Converging posterior distributions in space debris monitoring

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Abstract. Ground-based radars monitor the falling space debris in order to prevent collisions with space crafts and satelites. Experiments with European Incoherent Scatter (EISCAT) Scientific Association radars using new data acquisition equipment suitable for space debris detection have raised a question what happens to a Bayesian solution when the sampling frequency of the reflected signal is increased. Assuming slightly idealized measurements, we show that the posterior densities converge in this case. This shows that the sampling method suits well for the statistical inverse problem.

1. Introduction
Space debris consists of shattered parts of rockets and satellites remaining in the vicinity of Earth. These pieces can orbit Earth for centuries but are constantly falling down. Ground-based radars can monitor the space debris and the collected data helps to prevent harmful collisions in the orbit. In detecting space debris, the radar sends an electromagnetic signal that reflects from a hard object at the orbit and the echo is registered on the ground. Analysis of the echo reveals information about the size and the motion of the object.

Recently, EISCAT radars have been used in detecting the space debris [1]. The measuring equipment used in EISCAT radar experiments captures values that are nearly

\[
Z_n(t) = \frac{1}{\tau_n} \sum_{i=0}^{2^n-1} (s, 1_{(T_i, T_{i+1})}) L_2 1_{(T_i, T_{i+1})}(t) + \epsilon_n(t)
\]

on the observation time interval \((0, T)\). Here \(s\) is the reflected analog signal, \(\tau_n = 2^{-n}T\), \(1_{(T_i, T_{i+1})}\) is the indicator function of the interval \((T_i, T_{i+1})\), \(T_i = i2^{-n}T\), \(i = 0, ..., 2^n\), and \(\epsilon_n\) represents the measurement noise.

The inverse problem of finding information about unknown parameters from \(Z_n\) can be solved with statistical inverse theory [1],[2]. In statistical inverse problems one tries to find the probability of causes from inexact consequences of the cause. The unknown and the measurement are modeled as random vectors \(X\) and \(Z_n = s_n(X) + \epsilon_n\) with continuous probability densities representing respectively the probability density \(D_{pr}(x)\) of causes without knowing any consequences and the probability density \(D_n(z)\) of inexact consequences of those random causes. Assuming that \(X\) and \(\epsilon_n\) are statistically independent, we find that the conditional density \(D_n(z|x)\) of \(Z_n\), when \(X\) is known to be \(x\), is the density for \(s_n(x) + \epsilon_n\). By Bayes formula, the
solution to the statistical inverse problem is

\[ D_{\text{post}}(x|Z_n) = C_n(Z_n)D_n(Z_n|x)D_{\text{pr}}(x) \] (2)

whenever the norming constant \( C_n(Z_n) \) is finite. We study in Section 3 what happens to the solution when \( n \) increases. The sampled signal \( Z_n \) is described as a kind of a measurable projection of the analog signal \( Z = s(X) + \epsilon \). The main tool in the proof is the famous Cameron-Martin formula that gives the Radon-Nikodym derivative of a shifted Gaussian measure with respect to the original measure.

The convergence result of Theorem 4 shows that the statistical inverse theory plays well with the slightly idealized sampling method. As sampling speed is increased, also the solutions cumulate error to the solution. This suggests that putting more attention on the design of the approach to the solution of the analog problem. In this sense, the sampling method does not approach the solution of the analog problem. In this sense, the sampling method does not cumulate error to the solution. This suggests that putting more attention on the design of the digital data acquisition pays off as compatible Bayesian solutions.

2. The direct problem

Let \( \psi(t) \) be the transmitted signal. The equation of radial motion for the object is

\[ r(x_2, x_3, x_4; t) = x_2 + x_3 t + \frac{1}{2} x_4 t^2, \]

where \( x_2 \) is the range, \( x_3 \) is the radial velocity and \( x_4 \) is the radial acceleration of the object. The phase of the reflected signal is modeled by assuming that the unknown object is like a mirror that moves with a constant radial acceleration. We assume that the amplitude \( x_1 \) of the reflected signal stays constant up to our time window \( T \) which is about 300 ms. Then the reflected signal may be approximated as

\[ s(x; t) = x_1 \psi(t - \frac{2}{c} x_2) e^{-i \frac{2\pi}{c}(x_3 t + \frac{1}{2} x_4 t^2)} \]

The unknown parameters describing the object are modeled as a random vector \( X \in \mathbb{R}^4 \) whose components represent the amplitude of the reflected signal, range, radial velocity, and radial acceleration of the object. We model the reflected signal with \( s(X) \), where \( s : \mathbb{R}^n \to L^2([0, T]) \) is taken to be real-part of \( s(x; t) \) for simplicity. When we take the measurement noise into account, we arrive to the indirect and noisy analog observation

\[ Z = s(X) + \epsilon \] (3)

of \( X \). The noise \( \epsilon \) is assumed to be a Gaussian white noise defined as a Hilbert space process over \( L^2([0, T]) \). Namely, \( \epsilon(f) \) are square-integrable random variables which depend linearly on \( f \) and which have Gaussian distributions with zero mean and covariance \( \mathbb{E}[(\epsilon(f)\epsilon(g))] = \sigma^2(f, g) \) for all \( f, g \in L^2([0, T]) \). Recall, that this does not mean that \( \epsilon \in L^2([0, T]) \) but rather that \( \epsilon \) has values in the space of distributions [3]. Since \( s \) is a sequentially continuous mapping from \( \mathbb{R}^4 \) into \( L^2([0, T]) \), also \( s(X) \) can be treated as a Hilbert space process over \( L^2([0, T]) \).

Let \( [0, T] \) be divided to intervals \((T_i^{(n)}, T_{i+1}^{(n)})\) of length \( \tau_n = T 2^{-n} \), where \( T_0 = 0 \) and \( T_i^{(n)} = i 2^{-n} T \) for \( i = 1, ..., 2^n \). The measurements are

\[ Z_n(t) = \frac{1}{\tau_n} \sum_{i=0}^{2^n-1} (s(X), 1_{(T_i^{(n)}, T_{i+1}^{(n)})}) L^2 1_{(T_i^{(n)}, T_{i+1}^{(n)})}(t) + \epsilon(1_{(T_i^{(n)}, T_{i+1}^{(n)})}), \] (4)

where statistically independent random variables \( \epsilon(1_{(T_i^{(n)}, T_{i+1}^{(n)})}) \) obey the distribution \( N(0, \sigma^2 \tau_n) \).

Denote

\[ M_n := \{ f \in L^2([0, T]) : f = \frac{1}{\tau_n} \sum_{i=0}^{2^n-1} (f, 1_{(T_i^{(n)}, T_{i+1}^{(n)})}) L^2 1_{(T_i^{(n)}, T_{i+1}^{(n)})} \} \] (5)
the subspace of \( L^2([0, T]) \) with basis \( \frac{1}{\sqrt{\tau_n}}1_{(T_i^{(n)}, T_{i+1}^{(n)})}, \ i = 0, \ldots, 2^n - 1 \). Then the \( M_n \)-valued vector

\[
\epsilon_n := \frac{1}{\tau_n} \sum_{i=0}^{2^n-1} \epsilon \left( 1_{(T_i^{(n)}, T_{i+1}^{(n)})} \right) 1_{(T_i^{(n)}, T_{i+1}^{(n)})}
\]

(6)

has a Gaussian density with zero mean and covariance operator \( \sigma^2 I \) on \( M_n \). Especially, the conditional density of \( Z_n \) given \( X = x \) is

\[
D_n(Z_n|x) = Ce^{-\frac{1}{2\pi} ||Z_n-s_n(x)||^2_{L^2}}
\]

(7)

for every \( Z_n \in M_n \). Here

\[
s_n(x; t) := \frac{1}{\tau_n} \sum_{i=0}^{2^n-1} \left( s(x), 1_{(T_i^{(n)}, T_{i+1}^{(n)})} \right) 1_{(T_i^{(n)}, T_{i+1}^{(n)})}
\]

(8)

for all samples \( x \in \mathbb{R}^4 \) of \( X \).

The sampled signal \( Z_n \) lacks some details of the original analog signal \( Z \) but it is not overly distorted as is shown below. For issues in measure theory, we refer to [4].

**Theorem 1.** If \( f \in L^2([0, T]) \), then \( \lim_{n \to \infty} (Z_n, f) = Z(f) \) almost surely.

**Proof.** Due linearity of the Hilbert space processes, \( (Z_n, f) = Z(f_n) \) almost surely.

The well-known arguments for showing \( L^2 \)-convergence of \( f_n \) to \( f \) are the following. Since almost all points of the real line are Lebesgue points, then \( f_n(t) \to f(t) \) a.e. as \( n \to \infty \). Consequently, \( f_n \) converges to \( f \) in \( L^2([0, T]) \) for continuous functions \( f \) by the Lebesgue dominated convergence theorem. The density of continuous functions in \( L^2([0, T]) \) implies norm convergence for all \( L^2 \)-functions. Thus \( (s_n, f) \) converges to \( (s, f) \) as \( n \to \infty \).

Let us consider the noise terms \( \epsilon(f_n) \). These can be written as conditional expectations

\[
\epsilon(f_n) = \sum_{i=0}^{2^n-1} \frac{1}{\sigma^2 \tau_n} E[\epsilon(f) \epsilon(1_{(T_i, T_{i+1})})] \epsilon(1_{(T_i, T_{i+1})})
\]

\[
= E[\epsilon(f) \frac{1}{\sigma \sqrt{\tau_n}} \epsilon(1_{(T_i, T_{i+1})})], \ i = 0, \ldots, 2^n - 1.
\]

The subspaces spanned by the functions \( 1_{(T_i, T_{i+1})} \) increase, when new end points are added in the middle of the previous intervals. Therefore, the corresponding sigma algebras generated by \( \epsilon(1_{(T_i, T_{i+1})}) \), \( i = 0, \ldots, 2^n - 1 \) increase too. Moreover, \( E[\epsilon(f_n)^2] \) are uniformly bounded since sequence \( \{f_n\} \) converges in \( L^2([0, T]) \). By the martingale convergence theorem, \( \epsilon(f_n) \) converge a.s. and in \( L^2(P) \). Finally, we identify the limit with \( \epsilon(f) \).

Since \( E[(\epsilon_n(f) - \epsilon(f))^2] = \sigma^2 ||f - f_n||^2_{L^2([0,T])} \), we may choose a subsequence of \( \epsilon(f_n) \) that converges a.s. to \( \epsilon(f) \). Thus \( \lim_{n \to \infty} \epsilon_n(f) = \epsilon(f) \) almost surely.

Since countable unions of zero probability sets have zero probability, we obtain the almost sure convergence for countably many \( L^2 \)-function outside a common set of zero probability. But we need a stronger result, the a.s. convergence for the random function \( s(X) \).

We first clarify the meaning of the mapping \( \epsilon(s(X)) \). When \( \{e_i\} \) is an orthonormal basis of \( L^2([0, T]) \), we set

\[
\epsilon(s(X)) := \sum_{i=1}^{\infty} \epsilon(e_i)(s(X), e_i).
\]

(9)

whenever the limit exists. Here \( \{\epsilon(e_i)\}_{i=1}^{\infty} \) is a sequence of statistically independent identical normal random variables that are also independent from \( X \). For deterministic \( X \), this coincides a.s. with the original definition of \( \epsilon(s(X)) \) [5].
Theorem 2. Let $X$ be a $\mathbb{R}^4$-valued random vector statistically independent from a Gaussian white noise $\epsilon$ and $s: \mathbb{R}^4 \to L^2([0,T])$ be a continuous mapping such that $E[\|s(X)\|^2_{L^2([0,T])}] < \infty$. Then the series $\sum_{i=1}^{\infty} \epsilon(e_i)(s(X),e_i)$ converges almost surely.

Proof. The finite sums $\sum_{i=1}^{m} \epsilon(e_i)(s(X),e_i)$ form a martingale sequence with respect to sigma fields generated by $\epsilon(e_i)$ and $(s(X),e_i)$, $i = 1, \ldots, m$. The expectation of the square

$$\sum_{i,j=1}^{m} E[\epsilon(e_i)\epsilon(e_j)(s(X),e_i)(s,e_j)] \leq \left( \sup_{i} E[\epsilon(e_i)^2] \right) \sum_{i=1}^{\infty} E[(s(X),e_i)^2]$$

(10)

is uniformly bounded. Due the martingale convergence theorem, the finite sums converge almost surely.

Note that

$$E[\|s\|^2_{L^2([0,T])}] \leq CE[X_1^2] \sup |\psi| < \infty$$

(11)

when $X_1$ is square integrable.

Theorem 3. Let $X$, $\epsilon$ and $s$ be as in Theorem 2. Let $Z = s_0 + \epsilon$, where $s_0 \in L^2([0,T])$ is deterministic. Then $\lim_{n \to \infty} (Z_n, s_n(X)) = Z(s(X))$ almost surely.

Proof. Due continuity of the inner product $(s_0, s_n(X)) \to (s_0, s(X))$ as $n$ increases. Next, we note that $(\epsilon_n, s_n(X)) = (\epsilon_n, s(X))$. Moreover,

$$E[\epsilon_n(s(X))^2] = \sum_{i,j=0}^{2^n-1} \frac{1}{2^n} E[\epsilon((1_{[T_i,T_{i+1}]})\epsilon((1_{[T_j,T_{j+1}]}))E[(s(X), 1_{[T_i,T_{i+1}]})(s(X), 1_{[T_j,T_{j+1}]}))] < \infty.$$  

(12)

But $\epsilon_n(s(X))$ is a martingale with respect to the increasing sigma-algebra generated by $\epsilon((1_{[T_i,T_{i+1}]})$, $(s(X), 1_{[T_j,T_{j+1}]})$, $i = 0, \ldots, 2^n - 1$. By the martingale convergence theorem, $(Z_n, s_n(X))$ converges almost surely and the limit is necessarily $Z(s(X))$ since the convergence holds also in $L^2(P)$.

3. Convergence results

The Bayesian solution to the finite-dimensional inverse problem of finding the probability density function of causes $X$ from given $Z_n = s_n(X) + \epsilon_n$ is the conditional probability density function

$$D_n(x|Z_n) = C_n(Z_n)D_{pr}(x)D_n(Z_n|x)$$

(13)

for all $Z_n \in M_n$ such that the normal constant $C_n(Z_n)$ is finite. Here $D_{pr}(x)$ is a continuous prior probability density function of $X$ describing the probability of causes without knowing any consequences and $D_n(x|\cdot)$ is the conditional probability density of $Z_n$ given value $X = x \in \mathbb{R}^4$.

The density $D_n(\cdot|x)$ is obtained from the distribution of the Gaussian noise as

$$D_n(z|x) = C_n \exp \left( -\frac{1}{2\sigma^2} \|z - s_n(x)\|^2_{L^2([0,T])} \right), \ z \in M_n.$$  

(14)

We ask if the posterior distribution has the correct asymptotics so that as $n$ grows, it starts to resemble more and more the probability of causes given the analog signal $Z = s(X) + \epsilon$. As signal $Z$ includes additive white noise, it is a power signal and therefore the norming constants $C_n(Z_n)$ are almost surely unbounded. Fortunately, when the norm $\|Z_n - s_n\|^2$ is expanded to $\|Z_n\|^2 - 2(Z_n, s_n) + \|s_n\|^2$, the worst terms cancel in $D_n(x|Z_n)$, and we get

$$D_n(x|Z_n) = \frac{D_{pr}(x) \exp \left( \frac{2(Z_n, s_n) - \|s_n\|^2}{2\sigma^2} \right)}{\int D_{pr}(x) \exp \left( \frac{2(Z_n, s_n) - \|s_n\|^2}{2\sigma^2} \right) dx}.$$  

(15)
Theorem 4. Let $X$ be a random vector in $\mathbb{R}^4$ having continuous density function with compact support. Let $s : \mathbb{R}^4 \rightarrow L^2([0,T])$ be a continuous mapping such that $E[\|s(X)\|_{L^2([0,T])}^2] < \infty$. Let $Z = s(X) + \epsilon$, where $\epsilon$ is a zero mean Gaussian Hilbert space process on $L^2([0,T])$ that is statistically independent from $X$ and $E(\epsilon(f)) = \sigma^2(f,g)L^2([0,T])$ for all $f,g \in L^2([0,T])$. Denote $Z_n = \sum_{i=0}^{2^n-1} Z_i(T_i(n),T_{i+1}(n))$ where $T_i = i2^{-n}T$, $i = 0, \ldots, 2^n$. Let $s_0 \in L^2([0,T])$.

When $n$ grows, the posterior densities (15) converge pointwise to the density
\[
D(x|Z) = \frac{D_{pr}(x) \exp\left(\frac{2Z(s(x)) - \|s(x)\|^2}{2\sigma^2}\right)}{\int D_{pr}(x) \exp\left(\frac{2Z(s(x)) - \|s(x)\|^2}{2\sigma^2}\right) dx}
\] (16) for almost all $Z$ of the form $Z = s_0 + \epsilon$ such that $\inf_n \int D_{pr}(x) \exp\left(\frac{2Z(s_n(x)) - \|s_n(x)\|^2}{2\sigma^2}\right) dx > 0$. Moreover, the corresponding posterior probabilities of a Borel set $B$ converge for almost all $Z$ of the described form.

Remark, that it is enough to prove the convergence at those $Z$ that are of the form $s_0 + \epsilon$, where $s_0 \in L^2([0,T])$ is the “true” signal and $\epsilon$ is the white noise.

Below, we will consider a Gaussian Hilbert space process as a distribution-valued random variable [3], [6]. Let $D([0,T])$ be the space of all smooth functions whose support is contained in $[0,T]$ equipped with the usual topology and $D'(([0,T]))$ be its weak dual space. Then the image measure $\mu_{\epsilon} = P \circ \epsilon^{-1}$ is a Gaussian measure on the distribution space $D'(([0,T]))$ equipped with the Borel sigma algebra with respect to the weak topology. It has a characteristic function $H(\phi) := E[e^{i\langle\phi, x\rangle}]$, where $\phi \in D'(([0,T]))$.

By the Cameron-Martin formula [5], the shifted white noise measure $\mu_{\epsilon + s_0}$ for $s_0 \in L^2([0,T])$ is absolutely continuous with respect to the original measure $\mu_{\epsilon}$ and the corresponding Radon-Nikodym derivative is
\[
\frac{d\mu_{\epsilon + s_0}}{d\mu_{\epsilon}}(z) = \exp\left(\frac{2z(s_0) - \|s_0\|^2}{2\sigma^2}\right),
\] (17)
where the right-hand side appears also in (16). Moreover, the measures $\mu_{\epsilon + s_0}$ and $\mu_{\epsilon}$ have same sets of zero measure.

We start showing the convergence of the posterior distributions. We are nearly done, if we can prove the following theorem.

Theorem 5. Let $s_0 \in L^2([0,T])$. Then for $P$-almost every $Z$ of the form $Z = s_0 + \epsilon$
\[
\lim_{n \rightarrow \infty} \int_B D_{pr}(x) \exp\left(\frac{2(Z_n, s_n(x)) - \|s_n(x)\|^2}{2\sigma^2}\right) dx
\] (18)
for a Borel set $B \subset \mathbb{R}^4$

Indeed, then the normal constants converge which together with Theorem 3 shows that the posterior densities converge pointwisely. Moreover, also the posterior probabilities of Borel sets converge.

The following lemma will be helpful.

Lemma 1. Let $s_0 \in L^2([0,T])$ and $D_{pr}$ have compact support. Then
\[
\int D_{pr}(x) \sup_n \frac{1}{2\sigma^2} (2Z_n(s_n(x)) - \|s_n(x)\|^2_{L^2([0,T])}) dx < \infty
\] (19)
for almost every $Z$ of the form $Z = s_0 + \epsilon$. 

Proof. In Theorem 1, we showed that $\epsilon_n(s(x)) = \epsilon(s_n(x))$ is a martingale for fixed $x$. Then $e^{\frac{1}{2}a^2\epsilon(s(x))}$ is a non-negative submartingale that is square-integrable since

$$\int_A e^{\frac{1}{2}a^2\epsilon(s_n(x))} d\mu_\epsilon(z) = \int_A e^{\frac{1}{2}a^2(2\epsilon(s_n(x)) - ||s_n(x)||^2)} d\mu_\epsilon(z) \leq e^{\sup_n ||s_n(x)||^2/\sigma^2} \mu_1/\sqrt{2}\epsilon + s_n(x)(A)$$

for every Borel set $A$. By Doob’s submartingale inequality

$$\int_A \sup_n e^{\frac{1}{2}a^2\epsilon(s_n(x))} d\mu_\epsilon(z) \leq C \sup_n \int_A e^{\frac{1}{2}a^2\epsilon(s_n(x))} d\mu_\epsilon(z) \leq C e^{\sup_n ||s_n(x)||^2/\sigma^2} \sup_n E[1_A(\sqrt{2}^{-1} \epsilon + s_n(x))] \leq C'$$

on the compact support of $D_{pr}$. By Fubini theorem

$$\int_B \int_A \sup_n e^{\frac{1}{2}a^2(2\epsilon(s_n(x)) - ||s_n(x)||^2)} d\mu_\epsilon(z) D_{pr}(x) dx = \int_A \int_B \sup_n e^{\frac{1}{2}a^2(2\epsilon(s_n(x)) - ||s_n(x)||^2)} D_{pr}(x) dx d\mu_\epsilon(z)$$

which is finite due the Cauchy-Schwartz inequality so that also

$$\int_B \sup_n e^{\frac{1}{2}a^2(2\epsilon(s_n(x)) - ||s_n(x)||^2)} D_{pr}(x) dx < \infty$$

(20)

for $\mu_\epsilon$-a.e. $z$. \qed

Proof. (Theorem 5) By Theorem 3, the integrands

$$\exp\left(\frac{2(Z_n, s_n(X)) - ||s_n(X)||^2}{2\sigma^2}\right)$$

converge $P$-a.s. Due Lemma 1, we may use the Lebesgue’s dominated convergence theorem for conditional expectations in order to obtain that

$$\lim_{n \to \infty} \int_B D_{pr}(x) \exp\left(\frac{2(Z_n, s_n(X)) - ||s_n(X)||^2}{2\sigma^2}\right) dx = \int_B D_{pr}(x) \exp\left(\frac{2Z(s(x)) - ||s(x)||^2}{2\sigma^2}\right) dx$$

for $\mu_{\epsilon + s_0}$-a.e. $Z$. \qed

Finally, we verify that $D(x|z)$ is the posterior probability density of $X$ given the noisy analog signal $Z$.

Lemma 2. Let $B \subset \mathbb{R}^1$ be a Borel set. Then $\int_B D(x|Z) dx$ is the conditional probability of $X \in B$ given $Z = s(X) + \epsilon$ for almost all $Z$ such that $\inf_n \int D_{pr}(x) \exp\left(\frac{2Z(s_n(x)) - ||s_n(x)||^2}{2\sigma^2}\right) dx > 0$.

Proof. By definition, the posterior probability density $D_n(x|Z_n)$ satisfies

$$\int_A \int_B D_n(x|z) dx d\mu_{Z_n}(z) = P(X \in B \cap Z_n \in A).$$

(21)
for all Borel sets \( B \subset \mathbb{R}^k \) and Borel sets \( A \subset \mathcal{D}' \). By well-known arguments, we may restrict to Borel sets \( A \) with boundary of measure zero. The left-hand side can be written as an expectation

\[ E[1_A(Z_n)\mu_n(B|Z_n)] \]

of the posterior probability \( \mu_n(B|Z_n) = \int_B D_n(x|Z_n)dx \). (22)

In other words, the posterior probability \( \mu_n(B|Z_n) \) is a projection in probability of \( 1_B(X) \) onto certain linear space generated by all indicator functions \( 1_A(Z_n) \) of Borel sets \( A \subset \mathcal{D}' \). That is \( E[1_B(X)1_A(Z_n)] = E[\mu_n(B|Z_n)1_A(Z_n)] \). Taking the limits and applying Lebesgue’s dominated convergence theorem gives

\[ E[1_B(X)1_A(Z)] = E[\lim_{n\to\infty} \mu_n(B|Z_n)1_A(Z)] \]. (23)

Hence the limit measure \( \mu(B|Z) = \int_B D(x|Z)dx \) has the correct probabilities of the conditional probability measure. Moreover, the limit may be modified on a zero measurable Borel set so that convergence holds everywhere and the limit is a measurable function of \( Z \).

Acknowledgments

This work was supported by the Academy of Finland (application number 213476, Finnish Programme for Centres of Excellence in Research 2006-2011).

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