REPRESENTATION THEOREMS FOR OPERATORS ON FREE BANACH SPACES OF COUNTABLE TYPE

J. AGUAYO\(^1\), M. NOVA\(^2\) and J. OJEDA\(^1\)*

Abstract. This work will be centered in commutative Banach subalgebras of the algebra of bounded linear operators defined on a Free Banach spaces of countable type. The main goal of this work will be to formulate a representation theorem for these operators through integrals defined by spectral measures type. In order to get this objective, we will show that, under special conditions, each one of these algebras is isometrically isomorphic to some space of continuous functions defined over a compact set. Then, we will identify such compact developing the Gelfand space theory in the non-archimedean setting. This fact will allow us to define a measure which is known as spectral measure. As a second goal, we will formulate a matrix representation theorem for this class of operators whose entries of these matrices will be integrals coming from scalar measures.

1. Introduction and notation

Many researchers have tried to generalize the elemental studies of Banach algebras from classical case to vectorial structures over non-archimedean fields. The first big task was to find a result similar to the Gelfand-Mazur Theorem in this context. But, this theorem failed since every field \( \mathbb{K} \) with a "non-archimedean valuation" is contained in another field \( \tilde{\mathbb{K}} \) whose valuation is an extension of previous one and both fields are different.

One of the pioneers in the study of non-archimedean Banach algebras of linear operators and spectral theory in this context has been M. Vishik [8], especially in the class of linear operators which admit compact spectrum. We can also mention another important pioneer, V. Berkovick [3], who made a deep study of this subject on his survey.

This work will be centered in subalgebras commutative Banach of the algebra of bounded linear operators defined on a Free Banach spaces of countable type. The main goal of this work will be to formulate a representation theorem for these operators through integrals defined by spectral measures type. In order to get this objective, we will show that, under special conditions, each one of these algebras is isometrically isomorphic to some space of continuous functions defined over a compact. Then, we will identify such compact developing the Gelfand space...
theory in the non-archimedean setting. This fact will allow us to define a measure which is known as spectral measure. As a second goal, we will be to formulate a matrix representation theorem for this class of operators whose entries of these matrices will be integrals coming from scalar measures.

Throughout this paper, \( K \) denotes a complete, non-archimedean valued field and its residue class field is formally real.

In the classical situation we can distinguish two type of normed spaces: those spaces which are separable and those which are not. If \( E \) is a separable normed space over \( K \), then each one-dimensional subspace of \( E \) is homeomorphic to \( K \), so \( K \) must be separable too. Nevertheless, we know that there exist non-archimedean fields which are not separable. Thus, for non-archimedean normed spaces the concept of separability is meaningless if \( K \) is not separable. However, linearizing the notion of separability, we obtain a useful generalization of this concept. A normed space \( E \) over a non-archimedean valued field is said to be of countable type if it contains a countable subset whose linear hull is dense in \( E \). An example of a normed space of countable type is \( (c_0, \|\cdot\|_\infty) \), where \( c_0 \) is the Banach space of all sequences \( x = (a_n)_{n \in \mathbb{N}} \), \( a_n \in K \), for which \( \lim_{n \to \infty} a_n = 0 \) and its norm is given by \( \|x\|_\infty = \sup \{|a_n| : n \in \mathbb{N}\} \).

A non-archimedean Banach space \( E \) is said to be Free Banach space if there exists a family \( \{e_i\}_{i \in J} \) of non-null vectors of \( E \) such that any element \( x \in E \) can be written in the form of convergent sum \( x = \sum_{i \in J} x_i e_i \), \( x_i \in K \), and \( \|x\| = \sup_{i \in J} |x_i| \|e_i\| \). The family \( \{e_i\}_{i \in J} \) is called orthogonal basis of \( E \). If \( s : J \to (0, \infty) \), then an example of Free Banach space is \( c_0 (J, K, s) \), the collection of all \( x = (x_i)_{i \in J} \) such that for any \( \epsilon > 0 \), the set \( \{i \in J : |x_i| s(i) > \epsilon\} \) is, at most, finite and \( \|x\| = \sup_{i \in J} |x_i| s(i) \).

We already know that a Free Banach space \( E \) is isometrically isomorphic to \( c_0 (J, K, s) \), for some \( s : J \to (0, \infty) \). In particular if a Free Banach space is of countable type, then it is isometrically isomorphic to \( c_0 (\mathbb{N}, K, s) \), for some \( s : \mathbb{N} \to (0, \infty) \). Note that if \( s(i) \in K \), for each \( i \in \mathbb{N} \), then \( E \) is isometrically isomorphic to \( c_0 (\mathbb{N}, K) \) (or \( c_0 \) in short). For more details concerning Free Banach spaces, we refer the reader to [4].

Now, since residual class field of \( K \) is formally real, the bilinear form

\[
\langle \cdot, \cdot \rangle : c_0 \times c_0 \to K; \quad \langle x, y \rangle = \sum_{i=1}^{\infty} x_i y_i
\]

is an inner product, \( \|\cdot\| = \sqrt{\langle \cdot, \cdot \rangle} \) is a non-archimedean norm in \( c_0 \) and the supremum norm \( \|\cdot\|_\infty \) coincides with \( \|\cdot\| \), that is, \( \|\cdot\| = \|\cdot\|_\infty \) (see [6]). Therefore, to study Free Banach spaces of countable type it is enough to study the space \( c_0 \).

If \( E \) and \( F \) are \( K \)-normed spaces, then \( \mathcal{L} (E, F) \) will be the space consisting of all continuous linear maps from \( E \) into \( F \). If \( F = E \), then \( \mathcal{L} (E, E) = \mathcal{L} (E) \). For any \( T \in \mathcal{L} (E, F) \), \( N(T) \) will denote its Kernel and \( R(T) \) its range.

A linear operator \( T \) from \( E \) into \( F \) is said to be compact operator if \( T (B_E) \) is compactoid, where \( B_E = \{x \in E : \|x\| \leq 1\} \) is the unit ball of \( E \). It was proved in [7] that \( T \) is compact if and only if, for each \( \epsilon > 0 \), there exists a lineal operator of finite-dimensional range \( S \) in \( \mathcal{L} (E, F) \) such that \( \|T - S\| \leq \epsilon \).
Since $c_0$ is not orthomodular, there exist operators in $\mathcal{L}(c_0)$ which do not admit adjoint; for example, the linear operator $T : c_0 \to c_0$ defined by $T(x) = (\sum_{i=1}^{\infty} x_i) e_1$, $x = (x_i)_{i \in \mathbb{N}} \in c_0$. We will denote by $\mathcal{A}_0$ the collection of all elements of $\mathcal{L}(c_0)$ which admit adjoint. A characterization of the elements of $\mathcal{A}_0$ (see [1]) is the following:

$$\mathcal{A}_0 = \left\{ T \in \mathcal{L}(c_0) : \forall y \in c_0, \lim_{i \to \infty} \langle Te_i, y \rangle = 0 \right\}.$$  

Of course, $\mathcal{A}_0$ is a Banach algebra with unit.

We will understand by a normal projection to any projection $P : c_0 \to c_0$ such that $\langle x, y \rangle = 0$ for each pair $(x, y) \in N(P) \times R(P)$. An example of normal projection is $P(\cdot) = \langle \cdot, y \rangle \frac{y}{\langle y, y \rangle}$ for a fix $y \in c_0 \setminus \{\theta\}$.

Now, for each $a = (a_i)_{i \in \mathbb{N}} \in c_0$, the linear operator $M_a$, defined by $M_a(\cdot) = \sum_{i=1}^{\infty} a_i \langle \cdot, e_i \rangle e_i$, belongs to $\mathcal{A}_0$; moreover, 

$$\lim_{n \to \infty} \|M_a e_n\| = \lim_{n \to \infty} \|\sum_{i=1}^{\infty} a_i \langle e_n, e_i \rangle e_i\| = \lim_{n \to \infty} |a_n| = 0,$$

meanwhile, the identity map $Id$ is also an element of $\mathcal{A}_0$, but 

$$\lim_{n \to \infty} \|Id(e_n)\| = \lim_{n \to \infty} \|e_n\| = 1.$$

Let us denote by $\mathcal{A}_1$ the collection of all $T \in \mathcal{L}(c_0)$ such that $\lim_{n \to \infty} Te_n = \theta$, i.e.,

$$\mathcal{A}_1 = \left\{ T \in \mathcal{L}(c_0) : \lim_{n \to \infty} T e_n = \theta \right\}.$$  

From the fact that 

$$|\langle Te_n, y \rangle| \leq \|Te_n\| \|y\|,$$

we have that $\mathcal{A}_1 \subseteq \mathcal{A}_0$ since $Id \notin \mathcal{A}_1$.

If $T, S \in \mathcal{A}_1$, then 

$$\langle S, T \rangle = \sum_{n=1}^{\infty} \langle S(e_n), T(e_n) \rangle$$

is well-defined, is an inner product in $\mathcal{A}_1$ and 

$$\|T\| = \sqrt{|\langle T, T \rangle|}.$$  

By [4], we know that each $T \in \mathcal{L}(c_0)$ can be represented by $T = \sum_{i,j=1}^{\infty} \alpha_{i,j} e_i' \otimes e_j$, where $\lim_{i \to \infty} \alpha_{i,j} = 0$, for all $j \in \mathbb{N}$. Also, 

$$\|T\| = \sup \{\|T(e_i)\| : i \in \mathbb{N}\} = \sup \{|\langle T(e_i), e_j \rangle| : i, j \in \mathbb{N}\}$$

and $T$ is compact if and only if 

$$\lim_{j \to \infty} \sup \{|\alpha_{i,j}| : i \in \mathbb{N}\} = 0.$$
Now, note that
\[
\|Te_n\|_\infty = \left\| \left( \sum_{i,j=1}^{\infty} \alpha_{i,j} e'_j \otimes e_i \right) (e_n) \right\|_\infty = \left\| \sum_{i,j=1}^{\infty} \alpha_{i,j} e'_j (e_n) e_i \right\|_\infty
\]
\[
= \left\| \sum_{i=1}^{\infty} \alpha_{i,n} e_i \right\|_\infty = \sup \{|\alpha_{i,n}| : i \in \mathbb{N} \},
\]
thus,
\[
T \in A_1 \iff (T \in A_0 \text{ and } T \text{ is compact})
\]

For the rest of the paper, let us take a fix orthonormal sequence \(\{y^{(i)}\}_{i \in \mathbb{N}}\) in \(c_0\), that is, \(\langle y^{(i)}, y^{(j)} \rangle = 0, i \neq j\), and \(\|y^{(i)}\|_\infty = 1\).

The next theorem involves normal projections with compact and self-adjoint operators. The proof can be found in [2].

**Theorem 1.1.** If the linear operator \(T : c_0 \to c_0\) is compact and self-adjoint, then there exists an element \(\lambda = (\lambda_i)_{i \in \mathbb{N}} \in c_0\) and an orthonormal sequence \(\{y^{(i)}\}_{i \in \mathbb{N}}\) in \(c_0\) such that
\[
T = \sum_{i=1}^{\infty} \lambda_i P_i,
\]
where
\[
P_i (\cdot) = \frac{\langle \cdot, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)}
\]
is the normal projection defined by \(y^{(i)}\). Moreover, \(\|T\| = \|\lambda\|_\infty\).

**Remark 1.2.**

(1) This theorem gives us a characterization for compact and self-adjoint operators. In fact, it is not hard to see that if we take \(\lambda = (\lambda_i)_{i \in \mathbb{N}} \in c_0\) and an orthonormal sequence \(\{y^{(i)}\}_{i \in \mathbb{N}}\) in \(c_0\), the operator
\[
T_\lambda = \sum_{i=1}^{\infty} \lambda_i P_i,
\]
is compact, self-adjoint and \(\|T_\lambda\| = \|\lambda\|_\infty\), where \(P_i\) is as in the Theorem 1.1.

(2) The projection family \(\{P_i : i \in \mathbb{N}\}\) is orthonormal which implies that is orthogonal in the van Rooij’s sense. In fact, for \(i \neq j\)
\[
\langle P_i, P_j \rangle = \sum_{n=1}^{\infty} \langle P_i (e_n), P_j (e_n) \rangle = \langle y^i, y^j \rangle \sum_{n=1}^{\infty} \frac{y^i_n}{\langle y^i, y^j \rangle} \frac{y^j_n}{\langle y^j, y^j \rangle} = 0
\]
and, for \(i \in \mathbb{N}, \|P_i\| = 1\).

(3) It is not difficult to prove that \(\sum_{s=1}^{N} P_s\) and \(Id - \sum_{r=1}^{M} P_r\) are normal projection, when \(P_s\) and \(P_r\) are.
2. Algebra of operators

2.1. A commutative algebra. From now on, we will consider a fixed orthonormal family $Y = \{ y^{(i)} \}_{i \in \mathbb{N}}$ in $c_0$. We will denote by $\mathfrak{S}_Y(c_0)$ the collection of all compact operators $T_\mu$, $\mu \in c_0$, where

$$T_\mu = \sum_{i=1}^{\infty} \mu_i P_i$$

As we know, the adjoint $T_\mu^*$ of $T_\mu$ is itself and $\lim_{n \to \infty} T_\mu(e_n) = 0$. On the other hand, since $Y$ is orthonormal, $T_\mu(y^{(i)}) = \mu_i y^{(i)}$; in other words $\mu_i$, $i \in \mathbb{N}$, is an eigenvalue of $T_\mu$. Let us denote by $\sigma(T_\mu)$ the set of eigenvalues of $T_\mu$.

Now, the collection $\mathfrak{S}_Y(c_0)$ is a linear space with the operations

$$T_\lambda + T_\mu = T_{\lambda+\mu}; \quad \alpha T_\lambda = T_{\alpha \lambda}$$

On the other hand, since $c_0$ is a commutative algebra with the operation $\lambda \cdot \mu = (\lambda, \mu)$, we have

$$T_\lambda \circ T_\mu = T_{\lambda \cdot \mu} = T_\mu \circ T_\lambda.$$ 

In order to simply the notation, $T_\lambda \circ T_\mu$ will be denoted by $T_{\lambda T_\mu}$.

With the operations described above, $\mathfrak{S}_Y(c_0)$ becomes a commutative algebra without unit. Even more, by the fact that $T_\lambda = T_\mu$ implies $\lambda = \mu$, the map

$$\Lambda : c_0 \to \mathfrak{S}_Y(c_0); \quad \lambda \mapsto \Lambda(\lambda) = T_\lambda$$

is an isometric isomorphism of algebras.

As we know, each algebra $E$ without unit can be transformed in an algebra with unit by considering the collection $E^+ = \mathbb{K} \oplus E$ provided with the usual operations and the multiplication operation defined by

$$(\alpha, \mu) \circ (\beta, \nu) = (\alpha \beta, \alpha \nu + \beta \mu + \mu \cdot \nu).$$

The unit of this algebra is $(1, \theta)$, where $\theta$ is the null vector of $E$. If $E$ is, in particular, a normed space, then so is $E^+$ and

$$\| (\alpha, \mu) \| = \max \{ |\alpha|, \| \mu \|_{\infty} \}.$$ 

It is known that if $E$ is an algebra with power multiplicative norm, that is

$$\| \nu^n \| = \| \nu \|^n; \quad \nu \in E, \quad n \in \mathbb{N},$$

then the norm on $E^+$ is also power multiplicative. As, an example of algebra with power multiplicative norm is $c_0$.

Now, the commutative Banach algebra $(\mathfrak{S}_Y(c_0), +, \cdot, \circ, \| \cdot \|)$ can be transformed, as above, in a commutative Banach algebra $(\mathfrak{S}_Y(c_0)^+, +, \cdot, \circ, \| \cdot \|)$ with unit. By the fact that $c_0$ is isometrically isomorphic to $\mathfrak{S}_Y(c_0)$, $\mathfrak{S}_Y(c_0)^+$ is isometrically isomorphic to $c_0^+$.

We will denote by $\mathcal{S}_Y(c_0)$ the collection of all linear operators $\alpha I_\mu + T_\lambda$, where $\alpha \in \mathbb{K}$, $T_\lambda \in \mathfrak{S}_Y(c_0)$ and $I_\mu$ is the identity operator on $c_0$. $\mathcal{S}_Y(c_0)$ is a normed space and if we add the operation

$$(\alpha_1 I_\mu + T_\mu)(\alpha_2 I_\mu + T_\nu) = \alpha_1 \alpha_2 I_\mu + \alpha_1 T_\nu + \alpha_2 T_\mu + T_\mu T_\nu$$

$$= \alpha_1 \alpha_2 I_\mu + T_{\alpha_1 \nu + \alpha_2 \mu + \mu \nu}$$
then $\mathcal{S}_Y(c_0)$ is converted in a commutative algebra with unit.

**Theorem 2.1.** The algebra $\mathcal{S}_Y(c_0)$ is isometrically isomorphic to $\mathcal{S}_Y(c_0)^+$. As a consequence, $\mathcal{S}_Y(c_0)$ is a commutative Banach algebra with unit.

**Proof.** We define

$$\mathcal{S}_Y(c_0)^+ \rightarrow \mathcal{S}_Y(c_0),$$

$$(\alpha, T_\lambda) \mapsto \alpha \text{Id} + T_\lambda$$

Since $\alpha T_\mu + \beta T_\lambda + T_\lambda T_\mu = T_{\alpha \mu + \beta \lambda + \mu \lambda}$, the above transformation is a homomorphism of algebras. Obviously, this homomorphism is onto; hence it is enough to prove that it is an isometry. We claim that

$$\|\alpha \text{Id} + T_\lambda\| = \|(\alpha, T_\lambda)\|$$

If $\alpha = 0$ or $\|T_\lambda\| = 0$ or $\|\alpha \text{Id}\| \neq \|T_\lambda\|$, we are done. We only need to check it when

$$|\alpha| = \|\alpha \text{Id}\| = \|T_\lambda\| \neq 0.$$ 

Of course,

$$\|\alpha \text{Id} + T_\lambda\| \leq \max \{|\alpha|, \|T_\lambda\|\}.$$ 

Now, by the compactness of $T_\lambda$,

$$\lim_{n \to \infty} T_\lambda(e_n) = 0.$$ 

Thus, there exists $N \in \mathbb{N}$ such that

$$n \geq N \Rightarrow \|T_\lambda(e_n)\| < |\alpha|.$$ 

Therefore,

$$\|\alpha \text{Id} + T_\lambda\| = \sup \{\|\alpha e_n + T_\lambda(e_n)\| : n \in \mathbb{N}\}$$

$$= \max \{\|\alpha e_1 + T_\lambda(e_1)\|, \|\alpha e_2 + T_\lambda(e_2)\|, \ldots, \|\alpha e_{N-1} + T_\lambda(e_{N-1})\|, |\alpha|\}$$

$$= |\alpha| = \max \{|\alpha|, \|T_\lambda\|\} = \|(\alpha, T_\lambda)\|. \quad \Box$$

**Remark 2.2.** Since $c_0^+$ is isometrically isomorphic to $\mathcal{S}_Y(c_0)^+$, the above theorem says that $c_0^+$ is isometrically isomorphic to $\mathcal{S}_Y(c_0)$.

We claim that the usual norm in $\mathcal{S}_Y(c_0)$ is power multiplicative, that is,

**Proposition 2.3.** $\mathcal{S}_Y(c_0)$ is an algebra with power multiplicative norm.

**Proof.** It follows from the fact that $c_0^+$ is isometrically isomorphic to $\mathcal{S}_Y(c_0). \quad \Box$

**Definition 2.4.** A commutative Banach algebra $\mathcal{A}$ is called a C-algebra if there exists a locally compact zero-dimensional Hausdorff space $X$ such that $\mathcal{A}$ is isometrically isomorphic to $C_\infty(X)$, where $C_\infty(X)$ is the space of all continuous functions from $X$ into $\mathbb{K}$ which vanishes at infinity.
As we know \( \{ e_j = (\delta_{i,j})_{i \in \mathbb{N}} : j \in \mathbb{N} \} \), where \( \delta_{i,j} \) denotes the Kronecker symbol, is the canonical basis of \( c_0 \). Since
\[ e_j^2 = e_j \]
and
\[ \langle \{ e_j : j \in \mathbb{N} \} \rangle = c_0 \]
we conclude that the collection of all the idempotent elements of \( c_0 \) with norm less than or equal to 1 is dense in \( c_0 \). As a consequence, \( c_0 \) and \( c_0^+ \) are C-algebras (see [7]).

**Theorem 2.5.** \( S_Y (c_0) \) is a C-algebra with unity.

**Proof.** It follows from the fact that \( c_0^+ \) is isometrically isomorphic to \( S_Y (c_0) \). \( \square \)

**Remark 2.6.** We recall that the spectrum of a commutative Banach algebra \( \mathfrak{A} \) is the collection \( Sp (\mathfrak{A}) \) of all nonzero algebra homomorphisms defined from \( \mathfrak{A} \) into \( \mathbb{K} \), that is,
\[ Sp (\mathfrak{A}) = \{ \phi : \mathfrak{A} \to \mathbb{K} : \phi \text{ is a nonzero homomorphism} \} . \]

Note that the natural topology on \( Sp (\mathfrak{A}) \) is induced by the product topology on \( \mathbb{K}^\mathfrak{A} \) and also for each \( \phi \in Sp (\mathfrak{A}) \), \( \| \phi \| \leq 1 \). For any \( x \in \mathfrak{A} \), we define
\[ G_x : Sp (\mathfrak{A}) \to \mathbb{K}, \quad \phi \mapsto G_x (\phi) = \phi (x) \]
which is clearly continuous and bounded. Let us denote by
\[ \| x \|_{sp} = \sup \{ | \phi (x) | : \phi \in Sp (\mathfrak{A}) \} = \| G_x \|_\infty \]
the spectral norm of \( x \). Since
\[ | \phi (x) | \leq \| \phi \| \| x \| \leq \| x \| , \]
we have, in general, that
\[ \| x \|_{sp} \leq \| x \| . \]

Finally, let us denote by \( R (G_x) \) the range of \( G_x \). The closure of \( R (G_x) \) is called spectrum of \( x \).

L. Narici proved the following result (see [7]):

**Proposition 2.7.** A commutative Banach algebra \( \mathfrak{A} \) with unit is a C–algebra if and only if its spectrum \( Sp (\mathfrak{A}) \) is compact and its spectral norm \( \| x \|_{sp} \) is equal to \( \| x \| \), for every \( x \in \mathfrak{A} \).

**Remark 2.8.** As \( S_Y (c_0) \) satisfies the hypothesis of the Proposition 2.7, we conclude that \( Sp (S_Y (c_0)) \) is compact and, for every \( H \in S_Y (c_0) \),
\[ \| H \| = \sup_{i \in \mathbb{N}} \| H (e_i) \| = \| H \|_{sp} . \]

Under the conditions that \( S_Y (c_0) \) is a C-algebra commutative with unity and \( Sp (S_Y (c_0)) \) is compact, we conclude that \( S_Y (c_0) \) is isometrically isomorphic to the space of all continuous \( \mathbb{K} \)-valued functions defined on \( Sp (S_Y (c_0)) \) provided by the supremum norm, that is, there exists an isomorphism of algebras
\[ \Psi : S_Y (c_0) \to C (Sp (S_Y (c_0))) \]
such that, for all $H \in \mathcal{S}_Y (c_0)$, $\| H \| = \| \psi (H) \|_\infty$.

Now, since $\mathcal{S}_Y (c_0)$ is the closure of the span of the collection $\{ Id, P_1, P_2, \ldots \}$, we can define the homomorphism of algebra. Let $n \in \mathbb{N}$;

$\phi_n : \text{Span} \{ Id, P_1, P_2, \ldots \} \rightarrow \mathbb{K}$

by

$\phi_n (P_i) = \begin{cases} 1 & \text{if } n = i \\ 0 & \text{if } n \neq i \end{cases}$

and $\phi_0 (P_i) = 0$, for every $i \in \mathbb{N}$. Thus, for any $H = \alpha_0 Id + \sum_{i=1}^k \alpha_i P_i \in \text{Span} \{ Id, P_1, P_2, \ldots \}$, we have

$| \phi_n (H) | = \left| \alpha_0 Id + \sum_{i=1}^k \alpha_i \phi_n (P_i) \right| \leq \max \{ |\alpha_0|, |\alpha_1 \phi_n (P_1)|, \ldots, |\alpha_k \phi_n (P_k)| \} \leq \| H \|,$

that is, $\phi_n$ is continuous in $\text{Span} \{ Id, P_1, P_2, \ldots \}$. From this, $\phi_n$ can be uniquely extended to a continuous algebra homomorphism from $\mathcal{S}_Y (c_0)$ into $\mathbb{K}$.

Note that if $\phi \in \text{Sp}(\mathcal{S}_Y (c_0))$, then $\phi (P_i) = \phi_n (P_i)$ for some $n \in \mathbb{N} \cup \{0\}$. In other words, $\text{Sp}(\mathcal{S}_Y (c_0)) = \{ \phi_n : n \in \mathbb{N} \cup \{0\} \}$. Therefore, the function $\Gamma : \mathbb{N} \cup \{0\} \rightarrow \text{Sp}(\mathcal{S}_Y (c_0))$ defined by $\Gamma (n) = \phi_n$ is bijective.

If we equip $\mathbb{N} \cup \{0\} = \mathbb{N}^*$ with the one-point compactification topology of the discrete space $\mathbb{N}$, then $\mathbb{N}^*$ is homeomorphic to $\text{Sp}(\mathcal{S}_Y (c_0))$.

Since the $\mathcal{S}_Y (c_0) \cong C(\text{Sp}(\mathcal{S}_Y (c_0)))$ and the compact space $\text{Sp}(\mathcal{S}_Y (c_0))$ is unique up to homeomorphism, we have that $\mathcal{S}_Y (c_0) \cong C(\mathbb{N}^*)$.

Let us identify the isometric isomorphism $\Psi$. If we replace $\text{Sp}(\mathcal{S}_Y (c_0))$ by $\mathbb{N}^*$ and considering the map

$G_{T_\lambda} \overset{\text{notation}}{=} f_{T_\lambda} : \mathbb{N}^* \rightarrow \mathbb{K}; \ n \mapsto f_{T_\lambda} (n) = \lambda_n,$

then, for $H = \alpha_0 Id + T_\lambda$,

$G_H : \mathbb{N}^* \rightarrow \mathbb{K}; \ n \mapsto G_H (n) = \alpha_0 + \lambda_n = \alpha_0 + f_{T_\lambda} (n)$.

Note that, if $\lambda = e_n$, then $T_{e_n} = P_n$ and therefore $G_{P_n} = \eta_{(n)}$, where $\eta_{(n)}$ is the $\mathbb{K}$-characteristic function.

From this, we can define the transformation

$G : \mathcal{S}_Y (c_0) \rightarrow C(\mathbb{N}^*); \ H \mapsto G_H.$

Clearly, $G$ is an algebra homomorphism and, for any $H \in \mathcal{S}_Y (c_0)$, $\| H \|_{sp} = \| G_H \|$ and $\| H \|_{sp} = \| H \|$; in other words, $G$ is an isometry. This transformation is very-well known as the Gelfand transformation.

As a consequence, we have the following theorem which is analogous to the Gelfand-Naimark classical theorem in the non-archimedean setting:

**Theorem 2.9.** $\mathcal{S}_Y (c_0)$ is isometrically isomorphic to $C(\mathbb{N}^*)$ through $G$. 
2.2. Spectral measure. Let $\Omega(\mathbb{N}^*)$ be the Boolean ring of all clopen subsets of $\mathbb{N}^*$. The elements of this ring are classified in two classes of subcollections: the first one contains finite subsets of $\mathbb{N}$ that we will call them of type 1 and the second one contains the complement of finite subsets in $\mathbb{N}^*$ that we will call them of type 2.

For a $C \subset \mathbb{N}^*$, $\eta_C$ denotes the $\mathbb{K}$-characteristic function of $C$. If $C_1, C_2 \subset \mathbb{N}^*$, then

$$\eta_{C_1} \cdot \eta_{C_2} = \eta_{C_1 \cap C_2}; \quad \eta_{C_1}^2 = \eta_C \quad \eta_{C_1} + \eta_{C_2} = \eta_{C_1 \cup C_2}, \text{ if } C_1 \cap C_2 = \emptyset.$$ 

Of course, $\eta_C$ is continuous if and only if $C \in \Omega(\mathbb{N}^*)$.

Now, let us take $f \in C(\mathbb{N}^*)$ and $\epsilon > 0$. Since the subspace generated by $\{\eta_n : n \in \mathbb{N}\} \cup \{\eta_{\mathbb{N}^* \setminus \{n_1, n_2, \ldots, n_k\}} : \{n_1, n_2, \ldots, n_k\} \subset \mathbb{N}\}$ is $\|\cdot\|_\infty$-dense on $C(\mathbb{N}^*)$, there exists finite collection $\{\alpha_0, \alpha_1, \ldots, \alpha_n\} \subset \mathbb{K}$ such that

$$\|f - (\alpha_0 \eta_{\mathbb{N}^*} + \sum_{s=1}^{k} \alpha_s \eta_{\{n_s\}})\|_\infty = \|f - (\alpha_0 \eta_{\mathbb{N}^*} + \sum_{s=1}^{k} \lambda_s \eta_{\{n_s\}})\|_\infty < \epsilon.$$ 

Let us denote by $\Lambda$ the inverse transformation of $G$. By the isometry condition of $\Lambda$, we get that

$$\|\Lambda f - \Lambda (\alpha_0 \eta_{\mathbb{N}^*} + \sum_{s=1}^{k} \lambda_s \eta_{\{n_s\}})\| = \|\Lambda f - (\alpha_0 \text{Id} + \sum_{s=1}^{k} \lambda_s P_{n_s})\| < \epsilon.$$ 

Now, we define the set-function $m : \Omega(\mathbb{N}^*) \to \mathcal{S}_Y(e_0)$ by $m(C) = \Lambda(\eta_C)$. This set-function is finitely additive and satisfies

$$m(C) = \begin{cases} 
\theta & \text{if } C = \emptyset \\
\text{Id} & \text{if } C = \mathbb{N}^* \\
\sum_{i=1}^{k} P_{n_i} & \text{if } C = \{n_1, \ldots, n_k\} \\
\text{Id} - \sum_{i=1}^{k} P_{n_i} & \text{if } C = \mathbb{N}^* \setminus \{n_1, \ldots, n_k\}
\end{cases} \quad (2.1)$$

Also, by the fact that $\|m(D)\| = 1$, if $D \in \Omega(\mathbb{N}^*) \setminus \{\emptyset\}$, the set

$$\{m(B) : B \in \Omega(\mathbb{N}^*) \setminus \emptyset, B \subset C\} \quad (2.2)$$

is bounded for any $C \in \Omega(\mathbb{N}^*) \setminus \emptyset$.

On the other hand, if $\{C_\mu\}_{\mu \in \Gamma}$ is shrinking on $\Omega(\mathbb{N}^*)$ and $\bigcap_{\mu \in \Gamma} C_\mu = \emptyset$, then there exists $\mu_0 \in \Gamma$ such that for $\mu \geq \mu_0$, $C_\mu = \emptyset$ and then

$$\lim_{\mu \in \Gamma} m(C_\mu) = 0. \quad (2.3)$$

By (2.1), (2.2) and (2.3), $m$ is a vector measure in the Katsaras’s sense (see [5]).

Let us take a $C \in \Omega(\mathbb{N}^*)$, $C \neq \emptyset$ and denote by $D_C$ the collection of all $\alpha = \{C_1, C_2, \ldots, C_n, x_1, x_2, \ldots, x_n\}$, where $\{C_k : k = 1, \ldots, n\}$ is a clopen partition of $C$ and $x_k \in C_k$. We define a partial order in $D_C$ by $\alpha_1 \geq \alpha_2$ if and only if the clopen partition of $C$ in $\alpha_1$ is a refinement of the clopen partition of $C$ in $\alpha_2$. Thus, $(D_C, \geq)$ is a directed set.
Now, for \( f \in C(\mathbb{N}^*) \), \( C \in \Omega(\mathbb{N}^*) \) and \( \alpha = \{C_1, C_2, \ldots, C_n; x_1, x_2, \ldots, x_n\} \in \mathcal{D}_C \), we define

\[
\omega_\alpha(f, m, C) = \sum_{k=1}^{n} f(x_k) \Lambda(\eta_{C_k}) = \Lambda \left( \sum_{k=1}^{n} f(x_k) \eta_{C_k} \right).
\]

On the other hand, since \( f\eta_C \) can be reached by a net

\[
\left\{ \sum_{k=1}^{n} f(x_k) \eta_{C_k} \right\}_{\alpha \in \mathcal{D}_C}
\]

in \( (C(\mathbb{N}^*), \|\cdot\|_\infty) \) and \( \Lambda \) is continuous, \( \lim_{\alpha \in \mathcal{D}_C} \omega_\alpha(f, m, C) \) exists in \( \mathcal{S}_Y \). Therefore, the operator \( \Lambda(f\eta_C) \) can be interpreted as an integral as follows

\[
\Lambda(f\eta_C) = \int_{\mathbb{N}^*} f\eta_C dm = \int_C f dm = \lim_{\alpha \in \mathcal{D}_C} \omega_\alpha(f, m, C).
\]

In particular,

\[
\Lambda(\eta_C) = \int_{\mathbb{N}^*} \eta_C dm = m(C) = \begin{cases} \sum_{i=1}^{s} P_{n_i} & \text{if } C = \{n_1, \ldots, n_s\} \\ \text{Id} - \sum_{i=1}^{k} P_{n_i} & \text{if } C = \mathbb{N}^* \setminus \{n_1, \ldots, n_k\} \end{cases}.
\]

**Theorem 2.10.** Each operator in \( \mathcal{S}_Y \) is represented as an integral defined by the projection-valued measure.

### 2.3. Scalar measures and a matrix representation of an operator

In this subsection we will show that each operator of the \( C \)-algebra \( \mathcal{S}_Y \) can be associated to an infinite matrix where each entry is an integral defined by a scalar measures.

Let \( x, y \in c_0 \). We define \( m_{x,y}: \Omega(\mathbb{N}^*) \to \mathbb{K} \) by

\[
m_{x,y}(C) = \langle m(C)x, y \rangle.
\]

Clearly, \( m_{x,y} \) is a scalar measure in the van Rooij's sense and, following the same arguments given in the previous subsection,

\[
\Lambda_{x,y}(f) = \langle \Lambda(f)x, y \rangle
\]

can be interpreted as an integral, say

\[
\int_{\mathbb{N}^*} f dm_{x,y} = \left\langle \left( \int_{\mathbb{N}^*} f dm \right)(x), y \right\rangle.
\]

In fact, as in the previous subsection

\[
\omega_\alpha(f, m_{x,y}, \mathbb{N}^*) = \langle \omega_\alpha(f, m, \mathbb{N}^*)x, y \rangle = \sum_{k=1}^{n} f(x_k) m_{x,y}(\mathbb{N}^*)
\]

and then we can denote by

\[
\int_{\mathbb{N}^*} f dm_{x,y} = \lim_{\alpha \in \mathcal{D}_{\mathbb{N}^*}} \omega_\alpha(f, m_{x,y}, \mathbb{N}^*) = \lim_{\alpha \in \mathcal{D}_{\mathbb{N}^*}} \langle \omega_\alpha(f, m, \mathbb{N}^*)x, y \rangle = \langle H(x), y \rangle.
\]

where \( H = \int_{\mathbb{N}^*} f dm \in \mathcal{S}_Y(c_0) \).

A particular case is when \( x = e_i \) and \( y = e_j \). In such a case, the measure \( m_{e_i,e_j} \) will be denoted by \( m_{ij} \).
Note that, if $C \in \Omega(\mathbb{N}^*) \setminus \{\emptyset\}$, then
\[
\sup_{i,j \in \mathbb{N}} |m_{ij}(C)| = \sup_{i,j \in \mathbb{N}} |\langle m(C)e_i, e_j \rangle| = \|m(C)\| = 1 = \|\eta_C\|
\]
We define the linear functional
\[
\Lambda_{ij} : C(\mathbb{N}^*) \to \mathbb{K}; \quad f \to \Lambda_{ij}(f) = \int_{\mathbb{N}^*} f \, dm_{ij} = \langle H(e_i), e_j \rangle
\]
where $H = \int_{\mathbb{N}^*} f \, dm \in \mathcal{S}_Y(c_0)$. Moreover,
\[
\sup_{i,j \in \mathbb{N}} |\Lambda_{ij}(f)| = \sup_{i,j \in \mathbb{N}} |\langle H(e_i), e_j \rangle| = \|H\| = \|f\|
\]

Let us denote by $\mathcal{M}$ the space of all infinite matrices of the form $(\Lambda_{ij}(f))_{i,j \in \mathbb{N}}$, i.e.,
\[
\mathcal{M} = \left\{ A(f) = (\Lambda_{ij}(f))_{i,j \in \mathbb{N}} : f \in C(\mathbb{N}^*) \right\}
\]
Clearly, $\mathcal{M}$ is a vector space over $\mathbb{K}$ and the function $\|\cdot\|_{\mathcal{M}} : \mathcal{M} \to \mathbb{R}$ defined by $\|A(f)\|_{\mathcal{M}} = \sup_{i,j \in \mathbb{N}} |\Lambda_{ij}(f)|$ is a non-archimedean norm. On the other hand, if $f = \eta_{\{k\}}$ or $\eta_{\mathbb{N}^* \setminus \{k_1, k_2, \ldots, k_n\}}$, then
\[
A(\eta_{\{k\}}) = \left( \begin{array}{c}
\frac{y_i^k y_j^k}{\langle y^k, y^k \rangle} \\
\end{array} \right)_{(i,j) \in \mathbb{N} \times \mathbb{N}} = \left( \langle P_k(e_i), e_j \rangle \right)_{(i,j) \in \mathbb{N} \times \mathbb{N}},
\]
\[
A(\eta_{\mathbb{N}^* \setminus \{k_1, k_2, \ldots, k_n\}}) = \left( \left\langle \left( I - \sum_{s=1}^n P_{k_s} \right)(e_i), e_j \right\rangle \right)_{(i,j) \in \mathbb{N} \times \mathbb{N}}.
\]

Using the above remark, we state the following theorem:

**Proposition 2.11.** There exists an isometric isomorphism between $\mathcal{M}$ and $\mathcal{S}_Y$.

**Theorem 2.12.** Each operator in $\mathcal{S}_Y$ is represented as a matrix whose entries are integrals defined by scalar measures.

### 3. The subalgebra $\mathcal{L}_T$

In this section we study the smallest closed subalgebra with unity of $\mathcal{S}_Y(c_0)$ generated by a fixed element $T_\lambda \in \mathcal{S}_Y(c_0)$, where $\lambda = (\lambda_n) \in c_0$ and
\[
T_\lambda = \sum_{n=1}^{\infty} \lambda_n P_n,
\]
we also show that, as $\mathcal{S}_Y(c_0)$, this algebra is generated by a family of normal projections and, under certain conditions, both algebras are isometrically isomorphic.

Let us denote by $\mathcal{L}_{T_\lambda}$ the closure of $alg_{\mathcal{S}_Y(c_0)}\{I, T_\lambda\}$ with respect to the operator norm, that is, the closure of the space of polynomials in $T_\lambda$. Clearly, $\mathcal{L}_{T_\lambda}$ is a $C$-algebra since it is closed Banach subalgebra of $\mathcal{S}_Y(c_0)$.

By Proposition 2.7, $Sp(\mathcal{L}_{T_\lambda})$ is compact and, for each $H \in \mathcal{L}_{T_\lambda}$, where
\[
\|H\| = \sup_{i \in \mathbb{N}} \|H(e_i)\| = \|H\|_{sp},
\]
On the other hand, since $S_Y (c_0)$ has the power multiplicative norm property, $L_{T_\lambda}$ inherits such property.

Under the conditions that $L_{T_\lambda}$ is a $C$-algebra and $Sp \left( L_{T_\lambda} \right)$ is compact, we conclude that $L_{T_\lambda}$ is isometrically isomorphic to the space of all continuous functions $C \left( Sp \left( L_{T_\lambda} \right) \right)$ provided by the supremum norm, that is, there exists an isomorphism of algebras

$$\Psi : L_{T_\lambda} \to C \left( Sp \left( L_{T_\lambda} \right) \right)$$

such that, for all $H \in L_{T_\lambda}$, $\| H \| = \| \Psi (H) \|_\infty$.

Suppose, for instance, that the range of the sequence $\lambda$ is infinite and define the equivalence relation $n \sim m \iff \lambda_n = \lambda_m$. Observe that each equivalence class of a non-null entry is at most a finite set. Let us denote by $\{ \lambda_{n_1}, \lambda_{n_2}, \ldots \}$ the collection of all non-null representative of such classes. Of course, if $n_1 < n_2 < \ldots$, then $\lim_{n \to \infty} \lambda_{n} = \lambda_0$. Therefore, $\{ \lambda_{n_1}, \lambda_{n_2}, \ldots \} \cup \{ \lambda_0 \} = \sigma (T_\lambda)$, the sequence of all eigenvalues of $T_\lambda$.

Let us consider the unique homomorphism of algebra

$$\phi_i : alg_{S_Y (c_0)} \{ Id, T_\lambda \} \to \mathbb{K}, \quad i \in \mathbb{N} \cup \{ 0 \}$$

such that

$$\phi_i (T_\lambda) = \lambda_{n_i}.$$ 

Thus, for any $H = \sum_{m=0}^k \alpha_m T_{\lambda}^m \in alg_{S_Y (c_0)} \{ Id, T_\lambda \}$, we have

$$| \phi_i (H) | = \left| \alpha_0 + \sum_{m=1}^k \alpha_m \lambda_{n_i}^m \right| \leq \max \left\{ | \alpha_0 |, \left| \sum_{m=1}^k \alpha_m \lambda_{n_i}^m \right| \right\}$$

$$\leq \max \left\{ | \alpha_0 |, \left| \sum_{m=1}^k \alpha_m \lambda_{n_i}^m \right| \right\} = \left\| \left( \alpha_0, \sum_{m=1}^k \alpha_m \lambda_{n_i}^m \right) \right\|_{c_0^+}$$

$$= \left\| \alpha_0 Id + T_{\lambda} \sum_{m=0}^k \alpha_m \lambda_{n_i}^m \right\| = \left\| \sum_{m=0}^k \alpha_m \lambda_{n_i}^m \right\| = \| H \|,$$

that is, $\phi_i$ is continuous in $alg_{S_Y (c_0)} \{ Id, T_\lambda \}$. From this, $\phi_i$ can be uniquely extended to a continuous homomorphism of algebra from $L_{T_\lambda}$ into $\mathbb{K}$.

We claim that $\sigma (T_\lambda)$, equipped with a certain topology, is homeomorphic to $Sp \left( L_{T_\lambda} \right)$.

Note that the function

$$\Gamma : \sigma (T_\lambda) \to Sp \left( L_{T_\lambda} \right); \quad \lambda_{n_i} \mapsto \Gamma (\lambda_{n_i}) = \phi_i$$

is well-defined and is injective.

The next proposition tell us when an element of $S_Y (c_0)$ admits an inverse:

**Proposition 3.1.** Let $T \in S_Y (c_0)$. If $z \notin \sigma (T)$, then $z Id - T$ is invertible in $S_Y (c_0)$.

**Proof.** For $y \in R (z Id - T)$, there exists $x \in c_0$ such that

$$(z Id - T) (x) = y.$$
Since \( z \notin \sigma(T) \), we can solve the above equation for \( x \) and get

\[
x = \frac{1}{z} y + \frac{1}{z} Tx = \frac{1}{z} y + \frac{1}{z} \sum_{i=1}^{\infty} \lambda_i \frac{\langle x, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)}
\]  \( (3.1) \)

Applying the continuous functional \( \langle \cdot, y^{(k)} \rangle \) to \( x \), we have

\[
\langle x, y^{(k)} \rangle = \frac{1}{z} \langle y, y^{(k)} \rangle + \frac{1}{z} \lambda_k \langle x, y^{(k)} \rangle
\]

Now, solving the last equation for \( \langle x, y^{(k)} \rangle \), we obtain

\[
\left(1 - \frac{\lambda_k}{z}\right) \langle x, y^{(k)} \rangle = \frac{1}{z} \langle y, y^{(k)} \rangle
\]

\[
\langle x, y^{(k)} \rangle = \frac{1}{z - \lambda_k} \langle y, y^{(k)} \rangle
\]

Note that the sequence

\[
\left(\frac{\lambda_k}{z - \lambda_k}\right)_{k \in \mathbb{N}}
\]

is an element of \( c_0 \). In fact, for a given \( 0 < \epsilon < 1 \), there exists \( i_0 \in \mathbb{N} \) such that

\[
i \geq i_0 \Rightarrow |\lambda_i| < \epsilon |z|.
\]

Thus,

\[
i \geq i_0 \Rightarrow \left| \frac{\lambda_i}{z - \lambda_i} \right| = \frac{|\lambda_i|}{|z|} < \epsilon.
\]

Now, replacing in \( (3.1) \), we get

\[
x = \frac{1}{z} y + \frac{1}{z} \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} \frac{\langle y, y^{(i)} \rangle}{\langle y^{(i)}, y^{(i)} \rangle} y^{(i)}
\]

\[
= \frac{1}{z} y + \frac{1}{z} \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(y).
\]

Although \( y \) belongs to \( R(zI - T) \), the last expression holds for any \( y \in c_0 \). Thus, if we denote by

\[
R_z(T)(y) = \frac{1}{z} y + \frac{1}{z} \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(y),
\]

then \( R_z(T)(\cdot) \in \mathcal{S}_Y(c_0) \), since \( \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(\cdot) \) is compact and self-adjoint operator.
Let us show that, effectively, $R_z(T)(\cdot)$ is the inverse operator of $zId - T$

$$[(zId - T) \circ R_z(T)](y) = (zId - T) \left(\frac{1}{z} y + \frac{1}{z} \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(y)\right)$$

$$= y + \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(y) - \frac{1}{z} \sum_{i=1}^{\infty} \lambda_i P_i(y) - \frac{1}{z} \sum_{i=1}^{\infty} \frac{\lambda_i^2}{z - \lambda_i} P_i(y); \quad T(P_i(y)) = \lambda_i P_i(y)$$

$$= y + \sum_{i=1}^{\infty} \left[\frac{\lambda_i}{z - \lambda_i} - \frac{\lambda_i}{z} - \frac{\lambda_i^2}{z (z - \lambda_i)}\right]_{=0} P_i(y) = y = Id(y)$$

In the other direction, since

$$P_j \circ P_i(x) = \begin{cases} P_i(x) & \text{if } j = i \\ 0 & \text{if } j \neq i \end{cases},$$

we have

$$[R_z(T) \circ (zId - T)](x) = zR_z(T)(x) - R_z(T)(Tx)$$

$$= x + \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(x) - \sum_{i=1}^{\infty} \lambda_i R_z(T)(P_i(x))$$

$$= x + \sum_{i=1}^{\infty} \frac{\lambda_i}{z - \lambda_i} P_i(x) - \sum_{i=1}^{\infty} \lambda_i \left[\frac{1}{z} P_i(x) + \frac{1}{z} \sum_{j=1}^{\infty} \frac{\lambda_j}{z - \lambda_j} P_j(P_i(x))\right]$$

$$= x + \sum_{i=1}^{\infty} \left[\frac{\lambda_i}{z - \lambda_i} - \frac{\lambda_i}{z} - \frac{\lambda_i^2}{z (z - \lambda_i)}\right]_{=0} P_i(x) = x = Id(x)$$

Therefore, $R_z(T) = (zId - T)^{-1} \in S_Y(c_0)$.

**Corollary 3.2.** If $z \notin \sigma(T_\lambda)$, then $R_z(T_\lambda) = (zId - T_\lambda)^{-1} \in \mathcal{L}_{T_\lambda}$.

**Proof.** We already know that $S_Y(c_0)$ is a $C$-algebra with unity and $zId - T_\lambda$ is invertible in $S_Y(c_0)$. By Th. 6.10 in [7], we have that

$$R_z(T_\lambda) \in \overline{alg_{S_Y(c_0)}}\{Id, zId - T_\lambda\}$$

Now, since $\overline{alg_{S_Y(c_0)}}\{Id, zId - T_\lambda\}$ is the smallest closed subalgebra that contains to $zId - T_\lambda$, we conclude that $R_z(T_\lambda) \in \mathcal{L}_{T_\lambda}$.

**Proposition 3.3.** The function $\Gamma$ is bijective.

**Proof.** By above, $\Gamma$ is injective. If $\phi \in Sp(\mathcal{L}_{T_\lambda})$, then $\phi(T_\lambda) = z$, for some $z \in \mathbb{K}$. Suppose that $z \notin \sigma(T_\lambda)$, hence $zId - T_\lambda$ has an inverse and, by the previous corollary, $R_z(T_\lambda) \in \mathcal{L}_{T_\lambda}$. Since the function $\phi$ is a homomorphism between algebras with unities, we have

$$1 = \phi(Id) = \phi((zId - T_\lambda)^{-1} \circ (zId - T_\lambda)) = \phi((zId - T_\lambda)^{-1}) \phi(zId - T_\lambda),$$
but, by the linearity of $\phi$, the factor $\phi(z\text{Id} - T_\lambda)$ is null, which is a contradiction. Thus, if $\phi \in Sp(L_{T_\lambda})$, then there exists $\mu \in \sigma(T_\lambda)$ such that $\phi = \phi_\mu$ and therefore $\Gamma$ is bijective. □

Remark 3.4. We have identified $Sp(L_{T_\lambda})$ with $\sigma(T_\lambda)$ through the bijective function $\Gamma$. Let us consider the induced topology by $\mathbb{K}$ on $\sigma(T_\lambda)$. Note that $\sigma(T_\lambda)$ is compact.

Proposition 3.5. $\sigma(T_\lambda)$ is homeomorphic to $Sp(L_{T_\lambda})$

Proof. We claim first that $\Upsilon = \Gamma^{-1}$ is continuous. In fact, if $\phi_\alpha \rightarrow \phi$ in the induced topology on $Sp(L_{T_\lambda})$ by the product topology in $\mathbb{K}L_{T_\lambda}$, then

$$\phi_\alpha(H) \rightarrow \phi(H)$$

for each $H \in L_{T_\lambda}$. In particular,

$$\phi_\alpha(T_\lambda) \rightarrow \phi(T_\lambda)$$

or, equivalently,

$$\Upsilon(\phi_\alpha) \rightarrow \Upsilon(\phi).$$

Now, since $\Upsilon$ is bijective and continuous, $Sp(L_{T_\lambda})$ is compact and $\sigma(T_\lambda)$ is a Hausdorff space, we conclude that $\Upsilon$ is a homeomorphism. □

By this proposition and by the uniqueness of $X$ (up to homeomorphism) for which $L_{T_\lambda} \cong C(X)$, we have

$$L_{T_\lambda} \cong C(Sp(L_{T_\lambda})) \cong C(\sigma(T_\lambda)).$$

Let us identify the isometric isomorphism $\Psi$. Replacing $Sp(L_{T_\lambda})$ by $\sigma(T_\lambda)$ and considering the map

$$G_{T_\lambda} \; \text{notation} \; f_{T_\lambda} : \sigma(T_\lambda) \rightarrow \mathbb{K}; \; \lambda_{n_i} \mapsto f_{T_\lambda}(\lambda_{n_i}) = \lambda_{n_i},$$

we can get, for a fixed $H = \alpha_0 \text{Id} + \sum_{m=1}^{k} \alpha_m T_\lambda^m \in alg_{S_Y(c_0)} \{\text{Id}, T_\lambda\}$, the map $G_H : \sigma(T_\lambda) \rightarrow \mathbb{K}$ defined by

$$G_H(\lambda_{n_i}) = \left\{ \begin{array}{ll} \alpha_0 + \sum_{m=1}^{k} \alpha_m \lambda_{n_i}^m = \alpha_0 + \sum_{m=1}^{k} \alpha_m [f_{T_\lambda}(\lambda_{n_i})]^m & \text{if } i \in \mathbb{N} \\ \alpha_0 & \text{if } i = 0 \end{array} \right..$$

From this, we can define

$$G : alg_{S_Y(c_0)} \{\text{Id}, T_\lambda\} \rightarrow C(\sigma(T_\lambda)); \; H \mapsto G_H,$$

the well-known Gelfand transformation. Clearly, $G$ is a homomorphism of algebras, $\|H\|_{sp} = \|G_H\|$ for any $H \in alg_{S_Y(c_0)} \{\text{Id}, T_\lambda\}$, and since $L_{T_\lambda}$ satisfies the condition given by Proposition 2.7, we have

$$\|H\|_{sp} = \|H\|.$$

Thus, $G$ is an isometry and then it can be extended to the whole $L_{T_\lambda}$. Let us denote by the same capital letter $G$ such extension.

Proposition 3.6. $G$ is an isometric isomorphism of algebras.
**Proof.** It is enough to prove that $G$ is surjective. By the fact that $G$ is a homomorphism of algebras and the image of $T_\lambda$ by $G$ is the identity map $G_{T_\lambda} = f_{T_\lambda}$, the collection $\{1, f_{T_\lambda}, f_{T_\lambda}^2, \ldots\}$ is the image of $\{Id, T_\lambda, T_\lambda^2, \ldots\}$ by $G$. Now, by the compactness of $\sigma (T_\lambda)$, Theorem 5.28 in [7] guarantees that $algC(\sigma (T_\lambda))\{1, f_{T_\lambda}\}$ is dense in $C(\sigma (T_\lambda))$. Thus, if $f \in C(\sigma (T_\lambda))$, then there exists a sequence $\{g_n\}_{n \in \mathbb{N}}$ in $algC(\sigma (T_\lambda))\{1, f_{T_\lambda}\}$ such that $f = \lim_{n \to \infty} g_n$. Now, for each $n \in \mathbb{N}$, there exists $H_n \in \mathcal{L}_{T_\lambda}$ such that

$$G_{H_n} = g_n.$$ 

By the fact that $G$ is an isometry, the sequence $(H_n)$ is a Cauchy sequence in $\mathcal{L}_{T_\lambda}$ and then convergence to an $H \in \mathcal{L}_{T_\lambda}$. Since $G$ is continuous, we have that

$$G_H = \lim_{n \to \infty} G_{H_n} = \lim_{n \to \infty} g_n = f.$$ 

\[\hfill \square\]

**Remark 3.7.** By above proposition, the Gelfand transformation $G$ is an isometric isomorphism between $\mathcal{L}_{T_\lambda}$ and $C(Sp(\mathcal{L}_{T_\lambda}))$ or $C(\sigma (T_\lambda))$.

**Remark 3.8.** If we suppose that, for $\lambda, \mu \in c_0$, the corresponding sets $\{\lambda_n : n \in \mathbb{N}\}$ and $\{\mu_n : n \in \mathbb{N}\}$ are infinite, then both are connected by a bijective correspondence which is a homeomorphism if we provide them with the discrete topology. Therefore, the spaces $\sigma (T_\lambda), \sigma (T_\mu)$ and $\mathbb{N}^*$ are homeomorphic each other. Now, since any $C$-algebra with unity is isometrically isomorphic to $C(X)$, where $X$ is compact space and it is unique up to homeomorphism, we conclude that $\mathcal{L}_{T_\lambda} \cong \mathcal{L}_{T_\mu} \cong S_Y$.

### 3.1. **Spectral measure.**

By the previous section, there exists an isometric isomorphism of algebras $\Phi = G^{-1} : C(\sigma (T_\lambda)) \to \mathcal{L}_{T_\lambda}$. Let us denote by $\Omega(\sigma (T_\lambda))$ the Boolean ring of all clopen subsets of $\sigma (T_\lambda)$. Of course, $\eta_C$ is continuous if and only if $C \in \Omega(\sigma (T_\lambda))$.

Note that the elements of $\Omega(\sigma (T_\lambda))$ can be classified as follows: the first type are those which are finite subsets of $\sigma (T_\lambda) \setminus \{\lambda_m\}$ and the second type are those which are complement on $\sigma (T_\lambda)$ of the first type.

Now, since $\Phi$ is a homomorphism of algebras, we have

$$\Phi (\eta_C) = \Phi (\eta_C^2) = \Phi (\eta_C)^2.$$ 

In other words, $\Phi (\eta_C)$ is a projection in $\mathcal{L}_{T_\lambda}$ and if $C \in \Omega(\sigma (T_\lambda)) \setminus \{\emptyset\}$, then $\Phi (\eta_C)$ is a non-null.

On the other hand, by the fact that the linear hull of $\{\eta_C : C \in \Omega(\sigma (T_\lambda))\}$ is dense in $C(\sigma (T_\lambda))$, for any fixed $f \in C(\sigma (T_\lambda))$ and $\epsilon > 0$, there exists a finite clopen partition $\{C_k : k = 1, \ldots, s\}$ of $\sigma (T_\lambda)$ and a finite collection of scalars $\{\alpha_k : k = 1, \ldots, s\}$ such that

$$\left\| f - \sum_{k=1}^{s} \alpha_k \eta_{C_k} \right\|_\infty = \sup_{x \in \sigma (T_\lambda)} \left| f(x) - \sum_{k=1}^{s} \alpha_k \eta_{C_k} (x) \right| < \epsilon \quad (3.2)$$

Since $\{C_k : k = 1, \ldots, s\}$ is a clopen partition of $\sigma (T_\lambda)$, only one of these sets is of second type, say $C_1 = \sigma (T_\lambda) \setminus \{\lambda_{m_1}, \lambda_{m_2}, \ldots, \lambda_{m_n}\}$. From this, $\cup_{k=2}^{s} C_k =$
\{\lambda_m, \lambda_{m_2}, \ldots, \lambda_{m_n}\}$. Using the characteristic functions properties and the fact that the single subsets \{\lambda_k\} belong to $\Omega(\sigma(T_\lambda))$, we can rewrite (3.2) as follows:

$$
\left\| f - \left[ \alpha_1 \eta_{\sigma(T_\lambda)} + \sum_{l=1}^n (\alpha_l - \alpha_1) \eta_{\{\lambda_{m_l}\}} \right] \right\|_\infty
= \sup_{x \in \sigma(T_\lambda)} \left| f(x) - \left[ \alpha_1 \eta_{\sigma(T_\lambda)}(x) + \sum_{l=1}^n (\alpha_l - \alpha_1) \eta_{\{\lambda_{m_l}\}}(x) \right] \right| < \epsilon
$$

and without loss of generality, we can assume that

$$
\left\| f - \left[ \alpha_1 \eta_{\sigma(T_\lambda)} + \sum_{l=1}^n (\alpha_l - \alpha_1) \eta_{\{\lambda_{m_l}\}} \right] \right\|_\infty
= \sup_{x \in \sigma(T_\lambda)} \left| f(x) - \left[ f(\lambda_{m_0}) \eta_{\sigma(T_\lambda)}(x) + \sum_{l=1}^n [f(\lambda_{m_l}) - f(\lambda_{m_0})] \eta_{\{\lambda_{m_l}\}}(x) \right] \right| < \epsilon
$$

Using the isometry $\Phi$, we have

$$
\left\| \Phi(f) - \left[ f(\lambda_{m_0}) Id + \sum_{l=1}^n [f(\lambda_{m_l}) - f(\lambda_{m_0})] E_l \right] \right\| < \epsilon, \quad (3.3)
$$

where $E_l$ is the corresponding projection $\Phi(\eta_{\{\lambda_{m_l}\}})$. At the same time, (3.3) shows that the space generated by \{\lambda \in \mathcal{L}_{T_\lambda} : E^2 = E\} is dense in $\mathcal{L}_{T_\lambda}$.

Let us consider the following set-function:

$$
m_{T_\lambda} : \Omega(\sigma(T_\lambda)) \to \mathcal{L}_{T_\lambda}^2; \quad C \mapsto m_{T_\lambda}(C) = \Phi(\eta_C) = E_C.
$$

In similar way as in subsection 2.2, $m_{T_\lambda}$ is a finite additive measure valued-projection which is known as spectral measure associated to $\mathcal{L}_{T_\lambda}$.

Since $E$ is, in particular, an element of $\mathcal{S}_T$, there exists $\alpha \in \mathbb{K}$ and $\mu = (\mu_i) \in c_0$ such that $E = \alpha Id + T_\mu$. By the fact that $E$ is a projection, $E^2 = E$, that is, $(\alpha Id + T_\mu)(\alpha Id + T_\mu) = \alpha Id + T_\mu$ or equivalent to $(\alpha^2 - \alpha) Id + T_{(2\alpha - 1)\mu + \mu^2} = 0$. Taking the norm of this operator, we get

$$
0 = \|(\alpha^2 - \alpha) Id + T_{(2\alpha - 1)\mu + \mu^2}\| = \max\{ |\alpha(\alpha - 1)|, (2\alpha - 1)\mu + \mu^2 \}
$$

From this, if $\alpha = 0$, then $\mu_i = 0$ or $\mu_i = 1$. Since $\lim_{n \to \infty} \mu_n = 0$, $\mu_i = 0$ for all $i \in \mathbb{N}$, excepts for a finite collection \{\imath_1, \imath_2, \ldots, \imath_n\} for which $\mu_{\imath_s} = 1$, $s = 1, 2, \ldots, n$. Thus, $E = \sum_{s=1}^n P_{\imath_s}$.

On the other hand, if $\alpha = 1$, then we also get finite collection, say \{\j_1, \j_2, \ldots, \j_m\} for which $\mu_{\j_k} = -1$, $k = 1, 2, \ldots, m$, and the rest of the elements of the sequence $\mu$ are 0. Thus, $E = Id - \sum_{k=1}^m P_{\j_k}$.

If $\sum_{s=1}^n P_s \in \mathcal{L}_{T_\lambda}$, $n \neq 1$, then $P_s \notin \mathcal{L}_{T_\lambda}$, for any $s \in \{1, 2, \ldots, n\}$.

Remark 3.9. Note that the length of the sum in $E = \sum_{s=1}^n P_s$ depends on exclusively for the repetition of some non-null entries of the sequence $\lambda$. 

Following the same arguments as in subsection 2.2, for \( f \in C(\sigma(T_\lambda)) \) and 
\[ \alpha = \{C_1, C_2, \ldots, C_n; x_1, x_2, \ldots, x_n\} \in \mathcal{D}, \]
where \( \sigma(T_\lambda) = \bigsqcup_{k=1}^n C_k \), we define
\[ \omega_\alpha(f, m_{T_\lambda}, \sigma(T_\lambda)) = \sum_{k=1}^n f(x_k) m_{T_\lambda}(C_k) = \sum_{k=1}^n f(x_k) E_{C_k}. \]
and since the function \( f \) can be reached by a net
\[ \left\{ \sum_{k=1}^n f(x_k) \eta_{C_k} \right\}_{\alpha \in \mathcal{D}} \]
in \( C(\sigma(T_\lambda)) \), the isometry of \( \Phi \) allows us to get
\[ \lim_{\alpha \in \mathcal{D}} \omega_\alpha(f, m_{T_\lambda}, \sigma(T_\lambda)) = \Phi(f) \]
Therefore, the operator \( \Phi(f) \) is interpreted as an integral, that is,
\[ \Phi(f) = \int_{\sigma(T_\lambda)} f dm_{T_\lambda} = \lim_{\alpha \in \mathcal{D}} \omega_\alpha(f, m_{T_\lambda}, \sigma(T_\lambda)) \]
For example, for \( f_{T_\lambda}, \eta_{(\lambda_n)} \) or \( f \equiv 1 \), their respective integral are
\[ T_\lambda = \Phi(f_{T_\lambda}) = \int_{\sigma(T_\lambda)} f_{T_\lambda} dm_{T_\lambda}; \quad E_n = \int_{\sigma(T_\lambda)} \eta_{(\lambda_n)} dm_{T_\lambda} = \left\{ \begin{array}{ll} \sum_{s=1}^n P_s \\ \text{or} \\ \id - \sum_{s=1}^m P_s \end{array} \right. \]
\[ \id = \Phi(1) = \int_{\sigma(T_\lambda)} dm_{T_\lambda}. \]

**Remark 3.10.** Since \( \{P_n : n \in \mathbb{N}\} \) is a family of normal projections, we conclude that \( \{E_k : k \in \mathbb{N}\} \) is also a family of normal projections. Even more, using the inner product for operators in \( \mathcal{A}_1 \), \( \{E_k : k \in \mathbb{N}\} \) is an orthonormal family.

Now, since \( \{\eta_{(\lambda_k)} : k \in \mathbb{N}\} \) generates to \( C(\sigma(T_\lambda)) \), the isometry isomorphism \( \Phi \) tells us \( \mathcal{L}_{T_\lambda} \) is generated by \( \{\id, E_1, E_2, \ldots\} \). In other words, \( \mathcal{L}_{T_\lambda} = \text{Span} \{\id, E_1, E_2, \ldots\} \).

According to this, \( \mathcal{L}_{T_\lambda} \) has the same structure than \( \mathcal{S}_Y \) and, therefore, with the same arguments developed in subsection 2.3, each operator in \( \mathcal{L}_{T_\lambda} \) admits a matrix representation whose entries are integrals defined by scalar measures.

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\[\text{1Departamento de Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidade de Concepción, Casilla 160-C, Concepción, Chile.} \]

\[\text{E-mail address: jaguayo@udec.cl; jacqojeda@udec.cl} \]

\[\text{2Departamento de Matemática y Física Aplicadas, Facultad de Ingeniería, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile.} \]

\[\text{E-mail address: mnova@ucsc.cl} \]