AN INTRODUCTION TO NON-COMMUTATIVE
DIFFERENTIAL GEOMETRY ON QUANTUM GROUPS

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Abstract

We give a pedagogical introduction to the differential calculus on quantum groups by stressing at all stages the connection with the classical case ($q \to 1$ limit). The Lie derivative and the contraction operator on forms and tensor fields are found. A new, explicit form of the Cartan–Maurer equations is presented. The example of a bicovariant differential calculus on the quantum group $GL_q(2)$ is given in detail. The softening of a quantum group is considered, and we introduce $q$-curvatures satisfying $q$-Bianchi identities, a basic ingredient for the construction of $q$-gravity and $q$-gauge theories.
1 Introduction

Quantum groups \([1]-[4]\) have emerged as interesting non-trivial generalizations of Lie groups. The latter are recovered in the limit \(q \to 1\), where \(q\) is a continuous deformation parameter (or set of parameters). In the \(q \neq 1\) case, the \(q\)-group may have, and in general does have, more than one corresponding \(q\)-algebra, the \(q\)-analogue of the Lie algebra. This can be rephrased by saying that the differential calculus on \(q\)-groups is not unique \([5]\). On a classical Lie group one can define a left and right action of the group on itself, and these commute. By imposing this “bicovariance” to hold also in the \(q\)-deformation, one restricts the possible differential calculi (still to a number > 1 in general).

Our motivations for studying the differential geometry of quantum groups are twofold:

– the \(q\)-differential calculus offers a natural scenario for \(q\)-deformations of gravity theories, based on the quantum Poincaré group \([6]\). Space-time becomes non-commutative, a fact that does not contradict (Gedanken) experiments under the Planck length, and that could possibly provide a regularization mechanism \([7, 8]\).

– the quantum Cartan-Maurer equations define \(q\)-curvatures, and these can be used for constructing \(q\)-gauge theories \([9]\). Here space-time can be taken to be the ordinary commutative Minkowski spacetime, while the \(q\)-structure resides on the fibre, the gauge potentials being non-commuting. These theories could offer interesting examples of a novel way of breaking symmetry by \(q\)-deforming the classical one.

In this paper we intend to give an introductory review of the \(q\)-differential calculus. A discussion on Hopf structures will not be omitted: in Section 2 we recognize these structures in ordinary Lie groups and Lie algebras. Quantum groups and their non-commutative geometry are discussed in Sections 3 and 4, and Section 5 describes an explicit construction of a bicovariant calculus on quantum groups. Some new results of Section 5 include a formula that gives the commutations of the left-invariant one-forms, and (as a consequence) a new expression for the Cartan-Maurer equations.

The example of \(GL_q(2)\) [and its restrictions to \(U_q(2)\) and \(SU_q(2)\)] is systematically used to illustrate the general concepts. In Section 6 we study the quantum Lie derivative, an essential tool for the definition of \(q\)-variations. Section 7 extends the notion of “soft” group manifolds (see for example \([10]\)) to \(q\)-groups, by introducing \(q\)-curvatures.

After refs. \([1]\), there have been a number of papers treating the differential calculus on \(q\)-groups from various points of view \([11]-[20]\).

In this paper we use the formalism of refs. \([5]\) and \([17]\).
2 Hopf structures in ordinary Lie groups and Lie algebras

Let us begin by considering $Fun(G)$, the set of differentiable functions from a Lie group $G$ into the complex numbers $\mathbb{C}$. $Fun(G)$ is an algebra with the usual pointwise sum and product $(f + h)(g) = f(g) + h(g)$, $(f \cdot h) = f(g)h(g)$, $(\lambda f)(g) = \lambda f(g)$, for $f, h \in Fun(G)$, $g \in G$, $\lambda \in \mathbb{C}$. The unit of this algebra is $I$, defined by $I(g) = 1$, $\forall g \in G$.

Using the group structure of $g$, we can introduce on $Fun(G)$ three other linear mappings, the coproduct $\Delta$, the counit $\varepsilon$, and the coinverse (or antipode) $\kappa$:

\[
\Delta(f)(g, g') \equiv f(gg'), \quad \Delta : Fun(G) \rightarrow Fun(G) \otimes Fun(G) \tag{2.1}
\]

\[
\varepsilon(f) \equiv f(e), \quad \varepsilon : Fun(G) \rightarrow \mathbb{C} \tag{2.2}
\]

\[
(\kappa f)(g) \equiv f(g^{-1}), \quad \kappa : Fun(G) \rightarrow Fun(G) \tag{2.3}
\]

where $e$ is the unit of $G$. It is not difficult to verify the following properties of the co-structures:

\[
(id \otimes \Delta)\Delta = (\Delta \otimes id)\Delta \quad \text{(coassociativity of $\Delta$)} \tag{2.4}
\]

\[
(id \otimes \varepsilon)\Delta(a) = (\varepsilon \otimes id)\Delta(a) = a \tag{2.5}
\]

\[
m(\kappa \otimes id)\Delta(a) = m(id \otimes \kappa)\Delta(a) = \varepsilon(a)I \tag{2.6}
\]

and

\[
\Delta(ab) = \Delta(a)\Delta(b), \quad \Delta(I) = I \otimes I \tag{2.7}
\]

\[
\varepsilon(ab) = \varepsilon(a)\varepsilon(b), \quad \varepsilon(I) = 1 \tag{2.8}
\]

\[
\kappa(ab) = \kappa(b)\kappa(a), \quad \kappa(I) = I \tag{2.9}
\]

where $a, b \in A = Fun(G)$ and $m$ is the multiplication map $m(a \otimes b) \equiv ab$. The product in $\Delta(a)\Delta(b)$ is the product in $A \otimes A$: $(a \otimes b)(c \otimes d) = ab \otimes cd$.

In general a coproduct can be expanded on $A \otimes A$ as:

\[
\Delta(a) = \sum_i a_i^1 \otimes a_i^2 \equiv a_1 \otimes a_2, \tag{2.10}
\]

where $a_i^1, a_i^2 \in A$ and $a_1 \otimes a_2$ is a short-hand notation we will often use in the sequel. For example for $A = Fun(G)$ we have:

\[
\Delta(f)(g, g') = (f_1 \otimes f_2)(g, g') = f_1(g)f_2(g') = f(gg'). \tag{2.11}
\]

Using (2.11), the proof of (2.4)-(2.6) is immediate.

An algebra $A$ endowed with the homomorphisms $\Delta : A \rightarrow A \otimes A$ and $\varepsilon : A \rightarrow \mathbb{C}$, and the antimorphism $\kappa : A \rightarrow A$ satisfying the properties (2.4)-(2.9) is a Hopf
algebra. Thus $\text{Fun}(G)$ is a Hopf algebra.  

Note that the properties (2.4)-(2.9) imply the relations:

$$\Delta(\kappa(a)) = \kappa(a_2) \otimes \kappa(a_1)$$  \hspace{1cm} (2.12)

$$\varepsilon(\kappa(a)) = \varepsilon(a).$$  \hspace{1cm} (2.13)

Consider now the algebra $A$ of polynomials in the matrix elements $T^a_b$ of the fundamental representation of $G$. The algebra $A$ is said to be freely generated by the $T^a_b$.

It is clear that $A \subset \text{Fun}(G)$, since $T^a_b(g)$ are functions on $G$. In fact every function on $G$ can be expressed as a polynomial in the $T^a_b$ (the reason is that the matrix elements of all irreducible representations of $G$ form a complete basis of $\text{Fun}(G)$, and these matrix elements can be constructed out of appropriate products of $T^a_b(g)$), so that $A = \text{Fun}(G)$. The group manifold $G$ can be completely characterized by $\text{Fun}(G)$, the co-structures on $\text{Fun}(G)$ carrying the information about the group structure of $G$. Thus a classical Lie group can be “defined” as the algebra $A$ freely generated by the (commuting) matrix elements $T^a_b$ of the fundamental representation of $G$, seen as functions on $G$. This definition admits non-commutative generalizations, i.e. the quantum groups discussed in the next Section.

Using the elements $T^a_b$ we can write an explicit formula for the expansion (2.10) or (2.11): indeed (2.1) becomes

$$\Delta(T^a_b)(g, g') = T^a_b(g g') = T^a_c(g) T^c_b(g'),$$ \hspace{1cm} (2.14)

since $T$ is a matrix representation of $G$. Therefore:

$$\Delta(T^a_b) = T^a_c \otimes T^c_b.$$  \hspace{1cm} (2.15)

Moreover, using (2.2) and (2.3), one finds:

$$\varepsilon(T^a_b) = \delta^a_b$$  \hspace{1cm} (2.16)

$$\kappa(T^a_b) = (T^{-1})^a_b.$$  \hspace{1cm} (2.17)

Thus the algebra $A = \text{Fun}(G)$ of polynomials in the elements $T^a_b$ is a Hopf algebra with co-structures defined by (2.13)-(2.17) and (2.7)-(2.9).

Another example of Hopf algebra is given by any ordinary Lie algebra, or more precisely by the universal enveloping algebra of a Lie algebra, i.e. the algebra (with unit $I$) of polynomials in the generators $T_i$ modulo the commutation relations

$$[T_i, T_j] = C_{ij}^k T_k.$$  \hspace{1cm} (2.18)

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1To be precise, $\text{Fun}(G)$ is a Hopf algebra when $\text{Fun}(G \times G)$ can be identified with $\text{Fun}(G) \otimes \text{Fun}(G)$, since only then can one define a coproduct as in (2.1). This is possible for compact $G$.  

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Here we define the co-structures as:

$$\Delta(T_i) = T_i \otimes I + I \otimes T_i \quad \Delta(I) = I \otimes I$$  
(2.19)

$$\varepsilon(T_i) = 0 \quad \varepsilon(I) = 1$$  
(2.20)

$$\kappa(T_i) = -T_i \quad \kappa(I) = I$$  
(2.21)

The reader can check that (2.14)-(2.16) are satisfied.

In general the dual of a (finite-dimensional) Hopf algebra $A$ is a Hopf algebra $A'$, whose structures and co-structures are given, respectively, by the co-structures and structures of $A$, i.e.:

$$\chi_1 \chi_2(a) \equiv (\chi_1 \otimes \chi_2)\Delta(a), \quad \chi_1, \chi_2 \in A'$$  
(2.22)

$$I'(a) \equiv \varepsilon(a) \quad I' = \text{unit of } A'$$  
(2.23)

and:

$$\Delta'(\chi)(a \otimes b) \equiv \chi(ab)$$  
(2.24)

$$\varepsilon'(\chi) \equiv \chi(I)$$  
(2.25)

$$\kappa'(\chi)(a) \equiv \chi(\kappa(a))$$  
(2.26)

3 Quantum groups. The example of $GL_q(2)$

Quantum groups are introduced as non-commutative deformations of the algebra $A = \text{Fun}(G)$ of the previous section [more precisely as non-commutative Hopf algebras obtained by continuous deformations of the Hopf algebra $A = \text{Fun}(G)$]. In the following we consider quantum groups defined as the associative algebras $A$ freely generated by non-commuting matrix entries $T^a_{\phantom{a}b}$ satisfying the relation

$$R^{ab}_{\phantom{ab}ef} T^e_{\phantom{e}c} T^f_{\phantom{f}d} = T^b_{\phantom{b}f} T^a_{\phantom{a}c} R^{cf}_{\phantom{cf}cd}$$  
(3.1)

and some other conditions depending on which classical group we are deforming (see later). The matrix $R$ controls the non-commutativity of the $T^a_{\phantom{a}b}$, and its elements depend continuously on a (in general complex) parameter $q$, or even a set of parameters. For $q \to 1$, the so-called “classical limit”, we have

$$R^{ab}_{\phantom{ab}cd} \xrightarrow{q \to 1} \delta^a_c \delta^b_d, \quad (3.2)$$

i.e. the matrix entries $T^a_{\phantom{a}b}$ commute for $q = 1$, and one recovers the ordinary $\text{Fun}(G)$.

The associativity of $A$ implies a consistency condition on the $R$ matrix, the quantum Yang–Baxter equation:

$$R^{ab}_{\phantom{ab}c1} R^{ac_1}_{\phantom{ac_1}b2} R^{b_2c_2}_{\phantom{b_2c_2}b_3c_3} = R^{b_1c_1}_{\phantom{b_1c_1}b_2c_2} R^{a_1c_2}_{\phantom{a_1c_2}a_2c_3} R^{a_2b_2}_{\phantom{a_2b_2}a_3b_3}, \quad (3.3)$$
For simplicity we rewrite the "RTT" equation (3.1) and the quantum Yang–Baxter equation as

\[ R_{12} T_1 T_2 = T_2 T_1 R_{12} \] (3.4)

\[ R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \] (3.5)

where the subscripts 1, 2 and 3 refer to different couples of indices. Thus \( T_1 \) indicates the matrix \( T^a_{\ b} \), \( T_1 T_1 \) indicates \( T^a_{\ c} T^c_{\ b} \), \( R_{12} T_2 \) indicates \( R^{ab}_{\ cd} T^d_{\ e} \) and so on, repeated subscripts meaning matrix multiplication. The quantum Yang–Baxter equation (3.5) is a condition sufficient for the consistency of the RTT equation (3.4). Indeed the product of three distinct elements \( T^a_{\ b}, T^c_{\ d} \) and \( T^e_{\ f} \), indicated by \( T_1 T_2 T_3 \), can be reordered as \( T_3 T_2 T_1 \) via two different paths:

\[ T_1 T_2 T_3 \rightarrow T_3 T_1 T_2 \rightarrow T_3 T_2 T_1 \] (3.6)

by repeated use of the RTT equation. The relation (3.5) ensures that the two paths lead to the same result.

The algebra \( A \) ("the quantum group") is a non-commutative Hopf algebra whose co-structures are the same of those defined for the commutative Hopf algebra \( Fun(G) \) of the previous section, eqs. (2.15)-(2.17), (2.7)-(2.9).

Let us give the example of \( SL_q(2) \), the algebra freely generated by the elements \( \alpha, \beta, \gamma \) and \( \delta \) of the \( 2 \times 2 \) matrix

\[ T^a_{\ b} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \] (3.7)

satisfying the commutations

\[ \alpha \beta = q \beta \alpha, \quad \alpha \gamma = q \gamma \alpha, \quad \beta \delta = q \delta \beta, \quad \gamma \delta = q \delta \gamma \]

\[ \beta \gamma = \gamma \beta, \quad \alpha \delta - \delta \alpha = (q - q^{-1}) \beta \gamma, \quad q \in \mathbb{C} \] (3.8)

and

\[ \det_q T \equiv \alpha \delta - q \beta \gamma = I. \] (3.9)

The commutations (3.8) can be obtained from (3.1) via the \( R \) matrix

\[ R^{ab}_{\ cd} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix} \] (3.10)

where the rows and columns are numbered in the order 11, 12, 21, 22.
It is easy to verify that the “quantum determinant” defined in (3.9) commutes with \(\alpha, \beta, \gamma\) and \(\delta\), so that the requirement \(\det_q T = I\) is consistent. The matrix inverse of \(T^a_b\) is
\[
(T^{-1})^a_b = (\det_q T)^{-1} \begin{pmatrix} \delta & -q^{-1}\beta \\ -q\gamma & \alpha \end{pmatrix}
\]  
(3.11)

The coproduct, counit and coinverse of \(\alpha, \beta, \gamma\) and \(\delta\) are determined via formulas (2.15)-(2.17) to be:
\[
\Delta(\alpha) = \alpha \otimes \alpha + \beta \otimes \gamma, \quad \Delta(\beta) = \alpha \otimes \beta + \beta \otimes \delta
\]
\[
\Delta(\gamma) = \gamma \otimes \alpha + \delta \otimes \gamma, \quad \Delta(\delta) = \gamma \otimes \beta + \delta \otimes \delta
\]
(3.12)
\[
\varepsilon(\alpha) = \varepsilon(\delta) = 1, \quad \varepsilon(\beta) = \varepsilon(\gamma) = 0
\]
(3.13)
\[
\kappa(\alpha) = \delta, \quad \kappa(\beta) = q^{-1}\beta, \quad \kappa(\gamma) = -q\gamma, \quad \kappa(\delta) = \alpha
\]
(3.14)

Note 1: In general \(\kappa^2 \neq 1\), as can be seen from (3.14). The following useful relation holds [3]:
\[
\kappa^2(T^a_b) = D^c_a T^c_d (D^{-1})^d_b = d^a d^b T^a_b,
\]
(3.15)
where \(D\) is a diagonal matrix, \(D^a_b = d^a d^b\), given by \(d^a = q^{2a-1}\) for the \(q\)-groups \(A_{n-1}\).

Note 2: The commutations (3.8) are compatible with the coproduct \(\Delta\), in the sense that \(\Delta(\alpha \beta) = q \Delta(\beta \alpha)\) and so on. In general we must have
\[
\Delta(R_{12}T_1 T_2) = \Delta(T_2 T_1 R_{12}),
\]
(3.16)
which is easily verified using \(\Delta(R_{12}T_1 T_2) = R_{12} \Delta(T_1) \Delta(T_2)\) and \(\Delta(T_1) = T_1 \otimes T_1\). This is equivalent to proving that the matrix elements of the matrix product \(T_1 T'_1\), where \(T'\) is a matrix [satisfying (3.4)] whose elements commute with those of \(T^a_b\), still obey the commutations (3.4).

Note 3: \(\Delta(\det_q T) = \det_q T \otimes \det_q T\) so that the coproduct property \(\Delta(I) = I \otimes I\) is compatible with \(\det_q T = I\).

Note 4: Other conditions compatible with the RTT relation can be imposed on \(T^a_b\):

i) Unitarity condition: \(T'^{\dagger} = T^{-1} \Rightarrow \bar{\alpha} = \delta, \quad \bar{\beta} = -q\gamma, \quad \bar{\gamma} = -q^{-1}\beta, \quad \bar{\delta} = \alpha\), where \(q\) is a real number and the bar denotes an involution, the \(q\)-analogue of complex conjugation, satisfying \((\alpha \beta) = \bar{\beta} \bar{\alpha}\) etc. Restricts \(SL_q(2)\) to \(SU_q(2)\).

ii) Reality condition: \(T = T \Rightarrow \bar{\alpha} = \alpha, \quad \bar{\beta} = \beta, \quad \bar{\gamma} = \gamma, \quad \bar{\delta} = \delta, \quad |q| = 1\). Restricts \(SL_q(2)\) to \(SL_q(2,\mathbb{R})\).

iii) The \(q\)-analogue of orthogonal and symplectic groups can also be defined, see [3].
Note 5: The condition (3.9) can be relaxed. Then we have to include the central element \( \zeta = (\det_q T)^{-1} \) in \( A \), so as to be able to define the inverse of the \( q \)-matrix \( T^a_b \) as in (3.11), and the coinverse of the element \( T^a_b \) as in (2.17). The \( q \)-group is then \( GL_q(2) \), and the unitarity condition restricts it to \( U_q(2) \). The reader can deduce the co-structures on \( \zeta \): \( \Delta(\zeta) = \zeta \otimes \zeta \), \( \varepsilon(\zeta) = 1 \), \( \kappa(\zeta) = \det_q T \).

Note 6: More generally, the quantum determinant of \( n \times n \) \( q \)-matrices is defined by \( \det_q T = \sum_{\sigma} (-q)^{l(\sigma)} T^1_{\sigma(1)} \cdots T^n_{\sigma(n)} \), where \( l(\sigma) \) is the minimal number of inversions in the permutation \( \sigma \). Then \( \det_q T = 1 \) restricts \( GL_q(n) \) to \( SL_q(n) \).

Note 7: We recall the important relations [3] for the \( \hat{R} \) matrix defined by \( \hat{R}^{ab}_{\ cd} \equiv R^{ba}_{\ cd} \), whose \( q \to 1 \) limit is the permutation operator \( \delta^a_d \delta^b_c \):

\[
\hat{R}^2 = (q - q^{-1}) \hat{R} + I, \quad \text{for } A_{n-1} \quad \text{(Hecke condition)} \quad (3.17)
\]

\[
(\hat{R} - qI)(\hat{R} + q^{-1}I)(\hat{R} - q^{1-N}I) = 0, \quad \text{for } B_n, C_n, D_n, \quad (3.18)
\]

with \( N = 2n + 1 \) for the series \( B_n \) and \( N = 2n \) for \( C_n \) and \( D_n \). Moreover for all \( A, B, C, D \) \( q \)-groups the \( R \) matrix is lower triangular and satisfies:

\[
(R^{-1})^{ab}_{\ cd}(q) = R^{ab}_{\ cd}(q^{-1}) \quad (3.19)
\]

\[
R^{ab}_{\ cd} = R^{dc}_{\ ba}. \quad (3.20)
\]

4 Differential calculus on quantum groups

In this section we give a short review of the bicovariant differential calculus on \( q \)-groups as developed by Woronowicz [5]. The \( q \to 1 \) limit will constantly appear in our discussion, so as to make clear which classical structure is being \( q \)-generalized.

Consider the algebra \( A \) of the preceding section, i.e. the algebra freely generated by the matrix entries \( T^a_b \), modulo the relations (3.11) and possibly some reality or orthogonality conditions.

A first-order differential calculus on \( A \) is then defined by

i) a linear map \( d: A \to \Gamma \), satisfying the Leibniz rule

\[
d(ab) = (da)b + a(db), \quad \forall a, b \in A; \quad (4.1)
\]

\( \Gamma \) is an appropriate bimodule (see for example [21]) on \( A \), which essentially means that its elements can be multiplied on the left and on the right by elements of \( A \), and \( q \)-generalizes the space of 1-forms on a Lie group;

ii) the possibility of expressing any \( \rho \in \Gamma \) as

\[
\rho = a_k db_k \quad (4.2)
\]
for some $a_k, b_k$ belonging to $A$.

**Left- and right-covariance**

The first-order differential calculus $(\Gamma, d)$ is said to be **left- and right-covariant** if we can consistently define a left and right action of the $q$-group on $\Gamma$ as follows

\[
\begin{align*}
\Delta_L(ab) &= \Delta(a)(id \otimes d)\Delta(b), \quad \Delta_L : \Gamma \to A \otimes \Gamma \quad \text{(leftcovariance)} \quad (4.3) \\
\Delta_R(ab) &= \Delta(a)(d \otimes id)\Delta(b), \quad \Delta_R : \Gamma \to \Gamma \otimes A \quad \text{(rightcovariance)} \quad (4.4)
\end{align*}
\]

How can we understand these left and right actions on $\Gamma$ in the $q \to 1$ limit? The first observation is that the coproduct $\Delta$ on $A$ is directly related, for $q = 1$, to the pullback induced by left multiplication of the group on itself

\[L_{xy} \equiv xy, \quad \forall x, y \in G. \quad (4.5)\]

This induces the left action (pullback) $L^*_x$ on the functions on $G$:

\[L^*_x f(y) \equiv f(xy)|_y, \quad L^*_x : Fun(G) \to Fun(G) \quad (4.6)\]

where $f(xy)|_y$ means $f(xy)$ seen as a function of $y$. Let us introduce the mapping $L^*$ defined by

\[(L^* f)(x, y) \equiv (L^*_x f)(y) = f(xy)|_y \quad (4.7)\]

The coproduct $\Delta$ on $A$, when $q = 1$, reduces to the mapping $L^*$. Indeed, considering $T^a(b)(y)$ as a function on $G$, we have:

\[L^*(T^a_b)(x, y) = L^*_x T^a_b(y) = T^a_b(xy) = T^a_c(x)T^c_b(y), \quad (4.8)\]

since $T^a_b$ is a representation of $G$. Therefore

\[L^*(T^a_b) = T^a_c \otimes T^c_b \quad (4.9)\]

and $L^*$ is seen to coincide with $\Delta$, cf. (2.13).

The pullback $L^*_x$ can also be defined on 1-forms $\rho$ as

\[(L^*_x \rho)(y) \equiv \rho(xy)|_y \quad (4.10)\]

and here too we can define $L^*$ as

\[(L^* \rho)(x, y) \equiv (L^*_x \rho)(y) = \rho(xy)|_y. \quad (4.11)\]

In the $q = 1$ case we are now discussing, the left action $\Delta_L$ coincides with this mapping $L^*$ for 1-forms. Indeed for $q = 1$

\[
\begin{align*}
\Delta_L(ab)(x, y) &= [\Delta(a)(id \otimes d)\Delta(b)](x, y) = [(a_1 \otimes a_2)(id \otimes d)(b_1 \otimes b_2)](x, y) \\
&= [a_1 b_1 \otimes a_2 db_2](x, y) = a_1(x)b_1(x)a_2(y)db_2(y) = a_1(x)a_2(y)db_2(y) = L^*(a)(x, y)db_2(y)[L^*(b)(x, y)] = a(xy)db(x)|_y. \quad (4.12)
\end{align*}
\]
On the other hand:

\[ L^*(adb)(x, y) = a(xy)db(xy)|_y, \quad (4.13) \]

so that \( \Delta_L \to L^* \) when \( q \to 1 \). In the last equation we have used the well-known property \( L^*_x(adb) = L^*_x(a)L^*_x(db) = L^*_x(a)dL^*_x(b) \) of the classical pullback. A similar discussion holds for \( \Delta_R \), and we have \( \Delta_R \to R^* \) when \( q \to 1 \), where \( R^* \) is defined via the pullback \( R^*_x \) on functions (0-forms) or on 1-forms induced by the right multiplication:

\[ R_xy = yx, \quad \forall x, y \in G \quad (4.14) \]

\[ (R^*_x\rho)(y) = \rho(yx)|_y \quad (4.15) \]

\[ (R^*_x\rho)(y, x) \equiv (R^*_x\rho)(y) \quad (4.16) \]

These observations explain why \( \Delta_L \) and \( \Delta_R \) are called left and right actions of the quantum group on \( \Gamma \) when \( q \neq 1 \).

From the definitions (4.3) and (4.4) one deduces the following properties \([5]\):

\[ (\varepsilon \otimes id)\Delta_L(\rho) = \rho, \quad (id \otimes \varepsilon)\Delta_R(\rho) = \rho \quad (4.17) \]

\[ (\Delta \otimes id)\Delta_L = (id \otimes \Delta_L)\Delta_L, \quad (id \otimes \Delta)\Delta_R = (\Delta_R \otimes id)\Delta_R \quad (4.18) \]

**Bicovariance**

The left- and right-covariant calculus is said to be bicovariant when

\[ (id \otimes \Delta_R)\Delta_L = (\Delta_L \otimes id)\Delta_R, \quad (4.19) \]

which is the \( q \)-analogue of the fact that left and right actions commute for \( q = 1 \)

\( (L^*_xR^*_y = R^*_yL^*_x). \)

**Left- and right-invariant \( \omega \)**

An element \( \omega \) of \( \Gamma \) is said to be left-invariant if

\[ \Delta_L(\omega) = I \otimes \omega \quad (4.20) \]

and right-invariant if

\[ \Delta_R(\omega) = \omega \otimes I. \quad (4.21) \]

This terminology is easily understood: in the classical limit,

\[ L^*\omega = I \otimes \omega \quad (4.22) \]

\[ R^*\omega = \omega \otimes I \quad (4.23) \]

indeed define respectively left- and right-invariant 1-forms.
Proof: the classical definition of left-invariance is

\[(L^*_x \omega)(y) = \omega(y)\]  

(4.24)

or, in terms of \(L^*\),

\[(L^* \omega)(x, y) = L^*_x \omega(y) = \omega(y).\]  

(4.25)

But

\[(I \otimes \omega)(x, y) = I(x)\omega(y) = \omega(y),\]  

(4.26)

so that

\[L^* \omega = I \otimes \omega\]  

(4.27)

for left-invariant \(\omega\). A similar argument holds for right-invariant \(\omega\).

Consequences

For any bicovariant first-order calculus one can prove the following:

i) Any \(\rho \in \Gamma\) can be uniquely written in the form:

\[
\rho = a_i \omega^i \quad (4.28)
\]

\[
\rho = \omega^i b_i \quad (4.29)
\]

with \(a_i, b_i \in A\), and \(\omega^i\) a basis of \(\text{inv}^\Gamma\), the linear subspace of all left-invariant elements of \(\Gamma\). Thus, as in the classical case, the whole of \(\Gamma\) is generated by a basis of left invariant \(\omega^i\). An analogous theorem holds with a basis of right invariant elements \(\eta^i \in \Gamma_{\text{inv}}\). Note that in the quantum case we have \(a \omega^i \neq \omega^i a\) in general, the bimodule structure of \(\Gamma\) being non-trivial for \(q \neq 1\).

ii) There exist linear functionals \(f^i_j\) on \(A\) such that

\[
\omega^i b = (f^i_j * b) \omega^j \equiv (id \otimes f^i_j) \Delta(b) \omega^j \quad (4.30)
\]

\[
a \omega^i = \omega^j [(f^i_j \circ \kappa^{-1}) * a] \quad (4.31)
\]

for any \(a, b \in A\). In particular,

\[
\omega^i T^a_b = (id \otimes f^i_j)(T^a_c \otimes T^c_b) \omega^j = T^a_c f^i_j (T^c_b) \omega^j. \quad (4.32)
\]

Once we have the functionals \(f^i_j\), we know how to commute elements of \(A\) through elements of \(\Gamma\). The \(f^i_j\) are uniquely determined by (4.30) and for consistency must satisfy the conditions:

\[
f^i_j(ab) = f^i_k(a) f^k_j(b) \quad (4.33)
\]

\[
f^i_j(I) = \delta^i_j \quad (4.34)
\]

\[
(f^k_j \circ \kappa) f^i_j = \delta^k_i \epsilon; \quad f^k_j (f^i_j \circ \kappa) = \delta^i_k \epsilon, \quad (4.35)
\]
so that their coproduct, counit and coinverse are given by:

\[ \Delta'(f^i_j) = f^i_k \otimes f^k_j \]  
\[ \varepsilon'(f^i_j) = \delta^i_j \]  
\[ \kappa'(f^k_j f^j_i) = \delta^k_i \varepsilon = f^k_j \kappa'(f^j_i) \]  

(4.36)  
(4.37)  
(4.38)

cf. (2.24)-(2.26). Note that in the \( q = 1 \) limit \( f^i_j \rightarrow \delta^i_j \varepsilon \), i.e. \( f^i_j \) becomes proportional to the identity functional \( \varepsilon(a) = a(e) \), and formulas (4.30), (4.31) become trivial, e.g. \( \omega^i b = b \omega^i \) [use \( \varepsilon^* a = a \) from (2.5)].

iii) There exists an adjoint representation \( M_j^i \) of the quantum group, defined by the right action on the (left-invariant) \( \omega^i \):

\[ \Delta_R(\omega^i) = \omega^j \otimes M_j^i, \quad M_j^i \in A. \]  

(4.39)

It is easy to show that \( \Delta_R(\omega^j) \) belongs to \( \text{inv} \Gamma \otimes A \), which proves the existence of \( M_j^i \). In the classical case, \( M_j^i \) is indeed the adjoint representation of the group.

We recall that in this limit the left-invariant 1-form \( \omega^i \) can be constructed as

\[ \omega^i(y)T_i = (y^{-1} dy)^i T_i, \quad y \in G. \]  

(4.40)

Under right multiplication by a (constant) element \( x \in G : y \rightarrow yx \) we have:

\[
\omega^i(yx)T_i = [x^{-1} - 1 d(yx)]^i T_i = [x^{-1}(y^{-1} dy)x]^i T_i \\
= [x^{-1} T_j x]^i (y^{-1} dy)^j T_i = M_j^i(x) \omega^j(y) T_i,
\]

(4.41)  
(4.42)

so that

\[ \omega^i(yx) = \omega^j(y) M_j^i(x) \]  

(4.43)

or

\[ R^* \omega^i(y, x) = \omega^j \otimes M_j^i(y, x), \]  

(4.44)

which reproduces (4.39) for \( q = 1 \).

The co-structures on the \( M_j^i \) can be deduced:

\[ \Delta(M_j^i) = M_j^l \otimes M_l^i \]  
\[ \varepsilon(M_j^i) = \delta^i_j \]  
\[ \kappa(M_l^i) M_j^l = \delta^j_i = M_l^i \kappa(M_l^j). \]  

(4.45)  
(4.46)  
(4.47)

For example, in order to find the coproduct (4.45) it is sufficient to apply \((id \otimes \Delta)\) to both members of (1.39) and use the second of eqs.(1.18).

The elements \( M_j^i \) can be used to build a right-invariant basis of \( \Gamma \). Indeed the \( \eta^i \) defined by

\[ \eta^i \equiv \omega^j \kappa(M_j^i) \]  

(4.48)

\(^2\)Recall the \( q = 1 \) definition of the adjoint representation \( x^{-1} T_j x \equiv M_j^i(x) T_i \).
are a basis of $\Gamma$ (every element of $\Gamma$ can be uniquely written as $\rho = \eta^i b_i$) and their right-invariance can be checked directly:

$$\Delta_R(\eta^i) = \Delta_R(\omega^j)\Delta[\kappa(M_j^i)] =$$

$$[\omega^k \otimes M_k^j][\kappa(M_s^i) \otimes \kappa(M_j^s)] = \omega^k \kappa(M_s^i) \otimes \delta_s^i I = \eta^i \otimes I \quad (4.49)$$

It can be shown that the functionals $f^i_j$ previously defined satisfy:

$$\eta^i b = (b \ast f^i_j \circ \kappa^{-2}) \eta^j \quad (4.50)$$

$$a \eta^i = \eta^j[a \ast (f^i_j \circ \kappa^{-1})] \quad (4.51)$$

where $a \ast f \equiv (f \otimes id)\Delta(a), \; f \in A'$.

Moreover, from the last of these relations, using (4.48) and (4.31) one can prove the relation

$$M_j^i (a \ast f^i_k) = (f^j_k \ast a) M_i^k \quad (4.52)$$

with $a \ast f^i_j \equiv (f^i_k \otimes id)\Delta(a)$.

iv) An exterior product, compatible with the left and right actions of the $q$-group, can be defined by a bimodule automorphism $\Lambda$ in $\Gamma \otimes \Gamma$ that generalizes the ordinary permutation operator:

$$\Lambda(\omega^i \otimes \eta^j) = \eta^j \otimes \omega^i \quad (4.53)$$

where $\omega^i$ and $\eta^j$ are respectively left and right invariant elements of $\Gamma$. Bimodule automorphism means that

$$\Lambda(a \tau) = a \Lambda(\tau) \quad (4.54)$$

$$\Lambda(\tau b) = \Lambda(\tau)b \quad (4.55)$$

for any $\tau \in \Gamma \otimes \Gamma$ and $a, b \in A$. The tensor product between elements $\rho, \rho' \in \Gamma$ is defined to have the properties $\rho a \otimes \rho' = \rho \otimes a \rho'$, $a(\rho \otimes \rho') = (a \rho) \otimes \rho'$ and $(\rho \otimes \rho') a = \rho \otimes (\rho' a)$. Left and right actions on $\Gamma \otimes \Gamma$ are defined by:

$$\Delta_L(\rho \otimes \rho') \equiv \rho_1 \rho'_1 \otimes \rho_2 \otimes \rho'_2 \quad \Delta_L : \Gamma \otimes \Gamma \to A \otimes \Gamma \otimes \Gamma \quad (4.56)$$

$$\Delta_R(\rho \otimes \rho') \equiv \rho_1 \rho'_1 \otimes \rho_2 \rho'_2 \quad \Delta_R : \Gamma \otimes \Gamma \to \Gamma \otimes \Gamma \otimes A \quad (4.57)$$

where as usual $\rho_1, \rho_2, \text{etc.}$, are defined by

$$\Delta_L(\rho) = \rho_1 \otimes \rho_2, \; \rho_1 \in A, \; \rho_2 \in \Gamma \quad (4.58)$$

$$\Delta_R(\rho) = \rho_1 \otimes \rho_2, \; \rho_1 \in \Gamma, \; \rho_2 \in A \quad (4.59)$$

More generally, we can define the action of $\Delta_L$ on $\Gamma \otimes \Gamma \otimes \cdots \otimes \Gamma$ as

$$\Delta_L(\rho \otimes \rho' \otimes \cdots \otimes \rho'') \equiv \rho_1 \rho'_1 \cdots \rho_1'' \otimes \rho_2 \otimes \rho'_2 \otimes \cdots \otimes \rho_2'' \quad (4.60)$$

$$\Delta_L : \Gamma \otimes \Gamma \otimes \cdots \otimes \Gamma \to A \otimes \Gamma \otimes \Gamma \otimes \cdots \otimes \Gamma$$
\[ \Delta_R(\rho \otimes \rho' \otimes \cdots \otimes \rho'') \equiv \rho_1 \otimes \rho'_1 \otimes \rho''_1 \otimes \rho_2 \rho'_2 \otimes \cdots \]

\[ \Delta_R : \Gamma \otimes \Gamma \otimes \cdots \otimes \Gamma \rightarrow \Gamma \otimes \Gamma \otimes \cdots \otimes \Gamma \otimes A. \quad (4.61) \]

Left-invariance on \( \Gamma \otimes \Gamma \) is naturally defined as \( \Delta_L(\rho \otimes \rho') = I \otimes \rho \otimes \rho' \) (similar definition for right-invariance), so that, for example, \( \omega^i \otimes \omega^j \) is left-invariant, and is in fact a left-invariant basis for \( \Gamma \otimes \Gamma \).

- In general \( \Lambda^2 \neq 1 \), since \( \Lambda(\eta^i \otimes \omega^j) \) is not necessarily equal to \( \omega^i \otimes \eta^j \). By linearity, \( \Lambda \) can be extended to the whole of \( \Gamma \otimes \Gamma \).

- \( \Lambda \) is invertible and commutes with the left and right action of \( q \)-group \( G \), i.e. \( \Delta_L \Lambda(\rho \otimes \rho') = (id \otimes \Lambda) \Delta_L(\rho \otimes \rho') = \rho_1 \rho'_1 \otimes \Lambda(\rho_2 \otimes \rho'_2) \), and similar for \( \Delta_R \).

Then we see that \( \Lambda(\omega^i \otimes \omega^j) \) is left-invariant, and therefore can be expanded on the left-invariant basis \( \omega^k \otimes \omega^l \):

\[ \Lambda(\omega^i \otimes \omega^j) = \Lambda^{ij}_{kl} \omega^k \otimes \omega^l. \quad (4.62) \]

- From the definition \((4.53)\) one can prove that \( \Lambda^{ij}_{kl} = f^i_j (M^j_k) \); (4.63)

thus the functionals \( f^i_j \) and the elements \( M^j_k \in A \) characterizing the bimodule \( \Gamma \) are dual in the sense of eq. \((4.63)\) and determine the exterior product:

\[ \rho \wedge \rho' \equiv \rho \otimes \rho' - \Lambda(\rho \otimes \rho') \]

\[ \omega^i \wedge \omega^j \equiv \omega^i \otimes \omega^j - \Lambda^{ij}_{kl} \omega^k \otimes \omega^l. \quad (4.65) \]

Notice that, given the tensor \( \Lambda^{ij}_{kl} \), we can compute the exterior product of any \( \rho, \rho' \in \Gamma \), since any \( \rho \in \Gamma \) is expressible in terms of \( \omega^i \) [cf. \((4.28), (4.29)\)]. The classical limit of \( \Lambda^{ij}_{kl} \) is

\[ \Lambda^{ij}_{kl} \xrightarrow{q \rightarrow 1} \delta_i^j \delta_k^l \]

since \( f^i_j \xrightarrow{q \rightarrow 1} \delta_i^j \varepsilon \) and \( \varepsilon(M^k_j) = \delta_k^j \). Thus in the \( q = 1 \) limit the product defined in \((4.65)\) coincides with the usual exterior product.

From the property \((4.54)\) applied to the case \( \tau = \omega^i \otimes \omega^j \), one can derive the relation

\[ \Lambda^{nm}_{ij} f^i_q f^j_p = f^m_i f^n_j \Lambda^{ij}_{pq}. \quad (4.67) \]

Applying both members of this equation to the element \( M^s_r \) yields the quantum Yang–Baxter equation for \( \Lambda \):

\[ \Lambda^{nm}_{ij} \Lambda^{ik}_{rp} \Lambda^{js}_{kq} = \Lambda^{nk}_{ri} \Lambda^{ms}_{kj} \Lambda^{ij}_{pq}, \quad (4.68) \]

which is sufficient for the consistency of \((4.67)\).

Taking \( a = M^q_p \) in \((4.52)\), and using \((4.63)\), we find the relation

\[ M^{i}_r M^{q}_p \Lambda^{ir}_{pk} = \Lambda^{iq}_{rl} M^{r}_p M^{i}_k \quad (4.69) \]
and, defining
\[ R^{ij}_{kl} \equiv \Lambda^{ij}_{kl}, \] (4.70)
we see that \( M_{ij} \) satisfies a relation identical to the “RTT” equation (3.1) for \( T^a_{b} \), and that \( R^{ij}_{kl} \) satisfies the quantum Yang–Baxter equation (3.3), sufficient for the consistency of (4.69). The range of the indices is different, since \( i, j, \ldots \) are adjoint indices whereas \( a, b, \ldots \) are in the fundamental representation of \( G_q \).

v) Having the exterior product we can define the exterior differential
\[ d : \Gamma \rightarrow \Gamma \wedge \Gamma \] (4.71)
\[ d(a_k db_k) = da_k \wedge db_k, \] (4.72)
which can easily be extended to \( \Gamma^{\wedge n} \) \((d : \Gamma^{\wedge n} \rightarrow \Gamma^{\wedge(n+1)}, \Gamma^{\wedge n} \) being defined as in the classical case but with the quantum permutation operator \( \Lambda \) [5]) and has the following properties:
\[ d(\theta \wedge \theta') = d\theta \wedge \theta' + (-1)^k \theta \wedge d\theta' \] (4.73)
\[ d(d\theta) = 0 \] (4.74)
\[ \Delta_L(d\theta) = (id \otimes d)\Delta_L(\theta) \] (4.75)
\[ \Delta_R(d\theta) = (d \otimes id)\Delta_R(\theta), \] (4.76)
where \( \theta \in \Gamma^{\wedge k}, \theta' \in \Gamma^{\wedge n} \). The last two properties express the fact that \( d \) commutes with the left and right action of the quantum group, as in the classical case.

vi) The space dual to the left-invariant subspace \( \text{inv} \Gamma \) can be introduced as a linear subspace of \( A' \), whose basis elements \( \chi_i \in A' \) are defined by
\[ da = (\chi_i \ast a)\omega^i, \quad \forall a \in A. \] (4.77)
In order to reproduce the classical limit
\[ da = \frac{\partial}{\partial y^\mu}a(y)dy^\mu = \left(\frac{\partial}{\partial y^\mu}a\right)e^\mu_i(y)e^i_\nu(y)dy^\nu = \left(\frac{\partial}{\partial y^\mu}a\right)e^\mu_i(y)\omega^i(y), \] (4.78)
where \( e^i_\nu(y) \) is the vielbein of the group manifold \((\text{and } e^\mu_i \text{ is its inverse})\), we must require
\[ \chi_i(a) \xrightarrow{q \rightarrow 1} \frac{\partial}{\partial x^i}a(x)|_{x=e}. \] (4.79)
Indeed, for \( q = 1 \) we have
\[ (\chi_i \ast a)(y) = (id \otimes \chi_i)L^*(a)(y) = (id \otimes \chi_i)(a_1 \otimes a_2)(y) = a_1(y)\chi_i(a_2) = a_1(y)[\frac{\partial}{\partial x^i}a_2(x)]|_{x=e} = \frac{\partial}{\partial x^i}[a_1(y)a_2(x)]|_{x=e} = \frac{\partial}{\partial x^i}[(L^*a)(y, x)]|_{x=e} = \frac{\partial}{\partial x^i}[a(yx)]|_{x=e} = \frac{\partial}{\partial (yx)^\mu}[a(yx)]|_{x=e} \frac{\partial}{\partial x^i}(yx)^\mu|_{x=e} = (\frac{\partial}{\partial y^\mu}a(y))e^\mu_i(y) \] (4.80)
and we recover (4.78). In other words

\[
\chi_i * a \xrightarrow{q \to 1} \frac{\partial}{\partial y^\mu} a(y) e^\mu_i \equiv \partial_i a(y),
\]

so that \(\chi_i * a\) is the \(q\)-analogue of left-invariant vector fields, while \(\chi_i\) is the \(q\)-analogue of the tangent vector at the origin \(e\) of \(G\).

vii) The \(\chi_i\) functionals close on the “quantum Lie algebra”:

\[
\chi_i \chi_j - \Lambda_{kl}^{ij} \chi_k \chi_l = C_{ij}^k \chi_k,
\]

with \(\Lambda_{kl}^{ij}\) as given in (4.63). The product \(\chi_i \chi_j\) is defined by

\[
\chi_i \chi_j \equiv (\chi_i \otimes \chi_j) \Delta
\]

and sometimes indicated by \(\chi_i * \chi_j\). Note that this \(*\) product (called also convolution product) is associative:

\[
\chi_i * (\chi_j * \chi_k) = (\chi_i * \chi_j) * \chi_k
\]

(4.84)

\[
\chi_i * (\chi_j * a) = (\chi_i * \chi_j) * a, \quad a \in A.
\]

(4.85)

We leave the easy proof to the reader. The \(q\)-structure constants \(C_{ij}^k\) are given by

\[
C_{ij}^k = \chi_j (M^k_i).
\]

(4.86)

This last equation is easily seen to hold in the \(q = 1\) limit, since the \((\chi_j)_i^k \equiv C_{ij}^k\) are indeed in this case the infinitesimal generators of the adjoint representation:

\[
M^k_i = \delta^k_i + C_{ij}^k x^j + 0(x^2).
\]

(4.87)

Using \(\chi_j \xrightarrow{q \to 1} \frac{\partial}{\partial y^\mu}|_{x = e}\) indeed yields (4.86).

By applying both sides of (4.82) to \(M^s_r \in A\), we find the \(q\)-Jacobi identities:

\[
C_{ri}^n C_{nj}^s - \Lambda_{kl}^{ij} C_{rk}^n C_{nl}^s = C_{ij}^k C_{rl}^s,
\]

(4.88)

which give an explicit matrix realization (the adjoint representation) of the generators \(\chi_i\):

\[
(\chi_i)_k^l = \chi_i (M^l_k) = C_{ki}^l.
\]

(4.89)

Note that the \(q\)-Jacobi identities (4.88) can also be given in terms of the \(q\)-Lie algebra generators \(\chi_i\) as :

\[
[[\chi_r, \chi_i], \chi_j] - \Lambda_{kl}^{ij} [[\chi_r, \chi_k], \chi_l] = [\chi_r, [\chi_i, \chi_j]],
\]

(4.90)

where

\[
[\chi_i, \chi_j] \equiv \chi_i \chi_j - \Lambda_{kl}^{ij} \chi_k \chi_l
\]

(4.91)

is the deformed commutator of eq. (4.82).
viii) The left-invariant $\omega^i$ satisfy the $q$-analogue of the Cartan-Maurer equations:

$$d\omega^i + C_{jk}^i \omega^j \wedge \omega^k = 0,$$

where

$$C_{jk}^i \equiv \chi_j \chi_k(x^i)$$

$$\chi_i(x^k) \equiv \delta^i_k.$$  \hspace{1cm} (4.93)

The $x^k \in A$ defined in (4.94) are called the “coordinates of $G_q$” and satisfy $\varepsilon(x^i) = 0$, which is the $q$-analogue of the fact that classically they vanish at the origin of $G$ [recall that $\varepsilon(x^i) \xrightarrow{q \to 1} x^i(e)$]. Such $x^i$ can always be found \[4\]. Note that $\chi_i \ast x^j$ is the $q$-analogue of the inverse vielbein.

The structure constants $C$ satisfy the Jacobi identities obtained by taking the exterior derivative of (4.92):

$$(C_{jk}^i C_{rs}^j - C_{rj}^i C_{sk}^j) \omega^r \wedge \omega^s \wedge \omega^k = 0.$$ \hspace{1cm} (4.95)

In the $q = 1$ limit, $\omega^j \wedge \omega^k$ becomes antisymmetric in $j$ and $k$, and we have

$$C_{jk}^i \xrightarrow{q \to 1} = \frac{1}{2}(\chi_j \chi_k - \chi_k \chi_j)(x^i) = \frac{1}{2}C_{jk}^l \chi_l(x^i) = \frac{1}{2}C_{jk}^i,$$ \hspace{1cm} (4.96)

where $C_{jk}^l$ are now the classical structure constants. Thus when $q = 1$ we have $C_{jk}^i = \frac{1}{2}C_{jk}^i$ and (4.92) reproduces the classical Cartan-Maurer equations.

For $q \neq 1$, we find the following relation:

$$C_{jk}^i = C_{jk}^i - \Lambda_{rs}^j \Lambda_{jk}^i C_{rs}^i$$ \hspace{1cm} (4.97)

after applying both members of eq. (4.82) to $x^i$. Note that, using (4.97), the Cartan-Maurer equations (4.92) can also be written as:

$$d\omega^i + C_{jk}^i \omega^j \otimes \omega^k = 0.$$ \hspace{1cm} (4.98)

ix) Finally, we derive two operatorial identities that become trivial in the limit $q \to 1$. From the formula

$$d(h \ast \theta) = h \ast d\theta, \quad h \in A', \quad \theta \in \Gamma^\wedge n$$ \hspace{1cm} (4.99)

[a direct consequence of (4.76)] with $h = f^n_l$, we find

$$\chi_k f^n_l = \Lambda_{kl}^i f^n_i \chi_j.$$ \hspace{1cm} (4.100)

By requiring consistency between the external derivative and the bimodule structure of $\Gamma$, i.e. requiring that

$$d(\omega^i a) = d[(f^i_j \ast a) \omega^j],$$ \hspace{1cm} (4.101)
one finds the identity

\[ C_{mn}^i f^m_j f^n_k + f^i_j \chi_k = \Lambda^{pq}_{jk} \chi_p f^i_q + C_{jk}^l f^i_l. \]  

(4.102)

See Appendix A for the derivation of (4.100) and (4.102).

In summary, a bicovariant calculus on a \( G_q \) is characterized by functionals \( \chi_i \) and \( f^i_j \) on \( A \) ("the algebra of functions on the quantum group") satisfying

\[ \chi_i \chi_j - \Lambda^{kl}_{ij} \chi_k \chi_l = C_{ij}^k \chi_k \]  

(4.103)

\[ \Lambda^{mn}_{ij} f^i_p f^j_q = f^i_p f^m_j \Lambda^{ij}_{pq} \]  

(4.104)

\[ C_{mn}^i f^m_j f^n_k + f^i_j \chi_k = \Lambda^{pq}_{jk} \chi_p f^i_q + C_{jk}^l f^i_l \]  

(4.105)

\[ \chi_k f^n_i = \Lambda^{ij}_{kl} f^n_i \chi_j, \]  

(4.106)

where the \( q \)-structure constants are given by \( C_{jk}^i = \chi_k(M_j^i) \) and the braiding matrix by \( \Lambda^{ij}_{kl} = f^i_j(M_k^j) \). In fact, these four relations seem to be also sufficient to define a bicovariant differential calculus on \( A \) (see e.g. [12]). By applying them to the element \( M_s^r \) we express these relations (henceforth called bicovariance conditions) in the adjoint representation:

\[ C_{ri}^n C_{nj}^s - R_{ij}^{kl} C_{rk}^n C_{nl}^s = C_{ij}^k C_{rk}^s \]  

(4.107)

\[ \Lambda^{mn}_{ij} \Lambda^{ik}_{rp} \Lambda^{ls}_{kq} = \Lambda^{nk}_{ri} \Lambda^{ns}_{kj} \Lambda^{ij}_{pq} \]  

(4.108)

\[ C_{mn}^i \Lambda^{ml}_{rq} \Lambda^{ns}_{lk} + \Lambda^{il}_{rj} C_{lk}^s = \Lambda^{pq}_{jk} \Lambda^{il}_{rq} C_{lp}^s + C_{jk}^m \Lambda^{is}_{rm} \]  

(4.109)

\[ C_{rk}^m \Lambda^{ns}_{ml} = \Lambda^{ij}_{kl} \Lambda^{nm}_{ri} C_{mj}^s \]  

(4.110)

We conclude this section by giving the co-structures on the quantum Lie algebra generators \( \chi_i \) [those on the functionals \( f^i_j \) have been given in (4.36)-(4.38)]:

\[ \Delta'(\chi_i) = \chi_j \otimes f^j_i + I' \otimes \chi_i \]  

(4.111)

\[ \varepsilon'(\chi_i) = 0 \]  

(4.112)

\[ \kappa'(\chi_i) = -\chi_j \kappa'(f^j_i). \]  

(4.113)

which \( q \)-generalize the ones given in (2.19)-(2.21). These co-structures derive from (2.24)-(2.26). For example, using (4.73) and (4.30), eq. (2.24) yields (4.111). They are consistent with the bicovariance conditions (4.103)-(4.106).

In the next section, we describe a constructive procedure due to Jurčo [17] for a bicovariant differential calculus on any \( q \)-group of the \( A, B, C, D \) series considered in [3]. The procedure is illustrated on the example of \( GL_q(2) \), for which all the objects \( f^i_j, M_s^r, \Lambda^{ij}_{kl}, C_{jk}^i \) and \( C_{jk}^i \) are explicitly computed.
5 Constructive procedure and the example of $GL_q(2)$

The generic $q$-group discussed in Section 3 is characterized by the matrix $R^{ab}_{cd}$. In terms of this matrix, it is possible to construct a bicovariant differential calculus on the $q$-group. The general procedure is described in this section, and the results for the specific case of $GL_q(2)$ are collected in the table.

**The $L^\pm$ functionals**

We start by introducing the linear functionals $(L^\pm)^a_b$, defined by their value on the elements $T^a_b$:

$$(L^\pm)^a_b(T^c_d) = (R^\pm)^{ac}_{bd}, \quad (5.1)$$

where

$$(R^+)^{ac}_{bd} \equiv c^+ R^{ca}_{db}, \quad (5.2)$$

$$(R^-)^{ac}_{bd} \equiv c^- (R^{-1})^{ac}_{bd}, \quad (5.3)$$

where $c^+$, $c^-$ are free parameters (see later). The inverse matrix $R^{-1}$ is defined by

$$(R^{-1})^{ab}_{cd} R^{cd}_{ef} \equiv \delta^b_e \delta^a_f \equiv R^{ab}_{cd} (R^{-1})^{cd}_{ef}. \quad (5.4)$$

We see that the $(L^\pm)^a_b$ functionals are dual to the $T^a_b$ elements (fundamental representation) in the same way the $f^i_j$ functionals are dual to the $M^i_j$ elements of the adjoint representation. To extend the definition $(5.1)$ to the whole algebra $A$, we set:

$$(L^\pm)^a_b(ab) = (L^\pm)^a_g(a)(L^\pm)^g_b(b), \quad \forall a, b \in A \quad (5.5)$$

so that, for example,

$$(L^\pm)^a_b(T^c_d T^e_f) = (R^\pm)^{ac}_{gd}(R^\pm)^{ge}_{bf}. \quad (5.6)$$

In general, using the compact notation introduced in Section 3,

$$L^\pm_1(T_2 T_3 \cdots T_n) = R^\pm_{12} R^\pm_{13} \cdots R^\pm_{1n}. \quad (5.7)$$

Finally, the value of $L^\pm$ on the unit $I$ is defined by

$$(L^\pm)^a_b(I) = \delta^a_b. \quad (5.8)$$

Thus the functionals $(L^\pm)^a_b$ have the same properties as their adjoint counterpart $f^i_j$, and not surprisingly the latter will be constructed in terms of the former.

From $(5.7)$ we can also find the action of $(L^\pm)^a_b$ on $a \in A$, i.e. $(L^\pm)^a_b * a$. Indeed

$$(L^\pm)^a_b * (T^{c_1}_{d_1} T^{c_2}_{d_2} \cdots T^{c_n}_{d_n}) = [id \otimes (L^\pm)^a_b] \Delta(T^{c_1}_{d_1} T^{c_2}_{d_2} \cdots T^{c_n}_{d_n}) = \bigg[ id \otimes (L^\pm)^a_b \bigg] \Delta(T^{c_1}_{d_1}) \cdots \Delta(T^{c_n}_{d_n}) = \bigg[ id \otimes (L^\pm)^a_b \bigg] (T^{c_1}_{e_1} \cdots T^{c_n}_{e_n} \otimes T^{e_1}_{d_1} \cdots T^{e_n}_{d_n})$$

$$T^{c_1}_{e_1} \cdots T^{c_n}_{e_n} (R^\pm)^a_b(T^{e_1}_{d_1} \cdots T^{e_n}_{d_n}) = T^{c_1}_{e_1} \cdots T^{c_n}_{e_n} (R^\pm)^{ae_1}_{ge_1} (R^\pm)^{ge_2}_{gd_2} \cdots (R^\pm)^{ge_n}_{bd_n} \quad (5.9)$$
or, more compactly,

\[ L_1^\dagger \ast T_2 \ast \cdots \ast T_n = T_2 \ast \cdots \ast T_n R_{12}^\dagger R_{13}^\dagger \cdots R_{1n}^\dagger, \quad (5.10) \]

which can also be written as

\[ L_1^\dagger \ast T_2 = T_2 R_{12}^\dagger L_1^\dagger. \quad (5.11) \]

It is not difficult to find the commutations between \((L^\pm)^a \, _b\) and \((L^\pm)^c \, _d\):

\[ R_{12} L_2^\dagger L_1^\dagger = L_1^\dagger L_2^\dagger R_{12} \quad (5.12) \]

\[ R_{12} L_2^+ L_1^- = L_1^- L_2^+ R_{12}, \quad (5.13) \]

where as usual the product \(L_2^\dagger L_1^\dagger\) is the convolution product \(L_2^\dagger L_1^\dagger \equiv (L_2^\dagger \otimes L_1^\dagger) \Delta.\)

Consider

\[ R_{12}(L_2^\dagger L_1^\dagger)(T_3) = R_{12}(L_2^\dagger \otimes L_1^\dagger) \Delta(T_3) = R_{12}(L_2^\dagger \otimes L_1^\dagger)(T_3 \otimes T_3) = (c^+)^2 R_{12} R_{32} R_{31} \]

and

\[ L_1^\dagger L_2^\dagger (T_3) R_{12} = (c^+)^2 R_{31} R_{32} R_{12} \quad (5.15) \]

so that the equation (5.12) is proven for \(L^+\) by virtue of the quantum Yang–Baxter equation (5.3), where the indices have been renamed \(2 \rightarrow 1, 3 \rightarrow 2, 1 \rightarrow 3\). Similarly, one proves the remaining “RLL” relations.

**Note 1:** As mentioned in [3], \(L^+\) is upper triangular, \(L^-\) is lower triangular (this is due to the upper and lower triangularity of \(R^+\) and \(R^-\), respectively).

**Note 2:** When \(\det_q T = 1\), we have \(\det_q^{-1} L^\pm = (L^\pm)^1 \, _1 (L^\pm)^2 \, _2 \cdots (L^\pm)^n \, _n = \varepsilon\), and \((L^+, i)(L^-)\, ^j \, _j = (L^-)^i \, _j (L^+)\, ^i \, _j\) (no sum on repeated indices). Then \((\det_q^{-1} L^\pm)(\det_q T) = \det_q R^\pm = 1\), which requires \(c^+ = q^1/n, c^- = q^{-1/n}\) for the \(A_{n-1}\) series (and \(c^\pm = 1\) for the remaining \(B, C, D\) series) in (5.2) and (5.3). In the more general case of \(GL_q(n), c^\pm\) are extra free parameters, cf. [21]. In fact, they appear only in the combination \(s = (c^+)^{-1}c^-\). They do not enter in the \(\Lambda\) matrix, nor in the structure constants or the Cartan-Maurer equations (see the table). Different values of \(s\) lead to isomorphic differential calculi (in the sense of ref. [5]), so that \(s\) is not really an essential parameter.

The co-structures are defined by the duality (5.1):

\[ \Delta'((L^\pm)^a \, _b)(T^c \, _d \otimes T^e \, _f) \equiv (L^\pm)^a \, _b(T^c \, _d T^e \, _f) = (L^\pm)^a \, _g(T^c \, _d)(L^\pm)^g \, _b(T^e \, _f) \quad (5.16) \]

\[ \varepsilon'((L^\pm)^a \, _b) \equiv (L^\pm)^a \, _b(I) \quad (5.17) \]

\[ \kappa'((L^\pm)^a \, _b)(T^c \, _d) \equiv (L^\pm)^a \, _b(\kappa(T^c \, _d)) \quad (5.18) \]
cf. \([2.24]-(2.26)\), so that
\[
\Delta'((L^\pm)^a_b) = (L^\pm)^a_g \otimes (L^\pm)^g_b
\]
\[
\varepsilon'((L^\pm)^a_b) = \delta^a_b
\]
\[
\kappa'((L^\pm)^a_b) = (L^\pm)^a_b \circ \kappa
\]
and the \((L^\pm)^a_b\) generate the Hopf algebra dual to the quantum group. Note that
\[
(L^\pm)^a_b(\kappa(T^c_d)) = ((R^\pm)^{-1})^{ac}_{bd},
\]
(5.22)
since
\[
(L^\pm)^a_b(\kappa(T^c_d)T^d_e) = \delta^c_d(L^\pm)^a_b(I) = \delta^c_e \delta^a_b
\]
(5.23)
and
\[
(L^\pm)^a_b(\kappa(T^c_d)T^d_e) = (L^\pm)^a_f \kappa(T^c_d)(R^\pm)^f_d \delta^a_{be}.
\]
(5.24)

The space of quantum 1-forms

The bimodule \(\Gamma\) ("space of quantum 1-forms") can be constructed as follows. First we define \(\omega^a_b\) to be a basis of left-invariant quantum 1-forms. The index pairs \(a_b\) or \(a^b\) will replace in the sequel the indices \(i\) or \(i^\circ\) of the previous section. The dimension of \(\text{inv}_\Gamma\) is therefore \(N^2\) at this stage. The existence of this basis can be proven by considering \(\Gamma\) to be the tensor product of two fundamental bimodules, see refs. [17, 16]. Here we just assume that it exists. Since the \(\omega^a_b\) are left-invariant, we have:
\[
\Delta_L(\omega^a_b) = I \otimes \omega^a_b, \quad a, b = 1, ..., N.
\]
(5.25)
The left action \(\Delta_L\) on the whole of \(\Gamma\) is then defined by \((5.23)\), since \(\omega^a_b\) is a basis for \(\Gamma\). The bimodule \(\Gamma\) is further characterized by the commutations between \(\omega^a_b\) and \(a \in A\) [cf. eq. \((4.30)\)]:
\[
\omega^{a^2}_a b = (f^{a^2}_a b \ast b) \omega^b_b,
\]
(5.26)
where
\[
f^{a^2}_a b \equiv \kappa'(((L^+)_{a^1}^{a^2})(L^-)^{a^2}_{b^2}.
\]
(5.27)
Finally, the right action \(\Delta_R\) on \(\Gamma\) is defined by
\[
\Delta_R(\omega^{a^2}_a) = \omega^b_b \otimes M^{b^1}_{b^2 a^2}.
\]
(5.28)
where \(M^{b^1}_{b^2 a^2}\), the adjoint representation, is given by
\[
M^{b^1}_{b^2 a^2} \equiv T^{b^1}_{a^1} \kappa(T^{a^2}_{b^2}).
\]
(5.29)
It is easy to check that \( f_{a_2b_1}^{a_1b_2} \) fulfill the consistency conditions (4.33)-(4.35), where the \( i,j,... \) indices stand for pairs of \( a,b,... \) indices. Also, the co-structures are as given in (4.36)-(4.38).

The \( \Lambda \) tensor and the exterior product

The \( \Lambda \) tensor defined in (4.70) can now be computed:

\[
\Lambda_{a_1d_1|c_1b_1} = f_{a_2b_1}^{a_1b_2}(M_{a_2d_1}^{c_1} d_2) = \kappa'((L^+)^{b_1}_{a_1})(L^-)^{a_2}_{b_2} (T^c_{d_1} T^d_{d_2}) = [\kappa'((L^+)^{b_1}_{a_1}) \otimes (L^-)^{a_2}_{b_2}] \Delta(T^c_{d_1} \kappa(T^d_{d_2}))
\]

where we made use of relations (2.12), (2.26), (3.15), (5.1) and (5.2). The \( \Lambda \) tensor does the trick. Another useful relation gives a particular trace of the \( \Lambda \) matrix:

\[
\kappa = \kappa = \kappa = \kappa
\]

(5.30)

where we have used relations (2.12), (2.20), (3.15) and (5.22). The \( \Lambda \) tensor allows the definition of the exterior product as in (4.65). For future use we give here also the inverse \( \Lambda^{-1} \) of the \( \Lambda \) tensor, defined by:

\[
(\Lambda^{-1})_{a_2d_2|b_1c_1} = \delta_{a_2}^{a_1} \delta_{d_2}^{d_1} \delta_{b_1}^{b_2} \delta_{c_1}^{c_2}.
\]

(5.31)

It is not difficult to see that

\[
(\Lambda^{-1})_{a_2d_2|b_1c_1} = f_{d_1}^{d_2b_1}(T^{a_2}_{c_1} \kappa^{-1}(T^a_{b_1})) = R^{d_2b_1}_{a_1g_1} (R^{-1})^{a_2g_2}_{e_1d_1} (R^{-1})^{d_2e_2}_{g_2c_2} R^{g_2c_1}_{b_2f_1} (d^{-1})^{d_1}_{c_1} f_1.
\]

(5.32)

does the trick. Another useful relation gives a particular trace of the \( \Lambda \) matrix:

\[
\Lambda_{c_2b_2|a_1b_1} = \delta_{a_2}^{a_1} \delta_{b_2}^{b_1}.
\]

(5.33)

This identity is simply proven. Indeed:

\[
\Lambda_{c_2b_2|a_1b_1} = f_{c_2b_1}^{a_2b_2}(M_{a_2b_2}^{c_1}) = \kappa'((L^+)^{b_1}_{c_1})(L^-)^{a_2}_{b_2} (T^a_{b_1} \kappa(T^b_{a_2})) = \kappa'((L^+)^{b_1}_{c_1})(L^-)^{a_2}_{b_2} (\delta_{a_2}^{a_1} I) = \delta_{a_2}^{a_1} \kappa'((L^+)^{b_1}_{c_1}) \otimes (L^-)^{a_2}_{b_2} (I \otimes I) = \delta_{a_2}^{a_1} \delta_{b_2}^{b_1} \delta_{c_2}.
\]

(5.34)

The relations (3.17), (3.18) for the \( R \) matrix reflect themselves in relations for the \( \Lambda \) matrix (3.30). For example, the Hecke condition (3.17) implies:

\[
(\Lambda + q^2)(\Lambda + q^{-2})(\Lambda - I) = 0
\]

(5.35)
for the $A_{n-1}$ $q$-groups, and replaces the classical relation $(\Lambda - 1)(\Lambda + 1) = 0$, $\Lambda$ being for $q = 1$ the ordinary permutation operator, cf. \[(4.66)\).

With the help of \[(5.33)\] we can give explicitly the commutations of the left-invariant forms $\omega^i$. Indeed, reverting to the $i,j...$ indices, relation \[(5.35)\] implies:

\[
(\Lambda_{ij}^{kl} + q^2\delta^i_k\delta^j_l)(\Lambda_{kl}^{mn} + q^{-2}\delta^m_k\delta^n_l)(\Lambda_{mn}^{rs} - \delta^m_r\delta^n_s)\omega^r \otimes \omega^s = (\Lambda_{ij}^{kl} + q^2\delta^i_k\delta^j_l)(\Lambda_{kl}^{mn} + q^{-2}\delta^m_k\delta^n_l)\omega^m \wedge \omega^n = 0
\]

and it is easy to see that the last equality can be rewritten as

\[
\omega^i \wedge \omega^j = -Z_{ij}^{kl}\omega^k \wedge \omega^l
\]

\[
Z_{ij}^{kl} \equiv \frac{1}{q^2 + q^{-2}}[\Lambda_{ij}^{kl} + (\Lambda^{-1})_{ij}^{kl}].
\]

**The exterior differential**

The exterior differential on $\Gamma^\wedge k$ is defined by means of the bi-invariant (i.e. left- and right-invariant) element $\tau = \sum_a \omega_a^a \in \Gamma$ as follows:

\[
d\theta \equiv \frac{1}{\lambda} [\tau \wedge \theta - (-1)^k \theta \wedge \tau],
\]

where $\theta \in \Gamma^\wedge k$, and $\lambda$ is a normalization factor depending on $q$, necessary in order to obtain the correct classical limit. It will be later determined to be $\lambda = q - q^{-1}$. Here we can only see that it has to vanish for $q = 1$, since otherwise $d\theta$ would vanish in the classical limit. For $a \in A$ we have

\[
da = \frac{1}{\lambda} [\tau a - a\tau].
\]

This linear map satisfies the Leibniz rule \[(1.1)\], and properties \[(1.73)-(1.76)\], as the reader can easily check (use the definition of exterior product and the bi-invariance of $\tau$). A proof that also the property \[(1.2)\] holds can be obtained by considering the exterior differential of the adjoint representation:

\[
dM^i_j = (\chi_k \ast M^i_j)\omega^k = M^i_j C^i_{kl} \omega^k
\]

or

\[
\kappa(M^i_j) dM^i_j = C^i_{kl} \omega^k.
\]

Multiplying by $C^i_{nl}$, we have:

\[
C^i_{nl} \kappa(M^i_j) dM^i_j = C^i_{kl} \omega^k \equiv g_{nk} \omega^k,
\]

where $g_{nk}$ is the $q$-Killing metric. The explicit example of this paper being $GL_q(2)$, one may wonder what happens to the invertibility of the $q$-Killing metric, since its classical limit is no more invertible [$GL(2)$ being nonsemisimple]. The answer
is that for \( q \neq 1 \) the \( q \)-Killing metric of \( GL_q(2) \) is invertible, as can be checked explicitly from the values of the structure constants given in the table. Therefore \( GL_q(2) \) could be said to be “\( q \)-semisimple”. With an analogous procedure (using \( T_a^b \) instead of \( M_j^i \)) we have derived in the table the explicit expression of the \( \omega^i \) in terms of the \( dT_a^b \) for \( GL_q(2) \).

The \( q \)-Lie algebra

The “quantum generators” \( \chi^{a_1}_{a_2} \) are introduced as in \( (1.74) \):

\[
da = \frac{1}{\lambda}[\tau a - a \tau] = (\chi^{a_1}_{a_2} \ast a) \omega^{a_2}.
\]

Using \( (5.26) \) we can find an explicit expression for the \( \chi^{a_1}_{a_2} \) in terms of the \( L^\pm \) functionals. Indeed

\[
\tau a = \omega^b_{\{b}\} a = (f_b_{bc_1}^{\{a\}} \ast a) \omega^{c_2}_{c_1} = ([\kappa'((L^+)^{c_1}_{b})(L^-)^{b}_{c_2}] \ast a) \omega^{c_2}_{c_1}.
\]

Therefore

\[
da = \frac{1}{\lambda}[(\kappa'((L^+)^{c_1}_{b})(L^-)^{b}_{c_2} - \delta^{c_1}_{c_2} \varepsilon) \ast a) \omega^{c_2}_{c_1}]
\]

(recall \( \varepsilon \ast a = a \)), so that the \( q \)-generators take the explicit form

\[
\chi^{c_1}_{c_2} = \frac{1}{\lambda}[(\kappa'((L^+)^{c_1}_{b})(L^-)^{b}_{c_2} - \delta^{c_1}_{c_2} \varepsilon)].
\]

The commutations between the \( \chi \)'s can now be obtained by taking the exterior derivative of eq. \( (5.46) \). We find

\[
d^2(a) = 0 = d[(\chi^{c_1}_{c_2} \ast a) \omega^{c_2}_{c_1}] = (\chi^{d_1}_{d_2} \ast \chi^{c_1}_{c_2} \ast a) \omega^{d_2}_{d_1} \wedge \omega^{c_2}_{c_1} + (\chi^{c_1}_{c_2} \ast a)d\omega^{c_2}_{c_1}
\]

\[
= (\chi^{d_1}_{d_2} \ast \chi^{c_1}_{c_2} \ast a)(\omega^{d_2}_{d_1} \otimes \omega^{c_2}_{c_1} - \Lambda^{d_2}_{d_1} c_2 \otimes f_1^{e_1} f_2^{e_2} \omega^{e_1}_{e_2} \otimes \omega^{f_2}_{f_1})
\]

\[
+ \frac{1}{\lambda}(\chi^{c_1}_{c_2} \ast a)(\omega^b_{\{b\}} \wedge \omega^{c_2}_{c_1} + \omega^{c_2}_{c_1} \wedge \omega^b_{\{b\}}).
\]

Now we use the fact that \( \tau = \omega^b_{\{b\}} \) is bi-invariant, and therefore also right-invariant, so that we can write

\[
\omega^b_{\{b\}} \wedge \omega^{c_2}_{c_1} + \omega^{c_2}_{c_1} \wedge \omega^b_{\{b\}} = \omega^b_{\{b\}} \otimes \omega^{c_2}_{c_1} - \Lambda(\omega^b_{\{b\}} \otimes \omega^{c_2}_{c_1}) + \omega^{c_2}_{c_1} \otimes \omega^b_{\{b\}} - \Lambda(\omega^{c_2}_{c_1} \otimes \omega^b_{\{b\}}) =
\]

\[
\omega^{c_2}_{c_1} \otimes \omega^b_{\{b\}} - \Lambda(\omega^b_{\{b\}} \otimes \omega^{c_2}_{c_1}) = \omega^{c_2}_{c_1} \otimes \omega^b_{\{b\}} - \Lambda(b_{c_2}^{e_1} f_1^{e_2} f_2^{e_2} \omega^{e_1}_{e_2} \otimes \omega^{f_2}_{f_1}),
\]

where we have used \( \Lambda(\omega^{c_2}_{c_1} \otimes \tau) = \tau \otimes \omega^{c_2}_{c_1} \), cf. \( (1.53) \). After substituting \( (5.49) \) in \( (5.48) \), and factorizing \( \omega^{d_1}_{d_2} \otimes \omega^{c_2}_{c_1} \), we arrive at the \( q \)-Lie algebra relations:

\[
\chi^{d_1}_{d_2} \chi^{c_2}_{c_1} - \Lambda^{e_1}_{e_2} f_1^{d_2} e_2^{d_1} c_1^{c_2} \chi^{e_1}_{e_2} \chi^{f_2}_{f_2} = \frac{1}{\lambda}[-\delta^{c_2}_{c_2} \chi^{d_1}_{d_2} + \Lambda(b_{c_2}^{e_2} e_2^{d_1} c_1^{c_1} \chi^{e_2}_{e_2}].
\]
The structure constants are then explicitly given by:

\[ \mathbf{C}^{a_1 \ b_1 \mid c_1}_{\ a_2 \ b_2 \mid c_2} = \frac{1}{\lambda}[-\delta^{b_1}_{c_1} \mathbf{C}^{a_1 \ c_2}_{\ a_2 \ b_2} + \Lambda_{b \ c_1 \mid a_1 \ b_1 \ a_2 \ b_2}]. \]  

(5.51)

Here we determine \( \lambda \). Indeed we first observe that

\[ \Lambda_{a_1 \ d_1 \mid c_1 \ b_1} = \delta^{b_1}_{c_1} \delta^{d_1}_{b_1} + (q - q^{-1})U_{a_1 \ d_1 \mid c_1 \ b_1}. \]  

(5.52)

where the matrix \( U \) is finite and different from zero in the limit \( q = 1 \). This can be proven by considering the explicit form of the \( R \) and \( R^{-1} \) matrices. In the case of the \( A_{n-1} \) \( q \)-groups, for example, these matrices have the form \( 3 \):

\[ R^{ab}_{\ cd} = \delta^{a}_{c} \delta^{b}_{d} + (q - q^{-1}) \left[ \frac{q - 1}{q - q^{-1}} \delta^{a}_{c} \delta^{b}_{d} + \delta^{a}_{c} \delta^{b}_{d} \theta(a - b) \right] \]  

(5.53)

\[ (R^{-1})^{ab}_{\ cd} = \delta^{a}_{c} \delta^{b}_{d} - (q - q^{-1}) \left[ \frac{1 - q^{-1}}{q - q^{-1}} \delta^{a}_{c} \delta^{b}_{d} + \delta^{a}_{c} \delta^{b}_{d} \theta(a - b) \right], \]  

(5.54)

where \( \theta(x) = 1 \) for \( x > 0 \) and vanishes for \( x \leq 0 \). Substituting these expressions in the formula for \( \Lambda \) (5.30) we find (5.52). Using (5.52) in the expression (5.51) for the \( q \)-structure constants \( \mathbf{C} \), we find that the terms proportional to \( \frac{1}{\lambda} \) do cancel, and we are left with:

\[ \mathbf{C}^{a_1 \ b_1 \mid c_1}_{\ a_2 \ b_2 \mid c_2} = -\frac{1}{\lambda} (q - q^{-1}) U_{b \ c_1 \mid a_1 \ b_1 \ a_2 \ b_2}. \]  

(5.55)

A simple choice for \( \lambda \) is therefore \( \lambda = q - q^{-1} \), ensuring that \( \mathbf{C} \) remains finite in the limit \( q \to 1 \).

**The Cartan-Maurer equations**

The Cartan-Maurer equations are found as follows:

\[ d\omega^{c_2}_{\ c_1} = \frac{1}{\lambda} (\omega^{b \ c_2}_{\ b \ c_1} + \omega^{c_2}_{\ c_1} \wedge \omega^{b}) \equiv -\mathbf{C}^{a_1 \ b_1 \mid c_1}_{\ a_2 \ b_2 \mid c_2} \omega^{a_2}_{\ a_1 \ b_1} \wedge \omega_{b_2}. \]  

(5.56)

In order to obtain an explicit expression for the \( C \) structure constants in (5.54), we must use the relation (5.37) for the commutations of \( \omega^{a_2}_{\ a_1 \ b_1} \) with \( \omega_{b_2} \). Then the term \( \omega^{c_2}_{\ c_1} \wedge \omega_{b} \) in (5.56) can be written as \( -Z \omega \) via formula (5.37), and we find the \( C \)-structure constants to be:

\[ \mathbf{C}^{a_1 \ b_1 \mid c_1}_{\ a_2 \ b_2 \mid c_2} = \frac{1}{\lambda} (\delta^{a_1}_{a_2} \delta^{b_1}_{c_1} \delta^{c_2}_{b_2} - \frac{1}{q^2 + q^{-2}} [\Lambda^{c_2}_{\ c_1 \ b \mid a_1 \ b_1 \ a_2 \ b_2} + (\Lambda^{-1})^{c_2}_{\ c_1 \ b \mid a_1 \ b_1 \ a_2 \ b_2}]). \]  

(5.57)

where we have also used eq. (5.33). By considering the analogue of (5.52) for \( \Lambda^{-1} \), it is not difficult to see that the terms proportional to \( \frac{1}{\lambda} \) cancel, and the \( q \to 1 \) limit
of (5.57) is well defined. For a more detailed discussion, including also the $B_n, C_n$ and $D_n$ $q$-groups, we refer to [22].

In the table we summarize the results of this section for the case of $GL_q(2)$. The composite indices $^b_a$ are translated into the corresponding indices $^i$, $i = 1, +, -, 2$, according to the convention:

$$
^1_1 \rightarrow 1, \quad ^2_1 \rightarrow +, \quad ^1_2 \rightarrow -, \quad ^2_2 \rightarrow 2.
$$

(5.58)

A similar convention holds for $^a_b \rightarrow i$. 
Table

The bicovariant $GL_q(2)$ algebra

$R$ and $D$-matrices:

$$R^{ab}_{\ cd} = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & q^{-1} & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}$$

$$(R^-)^{ab}_{\ cd} \equiv c^-(R^{-1})^{ab}_{\ cd} = c^- \begin{pmatrix} q^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & -(q - q^{-1}) & 1 & 0 \\ 0 & 0 & 0 & q^{-1} \end{pmatrix}$$

$$(R^+)^{ab}_{\ cd} \equiv c^+R^{ba}_{\ dc} = c^+ \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad D^a_b = \begin{pmatrix} q & 0 \\ 0 & q^3 \end{pmatrix}$$

Non-vanishing components of the $\Lambda$ matrix:

\begin{align*}
\Lambda^{11}_{11} &= 1 & \Lambda^{1+}_{1+} &= q^{-2} & \Lambda^{1-}_{1-} &= q^2 & \Lambda^{12}_{21} &= 1 \\
\Lambda^{+1}_{1+} &= 1 & \Lambda^{+1+}_{1+} &= 1 - q^{-2} & \Lambda^{++}_{++} &= 1 & \Lambda^{++}_{11} &= 1 - q^2 \\
\Lambda^{+}_{-+} &= 1 & \Lambda^{+-}_{-1} &= 1 - q^{-2} & \Lambda^{-+}_{+-1} &= -1 + q^{-2} & \Lambda^{+2}_{2+} &= 1 \\
\Lambda^{-1}_{-1} &= 1 & \Lambda^{-1-}_{-1} &= 1 - q^2 & \Lambda^{-2}_{-2} &= -1 + q^2 & \Lambda^{-2}_{2-} &= 1 \\
\Lambda^{21}_{21} &= (q^2 - 1)^2 & \Lambda^{21}_{12} &= 1 & \Lambda^{21}_{+-} &= q^2 - 1 & \Lambda^{21}_{+-1} &= 1 - q^2 \\
\Lambda^{22}_{21} &= 2 - q^2 - q^{-2} & \Lambda^{2+}_{1+} &= -q^2 + q^4 & \Lambda^{2+}_{+2} &= q^2 & \Lambda^{2+}_{2+} &= 1 - q^2 \\
\Lambda^{2-}_{-1} &= 1 - q^2 & \Lambda^{2-}_{-1} &= q^{-2} - 1 - q^2 + q^4 & \Lambda^{2-}_{-2} &= q^{-2} & \Lambda^{2-}_{2-} &= 1 - q^{-2} \\
\Lambda^{22}_{11} &= -(q^2 - 1)^2 & \Lambda^{22}_{+-} &= 1 - q^2 & \Lambda^{22}_{-+} &= q^2 - 1 & \Lambda^{22}_{21} &= (q^{-1} - q)^2 \\
\Lambda^{22}_{22} &= 1
\end{align*}

Non-vanishing components of the $C$ structure constants:

\begin{align*}
C^{11}_{11} &= q(q^2 - 1) & C^{11}_{12} &= -q(q^2 - 1) & C^{1+}_{1+} &= q^3 & C_{1-} &= -q \\
C^{21}_{21} &= q^{-2} - q & C^{21}_{22} &= q - q^{-1} & C^{2+}_{2+} &= -q & C_{2-} &= q^{-1} \\
C^{+1}_{1+} &= -q^{-1} & C^{+2}_{+2} &= q & C^{+-}_{+-} &= q & C^{+-}_{+-2} &= -q \\
C^{-1}_{-1} &= q(q^2 + 1) - q^{-1} & C^{-2}_{-2} &= -q^{-1} & C_{-+} &= -q & C_{-+2} &= q
\end{align*}
The \( q \)-Lie algebra:

Commutation relations between left-invariant Cartan-Maurer equations:

Non-vanishing components of the \( C \) structure constants:

\[
\begin{align*}
C_{11}^- &= \frac{q(q^2-1)}{1+q^2}, & C_{11}^+ &= \frac{q^2(1-q^2)}{1+q^2}, & C_{1+}^- &= \frac{q^5}{1+q^2}, & C_{1-}^- &= \frac{-q^3}{1+q^2} \\
C_{12}^- &= \frac{q^3}{1+q^2}, & C_{12}^+ &= \frac{-q^3}{1+q^2}, & C_{2+}^- &= \frac{-q^3}{1+q^2}, & C_{2-}^- &= \frac{q^3}{1+q^2} \\
C_{+2}^- &= \frac{q}{1+q^2}, & C_{+2}^+ &= \frac{q}{1+q^2}, & C_{-+}^- &= \frac{q}{1+q^2}, & C_{-+}^+ &= \frac{q}{1+q^2} \\
C_{22}^- &= \frac{q(q^2-1)}{1+q^2}.
\end{align*}
\]

Cartan-Maurer equations:

\[
\begin{align*}
d\omega^1 + q\omega^+ \wedge \omega^- &= 0 \\
d\omega^+ + q\omega^+(-q^2\omega^1 + \omega^2) &= 0 \\
d\omega^- + q(-q^2\omega^1 + \omega^2)\omega^- &= 0 \\
d\omega^2 - q\omega^+ \wedge \omega^- &= 0
\end{align*}
\]

The \( q \)-Lie algebra:

\[
\begin{align*}
\chi_1\chi_+ - \chi_+\chi_1 - (q^4 - q^2)\chi_2\chi_+ &= q^3\chi_+ \\
\chi_1\chi_- - \chi_-\chi_1 + (q^2 - 1)\chi_2\chi_- &= -q\chi_- \\
\chi_1\chi_2 - \chi_2\chi_1 &= 0 \\
\chi_+\chi_- - \chi_-\chi_+ + (1 - q^2)\chi_2\chi_1 - (1 - q^2)\chi_2\chi_2 &= q(\chi_1 - \chi_2) \\
\chi_+\chi_2 - q^2\chi_2\chi_+ &= q\chi_+ \\
\chi_-\chi_2 - q^{-2}\chi_2\chi_- &= -q^{-1}\chi_-
\end{align*}
\]

Commutation relations between left-invariant \( \omega^i \) and \( \omega^j \):

\[
\begin{align*}
\omega^1 \wedge \omega^+ + \omega^+ \wedge \omega^1 &= 0 \\
\omega^1 \wedge \omega^- + \omega^- \wedge \omega^1 &= 0 \\
\omega^1 \wedge \omega^2 + \omega^2 \wedge \omega^1 &= (1 - q^2)\omega^+ \wedge \omega^- \\
\omega^+ \wedge \omega^- + \omega^- \wedge \omega^+ &= 0 \\
\omega^2 \wedge \omega^+ + q^2\omega^+ \wedge \omega^2 &= q^2(q^2 - 1)\omega^+ \wedge \omega^- \\
\omega^2 \wedge \omega^- + q^-2\omega^- \wedge \omega^2 &= (1 - q^2)\omega^- \wedge \omega^1 \\
\omega^2 \wedge \omega^2 &= (q^2 - 1)\omega^+ \wedge \omega^- \\
\omega^1 \wedge \omega^1 &= \omega^+ \wedge \omega^+ = \omega^- \wedge \omega^- = 0
\end{align*}
\]

Commutation relations between \( \omega^i \) and the basic elements of \( A \) (\( s = (c^+)^{-1}c^- \)):
\[\begin{align*}
\omega^1\alpha &= s q^{-2} \alpha \omega^1 & \omega^1\omega &= sq^{-1} \alpha \omega^+ \\
\omega^1\beta &= s \beta \omega^1 & \omega^1\beta &= sq^{-1} \beta \omega^+ + s(q^{-2} - 1) \alpha \omega^1 \\
\omega^1\gamma &= sq^{-2} \gamma \omega^1 & \omega^1\gamma &= sq^{-1} \gamma \omega^+ \\
\omega^1\delta &= s \delta \omega^1 & \omega^1\delta &= sq^{-1} \delta \omega^+ + s(q^{-2} - 1) \gamma \omega^1 \\
\omega^{-}\alpha &= sq^{-1} \alpha \omega^+ + s(q^{-2} - 1) \beta \omega^1 & \omega^{-}\alpha &= sa\omega^2 + s(q^{-1} - q) \beta \omega^+ \\
\omega^{-}\beta &= sq^{-1} \beta \omega^+ & \omega^{-}\beta &= s q^{-2} \beta \omega^2 + s(q^{-1} - q) \alpha \omega^+ + s(q^{-1} - q)^2 \beta \omega^1 \\
\omega^{-}\gamma &= sq^{-1} \gamma \omega^+ + s(q^{-2} - 1) \delta \omega^1 & \omega^{-}\gamma &= s \gamma \omega^2 + s(q^{-1} - q) \delta \omega^+ \\
\omega^{-}\delta &= sq^{-1} \delta \omega^+ & \omega^{-}\delta &= s q^{-2} \delta \omega^2 + s(q^{-1} - q) \gamma \omega^+ + s(q^{-1} - q)^2 \delta \omega^1
\end{align*}\]

Values and action of the generators on the q-group elements:

\[
\begin{align*}
\chi_1(\alpha) &= \frac{s q^{-2}}{q^3 - q} & \chi_1(\beta) &= 0 & \chi_1(\gamma) &= 0 & \chi_1(\delta) &= -\frac{q^2 + s(1 - q^2 + q^4)}{q^3 - q}
\end{align*}\]

\[
\begin{align*}
\chi_1(\alpha) &= \frac{sq^{-2}}{q^3 - q} & \chi_1(\beta) &= 0 & \chi_1(\gamma) &= 0 & \chi_1(\delta) &= 0
\end{align*}\]

\[
\begin{align*}
\chi_1(\alpha) &= \frac{sq^{-2}}{q^3 - q} & \chi_1(\beta) &= 0 & \chi_1(\gamma) &= 0 & \chi_1(\delta) &= 0
\end{align*}\]

\[
\begin{align*}
\chi_1(\alpha) &= \frac{sq^{-2}}{q^3 - q} & \chi_1(\beta) &= 0 & \chi_1(\gamma) &= 0 & \chi_1(\delta) &= 0
\end{align*}\]

Exterior derivatives of the basic elements of A:

\[
\begin{align*}
d\alpha &= \frac{s q^{-2}}{q^3 - q} \alpha \omega^1 - s \beta \omega^1 + \frac{s q^{-1}}{q^3 - q} \alpha \omega^2 \\
d\beta &= -\frac{q^2 + s(1 - q^2 + q^4)}{q^3 - q} \beta \omega^1 - s \alpha \omega^+ + \frac{s q^{-2}}{q^3 - q} \beta \omega^2 \\
d\gamma &= \frac{s q^{-2}}{q^3 - q} \gamma \omega^1 - s \delta \omega^1 + \frac{s q^{-1}}{q^3 - q} \gamma \omega^2 \\
d\delta &= \frac{s q^{-2} + s(1 - q^2 + q^4)}{q^3 - q} \delta \omega^1 - s \gamma \omega^1 + \frac{s q^{-2}}{q^3 - q} \delta \omega^2
\end{align*}\]

The \( \omega^i \) in terms of the exterior derivatives on \( \alpha, \beta, \gamma, \delta \):

\[
\begin{align*}
\omega^1 &= \frac{s q^{-2}}{q^3 - q}[(q^2 - s)(\kappa(\alpha) d \alpha + \kappa(\beta) d \gamma) + q^2(s - 1)(\kappa(\gamma) d \beta + \kappa(\delta) d \delta)] \\
\omega^+ &= -\frac{\kappa(\gamma) d \alpha + \kappa(\delta) d \gamma}{q^3 - q} \\
\omega^- &= -\frac{s \kappa(\delta) d \beta + \kappa(\beta) d \delta}{q^3 - q} \\
\omega^2 &= \frac{s q^{-2} + s(1 - q^2 + q^4)}{q^3 - q}[(s - q^2 - sq^2 + sq^4)((\kappa(\alpha) d \alpha + \kappa(\beta) d \gamma) + (q^2 - s)(\kappa(\gamma) d \beta + \kappa(\delta) d \delta)]
\end{align*}\]
Lie derivative on $\omega^i$:

\[
\begin{align*}
\chi_1 \ast \omega^1 &= q(q^2 - 1)\omega^1 + (q^{-1} - q)\omega^2 & \chi_+ \ast \omega^1 &= -q\omega^- \\
\chi_1 \ast \omega^+ &= -q^{-1}\omega^+ & \chi_+ \ast \omega^+ &= -q\omega^2 + q^3\omega^1 \\
\chi_1 \ast \omega^- &= [q(q^2 + 1) - q^{-1}]\omega^- & \chi_+ \ast \omega^- &= 0 \\
\chi_1 \ast \omega^2 &= -q(q^2 - 1)\omega^1 - (q^{-1} - q)\omega^2 & \chi_+ \ast \omega^2 &= q\omega^- \\
\chi_- \ast \omega^1 &= q\omega^+ & \chi_2 \ast \omega^1 &= 0 \\
\chi_- \ast \omega^+ &= 0 & \chi_2 \ast \omega^+ &= q\omega^+ \\
\chi_- \ast \omega^- &= q^{-1}\omega^2 - q\omega^1 & \chi_2 \ast \omega^- &= -q^{-1}\omega^- \\
\chi_- \ast \omega^2 &= -q\omega^+ & \chi_2 \ast \omega^2 &= 0
\end{align*}
\]
6 More $q$-geometry: the contraction operator and the Lie derivative

In section 4 we have seen that the $\chi_i$ defined by $da = (\chi_i * a) \omega^i$ are the quantum analogues of the tangent vectors at the origin of the group:

$$\chi_i \mapsto \left. \frac{\partial}{\partial x^i} \right|_{x=0}$$  \hspace{1cm} (6.1)

and that the left-invariant vector fields $t_i$ constructed from the $\chi_i$ are:

$$t_i = \chi_i * = (id \otimes \chi_i) \Delta$$  \hspace{1cm} (6.2)

$$t_i \mapsto e_i^\mu \frac{\partial}{\partial x^\mu}.$$  \hspace{1cm} (6.3)

There is a one-to-one correspondence $\chi_i \leftrightarrow t_i = \chi_i *$. In order to obtain $\chi_i$ from $\chi_i *$ we simply apply $\varepsilon$:

$$(\varepsilon \circ t_i)(a) = \varepsilon(id \otimes \chi_i)\Delta(a) = \varepsilon(a_1 \chi_i(a_2)) = \varepsilon(a_1)\chi_i(a_2) = \chi_i(\varepsilon \otimes id)\Delta(a) = \chi_i(a)$$  \hspace{1cm} (6.4)

[recall (2.5)].

Note 1: The vector space $T$ can also be defined intrinsically as the space of all linear functionals from $A$ to $\mathbb{C}$ such that $\chi(I) = 0$ and $\chi(a) = 0$ if $da = 0$; indeed from $0 = da = (\chi_i * a) \omega^i$ we have $\chi_i * a = 0$ and applying $\varepsilon$ we get $\chi_i(a) = 0$.

Note 2: The vector space $T$ is a quantum Lie algebra with Lie bracket $[\chi, \chi']$ as given in (4.82); the vector space $\text{inv} \Xi$ spanned by the left-invariant vector fields $t_i$ is also a Lie algebra with the induced Lie bracket $[t, t'] \equiv [\chi, \chi'] *$.

The * product of a functional with any $\tau \in \Gamma \otimes^n$ may be defined as

$$\chi * \tau \equiv (id \otimes \chi)\Delta_R(\tau),$$  \hspace{1cm} (6.5)

where the $\Delta_R$ acts on a generic element $\tau = \rho^1 \otimes \rho^2 \otimes \cdots \rho^n \in \Gamma \otimes^n$ as in (4.61).

Definition

We call quantum Lie derivative along the left-invariant vector field $t = (id \otimes \chi) \Delta$ the operator:

$$\ell_t \equiv \chi * ,$$  \hspace{1cm} (6.6)

that is

$$\ell_t(\tau) \equiv (id \otimes \chi)\Delta_R(\tau) = \chi * \tau \quad \ell_t : \Gamma \otimes^n \longrightarrow \Gamma \otimes^n .$$

For example:

$$\ell_t(a) = t(a), \quad a \in A,$$  \hspace{1cm} (6.7)
\[ \ell_{ti}(\omega^j) = (id \otimes \chi_i)\Delta_R(\omega^j) = \omega^k \chi_i(M^j_k) = C_{ki}^j \omega^k, \] (6.8)

the classical limit being evident.

The quantum Lie derivative has properties analogous to that of the ordinary Lie derivative:

i) it is linear in \( \tau \):
\[ \ell_t(\lambda \tau + \tau') = \lambda \ell_t(\tau) + \ell_t(\tau'); \] (6.9)

ii) it is linear in \( t \):
\[ \ell_{t+\lambda t'} = \lambda \ell_t + \ell_{t'}, \ \lambda \in C. \] (6.10)

By virtue of this last property we can just study \( \ell_{ti} \), where \( \{t_i\} \) is a basis of \( \text{inv} \Xi \).

**Theorem**

The following relation holds:
\[ \ell_{ti}(\tau \otimes \tau') = \ell_{tj}(\tau) \otimes f^{ji}_i \tau' + \tau \otimes \ell_{ti}(\tau') \] (6.11)

**Proof**

\[ \ell_{ti}(\tau \otimes \tau') =
= (id \otimes \chi_i)\Delta_R(\tau \otimes \tau')
= (id \otimes \chi_i)(\tau_1 \otimes \tau'_1) \otimes \tau_2 \tau'_2
= (\tau_1 \otimes \tau'_1)\chi_i(\tau_2 \tau'_2) = (\tau_1 \otimes \tau'_1)[\chi_j(\tau_2)f^{ji}_i(\tau'_2) + \varepsilon(\tau_2)\chi_i(\tau'_2)]
= \tau_1\chi_j(\tau_2) \otimes \tau'_1 f^{ji}_i(\tau'_2) + \tau_1 \varepsilon(\tau_2) \otimes \tau'_1 \chi_i(\tau'_2)
= \ell_{tj}(\tau) \otimes (id \otimes f^{ji}_i) \tau' + \tau \otimes \ell_{ti}(\tau') \]

[remember that \( \chi_j(a) \) and \( f^{ji}_i(a) \) are \( C \) numbers]. The same argument leads to:
\[ \ell_{tj}(a\omega^j) = \ell_{tk}(a)(f^{ki}_k \ast \omega^j) + a\ell_{ti}(\omega^j) \] (6.12)
\[ \ell_{tj}(\omega^j a) = \ell_{tk}(\omega^j)(f^{ki}_k \ast a) + \omega^j \ell_{ti}(a). \] (6.13)

The classical limit of (6.11) is easy to recover if we remember that \( \varepsilon \ast \tau = \tau \). Formulas (6.11), (6.7) and (6.8) uniquely define the quantum \( \ell_t \), which reduces, for \( q \to 1 \), to the classical Lie derivative.

**Theorem:**

The Lie derivative commutes with the exterior derivative:
\[ \ell_{ti}(d\vartheta) = d(\ell_{ti}\vartheta), \quad \vartheta: \text{generic form.} \] (6.14)

**Proof:**

\[ \ell_{ti}(d\vartheta) = (id \otimes \chi_i)\Delta_R(d\vartheta) = (id \otimes \chi_i)(d \otimes id)\Delta_R(\vartheta) =
(d \otimes \chi_i)\Delta_R(\vartheta) = d\vartheta_1 \chi_i(\vartheta_2) = d[\vartheta_1 \chi_i(\vartheta_2)] = d(\ell_{ti}\vartheta), \]

\[ \in C \]
where in the second equality we have used property (4.76).

**Theorem:**

The Lie derivative commutes with the left and right actions $\Delta_L$ and $\Delta_R$:

\[(id \otimes \ell_t)\Delta_L(\theta) = \Delta_L(\ell_t \theta)\]  
(6.15)

\[(id \otimes \ell_t)\Delta_R(\theta) = \Delta_R(\ell_t \theta), \quad \theta \in \Gamma^\otimes n.\]  
(6.16)

The proof is easy and relies on the fact that left and right actions commute, cf. eq. (4.19). In the classical limit, eq. (6.15) becomes:

\[\ell_t(L^*_x \theta) = L^*_x (\ell_t \theta).\]  
(6.17)

**Note 3:** It is not difficult to prove the associativity of the generalized $\ast$ product, for example that $(\chi \ast \chi') \ast \tau = \chi \ast (\chi' \ast \tau)$. From this property it follows that the $q$-Lie derivative is a representation of the $q$-Lie algebra:

\[[\ell_t, \ell_{t'}](\tau) = \ell_{[t,t']}(\tau),\]

where $[\ell_t, \ell_{t'}](\tau) \equiv [\chi, \chi'] \ast \tau$.

We now come to the construction of the contraction operator $i_t$ along the left-invariant vector field $t$.

**Definition**

The operator $i_t$ is characterized by:

\[\alpha) \quad i_t(a) = 0 \quad a \in A\]

\[\beta) \quad i_t(\omega^j) = \delta^j_i I\]

\[\gamma) \quad i_t(\omega^{i_1} \wedge \ldots \wedge \omega^{i_n}) = i_{t_j}(\omega^{i_1}) f^j_{i} (\omega^{i_2} \wedge \ldots \wedge \omega^{i_n}) - \omega^{i_1} \wedge i_{t_j}(\omega^{i_2} \wedge \ldots \wedge \omega^{i_n})\]

\[\delta) \quad i_t(a \vartheta + \vartheta') = ai_t(\vartheta) + i_t(\vartheta') \quad \vartheta, \vartheta' \text{ generic forms}\]

\[\varepsilon) \quad i_{\lambda t_i} = \lambda^i i_{t_i} \quad \lambda^i \in C\]

These relations uniquely define $i_t$, and its existence is ensured by the unicity of the expansion of a generic $n$-form on a basis of left-invariant 1-forms: $\vartheta = a_{i_1 i_2 \ldots i_n} \omega^{i_1} \wedge \ldots \wedge \omega^{i_n}$.

Relation $\delta$) expresses the $A$-linearity of $i_{t_i}$ (not just the $C$-linearity).

Relation $\gamma$) in the commutative limit reduces to the analog property of the classical contraction. This relation can be generalized by substituting $\otimes$ to $\wedge$. 

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Theorem:
With the above-defined contraction operator $i_t$, the Lie derivative can be expressed as:

$$\ell_{i_t} = i_t d + di_t. \quad (6.18)$$

A proof of this theorem is given in Appendix B, together with the proof of the property

$$i_t(\omega^{i_1} \wedge \ldots \wedge \omega^{i_n}) = i_{t_j}(\omega^{i_1} \wedge \ldots \wedge \omega^{i_s}) \wedge f^j_i \ast (\omega^{i_{s+1}} \wedge \ldots \wedge \omega^{i_n})$$

$$+ (-1)^s \omega^{i_1} \wedge \ldots \wedge \omega^{i_s} \wedge i_{t_i}(\omega^{i_{s+1}} \wedge \ldots \wedge \omega^{i_n}), \quad (6.19)$$

where $s$ and $n$ are integers such that $1 \leq s < n$.

By induction on $n$ one can also prove that

$$(id \otimes i_t)\Delta_L = \Delta_L i_t \quad (6.20)$$

holds on any $n$-form. This formula $q$-generalizes the classical commutativity of $i_t$ with the left action $\Delta_L$, when $t$ is a left- invariant vector field.
7 Softening the quantum group

As in the classical case, we may consider the softening of the $q$-group structure. The idea is to allow the right-hand side of the Cartan-Maurer equations (4.92) to be nonvanishing, i.e. to consider “deformations” $\mu^i$ of $\omega^i$ that are no longer left-invariant. The amount of “deformation” is measured, as in the classical case, by a $q$-curvature two-form $R^i$:

$$R^i = d\mu^i + C_{jk}^i \mu^j \wedge \mu^k$$  \hspace{1cm} (7.1)

For this definition to be consistent with $d^2 = 0$, the following $q$-Bianchi identities must hold:

$$dR^i - C_{jk}^i R^j \wedge \mu^k + C_{jk}^i \mu^j \wedge R^k = 0;$$  \hspace{1cm} (7.2)

these are easily obtained by taking the exterior derivative of (7.1) and using the $q$-Jacobi identities for the $C$ structure constants given in (4.95).

The bimodule structure of the deformed $\Gamma$ is assumed to be unchanged, i.e. the commutations between elements of $A$ and elements of the deformed $\Gamma$ are unchanged. Also, the definition (4.65) for the wedge product is still kept unaltered, so that the commutations between the $\mu^i$ are identical to those for the $\omega^i$ given in (5.37):

$$\mu^i \wedge \mu^j = -Z_{ij}^{kl} \mu^k \wedge \mu^l,$$  \hspace{1cm} (7.3)

with $Z$ given by (5.38). Note that by taking the exterior derivative of (7.3) we can infer the commutations of $R^i$ with $\mu^j$:

$$R^i \wedge \mu^j - \mu^i \wedge R^j = -Z_{ij}^{kl} (R^k \wedge \mu^l - \mu^k \wedge R^l).$$  \hspace{1cm} (7.4)

Indeed the terms trilinear in $\mu$ that arise after using $d\mu = R - C\mu\mu$ do cancel, since they cancel in the case $R^i = 0$, and the wedge products are unaltered.

In the constructive procedure of Section 5, we have defined the exterior derivative to act as:

$$da = \frac{1}{\lambda}[\tau a - a\tau]$$  \hspace{1cm} (7.5)

$$d\theta = \frac{1}{\lambda}[\tau \wedge \theta - (-1)^k \theta \wedge \tau]$$  \hspace{1cm} (7.6)

with $\theta \in \Gamma^k$. It is now clear that eq. (7.6) must be modified. Indeed this equation, with $\tau = \mu^b$, leads to the Cartan-Maurer equations (5.56), since the commutations between the $\mu^i$ just mimic those between the left-invariant $\omega^i$. Then we define the exterior differential as:

$$da = \frac{1}{\lambda}[sa - as]$$  \hspace{1cm} (7.7)

$$d\theta = \frac{1}{\lambda}[s \wedge \theta - (-1)^k \theta \wedge s],$$  \hspace{1cm} (7.8)

with

$$s = \tau + \phi.$$  \hspace{1cm} (7.9)
It is not difficult to see that this $d$ still satisfies the usual properties of the exterior derivative, provided
\[ s \wedge s = 0, \quad (7.10) \]
and that it can be extended over the whole “soft” exterior algebra in the same way as in the undeformed case.

From $s \wedge s = 0$ we find:
\[ \tau \wedge \phi + \phi \wedge \tau + \phi \wedge \phi = 0 \quad (7.11) \]
since $\tau \wedge \tau = 0$ still holds. It is easy to compute the curvatures, as defined in (7.1), in terms of $\phi$:
\[ R^i = \frac{1}{\lambda} [s \wedge \mu^i + \mu^i \wedge s] + C_{jk} i \mu^j \wedge \mu^k = \frac{1}{\lambda} [\phi \wedge \mu^i + \mu^i \wedge \phi]. \quad (7.12) \]
where the last equality is due to the fact that if $s = \tau$ the Cartan-Maurer equations hold ($\Rightarrow R^i = 0$). Similarly we find the curvature of $\tau$:
\[ R(\tau) = \frac{1}{\lambda} [\phi \wedge \tau + \tau \wedge \phi] = -\frac{1}{\lambda} [\phi \wedge \phi], \quad (7.13) \]
the last equality being due to (7.11).

A more detailed discussion on the differential calculus corresponding to this “soft” exterior derivative will be given in a later publication. Here we mention that the soft calculus allows the definition of quantum “diffeomorphisms”:
\[ \delta_t \mu^k \equiv \ell_t \mu^k = (i_t d + di_t) \mu^k = (\nabla t)^k + i_t R^k, \quad (7.14) \]
where $\nabla$ is the quantum covariant derivative whose definition can be read off the Bianchi identities (7.2) $\nabla R^k = 0$. The construction of an action, invariant under these diffeomorphisms, proceeds as in the classical case. We refer to [3, 9] for some preliminary applications of this formalism to the construction of $q$-gravity and $q$-gauge theories.
A The derivation of two equations

In this Appendix we derive the two equations (4.100) and (4.102). Consider the exterior derivative of eq. (4.30):

\[ d(\omega^i a) = d[(f^i \ast a) \omega^j]. \]  

(A.1)

The left-hand side is equal to:

\[ d(\omega^i a) =\]
\[= d\omega^i \wedge a - \omega^i \wedge da = -C_{jk}^i \omega^j \wedge \omega^k a - \omega^i \wedge (\chi_j \ast a) \omega^j =\]
\[= -C_{jk}^i \omega^j \wedge \omega^k a - (f^s \ast \chi_j \ast a) \omega^s \wedge \omega^j =\]
\[= -C_{jk}^i (f^j_p \wedge f^k_q \ast a) \omega^p \wedge \omega^q - (f^s \ast \chi_j \ast a) (\omega^s \wedge \omega^j - \Lambda^{s j}_{p q} \omega^p \wedge \omega^q) =\]
\[= \left[ -C_{jk}^i f^j_p f^k_q - f^i_p \chi_q + \Lambda^{s j}_{p q} f^s \chi_j \ast a \right] (\omega^p \wedge \omega^q). \]  

(A.2)

The right-hand side reads:

\[ d[(f^i \ast a) \omega^j] =\]
\[= d(f^i \ast a) \omega^j + (f^i \ast a) d\omega^j =\]
\[= (\chi_k \ast f^i \ast a) \omega^j \wedge \omega^k - (f^i \ast a) C_{pq}^j \omega^p \wedge \omega^q =\]
\[= (\chi_k \ast f^i \ast a) (\omega^k \wedge \omega^j - \Lambda^{k j}_{p q} \omega^p \wedge \omega^q) - (f^i \ast a) C_{pq}^j \omega^p \wedge \omega^q =\]
\[= \left[ (\chi_k f^i_p - \Lambda^{k j}_{p q} \chi_k f^i_j - C_{pq}^j f^i_j) \ast a \right] (\omega^p \wedge \omega^q), \]  

(A.3)

so that we deduce the equation

\[ -C_{jk}^i f^j_p f^k_q - f^i_p \chi_q + \Lambda^{s j}_{p q} f^s \chi_j =\]
\[= \chi_p f^i_q - \Lambda^{k j}_{p q} \chi_k f^i_j - C_{pq}^j f^i_j. \]  

(A.4)

We now need two lemmas.

Lemma 1

\[ f^n_r \ast a \theta = (f^n_r \ast a)(f^n_r \ast \theta), \quad a \in A, \; \theta \in \Gamma^{\otimes n}. \]  

(A.5)

Proof:

\[ f^n_r \ast a \theta =\]
\[= (id \otimes f^n_r) \Delta(a) \Delta_R(\theta) = a_1 \theta_1 f^n_r(a_2 \theta_2) =\]
\[= a_1 \theta_1 f^n_r(a_2) f^n_r(\theta_2) = a_1 f^n_r(a_2) \theta_1 f^n_r(\theta_2) =\]
\[= (f^n_r \ast a) \theta_1 f^n_r(\theta_2) = (f^n_r \ast a)(f^n_r \ast \theta). \]  

(A.6)
Lemma 2

\[ f^r_l \ast \omega^j = \Lambda^{rj}_{kl} \omega^k. \quad (A.7) \]

Proof:

\[ f^r_l \ast \omega^j = \]
\[ (id \otimes f^r_l) \Delta_R(\omega^j) = (id \otimes f^r_l)[\omega^k \otimes M^j_k] = \]
\[ = \omega^k f^r_l(M^j_k) = \Lambda^{rj}_{kl}. \quad (A.8) \]

Consider now eq. (4.99) with \( h = f^n_l \):

\[ d(f^n_l \ast a) = f^n_l \ast da. \quad (A.9) \]

The first member is equal to \( (\chi^*_k \ast f^n_l \ast a) \omega^k \), while the second member is:

\[ f^n_l \ast da = f^n_l \ast [(\chi^*_j \ast a) \omega^j] = (f^n_r \ast \chi^*_j \ast a)(f^r_l \ast \omega^j) = (f^n_r \ast \chi^*_j \ast a)(\Lambda^{rj}_{kl} \omega^k) \quad (A.10) \]

We have used here the two lemmas (A.5) and (A.7). Therefore the following equation holds:

\[ \chi^*_k \ast f^n_l = \Lambda^{rj}_{kl} f^n_r \ast \chi^*_j, \quad (A.11) \]

which is just eq. (4.100). Equation (4.102) is obtained simply by subtracting (A.11) from eq. (4.10).
B Two theorems on $i_t$ and $\ell_t$

Theorem

The contraction operator $i_t$ satisfies:

$$\forall n, \forall s : 1 \leq s < n, \quad i_t(\omega^{i_1} \land \omega^{i_2} \land \ldots \land \omega^{i_n}) = i_t(\omega^{i_1} \land \omega^{i_2} \land \ldots \land \omega^{i_s}) \land f^j_i \ast (\omega^{i_{s+1}} \land \ldots \land \omega^{i_n}) + (-1)^s \omega^{i_1} \land \ldots \land \omega^{i_s} \land i_t(\omega^{i_{s+1}} \land \ldots \land \omega^{i_n}).$$ (B.1)

Proof

For all $n$, when $s = 1$ (B.1) is just property $\gamma$ of the definition of $i_t$ (see Section 6). We prove the theorem by induction on $s$. Suppose that (B.1) be true for $s - 1$; then it is true for $s$. Indeed:

$$i_t(\omega^{i_1} \land \ldots \land \omega^{i_n}) =$$

$$= i_t(\omega^{i_1} \land \ldots \land \omega^{i_{s-1}}) \land f^j_i \ast (\omega^{i_s} \land \ldots \land \omega^{i_n}) + (-1)^{s-1} \omega^{i_1} \land \ldots \land \omega^{i_{s-1}} \land i_t(\omega^{i_s} \land \ldots \land \omega^{i_n}) =$$

$$= i_t(\omega^{i_1} \land \ldots \land \omega^{i_{s-1}}) \land f^j_i \land f^j_k \ast \omega^{i_s} \land f^k_i \ast (\omega^{i_{s+1}} \land \ldots \land \omega^{i_n}) + (-1)^{s-1} \omega^{i_1} \land \ldots \land \omega^{i_{s-1}} \land i_t(\omega^{i_s} \land \ldots \land \omega^{i_n}) + (-1)^{s-1} \omega^{i_1} \land \ldots \land \omega^{i_{s-1}} \land \omega^{i_s} \land i_t(\omega^{i_{s+1}} \land \ldots \land \omega^{i_n}) =$$

$$= i_t(\omega^{i_1} \land \ldots \land \omega^{i_{s-1}} \land \omega^{i_s}) \land f^j_i \ast (\omega^{i_{s+1}} \land \ldots \land \omega^{i_n}) + (-1)^{s-1} \omega^{i_1} \land \ldots \land \omega^{i_{s-1}} \land i_t(\omega^{i_s} \land \ldots \land \omega^{i_n}) + (-1)^{s-1} \omega^{i_1} \land \ldots \land \omega^{i_s} \land i_t(\omega^{i_{s+1}} \land \ldots \land \omega^{i_n});$$

in the last equality we have used the inductive hypothesis.

We can conclude that (B.1) is true for all $s : 1 \leq s < n$. Q.E.D.

Remembering the $A$-linearity of $i_t$, the subsequent generalization is straightforward:

$$i_t(a_{i_1 \ldots i_n} \omega^{i_1} \land \ldots \land \omega^{i_n}) = i_t(a_{i_1 \ldots i_n} \omega^{i_1} \land \ldots \land \omega^{i_s}) \land f^j_i \ast (\omega^{i_{s+1}} \land \ldots \land \omega^{i_n}) + (-1)^s a_{i_1 \ldots i_n} \omega^{i_1} \land \ldots \land \omega^{i_s} \land i_t(\omega^{i_{s+1}} \land \ldots \land \omega^{i_n})$$

with $a_{i_1 \ldots i_n} \in A$. 

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Theorem

\[ \ell_t = i_t d + d i_t \]

that is

\[ \forall a_{i_1 \ldots i_n} \omega^{i_1} \wedge \ldots \wedge \omega^{i_n} \in \Gamma^{\wedge n}, \]

\[ \ell_t(a_{i_1 \ldots i_n} \omega^{i_1} \wedge \ldots \wedge \omega^{i_n}) = (i_t d + d i_t)(a_{i_1 \ldots i_n} \omega^{i_1} \wedge \ldots \wedge \omega^{i_n}). \quad (B.2) \]

We will show this theorem by induction on the integer \( n \). To do this, we need the following:

Lemma

If \( n = 1 \), the theorem is true, i.e.

\[ \ell_t(b_k \omega^k) = (i_t d + d i_t)(b_k \omega^k). \quad (B.3) \]

First we show that:

\[ \ell_t(\omega^k) = (i_t d + d i_t)\omega^k. \quad (B.4) \]

We already know that \( \ell_t(\omega^k) = \omega^j C_{ji}^k \). The right-hand side of (B.4) yields:

\[
(i_t d + d i_t)(\omega^k) = i_t d \omega^k + d(i_t \omega^k) =
\]

\[
- C_{nj}^k i_t(\omega^j \wedge \omega^k) =
\]

\[
- C_{nj}^k \left( f^n_i * \omega^j - \omega^n \delta_i^j \right) =
\]

\[
= - C_{nj}^k \left[ (id \otimes f^n_i) \Delta_R(\omega^j) - \delta_i^j \omega^n \right] =
\]

\[
= - C_{nj}^k \left[ (id \otimes f^n_i) \left( \omega^\ell \otimes M_j^\ell \right) - \delta_i^j \omega^n \right] =
\]

\[
= - C_{nj}^k \left[ \omega^\ell \Lambda_{nj}^j \ell_i - \delta_i^j \omega^n \right] =
\]

\[
= + C_{nj}^k \left[ \delta_i^j \delta_j^k - \Lambda_{nj}^j \ell_i \right] \omega^\ell =
\]

\[ C_{\ell i}^k \omega^\ell \]

and (B.4) is thus proved.

The right-hand side of (B.3) gives:

\[
(i_t d + d i_t)(b_k \omega^k) =
\]

\[
= i_t \left( db_k \wedge \omega^k + b_k d \omega^k \right) + d \left( b_k i_t(\omega^k) \right) =
\]

\[
= i_t \left( db_k f^j_i * \omega^k - (db_k)i_t(\omega^k) \right) +
\]

\[
+ b_k i_t(d \omega^k) + \left( db_k \right)i_t(\omega^k) =
\]

\[
= i_t \left( X_n * b_k(\omega^k) \right) f^j_i * \omega^k + b_k i_t(d \omega^k) =
\]

\[
= \left( X_n * b_k \right) \delta^n_j f^j_i * \omega^k + b_k i_t(d \omega^k) =
\]

\[
= \left( X_n * b_k \right)f^n_i * \omega^k + b_k \ell_{i_t}(\omega^k) =
\]

\[
= \ell_{i_t}(b_k \omega^k),
\]

and the lemma is proved. We now finally prove the theorem.

Let us suppose it to be true for a \((n-1)\)-form:

\[ \ell_{i_t}(a_{i_2 \ldots i_n} \omega^{i_2} \wedge \ldots \wedge \omega^{i_n}) = (i_t d + d i_t)(a_{i_2 \ldots i_n} \omega^{i_2} \wedge \ldots \wedge \omega^{i_n}). \quad (B.5) \]
Then it holds also for an $n$-form. Indeed, the left-hand side of (B.2) yields

$$\ell_t(a_{i_1...i_n}\omega^{i_1} \wedge ... \wedge \omega^{i_n}) = \ell_t(a_{i_1...i_n}\omega^{i_1}) \wedge f^j_i * (\omega^{i_2} \wedge ... \wedge \omega^{i_n}) + a_{i_1...i_n}\omega^{i_1} \wedge \ell_t(\omega^{i_2} \wedge ... \wedge \omega^{i_n})$$

while the right-hand side of (B.2) is given by:

$$(i_t d + di_t)(a_{i_1...i_n}\omega^{i_1} \wedge ... \wedge \omega^{i_n}) = i_t[d(a_{i_1...i_n}\omega^{i_1}) \wedge \omega^{i_2} \wedge ... \wedge \omega^{i_n} - (a_{i_1...i_n}\omega^{i_1}) \wedge d(\omega^{i_2} \wedge ... \wedge \omega^{i_n})] +$$

$$d[i_t f_j(a_{i_1...i_n}\omega^{i_1}) f^j_i * (\omega^{i_2} \wedge ... \wedge \omega^{i_n}) - (a_{i_1...i_n}\omega^{i_1}) \wedge i_t(\omega^{i_2} \wedge ... \wedge \omega^{i_n})] =$$

$$i_t(d a_{i_1...i_n}\omega^{i_1}) \wedge f^j_i * (\omega^{i_2} \wedge ... \wedge \omega^{i_n}) + d(a_{i_1...i_n}\omega^{i_1}) \wedge i_t(\omega^{i_2} \wedge ... \wedge \omega^{i_n}) +$$

$$- i_t f_j(a_{i_1...i_n}\omega^{i_1}) f^j_i * d(\omega^{i_2} \wedge ... \wedge \omega^{i_n}) + a_{i_1...i_n}\omega^{i_1} \wedge i_t d(\omega^{i_2} \wedge ... \wedge \omega^{i_n}) +$$

$$+d i_t f_j(a_{i_1...i_n}\omega^{i_1}) f^j_i * (\omega^{i_2} \wedge ... \wedge \omega^{i_n}) + i_t f_j(a_{i_1...i_n}\omega^{i_1}) \wedge f^j_i * d(\omega^{i_2} \wedge ... \wedge \omega^{i_n}) +$$

$$-d(a_{i_1...i_n}\omega^{i_1}) \wedge i_t(\omega^{i_2} \wedge ... \wedge \omega^{i_n}) + a_{i_1...i_n}\omega^{i_1} \wedge d i_t(\omega^{i_2} \wedge ... \wedge \omega^{i_n}) =$$

$$[(i_t d + di_t)(a_{i_1...i_n}\omega^{i_1})] \wedge f^j_i * (\omega^{i_2} \wedge ... \wedge \omega^{i_n}) +$$

$$+a_{i_1...i_n}\omega^{i_1}(i_t d + di_t)(\omega^{i_2} \wedge ... \wedge \omega^{i_n}) =$$

$$\ell_t(a_{i_1...i_n}\omega^{i_1}) \wedge f^j_i * (\omega^{i_2} \wedge ... \wedge \omega^{i_n}) + a_{i_1...i_n}\omega^{i_1} \wedge \ell_t(\omega^{i_2} \wedge ... \wedge \omega^{i_n})$$

and the theorem is proved.
C  A collection of formulas

We list here some useful formulas. Most have been derived in the paper, or are particular cases of those.

\[ d\omega^i = -C_{jk}^i \omega^j \wedge \omega^k = -C_{jk}^i \omega^j \otimes \omega^k \]

\[ C_{jk}^i = (\chi_j \ast \chi_k)(x^i), \quad C_{jk}^i = [\chi_j, \chi_k](x^i) = \chi_k(M^i_j) \]

\[ f^i_j(M^k_i) = \Lambda^{ik} l_j \]

\[ \chi_i(ab) = \chi_j(a) f^j_i(b) + \varepsilon(a) \chi_i(b) \]

\[ x^j \in A : \quad \chi_i(x^j) = \delta^j_i \varepsilon(x^j) = 0 \]

\[ \chi_i(x^j b) = f^j_i(b) \]

\[ (\chi_i \ast ab) = (\chi_j \ast a)(f^j_i \ast b) + a(\chi_i \ast b) \]

\[ \chi_i \ast \omega^j = C_{ki}^j \omega^k, \quad f^i_j \ast \omega^k = \Lambda^{ik} l_j \omega^j \]

\[ \chi_i \ast M^j_k = C_{li}^j M^l_k, \quad f^i_j \ast M^l_k = \Lambda^{il} m_j M^m_k \]

\[ \chi_i \ast (a \theta) = (\chi_j \ast a)(f^j_i \ast \theta) + a(\chi_i \ast \theta) \]

\[ \chi_i \ast (\theta a) = (\chi_j \ast \theta)(f^j_i \ast a) + \theta(\chi_i \ast a) \]

\[ f^i_j \ast (a \theta) = (f^j_i \ast a)(f^k_j \ast \theta) \]

\[ f^i_j \ast (\theta a) = (f^j_i \ast \theta)(f^k_j \ast a) \]

\[ \ell_t(\tau \otimes \tau') = \ell_t(\tau) \otimes f^j_i \ast \tau' + \tau \otimes \ell_t(\tau') \]

\[ M^i_j(a \ast f^i_k) = (f^j_i \ast a)M^i_k \]

\[ [\chi_i, \chi_j] = C_{ij}^k \chi_k \]

\[ [[\chi_r, \chi_i], \chi_j] - \Lambda^{kl}_{ij} [[\chi_r, \chi_k], \chi_l] = [\chi_r, [\chi_i, \chi_j]] \]

\[ \Delta'(\chi_i) = \chi_j \otimes f^j_i + \varepsilon \otimes \chi_i, \quad \varepsilon'(\chi_i) = 0 \]

\[ C_{mn} f^m_j f^n_k + f^i_j \chi_k = \Lambda^{pq} j_k \chi_p f^i_q + C_{jk}^l f^i_l \]

\[ \chi_k f^n_l = \Lambda^{ij} k_l f^n_i \chi_j \]
\[ \Lambda_{ij}^{nm} f^i_p f^f_j q = f^n_i f^m_j \Lambda_{pq}^{ij} \]

\[ \Delta'(f^i_j) = f^{i_k} \otimes f^{k_j} \]

\[ \varepsilon'(f^i_j) = \delta^i_j \]

\[ \kappa'(f^i_j) f^f_j = f^i_i \kappa'(f^f_j) = \delta^i_j \varepsilon, \quad \kappa'^{-1}(f^f_i) f^j_i = f^i_i \kappa'^{-1}(f^f_i) = \delta^i_j \varepsilon \]

\[ M^{ij} M_{pk} q \Lambda_{ir}^{pq} = \Lambda_{ir}^{pq} r_i M_p^{r} M_k^{i} \]

\[ \Delta(M_i^j) = M_i^k \otimes M_k^j, \quad \varepsilon(M_i^j) = \delta^i_j \]

\[ \kappa(M_i^j)(M_j^l) = M_i^j \kappa(M_j^l) = \delta_i^l I, \quad \kappa^{-1}(M_i^j)(M_j^l) = M_j^l \kappa^{-1}(M_i^j) = \delta_i^l I \]

\[ \ell_t(d\tau) = d(\ell_t \tau) \]

\[ (id \otimes \ell_t) \Delta_L(\tau) = \Delta_L(\ell_t \tau) \]

\[ (id \otimes \ell_t) \Delta_R(\tau) = \Delta_R(\ell_t \tau) \]

\[ i_{\ell_t}(\tau \otimes \tau') = i_{\ell_t}(\tau) \otimes f^j_i(\tau') + \tau \otimes i_{\ell_t}(\tau') \]

\[ i_{\ell_t}(\tau \wedge \tau') = i_{\ell_t}(\tau) \wedge f^j_i(\tau') + \tau \wedge i_{\ell_t}(\tau') \]

\[ (id \otimes \ell_t) \Delta_L = \Delta_L \ell_t \]

\[ \ell_t(\theta) = [i_{\ell t} d + di_{\ell t}](\theta) \]
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