) \Psi_n \| + \sum_{j=1}^{m} \| \Psi_{\delta,n}^{(m)} - S^{(1)}_{h_j} \Psi_{\delta,n}^{(m)} \|.
\]

By the definition of the pointwise annihilation operator,
\[
\| \Psi_{\delta,n}^{(m)} - S^{(1)}_{h} \Psi_{\delta,n}^{(m)} \|^2 = \frac{1}{m} \sum_{\lambda=0}^{m-1} \int_{\mathbb{R}^3} |x_{\delta}^{-1} \left( (\chi_\delta a \Psi_n)^{(m-1)}(k, \lambda) - (\chi_\delta a \Psi_n)^{(m-1)}(k + h, \lambda) \right)|^2 \, dk,
\]
where the norm under the \( dk \)-integral is the one on \( L^2(\mathbb{R}^3 \times (\mathbb{R}^3 \times \{0,1\})^{(m-1)}) \) and \( \chi_\delta^{-1} \) is the multiplication operator associated with the function \( (k_1, \ldots, k_{m-1}) \mapsto \chi_\delta(k_1) \cdots \chi_\delta(k_{m-1}) \). Therefore, \( \sup_n \| \Psi_{\delta,n}^{(m)} - S^{(1)}_{h} \Psi_{\delta,n}^{(m)} \|^2 \leq \triangle_{\delta}(h) \). We finally observe that \( \| \Psi_n - S(y, h) \Psi_n \| \leq |y| \| \tilde{\Psi}_n \| \leq c'' |y| \), with the \( n \)-independent constant \( c'' \) appearing in (11.7). (Here \( \tilde{\Psi}_n \) is the \( \mathcal{F} \)-valued Fourier transform of \( \Psi_n \).) Putting these remarks together and applying Cor. 11.10 we deduce that
\[
\sup_{n \in \mathbb{N}} \| \Psi_{\delta,n}^{(m, \Delta)} - S(y, h_{(m)}) \Psi_{\delta,n}^{(m, \Delta)} \| \rightarrow_{(y, h_{(m)}) \rightarrow 0} 0.
\]

By the well-known characterization of relatively compact sets in \( L^2(\mathbb{R}^{3(m+1)}) \), this proves the assertion. \( \square \)

Finally, we can keep a promise that will also conclude the proof of Thm. 11.3

Proof of Prop. 11.3. Let \( 1 \leq n_1 < n_2 < \ldots \) be integers. Then \( \{ \Psi_{n_j} \}_{j \in \mathbb{N}} \) contains a weakly converging subsequence, say \( \{ \Psi_{n_j} \}_{j \in \mathbb{N}} \), whose weak limit we denote by \( \Psi_\infty \). We shall show that \( \| \Psi_\infty \| = 1 \), which will imply that \( \{ \Psi_{n_j} \}_{j \in \mathbb{N}} \) is actually norm convergent.
To this end, let $m_0 \in \mathbb{N}$ and $\delta \in (0, 1]$. By virtue of Lem. [11.11] we may successively, in a finite number of steps, select subsequences to find $\kappa_\ell \in \mathbb{N}$, $\ell \in \mathbb{N}$, with $\kappa_\ell \to \infty$, $\ell \to \infty$, such that every sequence $\{\Psi^{(m, \lambda)}_{\delta, \kappa_\ell}\}_{\ell \in \mathbb{N}}$ with $m \in \{0, \ldots, m_0\}$ and $\lambda \in \{0, 1\}^m$ converges strongly to its weak limit $\Gamma(\chi, \delta)\Psi^\lambda$. Let $p_{m_0}$ be the orthogonal projection onto the first $m_0 + 1$ direct summands is $\mathcal{H} = \bigoplus_{m=0}^\infty L^2(R^3, \mathcal{F}(m))$. Then

$$\|\Psi_\infty\| \geq \|p_{m_0} \Gamma(\chi, \delta)\Psi_\infty\| = \lim_{\ell \to \infty} \|p_{m_0} \Gamma(\chi, \delta)\Psi_{\kappa_\ell}\|$$

$$\geq \lim_{\ell \to \infty} \|p_{m_0} \Psi_{\kappa_\ell}\| - \sup_n \langle \Psi_n \rangle \{1 - \Gamma(\chi, \delta)\Psi_n\}^{1/2}$$

$$(11.16) \geq 1 - \sup_n \| (1 - p_{m_0})\Psi_n\| - \sup_n \|\Gamma(\chi, \delta)\Psi_n\|.$$  

Here Cor. [11.9] implies $\sup_n \| (1 - p_{m_0})\Psi_n\| \leq m_0^{-1/2} \sup_n \|\Gamma(\chi, \delta)\Psi_n\| \leq cm_0^{-1/2}$, while Cor. [11.10] entails

$$\|\Gamma(\chi, \delta)\Psi_n\|^2 = \sum_{\lambda=0}^1 \int_{R^3} |\chi_\delta(k)| a(k, \lambda)\Psi_n|^2 dk \leq F(0, 2\delta) \delta^{-1} \to 0,$$

where $c$ and $F(0, 2\delta)$ are $m$-independent. Since $m_0 \in \mathbb{N}$ was arbitrary large and $\delta \in (0, 1]$ arbitrary small in [11.10], we conclude that $\|\Psi_\infty\| = 1$. \qed

**APPENDIX A. MULTIPLE COMMUTATOR ESTIMATES**

In this appendix we derive some norm bounds on commutators between creation and annihilation operators and functions of second quantized multiplication operators in the boson Hilbert space. They are applied in Sect. 4 to estimate the norms of the operators in (3.7)–(3.9) and in particular of

$$2T_1(s) = \vartheta^{-1/2} \Theta^{-1}_s \{\text{ad}_{\varphi(G_{B})}^2(\Theta)^2\} \Theta^{-1}_s \vartheta^{-1/2}$$

$$= 2\{\vartheta^{-1/2} \Theta^{-1}_s \text{ad}_{\varphi(G_{B})} \Theta\} \{\text{ad}_{\varphi(G_{B})} \Theta\} \Theta^{-1}_s \vartheta^{-1/2}\}$$

$$+ \vartheta^{-1/2} \Theta^{-1}_s \text{ad}_{\varphi(G_{B})} \Theta\vartheta^{-1/2} + \vartheta^{-1/2} \text{ad}_{\varphi(G_{B})} \Theta\Theta^{-1}_s \vartheta^{-1/2},$$

(A.1)

where $\text{ad}_{\mathcal{S}} T := [S, T]$ and where we used the product rule $\text{ad}_S(TT') = T\text{ad}_S(T') + (\text{ad}_S T)T'$. The lower the power of $\Theta_s$, the weaker the conditions imposed on the coefficient vector $c$ in our bounds below, whence we wrote commutators with $\Theta^2_s$ as combinations of commutators involving only $\Theta_s$. Likewise,

$$T_2(s) = -i(\text{ad}_{\varphi(G_{B})} \Theta) \Theta^{-1}_s \vartheta^{-1/2}$$

$$- (\text{ad}_{\varphi(G_{B})} \Theta) \Theta^{-1}_s \vartheta^{-1/2} + \Theta^{-1}_s (\text{ad}_{\varphi(G_{B})} \Theta) \vartheta^{-1/2},$$

(A.2)

$$T(s) = -2(\text{ad}_{\varphi(G_{B})} \Theta) \Theta^{-1}_s \vartheta^{-1/2},$$

(A.3)

Let us also note for later reference that, for every $g \in \mathfrak{h}$,

$$\text{ad}_{\varphi(g)} \Theta^2_s \Theta^2_s = -\Theta^2_s \text{ad}_{\varphi(g)} \Theta^2_s,$$

(A.4)

$$\text{ad}_{\varphi(g)} \Theta^2_s \Theta^2_s = -\Theta^2_s \text{ad}_{\varphi(g)} \Theta^2_s + 2(\Theta^2_s \text{ad}_{\varphi(g)} \Theta^2_s)^2.$$
In view of \((2.4)\) all simple and double commutators appearing above can be written as linear combinations of terms of the form

\[
(1 + d\Gamma(\omega))^{-n/2} \Theta_s^{-\beta+\gamma} \left\{ \prod_{j=1}^N \text{ad}_{a(f_j)} \right\} \left( \prod_{\ell=1}^M \text{ad}_{a(g_{\ell})} \right) \Theta_s^{-\gamma-n}(1 + d\Gamma(\omega))^{-m/2}.
\]

with \(\kappa = 0\), \(\beta + \gamma = 1\), \(M + N \leq 2\), and \(m + n \leq M + N\). Here we used that, on account of \((A.1)\) and the Jacobi identity, the order of the \(M + N\) commutations with the creation or annihilation operators is immaterial. In all cases of our present interest \(\Theta_s\) is some function of a second quantized multiplication operator. Since we need bounds on simple and double commutators involving all combinations of the creation and annihilation operators and different functions of various second quantized operators, a systematic treatment seems to be in order. We shall consider the general case of multiple commutators with arbitrary \(M\) and \(N\) right away. The resulting bounds might also be useful elsewhere.

Let \(\mathcal{L}\) be some finite index set, \(M := \#\mathcal{L}\), and let \(p_{\mathcal{L}} := (p_\ell)_{\ell \in \mathcal{L}} \in M^{2^{\mathcal{L}}}\). For \(\psi \in \mathcal{D}(d\Gamma(1)^{M/2})\), or rather a representative of it, and for \(n \in \mathbb{N}_0\), we write

\[
(a(p_{\mathcal{L}}) \psi)^{(n)}(k_{[n]}) := (n + M)^{1/2} \ldots (n + 1)^{1/2} \phi^{(n+M)}(k_{[n]}, p_{\mathcal{L}}), \quad k_{[n]} \in \mathcal{M}^n.
\]

For almost every \(p_{\mathcal{L}}\), this defines a new element \(a(p_{\mathcal{L}}) \psi\) of \(\mathcal{F}\). Notice that the order of the variables \(p_\ell\) is immaterial because of the permutation symmetry of \(\phi^{(n+M)}\) and that \(a(p_{\mathcal{L}}) = \prod_{\ell \in \mathcal{L}} a(p_\ell)\); compare \((2.2)\). The identity

\[
(A.6) \quad a(p_{\mathcal{L}}) a^\dagger(f) \psi = a^\dagger(f) a(p_{\mathcal{L}}) \psi + \sum_{\ell \in \mathcal{L}} f(p_\ell) a(p_{\mathcal{L}} \setminus \ell) \psi
\]

holds almost everywhere, if \(\psi \in \mathcal{D}(d\Gamma(1)^{(M+1)/2})\) and \(f \in \mathfrak{h}\). If \(\nu : \mathcal{M} \to \mathbb{R}^L\) is a vector of multiplication operators and if \(F : \mathbb{R}^L \to \mathbb{R}\) is measurable, then both sides of the following \emph{pull-through formula},

\[
(A.7) \quad a(p_{\mathcal{L}}) F(d\Gamma(\nu)) \psi = F(d\Gamma(\nu) + \sum_{\ell \in \mathcal{L}} \nu(p_\ell)) a(p_{\mathcal{L}}) \psi,
\]

define the same elements of \(\mathcal{F}\), for almost every \(p_{\mathcal{L}}\), provided that \(\psi\) belongs to the domain of \(d\Gamma(1)^{M/2} F(d\Gamma(\nu))\). This is a tautological consequence of the definitions. Together with \((2.4)\) an \(M\)-fold repeated application of the pull-through formula for each \(a(p_\ell)\) gives a handy formula for multiple commutators with annihilation operators of wave functions \(g_\ell \in \mathfrak{h}, \ell \in \mathcal{L}\);

\[
(A.8) \quad \left\langle \phi \left| \left( \prod_{\ell \in \mathcal{L}} \text{ad}_{a(g_\ell)} \right) F(d\Gamma(\nu)) \right| \psi \right\rangle = \int \left( \prod_{\ell \in \mathcal{L}} g(p_\ell) \right) \left\langle \phi \right| \triangle_{p_{\mathcal{L}}} F(d\Gamma(\nu)) a(p_{\mathcal{L}}) \psi \right\rangle \, d\mu(M(p_{\mathcal{L}})),
\]

for \(\psi\) as above and all \(\phi \in \mathcal{F}\), where, for any \(\tilde{F} : \mathbb{R}^L \to \mathbb{R}\),

\[
\triangle_{p_{\mathcal{L}}} \tilde{F}(d\Gamma(\nu)) := \tilde{F}(d\Gamma(\nu) + \nu(p_\ell)) - \tilde{F}(d\Gamma(\nu)), \quad \triangle_{p_{\mathcal{L}}} := \prod_{\ell \in \mathcal{L}} \triangle_{p_\ell}.
\]

Let \(\mathcal{J}\) be another finite index set disjoint from \(\mathcal{L}, N := \#\mathcal{J}\), and pick additional wave functions, \(g_j \in \mathfrak{h}, j \in \mathcal{J}\). Applying \((A.8)\) once with \(\phi\) replaced by \(a(g_j)\) \(\phi\).
and another time with $\psi$ substituted by $a^i(g_j)$ $\psi$ using (A.6), and substracting the results we find, for all $\phi, \psi \in \mathcal{D}(d\Gamma(1) \frac{\partial}{\partial v} F(d\Gamma(v)))$,

$$
\langle \phi | \text{ad}_{a^i(g_j)} \left( \prod_{\ell \in \mathcal{L}} \text{ad}_{a(g_i)} \right) F(d\Gamma(v)) \psi \rangle \\
= - \int \left( \prod_{\ell \in \mathcal{L}} \overline{g(p\ell)} \right) \langle \text{ad}_{a(g_j)} \Delta_{p\mathcal{L}} F(d\Gamma(v)) \phi | a(p\mathcal{L}) \psi \rangle d\mu^M(p\mathcal{L}) \\
- \sum_{\ell \in \mathcal{L}} \int \left( \prod_{\ell \in \mathcal{L}} \overline{g(p\ell)} \right) g_{\ell}(p\ell) \langle \phi | \Delta_{p\mathcal{L}\setminus \{\ell'\}} F(d\Gamma(v)) a(p\mathcal{L}\setminus \{\ell'\}) \psi \rangle d\mu^M(p\mathcal{L}) \\
= - \int \left( \prod_{\ell \in \mathcal{L}} \overline{g(p\ell)} \right) g_{j}(p_j) \langle a(p_j) \phi | \Delta_{p\mathcal{L}\cup \{j\}} F(d\Gamma(v)) a(p\mathcal{L}\cup \{j\}) \psi \rangle d\mu^{M+1}(p\mathcal{L}\cup \{j\}) \\
- \sum_{\ell \in \mathcal{L}} \int \left( \prod_{\ell \in \mathcal{L}} \overline{g(p\ell)} \right) g_{j}(p_j) \langle \phi | \Delta_{p\mathcal{L}} F(d\Gamma(v)) a(p\mathcal{L}\setminus \{j\}) \psi \rangle d\mu^M(p\mathcal{L}).
$$

Repeating this procedure using $a(p_i) a(g_j) = a(g_j) a(p_i)$ and applying (A.7) to $F_1$ and $F_3$ in (A.9) below we obtain the following result:

**Lemma A.1.** For $\ell = 1, 2, 3$, let $F_\ell : \mathbb{R}^{L_\ell} \rightarrow \mathbb{R}$ be measurable and let $v_\ell : \mathcal{M} \rightarrow \mathbb{R}^{L_\ell}$ be a vector of multiplication operators. With the notation explained above we have, for all $\phi, \psi \in \mathcal{D}(d\Gamma(1) \frac{\partial}{\partial v} \prod_{\ell = 1}^3 F_\ell(d\Gamma(v_\ell)))$,

$$
\langle F_1(d\Gamma(v_1)) \phi \left| \left( \prod_{j \in \mathfrak{J}} \text{ad}_{a^i(g_j)} \right) \left( \prod_{\ell \in \mathcal{L}} \text{ad}_{a(g_i)} \right) F_2(d\Gamma(v_2)) \right| F_3(d\Gamma(v_3)) \psi \rangle \\
= (-1)^N \sum_{\mathfrak{J} \cup \mathcal{L} \neq \emptyset} \int \langle a(p_{\mathfrak{J}}) \phi | M_{\mathfrak{J}, \mathcal{L}}(p_{\mathfrak{J} \cup \mathcal{L}}) a(p_{\mathcal{L}}) \psi \rangle d\mu^{\mathfrak{J} \cup \mathcal{L}}(p_{\mathfrak{J} \cup \mathcal{L}}),
$$

(A.9)

with the following family of multiplication operators, parametrized by $\mathfrak{J}, \mathcal{L}, p_{\mathfrak{J} \cup \mathcal{L}}$, and the states $g_{\ell} \in \mathfrak{h}$ (which are dropped in the notation),

$$
M_{\mathfrak{J}, \mathcal{L}}(p_\mathfrak{J}, p_\mathcal{L}, p_{\mathfrak{J} \cup \mathcal{L}}) := \sum_{\pi \in \text{Bij}(\mathcal{B}, \mathcal{D})} \left( \prod_{\ell \in \mathcal{L}} \overline{g_{\ell}(p\ell)} \right) \left( \prod_{\ell \in \mathfrak{J}} g_{\ell}(p_\mathfrak{J}) \right) \left( \prod_{a \in \mathfrak{A}} g_a(p_a) \right) \times \\
\times F_1 \left( d\Gamma\left( v_1 + \sum_{a \in \mathfrak{A}} v_1(p_a) \right) \right) \left\{ \Delta_{p_{\mathfrak{J} \cup \mathcal{L}}} F_2(d\Gamma(v_2)) \right\} F_3 \left( d\Gamma\left( v_3 + \sum_{c \in \mathfrak{E}} v_3(p_c) \right) \right).
$$

Here $\text{Bij}(\mathcal{B}, \mathcal{D})$ denotes the set of bijections of $\mathcal{B}$ onto $\mathcal{D}$.

The next lemma reduces the problem to find a bound on the expression (A.9) to the estimation of real functions of several variables.

**Lemma A.2.** For fixed $p_\mathfrak{B}$, let $\mathcal{M}(p_\mathfrak{J}, p_\mathcal{E}) := M_{\mathfrak{J}, \mathcal{L}}(p_\mathfrak{J}, p_\mathcal{B}, p_\mathcal{E})$ denote one of the multiplication operator-valued functions in (A.10) (for any other reasonable family of operators parametrized by $p_\mathfrak{J}$ and $p_\mathcal{E}$). Let $\theta := 1 + d\Gamma(\omega)$, where we suppose that the measurable function $\omega : \mathcal{M} \rightarrow \mathbb{R}$ is strictly positive almost everywhere.
Then, if \( m, n \in \mathbb{N}_0 \) with \( n \leq \#\mathcal{A} \) and \( m \leq \#\mathcal{C} \),
\[
\int \left| \langle a(p_{\mathcal{A}}) \theta^{-m/2} \phi \rangle, \mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{C}}) a(p_{\mathcal{C}}) \theta^{-m/2} \psi \rangle \right| \frac{d\mu^\#(\mathcal{A}\cup\mathcal{C}) (p_{\mathcal{A}\cup\mathcal{C}})}{((\#\mathcal{A} - n)!(\#\mathcal{C} - m)!)^{1/2}} \\
\leq ||\phi|| ||\psi|| \sum_{\#\mathcal{A} \geq n} \sum_{\#\mathcal{C} \geq m} \left[ \int \frac{||\theta^{(\#\mathcal{A} - n)/2} \mathcal{M}(p_{\mathcal{A}}, p_{\mathcal{C}}) \theta^{(\#\mathcal{C} - m)/2}||^2}{(\prod_{a \in \mathcal{A}} \omega(p_a)) (\prod_{c \in \mathcal{C}} \omega(p_c))} d\mu^\#(\mathcal{A}\cup\mathcal{C}) (p_{\mathcal{A}\cup\mathcal{C}}) \right]^{1/2}.
\]

If \( n > \#\mathcal{A} \), then the sum over \( a \) reduces to only one summand with \( a = \mathcal{A} \), \( \theta^{(\#\mathcal{A} - \#\mathcal{A})/2} \) replaced by \( \theta + \sum_{a \in \mathcal{A}} \omega(p_a) \), \( (\#\mathcal{A} - n)! \) replaced by \( 1 \). Analogous replacements occur in the case \( m > \#\mathcal{C} \). (The latter remarks apply in particular when \( n > \#\mathcal{A} \) and \( m > \#\mathcal{C} \).)

**Proof.** Let us first assume that \( n \leq \#\mathcal{A} \) and \( m \leq \#\mathcal{C} \). For notational simplicity we may also assume that \( \mathcal{A} = [N] := \{1, \ldots, N \} \) and \( \mathcal{C} = [M] \). Moreover, we shall use the letter \( k \) to denote variables labeled by \( \mathcal{A} = [N] \). We choose a partition \( \mathcal{M}^N = \bigcup_{\pi \in Bij([N],[N])} D_\pi(N) \) into disjoint measurable sets \( D_\pi(N) \) such that
\[
k_{\pi|}[N] \in D_\pi(N) \quad \Rightarrow \quad \omega(k_{\pi(1)}) \geq \omega(k_{\pi(2)}) \geq \ldots \geq \omega(k_{\pi(N)}),
\]
for all \( \pi \in Bij([N],[N]) \). We choose a partition \( \mathcal{M}^M = \bigcup_{\pi \in Bij([M],[M])} D_\pi(M) \) with the analogous property. We now fix permutations \( \pi \in Bij([N],[N]) \) and \( \pi' \in Bij([M],[M]) \), write \( D_{\pi,\pi'} := D_\pi(N) \times D_{\pi'}(M) \), and consider
\[
I_{\pi,\pi'} := \int_{D_{\pi,\pi'}} \left| \langle a(k_{[N]}) \theta^{-n/2} \phi \rangle, \mathcal{M}(k_{[N]}, p_{[M]}) a(p_{[M]}) \theta^{-m/2} \psi \rangle \right| d\mu^{M+N}(k_{[N]}, p_{[M]}).
\]
We use the less space consuming notation \( k_\pi^\# := k_{\pi(1)} \). By the pull-through formula,
\[
a(k_1) \ldots a(k_N) \theta^{-n/2} = a(k_1^\#) \ldots a(k_N^\#) \theta^{-n/2} = (\theta + \omega(k_1^\#)) \ldots (\theta + \omega(k_N^\#)) \theta^{-n/2} \ldots a(k_{n+1}^\#)((\theta + \omega(k_1^\#)) \ldots (\theta + \omega(k_N^\#)))^{-1/2} \times \ldots \times (\theta + \omega(k_{n+1}^\#)(\theta + \omega(k_1^\#)) \ldots (\theta + \omega(k_{n+1}^\#)))^{-1/2} \omega(k_1^\#) \theta^{-n/2},
\]
where the second line has to be ignored when \( n = N \). If \( N > n > n \), then the multiplication operator in the second line can be bounded, pointwise on \( D_\pi(N) \), as
\[
(\theta + \omega(k_{n+1}^\#)) \ldots (\theta + \omega(k_N^\#)) \theta^{-n/2} \leq ((N - n)!)^{1/2} (\theta^{1/2} + \omega(k_{n+1}^\#)) \ldots (\theta^{1/2} + \omega(k_N^\#))^{1/2})
\]
where we repeatedly used \( a + b \) \( a^2 + b^2 \) and \( \omega(k_{n+1}^\#) \) \( \leq \omega(k_{n+1}^\#) \) on \( D_\pi(N) \). An analogous estimate certainly holds for the variables \( p_{[M]} \) and the permutation \( \pi' \). Applying the Cauchy-Schwarz inequality to each summand thus contributing to \( I_{\pi,\pi'} \) we now see that
\[
\sum_{\pi,\pi'} I_{\pi,\pi'} \leq ((N - n)!(M - m)!)^{1/2} J_{n,\pi} N(\phi)^{1/2} J_{m,\pi} M(\psi)^{1/2}
\]

\[
\left( \sum_{b \in ([N] \setminus \{n\})} \int_{D_{\pi,\pi'}} \frac{||\theta^{b/2} \mathcal{M}(k_{[N]}, p_{[M]}) \theta^{b/2}||^2}{(\prod_{b \in [N] \setminus \{n\}} \omega(k_{b}^\#)) (\prod_{d \in [N] \setminus \{n\}} \omega(p_d^\#))} d\mu^{M+N}(k_{[N]}, p_{[M]}) \right)^{1/2},
\]
with

\[ J_{n,N}(\phi) := \int \omega(k_1) \cdots \int \omega(k_N) |a(k_N) \theta^{1/2} \cdots a(k_{n+1}) \theta^{1/2} \times \]
\[ \times a(k_n)(\theta + \omega(k_1) + \cdots + \omega(k_{n-1}))^{-1/2} \cdots a(k_1) \theta^{-1/2} \phi |^2 \, d\mu(k_N) \cdots d\mu(k_1), \]

and a similar definition for \( J_{m,M}(\psi) \). Applying the following familiar consequence of Fubini’s theorem and the permutation symmetry of \( \eta \in \mathcal{D}(d\Gamma(\omega)^{1/2}) \) repeatedly,

\[ \int \omega(k) |a(k)\eta|^2 \, d\mu(k) = ||d\Gamma(\omega)^{1/2}\eta||^2 \leq ||(\theta + t)^{1/2}\eta||^2, \quad t \geq 0, \]

we see that \( J_{n,N}(\phi) \leq ||\phi||^2 \) and, analogously, \( J_{m,M}(\psi) \leq ||\psi||^2 \).

If \( n > N \), then the second line in the formula for \( a(k_N) \cdots a(k_1) \theta^{-n/2} \) on \( D_\pi(N) \) above is replaced by \( (\theta + \omega(k_1^\pi) + \cdots + \omega(k_N^\pi))^{-n-N/2} \) and \( n \) is substituted by \( N \) in the third line. It is then clear how to conclude in this case. \( \square \)

The multiplication operators we have to bound according to the previous lemma involve the higher order difference operations appearing in the formula (A.10). To deal with such expressions we introduce some more notation.

For any real function \( f : \mathbb{R} \to \mathbb{R} \), we set

\[ \Delta_s f := (\tau_s - \mathbb{1})f, \quad (\tau_s f)(t) := f(t + s), \quad s, t \in \mathbb{R}, \]

and we shall write

\[ \Delta_s \mathcal{L} := \prod_{\ell \in \mathcal{L}} \Delta_{s,\ell}, \quad \tau_s \mathcal{L} := \prod_{\ell \in \mathcal{L}} \tau_{s,\ell}, \quad s, t \in \mathbb{R}. \]

For \( f_1, f_2 : \mathbb{R} \to \mathbb{R} \), we notice that the product rule,

(A.11) \[ \Delta_s (f_1 f_2) = (\Delta_s f_1) \tau_s f_2 + f_1 (\Delta_s f_2), \]

and the translation invariance of \( \Delta_s \) entail

(A.12) \[ \Delta_s \mathcal{L} (f_1 f_2) = \sum_{\alpha \cup \beta = \mathcal{L}} (\Delta_{s,\alpha} f_1) \tau_{s,\beta} \Delta_{s,\beta} f_2, \]

where \( \mathcal{L} \) and \( \mathcal{B} \) are always disjoint and possibly empty. Furthermore, we readily verify by induction that

(A.13) \[ \Delta_{s,\mathcal{L}} t^n = \sum_{\mathcal{N} \subseteq \mathcal{L}} \binom{n}{\mathcal{N}_\mathcal{L}} t^{n - |\mathcal{N}_\mathcal{L}|} \prod_{\ell \in \mathcal{L} \setminus \mathcal{N}} s_{\ell}^{t_{\ell}}, \quad \binom{n}{\mathcal{N}_\mathcal{L}} := \frac{n!}{(n - |\mathcal{N}_\mathcal{L}|)! \prod_{\ell \in \mathcal{N}_\mathcal{L}} \mathcal{N}_\ell !}, \]

where empty sums are zero and empty products are one. Notice that \( \mathcal{N}_\ell \geq 1 \), for each \( \ell \in \mathcal{L} \), in (A.13).

One is inclined to estimate higher order difference quotients by repeatedly applying the mean value theorem and using \( L^\infty \)-bounds on derivatives. A too naive estimation of the derivatives would, however, not be sufficient for the function \( t^n \) to find the canonical conditions on the photon states \( g_j \) in Lemma A.3 below, whence we shall argue more carefully in the next lemma.

**Lemma A.3.** Let \( a, \alpha > 0 \), \( \varepsilon \in [0, 1] \), \( s_0 := 0 \), \( s_\ell > 0 \) and \( \delta_\ell \in [0, 1] \), for \( \ell \in \mathcal{L} \), and write \( |\delta_{\mathcal{L}}| := \sum_{\ell \in \mathcal{L}} \delta_\ell \), \( s_{\mathcal{L}}^{\delta_{\mathcal{L}}} := \prod_{\ell \in \mathcal{L}} s_{\ell}^{\delta_\ell} \), etc., and

(A.14) \[ F_\varepsilon(t) := t/(1 + \varepsilon t), \quad t > 0. \]
Then the following bound holds, for all $t > 0$,

(A.15) \[ |\triangle s_{\alpha} F_{\alpha}^{-\alpha}(t)| \leq c(\alpha, \#L, \delta_{L}) F_{\alpha}^{-\alpha}(t) t^{-|\delta_{L}|} s_{\alpha}^{\delta_{L}}, \]

(A.16) \[ |\triangle s_{\alpha} F_{\alpha}^{\alpha}(t)| \leq c(\alpha, \#L, \delta_{L}) \sum_{\ell \in \{0\} \cup L} F_{\alpha}^{\alpha}(t + s_{\ell})(t + s_{\ell})^{-|\delta_{L}|} s_{\alpha}^{\delta_{L}}, \]

(A.17) \[ |\triangle s_{\alpha} e^{aF_{\alpha}(t)}| \leq c(\#L) a^{\delta_{L}} e^{aF_{\alpha}(t)} s_{\alpha}^{\delta_{L}}, \quad \text{if } \varepsilon \leq a. \]

Proof. First, we consider (A.15) and (A.16) in the case $\varepsilon = 0$. In view of (A.17) it is convenient to represent $t^{-\alpha}$ as a superposition of exponentials,

(A.19) \[ t^{-\alpha} = \Gamma(\alpha)^{-1} \int_{0}^{\infty} e^{-rt} \frac{dr}{r^{1-\alpha}}. \]

Together with (A.18) the above formula yields

(A.20) \[ \triangle s_{\alpha} t^{-\alpha} = \Gamma(\alpha)^{-1} \int_{0}^{\infty} e^{-rt} \left( \prod_{\ell \in L} (e^{-s_{\ell}r} - 1) \right) \frac{dr}{r^{1-\alpha}}. \]

Estimating first $1 - e^{-s_{\ell}r} \leq (s_{\ell}r)^{\delta_{L}}$ and substituting $r \to r/t$ thereafter we arrive at (A.15) with $\varepsilon = 0$.

To prove (A.16) with $\varepsilon = 0$ we pick some some $n \in N$ such that $\alpha < n$ and write $t^{\alpha} = t^{n} t^{-(n-\alpha)}$. Together with (A.12) and (A.13) this yields

\[ \triangle s_{\alpha} t^{\alpha} = \sum_{\mathcal{A} \cup \mathcal{B} = L} \sum_{|\mathcal{A}| \leq n} \left( \begin{array}{c} n \\ |\mathcal{A}| \end{array} \right) t^{n - |\mathcal{A}|} s_{\mathcal{A}}^{\mathcal{A}} t^{(n-\alpha)} \triangle s_{\mathcal{A}}. \]

Applying (A.15) with $\varepsilon = 0$ and $\alpha$ replaced by $n - \alpha$ we see that

\[ t^{n - |\mathcal{A}|} s_{\mathcal{A}}^{\mathcal{A}} t^{(n-\alpha)} \leq c(n, \alpha, \delta_{L})(t + |\mathcal{A}|)^{\alpha} \frac{t^{n - |\mathcal{A}|} s_{\mathcal{A}}^{\mathcal{A}}}{(t + |\mathcal{A}|)^{n + |\delta_{L}|}} \]

\[ \leq c(n, \alpha, \delta_{L})(t + |\mathcal{A}|)^{\alpha - |\delta_{L}|} s_{\mathcal{A}}^{\delta_{L}} \sum_{\alpha \in \mathcal{A}} (t + s_{\alpha})^{\alpha - |\delta_{L}|} s_{\mathcal{A}}^{\delta_{L}}. \]

In the second step we also used that $\kappa_{\alpha} \geq 1$, for each component of $\kappa_{\alpha} = (\kappa_{\alpha})_{\alpha \in \mathcal{A}}$, that $t + |\mathcal{A}|$ is bigger than $t$ and each $s_{\alpha}$, and that $\mathcal{A} \cup \mathcal{B} = L$, which implies

\[ \frac{t^{n - |\mathcal{A}|} s_{\mathcal{A}}^{\mathcal{A}}}{(t + |\mathcal{A}|)^{n + |\delta_{L}|}} \leq \frac{s_{\mathcal{A}}^{\mathcal{A}}}{(t + |\mathcal{A}|)^{n + |\delta_{L}|}}. \]

This proves (A.19), for $\varepsilon = 0$.

Now, let $\tilde{\varepsilon} > 0$. If we replace $t$ by $1 + \tilde{\varepsilon} t$ in (A.15) and (A.16) with $\varepsilon = 0$, then each $s_{t}$ has to be replaced by $\tilde{\varepsilon} s_{t}$. Combining (A.15) & (A.16) with $\varepsilon = 0$ and the product rule (A.12) we thus obtain

\[ |\triangle s_{\alpha} t^{\alpha}(1 + \tilde{\varepsilon} t)^{-\alpha}| \leq c(\alpha, \#L, \delta_{L}) \sum_{\mathcal{A} \cup \mathcal{B} = L} \sum_{\alpha \in \mathcal{A}} \tilde{\varepsilon}^{\delta_{L}} (t + s_{\alpha})^{\alpha - |\delta_{L}|} s_{\mathcal{A}}^{\delta_{L}} \]

\[ + c(\alpha, \#L, \delta_{L}) \tilde{\varepsilon}^{\delta_{L}} t^{\alpha}(1 + \tilde{\varepsilon} t)^{-\alpha - |\delta_{L}|} s_{\mathcal{A}}^{\delta_{L}}. \]
Since \( \frac{\varepsilon}{1 + \varepsilon (t + |s^2|)} \leq \frac{1}{1 + \varepsilon a} \), for every \( a \in \mathcal{A} \), this yields (A.16) with \( \varepsilon = \bar{\varepsilon} \).

We may analogously extend (A.15) to positive \( \varepsilon \) using that \( (1 + \varepsilon t + \varepsilon s_a)/(t + |s^2|) \leq (1 + \varepsilon t)/t \) in addition.

To prove (A.17) we first recall Faà di Bruno’s formula,

\[
(g \circ f)^{(k)}(t) = \sum_{\ell=1}^{k} \frac{g^{(\ell)}(f(t))}{\ell!} \sum_{\alpha \in \mathbb{N}^r:|\alpha|=k} \frac{k!}{\alpha_1! \ldots \alpha_\ell!} f^{(\alpha_1)}(t) \ldots f^{(\alpha_\ell)}(t),
\]

for \( k \)-times differentiable functions \( f, g : \mathbb{R} \rightarrow \mathbb{R} \), which implies

\[
\frac{d^k}{dt^k} e^{a F_\varepsilon(t)} = e^{a F_\varepsilon(t)} \sum_{\ell=1}^{k} c_{k,\ell} \frac{a^\ell (-\varepsilon)^{k-\ell}}{(1 + \varepsilon t)^k} \left| u_{[k]} \right|^{k+\ell} du_1 \ldots du_k,
\]

where (with \( |k| = \{1, \ldots, k\} \), \( |u_{[k]}| = u_1 + \ldots + u_k \))

\[
\Delta_{s_{[k]}} e^{a F_\varepsilon(t)} = \sum_{\ell=1}^{k} c_{k,\ell} \int_0^{\varepsilon s_{1}} \ldots \int_0^{\varepsilon s_{1}} e^{a F_\varepsilon(t + \varepsilon |u_{[k]}|)} \frac{a^\ell (-\varepsilon)^{k-\ell}}{(1 + \varepsilon t + \varepsilon |u_{[k]}|)^{k+\ell}} du_1 \ldots du_k.
\]

Using \( F_\varepsilon(t + r) \leq F_\varepsilon(t) + r, r, t \geq 0 \), repeatedly and taking the condition \( \varepsilon \leq a \) into account we obtain

\[
|\Delta_{s_{[k]}} e^{a F_\varepsilon(t)}| \leq c_{k} e^{a F_\varepsilon(t)} \int_0^{\varepsilon s_{1}} \ldots \int_0^{\varepsilon s_{1}} e^{a |u_{[k]}|} du_1 \ldots du_k \leq c_{k} e^{a F_\varepsilon(t)} \int_0^{\varepsilon s_{1}} \ldots \int_0^{\varepsilon s_{1}} e^{a |u_{[k]}|} du_1 \ldots du_k.
\]

\[ \square \]

We are now prepared to derive bounds on the type of commutators appearing in our applications:

**Lemma A.4.** Let \( \mathcal{J} \) and \( \mathcal{L} \) be disjoint finite, possibly empty index sets and set \( N := \# \mathcal{J}, M := \# \mathcal{L} \). Let \( m, n \in \mathbb{N}_0 \) with \( n \leq N \) and \( m \leq M \), \( \alpha \geq 1/2 \), \( \beta, \gamma, \sigma, \tau \geq 0 \) with \( \alpha = \beta + \gamma \) and \( \sigma + \tau \leq (M + N)/2 \), and let \( \kappa \in \mathbb{R} \). For \( \varepsilon \in [0,1] \) and \( E \geq 1 \), define

\[
F_{\varepsilon,E}(t) := (E + t)/(1 + \varepsilon (E + t)), \quad t \geq 0.
\]

Pick \( g_\ell \in \mathfrak{h} \) satisfying \( v^{\mu+1/2} g_\ell \in \mathfrak{h} \), for all \( \ell \in \mathcal{J} \cup \mathcal{L} \), with

\[
\mu := (|\kappa| + \sigma + \tau) \vee (\alpha + |\kappa| + \sigma + \tau - \frac{M + N + m + n}{2}),
\]

where the measurable function \( v : \mathcal{M} \rightarrow \mathbb{R} \) is strictly positive almost everywhere and monotonically increasing in \( |k| \). Finally, set \( v_\varepsilon := v/(1 + \varepsilon v) \). Then the densely defined operator

\[
T := (1 + d\Gamma(v))^{-n/2} F_{\varepsilon,E}^{-\beta + \kappa} (d\Gamma(v)) \left\{ \left( \prod_{j \in \mathcal{J}} \text{ad}_{a_j(g_j)} \right) \left( \prod_{\ell \in \mathcal{L}} \text{ad}_{a_\ell(g_\ell)} \right) F_{\varepsilon,E}^{\alpha} (d\Gamma(v_\varepsilon)) \right\}
\]

\[
\times F_{\varepsilon,E}^{-\gamma - \kappa} (d\Gamma(v_\varepsilon)) (1 + d\Gamma(v))^{-m/2},
\]
extends uniquely to a bounded operator on $\mathcal{F}$ with

$$
\|T\| \leq c(\alpha, |\kappa|, \omega, |\sigma| + N, M) E^{-\frac{M + N + m + n}{2} + \sigma + \tau}
$$

$$
\cdot \left\{ \sum_{a, b \in \mathcal{J} \cup \mathcal{K}} \left\| v^{1/2}(1 + E)^{\mu a} g_a \right\| \left\| v^{1/2}(1 + E)^{\mu b} g_b \right\| \prod_{c \in \mathcal{J} \cup \mathcal{K}} \left\| v^{1/2} g_c \right\| 
+ \sum_{a \in \mathcal{J} \cup \mathcal{K}} \left\| v^{1/2}(1 + E)^{\mu a} g_a \right\| \prod_{c \in \mathcal{J} \cup \mathcal{K}} \left\| v^{1/2} g_c \right\| \right\}.
$$

Here empty sums should be read as 0 and empty products are 1.

Proof. We abbreviate $s_\ell := v_*(p_\ell)$, $\ell \in \mathcal{J} \cup \mathcal{K}$, and $|s_{\mathcal{A}}| := \sum_{a \in \mathcal{A}} s_a$, etc. According to (A.10) and Lem. (A.2) (which we apply with $\omega = v_e$ and $m = n = 0$) we have to find bounds on the norms

$$
N_{\mathcal{A}, \mathcal{C}}^{s_\ell, E} := \left\| (1 + \delta\Gamma(v_e)) \frac{\# a + \# c}{2} F_{E,E}^{-\beta + \gamma} (d\Gamma(v_e) + |s_{\mathcal{A}}|) \right\| \times (\Delta s_{\mathcal{A} \cup \mathcal{K}}^{s_\ell, E})(d\Gamma(v_e)) F_{E,E}^{-\gamma - \kappa} (d\Gamma(v_e) + |s_{\mathcal{C}}|),
$$

where $a \subset \mathcal{A}$, $c \subset \mathcal{C}$ satisfy $|a| \geq n$, $|c| \geq m$, and $\mathcal{A} \cup \mathcal{B} = \mathcal{J}$, $\mathcal{C} \cup \mathcal{D} = \mathcal{K}$, are partitions with $\# \mathcal{A} = \# \mathcal{B}$. We again set $s_0 := 0$, assuming without loss of generality that $0 \notin \mathcal{J} \cup \mathcal{K}$. Employing (A.16) and taking $\alpha = \beta + \gamma$ into account we obtain

$$
N_{\mathcal{A}, \mathcal{C}}^{s_\ell, E} \leq c_0 \sum_{\ell \in \{0\} \cup \mathcal{J} \cup \mathcal{K}} \sup_{t \geq 0} \left( \frac{E + t + s_\ell}{E + t + |s_{\mathcal{A}}|} \right)^{\beta} \left( \frac{E + t + s_\ell}{E + t + |s_{\mathcal{C}}|} \right)^{\gamma} \left( \frac{E + t + s_\ell}{E + t + |s_{\mathcal{A}}|} \right)^{\kappa}
\cdot \sup_{t \geq 0} \left( \frac{E + t + s_\ell}{E + t + |s_{\mathcal{A}}|} \right)^{\alpha} \left( \frac{E + t + s_\ell}{E + t + |s_{\mathcal{C}}|} \right)^{\sigma} (E + t + |s_{\mathcal{A}}|)^{\tau} \frac{\# a + \# c}{2} \left( \frac{E + t + s_\ell}{E + t + |s_{\mathcal{A}}|} \right)^{\delta_{\mathcal{A} \cup \mathcal{K}}} \left( \frac{E + t + s_\ell}{E + t + |s_{\mathcal{C}}|} \right)^{\delta_{\mathcal{C} \cup \mathcal{D}}},
$$

where the parameters $\delta_a \in [0, 1]$ may be chosen depending on $\mathcal{A}$, $\mathcal{C}$, $a$, and $c$ and where we used that $\varepsilon s_a \leq 1$ and

$$
\forall s, v \geq 0 : \sup_{t \geq 0} \left( \frac{E + t + v}{E + t + s} \right) = \begin{cases} 
(E + v - s)/E, & s < v, \\
1, & s \geq v.
\end{cases}
$$

Notice that the fractions containing the $v$'s in the previous inequality are all uniformly bounded because of $v_e \leq 1$. We now choose $\delta_a = 1$, if $\ell \in (\mathcal{A} \setminus a) \cup (\mathcal{D} \setminus c)$, and $\delta_a = 1/2$, if $\ell \in (\mathcal{A} \setminus a) \cup (\mathcal{D} \setminus c)$. Then $|\delta_{\mathcal{A} \cup \mathcal{K}}| - (|a| + |c|)/2 = (M + N)/2 \geq \sigma + \tau$, since $\# \mathcal{B} = \# \mathcal{D}$, and using also

$$
E + |s_{\mathcal{A} \cup \mathcal{K}}| \leq (E + s_\ell) \frac{E + |s_{\mathcal{A} \cup \mathcal{K}}|}{E},
$$

we conclude that

$$
\frac{(N_{\mathcal{A}, \mathcal{C}}^{s_\ell, E})^2}{(\prod_{a \in \mathcal{A} \cup \mathcal{K}} v(p_a))} \leq c_2 \frac{\mathcal{J}(p_{\mathcal{A} \cup \mathcal{K}})}{E^{M + N + m + n - 2(\sigma + \tau)}} \left( \prod_{\mathcal{A}} v(p_d)^2 \right) \prod_{a \in \mathcal{A} \cup \mathcal{K}} v(p_a),
$$
where the $(\alpha, \sigma, \tau, \kappa, \mathcal{A} \cup \mathcal{L}, N, M, n, m)$-dependent function $\mathcal{J}$ is given by

$$\mathcal{J}(p_{\mathcal{A} \cup \mathcal{L}}) := \sum_{\ell \in \mathcal{A} \cup \mathcal{L}} \left( \frac{E + v(p_\ell)}{E} \right)^{2\mu_\ell} \left( 1 + \sum_{s \in \mathcal{A} \cup \mathcal{L}} \left( \frac{E + v(p_s)}{E} \right) \right)^{2(\kappa + \sigma + \tau)}.$$ 

Putting everything together we arrive at

$$\|T\| \leq c_3 \sum_{s \in \mathcal{A} \cup \mathcal{L}} \sum_{\varrho \in \mathcal{B}(\mathcal{A}, \mathcal{B})} \int \left( \prod_{d \in \mathcal{D}} v(p_d)^{1/2} |g_d(p_d)| \right) \left( \prod_{b \in \mathcal{B}} v(p_{\tau(b)})^{1/2} |g_b(p_{\tau(b)})| \right) \cdot E^{-(M+N)/2+\sigma+\tau} \left[ \int \mathcal{J}(p_{\mathcal{A} \cup \mathcal{L}}) \prod_{\alpha \in \mathcal{A} \cup \mathcal{L}} \{ v(p_\alpha) |g_\alpha(p_\alpha)|^2 \} \right]^{1/2} \, \mu^{\# \mathcal{B}(p_{\mathcal{B}})},$$

from which we easily infer the asserted estimate. \hfill \Box

**Remark A.5.** If we additionally assume that $\mathcal{J} = \emptyset$ or $\mathcal{L} = \emptyset$, then the whole statement of Lem. A.4 still holds true with $\mu$ replaced by $\tilde{\mu} := (|\kappa| + \sigma + \tau) \lor (\beta \lor \gamma + |\kappa| + \sigma + \tau - M+N+m+n)$. In fact, in this case we always have $\mathcal{B} = \emptyset$, i.e. $\mathcal{A} = \mathcal{J}$ and $\mathcal{L} = \emptyset$ in the proof of Lem. A.4. This implies that $s_t \leq \varepsilon_{\mathcal{A}}$ or $s_t \leq \varepsilon_{\mathcal{B}}$, for all $t \in \{0\} \cup \mathcal{A} \cup \mathcal{L}$, so that $\alpha$ can be replaced by $\beta \lor \gamma$ in the estimate on $N^{a,b}_t$. Following the proof without any further changes we arrive at the assertion with $\tilde{\mu}$ in place of $\mu$.

**Example A.6.** Let $f \in \mathfrak{h}$ with $\omega^{1/2} \in \mathfrak{h}$ and set $\theta := 1 + d\Gamma(\omega)$. Choosing $\varepsilon = 0$, $E = 1$, $n = m = 0$, $M + N = 1$, $\beta = \sigma = \tau = \kappa = 0$, and $\alpha = \gamma = 1/2$ in Lem. A.4, we see that $Y(f) := [\theta^{1/2}, \varphi(f)]\theta^{-1/2}$, which is well-defined a priori on $D(d\Gamma(\omega)^{1/2})$, extends to a bounded operator on $\mathcal{H}$ with $\|Y(f)\| \leq c\|\omega^{1/2} f\|$. This implies that the map $\mathbb{R}^n \ni x \mapsto \varphi(G_x)^2 \in \mathcal{B}(D(d\Gamma(\omega)), \mathcal{H})$ is continuous. In fact, using (2.6) & (2.7) repeatedly, we obtain, for all $x, y \in \mathbb{R}^n$ and $\psi \in D(d\Gamma(\omega))$,

$$\|\varphi(G_x)^2 \psi - \varphi(G_y)^2 \psi\| = \|((\varphi(G_x) - \varphi(G_y)) \varphi(G_x + G_y) \psi)\|
\leq c' \|G_x - G_y\|_{\mathfrak{h}} \|\theta^{1/2} (Y(G_x + G_y) + \varphi(G_x + G_y))\| \|\theta^{-1/2} \psi\|
\leq c'' \|G_x - G_y\|_{\mathfrak{h}} \sup_x \|G_x\|_\psi \|\theta \psi\|,$$

and the claim follows from the continuity of $x \mapsto G_x \in \mathfrak{h}$ required in Hyp. 2.1.

**Example A.7.** Let us explain how to read off the proof of Lem. A.4 from Lem. A.3. First, we show that

$$\|T(s)\| \leq c_\alpha \left( \frac{v_{\alpha,s}(s)}{1 + v_{\alpha,s}(s)} \right)^{1/2} \left( 1 + \frac{v_{\alpha,s}(s)}{1 + v_{\alpha,s}(s)} \right)^{1/2} |G|^{1/2},$$

(A.22)

where $v_{\alpha,s}(s), \varpi, \kappa,$ and $v_{\alpha,s}$ are introduced in the paragraph preceding Lem. A.10. In fact, recalling (A.3) and (A.3) and using the notation of Lem. A.3 we may write

$$T(s) = -2F_{\varepsilon,E}(d\Gamma(v_{\alpha,s}))^{-\beta} \left( \text{ad}_{G(B_s^2)}^{[\alpha]}(d\Gamma(v_{\alpha,s})) \right) F_{\varepsilon,E}(d\Gamma(v_{\alpha,s}))^{-\gamma}$$

(A.23)

with $E := 1 + \varepsilon_{\alpha}(s)$ and either $(\beta, \gamma) = (|\alpha|, 0)$ or $(\beta, \gamma) = (0, |\alpha|)$. Of course, the two terms on the right hand side correspond to the choices $N = 1, M = 0,$
Let $\omega$ be measurable, and let $\omega$ be a measurable function on $\mathcal{M}$ which is strictly positive almost everywhere. Let $\mathcal{J}$ and $\mathcal{L}$ be disjoint finite, possibly empty index sets, $N := \# \mathcal{J}$, $M := \# \mathcal{L}$, let $m, n \in \mathbb{N}_0$ such that $m + n = M + N$, and let $\delta, \beta, \gamma \in [0, 1]$ satisfy $\varepsilon \leq \delta = \beta + \gamma$. Pick $g_\ell \in \mathfrak{h}$ satisfying

$$u^\beta \delta v g_\ell \in \mathfrak{h}, \text{ where } u := (\delta v) \vee (\delta v)^2 \omega,$$

for all $\ell \in \mathcal{J} \cup \mathcal{L}$. Finally, set $v_\varepsilon := v/(1 + \varepsilon v)$ and define $F_{\varepsilon, E}$ by (A.21). Then the densely defined operator

$$T := (1 + d\Gamma(\omega))^{-n/2} e^{-\varepsilon F_{\varepsilon, E}(d\Gamma(v_\varepsilon))} \left( \prod_{j \in \mathcal{J}} \text{ad}_{\alpha_\varepsilon(g_j)} \prod_{\ell \in \mathcal{L}} \text{ad}_{\alpha_\varepsilon(g_\ell)} e^{\delta F_{\varepsilon, E}(d\Gamma(v_\varepsilon))} \right) \times e^{-\varepsilon F_{\varepsilon, E}(d\Gamma(v_\varepsilon))}(1 + d\Gamma(\omega))^{-m/2},$$

extends uniquely to a bounded operator on $\mathcal{F}$ with

$$(A.24) \quad \|T\| \leq c_{m, n, M, N} \prod_{\ell \in \mathcal{J} \cup \mathcal{L}} \|u^\beta \delta v g_\ell\|.$$

Proof. Consider two partitions $\mathcal{A} \cup \mathcal{B} = \mathcal{J}$ and $\mathcal{C} \cup \mathcal{D} = \mathcal{L}$ with $\# \mathcal{B} = \# \mathcal{D}$. Since $m + n = M + N$ at least one of the conditions $n \geq \# \mathcal{A}$ or $m \geq \# \mathcal{C}$ is satisfied. Without loss of generality we may assume for the moment that $n \geq \# \mathcal{A}$. Setting $\omega_{\mathcal{A}} := \sum_{a \in \mathcal{A}} \omega(p_a)$, $s_\varepsilon := v_\varepsilon(p_x)$, $|s_{\mathcal{A}}| := \sum_{a \in \mathcal{A}} v_\varepsilon(p_a)$, etc., we then infer from Lem. A.3 and Lem. A.2 that it suffices to find a bound on the norms

$$
\mathcal{N}_{\mathcal{A}, \mathcal{C}}^\varepsilon := \left\| (1 + d\Gamma(\omega) + \omega_{\mathcal{A}})^{-\frac{n-\# \mathcal{A}}{2}} e^{-\varepsilon F_{\varepsilon, E}(d\Gamma(v_\varepsilon))} \right\| \times \left\| (\Delta_{s_{\mathcal{A}}, \mathcal{C}, \mathcal{D}} e^{\delta F_{\varepsilon, E}(d\Gamma(v_\varepsilon))} e^{-\varepsilon F_{\varepsilon, E}(d\Gamma(v_\varepsilon))} (1 + d\Gamma(\omega))^{\frac{m-\# \mathcal{C}}{2}} \right\|
$$

where $\mathcal{C} \subset \mathcal{G}$ satisfies $\# \mathcal{C} \geq m$. On account of $n - \# \mathcal{A} \geq \# \mathcal{C} - m$ we have

$$\|1 + d\Gamma(\omega) + \omega_{\mathcal{A}}\|^{-\frac{n-\# \mathcal{A}}{2}} \left(1 + d\Gamma(\omega)\right)^{\frac{m-\# \mathcal{C}}{2}} \leq 1,$$

and taking also Lem. A.3 into account we see that

$$
\mathcal{N}_{\mathcal{A}, \mathcal{C}}^\varepsilon \leq c_{\mathcal{A}, \mathcal{D}} \delta \varepsilon \delta_{s_{\mathcal{A}}, \mathcal{D}} e^{\delta |s_{\mathcal{A}}, \mathcal{D}|} \sup_{t \geq 0} e^{-\varepsilon F_{\varepsilon, E}(t + |s_{\mathcal{A}}| - \delta F_{\varepsilon, E}(t) - \gamma F_{\varepsilon, E}(t + |s_{\mathcal{A}}|) + \delta F_{\varepsilon, E}(t))},
$$

where we now choose $\delta = 1$, if $\ell \in \mathcal{A} \cup \mathcal{C} \cup \mathcal{D}$, and $\delta = 1/2$, if $\ell \in \mathcal{C} \setminus \mathcal{C}$. The supremum in the previous inequality is $\leq 1$ by the assumption $a = b + c$ and the fact that $F_{\varepsilon, E}$ is monotonically increasing. We thus arrive at

$$
\left( \prod_{a \in \mathcal{A} \cup \mathcal{C}} \omega(p_a) \right)^2 \leq c \left( \prod_{d \in \mathcal{D}} u_\delta(p_d) e^{2\delta v(p_d)} \prod_{a \in \mathcal{A} \cup \mathcal{C}} u_\delta(p_a) e^{2\delta v(p_a)} \right),
$$

and $N = 0$, $M = 1$, respectively. Hence, Lem. A.1 implies the first inequality of (A.22), and the second one follows from $v_{\alpha, s} \leq (1 + i\tau_\alpha(s))(\omega + \omega)$. In the same way, taking A.2 and A.4 into account, we obtain

$$\|T_2(s)\| \leq c_\alpha((\omega + \omega)^{1/2}(1 + \omega + \omega) |\alpha|^{-1/2}(9B_W^2, F_{B_W^2})\|.$$
In view of (A.9), (A.10), and Lem. A.2 and since $\delta \leq 1$, it finally follows that

$$\|T\| \leq c \sum_{\delta \in \text{Bij}(D, \mathcal{S})} \sum_{\pi \in \text{Bij}(D, \mathcal{S})} \left( \prod_{d \in \mathcal{S}} u_\delta(p_d)^{1/2} e^{\delta v(p_d)/2} g_d(p_d) \right)$$

$$\cdot \left( \prod_{b \in \mathcal{S}} u_\delta(p_\pi(b))^{1/2} e^{\delta v(p_{\pi(b)}/2} g_b(p_{\pi(b)}) \right)$$

$$\cdot \left( \prod_{a \in \mathcal{S}, \pi \in \text{Bij}(D, \mathcal{S})} \int \prod_{b \in \mathcal{S}} \left\{ u_\delta(p_a)e^{2\delta v(p_a)}|g_a(p_a)|^2 d\mu(p_a) \right\} \right)^{1/2} d\mu_{\# \mathcal{S}}(p_{\mathcal{S}}),$$

and we conclude that (A.24) holds true.

\[ \square \]

Example A.9. For positive $\delta$, the bound (4.11) is an easy consequence of Lem. A.8, if we choose $E = 1 + \iota\tau_1(s)$, $v = v_1,s$, so that $\Theta^{(1)}_{\gamma,s} = F_{\gamma,1+\iota\tau_1(s)}(d\Gamma(v_1,s))$, where $\iota\tau_1(s), v_1,s,$ and $\Theta^{(1)}_{\gamma,s}$ are defined in the beginning of Sect. 4. Similarly as in Ex. A.7 the case $\delta < 0$ can be reduced to the case of positive $\delta$ by means of (A.4) and (A.5).

Acknowledgements. It is a pleasure to thank Batu Güneysu and Jacob Schach Møller for many inspiring and helpful discussions.

This work has been supported by the Villum Foundation.

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