Quasioptical modeling of wave beams with and without mode conversion: I. Basic theory

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This work opens a series of papers where we develop a general quasioptical theory for mode-converting electromagnetic beams in plasma and implement it in a numerical algorithm. Here, the basic theory is introduced. We consider a general quasimonochromatic multi-component wave in a weakly inhomogeneous linear medium with no sources. For any given dispersion operator that governs the wave field, we explicitly calculate the approximate operator that governs the wave envelope \( \psi \) to the second order in the geometrical-optics parameter. Then, we further simplify this envelope operator by assuming that the gradient of \( \psi \) transverse to the local group velocity is much larger than the corresponding parallel gradient. This leads to a parabolic differential equation for \( \psi \) ("quasioptical equation") in the basis of the geometrical-optics polarization vectors. Scalar and mode-converting vector beams are described on the same footing. We also explain how to apply this model to electromagnetic waves in general. In the next papers of this series, we report successful quasioptical modeling of radiofrequency wave beams in magnetized plasma based on this theory.

I. INTRODUCTION

A. General idea

Describing the propagation of waves in inhomogeneous media is a classic problem with a long history [1, 2]. It is particularly important in fusion research, where quasistationary beams of electromagnetic (EM) radiation are commonly used for many purposes and need to be modeled with fidelity [3]. Full-wave modeling, which involves solving the complete Maxwell’s equations, can be impractical at short (cm and mm) wavelengths, especially when multi-dimensional simulations with complex geometries are required, such as those of tokamak and stellarator plasmas. Hence, reduced methods have been widely used in practice. These methods are rooted in geometrical optics (GO) [1] and include conventional ray tracing [3–7], complex ray tracing [8, 9], beam tracing [10–12], and variations of thereof [13]. There are also other quasioptical models, such as in Refs. [14, 14] that resolve the evolution of the beam transverse structure without adopting any particular ansatz for the intensity profile. Still, they assume that only one branch of the dispersion relation is excited in each given case [17]. This impedes applications of quasioptical algorithms to problems involving mode conversion [1, 2], which is simulated today almost exclusively with full-wave codes.

One may wonder whether full-wave codes are really a necessity in modeling mode conversion. The recent development of “extended geometrical optics” (XGO) [15–22] indicates that this may not be the case. XGO is a theory that calculates the leading-order correction \( U \) to the GO dispersion operator of a general vector wave and shows [20] that this correction is analogous to (and a generalization of) the Stern–Gerlach Hamiltonian of a quantum spin-1/2 electron. Accordingly, \( U \) is responsible for two effects simultaneously: (i) it modifies the ray equations just like spin-orbital interactions affect the electron motion, and (ii) it also governs mode conversion, which appears as a direct analog of spin up–down transitions [23]. An examination of the quasioptical algorithms such as those in Refs. [10, 14] shows that they already involve calculations of terms similar to \( U \). Hence, adding the mode-conversion capability to quasioptical codes should not be burdensome and is not expected to slow down the codes considerably. However, formulating the corresponding theory is easier to do using the abstract quantumlike formalism of XGO. One only needs to upgrade the existing XGO by adding diffraction to it, for which it also helps to cast the theory in a covariant form [24].

In this series of papers (Papers I–III), we propose such an upgrade of XGO and apply it to numerical simulations, namely, as follows. In Paper I, we introduce the basic theory of waves that diffract and mode-convert simultaneously. In Papers II and III, we report successful quasioptical modeling of radiofrequency wave beams in magnetized plasma based on this theory. In particular, we consider applications to EM waves in the electron cyclotron range in the Large Helical Device [25, 26], where mode conversion occurs at the plasma edge due to the magnetic shear [25, 27, 28].

B. Outline

In this first paper of our series, we consider an arbitrary quasimonochromatic multi-component wave in a weakly inhomogeneous linear medium. Supposing that \( \hat{D} \) is the dispersion operator governing the wave dynamics, we simplify \( \hat{D} \) in order to obtain an approximate operator that governs the wave envelope \( \psi \). Then, we obtain a parabolic differential equation for \( \psi \) ("quasioptical equation") by assuming that the gradient of the wave envelope
transverse to the local group velocity is much larger than the corresponding parallel gradient. The resulting theory applies to both scalar and mode-converting vector beams. At the end of the paper, we also discuss how this model can be applied to EM waves in particular. However, readers who are mainly interested in simulations as opposed to the general theory are encouraged to proceed straight to Paper II, where our key equations are overviewed in a simplified form and without derivations.

This paper is organized as follows. In Sec. III we introduce the general problem. In Sec. III we formalize the concept of the envelope dispersion operator. In Sec. IV we derive an approximation for the envelope dispersion operator for scalar waves, and we also derive its quasioptical approximation. In Sec. V we extend this model to vector waves. In Sec. VI we explain how to apply the quasioptical modeling will be overviewed in a simplified form and without derivations.

In Sec. VII we present our main conclusions. In Appendix A, we present our main conclusions. In Appendix A, we summarize some of our notations. In Appendix B, some auxiliary calculations are presented. Our paper also contains Supplementary Material [25]. There, we overview the Weyl calculus on a curved configuration space, which is used in this work.

II. GENERAL PROBLEM

Consider a wave propagating on an \( n \)-dimensional configuration space \( M^n \) with coordinates \( x \equiv \{ x^0, x^1, \ldots, x^{n-1} \} \) and some general metric \( g(x) \). For simplicity, assume that \( M^n \) is diffeomorphic to \( \mathbb{R}^n \), i.e., the \( n \)-dimensional Euclidean space or pseudo-Euclidean space with the same metric signature as \( M^n \). (For more information on why the diffeomorphism with \( \mathbb{R}^n \) is needed, see Sec. [V.A.3]). Suppose that the wave field \( \Psi = \Psi(x) \), which may have multiple components, is governed by a linear equation with no source terms,

\[
\hat{D}\Psi = 0, \tag{1}
\]

where \( \hat{D} \) is a differential or, most generally, an integral dispersion operator. (For vector waves, \( \hat{D} \) is a matrix whose elements are operators; see Sec. [V]). We shall assume that the GO parameter \( \epsilon \) is small, namely,

\[
\epsilon \doteq \lambda/L \ll 1 \tag{2}
\]

(the symbol \( \doteq \) denotes definitions), where \( \lambda \) is the characteristic wave period, or wavelength, and \( L \) is the least characteristic scale among those of the wave envelope and of the medium, including the metric \([30]\). Below, we propose a systematic reduction of Eq. (1) using the smallness of \( \epsilon \) and eventually obtain a quasioptical model based on this equation. The idea of quasioptical modeling will be formalized later (Secs. [IV.E] and [V.F]).

III. ENVELOPE DISPERION OPERATOR

As the first step, let us introduce a unitary variable transformation

\[
\Psi = \hat{U}\psi, \quad \hat{U} \doteq e^{i\theta(x)}. \tag{3}
\]

(Other \( \hat{U} \) may also be justified in some cases, e.g., for dealing with caustics or quasiperiodic media [31], but we shall not consider this possibility in the present work.) The phase \( \theta \), which we call the “reference phase”, serves as a gauge potential. It is a real function such that

\[
k(x) \doteq \nabla\theta(x) \tag{4}
\]

is the wave vector identical or close to that predicted by the GO approximation. This implies that the envelope \( \psi \) and also \( k \) are slow functions [and that the wavelength entering Eq. (2) is \( \lambda \approx 2\pi/k \)]. Then, Eq. (1) becomes

\[
\hat{D}\psi = 0, \tag{5}
\]

where the “envelope dispersion operator” is \( \hat{D} \doteq \hat{U}^\dagger\hat{D}\hat{U} \) (the dagger denotes the adjoint, as usual), or more explicitly,

\[
\hat{D} = e^{-i\theta}\hat{D}e^{i\theta}. \tag{6}
\]

The reference phase \( \theta \) is treated as a prescribed function. As will become clear later, knowing \( \theta \) per se is not actually needed for our purposes; instead, it is \( k \) that matters. The latter can be calculated on some “reference rays” using the conventional ray equations [1]

\[
\frac{dx^\alpha}{d\tau} = \frac{\partial H}{\partial k_\alpha}, \quad \frac{dk_\alpha}{d\tau} = -\frac{\partial H}{\partial x^\alpha} \tag{7}
\]

(\( \tau \) is any parameter along the ray), which also lead to

\[
H(x, k(x)) = \text{const}. \tag{8}
\]

Note that, in general, \( \theta \) and \( k \) are defined uniquely only to the leading order, so there exists some freedom in choosing reference rays and their Hamiltonian \( H \). This means that more than one \( \hat{D} \) is possible. Still, envelope equations that (slightly) differ in the choice of \( k \) are equivalent in the sense that the total field \( \Psi \) that they describe is the same by construction. We shall discuss this in more detail in Secs. [V.F] and [V.G].

Also note that our approach equally applies to stationary and nonstationary waves. In the case of a stationary wave, we assume that \( x \) is a coordinate in physical space (“spatial problem”), and the wave frequency \( \omega \) serves as a constant parameter. In the case of a nonstationary wave, we assume that \( x \) is a coordinate in spacetime (“spacetime problem”), and then \( \omega \) is a part of \( k \). In spacetime problems, we assume coordinates such that \( x^0 = ct \), where \( c \) is the speed of light, \( t \) is time, and the metric signature is \((-;+++; \ldots)\); then, \( k_0 = -\omega/c \) (in case of the Minkowski metric, this implies that \( k^0 = \omega/c \)), so \( \omega = -\partial_x\theta \), as usual. In other respects, spatial and spacetime problems are described on the same footing and will be distinguished only in Sec. [VII].

Having defined this terminology, we shall now discuss how \( \hat{D} \) can be expanded in \( \epsilon \) asymptotically for any \( \hat{D} \).
IV. SCALAR WAVES

A. Weyl calculus

1. Basic definitions

Until Sec. V, we shall assume that Ψ is a scalar function. Any given operator \( \hat{A} \) acting on it maps Ψ to a new scalar function \( \hat{A}\Psi \) that can be expressed as follows:

\[
(\hat{A}\Psi)(x) = \int d^n x' \sqrt{|g_n(x')|} A(x,x')\Psi(x').
\]

Here the integral is taken over \( \mathbb{R}^n \) (and so are all integrals below, up to dimension), \( g_n \equiv |\det g| \), and \( A \) is some kernel function that determines \( \hat{A} \). Consider also a family of all unitary operators \( \hat{A}_n \), which is a subset of all possible \( \hat{A} \). For a given \( \Psi \), all image functions \( \hat{A}_n\Psi \) are mutually equivalent up to an isomorphism, so their family \( \{\hat{A}_n\Psi\} \) can be viewed as a single object, a “state vector” \( |\Psi\rangle \), which belongs to a Hilbert space \( \mathcal{H} \), with inner product

\[
\langle\Psi|\Phi\rangle = \int d^n x \sqrt{|g_n(x)|} \Phi^*(x)\Psi(x).
\]

Then, Eq. (9) can be viewed as the “\( x \) representation” of \( \hat{A} \), while the operator itself can be understood more generally as a transformation of \( |\Psi\rangle \), i.e., of the whole family \( \{\hat{A}_n\Psi\} \). Using this invariant notation, one can formulate a machinery, called the Weyl calculus, that allows efficient asymptotic approximation of operators using the smallness of \( \epsilon \). Below, we overview the key theorems of the Weyl calculus that are used in our paper. Readers interested in more details and proofs of these theorems can find them in the Supplementary Material [20].

2. Coordinate and momentum operators

We start by defining the coordinate and momentum (wave-vector) operators

\[
\hat{x} = \{\hat{x}^0, \hat{x}^1, \ldots, \hat{x}^{n-1}\},
\]

\[
\hat{p} = \{\hat{p}_0, \hat{p}_1, \ldots, \hat{p}_{n-1}\}
\]

such that the \( x \) representations of \( \hat{x}^\mu \) and \( \hat{p}_\mu \) be as follows:

\[
\hat{x}^\mu \Psi = x^\mu \Psi,
\]

\[
\hat{p}_\mu \Psi = -ig_\mu^{-1/4} \partial_\mu (g_\mu^{1/4} \Psi).
\]

Here \( \partial_\mu \equiv \partial/\partial x^\mu \), \( x^\mu \) and \( p_\mu \) are the corresponding eigenvalues, and the factors \( g_\mu^{1/4} \) are introduced to keep \( \hat{p} \) self-adjoint under the inner product [10] [33]. (This would not be the case if \( M^n \) were not diffeomorphic to \( \mathbb{R}^n \) [34].) Since

\[
\hat{p}_\nu = -i\partial_\nu - iq_\nu(x), \quad q_\nu \equiv \frac{1}{4} \partial_\nu (\ln g_\nu),
\]

and \( q_\nu \) commutes with \( \hat{x}^\mu \), one arrives at the usual commutation relation [\( \hat{x}^\mu, \hat{p}_\nu \) = \( i\delta_\nu^\mu \)].

Let us consider the eigenvectors \( |x\rangle \) and \( |p\rangle \) of the coordinate and momentum operators, which are defined as

\[
\hat{x} |x\rangle = x |x\rangle, \quad \hat{p} |p\rangle = p |p\rangle.
\]

Since the operators are self-adjoint, these eigenvectors can be chosen as mutually orthogonal, and we shall assume the following normalization:

\[
\langle x_1|x_2 \rangle = \mathcal{G}(x_1, x_2) \delta(x_1 - x_2),
\]

\[
\langle p_1|p_2 \rangle = \mathcal{G}(p_1, p_2) \delta(p_1 - p_2).
\]

Here we introduced

\[
\mathcal{G}(x_1, x_2) = [g_0(x_1)g_0(x_2)]^{-1/4},
\]

\[
\mathcal{G}(p_1, p_2) = [g_0(p_1)g_0(p_2)]^{-1/4}.
\]

The function \( g_0 \) can be chosen arbitrarily as long as it is kept positive. It plays a role similar to that of \( g_0 \) in the Weyl calculus, but note that this is just a normalization factor, and we introduced it only to maintain the symmetry between \( \hat{x} \) and \( \hat{p} \). One can show [20] that our original function \( \Psi \) and the envelope \( \psi \) can be expressed through the corresponding state vectors as

\[
\Psi(x) = \langle x|\Psi\rangle, \quad \psi(x) = \langle x|\psi\rangle.
\]

The \( p \) representations of these state vectors are introduced similarly as \( \langle p|\Psi\rangle \) and \( \langle p|\psi\rangle \). One can also show [20] that \( \langle x_1|\hat{A}|x_2 \rangle = \hat{A}(x_1, x_2) \), and

\[
\langle x|p \rangle = \langle p|x \rangle^* = \frac{\exp(ip\cdot x)}{(2\pi)^{n/2}[g_0(x)g_0(p)]^{1/4}}.
\]

Here, \( p \cdot x \equiv p_\mu x^\mu \), and summation over repeating indices is assumed here and further.

3. Wigner–Weyl transform

The set of all eigenvalues of the coordinate and momentum operators form a 2n-dimensional “phase space” \( z \equiv (x, p) \). For every given \( z \), we introduce the so-called Wigner operator \( \hat{\Delta}_z \), which is self-adjoint and defined as

\[
\hat{\Delta}_z \equiv \int d^n s \ G(x, s) \ |x - s/2 \rangle \langle x + s/2| e^{-ip \cdot s},
\]

\[
G(x, s) = [g_0(x - s/2)g_0(x + s/2)]^{1/4}.
\]

(Note that \( G = 1 \) if the \( x \) space is Euclidean or pseudo-Euclidean.) Using \( \hat{\Delta}_z \), we define the Wigner–Weyl transform \( \mathcal{W}_z : A \mapsto \hat{A} \), which maps a given operator \( \hat{A} \) on \( \mathcal{H} \) to a function \( A \) (“Weyl symbol”) on the \( z \) space. Specifically, the Weyl symbol of a given operator \( \hat{A} \) is \( A(x, p) \equiv \text{tr}(\hat{\Delta}_z \hat{A}) \) (“\( \text{tr} \)” stands for trace); i.e.,

\[
A(x, p) = \int d^n s \ G(x, s) \ (x+s/2|\hat{A}|x-s/2) e^{-ip \cdot s}.
\]
As can be seen easily from this definition, if a given operator is self-adjoint, then its Weyl symbol is real. This also leads to the following corollary. Consider splitting a given \( \hat{A} \) as \( \hat{A} = \hat{A}_H + i\hat{A}_A \), where the subscripts denote the Hermitian and anti-Hermitian parts,

\[
\hat{A}_H = \frac{1}{2} (\hat{A} + \hat{A}^\dagger), \quad \hat{A}_A = \frac{1}{2i} (\hat{A} - \hat{A}^\dagger). \tag{26}
\]

Both \( \hat{A}_H \) and \( \hat{A}_A \) (not to be confused with \( i\hat{A}_A \)) are self-adjoint by definition. Thus, the corresponding Weyl images \( A_H \) and \( A_A \) are real.

We also define the inverse Wigner–Weyl transform \( \mathcal{W}^{-1} : A \mapsto \hat{A} \), which maps a given function \( A \) to the corresponding operator \( \hat{A} \) via

\[
\hat{A} = \frac{1}{(2\pi)^n} \int d^n x \, d^n p \, A(x, p) \hat{\Delta}_z. \tag{27}
\]

The direct and inverse transforms set the “Weyl correspondence” between operators and functions on the \((x, p)\) space, \( A \mapsto A(x, p) \). As can be checked by a direct calculation, for any function \( f \), one has

\[
f(\hat{x}) \iff f(x), \quad f(\hat{p}) \iff f(p). \tag{28}
\]

However, the Weyl symbols of operators that cannot be represented as \( f_1(\hat{x}) + f_2(\hat{p}) \) are generally more complicated. In particular, one can show that \[29\]

\[
f(\hat{x})\hat{p}_\alpha \iff p_\alpha f(x) + \frac{i}{2} \partial_\alpha f(x), \tag{29}
\]

\[
\hat{p}_\alpha f(\hat{x}) \iff p_\alpha f(x) - \frac{i}{2} \partial_\alpha f(x), \tag{30}
\]

and also \[29\]

\[
\hat{p}_\alpha f(\hat{x})\hat{p}_\beta \iff p_\alpha p_\beta f(x) + \frac{1}{4} \partial_\alpha \partial_\beta f(x)
\]

\[
+ \frac{i}{2} p_\alpha \partial_\beta f(x) - \frac{i}{2} p_\beta \partial_\alpha f(x). \tag{31}
\]

Overall, the Weyl symbol of an operator that is any combination \( f(\hat{x}, \hat{p}) \) of \( \hat{x} \) and \( \hat{p} \) approaches \( f(x, p) \) in the GO limit, when \([\hat{\imath}, \hat{p}_\mu] \) is negligible. However, in general, \( f(\hat{x}, \hat{p}) \) does not map simply to \( f(x, p) \).

B. Approximate \( \hat{D} \) in the invariant form

Using the notation introduced in Sec. IV A one can express Eq. (1) in the following invariant form:

\[
\hat{D} |\Psi\rangle = 0. \tag{32}
\]

Accordingly, Eq. (6) becomes

\[
\hat{D} |\psi\rangle = 0, \tag{33}
\]

where the invariant form of the envelope dispersion operator [Eq. (10)] is as follows:

\[
\hat{D} = e^{-i\theta(\hat{x})} \hat{D} e^{i\theta(\hat{x})}. \tag{34}
\]

We shall now approximate \( \hat{D} \) in three steps: (i) we map the right-hand side of Eq. (34) onto a function using the Wigner–Weyl transform; (ii) we approximate that function using the smallness of \( \epsilon \) [Eq. (2)]; and (iii) we produce an operator out of the approximated function using the inverse Wigner–Weyl transform.

Let us start by expressing \( \hat{D} \) through its Weyl symbol \( D \), which is a function of \((x, p)\):

\[
\hat{D} = \frac{1}{(2\pi)^n} \int d^n x \, d^n p \, d^n s \, G(x, s) \, |x - s/2| \, e^{-ip \cdot s} \, D(x, p) \, (x + s/2), \tag{35}
\]

where \( G \) is a metric factor given by Eq. (24). Using the fact that \( \theta(\hat{x}) \, |x \pm s/2| = \theta(x \pm s/2) \, |x \pm s/2| \), one obtains from Eq. (35) that

\[
\hat{D} = \frac{1}{(2\pi)^n} \int d^n x \, d^n p \, d^n s \, G(x, s) \, e^{i[\theta(x - s/2) - \theta(x - s/2) - s \cdot p]} \, |x - s/2| \, D(x, p) \, (x + s/2). \tag{36}
\]

Consider a formal Taylor expansion of the reference phase \( \theta \) in \( s \):

\[
\theta(x \pm s/2) = \theta(x) \pm \frac{s^\alpha}{2} \frac{\partial \theta(x)}{\partial x^\alpha} + \frac{s^\alpha s^\beta}{2} \frac{\partial^2 \theta(x)}{\partial x^\alpha \partial x^\beta} \pm \frac{1}{6} s^\alpha s^\beta s^\gamma \frac{\partial^3 \theta(x)}{\partial x^\alpha \partial x^\beta \partial x^\gamma} + \ldots \tag{37}
\]

This gives

\[
\theta(x + s/2) - \theta(x - s/2) = s^\alpha \frac{\partial \theta(x)}{\partial x^\alpha} + \frac{s^\alpha s^\beta s^\gamma}{24} \frac{\partial^3 \theta(x)}{\partial x^\alpha \partial x^\beta \partial x^\gamma} + \ldots = k(x) \cdot s + \frac{s^\alpha s^\beta s^\gamma}{24} \frac{\partial^3 k(x)}{\partial x^\alpha \partial x^\beta \partial x^\gamma} + \ldots, \tag{38}
\]

where \( k(x) \) is a function.
where we integrated by parts and introduced an "effective dispersion function" \( \Theta \). This leads to the following expression for the envelope dispersion operator:

\[
\hat{D} = \frac{1}{(2\pi)^n} \int d^n x d^n p d^n s \, G(x, s) \left\{ \left[ 1 + \frac{1}{24} \frac{\partial^2 k_\gamma(x)}{\partial p_\rho \partial p_\sigma} + \ldots \right] e^{i[k(x) - p]s} \right\} \langle x - s/2 \rangle D(x, p) \langle x + s/2 \rangle.
\]

The first term on the right-hand side of Eq. (41) is \( O(1) \) and becomes the GO dispersion function at \( \epsilon \to 0 \). At nonzero \( \epsilon \), the Weyl symbol \( D \) generally does not coincide with the dispersion function that governs the wave in a homogeneous medium; see Sec. [VA3]. The second term is \( O(\epsilon^2) \) and thus must be kept in general; however, it is often negligible, as discussed in Sec. [VII] for EM waves. Higher-order terms can also be calculated similarly if needed.

Note that the inverse Wigner–Weyl transform will turn the coordinate \( p \) into the operator \( \hat{p} \). The latter will act on the wave envelope \( \langle \psi \rangle \), which is considered slow in the coordinate representation, so \( \hat{p} \langle \psi \rangle = \langle \epsilon \rangle \). In this sense, \( p = O(\epsilon) \), so by Taylor-expanding \( D \) in \( p \), we effectively expanding \( \hat{D} \) in \( \epsilon \). It is sufficient for our purposes to adopt the second-order expansion,

\[
\hat{D} \approx \frac{1}{(2\pi)^n} \int d^n x d^n p d^n s \, G(x, s) e^{-ip\epsilon} \left\{ \right. \left[ 1 + \frac{1}{24} \frac{\partial^2 k_\gamma(x)}{\partial p^\rho \partial p^\sigma} + \ldots \right] D(x, p) \left\{ \right. \langle x - s/2 \rangle D(x, p) \langle x + s/2 \rangle.
\]

where we introduced \( D(x) \equiv D(x, k(x)) \) and

\[
\nu^\mu(x) = \frac{\partial D(x, k(x))}{\partial k^\mu}, \quad \Theta^{\mu\nu}(x) = \frac{\partial^2 D(x, k(x))}{\partial k^\mu \partial k^\nu}.
\]

By properties of the inverse Wigner–Weyl transform \( \mathcal{W}^{-1} \) (Sec. [VA3]), \( \hat{D} \) can be written as follows:

\[
\hat{D} \approx \mathcal{W}^{-1}[D(x)] + \mathcal{W}^{-1}[p_\mu \nu^\mu(x)] + \frac{1}{2} \mathcal{W}^{-1}[p_\mu p_\nu \Theta^{\mu\nu}(x)].
\]

The inverse Wigner–Weyl transforms here can be calculated using Eqs. (28)-(31) and \( \Theta^{\mu\nu} = \Theta^{\mu\nu} \). This leads to

\[
\hat{D} \approx D(\hat{x}) + \frac{1}{2} \left[ \hat{p}_\mu \nu^\mu(\hat{x}) + \nu^\mu(\hat{x}) \hat{p}_\mu \right] + \frac{1}{2} \left[ \hat{p}_\mu \Theta^{\mu\nu}(\hat{x}) \right] \hat{p}_\nu - \frac{1}{8} \left( \partial^2_{\mu\nu} \Theta^{\mu\nu}(\hat{x}) \right).
\]

As a reminder, \( \hat{D} \) is the approximate operator govern ing the dynamics of the wave envelope \( \psi \) [Eq. (6)], and it is accurate up to the second order in the GO parameter \( \epsilon \). (Below, we shall not emphasize the approximate nature of this equality, so \( \approx \) will be replaced with \( = \).) In particular, note that if \( \hat{D} \) is self-adjoint, then \( D \) is real (Sec. [VA3]), and thus \( \hat{D} \) is self-adjoint too.

C. Approximate envelope equation

In order to obtain an explicit form of Eq. (6), let us also introduce the \( x \) representation of \( \hat{D} \). From the \( x \) representation of \( \hat{D} \) and \( \hat{p} \) given by Eqs. (14) and (13), terms like \( \hat{p} f(\hat{x}) \) (for any \( f \)) must be interpreted as

\[
[\hat{p}_\mu f(\hat{x})] \psi = -ig_0^{-1/4} \partial_\mu [g_0^{1/4} f(\hat{x}) \psi],
\]

where \( g \) and \( \psi \) are also functions of \( x \). Then,

\[
\hat{D} \psi = D \psi - i \nu^\mu \partial_\mu \psi - \frac{i}{2g_0} \partial_\mu (\sqrt{g_0} \nu^\mu \psi) - \frac{1}{2g_0^{1/4}} \partial_\mu [\Theta^{\mu\nu}(g_0^{1/4} \psi)] - \frac{1}{8} (\partial^2_{\mu\nu} \Theta^{\mu\nu}) \psi,
\]

where
so the Hermitian and anti-Hermitian parts of $\hat{D}$ are

$$\hat{D}_H \psi = D_H \psi - iV^\mu \psi,_{\mu} - \frac{i}{2} V^{\mu \nu} \psi_{,\mu}$$

$$- \frac{1}{2g_{\mu \nu}} \left[ D_{H \mu} (g_{\nu \psi})_{,\nu},_{\mu} - \frac{1}{8} (D_{H \mu} )_{,\mu} \psi, \right] (49)$$

$$\hat{D}_A \psi = D_A \psi - iV^\mu \psi,_{\mu} + \frac{i}{2} V^{\mu \nu} \psi_{,\mu}$$

$$- \frac{1}{2g_{\mu \nu}} \left[ D_{A \mu} (g_{\nu \psi})_{,\nu},_{\mu} - \frac{1}{8} (D_{A \mu} )_{,\mu} \psi, \right] (50)$$

Here,

$$D_H = \text{Re} \, D, \quad D_A = \text{Im} \, D, \quad V^\mu = V^\mu_H \equiv D^\mu_H,$$  

and $\psi,_{\mu}$ denotes the covariant derivative with respect to $x^\mu$, so $X^\mu,_{\mu}$ (for any given $X^\mu$) is the divergence; namely,

$$X^\mu,_{\mu} = \frac{1}{\sqrt{g}} \sqrt{g} X^\mu,_{\mu}. \quad (52)$$

We have also assumed the standard notation

$$f^\mu = \frac{\partial f}{\partial x^\mu}, \quad f_{,\mu} = \frac{\partial^2 f}{\partial x^\mu \partial x^\nu}. \quad (53)$$

Note that $\psi,_{\mu}$ is the “full” derivative in the sense that, for any given $f(x, k(x))$, it applies both to the first and the second argument of $f$; namely, the chain rule leads to

$$f_{,\mu}(x, k(x)) = f_{\mu} + f^{\nu} k_{,\nu |\mu}. \quad (54)$$

The true partial derivatives are introduced as follows:

$$f_{,\mu} = \frac{\partial f(x, k)}{\partial x^\mu}, \quad f^{\mu \nu} = \frac{\partial^2 f(x, k)}{\partial x^\mu \partial x^\nu}, \quad (55)$$

$$f^{\mu \nu} = \frac{\partial f(x, k)}{\partial k_{\mu}}, \quad f^\mu_{,\nu} = \frac{\partial^2 f(x, k)}{\partial k_{\mu} \partial k_{\nu}}. \quad (56)$$

The derivatives $\psi,_{\mu}$ and $\psi_{,\mu}$ are equivalent for functions that depend only on $x$. In particular, since $k_{\nu} = \partial_{\nu} \theta(x)$, one has

$$k_{\nu,\mu} = k_{\nu,\mu} = k_{\mu,\nu} = \theta,_{\mu,\nu} = \theta,_{\nu,\mu}. \quad (57)$$

As a side remark, note that the quantities introduced with an upper index, such as $V^\mu$, are vectors and belong to the space tangent to $M^\mu$. (Such vectors must not be confused with multi-component fields denoted with Latin indices in Sec. 111) Likewise, the quantities introduced with a lower index, such as $k_{\mu}$, are covectors and belong to the space cotangent to $M^\mu$. Vectors and covectors are the same only if the metric is Euclidean. In the case of a general metric $g_{\mu \nu}$ (and its inverse $g^{\mu \nu}$), one has

$$X^\mu = g_{\mu \nu} X^\nu, \quad X^\mu = g^{\mu \nu} X^\nu. \quad (58)$$

In Appendix A we have included a summary of our notations involving index manipulations, various derivatives, and inner products.

D. Leading-order approximation: geometrical optics

For the envelope approximation to hold, i.e., for $\psi(x)$ to remain a slow function, $k(x)$ must be chosen such that $D(x, k(x)) \psi \lesssim O(\epsilon)$. [The other terms in Eq. (54) are automatically small on the account of Eq. (2).] Let us adopt the usual GO ordering

$$D_H = O(1), \quad D_A = O(\epsilon), \quad (59)$$

or more rigorously, $D_A \lesssim O(\epsilon)$; i.e., $D_A$ much smaller than $O(\epsilon)$ is also allowed. Then, one can define $k(x)$ such that

$$D_H(x, k(x)) = 0. \quad (60)$$

This can be achieved by calculating $k$ as discussed in Sec. 111 with the ray Hamiltonian $H = D_H$ and the initial condition [at any chosen $x^{(0)}$ and $k^{(0)}$] such that

$$D_H(x^{(0)}, k^{(0)}) = 0. \quad (61)$$

To the first order in $\epsilon$, Eq. (55) is

$$V^\mu \psi,_{\mu} + \frac{i}{2} V^{\mu \nu} \psi_{,\mu} = D_A \psi, \quad (62)$$

and as a corollary,

$$\mathcal{J}^{\mu \nu} = -2D_A |\psi|^2, \quad \mathcal{J}^{\mu} = -V^\mu |\psi|^2. \quad (63)$$

At zero $D_A$, the dynamics is conservative, and Eq. (63) reflects conservation of the wave action, or quanta [32]. Accordingly, $D_A$ determines the dissipation rate, $J^\mu$ can be identified (at least, up to a constant factor) as the action flux density, and $V^\mu$ is proportional to the group velocity (in space or in spacetime, depending on the problem). Equations (60)–63 coincide with the equations of the traditional GO theory [1, 35]. Below, we extend them by retaining the second-order terms neglected in Eq. (52).

E. Quasioptical model

1. Ray-based coordinates

Let us start by introducing ray-based coordinates as follows [35]. Suppose multiple rays launched with different $x^{(0)}$ within the beam. Their trajectories can be found using the ray equations as discussed in Sec. 111 this determines the path $\zeta$ along each ray as a function of the initial coordinate. We treat $\zeta$ as the longitudinal coordinate along the wave beam, and each of its isosurfaces is considered as the transverse space $M^{\mu=1}_n(\zeta)$. The coordinates on this space, $\mathbf{e} \equiv \{\theta^1, \theta^2, \ldots, \theta^{n-1}\}$, can be introduced arbitrarily (yet they will be specified below). Then, we define $n - 1$ independent vector fields $\mathbf{e}_\sigma$ via

$$\mathbf{e}_\sigma = \partial x / \partial \theta^\sigma. \quad (64)$$
and another vector $e_0$ such that it is linearly-independent from all $e_\alpha$. Hence, any $dx$ can be decomposed as follows:

$$dx = e_0 d\zeta + e_\alpha d\varphi^\alpha,$$  \hspace{1cm} (65)

and summation over repeating indices $\sigma$ (and $\tilde{\sigma}$) is henceforth assumed from 1 to $n-1$.

Let us also construct the dual basis $\{e^\alpha\}$. [We treat vectors and covectors on the same footing; namely, $X \cdot Y = X^\mu Y_\mu$, and $X = Y$ equally means $X^\mu = Y^\mu$ and $X_\mu = Y_\mu$.] By definition,

$$e^\mu \cdot e_\nu = \delta^\mu_\nu,$$  \hspace{1cm} (66)

so we adopt $e^0 \equiv \nabla \zeta$. Then, the general form of $e_0$ is

$$e_0 = \pm \alpha^2 e^0 + \beta,$$  \hspace{1cm} (67)

where the first sign is determined by the metric signature, $\alpha \equiv |e^0 \cdot e^0|^{-1/2}$, $\beta \equiv e_\alpha \beta^\alpha$, and $\beta^\alpha$ are arbitrary coefficients. This leads to the following metric representation in the coordinates $\{\zeta, \varphi\}$:

$$g = \left( \begin{array}{cc} \alpha^2 + \beta^T h \beta & (h \beta)^T \\ h \beta & h \end{array} \right),$$  \hspace{1cm} (68)

Here, $h$ is the matrix with elements $h_{\alpha\beta} \equiv e_\alpha \cdot e_\beta$, and $^T$ denotes transposition. By using a known theorem for the determinant of a block matrix, one obtains

$$g_0 = \alpha^2 h_0, \quad h_0 \equiv |\det h|.$$  \hspace{1cm} (69)

Let us also introduce the transverse projection of $d\varphi$,

$$d\varphi_\perp = (\mathbb{1} \pm \alpha^2 e^0 e^0) d\varphi, \quad d\varphi^\perp_\alpha = d\varphi^\alpha + \beta^\alpha d\zeta,$$  \hspace{1cm} (70)

where $\mathbb{1}$ is a unit matrix and $e^0 e^0$ is a dyad formed out of $e^0$. Then, one obtains from Eq. (63) that

$$dx \cdot dx = \pm \alpha^2 (d\zeta)^2 + h_{\sigma\beta} d\varphi^\sigma_\perp d\varphi^\beta_\perp,$$  \hspace{1cm} (71)

so $h_{\sigma\beta}$ serves as the transverse metric. Below, we assume $\beta = 0$, so $\varphi^\perp_\alpha$ and $\varphi^\perp\sigma$ do not need to be distinguished and we can define the inner product on $M_\perp^{n-1}$ as follows:

$$\langle \psi|\varphi \rangle \perp = \int d^{n-1} \varphi \sqrt{g_\perp(\zeta, \varphi)} \psi^*(\zeta, \varphi) \phi(\zeta, \varphi).$$  \hspace{1cm} (72)

We also choose transverse coordinates specifically such that $\varphi = \text{const}$. In other words, the coordinate $\varphi$ of a given point in space is the initial location of the ray that arrives at this point from the initial transverse surface $M_\perp^{n-1}(0)$. Then, the transverse group velocity is zero,

$$V^\perp \equiv e^\perp \cdot \frac{dx}{d\tau} = \frac{d\varphi^\perp}{d\tau} = 0,$$  \hspace{1cm} (73)

where we used Eqs. (65) and (66). This simplifies our equations below. A different definition of $\varphi$ is assumed in Papers II and III, leading to straightforward modifications of the equations, which are not discussed below.

### 2. Quasioptical equation

Suppose a wave beam such that its longitudinal scale $L_{||}$, which is defined via $\psi_\zeta \sim \psi/L_{||}$, is much larger than its perpendicular scale $L_\perp$, which is defined via $\psi_\varphi \sim \psi/L_\perp$ and may or may not be the beam global width. (We assume the notation $\psi_\zeta \equiv \partial \psi/\partial \zeta$ and $\psi_\varphi \equiv \partial \psi/\partial \varphi^\perp$. This typically implies that the characteristic scale of the medium, the metric included, is of order $L_{||}$ or even larger. Hence, we introduce two GO parameters,

$$\epsilon_{||} \equiv \lambda/L_{||}, \quad \epsilon_\perp \equiv \lambda/L_\perp, \quad \epsilon_{||} \ll \epsilon_\perp$$  \hspace{1cm} (74)

[so the original parameter $\epsilon$ is $\epsilon = \epsilon_\perp$, and we shall neglect terms smaller than $O(\epsilon_\parallel \epsilon_\perp)$ from now on. Also, we require that either $\alpha$ is chosen as independent of $\varphi$ or, more generally, the relative variation of $\alpha$ across the wave beam is small enough such that $\alpha_\sigma$ is negligible. Under these assumptions, Eq. (49) can be approximated as follows [assuming also Eq. (63)]:

$$\tilde{D}_H\psi = -iV\psi_\zeta - \frac{i}{2\sqrt{g_\perp}} (\sqrt{g_\perp} V, \zeta) \psi + \tilde{G}\psi.$$  \hspace{1cm} (75)

Here, $V \equiv V^0$, and we used that $V^\mu \psi_\mu = V \psi_\zeta$ due to Eq. (73). The operator $\tilde{G}$ is self-adjoint under the inner product (72) and given by

$$\tilde{G}\psi \equiv -\frac{1}{2h_0} [D_\sigma^\parallel (h_0^{1/4} \psi_\zeta), \sigma].$$  \hspace{1cm} (76)

The last term in Eq. (49) is neglected as it is of order $\epsilon_\perp^2$. Also, only the transverse derivatives are kept in the second-to-last term in Eq. (49), which is of order $\epsilon_\perp^2$.

Similarly, Eq. (50) becomes

$$\tilde{D}_A\psi = D_A - iV_A^\sigma \psi_\sigma - \frac{i}{2\sqrt{h_0}} (\sqrt{h_0} V_A, \sigma) \psi,$$  \hspace{1cm} (77)

where the higher-order terms are neglected because $D_A$ is assumed small [Eq. (51)]. Although the last term in Eq. (77) is negligible within the assumed accuracy, it is retained anyway to make $\tilde{D}_A$ (not to be confused with $i\tilde{D}_A$) exactly self-adjoint under the inner product (72). Then, Eq. (49) becomes

$$iV\psi_\zeta + \frac{i}{2\sqrt{g_\perp}} (\sqrt{g_\perp} V, \zeta) \psi - \tilde{G}\psi = i\tilde{D}_A\psi,$$  \hspace{1cm} (78)

and, as a corollary,

$$\frac{1}{\alpha} \frac{d}{d\zeta} \langle \psi|\alpha V|\psi \rangle \perp = 2 \langle \psi|\tilde{D}_A|\psi \rangle \perp,$$  \hspace{1cm} (79)

where we used the fact that $\tilde{G}$ and $\tilde{D}_A$ are self-adjoint. In particular, for zero $D_A$, Eq. (79) predicts conservation of the wave-action flux through the beam cross section,

$$\langle \psi|\alpha V|\psi \rangle \perp = \text{const}.$$  \hspace{1cm} (80)
Equation (79) indicates that $D_A$ cannot be much larger than $\partial/\partial \zeta$, so the correct scaling to assume for the dissipation term is $D_A \lesssim O(\epsilon_j)$. Then, Eq. (78) leads to the “quasioptical scaling”
\[ \epsilon_\parallel \sim \epsilon_j^2. \]
(81)

Hence, the difference between $D_A$ and $D_A$ in Eq. (77) is $\lesssim O(\epsilon^3)$, so it is beyond the accuracy of our theory, which is $O(\epsilon^2)$. For this reason, we henceforth adopt
\[ \hat{D}_A = D_A, \]
(82)
which will also simplify our notation. [Also, as a reminder, $\partial \mu \phi^\alpha = O(\epsilon_\parallel) \sim \epsilon_j^2$, and the parameters of the medium vary similarly.] This leads to
\[ iV\psi,\zeta + \frac{i}{2\sqrt{g_\phi}}(\sqrt{g_\phi}V)\psi,\zeta - \hat{\mathcal{G}}\psi = iD_A\psi. \]
(83)

Equation (83) is a general quasioptical equation for a scalar-wave beam in an inhomogeneous medium. Since it is a parabolic equation (it contains only the first-order derivative with respect to $\zeta$), Eq. (83) is much easier to solve than Eq. (5) with the original expressions for $\hat{D}_A$ and $D_A$ given by Eqs. (49) and (50).

### 3. Simplified equations

For any $f$, one has
\[ \frac{1}{\sqrt{g_\phi}}(\sqrt{g_\phi}f)\zeta = f,\zeta + \frac{f}{2}(\ln g_\phi)\zeta. \]
(84)

If the assumed metric is close to Euclidean (or pseudo-Euclidean), then $g_\phi \approx 1$, so $\ln g_\phi$ is small. Furthermore, $(\ln g_\phi)\zeta$ is even smaller; namely, $(\ln g_\phi)\zeta = \alpha(\epsilon_j)$. Likewise, derivatives of $g_\phi$ are negligible in Eq. (76). Hence, the metric factors in the quasioptical equation can be replaced with unity. Then, Eq. (83) becomes
\[ iV\psi,\zeta + \frac{i}{2}V\psi,\zeta + \frac{1}{2}(D_H^{[\xi]}\psi,\zeta)\sigma = iD_A\psi. \]
(85)

Using the variable transformation $\phi = \sqrt{V}\psi$, this equation can be further simplified as follows:
\[ i\phi,\zeta + \frac{1}{2}(\Sigma^{[\xi]}\phi,\zeta)\sigma = i\hat{\mathcal{T}}\phi, \]
(86)

which is just a dissipative Schrödinger equation with
\[ \Sigma^{[\xi]} = \pi^{[\xi]} / V, \quad \hat{\mathcal{T}} = D_A / V. \]
(87)

[In Eq. (86), we used that $V\phi,\sigma^\alpha$ is negligible compared to $V\phi,\sigma^\alpha = O(\epsilon_\parallel)$.] Also, Eq. (79) becomes
\[ \frac{d}{d\zeta} (\phi(\phi)\zeta) = 2 (\phi]\hat{\mathcal{Y}}|\phi)\zeta. \]
(88)

Note that $\Sigma$ can also be expressed alternatively as follows. Suppose one finds the group velocity and constructs a ray-based metric (as described in Sec. IV E 1) in the vicinity of a given $x$. Then, Eq. (64) can be considered as an equation for the local $k_\sigma$ as a function of the transverse wave-vector components $k_\sigma$ (and $x$) in this prescribed metric; i.e.,
\[ D_H(x, k_\sigma, k_0(x, k_\sigma)) = 0. \]
(89)

By differentiating this with respect to $k_\sigma$, we obtain
\[ k_0|k_\sigma = - \frac{1}{D_H} \left( D_H^{[\xi]} k_0|k_0 \sigma + D_H^{[\xi]} k_0 |k_0 \sigma \right) \]
(80)

Recall that $D_H^\sigma = V^\sigma$ and $V^\sigma = 0$ [Eq. (73)]. Then, $k_0|k_0 = 0$, and by comparing Eq. (80) with Eq. (84) and using Eq. (81), one finds that $k_0|k_\sigma = - \Sigma|k_\sigma$. Thus, Eq. (86) can be rewritten as
\[ i\phi,\zeta - \frac{1}{2} (k_0|k_\sigma \phi,\sigma) = i\hat{\mathcal{T}}\phi. \]
(91)

In the case of a spacetime problem, $k_0|k_\sigma$ can also be expressed through the wave frequency in the laboratory frame. Then, the familiar Schrödinger equation for the wave envelope $\psi$ can be reproduced. We do not discuss it further for this subject is not essential to our paper.

### V. VECTOR WAVES

#### A. Hilbert space for vector waves

Now, let us generalize the above results to vector waves. Suppose an $m$-component wave field $\Psi \equiv \Psi(x)$, so
\[ \Psi = \begin{pmatrix} \Psi^1 \\ \Psi^2 \\ \vdots \\ \Psi^m \end{pmatrix}, \quad |\Psi \rangle = \begin{pmatrix} |\Psi^1 \rangle \\ |\Psi^2 \rangle \\ \vdots \\ |\Psi^m \rangle \end{pmatrix}. \]
(92)

Here, $|\Psi \rangle$ is a vector on some Hilbert space $\mathcal{H}_m$, for which we assume the following general inner product:
\[ \langle \Psi|\Phi \rangle_m = \int d^m x \sqrt{g_\phi(x)} \gamma_{ab}(x) \Psi^a(x) \Phi^b(x). \]
(93)

Here, the Latin indices that characterize the field components span from 1 to $m$. (This is in contrast with the Greek indices introduced earlier, which characterize the components of $x$ and span from 0 to $n - 1$.) The matrix $\gamma(x)$ can be any $m \times m$ symmetric matrix. It can be considered as an additional metric. This metric does not have to be the same as $g$, as seen already from the fact that $m$ and $n$ do not have to be the same. (For example,
Ref. [21] deals with a six-dimensional field on a three-dimensional spacetime; also see Sec. VI B.) This means that the space to which \( \Psi(x) \) belongs is not necessarily the space tangent to \( M^n \). That is said, having \( g = g \) is possible as a special case (Secs. VI A and VI C). Using \( \gamma \) as a metric, we introduce the standard rules for manipulating the Latin indices,

\[
\Psi_a = \gamma_{ab} \Psi^b, \quad \Psi^a = \gamma^{ab} \Psi_b,
\]

where \( \gamma^{ab} \) are elements of \( \gamma^{-1} \). In particular, in the \( x \) representation, one has

\[
\Psi^\dagger(x) = (\Psi^\dagger_1, \Psi^\dagger_2, \ldots, \Psi^\dagger_m),
\]

and we shall also use the following local dot product:

\[
\Psi \cdot \Phi = \Psi^\dagger(x) \Phi(x) = \Psi^a_0(x) \Phi^a_0(x).
\]

Under this dot product, a matrix \( A \) with mixed-index elements \( A^a_b \) is self-adjoint \([\langle A \Psi \rangle \cdot \Phi = \Psi \cdot (A \Phi)]\) if the matrix with the corresponding lower-index elements \( A_{ab} = \gamma_{ac} A^c_b \) is Hermitian (\( A_{ab} = A_{ba}^\dagger \)). The difference between Hermitian and anti-Hermitian matrices can be ignored if the metric \( \gamma \) is Euclidean or pseudo-Euclidean.

### B. Operators on \( H^n_m \)

Any operator \( \hat{A} \) on \( H^n_m \) can be understood as an \( m \times m \) matrix with elements \( A^a_b \) which are operators on \( H^n_1 \). For any given \( A^a_b \), we also introduce \( A_{ab}^\dagger = \gamma_{ac} A^c_b \), which is also an operator on \( H^n_1 \). Correspondingly, two types of Weyl images can be introduced, namely, \( A^a_b \equiv \hat{W}_z[A^a_b] \) and \( A_{ab}^\dagger = \hat{W}_z[\hat{A}_{ab}] \). In general,

\[
A^a_b(x, p) \neq \gamma^{ac} A_{cb}(x, p),
\]

but \( p \)-independent matrices are an exception,

\[
A^a_b(x) = \gamma^{ac} A_{cb}(x).
\]

By Eq. (38), the following equality holds for any \( \Phi, \Psi \), and \( \hat{A} \) on \( H^n_m \):

\[
\langle \Psi \vert \hat{A} \Phi \rangle_m = \int d^n x \sqrt{g_0} \gamma_{ab} \Psi^a (\hat{A}^b_c \Phi^c)
\]

\[
= \int d^n x \sqrt{g_0} \Psi^a (\hat{A}_c^b \Phi^c)
\]

\[
= \langle \Psi \vert \hat{A}_c^b \Phi^c \rangle_m,
\]

where we invoked the definition of the inner product on \( H^n_1 \) [Eq. (11)]. By definition of the adjoint operator \( \hat{A}^\dagger \),

\[
\langle \Psi \vert \hat{A}^\dagger \Phi \rangle_m = (\hat{A}^\dagger \Phi)^c_a \Phi^a = \int d^n x \sqrt{g_0} \gamma_{cb} (\hat{A}^\dagger)^c_a \Phi^a \Phi^b
\]

\[
= \int d^n x \sqrt{g_0} \gamma_{cb} (\hat{A}^\dagger_{ba} \Phi^a) \Phi^b.
\]

In order to express \( \hat{A}^\dagger \) through \( \hat{A} \), let us represent \( \hat{A}_{ab}^\dagger \) for the corresponding Weyl image \( \hat{A}_{ab} \) and the Wigner operator \( \hat{A}_{ab} \) on \( H^n_1 \), with \( z = (x, p) \) (Sec. IV A 3). Then, Eq. (100) leads to

\[
\langle \Psi \vert \hat{A} \Phi \rangle_m = \int \frac{d^2 n z'}{(2\pi)^n} \hat{A}_{ab}(z') \langle \Psi^a \vert \hat{A}_{ba} \Phi^b \rangle
\]

\[
= \int \frac{d^2 n z'}{(2\pi)^n} \hat{A}_{ab}(z') \langle \Psi^a \vert \hat{A}_{ba} \Phi^b \rangle,
\]

where we used that \( \hat{A}_{ba} \) is self-adjoint on \( H^n_1 \). By comparing this with Eq. (100), one finds that, on one hand,

\[
(\hat{A}^\dagger)_{ba} = \int \frac{d^2 n z'}{(2\pi)^n} \hat{A}_{ab}(z') \hat{A}_{ba}.
\]

On the other hand, by Eq. (27) for the inverse Wigner–Weyl transform, one has

\[
(\hat{A}^\dagger)_{ba} = \int \frac{d^2 n z'}{(2\pi)^n} \hat{W}_z[\hat{A}^\dagger_{ab}] \hat{A}_{ba}.
\]

Hence, the Weyl image of \( \hat{A}^\dagger \) is simply the conjugate transpose of the Weyl image of \( \hat{A} \) in the sense that

\[
\hat{W}_z[\hat{A}^\dagger_{ab}] = \hat{W}_z[\hat{A}_{ab}]^\dagger.
\]

In other words, for operators with both indices lowered, the operations \( \hat{W}_z \) and \( \dagger \) commute.

Equation (114) implies that, if \( \hat{A} \) is self-adjoint on \( H^n_m \), then the Weyl image \( \hat{A} \) is a Hermitian matrix, and thus the corresponding matrix \( A \) is self-adjoint (Sec. IV A). In particular, this means that the Hermitian and anti-Hermitian parts [Eq. (26)] of an operator are determined by, respectively, the Hermitian and anti-Hermitian parts of its lower-index Weyl symbol. This also applies to any matrix function \( A(x) \) that has elements with mixed indices, \( A^a_b(x) \), because it can be considered as an operator too, namely, \( A = A(x) \). Clearly then, the Hermitian and anti-Hermitian parts of such \( A \) are the self-adjoint and anti-self-adjoint parts of \( A(x) \). Hence, one obtains

\[
(A_H)_{ab} = \frac{1}{2} \gamma^{ac} (A^c_{ab} + A^c_{ba}),
\]

\[
(A_A)_{ab} = \frac{1}{2} \gamma^{ac} (A^c_{ab} - A^c_{ba}).
\]

For lower-index matrices, the Hermitian and anti-Hermitian parts are defined as usual,

\[
(A_H)_{ab} = \frac{1}{2} (A_{ab} + A^\dagger_{ba}),
\]

\[
(A_A)_{ab} = \frac{1}{2} (A_{ab} - A^\dagger_{ba}).
\]

### C. Weyl expansion of the dispersion operator

Now, let us consider the dispersion operator \( \hat{D} \) in particular. According to the above definition, it is viewed
as an \( m \times m \) matrix with elements \( \hat{D}^a_b \). By lowering the index in the envelope equation (10), one can write this equation as follows:

\[
\hat{\cal D}_{ab}\psi^b = 0, \quad a = 1, 2, \ldots, m,
\]

where \( \hat{\cal D}_{ab} = \gamma_{ac} \hat{D}^c_b \). Equivalently, this can be written as the envelope equation

\[
\hat{\cal D}_{ab}\psi^b = 0, \quad \hat{\cal D}_{ab} = e^{-i\theta} \hat{D}_{ab} e^{i\theta}.
\]

By analogy with \( \mathcal{D} \) [Eq. (11)], we introduce the effective dispersion tensor

\[
\mathcal{D}_{ab}(x, p) \doteq \hat{\cal D}_{ab}(x, p) - \frac{1}{24} \frac{\partial^2 k_\nu(x)}{\partial x^\alpha \partial x^\beta} \frac{\partial p_\alpha \partial p_\beta \partial p_\gamma + \ldots}{\partial p_\gamma},
\]

where \( \hat{\cal D}_{ab} = \mathcal{W}[\hat{\cal D}_{ab}] \) are the corresponding Weyl symbols. Then, the approximate operators \( \hat{\cal D}_{ab} \) can be expressed just like \( \hat{D} \) for scalar waves (Sec. IVB3).

\[
\hat{\cal D}_{ab} \approx \mathcal{D}_{ab} + \frac{1}{2} \left[ \hat{p}_\mu \left( \mathcal{D}^\mu \right)_{ab} + \left( \mathcal{D}^\mu \right)_{ab} \hat{p}_\mu \right] + \frac{1}{2} \hat{p}_\mu \left( \Theta^{\mu\nu} \right)_{ab} \hat{p}_\nu - \frac{8}{3} \left( \Theta^{\mu\nu} \right)_{ab}.
\]

Here, the coefficients are the \( x \)-dependent matrices \( \mathcal{D}_{ab}(x) = \mathcal{D}_{ab}(x, k(x)) \) and

\[
\left[ \mathcal{D}^\mu \right]_{ab} = \frac{\partial \mathcal{D}_{ab}(x, k(x))}{\partial k_\mu},
\]

\[
\left[ \Theta^{\mu\nu} \right]_{ab} = \frac{\partial^2 \mathcal{D}_{ab}(x, k(x))}{\partial k_\mu \partial k_\nu}.
\]

Let us multiply Eq. (110) by \( \gamma \) in order to raise the first index. This leads to

\[
\hat{\cal D}^a_b \psi^b = 0, \quad \hat{\cal D}^a_b \doteq \gamma^{ac} \hat{\cal D}_{cb}.
\]

Note that if \( \hat{D} \) is self-adjoint, then \( \hat{\cal D} \) is Hermitian, and so is \( \hat{\cal D}_{ab} \); thus, \( \hat{\cal D}^a_b \) is self-adjoint too. Finally, assuming the same orderings as in the scalar-wave case (also see below), the following approximation is enough both for the first-order theory and for the quasioptical theory:

\[
\hat{\cal D} \approx \mathcal{D}_H + \hat{\cal G},
\]

\[
\hat{\cal G} = i\mathcal{D}_A + \hat{\cal D}_{H1} + \hat{\cal D}_{H2}.
\]

Here, \( \mathcal{D} \equiv \mathcal{D}(x) = \gamma^{-1}\mathcal{D}(x) \); i.e., \( \mathcal{D}^a_b(x) = \gamma^{ac} \mathcal{D}_{cb}(x) \), which notation is consistent with Eq. (45). Accordingly, the matrices \( \mathcal{D}_H \) and \( \mathcal{D}_A \) are self-adjoint, and so are the operators \( \hat{\cal D}_{H1} \) and \( \hat{\cal D}_{H2} \), which are given by

\[
\hat{\cal D}_{H1} \doteq \frac{1}{2} \gamma^{-1}(\hat{p}_\mu \mathcal{D}^\mu + \mathcal{D}^\mu \hat{p}_\mu),
\]

\[
\hat{\cal D}_{H2} \doteq \frac{1}{2} \gamma^{-1}\hat{p}_\mu \Theta^{\mu\nu} \hat{p}_\nu.
\]

Using \( \gamma^{-1}\hat{p}_\mu = i(\gamma^{-1})_\mu \gamma \) and \( (\gamma^{-1}_\gamma)_\mu = 0 \) to obtain

\[
(\gamma^{-1}_\gamma)_\mu \gamma = -\gamma^{-1}_\gamma, \quad \gamma = -\gamma^{-1}_\gamma, \equiv -\ell_\gamma,
\]

one can also express these as

\[
\hat{\cal D}_{H1} \doteq \frac{1}{2} \hat{p}_\mu \mathcal{V}^\mu(x) + \mathcal{V}^\mu(x) \hat{p}_\mu - i\ell_\gamma \mathcal{V}^\mu(x),
\]

\[
\hat{\cal D}_{H2} \doteq \frac{1}{2} \hat{p}_\mu \Theta^{\mu\nu} \hat{p}_\nu.
\]

(The term \( \ell_\gamma \) must be kept in \( \hat{\cal D}_{H1} \) but a similar term in \( \hat{\cal D}_{H2} \) can be neglected.) Also,

\[
\mathcal{V}^\mu(x) \doteq \gamma^{-1} \mathcal{V}^\mu(x), \quad \Theta^{\mu\nu}(x) \doteq \gamma^{-1} \Theta^{\mu\nu}(x),
\]

which leads to

\[
|\mathcal{V}^\mu(x)|^a_b \doteq \frac{\partial(D\mathcal{Y})^{ca}_{ab}(x, k(x))}{\partial k_\mu},
\]

\[
|\Theta^{\mu\nu}(x)|^a_b \doteq \frac{\partial^2(D\mathcal{Y})^{ca}_{ab}(x, k(x))}{\partial k_\mu \partial k_\nu}.
\]

Notably, if \( \hat{\cal D}_{H2} \) is neglected, the envelope equation (10) becomes a Dirac-type equation similar to those considered in the context of XGO [18–22] and also, for instance, in Refs. [40, 41]. We shall also discuss this limit in Sec. V E.

\[\text{D. Active and passive modes}\]

\[\text{1. Diabatic basis}\]

Let us assume the GO ordering as in Eq. (59), except that \( \mathcal{D}_H \) and \( \mathcal{D}_A \) are now matrices. Then, \( \hat{\cal D} = O(\epsilon) \), so the envelope equation [Eq. (59)] acquires the form

\[
\mathcal{D}_H(x) \psi = O(\epsilon).
\]

Hence, it is convenient to express \( \psi \) through the eigenvectors of \( \mathcal{D}_H(x) \). Such representations are called “diabatic” [42], not to be confused with adiabatic representations.

Let us denote the eigenvectors of \( \mathcal{D}_H(x) \) as \( \eta_s(x) \) and the corresponding eigenvalues as \( \Lambda_s(x) \), so

\[
\mathcal{D}_H \eta_s = \Lambda_s \eta_s.
\]

[Note that \( \eta_s(x) = \eta_s(x, k(x)) \), where \( \eta_s(x, p) \) are the eigenvectors of \( \mathcal{D}_H(x, p) \). Likewise, \( \Lambda_s(x) = \Lambda_s(x, k(x)) \), where \( \Lambda_s(x, p) \) are the eigenvalues of \( \mathcal{D}_H(x, p) \).] Since \( \mathcal{D}_H \) is self-adjoint, it has m eigenvectors \( \eta_s \) that form a complete basis \( \{\eta_s\} \), which we call the diabatic basis. Let us also introduce the corresponding dual basis \( \{\eta^s\} \) as usual, i.e., such that

\[
\eta^s \cdot \eta^{s'} = \delta^{s'} \delta_s.
\]

Then, we can represent \( \psi \) as

\[
\psi = \eta_s a^s, \quad \eta^s a^s = \eta^s \psi = (\eta^s)^a \psi^a,
\]

where \( a^s \) serve as the components of \( \psi \) in the basis \( \{\eta_s\} \). [Remember that the dot product \( \eta^s \psi^a \) includes conjugation.] Since \( \mathcal{D}_H \) is self-adjoint, it is always possible to
make the basis \( \{ \eta_s \} \) orthonormal, and this choice is assumed below. Then, \( \eta_h \cdot \eta_{s'} = \delta_{ss'} \); thus \( \eta_h = \eta^s \), i.e.,

\[
(\eta^s)_a = (\eta_h)_a = \gamma_{ab}(\eta_h)_b. \tag{130}
\]

Also, for any two fields \( \psi = \eta_h a^s \) and \( \phi = \eta_h b^s \), the inner product \( \langle \psi | \phi \rangle_m \) can be written as follows:

\[
\langle \psi | \phi \rangle_m = \int d^3 x \sqrt{g(x)} a^*_s(x) b^s(x), \tag{131}
\]

where \( a_s = \delta_{ss'} a^{s'} \). The matrix \( \delta \) with elements \( \delta_{ss'} \) serves as a Euclidean metric for manipulating the mode indices \( s \) and \( s' \). Those should not to be confused with the coordinate indices denoted with Greek letters and also with other Latin indices that are manipulated by the metric \( \gamma \) \[Eq. (130)].

2. Amplitude vectors

The physical meaning of the expansion coefficients \( a^s \) is understood as follows. Consider multiplying Eq. \[Eq. (126)\] by \( \eta^s \) from the left. That gives \( \Lambda_s a^s = O(\epsilon) \), where no summation over \( s \) is assumed. This shows that, for a given \( s \), there are two possibilities: either \( a^s \) is small or \( \Lambda_s \) is small. In the first case, the polarization \( \eta_s \) does not correspond to a propagating wave mode per se; the small nonzero projection of \( \psi \) on \( \eta_s \) is only due to the fact that the wave field is not strictly sinusoidal in inhomogeneous medium. We call such “modes” passive. In the second case, the local \( k(x) \) approximately satisfies the local dispersion relation \( \Lambda_s(x, k(x)) \approx 0 \). Then, \( a^s \) can be understood as the local scalar amplitude of an actual GO mode, so that \( a^s = O(1) \) is allowed. We call such modes active, and mode conversion occurs when more than one active mode exists. In other words, for a given boundary or initial conditions, active modes are those that are excited resonantly, while passive modes are those that are nonresonant and, thus, adiabatically isolated. (However, this does not mean that the passive-modes amplitudes are simply negligible; see below.)

Assuming that there are \( N \geq 1 \) active modes, we shall order them such that they correspond to \( s = 1, 2, \ldots, N \) and the remaining \( \bar{N} = m - N \) modes with \( s = (N + 1), (N + 2), \ldots, m \) are passive. Let us also adopt the notation

\[
\bar{a}^s = a^{s+N}, \quad \bar{\eta}_s = \eta_{s+N}, \quad \bar{\Lambda}_s = \Lambda_{s+N} \tag{132}
\]

(s = 1, 2, \ldots, \( \bar{N} \)) for the passive-mode amplitudes, polarizations, and eigenvalues. Then, it is convenient to introduce the following “amplitude vectors”

\[
a = \begin{pmatrix} a^1 \\
\vdots \\
a^N \end{pmatrix}, \quad \bar{a} = \begin{pmatrix} \bar{a}^1 \\
\vdots \\
\bar{a}^{\bar{N}} \end{pmatrix} \tag{133}
\]

and the corresponding row vectors that are dual to the amplitude vectors under the Euclidean complex dot product; namely,

\[
a^\dagger = (a_1^1, a_2^2, \ldots, a_N^N), \quad \bar{a}^\dagger = (\bar{a}_1^1, \bar{a}_2^2, \ldots, \bar{a}_{\bar{N}}^{\bar{N}}), \tag{134}
\]

\[
a_s = \delta_{ss'} a^{s'}, \quad \bar{a}_s = \delta_{ss'} \bar{a}^{s'}. \tag{135}
\]

Below, we seek to derive an approximate envelope equation in terms of these amplitude vectors.

3. Polarization matrices

Before we proceed, let us introduce the following notation. First, consider the “polarization matrices”

\[
\Xi = (\eta_1, \eta_2, \ldots, \eta_N), \quad \bar{\Xi} = (\bar{\eta}_1, \bar{\eta}_2, \ldots, \bar{\eta}_{\bar{N}}). \tag{136}
\]

These are non-square matrices that have active- and passive-mode polarizations as their columns; namely,

\[
\Xi^a = (\eta_h)^a, \quad \bar{\Xi}^a = (\bar{\eta}_h)^a. \tag{137}
\]

Using these, Eq. \[Eq. (129)\] can be rewritten compactly as

\[
\psi = \Xi a + \bar{\Xi} \bar{a}. \tag{138}
\]

Also consider the auxiliary polarization matrices

\[
\Xi^+ = \begin{pmatrix} \eta^1 \\
\vdots \\
\eta^N \end{pmatrix}, \quad \Xi^+ = \begin{pmatrix} \bar{\eta}^1 \\
\vdots \\
\bar{\eta}^N \end{pmatrix} \tag{139}
\]

which have the \( \eta^s \) as their rows,

\[
(\Xi^+)^a_s = (\eta^s)_a, \quad (\Xi^+)^s_a = (\bar{\eta}^s)_a. \tag{140}
\]

Since

\[
(\psi^\dagger)_a = (\eta_h)^a_s a^{ss*} + (\bar{\eta}_h)^a_s \bar{a}^{ss*} = (\eta^s)_a a^{ss*} + (\bar{\eta}^s)_a \bar{a}^{ss*}, \tag{141}
\]

one obtains

\[
\psi^\dagger = a^\dagger \Xi^+ + \bar{a} \bar{\Xi}^+. \tag{142}
\]

However, note that in general, \( + \) does not mean to represent the adjoint in the common sense \[cf. Eq. (147)\].

Next, notice that

\[
(\Xi^+ \Xi^s_a s') = (\eta^s)_a (\eta_{s'})^a = \eta^s \cdot \eta_{s'} = \delta^s_{s'}, \tag{143}
\]

where the indices \( s \) and \( s' \) span from 1 to \( N \). Analogous formulas apply to \( \Xi \). Then,

\[
(\Xi^\dagger \Xi)^s_a s' = (\bar{\eta}^s)_a (\eta_{s'})^a = \bar{\eta}^s \cdot \eta_{s'} = 0, \tag{144}
\]

and similarly for \( \Xi^\dagger \Xi \). In other words, one has

\[
\Xi^+ \Xi = \Xi \Xi^+ = 0, \quad \Xi^\dagger \Xi = \Xi \Xi^\dagger = 0. \tag{145}
\]
\[ a = \Xi^+ \psi, \quad \bar{a} = \bar{\Xi}^+ \psi. \]  

(146)

Another property that we shall use later on is

\[ (\Xi^+)^* a = \delta^{ss'} (\eta_{v'})^* a = \delta^{ss'} (\eta_v) b \gamma_{ab} = (\delta^{-1} \Xi^H \gamma)^* a, \]

and similarly for \( \bar{\Xi}^+ \). Here, \( \bar{\Xi}^+ \) denotes the conjugate transpose \( (\Xi^H \doteq \Xi^T) \), and \( \delta^{ss'} \) is the Kronecker matrix with upper-index elements \( \delta^{ss'} \). Hence,

\[ \Xi^+ = \delta^{-1} \Xi^H \gamma, \quad \bar{\Xi}^+ = \delta^{-1} \bar{\Xi}^H \gamma. \]  

(147)

The factor \( \delta^{-1} \) can be omitted if one ignores the difference between upper and lower indices \( s \) and \( s' \) that refer to the mode number. If \( \gamma \) is Euclidean, one can also ignore the difference between upper and lower coordinate indices; then, \( \Xi^+ = \Xi^H \).

As a side remark, note that \( \Xi \) and \( \bar{\Xi} \) are generally non-square, so \( \Pi \doteq \Xi \Xi^+ \) and \( \bar{\Pi} \doteq \bar{\Xi} \bar{\Xi}^+ \) are not unit matrices but rather projectors. Indeed, due to Eqs. (113), one has \( \Pi^2 = \Pi \) and \( \bar{\Pi}^2 = \bar{\Pi} \). Also, by applying \( \Pi \) and \( \bar{\Pi} \) to Eq. (135), one obtains

\[ \Xi a = \Pi \psi, \quad \bar{\Xi} a = \bar{\Pi} \psi. \]  

(148)

Thus, \( \Pi \psi \) is the projection of \( \psi \) on the active-mode space, and \( \bar{\Pi} \psi \) is the projection of \( \psi \) on the passive-mode space.

4. Equation for the active modes

Using the eigendecomposition theorem and Eq. (135), one obtains

\[ D_H = \eta_\lambda A \eta^\dagger = \Xi \Lambda \Xi^+ + \bar{\Xi} \bar{\Lambda} \bar{\Xi}^+, \]  

(149)

where \( \Lambda \) and \( \bar{\Lambda} \) are the diagonal eigenvalue matrices,

\[ \Lambda \doteq \text{diag} \{ \lambda_1, \lambda_2, \ldots, \lambda_N \} = O(\epsilon), \]  

(150)

\[ \bar{\Lambda} \doteq \text{diag} \{ \bar{\lambda}_1, \bar{\lambda}_2, \ldots, \bar{\lambda}_N \} = O(1). \]  

(151)

These and Eq. (135) also lead to

\[ D_H \Xi = \Xi \Lambda = O(\epsilon), \quad D_H \bar{\Xi} = \bar{\Xi} \bar{\Lambda} = O(1). \]  

(152)

Hence, the envelope equation (53) can be written as

\[ 0 = D_H \Xi a + D_H \bar{\Xi} \bar{a} + \hat{D} \Xi a + \hat{D} \bar{\Xi} \bar{a} \]

\[ = \Xi \Lambda a + \Xi \bar{\Lambda} \bar{a} + \hat{D} \Xi a + \hat{D} \bar{\Xi} \bar{a}, \]  

(153)

where we used Eq. (132). Let us multiply this by \( \Xi^+ \) and, separately, by \( \bar{\Xi}^+ \). Then, due to Eq. (113), one obtains

\[ \Lambda a + \Xi^+ \hat{D} \Xi a + \Xi^+ \hat{D} \bar{\Xi} \bar{a} = 0, \]  

(154)

\[ \bar{\Lambda} \bar{a} + \bar{\Xi}^+ \hat{D} \Xi a + \bar{\Xi}^+ \hat{D} \bar{\Xi} \bar{a} = 0. \]  

(155)

Let us also use Eq. (154) to express \( \bar{a} \) through \( a \) and substitute the result into Eq. (154). Since \( \bar{a} \) and \( \hat{D} \) are both of order \( \epsilon \), it is sufficient to solve Eq. (155) for \( \bar{a} \) approximately to the leading order,

\[ \bar{a} \approx -\Lambda^{-1} \Xi^+ \hat{D} \Xi a. \]  

(156)

Then, Eq. (154) becomes an equation for just the \( N \)-dimensional active-mode amplitude vector,

\[ (\Lambda + \Xi^+ \hat{D} \Xi - \Xi^+ \hat{D} \Xi \Lambda^{-1} \Xi^+ \hat{D} \Xi) a = 0. \]  

(157)

As a reminder, this equation is valid up to \( O(\epsilon^2) \).

E. Leading-order approximation: extended geometrical optics

Upon substituting Eqs. (117) into Eq. (157), we obtain the following equation to lowest (first) order in \( \epsilon \):

\[ (\Lambda + i \Gamma + \hat{K}) a = 0, \]  

(158)

where we introduced

\[ \Gamma \doteq \Xi^+ D_A \Xi, \]  

(159)

\[ \hat{K} \doteq \Xi^+ \hat{D} H I \Xi. \]  

(160)

As shown in Appendix B1

\[ \hat{K} a = -i V^\mu a, \quad \frac{i}{2} V^\mu_{\mu} a - U a, \]  

(161)

\[ V^\mu \Xi^+ \Xi^\mu \Xi \approx \Lambda^\mu, \]  

(162)

\[ U = \delta^{-1} (\Xi^H \Xi^\mu \Xi^\mu \Xi) A, \]  

(163)

where \( \Xi \) is considered as a function of \( \mathbf{x} \). [As a reminder, \( \Xi^H \doteq \Xi^T \) is the conjugate transpose of \( \Xi \); \( V^\mu \) is given by Eq. (113); and \( \delta^{-1} \) is a unit matrix that only raises the mode index.] Alternatively, \( \Xi \) can be considered as a function of \( (x, k) \). Then, as shown in Appendix B2

\[ U \approx \delta^{-1} (\Lambda^\mu \Xi^\mu \Xi^\nu \Xi^\nu \Xi - \Lambda^\nu \Xi^H \Xi^\nu \Xi^\nu \Xi + \Xi^H \Xi^\mu \Xi^\nu \Xi^\nu \Xi) A, \]  

(164)

where the partial derivatives \( \mu \) and \( \nu \) are defined in Sec. IV C The term \( U \) is the Stern–Gerlach Hamiltonian mentioned in the introduction (Sec. II), except here it is generalized to an arbitrary metric. This term causes polarization-driven bending of the ray trajectories, which is missed in traditional GO; also, it causes mode conversion, if more than one active mode is present [18 22]. These effects are discussed in further detail in Sec. V G.

More explicitly, Eq. (163) can be written as

\[ \Lambda a + i \Gamma a - i V^\mu a = \frac{i}{2} V^\nu_{\nu} a - U a = 0, \]  

(165)

which represents a generalization of the XGO equation for the polarization vector \( a \) [18 23], to the case where wave dissipation is included and where an arbitrary
spacetime metric is considered. It is similar to Eq. (62) for scalar waves and has a similar corollary,

\[ \mathcal{J}^\mu_{;\mu} = -2a^1 \Gamma a, \quad \mathcal{J}^\mu = -a^1 V^\mu a. \]  

(166)

Using Eqs. (69), (123), (138), (142), (148), and (162) one obtains

\[ - \mathcal{J}^\mu = a^1 \Xi^\nu \partial^\nu a = (\Pi \psi)^{1} \gamma^\mu (\Pi \psi) \approx \psi^* \gamma^\mu \psi = \psi^* \gamma^\mu \psi, \]  

(167)

so Eq. (166) can be viewed as a generalization of Eq. (68). At zero \( \Gamma \), the dynamics is conservative, and Eq. (166) reflects conservation of the wave action, or quanta \([35]\). Accordingly, \( \Gamma \) determines the dissipation rate, \( \mathcal{J}^\mu \) can be identified (up to a constant factor) as the action flux density summed over all active modes (for example, see Sec. V F), and the elements of the diagonal matrix \( \Lambda^\mu \) are proportional to the group velocities of the corresponding active modes. We shall call them the group velocities (without “proportional to”) for brevity. We also emphasize that this model applies even when the group velocities of the active modes are very different, unlike the quasioptical model discussed in Sec. V F.

For scalar waves studied in Sec. IV, we had \( \Lambda = D_H \) and we defined \( k(x) \) such that this term be zero [Eq. (69)]. Now, \( \Lambda \) is a diagonal matrix with \( N \geq 1 \) nonzero elements, so it cannot be zeroed entirely by imposing just one scalar constraint on \( k(x) \). Hence, there can be more than one natural way to define \( k(x) \). One aesthetically pleasing and convenient \([23]\) option is to require that \( \Lambda \) or \( \Lambda - U \) be traceless. This amounts to choosing the reference-ray Hamiltonian in Sec. III as \( H = \text{tr} \Lambda / N \) or \( H = \text{tr}(\Lambda - U) / N \), correspondingly. [Arbitrary constant factors can be introduced instead of \( N \), for that only redefines \( \tau \) in Eq. (71).] Another option is to adopt \( \Lambda_s = 0 \) for some single \( s \leq N \), since all active modes have close wave vectors anyway; then \( H = \Lambda_s \). As mentioned in Sec. III all such choices of \( k(x) \) are equally justified. Although they lead to slightly different envelope equations, those equations are equivalent in the sense that they all describe the same total field by construction. (One might call this a gauge freedom.) However, for the case \( N = 1 \), choosing \( H = \Lambda - U \) is preferable, as discussed in Sec. V G 2.

**F. Quasioptical model**

1. Quasioptical equation

Suppose that a wave propagates largely as a single beam. This implies that all active modes have group velocities close to their average group velocity \( \mathbf{V}_{\text{avr}} \), which can be defined via \( \mathbf{V}^\mu_{\text{avr}} = \text{tr} \Lambda^\mu \). Then,

\[ \mathbf{V}^\mu = \mathbf{V}^\mu_{\text{avr}} + \Delta \mathbf{V}^\mu, \quad \Delta \mathbf{V}^\mu = \mathbf{V}^\mu - \mathbf{V}^\mu_{\text{avr}}. \]  

(168)

where \( \Delta \mathbf{V}^\mu \) are matrices with eigenvalues much smaller than \( \mathbf{V}_{\text{avr}} \). Let us assume a ray-based coordinate system aligned with \( \mathbf{V}_{\text{avr}} \) with the inner product on the transverse space \( M_{\text{avr}}^{-1} \) defined similarly to Eq. (72) [cf. also Eq. (131)], namely,

\[ (a|b)_{\text{avr}} = \int d^{n-1}\Omega \sqrt{h_0}(\zeta, \varphi) \, a^*_\xi(\zeta, \varphi) b^\xi(\zeta, \varphi). \]  

(169)

Let us also adopt the quasioptical ordering as we did for scalar waves in Sec. [173]. In particular, we allow \( \Delta \mathbf{V}^\mu = O(\epsilon_\perp) \). Then, Eq. (157) becomes

\[ (\Lambda + i\Gamma + \vec{K} + \vec{G}) \mathbf{a} = 0. \]  

(170)

Here, \( \vec{K} \) is defined as in Eq. (160) and can be approximated as follows:

\[ \vec{K} \mathbf{a} = -i V_{\text{avr}}(\zeta) \, a + \Delta \vec{K} \mathbf{a}, \]  

(171)

\[ \Delta \vec{K} \mathbf{a} = -U \mathbf{a} - i \mathbf{V}^\sigma \mathbf{a}_{\sigma} - \frac{i}{2 \sqrt{h_0}} \, (\sqrt{h_0} \mathbf{V}^\sigma)_{\sigma} \mathbf{a}, \]  

(172)

\[ \Delta \mathbf{V}^\sigma = \Xi^\sigma D^\sigma_{\text{avr}} \Xi, \]  

(173)

where \( \mathbf{V} \equiv V^\sigma_{\text{avr}} \). (We used the fact that \( V^\sigma_{\text{avr}} = 0 \) by definition of the ray-based coordinates.) The last term in \( \Delta \vec{K} \) is beyond the accuracy of our theory but is retained to make the operator \( \Delta \vec{K} \) precisely self-adjoint under the scalar product \( (169) \). Also, \( U = O(\epsilon_\parallel) \) is given by Eq. (164) and \( \Gamma \) is given by Eq. (159). Finally,

\[ \hat{\mathbf{G}} = \Xi^\sigma \hat{\mathbf{D}}_{\text{avr}} \Xi - \Xi^\sigma \hat{\mathbf{D}}_{\text{avr}} \Xi = \Xi. \]  

(174)

As shown in Appendix 133 \( \hat{\mathbf{G}} \) can also be simplified as

\[ \hat{\mathbf{G}} \mathbf{a} \approx -\frac{1}{2 \sqrt{h_0}} \, \Xi^\sigma \mathbf{a}, \]  

(175)

so it is also self-adjoint under the inner product \( (169) \).

In summary then, the quasioptical equation for vector waves can be written as follows:

\[ \Lambda \mathbf{a} + i \Gamma \mathbf{a} - i V_{\text{avr}}(\zeta) \, a + \hat{\mathbf{G}} \mathbf{a} + \Delta \vec{K} \mathbf{a} = 0. \]  

(176)

This is the main result of our paper. As a reminder, \( \mathbf{a} \) is generally a vector (dim \( a = N \geq 1 \)), whose elements are the scalar amplitudes of active modes (Sec. V D 2); \( \Lambda \) is the diagonal matrix formed by the eigenvalues of \( D_H \); \( \Gamma \) is given by Eq. (165); \( V = N^{-1} \text{tr} \Lambda^\sigma \) is a scalar; \( \mathcal{K} \ominus \tau \) is defined by Eq. (175); \( \hat{\mathbf{G}} \) is given by Eq. (173), and \( \Delta \vec{K} \) is given by Eq. (172). There, \( \mathbf{a} \) is given by Eq. (164), and \( \Delta \mathbf{V} \) is given by Eq. (173). As a reminder, \( (\sqrt{h_0} \Delta \mathbf{V})_{\sigma} \) could be neglected within the assumed accuracy, but a purist might want to keep this term anyway in order to keep \( \Delta \vec{K} \) precisely self-adjoint under the inner product \( (169) \). The same applies to the factor \( h_0 \) in the expression for \( \hat{\mathbf{G}} \).
Like in Sec. [IV.E.2], we require the $\alpha$ parameter of the ray-based coordinates to be defined as $\varrho$-independent. Then, Eq. (176) has the following corollary:

$$
\frac{1}{\alpha} \frac{d}{d\zeta} \langle a|\alpha V|a\rangle_N^\perp = 2 \langle a|\Gamma|a\rangle_N^\perp,
$$

(177)

which is similar to Eq. (73). For the case of zero $\Gamma$, Eq. (177) predicts conservation of the wave-action flux through the beam cross section,

$$
\langle \psi|\alpha V|\psi\rangle_N^\perp = \text{const.}
$$

(178)

Unlike in Eq. (80), this is the flux of all active modes combined, and the fluxes of individual active modes may not be conserved separately.

2. Simplified equations

If the metric is nearly Euclidean (or pseudo-Euclidean), we can invoke the same argument as in Sec. [IV.E.3] to drop the metric factors and rewrite Eq. (169) as follows:

$$
\Lambda a + i\Gamma a - iVa,_{\zeta} - \frac{i}{2} V\zeta a - \frac{1}{2} \left( \Lambda|\sigma \, a,_{\sigma} \right),_{\sigma} \\
- Ua - i\Delta V^\sigma a,_{\sigma} - \frac{i}{2} \Delta V^\sigma a = 0.
$$

(179)

Using the variable transformation $\phi = \sqrt{V}a$, this equation can be further simplified as

$$
i\phi,_{\zeta} = \tilde{\Lambda} \phi + i\Upsilon \phi.
$$

(180)

Here, $\tilde{\Lambda}$ is an operator self-adjoint under the inner product (169) and given by

$$
\tilde{\Lambda} \phi \approx Q \phi - \frac{1}{2} \left( \Sigma|\sigma \, \phi,_{\sigma} \right),_{\sigma} - \frac{\Delta V^\sigma}{V} i\phi,_{\sigma} - \frac{i}{2} \frac{\left( \Delta V^\sigma \right)}{V} \phi.
$$

(181)

Also, $Q$, $\Sigma$, and $\Upsilon$ are Hermitian matrices given by

$$
Q \doteq (\Lambda - U)/V, \quad \Sigma|\sigma \, \phi,_{\sigma} \doteq \Lambda|\sigma \, \phi,_{\sigma}/V, \quad \Upsilon \doteq \Gamma/V.
$$

(182)

Accordingly, Eq. (177) becomes

$$
\frac{d}{d\zeta} \langle \phi|\phi\rangle_N^\perp = 2 \langle \phi|\Upsilon|\phi\rangle_N^\perp,
$$

(183)

so $\langle \phi|\phi\rangle_N^\perp$ is conserved if $\Upsilon$ is zero.

G. Summary and discussion

The quasioptical model proposed above is equally applicable to scalar beams, single-mode vector beams ($N = 1$), and multi-mode vector beams ($N > 1$).

1. Scalar beams

In the case of a scalar beam, one has $\Xi = 1$, so

$$
\Lambda = D_{H}, \quad \Gamma = D_{\alpha}, \quad U = 0, \quad \Delta V^\sigma = 0,
$$

(184)

where the latter is due to the fact that $V^\sigma$ are scalars. Then, the equations from Sec. [IV.E.3] are reproduced. In particular, $\Lambda$ is made zero by the choice of $k$ [Eq. (60)]. This implies that the ray Hamiltonian is $H = \Lambda$, which leads to the following ray equations:

$$
\frac{dx^\alpha}{d\tau} = \frac{\partial(\Lambda - U)}{\partial k_\alpha}, \quad \frac{dk_\alpha}{d\tau} = -\frac{\partial(\Lambda - U)}{\partial x^\alpha}.
$$

(185)

2. Single-mode vector beams

In the case of a single-mode vector beam ($N = 1$), one has $\Xi = \eta$, where $\eta$ is the polarization vector. Then,

$$
\Lambda = \eta^\dagger D_{H}\eta, \quad \Gamma = \eta^\dagger D_{\alpha}\eta,
$$

(186)

so they are scalars. Also, $\Delta V^\sigma = 0$, but $U$ is generally a nonzero scalar function. There are two natural ways to define $k$ in this case. One is to require that $\Lambda = 0$, i.e., to adopt $H = \Lambda$. In this case, $U$ is left in the envelope equation and can intensify the envelope inhomogeneity in the direction perpendicular to the group velocity. This may eventually result in the violation of the envelope approximation. (For multi-mode beams, this is less of a concern because they typically split into single-mode beams before the problem becomes significant.) Hence, it is potentially advantageous to define $k$ by requiring $\Lambda - U = 0$, i.e., by adopting $H = \Lambda - U$. In this case, the amplitude equation is identical to that of a scalar wave while the effect of $U$ is absorbed in the reference phase, leading to modified ray equations,

$$
\frac{dx^\alpha}{d\tau} = \frac{\partial(\Lambda - U)}{\partial k_\alpha}, \quad \frac{dk_\alpha}{d\tau} = -\frac{\partial(\Lambda - U)}{\partial x^\alpha}.
$$

(187)

The effect of $U$ on the ray trajectories is known as the (spin) Hall effect of light in optics [43–47]. In plasma physics, this effect is typically neglected. (To our knowledge, all existing ray-tracing codes ignore $U$ entirely.) That is roughly justified for stable trajectories but may lead to substantial errors otherwise; namely, the $U$-driven beam displacement accumulated over time $\tau = O(\epsilon^{-1})$ can be of order unity.

Finally, note that the ray dynamics can also be represented in an alternative form. Since $\Lambda$ is a scalar, one can rewrite Eq. (169) for $U(x,k)$ as follows:

$$
U = U_0 - \Lambda^{\mu\overline{\mu}} \text{Im}(\eta^\dagger \eta_{\mu\overline{\mu}}) + \Lambda^{\mu}_{\overline{\mu}} \text{Im}(\eta^\dagger \eta^{\mu\overline{\mu}}) \\
\approx U_0 - z^{\overline{\mu}} A_{\overline{\mu}} - \overline{k}_{\mu} A_{\mu}.
$$

(188)

Since there is only one mode, there is no mode index to manipulate, so the "metric" factor $\delta$ has been dropped.
Likewise, the anti-Hermitian parts are simply the imaginary parts in this case. We also substituted the GO ray equations (the dot denotes $d/dt$) and adopted

$$U_0 = \text{Im} (\eta^H \eta H \eta \mu),$$ (188)
$$A^{(x)}_{\mu} = \text{Im} (\eta^\dagger \eta_{\mu}), \quad A^{(x)}_{(k)} = \text{Im} (\eta^\dagger \eta^{(k)}).$$ (189)

Then, the phase-space Lagrangian of a ray, $L[\mathbf{x}, k] = k_{\mu} \dot{x}^{\mu} - H$, has a non-canonical structure, namely,

$$L = [k_{\mu} + A^{(x)}_{\mu}] \dot{x}^{\mu} + A^{(x)}_{(k)} \dot{k}^{\mu} - (\Lambda - U_0).$$ (190)

The equations originating from a special case of this non-canonical phase-space Lagrangian were used, for example, in Ref. [44] to study the Hall effect for light propagating in non-birefringent material. A comparison of the resulting non-canonical ray equations with the canonical ray equations [186] can be found in Ref. [21].

3. Multi-mode vector beams

In the case of multiple active modes ($N > 1$), $a$ is an $N$-dimensional vector, and the coefficients in the amplitude equation are matrices (except for $V$, which is a scalar). In particular, the matrices $\Gamma$, $\Delta V^{\sigma}$, and $U$ are generally nondiagonal. Accordingly, $a$ can cause mode conversion. In the simplest case, when both the transverse gradients and dissipation are negligible, this process is most transparently described by Eq. (180), which then becomes

$$i \dot{\phi}_x = Q \phi,$$ (191)

with $Q$ being a Hermitian matrix. The modes decouple when $Q$ is close to diagonal. More generally, the dynamics governed by Eq. (191) is similar to that of an $N$-level quantum system with a Hamiltonian $Q$ (and to the dynamics of $N$ coupled classical oscillators whose parameters do not change too rapidly with $\zeta$ [45]). Alternatively, it can be mapped to a precession equation for a real $(N^2 - 1)$-dimensional “spin” vector [20]. This approach is particularly intuitive at $N = 2$, when the spin is three-dimensional. A detailed analysis of mode conversion for this case can be found in Ref. [22].

As a reminder, the model presented here (Sec. V E) relies on the assumption that the group velocities of the active modes are close to each other. Otherwise, beam splitting occurs rapidly, and the assumption that $a_x \ll a_{\sigma}$ does not hold. The quasilinear description is inapplicable to such beams; however, the first-order XGO model described in Sec. V E can be used.

VI. ELECTROMAGNETIC WAVES

A. Covariant formulation

Here, we shall explain how the above theory applies to EM waves. We start with Maxwell’s equation [49]

$$\frac{1}{\sqrt{\gamma_0}} \partial_\sigma (\sqrt{\gamma_0} F^{\alpha \beta}) = \frac{4 \pi}{c} J^{\beta},$$ (192)

where $F^{\alpha \beta}$ is the EM tensor, namely,

$$F^{\alpha \beta} = g^{\alpha \mu} g^{\beta \nu} F_{\mu \nu}, \quad F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha,$$ (193)

$A$ is the vector potential, and $J$ is the current density. This equation can also be represented as follows:

$$\hat{D}^{(0)}_\alpha A = -\frac{4 \pi}{c} J.$$ (194)

Here $\hat{D}^{(0)}_\alpha$ is the vacuum dispersion operator,

$$[\hat{D}^{(0)}_\alpha]^{\alpha \beta} \approx g_{\alpha \gamma} \{ [\hat{D}^{(0)}_\alpha]^{\gamma \beta} \}.$$ (195)

Hence the theory developed in the previous sections readily applies. In particular, the dimension of the configuration space is $n = 4$, the dimension of the vector field $A$ is the same, $m = 4$, and $\gamma = g$.

First, we calculate $[\hat{D}^{(0)}]^{\alpha \beta} = g_{\alpha \gamma} [\hat{D}^{(0)}]^{\gamma \beta}$. It can be shown straightforwardly that

$$[\hat{D}^{(0)}_\alpha]^{\alpha \beta} \approx \hat{p}_\alpha \hat{p}_\beta - \hat{p}_\mu g^{\mu \nu} g_{\alpha \beta} \hat{p}_\nu + \Delta^{\alpha \beta},$$ (196)

$$\Delta^{\alpha \beta} = i(q_\alpha \hat{p}_\beta - q_\beta \hat{p}_\alpha)/2.$$ (197)

Here, we omitted second-order derivatives of $g$, which are negligible within the accuracy of our theory. Also, $q$ is defined as in Eq. (13), and $\ell_\mu = g^{-1} g_{\mu \nu}$, which is in agreement with Eq. (1) because $g = \gamma$.

The operator $\Delta^{\alpha \beta} \approx g^{\alpha \gamma} \Delta^{\gamma \beta}$ is small, so it can be treated as a perturbation. Then, the dispersion operator for $A$ can be adopted in a simplified form

$$[\hat{D}^{(0)}]^{\alpha \beta} = \hat{g}^{\alpha \gamma} (\hat{p}_\gamma \hat{p}_\beta - \hat{p}_\mu g^{\mu \nu} g_{\alpha \beta} \hat{p}_\nu) + 4 \pi \hat{p}^{\alpha \beta},$$ (198)

$$[\hat{D}^{(0)}_\alpha]^{\alpha \beta} \approx \hat{p}_\alpha \hat{p}_\beta - g_{\alpha \beta} \hat{p}_\mu g^{\mu \nu} g_{\alpha \beta} \hat{p}_\nu + 4 \pi \Delta^{\alpha \beta},$$

and $\Delta$ can be absorbed in the term $U$ (Sec. V E) as

$$U \rightarrow U - \Xi^\dagger \Delta \Xi,$$

$$\Delta^{\alpha \beta} \approx ig^{\alpha \gamma} (q_\gamma k_\beta - q_\beta k_\gamma) + ig^{\alpha \gamma} [(\ell_\beta)^\mu \gamma - (\ell_\gamma)^\mu \beta] k_\mu.$$ (199)

Applications of this formulation will be reported separately. Also note that, for a metric that differs from the Minkowski metric only by some $\mathbf{g} = o(1)$, one has $\partial_\mu \mathbf{g} = o(\epsilon)$ since $\partial_\mu = O(\epsilon)$, so $\Delta = o(\epsilon)$; then, $\Delta$ can be neglected entirely.
B. Non-covariant formulation

Suppose a metric of the form

\[ g = \begin{pmatrix} -1 & 0 \\ 0 & h \end{pmatrix} \]

with time-independent spatial metric \( h \). Then, the above equations can also be cast in a simplified form, namely, as follows. Let us assume the Weyl gauge \((A_0 = 0)\). Then, it is sufficient to consider just the spatial part of the four-dimensional Eq. (166) and replace the vector potential with the electric field

\[ E^a = F^{0a} = -i\tilde{\rho}_0 A^a. \]

(Assual, \( \tilde{\rho}_0 = -ic^{-1}\partial_t \), which is a self-adjoint operator. Assuming we work with wave fields that have zero time average, \( \tilde{\rho}_0 \) can also be considered reversible.) Unlike in the previous sections, we now use the Latin indices to denote spatial components, and we shall adopt the bold font for spatial vectors and matrices when using index-free notation. Specifically, one obtains

\[ \hat{D}^{(E)} \vec{E} = 0, \]

\[ \hat{D}^{(E)} \equiv (\tilde{\rho}_0)^{-1} \hat{D}^{(A)} + 4\pi\hat{\vartheta} \mid (\tilde{\rho}_0)^{-1}, \]

where we also multiplied the equation by \((\tilde{\rho}_0)^{-1}\). This has the form (1) with \( n = 4, m = 3 \), and \( \gamma = h \).

Let us introduce the conductivity operator \( \hat{\sigma} \) via \( J = \hat{\sigma} \vec{E} \) and notice that \( 4\pi\hat{\sigma} = ic\tilde{\rho}_0 \hat{\chi} \) by definition of the susceptibility operator \( \hat{\chi} \). Then, \( 4\pi\hat{\vartheta} = \tilde{\rho}_0 \hat{\chi} \tilde{\rho}_0 \), so \( \hat{D}^{(E)} \) can be expressed as follows:

\[ \hat{D}^{(E)} \equiv (\tilde{\rho}_0)^{-1} \hat{D}^{(A)}(\tilde{\rho}_0)^{-1} + \hat{\chi}, \]

or equivalently, \( \hat{D}^{(E)} = (\tilde{\rho}_0)^{-2} \hat{D}^{(A)} + \hat{\chi} \). The corresponding Weyl image is \( \hat{D}_{ab}^{(E)} = (\tilde{\rho}_0)^{-2} \hat{D}_{ab}^{(A)} + \hat{\chi}_{ab} \)

or

\[ \hat{D}_{ab}^{(E)} = (\tilde{\rho}_0)^{-2}(\tilde{\rho}_0p_{ab} - p_{ab}h^{cd}h_{cd} + \Delta_{ab}) + \hat{\chi}_{ab}. \]

where \( \hat{\chi}_{ab} \) serves as the dielectric tensor.

Using the fact that the dispersion operator is defined only up to a constant factor, let us introduce an additional factor \( 1/(16\pi) \). Then, Eq. (204) becomes

\[ \hat{D}_{ab}^{(E)} = \frac{1}{16\pi\tilde{\rho}_0^2}(\tilde{\rho}_0p_{ab} - p_{ab}h^{cd}h_{cd} + \Delta_{ab}) + \hat{\chi}_{ab}, \]

and \( \hat{J}^a \) \[ \hat{J}^a \equiv \hat{D}^a - \frac{1}{16\pi}(\tilde{\rho}_0)^{-2}(\tilde{\rho}_0p_{ab} - p_{ab}h^{cd}h_{cd} + \Delta_{ab}) + \hat{\chi}_{ab}, \]

becomes as the true action-flux density \( \hat{J}^a \). As a reminder, the action density \( I \equiv \hat{J}^0 \) can be written as

\[ I \approx \frac{E^a E^b}{16\pi\omega^2} \frac{\partial}{\partial\omega} \{ \omega^2 \hat{\omega}(t, x, \omega, k) \}_{ab}, \]

where we used \( \hat{D}^{(E)}_{ab}E^b = 0 \). Using the latter and Faraday’s law \( B \approx c \times E/\omega \) for the magnetic field \( B \), one can also rewrite \( I \) in another common form.

\[ I \approx \frac{E^a E^b}{16\pi\omega} \frac{\partial}{\partial\omega} \{ \omega \hat{\omega}(t, x, \omega, k) \}_{ab} + \frac{B^a B^b}{16\pi\omega}. \]

One can also show that the spatial component of the action flux density \( \hat{J}^a \) can be expressed as

\[ \hat{J} \approx \frac{S}{\omega} - \frac{E^a E^b}{16\pi} \frac{\partial}{\partial\omega} \{ \omega \hat{\omega}(t, x, \omega, k) \}_{ab}, \]

where \( S \) is the (time- or space-averaged) Poynting vector,

\[ S = \frac{c}{8\pi} \text{Re} (\vec{E} \times \vec{B}^*). \]

The GO equation (166) can be written as

\[ \frac{\partial I}{\partial t} + \frac{1}{\sqrt{h_0}} \frac{\partial}{\partial x^a} (\sqrt{h_0} J^a) = -\frac{E^a E^b}{8\pi} \{ \hat{\omega}(t, x, \omega, k) \}_{ab}. \]

The corresponding density of the wave energy is \( \hat{\mathcal{I}} \), and the density of the energy flux is \( \hat{\mathcal{J}} \), as flows from the variational principle.

Like in Sec. VI A, the effect of the metric inhomogeneity can be treated as a perturbation. In the latter case, the dispersion operator for \( E \) can be adopted in the form

\[ \hat{D}_{ab}^{(E)} = \frac{1}{16\pi}(\tilde{\rho}_0)^{-2}(\tilde{\rho}_0p_{ab} - p_{ab}h^{cd}h_{cd} + \Delta_{ab}) + \hat{\chi}_{ab}, \]

and the Stern–Gerlach potential is modified as follows:

\[ U \rightarrow U - \frac{\epsilon^2}{16\pi\omega^2} \Delta \hat{\chi}, \]

\[ \Delta_{ab} \approx i\hbar(c \times E_{ab} - q \rho_{ab}) + i\hbar(\pi_{ab}^c (\ell^c d)_{ab} - (\ell^c d)_{ab} k_d). \]

If \( h \) differs from the Euclidean metric only by \( o(1) \), then \( \Delta \) can be neglected entirely, like in Sec. VI A. Also, in the strictly Euclidean metric, upper and lower indices do not need to be distinguished, and the underlining does not need to be introduced. Then, the Weyl symbol of the dispersion operator can be written simply as follows:

\[ \hat{D}^{(E)} = \frac{1}{16\pi\rho_0^2} \{ \rho_0 \vec{p} - (\vec{p} \cdot \vec{p}) \}^a + \frac{1}{16\pi} \vec{\varepsilon}(t, \rho_0, \vec{x}, \vec{p}). \]

Note that other formulations of the dispersion operator have also been proposed in this case, e.g., for light in nondispersive dielectric media and waves in cold plasmas.

C. Stationary waves

For stationary waves with fixed frequency, one has \( \tilde{\rho}_0 = -\omega/c \) and \( \omega \) can be treated as a constant parameter. Such waves can be studied on the three-dimensional
configuration space, namely, the physical space $x$. In this case, $n = m = 3$ and $\gamma = h$. If there is no spatial dispersion, then $\varepsilon = \varepsilon(x)$, where the dependence on $\omega$ is assumed but not emphasized. Hence, the Weyl symbol $\Delta_{ab}$ is identical to the (lower-index) dielectric tensor of a homogeneous medium. Furthermore, the resulting $D^{(E)}_{ab}$ is quadratic in $p$, so $D_{ab} = D^{(E)}_{ab}$, as seen from Eq. (111).

The latter property can also be approximately extended to media with weak spatial dispersion, i.e., such that $\Delta_{ab} \approx \Delta^{(0)}_{ab}(x) + \Delta_{ab}(x, p)$ where $\Delta_{ab}$ is small. For example, in plasma, $\Delta_{ab}$ is proportional to the temperature, which is often ignorable for EM waves [4]. Although the Weyl symbol $\Delta_{ab}$ is, strictly speaking, different from the corresponding part of the dielectric tensor in homogeneous medium, the difference is of order $\epsilon \Delta_{ab}$ so it is much smaller than $\epsilon$ and thus can be neglected.

In principle, our general method is also applicable to waves in media with strong spatial dispersion. However, $\varepsilon$ may be hard to calculate in that case unless a medium is homogeneous, and no extrapolation of a known homogeneous-medium model is guaranteed to yield the true general $\varepsilon$.

VII. CONCLUSIONS

In summary, we propose a quasioptical theory of mode-converting wave beams in inhomogeneous media such as plasma. This includes the following. For any given dispersion operator $\hat{D}$ that governs the original wave field $\Psi$, we explicitly calculate the approximate operator $\hat{D}$ that governs the wave envelope $\psi$ to the second order in the GO parameter $\epsilon$. Then, we further simplify this envelope operator by assuming that the gradient of $\psi$ transverse to the local group velocity is much larger than the corresponding parallel gradient. This leads to a parabolic differential equation for $\psi$ (“quasioptical equation”) in the basis of the GO polarization vectors [Eq. (129)]. Our main results can be found in Sec. VII which includes the general quasioptical equation (170) and its special case (180). Scalar and mode-converting vector beams are described on the same footing (Sec. VII). We also explain how to apply this model to EM waves considered as a special case (Sec. VII). In the follow-up papers, we report successful quasioptical modeling of radiofrequency wave beams in magnetized plasma based on this theory.

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Appendix A: Summary of selected notations

Here, we present a summary of selected notations used in the main text.

1. Basic symbols

- $\dagger$ denotes a definition.
- $\hat{\cdot}$ denotes an operator.
- $\dagger$ denotes either a dual vector [Eqs. (95) and (134)] or an adjoint operator.
- $\hat{\cdot}$ denotes the matrix transpose.
- $\hat{\cdot}^*$ denotes the conjugate matrix transpose.
- $\hat{\cdot}^*$ is used only in $\Xi^+, \Xi^+$ defined in Eqs. (139).
- $H$ and $A$ denote the Hermitian and anti-Hermitian parts of an operator [Eqs. (26)] or those of a matrix (Sec. VI B). When applied to a scalar, $H$ and $A$ denote the real and imaginary parts, correspondingly.
- The underline notation $\underline{\cdot}$ is explained in Sec. VI B and is used for matrices with lower indices only.
- $\mathcal{U}$ and $\mathcal{U}^{-1}$ denote the Wigner–Weyl transform and its inverse (Sec. VI A 3), respectively.

2. Index manipulation

On the $n$-dimensional configuration space $M^n$, we assume a general metric tensor $g$ with components $g_{\mu\nu}$. For (co)vector fields on the space (co)tangent to $M^n$, the indices are manipulated as usual, namely, via
\begin{equation}
X_\mu = g_{\mu\nu}X^\nu, \quad \mu, \nu = 0, 1, \ldots, (n - 1). \tag{A1}
\end{equation}
(Summation over repeated indices is always assumed unless specified otherwise.) For vector waves, we also introduce an additional vector space that may or may not be the same as the space tangent to $M^n$ (Sec. VI A). On that space, an additional metric $\gamma$ is introduced, and the indices are manipulated as follows:
\begin{equation}
\Psi_a = \gamma_{ab}\Psi^b, \quad a, b = 1, 2, \ldots, m. \tag{A2}
\end{equation}
In special cases, Eqs. (A1) and (A2) can be equivalent, but that is not a generic situation.

We also introduce the Euclidean metric $\delta$ on the space of amplitude vectors $a$ (Sec. VI D 2),
\begin{equation}
a_s = \delta_{ss'}a^{s'}. \tag{A3}
\end{equation}
For active modes, which are of our primary interest, the mode indices $s$ and $s'$ range from 1 to $N$. The distinction between $s^2$ and $a_s$ is made only for aesthetic reasons (consistency of notation) and can be neglected otherwise.
3. Derivatives

For spatial derivatives, we use the notation that is standard, for example, in general relativity \[51\]. In particular, \(X^{\mu};\mu\) is the divergence; namely,

\[
X^{\mu};\mu = \frac{1}{\sqrt{g_0}} (\sqrt{g_0} X^{\mu})_{,\mu}.
\]

(A4)

Here, \(g_0 \equiv |\det g|\), and

\[
f_{\mu} \equiv \partial_\mu f = \frac{\partial f}{\partial x^{\mu}}, \quad f_{,\mu\nu} \equiv \frac{\partial^2 f}{\partial x^{\mu}\partial x^{\nu}}.
\]

(A5)

For functions of the form \(f(x,k)\), we also introduce the following partial derivatives:

\[
f_{\mu}(x,k) = \frac{\partial f(x,k)}{\partial x^{\mu}}, \quad f_{,\mu\nu}(x,k) = \frac{\partial^2 f(x,k)}{\partial x^{\mu}\partial x^{\nu}}.
\]

(A6)

(A7)

4. Inner products

The inner product for scalar waves on \(M^n\) is

\[
\langle \psi | \Phi \rangle \equiv \int \! d^n\! x \sqrt{g_0(x)} \psi(x) \Phi(x).
\]

(A10)

and for \(m\)-dimensional vector waves on \(M^n\),

\[
\langle \psi | \Phi \rangle_m \equiv \int \! d^n\! x \sqrt{g_0(x)} \gamma_{\alpha\beta}(x) \psi^\alpha(x) \Phi^\beta(x).
\]

(A11)

In particular, in the \(x\) representation,

\[
\Psi^\dagger(x) \equiv \langle \psi(x) | (\Psi^1*, \Psi^2*, \ldots, \Psi^m*) \rangle. \quad (A12)
\]

We also use the dot product

\[
\Psi \cdot \Phi \equiv \Psi^\dagger(x) \Phi(x) = \Psi^*_{\alpha}(x) \Phi^\alpha(x).
\]

(A13)

The inner product on the transverse space \(M^n_{\perp}\) is defined for scalar fields as

\[
\langle \psi | \phi \rangle_{\perp} \equiv \int \! d^n\! x \sqrt{h_0(\zeta, \varrho)} \psi^*(\zeta, \varrho) \phi(\zeta, \varrho).
\]

(A14)

and for the \(N\)-dimensional “active” parts of the vector fields (Sec. \[\ref{V}D\]2) as

\[
\langle a | b \rangle_N \equiv \int \! d^n\! x \sqrt{h_0(\zeta, \varrho)} a^*_\alpha(\zeta, \varrho) b^\alpha(\zeta, \varrho).
\]

(A15)

Appendix B: Auxiliary calculations

Here we present some auxiliary calculations whose results are used in Sec. \[\ref{V}\]

1. Calculation of \(\hat{K}\) and \(U(x)\)

We start with Eq. \((\ref{16})\) for \(\hat{K}\). This equation can be rewritten as follows:

\[
\hat{K} a = \frac{1}{2} \Xi^+ (\tilde{p}_\mu V^\mu_H + V^\mu_H \tilde{p}_\mu - i \ell_\mu V^\mu_H) \Xi a
\]

\[
= - \frac{i}{2 \sqrt{g_0}} \Xi^+ g_0^{1/4} V^\mu_H \Xi a
\]

\[
= - \frac{i}{\sqrt{g_0}} (g_0^{1/4} \Xi^+ V^\mu_H \Xi a - i \Xi^+ V^\mu_H a) - \frac{i}{2} [\Xi^+(V^\mu_H \Xi),_\mu] a - \frac{i}{2} \Xi^+ \ell_\mu V^\mu_H \Xi a
\]

\[
= - \frac{i}{2} (\ln \sqrt{g_0})_\mu V^\mu_H a - i V^\mu_H a - \frac{i}{2} [V^\mu_H, (\Xi^+)_\mu] \Xi a - \Xi^+ \ell_\mu V^\mu_H \Xi a
\]

\[
= - i V^\mu_H a - \Xi^+ (V^\mu_H \Xi)_\mu - \Xi^+ \ell_\mu V^\mu_H \Xi a = 0,
\]

(B1)
where we used Eq. (52) and introduced

\[ V^\mu = \Xi^+ \nabla_\mu \Xi = (\Xi^+ D_H \Xi)^\mu - (\Xi^+ D_H \Xi - \Xi^+ D_H \Xi)^\mu = \Lambda^\mu - \Xi^+ D_H \Xi = \Lambda^\mu - \Xi^+ D_H \Xi = \Lambda^\mu, \]  

(B2)

where, at the end, we used \( \Lambda \sim O(\epsilon) \) (Sec. [V D 3]), so that the last two terms in Eq. (B2) can be neglected.

From Eq. (147), we have \( (\Xi^+)_{\mu} = \delta^{-1} \Xi_{\mu}^\gamma + \Xi^+ \ell_\mu \). Using this and also Eq. (123), one obtains

\[ \bar{K} a = -iV^\nu a_{\mu} - \frac{i}{2} V^\nu a - U a, \]  

(B3)

\[ U = \frac{i}{2} \delta^{-1} (\Xi^H a^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi) = \delta^{-1} (\Xi^H a^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi) A, \]  

(B4)

where the subscript \( A \) denotes the anti-Hermitian part, as usual.

2. Calculation of \( U(x, k) \)

Let us also derive an alternative expression for \( U \) in terms of the partial derivatives \( \nabla^\mu \) instead of the “full” derivatives \( \partial^\mu \). Using Eq. (54), one obtains \( \Xi_{\mu} = \Xi^\nu k_{\nu,\mu} + \Xi_{\mu} \). Then, Eq. (163) can be rewritten as follows:

\[ U = \frac{i}{2} \delta^{-1} (\Xi^H a^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi) k_{\nu,\mu} + \frac{i}{2} \delta^{-1} (\Xi^H a^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi). \]  

(B5)

By differentiating Eq. (162) with respect to \( k_{\mu} \), we obtain

\[ \nabla_\mu \Xi = \Xi^\nu \Lambda_{\nu,\mu} - D_H \Xi^\nu = \Xi \Lambda_{\nu,\mu} - D_H \Xi^\nu, \]  

(B6)

where we used that \( \Lambda \) (but not its partial derivatives) is small. Let us multiply this by \( \gamma \) to obtain

\[ \nabla_\mu \Xi = \Xi^\nu \Lambda_{\nu,\mu} - D_H \Xi^\nu, \]  

(B7)

where the latter equality is the complex conjugate of the former. Using these and \( k_{\nu,\mu} = \theta_{\nu,\mu} = \theta_{\mu,\nu} = O(\epsilon) \), we further obtain, to the leading order,

\[ (\Xi^H a^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi) k_{\nu,\mu} \approx (\Lambda_{\nu,\mu} \Xi^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi) k_{\nu,\mu} \]

\[ \approx (\Lambda_{\nu,\mu} \Xi^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi) k_{\nu,\mu} \]

\[ \approx (\Lambda_{\nu,\mu} \Xi^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi) \]

\[ \approx -(\Lambda_{\nu,\mu} \Xi^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi) \]  

(B8)

where we neglected terms of the second order in \( \epsilon \). We also used the fact that \( \Lambda_\gamma (x, k(x)) \lesssim O(\epsilon) \) by definition of active modes and, in addition, \( \Lambda_\gamma \) are changing slowly, so \( \Lambda_{\mu,\nu} \lesssim O(\epsilon^2) \). (Note that this is only true for \( \Lambda_{\mu,\nu} \) but not for \( \Lambda_{\nu,\mu} \).) Using Eqs. (B8) , we also obtain similarly that

\[ \Xi^H \nabla^\mu \Xi_{\mu} - \Xi^H \nabla^\mu D_H \Xi_{\mu} \approx \Lambda_{\nu,\mu} \Xi^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi_{\mu} \]

\[ \approx \Lambda_{\nu,\mu} \Xi^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi_{\mu} \]  

(B9)

Then,

\[ U = \delta^{-1} (\Lambda_{\mu,\nu} \Xi^\nu \Xi_{\mu} - \Xi^H \nabla^\nu D_H \Xi_{\mu} + \Xi^H \nabla^\nu \Xi_{\mu} - \Xi^H \nabla^\nu \Xi_{\mu} / (2i) \]

\[ = \delta^{-1} (\Lambda_{\mu,\nu} \Xi^\nu \Xi_{\mu} - \Lambda_{\mu,\nu} \Xi^H \nabla^\nu \Xi_{\mu} + \Xi^H \nabla^\nu \Xi_{\mu} A \]  

(B10)

As a reminder, the subscript \( A \) denotes the anti-Hermitian part, and \( \delta^{-1} \) is a unit matrix that only raises the mode index. Equation (B10) is similar to the corresponding result derived in Refs. 18 20. Extra terms are included in those papers in the expressions for \( U \), but they are of the second order in \( \epsilon \) and could be neglected.
3. Calculation of $\hat{G}$

Consider Eq. (174) for $\hat{G}$ derived under the assumption of the quasioptical ordering. Here, we simplify that equation and derive the explicit formula (175). First, note that

$$\Xi^+ \hat{D}_H \Xi a \approx \frac{1}{2} \Xi^+ \tilde{p}_\mu \Theta^+_{m^*} \tilde{p}_\nu \Xi a \approx -\frac{1}{2h_0^{1/4}} [\Xi^+ D_H^{(\sigma \delta)} \Xi (h_0^{1/4} a),_\sigma,] \Xi,$$

(B11)

where we used Eq. (122). Also, using Eq. (121) for $\hat{D}_H$, one obtains

$$\Xi^+ \hat{D}_H \Xi a \approx -\Xi^+ V_H^{(\sigma \delta)} \Xi \approx -\Xi^+ V_H^{(\sigma \delta)} \Xi a,_{\sigma \delta}$$

$$\approx -h_0^{-1/2} [\Xi^+ V_H^{(\sigma \delta)} \Xi \Xi^+ V_H^{(\sigma \delta)} \Xi (h_0^{1/4} a),_\sigma,].$$

(B12)

Here, in the first line, we substituted Eq. (121) for $\hat{D}_H$ and also Eq. (14) for $\tilde{p}_\mu$. Then, we only kept the terms involving derivatives of $a$ along the perpendicular direction of the wave beam. The parenthesis in the indices denote symmetrization; namely, for any $A$ and $B$,

$$A^{(\sigma \beta)} = \frac{1}{2} (A^{\sigma \beta} + A^{\beta \sigma}).$$

(B13)

The metric factor $h_0$ has been added to keep the operator self-adjoint under the inner product (169). (Accounting for the inhomogeneity of $h$ in $\hat{G}$ is beyond the accuracy of our theory. However, keeping $\hat{G}$ self-adjoint is convenient and physically meaningful, for it is known that the exact operator is not responsible for dissipation.) Hence,

$$\hat{G} \approx -\frac{1}{2h_0^{1/4}} [\Phi^{(\sigma \delta)} (h_0^{1/4} a),_\sigma,] \Xi, \quad \Phi^{(\sigma \delta)} = \Xi^+ D_H^{(\sigma \delta)} \Xi - 2\Xi^+ D_H^{(\sigma \delta)} \Xi \Xi H \Xi^+ D_H^{(\sigma \delta)} \Xi.$$

(B14)

In order to simplify the expression for $\Phi^{(\sigma \delta)}$, note that

$$\Xi^+ D_H^{(\sigma \delta)} \Xi = (\Xi^+ D_H \Xi)^{(\sigma \delta)} - \Xi^+ \Xi D_H \Xi - \Xi^+ D_H \Xi \Xi^+ \Xi = (\Xi^+ \Xi \Lambda \Xi^+ \Xi - \Xi^+ \Xi \Lambda \Xi - \Xi^+ \Xi \Lambda \Xi - \Xi^+ \Xi \Lambda \Xi)^{(\sigma \delta)} \Xi,$$

(B15)

where we used $\Xi^+ \Xi = 0$ (and thus $\Xi^+ \Xi^+ \Xi = -\Xi^+ \Xi^+ \Xi$) and neglected terms proportional to $\Lambda = O(\epsilon^2)$. Hence,

$$\Xi^+ D_H^{(\sigma \delta)} \Xi \Xi^+ \Xi \Lambda \Xi = \Xi^+ \Xi \Xi \Lambda \Xi \Xi^+ \Xi \Lambda \Xi^+ \Xi = \Xi^+ \Xi \Xi \Lambda \Xi \Xi^+ \Xi \Lambda \Xi^+ \Xi = \Xi^+ \Xi \Xi \Lambda \Xi \Xi^+ \Xi \Lambda \Xi^+ \Xi.$$

(B16)

Using Eqs. (149) for $\Xi$ and $\Xi \Xi$ like we did above, Eq. (149) for $D_H$, and the fact that $\Lambda = O(\epsilon)$, we finally obtain

$$\Phi^{(\sigma \delta)} = \Xi^+ (\Xi \Lambda \Xi^+ \Xi + \Xi \Lambda \Xi^+ \Xi)^{(\sigma \delta)} \Xi - 2\Xi^+ \Xi \Lambda \Xi^+ \Xi \Xi = \Xi^+ (\Xi \Lambda \Lambda \Xi^+ \Xi + \Xi \Lambda \Lambda \Xi^+ \Xi + \Xi \Lambda \Lambda \Xi^+ \Xi + \Xi \Lambda \Lambda \Xi^+ \Xi)^{(\sigma \delta)} \Xi - 2\Xi^+ \Xi \Lambda \Xi^+ \Xi \Lambda \Xi^+ \Xi = \Lambda^{(\sigma \delta)} + \Xi^+ \Xi \Lambda^{(\sigma \delta)} + \Xi^+ \Xi \Lambda^{(\sigma \delta)} + \Xi^+ \Xi \Lambda^{(\sigma \delta)} \Xi = \Lambda^{(\sigma \delta)} + \Xi^+ \Xi \Lambda^{(\sigma \delta)} + \Xi^+ \Xi \Lambda^{(\sigma \delta)} - \Lambda^{(\sigma \delta)} \Xi^+ \Xi \Lambda^{(\sigma \delta)} - \Lambda^{(\sigma \delta)} \Xi^+ \Xi \Lambda^{(\sigma \delta)}.$$

(B17)

where we substituted $\Lambda^{(\sigma \delta)} \approx V^{(\sigma \delta)}$ [Eq. (B2)]. The quasioptical approximation implies that $V^{(\sigma \delta)}$ is close to a scalar matrix, so the commutator $[\Xi^+ \Xi^{(\sigma \delta)}], \Xi^+ \Xi \Lambda^{(\sigma \delta)}]$ can be neglected. Hence, $\Phi^{(\sigma \delta)} \approx \Lambda^{(\sigma \delta)}$. 

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This also allows applications beyond plasma physics, as will be discussed in the future publications.

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