Ostrowski type Inequalities for \( m \)- and \((\alpha, m)\)-geometrically convex functions via Riemann–Louville Fractional integrals

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Abstract In this paper, some new inequalities of Ostrowski type established for the class of \( m \)- and \((\alpha, m)\)-geometrically convex functions which are generalizations of geometric convex functions.

Keywords Ostrowski’s inequality · \( m \)- and \((\alpha, m)\)-geometrically convex functions

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1 Introduction

The following result is known in the literature as Ostrowski’s inequality [14].

**Theorem 1** Let \( f : [a, b] \to \mathbb{R} \) be a differentiable mapping on \((a, b)\) with the property that \( |f'(u)| \leq M \) for all \( u \in (a, b) \). Then the following inequality holds:

\[
\left| f(x) - \frac{1}{b-a} \int_a^b f(u) \, du \right| \leq M(b-a) \left[ \frac{1}{4} + \left( \frac{x - \frac{a+b}{2}}{b-a} \right)^2 \right]
\]

(1.1)

for all \( x \in [a, b] \). The constant \( 1/4 \) is best possible in the sense that it cannot be replaced by a smaller constant.

This inequality gives an upper bound for the approximation of the integral average

\[
\frac{1}{b-a} \int_a^b f(u) \, du
\]

by the value \( f(x) \) at point \( x \in [a, b] \). For recent results and generalizations concerning Ostrowski’s inequality, see [1–5, 7, 12, 15, 18, 21, 22] and the references therein.
The following notations is well known in the literature.

**Definition 1** A function \( f : I \to \mathbb{R}, \emptyset \neq I \subseteq \mathbb{R} \), where \( I \) is a convex set, is said to be convex on \( I \) if inequality

\[
 f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y)
\]

holds for all \( x, y \in I \) and \( t \in [0, 1] \).

In particular in [19], Toader introduced the class of \( m \)-convex functions as a generalizations of convexity as the following:

**Definition 2** The function \( f : [0, b] \to \mathbb{R} \) is said to be \( m \)-convex, where \( m \in [0, 1] \), if for every \( x, y \in [0, b] \) and \( t \in [0, 1] \) we have

\[
 f(tx + m(1 - t)y) \leq tf(x) + m(1 - t)f(y)
\]

(1.2)

Moreover, in [13], Miheşan introduced the class of \((\alpha, m)\)-convex functions as the following:

**Definition 3** The function \( f : [0, b] \to \mathbb{R} \) is said to be \((\alpha, m)\)-convex, where \((\alpha, m) \in [0, 1]^2\), if for every \( x, y \in [0, b] \) and \( t \in [0, 1] \) we have

\[
 f(tx + m(1 - t)y) \leq t^\alpha f(x) + m(1 - t^\alpha)f(y)
\]

(1.3)

In [23], Xi et al. introduced the class of \(m\)- and \((\alpha, m)\)-geometrically convex functions as the following:

**Definition 4** [23] Let \( f(x) \) be a positive function on \([0, b]\) and \( m \in (0, 1) \). If

\[
 f\left(x^ty^{m(1-t)}\right) \leq [f(x)]^t [f(y)]^{m(1-t)}
\]

(1.4)

holds for all \( x, y \in [0, b] \) and \( t \in [0, 1] \), then we say that the function \( f(x) \) is \( m \)-geometrically convex on \([0, b]\).

Obviously, if we set \( m = 1 \) in Definition 4, then \( f \) is just the ordinary geometrically convex on \([0, b]\).

**Definition 5** [23] Let \( f(x) \) be a positive function on \([0, b]\) and \((\alpha, m) \in (0, 1] \times (0, 1] \). If

\[
 f\left(x^ty^{m(1-t)}\right) \leq [f(x)]^{t\alpha} [f(y)]^{m(1-t^\alpha)}
\]

(1.5)

holds for all \( x, y \in [0, b] \) and \( t \in [0, 1] \), then we say that the function \( f(x) \) is \((\alpha, m)\)-geometrically convex on \([0, b]\).

Clearly, when we choose \( \alpha = 1 \) in Definition 5, then \( f \) becomes the \( m \)-geometrically convex function on \([0, b]\). A very useful inequality will be given as following:

**Lemma 1** [23] For \( x, y \in [0, \infty) \) and \( m, t \in (0, 1] \), if \( x < y \) and \( y \geq 1 \), then

\[
 x^t y^{m(1-t)} \leq tx + (1 - t)y.
\]

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.
Definition 6 Let $f \in L_1[a, b]$. The Riemann–Liouville integrals $J_{a^+}^\mu f$ and $J_{b^-}^\mu f$ of order $\mu > 0$ with $a \geq 0$ are defined by

$$J_{a^+}^\mu f (x) = \frac{1}{\Gamma(\mu)} \int_a^x (x-t)^{\mu-1} f(t) dt, \quad x > a$$

and

$$J_{b^-}^\mu f (x) = \frac{1}{\Gamma(\mu)} \int_x^b (t-x)^{\mu-1} f(t) dt, \quad x < b$$

respectively where $\Gamma(\mu) = \int_0^\infty e^{-tu} u^{\mu-1} du$. Here is $J_{a^+}^0 f (x) = J_{b^-}^0 f (x) = f (x)$.

In the case of $\mu = 1$, the fractional integral reduces to the classical integral. Several researchers have interested on this topic and several papers have been written connected with fractional integral inequalities see [6,8–11,16,17,20].

The aim of this study is to establish some Ostrowski type inequalities for the class of functions whose derivatives in absolute value are $m$- and $(\alpha, m)$-geometrically convex functions via Riemann–Liouville fractional integrals.

2 Ostrowski type inequalities for $m$- and $(\alpha, m)$-geometrically convex functions

In order to prove our main theorems, we need the following lemma that has been obtained in [18]:

Lemma 2 Let $f : [a, b] \to \mathbb{R}$ be a differentiable mapping on $(a, b)$ with $a < b$. If $f^' \in L [a, b]$, then for all $x \in [a, b]$ and $\mu > 0$ we have:

$$\frac{(x-a)^\mu + (b-x)^\mu}{b-a} f (x) - \frac{\Gamma(\mu + 1)}{b-a} \left[ J_{x^-}^\mu f (a) + J_{x^+}^\mu f (b) \right] = \frac{(x-a)^{\mu+1}}{b-a} \int_0^1 t^\mu f' (tx + (1-t) a) dt + \frac{(b-x)^{\mu+1}}{b-a} \int_0^1 t^\mu f' (tx + (1-t) b) dt$$

where $\Gamma(\mu) = \int_0^\infty e^{-tu} u^{\mu-1} du$.

Theorem 2 Let $I \supset [0, \infty)$ be an open interval and $f : I \to (0, \infty)$ is differentiable. If $f' \in L [a, b]$ and $|f'|$ is decreasing and $(\alpha, m)$-geometrically convex on $[\min \{1, a\}, b]$ for $a \in [0, \infty)$, $b \geq 1$ with $a < b$, and $|f' (x)| \leq M \leq 1$, and $(\alpha, m) \in (0, 1] \times (0, 1]$, then the following inequality for fractional integrals with $\mu > 0$ holds:

$$\left| \frac{(x-a)^\mu + (b-x)^\mu}{b-a} f (x) - \frac{\Gamma(\mu + 1)}{b-a} \left[ J_{x^-}^\mu f (a) + J_{x^+}^\mu f (b) \right] \right| \leq \left[ \frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{b-a} \right] \times K (\alpha, m, \mu; k (\alpha))$$

where

$$k (\alpha) = \begin{cases} M^m \int_0^1 t^\mu M^{\alpha(1-m)} dt, & M < 1 \\ \frac{1}{\mu+1}, & M = 1 \end{cases}$$
\begin{proof}
By Lemma 2 and since $|f'|$ is decreasing and $(\alpha, m)$-geometrically convex on $[\min \{1, a\}, b]$, we have
\begin{align*}
&\left| \frac{(x-a)^\mu + (b-x)^\mu}{b-a} f(x) - \frac{\Gamma(\mu + 1)}{b-a} \left[ J_x^\mu f(a) + J_x^\mu f(b) \right] \right| \\
&\leq \frac{(x-a)^{\mu+1}}{b-a} \int_0^1 t^\mu \left| f\left( t x + (1-t) a \right) \right| dt + \frac{(b-x)^{\mu+1}}{b-a} \int_0^1 t^\mu \left| f\left( t x + (1-t) b \right) \right| dt \\
&\leq \frac{(x-a)^{\mu+1}}{b-a} \int_0^1 t^\mu \left| f' \left( x a^{m(1-t)} \right) \right| dt + \frac{(b-x)^{\mu+1}}{b-a} \int_0^1 t^\mu \left| f' \left( x b^{m(1-t)} \right) \right| dt \\
&\leq \frac{(x-a)^{\mu+1}}{b-a} \int_0^1 t^\mu \left| f' \left( x \right) \right|^\alpha \left| f' \left( b \right) \right|^{m(1-\alpha)} dt \\
&\qquad + \frac{(b-x)^{\mu+1}}{b-a} \int_0^1 t^\mu \left| f' \left( x \right) \right|^\alpha \left| f' \left( a \right) \right|^{m(1-\alpha)} dt \\
&\leq \frac{(x-a)^{\mu+1}}{b-a} \int_0^1 t^\mu M^{m+\alpha}(1-m) dt + \frac{(b-x)^{\mu+1}}{b-a} \int_0^1 t^\mu M^{m+\alpha}(1-m) dt \\
&= \frac{M^{m}}{b-a} \int_0^1 t^\mu M^{(1-m)} dt \left[ (x-a)^{\mu+1} + (b-x)^{\mu+1} \right].
\end{align*}

If $0 < \lambda \leq 1 \leq \delta, 0 < u, v \leq 1$, then
\begin{equation}
\lambda^{\mu u} \leq \lambda^{\mu v}. \tag{2.2}
\end{equation}

When $M \leq 1$, by (2.2), we get that
\begin{align*}
&\left| \frac{(x-a)^\mu + (b-x)^\mu}{b-a} f(x) - \frac{\Gamma(\mu + 1)}{b-a} \left[ J_x^\mu f(a) + J_x^\mu f(b) \right] \right| \\
&\leq \frac{M^m}{b-a} \left[ (x-a)^{\mu+1} + (b-x)^{\mu+1} \right] \int_0^1 t^\mu M^{(1-m)} dt \\
&\leq \frac{M^m}{b-a} \left[ (x-a)^{\mu+1} + (b-x)^{\mu+1} \right] \int_0^1 t^\mu M^{(1-m)} dt. \tag{2.3}
\end{align*}

The proof is completed.
\end{proof}

\begin{corollary}
Let $I \supset [0, \infty)$ be an open interval and $f : I \rightarrow (0, \infty)$ is differentiable. If $f' \in L[a, b]$ and $|f'|$ is decreasing and $m$-geometrically convex on $[\min \{1, a\}, b]$ for $a \in [0, \infty)$, $b \geq 1$ with $a < b$, and $|f'(x)| \leq M \leq 1$, and $m \in (0, 1)$, then the following inequality for fractional integrals with $\mu > 0$ holds:
\begin{align*}
&\left| \frac{(x-a)^\mu + (b-x)^\mu}{b-a} f(x) - \frac{\Gamma(\mu + 1)}{b-a} \left[ J_x^\mu f(a) + J_x^\mu f(b) \right] \right| \\
&\leq \left[ \frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{b-a} \right] \times K(1, m, \mu; k(1))
\end{align*}

where
\begin{equation*}
k(1) = \begin{cases} 
\frac{1}{\mu+1}, & M = 1 \\
M^m \int_0^1 t^\mu M^{(1-m)} dt, & M \neq 1
\end{cases}
\end{equation*}
\end{corollary}
Proof We take $\alpha = 1$ in (2.1), we get the required result. \hfill $\square$

The corresponding version for powers of the absolute value of the first derivative is incorporated in the following result:

**Theorem 3** Let $I \supset [0, \infty)$ be an open interval and $f : I \to (0, \infty)$ is differentiable. If $f' \in L[a, b]$ and $|f'|^q$ is decreasing and $(\alpha, m)$-geometrically convex on $[\min \{1, a\}, b]$ for $a \in [0, \infty)$, $b \geq 1$ with $a < b$, $p, q > 1$ and $|f'(x)| \leq M < 1$, $x \in [\min \{1, a\}, b]$ and $(\alpha, m) \in (0, 1) \times (0, 1)$, then the following inequality for fractional integrals with $\mu > 0$ holds:

$$
\left| \frac{(x-a)^\mu + (b-x)^\mu}{b-a} f(x) - \frac{\Gamma (\mu + 1)}{b-a} \left[ J^\mu_{x^{-}} f(a) + J^\mu_{x^{+}} f(b) \right] \right| \\
\leq M^\mu \left( \frac{1}{p\mu + 1} \right)^{\frac{1}{p}} \left[ \frac{M^{q\alpha(1-m)} - 1}{q\alpha (1-m) \ln M} \right]^{\frac{1}{q}} \left[ \frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{b-a} \right]
$$

(2.4)

where $p^{-1} + q^{-1} = 1$.

**Proof** By Lemma 2 and since $|f'|^q$ is decreasing, and using the famous Hölder inequality, we have

$$
\left| \frac{(x-a)^\mu + (b-x)^\mu}{b-a} f(x) - \frac{\Gamma (\mu + 1)}{b-a} \left[ J^\mu_{x^{-}} f(a) + J^\mu_{x^{+}} f(b) \right] \right| \\
\leq \frac{(x-a)^{\mu+1}}{b-a} \int_0^1 t^\mu |f'(tx + (1-t)a)| dt + \frac{(b-x)^{\mu+1}}{b-a} \int_0^1 t^\mu |f'(tx + (1-t)b)| dt \\
\leq \frac{(x-a)^{\mu+1}}{b-a} \left( \int_0^1 t^{p\mu} dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f' \left( x' a^{m(1-t)} \right) \right|^q dt \right)^{\frac{1}{q}} + \frac{(b-x)^{\mu+1}}{b-a} \left( \int_0^1 t^{p\mu} dt \right)^{\frac{1}{p}} \left( \int_0^1 \left| f' \left( x' b^{m(1-t)} \right) \right|^q dt \right)^{\frac{1}{q}}
$$

Since $|f'|^q$ is $(\alpha, m)$-geometrically convex on $[\min \{1, a\}, b]$ and $|f'(x)| \leq M < 1$, we obtain that

$$
\int_0^1 \left| f' \left( x' a^{m(1-t)} \right) \right|^q dt \leq \int_0^1 |f'(x)|^{q\alpha} |f'(a)|^{mq(1-m)} dt \\
\leq \int_0^1 M^{q\alpha + mq(1-m)} dt \leq M^{mq} \int_0^1 M^{q\alpha(1-m)} dt \\
\leq M^{mq} \int_0^1 M^{q\alpha(1-m)} dt = M^{mq} \frac{M^{q\alpha(1-m)} - 1}{q\alpha (1-m) \ln M}
$$

and

$$
\int_0^1 \left| f' \left( x' b^{m(1-t)} \right) \right|^q dt \leq \int_0^1 |f'(x)|^{q\alpha} |f'(b)|^{mq(1-m)} dt \\
\leq M^{mq} \frac{M^{q\alpha(1-m)} - 1}{q\alpha (1-m) \ln M}
$$

and by simple computation

$$
\int_0^1 t^{p\mu} dt = \frac{1}{p\mu + 1}.
$$
Hence, we have
\[
\left| \frac{(x - a)^\mu + (b - x)^\mu}{b - a} f(x) - \frac{\Gamma(\mu + 1)}{b - a} \left[ J_{x^-} f(a) + J_{x^+} f(b) \right] \right|
\leq M^m \left( \frac{1}{p\mu + 1} \right)^{\frac{1}{\gamma}} \left( \frac{M^{q\alpha(1-m)} - 1}{q\alpha (1-m) \ln M} \right)^{\frac{1}{\gamma}} \left[ \frac{(x - a)^{\mu + 1} + (b - x)^{\mu + 1}}{b - a} \right]
\]
which completes the proof. \(\square\)

**Corollary 2** Let \( I \supset [0, \infty) \) be an open interval and \( f : I \to (0, \infty) \) is differentiable. If \( f' \in L[a, b] \) and \( |f'|^q \) is decreasing and \( m \)-geometrically convex on \([\min\{1, a\}, b]\) for \( a \in [0, \infty) \), \( b \geq 1 \) with \( a < b \), \( p, q > 1 \) and \( |f'(x)| \leq M < 1 \), \( x \in [\min\{1, a\}, b] \) and \( m \in (0, 1) \), then the following inequality for fractional integrals with \( \mu > 0 \) holds:
\[
\left| \frac{(x - a)^\mu + (b - x)^\mu}{b - a} f(x) - \frac{\Gamma(\mu + 1)}{b - a} \left[ J_{x^-} f(a) + J_{x^+} f(b) \right] \right|
\leq M^m \left( \frac{1}{p\mu + 1} \right)^{\frac{1}{\gamma}} \left( \frac{M^{q\alpha(1-m)} - 1}{q\alpha (1-m) \ln M} \right)^{\frac{1}{\gamma}} \left[ \frac{(x - a)^{\mu + 1} + (b - x)^{\mu + 1}}{b - a} \right]
\]
where \( 1/p + 1/q = 1 \).

**Proof** We take \( \alpha = 1 \) in (2.4), we get the required result. \(\square\)

A different approach leads to the following result.

**Theorem 4** Let \( I \supset [0, \infty) \) be an open interval and \( f : I \to (0, \infty) \) is differentiable. If \( f' \in L[a, b] \) and \( |f'|^q \) is decreasing and \( (\alpha, m) \)-geometrically convex on \([\min\{1, a\}, b]\) for \( a \in [0, \infty) \), \( b \geq 1 \) with \( a < b \), \( q \geq 1 \) and \( |f'(x)| \leq M < 1 \), \( x \in [\min\{1, a\}, b] \) and \( \alpha \in (0, 1) \), \( m \in (0, 1) \), then the following inequality for fractional integrals with \( \mu > 0 \) holds:
\[
\left| \frac{(x - a)^\mu + (b - x)^\mu}{b - a} f(x) - \frac{\Gamma(\mu + 1)}{b - a} \left[ J_{x^-} f(a) + J_{x^+} f(b) \right] \right|
\leq M^m \left( \frac{1}{(\mu + 1)} \right)^{\frac{1}{\gamma}} \left( \int_0^1 t^\mu M^{q\alpha(1-m)} dt \right)^{\frac{1}{\gamma}} \left[ \frac{(x - a)^{\mu + 1} + (b - x)^{\mu + 1}}{b - a} \right] \tag{2.5}
\]

**Proof** By Lemma 2 and since \( |f'|^q \) is decreasing, and using the power mean inequality, we have
\[
\left| \frac{(x - a)^\mu + (b - x)^\mu}{b - a} f(x) - \frac{\Gamma(\mu + 1)}{b - a} \left[ J_{x^-} f(a) + J_{x^+} f(b) \right] \right|
\leq \frac{(x - a)^{\mu + 1}}{b - a} \int_0^1 t^\mu \left| f'(tx + (1-t)a) \right| dt + \frac{(b - x)^{\mu + 1}}{b - a} \int_0^1 t^\mu \left| f'(tx + (1-t)b) \right| dt
\leq \frac{(x - a)^{\mu + 1}}{b - a} \left( \int_0^1 t^\mu dt \right)^{\frac{1}{1 - \frac{1}{\gamma}}} \left( \int_0^1 t^\mu \left| f'(tx + (1-t)a) \right|^q dt \right)^{\frac{1}{q}}
\leq \frac{(b - x)^{\mu + 1}}{b - a} \left( \int_0^1 t^\mu dt \right)^{\frac{1}{1 - \frac{1}{\gamma}}} \left( \int_0^1 t^\mu \left| f'(tx + (1-t)b) \right|^q dt \right)^{\frac{1}{q}}
\]
Since \( |f'|^q \) is \((\alpha, m)\)-geometrically convex and \( |f'(x)| \leq M < 1 \) and by (2.2), we obtain
\[
\int_0^1 t^{\mu} \left| f' \left( x \cdot a^{m(1-t)} \right) \right|^q dt \leq \int_0^1 t^{\mu} \left| f'(x)^q \right| \left| f'(a)^{(1-t)^q} \right| dt
\]
\[
\leq \int_0^1 t^{\mu} M^{q(m(1-t))} dt \leq M^{mq} \int_0^1 t^{\mu} M^{q(1-t)} dt
\]
and similarly
\[
\int_0^1 t^{\mu} \left| f' \left( x \cdot b^{m(1-t)} \right) \right|^q dt \leq M^{mq} \int_0^1 t^{\mu} M^{q(1-m)} dt
\]
Hence, we have
\[
\left| \frac{(x-a)^{\mu} + (b-x)^{\mu}}{b-a} f(x) - \frac{\Gamma(\mu+1)}{\Gamma(\mu+1)} \left( \int_0^1 t^{\mu} M^{qt(1-m)} dt \right)^{\frac{1}{q}} \left( \frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{b-a} \right) \right|
\]
which completes the proof.

**Corollary 3** Let \( I \supset [0, \infty) \) be an open interval and \( f : I \to (0, \infty) \) is differentiable. If \( f' \in L[a, b] \) and \( |f'|^q \) is decreasing and \( m\)-geometrically convex on \([\min \{1, a\}, b]\) for \( a \in [0, \infty), b \geq 1 \) with \( a < b, q \geq 1 \) and \( |f'(x)| \leq M < 1, x \in [\min \{1, a\}, b]\) and \( m \in (0, 1) \), then the following inequality for fractional integrals with \( \mu > 0 \) holds:
\[
\left| \frac{(x-a)^{\mu} + (b-x)^{\mu}}{b-a} f(x) - \frac{\Gamma(\mu+1)}{\Gamma(\mu+1)} \left( \int_0^1 t^{\mu} M^{qt(1-m)} dt \right)^{\frac{1}{q}} \left( \frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{b-a} \right) \right|
\]

**Proof** We take \( \alpha = 1 \) in (2.5), we get the required result.

**Corollary 4** Let \( I \supset [0, \infty) \) be an open interval and \( f : I \to (0, \infty) \) is differentiable. If \( f' \in L[a, b] \) and \( |f'|^q \) is decreasing and geometrically convex on \([a, b]\) for \( a \in [0, \infty), b \geq 1 \) with \( a < b, q \geq 1 \) and \( |f'(x)| \leq M < 1, x \in [a, b]\), then the following inequality for fractional integrals with \( \mu > 0 \) holds:
\[
\left| \frac{(x-a)^{\mu} + (b-x)^{\mu}}{b-a} f(x) - \frac{\Gamma(\mu+1)}{\Gamma(\mu+1)} \left( \int_0^1 t^{\mu} M^{qt(1-m)} dt \right)^{\frac{1}{q}} \left( \frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{b-a} \right) \right|
\]

then, the inequality in (2.6) is special version of Corollary 3 of [18].

**Proof** If we take \( \alpha = 1 \) and \( m \to 1 \) in (2.5), we get the required result.
Corollary 5  In Theorem 4, if we choose $\mu = 1$, then (2.5) reduce to the following inequality

$$
\left| f(x) - \frac{1}{b-a} \int_{a}^{b} f(y) \, dy \right| 
\leq M^{m} 2^{\frac{1}{q}} \left( \frac{M^{q\alpha(1-m)} - 1}{\ln M^{q\alpha(1-m)}} \left( 1 - \frac{1}{\ln M^{q\alpha(1-m)}} \right) \right)^{\frac{1}{q}} \left[ \frac{(x-a)^2 + (b-x)^2}{2 (b-a)} \right]
$$

Corollary 6  Let $f, g, a, b, \mu, q$ be as in Theorem 4, and $u, v > 0$ with $u + v = 1$. Then

$$
\left| \frac{(x-a)^\mu + (b-x)^\mu}{b-a} f(x) - \frac{\Gamma (\mu + 1)}{b-a} \left[ J_{x-}^{\mu} f(a) + J_{x+}^{\mu} f(b) \right] \right|
\leq M^{m} \left[ \frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{b-a} \right]
\times \left( \frac{1}{\mu + 1} \right)^{1 - \frac{1}{q}} \left( \frac{u^2}{\mu + u} + \frac{v^2 \left( M^{q\alpha(1-m)} - 1 \right)}{q \alpha (1-m) \ln M} \right)^{\frac{1}{q}}
$$

(2.7)

Proof  By Lemma 2 and since $|f'|^q$ is decreasing, and using the power mean inequality, we have

$$
\left| \frac{(x-a)^\mu + (b-x)^\mu}{b-a} f(x) - \frac{\Gamma (\mu + 1)}{b-a} \left[ J_{x-}^{\mu} f(a) + J_{x+}^{\mu} f(b) \right] \right|
\leq \frac{(x-a)^{\mu+1}+1}{b-a} \left( \int_{0}^{1} t^{\mu} \, dt \right)^{1 - \frac{1}{q}} \left( \int_{0}^{1} t^{\mu} \left| f'(x'a^{m(1-t)}) \right|^q \, dt \right)^{\frac{1}{q}}
+ \frac{(b-x)^{\mu+1}}{b-a} \left( \int_{0}^{1} t^{\mu} \, dt \right)^{1 - \frac{1}{q}} \left( \int_{0}^{1} t^{\mu} \left| f'(x'b^{m(1-t)}) \right|^q \, dt \right)^{\frac{1}{q}}
$$

Since $|f'|^q$ is $(\alpha, m)$-geometrically convex and $|f'(x)| \leq M < 1$ and by (2.2), we obtain

$$
\int_{0}^{1} t^{\mu} \left| f'(x'a^{m(1-t)}) \right|^q \, dt \leq M^{mq} \int_{0}^{1} t^{\mu} M^{q\alpha(1-m)} \, dt
\int_{0}^{1} t^{\mu} \left| f'(x'b^{m(1-t)}) \right|^q \, dt \leq M^{mq} \int_{0}^{1} t^{\mu} M^{q\alpha(1-m)} \, dt
$$

By using the well known inequality $cd \leq u c^{\frac{1}{u}} + v d^{\frac{1}{v}}$, we get that

$$
\int_{0}^{1} t^{\mu} M^{q\alpha(1-m)} \, dt \leq \int_{0}^{1} \left( ut^{\frac{u}{u+1}} + v M^{\frac{mq\alpha(1-m)}{q\alpha(1-m)}} \right) \, dt
= \frac{u}{\frac{u}{u+1} + \frac{v M^{\frac{mq\alpha(1-m)}{q\alpha(1-m)}}}{q\alpha(1-m) \ln M}} - 1
= \frac{u^2}{\mu + u} + \frac{v^2 \left( M^{\frac{mq\alpha(1-m)}{q\alpha(1-m)}} - 1 \right)}{q \alpha (1-m) \ln M}
$$
Hence, we have

\[
\left| \frac{(x-a)\mu + (b-x)\mu}{b-a} f(x) - \frac{\Gamma(\mu+1)}{b-a} \left[ J_{x-}^{\mu} f(a) + J_{x+}^{\mu} f(b) \right] \right|
\]

\[
\leq M^m \left( \frac{1}{\mu+1} \right)^{1-\frac{1}{q}} \left( \frac{u^2}{\mu+u} + \frac{v^2 \left( M^{\frac{q(1-m)}{\alpha(1-m)\ln M}} - 1 \right)}{\alpha(1-m)\ln M} \right)^{\frac{1}{q}} \left[ \frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{b-a} \right]
\]

which completes the proof. □

**Remark 1** In Corollary 6, if we choose \( q = 1 \), then inequality (2.7) reduce to the following inequality

\[
\left| \frac{(x-a)\mu + (b-x)\mu}{b-a} f(x) - \frac{\Gamma(\mu+1)}{b-a} \left[ J_{x-}^{\mu} f(a) + J_{x+}^{\mu} f(b) \right] \right|
\]

\[
\leq M^m \left[ \frac{(x-a)^{\mu+1} + (b-x)^{\mu+1}}{b-a} \right] \left( \frac{u^2}{\mu+u} + \frac{v^2 \left( M^{\frac{q(1-m)}{\alpha(1-m)\ln M}} - 1 \right)}{\alpha(1-m)\ln M} \right)
\]

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