The roles of yield function and plastic potential under non-associated flow rule for formability prediction with perturbation approach

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Abstract. In this study, the perturbation approach for predicting material’s forming limit strains under non-associated flow rule (non-AFR) is proposed. The influence of yield function and plastic potential function on the forming limit curve (FLC) evaluated by the perturbation approach are discussed through analyzing the normalized growth rate of a perturbation. In the framework of non-AFR, Hill’48 and Yld2000-2d are chosen for AA5754-O. The results show that the left side of FLCs predicted with the different forms of yield function and plastic potential nearly overlap. Hence, it is concluded that the yield function and plastic potential have a negligible influence on the forming limit strain under a negative strain path. However, the FLC under a positive strain path is principally dependent on the relationship between strain ratio $\beta$ and stress ratio $\alpha$, which can be determined by the plastic potential. Additionally, in comparison to the FLC under AFR, an increase in the forming limits strain is observed near the plain strain region, when the derivative of the normalized yield function concerning $\alpha$ is positive and vice versa.

1. Introduction

To improve the reliability of the process and the product quality, the instabilities caused by necking and wrinkling should be avoided during sheet metal forming. First of all, the compressive instability like wrinkling is influenced by many factors including mechanical properties of the sheet metal, geometry of the workpiece, and contact conditions. Various analytic methods on wrinkling have been developed based on the bifurcation theory including Kim et al. [1]. Some researchers used the neural network approach and numerical analysis investigated the wrinkling behaviour of sheet metal [2,3]. Recently, there was a good progress for the compressive instability theory for hydroforming process by Chu et al. [4-6]. Second, the tensile instability in sheet metal has been a long research topic. In order to evaluate sheet metal formability, forming-limit diagram (FLD) is generally used, which is usually obtained through Marciniak [7] and Nakazima tests [8]. Leotoing et al.[9] proposed an alternative method by using an in-plane biaxial tensile test for the prediction of FLD. However, these experimental methods are costly and time-consuming. To overcome these shortcomings, several phenomenological models including diffused and localized necking theories [10,11], bifurcation analyses [12,13], imperfection approach (M-K model) [7], perturbation theories [14-16] and modified maximum force criterion (MMFC) [17] have been established for the theoretical forming limit curve (FLC).
The influences of non-linear loading paths on materials’ FLCs have been studied by several researchers [15, 18-20] with different theoretical models. To improve the prediction accuracy of these models, significant improvements have been made under associated flow rule (AFR). For example, incorporating more accurate anisotropic yield criteria [21] and hardening models [19]. Stoughton [22] found that under non-associated flow rule (non-AFR), the quadratic yield criteria have a similar anisotropic characteristic predictive capability as the advanced non-quadratic associated anisotropic models. Shen [23] compared the FLCs predicted by the M-K model and MMFC with Hill’48 under AFR and non-AFR. They found that these theoretical models are more accurate under non-AFR than those under AFR. Recently, Hu et al. [15] developed an effective perturbation approach under AFR to predict the material’s formability. Hence, it is worth investigating FLC predicted by the perturbation approach under non-AFR.

2. Basic theoretical fundamentals of perturbation approach under non-AFR

The perturbation theory under non-AFR is based on the work of Hu et al. [15,16]. The perturbation approach proposed that when the normalized growth rate of a perturbation $\bar{\omega}$ is negative, the plastic deformation is stable, the perturbation growth accumulation starts at $\bar{\omega} = 0$. The failure criterion is proposed as Equation (1). When the perturbation growth accumulation reaches a certain value, the necking is assumed to be well developed and become observable and the strain level in the homogeneous region will be the limit strain as presented in Figure 1.

$$\int_{\bar{\omega}}^{n} \bar{\omega}(\beta, \bar{\epsilon}) d\bar{\epsilon} = W$$  \hspace{1cm} (1)

where $\bar{\sigma}_n$ is the equivalent plastic strain (EPS) at necking, $\bar{\epsilon}$ is the EPS at $\bar{\omega} = 0$, $W$ is perturbation growth accumulation, $\bar{\omega}$ is the function of strain path $\beta = \Delta \epsilon_{xx} / \Delta \epsilon_{xx}$ and $\bar{\epsilon}$.

According to the force balance, if a perturbation is applied during deformation, the governing equation for the onset of deformation instability becomes [15,16]:

$$\begin{align*}
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} &= 0 \\
\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{xy}}{\partial x} &= 0 \\
\frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yy}}{\partial x} &= 0
\end{align*}$$  \hspace{1cm} (2)

where $\bar{\sigma}_{ij}$ ($i,j=x,y$) are the corresponding perturbations of $\sigma_{ij}$, $\bar{\epsilon}_{ij}$ is the perturbation of strain along the thickness direction.

According to the [15,16], the perturbations of strain rate components are:

$$\begin{align*}
\bar{\epsilon}_{xx} &= \delta \nu'_{x} n_{1} (iq) \exp \left( iq (\mathbf{n} \cdot \mathbf{x}) + \omega T \right) \\
\bar{\epsilon}_{yy} &= \delta \nu'_{y} n_{1} (iq) \exp \left( iq (\mathbf{n} \cdot \mathbf{x}) + \omega T \right) \\
\bar{\epsilon}_{xy} &= - \left( \delta \nu'_{x} n_{1} + \delta \nu'_{y} n_{2} \right) (iq) \exp \left( iq (\mathbf{n} \cdot \mathbf{x}) + \omega T \right) \\
\bar{\epsilon}_{yy} &= \frac{1}{2} \left( \delta \nu'_{x} n_{2} + \delta \nu'_{y} n_{1} \right) (iq) \exp \left( iq (\mathbf{n} \cdot \mathbf{x}) + \omega T \right) = 0
\end{align*}$$  \hspace{1cm} (3)

The perturbations of the strains can be obtained by integrating the perturbations of strain rate.
\[ \ddot{e}_y = \int_{\Omega} \dot{e}_y d\Omega = \frac{1}{\omega} \ddot{e}_y \]  

(4)

According to the flow theory of plasticity, the plastic strain rate \( \dot{e}_y \) can be expressed as:

\[ \dot{e}_y = d\lambda \frac{\partial f_p}{\partial \sigma_y} \]  

(5)

where \( f_p \) is a plastic potential function.

Based upon the principle of plastic work equivalence, Equation (6) can be deduced:

\[ f_p \dot{\varepsilon} = \sigma_{xx} \dot{\varepsilon}_{xx} + \sigma_{yy} \dot{\varepsilon}_{yy} = \sigma_{xx} (1 + \alpha \beta) \dot{\varepsilon}_{xx} \]  

(6)

where \( f_p \) is yield function and \( \alpha = \sigma_{xx}/\sigma_{yy} \).

The power type of constitutive law is adopted for strain rate dependent materials as in Equation (7).

\[ \sigma = \sigma_m(\varepsilon) \dot{\varepsilon}^m \]  

(7)

where \( m \) is the strain rate sensitivity parameter and \( \sigma_m(\varepsilon) \) is a function of \( \varepsilon \).

Substituting Equations (3)-(7) into Equation (2), the normalized growth rate of a perturbation \( \vec{\sigma} \) can be deduced after mathematical manipulation.

\[ \vec{\sigma} = \frac{\omega}{\varepsilon} = \frac{1}{m} \left[ \varphi_r(\alpha) \frac{(\beta + 1)}{(\alpha \beta + 1)} \frac{\partial \ln \sigma_m(\varepsilon)}{\partial \varepsilon} \right], \]  

(8)

\[ \vec{\sigma} = \frac{\varphi_r(\alpha)}{(\dot{\varepsilon} \partial/\partial \varepsilon)^\varepsilon} \left[ \frac{1 - \beta (\dot{\varepsilon} \partial/\partial \varepsilon)}{\varphi_p(\alpha) \frac{\partial \ln \sigma_m(\varepsilon)}{\partial \varepsilon} + m (\dot{\varepsilon} \partial/\partial \varepsilon) \left( \beta - \frac{\varphi_p(\alpha) \frac{\partial \ln \sigma_m(\varepsilon)}{\partial \varepsilon}}{\varphi_p(\alpha) \frac{\partial \ln \sigma_m(\varepsilon)}{\partial \varepsilon}} \right) \right], \]  

(9)

Where \( \varphi_r(\alpha) = f_r/\sigma_{xx} \), \( \varphi_p(\alpha) = f_p/\sigma_{xx} \) and \( \theta \) is the necking angle in Figure 1.

3. Effects of \( f_p \) and \( f_r \) under non-AFR on perturbation approach

In Section 2, the expressions of \( \vec{\sigma} \) under non-AFR for positive and negative strain paths have been derived. To study the influence of yield function \( f_p \) and plastic potential \( f_r \) on the forming limit strains predicted by perturbation approach, the material AA5754-O [24] is adopted. The corresponding material property can be found in Table 1. By fitting the experimental forming limit strains around the plain strain with M-K model under AFR, the strain rate parameter for AA5754-O is set to 0.003. In the present study, the modified Voce hardening law is used as presented in Equation (10). As presented in Figure 2, by fitting the forming limit strains around plane strain with perturbation approach under AFR, the perturbation growth accumulation \( W \) is calculated as 2.2677.

\[ \vec{\sigma}_i(\varepsilon) = 247.53 - 149.6 \left[ 1 - \exp(-15.26\varepsilon) \right] + 150\varepsilon \]  

(10)

Table 1. Material parameters of AA5754-O [24].

| \( \sigma_0/\sigma_0 \) | \( \sigma_{45}/\sigma_0 \) | \( \sigma_{90}/\sigma_0 \) | \( \sigma_0/\sigma_0 \) | \( r_0 \) | \( r_{45} \) | \( r_{90} \) | \( r_0 \) |
|-----------------|-----------------|-----------------|-----------------|---------|---------|---------|---------|
| 1               | 0.955           | 0.964           | 1.012           | 0.692   | 0.859   | 0.934   | 1       |

Here, Hill’48 and Yld2000-2d are used under non-AFR. Hill’48 parameters determined by \( r \)-values and yield stresses will be denoted as Hill’48 \(_r\) and Hill’48 \(_s\), respectively. It is found that the term \( \beta (\dot{\varepsilon} \partial/\partial \varepsilon) \left( \beta - \frac{f_p \dot{\varepsilon} \partial/\partial \varepsilon \sigma_m}{f_r \dot{\varepsilon} \partial/\partial \varepsilon \sigma_m} \right) \) in Equation (9) is very small, which can be ignored and modified as in Equation (11). The difference between Equation (9) and Equation (11) is very small, which also can be found in Figure 3. Henceforth, Equation (11) is used for \( \vec{\sigma} \) calculations for \( \beta > 0 \).
\[
\sigma \approx \frac{\varphi_y(\alpha) - \ln \sigma(\varphi)}{m + \beta^2 (\varphi / \alpha) \frac{\varphi \varphi \varphi / \alpha}{\varphi \varphi / \alpha}} \quad \theta = 0 \quad (\beta > 0)
\]

Figure 2. FLC predicted by perturbation approach with Hill’48 and Yld2000-2d under AFR.

Figure 3. FLC predicted by perturbation approach under non-AFR: solid line by Eq. (9), chain-dotted line by Eq. (11).

Figure 4. FLCs predicted by perturbation approach with different \( f_y \): (a) \( f_y \) is Hill’48, (b) \( f_y \) is Yld2000.
Comparison of FLCs predicted by the perturbation approach with different $f_p$ is shown in Figure 4. It is found that if yield criterion is the same, the forming limit strains under negative strain path are almost the same for different plastic potential function. That means $f_p$ does not affect the left side of FLD. For the positive strain path, regardless of the yield function, the forming limit strains near the plain strain region predicted with Hill’48_r is smaller than that of Yld2000. On the contrary, the predictions by the two potential functions near equi-biaxial tension are opposite.

Figure 5 shows the predicted FLCs with different yield function for the same plastic potential function. It’s found that the yield function $f_y$ also has no influence on the forming limit strains under negative strain path. For AA5754-O, if $f_p$ is Hill’48_r, the predicted FLCs under a positive strain path decrease in sequence with yield functions of Hill’48_s, Hill’48_r and Yld2000 as presented Figure 5(a). If $f_p$ is Yld2000, the forming limit strains on the right side of FLD predicted with Hill’48_s yield function will be larger than that of the Yld2000.

![FLCs predicted by perturbation approach with different $f_y$: (a) $f_p$ is Hill’48_r, (b) $f_y$ is Yld2000.](image)

### Figure 5

To investigate the influence of $f_y$ and $f_p$ on perturbation approach, the effects on the normalized growth rate of a perturbation $\tilde{\omega}$ are analyzed. For negative strain path, Equation (8) will be used. Under uniaxial tension along the rolling direction, the stress ratio $\alpha=0$, Equation (8) will reduce to Equation (12), which has no relation with $f_y$. If $f_p$ accurately predicts the Lankford coefficient along the rolling direction, the strain ratio $\beta|_{\alpha=0}$ will be the same for different $f_p$. For the plain strain state, $\beta=0$, Equation (8) will reduce to Equation (13), which doesn’t depend on $f_p$. $\varphi(\alpha)$ is around 1 and it shows little difference for different yield criterion. Hence, for the given hardening law, $\tilde{\omega}$ under uniaxial tension and plain strain states will be almost the same for different $f_y$ and $f_p$. Therefore, $f_y$ and $f_p$ have no influences on the forming limit strains at uniaxial tension and plain strain states according to Equation (1). In addition, FLC on the left side of FLD is almost a straight line, the forming limit strain under negative strain path will be the same for different $f_y$ and $f_p$ as presented in Figure 4 and Figure 5.

\[
\tilde{\omega} = \frac{1}{m} \left[ \left( \beta|_{\alpha=0} + 1 \right) \frac{\partial \ln \sigma(\varepsilon)}{\partial \varepsilon} - \frac{\varphi_y(\alpha)}{\partial \varepsilon} \right] \tag{12}
\]

\[
\tilde{\omega} = \frac{1}{m} \left[ \varphi_y(\alpha) - \frac{\partial \ln \sigma(\varepsilon)}{\partial \varepsilon} \right] \tag{13}
\]

For a positive strain path, Equation (11) is used. As $\varphi_y(\alpha)$ is around 1, the numerator of Equation (11) is nearly similar for the given hardening law. The term $\beta^2 (\partial \beta/\partial \alpha)^{-1} \frac{\varphi_y}{\varphi_p} \frac{\partial \varphi_p}{\partial \alpha}$ in Equation (11) will play an important role. In the framework of AFR, the term will reduce to $\beta^2 (\partial \beta/\partial \alpha)^{-1}$. A larger value of $\partial \beta/\partial \alpha$ under positive strain path will result in larger denominator value in Equation (11).
That’s the reason why FLC on the right side of FLD predicted by perturbation approach with Hill’48_AFR is higher than that of the Yld2000_AFR as presented in Figure 2. In the framework of non-AFR, as presented in Figure 5, FLC under positive strain path still principally relies on the term \( \beta^i(\partial \beta / \partial \alpha)^{-1} \), i.e. plastic potential function. The term \( \Omega = \frac{\partial^2 \phi^i / \partial \alpha}{\partial^2 \phi^i / \partial \alpha} \) changes the shape of FLC compared to that of AFR. As presented in Figure 6, if \( f_p \) is Hill’48, the numerator of \( \Omega \) under non-AFR will be negative after the point A, i.e. around the plain strain region. A smaller denominator value of Equation (11) will result in larger value of \( \partial \phi \). Then the forming limit around the plain strain region will become smaller compared that under that of AFR as presented in Figure 5(a). The lowest forming limit in Figure 5(a) depends on the range of \( \partial^2 \phi^i / \partial \alpha = \partial^2 \phi^i / \partial \alpha \). If \( f_p \) is Yld2000, \( \Omega \) under non-AFR will be larger than 10 after the point B. A larger denominator of Equation (11) will result in smaller value of \( \partial \phi \), which finally leads to the higher forming limit strains around the plain strain region as present in Figure 5(b). From Figure 5, the results show that although normalized yield stress and r-values can be accurately predicted by non-AFR, the predicted FLC would be different with \( f_y \) and \( f_p \). The normal vectors of \( f_y \) and \( f_p \) under the plane stress state play an important role in the predicted FLC.

![Figure 6](image_url)  
*Figure 6. The relationship between \( \partial^i \phi / \partial \alpha \) and \( \alpha \).*

4. Conclusions
The perturbation approach under non-AFR has been deduced. The influences of yield function \( f_y \) and plastic potential \( f_p \) on FLC predicted by perturbation approach under non-AFR have been systematically studied. The results show that \( f_y \) and \( f_p \) almost have no influences on the left side of FLD. Although different forms of non-AFR can describe normalized yield stresses and r-values well, FLC under positive strain path still principally relies on the term \( \beta^i(\partial \beta / \partial \alpha)^{-1} \). A negative \( \partial \phi, / \partial \alpha \) around the plain strain state will decrease the forming limit strains and a positive \( \partial \phi, / \partial \alpha \) around plain strain will improve FLC.

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