Mesons with Beauty and Charm: New Horizons in Spectroscopy

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The $B_c^+$ family of $(c\bar{b})$ mesons with beauty and charm is of special interest among heavy quarkonium systems. The $B_c^+$ mesons are intermediate between $(c\bar{c})$ and $(b\bar{b})$ states both in mass and size, so many features of the $(c\bar{b})$ spectrum can be inferred from what we know of the charmonium and bottomonium systems. The unequal quark masses mean that the dynamics may be richer than a simple interpolation would imply, in part because the charmed quark moves faster in $B_c$ than in the $J/\psi$. Close examination of the $B_c^+$ spectrum can test our understanding of the interactions between heavy quarks and antiquarks and may reveal where approximations break down.

Whereas the $J/\psi$ and $\Upsilon$ levels that lie below flavor threshold are metastable with respect to strong decays, the $B_c$ ground state is absolutely stable against strong or electromagnetic decays. Its dominant weak decays arise from $\bar{b} \to \bar{c}W^+$, $c \to sW^+$, and $c\bar{b} \to W^+$ transitions, where $W^+$ designates a virtual weak boson. Prominent examples of the first category are quarkonium transmutations such as $B_c^+ \to J/\psi \pi^+$ and $B_c^0 \to J/\psi \ell^+\nu$, where $J/\psi$ designates the $(c\bar{c})$ $1S$ level.

The high data rates and extraordinarily capable detectors at the Large Hadron Collider give renewed impetus to the study of mesons with beauty and charm. Motivated by the recent experimental searches for the radially excited $B_c$ states, we update the expectations for the low-lying spectrum of the $B_c$ system. We make use of lattice QCD results, a novel treatment of spin splittings, and an improved quarkonium potential to obtain detailed predictions for masses and decays. We suggest promising modes in which to observe excited states at the LHC. The $3P$ and $3S$ states, which lie close to or just above the threshold for strong decays, may provide new insights into the mixing between quarkonium bound states and nearby two-body open-flavor channels. Searches in the $B^{(*)}D^{(*)}$ final states could well reveal narrow resonances in the $J^P = 0^+, 1^-$, and $2^+$ channels and possibly in the $J^P = 0^+$ and $1^+$ channels at threshold.

Looking further ahead, the prospect of very-high-luminosity $e^+e^-$ colliders capable of producing tera-$Z$ samples raises the possibility of investigating $B_c$ spectroscopy and rare decays in a controlled environment.

I. INTRODUCTION

Although the lowest-lying $(c\bar{b})$ meson has long been established, the spectrum of excited states is little explored. The ATLAS experiment at CERN’s Large Hadron Collider reported the observation of a radially excited $B_c$ state [1], but this sighting was not confirmed by the LHCb experiment [2]. The unsettled experimental situation and the large data sets now available for analysis make it timely for us to provide up-to-date theoretical expectations for the spectrum and decay patterns of narrow $(c\bar{b})$ states, and for their production in hadron colliders [3]. New work from the CMS Collaboration [4] shows the way toward exploiting the potential of $(c\bar{b})$ spectroscopy.

A. What we know of the $B_c$ mesons

The possibility of a spectrum of narrow $B_c$ states was first suggested by Eichten and Feinberg [5]. Anticipating the copious production of $b$-quarks at Fermilab’s Tevatron Collider and CERN’s Large Electron–Positron Collider (LEP), we presented a comprehensive portrait of the spectroscopy of the $B_c$ meson and its long-lived excited states [6], based on then-current knowledge of the interaction between heavy quarks derived from $(c\bar{c})$ and $(b\bar{b})$ bound states, within the framework of nonrelativistic quantum mechanics [7]. Surveying four representative potentials, we characterized the mass of the $J^P = 0^-$ ground state as $M(B_c) \approx 6258 \pm 20$ MeV. A small number of $B_c$ candidates appeared in hadronic $Z^0$ decays at LEP. The CDF Collaboration observed the decay $B_c^{\pm} \to J/\psi \ell^\pm\nu$ in 1.8-TeV $p\bar{p}$ collisions at the Fermilab Tevatron [8], estimating the mass as $M(B_c) \approx 6400 \pm 411$ MeV. (The generic lepton $\ell$ represents an electron or muon.) Subsequent work by the CDF [9], D0 [10], and LHCb [11] Collaborations has refined the mass to $M(B_c) = 6274.9 \pm 0.8$ MeV [12], with the most precise determinations coming from fully reconstructed final states such as $J/\psi \pi^+$.

Investigations based on the spacetime lattice formulation of QCD aim to provide ab initio calculations that incorporate the full dynamical content of the theory of strong interactions. Before the nonleptonic $B_c$
decays had been observed, a first unquenched lattice QCD prediction, incorporating $2 + 1$ dynamical quark flavors ($u/d, s$) found $M(B_c) = 6304 ± 12^{+3}_{-0}$ MeV [13], where the first error bar represents statistical and systematic uncertainties and the second characterizes heavy-quark discretization effects. Calculations incorporating $2 + 1 + 1$ dynamical quark flavors ([14] yield $M(1S_0) = 6278 ± 9$ MeV, in impressive agreement with the measured $B_c$ mass, and predict $M(2S_0) = 6894 ± 19 ± 8$ MeV [15].

Three distinct elementary processes contribute to the decay of $B_c$: the individual decays $b \to cW^+\bar{c}$ and $c \to sW^+\bar{c}$ of the two heavy constituents, and the annihilation $cb \to W^{**}$ through a virtual $W$-boson. Several examples of the $b \to c$ transition have been observed, including the final states $J/\psi \ell^+\nu_\ell, J/\psi\pi^+, J/\psi K^+, J/\psi \pi^+\pi^+\pi^-, J/\psi \pi^+ K^+ K^-, J/\psi \pi^+ \pi^+\pi^-\pi^-, J/\psi D^+, J/\psi D^{**},$ and $J/\psi (2S\pi^+)$. A single channel, $B_c^0\pi^+$, representing the $c \to s$ transition is known. The annihilation mechanism, which would lead to final states such as $\tau^+\nu_\tau$ and $pp\pi^+$, has not yet been established. The observed lifetime, $\tau(B_c) = (0.507 ± 0.009)$ ps [12], is consistent with theoretical expectations [15] [17]. Predictions for partial decay rates (or relative branching fractions) await experimental tests. Some recent theoretical works explore the potential of rare $B_c$ decays [13].

Until recently, the only evidence reported for a $(cb)$ excited state was presented by the ATLAS Collaboration [1] in $pp$ collisions at 7 and 8 TeV, in samples of 4.9 and 19.2 fb$^{-1}$. They observed a new state at $6842 ± 7$ MeV in the $M(B_c^0\pi^+) - M(B_c^0)$ mass difference, with $B_c^0$ detected in the $J/\psi \pi^\pm$ mode. The mass $527 ± 7$ MeV above $M(1S)$ and decay of this state are broadly in line with expectations for the second $s$-wave state, $B_c^+(2S)$. In addition to the nonrelativistic potential-model calculations cited above, the HPQCD Collaboration has presented preliminary results from a lattice calculation using $2 + 1 + 1$ dynamical fermion flavors and highly improved staggered quark correlators [19]. They report $M(2S_0) = 6892 ± 41$ MeV, which is $576.5 ± 41$ MeV above $M(1S))$. This result and the NRQCD prediction [14] lie above the ATLAS report by one and two standard deviations, respectively. The significance of the discrepancy is limited for the moment by lattice uncertainties. A plausible interpretation has been that ATLAS might have observed the transition $B_c^+(2S) \to B_c(1S)\pi^+\pi^-$, missing the low-energy photon from the subsequent $B_c^* \to B_c\gamma$ decay, and that the signal is an unresolved combination of $2S_1$ and $2S_0$ peaks. A search by the LHCB collaboration in 2 fb$^{-1}$ of 8-TeV $pp$ data yielded no evidence for either $B_c(2S)$ state [2]. As we prepared this article for publication, the CMS Collaboration provided striking evidence for both $B_c(2S)$ levels, in the form of well-separated peaks in the $B_c\pi^+\pi^-$ invariant mass distribution, closely matching the theoretical template [4]. We incorporate these new observations into the discussion that follows in [V.A].

### B. Analyzing the $(cb)$ bound states

The nonrelativistic potential picture, motivated by the asymptotic freedom of QCD [20], gave early insight into the nature of charmonium and generated a template for the spectrum of excited states [21]. For more than four decades, it has served as a reliable guide to quarkonium spectroscopy, including the states lying near or just above flavor threshold for fission into two heavy-light mesons that are significantly influenced by coupled-channel effects [22] [23].

We view the nonrelativistic potential-model treatment as a steppingstone, not a final answer, however impressive its record of utility. Potential theory does not capture the full dynamics of the strong interaction, and while the standard coupled-channel treatment is built on a plausible physical picture, it is not derived from first principles. Moreover, relativistic effects may be more important for $(cb)$ than for $(cc)$. The $c\bar{c}$-quark moves faster in the $B_c$ meson than in the $J/\psi$, because it must balance the momentum of a more massive $b$-quark. One developing area of theoretical research has been to explore methods more robust than nonrelativistic quantum mechanics [17] [24] [25].

Nonperturbative calculations on a spacetime lattice in principle embody the full content of QCD. This approach is yielding increasingly precise predictions for the masses of $(cb)$ levels up through $B_c^*(2S_1)$ state. It is not yet possible to extract reliable signals for higher-lying states from the lattice, so we rely on potential-model methods to construct a template for the $B_c$ spectrum through the $4S_1$ level. If experiments should uncover systematic deviations from the expectations we present, they may be taken as evidence of dynamical features absent from the nonrelativistic potential-model paradigm, including—or coupling to states above flavor threshold, which we neglect our calculations of the spectrum.

In the following [14] we develop the theoretical tools required to compute the $(cb)$ spectrum. In earlier work [9], we examined the Cornell Coulomb-plus-linear potential [22], a power-law potential [26], Richardson’s QCD-inspired potential [27], and a second QCD-inspired potential due to Buchmüller and Tye [28], which we took as our reference model. We used a perturbation-theory treatment of spin splittings. Using insights from lattice QCD and higher-order perturbative calculations, we construct a new potential that differs in detail from those explored in earlier work. We also use lattice results and rich experimental information on the $(cc)$ and $(bb)$ spectra to refine the treatment of spin splittings. We present our expectations for the spectrum of narrow states in Section [II].

We consider decays of the narrow states in section [IV] updating the results we gave in Ref. [9]. We compute differential and integrated cross sections for the narrow $B_c$ levels in proton–proton collisions at the Large Hadron Collider in [V]. Putting all these elements together, we show how to unravel the $2S$ levels and explore how higher levels might be observed. Prospects for a future $e^+e^- \to$ Tera-$Z$ machine appear in [VI]. We draw
II. THEORETICAL PRELIMINARIES

We take as our starting point a Coulomb-plus-linear potential (the “Cornell potential”\cite{22}),

\[ V(r) = -\frac{\kappa}{r} + \frac{r}{a^2}, \]  

(1)

where \( \kappa \equiv 4\alpha_s/3 = 0.52 \) and \( a = 2.34 \text{ GeV}^{-1} \) were chosen to fit the quarkonium spectra. Analysis of the \( J/\psi \) and \( \Upsilon \) families led to the choices

\[ m_c = 1.84 \text{ GeV} \quad m_b = 5.18 \text{ GeV}. \]  

(2)

This simple form has been modified to incorporate running of the strong coupling constant in Refs.\cite{27,28}, among others, using the perturbative-QCD evolution equation at leading order and beyond. At distances relevant for confinement, perturbation theory ceases to be a reliable guide. It is now widely held, following Gribov\cite{29}, that as a result of quantum screening \( \alpha_s \) approaches a critical, or frozen, value at long distances (low energy scales). In a light (\( q\bar{q} \)) system, Gribov estimated

\[ \alpha_s \to \bar{\alpha}_s = \frac{3\pi}{4} \left( 1 - \sqrt{2/3} \right) \approx 0.14\pi = 0.44. \]  

(3)

We incorporate the spirit of this insight into a new version of the Coulomb-plus-linear form that we call the frozen-\( \alpha_s \) potential.

The long-range part is the standard Cornell linear term. To obtain the Coulomb piece, we convert the four-loop running of \( \alpha_s(q) \) in momentum space \cite{30} to the behavior in position space using the method of \cite{31}, with an important modification. We set \( \alpha_s(q = 1.6 \text{ GeV}) = 0.338 \) and evolve with three active quark flavors. To enforce saturation of \( \alpha_s(r) \) at long distances, we alter the recipe of Ref.\cite{31}, replacing the identification \( q = 1/r \exp(\gamma_E) \), where \( \gamma_E = 0.57721 \ldots \) is Euler’s constant, with the damped form \( q = 1/[(r \exp(\gamma_E))^2 + \mu^2]^{1/2} \). For our reference potential, we have chosen the damping parameter \( \mu = 1.2 \text{ GeV} \). The consequent evolution of \( \alpha_s(r) \) is plotted as the solid red curve in Figure\textsuperscript{1}, where we also show an alternative choice of \( \mu = 0.8 \text{ GeV} \) (dashed gold curve), the constant \( \alpha_s \) of the original Cornell potential (dotted green curve) and \( \alpha_s(r) \) corresponding to the Richardson potential (dot-dashed blue curve).

We plot in Figure\textsuperscript{2} the frozen-\( \alpha_s \) potential for both our chosen example, \( \mu = 1.2 \text{ GeV} \), and the alternative, \( \mu = 0.8 \text{ GeV} \). There we also show the Richardson and Cornell potentials. All coincide at large distances. The Cornell potential is deeper at short distances than any of the potentials that take account of the evolution of \( \alpha_s \). For the convenience of others who may wish to apply the new potential, we present values of \( \alpha_s(r) \) suitable for interpolation in an Appendix.

We presented the general formalism for spin-dependent interactions as laid out by Eichten & Feinberg\cite{5} and Gromes\cite{32} in §II B of Ref.\cite{9}, where we took a perturbative approach to the spin–orbit and tensor interactions. In the intervening time, the charmonium and bottomonium spectra have been mapped in detail, as summarized in Table\textsuperscript{I}. This wealth of information leads us now to choose a more phenomenological approach.
We write the spin-dependent contributions to the \((c\bar{b})\) masses as
\[
\Delta = \sum_{k=1}^{4} T_k ,
\]
where the individual terms are
\[
T_1 = \frac{\langle \vec{S} \cdot \vec{s}_i \rangle}{m_i^2} \tilde{T}_1(m_i, m_j) + \frac{\langle \vec{S} \cdot \vec{s}_j \rangle}{m_j^2} \tilde{T}_1(m_j, m_i) \\
T_2 = \frac{\langle \vec{s}_i \cdot \vec{s}_j \rangle}{m_i m_j} \tilde{T}_2(m_i, m_j) + \frac{\langle \vec{s}_i \cdot \vec{s}_j \rangle}{m_i m_j} \tilde{T}_2(m_j, m_i) \\
T_3 = \frac{\langle S_{ij} \rangle}{m_i m_j} \tilde{T}_3(m_i, m_j) \\
T_4 = \frac{\langle S_{ij} \rangle}{m_i m_j} \tilde{T}_4(m_i, m_j) ,
\]
where we have introduced the phenomenological coefficients \(\tilde{c}_2\) and \(\tilde{c}_4\), which take the value unity in the perturbative approach.

We extract values of \(\tilde{T}_2\) and \(\tilde{T}_4\) for the observed levels that appear in Table II. These are shown as the underlined entries in Table II. Then, we combine the definitions in Eq. (10) with our calculated values of \(\langle \alpha_s/r^3 \rangle\) to determine \(\tilde{c}_2\) and \(\tilde{c}_4\) in the \((c\bar{c})\) and \((b\bar{b})\) families. The geometric mean of these values is our estimates for the coefficients in the \((c\bar{b})\) system. We insert these back into Eq. (6) to estimate the values of \(\tilde{T}_2\) and \(\tilde{T}_4\) for the \(B_c\) family. For completeness, we include our evaluations of \((1/r)dV/dr\) in the Table.

For the \(J/\psi\) and \(\Upsilon\) families, composed of equal-mass heavy quarks, the familiar \(LS\) coupling scheme, in which states are labeled by \(nL^p_{LJ}\), is apt. When the quark masses are unequal, as in the case at hand, spin-dependent terms in the Hamiltonian mix the spin-singlet and spin-triplet \(J = L\) states. We define
\[
|nL_L\rangle' = \cos \theta |nL_L\rangle + \sin \theta |n^3L_L\rangle \\
|nL_L\rangle = -\sin \theta |nL_L\rangle + \cos \theta |n^3L_L\rangle ,
\]
where
\[
\tan \theta = \frac{2A}{B + \sqrt{B^2 + 4A^2}},
\]
with
\[
A = \frac{1}{4} \sqrt{L(L + 1) \left( \frac{1}{m_c^2} - \frac{1}{m_b^2} \right)} \tilde{T}_1
\]
and
\[
B = \frac{1}{4} \left( \frac{1}{m_c^2} + \frac{1}{m_b^2} \right) \tilde{T}_1 + \frac{1}{m_c m_b} \tilde{T}_2 - \frac{1}{m_c m_b} \tilde{T}_4.
\]

Then our calculations of the \(\tilde{T}_k\) defined in Eq. (6) lead to these values for the mixing angle: \(\theta_{3P} = 18.7^\circ\), \(\theta_{3D} = -49.2^\circ\), \(\theta_{4P} = 21.2^\circ\), \(\theta_{4F} = -49.5^\circ\), \(\theta_{4D} = -40.3^\circ\). A Lattice calculation in quenched QCD [33] gave \(\theta_{2P} = 33 \pm 2^\circ\).

The masses of the mixed states are
\[
M(nL_L') = \langle M(nL) \rangle - \frac{1}{2} (B - \sqrt{4A^2 + B^2}) \\
M(nL_L) = \langle M(nL) \rangle - \frac{1}{2} (B + \sqrt{4A^2 + B^2}),
\]
where \(\langle M(nL) \rangle\) is the \(nL\) centroid.

At lowest order, the hyperfine splitting between \(s\)-wave states, arising from \(T_3\), is given by
\[
\Delta_{HFS}^{(s)} = M(n^3S_1) - M(n^3S_0) = \frac{8\alpha_s |R_{n0}(0)|^2}{9m_c m_b},
\]
which is susceptible to significant quantum corrections. Rather than make \textit{a priori} calculations of the hyperfine splitting, we adopt the lattice QCD result for the ground state and scale the splittings of excited states according to
\[
\frac{\Delta_{HFS}^{(n)}}{\Delta_{HFS}^{(1)}} = \frac{|R_{n0}(0)|^2}{|R_{10}(0)|^2}.
\]
TABLE III. Calculated excitation energies (in MeV) for (c\bar{b}) levels with respect to the $B_c (1S)$ centroid according to potential models and Lattice QCD simulations. The potential models have been aligned with the 1$S$ doublet centroid at 6315.5 MeV. Communication with states above flavor threshold is neglected.

| Level | EQ94 [14] | Lattice QCD | This Work |
|-------|------------|-------------|-----------|
| $1^3S_0$ | -54.8 | -40.5 [14] [34] [35] | -40.5 |
| $1^1S_1$ | 18.2 | 13.5 [14] [34] [35] | 13.5 |
| $2^1D_3/2$ | 381 | 393(17)(7) [35] | 377 |
| $2^3D_3/2$ | 411 | 417(18)(7) [35] | 415 |
| $2^3D_5/2$ | 417 | 446(30) [37] | 423 |
| $2^3S_1$ | 428 | 464(30) [35] | 435 |
| $2^5S_1$ | 537 | 561(18)(1) [14] | 551 |
| $3^3D_3/2$ | 580 | 601(19)(1) [14] | 582 |

3$D_2$ levels: 693 - 691, 3$D_2$ levels: 693 - 690, 3$D_3$ levels: 690 - 695, 3$D_3$ levels: 789 - 788, 3$D_3$ levels: 783 - 782, 3$D_3$ levels: 816 - 828, 3$D_3$ levels: 834 - 839, 3$S_0$ levels: 925 - 938, 3$S_1$ levels: 961 - 964.

3$P_2$ levels: 918 - 916, 3$P_2$ levels: 906 - 903, 3$P_2$ levels: 922 - 920, 3$P_2$ levels: 908 - 906, 3$P_2$ levels: 1031 - 1033, 3$P_2$ levels: 1033 - 1034, 3$P_2$ levels: 1040 - 1038, 3$P_2$ levels: 1038 - 1036, 3$P_2$ levels: 1121 - 1119, 3$P_2$ levels: 1158 - 1156, 3$P_2$ levels: 1167 - 1165, 3$S_0$ levels: 1243 - 1257, 3$S_1$ levels: 1276 - 1280.

III. THE $B_c$ SPECTRUM

The vector meson $B_c^*$, the $1^3S_1$ hyperfine partner of $B_c$ and analogue of $J/\psi$ and $\Upsilon$, has not yet been observed. Modern lattice calculations [14, 34, 35] give consistent values for the hyperfine splitting $M(B_c^*) - M(B_c) = (53 \pm 7.5, 54 \pm 7.5, 55 \pm 3$ MeV), so we take the mass of the vector state to be $M(B_c^*) = 6329$ MeV and fix the centroid ($M(1S)$) of the ground-state s-wave doublet at 6315.5 MeV for the lattice.

We summarize in Table III predictions for the spectrum of mesons with beauty and charm from our 1994 article [6], lattice QCD calculations, and the present work, expressed as excitations with respect to the 1$S$ centroid. Other potential-model calculations, some incorporating relativistic effects, may be found in the works cited in Ref. [7].

Our expectations for the spectrum of states are shown in the Grotrian diagram, Figure 3, along with several of the lowest-lying open-flavor thresholds. The thresholds for strong decays of excited (c\bar{b}) levels are known experimentally to high accuracy, as shown in Table IV.

Comparing with the model calculations summarized in Table III we conclude that two sets of narrow s-wave (c\bar{b}) levels will lie below the beauty+charm flavor threshold, in agreement with general arguments [30]. All of the potential models cited in Ref. [7] predict 3$^3S_1$ masses well above the 829-MeV $BD$ threshold. For the 3$^1S_0$ level, only the Ebert et al. prediction does not lie significantly above $B^*D$ threshold. Lattice QCD calculations do not yet exist for states beyond the 2$S$ levels.

IV. DECAYS OF NARROW (c\bar{b}) LEVELS

A. Electromagnetic transitions

The only significant decay mode for the 1$^3S_1$ ($B_c^*$) state is the magnetic dipole (spin-flip) transition to the ground state, $B_c$. The M1 rate for transitions between s-wave...
levels is given by
\[
\Gamma_{M1}(i \rightarrow f + \gamma) = \frac{16\alpha}{3} \mu^2 k^3 (2J_f + 1)|\langle f|J_0(kr/2)|i\rangle|^2 ,
\]
where the magnetic dipole moment is
\[
\mu = \frac{m_bc_c - m_c\epsilon_\gamma}{4m_cm_b}
\]
and \(k\) is the photon energy.

Apart from that M1 transition, only the electric dipole transitions are important for mapping the \((c\bar{b})\) spectrum. The strength of the electric-dipole transitions is governed by the size of the radiator and the charges of the constituent quarks. The E1 transition rate is given by
\[
\Gamma_{E1}(i \rightarrow f + \gamma) = \frac{4\alpha\langle e_Q\rangle^2}{27} k^3 (2J_f + 1)|\langle f|r|i\rangle|^2 S_{if} ,
\]
where the mean charge is
\[
\langle e_Q\rangle = \frac{m_bc_c - m_c\epsilon_\gamma}{m_b + m_c} ,
\]

\[\text{B. Hadronic transitions}\]

We evaluate the rates for hadronic transitions within \((c\bar{b})\) levels according to the prescription we detailed in §III B of Ref. [6]. The results are included in Table VI. Dipion cascades to the ground-state doublet are the dominant decay modes of \(2^3S_1\) and \(2^3S_0\), and will be key to characterizing those states, as we shall discuss in Table VII.

As observed long ago by Brown and Cahn [38], an amplitude zero imposed by chiral symmetry pushes the \(\pi^+\pi^-\) invariant mass distribution to higher invariant masses than phase-space alone would predict. In its simplest form, this analysis yields a universal form for the normalized dipion invariant mass distribution in quarkonium cascades \(\Phi' \rightarrow \Phi \pi^+\pi^-\),

\[
\frac{1}{\Gamma} \frac{d\Gamma}{dM} = \text{Constant} \times \frac{|K|^2}{M_{\Phi'}} (2x^2 - 1)^2 \sqrt{x^2 - 1} ,
\]
where \(x = M/2m_\pi\) and \(K\) is the three-momentum carried by the pion pair. The soft-pion expression describes the depletion of the dipion spectrum at low invariant masses observed in the transitions \(\psi(2S) \rightarrow \psi(1S)\pi\pi\), \(\Upsilon(2S) \rightarrow \Upsilon(1S)\pi\pi\), and \(\Upsilon(3S) \rightarrow \Upsilon(2S)\pi\pi\), but fails to account for structures in the \(\Upsilon(3S) \rightarrow \Upsilon(1S)\pi\pi\) spectrum [39]. We expect the \(3S\) levels to lie above flavor threshold in the \((c\bar{b})\) system, and so to have very small branching fractions for cascade decays (but see the final paragraph of Table VII).

\[\text{C. Properties of } (c\bar{b}) \text{ wave functions at the origin}\]

For quarks bound in a central potential, it is convenient to separate the Schrödinger wave function into radial and angular pieces, as \(\Psi_{n\ell m}(r) = R_{n\ell}(r)Y_{\ell m}(\theta, \phi)\), where \(n\) is the principal quantum number, \(\ell\) and \(m\) are the orbital angular momentum and its projection, \(R_{n\ell}(r)\) is the radial wave function, and \(Y_{\ell m}(\theta, \phi)\) is a spherical harmonic [40]. The Schrödinger wave function is normalized, \(\int dr |\Psi_{n\ell m}(r)|^2 = 1\), so that \(\int_0^\infty r^2dr|R_{n\ell}(r)|^2 = 1\). The value of the radial wave function, or its first nonvanishing derivative, at the origin,

\[
R_{n\ell}^{(0)}(0) \equiv \left. \frac{dR_{n\ell}(r)}{dr} \right|_{r=0} ,
\]
is required to evaluate pseudoscalar decay constants and production rates through heavy-quark fragmentation. Our calculated values of \(|R_{n\ell}^{(0)}(0)|^2\) are given in Table VII.

The pseudoscalar decay constant \(f_{B_c}\), which enters the calculations of annihilation decays such as \(c\bar{b} \rightarrow W^+ \rightarrow \pi^+ + \nu_\tau\), is defined by

\[
\langle 0|A_\mu(0)|B_c(q)\rangle = if_{B_c}V_{cb}q_\mu ,
\]
where \(A_\mu\) is the axial-vector part of the charged weak current, \(V_{cb}\) is an element of the Cabibbo-Kobayashi-Maskawa quark-mixing matrix, and \(q_\mu\) is the four-momentum of the \(B_c\). Its counterpart for the vector state is

\[
\langle 0|V_\mu(0)|B_c^*(q)\rangle = if_{B_c^*}V_{cb}^*q_\mu ,
\]
where \( c \) and related quantities (cf. Eq. (19)) for \( (c \bar{b}) \) will dominate over the tabulated decay modes for states above threshold.

| Decay Mode | \( k_0 \) [keV] | Branching Fraction (%) |
|------------|-----------------|------------------------|
| \( 1^1S_0 + \gamma \) | 54 | 100 |
| \( 1^1S_1 (6329) \) : \( \Gamma = 0.144 \) keV | | |
| \( 2^3P_0 (6692) \) : \( \Gamma = 53.1 \) keV | 354 | 100 |
| \( 1^3S_1 (6329) \) | 389 | 86.2 |
| \( 1^3S_0 (6275) \) | 140 | 13.7 |
| \( 2P_1^* (6738) \) : \( \Gamma = 72.5 \) keV | 448 | 92.4 |
| \( 1^3S_0 (6275) \) | 397 | 7.51 |
| \( 1^1S_1 (6329) \) : \( \Gamma = 79.7 \) keV | 409 | 100 |
| \( 1^1S_1 (6686) \) : \( \Gamma = 73.1 \) keV | | |
| \( 2P_1^* (6738) \) | 134 | 2.24 |
| \( 2^3S_1 (6897) \) : \( \Gamma = 76.8 \) keV | 65.0 |
| \( 2^1S_0 (6692) \) | 201 | 7.66 |
| \( 2^3P_1 (6730) \) | 165 | 11.5 |
| \( 2^3P_2 (6750) \) | 145 | 14.6 |
| \( 3^3D_1 (7005) \) : \( \Gamma = 93.7 \) keV | 12.2 |
| \( 1^3S_1 + \pi \pi \) | 270 | 9.0 |
| \( 1^3S_1 + \pi \pi \) | 262 | 29.1 |
| \( 2^3P_2 (6750) \) | 250 | 7.24 |
| \( 3^3D_1 (7006) : \Gamma = 117 \) keV | 17.0 |
| \( 2^3P_0 (6692) \) | 306 | 53.4 |
| \( 2^3P_1 (6730) \) | 270 | 25.3 |
| \( 2^3P_2 (6750) \) | 262 | 2.62 |
| \( 3^3D_3 (7010) : \Gamma = 87.2 \) keV | 22.9 |
| \( 2^3P_2 (6750) \) | 255 | 77.0 |
| \( 3^3D_3 (7010) : \Gamma = 92.1 \) keV | 9.2 |
| \( 1^1S_0 + \pi \pi \) | 12.4 |
| \( 2^3P_3 (6738) \) | 272 | 37.2 |
| \( 2^3P_1 (6730) \) | 279 | 41.0 |

Table VII. Squares of radial wave functions at the origin (cf. Eq. (23)) for \((c \bar{b})\) mesons.

| Level | \( |R_{m_s}^{(c)}(0)|^2 \) [GeV]\(^3\) |
|-------|-------------------------------|
| \( 1S \) | 1.994 GeV\(^3\) |
| \( 2P \) | 0.3083 GeV\(^5\) |
| \( 2S \) | 1.144 GeV\(^3\) |
| \( 3D \) | 0.0986 GeV\(^7\) |
| \( 3P \) | 0.3939 GeV\(^5\) |
| \( 3S \) | 0.9440 GeV\(^3\) |
| \( 4F \) | 0.0493 GeV\(^9\) |
| \( 4D \) | 0.1989 GeV\(^7\) |
| \( 4P \) | 0.4540 GeV\(^5\) |
| \( 4S \) | 0.8504 GeV\(^3\) |

\( \epsilon^* \) is the polarization vector of the \( B^*_c \). The ground-state pseudoscalar and vector decay constants are given in terms of the wave function at the origin by the Van Royen–Weisskopf formula [31], generally

\[
\mathcal{B}_{B^*} = \frac{3|R_{10}(0)|^2}{\pi M} \mathcal{O}^2(\alpha_s),
\]

(22)

where the leading-order QCD correction is given by [42]

\[
\mathcal{O}^2(\alpha_s) = 1 - \frac{\alpha_s}{\pi} \left( \delta^P V - \frac{m_c - m_b}{m_c + m_b} \ln \frac{m_c}{m_b} \right),
\]

(23)

and

\[
\delta^P = 2; \hspace{1cm} \delta^V = 8/3.
\]

(24)

Choosing the representative value \( \alpha_s = 0.38 \), and using
the quark masses given in Eq. (2), we find

\[
\mathcal{C}(\alpha_s) = \begin{pmatrix}
0.904, & P \\
0.858, & V \\
\end{pmatrix}
\]  

(25)

Consequently, we estimate the ground-state meson decay constants as

\[
f_{B_c} = 498 \text{ MeV}; \quad f_{B_c^*} = 471 \text{ MeV},
\]

(26)

so that \( f_{B_c^*}/f_{B_c} = 0.945 \). The compact size of the \((c\bar{b})\) system enhances the pseudoscalar decay constant relative to \( f_\pi \) and \( f_K \).

This is to be compared to a state-or-the-art lattice evaluation \[43\], \( f_{B_c} = 434 \pm 15 \text{ MeV} \), which entails improved NonRelativistic QCD for the valence \( b \) quark and the Highly Improved Staggered Quark (HISQ) action for the lighter quarks on gluon field configurations that include the effect of \( u/d, s \) and \( c \) quarks in the sea with the \( u/d \) quark masses going down to physical values.

The same calculation yields \( f_{B_c^*}/f_{B_c} = 0.988 \pm 0.027 \). A calculation in the framework of QCD sum rules gives \( f_{B_c} = 528 \pm 19 \text{ MeV} \) \[44\].

V. PRODUCTION OF \((c\bar{b})\) STATES AT THE LARGE HADRON COLLIDER

We present in Table VIII cross sections for the production of \( B_c \) states at the Large Hadron Collider, calculated using the framework of the BCVEGPY2.2 generator \[45\], which we have extended to include the production of \( 3P \) states. Cross sections for the physical \((2, 3)P_1^{(0)}\) states are appropriately weighted mixtures of the \( 3P_1 \) and \( 1P_1 \) cross sections.

TABLE VIII. Production rates (in nb) for \((c\bar{b})\) states in \( pp \) collisions at the LHC. The production rates were calculated using the BCVEGPY2.2 generator of Ref. \[45\], extended to include the production of \( 3P \) states. Color-octet contributions to s-wave production are small; we show them (following \(|\) only for the \( 1S \) states.

| \((c\bar{b})\) level | \(\sigma(\sqrt{s} = 8 \text{ TeV})\) | \(\sigma(\sqrt{s} = 13 \text{ TeV})\) | \(\sigma(\sqrt{s} = 14 \text{ TeV})\) |
|---------------------|-------------------------------|-------------------------------|-------------------------------|
| \(1^S_0\)           | 46.8 \(\pm\) 1.01             | 80.3 \(\pm\) 1.75             | 88.0 \(\pm\) 1.90             |
| \(1^S_1\)           | 123.0 \(\pm\) 4.08            | 219.1 \(\pm\) 6.97            | 237.0 \(\pm\) 7.55            |
| \(2^P_0\)           | 1.113                         | 1.959                         | 2.108                         |
| \(2^P_1\)           | 2.676                         | 4.783                         | 5.214                         |
| \(2^P_2\)           | 3.185                         | 5.702                         | 6.166                         |
| \(2^P_3\)           | 6.570                         | 11.57                         | 12.64                         |
| \(3^S_0\)           | 9.58                          | 16.94                         | 18.45                         |
| \(3^S_1\)           | 23.46                         | 41.72                         | 45.53                         |
| \(3^P_0\)           | 0.915                         | 1.642                         | 1.806                         |
| \(3^P_1\)           | 2.263                         | 4.082                         | 4.478                         |
| \(3^P_2\)           | 2.695                         | 4.817                         | 5.287                         |
| \(3^P_3\)           | 5.53                          | 9.98                          | 10.90                         |
| \(3^S_0\)           | 4.23                          | 7.53                          | 8.08                          |
| \(3^S_1\)           | 10.16                         | 18.21                         | 19.83                         |

FIG. 4. Rapidity distribution for production of \( B_c^* \) in \( pp \) collisions at \( \sqrt{s} = 8 \text{ TeV} \) (dotted blue curve), \( \sqrt{s} = 13 \text{ TeV} \) (solid black curve), and \( \sqrt{s} = 14 \text{ TeV} \) (dashed red curve), calculated using BCVEGPY2.2 \[45\]. The bin width is \( \Delta y = 0.5 \). The mild asymmetries are statistical fluctuations.

FIG. 5. Transverse momentum distribution of \( B_c \) produced in \( pp \) collisions at \( \sqrt{s} = 8 \text{ TeV} \) (dotted blue curve), \( \sqrt{s} = 13 \text{ TeV} \) (solid black curve), and \( \sqrt{s} = 14 \text{ TeV} \) (dashed red curve), calculated using BCVEGPY2.2 \[45\]. Small shape variations are statistical fluctuations.

The rapidity distributions (for \( B_c^* \) production, Figure 4) and transverse-momentum distributions (shown for \( B_c \) production, Figure 5) are similar in character for \( \sqrt{s} = 8, 13 \), and \( 14 \text{ TeV} \). The rapidity distributions for low-lying \((c\bar{b})\) states are shown in Figure 6. The acceptance of the CMS and ATLAS detectors covers central pseudorapidity \( |\eta| \leq 2.5 \), whereas the geometrical acceptance of the LHCb detector is characterized by \( 2 \leq \eta \leq 5 \).
From highest to lowest, the histograms in Figure 6 give the rapidity distributions for the production of low-lying states in pp collisions at $\sqrt{s} = 13$ TeV, calculated using BcVEGPY2.2~[15]. From highest to lowest, the histograms refer to production of the $1^3S_1$, $1^1S_0$, $2^3S_1$, $2^1S_0$, $2^3P_2$, $2^1P_1$, and $2^3P_0$ levels.

For comparison, approximately 68% of the $B_c^+$ cross section lies within $|y| \leq 2.5$, and approximately 22% is produced at forward rapidities $y > 2$. Similar fractions hold for all the $(c\bar{b})$ levels.

### A. Dipion cascades

The path to establishing excited states will proceed by resolving two separate peaks in the invariant mass distributions associated with the cascades $B_c^+ \to B_c \pi^+ \pi^-$ and $B_c^{(*)} \to B_c^+ \pi^+ \pi^-$, $B_c^0 \to B_c + \gamma$ (gamma unobserved). The splitting between the peaks is set by the difference of mass differences,

$$\Delta_{21} \equiv [M(B_c^{(*)}) - M(B_c^0)] - [M(B_c^+ \to B_c^0)] \equiv [M(B_c^{(*)}) - M(B_c^0)] - [M(B_c^+ \to B_c^0)],$$

(27)

generically expected to be negative~[16]. The corresponding quantity is approximately $-64$ MeV in the $(c\bar{c})$ family and $-37$ MeV in the $(b\bar{b})$ family~[12]. For the $(c\bar{b})$ system, a modern lattice simulation~[14] gives $\Delta_{21} = -15$ MeV, whereas the result of our potential-model calculation is $-23$ MeV. In these circumstances, the undetected four-momentum of the photon means that the reconstructed “$B_c^*$” mass should correspond to the lower peak.

We show an example of what is to be expected in Figure 7, taking the direct production cross sections (with no rapidity cuts) from Table VII and the branching fractions from Table VI. The relative heights of, relative number of events in, the peaks measures the ratio

$$R = \frac{\sigma(B_c^{(*)} \to X) B(\gamma \to B_c^0 \to B_c^+ \pi^+ \pi^-)}{\sigma(B_c^{(*)} \to X) B(\gamma \to B_c^+ \to B_c^+ \pi^+ \pi^-)}$$

(28)

At $\sqrt{s} = 13$ TeV, the ratio of cross sections is nearly 2.5. Taking account of the branching fractions, we estimate $R \approx 2$. If $B_c^*$ and $B_c$ were produced with equal frequency, we would find $R \approx 0.8$.

Now the CMS Collaboration~[3] at the Large Hadron Collider, analyzing $140$ fb$^{-1}$ of pp collisions at $\sqrt{s} = 13$ TeV, has observed a pattern that closely resembles the template of Figure 7. The observed $B_c$ mass is replaced, event by event, with the world-average value to sharpen resolution. The putative $B_c^{(2S)}$ lies within 5 MeV of our expectation for the $2^3S_0$ level, and the separation is to be compared with our expectation of 23 MeV. If we impose the scaling relation Eq. (13), we reproduce the observed 29-MeV separation with $R = (2.0 \pm 0.4) \times 10^{-4}$. The $B_c^+ \to B_c + \gamma$ photon momentum would be 68 MeV.

An unbinned extended maximum-likelihood fit to the CMS data returns $66 \pm 10$ events for the lower peak and $51 \pm 10$ for the upper. These yields are not yet corrected for detection efficiencies and acceptances, so they cannot be used to infer ratios of production cross sections times branching fractions. We look forward to the final result and to studies of the $\pi^+ \pi^-$ invariant mass distribution as next steps in $B_c$ spectroscopy.

Our calculations indicate that the $3S$ levels will lie above flavor threshold (see Table VII and the discussion surrounding Figures 9 and 10), but it is conceivable that coupled-channel effects might push one or both states lower in mass. For that reason, it is worth examining the $B_c \pi^+ \pi^-$ mass spectrum up through 7200 MeV for indications of $3^1S_0 \to B_c \pi^+ \pi^-$ and $3^3S_1 \to B_c^\pi^+ \pi^-$ lines. 

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**FIG. 6.** Rapidity distributions for the production of low-lying $(c\bar{b})$ states in pp collisions at $\sqrt{s} = 13$ TeV, calculated using BcVEGPY2.2~[15]. From highest to lowest, the histograms refer to production of the $1^3S_1$, $1^1S_0$, $2^3S_1$, $2^1S_0$, $2^3P_2$, $2^1P_1$, and $2^3P_0$ levels.

---

**FIG. 7.** Calculated positions and relative strengths of the two-pion cascades $B_c(2S) \to B_c(1S) \pi^+ \pi^-$, represented as Gaussian line shapes with standard deviation of 4 MeV. Production rates are given in Table VII and branching fractions in Table VI. We assume that the photon in the transition $B_c^+ \to B_c + \gamma$ is not included in the reconstruction. Rates confined to rapidity $|y| \leq 2.5$ are 0.68× those shown.
transitions to $B_c^*$ will be shifted downward because of the unobserved M1 photon. It is not possible to produce enriched samples of the 2S levels by tuning the energy of $e^+e^-$ collisions, as is done for $J/\psi$ and $\Upsilon$, so reconstruction of the left-hand group of $2S \rightarrow 2P$ transitions (blue lines in Figure 8) will be problematic.

In the far future, combining the photon transition energies and relative rates with expectations for production and decay may eventually make it possible to disentangle mixing of the spin-singlet and spin-triplet $J = L$ states.

C. States above open-flavor threshold

We estimate the strong decay rates for $(c\bar{b})$ states that lie above flavor threshold using the Cornell coupled-channel formalism [22] that we elaborated and applied to charmonium states in [23].

We expect both the $3^1S_0$ and $3^3S_1$ states to lie above threshold for strong decays. The $3^1S_0$ state can decay into the final state $B^*D$ and the $3^3S_1$ level has decays into both the $BD$ and $B^*D$ final states. The open decay channels as a function of the masses of these states is shown in Figures 9 and 10.

The $3^3P_2$ state might be observed as a very narrow (d-wave) $BD$ line near open-flavor threshold. Its decay width as a function of mass for the $2P$ states are given in Figure 11.

In the phenomenological models the remaining $3P$ states lie just below the thresholds for strong decays. However they are near enough to these thresholds that there might be interesting behavior at the threshold for $B^*D$ in the $3P_1^{(i)}$ cases and for the $BD$ threshold in the case of the $3P_0$ state. Figure 12 shows that the $3P_0$ width grows rapidly just above threshold. The strong

According to our estimate of the 3S hyperfine splitting, the $3^3S_1$ line would lie about 28 MeV below the $3^1S_0$ line (36 MeV if we reset the 1S splitting to 68 MeV). For orientation, note that $B(\Upsilon(3S) \rightarrow \Upsilon(1S)\pi^+\pi^-) = 4.37 \pm 0.08\%$, while 36\% of $\Upsilon(3S)$ decays proceed through the $ggg$ channel, which is not available to the $(c\bar{b})$ states. According to Table VIII, the 3S states are produced at approximately 44\% of the rate for their 2S counterparts.

### B. Electromagnetic transitions

It may in time become possible for experiments to detect some of the more energetic E1-transition photons that appear in Table VIII. As an incentive for the search, we show in Figure 8 the spectrum of E1 photons in decays of the $2^3S_1$ and $2^1S_0$ levels as well as the $2P \rightarrow 2S$ transitions, assuming as always a missing $B_c^* \rightarrow B_c + \gamma$ photon in the reconstruction. Here we include direct production of the $2P$ states as well as feed-down from $2S \rightarrow 2P$ transitions. The strong $B_c^* \rightarrow B_c$ line arising from direct production of $B_c^*$, for which we calculate $\sigma \cdot B \approx 225$ nb at $\sqrt{s} = 13$ TeV, is probably too low in energy to be observed. More promising are the $2P$ levels, which might show themselves in $B_c + \gamma$ invariant mass distributions. These lines make up the right-hand group (black lines) in Figure 8. The $2^3P_2(6750) \rightarrow B_c^*\gamma$ line is a particularly attractive target for experiment, because of the favorable production cross section, branching fraction, and 409-MeV photon energy. The $2P$ masses inferred from

---

**FIG. 8.** Photon energies $k$ and relative strengths of E1 transitions from $2S \rightarrow 2P$ (left group, blue curves) and $2P \rightarrow 1S$ (right group, black curves) $(c\bar{b})$ states. Production rates are taken from Table VIII and branching fractions from Table VI. We suppose that the photon transition $B_c^* \rightarrow B_c + \gamma$ goes unobserved in the cascade transitions. We assume Gaussian lineshapes with standard deviation 2 MeV.

**FIG. 9.** Strong decay widths of the $3^1S_0$ $(c\bar{b})$ level near open-flavor threshold. The shaded band on the mass axis indicates $\pm 20$ MeV around our nominal value for the mass of this state, 7253 MeV.
FIG. 10. Strong decay widths of the $3^3S_1$ ($c\bar{b}$) level near open-flavor threshold. The shaded band on the mass axis indicates $\pm 20$ MeV around our nominal value for the mass of this state, 7279 MeV.

FIG. 11. Strong decay widths of the $3^3P_2$ ($c\bar{b}$) level near open-flavor threshold. The shaded band on the mass axis indicates $\pm 20$ MeV around our nominal value for the mass of this state, 7154 MeV.

FIG. 12. Estimated strong decay widths of the $3^3P_0$ ($c\bar{b}$) level near open-flavor threshold. The shaded band on the mass axis indicates $\pm 20$ MeV around our nominal value for the mass of this state, 7104 MeV.

FIG. 13. Strong decay widths of the $3P_1$ or $3P_1'$ states. The shaded band on the mass axis indicates $\pm 20$ MeV around our nominal values for the masses of these state, 7135 and 7143 MeV.

decay widths as a function of mass for the $3P_1$ and $3P_1'$ states have a common behavior, displayed in Figure 13.

It is worth keeping in mind that while narrow $BD$ peaks may signal excited $(c\bar{b})$ levels, narrow $\bar{B}D$ peaks could indicate nearly bound $bc\bar{q}\bar{q}$ tetraquark states [47].

VI. TERA-Z PROSPECTS

In response to the discovery of the 125-GeV Higgs boson, $H(125)$ [48], plans for large circular electron–positron colliders (FCC-ee [49] and CEPC [50]) are being developed as $e^+e^- \to HZ^0$ “Higgs factories” to run at c.m. energy $\sqrt{s} \approx 240$ GeV. As now envisioned, these machines would have the added capability of high-luminosity running at $\sqrt{s} = M_Z$ that would accumulate $10^{12}$ examples of the reaction $e^+e^- \to Z^0$. With the observed branching fraction, $B(Z^0 \to b\bar{b}) = (15.12 \pm 0.05)%$ [12], the era-Z mode would produce some $3 \times 10^{11}$ boosted $b$-quarks, which would enable high-sensitivity searches for $(c\bar{b})$ states in a variety of decay channels. A recent computation suggests that $B(Z^0 \to (c\bar{b}) + X) \approx 6 \times 10^{-4}$ [51].

The largest existing $e^+e^- \to Z^0 \to$ hadrons data sets were recorded by experiments at CERN’s Large Electron–Positron collider (LEP) during the 1990s. In samples of (3.02, 3.9, and 4.2) million hadronic $Z^0$
state wave function, complementing what will be learned from the $B_c^*-B_c$ splitting.

Appendix: Strong coupling evolution

To make calculations with the frozen-\(\alpha_s\) potential, one must combine a linear term with a Coulomb term, \(-4\alpha_s(r)/3r\), for which \(\alpha_s(r)\) is characterized by the solid red curve of Figure 1. We present in Table IX numerical values of the strong coupling over the relevant range of distances, \(0 \leq r \leq 0.8\) fm. The entries advance in steps of \(\delta \ln r = 0.1\).

| \(r \text{ [fm]}\) | \(\alpha_s(r)\) |
|------------|----------|
| 0.0080      | 0.1706   |
| 0.0088      | 0.1742   |
| 0.0097      | 0.1780   |
| 0.0108      | 0.1819   |
| 0.0119      | 0.1862   |
| 0.0132      | 0.1908   |
| 0.0145      | 0.1957   |
| 0.0161      | 0.2007   |
| 0.0178      | 0.2061   |
| 0.0196      | 0.2116   |
| 0.0217      | 0.2174   |
| 0.0240      | 0.2235   |
| 0.0265      | 0.2299   |
| 0.0293      | 0.2365   |
| 0.0323      | 0.2434   |
| 0.0358      | 0.2505   |
| 0.0395      | 0.2579   |
| 0.0437      | 0.2659   |
| 0.0483      | 0.2743   |
| 0.0533      | 0.2829   |
| 0.0589      | 0.2915   |
| 0.0651      | 0.3001   |
| 0.0720      | 0.3087   |
| 0.0796      | 0.3171   |
| 0.0879      | 0.3252   |
| 0.0972      | 0.3330   |
| 0.1074      | 0.3403   |
| 0.1187      | 0.3471   |
| 0.1312      | 0.3533   |
| 0.1450      | 0.3590   |
| 0.1602      | 0.3640   |
| 0.1771      | 0.3685   |
| 0.1957      | 0.3723   |
| 0.2163      | 0.3757   |
| 0.2390      | 0.3786   |
| 0.2642      | 0.3811   |
| 0.2920      | 0.3832   |
| 0.3227      | 0.3849   |
| 0.3566      | 0.3864   |
| 0.3941      | 0.3876   |
| 0.4355      | 0.3886   |
| 0.4813      | 0.3895   |
| 0.5320      | 0.3902   |
| 0.5879      | 0.3908   |
| 0.6497      | 0.3913   |
| 0.7181      | 0.3917   |
| 0.7936      | 0.3920   |

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In an effective power-law potential $V(r) = \lambda r^\nu$, $\Delta_{21} < 0$ so long as $\nu < 1$. See §4.1.1 and §5.3.2 of C. Quigg and Jonathan L. Rosner, “Quantum Mechanics with Applications to Quarkonium,” Phys. Rept. 56, 167–235 (1979) particularly Eqs. (4.21, 4.22).

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