Adaptive model selection method for a conditionally Gaussian semimartingale regression in continuous time *

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Abstract

This paper considers the problem of robust adaptive efficient estimating of a periodic function in a continuous time regression model with the dependent noises given by a general square integrable semimartingale with a conditionally Gaussian distribution. An example of such noise is the non-Gaussian Ornstein–Uhlenbeck–Lévy processes. An adaptive model selection procedure, based on the improved weighted least square estimates, is proposed. Under some conditions on the noise distribution, sharp oracle inequality for the robust risk has been proved and the robust efficiency of the model selection procedure has been established. The numerical analysis results are given.

Key words: Improved non-asymptotic estimation, Least squares estimates, Robust quadratic risk, Non-parametric regression, Semimartingale noise, Ornstein–Uhlenbeck–Lévy process, Model selection, Sharp oracle inequality, Asymptotic efficiency.

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1 Introduction

Consider a regression model in continuous time
\[
dy_t = S(t)dt + d\xi_t, \quad 0 \leq t \leq n, \tag{1.1}
\]
where \( S \) is an unknown 1-periodic \( \mathbb{R} \to \mathbb{R} \) function, \( S \in L_2[0, 1] \); \( (\xi_t)_{t \geq 0} \) is an unobservable conditionally Gaussian semimartingale with the values in the Skorokhod space \( D[0, n] \) such that, for any cadlag \( [0, n] \to \mathbb{R} \) function \( f \) from \( L_2[0, n] \), the stochastic integral
\[
I_n(f) = \int_0^n f(s)d\xi_s \tag{1.2}
\]
is well defined and has the following properties
\[
E_Q I_n(f) = 0 \quad \text{and} \quad E_Q f_n^2(f) \leq \kappa_Q \int_0^n f^2(s)ds. \tag{1.3}
\]
Here \( E_Q \) denotes the expectation with respect to the distribution \( Q \) of the noise process \( (\xi_t)_{0 \leq t \leq n} \) on the space \( D[0, n] \); \( \kappa_Q > 0 \) is some positive constant depending on the distribution \( Q \). The noise distribution \( Q \) is unknown and assumed to belong to some probability family \( Q_n \) specified below. All necessary tools concerning the stochastic calculus can be found, for example, in [13].

The class of the disturbances \( \xi \) satisfying conditions (1.3) is rather wide and comprises, in particular, the Lévy processes which are used in different applied problems (see [4], for details). The models (1.1) with the Lévy’s type noise naturally arise in the nonparametric functional statistics problems (see, for example, [8, 15, 16]). Moreover, as is shown in Section 2, non-Gaussian Ornstein–Uhlenbeck-based models, introduced in [2], enter this class. It is well-known that in the filtration theory the assumption of the conditional gaussinity of the unobserved process with respect to the observed one led to the extension of the classical Kalman–Bucy problem with the closed form solution to a class of the stochastic models described by the equations including nonlinearly the process under observation [21].

The problem is to estimate the unknown function \( S \) in the model (1.1) on the basis of observations \((y_t)_{0 \leq t \leq n}\).

We define the error of an estimate \( \hat{S} \) (any real-valued function measurable with respect to \( \sigma\{y_t, 0 \leq t \leq n\} \)) for \( S \) by its integral quadratic risk
\[
\mathcal{R}_Q(\hat{S}, S) := E_{Q,S} \|\hat{S} - S\|^2, \tag{1.4}
\]
where $E_{Q,S}$ stands for the expectation with respect to the distribution $P_{Q,S}$ of the process (1.1) with a fixed distribution $Q$ of the noise $(\xi_t)_{0 \leq t \leq n}$ and a given function $S$; $\| \cdot \|$ is the norm in $L^2[0,1]$, i.e.

$$\| f \|^2 := \int_0^1 f^2(t) dt.$$  

Since in our case the noise distribution $Q$ is unknown, we will measure the quality of an estimate $\widehat{S}$ by the robust risk defined as

$$R^*_n(\widehat{S}, S) = \sup_{Q \in \mathcal{Q}_n} R_Q(\widehat{S}, S) \tag{1.5}$$  

which assumes taking supremum of the error (1.4) over the whole family of admissible distributions $\mathcal{Q}_n$ (see, for example, [9]).

We will study the stated problem from the standpoint of the model selection approach. It will be noted that the origin of this method goes back to papers by Akaike [1] and Mallows [22]. The further progress has been made by Barron, Birgé and Massart [3, 24], who developed a non-asymptotic model selection method which enables one to derive nonasymptotic oracle inequalities for nonparametric regression models with the i.i.d. Gaussian disturbances. Fourdrinier and Pergamenshchikov [7] extended the Barron-Birgé-Massart method to the models with the spherically symmetric dependent observations. The authors in [17] applied this method to the nonparametric problem of estimating a periodic function in a continuous time model with a Gaussian colored noise. Unfortunately, the oracle inequalities obtained in these papers can not provide the efficient estimation in the adaptive setting. For constructing adaptive procedures in our case one needs to use the approach based on the sharp oracle inequalities, proposed by Galtchouk and Pergamenshchikov [10, 11] for the heteroscedastic regression models in discrete time and which developed by Konev and Pergamenshchikov [18, 19] for nonparametric regression models in continuous time.

The goal of this paper is to develop the adaptive robust efficient model selection method for the regression (1.1) with dependent noises having conditionally Gaussian distribution using the improved estimation approach. This paper proposes the shrinkage least squares estimates which enable us to improve the non-asymptotic estimation accuracy. For the first time such idea was proposed by Fourdrinier and Pergamenshchikov in [7] for regression models in discrete time and by Konev and Pergamenshchikov in [17] for Gaussian regression models in continuous time. We develop these methods for the general semimartingale regression models in continuous time. It should be noted that for the conditionally Gaussian regression models we can not use
the well-known improved estimators proposed in [14] for Gaussian or spherically symmetric observations. To apply the improved estimation methods to the non-Gaussian regression models in continuous time one needs to use the modifications of the well-known James - Stein estimators proposed in [20, 28] for parametric problems. We develop the new analytical tools which allow one to obtain the sharp non-asymptotic oracle inequalities for robust risks under general conditions on the distribution of the noise in the model (1.1). This method enables us to treat both the cases of dependent and independent observations from the same standpoint, it does not assume the knowledge of the noise distribution and leads to the efficient estimation procedure with respect to the risk (1.5). The validity of the conditions, imposed on the noise in the equation (1.1) is verified for a non-Gaussian Ornstein–Uhlenbeck process.

The rest of the paper is organized as follows. In the next Section 2, we describe the Ornstein–Uhlenbeck process as the example of a semimartingale noise in the model (1.1). In Section 3 we construct the shrinkage weighted least squares estimates and study the improvement effect. In Section 4 we construct the model selection procedure on the basis of improved weighted least squares estimates and state the main results in the form of oracle inequalities for the quadratic risk (1.4) and the robust risk (1.5). In Section 5 it is shown that the proposed model selection procedure for estimating $S$ in (1.1) is asymptotically efficient with respect to the robust risk (1.5). In Section 6 we illustrate the performance of the proposed model selection procedure through numerical simulations. In Section 7 we establish some properties of the stochastic integrals with respect to the non-Gaussian Ornstein-Uhlenbeck process (2.1). Section 8 gives the proofs of the main results. In the Appendix some auxiliary lemmas are given.

\section{Ornstein-Uhlenbeck-Lévy process}

Now we consider the noise process $(\xi_t)_{t \geq 0}$ in (1.1) defined by a non-Gaussian Ornstein–Uhlenbeck process with the Lévy subordinator. Such processes are used in the financial Black–Scholes type markets with jumps (see, for example, [6] and the references therein). Let the noise process in (1.1) obeys the equation

$$d\xi_t = a\xi_t dt + du_t, \quad \xi_0 = 0,$$

(2.1)

where

$$u_t = \varrho_1 w_t + \varrho_2 z_t \quad \text{and} \quad z_t = \int_{\mathbb{R}} x * (\mu - \tilde{\mu})_t.$$

(2.2)

Here $(w_t)_{t \geq 0}$ is a standard Brownian motion, $\mu(ds dx)$ is the jump measure with the deterministic compensator $\tilde{\mu}(ds dx) = ds\Pi(dx)$, $\Pi(\cdot)$ is a Lévy
measure, i.e. some positive measure on $\mathbb{R} = \mathbb{R} \setminus \{0\}$, see, for example [5, 13], such that
\[
\Pi(x^2) = 1 \quad \text{and} \quad \Pi(x^8) < \infty.
\] (2.3)

We use the notation $\Pi(|z|^m) = \int_{\mathbb{R}} |z|^m \Pi(dz)$. Note that the Lévy measure $\Pi(\mathbb{R})$ could be equal to $+\infty$.

We assume that the nuisance parameters $a \leq 0$, $\varrho_1$ and $\varrho_2$ satisfy the conditions
\[
-a_{\max} \leq a \leq 0, \quad 0 < \varrho \leq \varrho_1^2 \quad \text{and} \quad \sigma_Q = \varrho_1^2 + \varrho_2^2 \leq \varsigma^*,
\] (2.4)
where the bounds $a_{\max}$, $\varrho$ and $\varsigma^*$ are functions of $n$, i.e. $a_{\max} = a_{\max}(n)$, $\varrho = \varrho_n$ and $\varsigma^* = \varsigma^*_n$, such that for any $\tilde{\delta} > 0$
\[
\lim_{n \to \infty} \frac{a_{\max}(n)}{n} = 0, \quad \liminf_{n \to \infty} n^\varepsilon \varrho_n > 0 \quad \text{and} \quad \lim_{n \to \infty} n^{-\varepsilon} \varsigma^*_n = 0.
\] (2.5)

We denote by $\mathcal{Q}_n$ the family of all distributions of process (1.1) – (2.1) on the Skorokhod space $D[0, n]$ satisfying the conditions (2.4) – (2.5).

It should be noted that in view of Corollary 7.2 the condition (1.3) for the process (2.1) holds with $\kappa_Q = 2 \varrho^*_n$.

Note also that the process (2.1) is conditionally-Gaussian square integrated semimartingale with respect to $\sigma$-algebra $\mathcal{G} = \sigma\{z_t, t \geq 0\}$ which is generated by jump process $(z_t)_{t \geq 0}$.

3 Shrinkage estimates

For estimating the unknown function $S$ in (1.1) we will consider it’s Fourier expansion. Let $(\phi_j)_{j \geq 1}$ be an orthonormal basis in $L_2[0, 1]$. We extend these functions by the periodic way on $\mathbb{R}$, i.e. $\phi_j(t) = \phi_j(t + 1)$ for any $t \in \mathbb{R}$.

B1) Assume that the basis functions are uniformly bounded, i.e. for some constant $\phi_* \geq 1$, which may be depend on $n$,
\[
\sup_{0 \leq j \leq n} \sup_{0 \leq t \leq 1} |\phi_j(t)| \leq \phi_* < \infty.
\] (3.1)

B2) Assume that there exist some $d_0 \geq 7$ and $\tilde{a} \geq 1$ such that
\[
\sup_{d \geq d_0} \frac{1}{d} \int_{0}^{1} \Phi^*_d(v) \, dv \leq \tilde{a},
\] (3.2)
where \( \Phi_d^*(v) = \max_{t \geq v} \left| \sum_{j=1}^{d} \phi_j(t) \phi_j(t-v) \right| \).

For example, we can take the trigonometric basis defined as \( Tr_1 \equiv 1 \) and for \( j \geq 2 \)
\[
Tr_j(x) = \sqrt{2} \begin{cases} 
\cos(\varpi_j x) & \text{for even } j; \\
\sin(\varpi_j x) & \text{for odd } j,
\end{cases}
\]
(3.3)

where the frequency \( \varpi_j = 2\pi[j/2] \) and \([x]\) denotes integer part of \( x \).

In Lemma A.1 we shown that these functions satisfy the condition \( B_2 \) with
\[
d_0 = \inf\{d \geq 7 : 5 + \ln d \leq \tilde{a}d\} \quad \text{and} \quad \tilde{a} = (1 - e^{-a_{\text{max}}})/(4a_{\text{max}}). \quad (3.4)
\]

We write the Fourier expansion of the unknown function \( S \) in the form
\[
S(t) = \sum_{j=1}^{\infty} \theta_j \phi_j(t),
\]

where the corresponding Fourier coefficients
\[
\theta_j = (S, \phi_j) = \int_{0}^{1} S(t) \phi_j(t) \, dt
\]
(3.5)
can be estimated as
\[
\hat{\theta}_{j,n} = \frac{1}{n} \int_{0}^{n} \phi_j(t) \, dy_t.
\]
(3.6)

We replace the differential \( S(t)dt \) by the stochastic observed differential \( dy_t \).

In view of (1.1), one obtains
\[
\hat{\theta}_{j,n} = \theta_j + \frac{1}{\sqrt{n}} \xi_{j,n}, \quad \xi_{j,n} = \frac{1}{\sqrt{n}} I_n(\phi_j)
\]
(3.7)

where \( I_n(\phi_j) \) is given in (1.2). As in [18] we define a class of weighted least squares estimates for \( S(t) \) as
\[
\hat{S}_{\gamma} = \sum_{j=1}^{n} \gamma(j)\hat{\theta}_{j,n} \phi_j,
\]
(3.8)

where the weights \( \gamma = (\gamma(j))_{1 \leq j \leq n} \in \mathbb{R}^n \) belong to some finite set \( \Gamma \) from \([0, 1]^n\) for which we set
\[
\nu = \text{card}(\Gamma) \quad \text{and} \quad |\Gamma|_* = \max_{\gamma \in \Gamma} \sum_{j=1}^{n} \gamma(j),
\]
(3.9)
where \( \text{card}(\Gamma) \) is the number of the vectors \( \gamma \) in \( \Gamma \). In the sequel we assume that all vectors from \( \Gamma \) satisfies the following condition.

**D_1)** Assume that for any vector \( \gamma \in \Gamma \) there exists some fixed integer \( d = d(\gamma) \) such that their first \( d \) components equal to one, i.e. \( \gamma(j) = 1 \) for \( 1 \leq j \leq d \) for any \( \gamma \in \Gamma \).

**D_2)** There exists \( n_0 \geq 1 \) such that for any \( n \geq n_0 \) there exists a \( \sigma \)-field \( \mathcal{G}_n \) for which the random vector \( \tilde{\xi}_{d,n} = (\xi_{j,n})_{1 \leq j \leq d} \) is the \( \mathcal{G}_n \)-conditionally Gaussian in \( \mathbb{R}^d \) with the covariance matrix

\[
\mathbf{G}_n = \left( \mathbb{E} \xi_{i,n} \xi_{j,n} | \mathcal{G}_n \right)_{1 \leq i,j \leq d}
\]

and for some nonrandom constant \( l^*_n > 0 \)

\[
\inf_{Q \in \mathcal{Q}_n} \left( \text{tr} \mathbf{G}_n - \lambda_{\max}(\mathbf{G}_n) \right) \geq l^*_n \quad \text{a.s.,} \quad (3.11)
\]

where \( \lambda_{\max}(A) \) is the maximal eigenvalue of the matrix \( A \).

As it is shown in Proposition 7.11 the condition \( \text{D}_2 \) holds for the non-Gaussian Ornstein–Uhlenbeck-based model \((1.1) - (2.1)\).

Further, for the first \( d \) Fourier coefficients in \((3.7)\) we will use the improved estimation method proposed for parametric models in [28]. To this end we set \( \tilde{\theta}_n = (\theta_{j,n})_{1 \leq j \leq d} \). In the sequel we will use the norm \( |x|_d^2 = \sum_{j=1}^d x_j^2 \) for any vector \( x = (x_j)_{1 \leq j \leq d} \) from \( \mathbb{R}^d \). Now we define the shrinkage estimators as

\[
\theta^*_{j,n} = (1 - g(j)) \tilde{\theta}_{j,n},
\]

where \( g(j) = (c_n/|\tilde{\theta}_n|_d) \mathbf{1}_{\{1 \leq j \leq d\}} \),

\[
c_n = \frac{l^*_n}{(r^*_n + \sqrt{d} \kappa_n/k_n)} \quad \text{and} \quad \kappa_n = \sup_{Q \in \mathcal{Q}_n} \kappa_Q.
\]

The positive parameter \( r^*_n \) is such that

\[
\lim_{n \to \infty} r^*_n = \infty \quad \text{and} \quad \lim_{n \to \infty} \frac{r^*_n}{n^\delta} = 0 \quad (3.13)
\]

for any \( \delta > 0 \).

Now we introduce a class of shrinkage weighted least squares estimates for \( S \) as

\[
S^*_\gamma = \sum_{j=1}^n \gamma(j) \theta^*_{j,n} \phi_j. \quad (3.14)
\]
We denote the difference of quadratic risks of the estimates (3.8) and (3.14) as
\[
\Delta Q(S) := \mathcal{R}_Q(S^*_\gamma, S) - \mathcal{R}_Q(\hat{S}_\gamma, S).
\]
For this difference we obtain the following result.

**Theorem 3.1.** Assume that the conditions \( D_1 \) – \( D_2 \) hold. Then for any \( n \geq n_0 \)
\[
\sup_{Q \in \mathcal{Q}_n} \sup_{\|S\| \leq r^*_n} \Delta Q(S) < -c^2_n. \tag{3.15}
\]

**Remark 3.1.** The inequality (3.15) means that non asymptotically, i.e. for any \( n \geq n_0 \) the estimate (3.14) outperforms in mean square accuracy the estimate (3.8).

## 4 Model selection method and oracle inequalities

This Section gives the construction of a model selection procedure for estimating a function \( S \) in (1.1) on the basis of improved weighted least square estimates and states the sharp oracle inequality for the robust risk of proposed procedure.

The model selection procedure for the unknown function \( S \) in (1.1) will be constructed on the basis of a family of estimates \((S^*_\gamma)_{\gamma \in \Gamma}\).

The performance of any estimate \( S^*_\gamma \) will be measured by the empirical squared error
\[
\text{Err}_n(\gamma) = \|S^*_\gamma - S\|^2.
\]
In order to obtain a good estimate, we have to write a rule to choose a weight vector \( \gamma \in \Gamma \) in (3.14). It is obvious, that the best way is to minimise the empirical squared error with respect to \( \gamma \). Making use the estimate definition (3.14) and the Fourier transformation of \( S \) implies
\[
\text{Err}_n(\gamma) = \sum_{j=1}^{n} \gamma(j)^2(\hat{\theta}_{j,n}^*)^2 - 2 \sum_{j=1}^{n} \gamma(j)\hat{\theta}_{j,n}^* \hat{\theta}_j + \sum_{j=1}^{n} \hat{\theta}_j^2. \tag{4.1}
\]
Since the Fourier coefficients \((\hat{\theta}_j)_{j \geq 1}\) are unknown, the weight coefficients \((\gamma_j)_{j \geq 1}\) can not be found by minimizing this quantity. To circumvent this difficulty one needs to replace the terms \( \hat{\theta}_{j,n}^* \hat{\theta}_j \) by their estimators \( \hat{\theta}_{j,n} \). We set
\[
\hat{\theta}_{j,n} = \hat{\theta}_{j,n}^* \hat{\theta}_{j,n} - \frac{\hat{\sigma}_n^2}{n}. \tag{4.2}
\]
where $\hat{\sigma}_n$ is the estimate for the limiting variance of $E_Q \xi_{j,n}^2$ which we choose in the following form

$$\hat{\sigma}_n = \sum_{j=\lceil \sqrt{n} \rceil + 1}^{n} \hat{t}_{j,n}^2, \quad \hat{t}_{j,n} = \int_{0}^{1} T_r(t) dy_t.$$  \hspace{1cm} (4.3)

For this change in the empirical squared error, one has to pay some penalty. Thus, one comes to the cost function of the form

$$J_n(\gamma) = \sum_{j=1}^{n} \gamma^2(j) (\theta_{j,n}^*)^2 - 2 \sum_{j=1}^{n} \gamma(j) \tilde{\theta}_{j,n} + \rho \hat{P}_n(\gamma)$$  \hspace{1cm} (4.4)

where $\rho$ is some positive constant, $\hat{P}_n(\gamma)$ is the penalty term defined as

$$\hat{P}_n(\gamma) = \frac{\hat{\sigma}_n|\gamma|^2}{n}.$$  \hspace{1cm} (4.5)

Substituting the weight coefficients, minimizing the cost function

$$\gamma^* = \arg\min_{\gamma \in \Gamma} J_n(\gamma),$$  \hspace{1cm} (4.6)

in (3.8) leads to the improved model selection procedure

$$S^* = S^*_{\gamma^*}.$$  \hspace{1cm} (4.7)

It will be noted that $\gamma^*$ exists because $\Gamma$ is a finite set. If the minimizing sequence in (4.6) $\gamma^*$ is not unique, one can take any minimizer.

To prove the sharp oracle inequality, the following conditions will be needed for the family $\mathcal{Q}_n$ of distributions of the noise $(\xi_t)_{t \geq 0}$ in (1.1).

We need to impose some stability conditions for the noise Fourier transform sequence $(\xi_{j,n})_{1 \leq j \leq n}$ introduced in [29]. To this end for some parameter $\sigma_Q > 0$ we set the following function

$$L_{1,n}(Q) = \sum_{j=1}^{n} \left| E_Q \xi_{j,n}^2 - \sigma_Q \right|.$$  \hspace{1cm} (4.8)

In [18] the parameter $\sigma_Q$ is called proxy variance.

**C1)** There exists a proxy variance $\sigma_Q > 0$ such that for any $\epsilon > 0$

$$\lim_{n \to \infty} \frac{L_{1,n}(Q)}{n^\epsilon} = 0.$$
Moreover, we define
\[ L_{2,n}(Q) = \sup_{|x| \leq 1} E_Q \left( \sum_{j=1}^{n} x_j \tilde{\xi}_{j,n} \right)^2 \]
and
\[ \tilde{\xi}_{j,n} = \xi_{j,n} - E_Q \xi_{j,n}^2. \]

\textbf{C}_2) Assume that for any \( \epsilon > 0 \)
\[ \lim_{n \to \infty} \frac{L_{2,n}(Q)}{n^\epsilon} = 0. \]

As is shown in Propositions 7.9 and 7.10, both conditions \( \textbf{C}_1 \) and \( \textbf{C}_2 \) hold for the model (1.1) with Ornstein-Uhlenbeck noise process (2.1).

**Theorem 4.1.** If the conditions \( \textbf{C}_1 \) and \( \textbf{C}_2 \) hold for the distribution \( Q \) of the process \( \xi \) in (1.1), then, for any \( n \geq 1 \) and \( 0 < \rho < 1/2 \), the risk (1.4) of estimate (4.7) for \( S \) satisfies the oracle inequality
\[ \mathcal{R}_Q(S^*, S) \leq 1 + \frac{5\rho}{1 - \rho} \min_{\gamma \in \Gamma} \mathcal{R}_Q(S^*_\gamma, S) + \frac{B_n(Q)}{n}, \tag{4.9} \]
where \( B_n(Q) = U_n(Q) \left( 1 + |\Gamma| E_Q |\hat{\sigma}_n - \sigma_Q| \right) \) and the coefficient \( U_n(Q) \) is such that for any \( \epsilon > 0 \)
\[ \lim_{n \to \infty} \frac{U_n(Q)}{n^\epsilon} = 0. \tag{4.10} \]

In the case, when the value of \( \sigma_Q \) in \( \textbf{C}_1 \) is known, one can take \( \hat{\sigma}_n = \sigma_Q \) and
\[ P_n(\gamma) = \frac{\sigma_Q |\gamma|^2}{n}, \tag{4.11} \]
and then we can rewrite the oracle inequality (4.9) with \( B_n(Q) = U_n(Q) \). Now we study the estimate (4.3).

**Proposition 4.2.** Let in the model (1.1) the function \( S(\cdot) \) is continuously differentiable. Then, for any \( n \geq 2 \),
\[ E_Q |\hat{\sigma}_n - \sigma_Q| \leq \frac{\kappa_n(Q)(1 + \|\dot{S}\|^2)}{\sqrt{n}}, \]
where the term \( \kappa_n(Q) \) possesses the property (4.10) and \( \dot{S} \) is the derivative of the function \( S \).
To obtain the oracle inequality for the robust risk (1.5) we need some additional condition on the distribution family \( Q_n \). We set
\[
\varsigma^* = \varsigma_n^* = \sup_{Q \in Q_n} \sigma_Q \quad \text{and} \quad L_n^* = \sup_{Q \in Q_n} (L_{1,n}(Q) + L_{2,n}(Q)).
\] (4.12)

**C\(_1^*\)** Assume that the conditions \( C_1^* \)–\( C_2^* \) hold and for any \( \epsilon > 0 \)
\[
\lim_{n \to \infty} \frac{L_n^* + \varsigma_n^*}{n^\epsilon} = 0.
\]

Now we impose the conditions on the set of the weight coefficients \( \Gamma \).

**C\(_2^*\)** Assume that the set \( \Gamma \) is such that for any \( \epsilon > 0 \)
\[
\lim_{n \to \infty} \frac{\nu_n}{n^\epsilon} = 0 \quad \text{and} \quad \lim_{n \to \infty} \frac{|\Gamma|^n}{n^{1/2+\epsilon}} = 0.
\]

**Theorem 4.3.** Assume that the conditions \( C_1^* \)–\( C_2^* \) hold. Then the robust risk (1.5) of the estimate (4.7) for continuously differentiable function \( S(t) \) satisfies for any \( n \geq 2 \) and \( 0 < \rho < 1/2 \) the oracle inequality
\[
R_n^*(S^*, S) \leq 1 + \frac{5\rho}{1 - \rho} \min_{\gamma \in \Gamma} R_n^*(S^*_{\gamma}, S) + \frac{1}{\rho n} B_n^*(1 + \|\dot{S}\|^2),
\]
where the term \( B_n^* \) satisfies the property (4.10).

Now we specify the weight coefficients \( \gamma(j) \), \( j \geq 1 \) in the way proposed in [10] for a heteroscedastic regression model in discrete time. Firstly, we define the normalizing coefficient which defined the minimax convergence rate
\[
v_n = \frac{n}{\varsigma^*},
\] (4.13)
where the upper proxy variance \( \varsigma^* \) is defined in (4.12). Consider a numerical grid of the form
\[
\mathcal{A}_n = \{1, \ldots, k^*\} \times \{r_1, \ldots, r_m\},
\]
where \( r_i = i\varepsilon \) and \( m = [1/\varepsilon^2] \). Both parameters \( k^* \geq 1 \) and \( 0 < \varepsilon \leq 1 \) are assumed to be functions of \( n \), i.e. \( k^* = k^*(n) \) and \( \varepsilon = \varepsilon(n) \), such that for any \( \delta > 0 \)
\[
\begin{align*}
\lim_{n \to \infty} k^*(n) &= +\infty, & \lim_{n \to \infty} \frac{k^*(n)}{\ln n} &= 0, \\
\lim_{n \to \infty} \varepsilon(n) &= 0 & \text{and} & \lim_{n \to \infty} n^{\delta} \varepsilon(n) &= +\infty.
\end{align*}
\]
One can take, for example,
\[ \varepsilon(n) = \frac{1}{\ln(n + 1)} \quad \text{and} \quad k^*(n) = \sqrt{\ln(n + 1)}. \]

For each \( \alpha = (\beta, r) \in \mathcal{A}_n \) we introduce the weight sequence \( \gamma_\alpha = (\gamma_\alpha(j))_{j \geq 1} \) as
\[ \gamma_\alpha(j) = 1_{\{1 \leq j \leq d\}} + \left(1 - \left(j/\omega_\alpha\right)^\beta\right) 1_{\{d < j \leq \omega_\alpha\}} \] (4.14)
where \( d = d(\alpha) = [\omega_\alpha/\ln(n + 1)] \), \( \omega_\alpha = \left(\tau_\beta r v_n\right)^{1/(2\beta + 1)} \) and
\[ \tau_\beta = \frac{(\beta + 1)(2\beta + 1)}{\pi^{2\beta}}. \]

We set
\[ \Gamma = \{\gamma_\alpha : \alpha \in \mathcal{A}_n\}. \] (4.15)

It will be noted that such weight coefficients satisfy the condition \( D_1 \) and in this case the cardinal of the set \( \Gamma \) is \( \nu = k^*m \). Moreover, taking into account that \( \tau_\beta < 1 \) for \( \beta \geq 1 \) we obtain for the set (4.15)
\[ |\Gamma| \leq 1 + \sup_{\alpha \in \mathcal{A}} \omega_\alpha \leq 1 + (v_n/\varepsilon)^{1/3}. \]

**Remark 4.1.** Note that the form (4.14) for the weight coefficients was proposed by Pinsker in [27] for the efficient estimation in the nonadaptive case, i.e. when the regularity parameters of the function \( S \) are known. In the adaptive case these weight coefficients are used in [18, 19] to show the asymptotic efficiency for model selection procedures.

## 5 Asymptotic efficiency

In order to study the asymptotic efficiency we define the following functional Sobolev ball
\[ W_{k,r} = \{f \in C^k_p[0,1] : \sum_{j=0}^{k} ||f^{(j)}||^2 \leq r\}, \] (5.1)
where \( r > 0 \) and \( k \geq 1 \) are some unknown parameters, \( C^k_p[0,1] \) is the space of \( k \) times differentiable 1 - periodic \( \mathbb{R} \to \mathbb{R} \) functions such that for any \( 0 \leq i \leq k - 1 \)
\[ f^{(i)}(0) = f^{(i)}(1). \]
In order to formulate our asymptotic results we define the well-known Pinsker constant which gives the lower bound for normalized asymptotic risks

$$l_k(r) = ((1 + 2k)r)^{1/(2k+1)} \left( \frac{k}{\pi(k+1)} \right)^{2k/(2k+1)} \left( \frac{k}{\pi(k+1)} \right)^{2k/(2k+1)}.$$  (5.2)

It is well known that for any $S \in W_{k,r}$ the optimal rate of convergence is $T^{-2k/(2k+1)}$ (see, for example, [11]). On the basis of the model selection procedure we construct the adaptive procedure $\hat{S}_n$ for which we obtain the following asymptotic upper bound for the quadratic risk.

Now we show that the parameter (5.2) gives a lower bound for the asymptotic normalized risks. To this end we denote by $\Sigma_n$ of all estimators $\hat{S}_n$ of $S$ measurable with respect to the process (1.1), i.e. $\sigma\{y_t, 0 \leq t \leq n\}$.

**Theorem 5.1.** The robust risk (1.5) admits the following lower bound

$$\liminf_{n \to \infty} \inf_{\hat{S}_n \in \Sigma_n} v_n^{2k/(2k+1)} \sup_{S \in W_{k,r}} \mathcal{R}_n^*(\hat{S}_n, S) \geq l_k(r).$$  (5.3)

We show that this lower bound is sharp in the following sense.

**Theorem 5.2.** The quadratic risk (1.2) for the estimating procedure $S^*$ has the following asymptotic upper bound

$$\limsup_{n \to \infty} v_n^{2k/(2k+1)} \sup_{S \in W_{k,r}} \mathcal{R}_n^*(S^*, S) \leq l_k(r).$$  (5.4)

It is clear that Theorem 5.2 and Theorem 5.1 imply

**Corollary 5.3.** The model selection procedure $S^*$ is efficient, i.e.

$$\lim_{n \to \infty} (v_n)^{2k/(2k+1)} \sup_{S \in W_{k,r}} \mathcal{R}_n^*(S^*, S) = l_k(r).$$  (5.5)

**Remark 5.1.** Note that the equality (5.5) implies that the parameter (5.2) is the Pinsker constant in this case (cf. [27]).

**Remark 5.2.** It should be noted that the equality (5.5) means that the robust efficiency holds with the convergence rate

$$(v_n)^{2k/(2k+1)}.$$

It is well known that for the simple risks the optimal (minimax) estimation convergence rate for the functions from the set $W_{k,r}$ is $n^{2k/(2k+1)}$ (see, for example, [27], [26]). So, if the upper bound of the distribution variance $\varsigma^* \to 0$ as $n \to \infty$ we obtain the more rapid rate, and if $\varsigma^* \to \infty$ as $n \to \infty$ we obtain the more slow rate. In the case when $\varsigma^*$ is constant the robust rate is the same as the classical non-robust convergence rate.
6 Monte Carlo simulations

In this section we give the results of numerical simulations to assess the performance and improvement of the proposed model selection procedure (4.6). We simulate the model (1.1) with 1-periodic function $S$ of the form

$$S(t) = t \sin(2\pi t) + t^2(1 - t) \cos(4\pi t)$$

(6.1)
on [0, 1] and the Lévy noise process $\xi_t$ is defined as

$$d\xi_t = -\xi_t dt + 0.5 dw_t + 0.5 dz_t.$$

Here $z_t$ is a compound Poisson process with intensity $\lambda = \Pi(x^2) = 1$ and a Gaussian $\mathcal{N}(0, 1)$ sequence $(Y_j)_{j \geq 1}$ (see, for example, [19]).

We use the model selection procedure (4.6) with the weights (4.14) in which $k^* = 100 + \sqrt{\ln(n + 1)}$, $r_i = i / \ln(n + 1)$, $m = [\ln^2(n + 1)]$, $\varsigma^* = 0.5$ and $\delta = (3 + \ln n)^{-2}$. We define the empirical risk as

$$R(S^*, S) = \frac{1}{p} \sum_{j=1}^{p} \hat{E}(S^*_n(t_j) - S(t_j))^2,$$

$$\hat{E}(S^*_n(\cdot) - S(\cdot))^2 = \frac{1}{N} \sum_{l=1}^{N} (S^*_n(\cdot) - S(\cdot))^2,$$

where the observation frequency $p = 100001$ and the expectations was taken as an average over $N = 1000$ replications.

| $n$ | 100  | 200  | 500  | 1000 |
|-----|------|------|------|------|
| $R(S^*_n, S)$ | 0.0289 | 0.0089 | 0.0021 | 0.0011 |
| $R(S^*_n, S)$ | 0.0457 | 0.0216 | 0.0133 | 0.0098 |
| $R(S^*_n, S) / R(S^*_n, S)$ | 1.6 | 2.4 | 6.3 | 8.9 |

Table 1 gives the values for the sample risks of the improved estimate (4.6) and the model selection procedure based on the weighted LSE (3.15) from [18] for different numbers of observation period $n$. Table 2 gives the values for the sample risks of the the model selection procedure based on the weighted LSE (3.15) from [18] and it’s improved version for different numbers of observation period $n$. 
Table 2: The sample quadratic risks for the same optimal $\hat{\gamma}$

| $n$   | 100  | 200  | 500  | 1000 |
|-------|------|------|------|------|
| $R(S^*_\gamma, S)$ | 0.0391 | 0.0159 | 0.0098 | 0.0066 |
| $R(\hat{S}_\gamma, S)$ | 0.0457 | 0.0216 | 0.0133 | 0.0098 |
| $R(\hat{S}_\gamma, S)/R(S^*_\gamma, S)$ | 1.2 | 1.4 | 1.3 | 1.5 |

Figure 1: Behavior of the regression function and its estimates for $n = 500$.

Figure 2: Behavior of the regression function and its estimates for $n = 1000$.

Remark 6.1. Figures 1–2 show the behavior of the procedures (3.8) and (4.6) depending on the values of observation periods $n$. The bold line is the function (6.1), the continuous line is the model selection procedure based on the least squares estimators $\hat{S}$ and the dashed line is the improved model selection.
procedure $S^*$. From the Table 2 for the same $\gamma$ with various observations numbers $n$ we can conclude that theoretical result on the improvement effect (3.15) is confirmed by the numerical simulations. Moreover, for the proposed shrinkage procedure, Table 1 and Figures 1–2, we can conclude that the benefit is considerable for non large $n$.

7 Stochastic calculus for Ornstein-Uhlenbeck–Lévy process

In this section we study the process (2.1).

**Proposition 7.1.** Let $f$ and $g$ be two nonrandom left continuous $\mathbb{R}_+ \to \mathbb{R}$ functions with the finite right limits. Then for any $t > 0$

$$
\mathbb{E} I_t(f) I_t(g) = \sigma_Q \tau_t(f, g),
$$

where $\tau_t(f, g) = \int_0^t (f(s)g(s) + \dot{\varepsilon}_s(f)g(s) + f(s)\dot{\varepsilon}_s(g)) \, ds$ and

$$
\dot{\varepsilon}_t(f) = a \int_0^t e^{a(t-s)} f(s) \left( \frac{1 + e^{2as}}{2} \right) ds.
$$

**Proof.** Taking into account the definitions (3.7) and (2.1) we obtain through the Ito formula that

$$
I_t(f) I_t(g) = \sigma_Q \int_0^t f(s)g(s)ds + a \int_0^t \Upsilon_s(f, g) \xi_s ds + M_t(f, g),
$$

where $\Upsilon_s(f, g) = f(s)I_s(g) + g(s)I_s(f)$,

$$
M_t(f, g) = \int_0^t \Upsilon_{s-}(f, g) \, du_s + \varrho^2 \int_0^t f(s) g(s) \, dm_s
$$

and $m_t = x^2 * (\mu - \tilde{\mu})_t$. Moreover, using the Ito formula we obtain

$$
\mathbb{E} I_t^2(1) = \mathbb{E} \xi_t^2 = \sigma_Q \frac{e^{2at} - 1}{2a}.
$$

Note now, that

$$
\mathbb{E} I_t^2 = \mathbb{E} \left( a \int_0^t f(s)\xi_s ds + \int_0^t f(s) \, du_s \right)^2 \\
\leq 2a^2 \int_0^t f^2(s)ds \int_0^t \mathbb{E} \xi_s^2 ds + 2\sigma_Q \int_0^t f^2(s)ds.
$$
So, from here

\[
\sup_{0 \leq t \leq n} E I^2_t(f) \leq 2\sigma_Q (|a| + 1) \int_0^n f^2(s) ds < \infty. \quad (7.4)
\]

This implies immediately that \( E M_t(f, g) = 0 \). Using this in (7.2) yields

\[
E I_t(f) I_t(g) = \sigma_Q \int_0^t f(s) g(s) ds + a \int_0^t (f(s) E \tilde{\zeta}_s(g) + g(s) E \tilde{\zeta}_s(f)) ds, \quad (7.5)
\]

where \( \zeta_t(f) = \xi_t I_t(f) = I_t(1) I_t(f) \). Therefore, putting \( g = 1 \) in (7.5), we obtain that

\[
E \zeta_t(f) = \sigma_Q \int_0^t f(s) ds + a \int_0^t (f(s) E \zeta_s(1) + E \zeta_s(f)) ds.
\]

Taking into account here, that \( \zeta_t(1) = \xi_t^2 \), we obtain that

\[
E \zeta_t(f) = \sigma_Q \int_0^t e^{a(t-s)} f(s) \frac{1 + e^{2as}}{2} ds = \sigma_Q \tilde{\varepsilon}_t(f).
\]

Therefore, using this in (7.5) we obtain (7.1). \( \square \)

**Corollary 7.2.** For any cadlag function \( f \) from \( L_2[0, n] \)

\[
E I_n^2(f) \leq 2\sigma_Q \int_0^n f^2(s) ds. \quad (7.6)
\]

**Proof.** Indeed, putting \( f = g \) in (7.1) we get

\[
E I_n^2(f) = \sigma_Q \int_0^n \left( f^2(t) + 2\tilde{\varepsilon}_t(f) f(t) \right) dt.
\]

Moreover, note that

\[
\int_0^n \tilde{\varepsilon}_t(f) f(t) dt = a \int_0^n e^{ax} \int_x^n \left( f(t) f(t-x) \frac{1 + e^{2a(t-x)}}{2} \right) dt dx.
\]

By the Bunyakovskii–Cauchy–Schwarz inequality

\[
\int_0^n \tilde{\varepsilon}_t(f) |f(t)| dt \leq |a| \int_0^n e^{ax} dx \int_0^n f^2(t) dt \leq \int_0^n f^2(t) dt.
\]
This implies immediately upper bound (7.6). Hence Corollary 7.2. □

Now we set

\[ \tilde{I}_t(f) = I_t^2(f) - \mathbf{E} I_t^2(f) \quad \text{and} \quad V_t(f) = \zeta_t(f) - \mathbf{E} \zeta_t(f). \]  

Using (7.2) with \( f = g \) we can obtain that

\[ d\tilde{I}_t(f) = 2a f(t) V_t(f) dt + d\tilde{M}_t(f), \]  

where \( \tilde{M}_t(f) = M_t(f, f) \). To study this process we need to introduce the following functions

\[ \tilde{\tau}_t(f, g) = f(t)g(t)\tau_t(1, 1) + f(t)\tau_t(1, g) + g(t)\tau_t(1, f) + \tau_t(f, g) \]  

and

\[ A_t(f) = \int_0^t e^{3a(t-s)} f(s) v(s) ds + 2\sigma_Q^2 \int_0^t e^{3a(t-s)} \tilde{\xi}_s(f) ds, \]  

where \( v(s) = a^2 \mathbf{E} \tilde{\xi}_s^2 + \sigma_Q^2 (e^{2as} - 1) + a\varrho_2, \tilde{\xi}_s = \xi_s - \mathbf{E} \xi_s^2 \) and \( \varrho_2 = \varrho_2^4 \Pi(x^4) \).

**Proposition 7.3.** For any left continuous functions with finite right limits \( f \) and \( g \)

\[ \mathbf{E} V_t(f) V_t(g) = \int_0^t e^{2a(t-s)} H_s(f, g) ds \]  

where \( H_t(f, g) = g(t)A_t(f) + f(t)A_t(g) + \sigma_Q^2 \tilde{\tau}_t(f, g) + \tilde{\varrho}_2 f(t)g(t). \)

**Proof.** Applying again (7.2) with \( g = 1 \) yields

\[ dV_t(f) = aV_t(f) dt + a f(t) \tilde{I}_t(1) dt + dL_t(f), \]  

where \( L_t(f) = \int_0^t \tilde{I}_{s-} f(s) d\mu_s + \varrho_2^2 \int_0^t f(s) d\lambda_s \) and \( \tilde{I}_s(f) = f(s)\xi_s + I_s(f) \). By the Ito formula we get

\[ dV_t(f)V_t(g) = 2a V_t(f) V_t(g) dt + a(g(t)V_t(f) + f(t)V_t(g)) \tilde{I}_t(1) dt \]
\[ + d[L(f), L(g)]_t + V_{t-}(f)dL_t(g) + V_{t-}(g)dL_t(f). \]

Now from Lemma A.2 we obtain that

\[ d\mathbf{E} V_t(f) V_t(g) = 2a \mathbf{E} V_t(f) V_t(g) dt + (g(t)A_t(f) + f(t)A_t(g)) dt \]
\[ + d\mathbf{E} [L(f), L(g)]_t, \]  

(7.13)
where \( A_t(f) = a E V_t(f) \tilde{I}_t(1) = a E V_t(f) V_t(1) \). Note that \( E \tilde{I}_s(f) \tilde{I}_s(g) = \sigma_Q \tilde{\tau}_s(f, g) \) and
\[
E[L(f), L(g)]_t = \varrho_1^2 \int_0^t E \tilde{I}_s(f) \tilde{I}_s(g) \, ds + E \sum_{0 \leq s \leq t} \Delta L_s(f) \Delta L_s(g) \\
= \sigma_Q^2 \int_0^t \tilde{\tau}_s(f, g) \, ds + \tilde{\varrho}_2 \int_0^t f(s) g(s) \, ds.
\]
To find the function \( A_t(f) \) we put \( g = 1 \) in (7.13). Taking into account that \( A_t(1) = \tilde{\xi}_t^2 \), we get
\[
E V_t(f) V_t(1) = \int_0^t e^{3a(t-s)} \left( a f(s) E \tilde{\xi}_s^2 + \sigma_Q \tilde{\tau}_s(f, 1) + \tilde{\varrho}_2 f(s) \right) \, ds.
\]
Using here that \( a \tau_t(1, 1) = (e^{2at} - 1)/2 \) and \( a \tau_t(1, f) = \tilde{\varepsilon}_t(f) \), (7.14)
we obtain the representation (7.10). Hence Proposition 7.3. \( \square \)

**Proposition 7.4.** For any left continuous function \( f \) with finite right limits
\[
E \tilde{I}_t(f) \tilde{I}_t(1) = \int_0^t e^{2a(t-s)} \tilde{\xi}_s(f) \, ds,
\]
where \( \tilde{\xi}_s(f) = 2f(s)A_s(f) + 4\sigma_Q^2 f(s) \tau_s(f, 1) + \tilde{\varrho}_2 f^2(s) \).

**Proof.** Using the Ito formula and Lemma A.2 we obtain that for any bounded nonrandom functions \( f \) and \( g \)
\[
\begin{align*}
&d E \tilde{I}_t(f) V_t(g) = a E \tilde{I}_t(f) V_t(g) \, dt + 2af(t) E V_t(f) V_t(g) \, dt \\
&\quad + a g(t) E \tilde{I}_t(f) \tilde{I}_t(1) \, dt + d E [\tilde{M}(f), L(g)]_t.
\end{align*}
\] (7.16)
Putting here \( g = 1 \) and taking into account that \( V_t(1) = \tilde{I}_t(1) \), we obtain that
\[
\begin{align*}
&d E \tilde{I}_t(f) V_t(1) = 2a E \tilde{I}_t(f) V_t(1) \, dt + 2af(t) E V_t(f) V_t(1) \, dt \\
&\quad + d E [\tilde{M}(f), L(1)]_t.
\end{align*}
\]
By the direct calculation we find
\[
E [\tilde{M}(f), L(1)]_t = \int_0^t \tilde{A}_s(f) \, ds.
\]
So, we get (7.15) and this proposition.

Further we need the following correlation measures for two integrated \([0, +\infty) \to \mathbb{R}\) functions \(f\) and \(g\)

\[
\varpi_n(f, g) = \max_{0 \leq v + t \leq n} \left( \left\| \int_0^t f(u + v)g(u)du \right\| + \left\| \int_0^t g(u + v)f(u)du \right\| \right) \tag{7.17}
\]

For any bounded \([0, \infty) \to \mathbb{R}\) function \(f\) we introduce the following uniform norm

\[
\|f\|_{*, n} = \sup_{0 \leq t \leq n} |f(t)|.
\]

**Proposition 7.5.** Let \(f\) and \(g\) be two left continuous bounded by \(\phi_*\) functions with finite right limits, i.e. \(\|f\|_{*, n} \leq \phi_*\) and \(\|g\|_{*, n} \leq \phi_*\). Then for any \(0 \leq t \leq n\)

\[
|aE \tilde{I}_t(f) V_t(g)| \leq u^*_1 \varpi_t(1, g) + u^*_2 \varpi_t(f, g) + u^*_3, \tag{7.18}
\]

where \(u^*_1 = 4\phi^2(a_{\text{max}})\tilde{\varrho}_2 + 3\sigma^2_Q\), \(u^*_2 = 44\phi_*\sigma^2_Q\) and \(u^*_3 = 3\phi^3_\varrho \tilde{\varrho}_2\).

**Proof.** First, note that from Ito formula we find

\[
aE \tilde{I}_t(f) V_t(g) = a^2 \int_0^t e^{a(t-s)} g(s) \left( E \tilde{I}_s(f) \tilde{I}_s(1) \right) ds \\
+ 2a^2 \int_0^t e^{a(t-s)} f(s) \left( E V_s(g) V_s(f) \right) ds \\
+ a \int_0^t e^{a(t-s)} dE [\tilde{M}(f), L(g)]_s. \tag{7.19}
\]

Using here Lemma A.4. and Lemma A.6 we can obtain that

\[
|aE V_t(g) V_t(f)| \leq 15\sigma^2_Q \varpi_t(f, g) + \tilde{\varrho}_2 \|f\|_{*, t} \|g\|_{*, t}. \tag{7.20}
\]

One can check directly that

\[
E [\tilde{M}(f), L(g)]_s = 2\sigma_Q \int_0^t g(s)f(s) \left( E I_s(f) I_s(1) \right) ds \\
+ 2\sigma_Q \int_0^t f(s) \left( E I_s(f) I_s(g) \right) ds + \tilde{\varrho}_2 \int_0^t f^2(s)g(s) ds.
\]
From (7.1) we find that
\[
E \left[ \tilde{M}(f), L(g) \right]_s = 2\sigma^2_Q \int_0^t g(s)f(s)\tau_s(f, 1) \, ds \\
+ 2\sigma^2_Q \int_0^t f(s)\tau_s(f, g) \, ds + \tilde{\varrho}_2 \int_0^t f^2(s)g(s) \, ds.
\]

Using the last equality in (7.14) we obtain that
\[
a \int_0^t e^{a(t-s)} \, dE \left[ \tilde{M}(f), L(g) \right]_s = 2\sigma^2_Q \int_0^t e^{a(t-s)} g(s)f(s)\xi_s(f) \, ds \\
+ 2\sigma^2_Q \int_0^t e^{a(t-s)} f(s)\tau_s(f, g) \, ds + \tilde{\varrho}_2 \int_0^t e^{a(t-s)} f^2(s)g(s) \, ds.
\]

Note now that
\[
\tilde{\xi}'(f) = a\xi_t(f) + af(t)(1 + e^{2at})/2,
\]
i.e. \( ||\tilde{\xi}'(f)||_{s,t} \leq 2|a||f||_{s,t} \). Therefore, in view of Lemma A.3 we get
\[
\left| \int_0^t e^{a(t-s)} g(s)f(s)\xi_s(f) \, ds \right| \leq 4\varpi_t(f, g)||f||_{s,t}.
\]

Moreover, by integrating by parts we can obtain directly that
\[
\left| \int_0^t g(s)\xi_s(f) \, ds \right| \leq \varpi_t(f, g),
\]
and, therefore,
\[
|\tau_t(f, g)| \leq 3\varpi_t(f, g).
\]

So, the last term in (7.19) can be estimated as
\[
\left| a \int_0^t e^{a(t-s)} \, dE \left[ \tilde{M}(f), L(g) \right]_s \right| \leq 14\sigma^2_Q\varpi_t(f, g)||f||_{s,t} + \tilde{\varrho}_2||f||_{s,t}^2||g||_{s,t}.
\]

Using Lemma A.5 in (7.19) we come to the bound (7.18). Hence Proposition 7.5.

**Proposition 7.6.** Let \( f \) and \( g \) be two left continuous bounded by \( \phi \) functions with finite right limits, i.e. \( ||f||_{s,n} \leq \phi \) and \( ||g||_{s,n} \leq \phi \). Then for any \( t > 0 \)
\[
\left| E \left[ \tilde{M}(f), \tilde{M}(g) \right]_t \right| \leq \left( 12\sigma^2_Q\phi^2\varpi_t(f, g) + \phi^4\tilde{\varrho}_2 \right) t.
\]
Proof. First of all note that from (7.1) we obtain that
\[
E \{ \tilde{M}(f), \tilde{M}(g) \}_{t} = 4\sigma_{Q}^{2} \int_{0}^{t} f(s)g(s)\tau_{s}(f, g) \, ds \\
+ \tilde{\varrho}_{2} \int_{0}^{t} f^{2}(s)g^{2}(s) \, ds.
\] (7.23)

Using here the bound (7.21) we obtain (7.22). Hence Proposition 7.6.

Corollary 7.7. Let \( f \) and \( g \) be two left continuous bounded by \( \phi_{*} \) functions with finite right limits, i.e. \( \|f\|_{*,n} \leq \phi_{*} \text{ and } \|g\|_{*,n} \leq \phi_{*} \). Then for any \( t > 0 \)
\[
E \tilde{I}_{t}(f)\tilde{I}_{t}(g) \leq \left( v_{1}^{*}(\varpi_{t}(1, f) + \varpi_{t}(1, g)) + v_{2}^{*}\varpi_{t}(f, g) + v_{3}^{*} \right) t,
\] (7.24)
where \( v_{1}^{*} = 8\phi_{2}^{2}\varphi_{\text{max}}\tilde{\varrho}_{2} + 6\sigma_{Q}^{2}, \ v_{2}^{*} = 100\phi_{2}^{2}\sigma_{Q}^{2} \) and \( v_{3}^{*} = 13\phi_{4}^{4}\tilde{\varrho}_{2}. \)

Proof. From (7.8) by the Ito formula one finds for \( t \geq 0 \)
\[
E \tilde{I}_{t}(f)\tilde{I}_{t}(g) = E \{ \tilde{M}(f), \tilde{M}(g) \}_{t} \\
+ 2a \int_{0}^{t} \left( f(s)E \tilde{I}_{s}(g)V_{s}(f) + g(s)E \tilde{I}_{s}(f)V_{s}(g) \right) \, ds.
\] (7.25)

Using here Proposition 7.5 and Proposition 7.6 we come to desire result.

Now we set
\[
\tilde{I}_{n}(x) = \sum_{j=1}^{n} x_{j} \tilde{I}_{n}(\phi_{j}).
\] (7.26)

For this we show the following proposition.

Proposition 7.8. Assume that \( \phi_{1} \equiv 1 \). Then for any \( n \geq 1 \)
\[
E \tilde{I}_{n}^{2}(x) \leq 4n^{2} \left( (2\varpi_{n}^{*} + 4\phi_{2}^{2})v_{1}^{*} + (\varpi_{n}^{*} + 8\phi_{*}^{2})v_{2}^{*} + (1 + 2\phi_{2}^{2})v_{3}^{*} \right),
\] (7.27)
where \( \varpi_{n}^{*} = \sup_{|i-j| \geq 2} \varpi_{n}(\phi_{i}, \phi_{j}) \).

Proof. We represent the sum as
\[
\sum_{j=1}^{n} x_{j} \tilde{I}_{n}(\phi_{j}) = J_{1,n} + J_{2,n},
\]
where \( J_{1,n} = x_1 \tilde{I}_n(\phi_1) + x_2 \tilde{I}_n(\phi_2) \) and \( J_{2,n} = \sum_{j=3}^n x_j \tilde{I}_n(\phi_j) \). From here we have
\[
E \tilde{I}_n^2(x) \leq 2 \left( E J_{1,n}^2 + E J_{2,n}^2 \right) .
\] (7.28)

By applying the Cauchy-Schwarz-Bounyakovskii inequality and noting that \( x_1^2 + x_2^2 \leq 1 \), one gets
\[
E J_{1,n}^2 \leq E \tilde{I}_n^2(\phi_1) + E \tilde{I}_n^2(\phi_2).
\]

Corollary 7.7 implies
\[
E_{Q,S} J_{1,n}^2 \leq 4 \phi_2^2 n^2 \left( 2 v_1^* + v_2^* + v_3^* \right) .
\]

Here we use that each \( \tilde{\varpi}_n(\phi_i, \phi_j) \leq 2 \phi_2^* n \).

Applying Corollary 7.7, one gets
\[
E J_{2,n}^2 = 2 \sum_{i,j=3}^n x_i x_j E \tilde{I}_n(\phi_i) \tilde{I}_n(\phi_j) \leq 2n \sum_{i,j=3}^n |x_i||x_j| \tilde{\kappa}_{i,j} ,
\] (7.29)

where \( \tilde{\kappa}_{i,j} = v_1^*(\tilde{\varpi}_n(1, \phi_i) + \tilde{\varpi}_n(1, \phi_j)) + v_2^* \tilde{\varpi}_n(\phi_i, \phi_j) + v_3^* . \) We can estimate the coefficient \( \tilde{\varpi}_{i,j} = \tilde{\varpi}_n(\phi_i, \phi_j) \) for any \( i \geq 3 \) as \( \tilde{\varpi}_{i,j} \leq 2 \phi_2^* n 1_{\{|i-j|\leq 1\}} + \phi^*_n 1_{\{|i-j|\geq 2\}} \). By making use of this estimate in (7.29) and taking into account that
\[
\sum_{i,j \geq 1} |x_i||x_j| \leq 1 \quad \text{and} \quad \sum_{i,j \geq 3} 1_{\{|i-j|\leq 1\}} |x_i||x_j| \leq 3 ,
\]

one gets
\[
\sum_{i,j \geq 3} |x_i||x_j| \tilde{\kappa}_{i,j} \leq n \left( 2 \phi^*_n v_1^* + (6 \phi^*_n) v_2^* + v_3^* \right) .
\]

From here and the inequalities (7.28)–(7.29) we come to the desired assertion.

Hence Proposition 7.8.

Now we check the conditions \( C_1 \) and \( C_2 \) for Ornstein–Uhlenbeck model. For this we will use the trigonometric basis (3.3).

Note that in this case the proxy variance \( \sigma_Q > 0 \) is defined in (2.4).

**Proposition 7.9.** Then for any \( Q \in Q_n \) and any \( n \geq 1 \)
\[
L_{1,n}(Q) \leq 2 \sigma_Q^2 (4 a^2 + 15 |a| + 2) .
\]

**Proof.** First we note that
\[
E_{Q,S} \xi_{j,n}^2 = \sigma_Q \left( 1 + b_{j,n} \right) ,
\] (7.30)
where \( b_{j,n} = n^{-1} a \int_0^n e^{av} \Upsilon_j(v) dv \) and
\[
\Upsilon_j(v) = \int_0^{n-v} \text{Tr}_j(t + v) \text{Tr}_j(t) \left(1 + e^{2at}\right) dt.
\]

If \( j = 1 \), one has
\[
|\mathbf{E}_{Q,S}^{2\xi_{1,n}^2} - \sigma_Q| \leq 2\sigma_Q. \tag{7.31}
\]

Since for the trigonometric basis (3.3) for \( j \geq 2 \)
\[
\text{Tr}_j(t + v) \text{Tr}_j(t) = \cos(\gamma_j v) + (-1)^j \cos(2t + v)
\]
where \( \gamma_j = 2\pi [j/2] \), therefore,
\[
\Upsilon_j(v) = \cos(\gamma_j v) F(v) + (-1)^j \Upsilon_0,j(v), \quad F(v) = \int_0^{n-v} \left(1 + e^{2at}\right) dt
\]
and
\[
\Upsilon_{0,j}(v) = \int_0^{n-v} \cos(\gamma_j(2t + v)) \left(1 + e^{2at}\right) dt.
\]
Integrating by parts one finds
\[
\Upsilon_{0,j}(v) = -\frac{2 + e^{2a(n-v)}}{2\gamma_j} \sin(v\gamma_j) + \frac{a}{2\gamma_j^2} \Upsilon_{1,j}(v)
\]
where
\[
\Upsilon_{1,j}(v) = \cos(v\gamma_j)(e^{2a(n-v)} - 1) - 2a \int_0^{n-v} e^{2at} \cos((2t + v)\gamma_j) dt.
\]

It is obvious that \(|\Upsilon_{1,j}(v)| \leq 2\). Further we calculate
\[
b_{j,n} = \frac{a}{n} \int_0^n e^{av} F(v) \cos(v\gamma_j) dv + \frac{a}{n} (-1)^j \int_0^n e^{av} \Upsilon_{0,j}(v) dv
\]
\[
: = aD_{1,j} + a(-1)^j D_{2,j}.
\]

Integrating by parts two times yields
\[
D_{1,j} = \frac{1}{n\gamma_j^2} \left(e^{an} \hat{F}(n) - \hat{F}(0) - aF(0) - \int_0^n e^{av} F_1(v) dv \right),
\]
where \( F_1(v) = a^2 F(v) + 2a \hat{F}(v) + \ddot{F}(v) \). Since \( \gamma_j \geq j \) for \( j \geq 2 \), we obtain
\[
|D_{1,j}| \leq \frac{1}{j^2} (4|a| + 10).
\]
Similarly, one gets \(|D_{2,j}| \leq 5/j^2\). Substituting these estimates in (7.30) and using the upper bound (7.31), we obtain for all \(j \geq 1\)

\[ |E_{Q,s} \xi^2_{j,n} - \sigma_Q| \leq \sigma_Q \left( \frac{4a^2 + 15|a| + 2}{j^2} \right). \] (7.32)

Thus we arrive at the inequality

\[ L_{1,n}(Q) \leq 2\sigma_Q(4a^2 + 15|a| + 2). \]

\[ \square \]

Proposition 7.9 and (2.4) – (2.5) imply that the condition \(C_1\) holds.

**Proposition 7.10.** For any \(n \geq 1\) and \(Q \in Q_n\)

\[ L_{2,n}(Q) \leq 8M_Q, \]

where \(M_Q = 48\sqrt{2}|a|\tilde{\varrho}_2 + 918\sigma_Q + 65\tilde{\varrho}_2\).

**Proof.** We note that for the trigonometric basis (3.3) \(\|\text{Tr}_j\|_{*,n} \leq \sqrt{2}\) and \(\varpi^*_n = 2\). Indeed, for any \(i \geq 3\),

\[ \text{Tr}_i(v + u) = \kappa_{1,i}(v)\text{Tr}_{i-1}(u) + \kappa_{2,i}(v)\text{Tr}_i(u) + \kappa_{3,i}(v)\text{Tr}_{i+1}(u), \]

where \(\kappa_{i,j}(\cdot)\) are bounded functions. From here in view of the orthonormality and the periodicity of the functions \((\text{Tr}_j)_{j \geq 1}\), it follows that for \(0 \leq t \leq n\) and \(|i - j| \geq 2\)

\[ \left| \int_0^t \text{Tr}_i(u + v)\text{Tr}_j(u)du \right| = \left| \int_0^{\{t\}} \text{Tr}_i(u + v)\text{Tr}_j(u)du \right| \leq \sqrt{\int_0^1 \text{Tr}_i^2(u + v)du} = 1, \]

where \(\{t\}\) is the fractional part of \(t\). Therefore \(\varpi^*_n \leq 2\) if \(|i - j| \geq 2\). Thus, we have that \(L_{2,n}(Q) \leq 8M_Q\). Hence Proposition 7.10. \(\square\)

Proposition 7.10 and (2.4) – (2.5) imply that the condition \(C_2\) holds.

**Proposition 7.11.** Let the noise \((\xi_t)_{t \geq 0}\) in equation (1.1) describes by non-Gaussian Ornstein–Uhlenbeck process (2.1). Assume that the basis function satisfy the conditions \(B_1\) and \(B_2\). Then for all \(d \geq d_0\) the condition \(D_2\) holds with \(l^*_n = \varrho_n(d - 6)/2\).
Proof. We have
\[ \xi_t = \rho_1 \xi_t^{(1)} + \rho_2 \xi_t^{(2)}, \quad 0 \leq t \leq n, \]
where \((\xi_t^{(1)})_{t \geq 0}\) and \((\xi_t^{(2)})_{t \geq 0}\) are independent Ornstein–Uhlenbeck processes obey the equations
\[ d\xi_t^{(1)} = a \xi_t^{(1)} dt + dw_t \quad \text{and} \quad d\xi_t^{(2)} = a \xi_t^{(2)} dt + dz_t. \]
Moreover, for any square integrated functions \(f\) we set
\[ I_t^{(1)}(f) = \int_0^t f(s)d\xi_s^{(1)} \quad \text{and} \quad I_t^{(2)}(f) = \int_0^t f(s)d\xi_s^{(2)}. \]
(7.33)
Then the matrix \(D_n\) can be rewritten as
\[ D_n = \rho_2 D_{1,n} + \rho_2^2 D_{2,n}, \]
where the \((i, j)\) element of the matrix \(D_{i,n}\) is defined as
\[ E(I_n^{(l)}(\phi_i)I_n^{(l)}(\phi_j)|G). \]
Using the celebrated inequality of Lidskii and Wieland (see, for example, in [23], G.3.a., p.334 ) we obtain
\[ \text{tr} D(G) - \lambda_{\text{max}}(D(G)) \geq \rho_2^2 (\text{tr} G_{1,n} - \lambda_{\text{max}}(G_{1,n})) \quad \text{a.s.} \quad (7.34) \]
Now, using Proposition 7.1 with \(\rho_1 = 1\) and \(\rho_2 = 0\) we obtain that
\[ \text{tr} G_n = \frac{1}{n} \sum_{j=1}^d \text{E}(I_n^{(1)}(\phi_j))^2 = d + \sum_{j=1}^d b_{j,n}, \]
(7.35)
where
\[ b_{j,n} = \frac{a}{n} \int_0^n \phi_j(t) \int_0^t e^{a(t-s)} \phi_j(s)(1 + e^{2as})dsdt. \]
Therefore, setting \(\Phi(t, v) = \sum_{j=1}^d \phi_j(t)\phi_j(t-v)\), we get
\[ \text{tr} \Phi_{1,n} = d + \frac{a}{n} \int_0^n e^{av} \left( \int_0^n \Phi(t, v) (1 + e^{2a(t-v)}) dt \right) dv \]
\[ \geq d - 2|a| \int_0^n e^{av} \Phi_d(v) dv, \]
where the function \(\Phi_d(v)\) is defined in the condition \(B_2\). Taking into account that this function is 1 - periodic we conclude that
\[ \text{tr} \Phi_{1,n} > d - 2|a| \sum_{k=1}^n e^{a(k-1)} \int_0^1 \Phi_d(v) dv \]
\[ > d - \frac{2a_{\text{max}}}{1 - e^{-a_{\text{max}}}} \int_0^1 \Phi_d(v) dv = d - \frac{1}{2a} \int_0^1 \Phi_d(v) dv, \]
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where  \( \bar{a} \) is given in the condition \( B_{2} \) which implies immediately

\[
\text{tr} \mathcal{D}_{1,n} > \frac{d}{2}.
\]

Now, note that

\[
\lambda_{\text{max}}(G_{1,n}) = \sup_{\|z\| = 1} n^{-1} \mathbb{E}(I_{n}^{(1)}(g_{z}))^{2},
\]

where \( g_{z} = \sum_{j=1}^{d} z_{j} \phi_{j} \). Using again Proposition 7.1 we find

\[
\frac{1}{n} \mathbb{E}(I_{n}^{(1)}(g_{z}))^{2} = \frac{2a}{n} \int g_{z}(t) \overline{g}_{z}(t) dt + \frac{a}{n} \int g_{z}^{2}(t) dt
\]

\[
= \int_{0}^{n} e^{av} G_{z}(v) dv + \int_{0}^{1} g_{z}^{2}(t) dt.
\]

where \( G_{z}(v) = \int_{v}^{n} g_{z}(t)g_{z}(t-v)(1+e^{2a(t-v)})dt \). Note now that for all \( a \leq 0 \)

\[
|G_{z}(v)| \leq 2 \int_{0}^{n} g_{z}^{2}(t) dt = 2n \int_{0}^{1} g_{z}^{2}(t) dt .
\]

Moreover, taking into account

\[
\int_{0}^{1} g_{z}^{2}(t) dt = \sum_{j=1}^{d} z_{j}^{2} = 1,
\]

we obtain that \( \lambda_{\text{max}}(G_{1,n}) \leq 3 \). Hence Proposition 7.11. \( \square \)

8 Proofs

8.1 Proof of Theorem 3.1

Consider the quadratic error of the estimate (3.14)

\[
\|S_{\gamma}^{*} - S\|^{2} = \sum_{j=1}^{n} (\gamma(j)\theta_{j,n}^{*} - \theta_{j})^{2} = \sum_{j=1}^{d} (\gamma(j)\theta_{j,n}^{*} - \theta_{j})^{2} + \sum_{j=d+1}^{n} (\gamma(j)\hat{\theta}_{j,n} - \theta_{j})^{2}
\]

\[
= \sum_{j=1}^{n} (\gamma(j)\hat{\theta}_{j,n} - \theta_{j})^{2} + c_{n}^{2} - 2c_{n} \sum_{j=1}^{d} (\hat{\theta}_{j,n} - \theta_{j}) \frac{\hat{\theta}_{j,n}}{\|\hat{\theta}_{n}\|_{d}}
\]

\[
= \|\hat{S}_{\gamma} - S\|^{2} + c_{n}^{2} - 2c_{n} \sum_{j=1}^{d} (\hat{\theta}_{j,n} - \theta_{j}) \psi_{j}(\hat{\theta}_{n}),
\]

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where \( t_j(x) = x_j/\|x\| \) for \( x = (x_j)_{1 \leq j \leq d} \in \mathbb{R}^d \). Therefore, we can represent the risk for the improved estimator \( S^*_\gamma \) as

\[
R_Q(S^*_\gamma, S) = R_Q(S^*_\gamma, S) + c_n^2 - 2c_n \mathbf{E}_{Q,S} \sum_{j=1}^{d} (\hat{\theta}_{j,n} - \theta_j) I_{j,n},
\]

where \( I_{j,n} = \mathbf{E}(t_j(\tilde{\theta}_n)(\hat{\theta}_{j,n} - \theta_j)|\mathcal{G}_n) \). Now, taking into account that the vector \( \tilde{\theta}_n = (\hat{\theta}_{j,n})_{1 \leq j \leq d} \) is the \( \mathcal{G}_n \) conditionally Gaussian vector in \( \mathbb{R}^d \) with mean \( \tilde{\theta} = (\theta_j)_{1 \leq j \leq d} \) and covariance matrix \( n^{-1} \mathbf{G}_n \), we obtain

\[
I_{j,n} = \int_{\mathbb{R}^d} t_j(x)(x - \theta_j)p(x|\mathcal{G}_n)dx.
\]

Here \( p(x|\mathcal{G}_n) \) is the conditional distribution density of the vector \( \tilde{\theta}_n \), i.e.

\[
p(x|\mathcal{G}_n) = \frac{1}{(2\pi)^{d/2} \sqrt{\det \mathbf{G}_n}} \exp \left( -\frac{(x - \theta)' \mathbf{G}_n^{-1}(x - \theta)}{2} \right).
\]

Changing the variables by \( u = \mathbf{G}_n^{-1/2}(x - \theta) \), one finds that

\[
I_{j,n} = \frac{1}{(2\pi)^{d/2}} \sum_{l=1}^{d} g_{j,l} \int_{\mathbb{R}^d} \tilde{t}_{j,n}(u)u_1 \exp \left( -\frac{||u||^2}{2} \right) du,
\]

where \( \tilde{t}_{j,n}(u) = t_j(\mathbf{G}_n^{1/2}u + \theta) \) and \( g_{ij} \) denotes the \((i,j)\)-th element of \( \mathbf{G}_n^{1/2} \). Furthermore, integrating by parts, the integral \( I_{j,n} \) can be rewritten as

\[
I_{j,n} = \sum_{l=1}^{d} \sum_{k=1}^{d} \mathbf{E} \left( g_{jl} g_{kl} \frac{\partial t_j}{\partial u_k} \bigg|_{u=\tilde{\theta}_n} |\mathcal{G}_n \right). \tag{8.1}
\]

Now taking into account that \( z'Az \leq \lambda_{\max}(A)||z||^2 \) and the condition \( D_2 \) we obtain that

\[
\Delta_Q(S) = c_n^2 - 2c_n n^{-1} \mathbf{E}_{Q,S} \left( \frac{\text{tr} \mathbf{G}_n}{\|\theta_n\|} - \frac{\tilde{\theta}_n \mathbf{G}_n \tilde{\theta}_n}{\|\theta_n\|^2} \right)
\]

\[
\leq c_n^2 - 2c_n \bar{a}_n n^{-1} \mathbf{E}_{Q,S} \frac{1}{\|\theta_n\|}.
\]

Recall, that the \( \bar{a} \) denotes the transposition. Moreover, in view of the Jensen inequality we can estimate the last expectation from below as

\[
\mathbf{E}_{Q,S} \left( \|\theta_n\| \right)^{-1} = \mathbf{E}_{Q,S} \left( \|\tilde{\theta} + n^{-1/2} \tilde{\xi}_n \| \right)^{-1} \geq (\|\theta\|_d + n^{-1/2} \mathbf{E}_{Q,S} \|\tilde{\xi}_n\|_d)^{-1}.
\]

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Note now that the condition through the inequality (1.3) we obtain
\[ E_{Q,S} \|\tilde{\xi}_n\|_d^2 \leq \kappa_Q d. \]
So, for \( \|S\|^2 \leq r_n^* \)
\[ E_{Q,S} \|\tilde{\theta}_n\|^{-1} \geq \left( r_n^* + \sqrt{d \kappa_Q / n} \right)^{-1} \]
and, therefore,
\[ \Delta_Q(S) \leq c_n^2 - 2c_n \frac{l_n^*}{(r_n^* + \sqrt{d \kappa_Q / n}) n} = -c_n^2. \]
Hence Theorem 3.1. \( \square \)

8.2 Proof of Theorem 4.1
Substituting (4.4) in (4.1) yields for any \( \gamma \in \Gamma \)
\[
\text{Err}_n(\gamma) = J_n(\gamma) + 2 \sum_{j=1}^{n} \gamma(j) \left( \theta_j^* \hat{\theta}_{j,n} - \hat{\sigma}_n - \theta_j \hat{\theta}_{j,n} \right) + \|S\|^2 - \rho \hat{P}_n(\gamma).
\]  
(8.2)

Now we set \( L(\gamma) = \sum_{j=1}^{n} \gamma(j), \)
\[
B_{1,n}(\gamma) = \sum_{j=1}^{n} \gamma(j)(E_{Q\xi_j,n} - \sigma_Q), \quad B_{2,n}(\gamma) = \sum_{j=1}^{n} \gamma(j)\tilde{\xi}_{j,n},
\]
\[
M(\gamma) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \gamma(j)\theta_j \hat{\theta}_{j,n}\tilde{\xi}_{j,n} \quad \text{and} \quad B_{3,n}(\gamma) = \frac{1}{\sqrt{n}} \sum_{j=1}^{n} \gamma(j)g(j)\hat{\theta}_{j,n}\tilde{\xi}_{j,n}.
\]
Taking into account the definition (4.5), we can rewrite (8.2) as
\[
\text{Err}_n(\gamma) = J_n(\gamma) + 2 \sigma_Q \hat{\sigma}_n L(\gamma) + 2 M(\gamma) + \frac{2}{n} B_{1,n}(\gamma)
\]  
\[ + 2\sqrt{P_n(\gamma)} \frac{B_{2,n}(\gamma)}{\sqrt{\sigma_Q n}} - 2B_{3,n}(\gamma) + \|S\|^2 - \rho \hat{P}_n(\gamma) \]  
(8.3)
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with \( \gamma = |\gamma|_n \). Let \( \gamma_0 = (\gamma_0(j))_{1 \leq n} \) be a fixed sequence in \( \Gamma \) and \( \gamma^* \) be as in (4.6). Substituting \( \gamma_0 \) and \( \gamma^* \) in (8.3), we consider the difference

\[
\text{Err}_n(\gamma^*) - \text{Err}_n(\gamma_0) \leq 2\frac{\sigma_Q - \hat{\sigma}_n}{\sqrt{n}} L(x) + 2M(x) + \frac{2}{n} B_{1,n}(x)
\]

\[
+ 2\sqrt{P_n(\gamma^*)} \frac{B_{2,n}(\gamma^*)}{\sqrt{\sigma_Qn}} - 2\sqrt{P_n(\gamma_0)} \frac{B_{2,n}(\gamma_0)}{\sqrt{\sigma_Qn}}
\]

\[
- 2B_{3,n}(\gamma^*) + 2B_{3,n}(\gamma_0) - \rho \hat{P}_n(\gamma^*) + \rho \hat{P}_n(\gamma_0),
\]

where \( x = \gamma^* - \gamma_0 \). Note that \( |L(x)| \leq 2|\Gamma|_x \) and \( |B_{1,n}(x)| \leq L_{1,n}(Q) \). Applying the elementary inequality

\[
2|\hat{a}b| \leq \varepsilon a^2 + \varepsilon^{-1}b^2 \quad \text{with any } \varepsilon > 0,
\]

we get

\[
2\sqrt{P_n(\gamma)} \frac{B_{2,n}(\gamma)}{\sqrt{\sigma_Qn}} \leq \varepsilon P_n(\gamma) + \frac{B_{2,n}(\gamma)}{\varepsilon \sigma_Qn} \leq \varepsilon P_n(\gamma) + \frac{B^*_{2,n}(\gamma)}{\varepsilon \sigma_Qn},
\]

where

\[
B^*_{2,n} = \max_{\gamma \in \Gamma} \left( B_{2,n}(\gamma) + B^2_{2,n}(\gamma^2) \right)
\]

with \( \gamma^2 = (\gamma^2_j)_{1 \leq j \leq n} \). Note that from definition the function \( L_{2,n}(Q) \) in the condition \( C_2 \) we obtain that

\[
E_Q B^*_{2,n} \leq \sum_{\gamma \in \Gamma} \left( E_Q B_{2,n}(\gamma) + E_Q B^2_{2,n}(\gamma^2) \right) \leq 2\nu L_{2,n}(Q). \quad (8.5)
\]

Moreover, by the same method we estimate the term \( B_{3,n} \). Note that

\[
\sum_{j=1}^n g^2(j) \hat{\theta}^2_j = c_n^2 \leq \frac{c^*_n}{n}, \quad (8.6)
\]

where \( c^*_n = n \max_{\gamma \in \Gamma} c^2_{\gamma,n} \). Therefore, through the Cauchy–Schwarz inequality we can estimate the term \( B_{3,n}(\gamma) \) as

\[
|B_{3,n}(\gamma)| \leq \frac{|\gamma|_n c_n \left( \sum_{j=1}^n \gamma^2(j) \xi_j^2 \right)^{1/2}}{\sqrt{n}} = \frac{|\gamma|_n c_n \left( \sigma_Q + B_{2,n}(\gamma^2) \right)^{1/2}}{\sqrt{n}}.
\]
So, applying the elementary inequality (8.4) with some arbitrary \( \varepsilon > 0 \), we have

\[
2|B_{3,n}(\gamma)| \leq \varepsilon P_n(\gamma) + \frac{c^*}{\varepsilon \sigma_Q n}(\sigma_Q + B^*_2).
\]

Using the bounds above, one has

\[
\text{Err}_n(\gamma^*) \leq \text{Err}_n(\gamma_0) + \frac{4|\Gamma|_n|\tilde{\sigma}_n - \sigma_Q|}{n} + 2M(x) + \frac{2}{n}L_{1,n}(Q)
\]

\[
+ \frac{2}{\varepsilon} \frac{c^*}{n\sigma_Q}(\sigma_Q + B^*_2) + \frac{2}{\varepsilon} \frac{B^*_2}{n\sigma_Q}
\]

\[
+ 2\varepsilon P_n(\gamma^*) + 2\varepsilon P_n(\gamma_0) - \rho \tilde{P}_n(\gamma^*) + \rho \tilde{P}_n(\gamma_0).
\]

The setting \( \varepsilon = \rho/4 \) and the estimating where this is possible \( \rho \) by 1 in this inequality imply

\[
\text{Err}_n(\gamma^*) \leq \text{Err}_n(\gamma_0) + \frac{5|\Gamma|_n|\tilde{\sigma}_n - \sigma_Q|}{n} + 2M(x) + \frac{2}{n}L_{1,n}(Q)
\]

\[
+ \frac{16(c^*_n + 1)(\sigma_Q + B^*_2)}{\rho n\sigma_Q} - \frac{\rho}{2} \tilde{P}_n(\gamma^*) + \frac{\rho}{2} P_n(\gamma_0) + \rho \tilde{P}_n(\gamma_0).
\]

Moreover, taking into account here that

\[
|\tilde{P}_n(\gamma_0) - P_n(\gamma_0)| \leq \frac{|\Gamma|_n|\tilde{\sigma}_n - \sigma_Q|}{n}
\]

and that \( \rho < 1/2 \), we obtain that

\[
\text{Err}_n(\gamma^*) \leq \text{Err}_n(\gamma_0) + \frac{6|\Gamma|_n|\tilde{\sigma}_n - \sigma_Q|}{n} + 2M(x) + \frac{2}{n}L_{1,n}(Q)
\]

\[
+ \frac{16(c^*_n + 1)(\sigma_Q + B^*_2)}{\rho n\sigma_Q} - \frac{\rho}{2} P_n(\gamma^*) + \frac{3\rho}{2} P_n(\gamma_0).
\]  

(8.7)

Now we examine the third term in the right-hand side of this inequality. Firstly we note that

\[
2|M(x)| \leq \varepsilon \|S_x\| + \frac{Z^*}{n\varepsilon},
\]

where \( S_x = \sum_{j=1}^n x_j \theta_j \phi_j \) and

\[
Z^* = \sup_{x \in \Gamma_1} \frac{nm^2(x)}{\|S_x\|^2}.
\]
We remind that the set $\Gamma_1 = \Gamma - \gamma_0$. Using Proposition 7.1 we can obtain that for any fixed $x = (x_j)_{1 \leq j \leq n} \in \mathbb{R}^n$

$$E M^2(x) = \frac{E \ell_n^2(S_x)}{n^2} = \frac{\sigma Q \|S_x\|^2}{n} = \frac{\sigma Q}{n} \sum_{j=1}^{n} x_j^2 \theta_j^2$$

(8.9)

and, therefore,

$$E_Q Z^* \leq \sum_{x \in \Gamma_1} \frac{nM^2(x)}{\|S_x\|^2} \leq \sigma Q \nu.$$  

(8.10)

Moreover, the norm $\|S^*_{\gamma} - S^*_{\gamma_0}\|$ can be estimated from below as

$$\|S^*_{\gamma} - S^*_{\gamma_0}\|^2 = \sum_{j=1}^{n} (x(j) + \beta(j))^2 \hat{\theta}_j^2$$

$$\geq \|\hat{S}_x\|^2 + 2 \sum_{j=1}^{n} x(j) \beta(j) \hat{\theta}_j,$$

where $\beta(j) = \gamma_0(j) g_j(\gamma_0) - \gamma(j) g_j(\gamma)$. Therefore, in view of (3.7)

$$\|S_x\|^2 - \|S^*_{\gamma} - S^*_{\gamma_0}\|^2 \leq \|S_x\|^2 - \|\hat{S}_x\|^2 - 2 \sum_{j=1}^{n} x(j) \beta(j) \hat{\theta}_j$$

$$\leq -2M(x^2) - 2 \sum_{j=1}^{n} x(j) \beta(j) \hat{\theta}_j \theta_j - \frac{2}{\sqrt{n}} \Upsilon(x),$$

where $\Upsilon(\gamma) = \sum_{j=1}^{n} \gamma(j) \beta(j) \hat{\theta}_j \xi_j$. Note that the first term in this inequality we can estimate as

$$2M(x^2) \leq \varepsilon \|S_x\|^2 + \frac{Z^*_1}{n \varepsilon} \quad \text{and} \quad Z^*_1 = \sup_{x \in \Gamma_1} \frac{nM^2(x^2)}{\|S_x\|^2}.$$ 

Note that, similarly to (8.10) we can estimate the last term as

$$E_Q Z^*_1 \leq \sigma Q \nu.$$ 

From this it follows that for any $0 < \varepsilon < 1$

$$\|S_x\|^2 \leq \frac{1}{1 - \varepsilon} \left( \|S^*_{\gamma} - S^*_{\gamma_0}\|^2 + \frac{Z^*_1}{n \varepsilon} \right.$$ 

$$\left. -2 \sum_{j=1}^{n} x(j) \beta(j) \hat{\theta}_j \theta_j - \frac{2 \Upsilon(x)}{\sqrt{n}} \right).$$

(8.11)
Moreover, note now that the property (8.6) yields
\[ \sum_{j=1}^{n} \beta^2(j) \hat{\theta}_j^2 \leq 2 \sum_{j=1}^{n} g_j^2 \beta_j^2 + 2 \sum_{j=1}^{n} \tilde{g}_{j_0}^2 \beta_j^2 \leq \frac{4c^*}{\varepsilon n}. \]  
(8.12)

Taking into account that \( |\beta(j)| \leq 1 \) and using the inequality (8.4), we get that for any \( \varepsilon > 0 \)
\[ 2 \left| \sum_{j=1}^{n} x(j) \beta(j) \hat{\theta}_j \right| \leq \varepsilon \| \mathbf{S}_x \|^2 + \frac{4c^*}{\varepsilon n}. \]

To estimate the last term in the right hand of (8.11) we use first the Cauchy–Schwarz inequality and then the bound (8.12), i.e.
\[ \frac{2}{\sqrt{n}} |\mathbf{Y}(\gamma)| \leq \frac{2|\gamma|}{\sqrt{n}} \left( \sum_{j=1}^{n} \beta^2(j) \hat{\theta}_j^2 \right)^{1/2} \left( \sum_{j=1}^{n} \tilde{g}_j^2 \xi_j^2 \right)^{1/2} \leq \varepsilon P_n(\gamma) + \frac{c^*}{n \varepsilon} \sum_{j=1}^{n} \tilde{g}_j^2 \xi_j^2 \leq \varepsilon P_n(\gamma) + \frac{c^*(\sigma_Q + B_2^* \varepsilon)}{n \varepsilon}. \]

Therefore,
\[ \frac{2}{\sqrt{n}} |\mathbf{Y}(x)| \leq \frac{2|\gamma|}{\sqrt{n}} |\mathbf{Y}(\gamma)| + \frac{2|\gamma|}{\sqrt{n}} |\mathbf{Y}(\gamma_0)| \leq \varepsilon P_n(\gamma) + \varepsilon P_n(\gamma_0) + \frac{2c^*(\sigma_Q + B_2^* \varepsilon)}{n \varepsilon}. \]

So, using all these bounds in (8.11), we obtain that
\[ \| \mathbf{S}_x \|^2 \leq \frac{1}{1-\varepsilon} \left( \frac{Z^*_1}{n \varepsilon} + \| \mathbf{S}_{\gamma^*}^* - \mathbf{S}_{\gamma_0}^* \|^2 + \frac{6c^*(\sigma + B_2^* \varepsilon)}{n \varepsilon} \right) \varepsilon P_n(\gamma) + \varepsilon P_n(\gamma_0). \]

Using the inequality (8.8) this bound and the estimate
\[ \| \mathbf{S}_{\gamma^*}^* - \mathbf{S}_{\gamma_0}^* \|^2 \leq 2(\text{Err}_n(\gamma^*) + \text{Err}_n(\gamma_0)), \]
we obtain
\[ 2|M(x)| \leq \frac{Z^* + Z^*_1}{n(1-\varepsilon) \varepsilon} + \frac{2\varepsilon(\text{Err}_n(\gamma^*) + \text{Err}_n(\gamma_0))}{(1-\varepsilon)} \]
\[ + \frac{6c^*(\sigma_Q + B_2^* \varepsilon)}{n \sigma_Q (1-\varepsilon)} + \frac{\varepsilon^2}{1-\varepsilon} \left( P_n(\gamma^*) + P_n(\gamma_0) \right). \]
Choosing here $\epsilon \leq \rho/2 < 1/2$ we obtain that
\[
2|M(x)| \leq \frac{2(Z^* + Z_1^*)}{n\epsilon} + \frac{2\epsilon (\text{Err}_n(\gamma^*) + \text{Err}_n(\gamma_0))}{(1 - \epsilon)} \\
+ \frac{12c^*_n(\sigma_Q + B^*_2)}{n\sigma_Q} + \epsilon (P_n(\gamma^*) + P_n(\gamma_0)).
\]
From here and (8.7), it follows that
\[
\text{Err}_n(\gamma^*) \leq \frac{1 + \epsilon}{1 - 3\epsilon} \text{Err}_n(\gamma_0) + \frac{6|\Gamma|_n}{n(1 - 3\epsilon)} |\hat{\sigma}_n - \sigma_Q| + \frac{2}{n(1 - 3\epsilon)} L_{1,n}(Q) \\
+ \frac{28(1 + c^*_n)(B^*_2 + \sigma_Q)}{\rho(1 - 3\epsilon)n\sigma_Q} + \frac{2(Z^* + Z_1^*)}{n(1 - 3\epsilon)} + \frac{2\rho P_n(\gamma_0)}{1 - 3\epsilon}.
\]
Choosing here $\epsilon = \rho/3$ and estimating $(1 - \rho)^{-1}$ by 2 where this is possible, we get
\[
\text{Err}_n(\gamma^*) \leq \frac{1 + \rho/3}{1 - \rho} \text{Err}_n(\gamma_0) + \frac{12|\Gamma|_n}{n} |\hat{\sigma}_n - \sigma_Q| + \frac{4}{n} L_{1,n}(Q) \\
+ \frac{56(1 + c^*_n)(B^*_2 + \sigma_Q)}{\rho n\sigma_Q} + \frac{4(Z^* + Z_1^*)}{n} + \frac{2\rho P_n(\gamma_0)}{1 - \rho}.
\]
Taking the expectation and using the upper bound for $P_n(\gamma_0)$ in Lemma A.7 with $\epsilon = \rho$ yields
\[
\mathcal{R}_Q(S^*, S) \leq \frac{1 + 5\rho}{1 - \rho} \mathcal{R}_Q(S^*_{\gamma_0}, S) + \frac{\hat{U}_{Q,n}}{n\rho} + \frac{12|\Gamma|_n E_Q|\hat{\sigma}_n - \sigma_Q|}{n},
\]
where $\hat{U}_{Q,n} = 4L_{1,n}(Q) + 56(1 + c^*_n)(2L_{2,n}(Q)\nu + 1) + 2c^*_n$. The inequality holds for each $\gamma_0 \in \Lambda$, this implies Theorem 4.1. \qed

### 8.3 Proof of Theorem 5.1
Firstly, note, that for any fixed $Q \in Q_n$
\[
\sup_{S \in W_{k,r}} \mathcal{R}_n^*(\hat{S}_n, S) \geq \sup_{S \in W_{k,r}} \mathcal{R}_Q(\hat{S}_n, S). \tag{8.13}
\]
Now for any fixed $0 < \epsilon < 1$ we set
\[
d = d_n = \left[ \frac{k + 1}{k} \sigma_n^{(2k+1)} l_k(r_\epsilon) \right] \quad \text{and} \quad r_\epsilon = (1 - \epsilon)r. \tag{8.14}
\]

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Next we approximate the unknown function by a trigonometric series with \( d = d_n \) terms, i.e. for any array \( z = (z_j)_{1 \leq j \leq d_n} \), we set

\[
S_z(x) = \sum_{j=1}^{d_n} z_j \phi_j(x).
\]

(8.15)

To define the Bayesian risk we choose a prior distribution on \( \mathbb{R}^d \) as

\[
\kappa = (\kappa_j)_{1 \leq j \leq d_n} \quad \text{and} \quad \kappa_j = s_j \eta_j,
\]

(8.16)

where \( \eta_j \) are i.i.d. Gaussian \( \mathcal{N}(0, 1) \) random variables and the coefficients

\[
s_j = \sqrt{s_j^*} \quad \text{and} \quad s_j^* = \left( \frac{d_n}{j} \right)^k - 1.
\]

Furthermore, for any function \( f \), we denote by \( p(f) \) its projection in \( L^2[0, 1] \) onto \( W_{k, r} \), i.e.

\[
p(f) = \Pr_{W_{k, r}}(f).
\]

Since \( W_{k, r} \) is a convex set, we obtain

\[
\|\hat{S} - S\|^2 \geq \|\hat{p} - S\|^2 \quad \text{with} \quad \hat{p} = p(\hat{S}).
\]

Therefore,

\[
\sup_{S \in W_{k, r}} \mathcal{R}(\hat{S}, S) \geq \int_{\left\{ z \in \mathbb{R}^d : S_z \in W_{k, r} \right\}} E_{S_z} \|\hat{p} - S_z\|^2 \mu_\kappa(dz).
\]

Using the distribution \( \mu_\kappa \) we introduce the following Bayes risk

\[
\tilde{\mathcal{R}}_Q(\hat{S}) = \int_{\mathbb{R}^d} \mathcal{R}_Q(\hat{S}, S_z) \mu_\kappa(dz).
\]

Taking into account now that \( \|\hat{p}\|^2 \leq r \) we obtain

\[
\sup_{S \in W_{k, r}} \mathcal{R}_Q(\hat{S}, S) \geq \tilde{\mathcal{R}}_Q(\hat{p}) - 2 R_{0, n}
\]

(8.17)

with

\[
R_{0, n} = \int_{\left\{ z \in \mathbb{R}^d : S_z \notin W_{k, r} \right\}} (r + \|S_z\|^2) \mu_\kappa(dz).
\]

Therefore, in view of (8.13)

\[
\sup_{S \in W_{k, r}} \mathcal{R}^*_n(\hat{S}_n, S) \geq \sup_{Q \in \mathcal{Q}_n} \tilde{\mathcal{R}}_Q(\hat{p}) - 2 R_{0, n}.
\]

(8.18)
In Lemma A.8 we studied the last term in this inequality. Now it is easy to see that
\[ \| \hat{p} - S_{z} \|^{2} \geq \sum_{j=1}^{d_{n}} (\hat{z}_{j} - z_{j})^{2}, \]
where \( \hat{z}_{j} = \int_{0}^{1} \hat{p}(t) \phi_{j}(t) dt \). So, in view of Lemma A.9 and reminding that \( v_{n} = n/\varsigma^{*} \), we obtain
\[
\sup_{Q \in \mathcal{Q}_{n}} \mathcal{R}_{Q}(\hat{p}) \geq \sup_{0 < \epsilon_{1}^{*} \leq \epsilon} \sum_{j=1}^{d_{n}} \frac{1}{n \epsilon_{1}^{*} + v_{n} (s_{j}^{*})^{-1}} = \frac{1}{v_{n}} \sum_{j=1}^{d_{n}} \frac{s_{j}^{*}}{s_{j}^{*} + 1} = \frac{1}{v_{n}} \sum_{j=1}^{d_{n}} \left( 1 - \frac{k_{j}}{d_{n}} \right) .
\]
Therefore, using now the definition (8.14), Lemma A.8 and the inequality (8.18) we obtain that
\[
\liminf_{n \to \infty} \inf_{\hat{S} \in \Sigma_{n}} v_{n}^{\frac{2k}{n+1}} \sup_{S \in W_{k,r}} \mathcal{R}_{n}^{*}(\hat{S}_{n}, S) \geq (1 - \epsilon) \frac{1}{n+1} l_{n}(r_{\epsilon}).
\]
Taking here limit as \( \epsilon \to 0 \) implies Theorem 5.1.

8.4 Proof of Theorem 5.2

This theorem follows from Theorems 4.3 and 3.1 and Theorem 3.1 in [16].

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A Appendix

A.1 Property of the trigonometric basis

Lemma A.1. The trigonometric basis (3.3) satisfies the conditions $B_1$ and $B_2$ with $d_0$ and $\tilde{a}$ defined in (3.4).

Proof. First, we set

$$
\Phi_d(t, v) = \sum_{l=1}^{d} \phi_l(t)\phi_l(t-v) \quad \text{and} \quad N = \lfloor d/2 \rfloor.
$$

Note now that for any $d \geq 3$ this sum can be represented as

$$
\Phi_d(t, v) = 1 + \sum_{j=1}^{N} \left( \phi_{2j}(t)\phi_{2j}(t-v) + \phi_{2j+1}(t)\phi_{2j+1}(t-v) \right) - \phi_{d+1}(t)\phi_{d+1}(t-v) 1_{\{d=2N\}}
$$

$$
= 1 + 2 \sum_{j=1}^{N} \cos(2\pi jv) - \phi_{d+1}(t)\phi_{d+1}(t-v) 1_{\{d=2N\}}.
$$

Therefore,

$$
\Phi^*_d(v) \leq 2 + \left| 1 + 2 \sum_{j=1}^{N} \cos(2\pi jv) \right| = 2 + \left| \frac{\sin(\pi(2N+1)v)}{\sin(\pi v)} \right|.
$$

So, taking into account that $|1 + 2 \sum_{j=1}^{N} \cos(2\pi jv)| \leq 2N + 1$, we obtain that for any fixed $0 < \delta < 1/2$

$$
\int_0^1 \Phi^*_d(v) \, dv \leq 2 + 2\delta(d+1) + \int_{\delta}^{1-\delta} \frac{1}{\sin(\pi v)} \, dv
$$

$$
= 2 + 2\delta(d+1) + 2 \int_{\delta}^{1/2} \frac{1}{\sin(\pi v)} \, dv.
$$

Using here that $\sin(\pi v) \geq 2v$ for any $0 < v < 1/2$ we obtain that

$$
\int_0^1 \Phi^*_d(v) \, dv \leq 4 + 2\delta - \ln(2\delta).
$$

Minimizing this upper bound, we obtain that (for $\delta = 1/(2d)$)

$$
\int_0^1 \Phi^*_d(v) \, dv \leq 5 + \ln d.
$$

Hence Lemma A.1. \qed

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A.2 Technical lemmas

In the following lemma we need the well-known Novikov inequalities [25] for purely discontinuous martingales. Namely, for any \( p \geq 2 \) and for any \( t > 0 \)

\[
\mathbb{E} \sup_{u \leq t} |J_u(h)|^p \leq \tilde{C}_p \mathbb{E} (|h|^2 * \nu_t)^{p/2} + \mathbb{E} |h|^p * \nu_t, \tag{A.1}
\]

where \( J_t(h) = g * (|\Gamma|_t - \nu)_t \).

**Lemma A.2.** Assume that \( \Pi(x^8) < \infty \). Then for any non random left continuous \( \mathbb{R}_+ \to \mathbb{R} \) functions \( f \) and \( g \) having the finite right limits and for any \( t > 0 \)

\[
\mathbb{E} \int_0^t V_{s-}(f) dL_s(g) = 0.
\]

**Proof.** By the definition we have

\[
\int_0^t V_{s-}(f) dL_s(g) = \int_0^t V_{s-}(f)(I_{s-}(g) + g(s)I_{s-}(1)) ds + \varrho^2 \int_0^t V_{s-}(f) g(s) dm_s.
\]

Note that in this case \( <m>_t = \Pi(x^4)t \). So, to prove this lemma we need to check that

\[
\int_0^t \mathbb{E} V^2_s(f) I^2_s(g) ds < \infty \quad \text{and} \quad \int_0^t \mathbb{E} V^2_s(f) ds < \infty.
\]

It is clear that to obtain these properties it suffices to show that for any bounded on the interval \([0, t]\) nonrandom function \( f \)

\[
\sup_{0 \leq s \leq t} \mathbb{E} I^8_s(f) < \infty. \tag{A.2}
\]

Taking into account the definitions in (7.33) we get

\[
I_s(f) = I^{(1)}_s(f) + I^{(2)}_s(f).
\]

Since the first term \( I^{(1)}_s(f) \) is the Gaussian random variable we need to show the inequality (A.2) only for \( I^{(2)}_s(f) \). But it follows immediately from the Novikov inequality (A.1). Hence lemma A.2.

\( \square \)
Lemma A.3. Let \( \nu \) be a continuously differentiable \( \mathbb{R} \rightarrow \mathbb{R} \) function. Then, for any \( t > 0 \), \( \alpha > 0 \) and for any integrated \( \mathbb{R} \rightarrow \mathbb{R} \) function \( h \),

\[
\left| \int_0^t e^{-\alpha(t-s)} h(s) \nu(s) \, ds \right| \leq \| \tilde{h} \|_{\ast,t} \left( 2\| \nu \|_{\ast,t} + \frac{\| \dot{\nu} \|_{\ast,t}}{\alpha} \right),
\]

where \( \tilde{h}_s = \int_0^t h(u) \, du \).

**Proof.** This Lemma A.3 follows immediately from the integrating parts.

Lemma A.4. For any measurable bounded \( [0, +\infty) \rightarrow \mathbb{R} \) functions \( f \) and \( g \), for any \( -\infty < a \leq 0 \) and for any \( t > 0 \),

\[
\bigg| a \int_0^t e^{2a(t-s)} g(s) A_s(f) \, ds \bigg| \leq 3 \varrho_2^2 \varpi_t^4(f, g) + \varrho_2 \| f \|_{\ast,t} \| g \|_{\ast,t},
\]

where \( \varrho_2 = \varrho_2^2 \Pi(x^4) \).

**Proof.** First note that

\[
A_t = J_t(f) + \varrho_2^2 \bar{J}_t(f) \quad \text{ (A.3)}
\]

where

\[
J_t(f) = \int_0^t e^{3a(t-u)} f(u) \nu(u) \, du \quad \text{and} \quad \bar{J}_t(f) = 2 \int_0^t e^{3a(t-u)} \tilde{\xi}_u(f) \, du.
\]

To study these integrals we need to calculate \( \mathbb{E} \tilde{\xi}_t^2 \). To this end through the equation (7.8) we can represent this expectation in the following integral form

\[
\mathbb{E} \tilde{\xi}_t^2 = \int_0^t e^{4a(t-s)} \, d\mathbb{E} [\tilde{M}(1), \tilde{M}(1)]_s.
\]

Moreover, using the definition of \( M_{s}(1,1) \) in (7.2) we obtain that

\[
\mathbb{E} [\tilde{M}(1), \tilde{M}(1)]_t = 4\varrho_2 \int_0^t \mathbb{E} \xi_s^2 \, ds + \varrho_2 t,
\]

where \( \varrho_2 = \varrho_2^4 \Pi(x^4) \). Therefore,

\[
\mathbb{E} \tilde{\xi}_t^2 = \int_0^t e^{4a(t-s)} \left( 4\varrho_2 \mathbb{E} \xi_s^2 + \varrho_2 \right) \, ds.
\]
Using here (7.3) we obtain that for $a < 0$

$$E \tilde{c}_{t}^{2} = e^{4at} \frac{2\tilde{\vartheta}_{*}^{2} + a\tilde{\vartheta}_{2}}{4a^{2}} - e^{2at} \frac{\vartheta_{*}^{2}}{a^{2}} + \frac{2\vartheta_{*}^{2} - a\tilde{\vartheta}_{2}}{4a^{2}}. \quad (A.4)$$

Note now that the function $v(\cdot)$ defined in (7.10) can be represented as

$$v(s) = av_{1}(s) + v_{2}(s) \quad (A.5)$$

with

$$v_{1}(s) = \frac{\bar{\vartheta}_{2}}{4} (e^{4at} + 3) \quad \text{and} \quad v_{2}(s) = \frac{\vartheta_{*}^{2}}{2} (e^{4at} - 1).$$

It is clear that

$$\|v_{1}\|_{*,n} \leq \bar{\vartheta}_{2} \quad \text{and} \quad 2\|v_{2}\|_{*,n} + \frac{\|\dot{v}_{2}\|_{*,n}}{2|a|} \leq 2\vartheta_{*}. \quad (A.6)$$

Now we have

$$J_{1}(f) = J_{1,t}(f) + J_{2,t}(f),$$

where $J_{1,t}(f) = a \int_{0}^{t} e^{3a(s-u)} f(u)v_{1}(u)du$ and $J_{2,t}(f) = \int_{0}^{t} e^{3a(s-u)} f(u)v_{2}(u)du$. It is clear that

$$|J_{1,t}(f)| \leq \bar{\vartheta}_{2}\|f\|_{*,n}/3.$$

So,

$$\left| a \int_{0}^{t} e^{2a(t-s)} g(s) J_{1,s}(f) \, ds \right| \leq \bar{\vartheta}_{2}\|f\|_{*,t} \|g\|_{*,t}/6.$$

Now we represent the corresponding integral for $J_{2,t}(f)$ as

$$\int_{0}^{t} e^{2a(t-s)} g(s) J_{2,s}(f) \, ds = \int_{0}^{t} e^{3au} \tilde{J}_{t-u,u}(f,g) \, du$$

and

$$\tilde{J}_{t,u}(f,g) = \int_{0}^{t} e^{2a(t-s)} g(s + u) f(s) v_{2}(s) \, ds.$$

Using Lemma A.3 and the last inequality in (A.6) we obtain that

$$\sup_{0 \leq u \leq t} \left| \tilde{J}_{t-u,u}(f) \right| \leq 2\vartheta_{*}^{2}\Omega_{t}(g,f),$$

where

$$\Omega_{t}(g,f) = \sup_{v \geq 0, u \geq 0, 0 \leq v + u \leq t} \left| \int_{0}^{v} g(s + u) f(s) ds \right|.$$
Therefore,

\[ \left| a \int_{0}^{t} e^{2a(t-s)} g(s) J_{2,s}(f) \, ds \right| \leq \frac{2 \varrho_{2}^{2}}{3} \Omega_{t}(g, f) \leq \frac{2 \varrho_{2}^{2}}{3} \varpi_{t}(f, g). \]

Similarly we can get

\[ \left| a \int_{0}^{t} e^{2a(t-s)} g(s) \tilde{J}_{s}(f) \, ds \right| \leq \frac{4}{3} \Omega_{t}(g, \tilde{\varepsilon}(f)). \]

Note now that for any fixed \( v > 0 \) and \( \theta \geq 0 \) with \( v + \theta \leq t \) we have

\[ \int_{0}^{v} g(u + \theta) \tilde{\varepsilon}_{u}(f) \, du = \frac{a}{2} \int_{0}^{v} e^{\alpha s} D_{s, \theta} \, ds, \]

where \( D_{s, \theta} = \int_{0}^{v-s} g(y + \theta + s) \tilde{f}(y)dy \) and \( \tilde{f}(y) = f(y)(1 + e^{2ay}). \) Integrating by parts yields

\[ D_{s, \theta} = (1 + e^{2a(t-s)}) \int_{0}^{v-s} g(z + \theta + s) f(z) \, dz \]

\[ + 2a \int_{0}^{v-s} e^{2a u} \int_{0}^{u} g(z + \theta + s) f(z) \, dz \, du. \]

This implies that

\[ |D_{s, \theta}(f, g)| \leq 3 \varpi_{t}(f, g), \]

i.e. for any \( v > 0 \) and \( \theta \geq 0 \)

\[ \left| \int_{0}^{v} g(u + \theta) \tilde{\varepsilon}_{u}(f) \, du \right| \leq 3 \varpi_{t}(f, g)/2. \quad (A.7) \]

Therefore,

\[ \Omega_{t}(g, \tilde{\varepsilon}(f)) \leq 3 \varpi_{t}(f, g)/2. \]

Hence Lemma A.4.

**Lemma A.5.** For any measurable bounded \([0, +\infty) \to \mathbb{R}\) functions \( f \) and \( g \), for any \(-a_{\text{max}} \leq a \leq 0\) and for any \( t > 0 \)

\[ a^{2} \left| \int_{0}^{t} e^{a(t-s)} g(s) E \tilde{I}_{s}(f) \tilde{I}_{s}(1) \, ds \right| \leq 4 \| f \|_{s,t}^{2} (a_{\text{max}} \varrho_{2} + 3 \varrho_{2}^{2}) \varpi_{t}(1, g). \]
Proof. Firstly, note that if \( a = 0 \) then this bound is obvious. Let now \( |a| > 0 \). Then, taking into account the representation (A.3) and the bound \( |\tilde{\varepsilon}_t(f)| \leq \|f\|_{*,t} \) we obtain that

\[
\|aA(f)\|_{*,t} \leq \|f\|_{*,t} \left( a_{\max} \bar{\theta}_2/3 + \varrho_2^2 \right). \tag{A.8}
\]

Thus, from the definition of \( \tilde{\kappa}_u(f) \) in (7.15) we obtain that

\[
\|a\tilde{\kappa}(f)\|_{*,t} \leq \|f\|_{*,t}^2 \left( 2a_{\max} \bar{\theta}_2 + 6\varrho_2^2 \right). \tag{A.9}
\]

Moreover, note now, that

\[
\int_0^t e^{a(t-s)} g(s) E \tilde{I}_s(f) \tilde{I}_s(1) \, ds = \int_0^t e^{a(t-u)} \tilde{\kappa}_u(f) G_{t-u,u} \, du,
\]

where \( G_{T,u} = \int_0^T e^{az} \, g(z+u) \, dz \). The integrating by parts yields

\[
G_{T,u} = \int_0^T g(z+u) \, dz + a \int_0^T e^{ay} \left( \int_0^y g(v+u) \, dv \right) \, dy.
\]

So, for any \( T + u \leq t \) we obtain that \( |G_{T,u}| \leq 2\varpi_t(1,g) \) and, therefore,

\[
\left| \int_0^t e^{a(t-u)} \tilde{\kappa}_u(f) G_{t-u,u} \, du \right| \leq \frac{4}{a^2} \|f\|_{*,t}^2 \left( a_{\max} \bar{\theta}_2 + 3\varrho_2^2 \right) \varpi_t(1,g).
\]

Hence Lemma A.5. \( \square \)

Lemma A.6. For any measurable bounded \([0, +\infty) \to \mathbb{R}\) functions \( f \) and \( g \), for any \( -\infty < a \leq 0 \) and for any \( t > 0 \)

\[
\left| a \int_0^t e^{2a(t-s)} \tilde{\varepsilon}_s(f,g) \, ds \right| \leq 9 \varpi_t^*(f,g). \tag{A.10}
\]

Proof. Firstly note, that using the bound (A.7) with \( \theta = 0 \) we obtain

\[
\left| \int_0^t g(s)\tilde{\varepsilon}_s(f) \, ds \right| \leq 3\varpi_t(f,g)/2. \tag{A.11}
\]

So, \( |\tau_t(f,g)| \leq 4\varpi_t(f,g) \). Moreover, through the bound (A.11) and Lemma A.3 we obtain that

\[
\left| \int_0^t e^{2a(t-s)} f(s)\tilde{\varepsilon}_s(g) \, ds \right| \leq 3\varpi_t(f,g).
\]

Using again Lemma A.3 and taking into account that \( a\tau_t(1,1) = (e^{2at} - 1)/2 \) we estimate

\[
\left| a \int_0^t e^{2a(t-s)} f(s)g(s)\tau_s(1,1) \, ds \right| \leq \varpi_t(f,g).
\]

Thus, from taking into account the definition (7.13) we obtain the bound (A.10). Hence Lemma A.6. \( \square \)
A.3 Property of Penalty term

Lemma A.7. For any \( n \geq 1, \gamma \in \Gamma \) and \( 0 < \varepsilon < 1 \)

\[
P_n(\gamma) \leq \frac{E \text{Err}_n(\gamma)}{1 - \varepsilon} + \frac{c^n}{n\varepsilon(1 - \varepsilon)}.
\] (A.12)

Proof. By the definition of \( \text{Err}_n(\gamma) \) one has

\[
\text{Err}_n(\gamma) = \sum_{j=1}^{n} (\gamma(j)\theta^*_{j,n} - \theta_j)^2 = \sum_{j=1}^{n} \left( \gamma(j)(\theta^*_{j,n} - \theta_j) + (\gamma(j) - 1)\theta_j \right)^2
\]

\[
\geq \sum_{j=1}^{n} \gamma(j)^2(\theta^*_{j,n} - \theta_j)^2 + 2\sum_{j=1}^{n} \gamma(j)(\gamma(j) - 1)\theta_j(\theta^*_{j,n} - \theta_j).
\]

Taking into account the condition \( B_2 \) and the definition (3.12) we obtain that the last term in tho sum can be replaced as

\[
\sum_{j=1}^{n} \gamma(j)(\gamma(j) - 1)\theta_j(\theta^*_{j,n} - \theta_j) = \sum_{j=1}^{n} \gamma(j)(\gamma(j) - 1)\theta_j(\theta^*_{j,n} - \theta_j),
\]

i.e. \( E \sum_{j=1}^{n} \gamma(j)(\gamma(j) - 1)\theta_j(\theta^*_{j,n} - \theta_j) = 0 \) and, therefore, taking into account the definition (4.11) we obtain that

\[
E \text{Err}_n(\gamma) \geq \sum_{j=1}^{n} \gamma(j)^2E(\theta^*_{j,n} - \theta_j)^2 = \sum_{j=1}^{n} \gamma(j)^2E \left( \frac{\xi_{j,n}}{\sqrt{n}} - g(\gamma(j))\hat{\theta}_{j,n} \right)^2
\]

\[
\geq P_n(\gamma) - \frac{2}{\sqrt{n}} \sum_{j=1}^{n} \gamma(j)^2g(\gamma(j))\hat{\theta}_{j,n} \xi_j
\]

\[
\geq (1 - \varepsilon) P_n(\gamma) - \frac{1}{\varepsilon} E \sum_{j=1}^{n} g^2(\gamma(j))\hat{\theta}_j^2.
\]

The inequality (8.6) implies the bound (A.12). Hence Lemma A.7. \( \square \)

Lemma A.8. For any \( m > 0 \) the term \( R_{0,n} \) introduced in (8.17) satisfies the following property

\[
\lim_{T \to \infty} n^m R_{0,n} = 0.
\] (A.13)
Proof. First, setting $\zeta_n = \sum_{j=1}^{d_n} \kappa_j^2 a_j$, we obtain that

$$\{S_\kappa \notin W_{k,r} \} = \left\{ \sum_{j=1}^{d_n} \kappa_j^2 \sum_{l=0}^{k} \| \phi_j^{(l)} \| > r \right\} = \{ \zeta_n > r \}.$$  

Moreover, note that one can check directly that

$$\lim_{n \to \infty} E \zeta_n = \lim_{n \to \infty} \frac{1}{v_n} \sum_{j=1}^{d_n} s_j^* a_j = r_\varepsilon = (1 - \varepsilon) r.$$  

So, for sufficiently large $n$ we obtain that

$$\{ S_\kappa \notin W_{k,r} \} \subset \{ \tilde{\zeta}_n > r_1 \},$$

where $r_1 = r \varepsilon / 2$,

$$\tilde{\zeta}_n = \zeta_n - E \zeta_n = \frac{1}{v_n} \sum_{j=1}^{d_n} s_j^* a_j \tilde{\eta}_j \quad \text{and} \quad \tilde{\eta}_j = \eta_j^2 - 1.$$  

Through the correlation inequality from \cite{12} we can get that for any $p \geq 2$ there exists some constant $C_p > 0$ for which

$$E \tilde{\zeta}_n^p \leq C_p \left( \frac{1}{v_n} \sum_{j=1}^{d_n} (s_j^*)^2 a_j^2 \right)^{p/2} \leq C v_n^{-\frac{p}{2(p+2)}},$$

i.e. the expectation $E \tilde{\zeta}_n^p \to 0$ as $n \to \infty$. Therefore, using the Chebychev inequality we obtain that for any $m > 1$

$$n^m P(\tilde{\zeta}_n > r_1) \to 0 \quad \text{as} \quad n \to \infty.$$  

Hence Lemma A.8. \hfill \qed

A.4 The van Trees inequality for the Levy processes.

In this section we consider the following continuous time parametric regression model

$$dy_t = S(t, \theta) dt + d\xi_t, \quad 0 \leq t \leq n, \quad \text{(A.14)}$$

where $\xi_t = \varrho_1 W_t + \varrho_2 z_t$ and

$$S(t, \theta) = \sum_{i=1}^{d} \theta_i \psi_i(t),$$

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with the unknown parameters \( \theta = (\theta_1, \ldots, \theta_d)' \). Here we assume that the functions \((\psi_j)_{1 \leq j \leq d}\) are 1 periodic and orthogonal functions.

Let us denote by \( \nu_\xi \) the distribution of the process \((\xi_t)_{0 \leq t \leq n}\) on the Skorokhod space \( D[0, n] \). One can check directly that in this space for any parameters \( \theta \in \mathbb{R}^d \), the distribution \( P_\theta \) of the process \((A.14)\) is absolutely continuous with respect to the \( \nu_\xi \) and the corresponding Radon-Nikodym derivative, for any function \( x = (x_t)_{0 \leq t \leq T} \) from \( D[0, n] \), is defined as

\[
f(x, \theta) = \frac{dP_\theta}{d\nu_\xi}(x) = \exp \left\{ \int_0^n S(t, \theta) \frac{dx_t}{\theta_1} - \int_0^n \frac{S^2(t, \theta)}{2\theta_1^2} \, dt \right\}, \quad (A.15)
\]

where

\[
x_t = \frac{1}{\theta_1} \left( x_t - \int_0^t \int_\mathbb{R} v (\mu_x(ds, dv) - \Pi(dv)ds) \right)
\]

and for any measurable set \( \Gamma \) in \( \mathbb{R} \) with \( 0 \notin \Gamma \)

\[
\mu_x([0, t] \times \Gamma) = \sum_{0 \leq s \leq t} 1_{\{\Delta \xi_s \in \varphi \Gamma\}}.
\]

Let \( \Phi \) be a prior density on \( \mathbb{R}^d \) having the following form:

\[
\Phi(\theta) = \Phi(\theta_1, \ldots, \theta_d) = \prod_{j=1}^d \varphi_j(\theta_j),
\]

where \( \varphi_j \) is some continuously differentiable density in \( \mathbb{R} \). Moreover, let \( g(\theta) \) be a continuously differentiable \( \mathbb{R}^d \to \mathbb{R} \) function such that, for each \( 1 \leq j \leq d, \)

\[
\lim_{|\theta_j| \to \infty} g(\theta) \varphi_j(\theta_j) = 0 \quad \text{and} \quad \int_{\mathbb{R}^d} |g_j'(\theta)| \Phi(\theta) \, d\theta < \infty, \quad (A.16)
\]

where

\[
g_j'(\theta) = \frac{\partial g(\theta)}{\partial \theta_j}.
\]

For any \( \mathcal{B}(\mathcal{X}) \times \mathcal{B}(\mathbb{R}^d) \) measurable integrable function \( H = H(x, \theta) \) we denote

\[
\mathbb{E} H = \int_{\mathbb{R}^d} \int_{\mathcal{X}} H(x, \theta) \, dP_\theta \, \Phi(\theta) \, d\theta = \int_{\mathbb{R}^d} \int_{\mathcal{X}} H(x, \theta) f(x, \theta) \Phi(\theta) \, d\nu_\xi(x) \, d\theta,
\]

where \( \mathcal{X} = D[0, n] \).
Lemma A.9. For any $\mathcal{F}_T^n$-measurable square integrable function $\hat{g}_T$ and for any $1 \leq j \leq d$, the following inequality holds

$$\tilde{E}(\hat{g}_n - g(\theta))^2 \geq \frac{\Lambda_j^2}{n\|\psi_j\|^2} + I_j,$$

where

$$\Lambda_j = \int_{\mathbb{R}^d} g'_j(\theta) \Phi(\theta) \, d\theta \quad \text{and} \quad I_j = \int_{\mathbb{R}} \frac{\hat{\varphi}_j^2(z)}{\varphi_j(z)} \, dz.$$

Proof. First of all note that, the density (A.15) on the process $\xi$ is bounded with respect to $\theta_j \in \mathbb{R}$ and for any $1 \leq j \leq d$

$$\limsup_{|\theta_j| \to \infty} f(\xi, \theta) = 0. \quad \text{a.s.}$$

Now, we set

$$\bar{\Phi}_j = \tilde{\Phi}_j(x, \theta) = \frac{\partial (f(x, \theta)\Phi(\theta))/\partial \theta_j}{f(x, \theta)\Phi(\theta)}.$$

Taking into account the condition (A.16) and integrating by parts yield

$$\tilde{E}\left((\hat{g}_n - g(\theta)) \bar{\Phi}_j\right) = \int_{X \times \mathbb{R}^d} (\hat{g}_n(x) - g(\theta)) \frac{\partial}{\partial \theta_j} (f(x, \theta)\Phi(\theta)) \, d\theta \nu_\xi(dx)$$

$$= \int_{X \times \mathbb{R}^{d-1}} \left( \int_{\mathbb{R}} g'_j(\theta) f(x, \theta)\Phi(\theta) \, d\theta_j \left( \prod_{i \neq j} d\theta_i \right) \right) \nu_\xi(dx) = \Lambda_j.$$

Now by the Bouniakovskii-Cauchy-Schwarz inequality we obtain the following lower bound for the quadratic risk

$$\tilde{E}(\hat{g}_T - g(\theta))^2 \geq \frac{\Lambda_j^2}{\mathbb{E}\varphi_j^2}.$$

To study the denominator in the left hand of this inequality note that in view of the representation (A.15)

$$\frac{1}{f(y, \theta)} \frac{\partial f(y, \theta)}{\partial \theta_j} = \frac{1}{\varphi_j} \int_0^\infty \psi_j(t) \, dw_t.$$

Therefore, for each $\theta \in \mathbb{R}^d$,

$$\mathbb{E}_\theta \frac{1}{f(y, \theta)} \frac{\partial f(y, \theta)}{\partial \theta_j} = 0$$

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and
\[
\mathbb{E}_\theta \left( \frac{1}{f(y, \theta)} \frac{\partial f(y, \theta)}{\partial \theta_j} \right)^2 = \frac{1}{\varrho_1^2} \int_0^n \psi_j^2(t) \, dt = \frac{n}{\varrho_1^2} \| \psi \|^2.
\]

Taking into account that
\[
\tilde{\Phi}_j = \frac{1}{f(x, \theta)} \frac{\partial f(x, \theta)}{\partial \theta_j} + \frac{1}{\Phi(\theta)} \frac{\partial \Phi(\theta)}{\partial \theta_j},
\]
we get
\[
\tilde{\mathbb{E}} \Psi_j^2 = \frac{n}{\varrho_1^2} \| \psi \|^2 + I_j.
\]

Hence Lemma A.9. \qed
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