The role of vector fields in modified gravity scenarios

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Abstract

Gravitational vector degrees of freedom typically arise in many examples of modified gravity models. We start to systematically explore their role in these scenarios, studying the effects of coupling gravitational vector and scalar degrees of freedom. We focus on set-ups that enjoy a Galilean symmetry in the scalar sector and an Abelian gauge symmetry in the vector sector. These symmetries, together with the requirement that the equations of motion contain at most two space-time derivatives, only allow for a small number of operators in the Lagrangian for the gravitational fields. We investigate the role of gravitational vector fields for two broad classes of phenomena that characterize modified gravity scenarios. The first is self-acceleration: we analyze in general terms the behavior of vector fluctuations around self-accelerating solutions, and show that vanishing kinetic terms of vector fluctuations lead to instabilities on cosmological backgrounds. The second phenomenon is the screening of long range fifth forces by means of Vainshtein mechanism. We show that if gravitational vector fields are appropriately coupled to a spherically symmetric source, they can play an important role for defining the features of the background solution and the scale of the Vainshtein radius. Our general results can be applied to any concrete model of modified gravity, whose low-energy vector and scalar degrees of freedom satisfy the symmetry requirements that we impose.

1 Introduction

General relativity is the unique theory describing the dynamics of an interacting spin-2, massless degree of freedom. Any consistent modification of general relativity (GR) introduces new light dynamical degrees of freedom (DOFs) of lower spins, typically scalars and vectors, which together with tensors constitute the low energy spectrum of modified gravity theories. These additional fields can change the strength of gravitational interactions, for example rendering gravity weaker at large scales, and provide an intriguing explanation for the observed present day acceleration of our universe. Moreover, theories of modified gravity are characterized by interesting environmental effects, since the strength of gravitational interactions depends on the particular background under consideration. For example nearby a massive source as the sun, screening mechanisms are capable to hide the additional DOFs, recovering with excellent accuracy the predictions of GR, and satisfying stringent solar system constraints on deviations from it. For a recent comprehensive review on modified gravity see \cite{1}.

Usually, a low energy effective description of modified gravity scenarios is made in terms of a Lagrangian controlling the dynamics of tensor and scalar modes. The dynamics of the scalar, in particular, is assumed to control with good approximation the effects of the modification of gravity. In this work we start to systematically explore the role of vectors in modified gravity, specifically studying the effects of couplings between vector and scalar DOFs. We take the point of view that vector DOFs are part of the set of fields that control gravitational interactions. Typically, they arise as Goldstone bosons associated with symmetry breaking effects that characterize
modified gravity scenarios. Representative examples are brane-world models, in which translational invariance in the extra dimensions is broken by the presence of branes; and massive gravity, in which the graviton mass breaks the diffeomorphism invariance of GR. In both of these cases, vectors appear as Goldstone bosons of broken symmetries. Moreover, we will also comment on a second, alternative perspective, in which vectors are part of the matter sector of the theory under consideration, and couple by means of higher order operators with the scalar DOF that mediate gravitational interactions in modified gravity.

Whatever the perspective we adopt, we are generally allowed to write a large number of effective operators that couple vectors with the gravitational scalar DOF. This number can be reduced by imposing appropriate symmetries and physical constraints on the theory. In this work, we consider two symmetries: a Galilean symmetry in the scalar sector, and an Abelian gauge symmetry in the vector sector. The Galilean symmetry demands that the action is invariant under $\pi \rightarrow \pi + c + b_\mu x^\mu$, where $\pi$ is the gravitational scalar DOF. The $U(1)$ Abelian gauge symmetry in the vector sector reads $A_\mu \rightarrow A_\mu + \partial_\mu \zeta$. Such a gauge symmetry is usually encountered in vector Lagrangians arising from symmetry breaking effects (for example, in massive gravity it is a remnant of broken diffeomorphism invariance [2]). As a physical requirement, we demand that our Lagrangian leads to equations of motion with at most two derivatives, so to automatically avoid Ostrogradsky instabilities. The theory we analyze is a generalization of scalar Galileons to the so-called $p$-form Galileons [3, 4]. Notice that our work on coupling vectors to scalars by means of derivative interactions, such to lead to equations of motion with at most two space-time derivatives, is complementary to the work of Hordenski [5], that found the most general vector-tensor Lagrangian satisfying the same requirement.

We will focus on investigating the role of vector fields for two broad classes of phenomena that characterize the most interesting versions of modified gravity scenarios. The first is self-acceleration: theories with higher order derivative scalar self-interactions admit cosmological solutions corresponding to accelerating universes in the vacuum. The most famous example is the DGP model [6] whose basic features have then been extended in [7] to ghost-free set-ups enjoying a Galilean symmetry. While the contributions of the scalar sector to self-acceleration is well understood, much less studied is the role of a vector sector derivatively coupled to the scalar. In the set-up we consider, vectors can acquire a non-trivial profile that preserves the symmetries of cosmological backgrounds (i.e. the Friedmann-Robertson-Walker symmetry of the metric) and, at the same time, contributes to determine the size of cosmological acceleration. Moreover, the behavior of vector perturbations around self-accelerating configurations can provide key constraints to characterize the stability of a given cosmological background. In our work, we will analyze in general terms possible instabilities that vectors induce on cosmological solutions in Galileon models, and provide criteria to avoid them.

The second topic we will consider is the role of vectors for characterizing screening mechanisms, in particular the Vainshtein effect. In Galileon theories, within a certain distance from a spherically symmetric source, the predictions of GR can be recovered despite the presence of additional light degrees of freedom besides tensors, thanks to the non-linear contributions to the equations of motion. Usually, in these theories one considers minimal couplings between tensor and scalar gravitational DOFs to the source. On the other hand, gravitational vector degrees of freedom are also allowed to couple directly to the source without violating the symmetries we impose on the theory. This happens if the source is characterized, besides its usual energy momentum tensor, by a gravitational vector current. In this case the source has a gravitational vector charge, and we will show that this fact can considerably influence the realization of the Vainshtein mechanism. We will also comment on how our findings can be applied to set-ups in which scalars enjoying Galileon symmetries couple to bodies that are charged under electromagnetic interactions. Also in this case the realization of the Vainshtein mechanism might be influenced by the vector charges.

This work is organized as follows. In Section 2 we present the Lagrangian for the theory we consider. In Section 3 we consider its maximally symmetric solutions in the vacuum, and analyze the role of vector fields for the stability of these space-times. In Section 4 we study how vectors contribute to the Vainshtein mechanism. We conclude with a section of Summary.

2 The scalar-vector-tensor Lagrangian

As described in the Introduction, we keep our analysis as general as possible without focusing on any particular modified gravity model. We consider a general Lagrangian in four dimensions for scalar, vector, and tensor DOFs that respects Galileon and Abelian gauge symmetries, and that leads to equations of motion containing
at most second derivatives on the fields. Our analysis encompasses the phenomenology of any modified gravity model whose low energy description obeys these requirements. Such Lagrangian contains non-linear derivative interactions involving scalar and vector DOFs, that we can interpret as the additional gravitational degrees of freedom characterizing modifications of GR, or alternatively as part of matter sector of the theory. The symmetries that we impose, and the physical requirements that we demand, only allow for a small number of terms in the effective Lagrangian \( \mathcal{L} \) describing the non-linear interactions for our system.

The Lagrangian we will investigate has the following structure:

\[
\mathcal{L} = \mathcal{L}_{st} + \mathcal{L}_{sv},
\]

where \( \mathcal{L}_{st} \) is the scalar tensor part, while \( \mathcal{L}_{sv} \) is the scalar vector contribution. Non-linear self-interactions involving tensors are described by GR and so their consequences are well known. For this reason we will neglect non-linear tensor contributions in what follows: technically this condition can be achieved by imposing that the Lagrangian \( \mathcal{L} \), up to total derivatives, is invariant under linearized diffeomorphism invariance \( h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu \).

In what follows we will discuss separately the two parts appearing in eq (1). In both cases, the starting point is the well known fact that, in \( d \) dimensions, the following combinations of \( n \) scalars is a total derivative:

\[
\mathcal{L}^{(n)}_{\text{der}} = \epsilon^{\alpha_1...\alpha_d...\mu_1...\mu_n} \epsilon_{\alpha_1...\alpha_d...\nu_1...\nu_n} \Pi_{\mu_1\nu_1} \cdots \Pi_{\mu_n\nu_n}
\]

(2)

where \( \epsilon_{\mu\nu...} \) is the Levi-Civita antisymmetric tensor, while \( \Pi_{\mu\nu} = \nabla_\mu \partial_\nu \pi \), with \( \pi \) a scalar field.

The Lagrangians in (2), expressed in terms of derivatives of scalars, can be rewritten as total derivative; on the other hand, starting from it one can obtain non-trivial field dynamics by substituting one or more of the \( \Pi_{\mu\nu} \) with other symmetric tensors, involving for example spin-2 and spin-1 fields. This is the method we will adopt to build the constituents of our \( \mathcal{L} \) in eq. (1).

### 2.1 The scalar-tensor contribution

The tensor scalar Lagrangian describes interactions between a tensor \( h_{\mu\nu} \) (the linearized perturbation of the metric \( \frac{h_{\mu\nu}}{M_{Pl}} = g_{\mu\nu} - \eta_{\mu\nu} \)) and a scalar \( \pi \). It has the following (generalized) Galileon structure, analyzed in [8]:

\[
\mathcal{L}_{st} = -\frac{1}{2} h^{\mu\nu} \mathcal{E}^{\alpha\beta}_{\mu\nu} h_{\alpha\beta} + h^{\mu\nu} \sum_{n=0}^{3} c_n X^{(n)}_{\mu\nu}(\Pi)
\]

(3)

where \( \mathcal{E}^{\alpha\beta}_{\mu\nu} \) is the operator acting on \( Z_{\alpha\beta} \) as

\[
\mathcal{E}^{\alpha\beta}_{\mu\nu} Z_{\alpha\beta} = -\frac{1}{2} \left( \Box Z_{\mu\nu} - \partial_\mu \partial_\alpha Z^\alpha_{\nu} - \partial_\nu \partial_\alpha Z^\alpha_{\mu} + \partial_\mu \partial_\nu Z^\alpha_{\alpha} - \eta_{\mu\nu} \Box Z^\beta_{\beta} + \eta_{\mu\nu} \partial_\alpha \partial_\beta Z^\alpha_{\beta} \right).
\]

(4)

Moreover,

\[
X^{(n)}_{\mu\nu}(\Pi) = \epsilon^{\alpha_1...\alpha_n \gamma_1...\gamma_3-n} \epsilon_{\beta_1...\beta_n \gamma_1...\gamma_3-n} \Pi_{\alpha_1\beta_1} \cdots \Pi_{\alpha_n\beta_n}
\]

(5)

and \( \Pi_{\mu\nu} = \nabla_\mu \partial_\nu \pi \), \( \Pi = \Pi_{\mu}^\mu \). The first term in eq. (3) is the Einstein-Hilbert (EH) Lagrangian expanded at quadratic order in perturbations around flat space. The remaining terms are obtained starting from the total derivative combinations of eq. (2), and substituting to one of the \( \Pi_{\mu\nu} \) a metric tensor \( h_{\mu\nu} \).

These terms respect the Galilean symmetry \( \pi \rightarrow \pi + h_\mu x^\mu + c \), and lead to equations of motion with at most two derivatives (thanks to the antisymmetric properties of the Levi-Civita tensor). The Lagrangian (3) additionally enjoys a linearized diffeomorphism invariance in the tensor sector, \( h_{\mu\nu} \rightarrow h_{\mu\nu} + \partial_\mu \zeta_\nu + \partial_\nu \zeta_\mu \). The resulting scalar-tensor Lagrangian contains higher order derivative interactions for the scalar field \( \pi \), that play a crucial role for characterizing the most interesting features of modified gravity scenarios. These interactions are controlled by four contributions, weighted by dimensionful coefficients \( c_n \)

\[
c_n = \frac{\hat{c}_n}{\Lambda^{3(n-1)}},
\]

(6)

where \( \hat{c}_n \) is dimensionless, and \( \Lambda \) some mass scale associated with the theory under examination.
Recall that the Levi-Civita tensor $\epsilon$ satisfies the following identity

$$\epsilon_{i_1\ldots i_k i_{k+1}\ldots i_n} \epsilon^{i_1\ldots i_k j_{k+1}\ldots j_n} = k! \delta_{i_{k+1}\ldots i_n}^{j_{k+1}\ldots j_n}$$

where sum over repeated indexes is assumed. The $\delta_{i_{k+1}\ldots i_n}^{j_{k+1}\ldots j_n}$ denotes antisymmetrization: for example $\delta_{cd}^{ab} = \delta_c^a \delta_d^b - \delta_c^b \delta_d^a$. The $X^{(n)}_{\mu\nu}$ tensors satisfy the following recursion relation

$$X^{(n)}_{\mu\nu} = -n \Pi^\alpha_{\mu} X^{(n-1)}_{\alpha\nu} + \Pi^{\alpha\beta} X^{(n-1)}_{\alpha\beta} \eta_{\mu\nu}$$

for $n > 1$, and are symmetric and identically conserved

$$\partial^\mu X^{(n)}_{\mu\nu} = 0.$$ (9)

One finds for the first ones

$$X^{(0)}_{\mu\nu} = 6 \eta_{\mu\nu},$$

$$X^{(1)}_{\mu\nu} = 2 \left[ \eta_{\mu\nu} \Pi - \Pi_{\mu\nu} \right],$$

$$X^{(2)}_{\mu\nu} = -2 \epsilon^{\alpha\beta} Z^{(1, 2)}_{\alpha\beta}$$

from which, using (8), the remaining ones can be obtained. Notice that the contribution $h^{\mu\nu} X^{(0)}_{\mu\nu}$ corresponds to a bare cosmological constant: we will not be interested in this and hence we will set $c_0 = 0$ in what follows.

It is also useful to observe that

$$X^{(1, 2)}_{\mu\nu} = -2 \epsilon^{\alpha\beta} Z^{(1, 2)}_{\alpha\beta}$$

with

$$Z^{(1)}_{\mu\nu} = \pi \eta_{\mu\nu},$$

$$Z^{(2)}_{\mu\nu} = \partial_\mu \pi \partial_\nu \pi$$

that can be used to de-mix the kinetic terms of scalars and tensors, and to express scalar Galileon contributions to the Lagrangian in their original form [7].

2.2 The scalar-vector contribution

Various works in the past have been dedicated to understand the effects of vector fields in cosmology, in particular during inflation to build models for primordial magnetogenesis (see e.g. [9] for a recent review). Also, non-minimal couplings between vectors and curvature can provide models for dark energy – see for example [10] – that are however often plagued by instabilities [11]. It has also been shown that Horndeski vector-tensor theory [5] leads to instabilities when applied to cosmology [12], although stable regimes can be found [13].

In this work, we would like analyze models that couple vectors with scalars in a way that preserve both Galileon and gauge symmetries. Both these symmetries might be useful to render the structure of the Lagrangian stable under quantum corrections. In order to build the scalar-vector contribution, we use the construction of $p$-form Galileons [3], and the results of [14]. The subject of couplings Galileons to gauge fields has also been investigated in [15]. Starting from eq. [2], we substitute to it one (or more) $\Pi$’s with one (or more) symmetric tensors built up with the vectors

$$S_{\mu\nu} = \nabla_\mu A_\nu + \nabla_\nu A_\mu = 2 \nabla_\mu A_\nu - F_{\mu\nu}.$$ (16)
on the four dimensional case, in which two of the II-tensors in eq (2) are substituted by the symmetric tensors $S_{\mu\nu}$. (We do not consider the additional case in which contractions of four $S_{\mu\nu}$ are involved, since we checked it does not qualitatively change the results we will discuss in what follows.) The general ghost-free vector Lagrangian that we consider, coupling scalars with vectors, is composed by three independent contributions:

$$L_{sv} = S^{\mu\nu} \sum_{n=1}^{3} e_n Z_{\mu\nu}^{(n)}(S_{\rho\sigma}, \Pi),$$

with

$$Z_{\mu\nu}^{(n)} = \epsilon_{\mu_1 \ldots \mu_n}^\gamma_1 \cdot \cdot \cdot \gamma_{n-2} \epsilon_{\nu_1 \cdot \cdot \cdot \nu_n}^\beta_1 \cdot \cdot \cdot \beta_{n-2} S_{\alpha_1 \beta_1} \Pi_{\alpha_2 \beta_2} \cdot \cdot \cdot \Pi_{\alpha_n \beta_n}.$$  

The expressions for the $Z^{(n)}$ can be made more explicit, and read

$$S^{\mu\nu} Z_{\mu\nu}^{(1)} = -2(S_{\mu\nu} S^\mu_{\nu} - S_\mu^\nu S_\nu^\mu),$$

$$S^{\mu\nu} Z_{\mu\nu}^{(2)} = -2\Pi_{\mu\nu} (S_{\mu\rho} S^\rho_{\nu} - S_{\nu\rho} S^\rho_{\mu}) - \Pi_\mu (S_{\mu\nu} S_\nu^\mu - S_\nu^\mu S_{\nu\mu}),$$

$$S^{\mu\nu} Z_{\mu\nu}^{(3)} = -2\Pi_{\mu\nu} \Pi_\rho S_{\mu\rho} S_{\nu\sigma} + 2\Pi_{\mu\nu} \Pi_{\sigma\rho} S_\mu S_\nu - 4\Pi_{\mu\nu} \Pi_{\sigma\rho} S_\sigma S_\rho + 4\Pi_{\mu\rho} S^\rho_{\nu\sigma} S_\nu S_\rho + 4\Pi_{\mu\nu} S_{\nu\rho} S_{\rho\sigma} - 4\Pi_{\mu\rho} S_{\nu\rho} S_{\sigma\nu} + (\Pi_\mu^2)^2 (S_\nu^\rho)^2 - \Pi_{\mu\rho} \Pi_{\nu\sigma} (S_\rho^\sigma)^2 - S_{\mu\rho} S_{\nu\rho} \Pi_{\mu\rho} \Pi_{\nu\sigma}.$$  

The resulting scalar-vector Lagrangian contains higher order derivative interactions between the vector and the scalar DOFs, that will play interesting roles in strong coupling regimes that we will examine in the next sections. The parameters $e_n$ appearing in the Lagrangian (17) are dimensionful, and can be expressed as

$$e_n = \hat{e}_n / \Lambda^{3(n-1)},$$

with dimensionless $\hat{e}_n$ and with $\Lambda$ corresponding to some mass scale of the theory under consideration.

As explained in the appendix of [14], by using the definition [16], the previous Lagrangian is equivalent to a Lagrangian in which the $S'$s are substituted by $F$'s, up to total derivative terms. For example, for what respect the structure of the first contribution $e_1 S^{\mu\nu} Z_{\mu\nu}^{(1)}$ to the vector Lagrangian, one finds

$$e_1 S^{\mu\nu} Z_{\mu\nu}^{(1)} = e_1 S^{\mu\nu} \epsilon_{\mu_1 \gamma_1} \cdot \cdot \cdot \epsilon_{\nu_1 \gamma_2} S_{\alpha_1 \beta_1} = -2 e_1 \left[ S^{\mu\nu} S_{\mu\nu} - (S_\mu^\nu)^2 \right]$$

$$\Rightarrow -2 e_1 F^{\mu\nu} F_{\mu\nu},$$

where to reach the second line we made integrations by parts, neglected total derivative terms, and used the fact that $F^\mu_\mu = 0$. Hence the standard kinetic terms for the vector are associated with the coupling $S^{\mu\nu} Z_{\mu\nu}^{(1)}$ in the Lagrangian. The remaining terms associated with $S^{\mu\nu} Z_{\mu\nu}^{(2)}$ and $S^{\mu\nu} Z_{\mu\nu}^{(3)}$ can be also rewritten in terms of $F_{\mu\nu}$, and lead to the combinations analyzed in [3]:

$$S^{\mu\nu} Z_{\mu\nu}^{(2)} = \text{tr} II F^2 - 2 \text{tr} III F^2,$$

$$S^{\mu\nu} Z_{\mu\nu}^{(3)} = -\text{tr} F^2 \left[ \text{tr} II^2 - (\text{tr} II)^2 \right] - 4 \text{tr} II \text{tr} III F^2 + 4 \text{tr} III F^2 + 2 \text{tr} II III F^2,$$

where we used a more synthetic notation in terms of traces $\text{tr} MN = M_\mu^\nu N_\mu^\nu$. Hence, one can work with a vector Lagrangian using $S_{\mu\nu}$ or $F_{\mu\nu}$ depending on her own convenience.

3 Maximally symmetric configurations in the vacuum and self-acceleration

After defining the Lagrangian that we will be working with, we start to study its consequences for cosmology. In this section, we would like to determine maximally symmetric solutions for this theory in the vacuum. Self-accelerating solutions are included in this class, and correspond to pure de Sitter configurations in the vacuum. Our aim is to obtain some general lessons on the role of vector fields for determining self-accelerating solutions in theories with Galileon symmetry, and characterize their stability.
An observer, embedded into a maximally symmetric space-time, locally experiences the following form of the metric (for $|\vec{x}| \ll H^{-1}$, with $|\vec{x}|$ the distance from his/her position)

$$ds^2 = \left(1 - \frac{H^2}{2} x^2\right) \eta_{\mu\nu} dx^\mu dx^\nu$$  \hspace{1cm} (26)

where $H$ is the (constant) Hubble parameter: the space-time is locally (A)dS space for (negative) positive $H^2$.

The approximation of working with a form of the metric valid only in the region near the observer is sufficient for our purposes. We would like to switch on a non-trivial profile for scalar, vector, and tensor modes so to obtain a background metric configuration as the above. In order to do so, we consider background profiles for the available fields as follows:

$$h_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{H^2}{2} x^2 \eta_{\mu\nu},$$ \hspace{1cm} (27)

$$\pi = \hat{\pi} + \frac{1}{2} q_0 x^2,$$ \hspace{1cm} (28)

$$A_\mu = \hat{A}_\mu + \frac{n_\mu}{2} x^2,$$ \hspace{1cm} (29)

where $H, q_0$ are constant numbers, while $n_\mu$ a constant vector (that can be space-like, time-like or light-like depending on the sign of $n^2$). The hat quantities can be interpreted as fluctuations around the background profiles after the appropriate shifts of the fields. In this section, we determine the conditions on the parameters $H, q_0,$ and $n_\mu$ to obtain a solution of the equations of motion, and the features of the dynamics of fluctuations around it. The field profiles (27)-(29) depend on the quadratic combination $x^2$, and are designed in such a way to generate in a simple way the maximally symmetric solutions with metric as in eq. (26). As we will see, with these choices of profiles the tadpole conditions for the effective Lagrangian for fluctuations, which determine the background solutions, will be satisfied by imposing simple algebraic conditions on the parameters of the theory.

We substitute the shifted configurations (27)-(29) into our Lagrangian: these background configurations are solutions of the background field equations if the tadpole terms in the Lagrangian for the hat fluctuations vanish.

### 3.1 Performing the shifts and obtaining the Lagrangian for fluctuations

In this section, we perform the shifts of eqs (27)-(29), and determine the effective Lagrangian for tensor, scalar and vector fluctuations $\hat{h}_{\mu\nu}, \hat{\pi}, \hat{A}_\mu$ around the maximally symmetric background of eq. (26).

**The $\pi$ shift**

We start considering the following shifted expression for the scalar $\pi$:

$$\pi = \tilde{\pi} + \frac{1}{2} q_0 x^2$$ \hspace{1cm} (30)

with $x^2 = \eta_{\mu\nu} x^\mu x^\nu$, and $q_0$ a constant. Hence, the $X^{(n)}$ tensors can be expressed as

$$X^{(n)}_{\mu\nu} = \sum_{m=0}^{n} \binom{n}{m} q_0^m \hat{X}^{(n-m)}_{\mu\nu},$$ \hspace{1cm} (31)

where $\hat{X}^{(n)}_{\mu\nu} = X^{(n)}_{\mu\nu}(\hat{\pi})$. After doing the shift the scalar-tensor part of the Lagrangian reads

$$\mathcal{L}_{st} = -\frac{1}{2} h^{\mu\nu} E_{\alpha\beta} h_{\alpha\beta} + h^{\mu\nu} \sum_{n=1}^{3} c_n \sum_{m=0}^{n} \binom{n}{m} \hat{X}^{(n-m)}_{\mu\nu} q_0^m.$$ \hspace{1cm} (32)

On the other hand, a straightforward calculation shows that the vector Lagrangian becomes

$$\mathcal{L}_{sv} = S^{\mu\nu} \left( \tilde{e}_1 Z^{(1)}_{\mu\nu} + \tilde{e}_2 Z^{(2)}_{\mu\nu} + \tilde{e}_3 Z^{(3)}_{\mu\nu} \right),$$ \hspace{1cm} (33)
where $\pi \to \hat{\pi}$ in $Z_{\mu\nu}^{(n)}$ and with the tilde couplings reading

\begin{align}
\hat{e}_1 &= e_1 + q_0 e_2 + q_0^2 e_3, \\
\hat{e}_2 &= e_2 + 2q_0 e_3, \\
\hat{e}_3 &= e_3.
\end{align}

The $h$-shift

We consider now the shift in the tensor degree of freedom; we express the metric $h$ tensor as

$$h_{\mu\nu} = \hat{h}_{\mu\nu} - \frac{H^2}{2} x^2 \eta_{\mu\nu}. \tag{37}$$

The Lagrangian expressed in terms of the variable $\hat{h}$ is

\begin{align}
L_{st} &= \frac{1}{2} \hat{h}^{\mu\nu} \varepsilon_{\alpha\beta} \hat{h}_{\alpha\beta} + 3H^2 \hat{h} + \left( \hat{h}^{\mu\nu} - \frac{H^2}{2} x^2 \eta_{\mu\nu} \right) \sum_{n=1}^{3} \sum_{m=0}^{n} \binom{n}{m} \hat{X}_{\mu\nu}^{(n-m)} q_0^m \\
&= \frac{1}{2} \hat{h}^{\mu\nu} \varepsilon_{\mu\nu} \hat{h}_{\alpha\beta} + 3H^2 \hat{h} + \hat{h}^{\mu\nu} \sum_{n=1}^{3} \sum_{m=0}^{n} \binom{n}{m} \hat{X}_{\mu\nu}^{(n-m)} q_0^m - H^2 \hat{\pi} \eta_{\mu\nu} \sum_{n=1}^{3} \sum_{m=0}^{n} \binom{n}{m} \eta_{\mu\nu} \hat{X}_{\mu\nu}^{(n-m-1)} q_0^m.
\end{align}

The sums involving the binomial coefficients appearing in the previous expression can be expanded and we obtain

\begin{align}
L_{st} &= \frac{1}{2} \hat{h}^{\mu\nu} \varepsilon_{\mu\nu} \hat{h}_{\alpha\beta} + 3H^2 \hat{h} \\
&+ \hat{h}^{\mu\nu} \left[ \hat{X}_{\mu\nu}^{(0)} (c_1 q_0 + c_2 q_0^2 + c_3 q_0^3) + \hat{X}_{\mu\nu}^{(1)} (c_1 + 2c_2 q_0 + 3c_3 q_0^2) + \hat{X}_{\mu\nu}^{(2)} (c_2 + 3c_3 q_0) + c_3 \hat{X}_{\mu\nu}^{(3)} \right] \\
&- H^2 \hat{\pi} \eta_{\mu\nu} \left[ \hat{X}_{\mu\nu}^{(0)} (c_1 + 2c_2 q_0 + 3c_3 q_0^2) + \hat{X}_{\mu\nu}^{(1)} (c_2 + 3c_3 q_0) + c_3 \hat{X}_{\mu\nu}^{(2)} \right]. \tag{39}
\end{align}

The scalar-vector part of the Lagrangian is not affected, since the vector does not directly couple to the tensor.

The $A_{\mu}$-shift

As a last step, we consider the vector shift $\eta_{\mu\nu}$, that reads

$$A_{\mu} = \frac{n_{\mu}}{2} x^2 + \hat{A}_{\mu}, \tag{40}$$

hence

$$S_{\mu\nu} = (n_\mu x_{\nu} + n_\nu x_\mu) + \hat{S}_{\mu\nu}, \tag{41}$$

where $\hat{S}_{\mu\nu} = 2\nabla_{\mu} \hat{A}_{\nu} - \hat{F}_{\mu\nu}$. This shift only changes the structure of the scalar-vector part of the Lagrangian. We analyze separately the three different contributions appearing in eq. (33):

- $S_{\mu\nu} Z_{\mu\nu}^{(1)}$. This leads (up to constant terms)

$$S_{\mu\nu} Z_{\mu\nu}^{(1)} = \hat{S}_{\mu\nu} \hat{Z}_{\mu\nu}^{(1)} + 24 n_\mu \hat{A}_\mu. \tag{42}$$

Notice that this piece generates a tadpole for $A^\mu$ depending on the vector $n_\mu$.

- $S_{\mu\nu} Z_{\mu\nu}^{(2)}$. This gives

$$S_{\mu\nu} Z_{\mu\nu}^{(2)} = \hat{S}_{\mu\nu} \hat{Z}_{\mu\nu}^{(2)} + 4 \left( n_\mu \hat{A}_\mu \square \hat{\pi} - n_\mu \hat{A}_\mu \hat{\Pi}_\mu \right) + 12 \hat{\pi} n_\mu n^\mu. \tag{43}$$

This piece generates a tadpole for $\pi$, and a quadratic coupling between scalar and vector.

- $S_{\mu\nu} Z_{\mu\nu}^{(3)}$. This gives

$$S_{\mu\nu} Z_{\mu\nu}^{(3)} = \hat{S}_{\mu\nu} \hat{Z}_{\mu\nu}^{(3)} - 2 \left( n^{\mu} \hat{A}_\mu + n^{\nu} \hat{A}_\mu \right) \hat{X}_{\mu\nu}^{(2)} + 4n^2 \hat{\pi} \square \hat{\pi} + 4 (n^\mu \partial_\mu \hat{\pi})^2. \tag{44}$$
The complete vector Lagrangian can then be obtained by plugging these different pieces into eq. (33). One finds

\[ L_{sv} = \tilde{e}_1 \left( S^{\mu\nu} \tilde{Z}^{(1)}_{\mu\nu} + 24 n_\mu \hat{A}^\mu \right) + \tilde{e}_2 \left( S^{\mu\nu} \tilde{Z}^{(2)}_{\mu\nu} + 4 \left( n_\mu \hat{A}^\mu \square \hat{\pi} - n_\mu \hat{A}_\mu \tilde{\Pi}^{\mu\nu} \right) + 12 \hat{\pi} n_\mu n_\mu \right) + \tilde{e}_3 \left( S^{\mu\nu} \tilde{Z}^{(3)}_{\mu\nu} - 2 \left( n_\mu \hat{A}^\nu + n_\nu \hat{A}^\mu \right) \tilde{X}_{\mu\nu}^{(2)} + 4 n_\mu \hat{\pi} n_\mu + 4 \left( n_\mu \partial_\mu \hat{\pi} \right)^2 \right) \]

(45)

Notice that the Lagrangian for fluctuations explicitly depends on the direction \( n^\mu \) along which we turn on the vector profile (see eq. (40)). Hence while the background configurations for the metric and the scalar are isotropic, the dynamics of scalar and vector fluctuations depend on the particular direction along the vector background.

**Imposing the tadpole conditions**

A necessary and sufficient condition to determine background solutions is to cancel the tadpole terms depending on tensor, scalar, and vector hat fluctuations. These read

\[
\begin{align*}
\text{(tensor)} & \quad H^2 + 2 \sum_{n=1}^{3} c_n q_0^n = 0 \quad \Rightarrow \quad -\frac{H^2}{2} = c_1 q_0 + c_2 q_0^2 + c_3 q_0^3, \\
\text{(scalar)} & \quad H^2 \sum_{n=1}^{3} c_n n q_0^{n-1} + 12 e_2 n^2 = 0 \quad \Rightarrow \quad H^2 \left( c_1 + 2 q_0 c_2 + 3 q_0^2 c_3 \right) = -12 e_2 n^2, \\
\text{(vector)} & \quad (e_1 + q_0 e_2 + q_0^2 e_3) n_\mu \hat{A}^\mu = 0 \quad \Rightarrow \quad \tilde{e}_1 = (e_1 + q_0 e_2 + q_0^2 e_3) = 0 \quad \text{or} \quad n_\mu = 0.
\end{align*}
\]

(46) (47) (48)

As anticipated above, these are algebraic equations between the quantities appearing in the Lagrangian for fluctuations. Choosing parameters such to satisfy these conditions, we find maximally symmetric space-times in the vacuum, around which the dynamics of perturbations preserve Galilean and gauge symmetries. The three conditions above fix the numerical quantities \( q_0, H_0 \) and \( n^2 \) completely. We can find different branches of solutions of the previous system of equations, that we will discuss in what follows. As we will see, although we can switch on a vector profile along a direction \( n^\mu \), nevertheless we will be able to find isotropic (and maximally symmetric) solutions for the metric. This is due to the particular derivative couplings of the scalar to the vector, that can allow us to solve eq. (48) with \( n_\mu \neq 0 \).

**3.2 The maximally symmetric background solutions**

The different branches of solutions to the set of equations (46)-(48) are:

1. **First branch:**
   Suppose that equation (48) admits real solutions for \( q_0 \) by imposing \( \tilde{e}_1 = 0 \). Then, it determines up to two real solutions for \( q_0 \). Plugging one of these solutions in (46) we determine \( H^2 \). Plugging these results in (47) one finally determines \( n^2 \). Notice that \( H^2 \) and \( n^2 \) can have either sign: if \( H^2 > 0 \) one obtains de Sitter space and a self-accelerating configuration in the vacuum. This branch is characterized by a background vector field turned on with \( n_\mu \neq 0 \), that however does not break the isotropy of three dimensional space-time, nor breaks the vector gauge symmetry. Let us emphasize that this branch is characterized by the condition \( \tilde{e}_1 = 0 \): this implies the vanishing of the standard kinetic term for the vector fluctuations \( \hat{A}_\mu \), that is proportional to \( S^{\mu\nu} \tilde{Z}^{(1)}_{\mu\nu} \) in the Lagrangian (43). On the other hand, the vector fluctuations can acquire dynamics through coupling with the scalar fluctuations \( \hat{\pi} \), as we will see below.

2. **Second branch:** We can also recover well-known scalar Galileon maximally symmetric solutions with vector field turned off. Choose \( n^\mu = 0 \), hence (48) does not give any constraint on \( q_0 \). If (47) admits at least one real solution for \( q_0 \), we can use its value in (46) to determine \( H^2 \). If \( H^2 > 0 \), we obtain the self-acceleration. In this case the kinetic terms for the vector fluctuations are generally not vanishing.

3. **Third branch:** The last option is to turn off the gauge field, \( n^\mu = 0 \), and choose the Minkowski space with \( H = 0 \). Hence (47) and (48) are automatically satisfied. Then (46), when admitting real solutions, fixes \( q_0 \).
The first branch of solutions is new, and specific to the case of having a vector field turned on (although similar configurations have been already studied in massive gravity \cite{[17]}). The other two branches were already known in the literature, at least for the specific set-up of massive gravity \cite{[16]} (while the case with \textit{c} = 0 had been already investigated in \cite{[17]}). Notice that we can have intermediate situations in which different branches are connected. Suppose that condition (48) is satisfied with \( \hat{e}_1 = (\hat{e}_1 + \hat{q}_0 \hat{e}_2 + \hat{q}_0^2 \hat{e}_3) = 0 \), and the value of \( \hat{q}_0 \) satisfying this condition also satisfies (47) with \( n^2 = 0 \) (but with the (null-like) vector \( n_\mu \) not necessarily vanishing). This configuration continuously connects the first and second branches. A similar situation can be realized, for example, in massive gravity \cite{[16] [17] [14]}.

### 3.3 An instability around the first branch of maximally symmetric solutions

Let us focus on the first branch with the vector fields turned on to study the dynamics of the fluctuations. After imposing the conditions to remove the tadpoles, the quadratic contributions to the complete Lagrangian result

\[
\mathcal{L}_{\text{quadr}} = -\frac{1}{2} \hat{h}^{\mu \nu} \mathcal{E}_{\alpha \beta} \hat{h}_{\alpha \beta} - \frac{12 n^2}{H^2} \hat{h}^{\mu \nu} \hat{X}_{\mu \nu}^{(1)} - 6H^2 (c_2 + 3c_3 \hat{q}_0) \hat{\pi} \hat{\square} \hat{\pi} + 4\hat{e}_3 \left( n^2 \hat{\pi} \hat{\square} \hat{\pi} + (n^\mu \partial_\mu \hat{\pi})^2 \right) + 4\hat{e}_2 \left( n_\mu \hat{A}^\mu \hat{\square} \hat{\pi} - n_\mu \hat{A}_\mu \hat{\Pi}^{\rho \nu} \right). \tag{49}
\]

Let us emphasize again that in this branch the vector tadpole cancelation, associated with condition (48), implies that the vector field has no standard kinetic term. On the other hand, the vector acquires a coupling with the scalar at quadratic order in perturbations (if \( \hat{e}_2 \) is non vanishing, as we will suppose from now on) that depends on the background vector profile \( n_\mu \). Notice that the previous quadratic contribution to the Lagrangian is linear on \( \hat{A}_\mu \). On the other hand, higher order contributions to the Lagrangian will also include terms quadratic in the vector field.

The quadratic Lagrangian for tensor and scalar can then be diagonalized with the standard field transformation of \( \hat{h}_{\mu \nu} \) to \( \hat{h}_{\mu \nu} \)

\[
\hat{h}_{\mu \nu} = \hat{h}_{\mu \nu} + \frac{6n^2}{H^2} \hat{\pi} \eta_{\mu \nu} \tag{50}
\]

finding

\[
\mathcal{L}_{\text{quadr}} = -\frac{1}{2} \hat{h}^{\mu \nu} \mathcal{E}_{\mu \nu} \hat{h}_{\alpha \beta} + \left[ \frac{108 (n^2)^2}{H^4} - 6H^2 (c_2 + 3c_3 \hat{q}_0) \right] \hat{\pi} \hat{\square} \hat{\pi} - 2\hat{e}_3 \left( n_\mu \partial_\nu \hat{\pi} - n_\nu \partial_\mu \hat{\pi} - \frac{\hat{e}_2}{2\epsilon_3} \hat{F}_{\mu \nu} \right)^2 + \frac{\hat{e}_2^2}{2\epsilon_3} \hat{F}_{\mu \nu}^2. \tag{51}
\]

The scalar-vector coupling at quadratic order (associated with the first parenthesis in the second line of the previous formula (51)) cannot be removed by a simple local field redefinition. But the structure of the Lagrangian is sufficiently simple to exhibit an instability.

For simplicity, let us focus on purely time dependent perturbations, with \( \hat{\pi} = \hat{\pi}(t) \), and \( \hat{A}_\mu = (0, A_r(t), 0, 0) \). This Ansatz for the fluctuations is very simple, but it is sufficient for our purpose. Focus on the scalar-vector part of Lagrangian (51); it can be rewritten as

\[
\mathcal{L}_{\text{quadr}} = \left[ \frac{108 (n^2)^2}{H^4} - 6H^2 (c_2 + 3c_3 \hat{q}_0) + 4\hat{e}_3 n^2 \right] \hat{\pi}^2 + 4\hat{e}_2 n_r \hat{A}_r \hat{\pi}. \tag{52}
\]

Calling \( \mathcal{Q} \) the part in square parenthesis of the previous equation, one finds

\[
\mathcal{L}_{\text{quadr}} = \mathcal{Q} \left( \hat{\pi} + \frac{2\hat{e}_2 n_r}{\mathcal{Q}} \hat{A}_r \right)^2 - \frac{4\hat{e}_2^2 n^2}{\mathcal{Q}} \hat{A}_r^2. \tag{53}
\]

Hence, when \( \hat{e}_2 \neq 0 \), one always finds an instability on the scalar, or on the vector sectors (depending on the sign of \( \mathcal{Q} \)) for the first branch of solutions \cite{[16]}.

*The case \( \hat{e}_2 = 0 \) can be studied using the Hamiltonian approach applied in \cite{[17]} for the special case of massive gravity.
The conclusion is that the first branch of maximally symmetric configurations for our theory, that admits a non-trivial profile for the vector field, is generically unstable. Vector or scalar fluctuations have the wrong sign for the kinetic terms. The same consideration holds for set-ups that interpolate between the first and second branches, as the ones discussed at the end of Section 3.2 in which a light-like vector field can be turned on.

The general lesson is that, when considering modified gravity scenarios that rely on Galileon symmetries, one has to pay extra care to the dynamics of vector fluctuations, in particular around the branches of self-accelerating solutions in which the standard vector kinetic terms vanish. Indeed, these configurations are generally plagued by instabilities associated with higher order Galileon interactions between vector and scalars. This fact has been pointed out in [4, 17] for the special case of massive gravity in decoupling limit: the results of the present paper generalize the analysis to a broader context and provide the tools to analyze this issue in more general set-ups respecting Galileon and gauge symmetries.

4 Spherically symmetric configurations: how vectors contribute to the Vainshtein mechanism

Vector degrees of freedom derivatively coupled to scalars can contribute to the screening mechanisms that characterize the most interesting modified gravity scenarios.

In this section, we will analyze spherically symmetric solutions around a given source in the theory we are considering. Usually, for simplicity in treating modified gravity scenarios one makes the hypothesis that the only degrees of freedom coupling to the source are the tensor, via a minimal coupling $h_{\mu\nu}T^{\mu\nu}$, and the scalar, which couples to the trace of the energy momentum tensor as $\pi T$. On the other hand, since vector degrees of freedom are normally contained in the low-energy spectrum of gravitational interactions, they can couple directly to the source in a way that respects the symmetries of the theory. We propose to consider the case in which, besides the usual energy momentum tensor, sources are also characterized by gravitational vector currents $J^\mu$, associated with a gravitational vector coupling of the form $A^\mu J^\mu$. If the vector currents are conserved, such couplings respect the Abelian gauge symmetry $A^\mu \rightarrow A^\mu + \partial^\mu \xi$ of the Lagrangian. As we will see later, the vector charge influences the realization of the Vainshtein effect, and provides new interesting environmental effects that change the gravitational interactions around a spherically symmetric source. (See also [19] for a discussion of a screening mechanism in a theory involving vectors.)

We look for static spherically symmetric configurations around Minkowski space. In order to study these configurations, we re-express the scalar and vector Lagrangians in a more convenient form. We set to zero the coupling $c_3$ of the scalar Galileon Lagrangian of eq. (3), since it has been shown in [20] that these couplings lead to instabilities around spherically symmetric solutions. After performing a proper diagonalization procedure using the relations (13)-(15), the scalar Lagrangian can be written as

$$L_{sc}^{tot} = \sqrt{-g} \left[ d_2 L_{sc}^{(2)} + d_3 L_{sc}^{(3)} + d_4 L_{sc}^{(4)} \right]$$

with

$$L_{sc}^{(2)} = \frac{1}{2} \partial \pi \cdot \partial \pi,$$

$$L_{sc}^{(3)} = \frac{1}{2} \left[ (\text{tr}[\Pi]) \partial \pi \cdot \partial \pi \right.,$$

$$L_{sc}^{(4)} = \frac{1}{4} \left( (\text{tr}[\Pi])^2 \partial \pi \cdot \partial \pi - 2 \text{tr}[\Pi] \partial \pi \cdot \partial \pi \cdot \partial \pi - \text{tr}[\Pi^2] \partial \pi \cdot \partial \pi + \partial \pi \cdot \Pi^2 \cdot \partial \pi \right),$$

where the $d_i$ are suitable linear combinations of the original parameters $c_i$ appearing in eq. (3). These couplings can be expressed as

$$d_i = \frac{d_i}{\Lambda^{r_i(r-2)}},$$

where the $d_i$ are dimensionless quantities while $\Lambda$ is a scale of dimension of a mass, whose value depends on the theory under consideration. $\Pi$ corresponds to the matrix $\Pi_{\mu\nu}$, and we indicate traces with $\text{tr}[\ldots]$. The vector

\textsuperscript{1}The same analysis holds also around the maximally symmetric second branch of solutions discussed in the previous section.
Lagrangian is expressed in terms of \( F_{\mu\nu} = \nabla_\mu A_\nu - \nabla_\nu A_\mu \) as

\[
\mathcal{L}^{tot}_{sv} = \sqrt{-g} \left[ e_2 \mathcal{L}^{(2)}_{sv} + e_3 \mathcal{L}^{(3)}_{sv} + e_4 \mathcal{L}^{(4)}_{sv} \right],
\]

(59)

with

\[
\mathcal{L}^{(2)}_{sv} = -\text{tr}[F^2],
\]

(60)

\[
\mathcal{L}^{(3)}_{sv} = -\left( \text{tr}[\Pi] \text{tr}[F^2] - 2 \text{tr}[\Pi F^2] \right),
\]

(61)

\[
\mathcal{L}^{(4)}_{sv} = -\frac{1}{4} \left( -\text{tr}[F^2] \left[ \text{tr}[\Pi^2] - (\text{tr}[\Pi])^2 \right] - 4\text{tr}[\Pi] \text{tr}[\Pi F^2] + 4\text{tr}[\Pi^2 F^2] + 2\text{tr}[\Pi F^2] \right).
\]

(62)

The parameters \( e_n \) can be associated with linear combinations of the parameters \( e_i \) appearing in the Lagrangian (17) that was expressed in terms of the tensor \( S_{\mu\nu} \). These parameters can be expressed as

\[
e_n = \frac{\hat{e}_n}{\Lambda^{3(n-2)}},
\]

(63)

where the \( \hat{e}_n \) are dimensionless while \( \Lambda \) is a scale of dimension of a mass.

To the previous Lagrangians we then add contributions that control couplings of tensor, scalar, and vectors to the source. We assume that a source is characterized by a conserved energy momentum tensor \( T_{\mu\nu} \), and a conserved vector current \( J_\mu \). The couplings we consider are

\[
\mathcal{L}_{\text{coupl}} = \sqrt{-g} \left[ \frac{1}{M_{Pl}} h_{\mu\nu} T^{\mu\nu} + \frac{1}{2 M_{Pl}} \pi T + A_\mu J^\mu \right].
\]

(64)

We parameterize flat Minkowski space in spherical coordinates: we will then include the overall factor \( \sqrt{-g} = r^2 \sin \theta \) to the previous Lagrangians. Focussing on static spherically symmetric configurations, we drop the explicit dependence on time, and hence focus on the component \( A_\mu = (A_0(r), 0, 0, 0) \) for the vector, and \( \pi(r) \) for the scalar.

The scalar Lagrangian (including the coupling with source) in this spherically symmetric case reads

\[
\mathcal{L}_{sc} = \frac{d_2}{2} r^4 \left( \frac{\pi'}{r} \right)^2 - \frac{d_3}{3} r^4 \left( \frac{\pi'}{r} \right)^3 - \frac{d_4}{2} r^4 \left( \frac{\pi'}{r} \right)^4 + r^2 \frac{\pi}{2 M_{Pl}} T.
\]

(65)

We choose the energy momentum tensor of a spherically symmetric, point-like source of mass \( M \). The trace of it is given by

\[
T = M \frac{\delta(r)}{r^2} = -M \frac{1}{r^2} \partial_r \left[ r^2 \partial_r \left( \frac{1}{r} \right) \right].
\]

(66)

Then the scalar Lagrangian, upon integration by parts, becomes

\[
\mathcal{L}_{sc} = -\frac{d_2}{2} r^4 \left( \frac{\pi'}{r} \right)^2 - \frac{d_3}{3} r^4 \left( \frac{\pi'}{r} \right)^3 - \frac{d_4}{2} r^4 \left( \frac{\pi'}{r} \right)^4 - \frac{\pi'}{r} \left( \frac{M r}{2 M_{Pl}} \right).
\]

(67)

As explained above, we assume that each source couples not only to the scalar, but also to the vector field via a current in the form \( A_\mu J^\mu \). Our point-like source is characterized by a non-vanishing dimensionless vector charge \( Q_0 \). For a spherically symmetric, static source the vector current reads \( J^0 = 2 Q_0 \delta^{(3)}(x) \).

The vector Lagrangian, after integrating by parts, is given by

\[
\mathcal{L}_{sv} = A'_0(r)^2 r^2 \left( 2e_2 + 4e_3 \frac{\pi'}{r} + e_4 \frac{\pi'^2}{r^2} \right) - 2 Q_0 A'_0(r).
\]

(68)

We can solve the equation of motion corresponding to \( A'_0(r) \):

\[
A'_0(r) = \frac{Q_0}{r^2 \left( 2e_2 + 4e_3 \frac{\pi'}{r} + e_4 \frac{\pi'^2}{r^2} \right)}.
\]

(69)
This is proportional to the ‘electric’ part of the vector field strength associated with a point-like charge $Q_0$. Since no time derivatives are involved, we can plug the result for $A'_0(r)$ back in the vector Lagrangian, finding

$$\mathcal{L}_{sv} = -\frac{Q_0^2}{r^2 (2e_2 + 4\epsilon_3 \frac{r}{r'} + \epsilon_4 \frac{\pi^2}{r'})}.$$  \hspace{1cm} (70)

When added to the scalar Galileon terms, we find the following algebraic equation of motion for the quantity $y = \pi'/r$ that controls the scalar field, in the presence of a source with mass and vector charge

$$d_2 y + 2d_3 y^2 + 2d_4 y^3 - \frac{(4\epsilon_3 + 2\epsilon_4 y) Q_0^2}{r^6 (2e_2 + 4\epsilon_3 y + \epsilon_4 y^2)^2} = \frac{M}{M_P r^3}.$$  \hspace{1cm} (71)

We study now the system in two different cases. The case in which only the cubic Galileon is included (setting the quartic couplings $d_4$ and $\epsilon_4$ to zero in the equation above) will be discussed in the next section. The case in which also the quartic Galileon is switched on is conceptually very similar, and its analysis is relegated to the Appendix. For simplicity, we will not consider here the effect of the quintic Galileon in this work.

### 4.1 The cubic Galileon

Let us start our discussion with the case of cubic Galileon: set $d_4 = \epsilon_4 = 0$ in eq. (71). The scalar equation is

$$d_2 y + 2d_3 y^2 - \frac{4\epsilon_3 Q_0^2}{r^6 (2e_2 + 4\epsilon_3 y)^2} = \frac{M}{M_P r^3}.$$  \hspace{1cm} (72)

Far from the source, $r \gg 1$, $y$ is small and the solution of the previous equation is $y = M/(M_P d_2 r^3)$, which implies $\pi' = M/(M_P d_2 r^2)$ and $A'_0 = Q_0/(2e_3 r^2)$. Hence scalar and vectors mediates fifth, long-range forces that lead to a modification with respect to GR predictions. Notice that the expression for $\pi$ does not depend on the vector charge $Q_0$ in this large $r$ limit.

More interesting to us is what happens in proximity of the source: hence $y$ becomes large, and eq. (72) admits two branches of solutions (recall the definition of the dimensionless hat quantities eqs (58), (63))

$$y = \sqrt[3]{\frac{M}{d_3 M_P r^3}} \left[ 1 \pm \sqrt{1 + \frac{2d_3 M_P^2 Q_0^2}{\epsilon_3 M^2 e_3}} \right]^{1/3},$$  \hspace{1cm} (73)

$$= \frac{\hat{d}_2 \Lambda^3}{d_3} (r^+_{V})^{3/2},$$  \hspace{1cm} (74)

with

$$r^+_{V} \equiv \frac{1}{\Lambda} \left[ \frac{\hat{d}_3 M}{4d_2^2 M_P} \left( 1 \pm \sqrt{1 + \frac{2d_3 M_P^2 Q_0^2}{\epsilon_3 M^2 e_3}} \right) \right]^{\frac{1}{2}},$$  \hspace{1cm} (75)

where the Vainshtein radius $r_V$ is defined in such a way to correspond to the scale at which the non-linear terms in the scalar equations of motion become important. Indeed when $r = r_V$, one finds that the value of $y$ is $y = \hat{d}_2 \Lambda^3/\hat{d}_3$. This is the scale at which the second term in the expression becomes comparable to the first term. This is why we included the additional factors in eq. (74).

At short distances $r \ll 1$ we get the following behavior for the scalar field $\pi$ and $A_0$,

$$\pi(r) = \frac{2\hat{d}_2 \Lambda^3}{\hat{d}_3} (r^+_{V})^{\frac{3}{2}} \sqrt{r},$$  \hspace{1cm} (76)

$$A_0(r) = \frac{Q_0 \hat{d}_3 \sqrt{r}}{2 \epsilon_3 \hat{d}_2 (r^+_{V})^{\frac{3}{2}}}.$$  \hspace{1cm} (77)

Since $\pi', A'_0 \propto 1/\sqrt{r}$ the scalar and vector contributions are much weaker than the usual gravitational one in proximity of the source: the Vainshtein mechanism is at work and GR is recovered nearby a spherically symmetric source.
As we will see explicitly in a moment, the requirement of stability of our configuration demands that the parameters \(d_2, d_3, \hat{e}_2, \hat{e}_3\) are non-negative. The obvious requirement of having \(r_V^2 > 0\) selects only the positive branch in the above choice \((75)\) (at least for \(M > 0\)) hence we will focus on this case from now on. Then, the value of the Vainshtein radius depends not only on the mass of the object, but also on how much the object is coupled to the vector fields. This fact could play an important role for analyzing the effective theory of fluctuations around a source and increase the effective cut-off for low energy theory of perturbations, as discussed for example in \([21]\). Hence higher order Galilean interactions involving vectors, as the ones we consider, render environmental effects richer and subtler.

Before briefly discussing some phenomenological consequences of these findings, let us analyze more in detail the stability under small fluctuations of the spherically symmetric backgrounds we have determined. We focus on spherically symmetric scalar and vector perturbations, which respect the spherically symmetric Ansatz in terms of the criteria used in \([17]\):

\[
\pi = \bar{\pi}(r) + \phi(t,r),
\]

\[
A_{\mu} = (A_0(r) + \delta A_0(t,r), \delta A_r(t,r), 0, 0),
\]

\[
F_{tr} = \delta A_t(t,r) - \delta A_0(t,r).
\]

The bars denote background quantities. We limit our attention to the spherically symmetric case to be able to treat fully analytically the system of coupled scalar and vector equations of motion for the fluctuations.

After a straightforward de-mixing procedure of scalar from vectors, that involves a field redefinition

\[
F_{tr} \to -\frac{2e_3 \hat{A}_0' \phi'}{e_2 r + 2e_3 \bar{\pi}'} + \tilde{F}_{tr},
\]

we obtain the following Lagrangian for spherically symmetric quadratic fluctuations around our spherically symmetric background

\[
S_\varphi = \frac{1}{2} \int d^4 x \sqrt{-g} \left[ K_t(r) \dot{\phi}^2 - K_r(r) \phi'^2 - K_A(r) \tilde{F}_{tr} \tilde{F}^{tr} \right].
\]

The functions \(K_i\) have to be positive in order to obtain a stable configuration. They read

\[
K_t = d_2 + 4d_3 \frac{\pi'}{r} + 2d_3 \bar{\pi}'',
\]

\[
K_r = d_2 + 4d_3 \frac{\pi'}{r} + \frac{4e_3^2 Q_0^2}{r^6 (e_2 + 2e_3 \pi'/r)^3},
\]

\[
K_A = e_2 + 2e_3 \frac{\pi'}{r}.
\]

Nearby the source, the quantity \(\pi'/r\) is large: in order to have positive \(K_i\), we demand that \(d_3, e_3\) are positive. Far from the source, instead, \(\pi'/r\) is small: we have to demand that \(d_2, e_2\) are positive. This implies that a vector charge increases the size of the function \(K_r\) above nearby the source, while it gives only negligible contributions far from it. Notice that these results, as anticipated, require that we can only take the positive sign in the option for the Vainshtein radius in eq. \((75)\).

Computing the speed of radial scalar fluctuations in proximity of the source we find

\[
c^2_r = \frac{K_r}{K_t} = \frac{4}{3} \left(1 + \frac{\mathcal{Y}}{(1 + \sqrt{1 + \mathcal{Y}})^2}\right)
\]

with

\[
\mathcal{Y} = \frac{2\hat{d}_3 M_{pl}^2 Q_0^2}{M^2 e_3}
\]

Hence, being \(\mathcal{Y} \geq 0\), the already superluminal speed of radial fluctuations is further increased by the presence of the vector charge. Far from the source, the vector charge gives negligible contributions and one recovers the well-known predictions of standard scalar Galileon models. We conclude that the vector charge does not help to solve the issue of superluminal propagation in cubic Galileon theories.

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4.2 Phenomenological considerations

Let us make some simple phenomenological considerations on the results obtained so far. Under the hypothesis that the vector fields are part of the spectrum of gravitational DOFs, a non-vanishing charge $Q_0$ would be associated with gravitational vector interactions. The vector charge modifies the expression for the Vainshtein radius for the scalar interaction, which is bounded from below no matter how small the mass of the source is:

$$r_V^3 \geq \frac{1}{\Lambda^3} \frac{d_3^{3/2} |Q_0|}{\sqrt{8} d_2^2 e_3^{1/2}}$$  

(88)

More in general, if the dimensionless quantity

$$\mathcal{Y} \equiv \frac{2d_3 M_P^2 Q_0^2}{e_3 M^2},$$  

(89)

is much larger than one, then the expression for the Vainshtein radius is sensitive to the vector charge $Q_0$, and saturates the inequality (88) in the limit $\mathcal{Y} \to \infty$. Notice that $\mathcal{Y}$ does not depend on the scale $\Lambda$, and is proportional to the (typically very large) ratio $M_P^2/M^2$.

The ratio of the Vainshtein radii, calculated respectively in the limits of large and small $\mathcal{Y}$, reads

$$\frac{r_{V\to\infty}}{r_{V\to0}} = \mathcal{Y}^{1/6}.$$  

(90)

Hence the Vainshtein radius can increase considerably in the presence of a vector charge, potentially changing the predictions of modified gravity scenarios based on the Vainsthein mechanism.

Until now, our considerations were made under the hypothesis that the vectors under consideration are part of the gravitational sector of the theory. In this case, we found that vector gravitational interactions, as well as scalar interactions, are screened well inside the Vainshtein radius, and GR predictions are well recovered nearby a source.

For the remaining part of this section we would like to consider the different, alternative perspective that the vectors we are analyzing are part of the matter sector, and not of the gravitational sector of the theory. We consider a purely scalar Galileon theory describing a theory of modified gravity, that couples with standard electromagnetism. Higher order interactions between the scalar $\pi$ and electromagnetism, with a structure described by Lagrangian (59), are allowed by the symmetries of the theory: the coefficient $\hat{e}_3$ and the scale $\Lambda$ should then be constrained in such a way to agree with the very accurate experimental tests of electromagnetic interactions and quantum electrodynamics. This is an interesting topic, that however we will not analyze in this work. Here, we would like only to point out how the electric charge of a body changes the size of its Vainshtein radius, using the results that we obtained above.

Consider for definiteness an electron: it has an electromagnetic vector charge, as well as a mass. In appropriate units, the ratio between electron charge $q_e$ and electron mass $m_e$ (including a Planck mass to render this quantity dimensionless) is

$$\frac{M_P^2 q_e^2}{m_e^2} \simeq 10^{42}$$  

(91)

and this huge number reflects the well known fact that the relative strength of the electromagnetic force is much larger than the one of gravitational interaction. The ratio (91) enters in the expression for the Vainshtein radius for a charged body. If the parameters $\hat{e}_i, \hat{d}_i$ are not exceedingly small, for the considerations we made above this implies that the size of Vainshtein radius for an electron is independent of the electron mass, and reads

$$r_V = \frac{1}{\Lambda} \left[ \frac{d_3^{3/2} q_e}{\sqrt{8} d_2^2 e_3^{1/2}} \right]^{1/2} \simeq 10^{10} \left[ \frac{d_3^{3/2} q_e}{\sqrt{8} d_2^2 e_3^{1/2}} \right]^{1/2} \left( \frac{m_e}{\Lambda} \right) r_e.$$  

(92)

where $r_e = 10^{-15}$ meters is the scale of the classical radius of the electron. Hence we learn that for scales $\Lambda$ of order of the electron mass and choosing the couplings not exceedingly small, the Vainshtein radius would be much larger than the classical electron radius.
On the other hand, we have to take into account that also the electromagnetic force is changed when coupling
the scalar to the electromagnetic field. Recall that the spherically symmetric electric field produced by the
electron, for the case of cubic Galileon we are focusing on in this section, is given by the expression (69) that
reads
\[ A'_0(r) = \frac{Q_0}{r^2(2e_2 + 4e_3 \pi r^2)}. \]  
A non-trivial background profile for the scalar can change the $1/r^2$-dependence of the previous expression nearby
the source. One easily check that the second term in the denominator of the previous formula becomes negligible
when $r$ larger than a scale $r_A$, given by
\[ r_A = \left[ \left( \frac{\hat{e}_3 \hat{d}_2}{\hat{e}_2 \hat{d}_3} \right) \right]^{2/3} r_V. \]  
We can express this quantity using the formulae above, and find
\[ r_A = 10^{10} \left( \frac{\frac{\hat{e}_3^2}{\hat{e}_2^2 \hat{d}_3^2}}{\Lambda} \right) \left( \frac{m_e}{\Lambda} \right) r_e. \]  
Hence we learn that $r_A$ is much larger than the electron radius, unless the coupling $\hat{e}_3$ is very small, or the
scale $\Lambda$ well larger than the electron mass. These simple considerations can lead to strong bounds on these
parameters, and show in a simple example that derivative couplings between scalars with Galilean symmetry and
electromagnetism can change considerably the behavior of electromagnetic and gravitational interactions. In this
section we focussed on the case of cubic Galileon interactions. A set-up based on a quartic Galileon Lagrangian
will be analyzed in the Appendix.

5 Summary

Vector degrees of freedom typically arise in many examples of modified gravity models. In this work, we started to
systematically explore their role in these scenarios, specifically studying the effects of derivatively coupled vectors
and scalars. To reduce the number of effective operators in the Lagrangian, we imposed appropriate symmetries
and physical constraints on the theory. We required that our Lagrangian is invariant under a Galilean symmetry in
the scalar sector, and an Abelian gauge symmetry in the vector sector. Moreover, in order to avoid Ostrogradsky
instabilities, we demanded that its associated equations of motion contain at most two space-time derivatives.
The resulting Lagrangian contains only a small number of terms. Starting from it, we investigated the role
of vector fields for two broad classes of phenomena that characterize modified gravity scenarios. The first is self-
acceleration. We analyzed in general terms the behavior of vector fluctuations around self-accelerating solutions.
We showed that it can provide key constraints to characterize instabilities of cosmological backgrounds, in cases in
which the kinetic terms of vector fluctuations vanish. The second phenomenon we studied is the screening of long
range fifth forces in modified gravity models, in particular for the so-called Vainshtein mechanism. In modified
gravity scenarios based on Galileon symmetries, the non-linearities of field equations allow to screen the effects
of light scalars within a certain distance (the Vainshtein radius) from a spherically symmetric source. We showed
that if a given source is characterized by a gravitational vector current, besides its usual energy momentum tensor,
vectors play an important role for defining the background solution and the scale corresponding to the Vainshtein
radius. We also commented on how our findings can also be applied to set-ups in which scalars enjoying Galileon
symmetries couple to bodies that are charged under electromagnetic interactions. Also in this case the realization
of the Vainhstein mechanism might be influenced by the vector charges.

Our general results can be applied to any concrete model of modified gravity that satisfy the requirements
that we imposed in this paper. It would be interesting to understand the dynamics of vectors also for space-times
that are less symmetrical than the ones we considered. For example, studying self-acceleration in cosmological
space-times that break the isotropy of the Friedmann-Robertson-Walker Ansatz; or studying the realization of
Vainshtein mechanism for stationary space-times associated to sources that rotate around a given axis. It can be
expected that vector degrees of freedom can have interesting roles also in these set-ups, since they can acquire
vacuum expectation values along a preferred spatial direction.
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A The case of quartic galileon

The case of quartic Galileon can be discussed similarly to the cubic case. After substituting the vector equation, the scalar equation to solve is

\[ \frac{d_2 y + 2 d_3 y^2 + 2 d_4 y^3}{r^6} - \frac{(4e_3 + 2e_4 y)}{2r^2 + 4e_3 y + e_4 y^2)^2} = \frac{M}{M_{Pl} r^3} \]

(96)

In proximity of the source, we find the following solution for \( y \)

\[ y = \frac{\hat{d}_2 A^3 r^\pm_v}{\hat{d}_3} \]

(97)

with

\[ r^\pm_v = \frac{\hat{d}_3}{\hat{d}_2 A} \left[ \frac{M}{4 \hat{d}_4 M_{Pl}} \left( 1 \pm \sqrt{1 + \frac{16\hat{d}_4 M_{Pl}^2 Q_0^2}{\epsilon_4 M^2}} \right) \right]^{\frac{1}{2}} \]

(98)

The Vainshtein radius corresponds to the scale at which non-linear terms in the scalar equation become important. Also in this case, a \( Q_0 \neq 0 \) leads to two solutions for the Vainshtein radius. As we will see, the requirement of stability of the configuration imposes that the parameters \( \hat{d}_4, \epsilon_4 \) are positive: only the positive branch \( r^+_v \) is then allowed, and we will focus on it from now on. The solution for the scalar and vector spherically symmetric configuration is

\[ \pi(r) = r_v \]

(99)

\[ A_0(r) = -\frac{Q_0 r}{\epsilon_4 r_v} \]

(100)

The scalar contribution is much weaker than the usual gravitational one: the Vainshtein mechanism is at work and GR is recovered nearby a source. Far from the source, instead, fifth forces become important.

Let us discuss also in this case the stability of these spherically symmetric configurations. As before, we focus on spherically symmetric scalar and vector perturbations, which respect the spherically symmetric Ansatz in terms of the criteria used in [17]:

\[ \pi = \bar{\pi}(r) + \varphi(t, r), \]

(101)

\[ A_\mu = (\bar{A}_0(r) + \delta A_0(t, r), \delta A_r(t, r), 0, 0), \]

(102)

\[ F_{tr} = \delta A_r(t, r) - \delta A_0(t, r). \]

(103)

The bars denote background quantities.

A de-mixing procedure of scalar from vectors requires a field redefinition

\[ \delta F_{tr} \to -\frac{2 (2 e_3 + e_4 \bar{\pi}')}{(2 e_2 r + 4 e_3 \bar{\pi}' + e_4 \bar{\pi}'^2)} + \bar{F}_{tr}, \]

(104)

hence we obtain the following Lagrangian for spherically symmetric quadratic fluctuations around our spherically symmetric background

\[ S_\varphi = \frac{1}{2} \int d^4x \sqrt{-g} \left[ K_{\varphi} \dot{\varphi}^2 - K_r(r) \varphi'^2 - K_A(r) \bar{F}_{tr} \right]. \]

(105)
The functions $K_i$ have to be positive in order to obtain a stable configuration. They read

$$K_t = d_2 + 4d_3 \frac{\bar{\pi}'}{r} + 4d_4 \frac{\bar{\pi}''}{r^2} + 2d_3 \frac{\bar{\pi}'''}{r} + 8d_4 \frac{\bar{\pi}'''}{r},$$  \hspace{1cm} (106)

$$K_r = d_2 + 4d_3 \frac{\bar{\pi}'}{r} + 6d_4 \frac{\bar{\pi}'}{r^2} + \frac{2Q_0}{r^6} \left(16e_4^2 - 2e_2e_4 + 12e_3e_4 \frac{\bar{\pi}'}{r} + 3e_4^2 \frac{\bar{\pi}'}{r^2}\right),$$  \hspace{1cm} (107)

$$K_A = e_2 + 2e_3 \frac{\bar{\pi}'}{r} + \frac{e_4}{2} \frac{\bar{\pi}'}{r^2}.$$  \hspace{1cm} (108)

Nearby the source, the quantity $\bar{\pi}'/r$ is large: in order to have positive $K_i$, we demand that $d_4, e_4$ are positive. Far from the source, instead, $\bar{\pi}'/r$ is small: we have to demand that $d_2, e_2$ are positive. This implies that a vector charge increases the size of the function $K_r$ above nearby the source, while it gives only negligible contributions far from it.

Computing the speed of radial scalar fluctuations in proximity of the source we find

$$c_{rad}^2(r \to 0) = \frac{K_r}{K_t}(r \to 0) = \frac{2}{3} \left(1 + \frac{Q_0^2}{d_4 e_4 r^6}\right)$$  \hspace{1cm} (109)

Hence the superluminal speed of radial fluctuations is increased by the presence of the vector charge as in the cubic case.

References

[1] T. Clifton, P. G. Ferreira, A. Padilla and C. Skordis, Phys. Rept. 513 (2012) 1 [arXiv:1106.2476 [astro-ph.CO]].
[2] N. Arkani-Hamed, H. Georgi and M. D. Schwartz, Annals Phys. 305 (2003) 96 [hep-th/0210184].
[3] C. Deffayet, S. Deser and G. Esposito-Farese, Phys. Rev. D 82 (2010) 061501 [arXiv:1007.5278 [gr-qc]].
[4] G. Tasinato, K. Koyama and G. Niz, Phys. Rev. D 87 (2013) 064029 [arXiv:1210.3627 [hep-th]].
[5] G. W. Horndeski, J. Math. Phys. 17 (1976) 1980.
[6] G. R. Dvali, G. Gabadadze and M. Porrati, Phys. Lett. B 485 (2000) 208 [hep-th/0005016]; C. Deffayet, G. R. Dvali and G. Gabadadze, Phys. Rev. D 65 (2002) 044023 [astro-ph/0105068].
[7] A. Nicolis, R. Rattazzi, E. Trincherini and , Phys. Rev. D 79 (2009) 064036 [arXiv:0811.2197 [hep-th]].
[8] C. de Rham and G. Gabadadze, Phys. Rev. D 82 (2010) 044020 [arXiv:1007.0443 [hep-th]].
[9] R. Durrer and A. Neronov, [arXiv:1303.7121 [astro-ph.CO]].
[10] T. Koivisto and D. F. Mota, JCAP 0808 (2008) 021 [arXiv:0805.4229 [astro-ph]]; A. Golovnev, V. Mukhanov and V. Vanchurin, JCAP 0806 (2008) 009 [arXiv:0802.2068 [astro-ph]]; J. Beltran Jimenez and A. L. Maroto, Phys. Rev. D 78 (2008) 063005 [arXiv:0801.1486 [astro-ph]].
[11] B. Himmetoglu, C. R. Contaldi and M. Peloso, Phys. Rev. Lett. 102 (2009) 111301 [arXiv:0809.2779 [astro-ph]]; S. M. Carroll, T. R. Dulaney, M. I. Gresham and H. Tam, Phys. Rev. D 79 (2009) 065011 [arXiv:0812.1049 [hep-th]]; T. S. Koivisto, D. F. Mota and C. Pitrou, JHEP 0909 (2009) 092 [arXiv:0903.4158 [astro-ph.CO]]; B. Himmetoglu, C. R. Contaldi and M. Peloso, Phys. Rev. D 80 (2009) 123530 [arXiv:0909.3524 [astro-ph.CO]].
[12] G. Esposito-Farese, C. Pitrou and J. -P. Uzan, Phys. Rev. D 81 (2010) 063519 [arXiv:0912.0481 [gr-qc]].
[13] J. D. Barrow, M. Thorsrud and K. Yamamoto, JHEP 1302 (2013) 146 [arXiv:1211.5403 [gr-qc]].
[14] M. Mirbabayi, Phys. Rev. D 86 (2012) 084006 [arXiv:1112.1435 [hep-th]].
[15] S. -Y. Zhou and E. J. Copeland, Phys. Rev. D 85 (2012) 065002 [arXiv:1112.0968 [hep-th]]; G. Goon, K. Hinterbichler, A. Joyce and M. Trodden, Phys. Lett. B 714 (2012) 115 [arXiv:1201.0015 [hep-th]].
[16] C. de Rham, G. Gabadadze, L. Heisenberg and D. Pirtskhalava, Phys. Rev. D 83 (2011) 103516 [arXiv:1010.1780 [hep-th]].
[17] K. Koyama, G. Niz and G. Tasinato, JHEP 1112 (2011) 065 \texttt{arXiv:1110.2618} [hep-th].

[18] G. Tasinato, K. Koyama and G. Niz, \texttt{arXiv:1304.0601} [hep-th].

[19] J. B. Jimenez, A. L. D. Froes and D. F. Mota, \texttt{arXiv:1212.1923} [astro-ph.CO].

[20] K. Koyama, G. Niz and G. Tasinato, \texttt{arXiv:1305.0279} [hep-th]; L. Bereziani, G. Chkareuli, C. de Rham, G. Gabadadze and A. J. Tolley, \texttt{arXiv:1305.0271} [hep-th].

[21] C. Burrage, N. Kaloper and A. Padilla, \texttt{arXiv:1211.6001} [hep-th].