Singular solutions of Navier Stokes equations with time-dependent external force terms in $L^2$

Jörg Kampen

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Abstract
It is shown that Navier Stokes equation models of dimension $n \geq 3$ with time dependent external forces in $L^2$ can have singular solutions.

1 Introduction

We consider incompressible Navier Stokes equation models of the form

$$\frac{\partial v}{\partial t} - \nu \sum_{j=1}^{n} \frac{\partial^2 v_j}{\partial x_j^2} + \sum_{j=1}^{n} v_j \frac{\partial v_j}{\partial x_j} = f_i + \sum_{j,m=1}^{n} \int_{\mathbb{R}} (\frac{\partial}{\partial x_i} K_n(x-y)) \sum_{j,m=1}^{n} \left( \frac{\partial v_m}{\partial x_j} \frac{\partial v_j}{\partial x_m} \right)(t,y)dy,$$

where $v(0,.) = h$.

Singular solution construction for $L^2$-data of external forces

In [1] there has been a discussion of a proposed strong solution of a Navier Stokes equation model on the 3-dimensional torus with force terms which are just in $L^2$. We show that there are singular solutions for these types of models, implying that a global regular existence and uniqueness result cannot be obtained for this general class of models. As remarked, we consider the whole domain with spatial
part $\mathbb{R}^n$ here, where an analogous reasoning is possible if the spatial part of the domain is a three-dimensional torus. We observe that time dependent external forces of low regularity, e.g., $f_i \in L^2$ for all $1 \leq i \leq n$ can consume the damping by the viscosity term such that we have some analogy with the incompressible Euler equation. We denote velocity component functions of the incompressible Euler equation by $v_i^E$, $1 \leq i \leq n$. The domain of regular existence of the constructed singular solution functions will be $[0, \rho) \times \mathbb{R}^n$, where $\rho > 0$ is a positive real number. We construct singularities at the tip of a cone, where some features of the Euler equation can be transferred to the incompressible Navier Stokes equation with weak force terms. Next we implement this idea. For the convenience of the reader we derive the transformation rather explicitly. First we define functions $w_i^E$, $1 \leq i \leq n$ in terms of the velocity components $v_i^E$, $1 \leq i \leq n$ (of a solution of the incompressible Euler equation) by

$$w_i^E(s, y) = \frac{v_i^E(t, x)}{\rho - t}, \quad s = \frac{t}{\sqrt{\rho^2 - t^2}}, \quad y_i = (\rho - t) \arctan(x_i), \quad 1 \leq i \leq n.$$ (2)

Note that

$$t = t(s) = \frac{\rho s}{\sqrt{1 + s^2}}.$$ (3)

We also use the abbreviations

$$y_i = y_i(t, x_i) = (\rho - t) \arctan(x_i)$$ (4)

in the following. For the initial data we note that

$$w_i^E(0, \cdot) = h_i^\rho(\cdot),$$ (5)

where $h_i^\rho(\cdot)$, $1 \leq i \leq n$ denote the transformed data, i.e., $h_i^\rho(y) = h_i(x)$ for all $1 \leq i \leq n$. The idea of the following singular solution construction is that for certain regular data $h_i^\rho$, $1 \leq i \leq n$ corresponding to data $h_i$, $1 \leq i \leq n$ in original coordinates with strong polynomial decay at spatial infinity and small $\rho > 0$ we have global solution branches

$$w_i^E : K_\rho \rightarrow \mathbb{R}, \quad 1 \leq i \leq n$$ (6)

in time $s$-coordinates, which correspond to local-time solutions on a small time interval $[0, \rho)$ in original time $t$-coordinates, where for some regular data $h_i^\rho$, $1 \leq i \leq n$ we have in addition

$$w_i^E(0, 0) = h_i^\rho(0) \neq 0, \quad \lim_{t(s) \uparrow \rho} w_i^E(s, 0) \neq 0, \quad \text{for some } i_0.$$ (7)

Here the domain $K_\rho$ is the image of the domain $[0, \rho) \times \mathbb{R}^n$ under the transformation $(t, x) \rightarrow (s, y)$ in (2). The solutions are viscosity limits of local solutions of a family of related equations closely related to incompressible Navier Stokes equations (a family parameterized by the viscosity $\nu > 0$). The function $w_i^E$, $1 \leq i \leq n$ is well defined at the tip of an infinite cone corresponding to the point $(\rho, 0)$ in original coordinates, and in addition we can prove that for some $\rho > 0$ and for $s = s(t) \uparrow \infty$ or $t \uparrow \rho$ we have that $\lim_{t \uparrow \rho} w_i(s(t), 0) \neq 0$ exists for some $1 \leq i \leq n$. This corresponds then to a singularity at the tip of a cone.
Let $(\rho,0)$ of the corresponding velocity function component $v_i^E$ evaluated at that point, since

$$v_i^E(t,0) = \frac{w_i^E(s(t),0)}{\rho - t} \quad \text{for all} \ t < \rho. \quad (8)$$

This construction is only possible as the transformed function $w_i^E, \ 1 \leq i \leq n$ is supported on a spatially finite cone (the support of the cone is finite with respect to the spatial variables as we use the arctan-function in the transformation).

Note that for the change of the spatial measure we have

$$dy_1dy_2\cdots dy_n = (\rho - t)^n \Pi_{i=1}^n \frac{dx_i}{1 + x_i^2}. \quad (9)$$

We shall observe that a local contraction result for the function $w_i^E, \ 1 \leq i \leq n$ can be obtained via an iterative fixed point scheme of viscosity approximations $w_i^{v,E}, \ 1 \leq i \leq n$, where the latter function essentially satisfies a related equation with additional viscosity term $-\nu \Delta w_i^{v,E}$ on the left side of the equation ($\nu > 0$ is a diffusion constant). The measure in (9) ensures that the potential damping term created by a transformation as a viscosity approximation

$$-\frac{\sqrt{\rho^2 - t^2}}{\rho^2(\rho - t)} w_i^E, \quad (10)$$

can be integrated in $(s,y)$-time coordinates globally. Note that otherwise, the condition $\lim_{t \to \rho}s(t),0 \neq 0$ cannot be guaranteed. Moreover, the latter term turns out to be relatively strong compared to the nonlinear Euler terms for this transformation on the small time interval $[0,\rho]$ (cf. below). Note that without the measure change the coefficient $\frac{\sqrt{\rho^2 - t^2}}{\rho^2(\rho - t)}$ of the potential term integrates over

$$t \in [0,\rho] \ \text{corresponding to} \ s \in \left[0, \frac{\rho - t}{\sqrt{\rho^2 - t^2}} \right] \ \text{for} \ 0 < \epsilon < \rho \ \text{as} \ \text{(cf. below)}$$

$$\int_{t(s) \in [0,\rho - \epsilon]} \frac{\sqrt{\rho^2 - t(s)^2}}{\rho^2(\rho - t(s))} \, ds = \int_{0}^{\rho - \epsilon} \frac{\sqrt{\rho^2 - t^2}}{\rho^2(\rho - t)} \, dt \quad (11)$$

as $\epsilon$ goes to 0 and $\rho > 0$ is fixed. Hence we need a spatial transformation which ensures that the volume of the cone of support is small enough such that $w_i^E(s(t),0)$ becomes well defined as $t \uparrow \rho$ while it is different from zero for at least one index $1 \leq i \leq n$.

Next we derive the transformation in detail for convenience of the reader. Note that for given $t$ the transformed spatial variable $y_i \in (\rho - t, -\rho - t + \pi \frac{2}{t})$ corresponds to $x_i \in \mathbb{R}$. More precisely, the transformed functions are supported on a cone

$$K_\rho := \left\{ (s,y) \mid 0 \leq t(s) \leq \rho \ \& \ (\rho - t(s)\pi \frac{2}{t})^2 \leq y_i \leq (\rho - t(s)\pi \frac{2}{t}) \right\}. \quad (12)$$

A time section of the cone $K_\rho$ at time $s$ is denote by $K_\rho^s$, i.e., we write

$$K_\rho^s := \{(s,y) \in K_\rho | \sigma = s\}. \quad (13)$$
The cone is of infinite 'height' with respect to $s$-coordinates and of height $\rho$ with respect to $t$-coordinates, where the basis (at $s = t = 0$) is the cube $\left]-\rho \frac{\pi^2}{2}, \rho \frac{\pi^2}{2}\right[$. We get
\[ v^E_{i,t} = \frac{w^E_{i,s}(s, \cdot)}{\rho - t} \int_0^1 w^E_{i,s}(s, \cdot) \, ds + \frac{n}{\rho - t} \sum_{j=1}^n w^E_{j}(s, y) \arctan(x_i), \quad (14) \]
where
\[ \frac{ds}{dt} = \frac{1}{\sqrt{\rho^2 - t^2}} + \frac{\frac{1}{\sqrt{\rho^2 - t^2}}(\rho - 2t)}{\sqrt{\rho^2 - t^2}} = \frac{\rho^2}{\sqrt{\rho^2 - t^2}}. \quad (15) \]
Furthermore,
\[ v^E_{i,j} = \frac{w^E_{i,j}(s, y)}{\rho - t} \int_0^1 \frac{w^E_{i,j}(s, y)}{\rho - t} \, dx_j = \frac{n}{\rho - t} \sum_{j=1}^n w^E_{i,j}(s, y) \frac{\rho - t}{1 + x_j^2} = \frac{n}{\rho - t} w^E_{i,j}(s, y) \quad (16) \]
such that Burgers terms transform like $v^E_{i,j} v^E_{i,j} = \frac{1}{(\rho - t)(1 + x^2)} w^E_{i,j} w^E_{i,j}$ and Leray data, i.e., data of the Poisson equation for the pressure, transform like $v^E_{i,j} v^E_{k,j} = \frac{1}{(1 + x^2)(1 + x^2)} w^E_{i,j} w^E_{k,j}$. Multiplying the inverse of the coefficient of the time derivative we get an additional factor $\frac{\rho^2}{\rho - t} \sqrt{\rho^2 - t^2}$ for all these terms. Hence, if $v^E_i, 1 \leq i \leq n$ satisfies the incompressible Euler equation on the interval $[0, \rho)$, then $w^E_i, 1 \leq i \leq n$ satisfies
\[ w^E_{i,s} + \sum_{j=1}^n \frac{\sqrt{\rho^2 - t^2}}{\rho(1 + x^2)} w^E_{i,j} \frac{\partial w^E_{i,j}}{\partial y_j} = \sum_{j=1}^n \frac{\rho - t}{\rho(1 + x^2)} \arctan(x_j) w^E_{i,j}(s, y) \]
\[ = -\frac{\sqrt{\rho^2 - t^2}}{\rho(\rho - t)} w^E_i + \int_{K^*} \sum_{j=1}^n \frac{\rho - t}{\rho(1 + x^2)} \arctan(x_j) w^E_{i,j}(s, y) \]
\[ \times \left(K^*_{n,i}(, - z)\right) \left(\frac{\partial w^E_{i,j}}{\partial y_j}, \frac{\partial w^E_{i,j}}{\partial y_n}\right) (s, z) \Pi_{m=1}^n \frac{(1 + x^2)}{(\rho - t)^n} \, dz, \quad (17) \]
where for $n \geq 3$
\[ K^*_{n,i}(z) = c \frac{x_i}{|x|}, \quad (18) \]
(with a well-known dimension dependent constant $c$) and $K^*_{n,i}$ is obtained form $K_{n,i}$ by transformation $x_i = \tan \left(\frac{w_{n,i}}{\rho - t}\right)$. Note that the equation in (17) is with respect to $y_i$-coordinates, where we understand that
\[ x_i = x_i(s, y_i) \quad (19) \]
according to the transformation above. The integrals are -strictly speaking- on the domain $K^*$, but we shall use the convenience of classical representations of solutions of approximating equations which are supported on the whole domain. Note that in the Leray projection term $x_i$ is a function which is convoluted, and the support for the solution is the domain $K^*$. We always understand $t = t(s)$ according to the transformation above. Note that
\[ \int_{n=1}^n \frac{\Pi_{m=1}^n (1 + x^2)}{(\rho - t)^n} \, dy_1 dy_2 \cdots dy_n = \Pi_{n=1}^n dx_i. \quad (20) \]
The equation in (17) is studied along with a family of incompressible Navier-Stokes equations for functions \( w^{E}_{\nu,i}, 1 \leq i \leq n \), which solve the same equation as in (17), but with an additional viscosity term

\[- \nu \Delta w^{E}_{\nu,i}. \tag{21}\]

Note that for any \( \rho > 0 \) the time dependent coefficients are bounded on the interval \([0, \rho]\) which follows from the observation

\[\lim_{\rho \downarrow 0} \sup_{t \in [0, \rho]} \sqrt{\rho^2 - t^2} = 0, \tag{22}\]

for a coefficient of the Burgers term, the Leray projection term and the artificial convection term, while

\[\lim_{\rho \downarrow 0} \sup_{t \in [0, \rho]} \frac{\sqrt{\rho^2 - t^2}}{\rho^2 (\rho - t)} = 1 \tag{23}\]

for the potential damping term. Moreover, simple observations as in (22) show that the coefficients become arbitrarily small for small \( \rho \), while the observation in (23) shows that the coefficient of the damping term is comparatively large (close to 1 for small \( \rho > 0 \)). Although the problem for the function \( w^{E}_{\nu,i}, 1 \leq i \leq n \) is defined on a global time interval in \( s\)-time coordinates we can get global existence via a local contraction argument (for small \( \rho \)), i.e., a local contraction argument leads to local regular existence in the time interval \( t \in [0, \rho] \) for \( \rho > 0 \) small enough corresponding to the global time interval \( s \in [0, \infty) \). In order to ensure local contraction it is useful to have strong polynomial decay at infinity of the data functions \( h_{i} \), i.e., for some \( m \geq 2 \) we have

\[h_{i} \in \mathcal{E}^{m(n+1)}_{pol,m}, 1 \leq i \leq n, \tag{24}\]

where

\[\mathcal{E}^{m}_{pol,m} = \left\{ f : \mathbb{R}^{D} \to \mathbb{R} : \exists c > 0 \ \forall x \geq 1 \ \forall 0 \leq |\gamma| \leq m \ |D^{\gamma} f(x)| \leq \frac{c}{1 + |x|^{m}} \right\}. \tag{25}\]

Remark 2.1. In this article we use the function space \( \mathcal{E}^{m(n+1)}_{pol,m} \) for a local time iterative scheme of approximative solutions for \( w^{E}_{\nu,i}, 1 \leq i \leq n \) (and then \( w^{E}_{\nu}, 1 \leq i \leq n \) in the viscosity limit), where the mass is concentrated almost on a (spatially compact) cone. The choice of the strong function space \( \mathcal{E}^{m(n+1)}_{pol,m} \) can illustrate how some iterative schemes 'can be tamed' by consideration of strong function spaces although a convolution with Gaussians may transport mass to high frequencies. This taming holds for local time schemes, where an additional argument shows that it holds even for global time schemes. Consider first local time iterative solution schemes \( v^{(0)}_{i}, 1 \leq i \leq n, k \geq 0 \) of (1) with \( f_{i} \equiv 0, 1 \leq i \leq n \) in terms of classical representations with the fundamental solution \( G_{\nu} \) of the equation \( \frac{\partial u}{\partial t} - \nu \sum_{j=1}^{n} \frac{\partial^{2} u}{\partial x_{j}^{2}} = 0 \). We define for all \( t \geq 0 \) and \( x \in \mathbb{R}^{n} \)

\[v^{(0)}_{i}(t,x) := \int_{\mathbb{R}^{n}} h_{i}(y) G_{\nu}(t,x;0,y) dy = h_{i} *_{sp} G_{\nu}, \tag{26}\]
and for $k \geq 1$

$$v_i^{(k)} = h_i *_{sp} G_{\nu} + \sum_{j=1}^{n} \left( v_j^{(k-1)} \frac{\partial v_j^{(k-1)}}{\partial x_j} \right) * G_{\nu}$$

$$+ \sum_{j,m=1}^{n} \int_{\mathbb{R}^{n}} \left( \frac{\partial}{\partial x_j} K_n(\cdot - y) \right) \sum_{j,m=1}^{n} \left( \frac{\partial v_j^{(k-1)}}{\partial x_j} \partial v_j^{(k-1)} \right)(\cdot, y) dy * G_{\nu},$$

(27)

where $* \text{ denotes convolution with respect to space and time.}$

In the computation of the increment

$$\delta v_i := v_i - h_i *_{sp} G_{\nu}$$

(28)

the effect of the initial data smoothing can be eliminated by the consideration of a related iterative solution scheme. Define

$$\delta v_i^{(0)}(t, x) := 0,$$

(29)

and for $k \geq 1$

$$\delta v_i^{(k)} := v_i^{(k)} - v_i^{(k-1)}.$$  

(30)

The nonlinear quadratic terms have the effect that the effect of decreasing the degree of spatial decay caused by the convolutions with the Gaussians or first order spatial derivatives of the Gaussian and with first order derivatives of the Laplacian kernel is offset by the effect quadratic powers of functions with strong spatial decay. You prove straightforwardly that for local time $t \geq 0$ and $m \geq 2$

$$\delta v^{(k)}(t, \cdot) \in \mathcal{C}^{m(n+1)}_{pol,m}, \quad v^{(k)}(t, \cdot) \in \mathcal{C}^{m(n+1)-\mu}_{pol,m},$$

(31)

for some $\mu \in (0, 1)$ which accounts for the convolution effects of the linear term (in this case the convoluted initial data). For local time you straightforwardly show that the functions $v^{(k)}(t, \cdot)$ are uniformly bounded in $\mathcal{C}^{m(n+1)}_{pol,m}$, and that local time contraction holds. This leads to a local time solution representation

$$v_i := v_i^{(0)} + \sum_{k \geq 1} \delta v_i^{(k)} = h_i *_{sp} G_{\nu} + \sum_{k \geq 1} \delta v_i^{(k)} \in \mathcal{C}^{m(n+1)-\mu}_{pol,m},$$

(32)

where we have some loss of spatial decay of the solution compared to the initial data. However, this does not mean that this loss of spatial decay explodes as we extend the scheme based on the semigroup property, damping effect of the convoluted initial data term $h_i *_{sp} G_{\nu}$ (or $v_i(t_0, \cdot) *_{sp} G_{\nu}$) and/or an appropriate external control. However, we not dwell on this in this remark as we need only time local considerations for the purposes of this article. Furthermore in this article we have a localized equation where local time iterative approximative solution have exponential decay for data with strong polynomial decay.

**Remark 2.2**. We mention here that the considerations of the last remark can be extended to global schemes (although the following observation is not needed for the local time argument of this article). We have define global schemes elsewhere where the spatial dependence is in the $H^m \cap C^m$. However in more general models (such as highly degenerated diffusions which satisfy a Hörmander condition), the assumption of strong spatial polynomial decay is useful in controlled schemes. There may be some misunderstandings about this- so let us add some remarks. The semigroup property of the Gaussian, the preservation of
spatial polynomial decay of the functional increments for local time, combined with an external control, or with local damping of the convoluted initial data term \( h_i \ast_{sp} G_{\nu} \) (or \( v_i(t_0, \cdot) \ast_{sp} G_{\nu} \)) lead to global schemes. We only add some remarks concerning spatial polynomial decay here. Assume that

\[
v_i(t_0, \cdot) = h_i \ast_{sp} G_{\nu} + \sum_{k \geq 1} \delta v_i^{(k)}(t_0, \cdot) \in C_{pol,m}^{m(n+1) - \mu},
\]

has been proved for some \( t_0 > 0 \). Furthermore assume that

\[
\sum_{k \geq 1} \delta v_i^{(k)}(t_0, \cdot) \in C_{pol,m}^{m(n+1)}
\]

We then define a local time iteration scheme on an interval \([t_0, t_1]\) for \( t_1 > t_0 \). First define

\[
v_i^{t_0, (0)} := v_i(t_0, \cdot) \ast_{sp} G_{\nu}.
\]

Note that we have

\[
v_i^{t_0, (0)} = v_i(t_0, \cdot) \ast_{sp} G_{\nu}
\]

\[
= (h_i \ast_{sp} G_{\nu}(t_0, \cdot)) G_{\nu}(t_1 - t_0, \cdot) + \left( \sum_{k \geq 1} \delta v_i^{(k)} \right) \ast_{sp} G_{\nu}(t_1 - t_0, \cdot)
\]

\[
= h_i \ast_{sp} G_{\nu}(t_1, \cdot) + \left( \sum_{k \geq 1} \delta v_i^{(k)}(t_0, \cdot) \right) \ast_{sp} G_{\nu}(t_1 - t_0) \in C_{pol,m}^{m(n+1) - \mu},
\]

where for the second summand we may use (34). For \( k \geq 1 \) define on \([t_0, t_1]\]

\[
v_i^{t_0, (k)} = v_i(t_0, \cdot) \ast_{sp} G_{\nu} + \sum_{j=1}^{n} \left( v_j^{t_0, (k-1)} \ast \frac{\partial v_j^{t_0, (k-1)}}{\partial x_j} \right) \ast G_{\nu}
\]

\[
+ \sum_{j,m=1}^{n} \int_{\mathbb{R}^n} \left( \frac{\partial}{\partial x_j} K_{\nu}(\cdot - y) \right) \sum_{j=1}^{n} \left( \frac{\partial v_j^{t_0, (k-1)}}{\partial x_j} \right) (\cdot, y) dy \ast G_{\nu}.
\]

Define

\[
\delta v_i^{t_0, (0)}(t, x) := 0,
\]

and for \( k \geq 1 \)

\[
\delta v_i^{t_0, (k)} := v_i^{t_0, (k)} - v_i^{t_0, (k-1)}.
\]

You may still prove that for local time \( t \geq 0 \) and \( m \geq 2 \)

\[
\delta v_i^{t_0, (k)}(t, \cdot) \in C_{pol,m}^{m(n+1)}, \quad v_i^{t_0, (k)}(t, \cdot) \in C_{pol,m}^{m(n+1) - \mu}
\]

for some \( \mu \in (0, 1) \). This observation can be used to designing global schemes using damping effects of the convoluted initial data terms or/ and external control.

Remark 2.3. In this article we add the additional assumption of strong polynomial decay in order to ensure strong spatial polynomial decay of local solutions. In a former version of this paper we omitted this additional assumption, and it may be optional indeed. In any case the additional assumption simplifies the proof. Here, note that strong spatial polynomial decay is inherited in local time fixed point iteration scheme for a local solution such that the terms \( w_{m,j}^{E} \).
can compensate polynomial growth of the rational function \( \frac{\Pi_n^+(1+x^2)}{(1+x^2)(1+z_m^2)} \) in the crucial Leray projection term

\[
+ \int_{K^*_{0}} \sum_{j,m=1}^{N} \frac{(p-t)\sqrt{x^2-y^2}}{x^2+y^2} \times \\
\times \left( K_{n,i}(.-z) \right) \left( \frac{\partial w_{j}^E}{\partial z_j} \frac{\partial w_{m}^E}{\partial z_m} \right) (s,z) \frac{\Pi_{n+1}^+(1+x^2)}{(p-t)^2} dz.
\]

We have

**Theorem 2.4.** Let \( m \geq 2 \) be an integer number and let \( \epsilon > 0 \) be any small positive real number. We assume \( n \geq 3 \). Given data \( h_i \in H^m \cap C_{pol,m}^{(n+1)} \) for \( m > \frac{n}{2} + 1 \), for all \( 1 \leq i \leq n \) there is a \( \rho > 0 \) such that there exists a regular solution \( w_i^E \in C^1 \), \((0,\infty), H')\), \( 1 \leq i \leq n \) of the equation (17) for \( r = m - \epsilon \) with initial data \( h_i^0, 1 \leq i \leq n \) corresponding to data \( h_i \in H^m \cap C_{pol,m}^{(n+1)} \) for \( m > \frac{n}{2} + 1 \) in original coordinates, such that \( h_i^0(0) \neq 0 \) for some \( 1 \leq i \leq n \), and such that there is a \( \rho > 0 \) such that for small \( \epsilon > 0 \), there is a solution \( w_i^E \in C^1 \), \((0,\infty), H')\), \( 1 \leq i \leq n \), \( r = m - \epsilon \) of the associated Cauchy problem in (17) which satisfies \( \lim_{t \rightarrow \infty} w_i^E(s,0) \neq 0 \) at time \( t(s) = \rho > 0 \) (corresponding to \( s = \infty \)). Hence, for \( m \geq 3 \) the corresponding local solution function \( v_i^E \), \( 1 \leq i \leq n \) is a local classical solution of the incompressible Euler equation and has a singularity at the point \((\rho,0)\), in the sense that

\[
t \uparrow \rho \Rightarrow |v_i^E(t,0)| \uparrow \infty,
\]

where the singularity is at most of order \(-1\), i.e.,

\[
\sup_{0 \leq t \leq \rho} |v_i^E(t,0)(\rho-t)| \leq C < \infty.
\]

**Remark 2.5.** We note that the limit \( \lim_{t \uparrow \infty} w_i^E(s,0) \) exists for all \( 1 \leq i \leq n \) and corresponds to the limit \( \lim_{t \uparrow \rho} w_i^{*,E}(t,0) \), where

\[
w_i^{*,E}(t,0) = w_i^E(s,0)
\]

for all \( 1 \leq i \leq n \) with \( s = \frac{t}{\sqrt{x^2-y^2}} \in [0,\infty) \) and \( t \in [0,\rho] \). Due to existing limits the function \( w_i^{*,E}(t,0), 1 \leq i \leq n \) can be extended to the domain with time interval \([0,\rho]\).

**Proof.** A direct local iteration scheme seems to be inappropriate in order to prove contraction and existence, since the order of spatial derivatives increases at each iteration step for such schemes. Therefore we add a Laplacian operator \(-\nu\Delta\) (times viscosity \( \nu > 0 \)) to the left side of the equation in (17). We proceed in three steps. First a) we prove existence for the family of equations (a family
with parameter \( \nu > 0 \)

\[
w_{i,s}^{\nu,E} = \nu \Delta w_{i}^{\nu,E} + \sum_{j=1}^{n} \frac{\sqrt{\rho^{2} - \xi_{j}^{2}}}{\rho^{2}} w_{j}^{\nu,E} \frac{\partial w_{j}^{\nu,E}}{\partial y_{j}}
\]

\[
- \sum_{j=1}^{n} \frac{(\rho-t)\sqrt{\rho^{2} - \xi_{j}^{2}}}{\rho^{2}} \arctan(x_{j}) w_{j}^{\nu,E}(s, y) = - \frac{\sqrt{\rho^{2} - \xi_{j}^{2}}}{\rho^{2}(\rho-t)} w_{j}^{\nu,E}
\]

\[
+ \sum_{j,m=1}^{n} \int \frac{(\rho-t)\sqrt{\rho^{2} - \xi_{j}^{2}}}{\rho^{2}(1+x_{j}^{2}(s,z))(1+x_{m}^{2}(s,z))} (K_{n,i}^{+}(. - z)) \left( \frac{\partial w_{j}^{\nu,E}}{\partial y_{j}} \frac{\partial w_{m}^{\nu,E}}{\partial y_{m}} \right) (s, z) \Pi_{i=1}^{n}(1+x_{j}^{2}) dz,
\]

for strong initial initial data \( w_{1}^{\nu,E}(0, \cdot) = h_{i}^{\nu}(.), 1 \leq i \leq n \). More precisely we prove existence for closely related problems defined on the whole space (cf. item a) below). In a second step b) we prove that there exists a \( \rho > 0 \) and data \( h_{i}^{\nu}, 1 \leq i \leq n \) with \( w_{i}^{\nu,E}(0, 0) = \rho h_{i}(0) = h_{i}^{\nu}(0) \neq 0 \) for some 1 \( \leq i \leq n \) such that

\[
w_{i}^{\nu,E}(\rho, 0) \neq 0,
\]

where we recall that \( w_{i}^{\nu,E}(t, \cdot) := w_{i}^{\nu,E}(s, \cdot) \). In a third step c) we prove that the properties of the solution \( w_{i}^{\nu,E}, 1 \leq i \leq n \) described in a) and b), i.e., existence of the Cauchy problem in [15] and the property in [16] are preserved in the viscosity limit \( \nu \downarrow 0 \), and conclude that the corresponding solution of the incompressible Euler equation has a singularity at \((\rho,0)\).

a) We define a fixed point iteration scheme based on the equation in [15]. Recall that in [15] we understand \( x_{i} = x_{i}(s, y_{i}) \) as functions of the convolution, i.e., the Leray projection term reads

\[
\sum_{j,m=1}^{n} \int \frac{(\rho-t)\sqrt{\rho^{2} - \xi_{j}^{2}}}{\rho^{2}(1+x_{j}^{2}(s,z))(1+x_{m}^{2}(s,z))} (K_{n,i}^{+}(. - z)) \left( \frac{\partial w_{j}^{\nu,E}}{\partial y_{j}} \frac{\partial w_{m}^{\nu,E}}{\partial y_{m}} \right) (s, z) \times
\]

\[
\times \frac{\Pi_{i=1}^{n}(1+x_{i}^{2}(s,z))}{(\rho-t)^{n}} dz,
\]

where \( x_{i}^{2}(s, z_{i}) \) denote the squared value of \( x_{i}(s, z_{i}) \). Note that the integral in [17] is with respect to the time section \( K_{n} \) of the cone \( K_{n} \) at each time \( s \). We suppress this reference of the integral in the following for simplicity of notation (if the reference is clear from the context). We remark that at each time \( s \) the spatial domain of the integral is

\[
- (\rho-t)\frac{\pi}{2} \leq z_{i} \leq (\rho-t)\frac{\pi}{2}, 1 \leq i \leq n,
\]

such that \( dz = (\rho-t)^{n}dz^{*} \) for \( z_{i}^{*} = \frac{z_{i}}{\rho-t} \). Hence, the factor \( \left( \frac{1}{\rho-t} \right)^{n} \) cancels if we consider a spatial transformation to a ball at each time. Furthermore, the functions \( w_{i}^{\nu,E}, 1 \leq i \leq n \) and their derivatives have a strong spatial decay such that the spatial factor

\[
\Pi_{i=1}^{n}(1+x_{i}^{2}(s,z_{i}))
\]

in [17] is compensated by the convolution of the Laplacian kernel with these functions \( w_{i}^{\nu,E}, 1 \leq i \leq n \), and by the factor \( \frac{1}{\rho^{2}(1+x_{j}^{2}(s,z))(1+x_{m}^{2}(s,z))} \).
in \([17]\). Note that the functions \(w^{E}_{i}\) are supported on the cone \(K_{\rho}\). We may extend \(w^{E}_{i}\) trivially assuming that
\[
w^{E}_{i}(s, y) = 0 \text{ for } (s,y) \in [0, \infty) \times \mathbb{R}^{n} \setminus K_{\rho},
\] (50)
where we use the same symbol for these trivial extensions. The reason for this extension is that approximations \(w^{\nu, E}_{i}\) of the functions \(w^{E}_{i}\) based on convolutions with the Gaussian are naturally defined on the whole space. They have exponential decay outside the cone and in the limit \(\nu \downarrow 0\) the support is the cone \(K_{\rho}\) of course. We prove that there exist solutions \(w^{\nu, E}_{i}, 1 \leq i \leq n, \nu > 0\) of \([15]\) (as an equation on the whole domain) with strong data \(h_{i}, 1 \leq i \leq n\) at time \(s = 0\) such that
\[
w^{\nu, E}_{i} \in C^{1}\left([0, \infty), H^{m-\epsilon}(\mathbb{R}^{n})\right) \cap C^{1}, m - \epsilon > \frac{n}{2} + 1.
\] (51)
Here, we understand \(H^{m-\epsilon} = H^{m-\epsilon}(\mathbb{R}^{n})\). We work with classical representations of solutions of the equation in \([14]\) in terms of convolutions with the Gaussian. Let \(G_{\nu}\) be the fundamental solution of the equation
\[
\frac{\partial G_{\nu}}{\partial t} - \nu \Delta G_{\nu} = 0.
\] (52)
We have
\[
G_{\nu}(t,x; s,y) = \frac{1}{\sqrt{4\pi \nu (t - s)}} \exp\left(-\frac{(x - y)^{2}}{4\nu (t - s)}\right).
\] (53)
We consider the Gaussian \(G_{\nu}\) on the whole domain such that \(G_{\nu}\) is defined for all \((t, x) \in (0, \infty) \times \mathbb{R}^{n}\) and
\[
(s,y) \in \{(s', y') | (s', y') \in [0, \infty) \times \mathbb{R}^{n} \& \ s < t\}.
\]
We note again: in order to work with convolutions with Gaussians on the whole space while the convoluted functions are defined only on a cone we (trivially) extend the latter functions to the whole space, and work with approximations of solutions to \([14]\). We may do this as we are interested in the viscosity limit \(\nu \downarrow 0\), and this viscosity limit is supported on the cone \(K_{\rho}\). In the following we denote spatial convolutions with \(G_{\nu}\) by \(*_{sp}\) and convolutions with respect to space and time by \(*\). An approximative classical solution representation of the equation in \([14]\) (while abbreviating \(t = t(s)\) for \(0 < s < \infty\) corresponding to \(0 < t < \rho\) is
\[
w^{\nu, E, \epsilon}_{i}(s,.) = w_{i}^{\nu, E, \epsilon}(0,.) \ast_{sp} G_{\nu} - \sum_{j=1}^{n} \left(\frac{\sqrt{\rho^{2} - t^{2}}}{\rho^{2} (1 + x_{j}^{2})}\right) \left(\frac{\partial w^{\nu, E, \epsilon}_{j}}{\partial y_{j}}\right) \ast_{sp} G_{\nu}
\]
\[
+ \sum_{j=1}^{n} \frac{(\rho - t) \sqrt{\rho^{2} - t^{2}}}{\rho^{2} \arctan (x_{j})} \partial_{y_{j}} w^{\nu, E, \epsilon}_{j} \ast G_{\nu} = - f_{0}^{s} \left(\frac{\sqrt{\rho^{2} - t^{2}}}{\rho^{2} (1 + x_{j}^{2})}\right) \left(\frac{\partial w^{\nu, E, \epsilon}_{j}}{\partial y_{j}}\right) \ast_{sp} \partial_{y_{j}} G_{\nu} +
\]
\[
\sum_{j=1}^{n} \frac{(\rho - t) \sqrt{\rho^{2} - t^{2}}}{\rho^{2} (1 + x_{j}^{2}) (1 + z_{j}^{2})} \left(\frac{\partial w^{\nu, E, \epsilon}_{j}}{\partial y_{j}}\right) \ast_{sp} \left(\frac{\partial w^{\nu, E, \epsilon}_{j}}{\partial y_{j}}\right) \ast_{sp} G_{\nu} \times
\]
\[
\times \frac{1}{(\rho - t)} \int_{(\rho - t)}^{\infty} dz \ast G_{\nu},
\] (54)
where $w_i^{\nu,E,\epsilon}(0,.) = w_i^{\nu,E}(0,.)$, $1 \leq i \leq n$. The parameter $\epsilon > 0$ is a dummy parameter which reminds us that we are working with approximative solutions defined on the whole domain with exponential decay outside the cone. For small $\rho > 0$ this function is defined by local contraction in strong function spaces which leads to the fixed point in (54).

The function $w_i^{\nu,E,\epsilon}$, $1 \leq i \leq n$ is an approximative solution of (55) as we have spatial exponential decay outside the cone $K_\rho$ which becomes stronger as the parameter $\nu$ goes to zero. For $m \geq \frac{1}{2}n + 1$ and for each $\nu > 0$, $1 \leq |\alpha| \leq m$, $1 \leq |\beta| + 1 = |\alpha|$ with $\beta_j + 1 = \alpha_j \neq 0$ and $\beta_l = \alpha_l$ for $l \neq j$ a regular solution $w_i^{\nu,E,\epsilon}$, $1 \leq i \leq n$ has the representation

$$D_y^\alpha w_i^{\nu,E,\epsilon} = D_y^\alpha w_i^{\nu,E,\epsilon}(0,.) *_{sp} G_\nu + \sum_{j=1}^n D_y^\beta \left( \frac{\sqrt{\rho^2 - t^2}}{\rho^2} \left( w_j^{\nu,E,\epsilon} \frac{\partial w_j^{\nu,E,\epsilon}}{\partial y_j} \right) \right) * G_{\nu,j}$$

$$+ \sum_{j=1}^n \frac{(\rho-t)\sqrt{\rho^2 - t^2}}{\rho^2} D_y^\beta \left( \arctan(x_i) w_i^{\nu,E,\epsilon}(s,y) \right) * G_{\nu,j}$$

$$- \frac{\sqrt{\rho^2 - t^2}}{\rho^2} D_x^\alpha w_i^{\nu,E,\epsilon} * G_\nu + \sum_{j,m=1}^n D_y^\beta \left( \frac{(\rho-t)\sqrt{\rho^2 - t^2}}{\rho^2(1+s_j^2)(1+s_m^2)} \right) \times$$

$$\times \left( K_{n,i}(y) \left( \frac{\partial w_m^{\nu,E,\epsilon}}{\partial y_m} \frac{\partial w_j^{\nu,E,\epsilon}}{\partial y_j} \right)(s,z) \right) \frac{\partial w_n^{\nu,E,\epsilon}}{\partial y_n} d\tau \right) * G_{\nu,j}.$$ (55)

Such classical representations (and the existence of a solution) can be justified by a contraction principle for an iteration scheme

$$\left( w_i^{\nu,E,\epsilon,k} \right)_{k \geq 0}, 1 \leq i \leq n$$

of successive approximations of $w_i^{\nu,E}$, $1 \leq i \leq n$, which are defined recursively. At step $k = 0$ we define

$$w_i^{\nu,E,\epsilon,0} = w_i^{\nu,E,0} = h_i *_{sp} G_\nu, 1 \leq i \leq n,$$ (56)

and for $k \geq 1$, the function $w_i^{\nu,E,\epsilon,k}$, $1 \leq i \leq n$ is defined recursively as the approximative (approximative solution in the sense of classical representations as outlined above) solution of the family of Cauchy problems

$$w_i^{\nu,E,\epsilon,k} + \sum_{j=1}^n \frac{(\rho-t)\sqrt{\rho^2 - t^2}}{\rho^2(1+s_j^2)(1+s_m^2)} \left( K_{n,i}(x,y) \right) \times$$

$$\times \left( \frac{\partial w_m^{\nu,E,\epsilon,k-1}}{\partial y_m} \frac{\partial w_j^{\nu,E,\epsilon,k-1}}{\partial y_j} \right)(s,z) \right) \frac{\partial w_n^{\nu,E,\epsilon,k-1}}{\partial y_n} d\tau \right) * G_{\nu,j}.$$ (57)

For $k \geq 1$ and $1 \leq i \leq n$ we define the functional increments

$$\delta w_i^{\nu,E,\epsilon,k} := w_i^{\nu,E,\epsilon,k} - w_i^{\nu,E,\epsilon,k-1},$$ (58)
where we have
\[
\delta w_i^{\nu,E,\varepsilon,k}(0,.) \equiv 0,
\] (59)
and
\[
\delta w_i^{\nu,E,\varepsilon,k} = - \sum_{j=1}^{n} \frac{\sqrt{\rho^2 - t^2}}{\rho^2 (1 + x_j^2)} \delta \delta w_j^{\nu,E,\varepsilon,k-1} \frac{\partial}{\partial y_j} \star G \nu
\]
\[
- \sum_{j=1}^{n} \frac{\sqrt{\rho^2 - t^2}}{\rho^2 (1 + x_j^2)} w_j^{\nu,E,\varepsilon,k-1} \frac{\partial w_j^{\nu,E,\varepsilon,k-1}}{\partial y_j} \star G \nu - \frac{\sqrt{\rho^2 - t^2}}{\rho^2 (p - t)} \delta w_i^{\nu,E,\varepsilon,k} \star G \nu
\]
\[
+ \sum_{j=1}^{n} \frac{\sqrt{\rho^2 - t^2}}{\rho^2} y_j \delta w^{\nu,E,\varepsilon,k-1} (s, y) \star G \nu + 2 \int \sum_{j,m=1}^{n} \frac{(p - t) \sqrt{\rho^2 - t^2}}{\rho^2 (1 + x_j^2)(1 + x_m^2)} \times
\]
\[
\times (K_{n,i} (., - z)) \left( \frac{\partial \delta w_j^{\nu,E,\varepsilon,k-1} \partial w_j^{\nu,E,\varepsilon,k-1}}{\partial y_m} \right) (., z) \frac{\Pi_n^a (1 + x_j^2)}{p - t} \frac{dz}{G \nu},
\] (60)
and where the spatial integral of the Leray projection term is with respect to the cone section \( K_{n,i} \) at each time \( s \). For the derivatives of order \( \alpha \neq 0 \) we get analogous representations (which are easily derived from (55)). There are two linear terms in the representation (61) and similar representations for spatial derivatives of order \( \alpha \). One is the damping term of the form
\[
- \frac{\sqrt{\rho^2 - t^2}}{\rho^2 (p - t)} D \delta w_i^{\nu,E,\varepsilon,k} \star G \nu
\] (61)
which lowers the value function and its derivatives pointwise. For \( \alpha = 0 \) we have
\[
\lim_{\nu \downarrow 0} \int_{K_{n,i}} \frac{\sqrt{\rho^2 - t^2}}{\rho^2 (p - t)} |w_i^{\nu,E} (s, y)| \, ds dy
\]
\[
= \lim_{\nu \downarrow 0} \int_{Z} \frac{\sqrt{\rho^2 - t^2}}{\rho^2 (p - t)} |w_i^{\nu,E} (t, .)| \frac{\rho^2}{\sqrt{\rho^2 - t^2}} (p - t)^n dt dz
\] (62)
\[
= \lim_{\nu \downarrow 0} \int_{Z} \frac{(p - t)^n}{(p - t)^n} |w_i^{\nu,E} (t, .)| \, dt dz \leq C \text{vol}(Z) \frac{(p - t)^n}{n} \downarrow 0
\]
as \( \rho \downarrow 0 \), where \( C > 0 \) is an upper bound of \( \sup_{0 \leq t \leq \rho} |w_i^{\nu,E} (t, .)| \) and \( \text{vol}(Z) \) is the volume of the cylinder
\[
Z = \{ (t, z) | (s, y) \in K_{n,i} \}.
\]
Note that \( t \in [0, \rho] \) and \( z \in (-\frac{\rho}{2}, \frac{\rho}{2}) \) for elements of the cylinder. Here
\[
\frac{ds}{dt} = \frac{\rho^2}{\sqrt{\rho^2 - t^2}}, \quad z_i = \frac{y_i}{(p - t)} = \arctan(x_i), \quad 1 \leq i \leq n,
\] (63)
such that
\[
dy = \Pi_n^a dy_i = \Pi_i^a \frac{dy_i}{dz_i} dz_i = (p - t)^n dz.
\] (64)
Hence, for small \( \rho \) the damping is not strong enough such it can force that \( w_i^{\nu,E} (\rho, 0) = 0 \). The change of measure and the cylinder are considered above.
The other linear term in (60) is
\[ \sum_{j=1}^{n} \frac{\sqrt{\rho^2 - t^2}}{\rho^2} y_j \partial w_{\nu,j}^{\nu,E,\epsilon,k-1}(s,y) * G_{\nu}. \] (65)

Here we note that we understand \( y_j = (\rho - t) \arctan(x_j) \) is bounded and can be kept bounded as we interpret \( \partial w_{\nu,j}^{\nu,E,\epsilon,k-1} \) to be defined on the whole domain. Exponential spatial decay is preserved at each iteration step \( k \) of the scheme by the functional increments which have almost all their mass on the cone for data \( h_{\nu}^{\rho} \). Note that nonlinear terms preserve exponential spatial decay a fortiori as products of functions with exponential spatial decay have stronger exponential spatial decay than their factors and this additional decay is stronger as the lowering effect of exponential decay caused by convolutions with the Gaussian or by convolutions with first order derivatives of the Gaussian. In the Leray projection term the factor \( \Pi_{\nu}^{n}(1 + x_t^2) \) is compensated by the exponential spatial decay of the convolution \( (K_{\nu}^{\rho}, -z) \) \( \left( \frac{\partial \delta w_{\nu,s}^{\nu,E,\epsilon,k-1}}{\partial y_{\nu,s}} \right) \left( (s,y) \right) \) (and also by the factor \( \frac{1}{\rho(1 + x_t^2)(1 + x_{t,0}^2)} \)). We have we have for all \( k \geq 1 \) and \( s \geq 0 \)
\[ \delta w_{\nu,s}^{\nu,E,\epsilon,k}(s,,) \in \mathbb{C}^{m(n+1)}_{\rho(1 + x_{t,0}^2)}. \] (66)

For the Leray projection term we remark that a transformation of the cone \( K_{\nu}^{\rho} \) to the cylinder \( Z \) at each time \( s \) ensures that the factor \( \frac{1}{(1 + x_t^2)} \) cancels such that the integrated Leray projection term can have an upper bound. The convolutions with the Gaussian, the first order spatial derivatives of the Gaussian, and the Laplacian kernel can be estimated by Young inequalities using the classical representations for \( \delta w_{\nu,s}^{\nu,E,\epsilon,k}(s,,) \). The integrals over time \( s \) are splitted into local time integrals for \( s \in [0,1] \) and global time integrals for \( s \geq 1 \). The local time integrals \( s \in [0,1] \) are splitted into local spatial integrals with factor \( 1_{B_1} \) and their complements. For the local time and local spatial intergals we may use local standard estimates for the Gaussian and first order derivatives of the Gaussian, which are locally \( L^1 \). First for \( \nu > 0 \) we have for \( t \neq s \) and \( x \neq y \) and \( \mu \in (0,1) \) an upper bound for \( G_{\nu} \) of the form
\[ \left| \frac{1}{\sqrt{4\pi\nu(t-s)}} \left( \frac{(x-y)}{2\sqrt{\nu(t-s)}} \right)^{2\mu-n} \left( \frac{(x-y)}{2\sqrt{\nu(t-s)}} \right)^{n-2\mu} \exp \left( -\frac{(x-y)^2}{4\nu(t-s)} \right) \right| \leq C \frac{1}{\sqrt{\nu(t-s)}} \frac{C'}{\mu(t-s)}, \] (67)
where
\[ C = \frac{1}{\sqrt{\pi}} \sup_{z > 0} z^{n-2\mu} \exp(-z^2). \] (68)

Note that the upper bound in (67) is integrable for \( \mu \in (0,1) \). For the first spatial derivatives \( G_{\nu,j} \) we have an additional factor which we may estimate by
\[ \left| \frac{(x_j - y_j)}{\mu(t-s)} \right| \leq \frac{1}{|x_j - y_j|} \frac{(x_j - y_j)^2}{\mu(t-s)} \] such that the upper bound of \( G_{\nu,j} \) of the form
\[ C' \frac{1}{\sqrt{\nu(t-s)}} |x - y|^{n+1-2\mu} \] (69)
holds for \( t \neq s \) and \( x \neq y \) and becomes locally integrable for \( \mu \in (0,1) \). The functions \( D^\alpha \omega_{i}\nu,E,k}(s,.) \) are \( L^2 \cap C \) for all \( 0 \leq |\alpha| \leq m \) inductively for all \( k \geq 0 \) where products of functions can be estimated by suprema of one factor (we do not even need the product rule for regular Sobolev norms). For the complementary time local \((s \in [0,1])\) and spatially global estimates we may use the exponential decay of the truncated Gaussian

\[
\left| \mathbb{1}_{B_1} \mathbb{1}_{B_1} \frac{1}{\sqrt{4\pi\rho(t-s)}} \exp \left( -\frac{(x-y)^2}{4\rho(t-s)} \right) \right|. \tag{70}
\]

Here in any case \( B_1 \) is the characteristic function which is 1 on the ball of radius 1 and zero elsewhere, and \( \mathbb{1}_{B_1} \) is the complementary characteristic function. For the Leray projection term we note that the truncated kernel \( \mathbb{1}_{B_1}K_{n,i} \) is in \( L^1 \) while the complement \( \mathbb{1}_{B_1^c}K_{n,i} \) is in \( L^\infty \), and the Gaussian estimates above , Plancherel’s identity and Young inequalities ensure boundedness of the Leray projection term in local time. We have given the details of this local time part of contraction results elsewhere, and need not to repeat every detail here. For large time estimates \((s \in (1,\infty))\) we may use a simple Gaussian upper bound based on the factor \( \sqrt{s^{-n}} \), i.e.,

\[
\left| \frac{1}{\sqrt{4\pi\rho s}} \exp \left( -\frac{(x-y)^2}{4\rho s} \right) \right| \leq \frac{1}{\sqrt{4\pi\rho s}}, \tag{71}
\]

where the right side is global integrable for \( n \geq 3 \). Summing up we can extract a factor \( \rho \) from all terms on the right side of (60) such that

\[
\max_{1 \leq i \leq n} |\delta w_{i}^\nu,E,k|_{H^m \cap C^m} \leq \rho c^* \max_{1 \leq i \leq n} |\delta w_{i}^\nu,E,k|_{H^m \cap C^m}, \tag{72}
\]

where for \( \rho > 0 \) small we choose

\[
c^* := 42^m C_2^2 C_1^2 (1 + C_{K_n}), \tag{73}
\]

along with

\[
|h_i^\rho|_{H^m \cap C^m} \leq C_{h_i^\rho}, \tag{74}
\]

and

\[
C_{K_n} = |K_i|_{L^1(B_1)} + |K_i|_{L^2(\mathbb{R}^n \setminus B_1)}, \tag{75}
\]

and

\[
C = \max \{C, C'\}. \tag{76}
\]

The factor \( 2^m \) is due to the number of terms in the expansion of derivatives. Note that we can choose

\[
\max_{1 \leq i \leq n} |h_i^\rho|_{H^m \cap C^m} \text{ close to } 1, \tag{77}
\]

for transformed data for convenience.

b) We consider a family of data functions \( h_i^\rho \), \( 1 \leq i \leq n \), \( \rho > 0 \) such that

\[
\rho h_{i}(0) = h_{i}^\rho(0) = 1, \ h_i \in C^m \cap H^m \cap \mathcal{C}^{m(n+1)}, \ 1 \leq i \leq n, \tag{78}
\]

where \( h_i \) and \( h_i^\rho \), \( 1 \leq i \leq n \) are the data in the respective coordinates and \( \text{supp}(h_i) \subseteq [-\rho \pi, \rho \pi]^n \). We show that there exists a \( \rho > 0 \) such that for
data $h_i^0, 1 \leq i \leq n$ and $w_i^{\nu,E}(0,0) = \rho h_i^0(0) = 1$ for some $1 \leq i_0 \leq n$ we have
\[ w_i^{*,\nu,E}(\rho,0) = \lim_{s \to \infty} w_i^{\nu,E}(s,0) \neq 0. \tag{79} \]
We have for all $1 \leq i \leq n$ and $s \in (0,\infty)$
\[ w_i^{\nu,E}(s,.) = h_i^0(.) + \sum_{l=1}^{\infty} \delta w_i^{\nu,E,l}(s,.) \tag{80} \]
where the contraction result with respect to the $\sup_{s \in [0,\infty]} |.|_{H^m \cap C^m}$-norm (with $m \geq 2$) implies that this representation holds pointwise for the functions $w_i^{\nu,E}, 1 \leq i \leq n$ and for spatial derivatives of these functions up to order $m$. Next, since for $k \geq 2$, some $m \geq 2$, and all $s \in [0,\infty)$
\[ \max_{1 \leq i \leq n} \left| \delta w_i^{\nu,E,k}(s,.) \right|_{H^m \cap C^m} \leq \sqrt{\rho} - \epsilon \max_{1 \leq i \leq n} |\delta w_i^{\nu,E,k-1}(s,.)|_{H^m \cap C^m} \tag{81} \]
holds, we have
\[ \sum_{i=1}^{\infty} \left| \delta w_i^{\nu,E,k}(s,.) \right|_{H^m \cap C^m} \leq \frac{\sqrt{\rho} - \epsilon}{1 - \sqrt{\rho} - \epsilon} |\delta w_i^{\nu,E,k-1}(s,.)|_{H^m \cap C^m} \tag{82} \]
Next we determine an upper bound for $\sup_{s \geq 0} |\delta w_i^{\nu,E,k}(s,.)|_{H^m \cap C^m}$. From (57) and $\delta w_i^{\nu,E,k} := w_i^{\nu,E,k} - w_i^{\nu,E,k-1} = w_i^{\nu,E,k} - h_1 * \rho G_\nu$ we have
\[ |\delta w_i^{\nu,E,k}(s,.)|_{H^m \cap C^m} \leq \left| \frac{\sqrt{\rho} - \epsilon}{1 - \sqrt{\rho} - \epsilon} \right| |\delta w_i^{\nu,E,k-1}(s,.)|_{H^m \cap C^m} \tag{83} \]
\[ \times \left( K_{n,i}(.,z) \right) \left( \frac{\partial h_i^0}{\partial y_j} \frac{\partial h_i^0}{\partial y_m} \right) (.,z) \frac{1}{(\rho - t)^{1/2}} dz * G_\nu. \]
Hence,
\[ \sup_{s \geq 0} |\delta w_i^{\nu,E,k}(s,.)|_{H^m \cap C^m} \leq \rho c^*, \tag{84} \]
with $c^*$ as in item a). Hence, for $\rho$ small enough such that
\[ \rho c^* \left( 1 + \frac{\sqrt{\rho} - \epsilon}{1 - \sqrt{\rho} - \epsilon} \right) \leq \frac{1}{2} h_i^0(0) \neq 0 \tag{85} \]
for some small $\epsilon > 0$ (cf. item a)), we conclude that
\[ w_i^{*,\nu,E}(\rho,.) := \lim_{s \to \infty} w_i^{\nu,E,k}(s,.) \neq 0. \tag{86} \]

**Remark 2.6.** Note that the conclusion in item b) can also be obtained easily from
\[ w_i^{*,\nu,E} \in C^1([0,\rho], H^m \cap C^m), \text{ for some } m \geq 2. \tag{87} \]
However, using the contraction result explicitly we have an explicit upper bound for $\rho$. 

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c) First we mention that the contraction constant in item a) can be chosen independently of the viscosity $\nu$. The reason for this is that in the classical representations for $D^\alpha w_{i}^{\nu,E,\epsilon}$, $1 \leq i \leq n$ for $0 \leq |\alpha| \leq$ are of the form

$$H \ast G_{\nu} \quad (\text{for } |\alpha| = 0), \quad F \ast G_{\nu,i} \quad (\text{for } |\alpha| > 0), \quad (88)$$

for regular functions $H$ and $F$. The convolutions $H \ast G_{\nu}$ have a natural upper bound such that for sequences of function with strong spatial polynomial decay we may use a transformation and compactness arguments (related to Rellich’s theorem) and obtain a regular limit with a small loss of regularity (cf. below for the use of strong spatial polynomial decay in this context). For the functionals $F \ast G_{\nu,i}$ we may use Lipschitz continuity of the functional $F(t,\cdot) \in L^2 \cap C$ (the convoluted Burgers and Leray projection functionals and their derivatives at each iteration step)

$$|F(x - y) - F(x - y')| =: |F_x(y) - F_x(y')| \leq |y - y'| \quad (89)$$

for some constant $L > 0$, where for $y^{-1} = (y_1^{-1}, \cdots, y_n^{-1})$ with $y^{-1}_k = y_k$ for $k \neq i$ and $y^{-1}_i = -y_i$ we have

$$\left|F \ast G_{\nu,i}\right| = \left|\int F(x - y) \frac{2\nu}{y_0} G_{\nu}(t,y)dy\right|$$

$$\left|\int_{y_0 \geq 0} F_x(y) - F_x(y^{-1}) \frac{2\nu}{y_0} G_{\nu}(t,y)dy\right| \quad (90)$$

which leads to natural $\nu$-independent estimates in item a).

Next we prove c1) that the viscosity limit exists for the function $w_{i}^{\nu,E,\epsilon}(s,\cdot)$, $1 \leq i \leq n$, i.e., for $m \geq 2$ and $h_{\nu}^i \in H^m \cap C^m$, $1 \leq i \leq n$ we prove

$$w_{i}^{*\nu,E,\epsilon} := \lim_{\epsilon \to 0} w_{i}^{\nu,E,\epsilon} \in C^1 \left([0,\rho], H^{m-\epsilon} (\mathbb{R}^n)\right), \quad (91)$$

and where for some $m > \frac{3}{2} + 1$ we have for some finite constant

$$\sup_{0 \leq t \leq \rho} \max_{1 \leq i \leq n} |w_{i}^{*,\nu,E,\epsilon}(t,\cdot)|_{H^r} \leq C. \quad (92)$$

Note that this implies

$$\sup_{0 \leq t \leq \rho} |w_{i}^{*,\nu,E,\epsilon}(t,\cdot)|_{C^1} = \sup_{0 \leq s < \infty} |w_{i}^{\nu,E,\epsilon}(s,\cdot)|_{C^1} \leq \tilde{C}. \quad (93)$$

for some finite constant $\tilde{C} > 0$ by the Sobolev lemma, and where (as usual) we denote $|f|_{C^1} = \sum_{0 \leq |\alpha| \leq 1} \sup_{x \in \mathbb{R}^n} |D^\alpha f(\cdot)|$. Here recall that the dummy upper script $\epsilon > 0$ reminds us that this solution is the viscosity limit of a problem which is defined on the whole domain. The we shall conclude that a corresponding short time classical solution $v_{i}^{E}$, $1 \leq i \leq n$ of the incompressible Euler equation exists. In item a) we have constructed regular upper bounds of $w_{i}^{\nu,E,\epsilon}$ $1 \leq i \leq n$ which are independent of the viscosity $m$, i.e., for each $m \geq 2$, regular data $h_i$, $1 \leq i \leq n$ (as assumed)
and corresponding regular data \( h_i^0 \), \( 1 \leq i \leq n \) we have a finite constant \( C_m \) which is independent of \( \nu \) such that
\[
\max_{1 \leq i \leq n} \sup_{s \geq 0} \| w_{i}^{\nu,E,\epsilon}(s,) \|_{H^{m,C_m}} \leq C_m, \tag{94}
\]
and where in addition
\[
w_{i}^{\nu,E,\epsilon}(s,) \in C_{pol,m}^{m(n+1)}, 1 \leq i \leq n, \text{ for } s \geq 0. \tag{95}
\]

The existence of a viscosity limit is based on compactness arguments and the Lipschitz-continuity of the terms in \( 5[3] \), which are convoluted with the Gaussian.

Here, we note that convergence in strong norms implies that compactness arguments are available. Indeed, we can apply Rellich’s embedding as follows - note that Rellich’s theorem is formulated on bounded domains. Recall that Rellich’s embedding is with respect to spaces \( H_{s}^{a}(\Omega) \) which are closures in \( H^{s} \) for \( s > 0 \) of \( C_{c}^{\infty}(\Omega) \), i.e., the space of smooth function with compact support. These are the functions of \( H^{a} \) which are supported in the closure of \( \Omega \) of \( \Omega \). For simplicity let is consider again the integer value \( m \geq 2 \). For \( 0 \leq |\alpha| \leq m \) we consider the functions
\[
\frac{\pi}{2} \sup_{\Omega} \sup_{\rho} \sup_{r} \| y \to \left( D_{x}^{\alpha} w_{i}^{\nu,E,\epsilon} \right)^{*}(s,z) = D_{x}^{\alpha} w_{i}^{\nu,E,\epsilon}(s, \tan \left( \frac{y}{\rho - s} \right)) , \tag{96}
\]
where \( \tan(y) = (\tan(y_1), \cdots, \tan(y_n))^T \). Since \( 93 \) holds, and
\[
| \left( D_{x}^{\alpha} w_{i}^{\nu,E,\epsilon} \right)^{*}(s,z) | = c \left( 1 + |x|^{2m} \right) | D_{x}^{\alpha} w_{i}^{\nu,E,\epsilon}(s,x) | , \tag{97}
\]
it follows that
\[
w_{i}^{E} := \lim_{\nu \downarrow 0} w_{i}^{\nu,E,\epsilon} \in C^{1}_{r}([0,\infty), H^{r}(\mathbb{R}^{n})) \text{ for } \frac{n}{2} < r < m. \tag{98}
\]
Note that for \( m \geq 3 \) we have \( \nu \Delta w_{i}^{\nu,E,\epsilon} \downarrow 0 \) as \( \nu \downarrow 0 \) while \( w_{i}^{\nu,E} \), \( 1 \leq i \leq n \) is a classical solution of the dampened incompressible Navier Stokes equation with viscosity \( \nu > 0 \). It follows that the viscosity limit \( w_{i}^{E} \), \( 1 \leq i \leq n \) is a regular solution of a dampened incompressible Euler type equation corresponding to a local regular solution \( v_{i}^{E} \), \( 1 \leq i \leq n \) of the original Euler equation on the time interval \([0,\rho)\). Together with local contraction this leads to
\[
\sup_{0 \leq t(s) \leq \rho} \| w_{i}^{E}(s,) \|_{H^{r}} := \sup_{0 \leq t(s) \leq \rho \nu \downarrow 0} \| w_{i}^{\nu,E}(s,) \|_{H^{r}} \leq C \tag{99}
\]
for some constant \( C > 0 \). Note that the existence of a classical solution of the incompressible Euler equation is implied. Indeed, iterating spatial derivatives starting from \( 143 \) we get for all \( t = t(s) < \rho \) and corresponding \( s \geq 0 \)
\[
| D_{x}^{\alpha} v_{i}^{E}(t,) | = c \left( 1 + |x|^{2m} \right) | D_{x}^{\alpha} w_{i}^{E}(.,.) | \tag{100}
\]
by induction, and local classical existence of a solution \( v_{i}^{E} \), \( 1 \leq i \leq n \) on the time interval \([0,\rho)\) follows from compactness. Finally \( C3 \) we show that the property \( w_{i}^{\star,E}(\rho,0) \neq 0 \) can be derived from
\[
\text{for all } \nu > 0 w_{i}^{\star,E}(\rho,0) \neq 0. \tag{101}
\]
This follows from the fact that in estimate in b) the finite constant $c^*$ can be chosen independently of the viscosity $\nu$. In addition, we observe that for $\rho$ small enough we have

\[ \rho c^* \left(1 + \frac{\sqrt{\rho^{1-\epsilon}}}{1 - \sqrt{\rho^{1-\epsilon}}} \right) \leq \frac{1}{2} h_{i_0}(0) \neq 0 \quad (102) \]

for some small $\epsilon > 0$ (cf. item a)). We conclude that for he converging subsequence $w_{i_0}^{\nu_k, E, \epsilon}(\rho_\ast)$, $k \geq 1$ we have

\[ w_{i_0}^{\nu_k, E, \epsilon}(\rho_\ast) := \lim_{s \uparrow \infty} w_{i_0}^{\nu_k, E, \epsilon}(s, \cdot) \geq \frac{h_{i_0}(0)}{2}. \quad (103) \]

Hence,

\[ \lim_{k \uparrow \infty} w_{i_0}^{\nu_k, E, \epsilon}(\rho_\ast) := \lim_{s \uparrow \infty} w_{i_0}^{\nu_k, E, \epsilon}(s, \cdot) \geq \frac{h_{i_0}(0)}{2} \neq 0. \quad (104) \]

It follows that $v_1^E(t, 0) = \frac{w_{E}^{E}((0, 0))}{\rho - t}$ has a singularity at $t = \rho$.

The proof in Theorem 2.4 goes through for strong data $h_i \in H^n \cap C^n \cap \mathcal{C}^{m(n+1)}$ for all $m \geq 2$, where the transformed data satisfy $h_i(0) = 1 \neq 0$ for small $\rho$ and $w_{i_0}^{E}(\rho, 0) \neq 0$. We conclude

**Corollary 2.7.** In the situation of the last statement of Theorem 2.4, there are data $h_i$, $1 \leq i \leq n$ such that $h_i \in C^\infty$ and such that the corresponding solution function $v_1^E$ of the incompressible Euler equation defined by $v_i^E(t, x) = \frac{1}{\rho - t} w_{E}^{E}(s, y)$, $1 \leq i \leq n$ has a singularity at the point $(\rho, 0)$.

Next we consider extended models with time-dependent external force data $f_i \in L^2$ for $1 \leq i \leq n$. Note that in any case such models include models with nonzero initial data implicitly, i.e., schemes with regular initial data $h_i \in H^n$, $1 \leq i \leq n$ for $s > \frac{n}{2} + 1$ can essentially be always written in the form with zero initial data and an adjusted external force term. Just consider $v_i' = v_i - h_i$ instead of the velocity function $v_i$. This leads to additional linear first order derivative term in the equation for $v_i'$, $1 \leq i \leq n$, but this is not essential. Here ‘essential’ means that the additional terms in the equation for $v_i'$, $1 \leq i \leq n$ do not alter the analysis of the equation essentially. Therefore, we may consider models with nonzero regular initial data in the following without loss of generality in the sense that the following considerations can also be applied to zero initial data and related external force terms in $L^2$. Recall from (16) that the singularity factor $\frac{1}{\rho - t}$ in $v_i^E = \frac{w_{E}^{E}}{\rho - t}$ cancels for the first order spatial derivatives, i.e.,

\[ v_{i,j}^E = \frac{w_{i,j}^{E}(s, y)}{1 + x_j^2}, \quad (105) \]

hence we get $(x_j \equiv x_j(y_j))$

\[ \Delta v_i^E = \sum_{j=1}^n v_{i,j,j}^E = \sum_{j=1}^n \frac{w_{i,j,j}^{E}(s, y)}{1 + x_j^2} - \frac{w_{i,j}^{E}(s, y)}{(1 + x_j^2)^2} 2x_j \]

\[ = \sum_{j=1}^n \frac{w_{i,j}^{E}(s, y)}{(1 + x_j^2)^2} 2x_j \quad (106) \]
This means that $\Delta w_i^E \in L^2$ implies $\Delta v_i^E \in L^2$. Note that for data $h_i \in H^s$ with $s > \frac{n}{r} + 2$ we even get solutions $w_i^E$, $1 \leq i \leq n$ with $w_i^E \in H^r$ for some order $\frac{n}{r} + 2 < r < s$, which implies that $w_{i,j}^E$ in (106) is of bounded modulus, which makes the latter conclusion even more obvious. In the following statement we refer to the latter observation as a 'situation of data'. As a consequence of Lemma 2.4 and Corollary 2.7 we get

**Theorem 2.8.** In the situation of Theorem 2.4 or in a situation of data with higher regularity consider a solution function $w_i^E$, $1 \leq i \leq n$ corresponding to a solution $v_i^E$, $1 \leq i \leq n$ of the incompressible Euler equation as described in Corollary 2.4. Define

$$f_i^{w,E} = -\nu \Delta w_i^E.$$  

According to lemma 2.4 there exists a $\rho > 0$ such that the function $w_i^E$, $1 \leq i \leq n$ satisfies (with $t = t(s)$)

$$w_{i,s}^E - \nu \Delta w_i^E + \sqrt{\rho^2 - t^2} \sum_{j=1}^n w_j^E \frac{\partial w_j^E}{\partial y_j} + \sum_{j=1}^n \sqrt{(\rho - t) - \rho^2} \sqrt{\rho^2 - t^2} w_j(s, y)$$

$$= -\sqrt{\rho^2 - t^2} \rho \frac{\partial w_i^E}{\partial s} + f_i^{w,E}(s, \cdot) + \sqrt{\rho^2 - t^2} \sum_{j,m=1}^n \int_{K^\rho} \left(K_{i,j,m}^\rho(-z)\right) \sum_{j,m=1}^n \left(\frac{\partial w_i^E}{\partial x_j} \frac{\partial w_j^E}{\partial x_m}\right) \Pi_n(1+4\rho^2) dz(s, z) dz,$$

with the initial data

$$w_i^E(0, \cdot) = h_i, \ 1 \leq i \leq n$$

on the time interval $t \in [0, \rho]$ corresponding to the time interval $s \in [0, \infty)$, and where the solution is a solution in the classical sense. Furthermore the solution can be extended to $t = \rho$ at $y = 0$, where $\lim_{s\rightarrow \infty} w_i^E(s, 0) = \lim_{t\rightarrow \rho} w_i^E(s, 0) \neq 0$. The corresponding functions $v_i^E$, $1 \leq i \leq n$ satisfy

$$v_{i,t}^E - \nu \Delta v_i^E + \sum_{j=1}^n v_j^E \frac{\partial v_j^E}{\partial y_j} + \sum_{j,m=1}^n \int_{\mathbb{R}^n} \left(\frac{\partial}{\partial x_j} K_n(x - y)\right) \sum_{j,m=1}^n \left(\frac{\partial v_i^E}{\partial x_j} \frac{\partial v_j^E}{\partial x_m}\right) (t, y) dy + f_i^{v,E}(t, \cdot),$$

with the initial data

$$v_i^E(0, \cdot) = h_i, \ 1 \leq i \leq n,$$

and where the relation of $f_i^{v,E}(t, \cdot)$ and $f_i^{w,E}(s, \cdot)$ is given via (107) and (108). Here, we have that $f_i^{v,E} \in L^2$ and $v_i^E$ has a singularity at $(\rho, 0)$ for an index $i$ with $\lim_{s\rightarrow \infty} w_i(s, 0) \neq 0$.

### 3 Conclusion

The preceding considerations show that the argument discussed in [1] cannot lead to a unique solution in the spaces chosen there (even if the error outlined in [1] can be corrected). For that purpose it is necessary that the external force terms are located in more specific spaces, which satisfy additional conditions (e.g., are independent of time etc.). Hence, there cannot be a global solution.
branch in spaces where uniqueness is established while time-dependent external forces are just assumed to be in $L^2$. The argument holds only for dimension $n \geq 3$ (as the Leray projection term has not the same bound in lower dimension), and, in its present form, it makes no specific prediction about the type of singularity at the tip of the cone.

4 Comment to literature

In [2] it is shown that it seems unlikely that an argument based on the energy identity can succeed in proving uniqueness and global regular existence for the Navier Stokes equation. So it seems unlikely that an argument as cited in [1] can succeed - at least if uniqueness is included in the statement. The argument presented here does not depend on the energy identity, and it shows that there are singular solution branches of the incompressible Euler equation.

References

[1] math.stackexchange.com/questions/634890

[2] Tao, T. *Finite time blowup for an averaged three-dimensional Navier Stokes equation*, arXiv.1402.0290v2, Feb. 2014.