Exact Propagator for \( SU(N) \) Coherent State

Phillial Oh

Department of Physics, Sung Kyun Kwan University, Suwon 440-746, KOREA

Abstract

We present a classical integrable model of \( SU(N) \) isospin defined on complex projective phase space in the external magnetic field and solve it exactly by constructing the action-angle variables for the system. We quantize the system using the coherent state path integral method and obtain an exact expression for quantum mechanical propagator by solving the time-dependent Schrödinger equations.

PACS numbers: 02.40.Hw, 03.65.-w
There exist a number of integrable systems [1,2] which can be solved exactly. Classical solvability is a question of finding the action-angle variables of the system. Quantum mechanically, one way of solving the system would be to find an exact expression for the propagator by solving the time-dependent Schrödinger equation.

The classical models of point particles carrying isospin charge in interaction with external gauge field have been investigated for over two decades [3]. Recently, attention has been paid to them in relation to the non-Abelian Chern-Simons quantum mechanics [4–8] which may have some application in condensed matter physics [9]. In this Letter, we construct a completely integrable classical model of non-relativistic $SU(N+1)(N \geq 1)$ isospin particle interacting with constant external magnetic field and show that it can be solved exactly in the above way. We will be concerned with only isospin degrees of freedom and drop the spatial dependence altogether.

Classical $SU(N+1)$ symmetry can be well described on complex projective space $\mathcal{M} = CP(N)$ which is a symplectic manifold and therefore could be considered to be the phase space of classical mechanics. The symplectic structure is given by the Fubini-Study metric [10] with a constant $J$

$$\Omega = 2iJ \left[ \frac{d\xi \wedge d\bar{\xi}}{1 + |\xi|^2} - \frac{(\xi d\bar{\xi}) \wedge (\bar{\xi} d\xi)}{(1 + |\xi|^2)^2} \right], \quad (1)$$

where $\xi = (\xi_1, \cdots, \xi_N)$ is the coordinates of $CP(N)$ and $|\xi|^2 = \sum_{m=1}^{N} \xi_m^* \xi_m \equiv \bar{\xi} \xi$. We introduce the notation $\xi^A = (\xi_m^*, \xi_n) \ (m, n = 1, \cdots, N)$ and write the symplectic two form Eq.(1) as $\Omega = \frac{1}{2} \Omega_{AB} d\xi^A \wedge d\xi^B$. Then, the Poisson bracket between $F, H \in C^\infty(\mathcal{M})$ is defined by

$$\{F, H\} = \Omega^{AB} \partial_A F \partial_B H \quad (2)$$

where $\Omega^{AB}$ is the inverse matrix of $\Omega_{AB}$. The use of Eq.(1) results in the following expression for Poisson bracket:

$$\{F, H\} = -i \sum_{m,n} g^{mn} \left( \frac{\partial F}{\partial \xi_m^*} \frac{\partial H}{\partial \xi_n} - \frac{\partial F}{\partial \xi_n} \frac{\partial H}{\partial \xi_m^*} \right), \quad (3)$$
where $g^{mn}$ is the inverse of the Fubini-Study metric given by

$$g^{mn} = \frac{1}{2J}(1 + |\xi|^2)(\delta_{mn} + \xi^*_m \xi_n). \quad (4)$$

Now, the group $SU(N+1)$ acts transitively on $CP(N)$ and leaves the symplectic two form

Eq.(1) invariant. It can be shown \[8\] that the group action $g = \exp(itT^a) \in SU(N+1), t \in \mathbb{R}$ generates the following Hamiltonian vector field on $M$

$$X_a = \left[ (T^a)_{mn} \xi_n - (T^a)_{0n} \xi_m - (T^a)_{00} \xi_n \right] \frac{\partial}{\partial \xi_m} + \text{(c.c.)}. \quad (5)$$

Here $T^a$’s form the generators of the Lie algebra $G = su(N+1)$:

$$[T^a, T^b] = if^{abc}T^c \quad (6)$$

We use the normalization $Tr(T^a T^b) = (1/2)\delta_{ab}$ and the notation that $T^a$ is $(N+1) \times (N+1)$ matrix with element $(T^a)_{ij} (i, j = 0, 1, \cdots, N)$. The above Hamiltonian vector field defines the isospin function (so-called momentum mapping function \[3\]) $Q^a$ on $M$ by

$$X_a \Omega + dQ^a = 0. \quad (7)$$

One can show that $Q^a$ can be expressed as follows \[8\]

$$Q^a = -2J \sum_{i,j=0}^{N} u_i^a (T^a)_{ij} u_j |_{u_0 = \frac{1}{\sqrt{1+|\xi|^2}}, u_m = u_0 \xi_m}. \quad (8)$$

It can be easily checked that $X'_a$s form $su(N+1)$ Lie algebra $[X_a, X_b] = f^{abc}X_c$ and $Q^a$ satisfy $su(N+1)$ Poisson-Lie algebra:

$$\{Q^a, Q^b\} = f^{abc}Q^c. \quad (9)$$

To define our integrable model with the use of isospin functions Eq.(8) and (9), we need an explicit representation for the generator $T^a$’s. Recall that $T^a$’s can be decomposed into a maximally commuting set $\{H^m\}(m = 1, \cdots, N)$ which is called the Cartan subalgebra and a set of ladder operators $E_\alpha$ one for each root vectors $\alpha$. We use the defining representation of $SU(N+1)$ where the diagonal $H^m$s are defined as follows \[11\]: $H^m$ has $m$ 1’s along
the diagonal from the upper left-hand corner. The next diagonal element is $-m$ to make it traceless. The rest of the diagonal element (if any) are zero. So with the normalization defined previously, we have

$$
(H^m)_{ij} = \left( \sum_{k=0}^{m-1} \delta_{ik} \delta_{jk} - m \delta_{i,m} \delta_{j,m} \right) / \sqrt{2m(m+1)} \tag{10}
$$

The vector field generated by $\exp(itH^m)$ can be expressed using Eq.(5) as

$$
X_m = \frac{-i}{\sqrt{2m(m+1)}} \left[ (m+1)\xi_m \frac{\partial}{\partial \xi_m} + \left( \sum_{k=m+1}^{N} \xi_k \frac{\partial}{\partial \xi_k} \right) \right] + (c.c) \tag{11}
$$

In the above, the index $m$ is not summed. The corresponding isospin functions Eq.(8) are given by

$$
Q_m = \frac{-2J}{\sqrt{2m(m+1)}} \left( \sum_{k=0}^{m-1} u_k^* u_k - m u_m^* u_m \right) \bigg|_{u_0 = \frac{1}{\sqrt{1+|\xi|^2}}, u_m = u_0 \xi_m} \tag{12}
$$

We define an integrable model on $\mathcal{M}$ with the Hamiltonian given by a linear combination of the above isospin functions

$$
H = \sum_{m=1}^{N} \mu_m Q^m \tag{13}
$$

with real constants $\mu_m$. When restricted to $SU(2)$ case with $N = 1$, this model describes a classical (iso)spin in the constant external magnetic field as can be readily checked in terms of stereo graphical projection $\xi = \tan(\theta/2) e^{-i\phi}$:

$$
H = \mu Q, \quad Q = -J \cos \theta. \tag{14}
$$

with the magnetic field given by $\mu$ and $J$ represents the magnitude of classical (iso)spin.

By construction, there are $N$ conserved quantities $Q^m$ which are in involution

$$
\{H, Q^m\} = 0, \quad \{Q^m, Q^n\} = 0 \tag{15}
$$

and the system is completely integrable [1,2]. The Liouville’s theorem states that the manifold $\mathcal{M}_Q$ defined by the level set of the functions $Q^m$,

$$
\mathcal{M}_Q = \{x : Q^m(x) = q^m, x = (\xi, \xi)\} \tag{16}
$$
with \( q^m = \text{constant} \), is a smooth \( N \)-dimensional manifold diffeomorphic to the torus

\[
T^N = \{ (\phi_1, \ldots, \phi_N) \mod 2\pi \}.
\]

and invariant with respect to the phase flow with the Hamiltonian. In this case, we can find the action variables \((I_1, \ldots, I_N)\) conjugate to the angle \((\phi_1, \ldots, \phi_N)\) so that the original symplectic structure Eq.(1) is expressed by the canonical two form

\[
\Omega = \sum_{m=1}^{N} dI_m \wedge d\phi_m.
\]

The explicit form of the action variables can be found by the use of stereo graphical projection. Let us introduce the polar angles \((\theta_1, \ldots, \theta_N)(0 \leq \theta \leq \pi)\) and \((\phi_1, \ldots, \phi_N)\) via

\[
\begin{align*}
\xi_1 &= \tan(\theta_1/2) \cos(\theta_2/2) e^{-i\phi_1} \\
\xi_2 &= \tan(\theta_1/2) \sin(\theta_2/2) \cos(\theta_3/2) e^{-i\phi_2} \\
&\quad \cdots \\
\xi_{N-1} &= \tan(\theta_1/2) \sin(\theta_2/2) \cdots \sin(\theta_{N-1}/2) \cos(\theta_N/2) e^{-i\phi_{N-1}} \\
\xi_N &= \tan(\theta_1/2) \sin(\theta_2/2) \cdots \sin(\theta_{N-1}/2) \sin(\theta_N/2) e^{-i\phi_N}
\end{align*}
\]

It is interesting to find that the action variables \(I_m\) are given by the following formula:

\[
I_m = 2J \sin^2(\theta_1/2) \cdots \sin^2(\theta_m/2) \cos^2(\theta_{m+1}/2) \quad (m < N)
\]

\[
I_N = 2J \sin^2(\theta_1/2) \cdots \sin^2(\theta_{N-1}/2) \sin^2(\theta_N/2)
\]

It can be expressed more compactly as \(I_m = 2J \xi_m^* \xi_m/(1 + |\xi|^2)\). One can check that substitution of Eq.(19) into Eq.(11) produces Eq.(18). The above action variables enables \(Q^m\) to be expressed as a linear combination of them:

\[
Q^m = \frac{1}{\sqrt{2m(m+1)}} \left[ (m+1)I_m + \sum_{k=m+1}^{N} I_k - 2J \right]
\]

Substituting into Eq.(13), we obtain

\[
H = \sum_{n=1}^{N} \omega_n I_n + H_0
\]
where $H_0$ is a constant and the Larmor frequency $\omega_n$ is given by

$$\omega_n = \sum_{m=1}^{N} \frac{\mu_m}{\sqrt{2m(m+1)}} [(m+1)\delta_{nm} + \theta_{nm}]$$

(23)

with the step symbol $\theta_{nm}$,

$$\theta_{nm} = \begin{cases} 
0, & n \leq m \\
1, & n > m.
\end{cases}$$

(24)

With Eq.(22), we have completely solved the system and the solution is given by

$$\theta_m(t) = \theta_m(0), \quad \phi_m(t) = \phi_m(0) + \omega_m t.$$ 

(25)

Instead of performing the path integral in terms of the above Darboux variables [12,13], we quantize the above classical model using the coherent state path integral method [14–16] and obtain the exact quantum mechanical propagator by solving the time-dependent Schrödinger equations. We first construct coherent states on $CP(N)$. Let us consider $|0\rangle$, the highest weight state annihilated by all positive roots of $SU(N+1)$ algebra in Cartan basis. Then for $CP(N)$ with given $P \equiv 2J$ ($P \in \mathbb{Z}^+$) we have an irreducible representation $(P, 0, \cdots, 0)$ of $SU(N+1)$ group [17] and there are precisely $N$ negative roots $E_\alpha, \alpha = 1, 2, \cdots, N$ such that $E_\alpha |0\rangle \neq |0\rangle$. Let us label $\{E_\alpha\} = \{E_m\}$. We define a coherent state on $CP(N)$ corresponding to the point $\xi = (\xi_1, \cdots, \xi_N)$ by [15,16]

$$|P, \xi\rangle = \exp \left( \sum_m \xi_m E_m \right) |0\rangle$$

(26)

Notice that this definition differs from the usual one by the normalization factor. We have chosen this definition here because in the subsequent analysis, $\bar{\xi}$ and $\xi$ can be treated independently and the overspecification problem can be side-stepped [18–20]. We denote $|P, \xi\rangle = |\xi\rangle$ from now on. The coherent states which we have defined on $CP(N)$ have the following two properties which are essential in the path integral formulation. One is the resolution of unity,

$$\int D\mu(\bar{\xi}, \xi) \frac{|\xi\rangle \langle \xi|}{(1 + |\xi|^2)^{2J}} = I,$$

(27)
where \( D\mu(\bar{\xi}, \xi) = c d\bar{\xi}d\xi/(1 + |\xi|^2)^{N+1} \) with a constant \( c \) is the Liouville measure \([10]\). The other is reproducing kernel,

\[
\langle \xi''|\xi' \rangle = (1 + \bar{\xi}''\xi')^{2J}.
\] (28)

We are interested in evaluating the propagator

\[
G(\bar{\xi}''', \xi'; t) = \langle \xi''|e^{-i\hat{H}t}|\xi' \rangle.
\] (29)

Inserting Eq.(27) and using Eq.(28) repeatedly, we obtain the following expression

\[
G(\bar{\xi}''', \xi'; t) = \int D\mu \exp \left\{ 2J \log(1 + \bar{\xi}'''\xi'(t_f)) + i \int_{t_i}^{t_f} dt \left[ i \frac{2J\bar{\xi}'\xi'}{1 + |\xi|^2} - H(\bar{\xi}, \xi) \right] \right\}
\] (30)

where \( H(\bar{\xi}, \xi) = \langle \xi|\hat{H}(\bar{\xi}, \xi)|\xi \rangle/\langle \xi|\xi \rangle \) is the classical Hamiltonian given by the Eq.(13). The boundary conditions in the path integral is given by \( \xi(t_i) = \xi' \) and \( \bar{\xi}(t_f) = \bar{\xi}''' \). We introduced \( \xi(t_f) \) which is only a superfluous variable because the result of path integral Eq.(30) does not depend on this variable. It depends only on \( \bar{\xi}''' \) and \( \xi' \).

The equations of motion

\[
\begin{align*}
    i\dot{\xi}_m &= g_{mn}^* \frac{\partial H(\bar{\xi}, \xi)}{\partial \xi_n^*}, & i\dot{\xi}_m^* &= -g_{mn} \frac{\partial H(\bar{\xi}, \xi)}{\partial \xi_n},
\end{align*}
\] (31)

are those of collections of \( N \) harmonic oscillator although the Hamiltonian Eqs.(13) and (12) appears to be highly nonlinear:

\[
\begin{align*}
    \dot{\xi}_m^* - i\omega_m \xi_m^* &= 0, & \dot{\xi}_m + i\omega_m \xi_m &= 0.
\end{align*}
\] (32)

Here \( \omega_m \) is given by Eq.(23). The solutions are given by

\[
\begin{align*}
    \xi_m^*(t) &= \xi_m^{*\prime} e^{i\omega_m(t-t')} & \xi_m(t) &= \xi_m' e^{-i\omega_m(t-t')}
\end{align*}
\] (33)

Using the above fact, we can evaluate the propagator by the semiclassical approximation method \([21]\) and the result agrees \([22]\) with the exact propagator which is obtained by solving the time-dependent Schrödinger equation as is adopted in this paper.

In order to have explicit operator form for the Hamiltonian to set up the Schrödinger equation, we resort to geometric quantization \([23]\). In the geometric quantization of classical
phase space $\mathcal{M} = CP(N)$ with symplectic structure $\Omega$, we quantize classical observables $F \equiv F(Q^a)$ which are functions of only $Q^a$'s satisfying the Poisson-Lie algebra Eq.(34). The prequantum operator corresponding $Q^a$ is given by

$$\hat{Q}^a = -iX_a - X_a|\Theta + Q^a$$  \hspace{1cm} (34)$$

where $\Theta$ is the canonical one form $\Omega = d\Theta$ given by

$$\Theta = iJ\bar{\xi}d\xi - d\bar{\xi}\xi$$  \hspace{1cm} (35)$$

Since our Hamiltonian Eq.(13) is linear in $Q^a$'s, there is no normal ordering ambiguity. Also the polarization is chosen such that $G(\bar{\xi}'', \xi'; t)$ is a function of $\bar{\xi}''$ but not of $\xi''$. Hence we get the following time-dependent Schrödinger equation for the propagator:

$$i\frac{\partial}{\partial t}G(\bar{\xi}'', \xi'; t) = \sum_{m=1}^{N} \frac{\mu_m}{\sqrt{2m(m+1)}} \left[ (m+1)\xi''_m \frac{\partial}{\partial \xi''_m} + \left( \sum_{k=m+1}^{N} \xi''_k \frac{\partial}{\partial \xi''_k} \right) - 2J \right] G(\bar{\xi}'', \xi'; t)$$  \hspace{1cm} (36)$$

We note that the above equation is invariant under the gauge transformation of the canonical one form by $\Theta \rightarrow \Theta + d\Lambda(\bar{\xi}, \xi)$. The boundary condition is given by

$$G(\bar{\xi}'', \xi'; t)\big|_{t\rightarrow 0} = (1 + \bar{\xi}''\xi')^{2J}$$  \hspace{1cm} (37)$$

It is remarkable that the solution can be obtained in a closed simple expression by

$$G(\bar{\xi}'', \xi'; t) = \left( 1 + \sum_{m=1}^{N} \xi''_m \xi'_m \exp\{-i\omega_m t\} \right)^{2J} \exp\{iJ\sum_{m=1}^{N} \mu_m/\sqrt{m(m+1)/2t}\}.$$  \hspace{1cm} (38)$$

The above results reduces to the exact propagator and reproduces the Weyl character formula for $SU(2)$ case [24,26].

In conclusion, we presented a classical integrable model of $SU(N)$ isospin in interaction with external constant magnetic field neglecting the spatial dependence completely and solved it exactly both classically and quantum mechanically. Especially, we obtained the exact expression for the quantum mechanical propagator by solving the time-dependent Schrödinger equation set up by the method of geometric quantization. This method could
be applied to other integrable models such as the one in which the Hamiltonian is a sum of quadratic functions of $Q^m$ of Eq. (12) and other coadjoint orbits [17] of Lie group including the non-compact case. These and related issues will be reported elsewhere [22].

ACKNOWLEDGMENTS

This work is supported by the KOSEF through C.T.P. at S.N.U. and Ministry of Education through the Research Institute of Basic Science.
REFERENCES

∗ E-mail address:ploh@yurim.skku.ac.kr

[1] V. I. Arnold, Mathematical Methods of Classical Mechanics (Springer-Verlag, Berlin, 1978).

[2] R. Abraham and J. E. Marsden, Foundations of Mechanics (Addison Wesley, 1978).

[3] S. K. Wong, Nuovo Cimento A 65, 689 (1970).

[4] E. Verlinde, in Modern Quantum Field Theory (World Scientific, Singapore, 1991).

[5] T. Lee and P. Oh, Phys. Rev. Lett. 72, 1141 (1994); Ann. Phys. (N. Y.) in press.

[6] T. Lee and P. Oh, Phys. Lett. B 319, 497 (1993).

[7] D. Bak, R. Jackiw and S.-Y. Pi, Preprint MIT-CTP-# 2276; W. T. Kim and C. Lee, SNUTP 94-14.

[8] M. Kim and P. Oh, submitted to J. Math. Phys.

[9] M. Stone, ed., Quantum Hall Effect (World Scientific, Singapore, 1992).

[10] P. Griffiths and J. Harris, Principles of Algebraic Geometry (Wiley, New York, 1978).

[11] H. Georgi, Lie Algebra in Particle Physics (Benjamin/Cummings, 1982) p.115.

[12] H. B. Nielsen and D. Rohrlich, Nucl. Phys. B 299, 471 (1988).

[13] A. Alekseev, L.D. Faddeev and S. L. Shatashvili, J. Geom. Phys. 3, 1 (1989).

[14] J. R. Klauder and B. S. Skagerstam, Coherent States: Applications in Physics and Mathematical Physics (World Scientific, Singapore, 1985).

[15] A. Perelomov, Generalized Coherent States and Their Applications (Springer-Verlag, Berlin, 1986).

[16] W. M. Zhang, D. H. Feng and R. Gilmore, Rev. Mod. Phys. 62, 867 (1990).
[17] A. A. Kirillov, *Elements of the Theory of Representations* (Springer-Verlag, 1976).

[18] J. R. Klauder, Phys. Rev. D 19, 2349 (1979).

[19] L. D. Faddeev and A. A. Slavnov, *Gauge Fields: Introduction to Quantum Theory* (Benjamin/Cummings Pub., MA, 1980).

[20] L. S. Brown, *Quantum Field Theory* (Cambridge Univ. Press, 1992).

[21] J. Zinn-Justin, *Quantum Field Theory and Critical Phenomena* (Clarendon Press, Oxford, 1989).

[22] P. Oh, in preparation.

[23] N. Woodhouse, *Geometric Quantization* (Claredon Press, 1980).

[24] M. Stone, Nucl. Phys. B 314, 557 (1989).

[25] O. Alvarez, I. M. Singer and P. Windey, Nucl. Phys. B 337, 467 (1990).

[26] E. Keski-Vakkuri, A. J. Niemi, G. Semenoff and O. Tirkkonen, Phys. Rev. D 44, 3899 (1991).