GENERALIZATIONS OF MINKOWSKI AND BECKENBACH–DRESHER INEQUALITIES AND FUNCTIONALS ON TIME SCALES

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Abstract. We generalize integral forms of the Minkowski inequality and Beckenbach–Dresher inequality on time scales. Also, we investigate a converse of Minkowski’s inequality and several functionals arising from the Minkowski inequality and the Beckenbach–Dresher inequality.

1. Introduction and Preliminaries

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers. The theory of time scales was introduced by Stefan Hilger [7] in order to unify the theory of difference equations and the theory of differential equations. For an introduction to the theory of dynamic equations on time scales, we refer to [3,8]. Martin Bohner and Gusein Sh. Guseinov [4,5] defined the multiple Riemann and multiple Lebesgue integration on time scales and compared the Lebesgue $\Delta$-integral with the Riemann $\Delta$-integral.

Let $n \in \mathbb{N}$ be fixed. For each $i \in \{1, \ldots, n\}$, let $T_i$ denote a time scale and

$$\Lambda^n = T_1 \times \ldots \times T_n = \{t = (t_1, \ldots, t_n) : t_i \in T_i, 1 \leq i \leq n\}$$
an \( n \)-dimensional time scale. Let \( \mu_\Delta \) be the \( \sigma \)-additive Lebesgue \( \Delta \)-measure on \( \Lambda^n \) and \( \mathcal{F} \) be the family of \( \Delta \)-measurable subsets of \( \Lambda^n \). Let \( E \in \mathcal{F} \) and \( (E, \mathcal{F}, \mu_\Delta) \) be a time scale measure space. Then for a \( \Delta \)-measurable function \( f : E \to \mathbb{R} \), the corresponding \( \Delta \)-integral of \( f \) over \( E \) will be denoted according to \([5, (3.18)]\) by

\[
\int_E f(t_1, \ldots, t_n) \Delta t_1 \ldots \Delta t_n, \quad \int_E f(t) \Delta t, \quad \int_E f \, d\mu_\Delta, \quad \text{or} \quad \int_E f(t) d\mu_\Delta(t).
\]

By [5, Section 3], all theorems of the general Lebesgue integration theory, including the Lebesgue dominated convergence theorem, hold also for Lebesgue \( \Delta \)-integrals on \( \Lambda \). Here we state Fubini's theorem for integrals on time scales. It is used in the proofs of our main results.

**Theorem 1.1** (Fubini's theorem). Let \((X, \mathcal{M}, \mu_\Delta) \) and \((Y, \mathcal{L}, \nu_\Delta) \) be two finite-dimensional time scale measure spaces. If \( f : X \times Y \to \mathbb{R} \) is a \( \Delta \)-integrable function and if we define the functions

\[
\varphi(y) = \int_X f(x, y) d\mu_\Delta(x) \quad \text{for a.e.} \quad y \in Y
\]

and

\[
\psi(x) = \int_Y f(x, y) d\nu_\Delta(y) \quad \text{for a.e.} \quad x \in X,
\]

then \( \varphi \) is \( \Delta \)-integrable on \( Y \) and \( \psi \) is \( \Delta \)-integrable on \( X \) and

\[
\int_X d\mu_\Delta(x) \int_Y d\nu_\Delta(y) = \int_X f(x, y) d\mu_\Delta(x) \int_Y f(x, y) d\nu_\Delta(y).
\]  

(1.1)

Hölder’s inequality and Minkowski’s inequality and their converses for multiple integration on time scales were investigated in \([1]\). These inequalities hold for both Riemann integrals and Lebesgue integrals on time scales. For completeness, let us recall these inequalities from \([1]\).

**Theorem 1.2** (Hölder’s inequality \([1, \text{Theorem 6.2}]\)). For \( p \neq 1 \), define \( q = p/(p - 1) \). Let \((E, \mathcal{F}, \mu_\Delta) \) be a time scale measure space. Assume \( w, f, g \) are nonnegative functions such that \( wf^p, wg^q, wfg \) are \( \Delta \)-integrable on \( E \). If \( p > 1 \), then

\[
\int_E w(t)f(t)g(t)d\mu_\Delta(t) \leq \left( \int_E w(t)f^p(t)d\mu_\Delta(t) \right)^{1/p} \times \left( \int_E w(t)g^q(t)d\mu_\Delta(t) \right)^{1/q}.
\]  

(1.2)

If \( 0 < p < 1 \) and \( \int_E wg^q d\mu_\Delta > 0 \), or if \( p < 0 \) and \( \int_E w f^p d\mu_\Delta > 0 \), then (1.2) is reversed.

**Theorem 1.3** (Minkowski’s inequality \([1, \text{Theorem 7.2}]\)). Let \((E, \mathcal{F}, \mu_\Delta) \) be a time scale measure space. For \( p \in \mathbb{R} \), assume \( w, f, g \), are nonnegative functions such that \( wf^p, wg^p, w(f + g)^p \) are \( \Delta \)-integrable on \( E \). If \( p \geq 1 \), then

\[
\left( \int_E w(t)(f(t) + g(t))^p d\mu_\Delta(t) \right)^{1/p} \leq \left( \int_E w(t)f^p(t)d\mu_\Delta(t) \right)^{1/p} + \left( \int_E w(t)g^p(t)d\mu_\Delta(t) \right)^{1/p}.
\]  

(1.3)

If \( 0 < p < 1 \) or \( p < 0 \), then (1.3) is reversed provided each of the two terms on the right-hand side is positive.
If \( p \neq 1 \), define \( q = p/(p - 1) \). Let \((E, F, \mu_\Delta)\) be a time scale measure space. Assume \( w, f, g \) are nonnegative functions such that \( w^p, wg^q, wfg \) are \( \Delta \)-integrable on \( E \). Suppose

\[
0 < m \leq f(t)g^{-q/p}(t) \leq M \quad \text{for all} \quad t \in E.
\]

If \( p > 1 \), then

\[
\int_E w(t)f(t)g(t)d\mu_\Delta(t) \geq K(p, m, M) \left( \int_E w(t)f^p(t)d\mu_\Delta(t) \right)^{1/p} \times \left( \int_E w(t)g^q(t)d\mu_\Delta(t) \right)^{1/q},
\]

where

\[
K(p, m, M) = |p|^{1/p}|q|^{1/q} \frac{(M - m)^{1/p}|mM^p - Mm^p|^{1/q}}{|M^p - m^p|}.
\]

If \( 0 < p < 1 \) or \( p < 0 \), then (1.4) is reversed provided either \( \int_E wg^q d\mu_\Delta > 0 \) or \( \int_E w^p d\mu_\Delta > 0 \).

In [2] Bibi et al., obtain integral forms of Minkowski’s and Beckenbach–Dresher inequality on time scales. In this paper we generalize these inequalities and investigate functional obtained from our new inequalities.

2. Minkowski Inequalities

Let \( U_l(x_1, x_2, \ldots, x_l), V_m(x_1, x_2, \ldots, x_m), G_k(x_1, x_2, \ldots, x_k) \), are real valued functions of \( l, m, \) and \( k \) variables, respectively. Let \((X, M, \mu_\Delta)\) and \((Y, L, \nu_\Delta)\) be time scale measure spaces. Then, throughout in the following sections, we use the following notations:

\[
U_l = U_l(x) = U_l(u_1(x), u_2(x), \ldots, u_l(x)),
\]

\[
V_m = V_m(y) = V_m(v_1(y), v_2(y), \ldots, v_m(y)),
\]

\[
F_k = F_k(x, y) = F_k(f_1(x, y), f_2(x, y), \ldots, f_k(x, y)),
\]

where \( \{u_i(x)\}_{i=1}^l, \{v_i(y)\}_{i=1}^m, \{f_i(x, y)\}_{i=1}^k \), are defined on \( X, Y, \) and \( X \times Y, \) respectively. In the sequel, we assume that all occurring integrals are finite.

**Theorem 2.1** (Integral Minkowski inequality). If \( p \geq 1 \), then

\[
\left[ \int_X \left( \int_Y F_k(x, y)V_m(y)d\nu_\Delta(y) \right)^p U_l(x)d\mu_\Delta(x) \right]^{\frac{1}{p}} \leq \int_Y \left( \int_X F_k^p(x, y)U_l(x)d\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)d\nu_\Delta(y)
\]

holds provided all integrals in (2.2) exists. If \( 0 < p < 1 \) and

\[
\int_X \left( \int_Y F_kV_m d\nu_\Delta \right)^p U_l d\mu_\Delta > 0, \quad \int_Y F_k V_m d\nu_\Delta > 0
\]

holds, then (2.2) is reversed. If \( p < 0 \) and (2.3) and

\[
\int_X F_k^pU_l d\mu_\Delta > 0,
\]

hold, then (2.2) is reversed as well.
Proof. Let \( p \geq 1 \). Put

\[
H(x) = \int_Y F_k(x,y)V_m(y)d\nu(y).
\]

Now, by using Fubini’s theorem (Theorem 1.1) and Hölder’s inequality (Theorem 1.2) on time scales, we have

\[
\int_X H^p(x)U_l(x)d\mu(x) = \int_X H(x)H^{p-1}(x)U_l(x)d\mu(x)
\]

\[
= \int_Y \left( \int_X F_k(x,y)V_m(y)d\nu(y) \right) H^{p-1}(x)U_l(x)d\mu(x)
\]

\[
= \int_Y \left( \int_X F_k(x,y)H^{p-1}(x)U_l(x)d\mu(x) \right) V_m(y)d\nu(y)
\]

\[
\leq \int_Y \left( \int_X F^p_k(x,y)U_l(x)d\mu(x) \right) \left( \int_X H^p(x)U_l(x)d\mu(x) \right)^{p-1} V_m(y)d\nu(y)
\]

\[
= \int_Y \left( \int_X F^p_k(x,y)U_l(x)d\mu(x) \right) \frac{1}{p} V_m(y)d\nu(y) \left( \int_X H^p(x)U_l(x)d\mu(x) \right)^{\frac{p-1}{p}}
\]

and hence

\[
\left( \int_X H^p(x)U_l(x)d\mu(x) \right)^\frac{1}{p} \leq \int_Y \left( \int_X F^p_k(x,y)U_l(x)d\mu(x) \right)^\frac{1}{p} V_m(y)d\nu(y).
\]

For \( p < 0 \) and \( 0 < p < 1 \), the corresponding results can be obtained similarly. \( \square \)

**Theorem 2.2** (Converse of integral Minkowski inequality). Suppose

\[
0 < m \leq \frac{F_k(x,y)}{\int_Y F_k(x,y)V_m(y)d\nu(y)} \leq M \quad \text{for all} \quad x \in X, \ y \in Y.
\]

If \( p \geq 1 \), then

\[
\left[ \int_X \left( \int_Y F_k(x,y)V_m(y)d\nu(y) \right)^p U_l(x)d\mu(x) \right]^\frac{1}{p} \geq K(p,m,M) \int_Y \left( \int_X F^p_k(x,y)U_l(x)d\mu(x) \right)^\frac{1}{p} V_m(y)d\nu(y)
\]

(2.5)

provided all integrals in (2.5) exist, where \( K(p,m,M) \) is defined by (1.5). If \( 0 < p < 1 \) and (2.3) holds, then (2.5) is reversed. If \( p < 0 \) and (2.3) and (2.4) hold, then (2.5) is reversed as well.

Proof. Let \( p \geq 1 \). Put

\[
H(x) = \int_Y F_k(x,y)V_m(y)d\nu(y).
\]
Then by using Fubini’s theorem (Theorem 1.1) and the converse Hölder inequality (Theorem 1.4) on time scales, we get
\[
\int_X H^p(x)U_t(x)d\mu_\Delta(x) = \int_X \left( \int_Y F_k(x,y)V_m(y)d\nu_\Delta(y) \right) H^{p-1}(x)U_t(x)d\mu_\Delta(x)
\]
\[
= \int_Y \left( \int_X F_k(x,y)H^{p-1}(x)U_t(x)d\mu_\Delta(x) \right) V_m(y)d\nu_\Delta(y)
\]
\[
\geq K(p, m, M) \int_Y \left( \int_X F_k^p(x,y)U_t(x)d\mu_\Delta(x) \right)^{1/p}
\]
\[
\times \left( \int_X H^p(x)U_t(x)d\mu_\Delta(x) \right)^{p-1} V_m(y)d\nu_\Delta(y).
\]
Dividing both sides by \( (\int_X H^p(x)U_t(x)d\mu_\Delta(x))^{\frac{p-1}{p}} \), we obtain (2.5). For \( 0 < p < 1 \) and \( p < 0 \), the corresponding results can be obtained similarly.

Now we define the \( r \)th power mean \( M^{[r]}(F_k, \mu_\Delta) \) of the function \( F_k \) with respect to the measure \( \mu_\Delta \) by
\[
M^{[r]}(F_k, \mu_\Delta) = \begin{cases} 
\left( \frac{\int_X F_k^r(x,y)U_t(x)d\mu_\Delta(x)}{\int_X U_t(x)d\mu_\Delta(x)} \right)^{\frac{1}{r}} & \text{if } r \neq 0, \\
\exp \left( \frac{\int_X \log F_k(x,y)V_m(y)d\mu_\Delta(x)}{\int_X V_m(y)d\mu_\Delta(x)} \right) & \text{if } r = 0,
\end{cases}
\]
(2.6)

where \( \int_X U_t d\mu_\Delta > 0 \).

Corollary 2.1. Let \( 0 < s \leq r \). Then
\[
M^{[r]}(M^{[s]}(F_k, d\nu_\Delta), d\mu_\Delta) \geq K \left( \frac{r}{s}, m, M \right) M^{[s]}(M^{[r]}(F_k, d\mu_\Delta), d\nu_\Delta).
\]

Proof. By putting \( p = r/s \) and replacing \( F_k \) by \( F_k^s \) in (2.5), raising to the power of \( \frac{1}{s} \) and dividing by
\[
\left( \int_X U_t(x)d\mu_\Delta(x) \right)^{\frac{1}{s}} \left( \int_Y V_m(y)d\nu_\Delta(y) \right)^{\frac{1}{s}},
\]
we get the above result.

3. Minkowski Functionals

In this section, we will consider some functionals which arise from the Minkowski inequality. Similar results (but not for time scales measure spaces) can be found in [9].

Let \( F_k \) and \( V_m \) be fixed functions satisfying the assumptions of Theorem 2.1. Let us consider the functional \( M_1 \) defined by
\[
M_1(U_t) = \left[ \int_Y \left( \int_X F_k^p(x,y)U_t(x)d\mu_\Delta(x) \right)^{\frac{p}{s}} V_m(y)d\nu_\Delta(y) \right]^p - \int_X \left( \int_Y F_k(x,y)U_t(x)d\mu_\Delta(x) \right)^p U_t(x)d\mu_\Delta(x),
\]
where $U_i$ is a nonnegative function on $X$ such that all occurring integrals exist. Also, if we fix the functions $F_k$ and $U_i$, then we can consider the functional

$$M_2(V_m) = \int_Y \left( \int_X F_k^p(x,y)U_i(x) d\mu_\Delta(x) \right)^\frac{1}{p} V_m(y) d\nu_\Delta(y)$$

$$- \left[ \int_X \left( \int_Y F_k(x,y)V_m(y) d\nu_\Delta(y) \right)^p U_i(x) d\mu_\Delta(x) \right]^\frac{1}{p},$$

where $V_m$ is a nonnegative function on $Y$ such that all occurring integrals exist.

**Remark 3.1.**

(i) It is obvious that $M_1$ and $M_2$ are positive homogeneous, i.e., $M_1(aU_i) = aM_1(U_i)$, and $M_2(aV_m) = aM_2(V_m)$, for any $a > 0$.

(ii) If $p \geq 1$ or $p < 0$, then $M_1(U_i) \geq 0$, and if $0 < p < 1$, then $M_1(U_i) \leq 0$.

(iii) If $p \geq 1$, then $M_2(V_m) \geq 0$, and if $p < 1$ and $p \neq 0$, then $M_2(V_m) \leq 0$.

**Theorem 3.1.**

(i) If $p \geq 1$ or $p < 0$, then $M_1$ is superadditive. If $0 < p < 1$, then $M_1$ is subadditive.

(ii) If $p \geq 1$, then $M_2$ is superadditive. If $p < 1$ and $p \neq 0$, then $M_2$ is subadditive.

(iii) Suppose $U_{i1}$ and $U_{i2}$ are nonnegative functions such that $U_{i2} \geq U_{i1}$. If $p \geq 1$ or $p < 0$, then

$$0 \leq M_1(U_{i1}) \leq M_1(U_{i2}),$$

and if $0 < p < 1$, then (3.1) is reversed.

(iv) Suppose $V_{m1}$ and $V_{m2}$ are nonnegative functions such that $V_{m2} \geq V_{m1}$. If $p \geq 1$, then

$$0 \leq M_2(V_{m1}) \leq M_2(V_{m2}),$$

and if $p < 1$ and $p \neq 0$, then (3.2) is reversed.

**Proof.** First we show (i). We have

\[
M_1(U_{i1} + U_{i2}) - M_1(U_{i1}) - M_1(U_{i2})
= \left[ \int_Y \left( \int_X f^p(x,y)(U_{i1} + U_{i2})(x) d\mu_\Delta(x) \right)^\frac{1}{p} V_m(y) d\nu_\Delta(y) \right]^p
- \int_X \left( \int_Y F_k(x,y)V_m(y) d\nu_\Delta(y) \right)^p (U_{i1} + U_{i2})(x) d\mu_\Delta(x)
- \left[ \int_X \left( \int_Y F_k^p(x,y)U_{i1}(x) d\mu_\Delta(x) \right)^\frac{1}{p} V_m(y) d\nu_\Delta(y) \right]^p
+ \int_X \left( \int_Y F_k(x,y)V_m(y) d\nu_\Delta(y) \right)^p U_{i1}(x) d\mu_\Delta(x)
- \left[ \int_X \left( \int_Y F_k^p(x,y)U_{i2}(x) d\mu_\Delta(x) \right)^\frac{1}{p} V_m(y) d\nu_\Delta(y) \right]^p
\]
\[
+ \int_X \left( \int_Y F_k(x, y)V_m(y)d\nu_\Delta(y) \right)^p U_{i2}(x)d\mu_\Delta(x)
\]
\[
= \left[ \int_Y \left( \int_X F_k^p(x, y)U_{i1}(x)d\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)d\nu_\Delta(y) \right]^p
\]
\[
- \left[ \int_Y \left( \int_X F_k^p(x, y)U_{i1}(x)d\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)d\nu_\Delta(y) \right]^p
\]
\[
- \left[ \int_Y \left( \int_X F_k^p(x, y)U_{i2}(x)d\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)d\nu_\Delta(y) \right]^p.
\]

Using the Minkowski inequality (1.3) for integrals (Theorem 1.3) with \( p \) replaced by \( 1/p \), we have
\[
M_1(U_{i1} + U_{i2}) - M_1(U_{i1}) - M_1(U_{i2}) \begin{cases} 
\geq 0 & \text{if } p \geq 1 \text{ or } p < 0, \\
\leq 0 & \text{if } 0 < p \leq 1.
\end{cases}
\]
(3.3)

So, \( M_1 \) is superadditive for \( p \geq 1 \) or \( p < 0 \), and it is subadditive for \( 0 < p \leq 1 \). The proof of (ii) is similar:

After a simple calculation, we have
\[
M_2(V_{m1} + V_{m2}) - M_2(V_{m1}) - M_2(V_{m2})
\]
\[
= \left[ \int_X \left( \int_Y F_k(x, y)V_{m1}(y)d\nu_\Delta(y) \right)^p U_{i1}(x)d\mu_\Delta(x) \right]^{\frac{1}{p}}
\]
\[
+ \left[ \int_X \left( \int_Y F_k(x, y)V_{m2}(y)d\nu_\Delta(y) \right)^p U_{i1}(x)d\mu_\Delta(x) \right]^{\frac{1}{p}}
\]
\[
- \left[ \int_X \left( \int_Y F_k(x, y)(V_{m1}(y) + V_{m2}(y))d\nu_\Delta(y) \right)^p U_{i1}(x)d\mu_\Delta(x) \right]^{\frac{1}{p}}.
\]

Using the Minkowski inequality (2.2) for integrals (Theorem 2.1), we have that this is nonnegative for \( p \geq 1 \) and nonpositive for \( p < 1 \) and \( p \neq 0 \). Now we show (iii). If \( p \geq 1 \) or \( p < 0 \), then using superadditivity and positivity of \( M_1, U_{i2} \geq U_{i1} \) implies
\[
M_1(U_{i2}) = M_1(U_{i1} + (U_{i2} - U_{i1})) \geq M_1(U_{i1}) + M_1(U_{i2} - U_{i1}) \geq M_1(U_{i1}),
\]
and the proof of (3.1) is established. If \( 0 < p < 1 \), then using subadditivity and negativity of \( M_1, U_{i2} \geq U_{i1} \) implies
\[
M_1(U_{i2}) \leq M_1(U_{i1}) + M_1(U_{i2} - U_{i1}) \leq M_1(U_{i1}).
\]

The proof of (iv) is similar. \( \square \)

**Remark 3.2.** Put \( X, Y \subseteq \mathbb{N} \), then for fixed \( F_k \) and \( U_i \), the function \( M_2 \) has the form
\[
M_2(V_{m1}) = \sum_{j \in Y} V_{m1}(j) \left( \sum_{i \in X} U_i(i) F_k(i, j)^p \right)^{1/p} - \left( \sum_{i \in X} U_i(i) \left( \sum_{j \in Y} V_{m1}(j) F_k(i, j) \right)^p \right)^{1/p},
\]
where \( f(i, j) = F_k(i, j) \geq 0 \). If \( p \geq 1 \), then the mapping \( M_2 \) is superadditive, and \( V_{m_2}(j) \geq V_{m_1}(j) \) for all \( j \in Y \) implies

\[
0 \leq \sum_{j \in Y} V_{m_1}(j) \left( \sum_{i \in X} U_i(i) F_k(i, j)^p \right)^{1/p} - \left( \sum_{i \in X} U_i(i) \left( \sum_{j \in Y} V_{m_1}(j) F_k(i, j) \right)^p \right)^{1/p} \leq \sum_{j \in Y} V_{m_2}(j) \left( \sum_{i \in X} U_i(i) F_k(i, j)^p \right)^{1/p} - \left( \sum_{i \in X} U_i(i) \left( \sum_{j \in Y} V_{m_2}(j) F_k(i, j) \right)^p \right)^{1/p}
\]

provided all occurring sums are finite.

**Corollary 3.1.**  
(i) Suppose \( U_{11} \) and \( U_{12} \) are nonnegative functions such that \( CU_{12} \geq U_{11} \geq cU_{12} \), where \( c, C \geq 0 \). If \( p \geq 1 \) or \( p < 0 \), then

\[
cM_1(U_{12}) \leq M_1(U_{11}) \leq CM_1(U_{12}),
\]

and if \( 0 < p < 1 \), then the above inequality is reversed.

(ii) Suppose \( V_{m_1} \) and \( V_{m_2} \) are nonnegative functions such that \( CV_{m_2} \geq V_{m_1} \geq cV_{m_2} \), where \( c, C \geq 0 \). If \( p \geq 1 \), then

\[
cM_2(V_{m_2}) \leq M_2(V_{m_1}) \leq CM_2(V_{m_2}),
\]

and if \( p < 1 \) and \( p \neq 0 \), then the above inequality is reversed.

**Corollary 3.2.** If \( V_{m_1} \) and \( V_{m_2} \) are nonnegative functions such that \( V_{m_2} \geq V_{m_1} \), then

\[
M_1^0 \left( \int_Y F_k(x, y) V_{m_1}(y) d\nu(y), \mu_\Delta \right) - \int_Y M_1^0(F_k, \mu_\Delta) V_{m_1}(y) d\nu(y) \leq M_1^0 \left( \int_Y F_k(x, y) V_{m_2}(y) d\nu(y), \mu_\Delta \right) - \int_Y M_1^0(F_k, \mu_\Delta) V_{m_2}(y) d\nu(y),
\]

where \( M_1^0(F_k, \mu_\Delta) \) is defined in (2.6).

The next result gives another property of \( M_1 \), but a similar result can also be stated for \( M_2 \).

**Theorem 3.2.** Let \( \varphi : [0, \infty) \rightarrow [0, \infty) \) be a concave function. Suppose \( U_{11} \) and \( U_{12} \) are nonnegative functions such that

\[
\varphi \circ U_{11}, \quad \varphi \circ U_{12}, \quad \varphi \circ (\alpha U_{11} + (1 - \alpha) U_{12})
\]

are \( \Delta \)-integrable for \( \alpha \in [0, 1] \). If \( p \geq 1 \), then

\[
M_1(\varphi \circ (\alpha U_{11} + (1 - \alpha) U_{12})) \geq \alpha M_1(\varphi \circ U_{11}) + (1 - \alpha) M_1(\varphi \circ U_{12}),
\]

and if \( 0 < p < 1 \), then the above inequality is reversed.
Proof. We show this only for $p \geq 1$ as the other case follows similarly. Since $\varphi$ is concave, we have
\[
\varphi(\alpha U_{11} + (1-\alpha)U_{12}) \geq \alpha \varphi(U_{11}) + (1-\alpha)\varphi(U_{12}).
\]

Now, from (3.1) and (3.3), we have
\[
M_1(\varphi \circ (\alpha U_{11} + (1-\alpha)U_{12})) \geq M_1(\alpha(\varphi \circ U_{11}) + (1-\alpha)(\varphi \circ U_{12}))
\]
\[
\geq M_1(\alpha(\varphi \circ U_{11})) + M_1((1-\alpha)(\varphi \circ U_{12}))
\]
\[
\geq \alpha M_1(\varphi \circ U_{11}) + (1-\alpha)M_1(\varphi \circ U_{12}),
\]
and the proof is established. \( \square \)

Let $F_k$, $U_1$ and $V_m$ be fixed functions satisfying the assumptions of Theorem 2.1. Let us define functionals $M_3$ and $M_4$ by
\[
M_3(A) = \left[ \int_Y \left( \int_A F_k^p(x,y)U_1(x)d\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)d\nu_\Delta(y) \right]^{p} U_1(x)d\mu_\Delta(x)
\]
and
\[
M_4(B) = \left[ \int_X \left( \int_B F_k^p(x,y)U_1(x)d\mu_\Delta(x) \right)^{\frac{1}{p}} V_m(y)d\nu_\Delta(y) \right]^{p} U_1(x)d\mu_\Delta(x)
\]
where $A \subseteq X$ and $B \subseteq Y$.

The following theorem establishes superadditivity and monotonicity of the mappings $M_3$ and $M_4$.

**Theorem 3.3.**

(i) Suppose $A_1, A_2 \subseteq X$ and $A_1 \cap A_2 = \emptyset$. If $p \geq 1$ or $p < 0$, then
\[
M_3(A_1 \cup A_2) \geq M_3(A_1) + M_3(A_2),
\]
and if $0 < p < 1$, then the above inequality is reversed.

(ii) Suppose $A_1, A_2 \subseteq X$ and $A_1 \subseteq A_2$. If $p \geq 1$ or $p < 0$, then
\[
M_3(A_1) \leq M_3(A_2),
\]
and if $0 < p < 1$, then the above inequality is reversed.

(iii) Suppose $B_1, B_2 \subseteq Y$ and $B_1 \cap B_2 = \emptyset$. If $p \geq 1$, then
\[
M_4(B_1 \cup B_2) \geq M_4(B_1) + M_4(B_2),
\]
and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.

(iv) Suppose $B_1, B_2 \subseteq Y$ and $B_1 \subseteq B_2$. If $p \geq 1$, then
\[
M_4(B_1) \leq M_4(B_2),
\]
and if $p < 1$ and $p \neq 0$, then the above inequality is reversed.
The proof of Theorem 3.3 is omitted as it is similar to the proof of Theorem 3.1.

**Remark 3.3.** For \( p \geq 1 \), if \( S_m \) is a subset of \( Y \) with \( m \) elements and if \( S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2 \), then we have

\[
M_4(S_m) \geq M_4(S_{m-1}) \geq \ldots \geq M_4(S_2) \geq 0
\]

and \( M_4(S_m) \geq \max\{M_4(S_2) : S_2 \text{ is any subset of } S_m \text{ with 2 elements}\} \).

4. Beckenbach–Dresher Inequalities

Let \( U_l, V_m, F_k \) be defined as in (4.1). Let \( F_n(x_1, x_2, \ldots, x_n), G_t(x_1, x_2, \ldots, x_t) \) are real valued functions of \( n \), and \( t \) variables, respectively. Let \( (X, M, \mu_\Delta) \), \( (X, M, \lambda_\Delta) \) and \( (Y, L, \nu_\Delta) \) be time scale measure spaces. Then, throughout in the following sections, we use the following notations:

\[
W_n = W_n(x) = W_n(w_1(x), w_2(x), \ldots, w_n(x)),
\]

\[
G_t = G_t(x, y) = G_t(g_1(x, y), g_2(x, y), \ldots, g_t(x, y)),
\]

where \( U_l \) and \( W_n \) are nonnegative functions on \( X \), \( V_m \) is a nonnegative function on \( Y \), \( F_k \) is a nonnegative function on \( X \times Y \) with respect to the measure \( (\mu_\Delta \times \nu_\Delta) \), and \( G_t \) is a nonnegative function on \( X \times Y \) with respect to the measure \( (\lambda_\Delta \times \nu_\Delta) \). In the sequel, we assume that all occurring integrals are finite.

**Theorem 4.1.** If

\[
s \geq 1, \quad q \leq 1 \leq p, \quad \text{and} \quad q \neq 0
\]

or

\[
s < 0, \quad p \leq 1 \leq q, \quad \text{and} \quad p \neq 0,
\]

then

\[
\left[ \int_X \left( \int_Y F_k(x, y)V_m(y)d\nu_\Delta(y) \right)^p U_l(x)d\mu_\Delta(x) \right]^{\frac{1}{p}} \leq \int_Y \left( \int_X G_t(x, y)V_m(y)d\nu_\Delta(y) \right)^{\frac{1}{q}} W_n(x)d\lambda_\Delta(x)
\]

provided all occurring integrals in (4.4) exist. If

\[
0 < s \leq 1, \quad p \geq 1, \quad q \leq 1, \quad \text{and} \quad q \neq 0,
\]

then (4.4) is reversed.
Proof. Assume (4.2) or (4.3). By using the integral Minkowski inequality (2.2) and Hölder’s inequality (1.2), we have

$$\frac{\left[ \int_Y \left( \int_X F_k(x,y)V_m(y) d\nu(x) \right)^p U_i(x) d\mu(x) \right]^\frac{1}{p}}{\left[ \int_Y \left( \int_X G_i(x,y)V_m(y) d\nu(x) \right)^q W_n(x) d\lambda(x) \right]^\frac{1}{q}}$$

$$\leq \frac{\left[ \int_Y \left( \int_X F_k(x,y) U_i(x) d\mu(x) \right)^{\frac{1}{p} \cdot \frac{s}{s-1}} V_m(y) d\nu(x) \right]^s}{\left[ \int_Y \left( \int_X G_i(x,y) W_n(x) d\lambda(x) \right)^{\frac{s}{s-1}} V_m(y) d\nu(x) \right]^{s-1}}$$

$$= \left[ \int_Y \left( \left( \int_X F_k(x,y) U_i(x) d\mu(x) \right)^{\frac{1}{p}} \right)^{\frac{s}{s-1}} V_m(y) d\nu(x) \right]^s \times \left[ \int_Y \left( \left( \int_X G_i(x,y) W_n(x) d\lambda(x) \right)^{\frac{1}{s}} \right)^{s-1} V_m(y) d\nu(x) \right]^{s-1}.$$

If (4.5) holds, then the reversed inequality in (4.4) can be proved in a similar way. □

5. BECKENBACH–DRESHER FUNCTIONALS

Let $F_k, G_i, U_i, W_n$ be fixed functions satisfying the assumptions of Theorem 4.1. We define the Beckenbach–Dresher functional $BD(V_m)$ by

$$BD(V_m) = \int_Y \left( \frac{\left( \int_X F_k(x,y) U_i(x) d\mu(x) \right)^{\frac{1}{p}}}{\left( \int_X G_i(x,y) W_n(x) d\lambda(x) \right)^{\frac{1}{q}}} V_m(y) d\nu(x) \right)^{\frac{1}{s}} - \frac{\left[ \int_X F_k(x,y) V_m(y) d\nu(x) \right]^p U_i(x) d\mu(x)}{\left[ \int_X G_i(x,y) V_m(y) d\nu(x) \right]^q W_n(x) d\lambda(x)}^{\frac{1}{q}}.$$

where we suppose that all occurring integrals exist.

Theorem 5.1. If (4.2) or (4.3) holds, then

$$BD(V_m + V_{m_2}) \geq BD(V_m) + BD(V_{m_2}).$$

(5.1)

If $V_{m_2} \geq V_m$, then

$$BD(V_m) \leq BD(V_{m_2}).$$

(5.2)

If $C, c \geq 0$ and $CV_m \geq V_m \geq cV_{m_2}$, then

$$CBD(V_{m_2}) \geq BD(V_m) \geq cBD(V_m).$$

(5.3)

If (4.5) holds, then (5.1), (5.2) and (5.3) are reversed.
Proof. Assume (4.2) or (4.3). Then we have

\[
\begin{align*}
\text{BD}(V_{m_1} + V_{m_2}) - \text{BD}(V_{m_1}) - \text{BD}(V_{m_2}) & = \frac{\left[ \int_X \left( \int_Y F_k(x,y) V_{m_1}(y) d\nu(y) \right)^p U_l(x) d\mu(x) \right]^q}{\left( \int_X \left( \int_Y G_t(x,y) V_{m_1}(y) d\nu(y) \right)^q W_n(x) d\lambda(x) \right)^{\frac{q}{q-1}}} \\
& + \left[ \int_X \left( \int_Y F_k(x,y) V_{m_2}(y) d\nu(y) \right)^p U_l(x) d\mu(x) \right]^q \left[ \int_X \left( \int_Y G_t(x,y) V_{m_2}(y) d\nu(y) \right)^q W_n(x) d\lambda(x) \right]^{\frac{q}{q-1}} \\
& - \frac{\left[ \int_X \left( \int_Y F_k(x,y) V_{m_1}(y) d\nu(y) \right)^p U_l(x) d\mu(x) \right]^q}{\left( \int_X \left( \int_Y G_t(x,y) V_{m_1}(y) d\nu(y) \right)^q W_n(x) d\lambda(x) \right)^{\frac{q}{q-1}}} \\
& \geq 0,
\end{align*}
\]

where in the last inequality we used (4.4) from Theorem 4.1. Using Theorem 4.1 again, \( V_{m_2} \geq V_{m_1} \) implies

\[
\text{BD}(V_{m_2}) = \text{BD}(V_{m_1} + (V_{m_2} - V_{m_1})) \geq \text{BD}(V_{m_1}) + \text{BD}(V_{m_2} - V_{m_1}) \geq \text{BD}(V_{m_1}).
\]

The proof of (5.3) is similar. If (4.5) holds, then the reversed inequalities of (5.1), (5.2) and (5.3) can be proved in a similar way. \( \square \)

Let \( F_k, G_t, U_l, V_m, W_n \) be fixed functions. We define a functional \( \text{BD}_1 \) by

\[
\text{BD}_1(A) = \int_A \frac{\left( \int_X F^p_k(x,y) U_l(x) d\mu(x) \right)^q}{\left( \int_X G^q_t(x,y) W_n(x) d\lambda(x) \right)^{\frac{q}{q-1}}} V_m(y) d\nu(y)
\]

where \( A \subseteq Y \).

For \( \text{BD}_1 \), the following result holds.

**Theorem 5.2.**

(i) Suppose \( A_1, A_2 \subseteq Y \) and \( A_1 \cap A_2 = \emptyset \). If (4.2) or (4.3) holds, then

\[
\text{BD}_1(A_1 \cup A_2) \geq \text{BD}_1(A_1) + \text{BD}_1(A_2),
\]

and if (4.5) holds, then the above inequality is reversed.

(ii) Suppose \( A_1, A_2 \subseteq Y \) and \( A_1 \subseteq A_2 \). If (4.2) or (4.3) holds, then

\[
\text{BD}_1(A_1) \leq \text{BD}_1(A_2),
\]

and if (4.5) holds, then the above inequality is reversed.

The proof of Theorem 5.2 is omitted as it is similar to the proof of Theorem 5.1.
Remark 5.1. If $S_k \subseteq X$ has $k$ elements and if $S_m \supseteq S_{m-1} \supseteq \ldots \supseteq S_2$, then (4.2) or (4.3) implies

$$\text{BD}_1(S_m) \geq \text{BD}_1(S_{m-1}) \geq \ldots \geq \text{BD}_1(S_2) \geq 0$$

and $\text{BD}_1(S_m) \geq \max\{\text{BD}_1(S_2) : S_2 \text{ is any subset of } S_m \text{ with 2 elements}\}$, while (4.5) implies the reversed inequalities with max replaced by min.

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