STRICT AND NON STRICT POSITIVITY OF DIRECT IMAGE BUNDLES.

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ABSTRACT. This paper is a sequel to [2]. In that paper we studied the vector bundle associated to the direct image of the relative canonical bundle of a smooth Kähler morphism, twisted with a semipositive line bundle. We proved that the curvature of such a vector bundle is always semipositive (in the sense of Nakano). Here we address the question if the curvature is strictly positive when the Kodaira-Spencer class does not vanish. We prove that this is so provided the twisting line bundle is strictly positive along fibers, but not in general.

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1. INTRODUCTION

Let \( p : \mathcal{X} \rightarrow Y \) be a smooth proper holomorphic fibration of complex manifolds of relative dimension \( n \), and let \( \mathcal{L} \rightarrow \mathcal{X} \) be a holomorphic line bundle equipped with a smooth metric of semipositive curvature. The direct image sheaf of the relative canonical bundle twisted with \( \mathcal{L} \),

\[ p_*(\mathcal{L} + K_{\mathcal{X}/Y}), \]

is then associated to a vector bundle, \( E \), over \( Y \) with fibers

\[ E_y = H^0(\mathcal{X}_y, K_{\mathcal{X}_y} + \mathcal{L}|_{\mathcal{X}_y}), \]

see e.g. [2]. In [2] we have shown that if \( \mathcal{X} \) is Kähler, the natural \( L^2 \)-metric on \( E \) has nonnegative curvature in the sense of Nakano. In this paper we shall, in two special cases, discuss more explicit formula for the curvature, which enables us to determine when the curvature is strictly positive. We will all the time consider only the case of a one dimensional base, but the computations generalize to the case of a base of higher dimension. For the moment however, these higher dimensional computations seem to give little more with regard to Nakano positivity than what is already contained in [2], so we omit them here. (The curvature in the sense of Griffiths can be obtained from the case of one dimensional base.)

We concentrate on two particular cases that are somewhat opposite. The first case is when \( \mathcal{L} \) is trivial, so that we are dealing with the direct image of the relative canonical bundle itself. The second case is when \( \mathcal{L} \) is strictly positive on all fibers.

In the first case the semipositivity theorem is already contained in the work of Griffiths, see [8], p 34, and further developed by Fujita, [7].

When the fibration is trivial it is clear that the curvature vanishes, so one expects that the positivity of \( E \) will depend on how far from being trivial the fibration is. This is measured by the Kodaira-Spencer class. We recall its definition in the next section; for the moment it is enough to remember that, at a point \( t \) in the base, it is given by an element in \( H^{0,1}(\mathcal{X}_t, T^{1,0}(\mathcal{X}_t)) \), which vanishes if the fibration is trivial (to first order) at \( t \). Let us denote this class by \( K_t \); it is represented by \( \partial \bar{\partial} \)-closed \((0,1)\)-forms with values in the holomorphic tangent bundle of the fiber.
An element in $E_t$ is a holomorphic $(n,0)$-form, $u$, on $X_t$. The Kodaira-Spencer class acts on $u$ in a natural way: If $k_t$ is a vectorvalued $(0,1)$ form in $\mathcal{K}_t$, which locally decomposes as

$$k_t = w \otimes v,$$

where $w$ is scalar valued and $v$ is a vector field, then first we let the vector field part of $k_t$ act on $u$ by contraction

$$k_t . u := w \wedge \delta_v u.$$

This gives a globally defined $\overline{\partial}$-closed form of bidegree $(n-1,1)$ and

$$\mathcal{K}_t . u := [k_t . u],$$

an element in $H^{n-1,1}(X_t)$. The following theorem is due to Griffiths, see [8] and further references there, but we shall also discuss how it follows from the formalism in [2] in sections 3 and 4 of this paper.

**Theorem 1.1.** Let $\Theta^E$ be the curvature of $E$ with the natural $L^2$-metric. Then

$$\langle \Theta^E u, u \rangle = \| \mathcal{K}_t . u \|^2. \quad (1.1)$$

The right hand side is the norm of the class $\mathcal{K}_t . u$ with respect to the given Kähler metric, i.e. the norm of its unique harmonic representative. It does not depend on the choice of Kähler metric.

It is clear from Theorem 1.1 that if the Kodaira-Spencer class vanishes, then $\Theta^E u = 0$ for any $u$ in $E_t$. It may however very well happen that $\mathcal{K}_t . u$, and hence $\Theta^E u$, vanish for some choice of $u$ even if $\mathcal{K}_t \neq 0$. This happens precisely when the class $\mathcal{K}_t$ contains a current, not cohomologous to zero, supported on the zero divisor of $u$. In section 3 we will give explicit examples of this, when the fibers of $p$ are Riemann surfaces of genus at least 2. We shall see that any compact Riemann surface of genus at least 2 can be put as the central fiber of some fibration as above, in such a way that the curvature is degenerate (i.e. not strictly positive), although the Kodaira-Spencer class is not zero. We will also give examples where the Kodaira-Spencer class is not zero at the central fiber, but the curvature $\Theta^E$ is zero applied to any $u$ in $E_0$. This is more exceptional; when fibers are compact Riemann surfaces this can be done precisely when the central fiber is hyperelliptic of genus at least three.

The other particular case that we study is when $\mathcal{L}$ is nontrivial, and has a metric $\phi$ of strictly positive curvature along the fibers of $\mathcal{X}$; we will call this the relatively ample case. Of course we do not assume that the curvature is strictly positive on the total space $\mathcal{X}$, but just on the fibers. We will see that in this case the moral is opposite to when $\mathcal{L}$ is trivial: If the Kodaira-Spencer class is not zero, then the curvature $\Theta^E$ is strictly positive. We emphazise again that this holds as soon as the metric on $\mathcal{L}$ is positive fiberwise, and no assumption is made on positivity of $\mathcal{L}$ in 'horizontal' directions. Thus the main conclusion of this paper is that degeneracy of $\Theta^E$ implies that the fibration is (infinitesimally) trivial, provided $\mathcal{L}$ is relatively ample, whereas this does not hold in general.

In the relatively ample case we have already discussed strict positivity in [2] and [3] for the case when the fibration is trivial. We shall now extend this to nontrivial fibrations.

To formulate our result in that case we first recall a few well known facts. On $\mathcal{X}$ there is a certain smooth vector field, $V_\phi$ depending on the metric, see [14],[11],[3] and [1]. It is defined
as follows. Choose local coordinates \((t, z)\) on \(\mathcal{X}\) that respect the fibration so that \(p(t, z) = t\). Let the metric \(\phi\) be represented by a local function \(\varphi\), with respect to some local trivialization of \(\mathcal{L}\) over the coordinate neighbourhood. Put

\[ \dot{\varphi}_t = \frac{\partial \varphi}{\partial t}, \]

and define on each fiber \(X_t\) a vector field \(W_\varphi\) by

\[ \delta W_\varphi i(\partial \bar{\partial}) z \varphi = \bar{\partial} \dot{\varphi}_t. \]

This defines \(W_\varphi\) uniquely if \(i\partial \bar{\partial} \varphi > 0\) on fibers; \(W_\varphi\) is the complex gradient of \(\dot{\varphi}_t\). There is no reason why \(W_\varphi\) should be independent of the choices we made, but as proved in [11], the field

\[ V_\phi := \frac{\partial}{\partial t} - W_\varphi\]

is a well defined smooth field on \(\mathcal{X}\) if we let \(t\) vary (we will reprove this in Lemma 4.1). Moreover, \(\bar{\partial}_t V_\phi, \partial V_\phi\) restricted to a fiber \(X_t\), is a representative of the Kodaira-Spencer class \(K_t\). We denote this representative \(k_\phi^t\); it vanishes precisely when \(V_\phi\) is holomorphic on \(X_t\).

We need one more ingredient in order to state our next result. Let

\[ (1.2) \quad c(\phi) := \frac{\partial \dot{\varphi}_t}{\partial t} - |\bar{\partial} \dot{\varphi}_t|^2, \]

where we measure \(\bar{\partial} \dot{\varphi}_t\) with respect to the metric \(\omega^t = i\partial \bar{\partial} \varphi|_{X_t}\). As proved in [13]

\[ (1.3) \quad (i\partial \bar{\partial} \varphi)_{n+1} = c(\phi)(i\partial \bar{\partial} \varphi)_n \wedge dt \wedge d\bar{t}. \]

Hence \(c(\phi)\) is a globally well defined function on \(\mathcal{X}\) which measures the (lack of) strict positivity of \(i\partial \bar{\partial} \varphi\) on \(\mathcal{X}\).

**Theorem 1.2.** Assume \(\phi\) is a metric on \(\mathcal{L}\) with \(i\partial \bar{\partial} \varphi\) strictly positive on fibers. The curvature of the \(L^2\)-metric on \(E\) is then given at \(t\) by

\[ \langle \Theta^E u, u \rangle = \int_{X_t} c(\phi)|u|^2 e^{-\varphi} + \langle (\Box' + 1)^{-1} \eta, \eta \rangle, \]

with \(\eta = k_\phi^t u\). Here \(\Box' = \nabla'(\nabla')^* + (\nabla')^* \nabla'\) is the Laplacian on \(\mathcal{L}|_{X_t}\)-valued forms on \(X_t\) defined by the \((1, 0)\)-part of the Chern connection on \(\mathcal{L}|_{X_t}\).

Several conclusions can be drawn from this. First we note that if \(\Theta^E u = 0\) for some \(u\) in \(E_t\), then \(c(\phi) = 0\) and \(k_\phi^t u = 0\). The second condition implies that \(k_\phi^t = 0\). (This is because \(k_\phi^t\) is now one fixed smooth form, and not a cohomology class, so if it vanishes outside the zero divisor of \(u\), it vanishes identically.) Hence the Kodaira-Spencer class vanishes, which as we saw is not necessarily the case for \(L\) trivial. It also follows that \(\Theta^E u = 0\) for all \(u\) in \(E_t\). Moreover \(V_\phi\) is holomorphic on \(X_t\) and we shall see in the last section that even

\[ \frac{\partial}{\partial t} V_\phi \]

vanishes at \(t\), so \(V_\phi\) is holomorphic to first order also in directions transverse to the fiber. Finally, \(V_\phi\) can be lifted to a field on \(\mathcal{L}|_{X_t}\) with the same properties, and the infinitesimal automorphism...
of $\mathcal{L}$ defined by the lift preserves the metric on $\mathcal{L}$. All in all, if the curvature is degenerate in the relatively ample case, the fibers and the line bundle over it just move by an infinitesimal automorphism as we vary the point in the base.

In the special case when we have a fibration where the fibers are Riemann surfaces of genus at least 2, $\mathcal{L} = K_{X/\Delta}$ and $\phi$ restricts to the Kähler-Einstein metric on the fibers $X_t$, the metric on $E$ is dual to the Weil-Petersson metric. Explicit formulas for the curvature of $E$ were in this case found by Wolpert, see [17], and also Liu-Sun-Yau, [18] for much more on this area. In section 5 we show how Wolpert’s formula follows from (1.4), via a theorem of Schumacher.

One might also note that since $\square' \geq 0$, (1.4) also gives an estimate from above of the curvature

$$\langle \Theta E u, u \rangle \leq \int_{X_t} \left( c(\phi) + |\bar{\partial}V_\phi|^2 \right) |u|^2 e^{-\phi}. \quad (1.5)$$

The plan of the rest of the paper is as follows. In the next section we summarize some background material, including the curvature formula from [2] that we will use. Although brief, the discussion here is essentially self contained, and somewhat easier than in [2], since we only deal with the case of onedimensional base. In section 3 we prove Theorem 1 and give the examples when the curvature is degenerate (or even 0) even though the Kodaira-Spencer class does not vanish. (The reader who is only interested in the examples can go directly to section 3.1.) In section 3.2 we discuss the relation between the Chern connection on $E$ (for $\mathcal{L} = 0$) and the Gauss-Manin connection on the Hodge bundle of the fibration. This gives an alternate proof of Theorem 1.1 and also proves Griffiths theorem that $E$ is a holomorphic subbundle of the Hodge bundle. The last section contains the proof of Theorem 1.2 and the consequences of it mentioned above.

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2. Background material

2.1. The bundle $E$ and its metric. We consider the general setting from the introduction and recall the setup from [2]. In particular, we assume throughout that $\mathcal{X}$ is Kähler, and we let $\omega$ denote some choice of Kähler form on $\mathcal{X}$. Since the discussion is local we take the base to be $\Delta$, the unit disk in $\mathbb{C}$. Let us first note that it follows from the Ohsawa-Takegoshi extension theorem that $E$ is a holomorphic vector bundle in this case. A section of the bundle $E$ is a function that maps $t$ in $\Delta$ to an element of $E_t$, i.e. to a global holomorphic $(n, 0)$-form, $u_t$ with values in $\mathcal{L}$ on $X_t$. (From now we will denote the fibers $X_t$ instead of $\mathcal{X}_t$.) $E$ has a holomorphic structure such that $u_t$ is a holomorphic section if and only if the $(n + 1, 0)$-form, defined fiberwise by $u_t \wedge dt$, is a holomorphic section of $K_{\mathcal{X}} + \mathcal{L}$. The computations in [2] are based on the notion of a representative of $u = u_t$. We say that $u$ is a representative of $u$ if $u$ is an $(n, 0)$-form on $\mathcal{X}$, with values in $\mathcal{L}$ such that $u$ restricts to $u_t$ on fibers $X_t$. This means that

$$i^*_t(u) = u_t$$
where \( i_t \) is the natural inclusion map from \( X_t \) to \( X \). Representatives are not unique; any two differ by a term
\[
dt \wedge v
\]
where \( v \) is an \((n - 1, 0)\)-form.

If \( \phi \) is a metric on \( L \) we get a natural \( L^2 \)-metric on \( E \) by
\[
\|u_t\|^2_t = \int_{X_t} |u_t|^2 e^{-\phi}.
\]
The expression \( |u|^2 e^{-\phi} \) should here be interpreted as
\[
c_n u \wedge \bar{u} e^{-\phi};
\]
it is a well defined volume form and can be integrated over fibers. Here and in the sequel, \( c_n = i^{n^2} \) is a unimodular constant chosen so that we get a positive form.

The bundle \( E \) is thus a hermitian holomorphic vector bundle and as such has a Chern connection
\[
D = D' + D''.
\]
In terms of a representative of the section, the connection can be described as follows. First (locally), since \( \bar{\partial} u = 0 \) on fibers,
\[
\bar{\partial} u = dt \wedge \nu + dt \wedge \eta.
\]
(We use here a different sign convention from the one in [2].) Here \( \nu \) is of bidegree \((n, 0)\) and \( \eta \) is of bidegree \((n - 1, 1)\). The forms \( \nu \) and \( \eta \) are not uniquely determined but their restrictions to fibers are, and since
\[
dt \wedge d\bar{t} \wedge \bar{\partial} \nu = dt \wedge d\bar{\partial}^2 u = 0
\]
\( \nu \) is holomorphic on fibers. Then
\[
D'' u = \nu dt.
\]
In particular, \( u \) is a holomorphic section of \( E \) if and only if, \( \bar{\partial} u \wedge dt = 0 \), i.e., if and only if
\[
\bar{\partial} u = dt \wedge \eta
\]
for some \( \eta \). Changing representative to \( u - dt \wedge v \) changes \( \eta \) to \( \eta + \bar{\partial} v \). Thus the cohomology class of \( \eta \) in \( H^{n-1,1}(X_t, L) \) is well defined, and any element in this cohomology class can be obtained from some representative of the section. We shall see in Lemma 2.2 that this class is \(-\mathcal{K}_t . u\).

Given a representative, we can express the norm of a section by
\[
\|u_t\|^2_t = p_*(c_n u \wedge \bar{u} e^{-\phi}),
\]
the push forward of an \((n, n)\)-form on \( X \). Using this, and the definition of Chern connection, one finds that
\[
(D' u)_t = P(\mu) dt,
\]
where
\[
\partial^\phi u = e^\phi \partial e^{-\phi} u = dt \wedge \mu
\]
and \( P(\mu) \) is the orthogonal projection of the \((n, 0)\)-form \( \mu \) on the space of holomorphic \((n, 0)\)-forms.
As in [2] we can now compute the Laplacian of \( \|u_t\|^2_t \) with respect to \( t \) by computing \( i\partial \bar{\partial} p_*(c_n u \wedge \bar{u} \wedge e^{-\phi}) \). This uses that \( \partial \) and \( \bar{\partial} \) commute with the pushforward operator, and the result is

\[
i\partial \bar{\partial} p_*(c_n u \wedge \bar{u} \wedge e^{-\phi}) = -p_*(c_n i\partial \bar{\partial} \phi \wedge u \wedge \bar{u} \wedge e^{-\phi}) + c_n \int_{X_t} \eta \wedge \bar{\eta} e^{-\phi} dV_t + \|\mu\|^2_t dV_t,
\]

if \( u_t \) is a holomorphic section of \( E \).

Under this assumption we moreover have the standard formula

\[
\Delta \|u_t\|^2_t = -\langle \Theta^E u_t, u_t \rangle + \|D' u_t\|^2_t.
\]

Combining the two we get for the curvature of \( E \) \((dV_t = dt \wedge d\bar{t})\)

\[
\langle \Theta^E u_t, u_t \rangle_t = \|D' u_t\|^2_t - \|\mu\|^2_t + \frac{1}{c_n} \int_{X_t} \eta \wedge \bar{\eta} e^{-\phi} - \frac{1}{c_n} \int_X \eta \wedge \bar{\eta} e^{-\phi} \]

where \( P_\perp \) is the orthogonal projection on the orthogonal complement of holomorphic forms. Notice that every term in the right hand side here depends on the choice of representative, whereas the left hand side does not. The following lemma from [2] tells us that we can choose representative so that the right hand side becomes manifestly nonnegative.

**Lemma 2.1.** Let \( u \) be a holomorphic section of \( E \), and \( t \) a point in the base \( \Delta \). Then there is a representative \( u \) such that \( \mu \) is holomorphic on \( X_t \) and \( \eta \) is primitive on \( X_t \) (i.e. \( \eta \wedge \omega = 0 \) on \( X_t \), where \( \omega \) is the Kähler form on \( X' \)).

**Proof.** We will sketch the proof here since the proof in [2] is carried out in the general case of a higher dimensional base and therefore is more complicated. Take \( t = 0 \)

Let \( u \) be an arbitrary representative and recall that \( \partial u = dt \wedge \eta \) and \( \partial^\phi u = dt \wedge \mu \). Changing to a different representative \( u + dt \wedge v \), where \( v \) is a \((n - 1, 0)\)-form changes \( \eta \) and \( \mu \) to \( \eta - \partial v \) and \( \mu - \partial^\phi v \) respectively. We want to choose \( v \) in such a way that

\[
\omega \wedge (\eta - \partial v) = 0
\]
on \( X_0 \), and

\[
\mu - \partial^\phi v
\]
is holomorphic on \( X_0 \). Let \( \alpha = v \wedge \omega \) on \( X_0 \), so that \( \alpha \) is an \((n, 1)\)-form on \( X_0 \). Then the first equation becomes

\[
\eta \wedge \omega = \bar{\partial} \alpha.
\]

Let us first see that this equation is solvable, or in other words that the cohomology class of \( \eta \) is primitive: Since \( u \wedge \omega \) is of bidegree \((n + 1, 1)\) we can write

\[
u \wedge \omega = dt \wedge u'
\]
where \( u' \) is a well defined \((n-1,1)\)-form on fibers. Applying the \(\bar{\partial}\)-operator on \(X\) we get
\[
dt \wedge \eta \wedge \omega = \bar{\partial}u \wedge \omega = -dt \wedge \bar{\partial}u'.
\]
Hence \( \eta \wedge \omega = -\bar{\partial}u' \) on fibers, so \( \eta \wedge \omega \) is \(\bar{\partial}\)-exact on all fibers. Hence (2, 2) is solvable.

The second equation, (2.3), is satisfied if \(\bar{\partial}^* \alpha = \mu_{\perp} \), where \(\mu_{\perp} \) is the projection of \(\mu\) on the orthogonal complement of holomorphic \((n,0)\)-forms. Since this space is precisely the range of \(\bar{\partial}^*\), (2.3) is solvable as well.

Let \(\alpha_1\) solve \(\bar{\partial}\alpha_1 = \eta \wedge \omega\) on \(X_0\) and take \(\alpha_1\) to be orthogonal to the kernel of \(\bar{\partial}\). Then \(\alpha_1\) is orthogonal to the range of \(\bar{\partial}\) so \(\bar{\partial}^*\alpha_1 = 0\). Let \(\alpha_2\) solve \(\bar{\partial}^*\alpha_2 = \mu_{\perp}\) and take \(\alpha_2\) orthogonal to the kernel of \(\bar{\partial}^*\). Then \(\alpha_2\) is orthogonal to the range of \(\bar{\partial}^*\) so \(\bar{\partial}\alpha_2 = 0\). Thus \(\alpha = \alpha_1 + \alpha_2\) solves both equations, and we are done.

\[\square\]

Since
\[
-c_n \int_{X_t} \eta \wedge \bar{\eta} e^{-\phi} = \|\eta\|^2
\]
for \(\eta\) primitive it follows that
\[
\langle \Theta^E u_t, u_t \rangle_t = p(c_n i \bar{\partial} \partial e^{-\phi} \wedge u_t \wedge \bar{\partial} e^{-\phi})/dV_t + \|\eta\|^2 \geq 0.
\]
This choice of representative will be used again in the proof of Theorem 1.1 in section 3. The proof of Theorem 1.2 however, is based on a different choice of representative, using the extra structure provided by the line bundle \(L\).

2.2. The Kodaira-Spencer class. We first recall the definition of the K-S-class in a suitable form. Let \(V\) be a smooth vector field of type \((1,0)\) on \(X\) which maps to the field \(\partial/\partial t\) on \(\Delta\) under the derivative of the map \(p\) from \(X\) to \(\Delta\). Then \(\partial V\) is a \((0,1)\) form on \(X\) with values in the bundle of tangent vectors tangential to the fiber. Its restriction to a fiber \(X_t\) (\(i_t^*(\partial V)\)), \(\kappa_t\) is a \(\bar{\partial}\)-closed form with values in \(T^{1,0}\) of the fiber. The cohomology class it defines is the Kodaira-Spencer class, \(K_t\) at \(t\).

If \(u_t\) is an element in \(E_t\), we can let the Kodaira-Spencer class operate on \(u_t\) to obtain \(K_t u_t\). This class is defined as the cohomology class of \(\kappa_t u_t\), where we let the vector valued form \(\kappa_t\) operate on \(u_t\) by contraction as described in the introduction. This is then a cohomology class in \(H^{n-1,1}(X_t, L|_{X_t})\). Notice that when \(n = 1\) and (say) \(L\) is trivial, \(\kappa_t\) is a \((0,1)\)-form with values in \(-K_{X_t}\), and if \(u\) is a holomorphic 1-form on \(X_t\) then \(\kappa_t u\) is just the product, defining a scalar \((0,1)\)-form.

Lemma 2.2. Let \(u\) be any representative of a holomorphic section \(u\) and let \(\eta\) be defined by \(\bar{\partial}u = dt \wedge \eta\) as above. Then
\[
\eta + \kappa_t u = \bar{\partial}(\delta V u)
\]
on fibers. In particular, \(\eta\) and \(-\kappa_t u\) define the same class in \(H^{n-1,1}(X_t, L|_{X_t})\).

Proof. On \(X\) we have
\[
\bar{\partial}(\delta V u) = \delta_{\bar{\partial} V} u + \delta V (dt \wedge \eta) = \delta_{\bar{\partial} V} u + \eta - dt \wedge \delta V \eta.
\]
When we restrict to a fiber $X_t$, the last term disappears and we get
\[ \bar{\partial}(\delta V u) = \kappa_t u + \eta. \]

\[ \square \]

2.3. The Gauss-Manin connection. The content of this subsection will only be used in section 3.2 and is not necessary to understand the proofs of theorems 1.1 and 1.2.

The fibration $p$ of $\mathcal{X}$ over $\Delta$ is smoothly (locally) trivial, so there is a map $F = (p, f)$ from $\mathcal{X}$ to $\Delta \times X_0$ which is fiber preserving, diffeomorphic and equals the identity map on the central fiber. If $F'$ is another such map, it is related to $F$ by
\[ F' = G \circ F \]
where $G$ is a fiberpreserving map from $\Delta \times X_0$ to itself. Hence $G$ is given as $G(t, x) = (t, G_t(x))$ where $G_t$ is a smooth family of diffeomorphisms of $X_0$, which are moreover homotopic to $G_0 = \text{Id}$.

Let $H$ be the Hodge bundle over $\Delta$, i.e. the vector bundle whose fiber over a point $t$ is
\[ H_t = H^n(X_t, \mathbb{C}), \]
the $n$th de Rham cohomology of $X_t$. Then $H$ is a trivial vector bundle with a trivialization given by
\[ e_j^*(t) = i_t^* \circ f^*(e_j^0), \]
e_j^0 being some arbitrary basis of $H_0$. If we replace $F = (p, f)$ by $F' = (p, f')$ as above,
\[ i_t^* \circ (f')^* = i_t^* \circ f^* \circ G_t^* = i_t^* \circ f^*, \]
where the last equality follows since $G_t$ homotopic to the identity map implies that $G_t^* = \text{id}$ on the cohomology. Our trivialization is therefore independent of the choice of $F$.

We define the Gauss-Manin connection on $H$, by
\[ D_{GM} e_j = 0; \]
this is clearly independent of the choice of basis $e_j$. Clearly $D_{GM}^2 = 0$, so the Gauss-Manin connection is flat.

Next there is a quadratic hermitian form on each $H_t$ by
\[ \langle v, v \rangle_t = c_n \int_{X_t} v \wedge \overline{v}. \]
This form is indefinite, but non degenerate (and well defined) on the cohomology groups $H_t$. If $v_t$ is a smooth section of $H$ we can as before represent it by an $n$-form on the total space, $v$ such that $v$ restricts to (an element of the cohomology class) $v_t$ on $X_t$. By construction, a section satisfying $D_{GM} v = 0$ can be represented by a form of the type
\[ v = f^*(v_0) \]
where $v_0$ is a closed form on $X_0$, so that $v$ is closed on $\mathcal{X}$. Moreover, for any smooth section,
\[ \langle v, v \rangle = c_n p_*(v \wedge \overline{v}). \]
It follows that if $D_{GM}v = 0$ and $D_{GM}u = 0$, then
\[ d\langle v, u \rangle = 0, \]
so more generally
\[ d\langle v, u \rangle = \langle D_{GM}v, u \rangle + \langle v, D_{GM}u \rangle. \]
In other words, the Gauss-Manin connection is compatible with our hermitian form, so it is the Chern connection for the complex structure $D'_{GM}$ and the hermitian form $\langle , \rangle$.

3. The untwisted case

In this section we discuss the case when $L$ is trivial so that $\phi = 0$. Then the curvature formula (2.1) becomes

\[ \langle \Theta^E u_t, u_t \rangle_t = -\|P_\perp \mu \|_t^2 - c_n \int_{X_t} \eta \wedge \bar{\eta} \]

We decompose $\eta = \eta_p + \eta_\perp$, where $\eta_p$ is primitive on $X_t$ and $\eta_\perp$ is orthogonal to the space of primitive forms (all with respect to a Kähler metric on $X$ restricted to $X_t$). Then

\[ -c_n \int_{X_t} \eta \wedge \bar{\eta} = \|\eta_p\|_t^2 - \|\eta_\perp\|_t^2 = \|\eta\|_t^2 - 2\|\eta_\perp\|_t^2. \]

Inserting this in (3.1) we find

\[ \langle \Theta^E u_t, u_t \rangle_t = -\|P_\perp \mu \|_t^2 + \|\eta\|_t^2 - 2\|\eta_\perp\|_t^2 \leq \|\eta\|_t^2. \]

This holds for any choice of representative $u$. When we let the representative vary, the corresponding $\eta$ ranges through an entire cohomology class. By Lemma 2.2 that cohomology class is $-K_t. u$. By Lemma 2.1 we can get equality in (3.2) by choosing $u$ so that $\mu$ is holomorphic and $\eta$ is primitive. This choice must thus give us the representative of $-K_t. u$ of minimum norm, i.e. the harmonic representative of the class. Hence

\[ \langle \Theta^E u_t, u_t \rangle_t = \|K_t. u\|_t^2 \]

so Theorem 1.1 is proved.

The conclusion of this is that the curvature of $E$ degenerates in a certain direction $u_t$ over a point $t$ if the Kodaira-Spencer class vanishes after multiplication by $u_t$. This may well happen even if $K_t$ itself is nonzero, as we see in the next section.

3.1. Examples of degenerate curvature. The construction of the examples is a direct consequence of the following (well known) lemmas.

**Lemma 3.1.** Let $X$ be a compact Riemann surface of genus at least 2 and let $\mu$ be a class in $H^{0,1}(X, T^{1,0}(X)) = H^{0,1}(X, -K_X)$. Then there is a smooth proper fibration $X \to \Delta$ over a disk $\Delta$ such that $X_0 = X$ and the Kodaira-Spencer class $K_0 = \mu$. 
Proof. This is a consequence of basic properties of Teichmüller space. Teichmüller space \( T_g \) is a complex manifold of dimension \( 3g - 3 \) consisting of equivalence classes of Riemann surfaces of genus \( g \). Over \( T_g \) there is a smooth proper fibration \( C_g \to T_g \) such that the fiber over a point \( t \) in \( T_g \) is a compact Riemann surface in the class \( t \), and all compact Riemann surfaces of genus \( g \) appear as fibers over some point(s). (This is a fundamental result of Earle and Eells, [5].) The tangent space to \( T_g \) at \( t \) is (isomorphic to) \( H^{0,1}(C_g, -K(C_g)) \). We need only take a disk in \( T_g \), centered at a point representing \( X \) with tangent vector at the origin equal to \( \mu \), and let \( \mathcal{X} \) be \( C_g \) restricted to that disk.

**Lemma 3.2.** Let \( X \) be a compact Riemann surface of genus at least 2. Then for any \( u \) in \( H^0(X, K_X) \) there is a nonzero class \( \mu \) in \( H^{0,1}(X, -K_X) \) such that \( \mu u = 0 \) in \( H^{0,1}(X) \).

**Proof.** This follows from a simple count of dimensions. Given \( u \) multiplication by \( u \) defines a linear map from \( H^{0,1}(X, -K_X) \) to \( H^{0,1}(X) \). Since the dimension of \( H^{0,1}(X, -K_X) \) is \( 3g - 3 \) by the Riemann-Roch theorem, while the dimension of \( H^{0,1}(X) \) is \( g \), the map cannot be injective if \( g \) is greater than 1.

This can also be seen more concretely in the following way. (We denote by \( \mu \) also a smooth form representing the cohomology class \( \mu \).) That \( \mu u \) vanishes in cohomology means that

\[
\mu u = \bar{\partial} v
\]

for some smooth function \( v \). This implies

\[
\bar{\partial} \frac{v}{u} = \mu + v \bar{\partial} \frac{1}{u}.
\]

Hence \( \mu \) is cohomologous to the \( K_X \)-valued current \(-v \bar{\partial} 1/u \), supported in the zero set of \( u \). To find a \( \mu \) which is not zero in cohomology, but is annihilated by multiplication by \( u \) it suffices conversely to find a \( K_X \)-valued current with measure coefficients, \( \nu \), supported in the zeroset of \( u \), which is not exact. For this it is enough to take \( \nu = \bar{\partial} w \) supported in one single point. If \( \nu = \bar{\partial} w \) then \( w \) is a meromorphic section of \(-K_X \) with exactly one pole. But such sections do not exist since the number of zeros minus the number of poles of any section of \(-K_X \) equals the degree of \(-K_X \) which is \( 2 - 2g < -1 \).

To get curvature that is not only degenerate but vanishes completely, we need a final lemma.

**Lemma 3.3.** Let \( X \) be a compact Riemann surface. Then there is a nonzero class \( \mu \) in \( H^{0,1}(X, -K_X) \) such that \( \mu u = 0 \) in \( H^{0,1}(X) \) for any \( u \) in \( H^0(X, K_X) \) if and only if \( X \) is hyperelliptic of genus at least 3.

**Proof.** That \( \mu u = 0 \) in \( H^{0,1}(X) \) means that the integral

\[
\int_X (\mu u) \wedge v = 0
\]

for any \( v \) in \( H^0(X, K_X) \). This means that in the duality pairing between \( H^{0,1}(X, -K_X) \) and \( H^0(X, 2K_X) \), \( \mu \) is annihilated by any section of \( 2K_X \) that factors as a product of two sections of \( K_X \), and also of course by any linear combination of such sections. On the other hand, \( \mu = 0 \) means that \( \mu \) is annihilated by any section of \( 2K_X \). The question is therefore if any section of
$2K_X$ is a linear combination of products of sections of $K_X$. By a theorem of M Noether this is the case if $X$ is not hyperelliptic of genus greater than 2 (see [6] p 149), but not the case if $X$ is hyperelliptic of genus at least 3 (see [6] p 98).

**Proposition 3.4.** 1. Let $X$ be a compact Riemann surface of genus at least 2. Then there is a smooth proper fibration $X'$ over a disk $\Delta$, having $X$ as central fiber, such that its Kodaira-Spencer class does not vanish at 0, but the curvature of the direct image bundle of $K_{X/\Delta}$ is degenerate (i.e. not strictly positive) at 0.

2. Assume $X$ is hyperelliptic of genus at least 3. Then there is a smooth proper fibration $X'$ over a disk $\Delta$, having $X$ as central fiber, such that its Kodaira-Spencer class at 0 does not vanish, but the curvature of the direct image bundle of $K_{X/\Delta}$ is zero at 0.

**Proof.** 1. By Lemma 3.2 there is a non zero class $\mu$ in $H^{0,1}(X, -K_X)$ that is annihilated by multiplication by a canonical form $u$ on $X$. By Lemma 3.1 $\mu$ is the Kodaira-Spencer class of some fibration, and by Theorem 1.1, $\Theta_Eu = 0$.

2. Follows from Lemma 3.3 in the same way.

**Remark:** A reader more knowledgeable than the author can probably compare this section with the discussion of the local period mapping for Riemann surfaces in [9], p 842-843.

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### 3.2. The relation to the Gauss-Manin connection

A holomorphic $(n, 0)$-form on a fiber defines a unique element in $H^n$, so our bundle $E$ is a smooth subbundle of $H$. By a theorem of Griffiths (that we will reprove below) $E$ is in fact a holomorphic subbundle ([16], p 250), if we give $H$ the holomorphic structure induced by the Gauss-Manin connection (so that $\bar{\partial}$ on section of $H$ is the $(0, 1)$-part of $D_{GM}$).

If we have a smooth section of $E$ and $V$ is a vector field on $\Delta$ we may thus take $D_Vu$, where $D$ is the connection defined in section 2, at a point $t$ and obtain a holomorphic $(n, 0)$-form on $X_t$, or we can apply $(D_{GM})_Vu$ and obtain a cohomology class in $H^n(X_t, \mathbb{C})$. The relation between the two is given by the next theorem.

**Theorem 3.5.** Let $u$ be a smooth section of $E$. Then

\begin{equation}
D_{GM}u = [Du] - [\mathcal{K}_t.u]dt.
\end{equation}

This statement should be read as follows: The class $[\mathcal{K}_t.u]$ lies in $H^{n-1,1}$, which by Hodge theory can be thought of as a subspace of $H^n$. So the right hand side of (4.1) should be read as 'the $\hat{H}^n$-class defined by $Du$ minus the $H^n$-class defined by $[\mathcal{K}_t.u]$, multiplied by $dt$'.

**Proof.** Let $v$ be a section of $H$ with $D_{GM}v = 0$. We represent $v$ by $v$, a closed form on $X$. Then, if $u$ is a smooth section of $E$, represented by $u$,

\begin{equation}
\langle D_{GM}u, v \rangle = d\langle u, v \rangle = dp_*(c_nu \wedge \bar{v}) = c_n p_*(du \wedge \bar{v}) = c_n (\mu + \eta) \wedge \bar{v})dt + p_*(\nu, \bar{v})d\bar{t}.
\end{equation}

Since $\nu$ and $\mu + \eta$ are closed, and the form $\langle , \rangle$ is nondegenerate on the cohomology groups, we see that $D'_{GM}u = [\mu + \eta]dt$ and $D''_{GM}u = \nu d\bar{t} = D''u$. This formula holds for any choice of representative. By Lemma 2.1, we can always choose our representative in such a way that $\mu$ is holomorphic, and then $\mu dt = D'u$. By Lemma 1.3, $[\eta] = -[\mathcal{K}_t.u]$. This completes the proof of Theorem 1.3.
Formula (4.1) contains many properties of the direct image bundle \( E \). First, we note that, with the convention described above, \( D'_{GM} = D' \), i.e. the \((0, 1)\) part of the two connections agree. This is what lies behind Griffiths theorem, [16] p 250 that \( E \) is a holomorphic subbundle of \( H \). Since \( D'^2_{GM} = 0 \), the equation implies that \( (D')^2 = 0 \), so \( D' \) defines an integrable complex structure on \( E \), [4]. Thus \( E \) has a (local) frame of holomorphic sections which are also holomorphic for the Gauss-Manin connection, so \( E \) is a holomorphic subbundle of \( H \).

Looking at the \((1, 0)\)-part of (4.1), we see that the term \([K_t, u]\) is 'orthogonal' to the fibers of \( E \) for the quadratic form \( \langle . , . \rangle \) on \( H \), simply for reasons of bidegree. By the Griffiths formula for the curvature of a holomorphic subbundle (see below) this means that

\[
\langle \Theta^{GM} u, u \rangle = \langle \Theta^E u, u \rangle + \langle K_t, u, K_t, u \rangle
\]

for sections of \( E \). In the standard situation, when the quadratic form is positive definite, this formula implies that the curvature of the holomorphic subbundle is smaller than the curvature of \( H \). In our case, the form is not positive definite, which changes things completely.

Note first that \( K_t, u \) is a primitive class. This follows from Lemmas 2.2 and 2.1; by Lemma 2.2 \(-K_t, u \) is cohomologous to \( \eta \) which is primitive by (the proof of) Lemma 2.1.

This implies that the second term on the right hand side of (3.4) is equal to the negative of the norm of the cohomology class with respect to any Kähler metric we choose on the fiber. (This quantity is independent of the choice of Kähler metric.) Since we also have that \( G^{GM} = 0 \), (3.4) implies

\[
\langle \Theta^E u, u \rangle = \|K_t, u\|^2,
\]

so we get another proof of Theorem 1.1

**Remark:** Formula (3.4) can be obtained as follows. Choose a holomorphic section of \( E \), such that \( D' u = 0 \) at a given point. Then, at that point,

\[
\Delta \langle u, u \rangle = -\langle \Theta^E u, u \rangle.
\]

On the other, we can also compute the Laplacian using the Gauss-Manin connection:

\[
\Delta \langle u, u \rangle = -\langle \Theta^{GM} u, u \rangle + \langle D'_{GM} u, D'_{GM} u \rangle = \langle K_t, u, K_t, u \rangle,
\]

by (3.3). Comparing these two formulas we get (3.4). \( \square \)

### 4. The twisted case

In this section we consider a semiample line bundle \( \mathcal{L} \) over \( \mathcal{X} \), equipped with a semipositive metric \( \phi \) which is assumed to be strictly positive when restricted to any fiber. We can then take our underlying Kähler metric to be

\[
i \partial \overline{\partial} (\phi + |t|^2) =: \omega,
\]

so in particular \( \omega \) restricts to \( i \partial \overline{\partial} \phi \) on any fiber. As in the introduction, we choose local coordinates \((t, z)\) on \( \mathcal{X} \) that respect the fibration, a local representative \( \varphi \) of the metric \( \phi \) with respect to some trivialization of \( \mathcal{L} \), and define a local vector field \( W_\varphi \) as the complex gradient of the local function \( \varphi t \). We then let

\[
V_\varphi := \frac{\partial}{\partial t} - W_\varphi.
\]
By a theorem of Schumacher, [11], the field \( V_\phi \) is globally defined on \( X \). This is also a consequence of the following lemma, that we will use repeatedly.

**Lemma 4.1.**

\[ (4.1) \]
\[ \delta_{V_\phi} \partial \bar{\partial} \phi = c(\phi) d\bar{t}. \]

*(The function \( c(\phi) \) is defined in (1.2) and (1.3).)*

**Proof.** Recall that
\[ \dot{\phi}_t = \frac{\partial \phi}{\partial t} \]
and that the local vector field \( W_\phi \) was defined by
\[ \delta_{W_\phi} \omega = \bar{\partial} \dot{\phi}_t \]
\[ \text{i.e.} \]
\[ \delta_{W_\phi} (\partial \bar{\partial} \phi)_z = \bar{\partial} \dot{\phi}_t \]
on fibers. We have
\[ \partial \bar{\partial} \phi = (\partial \bar{\partial} \phi)_z + dt \wedge \bar{\partial} \dot{\phi}_t + \partial_z \bar{\partial} \phi_t + d\bar{t} + \frac{\partial^2 \phi}{\partial t \partial \bar{t}} dt \wedge d\bar{t}. \]
Contracting with \( V_\phi \) we get
\[ \delta_{V_\phi} \partial \bar{\partial} \phi = -\delta_{W_\phi} (\partial \bar{\partial} \phi)_z + \bar{\partial} \dot{\phi}_t - |\bar{\partial} \dot{\phi}_t|^2 d\bar{t} + \frac{\partial^2 \phi}{\partial t \partial \bar{t}} dt \wedge d\bar{t} = (\frac{\partial^2 \phi}{\partial t \partial \bar{t}} - |\bar{\partial} \dot{\phi}_t|^2) d\bar{t}. \]
By formula (1.2) this equals \( c(\phi) d\bar{t}. \)

\( \square \)

The lemma shows in particular that, when \( \phi \) is relatively positive, \( V_\phi \) is globally well defined. This is so, because either \( c(\phi) \) is nonzero, and then (4.1) determines \( V_\phi \) directly, or \( i \partial \bar{\partial} \phi \) has one zero eigenvalue. Then (4.1) shows that \( V_\phi \) lies in the eigenspace corresponding to the eigenvalue zero. This determines \( V_\phi \) up to a multiplicative constant, which must be the same for all local definitions, since \( dp \) maps \( V_\phi \) to \( \partial / \partial t \).

Let \( u = u_t \) be a holomorphic section of \( E \). Then \( u \wedge dt \) is a holomorphic \((n+1,0)\)-form on \( X \), with values in \( L \). We now choose a representative of \( u \) by,

\[ (4.2) \]
\[ u := \delta_{V_\phi} (dt \wedge u). \]

This choice of \( u \) is the main point of the argument. If with respect to our local coordinates
\[ u = u^0 = \hat{u}^0 dz, \]
where \( \hat{u}^0 \) is a function, we get
\[ u = u^0 + dt \wedge v, \]
with
\[ v = \delta_{W_\phi} u^0 \]
on fibers.

Hence \( \eta \), defined by \( \bar{\partial} u = dt \wedge \eta \) is given by
\[ \eta = -\bar{\partial} v = -k^\phi_t u^0 \]
on fibers. Notice that since on fibers
\[ 0 = \delta_{u_0}(i\partial \bar{\partial} \phi \wedge u^0) = i\bar{\partial} \phi_t \wedge u^0 + i\partial \bar{\partial} \phi \wedge v, \]
we get
\[ \partial \bar{\partial} \phi \wedge v = -\bar{\partial} \phi_t \wedge u^0. \]
It follows that
\[ \eta \wedge \omega = \bar{\partial} \partial \phi_t \wedge u^0 = 0, \]
so \( \eta \) is primitive on fibers. Since \( \eta \) is primitive we get for the fiberwise Hodge \(*\)-operator that
\[ *\eta = -\eta, \]
so since \( \eta \) is always \( \bar{\partial} \)-closed on fibers, \( \bar{\partial} * \eta = 0 \) too.

Recall that \( \bar{\partial} \phi u = dt \wedge \mu \). Fix a point in \( \Delta \) that we take to be 0. Assume the section satisfies \( D'u = 0 \) at 0. Since \( D'u = P(\mu)dt \) this means that \( \mu \) is orthogonal to the space of holomorphic forms. By the curvature formula (2.1)
\[ \langle \Theta^F u_0, u_0 \rangle_0 = \|\eta\|^2_0 - \|\mu\|^2_0 + p_*(c_n i\bar{\partial} \partial \phi \wedge u \wedge \bar{u})/dV_t \]
since \( \eta \) is primitive and \( D'u = 0 \). Here we have an apparently negative contribution from the norm of \( \mu \), but we know from the general curvature formula in section 2.1 that it must be possible to absorb it in the norm squared of \( \eta \).

**Lemma 4.2.**
\[ \partial \bar{\partial} \phi \wedge u = c(\phi)u \wedge dt \wedge d\bar{t}. \]

**Proof.** For degree reasons \( \partial \bar{\partial} \phi \wedge dt \wedge u = 0 \) so
\[ 0 = \delta_V(\partial \bar{\partial} \phi \wedge dt \wedge u) = c(\phi)dt \wedge dt \wedge u + \partial \bar{\partial} \phi \wedge u. \]

From this, the next lemma follows immediately.

**Lemma 4.3.** At \( t = 0 \)
\[ p_*(c_n i\bar{\partial} \partial \phi \wedge u \wedge \bar{u})/dV_t = \int_{X_0} c(\phi)|u|^2 e^{-\phi}. \]

**Lemma 4.4.** On fibers
\[ \bar{\partial} \mu = -\partial^\phi \eta. \]

**Proof.** By definition
\[ \bar{\partial} u = dt \wedge \eta. \]
Hence
\[ \partial^\phi \bar{\partial} u = -dt \wedge \partial^\phi \eta. \]
But the left hand side here is
\[ -\bar{\partial} \partial^\phi u - \partial \bar{\partial} \phi \wedge u, \]
and \( \partial \bar{\partial} \phi \wedge u \) vanishes on fibers by Lemma 5.2. Hence
\[ \bar{\partial}(dt \wedge \mu) = dt \wedge \partial^\phi \eta, \]
which proves the lemma.
Since we have also seen that $\mu$ is orthogonal to holomorphic forms, $\mu$ is the $L^2$-minimal solution to
\[ \bar{\partial}\mu = -\bar{\partial}^{\phi}\eta = -\nabla'\eta \]
on $X_0$. Hence
\[ \mu = -\bar{\partial}^*(\Box'')^{-1}\nabla'\eta. \]
Here $\nabla'$ is the $(1, 0)$-part of the Chern connection for $L$ restricted to $X_0$,
\[ \Box'' = \bar{\partial}\partial + \partial^*\bar{\partial}, \]
and we will also use
\[ \Box' = \nabla'(\nabla')^* + (\nabla')^*\nabla'. \]
Then, on $(n + 1)$-forms $\Box' = \Box'' + 1$ by the Kodaira-Nakano formula. Hence
\[ \|\mu\|^2 = \langle -\bar{\partial}^*(\Box'')^{-1}\nabla'\eta, \mu \rangle = \langle (\Box'')^{-1}\nabla'\eta, \nabla'\eta \rangle = \langle (\Box' + 1)^{-1}\nabla'\eta, \nabla'\eta \rangle = \langle (\Box' + 1)^{-1}\eta, (\nabla')^*(\nabla')^*\nabla'\eta \rangle, \]
since $\nabla'$ commutes with $(\Box' + 1)^{-1}$. But $(\nabla')^*\eta = *\bar{\partial}^*\eta = 0$ so
\[ (\nabla')^*\nabla'\eta = \Box'\eta. \]
Hence
\[ \|\eta\|^2 - \|\mu\|^2 = \|\eta\|^2 - \langle (\Box' + 1)^{-1}\eta, \Box'\eta \rangle = \langle (\Box' + 1)^{-1}\eta, \eta \rangle. \]
Inserting this in (4.3), using lemma 4.3, we get Theorem 1.2.

Remark: The last part of the proof amounts to a calculation of
\[ \|\eta\|^2 - \|\mu\|^2, \]
if $\mu$ is the $L^2$-minimal solution to
\[ \bar{\partial}\mu = \bar{\partial}^{\phi}\eta, \]
showing in particular that this quantity is nonnegative. This is formally similar to a classical $L^2$-estimate for the Beurling transform of a function in the plane. It is different from our earlier expression for the curvature in the case of a trivial fibration, [3], where we estimated instead the $L^2$-minimal solution to the equation
\[ \bar{\partial}u = \bar{\partial}\hat{\phi}_t \wedge u. \]
The two expressions turn out to be identical for a trivial fibration, but the formula from [3] cannot be used here, since $\bar{\partial}\hat{\phi}_t$ has no global meaning.
4.1. **Infinitesimal triviality.** We see from Theorem 1.2 that if \( \Theta^E u = 0 \) for some \( u \) in \( E^0 \), then \( c(\phi) = 0 \) on \( X_0 \) and \( \eta = -k^0_0 u = 0 \) so \( k^0_0 = 0 \) on \( X_0 \). This last condition says that \( V_\phi \) is holomorphic along \( X_0 \). By Lemma 2.5 in \([3]\) it also follows from \( c(\phi) = 0 \) that \[
\frac{\partial}{\partial t}\big|_{t=0} V_\phi = 0,
\]
so \( V_\phi \) is holomorphic to first order also in directions tranverse to the fiber. If moreover \( \Theta^E \) is degenerate not only for \( t = 0 \) but for all \( t \) in \( \Delta \), then \( V_\phi \) is holomorphic on \( \mathcal{X}' \) in the sequel. We shall then see that \( V_\phi \) lifts to a holomorphic vector field on \( \mathcal{L}, \hat{V}_\phi \), the flow of which acts linearly on the fibers of \( \mathcal{L} \) and moreover satisfies \[
\hat{V}_\phi|_\xi|_\phi^2 = 0.
\]

**Lemma 4.5.** Assume \( \Theta^E = 0 \) (or is just degenerate) on \( \Delta \). Then, near any point in \( \mathcal{X} \) there is a local trivialization of \( \mathcal{L} \) with respect to which the metric \( \phi \) is represented by a local function \( \varphi \) such that \( V_\phi(\varphi) = 0 \).

**Proof.** Let \( \varphi \) be any local function, representing \( \phi \) in some holomorphic frame. Any other representative near the point is obtained by subtracting a pluriharmonic function. By definition \[
V_\phi(\varphi) = \delta V_\phi \partial \varphi.
\]
Taking \( \bar{\partial} \) we get, since \( V_\phi \) is holomorphic,
\[
\bar{\partial} V_\phi(\varphi) = -\delta V_\phi \bar{\partial} \partial \varphi.
\]
By Lemma 5.1 this equals \( c(\phi) d\bar{t} = 0 \). Hence \( V_\phi(\varphi) := \gamma \) is holomorphic. Locally, we can write \( \gamma = V_\phi(\Gamma) \), with \( \Gamma \) holomorphic, and then it is enough to replace \( \varphi \) by \( \varphi - 2 \text{Re} \Gamma \).

By the lemma we get a covering of \( \mathcal{X}' \) by open sets \( U_i \) over which there are local frames \( e_i \) of \( \mathcal{L} \), such that \[
V_\phi \log |e_i|_\phi = 0.
\]
Then the transition functions \( g_{ij} = e_i / e_j \) satisfy
\[
V_\phi \log |g_{ij}|^2 = 0
\]
so
\[
V_\phi g_{ij} = 0.
\]
We can now define \( \hat{V}_\phi \), by letting it be horizontal with respect to these local frames.

4.2. **The Weil-Petersson metric.** In this subsection we will rewrite formula (1.4) in the case when the fibers are Riemann surfaces, \( \mathcal{L} \) is equal to the relative canonical bundle \( K_{\mathcal{X}/Y} \) and the metric \( \phi \) is Kähler-Einstein on each fiber. We shall see that Theorem 1.2 in this case, together with results of Schumacher, \([11]\) implies the formula of Wolpert, \([17]\), for the curvature of the Weil-Petersson metric.

Notice that in this case, \( k^0_t = i^*_t (\bar{\partial} V_\phi) \) is a \((0, 1)\)-form on the fiber \( X_t \), with values in \( T^{1,0}(X_t) = -K_{X_t} \). This means that the pointwise norm squared
\[
|k^0_t|^2
\]
is a well defined function on $X_t$. By Schumacher’s formula, [11] Proposition 2, [12] Proposition 3, it is related to the function $c(\phi)$ by

$$c(\phi) = (1 + \Box)^{-1} |k'^t|^2.$$  

(Schumacher proves the same formula in any dimension if $\phi$ is Kähler-Einstein on fibers.) Schumacher also proves that the form $k'^t$ is harmonic on $X_t$. Moreover, the holomorphic section $u$ of $K_{X_t} + \mathcal{L} = 2K_{X_t}$ is Hodge dual to another harmonic $(0, 1)$-form with values in $-K_{X_t}$

$$k^u_t := \bar{u}_t e^{-\phi}.$$  

The first term in the right hand side of (1.4) can therefore be rewritten as

$$(4.7) \quad \int_{X_t} (1 + \Box)^{-1} \left( |k'^t|^2 \right) \cdot |k^u|^2 e^\phi.$$  

The second term in the right hand side of (1.4) depends on a $K_{X_t}$-valued $(0, 1)$-form $\eta = k'^t u_t = k'^t k^u\bar{e} e^{\phi}$.

It is easy to check that if $\xi$ is any such form, $\xi e^{-\phi}$ is a function. The $\Box'$-Laplacian of $\xi$ and the $\Box$-Laplacian on functions are related by

$$\Box(\xi e^{-\phi}) = e^{-\phi} \Box' \xi.$$  

Hence

$$\Box' = (1 + \Box)^{-1} e^\phi (1 + \Box)^{-1} (k'^t k^u).$$  

Therefore the second term in (1.4) equals

$$(1 + \Box)^{-1} \int_{X_t} (1 + \Box)^{1-1} (k'^t k^u) \cdot (k^u u_t \bar{e}) e^\phi,$$  

so altogether

$$(4.8) \quad \langle \Theta^E u, u \rangle = \int_{X_t} \left( (1 + \Box)^{-1} |k'^t|^2 \cdot |k^u|^2 + (1 + \Box)^{-1} (k'^t k^u) \cdot (k^u \bar{e}) \right) e^\phi.$$  

Let us now compare this to Wolpert’s formula. Let $u_1, \ldots u_N$ be a basis for $H^0(\Delta^N, 2K_{X_0})$ and consider a fibration over the polydisk $\Delta^N$. We can then perform the earlier construction with respect to each of the coordinates $t_i$, and obtain corresponding $-K_{X_0}$-values $(0, 1)$-forms

$$k'^t_{x_i} := B_i.$$  

Then put (following [18])

$$f_{ij} = B_i \bar{B}_j$$  

and

$$e_{ij} = (1 + \Box)^{-1} f_{ij}.$$  

We now assume the fibration is such that the Hodge dual of $B_i$ is $u_i$, so that the fibration contains all possible infinitesimal deformations of $X_0$. Choosing $u = u_k$ and $k'^t_k = B_i$, (5.8) becomes

$$(\Theta^E)_{i,k} = \int_{X_0} (e_{ik} f_{k,i} + e_{ik} f_{k,i}) e^\phi,$$  

where $e_{ik}$ is the $i$-th component of the (1,1)form $e$.
which is equivalent to formula (2.2) from [18] if we use that the $L^2$ metric on $E$ (the bundle of quadratic differentials) is dual to the Weil-Petersson metric, see [10] p 328.

REFERENCES

[1] ROBERT J. Berman: Relative Kähler-Ricci flows and their quantization, arXiv:1002.3717.
[2] BERNDTSSON, B: Curvature of vector bundles associated to holomorphic fibrations, Ann Math 169 2009, pp 531-560, or arXiv:math/0511225.
[3] BERNDTSSON, B: Positivity of direct image bundles and convexity on the space of Kähler metrics, J Differential geom 81.3, 2009, pp 457-482.
[4] DONALDSON, S. K. AND KRONHEIMER, P.B.: The Geometry of Four-Manifolds, Oxford Mathematical Monographs, Clarendon Oxford 1990.
[5] EARLE, C AND EELLS, J: A fibre bundle description of Teichmüller theory, J. Differential Geometry 3 1969 19–43.
[6] FARKAS, H AND KRA, I: Riemann Surfaces, Graduate Texts in Mathematics, Springer 1980.
[7] FUJITA, T: On Kähler fiber spaces over curves, J. Math. Soc. Japan 30 (1978), no. 4, 779–794.
[8] GRIFFITHS, P A: Curvature properties of the Hodge bundles (Notes written by Loring Tu), Topics in Transcendental Algebraic Geometry, Annals of Mathematics Studies, Princeton University Press 1984.
[9] GRIFFITHS, P A: Integrals on algebraic manifolds II, American Journal of Mathematics 90, 1968, pp 805-865.
[10] HUBBARD, J: Teichmüller Theory, Vol I, Matrix Editions, 2006.
[11] SCHUMACHER, G: Curvature of higher direct images and applications, arXiv:0808.3259.
[12] SCHUMACHER, G: Positivity of relative canonical bundles for families of canonically polarized manifolds, arXiv:1002.4858.
[13] SEMMES, S: Interpolation of Banach spaces, differential geometry and differential equations, Rev. Mat. Iberoamericana 4 (1988), no. 1, 155–176.
[14] SIU, Y-T: Curvature of the Weil-Petersson metric in the moduli space of compact Kähler-Einstein manifolds of negative first Chern class, In: Contributions to several complex variables, ed A Howard and P-M Wong, Vieweg 1986.
[15] TODOROV, A: The Weil-Petersson geometry of the moduli space of SU$\geq3$ (Calabi-Yau) manifolds, Comm. Math. Phys. 126 (1989), 325-346.
[16] VOISIN, C: Hodge theory and complex algebraic geometry. I, Translated from the French original by Leila Schneps. Cambridge Studies in Advanced Mathematics, 76. Cambridge University Press, Cambridge, 2002.
[17] WOLPERT, S: Understanding Weil-Petersson curvature, arXiv:0809.3699.
[18] LIU, K; SUN, X AND YAU S-T: Recent Development on the Geometry of the Teichmüller and Moduli Spaces of Riemann Surfaces and Polarized Calabi-Yau Manifolds, arXiv:0912.5471.

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