Generalized Quantization Scheme for Two-Person Non-Zero-Sum Games

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Abstract
We have proposed a generalized quantization scheme for non-zero sum games which can be reduced to two existing quantization schemes under appropriate set of parameters. Some other important situations are identified which are not apparent in the exiting two quantizations schemes.

1 Introduction
Game theory stepped into quantum domain with the success of a hypothetical quantum player over a classical player in a Quantum Penny Flip game [1, 2]. Later Eisert et. al. [3] introduced an elegant scheme to deal with non-zero sum games quantum mechanically. In this quantization scheme the strategy space of the players is a two parameter set of unitary $2 \times 2$ matrices. Starting with maximally entangled initial state they analyzed a well-known Prisoner Dilemma game and showed that for a suitable quantum strategy the dilemma disappears. They also pointed out a quantum strategy which always wins over all the classical strategies. Later on, Marinatto and Weber [4] introduced another interesting and simple scheme for the analysis of non-zero sum games in quantum domain. They gave Hilbert structure to the strategic spaces of the players. They used maximally entangled initial state and allowed the players to play their tactics by applying the probabilistic choice of unitary operators. They applied their scheme to an interesting game of Battle of Sexes and found out the strategy for which both the players can achieve equal payoffs.

Both Eisert’s and Marinatto and Weber’s schemes give interesting results for various quantum versions of the games [5, 6, 7, 8, 9, 10]. It seems natural to look for a relationship between these two apparently different quantization schemes. In this papers we have developed a generalized quantization scheme for non-
zero sum games. The game of Battle of Sexes has been used as an example to introduce this quantization scheme which is applicable to other games as well. Separate set of parameters are identified for which our scheme reduces to that of Marinatto and Weber [4] and Eisert et al [3] schemes. Furthermore we have identified other interesting situations which are not apparent within the exiting two quantizations schemes. After a brief introduction to Battle of Sexes we have extended Marinatto and Weber’s mathematical framework by redefining unitary operators for our generalized quantization scheme.

2 Generalized Quantization Scheme

Battle of sexes is an interesting static game of complete information. In its usual exposition two players Alice and Bob are trying to decide a place to spend Saturday evening. Alice wants to attend Opera while Bob is interested in watching TV at home and both would prefer to spend the evening together. The game is represented by the following payoff matrix:

$$
\begin{array}{cc}
O & T \\
\text{Alice} & O & (\alpha, \beta) & (\sigma, \sigma) \\
& T & (\sigma, \sigma) & (\beta, \alpha)
\end{array}
$$

where \( O \) and \( T \) represent Opera and TV, respectively, and \( \alpha, \beta, \sigma \) are the payoffs for players for different choices of strategies with \( \alpha > \beta > \sigma \). There are two Nash equilibria \( (O, O) \) and \( (T, T) \) existing in the classical form of the game. In absence of any communication between Alice and Bob, there exists a dilemma as Nash equilibria \( (O, O) \) suits Alice whereas Bob prefers \( (T, T) \). As a result both the players could end up with worst payoff in the case they play mismatched strategies. Marinatto and Weber [4] presented the quantum version of the game to resolve this dilemma. In our earlier paper we have further extended their work to remove the worst case payoff situation in Battle of Sexes [10]. On the other hand Eisert et. al. [3] presented a different scheme to remove dilemma in the game of Prisoner’s Dilemma through quantization of the game.

Here we present generalized quantization scheme by redefining unitary operators in the Marinatto and Weber scheme. Let Alice and Bob are given the following initial state

$$
|\psi_{in}\rangle = \cos \frac{\gamma}{2} |OO\rangle + i \sin \frac{\gamma}{2} |TT\rangle.
$$

(1)

Here \( |O\rangle \) and \( |T\rangle \) represent the vectors in the strategy space corresponding to Opera and TV, respectively and \( \gamma \in [0, \frac{\pi}{2}] \). The strategy of each of the players is represented by the unitary operator \( U_i \) of the form

$$
U_i = \cos \frac{\theta_i}{2} R_i + \sin \frac{\theta_i}{2} C_i,
$$

(2)

where \( i = 1 \) or \( 2 \) and \( R, C \) are the unitary operators defined as:
Here we restrict our treatment to two parameter set of strategies for mathematical simplicity in accordance with Ref. [3]. After the application of the strategies, the initial state (1) transforms into

\[ |\psi_f\rangle = (U_1 \otimes U_2) |\psi_{in}\rangle. \]  

and using eqs. (2) and (3) the above expression becomes

\[ |\psi_f\rangle = \cos \gamma\frac{\gamma}{2} \cos\frac{\theta_1}{2} e^{i(\phi_1+\phi_2)} |OO\rangle - \cos\frac{\theta_1}{2} \sin\frac{\theta_2}{2} e^{i\phi_1} |OT\rangle \\
- \cos\frac{\theta_2}{2} \sin\frac{\theta_1}{2} e^{i\phi_2} |TO\rangle + \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} |TT\rangle \\
+ i \sin \frac{\gamma}{2} \cos\frac{\theta_1}{2} \cos\frac{\theta_2}{2} e^{-i(\phi_1+\phi_2)} |TT\rangle + \cos\frac{\theta_1}{2} \sin\frac{\theta_2}{2} e^{-i\phi_1} |TO\rangle \\
+ \cos\frac{\theta_2}{2} \sin\frac{\theta_1}{2} e^{-i\phi_2} |OT\rangle + \sin\frac{\theta_1}{2} \sin\frac{\theta_2}{2} |OO\rangle. \]  

The payoff operators for Alice and Bob are

\[ P_A = \alpha P_{OO} + \beta P_{TT} + \sigma (P_{OT} + P_{TO})\]
\[ P_B = \alpha P_{TT} + \beta P_{OO} + \sigma (P_{OT} + P_{TO}) \]  

where

\[ P_{OO} = |\psi_{OO}\rangle \langle \psi_{OO}|, \quad |\psi_{OO}\rangle = \cos\frac{\delta}{2} |OO\rangle + i \sin\frac{\delta}{2} |TT\rangle, \]  

\[ P_{TT} = |\psi_{TT}\rangle \langle \psi_{TT}|, \quad |\psi_{TT}\rangle = \cos\frac{\delta}{2} |TT\rangle + i \sin\frac{\delta}{2} |OO\rangle, \]  

\[ P_{TO} = |\psi_{TO}\rangle \langle \psi_{TO}|, \quad |\psi_{TO}\rangle = \cos\frac{\delta}{2} |TO\rangle - i \sin\frac{\delta}{2} |OT\rangle, \]  

\[ P_{OT} = |\psi_{OT}\rangle \langle \psi_{OT}|, \quad |\psi_{OT}\rangle = \cos\frac{\delta}{2} |OT\rangle - i \sin\frac{\delta}{2} |TO\rangle. \]  

with \( \delta \in \left[0, \frac{\pi}{2}\right]. \) Above payoff operators reduce to that of Eisert’s scheme for \( \delta \) equal to \( \gamma \), which represents the entanglement of the initial state. And for \( \delta = 0 \) above operators transform into that of Marinatto and Weber’s scheme. In generalized quantization scheme payoff for the players are calculated as

\[ \$A(\theta_1, \phi_1, \theta_2, \phi_2) = \text{Tr}(P_A \rho_f), \]
\[ \$B(\theta_1, \phi_1, \theta_2, \phi_2) = \text{Tr}(P_B \rho_f), \]  

\[ 3 \]
where $\rho_f = |\psi_f\rangle \langle \psi_f|$ is the density matrix for the quantum state given by (5) and $\text{Tr}$ represents the trace of a matrix. Using eqs. (5, 6, 8) the payoffs for players are obtained as

$$A(\theta_1, \phi_1, \theta_2, \phi_2) = \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \left[ \eta \sin^2 \frac{\gamma}{2} + \xi \cos^2 \frac{\gamma}{2} + \chi \cos 2(\phi_1 + \phi_2) \sin \gamma ight. \\
\left. - \sigma \right] + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} (\eta \cos^2 \frac{\gamma}{2} + \xi \sin^2 \frac{\gamma}{2} - \chi \sin \gamma - \sigma) + \\
\frac{(\alpha + \beta - 2\sigma) \sin \gamma}{4} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) + \sigma$$

(9a)

$$B(\theta_1, \phi_1, \theta_2, \phi_2) = \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \left[ \xi \sin^2 \frac{\gamma}{2} + \eta \cos^2 \frac{\gamma}{2} - \chi \cos 2(\phi_1 + \phi_2) \sin \gamma ight. \\
\left. - \sigma \right] + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} (\xi \cos^2 \frac{\gamma}{2} + \eta \sin^2 \frac{\gamma}{2} + \chi \sin \gamma - \sigma) + \\
\frac{(\alpha + \beta - 2\sigma) \sin \gamma}{4} \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) + \sigma$$

(9b)

where $\xi = \alpha \cos^2 \frac{\delta}{2} + \beta \sin^2 \frac{\delta}{2}$, $\eta = \alpha \sin^2 \frac{\delta}{2} + \beta \cos^2 \frac{\delta}{2}$, and $\chi = \frac{(\alpha - \beta)}{2} \sin \delta$.

Classical results can easily be found from eqs (9a,9b) by simply unentangling, the initial quantum state of the game i.e. letting $\gamma = 0$. Furthermore all the results found by Marinatto and Weber [4] and Eisert et. al. [3] are also embedded in these payoffs. For different combinations of $\delta$ and $\phi$'s there arise the following possibilities

**Case (a):** When $\delta = 0$ and

(i) $\phi_1 = 0, \phi_2 = 0$, then the payoffs for the players from eqs (9a,9b) reduce to

$$A(\theta_1, \phi_1, \theta_2, \phi_2) = \cos^2 \frac{\theta_1}{2} \left[ \alpha \sin^2 \frac{\gamma}{2} - \beta \cos^2 \frac{\gamma}{2} + \sigma \right] \\
+ \cos^2 \frac{\theta_2}{2} ( - \alpha \sin^2 \frac{\gamma}{2} - \beta \cos^2 \frac{\gamma}{2} + \sigma) + \alpha \sin^2 \frac{\gamma}{2} + \beta \cos^2 \frac{\gamma}{2}$$

(10a)

$$B(\theta_1, \phi_1, \theta_2, \phi_2) = \cos^2 \frac{\theta_1}{2} \left[ \alpha \sin^2 \frac{\gamma}{2} - \beta \cos^2 \frac{\gamma}{2} + \sigma \right] \\
+ \cos^2 \frac{\theta_2}{2} ( - \beta \sin^2 \frac{\gamma}{2} - \alpha \cos^2 \frac{\gamma}{2} + \sigma) + \beta \sin^2 \frac{\gamma}{2} + \alpha \cos^2 \frac{\gamma}{2}$$

(10b)

These payoffs are the same as found by Marinatto and Weber [4] when the players are applying the identity operators $I_1$ and $I_2$ with probabilities $\cos^2 \frac{\theta_1}{2}$ and $\cos^2 \frac{\theta_2}{2}$ respectively for the given initial quantum state of the form (1).

(ii) $\phi_1 + \phi_2 = \frac{\pi}{2}$, eqs (9a,9b) reduce to
\[ S_A(\theta_1, \phi_1, \theta_2, \phi_2) = \cos^2 \frac{\theta_1}{2} \left[ \cos^2 \frac{\theta_2}{2} \left( \alpha + \beta - 2\sigma \right) - \alpha \sin^2 \frac{\gamma}{2} - \beta \cos^2 \frac{\gamma}{2} + \sigma \right] \]
\[ + \cos^2 \frac{\theta_2}{2} \left( -\alpha \sin^2 \frac{\gamma}{2} - \beta \cos^2 \frac{\gamma}{2} + \sigma \right) + \alpha \sin^2 \frac{\gamma}{2} + \beta \cos^2 \frac{\gamma}{2} \]
\[ + \frac{(\alpha + \beta - 2\sigma)}{4} \sin \gamma \sin \theta_1 \sin \theta_2 \] (11a)

\[ S_B(\theta_1, \phi_1, \theta_2, \phi_2) = \cos^2 \frac{\theta_1}{2} \left[ \cos^2 \frac{\theta_1}{2} \left( \alpha + \beta - 2\sigma \right) - \alpha \sin^2 \frac{\gamma}{2} - \beta \cos^2 \frac{\gamma}{2} + \sigma \right] \]
\[ + \cos^2 \frac{\theta_1}{2} \left( -\beta \sin^2 \frac{\gamma}{2} - \alpha \cos^2 \frac{\gamma}{2} + \sigma \right) + \beta \sin^2 \frac{\gamma}{2} + \alpha \cos^2 \frac{\gamma}{2} \]
\[ + \frac{(\alpha + \beta - 2\sigma)}{4} \sin \gamma \sin \theta_1 \sin \theta_2 \] (11b)

These payoffs are equivalent to as if the players are using a linear combination of operators I and flip operator \( \sigma_z \) of the form \( O_i = \sqrt{p_i}I + \sqrt{1 - p_i}\sigma_z \) where \( p_i = \cos^2 \frac{\theta_i}{2}, i = 1 \) or 2 using Marinatto and Weber scheme [4, 14], for the initial entangled state of the form of eq. \[ ]

**Case (b)** When \( \delta = \gamma \) and

(i) \( \phi_1 \neq 0, \phi_2 \neq 0 \) the payoffs given by the eqs (11a,11b) very interestingly change to the payoffs as if the game has been quantized using Eisert et. al. [3] scheme for the initial quantum state of the form \[ ] In this situation the payoffs for both the players are

\[ S_A(\theta_1, \phi_1, \theta_2, \phi_2) = \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \left[ \eta_1 \sin^2 \frac{\gamma}{2} + \xi_1 \cos^2 \frac{\gamma}{2} + \chi_1 \cos 2(\phi_1 + \phi_2) \right. \]
\[ - \sigma] + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \left( \eta_1 \cos^2 \frac{\gamma}{2} + \xi_1 \sin^2 \frac{\gamma}{2} - \chi_1 - \sigma \right) \]
\[ + \frac{(\beta - \sigma)}{2} \sin \gamma \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) + \sigma \] (12a)

\[ S_B(\theta_1, \phi_1, \theta_2, \phi_2) = \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \left[ \xi_1 \sin^2 \frac{\gamma}{2} + \eta_1 \cos^2 \frac{\gamma}{2} - \chi_1 \cos 2(\phi_1 + \phi_2) \right. \]
\[ - \sigma] + \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2} \left( \xi_1 \cos^2 \frac{\gamma}{2} + \eta_1 \sin^2 \frac{\gamma}{2} + \chi_1 - \sigma \right) \]
\[ + \frac{(\alpha - \sigma)}{2} \sin \gamma \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) + \sigma \] (12b)

where \( \xi_1 = \alpha \cos^2 \frac{\gamma}{2} + \beta \sin^2 \frac{\gamma}{2}, \eta_1 = \alpha \sin^2 \frac{\gamma}{2} + \beta \cos^2 \frac{\gamma}{2}, \) and \( \chi_1 = \frac{(\alpha - \beta)}{2} \sin^2 \gamma. \)

To draw a better comparison we take \( \delta = \gamma = \frac{\pi}{2} \) then the payoffs given by eqs
reduce to

$$S_A(\theta_1, \phi_1, \theta_2, \phi_2) = (\alpha - \sigma) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \sin^2 (\phi_1 + \phi_2)$$

$$+ (\beta - \sigma) \left[ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin (\phi_1 + \phi_2) + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right]^2 + \sigma$$

(13a)

$$S_B(\theta_1, \phi_1, \theta_2, \phi_2) = (\alpha - \sigma) \left[ \cos \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \sin (\phi_1 + \phi_2) + \sin \frac{\theta_1}{2} \sin \frac{\theta_2}{2} \right]^2$$

$$+ (\beta - \sigma) \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} \sin^2 (\phi_1 + \phi_2) + \sigma$$

(13b)

The payoffs given in eqs (13) have already been found by J. Du et. al. [15] through Eisert et. al. scheme [3].

(ii) $\phi_1 = \phi_2 = 0$ As shown by Eisert et. al. [3, 16] that one gets classical payoffs with mixed strategies when one parameter set of strategies is used for the quantization of the game. For a better comparison putting $\gamma = \delta = \frac{\pi}{2}$ and $\phi_1 = \phi_2 = 0$ in eqs (12a) and (12b) the same situation occurs and the payoffs reduce to

$$S_A(\theta_1, \phi_1, \theta_2, \phi_2) = \alpha \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \beta \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2}$$

$$+ \sigma \left( \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} + \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)$$

(14a)

$$S_B(\theta_1, \phi_1, \theta_2, \phi_2) = \beta \cos^2 \frac{\theta_1}{2} \cos^2 \frac{\theta_2}{2} + \alpha \sin^2 \frac{\theta_1}{2} \sin^2 \frac{\theta_2}{2}$$

$$+ \sigma \left( \cos \frac{\theta_1}{2} \sin \frac{\theta_2}{2} + \sin \frac{\theta_1}{2} \cos \frac{\theta_2}{2} \right)$$

(14b)

In this case the game behaves just like classical game where the players are playing mixed strategies with probabilities $\cos^2 \frac{\theta_1}{2}$ and $\cos^2 \frac{\theta_2}{2}$ respectively.

Case (c) when $\delta \neq \gamma$ and $\phi_1 = 0$, $\phi_2 = 0$ the payoffs given by the eqs (13a, 13b) reduce to

$$S_A(\theta_1, \phi_1, \theta_2, \phi_2) = \cos^2 \frac{\theta_1}{2} \left[ \cos^2 \frac{\theta_2}{2} (\alpha + \beta - 2\sigma) - \alpha \sin^2 \frac{\gamma - \delta}{2} \right]$$

$$- \beta \cos^2 \frac{\gamma - \delta}{2} + \alpha \sin^2 \frac{\gamma - \delta}{2}$$

$$- \beta \cos^2 \frac{\gamma - \delta}{2} + \alpha \sin^2 \frac{\gamma - \delta}{2}$$

$$+ \sin^2 \frac{\gamma - \delta}{2} + \beta \cos^2 \frac{\gamma - \delta}{2}$$

(15a)

$$S_B(\theta_1, \phi_1, \theta_2, \phi_2) = \cos^2 \frac{\theta_1}{2} \left[ \cos^2 \frac{\theta_2}{2} (\alpha + \beta - 2\sigma) - \beta \sin^2 \frac{\gamma - \delta}{2} \right]$$

$$- \alpha \cos^2 \frac{\gamma - \delta}{2} + \beta \sin^2 \frac{\gamma - \delta}{2}$$

$$- \alpha \cos^2 \frac{\gamma - \delta}{2} + \beta \sin^2 \frac{\gamma - \delta}{2}$$

$$+ \alpha \cos^2 \frac{\gamma - \delta}{2}$$

(15b)
These payoffs are equivalent to Marinatto and Weber [4] when $\gamma$ replaced with $\gamma - \delta$.

**Case (d)** When $\delta \neq 0$ and $\gamma = 0$ then from eqs (16a,16b) the payoffs of the players reduce to

$$
A(\theta_1, \phi_1, \phi_2, \theta_2) = \cos^2 \frac{\theta_1}{2} \left[ \cos^2 \frac{\theta_2}{2} (\alpha + \beta - 2\sigma) - \alpha \sin^2 \frac{\delta}{2} - \beta \cos^2 \frac{\delta}{2} + \sigma \right]
$$

$$
+ \cos^2 \frac{\theta_2}{2} \left( -\alpha \sin^2 \frac{\delta}{2} - \beta \cos^2 \frac{\delta}{2} + \sigma \right) + \alpha \frac{\delta}{2} + \beta \frac{\delta}{2}
$$

$$
- \frac{(\alpha - \beta)}{2} \sin \delta \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \quad (16a)
$$

$$
B(\theta_1, \phi_1, \phi_2, \theta_2) = \cos^2 \frac{\theta_2}{2} \left[ \cos^2 \frac{\theta_1}{2} (\alpha + \beta - 2\sigma) - \beta \sin^2 \frac{\delta}{2} - \alpha \cos^2 \frac{\delta}{2} + \sigma \right]
$$

$$
+ \cos^2 \frac{\theta_1}{2} \left( -\beta \sin^2 \frac{\delta}{2} - \alpha \cos^2 \frac{\delta}{2} + \sigma \right) + \beta \frac{\delta}{2} + \alpha \frac{\delta}{2}
$$

$$
+ \frac{(\alpha - \beta)}{2} \sin \delta \sin \theta_1 \sin \theta_2 \sin (\phi_1 + \phi_2) \quad (16b)
$$

This shows that the measurement plays a crucial role in quantum games as if initial state is unentangled, i.e., $\gamma = 0$, arbiter can still apply entangled basis for the measurement to obtain quantum mechanical results. Above payoffs are similar to that of Marinatto and Weber for the Battle of Sexes games if $\delta$ is replaced by $\gamma$.

### 3 Conclusion

A generalized quantization scheme for non zero sum games is proposed. The game of Battle of Sexes has been used as an example to introduce this quantization scheme. However our quantization scheme is applicable to other games as well. This new scheme reduces to Eisert’s et al [3] scheme under the condition

$$
\delta = \gamma, \phi_1 + \phi_2 = \pi/2
$$

and to Marinatto and Weber [4] scheme when

$$
\delta = 0, \phi_1 = 0, \phi_2 = 0.
$$

In the above conditions $\gamma$ is a measure of entanglement of the initial state. For $\gamma = 0$, classical results are obtained when $\delta = 0, \phi_1 = 0, \phi_2 = 0$. Furthermore, we have identified some interesting situations which are not apparent within the exiting two quantizations schemes. For example, with $\delta \neq 0$, nonclassical results are obtained for initially unentangled state. This shows that the measurement plays a crucial role in quantum games.
References

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