On the Lengths of Curves Passing Through Boundary Points of a Planar Convex Shape

Arseniy Akopyan and Vladislav Vysotsky

Abstract. We study the lengths of curves passing through a fixed number of points on the boundary of a convex shape in the plane. We show that, for any convex shape $K$, there exist four points on the boundary of $K$ such that the length of any curve passing through these points is at least half of the perimeter of $K$. It is also shown that the same statement does not remain valid with the additional constraint that the points are extreme points of $K$. Moreover, the factor $\frac{1}{2}$ cannot be achieved with any fixed number of extreme points. We conclude the paper with a few other inequalities related to the perimeter of a convex shape.

1. INTRODUCTION. We study the lengths of curves passing through a fixed number of points on the boundary of a convex shape in the plane. All the curves considered are supposed to be rectifiable, i.e., have finite length. By convex shapes, we mean compact convex subsets of the plane, and we assume that a convex shape has a nonempty interior to avoid trivialities.

We first show that, for any convex shape $K$, there exist four points on the boundary of $K$ such that the length of any curve passing through these points is at least half of the perimeter of $K$; see Theorem 1. It turns out that this statement is optimal: The lower bound $\frac{1}{2}$ per $K$ (where per $K$ denotes the perimeter of $K$) cannot hold for three points, and we cannot exceed the factor $\frac{1}{2}$ even by increasing the number of points. Moreover, if we additionally require that these boundary points are extreme, then it does not suffice to take four points, and, in fact, the factor $\frac{1}{2}$ cannot be achieved with any fixed number of extreme points; see Theorem 3. By convention, in the statements concerning a number of extreme points, we do not require that the points be distinct; for example, we can choose five extreme points in a triangle.

We conclude the paper by considering curves whose convex hulls cover $K$ and, in particular, curves that pass through all extreme points of $K$. It is well known that the length of such curves is at least $\frac{1}{2}$ per $K$, which explains the factor $\frac{1}{2}$ appearing above. This consideration is related to the question of H. T. Croft [5] on the minimal length of a curve such that its convex hull contains a unit disk.

Our results can be regarded as upper estimates for the perimeter of a convex shape. The approach presented was motivated by our studies [1] of a problem in probability theory concerning the trajectories of planar random walks whose convex hulls have atypically large perimeters.

A similar question was considered by A. Zirakzadeh [19], who proved that any triangle with its vertices dividing the boundary of a convex shape $K$ into three arcs of equal lengths has perimeter at least $\frac{1}{2}$ per $K$. This was extended by B. Bollobás [3] who proved that the perimeter of any inscribed $n$-gon with its vertices dividing the boundary of $K$ into $n$ equal arcs is at least $(1 - \frac{2}{n})$ per $K$, which is a tight bound for even $n$.

The related works of A. Glazyrin and F. Morić [11], Z. Lángi [14], and R. Pinchasi [15] concern inequalities for the perimeters of a convex body and one or several

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disjoint polygons covered by the body. There is an impressive survey by P. Scott and P. W. Awyong [17] of inequalities relating the perimeter and other characteristics (area, width, etc.) of convex shapes. 

One may consider the results presented in the context of the open tour Euclidean traveling salesman problem of finding the shortest path connecting certain points in the plane. This problem is usually considered in the asymptotic setting for large numbers of points. Thus, L. Fejes Tóth [9] posed the question of finding the asymptotics of the length of this path through \( n \) points in a unit square and proved the lower bound \( (4/3)^{1/4} \sqrt{n} + o(\sqrt{n}) \). The best current upper bound \( 1.391 \sqrt{n} + o(\sqrt{n}) \) was obtained by H. J. Karloff [13]. A probabilistic version of this problem was studied by J. Beardwood, J. H. Halton, and J. M. Hammersley [2], who considered points in an arbitrary body in any dimension.

2. CURVES THROUGH FOUR POINTS ON THE BOUNDARY. For any convex shape \( K \) and any number \( \varepsilon > 0 \), one can choose several points on the boundary \( \partial K \) of \( K \) such that the length of any curve passing through these points is at least \( (1 - \varepsilon) \text{per} K \); we leave this statement without a proof. The number of points required for this approximation depends on both \( K \) and \( \varepsilon \).

It is clear that we cannot control \( \varepsilon \) (uniformly in \( K \)) by increasing the number of points. Indeed, fix \( n \) and consider a thin \( 1 \times L \) rectangle with \( L \gg n \). Any \( n \) points on its boundary can be connected by a curve of length exactly \( L + n \approx \frac{1}{2} \text{per} P \) (see Figure 1). Actually, \( \varepsilon = 1/2 \) is the threshold.

![Figure 1](image1.png)

**Figure 1.** An \( 1 \times L \) rectangle with a path of length \( L + n \) through \( n \) points.

**Theorem 1.** Let \( K \) be a compact convex shape in the plane. There exist four points on the boundary of \( K \) such that the length of any path connecting these points is at least \( \frac{1}{2} \text{per} K \).

![Figure 2](image2.png) ![Figure 3](image3.png)

**Figure 2.** The optimal configuration of three points in a thin lens. **Figure 3.** A nonoptimal configuration of three points in a thin lens.

It will not suffice to take three points. Indeed, consider a thin lens formed by two identical circular segments; see Figure 2. Let us show that the triple \( \{a, m, b\} \) consisting of the endpoints \( a \) and \( b \) and the midpoint \( m \) of either of the arcs maximizes the length of the shortest path through any triple of points on the boundary of the lens. Clearly, the length of the path \( amb \) is less than the length of the arc.

Consider a triple of points \( \{a', m', b'\} \) on the boundary of the lens, which are denoted in the order of increasing \( x \)-coordinates. Since \( |aa'| \leq |am'| \) and \( \angle a'am' < 60^{\circ} \), if the lens is thin enough, we see that \( a'm' \) is not a longest side in the triangle \( aa'm' \), and therefore \( |am'| \geq |a'm'| \) (and equality holds only if \( a' \) coincides with \( a \)). By the same argument, we have \( |m'b| \geq |m'b'| \). Hence, the path \( a'm'b' \) (which is not required to be a shortest path connecting \( a', m', b' \)) is not longer than \( am'b \). It is left to note that
the path \(am'b\) has maximal length if and only if \(m'\) is the midpoint of one of the arcs (apply Lemma 2 for the angle \(\varphi = \angle am'b\), which does not depend on \(m'\)). In the proof of Theorem 1, we will use the following statement, which, in our opinion, is interesting by itself.

**Theorem 2.** Let \(K\) be a compact convex shape in the plane, and let \(ab\) be one of its diameters. Suppose that the perpendicular bisector of \(ab\) intersects the boundary of \(K\) at two points \(c\) and \(d\). Then

\[
\per K < 2|ab| + 2|cd|.
\]

**Proof.** The diameter \(ab\) divides the boundary of \(K\) into two parts \(K_c\) and \(K_d\) that contain the points \(c\) and \(d\), respectively. We will show that if \(c \notin \overline{ab}\), then the length of \(K_c\) is greater than \(|ab| + 2|co|\), where \(o\) is the midpoint of \(ab\). Since at least one of the points \(c\) and \(d\) does not lie on \(ab\), this inequality with the analogous inequality for \(K_d\) implies the statement of the theorem.

![Figure 4. Illustration for the proof of Theorem 2.](image)

Let us construct two circles of radius \(ab\) centered at \(a\) and \(b\); see Figure 4. It is clear that \(K\) lies in the intersection of the corresponding disks. We draw a support line to \(K\) at the point \(c\) and denote its intersection points with the circles centered at \(a\) and \(b\) by \(q\) and \(p\), respectively. Without loss of generality, we assume that \(q\) is farther from \(ab\) than \(p\). Then \(K_c\) lies inside the region bounded by the closed convex curve \(apqba\) (\(ap\) and \(qb\) are circular arcs), hence the length of \(K_c\) does not exceed the length of \(apqb\). We seek to bound the length of \(apqb\).

Denote by \(p_1\) and \(q_1\) the points of intersection of \(pq\) with the respective perpendiculars to \(ab\) at the points \(a\) and \(b\); see Figure 4. Since \(2|co| = |p_1a| + |q_1b|\), it suffices to show that the length of \(apqb\) is less than \(|ab| + |p_1a| + |q_1b|\).

The line through the point \(a\) parallel to \(pq\) intersects the arc \(qb\) and the segment \(q_1b\) at the points \(l\) and \(m\), respectively. Denote by \(k\) the point on \(p_1q_1\) such that the quadrilaterals \(ap_1kl\) and \(lkq_1m\) are parallelograms.

The length of the arc \(lb\) satisfies \(|lb| = |ab| \cdot \angle(bal)|, and we have that \(|bm| = |ab| \tan(\angle bal)|. Hence, since \(\tan(x) \geq x\), we have

\[
|bm| \geq |lb|.
\]

(1)

Since the closed curve \(apqla\) is convex, the length of \(apql\) is less than the length of the path \(ap_{1}kl\). Since \(|kl| = |p_1a| = |q_1m|\) and \(|p_1k| = |al| = |ab|\), we have

\[
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\]
Combining (1) and (2), we obtain the required inequality for the length of the path $apqb$.

**Proof of Theorem 1.** As in Theorem 2, denote by $ab$ a diameter of $K$, by $o$ the midpoint of $ab$, and by $c$ and $d$ the points of intersection of the perpendicular bisector of $ab$ with the boundary of $K$; see Figure 5. Without loss of generality, we can assume that $|co| \geq |do|$ and $|ao| = 1$.

![Figure 5. Illustration for the proof of Theorem 1.](image)

It is clear that the shortest path connecting the points $a, b, c, d$ is $acdb$ (if $|ad| \geq |cd|$) or $cadb$ (if $|ad| < |cd|$). In the former case, the length of the path is greater than $|ab| + |cd|$, which exceeds $\frac{1}{2}$ per $K$ by Theorem 2.

In the latter case, we note that $|cd| > |ad| \geq 1$. For a fixed $|cd|$, $|ad| + |ac|$ attains its minimum value when $acd$ is an isosceles triangle. Therefore, $|ad| + |ac| > \sqrt{5} > 2.2$. Since $|ad| \geq 1$, we conclude that the length of the path $cadb$ is at least $3.2 > \pi$. It remains to use the fact that the perimeter of a convex set of diameter 2 is at most $2\pi$; this fact follows from the Crofton formula (10).

**3. CURVES THROUGH EXTREME POINTS.** We now consider curves that are required to pass through extreme points of convex shapes. Let us recall that a point of a convex shape $K$ is called extreme if it does not belong to any open line segment with end points in $K$. In the case that $K$ is a convex polygon, the set of its extreme points coincides with its vertices.

**Theorem 3.** For any $n \geq 2$, there exists a convex shape $K_n$ such that any $n$ extreme points of $K_n$ can be connected by a path of length less than $\frac{1}{2}$ per $K_n$.

Since any convex shape can be approximated by a convex polygon, we obtain the following corollary.

**Corollary 4.** For any $n \geq 3$, there exists a convex polygon $P_n$ such that any $n$ vertices of $P_n$ can be connected by a path of length less than $\frac{1}{2}$ per $P_n$.

**Proof of Theorem 3.** Let $E_k$ be a half of an elongated ellipse with semi-axes 1 and $k$, where $k$ is sufficiently small (to be chosen later), bisected through its major axis $ab$ of length two, as shown in Figure 6. Since the set of extreme points of $E_k$ is the arc $\overline{ab}$, it suffices to prove that for any $n$ points $m_1, \ldots, m_n$ on the arc enumerated in the direction from $a$ to $b$, we have

$$|m_1m_2| + \cdots + |m_{n-1}m_n| < \frac{1}{2} \text{ per } E_k.$$  \hspace{1cm} (3)

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Note that we do not require that the polygonal path \( m_1m_2 \ldots m_n \) be the shortest among the paths connecting the points \( m_i \), and although it is not hard to verify this statement, we will not prove it.

We use the following lemma, which is proved right after the proof of Theorem 3.

**Lemma 1.** For any three points \( m_{i-1}, m_i, m_{i+1} \) on the arc \( \overline{ab} \) (such that \( m_i \in \overline{m_{i-1}m_{i+1}} \)),

\[
|m_{i-1}m_i| + |m_im_{i+1}| - |m_{i-1}m_{i+1}| \leq 2\sqrt{1+k^2} - 2. \tag{4}
\]

Applying the lemma \( n - 2 \) times to the triples of points \( m_1, m_i, m_{i+1} \), we find that the length of the polygonal line \( m_1 \ldots m_n \) satisfies

\[
|m_1m_2| + \cdots + |m_{n-1}m_n| \leq 2(n-2)(\sqrt{1+k^2} - 1) + 2 \tag{5}
\]

since \( |m_1m_n| \leq |ab| \).

On the other hand, the perimeter of an elongated ellipse with semi-axes 1 and \( k \) has the asymptotics \( 4 + 2k^2 \log \frac{1}{k} + O(k^2) \) as \( k \to 0 \), which corresponds to the first two terms of the so-called Cayley series (see [4, Ch. III.78] or [12, Eq. 8.114.3]). Hence,

\[
\frac{1}{2} \text{ per } E_k = 2 + \frac{1}{2} k^2 \log \frac{1}{k} + O(k^2), \quad k \to 0. \tag{6}
\]

The statement of Theorem 3 now follows by (3), (5), and (6) since \( n \) is fixed and \( \log \frac{1}{k} \to \infty \) as \( k \to 0 \): We choose \( k \) small enough and put \( K_n := E_k \).

**Proof of Lemma 1.** We claim that

\[
|m_{i-1}m_i| + |m_im_{i+1}| - |m_{i-1}m_{i+1}| < |m_{i-1}m_i| + |m_ib| - |m_{i-1}b|; \tag{7}
\]

see Figure 6. This is equivalent to

\[
|m_im_{i+1}| + |m_{i-1}b| < |m_ib| + |m_{i-1}m_{i+1}|,
\]

which follows from the fact that for any convex quadrilateral, the sum of the lengths of its diagonals exceeds the sum of the lengths of either pair of opposite sides. This fact holds by the triangle inequality applied to the two triangles formed by the intercepts of the diagonals and the corresponding side.

![Figure 6. Illustration for the proof of Lemma 1.](image)

Analogously to (7), we have

\[
|m_{i-1}m_i| + |m_ib| - |m_{i-1}b| < |am_i| + |m_ib| - |ab|. \tag{8}
\]
It remains to use the fact that $|am_i| + |m_i b|$ reaches its maximum if $m_i$ is the midpoint $m$ of the arc $ab$: Then (4) follows since $|am| + |mb| - |ab| = 2\sqrt{1 + k^2} - 2$.

Indeed, if the maximum is attained at some other point $m'$, then the ellipse $E'$ with foci $a$ and $b$ and major axis of length $|am'| + |bm'|$ touches the half-ellipse $E_k$ (with semi-axes 1 and $k$) at two points, namely, $m'$ and its symmetric image about the minor axis of $E_k$. By taking the symmetric image of $E_k$ about its major axis, we obtain the complete ellipse inscribed in $E'$ and touching it at four points, which is impossible for two conic curves.

The positive result here is that any fraction less than half of the perimeter can be reached by increasing the number of vertices.

**Theorem 5.** For any $\varepsilon > 0$, there exists a positive integer $n$ such that for any convex shape $K$ one can choose $n$ extreme points that cannot be connected by a curve of length less than $\frac{1 - \varepsilon}{2}$ per $K$.

For the proof, we will need the following statement.

**Lemma 2.** For any triangle $abc$ with the angle $\angle bac = \varphi$, we have

$$\frac{|bc|}{|ab| + |ac|} \geq \sin \frac{\varphi}{2}. \quad (9)$$

**Proof.** Note that $|ab| \sin \frac{\varphi}{2}$ and $|ac| \sin \frac{\varphi}{2}$ are the distances to the angle bisector of the angle $\angle bac$ from $b$ and $c$, respectively. The sum of these distances is at most $|bc|$. ■

**Proof of Theorem 5.** Choose $n$ such that $\cos \frac{\pi}{n} > 1 - \varepsilon$. For each $1 \leq i \leq n$, choose an extreme point $v_i$ of $K$ that admits a support line $\ell_i$ through it with the outer normal vector at the angle $2\pi \frac{i}{n}$ with some fixed direction (see Figure 7). If there are two such points, choose either of them; some extreme points can correspond to several $i$.

Now let us show that perimeter of the convex polygon $V = v_1 v_2 \ldots v_n$ is at least $(1 - \varepsilon)$ per $K$. Denote by $o_i$ the intersection of the support lines $\ell_i$ and $\ell_{i+1}$ (we assume that $\ell_{n+1} = \ell_1$ and $v_{n+1} = v_1$). Note that the part of the perimeter of $K$ lying between $v_i$ and $v_{i+1}$ has length at most $|v_i o_i| + |o_i v_{i+1}|$, which by Lemma 2 is at most $\frac{|v_i v_{i+1}|}{\cos \frac{\pi}{n}}$. Applying this inequality for all the arcs $\overline{v_i v_{i+1}}$ of the perimeter of $K$, we obtain the inequality per $K \leq \frac{\text{per } V}{\cos \frac{\pi}{n}}$. Therefore, $(1 - \varepsilon)$ per $K < \text{per } V$.

![Figure 7. Illustration for the proof of Theorem 5.](image-url)
By Theorem 6, which is an independent statement presented below, the length of any curve passing through all vertices of $V$ should be at least $\frac{1}{2}$ per $V > \frac{1-\varepsilon}{2}$ per $K$.

Note that in [16] R. Schneider applied a similar construction for his solution of the problem of L. Fejes Tóth on the $n$-gon of the maximum (minimum) perimeter inscribed (resp., circumscribed) in a convex shape [10, p. 39].

4. BARRIERS FOR CONVEX SHAPES. The problem of finding a shortest curve whose convex hull covers a unit disk was posed by H. T. Croft in [5] and solved by V. Faber, J. Mycielski, and P. Pedersen in [8]. Following [6], let us call such a curve a barrier. Not much is known if instead of the unit disk we consider a general convex shape. In [7], V. Faber and J. Mycielski give examples of plausibly optimal barriers for regular $n$-gons, $n \leq 6$, a half disk, and a certain parallelogram.

The following statement is widely known and even mentioned to be “folklore.”

**Theorem 6 (See [6] or [8]).** Let $\gamma$ be a curve such that its convex hull covers a planar convex shape $K$. Then

$$\text{length } \gamma \geq \frac{1}{2} \text{per } K.$$  

Let us prove a similar statement.

**Theorem 7.** Let $K$ be a convex shape on a plane, and let $\gamma$ be a curve passing through all extreme points of $K$. Then

$$\text{length } \gamma \geq \text{per } K - \text{diam } K.$$  

**Proof:** Let $a$ and $b$ be the first and the last points (with respect to any parametrization of $\gamma$) of the intersection of $\gamma$ and $\partial K$; see Figure 8. Define the closed curve $\gamma'$ formed by the part of $\gamma$ between points $a$ and $b$ and the line segment $ab$. Since $\gamma'$ passes through all the extreme points of $K$, its convex hull covers $K$. Since $|ab| \leq \text{diam } K$, the claim will follow if we show that the length of $\gamma'$ is at least per $K$.

![Figure 8. Illustration for the proof of Theorem 7.](image)

Let us use the Crofton formula from integral geometry (see e.g., S. Tabachnikov [18]):

$$\text{length}(\gamma') = \frac{1}{2} \iint_{S^1 \mathbb{R}^+} n_{\gamma'}(\phi, p) d\phi dp, \quad \text{per}(K) = \frac{1}{2} \iint_{S^1 \mathbb{R}^+} n_{\partial K}(\phi, p) d\phi dp, \quad (10)$$

where $n_{\nu}(\phi, p)$ denotes the number of intersections of a curve $\nu$ with the line perpendicular to the direction $\phi$ passing at the distance $p$ from the origin. We have that

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\( n_{\gamma'}(\phi, p) \geq n_{\partial K}(\phi, p) \) for almost every pair \((\phi, p)\). Indeed, each line intersecting \( K \) intersects \( \gamma' \). Since \( \gamma' \) is closed, almost every line that intersects it has at least two points of intersection with \( \gamma' \), while almost every line that intersects \( \partial K \) has exactly two points of intersection with \( \partial K \) since \( K \) is convex.

The authors believe that the following generalization is true.

**Conjecture.** Let \( \gamma \) be a curve such that its convex hull covers a planar convex shape \( K \). Then

\[
\text{length } \gamma \geq \text{per } K - \text{diam } K.
\]

Note that the proof of Theorem 7 does not work if the convex hull of \( \gamma \) does not cover \( K \) or if the distance between the endpoints is greater than \( \text{diam } K \).

It looks plausible that the function \( \text{per } K - \text{diam } K \) is inclusion monotone. But this is not true. A counterexample is shown in Figure 9. Let \( abcd \) and \( abcd' \) be deltoids containing an equilateral triangle \( abc \) of the side length 1, with their axes of respective lengths 1 (the angle \( d \) equals 150°) and \( 2/\sqrt{3} \) (the angle \( d' \) equals 120°). The diameters of the deltoids are \( bd \) and \( bd' \). Then the value of \( \text{per } K - \text{diam } K \) equals \((2 + 4 \sin 15°) - 1 \approx 2.035 \) for the quadrilateral \( abcd \) and \((2 + 2/\sqrt{3}) - 2/\sqrt{3} = 2 \) for \( abcd' \).

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