Matrix product states for interacting particles without hardcore constraints

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Abstract
We construct matrix product steady states for a class of interacting particle systems where particles do not obey hardcore exclusion, meaning each site can occupy any number of particles subjected to the global conservation of the total number of particles in the system. To represent the arbitrary occupancy of the sites, the matrix product ansatz here requires an infinite set of matrices which in turn leads to an algebra involving an infinite number of matrix equations. We show that these matrix equations, in fact, can be reduced to a single functional relation when the matrices are parametric functions of the representative occupation number. We demonstrate this matrix formulation in a class of stochastic particle hopping processes on a one dimensional periodic lattice where hop rates depend on the occupation numbers of the departure site and its neighbors within a finite range; this includes some well known stochastic processes like, the totally or partially asymmetric zero range process, misanthrope process and finite range process.

Keywords: matrix product ansatz, stochastic process, finite range processes, exact results

1. Introduction
Non-equilibrium systems [1–3] are quite common in nature. Driven systems naturally attain a non-equilibrium steady-state (NESS) with a nonzero probability current, in contrast to equilibrium systems which satisfy detailed balance condition, ensuring a zero-current stationary state given by the well known Gibbs measure [4, 5]. There is no such generic invariant measure for NESS [1, 2]; in general the structure of NESS is more complex, yet very interesting, as they exhibit nontrivial correlations [6–8], current reversal [9], current fluctuations [10] and many other novel phenomena like phase transitions [3, 11–16], even in one dimension. In absence of any generic formalism, finding and characterizing the stationary measure of non-equilibrium
dynamics is usually a non-trivial task. In spite of these technical difficulties, many exactly solvable models have been found using special techniques like Bethe ansatz [17–19], matrix product ansatz [20], transfer matrix methods [21] and methods of large deviation [22] etc. One of the much celebrated examples of solvable open non-equilibrium system is the totally asymmetric simple exclusion process (TASEP) [23, 24] where particles following hardcore repulsion, enter to the system from left, move to their rightward vacant neighbor with unit rate and finally exit through the right boundary with certain rate. The steady state of this system has been obtained, exactly using matrix product ansatz (MPA) [20] where steady state weight of any configuration is represented by matrix string containing two non-commuting matrices, one for occupied site and the other for the vacant site. Although the dynamics of TASEP is quite simple, it exhibits three different phases and transitions among them depending on the entry and exit rates of particles. Generalization of TASEP and related models [6, 25–32] have been very helpful in understanding NESS in general.

Matrix product ansatz (MPA) for interacting particle systems following hardcore constraint is known to be one of the most useful and elegant analytical tool for finding NESS. Soon after being introduced in context of TASEP [20], MPA has found enormous applications in different branches of physics. MPA [33] has been very helpful in calculating spatial correlation functions for exclusion processes with point objects [8] as well as for extended objects [34] in one dimension. Study of relations between algebraic Bethe ansatz [35] and matrix product states for stochastic Markovian models in 1D [36] and the same for spin-$\frac{1}{2}$ Heisenberg chains [37] brought calculational convenience and also gave good physical insight to the problems. In connection to correlated non-equilibrium systems, MPA can describe asymptotic distributions of the sum of correlated random variables [38]. Moreover, as a natural extension of MPA on discrete lattice, continuous matrix product states (cMPS) have been introduced as variational states for 1D continuum models [39] and cMPS have already proved to be convenient in studying Bose gas in 1D [40], interacting spin-$\frac{1}{2}$ systems [41] etc. In a nutshell, MPA has been attracting interest in vast research areas starting from condensed matter physics to quantum information [42].

In matrix product ansatz (MPA), any configuration $\{n_i\}$ ($n_i = 0$ or 1 for exclusion processes) in the configuration space is represented by a matrix string $\{A_\alpha^i\}$, where each matrix $A_\alpha^i$ represents either a vacancy ($\alpha = 0$) or a particle of any one of the species ($\alpha = 1, 2, \ldots$) present at site $i$. Generally the representation of the matrices $A_\alpha^i$ do not depend on the site index $i$. But, notably, particles of different species and vacancies are denoted by matrices that are in general non-commuting $[A_\alpha^i, A_\beta^j] \neq 0$ for $\alpha \neq \beta$. If the system is open, one needs additional vectors (say $\langle W |$ and $| V \rangle$) to represent the boundaries. The MPA assumes that the steady state weights of the configuration $\{n_i\}$ to be,

$$
P(\{n_i\}) \propto \begin{cases} 
\text{Tr}[\prod_{i=1}^L A_\alpha^i] & \text{periodic} \\
\langle W | \prod_{i=1}^L A_\alpha^i | V \rangle & \text{open}
\end{cases}
$$

A specific stochastic dynamics on the lattice insists the matrices and vectors (if present) to satisfy a set of equations, commonly known as matrix algebra. Any representation (of the matrices) that satisfy this matrix algebra provide a steady state solution of the respective dynamics. An important point to note is, in all these systems particles are constrained by hard-core interactions, that lead to a finite number of algebraic equations to be satisfied.

In this article, we study interacting particle systems without hard-core interaction, where each lattice site can be occupied by any number of particles. To form a matrix product state, thus, we require infinitely many matrices; any given dynamics of the system would then insist on a algebra containing infinitely many matrix equations. It is not a priori clear whether
writing the steady state in such a matrix product form is at all possible. Here we show that, if the matrices are parametric function of the occupation numbers, the matrix algebra, for a class of models, reduce to a single functional relation which is easier to deal with. In fact, a solution to this functional relation eventually leads to an exact steady state weights of the model. We demonstrate this in a class of interacting particle system where particles hop to one of the nearest neighbors with rates that depend on the occupation of the departure site and its neighbors within a finite range.

The paper is organized at follows. In section 2 we describe a stochastic process where hop rates are totally asymmetric, i.e. particles hop only to the rightward neighbor, and formulate the matrix product ansatz for the steady state of this model in details. Section 3 deals with possible matrix product states for some well known models like zero range process (ZRP) [43, 44], misanthrope process (MAP) [45] and finite range process (FRP) [46]. Further, in section 4 we study the asymmetric particle transfer process where the functional form of rate function for rightward hop is different from that of the leftward hop. Finally, the summary of our results and some discussions are given in section 5.

2. Matrix product ansatz in absence of exclusion

In this section we introduce the matrix product formulation for interacting particle systems in absence of hardcore exclusion, i.e. the systems allow multiple occupancy at any lattice site. We first consider a generic stochastic process where particles execute directed motion on a periodic lattice in one dimension (1D). The dynamics of the model is totally asymmetric in a sense that particles here hop along a specified direction with hop rates depending on the occupancy of several sites, namely the departure site, its left neighbors within a range $R_l$ and right neighbors within a range $R_r$. Below we describe the model in details.

Let the sites of the periodic lattice be labeled by $i = 1, 2, ..., L$. With each site $i$, is associated a nonnegative integer variable $n_i (\geq 0)$ representing the number of particles at that site (for a vacant site $n_i = 0$). The dynamics is as follows. A particle from a randomly chosen site $i$ hops to its right neighbor $(i + 1)$ with rate $u(n_{i-R_l}, ..., n_{i-1}, n_i, n_{i+1}, ..., n_{i+R_r})$. Formally,

$$\{\ldots, n_{i-1}, n_i, n_{i+1}, \ldots\} \rightarrow \{\ldots, n_{i-1}, n_i - 1, n_{i+1} + 1, \ldots\}$$

with rate $u(n_{i-R_l}, ..., n_i, ..., n_{i+R_r})$. (2)

Clearly, this driven non-equilibrium dynamics conserves the total number of particles $N$ in the system. The Master equation dictating the evolution of probability $P(\{n_i\})$ of configuration $\{n_i\}$ of the system reads

$$\frac{d}{dt} P(\{n_i\}) = \sum_{i=1}^{L} u(n_{i-R_l}, \ldots, n_i + 1, n_{i+1} - 1, \ldots, n_{i+R_r})$$

$$\times P(\ldots, n_i + 1, n_{i+1} - 1, \ldots)$$

$$- \sum_{i=1}^{L} u(n_{i-R_l}, \ldots, n_i, \ldots, n_{i+R_r}) P(\{n_i\}).$$

(3)

In steady state, the net probability flux must vanish for each configuration $\{n_i\}$, i.e. the total in-flux (the first sum on the right hand side of equation (3)) must balance the out-flux (the second sum). This cancellation may occur in several different ways, with detailed balance being one of the special cases which, if exists, guarantees equilibrium. Pairwise balance is another special condition giving rise to non-equilibrium steady states.
For the dynamics in equation (2), to ensure that the in-flux is balanced by the out-flux we first make an ansatz that the steady state weight \( P(\{n_i\}) \) can be written in the matrix product form

\[
P(\{n_i\}) = \frac{1}{Z_{L,N}} \text{Tr} \left[ \prod_{i=1}^{L} A(n_i) \right] \delta \left( \sum_i n_i - N \right),
\]

(4)

where any configuration is represented by a string of \( L \) matrices, \( A(n_i) \) being the matrix associated with \( k \)-th site containing \( n_i \) particles. The \( \delta \) function here ensures the particle number conservation and \( Z_{L,N} \) is the canonical partition function. Now for the ansatz to be a valid one, we must ensure that the matrices in equation (4) satisfy equation (3) in steady state. This can be achieved by constructing a suitable cancellation scheme involving additional auxiliary matrices. In this context we propose the following cancellation scheme,

\[
u(n_{-R}, \ldots, n_i, n_{i+1}, \ldots, n_{+R}) A(n_{-R}) \ldots A(n_i)A(n_{i+1}) \ldots A(n_{+R})
\]

\[
- u(n_{-R}, \ldots, n_i + 1, n_{i+1} - 1, \ldots, n_{+R})A(n_{-R}) \ldots A(n_{i+1} - 1)A(n_{+R})
\]

\[
= A(n_{-R}) \tilde{A}(n_i)A(n_{i+1}) \ldots A(n_{-R})A(n_{i+1} - 1)A(n_{+R})
\]

\[
= A(n_{-R}) \ldots A(n_{i-1})[\tilde{A}(n_i)A(n_{i+1}) - A(n_{i})\tilde{A}(n_{i+1})A(n_{i+2}) \ldots A(n_{+R})]
\]

(5)

where we have introduced a new set of matrices \( \tilde{A}(n) \)-which, in the language of matrix product ansatz, are generally known as the auxiliary matrices [33]. It is straightforward to check that the above cancellation-scheme satisfies the Master equation (3) in steady state. What remains, is to find suitable representation of the set of matrices \( \{A(n_i)\} \) and the auxiliary matrices \( \{\tilde{A}(n_i)\} \) which follow the matrix-algebra given by equation (5).

A sufficient condition (though not necessary) to satisfy equation (5) is

\[
u(n_{-R}, \ldots, n_i, n_{i+1}, \ldots, n_{+R}) A(n_{-R}) \ldots A(n_i)A(n_{i+1}) \ldots A(n_{+R})
\]

\[
= A(n_{-R}) \ldots A(n_{i-1})\tilde{A}(n_i)A(n_{i+1}) \ldots A(n_{+R})
\]

(6)

\[
u(n_{-R}, \ldots, n_i + 1, n_{i+1} - 1, \ldots, n_{+R})A(n_{-R}) \ldots A(n_{i+1} - 1)A(n_{+R})
\]

\[
= A(n_{-R}) \ldots A(n_{i-1})A(n_{i})\tilde{A}(n_{i+1}) \ldots A(n_{+R}).
\]

(7)

Now both the equations (6) and (7) are satisfied consistently if we choose the auxiliary matrix \( \tilde{A}(n) \) to be

\[
\tilde{A}(n) = A(n - 1) \quad \text{for} \quad n \geq 1 \quad \text{and} \quad \tilde{A}(0) = 0.
\]

(8)

With this choice, equations (6) and (7) reduce to

\[
u(n_{-R}, \ldots, n_i, n_{i+1}, \ldots, n_{+R}) A(n_{-R}) \ldots A(n_i)A(n_{i+1}) \ldots A(n_{+R})
\]

\[
= A(n_{-R}) \ldots A(n_{i-1})A(n_{i})A(n_{i+1} - 1)A(n_{i+2}) \ldots A(n_{+R}).
\]

(9)

Thus the matrix product ansatz, formulated here for systems with multiple site occupancy, finally leads to a unique set of equations, as above. For a given model, with totally asymmetric hop rate \( u(.) \), we have to solve the matrix algebra (9) to find a possible representation of \( \{A(0), A(1), \ldots\} \). Practically this is a difficult task as we need to solve infinitely many matrix equations to be satisfied by infinitely large set of matrices \( \{A(0), A(1), \ldots\} \) which are non commuting and, in principle, independent and unrelated to each other. However for a generic class of models, which we discuss in the following sections, it is possible to find a matrix representation where \( A(n) \) is a parametric function of \( n \); i.e. all elements of the matrix \( A(n) \) are
specific functions of \( n \) i.e. \( A(n)_{ij} = f_{ij}(n) \). More precisely, the matrices \( A(0), A(1), A(2), \ldots \) are not unrelated, rather they are only instances of a general representative matrix function \( A(n) \). In that case, we do not need to solve the matrix algebra each time separately to obtain \( A(0), A(1), A(2), \ldots \), rather we should obtain the general matrix function \( A(n) \) by treating the algebra (9) as a single equation of the matrix function \( A(n) \). Once we find such a matrix function \( A(n) \), any desired matrix \( A(k) \) can be obtained just by putting the desired value \( n = k \). This technique of considering \( A(n) \) as a matrix function of the occupation number \( n \) and solving the matrix algebra just once to find \( A(n) \) for general \( n \), is indeed possible for a large class of hop rates which we are going to discuss in details in the following sections. Also, since \( A(n) \) is now a general matrix function of the site occupation variable \( n \), we call it the site occupation matrix.

In order to proceed further, we need to be specific about the dynamics as the matrix algebra (9) explicitly depend on the hop rates. In the next section we will consider some stochastic processes like finite range processes (FRP) [46] (which includes as a special case zero range process (ZRP) [43, 44]), misanthrope process (MAP) [45] etc.

3. MPA for finite range process

In this section we consider finite range process (FRP) where particles hop to right with a hop rate that depends on occupation of the departure site and its neighbor within a range \( R_l \) to left and \( R_r \) to right. In particular we discuss different cases, \( R_l = 0 = R_r \) (ZRP), \( R_l = 0, R_r = 1 \) (namely misanthrope process), \( R_l = 1 = R_r \) (systems having pair factorized steady states) and the generic scenario with \( R_l = R = R_r \). For \( R_l = R_r = R \), the hop rate \( u(n_{i-R}, \ldots, n_i, \ldots, n_{i+R}) \) in FRP depends on same number \( R \) of neighbors in both directions with respect to the departure site \( i \). This special case was studied earlier [46] and it was shown that, when the hop-rates obey certain conditions, FRP leads to a \((R+1)\)-cluster factorized steady state (CFSS), \( P\{n_i\} \sim \prod_{i}^{R} g(n_i, n_{i+1}, \ldots n_{i+R}) \). For \( R = 1 \) we have a 2-cluster factorized state, commonly known as the pair factorized state (PFSS) where the steady state weights are \( P\{n_i\} \sim \prod_{i} g(n_i, n_{i} + 1) \). Clearly, a PFSS can equivalently be written as a matrix product state as \( g(n_i, n_{i} + 1) \) can directly be considered as the elements of an infinite matrix \( T \), i.e. \( T_{n_i, n_{i+1}} = g(n_i, n_{i+1}) \). In the following we show that, whenever a dynamics leads to PFSS, the matrix product ansatz also naturally leads to the same steady state. For \( R > 1 \), however, existence of a cluster factorized state does not ensure that it can also be written as matrix product state. In this section, we show that, even for \( R > 1 \), one can indeed construct matrix product states through the matrix formulation developed in the previous section. Below, some examples of totally asymmetric finite range processes for which one obtains steady states in matrix product form, are discussed in details.

3.1. Zero range process \((R_l = 0 = R_r)\)

Zero range process (ZRP) [43, 44] is a very familiar stochastic process where hop rate of particles depends only on the occupation of the departure site; thus FRP reduces to ZRP when \( R_l = R_r = 0 \). One important feature of ZRP is that its steady state has a simple factorized form irrespective of the functional form of the hop rates, lattice geometry or spatial dimension. In spite of having a rather simple dynamics, ZRP shows condensation transition for specific choice of rates - the condensation transition can be mapped to a phase separation transition in an equivalent exclusion process. Interestingly related phenomena like wealth redistribution...
[47] in agent based models, jamming in traffic flow [48] can be related to condensation transition in ZRP.

Clearly ZRP fits into the generic matrix product formulation discussed in previous section, as the hop rate \( u(n_i, n_{i-1}) \) here is equivalent to \( u(n_i) \). Thus for ZRP, the matrix algebra in equation (9) reduces to
\[
\begin{align*}
    u(n_i) A(n_{i-1}) \ldots A(n_i) A(n_{i+1}) \ldots A(n_{i+R}) &= A(n_{i-1}) \ldots A(n_i) A(n_i - 1) A(n_{i+1}) \ldots A(n_{i+R}),
\end{align*}
\]
along with the auxiliary matrix \( \tilde{A}(n) = A(n - 1) \), as in equation (8). First we try for a scalar solution by setting \( A(n) = a(n) \), with \( a(n) \) being a positive function for \( n \geq 0 \). This particular choice implies that the auxiliary matrices for ZRP are also scalar, \( \tilde{A}(n) = a(n - 1) \). So equation (10) simplifies to
\[
    u(n_i) a(n_i) = a(n_i - 1) \Rightarrow a(n) = \frac{a(n - 1)}{u(n)} = a(0) \prod_{k=1}^{n} \frac{1}{u(k)}.
\]
Thus, the steady state weight is
\[
    P(\{n_i\}) \sim \text{Tr} \left[ \prod_{i=1}^{L} A(n_i) \right] = \prod_{i=1}^{L} a(n_i)
\]
which is the familiar factorized steady state we know for ZRP [44]. The matrices \( A(n) \) being scalar states that there is no spatial correlation between occupation at different lattice sites, apart from the global conservation of the total number of particles.

3.2. Misanthrope process (\( R_l = 0, R_r = 1 \))

Misanthrope process [45] is a special case of FRP with \( R_l = 0 \) and \( R_r = 1 \), i.e. the hop rate \( u(n_i, n_{i+1}) \) here is no longer departure-site symmetric, it depends on the occupation number of the departure site \( i \) and the arrival site \( (i + 1) \) only. For certain choice of hop rates, misanthrope process is known to have a factorized steady state as studied in [45, 49]. Here we will show that the same factorized state can be obtained starting from matrix product ansatz.

Note, that the generic choice of auxiliary matrices \( \tilde{A}(n) = a(n - 1) \) along with \( A(n) = a(n) \), given by equations (8) and (9), would result in \( u(n_i, n_{i+1}) = \frac{a(n_i - 1)}{a(n_i)} \) which is inconsistent as this choice does not allow the hop rate to depend on the occupation of the arrival site. We now proceed with a scalar choice \( A(n) = a(n), \tilde{A}(n) = \tilde{a}(n) \), where both functions \( a(n) \) and \( \tilde{a}(n) \) are yet to be determined. The cancellation scheme in equation (5) now becomes
\[
    u(n_i, n_{i+1}) - u(n_i + 1, n_{i+1} - 1) \frac{a(n_i + 1) a(n_{i+1} - 1)}{a(n_i) a(n_{i+1})} = \frac{\tilde{a}(n_i)}{a(n_i)} - \frac{\tilde{a}(n_{i+1})}{a(n_{i+1})}.
\]
Since the hop rates \( u(n, 0) = 0 \) for \( m < 1 \) or for \( n < 0 \), the above equation, for \( n_{i+1} = 0 \) and for \( n_i = 0 \) reduces to,
\[
    u(n, 0) = \frac{\tilde{a}(n)}{a(n)} - \frac{\tilde{a}(0)}{a(0)} = u(1, n - 1) \frac{a(1)}{a(0)} \frac{a(n - 1)}{a(n)}.
\]
These equations further result in,
\[
    a(n) = \left( \frac{a(1)}{a(0)} \right) \frac{u(1, n - 1)}{u(0, n)} a(n - 1) = a(0) \left( \frac{a(1)}{a(0)} \right)^n \prod_{k=1}^{n} \frac{u(1, k - 1)}{u(k, 0)} \tag{15}
\]
\[
    \tilde{a}(n) = \left[ \frac{\tilde{a}(0)}{a(0)} + u(n, 0) \right] a(n).
\tag{16}
\]

It appears from equation (15) that we have a factorized steady state \( P(\{n_i\}) \propto \prod_{i=1}^{L} a(n_i) \) for any hop rate \( u(m, n) \) which is certainly not true, because these equations (14) and (16), derived by using specific boundary conditions \( (n_i = 0 \text{ or } n_{i+1} = 0) \), must also respect equation (13) for all \( n_i > 0, n_{i+1} > 0 \). Using equation (16) in equation (13) we get
\[
    u(n_i, n_{i+1}) - u(n_i + 1, n_{i+1} - 1) \frac{a(n_i + 1)a(n_{i+1} - 1)}{a(n_i)a(n_{i+1})} = u(n_i, 0) - u(n_{i+1}, 0).
\tag{17}
\]

Thus, in MAP, we have a factorized steady state \( P(\{n_i\}) \propto \prod_{i=1}^{L} a(n_i) \) only when the hop rate \( u(n_i, n_{i+1}) \) satisfies equation (17), which is the familiar constraint that has been reported earlier [49]. The steady state weights given by equation (15) is also identical to the one which is already known for MAP [49].

3.3. FRP with \( R_l = R_r = R = 1 \)

If \( R_l = R_r = R = 1 \), particle from an occupied site \( i \) hops to site \( (i + 1) \) with rate \( u(n_i-1, n_i, n_{i+1}) \) that depends on the occupancies of the departure site \( i \), its left nearest and right nearest neighbors \((i-1)\) and \((i+1)\) respectively. For a special class of hop rates, this finite range process has a pair factorized steady state (PFSS) [50] given by
\[
    P(\{n_i\}) = \frac{1}{Z_{L,N}} \prod_{i=1}^{L} g(n_i, n_{i+1}) \delta(\sum_i n_i - N).
\tag{18}
\]

Obviously \( g(n_i, n_{i+1}) \) itself can be considered as elements of an infinite dimensional matrix, infinite dimensional because \( n_i \) and \( n_{i+1} \) can take arbitrarily large positive integer values. In other words, PFSS is a matrix product state represented by infinite dimensional matrices. We show below that for a class of models one can obtain finite dimensional representation.

Let us consider hop rates of the form
\[
    u(n_{i-1}, n_i, n_{i+1}) = \frac{\langle \alpha(n_{i-1}) | \beta(n_{i-1}) \rangle \langle \alpha(n_i - 1) | \beta(n_{i+1}) \rangle}{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle \langle \alpha(n_i) | \beta(n_{i+1}) \rangle},
\tag{19}
\]
\( \alpha(n) \) and \( \beta(n) \) are arbitrary positive functions with \( \alpha(-1) = \beta(-1) = 0 \). It is easy to see that the matrix algebra (9), along with the choice of auxiliary \( A(n) = A(n - 1) \) as in equation (8), can be satisfied if,
\[
    A(n) = |\beta(n) \rangle \langle \alpha(n)|.
\tag{20}
\]

Correspondingly, the steady state probability of configurations are \( P(\{n_i\}) \sim \text{Tr} \left[ \prod_i A(n_i) \right] \delta(\sum_i n_i - N) \). The grand canonical partition function is then
\[
    Z_L(z) = \sum_{\{n_i\}} z^n \text{Tr} \left[ \prod_i A(n_i) \right] = \text{Tr} \left[ T(z)^L \right] ; T(z) = \sum_n z^n |\beta(n) \rangle \langle \alpha(n)|.
\tag{21}
\]
Now one can conveniently calculate steady state average of desired observables in the steady state, like spatial correlations, density fluctuations, particle current etc. For example, since the particle hops only towards right, the average steady state current of the system is

\[ J = \langle u(n_{i-1}, n_i, n_{i+1}) \rangle = \frac{1}{Z_L(z)} \sum_{\{n_i\}} u(n_{i-1}, n_i, n_{i+1}) z^{n_i} \text{Tr} \left[ \prod_{i=1}^{L} A(n_i) \right] = z. \]  

(22)

To find the dependence of \( J \) on the average particle density \( \rho \), one must calculate \( \rho(z) = \frac{\partial}{\partial z} \frac{1}{Z_L} \) and then invert this relation.

### 3.4. Finite range process with \( R_l = R_r = R > 1 \)

For a more general finite range process (FRP) corresponding to \( R_l = R_r = R > 1 \) the hop rate \( u(n_{i-1}, \ldots, n_i, \ldots, n_{i+k}) \) is a function of \( (2R + 1) \) site variables, namely the occupation number of the departure site and that of \( R \) neighbors to its left and to right. This model was introduced earlier in Ref. [46] where it has been shown that the steady state of the system is cluster factorized when the hop rates \( u(\cdot) \) satisfy certain specific conditions. For a cluster factorized steady state (CFSS), the probability of configurations are given by,

\[ P(\{n_i\}) = \frac{1}{Z_{L,N}} \prod_{i=1}^{L} g(n_i, n_{i+1}, \ldots, n_{i+k}) \delta(\sum_i n_i - N), \]  

(23)

where \( g(n_i, n_{i+1}, \ldots, n_{i+k}) \) is a function of \( (R + 1) \) variables, as known as the cluster weight function, and \( Z_{L,N} \) is the canonical partition function. The authors in [46] have restricted their study to FRP where the cluster weight function has a ‘sum-form’ \( g(n_i, n_{i+1}, \ldots, n_{i+k}) = \sum_{k=0}^{R} f_k(n_{i+k}) \).

For example, when \( R = 2 \), FRP has a 3-cluster factorized steady state with weight function \( g(n_i, n_{i+1}, n_{i+2}) = \gamma_0(n_i) + \gamma_1(n_{i+1}) + \gamma_2(n_{i+2}) \) if the hop rate (that satisfies the required condition) is

\[ u(n_{i-2}, n_{i-1}, n_i, n_{i+1}, n_{i+2}) = \prod_{k=0}^{2} \frac{\gamma_0(n_{i-2+k}) + \gamma_1(n_{i-1+k}) + \gamma_2(n_{i+k-1})}{\gamma_0(n_{i-2+k}) + \gamma_1(n_{i-1+k}) + \gamma_2(n_{i+k})}. \]

(24)

Clearly \( g(\cdot) \) being a function of \( (R + 1) \) variables, unlike for \( R = 1 \) case, it can not be considered directly as matrix when \( R > 1 \). Thus, for \( R > 1 \), rewriting a cluster factorized steady state as a matrix product state is already challenging. Moreover, here we will discuss more generalized forms of the hop rates which does not necessarily lead to the ‘sum-form’ of the cluster weight function. Let us consider an example \( R_l = R_r = R = 2 \), where a particle from a randomly chosen site \( i \) hops to its right neighbor \( (i + 1) \) with a rate

\[ u(n_{i-2}, n_{i-1}, n_i, n_{i+1}, n_{i+2}) = \prod_{k=0}^{2} \frac{\langle f_0(n_{i-2+k}) | f_1(n_{i-1+k}) \rangle + \langle f_2(n_{i-1+k}) | f_3(n_{i+k-1}) \rangle}{\langle f_0(n_{i-2+k}) | f_1(n_{i-1+k}) \rangle + \langle f_2(n_{i-1+k}) | f_3(n_{i+k}) \rangle}. \]

(25)

where \( \langle f_\nu(n) \rangle = (h_1^\nu(n), h_2^\nu(n), \ldots, h_d^\nu(n))^T \) (here \( \nu = 0, 1, 2, 3 \)). In fact the rates here satisfy the conditions required [46] for a system to have 3-cluster factorized steady state \( P(\{n_i\}) \sim \prod g(n_i, n_{i+1}, n_{i+2}) \) with

\[ g(l, m, n) = \langle f_0(l) | f_1(m) \rangle + \langle f_2(m) | f_3(n) \rangle. \]

(26)

Although we have exact steady state weights for these rates, it is not very useful in calculating the partition function or other physical observables. This is because, any occupation variable
\( \langle n_i \rangle \) appears thrice in the cluster factorized state and carrying out the sum over the all possible values of \( n_i \) is not straightforward. In this regards, the matrix formulation, where the matrices are parametrized by the local occupation number, is very helpful. In the following we proceed with the MPA and use the auxiliary matrices \( \tilde{A}(n) = A(n - 1) \), as in equation (8). The matrices \( A(n) \) should then follow the matrix algebra given by equation (9) with hop rate there replaced by equation (25). We find that this algebra is satisfied by the following representation of matrices,

\[
A(n) = (| \beta(n) \rangle \otimes I) \Gamma(n) (I \otimes | \alpha(n) \rangle)
\]

where,

\[
| \beta(n) \rangle = \begin{pmatrix} 1 \\ | f_3(n) \rangle \end{pmatrix}; \quad \langle \alpha(n) | = \begin{pmatrix} | f_0(n) \rangle \\ 1 \end{pmatrix}
\]

are \((d + 1)\)-dimensional vectors and

\[
\Gamma(n) = \begin{pmatrix} | f_1(n) \rangle & 0_{d \times d} \\ 0 & | f_2(n) \rangle \end{pmatrix}
\]

is a \((d + 1)\)-dimensional matrix. Also, \( I \) is the identity matrix in \((d + 1)\) dimension. The operation \( \otimes \) is the familiar direct product. Note that \(| \beta(n) \rangle \otimes I \) and \( I \otimes | \alpha(n) \rangle \) are not square matrices; their dimensions are respectively \((d + 1)^2 \times (d + 1)\) and \((d + 1) \times (d + 1)^2\). Thus, the dimension of the matrices \( A(n) \) that represent the steady state weights is \((d + 1)^2\). In the appendix we have discussed how to generate the matrix representation systematically for a dynamics (25) or equivalently for a model which has a cluster factorized steady state with cluster weight function \( g(\cdot) \) given by equation (26).

Let us illustrate the dynamics and the steady state weights for a specific example where the hop rates are given by equation (25) with scalar choice of \( \{ f_i(n) \} \), i.e. \( f_i(n) = f_i(\cdot) \). Explicitly, the hop rates are now

\[
u(n_{i-2}, n_{i-1}, n_i, n_{i+1}, n_{i+2}) = \sum_{k=0}^{2} \frac{f_0(n_{i-2+k}) f_1(n_{i-1+k}) + f_2(n_{i-1+k}) f_3(n_{i+k} - 1)}{f_0(n_{i-2+k}) f_1(n_{i-1+k}) + f_2(n_{i-1+k}) f_3(n_{i+k})}.
\]

For this simple choice of hop rate,

\[
| \beta(n) \rangle = \begin{pmatrix} 1 \\ f_3(n) \end{pmatrix}; \quad \langle \alpha(n) | = \begin{pmatrix} f_0(n) \\ 1 \end{pmatrix}; \quad \Gamma(n) = \begin{pmatrix} f_1(n) & 0 \\ 0 & f_2(n) \end{pmatrix},
\]

and correspondingly the steady state matrix \( A(n) \), from equation (27), reduces to a \(4\)-dimensional matrix

\[
A(n) = \begin{pmatrix} f_0(n) f_1(n) & f_1(n) & 0 & 0 \\ 0 & 0 & f_0(n) f_2(n) & f_2(n) \\ f_0(n) f_1(n) f_3(n) & f_1(n) f_3(n) & 0 & 0 \\ 0 & 0 & f_0(n) f_2(n) f_3(n) & f_2(n) f_3(n) \end{pmatrix}
\]

Thus, we obtain the matrix product steady state \( P(\{ n_i \}) \sim \text{Tr}[\prod_{i=1}^L A(n_i)] \) for the dynamics (30). As we have already mentioned, the steady state of this dynamics has 3-cluster factorized form \( P(\{ n_i \}) \sim \prod_{i} g(n_i, n_{i+1}, n_{i+2}) \). Finally, once the representation of matrices \( A(n) \) as in (27) are known, it is quite straightforward to calculate the partition function and any desired observable.
4. MPA for finite range process with asymmetric rate functions

In the previous sections, we have studied finite range processes where particles hop only to the right. In fact, the steady state measure of FRP remains invariant if we introduce a parameter \( p \), the probability that a particle chooses the right neighbor as a target site and moves there with rate \( u(\cdot) \) or with probability \( (1-p) \) it decides to hop to left and moves there with the same rate \( u(\cdot) \). A non-trivial situation is when the functional form of rate functions for right hop is different from that of the left hop. A class of such asymmetric motion of particles without hardcore constraints has recently been introduced and studied in [9] in context of asymmetric zero range process (AZRP), asymmetric misanthrope process (AMAP) and asymmetric finite range process (AFRP); each of them having exactly solvable non-equilibrium invariant measures -factorized steady states (FSS) for AZRP, AMAP and cluster factorized steady states (CFSS) for AFRP. AZRP, AMAP show interesting features like density dependent current reversal (keeping the external bias fixed), condensation (tuned by the proportion of right and left moves executed by the particles)- phenomena solely induced by different functional forms of the left and right rates. Now AZRP and AMAP, having FSS, would not be of much interest in context of matrix product states since in the previous sections we have already discussed how the matrices and the auxiliaries reduce to scalars for a steady state to have a factorized form. So we would like to explore only the possibility of obtaining a matrix product state for AFRP. In this section we will first introduce a very general dynamics for asymmetric hopping process in one dimension which includes AFRP as a special case. We will then illustrate the matrix formulation with some examples.

4.1. General asymmetric hopping dynamics

Let us consider an interacting particle system on a one dimensional periodic lattice where particles (without hardcore exclusion) can hop in both directions with respective forward and backward rates; the rate functions depend on the occupation of several lattice sites as well as on the direction of motion of the particles, i.e. the right and left hop rates can have different functional forms. The model is defined on a one dimensional periodic lattice with \( L \) sites where each site \( i \) contains \( n_i \) particles with \( n_i \geq 0 \) being a nonnegative integer. A particle from site \( i \) (with \( n_i > 0 \)), can move either to its immediate right neighbor \((i+1)\) with rate \( u_R(n_{i-1},n_i,...,n_i,R) \) or it can hop to its immediate left neighbor \((i-1)\) with rate \( u_L(n_{i-1},R,...,n_{i+1},n_i) \). Note that, the model is different from the one discussed in Ref. [9] as not only the forward and backward rates have different functional forms \( u_R(\cdot) \) and \( u_L(\cdot) \), also, they have different number of arguments; the right hop rate depends on \( R_l \) left neighbors and \( R_r \) right neighbors in contrast to \( R_l' \) left and \( R_r' \) right neighbors for the left hop rate. In general, all four numbers \( R_l, R_r, R_l', R_r' \) can be different. We ask if this stochastic process can lead to a non-equilibrium steady state, in matrix product form. Solving the matrix algebra to find out a matrix product state for arbitrary values of \( R_l, R_r, R_l', R_r' \) appears to be quite complex. We restrict our selves to some special cases and study below two specific examples.

4.1.1. Example 1. Our first example is \( R_l = 1 \neq R_l' = 2 \) and \( R_r = 2 \neq R_r' = 0 \), i.e. a particle from site \( i \) hops to the right neighbor with rate \( u_R(n_{i-1},n_i,...,n_{i+1},n_{i+2}) \) and it hops to the left with rate \( u_L(n_{i-2},n_{i-1},n_i) \). This dynamics has not been studied earlier in context of particle or mass transfer processes and clearly the criteria for having a factorized or cluster-factorized steady state is not known. In the following we show, using a specific example, that one can use MPA to obtain an exact steady state weights of these models in some special cases.
Let us choose the rate functions in the following form

\[ u_R(n_{i-1}, n_i, n_{i+1}, n_{i+2}) = u(n_{i-1}, n_i, n_{i+1}) + v(n_i, n_{i+1}, n_{i+2}) \]

\[ u_L(n_{i-2}, n_{i-1}, n_i) = v(n_{i-2}, n_{i-1}, n_i). \]  

(31)

Here, the right hop rate \( u_R(\cdot) \) is a sum of two independent functions- the first part \( u(\cdot) \) is symmetric with respect to the departure site \( i \) and the rest \( v(\cdot) \) is symmetric with respect to the arrival site \((i + 1)\). On the other hand, the left hop rate \( u_L(\cdot) \equiv v(\cdot) \) is symmetric with respect to the departure site. Assuming that the steady state of the model can be written in a matrix product form \( P(n_i) \sim Tr \left[ \prod_i A(n_i) \delta(\sum_i n_i - N) \right] \), the Master equation for dynamics (31) in steady state reduces to,

\[
\sum_{i=1}^{L} \left[ u(n_{i-1}, n_i, n_{i+1}) + v(n_i, n_{i+1}, n_{i+2}) + v(n_{i-2}, n_{i-1}, n_i) \right]
\]

\[ \text{Tr}[\ldots A(n_{i-2})A(n_{i-1})A(n_i)A(n_{i+1})A(n_{i+2}) \ldots ] \]

\[ - \sum_{i=1}^{L} \left[ u(n_{i-2}, n_{i-1} + 1, n_i - 1)\text{Tr}[\ldots A(n_{i-2})A(n_{i-1} + 1)A(n_i - 1)\ldots ] \right] \]

\[ + v(n_{i-1} + 1, n_i - 1, n_{i+1})\text{Tr}[\ldots A(n_{i-1} + 1)A(n_i - 1)A(n_{i+1})\ldots ] \]

\[ + v(n_{i-1} - n_i - 1, n_{i+1} + 1)\text{Tr}[\ldots A(n_{i-1})A(n_i - 1)A(n_{i+1} + 1)\ldots ] = 0. \]

(32)

The above equation can be equivalently written as

\[
\sum_{i=1}^{L} \text{Tr}[\ldots A(n_{i-2})F(n_{i-1}, n_i, n_{i+1})A(n_{i+2}) \ldots ] = 0, \]

(33)

where,

\[
F(n_{i-1}, n_i, n_{i+1}) = [u(n_{i-1}, n_i, n_{i+1})A(n_{i-1})A(n_i)A(n_{i+1}) - u(n_{i-1}, n_i + 1, n_{i+1} - 1)A(n_{i-1})A(n_i + 1)A(n_{i+1} - 1)]
\]

\[ + [v(n_{i-1}, n_i, n_{i+1})A(n_{i-1})A(n_i)A(n_{i+1}) - v(n_{i-1} + 1, n_i, n_{i+1} + 1)A(n_{i-1})A(n_i + 1)A(n_{i+1})]
\]

\[ + [v(n_{i-1}, n_i, n_{i+1})A(n_{i-1})A(n_i)A(n_{i+1}) - v(n_{i-1}, n_i - 1, n_{i+1} + 1)A(n_{i-1})A(n_i - 1)A(n_{i+1} + 1)]. \]

Equation (33) is a sum of \( L \) similar terms where each term carries a three site function \( F(x, y, z) \) that contains the relevant information about the dynamics, i.e. the in-flux and out-flux for a given configuration. So it would be reasonable to find a local three site cancellation scheme for \( F(x, y, z) \) that would make the sum of \( L \) terms in equation (33) equal to zero. We propose the following cancellation scheme,

\[
F(n_{i-1}, n_i, n_{i+1}) = [A(n_{i-1})\tilde{A}(n_i)A(n_{i+1}) - A(n_{i-1})\tilde{A}(n_i)A(n_{i+1})]
\]

\[ + [\tilde{A}(n_{i-1})A(n_i)\tilde{A}(n_{i+1}) - \tilde{A}(n_{i-1})A(n_i)\tilde{A}(n_{i+1})]
\]

\[ + [A(n_{i-1})\tilde{A}(n_i)A(n_{i+1}) - \tilde{A}(n_{i-1})A(n_i)A(n_{i+1})]. \]

(34)

It is easy to check that this form of \( F(n_{i-1}, n_i, n_{i+1}) \) indeed serves the purpose. Note that, unlike the previous cases where we had only one kind of auxiliary matrix \( \tilde{A}(n) \), here we have used three different auxiliary matrices \( \tilde{A}(n), \tilde{A}(n), \tilde{A}(n) \). In fact, if all three auxiliaries were same i.e. \( \tilde{A}(n) = \tilde{A}(n) = \tilde{A}(n) \), then (34) reduces to the familiar cancellation scheme studied here in (5) with \( R_L = R_R = 1 \) and correspondingly one obtains a matrix product steady state for totally asymmetric hoping model with hop rate \( u_R = u(n_{i-1}, n_i, n_{i+1}) \) and \( u_L = 0 \), a model which we have already discussed in the previous section.
To proceed further with the asymmetric hopping model we need to be more specific about the dynamics, that is, we must be specific about functional forms of \( u(\cdot) \) and \( v(\cdot) \). If we consider the functions \( u(\cdot) \) and \( v(\cdot) \) as

\[
\begin{align*}
 u(n_{i-1}, n_i, n_{i+1}) &= \frac{\langle \alpha(n_{i-1}) | \beta(n_1) \rangle \langle \alpha(n_i-1) | \beta(n_i) \rangle}{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle \langle \alpha(n_1) | \beta(n_1) \rangle}
 v(n_{i-1}, n_i, n_{i+1}) &= \frac{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle \langle \alpha(n_i+1) | \beta(n_i) \rangle}{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle \langle \alpha(n_i) | \beta(n_i+1) \rangle},
\end{align*}
\]

then, equation (34) results in the following solution:

\[
\hat{A}(n) = A(n - 1); \hat{A}(n) = A(n + 1); \hat{A}(n) = \theta(n)A(n)
\]

where \( \theta(n) \) is the Heaviside step function.

So, to summarize, if particles on a one dimensional periodic lattice undergo asymmetric hopping with different right and left rate functions (constructed below by substituting equation (35) in (31))

\[
\begin{align*}
 u_L(n_{i-2}, n_{i-1}, n_i) &= \frac{\langle \alpha(n_{i-2}) | \beta(n_i+1) \rangle \langle \alpha(n_{i-1}) + 1 | \beta(n_i) \rangle}{\langle \alpha(n_{i-2}) | \beta(n_i) \rangle \langle \alpha(n_{i-1}) | \beta(n_i) \rangle}
 u_R(n_{i-1}, n_i, n_{i+1}, n_{i+2}) &= \frac{\langle \alpha(n_{i-1}) | \beta(n_i+1) \rangle \langle \alpha(n_i-1) + 1 | \beta(n_i) \rangle}{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle \langle \alpha(n_i) | \beta(n_i+1) \rangle}
 u_L(n_{i-1}, n_i, n_{i+1}) &= \frac{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle \langle \alpha(n_i-1) | \beta(n_i) \rangle}{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle \langle \alpha(n_i) | \beta(n_i+1) \rangle}
\end{align*}
\]

along with \( u_L(x, 0, z, w) = 0 \) and \( u_L(x, y, 0) = 0 \), the steady state of the model has a matrix product form \( P(\{n_i\}) \sim \text{Tr} \left[ \prod_n A(n) \right] \delta(\sum_n n_i - N) \) with matrices \( A(n) = |\beta(n)\rangle \langle \alpha(n)| \) and the auxiliary matrices \( \hat{A}(\cdot), \hat{A}(\cdot) \) and \( \hat{A}(\cdot) \) given by equation (36).

We conclude this subsection with the following remark. Matrices \( A(n) \) we obtain for the asymmetric hopping dynamics (37) are same as those we obtain for dynamics (19). The auxiliary matrices in two cases are different, but they do not explicitly appear in the steady state weights. This indicates that these two very different dynamics lead to the same steady state measure.

4.1.2. Example 2. In this example we study an asymmetric finite range process where \( R_L = R_R = R_L' = R_R' = 1 \). In details, we consider a one dimensional periodic lattice with \( L \) sites with each site \( i \) containing \( n_i (\geq 0) \) particles and a particle from a randomly chosen site \( i \) (if not vacant) jumps either to its right neighbor \( (i + 1) \) with a hop rate \( u_L(n_{i-1}, n_i, n_{i+1}) \) or to its left neighbor \( (i - 1) \) with rate \( u_L(n_{i-1}, n_{i-1}, n_{i+1}) \). In this model both the right and left rate functions are symmetric with respect to the departure site \( i \). Let us assume that the steady state probability of any configuration \( \{n_i\} \) of this stochastic process can be expressed as a product of matrices in the form \( P(\{n_i\}) \sim \text{Tr} \left[ \prod_n A(n) \right] \delta(\sum_n n_i - N) \) where \( A(n) \) is the site occupation matrix corresponding to site \( i \) containing \( n_i \) particles. The steady state Master equation for this interacting particle system reads as

\[
\sum_{i=1}^{L} [u_L(n_{i-1}, n_i, n_{i+1}) + u_L(n_{i-1}, n_{i+1}, n_{i+1})] \text{Tr}[\ldots A(n_i)A(n_i)A(n_{i+1})\ldots] - \sum_{i=1}^{L} [u_L(n_{i-2}, n_{i-1}, n_{i-1} + 1, n_{i-1}) \text{Tr}[\ldots A(n_{i-2})A(n_{i-1} + 1)A(n_{i-1} + 1)\ldots]] + u_L(n_i - 1, n_{i+1} + 1, n_{i+2}) \text{Tr}[\ldots A(n_i - 1)A(n_{i+1} + 1)A(n_{i+2})\ldots] = 0.
\]
Shifting the sum indexes in the above equation and rearranging them suitably, we arrive at
\[ \sum_{i=1}^{L} \text{Tr}[(\ldots A(n_{i-2}) F(n_{i-1}, n_i, n_{i+1}) A(n_{i+2}) \ldots)] = 0, \]
where
\[ F(n_{i-1}, n_i, n_{i+1}) = [u_R(n_{i-1}, n_i, n_{i+1}) + u_L(n_{i-1}, n_i, n_{i+1})] A(n_{i-1}) A(n_i) A(n_{i+1}) \]
\[ - u_R(n_{i-1}, n_i + 1, n_{i+1} - 1) A(n_{i-1}) A(n_i + 1) A(n_{i+1} - 1) \]
\[ - u_L(n_{i-1} - 1, n_i + 1, n_{i+1}) A(n_{i-1} - 1) A(n_i + 1) A(n_{i+1}). \]

(38)

So, just like the previous example, the Master equation in steady state has been written as a sum of \( L \) terms each containing a three site function \( F(x,y,z) \), which we must write in a way using auxiliaries so that the terms within the sum cancel with each other. To this end, we further specify the rate functions \( u_{R,L}(\cdot) \) as
\[ u_R(n_{i-1}, n_i, n_{i+1}) = \gamma \frac{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle \langle \alpha(n_i) | \beta(n_{i+1}) \rangle}{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle} + \delta \frac{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle \langle \alpha(n_i) | \beta(n_{i+1} + 1) \rangle}{\langle \alpha(n_{i-1}) | \beta(n_i) \rangle} \]
\[ u_L(n_{i-1}, n_i, n_{i+1}) = \delta \frac{\langle \alpha(n_{i-1} + 1) | \beta(n_i) \rangle \langle \alpha(n_i) | \beta(n_{i+1}) \rangle}{\langle \alpha(n_i) | \beta(n_{i+1}) \rangle}. \]

(39)

These hop rates resemble the rate functions considered by the authors in [9] in context of asymmetric finite range process. Here too, we use three auxiliary matrices \( \hat{A}, \tilde{A} \) and \( \bar{A} \), but now the last two auxiliary matrices are functions of two arguments whereas \( \hat{A} \) has one argument as in earlier cases. Explicitly, the cancellation scheme reads as,
\[ F(n_{i-1}, n_i, n_{i+1}) = [A(n_{i-1}) \hat{A}(n_i) A(n_{i+1}) - A(n_{i-1}) A(n_i) \bar{A}(n_{i+1})] \]
\[ + [A(n_{i-1}) \tilde{A}(n_i, n_{i+1}) A(n_{i+1}) - \tilde{A}(n_{i-1}, n_i) A(n_i) A(n_{i+1})] \]
\[ + [A(n_{i-1}) \bar{A}(n_i, n_{i+1}) A(n_{i+1}) - A(n_{i-1}) A(n_i) \bar{A}(n_i, n_{i+1})]. \]

(40)

One can easily check that equation (40) satisfies the steady state condition (38) and it results in a matrix product state with matrices \( A(n) \) in the familiar form
\[ A(n) = |\beta(n)\rangle \langle \alpha(n)\rangle. \]

(41)

The corresponding choice of auxiliary matrices are then
\[ \hat{A}(n) = \gamma A(n - 1), \quad \tilde{A}(m,n) = \delta A(m - 1) |\beta(n + 1)\rangle \langle \alpha(m)\rangle, \]
\[ \bar{A}(m,n) = \delta |\beta(n)\rangle \langle \alpha(m + 1)\rangle A(n - 1). \]

(42)

So, if we have an asymmetric particle transfer process with right and left rate functions expressed by (39) we have a matrix product steady state, same as the one obtained for dynamics (37) or for (19).

However, it should be mentioned that the cancellation scheme used here in equation (40) is again very much distinct from the schemes used in the previous examples.

5. Conclusion

We have introduced a matrix product ansatz for systems of interacting particles without any hardcore constraints. In these class of models particles on a one dimensional lattice jump to their neighboring sites with some rate that depends on the occupation of the departure site and its neighbors within a specified range. In case of MPA for exclusion processes, where particles
obey hard core constraints, we need only a finite number of matrices to represent each species of particle. For systems without hardcore constraints, the sites can either be vacant or occupied by arbitrary number of particles and thus a matrix product state that describe these systems would require infinite number of matrices (in contrast to the hardcore exclusion processes), each corresponding to a specific occupation number. Further, any given dynamics would insist the matrices to follow an algebra, consisting of infinitely many matrix-relations. Finding specific representation of these infinite set of matrices that follow the algebra, appears to be complex, but here, in this article, for a generic class of models, we show that the matrices can be parametrized by the occupation number (which essentially leads to the name site occupancy matrices of the matrices \( A(n) \)), i.e. the elements of the matrix are functions of the occupation number. This parametrization actually helps to treat the infinite set of matrix algebra as a single equation of the matrix function \( A(n) \) which can be solved once and for all for any general \( n \), so that one no more has to solve for the matrices \( A(0), A(1), A(2) \ldots \) separately.

The class of hopping models we studied here is very general; many well known models, like zero range process, misanthrope process, models with pair factorized steady state, and finite range processes are only some of the special cases, for which the exact steady state weights are already known. In this article, first we re-derive the steady state weights of these models using matrix product formulation.

We also study FRP for very general rates which has not been studied earlier, and show that their steady state can be expressed in matrix product form. A specific example is FRP with \( R = 2 \), which leads to a 3-cluster factorized steady state with weights

\[
P_i(n_l) = \prod_{i=1}^{L} g(n_{i-1}, n_i, n_{i+1})
\]

when the hop-rates satisfy a specific condition. Even when the steady state is known exactly, there are practical difficulties in calculating the partition function or average steady state values of the observables; this is because any particular occupation variable \( n_l \) appears thrice in the product and carrying out sum of \( n_l \) for all possible values is non-trivial. For some special cases, like when the weight function has a sum form \( g(k, l, m) = f_0(k) + f_1(l) + f_2(m) \) one can write the steady state in a matrix product form, where matrices depend on only a single occupation variable \( n_l \) which enables us to carry out the corresponding sum over \( n_l \). Such a matrix product solution has been known for totally asymmetric finite range process [46]. In [46], however, re-writing the 3-cluster factorized steady state in a matrix product form was only a mathematical trick, a relation between the matrices and dynamics of the system were not established. When \( g(k, l, m) \) has a ‘sum-form’, the matrix product ansatz formulated here leads to a matrix algebra which is naturally satisfied by the matrices constructed in [46]. Moreover we explicitly derive matrix representations for certain other class of weight functions \( g(k, l, m) = f_0(k) f_1(l) + f_2(l) f_3(m) \) and more generally for \( g(l, m, n) = (f_0(l)f_1(m)) + (f_2(m)f_3(n)) \). In general there are no well defined methods to obtain matrix representation from a given matrix algebra. Fortunately for systems having a cluster factorized steady state, the matrix representations that describe a matrix product state can be derived systematically. In the appendix we have discussed this in details.

We further study asymmetric finite range processes where the rate functions for right and left hops are different in the sense that they have different number of arguments and/or different functional forms. In particular, we introduce a model where the hop rate for right move \( u_R(\cdot) \) depends on occupation of departure sites, \( R_l \) neighbors to its left and \( R_r \) neighbors to the right. Whereas the left hop rate \( u_L(\cdot) \) depends on the departure site and \( R_{l,r} \) sites to its left and right respectively. We obtain a matrix product steady state for two specific cases (i) \( R_l = 1 \neq R_r = 2 \) and \( R_l = 2 \neq R_r = 0 \), (ii) \( R_l = R_r = R_{l,r} = R_{l,r} = 1 \). Interestingly, both models lead to same MPS, but the auxiliaries, used in the cancellation scheme to satisfy the Master equation in steady state, turns out to be very different.
There are many other interesting directions to pursue in the study of matrix product formulation for interacting particles in absence of hardcore constraints. One important direction is to investigate the open systems, where particles can enter from left boundary and exit from right boundary of the system. It is well known that open exclusion processes (EP), where particles obey hardcore constraints, give rise to interesting results; even the simplest case, namely totally asymmetric simple exclusion process (TASEP) which is exactly solved through MPA [20], shows rich variety of phases and transitions among them as the entry and exit rate of particles are varied. One can also study exclusion processes that can be mapped to a particular finite range process. It is well known that, steady state weight of exclusion processes can always be written in matrix product form if they can be mapped to zero-range process [8]; in this situation explicit representations can be obtained from the known steady state weights of the corresponding zero range process, which helps in finding spatial correlation in EP. In a similar fashion, using matrix product formulation, one can study the spatial correlation functions in exclusion processes which can be mapped to finite range processes.

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Appendix. Matrix product form of cluster factorized steady states

We have seen in sections 3 and 4 that the matrix product ansatz naturally leads to a cluster factorized steady state if the dynamics of the system allows one. Depending on the dynamics of the model, MPA results in a specific matrix-algebra, but there are no systematic methods to obtain matrix representation from a given algebra. Thus for models that have a cluster factorized steady state, it is useful to construct the matrices from the known steady state, whenever possible. We must remind that, for FRP with $R \geq 2$, calculating the partition function or average value of observables is not straightforward even when the exact steady state weights are known in cluster factorized form; in such situations the matrix formulation is certainly a relief.

To this end we construct the matrices for a 3-cluster factorized steady state; it is straightforward to generalize this for larger clusters. Let us consider a specific CFSS,

$$P((n_i)) \sim \prod_i g(n_{i-1}, n_i, n_{i+1})$$

with

$$g(k, l, m) = \langle f_0(k) | f_1(l) \rangle + \langle f_2(l) | f_3(m) \rangle.$$  \hspace{1cm} (A.1)

where $\langle f_\nu(n) | = (h_{1\nu}(n), h_{2\nu}(n), h_{3\nu}(n), \ldots h_{d\nu}(n))$ are $d$-dimensional row-vectors and $| f_\nu(n) \rangle = (f_{\nu}(n))$ (here $\nu = 0, 1, 2, 3$). These cluster weight functions can be rewritten as inner product of vectors and matrices, each of which now depends on a single individual occupation number. More precisely,

$$g(k, l, m) = \langle \alpha(k) | \Gamma(l) | \beta(m) \rangle,$$

where $\langle \alpha(k) | = \langle f_0(k) | 1 \rangle$; $| \beta(m) \rangle = \langle 1 | f_3(m) \rangle$

and $\Gamma(l) = \begin{pmatrix} f_1(l) \langle f_2(l) | \end{pmatrix}$.

(A.2)

(A.3)
Now the steady state weights can be written as
\[ P(\{n_i\}) \sim \prod_i g(n_{i-1}, n_i, n_{i+1}) \]
\[ = \langle \alpha|\Gamma(l)|\beta(m)\rangle \langle \alpha|\Gamma(m)|\beta(n)\rangle \langle \alpha|m|\beta(p)\rangle \langle \alpha|\Gamma(p)|\beta(q)\rangle \ldots \]
\[ = \text{Tr} [\Gamma(l)|\beta(m)\rangle \langle \alpha|l|\Gamma(m)|\beta(n)\rangle \langle \alpha|m|\Gamma(n)|\beta(p)\rangle \langle \alpha|n|\Gamma(p)|\beta(q)\rangle \ldots ] \]
\[ = \text{Tr} [G(l, m) G(m, n) G(n, p) \ldots ] . \quad (A.4) \]

Thus we have transformed the 3–cluster weight functions to a matrix product form with matrices \( G(l, m) = \Gamma(l)|\beta(m)\rangle \langle \alpha|l| \) depending on occupancy of two neighboring sites. To get the site occupation matrices \( A(n) \) which depend only on a single site occupation number, as in the matrix product ansatz (4), we proceed as follows. Since the outer product of any two vectors \(|b\rangle\) and \(|a\rangle\) can be written as
\[ |b\rangle\langle a| = (I \otimes |a\rangle\langle b| \otimes I) \]
with \( I \) being the identity matrix of same dimension as that of \(|b\rangle\) and \(|a\rangle\), we rewrite \( G(l, m) \) as
\[ G(l, m) = \Gamma(l)|\beta(m)\rangle \langle \alpha|l| = \Gamma(l) (I \otimes |\alpha|l\rangle\langle \beta(m)| \otimes I). \quad (A.6) \]

Using this in equation (A.4) we get
\[ P(\{n_i\}) \sim \text{Tr} \prod_i A(n_i) \]
\[ \text{with } A(n) = (|\beta(n)\rangle \otimes I) \Gamma(n) (I \otimes |\alpha(n)\rangle). \quad (A.7) \]

In the appendix, we have demonstrated how to obtain a matrix product form from a known 3-cluster factorized steady state. There is no particular difficulty in extending this formulation to systems with larger cluster factorized steady state (like the steady states of FRP \([46]\) with \( R > 2 \)).

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