Rigorous computer-assisted proof for existence of period doubling
renormalisation fixed points in maps with critical point of degree 4

Andrew D Burbanks, Andrew H Osbaldestin and Judi A Thurlby
School of Mathematics and Physics, University of Portsmouth, UK
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We gain tight rigorous bounds on the renormalisation fixed point for period doubling in families of unimodal maps with degree 4 critical point. We prove that the fixed point is hyperbolic and use a contraction mapping argument to bound essential eigenfunctions and eigenvalues for the linearisation and for the scaling of additive noise. We find analytic extensions of the fixed point function to larger domains. We use multi-precision arithmetic with rigorous directed rounding to bound operations in a space of analytic functions yielding tight bounds on power series and universal constants.

Keywords: Dynamical systems; Renormalisation group; Universality; Period-doubling; Bifurcations; Computer-assisted proofs

I. INTRODUCTION

A. Background

An explanation for the remarkable universality observed in period-doubling cascades for families of unimodal maps of the interval with quadratic critical point was offered by Feigenbaum [1–3] and Coullet and Tresser [4] in terms of a renormalisation operator acting on a suitable space of functions.

The explanation rests on the following conjectures: There exists a nontrivial hyperbolic renormalisation fixed point. The spectrum of the linearisation of the operator has a single essential expanding eigenvalue. The associated one-dimensional unstable manifold crosses the manifold corresponding to functions with superstable period 2\(^n\) orbits transversally for sufficiently large \(n\).

Lanford [5] established the existence of a nontrivial locally-unique hyperbolic fixed point of the operator by rigorous computer-assisted means. He established that a certain quasi-Newton operator is a contraction mapping on a carefully chosen ball in a suitable space of functions and then bounded the spectrum of the derivative of the operator at the fixed point in order to establish hyperbolicity.

The efficacy of rigorous computer-assisted proofs in this area is apparent in the body of work that followed. Eckmann et al [6, 7] proved the existence of a fixed point of the corresponding renormalisation operator for period doubling in area-preserving maps, providing a detailed framework for rigorous computation in Banach spaces of multivariate analytic functions. Eckmann and Wittwer [8] examined universality in period doubling for families of unimodal maps in the limit of large even integer degree at the critical point.

These techniques have also proved effective in establishing universal scaling results concerning the breakup of quasiperiodicity in various scenarios. Mestel [9] proved the existence and hyperbolicity of a renormalisation fixed point for the breakup of quasiperiodicity in circle maps with golden mean rotation number. MacKay [10] examined critical scaling in the breakup of invariant tori in area-preserving maps, and Stirnemann [11] proved the existence of the corresponding critical fixed point for the breakup of conjugacy to rigid rotation taking place on the boundary of Siegel discs in iterated complex maps.

Analytical proofs of universality for critical scaling in the period doubling of families of unimodal maps have been harder to come by. Campanino et al [12] proved existence of the nontrivial renormalisation fixed point for period doubling in the case of unimodal maps with degree 2 at the critical point. Epstein [13] established that solutions to the corresponding functional equation exist within the class of even functions of general degree at the critical point providing another proof that did not require a computer. Eckmann and Wittwer [14] re-cast the problem in terms of an extended renormalisation group operator, written in a form that includes the bifurcation parameter itself, and hence established existence and hyperbolicity of the fixed point for maps with degree 2 at the critical point, together with transversal crossing of the manifold of superstable period two functions by the corresponding unstable manifold, thus providing a full proof of the Feigenbaum conjectures in the case of critical exponent 2. The reader is referred to Cvitanovic [15] for a thorough compendium of results in this area.

The work of Douady and Hubbard in complexifying the operator, together with Sullivan’s program to find the fixed point [16,17], enriched the field with
ideas from holomorphic dynamics, Teichmueller theory, and hyperbolic geometry. McMullen [18, 19] developed the approach of quasiconformal rigidity and hence established global uniqueness of the nontrivial renormalisation fixed point. Lyubich [20] and Avila and Lyubich (see, in particular, [21]) extended global uniqueness and hyperbolicity of the fixed point to arbitrary even integer degree, establishing the existence of a renormalisation horseshoe. Faria et al [22] have extended global hyperbolicity from analytic to $C^r$ mappings in the degree 2 case. A survey of four decades of research in the area is provided by [23]. More recently, Gorbovickis and Yampolsky [24] have broadened the reach to certain maps with non-integer critical exponent.

B. Overview

In this note, we focus on universality in period-doubling of unimodal maps of degree 4 at the critical point and note that maps with other even integer degrees are amenable to the same treatment. We adapt the methods of proof of [5,11], using rigorous computer-assisted means (‘function-ball algebra’) to gain tight bounds on the nontrivial fixed point of the renormalisation operator, by showing that a quasi-Newton operator for the fixed-point problem is a contraction map on a suitable ball in a Banach space of analytic functions (Sections I, II, III).

We bound the spectrum of the derivative of the operator at the fixed point, establishing the hyperbolic structure. By recasting the resulting eigenproblem for the derivative operator in nonlinear form, we use a novel contraction mapping argument to gain tight rigorous bounds on eigenfunctions and their corresponding eigenvalues. In particular, we gain tight bounds on the eigenfunction corresponding to the essential expanding eigenvalue delta (Section IV). By adapting the method to the relevant operator, and dealing with the corresponding dependency problem in the rigorous function ball framework, we bound the eigenfunction and eigenvalue that govern the universal scaling of additive uncorrelated noise (Section V). By using a recursive scheme based on the fixed-point equation, we gain rigorous bounds on the domain of analyticity of the renormalisation fixed point (Section VI).

Our computations use multi-precision arithmetic with rigorous directed rounding modes to bound tightly the coefficients of the relevant power series (including the polynomial parts taken to high truncation degree alongside rigorous bounds on all high-order terms). Indeed, we are able to obtain bounds that are tight, in the $\ell_1$-sense, on the power series coefficients of the critical fixed point, on the eigenfunctions corresponding to critical scaling in both the dynamical space and the parameter space, and on the eigenfunction corresponding to the critical scaling of additive noise, together with their accompanying universal scaling constants.

Working to degree 2560 (reduced to 640 via symmetry), we are able to bound the fixed point within a ball of analytic functions of $\ell_1$ radius $10^{-331}$. Similarly, we bound the eigenfunction corresponding to the parameter-scaling eigenvalue within radius $10^{-325}$ and the eigenfunction controlling the scaling of additive noise within radius $10^{-323}$. We note that the individual power series coefficients of these functions are therefore constrained within intervals having those same radii. This yields bounds on universal scaling constants in both the dynamical and the parameter space, and on the eigenvalue for scaling of additive noise: we are able to prove 331, 325, and 323, digits of these correct, respectively.

II. THE RENORMALISATION FIXED POINT

A. The renormalisation operator

We consider the operator $Rg(x) \triangleq a^{-1}g(ay)$, where $a = a_d \triangleq g(1)$ is chosen to preserve the normalisation $g(0) = 1$. (We note that other choices for $a$, also preserving this normalisation, may be taken and that, as is well-known, the particular variant will later affect the spectrum of $DR(g)$ only up to coordinate-change eigenvalues.)

We seek a nontrivial fixed point of $R$, with a critical point of even integer degree $d$ at the origin, in a Banach space $A \triangleq \mathcal{A}(\Omega)$ of functions analytic on an open disc $\Omega = D(c,r) \triangleq \{z \in \mathbb{C} : |z - c| < r\}$ and continuous on its closure, $\overline{\Omega}$, with (finite) $\ell_1$-norm. Specifically, we let $X = Q(x) \triangleq x^d$ and write

$$g(x) = G(Q(x)) = G(X).$$

We then seek a fixed point of the corresponding operator $T$ defined by

$$TG(X) \triangleq a^{-1}G(Q(Q(a)X)),$$

where $a \triangleq G(1)$.

In what follows, we focus exclusively on the case $d = 4$, the case $d = 2$ having been studied exhaustively. We deal with the cases of more general even integer degree and, separately, odd degree, at greater depth in forthcoming publications.
B. The disc algebra

We write \( G \in \mathcal{A}(\Omega) \) as

\[
G = G^u \circ \psi,
\]

where \( \psi : \Omega \to \mathbb{D} \) is the affine map from the domain \( \Omega \) to the unit disc given by

\[
\psi : x \mapsto \frac{x - c}{r}.
\]

We then take \( G^u \in \mathcal{A}(\mathbb{D}) \), the disc algebra: the set of functions analytic on the open unit disc \( \mathbb{D} \) and continuous on its closure, \( \overline{\mathbb{D}} \), with (finite) \( \ell^1 \)-norm. Equipped with the usual addition and scalar multiplication, viz. \((f + g)(x) = f(x) + g(x)\) and \((\alpha f)(x) = \alpha f(x)\), and with the \( \ell^1 \)-norm, \( \mathcal{A}(\mathbb{D}) \) (and, hence, \( \mathcal{A}(\Omega) \)) is a Banach space (moreover, when equipped with the product \((f \cdot g)(x) = f(x) \cdot g(x)\), it is a commutative unital Banach algebra) isometrically isomorphic to the sequence space \( \ell^1 \): functions \( f \in \mathcal{A}(\Omega) \) may be written as power series expansions

\[
f(x) = \sum_{k=0}^{\infty} a_k \left(\frac{x - c}{r}\right)^k,
\]

convergent on \( \Omega \).

C. Nonrigorous calculation

Firstly, we compute approximate fixed points of the renormalisation operator, \( T \), by working in the space of truncated power series of some fixed degree \( N \) expanded on the disc \( \Omega \). To this end, we write \( \ell_1 \) as the direct sum,

\[
\ell_1 \cong \mathbb{R}^{N+1} \oplus \ell_1,
\]

and let \( PA \) and \( HA = (I-P)A \) denote the canonical projections onto the polynomial part and high-order part of the space, respectively. Thus we may write \( f \in A \) as

\[
f = f_P + f_H,
\]

with \( f_H \in HA \) and \( f_P \in PA \) where

\[
f_P(x) = \sum_{k=0}^{N} a_k \left(\frac{x - c}{r}\right)^k.
\]

As a starting point, we consider the one-parameter family of maps given by

\[
f_\mu(x) = 1 - \mu x^d,
\]

and choose a parameter value \( \mu \) close to the accumulation \( \mu_\infty \) of the first period-doubling cascade for the family. (The intention is to find a function that lies close to the stable manifold of the critical renormalisation fixed point.) In the case \( d = 4 \), we establish, by locating superstable periodic orbits of periods \( 2^k \) for \( 1 \leq k \leq 32 \), that \( \mu_\infty \approx 1.594901356228820564497828 \). Writing \( f_\mu = G \circ Q \) and then applying the (truncated) renormalisation operator iteratively until we no longer observe an improvement in the residue \( \|T^{n+1}(G) - T^n(G)\| \) (when working with our chosen truncation degree and precision) then provides an initial approximate fixed point.

D. Newton operator

We note that fixed points of \( T \) are zeros of the operator \( F = T - I \), and perform Newton iterations, in the space of power series truncated to degree \( N \), to approximate such a zero. The one-step Newton operator is given by

\[
\phi : G \mapsto G - [DF(G)]^{-1} F(G) = G - [DT(G) - I]^{-1}(T(G) - G),
\]

in which \( DT(G) \in \mathcal{B}(A, A) \) denotes the tangent map of \( T \) at \( G \), given formally by the Frechet derivative

\[
DT(G) \colon \delta G \mapsto -a^{-2} \delta a G(Q(G(Q(a)X))) + a^{-1} \left( \delta G(Q(G(Q(a)X))) + G'(Q(G(Q(a)X))) \cdot Q'(G(Q(a)X)) \cdot [\delta G(Q(a)X) + G'(Q(a)X) \cdot Q'(a) \delta a \cdot X] \right),
\]

(4)
where $\delta a = \delta G(1)$. After the Newton iterations converge to our chosen precision, we denote the resulting approximate fixed point by $G^0$. (See Fig. 1)

Our goal is then to appeal to the contraction mapping theorem to prove that the operator $T$ has a locally-unique fixed point in a ball $B^1$ of functions centered on $G^0$ in the space $\mathcal{A}(\Omega)$. The operator $T$ is not itself contractive at the fixed point (indeed, we later bound the spectrum of the derivative there and obtain the eigenfunctions corresponding to the expanding eigenvalues). However, we can find a quasi-Newton operator $\Phi$ that has the same fixed points as $T$ and establish instead that $\Phi$ is a contraction mapping on $B^1$.

III. EXISTENCE OF THE FIXED POINT

A. Rigorous computations in the function space

We bound operations in the function space $\mathcal{A}(\mathbb{D})$ (and hence $\mathcal{A}(\Omega)$) by maintaining careful control over the coefficients of truncated power series along with all high-order terms. In order to maintain rigour, we work with interval arithmetic using high-precision computer-representable bounds with directed rounding modes, conforming to the relevant industry standards. To this end, we define a ball of functions, centered on a polynomial $f_P \in P\mathcal{A}(\mathbb{D})$, with high-order bound $v_H \geq 0$ and general bound $v_G \geq 0$, as follows

$$B(f_P; v_H, v_G) \triangleq \{ f \in \mathcal{A}(\mathbb{D}) : f = f_P + f_H + f_G, \\
f_H \in H\mathcal{A}(\mathbb{D}), \| f_H \| \leq v_H, \\
f_G \in \mathcal{A}(\mathbb{D}), \| f_G \| \leq v_G \}.$$ 

Following [7, 8], we extend the definition slightly, to the case where the function $f_P$ is not known exactly, but rather has coefficients $a_k$ confined within intervals. Let $v_P = ([b_0, c_0], \ldots, [b_N, c_N]) \in J^{N+1}$ be a vector of intervals (here, $J$ denotes $[a, b : a, b \in \mathbb{R}, a \leq b]$). Given the bounds $v = (v_P, v_H, v_G)$, we define the standard function ball $B(v) \subset \mathcal{A}(\mathbb{D})$ by

$$B(v_P, v_H, v_G) \triangleq \{ f \in \mathcal{A}(\mathbb{D}) : \\
f = f_P + f_H + f_G, \\
f_P \in P\mathcal{A}(\mathbb{D}), \\
f_P(x) = \sum_{k=0}^{N} a_k x^k, a_k \in [b_k, c_k], \\
f_H \in H\mathcal{A}(\mathbb{D}), \| f_H \| \leq v_H, \\
f_G \in \mathcal{A}(\mathbb{D}), \| f_G \| \leq v_G \}.$$ 

The resulting set of functions is convex and closed. The definition extends in a natural way to function balls $B(\mathbb{D}(v_P, v_H, v_G))$, for a general disc $\Omega$, by writing $f = f^u \circ \psi$ where $f^u \in B(v_P, v_H, v_G)$.

We bound operations on the function space $\mathcal{A}(\Omega)$ by first choosing computer-representable numbers for the quantities in $v \triangleq (v_P, v_H, v_G)$. For each binary operation $\oplus$, we then define a version, $\oplus_b$, act-

FIG. 1. Nonrigorous approximations of the function $G^*$ (topmost) and $g^*(z)$ for $z \in [-|\alpha|^k, |\alpha|^k]$ for $k = 3, 5, 9$.
ing on bounds \( v, w \) such that \( \forall f \in B(v), \forall g \in B(w), \)
\[
f \oplus g \in B(v) \oplus B(w) \subseteq B(v \oplus w).\]

The operation \( v \oplus_b w \), on bounds, is constructed carefully in order to guarantee that the above inclusion holds even when implemented using finite-precision arithmetic. In this way, all vector space operations, together with the product, \( f \cdot g \), composition of functions, \( f \circ g \), differentiation followed by composition, \( f' \circ g \), and the norm \( \|f\| \), may be bounded. For an exhaustive exposition, in the case of maps of two variables, see [2].

\[\text{B. Quasi-Newton operator}\]

The Newton operator for the fixed-point problem was shown in equation [3]. However, in order to establish contractivity, we would need to work with its derivative, which would involve taking the second Frechet derivative of \( T \). While possible, this proves to be inconvenient in practice. Instead, we note that if \( \Lambda \) is any invertible linear operator, then the fixed points of the quasi-Newton method given by
\[
\Phi : G \mapsto G - \Lambda(T(G) - G), \tag{5}
\]
are exactly the fixed points of \( T \). We choose
\[
\Lambda \simeq [DT(G) - I]^{-1}.
\]
Specifically, we approximate the Frechet derivative \( DT(G) \) by a fixed linear operator \( \Delta \simeq DT(G^0) \) with action zero on high-order terms. For the polynomial terms, we evaluate the expression for the Frechet derivative at Schauder basis elements forming the sequence of monomials
\[
e_j(x) = \left( \frac{x - c}{r} \right)^j,
\]
for \( j = 0, \ldots, N \) and bound the resulting matrix elements by trivial intervals to give a real interval matrix denoted \( \Delta_{PP} \). We compute an interval matrix \( \Lambda_{PP} \) guaranteed to bound the inverse \( (\Delta_{PP} - I)^{-1} \). Thus the corresponding linear operator \( \Lambda \) has action on the polynomial part of the space, and action \( -I \) on the high-order part.

\[\text{C. Bound 1: distance moved by the approximate fixed point}\]

In order to use the contraction mapping principle, we need to prove that a certain ball in \( A(\Omega) \) is mapped into itself contractively by \( \Phi \). We achieve this by establishing two bounds: a bound on how far the approximate fixed point \( G^0 \) moves under the operator \( \Phi \), and a bound on the derivative \( D\Phi \) that we will use in order to show that \( \Phi \) is contractive and that \( \Phi \) maps the ball to itself.

To this end, we define a ball of functions \( B^0 = B_{\Omega}(G^0, 0, 0) \) of radius zero; the singleton \( \{G^0\} \). By applying \( \Phi \) to \( B^0 \), in the sense of using the corresponding function ball operations to contain the result, we gain a rigorous bound on how far \( G^0 \) moves under \( \Phi \):
\[
\|\Phi(G^0) - G^0\| < \varepsilon. \tag{6}
\]
We now choose a radius \( \rho > \varepsilon \) and form the function ball \( B^1 = B_{\Omega}(G^0, 0, \rho) \), on which we need to prove that \( \Phi \) is a contraction mapping.

\[\text{D. Domain extension}\]

![Figure 2](image)

FIG. 2. Verification of the domain extension conditions computed using the function ball \( B^1 \), using a rigorous covering of the boundary \( \partial \Omega \) by 256 rectangles.

The first step in what follows is to show that \( T \) is well-defined and differentiable, with compact derivative, on \( B^1 \). We do this by establishing the ‘domain extension’ or ‘analyticity improving’ property [9][10]: for all \( G \in B^1 \) we demand that
\[
Q(a) \Omega \subset \Omega, \tag{7}
Q(G(Q(a) \Omega)) \subset \Omega. \tag{8}
\]
In the above, the overline denotes topological closure. Recall that we take \( a \equiv G(1) \). Thus the
universal quantifier is not vacuous for equation \( 7 \). Systematic experimentation is used to find a suitable domain \( \Omega = D(c, r) \). For \( d = 4 \), we may choose \( \Omega = D(0.5754, 0.8) \). The domain may be improved further by choosing \( c \) so as to minimise the absolute value of the constant term on \( G'' \) where \( G'' = G'' \circ \psi \). Doing so reduces the dominant contribution to the error bounds involved in composition. Fig. 2 illustrates domain extension for a rigorous covering of the boundary \( \partial \Omega \).

We note that the space \( \mathcal{A}(\Omega) \) is infinite-dimensional and has the bounded approximation property. It follows that the spectrum of a compact operator consists of 0 together with only isolated eigenvalues of finite multiplicity. The spectrum of finite-rank approximations converges to the spectrum of the operator itself; if \( L \) is compact and \( \| L' - L \| \to 0 \), then the spectrum of \( L' \) (and, indeed, the corresponding eigenfunctions) converges to that of \( L \) apart from at 0. (In the case of complex domains, one can prove that domain extension yields compactness by appealing to the Cauchy estimates on suitable discs \( \Delta_w = D(c, w) \) for \( w \leq r' < r \) to provide uniform continuity, and hence establish normality. Montel’s theorem then implies the result [10, 11].) Compactness will prove crucial in bounding the spectrum of the linearisation \( DT(G) \) at the fixed point in section IV.C.

E. Bound 2: uniform contractivity

Our final goal is to find a uniform bound on the contractivity of \( \Phi \) on \( B^1 \). We do this by bounding

\[
\| D\Phi(G) \| \leq \kappa < 1, \quad \forall G \in B^1, \tag{9}
\]

for a suitable norm, and then appealing to the mean value theorem (that this yields uniform contractivity may be seen by considering the line segment joining any two points in the convex set \( B^1 \) and noting that a bound on the norm of \( D\Phi(G) \) valid for all \( G \in B^1 \) provides an upper bound on all of the corresponding pairwise contractivities).

The Frechet derivative of the quasi-Newton operator \( \Phi \) (from equation 5) is given by

\[
D\Phi(G) : \delta G \mapsto \delta G - \Lambda [DT(G)\delta G - \delta G], \tag{10}
\]

We bound \( D\Phi(G) \) in the maximum column sum norm. That is, we bound the norms \( \| D\Phi(G)e_k \| \) for all basis elements \( e_k \) and then take the supremum. To do this, we bound the action of the Frechet derivative of \( \Phi \) at \( B^1 \) on function balls containing the \( e_k \). Firstly, we let \( E_k = B(c_k; 0, 0) \) for \( k = 0, \ldots, N \), i.e., we consider singletons containing each of the polynomial basis elements. The problem of capturing the (infinitely-many) norms that remain is reduced to a finite computation by taking the single ball \( E_H = B(0; 1, 0) \), i.e., the convex hull of all high-order basis elements, and bounding \( \| D\Phi(B^1)E_H \| \), i.e., \( \| D\Phi(G)\delta G \| \forall G \in B^1, \forall \delta G \in E_H \). Bounding the supremum of these norms yields

\[
\kappa \geq \sup \{ \| D\Phi(G)E \| \},
\]

taken for all \( G \in B^1 \) and all \( E \in \{ E_0, \ldots, E_N, E_H \} \), from which, for \( \kappa < 1 \), the mean value theorem delivers the uniform bound on contractivity

\[
\| T(f) - T(g) \| \leq \kappa \| f - g \| \quad \forall f, g \in B^1.
\]

F. Dependency problems

It is crucial, for the case where \( \delta G \) is a high-order perturbation, i.e., \( \delta G \in H\mathcal{A}(\Omega) \), to mitigate the function ball analogue of the dependency problem, well-known in interval arithmetic [28, 29]. In the expression for \( D\Phi \) (equation 10), the action of \( \Lambda \) on high-order terms is \(-I\), thus the action of \( D\Phi \) on a high-order perturbation \( \delta b_H \) is given by:

\[
D\Phi(B^1)\delta b_H = \delta b_H - \Lambda [DT(B^1)\delta b_H - \delta b_H] \tag{11}
\]

\[
= \delta b_H - \Lambda [DT(B^1)\delta b_H - \delta b_H] \tag{12}
\]

\[
= - \Lambda [DT(B^1)\delta b_H]. \tag{13}
\]

Computing the norm \( \| D\Phi(B^1)E_H \| \) naively by performing function ball operations based on expression 11 instead of expression 13 would result in an upper bound on contractivity larger than 2, even in the case where \( D\Phi(B^1) \) is indeed contractive, due to the implicit presence of uncancelled terms \( \delta b_H - \delta b_H \) in 12. The operands in an expression of the form \( \| f - g \| \) are treated as independent (high-order) functions, here, subject only to the bounds \( \| f \|, \| g \| \leq 1.\)
G. Existence and local uniqueness

Finally, using the bounds obtained in equations 6 and 9 we verify the inequality

\[ \varepsilon < \rho(1 - \kappa), \]

to ensure that \( \Phi(B^1) \subset B^1 \), which establishes that \( \Phi \) is a contraction mapping on \( B^1 \). Fig. 3 illustrates the situation schematically. Hence, \( \Phi \) (and, therefore, \( T \)) has a locally unique fixed point, \( G^* \in B^1 \).

For degree \( d = 4 \), for example, and using our chosen disc \( \Omega \), we are able to complete the proof by choosing truncation degree \( N = 40 \), thus \( g \) has degree 160. Working with precision equivalent to 40 digits in the significand, we obtain \( \varepsilon = 1.59 \times 10^{-21} \), and choosing \( \rho = 10^{-20} \) gives \( \kappa = 6.88 \times 10^{-3} \).

![Rigorous coverings of the functions G*](image1)

![Rigorous coverings of the functions g*](image2)

**FIG. 4.** Rigorous coverings of the functions \( G^* \) (top) and \( g^* \) (bottom) using 10 (dashed lines), 50, and 500 rectangles computed using the function ball \( B^1 \supset G^* \) together with the fixed-point equation.

H. Tight bounds on the fixed point

In proving the existence of the fixed point, a relatively low truncation degree for \( G \) and a relatively low precision is adequate (indeed, one could even have used standard 64-bit double precision numbers, with careful control over directed rounding modes). The resulting function ball radius \( \rho \) gives an \( \ell^1 \)-bound on the accuracy of the intervals bounding the coefficients of \( G \).

We improve these bounds significantly by both increasing the truncation degree and by using rigorous multi-precision arithmetic. Table I shows parameters and bounds proven valid for establishing the existence of \( G^* \) and hence \( g^* \). Table II lists the digits of the relevant universal constants, including \( a_4 \overset{\Delta}{=} g^*(1) \), that we have been able to prove correct as a result (for comparable numerical results, see [30, 31]).

Figure 4 demonstrates a rigorous covering of the fixed-point functions \( G^* \) (resp. \( g^* \)). These were computed by using the function ball \( B^1 \) (resp. \( B^1 \circ Q \)) with truncation degree 40 on the domain \( \Omega \) (resp. on the preimage \( Q^{-1}(\Omega) \)) together with recurrences derived from the corresponding fixed-point equations in order to bound the functions on larger subsets of \( \mathbb{R} \setminus \Omega \) (resp. on its preimage under \( Q \)).

IV. SPECTRAL THEORY

A. The spectrum

We now address hyperbolicity of the fixed point. The space \( A \) is infinite-dimensional. Thus, compactness of a bounded linear operator \( L \in \mathcal{B}(A, A) \) implies that the spectrum of \( L \) consists of the origin together with a countable set of isolated eigenvalues of finite multiplicity (which accumulate at 0) [25].

We note that the spectrum of \( DT(G) \) and that of \( DR(g) \) are related in the following manner. Consider \( G \in A \) and \( \delta G \in A \) and let \( g = G \circ Q \) and \( \delta g = \delta G \circ Q \), then we have

\[
(DT(G) \delta G) \circ Q = DR(g) \delta g.
\]

Then \( \lambda \in \sigma(DT(G)) \) with \( DT(G) V = \lambda V \) implies that \( \lambda \in \sigma(DR(g)) \) with \( DR(g) u = \lambda u \) where \( u = V \circ Q \).

The spectrum of \( DT(G^*) \) has 2 eigenvalues in the complement of the closed unit disc,

\[ \alpha^4_4, \delta_4, \]

whereas the spectrum of \( DR(g^*) \) has 5 eigenvalues in the complement of the closed unit disc,

\[ \alpha^4_4, \delta_4, \alpha^3_4, \alpha^2_4, \alpha_4. \]
TABLE I. Parameters and bounds valid for rigorous proofs of existence for the renormalisation fixed point, $G^*$, the eigenfunction, $V^*$, corresponding to $\delta$, and the noise eigenfunction, $W^*$, corresponding to the scaling of additive noise. In all cases, the number of digits $P$ in the significant, for the decimal floating-point versions of the proofs, was chosen to be equal to the truncation degree $N$. (The table also indicates the corresponding number of bits chosen in the significant for the independent floating point versions of the proofs. Experimentation reveals that we may reduce $P$ at least as far as $\lceil 2N/3 \rceil$, for the computations shown, and still gain rigorous bounds of the same orders of magnitude.)

| $N$ | #bits | $\varepsilon$ | $\rho$ | $\kappa$ | $\bar{\varepsilon}$ | $\bar{\rho}$ | $\bar{\kappa}$ |
|-----|--------|---------------|--------|----------|-----------------|-------------|-------------|
| 40  | 132    | $1.59 \cdot 10^{-21}$ | $10^{-20}$ | $6.88 \cdot 10^{-3}$ | $3.17 \cdot 10^{-16}$ | $10^{-15}$ | $1.17 \cdot 10^{-3}$ | $2.35 \cdot 10^{-16}$ | $10^{-15}$ | $7.85 \cdot 10^{-3}$ |
| 90  | 265    | $3.75 \cdot 10^{-42}$ | $10^{-41}$ | $1.01 \cdot 10^{-6}$ | $4.88 \cdot 10^{-37}$ | $10^{-36}$ | $1.39 \cdot 10^{-7}$ | $3.3 \cdot 10^{-37}$ | $10^{-36}$ | $1.9 \cdot 10^{-7}$ |
| 160 | 531    | $7.84 \cdot 10^{-84}$ | $10^{-83}$ | $1.36 \cdot 10^{-12}$ | $8.37 \cdot 10^{-29}$ | $10^{-28}$ | $1.87 \cdot 10^{-13}$ | $8.57 \cdot 10^{-29}$ | $10^{-28}$ | $4.32 \cdot 10^{-14}$ |
| 320 | 1063   | $2.89 \cdot 10^{-166}$ | $10^{-165}$ | $3.01 \cdot 10^{-24}$ | $1.52 \cdot 10^{-160}$ | $10^{-159}$ | $4.12 \cdot 10^{-25}$ | $6.24 \cdot 10^{-160}$ | $10^{-159}$ | $9.56 \cdot 10^{-26}$ |
| 480 | 1594   | $4.14 \cdot 10^{-249}$ | $10^{-248}$ | $7.28 \cdot 10^{-36}$ | $2.21 \cdot 10^{-242}$ | $10^{-242}$ | $9.99 \cdot 10^{-37}$ | $5.31 \cdot 10^{-242}$ | $10^{-242}$ | $2.32 \cdot 10^{-37}$ |
| 640 | 2126   | $5.01 \cdot 10^{-332}$ | $10^{-331}$ | $1.85 \cdot 10^{-47}$ | $2.90 \cdot 10^{-326}$ | $10^{-325}$ | $2.53 \cdot 10^{-48}$ | $1.36 \cdot 10^{-324}$ | $10^{-323}$ | $5.87 \cdot 10^{-49}$ |

TABLE II. Digits proven correct of $a_4 = G^*(1)$ (331 digits), $\alpha_4 = 1/a_4$ (331 digits), $\delta_4 = \varphi(V^*)$ (325 digits), and $\gamma_4 = \varphi(W^*)$ (323 digits) obtained from the proof with truncation degree $N = 640$ for $G^*, V^*, W^*$ (corresponding to degree $4N = 2560$ for $g^*, v^*, w^*$).

(Each of the three correspond to perturbations ruled-out for $DT(G)$ on symmetry grounds) with the others in the open unit disc. Note that $\alpha_4^2$ is a coordinate-change eigenvalue and that $\alpha_4^2, \alpha_4^2, \alpha_4^2$ correspond to perturbations that destroy the symmetry of the quartic critical point. The eigenvalue $\alpha_4^2$ plays a role in tricritical vector scaling for locally bimodal maps in which one quadratic extremum is mapped to another, corresponding to an additional solution $q_2(x) = g^*(v^2) = G^*(x^2)^2$ of the functional equation $R(g) = g$ with universal scaling constant $g^*(1)^2 = \alpha_4^2 [92] [33]$. We note also that the choice of a particular normalisation fixing $g(0) = 1$ affects the spectrum only up to coordinate-change eigenvalues.

B. Establishing hyperbolicity

We are interested, here, primarily in establishing the hyperbolic structure of $DT(G^*)$ and $DR(g^*)$, and on bounding eigenvalues for the purpose of
matching with eigenfunction-eigenvalue pairs below. Apart from non-essential eigenvalues, the only part of the spectrum of $DT(G^*)$ outside the unit disc is the eigenvalue $\delta_4$ associated with critical scaling in the parameter space for the period-doubling cascade. All other eigenvalues are contained in the interior of the unit disc. Subject to a projection removing coordinate-change directions and their corresponding eigenvalues, this helps to establish the picture conjectured by Feigenbaum, in which $G^*$ has one-dimensional unstable manifold with eigenvalue $\delta_4$ and co-dimension one stable manifold. Transverse intersection of the unstable manifold with manifolds of superstable periodic functions has been established elsewhere. The conclusion is that families of maps with critical point of degree 4 that exhibit a period doubling cascade, and so (generically) cross maps with critical point of degree $\delta_4$ associated with critical scaling in $\mathbb{C}$, has one-dimensional unstable manifold.

Apart from non-essential eigenvalues, the only part matching with eigenfunction-eigenvalue pairs below.

The eigenvalue $\delta_4$ in colour copy) and $\delta_{f, \Omega} = \delta_{f, \Omega}(\Omega)$, that satisfies the corresponding domain maps are $\psi_0, \psi_1$, where $\psi_k : x \mapsto (x - c_k)/r_k$. The power series that we work with are therefore those for $g_0^k \oplus g_1^k \in \mathcal{A}(\mathbb{D}(0,1))^2$, where $g_k = g_0^k \circ \psi_k$.

Choosing, for example, $\Omega = \mathbb{D}(-0.1,0.7) \cup \mathbb{D}(0.85,0.3)$ and noting that, in the operator, we have $a \triangleq g(1) = g_1(1)$, we are able to prove that

$$\begin{align*}
a \Omega & \subset \Omega, \\
g(a \Omega) & \subset \Omega,
\end{align*}$$

which yields domain extension (Fig. 5); thus $R$ is well-defined on the resulting space, differentiable, and the derivative is compact.

We may complete the proof of existence of the fixed point for $R$ directly by using a ball around an approximate fixed point in the space of pairs of maps and, by choosing a suitable basis for the space, we may then bound the spectrum of $DR(g)$ at the fixed point directly, allowing perturbations that destroy the symmetry $g = G \circ Q$ (albeit at the cost of working in the space of pairs of maps).

C. Bounding the spectrum

We establish firstly that the spectrum has the form described above. For brevity, we demonstrate this for $DT(G^*)$ (and apply a similar procedure directly to $DR(g^*)$). To do this, we make an invertible change of coordinates that puts $DT(G)$ into a form $C^{-1}DT(G)C$ close to diagonal, for all $G \in B_1$. We then bound the resulting operator by a so-called contracted matrix $M$. This is an $(m+1) \times (m+1)$ matrix of rectangles, $[a, b] + i[c, d] \subset \mathbb{C}$, with $m \leq N$ with the property that if $\Lambda = [e, f] + i[g, h] \subset \mathbb{C}$ is a rectangle containing an eigenvalue of $C^{-1}DT(G)C$, then taking the determinant $\det(M - \Lambda I)$ using rectangle arithmetic (a natural complex analogue of interval
arithmetic) yields a rectangle containing zero. Thus, if the determinant is bounded away from zero, then we conclude that the rectangle λ does not contain an eigenvalue.

We then consider a smooth one-parameter family of linear operators \( \mu \mapsto L_\mu \) with \( L_1 = M \) and \( L_0 = D \), a diagonal operator whose spectrum can therefore be determined trivially to have the correct form. We may then identify disjoint circles \( \Gamma_1, \Gamma_2, \Gamma_3 \) chosen so that \( \Gamma_1, \Gamma_2 \) surround the expansion eigenvalues \( \alpha_4^\dagger \) and \( \delta_4 \) respectively, while \( \Gamma_3 \) surrounds the rest of the spectrum within the interior of the unit disc; see Fig. 6. We note that the determinant is continuous in the linear operator and, by prov-

![Graph showing the spectrum bounded within each of the circles \( \Gamma \)]

FIG. 6. Rigorous coverings of the circles \( \Gamma_1, \Gamma_2, \Gamma_3 \) (by 16, 16, and 1000 rectangles, respectively) used to bound the determinant \( \det(L_\mu - \lambda I) \) away from zero \( \forall \mu \in [0, 1] \), \( \forall \lambda \in \Gamma_{1,2,3} \) and hence establish that no eigenvalues of \( L_\mu \) may intersect \( \Gamma_{1,2,3} \). The unit circle is shown (dashed) for comparison.

D. Bounding eigenfunctions and their eigenvalues

Next, we are able to find tight rigorous bounds on eigenfunction-eigenvalue pairs \((V, \lambda)\) by adapting the method used in the proof of existence of \( G^* \) to the corresponding eigenproblem,

\[
(DT(G^*) - \lambda I)V = 0.
\]

Specifically, sticking with the sequence of monomials (expanded with respect to \( \Omega \)) as Schauder basis, we take \( k \) to be the coordinate index of the first nonzero coefficient of the eigenfunction corresponding to \( \delta_4 \) (resp. \( \alpha_4^\dagger \)), and define \( \varphi \) to be the corresponding linear coordinate functional. We choose a normalisation for the eigenfunctions that fixes the corresponding eigenvalue as the coefficient \( a_k \) of \( V \),

\[
(V, \lambda) \mapsto \lambda \frac{V}{\varphi(V)},
\]

and solve the corresponding (nonlinear in \( V \)) eigenproblem

\[
F(V) \triangleq (DT(G^*) - \varphi(V))V = 0,
\]

An initial guess, \( V^0 \), for the eigenfunction \( V \) may be found by computing the corresponding normalised eigenvector for the truncated problem nonrigorously and then employing a nonrigorous newton iteration to improve the initial guess.

E. Newton’s method for eigenfunctions

Following the method used for the existence proof, we then form a quasi-Newton operator, \( \Psi \), whose fixed points are the relevant zeros. We first note that \( F \) has Frechet derivative given formally by

\[
DF(V)\delta V = DT(G^*)\delta V - \varphi(\delta V)V - \varphi(V)\delta V.
\]

The quasi-Newton operator for this problem is given by

\[
\Psi : V \mapsto V - \hat{\Lambda} \left[ DT(G^*)V - \varphi(V)V \right],
\]

in which we choose a fixed invertible linear operator \( \hat{\Lambda} \) such that for all \( f \in B^3 \triangleq B(V^0, 0, \rho) \), we have

\[
\hat{\Lambda} \delta V \simeq \left[ DT(G^*) \delta V - \varphi(\delta V)V^0 - \varphi(V^0)\delta V \right]^{-1}.
\]

The Frechet derivative of the quasi-Newton operator is thus given by

\[
D\Psi(V)\delta V = \delta V - \hat{\Lambda} \left[ DT(G^*)\delta V - \phi(\delta V)V - \phi(V)\delta V \right].
\]

1. Choosing the fixed linear operator

Following sections IIIC and IIIE we aim to bound \( \|\Psi(V^0) - V^0\| \leq \epsilon \) via function ball operations on a singleton ball \( B^2 \triangleq B(V^0; 0, 0) \). We
must then bound \( \|DF(V)(e_j)\| \leq \kappa < 1 \) for all \( V \in B^3 = B(V^0; 0, \bar{\rho}) \) and all \( j \geq 0 \).

Anticipating a dependency problem of the sort encountered in section [III], we examine the linear operator, \( \hat{\Lambda} \) more closely. We have

\[
DF(V)\delta V = DT(G)\delta V - \varphi(\delta V)V - \varphi(V)\delta V = (DT(G) - V e_k^* - V_k I) \delta V,
\]

\[
DF(V) \simeq \Delta - V^0 e_k^* - V_k^0 I,
\]

where \( V^0 \) is a suitable approximate eigenfunction and \( e_k^* \) denotes the adjoint of the basis element \( e_k \), and the subscript on \( V \) and \( V^0 \) denotes the relevant power series coefficient. Recall that \( \Delta \simeq DT(G^0) \) is chosen so that its action on \( H_s^0(\Omega) \) is zero. In order to implement \( \Lambda \) (which we choose to be the inverse of the above operator) we need to think about the action of the operator on the polynomial and high-order parts of the space.

Assume, without loss of generality, that \( k = 0 \) so that \( \varphi(V) = V_0 \) then, for a suitable \( V^0 \) (chosen with \( HV^0 = 0 \)), we may then take the (block diagonal) operator specified by

\[
\Gamma = \Delta - V^0 e_k^* - V_0^0 I = \begin{pmatrix}
\Delta_{00} - 2V_0^0 & \Delta_{01} & \cdots & \Delta_{0N} & 0 \\
\Delta_{10} - V_1^0 & \Delta_{11} - V_0^0 & \cdots & \Delta_{1N} & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\Delta_{N0} - V_N^0 & \Delta_{N1} & \cdots & \Delta_{NN} - V_0^0 & 0 \\
0 & 0 & \cdots & 0 & -V_0^0 I
\end{pmatrix}.
\]

2. Overcoming the dependency problem

Recall that

\[
\Psi : V \mapsto V - \hat{\Lambda}[DT(G)V - \varphi(V)V],
\]

with Frechet derivative

\[
D\Psi(V) : \delta V \mapsto \delta V - \hat{\Lambda}[DT(G)\delta V - \varphi(\delta V)V - \varphi(V)\delta V].
\]

We recall that multiple occurrences of the perturbation \( \delta V \) in an expression are treated as functions varying independently within function balls in the rigorous computational framework, each contributing separately to the resulting norm. There is therefore a dependency problem due to the terms \( \delta V \) and \( \hat{\Lambda}\varphi(V)\delta V \) in the above.

To resolve this, consider the action of \( D\Psi(V) \) on a high-order perturbation \( \delta V_H \in H_s^0(\Omega) \):

\[
D\Psi(V)\delta V_H = \delta V_H - \hat{\Lambda}[DT(G)\delta V_H - \varphi(V)\delta V_H] = \left(1 - \frac{\varphi(V)}{\varphi(V^0)}\right)\delta V_H - \hat{\Lambda}DT(G)\delta V_H,
\]

since \( \varphi(\delta V_H) = 0 \) and the action of \( \hat{\Lambda} \) on the high-order part of the space is given by \(-1/V_0^0 I\). Note that for \( V \) close to \( V^0 \), the contribution from the first term in the above expression is close to zero.

In order to avoid a bound on \( \|D\Psi(B(V^0; 0, \bar{\rho}))(E_H)\| \) exceeding 2, we therefore use the latter expression given above for \( D\Psi(V)\delta V_H \), with \( V \) ranging over the ball \( B(V^0; 0, \bar{\rho}) \), when computing \( D\Psi(V)E_H \).

Using the parameters from the proof of existence for \( G^* \), given in the first row of Table [IV], we obtain a rigorous bound \( \|\Psi(V^0) - V^0\| < \bar{\varepsilon} = 3.17 \times 10^{-16} \), then choosing \( \hat{\rho} = 10^{-15} \) yields \( \|D\Psi(B(V^0; 0, \hat{\rho}))\| < \bar{\kappa} = 1.17 \times 10^{-3} \), which establishes that \( \Psi \) is indeed a contraction mapping on \( B(V^0; 0, \hat{\rho}) \). The eigenvalue satisfies \( \delta_4 \in [7.28468621706, 7.28468621709] \). We use high precision and high truncation degree to obtain much tighter rigorous bounds on both the eigenvalue and on the coefficients of the corresponding eigenfunction \( V^* \), as shown in Table [IV] and Table [III].

F. Evaluating the eigenfunction on larger intervals

We note that the eigenfunction \( V \) satisfies the equation
\[ V(X) = \delta^{-1} DT(G)V(X) \]
\[ = \delta^{-1}\left[ -a^{-2}V(1) \cdot G(Q(G(Q(X)))) \right. \]
\[ + a^{-1} \cdot V(Q(G(Q(X)))) \]
\[ + a^{-1} \cdot G'(Q(G(Q(X)))) \cdot Q'(G(Q(X))) \cdot V(Q(X)) \]
\[ + a^{-1} \cdot G'(Q(G(Q(X)))) \cdot Q'(G(Q(X))) \cdot G'(Q(a)X) \cdot Q'(a)V(1) \cdot X, \]

where \( a = G(1) \) and \( \delta = \varphi(V) \). This allows us to evaluate the eigenfunction \( V(X) \) of \( DT(G^*) \), and hence \( v(x) = V(Q(x)) \), the corresponding eigenfunction for \( DR(g^*) \), over larger subintervals of the real line, by constructing recurrence relations that utilise the function balls \( B^1 \ni G^* \) and \( B^3 \ni V \), already computed, as a base case. See Fig. 7

Specifically, we first make use of the fixed-point equation in order to bound \( G' \) over larger domains: let \( G = G^* \), then
\[ G(X) = a^{-1}G(Q(G(Q(a)X))). \quad (20) \]

Differentiating gives
\[ G'(X) = a^{-1}G'(Q(G(Q(a)X))) \cdot Q'(G(Q(a)X)) \cdot G'(Q(a)X) \cdot Q(a). \]

Using the above expression (together with the fixed-point equation for \( G \)) recursively allows us to bound \( G' \) and hence, in combination with the above, \( V \), over larger intervals extending outside \( \Omega \cap \mathbb{R} \).

V. CRITICAL SCALING OF ADDITIVE NOISE

We now find tight rigorous bounds on the eigenfunction, \( w \), and eigenvalue, \( \gamma \), controlling the universal scaling of additive uncorrelated noise. The iteration of a prototypical one-parameter family, \( f_\mu \), is modified to give \( x_{n+1} = f_\mu,\nu(x_n) \triangleq f_\mu(x_n) + \xi_n \), where, in the simplest case, the \( \xi_n \) are i.i.d. random variables, independent of the \( x_n \). Adapting the arguments presented in \[34, 35\], and retaining the deterministic scaling \( a = f(1) \) in the definition of the renormalisation operator, we write \( w = W \circ Q \) and consider the eigenproblem
\[ \gamma^2 W = L W, \]
in which we define the linear operator \( L \) by
\[ L W \triangleq L_1^2 \cdot W(Q(G(Q(a)X))) + L_2^2 \cdot W(Q(a)X), \]
where we define
\[ L_1 \triangleq a^{-1}, \]
\[ L_2 \triangleq a^{-1}G'(Q(G(Q(a)X))) \cdot Q'(G(Q(a)X)). \]

\[ \text{FIG. 7. (Top) The eigenfunction } V \text{ corresponding to the essential expanding eigenvalue } \delta_1. \text{ (Bottom) } v(x) = V(Q(x)). \]

In the above, \( G = G^* \). We note that the expressions \( L_1, L_2 \) are those prefactors in the Frechet derivative, \( DT(G) \), of equation \[4\] that do not correspond to variations in \( a \) (equivalently, those in the terms \[14, 15\] of \( DR(g) \)). We note also that the corresponding operator acting on \( g \) emerges as a special case of the analysis presented in \[36\] for the correlated case. Following our treatment for the eigenfunctions of \( DT(G) \), we encode the eigenvalue within \( W \) by defining \( \gamma = \varphi(W) \) and expressing the problem as
\[ F(W) \triangleq (L - \varphi(W)^2 I) W = 0. \]

The operator \( F \) has Frechet derivative
\[ DF(W) : \delta W \mapsto L\delta W - 2\varphi(W)\varphi(\delta W)W - \varphi(W)^2 \delta W. \]
Using the parameters from the proof of existence for $G^*$ given in the first row of Table I, we obtain $||\Theta(W^0) - W^0|| < \tilde{\varepsilon} = 2.35 \times 10^{-10}$; choosing $\tilde{\rho} = 10^{-15}$ then gives $\|D\Theta(B(W^0, 0, \tilde{\rho}))\| < \tilde{\kappa} = 7.85 \times 10^{-3}$, establishing that $\Theta$ is a contraction on $B(W^0, 0, \tilde{\rho})$. Table I demonstrates that these bounds may be improved significantly. Fig. 8 shows the corresponding eigenfunction. Working with truncation degree 40 and 40 digits in the significand yields the crude bound $\gamma \in [8.24391085424, 8.24391085427]$ for the noise eigenvalue, which we again improve by taking higher truncation degree and by using multiprecision arithmetic. Table II shows 323 digits proven correct. These bounds agree with the initial digits presented in [37].

In the above examples, the bound on $\kappa$ corresponds to the high-order bound from $D\Phi(B^1)E_H$ for $N = 40, 80, 160, 320$. However, for $N = 480, 640$, the supremum is achieved by one of the $D\Phi(B^1)E_k$ for $0 \leq k \leq N$, indicating that a sufficiently high truncation degree has been taken such that the loss of information concerning the distribution of the high-order bound amongst high-order coefficients no longer provides the dominant obstacle to improving the bound on contractivity.

**VI. DOMAIN OF ANALYTICITY**

We now use the fixed-point equations

$$g^*(z) = a^{-1}g^*(g^*(az)),$$

and

$$G^*(Z) = a^{-1}G^*(Q(G^*(Q(a)Z))),$$

to find analytic continuations of $G^*$ and $g^* = G^* \circ Q$ to larger domains than $\Omega$ and $Q^{-1}(\Omega)$ respectively (here, $Q^{-1}(\Omega)$ denotes the preimage of $\Omega$ under $Q$).

We note firstly that $G^*$ has a real-analytic extension to $\mathbb{R}$ and thus that $g^*$ is well defined on rays $\{z = r e^{i\pi n/2} : r \geq 0, n = 0, 1, 2, 3\}$ corresponding to $G^*(X)$ with $X \in \mathbb{R}$, $X \geq 0$, and on rays $\{z = r e^{i(2n+1)\pi/4} : r \geq 0, n = 0, 1, 2, 3\}$ corresponding to $G^*(X)$ for $X < 0$.

We further categorise points $z \in \mathbb{C}$ according to whether $g^*(z)$ may be evaluated by using the power series obtained for $G^*$ directly for $Z = Q(z) = z^d \in \Omega$, and indirectly for $Z \notin \Omega$ by using the fixed point equation recursively. The latter results in a binary tree of recursive evaluations of $g$, which we terminate after a maximum recursion depth is reached.

Specifically, we define the following recursive func-
FIG. 9. (Top) Part of the domain of analyticity of $G^*$ plotted by using the function ball $B^1$ and the fixed-point equation recursively. (Bottom) For comparison, the complement of the domain of analyticity of $G^*$ approximated by adapting a method from [38]: we compute contours of the escape time $m$ for the condition $|(G \circ Q)^{2m-1}(G(Q(a)^mZ))| > s$, where $G$ corresponds to $f_{\mu_{\infty}}$ and $s$ is chosen large.

\[ \mathcal{G}(Z, d) \triangleq \begin{cases} (0, \infty) & \text{if } d < 0, \\ (G^*(Z), 0) & \text{if } d \geq 0, Z \in \Omega, \\ (a^{-1}Y, \max(d_1, d_2) + 1) & \text{otherwise,} \end{cases} \]

where

\[ (X, d_1) \triangleq \mathcal{G}(Q(a)Z, d - 1), \]
\[ (Y, d_2) \triangleq \mathcal{G}(Q(X), d - 1). \]

FIG. 10. Part of the domain of analyticity of $g^*$ plotted by using the fixed-point equation recursively.

Pieces of the domains of analyticity of $G^*$ and $g^*$, thus obtained, are shown in Figs. 9 and 10 respectively.

Secondly, for those points at which $g^*$ may indeed be evaluated, subject to that maximum recursion depth, we classify points according to whether $g^*(z) \in \mathbb{H}_-$ or $g^*(z) \in \mathbb{H}_+$ so that the resulting tiling, shown in Fig. 11 may be compared with that for the corresponding Yoccoz puzzle pieces in the case of maps with degree $2$ at the critical point [38, 39].

VII. CONCLUSIONS

We have obtained tight bounds on the renormalisation fixed point function for period doubling in unimodal maps with critical point of degree $4$, by means of a rigorous computer-assisted existence proof using the contraction mapping theorem on a suitable space of analytic functions. We have established the structure of the spectrum of the linearised operator at the fixed point, proving hyperbolicity and providing bounds on expanding eigenvalues. By expressing the corresponding eigenproblem in nonlinear form, we have adapted the contraction mapping argument to provide rigorous bounds on eigenfunction-eigenvalue pairs, and have adapted the technique to bound the eigenfunction and eigenvalue controlling the universal scaling of additive noise in the case of a deterministic choice of normalisation in the renormalisation operator. These techniques deliver tight bounds on the relevant analytic functions.
FIG. 11. Part of the domain of analyticity of $G^*$ (top) and $g^*$ (bottom) plotted by using the fixed-point equation recursively. Dark and light tiles indicate where $f(z) \in \mathbb{H}_+$ and $f(z) \in \mathbb{H}_-$, for $f = G^*, g^*$.

and the corresponding universal constants. We have computed analytic extensions of the relevant functions to larger domains.

The method may be adapted to unimodal maps with general integer critical exponent. In the case of general even degree critical points, this relies on finding suitable function domains. In the case of odd degree critical points, the method may also be applied by recourse to a suitably-modified functional equation. Increasing the degree will inevitably lead to challenges in the rigorous numerics. We examine both cases in forthcoming publications.

A. Computational issues

All computations are verified independently by two different implementations of the function ball algebra: the first is written in the high-performance language Julia [40] and utilises multi-precision binary floating-point arithmetic with rigorous directed rounding modes [29] conforming to the relevant subset of standard IEEE754-2008. The second is written in the language Python and utilises multi-precision decimal floating-point arithmetic with rigorous directed rounding modes conforming to the relevant subsets of standards ANSI X3.274-1996, IEEE754-2008, and ISO/IEC/IEEE60559:2011.

The framework for rigorous function ball operations is adapted from that of [6, 8] with the addition of optimisations for the computation of the high-order bound on products.

The integrity of the frameworks is verified with the aid of over 1200 unit tests and functional tests. Where parallel computation has been used, care was taken to protect the integrity of rounding modes across processes; the results of all computations have been verified as bit-for-bit identical against the corresponding serial code.

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