Physics in non-fixed spatial dimensions

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Abstract. We study the quantum statistical electronic properties of random networks which inherently lack a fixed spatial dimension. We use tools like the density of states (DOS) and the inverse participation ratio (IPR) to uncover various phenomena, such as unconventional properties of the energy spectrum and persistent localized states (PLS) at various energies, corresponding to quantum phases with zero-dimensional (0D) and one-dimensional (1D) order. For small ratio of edges over vertices in the network \(RT\) we find properties resembling graphene/honeycomb lattices, like a similar DOS containing a linear dispersion relation at the band center at energy \(E=0\). In addition we find PLS at various energies including \(E\approx-1,0,1\) and others, for example related to the golden ratio. At \(E=0\) the PLS lie at disconnected vertices, due to partial bipartite symmetries of the random networks (0D order). At \(E\approx-1,1\) the PLS lie mostly at pairs of vertices (bonds), while the rest of the PLS at other energies, like the ones related to the golden ratio, lie at lines of vertices of fixed length (1D order), at the spatial boundary of the network, resembling the edge states in confined graphene systems with zig-zag edges. As the ratio \(RT\) is increased the DOS of the network approaches the Wigner semi-circle, corresponding to random symmetric matrices (Hamiltonians) and the PLS are reduced and gradually disappear as the connectivity in the network increases. Finally we calculate the spatial dimension \(D\) of the network and its fluctuations. We obtain both integer and non-integer \(D\) and a logarithmic dependence on \(RT\). In addition we examine the relation of \(D\) and its fluctuations to the electronic properties derived. Our results imply that universal physics can manifest in physical systems irrespectively of their spatial dimension. Relations to emergent spacetime in quantum and emergent gravity approaches are also discussed.

1. Introduction
The concept of spatial dimension has been central in physics for many centuries. Physics problems are usually formulated mathematically on geometrical manifolds with a well defined spatial dimension, which is taken as an input. However in the last decades problems like quantum and emergent gravity and geometry, have shown that a reconsideration of this concept is required. For example models like, string theory formulated on continuous manifolds (membranes) of variable dimensions higher than three [1, 2], or discrete/network/graph models like loop quantum gravity [3, 4], causal set cosmology [5, 6, 7, 8] the Wolfram-Gorard-Piskunov model [9, 10] and others [11, 12, 13, 14], hint that spatial dimension as a fundamental concept has to be revised, if spacetime and its dimensionality is to be recovered as emergent from other more fundamental structures. Also, problems like the small value of the cosmological constant, which could be an intrinsic property of spacetime as it emerges from other structures is a major related issue. All the above are strong suggestions that a theory reproducing the properties of spacetime in our universe, gravity and quantum effects could be spatio-dimensionless in its nature. Apart from...
quantum and emergent gravity, physics in non-fixed spatial dimensions can be useful in characterizing
quantum and other phases emerging in many-body networks\cite{15, 16, 17, 18}, that could be also relevant
to emergent spacetime and its dimensionality, through entanglement.

In this paper we consider spatio-dimensionless physical systems, modelled via the most structureless
models, random networks\cite{19, 20}, which are discrete mathematical models that do not inherently have
a fixed spatial dimension. We consider uniform networks and study their quantum statistical electronic
properties, using tools like the density of states (DOS) and the inverse participation ratio (IPR). We find
that for small ratio of edges over vertices in the network $RT$ its electronic properties resemble those of
graphene/honeycomb lattices with properties like linear dispersion relation at the band center at energy
$E=0$ and edge states concentrated at the spatial boundary of the network\cite{21, 22, 23, 24, 25}. We find
persistent localized states (PLS) at various energies including $E=\pm 1, 0, 1$ and others, for example related
to the golden ratio, via the study of the DOS and the IPR. These states comprise primarily of single
unconnected vertices (0D order) at $E=0$ and one-dimensional (1D) clusters (1D order) for the other
energies, where the wavefunction is localized. The energy of these 1D ordered states is determined
by the dispersion of 1D tight-binding chains formed by the 1D clusters of various lengths. The 0D
ordered states at $E=0$ spread over the whole network, while the 1D ordered states are concentrated at
the spatial boundary of the network, resembling the edge states in confined graphene systems with zig-
zag edges\cite{21, 22, 25}. As the number of edges between the vertices in the network increases, the DOS
approaches the Wigner semicircle, corresponding to ensembles or random symmetric matrices, and the
number of PLS is gradually reduced until they disappear for large $RT$. Finally we calculate the spatial
dimension $D$ of the networks, i.e. the dimension of the manifold in which they can be embedded. We
find that the networks which resemble graphene have $D \approx 2$, fitting into a 2D plane. The dimension
is increased logarithmically as $RT$ increases and the network becomes more dense as more edges are
added between its vertices. Additionally $D$ reaches integer values like $D \approx 4$ apart from non-integer
ones. Our results show that universal phenomena can manifest in physical systems irrespectively of their
spatial dimension. In addition we briefly discuss the relation of our results to emergent spacetime and
relativity/gravity from discrete mathematical models.

2. Network model

Random networks are discrete mathematical models comprising of many vertices randomly connected
with edges\cite{19, 20}. They can be used to express relations between different quantities, which is
useful for example in simulating various real world behaviors such as propagation of behavioral
patterns in social networks or in communications technology. Random networks are also useful in
studying various localization phenomena in mesoscopic and random waveguide systems, theoretically
and experimentally\cite{26}. The probability distribution of the number of edges at each vertex (degree $deg$)
determines the type of network. In the current paper we choose the most simple type, uniform networks
whose degree at each vertex follows a hypergeometric function. Essentially we have a fixed number of
vertices $n$ and edges $m$ between them randomly distributed forming a random network $G(n, m)$. All
configurations of the edges among the vertices, whose number is $\binom{m}{n}$, have an equal probability to
appear. The mean degree for each vertex in the uniform network is given by $<deg>=\frac{2m}{n}$ which can
be interpreted as the average connectivity in the network. Note, there are no multi-edges or loops namely
the network is a simple undirected graph. We define the ratio of vertices over the edges in the network
$RT = \frac{m}{n}$, which is half its average connectivity $RT = \frac{<deg>}{2}$.

We examine how a quantum particle behaves as it propagates through the tight-binding lattice formed
by the random network. i.e. the electronic properties of the random network. The Hamiltonian of the
system can be written as

$$H = \sum_{<i,j>} (c^\dagger_i c_j + h.c.)$$

where $c^\dagger_i (c_i)$ is the creation (annihilation) operation for a particle at vertex $i$ in the random network.
Figure 1. Various random networks for different values of the ratio between the number of edges over the number of vertices $RT$. The number of vertices is $n=3000$ for all the cases shown. For $RT = 0.67$ the network is almost planar with few overlapping edges, meaning that it can be fitted approximately into a 2D plane. As more edges are added $RT$ increases and the network becomes spatially more dense.

The indexes $i,j$ are randomly sampled and create $m$ pairs, representing the edges between the vertices in the network. The Eq. 1 is a random matrix with a fixed number of elements $m$ (the number of edges) with the value of one, randomly distributed inside the matrix. In certain cases, such random network Hamiltonians have been shown to belong to the same universality class as models used to study Anderson localization phenomena\[26\]. For example regular tight-binding lattices like the square or the cubic lattice with a random on-site potential (Anderson model). The randomly distributed elements in the Hamiltonian matrix Eq. 1 induce interference effects leading to localization of the wavefunctions as in the Anderson model or in random matrix theory (RMT) models.

3. Energy spectrum
In this section we examine the properties of the energy spectrum of the random networks via the distribution of eigenvalues of Eq. 1 i.e. the density of states (DOS) of the random network. Firstly there are two limiting cases worth mentioning. When all the $n$ vertices in the network are disconnected ($m=0$) then all the elements of the Hamiltonian Eq. 1 are zero resulting in $n$ zero eigenvalues and a DOS localized at energy $E=0$. On the other hand when all the vertices in the network are connected
Figure 2. The distribution of eigenvalues for the networks, i.e. the density of states (DOS) for various values of the ratio of edges over vertices $RT$, represented by the black boxes. We have considered 100 different configurations/realizations of the network (runs). The DOS in the upper panels contain features from graphene/honeycomb lattice systems, as shown from the comparison with the graphene DOS represented by the blue dashed curve. A linear behavior of the DOS is shown in the insets as for graphene. As $RT$ increases and the network becomes more dense the DOS starts resembling the distribution of eigenvalues of random symmetric matrices, the Wigner semi-circle represented by the red curves. A prominent peak at $E=0$ is present in all cases coming from many states at this energy persistent for all networks.

with each other, the Hamiltonian is full apart from its diagonal which has all its elements equal to zero. In the intermediate case, for an arbitrary number of vertices $n$ and edges $m$ in the network we have found two major forms for the DOS. The results are shown in Fig. 2. In the plots we have removed the eigenvalues $E=0$ coming from disconnected vertices in the network which lead to lines/columns of zeros in the Hamiltonian. For low ratio $RT$ in the two upper panels of Fig. 2 the DOS of the random network ($\rho(E)$) resembles that of a graphene/honeycomb lattice as can be seen from the comparison between the blue dashed line and the histogram. The DOS of graphene is given by

$$
\rho(E) = \begin{cases} 
\frac{1}{2\pi^2} \frac{4E}{(E+1)^2(3-E)} K \left( \sqrt{\frac{16}{(E+1)^2(3-E)}} \right), & 0 \leq E < 1 \\
\frac{1}{2\pi^2} K \left( \sqrt{\frac{16}{(E+1)^2(3-E)}} \right), & 1 \leq E \leq 3
\end{cases}
$$

(2)

for positive $E$, where $K$ denotes the elliptic integral. The DOS for negative $E$ is produced by applying $E \rightarrow -E$ in the above equation. The main features are two peaks at energies $E=-1,1$ appearing in both cases and a linear behavior $\rho(E) = \alpha |E|$ near $E=0$ shown in the insets of the upper panels. The DOS
Figure 3. The inverse participation ratio (IPR) for various values of the ratio of the number of edges over the number of vertices in the network, for 100 runs. The states across the whole energy spectrum become in average less localized as the network becomes more dense, with increasing \( RT \). Various persistent localized states (PLS) can be distinguished at various energies including \( E=-1,0,1 \) where IPR forms vertical lines with high values. The number of PLS at \( E=-1,1 \) is gradually decreased with increasing \( RT \).

of graphene also reduces to a linear form \( \rho(E) = \frac{1}{\pi \hbar v_f |E|} \) near \( E=0 \), where \( v_f = \frac{\sqrt{3}a}{2\hbar} \) is the Fermi velocity of the electrons which is represented by the blue dashed curve in the insets of Fig. 2 (\( a \) is the lattice constant). We note that for low \( RT \) there is a large probability that the network contains many disconnected components as shown in Fig. 1. We have found that these components largely contribute in the number of states appearing at \( E=-1,1 \). Another main feature is a peak at \( E=0 \) which appears also for confined graphene systems with zig-zag edges like flakes and nanoribbons, due to edge states at \( E=0 \) [21, 22, 25]. We examine the localization properties of these states in the following sections. The resemblance to graphene can be seen schematically by the network presented in the upper left panel of Fig. 1. The network consists primarily of polygons (cycles) whose corners consist of vertices that are connected mostly to three or four neighboring vertices with \( < deg > \approx 3 \). In addition there are very few crossing edges, i.e. the network is approximately planar, meaning that it can be fitted into a two-dimensional (2D) plane. This structure is very similar to a graphene/honeycomb lattice which is a 2D plane of hexagons with every lattice site having three neighboring sites, i.e. the connectivity is three.

A linear dispersion relation \( E \sim k \) can be derived by assuming a spherical symmetry in the Brillouin zone [27]. Since the network for \( RT=0.67 \) resembles a 2D manifold we can assume a coordinate system with two parameters \( x, y \) to describe it. We can define two wavevectors \( k_x, k_y \) for a particle in a two-
Figure 4. Some examples of small networks demonstrating the partial bipartite symmetry. Red vertices form a sublattice of disconnected vertices (A) while the rest of the vertices form another sublattice (B). The difference in the number of vertices/sites between the two sublattices \( n_A - n_B \) leads to an equal number of \( E=0 \) states. The wavefunction of these states is zero on the B sublattice which has the least number of sites.

Dimensional box

\[
\begin{align*}
  k_x &= \frac{\pi n_x}{L} \\
  k_y &= \frac{\pi n_y}{L}
\end{align*}
\]  

(3)

where \( L \) is the size of the box along the \( x \) and \( y \) directions. The total number of states \( N \) in a volume/area inside the Brillouin zone is

\[
N = \sum_{n_x,n_y} \Delta n_x \Delta n_y = \frac{L^2}{\pi^2} \sum_{k_x,k_y} \Delta k_x \Delta k_y.
\]  

(4)

Assuming a circular(rotational) symmetry in the Brillouin zone and using polar coordinates we can write

\[
\sum_{k_x,k_y} \Delta k_x \Delta k_y = \int_0^k kdk \int_0^{2\pi} d\phi = \pi k^2,
\]  

(5)

where \( k^2 = k_x^2 + k_y^2 \). Then we have

\[
N = \frac{L^2 k^2}{\pi}.
\]  

(6)

The total number of states per system area is

\[
n = \frac{N}{L^2} = \frac{k^2}{\pi} \Rightarrow \frac{dn}{dk} = \frac{2k}{\pi}.
\]  

(7)

Then we can derive a linear energy dispersion relation \( E(k) \) by taking account of the linear behavior of the DOS near \( E=0 \) \( (\rho(E) = \alpha |E|) \) as follows,

\[
\rho(E) = \frac{dn}{dE} \Rightarrow \rho(E)dE = \frac{dn}{dk}dk \Rightarrow \pi \int \alpha |E|dE = \int 2kdE \Rightarrow E = \frac{1}{\sqrt{\pi \alpha}} |k|.
\]  

(8)
Figure 5. Upper left panel: The process of calculating the spatial dimension $D$ of the network by counting how the number of vertices contained into a spherical volume of radius $L$ grows with $L$. The number of vertices added at each step by increasing $L$ are determined by the edges/connections between the vertices. Upper right panel: The linear scaling of $\ln(V)$ with $\ln(L)$ gives the value of the dimension $D$ from the relation $\ln(V) = \beta + D\ln(L)$ where $\beta$ is a fitting parameter. Lower left panel: The mean value of $D$ (open circles) for 1000 configurations/realizations (runs) of the network versus the ratio $RT$. Integer and non-integer values like $D \approx 4$ are reached. The dimension follows a logarithmic behavior $D = \gamma + \delta\ln(RT = \text{deg})$ represented by the red curve with fitting parameters $\gamma, \delta$. Lower right panel: The probability distribution of $D$ follows the normal (Gaussian) distribution whose variance (fluctuations) is decreased with increasing $RT$ as shown in the inset and is minimized at $RT = 2$ where $D \approx 4$.

The above dispersion is analogous to the linear energy dispersion for relativistic massless particles $E = \hbar c k$, followed also by graphene near $E=0$, where $c$ is replaced by the Fermi velocity of the electrons $v_f = \frac{\sqrt{3}a}{2\hbar}$. For the networks from Eq. 8 we get an effective speed of light $v_{nt} = \frac{1}{\hbar\sqrt{\alpha}}$ where $\alpha$ can be obtained by the linearity of the DOS near $E=0$ from $\rho(E) = \alpha|E|$. The velocity $v_{nt}$ could represent an upper limit for the transmission speed of information inside the network. Notice that the same relation $v_f = \frac{1}{\hbar\sqrt{\alpha}}$ is followed also by graphene, meaning that $v_{nt}$ and $v_f$ will have the same order of magnitude ($10^6\text{ m/s}$), assuming that the hopping $t$ and the lattice constant $a$ are the same for both systems.

The peaks at $E=\pm 1,0,1$ and the linear dispersion near $E=0$, gradually disappear as $RT$ increases as can be seen in the lower panels of Fig. 2. As more edges are added in the network its Hamiltonian starts to resemble ensembles of random symmetric matrices, from random matrix theory (RMT), whose distribution of eigenvalues follows the Wigner semicircle, as the size of the matrices approach infinity.
The Wigner semicircle function is defined as

$$f(x) = \begin{cases} \frac{2}{\pi R^2} \sqrt{R^2 - x^2} & -R \leq x \leq R \\ 0 & \text{otherwise} \end{cases}$$  \tag{9}$$

The above equation for various $R$ is represented by the red continuous curve in Fig. 2 and describes sufficiently the DOS for the cases shown in the lower panels. Some features of the DOS at its edges are also captured by the Wigner semicircle in the upper panels for low $RT$. In overall we see that the DOS of the random network contains features from both graphene/honeycomb lattices and disordered systems described by RMT. The contribution of these two systems on the electronic properties of the network depends on the value of $RT$, which determines the spatial density of the network. Sparse networks for low $RT$ resemble graphene, while dense networks for large $RT$ resemble disordered systems described by RMT.

The main difference with regular/periodic lattices such as the square, the cubic and the hypercubic is that their DOS reaches gradually a Gaussian distribution as the lattice connectivity, i.e. the number of neighbors at each site in the lattice goes to infinity.

4. Localization properties

In Fig. 3 we show the inverse participation ratio (IPR) for the wavefunctions of Eq. 1 defined as

$$\text{IPR}(E) = \sum_{i=1}^{n} |\Psi_i(E)|^4$$  \tag{10}$$

where $i$ runs over the all the vertices and $\Psi$ is the corresponding wavefunction amplitude. We show IPR for 100 different configurations/realizations of the network (runs). There are two major features that we can observe. As $RT$ increases and the network becomes spatially more dense, IPR decreases in average. This means that the corresponding wavefunctions become in average less localized with increasing $RT$. We have verified that these wavefunctions corresponding to low values of IPR spread over the whole network with random amplitudes at each vertex, resembling chaotic wavefunctions encountered in disordered tight-binding lattices\[28, 29\]. Another major feature that we can observe in Fig. 3 is persistent localized states (PLS) appearing at various energies corresponding to high values of IPR, forming persistent vertical lines for all $RT$. We can clearly distinguish PLS at energies $E=\pm 1, 0, 1$ which correspond to peaks in the DOS shown in Fig. 2. As shown in the inset of Fig. 3 the number of these states are reduced as $RT$ increases. We have found that the states at $E=0$ are localized at vertices along the whole network which are not connected by edges. All the $E=0$ states are localized on the same vertices albeit with different amplitudes. This is one main feature of the $E=0$ states for all $RT$ and differs from confined graphene systems with zig-zag edges which contain edge states at $E=0$ concentrated at the boundary of the system\[21, 22, 25\]. We can consider that the PLS at $E=0$ in the network, form a quantum phase of zero-dimensional (0D) order, since the wavefunction lies only on disconnected vertices. On the other hand we have found that the PLS at $E=\pm 1, 1$ are concentrated only at the spatial boundary of the network and localize mostly on pairs of vertices connected with an edge (bonds). This bond-ordered localized phase disappears for large $RT$ indicating a phase transition at some critical value of $RT$. On the other hand the $E=0$ states persist even for large $RT$ as shown by the peaks in the DOS in Fig. 2. Other cases of PLS appear also at other energies for example at $E = -\phi, -1 + \phi, 1 - \phi, \phi$ where $\phi = \frac{1 + \sqrt{5}}{2}$ is the so called golden ratio. The wavefunctions of these states are localized primarily on lines of vertices (1D clusters) with fixed length, for example lines of four vertices for the energies related to the golden ratio, instead of bonds, and disappear for large $RT$ as for the PLS at $E=\pm 1, 1$. All the phases expect those at $E=0$ can be considered as quantum phases of one-dimensional (1D) order. We have found that the energies of these 1D ordered states are coming from tight-binding chains whose length is determined by
the length of the 1D clusters contained in the 1D ordered phases. The dispersion of a tight-binding chain with \( N \) sites is given by

\[
E = 2 \cos \frac{n\pi}{N + 1}, \quad n = 1, 2, \ldots, N.
\]  

(11)

The networks contain mostly the eigenvalues of the above equation for low \( N \) corresponding to short chains. For example some of the \( E=1,1 \) eigenvalues in the network come from Eq. (11) for \( N=2 \) (chain with two sites) and the corresponding wavefunctions are localized along bonds of vertices. The case \( N=4 \) gives the energies related to the golden ratio. In general, the wavefunctions of the states in the network coming from Eq. (11) are localized along lines of vertices of length \( N \) and lie at the spatial boundary of the network, connected with one edge to the rest of the vertices, resembling dangling bonds. In this sense these 1D ordered states at energies coming from Eq. (11) bear similarity to the edge states appearing in confined graphene systems, like flakes and ribbons, which are concentrated along the zig-zag edges of these systems[21][22][25] and persist even with disorder[29]. Also, a large contribution to the eigenvalues Eq. (11) comes from disconnected chains in the network of length \( N \).

5. Partial bipartite symmetry

The appearance of the \( E=0 \) wavefunctions only on vertices of the network that are not directly connected with an edge can be explained as a consequence of a partial bipartite symmetry of the random networks[30][31][32]. The random network if seen as a lattice can always be split into two sublattices, say A and B with \( n_A \) and \( n_B \) sublattice sites respectively. Additionally it can always be splitted in such a way that there are no edges between the sites in one of the sublattices, say A. Then the Hamiltonian of the system can be simplified if written in the basis of A and B sites, as

\[
H = \begin{bmatrix}
0 & H_{AB} \\
H_{AB}^\dagger & H_{BB}
\end{bmatrix}.
\]  

(12)

We can write the Schrödinger difference equations centered on A and B sites as

\[
E\Psi_{A,i} = \sum_j \Psi_{B,j}
\]

\[
E\Psi_{B,i} = \sum_j \Psi_{A,j} + \sum_j \Psi_{B,j}.
\]  

(13)

By setting \( E=0 \) we can see that the upper equation in Eq. (13) transforms to \( \sum_j \Psi_{B,j} = 0 \) which is a set of \( n_A \) equations with \( n_B \) unknowns. If \( n_A > n_B \) then this set of equations can only be satisfied by setting \( \Psi_{B,j} = 0 \) since there are more equations than unknowns i.e. the system of equations is overdetermined. Then the amplitudes on A sublattice \( \Psi_{A,j} \) can be calculated by the lower equation in Eq. (13) which is a system of \( n_B \) equations with \( n_A \) unknowns and gives \( n_A - n_B \) linearly independent solutions. Consequently if the random network is split in two sublattices A and B, with one of them having no edges between its vertices, say A, whose number of vertices is larger than the number of vertices for the other sublattice B, then there will always exist at least \( n_A - n_B \) states at \( E=0 \). In addition the wavefunction of these states will have zero amplitudes on the sublattice with the smallest number of vertices, sublattice B. We remark that for periodic lattices like the square lattice which satisfy the full bipartite symmetry the value \( n_A - n_B \) will depend on the shape of the boundary. For example for a square sample of square lattice we will always have \( n_A - n_B = 1 \)[30] whereas in graphene flakes \( n_A - n_B \) depends on the size of the system[32]. Depending on the network configuration(run) there exist in general many different choices of the sublattice whose vertices are disconnected, corresponding to different partial bipartite symmetries. Note that in rare cases and more frequently in small networks with a low number of vertices and edges, some of equations for the sublattice A might be identical reducing the total number of equations for A. If the number of these equations becomes smaller than the number of B sites then the argument presented above breaks down and the wavefunctions spreads on both sublattices for the \( E=0 \) states. Nevertheless the partial bipartite symmetry is present for any type of random network and will lead to \( E=0 \) states with the properties described above in most cases of networks with low spatial density corresponding to low ratio of edges over vertices \( RT \).
6. Spatial dimension

In this section we examine the spatial dimension of the random network, i.e. the dimension $D$ of the ambient space in which it can be embedded so that it contains no crossing edges. This $D$ is essentially the minimum dimension of the space in which the network can be embedded. The calculation can be achieved by examining how the connected vertices fill an ambient space of dimension $D$, which can take integer or non-integer values. One example of this process can be seen schematically in the upper left panel of Fig. 5. At each step in the scaling characterized by the linear length scale $L$ we count the number of vertices that are connected directly with edges to the previous layer with $L-1$. This number is added in the overall number $V$, which counts the total number of vertices contained in a spherical volume of radius $L$. Therefore by calculating the growth(scaling) of $V$ with $L$ we can calculate the dimension $D$ of the network via

$$V \sim L^D.$$  \hspace{1cm} (14)

Some results of this process for different $RT$ can be seen in the upper right panel of Fig. 5. In addition in the lower left panel we plot the average $<D>$ for 1000 runs (configurations/realizations of the random network). For $RT = 0.67$ the network is almost planar as $D \approx 2$, fitting into a 2D plane. This is one of the reasons that the sparse networks resemble graphene/honeycomb lattices and have similar electronic properties, as we have shown in the previous sections by the comparison between the DOS of the two systems. As the network becomes spatially more dense due to the increased connectivity, $D$ grows logarithmically with $RT$ represented by the fitting red curve in the lower left panel of Fig. 5. Integer and non-integer dimensions like $D \approx 2$ and $D \approx 4$ are reached. However the electronic properties of the networks, as we have shown in the previous sections, do not resemble the respective regular lattices of integer dimension $D$, like hypercubic lattices, whose DOS approaches a Gaussian distribution as $D$ increases. Instead the DOS of the random networks follows the Wigner semicircle, resembling disordered systems described by random symmetric Hamiltonians. A linear dependence of the spatial dimension on the connectivity is followed by regular lattices like the chain, square, cube and hypercube. The connectivity $deg$ at each site in these lattices is two,four, six and eight giving the linear dependence $deg = 2D \Rightarrow D = \frac{deg}{2}$. Since $RT = \frac{<deg>}{2}$ our result in the lower left panel of Fig. 5 shows that the dimension of the random network depends logarithmically on its average connectivity at each vertex, as $D \sim ln(<deg>)$, unlike the regular lattices. In the lower right panel of Fig. 5 the probability distribution of $D$ is shown along with the variance $\sigma^2$(fluctuations) in the inset. All the distributions can be fitted by normal (Gaussian) functions represented by the solid curves in the lower right panel of Fig. 5. As can be seen in the inset, the dimensional fluctuations are reduced as $RT$ increases, corresponding to higher dimension and spatially denser networks and are minimized at $RT=2$ where $D \approx 4$.

7. Emergent spacetime and relativity/gravity

In this section we discuss briefly the relation of our model with the notion of emergent spacetime and relevant relativistic and gravitational effects. A natural way, in the sense of being the most structureless lacking an inherent spatial dimension, would be to consider spacetime or space as a collection of random relations between abstract objects, sometimes called atoms of space\textsuperscript{[3, 4, 5, 6, 7, 8, 9, 10, 12, 13, 14]}, with various constraints. These relations can be naturally modeled by random networks, with the abstract objects(relations) represented by the vertices(edges), and the constraint in our case being the fixed ratio defined as the number of edges over the number of vertices $RT$. Essentially space could be represented as a quantum mechanical superposition of the different realizations of the random networks for fixed $RT$. Then one of the main issues would be if the network can be reduced to a continuous manifold at a limit of large number of vertices and whether geometry can emerge in such a system. For instance, when $RT = 0.67$ in the random network that we studied, its spatial dimension is $D \approx 2$ resembling a 2D plane. Moreover its electronic properties, as indicated by the DOS, resemble those of graphene, which is a 2D honeycomb lattice with the electrons following the Dirac equation near low energies $E=0$. Electrons in graphene, at $E=0$, behave relativistically as massless particles with the speed of light $c$ being replaced by the Fermi velocity of the electrons $v_f$, with a linear energy dispersion $E = \hbar v_f k$. The
similarities of the electronic properties between graphene and spatially sparse random networks, pose an
intriguing question, whether relativistic effects are also encountered in those random networks. As we
have shown in the previous sections the effective speed of light in the networks is \( v_{nt} = \frac{1}{\hbar \sqrt{\pi \alpha}} \) where \( \alpha \) is determined by the DOS near \( E=0 \) from \( \rho(E) = \alpha |E| \). The value of \( v_{nt} \) could represent an upper
limit for the transmission speed of information inside the network in the same way that \( c \) sets an upper
limit for velocities in our universe. We remark also that \( D \approx 4 \) is reached at \( RT=2 \), when the number of
edges is double the number of vertices, where the dimensional fluctuations are minimized. This result
would be tempting to relate to the 4D spacetime manifold in the models describing our universe, like
in the Einstein field equations (EFE). In addition such a universe would be physically more stable, if its
dimensional fluctuations are minimized, which we observe at \( D \approx 4 \). We note that time in network/graph
models can be integrated either in the network itself\([6, 7, 8]\) or can be treated as an external updating
rule\([9, 10]\).

In addition gravitational effects could potentially emerge in a random network approach to space.
Various definitions of the curvature at each vertex in the network exist\([33, 34, 35, 36, 37, 38]\), the simplest
one being for tree and grid graphs

\[
K(i) = 1 - \frac{\text{deg}(i)}{2},
\]

(15)

where \( \text{deg}(i) \) is the number of connected neighbors at vertex \( i \) in the network (degree). In the above
definition triangles, tetrahedra and other higher \( n \)-dimensional cell complexes formed in the network
structure are ignored. Since the average degree for a uniform network is \( < \text{deg} > = 2m/n \), using
Eq. [15] we have \( < K > = 1 - m/n \) for the curvature averaged over all the vertices in the network.
We notice that \( < K > \) transitions from a positive to negative value at \( m/n = RT = 1 \), where the
dimension of the network is close to \( D = 3 \) and \( < K > = 0 \). This implies that if space is modeled
with uniform random networks, having a zero or a slightly positive/negative curvature, which is the case
for our current universe, would require a dimensionality close to \( D = 3 \). Moreover the Ricci curvature
tensor can be calculated in graph/network models\([33, 35]\). In addition geodesics can be defined in the
network as simply the path between two vertices that has the least number of edges. Again an intriguing
question arises, whether the curvature in the network can be related to the Riemannian curvature used in
Einstein field equations (EFE) to describe how the geometry of spacetime, influenced by the mass-energy
g distribution, gives rise to gravitational effects. Collections of vertices in the network could represent
mass while edges can be related to the energy\([9, 10]\). In this sense the PLS that we have found in the
networks at various energies, comprising of lines of vertices (1D order), could be loosely interpreted as
emergent particles, whose mass is determined by the number of vertices in these 1D structures. Finally
a random network approach to space would yield a vertex density, which could be connected to the
cosmological constant \( \Lambda \) in EFE. Loosely speaking, \( \Lambda \) can be interpreted as an intrinsic property of
space (energy density of the vacuum \( \rho_{\text{vacuum}} \)) as it emerges from a random network model, like the
uniform network that we have considered in the current paper. Since we have shown that the number of
vertices \( V \) grows approximately as \( V = \epsilon + \zeta L^D \) then the density of vertices \( \rho_{nt} = \frac{V}{L^D} \) would be
simply \( \rho_{nt}(L, D) = \epsilon L^{-D} + \zeta \). The constants \( \epsilon \) and \( \zeta \) could be related to \( \Lambda \) or \( \rho_{\text{vacuum}} \), for example
near \( RT = 1 \) where \( D \approx 3 \), with the necessary substitutions of fundamental constants, the vertex mass
corresponding to the Planck mass and the edge length to the Planck length.

The general question of how geometry arises from discrete mathematical models is a fundamental
issue in both mathematics and physics. Another related problem is how to construct mathematics and
physical theories in non-integer/fractional spatial dimensions or in the complete absence of them, like
the approach that we follow in the current manuscript, by using random networks.

8. Summary and Conclusions

We have studied the quantum statistical electronic properties of random networks which inherently lack
a fixed spatial dimension. We have found that the electronic properties of network contain features
from graphene/honeycomb lattice systems and disordered systems described by random symmetric
Hamiltonians. The similarities to graphene occur for spatially sparse networks with low ratio of edges over vertices $RT$ and include features like linear energy dispersion relation at the band center at $E=0$ and various persistent localized states (PLS) concentrating at the spatial boundary of the networks, resembling the edge states in graphene systems with zig-zag edges. The PLS occur at various energies for example at $E=\pm 1, 0, 1$ and others related for example to the golden ratio, and are localized either at single unconnected vertices (0D order) or lines of vertices of fixed length which determines their energy (1D order). As the network becomes spatially more dense its electronic properties, indicated for example by the DOS start to resemble disordered systems described by random symmetric Hamiltonians whose distribution of eigenvalues follows the Wigner semi-circle. The PLS apart from those at $E=0$ gradually disappear as $RT$ increases and the network becomes more dense, with more edges between its vertices. Finally we have calculated the dimension $D$ of the ambient space in which the networks can be embedded. We have found a logarithmic growth of $D$ with $RT$, reaching integer and non-integer values, implying an exponential dependence of the average connectivity in the network to $D$, unlike regular lattices. In summary we have studied quantum mechanics in physical systems lacking a fixed spatial dimension, demonstrating various unconventional electronic properties. These properties could be universal for quantum mechanical systems irrespectively of their spatial dimension. Finally we have discussed the relation of our results to spacetime and relativistic/gravity effects that could emerge from discrete mathematical models, like the random networks that we considered.

Acknowledgements
We acknowledge resources and financial support provided by the National Center for Theoretical Sciences of R.O.C. Taiwan and the Department of Physics of Ben-Gurion University of the Negev in Israel. Also we acknowledge support by the Project HPC-EUROPA3, funded by the EC Research Innovation Action under the H2020 Programme. In particular, we gratefully acknowledge the computer resources and technical support provided by ARIS-GRNET and the hospitality of the Department of Physics at the University of Ioannina in Greece.

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