Explaining Fast Improvement in Online Policy Optimization

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Abstract

Online policy optimization (OPO) views policy optimization for sequential decision making as an online learning problem. In this framework, the algorithm designer defines a sequence of online loss functions such that the regret rate in online learning implies the policy convergence rate and the minimal loss witnessed by the policy class determines the policy performance bias. This reduction technique has been successfully applied to solving various policy optimization problems, including imitation learning, structured prediction, and system identification. Interestingly, the policy improvement speed observed in practice is usually much faster than existing theory suggests. In this work, we provide an explanation of this fast policy improvement phenomenon. Let $\epsilon$ denote the policy class bias and assume the online loss functions are convex, smooth, and non-negative. We prove that, after $N$ rounds of OPO with stochastic feedback, the policy converges in $\tilde{O}(1/N + \sqrt{\epsilon/N})$ in both expectation and high probability. In other words, we show that adopting a sufficiently expressive policy class in OPO has two benefits: both the convergence rate increases and the performance bias decreases, as the policy class becomes reasonably rich. This new theoretical insight is further verified in an online imitation learning experiment.

1 Introduction

Viewing policy optimization as no-regret online learning [1] has recently gained traction in the machine learning community. Pioneered by Ross et al. [2], this reduction was designed to mitigate the compounding error resulting from multi-step action executions in sequential decision problems. Since this work was published, significant progress has been made in both theory and practice: Online imitation learning (IL) has been validated on physical robot control tasks [3, 4], and high-performance algorithms have been developed for structured prediction [5–7]. Similar ideas have also been applied in reinforcement learning [8] and system identification [9, 10].

Here we collectively call these algorithms online policy optimization (OPO) in order to capture the breadth of this framework. The main idea of OPO is to treat each policy in the sequential decision problem as an online decision $^1$ in online learning: the designer defines a sequence of online losses, such that the regret rate in the online learning problem implies the speed of policy improvement, and the minimal loss witnessed by the policy class determines the policy performance bias in the

$^1$The online decision in the iterative process of online learning should not be confused with the decisions made at each time step in sequential decision making.
sequential decision making problem. When this loss qualification holds and the aforementioned performance bias is small, running a no-regret online algorithm in this online learning problem can generate policies with performance guarantees.

Because the online losses in OPO are designed to satisfy the performance relationship with respect to the given sequential decision making problem, the resulting online learning problem has a mixture of different properties, such as predictability, continuity, and stochasticity [11]. The interactions of these properties make the classic adversary-style online learning analysis taken by Ross et al. [2] overly conservative, creating a mismatch between provable theoretical guarantees and the learning phenomena observed in practice. This reality gap has motivated researchers to study deeper the theoretical underpinnings of OPO [12–14].

In this work, we are interested in explaining the fast policy improvement of OPO observed in practice, which existing OPO theory fails to capture. When the online loss functions are convex and Lipschitz, typical analyses of regret and martingale concentration [2, 15] suggest an on-average convergence rate in $O(1/\sqrt{N})$ after $N$ rounds. However, empirically, OPO algorithms learn much faster; for example, the online IL algorithm DAgger [2] learned to mimic a model predictive control (MPC) policy for autonomous off-road driving in only three rounds in [4]. Although the convergence rate improves to $\tilde{O}(1/N)$ when the online losses are strongly convex [12], this condition can be difficult to satisfy especially when the policy class is large, such as a linear function class built on high-dimensional features. The empirical effectiveness and sample efficiency of OPO demand alternative explanations.

We prove a new problem-dependent convergence rate for OPO that is adaptive to the performance bias from using a limited policy class. Interestingly, we show that an OPO algorithm can learn faster as this performance bias becomes smaller. In other words, adopting a sufficiently expressive policy class in OPO has two benefits: as the policy class becomes reasonably rich, both the learning speed increases and the performance bias decreases at the same time. Concretely, let $\epsilon$ denote the policy class bias. Under the assumptions that the online losses are convex, smooth, and non-negative, we give a convergence rate in $\tilde{O}(1/N + \sqrt{\epsilon/N})$ both in expectation and in high probability for OPO algorithms using stochastic feedback. This new result shows a transition from the faster rate of $\tilde{O}(1/N)$ to the usual rate of $\tilde{O}(1/\sqrt{N})$ as the policy class bias increases.

This type of problem-dependent convergence rate has been studied in many other learning settings. When the loss functions are convex, smooth, and non-negative, similar rates have been shown in statistical learning [16], stochastic optimization [17, 18], and online learning [16]. Inspired by these results, we derive the bias-dependent improvement rate for OPO. The mix of non-stationarity and stochasticity in the online loss functions of OPO makes the analysis here challenging; indeed, previous analyses tackle only one of these two properties. Our fast high-probability bound is made possible by resorting to a recent martingale concentration result that depends only on path-wise statistics [19].

We conclude by corroborating the new theoretical findings with experimental results of online IL. The detailed proofs for this paper can be found in the Appendix.

2 Background: Online Policy Optimization

2.1 Policy Optimization

Policy optimization aims to find a high-performance policy in a policy class $\Pi$ for sequential decision making problems. Typically, it models the world as a Markov decision process (MDP), defined by an initial state distribution, transition dynamics, and an instantaneous state-action cost function [20]. This MDP is often assumed to be unknown to the learning agent; therefore the learning algorithm for policy optimization needs to perform systematic exploration in order to discover good policies in $\Pi$. Concretely, let us consider a policy class $\Pi$ that has a one-to-one mapping to a parameter space $\Theta$, and let $\pi_\theta$ denote the policy associated with the parameter $\theta \in \Theta$. That is, $\Pi = \{\pi_\theta : \theta \in \Theta\}$. The goal of policy optimization is to find a policy $\pi_\theta \in \Pi$ that minimizes the expected cost,

$$J(\pi) := E_{s \sim d_\pi} E_{a \sim \pi_s}[c(s, a)],$$

(1)

where $s$ and $a$ are the state and the action, respectively, $c$ is the instantaneous cost function and $d_\pi$ denotes the average state distribution over the problem horizon induced by executing policy $\pi_\theta$ starting from a state sampled from the initial state distribution. The problem formulation in (1) applies to various settings on problem horizon and discount rate, where the main difference is how the average
Approximate execute policy for define the online loss function. Pass initialize θ where whereas d

Algorithm 1: Online Policy Optimization (OPO)

**Input:** Initial policy πθ₁, and an online algorithm A

**Output:** The best policy in the sequence of policies \{πθₙ\}_n=1^N

1. Initialize A with the initial policy πθ₁

2. for n from 1 to N do

   3. Define the online loss function \hat{l}_n based on πθₙ

   4. Execute policy πθₙ in the MDP to gather samples

   5. Approximate \hat{l}_n using an unbiased sample-based estimate \tilde{l}_n, which satisfies E[\tilde{l}_n] = l_n

   6. Pass \tilde{l}_n to A and use the return of A to update policy to πθₙ₊₁

state distribution is defined [20]; e.g., for a discounted problem, dπₙ is defined by a geometric mean, whereas dπₙ corresponds to the stationary state distribution for average infinite-horizon problems.

2.2 Reducing Policy Optimization to Online Learning

OPO works by devising a sequence of online loss functions \{lₙ\} such that no regret and small policy class bias imply good policy performance in the original sequential decision problem. Below we use online imitation learning (IL) as a concrete example of OPO to illustrate this idea.

**Illustrative Example** Online IL performs policy optimization using an interactive expert policy πₑ. Instead of minimizing (1) directly, online IL minimizes an upper bound of the performance difference between the policy πθ and the expert πₑ:

\[
J(\pi) - J(\piₑ) \leq O \left( \mathbb{E}_{s \sim d_{πₑ}} \mathbb{E}_{a \sim πₑ} [D_{πₑ}(s, a)] \right).
\] (2)

The loss \(D_{πₑ}(s, a)\) represents whether an action \(a\) is similar to the action taken by expert policy \(πₑ\) at state \(s\), which can be constructed as statistical distances (e.g., Wasserstein distance and KL divergence) or their upper bounds [2, 5, 7, 13]. Although the surrogate function on the right-hand side of (2) resembles (1), it has an additional nice property [12]: if the policy class Π has enough capacity to contain the expert policy πₑ, then there is a policy πθ ∈ Π such that, for all the states,

\[
\mathbb{E}_{a \sim πₑ} [D_{πₑ}(s, a)] = 0.
\] (3)

Leveraging the realizability property in (3), online IL minimizes the surrogate function in (2) by solving an online learning problem: Let parametric space Θ be the decision set (i.e., policies) in online learning; we can define the online loss in round \(n\) as

\[
lₙ(θ) = \mathbb{E}_{s \sim d_{πₑₙ}} \mathbb{E}_{a \sim πₑₙ} [D_{πₑₙ}(s, a)],
\] (4)

where \(θₙ \in Θ\) is the online decision made by the learning algorithm in round \(n\). The main benefit of this indirect approach is that, e.g., the sampled gradient of \(lₙ(θ)\) in (4) is less noisy than that of the surrogate problem in (2), because the average state distribution \(d_{πₑₙₙ}\) in (4) is not considered as a function of the policy parameter \(θ\). The influence of the policy parameter on the change in the average state distribution can be ignored here, because of the realizable property in (3). When the expert policy \(πₑ\) is only nearly realizable by the policy class Π (that is, (3) can only be satisfied up to a certain error), optimizing the policy with this online learning reduction would suffer from an extra performance bias due to using a limited policy class, as we will later discuss in Section 2.3.

**General Learning Protocol** OPO algorithms in general closely follow the design idea we showed in the online IL example. First, an online loss is selected for the specific domain to satisfy conditions similar to (2) and (3) (or their approximations). An online algorithm A is then selected to optimize the policy with respect to the online loss functions. As a summary, Algorithm 1 illustrates the iterative process of a general OPO algorithm, where we take into account that in practice the MDP is unknown and therefore the online loss \(lₙ\) needs to be further approximated by finite samples as \(\hat{l}_n\). At the end, the online algorithm A would generate a sequence of policies \{πθₙ\}_n=1^N. By this reduction, performance guarantees can be obtained for the best policy in this sequence, as we will show next.
2.3 Performance Guarantees in Online Policy Optimization

Now that we have reviewed the algorithmic aspects of OPO, we provide a brief tutorial of the theoretical foundation of OPO and the known convergence results that show exactly how regret and policy class bias are related to the performance in the original sequential decision problem. To this end, let us formally define the regret and the policy class bias: For a sequence of online loss functions \( \{f_n\}_{n=1}^N \) and decisions \( \{\theta_n\}_{n=1}^N \), the regret in online learning is defined as

\[
\text{Regret}(f_n) = \sum_{n=1}^{N} f_n(\theta_n) - \min_{\theta \in \Theta} \sum_{n=1}^{N} f_n(\theta).
\]  

(5)

Note that, for brevity, the range in \( \sum_{n=1}^{N} \) is omitted in (5) and we will continue to do so in the following as long as the range is clear from the context. In addition to the regret, we introduce two problem-dependent biases of the decision set \( \Theta \) (the equivalence of the policy class \( \Pi \)).

**Definition 1.** For the sampled loss functions \( \hat{l}_n \) experienced by running Algorithm 1, we define \( \hat{\epsilon} = \frac{1}{N} \min_{\theta \in \Theta} \sum_{n=1}^{N} \hat{l}_n(\theta) \) and \( \epsilon = \frac{1}{N} \min_{\theta \in \Theta} \sum_{n=1}^{N} l_n(\theta) \), where \( l_n(\theta) = \mathbb{E}[\hat{l}_n(\theta)] \) for a given \( \theta \).

A typical OPO analysis uses the regret and the policy class biases \( \epsilon \) and \( \hat{\epsilon} \) to decompose the cumulative loss \( \sum l_n(\theta_n) \) to provide policy performance guarantees. Specifically, define \( \theta^* \in \arg \min_{\theta \in \Theta} \sum l_n(\theta) \). By (5) and Definition 1, we can write

\[
\sum_{n=1}^{N} l_n(\theta_n) = \text{Regret}(\hat{l}_n) + \left( \sum_{n=1}^{N} l_n(\theta_n) - \hat{l}_n(\theta_n) \right) + N\hat{\epsilon}
\]

(6)

\[
\leq \text{Regret}(\hat{l}_n) + \left( \sum_{n=1}^{N} l_n(\theta_n) - \hat{l}_n(\theta_n) \right) + \left( \sum_{n=1}^{N} \hat{l}_n(\theta^*) - l_n(\theta^*) \right) + N\epsilon
\]

(7)

where, in both (6) and (7), the first term is the online learning regret, the middle term(s) are the generalization error(s), and the last term is the policy class bias. Because \( l_n(\theta_n) \) in OPO provides an upper bound on the policy performance (see (2)), making the cumulative loss \( \sum l_n(\theta_n) \) small implies performance guarantees on the best policy in the sequence \( \{\pi_n\}_{n=1}^N \). For simplicity, hereafter, we will abstract away the domain details of the sequential decision problem and focus on the size of \( \sum l_n(\theta_n) \), as it directly implies the policy performance.

In a nutshell, existing convergence results of OPO are applications of (6) and (7) with different upper bounds on the regret and the generalization errors [2, 5, 7, 9]. For example, when the sampled loss functions \( \{\hat{l}_n\} \) are bounded, the generalization error(s) (i.e., the middle term(s) in (6) and (7)) can be bounded by \( \tilde{O}(\sqrt{N}) \) with high probability by Azuma’s inequality (see [15] or Chapter 9 in [21]). Together with an \( O(\sqrt{N}) \) bound on the regret (which is standard for online convex losses) [21, 22], it implies that the average performance \( \frac{1}{N} \sum l_n(\theta_n) \) and the best performance \( \min_{\theta \in \Theta} l_n(\theta_n) \) converge to \( \epsilon \) or \( \epsilon \) at the speed of \( \tilde{O}(1/\sqrt{N}) \).

However, the rate above does not fully justify the fast improvement of OPO observed in practice [4, 7, 23, 24], as we will also show experimentally in Section 5. While faster rates in \( \tilde{O}(1/N) \) can be obtained for strongly convex loss functions [2, 12], these results are not very assuring either: the strong convexity assumption does not usually hold, especially when an expressive policy class \( \Pi \) is used to reduce the policy class bias. Thus, alternative explanations are needed.

3 Bias-dependent Convergence Rates

In this section, we present new policy convergence rates that are adaptive to the performance biases in Definition 1. The full proof of these theorems are provided in the Appendix.

3.1 Setup and Assumptions

We suppose the parameter space of the policy class \( \Theta \) is a closed convex subset of a Hilbert space \( \mathcal{H} \) that is equipped with norm \( \| \cdot \| \). Note that \( \| \cdot \| \) is not necessarily the norm induced by the inner product. We will denote its dual norm as \( \| \cdot \|_* \), which is defined as \( \| x \|_* = \max_{\| y \| = 1} (x, y) \).

We define admissible algorithms to broaden the scope of OPO algorithms that our analysis covers.

**Definition 2.** We say an online algorithm \( \mathcal{A} \) is admissible if there exists \( R_{\mathcal{A}} \in [0, \infty) \) such that given any \( \eta > 0 \) and any sequence of differentiable convex functions \( \{f_n\} \), \( \mathcal{A} \) can achieve \( \text{Regret}(f_n) \leq R_{\mathcal{A}} + \frac{\eta}{2} \sum_{n=1}^{N} \| \nabla f_n(\theta_n) \|_*^2 \), where \( \theta_n \) is the decision made by \( \mathcal{A} \) in round \( n \).
We will assume that Algorithm 1 is realized by an admissible online learning algorithm $\mathcal{A}$. This assumption is satisfied by common online algorithms, such as mirror descent [25] and Follow-The-Regularized-Leader [22] under first-order or full-information feedback, where $\eta$ in Definition 2 corresponds to a constant stepsize that’s chosen before seeing the online losses, and $R_\mathcal{A}$ measures that size of the decision set $\Theta$.

Finally, we formally define convex, smooth, and non-negative (CSN) functions; we will assume the online loss $l_n$ in OPO and its sampled version $\hat{l}_n$ belong to this class.

**Definition 3.** A function $f : \mathcal{H} \to \mathbb{R}$ is CSN if $f$ is convex, $\beta$-smooth, and non-negative.

Several popular loss functions used in OPO (e.g., squared $\ell_2$-loss and KL-divergence) are indeed CSN (Definition 3) (see Section 4 for examples). If the losses are not smooth, several smoothing techniques in the optimization literature are available to smooth the losses locally, e.g., Nesterov’s smoothing [26], Moreau-Yosida regularization [27], and randomized smoothing [28].

### 3.2 Convergence Rate in Expectation

Our first contribution is a bias-dependent convergence rate in expectation.

**Theorem 1.** In Algorithm 1, suppose $\{\hat{l}_n\}$ is CSN and the online algorithm $\mathcal{A}$ is admissible. Let $\hat{\epsilon} = \frac{1}{N} \min_{\theta \in \Theta} \sum l_n(\theta)$ be the bias, and let $\hat{E}$ be such that $\hat{E} \geq \hat{\epsilon}$ almost surely. Choose $\eta$ for $\mathcal{A}$ to be $\frac{1}{2\beta + 4\beta^2 + 2\beta N R_\mathcal{A}^2}$. Then in expectation

$$
\frac{1}{N} \sum l_n(\theta_n) - \hat{\epsilon} \leq \frac{8\beta R_\mathcal{A}^2}{N^2} + \sqrt{\frac{8\beta R_\mathcal{A}^2 \hat{E}}{N}}
$$

(8)

**Proof Sketch.** The rate (8) follows from analyzing the regret and the generalization error in the decomposition in (6). First, under the assumption of CSN loss functions and admissible online algorithms, the online regret can be bounded by an extension of the bias-dependent regret bound that is stated for mirror descent in [16, Theorem 2], whose average gives the rate in (8). Second, the generalization error in (6) vanishes in expectation because it is a martingale difference sequence.

The rate in (8) suggests that an OPO algorithm can learn faster as the policy class bias becomes smaller; this is reflected in the transition from the usual rate $O(1/\sqrt{N})$ to the faster rate $O(1/N)$ when the bias goes to zero. Notably, the rate in (8) does not depend on the dimensionality of $\mathcal{H}$ but only on $R$, which one can roughly think of as the largest norm in $\Theta$. Therefore, we can increase the dimension of the policy class to reduce the bias (e.g. by using reproducing kernels [29]) as long as the diameter of $\Theta$ measured by norm $\| \cdot \|$ stays controlled.

### 3.3 Convergence Rate in High Probability

We show that a similar bias-dependent convergence rate to (8) also holds with high probability.

**Theorem 2.** Under the assumptions and setup of Theorem 1, further assume that there is $G \in [0, \infty)$ such that, for any $\theta \in \Theta$, $\|\nabla \hat{l}_n(\theta)\|_* \leq G$ almost surely. For $\delta < 1/e$, with probability at least $1 - \delta$,

$$
\frac{1}{N} \sum l_n(\theta_n) - \epsilon = O\left( \frac{C \beta R^2}{N} + \sqrt{\frac{C \beta R^2 (\hat{E} + \epsilon)}{N}} \right)
$$

(9)

where $R_\Theta = \max_{\theta \in \Theta} \|\theta\|$, $R = \max(1, R_\Theta, R_\mathcal{A})$, and $C = \log(1/\delta) \log(GRN)$.

We remark that the uniform bound $G$ on the norm of the gradients only appears in logarithmic terms. Therefore, this rate stays reasonable when the loss functions have gradients whose norm grows with the size of $\Theta$, such as the popular the squared loss.

To prove Theorem 2, one may attempt to apply basic martingale concentration properties on the martingale difference sequences (MDSs) in (6) and (7), as in the proof of Theorem 1. However, taking this direct approach will bring back the slow rate of $O(1/\sqrt{N})$. To the best of our knowledge,
sharp concentration inequalities for the counterparts of MDS in other learning settings cannot be adapted here in a straightforward way: for example, [16] relies on an argument on local Rademacher complexities (which does not have obvious extension to nonstationary losses) and [30] assumes that the losses are both Lipschitz and strongly convex (our goal is to relax these assumptions).

3.4 Proof Sketch for Theorem 2

The key to avoid the above slow rate due to the direct application of martingale concentration analyses on the MDSs in (6) and (7) is to take a different decomposition of the cumulative loss. Here we construct two new MDSs in terms of the gradients: recall $\epsilon = \min_{\theta \in \Theta} \sum l_n(\theta)$ and $\theta^* \in \text{arg min}_{\theta \in \Theta} \sum l_n(\theta)$. Then by convexity of $l_n$, we can derive

$$\sum l_n(\theta_n) - N\epsilon \leq \sum (\nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n), \theta_n) - \sum (\nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n), \theta^*) + \text{Regret}(\langle \nabla \hat{l}_n(\theta_n), \cdot \rangle) \quad (10)$$

Our proof is based on analyzing these three terms. For the MDSs in (18), we notice that, for smooth and non-negative functions, the squared norm of the gradients can be bounded by its function value.

Lemma 1 (Lemma 3.1 [16]). Suppose a function $f : \mathcal{H} \to \mathbb{R}$ is $\beta$-smooth and non-negative, then for any $x \in \mathcal{H}$, $\|\nabla f(x)\|^2 \leq 4\beta f(x)$.

Lemma 1 enables us to properly control the second-order statistics of the MDSs in (18). By a recent vector-valued martingale concentration inequality that depends only on second-order statistics [19], we obtain a self-bounding property for (18) to get fast concentration rate.

Besides analyzing the MDSs, we need to bound the regret to the linear functions defined by the gradients (the last term in (18)). Since this last term is linear, not CSN, the bias-dependent online regret bound in the proof of Theorem 1 does not apply. Nonetheless, because these linear functions are based on the gradients of CSN functions, we discover that their regret rate actually obeys the exact same rate as the regret to the CSN loss functions. This is notable because the regret to these linear functions upper bounds the regret to the CSN loss functions.

Combining the bounds on the MDSs and the regret, we obtain the rate in (9).

4 Case Studies

In this section, we use two concrete applications of OPO to show how the new theoretical results in Section 3 improve existing understanding of the policy improvement speed.

4.1 Online Imitation Learning

Online IL [2] has demonstrated successes in solving many real-world sequential decision making problems [4, 23, 31]. When the action space is discrete, a popular design choice is to set $D_{\pi_e}(s, a)$ in (4) as the hinge loss [2] (i.e., the loss function used in SVM [29]). For continuous domains, $\ell_1$-loss becomes a natural alternative for defining $D_{\pi_e}(s, a)$, which, e.g., is adopted by Pan et al. [4] for high-speed autonomous off-road driving. When the policy is linear in the parameters, one can verify that these loss functions are convex and non-negative, though not strongly convex. Therefore, existing convergence results can only guarantee an $O(1/\sqrt{N})$ policy improvement rate, which does not reflect the fast convergence observed in the experiments [2, 4].

Although our new theorems are not directly applicable to these non-smooth loss functions, we can apply our new results to a smoothed version of these non-negative convex loss functions. For instance, applying the Huber approximation (an instantiation of Nesterov’s smoothing) [26] to “smooth the tip” of these $\ell_1$-like losses yields a globally smooth function with respect to the $\ell_2$-norm. Because the smoothing mainly changes where the loss is close to zero, our new theorems suggest that, when the policy class is expressive enough, learning with these $\ell_1$-like losses would converge in a $\tilde{O}(1/N)$ rate before the policy gets very close to the expert policy during policy optimization.
We verify the change of rates due to policy class capacity by running an online IL experiment in the
Although the main focus of this paper is the new theoretical insights, we conduct experiments to
As these online losses are CSN, our theoretical results apply and suggest a convergence rate in
state space and a 1-dimensional continuous action space.

The goal of the CartPole task is to keep the pole upright by controlling the acceleration

Figure 1: The convergence rates of online IL when the policy class has zero bias (Fig. 1a) and an
additional bias due to an \( \ell_2 \)-norm constraint on the weights (Fig. 1b). The rates are obtained by fitting
the curve of the average loss \( \frac{1}{N} \sum_{n=1}^{N} l_n(\theta_n) \) with parametric \( O(1/N) \) and \( O(1/\sqrt{N}) \) upper bounds
to minimize \( \ell_1 \)-error. The average loss curve is the median, and the shaded region represents 10% and 90% percentile, over 4 random seeds, due to the randomness in the initial state of the MDP and the initialization of the policy.

4.2 Interactive System Identification for Model-based RL
Interactive system identification (ID) is a technique that interleaves data collection and dynamics
model learning for robust model-based RL. Ross and Bagnell [9] show that interactive system ID
can be analyzed under the OPO framework, where the regret guarantee implies learning a dynamics
model that mitigates the train-test distribution shift problem [9, 32]. Let \( T \) and \( T_\theta \) denote the
true and the learned dynamics, respectively. A common online loss for interactive system ID is
\( l_n(\theta) = E_{(s,a) \sim i \sim d_{T_{\theta n}} + \frac{1}{\nu} |D_{s,a}(T_\theta||T)|} \), where \( D_{s,a}(T_\theta||T) \) is some distance between \( T \) and \( T_\theta \)
under state \( s \) and action \( a \), \( \nu \) is the state-action distribution of an exploration policy, and \( d_{T_{\theta n}} \) is the state-action distribution induced by running an optimal policy with respect to the model \( T_{\theta n} \). When the model class is expressive enough to contain the \( T \), it holds \( l_n(\theta) = 0 \) for some \( \theta \in \Theta \) (cf. (3)).

Suppose that the states and actions are continuous. A common choice for \( D_{s,a}(T_\theta||T) \) in learning
deterministic dynamics is the squared error \( D_{s,a}(T_\theta||T) = \left\| T_\theta(s,a) - s' \right\|_2^2 \) [9], where the \( s' \) is the
next state in the true transition of \( T \). If \( T_\theta \) is linear in \( \theta \) or belongs to a reproducing kernel Hilbert
space [29], the sampled loss function \( \hat{l}_n \) is CSN. Alternatively, when learning a probabilistic model,
\( D_{s,a} \) can be selected as the KL-divergence [9]; it is known that if \( T_\theta \) belongs to the exponential
family of distributions, the KL divergence, and hence \( \hat{l}_n \), are smooth and convex [33]. If the sample
size is large enough, \( \hat{l}_n \) becomes non-negative in high probability.

As these online losses are CSN, our theoretical results apply and suggest a convergence rate in
\( \tilde{O}(1/N) \). On the contrary, the finite sample analysis conducted in [9] uses the standard online-
to-batch techniques [15] and can only give a rate of \( O(1/\sqrt{N}) \). Our new results provide a better
explanation to justify the fast policy improvement speed observed empirically, e.g., Figure 2 of [9].

5 Experimental Results
Although the main focus of this paper is the new theoretical insights, we conduct experiments to
provide evidence that the fast policy improvement phenomena indeed exist, as our theory predicts.
We verify the change of rates due to policy class capacity by running an online IL experiment in the
CartPole balancing task in OpenAI Gym [34] with DART physics engine [35].

MDP setup. The goal of the CartPole task is to keep the pole upright by controlling the acceleration
of the cart. The start state is a configuration with a small uniformly sampled offset from being static
and vertical, and the dynamics is deterministic. In each time step, if the pole is maintained within
a threshold from being upright, the learner receives an instantaneous reward of one; otherwise, the
learner receives zero reward and the episode terminates. This MDP has a 4-dimensional continuous
state space and a 1-dimensional continuous action space.
Table 1: Comparison of different learning settings.

| Setting       | Info. | Stochastic | Stationary | Estimator | Excess loss |
|---------------|-------|------------|------------|-----------|-------------|
| OPO           | $\hat{1}_n$ | Yes        | No         | online    | $\sum_n l_n(\theta_n) - \min \sum_n l_n(\theta)$ |
| Online learning | $l_n$    | No         | No         | online    | $\sum_n l_n(\theta_n) - \min \sum_n l_n(\theta)$ |
| Statistical learning | $\hat{l}$ | Yes       | Yes        | ERM       | $l(\theta_{ERM}) - \min l(\theta)$ |
| Online-to-batch | $\hat{l}$ | Yes       | Yes        | online    | $\sum_n l(\theta_n) - N \min l(\theta)$ |
| Stochastic bandits | $l(\theta_n)$ | Yes   | Yes        | online    | $\sum_n l(\theta_n) - N \min l(\theta)$ |

**Expert and learner policies** To simulate the online IL task, we consider a neural network expert policy (with one hidden layer of 64 units and tanh activation). The expert policy is trained with additional Gaussian noise (with zero mean and a learnable variance) on the actions using a model-free policy gradient method (ADAM [36] with GAE [37]). We let the learner policy be another neural network that has exactly the same architecture as the expert policy; we copy the weights for the hidden layer from those of the expert policy and randomly initialized the weights of the output layer. During training, only the weights of the learner’s output layer were updated. In this way, we can view the learner as a linear policy using the representation of the expert policy.

**Online IL setup** We emulate online IL with unbiased and biased policy classes. We define the unbiased class as all the policies satisfying the above architecture, whereas we define the biased policy class by imposing an additional $\ell_2$-norm constraint on the learner’s weights in the second layer so that the learner cannot perfectly mimic the expert policy. We select $l_n(\theta) = E_{s \sim d_{\pi_\theta}} [H_{\mu}(\pi_{\theta}(s) - \pi_{\epsilon}(s))]$ as the online loss in IL (see Section 2.2), where $H_{\mu}$ is the Huber function defined as $H_{\mu}(x) = \frac{1}{2} x^2$ for $|x| \leq \mu$ and $\mu |x| - \frac{1}{2} \mu^2$ for $|x| > \mu$. In the experiments, $\mu$ is set to 0.05; as a result, $H_{\mu}$ is linear when its function value is larger than 0.00125. Because the learner’s policy is linear, this online loss is CSN in the unknown weights of the learner. We use ADAM [36] to optimize the learner policy with constant stepsize 0.01.

**Simulation results** We compare the results in the unbiased and the biased settings, in terms of how the average loss $\frac{1}{N} \sum_{n=1}^{N} l_n(\theta_n)$ changes as the number of rounds $N$ in online learning increases. To see whether the rate of the average loss is $O(1/N)$ or $O(1/\sqrt{N})$, we fit the curves using the parametric functions $f_1(N) = a \frac{1}{N} + b$ and $f_2(N) = a \frac{1}{\sqrt{N}} + b$. The parameters $a, b$ are obtained by solving a constrained convex program that minimizes the $\ell_1$-loss with a constraint that the graphs of the parametric functions lie above the curves of the average loss. The experimental results are depicted in Fig. 1. In the unbiased setting (Fig. 1a), the curve fits well with the parametric function $f_1$. By contrast, in the biased setting (Fig. 1b), the curve aligns better with the parametric function $f_2$.

## 6 Related Work and Discussion

In this paper, we prove new expected and high-probability convergence rates that depend on the policy class capacity for OPO problems. Our results are closely related to the problem-dependent rates studied in several more typical learning settings. We summarize the relationship between our work and other related results in Table 1, where we compare different learning settings in terms of the information available to the learner, the properties of the loss functions, and the form of excess loss.

In statistical learning, there is a new trend studying how the generalization of the empirical risk minimizer (ERM) depends on the properties of loss functions. Rates dependent on the bias due to the hypothesis class are shown for Lipschitz loss functions [38], and smooth loss functions [16]. These results are extended to ERM in general stochastic optimization in [17, 18].

Another line of research on problem-dependent convergence rates focus on online learning with adversarial loss sequences. Although online learning does not impose the i.i.d. assumption on loss functions, comparator-dependent rates on the regret can also be proved for smooth [16] and smooth plus log-concave [39] online loss functions.
Finally, the research on online-to-batch conversion [40] studies the generalization in statistical learning through analyzing online learning problems with loss functions that are i.i.d. sampled. Cesa-Bianchi et al. [15] establish a fundamental connection between the regret in such online learning problems and the generalization error in statistical learning. Cesa-Bianchi and Gentile [41] relate the cumulative loss in online learning to the generalization in statistical learning, showing that a faster rate of generalization can be achieved if the cumulative loss is small.

In comparison, OPO concerns loss functions that are both stochastic and online; that is, we can view statistical and online learning as special cases of OPO. The interactions between noises and non-stationarity make the analysis of OPO especially interesting. We tackle these challenges by joining analysis techniques from stochastic optimization [17] and online learning [16]. As a consequence, we are able to develop new insights to explain certain fast convergence phenomena of OPO, which existing OPO theory fails to capture.

However, our current results cannot explain all the fast improvements of OPO observed in practice. The analyses here are based on the assumption of using convex and smooth loss functions. This assumption would be violated, for example, with a deep neural network policy with ReLU activation; yet Pan et al. [4] show fast empirical convergence rates of these networks in OPO. Nonetheless, we envision that the insights from this paper can provide a promising starting point to better understanding the behaviors of OPO, and to suggest directions for designing new OPO algorithms that proactively leverage these self-bounding regret properties to achieve faster learning.

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Appendix for Paper: Explaining Fast Improvement in Online Policy Optimization

A Proof of Tool Lemmas

A.1 Proof of Lemma 1

For completeness, we provide the proof for the basic inequality that upper bounds the norm of gradients by the function values, for smooth and nonnegative functions. This is essential for obtaining the self-bounding properties for proving Lemma 2 and Theorem 2 later on.

**Lemma 1** (Lemma 3.1 [16]). Suppose a function $f : \mathcal{H} \to \mathbb{R}$ is $\beta$-smooth and non-negative, then for any $x \in \mathcal{H}$, $\| \nabla f(x) \|_*^2 \leq 4\beta f(x)$. 

**Proof.** Fix any $x \in \mathcal{H}$. And fix any $y \in \mathcal{H}$ satisfying $\| y - x \| \leq 1$. Let $g(u) = f(x + u(y - x))$ for any $u \in \mathbb{R}$. Fix any $u, v \in \mathbb{R}$,

$$ |g'(v) - g'(u)| = |\langle \nabla f(x + v(y - x)) - \nabla f(x + u(y - x)), y - x \rangle| 
\leq \| \nabla f(x + v(y - x)) - \nabla f(x + u(y - x)) \| \| y - x \| 
\leq \beta \| v - u \| \| y - x \|^2 
\leq \beta \| v - u \|$$

Hence, $g$ is $\beta$-smooth. By the mean-value theorem, for any $u, v \in \mathbb{R}$, there exists $w \in (u, v)$, such that $g(v) = g(u) + g'(w)(v - u)$. Hence

$$ 0 \leq g(v) = g(u) + g'(u)(v - u) + (g'(w) - g'(u))(v - u) 
\leq g(u) + g'(u)(v - u) + \beta \| w - u \| \| v - u \| 
\leq g(u) + g'(u)(v - u) + \beta \| v - u \|^2 $$

Setting $v = u - \frac{g'(u)}{2\beta}$ yields that $|g'(u)| \leq \sqrt{4\beta g(u)}$. Therefore, we have

$$ |g'(0)| = |\langle \nabla f(x), y - x \rangle| \leq \sqrt{4\beta g(0)} = \sqrt{4\beta f(x)} $$

Therefore, by the definition of dual-norm,

$$ \| \nabla f(x) \|_* = \sup_{y \in \mathcal{B}, \| y - x \| \leq 1} |\langle \nabla f(x), y - x \rangle| \leq \sqrt{4\beta f(x)} $$

It’s worthy to note that $f$ needs to be smooth and non-negative on the entire Hilbert space $\mathcal{H}$.

B Proof of Theorem 1

**Theorem 1.** In Algorithm 1, suppose $\{ \hat{t}_n \}$ is CSN and the online algorithm $A$ is admissible. Let $\hat{\epsilon} = \frac{1}{N} \min_{\theta \in \Theta} \sum_{n} l_n(\theta)$ be the bias, and let $\bar{E}$ be such that $\bar{E} \geq \hat{\epsilon}$ almost surely. Choose $\eta$ for $A$ to be

$$ \eta = \frac{1}{2 \beta + \sqrt{4\beta^2 + 2\beta N E R_A^2}}. $$

Then in expectation

$$ \frac{1}{N} \sum l_n(\theta_n) - \hat{\epsilon} \leq \frac{8\beta R_A^2}{N} + \sqrt{\frac{8\beta R_A^2 \bar{E}}{N}} \quad (8) $$

The rate (8) follows from analyzing the regret and the generalization error in the decomposition in (6). First, under the assumption of CSN loss functions and admissible online algorithms, the online regret can be bounded by an extension of the bias-dependent regret bound that is stated for mirror descent in [16, Theorem 2], whose average gives the rate in (8) (see Appendix B.1). Second, the generalization error in (6) vanishes in expectation because it is a martingale difference sequence (see Appendix B.2).
B.1 Upper Bound of Online Regret

We show a bias-dependent regret bound for admissible online algorithms (Definition 2) with CSN functions (Definition 3) by extending Theorem 2 of [16] as follows.

Lemma 2. Consider running an admissible online algorithm \( A \) on a sequence of CSN loss functions \( \{ f_n \} \). Let \( \{ \theta_n \} \) denote the online decisions made in each round, and let \( \hat{\epsilon} = \frac{1}{\min_{\theta \in \Theta} \sum f_n(\theta)} \) be the bias, and let \( \tilde{E} \) be such that \( \tilde{E} \geq \hat{\epsilon} \) almost surely. Choose \( \eta \) for \( A \) to be \( \frac{1}{2} \left( \beta + \sqrt{\beta^2 + \frac{\beta N \tilde{E}}{R_A}} \right) \). Then the following holds

\[
\text{Regret}(f_n) \leq 8\beta R_A^2 + \sqrt{8\beta R_A^2 N \tilde{E}}. 
\]

Proof. Because the online algorithm \( A \) is admissible, we have

\[
\text{Regret}(f_n) \leq \frac{1}{\eta} R_A^2 + \frac{\eta}{2} \sum \| \nabla f_n(\theta_n) \|_*^2 
\]

Let \( \lambda = \frac{1}{2\eta} \) and \( r^2 = 2R_A^2 \), then

\[
\frac{1}{\eta} R_A^2 + \frac{\eta}{2} \sum \| \nabla f_n(\theta_n) \|_*^2 = \lambda r^2 + \sum \frac{1}{4\lambda} \| \nabla f_n(\theta_n) \|_*^2 
\]

Using Lemma 1 yields a self-bounding property for \( \text{Regret}(f_n) \):

\[
\text{Regret}(f_n) \leq \lambda r^2 + \frac{\beta}{\lambda} \sum f_n(\theta_n) \leq \lambda r^2 + \frac{\beta}{\lambda} \text{Regret}(f_n) + \frac{\beta}{\lambda} N \tilde{E} 
\]

By rearranging the terms, we have a bias-dependent upper bound

\[
\text{Regret}(f_n) \leq \frac{\beta}{\lambda - \beta} N \tilde{E} + \frac{\lambda^2}{\lambda - \beta} r^2 
\]

The upper bound can be minimized by choosing an optimal \( \lambda \). Setting the derivative of the right-hand side to zero, and computing the optimal \( \lambda \) \((\lambda > 0)\) gives us

\[
\lambda^2 - 2\beta r^2 \lambda - \beta N \tilde{E} = 0, \quad \lambda > 0 \quad \text{and} \quad \lambda = \beta + \sqrt{\beta^2 + \frac{\beta N \tilde{E}}{r^2}} 
\]

which implies that the optimal \( \eta \) is \( \frac{1}{2} \left( \beta + \sqrt{\beta^2 + \frac{\beta N \tilde{E}}{r^2}} \right) \). Since the optimal \( \lambda \) satisfies \( \beta N \tilde{E} = r^2 \lambda^2 - 2\beta r^2 \lambda \) implied from (15), (14) can be simplified into:

\[
\text{Regret}(f_n) \leq \frac{1}{\lambda - \beta} \beta N \tilde{E} + \frac{\lambda^2}{\lambda - \beta} r^2 = \frac{1}{\lambda - \beta} \left( r^2 \lambda^2 - 2\beta r^2 \lambda \right) + \frac{\lambda^2}{\lambda - \beta} r^2 
\]

\[
= \frac{2\lambda^2 r^2 - 2\beta \lambda r^2}{\lambda - \beta} = 2\lambda r^2 
\]

Plugging in the optimal \( \lambda \) yields

\[
\text{Regret}(f_n) \leq 2\lambda r^2 = 2 \left( \beta + \sqrt{\frac{\beta^2 + \beta N \tilde{E}}{r^2}} \right) r^2 
\]

\[
= 2\beta r^2 + 2\beta r^2 \sqrt{2 \beta r^2 + 2 N \tilde{E}} 
\]

\[
\leq 4\beta r^2 + 2 \sqrt{2 \beta r^2 N \tilde{E}} 
\]

\[
= 8\beta R_A^2 + \sqrt{8\beta R_A^2 N \tilde{E}} 
\]

where the last inequality uses the basic inequality: \( \sqrt{a + b} \leq \sqrt{a} + \sqrt{b} \). \( \square \)
Notably, the admissibility defined in Definition 2 is satisfied by common online algorithms, such as mirror descent [25] and Follow-The-Regularized-Leader [22] under first-order or full-information feedback, where $\eta$ in Definition 2 corresponds to a constant stepsize, and $R_A$ measures the size of the decision set $\Theta$. More concretely, assume that the loss functions $\{f_n\}$ are convex. Then for mirror descent, with constant stepsize $\eta$, i.e., $\theta_{n+1} = \arg \min_{\theta \in \Theta} f_n(\theta) + \frac{1}{\eta} D_h(\theta\|\theta_n)$, where $h$ is 1-strongly convex and $D_h$ is the Bregman distance generated by $h$ defined by $D_h(x\|y) = h(x) - h(y) - \langle \nabla h(y), x - y \rangle$ [42], $R_A^2$ can be set to $\max_{x,y \in \Theta} D_h(x\|y)$. And for FTRL with constant stepsize $\eta$, i.e., $\theta_{n+1} = \arg \min_{\theta \in \Theta} \sum f_n(\theta) + \frac{1}{\eta} h(\theta)$, where $h$ is 1-strongly convex and non-negative, $R_A^2$ can be set to $\max_{\theta \in \Theta} h(\theta)$ [22, Theorem 1].

B.2 The Generalization Error Vanishes in Expectation

The generalization error in (6) vanishes in expectation because it is a martingale difference sequence.

Lemma 3. For Algorithm 1, the following holds: $E[\sum l_n(\theta_n) - \sum \hat{l}_n(\theta_n)] = 0$.

Proof. We show this by working from the end of the sequence:

$$\mathbb{E}_{l_{1:N}} \left[ \sum_{t=1}^{N} l_n(\theta_n) \right] = \mathbb{E}_{l_{1:N-1}} \left[ \sum_{t=1}^{N-1} l_n(\theta_n) + l_N(\theta_N) \right]$$

$$= \mathbb{E}_{l_{1:N-1}} \left[ \sum_{t=1}^{N-1} l_n(\theta_n) + \mathbb{E}_{l_N|l_{1:N-1}} \left[ \hat{l}_N(\theta_N) \right] \right]$$

$$= \mathbb{E}_{l_{1:N-2}} \left[ \sum_{t=1}^{N-2} l_n(\theta_n) + l_{N-1}(\theta_{N-1}) + \mathbb{E}_{l_{N-1}|l_{1:N-2}} \left[ \hat{l}_{N-1}(\theta_{N-1}) \right] \right]$$

$$= \mathbb{E}_{l_{1:N-2}} \left[ \sum_{t=1}^{N-2} l_n(\theta_n) + \mathbb{E}_{l_{N-1}|l_{1:N-2}} \left[ \hat{l}_{N-1}(\theta_{N-1}) \right] + \mathbb{E}_{l_{N-1}|l_{1:N-2}} \left[ \hat{l}_N(\theta_N) \right] \right]$$

$$= \mathbb{E}_{l_{1:N-2}} \left[ \sum_{t=1}^{N-2} l_n(\theta_n) + \mathbb{E}_{l_{N-1}|l_{1:N-2}} \left[ \sum_{\ell=N-1}^{N} \hat{l}_\ell(\theta_\ell) \right] \right]$$

By applying the steps above repeatedly, the desired equality can be obtained. \hfill \Box

B.3 Putting Together

Finally, plugging Lemma 2 and Lemma 3 into (6) yields (8).

C Proof of Theorem 2

Theorem 2. Under the assumptions and setup of Theorem 1, further assume that there is $G \in [0, \infty)$ such that, for any $\theta \in \Theta$, $\|\nabla l_n(\theta)\|_* \le G$ almost surely. For $\delta < 1/\epsilon$, with probability at least $1 - \delta$,

$$\frac{1}{N} \sum l_n(\theta_n) - \epsilon = O \left( \frac{C^2 R^2}{N} + \sqrt{\frac{C^2 R^2 (E + \epsilon)}{N}} \right) \tag{9}$$

where $R_\Theta = \max_{\theta \in \Theta} \|\theta\|$, $R = \max(1, R_\Theta, R_A)$, and $C = \log(1/\delta) \log(GR\Theta)$.

C.1 Decomposition

The key to avoid the slow rate due to the direct application of martingale concentration analyses on the MDSs in (6) and (7) is to take a different decomposition of the cumulative loss. Here we construct two new MDSs in terms of the gradients: recall $\epsilon = \min_{\theta \in \Theta} \sum l_n(\theta)$ and let $\theta^* = \arg \min_{\theta \in \Theta} \sum l_n(\theta)$. Then by convexity of $l_n$, we can derive

$$\sum l_n(\theta_n) - N\epsilon$$

14
\[
\sum_{\text{MDS}} (\nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n), \theta_n)
\]

\[
\sum_{\text{MDS}} (\nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n), \theta_n) - \sum_{\text{MDS}} (\nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n), \theta_n) + \text{Regret}(\langle \nabla \hat{l}_n(\theta_n), \cdot \rangle)
\]

(18)

Our proof is based on analyzing these three terms. The two MDSs are analyzed in Appendix C.2 and the regret is analyzed in Appendix C.3.

C.2 Upper Bound of the Martingale Concentration

For the MDSs in (18), we notice that, for smooth and non-negative functions, the squared norm of the gradient can be bounded by the corresponding function value through Lemma 1. This enables us to properly control the second-order statistics of the MDSs in (18). By a recent vector-valued martingale concentration inequality that depends only on the second-order statistics [19], we obtain a self-bounding property for (18) to get fast concentration rate. The martingale concentration inequality is stated in the following lemma.

**Lemma 4** (Theorem 3 [19]). Let \( K \) be a Hilbert space with norm \( \| \cdot \| \) whose dual is \( \| \cdot \|_* \). Let \( \{ z_t \} \) be a \( K \)-valued martingale difference sequence with respect to \( \{ y_t \} \), i.e., \( \mathbb{E}[z_t|y_1, \ldots, y_{t-1}] = 0 \), and let \( h \) be a 1-strongly convex function with respect to norm \( \| \cdot \| \) and let \( B^2 = \sup_{x, y \in K, \| x \| = 1, \| y \| = 1} D_h(x||y) \). Then for \( \delta \leq 1/\epsilon \), with probability at least \( 1 - \delta \)

\[
\left\| \sum z_t \right\|_*^2 \leq 2B \sqrt{V} + \sqrt{2\log(1/\delta)} \sqrt{1 + 1/2\log(2V + 2W + 1)} \sqrt{2V + 2W + 1}
\]

where \( V = \sum \| z_t \|_*^2 \) and \( W = \sum \mathbb{E}[z_t|y_1, \ldots, y_{t-1}] \| z_t \|_*^2 \).

In order to apply Lemma 4 to the MDSs in (18), the key is to properly upper bound the statistics \( V \) and \( W \) in Lemma 4 for these MDSs.

C.2.1 Upper Bound of the Concentration for MDS \( (\nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n), \theta_n) \)

Suppose that the decision set \( \Theta \) is inside a ball centered at the origin in \( \mathcal{H} \) with radius \( R_\Theta \).

**Assumption 1.** There exists \( R_\Theta \in [0, \infty) \), such that \( \max_{\theta \in \Theta} \| \theta \| \leq R_\Theta \).

Then by the definition of \( V \) and \( W \) in Lemma 4, and the definitions of the two problem-dependent policy class biases \( \epsilon \) and \( \hat{\epsilon} \) (see Definition 1), one can obtain

\[
V = \sum |\langle \nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n), \theta_n \rangle|^2 \\
\leq \sum R_\Theta^2 \| \nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n) \|_*^2
\]  
**Assumption 1**

\[
\leq \sum R_\Theta^2 \left( 2\| \nabla l_n(\theta_n) \|_*^2 + 2\| \nabla \hat{l}_n(\theta_n) \|_*^2 \right)
\]  
triangle inequality

\[
\leq \sum R_\Theta^2 \left( 8\beta l_n(\theta_n) + 8\beta \hat{l}_n(\theta_n) \right)
\]  
**Lemma 1**

\[
= 8\beta R_\Theta^2 (\text{Regret}(l_n) + \text{Regret}(\hat{l}_n) + N\epsilon + N\hat{\epsilon})
\]  
**Definition 1**

Similarly for \( W \), we have

\[
W = \sum \mathbb{E}[z_t|\theta_n] |\langle \nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n), \theta_n \rangle|^2 \\
\leq \sum R_\Theta^2 \mathbb{E}[z_t|\theta_n] \| \nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n) \|_*^2
\]  
**Assumption 1**

\[
\leq \sum R_\Theta^2 \left( 2\| \nabla l_n(\theta_n) \|_*^2 + 2\mathbb{E}[z_t|\theta_n] \| \nabla \hat{l}_n(\theta_n) \|_*^2 \right)
\]  
triangle inequality

\[
\leq \sum R_\Theta^2 \left( 8\beta l_n(\theta_n) + 8\mathbb{E}[z_t|\theta_n] \| \beta \hat{l}_n(\theta_n) \|_*^2 \right)
\]  
**Lemma 1**

\[
= \sum R_\Theta^2 \left( 8\beta l_n(\theta_n) + 8\beta \hat{l}_n(\theta_n) \right)
\]  
**Definition 1**

\[
= 16\beta R_\Theta^2 (\text{Regret}(l_n) + N\epsilon)
\]  
**Definition 1**
Therefore,
\[
V + W \leq 24\beta R^2_\Theta (\text{Regret}(l_n) + \text{Regret}(\hat{l}_n) + N\epsilon + N\hat{\epsilon}) \tag{23}
\]
Further suppose that the gradient of the sampled loss can be uniformly bounded:

**Assumption 2.** For any loss sequence \(\{\hat{l}_n\}\) that can be experienced by Algorithm 1, suppose that there is \(G \in [0, \infty)\) such that, for any \(\theta \in \Theta\), \(\|\nabla l_n(\theta)\|_* \leq G\).

Then due to (23), \(V \leq 4G^2 R^2_\Theta N\) and \(W \leq 4G^2 R^2_\Theta N\). Now we are ready to invoke Lemma 4 by letting the Hilbert space \(K\) in Lemma 4 be \(\mathbb{R}\), and denoting the corresponding \(B\) in Lemma 4 by \(B_{\mathbb{R}}\). Then, for \(\delta > 1/e\), with probability at least \(1 - \delta\), the following holds
\[
\left| \sum \langle \nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n), \theta_n \rangle \right| \\
\leq 2B_{\mathbb{R}} \sqrt{8\beta R^2_\Theta \text{Regret}(l_n) + \text{Regret}(\hat{l}_n) + N\epsilon + N\hat{\epsilon}} + \\
\sqrt{96\beta R^2_\Theta \log(1/\delta) \left(1 + 1/2 \log(16G^2 R^2_\Theta N + 1)\right)} \sqrt{\text{Regret}(l_n) + N\epsilon + \text{Regret}(\hat{l}_n) + N\hat{\epsilon} + 1/(48\beta R^2_\Theta)} \tag{24}
\]

### C.2.2 Upper Bound of the Concentration for MDS \(\nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n)\)

To bound \(\| \sum \nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n) \|_*\) that appears in
\[
\sum \langle \nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n), \theta_n \rangle \leq R_\Theta \| \sum \nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n) \|_* \tag{25}
\]

We use Lemma 4 again in a similar way of deriving (24), except that this time the MDS \(\nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n)\) is vector-valued. Akin to showing (20) and (22), the statistics \(V\) can be bounded as
\[
V \leq \sum \left(2\|\nabla l_n(\theta_n)\|^2 + 2\|\nabla \hat{l}_n(\theta_n)\|^2\right) \leq 8\beta(\text{Regret}(l_n) + \text{Regret}(\hat{l}_n) + N\epsilon + N\hat{\epsilon}) \tag{26}
\]
and similarly for \(W\):
\[
W \leq \sum \left(2\|\nabla l_n(\theta_n)\|^2 + 2\mathbb{E}_{l_n|\theta_n} \|\nabla \hat{l}_n(\theta_n)\|^2\right) \leq 16\beta(\text{Regret}(l_n) + N\epsilon) \tag{28}
\]
Therefore,
\[
V + W \leq 24\beta(\text{Regret}(l_n) + \text{Regret}(\hat{l}_n) + N\epsilon + N\hat{\epsilon}) \tag{30}
\]

Furthermore, by Assumption 2, it can be shown from (26) and (28) that \(V \leq 4G^2 N\) and \(W \leq 4G^2 N\). To invoke Lemma 4, let \(K\) in Lemma 4 be \(\mathcal{H}\), and denote the corresponding \(B\) in Lemma 4 by \(B_{\mathcal{H}}\). Then, for \(\delta < 1/e\), with probability at least \(1 - \delta\), the following holds
\[
\| \sum \nabla l_n(\theta_n) - \nabla \hat{l}_n(\theta_n) \|_* \leq 2B_{\mathcal{H}} \sqrt{8\beta \text{Regret}(l_n) + \text{Regret}(\hat{l}_n) + N\epsilon + N\hat{\epsilon}} + \\
\sqrt{96\beta \log(1/\delta) \left(1 + 1/2 \log(16G^2 N + 1)\right)} \sqrt{\text{Regret}(l_n) + N\epsilon + \text{Regret}(\hat{l}_n) + N\hat{\epsilon} + 1/(48\beta)} \tag{31}
\]

### C.3 Upper Bound of the Regret

Besides analyzing the MDSs, we need to bound the regret to the linear functions defined by the gradients (the last term in (18)). Since this last term is linear, not CSN, the bias-dependent online regret bound in the proof of Theorem 1 does not apply. Nonetheless, because these linear functions are based on the gradients of CSN functions, we discover that their regret rate actually obeys the exact same rate as the regret to the CSN loss functions. This is notable because the regret to these linear functions upper bounds the regret to the CSN loss functions.
We now have all the pieces to prove Theorem 2. Plugging (24), (25), and (31) into the decomposition (18), we have, for \( \delta \leq \frac{1}{2\eta} \) constant \( \eta \)
This self-bounding property is exactly like what we have seen in the self-bounding property for
To proceed, as in Lemma 2, let \( \hat{\lambda} \)
Plugging them into the above upper bound on Regret
Almost surely. Using Lemma 1 and the admissibility of online algorithm \( A \)
Equations (13) through algebraic manipulations. As in
Lemma 5.

Under the same assumptions and setup in Lemma 2,
\[
\text{Regret}(\langle \nabla f_n(\theta_n), \cdot \rangle) \leq 8\beta R^2_A + 8\beta R_A^2 N \hat{E}.
\]  
(32)
Proof. It suffices to show a self-bounding property for \( \text{Regret}(\langle \nabla f_n(\theta_n), \cdot \rangle) \) as (13). Once this is established, the rest resembles how (17) follows from (13) through algebraic manipulations. As in
Lemma 2, define \( \lambda = \frac{1}{2\eta} \) and \( r^2 = 2R_A^2 \). Due to the property of admissible online algorithms, one can obtain
\[
\text{Regret}(\langle \nabla f_n(\theta_n), \cdot \rangle) \leq \frac{1}{\eta} R^2_A + \frac{1}{\eta} \sum_1^\eta \| \nabla f_n(\theta_n) \|^2 = \lambda r^2 + \sum_1^\lambda \frac{1}{4\lambda} \| \nabla f_n(\theta_n) \|^2
\]  
(33)
To proceed, as in Lemma 2, let \( \hat{\epsilon} = \frac{1}{\eta} \min_{\theta \in \Theta} \sum_1^\theta \hat{f}_n(\theta) \) be the bias, and let \( \hat{E} \) be such that \( \hat{E} \geq \hat{\epsilon} \) almost surely. Using Lemma 1 and the admissibility of online algorithm \( A \) yields a self-bounding property for \( \text{Regret}(\langle \nabla f_n(\theta_n), \cdot \rangle) \):
\[
\text{Regret}(\langle \nabla f_n(\theta_n), \cdot \rangle) \leq \lambda r^2 + \frac{\beta}{\lambda} \sum_1^\lambda \| \nabla f_n(\theta_n) \|
\leq \lambda r^2 + \frac{\beta}{\lambda} \text{Regret}(f_n) + \frac{\beta}{\lambda} N \hat{E}
\leq \lambda r^2 + \frac{\beta}{\lambda} \text{Regret}(\langle \nabla f_n(\theta_n), \cdot \rangle) + \frac{\beta}{\lambda} N \hat{E}
\]  
This self-bounding property is exactly like what we have seen in the self-bounding property for
Regret \( (\hat{f}_n) \). After rearranging and computing the optimal \( \lambda \) (which coincides with the optimal \( \lambda \) in
Lemma 2), (32) follows.

Lemma 5 provides a bias-dependent regret to the linear functions defined by the gradients when the (stepsize) constant \( \eta \) is set optimally in the online algorithm \( A \) (used in Algorithm 1). Interestingly, the optimal \( \eta \) that achieves the bias-dependent regret coincides with the one for achieving a bias-dependent regret to CSN functions. Therefore, a bias-dependent bound for \( \text{Regret}(\hat{f}_n) \) and \( \text{Regret}(\langle \nabla \hat{f}_n(\theta_n), \cdot \rangle) \) can be achieved simultaneously.

C.4 Putting Things Together

We now have all the pieces to prove Theorem 2. Plugging (24), (25), and (31) into the decomposition (18), we have, for \( \delta < 1/e \), with probability at least \( 1 - 2\delta \)
\[
\text{Regret}(l_n) \leq 2B_R \sqrt{8\beta R^2_A} \sqrt{\text{Regret}(l_n) + \text{Regret}(\hat{l}_n) + N\epsilon + N\hat{\epsilon}}
\]
\[
+ \sqrt{96\beta R^2_A \log(1/\delta)} \sqrt{1 + 1/2 \log(16G^2 R^2_A N + 1)} \sqrt{\text{Regret}(l_n) + N\epsilon + \text{Regret}(\hat{l}_n) + N\hat{\epsilon} + 1/(48\beta R^2_A)}
\]
\[
+ 2B_H \sqrt{8\beta R^2_A} \sqrt{\text{Regret}(l_n) + \text{Regret}(\hat{l}_n) + N\epsilon + N\hat{\epsilon}}
\]
\[
+ \sqrt{96\beta \log(1/\delta)} \sqrt{1 + 1/2 \log(16G^2 N + 1)} \sqrt{\text{Regret}(l_n) + N\epsilon + \text{Regret}(\hat{l}_n) + N\hat{\epsilon} + 1/(48\beta)}
\]
\[
+ \text{Regret}(\langle \nabla \hat{l}_n(\theta_n), \cdot \rangle)
\]  
To simplify it, we denote
\[
A_1 = 8 \max(B_R, B_H) \sqrt{2\beta R^2_A},
\]
\[
A_2 = 8 \sqrt{6\beta R^2_A \log(1/\delta)} \sqrt{1 + 1/2 \log(16G^2 \max(1, R^2_A) N + 1)},
\]
\[
\tilde{R} = \min(1, R_A)
\]  
Plugging them into the above upper bound on \( \text{Regret}(l_n) \) and using the basic inequality \( \sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \) yield
\[
\text{Regret}(l_n) \leq (A_1 + A_2) \sqrt{\text{Regret}(l_n) + \text{Regret}(\hat{l}_n) + (A_1 + A_2) \sqrt{N\epsilon + N\hat{\epsilon}}}
\]
To further simplify, using the basic inequality $\sqrt{ab} \leq (a + b)/2$ yields

$$\text{Regret}(l_n) \leq \frac{\text{Regret}(\hat{l}_n)}{2} + \frac{\text{Regret}(\hat{l}_n)}{2} + (A_1 + A_2)\sqrt{N\epsilon' + N\epsilon}$$

$$+ \frac{(A_1 + A_2)^2}{2} + \frac{A_2}{\sqrt{48\beta R^2}} + \text{Regret}(\langle \nabla \hat{l}_n(\theta_n), \cdot \rangle)$$

Rearranging terms and invoking the bias-dependent rate in Lemma 2 and Lemma 5 give

$$\text{Regret}(l_n) \leq \text{Regret}(\hat{l}_n) + 2(A_1 + A_2)\sqrt{N\epsilon' + N\epsilon} + (A_1 + A_2)^2 + \frac{A_2}{\sqrt{12\beta R^2}} + 2\text{Regret}(\langle \nabla \hat{l}_n(\theta_n), \cdot \rangle)$$

$$\leq 2(A_1 + A_2)\sqrt{N\epsilon' + N\epsilon' + 6\sqrt{2\beta R^2 N \hat{E}} + (A_1 + A_2)^2 + \frac{A_2}{\sqrt{12\beta R^2}}} + 24\beta R^2$$

Finally, to derive a big-$O$ bound, denote

$$R = \max(1, R_\Theta, R_A), \quad C = \log(1/\delta) \log(GRN)$$

then one can obtain the rate in terms of $N$ in big-$O$ notation, while keeping $\hat{R}, R, B_R, B_H, \log(1/\delta), G, \epsilon,$ and $\hat{E}$ as multipliers:

$$\text{Regret}(l_n) = O\left(C\beta R^2 N(\hat{E} + \epsilon)\right)$$

Therefore

$$\frac{1}{N} \sum l_n(\theta_n) - \epsilon = O\left(\frac{C\beta R^2}{N} + \sqrt{\frac{C\beta R^2(\hat{E} + \epsilon)}{N}}\right)$$

### D Experiment Details

Although the main focus of this paper is the new theoretical insights, we conduct experiments to provide evidence that the fast policy improvement phenomena indeed exist, as our theory predicts. We verify the change of rates due to policy class capacity by running an online IL experiment in the CartPole balancing task in OpenAI Gym [34] with DART physics engine [35].

#### D.1 MDP Setup

The goal of the CartPole balancing task is to keep the pole upright by controlling the acceleration of the cart. This MDP has a 4-dimensional continuous state space (the position and the velocity of the cart and the pole), and 1-dimensional continuous action space (the acceleration of the cart). The initial state is a configuration with a small uniformly sampled offset from being static and vertical, and the dynamics is deterministic. This task has a maximum horizon of 1000. In each time step, if the pole is maintained within a threshold from being upright, the learner receives an instantaneous reward of one; otherwise, the learner receives zero reward and the episode terminates. Therefore, the maximum sum of rewards for an episode is 1000.

#### D.2 Expert Policy Representation and Training

To simulate the online IL task, we consider a neural network expert policy (with one hidden layer of 64 units and tanh activation), and the inputs to the neural network is normalized using a moving average over the samples. The expert policy is trained using a model-free policy gradient method (ADAM [36] with GAE [37]). And the value function used by GAE is represented by a neural network with two hidden layers of 128 units and tanh activation. To compute the policy gradient
during training, additional Gaussian noise (with zero mean and a learnable variance that does not depend on the state) is added to the actions, and the gradient is computed through log likelihood ratio. After 100 rounds of training, the expert policy can consistently achieve the maximum sum of rewards both with and without the additional Gaussian noise. After the expert policy is trained, during online IL, Gaussian noise is not added in order to reduce the variance in the experiments.

D.3 Learner Policy Representation

We let the learner policy be another neural network that has exactly the same architecture as the expert policy with no Gaussian noise added; we copy the weights for the hidden layer and the input normalizer from those of the expert policy and randomly initialized the weights of the output layer. During training, only the weights of the learner’s output layer were updated. In this way, we can view the learner as a linear policy using the representation of the expert policy.

D.4 Online IL Setup

Policy class We conduct online IL with unbiased and biased policy classes. One one hand, we define the unbiased class as all the policies satisfying the representation in Appendix D.3. On the other hand, we define the biased policy class by imposing an additional $\ell_2$-norm constraint on the learner’s weights in the second layer so that the learner cannot perfectly mimic the expert policy. More concretely, in the experiments, the $\ell_2$-norm constraint is set to 0.1, whereas the $\ell_2$-norm of the final policy trained without the constraint is 0.56.

Loss functions We select $l_n(\theta) = \mathbb{E}_{s \sim d_{\pi_e}} [H_\mu(\pi_\theta(s) - \pi_e(s))]$ as the online IL loss (see Section 2.2), where $H_\mu$ is the Huber function defined as $H_\mu(x) = \frac{1}{2} x^2$ for $|x| \leq \mu$ and $\mu |x| - \frac{1}{2} \mu^2$ for $|x| > \mu$. In the experiments, $\mu$ is set to 0.05; as a result, $H_\mu$ is linear when its function value is larger than 0.00125. Because the learner’s policy is linear, this online loss is CSN in the unknown weights of the learner.

Policy update rule We choose ADAM [36], which is a first-order mirror descent algorithm, as the online algorithm in Algorithm 1. When the $\ell_2$-norm constraint is imposed, an additional projection step is taken after taking a gradient step using ADAM. The final algorithm is a special case of the DAgger algorithm [2] (called DAggereD in [24]) with only first-order information and continuous actions [24]. In the experiments, the stepsize is set to 0.01. In each round, for updating the learner policy, 1000 samples, i.e., state and expert action pairs, are gathered, and for computing the loss $l_n(\theta_n)$, more samples (5000 samples) are used due to the randomness in the initial state of the MDP.

Hyperparameter tuning The hyperparameters are tuned in a very coarse manner. We eliminated the ones that are obviously not proper. Here are the hyperparameters we have experimented. The
stepsizes in online IL: 0.1, 0.01, 0.001. The $\ell_2$-norm constraint for biased policies: 0.1, 0.4. The Huber function parameter $\mu$: 0.05.

D.5 Curve Fitting and Simulation results

We compare the learning results in the unbiased and biased settings, in terms of how the average loss $\frac{1}{N} \sum_{n=1}^{N} l_n(\theta_n)$ changes as the number of rounds $N$ in online learning increases. Due to the randomness in the initial state of the MDP, we used the median of the average loss from 4 random seeds. To see whether the rates of the learning curves are $O(1/N)$ or $O(1/\sqrt{N})$, we fit the curves of the average loss using the parametric functions $f_1(N) = a \frac{1}{N} + b$ and $f_2(N) = a \frac{1}{\sqrt{N}} + b$.

The parameters $a, b$ in the parametric functions are obtained by solving a constrained convex program using the python package CVXPY [43, 44]. We chose the loss function of the convex program to be the $\ell_1$-loss, in order to avoid over-penalizing the error at first several rounds and to capture the overall rate of the curves. The constraint of the convex program is that the graphs of the parametric functions must lie above the learning curves. The constraint is imposed because the rate predicted by the new theory is an upper bound.

The experimental results are depicted in Fig. 2. In the unbiased setting (Fig. 2a), the curve fits well with the parametric function $f_1$ in $O(1/N)$. By contrast, in the biased setting (Fig. 2b), the curve aligns better with the parametric function $f_2$ in $O(1/\sqrt{N})$.

D.6 Other Details

Computing infrastructure All the experiments were conducted on a desktop with Intel(R) Core(TM) i7-4770 CPU @ 3.40GHz, 32GB memory, and no GPU. The operating system is Ubuntu 16.04.

Average runtime On the aforementioned desktop, it took 15 min to train the expert, 45 min to do online IL, and 3 sec to fit and plot the curves.