Lossless Polariton Solitons

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Abstract. Photons and excitons in a semiconductor microcavity interact to form exciton-polariton condensates. These are governed by a nonlinear quantum-mechanical system involving exciton and photon wavefunctions. We calculate all non-traveling harmonic soliton solutions for the one-dimensional lossless system. There are two frequency bands of bright solitons when the inter-exciton interactions produce a repulsive nonlinearity and two frequency bands of dark solitons when the nonlinearity is attractive. In addition, there are two frequency bands for which the exciton wavefunction is discontinuous at its symmetry point, where it undergoes a phase jump of $\pi$. A band of continuous dark solitons merges with a band of discontinuous dark solitons, forming a larger band over which the soliton far-field amplitude varies from 0 to $\infty$; the discontinuity is initiated when the operating frequency exceeds the free exciton frequency. The far fields of the solitons in the lowest and highest frequency bands (one discontinuous and one continuous dark) are linearly unstable, whereas the other four bands have linearly stable far fields, including the merged band of dark solitons.

Key words: polariton, soliton, exciton, photon, nonlinear, semiconductor microcavity

1 Introduction

Exciton-polaritons arise from the coupling of photons with excitons, a type of electric dipole. An exciton is a quasiparticle generated in a semiconductor when an electron absorbs a photon and jumps from the valence to the conduction band thus leaving a hole in the first band. The electron and hole are attracted to each other by an effective electrostatic Coulomb
force, resulting in the excitation of an electron-hole pair, viewed as a quantum-mechanical quasiparticle. Excitons can be trapped in a planar microcavity containing a semiconductor material, that is, they live in a two-dimensional quantum well.

Exciton-polaritons can form Bose-Einstein condensates (BEC) at relatively high temperatures [4, 5, 12, 14], sustained by continuous laser pumping of photons. The condensate wavefunctions produce a rich variety of localised quantum states in the micrometer scale: dark solitons [2, 10, 11, 16, 19], bright solitons [7, 8, 16, 20], and vortices [9, 17]. Solitons in polaritonic condensates have potential for applications in ultrafast information processing [1] due to picosecond response times and strong nonlinearities [8, 20].

In a mean-field approximation, the excitons and photons are described by separate wavefunctions $\psi_X$ (excitons) and $\psi_C$ (photons) of spatial coordinates $x = (x_1, x_2)$ and time $t$. A continuous absorption and emission of photons by atoms in the semiconductor (Rabi oscillation) is represented by a coupling of the two equations. The kinetic term (Laplacian) is typically neglected for the excitons due to their significantly larger mass. On the other hand, exciton-exciton interaction is significant, so a nonlinear term arises in the equation for the exciton field. The system of equations reads [3, 6, 13, 18, 22]

$$i\partial_t \begin{pmatrix} \psi_X \\ \psi_C \end{pmatrix} = \begin{pmatrix} \omega_X - i\kappa_X + g|\psi_X|^2 \\ \gamma \omega_C - i\kappa_C - \frac{\hbar}{2m_C} \Delta \end{pmatrix} \begin{pmatrix} \psi_X \\ \psi_C \end{pmatrix} + \begin{pmatrix} 0 \\ F \end{pmatrix}.$$ (1.1)

The frequency of a free exciton is $\omega_X$, and $\omega_C$ is the frequency of the free, zero-momentum photon; $\kappa_X$ and $\kappa_C$ are the attenuation constants of the exciton and photon and account for losses; $m_C$ is the mass associated with the photons. The Laplacian is denoted by $\Delta = \partial^2_{x_1} + \partial^2_{x_2}$. The forcing $F$ represents a pumping of photons into the microcavity. The coupling associated with the Rabi oscillations enters through the frequency parameter $\gamma$, which is half the Rabi frequency. The nonlinearity $g|\psi_X|^2$ is attractive when $g > 0$ and repulsive when $g < 0$.

In this work, we study polariton fields that are lossless ($\kappa_X$ and $\kappa_C$ are zero) and unforced ($F = 0$). In turning off both pumping and losses, which are due to radiation and thermalization, we focus on the synergy of exciton interaction (nonlinearity) and photon dispersion. We consider fields that depend on only one spatial variable, say $x_1$, and we use the notation $x = x_1$ below. Under these conditions, we discover and analyze three families of stationary cavity exciton-polariton solitons. We use the term “soliton” in a broad sense to refer to a field with amplitude that tends to a constant value as $|x| \to \infty$, which is typical in the physics literature on polaritons. See [8] for a discussion of the passage from the 2D system to the 1D one by creating a polariton waveguide through detuning of the microcavity in the $x_2$ direction.

For each of the three families, the solitons exist on two frequency bands, all six bands being mutually disjoint (Fig. 1). There is one family of dark solitons for $g > 0$, one family of bright solitons for $g < 0$, and one family of solitons whose exciton wavefunction exhibits a spatial jump discontinuity. The discontinuity is made physically possible by the vanishing of the photon field, which brings dispersion to the system, at the point of discontinuity of the exciton field. Because one field vanishes where the other is nonzero, this phenomenon lies...
outside the parameter regime of validity of the Gross-Pitaevskii model of polaritons, which represents the full polariton field using a single wavefunction.

The stationarity property (harmonic with a non-traveling envelope) permits a reduction of the polariton equations to a real first-order ordinary differential equation. This allows symbolic integration of the polariton equations, resulting in exact analytical formulae for stationary polariton solitons.

Given that losses and pumping must be present in any true physical system, we interpret the solitons we derive as applying for a short time after the pumping is removed and the losses have not significantly manifested themselves or the solitons lie outside the pump spots (where \( F \) is nonzero). For example, in [2, 10], quasi-one-dimensional polariton structures are observed outside the pump spots. Polariton condensates can be created at two pump spots [21] and localized structures can be sustained in the region between the two spots where there is no pumping.

The polariton equations (1.1) with \( \kappa_{X,C} = 0 \) and \( F = 0 \) admit two quantities that are conserved in time,

\[
N = \int \left( |\psi_X|^2 + |\psi_C|^2 \right) dx, \tag{1.2}
\]
\[
H = \int \left( \frac{\hbar^2}{2m_C} |\nabla \psi_X|^2 + \omega_C |\psi_C|^2 + \omega_X |\psi_X|^2 + \frac{g}{2} |\psi_X|^4 + 2\gamma \text{Re}(\psi_X \psi_C^*) \right) dx. \tag{1.3}
\]

For stationary solitons, \( \int |\nabla \psi_X|^2 dx \) and \( \int |\nabla \psi_C|^2 dx \) are also conserved, although these quantities are not conserved for general solutions of the polariton equations.

Figure 1: The lossless, unforced, one-dimensional polariton equations admit six frequency bands of stationary soliton-type solutions, whose graphs are shown in section 3.2. Solitons in bands 1.1, 1.2, and 3.2 are all dark and can coexist in a system with \( g > 0 \). Solitons in bands 2.1, 2.2, and 3.1 can coexist in a system with \( g < 0 \). The linear stability of the far-field value of the soliton as a constant-amplitude solution of the polariton equations is indicated. The endpoint frequencies \( \omega_i \) of the bands are defined as follows: With \( p(\omega) := (\omega - \omega_X)(\omega - \omega_C) \), \( p(\omega_1) = p(\omega_5) = \frac{3}{2} \gamma^2 \) and \( p(\omega_2) = \frac{9}{8} \gamma^2 \) and \( p(\omega_3) = p(\omega_4) = \gamma^2 \).

Finding traveling soliton-like solutions of the polariton equations, even in one spatial dimension, is not a simple matter and remains to be investigated. Unlike the nonlinear Schrödinger equation, the polariton system with \( \kappa_{X,C} = 0 \) and \( F = 0 \) is not invariant under
the Galilean transformation

\[
(\psi_X(x, t), \psi_C(x, t)) \mapsto (\psi_X(x - \xi t, t), \psi_C(x - \xi t, t))e^{i(kx - \omega t)},
\]

(1.4)
in which \(\xi = \frac{\hbar}{m_C}k\) and \(\omega = \frac{\hbar}{2m_C}|k|^2\). The upper left entry of the matrix in (1.1) gains a transport term \(-i\xi \cdot \nabla \psi_X\) and a frequency shift \(\omega_X \mapsto \omega_X - \omega\). The scaling transformation

\[
(\psi_X(x, t), \psi_C(x, t)) \mapsto \lambda(\psi_X(\lambda^2 t, \lambda x), \psi_C(\lambda^2 t, \lambda x)),
\]

(1.5)
which preserves the cubic nonlinear Schrödinger equation, effects a transformation of the polariton equations through a scaling of the frequency parameters,

\[
(\omega_X, \omega_C, \gamma) \mapsto \lambda^2(\omega_X, \omega_C, \gamma).
\]

(1.6)
The result is a simple scaling by \(\lambda^2\) of both the \(\omega\) and the \(\zeta\) axes in the depiction of the band structure of solitons in Fig. 1.

## 2 Lossless harmonic polaritons

Consider a lossless, unforced, one-dimensional polariton field, consisting of a photon wavefunction \(\psi_C(x, t)\) and an exciton wavefunction \(\psi_X(x, t)\) dynamically coupled through their standard quantum-mechanical equations,

\[
i\partial_t \psi_X = (\omega_X + g|\psi_X|^2)\psi_X + \gamma \psi_C,
\]

(2.7)

\[
i\partial_t \psi_C = (\omega_C - \frac{1}{2} \partial_x^2)\psi_C + \gamma \psi_X.
\]

(2.8)

obtained by restricting (1.1) to one spatial dimension and setting \(\kappa_{X,C} = 0\) and \(F = 0\). The time variable \(t\) is normalized to an arbitrary unit of time \(T\), frequencies (including \(\omega_X, \omega_C, \gamma, \) and \(g\)) are normalized to \(1/T\), and the spatial variable \(x\) is normalized to \(\sqrt{T\hbar/m_C}\). Thus all variables and parameters are non-dimensional.

A traveling-wave polariton field with carrier frequency \(\omega\), modulated by an envelope has the form

\[
\phi_X(x, t) = \phi_X(x - ct)e^{i(kx - \omega t)},
\]

(2.9)
\[
\phi_C(x, t) = \phi_C(x - ct)e^{i(kx - \omega t)}.
\]

(2.10)

Under this ansatz, the polariton equations are equivalent to the pair

\[
-ic\phi_X' = (\omega_X - \omega + g\phi_X^2)\phi_X + \gamma \phi_C,
\]

(2.11)
\[
i(k - c)\phi_C' = (\omega_C - \omega + \frac{k^2}{2})\phi_C - \frac{1}{2}\phi_C'' + \gamma \phi_X,
\]

(2.12)
in which the prime denotes the derivative with respect to the argument.

This system of two complex ODEs reduces to a system of two real ODEs when the polariton envelope depends only on the spatial variable (speed of travel \(c = 0\)) and the polariton carrier phase is spatially invariant (wavenumber \(k = 0\)):

\[
(\psi_X(x, t), \psi_C(x, t)) = (\phi_X(x), \phi_C(x))e^{-i\omega t}.
\]

(2.13)
Figure 2: The relation (2.21) that gives the photon envelope value $\phi_C$ vs. the exciton envelope value $\phi_X$ of a harmonic solution of the form $(\psi_X(x,t),\psi_C(x,t)) = (\phi_X(x),\phi_C(x)) e^{-i\omega t}$ of the polariton equations (2.7,2.8). Its shape depends on the signs of $g$ and $\varpi_X = \omega - \omega_X$. A solution cannot cross the vertical dotted lines through the critical points.

Under this assumption, and with the notation

$$\varpi_X = \omega - \omega_X, \quad \varpi_C = \omega - \omega_C,$$

the pair $(\phi_X, \phi_C)$ satisfies the equations

$$(g\phi_X^2 - \varpi_X) \phi_X + \gamma \phi_C = 0, \quad (2.14)$$

$$-\frac{1}{2} \phi''_C - \varpi_C \phi_C + \gamma \phi_X = 0. \quad (2.15)$$

The first equation fully determines the photon field $\phi_C$ as an odd cubic polynomial function of the exciton field $\phi_X$, illustrated in Fig. 2. Thus the field value pair $(\phi_X(x), \phi_C(x))$ runs along the graph of the cubic as the spatial variable $x$ varies. For all the solitons derived in this paper, the field pair lies on this cubic between the two nonzero equilibrium points $(\phi_X^\pm, \phi_C^\pm)$ of the system of equations (2.14,2.15), namely

$$\phi_X^\pm = \pm \sqrt{\frac{1}{g} \left( \varpi_X - \frac{\gamma^2}{\varpi_C} \right)}, \quad (2.16)$$

$$\phi_C^\pm = \frac{\gamma}{\varpi_C} \phi_X^\pm. \quad (2.17)$$

Set the nonlinearity of the system of equations (2.14,2.15) equal to a new variable $\zeta$,

$$\zeta = g\phi_X^2, \quad (2.18)$$

a quantity that will be crucial in the analysis. Notice that the sign of $\zeta$ coincides with the sign of the nonlinearity parameter $g$. In terms of $\zeta$, the equilibrium points are expressed as $g\phi_X^{\pm\infty} = \pm \zeta_{\infty}$, where $\zeta_{\infty}$ is the non-dimensional frequency

$$\zeta_{\infty} = \zeta_{\infty}(\omega) := \varpi_X - \frac{\gamma^2}{\varpi_C} = \omega - \omega_X - \frac{\gamma^2}{\omega - \omega_C}. \quad (2.19)$$

The system (2.14,2.15) for the envelopes $\phi_X$ and $\phi_C$ can be converted into a first-order ODE for $\zeta(x)$, in which $g$ is no longer present; this is (2.20) in the following theorem. The sign of $g$ is determined by the (constant) sign of $\zeta(x)$. 

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Theorem 1. (a) Whenever \( \phi'_X(x)\phi'_C(x) \neq 0 \), the pair of equations (2.14,2.15) is equivalent to the pair

\[
(3\zeta - \varpi_X)^2 \zeta^2 = 8\zeta Q(\zeta), \tag{2.20}
\]
\[
\phi_C = \frac{1}{\gamma} (\varpi_X - g\phi_X^2)\phi_X, \tag{2.21}
\]

in which \( \zeta(x) = g\phi_X^2(x) \) and \( Q(\zeta) \) is a cubic polynomial in \( \zeta \),

\[
Q(\zeta) = Q(\zeta; \omega) = -\varpi_C \left( \zeta^3 - \frac{1}{2}(3\zeta_\infty + \varpi_X)\zeta^2 + \zeta_\infty\varpi_X \zeta + K \right) \tag{2.22}
\]

and \( K \) is a constant of integration.

(b) Whenever \( g \) and \( \varpi_X \) have the same sign, the local extrema of \( \phi_C = \gamma^{-1}(\varpi_X - g\phi_X^2)\phi_X \) occur when \( \zeta = g\phi_X^2 = \varpi_X/3 \). The roots of \( Q' \) are \( \zeta_\infty \) and \( \varpi_X/3 \). Thus, whenever \( \varpi_X/3 \) is a root of \( Q \), the factor \( (3\zeta - \varpi_X)^2 \) appears on both sides of (2.20).

Proof. To prove these statements, the structure of equations (2.14,2.15) is illuminated by writing them as

\[
f_1(\phi_X) + \gamma\phi_C = 0, \tag{2.23}
\]
\[
f_2(\phi_C) + \gamma\phi_X = \frac{1}{2}\phi''_C, \tag{2.24}
\]

in which \( f_1 \) and \( f_2 \) are odd polynomials.

Assume that \( \phi'_X(x)\phi'_C(x) \neq 0 \) in a given \( x \)-interval. By multiplying the first equation by \( \phi'_X \) and the second by \( \phi'_C \), adding, and then taking antiderivatives, one obtains

\[
K' + \tilde{f}_1(\phi_X) + \tilde{f}_2(\phi_C) + \gamma\phi_X\phi_C = \frac{1}{4}(\phi'_C)^2, \tag{2.25}
\]

in which the even polynomials \( \tilde{f}_{1,2} \) are primitives of \( f_{1,2} \) and \( K' \) is an arbitrary constant. Equation (2.25) expresses \( \phi_C \) as an odd polynomial function of \( \phi_X \), so the left-hand side of (2.25) is an even polynomial function of \( \phi_X \), say \( P(\phi_X) \). Thus (2.23,2.24) is equivalent to the validity of the pair

\[
P(\phi_X) = \frac{1}{4}(\phi'_C)^2, \tag{2.26}
\]
\[
f_1(\phi_X) + \gamma\phi_C = 0, \tag{2.27}
\]

for some constant \( K' \) in the definition of \( P \).

Differentiating (2.26) with respect to \( x \) yields

\[
P'(\phi_X)\phi'_X = \frac{1}{2}\phi''_C\phi'_C, \tag{2.28}
\]

then rewriting the right-hand side using (2.24) and the \( x \)-derivative of (2.27) results in

\[
P'(\phi_X) = -f'_1(\phi_X) \left( \phi_X + \gamma^{-1}f_2(\phi_C) \right). \tag{2.29}
\]

This leads to the observation that any root of the polynomial \( f'_1 \) is also a root of \( P' \). But \( f'_1(\phi_X) = 3g\phi_X^2 - \varpi_X \), so

\[
P'(\phi_X) = 0 \quad \text{when} \quad g\phi_X^2 = \varpi_X/3. \tag{2.30}
\]
Equation (2.26) can be written equivalently in terms of $\phi_X$ alone by differentiating (2.27) with respect to $x$ and substituting the resulting expression for $\phi'_C$ into the right-hand side of (2.26),

$$4\gamma^2 P(\phi_X) = f'_1(\phi_X)^2 (\phi'_X)^2. \quad (2.31)$$

It is straightforward to compute the sixth-degree polynomial $P(\phi_X)$ in (2.31). Since it is even, it is a cubic polynomial function of $\zeta = g\phi^2_X$, and a calculation converts (2.31) into the differential equation stated in the theorem,

$$(3\zeta - \varpi X)^2 \zeta'' = 8 \zeta Q(\zeta), \quad (2.32)$$

in which the cubic polynomial $Q$ is related to $P$ through $2g\gamma^2 P(\phi_X) = Q(\zeta)$ and is given by

$$Q(\zeta) = 2g\gamma^2 K' + \zeta \varpi X (\gamma^2 - \varpi X \varpi_C) - \zeta^2 (\frac{3}{2}\gamma^2 - 2\varpi X \varpi_C) - \zeta^3 \varpi_C. \quad (2.33)$$

Notice that $g$ has disappeared from the differential equation (2.20) except where it is multiplied by the constant of integration in $Q$. By the observation that $P'$ vanishes when $\zeta = g\phi^2_X = \varpi X/3$, one finds that

$$Q'(\varpi X/3) = 0 \quad \text{(if } \varpi X \neq 0). \quad (2.34)$$

The other root of $Q'$ is found to be $\zeta_\infty = \varpi X - \frac{\gamma^2}{\varpi_C}$, and one can write $Q$ as stated in the theorem, with

$$K = -\frac{2g\gamma^2 K'}{\varpi_C}. \quad (2.35)$$

### 3 Polariton solitons

A solution of the ODE (2.20) corresponds to a stationary, or non-traveling, time-harmonic solution of the polariton system (2.7,2.8). Our interest is in soliton solutions, for which the spatial envelope has a limiting value as $|x| \to \infty$. All solitons and their frequency bands are described in Theorem 2 below, and proved in section 4. The system admits bands of dark and bright solitons.

In addition, we find solitons for which the exciton field is discontinuous at its point of symmetry where the photon field vanishes. These are distributional solutions of the soliton equations. The physical origin of the discontinuity is that the vanishing of the photon field at a point in space turns off the interaction between neighboring excitons because this interaction is mediated only by the coupling of the exciton field to the dispersive photon field. When $\omega_X < \omega_C$, a band of continuous dark solitons and a band of discontinuous dark solitons can be unified into a single band of dark solitons, as described in section 3.3.
3.1 Main theorem: description of all solitons

The reduction of the polariton system (2.7,2.8) to an ODE (2.20) under the assumption of harmonic solutions allows a complete derivation of all stationary soliton solutions of (2.7,2.8). By a stationary soliton solution, we mean a harmonic solution \((\psi_X(x,t),\psi_C(x,t)) = (\phi_X(x),\phi_C(x))e^{-i\omega t}\) for which the envelopes \((\psi_X(x),\psi_C(x))\) tend to far-field values as \(x \to \pm\infty\). The form of the ODE, \(\zeta' = f(\zeta)\) guarantees that \(|\zeta(x)|\) and therefore also \(|\phi_X(x)|\) exhibit a single maximum at the soliton peak or minimum at the soliton nadir.

**Theorem 2.** There are bright, dark, and discontinuous stationary soliton solutions of the lossless, unforced polariton equations (2.7,2.8) for frequencies within certain bands that depend on \(\omega_X, \omega_C\), and \(\gamma\). The three soliton classes described below exhaust all solutions of the form

\[ (\psi_X(x,t),\psi_C(x,t)) = (\phi_X(x),\phi_C(x))e^{-i\omega t} \]

for which \(\phi_X(x)\) and \(\phi_C(x)\) have limits (far-field values) as \(x \to \pm\infty\).

1. **Dark solitons.** (Red bands in Figs. 3 and 6) Equations (2.7,2.8), for \(g > 0\), admit solutions of the form (2.13) for which \(\phi_X(x)\) and \(\phi_C(x)\) are antisymmetric, monotonic, and bounded. The frequency bands for which these solutions exist are given by

\[
\begin{align*}
0 < \omega_X \omega_C < \gamma^2 & \quad \text{if} \quad \omega < \min\{\omega_C,\omega_X\}, \quad \text{band 1.1} \\
\gamma^2 < \omega_X \omega_C < \frac{3}{2}\gamma^2 & \quad \text{if} \quad \omega > \max\{\omega_C,\omega_X\}. \quad \text{band 1.2}
\end{align*}
\]

The far-field (suprimal) value of \(|\phi_X(x)|\) is

\[
\lim_{|x| \to \infty} |\phi_X(x)| = \frac{1}{\sqrt{|g|}} \left| \omega_X - \frac{\gamma^2}{\omega_C} \right|^{1/2}.
\]

2. **Bright solitons.** (Green bands in Figs. 4 and 6) Equations (2.7,2.8), for \(g < 0\), admit solutions of the form (2.13) for which \(\phi_X(x)\) and \(\phi_C(x)\) are symmetric and bounded and have a unique local maximum. The frequency bands for which these solutions exist are given by

\[
\begin{align*}
\gamma^2 < \omega_X \omega_C < \frac{9}{8}\gamma^2 & \quad \text{if} \quad \omega < \min\{\omega_C,\omega_X\}, \quad \text{band 2.1} \\
0 < \omega_X \omega_C < \gamma^2 & \quad \text{if} \quad \omega > \max\{\omega_C,\omega_X\}. \quad \text{band 2.2}
\end{align*}
\]

The solitons vanish at the far field (\(|x| \to \infty\)) and \(|g|\phi_X(x)^2\) attains a maximal value of

\[
\lim_{|x| \to \infty} |g|\phi_X(x)^2 = -\omega_X + \frac{\gamma^2}{\omega_C} \left[ \frac{3}{4} + \sqrt{\frac{9}{16} - \frac{\omega_X \omega_C}{2\gamma^2}} \right].
\]

3. **Discontinuous solitons.** (Orange bands in Figs. 5 and 6) There are solitons in which \(\phi_X(x)\) is discontinuous at \(x = 0\) but \(\phi_C(x)\) is continuous. These are antisymmetric, monotonic, and bounded. They satisfy the polariton equations in the distributional sense, and away from the point of discontinuity they satisfy the equations classically.
(a) For \( g < 0 \), and all frequencies satisfying \( \omega < \min\{\omega_C, \omega_X\} \) and \( \omega_X \omega_C > \frac{3}{2} \gamma^2 \), \( \text{(band 3.1)} \)

there is a soliton such that \( |\phi_X(x)| \) decreases monotonically

\[
\text{from } \left| \frac{\omega_X}{g} \right|^{1/2} \text{ down to } \frac{1}{\sqrt{|g|}} \left| \omega_X - \frac{\gamma^2}{\omega_C} \right|^{1/2}.
\]

as \( x \) runs from the peak of \( |\phi_X(x)| \) to \( \infty \).

(b) For \( g > 0 \) and all frequencies satisfying \( \omega_X < \omega < \omega_C \), \( \text{(band 3.2)} \)

there is a dark soliton such that \( |\phi_X(x)| \) increases monotonically

\[
\text{from } \left| \frac{\omega_X}{g} \right|^{1/2} \text{ up to } \frac{1}{\sqrt{|g|}} \left| \omega_X - \frac{\gamma^2}{\omega_C} \right|^{1/2}.
\]

as \( x \) runs from the nadir of \( |\phi_X(x)| \) to \( \infty \). In the “negative detuning” case, \( \omega_C < \omega_X \), this band is absent.

The far-field values of the dark solitons and the discontinuous solitons are expressed in the \( \zeta \) variable as

\[
\lim_{|x| \to \infty} g \phi_X(x)^2 = \zeta_\infty(\omega), \quad (3.40)
\]

with \( \zeta_\infty \) given by (2.19). Thus the far-field limit of the soliton is an equilibrium solution of the polariton equations. The proof of the theorem (section 4) reveals that \( \zeta_\infty(\omega) \) is one of the stationary points of \( Q(\zeta; \omega) \) (i.e., \( \partial Q(\zeta; \omega)/\partial \zeta = 0 \)).

The peak values of the bright solitons are expressed as a root of \( Q(\zeta) \) when \( K = 0 \), as will be demonstrated in the proof below. Since \( g < 0 \), \( \zeta(x) \) is negative and, according to (3.39), attains a minimal value of

\[
\zeta_0(\omega) := \omega_X - \frac{\gamma^2}{\omega_C} \left[ \frac{3}{4} + \sqrt{\frac{9}{16} - \frac{\omega_X \omega_C}{2 \gamma^2}} \right]. \quad (3.41)
\]

Band 3.1 of discontinuous solitons for \( g < 0 \) is unusual in that the exciton field exhibits a peak whereas the photon field exhibits a dip at the symmetry point (Fig. 5).

### 3.2 Graphical depiction of solitons

The figures in this section depict the soliton solutions of the form (2.13) for the polariton equations (2.7, 2.8). The three types of solitons announced in Theorem 2 are depicted in three separate figures below.

In Figures 3, 4, and 5, assume that the symmetry point of each soliton (peak or nadir) is at \( x = 0 \). In each figure:
The leftmost diagram shows two frequency bands of solitons on the $\omega$-axis of the $\omega\zeta$-plane. At a chosen frequency in each band, an arrow spans the range of $\zeta$-values of a soliton, pointing toward the far-field value $\lim_{|x| \to \infty} \zeta(x)$. The tail of the arrow, indicated by a solid dot, has its ordinate at $\zeta(0)$. The point of the arrow, indicated by an open circle, has its ordinate at the far-field value of $\zeta(x)$.

The sign of $g$ coincides with the sign of $\zeta = g\phi_x^2$.

The middle and rightmost diagrams depict the exciton and photon envelopes $\phi_X(x)$ and $\phi_C(x)$ for each frequency corresponding to the arrows in the leftmost diagram.

The upper graphs depict the trajectory of the point $(\phi_X(x), \phi_C(x))$ along the cubic relation (2.21) as $x$ traverses the real line. The solid dots mark the central point $(\phi_X(0), \phi_C(0))$, and the open circles mark the pair of far-field values $\lim_{x \to \pm \infty} (\phi_X(x), \phi_C(x))$.

The lower graphs depict the exciton and photon envelopes vs. the spatial variable $x$. When one passes from $\zeta = g\phi_x^2$ to $\phi_X$, the extraction of square roots results in two solitons, which are minuses of each other. One choice of square root is shown in the lower graphs.

In each figure, $\gamma = 1$ and $\omega_C - \omega_X = 1$. The inequality $\omega_C > \omega_X$ is referred to as “positive detuning”.

Figure 3: Dark solitons. Anti-symmetric dark solitons for nonlinearity coefficient $g > 0$. (See the bullet points above in section 3.2 for general explanation.) The far-field value $\zeta_\infty$ of $\zeta = g\phi_x^2$, indicated by the open dots in the leftmost diagram and given by (2.19), is equal to a double root of the cubic $Q(\zeta) = Q(\zeta; \omega)$ (see (2.22)) created by the appropriate choice of constant $K = K(\omega)$. In the upper graphs (middle and right), the pair $(\phi_X(x), \phi_C(x))$ travels from one open circle to the other as $x$ travels from $-\infty$ to $\infty$. 
Figure 4: **Bright solitons.** Symmetric bright solitons for nonlinearity coefficient $g < 0$. (See the bullet points above in section 3.2 for general explanation.) The minimal value $\zeta_0$ of $\zeta(x)$, indicated by the solid dot in the leftmost diagram and given by (3.41), is at a simple root of $Q(\zeta) = Q(\zeta; \omega)$ when $K = 0$ so that $\zeta Q(\zeta)$ has a double root at $\zeta = 0$. In the upper graphs (middle and right), the pair $(\phi_X(x), \phi_C(x))$ travels from the open circle to one of the solid dots and back as $x$ travels from $-\infty$ to $\infty$.

Figure 5: **Discontinuous solitons.** Anti-symmetric solitons for which the exciton envelope $\phi_X(x)$ is discontinuous at its point of symmetry. (See the bullet points above in section 3.2 for general explanation.) The far-field value $\zeta_\infty$ of $\zeta = g\phi_X^2$, indicated by the open dots in the leftmost diagram and given by (2.19), is equal to a double root of the cubic $Q(\zeta) = Q(\zeta; \omega)$ (see (2.22)) created by the appropriate choice of constant $K = K(\omega)$. In the upper graphs (middle and right), as $x$ travels from $-\infty$ to $\infty$, the pair $(\phi_X(x), \phi_C(x))$ travels along the cubic from an open circle to a solid dot, then jumps to the other solid dot, and then travels along the cubic to the other open circle.
Figure 6: **Left:** A superposition of the leftmost diagrams of Figures 3, 4, and 5, showing all bands simultaneously. This is the “positive detuning” case, in which $\omega_C > \omega_X$. A given polariton system admits either bands 1.1, 1.2, and 3.2 if $g > 0$ or bands 2.1, 2.2 and 3.1 if $g < 0$. **Right:** In the “negative detuning” case $\omega_C < \omega_X$, the band 3.2 of discontinuous solitons is absent.

### 3.3 A band of continuous and discontinuous dark solitons

Bands 1.1 and 3.2 merge to form a larger band of dark solitons for $g > 0$, which we call Band D. This band was reported by the same authors in [15]. It consists of the interval $(\omega_{LP}, \omega_C)$, where $\omega_{LP}$ is defined by

$$
(\omega_{LP} - \omega_X)(\omega_{LP} - \omega_C) = \gamma^2, \quad \omega_{LP} < \min\{\omega_X, \omega_C\},
$$

(3.42)

and coincides with the lower endpoint of a well-known band of homogeneous (constant in $x$) “lower polaritons” [18, Fig. 1] for the associated linear system obtained by setting $g = 0$ and keeping all other parameters unchanged. The far-field amplitude of the soliton is given by $\zeta(x) \to \zeta_\infty$ as $|x| \to \infty$, or

$$
g\phi_X^2 \to \omega - \omega_X - \frac{\gamma^2}{\omega - \omega_C} \quad \text{as} \quad |x| \to \infty,
$$

(3.43)

and ranges from 0 to $\infty$ as $\omega$ traverses the band $(\omega_{LP}, \omega_C)$. In the case of “negative detuning” ($\omega_C < \omega_X$), the discontinuous band 3.2 vanishes and the dark soliton is continuous on the entire band D. In the case of “positive detuning” ($\omega_X < \omega_C$), the exciton frequency $\omega_X$ lies within band D and marks the transition from band 1.1 to band 3.2, where the soliton becomes discontinuous.
Thus, in the positive-detuning case, the frequencies $\omega_{LP}$, $\omega_X$, and $\omega_C$ have the following significance for soliton band D:

- The value $\omega_{LP}$ is the threshold frequency that marks the onset of a soliton. For frequencies just above this threshold ($0 < \omega - \omega_{LP} \ll 1$), the soliton amplitude is small. This can be seen from equation (3.43), which shows that the soliton amplitude vanishes when $\omega = \omega_{LP}$. Thus, solitons at frequencies near the lower edge of the band are in the linear regime because the nonlinearity $g|\phi_X|^2$ is negligible.

- The exciton frequency $\omega_X$ is the transition frequency, at which the exciton field of the soliton becomes discontinuous, as shown in Fig. 7. As $\omega$ exceeds $\omega_X$, the quantity $\omega_X$ changes from negative to positive and the cubic relation between $\phi_C$ and $\phi_X g$ gains two nonzero roots at $\phi_X = \pm \sqrt{\omega_X/g}$ (Fig. 2, second row). The exciton field jumps between these two roots exactly when the photon field vanishes, as shown in the rightmost graphs of Fig. 7.

- The photon frequency $\omega_C$ is the blowup frequency: as $\omega$ goes up to $\omega_C$, the far-field amplitude (3.43) tends to infinity.

In the negative-detuning case, the threshold and blowup frequencies persist, but the soliton undergoes no transition to discontinuity.

Figure 7: When $\omega_C > \omega_X$ (positive detuning), band 1.1 of continuous dark solitons and band 3.2 of discontinuous dark solitons merge to form the single band D of dark solitons. Their far-field value of $g\phi_X^2 = \zeta_\infty$ is represented by the single expression $Q(\zeta_\infty) = 0$ when the constant $K$ in (2.22) is chosen so that $Q$ has a double root at $\zeta_\infty$; it is given explicitly by (3.43). The frequency $\omega_{LP}$ is the threshold frequency, marking the onset of the soliton; the exciton frequency $\omega_X$ marks the transition from continuous to discontinuous exciton field; and the far-field amplitude of the soliton blows up as $\omega$ goes up to the photon frequency $\omega_C$. 

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4 Proof of solitons

The derivation of all solitons is simplified by passing to a normalized frequency variable $\eta$ that conveniently parameterizes the operating frequency $\omega$,

$$\eta = \frac{1}{\gamma^2} (\omega - \omega_X)(\omega - \omega_C) = \frac{\omega_X \omega_C}{\gamma^2}.$$  \hspace{1cm} (4.44)

This expression, which is quadratic in $\omega$, produces generically two frequencies for the same value of $\eta$, a first indication of the fact that exciton-polariton soliton solutions typically come in pairs. One only needs to consider values $\eta \geq \eta_{\text{min}} = \frac{(\omega_X - \omega_C)^2}{4\gamma^2}$, \hspace{1cm} (4.45)

that are at or above the minimum $\eta_{\text{min}}$ of the quadratic and thus produce real frequencies.

The non-dimensional frequency $\varpi_X$ is a natural scaling factor for $\zeta$:

$$\zeta = \varpi_X u, \quad \zeta_{\infty} = \varpi_X u_{\infty}, \quad u_{\infty} = 1 - \frac{1}{\eta}, \quad \eta = \frac{\varpi_X \varpi_C}{\gamma^2}.$$ \hspace{1cm} (4.46)

Using $u$ instead of $\zeta$ greatly simplifies the algebraic computations of solitons. In the new variables, the ODE (2.20) for $\zeta$ becomes an ODE for $u$,

$$(3u - 1)^2 u' = -8\varpi_C \left[ u^3 - \left(2 - \frac{3}{2\eta}\right) u^2 + \left(1 - \frac{1}{\eta}\right) u + \tilde{K} \right], \text{ when } \varpi_X \neq 0. \hspace{1cm} (4.47)$$

The phase space of this ODE is one-dimensional (the $u$-axis). Because $u'$ appears squared, a soliton solution of the ODE is obtained through standard phaseline analysis by connecting a double root of the quartic polynomial $uB(u)$ on the right side of the ODE with a simple root of $uB(u)$. A solution connecting these consecutive roots is possible provided that the interval between the two roots does not contain the singular value $u = 1/3$; we call this the connectivity condition. In addition, the sign condition requires the positivity of the right side of the equation over the interval between the two roots; it can be expressed as

$$- \varpi_C (uB(u))''|_{u=\text{double root}} \geq 0. \hspace{1cm} (4.48)$$

The solution $u(x)$ approaches the double root exponentially slowly as $x$ approaches $\pm \infty$; thus the double root signifies the far-field amplitude of the soliton and is thus equal to $u_{\infty}$. The simple root $u_0$ is the extremal value (maximum or minimum) of $u(x)$ and is attained at a finite value $x_*$. The solution $u(x)$ is symmetric about $x_*$, and has local quadratic behavior there. Because of the invariance of the polariton system under a shift $x \mapsto x - x_*$, we will henceforth take $x_* = 0$.

Since $uB(u)$ vanishes at $u = 0$, it is guaranteed that the interval between two consecutive roots is either positive or negative, so that the solution $u(x)$ is of one sign. This allows one to choose the sign of $g$ appropriately so that $(\varpi_X/g)u > 0$ and $\phi_X = \pm \sqrt{(\varpi_X/g)u}$. If the simple root $u_0$ is nonzero, then the exciton envelope field is symmetric and of one sign,

$$\phi_X(x) = \sqrt{\frac{\varpi_X}{g}} u(x), \quad \text{if } u_0 \neq 0.$$
If the simple root \( u_0 \) is equal to zero, then the exciton envelope field is anti-symmetric,

\[
\phi_X(x) = \text{sgn}(x) \sqrt{\frac{\varpi_X}{g}} u(x), \quad \text{if } u_0 = 0.
\]

The rest of this section is dedicated to a proof of Theorem 2.

1. **Dark solitons.** These are solutions that connect a double nonzero root of \( uB(u) \), (far-field value) with the zero root (nadir).

   The following factorization is key to the analysis;

\[
B(u) = (u - u_\infty)^2(u - u_0), \quad \begin{cases} u_\infty = 1 - \frac{1}{\eta} \\ u_0 = \frac{1}{2\eta} \end{cases}, \quad \tilde{K} = -u_\infty^2u_0. \tag{4.49}
\]

One observes that the singular value \( u = 1/3 \) of the ODE lies between the double root \( u_\infty \) and the simple root \( u_0 \) and thus obstructs a soliton connection between them:

\[
\begin{cases} u_\infty = \frac{1}{3} + 2 \left( \frac{1}{3} - \frac{1}{2\eta} \right) \\ u_0 = \frac{1}{3} - \left( \frac{1}{3} - \frac{1}{2\eta} \right) \end{cases}. \tag{4.50}
\]

In order for \( u_\infty \) to be connectible to the other root \( u = 0 \) of \( uB(u) \), the following two mutually equivalent conditions must hold:

\[
u_\infty < \frac{1}{3}, \quad \text{i.e.,} \quad 0 < \eta < \frac{3}{2}. \tag{connectivity condition} \tag{4.51}
\]

Furthermore, one obtains from (4.48), the sign condition

\[
- \varpi_C u_\infty(u_\infty - \frac{1}{3}) > 0. \tag{sign condition} \tag{4.52}
\]

As a result of the positivity of \( \eta \), the frequencies \( \varpi_X \) and \( \varpi_C \) must have the same sign. There are therefore two cases (\( \varpi_X < 0, \varpi_C < 0 \)) and (\( \varpi_X > 0, \varpi_C > 0 \)).

In the case that \( \varpi_X \) and \( \varpi_C \) are both negative, or \( \omega < \min\{\varpi_X, \varpi_C\} \), conditions (4.51, 4.52) necessitate \( u_\infty < 0 \), which by \( u_\infty = 1 - \eta^{-1} \) is equivalent to \( 0 < \eta < 1 \). This yields the band

\[
\{0 < \eta < 1, \omega < \min\{\varpi_X, \varpi_C\}\}. \tag{Band 1.1} \tag{4.53}
\]

Thus for each \( \eta \) between 0 and 1 the ODE (4.47) has a soliton solution \( u(x) \) with nadir equal to the simple root \( u = 0 \) and far-field value equal to the double root \( u_\infty \).

When converting \( \eta \) back to the variable \( \omega \) through \( \eta = \varpi_X \varpi_C / \gamma^2 \), the condition \( \omega < \min\{\varpi_X, \varpi_C\} \) determines the choice of frequency \( \omega \) as the lower of the two solutions of \( \eta \gamma^2 = (\omega - \varpi_X)(\omega - \varpi_C) \). This results in Band 1.1 stated in Theorem 2 (see 3.36) and depicted in Fig. 3. The interval \( 0 < \eta < 1 \) corresponds to the condition \( 0 < \varpi_X \varpi_C < \gamma^2 \) in (3.36).

In converting \( u \) back to the variable \( \zeta \), notice that \( u_\infty \) and \( \varpi_X \) are both negative so that \( \zeta_\infty = \varpi_X u_\infty > 0 \), and thus \( \zeta(x) = \varpi_X u(x) > 0 \) for all \( x \) since \( u \) is of one sign on the
interval \((u_\infty, 0)\). The sign of \(g\) is determined by \(0 < \zeta = g\phi_X^2\), so \(g > 0\) for these solutions. The far-field value \((3.37)\) is obtained from the expression \((2.19)\) for \(\zeta_\infty\).

In the case that \(\varpi_X\) and \(\varpi_C\) are both positive, or \(\omega > \max\{\omega_X, \omega_C\}\), conditions \((4.51, 4.52)\) necessitate \(0 < u_\infty < \frac{1}{3}\), which by \(u_\infty = 1 - \eta^{-1}\) is equivalent to \(1 < \eta < \frac{3}{2}\). This yields the band
\[
\{1 < \eta < \frac{3}{2}, \ \zeta = g\varphi_X^2, g > 0\}.
\]
This results in Band 1.2 stated in Theorem 2 (see 3.36) and depicted in Fig. 3. The interval \(1 < \eta < \frac{3}{2}\) corresponds to the condition \(\gamma^2 < \varpi_X\varpi_C < \frac{3}{2}\gamma^2\) in \((3.36)\). Since again \(\zeta_\infty = \varpi_Xu_\infty > 0\), one has \(g\phi_X(x)^2 = \zeta(x) > 0\) for all \(x\), and again \(g > 0\).

2. Bright solitons. These are solutions that connect zero as a double root of \(uB(u)\) (far-field value) to a simple root (peak).

Assuming \(uB(u)\) has a double root at 0, \((4.47)\) takes the form
\[
(3u - 1)^2u^2 = -8\varpi_Cu^2\left[u^2 - \left(2 - \frac{3}{2\eta}\right)u + 1 - \frac{1}{\eta}\right], \quad \text{when} \ \varpi_X \neq 0.
\]
We are interested in the ranges of \(\eta\) for which roots of the quadratic \(A(u)\) are to the left of the singular point \(u = \frac{1}{3}\) and thus can be connected with the double root at \(u = 0\) (connectivity condition). By a simple argument\(^1\) there is either one root \(u_0\) or no root in the half-line \(u < \frac{1}{3}\). The range of \(\eta\) that for which this root is present is
\[
0 < \eta < \frac{9}{8}, \quad \begin{cases} 0 < u_0 < \frac{1}{3}, & \text{when} \ 1 < \eta < \frac{9}{8}; \\ u_0 < 0, & \text{when} \ 0 < \eta < 1. \end{cases}
\]

The other root is above \(\frac{1}{3}\), so one computes
\[
u_0 = 1 - \frac{1}{\eta}\left[\frac{3}{4} + \sqrt{\frac{9}{16} - \frac{\eta}{2}}\right].
\]
As \(\eta\) decreases from \(\eta = \frac{9}{8}\) to \(\eta = 1\), the root \(u_0\) descends from \(u = \frac{1}{3}\) to \(u = 0\), then turning negative as \(\eta\) decreases from the value 1. In the limit \(\eta \to +0\), \(u_0 \to -\infty\).

The condition \(\eta > 0\) implies \(\varpi_X\varpi_C > 0\), so that, as before, either \((\varpi_X < 0, \ \varpi_C < 0)\) or \((\varpi_X > 0, \ \varpi_C > 0)\). We must consider these cases in conjunction with the sign condition discussed above, which requires the right side of \((4.55)\) to be positive in the neighborhood of the double root \(u = 0\),
\[
\varpi_C(1 - \frac{1}{\eta}) < 0. \quad \text{(sign condition)}
\]
Putting these requirements together, we obtain two soliton bands,
\[
\begin{cases} \text{Band 2.1 = } \{1 < \eta < \frac{9}{8}, \ \varpi_X < 0, \ \varpi_C < 0\} \\ u_0 > 0, \quad \zeta_0 = \varpi_Xu_0 < 0, \quad g < 0, \end{cases}
\]
\(^1\)Write \((u - 1)^2 = -\frac{3}{2\eta}(u - \frac{3}{4})\) and examine how the line cuts the quadratic, as \(\eta\) is varied.
\[
\begin{align*}
\text{Band } 2.2 & = \{0 < \eta < 1, \quad \varpi_X > 0, \quad \varpi_C > 0\} \\
u_0 < 0, \quad \zeta_0 = \varpi_X u_0 < 0, \quad g < 0.
\end{align*}
\] (4.60)

As in the previous case of dark solitons, the endpoints of the frequency bands are imposed by the bounds of \(\eta\) and the relation \(\eta \gamma^2 = \varpi_X \varpi_C\), and the sign of \(g\) coincides with the sign of \(\zeta\), which is negative in these cases.

The minimal (negative) value \(\zeta_0\) of \(\zeta(x) = \varpi_X u(x)\) is equal to \(\varpi_X u_0\), which from (4.57) is equal to
\[
\zeta_0 := \varpi_X u_0 = \varpi_X - \frac{\gamma^2}{\omega_C} \left[ \frac{3}{4} \left( \frac{9}{16} - \frac{\varpi_X \omega_C}{2\gamma^2} \right) \right],
\] (4.61)
and since \(g < 0\), one obtains the peak value of \(|g|\varphi_X^2\) stated in the theorem.

3. Discontinuous solitons.

In the derivation of the dark solitons, the case \(u_\infty > \frac{1}{3}\) was excluded because the connectivity of \(u_\infty\) to the zero root was broken by the singularity at \(u = \frac{1}{3}\). Consider instead the \(u\)-interval between \(u_\infty\) and \(u = 1\), which does not contain \(1/3\) or any root (besides \(u_\infty\)) of \(B(u)\). The corresponding \(\zeta\)-interval connects \(\zeta_\infty\) an \(\varpi_X\) and does not contain \(\varpi_X/3\) or any other roots (besides \(\zeta_\infty\)) of \(\zeta Q(\zeta)\).

This \(\zeta\)-interval corresponds to two \(\phi_X\)-intervals, connecting \(\pm \phi_X^\infty\) with \(\pm \phi_0 := \pm \sqrt{\varpi_x/g}\) and not containing \(\pm \sqrt{\varpi_X/(3g)}\), where \(\phi_X^\infty > 0\) is defined through \(g(\phi_X^\infty)^2 = \zeta_\infty\). Naturally, \(g\) must take the sign of \(\zeta_\infty\), and thus the cubic (2.21) giving \(\phi_C\) as a function of \(\phi_X\) vanishes when \(\phi_X = \pm \phi_0\).

Given that the sign condition holds, a discontinuous soliton is constructed by taking a solution of (2.20) for which \(\zeta(x)\) travels from \(\phi_0\) to \(\phi_X^\infty\) as \(x\) travels from \(0\) to \(\infty\), then setting
\[
\begin{align*}
\phi_X(x) &= \sqrt{\zeta(x)/g} \quad \text{for} \quad x > 0 \\
\phi_X(x) &= -\sqrt{\zeta(-x)/g} \quad \text{for} \quad x < 0 \\
\phi_C(x) &= \gamma^{-1} \phi_X \left( \varpi_X - g \varphi_X^2 \right) \quad \text{for} \quad x \in \mathbb{R}.
\end{align*}
\]
The polariton field \((\phi_X(x), \phi_C(x))\) is antisymmetric about \(x = 0\) and satisfies the pair (2.14 2.15) for \(x \neq 0\). Since \(\phi_C\) vanishes when \(\phi_X = \pm \phi_0\), setting \(\phi_C(0) = 0\) makes the field \(\phi_C(x)\) continuous; this together with antisymmetry makes \(\phi_C(x)\) continuously differentiable at \(x = 0\). Thus (2.15) is satisfied in the sense of distributions, even through \(x = 0\), and the jump of \(\phi_C''(x)\) across \(x = 0\) is computed from the ODE:
\[
[\phi_C''(x)]_{x=0} = 2\gamma [\phi_X(x)]_{x=0} = 4\gamma \sqrt{\varpi_X/3}.
\] (4.62)

Violation of the connectivity condition means
\[
u_\infty > \frac{1}{3}, \quad \text{i.e.,} \quad \eta > \frac{3}{2} \quad \text{or} \quad \eta < 0. \quad \text{(no-connectivity condition)}
\] (4.63)
The sign condition (4.52) still applies and reduces to
\[
\varpi_C < 0,
\] (4.64)
as a result of $u_\infty > \frac{1}{3}$. The sign of $\varpi_X$ is opposite to the sign of $\eta$, as follows from the definition of $\eta$ (4.44). From the relation $u_\infty = 1 - \eta^{-1}$, we obtain two frequency bands, one for $\eta > \frac{3}{2}$, and one for $\eta < 0$. For $\eta > 3/2$, \[
\begin{align*}
\text{Band 3.1} &= \{ \eta > \frac{3}{2}, \omega < \min\{\omega_X, \omega_C\} \} \\
u_\infty > \frac{1}{3}, \quad \zeta_\infty = \varpi_X u_\infty < 0, \quad g < 0.
\end{align*}
\] (4.65)

In this band, the far-field value of $u$ is $u_\infty = 1 - \eta^{-1}$, so the range of $u(x)$ is $(u_\infty, 1)$ and thus $\zeta(x) = \varpi_X u(x)$ has a far-field value of $\varpi_X - \gamma^2 / \omega_C$ and, since $\varpi_X < 0$, its range is equal to negative interval $(\varpi_X, \zeta_\infty)$. The corresponding discontinuous soliton is bright since $|\zeta_\infty| < |\varpi_X|$.

In the case $\eta < 0$, one obtains \[
\begin{align*}
\text{Band 3.2} &= \{ \eta < 0, \omega_X < \omega < \omega_C \} \\
u_\infty > \frac{1}{3}, \quad \zeta_\infty = \varpi_X u_\infty > 0, \quad g > 0.
\end{align*}
\] (4.66)

The far-field amplitude of $u$ is again $u_\infty = 1 - \eta^{-1}$, so the range of $u(x)$ is $(1, u_\infty)$ since $\varpi_X > 0$, the range of $\zeta(x)$ is the positive interval $(\varpi_X, \zeta_\infty)$.

5 Stability analysis of the soliton far-field solutions

We carry out linear stability analysis of the far-field values of the solitons derived in the previous section. The far-field behavior of a soliton is asymptotic (as $|x| \to \infty$) to a solution of the pair (2.7,2.8) that is $x$-independent and oscillates in time with frequency $\omega$, i.e., is of the form $(\psi_X, \psi_C) e^{-i\omega t}$. The far-field value $(\phi_X^\infty, \phi_C^\infty)$ of the soliton envelope constitutes an equilibrium solution of the pair (2.14,2.15). We refer to $(\phi_X^\infty, \phi_C^\infty) e^{-i\omega t}$ as a soliton far-field solution. These of course exist only over values of the frequency $\omega$ for which a soliton exists. More generally, we refer to all equilibrium solutions of (2.20), including those at frequencies outside the soliton bands, as homogeneous solutions.

In the variable $\zeta = g\varphi_X^2$, a soliton far-field solution takes on either the value $\zeta = 0$ or the value $\zeta = \zeta_\infty = \zeta_\infty(\omega)$. The value $\zeta = 0$ is asymptotic to the far-field solution of the solitons in frequency bands 2.1 and 2.2, whereas the values $\zeta = \zeta_\infty$ are asymptotic to the far-fields of the solitons in frequency bands 1.1, 1.2, 3.1, 3.2. Additional equilibrium solutions of (2.20), correspond to the simple roots of the quartic polynomial on the right of this equation and are inadmissible, as they do not satisfy the original system (2.14,2.15). These extraneous solutions are generated by the fact that deriving (2.20) involves multiplying (2.14) and (2.15) by $\varphi_X'$ and $\varphi_C'$, which vanish when $\varphi_X$ and $\varphi_C$ are constant.

Theorem 3. Let $(\psi_X(x,t), \psi_C(x,t)) = (\phi_X(x), \phi_C(x)) e^{-i\omega t}$ be a solution of the nonlinear polariton system (2.7,2.8) such that \[
(\phi_X(x), \phi_C(x)) \to (\phi_X^\infty, \phi_C^\infty) \quad \text{as} \quad |x| \to \infty.
\]
The function pair \((\phi^\infty_X, \phi^\infty_C)e^{-i\omega t}\) is a “homogeneous” solution to (2.7,2.8) that is linearly stable if \(\omega \in \) Band 1.1, 2.1, 2.2, or 3.2, unstable if \(\omega \in \) Band 1.2 or 3.1.

In bands 2.1 and 2.2, the homogeneous solution is equal to \((0,0)\). In the other bands, one has \(g(\phi^\infty_X)^2 = \zeta_\infty\).

The rest of this section is devoted to a proof of this theorem. Inserting \(\psi_X = \tilde{\psi}_X e^{-i\omega t}\) and \(\psi_C = \tilde{\psi}_C e^{-i\omega t}\) into (2.7,2.8) yields
\[
\begin{align*}
i\partial_t \tilde{\psi}_X &= \left(-\omega_X + g|\psi_X|^2\right) \tilde{\psi}_X + \gamma \tilde{\psi}_C, \quad (5.67) \\
i\partial_t \tilde{\psi}_C &= \left(-\omega_C - (2m_C)^{-1}\partial_{xx}\right) \tilde{\psi}_C + \gamma \tilde{\psi}_X. \quad (5.68)
\end{align*}
\]

We take \(\tilde{\psi}_X\) and \(\tilde{\psi}_C\) to be the perturbed envelope functions: Setting \(\tilde{\psi}_X = \phi_X + \xi_X\) and \(\tilde{\psi}_C = \phi_C + \xi_C\), one obtains equations for the perturbations \((\xi_X, \xi_C)\),
\[
\begin{align*}
i\partial_t \xi_X &= -\omega_X \xi_X + g\phi_X^2 \left(2\xi_X + \hat{\xi}_X\right) + \gamma \xi_C + \text{h.o.t.}, \quad (5.69) \\
i\partial_t \xi_C &= \left(-\omega_C - (2m_C)^{-1}\partial_{xx}\right) \xi_C + \gamma \xi_X \quad (5.70)
\end{align*}
\]
in which the omitted terms are of higher order in \(\xi_X\) and \(\hat{\xi}_X\). By taking the Laplace transform of these equations and their conjugates, one obtains the system
\[
\begin{align*}
(is + \omega_X - 2g\phi_X^2) \hat{\xi}_X - g\phi_X^2 \hat{\xi}_X - \gamma \hat{\xi}_C &= 0 \quad (5.71) \\
(-is + \omega_X - 2g\phi_X^2) \hat{\xi}_X - g\phi_X^2 \hat{\xi}_X - \gamma \hat{\xi}_C &= 0 \quad (5.72) \\
(is + \omega_C + (2m_C)^{-1}\partial_{xx}) \hat{\xi}_C - \gamma \hat{\xi}_X &= 0 \quad (5.73) \\
(-is + \omega_C + (2m_C)^{-1}\partial_{xx}) \hat{\xi}_C - \gamma \hat{\xi}_X &= 0 \quad (5.74)
\end{align*}
\]
or, in matrix form,
\[
\begin{bmatrix}
2g\phi_X^2 - \omega_X - is & g\phi_X^2 & \gamma & 0 \\
g\phi_X^2 & 2g\phi_X^2 - \omega_X + is & 0 & \gamma \\
\gamma & 0 & -\partial_{xx} - \omega_C - is & 0 \\
0 & \gamma & 0 & -\partial_{xx} - \omega_C + is 
\end{bmatrix}
\begin{bmatrix}
\hat{\xi}_X \\
\hat{\xi}_C \\
\hat{\xi}_C \\
\hat{\xi}_C 
\end{bmatrix} = 0.
\]

Notice that the field \((\phi_X, \phi_C)\) occurs in the matrix only through \(\zeta = g\phi_X^2\). To analyze linear stability of soliton far-field solutions, we take \(g\phi_X^2\) equal to \(\zeta_\infty\) or 0.

Introduce the variable
\[
\alpha := 2g\phi_X^2 - \omega_X = 2\zeta - \omega_X \quad (5.76)
\]
which thus equals either \(2\zeta_\infty - \omega_X\) or \(-\omega_X\). Through the Fourier transform in the spatial variable \(x\), we make the replacement \(-\partial_{xx} \rightarrow k^2\) and introduce the variable
\[
\beta := k^2 - \omega_C. \quad (5.77)
\]
The matrix equation above becomes
\[
\begin{bmatrix}
\alpha - is & \zeta & \gamma & 0 \\
\zeta & \alpha + is & 0 & \gamma \\
\gamma & 0 & \beta - is & 0 \\
0 & \gamma & 0 & \beta + is
\end{bmatrix}
\begin{bmatrix}
\hat{\xi}_X \\
\hat{\xi}_X \\
\hat{\xi}_C \\
\hat{\xi}_C
\end{bmatrix} = 0.
\]

(5.78)

in which we have take the liberty to retain the notation in the column vector despite having passed to the Fourier transform. The determinant of the matrix is
\[
D := s^4 + (\alpha^2 + \beta^2 + 2\gamma^2 - \zeta^2)s^2 + \gamma^4 - 2\alpha\beta\gamma^2 + \alpha^2\beta^2 - \zeta^2\beta^2.
\]

(5.79)

Linearized stability at all modes \(k\) requires that the two roots of the determinant \(D\), considered as a quadratic in the variable \(s^2\), be negative or zero for all real values of \(k\), i.e. for all values of \(\beta\) that satisfy \(\beta \geq -\omega_C\). This is equivalent to the following three conditions:

1. The product of the roots is positive or zero
\[
\gamma^4 - 2\alpha\beta\gamma^2 + \alpha^2\beta^2 - \zeta^2\beta^2 \geq 0, \quad \text{for all } \beta \geq -\omega_C. \tag{5.80}
\]

2. Their sum of the roots is negative or zero
\[
\alpha^2 + \beta^2 + 2\gamma^2 - \zeta^2 \geq 0, \quad \text{for all } \beta \geq -\omega_C. \tag{5.81}
\]

3. The discriminant is positive or zero
\[
(\alpha^2 + \beta^2 + 2\gamma^2 - \zeta^2)^2 - 4(\gamma^4 - 2\alpha\beta\gamma^2 + \alpha^2\beta^2 - \zeta^2\beta^2) \geq 0, \quad \text{for all } \beta \geq -\omega_C. \tag{5.82}
\]

Inequality (5.82) is the hardest of the three conditions to analyze. Through algebraic manipulation, it is recast as
\[
(\alpha^2 - \beta^2 - \zeta^2)^2 + 4\gamma^2(\alpha^2 + \beta^2 + 2\alpha\beta - \zeta^2) \geq 0, \quad \text{for all } \beta \geq -\omega_C. \tag{5.83}
\]

The three inequalities together constitute necessary and sufficient conditions for the asymptotic values \(\zeta = 0\) or \(\zeta = \zeta_\infty\) of a soliton solution to be a linearly stable homogeneous solution. We refer to these inequalities below as the first, second, and third stability conditions.

### 5.1 Stability of the far-field solution \(\zeta = 0\).

The left side of each of the three inequalities above is either a perfect square or a sum of squares (see the third inequality in its recast form (5.83)). Thus, they are all satisfied, and so the soliton far-field solutions for bands 2.1 and 2.2 are stable.
5.2 Stability of the far-field solution $\zeta = \zeta_\infty$.

First stability condition. 

For $\zeta = \zeta_\infty$, the left side of the inequality (5.80) factors to 

$$
(\gamma^2 - (\alpha_\infty + \zeta_\infty)\beta) (\gamma^2 - (\alpha_\infty - \zeta_\infty)\beta) \geq 0,
$$

(5.84)
in which $\alpha_\infty$ is the value of $\alpha$ at $\zeta = \zeta_\infty$. The definition of $\eta$ gives directly 

$$
\varpi_X = \frac{\eta \gamma^2}{\varpi_C},
$$

(5.85)
from which one obtains easily 

$$
\alpha_\infty + \zeta_\infty = \frac{\gamma^2}{\varpi_C}(2\eta - 3), \quad \alpha_\infty - \zeta_\infty = -\frac{\gamma^2}{\varpi_C}.
$$

(5.86)

Inserting these into the above inequality and recalling that $\beta = k^2 - \varpi_C$, yields 

$$
\left[ (\eta - \frac{3}{2}) \frac{k^2}{\varpi_C} + \eta - 1 \right] \frac{k^2}{\varpi_C} \geq 0 \quad \text{for all real } k. \tag{5.87}
$$

The sign distribution of the left of the inequality reveals that the inequality is satisfied in exactly two regimes

$$
\begin{cases}
\eta_{\text{min}} \leq \eta \leq 1, & \varpi_C < 0 \\
1 \leq \eta \leq \frac{3}{2}, & \varpi_C > 0.
\end{cases} \tag{5.88}
$$

The homogeneous solutions corresponding to the far-field values of the solitons in bands 1.1 and 3.2 are in the first regime. Those corresponding to band 1.2 are in the second regime. Thus, the far-field values of all these dark solitons pass the first test for linear stability. On the other hand, the homogeneous solutions corresponding to the far-field values of the solitons in band 3.1 are outside these two regimes and therefore are not linearly stable.

The first stability condition (5.80) poses a simple restriction on the homogeneous solutions $\zeta = \zeta_\infty$, as one observes that it is a quadratic inequality in the variable $\beta/\gamma^2$,

$$
(\alpha^2 - \zeta^2) \left( \frac{\beta}{\gamma^2} \right)^2 - 2\alpha \left( \frac{\beta}{\gamma^2} \right) + 1 \geq 0, \quad \text{for all } \beta \geq -\varpi_C. \tag{5.89}
$$

Necessarily, 

$$
\alpha^2 - \zeta^2 \geq 0. \tag{5.90}
$$

This is an interesting inequality. Factoring and recalling that $\alpha = 2\zeta - \varpi_X$, it becomes,

$$
(\zeta - \varpi_X)(\zeta - \frac{1}{3}\varpi_X) \geq 0 \tag{5.91}
$$

Thus, stable homogeneous solutions takes values $\zeta = \zeta_\infty$ that lie outside the open interval between $\varpi_X$ and $\varpi_X/3$. To the right of this interval $\alpha > 0$, while $\alpha < 0$ holds when $\zeta$ is to the left of the interval.
Second stability condition.

Inequality in (5.81) follows immediately from the obtained requirement of the first stability condition, \( \alpha^2 - \zeta^2 \geq 0 \).

Third stability condition.

The far-field solutions for bands 1.1 and 3.2 satisfy the first two stability conditions. We show now that they also satisfy the third condition. It suffices to show that the second term in parentheses (call it \( A \)) in (5.83) is positive or zero. The proof is based on the fact that both solutions have \( \varpi_C < 0 \).

\[
A = (\alpha + \beta)^2 - \zeta^2 = (\alpha + \beta + \zeta)(\alpha + \beta - \zeta). \tag{5.92}
\]

Inserting the expression (5.87) for \( \alpha \pm \beta \), we obtain

\[
A = \left( \frac{\gamma^2}{\varpi_C} (2\eta - 3) + k^2 - \varpi_C \right) \left( -\frac{\gamma^2}{\varpi_C} + k^2 - \varpi_C \right). \tag{5.93}
\]

The term \( 2\eta - 3 \) is positive or zero by (5.88). With \( \varpi_C < 0 \), every term in each of the two parenthesis is positive or zero.

The expression for \( A \) above is quadratic in \( k^2 \) with roots \( -\frac{\gamma^2}{\varpi_C} (2\eta - 3) + \varpi_C \) and \( \frac{\gamma^2}{\varpi_C} + \varpi_C \). For the far-field solutions for band 1.2, necessarily \( \varpi_C > 0 \), and thus both roots are positive. Giving \( k^2 \) a value between these roots makes the quadratic expression negative. The third stability condition is thus violated, so the far-field solution for band 1.2 is unstable. Table 1 gives a summary of linear far-field stability of all solitons.

| Band | Bright/Dark | Linearly stable far field | \( \eta \) domain |
|------|-------------|--------------------------|-------------------|
| 3.1  | Neither     | no                       | \( \frac{5}{2} < \eta < \infty \) |
| 2.1  | Bright      | yes                      | \( 1 < \eta < \frac{9}{5} \) |
| 1.1  | Dark        | yes                      | \( 0 < \eta < 1 \) |
| 3.2  | Dark        | yes                      | \( \eta_{\text{min}} < \eta < 0 \) |
| 2.2  | Bright      | yes                      | \( 0 < \eta < 1 \) |
| 1.2  | Dark        | no                       | \( 1 < \eta < \frac{3}{2} \) |

Acknowledgment. This work was partially supported by the European Union’s FP7-REGPOT-2009-1 project “Archimedes Center for Modeling, Analysis and Computation” (grant agreement n. 245749) and by the US National Science Foundation under grants NSF DMS-0707488 and NSF DMS-1211638. We acknowledge discussions with P. Savvidis, G. Christmann, F. Marchetti, G. Kavoulakis, A. Gorbach.
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