ANALYTIC SOLUTIONS FOR THE TWO-PHASE NAVIER-STOKES EQUATIONS WITH SURFACE TENSION AND GRAVITY

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Dedicated to Herbert Amann on the occasion of his 70th birthday

Abstract. We consider the motion of two superposed immiscible, viscous, incompressible, capillary fluids that are separated by a sharp interface which needs to be determined as part of the problem. Allowing for gravity to act on the fluids, we prove local well-posedness of the problem. In particular, we obtain well-posedness for the case where the heavy fluid lies on top of the light one, that is, for the case where the Rayleigh-Taylor instability is present. Additionally we show that solutions become real analytic instantaneously.

1. Introduction and Main Results

We consider a free boundary problem describing the motion of two immiscible, viscous, incompressible capillary fluids, fluid_1 and fluid_2, occupying the regions

\[ \Omega_i(t) = \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : (-1)^i(y - h(t,x)) > 0, \ t \geq 0 \}, \quad i = 1, 2. \]

The fluids, thus, are separated by the interface

\[ \Gamma(t) := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} : y = h(t,x) : x \in \mathbb{R}^n, \ t \geq 0 \}, \]

called the free boundary, which needs to be determined as part of the problem. The motion of the fluids is governed by the incompressible Navier-Stokes equations where surface tension on the free boundary is included. In addition, we also allow for gravity to act on the fluids. The governing equations then are given by the system

\[
\begin{cases}
\rho(\partial_t u + (u|\nabla)u) - \mu \Delta u + \nabla q = 0 & \text{in } \Omega(t) \\
\text{div } u = 0 & \text{in } \Omega(t) \\
-\|S(u,q)\nu\| = \sigma \kappa \nu + [\rho]_\gamma a y & \text{on } \Gamma(t) \\
[u] = 0 & \text{on } \Gamma(t) \\
V = (u|\nu) & \text{on } \Gamma(t) \\
u(0) = u_0 & \text{in } \Omega_0 \\
\Gamma(0) = \Gamma_0.
\end{cases}
\]

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Here $\rho$ and $\mu$ are given by
\[
\rho = \rho_1 \chi \Omega_1(t) + \rho_2 \chi \Omega_2(t), \quad \mu = \mu_1 \chi \Omega_1(t) + \mu_2 \chi \Omega_2(t),
\]
with $\chi$ the indicator function, where the constants $\rho_i$ and $\mu_i$ denote the densities and viscosities of the respective fluids. The constant $\sigma > 0$ denotes the surface tension, and $\gamma_a$ is the acceleration of gravity. Moreover, $S(u, q)$ is the stress tensor defined by
\[
S(u, q) = \mu_i (\nabla u + (\nabla u)^T) - qI \quad \text{in} \quad \Omega_i(t),
\]
where $q = \tilde{q} + \rho \gamma_a y$ denotes the modified pressure incorporating the potential of the gravity force, and
\[
[v] = (v_{\Omega_2(t)} - v_{\Omega_1(t)})|_{\Gamma(t)}
\]
denotes the jump of the quantity $v$, defined on the respective domains $\Omega_i(t)$, across the interface $\Gamma(t)$. Finally, $\kappa = \kappa(t, \cdot)$ is the mean curvature of the free boundary $\Gamma(t)$, $\nu = \nu(t, \cdot)$ is the unit normal field on $\Gamma(t)$, and $V = V(t, \cdot)$ is the normal velocity of $\Gamma(t)$. Here we use the convention that $\nu(t, \cdot)$ points from $\Omega_1(t)$ into $\Omega_2(t)$, and that $\kappa(x, t)$ is negative when $\Omega_1(t)$ is convex in a neighborhood of $x \in \Gamma(t)$.

Given are the initial position $\Gamma_0 = \text{graph} (h_0)$ of the interface, and the initial velocity
\[
u_0 : \Omega_0 = \Omega_1(0) \cup \Omega_2(0).
\]
The unknowns are the velocity field $u(t, \cdot) : \Omega(t) \to \mathbb{R}^{n+1}$, the pressure field $q(t, \cdot) : \Omega(t) \to \mathbb{R}$, and the free boundary $\Gamma(t)$, where $\Omega(t) := \Omega_1(t) \cup \Omega_2(t)$.

Our main result shows that problem (1.1) admits a unique local smooth solution, provided that $\|\nabla h_0\|_{\infty} := \sup_{x \in \mathbb{R}^n} |\nabla h_0(x)|$ is sufficiently small.

**Theorem 1.1.** Let $p > n + 3$. Then given $\beta > 0$, there exists $\eta = \eta(\beta) > 0$ such that for all initial values
\[
(u_0, h_0) \in W^{2-2/p}_p(\Omega_0, \mathbb{R}^{n+1}) \times W^{3-2/p}_p(\mathbb{R}^n), \quad [u_0] = 0,
\]
satisfying the compatibility conditions
\[
[\mu D(u_0) v_0 - \mu(v_0) D(u_0)v_0] = 0, \quad \text{div} u_0 = 0 \quad \text{on} \quad \Omega_0,
\]
with $D(u_0) := (\nabla u_0 + (\nabla u_0)^T)$, and the smallness-boundedness condition
\[
\|\nabla h_0\|_{\infty} \leq \eta, \quad \|u_0\|_{\infty} \leq \beta,
\]
there is $t_0 = t_0(u_0, h_0) > 0$ such that problem (1.1) admits a classical solution $(u, q, \Gamma)$ on $(0, t_0)$. The solution is unique in the function class described in Theorem 4.2. In addition, $\Gamma(t)$ is a graph over $\mathbb{R}^n$ given by a function $h(t)$ and $M = \bigcup_{t \in (0, t_0)} \{t \times \Gamma(t)\}$ is a real analytic manifold, and with
\[
\mathcal{O} := \{(t, x, y) : t \in (0, t_0), \quad x \in \mathbb{R}^n, \quad y \neq h(t, x)\},
\]
the function $(u, q) : \mathcal{O} \to \mathbb{R}^{n+2}$ is real analytic.

**Remarks 1.2.** (a) More precise statements for the transformed problem will be given in Section 4. Due to the restriction $p > n + 3$ we obtain
\[
h \in C(J; \text{BUC}^2(\mathbb{R}^n)) \cap C^1(J; \text{BUC}^1(\mathbb{R}^n)),
\]
where $J = [0, t_0]$. In particular, the normal of $\Omega_1(t)$, the normal velocity of $\Gamma(t)$, and the mean curvature of $\Gamma(t)$ are well-defined and continuous, so that (1.1) makes sense pointwise. For $u$ we obtain
\[
u \in \text{BUC}(J \times \mathbb{R}^{n+1}, \mathbb{R}^{n+1}), \quad \nabla u \in \text{BUC}(\mathcal{O}, \mathbb{R}^{(n+1)^2}).
Also interesting is the fact that the surface pressure jump is analytic on $M$ as well.

(b) It is possible to relax the assumption $p > n + 3$. In fact, $p > (n + 3)/2$ turns out to be sufficient. In order to keep the arguments simple, we impose here the stronger condition $p > n + 3$.

(c) It is well-known that the situation where gravity is acting on two superposed immiscible fluids - with the heavier fluid lying above a fluid of lesser density - leads to an instability, the Rayleigh-Taylor instability. In this case, small disturbances of the equilibrium situation $(u, h) = (0, 0)$ can cause instabilities, where the heavy fluid moves down under the influence of gravity, and the light material is displaced upwards, leading to vortices. Our results show that problem (1.1) is also well-posed in this case, provided $\|\nabla h_0\|_\infty$ is small enough, yielding smooth solutions for a short time. In the forthcoming publication [29] we will give a rigorous proof showing that the equilibrium solution $(u, h) = (0, 0)$ is $L_p$-unstable. To the best of our knowledge these are the first rigorous results concerning the Navier-Stokes equations subject to the Rayleigh-Taylor instability.

(d) If $\gamma_a = 0$ then it is shown in [28] that problem (1.1) admits a solution with the same regularity properties on an arbitrary fixed time interval $[0, t_0]$, provided that $\|u_0\|_{W^{2-2/p}_p(\Omega_0)}$ and $\|h_0\|_{W^{3-2/p}_p(\mathbb{R}^n)}$ are sufficiently small (depending on $t_0$).

(e) We point out that in Theorem 1.1 we only need a smallness condition on the sup-norm of $\nabla h_0$ (relative to the vertical component of the velocity). In case of a more general geometry, this condition can always be achieved by a judicious choice of a reference manifold.

The motion of a layer of viscous, incompressible fluid in an ocean of infinite extent, bounded below by a solid surface and above by a free surface which includes the effects of surface tension and gravity (in which case $\Omega_0$ is a strip, bounded above by $\Gamma_0$ and below by a fixed surface $\Gamma_b$) has been considered by Allain [1], Beale [7], Beale and Nishida [8], Tani [35], by Tani and Tanaka [36], and by Shibata and Shimizu [32]. If the initial state and the initial velocity are close to equilibrium, global existence of solutions is proved in [7] for $\sigma > 0$, and in [36] for $\sigma \geq 0$, and the asymptotic decay rate for $t \to \infty$ is studied in [8]. We also refer to [9], where in addition the presence of a surfactant on the free boundary and in one of the bulk phases is considered.

In case that $\Omega_1(t)$ is a bounded domain, $\gamma_a = 0$, and $\Omega_2(t) = \emptyset$, one obtains the one-phase Navier-Stokes equations with surface tension, describing the motion of an isolated volume of fluid. For an overview of the existing literature in this case we refer to the recent publications [28, 31, 32, 33]. Results concerning the two-phase problem (1.1) with $\gamma_a = 0$ in the 3D-case are obtained in [11, 12, 13, 34]. In more detail, Densiova [12] establishes existence and uniqueness of solutions (of the transformed problem in Lagrangian coordinates) with $v \in W^{s, s/2}_2$ for $s \in (5/2, 3)$ in case that one of the domains is bounded. Tanaka [34] considers the two-phase Navier-Stokes equations with thermo-capillary convection in bounded domains, and he obtains existence and uniqueness of solutions with $(v, \theta) \in W^{s, s/2}_2$ for $s \in (7/2, 4)$, with $\theta$ denoting the temperature.

In order to prove our main result we transform problem (1.1) into a problem on a fixed domain. The transformation is expressed in terms of the unknown height
function \( h \) describing the free boundary. Our analysis proceeds with establishing maximal regularity results for an associated linear problem, relying on the powerful theory of maximal regularity, in particular on the \( H^\infty \)-calculus for sectorial operators, the Dore-Venni theorem, and the Kalton-Weis theorem, see for instance \( [2, 11, 16, 22, 23, 26, 30] \).

Based on the linear estimates we can solve the nonlinear problem by the contraction mapping principle. Analyticity of solutions is obtained as in \( [28] \) by the implicit function theorem in conjunction with a scaling argument, relying on an idea that goes back to Angenent \( [4, 5] \) and Masuda \( [24] \); see also \( [17, 18, 20] \).

The plan for this paper is as follows. Section 2 contains the transformation of the problem to a half-space and the determination of the proper underlying linear problem. In Section 3 we analyze this linearization and prove the crucial maximal regularity result in an \( L_p \)-setting. Section 4 is then devoted to the nonlinear problem and contains the proof of our main result. Finally we collect and prove in an appendix some of the technical results used in order to estimate the nonlinear terms.

2. The transformed problem

The nonlinear problem (1.1) can be transformed to a problem on a fixed domain by means of the transformations

\[
 v(t, x, y) := (u_1, \ldots, u_n)(t, x, y + h(t, x)), \\
 w(t, x, y) := u_{n+1}(t, x, y + h(t, x)), \\
 \pi(t, x, y) := q(t, x, y + h(t, x)),
\]

where \( t \in J = [0, a], x \in \mathbb{R}^n, y \in \mathbb{R}, y \neq 0 \). With a slight abuse of notation we will in the sequel denote the transformed velocity again by \( u \), that is, we set \( u = (v, w) \).

With this notation we obtain the transformed problem

\[
 \begin{array}{l}
 \rho \partial_t u - \mu \Delta u + \nabla \pi = F(u, \pi, h) \quad \text{in } \mathbb{R}^{n+1} \\
 \text{div } u = F_d(u, h) \quad \text{in } \mathbb{R}^{n+1} \\
 - [\mu \partial_y v] - [\mu \nabla_x w] = G_v(u, \|\pi\|, h) \quad \text{on } \mathbb{R}^n \\
 -2[\mu \partial_y w] + [\pi] - \sigma \Delta h - [\rho] \gamma \alpha h = G_w(u, h) \quad \text{on } \mathbb{R}^n \\
 [u] = 0 \quad \text{on } \mathbb{R}^n \\
 \partial_t h - \gamma w = - (\gamma v | \nabla h) \quad \text{on } \mathbb{R}^n \\
 u(0) = u_0, \ h(0) = h_0,
\end{array}
\] (2.1)

for \( t > 0 \), where \( \mathbb{R}^{n+1} = \{ (x, y) \in \mathbb{R}^n \times \mathbb{R} : y \neq 0 \} \).

The nonlinear functions have been computed in \( [28] \) and are given by:

\[
 F_v(v, w, \pi, h) = \mu \left( -2(\nabla h | \nabla_x) \partial_y v + |\nabla h|^2 \partial_y^2 v - \Delta h \partial_y v \right) + \partial_y \pi \nabla h \\
 + \rho \left( -(v | \nabla_x) v + (\nabla h | v) \partial_y v - w \partial_y v \right) + \rho \partial_t h \partial_y v,
\]

\[
 F_w(v, w, h) = \mu \left( -2(\nabla h | \nabla_x) \partial_y w + |\nabla h|^2 \partial_y^2 w - \Delta h \partial_y w \right) \\
 + \rho \left( -(v | \nabla_x) v + (\nabla h | v) \partial_y w - w \partial_y w \right) + \rho \partial_t h \partial_y w,
\]

\[
 F_d(v, h) = (\nabla h | \partial_y v)
\] (2.2)

for \( t > 0 \).
and
\[
G_v(v, w, [\pi], h) = -[\mu(\nabla_v + (\nabla_v)^T)]\nabla h + |\nabla h|^2[\mu\partial_y v] + (\nabla h)[\mu\partial_y v])\nabla h
- [\mu\partial_y w]\nabla h + (\{\pi\} - \sigma(\Delta h - G_\kappa(h)))\nabla h, \tag{2.3}
\]
\[
G_w(v, w, h) = -(\nabla h)[\mu\nabla x w]) - (\nabla h)[\mu\partial_y v]) + |\nabla h|^2[\mu\partial_y w] - \sigma G_\kappa(h)
\]
with
\[
G_\kappa(h) = \frac{|\nabla h|^2\Delta h}{(1 + \sqrt{1 + |\nabla h|^2})\sqrt{1 + |\nabla h|^2}} + \frac{(\nabla h)\nabla^2 h\nabla h}{(1 + |\nabla h|^2)^{3/2}}, \tag{2.4}
\]
where \(\nabla^2 h\) denotes the Hessian matrix of all second order derivatives of \(h\).

Before studying solvability results for problem (2.1) let us first introduce suitable function spaces. Let \(\Omega \subseteq \mathbb{R}^m\) be open and \(X\) be an arbitrary Banach space. By \(L_p(\Omega; X)\) and \(H^s_p(\Omega; X)\), for \(1 \leq p < \infty\), \(s \in \mathbb{R}\), we denote the \(X\)-valued Lebesgue and the Bessel potential spaces of order \(s\), respectively. We will also frequently make use of the fractional Sobolev-Slobodeckij spaces \(W^s_p(\Omega; X)\), \(1 \leq p < \infty\), \(s \in \mathbb{R}\setminus\mathbb{Z}\), with norm
\[
\|g\|_{W^s_p(\Omega; X)} = \|g\|_{W^s_p(\Omega; X)} + \sum_{|\alpha| = [s]} \left( \int_\Omega \int_\Omega \frac{||\partial^\alpha g(x) - \partial^\alpha g(y)||^p}{|x-y|^{m+[s]-|\alpha|}p} \, dx \, dy \right)^{1/p}, \tag{2.5}
\]
where \([s]\) denotes the largest integer smaller than \(s\). Let \(a \in (0, \infty)\) and \(J = [0, a]\). We set
\[
oW^s_p(J; X) := \begin{cases} \{ g \in W^s_p(J; X) : g(0) = g'(0) = \ldots = g^{(k)}(0) = 0 \}, & \text{if } k + \frac{1}{p} < s < k + 1 + \frac{1}{p}, \, k \in \mathbb{N} \cup \{0\}, \\ W^s_p(J; X), & \text{if } s < \frac{1}{p}. \end{cases}
\]
The spaces \(oW^s_p(J; X)\) are defined analogously. Here we remind that \(H^s_p(\Omega; X) = W^s_p(\Omega; X)\) for \(k \in \mathbb{Z}\) and \(1 < p < \infty\), and that \(W^s_p(\Omega; X) = B^s_{pp}\) for \(s \in \mathbb{R}\setminus\mathbb{Z}\).

For \(\Omega \subseteq \mathbb{R}^m\) open and \(1 \leq p < \infty\), the homogeneous Sobolev spaces \(\dot{H}_p^1(\Omega)\) of order 1 are defined as
\[
\dot{H}_p^1(\Omega) := \{ g \in L^1_{1,loc}(\Omega) : \|\nabla g\|_{L_p(\Omega)} < \infty \}, \| \cdot \|_{\dot{H}_p^1(\Omega)} \tag{2.6}
\]
\[
\|g\|_{\dot{H}_p^1(\Omega)} := \left( \sum_{j=1}^m \|\partial_j g\|_{L_p(\Omega)}^p \right)^{1/p}.
\]
Then \(\dot{H}_p^1(\Omega)\) is a Banach space, provided we factor out the constant functions and equip the resulting space with the corresponding quotient norm, see for instance [21] Lemma II.5.1. We will in the sequel always consider the quotient space topology without change of notation. In case that \(\Omega\) is locally Lipschitz, it is known that \(\dot{H}_p^1(\Omega) \subset H^1_{\text{loc}}(\Omega)\), see [21] Remark II.5.1, and consequently, any function in \(\dot{H}_p^1(\Omega)\) has a well-defined trace on \(\partial\Omega\).

For \(s \in \mathbb{R}\) and \(1 < p < \infty\) we also consider the homogeneous Bessel-potential spaces \(\dot{H}_p^s(\mathbb{R}^n)\) of order \(s\), defined by
\[
\dot{H}_p^s(\mathbb{R}^n) := \{ g \in S'(\mathbb{R}^n) : \dot{I}^s g \in L_p(\mathbb{R}^n) \}, \| \cdot \|_{\dot{H}_p^s(\mathbb{R}^n)} \tag{2.7}
\]
\[
\|g\|_{\dot{H}_p^s(\mathbb{R}^n)} := \|\dot{I}^s g\|_{L_p(\mathbb{R}^n)}.
\]
where \( S'(\mathbb{R}^n) \) denotes the space of all tempered distributions, and \( \hat{I}^s \) is the Riesz potential given by

\[
\hat{I}^s g := (-\Delta)^{s/2} g := \mathcal{F}^{-1}(|\xi|^s \mathcal{F} g), \quad g \in S'(\mathbb{R}^n).
\]

By factoring out all polynomials, \( \hat{H}^s_p(\mathbb{R}^n) \) becomes a Banach space with the natural quotient norm. For \( s \in \mathbb{R} \setminus \mathbb{Z} \), the homogeneous Sobolev-Slobodecki\j spaces \( \hat{W}^s_p(\mathbb{R}^n) \) of fractional order can be obtained by real interpolation as

\[
\hat{W}^s_p(\mathbb{R}^n) := (\hat{H}^k_p(\mathbb{R}^n), \hat{H}^{k+1}(\mathbb{R}^n))_{s-k, p}, \quad k < s < k + 1,
\]

where \( (\cdot, \cdot)_{s, p} \) is the real interpolation method. It follows that

\[
\hat{I}^s \in \text{Isom}(\hat{H}^{t+s}_p(\mathbb{R}^n), \hat{H}^t_p(\mathbb{R}^n)) \cap \text{Isom}(\hat{W}^{t+s}_p(\mathbb{R}^n), \hat{W}^t_p(\mathbb{R}^n)), \quad s, t \in \mathbb{R}, \quad (2.8)
\]

with \( \hat{W}^s_p = \hat{H}^s_p \) for \( k \in \mathbb{Z} \). We refer to [6, Section 6.3] and [37, Section 5] for more information on homogeneous functions spaces. In particular, it follows from parts (ii) and (iii) in [37, Theorem 5.2.3.1] that the definitions \((2.6)\) and \((2.7)\) are consistent if \( \Omega = \mathbb{R}^n \), \( s = 1 \), and \( 1 < p < \infty \). We note in passing that

\[
\left( \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|g(x) - g(y)|^p}{|x - y|^{n+sp}} \, dx \, dy \right)^{1/p}, \quad \left( \int_{0}^{\infty} t^{(1-s)p} \left\| P(t)g \right\|_{L^p(\mathbb{R}^n)}^p \frac{dt}{t} \right)^{1/p} \quad (2.9)
\]

define equivalent norms on \( \hat{W}^s_p(\mathbb{R}^n) \) for \( 0 < s < 1 \), where \( P(\cdot) \) denotes the Poisson semigroup, see [37, Theorem 5.2.3.2 and Remark 5.2.3.4]. Moreover,

\[
\gamma_\pm \in \mathcal{L}(\hat{W}^1_p(\mathbb{R}^{n+1}), \hat{W}^{1-1/p}_p(\mathbb{R}^n)), \quad (2.10)
\]

where \( \gamma_\pm \) denotes the trace operators, see for instance [21, Theorem II.8.2].

### 3. The Linearized Two-Phase Stokes Problem with Free boundary

It turns out that, unfortunately, the nonlinear term \( (\gamma v \nabla h) \) occurring in \((2.1)\) cannot be made small in the norm of \( F_4(a) \), defined below in \((4.4)\), by merely taking \( \|\nabla h\|_\infty \) small. This can, however, be achieved for the modified term \( (b - \gamma v \nabla h) \), provided \( b \) is properly chosen so that \( b(0) = \gamma v_0 \). As a consequence, we now need to consider the modified linear problem

\[
\begin{cases}
\rho \partial_t u - \mu \Delta u + \nabla \pi = f & \text{in } \mathbb{R}^{n+1} \\
\text{div } u = f_d & \text{in } \mathbb{R}^{n+1} \\
- [\mu \partial_y v] - [\mu \nabla_x w] = g_v & \text{on } \mathbb{R}^n \\
-2[\mu \partial_y w] + [\pi] = g_w + \sigma \Delta h + [\rho] \gamma_a h & \text{on } \mathbb{R}^n \\
[u] = 0 & \text{on } \mathbb{R}^n \\
\partial_t h - \gamma w + (b(t, x) \nabla) h = g_h & \text{on } \mathbb{R}^n \\
u(0) = u_0, \ h(0) = h_0.
\end{cases} \quad (3.1)
\]

Here we mention that the simpler case where \( b = 0 \) and \( \gamma_a = 0 \) was studied in [28, Theorem 5.1]. We obtain the following maximal regularity result.
Theorem 3.1. Let $p > n + 3$ be fixed, and assume that $\rho_j$ and $\mu_j$ are positive constants for $j = 1, 2$, and set $J = [0, a]$. Suppose

$$b_0 \in \mathbb{R}^n, \quad b_1 \in W_p^{1-2/p}(J; L_p(\mathbb{R}^n, \mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n, \mathbb{R}^n)),$$

and set $b(\cdot) = b_0 + b_1(\cdot)$. Then the Stokes problem with free boundary (3.1) admits a unique solution $(u, \pi, h)$ with regularity

$$u \in H_p^1(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; \dot{H}_p^2(\mathbb{R}^n, \mathbb{R}^{n+1})),
\pi \in L_p(J; H_p^1(\mathbb{R}^{n+1})),
[\pi] \in W_p^{1-2/p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)),
h \in W_p^{2-1/p}(J; L_p(\mathbb{R}^n)) \cap H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n))$$

if and only if the data $(f, f_d, g, g_n, u_0, h_0)$ satisfy the following regularity and compatibility conditions:

(a) $f \in L_p(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1}))$,
(b) $f_d \in H_p^1(J; H_p^1(\mathbb{R}^n)) \cap L_p(J; H_p^1(\mathbb{R}^n))$,
(c) $g = (g_v, g_w) \in W_p^{1-2/p}(J; L_p(\mathbb{R}^n, \mathbb{R}^{n+1})) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n, \mathbb{R}^{n+1}))$,
(d) $g_h \in W_p^{1-2/p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n))$, 
(e) $u_0 \in W_p^{2-2/p}(\mathbb{R}^n)$, $h_0 \in W_p^{3-2/p}(\mathbb{R}^n)$,
(f) $\text{div } u_0 = f_d(0) \text{ in } \mathbb{R}^{n+1}$ and $[u_0] = 0$ on $\mathbb{R}^n$ if $p > 3/2$,
(g) $-\mu \partial_y v_0 - \mu \nabla w_0 = g_0(0)$ on $\mathbb{R}^n$ if $p > 3$.

The solution map $[(f, f_d, g, g_n, u_0, h_0) \mapsto (u, \pi, h)]$ is continuous between the corresponding spaces.

If $b_1 \equiv 0$ then the result is true for all $p \in (1, \infty)$, $p \neq 3/2, 3$.

Proof. (i) Since $\mathcal{F}_4(a)$, defined by

$$\mathcal{F}_4(a) := W_p^{1-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{2-1/p}(\mathbb{R}^n)),$$

is a multiplication algebra for $p > n + 3$, the operator $[h \mapsto (b \nabla) h]$ maps the space

$$\mathcal{E}_4(a) := W_p^{2-1/2p}(J; L_p(\mathbb{R}^n)) \cap H_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n))$$

continuously into $\mathcal{F}_4(a)$ with bound $\|b_0| + C_\mu \|b_1\|_{\mathcal{F}_4(a)}$, see Lemma 5.5 (a).

As in the proof of [28, Theorem 5.1] it suffices to consider the reduced problem

$$\rho \partial_t u - \mu \Delta u + \nabla \pi = 0 \quad \text{in } \mathbb{R}^{n+1}
\text{div } u = 0 \quad \text{in } \mathbb{R}^{n+1}
-[[\mu \partial_y v] - [[\mu \nabla_x w] = 0 \quad \text{on } \mathbb{R}^n
-2[[\mu \partial_y w] + [[\pi] = \sigma \Delta h + [[\rho] \gamma_n h \quad \text{on } \mathbb{R}^n
[\mu] = 0 \quad \text{on } \mathbb{R}^n
\partial_t h - \gamma w + (b(t, x) \nabla) h = \tilde{g}_h \quad \text{on } \mathbb{R}^n
u(0) = 0, \quad h(0) = 0,$$

where the function $\tilde{g}_h \in _0 \mathcal{F}_4(a)$ is defined in a similar way as in formula (5.5) in [28]. This can be accomplished by choosing $h_1 := h_{1,b} \in \mathcal{E}_4(a)$ such that

$$h_1(0) = h_0, \quad \partial_t h_1(0) = g_b(0) + \gamma w_0 - (b(0) \nabla) h_0,$$
and then setting \( \bar{g}_h := \bar{g}_{h, b} := g_h + \gamma w_1 - (b \nabla h_1) - \partial_h h_1 \), where \( w_1 \) has the same meaning as in step (i) of the proof of [28] Theorem 5.1.

(ii) We first consider the reduced problem (3.3) for the case where \( b \equiv b_0 \) is constant. The corresponding boundary symbol \( s_{b_0}(\lambda, \xi) \) is given by

\[
s_{b_0}(\lambda, \xi) = \lambda + (\sigma |\xi| - [\rho] \gamma_a / |\xi|) k(z) + i(b_0 |\xi|), \tag{3.4}
\]

where we use the same notation as in the proof of [28] Theorem 5.1. Here we remind that \( k \) has the following properties: \( k \) is holomorphic in \( \mathbb{C} \setminus \mathbb{R}_- \) and

\[
k(0) = \frac{1}{2(\mu_1 + \mu_2)}, \quad zk(z) \to \frac{1}{\rho_1 + \rho_2} \text{ for } |z| \to \infty, \tag{3.5}
\]

uniformly in \( z \in \Sigma_\theta \) for \( \theta \in [0, \pi) \) fixed. In particular there is a constant \( N = N(\theta) \) such that

\[
|k(z)| + |zk(z)| \leq N, \quad z \in \Sigma_\theta. \tag{3.6}
\]

In the following we fix \( \beta > 0 \). For further analysis it will be convenient to introduce the related extended symbol

\[
\tilde{s}(\lambda, \tau, \zeta) := \lambda + \sigma \tau k(z) + i \tau \zeta - [\rho] \gamma_a k(z)/\tau, \tag{3.7}
\]

where \( (\lambda, \tau) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \) with \( \eta \) sufficiently small, \( z := \lambda/\tau^2 \), and \( \zeta \in U_{\beta, \delta} \) with \( U_{\beta, \delta} := \{ \zeta \in \mathbb{C} : \text{Re} \zeta < \beta + 1, \text{Im} \zeta < \delta \} \) and \( \delta \in (0, 1) \). Clearly \( \tilde{s}(\lambda, |\xi|, (b_0 |\xi|/|\xi|)) = s_{b_0}(\lambda, \xi) \) for \( (\lambda, \xi) \in \Sigma_\eta \times \mathbb{R}^n \).

We are going to show that for every fixed \( \beta > 0 \) there are positive constants \( \lambda_0, \delta, \eta = \eta(\beta), \) and \( c_j = c_j(\beta, \lambda_0, \delta, \eta) \) such that

\[
c_0 ([|\lambda| + |\tau|] \leq |\tilde{s}(\lambda, \tau, \zeta)| \leq c_1 ([|\lambda| + |\tau|], \tag{3.8}
\]

for all \( (\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta, \delta} \) with \( |\lambda| \geq \lambda_0 \). The upper estimate is easy to obtain: fixing \( \theta \in (\pi/2, \pi) \) and \( \lambda_0 > 0 \), it follows from (3.6) and the identity

\[
k(z) = zk(z) \tau/\lambda \text{ that}
\]

\[
|\tilde{s}(\lambda, \tau, \zeta)| \leq |\lambda| + (\sigma N + (\beta + 2) + |[\rho]| \gamma_a N/\lambda_0)|\tau| \leq c_1 ([|\lambda| + |\tau|], \tag{3.9}
\]

for all \( (\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta, \delta} \), where \( |\lambda| \geq \lambda_0 \) and \( \eta \in (0, \eta_0) \) with \( \eta_0 := (\theta - \pi/2)/3 \).

In order to obtain a lower estimate we proceed as follows. Suppose first that \( \beta, \lambda_0 > 0 \) are fixed and \( \eta_0 \) as above. Then we obtain

\[
|\tilde{s}(\lambda, \tau, \zeta)| \geq |\lambda| - (\sigma N + (\beta + 2) + |[\rho]| \gamma_a N/\lambda_0)|\tau|
\]

\[
\geq (1/2)|\lambda| + (m/4)|\tau| = c_0(\beta, \lambda_0)|[|\lambda| + |\tau||,
\]

provided \( (\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta, \delta}, \eta \in (0, \eta_0) \), and \( |\lambda| \geq \lambda_0 \) as well as \( |\lambda| \geq m|\tau| \) with

\[
(m/4) \geq \sigma N + (\beta + 2) + |[\rho]| \gamma_a N/\lambda_0.
\]

Next we will derive an estimate from below in case that \( |\lambda| \leq M|\tau|^2 \) with \( M \) a positive constant. From (3.5) follows that there are constants \( H, L, R > 0 \), depending on \( M \), such that

\[
L \leq \text{Re}(\sigma k(z)) \leq R, \quad |\text{Im}(\sigma k(z))| \leq H, \tag{3.11}
\]

whenever \( (\lambda, \tau) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \), for \( \eta \in (0, \eta_0) \) and \( |\lambda| \leq M|\tau|^2 \), where \( z = \lambda/\tau^2 \).

By choosing \( \delta \) small enough we obtain from (3.11) and the definition of \( U_{\beta, \delta} \)

\[
0 < L - \delta \leq \text{Re}(\sigma k(z) + i\zeta) \leq R + \delta, \quad |\text{Im}(\sigma k(z) + i\zeta)| \leq H + (\beta + 1)
\]
provided \((\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_{\eta} \times U_{\beta, \delta}, \eta \in (0, \eta_0)\) and \(|\lambda| \leq M|\tau|^{2}\), where \(z = \lambda/\tau^{2}\).

By choosing \(\eta\) small enough we conclude that there is \(\alpha = \alpha(M, \beta, \delta, \eta) \in (0, \pi/2)\) such that

\[
\tau(\sigma k(z) + i\zeta) \in \Sigma_{\alpha}
\]

whenever \((\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_{\eta} \times U_{\beta, \delta}\) and \(|z| \leq M\) with \(z = \lambda/\tau^{2}\). We can additionally assume that \(\eta\) is chosen so that \(\psi := \pi/2 - \alpha - \eta > 0\). This implies

\[
|\tilde{s}(\lambda, \tau, \zeta)| \geq c(\psi) |\lambda| + |\tau| |\sigma k(z) + i\zeta| - |\tau||[\rho]| \gamma_{a} N/\lambda_{1}
\]

\[
\geq c(\psi) \min(1, |L - \delta|) |\lambda| + |\tau| - |\tau||[\rho]| \gamma_{a} N/\lambda_{1}
\]

\[
\geq c_{0}(M, \beta, \lambda_{1}) |\lambda| + |\tau|,
\]

provided \((\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_{\eta} \times U_{\beta, \delta}, \lambda \leq M|\tau|^{2}\) and \(|\lambda| \geq \lambda_{1}\), where \(\lambda_{1}\) is chosen big enough.

Noting that the curves \(|\lambda| = m|\tau|\) and \(|\lambda| = M|\tau|^{2}\) intersect at \((m/M, m^{2}/M)\) we obtain \((3.12)\) by choosing \(\lambda_{0} := \max(\lambda_{1}, m^{2}/M)\).

(iii) In the following, we fix \(\beta > 0\) and we assume that \(b_{0} \in \mathbb{R}^{n}\) with \(|b_{0}| \leq \beta\). Let then \(S_{b_{0}}\) be the operator corresponding to the symbol \(s_{b_{0}}\). It is clear that \(S_{b_{0}}\) is bounded from \(0E_{\alpha}(a)\) to \(0F_{\alpha_{0}}(a) = : X\) and it remains to prove that it is boundedly invertible. For this we use the \(H^{\infty}\)-calculus and similar arguments as in [21] Section 4 and [22] Section 5]. First we note that \(D_{n}\) admits an \(R\)-bounded \(H^{\infty}\)-calculus in \(X\) with angle \(0\); this follows from [14] Theorem 4.11]. Therefore by the estimates obtained in \((3.13)\), the operator family

\[
\{(\lambda + D_{n}^{1/2})\tilde{s}^{-1}(\lambda, D_{n}^{1/2}, \zeta) : (\lambda, \zeta) \in \Sigma_{\pi/2+\eta} \times U_{\beta, \delta}, \lambda \geq \lambda_{0}\}
\]

is \(R\)-bounded. Since \(G = \partial_{\lambda}\) is in \(H^{\infty}(X)\) with angle \(\pi/2\), the theorem of Kalton and Weis [22] Theorem 4.4] implies that the operator family

\[
\{(G + D_{n}^{1/2})\tilde{s}^{-1}(G, D_{n}^{1/2}, \zeta) : \zeta \in U_{\beta, \delta}\}
\]

is bounded and holomorphic on \(U_{\beta, \delta}\). Finally, we employ the Dunford calculus for the bounded linear operator \(R_{b_{0}} := (b_{0}|R)\), where \(R\) denotes the Riesz operator with symbol \(\xi/|\xi|\), \(\xi \in \mathbb{R}^{n}\). The operator \(R_{b_{0}}\) is bounded and its spectrum is \(\sigma(R_{b_{0}}) = [-|b_{0}|, |b_{0}|]\), as e.g. the Mikhlin theorem shows. Since the operator family

\[
\{(G + D_{n}^{1/2})\tilde{s}^{-1}(G, D_{n}^{1/2}, \zeta) : \zeta \in U_{\beta, \delta}\}
\]

is bounded and holomorphic in a neighborhood of \(\sigma(R_{b_{0}})\), the classical Dunford calculus shows that the operator

\[
(G + D_{n}^{1/2})\tilde{s}^{-1}(G, D_{n}^{1/2}, R_{b_{0}})
\]

is bounded in \(X\), uniformly for all \(b_{0} \in \mathbb{R}^{n}\) with \(|b_{0}| \leq \beta\). This shows that \(S_{b_{0}} : 0E_{\alpha}(a) \to 0F_{\alpha}(a)\) is boundedly invertible, uniformly for all \(b_{0} \in \mathbb{R}^{n}\) with \(|b_{0}| \leq \beta\).

We emphasize that the bound for the operator \(S_{b_{0}}^{-1} : 0F_{\alpha}(a) \to 0E_{\alpha}(a)\) depends only on the parameters \(\rho_{j}, \mu_{j}, \sigma, \gamma_{a}, p\) and \(\beta\), for \(|b_{0}| \leq \beta\).

(iv) By means of a perturbation argument the result for constant \(b\) can be extended to variable \(b = b_{0} + b_{1}(t, x)\). In fact, given \(\beta > 0\) there exists a number \(\eta > 0\) such that the solution operator \(S_{b_{0}}^{-1}\) exists and is boundedly uniformly, provided \(|b_{0}| \leq \beta\) and \(\|b_{1}\|_{\infty} + \|b_{1}\|_{\mathcal{M}(4)(a)} \leq 2\eta\). This follows easily from the estimate

\[
\|b_{1}(\nabla h)\|_{0F_{\alpha}(a)} \leq c_{0}(\|b_{1}\|_{\infty} + \|b_{1}\|_{\mathcal{M}(4)(a)})\|\tilde{h}\|_{0E_{\alpha}(a)}
\]

see Lemma [5.5](c).
(v) In the general case we use a localization technique, similar to [3] Section 9. For this purpose we first decompose $J$ into subintervals $J_k = [k\delta, (k + 1)\delta]$ of length $\delta > 0$ and solve the problem successively on these subintervals. Since $b \in BUC(J; C_0(\mathbb{R}^n, \mathbb{R}^n))$, given any $\eta > 0$ we may choose $\delta > 0$ and $\varepsilon > 0$ such that

$$|b(t, x) - b(s, y)| \leq \eta \quad \text{for all} \quad (t, x), (s, y) \in J \times \mathbb{R}^n$$

with $|t - s| \leq \delta$ and $|x - y|_\infty \leq \varepsilon$. Let $\{U_j := x_j + (\varepsilon/2)Q : j \in \mathbb{N}\}$ be an enumeration of the open covering \{$(z/2 + Q) : z \in \mathbb{Z}^n$\} of $\mathbb{R}^n$, where $Q = (-1, 1)^n$. Clearly,

$$(3.14) \quad \|b(t, x) - b(s, y)\| \leq \eta, \quad s, t \in J_k, \quad x, y \in U_j.$$ 

Let $\phi$ be a smooth cut-off function with support contained in $(\varepsilon/2)Q$ such that $\phi \equiv 1$ on $(\varepsilon/4)Q$. Define

$$\phi_j := (\tau_{x_j, \phi}) \left( \sum_{k \in \mathbb{N}} (\tau_{x_k, \phi})^2 \right)^{-1/2}, \quad j \in \mathbb{N},$$

where $(\tau_{x, \phi})(x) := \phi(x - x_j)$. Consequently, $\phi_j$ is a smooth cut-off function with $\text{supp}(\phi_j) \subset U_j$ and $\sum_j \phi_j \equiv 1$. For a function space $\mathfrak{F}(J; \mathbb{R}^n) \subset L_p(J; \mathbb{R}^n)$ we define

$$r(h_j) := \sum_j \phi_j h_j, \quad (h_j) \in \mathfrak{F}(J; \mathbb{R}^n),$$

$$r^c h := (\phi_j h), \quad h \in \mathfrak{F}(J; \mathbb{R}^n).$$

Similarly as in [3] Section 9 one shows that

$$r \in \mathcal{L}(\ell_p(\mathfrak{F}(J; \mathbb{R}^n)), \mathfrak{F}(J; \mathbb{R}^n)), \quad r^c \in \mathcal{L}(\mathfrak{F}(J; \mathbb{R}^n), \ell_p(\mathfrak{F}(J; \mathbb{R}^n))), \quad rr^c = I, \quad (3.15)$$

for $\mathfrak{F}(J; \mathbb{R}^n) \in \{\mathbb{F}_4(a), \mathbb{F}_4(a)\}$. Let $\theta$ be a smooth cut-off function with $\text{supp}(\theta) \subset (\varepsilon/2)Q$ such that $\theta \equiv 1$ on $\text{supp}(\phi)$ and let $\theta_j := \tau_{x_j, \theta}$. Define

$$b_{j, k}(t, x) := \theta_j (x) (b(t, x) - b(k\delta, x_j)), \quad (t, x) \in J \times \mathbb{R}^n.$$

It follows that

$$\|b_{j, k}\|_{BC(J_k \times \mathbb{R}^n)} + \|b_{j, k}\|_{\mathbb{F}_4(J_k)} \leq c_0 \eta, \quad k = 0, \ldots, m, \quad j \in \mathbb{N}, \quad (3.16)$$

provided $\delta$ is chosen small enough. Indeed, the estimates for $\|b_{j, k}\|_{BC(J_k, \mathbb{R}^n)}$ follow immediately from (3.14), while the estimates for $\|b_{j, k}\|_{\mathbb{F}_4(J_k)}$ can be shown by approximating $b$ by functions that have better time regularity and by carefully estimating the products $\|\theta_j (b - b(k\delta, x_j))\|_{\mathbb{F}_4(J_k)}$.

We now concentrate on the first interval $J_0 = [0, \delta]$. Let $L \in \mathcal{L}([0, \mathbb{E}_4(a), 0, \mathbb{E}_4(a)])$ denote the operator with symbol $\sigma \tau k(z)$, i.e.

$$L := (\sigma D_{\delta}^{1/2} - [\rho \gamma a D_n^{-1/2}]) k(GD_n^{-1}) := L_1 + L_2.$$ 

If follows from (3.14) and step (iii) that the operator

$$S_j := G + L + (b(0, x_j) + b_{j, 0} \nabla) : 0 \mathbb{E}_4(\delta) \to 0 \mathbb{F}_4(\delta)$$

is invertible. Moreover, there is a constant $C_0$, depending only on $\text{sup} \|b(0, x_j)\| -$ and therefore only on $\|b\|_{BC(J \times \mathbb{R}^n)}$ - such that $\|S_j^{-1}\|_{\mathcal{L}(0 \mathbb{F}_4(\delta), 0 \mathbb{E}_4(\delta))} \leq C_0, \quad j \in \mathbb{N}.$

(vi) Suppose that for a given $g \in 0 \mathbb{F}_4(\delta)$ we have a solution $h \in 0 \mathbb{E}_4(\delta)$ of

$$Gh + Lh + (b(\nabla) h = g.$$
Multiplying this equation by \( \phi_j \), using that \( b \partial^\alpha \phi_j = (b(0,x_j) + b_{j,0}) \partial^\alpha \phi_j \) and \( r^c = 1 \) this yields
\[
S_j \phi_j h - [L, \phi_j] h - (b |\nabla \phi_j|) h = (S_j - [L, \phi_j]) r - (b |\nabla \phi_j|) r^c h = r^c g,
\]
where \([.,.]\) denotes the commutator. We now interpret this equation as an equation in \( \ell_p(\alpha \mathcal{F}^4(\delta)) \). It follows from step (iv) that \( (S_j) \in \text{Isom}(\ell_p(\alpha \mathcal{F}^4(\delta)), \ell_p(\alpha \mathcal{F}^4(\delta))) \) and
\[
\|(S_j)^{-1}\|_{\ell_p(\alpha \mathcal{F}^4(\delta)), \ell_p(\alpha \mathcal{F}^4(\delta))} \leq C_0. \tag{3.17}
\]
We shall show below in step (vi) that the commutators satisfy
\[
((L, \phi_j) + (b |\nabla \phi_j|)) \in \mathcal{L}(\alpha \mathcal{F}^a(a), \ell_p(\alpha \mathcal{F}^4(a))). \tag{3.18}
\]
Assuming this property, it follows from (3.16) that
\[
\|((L, \phi_j) + (b |\nabla \phi_j|) r(h_j))\|_{\ell_p(\alpha \mathcal{F}^4(\delta))} \leq C\|h_j\|_{\ell_p(\alpha \mathcal{F}^4(\delta))} \leq C\delta^\alpha \|h_j\|_{\ell_p(\alpha \mathcal{F}^4(\delta))}
\]
for some \( \alpha \) depending only on \( p \) and \( n \). Therefore, choosing \( \delta \) small enough we can conclude that \( (S_j - (L, \phi_j) + (b |\nabla \phi_j|) r) \in \text{Isom}(\ell_p(\mathcal{F}^4(\delta)), \ell_p(\alpha \mathcal{F}^4(\delta))) \) with
\[
\| (S_j - (L, \phi_j) + (b |\nabla \phi_j|) r)^{-1} r^c \| \leq 2C_0.
\]
Let \( T_b := r(S_j - (L, \phi_j) + (b |\nabla \phi_j|) r)^{-1} r^c. \) Then \( T_b \in \mathcal{L}(\alpha \mathcal{F}^4(\delta), \alpha \mathcal{F}^4(\delta)) \) is a left inverse of \( S_b := G + L + (b |\nabla \phi_j|) r^c. \) Hence
\[
\|h\|_{\alpha \mathcal{F}^4(\delta)} = \|T_b S_b h\|_{\alpha \mathcal{F}^4(\delta)} \leq 2C_0 \|r\| \|r^c\| \|S_b h\|_{\alpha \mathcal{F}^4(\delta)}, \quad h \in \alpha \mathcal{F}^4(\delta). \tag{3.19}
\]
Replacing \( b \) by \( \rho b \), \( \rho \in [0,1] \), we have a continuous family \( \{S_{\rho b}\} \) of operators \( S_{\rho b} \) which all satisfy the a-priori estimate (3.19) uniformly in \( \rho \in [0,1] \). Since \( S_0 \) is an isomorphism, we can infer from a homotopy argument that \( S_b \) is an isomorphism as well. Repeating successively these arguments for the intervals \( J_k \), including the reduction from step (i), proves the assertion of the corollary.

(vii) We still have to verify the estimate in (3.18). Since the covering \( \{U_j : j \in \mathbb{N}\} \) has finite multiplicity, one obtains
\[
\|((\partial^\alpha \phi_j) g)\|_{\ell_p(\alpha \mathcal{F}^4(a))} \leq C(\alpha) \|g\|_{\alpha \mathcal{F}^4(a)}, \quad g \in \alpha \mathcal{F}^4(a). \tag{3.20}
\]
This together with Proposition 5.3 (b) shows that
\[
\|b(\nabla \phi_j) h\|_{\ell_p(\alpha \mathcal{F}^4(a))} \leq C\|bh\|_{\alpha \mathcal{F}^4(a)} \leq C_0(\|b\|_\infty + \|b\|_{\mathcal{F}^4(a)})\|h\|_{\alpha \mathcal{F}^4(a)}.
\]
The estimates for the commutators \([L, \phi_j]\) are more involved. The operator \( A = GD_n^{-1} \) is sectorial and admits a bounded \( \mathcal{H}\)-calculus with angle \( \pi/2 \) in \( \alpha H_p^s(J; K^r_p(\mathbb{R}^n)) \), for \( K \in \{H, W\} \), and also in \( \alpha W^s_p(J; K^r_p(\mathbb{R}^n)) \) by real interpolation. Hence fixing \( \theta \in (0, \pi/2) \), the following resolvent estimate holds in these spaces:
\[
\|z(\mathbb{I} - A)^{-1}\| \leq M, \quad \text{for all } z \in -\Sigma_\theta.
\]
The function \( k(z) \) is holomorphic in \( \mathbb{C} \setminus (-\infty, -2\delta_0) \) for some \( \delta_0 > 0 \) and behaves like \( 1/z \) as \( |z| \to \infty \). Choose the contour
\[
\Gamma = (\infty, \delta_0)e^{i\psi} \cup \delta_0e^{i(\psi/2 + \pi)} \cup [\delta_0, \infty) e^{-i\psi},
\]
where \( \pi > \psi > \pi - \theta \). Then we have the Dunford integral
\[
k(A) = \frac{1}{2\pi i} \int_\Gamma k(z)(z - A)^{-1} dz,
\]
which is absolutely convergent. This shows that \( k(A) \) is bounded, as is \( Ak(A) \) thanks to \( A \in \mathcal{H}\), thus \( A^{1/2}k(A) \) is bounded as well. Therefore the identity
\[ k(A)D_n^{-1/2} = G^{-1/2}A^{1/2}k(A) \] shows that \( L_2 \) is bounded since \( G^{-1/2} \) is, and follows for \([L_2, \phi_j] \). For the commutator \([L_1, \phi_j] \) we obtain

\[ [L_1, \phi_j] = \sigma[k(A)D_n^{1/2}, \phi_j] = \sigma[k(A), \phi_j]D_n^{1/2} + \sigma k(A)[D_n^{1/2}, \phi_j]. \]

Using the Dunford integral for \( k(A) \) this yields

\[ [k(A), \phi_j] = \frac{1}{2\pi i} \int_{\gamma} k(z)[(z-A)^{-1}, \phi_j]dz = \frac{1}{2\pi i} \int_{\gamma} k(z)(z-A)^{-1}[A, \phi_j](z-A)^{-1}dz, \]

hence with

\[ [A, \phi_j] = GD_n^{-1}[\phi_j, D_n]D_n^{-1} = A(\Delta \phi_j + 2(\nabla \phi_j | \nabla))D_n^{-1} \]

\[ = A(\Delta \phi_j D_n^{-1} + 2i(\nabla \phi_j | R)D_n^{-1/2}), \]

we have

\[ [k(A), \phi_j]D_n^{1/2} = \frac{1}{2\pi i} \int_{\gamma} k(z)A(z-A)^{-1}\{-\Delta \phi_j G^{-1/2}A^{1/2} + 2i(\nabla \phi_j | R)\}(z-A)^{-1}dz. \]

Let \( h \in \mathfrak{g}\mathfrak{F}_4 \) be given. Then we obtain from

\[ \|k(z)A(z-A)^{-1}\|_{L(\mathfrak{g}\mathfrak{F}_4)} \leq C/|z|, \quad \|A^{1/2}(z-A)^{-1}\|_{L(\mathfrak{g}\mathfrak{F}_4)} \leq C/|z|^{1/2}, \quad z \in \Gamma, \]

from \([3.20]\), and from Minkowski’s inequality for integrals

\[
\left\| \left( \int_{\Gamma} k(z)A(z-A)^{-1}\Delta \phi_j G^{-1/2}A^{1/2}(z-A)^{-1}h \, dz \right) \right\|_{L(\mathfrak{g}\mathfrak{F}_4)} \\
\leq C \int_{\Gamma} \frac{1}{|z|} \|\Delta \phi_j G^{-1/2}A^{1/2}(z-A)^{-1}h\|_{L(\mathfrak{g}\mathfrak{F}_4)} \, |dz| \\
\leq C \int_{\Gamma} \frac{1}{|z|^{1/2}} \|h\|_{\mathfrak{g}\mathfrak{F}_4} \, |dz| \leq C \|h\|_{\mathfrak{g}\mathfrak{F}_4}
\]

where we also used that \( G^{-1/2} \) is bounded on compact intervals. In the same way we can estimate the second term in the integral representation of \([k(A), \phi_j]D_n^{1/2} \), this time using the fact that \( R \) is bounded.

To estimate the commutators \([D_n^{1/2}, \phi_j] \) in \( \mathfrak{g}\mathfrak{F}_4 \) note that

\[
(D_n)^{1/2} = D_n(D_n)^{-1/2} = \frac{1}{\sqrt{\pi}} D_n \int_{0}^{\infty} e^{-D_n t} t^{-\frac{1}{2}} dt \\
= \frac{1}{\sqrt{\pi}} \left( D_n \int_{0}^{1} e^{-D_n t} t^{-\frac{1}{2}} dt + D_n \int_{1}^{\infty} e^{-D_n t} t^{-\frac{1}{2}} dt \right) \\
= \frac{1}{\sqrt{\pi}} (T_1 + T_2),
\]

with \( e^{-D_n t} \) denoting the bounded analytic semigroup generated by the Laplacian in \( H^s_p(\mathbb{R}^n) \) which extends by real interpolation to \( W^s_p(\mathbb{R}^n) \), and then canonically to
denote the Gaussian kernel. Then for fixed 
$H$ then extends to 

Choosing a cut-off function $\chi$ there are constants $c$ and $0$

This shows that it is enough to estimate the commutator $\left[ T, H \right]$. We consider next the commutator $\left[ T, \phi \right]$. Let $k_t(x) = (2\pi t)^{-n/2} \exp(-|x|^2/4t)$ denote the Gaussian kernel. Then for fixed $t > 0$, the operator $D_t e^{-\Delta t}$ is the convolution with kernel $-\Delta k_t(x)$, which is of class $C^\infty$. It is not difficult to see that there are constants $C, c > 0$ such that

$$|\Delta k_t(x)| \leq Ct^{-(n+2)/2} e^{-c|x|^2/t}, \quad x \in \mathbb{R}^n, \quad t > 0. \quad (3.21)$$

Choosing a cut-off function $\chi \in C^\infty(\mathbb{R}^n)$ with $\chi \equiv 1$ in $B_0(0)$, supp $\chi \subset B_2(0)$ and $0 \leq \chi \leq 1$ elsewhere, we set

$$-\Delta k_t(x) = -(1 - \chi(x))\Delta k_t(x) - \chi(x)\Delta k_t(x) =: k_{3,t}(x) + k_{4,t}(x), \quad x \in \mathbb{R}^n, \quad t > 0,$$

and we denote by $T_l$ the convolution operators with kernels $\int_0^1 k_{l,t} t^{-1/2} dt$, $l = 3, 4$. For the kernel of $T_3$ we obtain from $3.21$ the estimate

$$\left| \int_0^1 k_{3,t}(x) t^{-1/2} dt \right| \leq C \int_0^1 e^{-c|x|^2/t} t^{-(n+3)/2} dt \leq C e^{-c_1|x|^2} \int_1^\infty e^{-c_2 s^{n-1/2}} ds \leq C e^{-c_1|x|^2},$$

as $k_{3,t}(x) = 0$ for $|x| \leq \rho$. Thus this kernel is in $L_1(\mathbb{R}^n)$ and hence we may estimate the commutator $\left[ T_3, \phi \right]$ in the same way as $\left[ T_2, \phi \right]$.

For the remaining commutator $\left[ T_4, \phi \right]$ note that

$$\partial^\alpha [T_4, \phi] = \sum_{\beta \leq \alpha} \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) [T_4, \partial^\beta \phi] \partial^{\alpha-\beta}.$$

This shows that it is enough to estimate the commutator $\left[ T_4, \phi \right]$ in $L_p(\mathbb{R}^n)$, as it then extends to $H^m_{\alpha} \mathbb{R}^n$ and by interpolation to $W^s_p(\mathbb{R}^n)$, and then canonically to $0F_4$. Next we observe that for $x, y \in \mathbb{R}^n$

$$\partial^\alpha \phi_j(y) - \partial^\alpha \phi_j(x) = \partial^\alpha \phi_j(x)(y - x) + r_{j,\alpha}(x, y),$$
where $|r_{j,\alpha}(x, y)| \leq C|x - y|^2$, with some constant $C$ independent of $j$ and $|\alpha| \leq 2$. Therefore

$$
[T_4, \phi_j]h(x) = \int_0^1 \int_\mathbb{R}^n (\phi_j(y) - \phi_j(x)) k_{4,t}(x - y) h(y) dy \, dt t^{-\frac{1}{2}} dt
$$

$$
= -\phi_j'(x) \int_\mathbb{R}^n \int_0^1 (y - x) k_{4,t}(x - y) t^{-\frac{1}{2}} dt \, h(y) dy + \int_\mathbb{R}^n \int_0^1 r_j(x, y) k_{4,t}(x - y) t^{-\frac{1}{2}} dt \, h(y) dy
$$

$$
=: T_{5,j} h(x) + T_{6,j} h(x).
$$

We observe that the support of the kernel $k_{4,t}$ is contained in $B_{2\rho}(0)$, and consequently we may replace $h$ by $\psi_j h$, where $\psi_j$ is a cut-off function which equals 1 on $\text{supp}(\phi_j) + B_{2\rho}(0)$. In the following we fix a smooth cut-off function $\psi$ which equals 1 on $\text{supp}(\phi) + B_{2\rho}(0)$ and then set $\psi_j := \tau_{\epsilon_j} \psi$. We then have

$$
||(T_i,j)h||_{\ell_p(L_\rho)} = ||(T_i,j)\psi_j h)||_{\ell_p(L_\rho)} \leq \sup_k ||T_{i,k}||_{\ell_p(L_\rho)} ||(\psi_j h)||_{\ell_p(L_\rho)} \leq C ||h||_{L_p},
$$

provided we can show that the operators $T_{i,k}$ are $L_p$-bounded with bound independent of $k \in \mathbb{N}$ for $l = 5, 6$.

The operators $T_{i,j}$ satisfy

$$
T_{5,j} h = \phi_j'(q * h) \quad \text{with} \quad q(x) = \chi(x) \int_0^1 x \Delta k_t(x) t^{-\frac{1}{2}} dt, \quad x \in \mathbb{R}^n
$$

$$
|T_{6,j} h| \leq r * |h| \quad \text{with} \quad r(x) = C \chi(x) \int_0^1 |x|^2 \Delta k_t(x) t^{-\frac{1}{2}} dt, \quad x \in \mathbb{R}^n.
$$

The Fourier transform of $q$ is given by $\hat{q}(\xi) = C \hat{\chi} * \int_0^1 \nabla \xi^i (|\xi|^2 e^{-t(|\xi|^2)}) t^{-1/2} dt$ and we verify that

$$
\sup_\alpha \sup_{(1, \ldots, 1)} \left| \int_{\mathbb{R}^n} |q|^\alpha \partial^{\alpha} \hat{q}(\xi) \right| \leq M
$$

for some $M < \infty$. It thus follows from Mikhlin’s multiplier theorem that

$$
||T_{5,j} h||_{L_p} \leq C ||\phi_j'||_{\infty} ||h||_{L_p} \leq C ||h||_{L_p}.
$$

Finally, in order to estimate $T_{6,j}$ we infer from (3.21) that

$$
r(x) \leq C \int_0^1 |x|^2 e^{-c_1|x|^2/t} t^{-\frac{1}{2}} dt \leq C e^{-c_1|x|^2} \int_1^\infty e^{-c_2s^2(n-1)/2} ds
$$

for $x \in \mathbb{R}^n$. It follows that $r \in L_1(\mathbb{R}^n)$ which implies by Young’s inequality $||T_{6,j} h||_p \leq C ||h||_p$ with a uniform constant $C$. \hfill \Box

Remarks 3.2. (a) We mention that the proof for the estimate of $[D_n^{1/2}, \phi_j]$ follows the ideas of [15, Lemma 6.4].

(b) If $\rho_2 \leq \rho_1$, i.e. the light fluid lies above the heavy one, then the estimate (3.8) can be improved in the following sense: for every $\beta > 0$ and $\lambda_0 > 0$ there are positive constants $\delta$, $\eta = \eta(\beta)$ and $c_j = c_j(\beta, \lambda_0, \delta, \eta)$ such that

$$
c_0 [\lambda + |\tau|] \leq \tilde{s}(\lambda, \tau, z) \leq c_1 [\lambda + |\tau|]
$$

(3.22)

for all $(\lambda, \tau, z) \in \Sigma_{\epsilon/2+\epsilon} \times \Sigma_\eta \times U_{\beta, \delta}$ and $|\lambda| \geq \lambda_0$. For this we observe that estimates (3.3) and (3.10) certainly also hold in case that $\rho_2 \leq \rho_1$. On the other hand, given $M > 0$ we conclude as in (5.11) that $L \leq \Re ((\rho_1 - \rho_2) g(a, \kappa(z))) \leq R$.
and \(|\text{Im}((\rho_1 - \rho_2)\gamma_0 k(z))| \leq H\), with appropriate positive constants \(L, R, H\). This shows that there exists \(\alpha = \alpha(M, \eta) \in (0, \pi/2)\) such that

\[
(\rho_1 - \rho_2)\gamma_0 k(z)/\tau \in \Sigma_\alpha, \quad (\lambda, \tau) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta, \quad |z| \leq M
\]

(3.23)

with \(\eta \in (0, \eta_0)\) chosen small enough, where we can assume that \(\alpha\) coincides with the angle in (3.12). Combining (3.12) and (3.23) yields

\[
|\hat{s}(\lambda, \tau, \zeta)| \geq c(\psi) \left| |\lambda| + |\tau(\sigma k(z) + i\zeta) + (\rho_1 - \rho_2)\gamma_0 k(z)/\tau| \right|
\]

\[
\geq c(\psi) c(\alpha) \left| |\lambda| + |\tau(\sigma k(z) + i\zeta)| + |(\rho_1 - \rho_2)\gamma_0 k(z)/\tau| \right|
\]

\[
\geq c_0(M, \beta, \delta, \eta) |\lambda| + |\tau|
\]

provided \((\lambda, \tau, \zeta) \in \Sigma_{\pi/2+\eta} \times \Sigma_\eta \times U_{\beta, \delta}\) and \(|\lambda| \leq M|\tau|^2\). Noting again that the curves \(|\lambda| = m|\tau|\) and \(|\lambda| = M|\tau|^2\) intersect at \((m/M, m^2/M)\) we obtain (3.22) by choosing \(M\) big enough.

(c) If \(\rho_2 \leq \rho_1\) we can conclude from the lower estimate in (3.22) that the function \(\hat{s}\) does not have zeros in \(\Sigma_{\pi/2} \times \mathbb{R}_+ \times [-\beta, \beta]\). This holds in particular true for the symbol \(s(\lambda, \tau) := \hat{s}(\lambda, \tau, 0)\), indicating that there are no instabilities in case that the light fluid lies on top of the heavy one.

(d) If \(\rho_2 > \rho_1\) then it is shown in [29] that the symbol \(s\) has for each \(\tau \in (0, \tau_*)\) with \(\tau_* := ((\rho_2 - \rho_1)\gamma_0/\sigma)^{1/2}\) a zero \(\lambda = \lambda(\tau) > 0\) pertinent to the Rayleigh-Taylor instability.

(e) Further mapping properties of the boundary symbol \(s(\lambda, \tau) := \hat{s}(\lambda, \tau, 0)\) and the associated operator \(S\) in case that \(\gamma_0 = 0\) have been derived in [27]. In particular, we have investigated the singularities and zeros of \(s\), and we have studied the mapping properties of \(S\) in case of low and high frequencies, respectively.

4. THE NONLINEAR PROBLEM

In this section we prove existence and uniqueness of solutions for the nonlinear problem (2.1), and we show additionally that solutions immediately regularize and are analytic in space and time. In order to facilitate this task, we first introduce some notation. We set

\[
\mathcal{E}_1(a) := \{u \in H_p^1(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \cap L_p(J; H_p^2(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) : [u] = 0\},
\]

\[
\mathcal{E}_2(a) := L_p(J; \dot{H}_p^1(\mathbb{R}^{n+1})),
\]

\[
\mathcal{E}_3(a) := W_p^{1/2-1/2p}(J; L_p(\mathbb{R}^n)) \cap L_p(J; W_p^{1-1/p}(\mathbb{R}^n)),
\]

\[
\mathcal{E}_4(a) := W_p^{2-1/2p}(J; L_p(\mathbb{R}^n)) \cap \dot{H}_p^1(J; W_p^{2-1/p}(\mathbb{R}^n)) \cap W_p^{1/2-1/2p}(J; H_p^2(\mathbb{R}^n)) \cap L_p(J; W_p^{3-1/p}(\mathbb{R}^n)),
\]

\[
\mathcal{E}(a) := \{(u, \pi, q, h) \in \mathcal{E}_1(a) \times \mathcal{E}_2(a) \times \mathcal{E}_3(a) \times \mathcal{E}_4(a) : [\pi] = q\}.
\]

The space \(\mathcal{E}(a)\) is given the natural norm

\[
\|(u, \pi, q, h)\|_{\mathcal{E}(a)} = \|u\|_{\mathcal{E}_1(a)} + \|\pi\|_{\mathcal{E}_2(a)} + \|q\|_{\mathcal{E}_3(a)} + \|h\|_{\mathcal{E}_4(a)}
\]
which turns it into a Banach space. Moreover, we set
\[ F_1(a) := L_p(J; L_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) , \]
\[ F_2(a) := H^1_p(J; H^1_p(\mathbb{R}^{n+1})) \cap L_p(J; H^1_p(\mathbb{R}^{n+1})) , \]
\[ F_3(a) := W^{1,2/p}(J; L_p(\mathbb{R}^{n}, \mathbb{R}^{n+1})) \cap L_p(J; W^{1,1/p}(\mathbb{R}^{n}, \mathbb{R}^{n+1})) , \]
\[ F_4(a) := W^{1,2/p}(J; L_p(\mathbb{R}^{n})) \cap L_p(J; W^{2,1/p}(\mathbb{R}^{n})) , \]
\[ F(a) := F_1(a) \times F_2(a) \times F_3(a) \times F_4(a) . \]
The generic elements of \( F(a) \) are the functions \( (f, f_d, g, g_h) \).

Let \( b \in F_4(a)^n \) be a given function. Then we define the nonlinear mapping
\[ N_b(u, \pi, q, h) := (F(u, \pi, h), F_d(u, h), G(u, q, h), (b - \gamma \nabla h)) \]
for \( (u, \pi, q, h) \in \mathbb{E}(a) \), where, as before, \( u = (v, w) \), \( F = (F_v, F_w) \) and \( G = (G_v, G_w) \).

We will now study the mapping properties of \( N_b \) and we will derive estimates for the Fréchet derivative of \( N_b \).

**Proposition 4.1.** Suppose \( p > n + 3 \) and \( b \in F_4(a)^n \). Then
\[ N_b \in C^\infty(\mathbb{E}(a), F(a)) , \quad a > 0 . \]

Let \( DN_b(u, \pi, q, h) \) denote the Fréchet derivative of \( N_b \) at \( (u, \pi, q, h) \in \mathbb{E}(a) \). Then \( DN_b(u, \pi, q, h) \in L_0(\mathbb{E}(a), aF(a)) \), and for any number \( a_0 > 0 \) there is a positive constant \( M_0 = M_0(a_0, p) \) such that
\[ ||DN_b(u, \pi, q, h)||_{L_0(\mathbb{E}(a), aF(a))} \]
\[ \leq M_0([||b - \gamma \nabla h||_{BC(J; BC)} + ||u, \pi, q, h||_{\mathbb{E}(a)}] \]
\[ + M_0([||\nabla h||_{BC(J; BC)} + ||u||_{BC(J; BC)}] ||h||_{\mathbb{E}(a)}) \]
\[ + M_0(P(||\nabla h||_{BC(J; BC)}) ||h||_{BC(J; BC)}) + Q(||\nabla h||_{BC(J; BC)}, ||h||_{\mathbb{E}(a)}) ||h||_{\mathbb{E}(a)}) \]
for all \( (u, \pi, q, h) \in \mathbb{E}(a) \) and all \( a \in (0, a_0) \). Here, \( P \) and \( Q \) are fixed polynomials with coefficients equal to one.

**Proof.** The proof of the proposition is relegated to the end of the appendix. \( \square \)

Given \( h_0 \in W^{3,2/p}(\mathbb{R}^n) \) we define
\[ \Theta_{h_0}(x, y) := (x, y + h_0(x)) , \quad (x, y) \in \mathbb{R}^n \times \mathbb{R} . \]

Letting \( \Omega_{h_0, i} := \{(x, y) \in \mathbb{R}^n \times \mathbb{R} : (-1)^i(y - (h_0(x))) > 0 \} \) and \( \Omega_{h_0} := \Omega_{h_0, 1} \cup \Omega_{h_0, 2} \) we obtain from Sobolev’s embedding theorem that
\[ \Theta_{h_0} \in Diff^1(\mathbb{R}^{n+1}, \Omega_{h_0}) \cap Diff^1(\mathbb{R}^{n+1}, \Omega_{h_0, 1}) \cap Diff^2(\mathbb{R}^{n+1}, \Omega_{h_0, 2}) , \]
i.e., \( \Theta_{h_0} \) yields a \( C^2 \)-diffeomorphism between the indicated domains. The inverse transformation obviously is given by \( \Theta_{h_0}^{-1}(x, y) = (x, y - h_0(x)) \). It then follows from the chain rule and the transformation rule for integrals that
\[ \Theta_{h_0}^* \in Isom(H_p^k(\mathbb{R}^{n+1}), H_p^k(\Omega_{h_0})), \quad [\Theta_{h_0}^*]^{-1} = \Theta_{h_0}^* , \quad k = 0, 1, 2 , \]
where we use the notation
\[ \Theta_{h_0}^* u := u \circ \Theta_{h_0} , \quad u : \Omega_{h_0} \rightarrow \mathbb{R}^m , \]
\[ \Theta_{h_0}^* v := v \circ \Theta_{h_0}^{-1} , \quad v : \mathbb{R}^{n+1} \rightarrow \mathbb{R}^m . \]

We are now ready to prove our main result of this section.
Theorem 4.2. (Existence of solutions for the nonlinear problem (2.1)).

(a) For every \( \beta > 0 \) there exists a constant \( \eta = \eta(\beta) > 0 \) such that for all initial values

\[
(u_0, h_0) \in W_p^{2-2/p}(\mathbb{R}^{n+1}, \mathbb{R}^n) \times W_p^{3-2/p}(\mathbb{R}^n) \quad \text{with} \quad \|u_0\| = 0,
\]

satisfying the compatibility conditions

\[
[\mu D(\Theta_1^h u_0) v_0 - \mu(v_0) D(\Theta_1^h u_0) v_0] = 0, \quad \text{div}(\Theta_1^h u_0) = 0,
\]

and the smallness-boundedness condition

\[
\|\nabla h_0\|_{\infty} \leq \eta, \quad \|u_0\|_{\infty} \leq \beta,
\]

there is a number \( t_0 = t_0(u_0, h_0) \) such that the nonlinear problem (2.1) admits a unique solution \((u, \pi, [\pi], h) \in E_1(t_0)\).

(b) The solution has the additional regularity properties

\[
(u, \pi) \in C^\omega((0, t_0) \times \mathbb{R}^{n+1}, \mathbb{R}^{n+2}), \quad [\pi], h \in C^\omega((0, t_0) \times \mathbb{R}^n).
\]

In particular, \( \mathcal{M} = \bigcup_{t \in (0, t_0)} \{\{t\} \times \Gamma(t)\} \) is a real analytic manifold.

Proof. The proof of this result proceeds in a similar way as the proof of Theorem 6.3 in [28].

For a given function \( b \in F_4(a)^n \) we consider the nonlinear problem

\[
\begin{aligned}
\rho \partial_t u - \mu \Delta u + \nabla \pi &= F(u, \pi, h) \quad \text{in} \ \mathbb{R}^{n+1} \\
\text{div} u &= F_d(u, h) \quad \text{in} \ \mathbb{R}^{n+1} \\
- [\mu \partial_y v] - [\mu \nabla_x w] &= G_v(u, [\pi], h) \quad \text{on} \ \mathbb{R}^n \\
- 2[\mu \partial_y w] + [\pi] - \sigma \Delta h &= G_w(u, h) \quad \text{on} \ \mathbb{R}^n \\
\|u\| &= 0 \quad \text{on} \ \mathbb{R}^n \\
\partial_t h - \gamma w + (b|\nabla h) &= (b - \gamma v|\nabla h) \quad \text{on} \ \mathbb{R}^n \\
\quad u(0) = u_0, \ h(0) = h_0,
\end{aligned}
\]

which clearly is equivalent to (2.1).

In order to economize our notation we set \( z := (u, \pi, q, h) \) for \((u, \pi, q, h) \in E(a)\). With this notation, the nonlinear problem (2.1) can be restated as

\[
L_b z = N_b(z), \quad (u(0), h(0)) = (u_0, h_0),
\]

where \( L_b \) denotes the linear operator on the left-hand side of (4.9), and \( N_b \) correspondingly denotes the nonlinear mapping on the right-hand site of (4.9).

It is convenient to first introduce an auxiliary function \( z^* = z^*_b \in E(a) \) which resolves the compatibility conditions and the initial conditions in (4.10), and then to solve the resulting reduced problem

\[
L_b z = N_b(z + z^*) - L_b z^* =: K_b(z), \quad z \in \mathcal{O}(a),
\]

by means of a fixed point argument.

(i) Suppose that \((u_0, h_0)\) satisfies the (first) compatibility condition in (4.6), and let

\[
[\pi_0] := \theta_h^\alpha \{[\mu(v_0) D(\Theta_1^h u_0)] + \sigma \kappa\},
\]
where \( \theta_{h_0} := \Theta_{h_0} |_{\mathbb{R}^{n+1} \times \{0\}} \). Here we observe that \( \theta_{h_0}^* [\omega] = [\Theta_{h_0}^* \omega] \) for any function \( \omega : \Omega_{h_0} \to \mathbb{R}^m \) which has one-sided limits. It is then clear from the definition in (2.3)–(2.4) that the following compatibility conditions hold:

\[
\begin{align*}
-\mu \partial_y v_0 - \mu \nabla_x w_0 &= G_v(u_0, \|\pi_0\|, h_0) & \text{on } \mathbb{R}^n \\
-2\mu \partial_y w_0 + \|\pi_0\| - \sigma \Delta h_0 &= G_w(u_0, h_0) & \text{on } \mathbb{R}^n
\end{align*}
\] (4.12)

where, as before, \( u_0 = (v_0, w_0) \). Next we introduce special functions \((0, f_d^*, g^*, g_b^*) \in \mathbb{F}(a)\) which resolve the necessary compatibility conditions. First we set

\[
c^*(t) := \begin{cases} 
R^+ e^{-tD_{n+1}} \mathcal{E}_+(v_0|\nabla h_0) & \text{in } \mathbb{R}^{n+1}^+, \\
R^- e^{-tD_{n+1}} \mathcal{E}_-(v_0|\nabla h_0) & \text{in } \mathbb{R}^{n+1}^-,
\end{cases}
\] (4.13)

where \( \mathcal{E}_\pm \in \mathcal{L}(W^{2-2/p}(\mathbb{R}^{n+1}), W^{2-2/p}(\mathbb{R}^{n+1})) \) is an appropriate extension operator and \( R_\pm \) is the restriction operator. Due to \((v_0|\nabla h_0) \in W^{2-2/p}(\mathbb{R}^{n+1}) \) we obtain

\[
c^* \in H^1_p(J; L_p(\mathbb{R}^{n+1})) \cap L_p(J; H^2_p(\mathbb{R}^{n+1})).
\]

Consequently, \( f_d^* := \partial_y c^* \in \mathbb{F}_d(a) \) and \( f_d^*(0) = F_d(v_0, h_0) \). (4.14)

Next we set

\[
g^*(t) := e^{-D_{n+1}t}G(u_0, \|\pi_0\|, h_0), \quad g_b^*(t) := e^{-D_{n+1}t}(b(0) - \gamma v_0|\nabla h_0).
\] (4.15)

It then follows from (4.13) and (19) Lemma 8.2 that \((0, f_d^*, g^*, g_b^*) \in \mathbb{F}(a)\). (4.12) and the second condition in (4.6) show that the necessary compatibility conditions of Theorem 3.1 are satisfied and we can conclude that the linear problem

\[
L_b z^* = (0, f_d^*, g^*, g_b^*), \quad (u^*(0), h^*(0)) = (u_0, h_0),
\] (4.16)

has a unique solution \( z^* = z_b^* \in \mathbb{E}(a) \). With the auxiliary function \( z^* \) now determined, we can focus on the reduced equation (4.11), which can be converted into the fixed point equation

\[
z = L_b^{-1} K_b(z), \quad z \in \mathbb{E}(a).
\] (4.17)

Due to the choice of \((f_d^*, g^*, g_b^*)\) we have \( K_b(z) \in \mathbb{F}(a) \) for any \( z \in \mathbb{E}(a) \), and it follows from Proposition 3.2 that

\[
K_b \in C^\omega(a \mathbb{E}(a), \mathbb{E}(a)).
\]

Consequently, \( L_b^{-1} K_b : \mathbb{E}(a) \to \mathbb{E}(a) \) is well defined and smooth.

(ii) An inspection of the proof of Theorem 3.1 shows that given \( \beta > 0 \) we can find a positive number \( \delta_0 = \delta_0(b) \) such that

\[
L_b^{-1} \in \mathcal{L}(a \mathbb{E}(a), \mathbb{F}(a)), \quad \|L_b^{-1}\|_{\mathcal{L}(a \mathbb{E}(a), \mathbb{F}(a))} \leq M, \quad a \in [0, \delta_0],
\] (4.18)

whenever \( b \in \mathbb{F}(a)^n \) and \( \|b\|_{BC[0,a],BC(\mathbb{R}^n)} \leq \beta \). It should be pointed out that the bound \( M \) is universal for all functions \( b \in \mathbb{F}(a)^n \) with \( \|b\|_{\infty} \leq \beta \), whereas the number \( \delta_0 = \delta(b) \) may depend on \( b \).

(iii) We will now fix a pair of initial values \((u_0, h_0) \in W^{2-2/p}(\mathbb{R}^{n+1}) \times W^{3-2/p}(\mathbb{R}^n)\) satisfying (4.6) and (4.7) with

\[
\eta := 1/(16 M_0 M),
\] (4.19)

where the constants \( M_0 \) and \( M \) are given in (4.11) and (4.18), respectively. We choose

\[
b(t) := e^{-D_{n+1}t}\gamma v_0, \quad t \geq 0.
\] (4.20)
Then \( b \in F_4(a)^n \) and \( \| b \|_{BC([0,a];BC(\mathbb{R}^n))} \leq \| \gamma v_0 \|_{BC(\mathbb{R}^n)} \leq \beta \) for any \( a > 0 \), as \( \{ e^{-D_n \cdot t} : t \geq 0 \} \) is a contraction semigroup on \( BC(\mathbb{R}^n) \). Hence the estimate (4.13) holds true for this (and any other choice) of initial values. It should be pointed out once more that the bound \( M \) is universal for all initial values \( u_0 \) with \( \| v_0 \| \leq \beta \) and hence for \( b(t) := e^{-D_n \cdot t} \gamma v_0 \) whereas the number \( \delta_0 \) may depend on \( \gamma v_0 \).

We note in passing that \( g^*_b = 0 \) for this particular choice of the function \( b \). Without loss of generality we can assume that \( M_0, M \geq 1 \). We shall show that \( L_b^{-1}K_b \) is a contraction on a properly defined subset of \( _0E(a) \) for \( a \in (0, \delta_0] \) chosen sufficiently small. For \( r > 0 \) and \( a \in (0, \delta_0] \) we set

\[
_0B_E(a)(z^*, r) := \{ z \in E_4(a) : z - z^* \in _0E(a), \| z - z^* \|_{E(a)} < r \}.
\]

We remark that \( a \) and \( r \) are independent parameters that can be chosen as we please. Let then \( r_0 > 0 \) be fixed. It is not difficult to see that there exists a number \( R_0 = R_0(u_0, h_0, \delta_0, r_0) \) such that

\[
\| D(H + h^*)\|_{BC(J;BC)} + \| h + h^*\|_{E_4(a)} + \| u + u^*\|_{BC(J;BC)} + Q(\| \gamma^* \|_{BC(J;BC)}) \| \gamma^* \|_{BC(J;BC)} \leq R_0
\]

for all \( u \in _0B_E(a)(0, r_0) \) and \( h \in _0B_E(a)(0, r_0) \), with \( a \in (0, \delta_0] \) and \( r \in (0, r_0] \) arbitrary, where \( z^* = (u^*, \pi^*, q^*, h^*) \) is the solution of equation (4.10) and where \( Q \) is defined in Proposition 4.1. Let \( M_1 := M_0(1 + R_0) \). It then follows from Proposition 4.1 and (4.18) that

\[
\| D(L_b^{-1}K_b)(z) \|_{E(a)} \leq M_1 M \| b - \gamma(v + v^*) \|_{BC(J;BC)} + \| z + z^* \|_{E(a)} \] (4.21)

\[
+ M_0 \| P(\| \gamma \|_{BC(J;BC)} \| \gamma \|_{BC(J;BC)}) \| \gamma \|_{BC(J;BC)} \]

for all \( z \in _0B_E(a)(0, r_0) \) and \( a \in (0, \delta_0] \).

(iv) For \( (u_0, h_0) \) fixed, the norm of \( z^* \in E(a) \) (which involves various integral expressions evaluated over the interval \( (0, a) \)) can be made as small as we like by choosing \( a \in (0, \delta_0] \) small. Let then \( a_1 \in (0, \delta_0] \) be fixed so that

\[
\| \gamma h \|_{BC([0,a_1],BC(\mathbb{R}^n))} \leq 2\eta, \quad M_1 M \| b - \gamma v^* \|_{BC([0,a_1];BC) \cap E_4(a_1)} + \| z^* \|_{E(a_1)} \leq 1/8. \] (4.22)

Since \( \| \gamma h \|_{BC([0,a_1],BC(\mathbb{R}^n))} \) and \( \| \gamma h \|_{BC([0,a_1],BC(\mathbb{R}^n))} \leq \eta \), the estimates in (4.22) certainly hold for \( a_1 \) sufficiently small.

In a next step we choose \( 2r_1 \in (0, r_0] \) so that

\[
\| \gamma\|_{BC([0,a_1],BC(\mathbb{R}^n))} \leq \eta, \quad M_1 M \| b - \gamma v^* \|_{BC([0,a_1];BC) \cap E_4(a_1)} + \| z^* \|_{E(a_1)} \leq 1/8, \] (4.23)

for all \( h \in _0B_E(a_1)(0, 2r_1), v \in _0B_E(a_1)(0, 2r_1) \), and \( z \in _0B_E(a_1)(0, 2r_1) \). It follows from Proposition 5.1 that (4.23) can indeed be achieved. Combining (4.19)–(4.23) gives

\[
\| D(L_b^{-1}K_b)(z) \|_{E(a)} \leq 1/2, \quad z \in _0B_E(a_1)(0, 2r_1)
\]

showing that \( L_b^{-1}K_b : _0B_E(a_1)(0, r_1) \rightarrow _0E(a_1) \) is a contraction, where \( _0B_E(a_1)(0, r_1) \) denotes the closed ball in \( _0E(a_1) \) with center at 0 and radius \( r_1 \).
It remains so show that \( L_b^{-1}K_b \) maps \( \mathcal{O}(0, r_1) \) into itself. From \([4.24]\) and the mean value theorem follows

\[
\|L_b^{-1}K_b(z)\|_{\mathcal{E}(a)} \leq \|L_b^{-1}K_b(z) - L_b^{-1}K_b(0)\|_{\mathcal{E}(a)} + \|L_b^{-1}K_b(0)\|_{\mathcal{E}(a)} \\
\leq r_1/2 + \|L_b^{-1}K_b(0)\|_{\mathcal{E}(a)} \quad z \in \mathcal{O}(0, r_1).
\]

Here we observe that the norm of \( L_b^{-1}K_b(0) = L_b^{-1}(K(z^*)-(0,f_{\delta^*}^*,g_{\delta^*}^*,g_{\delta}^*)) \) in \( \mathcal{E}(a) \) can be made as small as we wish by choosing \( a_1 \) small enough. We may assume that \( a_1 \) was already chosen so that \( \|L_b^{-1}K_b(0)\|_{\mathcal{E}(a)} \leq r_1/2 \).

(v) We have shown in (iv) that the mapping

\[
L_b^{-1}K_b : \mathcal{O}(0, r_1) \to \mathcal{O}(0, r_1)
\]

is a contraction. By the contraction mapping theorem \( L_b^{-1}K_b \) has a unique fixed point \( \hat{z} \in \mathcal{O}(0, r_1) \subset \mathcal{E}(a) \) and it follows immediately from \([4.10] - [4.11]\) that \( \hat{z} + z^* \) is the (unique) solution of the nonlinear problem \((2.1)\) in \( \mathcal{O}(0, r_1) \). Setting \( t_0 = a_1 \) gives the assertion in part (a) of the Theorem.

(vi) The proof that \((u, \pi, q, h)\) is analytic in space and time proceeds exactly in the same way as in steps (vi)–(vii) of the proof of Theorem 6.3 in \([28]\), with the only difference that here \( g_h^* = g_{h,\lambda,\nu} = 0 \), and that the operator \( D_{\lambda,\nu} \) in formula \((6.30)\) of \([28]\) is to be replaced by \( D_{\lambda,\nu} \), defined by

\[
D_{\lambda,\nu}h := (\lambda b_{\lambda,\nu} - \nu|\nabla h), \quad b_{\lambda,\nu}(t, x) := b(\lambda t, x + t\nu).
\]

We note that \( D_{1,0} = (b|\nabla \cdot) \). In the same way as in \([28], \text{Lemma 8.2}\) one obtains that

\[
[(\lambda, \nu) \mapsto b_{\lambda,\nu}] : (1 - \delta, 1 + \delta) \times \mathbb{R}^n \to \mathcal{F}(a)
\]

is real analytic. The remaining arguments are now the same as in \([28]\), and this completes the proof of Theorem \((1.2)\). □

**Proof of Theorem 1.1:** Clearly, the compatibility conditions of Theorem \((1.1)\) are satisfied if and only if \((1.6)\) is satisfied. Moreover, the smallness-boundedness condition of Theorem \((1.1)\) is equivalent to \((1.7)\), where we have slightly abused notation by using the same symbol for \( u_0 \) and its transformed version \( \Theta_{\nu_0}u_0 \).

Theorem \((1.2)\) yields a unique solution \((v, w, \pi, [\pi], h) \in \mathcal{E}(t_0) \) which satisfies the additional regularity properties listed in part (b) of the theorem. Setting

\[
(u, q)(t, x, y) = (v, w, \pi(t, x, y - h(t, x)), \quad (t, x, y) \in \mathcal{O},
\]

we then conclude that \((u, q) \in C^\omega(\mathcal{O}, \mathbb{R}^{n+2}) \) and \([q] \in C^\omega(\mathcal{M}) \). The regularity properties listed in Remark \((1.2)\) are implied by Proposition \((1.1)\), (c). Finally, since \( \pi(t, x, y) \) is defined for every \((t, x, y) \in \mathcal{O} \), we can conclude that

\[
q(t, \cdot) \in \dot{H}_p^1(\Omega(t)) \subset UC(\Omega(t))
\]

for every \( t \in (0, t_0) \). □
5. Appendix

In this section we state and prove some technical results that were used above.

**Proposition 5.1.** Suppose $p > n + 3$. Then the following embeddings hold:

(a) $\mathbb{E}_1(a) \hookrightarrow BC(J; W^{2-2/p}_p(\mathbb{R}^{n+1}, \mathbb{R}^{n+1})) \hookrightarrow BC(J; BC^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}))$ and there is a constant $C_0 = C_0(p)$ such that
\[
\|u\|_{aBC(J; W^{2-2/p}_p)} + \|u\|_{aBC(J; BC^1)} \leq C_0\|u\|_{\mathbb{E}_1(a)}
\]
for all $u \in \mathbb{E}_1(a)$ and all $a \in (0, \infty)$.

(b) $\mathbb{E}_0(a) \hookrightarrow BC(J; BC(\mathbb{R}^n))$ and there exists a constant $C_0 = C_0(p)$ such that
\[
\|g\|_{aBC(J; BC)} \leq C_0\|g\|_{\mathbb{E}_0(a)}
\]
for all $g \in \mathbb{E}_0(a)$ and all $a \in (0, \infty)$.

(c) $\mathbb{F}_4(a) \hookrightarrow BC(J; W^{1,p}_p(\mathbb{R}^n)) \cap BC(J; BC^1(\mathbb{R}^n))$ and there exists a constant $C_0 = C_0(p)$ such that
\[
\|g\|_{aBC(J; W^{1,p}_p)} + \|g\|_{aBC(J; BC^1)} \leq C_0\|g\|_{\mathbb{F}_4(a)}
\]
for all $g \in \mathbb{F}_4(a)$ and all $a \in (0, \infty)$.

(d) $\mathbb{E}_0(a) \hookrightarrow BC^1(J; BC^1(\mathbb{R}^n)) \cap BC(J; BC^2(\mathbb{R}^n))$ and there exists a constant $C_0 = C_0(p)$ such that
\[
\|h\|_{aBC^1(J; BC^1)} + \|h\|_{aBC(J; BC^2)} \leq C_0\|h\|_{\mathbb{E}_0(a)}
\]
for all $h \in \mathbb{E}_0(a)$ and all $a \in (0, \infty)$.

(e) $\partial_j \in \mathcal{L}(\mathbb{E}_4(a), \mathbb{E}_3(a)) \cap \mathcal{L}(\mathbb{E}_4(a), \mathbb{F}_4(a))$ for $j = 1, \ldots, n$. Moreover, for every given $a_0 > 0$ there is a constant $C_0 = C_0(a_0, p)$ such that
\[
\|\partial_j h\|_{\mathbb{E}_3(a)} + \|\partial_j h\|_{\mathbb{F}_4(a)} \leq C_0\|h\|_{\mathbb{E}_4(a)}
\]
for all $h \in \mathbb{E}_4(a)$ and all $a \in (0, a_0]$.

**Proof.** We refer to [25, Proposition 6.2] for a proof of (a)-(b). The assertion in (c) can established in the same way, using that $\mathbb{F}_4(a) \hookrightarrow BC(J; W^{1,3/p}_p(\mathbb{R}^n))$, see [19, Remark 5.3(d)]. In order to show that the embedding constant in (d) does not depend on $a \in (0, a_0]$ we define an extension operator in the following way: for $h \in \mathbb{E}_0(0, a; X)$, with $X$ an arbitrary Banach space, we first set $\hat{h}(t) := 0$ for $t \leq 0$, so that $\hat{h} \in BC^1((-\infty, a]; X)$, and then define
\[
(\mathcal{E}h)(t) := \begin{cases} 
\hat{h}(t) & \text{if } 0 \leq t \leq a, \\
3\hat{h}(2a - t) - 2\hat{h}(3a - 2t) & \text{if } a \leq t.
\end{cases}
\] (5.1)
A moment of reflection shows that $\mathcal{E}h \in \mathbb{E}_0(0, a; X)$, and that $\mathcal{E}h$ is an extension of $h$. It is evident that the norm of $\mathcal{E} : \mathbb{E}_0(0, a; X) \rightarrow \mathbb{E}_0(0, a; X)$ is independent of $a \in (0, a_0]$. The assertion follows now by the same arguments as in the proof of [25, Proposition 6.2].

Let $a_0 > 0$ be fixed. In order to establish part (e) it suffices to show that there is a constant $C_0 = C(a_0, p, r)$ such that
\[
\|g\|_{W^r_p(0, a; X)} \leq C_0\|g\|_{H^r_p(0, a; X)}, \quad a \in (0, a],
\] (5.2)
where $X$ is an arbitrary Banach space and $r \in [0,1]$. This follows from Hardy’s inequality as follows: for $r \in (0,1)$ we have

$$
\frac{1}{2} \langle g \rangle_{L^p([0,a];X)}^p = \int_0^a \int_0^a \frac{\|g(t) - g(s)\|_X^p}{(t-s)^{2r+2}} \, dt \, ds
$$

where

$$
\frac{\|g(t) - g(s)\|_X^p}{(t-s)^{2r+2}} = \int_0^a \frac{1}{(t-s)^{2r+2}} \left( \int_0^r \|\partial g(s+\sigma)\|_X \, d\sigma \right)^p \, d\tau \, ds
$$

$$
\leq c(r,p) \int_0^a \int_0^a \frac{1}{(t-s)^{2r+2}} \left( \int_0^r \|\partial g(s+\sigma)\|_X \, d\sigma \right)^p \, d\tau \, ds
$$

where $\partial g$ is the derivative of $g$, and this readily yields (5.2). □

Our next result will be important in order to derive estimates for the nonlinearities in (2.1).

**Lemma 5.2.** Suppose $p > n + 3$. Let $a_0 \in (0,\infty)$ be given. Then

(a) $E_3(a)$ is a multiplication algebra and we have the following estimate

$$
\|g_1 g_2\|_{E_3(a)} \leq (\|g_1\|_\infty + \|g_1\|_{E_3(a)}) (\|g_2\|_\infty + \|g_2\|_{E_3(a)})
$$

for all $(g_1, g_2) \in E_3(a) \times E_3(a)$ and all $a > 0$.

(b) There exists a constant $C_0 = C_0(a_0,p)$ such that

$$
\|g_1 g_2\|_{E_3(a)} \leq C_0 (\|g_1\|_\infty + \|g_1\|_{E_3(a)}) \|g_2\|_{a E_3(a)}
$$

for all $(g_1, g_2) \in E_3(a) \times a E_3(a)$ and all $a \in (0,a_0]$.

(c) There exists a constant $C_0 = C_0(a_0,p)$ such that

$$
\|g \partial_j h\|_{a E_3(a)} \leq C_0 (\|g\|_{E_3(a)} \|h\|_{a E_3(a)}, \quad j = 1, \ldots, n,
$$

for all $(g, h) \in E_3(a) \times a E_4(a)$ and $a \in (0,a_0]$.

(d) Suppose $(g, \psi) \in E_3(a) \times E_3(a)$ and let $\beta(t,x) := \sqrt{1 + \psi^2(t,x)}$. Then $\frac{g}{\beta^k} \in E_3(a)$ for $k \in \mathbb{N}$ and the following estimate holds

$$
\left\| \frac{g}{\beta^k} \right\|_{E_3(a)} \leq (1 + \|\psi\|_{E_3(a)})^k (\|g\|_\infty + \|g\|_{E_3(a)}).
$$

**Proof.** The assertions in (a)-(b) follow from (the proof of) Proposition 6.6.(ii) and (iv) in [25].

(c) To economize our notation we set $r = 1/2 - 1/2p$ and $\theta = 1 - 1/p$. Suppose that $(g, h) \in E_3(a) \times a E_4(a)$. We first observe that

$$
\|g \partial_j h\|_{L^p[J;L_p]} \leq (\|g\|_{L^p[J;L_p]} + \|g\|_{L^p[J;L_p]}) \|\partial_j h\|_{a BC(J;L_\infty)}
$$

$$
+ \left( \int_0^a \int_0^a \|g(s) (\partial_j h(t) - \partial_j h(s))\|_{L_p} \frac{dt \, ds}{|t-s|^{2r+2}} \right)^{1/p}.
$$
Using Hölder’s inequality, and the fact that $(1 - r - 1/p) = r > 0$, we obtain the estimate
\[
\int_0^a \int_0^a \|g(s)(\partial_t h(t) - \partial_t h(s))\|_{L^p_x} \frac{dt \, ds}{|t - s|^{1+rp}} 
\leq \int_0^a \int_0^a \|g(s)\|_{L^p_x} \left( \left| \int_s^t \|\partial_t \partial_x h(\tau)\|_{L^\infty} \, d\tau \right| \right)^p \frac{dt \, ds}{|t - s|^{1+rp}} 
\leq \int_0^a \int_0^a \|g(s)\|_{L^p_x} \left( \int_0^a \frac{dt}{|t - s|^{1+rp}} \right) ds \int_0^a \|\partial_t \partial_x h(\tau)\|_{L^p_x} \, d\tau 
\leq C_0(a_0, p) \|g\|_{L^p_x(J:L^p)} \|\partial_t \partial_x h\|_{L^p_x(J:L^\infty)} 
\] (5.7)
for $a \in (0, a_0]$. Hence we conclude that
\[
\|g\partial_t h\|_{W^r_p(J:L^p)} \leq C_0 \|g\|_{W^r_p(J:L^p) \cap L^\infty(J:L^\infty)} (\|\partial_t h\|_{BC(J:L^\infty)} + \|\partial_t \partial_x h\|_{L^p_x(J:L^\infty)}) 
\leq C_0 \|g\|_{\mathbb{E}_3(a)} \|h\|_{\mathbb{E}_3(a)} 
\] (5.8)
uniformly in $a \in (0, a_0]$. It is easy to verify that
\[
\|g\partial_t h\|_{L^p_x(J:W^r_p)} \leq \|g\|_{L^p_x(J:W^r_p)} \|\partial_t h\|_{BC(J:L^\infty)} + \|g\|_{L^p_x(J:L^\infty)} \|\partial_t \partial_x h\|_{BC(J:W^r_p)} 
\leq C_0 \|g\|_{\mathbb{E}_3(a)} \|h\|_{\mathbb{E}_3(a)} 
\] (5.9)
Combining the estimates (5.8), (5.9) yields (5.10).

(d) As in the proof of Proposition 6.6.(v) in [25] we obtain
\[
\|g/\beta\|_{W^r_p(J:L^p)} \leq \|1/\beta\|_{\infty} (\|g\|_{L^p_x(J:L^p)} + \|g\|_{W^r_p(J:L^p)}) 
\leq (1 + \|1/\beta\|_{W^r_p(J:L^p)}) (\|g\|_{\infty} + \|g\|_{W^r_p(J:L^p)}). 
\]
Thus it remains to estimate the term $\langle 1/\beta \rangle W^r_p(J:L^p)$. Using that $\beta^2(t, x) - \beta^2(s, x) = \psi^2(t, x) - \psi^2(s, x)$ one easily verifies that
\[
\left| \frac{1}{\beta(s, x)} - \frac{1}{\beta(t, x)} \right| = \left| \frac{\beta^2(t, x) - \beta^2(s, x)}{\beta^2(s, x) - \beta^2(t, x) + \beta^2(s, x)} \right| \leq |\psi(t, x) - \psi(s, x)| 
\]
and this yields $\langle 1/\beta \rangle W^r_p(J:L^p) \leq \langle \psi \rangle W^r_p(J:L^p)$. Consequently,
\[
\|g/\beta\|_{W^r_p(J:L^p)} \leq (1 + \|\psi\|_{W^r_p(J:L^p)}) (\|g\|_{\infty} + \|g\|_{W^r_p(J:L^p)}). 
\]
A similar argument shows that
\[
\|g/\beta\|_{L^p_x(J:W^r_p)} \leq (1 + \|\psi\|_{L^p_x(J:W^r_p)}) (\|g\|_{\infty} + \|g\|_{L^p_x(J:W^r_p)}). 
\]
Combining the last two estimates gives (5.6) for $k = 1$. The general case then follows by induction. \hfill $\square$

**Corollary 5.3.** Suppose $p > n + 3$. Let $a_0 \in (0, \infty)$ and $k \in \mathbb{N}$ with $k \geq 1$ be given.

(a) There exists a constant $C_0 = C_0(a_0, p, k)$ such that
\[
\|g_1 \cdots g_k \tilde{g}\|_{\mathbb{E}_3(a)} \leq C_0 \prod_{i=1}^k (\|g_i\|_{\infty} + \|g_i\|_{\mathbb{E}_3(a)}) \|\tilde{g}\|_{\mathbb{E}_3(a)} 
\]
for all functions $g_i \in \mathbb{E}_3(a)$, $1 \leq i \leq k$, $\tilde{g} \in \mathbb{E}_3(a)$, and all $a \in (0, a_0]$. 

Proposition 6.1(d) now implies the assertion for Remark 5.4 since Sobolov’s embedding theorem.

Next we note that

$$\| \partial_t h \cdots \partial_t h \partial_j \tilde{h} \|_{L_p(J; L_\infty)} \leq \| \partial_t \partial_j \tilde{h} \|_{L_p(J; L_\infty)},$$

and

$$\| \partial_t h \cdots (\partial_t \partial_t h) \partial_j \tilde{h} \|_{L_p(J; L_\infty)} \leq \| \partial_t \partial_j \tilde{h} \|_{L_p(J; L_\infty)} \| \partial_t \partial_t h \|_{L_p(J; L_\infty)} \| \partial_t h \|_{L_p(J; L_\infty)} \| \partial_j \tilde{h} \|_{L_p(J; L_\infty)}.$$

Proposition 6.1(d) now implies the assertion for \( \| \cdot \|_{W^p_p(J; L_p)} \). On the other hand we have by (5.9) for \( \theta = 1 - 1/p \)

$$\| g(\partial_t h \cdots \partial_t h \partial_j \tilde{h}) \|_{L_p(J; W^p_p)} \leq \| g \|_{L_p(J; W^p_p)} \| \partial_t h \cdots \partial_t h \partial_j \tilde{h} \|_{L_p(J; L_\infty)} + \| g \|_{L_p(J; L_\infty)}\| \partial_t h \cdots \partial_t h \partial_j \tilde{h} \|_{L_p(J; W^p_p)} \leq C_0 \| g \|_{E_3(a)} \left( \| \nabla h \|_{L_\infty}^k + \| \nabla h \|_{L_\infty}^k \right) \| h \|_{L_\infty} \| h \|_{L_\infty}$$

since \( W^p_p(\mathbb{R}^n) \) is a multiplication algebra. The last inequality then follows from Sobolov’s embedding theorem. \( \square \)

Remark 5.4. It can be shown that the estimate in (5.5) can be improved as follows: For every \( a_0 \in (0, \infty) \) there is a constant \( C_0 = C_0(a_0, p) > 0 \) and a constant \( \theta = \theta(p) > 0 \) such that

$$\| g \partial_t \tilde{h} \|_{L_\infty} \leq C_0 a_0^{\theta} \| g \|_{E_3(a)} \| \tilde{h} \|_{L_\infty}$$

holds for all \( (g, h) \in E_3(a) \times aE_4(a) \) and \( a \in (0, a_0) \). In the same way, the constant \( C_0 \) in Corollary 5.3(b) can be replaced by \( C_0a^\theta \).

Lemma 5.5. Suppose \( p > n + 3 \). Let \( a_0 \in (0, \infty) \) be given. Then

(a) \( F_4(a) \) is a multiplication algebra and we have the estimate

$$\| g_1 g_2 \|_{F_4(a)} \leq C_0 \| g_1 \|_{F_4(a)} \| g_2 \|_{F_4(a)}$$

for all \( (g_1, g_2) \in F_4(a) \times F_4(a) \), where the constant \( C_0 \) depends on \( a \).

(b) There exists a constant \( C_0 = C_0(a_0, p) \) such that

$$\| g_1 g_2 \|_{F_4(a)} \leq C_0 \left( \| g_1 \|_{F_4(a)} + \| g_1 \|_{F_4(a)} \right) \| g_2 \|_{F_4(a)}$$

for all \( (g_1, g_2) \in F_4(a) \times aF_4(a) \) and \( a \in (0, a_0) \).
(c) There exists a constant $C_0 = C_0(a_0, p)$ such that
\[ \| g_j h \|_{\mathcal{F}_4(a)} \leq C_0(\| g \|_{\infty} + \| g \|_{\mathcal{F}_4(a)}) \| h \|_{\mathcal{F}_4(a)}, \quad j = 1, \ldots, n, \] (5.11)
for all $(g, h) \in \mathbb{F}_4(a) \times \mathbb{F}_4(a)$ and $a \in (0, a_0]$.

Proof. Here we equip $\mathbb{F}_4(a)$ with the (equivalent) norm
\[ \| g \|_{\mathcal{F}_4(a)} = \| g \|_{W^{1-1/2p}(J; L_p)} + \sum_{i=1}^n \| \partial_i g \|_{L_p(J; W^{1-1/2p})}. \] (5.12)

(a) This follows from Proposition 5.1(c) by similar arguments as in the proof of Proposition 6.6(ii) and (iv) in [25].

(b) It follows from part (a) and Proposition 5.1(c) that
\[ \| g_1 g_2 \|_{W_p^r(J; L_p)} \leq C_0(\| g_1 \|_{\infty} + \| g_1 \|_{W_p^r(J; L_p)}) \| g_2 \|_{\mathcal{F}_4(a)}, \quad (g_1, g_2) \in \mathbb{F}_4(a) \times \mathbb{F}_4(a) \] where $r = 1 - 1/2p$. Next, observe that again by Proposition 5.1(c)
\[ \| \partial_t g_1 g_2 \|_{L_p(J; W_p^r)} \leq \| \partial_t g_1 \|_{L_p(J; W_p^r)} \| g_2 \|_{A^BC(J; L_{\infty})} + \| \partial_t g_1 \|_{L_p(J; L_{\infty})} \| g_2 \|_{A^BC(J; W_p^r)} \leq C_0(\| g_1 \|_{\mathcal{F}_4(a)} \| g_2 \|_{\mathcal{F}_4(a)}), \] where $\theta = 1 - 1/p$. Moreover,
\[ \| g_1 \partial_t g_2 \|_{L_p(J; W_p^r)} \leq \| g_1 \|_{L_p(J; W_p^r)} \| \partial_t g_2 \|_{A^BC(J; L_{\infty})} + \| g_1 \|_{\infty} \| \partial_t g_2 \|_{L_p(J; W_p^r)} \leq C_0(\| g_1 \|_{\infty} + \| g_1 \|_{\mathcal{F}_4(a)}) \| g_2 \|_{\mathcal{F}_4(a)}. \] The estimates above in conjunction with (5.12) yields (5.10).

(c) follows from (b) by setting $g_2 = \partial_j h$ and from Proposition 6.1(e), which certainly is also true for $0 \mathcal{F}_4(a)$.

Proof of Proposition 4.1:
It follows as in the proof of [28, Proposition 6.2] that $N_h \in C^\omega(\mathcal{E}(a), \mathcal{F}(a))$, and moreover, that $DH_h(z) \in \mathcal{L}(\mathcal{E}(a), \mathcal{F}(a))$ for $z \in \mathcal{E}(a)$. It thus remains to prove the estimates stated in the proposition.

Without always writing this explicitly, all the estimates derived below will be uniform in $a \in (0, a_0]$, for $a_0 > 0$ fixed. Moreover, all estimates will be uniform for $(\bar{u}, \bar{v}, \tilde{q}, \tilde{h}) \in 0 \mathcal{E}(a)$.

(i) Without changing notation we consider here the extension of $h$ from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ defined by $h(t, x, y) = h(t, x)$ for $(x, y) \in \mathbb{R}^n \times \mathbb{R}$ and $t \in J$. With this interpretation we have
\[ \| \partial h \|_{\infty, J \times \mathbb{R}^{n+1}} = \| \partial h \|_{\infty, J \times \mathbb{R}^n}, \quad \tilde{h} \in \mathcal{E}(a), \quad \partial \in \{ \partial_j, \Delta, \partial_t \}, \] (5.13)
where $\| \cdot \|_{\infty, U}$ denotes the sup-norm for the set $U \subset J \times \mathbb{R}^{n+1}$. Next we observe that
\[ BC(J; BC(\mathbb{R}^{n+1})) \cdot L_p(J; L_p(\mathbb{R}^{n+1})) \hookrightarrow L_p(J; L_p(\mathbb{R}^{n+1})), \]
\[ BC(J; L_p(\mathbb{R}^{n+1})) \cdot L_p(J; BC(\mathbb{R}^{n+1})) \hookrightarrow L_p(J; L_p(\mathbb{R}^{n+1})), \]
\[ BC(J; BC(\mathbb{R}^{n+1})) \cdot BC(J; BC(\mathbb{R}^{n+1})) \hookrightarrow BC(J; BC(\mathbb{R}^{n+1})), \] (5.14)
that is, multiplication is continuous and bilinear in the indicated function spaces (with norm equal to 1).
Let us first consider the term \( F_1(u, h) := |\nabla h|^2 \partial_y^2 u \) appearing in the definition of \( F \). Its Fréchet derivative at \((u, h)\) is given by
\[
DF_1(u, h)[\bar{u}, \bar{h}] = |\nabla h|^2 \partial_y^2 \bar{u} + 2(\nabla h | \nabla \bar{h}) \partial_y^2 u.
\]

Suppose \((\bar{u}, \bar{h}) \in aE_2(a) \times aE_4(a)\). From \((5.13)\), the first and third line in \((5.14)\) and Proposition \((5.1)\) that for all \((u, h)\) the estimate
\[
\|DF_1(u, h)[\bar{u}, \bar{h}]\|_{\mathcal{A}(a)} \leq C_0 \|\nabla h\|_{\infty} (\|\nabla h\|_{\infty} + \|u\|_{E_1(a)}) (\|\bar{u}\|_{E_2(a)} + \|\bar{h}\|_{E_4(a)})
\]
for all \((u, h) \in E_1(a) \times E_4(a)\). It is important to note that the constant \(C_0\) does not depend on the length of the interval \(J = (0, a)\) for \(a \in (0, a_0]\).

Next, let us take a closer look at the term \( F_2(u, h) := \Delta h \partial_y u \) in the definition of \( F \). The Fréchet derivative is \( DF_2(u, h)[\bar{u}, \bar{h}] = \Delta \bar{h} \partial_y \bar{u} + \Delta \bar{h} \partial_y u \). We infer from \((5.13)\), the first and second line in \((5.14)\), and Proposition \((5.1)\) that
\[
\|DF_2(u, h)[\bar{u}, \bar{h}]\|_{\mathcal{A}(a)} \leq (\|\nabla h\|_{L_p(J; L_\infty)} + \|\partial_y u\|_{L_p(J; L_p)}) \cdot (\|\partial_y \bar{u}\|_{\partial BC(J; L_p)} + \|\Delta \bar{h}\|_{\partial BC(J; L_\infty)})
\]
for all \((u, h) \in E_1(a) \times E_4(a)\).

The derivative of \( F_3(u, h) := (u|\nabla h) \partial_y u \), where \(\nabla h := (\nabla h, 0)\), is given by
\[
DF_3(u, h)[\bar{u}, \bar{h}] = (\bar{u} | \nabla h) \partial_y \bar{u} + (u | \nabla h) \partial_y u + (u | \nabla \bar{h}) \partial_y u
\]
and it follows from \((5.13) - (5.14)\) and Proposition \((5.1)\) that there is a constant \(C_0 > 0\) such that
\[
\|DF_3(u, h)[\bar{u}, \bar{h}]\|_{\mathcal{A}(a)} \leq C_0 (\|\nabla h\|_{\infty} + \|u\|_{\infty}) \|\bar{u}\|_{E_2(a)} + \|\bar{h}\|_{E_4(a)}
\]
for all \((u, h) \in E_1(a) \times E_4(a)\).

Let us also consider the term \( F_4(u, h) := \partial \partial_y \partial_y u \). Observing that
\[
DF_4(u, h)[\bar{u}, \bar{h}] = \partial \partial_y \partial_y \bar{u} + \partial \partial_y \partial_y u,
\]
that \(\partial_\pi : E_4(a) \to E_4(a)\) is linear and continuous and \(E_4(a) \hookrightarrow L_p(J; BC^1(\mathbb{R}^n) \cap BC(J; BC^1(\mathbb{R}^n)))\) \((5.15)\), we conclude from \((5.13) - (5.15)\) and Proposition \((5.1)\) that there is a constant \(C_0 = C_0(a_0)\) such that
\[
\|DF_4(u, h)[\bar{u}, \bar{h}]\|_{\mathcal{A}(a)} \leq C_0 \|\nabla h\|_{\infty} \|u\|_{E_1(a)} \|\bar{u}\|_{E_2(a)} + \|\bar{h}\|_{E_4(a)}
\]
for all \((u, h) \in E_1(a) \times E_4(a)\).

The derivative of \( F_5(\pi, h) := \partial \pi \nabla h \) is given by
\[
DF_5(\pi, h)[\bar{\pi}, \bar{h}] = \partial \pi \nabla \bar{h} + \partial \pi \nabla \bar{h}.
\]

It follows from \((5.13) - (5.14)\) and Proposition \((5.1)\) that there exists \(C_0\) such that
\[
\|DF_5(\pi, h)[\bar{\pi}, \bar{h}]\|_{\mathcal{A}(a)} \leq C_0 \|\nabla h\|_{\infty} \|\bar{\pi}\|_{L_p(J; L_p)} \|\bar{\pi}\|_{L_p(J; L_p)} \|\nabla \bar{h}\|_{a BC(J; L_\infty)}
\]
for all \((u, h) \in E_1(a) \times E_4(a)\).
for all \((\pi, h) \in \mathbb{E}_2(a) \times \mathbb{E}_4(a)\). The remaining terms in the definition of \(F\) can be analyzed in the same way. Summarizing we have shown that there is a constant \(C_0\) such that

\[
\|DF(u, \pi, h)[\tilde{u}, \tilde{\pi}, \tilde{h}]\|_{\mathcal{D}_4(a)} \leq C_0 \left[\|\nabla h\|_\infty + \|\nabla h\|_\infty^2 + \|\nabla u\|_\infty + \|\nabla u\|_\infty \|\bar{u}\|_{\mathcal{D}_{4}(a)}\right] (5.16)
\]

for all \((u, \pi, h) \in \mathcal{D}_4(a)\) and all \(a \in (0, a_0]\).

(ii) We will now consider the nonlinear function \(F_d(u, h) = (\nabla h|\partial_y v)\), stemming from the transformed divergence. Since \(h(x, y) := h(x)\) does not depend on \(y\) we have

\[
F_d(u, h) = (\nabla h|\partial_y u) = \partial_y (\nabla h|u). \tag{5.17}
\]

We note that

\[
\partial_y \in \mathcal{L}(H^1_p(J; \mathbb{L}(\mathbb{R}^{n+1})), H^1_p(J; \mathbb{H}^{-1}(\mathbb{R}^{n+1}))) \tag{5.18}
\]

The norm of this linear mapping does not depend on the length of the interval \(J = [0, a]\). It is easy to see that multiplication is continuous in the following function spaces:

\[
H^1_p(J; \mathcal{L}(\mathbb{B}(\mathbb{R}^{n+1})), \mathcal{L}(\mathbb{R}^{n+1})) \rightarrow H^1_p(J; \mathcal{L}(\mathbb{R}^{n+1})), \mathcal{L}(\mathbb{B}(\mathbb{R}^{n+1}))) \rightarrow L_p(J; \mathcal{L}(\mathbb{R}^{n+1}))). \tag{5.19}
\]

The derivative of \(F_d\) at \((u, h) \in \mathbb{E}_4(a)\) is given by

\[
DF_d(u, h)[\tilde{u}, \tilde{h}] = (\nabla h|\partial_y \tilde{u}) + (\nabla \tilde{h}|\partial_y u) \in \partial_y ((\nabla h|\tilde{u}) + (\nabla \tilde{h}|u)).
\]

We want to derive a uniform estimate for \(DF_d(u, h)[\tilde{u}, \tilde{h}]\) which does not depend on the length of the interval \(J = [0, a]\). We conclude from \((5.13)-(5.15)\) that

\[
\left\|\nabla h|\partial_y \tilde{u}\right\|_{\mathbb{E}^+(J; \mathbb{L}_p)} \sim \left\|\nabla h|\tilde{u}\right\|_{\mathbb{E}^+(J; \mathbb{L}_p)} + \left\|\partial_y \nabla h|\tilde{u}\right\|_{\mathbb{E}^+(J; \mathbb{L}_p)} + \left\|\nabla \tilde{h}|\partial_y \tilde{u}\right\|_{\mathbb{E}^+(J; \mathbb{L}_p)} \leq C_0\left\|\nabla h|\tilde{u}\right\|_{\mathcal{D}_4(a)} + \left\|\nabla \tilde{h}|\tilde{u}\right\|_{\mathcal{D}_4(a)} \leq C_0\left\|\nabla h|\tilde{u}\right\|_{\mathcal{D}_4(a)} + \left\|\nabla \tilde{h}|\tilde{u}\right\|_{\mathcal{D}_4(a)}
\]

for all \((u, h) \in \mathbb{E}_4(a)\) and \(a \in (0, a_0]\). Observing that \(\nabla h|\partial_y \tilde{u}\) defines an equivalent norm for \(\nabla h|\partial_y \tilde{u}\), we infer once more from \((5.13)-(5.14)\) and Proposition \((5.1)\) that

\[
\left\|\nabla h|\partial_y \tilde{u}\right\|_{\mathbb{E}^+(J; \mathbb{L}_p)} \leq C_0\left\|\nabla h\right\|_{\mathbb{E}_4(a)} + \left\|\nabla \tilde{h}\right\|_{\mathbb{E}_4(a)} \leq C_0\left\|\nabla h\right\|_{\mathcal{D}_4(a)} + \left\|\nabla \tilde{h}\right\|_{\mathcal{D}_4(a)}
\]

Summarizing we have shown that there exists a constant \(C_0\) such that

\[
\|DF_d(u, h)[\tilde{u}, \tilde{h}]\|_{\mathbb{E}_4(a)} \leq C_0\left\|\nabla h\right\|_{\mathcal{D}_4(a)} + \left\|\nabla \tilde{h}\right\|_{\mathcal{D}_4(a)} \tag{5.20}
\]

for all \((u, h) \in \mathbb{E}_4(a)\) and \(a \in (0, a_0]\).
are multilinear and continuous. Let us now take a closer look at the term 
\[ \[ \mu \partial_i u \] \partial_j h, \] \[ \[ \mu \partial_i u \] \partial_j h \partial_j h, \] \[ q \partial_j h, \] \[ \Delta h \partial_j h, \] \[ G_k(h), \] \[ G_k(h) \partial_j h \]
where \( u_k \) denotes the \( k \)-th component of a function \( u \in E_1(a) \). It follows from Lemma \( 5.2(a) \) and \( 5.21 \) that the mappings
\[ (u, h) \mapsto [\mu \partial_i u \] \partial_j h, \] \[ [\mu \partial_i u \] \partial_j h \partial_j h : E_1(a) \times E_4(a) \to E_3(a), \]
\[ (q, h) \mapsto q \partial_j h : E_3(a) \times E_4(a) \to E_3(a), \]
are multilinear and continuous. Let us now take a closer look at the term \( G_1(u, h) := [\mu \partial_i u \] \partial_j h. Its Fréchet derivative is given by
\[ DG_1(u, h)[\bar{u}, \bar{h}] = \partial_j h [\mu \partial_i u \] \partial_j \bar{h}. \]
Setting \( g_1 = \partial_j h \) and \( g_2 := [\mu \partial_i u \] \partial_j \bar{h} \) we obtain from \( 5.4 \) and \( 5.21 \) the estimate
\[ \| \partial_j h [\mu \partial_i u \] \partial_j \bar{h} \|_{E_3(a)} \leq C_0 (\| \nabla h \|_\infty + \| \nabla h \|_{E_3(a)} ) \| \bar{u} \|_{aE_1(a)}. \]
On the other hand, setting \( g := [\mu \partial_i u \] \partial_j h \) we conclude from \( 5.5 \) and \( 5.21 \) that
\[ \| [\mu \partial_i u \] \partial_j \bar{h} \|_{aE_3(a)} \leq C_0 \| u \|_{E_1(a)} \| \bar{h} \|_{aE_4(a)}. \]
Consequently,
\[ \| DG_1(u, h)[\bar{u}, \bar{h}] \|_{aE_3(a)} \]
\[ \leq C_0 (\| \nabla h \|_\infty + \| \nabla h \|_{E_3(a)} + \| u \|_{E_1(a)} ) \| \bar{u} \|_{aE_1(a)} + \| \bar{h} \|_{aE_4(a)} \] (5.22)
for all \( (u, h) \in E_1(a) \times E_4(a) \), and all \( a \in (0, a_0) \).

Given \( (u, h) \in E_1(a) \times E_4(a) \) let \( G_2(u, h) := [\mu \partial_i u \] \partial_j h \partial_j h. \) The Fréchet derivative of \( G_2 \) is given by
\[ DG_2(u, h)[\bar{u}, \bar{h}] = \partial_j h [\mu \partial_i u \] \partial_j \bar{h} + [\mu \partial_i u \] \partial_j h [\partial_j h \partial_j h + [\mu \partial_i u \] \partial_j h \partial_j \bar{h}. \]
From Corollary \( 5.3(a),(b) \) and \( 5.21 \) follows that there is a constant \( C_0 \) such that
\[ \| DG_2(u, h)[\bar{u}, \bar{h}] \|_{aE_3(a)} \leq C_0 (\| \nabla h \|_\infty + \| \nabla h \|_{E_3(a)}^2 \| \bar{u} \|_{aE_2(a)} + C_0 (\| \nabla h \|_{BC(J; BC^*)} + \| h \|_{E_4(a)} ) \| u \|_{E_1(a)} \| \bar{h} \|_{aE_4(a)} ) \]
(5.23)
for all \( (u, h) \in E_1(a) \times E_4(a) \) and all \( a \in (0, a_0) \).

The terms \( G_3(q, h) := q \partial_j h \) and \( G_4(h) := \Delta h \partial_j h \) can be analyzed in the same way as the term \( G_1 \), yielding the following estimates
\[ \| DG_3(q, h)[\bar{q}, \bar{h}] \|_{aE_3(a)} \]
\[ \leq C_0 (\| \nabla h \|_\infty + \| \nabla h \|_{E_3(a)} + \| q \|_{E_3(a)} ) \| \bar{q} \|_{aE_3(a)} + \| \bar{h} \|_{aE_4(a)} \] (5.24)
as well as
\[ \| DG_4(h)\bar{h} \|_{aE_3(a)} \leq C_0 (\| \nabla h \|_\infty + \| \nabla h \|_{E_3(a)} + \| \nabla^2 h \|_{E_3(a)} ) \| \bar{h} \|_{aE_4(a)}. \] (5.25)
Let us now consider the term
\[ G_5(h) = \frac{1}{(1 + \beta)^2} (\partial_j h)^2 \Delta h, \]
\( \beta(t, x) := \sqrt{1 + |\nabla h(t, x)|^2}, \)
appearing in the definition of \( G_\kappa \). The Fréchet derivative of \( G_5 \) at \( h \) is given by
\[
DG_5(h)\bar{h} = -\left(\frac{1}{(1+\beta)^2\beta^2} + \frac{1}{(1+\beta)^3}\right)(\partial_j h)^2 \Delta h \partial_k \partial_l \bar{h} \times (\partial_j h)^2 \Delta h \partial_k \partial_l \bar{h} + \frac{1}{(1+\beta)^3}(2\partial_j h \Delta h \partial_k \bar{h} + (\partial_j h)^2 \Delta h).
\]

Before continuing, we note that the term \( 1/(1+\beta) \) can be treated in exactly the same way as \( 1/\beta \), as a short inspection of the proof of Lemma 5.2(d) shows. It follows then from Corollary 5.3(a)–(b) and from (5.6) that there is a constant \( C_0 \) such that
\[
\|DG_5(h)\bar{h}\|_{E_5(a)} \leq C_0 \left[P(\|\nabla h\|_\infty) + Q(\|\nabla h\|_{BC(J;BC^1)} , \|h\|_{E_4(a)}) \right]\|\bar{h}\|_{E_4(a)} \quad (5.26)
\]
for all \( h \in E_4(a) \) and all \( a \in (0,\alpha] \), where \( P \) and \( Q \) are polynomials with coefficients equal to one and vanishing zero-order terms. Analogous arguments can be used for the remaining terms \( \langle \nabla h \nabla^2 h \rangle / \beta^3 \) and \( G_\kappa(h) \partial_j h \) appearing in \( G \), yielding the same estimate as in (5.26).

(iv) It remains to consider the nonlinear term \( H_b(v, h) := (b - \gamma v \nabla h) \). The Fréchet derivative is given by \( D H_b(v, h) \bar{v} = -(\nabla h) \gamma \bar{v} + (b - \gamma v) \Delta \bar{h} \). From Lemma 5.5(b) with \( g_1 = \partial_j h \) and \( g_2 = \gamma \bar{v}_k \), where \( \bar{v}_k \) denotes the \( k \)-th component of \( \bar{v} \), follows
\[
\|\nabla h\gamma \bar{v}\|_{F_4(a)} \leq C_0(\|\nabla h\|_\infty + \|h\|_{E_4(a)}) \|\bar{v}\|_{E_4(a)}.
\]
Lemma 5.5(c) with \( g = (b - \gamma v)_k \) and \( h = \bar{h} \) implies
\[
\|b - \gamma v \nabla \bar{h}\|_{F_4(a)} \leq C_0(\|b - \gamma v\|_\infty + \|b - \gamma v\|_{E_4(a)}) \|\bar{h}\|_{E_4(a)}.
\]
We have, thus, shown that
\[
\|DH_b(v, h)\|_N \leq C_0(\|\nabla h\|_\infty + \|h\|_{E_4(a)} + \|b - \gamma v\|_{BC(J;BC)^3} \cap F_4(a)). \quad (5.27)
\]
Combining the estimates in (5.26), (5.28) and (5.27) yields the assertions of Proposition 4.1. \( \square \)

REFERENCES
[1] G. Allain, Small-time existence for the Navier-Stokes equations with a free surface. Appl. Math. Optim. 16 (1987), 37–50.
[2] H. Amann, Linear and Quasilinear Parabolic Problems. Vol. I. Abstract Linear Theory. Monographs in Mathematics 89, Birkhäuser, Boston, 1995.
[3] J.T. Beale, Large-time regularity of viscous surface waves. Arch. Rational Mech. Anal. 84, (1983/84), 304–352.
[4] J.T. Beale, J. Prüss, G. Simonett, Well-posedness of a two-phase flow with soluble surfactant. Nonlinear elliptic and parabolic problems, Progress Nonlinear Differential Equations Appl., 64, Birkhäuser, Basel, 2005, 37–61.
[5] J.T. Beale, J. Prüss, Lp-theory for a class of non-Newtonian fluids. SIAM J. Math. Anal. 39 (2007), 379–421.
[11] I.V. Denisova, A priori estimates for the solution of the linear nonstationary problem connected with the motion of a drop in a liquid medium. (Russian) Trudy Mat. Inst. Steklov 188 (1990), 3–21. Translated in Proc. Steklov Inst. Math. 1991, no. 3, 1–24.

[12] I.V. Denisova, Problem of the motion of two viscous incompressible fluids separated by a closed free interface. Mathematical problems for Navier-Stokes equations (Centro, 1993). Acta Appl. Math. 37 (1994), 31–40.

[13] I.V. Denisova, V.A. Solonnikov, Classical solvability of the problem of the motion of two viscous incompressible fluids. (Russian) Algebra i Analiz 7 (1995), no. 5, 101–142. Translation in St. Petersburg Math. J. 7 (1996), no. 5, 755–786.

[14] R. Denk, M. Hieber, J. Prüss, $\mathcal{R}$-boundedness, Fourier multipliers, and problems of elliptic and parabolic type. AMS Memoirs 788, Providence, R.I. (2003).

[15] R. Denk, G. Dore, M. Hieber, J. Prüss, A. Venni, Some new thoughts on old results of R.T. Seeley. Math. Annalen 328 (2004) 545–583.

[16] G. Dore, A. Venni, On the closedness of the sum of two closed operators. Math. Z. 196 (1987), no. 2, 189–201.

[17] J. Escher, G. Simonett, Analyticity of the interface in a free boundary problem. Math. Ann. 305 (1996), no. 3, 439–459.

[18] J. Escher, G. Simonett, Analyticity of solutions to fully nonlinear parabolic evolution equations on symmetric spaces. Dedicated to Philippe Bénilan. J. Evol. Equ. 3 (2003), no. 4, 549–576.

[19] J. Escher, J. Prüss, G. Simonett, Analytic solutions for a Stefan problem with Gibbs-Thomson correction. J. Reine Angew. Math. 563 (2003), 1–52.

[20] J. Escher, J. Prüss, G. Simonett, A new approach to the regularity of solutions for parabolic equations. Evolution equations, 167–190, Lecture Notes in Pure and Appl. Math., 234, Dekker, New York, 2003.

[21] G.P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations. Vol. I. Linearized steady problems. Springer Tracts in Natural Philosophy, 38. Springer-Verlag, New York, 1994.

[22] N. Kalton, L. Weis, The $H^\infty$-calculus and sums of closed operators. Math. Ann. 321 (2001), 319–345.

[23] P.C. Kunstmann, L. Weis, Maximal $L_p$-regularity for parabolic equations, Fourier multiplier theorems and $H^\infty$-functional calculus. Functional analytic methods for evolution equations, 65–311, Lecture Notes in Math., 1855, Springer, Berlin, 2004.

[24] K. Masuda, On the regularity of solutions of the nonstationary Navier-Stokes equations, in: Approximation Methods for Navier-Stokes Problems, 360–370, Lecture Notes in Mathematics 771, Springer-Verlag, Berlin, 1980.

[25] J. Prüss, J. Saal, G. Simonett, Existence of analytic solutions for the classical Stefan problem. Math. Ann. 338 (2007), 703–755.

[26] J. Prüss, G. Simonett, $H^\infty$-calculus for the sum of noncommuting operators. Trans. Amer. Math. Soc. 359 (2007), no. 8, 3549–3565.

[27] J. Prüss, G. Simonett, Analysis of the boundary symbol for the two-phase Navier-Stokes equations with surface tension. Banach Center Publ. 86 (2009), 265–285.

[28] J. Prüss, G. Simonett, On the two-phase Navier-Stokes equations wit surface tension. Submitted for publication.

[29] J. Prüss, G. Simonett, On the Rayleigh-Taylor instability for the two-phase Navier-Stokes equations. Preprint.

[30] J. Prüss, H. Sohr, On operators with bounded imaginary powers in Banach spaces. Math. Z. 203 (1990), 429–452.

[31] Y. Shibata, S. Shimizu, On a free boundary problem for the Navier-Stokes equations. Differential Integral Equations 20 (2007), no. 3, 241–276.

[32] Y. Shibata, S. Shimizu, Local solvability of free boundary problems for the Navier-Stokes equations with surface tension. Preprint.

[33] V.A. Solonnikov, Lectures on evolution free boundary problems: classical solutions. Mathematical aspects of evolving interfaces (Funchal, 2000), 123–175, Lecture Notes in Math., 1812, Springer, Berlin, 2003.

[34] N. Tanaka, Two-phase free boundary problem for viscous incompressible thermo-capillary convection. Japan J. Mech. 21 (1995), 1–41.
[35] A. Tani, Small-time existence for the three-dimensional Navier-Stokes equations for an incompressible fluid with a free surface. Arch. Rational Mech. Anal. 133 (1996), 299–331.

[36] A. Tani, N. Tanaka, Large-time existence of surface waves in incompressible viscous fluids with or without surface tension. Arch. Rat. Mech. Anal. 130 (1995), 303–304.

[37] H. Triebel, Theory of function spaces. Monographs in Mathematics, 78. Birkhäuser Verlag, Basel, 1983.

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