Quenched Averaged Correlation Functions of the Random Magnets

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PACS: 05.70.jk, 11.25.Hf, 64.60.Ak

It is shown that the ratios of the quenched averaged three and four-point correlation functions of the local energy density operator to the connected ones in the random-bond Ising model approach asymptotically to some universal functions. We derive the explicit expressions of these universal functions. Moreover it is shown that the individual logarithmic operators have not any contribution to the connected correlation functions of the disordered Ising model.

Random systems represent the spatial inhomogeniuty where scale invariance is only preserved on average but not for specific disorder realization. The understanding of the role played by quenched impurities of the nature of phase transition is one of the significant subjects in statistical physics and has attracted a great deal of attention [1]. According to the Harris criterion [2], quenched randomness is a relevant perturbation at the second-order critical point for systems of dimension \( d \), when its specific heat exponent \( \alpha \) of the pure system is positive. Concerning the effect of randomness on the correlation functions, it is known that the presence of randomness induces a logarithmic factor to the correlation functions of pure system [3]. Theoretical treatment of the quenched disordered systems is a non-trivial task in view of the fact that, one has to average the logarithm of the partition function over various realization of the disorder in the statistical ensemble and therefore find physical results. There are two standard methods to perform this averaging, the supersymmetry (SUSY) approach, and the well-known replica approach. Recently using the replica approach it has been shown by Cardy [4], that the logarithmic factor multiplying power law behavior are to be expected in the scaling behavior near non-mean field critical points. It is shown also that the results are valid for systems with short-range interactions and in an arbitrary number of dimensions. He concludes that in the limit of \( n \to 0 \) of replicas the theory posses of a set of fields which are degenerate (they have the same scaling dimensions) and finds a pair of fields which form a Jordan cell structure for dilatation operator and derives logarithmic operator in such disordered systems. Cardy proves that the quenched disordered theory with \( Z = 1 \) can be described by logarithmic conformal field theory as well. The logarithmic conformal field theories (LCFT) are extensions of conventional conformal field theories, which have emerged in recent years in a number of interesting physical problems of condensed matter physics [5], string theory [6], and nonlinear dynamical systems [7]. The LCFT are characterized by the fact that their dilatation operator \( L_0 \) are not diagonalized and admit a Jordan cell structure. The non-trivial mixing between these operators leads to logarithmic singularities in their correlation functions. It has been shown [8] that the correlator of two fields in such field theories, has a logarithmic singularity.

\begin{equation}
< \psi(r)\psi(r') > \sim |r-r'|^{-2\Delta} \log |r-r'| + \ldots \tag{1}
\end{equation}

In this direction we show that the quenched averaged connected correlation functions of local energy density field can be written in terms of ordinary scaling operators which can be constructed by the difference of energy operators in two different replicas. We write the connected 3 and 4-point correlation functions of energy density explicitly in terms of such ordinary operators. Furthermore we prove that the logarithmic operators have no contribution in the quenched averaged connected correlation functions of the local density operator. However, these operators play a considerable role on the disconnected ones and produce some logarithmic factors in the correlation functions. We calculate the various types of quenched averaged 3 and 4-point correlation functions of the local energy density and show that the ratios of these correlation functions to the connected ones have the specific universal asymptotic and write down these universal functions explicitly.

\textbf{2-Connected Correlation Functions of Random Magnets}

We consider a quenched random ferromagnet, for instance an Ising model, with random-bond disorder. Let us describe this disordered system in the continuum limit by the following Hamiltonian,

\begin{equation}
H = H^0 + \int J(r)E(r)d^dr \tag{2}
\end{equation}

where \( H^0 \) is the Hamiltonian of the renormalization group at fixed point describing the pure Ising model. The field \( J(r) \) is a quenched random variable coupled to
the local energy density \( E(r) \). When the coupling \( J(r) \) is independent of \( x \) and not random, the above Hamiltonian describes the behavior of the statistical model near its critical point. For simplicity we assume that the random variable \( J(r) \) is a gaussian variable which is characterized by its two moments \( < J(r) >= 0 \) and \( < J(r)J(r') >= g\delta(r-r') \). The standard procedure of averaging over disorder is to introduce replicas, i.e., \( n \) identical copies of the same model for which

\[
Z^n = Tr \exp \left\{ - \sum_{a=1}^{n} H_a^0 - \int d^d r J(r) \sum_{a=1}^{n} E_a(r) \right\}
\]

(3)

averaging over the disorder gives rise to the following effective replical Hamiltonian.

\[
H_R = \sum_{a=1}^{n} H_a^0 - g \int \sum_{a \neq b} E_a(r)E_b(r)d^d r
\]

(4)

we keep only the non-diagonal terms, since using the operator algebra of the pure system one can absorb the diagonal terms into \( H_a^0 \). The replicas are now coupled via the disorder. The scaling dimension of coupling \( g \) is \( y_g = d - 2\Delta_E \) and is relevant at the pure fixed point if \( y_g > 0 \). For small \( y_g \) it is possible to use standard perturbation theory and find the possible random fixed point. It is noted by Cardy that the \( n \) operator \( E_a \) are degenerate at the pure fixed point and one can decompose them into irreducible representation of permutation group \( S_n \). It has been shown that the combination \( E_t = \sum_{a=1}^{n} E_a \) is a singlet (symmetric under the permutation of the replica group) and \( \tilde{E}_a = E_a - \frac{1}{n} \sum_{b=1}^{n} E_b \) transforms according to an \((n-1)\)-dimensional representation of \( S_n \). The fields \( \tilde{E}_a \) satisfy the condition \( \sum_{a=1}^{n} \tilde{E}_a = 0 \). The important observation is that the fields \( E_t \) and \( \tilde{E}_a \) have the proper scaling dimensions close to \( n \to 0 \) as \( \Delta_{E_t} = \Delta_{E_t}^{(0)} + \frac{1}{2}(1-n)y_g + O(y_g^2) \) and \( \Delta_{\tilde{E}} = \Delta_{\tilde{E}}^{(0)} + \frac{1}{2}y_g + O(y_g^2) \) respectively. It is clear that the singlet field \( E_t \) becomes degenerate with the \((n-1)\) operators \( \tilde{E}_a \). This is true to all orders \([4]\). However they do not form the basis of the Jordan cell for the dilatation operator. To find the logarithmic pair according to \([4]\) we define the correlation function of \( < E_t(0)E_t(r) >= A_1 \) and \( < \tilde{E}_a(0)\tilde{E}_a(r) >= B_1 \) and find the following relations for \( A_1 \) and \( B_1 \).

\[
A_1 = n(a - (n-1)b) \equiv nA(n)r^{-2\Delta_E(n)}
\]

\[
B_1 = (1 - \frac{1}{n})(a-b) \equiv (1 - \frac{1}{n})B(n)r^{-2\Delta_E(n)}
\]

(5)

where \( a = < E_t(0)E_t(r) > \) and \( b = < E_i(0)E_j(r) > \) with \( i \neq j \). the above equations enable us to write the quenched averaged connected two-point correlation functions of energy density operator in terms of \( a \) and \( b \) in the limit of \( n \to 0 \) as:

\[
< E(0)E(r) > \sim a - b \quad \text{which is equal to} \quad B(0)r^{-2\Delta_E}
\]

and it has a pure scaling behavior. However, the correlation functions \( a \) and \( b \) have the logarithmic singularities and behave as:

\[
< E_t(0)E_t(r) > = (A'(0) - B'(0) + B(0) - \frac{B(0)y_g}{2} \ln r)r^{-2\Delta_E}
\]

\[
< E_a(0)E_a(r) > = (A'(0) - B'(0) - A(0)\frac{y_g}{2} \ln r)r^{-2\Delta_E}
\]

(6)

where \( A(0) = B(0) \). The prime sign in the eq. (6) means differentiating with respect to \( n \). This means that in the limit \( n \to 0 \) the fields \( E_t \) and \( E_a \) form a basis of Jordan cell, i.e. their two point correlation functions behave as:

\[
< E_t(0)E_t(r) >= 0, \quad < E_0(0)E_a(r) >= a_1r^{-2\Delta_E} \quad \text{and} \quad < E_a(0)E_b(r) >= (-2a_1 \ln r + D_{a,b})r^{-2\Delta_E}, \quad \text{where} \quad a_1 \quad \text{and} \quad D_{a,b} \quad \text{are some constants}.
\]

As noted by Cardy the ratio of quenched averaged two-point correlators of the energy density operator to the connected one has a universal \( r \)-dependence as:

\[
\frac{< E(0)E(r) >}{< E(0)E(r) >_c} \sim \frac{< E(0)E(r) >}{< E(0)E(r) >_c} \sim \ln r
\]

(7)

To understand the structure of Jordan cell, we note that in 2D one can define the operator \( L_0 \) as

\[
L_0 = \begin{pmatrix} \Delta_E & 0 \\ 1 & \Delta_E \end{pmatrix}
\]

(8)

so that, \( L_0E_t = \Delta_EE_t \) and \( L_0E_a = \Delta_EE_a + E_t \) in the limit of \( n \to 0 \). Using this representation for \( L_0 \) one can show that the field \( E_t \) with its logarithmic partner \( E_a \) have the standard logarithmic correlation functions \([9]\), (see the correlation above the eq.(8) ). We note that in 2D we have dealt with two-dimensional conformal field theory, relying heavily on the underlying Virasoro algebra. For an extention to \( D \) dimensions one has to modify the representation of the Virasoro algebra to higher dimensions \([10]\). We consider a doublet of fields (Jordan cell) \( \Phi = \left( \begin{array}{c} E_t \\ E_a \end{array} \right) \) and note that under \( D \)-dimensional conformal transformation \( x \to x' \), we have, \( \Phi(x) \to \Phi'(x') = G^T \Phi(x) \) where \( T \) is a two dimensional matrix which has Jordan form and \( G = \frac{\partial x'}{\partial x} \) is the Jacobian. For our particular case \( T \) has the following Jordan form:

\[
T = \begin{pmatrix} -\frac{2\Delta_E}{D} & 0 \\ 1 & -\frac{2\Delta_E}{D} \end{pmatrix}
\]

(9)

and one can show that the two fields \( E_t \) and \( E_a \) transform as:

\[
E_t(x') = G^{-\frac{2\Delta_E}{D}}E_t(x)
\]

\[
E_a(x') = G^{-\frac{2\Delta_E}{D}}(\ln(G)E_t(x) + E_a(x))
\]

(10)
This expresses that the top-field $E_t$ always transforms as an ordinary scaling operator. It can be verified that the correlation functions of fields $E_t$ and $E_a$ have the standard D- dimensional logarithmic conformal field theory structure [9,10]. Using the above results, it is evident that the dimension of field-difference $E_a − E_b$ with $a ̸= b$ is $\Delta_\phi$ and it transforms as an ordinary operator under the scaling transformation. The interesting observation is that the connected averaged correlation functions depend on the difference fields $E_a − E_b$ only and therefore they behave as the ordinary correlation functions. For instance in the following we write the connected quenched averaged 1, 3- and 4-point functions of local energy density in terms of the field-difference operators explicitly,

$$<E(1)E(2)E(3)>_c = \frac{1}{2} <(E_a - E_b)(E_a - E_b)(E_a - E_b)(E_a - E_b)> \quad (11)$$

$$<E(1)E(2)E(3)>_c = <(E_a - E_b)(E_a - E_c)(E_a - E_d)(E_a - E_b)> \quad (12)$$

$$<E(1)E(2)E(3)>_c =$$

$$<E_a - E_b)(E_a - E_c)(E_a - E_d)(E_a - E_b) + \frac{1}{2} <E_a - E_b)(E_a - E_d)(E_a - E_b)(E_a - E_b) > \quad (13)$$

where the last equation has only 15 independent terms.

To confirm this prediction also one can directly show that the quenched averaged connected correlation functions have a pure scaling behaviour which is determined by ordinary scaling operators and the logarithmic operators $E_a$ do not change its behavior. This can be verified directly for the quenched averaged connected 3-point correlation function of energy density.

We are interested in deriving exactly the various 3-point quenched averaged functions as $<E(1)E(2)E(3)>$, $<E(1)E(2)E(3)>$ and $<E(1)E(2)E(3)>$, which can be written in terms of the replica correlation functions $<E(1)(E(2)E(3)) = a <E(1)(E(2)E(2)(3)) = b$ and $<E(1)(E(2)(3)) = c$, respectively. One can derive the correlation functions $a$, $b$ and $c$ by means of 3-point functions of $E_t$ and $\tilde{E}_a$ as follows:

$$<\tilde{E}_a(1)\tilde{E}_a(2)E(3)> = n_1a + (n_2^2(n-1))$$

$$-4n_1^2 + \frac{1}{n^2}(n-1)^2 + \frac{2}{n^2}(n-1)(n-2)b$$

$$+ (-\frac{2}{n}n_1(n-1)(n-2) + \frac{1}{n^2}(n-2)^2(n-1))c$$

$$\equiv (1 - \frac{1}{n})B_1 \quad (14)$$

and finally,

$$<\tilde{E}_a(1)\tilde{E}_a(2)E(3)> = (n_1^2 + \frac{n-1}{n^3})a$$

$$(-3n_1^2n_2 - \frac{3}{n^3}(n-1)(n-2) + \frac{3}{n^2}n_1(n-1))b$$

$$+ \frac{3}{n^2}n_1(n-1)(n-2) - \frac{1}{n^3}(n-1)(n-2)(n-3)c$$

$$\equiv (1 - \frac{1}{n})(1 - \frac{2}{n})C_1 \quad (15)$$

where $n_1 = (1 - \frac{1}{n})$ and $A_1$, $B_1$ and $C_1$ are pure scaling functions of variables $r_{i,j}$. To derive the above equations we use the replica symmetry and symmetry of the various types of 3-point correlation functions under interchanging of positions. We note that replica symmetry leads to have $<\tilde{E}_a(1)E_t(2)E_t(3)> = 0$ and therefore, does not give any new relationship between $a$, $b$ and $c$. Using the above equations, it can be found that the correlation functions $a$, $b$ and $c$ are as follows:

$$a = \frac{3nB_1 - 3nC_1 + n^2C_1 + A_1 - 3B_1 + 2C_1}{n^2}$$

$$b = \frac{nB_1 - nC_1 + A_1 - 3B_1 + 2C_1}{n^2}$$

$$c = \frac{A_1 - 3B_1 + 2C_1}{n^2} \quad (16)$$

Using the above equations we can show that the connected quenched averaged 3-point function behaves as:

$$<E(1)E(2)E(3)> = 2c + a - 3b$$

$$= C_1 \quad (17)$$

which is a scaling function and confirms the observation that the logarithmic operators (individually) have no role in the connected quenched averaged correlation functions. In addition one can derive the correlation functions $<E_i(1)E_j(2)E_k(3)>$ for given $i$, $j$ and $k$ in the limit of $n \rightarrow 0$ and show that they have the following form:

$$<E_i(1)E_j(2)E_k(3)> = [\alpha_{ijk} - \beta_{ijk}D_1]$$

$$+ \gamma_{ijk}(4D_2 - D_1^2)|f(1,2,3) \quad (18)$$
where \( f(1,2,3) = (r_{12}r_{13}r_{23})^{-2\Delta_E} \), \( D_1 = \ln(r_{12}r_{13}r_{23}) \) and \( D_2 = \ln r_{23} + \ln r_{13} + \ln r_{12} + \ln r_{23} \ln r_{12} \). It can also be shown that the ratio of various symmetrized 3-point functions to the connected one behaves asymptotically as a universal function

\[
\frac{1}{3}(4D_2 - D_1^2). \tag{19}
\]

We generalize the above calculations to derive the various type of 4-point correlation functions and show that the ratio of the various disconnected to the connected one have the following universal asymptotic:

\[
\sim \frac{1}{36}[O_1^3 - 6O_2 - 3O_3 - 12O_4 - 18O_5] \tag{20}
\]

where \( O_1 = \ln(r_{12}r_{13}r_{14}r_{23}r_{24}r_{34}) \), \( O_2 = (\ln r_{ij} \ln r_{kl}^2 + \cdots) \) with \( i \neq j \neq k \neq l \), \( O_3 = (\ln r_{ij} \ln r_{kl}^2 + \cdots) \) with \( i \neq j \neq k \neq l \), and finally \( O_5 = (\ln r_{ij} \ln r_{ik} \ln r_{il} + \cdots) \) with \( i \neq j \neq k \neq l \).

In summary, in this paper we have studied the correlation functions of disordered random magnets [11] and obtain the various types of 3 and 4-point quenched averaged correlation functions. One can check directly that these different types of the 3 and 4-point correlation functions have the general property of a logarithmic conformal field theory that the logarithmic partner can be regarded as the formal derivative of the ordinary fields (top field) with respect to their conformal weight [9]. In this case, one can consider the \( E_n \) fields as the derivative of \( E_i \) with respect to \( n \). We emphasise that the derivative with respect to scaling weight can be written in terms of the derivative with respect to \( n \). These properties enable us to calculate any \( N \)-point correlation function containing the logarithmic field \( E_n \) in terms of the correlation functions of the top-fields.

The general expression of the correlation functions of the LCFT’s are given in ref.[9] and here we determine the unknown constants in the logarithmic correlation functions in terms of details of the random-bond Ising model. It is noted that the formal derivations with respect to scaling dimensions can not predict the unknown constants in the quenched averaged correlation functions of the local energy density operators. The constant depends on the details of the statistical model. We have shown that the individual logarithmic operators \( E_n \) do not have any contribution to the quenched averaged connected correlation functions of the energy density. We also obtain that the connected correlation functions can be written in terms of the difference fields which transform as an ordinary scaling operator. However they will play a crucial role to the disconnected averaged correlation functions. Also we find that the ratio of the various types of 3 and 4-point quenched averaged correlation functions to the connected ones have a universal asymptotic behavior and give their explicit form. These predictions can also be investigated numerically. Our analysis are valid in all dimensions as long as the dimension is below the upper critical dimensions. To derive the above results we have used the replica symmetry. Any attempt towards the breaking of this symmetry will change completely the above picture and produces more than one logarithmic fields in the block and produces higher order logarithmic singularities [9].

These results can be easily generalized to other problem such as polymer statistics, percolation and random phase sine-Gordon model etc.

We would like to thank John Cardy for his useful comments and A. Aghamohammadi and R. Asgari, B. Davoudi and J. Davoudi for their useful discussions. This paper is dedicated to Dr. A. M. Zaker, associate professor of Physics Department, who deceased last may 2000.

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