Toward Second-Quantization of $D5$-Brane

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Abstract

A framework of second-quantization of $D5$-branes is proposed. It is based on the study of topology of the moduli space of their low energy effective worldvolume theory. Among the topological cycles which resolve singularities caused by overlapping $D5$-branes, it is introduced those cycles which duals, constituting a subspace of cohomology group of the moduli space, turn out to define the Fock space of the second-quantized $D5$-branes. The second-quantized operators are given by creation and annihilation operators of those cycles or their duals.
1 Introduction and Summary

The discovery of the Dirichlet branes (D-branes) [1] provides us a route to the quantization of solitonic objects [2] in string theory. The quantum fluctuations of these solitonic objects are described by open strings with one or both of their boundaries constrained on them, for which they are named Dirichlet branes.

Consider the Dirichlet five-branes (D5-branes) in Type IIB theory with open strings having the $U(n)$ Chan-Paton factors or Type IIB theory under the background of $n$ coincident D9-branes. When there are $k$ coincident D5-branes at $x^6 = \cdots = x^9 = 0$, their low energy effective worldvolume theory is given by a six-dimensional supersymmetric $U(k)$ gauge theory with global $U(n)$-symmetry [3],[4]. In this article we investigate this six-dimensional theory including the Fayet-Iliopoulos D-terms, $\zeta_{A\bar{B}} \int d^6x \ Tr D_{\bar{A}A}$. The degenerate vacua (or moduli space), which is denoted by $\mathcal{M}(k)$, are determined by the D-flat conditions. $\mathcal{M}(k)$ is the moduli space of $k$ D5-branes with open strings having the $U(n)$ Chan-Paton factors. Regarding auxiliary $D$-fields as a hyperkähler momentum map, the moduli space is a $4nk$-dimensional hyperkähler manifold obtained by the hyperkähler quotient construction [5],

$$\mathcal{M}(k) = \mu^{-1}_{A\bar{B}}(\zeta_{A\bar{B}})/U(k),$$

where $\mu_{A\bar{B}}$ is the hyperkähler momentum map of $U(k)$-symmetry (a global part of the $U(k)$ gauge symmetry).

Due to the coupling with open string D-branes are identified with the BPS states which have the Ramond-Ramond charges. These BPS states are allowed to have the bound states which are marginally stable [3],[7]. For the case of D5-branes it is discussed in [8],[9] the possibility of identifying these BPS states with cohomology elements of the moduli space of D5-branes and thereby the second-quantized five-branes are proposed. The purpose of this article is to provide a general framework of the second-quantization of D5-brane argued in [8].

In the next section we study topology of the moduli space utilizing the techniques developed by Nakajima [10]. Due to D5-branes the Lorentz group $SO(9,1)$ reduces, at least, to $SO(5,1) \times SO(4)$. $SO(4)$, which originates from the rotations in the four-
dimensions \((x^0, \cdots, x^n)\), is a global symmetry of the worldvolume theory. To study the vacua it turns out useful to fix a complex structure of \(\mathcal{M}(k)\). One can fix it by giving a complex structure of the four-dimensions. The hyperkähler quotient acquires the form,

\[
\mathcal{M}(k) = \frac{\mu_C^{-1}(0) \cap \mu_R^{-1}(\eta)}{U(k)},
\]

where the hyperkähler momentum map \(\mu_{\hat{A}\hat{B}}\) is decomposed into \(\mu_C\), the complex (holomorphic) part, and \(\mu_R\), the real part. The three constants \(\zeta_{\hat{A}\hat{B}}\) (coupled with the Fayet-Iliopoulos terms) are rotated equal to zero except the only one component, which is denoted by \(\eta\). \(\eta\) is assumed to be positive. In the case of \(\eta\) being positive, the moduli space \(\mathcal{M}(k)\) turns out to be a smooth hyperkähler manifold.

In order to preserve this complex structure, the structure group \(SO(4)\) of the four-dimensions reduces to \(U(2)\). An abelian part of \(U(2)\) rotates \((z_1, z_2)\), the holomorphic coordinates of \(C^2\), by phases and also acts on the moduli space \(\mathcal{M}(k)\). Its fixed points in \(\mathcal{M}(k)\) are related with overlapping \(D5\)-branes. Any fixed point turns out to become zero as \(\eta \to 0\). “zero” is a singularity of the moduli space obtained by setting all the components \(\zeta_{\hat{A}\hat{B}}\) being zero. This singularity is caused by \(k\) \(D5\)-branes overlapping at the origin \((z_1, z_2) = (0, 0)\). From the perspective of four-dimensional gauge theory one can say it corresponds to the small size limit of \(k\) \(SU(n)\)-instantons sitting at the origin.

It is shown that one can associate an appropriate set of \(n\) Young tableaux with each fixed point. This correspondence is not one-to-one because of degeneracies of the fixed points. Strictly speaking, each set of \(n\) Young tableaux satisfying conditions (2.42) corresponds to a fixed submanifold of the moduli space. Taking the Morse theoretical viewpoint \([10]\) topology of \(\mathcal{M}(k)\) is described from considerations on these fixed submanifolds. The action of the abelian part of \(U(2)\) is hamiltonian and its momentum map can be regarded, with an appropriate combination, as a perfect Morse function on the moduli space. The Poincaré polynomial of the moduli space turns out to have form (2.46)

\[
P_t(\mathcal{M}(k)) = \sum_{(\Gamma_1, \cdots, \Gamma_n)} t^{2\left(n^{(k-\sum_{j=1}^n l(\Gamma_j)) + \sum_{i<j} l(\Gamma_i \setminus \Gamma_j) - \sum_{i<j} \nu(\Gamma_i \setminus \Gamma_j)}\right)} P_t(\mathcal{F}(\Gamma_1, \cdots, \Gamma_n))
\]

where the summation is performed with respect to those set of \(n\) Young tableaux satisfying conditions (2.42). To each set of \(n\) Young tableaux \((\Gamma_1, \cdots, \Gamma_n)\), the corresponding
fixed submanifold is denoted by $\mathcal{F}_{(\Gamma_1, \ldots, \Gamma_n)}$. Note that $l(\Gamma_i), l(\Gamma_i/\Gamma_j)$ and $\nu(\Gamma_i/\Gamma_j)$ are positive integers introduced in the text. In this expression of the Poincaré polynomial, terms related with a given set of $n$ Young tableaux $(\Gamma_1, \ldots, \Gamma_n)$ can be read as

$$
2 \left\{ n(k-\sum_{j=1}^n l(\Gamma_j)) + \sum_{i<j} l(\Gamma_i/\Gamma_j) - \sum_{i<j} \nu(\Gamma_i/\Gamma_j) \right\} P_t(\mathcal{F}_{(\Gamma_1, \ldots, \Gamma_n)})
$$

$$
= 2 \left\{ n(k-\sum_{j=1}^n l(\Gamma_j)) + \sum_{i<j} l(\Gamma_i/\Gamma_j) - \sum_{i<j} \nu(\Gamma_i/\Gamma_j) \right\} + \dim \mathcal{F}_{(\Gamma_1, \ldots, \Gamma_n)} \left( 1 + O(t^{-2}) \right).
$$

A cycle which gives the leading will be called the maximal dimensional cycle labelled by $(\Gamma_1, \ldots, \Gamma_n)$ and denoted by $C_{(\Gamma_1, \ldots, \Gamma_n)}$. Among these topological cycles, those labelled by $(\Gamma, \emptyset, \ldots, \emptyset)$ are studied in Section 3 from the viewpoint of $D5$-branes. $\Gamma$ is an arbitrary Young tableau of $k$ boxes. One can write it explicitly by $\Gamma = [k_1, \ldots, k_l]$ where $k_i \ (1 \leq i \leq l)$ are non-decreasing positive integers satisfying $k_1 + \cdots + k_l = k$. We begin Section 3 by examining the case of $\Gamma = [l]$. An realization of the cycle $C_{([k], \emptyset, \ldots, \emptyset)}$ is presented. Any point of this cycle is shown to describe the vacuum of overlapping $k$ $D5$-branes which admits $2(nk - 1)$ additional degrees of freedom. (For the explicit form presented in the text, $k$ $D5$-branes are degenerate at the origin.) These parameters give a parametrization of the cycle $C_{([k], \emptyset, \ldots, \emptyset)}$. They turn out to disappear when $\eta$ goes to zero. It means that this topological cycle is added to resolve the aforementioned singularity of the moduli space at $\zeta_{AB} = 0$. By changing the center-of-mass of $k$ $D5$-branes from the origin to an arbitrary point, say, $P$ in the four-dimensions one can construct a cycle which is same as $C_{([k], \emptyset, \ldots, \emptyset)}$ except that $k$ $D5$-branes are degenerate at $P$, not at the origin. In order to make the position of $k$ coincident $D5$-branes, this cycle will be denoted by $C_{[k]}(P)$. It is isomorphic to $C_{([k], \emptyset, \ldots, \emptyset)}$.

Nextly we consider vacua of $k = k_1 + \cdots + k_l$ $D5$-branes ($k_1 \geq \cdots \geq k_l$) in which each $k_i$ pieces are degenerate at $P_i$. In particular we concentrate on the vacuum obtained by superposing the vacua of $k_i$ $D5$-branes which belong to $C_{[k_i]}(P_i)$. The additional degrees of freedom of each constituent are now regarded as parameters of the superposed vacuum. They constitute a $2(nk - l)$-dimensional cycle of $\mathcal{M}(k)$, which is shown to be identified with the maximal cycle $C_{([k_1, \ldots, k_l], \emptyset, \ldots, \emptyset)}$. In order to make the positions of overlapping $k$ $D5$-branes manifest this $2(nk - l)$-dimensional cycle will be denoted by $C_{[k_1, \ldots, k_l]}(P_1, \cdots, P_l)$. 

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In Section 4 we investigate on superposition of two vacua one of which is the vacuum of $k$ $D5$-branes belonging to $C_{[k_1, \ldots, k_l]}(P_1, \cdots, P_l)$ while the other is a generic vacuum of $\tilde{k}$ $D5$-branes. This superposition defines an inclusion of $C_{[k_1, \ldots, k_l]}(P_1, \cdots, P_l) \times \mathcal{M}(k)$ to $\mathcal{M}(k + \tilde{k})$. Its image consists of vacua of $k + \tilde{k}$ $D5$-branes in which the configurations of $k$ pieces, regarded as vacua of $k$ $D5$-branes, belong to $C_{[k_1, \ldots, k_l]}(P_1, \cdots, P_l)$. By letting the $l$-positions of the overlapping $k$ $D5$-branes free in the four-dimensions we obtain a noncompact submanifold of $\mathcal{M}(k + \tilde{k})$. It is denoted by $C_{[k_1, \ldots, k_l]}$. For each $i$, considering the inclusion of $C_{[k_i]}(P_i) \times \mathcal{M}(\tilde{k} + \sum_{j \neq i}^l k_j)$ to $\mathcal{M}(k + \tilde{k})$ and letting the position of the overlapping $k_i$ $D5$-branes free, we also obtain the noncompact submanifold $C_{[k_i]}$. The intersection of these submanifolds turns out to be $C_{[k_1, \ldots, k_l]} :$

$$\cap_{i=1}^l C_{[k_i]} = C_{[k_1, \ldots, k_l]}.$$

The Poincaré duals of the noncompact submanifolds $C_{[k_1, \ldots, k_l]}$ and $C_{[k_i]}$ ($1 \leq i \leq l$) are introduced, which are denoted respectively by $\mathcal{O}_{[k_1, \ldots, k_l]}$ and $\mathcal{O}_{[k_i]}$. Their degrees (or ghost numbers) are $2(nk - l)$ and $2(nk_i - 1)$. It is argued in the text that they can be made depend only on the local data of the resolutions of the singularities caused by overlapping $D5$-branes. The above intersection formula can be rephrased as

$$\mathcal{O}_{[k_1, \ldots, k_l]} = \wedge_{i=1}^l \mathcal{O}_{[k_i]}.$$

It is also possible to interpret $\mathcal{O}_{[k_1, \ldots, k_l]}$ as an element of the cohomology group $H^\ast(\mathcal{M}(k))$. It is dual to the cycle $C_{([k_1, \ldots, k_l],[\emptyset, \ldots, \emptyset])}$. With this identification we define the subspace of $H^\ast(\mathcal{M}(k))$,

$$\bigoplus_{k_1 + \cdots + k_l = k} \mathbb{C}\mathcal{O}_{[k_1, \ldots, k_l]}.$$

Due to the intersection formula it can be written as

$$\bigoplus_{k_1 + \cdots + k_l = k} \mathbb{C}\mathcal{O}_{k_1} \wedge \cdots \wedge \mathcal{O}_{k_l} = \left(\mathcal{O}_{k_i} \equiv \mathcal{O}_{[k_i]}\right).$$

Therefore, as a physical Hilbert space of the topological field theory, this subspace of $H^\ast(\mathcal{M}(k))$ admits to have a structure analogous to the Fock space. By considering the direct sum of the moduli spaces the Fock space structure is fully recovered. This
extension will be reasonable from the viewpoint of $D5$-branes since the number of $D5$-branes does not suffer apriori any restriction. (The four-dimensions is not compactified.) So, finally we obtain the following subspace of $\oplus_k H^*(\mathcal{M}(k))$ :

$$
\mathcal{H}_{total} \equiv \bigoplus_l \bigoplus_{k_1,\ldots,k_l} C O_{k_1} \wedge \cdots \wedge O_{k_l}.
$$

Our proposal can be summarized as follows : $\mathcal{H}_{total}$ is the Fock space of the second-quantized $D5$-branes which allow the $U(n)$ Chan-Paton factors, in which $O_m$ can be identified with a marginally stable bound state of $m$ $D5$-branes. The second-quantized operators are introduced as the creation and annihilation operators of these bound states.

2 Moduli Space of D5-Branes

The goal of this section is to describe topology of the moduli space of $D5$-branes which is introduced as the degenerate vacua of their effective worldvolume theory. We study it utilizing the techniques developed by Nakajima [10],[11].

We start by giving a brief review on the effective worldvolume theory of $D5$-brane. Let us consider Type IIB theory with open strings having the $U(n)$ Chan-Paton factor (or Type IIB theory under the background of $n$ coincident $D9$-branes). When there is a $D5$-brane an open string can have either Neumann boundary with the (anti-) fundamental representation of $U(n)$ or Dirichlet boundary on the $D5$-brane. The Dirichlet boundary has the index of the fundamental representation of $U(k)$ when $k$ $D5$-branes overlap [3]. The combinations of these boundary conditions correspond to three different open string sectors : Neumann-Neumann (NN), Dirichlet-Dirichlet (DD) and Dirichlet-Neumann (DN). The quantization of DD and DN strings leads to the (first) quantization of five-brane [1],[3],[4].

Suppose that there are $k$ coincident $D5$-branes at $x^6 = \cdots = x^9 = 0$. Their low energy effective worldvolume theory can be described by massless modes of the DD and DN strings. It is a six-dimensional supersymmetric $U(k)$ gauge theory with global $U(n)$-symmetry [3],[4].
There are two kinds of massless bosonic modes. \( A_\mu (\mu = 0, \cdots, 5) \) give a \( U(k) \) gauge field on the worldvolume. \( X^i (i = 6, \cdots, 9) \) are scalar fields which belong to the adjoint representation of \( U(k) \). Their \( U(k) \) gauge transformations are

\[
A_\mu (x) \mapsto g(x)A_\mu (x)g(x)^{-1} - i\partial_\mu g(x)g(x)^{-1}, \\
X^i (x) \mapsto g(x)X^i (x)g(x)^{-1},
\]

(2.1)

where \( g(x) \in U(k) \). These fields are invariant under the global \( U(n) \)-rotation since the DD string has no Neumann boundary. The vacuum expectation values of \( X^i \) become the collective coordinates of D5-branes. The \( SO(4) \simeq SU(2) \times SU(2)_R \) which originates from the rotations in the four-dimensions \((x^6, \cdots, x^9)\) is a global symmetry group of the worldvolume theory. \( SU(2)_R \) will be identified with the \( SU(2) \) R-symmetry. In order to make it clear, it is convenient to rewrite \( X^i \) as

\[
X_{A\dot{A}} = X^i \sigma^i_{A\dot{A}},
\]

(2.2)

where \( \sigma^i_{A\dot{A}} = (i\tau^1, i\tau^2, i\tau^3, 1_2) \). \( \tau^{1,2,3} \) are the Pauli matrices and \( 1_2 \) is a \( 2 \times 2 \) identity matrix. \( A (= 1, 2) \) and \( \dot{A} (= \dot{1}, \dot{2}) \) are respectively the \( SU(2) \) and \( SU(2)_R \) indices. In the DN sector, there is a \( SU(2)_R \) doublet complex scalar \( H_{\dot{A}} \). Since the DN string has both Neumann and Dirichlet boundaries, each \( H_{\dot{A}} \) transforms as \((k, \bar{n})\) representation under the action of \( U(k) \times U(n) \):

\[
H_{\dot{A}}(x) \mapsto g(x)H_{\dot{A}}(x)h^{-1},
\]

(2.3)

where \( (g(x), h) \in U(k) \times U(n) \). \( H_{\dot{A}} \) is represented by a \( k \times n \) complex matrix while \( A_\mu \) and \( X^i \) are represented by \( k \times k \) hermitian matrices.

The bosonic part of the effective action will be given by

\[
S_{boson} = \int d^6x \text{ Tr } \left\{ \frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} D^\mu X^{A\dot{A}} D_\mu X_{A\dot{A}} + D^\mu H_{\dot{A}} D_\mu \bar{H}_{\dot{A}} \\
- \frac{1}{2} D\dot{A}\dot{B} D_{\dot{A}\dot{B}} + D\dot{A}\dot{B} (X_{A\dot{A}} X^{A\dot{B}} + H_{\dot{A}} \bar{H}_{\dot{B}}) \right\},
\]

(2.4)

where \( D_\mu = \partial_\mu + i A_\mu \) and \( \bar{H}_{\dot{A}} = \epsilon_{\dot{A}\dot{B}} (H_{\dot{B}}) \). The degenerate vacua (or moduli space) are determined by the \( D \)-flat conditions obtained from (2.4),

\[
X_{A(A) X^A_{B\dot{B}}} + H_{\dot{A}} \bar{H}_{\dot{B}} = 0,
\]

(2.5)
which coincides with the ADHM equation \[12\] of $SU(n)$-instantons on $\mathbb{R}^4$.

One may modify effective action \((2.4)\) by adding the Fayet-Iliopoulos $D$-terms,

$$S_{F.I.} = -\zeta A \int d^6x \, Tr \, D^{\hat{A} \hat{B}} ,$$

which change the $D$-flat conditions to

$$X_{\hat{A}}(\, X^{\hat{A}} \, ) + H(\, \bar{\hat{H}} \, ) = \zeta A \, 1_k ,$$

where $1_k$ is a $k \times k$ identity matrix. Notice that equation \((2.7)\) can be regarded as the ADHM equation \[13\] of $SU(n)$-instantons in the gravitational instanson background $^1$.

**D5-Brane Vacua as Hyperkähler Quotient**

Let us introduce the hyperkähler structure on the space of $X_{\hat{A}}$ and $H_{\hat{A}}$. The triplet of symplectic forms are given by

$$\omega_{\hat{A} \hat{B}} = Tr \left\{ \frac{1}{2} dX_{\hat{A}} \wedge dX^{\hat{A}}_B + dH_{\hat{A}} \wedge d\bar{H}_B \right\} ,$$

where, since we are now considering the classical vacua, the field variables $X_{\hat{A} \hat{A}}$ and $H_{\hat{A}}$ are $c$-number matrices. The hyperkähler structure may be extractable from the expansion

$$\omega_{\hat{A} \hat{B}} = \omega_I I_{\hat{A} \hat{B}} + \omega_J J_{\hat{A} \hat{B}} + \omega_K K_{\hat{A} \hat{B}} ,$$

where $I_{\hat{A}} \hat{B} \equiv 2\bar{\sigma} \sigma_{12} \hat{A} \hat{B}$, $J_{\hat{A}} \hat{B} \equiv 2\bar{\sigma} \sigma_{13} \hat{A} \hat{B}$ and $K_{\hat{A}} \hat{B} \equiv 2\bar{\sigma} \sigma_{14} \hat{A} \hat{B}$ are the bases of $SU(2)_R$ which satisfy $I^2 = J^2 = K^2 = -1$ and $IJ = -JI = K$.

Under the global $U(k)$-symmetry (the global part of the $U(k)$ gauge symmetry of the worldvolume theory) $X_{\hat{A} \hat{A}}$ and $H_{\hat{A}}$ transform adjointly and vectorially. Their infinitesimal transforms are

$$\delta X_{\hat{A}} = [\Omega, X_{\hat{A} \hat{A}}] ,$$

$$\delta H_{\hat{A}} = \Omega H_{\hat{A}} ,$$

$$\left( \delta \bar{H}_{\hat{A}} = -\bar{H}_{\hat{A}} \Omega \right) ,$$

\[2.10\]

\[^1\]It is discussed in [14] from the viewpoint of D-brane.

\[^2\] $\bar{\sigma}_{ij} \hat{A} \hat{B} = \frac{1}{4} (\bar{\sigma}_i \sigma_j - \bar{\sigma}_j \sigma_i) \hat{A} \hat{B}$, where $\bar{\sigma}_i = \sigma_i^\dagger$. 

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where $\Omega \in u(k)$. (\(\Omega^\dagger = -\Omega\)). These infinitesimal transforms define a vector field $\xi_\Omega$ on the space of the field variables. Notice that symplectic forms (2.8) are invariant under $U(k)$-action (2.10). Moreover they satisfy

$$i_{\xi_\Omega} \omega_{\dot{A}\dot{B}} = Tr \{ \Omega d\mu_{\dot{A}\dot{B}} \}, \quad (2.11)$$

where

$$\mu_{\dot{A}\dot{B}} \equiv X_{A\dot{A}}X_{\dot{B}}^A + H(\dot{A}\dot{B}). \quad (2.12)$$

“$i_{\xi_\Omega}$” in (2.11) means taking an inner product by the vector field $\xi_\Omega$. Equation (2.11) besides the invariance of $\omega_{\dot{A}\dot{B}}$ show that the $U(k)$-action is hamiltonian with respect to the symplectic structures $\omega_{IJK}$ and that its momentum map is given by $\mu_{\dot{A}\dot{B}} \equiv \mu_I I_{\dot{A}\dot{B}} + \mu_J J_{\dot{A}\dot{B}} + \mu_K K_{\dot{A}\dot{B}}$, where $\mu_{IJK}$ are $u(k)$ (or $u(k)^*$)-valued.

The hyperkähler momentum map $\mu_{\dot{A}\dot{B}}$ is directly related with the $D$-flat conditions. Adding the Fayet-Iliopoulos $D$-terms, $\zeta_{\dot{A}\dot{B}} \int d^nx Tr D\dot{A}\dot{B} (\zeta_{\dot{A}\dot{B}} \in su(2)_R)$, $D$-flat conditions (2.7) can be written as

$$\mu_{\dot{A}\dot{B}} = \zeta_{\dot{A}\dot{B}} 1_k. \quad (2.13)$$

The classical moduli space of the low energy effective worldvolume theory is now given by the hyperkähler quotient [4]

$$\mathcal{M}(k) = \{ (X_{A\dot{A}}, H_{\dot{A}}) \mid \mu_{\dot{A}\dot{B}} = \zeta_{\dot{A}\dot{B}} 1_k \} / U(k). \quad (2.14)$$

(2.14) can be also regarded as the moduli space of $k$ D5-branes. As for the dimensionality, by simply counting up the degrees of freedom, it turns out to be

$$dim \mathcal{M}(k) = 4nk. \quad (2.15)$$

**Complex Structure of D5-Brane Vacua**

Due to D5-branes the Lorentz group $SO(9,1)$ reduces to $SO(5,1) \times SO(4)$. The $SO(4) \simeq SU(2) \times SU(2)_R$ which originates from the rotations in the four-dimensions is a global symmetry group of the worldvolume theory. ($SU(2)_R$ is identified with the
SU(2) R-symmetry. To investigate moduli space (2.14) it turns out useful to fix a complex structure in the four-dimensions. Let us introduce the complex structure by

$$x_{AA} = \begin{pmatrix} z_1 & \bar{z}_2 \\ -z_2 & \bar{z}_1 \end{pmatrix}.$$  

(2.16)

To preserve it the structure group $SO(4) \simeq SU(2) \times SU(2)_R$ must be restricted to $U(2) \simeq SU(2) \times U(1)_R$. (2.16) also fixes a complex structure of the space of field variables. One may introduce the complex $k \times k$ matrices $B_a (a = 1, 2)$ instead of the hermitian matrices $X^i$

$$X_{AA} = \begin{pmatrix} B_1 & B_2^\dagger \\ -B_2 & B_1^\dagger \end{pmatrix}.$$  

(2.17)

With these complex matrices one can write down the symplectic forms $\omega_{AB}$ as the combination of $(1, 1)$ and $(2, 0)$ $((0, 2))$ forms

$$\omega_{AB} = \omega_R \left( \begin{array}{c} \frac{1}{2} \\ -\frac{1}{2} \end{array} \right) + \omega_C \left( \begin{array}{c} 1 \\ 0 \end{array} \right) + \bar{\omega}_C \left( \begin{array}{c} 0 \\ 1 \end{array} \right),$$  

(2.18)

where $\omega_R$ and $\omega_C$ $($$\bar{\omega}_C$$)$ are respectively $(1, 1)$ and $(2, 0)$ $((0, 2))$ forms on the space of field variables. As regards $D$-flat conditions (2.13) they acquire the following form:

$$\mu_C = 0,$$  

(2.19)

$$\mu_R = \eta I_k,$$  

(2.20)

where $\mu_C$, $\mu_R$ are given by

$$\mu_C = [B_1, B_2] + 2H_1 H_2^\dagger,$$

$$\mu_R = [B_1, B_1^\dagger] + [B_2, B_2^\dagger] + 2H_1 H_1^\dagger - 2H_2 H_2^\dagger.$$  

(2.21)

$\mu_R$ and $\mu_C$ are the momentum maps of the $U(k)$-action constructed respectively from $\omega_R$ and $\omega_C$. Notice that, by using the $SU(2)_R$-rotations, we have set in (2.19) and (2.20) the three constants $\zeta_{AB}$ equal to zero except the only one component which is denoted by $\eta$. $\eta$ is assumed to be positive.
Throughout this section it is convenient to regard the $k \times k$ matrices $B_a$ and the $k \times n$ matrices $H_{\hat{A}}$ as elements of $\text{Hom}(V,V)$ and $\text{Hom}(W,V)$ where $V$ and $W$ respectively denote $\mathbb{C}^k$ and $\mathbb{C}^n$.

The positivity of $\eta$ ensures the following stability 10 for any solution of $D$-flat conditions (2.19,2.20):

- Any subspace $S$ of $V$ which satisfies $B_a(S) \subset S$ and $H_{\hat{A}}(W) \subset S$ is equal to $V$. (2.22)

Let us derive stability condition (2.22): Suppose (2.22) does not hold. Then there exists a subspace $S (\neq \emptyset)$ of $V$ such that $B_a(S) \subset S$ and $H_{\hat{A}}(W) \subset S$. Let $S_\perp$ be the subspace of $V$ which is orthogonal to $S$. Notice that the actions of $B_a$ and $B_a^\dagger$ are closed respectively on $S$ and $S_\perp$. Let $B_a|_S$ and $B_a^\dagger|_{S_\perp}$ denote their restrictions on $S$ and $S_\perp$. $B_a$ will acquire the form

$$B_a = \begin{pmatrix} B_a|_S & D_a \\ 0 & (B_a^\dagger|_{S_\perp})^\dagger \end{pmatrix}, \quad B_a^\dagger = \begin{pmatrix} (B_a|_S)^\dagger & 0 \\ D_a^\dagger & B_a^\dagger|_{S_\perp} \end{pmatrix}.$$ (2.23)

Equation (2.20), if one restrict it on the subspace $S_\perp$, can be written as

$$\sum_a [(B_a^\dagger|_{S_\perp})^\dagger, B_a^\dagger|_{S_\perp}] - \sum_a D_a^\dagger D_a - 2H_2|_{S_\perp} (H_2|_{S_\perp})^\dagger = \eta 1_{S_\perp},$$ (2.24)

where $H_{\hat{A}}|_{S_\perp}$ are the projections of $H_{\hat{A}}$ onto $S_\perp$. ($H_1|_{S_\perp} = 0$.) Taking the trace of this equation leads to a contradiction because we set $\eta > 0$. Therefore $S = \emptyset$.

At this stage it might be convenient to remark on the $U(k)$-quotient which appears in (2.14). It originates in the $U(k)$ gauge symmetry of the worldvolume theory. Any fixed point of the $U(k)$-action, if it exists, cause a singularity in moduli space (2.14). This singularity relates to symmetry enhancement of the theory. So it is important to ask whether such a fixed point does appear or not. Let $(B_a, H_{\hat{A}})$ be a fixed point of the $U(k)$-action. There exists $g \in U(k)$ which satisfies $gB_ag^{-1} = B_a$ and $gH_{\hat{A}} = H_{\hat{A}}$. Notice that $(B_a, H_{\hat{A}})$ satisfies the stability condition. Namely any vector $v$ of $V$ can be written in the form, $v = f(B)H_1(w)$, where $f(B)$ is an appropriate polynomial of $B_a$ and $w$ is an element of $W$. The action of $g$ on this vector can be evaluated as

$$gv = gf(B)H_1(w) = f(gBg^{-1})gH_1(w),$$ (2.25)
which turns out equal to \( v \). \( gv = v \). Since \( v \) is an arbitrary vector of \( V \), it shows \( g = 1 \). Therefore, owing to stability condition (2.22), the \( U(k) \)-action is transitive. The \( U(k) \)-quotient gives a smooth hyperkähler manifold.

**Fixed Points of \( T^2 \)-Action**

An abelian subgroup of the residual \( U(2) \cong SU(2) \times U(1)_R \) is interesting in the sense that its fixed points will be related with overlapping D5-branes. This \( U(1) \times U(1)_R \) (or \( T^2 \)) symmetry will rotate the complex coordinates \( z_a \) of the four-dimensions by the phases

\[
\begin{align*}
    z_1 & \mapsto e^{i\phi} z_1, \\
    z_2 & \mapsto e^{i\theta} z_2.
\end{align*}
\]

(2.26)

As regards \( B_a \) and \( H^A \), the \( T^2 \)-action has the forms

\[
\begin{align*}
    B_1 & \mapsto e^{i\phi} B_1, & H_1 & \mapsto e^{i\phi} H_1, \\
    B_2 & \mapsto e^{i\theta} B_2, & H_2 & \mapsto e^{-i\theta} H_2.
\end{align*}
\]

(2.27)

Notice that the transforms of \( H = (H_1, H_2) \) are modified in (2.27) by an right \( U(n) \)-action of the Chan-Paton symmetry. The \( U(1)_R \)-symmetry rotates \( H \) into \((e^{i(\theta+\phi)/2} H_1, e^{-i(\theta+\phi)/2} H_2)\). Multiplying it by \( e^{-i(\theta-\phi)/2} \in U(n) \), we can identify the transform with \((e^{i\phi} H_1, e^{-i\theta} H_2)\).

From the adjoint action of \( SU(2)_R \) on \( \omega^{A}_{\ B} \), one can find that \( U(1)_R \), the abelian part, rotates the symplectic forms as follows:

\[
\omega_R \mapsto \omega_R, \quad \omega_C \mapsto e^{i(\theta+\phi)} \omega_C.
\]

(2.28)

Notice that \( \omega_R \) is invariant under the \( U(1)_R \)-action. In fact the \( T^2 \)-action is hamiltonian with respect to \( \omega_R \). The corresponding kähler momentum map \( \mu_{\phi,\theta} \) are given by

\[
\begin{align*}
    \mu_{\phi} & = Tr\{B_1B_1^\dagger + 2H_1H_1^\dagger\}, \\
    \mu_{\theta} & = Tr\{B_2B_2^\dagger + 2H_2H_2^\dagger\}.
\end{align*}
\]

(2.29)
The action of $U(1) \times U(1)_R$ preserves $D$-flat conditions (2.13) and also commutes with that of $U(k)$. It means that $U(1) \times U(1)_R$ can act on the moduli space $\mathcal{M}(k)$. Let us give a closer look on the fixed points of this $T^2$-action. Notice that any two solutions of $D$-flat conditions (2.13) which differ from each other by the $U(k)$-action should be identified in moduli space (2.14). Therefore, for any fixed point $(B_a, H_\dot{A}) \in \mathcal{M}(k)$ there exists a homomorphism $\gamma \colon U(1) \times U(1)_R \to U(k)$ which satisfies

$$
e^i \phi B_1 = \gamma(\phi, \theta) B_1 \gamma(\phi, \theta)^{-1},$$
$$
e^i \theta B_2 = \gamma(\phi, \theta) B_2 \gamma(\phi, \theta)^{-1},$$
$$
e^\phi H_1 = \gamma(\phi, \theta) H_1,$$
$$
e^{-\theta} H_2 = \gamma(\phi, \theta) H_2. \quad (2.30)$$

Since $\gamma(\phi, \theta)$ is diagonalizable, we may decompose $V$ into the sum of the eigenspaces of $\gamma(\phi, \theta)$:

$$V = \bigoplus_{p,q \in \mathbb{Z}} V(p, q), \quad (2.31)$$

where $V(p, q)$ is an eigenspace of $\gamma$ with its eigenvalue equal to $e^{i(p\phi + q\theta)}$, that is, $\gamma(\phi, \theta)|_{V(p, q)} = e^{i(p\phi + q\theta)} 1_{V(p, q)}$.

It is important to note that these eigenvalues $e^{-i(p\phi + q\theta)}$ besides eigenspaces $V(p, q)$ are restricted by $D$-flat conditions (2.13), especially by the positivity of $\eta$. $B_a$ and $H_\dot{A}$ will be shown to satisfy the following properties:

- $B_1(V(p, q)) = V(p + 1, q), \quad B_2(V(p, q)) = V(p, q + 1). \quad (2.32)$
- $Im \ H_1 = V(1, 0), \quad H_2 = 0. \quad (2.33)$

And the allowed eigenvalues which appear in (2.31) will be restricted to

$$p \geq 1, \quad q \geq 0. \quad (2.34)$$

Therefore the decomposition has the form

$$V = \bigoplus_{p \geq 1, q \geq 0} V(p, q). \quad (2.35)$$
Let us derive these properties: i) We remark that relations (2.30) imply \( B_1(V(p, q)) \subseteq V(p+1, q) \), \( B_2(V(p, q)) \subseteq V(p, q+1) \) as for \( B_a \) and \( H_1(W) \subseteq V(1, 0) \), \( H_2(W) \subseteq V(0, -1) \) as for \( H_A \). Because \((B_a, H_A)\) satisfies stability condition (2.22), eigenspaces \( V(p, q) \) with \( p \leq 0 \) or \( q \leq -1 \) can not appear in (2.31), which means (2.34). In particular it implies \( V(0, -1) = \emptyset \). Thus we obtain \( H_2 = 0 \). ii) Nextly we will derive surjectivity (2.33) of \( H_1 \). Suppose there exists a vector \( v \) of \( V(1, 0) \) such that \( B_1 v = 0 \). Notice that condition (2.34) implies \( V(0, 0) = V(1, 1) = \emptyset \), which tells us \( B_a v = 0 \). The multiplication of \( v \) by \( \eta \) may be evaluated using D-flat condition (2.20),

\[
\eta v = \left( \sum_a [B_a, B_a^\dagger] + 2H_1 H_1^\dagger - 2H_2 H_2^\dagger \right) v
\]

which gives

\[
\eta |v|^2 = - \sum_a |B_a v|^2.
\]

This cause a contradiction except for the case of \( v = 0 \) since the L.H.S. of the equation is non-negative while the R.H.S. is non-positive. Thus we obtain \( v = 0 \). It means \( Im H_1 = V(1, 0) \). iii) As regards (2.32) we shall derive the first equation. Let us begin by considering the case of \( q = 0 \). Suppose there exists a vector \( v \) of \( V(p+1, 0) \) such that \( B_1 v = 0 \). Since we can now assume \( p \geq 1 \), both \( B_2 v \) and \( H_1^\dagger v \) vanish. The multiplication of \( v \) by \( \eta \) can be also evaluated using D-flat condition (2.20). It gives us the equality,

\[
\eta |v|^2 = - \sum_a |B_a v|^2,
\]

which means \( v = 0 \). Thus we obtain \( B_1(V(p, 0)) = V(p+1, 0) \). For the case of \( q > 0 \), by using the induction on \( q \), one can derive the equation, \( B_1(V(p, q)) = V(p+1, q) \). One can also prove the second equation of (2.32) by repeating the same argument as above.

What is the physical or geometrical meaning of these fixed points of the \( U(1) \times U(1)_R \)-symmetry? To answer this question it is convenient to consider their behavior under the limit of \( \eta \) being zero. Let \((B_a, H_A)\) be a fixed point of the \( T^2 \)-action. Owing to properties (2.32) we may regard \( B_a \) as upper triangular matrices. \( B_a = (b_{a_{ij}}) \) where \( b_{a_{ij}} = 0 \) for \( i \geq j \). The \((i, i)\)-component of equation (2.20) can be written as follows:

\[
\sum_a \left( \sum_{j=i+1}^k |b_{a_{ij}}|^2 - \sum_{j=1}^{i-1} |b_{a_{ji}}|^2 \right) + 2 \left( H_1 H_1^\dagger \right)_{ii} = \eta.
\]
Let us consider an implication of equation (2.39). For the case of $i = 1$, setting $\eta = 0$ in (2.39), we obtain $b_{a\ ij} = 0$ ($1 \leq j \leq k$) and $(H_{11}H_{11}^\dagger)_{ij} = 0$. For the case of $i \geq 2$, it is shown recursively that equations (2.39) with $\eta$ being zero gives the conditions, $b_{a\ ij} = 0$ ($1 \leq j \leq k$) and $(H_{ii}H_{ji}^\dagger)_{ij} = 0$. Therefore we can conclude that any fixed point $(B_a, H_A)$ goes to zero as $\eta \rightarrow +0$. Notice that “zero” is a singularity of the moduli space at $\eta = 0$. It is caused by $k$ D5-branes overlapping at the origin. From the perspective of four-dimensional gauge theory one can also say that it corresponds to the small size limit of $k\ SU(n)$-instantons sitting at the origin. The above behavior of the fixed points shows that these fixed points and their associated cycles (which we will discuss in the next section) are those appearing by the resolution of the singularity which exists in the case of $\eta$ being zero.

With each fixed point of the $T^2$-action it is possible to associate a set of $n$ Young tableaux by taking the following procedure. We first notice that, due to relation (2.32), we can draw the following flow diagram among the eigenspaces $V(p, q)$ in (2.33):

$$
\begin{align*}
\vdots & \quad \vdots \\
\uparrow & \quad \uparrow \\
V(2, 0) & \rightarrow V(2, 1) \rightarrow \cdots \\
\uparrow & \quad \uparrow \\
V(1, 0) & \rightarrow V(1, 1) \rightarrow \cdots
\end{align*}
$$

(2.40)

In this flow diagram the up- and right-arrows denote respectively the actions of $B_1$ and $B_2$. We also note that relations (2.32) and (2.33) give the inequalities on the dimensions of the eigenspaces in (2.33)

- $\dim V(p, q) \geq \dim V(p+1, q)$, $\dim V(p, q) \geq \dim V(p, q+1)$.
- $\dim V(p, q) \leq n$ ($= \dim W$).

(2.41)

Due to these inequalities we may introduce $n$ subdiagrams among the eigenspaces. Let us draw a flow diagram among the eigenspaces which dimensions are not less than $d$ ($1 \leq d \leq n$). (Note that we obtain the original diagram in the case of $d = 1$.) For each $d$, replacing each eigenspace in the diagram by a box, we will obtain a Young tableau $\Gamma_d$. Dimensional inequalities (2.41) can be rephrased as the conditions on these $n$ Young
tableaux $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$:

\begin{itemize}
  \item $\Gamma_1 \supseteq \Gamma_2 \supseteq \cdots \supseteq \Gamma_n$ .
  \item $|\Gamma_1| + |\Gamma_2| + \cdots + |\Gamma_n| = k$ .
\end{itemize}

(2.42)

Here, for given two Young tableaux $\Gamma$ and $\tilde{\Gamma}$, we say $\Gamma \supseteq \tilde{\Gamma}$ when the Young tableau $\tilde{\Gamma}$ can be obtained from the Young tableau $\Gamma$ by removing some boxes of $\Gamma$. The total number of boxes of $\Gamma$ is denoted by $|\Gamma|$.

The correspondence between the fixed points of the $T^2$-action and the sets of the $n$ Young tableaux is not one-to-one. The fixed points will be degenerate. This degeneracy is due to the commutativity of the $U(1) \times U(1)_R$-symmetry and the $U(n)$ Chan-Paton symmetry. One can say that each set of $n$ Young tableaux which satisfy conditions (2.42) corresponds to a fixed submanifold of the $T^2$-action.

**Topology of D5-Brane Vacua**

The fixed points of the $T^2$-action may be considered as the critical points of the following Morse function on the moduli space \[ \mu, [11] \]

$$\mu_{T^2} \equiv \mu_\phi + \epsilon \mu_\theta, \quad (2.43)$$

where $\epsilon(>0)$ is a perturbation parameter. Notice that the $T^2$-action is hamiltonian with respect to $\omega_R$ which is regarded as the kähler form of the moduli space. $\mu_{\phi,\theta}$ (2.29) are now the kähler momentum map on the moduli space.

Since the critical submanifolds of Morse function (2.43) are classified by the sets of $n$ Young tableaux $(\Gamma_1, \cdots, \Gamma_n)$ which satisfy conditions (2.42), let us write the corresponding critical submanifold by $\mathcal{F}(\Gamma_1, \cdots, \Gamma_n)$. To describe the Morse indices besides the dimensions of these critical submanifolds, we shall prepare a few notations on Young tableaux: One can realize a Young tableau $\Gamma$ by a set of non-increasing positive integers, $\Gamma = [k_1, k_2, \cdots, k_l]$. ($k_1 \geq k_2 \geq \cdots \geq k_l \geq 1$.) We introduce $l(\Gamma)$, the length of $\Gamma$, by $l(\Gamma) = l$. (Notice that the total number of boxes of $\Gamma$, which is denoted by $|\Gamma|$, is $k_1 + k_2 + \cdots + k_l$.) Let $\Gamma = [k_1, k_2, \cdots, k_l]$ and $\tilde{\Gamma} = [\tilde{k}_1, \tilde{k}_2, \cdots, \tilde{k}_\tilde{l}]$ be two Young tableaux which satisfy $\Gamma \supseteq \tilde{\Gamma}$. It means that $l \geq \tilde{l}$ and $k_i \geq \tilde{k}_i$ for $1 \leq i \leq \tilde{l}$. In such a situation
it turns out useful to extend $\tilde{\Gamma} = [\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_l]$ by writting $[\tilde{k}_1, \tilde{k}_2, \ldots, \tilde{k}_l]$ setting $\tilde{k}_i = 0$ for $\tilde{l} + 1 \leq i \leq l$. With this understanding let us introduce $l(\Gamma \setminus \tilde{\Gamma})$ as the number of $k_i \ (1 \leq i \leq l)$ which satisfy $k_i > \tilde{k}_i$. And also we introduce $\nu(\Gamma \setminus \tilde{\Gamma})$ as the number of $\tilde{k}_i \ (1 \leq i \leq l)$ which satisfy $\tilde{k}_i < k_i$ and $\tilde{k}_i = k_{i+1}. \ (k_{l+1} \equiv 0.)$

The dimensions of the critical submanifold $\mathcal{F}_{(\Gamma_1, \ldots, \Gamma_n)}$ turn out to be

$$\text{dim} \mathcal{F}_{(\Gamma_1, \ldots, \Gamma_n)} = 2 \sum_{i<j} \nu(\Gamma_i \setminus \Gamma_j), \quad (2.44)$$

and the Morse index at $\mathcal{F}_{(\Gamma_1, \ldots, \Gamma_n)}$ is given by

$$2 \left\{ n(k - \sum_{j=1}^{n} l(\Gamma_j)) + \sum_{i<j} l(\Gamma_i \setminus \Gamma_j) - \sum_{i<j} \nu(\Gamma_i \setminus \Gamma_j) \right\}. \quad (2.45)$$

Since the critical submanifolds have no odd dimensional cycles the Morse function is perfect and the Poincaré polynomial of the moduli space has the form

$$P_t(\mathcal{M}(k)) = \sum_{(\Gamma_1, \ldots, \Gamma_n)} t^{2 \left\{ n(k - \sum_{j=1}^{n} l(\Gamma_j)) + \sum_{i<j} l(\Gamma_i \setminus \Gamma_j) - \sum_{i<j} \nu(\Gamma_i \setminus \Gamma_j) \right\} P_t(\mathcal{F}_{(\Gamma_1, \ldots, \Gamma_n)})}$$

(2.46)

Formulae (2.44) and (2.45) can be derived by using the techniques developed by Nakajima [10], [11]. It is important to remark that one can also introduce [10] moduli space (2.14) by the complex symplectic quotient

$$\mathcal{M}(k) = \mu_C^{-1}(0)^s/GL(k : \mathbb{C}), \quad (2.47)$$

where

$$\mu_C^{-1}(0)^s = \left\{ (B_a, H_A) \mid \begin{array}{c} \mu_C = 0 \\ \text{stability condition (2.22)} \end{array} \right\}. \quad (2.48)$$

The action of $GL(k : \mathbb{C})$ in (2.47) is given by

$$B_a \mapsto gB_ag^{-1}, \quad H_1 \mapsto gH_1, \quad H_2^\dagger \mapsto H_2^\dagger g^{-1}, \quad (2.49)$$

where $g \in GL(k : \mathbb{C})$. Roughly speaking, the quotient by $GL(k : \mathbb{C})$ means that we are considering not the unitary group but its complexification as the gauge symmetry.
Taking complex symplectic description (2.47) of the moduli space let us consider an infinitesimal deformation \((B_a + \delta B_a, H_A + \delta H_A)\) from a given vacuum \((B_a, H_A)\). In order that the infinitesimal deformation still describes a vacuum configuration, \((\delta B_a, \delta H_A)\) necessarily satisfies the equation,

\[
\left[\delta B_1, B_2\right] + \left[B_1, \delta B_2\right] + 2\delta H_1 H_2^\dagger + 2H_1 \delta H_2^\dagger = 0 .
\] (2.50)

The L.H.S. of (2.50) is nothing but the infinitesimal deviation of \(\mu_C\), which will be denoted by \(\alpha(\delta B_a, \delta H_1, \delta H_2^\dagger)\). Not every solution of equation (2.50) gives an independent vacuum. One should take account of gauge symmetry (2.49). The infinitesimal \(GL(k : C)\)-transform can be read as

\[
\delta_Y \begin{pmatrix} B_a \\ H_1 \\ H_2^\dagger \end{pmatrix} = \begin{pmatrix} [Y, B_a] \\ Y H_1 \\ -H_2^\dagger Y \end{pmatrix},
\] (2.51)

where \(Y \in gl(k : C)(\equiv Hom(V, V))\). The R.H.S. of equations (2.51) gives a matrix-valued function of \(Y\), which will be denoted by \(\beta(Y)\). Now the tangent space of the moduli space at a given vacuum \((B_a, H_A)\) can be realized using these \(\alpha\) and \(\beta\). Namely let us introduce the complex :

\[
\begin{array}{c}
Hom(V, V)^{\oplus 2} \\
\oplus \\
\overset{\beta}{\rightarrow} Hom(W, V) \overset{\alpha}{\rightarrow} Hom(V, V) \\
\oplus \\
Hom(V, W)
\end{array}
\] (2.52)

where \(\alpha \cdot \beta = 0\) holds. Then we can identify the (holomorphic) tangent space at the vacuum \((B_a, H_A)\) with \(Ker \alpha/Im \beta\). Notice that, owing to the stability of \((B_a, H_A)\), \(\alpha\) and \(\beta\) in (2.52) turn out to be respectively surjective and injective.

Let \((B_a, H_A)\) be a fixed point of the \(U(1) \times U(1)_R\)-symmetry. In such a case there exists a homomorphism \(\gamma\) (cf.(2.30)) and we can decompose \(V\) into the direct sum of the eigenspaces of \(\gamma\). (These eigenspaces have the corresponding \(U(1) \times U(1)_R\)-charges (cf.(2.33)).) The spaces of matrices appearing in complex (2.52) can be also decomposed
into the eigenspaces of $\gamma$ with the definite $U(1) \times U(1)_R$-charges $[3]$. Since $\alpha$ and $\beta$ in (2.52) are respectively surjective and injective, it is possible to obtain the character of $U(1) \times U(1)_R$ on the tangent space at the fixed point, by identifying it with $\text{Ker } \alpha/\text{Im } \beta$, from the characters on the spaces of matrices appearing in (2.52) $[3]$. Counting the non-positive charges in the character on the tangent space we obtain formulae (2.44) and (2.45).

To end this section it may be convenient to comment on the fixed submanifolds, especially those of the type $(\Gamma_1, \Gamma_2, \cdots, \Gamma_n) = (\Gamma, \emptyset, \cdots, \emptyset)$. $\Gamma$ is a Young tableau of $k$ boxes. In these cases all the eigenspaces of $\gamma$ is one-dimensional, $\text{dim} V(p, q) = 1$, and the rank of $H_1$ is equal to one. From the corresponding flow diagrams one can normalize the $k \times n$ complex matrices $H_A$ as $[5]

$$H_1 = \begin{pmatrix}
0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 \\
\sqrt{k\eta/2} & 0 & \cdots & 0
\end{pmatrix}, \quad H_2 = 0. \tag{2.53}$$

Notice that the $U(n)$ Chan-Paton symmetry which commutes with the $U(1) \times U(1)_R$-symmetry can rotates $H_A$ to $H_A h$ where $h \in U(n)$. The critical submanifolds $\mathcal{F}_{(\Gamma, \emptyset, \cdots, \emptyset)}$ will be generated by this $U(n)$-action. From the matrix forms of (2.53) one can see that the action of $U(n-1) \times U(1)$ (of $U(n)$) is irrelevant. (The action of $U(1)$, which gives the phase of $H_1$, can be absorbed into the $U(k)$ gauge symmetry.) Therefore the critical submanifolds are

$$\mathcal{F}_{(\Gamma, \emptyset, \cdots, \emptyset)} \simeq \frac{U(n)}{U(n-1) \times U(1)} = CP^1_{n-1}. \tag{2.54}$$

---

[3] $\text{Hom}(V, V) \simeq V^* \otimes V$ and $\text{Hom}(W, V) \simeq W^* \otimes V$. In these identifications we should count the $U(1) \times U(1)_R$-charges of the eigenspace $V(p, q)$ as $(-p, -q)$ (cf. (3.4)).

[4] A little modification is needed since $\alpha$ and $\beta$ in (2.52) have the $U(1) \times U(1)_R$-charges. For the exact treatment we refer [10], in which $n = 1$ case is studied.

[5] Here we use hyperkähler description (2.14) of the moduli space.
3 Cycles of D5-Brane Vacua

There exist several nontrivial cycles in the moduli space $\mathcal{M}(k)$. Apart from topologies of the critical submanifolds of Morse function (2.43) these nontrivial cycles will be labelled, first of all, by the sets of $n$ Young tableaux which satisfy condition (2.42). In the Poincaré polynomial (2.46), terms related with a given set of $n$ Young tableaux $(\Gamma_1, \ldots, \Gamma_n)$ are

$$t^2 \left\{ n(k - \sum_{j=1}^{n} l(\Gamma_j)) + \sum_{i<j} l(\Gamma_i \setminus \Gamma_j) - \sum_{i<j} \nu(\Gamma_i \setminus \Gamma_j) \right\} P_t(\mathcal{F}(\Gamma_1, \ldots, \Gamma_n))$$

$$= t^2 \left\{ n(k - \sum_{j=1}^{n} l(\Gamma_j)) + \sum_{i<j} l(\Gamma_i \setminus \Gamma_j) - \sum_{i<j} \nu(\Gamma_i \setminus \Gamma_j) \right\} + \dim \mathcal{F}(\Gamma_1, \ldots, \Gamma_n) \left(1 + O(t^{-2})\right).$$

(3.1)

A cycle which gives the leading of (3.1) will be called the maximal dimensional cycle labelled by $(\Gamma_1, \ldots, \Gamma_n)$. It will be denoted by $C(\Gamma_1, \ldots, \Gamma_n)$. The goal of this section is the interpretation of these topological cycles in terms of D5-branes. In particular, we will investigate the maximal dimensional cycles which are labelled by $(\Gamma_1, \Gamma_2, \ldots, \Gamma_n) = (\Gamma, \emptyset, \ldots, \emptyset)$.

In order to proceed further we will need explicit forms of these cycles. These explicit forms may be handled by using another equivalent description (2.47) of the moduli space.

Let us start by giving a simple example of these maximal cycles. Consider the case of $l(\Gamma) = 1$. Namely $\Gamma = [k]$. The dimensions of the maximal cycle $C([k], \emptyset, \ldots, \emptyset)$ can be read from the dimensions of the critical submanifold $\mathcal{F}([k], \emptyset, \ldots, \emptyset)$ and the corresponding Morse index

$$\dim C([k], \emptyset, \ldots, \emptyset) = 2(n - 1) + 2n(k - 1)$$

$$= 2nk - 2. \quad (3.2)$$

Taking complex symplectic description (2.47) of the moduli space, an explicit form of
the vacuum which belongs to this maximal cycle will be given by

\[
B_1 = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 0 \\
\vdots & \vdots & \ddots & \ddots & 1 \\
0 & 0 & \ldots & \ldots & 0 
\end{pmatrix},
\]

\[
B_2 = \begin{pmatrix}
0 & a_1 & a_2 & \ldots & a_{k-1} \\
\vdots & 0 & a_1 & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & a_2 \\
\vdots & \vdots & \ddots & \ddots & a_1 \\
0 & 0 & \ldots & \ldots & 0 
\end{pmatrix},
\]

\[
H_1 = \begin{pmatrix}
0 & h_{11} & \ldots & h_{1n-1} \\
\vdots & \vdots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \vdots \\
0 & h_{k-11} & \ldots & h_{k-1n-1} \\
1 & h_{k1} & \ldots & h_{kn-1} 
\end{pmatrix},
\]

\[
H_2 = 0.
\]

\[
(3.3)
\]

\(a_i\) and \(h_{ij}\) in (3.3) are the \(nk - 1\) complex parameters which constitute the maximal cycle \(C([k],\emptyset,\ldots,\emptyset)\). Explicit form (3.3) may be obtained as follows. Let \((B_a, H_{\dot{A}})\) be a fixed point of \(T^2\)-action (2.27). There exist \(n\) Young tableaux \((\Gamma_1, \Gamma_2, \cdots, \Gamma_n)\) by which \((B_a, H_{\dot{A}})\) is labelled. Consider an infinitesimal deformation \((B_a + \delta B_a, H_{\dot{A}} + \delta H_{\dot{A}})\) from this fixed point. \((\delta B_a, \delta H_{\dot{A}})\) is a tangent vector of the moduli space at \((B_a, H_{\dot{A}})\). The \(U(1) \times U(1)_R\)-symmetry rotates tangent vectors at the fixed point. The \(T^2\)-action has the form

\[
\delta B_1 \quad \mapsto \quad e^{i\phi} \gamma(\phi, \theta)^{-1} \delta B_1 \gamma(\phi, \theta),
\]

\[
\delta B_2 \quad \mapsto \quad e^{i\theta} \gamma(\phi, \theta)^{-1} \delta B_2 \gamma(\phi, \theta),
\]

\[
\delta H_1 \quad \mapsto \quad e^{i\phi} \gamma(\phi, \theta)^{-1} \delta H_1,
\]

\[
\delta H_2 \quad \mapsto \quad e^{-i\theta} \gamma(\phi, \theta)^{-1} \delta H_2,
\]

\[
(3.4)
\]

where \(\gamma\) is the homomorphism from \(U(1) \times U(1)_R\) to \(U(k)\) which is associated with the fixed point (cf.(2.30)). It is possible to diagonalize the tangent space with respect to \(T^2\)-action (3.4), which also provides the diagonalization of the hessian of Morse function (2.43) at the critical point. The eigenspaces with the non-positive eigenvalues will

---

\(\footnote{The configuration given in (3.3) is a representative of the vacuum. Any configuration which can be obtained from (3.3) by \(GL(k : C)\)-action (2.49) describes the same vacuum.}

---
generate the maximal cycle $C_{(\Gamma_1, \Gamma_2, \ldots, \Gamma_n)}$. In the case of $(\Gamma_1, \Gamma_2, \ldots, \Gamma_n) = ([k], \emptyset, \ldots, \emptyset)$ we obtain (3.3). Each one of $a_i$ and $h_{ij}$ in (3.3) parametrizes the eigenspace with the non-positive eigenvalue in the (holomorphic) tangent space at the corresponding fixed point. Notice that configuration (3.3) itself satisfies $D$-flat condition (2.19) besides stability condition (2.22) for any values of $a_i$ and $h_{ij}$.

Taking the viewpoint of D5-branes the eigenvalues of $X^i$ (or $B_a$) will describe their positions in the four-dimensions. Therefore, $(B_a, H_A)$ in (3.3) describes the classical vacuum of $k \ D5$-branes degenerate at $(z_1, z_2) = (0, 0)$. It admits the extra $nk - 1$ complex parameters, $a_i$ and $h_{ij}$, which constitute the maximal cycle $C_{([k], \emptyset, \ldots, \emptyset)}$. This means that any point of the cycle $C_{([k], \emptyset, \ldots, \emptyset)}$ describes the vacuum of $k \ D5$-branes degenerate at the origin. Notice that, though we are describing the vacuum configuration $(B_a, H_A)$ using complex symplectic quotient (2.47), we can always find an element $g$ of $GL(k : \mathbb{C})$ so that $(gB_ag^{-1}, gH_1, H_2^\dagger g^{-1})$ satisfies $D$-flat condition (2.20). Without loss of generality we can choose $g$ such that its lower triangular part are zero and therefore $gB_ag^{-1}$ can be regarded as upper triangular matrices. So, the vacuum $(gB_ag^{-1}, gH_1, H_2^\dagger g^{-1})$, according to our previous argument, goes to zero as $\eta \rightarrow 0$, which shows that this topological cycle disappears at $\eta = 0$.

To study the vacua which belong to the maximal cycles it is also useful to describe their characteristics in a convenient form. For instance, $B_a$ in (3.3) satisfy the relations

$$
B_1 B_2 = B_2 B_1, \\
B_1^k = 0, \\
B_2 = \sum_{i=1}^{k-1} a_i B_1^i .
$$

(3.5)

Similar characterizations are also possible for $H_A$ (3.3). In order to describe them we first remark that moduli space (2.14) can be considered as the moduli space of the torsion free sheaves of rank $n$ on $\mathbb{C}P_2$ [15], [10]. (The complex coordinates $(z_1, z_2)$ of $\mathbb{C}^2$ are identified with the inhomogeneous coordinates of $\mathbb{C}P_2$.) The idea is the monad
construction of torsion free sheaf. Consider the following monad complex:

\[
V \otimes \mathcal{O}_{\mathbb{C}^2} \\
\oplus \\
V \otimes \mathcal{O}_{\mathbb{C}^2} \xrightarrow{a} V \otimes \mathcal{O}_{\mathbb{C}^2} \xrightarrow{b} V \otimes \mathcal{O}_{\mathbb{C}^2} \\
\oplus \\
W \otimes \mathcal{O}_{\mathbb{C}^2}
\]    

(3.6)

where

\[
a \equiv \begin{pmatrix}
B_1 - z_1 \\
B_2 - z_2 \\
\sqrt{2} H_1^2
\end{pmatrix}, \quad b \equiv \begin{pmatrix}
-B_2 + z_2, \\
B_1 - z_1, \\
\sqrt{2} H_1
\end{pmatrix}.
\]

(3.7)

Notice that it holds \(b \cdot a = \mu_{\mathbb{C}} = 0\) for any vacuum configuration. The corresponding torsion free sheaf \(E\) is given by

\[
E = \text{Ker } b/\text{Im } a.
\]

(3.8)

\(GL(k : \mathbb{C})\)-action (2.49) on \((B_a, H_\hat{A})\) is noting but the linear transform of the bases of \(V\). The torsion free sheaf \(E\) given by (3.8) is invariant under this transform. Therefore, to each vacuum of \(k\) D5-branes one can attach a torsion free sheaf of rank \(n\) uniquely.

Let \(B_a\) and \(H_\hat{A}\) in complex (3.6) be those given by equations (3.3). In such a situation an element

\[
\begin{pmatrix}
v_1(z) \\
v_2(z) \\
w(z)
\end{pmatrix}
\]

of \(\text{Ker } b/\text{Im } a\) satisfies the equation,

\[
\sqrt{2} \sum_{j=1}^{n} w_j(z) H_1(e_j) = (B_2 - z_2)v_1(z) - (B_1 - z_1)v_2(z),
\]

(3.9)

where \(w(z) = \sum_{j=1}^{n} w_j(z)e_j\). \(e_j(1 \leq j \leq n)\) are the bases of \(W(= \mathbb{C}^n)\):

\[
e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad e_n = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix}.
\]

(3.10)

\(^7\mathcal{O}_{\mathbb{C}^2}\) is the structure sheaf of \(\mathbb{C}^2\).
Since it holds that

$$H_1(e_j) = \sum_{i=1}^{k} h_{ij}^{-1} B_i^{-1} H_1(e_1), \quad (3.11)$$

for \( j \geq 2 \), we can find out \( g_j(z) \in \mathcal{O}_{C^2} \) such that \( H_1(e_j) = g_j(B) H_1(e_1). \) ( \( g_1(z) \equiv 1 \)).

Therefore equation \((3.9)\) reduces to

$$\sqrt{2} \sum_{j=1}^{n} w_j(z) g_j(B) H_1(e_1) = (B_2 - z_2)v_1(z) - (B_1 - z_1)v_2(z), \quad (3.12)$$

which, by replacing \( z_a \) with \( B_a \), implies

$$\sum_{j=1}^{n} w_j(B) g_j(B) H_1(e_1) = 0. \quad (3.13)$$

Because \( H_1(e_1) \) is cyclic with respect to \( B_a \) and \( B_a \) themselves are commutative, we can conclude that

$$\sum_{j=1}^{n} w_j(B) g_j(B) = 0. \quad (3.14)$$

Equation \((3.14)\) may be simplified by introducing the analogue of gauge transformation

$$w = \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix} \mapsto \tilde{w} = \begin{pmatrix} \tilde{w}_1 \\ \vdots \\ \tilde{w}_n \end{pmatrix} = Gw \quad , \quad (3.15)$$

where the “gauge transform” \( G(z) \) has the form

$$G(z) = \begin{pmatrix} 1 & g_2(z) & g_3(z) & \ldots & g_n(z) \\ 0 & 1 & 0 & \ldots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \ldots & \ldots & 0 & 1 \end{pmatrix} . \quad (3.16)$$

Notice that each \( g_j(z) \in \mathcal{O}_{C^2} \) is given by the relation, \( H_1(e_j) = g_j(B) H_1(e_1). \) The data of \( H_\tilde{A} \) are now encoded into the “gauge transform” \( G \). By this transformation, equation \((3.14)\) becomes equivalent to the following constraint on the first component of \( \tilde{w} \),

$$\tilde{w}_1(B) = 0. \quad (3.17)$$
Due to relations (3.5) any polynomial which satisfies this constraint can be given by the combinations

\[ r(z)z_1^k + s(z)(z_2 - \sum_{i=1}^{k-1} a_i z_1^i), \]

where \( r(z), s(z) \in \mathcal{O}_C \). Therefore it seems very plausible that one can distinguish \( B_a \) and \( H_A \) by relations (3.5) and “gauge transform” (3.16).

\( (B_a, H_A) \) given in (3.3) describes the vacuum of \( k \) D5-branes degenerate at the origin \((0, 0)\). These vacua form the maximal cycle \( C_{(k,\emptyset,\ldots,\emptyset)} \). By changing the eigenvalues of \( B_a \) from 0 to \( z_a \) in (3.3) it provides the vacuum configuration that \( k \) D5-branes are overlapping at \( P = (z_1, z_2) \). The additional parameters in (3.3) also constitute a cycle of the moduli space but it can be topologically identified with \( C_{(k,\emptyset,\ldots,\emptyset)} \). Let us consider the vacua of \( k = k_1 + \cdots + k_l \) D5-branes \((k_1 \geq k_2 \geq \cdots \geq k_l)\) in which each \( k_i \) pieces are degenerate at \( P_i = (z_1^{(i)}, z_2^{(i)}) \). Among them we shall concentrate on the configuration obtained by superposing the vacua of \( k_i \) D5-branes \((1 \leq i \leq l)\) having forms (3.3) with their eigenvalues shifted. Explicitly such a configuration may be written as

\[
B_a = \begin{pmatrix} B_a^{(1)} & \cdots & B_a^{(l)} \end{pmatrix}, \quad H_A = \begin{pmatrix} H_A^{(1)} \\ \vdots \\ H_A^{(l)} \end{pmatrix}.
\]

\( B_a^{(i)} \) and \( H_A^{(i)} \) are respectively \( k_i \times k_i \) and \( k_i \times n \) complex matrices such that \( (B_a^{(i)}, H_A^{(i)}) \) describes the vacuum of \( k_i \) D5-branes overlapping at \( P_i = (z_1^{(i)}, z_2^{(i)}) \) which is given (cf. (3.3)) by

\[
B_1^{(i)} = \begin{pmatrix} z_1^{(i)} & 1 & 0 & \cdots & 0 \\ 0 & z_1^{(i)} & 1 & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 1 \\ 0 & 0 & \cdots & 0 & z_1^{(i)} \end{pmatrix}, \quad B_2^{(i)} = \begin{pmatrix} z_2^{(i)} & a_1^{(i)} & a_2^{(i)} & \cdots & a_{k_i-1}^{(i)} \\ 0 & z_2^{(i)} & a_1^{(i)} & \cdots & \vdots \\ \vdots & 0 & \ddots & \ddots & a_2^{(i)} \\ \vdots & \vdots & \ddots & \ddots & a_1^{(i)} \\ 0 & 0 & \cdots & 0 & z_2^{(i)} \end{pmatrix}
\]
\[ H_1^{(i)} = \begin{pmatrix} 0 & h_{11}^{(i)} & \cdots & h_{1n-1}^{(i)} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & h_{k_1-11}^{(i)} & \cdots & h_{k_1-1n-1}^{(i)} \\ 1 & h_{k_1}^{(i)} & \cdots & h_{k_1n-1}^{(i)} \end{pmatrix}, \quad H_2^{(i)} = 0. \]  

For each \( i \), vacuum configuration (3.20) admits the additional \( nk_i - 1 \) complex parameters, \( a^{(i)} \) and \( h^{(i)} \). They form the maximal cycle \( C([k_1, \emptyset, \ldots, \emptyset]) \) in the moduli space \( \mathcal{M}(k_i) \). One can expect that these parameters, considering them as the additional parameters for the vacuum of \( k = k_1 + \cdots + k_i \) D5-branes, also constitute an appropriate cycle in the moduli space \( \mathcal{M}(k) \).

It is important to note that explicit form (3.19) does not always describe a vacuum when \( P_i \) coincides with \( P_j \) for some \( i \) and \( j \). This is because the first column vector of \( H_1 \) in (3.19) is no longer a cyclic vector when \((z_1^{(i)}, z_2^{(i)}) = (z_1^{(j)}, z_2^{(j)})\). Notice that, as far as all the \( l \)-points \( P_1, \cdots, P_l \) are different from one another, the first column vector of \( H_1 \) is cyclic and therefore configuration (3.19) describes the vacuum of \( k \) D5-branes for any values of the additional complex parameters. In the case when some of the \( l \)-points coincide, to guarantee stability (2.22) of configuration (3.19), it may become necessary to introduce a subspace of \( \text{Im} \ H_1 \) spanned by not only the first column vector but also the other column vectors of \( H_1 \) and then to consider the possibility whether the vectors of this subspace besides the vectors obtained from them by the successive actions of \( B_a \) can span \( V \). Though this prescription for restoring the stability of the configuration seems to work, the introduction of the column vectors of \( H_1 \) other than the first one turns out to freeze some of the additional complex parameters in (3.19). (In fact some of these parameters are absorbed into the \( GL(k : \mathbb{C}) \)-symmetry.)

There is another prescription to restore the stability condition. Take an element \( q \) of \( GL(k : \mathbb{C}) \) which depends on the \( l \)-points \( P_1, \cdots, P_l \) and becomes singular when some of these points coincide. Transform (2.49) of vacuum (3.19) by \( q \) is gauge-equivalent to the original configuration as far as these \( l \)-points are different from one another. It might be possible to choose \( q \) of \( GL(k : \mathbb{C}) \) so that the transform by \( q \) is still well-defined without missing any complex parameters in (3.19) even when some of the \( l \)-points coincide. The
complex parameters will be rescaled such that the singularities of $q$ are absorbed into them. One may say that such an element $q$ of $GL(k : C)$ gives the change of coordinates of the cycle.

These modified forms (which are obtained by appropriate singular gauge transformations) will appear naturally in our approach. It is because our description of the topological cycles of the moduli space is based on the consideration of the fixed points of the $T^2$-action. Since the $U(1) \times U(1)$-symmetry rotates $P_i = (z_1^{(i)}, z_2^{(i)})$ in the way given by (2.26), the cycle formed by the complex parameters in (3.19) will be captured in our treatment by examining the limit that the $l$-points in (3.19) approach to one another and go to the origin $(0, 0)$. In fact, we will show that the vacuum which belongs to the maximal cycle $C(\Gamma, \emptyset, \ldots, \emptyset)$ with $\Gamma = [k_1, \ldots, k_l]$ can be identified with (3.19) under the limit that all the $l$-different points $P_1, \ldots, P_l$ in (3.19) go to the origin $(0, 0)$. For this identification it is necessary to know how the vacuum configuration in which each $k_i$ D5-branes are degenerate at $P_i$ behaves when all these $l$-points approach to the origin and then to compare it with the configuration which belongs to the maximal cycle mentioned above.

It may be enough to consider the vacuum in which $m$ D5-branes are overlapping at a point $P$ in the four-dimensions while the other D5-branes are degenerate at another point, say $Q$, and then to compare this configuration, when $P$ goes to $Q$, with the vacuum which belongs to the corresponding maximal cycle. For definiteness let us suppose that $m$ and $m + m'$ D5-branes are degenerate respectively at $P = (0, \lambda)$ and $Q = (0, 0), \lambda \neq 0$. Their vacuum is denoted by $(\hat{B}_a, \hat{H}_A)$. $\hat{B}_a$ have the block-diagonal form

$$
\begin{pmatrix}
\hat{B}^{(1)}_a & 0 \\
0 & \hat{B}^{(2)}_a
\end{pmatrix}
$$

where $\hat{B}^{(1)}_a$ and $\hat{B}^{(2)}_a$ are respectively $m \times m$ and $(m + m') \times (m + m')$ complex matrices. $\hat{H}_A$ also have the form

$$
\begin{pmatrix}
\hat{H}^{(1)}_A \\
\hat{H}^{(2)}_A
\end{pmatrix}
$$

where $\hat{H}^{(1)}_A$ and $\hat{H}^{(2)}_A$ are respectively $m \times n$ and $(m + m') \times n$ complex matrices. The constituent $(\hat{B}_a^{(1)}, \hat{H}_A^{(1)})$ ( $(\hat{B}_a^{(2)}, \hat{H}_A^{(2)})$ ) describes the contribution of $m$ ( $m + m'$ ) D5-branes degenerate at $P$ ( $Q$ ). The explicit form is given in Table 1. $(\hat{B}_a, \hat{H}_A)$ in Table 1 admits $n(2m + m') - 2$ complex
parameters \( \hat{a}_j, \hat{b}_j, \hat{d}_j \) and \( \hat{h}_{ij} \). Among them, \( \hat{b}_j \) and \( \hat{h}_{ij} \) \((1 \leq i \leq m)\) are the additional parameters associated with the configuration of the first \( m \) D5-branes while \( \hat{a}_j, \hat{d}_j \) and \( \hat{h}_{ij} \) \((m + 1 \leq i \leq 2m + m')\) are those associated with the \( m + m' \) D5-branes at \( Q \).

For this vacuum one can provide a characterization similar to \((3.7)\). From the form given in Table 1 the following relations among \( \hat{B}_a \) can be found out:

\[
\hat{B}_{m + m'}^m = 0, \\
\hat{B}_1^m \left( \hat{B}_2 - \sum_{j=1}^{m'-1} \hat{a}_j \hat{B}_1^j \right) = 0, \\
\left( \hat{B}_2 - \sum_{j=1}^{m-1} \hat{b}_j \hat{B}_1^j - \lambda \right) \left( \hat{B}_2 - \sum_{j=1}^{m'-1} \hat{a}_j \hat{B}_1^j \right) = \sum_{j=1}^{m} \hat{c}_j \hat{B}_{m' + j - 1}^m, \\
\hat{B}_1 \hat{B}_2 = \hat{B}_2 \hat{B}_1 \tag{3.21}
\]

where \( \hat{c}_j \) \((1 \leq j \leq m)\) are introduced by

\[
\hat{c}_j \equiv -\lambda \hat{d}_j + \sum_{r+s=j} (\hat{a}_r - \hat{b}_r) \hat{d}_s. \tag{3.22}
\]

Since \( \lambda \) is not equal to zero, these \( \hat{c}_j \) are equivalent to the parameters \( \hat{d}_j \).

Let us examine the limit that \( P = (0, \lambda) \) goes to \( Q = (0, 0) \). The configuration obtained from \((\hat{B}_a, \hat{H}_A)\) in Table 1 by setting \( \lambda \) equal to zero does not always describe a vacuum of D5-branes. This is because, when \( \lambda = 0 \), the first column vector of \( \hat{H}_1 \) in Table 1 is not cyclic with respect to \( \hat{B}_a \) and it becomes necessary to compensate it with the other column vectors of \( \hat{H}_1 \) in order to satisfy stability condition \((2.22)\). With this compensation some of \( h_{ij} \) in \( \hat{H}_1 \) are frozen or absorbed into the \( GL(2m + m' : C) \)-symmetry. The number of the additional complex parameters will decrease from \((2m + m')n - 2\). To avoid such a decrease of parameters we need to change our perspective. We begin by describing the maximal cycle \( C_{(m + m', m, \emptyset, \ldots, \emptyset)} \). The dimensions of this cycle can be read as follows

\[
dim C_{(m + m', m, \emptyset, \ldots, \emptyset)} = 2(n - 1) + 2 \{n(2m + m' - 2) + (n - 1)\} = 2n(2m + m') - 4. \tag{3.23}
\]

The vacuum configuration \((B_a, H_A)\) in \( C_{(m + m', m, \emptyset, \ldots, \emptyset)} \) is given in Table 2. It has \( n(2m + m') - 2 \) complex parameters \( a_j, b_j, c_j \) and \( h_{ij} \). These parameters constitute the
maximal cycle $C([m + m', m, \emptyset, \emptyset, \ldots])$. After some manipulation we can find out $B_a$ in Table 2 satisfy the relations

$$B_1^{m+m'} = 0,$$

$$B_1^m \left( B_2 - \sum_{j=1}^{m'-1} a_j B_1^j \right) = 0,$$

$$\left( B_2 - \sum_{j=1}^{m-1} b_j B_1^j \right) \left( B_2 - \sum_{j=1}^{m'-1} a_j B_1^j \right) = \sum_{j=1}^{m} c_j B_1^{m'+j-1},$$

$$B_1 B_2 = B_2 B_1. \quad (3.24)$$

Now we can investigate the behavior of the vacuum $(\hat{B}_a, \hat{H}_A)$ when $P$ goes to $Q$. As we have mentioned, we should not set $\lambda = 0$ for the explicit form in Table 1. But relations $(3.21)$ themselves might be handled under this limit. We first rescale $\hat{c}_j$ $(3.22)$ such that they do not depend on $\lambda$ and then introduce the new parameters in stead of $\hat{a}_j, \hat{b}_j$ and $\hat{d}_j$:

$$a_j = \hat{a}_j,$$

$$b_j = \hat{b}_j,$$

$$c_j = \hat{c}_j(\hat{a}, \hat{b}, \hat{d} : \lambda). \quad (3.25)$$

With fixing these new parameters, relations $(3.21)$ become $(3.24)$ as $\lambda$ goes to zero! Notice that equations $(3.25)$ can be regarded as the change of coordinates as far as $\lambda \neq 0$. These new parameters themselves will survive even at $\lambda = 0$. The payoff for this transmutation of the parameters is the change of realization of the vacuum from those given in Table 1 to those given in Table 2. To complete the discussion one should also introduce similar rescalings of the parameters $\hat{h}_{ij}$. Though the investigation of “gauge transform” (cf. $(3.16)$) provides a nice description of the rescalings for these parameters, it becomes complicated and hard to give a general theory. So we convince the reader by giving some examples.

**Example 1 :** $(m, m') = (1, 0)$

Let us consider the case of two D5-branes located at two different points, say $P$ and $Q$. For definiteness we set $P = (0, \lambda)$ and $Q = (0, 0)$. $\lambda \neq 0$. The corresponding vacuum
configuration, which is denoted by \((\hat{B}_a, \hat{H}_A)\), has the form
\[
\hat{B}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \hat{B}_2 = \begin{pmatrix} \lambda & 0 \\ 0 & 0 \end{pmatrix},
\]
\[
\hat{H}_1 = \begin{pmatrix} 1 & \hat{h}_{11} & \cdots & \hat{h}_{1n-1} \\ 1 & \hat{h}_{21} & \cdots & \hat{h}_{2n-1} \end{pmatrix}, \quad \hat{H}_2 = 0.
\] (3.26)

\(\hat{h}_{1j}\) and \(\hat{h}_{2j}\) are respectively the additional parameters associated with the D5-brane located at \(P\) and \(Q\). Notice that \(\hat{B}_2\) in (3.26) satisfies the relation
\[
\hat{B}_2(\hat{B}_2 - \lambda) = 0.
\] (3.27)

As far as \(\lambda \neq 0\), \(\hat{H}_1(e_j) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}\) is cyclic with respect to \(\hat{B}_2\). In particular \(\hat{H}_1(e_j)\) \((j \geq 2)\) are obtainable from \(\hat{H}_1(e_1)\) by appropriate actions of \(\hat{B}_2\) :
\[
\hat{H}_1(e_j) = \left\{ \frac{\hat{h}_{1j-1} - \hat{h}_{2j-1}}{\lambda} \hat{B}_2 + \hat{h}_{2j-1} \right\} \hat{H}_1(e_1).
\] (3.28)

With this expression of \(\hat{H}_1(e_j)\) one can see that the corresponding “gauge transform” (cf. (3.16)), which we write as \(\hat{G}\), is given by
\[
\hat{G}(z) = \begin{pmatrix} 1 & \hat{g}_2(z) & \hat{g}_3(z) & \cdots & \hat{g}_n(z) \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix},
\] (3.29)

where \(\hat{g}_j(z)\) \((j \geq 2)\) are
\[
\hat{g}_j(z) = \frac{\hat{h}_{1j-1} - \hat{h}_{2j-1}}{\lambda} z_2 + \hat{h}_{2j-1}.
\] (3.30)

At this stage let us remark on the configuration obtained from (3.26) by setting \(\lambda = 0\). The first column vector of \(\hat{H}_1\), \(\begin{pmatrix} 1 \\ 1 \end{pmatrix}\), is no longer cyclic. In order that this configuration describes a vacuum one must impose an appropriate constraint on the parameters \(\hat{h}_{ij}\) so that it satisfies stability condition (2.22). It is sufficient if \(\begin{pmatrix} \hat{h}_{11} \\ \hat{h}_{21} \end{pmatrix}\)
besides \( \begin{pmatrix} 1 \\ 1 \end{pmatrix} \) span \( \mathbb{C}^2(\equiv V) \). In such a situation one can take these two vectors as the bases of \( V \). In terms of these new bases, vacuum (3.26), setting \( \lambda = 0 \), acquires the form, \( \hat{B}_a = \hat{H}_2 = 0 \) and \( \hat{H}_1 = \begin{pmatrix} 0 & 1 & * \\ 1 & 0 & * \end{pmatrix} \), which means the parameters \( \hat{h}_{11,21} \) are absorbed into the \( GL(2 : \mathbb{C}) \)-symmetry.

Nextly we describe the vacuum which belongs to the maximal cycle \( C([1,1],\emptyset,\cdots,\emptyset) \). We shall write it \( (B_a,H_A) \). It can be realized by

\[
B_1 = 0 , \quad B_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} , \\
H_1 = \begin{pmatrix} 0 & h_{11} & \cdots & h_{1n-1} \\ 1 & h_{21} & \cdots & h_{2n-1} \end{pmatrix} , \quad H_2 = 0,
\]

(3.31)

where \( 2(n - 1) \) complex parameters \( h_{ij} \) constitute the maximal cycle. Notice that \( B_2 \) satisfy the relation,

\[
B_2^2 = 0 .
\]

(3.32)

As for “gauge transform” \( G \) in (3.16), the elements of \( G \) turn out to be given by

\[
g_j(z) = h_{1j-1}z_2 + h_{2j-1}.
\]

(3.33)

Let us study the behavior of configuration (3.26) when \( P \) goes to \( Q \) by considering rescalings of the complex parameters in (3.26). Namely we rescale \( \hat{h}_{ij} \) such that \( \hat{h}_{2j} \) as well as \( (\hat{h}_{1j} - \hat{h}_{2j})/\lambda \) do not depend on \( \lambda \). We may introduce the new parameters in stead of \( \hat{h}_{ij} : \)

\[
h_{1j} = \frac{\hat{h}_{1j} - \hat{h}_{2j}}{\lambda} , \\
h_{2j} = \hat{h}_{2j}.
\]

(3.34)

Equations (3.34) can be also regarded as the change of coordinates as far as \( \lambda \neq 0 \). Taking the element \( \hat{q}(\lambda) \) of \( GL(2 : \mathbb{C}) \),

\[
\hat{q}(\lambda) = \begin{pmatrix} 1/\lambda & -1/\lambda \\ 0 & 1 \end{pmatrix}
\]

(3.35)
the transform of (3.26) by \( \hat{q}(\lambda) \) becomes

\[
\hat{q}(\lambda) \hat{B}_1 \hat{q}(\lambda)^{-1} = 0 \, , \quad \hat{q}(\lambda) \hat{B}_2 \hat{q}(\lambda)^{-1} = \begin{pmatrix} \lambda & 1 \\ 0 & 0 \end{pmatrix} ,
\]

\[
\hat{q}(\lambda) H_1 = \begin{pmatrix} 0 & \hat{h}_{11} - \hat{h}_{21} & \ldots & \hat{h}_{1n-1} - \hat{h}_{2n-1} \\ 1 & \hat{h}_{21} & \ldots & \hat{h}_{2n-1} \end{pmatrix} .
\] (3.36)

Rewriting configuration (3.36) by rescaled parameters (3.34), we can find that it coincides with (3.31) as \( \lambda \) goes to zero. This shows that the rescaled parameters survive even at \( \lambda = 0 \) and that, due to the singularity at \( \lambda = 0 \) in \( \hat{q}(\lambda) \), one should change the realization from (3.26) to (3.31), at least, at \( P = Q \).

**Example 2 :** \((m, m') = (1, 1)\)

As the second example we shall consider the case of three D5-branes one of which is located at \( P = (0, \lambda) \) and the other two are overlapping at \( Q = (0, 0) \). \( \lambda \neq 0 \). The corresponding vacuum \((\hat{B}_a, \hat{H}_A)\) has the form

\[
\hat{B}_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} , \quad \hat{B}_2 = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & 0 & \hat{a} \\ 0 & 0 & 0 \end{pmatrix} ,
\]

\[
\hat{H}_1 = \begin{pmatrix} 1 & \hat{h}_{11} & \ldots & \hat{h}_{1n-1} \\ 0 & \hat{h}_{21} & \ldots & \hat{h}_{2n-1} \\ 1 & \hat{h}_{31} & \ldots & \hat{h}_{3n-1} \end{pmatrix} , \quad \hat{H}_2 = 0 ,
\] (3.37)

where \( \hat{a} \) and \( \hat{h}_{ij} \) are \( 3n - 2 \) complex parameters. Taking the diagonal elements of \( \hat{B}_a \) into account one can say \( \hat{h}_{1j} \) are the additional parameters associated with the first D5-brane located at \( P \) while \( \hat{a} \) besides \( \hat{h}_{2j} \) and \( \hat{h}_{3j} \) are those for the two D5-branes overlapping at \( Q \). It is easy to check that \( \hat{B}_a \) in (3.37) satisfy the relations

\[
\hat{B}_1^2 = 0 ,
\]

\[
(\hat{B}_2 - \lambda)(\hat{B}_2 - \hat{a} \hat{B}_1) = 0 ,
\]

\[
\hat{B}_1 \hat{B}_2 = \hat{B}_2 \hat{B}_1 = 0 .
\] (3.38)
Notice that \( \hat{H}_1(e_1) \) is cyclic with respect to \( \hat{B}_a \). In particular one can write down \( \hat{H}_1(e_j) \) \((j \geq 2)\) in the form
\[
\hat{H}_1(e_j) = \left( \frac{\hat{h}_{1j-1} - \hat{h}_{3j-1}}{\lambda} \right) \hat{B}_2 + \left( \frac{\hat{h}_{2j-1} - \hat{a}(\hat{h}_{1j-1} - \hat{h}_{3j-1})}{\lambda} \right) \hat{B}_1 + \hat{h}_{3j-1} \hat{H}_1(e_1).
\]
(3.39)

The “gauge transform” \( \hat{G} \), which has a form similar to (3.29), is now given by the polynomials \( \hat{g}_j(z) \) \((j \geq 2)\) determined from (3.39):
\[
\hat{g}_j(z) = \frac{\hat{h}_{1j-1} - \hat{h}_{3j-1}}{\lambda} z_2 + \left( \frac{\hat{h}_{2j-1} - \hat{a}(\hat{h}_{1j-1} - \hat{h}_{3j-1})}{\lambda} \right) z_1 + \hat{h}_{3j-1}.
\]
(3.40)

Let us remark on the configuration obtained from (3.37) by setting \( \lambda = 0 \). In order that this configuration describes a vacuum it is sufficient that the second column vector of \( \hat{H}_1 \) besides \( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) and \( \hat{B}_1 \) \( \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \) span \( \mathbb{C}^3(\equiv V) \). This condition gives the constraint, \( \hat{h}_{11} \neq \hat{h}_{31} \). If it is satisfied, one can take a new bases of \( V \) so that vacuum (3.37) (with setting \( \lambda = 0 \)) acquires the form
\[
\hat{B}_1 = \begin{pmatrix} 0 & 1 & \hat{h}_{31} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{B}_2 = \begin{pmatrix} 0 & \hat{a} & \hat{a}\hat{h}_{31} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
\hat{H}_1 = \begin{pmatrix} 0 & 0 & \ast & \ast \\ 0 & 1 & \ast & \ast \\ 1 & 0 & \ast & \ast \end{pmatrix}, \quad \hat{H}_2 = 0,
\]
(3.41)

which means that the parameters \( \hat{h}_{11,21} \) are absorbed into the \( GL(3: \mathbb{C}) \)-symmetry.

Nextly we describe the vacuum \((B_a, H_A)\) which belongs to the maximal cycle \( C_{([2,1],\emptyset,\ldots,\emptyset)} \).

It can be realized by the form
\[
B_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 1 \\ a & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},
\]
\[
H_1 = \begin{pmatrix}
0 & h_{11} & \cdots & h_{1n-1} \\
0 & h_{21} & \cdots & h_{2n-1} \\
1 & h_{31} & \cdots & h_{3n-1}
\end{pmatrix}, \quad H_2 = 0,
\] (3.42)

where \(3n-2\) complex parameters \(a\) and \(h_{ij}\) constitute the maximal cycle \(C([2,1],\emptyset,\ldots,\emptyset)\).

One can check \(B_a\) satisfy the relations
\[
B_1^2 = 0, \\
B_2^2 - aB_1 = 0, \\
B_1B_2 = B_2B_1 = 0.
\] (3.43)

As for “gauge transform” \(G\) (3.16), its elements turn out to be given by
\[
g_j(z) = h_{1j-1}z_2 + h_{2j-1}z_1 + h_{3j-1}.
\] (3.44)

The behavior of configuration (3.37) when \(P\) goes to \(Q\) will be studied taking account of the rescaling of parameters. We first rescale \(\hat{a}\) such that \(\lambda\hat{a}\) does not depend on \(\lambda\). We may introduce the new parameter by
\[
a = -\lambda\hat{a}.
\] (3.45)

Notice that relations (3.38), with fixing \(a\), become (3.43) as \(\lambda\) goes to 0. One can also rescale \(\hat{h}_{ij}\) such that
\[
h_{1j} = \frac{\hat{h}_{1j} - \hat{h}_{3j}}{\lambda}, \\
h_{2j} = \hat{h}_{2j} + \frac{a(\hat{h}_{1j} - \hat{h}_{3j})}{\lambda^2}, \\
h_{3j} = \hat{h}_{3j}
\] (3.46)
do not depend on \(a\) and \(\lambda\). Elements (3.40) in the gauge transform \(\hat{G}\), expressing them in terms of these rescaled \(h_{ij}\), coincide with (3.44). Equations (3.43) and (3.46) can be also regarded as the change of coordinates as far as \(\lambda \neq 0\). Introduce the element \(\hat{q}(\lambda)\) of \(GL(3 : C)\) by
\[
\hat{q}(\lambda) = \begin{pmatrix}
1/\lambda & 0 & -1/\lambda \\
-\hat{a}/\lambda & 1 & \hat{a}/\lambda \\
0 & 0 & 1
\end{pmatrix}.
\] (3.47)
The transform of (3.37) by $\hat{q}(\lambda)$ becomes

$$\hat{q}(\lambda) \hat{B}_1 \hat{q}(\lambda)^{-1} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{q}(\lambda) \hat{B}_2 \hat{q}(\lambda)^{-1} = \begin{pmatrix} \lambda & 0 & 1 \\ -\lambda \hat{a} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$\hat{q}(\lambda) \hat{H}_1 = \begin{pmatrix} 0 & \frac{\hat{h}_{11} - \hat{h}_{33}}{\lambda} & \ldots & \frac{\hat{h}_{1n-1} - \hat{h}_{3n-1}}{\lambda} \\ 0 & \hat{h}_{21} - \frac{\hat{a}(\hat{h}_{11} - \hat{h}_{33})}{\lambda} & \ldots & \hat{h}_{2n-1} - \frac{\hat{a}(\hat{h}_{1n-1} - \hat{h}_{3n-1})}{\lambda} \\ 1 & \hat{h}_{31} & \ldots & \hat{h}_{3n-1} \end{pmatrix}.$$  \hspace{1cm} (3.48)

Rewriting (3.48) in terms of rescaled parameters (3.45) and (3.46) it exactly coincides with (3.42) at $P = Q$.

**Example 3 :** $(m, m') = (2, 0)$

As the last example we shall investigate the case of four D5-branes two of which overlap at $P = (0, \lambda)$ and the other two are degenerate at $Q = (0, 0)$. $\lambda \neq 0$. The corresponding vacuum $(\hat{B}_a, \hat{H}_A)$ has the form

$$\hat{B}_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{B}_2 = \begin{pmatrix} \lambda & \hat{a} & 0 & 0 \\ 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \hat{b} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{H}_1 = \begin{pmatrix} 0 & \hat{h}_{11} & \ldots & \hat{h}_{1n-1} \\ 1 & \hat{h}_{21} & \ldots & \hat{h}_{2n-1} \\ 0 & \hat{h}_{31} & \ldots & \hat{h}_{3n-1} \\ 1 & \hat{h}_{41} & \ldots & \hat{h}_{4n-1} \end{pmatrix}, \quad \hat{H}_2 = 0,$$  \hspace{1cm} (3.49)

where $\hat{a}, \hat{b}$ and $\hat{h}_{ij}$ are $4n - 2$ complex parameters in which, taking the diagonal parts of $\hat{B}_a$ into account, $\hat{a}, \hat{h}_{1j}$ and $\hat{h}_{2j}$ are the additional parameters associated with the two D5-branes degenerate at $P$ while $\hat{b}, \hat{h}_{3j}$ and $\hat{h}_{4j}$ are those for the two D5-branes at $Q$.

It is easy to see that $\hat{B}_a$ in (3.49) satisfy the relations,

$$\hat{B}_1^2 = 0,$$

$$\left(\hat{B}_2 - \hat{a}\hat{B}_1 - \lambda\right) \left(\hat{B}_2 - \hat{b}\hat{B}_1\right) = 0,$$

$$\hat{B}_1\hat{B}_2 = \hat{B}_2\hat{B}_1.$$  \hspace{1cm} (3.50)
Since $\hat{H}_1(e_1)$ is cyclic with respect to $\hat{B}_a$ one can write down $\hat{H}_1(e_j)$ ($j \geq 2$) in the form,

$$\hat{H}_1(e_j) = \left\{ \left( \frac{\hat{h}_{1j} - \hat{h}_{3j}}{\lambda} - \frac{(\hat{a} - \hat{b})(\hat{h}_{2j} - \hat{h}_{4j})}{\lambda^2} \right) \hat{B}_1 \hat{B}_2 + \left( \frac{\hat{h}_{3j} - \frac{\hat{b}(\hat{h}_{2j} - \hat{h}_{4j})}{\lambda}}{\lambda} \right) \hat{B}_1 \right. $$

$$\left. + \frac{\hat{h}_{2j} - \hat{h}_{4j}}{\lambda} \hat{B}_2 + \hat{h}_{4j} \right\} \hat{H}_1(e_1) \ . \quad (3.51)$$

The corresponding “gauge transform” $\hat{G}$, which has a similar form as (3.29), can be described by the following polynomials $\hat{g}_j(z)$ ($j \geq 2$):

$$\hat{g}_j(z) = \left( \frac{\hat{h}_{1j} - \hat{h}_{3j}}{\lambda} - \frac{(\hat{a} - \hat{b})(\hat{h}_{2j} - \hat{h}_{4j})}{\lambda^2} \right) \frac{z_1 z_2}{\lambda} + \left( \frac{\hat{h}_{3j} - \frac{\hat{b}(\hat{h}_{2j} - \hat{h}_{4j})}{\lambda}}{\lambda} \right) \frac{z_1}{\lambda}$$

$$+ \frac{\hat{h}_{2j} - \hat{h}_{4j}}{\lambda} z_2 + \hat{h}_{4j} \ . \quad (3.52)$$

As regards the configuration obtained from (3.49) by setting $\lambda = 0$, the first column vector of $\hat{H}_1$ is not cyclic. For this configuration to be a vacuum of D5-branes it may be enough, say, if, in addition to the first column vector $\hat{H}_1(e_1)$ and its descendant $\hat{B}_1 \hat{H}_1(e_1)$, the second column vector $\hat{H}_1(e_2)$ and its descendant $\hat{B}_1 \hat{H}_1(e_2)$ span $\mathbb{C}^4$ ($\equiv V$). This condition leads the constraint, $\hat{h}_{21} \neq \hat{h}_{41}$. If it is satisfied, one can introduce the new bases of $V$ with which vacuum (3.49) (with setting $\lambda = 0$) acquires the form

$$\hat{B}_1 = \begin{pmatrix} 0 & 0 & 1 & \hat{h}_{21} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \hat{B}_2 = \begin{pmatrix} 0 & 0 & \hat{b} \hat{h}_{21} - \frac{\hat{a} - \hat{b}}{\hat{h}_{21} - \hat{h}_{41}} \\ 0 & 0 & 0 & \hat{a} \\ 0 & 0 & 0 & \hat{b} \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\hat{H}_1 = \begin{pmatrix} 0 & 0 & * & * \\ 0 & 0 & * & * \\ 0 & 0 & * & * \\ 1 & 0 & * & * \end{pmatrix}, \quad \hat{H}_2 = 0 . \quad (3.53)$$

This means that parameters $\hat{h}_{11,31}$ are absorbed into the $GL(4 : \mathbb{C})$-symmetry.

Nextly let us describe the vacuum $(B_a, H_A)$ which belongs to the maximal cycle
$C_{([2,2],0,\ldots,0)}$. It has the realization:

$$B_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad B_2 = \begin{pmatrix} 0 & a & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & b & 0 & a \\ 0 & 0 & 0 & 0 \end{pmatrix} ,$$

$$H_1 = \begin{pmatrix} 0 & h_{11} & \cdots & h_{1n-1} \\ 0 & h_{21} & \cdots & h_{2n-1} \\ 0 & h_{31} & \cdots & h_{3n-1} \\ 1 & h_{41} & \cdots & h_{4n-1} \end{pmatrix} , \quad H_2 = 0 , \quad (3.54)$$

where complex parameters $a$, $b$ and $h_{ij}$ constitute the maximal cycle. $B_a$ in (3.54) satisfy the relations,

\[
\begin{align*}
B_1^2 &= 0 , \\
B_2^2 - 2aB_1B_2 - bB_1 &= 0 , \\
B_1B_2 &= B_2B_1 . 
\end{align*}
\]  

(3.55)

The elements of the “gauge transform” $G$ in (3.16) is now given by

$$g_j(z) = h_{1j}z_1z_2 + (h_{3j} - ah_{2j})z_1 + h_{2j}z_2 + h_{4j} . \quad (3.56)$$

The behavior of configuration (3.49) when $P$ goes to $Q$ can be studied by appropriate rescalings of the additional parameters. Let us first rescale $\hat{a}$ and $\hat{b}$ in (3.49) such that $\lambda(\hat{a} - \hat{b})$ and $\hat{a} + \hat{b}$ do not depend on $\lambda$. Introduce the new parameters by

$$a = \frac{\hat{a} + \hat{b}}{2} , \quad b = \frac{\lambda(\hat{a} - \hat{b})}{2} . \quad (3.57)$$

With fixing them one can see relations (3.50) become (3.55) as $\lambda$ goes to zero. As regards the other additional parameters in (3.49) we shall rescale them such that

\[
\begin{align*}
h_{1j} &= \frac{\hat{h}_{1j} - \hat{h}_{3j}}{\lambda} - \frac{(\hat{a} - \hat{b})(\hat{h}_{2j} - \hat{h}_{4j})}{\lambda^2} , \\
h_{2j} &= \frac{\hat{h}_{2j} - \hat{h}_{4j}}{\lambda} ,
\end{align*}
\]
\[ h_{3j} = \hat{h}_{3j} + \frac{(\hat{a} - \hat{b})(\hat{h}_{2j} - \hat{h}_{4j})}{2\lambda}, \]
\[ h_{4j} = \hat{h}_{4j} \]  
(3.58)

become independent of \(\hat{a}, \hat{b}\) and \(\lambda\). The elements \(\hat{g}_j(z)\) of the gauge transform \(\hat{G}\), expressing them in terms of these rescaled parameters, coincide with (3.56). Notice that equations (3.57) and (3.58) can be also regarded as the change of coordinates as far as \(\lambda \neq 0\). Introduce the element \(\hat{q}(\lambda)\) of \(GL(4 : \mathbb{C})\)

\[
\hat{q}(\lambda) = \begin{pmatrix}
1/\lambda & -(\hat{a} - \hat{b})/\lambda^2 & -1/\lambda & (\hat{a} - \hat{b})\lambda^2 \\
0 & 1/\lambda & 0 & -1/\lambda \\
0 & -(\hat{a} - \hat{b})/2\lambda & 1 & -(\hat{a} - \hat{b})/2\lambda \\
0 & 0 & 0 & 1
\end{pmatrix}.
\]  
(3.59)

The transform of (3.49) by \(\hat{q}(\lambda)\) becomes

\[
\hat{q}(\lambda)\hat{B}_1\hat{q}(\lambda)^{-1} = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{q}(\lambda)\hat{B}_2\hat{q}(\lambda)^{-1} = \begin{pmatrix}
\lambda & (\hat{a} + \hat{b})/2 & 1 & 0 \\
0 & \lambda & 0 & 1 \\
0 & \lambda(\hat{a} - \hat{b})/2 & (\hat{a} + \hat{b})/2 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

\[
\hat{q}(\lambda)\hat{H}_1 = \begin{pmatrix}
0 & \frac{h_{11} - h_{31}}{\lambda} & -\frac{(\hat{a} - \hat{b})(h_{21} - h_{41})}{\lambda^2} & \ldots & \frac{h_{1n-1} - h_{3n-1}}{\lambda} & -\frac{(\hat{a} - \hat{b})(h_{2n-1} - h_{4n-1})}{\lambda^2} \\
0 & \frac{h_{21} - h_{41}}{\lambda} & \ldots & \frac{h_{2n-1} - h_{4n-1}}{\lambda} & \frac{h_{2n-1} - h_{4n-1}}{\lambda} & \frac{h_{2n-1} - h_{4n-1}}{\lambda} \\
0 & \frac{h_{31} + (\hat{a} - \hat{b})(h_{21} - h_{41})}{2\lambda} & \ldots & \frac{h_{3n-1} + (\hat{a} - \hat{b})(h_{2n-1} - h_{4n-1})}{2\lambda} & \frac{h_{3n-1} + (\hat{a} - \hat{b})(h_{2n-1} - h_{4n-1})}{2\lambda} & \frac{h_{3n-1} + (\hat{a} - \hat{b})(h_{2n-1} - h_{4n-1})}{2\lambda} \\
1 & \hat{h}_{41} & \ldots & \hat{h}_{4n-1} & \hat{h}_{4n-1} & \hat{h}_{4n-1}
\end{pmatrix}.
\]  
(3.60)

Rewriting (3.60) in terms of the rescaled parameters we can see that it becomes (3.54) as \(P\) goes to \(Q\).

4 Toward Second-Quantization of D5-Brane
Superposition of D5-Brane Vacua

In the previous section it is shown that the additional complex parameters of vacuum (3.19) of \( k = k_1 + k_2 + \cdots + k_l \) D5-branes form a cycle in the moduli space \( \mathcal{M}(k) \) which can be identified with the maximal cycle \( C_{[\Gamma, \emptyset, \ldots, \emptyset]} \) with \( \Gamma = [k_1, k_2, \ldots, k_l] \). Vacuum (3.19) is described by the superposition of vacua of \( k_i \) pieces \((1 \leq i \leq l)\). Each constituent, that is, vacuum (3.20) of \( k_i \) D5-branes degenerates at \( P_i \), admits the additional degrees of freedom, which contributes to the parameters of (3.19). Even when some of the positions of these overlapping D5-branes coincide, say, \( P_i = P_j \), this superposed vacuum of \( k \) D5-branes is shown to be still well-defined with an appropriate change of realization.

In order to make the positions of these overlapping D5-branes explicit let us write the above topological cycle by \( C_{[k_1, k_2, \ldots, k_l]}(P_1, P_2, \ldots, P_l) \) where \( P_i = (z_{1i}^{(i)}, z_{2i}^{(i)}) \) is the position of \( k_i \) coincident D5-branes. \( C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l) \simeq C_{([k_1, \ldots, k_l], \emptyset, \ldots, \emptyset)} \).

It is physically reasonable to consider such a vacuum of \( k + \tilde{k} \) D5-branes in which the configuration of \( k \) pieces can be regarded as the vacuum given by (3.19) while the configuration of the other \( \tilde{k} \) pieces is an arbitrary vacuum of \( \tilde{k} \) D5-branes. Taking it in the reverse order one might say it should be possible, at least from the viewpoint of five-branes, to construct a new vacuum of \( k + \tilde{k} \) D5-branes from two vacua of \( k \) and \( \tilde{k} \) pieces. In spite of this physical speculation we should note that the superposition of two vacua, though it works directly when both vacua of \( k \) and \( \tilde{k} \) D5-branes admit to have forms similar to (3.19), does not work directly if either vacuum of \( k \) or \( \tilde{k} \) pieces is generic.

So, it is necessary to explain how one can “superpose” the vacuum of \( k \) D5-branes which belongs to the cycle \( C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l) \) with a vacuum of \( \tilde{k} \) D5-branes without restricting the latter.

Let \((B_a^{(1)}, H_A^{(1)})\) be the vacuum of \( k \) D5-branes which has form (3.19). It belongs to the cycle \( C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l) \). Take a generic vacuum of \( \tilde{k} \) D5-branes, which is denoted by \((B_a^{(2)}, H_A^{(2)})\). It may be convenient to handle these vacua by using hyperkähler description (2.14) (or (2.19) and (2.20)) of the moduli spaces. Notice that there exist \( g(\eta)^{(1)} \in GL(k : \mathbb{C}) \) and \( g(\eta)^{(2)} \in GL(\tilde{k} : \mathbb{C}) \) such that the transforms of \( H_2 = 0 \) for all the vacua \((B_a, H_A)\).
by \( g^{(1)}(\eta) \) and \( (B^{(2)}_a, H^{(2)}_A) \) by \( g^{(2)}(\eta) \) both satisfy \( D \)-flat conditions (2.19) and (2.20). These transforms will be denoted by \((B^{(1,2)}_a(\eta), H^{(1,2)}_A(\eta))\) in order to make their dependence on \( \eta \) explicit. For the same reason the corresponding moduli spaces, regarded as the hyperkähler quotients, will be denoted by \( \mathcal{M}(k)_\eta \) and \( \mathcal{M}(\tilde{k})_\eta \). Notice that \((B^{(1)}_a(\eta), H^{(1)}_A(\eta))\) ∈ \( \mathcal{M}(k)_\eta \) and \((B^{(2)}_a(\eta), H^{(2)}_A(\eta))\) ∈ \( \mathcal{M}(\tilde{k})_\eta \). We first consider their behaviors as \( \eta \) goes to zero. For the case of \( \eta \) being positive the vacuum \((B^{(1)}_a(\eta), H^{(1)}_A(\eta))\) admits additional parameters as it has in the complex symplectic description. But, as \( \eta \) goes to zero, these additional degrees of freedom vanish. It will become

\[
B^{(1)}_a(0) \equiv \lim_{\eta \to 0} B^{(1)}_a(\eta) = \begin{pmatrix} z^{(1)}_a \mathbf{1}_{k_1} & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & z^{(l)}_a \mathbf{1}_{k_l} \end{pmatrix},
\]

\[
H^{(1)}_A(0) \equiv \lim_{\eta \to 0} H^{(1)}_A(\eta) = 0 .
\] (4.1)

Notice that \( P_i = (z^{(i)}_1, z^{(i)}_2) \) (1 \( \leq \) \( i \) \( \leq \) \( l \)) are the positions of overlapping \( k_i \) D5-branes. Configuration (4.1) can be regarded as a vacuum at \( \eta = 0 \). In the terminology of four-dimensional gauge theory, it describes the small size limit of \( k \) \( SU(n) \)-instantons. These ideal instantons are located at \( P_i \). As regards the vacuum \((B^{(2)}_a(0), H^{(2)}_A(0))\) \( \equiv \lim_{\eta \to +0}(B^{(2)}_a(\eta), H^{(2)}_A(\eta))\), since it is a generic point of the moduli space of \( \tilde{k} \) D5-branes, it describes non-singular \( \tilde{k} \) \( SU(n) \)-instantons. At \( \eta = 0 \) one can easily superpose these two vacua. Namely let us introduce

\[
B_a(0) \equiv \begin{pmatrix} B^{(1)}_a(0) & 0 \\ 0 & B^{(2)}_a(0) \end{pmatrix}, \quad H_A(0) \equiv \begin{pmatrix} H^{(1)}_A(0) \\ H^{(2)}_A(0) \end{pmatrix} .
\] (4.2)

Then configuration (4.2) can be regarded as a point of the moduli space \( \mathcal{M}(k + \tilde{k})_{\eta=0} \). The contribution of the first \( k \) D5-branes makes it a singular point.

It is convenient to remark on the relation between the moduli spaces \( \mathcal{M}(k + \tilde{k})_{\eta>0} \) and \( \mathcal{M}(k + \tilde{k})_{\eta=0} \). As was explained in Section.2, the moduli space \( \mathcal{M}(k + \tilde{k})_{\eta>0} \) is smooth.
It has no singularity. Under the limit that \( \eta \) goes to zero singularities do appear. These singularities correspond to the small size limits of \( SU(n) \)-instantons. This phenomenon shows that \( \mathcal{M}(k + \tilde{k})_{\eta>0} \) can be regarded as a resolution of \( \mathcal{M}(k + \tilde{k})_{\eta=0} \). The projection \( \pi \) from \( \mathcal{M}(k + \tilde{k})_{\eta>0} \) to \( \mathcal{M}(k + \tilde{k})_{\eta=0} \) is defined by taking the limit of \( \eta \) being zero in the former moduli space. As regards configuration (4.2), which is a singular point of \( \mathcal{M}(k + \tilde{k})_{\eta=0} \), the inverse image of (4.2), that is, \( \pi^{-1}((B_a(0), H_A(0))) \) gives us the vacua which admit additional complex parameters by which the singularity is resolved. Among these vacua there exists a configuration \((B_a(\eta), H_A(\eta))\) which corresponds to a superposition of the two vacua \((B_a^{(1)}(\eta), H_A^{(1)}(\eta))\) and \((B_a^{(2)}(\eta), H_A^{(2)}(\eta))\). It realizes the “superposition” of the vacuum of \( k \) D5-branes which belongs to the cycle \( C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l) \) and a generic vacuum of \( \tilde{k} \) D5-branes.

Let us comment briefly on the case of \((B_a^{(2)}, H_A^{(2)})\) being a vacuum which causes a singularity at \( \eta = 0 \). Suppose \((B_a^{(2)}, H_A^{(2)}) \in C_{\tilde{\Gamma}}(P_{\tilde{1}}, \ldots, P_{\tilde{l}})\). (\(|\tilde{\Gamma}| = \tilde{k} \) and \(|\tilde{\Gamma}| = \tilde{l} \).) When all the \( \tilde{l} \)-positions \( P_{\tilde{1}}, \ldots, P_{\tilde{l}} \) are different from the \( l \)-positions \( P_1, \ldots, P_l \) of the first \( k \) D5-branes, their superposition will be given\(^9\)

\[
B_a \equiv \begin{pmatrix} B_a^{(1)} & 0 \\ 0 & B_a^{(2)} \end{pmatrix}, \quad H_A \equiv \begin{pmatrix} H_A^{(1)} \\ H_A^{(2)} \end{pmatrix}.
\]

Clearly configuration (4.3) belongs to the cycle \( C_{\Gamma \cup \tilde{\Gamma}}(Q_1, \ldots, Q_{l+\tilde{l}}) \) of the moduli space \( \mathcal{M}(k + \tilde{k}) \) where \( \Gamma \cup \tilde{\Gamma} \) is the Young tableau of \( k + \tilde{k} \) boxes obtained from \( \Gamma = [k_1, \ldots, k_l] \) and \( \tilde{\Gamma} = [\tilde{k}_1, \ldots, \tilde{k}_{\tilde{l}}] \) by reordering \( k_i \) and \( \tilde{k}_j \). \( Q_1, \ldots, Q_{l+\tilde{l}} \) are the corresponding re-arrangement of \( P_1, \ldots, P_l \) and \( P_{\tilde{1}}, \ldots, P_{\tilde{l}} \). In the case that some of the positions of \( \tilde{k} \) D5-branes coincide with those of the first \( k \) D5-branes, owing to the discussion given in the previous section, it is still possible to give their superposition in the cycle \( C_{\Gamma \cup \tilde{\Gamma}}(Q_1, \ldots, Q_{l+\tilde{l}}) \).

The “superposition” defines an inclusion \( \iota \) of \( C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l) \times \mathcal{M}(\tilde{k}) \) to \( \mathcal{M}(k + \tilde{k}) \):

\[
\iota: C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l) \times \mathcal{M}(\tilde{k}) \hookrightarrow \mathcal{M}(k + \tilde{k}). \tag{4.4}
\]

The image of \( \iota \), that is, \( \iota \left( C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l) \times \mathcal{M}(\tilde{k}) \right) \), is a noncompact submanifold.

\(^9\) Here we use the complex symplectic description.
of the moduli space $\mathcal{M}(k + \tilde{k})$, which will be denoted by $C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l)$. The dimensions of $C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l)$ is equal to the sum of the dimensions of the moduli space $\mathcal{M}(\tilde{k})$ and the cycle $C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l)$,

$$
\dim C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l) = 4n\tilde{k} + 2(nk - l).
$$

Notice that $k_1 + \cdots + k_l = k$. Physically speaking, any point of the submanifold $C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l)$ describes the vacuum of $k + \tilde{k}$ D5-branes in which the configuration of $k$ pieces, considering it as a vacuum of $k$ D5-branes, belongs to the cycle $C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l)$ of the moduli space $\mathcal{M}(k)$. Each point $P_i$ denotes the position of overlapping $k_i$ D5-branes. By letting these $l$-positions of $k = k_1 + k_2 + \cdots + k_l$ D5-branes free in the four-dimensions, we may introduce the noncompact submanifold $C_{[k_1, \ldots, k_l]}$ of $\mathcal{M}(k + \tilde{k})$ by

$$
C_{[k_1, \ldots, k_l]} \equiv \left\{ C_{[k_1, \ldots, k_l]}(P_1, \ldots, P_l) \mid P_1, \ldots, P_l \in \mathbb{C}^2 \right\}.
$$

The dimensions of this submanifold is

$$
\dim C_{[k_1, \ldots, k_l]} = 4n\tilde{k} + 2(nk + l).
$$

Now let us exchange the roles of $k$ and $\tilde{k}$ D5-branes in the construction of inclusion map (4.4). Consider the topological cycle $C_{\tilde{\Gamma}}(\tilde{P}_1, \ldots, \tilde{P}_l)$ of the moduli space $\mathcal{M}(\tilde{k})$. $\tilde{\Gamma} = [\tilde{k}_1, \ldots, \tilde{k}_l]$ with $|\tilde{\Gamma}| = \tilde{k}$. Each point $\tilde{P}_i$ denotes the position of overlapping $\tilde{k}_i$ D5-branes. By the superposition of vacua we will obtain the inclusion map,

$$
\mathcal{M}(k) \times C_{[\tilde{k}_1, \ldots, \tilde{k}_l]}(\tilde{P}_1, \ldots, \tilde{P}_l) \hookrightarrow \mathcal{M}(k + \tilde{k})
$$

which provides, as its image, the noncompact submanifold $C_{[k_1, \ldots, \tilde{k}_l]}(\tilde{P}_1, \ldots, \tilde{P}_l)$ of the moduli space $\mathcal{M}(k + \tilde{k})$. Letting the $\tilde{l}$-points $\tilde{P}_1, \ldots, \tilde{P}_\tilde{l}$ free we also obtain the noncompact submanifold $C_{[k_1, \ldots, \tilde{k}_l]}$. The dimensions of this submanifold is equal to $4nk + 2(n\tilde{k} + \tilde{l})$. Notice that any point of the submanifold $C_{[k_1, \ldots, \tilde{k}_l]}$ describes the vacuum of $k + \tilde{k}$ D5-branes in which the configuration of $\tilde{k}$ pieces, considering it as a vacuum of $\tilde{k}$ D5-branes, belongs to the cycle $C_{[k_1, \ldots, \tilde{k}_l]}(\tilde{P}_1, \ldots, \tilde{P}_l)$ of the moduli space $\mathcal{M}(\tilde{k})$. By examining two vacua of $k + \tilde{k}$ D5-branes which respectively belong to $C_{[k_1, \ldots, k_l]}$ and
\( \mathcal{C}_{[k_1, \ldots, k_l]} \) one can find that their intersection in the moduli space \( \mathcal{M}(k + \tilde{k}) \) has the form,

\[
\mathcal{C}_{[k_1, \ldots, k_l]} \cap \mathcal{C}_{[\tilde{k}_1, \ldots, \tilde{k}_l]} = \mathcal{C}_{[k_1, \ldots, k_l] \cup [\tilde{k}_1, \ldots, \tilde{k}_l]} \ .
\]

The noncompact submanifold \( \mathcal{C}_{[k_1, \ldots, k_l] \cup [\tilde{k}_1, \ldots, \tilde{k}_l]} \) is simply realized as the set of the cycles \( \mathcal{C}'_{[k_1, \ldots, k_l] \cup [\tilde{k}_1, \ldots, \tilde{k}_l]}(Q_1, \cdots, Q_{l+\tilde{l}}) \) of the moduli space \( \mathcal{M}(k + \tilde{k}) \):

\[
\mathcal{C}_{[k_1, \ldots, k_l] \cup [\tilde{k}_1, \ldots, \tilde{k}_l]} = \left\{ \mathcal{C}_{[k_1, \ldots, k_l] \cup [\tilde{k}_1, \ldots, \tilde{k}_l]}(Q_1, \cdots, Q_{l+\tilde{l}}) \mid Q_1, \cdots, Q_{l+\tilde{l}} \in \mathbb{C}^2 \right\} .
\]

Since \( \mathcal{C}'_{[k_1, \ldots, k_l] \cup [\tilde{k}_1, \ldots, \tilde{k}_l]}(Q_1, \cdots, Q_{l+\tilde{l}}) \) is isomorphic to \( \mathcal{C}'_{([k_1, \ldots, k_l] \cup [\tilde{k}_1, \ldots, \tilde{k}_l], 0, \ldots, 0)} \), we can identify the submanifold \( \mathcal{C}_{[k_1, \ldots, k_l] \cup [\tilde{k}_1, \ldots, \tilde{k}_l]} \) with \( (\mathbb{C}^2)^{\oplus (l+\tilde{l})} \times (\mathbb{C}^2)^{\oplus (l+\tilde{l})} \times C'_{([k_1, \ldots, k_l] \cup [\tilde{k}_1, \ldots, \tilde{k}_l], 0, \ldots, 0)} \). Here “\( (\mathbb{C}^2)^{\oplus (l+\tilde{l})} \)” parametrize the \( (l+\tilde{l}) \)-positions where \( k + \tilde{k} \) D5-branes are overlapping.

One can also generalize inclusion map \((4.4)\) to the following direction. For each \( i \), consider the inclusion map

\[
\mathcal{C}_{[k_i]}(P_i) \times \mathcal{M}(\tilde{k} + \sum_{j \neq i} k_j) \hookrightarrow \mathcal{M}(k + \tilde{k}) ,
\]

which now gives the submanifold \( \mathcal{C}_{[k_i]} \) of the moduli space \( \mathcal{M}(k + \tilde{k}) \). The dimensions of this submanifold is

\[
dim \mathcal{C}_{[k_i]} = 4n(\tilde{k} + \sum_{j \neq i} k_j) + 2(nk_i + 1).
\]

The intersection of these submanifolds clearly turns out to be \((4.6)\):

\[
\mathcal{C}_{[k_1]} \cap \mathcal{C}_{[k_2]} \cap \cdots \cap \mathcal{C}_{[k_i]} = \mathcal{C}_{[k_1, \ldots, k_i]} = \mathbb{C}(k_{i+1}^{j+1}[k_i]) .
\]

**Physical Observables of Worldvolume Topological Field Theory**

Let us concentrate on the moduli space \( \mathcal{M}(k) \), that is, the classical vacua of \( k \) D5-branes having open strings with the \( U(n) \) Chan-Paton factors. In \( [9] \) worldvolume topological \( U(k) \) gauge theory has been constructed such that its physical content can be identified with cohomology theory of the moduli space \( \mathcal{M}(k) \). In particular, the physical Hilbert space is realized by the cohomology group \( H^*(\mathcal{M}(k)) \). In this subsection, taking this
topological field theoretical viewpoint, we introduce a subclass of $H^*(\mathcal{M}(k))$, which admits, as a class of the physical observables of topological field theory, to have a structure analogous to the Fock space.

We begin by considering the noncompact submanifold $\mathcal{C}_\Gamma$ with $|\Gamma| = k$ and $l(\Gamma) = l$. It is the set of the cycles $C_\Gamma(P_1, \cdots, P_l)$ with $P_i$ being arbitrary in the four-dimensions. Since each cycle $C_\Gamma(P_1, \cdots, P_l)$ can be topologically identified with the maximal cycle $C_{(\Gamma, \emptyset, \cdots, \emptyset)}$, it holds that $C_\Gamma \simeq (\mathbb{C}^2)^{\otimes l} \times C_{(\Gamma, \emptyset, \cdots, \emptyset)}$. This isomorphism implies that the noncompact directions of the submanifold $\mathcal{C}_\Gamma$ can be identified with "$(\mathbb{C}^2)^{\otimes l}$", which simply parametrize the $l$-positions $P_1, \cdots, P_l$ where $k$ D5-branes overlap. Let us take the Poincaré dual of $\mathcal{C}_\Gamma$ in the moduli space $\mathcal{M}(k)$. It will be denoted by $\mathcal{O}_\Gamma$. The support of $\mathcal{O}_\Gamma$ is on the tubular neighborhood of $\mathcal{C}_\Gamma$ and therefore it is noncompact. Taking account of the isomorphism $\mathcal{C}_\Gamma \simeq (\mathbb{C}^2)^{\otimes l} \times C_{(\Gamma, \emptyset, \cdots, \emptyset)}$ one may modify $\mathcal{O}_\Gamma$ by adding or subtracting an appropriate exact form so that it does not depend on the noncompact directions. This means that one can make $\mathcal{O}_\Gamma$ independent of the coordinates of the $l$-positions $P_1, \cdots, P_l$ on $C_\Gamma(P_1, \cdots, P_l)$. (It will require a special care when some of these points coincide.) As regards the degrees of $\mathcal{O}_\Gamma$ it is given by

$$\deg \mathcal{O}_\Gamma = \dim \mathcal{M}(k) - \dim \mathcal{C}_\Gamma = 2(nk - l), \quad (4.14)$$

which is equal to the dimensions of the maximal cycle $C_{(\Gamma, \emptyset, \cdots, \emptyset)}$.

Suppose that $\Gamma = [k_1, \cdots, k_l]$. Consider the noncompact submanifold $\mathcal{C}_{[k_i]}$ of the moduli space $\mathcal{M}(k)$. $\mathcal{C}_{[k_i]}$ is the set of the vacua of $k$ D5-branes in which the configurations of $k_i$ pieces, considering it as vacua of $k_i$ five-branes, belong to the cycle $C_{[k_i]}(P)$ of the moduli space $\mathcal{M}(k_i)$. $P$ is arbitrary in the four-dimensions. Let us also take the Poincaré dual of this submanifold in the moduli space $\mathcal{M}(k)$, which will be denoted by $\mathcal{O}_{[k_i]}$. The support of $\mathcal{O}_{[k_i]}$ is on the tubular neighbourhood of $\mathcal{C}_{[k_i]}$. Through the inclusion map, $C_{[k_i]}(P) \times \mathcal{M}(k - k_i) \hookrightarrow \mathcal{M}(k)$, besides the identification of $C_{[k_i]}(P)$ with $C_{([k_i], \emptyset, \cdots, \emptyset)}$ in the moduli space $\mathcal{M}(k_i)$, the submanifold $\mathcal{C}_{[k_i]}$ can be regarded as $(\mathbb{C}^2) \times \mathcal{M}(k - k_i) \times C_{([k_i], \emptyset, \cdots, \emptyset)}$, where "$\mathbb{C}^2$" parametrize the position of the overlapping $k_i$ D5-branes. Under this identification the noncompact directions of $\mathcal{C}_{[k_i]}$
are \((C^2) \times M(k - k_i)\). By adding or subtracting an appropriate exact form one may adjust \(O_{[k_i]}\) such that it does not depend on these noncompact directions. Recalling \(C_{([k_i],\emptyset,\cdots,\emptyset)}\) is the cycle which is added in order to resolve the singularity of (overlapping) \(k_i\) ideal \(SU(n)\)-instantons, this means that \(O_{[k_i]}\) can be taken such that it only depends on the local data of the resolution. The degrees of \(O_{[k_i]}\) turns out to be

\[
deg O_{[k_i]} = \dim M(k) - \dim C_{[k_i]} \\
= 2(nk_i - 1) ,
\]

which is independent of \(k\) and equals to the dimensions of the cycle \(C_{([k_i],\emptyset,\cdots,\emptyset)}\) of the moduli space \(M(k_i)\).

Now let us apply formula (4.13). Taking the Poincaré dual of (4.13) one can obtain the relation

\[
O_\Gamma = \bigcup_{i=1}^l [k_i] = O_{[k_1]} \wedge \cdots \wedge O_{[k_l]} .
\]

One of the implications of (4.16) is as follows: As was explained in Section 2, several nontrivial cycles appear in the moduli space \(M(k)\), which one may classify by using the set of \(n\) Young tableaux \((\Gamma_1, \Gamma_2, \cdots, \Gamma_n)\) satisfying conditions (2.42). Among them we have investigated the maximal cycles \(C_{(\Gamma,\emptyset,\cdots,\emptyset)}\) with \(|\Gamma| = k\). \(O_\Gamma\) are introduced as the Poincaré duals of the submanifolds \(C_\Gamma\) which can be identified with \((C^2)^{\otimes l(\Gamma)} \times C_{(\Gamma,\emptyset,\cdots,\emptyset)}\). The noncompact degrees of freedom simply measure the positions where \(k\) D5-branes are overlapping. One can regard the vector space spanned by these \(O_\Gamma\) as the cohomological version of the subspace of the homology group \(H_* (M(k))\) spanned by the cycles \(C_{(\Gamma,\emptyset,\cdots,\emptyset)}\). Formula (4.16) implies that the subspace \(\oplus_{|\Gamma| = k} CO_\Gamma\) of \(H^* (M(k))\) admits a structure analogous to the Fock space

\[
\bigoplus_{|\Gamma| = k} CO_\Gamma = \bigoplus_{m_1 + \cdots + m_l = k} CO_{m_1} \wedge \cdots \wedge O_{m_l} , \tag{4.17}
\]

where we simplify the notation as follows:

\[
O_m \equiv O_{[m]} . \tag{4.18}
\]
Second-Quantization of D5-Brane

To recover the Fock space structure in (4.17) it is enough to consider the direct sum of these moduli spaces in stead of the specific moduli space. This extension is quite reasonable from the viewpoint of five-branes because the number of five-branes does not suffer apriori any restriction. (Notice that we do not compactify the four-dimensions.) So, summing up $k$ in (4.17) we obtain

\[ H_{\text{total}} \equiv \bigoplus_{l} \bigoplus_{m_1, \ldots, m_l} \mathcal{C} \mathcal{O}_{m_1} \wedge \cdots \wedge \mathcal{O}_{m_l}, \quad (4.19) \]

which is a subspace of $\bigoplus_k H^*(\mathcal{M}(k))$. One might suspect that our definition of $\mathcal{O}_m$ given in the previous subsection seems to depend on each moduli space $\mathcal{M}(k)$ since it is introduced as the Poincaré dual of the submanifold $\mathcal{C}_{[m]}$. But, by the closer look on the submanifold $\mathcal{C}_{[m]}$, we can expect that $\mathcal{O}_m$ will be taken to depend only on the local data of the resolution of the singularity at $\eta = 0$ caused by overlapping $m$ D5-branes and therefore it will acquire the form independent of $k$.\[\text{Footnote 10}\]

We would like to propose that $H_{\text{total}}$ is the Fock space of the second-quantized D5-brane which allows the $U(n)$ Chan-Paton factors. $\mathcal{O}_m$ will be identified with a marginally stable bound state of $m$ D5-branes. This identification seems to be reasonable since $\mathcal{O}_m$ will be regarded to be constructed from the local data of the resolution of the singularity caused by overlapping $m$ D5-branes. We may introduce the creation and annihilation operators of these bound states. The creation and annihilation operators of $\mathcal{O}_m$ are denoted respectively by $\alpha_{-m}$ and $\alpha_m$. These operators can be normalized to satisfy the relations

\[ [ \alpha_{m_1}, \alpha_{m_2} ] = \delta_{m_1+m_2,0} . \quad (4.20) \]

Notice that they are bosonic operators since the degrees of $\mathcal{O}_m$, which count the fermionic contributions to $\mathcal{O}_m$, are even.

\[\text{Footnote 11}\]

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10 This property is discussed in [4] as the universality of the physical observables.
11 From the topological field theoretical viewpoint the degrees of $\mathcal{O}_m$ can be interpreted as the ghost numbers.
Acknowledgements

We would like to thank H.Nakajima for sending us his beautiful lecture note [11]. We benefited from discussions with H.Kunitomo and K.Furuuchi. We also thank J.A.Harvey for letting us know the paper [8] in which Nakajima’s results are nicely reviewed from the viewpoint of string dualities.

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Table 1 \((\hat{\mathbf{B}}_a, \hat{\mathbf{H}}_\lambda)\)

\[
\hat{\mathbf{B}}_1 = \begin{pmatrix}
0 & 1 & 0 & \ldots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & 1 & 0 \\
0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 0 \\
0 & \ldots & 0 & \ldots & 0 \\
\end{pmatrix}
\]

\[
\hat{\mathbf{B}}_2 = \begin{pmatrix}
\lambda & \hat{b}_1 & \ldots & \hat{b}_{m-2} & \hat{b}_{m-1} \\
0 & \lambda & \hat{b}_1 & \ldots & \hat{b}_{m-2} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \hat{b}_1 & \vdots \\
0 & \ldots & 0 & \hat{b}_1 & \vdots \\
0 & \ldots & 0 & \hat{b}_1 & \vdots \\
0 & \ldots & 0 & \hat{b}_1 & \vdots \\
0 & \ldots & 0 & \hat{b}_1 & \vdots \\
0 & \ldots & 0 & \hat{b}_1 & \vdots \\
\end{pmatrix}
\]
\[
\begin{align*}
\hat{H}_1 &= \begin{pmatrix}
0 & \hat{h}_{11} & \ldots & \hat{h}_{1n-1} \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
0 & \hat{h}_{m-11} & \ldots & \hat{h}_{m-1n-1} \\
1 & \hat{h}_{m1} & \ldots & \hat{h}_{mn-1} \\
0 & \hat{h}_{m'1} & \ldots & \hat{h}_{m'n-1}
\end{pmatrix} \\
\hat{H}_2 &= 0
\end{align*}
\]
Table 2 \((B_a, H_{A})\)

\[
B_1 = \begin{pmatrix}
0 & 1 & 0 & \cdots & 0 \\
0 & 0 & 1 & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 1 & 0 \\
0 & \cdots & 0 & 0 & 1 \\
0 & \cdots & 0 & \cdots & \cdots \\
0 & \cdots & \cdots & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & \cdots \\
0 & \cdots & 0 & \cdots & 1
\end{pmatrix}
\]

\[
B_2 = \begin{pmatrix}
0 & b_1 & \cdots & b_{m-2} & b_{m-1} \\
0 & 0 & b_1 & \cdots & b_{m-2} \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 1 \\
0 & 0 & \cdots & 0 & 0 \\
0 & c_1 & \cdots & c_{m-1} & c_m \\
0 & 0 & \cdots & 0 & a_1 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 \\
0 & 0 & 0 & \cdots & \cdots \\
0 & 0 & 0 & \cdots & a_1 \\
0 & 0 & 0 & \cdots & \cdots \\
0 & 0 & 0 & \cdots & 0
\end{pmatrix}
\]
\[
H_1 = \begin{pmatrix}
0 & h_{11} & \ldots & h_{1n-1} \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
0 & h_{m-11} & \ldots & h_{m-1n-1} \\
0 & h_{m1} & \ldots & h_{mn-1} \\
0 & h_{m+11} & \ldots & h_{m+1n-1} \\
\vdots & \vdots & & \vdots \\
\vdots & \vdots & & \vdots \\
0 & h_{2n+m'-11} & \ldots & h_{2n+m'-1n-2} \\
1 & h_{2n+m'1} & \ldots & h_{2n+m'n-1}
\end{pmatrix}
\]

\[H_2 = 0\]