Abstract. We consider Laplacians on $\mathbb{Z}^2$-periodic discrete graphs. The following results are obtained: 1) The Floquet-Bloch decomposition is constructed and basic properties are derived. 2) The estimates of the Lebesgue measure of the spectrum in terms of geometric parameters of the graph are obtained. 3) The spectrum of the Laplacian is described, when the so-called fundamental graph consists of one or two vertices and any number of edges. 4) We consider the hexagonal lattice perturbed by adding one edge to the fundamental graph. There exist two cases: a) if the perturbed hexagonal lattice is bipartite, then the spectrum of the perturbed Laplacian coincides with the spectrum $[-1,1]$ for the unperturbed case, b) if the perturbed hexagonal lattice is not bipartite, then there is a gap in the spectrum of the perturbed Laplacian. Moreover, some deeper results are obtained for the perturbation of the square lattice.

1. Introduction and main results

1.1. Introduction. We discuss the spectrum of Laplacians on periodic discrete graphs. Laplacians on periodic graphs are of interest due to their applications to problems of physics and chemistry. They are used to describe and to study properties of different periodic media, including nanomedia, see [NG04], [Ha85], [SDD98].

There are a lot of papers, and even books, on the spectrum of the discrete Laplacian on an infinite graph. One of the main problems is to describe the spectral properties of the Laplacian in terms of geometric parameters of the graph. Useful sources concerning operators acting on infinite graphs are the books [Ba98], [Bo08], [BH12], [CDS95], [CDGT88], [P12] and the papers [HN09], [Mc04], [MW89], [MRA07]. See also the references therein.

There are papers about the spectrum of discrete Laplacians on periodic graphs. Higuchi and Shirai [HS04] (see also [RR07]) obtain the decomposition of the Laplacian into a constant fiber direct integral. Higuchi and Nomura [HN09] prove that the spectrum of the Laplacian consists of an absolutely continuous part and a finite number of flat bands (i.e., eigenvalues with infinite multiplicity). The absolutely continuous spectrum consists of a finite number of intervals (spectral bands) separated by gaps. Moreover, they also show that for each flat band there exists a finitely supported eigenfunction.

There are results about spectral properties of the discrete Laplace and Schrödinger operators on specific periodic graphs. The hexagonal lattice can be viewed as a discrete model of graphene, which is two-dimensional single-layered carbon sheet with honeycomb structure. It was recently discovered by Geim and Novoselov [NG04]. Graphene is a very hot subject in physics, where tight binding models for the Schrödinger operator are standard and give interesting band structures. Ando [A12] considers the spectral theory for the discrete Schrödinger...
operators with finitely supported potentials on the hexagonal lattice and their inverse scattering problem. Korotyaev and Kutsenko [KK10] – [KK10b] study the spectra of the discrete Schrödinger operators on graphene nano-tubes and nano-ribbons in external fields. See more about graphene in Section 5. Schrödinger operators with decreasing potentials on the lattice $\mathbb{Z}^d$ are considered by Boutet de Monvel-Sahbani [BS99], Isozaki-Korotyaev [IK12], Rosenblum-Solomjak [RoS09]. Gieseker-Knörrer-Trubowitz [GKT93] consider Schrödinger operators with periodic potentials on the lattice $\mathbb{Z}^2$, the simplest example of $\mathbb{Z}^2$-periodic graphs. They study its Bloch variety and its integrated density of states.

We describe our main goals of the paper:

1) to construct the Floquet theory for Laplacians on periodic graphs.
2) to estimate the Lebesgue measure of the spectrum of the Laplacian in terms of geometric parameters of the graph.
3) to describe the spectrum of the Laplacian under the perturbation of the square lattice and the hexagonal lattice.
4) It is known [C97] that the investigation of the spectrum of the Laplacian on an equilateral quantum graph (i.e., a graph consisting of identical segments) can be reduced to the study of the spectrum of the discrete Laplacian. Thus, we have to do the needed spectral analysis of the discrete Laplacian given in this paper, in order to describe spectral properties (including the Bethe-Sommerfeld conjecture) of the Laplacians on quantum graphs [KS].

It should be noted that the results obtained in this work can be generalized to the case of $\mathbb{Z}^d$, $d \geq 3$, periodic graphs [KS1].

1.2. The definitions of periodic graphs and fundamental graphs. Let $\Gamma = (V, E)$ be a connected graph, possibly having loops and multiple edges, where $V$ is the set of its vertices and $E$ is the set of its unoriented edges. The graphs under consideration are embedded into $\mathbb{R}^2$. Considering each edge in $E$ to have two orientations, we can introduce the set $A$ of all oriented edges. The inverse edge of $e \in A$ is denoted by $\bar{e}$. The oriented edge starting at $u \in V$ and ending at $v \in V$ will be denoted as the ordered pair $(u, v)$. Vertices $u, v \in V$ will be called adjacent and denoted by $u \sim v$, if $(u, v) \in A$. We define the degree $\kappa_v$ of the vertex $v \in V$ as the number of all oriented edges from $A$ starting at $v$. Below we consider $\mathbb{Z}^2$-periodic graphs $\Gamma$, satisfying the following conditions:

1) the number of vertices from $V$ in any bounded domain $\subset \mathbb{R}^2$ is finite;
2) the degree of each vertex is finite;
3) there exists a basis $a_1, a_2$ in $\mathbb{R}^2$ such that $\Gamma$ is invariant under translations through the vectors $a_1$ and $a_2$:

$$\Gamma + a_1 = \Gamma, \quad \Gamma + a_2 = \Gamma.$$ 

The vectors $a_1, a_2$ are called the periods of $\Gamma$.

In the plane $\mathbb{R}^2$ we consider a coordinate system with the origin at some point $O$. The coordinate axes of this system are directed along the vectors $a_1$ and $a_2$. Below the coordinates of all vertices of $\Gamma$ will be expressed in this coordinate system. Then it follows from the definition of $\mathbb{Z}^2$-periodic graph that $\Gamma$ is invariant under translations through any integer vector:

$$\Gamma + p = \Gamma, \quad \forall p \in \mathbb{Z}^2.$$ 

Examples of periodic graphs are shown in Figures 4, 7.
We define the fundamental graph $\Gamma_0$ of the periodic graph $\Gamma$ as a graph on the surface $\mathbb{R}^2/\mathbb{Z}^2$ by
\[
\Gamma_0 = \Gamma/\mathbb{Z}^2 \subset T^2, \quad \text{where} \quad T^2 = \mathbb{R}^2/\mathbb{Z}^2.
\]

The vertex set $V_0$, the set $E_0$ of unoriented edges and the set $A_0$ of oriented edges of $\Gamma_0$ are finite (see Proposition 6.2).

We introduce an edge "index", which is important in the study of the Laplace operator. We identify the vertices of the fundamental graph $\Gamma_0 = (V_0, E_0)$ with the vertices of the periodic graph $\Gamma = (V, E)$ in the set $[0, 1)^2$. Then for any $v \in V$ the following unique representation holds true:
\[
v = [v] + \tilde{v}, \quad [v] \in \mathbb{Z}^2, \quad \tilde{v} \in V_0 \subset [0, 1)^2.
\]

In other words, each vertex $v$ can be represented uniquely as the sum of an integer part $[v] \in \mathbb{Z}^2$ and a fractional part $\tilde{v}$ that is a vertex of the fundamental graph $\Gamma_0$. For any oriented edge $e = (u, v) \in A$ we define the edge "index" $\tau(e)$ as the integer vector
\[
\tau(e) = [v] - [u] \in \mathbb{Z}^2,
\]
where
\[
u = [v] + \tilde{v}, \quad \nu = [v] + \tilde{v}, \quad [u], [v] \in \mathbb{Z}^2, \quad \tilde{u}, \tilde{v} \in V_0.
\]

If $e = (u, v)$ is an oriented edge of the graph $\Gamma$, then by the definition of the fundamental graph there is an oriented edge $\tilde{e} = (\tilde{u}, \tilde{v})$ on $\Gamma_0$. For an edge $\tilde{e} \in A_0$ we define the edge index $\tau(\tilde{e})$ by
\[
\tau(\tilde{e}) = \tau(e).
\]

In other words, indices of periodic graph edges are inherited by edges of the fundamental graph. The edge indices, generally speaking, depend on the choice of the coordinate origin $O$. But in a fixed coordinate system the index of the fundamental graph edge is uniquely determined by (1.4), since due to Proposition 6.3.iii we have
\[
\tau(e + p) = \tau(e), \quad \forall (e, p) \in A \times \mathbb{Z}^2.
\]

Edges with nonzero indices will be called bridges. They are important, when we describe the spectrum of the Laplacian. The bridges provide the connectivity of the periodic graph and the removal of all bridges disconnects the graph into infinitely many connected components.

1.3. Laplace operators on graphs. Let $\ell^2(V)$ be the Hilbert space of all square summable functions $f : V \to \mathbb{C}$, equipped with the norm
\[
\|f\|_{\ell^2(V)} = \sum_{v \in V} |f(v)|^2 < \infty.
\]

Recall that the degree $\kappa_v$ of the vertex $v \in V$ is the number of oriented edges starting at the vertex $v$. We define the Laplacian (or the Laplace operator) $\Delta$ acting on the Hilbert space $\ell^2(V)$ by
\[
(\Delta f)(v) = \frac{1}{\sqrt{\kappa_v}} \sum_{(v, u) \in A} \frac{1}{\sqrt{\kappa_u}} f(u), \quad \forall f \in \ell^2(V).
\]

Note that sometimes $1 - \Delta \geq 0$ is also called the Laplace operator.

A graph is called bipartite if its vertex set is divided into two disjoint sets (called parts of the graph) such that each edge connects vertices from distinct sets (see p.105 in [Or62]). Examples of bipartite graphs are the square lattice (Figure 7a) and the hexagonal lattice (Figure 4a). The triangular lattice (Figure 9b) is non-bipartite. Note that for a bipartite
periodic graph there exists a bipartite fundamental graph (see Lemma 6.4.ii), but not every fundamental graph is bipartite. Indeed, the square lattice (Figure 7a) is bipartite, but its fundamental graph shown in Figure 7b is non-bipartite.

We recall well-known properties of the Laplacian $\Delta$, which hold true for finite and infinite graphs (in particular, for periodic graphs) (see [Ch97], [Me94], [M92], [MW89]):

Main properties of the Laplacian:
1) The operator $\Delta$ is self-adjoint and bounded.
2) The spectrum $\sigma(\Delta)$ is contained in $[-1, 1]$.
3) The point 1 belongs to the spectrum $\sigma(\Delta)$.
4) The graph is bipartite $\iff$ the point $-1 \in \sigma(\Delta) \iff$ the spectrum $\sigma(\Delta)$ is symmetric with respect to the point zero.

Denote by $v_1, \ldots, v_\nu$ the vertices of $V_0$, where $\nu < \infty$ is a number of vertices of $V_0$. Thus we have

$$V_0 = \{v_j : j \in \mathbb{N}_\nu\}, \quad \mathbb{N}_\nu = \{1, \ldots, \nu\}.$$ 

We present our first results about the decomposition of the Laplacian $\Delta$ into a direct integral.

Theorem 1.1. i) The operator $\Delta$ acting on $\ell^2(V)$ has the decomposition into a constant fiber direct integral

$$\ell^2(V) = \frac{1}{(2\pi)^2} \int_{T^2}^\oplus \ell^2(V_0) d\vartheta, \quad T^2 = \mathbb{R}^2/(2\pi \mathbb{Z})^2 = [-\pi; \pi]^2,$$

$$U \Delta U^{-1} = \frac{1}{(2\pi)^2} \int_{T^2}^\oplus \Delta(\vartheta) d\vartheta,$$

for some unitary operator $U$. Here the fiber space $\ell^2(V_0) = \mathbb{C}^\nu$ and $\Delta(\vartheta) = \{\Delta_{jk}(\vartheta)\}, \vartheta \in T^2$, is a Floquet $\nu \times \nu$ matrix given by

$$\Delta_{jk}(\vartheta) = \begin{cases} \frac{1}{\sqrt{\kappa_j \kappa_k}} \sum_{e=(v_j, v_k) \in A_0} e^{i\langle \tau(e), \vartheta \rangle}, & \text{if } (v_j, v_k) \in A_0 \\ 0, & \text{if } (v_j, v_k) \notin A_0 \end{cases},$$

where $\kappa_j$ is the degree of $v_j$ and $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbb{R}^2$. In particular,

$$\Delta_{jj}(\vartheta) = \frac{1}{\kappa_j} \sum_{e=(v_j, v_j) \in A_0} \cos(\tau(e), \vartheta), \quad \text{if a loop } (v_j, v_j) \in A_0,$$

$$0, \quad \text{if } (v_j, v_j) \notin A_0.$$

ii) Let $\Delta^{(1)}(\vartheta)$ be the Floquet matrix for $\Delta$ defined by (1.7) in another coordinate system with the origin $O_1$. Then the matrices $\Delta^{(1)}(\vartheta)$ and $\Delta(\vartheta)$ are unitarily equivalent for each $\vartheta \in T^2$.

iii) The spectrum $\sigma(\Delta(\vartheta))$ is contained in $[-1, 1]$ for each $\vartheta \in T^2$.

iv) The entry $\Delta_{jk}(\cdot)$ is constant iff there is no bridge $(v_j, v_k) \in A_0$.

v) The point 1 is never a flat band of $\Delta$.

vi) The Floquet matrix $\Delta(\cdot)$ has at least one non-constant entry $\Delta_{jk}(\cdot)$ for some $j \leq k$.

vii) A fundamental graph $\Gamma_0$ is bipartite iff the spectrum $\sigma(\Delta(\vartheta))$ is symmetric with respect to the point zero for each $\vartheta \in T^2$. 

for each multiplicity. We will call if the operator $\Delta$ has

They can be enumerated in decreasing order (counting multiplicities)

and they are labeled by

$\Delta(\vartheta)$ for any $\vartheta \in T$. Due to [HN09] we have that

Thus, we can define the multiplicity of a flat

because they connect vertices with different integer parts in the sense of the identity (1.2).

3) There are always some bridges on a graph $\Gamma$, if $\Gamma$ is connected. Otherwise, if there are no bridges on some graph $\Gamma$, then this graph is not connected and is a union of infinitely many connected components. In this case the spectrum of the Laplacian on $\Gamma$ consists of only a finite number of eigenvalues with infinite multiplicity. Due to Theorem 1.4 i, it is impossible. The presence of bridges on the graph ”enlarges” the spectrum of the Laplacian.

Remark. 1) The representations (1.7), (1.8) are new. Recall that the existence of $\Delta(\vartheta)$ was proved in [HS03].

2) The graph $\Gamma$ with $\nu = 5$ on Figure 1 has the following bridges

$(v_1, v_2 + a_1), (v_1, v_3 + a_2), (v_3, v_2 + a_1), (v_3, v_4 + a_1 + a_2)$

because they connect vertices with different integer parts in the sense of the identity (1.2).

1.4. Spectral properties of Laplacians. Theorem 1.1 and standard arguments (see Theorem XIII.85 in [RS78]) describe the spectrum of the Laplacian. Each Floquet $\nu \times \nu$ matrix $\Delta(\vartheta), \vartheta \in T^2$, has $\nu$ eigenvalues $\lambda_n(\vartheta), n = 1, \ldots, \nu$. They are real and locally analytic functions on the torus $T^2$, since $\Delta(\vartheta)$ is self-adjoint and analytic in $\vartheta$ on the torus $T^2$. The parameter $\vartheta$ is called quasimomentum. If some $\lambda_n(\cdot) = C_n = \text{const}$ on some set $B \subset T^2$ of positive Lebesgue measure, then the operator $\Delta$ on $\Gamma$ has the eigenvalue $C_n$ with infinite multiplicity. We will call $C_n$ a flat band. Due to [HN09] we have that $\lambda_\nu$ is an eigenvalue of $\Delta$ iff $\lambda_\nu$ is an eigenvalue of $\Delta(\vartheta)$ for any $\vartheta \in T^2$. Thus, we can define the multiplicity of a flat band by: an eigenvalue $\lambda_\nu$ of $\Delta$ has the multiplicity $m$ iff $\lambda_\nu = \text{const}$ is an eigenvalue of $\Delta(\vartheta)$ for each $\vartheta \in T^2$ with the multiplicity $m$ (except maybe for a finite number of $\vartheta \in T^2$). Thus if the operator $\Delta$ has $r \geq 0$ flat bands, then we denote them by

$$
\mu_j = \lambda_{\nu-j+1}(\vartheta), \quad j \in \mathbb{N}_r,
$$

and they are labeled by

$$
\mu_1 \leq \mu_2 \leq \ldots \leq \mu_r, \tag{1.9}
$$

counting multiplicities. Thus, all other eigenvalues $\lambda_n(\vartheta), n = 1, \ldots, \nu - r$ are not constant. They can be enumerated in decreasing order (counting multiplicities)

$$
\lambda_{\nu-r}(\vartheta) \leq \lambda_{\nu-r-1}(\vartheta) \leq \ldots \leq \lambda_1(\vartheta), \quad \forall \vartheta \in T^2. \tag{1.10}
$$
Each $\lambda_n(\vartheta)$ is a piecewise analytic function on the torus $\mathbb{T}^2$ and defines a dispersion relation. Its plot is a dispersion curve. Define the spectral bands $\sigma_n$ by

$$\sigma_n = [\lambda_n^-, \lambda_n^+] = \lambda_n(\mathbb{T}^2), \quad \text{where} \quad \lambda_n^- = \min_{\vartheta \in \mathbb{T}^2} \lambda_n(\vartheta), \quad \lambda_n^+ = \max_{\vartheta \in \mathbb{T}^2} \lambda_n(\vartheta), \quad \forall n \in \mathbb{N}_\nu. \quad (1.11)$$

For each $n = 1, \ldots, \nu - r$ we have $\lambda_n^+ < \lambda_n^-$ and the spectral band $\sigma_n$ is open (non-degenerate). For each $n = \nu - r + 1, \ldots, \nu$ we have $\lambda_n^+ = \mu_r$ and the spectral band $\sigma_n$ is close (degenerate). If $\lambda_{n+1}^+ < \lambda_n^-$ for some $n \in \mathbb{N}_{\nu-r-1}$, then the interval $(\lambda_{n+1}^+, \lambda_n^-)$ is called a gap.

Thus the spectrum of the operator $\Delta$ on the periodic graph $\Gamma$ has the form

$$\sigma(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta),$$

$$\sigma_{ac}(\Delta) = \bigcup_{n=1}^{\nu-r} \sigma_n, \quad \sigma_{fb}(\Delta) = \{\mu_1, \ldots, \mu_r\}, \quad (1.12)$$

where $\sigma_{ac}(\Delta)$ is the absolutely continuous spectrum and $\sigma_{fb}(\Delta)$ is a set of all flat bands (eigenvalues with infinite multiplicity). We now describe precisely all bands for specific graphs.

**Theorem 1.2.** i) Let all bridges of the fundamental graph $\Gamma_0$ be loops, i.e., the indices of all edges $(v_j, v_k)$, $1 \leq j < k \leq \nu$, of $\Gamma_0$ be zero. Then

$$\lambda_n^+ = \lambda_n(0), \quad \forall n \in \mathbb{N}_\nu. \quad (1.13)$$

Moreover, the eigenvalue $\lambda_n(0)$ is a flat band of $\Delta$ iff $\lambda_n^+ = \lambda_n^-$. The multiplicity of the flat band $\mu = \lambda_n^+$ of the Laplacian $\Delta$ is the multiplicity of $\lambda_n^+$ as the eigenvalue of the operator $\Delta(0)$.

ii) Let, in addition, $\cos(\tau(e), \vartheta_0) = -1$ for all bridges $e \in E_0$ and some $\vartheta_0 \in \mathbb{T}^2$. Then

$$\sigma_n = [\lambda_n^-, \lambda_n^+], \quad \lambda_n^- = \lambda_n(\vartheta_0), \quad \forall n \in \mathbb{N}_\nu. \quad (1.14)$$

iii) Let $\Gamma$ be a bipartite periodic graph, satisfying the condition i) ($\Gamma_0$ is not bipartite, since there is a loop on $\Gamma_0$). Then $\lambda_n^\pm$ are the eigenvalues of the matrix $\pm \Delta(0)$.

**Remark.** 1) $\lambda_n^+, n \in \mathbb{N}_\nu$, are the eigenvalues of the operator $\Delta(0)$ defined by (1.7) that is the Laplacian on the fundamental graph $\Gamma_0$. The case of item ii) is similar to the case of $N$-periodic Jacobi matrices on the lattice $\mathbb{Z}$ (and for Hill operators). The spectrum of these operators is absolutely continuous and is a union of spectral bands, separated by gaps. The endpoints of the bands are so-called $2N$-periodic eigenvalues.

2) For some classes of graphs one can easily determine $\vartheta_0$ such that $\cos(\tau(e), \vartheta_0) = -1$ for all bridges $e \in A_0$. Let $\tau(e) = (\tau_1(e), \tau_2(e)) \in \mathbb{Z}^2$ be the index of a bridge $e$ of $\Gamma_0$. Then

$$\vartheta_0 = \begin{cases} 
(\pi, 0), & \text{if } \tau_1(e) \text{ is odd for all bridges } e \in A_0 \\
(0, \pi), & \text{if } \tau_2(e) \text{ is odd for all bridges } e \in A_0 \\
(\pi, \pi), & \text{if } \tau_1(e) + \tau_2(e) \text{ is odd for all bridges } e \in A_0
\end{cases}. \quad (1.15)$$

3) The fundamental graph $\Gamma_0$ on Figure 2 has only 4 oriented bridges, which are loops with indices $(0, \pm 1)$ and $(\pm 1, 0)$. All other edges of $\Gamma_0$ have zero indices. Thus, $\tau_1(e) + \tau_2(e)$ is odd for all bridges $e \in A_0$ and according to Theorem 1.2 i–ii and the identity (1.14) the spectrum of the Laplacian on $\Gamma$, see Figure 2, has the form $\sigma(\Delta) = \bigcup_{n=1}^r \sigma_n, \quad \sigma_n = [\lambda_n(\pi, \pi), \lambda_n(0)]$.

We estimate the spectrum $\sigma(\Delta)$ of the Laplacian.
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Figure 2. a) The fundamental graph $\Gamma_0$; b) the periodic graph $\Gamma$.

Theorem 1.3. i) The Lebesgue measure $|\sigma(\Delta)|$ of the spectrum of the Laplacian $\Delta$ satisfies

$$|\sigma(\Delta)| \leq \sum_{n=1}^{\nu-r} |\sigma_n| \leq 2 \sum_{j,k=1}^{\nu} \frac{b_{jk}}{\zeta_j \zeta_k},$$

(1.16)

where $b_{jk}$ is the number of bridges $(v_j, v_k)$ on the fundamental graph $\Gamma_0$, $\zeta_j$ is the degree of $v_j$. Moreover, this estimate is sharp, i.e., estimates (1.16) become identities for some graphs.

ii) For any $\nu \geq 2$ there exists a periodic graph such that $|\sigma(\Delta)| = \frac{8}{\nu+3}$. In particular, $|\sigma(\Delta)| \to 0$ as $\nu \to \infty$.

iii) Let a fundamental graph $\Gamma_0$ be bipartite. If $\nu > 1$ is odd, then $\mu = 0$ is a flat band of $\Delta$.

Remark. 1) The estimate (1.16) is effective for the case

$$\sum_{j,k=1}^{\nu} \frac{b_{jk}}{\zeta_j \zeta_k} < 1.$$ (1.17)

If the inequality (1.17) does not hold true, then (1.16) yields only a simple estimate $|\sigma(\Delta)| \leq 2$, which follows from the basic property $\sigma(\Delta) \subset [-1, 1]$. The inequality (1.17) holds true when for each vertex $v \in V_0$ the ratio of the number of bridges starting at $v$ to the degree of $v$ is rather small. Increasing the degree of each vertex and fixing the bridges we can construct a graph such that the Lebesgue measure of the spectrum of the Laplacian is arbitrarily small.

2) The total length of spectral bands depends essentially on the bridges on $\Gamma_0$. If we remove the coordinate system, then the number of bridges on the corresponding fundamental graph $\Gamma_0$ is changed in general.

Figure 3. The cycle that is the support of the eigenfunction with the eigenvalue 0. The values of the eigenfunction in the vertices of the support are $0, \pm 1$. 

Figure 3. The cycle that is the support of the eigenfunction with the eigenvalue 0. The values of the eigenfunction in the vertices of the support are 0, ±1.
3) Recall that for each flat band there exists a finitely supported eigenfunction. It is well-known that for some specific case these eigenfunctions can easily be determined using some analysis of graph cycles. For example, if the graph has a cycle of length $4n$, $n \in \mathbb{N}$, and the degree of each second vertices of the cycle is 2, then one can easily construct an eigenfunction supported on this cycle (see Figure 3) with the eigenvalue 0. Theorems 1.2 and 1.3 give the simple sufficient conditions for the existence of flat bands, not based on an analysis of cycles. This is the only test for the existence of flat bands, that we know of.

Now we describe possible positions of flat bands.

**Theorem 1.4.** i) The first spectral band $\sigma_1 = [\lambda_1^-, 1]$ is open, i.e., $\lambda_1^- < 1$. Moreover, the number of flat bands $r < \nu$ and the spectrum of the operator $\Delta$ has at most $\nu - r - 1$ gaps.

ii) Let $\nu \geq 2$. Then

1) There is a graph $\Gamma$, such that the spectrum of the Laplacian on $\Gamma$ has exactly 2 open spectral bands and between them, in the gap, $\nu$ flat bands, counting multiplicity.

2) There is a graph $\Gamma$, such that the spectrum of the Laplacian on $\Gamma$ has $\lfloor \frac{\nu}{2} \rfloor$ different simple flat bands embedded in the absolutely continuous spectrum $[-1, 1]$.

3) There is a graph $\Gamma$, such that a point $-\frac{1}{2}$ is a flat band of the Laplacian on $\Gamma$ (the Kagome lattice, see subsection 5.3) and lies at the endpoint of the absolutely continuous spectrum $\sigma_{ac}(\Delta) = [-\frac{1}{2}, 1]$.

There is an open problem: does there exist a graph with any $\nu \geq 2$ vertices in the fundamental graph such that the spectrum of the Laplacian on $\Gamma$ has only 1 spectral band and $\nu - 1$ flat bands, counting multiplicity?

1.5. **Perturbations of the hexagonal lattice.** Let $G = (V, E)$ be the hexagonal lattice (Figure 4a). The periods of $G$ are the vectors $a_1 = (3/2, \sqrt{3}/2)$, $a_2 = (0, \sqrt{3})$ (the coordinates of $a_1$, $a_2$ are taken in the orthonormal basis $e_1, e_2$). The vertex set and the edge set are given by

$$V = \mathbb{Z}^2 \cup \left( \mathbb{Z}^2 + \left(\frac{1}{3},\frac{1}{3}\right) \right),$$

$$E = \left\{ (p,p + \left(\frac{1}{3},\frac{1}{3}\right)), (p,p + \left(-\frac{2}{3},\frac{1}{3}\right)), (p,p + \left(\frac{1}{3},-\frac{2}{3}\right)) \mid p \in \mathbb{Z}^2 \right\}.$$ 

Recall that the coordinates of all vertices are taken in the basis $a_1, a_2$. The fundamental graph $G_0$ consists of two vertices $v_1$, $v_2$, multiple edges $e_1 = e_2 = e_3 = (v_1, v_2)$ (Figure 4b) and $e_1 = e_2 = e_3$ with the indices

$$\tau(e_1) = \tau(e_1^*) = (0, 0), \quad \tau(e_2) = -\tau(e_2^*) = (1, 0), \quad \tau(e_3) = -\tau(e_3^*) = (0, 1).$$

It is known that the spectrum of the Laplacian $\Delta$ on $G$ has the form $\sigma(\Delta) = \sigma_{ac}(\Delta) = [-1, 1]$.

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**Figure 4.** a) Graphene $G$; b) the fundamental graph $G_0$ of the graphene.
We describe the spectrum under the perturbation of the graphene $G$ by adding an edge to its fundamental graph.

**Theorem 1.5** (Perturbations of the graphene). Let $G'$ be a perturbed graph obtained from the hexagonal lattice $G$ by adding one edge (i.e., two oriented edges $e, \bar{e}$ with the indices $\tau(e) = -\tau(\bar{e}) = (\tau_1, \tau_2)$) to its fundamental graph $G_0$. Then the spectrum $\sigma(\Delta')$ of the Laplacian $\Delta'$ on the perturbed graph $G'$ satisfies:

i) $\sigma(\Delta') = \sigma_{ac}(\Delta')$, i.e., the Laplacian on $G'$ has no flat bands.

ii) $\sigma(\Delta') = [-1, 1]$ iff $G'$ is bipartite.

iii) Let $G'$ be non-bipartite. Then

$$\sigma(\Delta') = [\lambda_2^-, 0] \cup [\lambda_1^-, 1], \quad 0 < \lambda_1^+ \leq \frac{2}{5}, \quad \text{if } \tau_1 - \tau_2 \in 3\mathbb{Z},$$

$$\sigma(\Delta') = [\lambda_2^-, \lambda_2^+] \cup [0, 1], \quad -\frac{1}{5} \leq \lambda_2^+ < 0, \quad \text{if } \tau_1 - \tau_2 \notin 3\mathbb{Z},$$

where

$$-1 < \lambda_2^- \leq -\frac{3}{5}.$$ (1.20)

Moreover, the Lebesgue measure of the spectrum satisfies

$$|\sigma(\Delta')| \geq \frac{6}{5}, \quad \text{if } \tau_1 - \tau_2 \in 3\mathbb{Z};$$

$$|\sigma(\Delta')| \geq \frac{7}{5}, \quad \text{if } \tau_1 - \tau_2 \notin 3\mathbb{Z}. \quad (1.21)$$

**Remark.** This theorem shows that if the perturbation keeps the bipartition of the graph, then the spectrum of the operator does not change. If the added edge breaks the bipartition of the graph, then there appears a gap in the spectrum and the lower point of the spectrum moves to the right.

We present the plan of our paper. In section 2 we prove Theorem 1.1 about the decomposition of the Laplacian into a direct integral and prove some basic properties of the Floquet operator. In section 3 we estimate the Lebesgue measure of the spectrum in terms of geometric parameters of the graph in Theorem 1.3. In sections 4 and 5 we consider the periodic graphs with one, two and three vertices in the fundamental graph. In particular, we describe the spectrum of the Laplacian on the square lattice and the hexagonal lattice, perturbed by adding edges to its fundamental graphs. In Appendix we recall some well-known properties of matrices and give some auxiliary statements, needed to prove our main results.

2. **Proof of Theorem 1.1**

In Appendix we recall some well-known properties of matrices, which will be used below. Also we prove that the fundamental graph of a periodic graph is finite and show that a bipartite periodic graph has a bipartite fundamental graph. Properties of an edge index, needed to prove our main results, are given in Appendix.

**Proof of Theorem 1.1**

Let $\mathcal{H}' = L^2(T^2, \frac{d\rho}{(2\pi)^2}, \mathcal{H}) = \int_{T^2}^{\mathbb{H}} \frac{d\rho}{(2\pi)^2}$ be a constant fiber direct integral, where $\mathcal{H} = \mathbb{C}^\nu$. It is the Hilbert space of square integrable functions $f : T^2 \to \mathbb{C}^\nu$, 
equipped with the norm
\[ \|f\|_{\mathcal{H}}^2 = \int_{T^2} \|f(\vartheta)\|_{C^0}^2 \frac{d\vartheta}{(2\pi)^2}. \]

Let \( \mathcal{S}(V) \) be the set of all finitely supported functions \( f \in \ell^2(V) \). Recall that \( v_1, \ldots, v_{\nu} \) are the vertices of the fundamental graph \( \Gamma_0 \). We identify the vertices of \( \Gamma_0 \) with the vertices of the periodic graph \( \Gamma \) from the set \([0,1]^2\). We define the operator \( U : \ell^2(V) \to \mathcal{H} \) by
\[ (Uf)_n(\vartheta) = \sum_{p \in \mathbb{Z}^2} e^{-i(p,\vartheta)} f(v_n + p), \quad (\vartheta, n) \in T^2 \times \mathbb{N}_\nu. \] (2.1)

Using standard arguments (see pp. 290–291 in [RS78]), we will show that \( U \) is well defined for \( f \in \mathcal{S}(V) \) and uniquely extendable to a unitary operator.

For \( f \in \mathcal{S}(V) \) the sum (2.1) is clearly convergent. For such functions \( f \) we compute
\[ \|Uf\|_{\mathcal{H}}^2 = \int_{T^2} \|(Uf)(\vartheta)\|_{C^0}^2 \frac{d\vartheta}{(2\pi)^2} \]
\[ = \int_{T^2} \sum_{n=1}^{\nu} \left( \sum_{p \in \mathbb{Z}^2} e^{-i(p,\vartheta)} f(v_n + p) \right)^2 \left( \sum_{q \in \mathbb{Z}^2} e^{i(q,\vartheta)} \bar{f}(v_n + q) \right) \frac{d\vartheta}{(2\pi)^2} \]
\[ = \sum_{n=1}^{\nu} \left( f(v_n + p) \int_{T^2} e^{-i(p,q,\vartheta)} \frac{d\vartheta}{(2\pi)^2} \right)^2 \]
\[ = \sum_{n=1}^{\nu} |f(v_n + p)|^2 = \sum_{v \in V} |f(v)|^2 = \|f\|_{\ell^2(V)}^2. \]

Here we have used the identity \( V = \{ v_n + p : (n, p) \in \mathbb{N}_\nu \times \mathbb{Z}^2 \} \). Then \( U \) is well defined and has a unique extension to an isometry. To see that \( U \) is onto \( \mathcal{H} \) we compute \( U^* \). We note that each vertex \( v \in V \) can be written as
\[ v = v_n + p \quad \text{for some} \quad (n, p) \in \mathbb{N}_\nu \times \mathbb{Z}^2 \] (2.2)
and this representation is unique. Let \( g = (g_n(\vartheta) : T^2 \to \mathbb{C})_{n \in \mathbb{N}_\nu} \in \mathcal{H} \). Then we define
\[ (U^*g)(v) = \int_{T^2} e^{i(p,\vartheta)} g_n(\vartheta) \frac{d\vartheta}{(2\pi)^2}, \quad v \in V, \]
where \( p \) and \( n \) are defined by (2.2). A direct computation shows that this is indeed the formula for the adjoint of \( U \). Moreover,
\[ \|U^*g\|_{\ell^2(V)}^2 = \sum_{v \in V} |(U^*g)(v)|^2 = \sum_{n=1}^{\nu} \sum_{p \in \mathbb{Z}^2} |(U^*g)(v_n + p)|^2 = \sum_{n=1}^{\nu} \sum_{p \in \mathbb{Z}^2} \left| \int_{T^2} e^{i(p,\vartheta)} g_n(\vartheta) \frac{d\vartheta}{(2\pi)^2} \right|^2 \]
\[ = \sum_{n=1}^{\nu} \int_{T^2} |g_n(\vartheta)|^2 \frac{d\vartheta}{(2\pi)^2} = \sum_{n=1}^{\nu} \sum_{p \in \mathbb{Z}^2} \int_{T^2} |g_n(\vartheta)|^2 \frac{d\vartheta}{(2\pi)^2} = \|g\|_{\mathcal{H}}^2, \]
where we have used the Parseval relation for the Fourier series.
For $f \in \mathcal{S}(V)$ and $j \in \mathbb{N}_\nu$ we obtain

$$(U \Delta f)_j(\vartheta) = \sum_{p \in \mathbb{Z}^2} e^{-i(\vartheta, \vartheta)} (\Delta f)(v_j + p) = \sum_{p \in \mathbb{Z}^2} e^{-i(\vartheta, \vartheta)} \frac{1}{\sqrt{\kappa_j}} \sum_{(v_j + p, u) \in \mathcal{A}} \frac{1}{\sqrt{\kappa_u}} f(u)$$

$$= \sum_{p \in \mathbb{Z}^2} e^{-i(\vartheta, \vartheta)} \frac{1}{\sqrt{\kappa_j}} \sum_{(v_j + p, u) \in \mathcal{A}} \frac{1}{\sqrt{\kappa_u}} f(v_j + p + \tau(e))$$

$$= \sum_{k=1}^{\nu} \sum_{e=(v_j, v_k) \in \mathcal{A}_0} \frac{1}{\sqrt{\kappa_j \kappa_k}} e^{i(\vartheta(e), \vartheta)} \sum_{p \in \mathbb{Z}^2} e^{-i(p + \tau(e), \vartheta)} f(v_k + p + \tau(e))$$

$$= \sum_{k=1}^{\nu} \sum_{e=(v_j, v_k) \in \mathcal{A}_0} \frac{1}{\sqrt{\kappa_j \kappa_k}} e^{i(\vartheta(e), \vartheta)} (\Delta f)(v_k) = \sum_{k=1}^{\nu} \Delta_{jk}(\vartheta)(U f)(\vartheta), \quad (2.3)$$

where $\Delta_{jk}(\vartheta)$ are defined by (1.7). The identity (2.3) yields $(U \Delta f)(\vartheta) = \Delta(\vartheta)(U f)(\vartheta)$, where $\Delta(\vartheta) = \{\Delta_{jk}(\vartheta)\}$. Thus, we obtain

$$U \Delta U^{-1} = \frac{1}{(2\pi)^2} \int_{T^2} \Delta(\vartheta) \, d\vartheta.$$ 

Using Proposition 6.3.i we can write the entries $\Delta_{jj}$ of the matrix $\Delta(\vartheta)$ in the form (1.8). Thus, the statement i) has been proved.

In order to prove ii) – vii), we discuss needed properties of the Floquet matrix $\Delta(0)$.

**Proposition 2.1.** The matrix $\Delta(0)$ is the Laplacian on the fundamental graph $\Gamma_0$ and has the form

$$\Delta(0) = \{\Delta_{jk}(0)\}, \quad \Delta_{jk}(0) = \frac{\zeta_{jk}}{\sqrt{\kappa_j \kappa_k}}, \quad \forall (j, k) \in \mathbb{N}_\nu^2, \quad (2.4)$$

where $\zeta_{jk} \geq 1$ is the multiplicity of the edge $(v_j, v_k) \in \mathcal{A}_0$ and $\zeta_{jk} = 0$ if $(v_j, v_k) \notin \mathcal{A}_0$. Moreover, they satisfy

$$\zeta_j = \sum_{k=1}^{\nu} \zeta_{jk} \geq 1, \quad \forall j \in \mathbb{N}_\nu. \quad (2.5)$$

**Proof.** The identity (2.4) for the matrix $\Delta(0)$ is obtained by a direct calculation of its entries using the formula (1.7). From the definition (1.5) we deduce that the Laplacian on $\Gamma_0$ has the form (2.4). The definition of the degree $\zeta_j$ of the vertex $v_j$ as the number of oriented edges starting at $v_j$ gives (2.5). Since the graph $\Gamma$ is connected, $\zeta_j \geq 1$. □

**Proof of Theorem 1.1.** ii) – vii). ii) Recall that we identify the vertices $v_1, \ldots, v_\nu$ of the fundamental graph $\Gamma_0$ with the vertices of the periodic graph $\Gamma$ from the set $[0, 1)^2$ in the coordinate system with the origin $O$. Proposition 6.3.iv gives that for each $(v_j, v_k) \in \mathcal{A}_0$

$$\tau(1)(v_j, v_k) = \tau(v_j, v_k) + p_k - p_j, \quad p_k = [v_k - b], \quad p_j = [v_j - b], \quad (2.6)$$

where $\tau(1)(v_j, v_k)$ is the index of the edge $(v_j, v_k)$ in the coordinate system with the origin $O_1$, $b = O\tilde{O}_1$. Using (2.6) we rewrite the entries of $\Delta^{(1)}(\vartheta)$ defined by (1.7) in the form

$$\Delta^{(1)}_{jk}(\vartheta) = \frac{1}{\sqrt{\zeta_j \zeta_k}} \sum_{e=(v_j, v_k) \in \mathcal{A}_0} e^{i(\tau(1)(e), \vartheta)} = \frac{e^{i(p_k - p_j, \vartheta)}}{\sqrt{\zeta_j \zeta_k}} \sum_{e=(v_j, v_k) \in \mathcal{A}_0} e^{i(\tau(e), \vartheta)} = e^{i(p_k - p_j, \vartheta)} \Delta_{jk}(\vartheta).$$
We define the diagonal $\nu \times \nu$ matrix
$$
\mathcal{U}(\vartheta) = \text{diag} \left( e^{-i|\mu, \vartheta|}, \ldots, e^{-i|\nu, \vartheta|} \right), \quad \forall \vartheta \in \mathbb{T}^2.
$$
A direct calculation yields
$$
\mathcal{U}(\vartheta) \Delta(\vartheta) \mathcal{U}^{-1}(\vartheta) = \Delta^{(1)}(\vartheta), \quad \forall \vartheta \in \mathbb{T}^2.
$$
Thus, for each $\vartheta \in \mathbb{T}^2$ the matrices $\Delta^{(1)}(\vartheta)$ and $\Delta(\vartheta)$ are unitarily equivalent.

iii) Since $\Delta(0)$ is the Laplacian on $\Gamma_0$, $\|\Delta(0)\| = 1$. From this fact and the formula $(1.7)$ it follows that the entries of each matrix $\Delta(\vartheta)$, $\vartheta \in \mathbb{T}^2$, satisfy
$$
|\Delta_{jk}(\vartheta)| \leq \Delta_{jk}(0) \leq 1, \quad \forall (j, k) \in \mathbb{N}_\nu^2.
$$
Then, Proposition 6.1.i implies that the spectral radius $\rho(\Delta(\vartheta))$ satisfies
$$
\rho(\Delta(\vartheta)) \leq \rho(\Delta(0)) = \|\Delta(0)\| = 1,
$$
which yields $\sigma(\Delta(\vartheta)) \subset [-1, 1]$ for each $\vartheta \in \mathbb{T}^2$.

iv) This statement follows immediately from $(1.7)$ and the definition of the bridge.

v) We prove by the contradiction. Let the point 1 be an eigenvalue of the Laplacian on a graph $\Gamma$. Then there exists an eigenfunction $0 \neq f \in l^2(V)$ with eigenvalue 1 and with a finite support $B \subset V$ (see Theorem 3.2 in [HN09]). Let $M = \max_{v \in B} f(v) = \frac{f(\tilde{v})}{\sqrt{\kappa_{\tilde{v}}}}$ for some $\tilde{v} \in B$. Thus, we have
$$
f(\tilde{v}) = (\Delta f)(\tilde{v}) = \frac{1}{\sqrt{\kappa_{\tilde{v}}}} \sum_{(\tilde{v}, u) \in A_0} \frac{1}{\sqrt{\kappa_u}} f(u) \leq \frac{1}{\sqrt{\kappa_{\tilde{v}}}} \sum_{(\tilde{v}, u) \in A_0} \frac{1}{\sqrt{\kappa_u}} f(\tilde{v}) = M \sqrt{\kappa_{\tilde{v}}} = f(\tilde{v}).
$$
We conclude that the inequality has to be an equality, and therefore
$$
\frac{f(u)}{\sqrt{\kappa_u}} = \frac{f(\tilde{v})}{\sqrt{\kappa_{\tilde{v}}}}, \quad \forall u \sim \tilde{v}.
$$
Repeating this argument until we reach a vertex from $V \setminus B$, we conclude that $f = 0$. We have a contradiction.

vi) We prove by the contradiction. Let all entries $\Delta_{jk}(\cdot)$, $1 \leq j \leq k \leq \nu$, be constant. Due to the self-adjointness of the matrix $\Delta(\vartheta)$ all its entries are constants. Then the point 1 is an eigenvalue of $\Delta(\vartheta) = \Delta(0)$ for each $\vartheta \in \mathbb{T}^2$, i.e., the point 1 is an eigenvalue of $\Delta$ with infinite multiplicity. This contradicts item v).

vii) Let $\Gamma_0$ be a bipartite fundamental graph with the parts $V_1 = \{v_1, \ldots, v_k\}$ and $V_2 = \{v_{k+1}, \ldots, v_{\nu}\}$. Since vertices from the same part of $\Gamma_0$ are not adjacent, each matrix $\Delta(\vartheta)$, $\vartheta \in \mathbb{T}^2$, has the form
$$
\Delta(\vartheta) = \begin{pmatrix}
O_{kk} & A(\vartheta) \\
A^*(\vartheta) & O_{\nu-k, \nu-k}
\end{pmatrix},
$$
for some $k \times (\nu - k)$ matrix $A(\vartheta)$. Here $O_{jk}$ is the zero $j \times k$ matrix. We define the matrix
$$
\mathcal{U} = \begin{pmatrix}
I_k & O_{k, \nu-k} \\
O_{\nu-k, k} & -I_{\nu-k}
\end{pmatrix} = \mathcal{U}^{-1},
$$
where $I_k$ is the identity $k \times k$ matrix. By a direct calculation, one can verify that
$$
\mathcal{U}\Delta(\vartheta)\mathcal{U}^{-1} = -\Delta(\vartheta),
$$
which yields that $\sigma(\Delta(\vartheta))$ is symmetric with respect to 0.
Conversely, since $\Delta(0)$ is the Laplacian on $\Gamma_0$ and its spectrum is symmetric with respect to 0, $\Gamma_0$ is bipartite due to the main property 4) of the Laplacian. ■

In the following theorem we show unitary equivalence of Laplacians on graphs with multiple indices.

**Theorem 2.2.** Let $\Gamma_0 = (V_0, E_0)$ and $\Gamma'_0 = (V'_0, E'_0)$ be fundamental graphs of periodic graphs $\Gamma$ and $\Gamma'$, respectively, satisfying the following conditions:

1) $\Gamma_0$ and $\Gamma'_0$ are isomorphic, i.e., there exists a bijection $\varphi : V_0 \rightarrow V'_0$ that preserves the adjacency of vertices;

2) there exists $n \in \mathbb{N}$ such that for any $e = (u, v) \in A_0$ $\tau(e) = n \tau(e')$, where $e' = (\varphi(u), \varphi(v)) \in A'_0$.

Then the Laplace operators on $\Gamma$ and $\Gamma'$ are unitarily equivalent.

**Proof.** Denote the Laplace operators on $\Gamma$ and $\Gamma'$ by $\Delta$ and $\Delta'$, respectively, and the Floquet matrices for $\Gamma$ and $\Gamma'$ by $\Delta(\vartheta)$ and $\Delta'(\vartheta')$, $\vartheta \in \mathbb{T}^2$, respectively. Since $\Gamma_0$ and $\Gamma'_0$ are isomorphic and the indices of their edges satisfy the identity (2.9), $\Delta(\vartheta) = \Delta'(n \vartheta)$, $\forall \vartheta \in \mathbb{T}^2$.

Then using (1.6) we obtain

$$U \Delta U^{-1} = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \Delta(\vartheta) d\vartheta = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \Delta'(n \vartheta) d\vartheta.$$ 

If we make the change of variables $\vartheta' = n \vartheta$, $\vartheta' \in \mathbb{T}^2 = \mathbb{R}^2/(2\pi n \mathbb{Z})^2 = [-\pi n; \pi n]^2$, then we rewrite the last identity in the form

$$U \Delta U^{-1} = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \Delta'(n \vartheta) d\vartheta = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \Delta'(n \vartheta) d\vartheta = U' \Delta' U'^{-1}$$

for some unitary operator $U'$. Thus, $\Delta$ and $\Delta'$ are unitarily equivalent. ■

### 3. Spectral analysis of Laplacians

Below we need the following representation of the Floquet matrix $\Delta(\vartheta)$, $\vartheta \in \mathbb{T}^2$:

$$\Delta(\vartheta) = \Delta_0 + \tilde{\Delta}(\vartheta), \quad \Delta_0 = \frac{1}{(2\pi)^2} \int_{\mathbb{T}^2} \Delta(\vartheta) d\vartheta.$$ (3.1)

From (3.1), (1.7) it follows that the entries of the matrix $\tilde{\Delta}(\vartheta) = \{\tilde{\Delta}_{jk}(\vartheta)\}$ have the form

$$\tilde{\Delta}_{jk}(\vartheta) = \frac{1}{\sqrt{\kappa_j \kappa_k}} \sum_{e \in \mathcal{A}_0, \varphi(e) \neq 0} e^{i \langle \tau(e), \vartheta \rangle}.$$ (3.2)

**Proof of Theorem 1.2.** i)-ii) Due to (3.1) we have $\Delta(\vartheta) = \Delta_0 + \tilde{\Delta}(\vartheta)$, where the matrix $\tilde{\Delta}(\vartheta)$, $\vartheta \in \mathbb{T}^2$, has the form

$$\tilde{\Delta}(\vartheta) = \text{diag}(\tilde{\Delta}_{11}, \ldots, \tilde{\Delta}_{\nu\nu})(\vartheta).$$
The identity (3.2) for $\tilde{\Delta}_{jj}$ has the form
\[
\tilde{\Delta}_{jj}(\vartheta) = \frac{1}{K_j} \sum_{e \in \mathcal{A}_j \setminus \{\vartheta(\mathbf{e})\}} \cos(\tau(\mathbf{e}), \vartheta),
\]
which yields
\[
\tilde{\Delta}_{jj}(\vartheta_0) \leq \tilde{\Delta}_{jj}(\vartheta) \leq \tilde{\Delta}_{jj}(0), \quad \forall j \in \mathbb{N}_\nu.
\]
Then $\Delta(\vartheta_0) \leq \Delta(\vartheta) \leq \Delta(0)$ and we have
\[
\Delta(\vartheta_0) = \Delta_0 + \Delta(\vartheta_0) \leq \Delta(\vartheta) = \Delta_0 + \Delta(\vartheta) \leq \Delta_0 + \Delta(0) = \Delta(0).
\]
Then Proposition 6.1.ii gives
\[
\lambda_n(0) \leq \lambda_n(\vartheta) \leq \lambda_n(0), \quad \forall (\vartheta, n) \in \mathbb{T}^2 \times \mathbb{N}_\nu.
\]
Thus, $\lambda_n^+ = \max_{\vartheta \in \mathbb{T}^2} \lambda_n(\vartheta) = \lambda_n(0)$, $\lambda_n^- = \min_{\vartheta \in \mathbb{T}^2} \lambda_n(\vartheta) = \lambda_n(\vartheta_0)$. The last statement of the item i) follows from the definition of flat bands.

iii) Since $\Gamma$ is bipartite, the spectrum of the Laplacian on $\Gamma$ is symmetric with respect to zero. From item i) it follows that $\lambda_1(0) \geq \ldots \geq \lambda_\nu(0)$ are the upper endpoints of the spectral bands. Then $-\lambda_1(0) \leq \ldots \leq -\lambda_\nu(0)$ are the lower endpoints of the spectral bands. Thus, the endpoints of the spectral bands $\lambda_n^\pm$, $n \in \mathbb{N}_\nu$, are the eigenvalues of the matrix $\pm \Delta(0)$.

Lemma 3.1. Let all edges $(v_j, v_k) \in \mathcal{A}_0$, $1 \leq j, k \leq \nu - 1$, of the fundamental graph $\Gamma_0$ have zero indices. Then

i) The Floquet matrix $\Delta(\vartheta)$ has the form
\[
\Delta(\vartheta) = \left( \begin{array}{cc} A & y(\vartheta) \\ y^*(\vartheta) & a(\vartheta) \end{array} \right),
\]
where for each $\vartheta \in \mathbb{T}^2$ the entry $y(\vartheta) \in \mathbb{C}^{\nu-1}$ is a vector and $a(\vartheta)$ is a real number, $A$ is a self-adjoint $(\nu - 1) \times (\nu - 1)$ matrix not depending on $\vartheta$.

ii) If $A$ has an eigenvalue $\mu$ with multiplicity $\geq 2$, then $\mu$ is a flat band of the Laplacian $\Delta$ on the periodic graph $\Gamma$.

Proof. i) This follows from Theorem 1.1.iv and the self-adjointness of $\Delta(\vartheta)$.

ii) Let the matrix $A$ have a multiple eigenvalue $\mu$. Due to Proposition 6.1.iii, there exists an eigenvalue $\lambda(\vartheta)$ of $\Delta(\vartheta)$ satisfying $\mu \leq \lambda(\vartheta) \leq \mu$ for all $\vartheta \in \mathbb{T}^2$, which yields $\lambda(\cdot) = \mu = \text{const}$, i.e., $\mu$ is a flat band of the Laplacian $\Delta$ on $\Gamma$.

Proposition 3.2. Let $\Gamma_0$ be obtained from the fundamental graph $\Gamma_0$ of the square lattice $S$ by adding $\nu - 1$ ($\nu \geq 2$) vertices $v_1, \ldots, v_{\nu-1}$ and $\nu - 1$ unoriented edges $(v_1, v_\nu), \ldots, (v_{\nu-1}, v_\nu)$ with zero indices (see Figure 3), $v_\nu$ is a single vertex of $\Gamma_0$. Then the spectrum of the Laplacian on $\Gamma$ has the form
\[
\sigma(\Delta) = \sigma_{\text{ac}}(\Delta) \cup \sigma_{\text{fb}}(\Delta), \quad \sigma_{\text{fb}}(\Delta) = \{0\},
\]
where the point 0 is a flat band of multiplicity $\nu - 2$ and the absolutely continuous part $\sigma_{\text{ac}}(\Delta)$ has only two spectral bands $\sigma_1$ and $\sigma_2$ given by
\[
\sigma_{\text{ac}}(\Delta) = \sigma_2 \cup \sigma_1, \quad \sigma_1 = -\sigma_2 = [\frac{\nu-1}{\nu+3}, 1].
\]
Proof. The fundamental graph $\Gamma_0$ consists of $\nu \geq 2$ vertices $v_1, v_2, \ldots, v_\nu; \nu - 1$ unoriented edges $(v_1, v_\nu), \ldots, (v_{\nu-1}, v_\nu)$ with zero indices and 2 unoriented loops in the vertex $v_\nu$ with the indices $(\pm 1, 0), (0, \pm 1)$. All bridges of $\Gamma_0$ are loops and the graph $\Gamma$ is bipartite. Then, by Theorem 1.2.iii, the spectrum of the Laplacian is completely defined by the eigenvalues of $\Delta(0)$. According to (2.4) we have

$$\Delta(0) = \begin{pmatrix}
0 & 0 & \cdots & \frac{1}{\sqrt{\nu+3}} \\
0 & 0 & \cdots & \frac{1}{\sqrt{\nu+3}} \\
\cdots & \cdots & \cdots & \cdots \\
\frac{1}{\sqrt{\nu+3}} & \frac{1}{\sqrt{\nu+3}} & \cdots & \frac{4}{\nu+3}
\end{pmatrix}.$$ 

Using the formula (6.1) we obtain

$$\det (\Delta(0) - \lambda I_\nu) = \frac{(-1)\nu \lambda^{\nu-2}}{\nu + 3}((\nu + 3)\lambda^2 - 4\lambda + 1 - \nu).$$

Then the eigenvalues of $\Delta(0)$ have the form

$$\lambda_1(0) = 1, \quad \lambda_2(0) = \ldots = \lambda_{\nu-1}(0) = 0, \quad \lambda_\nu(0) = -\frac{\nu - 1}{\nu + 3}.$$ 

Thus, due to Theorem 1.2.iii, the spectrum of the Laplacian on $\Gamma$ has the form (3.6), (3.7).

Figure 5. (a) The periodic graph $\Gamma$; (b) the fundamental graph $\Gamma_0$; only two loops in the vertex $v_\nu$ are bridges; (c) the spectrum of the Laplacian.

Figure 6. (a) Graph $\Gamma$ obtained by adding two vertices on all edges of the square lattice $S$; (b) the spectrum of the Laplacian.
Proposition 3.3. Let $\Gamma$ be the graph obtained from the square lattice $S$ by adding $N$ vertices on each edge of $S$ (for $N = 2$ see Figure 6). Then the fundamental graph of $\Gamma$ has $\nu = 2N + 1$ vertices and the spectrum of the Laplacian on $\Gamma$ has the form

$$\sigma(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta),$$

where the absolutely continuous part $\sigma_{ac}(\Delta) = [-1, 1]$ consists of $N + 1$ non-degenerate spectral bands $\sigma_1, \ldots, \sigma_{N+1}$,

$$\sigma_{fb}(\Delta) = \left\{ \cos \frac{\pi n}{N + 1} : n = 1, \ldots, N \right\},$$

where each flat band is simple. There are no other flat bands.

Proof. The case $N = 1$ will be considered in Corollary 5.4.

Let $N \geq 2$. The fundamental graph $\Gamma_0$ has $\nu = 2N + 1$ vertices. The matrix $\Delta(\vartheta)$ is given by (3.10), where the $2N \times 2N$ matrix $A$ and the vector $y(\vartheta)$ have the form

$$A = \begin{pmatrix} A_N & \mathbb{O}_{NN} \\ \mathbb{O}_{NN} & A_N \end{pmatrix}, \quad A_N = \frac{1}{2} \begin{pmatrix} 0 & 1 & 0 & \ldots \\ 1 & 0 & 1 & \ldots \\ 0 & 1 & 0 & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

$$y(\vartheta) = (y_1(\vartheta_1), y_2(\vartheta_2))^T, \quad y_s(\vartheta_s) = \frac{1}{2\sqrt{2}} (1, 0, \ldots, e^{i\vartheta_s})^T, \quad s = 1, 2,$$

$$a(\vartheta) = 0.$$ According to (6.1) all eigenvalues of $A_N$ have the form

$$\mu_n = \cos \frac{\pi n}{N + 1}, \quad n = 1, \ldots, N,$$

and they are different. Then the matrix $A$ has $N$ different eigenvalues $\mu_n$ of the multiplicity 2. Thus, due to Lemma 3.1.iii, the Laplacian on $\Gamma$ has at least $N = \left\lfloor \frac{\nu}{2} \right\rfloor$ flat bands $\mu_n, n \in \mathbb{N}_N$.

We describe $\sigma_{ac}(\Delta)$. Identity (6.1) yields

$$\det \left( \Delta(\vartheta) - \lambda I_\nu \right) = \det \left( A - \lambda I_{2N} \right) \left( -\lambda - y^*(\vartheta)(A - \lambda I_{2N})^{-1} y(\vartheta) \right).$$

By a direct calculation we get

$$\det \left( A - \lambda I_{2N} \right) = \mathcal{D}_N^2(\lambda), \quad \mathcal{D}_N(\lambda) = \det \left( A_N - \lambda I_N \right),$$

$$\left( A - \lambda I_{2N} \right)^{-1} = \begin{pmatrix} B & \mathbb{O}_{NN} \\ \mathbb{O}_{NN} & B \end{pmatrix}, \quad B = (A_N - \lambda I_N)^{-1},$$

$$y^*(\vartheta) (A - \lambda I_{2N})^{-1} y(\vartheta) = y_1^*(\vartheta_1) B y_1(\vartheta_1) + y_2^*(\vartheta_2) B y_2(\vartheta_2),$$

$$y_s^*(\vartheta_s) B y_s(\vartheta_s) = \frac{1}{4 \mathcal{D}_N(\lambda)} \left[ \mathcal{D}_{N-1}(\lambda) + \left( -\frac{1}{2} \right)^{N-1} \cos \vartheta_s \right], \quad s = 1, 2.$$

Substituting (3.13), (3.15) and (3.16) into (3.12) we obtain

$$\det \left( \Delta(\vartheta) - \lambda I_\nu \right) = \mathcal{D}_N(\lambda) \left( -\lambda \mathcal{D}_N(\lambda) - \frac{1}{2} \mathcal{D}_{N-1}(\lambda) - \left( -\frac{1}{2} \right)^{N+1} A(\vartheta) \right),$$

where

$$A(\vartheta) = \cos \vartheta_1 + \cos \vartheta_2.$$ 

From the identity (6.2) it follows that

$$\mathcal{D}_N(\lambda) = -\lambda \mathcal{D}_{N-1}(\lambda) - \frac{1}{4} \mathcal{D}_{N-2}(\lambda).$$
Using this formula we rewrite (3.17) in the form
\[
\det \left( \Delta(\vartheta) - \lambda I_\nu \right) = D_N(\lambda) \left( D_{N+1}(\lambda) - \frac{1}{4} D_{N-1}(\lambda) - \left( -\frac{1}{2} \right)^{N+1} A(\vartheta) \right). 
\] (3.18)

The identity (6.2) gives
\[
D_N(\lambda) = \frac{(-1)^N}{2^N} \sin(N + 1) \varphi, 
\] (3.19)
where \( \varphi \) is given by
\[
-\lambda = 2 \cos \varphi. 
\]

The formulas (3.18) and (3.19) give
\[
\det \left( \Delta(\vartheta) - \lambda I_\nu \right) = D_N(\lambda) \left( \frac{1}{2} \right)^{N+1} \left( \frac{\sin(N + 2) \varphi}{\sin \varphi} - \frac{\sin N \varphi}{\sin \varphi} - A(\vartheta) \right) 
\] (3.20)

Then the eigenvalues of the matrix \( \Delta(\vartheta) \) are defined by
\[
D_N(\lambda) = 0, \quad 2 \cos(N + 1) \varphi = A(\vartheta). 
\]
The first identity gives all flat bands of \( \Delta \) defined by (3.11). From the second identity and the fact that the range of the function \( A(\vartheta) \) is \([-2, 2]\) it follows that for any \( \lambda \in [-1, 1] \) there exists \( \vartheta \in \mathbb{T}^2 \) such that
\[
2 \cos(N + 1) \varphi = A(\vartheta), 
\] (3.21)
i.e., each \( \lambda \in [-1, 1] \) is an eigenvalue of \( \Delta(\vartheta) \) for some \( \vartheta \in \mathbb{T}^2 \). Thus, \( \sigma_{ac}(\Delta) = [-1, 1] \).

In order to prove Theorem 1.3, we need the following lemma.

**Lemma 3.4.** Let \( V = \{V_{jk}\} \) be a self-adjoint \( \nu \times \nu \) matrix such that \( \sum_{j,k=1}^{\nu} |V_{jk}| < \infty \). Then the following estimate holds true:
\[
-B \leq V \leq B, 
\]
where \( B = \text{diag}\{B_1, \ldots, B_\nu\}, \quad B_j = \sum_{k=1}^{\nu} |V_{jk}|. \) (3.22)

**Proof.** Let \( a = (j, k), j, k = 1, \ldots, \nu \). We have the identity
\[
V = \sum_{a=(j,k)} I(a) V_a = \sum_{a=(j,k)} Q(a), \quad Q(a) = \frac{1}{2} (I(a) V_a + I^*(a) V_a), \quad (3.23)
\]
where the matrix \( I(a) = \{I_{mn}(a)\}, a = (j,k) \) is given by
\[
I_{mn}(a) = \begin{cases} 1, & \text{if } (m,n) = a \\ 0, & \text{if } (m,n) \neq a. \end{cases}
\]
For any $a = (j, k)$ the following estimate holds true:

$$Q(a) = \frac{1}{2} (I(a) V_a + I^*(a) \tilde{V}_a) \leq |Q(a)|,$$

where

$$|Q(a)| = \sqrt{Q(a) Q^*(a)} = \frac{|V_a|}{2} (I(j, j) + I(k, k)).$$

(3.24)

Summing (3.24), we obtain

$$V = \sum_{a=(j,k)} Q(a) \leq \sum_{a=(j,k)} |Q(a)| = \sum_{a=(j,k)} \frac{1}{2} |V_a| (I(j, j) + I(k, k)) = \sum_{j=1}^{\nu} I(j, j) \sum_{k=1}^{\nu} |V_{jk}| = B,$$

which yields $V \leq B$. Moreover, this yields $-B \leq V$, since $-V \leq B$.  

\[ \blacksquare \]

**Proof of Theorem 1.3.** i) For each $\vartheta \in \mathbb{T}^2$ the estimate (3.22) yields

$$-B(\vartheta) \leq \tilde{\Delta}(\vartheta) \leq B(\vartheta), \quad B(\vartheta) = \text{diag}\{B_1(\vartheta), \ldots, B_\nu(\vartheta)\},$$

(3.25)

$$B_j(\vartheta) = \sum_{k=1}^{\nu} |\tilde{\Delta}_{jk}(\vartheta)|.$$

(3.26)

For all $(j, k, \vartheta) \in \mathbb{N}_0^3 \times \mathbb{T}^2$ it follows from (3.2) that

$$|\tilde{\Delta}_{jk}(\vartheta)| \leq \tilde{\Delta}_{jk}(0) = \frac{b_{jk}}{\sqrt{\kappa_j \kappa_k}}.$$

(3.27)

The estimate (3.27) yields that the entries $B_j(\vartheta)$ defined by (3.26) satisfy

$$B_j(\vartheta) \leq B_j(0), \quad \forall (j, \vartheta) \in \mathbb{N}_\nu \times \mathbb{T}^2,$$

and then

$$B(\vartheta) \leq B(0), \quad \forall \vartheta \in \mathbb{T}^2.$$

Using this estimate we rewrite (3.25) in the form

$$-B(0) \leq \tilde{\Delta}(\vartheta) \leq B(0), \quad B(0) = \text{diag}\{B_1(0), \ldots, B_\nu(0)\}, \quad B_j(0) = \sum_{k=1}^{\nu} \tilde{\Delta}_{jk}(0).$$

(3.28)

We use some arguments from [Ku10]. Combining (3.1) and (3.28), we obtain

$$\Delta_0 - B(0) \leq \Delta(\vartheta) \leq \Delta_0 + B(0).$$

(3.29)

Thus, Proposition 6.1ii implies

$$\lambda_n(\Delta_0 - B(0)) \leq \lambda_n^- \leq \lambda_n(\vartheta) \leq \lambda_n^+ \leq \lambda_n(\Delta_0 + B(0)), \quad \forall \vartheta \in \mathbb{T}^2$$

(3.30)

and then

$$|\sigma(\Delta)| \leq \sum_{n=1}^{\nu} (\lambda_n^+ - \lambda_n^-) \leq \sum_{n=1}^{\nu} (\lambda_n(\Delta_0 + B(0)) - \lambda_n(\Delta_0 - B(0))) = 2 \text{Tr} B(0).$$

(3.31)

Identity in (3.27) and relations (3.28) and (3.31) give

$$|\sigma(\Delta)| \leq 2 \text{Tr} B(0) = 2 \sum_{j=1}^{\nu} B_j(0) = 2 \sum_{j,k=1}^{\nu} \tilde{\Delta}_{jk}(0) = 2 \sum_{j,k=1}^{\nu} \frac{b_{jk}}{\sqrt{\kappa_j \kappa_k}},$$

which yields (1.16). In the proof of ii) we show that the estimates (1.16) are sharp.
ii) Consider the graph $\Gamma$ shown in Figure 5. Proposition 3.2 gives

$$|\sigma(\Delta)| = 2\left(1 - \frac{\nu - 1}{\nu + 3}\right) = \frac{8}{\nu + 3}.$$

(3.32)

On the other hand, we estimate $|\sigma(\Delta)|$ using (1.16). For the graph $\Gamma$ we have

$$\kappa_1 = \ldots = \kappa_{\nu-1} = 1, \quad \kappa_\nu = \nu + 3.$$ 

The fundamental graph $\Gamma_0$ has only 4 oriented bridges, which are the loops in the vertex $v_\nu$. Thus,

$$b_{\nu\nu} = 4; \quad b_{jk} = 0, \quad \forall (j, k) \in \mathbb{N}_\nu \setminus (\nu, \nu).$$

Then the estimate (1.16) for the graph $\Gamma$ has the form

$$|\sigma(\Delta)| \leq \frac{8}{\nu + 3}.$$

(3.33)

Thus, (3.32) and (3.33) show that for the graph $\Gamma$ the estimate (1.16) becomes an identity. The last statement of the item is a direct consequence of (3.32).

iii) For each $\vartheta \in T^2$ the matrix $\Delta(\vartheta)$ has $\nu$ eigenvalues, where $\nu$ is odd. Theorem 1.1.vii gives that the spectrum $\sigma(\Delta(\vartheta))$ is symmetric with respect to 0. Then $0 \in \sigma(\Delta(\vartheta))$ for any $\vartheta \in T^2$. Therefore, $\mu = 0$ is a flat band of $\Delta$.

**Proof of Theorem 1.4.** i) This follows from the facts that the point 1 is never a flat band (see Theorem 1.1.v) and $1 \in \sigma(\Delta)$. The number of open spectral bands of the operator is $\nu - r$. Some of them may overlap. Then the number of gaps between them is at most $\nu - r - 1$.

ii) This statement is a direct consequence of Propositions 3.2, 3.3, 5.5.

4. Perturbations of square lattice

4.1. Fundamental graphs with one vertex. Let a fundamental graph $\Gamma_0 = (V_0, E_0)$ consist of one vertex $v$ and any number of edges. We note that in this case the index $\tau(e)$ of each edge $e \in A$ coincides with the vector of its coordinates in the basis $a_1, a_2$ (the periods of $\Gamma$) and all edges of $\Gamma_0$ are loops. We consider the spectrum of the Laplacian $\Delta$ and describe all isospectral graphs (with one vertex in the fundamental graph) on which the spectrum $\sigma(\Delta) = \sigma_{ac}(\Delta) = [-1, 1]$.

Since $\nu = 1$, we deduce that $\Delta(\vartheta)$ is a scalar function given by

$$\Delta(\vartheta) = \frac{1}{\kappa_v} \sum_{e \in A_0} \cos(\tau(e), \vartheta), \quad \vartheta \in T^2,$$

(4.1)

where $\tau(e) = (\tau_1(e), \tau_2(e))$ is the vector of the coordinates of $e$ in the basis $a_1, a_2$. The spectrum of the operator $\Delta$ on the graph $\Gamma$ has the form

$$\sigma(\Delta) = \sigma_{ac}(\Delta) = [\lambda^-, 1], \quad \lambda^- = \min_{\vartheta \in T^2} \Delta(\vartheta) < 1.$$

(4.2)

**Theorem 4.1.** Let the fundamental graph $\Gamma_0 = (V_0, E_0)$ consist of one vertex $v$ of the degree $\kappa_v$. Then

i) $\sigma(\Delta) = [-1, 1] \Leftrightarrow$ the graph $\Gamma$ is bipartite.

ii) $\sigma(\Delta) = [-1, 1]$ if one of the following conditions holds true:

a) $\tau_1(e)$ is odd for all $e \in A_0$;

b) $\tau_2(e)$ is odd for all $e \in A_0$. 


c) \( \tau_1(e) + \tau_2(e) \) is odd for all \( e \in A_0 \).

**Proof.**  i) The spectrum of the Laplacian consists of one spectral band. The point \(-1 \in \sigma(\Delta)\) iff the graph is bipartite (see the main property 4) of the Laplacian). It gives the required statement.

Item ii) can be proved using Theorem 1.2 and the formula (1.15). But we give another proof (by contradiction). A graph is bipartite iff there are no cycles of odd length in it (see p.105 in [Or62]). Let one of the conditions a) – c) hold true and let the graph \( \Gamma \) be non-bipartite. Then there is a cycle with edges \( e_1, \ldots , e_J \in A \) of odd length \( J \) in it. It gives the identities

\[
\tau_s(e_1) + \ldots + \tau_s(e_J) = 0, \quad s = 1, 2.
\]

But it contradicts all conditions a) – c), because the sum of an odd number \( J \) of odd terms is not 0. Thus, the graph is bipartite and item i) gives \( \sigma(\Delta) = [-1, 1] \).

We consider the spectrum of the Laplacian on the square lattice \( S = (V, E) \), where the vertex set and the edge set are given by

\[
V = \mathbb{Z}^2, \quad E = \{(p, p + e_1), (p, p + e_2) \ \forall p \in \mathbb{Z}^2\},
\]

the orthonormal basis \( e_1, e_2 \) coincides with the periods \( a_1, a_2 \) of \( S \), see Figure 7a. The fundamental graph \( S_0 \) of the square lattice \( S \) consists of one vertex \( v \) and two unoriented edges-loops \( e_1 = e_2 = (v, v) \), see Figure 7b. It is known that the spectrum of the Laplacian \( \Delta \) on \( S \) has the form \( \sigma(\Delta) = \sigma_{ac}(\Delta) = [-1, 1] \).

![Figure 7](image)

**Figure 7.** a) The square lattice \( S \); b) the fundamental graph \( S_0 \).

We describe the spectrum **under the perturbation** of the graph \( S \) by adding some edges to its fundamental graph.

**Theorem 4.2. (Perturbations of the square lattice).** Let \( S' = (\mathbb{Z}^2, E') \) be a perturbed graph obtained from the square lattice \( S \) by adding \( N \) unoriented edges to its fundamental graph \( S_0 \) and let \( A'_0 \) be the set of all oriented edges of the fundamental graph of \( S' \). Then the spectrum \( \sigma(\Delta') \) on the perturbed graph \( S' \) satisfies:

i) \( \sigma(\Delta') = [-1, 1] \; \iff \; \tau_1(e) + \tau_2(e) \) is odd for all \( e \in A'_0 \), where \( (\tau_1(e), \tau_2(e)) \) is the vector of the coordinates of the edge \( e \).

ii) Let we add one oriented edge, i.e., two oriented edges \( e, \bar{e} \) and let \( \tau = (\tau_1, \tau_2) \) be the vector of the coordinates of the edge \( e \), where \( \tau_1 + \tau_2 \) is even. Then \( \sigma(\Delta') = [\lambda^-(\tau), 1] \), where \( \lambda^-(\tau) \) satisfies

\[
-1 < \lambda^-(\tau) \leq -\cos \frac{\pi}{|\tau_1| + |\tau_2| + 1} \leq -\frac{1}{2}.
\]

(4.3)
Moreover, the following asymptotics holds true:

$$\lambda^{-}(\tau) = -1 + \frac{\pi^2}{6|\tau|^2} + \frac{O(1)}{|\tau|^4} \quad \text{as} \quad |\tau| \to \infty.$$  \hspace{1cm} (4.4)

In particular, if $\tau_1 = \tau_2$, then

$$\lambda^{-}(\tau) = \frac{1}{3} \min_{\varphi \in [0, \pi]} (2 \cos \varphi + \cos 2\tau_1 \varphi).$$  \hspace{1cm} (4.5)

**Proof.** i) Let the edge set $A'_0$ consists of edges $e_1, e_2, \bar{e}_1, \bar{e}_2$ of the fundamental graph of the square lattice and $N$ additional unoriented edges, i.e., $2N$ oriented edges each of which also has an odd sum of the coordinates. Then by Theorem 4.1.ii (the condition c), $\sigma(\Delta') = [-1, 1]$.

In order to prove the converse we will use the proof by contradiction. Let $\tau_1(e) + \tau_2(e)$ be even for some edge $e \in A'_0$. Then there exists the cycle $e_1, \ldots, \bar{e}_1, e_2, \ldots, \bar{e}_2$ of odd length in the graph $S'$. Thus, the graph is non-bipartite and Theorem 4.1.i yields $\sigma(\Delta') \neq [-1, 1]$.

ii) Without loss of generality we may assume that $0 \leq \tau_1 \leq \tau_2$. From (4.1) and the fact that $\kappa_v = 6$ we deduce that

$$\Delta'(\vartheta) = \frac{1}{3} \left( \cos \vartheta_1 + \cos \vartheta_2 + \cos(\tau_1 \vartheta_1 + \tau_2 \vartheta_2) \right).$$  \hspace{1cm} (4.6)

We will show (4.3). Using (4.6) we have

$$\lambda^{-}(\tau) = \min_{\vartheta \in \mathbb{T}^2} \Delta'(\vartheta) \leq \min_{\vartheta_1 = \vartheta_2 = \varphi} \Delta'(\vartheta) = \frac{1}{3} \min_{\varphi \in [-\pi, \pi]} \left( 2 \cos \varphi + \cos(\tau_1 + \tau_2) \varphi \right) \leq \cos \varphi_0,$$  \hspace{1cm} (4.7)

where $\varphi_0$ is a solution of the equation

$$\cos \varphi_0 = \cos(\tau_1 + \tau_2) \varphi_0, \quad \varphi_0 \in \mathbb{R}. \hspace{1cm} (4.8)$$

The solutions of this equation have the form

$$\varphi_0 = \frac{2\pi n}{\tau_1 + \tau_2 \pm 1}, \quad n \in \mathbb{Z}.$$  \hspace{1cm} (4.9)

We take the solution $\varphi_0$ given by

$$\varphi_0 = \frac{\pi(\tau_1 + \tau_2)}{\tau_1 + \tau_2 + 1} = \pi - \frac{\pi}{\tau_1 + \tau_2 + 1},$$

i.e., the nearest to $\pi$. Then this identity and (4.7) give

$$\lambda^{-}(\tau) \leq \cos \varphi_0 = -\cos \frac{\pi}{\tau_1 + \tau_2 + 1} \leq -\frac{1}{2}.$$  \hspace{1cm} (4.10)

Thus, the inequality (4.3) has been proved.

If $\tau_1 = \tau_2 = 0$, then (4.6) gives

$$\lambda^{-}(0) = \min_{\vartheta \in \mathbb{T}^2} \Delta'(\vartheta) = -\frac{1}{3}.$$  \hspace{1cm} (4.11)
Let now $\tau_1 = \tau_2 \neq 0$. Differentiating $\Delta'(\vartheta)$ given by (4.6) at $\tau_1 = \tau_2$ we obtain the necessary conditions for a minimum of $\Delta'$:

\[
\begin{cases}
\sin \tau_1 (\vartheta_1 + \vartheta_2) = \frac{\sin \vartheta_2}{\tau_1}, \\
\sin \vartheta_1 = \sin \vartheta_2.
\end{cases}
\] (4.10)

Using the second condition in (4.10) we obtain two cases.

Firstly, if $\vartheta_1 = \vartheta_2$, then the function $\Delta'(\vartheta)$ has the form

\[
\Delta'(\vartheta) = \frac{1}{3} (2 \cos \vartheta_1 + \cos 2 \tau_1 \vartheta_1).
\] (4.11)

Secondly, if $\vartheta_1 + \vartheta_2 = \pi$, then $\Delta'(\vartheta) = \frac{(-1)^{\tau_1}}{3 \vartheta_1}$. But inequality (4.3) gives that $\min_{\vartheta \in \mathbb{T}^2} \Delta'(\vartheta) \leq \frac{1}{2}$.

Therefore the minimum point of the function $\Delta'(\vartheta)$ is on the line $\vartheta_1 = \vartheta_2$ and thus,

\[
\lambda^{-}(\tau) = \min_{\vartheta \in \mathbb{T}^2, \vartheta_1 = \vartheta_2} \Delta'(\vartheta) = \frac{1}{3} \min_{\vartheta_1 \in [0, \pi]} (2 \cos \vartheta_1 + \cos 2 \tau_1 \vartheta_1).
\]

This and (4.9) yield (4.11).

We determine the asymptotics (4.11). We have $\tau_2 \to \infty$ as $|\tau| \to \infty$, since $0 \leq \tau_1 \leq \tau_2$. Note that $\Delta'(\vartheta) = \Delta'(\vartheta)$. Hence

\[
\lambda^{-}(\tau) = \min_{\vartheta \in \mathbb{T}^2} \Delta'(\vartheta) = \min_{\vartheta \in \Theta} \Delta'(\vartheta), \quad \Theta = \{-\pi \leq \vartheta_1 \leq \pi, \; \vartheta_1 \leq \vartheta_2 \leq \pi\}.
\] (4.12)

We introduce the local coordinates $\varepsilon$ (see Figure 8) by

\[
\vartheta = (\pi, \pi) - \varepsilon, \quad \varepsilon = (\varepsilon_1, \varepsilon_2) \in \tilde{\Omega} = \{0 \leq \varepsilon_1 \leq 2 \pi, \; 0 \leq \varepsilon_2 \leq \varepsilon_1\}.
\]

Then, using that $\tau_1 + \tau_2$ is even, from (4.6) we obtain

\[
\Delta'(\vartheta) = \Delta'(\vartheta, \tau) = \frac{1}{3} (1 - \cos \varepsilon_1 - \cos \varepsilon_2 + \cos \beta(\varepsilon)),
\] (4.13)

where

\[
\beta(\varepsilon) = \tau_1 \varepsilon_1 + \tau_2 \varepsilon_2.
\]

Firstly, we show that the function $\Delta'(\vartheta, \tau)$ achieves its global minimum on the torus $\mathbb{T}^2$ in the domain (see Figure 8)

\[
\Omega = \{\varepsilon = (\varepsilon_1, \varepsilon_2) \in [0, \pi]^2 : 0 \leq \beta(\varepsilon) \leq \pi \leq \pi\}.
\] (4.14)

The function $\Delta'(\vartheta, \tau)$ at the minimum point $(\varepsilon_1, \varepsilon_2)$ satisfies

\[
\begin{cases}
\frac{\partial \Delta'}{\partial \varepsilon_1} = \frac{1}{3} (\sin \varepsilon_1 - \tau_1 \sin \beta(\varepsilon)) = 0, \\
\frac{\partial \Delta'}{\partial \varepsilon_2} = \frac{1}{3} (\sin \varepsilon_2 - \tau_2 \sin \beta(\varepsilon)) = 0,
\end{cases}
\]

that is equivalent to

\[
\frac{\sin \varepsilon_1}{\tau_1} = \frac{\sin \varepsilon_2}{\tau_2} = \sin \beta(\varepsilon).
\] (4.15)

From the fact that $0 \leq \tau_1 \leq \tau_2$ and the first identity in (4.15) it follows that $\sin \varepsilon_1$ and $\sin \varepsilon_2$ at the minimum point $(\varepsilon_1, \varepsilon_2)$ have the same sign. Thus, the minimum point $(\varepsilon_1, \varepsilon_2) \in [0, \pi]^2$. 
We show that in the domain $\Omega$ there exists a global minimum point of the function $\Delta'((\theta, \tau))$. Let $\Delta'((\theta, \tau))$ achieve its global minimum at a point $\tilde{\varepsilon} = (\tilde{\varepsilon}_1, \tilde{\varepsilon}_2) \in [0, \pi]^2 \setminus \Omega$ (see Figure 8). Show that there exists a point $\varepsilon^0 \in \Omega$ such that

$$\Delta'(\varepsilon^0, \tau) \leq \Delta'(\tilde{\varepsilon}, \tau). \quad (4.16)$$

We denote

$$\tilde{\beta} = \beta(\tilde{\varepsilon}), \quad \beta^0 = \arccos(\cos \tilde{\beta}) \in [0, \pi], \quad (4.17)$$

where the branch of $\arccos$ is fixed by the condition $\arccos(0) = \frac{\pi}{2}$. Note that $\tilde{\beta} > \pi$, since $\tilde{\varepsilon} \in [0, \pi]^2 \setminus \Omega$. Thus,

$$q = \frac{\beta^0}{\beta} \in [0, 1). \quad (4.18)$$

We define the point $\varepsilon^0$ (the point $A$ in Figure 8) by

$$\varepsilon^0 = (\varepsilon^0_1, \varepsilon^0_2) = q \tilde{\varepsilon}. \quad (4.19)$$

Using (4.17) – (4.19), we have

$$0 \leq \varepsilon^0_s = q \tilde{\varepsilon}_s < \tilde{\varepsilon}_s \leq \pi, \quad s = 1, 2,$$

and hence $\varepsilon^0 \in \Omega$. Furthermore, the identity (4.13) gives

$$\Delta'(\varepsilon^0, \tau) = \frac{1}{3}(-\cos \varepsilon^0_1 - \cos \varepsilon^0_2 + \cos \beta(\varepsilon^0)) = \frac{1}{3}(-\cos(q \tilde{\varepsilon}_1) - \cos(q \tilde{\varepsilon}_2) + \cos \beta^0) \leq \frac{1}{3}(-\cos \tilde{\varepsilon}_1 - \cos \tilde{\varepsilon}_2 + \cos \beta^0) = \Delta'(\tilde{\varepsilon}, \tau).$$

Here we have used the second identity in (4.17), (4.20) and the following simple inequality

$$\cos x \leq \cos(qx), \quad \forall (x, q) \in [0, \pi] \times [0, 1],$$

since $\cos$ is monotonic on the segment $[0, \pi]$. Thus, (4.16) holds true, i.e., the function $\Delta'((\theta, \tau))$ also achieves its global minimum at the point $\varepsilon^0 \in \Omega$. Then we rewrite (4.12) as

$$\lambda^-(\tau) = \min_{\varepsilon \in \Omega} \Delta'((\theta, \tau)). \quad (4.21)$$
Secondly, we show that at the global minimum point \( \varepsilon = (\varepsilon_1, \varepsilon_2) \in \Omega \) of the function \( \Delta'(\theta, \tau) \)
\[
(\varepsilon_1, \varepsilon_2) = \frac{O(1)}{|\tau|} \quad \text{as} \quad |\tau| \to \infty. \tag{4.22}
\]
The minimum point \( \varepsilon \in \Omega \), hence
\[
\beta(\varepsilon) \leq \pi, \quad 0 \leq \varepsilon_s, \quad s = 1, 2.
\]
Since \( 0 \leq \tau_1 \leq \tau_2 \), the last inequalities give
\[
\tau_2 \varepsilon_2 \leq \tau_1 \varepsilon_1 + \tau_2 \varepsilon_2 = \beta(\varepsilon) \leq \pi, \quad 0 \leq \varepsilon_2 \leq \frac{\pi}{\tau_2},
\]
which yields the asymptotics for the second component in (4.22). From this asymptotics and the first identity in (4.15) it follows that
\[
\sin \varepsilon_1 = \frac{\tau_1}{\tau_2} \sin \varepsilon_2 = O(1),
\]
and hence \( \varepsilon_1 \to 0 \) or \( \varepsilon_1 \to \pi \) as \( |\tau| \to \infty \). But in the second case\[\lambda^- (\tau) = \frac{1}{3} \left( - \cos \varepsilon_1 - \cos \varepsilon_2 + \cos (\tau_1 \varepsilon_1 + \tau_2 \varepsilon_2) \right) \to \frac{(-1)^{\tau_1}}{3} \cos \tau_2 \varepsilon_2 \quad \text{as} \quad |\tau| \to \infty.\]Since this contradicts the estimate (4.3), we conclude that \( \varepsilon_1 \to 0 \) as \( |\tau| \to \infty \) and the asymptotics for the first component in (4.22) holds true. Thus, (4.22) has been proved.

Thirdly, we obtain the asymptotics for \( \beta(\varepsilon) \) at the global minimum point \( \varepsilon \) as \( |\tau| \to \infty \).

The second identity in (4.15) and the asymptotics for the second component in (4.22) give
\[
\sin \beta(\varepsilon) = O(1), \tag{4.23}
\]
which yields
\[
\beta(\varepsilon) = \frac{\tau_1 \varepsilon_1 + \tau_2 \varepsilon_2 = \pi n(\tau) + \frac{O(1)}{|\tau|^2}}{\text{for some } n(\tau) \in \mathbb{Z}}.
\]
Since the minimum point \( \varepsilon \in \Omega \), \( n(\tau) = 0 \) or \( n(\tau) = 1 \). If \( n(\tau) = 0 \), then
\[
\lambda^- (\tau) = \frac{1}{3} \left( - \cos \varepsilon_1 - \cos \varepsilon_2 + \cos (\tau_1 \varepsilon_1 + \tau_2 \varepsilon_2) \right) \to -\frac{1}{3} \quad \text{as} \quad |\tau| \to \infty.
\]
This again contradicts the estimate (4.3). Thus, the function \( \Delta'(\theta, \tau) \) achieves its global minimum on the curve
\[
\beta(\varepsilon) = \tau_1 \varepsilon_1 + \tau_2 \varepsilon_2 = \pi + \frac{O(1)}{|\tau|^2}. \tag{4.24}
\]
This curve comes arbitrarily close to the line \( \tau_1 \varepsilon_1 + \tau_2 \varepsilon_2 = \pi \) when \( |\tau| \) is rather large.

Finally, we obtain the asymptotics for the value of \( \Delta'(\theta, \tau) \) at the minimum point \( \varepsilon = (\varepsilon_1, \varepsilon_2) \) as \( |\tau| \to \infty \). The identity (4.24) yields
\[
\cos (\tau_1 \varepsilon_1 + \tau_2 \varepsilon_2) = \cos \left( \pi + \frac{O(1)}{|\tau|^2} \right) = -1 + \frac{O(1)}{|\tau|^4}. \tag{4.25}
\]
Using the asymptotics (4.22) and (4.25) we rewrite the identity (4.13) as
\[
\Delta'(\theta, \tau) = \frac{1}{3} \left( -1 + \frac{\varepsilon_1^2}{2} - 1 + \frac{\varepsilon_2^2}{2} - 1 + \frac{O(1)}{|\tau|^4} \right) = -1 + \frac{\varepsilon_1^2 + \varepsilon_2^2}{6} + \frac{O(1)}{|\tau|^4}. \tag{4.26}
\]
This function achieves the global minimum at the point \( \varepsilon = (\varepsilon_1, \varepsilon_2) \) of the curve (4.24), such that the square of the distance \( \varepsilon_1^2 + \varepsilon_2^2 \) from the point \((0, 0)\) to \(\varepsilon\) is minimal. Therefore, \( \varepsilon_1^2 + \varepsilon_2^2 \) is equal to the distance from the point \((0, 0)\) to the curve (4.24). The distance \( d \) from a point \((x_0, y_0)\) to a line \(Ax + By + C = 0, A, B, C \in \mathbb{R}\), is given by

\[
d = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}.
\]

(4.27)

Using this formula we obtain

\[
\varepsilon_1^2 + \varepsilon_2^2 = \left(\frac{\pi}{|\tau|} + \frac{O(1)}{|\tau|^4}\right)^2 = \frac{\pi^2}{|\tau|^2} + \frac{O(1)}{|\tau|^5}.
\]

(4.28)

Substituting (4.28) into (4.26), we have

\[
\lambda^{-}(\tau) = \min_{\varepsilon \in \Omega} \Delta'(\theta, \tau) = -1 + \frac{\pi^2}{6|\tau|^2} + \frac{O(1)}{|\tau|^4}.
\]

Thus, the asymptotics (4.4) has been proved.

**Remark.** 1) Item i) describes all iso-spectral perturbations.

2) In ii) we estimate the end point \( \lambda^{-}(\tau) \) of the spectrum of the operator on the perturbed square lattice. In the case when the added edges has equal coordinates we determine \( \lambda^{-}(\tau) \).

3) The asymptotics (4.4) shows that if we add a “very long” edge to the fundamental graph of the square lattice, then the spectrum of the Laplacian is almost unchanged.

4) For the graph \( S' \) (on Figure 9a) \( \sigma(\Delta') = \sigma_{ac}(\Delta') = [-1, 1] \), since \( S' \) is obtained from the square lattice \( S \) by adding the loop with the coordinates \((2, 1)\) (and its inverse loop) to the fundamental graph \( S_0 \), \( 2 + 1 = 3 \) is odd.

The graph \( S' \), shown on Figure 9b, is obtained from \( S \) by adding the loop with the coordinates \((1, 1)\) to \( S_0 \). Then Theorem 4.2.ii yields

\[
\lambda^{-}(\tau) = \frac{1}{3} \min_{\varphi \in [0, \pi]} (2 \cos \varphi + \cos 2\varphi) = -\frac{1}{2} \quad \text{and} \quad \sigma(\Delta') = \sigma_{ac}(\Delta') = [-1/2, 1].
\]
5. Perturbations of Hexagonal Lattice

5.1. Fundamental graphs with two vertices. We consider a bipartite periodic graph $\Gamma$. By Lemma 6.4, there exists a bipartite fundamental graph $\Gamma_0 = (V_0, E_0)$. Assume that $\Gamma_0$ consists of two vertices $v_1, v_2$ and any number of edges. In this case the vertices $v_1$ and $v_2$ has the same degree $\kappa$ and there are no loops in $\Gamma_0$.

For each $\vartheta \in \mathbb{T}^2$ the $2 \times 2$ matrix $\Delta(\vartheta)$ defined by (1.7) is given by

$$
\Delta(\cdot) = \begin{pmatrix} 0 & \Delta_{12} \\ \bar{\Delta}_{12} & 0 \end{pmatrix}, \quad \Delta_{12}(\vartheta) = \frac{1}{\kappa} \sum_{e=(v_1,v_2) \in A_0} e^{i(\tau(e),\vartheta)}. \quad (5.1)
$$

The diagonal entries of $\Delta(\vartheta)$ are zeroes, since there are no loops on $\Gamma_0$. For each $\vartheta \in \mathbb{T}^2$ the eigenvalues $\lambda_1(\vartheta), \lambda_2(\vartheta)$ of the matrix $\Delta(\vartheta)$ have the form

$$
\lambda_1(\vartheta) = -\lambda_2(\vartheta) = |\Delta_{12}(\vartheta)|.
$$

Recall that the spectrum $\sigma(\Delta)$ of the Laplace operator on a bipartite graph with two vertices in the fundamental graph is symmetric with respect to 0 and consists of two spectral bands. Moreover, the point $1 \in \sigma(\Delta)$, Thus we obtain

$$
\begin{align*}
&i) \quad \sigma(\Delta) = \sigma_{ac}(\Delta) = [-1, -\lambda_0] \cup [\lambda_0, 1], \quad \text{where} \quad \lambda_0 = \min_{\vartheta \in \mathbb{T}^2} |\Delta_{12}(\vartheta)|, \\
&ii) \quad \sigma(\Delta) = [-1, 1] \Leftrightarrow \Delta_{12}(\vartheta_0) = 0, \quad \text{for some} \quad \vartheta_0 \in \mathbb{T}^2. \quad (5.2)
\end{align*}
$$

The following statement gives the method for constructing of periodic graphs with $\lambda_0 > 0$.

**Proposition 5.1.** Let the bipartite fundamental graph $\Gamma_0$ consist of two vertices $v_1, v_2$ and $N^2$ multiple oriented edges $(v_1, v_2)$ with indices running over all values in the set

$$
\mathcal{D} = \{(\tau_{1j}, \tau_{2j}) \in \mathbb{Z}^2 : j = 1, \ldots, N\}
$$

and their inverse edges. Then

i) The function $\Delta_{12}$ defined by (5.1) has the form

$$
\Delta_{12}(\vartheta) = \frac{1}{N^2} P_1(e^{i\vartheta_1}) P_2(e^{i\vartheta_2}),
$$

where

$$
P_s(z) = \sum_{j=1}^N z^{\tau_{sj}}, \quad |z| = 1, \quad s = 1, 2.
$$

ii) Let $P_s(z) \neq 0$ for each $(z, s) \in \{z \in \mathbb{C} : |z| = 1\} \times \{1, 2\}$. Then the spectrum $\sigma(\Delta)$ is symmetric with respect to 0 and consists of exactly two spectral bands separated by a gap $(-\lambda_0, \lambda_0)$, where $\lambda_0 = \min_{\vartheta \in \mathbb{T}^2} |\Delta_{12}(\vartheta)| > 0$.

**Proof.** i) The vertices $v_1$ and $v_2$ have the same degree $\kappa = N^2$. For the function $\Delta_{12}$ defined by (5.1) we have

$$
\begin{align*}
\Delta_{12}(\vartheta) &= \frac{1}{\kappa} \sum_{e=(v_1,v_2) \in A_0} e^{i(\tau(e),\vartheta)} = \frac{1}{N^2} \sum_{(\tau_1, \tau_2) \in \mathcal{D}} e^{i(\tau_1 \vartheta_1 + \tau_2 \vartheta_2)} \\
&= \frac{1}{N^2} \sum_{j=1}^N e^{i\tau_{1j} \vartheta_1} \sum_{k=1}^N e^{i\tau_{2k} \vartheta_2} = \frac{1}{N^2} P_1(e^{i\vartheta_1}) P_2(e^{i\vartheta_2}).
\end{align*}
$$
ii) Since the functions $P_1, P_2$ have no zeroes, $\Delta_{12}(\vartheta) = \frac{1}{N^2} P_1(e^{i\vartheta_1}) P_2(e^{i\vartheta_2}) \neq 0$ for any $\vartheta \in \mathbb{T}^2$. Due to (5.2), the spectrum $\sigma(\Delta)$ is symmetric with respect to 0 and consists of exactly two spectral bands separated by a gap $(-\lambda_0, \lambda_0)$, where $\lambda_0 = \min_{\vartheta \in \mathbb{T}^2} |\Delta_{12}(\vartheta)| > 0$.

![Figure 10](image)

**Figure 10.** A graph $\Gamma$; only edges of the fundamental graph $\Gamma_0$ are shown.

**Example of a graph such that $\sigma(\Delta)$ consists of two spectral bands separated by a gap.** The functions $P_1(z) = P_2(z) = z^{-1} + 1 + z^2$ have no zeroes on the unit circle. Let the bipartite fundamental graph $\Gamma_0$ consist of two vertices $v_1, v_2$ and $3^2 = 9$ oriented edges $(v_1, v_2)$ with the indices

$(0, 0), \quad (-1, 0), \quad (2, 0), \quad (0, -1), \quad (-1, -1), \quad (2, -1), \quad (0, 2), \quad (-1, 2), \quad (2, 2)$

and their inverse edges (see Figure 10). Then by Proposition 5.1, the function $\Delta_{12}$ defined by (5.1) has the form

$$\Delta_{12}(\vartheta) = \frac{1}{N^2} P_1(e^{i\vartheta_1}) P_2(e^{i\vartheta_2}) = \frac{1}{N^2} \left(1 + e^{-i\vartheta_1} + e^{2i\vartheta_1}\right) \left(1 + e^{-i\vartheta_2} + e^{2i\vartheta_2}\right),$$

and the spectrum $\sigma(\Delta)$ on the graph $\Gamma$ is symmetric with respect to 0 and consists of exactly two spectral bands separated by a gap $(-\lambda_0, \lambda_0)$, where $\lambda_0 > 0$. Direct calculations yield

$$\lambda_0 = \min_{\vartheta \in \mathbb{T}^2} |\Delta_{12}(\vartheta)| = |\Delta_{12}(\vartheta_1^*, \vartheta_2^*)| \approx 0.04;$$

where $\vartheta_s^* \in [0, \pi]$ is defined by $\cos \vartheta_s^* = \frac{-1 + \sqrt{7}}{6}$, $s = 1, 2$.

5.2. **Hexagonal lattice.** We discuss the Laplacian on the hexagonal lattice.

**Proposition 5.2.** The Laplacian on the hexagonal lattice $G$ satisfies

$$\Delta(\vartheta) = H_D(t) + O(|t|^2) \quad \text{as} \quad |t| \to 0,$$

$$t = (t_1, t_2) \in \mathbb{R}^2, \quad t_1 = -\frac{1}{6} (\vartheta_1 + \vartheta_2), \quad t_2 = \frac{\sqrt{3}}{6} \left(-\vartheta_1 + \vartheta_2 + \frac{4\pi}{3}\right),$$

where $H_D(t)$ is the 2D Dirac operator given by

$$H_D(t) = \sigma_1 t_1 + \sigma_2 t_2, \quad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}.$$
Proof. The Floquet matrix $\Delta(\vartheta)$ for the hexagonal lattice has the form

$$
\Delta(\vartheta) = \begin{pmatrix}
0 & \Delta_{12}(\vartheta) \\
\Delta_{12}(\vartheta) & 0
\end{pmatrix},
\quad
\Delta_{12}(\vartheta) = \frac{1}{3} (1 + e^{i\vartheta_1} + e^{i\vartheta_2}),
\quad
\vartheta = (\vartheta_1, \vartheta_2).
$$

It is easy to show that

$$
\Delta_{12}(\vartheta) = 0 \iff \vartheta = \pm \vartheta^0,
\quad
\vartheta^0 = (\vartheta_1^0, \vartheta_2^0) = \left( \frac{2\pi}{3}, -\frac{2\pi}{3} \right) \in T^2.
$$

The Taylor expansion for the entry $\Delta_{12}(\vartheta)$ about the point $\vartheta^0$ is given by

$$
\Delta_{12}(\vartheta) = \frac{1}{3} (1 + e^{i\vartheta_1} + e^{i\vartheta_2}) = \frac{1}{3} (1 + e^{i\vartheta_1^0}(1+\vartheta_1-\vartheta_1^0) + e^{i\vartheta_2^0}(1+\vartheta_2-\vartheta_2^0)) + O(|\vartheta - \vartheta^0|^2). \quad (5.5)
$$

Using the identity $e^{\pm i\frac{2\pi}{3}} = \frac{1}{2}(-1 \pm i\sqrt{3})$, we rewrite (5.5) in the form

$$
\Delta_{12}(\vartheta) = \frac{1}{3} + \frac{1}{6} \left[ (-1 + i\sqrt{3})(1+\vartheta_1-\vartheta_1^0) - (1-i\sqrt{3})(1+\vartheta_2-\vartheta_2^0) \right] + O(|\vartheta - \vartheta^0|^2) = t_1 - it_2 + O(|t|^2).
$$

Thus,

$$
\Delta(\vartheta) = \begin{pmatrix}
0 & t_1 - it_2 \\
t_1 + it_2 & 0
\end{pmatrix} + O(|t|^2),
$$

which yields (5.3). Finally, we note that the Taylor expansion for $\Delta_{12}(\vartheta)$ about the point $-\vartheta^0$ is given by the same asymptotics (5.6), but $t_2$ is defined by $t_2 = -\vartheta_1 + \vartheta_2 - \frac{4\pi}{3}$.

The result similar to the asymptotics (5.3) for the 2D hexagonal lattice was described earlier by Wallace [W47] and Slonczewski-Weiss [SW58]. Namely, they predicted that such a 2D monolayer material should present the branching points of the electron spectrum ($K$-points), where a degeneracy of the valence and conductivity band states takes place, the electronic state dispersion law asymptotically has a form of the double-napped cone, while the quasi-wave vector approaches the $K$-point. The equation set for the electron states in the vicinity of the $K$-point is mathematically similar to the Dirac equation for a zero-mass particle. The features stem from the specific symmetry of the hexagon lattice, which has two Bravais sublattices, and all atoms of it are situated in identical positions. These properties of this 2D material do not depend on approximate procedures used in calculation of the electronic spectrum. The mentioned above similarity to the Dirac equation was used by many authors in the study of the graphene electronic properties: bound and resonance states, electron scattering, conductivity and other transport coefficients, see [FKP09], [FKI10] and references therein.

5.3. **Perturbed Hexagonal lattice.** In order to prove Theorem 1.5 we consider the function $\mathcal{F}$ given by

$$
\mathcal{F}(\vartheta) = \left| 1 + e^{i\vartheta_1} + e^{i\vartheta_2} \right|^2,
\quad
\vartheta = (\vartheta_1, \vartheta_2) \in T^2.
$$

We have the following

$$
\min_{\vartheta \in T^2} \mathcal{F}(\vartheta) = \mathcal{F}(\pm \vartheta^0) = 0,
\quad
\text{where } \vartheta^0 = (2\pi/3, -2\pi/3).
$$

We rewrite $\mathcal{F}$ in the form

$$
\mathcal{F}(\vartheta) = 1 + 8 \cos \frac{\vartheta_1}{2} \cos \frac{\vartheta_2}{2}.
$$
which yields
\[ \max_{\vartheta \in T^2} F(\vartheta) = F(0) = 9. \]  
(5.9)

**Proof of Theorem 1.5.** Item i) is a direct consequence of ii) and iii).

We show ii). Recall that the fundamental graph \( G_0 \) of the hexagonal lattice \( G \) consists of two vertices \( v_1, v_2 \), three multiple oriented edges \( e_1 = e_2 = e_3 = (v_1, v_2) \) (Figure 4(b) with the indices \( \tau(e_1) = (0, 0), \tau(e_2) = (1, 0), \tau(e_3) = (0, 1) \) and their inverse edges.

The graph \( G' \) remains bipartite iff the adding edge \( e \) connects the vertices \( v_1 \) and \( v_2 \). Let us add the edge \( e = (v_1, v_2) \) with an index \( \tau = (\tau_1, \tau_2) \in \mathbb{Z}^2 \) (and its inverse edge) to \( G_0 \). For each \( \vartheta = (\vartheta_1, \vartheta_2) \in T^2 \) the matrix \( \Delta(\vartheta) \) defined by (1.7) has the form
\[ \Delta(\vartheta) = \begin{pmatrix} 0 & \Delta_{12}(\vartheta) \\ \bar{\Delta}_{12}(\vartheta) & 0 \end{pmatrix}, \quad \Delta_{12}(\vartheta) = \frac{1}{4} (1 + e^{i \vartheta_1} + e^{i \vartheta_2} + e^{i(\tau, \vartheta)}). \]
We define the point
\[ \vartheta_0 = \begin{cases} (0, \pi), & \text{if } \tau_2 \text{ is odd} \\ (\pi, 0), & \text{if } \tau_1 \text{ is odd} \\ (\pi, \pi), & \text{otherwise} \end{cases} \]

One can verify by a direct calculation that \( \Delta_{12}(\vartheta_0) = 0 \). From ii) in (5.2), it follows that \( \sigma(\Delta') = [-1, 1] \). Thus, we have proved that if \( G' \) is bipartite then \( \sigma(\Delta') = [-1, 1] \). The converse follows from item iii) of this theorem.

iii) The graph \( G' \) is non-bipartite iff we add a loop with an index \( \tau = (\tau_1, \tau_2) \in \mathbb{Z}^2 \) (and its inverse loop) to \( G_0 \). Without loss of generality we may assume that we add a loop \( e = (v_2, v_2) \). For the graph \( G' \) the matrix \( \Delta(\vartheta) \) has the form
\[ \Delta(\vartheta) = \begin{pmatrix} 0 & \Delta_{12}(\vartheta) \\ \bar{\Delta}_{12}(\vartheta) & \Delta_{22}(\vartheta) \end{pmatrix}, \]
where
\[ \Delta_{12}(\vartheta) = \frac{1}{\sqrt{15}} (1 + e^{i \vartheta_1} + e^{i \vartheta_2}), \quad \Delta_{22}(\vartheta) = \frac{2}{5} \cos(\tau, \vartheta), \quad \vartheta \in T^2. \]
The eigenvalues of the matrix \( \Delta(\vartheta) \) are given by
\[ \lambda_s(\vartheta) = \frac{1}{5} \cos(\tau, \vartheta) + (-1)^{s+1} \sqrt{\frac{1}{25} \cos^2(\tau, \vartheta) + \frac{1}{15} F(\vartheta)}, \quad s = 1, 2, \]
(5.10)
where \( F(\vartheta) \) is defined by (5.7). Thus, the spectrum of the Laplacian on the non-bipartite perturbed graphene has the form
\[ \sigma(\Delta) = [\lambda_2^-, \lambda_2^+] \cup [\lambda_1^-, \lambda_1^+], \quad \lambda_s^- = \min_{\vartheta \in T^2} \lambda_s(\vartheta), \quad \lambda_s^+ = \max_{\vartheta \in T^2} \lambda_s(\vartheta), \quad s = 1, 2. \]
Using (5.10) and (5.8), (5.9), we have
\[ \lambda_1^+ = \max_{\vartheta \in T^2} \lambda_1(\vartheta) = \lambda_1(0) = 1, \quad \lambda_2^- = \min_{\vartheta \in T^2} \lambda_2(\vartheta) \leq \lambda_2(0) = -\frac{3}{5}. \]
The last inequality and non-bipartition of the graph \( G' \) yield (1.20).

We will show (1.18), (1.19). From (5.10) and (5.8), (5.9), it follows that
\[ \lambda_1(\vartheta) \geq 0, \quad \lambda_2(\vartheta) \leq 0, \quad \forall \vartheta \in T^2, \]
(5.11)
and
\[ \lambda_s(\vartheta^0) = \frac{1}{5} \cos \left( \frac{2\pi}{3} (\tau_1 - \tau_2) \right) + \frac{(-1)^{s+1}}{5} \left| \cos \left( \frac{2\pi}{3} (\tau_1 - \tau_2) \right) \right|, \quad s = 1, 2, \]  
(5.12)
where \( \vartheta^0 = \left( \frac{2\pi}{3}, -\frac{2\pi}{3} \right) \). If \( \tau_1 - \tau_2 \in 3\mathbb{Z} \), then (5.11), (5.12) give
\[ \lambda_1^- = \min_{\vartheta \in \mathbb{T}^2} \lambda_1(\vartheta) \leq \lambda_1(\vartheta^0) = \frac{2}{5}, \quad 0 \geq \lambda_2^+ = \max_{\vartheta \in \mathbb{T}^2} \lambda_2(\vartheta) \geq \lambda_2(\vartheta^0) = 0. \]
If \( \tau_1 - \tau_2 \notin 3\mathbb{Z} \), then
\[ 0 \leq \lambda_1^- = \min_{\vartheta \in \mathbb{T}^2} \lambda_1(\vartheta) \leq \lambda_1(\vartheta^0) = 0, \quad \lambda_2^+ = \max_{\vartheta \in \mathbb{T}^2} \lambda_2(\vartheta) \geq \lambda_2(\vartheta^0) = -\frac{1}{5}. \]
This yields (1.18), (1.19).

The estimates (1.21) follow directly from (1.18) – (1.20).

**Example of a non-bipartite perturbed graphene.** Let the graph \( G' \) be obtained from the hexagonal lattice \( G \) by adding the loop \( e = (v_2, v_2) \) with the index \( \tau(e) = (\tau_1, \tau_2) = (1, 0) \) (and its inverse loop) to the fundamental graph \( G_0 \) (Figure 11).

![Figure 11. The non-bipartite perturbed graphene G'.](image)

Since \( \tau_1 - \tau_2 \notin 3\mathbb{Z} \), by Theorem 1.5 we have
\[ \sigma(\Delta') = \sigma_{ac}(\Delta') = [\lambda_2^-, \lambda_2^+] \cup [0, 1]. \]

We obtain numerically that
\[ \lambda_2^- \approx -0.627; \quad \lambda_2^+ \approx -0.106. \]

5.4. **Bipartite fundamental graphs with three vertices.** Consider a bipartite graph \( \Gamma \). It has a bipartite fundamental graph \( \Gamma_0 = (V_0, E_0) \) with parts \( V_1 \) and \( V_2 \). Let \( \Gamma_0 \) consist of three vertices \( v_1, v_2, v_3 \) with degrees \( \kappa_1, \kappa_2, \kappa_3 \), respectively, and any number of edges. Without loss of generality we may assume that \( v_1 \in V_1 \) and \( v_2, v_3 \in V_2 \).

For each \( \vartheta \in \mathbb{T}^2 \) the matrix \( \Delta(\vartheta) \) defined by (1.7) is the following 3 \( \times \) 3 matrix
\[ \Delta(\vartheta) = \begin{pmatrix} 0 & \Delta_{12}(\vartheta) & \Delta_{13}(\vartheta) \\ \Delta_{12}(\vartheta) & 0 & 0 \\ \Delta_{13}(\vartheta) & 0 & 0 \end{pmatrix}, \]  
(5.13)
where
\[ \Delta_{1s}(\vartheta) = \frac{1}{\sqrt{\kappa_1 \kappa_s}} \sum_{e = (v_1, v_s) \in E_0} e^{i(\tau(e), \vartheta)}, \quad s = 2, 3. \]  
(5.14)

In the following proposition we determine the spectrum of the operator \( \Delta \) on the graph \( \Gamma \) and formulate the necessary and sufficient conditions when \( \sigma(\Delta) = [-1, 1] \).
Proposition 5.3. Let the bipartite fundamental graph $\Gamma_0$ consist of three vertices and some number of edges. Then
\[
\sigma(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta), \quad \sigma_{ac}(\Delta) = [-1, -\lambda_0] \cup [\lambda_0, 1], \quad \sigma_{fb} = \{0\},
\]
where the eigenvalues $\lambda_1(\vartheta), \lambda_2(\vartheta), \lambda_3(\vartheta)$ of the matrix $\Delta(\vartheta)$ given by (5.13) have the form
\[
\lambda_1(\vartheta) = -\lambda_2(\vartheta) = \sqrt{|\Delta_{13}(\vartheta)|^2 + |\Delta_{12}(\vartheta)|^2}, \quad \lambda_3(\cdot) = \mu = 0,
\]
which yields (5.15).

Proof. For each $\vartheta \in \mathbb{T}^2$ the eigenvalues $\lambda_1(\vartheta), \lambda_2(\vartheta), \lambda_3(\vartheta)$ of the matrix $\Delta(\vartheta)$ given by (5.13) have the form
\[
\lambda_1(\vartheta) = -\lambda_2(\vartheta) = \sqrt{|\Delta_{13}(\vartheta)|^2 + |\Delta_{12}(\vartheta)|^2}, \quad \lambda_3(\cdot) = \mu = 0,
\]
which yields (5.15). \qed

Corollary 5.4. Let $\Gamma$ be the graph obtained from the square lattice $S$ by adding one vertex on each edge of $S$. Then the spectrum of the Laplacian on $\Gamma$ has the form
\[
\sigma(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta), \quad \sigma_{ac}(\Delta) = [-1, 1], \quad \sigma_{fb}(\Delta) = \{0\}.
\]

Proof. The graph $\Gamma$ is a bipartite periodic graph with three vertices on the fundamental graph. Then by Proposition 5.3 the spectrum has the form
\[
\sigma(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta), \quad \sigma_{ac}(\Delta) = [-1, -\lambda_0] \cup [\lambda_0, 1], \quad \sigma_{fb}(\Delta) = \{0\},
\]
where $\lambda_0 = \min_{\vartheta \in \mathbb{T}^2} \sqrt{|\Delta_{13}(\vartheta)|^2 + |\Delta_{12}(\vartheta)|^2}$. The functions $\Delta_{12}, \Delta_{13}$ defined by (5.14) have the form
\[
\Delta_{1\pm}(\vartheta) = \frac{1}{2\sqrt{2}} (1 + e^{-i\vartheta_i}).
\]
Due to identities $\Delta_{12}(\vartheta_0) = \Delta_{13}(\vartheta_0) = 0$, $\vartheta_0 = (\pi, \pi)$, Proposition 5.3 gives $\lambda_0 = 0$ and $\sigma_{ac}(\Delta) = [-1, 1]$. \qed

5.5. The Kagome lattice. As an example of non-bipartite graph with three vertices in the fundamental graph consider the graph $\Gamma$ shown in Figure 12b. This graph is called the Kagome lattice. It is a lattice structure found in many natural minerals’ molecular arrangements.

The fundamental graph of the Kagome lattice consists of three vertices $v_1, v_2, v_3$ each of which has the degree 4, six oriented edges
\[
e_1 = e_2 = (v_1, v_2), \quad e_3 = e_4 = (v_1, v_3), \quad e_5 = e_6 = (v_2, v_3)
\]
and their inverse edges. The indices of the fundamental graph edges in the coordinate system with the origin $O$ (Figure 12b) are given by
\[
\tau(e_1) = (0, 0), \quad \tau(e_2) = (-1, 0), \quad \tau(e_3) = (0, 0),
\tau(e_4) = (0, -1), \quad \tau(e_5) = (0, 0), \quad \tau(e_6) = (1, -1).
\]

Proposition 5.5. The spectrum of the Laplace operator on the Kagome lattice has the form
\[
\sigma(\Delta) = \sigma_{ac}(\Delta) \cup \sigma_{fb}(\Delta), \quad \sigma_{ac}(\Delta) = [-1/2; 1], \quad \sigma_{fb}(\Delta) = \{-1/2\}.
\]
The eigenfunctions corresponding to the eigenvalue \( \lambda \) for some self-adjoint \( \nu \) are pointed out.

**Remark.** 1) The eigenfunctions corresponding to the eigenvalue \( \mu_1 = -1/2 \) have the finite support shown in Figure 12b. The values of the eigenfunction in the vertices of the support are pointed out.

2) As sets the spectra of the Laplacians on the Kagome lattice and on the triangular lattice (Figure 12b) are the same.

### 6. Appendix

**6.1. Properties of matrices.** We recall some well-known properties of matrices (see e.g., [HJ85] and [HC96]), which will be used below. Let \( \rho(A) \) be the spectral radius of \( A \).

**Proposition 6.1.**

i) Let \( A = \{A_{jk}\} \) and \( B = \{B_{jk}\} \) be \( \nu \times \nu \) matrices. If \( |A_{jk}| \leq B_{jk} \) for all \( j, k \in \mathbb{N}_\nu \), then \( \rho(A) \leq \rho(B) \) (see Theorem 8.1.18 in [HJ85]).

ii) Let \( A, B \) be \( \nu \times \nu \) self-adjoint matrices and let \( B \geq 0 \). Then the eigenvalues \( \lambda_n(A) \leq \lambda_n(A + B) \) for all \( n \in \mathbb{N}_\nu \) (see Corollary 4.3.3 in [HJ85]).

iii) Let \( B \) be the self-adjoint \( (\nu + 1) \times (\nu + 1) \) matrix given by

\[
B = \begin{pmatrix}
A & y \\
y^* & a
\end{pmatrix}
\]

for some self-adjoint \( \nu \times \nu \) matrix \( A \), some real number \( a \) and some vector \( y \in \mathbb{C}^\nu \).
Let the eigenvalues of $A$ and $B$ be denoted by $\{\mu_j\}$ and $\{\lambda_j\}$, respectively, and assume that they have been arranged in decreasing order

$$\mu_\nu \leq \ldots \leq \mu_1, \quad \lambda_{\nu+1} \leq \mu_\nu \leq \ldots \leq \lambda_1.$$ 

Then

$$\lambda_{\nu+1} \leq \mu_\nu \leq \lambda_\nu \leq \ldots \leq \lambda_2 \leq \mu_1 \leq \lambda_1,$$

(see Theorem 4.3.8 in [HJ85]).

iv) Let $M$ be a $\nu \times \nu$ matrix having the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

for some square matrices $A, D$ and some matrices $B, C$. Then

$$\det M = \det A \cdot \det (D - CA^{-1}B)$$

(see pp. 21–22 in [HJ85]).

v) Let $M_\nu$ be a $\nu \times \nu$ finite Jacobi matrix given by

$$M_\nu = \begin{pmatrix} b & 1 & 0 & \ldots \\ 1 & b & 1 & \ldots \\ 0 & 1 & b & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{pmatrix},$$

where $b = 2 \cos \varphi \in (-2, 2)$. Then

$$\det M_\nu = (-1)^\nu \frac{\sin (\nu + 1) \varphi}{\sin \varphi},$$

which satisfies

$$\det M_{\nu+1} = b \det M_\nu - \det M_{\nu-1}, \quad \forall \ \nu \in \mathbb{N},$$

with the initial conditions $\det M_0 = 1$, $\det M_1 = b$ (see pp. 1511–1512 in [HC96]).

Moreover, the eigenvalues of the matrix $M_\nu$ have the form

$$\lambda_\nu = b + 2 \cos \frac{\pi n}{\nu + 1}, \quad \forall \ \nu \in \mathbb{N}_\nu.$$ 

6.2. The basic properties of periodic graphs and fundamental graphs. In this subsection we discuss main properties of the periodic graphs.

**Proposition 6.2.** The fundamental graph of a $\mathbb{Z}^2$-periodic graph is finite.

**Proof.** Recall that we identify the vertices of the fundamental graph $\Gamma_0$ with the vertices of the periodic graph from the bounded domain $[0, 1)^2$. So the number of vertices of $\Gamma_0$ is finite (see item 1 of the definition of a $\mathbb{Z}^2$-periodic graph). The number of edges of $\Gamma_0$ is also finite, because $\Gamma_0$ contains a finite number of vertices, each of which has a finite degree (see item 2 of the definition). 

Below we need the following properties of an edge index.
Proposition 6.3. Let \((u, v) \in \mathcal{A}\) and let \(p, q \in \mathbb{Z}^2\). Then the following identities hold true.

i) \(\tau(u, v) = -\tau(v, u)\).

ii) Let \((v + p, v + q) \in \mathcal{A}\). Then the index \(\tau(v + p, v + q) = q - p\).

iii) The periodicity of the graph yields that the edge \((u + p, v + p) \in \mathcal{A}\) and its index \(\tau(u + p, v + p) = \tau(u, v)\).

iv) Let \(\tilde{e}\) be the index of an edge \(e = (u, v) \in \mathcal{A}_0\) in the coordinate system with the origin \(O_1\). Then

\[
\tau^{(1)}(\tilde{e}) = \tau(e) + [v - b] - [u - b], \quad \text{where} \quad b = \overrightarrow{O_1O}.
\]

Proof. Using the definition \((1.3)\) of an edge index, we have

\[
i) \quad \tau(u, v) = [v] - [u] = -([u] - [v]) = -\tau(v, u),
\]

\[
ii) \quad \tau(v + p, v + q) = [v] + q - [v] - p = q - p.
\]

iii) The periodicity of the graph yields that the edge \((u + p, v + p) \in \mathcal{A}\) and

\[
\tau(u + p, v + p) = [v] + p - [u] - p = [v] - [u] = \tau(u, v).
\]

\[\text{Figure 13.} \quad \text{The edge } e = (u + p, v + q) \text{ in the coordinate system with the origin } O; \text{ } e = (u + p - b, v + q - b) \text{ in the coordinate system with the origin } O_1; \text{ } a_1, a_2 \text{ are the periods of the graph.}\]

iv) Let \(\tilde{e} = (u, v)\) be an oriented edge of the fundamental graph with an index \(\tau(\tilde{e})\). Then by the definition of the fundamental graph and the formulas \((1.3), (1.4)\) there is an edge \(e = (u + p, v + q) \in \mathcal{A}\), where \(p, q \in \mathbb{Z}^2\) are some integer vectors such that

\[
\tau(e) = q - p = \tau(\tilde{e})
\]

(Figure 13). Recall that we identify the vertices of the fundamental graph with the vertices of the periodic graph from the set \([0, 1)^2\) in the coordinate system with the origin \(O\). In the coordinate system with the origin \(O_1\) the edge \(e\) has the form

\[e = (u + p - b, v + q - b).\]

Using \((1.3), (1.4)\) and \((5.6)\) we obtain

\[
\tau^{(1)}(\tilde{e}) = \tau^{(1)}(e) = q + [v - b] - p - [u - b] = \tau(\tilde{e}) + [v - b] - [u - b].
\]

Thus, the identity \((6.3)\) has been proved. \(\blacksquare\)

Remark. 1) From item iv) it follows that an edge index, generally speaking, depends on the choice of the coordinate origin. Indeed, let the oriented edge \(e = (u, v)\) in the coordinate system with the origin \(O\) and the same edge \(e = (u_1, v_1)\) in the system with the origin \(O_1\) (Figure 13). Then the index of the edge \(e = (u, v)\) in the first system is defined by

\[
\tau(e) = [v] - [u] = (0, 0) - (0, 0) = (0, 0)
\]
and the index of the same edge \( e = (u, v) \) in the second system has the form
\[
\tau^{(1)}(e) = [v_1] - [u_1] = (1, 1) - (0, 0) = (1, 1).
\]

Item ii) shows that the index of the edge \((v + p, v + q) \in A\) does not depend on the choice of the coordinate origin \(O\). It also means that the indices of all loops on the fundamental graph do not depend on the choice of the point \(O\).

2) Under the group \(\mathbb{Z}^2\) action the set \(A\) of oriented edges of the graph \(\Gamma\) is divided into equivalence classes. Each equivalence class is an oriented edge \(e \in A_0\) of the fundamental graph \(\Gamma_0\). From item iii) it follows that all edges from one equivalence class \(e\) have the same index that is also the index of the fundamental graph edge \(e\).

Now we show that a bipartite periodic graph has a bipartite fundamental graph. Recall that a graph \(\Gamma = (V, E)\) is bipartite if its vertex set \(V\) can be divided into two disjoint sets \(V_1\) and \(V_2\) (called parts of the graph) such that every edge connects vertices from different sets.

**Lemma 6.4.** i) If \(v\) belongs to some part \(V_1\) of the graph, then \(v + 2p \in V_1\) for each \(p \in \mathbb{Z}^2\).

ii) For a bipartite periodic graph there exists a bipartite fundamental graph.

**Proof.** i) The distance between two vertices in a graph is the number of edges in the shortest path connecting these vertices. It is known that the distance between two vertices from one part of a bipartite graph is even and the distance between two vertices from the different parts is odd (see p.105 in [Or62]).

For the vertex \(v + p\) we have two cases.

Firstly, let \(v + p \in V_1\). Then the distance \(d(v, v + p)\) between \(v\) and \(v + p\) is even. Due to the periodicity of the graph the shortest path connecting \(v + p\) and \(v + 2p\) is obtained by the translation of the shortest path between \(v\) and \(v + p\) through the vector \(p\). Thus, the distance \(d(v + p, v + 2p)\) is also even and hence \(v + 2p \in V_1\).

Secondly, if \(v + p \in V_2\), then \(d(v, v + p)\) is odd. The distance \(d(v + p, v + 2p)\) is also odd and hence \(v + 2p \in V_1\). Thus, in both case \(v + 2p \in V_1\).

ii) Let \(\Gamma_0 = (V_0, E_0)\) be a fundamental graph of a bipartite periodic graph \(\Gamma\). If it is non-bipartite, then we consider another fundamental graph \(\Gamma'_0 = (V'_0, E'_0) = \Gamma/(2\mathbb{Z})^2\). We will show that \(\Gamma'_0\) is bipartite. We identify the vertices of \(\Gamma'_0\) with the vertices of the periodic graph \(\Gamma\) from the set \([0, 2]^2\). Let us divide \(V'_0\) into two disjoint sets \(V'_1\) and \(V'_2\) in the following way:
\[
v \in V'_s \Leftrightarrow v \in V_s \quad \text{for each } s = 1, 2. \tag{6.7}
\]
We show that each edge of the fundamental graph \(\Gamma'_0\) connects vertices from the different sets \(V'_1\) and \(V'_2\). Assume the contrary, that there exists an edge \(e = (u, v) \in E'_0\) connecting the
vertices $u, v$ from one set $V'_1$ or $V'_2$. Without loss of generality suppose that $u, v \in V'_1$. Then by the definition of the fundamental graph $\Gamma'_0$ there is an edge $(u + p, v + q) \in \mathcal{E}$ for some $p, q \in 2\mathbb{Z}$. Since $u, v \in V'_1$, by (6.7) and item i) we have $u + p, v + q \in V_1$. Thus, the edge $(u + p, v + q)$ on $\Gamma$ connects the vertices from one part of $\Gamma$. This contradicts the bipartition of $\Gamma$. Thus, there is no edge connecting vertices from the same part of $\Gamma'_0$ and hence $\Gamma'_0$ is a bipartite fundamental graph of $\Gamma$.

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