Time-inconsistent Risk-sensitive Equilibrium for Countable-stated Markov Decision Processes

Hongwei Mei∗

September 17, 2019

Abstract

This paper is devoted to solving a time-inconsistent risk-sensitive control problem with parameter \(\varepsilon\) and its limit case \((\varepsilon \to 0^+)\) for countable-stated Markov decision processes (MDPs for short). Since the cost functional is time-inconsistent, it is impossible to find a global optimal strategy for both cases. Instead, for each case, we will prove the existence of time-inconstant equilibrium strategies which verify the so-called step-optimality. Moreover, we prove the convergence of \(\varepsilon\)-equilibriums and the corresponding value functions as \(\varepsilon \to 0^+\).

1 Introduction

A Markov decision process (MDP for short) is a five-tuple \((X, U, \{U(x) : x \in X\}, Q, c)\) where \(X\) is the state space, \(U\) is the action set, \(\{U(x) : x \in X\}\) is feasible actions, \(Q\) is the transition kernel and \(c\) is the cost-per-stage function. For its wide application in different areas, it has been well studied in the last few decades.

To measure different types of risk in different real models, people have raised different cost functionals (i.e. \(c\)) for MDPs. In this paper we are interested in the risk-sensitive cases, i.e. given an appropriate policy \(\pi = \{u_t\}\), the cost functional parameterized by \(\varepsilon\) is defined as

\[
J^\varepsilon(x; \pi) = \varepsilon \log \mathbb{E}_x^{\varepsilon, \pi}[\exp(\varepsilon^{-1}J)] \quad \text{and} \quad J := \sum_{k=1}^{N} c_k(X_k, u_k) + c_N(X_{N+1}).
\]

Classical risk-sensitive MDPs have been intensively studied since the seminal paper [15]. In particular the average cost criterion has attracted a lot of researchers since it is quite different from the classical risk neutral average cost problem (e.g. see [7, 8, 17, 9, 14]). As far as applications are concerned, for example, where portfolio management is considered in [5], where revenue problems are treated in [3] and where the application of risk-sensitive control in finance can be found in [1]. In recent years, some partially observable risk-sensitive MDPs are considered in [2] and a class of risk sensitive MDPs with some certain costs are investigated in [4].

For general finite \(\varepsilon > 0\) case, dynamic programming is an efficient method to find the optimal control and derive the equation for the cost functional under the optimal control.

If \(\varepsilon \to \infty\), one can see that

\[
J^\varepsilon(x, \pi) = \mathbb{E}_x^{\varepsilon, \pi}(J) + \frac{\varepsilon^{-1}}{2} V_x^{\varepsilon, \pi}(J) + O(\varepsilon^{-2})
\]

∗Department of Mathematics, The University of Kansas, Lawrence, KS 66045, U.S. (hongwei.mei@ku.edu).
where $\mathbb{V}_{\epsilon,\pi}^{x}(J)$ is the variance of $J$ under $\mathbb{P}_{\epsilon,\pi}^{x}$. Thus for $|\epsilon|$ being large, the control problem is approximately to minimize a weighted combination of the mean and variance of $J$.

While the case for $\epsilon \to 0^+$ becomes totally different. If the transition $Q$ is independent of $\epsilon$, $J^\epsilon$ converges to the esssup $J$. Thus to make the problem well-posed, it is required that the transition of $X_n$ will depend on $\epsilon$ as well, i.e. the transition becomes 'slower' as $\epsilon$ decreases. The similar model for continuous-time processes (mostly stochastic differential equations) rather than MDPs, is the so-called risk-sensitive control with small noise (e.g. see [13]). It has been proved in [13], that limit value function ($\epsilon = 0$) is a viscosity solution of non-linear PDE which can be obtained by vanishing viscosity (letting $\epsilon \to 0$ for $\epsilon$-models). It is also proved that the $\epsilon = 0$ case is essentially a min-max optimization (or game) problem (see Chapter VI in [13]). In this paper, we mainly focus on such case, i.e. $\epsilon \to 0^+$. By the form of the risk-sensitive cost functional, the large deviation theory (see [10]) plays an important role in investigating the convergence as $\epsilon \to 0^+$. We will review some important results on the Large Deviation Principle (LDP in short) later.

In our paper, we will assume that the transition kernel of controlled Markov process depends on the parameter $\epsilon$. Different from classical risk-sensitive cost, the cost functional is parametrized by an additional non-exponential discounting factor $\tau$, i.e.

$$J_{\tau,t}^\epsilon(x; \pi) = \epsilon \log \mathbb{E}_{t,x}^{\epsilon,\pi}[\exp(\epsilon^{-1}J_{\tau,t})] \text{ and } J_{\tau,t} := \sum_{k=t}^{N} c_{\tau,k}(X_{k}, u_{k}) + c_{\tau}(X_{N+1}).$$

and the corresponding value function is $V_{t}^\epsilon(x) = J_{\tau,t}^\epsilon(x)$ where $\mathbb{E}_{t,x}^{\epsilon,\pi}$ is the conditional expectation on $X_{t} = x$ under the policy $\pi$.

The appearance of the non-exponential discounting factor $\tau$ makes the problem time-inconsistent, i.e. the optimal control now which minimizes the value function $V_{t}^\epsilon(x)$ doesn’t stay optimal in future, i.e. it doesn’t minimize $V_{s}^\epsilon(x)$ for $s > t$. Therefore it is impossible to find a global optimal control. To deal with time-inconsistency, we have to find a so-called time-inconsistent equilibrium which is locally optimal only in some appropriate sense. After the breakthrough in [24] and [12, 11], there are a lot works on time-inconsistent control concerning MDPs and continuous-time models in the last decade (e.g. see [16, 21, 23, 24, 22, 6, 18]). Using a similar idea from [24], we will try to find the time-inconsistent equilibrium strategy. Since we are dealing with a MDP instead of a continuous diffusion, our main goal is to derive the time-inconsistent recursive Hamiltonian sequences for the cost functionals with parameter $\epsilon$ instead of time-inconsistent HJB equations for continuous diffusions. To the best knowledge of the authors, there are few papers concerning on time-inconsistent risk-sensitive MDPs, especially for the convergence of $\epsilon \to 0^+$ case. This paper is to fill this gap.

There are two main differences between time-inconsistent risk-sensitive MDPs and time-consistent continuous stochastic processes. Instead of deriving a class of PDEs for the value function in continuous-time models, the value function will be in terms of a time-inconsistent Hamiltonian recursions for time-inconsistent MDPs. The second difference lies in the argument on the convergence of value functions as $\epsilon \to 0^+$. For time-consistent continuous-time models, the convergence of the value functions can be obtained by the well-known vanishing viscosity method (e.g. see [13]) where the convergence of optimal strategies is not necessary. While for time-inconsistent risk-sensitive MDPs, the convergence of time-inconsistent equilibriums is required to show the convergence of value functions. Since the equilibriums might not be unique and the convergence of equilibriums
requires strong regularity of value functions for general state space. Therefore we assume the state space is countable in this paper.

In our paper, we assume that the state space $X$ has countable many states and the control space $U$ is a complete metric space with metric $|.|_{U}$. Without loss of generality, we suppose that $X$ be the set of integers. Let $M(X)$ be the set of all measurable functions on $X$. $B(X)$ is the set of all bounded functions on $X$. $BB(X)$ is the set of functions bounded from below. Write $P(X)$ be the set of all probability measures on $X$. A function $f \in M(X)$ is called inf-finite if the set $\{x \in X : f(x) \leq K\}$ has finite elements for all $K \in \mathbb{R}$. Let $C(U)$ be the set of continuous functions on $U$. A function measurable $f$ on $U$ is called inf-compact if the set $\{u \in U : f(u) \leq K\}$ is compact for all $K \in \mathbb{R}$ (i.e. the set of real numbers).

The set of admissible policies $\Pi$ is assumed to be the collection of all deterministic Markov policies, i.e.

$$
\Pi = \{\pi = \tilde{u}_1 \oplus \cdots \oplus \tilde{u}_T : \tilde{u}_t = u_t(\cdot) \text{ is a measurable function from } X \text{ to } U\}.
$$

Write $\mathbb{T} := \{1, \cdots, T\}$ and $\pi_t := \tilde{u}_t \oplus \cdots \oplus \tilde{u}_T$. Here the notation $\tilde{u}$ means the strategy $u(\cdot) \in U$.

Given a deterministic policy $\pi \in \Pi$, the transition probability is

$$
\mathbb{P}^{\epsilon, \pi}(X_{t+1} = j|X_t = i, X_{t-1}, \cdots, X_1) = q^\pi_t(j; i, u_t(i)).
$$

where $q^\pi_t(j; i, u) \geq 0$ and $\sum_{j \in X} q^\pi_t(j; i, u) = 1$.

For each $(\tau, t) \in \mathbb{T} \times \mathbb{T}$, let $f_{\tau,t} : X \times U \mapsto \mathbb{R}$ and $g_{\tau} : X \mapsto \mathbb{R}$. Define the time-inconsistent risk-sensitive $\epsilon$-cost functional by

$$
J^\epsilon_{\tau,t}(x; \pi_t) = \epsilon \log \mathbb{E}^{\epsilon, \pi_t}[\exp \left\{\epsilon^{-1} \left(\sum_{s=t}^{T} f_{\tau,s}(X_s, u_s(X_s)) + g_{\tau}(X_{T+1})\right)\right\}]
$$

and the value function at $t \in \mathbb{T}$ is

$$
V^\epsilon_t(x; \pi_t) := J^\epsilon_{t,t}(x; \pi_t).
$$

We define the limit cost and value function as $\epsilon \to 0^+$ by

$$
J_{\tau,t}(x; \pi_t) = \limsup_{\epsilon \to 0^+} J^\epsilon_{\tau,t}(x; \pi_t)
$$

and

$$
V_t(x; \pi_t) := J_{t,t}(x; \pi_t).
$$

As we mentioned before, the dependence of the cost functions $f$ and $g$ on non-exponential discounting factor $\tau$ makes the problem time-inconsistent generally. Thus we will find a time-inconsistent equilibrium which satisfies some local optimality. The following is the definition for a time-inconsistent risk-sensitive equilibrium.

**Definition 1.1.** (1) A $T$-step strategy $\pi^{\epsilon,*}_t \in \Pi$ is called a time-inconsistent risk-sensitive $\epsilon$-equilibrium if the following step-optimality holds

$$
J^\epsilon_{t,t}(x; \pi^{\epsilon,*}_t) \leq J^\epsilon_{t,t}(x; \tilde{u} \oplus \pi^{\epsilon,*}_{t+1}) \text{ for any } t \in \mathbb{T}, \ \tilde{u} \in U.
$$
Recall \( \tilde{u} = u(\cdot) \in \mathcal{U} \).

(2) A \( T \)-step strategy \( \pi^* \in \Pi \) is called a time-inconsistent risk-sensitive equilibrium if the following step-optimality holds

\[
J_{t,t}(x; \pi_t^*) \leq J_{t,t}(x; \tilde{u} \oplus \pi_{t+1}^*) \quad \text{for any } t \in T, \; \tilde{u} \in \mathcal{U}.
\]

Our main goal in the paper is to derive the time-inconsistent risk-sensitive \( \varepsilon \)-equilibrium and time-inconsistent risk-sensitive equilibrium. Moreover, we will prove that the convergence of time-inconsistent risk-sensitive \( \varepsilon \)-equilibriums to time-inconsistent risk-sensitive equilibrium as \( \varepsilon \to 0^+ \).

The paper is arranged as follows. In Section 2, we will review some well-known results for LDP and present some preliminary results. In Section 3, we will derive the time-inconsistent risk-sensitive equilibriums and the corresponding recursive Hamiltonian sequences for both cases. Then in Section 6, we prove the convergence of \( \varepsilon \)-equilibriums as \( \varepsilon \to 0^+ \). Finally, an illustrative example is presented in Section 5 and some concluding remarks are made in Section 6.

## 2 Preliminary Results

### 2.1 Large Deviation Principle

In this subsection, we will review some well-known results on large deviation principle. For more details and their proofs, one can check [10].

On a complete separable space \( \mathcal{Y} \), \( I : \mathcal{Y} \to [0, \infty] \) is called a (good) rate function if it is inf-compact. Let \( Y_n \) be a sequence of \( \mathcal{Y} \)-valued random variables on some appropriate probability space. \( \{Y_n\} \) is said to satisfy the LDP with rate function \( I \) if

1. for any closed subset \( C \) of \( \mathcal{Y} \),
   \[
   \limsup_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n \in C) \leq - \inf_C I.
   \]
2. for any open subset \( O \) of \( \mathcal{Y} \),
   \[
   \liminf_{n \to \infty} \frac{1}{n} \log \mathbb{P}(Y_n \in O) \leq - \inf_O I.
   \]

In essential, the large deviation principle gives the rate of probability for rare events. Thus the corresponding so-called risk-sensitive control problem is a certain type of robustness control problems, i.e. take actions concerning on rare events (for example, the worst case). Thus let’s recall some well-known results on LDP which are used in our paper.

**Theorem 2.1.** (1) \( \{Y_n\} \) satisfies the LDP with rate function \( I \) if and only if \( I \) is a rate function (i.e. inf-compact) and for any \( h \in C_b(\mathcal{Y}) \) (i.e. bounded continuous functions on \( \mathcal{Y} \)),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left( \exp \left[ nh(Y_n) \right] \right) = \sup_{\mathcal{Y}} [h - I]
\]

(2) \( \{Y_n\} \) satisfies LDP with rate function \( I \) if and only if \( \{Y_n\} \) is exponential tight, i.e. for any \( a > 0 \), there exists a compact subset \( K_a \) of \( \mathcal{Y} \) such that

\[
\frac{1}{n} \log \mathbb{P}(Y_n \in K_a^c) \leq -a
\]
and for any bounded continuous function \( h \) on \( \mathcal{Y} \),

\[
\limsup_{n \to \infty} \frac{1}{n} \log \mathbb{E} \left( \exp[nh(Y_n)] \right) = \sup_{\mathcal{Y}} [h - I]
\]

(3) If there exists a positive, inf-compact function \( \mathcal{V} \) on \( \mathcal{Y} \) (i.e. Lyapunov function) satisfying

\[
\sup_{n} \frac{1}{n} \log \mathbb{E} \left( \exp[n\mathcal{V}(Y_n)] \right) < \infty,
\]

then \( \{Y_n\} \) is exponential tight.

(4) Let \( \mathcal{P}(\mathcal{Y}) \) be the set of probability measures on \( \mathcal{Y} \). The following variational equality (i.e. Varadhan’s equality) holds,

\[
\log \int_{\mathcal{Y}} e^{\mathcal{V}} d\mu = \sup_{\nu \in \mathcal{P}(\mathcal{Y})} \left( \int_{\mathcal{Y}} h d\nu - \mathcal{R}(\nu \| \mu) \right), \quad \text{for any } h \in C_{b}(\mathcal{Y})
\]

where the relative entropy \( \mathcal{R}(\cdot \| \cdot) \) is defined by

\[
\mathcal{R}(\nu \| \mu) := \int_{\mathcal{Y}} \log \left( \frac{d\nu}{d\mu} \right) d\nu, \quad \mu, \nu \in \mathcal{P}(\mathcal{Y}).
\]

Moreover, if (2.1) holds, then (2.2) holds for any \( h \in o(\mathcal{V}) \) (i.e. \( \lim_{|\mathcal{V}| \to \infty} |h(\mathcal{V})|/\mathcal{V}(\mathcal{V}) = 0 \)).

### 2.2 Preliminaries

Let’s recall the transition probability

\[
P^{\rho, \pi}(X_{t+1} = j | X_t = i, X_{t-1}, \cdots, X_1) = q^\rho_t(j; i, u_t(i)).
\]

For each \( t \in \mathbb{T} \) and \( \varepsilon > 0 \), define \( \Lambda^\varepsilon_t, \Lambda_t : \mathcal{X} \times \mathcal{U} \times B(\mathcal{X}) \mapsto \mathcal{X} \) by

\[
\Lambda^\varepsilon_t(x, u; h) := \varepsilon \log \left( \sum_{z \in \mathcal{X}} \exp \left\{ \varepsilon^{-1} h(z) \right\} q^\rho_t(z; x, u) \right) \quad \text{and} \quad \Lambda_t(x, u; h) = \lim_{\varepsilon \to 0^+} \Lambda^\varepsilon_t(x, u; h).
\]

Note that \( h \) is bounded, \( \Lambda^\varepsilon_t(x, u; h) \) is well-defined. \( \Lambda_t(x, u; h) \) is well-defined because of the following assumption.

**Assumption (A):** (A1) There exists an inf-finite, positive function \( \mathcal{V} : \mathcal{X} \mapsto \mathbb{R} \) such that for each \( (x, u) \in \mathcal{X} \times \mathcal{U} \),

\[
\limsup_{\varepsilon \to 0^+} \Lambda^\varepsilon_t(x, u; \lambda_0 \mathcal{V}) < \infty, \quad \text{for some } \lambda_0 > 0.
\]

(A2) Given any \( t \in \mathbb{T} \) and \( h \in B(\mathcal{X}) \), \( \Lambda^\varepsilon_t(x, \cdot; h) \) is a continuous function of \( u \in \mathcal{U} \). Moreover for each \( (x, u) \in \mathcal{X} \times \mathcal{U} \), there exists a rate function \( I_t(\cdot; x, u) : \mathcal{X} \mapsto \mathbb{R} \) such that for any \( h \in B(\mathcal{X}) \), and \( u^\varepsilon \to u \) in \( \mathcal{U} \)

\[
\lim_{\varepsilon \downarrow 0} \Lambda^\varepsilon_t(x, u^\varepsilon; h) = \sup_{z \in \mathcal{X}} [h(z) - I_t(z; x, u)] = \Lambda_t(x, u; h).
\]

(A3) There exists a \( \lambda_0 > 0 \) and a constant \( K_u \) depending on \( u \) only such that for any \( \lambda \in (0, \lambda_0) \) and each \( u \in \mathcal{U} \)

\[
\limsup_{|x| \to \infty} \frac{\sup_{0 \leq \varepsilon \leq \epsilon_0} \Lambda^\varepsilon_t(x, u; \lambda \mathcal{V})}{\lambda \mathcal{V}(x)} < K_u.
\]
For the positive function $V$ on $X$ in (A1), we define a subset $BB_V(X)$ of $BB(X)$ by
\[ BB_V(X) := \{ h \in BB(X) : \limsup_{|x| \to \infty} \frac{h(x)}{V(x)} = 0 \}. \]

We also write
\[ UBB_V := \left\{ \{ h^\varepsilon \} \subset BB_V(X) : h^\varepsilon \text{ is uniformly bounded below and } \sup_{\varepsilon} h^\varepsilon \in BB_V(X) \right\}. \]

**Remark 2.2.** (1) By Theorem 2.1, (A1) and (A2) are sufficient for that $\{ X^\varepsilon_{i+1} | X^\varepsilon_i = x, u \}$ satisfies LDP with rate function $I_t(\cdot ; x, u)$. Moreover for $h \in BB_V(X)$, $\Lambda^\varepsilon_t(x, u; h)$ and $\Lambda_t(x, u; h)$ are well-defined and (2.4) holds as well.

(2) (A2) says that the rate function $I_t$ is uniform on any compact subset of $U$. We can conclude that $\Lambda^\varepsilon_t(x, u)$ converges to $\Lambda_t(x, u; h)$ uniformly on any compact set of $U$. Moreover, $\Lambda_t(x, u; h)$ is continuous on any compact subset of $U$ given fixed $x$ and $h$ (See Proposition 1.2.7 in [10]).

(3) If (A1) and (A2) hold, the definition of $\Lambda^\varepsilon_t(x, u; h)$ and $\Lambda_t(x, u; h)$ can be extended to all $h \in BB_V(X)$ and (A2) is true for all $h \in BB_V(X)$.

In this paper, $BB_V(X)$ is equipped with the following metric,
\[ w(h, h') := \sup_X \frac{|h - h'|}{V}. \]

The following lemma says that $(BB_V(X), w)$ is a complete metric space.

**Lemma 2.3.** Given $V$ defined in (A1), the followings hold.

(1) If $h_n \in BB_V(X)$ with $w(h_n, h_m) \to 0$ for any $n, m \to \infty$, then there exists a $h \in BB_V(X)$ such that $w(h_n, h) \to 0$, i.e. $(BB_V(X), w)$ is a complete metric space.

(2) If $h_n$ is uniformly bounded below with $\sup_n h_n \in BB_V(X)$, then $\{ h_n \}$ has a convergent subsequence in $(BB_V(X), w)$. As a result, if $h_n$ is uniformly bounded below with $\sup_n h_n \in BB_V(X)$ and converges to $h$ point-wisely, then $h_n$ converges to $h$ in $(BB_V(X), w)$.

**Proof.** (1) For such $\{ h_n \}$, it is easy to see that there exists a $h \in M(X)$ such that $h_n$ converges to $h$ point-wisely. Now we show that the convergence is in metric sense as well.

For any $\delta > 0$, there exists a $N_\delta > 0$ such that
\[ w(h_n, h_m) < \delta, \text{ for any } n, m \geq N_\delta. \]

Note that for any $m \geq N_\delta$,
\[ \limsup_{|x| \to \infty} \frac{|h(x)|}{V(x)} \leq \limsup_{|x| \to \infty} \left( \frac{|h_m(x)|}{V(x)} + \lim_{n \to \infty} \frac{|h_n(x) - h_m(x)|}{V(x)} \right) \leq \limsup_{n \to \infty, x \in X} \frac{|h_n(x) - h_m(x)|}{V(x)} < \delta. \]

By the arbitrariness of $\delta > 0$, we have $h \in BB_V(X)$.

For any fixed $\delta > 0$, let $n_k$ satisfy
\[ w(h_n, h_m) < \frac{\delta}{2^k}, \text{ for any } n, m \geq n_k. \]
Then one can easily see that
\[ \sum_{k=1}^{\infty} w(h_{nk+1}, h_{nk}) < \delta. \]

It follows that for any \( n > N_{\delta}. \)
\[ \sup_{x \in X} \frac{|h(x) - h_n(x)|}{\mathcal{V}(x)} \leq \sup_{x \in X} \sum_{k=1}^{\infty} \frac{|h_{nk+1}(x) - h_{nk}(x)|}{\mathcal{V}(x)} + \sup_{x \in X} \frac{|h_n(x) - h_{n+1}(x)|}{\mathcal{V}(x)} \]
\[ \leq \sum_{k=1}^{\infty} w(h_{nk+1}, h_{nk}) + w(h_{n+1}, h_n) \leq 2\delta. \]

It is equivalent to say
\[ \lim_{n \to \infty} w(h_n, h) = 0. \]

(2) By the hypothesis, one can easily see that \( \{h_n\} \) has a point-wisely convergent subsequence with limit \( h. \) We still write the subsequence as \( \{h_n\}. \) Obviously we have \( h \in BB_{\mathcal{V}}(X) \) since \( h_n \) is uniformly bounded below and \( h \leq \sup_n h_n \in BB_{\mathcal{V}}(X). \)

Note that for any \( \delta > 0, \) there exists a \( x_\delta > 0 \) such that
\[ \frac{\sup_h h_{\delta}(x)}{\mathcal{V}(x)} \leq \delta \text{ for } x \geq x_\delta. \]

Then by the point-wise convergence, it follows that
\[ \lim_{n \to \infty} \sup_{x \in X} \frac{|h_n(x) - h(x)|}{\mathcal{V}(x)} \leq \lim_{n \to \infty} \sup_{|x| \leq x_\delta} \frac{|h_n(x) - h(x)|}{\mathcal{V}(x)} + 2\delta = 2\delta. \]

By the arbitrariness of \( \delta > 0, \) we have
\[ \lim_{n \to \infty} w(h_n, h) = 0. \]

Now we first prove that well-posedness of \( \Lambda \) and \( \Lambda^\varepsilon \) on the space \( BB_{\mathcal{V}}(X). \)

**Lemma 2.4.** Under Assumption (A), for any \( \{h^\varepsilon\} \in UBB_{\mathcal{V}}(X) \) and each \( u \in U, \) \( \{\Lambda^\varepsilon(\cdot, u; h^\varepsilon)\} \in UBB_{\mathcal{V}}. \) Therefore, for any \( h \in BB_{\mathcal{V}}(X), \Lambda^\varepsilon(\cdot, u; h) \in BB_{\mathcal{V}}(X) \) for each \( u \in U. \)

**Proof.** It is easy to see that \( \Lambda^\varepsilon(\cdot, u; h^\varepsilon) \) is uniformly bounded below. Note that by (2.2),
\[ \Lambda^\varepsilon_{\varepsilon}(x; u; h^\varepsilon) = \sup_{\nu \in P(X)} \left( \int_X h^\varepsilon d\nu - \varepsilon \mathcal{R}(\nu\|q^\varepsilon(\cdot; x, u)) \right) \]
\[ \leq \sup_{\nu \in P(X)} \left( \int_X \lambda \mathcal{V} d\nu - \varepsilon \mathcal{R}(\nu\|q^\varepsilon(\cdot; x, u)) \right) + \sup_{\nu \in P(X)} \int_X (h^\varepsilon - \lambda \mathcal{V}) d\nu \]
\[ \leq \Lambda^\varepsilon(x, u; \lambda \mathcal{V}) + \sup_X |h^\varepsilon - \lambda \mathcal{V}| \]

By (A3),
\[ \lim_{|x| \to \infty} \sup_{\varepsilon} \frac{\Lambda^\varepsilon_{\varepsilon}(x; u; h^\varepsilon)}{\mathcal{V}(x)} \leq \lim_{|x| \to \infty} \sup_{\varepsilon} \frac{1}{\mathcal{V}(x)} \left( \sup_{\varepsilon} \Lambda^\varepsilon_{\varepsilon}(x, u; \lambda \mathcal{V}) + \sup_X |h^\varepsilon - \lambda \mathcal{V}| \right) \leq K_u \]

By the arbitrariness of \( \lambda > 0, \) it follows that \( \{\Lambda^\varepsilon(\cdot, u; h^\varepsilon)\} \in UBB_{\mathcal{V}} \) for each \( u \in U. \)
Now we are ready to present the Hamiltonians used in our paper. Define $\mathcal{A}_t^i[\cdot], \mathcal{A}_t[\cdot] : BB_B(X) \to M(X)$ by
\[
\mathcal{A}_t^i[h](x) := \inf_{u \in U} \left[ f_{t,t}(x, u) + \Lambda_t^i(x, u; h) \right] \quad \text{and} \quad \mathcal{A}_t[h](x) := \inf_{u \in U} \left[ f_{t,t}(x, u) + \Lambda_t(x, u; h) \right].
\]

The following lemma will guarantee that $\mathcal{A}_t^i$ and $\mathcal{A}_t$ map $BB_B(X)$ into $BB_B(X)$ under the following assumption.

**Assumption (B):** For each fixed $u \in U$, $f_{t,t}(\cdot, u)$, $g_t(\cdot) \in BB_B(X)$. For each fixed $i \in X$, $f_{t,t}(i, \cdot)$ is continuous and inf-compact.

**Lemma 2.5.** Under Assumptions (A) and (B), for any $h \in BB_B(X)$, $\mathcal{A}_t^i[h]$, $\mathcal{A}_t[h] \in BB_B(X)$.

**Proof.** Since $f_{t,t}$ and $h$ are bounded below, so are $\mathcal{A}_t[h]$ and $\mathcal{A}_t^i[h]$ by their definitions. Since
\[
\mathcal{A}_t[h](x) \leq f_{t,t}(x, u_0) + \Lambda_t(x, u_0; h), \quad \text{for some } u_0 \in U,
\]
by Lemma 2.4 and Assumption (B), $\mathcal{A}_t[h](x) \in BB_B(X)$. Similarly we have $\mathcal{A}_t^i[h](x) \in BB_B(X)$. Moreover the infimums can be attained by Assumptions (A) and (B).

Given any $h \in BB_B(X)$, define
\[
\square \eta_t^i(\cdot; h) : x \mapsto \arg\min_{u \in U} \left[ f_{t,t}(x, u) + \Lambda_t^i(x, u; h) \right] \subset U.
\]
If $\square \eta_t^i(x; h) \neq \emptyset$ for any $x \in X$, we say $\eta_t^i(\cdot; h)$ is a choice of $\square \eta_t^i(\cdot; h)$ if
\[
\eta_t^i(x; h) \in \square \eta_t^i(\cdot; h), \quad \text{for any } x \in X.
\]
We write it as $\eta_t(\cdot; h) \in \square \eta_t^i(\cdot; h)$. Since $X$ is a countable-stated space, $\eta_t^i(\cdot; h)$ is naturally measurable. Similarly we can define $\square \eta_t$ and its one choice $\eta_t$.

Define $\mathcal{H}_{t,t}^\varepsilon[\cdot], \mathcal{H}_{t,t}[\cdot] : BB_B(X) \to M(X \times U)$ by
\[
\mathcal{H}_{t,t}^\varepsilon[h](x, u) := f_{t,t}(x, u) + \Lambda_t^\varepsilon(x, u; h) \quad \text{and} \quad \mathcal{H}_{t,t}[h](x, u) := f_{t,t}(x, u) + \Lambda_t(x, u; h).
\]
It is easy to see that
\[
\mathcal{A}_t^i[h](x) = \inf_{u \in U} \mathcal{H}_{t,t}^\varepsilon[h](x, u) \quad \text{and} \quad \mathcal{A}_t[h](x) = \inf_{u \in U} \mathcal{H}_{t,t}[h](x, u).
\]

From their definition, we know that $\mathcal{H}_{t,t}^\varepsilon$, $\mathcal{H}_{t,t}$ will map $BB_B(X)$ into $M(X)$. We raise the following assumption to guarantee $\mathcal{H}_{t,t}^\varepsilon[h]$, $\mathcal{H}_{t,t}[h] \in BB_B(X)$ for any $h \in BB_B(X)$ and fixed $u \in U$.

**Assumption (C)** Let
\[
B_{t,\lambda}(x) := \{ u \in U : f_{t,t}(x, u) \leq \lambda \mathcal{V}(x) \}.
\]
There exists constants $\lambda_0 > 0$ and $K_1$ such that for any $\lambda \in (0, \lambda_0)$ and $h \in BB_B(X)$,
\[
\limsup_{|x| \to \infty} \sup_{u \in B_{t,\lambda}(x)} \frac{f_{t,t}(x, u)}{\lambda \mathcal{V}(x)} \leq K_1
\]
and
\[
\limsup_{|x| \to \infty} \sup_{0 \leq \varepsilon \leq \varepsilon_0} \sup_{u \in B_{t,\lambda}(x)} \frac{\Lambda_t^\varepsilon(x, u; \lambda \mathcal{V})}{\lambda \mathcal{V}(x)} \leq K_1.
\]
Remark 2.6. We can see that

\[
\limsup_{|x| \to \infty} \sup_{u \in B_{t,\lambda}(x)} \Lambda_t(x, u; \lambda V) \leq K_1
\]

Moreover, since \( f_{t,t} \in BB_V(X) \), any \( u_0 \in X \) belongs to \( B_{t,\lambda}(x) \) if \( |x| \) is large. Thus (A3) is a consequence of Assumptions (B) and (C).

Lemma 2.7. Under Assumptions (A), (B) and (C), the followings are true.

1. For any \( h \in BB_V(X) \), \( \eta_t(\cdot, h) \in \square \eta_t(\cdot, h) \) and \( \eta_f^\epsilon(\cdot, h) \in \square \eta_f^\epsilon(\cdot, h) \),

\[
f_{t,t}(\cdot, \eta_t(\cdot, h)), f_{t,t}(\cdot, \eta_f^\epsilon(\cdot, h)) \in BB_V(X).
\]

2. For any \( h_1, h_2 \in BB_V(X) \), \( \eta_t(\cdot, h_2) \in \square \eta_t(\cdot, h_2) \) and \( \eta_f^\epsilon(\cdot, h_2) \in \square \eta_f^\epsilon(\cdot, h_2) \),

\[
\Lambda_t(\cdot, \eta_f^\epsilon(\cdot, h_2); h_1), \Lambda_t(\cdot, \eta_f^\epsilon(\cdot, h_2); h_1) \in BB_V(X).
\]

Proof. (1) Recall the definitions

\[
B_{t,\lambda}(x) := \{ u \in U : f_{t,t}(x, u) \leq \lambda V(x) \}
\]

and

\[
B_t(x) := \{ u \in U : f_{t,t}(x, u) \leq f_{t,t}(x, u_0) + \Lambda_t(x, u_0; h) - \inf_{h \in X} h + \delta \}.
\]

Since for each \( u \in U \), \( f_{t,t}(\cdot, u) \in BB_V(X) \), \( B_t(x) \subset B_{t,\lambda}(x) \) for large \( |x| \). By the definition of \( \eta_t \) and \( \eta_f^\epsilon \), it follows that

\[
\eta_t(x; h), \eta_f^\epsilon(x; h) \in B_{t,\lambda}(x) \text{ when } x \text{ is large}.
\]

As a consequence, for each \( \lambda \in (0, \lambda_0) \),

\[
\limsup_{|x| \to \infty} \sup_{V(x)} \frac{f_{t,t}(x, \eta_t(x; h))}{V(x)} \leq \limsup_{|x| \to \infty} \sup_{U(x)} \frac{f_{t,t}(x, u)}{V(x)} \leq \lambda K_1.
\]

By the arbitrariness of \( \lambda \in (0, \lambda_0) \), it follows that \( f_{t,t}(x, \eta_t(x; h)) \in BB_V(X) \). Similarly, we can prove that \( f_{t,t}(x, \eta_f^\epsilon(x; h)) \in BB_V(X) \).

(2) Let \( h_1, h_2 \in BB_V(X) \). Then

\[
\eta_t(x; h_2), \eta_f^\epsilon(x; h_2) \in B_{t,\lambda}(x) \text{ when } |x| \text{ is large}.
\]

By the definition of \( \Lambda_t(\cdot, u; \lambda V) \), we have \( -I_t(z; x, u) \leq \Lambda_t(x, u; \lambda V) - \lambda V(z) \). Note that

\[
(2.6) \quad \Lambda_t(x, u; h) = \sup_{z \in X} [h(z) - I_t(z; x, u)] \leq \sup_{z \in X} [h(z) - \lambda V(z)] + \Lambda_t(x, u; \lambda V).
\]

Therefore,

\[
\Lambda_t(x, \eta_t(x; h_2); h_1) \leq \sup_{z \in X} [h_1(z) - \lambda V(z)] + \Lambda_t(x, \eta_t(x; h_2); \lambda V)
\]

\[
\leq \sup_{z \in X} [h_1(z) - \lambda V(z)] + \sup_{u \in B_{t,\lambda}(x)} \Lambda_t(x, u; \lambda V)
\]

and for any \( \lambda \in (0, \lambda_0) \), by Assumption (C),

\[
\limsup_{|x| \to \infty} \frac{\Lambda_t(x, \eta_t(x; h_2); h_1)}{V(x)} \leq \lambda K_1.
\]

By the arbitrariness of \( \lambda \in (0, \lambda_0) \), it follows that \( \Lambda_t(x, \eta_t(x; h_2); h_1) \in CBB_V(X) \) for any \( h_1, h_2 \in CBB_V(X) \). Similarly, the result holds for \( \Lambda_f^\epsilon(x, \eta_f^\epsilon(x; h_2); h_1) \in BB_V(X) \).
3 Time-inconsistent Equilibrium

In this section, we will derive the time-inconsistent equilibrium strategy step by step. The section will be divided into several subsections.

3.1 Optimal Control for 1-step Transition

In this subsection, we will review the 1-step optimal control problem with risk-sensitive cost. Consider \( \{X_1^\varepsilon, X_2^\varepsilon\} \) with controlled transition probability

\[
P(X_2^\varepsilon = j|X_1^\varepsilon = i; u) = q_t^\varepsilon(j; i, u)
\]

Let

\[
\Lambda^\varepsilon(x, u; h) = \varepsilon \log \mathbb{E}(\exp[\varepsilon^{-1}h(X_2)]|X_1 = x; u).
\]

Given some function \( \tilde{f} : X \times U \mapsto \mathbb{R} \) and \( \tilde{g} : X \mapsto \mathbb{R} \), define the cost function

\[
\tilde{V}(x) = \tilde{J}(x; \tilde{u}) := \varepsilon \log \mathbb{E}\left(\exp\left[\varepsilon^{-1}(\tilde{f}(X_1, u(X_1)) + \tilde{g}(X_2))\right]|X_1 = x\right).
\]

**Problem-(CON):** to find a \( \tilde{u}^* \in U \) such that

\[
\tilde{J}(x; \tilde{u}^*) = \inf_{\tilde{u} \in U} \tilde{J}(x; \tilde{u}).
\]

By the definition of \( \Lambda^\varepsilon \), we have

\[
\tilde{J}(x; \tilde{u}^*) = \tilde{f}(x, u(x)) + \Lambda^\varepsilon(x, u(x); \tilde{g}).
\]

As a result,

\[
\tilde{V}(x) = \inf_{u \in U} [\tilde{f}(x, u) + \Lambda^\varepsilon(x, u; \tilde{g})] \text{ and } u^*(x) \in \text{argmin}_{u \in U} [\tilde{f}(x, u) + \Lambda^\varepsilon(x, u; \tilde{g})]
\]

Note that the optimal strategy \( u^*(\cdot) \) might not be unique. The existence of \( \tilde{u}^* = u^*(\cdot) \) will be guaranteed by the assumptions in the proof.

3.2 Time-inconsistent Strategy

Now we are ready to introduce the recursion process of finding the time-inconsistent equilibriums. We start with the last step first and move backward to the first step.

**T-th step strategy.** In the last step, the control is determined by solving a classical optimal control problem with discounting fact being \( \tau = T \).

**Problem-T:** to find \( \tilde{u}^T_{T-1} \in U \) such that

\[
J_{T,T}^\varepsilon(x; \tilde{u}^T_{T-1}) = \inf_{\tilde{u} \in U} J_{T,T}^\varepsilon(x; \tilde{u}).
\]

By the definition of \( \Lambda_T^\varepsilon \), one can see

\[
J_{T,T}^\varepsilon(x; \tilde{u}) = f_{T,T}(x, u) + \Lambda_T^\varepsilon(x, u; g_T).
\]
Thus the optimal control in this step is in the following feedback form

\[(3.1) \quad u_T^{\varepsilon^*}(x) \in \operatorname{argmin}_{u \in U} \{f_T(x, u) + \Lambda_T^\varepsilon(x, u; g_T)\} = \square \eta_T^\varepsilon(x; g_T).\]

By Assumption (B), the optimal feedback control must exist. The value function is

\[V_T^\varepsilon(x) = J_T^\varepsilon(x; \tilde{u}_T^\varepsilon) = \inf_{u \in U} \{f_T(x, u) + \Lambda_T^\varepsilon(x, u; g_T)\} = \mathcal{A}_T^\varepsilon[g_T](x).\]

While the minimum point is not unique, let \(\eta_T^\varepsilon(\cdot; g_T)\) be a choice of \(\square \eta_T^\varepsilon(\cdot; g_T)\). We choose

\[(3.2) \quad u_T^{\varepsilon^*}(x) = \eta_T^\varepsilon(x; g_T).\]

Given the optimal control we find this step, now for any \(\tau \in \mathbb{T}\), let

\[(3.3) \quad \Theta_{\tau, T}^\varepsilon(x) := f_{\tau, T}(x, \eta_T^\varepsilon(x; g_T)) + \Lambda_{\tau, T}^\varepsilon(x, \eta_T^\varepsilon(x; g_T); g_T) = \mathcal{H}_{\tau, T}[g_T](x, \eta_T^\varepsilon(x; g_T)).\]

It is easy to see that

\[(3.4) \quad \Theta_{T, T}^\varepsilon(x) = J_{T, T}^\varepsilon(x; \eta_T^\varepsilon(x; g_T)),\]

i.e. \(\Theta_{\tau, T}^\varepsilon\) is the value of the cost function at time \(T\) if we use the discounting factor \(\tau\) and the feed-back control \(\eta_T^\varepsilon(x; g_T)\). Note that

\[\Theta_{T, T}^\varepsilon(x) = \inf_{u \in \mathcal{U}} J_{T, T}^\varepsilon(x; u).\]

\((T - 1)\)-th step strategy. In the \((T - 1)\)th step, we know \(T\)-step strategy is \(\tilde{u}_T^\varepsilon = \eta_T^\varepsilon(\cdot; g_T)\) defined by (3.2) under discounting factor \(\tau = T\). While in this step, the strategy is based on the new discounting factor \(\tau = T - 1\). Thus we are solving the following optimal control problem.

**Problem**-(\(T - 1\)): to find \(\tilde{u}_{T-1}^{\varepsilon^*} \in \mathcal{U}\) such that

\[J_{T-1, T-1}^\varepsilon(x; \tilde{u}_{T-1}^{\varepsilon^*} \oplus \tilde{u}_{T-1}^{\varepsilon^*}) = \inf_{u \in \mathcal{U}} J_{T-1, T-1}^\varepsilon(x; \tilde{u} \oplus \tilde{u}_{T-1}^{\varepsilon^*}).\]

Note that

\[J_{T-1, T-1}^\varepsilon(x; \tilde{u} \oplus \tilde{u}_{T-1}^{\varepsilon^*}) = f_{T-1, T-1}(x, u(x)) + \Lambda_{T-1, T-1}^\varepsilon(x, u(x); J_{T-1, T-1}^\varepsilon(x; \tilde{u}_{T-1}^{\varepsilon^*})).\]

and by (3.4),

\[\Theta_{T-1, T-1}^\varepsilon(x) = J_{T-1, T-1}^\varepsilon(x, \tilde{u}_{T-1}^{\varepsilon^*}).\]

Similarly we can take \(\eta_{T-1}^\varepsilon(\cdot; \Theta_{T-1, T-1}^\varepsilon)\), a possible choice of \(\square \eta_{T-1}^\varepsilon(\cdot; \Theta_{T-1, T-1}^\varepsilon)\) and let

\[(3.5) \quad u_{T-1}^{\varepsilon^*}(x) = \eta_{T-1}^\varepsilon(x; \Theta_{T-1, T-1}^\varepsilon).\]

The value function

\[V_{T-1}^\varepsilon(x) = J_{T-1, T-1}^\varepsilon(x; \tilde{u}_{T-1}^{\varepsilon^*} \oplus \tilde{u}_{T-1}^{\varepsilon^*}) = \inf_{u \in \mathcal{U}} \{f_{T-1, T-1}(x, u) + \Lambda_{T-1, T-1}^\varepsilon(x, u; \Theta_{T-1, T-1}^\varepsilon)\} = \mathcal{A}_{T-1}^\varepsilon[\Theta_{T-1, T-1}^\varepsilon](x).\]

Here \(\Lambda_{T-1, T-1}^\varepsilon(x, u; \Theta_{T-1, T-1}^\varepsilon)\) is well-defined since \(\Theta_{T-1, T-1}^\varepsilon \in BB\mathcal{P}(\mathcal{X})\) by Lemma 2.7.
Now for any $\tau \in \mathbb{T}$, let

\[
\Theta^\varepsilon_{\tau,T-1}(x) := f_{\tau,T-1}(x, \eta^\varepsilon_{\tau-1}(x; \Theta^\varepsilon_{\tau-1,T})); \Lambda^\varepsilon_{T-1}(x, \eta^\varepsilon_{\tau-1}(x; \Theta^\varepsilon_{\tau-1,T})); \Theta^\varepsilon_{T-1,T})
\]

\[
= \mathcal{H}^\varepsilon_{\tau,T-1}[\Theta^\varepsilon_{\tau,T}](x, \eta^\varepsilon_{\tau-1}(x; \Theta^\varepsilon_{\tau-1,T})).
\]

It is easy to see that

\[
\Theta^\varepsilon_{\tau,T-1}(x) = J^\varepsilon_{\tau,T-1}(x; \tilde{u}^\varepsilon_{T-1} \oplus \tilde{u}^\varepsilon_T).
\]

**t-th step strategy.** Before the $t$th step, it has been already identified that $\tilde{u}^\varepsilon_{t+1,T} = \eta^\varepsilon_{t+1}(:, \Theta^\varepsilon_{t+1,k+1}) \oplus \cdots \oplus \eta^\varepsilon_{T}(:, g_T)$. In this step, we are using the new discounting factor $\tau = t$. Thus we are solving the following optimal control problem.

**Problem-(CON)-t:** to find $\tilde{u}^\varepsilon_{t+1,T} \in \mathcal{U}$ such that

\[
J^\varepsilon_{t,t}(x; \tilde{u}^\varepsilon_{t} \oplus \tilde{u}^\varepsilon_{t+1,T}) = \inf_{u \in \mathcal{U}} J^\varepsilon_{t,t}(x; \tilde{u} \oplus \tilde{u}^\varepsilon_{t+1,T})
\]

Similarly we can take one choice among the possible multiple choices that

\[
u^\varepsilon_t(x) = \eta^\varepsilon_t(x; \Theta^\varepsilon_{t,t+1})
\]

and the value function

\[
V^\varepsilon_t(x) = J^\varepsilon_{t,t}(x; \tilde{u}^\varepsilon_{t} \oplus \tilde{u}^\varepsilon_{t+1,T}) = \inf_{u \in \mathcal{U}} \{f_{t,t}(x,u) + \Lambda^\varepsilon_{t+1}(x,u; \Theta^\varepsilon_{t,t+1})\} = A^\varepsilon_t[\Theta^\varepsilon_{t,t+1}].
\]

Now for any $\tau \in \mathbb{T}$, let

\[
\Theta^\varepsilon_{\tau,t}(x) := f_{\tau,t}(x, \eta^\varepsilon_{\tau}(x; \Theta^\varepsilon_{\tau,t+1})); \Lambda^\varepsilon_{T-1}(x, \eta^\varepsilon_{\tau}(x; \Theta^\varepsilon_{\tau,t+1})); \Theta^\varepsilon_{\tau,t+1}) = \mathcal{H}^\varepsilon_{\tau,T}[\Theta^\varepsilon_{\tau,t+1}](x, \eta^\varepsilon_{\tau}(x; \Theta^\varepsilon_{\tau,t+1})).
\]

It is easy to see that

\[
\Theta^\varepsilon_{\tau,t}(x) = J^\varepsilon_{\tau,t}(x; \tilde{u}^\varepsilon_{t} \oplus \cdots \oplus \tilde{u}^\varepsilon_{T})
\]

By recursively repeating such process until the first step, we get a $T$-step strategy $\eta^\varepsilon_T = \eta^\varepsilon_1 \oplus \cdots \oplus \eta^\varepsilon_T$ and a sequence of functions $\{\Theta^\varepsilon_{\tau,t} : (\tau,t) \in \mathbb{T} \times \mathbb{T} \}$ by the following recursions,

\[
\begin{cases}
\Theta^\varepsilon_{\tau,t}(x) = \mathcal{H}^\varepsilon_{\tau,T}[\Theta^\varepsilon_{\tau,t+1}](x, \eta^\varepsilon_{\tau}(x; \Theta^\varepsilon_{\tau,t+1})), & \tau,t \in \mathbb{T} \\
\eta^\varepsilon_1(:, \Theta^\varepsilon_{\tau,t+1}) \in \Box \eta^\varepsilon_1(:, \Theta^\varepsilon_{\tau,t+1}) \\
\Theta^\varepsilon_{\tau,T+1}(x) = g_{\tau}(x).
\end{cases}
\]

Similarly, we can construct $T$-step strategy $\eta_T = \eta_1 \oplus \cdots \oplus \eta_T$ and a sequence of functions $\{\Theta_{\tau,t} : (\tau,t) \in \mathbb{T} \times \mathbb{T} \}$ by the following recursions,

\[
\begin{cases}
\Theta_{\tau,t}(x) = \mathcal{H}_{\tau,T}[\Theta_{\tau,t+1}](x, \eta_\tau(x; \Theta_{\tau,t+1})), & \tau,t \in \mathbb{T} \\
\eta_\tau(:, \Theta_{\tau,t+1}) \in \Box \eta_\tau(:, \Theta_{\tau,t+1}) \\
\Theta_{\tau,T+1}(x) = g_{\tau}(x).
\end{cases}
\]
Remark 3.1. (1) One can see that the construction of \( \eta^c_T (\eta^c_t) \) is in a reverse order. Moreover, if the choices \( \eta^c_T (\eta^c_t) \) changes, \( \eta^c_s (\eta^c_s) \) for \( s < t \) have to change correspondingly.

(2) If \( f_{\tau,t} \) and \( g_t \) is independent of \( \tau \), i.e. the time-consistent case, then \( H^c_{\tau,t} [h] (x, u) = A^c_{\tau} [h] (x) \) for any \( u \in \Box \eta^c_t (x; \Theta^c_{\tau,t+1}) \). Thus \( \Theta^c_{\tau,t} = \Theta^c_{t,t} \) for any \( \tau \in \mathbb{T} \) and the recursion for the value function is \( V^c_t (x) = \Theta^c_{t,t} (x) \)

\[
V^c_t = A^c_t [V^c_{t+1}], \quad \text{with} \quad V^c_{T+1} = g.
\]

One can see that the Hamiltonian recursion is independent of the choice of the optimal control in each step now.

Now we are ready to introduce our first main theorem.

**Theorem 3.2.** Under Assumptions (A),(B) and (C), the followings hold.

1. For any choice of \( \eta^c_T := \eta^c_1 \oplus \cdots \oplus \eta^c_T \) constructed in Section 3.2, the recursive sequence \( \{ \Theta^c_{\tau,t} (\cdot) : (\tau, t) \in \mathbb{T} \times \mathbb{T} \} \) from (3.9) is well-defined in \( BB_Y (X) \). Moreover \( \eta^c_T \) is a time-inconsistent risk-sensitive \( \varepsilon \)-equilibrium.

2. For any choice of \( \eta^c_T := \eta_1 \oplus \cdots \oplus \eta_T \) constructed in Section 3.2, the recursive sequence \( \{ \Theta_{\tau,t} (\cdot) : (\tau, t) \in \mathbb{T} \times \mathbb{T} \} \) from (3.10) is well-defined in \( BB_Y (X) \). Moreover \( \eta^c_T \) is a time-inconsistent risk-sensitive \( \varepsilon \)-equilibrium.

3. Any time-inconsistent risk-sensitive \( \varepsilon \)-equilibrium \( \eta^c_T \), coupled with \( \Theta^c_{\tau,t} (x) = J^c_{\tau,t} (x; \eta_t; \Theta^c_{t,t+1}) \), solves (3.9).

4. Any time-inconsistent risk-sensitive equilibrium \( \eta^c_T \), coupled with \( \Theta_{\tau,t} (x) = J_{\tau,t} (x; \eta_t; \Theta_{t,t+1}) \), solves (3.10).

**Proof.** (1) and (2). By Lemma 2.7, \( \{ \Theta^c_{\tau,t} (\cdot) : (\tau, t) \in \mathbb{T} \times \mathbb{T} \} \) and \( \{ \Theta_{\tau,t} (\cdot) : (\tau, t) \in \mathbb{T} \times \mathbb{T} \} \) are well-defined in \( BB_Y (X) \).

By the construction process of \( \eta^c_T \), one can see that

\[
\Theta^c_{t,t} (x) = H^c_{t,t} [\Theta^c_{t,t+1} (x, \eta^c_t (x; \Theta^c_{t,t+1})),
= A^c_t [\Theta^c_{t,t+1} (x)],
= \inf_{u \in U} [f_{t,t} (x, u) + \Lambda^c_t (x, u; \Theta^c_{t,t+1})].
\]

and \( \Theta^c_{t,t} (x) = J^c_{t,t} (x; \tilde{u}^{c}_{t,t}) \). The optimality (1.6) holds directly. Thus \( \eta^c_T \) is a time-inconsistent risk-sensitive \( \varepsilon \)-equilibrium. Similar argument can be applied to \( \eta^c_T \) as well.

(3) and (4). If \( \eta^c_T \) is a time-inconsistent risk-sensitive equilibrium strategy, by the optimality (1.7),

\[
u^c_t (\cdot) = \eta_t (\cdot; \Theta^c_{t,t+1}) \in \Box \eta_t (\cdot; \Theta^c_{t,t+1}).
\]

By \( \Theta^c_{\tau,t} (x) = J^c_{\tau,t} (x; \eta_t; \Theta^c_{t,t+1}) \), it is easy to see that

\[
\Theta^c_{\tau,t} (x) = H^c_{\tau,t} [\Theta^c_{\tau,t+1} (x, \eta_t (x; \Theta^c_{t,t+1})), \tau, t \in \mathbb{T}.
\]

Thus \( \{ \eta_t, \Theta^c_{\tau,t} \} \) solves (3.10). The similar results holds for \( \eta^c_T \).
4 The Convergence of $\varepsilon$-equilibriums

In this section, we focus on the convergence of $\varepsilon$-equilibrium as $\varepsilon \to 0^+$, i.e. whether the solutions of (3.9) converges to some solution of (3.10) as $\varepsilon \to 0^+$. We need the following two lemmas.

**Lemma 4.1.** Under Assumptions (A), if $\{h^\varepsilon\} \in \text{UBB}_V$ and $h^\varepsilon \to h$ point-wisely, then $\Lambda^\varepsilon_t(x, u; h^\varepsilon)$ converges to $\Lambda_t(x, u; h)$ uniformly on any compact compact set of $U$.

**Proof.** Let $u^\varepsilon \to u$. By Assumption (A), for any $\delta > 0$, there exists a compact subset $K_{\delta,x} \subset X$ (depending on $\delta$ and $x$ only) such that

$$\left| \varepsilon \log \left( \sum_{z \in X/K_{\delta,x}} \exp \{ \varepsilon^{-1}V(z) \} q^\varepsilon_t(z; x, u^\varepsilon) \right) \right| < \delta,$$

$$|\Lambda^\varepsilon_t(x, u^\varepsilon; h) - \Lambda_t(x, u^\varepsilon; h)| \leq \delta,$$

and

$$\sup_{\varepsilon} |h^\varepsilon(z)| \leq V(z) \text{ for any } z \in X/K_{\delta,x}.$$

Since $h^\varepsilon \to h$ uniformly on $K_{\delta,x}$, there exists an $\varepsilon_{\delta,x} > 0$ such that for any $0 \leq \varepsilon \leq \varepsilon_{\delta,x}$,

$$\sup_{K_{\delta,x}} |h^\varepsilon - h| \leq \delta.$$

Note that

$$\Lambda^\varepsilon_t(x, u^\varepsilon; h^\varepsilon) = \varepsilon \log \left( \sum_{z \in X} \exp \{ \varepsilon^{-1}h^\varepsilon(z) \} q^\varepsilon_t(z; x, u^\varepsilon) \right)$$

$$= \varepsilon \log \left( \sum_{z \in X/K_{\delta,x}} \exp \{ \varepsilon^{-1}h^\varepsilon(z) \} q^\varepsilon_t(z; x, u^\varepsilon) \right) + \varepsilon \log \left( \sum_{z \in K_{\delta,x}} \exp \{ \varepsilon^{-1}h^\varepsilon(z) \} q^\varepsilon_t(z; x, u^\varepsilon) \right)$$

Thus for $0 \leq \varepsilon \leq \varepsilon_{\delta,x}$,

$$|\Lambda^\varepsilon_t(x, u^\varepsilon; h^\varepsilon) - \Lambda_t(x, u; h)|$$

$$\leq |\Lambda^\varepsilon_t(x, u^\varepsilon; h^\varepsilon) - \Lambda^\varepsilon_t(x, u^\varepsilon; h)| + |\Lambda^\varepsilon_t(x, u^\varepsilon; h) - \Lambda_t(x, u^\varepsilon; h)|$$

$$\leq \varepsilon \log \left[ \sum_{z \in X/K_{\delta,x}} \exp \{ \varepsilon^{-1}h^\varepsilon(z) \} q^\varepsilon_t(z; x, u^\varepsilon) \right]$$

$$+ \varepsilon \log \left[ \sum_{z \in X/K_{\delta,x}} \exp \{ \varepsilon^{-1}h^\varepsilon(z) \} q^\varepsilon_t(z; x, u^\varepsilon) \right]$$

$$+ \varepsilon \log \left[ \sum_{z \in K_{\delta,x}} \exp \{ \varepsilon^{-1}h^\varepsilon(z) \} q^\varepsilon_t(z; x, u^\varepsilon) \right]$$

$$- \log \left[ \sum_{z \in K_{\delta,x}} \exp \{ \varepsilon^{-1}h^\varepsilon(z) \} q^\varepsilon_t(z; x, u^\varepsilon) \right]$$

$$+ |\Lambda^\varepsilon_t(x, u^\varepsilon; h) - \Lambda_t(x, u^\varepsilon; h)| < 3\delta.$$

By the arbitrariness of $u^\varepsilon \to u$, for each fixed $x$,

$$\lim_{\varepsilon \downarrow 0} \Lambda^\varepsilon_t(x, u; h^\varepsilon) = \Lambda_t(x, u; h), \quad \text{uniformly on any compact set of } U.$$
The following lemma concerns with the stability result of the Hamiltonians.

**Lemma 4.2.** Under Assumptions (A), (B) and (C), the followings hold.

1. Suppose \( \{h^\varepsilon\} \in UBBV \). Then \( \{\eta^\varepsilon_t(\cdot; h^\varepsilon)\}_{0 < \varepsilon < \varepsilon_0} \) is compact in point-wise convergence sense and the limit of any convergent subsequence (as \( \varepsilon \to 0 \)) belongs to \( \square\eta_t(\cdot ; h) \).

2. Let \( \{h^\varepsilon_1\}, \{h^\varepsilon_2\} \in UBBV \) and \( h^\varepsilon_1 \to h_1 \) and \( h^\varepsilon_2 \to h_2 \) point-wisely. For any convergent subsequence \( \{\eta_t^{\varepsilon_n}(\cdot; h_2^{\varepsilon_n})\} \) \((\varepsilon_n \to 0^+) \) with limit \( \eta^\varepsilon_t(\cdot; h_2) \)

\[
\lim_{n \to \infty} H_{t,t}^{\varepsilon_n}(x; \eta_t^{\varepsilon_n}(x; h_2^{\varepsilon_n})) = H_{t,t}(h_1)(x; \eta_t(x; h_2)) \text{ for any } x \in X.
\]

Moreover \( \{H_{t,t}^{\varepsilon_n}(\cdot; \eta^\varepsilon_t(\cdot; h_2))\} \in UBBV \).

**Proof.** (1) Recall

\[
\square\eta^\varepsilon_t(x; h^\varepsilon) = \arg\min_{u \in U} [f_{t,t}(x, u) + \Lambda^\varepsilon_t(x, u; h^\varepsilon)].
\]

Let

\[
B_t(x) := \{u \in U : f_{t,t}(x, u) \leq f_{t,t}(x, u_0) + \sup_{\varepsilon \in \varepsilon} \Lambda^\varepsilon_t(x, u_0; \lambda V) + \sup_{\varepsilon \in \varepsilon} [\sup_{\varepsilon \in \varepsilon} h^\varepsilon(x) - \inf_{\varepsilon \in \varepsilon} h^\varepsilon(x)]\}
\]

By (4.1) and (2.5), for any fixed \( x \), we can see that \( \square\eta^\varepsilon_t(x; h^\varepsilon) \subset B_t(x) \) and \( B_t(x) \) is compact. Thus for any sequence of choices \( \{\eta^\varepsilon_t(x; h^\varepsilon)\} \) \((\varepsilon_n \to 0)\), there exists a convergent subsequence with limit \( \eta^\varepsilon_t(x; h) \). Note that

\[
|A^\varepsilon_t[h^\varepsilon](x) - A_t[h](x)| \leq \sup_{u \in B_t(x)} |A^\varepsilon_t(x, u; h^\varepsilon) - \Lambda(x, u; h)|
\]

Therefore by (4.1),

\[
\lim_{\varepsilon_n \to \infty} A^\varepsilon_n [h^\varepsilon_n](x) = A_t[h](x).
\]

Moreover,

\[
A_t[h](x, u) = \lim_{n \to \infty} A^\varepsilon_n[h^\varepsilon_n](x, u) \]

\[
= \lim_{n \to \infty} \left[ f_{t,t}(x, \eta^\varepsilon_n(x; h^\varepsilon_n)) - f_{t,t}(x, \eta_t(x; h)) + \Lambda^\varepsilon_t(x, \eta^\varepsilon_n(x; h^\varepsilon_n); h^\varepsilon_n) - \Lambda_t(x, \eta_t(x; h); h) + f_{t,t}(x, \eta_t(x; h)) + \Lambda_t(x, \eta_t(x; h); h) \right]
\]

The last step holds since (4.1) and \( f_{t,t} \) is continuous. Thus \( \eta_t(x; h) \) the minimum point of \( A_t[h](x, \cdot) \) for fixed \( x \in X \).

Since \( X \) has only countable many states, by the classical diagonalization method, one can extract a convergent subsequence \( \eta^\varepsilon_n(x) \) such that the convergence is true for any \( x \in X \), i.e.

\[
\lim_{\varepsilon_n \to 0} \eta^\varepsilon_n(x) = \eta_t(x; h), \text{ for any } x \in X.
\]

(2) Note that

\[
H_{t,t}^{\varepsilon_n}(x; \eta_t^{\varepsilon_n}(x; h_2^{\varepsilon_n})) = f_{t,t}(x, \eta_t^{\varepsilon_n}(x; h_2^{\varepsilon_n})) + \Lambda_t^{\varepsilon_n}(x, \eta_t^{\varepsilon_n}(x; h_2^{\varepsilon_n}); h_1^{\varepsilon_n})
\]

\[
= f_{t,t}(x, \eta_t^{\varepsilon_n}(x; h_2^{\varepsilon_n})) + \Lambda_t(x, \eta_t^{\varepsilon_n}(x; h_2^{\varepsilon_n}); h_1) + \Lambda_t^{\varepsilon_n}(x, \eta_t^{\varepsilon_n}(x; h_2^{\varepsilon_n}); h_1^{\varepsilon_n}) - \Lambda_t(x, \eta_t^{\varepsilon_n}(x; h_2^{\varepsilon_n}); h_1)
\]
Since \( f_{\tau,t} \) is continuous,
\[
\lim_{n \to 0} f_{\tau,t}(x, \eta_t^n(x; h_2^n)) = f_{\tau,t}(x, \eta_t(x; h_2)).
\]

By (A2),
\[
\lim_{n \to \infty} \Lambda_t(x, \eta_t^n(x; h_2^n); h_1) = \Lambda_t(x, \eta_t(x; h_2); h_1).
\]

By (4.1),
\[
\lim_{n \to 0} |\Lambda_t^n(x, \eta_t^n(x; h_2^n); h_1^n) - \Lambda_t(x, \eta_t^n(x; h_2^n); h_1)|
\leq \lim_{n \to 0} \sup_{u \in B_t(x)} |\Lambda_t^n(x, u; h_1^n) - \Lambda_t(x, u; h_1)| = 0.
\]

Therefore, thus (4.2) holds.

It is easy to see that \( \{\sup \mathcal{H}_{\tau,t}[h_1^n](x; \eta_t^n(x; h_2^n))\}_{\varepsilon} \) is uniformly bounded below. Now we will prove that \( \sup \mathcal{H}_{\tau,t}[h_1^n](x; \eta_t^n(x; h_2^n)) \in \mathcal{BB}_V(X) \).

By (A3), \( \eta_t^n(x; h_2^n) \in B_1^n(x) \) where
\[
B_1^n(x) := \{ u \in U : f_{\tau,t}(x, u) \leq f_{\tau,t}(x, u_0) + \sup \Lambda_t^n(x, u_0; \lambda \nu) + \sup \sup h_2^n - \lambda \nu - \inf \inf h_2^n \}
\]
and \( B_1^n(x) \subset B_{t,\lambda}(x) \) when \( x \) is large, where \( B_{t,\lambda}(x) \) is defined from Assumption (C)
\[
B_{t,\lambda}(x) = \{ u \in U : f_{\tau,t}(x, u) \leq \lambda \nu(x) \}.
\]

Simple calculation yields
\[
\mathcal{H}_{\tau,t}[h_1^n](x; \eta_t^n(x; h_2^n)) \leq \sup_{u \in B_{t,\lambda}(x)} \left( f_{\tau,t}(x, u) + \Lambda_t^n(x, u; h_1^n) \right)
\leq \sup_{u \in B_{t,\lambda}(x)} \left( f_{\tau,t}(x, u) + \Lambda_t^n(x, u; \lambda \nu) + \sup \sup h_1^n - \lambda \nu \right)
\]

By Assumption (B),
\[
\limsup_{|x| \to \infty} \frac{1}{\nu(x)} \sup_{\varepsilon} \mathcal{H}_{\tau,t}[h_1^n](x; \eta_t^n(x; h_2^n)) \leq K_1 \lambda.
\]

By the arbitrariness of \( \lambda \), we have \( \{\mathcal{H}_{\tau,t}[h_1^n](x; \eta_t^n(x; h_2^n))\}_{\varepsilon} \in \mathcal{UBB}_V \).

Now we are ready to establish the convergence of time-inconsistent risk-sensitive \( \varepsilon \)-equilibria to time-inconsistent risk-sensitive equilibrium as \( \varepsilon \to 0^+ \).

**Theorem 4.3.** Under Assumptions (A),(B) and (C), as \( \varepsilon \to 0^+ \), the sequence of time-inconsistent risk-sensitive \( \varepsilon \)-equilibria \( \{\eta_t^n\}_{\varepsilon} \) is compact (in pointwise convergence sense) and the limit \( \eta_T \) of any convergent subsequence \( \{\eta_t^n\}_{\varepsilon} \) is a time-inconsistent risk-sensitive equilibrium strategy. Moreover, \( \{\Theta_t^n\}_{\varepsilon} \) defined in (1.6) using \( \eta_t^n \) converges to \( \{\Theta_t\}_{\varepsilon} \) defined in (1.7) using \( \eta_T \) in \( \mathcal{BB}_V(X), w \).

**Proof.** At Nth step, we take a subsequence \( \{\eta_T^n(\cdot; g_N)\} \) with limit \( \eta_N \). Note that
\[
\Theta_t^{\varepsilon_n}_N(x) = \mathcal{H}_{\tau,t}^{\varepsilon_n}[g_T](x; \eta_T^n(x; g_N)).
\]

By Lemma 4.2, we know that \( \Theta_t^{\varepsilon_n}_N(x) \) is uniformly bounded below and \( \sup_{\varepsilon_n} \Theta_t^{\varepsilon_n}_N \in \mathcal{BB}_V(X) \) with limit \( \Theta_t^0 \) in point-wise sense. By Lemma 2.3, \( \Theta_t^{\varepsilon_n}_N \) converges to \( \Theta_t^0 \) in \( \mathcal{BB}_V(X), w \).
At \((N - 1)\)th step, we take a subsequence of \(\{\eta^n_{N-1}(\cdot; \Theta^\varepsilon_{N-1,N})\}\) (still written as the same sequence) with limit \(\eta_{N-1}\). Note that
\[
\Theta^\varepsilon_{\tau,N}(x) = \mathcal{H}^\varepsilon_{\tau,N}(\Theta^\varepsilon_{\tau,N}(x; \eta^n_{N-1}(x; \Theta^\varepsilon_{N-1,N}))).
\]
By Lemma 4.2, \(\{\Theta^\varepsilon_{\tau,N}(x) \in BB_\varepsilon(X)\}\) \(\in UB_\varepsilon\) and it converges to \(\Theta_{\tau,N}(x)\) point-wisely as \(n \to \infty\). Thus \(\Theta^\varepsilon_{\tau,N}(x)\) converges in \((BB_\varepsilon(X), w)\). We repeat such process until the first step. Then the proof is complete.

The following corollary is obvious.

**Corollary 4.4.** (1) Under Assumptions (A), (B) and (C), if the solution \(\Theta\) of (1.7) is unique, then \(\Theta^\varepsilon_{\tau,t}\) converges to \(\Theta_{\tau,t}\) in \((BB_\varepsilon(X), w)\), i.e.
\[
\lim_{\varepsilon \to 0^+} w(\Theta^\varepsilon_{\tau,t}, \Theta_{\tau,t}) = 0.
\]

(2) Under Assumptions (A) and (B), if the cost functional is independent of the non-exponential discounting factor \(\tau\), i.e. time-consistent case, the solution \(\Theta^\varepsilon\) (resp.) is independent of the choices \(\eta\) (resp.) as well. As a result the solution \(\Theta\) of (1.7) is unique and \(V^\varepsilon_t = \Theta^\varepsilon_{\tau,t}\) converges to \(V_t = \Theta_{\tau,t}\) in \((BB_\varepsilon(X), w)\) for each fixed \(t\).

This corollary essentially repeats the vanishing viscosity procedure in [13] for continuous-time stochastic differential equations. While in our case, we are dealing with a discrete-time, countable-stated MDP. Compared to that of [13], our result has its own interesting feature because our problem is time-inconsistent.

### 5 An Illustrative Example

In this section, we will give an illustrative example to see that how the calculation works.

**Example 5.1.** Consider a sequence of random variables defined by
\[
X^\varepsilon_{t+1} = X^\varepsilon_t + u + \xi_t^\varepsilon.
\]
where the control \(u\) is taken in \(U = \{-1, 1\}\) and the distribution function of \(\xi_t^\varepsilon\) is
\[
\mathbb{P}(\xi_t^\varepsilon = x) = \begin{cases} \kappa \exp\{-\varepsilon^{-1}|x|^2\}, & \text{if } x \neq 0 \\ 1 - \kappa \sum_{z \neq 0} \exp\{-\varepsilon^{-1}|z|^2\}, & \text{if } x = 0. \end{cases}
\]
for some small \(\kappa > 0\). Simple calculation yields that
\[
I(z; x, u) = (z - x - u)^2 \text{ and } \Lambda(x, u; h) = \sup_{z \in X} [h(z) - (z - x - u)^2].
\]
Let \(V(x) = |x|^2\). Take \(\varepsilon_0\) small, (A1) holds. Since \(U\) is compact, (A2) holds. Note that
\[
\Lambda^\varepsilon(x, u; \lambda V) = \varepsilon \log \left\{ \kappa \sum_{z \neq 0} \exp\{-\varepsilon^{-1}(\lambda|x + z|^2 - |z|^2)\} \right\} \\
\leq \max \left\{ \sup_z [\lambda|x + z|^2 - |z|^2], \lambda|x|^2 \right\} \leq \lambda K_1 V(x)
\]
Therefore (A3) holds.

Let \( f_{\tau,t}(\cdot, u), g_{\tau}(\cdot) \in BB_{V}(X) \). Since \( U \) is compact, Assumption (B) and (C) are trivial because of (A3). Because the infimum or supremum can be attained, simple calculation shows that the Hamiltonians are

\[
\begin{align*}
H_{\tau,t} & [h](x, u) = f_{\tau,t}(x, u) + \max_{z \in X} [h(z) - (z - x - u)^2] \\
A_{t} & [h](x) = \min_{u \in U} \left( f_{t,t}(x, u) + \max_{z \in X} [h(z) - (z - x - u)^2] \right).
\end{align*}
\]

We can easily get the recursion sequence defined in (3.10).

If \( f \) and \( g \) are independent of the discounting factor \( \tau \). Then the value function \( V_{t} \) satisfies

\[
\begin{align*}
V_{t}(x) + \min_{u \in U} \left[ f_{t}(x, u) + \max_{z \in X} [V_{t+1}(z) - (z - x - u)^2] \right] & = 0, \\
V_{T}(x) & = g(x).
\end{align*}
\]

This is the time-inconsistent case which is equivalent to discrete min-max control problem.

6 Concluding Remarks

We have explored the time-inconsistent risk-sensitive MDPs with countable-stated state space. Due to the time-inconsistency of the risk-sensitive cost function, the theory on the time-inconsistent equilibriums and the convergence of value function as \( \varepsilon \rightarrow 0^+ \) have some unique interesting features, e.g. the convergence of \( \varepsilon \)-equilibriums are required for the convergence of value functions. Therefore, our results enrich the general theory of risk-sensitive MDPs and the time-inconsistent control problems. For our time-inconsistent risk-sensitive MDPs, a Hamiltonian recursion for each \( \varepsilon > 0 \) has been derived and the convergence for the solution sequences as \( \varepsilon \rightarrow 0^+ \) has been proved. An example is presented to show our assumptions are general.

We still can see that the theory is in its infancy and it is possible to be improved in several aspects. For example, can we conclude the similar results for general state space like \( X = \mathbb{R}^{d} \)? The main difficulty lies in the first-order regularity of the viscosity solutions of non-linear PDEs. We might deal with it in the other papers.

Acknowledgements The author would like to thank Professor François Dufour for his valuable comments on the manuscript.

References

[1] N. Bäuerle and U. Rieder, Markov Decision Processes with Applications to Finance. Springer-Verlag, Berlin Heidelberg, (2011)

[2] Bäuerle, N., Rieder, U. (2017). Partially observable risk-sensitive Markov decision processes. Mathematics of Operations Research, 42(4), 1180-1196.

[3] Barz, C., Waldmann, K.-H. (2007). Risk-sensitive capacity control in revenue management. Math. Methods Oper. Res. 65,
Bäuerle, N., Rieder, U. (2013). More risk-sensitive Markov decision processes. Mathematics of Operations Research, 39(1), 105-120.

Bielecki, T., Hernández-Hernández, D. ,Pliska, S. R. (1999). Risk sensitive control of finite state Markov chains in discrete time, with applications to portfolio management. Math. Methods Oper. Res. 50, 167–188. Financial optimization.

Björk, T., Khapko, M., & Murgoci, A. (2017). On time-inconsistent stochastic control in continuous time. Finance and Stochastics, 21(2), 331-360.

Cavazos-Cadena, R., Fernández-Gaucherand, E. (2000). The vanishing discount approach in Markov chains with risk-sensitive criteria. IEEE Trans. Automat. Control 45, 1800–1816.

Cavazos-Cadena, R. , Hernández-Hernández, D. (2011). Discounted approximations for risksensitive average criteria in Markov decision chains with finite state space. Math. Oper. Res. 36, 133–146.

Di Masi, G. B., Stettner, L. (1999). Risk-sensitive control of discrete-time Markov processes with infinite horizon. SIAM J. Control Optim. 38, 61–78.

Dupuis, P., Ellis, R. S. (2011). A weak convergence approach to the theory of large deviations (Vol. 902). John Wiley & Sons.

I. Ekeland and A. Lazrak, The golden rule when preferences are time inconsistent, Math. Finan. Econ., 4 (2010), 29–55.

I. Ekeland and T. A. Pirvu, Investment and consumption without commitment, Math. Finan. Econ., 2 (2008), 57–86.

Fleming, W. H., & Soner, H. M. (2006). Controlled Markov processes and viscosity solutions (Vol. 25). Springer Science & Business Media.

Ghosh, M. K., Saha, S. (2014). Risk-sensitive control of continuous time Markov chains. Stochastics An International Journal of Probability and Stochastic Processes, 86(4), 655-675.

Howard, R., Matheson, J. (1972). Risk-sensitive Markov Decision Processes. Management Science 18, 356–369.

Hu, Y., Jin, H., Zhou, X. Y. Time-inconsistent stochastic linear–quadratic control. SIAM journal on Control and Optimization, 50(3)(2012), 1548-1572.

Jaśkiewicz, A. (2007). Average optimality for risk-sensitive control with general state space. Ann. Appl. Probab. 17, 654–675.

Mei, H., Yong, J. (2018). Equilibrium strategies for time-inconsistent stochastic switching systems. arXiv preprint arXiv:1712.09505.

Marcus, S. I., Fernández-Gaucherand, E., Hernández-Hernandez, D., Coraluppi, S., Fard, P. (1997). Risk sensitive Markov decision processes. In Systems and control in the twenty-first century (pp. 263-279). Birkhäuser, Boston, MA.
[20] S. R. Grenadier and N. Wang, *Investment under uncertainty and time-inconsistent preferences*, J. Finan. Econ., 84 (2007), 2–39.

[21] Qi, Qingyuan, and Huanshui Zhang. *Time-inconsistent stochastic linear quadratic control for discrete-time systems*. Science China Information Sciences 60, no. 12 (2017): 120–204.

[22] Q. Wei, J. Yong, and Z. Yu, *Time-inconsistent recursive stochastic optimal control problems*, SIAM J. Control Optim., 55 (2017), 4156–4201.

[23] J. Yong, *Deterministic time-inconsistent optimal control problems—An essentially cooperative approach*, Acta Appl. Math. Sinica, 28 (2012), 1–20.

[24] J. Yong, *Time-inconsistent optimal control problems and the equilibrium HJB equation*, Math. Control Relat. Fields, 2 (2012), no. 3, 271-329.