Analytic integrability for Newton Cartan $D1$ branes

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Abstract

We probe torsional Newton Cartan (TNC) geometry with $D1$ branes and explore classical integrability criteria following two traditional approaches. In the first part of our analysis, we use Kovacic’s algorithm of classical (non)integrability to analyze dynamical phase space configurations of torsional Newton-Cartan (TNC) $D1$ branes on $R \times S^2$. We chose to work with two different $D1$ brane embeddings and obtain the corresponding Normal Variational Equations (NVEs). In both cases the NVEs are trivially satisfied yielding Liouvillian form of solutions. This ensures that the associated phase space configurations are classically integrable. Finally, based on the notion of Killing generators that span the $su(2)$ Lie algebra of the reduced space time, we explicitly construct the so called Lax connection and compute the infinite tower of conserved charges associated with the 2D world-volume theory. This finally confirms our claim on classical integrability of $D1$ branes over TNC geometries.

1 Overview and Motivation

The extension of non relativistic (NR) string sigma models [1]-[3] to arbitrary backgrounds and understanding two of its primary aspects namely, (i) the UV completion and (ii) the underlying integrable structure (if any) stands extremely important in its own right. The target space geometry corresponding to NR propagating strings could be classified into two different categories. One of these goes under the name of string Newton Cartan geometry obtained via gauging the centrally extended string Galilean algebra [4]-[9]. The other is obtained via null reduction of (relativistic) Lorentzian manifolds giving rise to what is known as torsional Newton Cartan (TNC) geometry [10]-[18]. A recent analysis of [16] reveals that under certain specific assumptions, these two seemingly different string theories could in principle be mapped into each other in a consistent manner.

The exciting evidence behind the existing integrable structure at the tree level of the Newton Cartan (closed string) sigma models [8],[15] has opened up a tremendous possibility of analyzing the NR stringy dynamics using the standard techniques of integrable models. This is therefore quite similar in spirit to that of its relativistic counterpart [19]-[20]. However, the understanding of similar questions in the corresponding open string sector still remains as a challange. The present article therefore aims to fill up some of these gaps and widen our current understanding beyond the closed string sector by taking

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into account the dynamics of extended objects like $D1$ branes [21]-[22] those probing the Galilean invariant manifolds. The corresponding target space geometry that we choose to work with happens to be a $2 + 1$ dimensional TNC spacetime (those are obtained via null reduction of $3 + 1$ dimensional Lorentzian manifolds [12] with $R \times S^2$ topology.

We address the above issue of classical integrability following two traditional paths. The first part of the paper deals with the so called Kovacic’s algorithm [23]-[24] of classical (non)integrability which has been applied with remarkable success in various examples of relativistic sigma models [25]-[31] with or without supersymmetries. However, in spite of its elegance, it is in fact quite difficult to claim integrability using Kovacic’s prescription alone as it is not the most generic way of proving integrability instead is a case by case study of different phase space configurations associated with the $D1$ brane configuration.

We confirm our claim in the second part of the paper where we explicitly construct the so called Lax connection [20] and establish its flatness following the equations of motion. This further allows us to compute the infinite tower of conserved charges associated with the $2D$ world-volume theory and thereby proving the integrability. In summary, we show that both of the above methods are mutually complementary and confirms the classical integrability of Newton Cartan $D1$ branes over $R \times S^2$. Finally, we conclude in Section 3.

2 Road to integrability

2.1 Kovacic’s method

For the sake of comprehensiveness, we briefly outline the essentials of Kovacic’s algorithm that was proposed originally in [23]. The algorithm essentially provides road to explore the classical (non)integrability criteria associated with dynamical phase space configurations. The steps are in fact quite straightforward to follow: (1) choose an invariant plane in the dynamical phase space and (2) consider fluctuations normal to this plane. These fluctuations generally obey differential equations,

$$a(\tau)\ddot{\eta}(\tau) + b(\tau)\dot{\eta}(\tau) + c(\tau)\eta(\tau) = 0$$

known as Normal Variational Equations (NVEs) [26]. Here, $a$, $b$ and $c$ are in general complex rational functions. The associated phase space configuration is said to be classically integrable if there exists simple algebraic/logarithmic/exponential solutions to (1) known as Liouvillian solutions [23]-[24], [25]-[26]. In summary, the algorithm sets rules to check whether NVE (1) admits Liouvillian solutions or not.

To check this explicitly, it is customary first to note down an equivalent representation [23]-[24] of (1),

$$\ddot{\xi} = V(\tau)\xi(\tau) ; \quad V(\tau) = \frac{2ba - 2b\dot{a} + b^2 - 4ac}{4a^2}.$$  

Substituting, $\xi(\tau) \sim e^{\int w(\tau)d\tau}$ into (2) finally yields,

$$\ddot{w}(\tau) + w^2(\tau) = V(\tau)$$  

where $w(\tau)$ is a (complex) rational function of the form, $\frac{P(\tau)}{Q(\tau)}$. Following the algorithm [23]-[24], the NVE (1) allows Liouvillian form of solutions $w(\tau)$ turns out to be a polynomial of degree 1, 2, 4, 6 or 12.
Interestingly enough, we discover that for extended objects like nonrelativistic $D1$ branes (those probing TNC geometries) it is indeed possible to find a very special form of NVEs \( \mathbf{1} \) with \( a \neq 0 \) together with \( b = c = 0 \) which therefore uniquely sets the potential \( V(\tau) = 0 \) as well as the rational polynomial \( w(\tau) \sim \frac{1}{\tau} \) with degree 1. The most general expression for these Polynomials goes under the name of Mobius transformations that generate the group of automorphisms of the Riemann sphere.

### 2.1.1 $D1$ brane embedding I

We probe torsional Newton Cartan (TNC) geometry with $Dp$ branes (with $p = 1$) in the presence of background NS-NS fluxes ($B_{MN}$). For technical simplicity, we set the dilaton as well as the background RR fluxes to zero. The resulting DBI action \([21]\) is given by,

\[
S_{Dp} = -T_1 \int d^2 \xi \sqrt{|\det A_{\alpha\beta}|} = -T_1 \int d^2 \xi \mathcal{L}_{D1} \tag{4}
\]

where we identify,

\[
A_{\alpha\beta} = G_{MN}(X^P)\partial_\alpha X^M \partial_\beta X^N + l_s^2 \mathcal{F}_{\alpha\beta} + B_{MN}(X^P)\partial_\alpha X^M \partial_\beta X^N. \tag{5}
\]

Here, \( T_1 = l_s^{-2} \) stands for the $D1$ brane tension together with \( \xi^\alpha (\alpha = 0, 1) \) as world-volume directions. Moreover, we identify \( \mathcal{F}_{\alpha\beta} = \partial_\alpha a_\beta - \partial_\beta a_\alpha \) as being the world-volume field strength tensor where \( a_\alpha \) is the corresponding $U(1)$ gauge field.

To proceed further, we consider TNC geometry \([10]-[11]\) with \( R \times S^2 \) topology \([12],[15]\)

\[
ds^2 = 2 \tau_\mu dX^\mu du - 2m_\varphi \tau_\mu dX^\mu d\varphi + \frac{1}{4}(d\theta^2 + \sin^2 \theta d\varphi^2) \tag{6}
\]

where we identify individual metric functions \([12],[15]\)

\[
\tau_\mu dX^\mu = \frac{1}{2} d\psi + dt - \frac{1}{2} \cos \theta d\varphi \quad m_\varphi = \frac{1}{4} \cos \theta. \tag{7}
\]

Notice that, here \( X^u \equiv u \) is the null isometry direction along which the $D1$ brane is presumed to be spatially extended,

\[
u = \xi^1 = \sigma \tag{8}
\]

where rest of the world-volume d.o.f. are taken to be independent of \( \xi^1 \).

Using \([6]\), it is therefore trivial to show

\[
A_{\alpha\beta} = 2\partial_\alpha t \partial_\beta u - \frac{1}{2} \cos \theta \partial_\alpha t \partial_\beta \varphi + \partial_\alpha \psi \partial_\beta u - \cos \theta \partial_\alpha \varphi \partial_\beta u
\]

\[
- \frac{1}{4} \cos \theta \partial_\alpha \psi \partial_\beta \varphi + \frac{1}{4} (\partial_\alpha \theta \partial_\beta \theta + \partial_\alpha \varphi \partial_\beta \varphi) + l_s^2 \mathcal{F}_{\alpha\beta} + B \sin \theta \partial_\alpha \theta \partial_\beta \varphi \tag{9}
\]

where we choose to work with NS-NS two form \( B_{\theta \varphi} = B \sin \theta \) \([18]\) that corresponds to some specific values of the page charge, \( Q_D \sim \int_{S^2} B_2 \) which takes quantized values on the world-volume of the $D1$ brane. This is related to the underlying mechanism known as flux stabilization which states that the background NS fluxes sort of prevents $D1$ branes (wrapping $S^2$) from shrinking to zero size.
Below we enumerate different elements of $2 \times 2$ matrix $A_{\alpha \beta}$,

\begin{align}
A_{\tau \tau} &= -\frac{1}{2} \cos \theta \dot{\tau} \dot{\phi} - \frac{1}{4} \cos \theta \dot{\psi} \dot{\phi} + \frac{1}{4} (\dot{\theta}^2 + \dot{\phi}^2) + B \sin \theta \dot{\phi} \\ 10 \\
A_{\tau \sigma} &= 2 \dot{t} + \dot{\psi} - \cos \theta \dot{\phi} + l_s^2 \dot{a}_\sigma \\ 11 \\
A_{\sigma \tau} &= -l_s^2 \dot{a}_\sigma \\ 12 \\
A_{\sigma \sigma} &= 0 \quad \text{(13)}
\end{align}

where we set, $\xi^0 = \tau$ throughout the analysis of this paper.

### 2.1.2 Conserved charges

Given (10)-(13), the corresponding Lagrangian density is given by,

\[ L_{D1} = l_s \sqrt{\dot{a}_\sigma (2 \dot{t} + \dot{\psi} - \cos \theta \dot{\phi} + l_s^2 \dot{a}_\sigma)} \equiv l_s \sqrt{\Lambda} \quad \text{(14)} \]

where we set,

\[ \Lambda(\tau) = \dot{a}_\sigma (2 \dot{t} + \dot{\psi} - \cos \theta \dot{\phi} + l_s^2 \dot{a}_\sigma). \quad \text{(15)} \]

The corresponding (conserved) charge densities associated with the $D1$ brane configuration are given by,

\[ E = \frac{\delta L_{D1}}{\delta \dot{t}} = \frac{l_s \dot{a}_\sigma}{\sqrt{\dot{a}_\sigma (2 \dot{t} + \dot{\psi} - \cos \theta \dot{\phi} + l_s^2 \dot{a}_\sigma)}} \quad \text{(16)} \]

\[ \Pi_\phi = \frac{\delta L_{D1}}{\delta \dot{\phi}} = \frac{-l_s \dot{a}_\sigma \cos \theta}{\sqrt{\dot{a}_\sigma (2 \dot{t} + \dot{\psi} - \cos \theta \dot{\phi} + l_s^2 \dot{a}_\sigma)}} \quad \text{(17)} \]

### 2.1.3 Equations of motion

Next, we note down equations of corresponding to different world-volume fields. To proceed further, we set $t = \tau$. Below we enumerate equations of motion for world volume fields,

\[ \dot{\phi} \dot{a}_\sigma \sin \theta = 0 \quad \text{(18)} \]

\[ 2\Lambda \ddot{a}_\sigma - \dot{a}_\sigma \frac{d\Lambda}{d\tau} = 0 \quad \text{(19)} \]

\[ \left(2\Lambda \ddot{a}_\sigma - \dot{a}_\sigma \frac{d\Lambda}{d\tau}\right) \cos \theta - 2l_s^2 \Lambda \dot{a}_\sigma \dot{\theta} \sin \theta = 0 \quad \text{(20)} \]

\[ \left(2\Lambda \ddot{a}_\sigma - \dot{a}_\sigma \frac{d\Lambda}{d\tau}\right) \left(l_s^2 \dot{a}_\sigma^2 - \Lambda\right) = 0. \quad \text{(21)} \]

### 2.1.4 Normal variational equations

The above set of equations (18)-(21) are essentially the basic ingredients of what we call Normal Variational Equations (NVEs) [25]-[26]. To start with, we choose to work with the dynamical phase space configuration with $F_{\tau \sigma} = \dot{a}_\sigma \neq 0 (= \text{constant})$ together with $\psi = \dot{\theta} = \ddot{\theta} = \dot{\phi} = \ddot{\phi} = 0$. 

\[ \dot{\theta} = \ddot{\theta} = \dot{\phi} = \ddot{\phi} = 0. \]

Here, we have set the external field as $\xi^0 = \tau$ throughout the analysis of this paper.
constant. Notice that, both the invariant planes that we choose below belong to the \( \psi = \text{constant} \) and \( \Pi_\psi = \text{constant} \) subspace of the full dynamical phase space configuration. This further simplifies (18)-(21),

\[
\dot{\psi} \cos \theta + \dot{\varphi} \sin \theta - 2l^2 \Lambda \ddot{\psi} \sin \theta - 2l^2 \Lambda \ddot{\varphi} \cos \theta = 0
\]

(22)

\[
\sin \theta \dot{\psi} \dot{\varphi} - \cos \theta \ddot{\varphi} = 0
\]

(23)

where the second equation (23) follows from setting \( \frac{d\Lambda}{d\tau} = 0 \).

In order to implement Kovacic’s algorithm, we further choose to work with the following invariant plane \( \theta = \dot{\theta} = \ddot{\theta} = 0 \) which identically satisfies (22). The invariant plane one might wish to think of as a submanifold with, \( \theta = 0 \), \( \Pi_\theta = 0 \) and \( \Pi_\varphi = \text{constant} \) within the subspace of the full dynamical phase space configuration.

Upon substitution into (23) this further yields,

\[
\dot{\varphi} \bigg|_{\theta \sim \dot{\theta} \sim 0} \approx 0
\]

(24)

which has a solution,

\[
\varphi(\tau) \approx 2l^2 \Lambda \tau.
\]

(25)

Substituting (25) into (22) and considering infinitesimal fluctuations \( \delta \theta(\tau) \sim \eta(\tau) \) normal to the invariant plane, we arrive at the following NVE

\[
\ddot{\eta}(\tau) \approx 0
\]

(26)

where we retain ourselves up to leading order in the fluctuations (\( \eta(\tau) \)) and also take into account of the fact \( \frac{\Delta \eta}{\eta} \ll 1 \). Under the above set of assumptions, the NVE (26) allows Liouvillian solution of the following form,

\[
\eta(\tau) \sim \tau + c
\]

(27)

which ensures the integrability of the associated phase space configuration.

The second phase space configuration that one might choose to work with is to set the invariant plane as, \( \varphi = \dot{\varphi} = \ddot{\varphi} = 0 \) which trivially solves (23). This is a submanifold that satisfies, \( \varphi = 0 \) and \( \Pi_\varphi = 0 \). Substituting this into (22) yields,

\[
\dot{\theta} \sin \theta \bigg|_{\varphi \sim \dot{\varphi} \sim 0} \approx 0
\]

(28)

which thereby sets, \( \theta(\tau) = \theta_c = \text{constant} \). Substituting (28) into (23) and considering fluctuations \( \delta \varphi(\tau) \sim \bar{\eta}(\tau) \) normal to the invariant plane in the phase space we find,

\[
\ddot{\bar{\eta}}(\tau) \approx 0
\]

(29)

which admits the Liouvillian solution of the following form,

\[
\bar{\eta}(\tau) \sim \tau + \bar{c}.
\]

(30)

The above analysis confirms that the second phase space configuration is also classically integrable in the sense of Kovacic.
2.1.5  

**D1 brane embedding II**

This time we consider a different $D1$ brane embedding namely the $D1$ brane is placed at $X^u = \text{constant}$ along the axis of symmetry. The ansatz that we choose to work with is that of a $D1$ brane wrapping the azimuthal direction of $S^2$,

$$t = t(\tau) = \tau ; \ \theta = \theta(\tau) ; \ \psi = \psi(\tau) ; \ \varphi = \kappa \sigma ; \ a_\sigma = a_\sigma(\tau)$$

(31)

where $\kappa$ is the corresponding winding number.

The resulting matrix elements $A_{\alpha\beta}$ are given by,

$$A_{\tau\tau} = \frac{\dot{\theta}^2}{4}$$

(32)

$$A_{\tau\sigma} = -\frac{\kappa}{2} \cos \theta (\dot{t} + \dot{\psi}) + l_s^2 \dot{a}_\sigma + \kappa B \dot{\theta} \sin \theta$$

(33)

$$A_{\sigma\tau} = -l_s^2 \dot{a}_\sigma$$

(34)

$$A_{\sigma\sigma} = 0$$

(35)

where we define, $\bar{\psi} = \frac{\psi}{2}$.

2.1.6  

**Equations of motion**

The corresponding Lagrangian density is given by,

$$L_{D1} = l_s \sqrt{\dot{a}_\sigma (-\frac{\kappa}{2} \cos \theta (\dot{t} + \dot{\psi}) + l_s^2 \dot{a}_\sigma + \kappa B \dot{\theta} \sin \theta)} \equiv l_s \sqrt{\Gamma}$$

(36)

where we choose,

$$\Gamma(\tau) = \dot{a}_\sigma (-\frac{\kappa}{2} \cos \theta (\dot{t} + \dot{\psi}) + l_s^2 \dot{a}_\sigma + \kappa B \dot{\theta} \sin \theta).$$

(37)

The resulting equations of motion could be formally expressed as,

$$\Gamma(\kappa \dot{a}_\sigma \sin \theta (1 + \dot{\psi}) - 2B \kappa l_s \dot{\dot{a}}_\sigma \sin \theta) + \kappa B \ddot{a}_\sigma \sin \theta \frac{d\Gamma}{d\tau} = 0$$

(38)

$$\Gamma(\kappa \ddot{a}_\sigma \cos \theta - \dot{a}_\sigma \dot{\theta} \cos \theta) - \frac{\kappa}{2} \dot{a}_\sigma \cos \theta \frac{d\Gamma}{d\tau} = 0$$

(39)

$$\left(2\dot{\Gamma} \ddot{a}_\sigma - \dot{a}_\sigma \frac{d\Gamma}{d\tau} \right) (l_s^2 \dot{a}_\sigma^2 - \Gamma) = 0.$$  

(40)

2.1.7  

**Normal variational equations**

Like in the previous example, we set $\ddot{a}_\sigma = 0$ which yields the reduced set of equations,

$$\cos \theta \dot{\theta}(1 + \dot{\psi} - \dot{\theta}) + \sin \theta (\ddot{\psi} - \dot{\theta}) = 0$$

(41)

$$\sin \theta \dot{\theta}(1 + \dot{\psi}) - \dot{\psi} \cos \theta + 2B(\ddot{\theta} \sin \theta + \dot{\theta}^2 \cos \theta) = 0.$$  

(42)

Notice that, the second equation (42) is a direct consequence of setting $\frac{d\Gamma}{d\tau} = 0$. In order to obtain NVE we set, $\theta = \dot{\theta} = \ddot{\theta} = 0$ which trivially solves (41). This automatically
fixes the corresponding invariant plane as \( \{ \theta = 0, \Pi_\theta = 0 \} \). Our aim would be to solve fluctuations \( (\delta \theta(\tau)) \) normal to this plane.

Substituting the above ansatz into (42) we find,
\[
\ddot{\psi} \bigg|_{\theta \sim \Pi_\theta \sim 0} \approx 0
\]
which thereby yields, \( \psi(\tau) \sim \tau + c \). Substituting this back into (41) and considering fluctuations \( \delta \theta \sim \eta(\tau) \) at leading order, we arrive at the following NVE
\[
\ddot{\eta}(\tau) \approx 0 \quad (44)
\]
as found in the previous examples. Like before, we therefore conclude that the associated dynamical phase space configuration is classically integrable.

### 2.2 Lax pairs and integrability

In this Section, we look forward towards identifying the \( D1 \) brane dynamics in terms of a proper formulation of Lax connections \([20]\) over \( X^u = \) constant sub-manifold of the full relativistic/Lorentz invariant 3 + 1 dimensional manifold \([6]\). The corresponding world-volume theory turns out to be,
\[
S_{D1} = -T_1 \int d^2 \xi \sqrt{|\det \mathcal{A}_{\alpha\beta}|} \quad (45)
\]
where we identify,
\[
\mathcal{A}_{\alpha\beta} = -\frac{1}{2} \cos \theta \partial_\alpha v \partial_\beta \varphi + \frac{1}{4} (\partial_\alpha \theta \partial_\beta \varphi + \partial_\alpha \varphi \partial_\beta \varphi) + B \sin \theta \partial_\alpha \varphi \partial_\beta \varphi + l^2_\text{s} \mathcal{F}_{\alpha\beta}
\]
\[
= \mathcal{G}_{\alpha\beta} + l^2_\text{s} \mathcal{F}_{\alpha\beta} \quad (46)
\]
together with \([12]\), \( v = \frac{\psi}{2} + t \) which we collectively identify as time.

#### 2.2.1 The 2D world-volume current

Our starting point is the consideration of the \( D1 \) brane dynamics over group manifold \( G \sim S^2 \) with \( SO(3) \) isometries. The Killing generators that span the \( \mathfrak{so}(3) \sim \mathfrak{su}(2) \) Lie algebra could be schematically expressed as \([12]\),
\[
\mathfrak{K}_a = \mathfrak{e}_a \cdot M(X^N) \partial_M; \ a = 1, 2, 3 \quad (47)
\]
where \( \mathfrak{e}_a \cdot M(X^M; M = v, \theta, \varphi) \) are the expansion coefficients that could be fit into the following \( 3 \times 3 \) matrix as,
\[
[\mathfrak{e}]_{3 \times 3} = \begin{pmatrix}
\frac{\cos \varphi}{2 \sin \theta} & \sin \varphi & \cos \varphi \cot \theta \\
-\frac{\sin \varphi}{2 \sin \theta} & \cos \varphi & -\sin \varphi \cot \theta \\
0 & 0 & 1
\end{pmatrix} \quad (48)
\]

In the following, we introduce 2D world-volume currents as
\[
\mathfrak{J} = g^{-1} dg \simeq \ell^{(M)} \mathfrak{e}^a_M \mathfrak{K}_a dX^M; \ g \in G \quad (49)
\]
subjected to the realization that \( e^a_M \) are the elements of the inverse matrix \([e^{-1}]_{3 \times 3}\) such that, \( e_b^M e^a_M = \delta^a_b \).

An explicit computation further reveals,
\[
[e^{-1}]_{3 \times 3} = \begin{pmatrix} 2 \cos \varphi \sin \theta & -2 \sin \theta \sin \varphi & -2 \cos \theta \\ \sin \varphi & \cos \varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}.
\] (50)

To proceed further, next we note down
\[
d_3 = d(g^{-1} dg) = -g^{-2} dg \wedge dg = -\mathcal{J} \wedge \mathcal{J}
\] (51)
which thereby yields the identity of the following form,
\[
d_3 + \mathcal{J} \wedge \mathcal{J} = 0.
\] (52)

Substituting (49) into (52) we find,
\[
\partial_N \tilde{e}^a_M - \partial_M \tilde{e}^a_N + \tilde{e}^b_M \epsilon_{bc}^N a = 0
\] (53)
where, \( \epsilon_{abc} \) is the structure constant of the underlying \( \mathfrak{so}(3) \) Lie algebra. Moreover, we absorb the proportionality constant into the definition of \( e^a_M \) namely, \( \tilde{e}^a_M = \ell(M) e^a_M \).

On the other hand, expressing the current explicitly into its components we find,
\[
\mathcal{J}_\alpha = \mathcal{J}_\beta^c \mathcal{J}_\alpha
\] (54)
\[
\mathcal{J}_\alpha^a = \ell^{(i)} e^{a_i} \partial_i X^i + \ell^{(\varphi)} e^{a_\varphi} \varphi \partial_\varphi ; \ i = v, \theta.
\] (55)

Using (55), it is in fact trivial to show
\[
\tilde{\mathcal{J}}^a_\alpha \tilde{\mathcal{J}}^b_\beta \Omega_{ab} = \ell^{(i)} \ell^{(j)} e^{a_i} e^{b_j} \partial_i \partial_j \Omega_{\alpha \beta} X^i \partial_\beta X^j + \ell^{(i)} \ell^{(\varphi)} e^{a_i} e^{b_\varphi} \varphi \partial_\varphi \partial_\beta \varphi
\] (56)
In order to map (56) into (46) we impose the following set of constraints,
\[
\ell^{(i)} e^{a_i} e^{b_j} \Omega_{ab} = \delta^{a_i}_{b_j} ; \ e^{a_\varphi} e^{b_\varphi} \Omega_{ab} = 0
\] (57)
\[
e^{a_\varphi} e^{b_\varphi} \varphi \Omega_{ab} = 1 ; \ e^{a_\theta} e^{b_\varphi} \varphi \Omega_{ab} = -\cos \theta
\] (58)
\[
e^{a_\theta} e^{b_\varphi} \Omega_{ab} = 4B \sin \theta
\] (59)

together with the coefficients, \( \ell^{(\theta)} = \ell^{(\varphi)} = \frac{1}{2} \) and \( \ell^{(\psi)} = 1 \).

The above set of equations (57)-(59) could be decomposed into a set of six linear algebraic equation in \( \Omega_{ab}(\theta, \varphi) \) which in principle can be solved for nine different elements in \( \mathcal{J}_{3 \times 3} \). Below we enumerate the set of solutions satisfying (57)-(59),
\[
\Omega_{11} = \frac{1}{4 \sin^2 \theta \sin^2 \varphi} ; \ \Omega_{22} = \frac{1}{\cos^2 \varphi}
\] (60)
\[
\Omega_{12} = \Omega_{21} = \frac{1}{4 \sin \theta \sin \varphi \cos \varphi}
\] (61)
\[
\Omega_{32} = \frac{\cot \theta}{2 \sin \varphi \cos \varphi} ; \ \Omega_{31} = \frac{\cot \theta}{2 \sin \theta \sin^2 \varphi}
\] (62)
\[
\Omega_{23} = -\sin \varphi \cos \theta + 4B \cos \varphi \sin \theta + \frac{\cot \theta}{2 \sin \varphi \cos \varphi}
\] (63)
\[
\Omega_{13} = -\frac{1}{2} \cos \varphi \cot \theta - 2B \cos \varphi + \frac{\cos \theta}{2 \sin^2 \varphi \sin^2 \theta}
\] (64)
\[
\Omega_{33} = 1 + 2 \cos \theta \Omega_{13}.
\] (65)
2.2.2 Equations of motion

With the above set-up in hand, the D1 world-volume theory (45) could be formally expressed as,

\[ S_{D1} = - T_1 \int d^2 \xi \sqrt{| \det A_{\alpha \beta} |} \]

where we introduce the notation,

\[ J_{\alpha} \cdot J_{\beta} \equiv J_{\alpha}^a J_{\beta}^b \Omega_{ab}(\theta, \phi) + l_s^2 F_{\alpha \beta} \]

\[ A_{\alpha \beta} = J_{\alpha} \cdot J_{\beta} + l_s^2 F_{\alpha \beta} \]

where \( A_{\alpha \beta} \) is a symmetric tensor filed on the world-volume of the D1 brane. Moreover, here \( A_{\alpha \beta} \) has been introduced as an inverse of the induced world-volume metric namely, \( A_{\alpha \beta} A_{\beta \gamma} = \delta_{\alpha \gamma} \).

On the other hand, the dynamics associated to world-volume gauge fields reveals,

\[ \partial_\tau \left( b_{\tau \sigma} \left( \sqrt{| \det A_{\alpha \beta} |} g^{\alpha \beta} \right) \right) = 0 \]

\[ \partial_\sigma \left( b_{\tau \sigma} \left( \sqrt{| \det A_{\alpha \beta} |} g^{\alpha \beta} \right) \right) = 0 \]

where \( b_{\tau \sigma} = \frac{1}{2} (J_{\tau}^T \Omega J_{\sigma} - J_{\sigma}^T \Omega J_{\tau}) = -b_{\sigma \tau} \).

The above set of equations (70)-(71) could be combined into a single equation,

\[ \varepsilon^{\alpha \beta} \partial_\beta \varpi_{\alpha \lambda} = 0 \]

where, \( \varpi_{\alpha \beta} = \frac{b_{\alpha \beta} + l_s^2 F_{\alpha \beta}}{\sqrt{| \det A_{\alpha \beta} |}} \) is the antisymmetric two form on the world-volume. Furthermore, as a natural consequence of (70)-(71) it is in fact trivial to see, \( \varpi_{\tau \sigma} = \Pi = \text{constant} \).

2.2.3 The flat connection

Given the above dynamics, we propose the Lax connection of the following form

\[ \mathfrak{L}_\alpha = \varphi_1 J^a + \varphi_2 \varepsilon_{\alpha \gamma} \sqrt{| \det A_{\alpha \beta} |} g^{\gamma \lambda} J_{\lambda b} \delta^{ba} + \varphi_3 \varpi_{\alpha \gamma} \varepsilon^{\gamma \lambda} J^a \]
where, \( \varphi_1, \varphi_2 \) and \( \varphi_3 \) are arbitrary constants that will be fixed from the flatness condition [20] of the Lax connection. Moreover, here \( \varepsilon^{\sigma\tau} = -\varepsilon^{\tau\sigma} = 1 \) is the 2D Levi-Civita symbol together with its inverse \( \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \) which is defined through the relation, \( \varepsilon^{\alpha\beta}\varepsilon_{\alpha\gamma} = \delta^\beta_\gamma \).

Using (53), (69), (70) and (71) it is now quite straightforward to show,

\[
\varepsilon^{\alpha\beta} \partial_\beta L^a_\alpha = -\partial_\tau L^a_\sigma + \partial_\sigma L^a_\tau = (\varphi_1 - \varphi_3 \Pi) \mathring{e}^b_b M \mathring{e}^c_c N \varepsilon_{bc} a \partial_\tau X^N \partial_\sigma X^M.
\] (74)

On the other hand, after some trivial algebra we find

\[
L^b_b \partial_\sigma L^c_c \varepsilon_{bc} a = -((\varphi_1 - \varphi_3 \Pi)^2 - 8 \varphi_2^2) \mathring{e}^b_b M \mathring{e}^c_c N \varepsilon_{bc} a \partial_\tau X^N \partial_\sigma X^M.
\] (75)

Combining (74) and (75) together, we arrive at the following relation

\[
\partial_\tau L^a_\sigma - \partial_\sigma L^a_\tau - L^b_b c^c_a e_{bc} = (\varphi_1 - \varphi_3 \Pi)(\varphi_1 - \varphi_3 \Pi - 1) \mathring{e}^b_b M \mathring{e}^c_c N \varepsilon_{bc} a \partial_\tau X^N \partial_\sigma X^M + 8 \varphi_2^2 \mathring{e}^b_b M \mathring{e}^c_c N \varepsilon_{bc} a \partial_\tau X^N \partial_\sigma X^M.
\] (76)

The flatness condition [20] of Lax implies that the R.H.S. of (76) must vanish identically. This is naturally achieved by setting the following constraint,

\[
(\varphi_1 - \varphi_3 \Pi)(\varphi_1 - \varphi_3 \Pi - 1) + 8 \varphi_2^2 = 0.
\] (77)

A non trivial choice that solves (77) could be of the form,

\[
\varphi_1 = \varphi_3 = \frac{1}{1 - \Pi^2}; \quad \varphi_2 = \frac{\sqrt{\Pi}}{2\sqrt{2}(1 + \Pi)}.
\] (78)

Using (78), the flat connection (73) could be formally expressed as,

\[
\mathcal{L}_a = \frac{1}{1 - \Pi^2} \left( \mathcal{A}_a + \omega_{\alpha\gamma} \varepsilon^\gamma\lambda \mathcal{J}_\lambda^a + \frac{\sqrt{\Pi}(1 - \Pi)}{2\sqrt{2}} \varepsilon_{\alpha\gamma} \sqrt{|\text{det} A_{\alpha\beta}| g^\gamma\lambda \mathcal{J}_\lambda^b \delta^b_\sigma} \right)
\] (79)

where, we identify \( \Pi \) as the spectral parameter associated with the 2D integrable model.

2.2.4 Conserved charges

The flat connection (79) estimated above could be used to define tower of conserved charges associated with the 2D world-volume theory. The first step is to introduce the so called monodromy matrix [19]-[20],

\[
\mathcal{T}(\Pi) = \mathcal{P} \exp \int_0^{2\pi} d\sigma \mathcal{L}_\sigma(\Pi) := \exp \mathfrak{Z}(\Pi)
\] (80)

where we presume that one of the world-volume directions of \( D1 \) brane is is compact where the world-volume fields obey periodic boundary conditions.

Using (80), it is in fact straightforward to show,

\[
\partial_\tau \mathcal{T} = [\mathcal{L}_\tau, \mathcal{T}(\Pi)].
\] (81)
The next step would be to introduce the \textit{transfer} matrix,

\[
T(\Pi) = trT(\Pi) = \sum_{n=0}^{\infty} Q_n \Pi^n
\]

which by means of (81) yields,

\[
\partial_\tau Q_n = 0 ; \forall n
\]

an infinite tower of conserved charges,

\[
Q_n = \frac{1}{n!} \frac{\partial^n}{\partial \Pi^n} tr \sum_{m,k=0}^{\infty} \frac{1}{k!} \left( \int_0^{2\pi} d\sigma \frac{\Pi^m}{m!} \frac{\partial^m}{\partial \Pi^m} \Omega_\sigma(\Pi) \right)^k \bigg|_{\Pi=0}
\]

associated with the \(D1\) brane dynamics over TNC geometry with \(R \times S^2\) topology.

\section{Summary and final remarks}

The purpose of the present paper was to show classical integrability of Newton Cartan \(D1\) branes following two traditional approaches - (i) the Kovacic’s algorithm of showing classical (non)integrability and (ii) the standard formulation of Lax connections. Although the former approach does not establish classical integrability rather it partially convinces about it, the later stands as a formal proof of it at the classical level. In this paper, we show that both of these approaches produce mutually convincing results. It has been shown that \(D1\) branes probing torsional Newton Cartan geometry are classically integrable in the sense that the world-volume theory allows an infinite tower of conserved charges that follow from the standard formulation of flat (Lax) connections. It would be really nice to extend these results in the presence of dilaton and the RR fluxes.

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\section*{References}

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