A Necessary Condition for a Nontrivial Zero of the Riemann Zeta Function via the Polylogarithmic Function

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We provide a new series expansion of the polylogarithm of complex argument \( \text{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s} \). From the new series, we define a new entire function \( Z(s, x) \) which is related to \( \text{Li}_s(x) \) but processes several advantages over the initial polylogarithmic series. For example, the limit of \( Z(s, x) \) when \( x \to 1 \) is convergent to \( (s-1)\zeta(s) \) for all complex numbers \( s \) while the limit of \( \text{Li}_s(x) \) converges only when \( \text{Re}(s) > 1 \). As an application of the expansion of \( Z(s, x) \), we derive a necessary condition for a non-trivial zero of the Riemann zeta function.

**Keywords**: Number Theory; Polylogarithm function; Riemann Zeta function; Riemann nontrivial zeros

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1. Appel and Jonquiére Integrals

The polylogarithm \( \text{Li}_s(x) \) is defined by the power series

\[
\text{Li}_s(x) = \sum_{n=1}^{\infty} \frac{x^n}{n^s}.
\]  

(1.1)

The definition is valid for all complex values \( s \) and all complex values of \( x \) such that \( |x| < 1 \). The series is convergent for \( x = 1 \) only when \( \text{Re}(s) > 1 \).

Using the identity

\[
\frac{1}{n^s} = \frac{1}{\Gamma(s)} \int_0^{\infty} e^{-nt} t^{s-1} dt,
\]

(1.2)

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equation (1.1) can be rewritten as

\[ L_i(x) = \frac{x}{\Gamma(s)} \int_0^\infty \frac{t^{s-1}}{e^t - x} \, dt. \]  

(1.3)

The integral in (1.3) is called Appell’s integral or Jonquièr’s integral. It defines \( L_i(x) \) not only in the unit circle but also in the whole slit plane \( \mathbb{C} \setminus [1, \infty) \) provided that \( \text{Re}(s) > 0 \).

To obtain a formula valid for practically every complex number \( s \), we use Hankel’s device which consists in replacing the real integral by a contour integral. The contour is denoted by \( C \) and is called Hankel contour. It consists of the three parts \( C = C_+ \cup C_\epsilon \cup C_- \): a path which extends from \( (\infty, \epsilon) \), around the origin counter clockwise on a circle of center the origin and of radius \( \epsilon \) and back to \( (\epsilon, \infty) \), where \( \epsilon \) is an arbitrarily small positive number. The integral (1.3) becomes

\[ L_i(x) = e^{-i\pi s} \frac{\Gamma(1 - s)}{2\pi i} \int_C \frac{xt^{s-1}}{e^t - x} \, dt. \]  

(1.4)

Equation (1.4) now defines \( L_i(x) \) for any \( x \) in the cut plane and any \( s \) not a positive integer.

2. A New Expansion of \( L_i(x) \)

The integral in (1.3) can be rewritten as

\[ L_i(x) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{xt^{s-1}}{1 - xe^{-t}} \, dt. \]  

(2.1)

By observing that

\[ \frac{d}{dt} \left( \frac{-xte^{-t}}{1 - xe^{-t}} \right) = \left( \frac{t}{(1 - xe^{-t})^2} - \frac{1}{1 - xe^{-t}} \right) xe^{-t}, \]  

(2.2)

we may integrate by parts (2.1) to obtain

\[ L_i(x) = \frac{1}{(s-1)\Gamma(s)} \int_0^\infty \left( \frac{t}{(1 - xe^{-t})^2} - \frac{1}{1 - xe^{-t}} \right) xe^{-t} t^{s-1} \, dt, \]  

(2.3)

where .

If we define the new variable \( X \) by

\[ X = 1 - xe^{-t}, \]  

(2.4)

\[ t = -\log(1 - X) + \log x, \]  

(2.5)

where \( \log(1 - X) \) and \( \log x \) are both real when \( X < 1 \) and \( x > 0 \) respectively, the function between the parenthesis inside the integral (2.3) becomes
\[ t \frac{1}{(1 - xe^{-t})^2} - \frac{1}{1 - xe^{-t}} = -\frac{\log(1 - X)}{X^2} - \frac{1}{X} + \frac{\log x}{X^2} = \sum_{n=1}^{\infty} \frac{X^{n-1}}{n + 1} + \frac{\log x}{X^2}, \quad (2.6) \]

provided of course that the infinite series on the right hand side of (2.6) is convergent. The radius of convergence of the series is 1, so we require that \(|X| = |1 - xe^t| < 1\). When \(|1 - x| < 1\), the condition \(|1 - xue^{-r}| < 1\) is trivially satisfied for all \(t > 0\).

The set \(|1 - x| < 1\) is a disk of center 1 and radius 1. If we want \(x\) to avoid the cut \([1, \infty)\), then it is judicious to restrict \(x\) to the set \(D\) defined by

\[ D = \{ x \in \mathbb{C} : |x| < 1 \quad \text{and} \quad |1 - x| < 1 \}. \quad (2.7) \]

In this paper, we will further restrict \(x\) to the real interval \((0, 1)\) which is a subset of \(D\) since the consideration of complex values of \(x\) does not offer any advantages to our analysis.

Finally, if we go back to the original variables, (2.3) simplifies to

\[ (s - 1)Li_s(x) = \frac{x}{\Gamma(s)} \int_0^\infty \sum_{n=1}^{\infty} \frac{(1 - xe^{-t})^{n-1}}{n + 1} xe^{-ts-1} dt + \frac{\log x}{\Gamma(s)} \int_0^\infty \frac{e^{-t}}{(1 - xe^{-t})^2} ts^{-1} dt. \quad (2.8) \]

Now, can we interchange the sum and the integral in (2.8)? The answer is affirmative if we can show that the series

\[ \sum_{n=1}^{\infty} \frac{1}{\Gamma(s)} \int_0^\infty \frac{(1 - xe^{-t})^{n-1}}{n + 1} xe^{-ts-1} dt \quad (2.9) \]

converges absolutely and uniformly for all \(t > 0\). For this purpose, we define

\[ \sigma_n(s, x) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} x^{k+1} (k+1)^{-s}, \quad (2.10) \]

which, using (1.2), can be rewritten as

\[ \sigma_n(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty (1 - xe^{-t})^{n-1} xe^{-ts-1} dt \quad (2.11) \]

when \(\text{Re}(s) > 0\).

The series (2.10) can then be rewritten as the function \(Z(s, x)\) defined by
Definition 2.1. For \( x \in (0, 1) \) and for \( s \in \mathbb{C} \), we define the function

\[
Z(s, x) \triangleq \sum_{n=1}^{\infty} \frac{\sigma_n(s, x)}{n+1}.
\]

(2.12)

Uniform and absolute convergence of (2.9) or (2.12) is a direct consequence of the asymptotic estimate of \( \sigma_n(s, x) \) in Proposition 3.6 to be proved later in Section 3. However, we will prove the result using simpler but characteristic estimates when \( \Re(s) > 0 \).

To prove absolute and uniform convergence, it suffices to prove absolute and uniform convergence for the dominant series

\[
\sum_{n=1}^{\infty} \int_0^\infty \frac{(1 - xe^{-t})^{n-1}}{n+1} e^{-t \sigma^{-1}} dt.
\]

(2.13)

A straightforward calculation of the derivative shows that the function \((1 - xe^{-t})^{n-1}e^{-t/2} \) is maximized when \( e^{-t} = \frac{1}{x(2n-1)} \) and that the maximum value is equal to

\[
K = (1 - \frac{1}{2n-1})^{n-1} \frac{1}{\sqrt{x(2n-1)}}.
\]

(2.14)

Hence, for \( n \geq 2 \),

\[
\int_0^\infty (1 - xe^{-t})^{n-1}xe^{-t \sigma^{-1}} dt = \int_0^\infty (1 - xe^{-t})^{n-1}e^{-t/2(1 - xe^{-t}2\sigma^{-1})} dt
\]

\[
\leq K \int_0^\infty xe^{-t/2} \sigma^{-1} dt
\]

\[
= \left( 1 - \frac{1}{x(2n-1)} \right)^{n-1} \frac{x\Gamma(\sigma)}{\sqrt{x(2n-1)(1/2)^\sigma}}
\]

\[
\leq K' \frac{\sqrt{x}}{\sqrt{2n-1}}.
\]

(2.15)

The last inequality implies that each term of the dominating series is bounded by \( K'\sqrt{x}/(n+1)\sqrt{2n-1} \). Therefore, by the the comparison test the series (2.8) is absolutely and uniformly convergent, and can be rewritten as

\[
(s - 1) \operatorname{Li}_s(x) = Z(s, x) + \log x \frac{1}{\Gamma(s)} \int_0^\infty \frac{xe^{-t}}{(1 - xe^{-t})^2} t^{s-1} dt.
\]

(2.16)

The last equation is our new expansion of \( \operatorname{Li}_s(x) \) when \( \Re(s) > 0 \). To extend the definition of (3.11) to all complex numbers \( s, s \neq 1, 2, \cdots \), we can still use Hankel’s contour defined previously to obtain our first result:

Proposition 2.2. For \( x \in (0, 1) \) and for \( s \notin \{1, 2, \cdots\} \),

\[
(s - 1) \operatorname{Li}_s(x) = Z(s, x) + e^{-i\pi s} \log x \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{xe^{-t}}{(1 - xe^{-t})^2} t^{s-1} dt.
\]

(2.17)
3. Some properties of the functions $\sigma_n(s, x)$ and $Z(s, x)$

3.1. The function $\sigma_n(s, x)$

The function $\sigma_n(s, x)$ has been defined in section 2 by the following alternating sum which is valid for any $s \in \mathbb{C}$ and any $x \in \mathbb{C}$:

$$\sigma_n(s, x) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} x^{k+1}(k+1)^{-s}. \quad (3.1)$$

As a function of $x$, $\sigma_n(s, x)$ is a polynomial in $x$ of degree $n$ for any fixed $s$; however, as a function of $s$, it is entire.

We can express the polynomials $\sigma_n(s, x)$ in terms the generalized Stirling numbers of the second kind:

**Proposition 3.1.**

$$\sigma_n(s, x) = -\frac{1}{n} \sum_{j=1}^{n} (-1)^j j! \left\{ \frac{1-s}{j} \right\} \binom{n}{j} x^j (1-x)^{n-j}, \quad (3.2)$$

where where $\left\{ \frac{\alpha}{j} \right\}$ are the generalized Stirling numbers of the second kind defined by

$$\left\{ \frac{\alpha}{j} \right\} = \frac{1}{j!} \sum_{m=1}^{j} \binom{j}{m} m^\alpha. \quad (3.3)$$

**Proof.** We have by definition

$$\sigma_n(s, x) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} x^{k+1}(k+1)^{-s}. \quad (3.4)$$

By making the change of variable, $k = m - 1$, the sum can obviously be put into the form

$$\sigma_n(s, x) = -\frac{1}{n} \sum_{m=1}^{n} \binom{n}{m} (-1)^m x^m m^{1-s}. \quad (3.5)$$

In [1], the following identity was proved for every integer $n$ and every complex numbers $\alpha, x, \alpha \neq 0$

$$\sum_{m=1}^{n} \binom{n}{m} x^m m^\alpha = \sum_{j=1}^{n} \binom{n}{j} j! \left\{ \frac{\alpha}{j} \right\} x^j (1+x)^{n-j}, \quad (3.6)$$

where $\left\{ \frac{\alpha}{j} \right\}$ are the generalized Stirling numbers of the second kind. Applying the identity to (3.5), we get
\[ \sigma_n(s, x) = \frac{-1}{n} \sum_{j=1}^{n} (-1)^j j! \left\{ \frac{1-s}{j} \right\} \binom{n}{j} x^j (1-x)^{n-j}. \] (3.7)

**Remark 3.2.** According to proposition 3.1, \( \sigma_n(s, x) \) can be viewed as a Bernstein polynomial \( B_n(f)(x) \) if one sets the function \( f \) to be such that

\[ f \left( \frac{j}{n} \right) = \frac{-1}{n} (-1)^j j! \left\{ \frac{1-s}{j} \right\}. \] (3.8)

**Remark 3.3.** Obviously, one has

\[ \sigma_n(s, 0) = 0 \] (3.9)
\[ \sigma_n(s, 1) = (-1)^n (n-1)! \left\{ \frac{1-s}{n} \right\}. \] (3.10)

The next proposition gives the asymptotic estimates of \( \sigma_n(s, x) \) when \( n \) is large:

**Proposition 3.4.** For \( \text{Re}(s) > 0 \), \( x \in (0, 1) \) and \( n \) large enough

\[ \sigma_n(s, x) \sim \frac{1}{n \log n} \frac{1}{1-s} \Gamma(s). \] (3.11)

**Proof.** When \( \text{Re}(s) > 0 \), we know from the previous section that \( \sigma_n(s, x) \) can be written as an integral:

\[ \sigma_n(s, x) = \frac{1}{\Gamma(s)} \int_0^\infty (1-xe^{-t})^{n-1} e^{-t} t^{s-1} dt \] (3.12)

when \( \text{Re}(s) > 0 \). Thus, the problem is reduced to find the asymptotic estimates of the following integral

\[ I(n, s) = \int_0^\infty (1-xe^{-t})^{n-1} xe^{-t} t^{s-1} dt. \] (3.13)

We prove the proposition by the method of Laplace. Put \( u = xe^{-t} \), the integral becomes

\[ I(n, s) = \int_0^1 e^{(n-1) \log(1-u)} (- \log u + \log x)^{s-1} du, \] (3.14)

Define \( h(u) = - \log(1-u) \), then \( h'(u) = \frac{1}{1-u} \) and \( h(0) = 0 \). To put the integral in a Laplace integral format, we let \( w = h(u) \). Since \( h(u) \) is increasing on \((0, 1)\) and \( h' > 0 \), then
\[ \int_0^1 e^{-(n-1)(-\log(1-u))} (-\log u + \log x)^{s-1} du = \int_0^{-\log(1-x)} f(w)e^{-(n-1)w} dw, \]  
(3.15)

where

\[ f(w) = \frac{(-\log u + \log x)^{s-1}}{h'(u)} = e^{-w} (-\log(1 - e^{-w}) + \log x)^{s-1} \]  
(3.16)

Now, using the generating function of Bernoulli numbers, we find

\[ \log(1 - e^{-w}) = \log(1 - e^{-1}) - \log w - \frac{1}{2}(1 - w) - \frac{1}{6.2}(1 - w^2) + \cdots \]

\[ = -\log w + O(1), \quad \text{as } w \to 0, \]  
(3.17)

and since \( e^{-w} = O(1) \) as \( w \to 0 \), then we have

\[ f(w) = (-\log w)^{s-1} + O\left(\frac{1}{\log w}\right) \quad \text{as } w \to 0. \]  
(3.18)

There are two cases to consider: \( x \geq 1 - e^{-1} \) and \( x < 1 - e^{-1} \). Let’s first suppose that \( x \geq 1 - e^{-1} \), then the integral (3.15) can be split into two parts:

\[ I(n, s) = \int_0^{1-\epsilon} f(w)e^{-(n-1)w} dw + \int_{1-\epsilon}^{-\log(1-x)} f(w)e^{-(n-1)w} dw, \]  
(3.19)

where \( \epsilon \) is an arbitrarily small positive number.

By the general properties of Laplace integrals, and the fact that \( f(w) = O(e^{-w}) \) for large \( w \), the second integral is exponentially small and verifies

\[ \int_{1-\epsilon}^{-\log(1-x)} f(w)e^{-(n-1)w} dw = O\left(e^\delta(1-\epsilon)^n\right), \]  
(3.20)

where \( \delta \) is an appropriate positive number.

Finally, replacing \( f(w) \) by the estimate (3.18), we get

\[ I(n, s) = \int_0^{1-\epsilon} (-\log w)^{s-1} e^{-(n-1)w} dw + O\left(\frac{1}{(n-1)\log(n-1)}\right). \]  
(3.21)

To obtain an expansion of the first integral, we use the following theorem which has been proved in [12]:

**Theorem 3.5 ([12] Theorem 2, p. 70).** Let \( \mu \) and \( \lambda \) be any complex numbers with \( \text{Re}(\mu) > -1 \) and \( \text{Re}(\lambda) > 0 \) and let \( c = |e|e^{iy}, \quad 0 < |c| < 1 \), then

\[ \int_0^\infty t^{\lambda-1} (\log t)^\mu e^{-zt} dt \sim z^{-\lambda} (\log z)^\mu \sum_{r=0}^\infty (-1)^r \left(\frac{\mu}{r}\right) \Gamma(r) (\log z)^{-r}, \]  
(3.22)
uniformly in \arg(z) as \( z \to \infty \) in \( |\arg(ze^{i\gamma})| \leq \pi/2 - \Delta \) where \( \Delta \) is a small positive number and the path of integration is a straight line joining \( t = 0 \) to \( t = c \).

In our case, \( \lambda = 1, \mu = s - 1 \) and \( z = n - 1 \), therefore the final result is

\[
I(n, s) \sim \frac{(\log n)^{s-1}}{n},
\]

for \( n \) large enough.

The case \( x < 1 - e^{-1} \) can be dealt with in a similar fashion. In this case, the integral (2.16) is equal to

\[
I(n, s) = \int_{c}^{0} f(w)e^{-(n-1)w} \, dw,
\]

with \( c = -\log(1 - x) < 1 \). Again, using the estimate of \( f(w) \) as \( w \to 0 \) and Theorem 3.5, we get

\[
I(n, s) \sim \frac{(\log n)^{s-1}}{n}.
\]

This completes the proof of the proposition.

**Proposition 3.6.** Let \( x \) be in \((0, 1)\).

1. For \( s = -k, k \) a positive integer, \( \sigma_n(-k, x) \) is a polynomial in \( x \) of degree \( n \). Moreover, \( |\sigma_n(-k, x)| \) is bounded above by a fixed constant for all \( n \).
2. For \( s \notin \{0, -1, -2, \cdots\} \) and \( n \) large enough

\[
\sigma_n(s, x) \sim \frac{1}{n(\log n)^{1-s} \Gamma(s)}.
\]

**Proof.**

By looking at the definition of \( \sigma_n(s, x) \), we can see that when \( s = -k, k \) a positive integer, \( \sigma_n(s, x) \) are polynomials in \( x \) of degree \( n \). Now, using proposition 3.1 these polynomials can be rewritten as

\[
\sigma_n(-k, x) = \frac{-1}{n} \sum_{j=1}^{n} (-1)^j j! \binom{k+1}{j} \binom{n}{j} x^j (1 - x)^{n-j}.
\]

The last equation shows that the coefficients of \( \sigma_n(s, x) \) are multiples of Stirling numbers of the second kind. It is well known that \( \{\binom{k+1}{j}\} \) is zero for \( n > k + 1 \). Thus, \( |\sigma_n(-k, x)| \) is bounded above by a fixed constant for every \( n > k + 1 \) and for every bounded \( x \). This proves the first statement.

The second statement of the proposition has been proved in Proposition 3.4 when \( \text{Re}(s) > 0 \). For the remaining values of \( s \), we can either use the method of
Laplace as in our proof of proposition 3.5 or use an elegant result of Flajolet et al. [6] regarding asymptotic expansions of sums of the form \((3.5)\).

The function that defines the alternating sums \(\sigma_n(-k, x)\) is \(x^s z^{1-s}\). Since \(x^s z^{1-s}\) has a non-integral algebraic singularity at \(s_0 = 0\), the proof of [6, Theorem 3] remains valid in its entirety, the only changes that are needed concern the change of variables immediately after equation (15) in [6, p. 119]. Instead of the change of variable \(\zeta = s \log n\) carried out in [6], one needs the change of variable \(\zeta = s \log(nx)\).

Consequently, when \(s\) is nonintegral, \(\sigma_n(s, x)\) has the following asymptotics when \(n\) is large

\[
\sigma_n(s, x) \sim \frac{(\log n + \log x)^{s-1}}{n \Gamma(s)} - \frac{1}{n(\log n)^{1-s} \Gamma(s)} + \log x \frac{s - 1}{n(\log n)^{2-s} \Gamma(s)}. \tag{3.28}
\]

When \(s = k \in \{1, 2, \cdots\}\), the following expansion applies

\[
\sigma_n(s, x) \sim \frac{(\log n)^{k-1}}{n(k-1)!} + \log x \frac{(\log n)^{k-2}}{n(k-2)!}. \tag{3.29}
\]

The asymptotic estimates (3.28) and (3.29) are valid for \(n\) large enough and for all \(s \notin \{0, -1, -2, \cdots\}\). These estimates are in accordance with the estimates of Proposition 3.4.

\[\square\]

3.2. The Function \(Z(s, x)\)

The function \(Z(s, x)\) is a function of two variables defined by

\[
Z(s, x) \triangleq \sum_{n=1}^{\infty} \frac{\sigma_n(s, x)}{n + 1}, \tag{3.30}
\]

where \(s \in \mathbb{C}\), \(x \in (0, 1)\), and

\[
\sigma_n(s, x) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} x^{k+1}(k+1)^{-s}. \tag{3.31}
\]

Clearly, \(Z(s, x)\) is an infinite series of functions that should be uniformly convergent in order to have a useful function. If the variable \(x\) is allowed to lie in the unit interval \(0 < x < 1\) and the variable \(s\) is fixed but allowed to be anywhere in \(\mathbb{C}\), then the series, viewed as a function of \(s\), is uniformly convergent by the Weierstrass M-test. Indeed, we have

**Proposition 3.7.** Suppose that \(0 < x < 1\), then

1. \(Z(s, x)\) is an entire function of \(s\).
2. \(\lim_{x \to 1} Z(s, x) = (s - 1)\zeta(s)\)
Proof.

By Proposition 3.6, the asymptotic estimates of \( \sigma_n(s, x) \) are valid for \( n \) large enough and for all \( s \notin \{0, -1, -2, \ldots \} \). Moreover, for \( s = -k, k \) a positive integer, \( |\sigma_n(-k, x)| \) is bounded above by a fixed constant for all \( n \) and all \( 0 < x < 1 \).

We first prove that \( Z(s, x) \) is well-defined and does not have any singularity when \( \Re(s) > 0 \). Indeed, by the logarithmic test of series our series is dominated by a uniformly convergent series for all finite \( s \) such that \( \Re(s) > 0 \) and \( 0 < x < 1 \). Moreover, \( Z(s, x) \) is finite for finite \( s \) and finite \( x \); therefore, \( Z(s, x) \) does not have any singularity when \( \Re(s) > 0 \). To extend \( Z(s, x) \) outside the domain \( \Re(s) > 0 \), we use Weierstrass theorem of the uniqueness of analytic continuation and repeat the same process for \( \Re(s) > -k, k \in \mathbb{N} \). The final result yields a well-defined function \( Z(s, x) \) with no finite singularity for all \( s \in \mathbb{C} \); hence, \( Z(s, x) \) is an entire function of \( s \).

To prove the second property, we take the limit \( x \to 1 \) in the identity defining \( \sigma_n(s, x) \). It yields the series representation of the Riemann zeta function [3,4]:

\[
(s - 1)\zeta(s) = \sum_{n=1}^{\infty} \frac{S_n(s)}{n+1},
\]

where

\[
S_n(s) = \sum_{k=0}^{n-1} (-1)^k \binom{n-1}{k} (k+1)^{-s}.
\]

This finishes the proof of the proposition.

Remark 3.8. It is the second statement of Proposition 3.7 that makes \( Z(s, x) \) more interesting that \( \text{Li}_s(x) \) in the analysis of the present paper. The reason is simple: \( Z(s, x) \) is convergent when \( x \to 1 \) for all values of \( s \) while \( \text{Li}_s(x) \) is convergent as \( x \) tends 1 only when \( \Re(s) > 1 \). In particular, \( \text{Li}_s(x) \) is divergent when \( x \to 1 \) for any nontrivial zero of the Riemann zeta function.

4. An Expansion of \( Z(s, x) \)

In this section, we suppose again that \( 0 < x < 1 \). To obtain an expansion of

\[
Z(s, x) = (s - 1)\text{Li}_s(x) - \log xe^{-\pi s} \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{xe^{-t}}{(1-xe^{-t})^{s-1}} dt,
\]

we replace \( \text{Li}_s(x) \) by its the integral expression [4] and we expand the contour integrals:

\[
Z(s, x) = (s - 1)e^{-\pi s} \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{xt^{s-1}}{e^t - x} dt - \log xe^{-\pi s} \frac{\Gamma(1-s)}{2\pi i} \int_C \frac{xe^{-t}}{(1-xe^{-t})^{s-1}} dt.
\]
We may first suppose that \( \text{Re}(s) < 0 \). To carry out the integration, we use Riemann’s trick of folding back the contour and integrating the function over the entire plane outside the contour. Of course, in the integration process the poles must be avoided. We leave out the details which can be found in [7].

The first integrand has simple poles at the points \( t = \log x + 2\pi ni, n = 0, \pm 1, \pm 2, \cdots \) and the second integrand has double poles at the points \( t = \log x + 2\pi ni, n = 0, \pm 1, \pm 2, \cdots \). By Cauchy’s theorem, (4.2) becomes

\[
Z(s, x) = \Gamma(1-s) \sum_{n=-\infty}^{\infty} -e^{-i\pi s} \text{Res}_{t=\log x+2\pi ni} \left( (s-1)\frac{xt^{s-1}}{e^t-x} - \log x \frac{xe^{t}s^{-1}}{(e^t-x)^2} \right). \tag{4.3}
\]

Regarding the first residue evaluation without the factor \( s-1 \), we have

\[
\text{Res}_{t=\log x+2\pi ni} \frac{xt^{s-1}}{e^t-x} = \lim_{t \to \log x+2\pi ni} \frac{(t-\log x-2\pi ni)t^{s-1}}{e^t-x} = (\log x + 2\pi ni)^{s-1}. \tag{4.4}
\]

To evaluate the second residue, we appeal to the following lemma

**Lemma 4.1.** Let \( f(z) \) and \( g(z) \) be two analytic functions. Let \( z = a \) be a simple zero of \( g(z) \) and suppose that \( f(a) \neq 0 \). Then,

\[
\text{Res}_{z=a} \left\{ \frac{f(z)}{g(z)^2} \right\} = \frac{f'(a)g'(a) - f(a)g''(a)}{g'''(a)}. \tag{4.5}
\]

**Proof.** Expand the numerator and denominator of \( \frac{f(z)}{g(z)^2} \) into Taylor series around \( z = a \) and identify the coefficient of \( \frac{1}{z-a} \).

In our case, we have \( g(t) = e^t - x, f(t) = xe^{t}s^{-1} \) and \( a = \log x + 2\pi ni \). Lemma 4.1 gives

\[
\text{Res}_{z=a} \left\{ \frac{xe^{t}s^{-1}}{(e^t-x)^2} \right\} = x(s-1)a^{s-2}e^a = x^2(s-1)(\log x + 2\pi ni)^{s-2}. \tag{4.6}
\]

Finally, using the fact that

\[
-e^{-i\pi s}(\log x + 2\pi ni)^{s-1} = (-\log x + 2\pi ni)^{s-1}, \tag{4.7}
\]
\[
-e^{-i\pi s}(\log x + 2\pi ni)^{s-2} = (-\log x + 2\pi ni)^{s-2}, \tag{4.8}
\]

the expansion for \( Z(s, x) \) can be summarized in the following theorem

**Theorem 4.2.** For \( 0 < x < 1 \) and \( s \notin \{1, 2, 3, \cdots \} \),

\[
Z(s, x) = (s-1)\Gamma(1-s) \sum_{n=-\infty}^{\infty} \left[ (-\log x + 2\pi ni)^{s-1} + x^2 \log x(-\log x + 2\pi ni)^{s-2} \right]. \tag{4.9}
\]
To further simplify (4.9), we substitute the following two identities from [2, eq. (8), p. 29]

\[\sum_{n=-\infty}^{\infty} (-\log x - 2\pi ni)^{s-1} = (-\log x)^{s-1}\]

\[\sum_{n=0}^{\infty} \frac{(\log x)^n}{n!} \zeta (s-n),\] (4.10)

\[\sum_{n=-\infty}^{\infty} (-\log x - 2\pi ni)^{s-2} = (-\log x)^{s-2}\]

\[-\frac{1}{(s-1)\Gamma(1-s)} \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!} \zeta (s-n-1)\] (4.11)

into the equation of Theorem 4.2. The expansion of \(Z(s, x)\) simplifies to

\[Z(s, x) = (s-1)\Gamma(1-s)(-\log x)^{s-1}(1-x^2) + (s-1)\sum_{n=0}^{\infty} \frac{(\log x)^n}{n!} \zeta (s-n)\]

\[-x^2 \log x \sum_{n=0}^{\infty} \frac{(\log x)^n}{n!} \zeta (s-n-1).\] (4.12)

We have proved the identity when \(\text{Re}(s) < 0\). The identity is however still valid by the principle of analytic continuation for every \(s\) which is not a pole of \(\Gamma(1-s)\). Put in a more convenient form, expansion (4.12) is summarized in the following theorem:

**Theorem 4.3.** For \(0 < x < 1\) and \(s \notin \{1, 2, 3, \ldots\}\),

\[Z(s, x) = (s-1)\zeta (s) + (s-1)\Gamma(1-s)(1-x^2)(-\log x)^{s-1} - x^2 \log x \zeta (s-1)\]

\[+ \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!} \left[(s-1)\zeta (s-n) - x^2 \log x \zeta (s-n-1)\right].\] (4.13)

When \(s = 1\), Definition 2.1 provides

\[Z(1, x) = \frac{1}{n(n+1)} - \sum_{n=1}^{\infty} \frac{(1-x)^n}{n(n+1)}\]

\[= \frac{x \log x}{1-x},\] (4.14)
and when $s$ is a positive integer different from 1, a similar expansion is still valid as a consequence of the previous theorem:

**Corollary 4.4.** For $0 < x < 1$ and $s = k \in \{2, 3, \ldots\}$,

$$Z(k, x) = (k - 1) \zeta(k) + (1 - x^2) \left( \frac{\log x}{k-1} \right)^{k-1} \left[ H_{k-1} - \log(-\log x) \right]$$

$$+ (k - 1) \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!} \zeta(k - n)$$

$$+ x^2 \sum_{n=1}^{\infty} \frac{(\log x)^{n+1}}{n!} \zeta(k - n - 1),$$

(4.15)

where $H_n$ is the $n$-th harmonic number.

**Proof.**

Let $s = k + \epsilon$, $k \in \{1, 2, 3, \ldots\}$ and $\epsilon$ small positive real number. Theorem 4.2 gives the expansion

$$Z(k + \epsilon, x) = (k - 1 + \epsilon) \zeta(k + \epsilon) + (k - 1 + \epsilon) \Gamma(1 - k - \epsilon) (1 - x^2)(-\log x)^{k-1+\epsilon}$$

$$- x^2 \log x \zeta(k - 1 + \epsilon) + (k - 1 + \epsilon) \zeta(1 + \epsilon) \left( \frac{\log x}{k-1} \right)^{k-1}$$

$$+ (k - 1 + \epsilon) \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!} \zeta(k - n + \epsilon)$$

$$- x^2 \log x \zeta(1 + \epsilon) \left( \frac{\log x}{k-2} \right)^{k-2}$$

$$- x^2 \log x \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!} \zeta(k - n - 1 + \epsilon).$$

(4.16)

Now, we replace $\Gamma(1 - k - \epsilon)$ and $\zeta(1 + \epsilon)$ by their well-known Laurent expansions:

$$\Gamma(1 - k - \epsilon) = -\frac{(-1)^{k-1}}{(k-1)!} \frac{1}{\epsilon} + \frac{(-1)^{k-1}}{(k-1)!} (H_{k-1} - \gamma) + o(\epsilon)$$

(4.17)

$$\zeta(1 + \epsilon) = \frac{1}{\epsilon} + \gamma + o(\epsilon),$$

(4.18)

where

$$H_{k-1} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{k-1},$$

(4.19)

$$\gamma = 0.577215665,$$

(4.20)

we obtain
\[ Z(k + \epsilon, x) = (k - 1)\zeta(k) + (1 - x^2)^{(\log x)^{k-1}} \left[ H_{k-1} - \frac{(-\log x)^\epsilon - 1}{\epsilon} \right] \\
+ (k - 1) \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!} \zeta(k - n) \\
+ x^2 \sum_{n=1}^{\infty} \frac{(\log x)^{n+1}}{n!} \zeta(k - n - 1) + o(\epsilon). \]  

Finally, making \( \epsilon \) tend to zero and recalling that
\[ \lim_{\epsilon \to 0} \frac{(-\log x)^\epsilon - 1}{\epsilon} = \log(-\log x), \]  
we get
\[ Z(k, x) = (k - 1)\zeta(k) + (1 - x^2)^{(\log x)^{k-1}} \left[ H_{k-1} - \log(-\log x) \right] \\
+ (k - 1) \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!} \zeta(k - n) \\
+ x^2 \sum_{n=1}^{\infty} \frac{(\log x)^{n+1}}{n!} \zeta(k - n - 1). \]  

The expansion of \( Z(s, x) \) in Theorem 4.2 or Corollary 4.4 is the counterpart of Lindelöf’s series expansion of the polylogarithm \[8\]. The applications of the expansion of \( Z(s, x) \) are numerous. One important application regarding the zeros of the Riemann zeta function is the subject of the next section.

5. A Necessary Condition for a Nontrivial Zero of the Riemann Zeta Function

Suppose that \( s \) is a nontrivial zero of \( \zeta(s) \). We necessarily have \( 0 < \text{Re}(s) < 1 \). The expansion of Theorem 4.3 becomes
\[ Z(s, x) = (s - 1)\Gamma(1 - s)(1 - x^2)(-\log x)^{s-1} - x^2 \log x \zeta(s - 1) \\
+ \sum_{n=1}^{\infty} \frac{(\log x)^n}{n!} \left[ (s - 1)\zeta(s - n) - x^2 \log x \zeta(s - n - 1) \right]. \]  

We immediately obtain
\[ \lim_{x \to 1} \frac{Z(s, x)}{(1 - x^2)(-\log x)^{s-1}} = (s - 1)\Gamma(1 - s). \]
Thus, we have proved the following important theorem:

**Theorem 5.1.** Let $s$ be a nontrivial zero of $\zeta(s)$ and let $Z(s, x)$ be the function defined by (3.31), then

$$
\lim_{x \to 1} \frac{Z(s, x)}{(1-x)(-\log x)^{s-1}} = 2(s-1)\Gamma(1-s).
$$

We also have an immediate corollary:

**Corollary 5.2.** Let $s$ be a nontrivial zero of $\zeta(s)$ and let $Z(s, x)$ be the function defined by (3.31), then

$$
\lim_{x \to 1} \frac{Z(s, x)}{(1-x)^s} = 2(s-1)\Gamma(1-s).
$$

**Proof.** The proof follows from the fact that

$$
\lim_{x \to 1} \frac{(1-x)^{s-1}}{(-\log x)^{s-1}} = 1,
$$

and Theorem 5.1.

Recalling that $\lim_{x \to 1} Z(s, x) = (s-1)\zeta(s)$, we may conclude that Theorem 5.1 and Corollary 5.2 can have important computational and theoretical applications to the theory of the Riemann zeta function. For example, when $s$ is a nontrivial zero, we can write

$$
Z(s, x) \approx 2(s-1)\Gamma(1-s)(1-x)^s
$$

when $x$ is near 1. Equation (5.6) is a measure of the rate of approach of $Z(s, x)$ to the zero $(s-1)\zeta(s)$ when $x$ is near one. The rate of approach to zero is equal to $2(s-1)\Gamma(1-s)(1-x)^s$.

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