Quadrupole properties of the eight $SU(3)$ algebras in $(sdgi)$ space

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Abstract. With nucleons occupying an oscillator shell $\eta$, there are $2^\lfloor \frac{\eta}{2}\rfloor$ number of $SU(3)$ algebras; $\lfloor \frac{\eta}{2}\rfloor$ is the integer part of $\eta/2$. Analyzing the first non-trivial situation with four $SU(3)$ algebras in $(sdg)$ space, demonstrated recently is that they generate quite different quadrupole properties though they all generate the same spectrum. More complex situation is with eight $SU(3)$ algebras in $(sdgi)$ space. In the present work, quadrupole properties generated by these eight algebras are analyzed first using the more analytically tractable interacting boson model. In addition, shell model and the closely related deformed shell model are used with three examples of nucleons in $(sdgi)$ space. It is found that in general six of the $SU(3)$ algebras generate prolate shape and two oblate shape. Out of all these, one of the $SU(3)$ algebras generates quite small quadrupole moments for the low-lying states.

1 Introduction

Shell model (SM) and the interacting boson model (IBM) admit $SU(3)$ algebra generating rotational spectra in nuclei. Going beyond the introduction of $SU(3)$ in SM by Elliott [1,2] that is applicable only to light nuclei, the scope of $SU(3)$ in nuclei is enlarged, spreading its applicability all across the periodic chart, by the developments in the pseudo-$SU(3)$ model [3,4], $SU(3)$ of fermion dynamical symmetry model [5], proxy-$SU(3)$ model [6] and the $Sp(6, R)$ model containing $SU(3)$ [7–9] all within the shell model on one hand and by various extended IBM’s such as IBM-2,3,4 [10], $sdg$IBM [11,12], $sdpf$IBM [13,14], IBFM (interacting boson-fermion model) and IBFFM (interacting boson–fermion–fermion model) [15–17] on the other. In addition, there are the algebraic cluster model [18,19] and many other models that employ $SU(3)$ symmetry. A new paradigm in the applications of $SU(3)$ in nuclei is the recent recognition that for nucleons in a given oscillator shell $\eta$, there will be multiple $SU(3)$ algebras [20,21]. For a recent overview of $SU(3)$ symmetry in atomic nuclei, see [22].

Multiple $SU(3)$ algebras corresponds to different realizations or different ways to generate the algebra of $SU(3)$. However, mathematically in all the cases we have the (only one) abstract algebra of the $SU(3)$ group uniquely defined by the commutation relations of its generators with no matter how they are constructed. As explained in

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reference [21], this situation appears as the result of a +/− phase indeterminacy in the radial matrix elements of the quadrupole operator between the wave functions of the 3D oscillator differing by two units of orbital angular momentum. This leads to different sets of SU(3) generators (see Eq. (1) and the related discussion in Sect. 2) characterized by the different combinations of (+) and (−) phase factors. Let us add that the concept of multiple SU(3) algebras extends to other algebras. For example, it is possible to have multiple pairing or quasi-spin algebras in SM and IBM spaces as described in detail in reference [20].

Simplest example of two SU(3) algebras in the \( \eta = 2 \) shell is well known [10,23] and this is the situation with (2s1d) shell in SM and also sdIBM. The prolate shape generated by one of the SU(3) algebras and oblate shape by the other are found to play an important role in quantum phase transition studies (QPT) [24]. However, the first non-trivial example of four SU(3) algebras in (sdg) space was analyzed only recently [21]. Used in this study are (sdg)\(^{6p}\), (sdg)\(^{6p,2n}\) and (sdg)\(^{6p,6n}\) examples in SM and a 10-boson system in sdgIBM with sdgIBM giving simple analytical results in the large boson number limit (note that \( p \) stands for protons and \( n \) for neutrons). Let us mention that in the (sdg)\(^{6p,6n}\) example, as the shell model matrix dimensions are very large, employed is the deformed shell model (DSM) based on Hartree–Fock single particle states [25]. It is found that the four SU(3) algebras in the (sdg) space exhibit quite different properties with regard to the quadrupole collectivity as brought out by the quadrupole moments \( Q(J) \) and \( B(E2) \)'s in the ground \( K = 0 \) band in the even–even nuclei studied. The general structure observed in the sdg examples is that one of the SU(3) algebras generates prolate shape, one oblate and the other two also generate prolate shape but one of them gives very small quadrupole moments for the low-lying levels. Thus, with multiple SU(3) algebras, it is possible to have rotational spectra (e.g., a \( K = 0 \) band with \( J = 0, 2, 4, \ldots \)) with very small quadrupole transition matrix elements. Our purpose in the present paper is to analyze the more complex situation of (sdgi) space that admits eight SU(3) algebras. Now we will give a preview.

In Section 2, first introduced are the multiple SU(3) algebras for bosons in an oscillator shell \( \eta \) and then restricting to (sdgi) space, the structure of the ground \( K = 0 \) bands generated by the eight algebras in the (sdgi) space is studied. Section 3 gives the formulas for quadrupole moments (\( Q(L) \)) and \( B(E2) \)'s for the ground \( K = 0 \) band and using them results generated for the quadrupole properties by the eight SU(3) algebras are presented. In Section 4, results are presented for the shell model (sdgi)\(^{6p}\), (sdg)\(^{6p,6n}\) and (sdgi)\(^{12p,6n}\) systems using SM codes for the first and DSM for all the three examples. Finally, Section 5 gives conclusions of the present work and also future outlook with a list of further investigations that need to be carried out for deeper understanding and applications of multiple SU(3) algebras in nuclei.

# 2 Eight SU(3) algebras in sdgiIBM and structure of \( K = 0 \) bands

Let us consider the situation with say \( N \) number of bosons occupying an oscillator shell with major shell number \( \eta \). Then, the spectrum generating algebra (SGA) is \( U(\mathcal{N}) \) with \( \mathcal{N} = (\eta + 1)(\eta + 2)/2 \). Also, for a given \( \eta \) the single particle orbital angular momentum \( \ell \) takes values \( \ell = \eta, \eta - 2, \ldots, 0 \) or 1. Now, as Elliott has established, \( U(\mathcal{N}) \supset SU(3) \supset SO(3) \) where SO(3) generates orbital angular momentum. The eight generators of \( SU(3) \) are the three angular momentum operators \( L^\ell_\eta \) with \( q = -1 \), 0 and +1 and the five quadrupole moment operators \( Q^\mu_\eta \) with \( \mu = -2, -1, 0, 1 \) and 2.
In terms of boson creation \((b^\dagger)\) and annihilation \((b)\) operators,

\[
Q^2 = \sum_{\ell_i, \ell_f} \frac{\langle \eta, \ell_f \mid \mid Q^2 \mid \mid \eta, \ell_i \rangle}{\sqrt{5}} \left( b^\dagger_{\ell_f} b^\dagger_{\ell_i} \right)^2.
\]

Note that \(\tilde{b}_{\ell,-m} = (-1)^{\ell-m} b_{\ell m}\) where \(m\) is \(L_z\) quantum number for a single boson. For a single oscillator shell \(\eta\), the quadrupole operator with oscillator parameter \(b = 1\) is equivalent to \(Q^2 = \sqrt{\frac{16\pi}{5}} r^2 Y^2_{\mu}(\theta, \phi)\). With this the reduced matrix element decomposes into the angular part that is easy to write down and the radial part. The \(\ell \rightarrow \ell \) and \(\ell \rightarrow \ell \pm 2\) radial matrix elements are \((2\eta + 3)/2\) and \(\alpha_{\ell,\ell+2} \sqrt{(\eta - \ell)(\eta + \ell + 3)}\) with \(\alpha_{\ell,\ell+2} = \alpha_{\ell+2,\ell} = \pm 1\). The phase factor \(\alpha_{\ell,\ell+2}\) arises as there is freedom in choosing the phases of the radial wave functions of a 3D oscillator. With this, in general we have multiple \(SU(3)\) algebras generated by the angular momentum operators \(L^1\) and quadrupole moment operators \(Q^2(\alpha)\) given by [21],

\[
L^1_q = \sum_{\ell} \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{3}} \left( b^\dagger_{\ell} b^\dagger_{\ell} \right)_q \quad ;
\]

\[
Q^2(\alpha) = -(2\eta + 3) \sum_{\ell} \sqrt{\frac{\ell(\ell+1)(2\ell+1)}{5(2\ell+3)(2\ell-1)}} \left( b^\dagger_{\ell} b^\dagger_{\ell} \right)_\mu \quad + \sum_{\ell<\eta} \alpha_{\ell,\ell+2} \sqrt{\frac{6(\ell+1)(\ell+2)(\eta - \ell)(\eta + \ell + 3)}{5(2\ell+3)}} \left( \left( b^\dagger_{\ell+2} b^\dagger_{\ell+2} \right)^2 + \left( b^\dagger_{\ell+2} b^\dagger_{\ell+2} \right)^2 \right);
\]

\[
\alpha = (\alpha_{0,2}, \alpha_{2,4}, \ldots, \alpha_{\eta-2,\eta}) \quad for \quad \eta \quad even,
\]

\[
\alpha = (\alpha_{1,3}, \alpha_{3,5}, \ldots, \alpha_{\eta-2,\eta}) \quad for \quad \eta \quad odd,
\]

\[
\alpha = (\pm 1, \pm 1, \ldots).
\]

Note that \(\ell\) in equation (1) takes values \(\eta, \eta - 2, \ldots, 0\) or 1. Now, the most important result that can be proved by using the straight forward angular momentum algebra is that the eight operators \([L^1_q, Q^2(\alpha)]\) in equation (1) commute among themselves generating \(SU(3)\) algebras independent of \(\alpha\)’s (for arbitrary values of \(\alpha_{\ell,\ell+2}\) the eight operators will not form a \(SU(3)\) algebra). Thus, for each choice of \(\alpha\) in equation (1) there is a \(SU(3)\) algebra. Clearly, given a \(\eta\), the number of \(SU(3)\) algebras is \(2[\frac{\eta}{2}]\) where \([\frac{\eta}{2}]\) is the integer part of \(\eta/2\). Then, there will be two \(SU(3)\) algebras for \(\eta = 2\) shell \((sd\) space\), four in \(\eta = 4\) shell \((sdg\) space\), eight in \(\eta = 6\) shell \((sdgi\) space\) and so on (if we have two different shells as in \(sdpfIBMiBM\) or bosons with internal degrees of freedom as in IBM-2,3,4 models or Bose-Fermi systems, the number of \(SU(3)\) algebras will be in general many more and these are not considered in the present paper). Standard choice for the \(\alpha\) as considered by Elliott and used in IBM studies is \(\alpha_{\ell,\ell+2} = +1\) independent of \(\ell\) value. In this article, we will focus on \((sdgi)\) space and consider the quadrupole properties generated by the eight \(SU(3)\) algebras in this space for a \(N\) boson system. With \(SU(3)\) operating, the \((sdgi)^N\) space decomposes into irreducible subspaces labeled by the irreducible representations (irreps) of \(SU(3)\) and they are denoted by \((\lambda\mu)\). The decomposition of \((\lambda\mu)\)’s into \(L\) is well known giving the \(K\) quantum number. For example, a \((\lambda,0)\) irrep gives \(K = 0\) with \(L = \lambda\), \(\lambda - 2, \ldots, 0\) or 1. Similarly, a \((\lambda,2)\) irrep gives \(K = 0\) and \(K = 2\) bands, a \((\lambda,4)\) irrep gives \(K = 0\), 2 and 4 bands and so on.
In $sdgiIBM$ with $\eta = 6$, $(\alpha) = (\alpha_{02}, \alpha_{24}, \alpha_{46})$ and with this there will be eight $SU^{(\alpha_{02}, \alpha_{24}, \alpha_{46})}(3)$ algebras. With $b_0^\dagger = s^\dagger$, $b_{2m}^\dagger = d_m^\dagger$, $b_{4m}^\dagger = g_m^\dagger$, the eight algebras are generated by the operators,

$$L_q = \sqrt{10}\left(b_2^\dagger b_2\right)_q + 2\sqrt{15}\left(b_1^\dagger b_1\right)_q + \sqrt{182}\left(b_0^\dagger b_6\right)_q,$$

$$Q_\mu^2(\alpha_{02}, \alpha_{24}, \alpha_{46}) = \sum_{\ell, \ell'} (\alpha_{\ell 0}, \alpha_{\ell 24}, \alpha_{\ell 46}) \left(b_\ell^\dagger b_{\ell'}\right)_\mu^2;$$

$$t_{2,2} = -15\sqrt{2}\sqrt{11}, \quad t_{4,4} = -\frac{90}{\sqrt{11}}, \quad t_{6,6} = -3\frac{\sqrt{182}}{11}, \quad t_{0,2} = t_{2,0} = \alpha_{02} 6\sqrt{\frac{6}{5}},$$

$$t_{2,4} = t_{4,2} = \alpha_{24} 12\sqrt{\frac{22}{35}}, \quad t_{4,6} = t_{6,4} = \alpha_{46} 6\sqrt{\frac{26}{11}}.$$

Note that $(\alpha_{02}, \alpha_{24}, \alpha_{46}) = (\alpha_{sd}, \alpha_{dq}, \alpha_{gq}) = (++, +, +), (+, +, -), (+, +, +), (+, +, -), (-, +, +), (-, +, -), (-, - , +)$ and $(-, - , -)$ where $+$ stands for $+1$ and $-$ stands for $-1$. Correspondingly, there are eight quadrupole-quadrupole Hamiltonians

$$H_Q^{(\alpha_{02}, \alpha_{24}, \alpha_{46})} = -\frac{1}{4} Q^2(\alpha_{02}, \alpha_{24}, \alpha_{46}) \cdot Q^2(\alpha_{02}, \alpha_{24}, \alpha_{46}).$$

As $H_Q^{(\alpha_{02}, \alpha_{24}, \alpha_{46})} = -C_2(SU^{(\alpha_{02}, \alpha_{24}, \alpha_{46})}(3)) + (3/4)L^2$, their eigenvalues over an $SU(3)$ state $| (\lambda \mu) KL \rangle$ are $-|\lambda^2 + \mu^2 + \lambda \mu + 3(\lambda + \mu)| + \frac{3}{4} L(L + 1)$. Therefore, all these eight $H_Q$'s generate the same spectrum and the ground band belongs to the $SU(3)$ irrep $(6N, 0)$ for a $N$ boson system. This is generated by the intrinsic state

$$|N; x_0, x_2, x_4, x_6 \rangle = (N!)^{-1/2} \left( x_0 s_0^\dagger + x_2 d_0^\dagger + x_4 g_0^\dagger + x_6 t_0^\dagger \right)^N |0\rangle$$

$$\Rightarrow |N; \beta_2, \beta_4, \beta_6 \rangle = [N! (1 + \beta_2^2 + \beta_4^2 + \beta_6^2)]^{-1/2} \left( s_0^\dagger + \beta_2 d_0^\dagger + \beta_4 g_0^\dagger + \beta_6 t_0^\dagger \right)^N |0\rangle.$$

Here, $x_\ell, \ell = 0, 2, 4, 6$ are free parameters determined by minimizing the $H_Q$ expectation value in the intrinsic state. Note that $x_0^2 + x_2^2 + x_4^2 + x_6^2 = 1$ and the choice $x_0 = 1/x, x_2 = \beta_2/x, x_4 = \beta_4/x$ and $x_6 = \beta_6/x$ gives the second form above; without loss of generality, choosing $x$ to be positive, we have $x = (1 + \beta_2^2 + \beta_4^2 + \beta_6^2)^{1/2}$. It is important to recognize that to a good approximation the $\beta_\ell$, $\ell = 2, 4, 6$ are essentially deformation parameters $[10,11]$. Now, the energy functional or the $H_Q$ expectation value is given by

$$E_{SU^{(\alpha_{02}, \alpha_{24}, \alpha_{46})}(3)}(N; \beta_2, \beta_4, \beta_6) = \left\langle N; \beta_2, \beta_4, \beta_6 \mid H_Q^{(\alpha_{02}, \alpha_{24}, \alpha_{46})} \mid N; \beta_2, \beta_4, \beta_6 \right\rangle$$

$$= -\frac{9}{29645} \left( 1 + \beta_2^2 + \beta_4^2 + \beta_6^2 \right) \left\{ 15125\beta_2^4 + 2420\beta_2^3 \left( 7\sqrt{30}\alpha_{02} + 4\sqrt{55}\beta_4\alpha_{24} \right) + 22\beta_2^2 \left( 6468 + 5122\beta_4^2 + 1225\beta_6^2 + 1232\sqrt{66}\beta_2\alpha_{02}\alpha_{24} + 1750\sqrt{2}\beta_4\beta_6\alpha_{46} \right) + 5 \left( 2500\beta_4^4 + 14700\beta_2^2\beta_6^2 + 2401\beta_4^4 + 7000\sqrt{2}\beta_2^2\beta_6\alpha_{46} + 6860\sqrt{2}\beta_4\beta_6^3\alpha_{46} \right) \right. + 44\beta_2 \left( 7\sqrt{15}\alpha_{02} \left( 50\sqrt{2}\beta_2^2 + 49\sqrt{2}\beta_6^2 + 140\beta_4\beta_6\alpha_{46} \right) + 4\sqrt{55}\beta_4\alpha_{24} \left( 50\beta_2^2 + 49\beta_6^2 + 70\sqrt{2}\beta_4\beta_6\alpha_{46} \right) \right\}.$$
Minimizing the energy functional with respect to $\beta_2$, $\beta_4$ and $\beta_6$ will give the parameters $(x_0, x_2, x_4, x_6)$ that define the intrinsic state for the ground $K = 0$ band for the eight $SU(3)$ algebras. For a given $(\alpha_{02}, \alpha_{24}, \alpha_{46})$, the parameters $x_\ell$ defining the intrinsic state in equation (4), after simplifying the results in [22], are given by

$$x_0 = \sqrt{\frac{33}{231}}, \quad x_2 = \alpha_{02} \sqrt{\frac{110}{231}}, \quad x_4 = \alpha_{02} \alpha_{24} \sqrt{\frac{72}{231}}, \quad x_6 = \alpha_{02} \alpha_{24} \alpha_{46} \sqrt{\frac{16}{231}}. \quad (6)$$

As seen from equation (6), the intrinsic states for the ground $K = 0$ band for the eight $SU(3)$’s differ only in the phases of $x_\ell$’s and this is similar to the situation with $sdgIBM$ [21]. In addition, the intrinsic state for $SU(\pm,\pm) (3)$ has the same structure as the Nilsson orbit [660] in orbital space. Substituting these values of $x_\ell$’s in equation (5) gives $E_{SU,sdg}(3)^{0} = -36N^2$ for all the eight $SU(3)$ algebras in agreement with the large-$N$ limit result for the ground $SU(3)$ irrep $(6N,0)$. However, the quadrupole properties (quadrupole moments and $B(E2)$’s) generated by them, as obtained using a quadrupole transition operator, will be different. We will turn to this now.

3 Quadrupole properties from the eight $SU(3)$ algebras in $sdgIBM$

In order to study the quadrupole moments and $B(E2)$ values in the ground $K = 0$ bands generated by the eight $SU(\alpha_{02},\alpha_{24},\alpha_{46}) (3)$ algebras in $sdgIBM$, the $E2$ transition operator is chosen to be

$$T^{E2} = q_2 Q_q^{2}(\pm,\pm,+)$$

where $q_2$ is a parameter. As stated before, in IBM the standard choice for the quadrupole operator is $Q_q^{2}(\alpha)$ with $\alpha_{\ell,\ell+2} = +1$ for all $\ell$. Now, as $Q_q^{2}(\pm,\pm,\pm)$ is a generator of $SU(\pm,\pm)(3)$, formulas for the quadrupole moments $Q(L)$ and $B(E2)$’s along the $K = 0$ ground band follow easily from equation (10) given ahead as the eigenstates obtained from $H^{(\pm,\pm,\pm)}_{Q}$ belong to $SU(\pm,\pm)(3)$. However, for the other $H_Q$ operators, the ground bands belong to the $(6N,0)$ irrep of the corresponding $SU(\alpha_{02},\alpha_{24},\alpha_{46}) (3)$ algebras. Therefore, the $T^{E2}$ chosen is no longer a generator of these algebras. Hence, the simple $SU(3)$ formulas given by equation (10) are not valid for these and we have to use a much more detailed $SU(3)$ algebra. Instead of this, large-$N$ limit formulas (order $1/N^2$) are used in the analysis.

Starting with the intrinsic state given by equation (4), it is easy to construct the angular momentum projected states $|N;K = 0, L, M\rangle$. Using these, formulas for the quadrupole moments $Q(L)$ and $B(E2; L \rightarrow L - 2)$ for the ground band are derived by Kuyucak and Morrison [26] that are valid for any one-body $T^{E2}$ operator. These formulas, valid to order $1/N^2$, are

$$Q(L) = q_2 \langle LL | Q_q^{2} | LL \rangle = q_2 \frac{\langle LL 20 | LL \rangle}{\sqrt{2L + 1}} \frac{\langle L || Q^2 || L \rangle}{},$$

$$B(E2; L \rightarrow L - 2) = \langle q_2 \rangle^2 \frac{5}{16\pi} \left( \frac{\langle L - 2 || Q^2 || L \rangle^2}{(2L + 1)} \right).$$
Table 1. Quadrupole moments \(Q(L)\) and \(B(E2; L \rightarrow L - 2)\) values for the low-lying states in the ground band for a 15-boson system generated by the eight \(SU^{(3)}(3)\) algebras in \(sdgi\)IBM. Note that in the table \(Q(L)\) (in units of \(q_2\)) and \(B(E2; L \rightarrow L - 2)\) (in units of \((q_2)^2\)) are given for \(L = 2, 4, 6, 8\) and 10 in columns 3 to 7 respectively.

| \(\alpha\)    | \(Q\) | \(2\) | \(4\) | \(6\) | \(8\) | \(10\) |
|----------------|-------|------|------|------|------|-------|
| (+,+,+)        | \(B(E2)\) | 666  | 951  | 1045 | 1090 | 1115  |
| (+,+,−)        | \(B(E2)\) | 460  | 654  | 714  | 739  | 747   |
| (+,−,+)        | \(B(E2)\) | 82   | 116  | 127  | 131  | 132   |
| (+,−,−)        | \(B(E2)\) | 22   | 30   | 32   | 31   | 29    |
| (−,+,+)        | \(B(E2)\) | 29   | 34   | 37   | 39   | 40    |
| (−,+,−)        | \(B(E2)\) | 51   | 56   | 58   | 59   |       |
| (−,−,+)        | \(B(E2)\) | 36   | 38   | 40   | 42   |       |
| (−,−,−)        | \(B(E2)\) | 41   | 58   | 64   | 66   |       |
|                | \(B(E2)\) | 22   | 27   | 30   | 30   | 30    |

\[
\langle N; K = 0, L_f \mid Q^2 \mid N; K = 0, L_i \rangle = \sum_{\ell, \ell'} \langle \ell' \mid (\ell' + 1) | \ell 0 | 0 \rangle t_{\ell', \ell} x_{\ell'} x_{\ell}, \quad F_1 = B_{20} - B_{11} - 10B_{10} + 12B_{00}, \quad F_2 = B_{20} - B_{11} + 6B_{10} - 12B_{00},
\]

\[
a = \sum_{\ell} (\ell + 1) (x_{\ell})^2.
\]

Note that \(\ell = 0, 2, 4\) and 6 for \(sdgi\)IBM and the \(t_{\ell', \ell}\) are the coefficients in the \(E2\) transition operator as given in equation (2). Using \(T^{E2}\) given above, the solutions for \(x_{\ell}\) given in equation (6) and the formulas in equation (7), calculated are \(Q(L)\) and \(B(E2; L \rightarrow L - 2)\) with \(L = 2, 4, 6, 8\) and 10 for a 15-boson system and the results are given in Table 1. Let us mention that for \(SU^{(+,+)}(3)\), exact \(SU(3)\) formulas are given by equation (10) ahead with the replacements \(\lambda = 6N\), \(J = L\), \(X_{eff}b^2 = q_2\).

Firstly, it can be verified that the results for \(SU^{(+,+)}(3)\) in Table 1 are essentially same as those from the exact \(SU(3)\) formulas and this in turn is a good test of the formulas in equation (7). More importantly, it is seen that only \(SU^{(+,+)}(3)\) and \(SU^{(+,+)}(3)\) generate oblate shapes and all other six \(SU(3)\)'s generate prolate shapes. Also, out of these six, only \(SU^{(+,+)}(3)\) and \(SU^{(+,+)}(3)\) generate large quadrupole moments and strong \(B(E2)\)'s. Similarly, \(SU^{(+,+)}(3)\) generates quite small quadrupole moments (less by a factor 5 compared to the largest). Thus, the eight \(SU(3)\) algebras generate quite different quadrupole properties for ground \(K = 0\)
bands. For further understanding of the quadrupole properties generated by the eight $SU(3)$ algebras in $(sdgi)$ space, we will now consider shell model examples.

4 Shell model and deformed shell model analysis in $(sdgi)$ space

4.1 Preliminaries

In the shell model with valence nucleons in $(sdgi)$ orbits, we have eight $SU^\alpha(3)$ algebras and the generators of these, in LST coupling, follow from equation (1) by replacing $(\hat{b}_{\ell f}^\dagger \hat{b}_{\ell i})^2_q$ by $2 (a_{\ell f}^\dagger \hat{a}_{\ell i}^{1\frac{1}{2}})_{q \ell}^L=2,S_0=0,T_0=0$. Then, we have

$$Q^2_\beta(\alpha) = 2 \sum_{\ell f, \ell i} t_{\ell f, \ell i}(\alpha) \left( a_{\ell f}^\dagger \hat{a}_{\ell i}^{1\frac{1}{2}} \right)_{q \ell}^L = 2,0,0.$$

(8)

Note that $t_{\ell f, \ell i}(\alpha)$ are given in equation (2) and $\alpha$ takes eight values as given before. In the shell model analysis used are the examples $(sdgi)^{6p}$, $(sdgi)^{6p,6n}$ and $(sdgi)^{12p,6n}$ systems giving the lowest $SU(3)$ irreps to be $(30,0)$, $(60,0)$ and $(78,0)$ respectively

[27]. Again, choosing

$$H_Q^{(\alpha)} = -(1/4)Q^2(\alpha) \cdot Q^2(\alpha),$$

for the eight $SU(3)$ algebras studied are the energies of the yrast ($K = 0$ band) levels, quadrupole moments $Q_2(J)$ of these levels and the $B(E2)$’s along the yrast line for $J$ up to 10. Used for this purpose are DSM and also the Antoine shell model code [28]. Note that DSM brings out shape information in a transparent manner and also it is useful for larger particle numbers where SM calculations are impractical [25]. For easy reference, in Appendix A given are the formulas for the single particle energies (spe) and two-body matrix elements (TBME) defining the $Q,Q$ operators and these are the inputs for both SM and DSM calculations.

In the SM (also DSM) studies, the $E2$ transition operator is taken to be

$$T^{E2} = \left[ e_{eff}^p Q^2_\beta(-,-,p) + e_{eff}^n Q^2_\beta(-,-,n) \right] b^2$$

(9)

where $b$ is the oscillator length parameter and $e_{eff}^p$ and $e_{eff}^n$ are proton and neutron effective charges. This choice follows from the fact that in SM it is standard to use $\alpha = (\alpha_{sd}, \alpha_{dg}, \alpha_{gi}) = (-, -, -)$; see for example [29–31]. In the situation the eigenstates obtained for $H_Q^{(-,-,-)}$ for the $K = 0$ band are of the form

$$|\lambda_p(0), \lambda_n(0), \lambda_p + \lambda_n, 0) K = 0, L, S = 0, J = L \rangle,$$

formulas for the $Q(J)$ and $B(E2)$’s generated by our choice of $T^{E2}$ are given by,

$$Q((0)J = -\frac{J}{2J + 3} (2\lambda + 3) X_{eff} b^2,$$

$$B(E2; (0)J \to J - 2) = \frac{5}{16\pi} \left( \frac{6J(J-1)(\lambda - J + 2)(\lambda + J + 1)}{(2J-1)(2J+1)} \right) \left( X_{eff} \right)^2 b^4;$$

$$X_{eff} = \frac{e_{eff}^p (\lambda_p^2 + 3\lambda_p + \lambda_p \lambda_n) + e_{eff}^n (\lambda_n^2 + 3\lambda_n + \lambda_n \lambda_p)}{(\lambda^2 + 3\lambda)}, \lambda = \lambda_p + \lambda_n.$$

(10)
These formulas are not valid for $H_Q^{(\alpha)} \neq H_Q^{(-,-,-)}$. Thus, numerical SM and DSM results are needed for the analysis of the eight algebras.

4.2 Results for $({sdgi})^{6p}$ system

In the example with six protons in $\eta = 6$ shell, SM matrix dimension in the $m$-scheme $\sim 18 \times 10^5$. For this system, the leading $SU(3)$ irrep is $(30,0)$ with $S = 0$ and $T = 3$ giving clearly $J = L$. It is seen that the SM calculations reproduce the $SU(3)$ results $E_{gs} = -990$ and excitation energies $0.75J(J+1)$ for all the eight $H$’s (note that the $H_Q$ matrix elements are unit less and hence $E$ are unit less – in practical applications we have to put back appropriately the unit MeV). Thus, all the eight $H_Q$’s give $SU(3)$ symmetry. Though the energy spectra are same, the wave functions of the yrast $J$ states are different. This is established by calculating $Q(J)$ and $B(E2)$’s for the ground band members. Choosing $e_{eff}^p = 1e$ and $b^2 = 4.644 fm^2$, the calculated results for $Q(2^+_1)$ in $e fm^2$ unit are $-84, -51, -19, 14, 30, -3, -35$ and $-68$ for $(\alpha) = (-, -, -), (-, -, +), (-, +, +), (-, +, -), (+, +, +), (+, +, -), (+, -, +), (+, -, -)$ respectively. Similarly, for $Q(4^+_1)$ they are $-106, -65, -25, 16, 38, -3, -43, -84$ respectively. The $B(E2)$’s also follow the same trend and for example $B(E2; 2^+_1 \rightarrow 0^+_1)$ values in $e^2 fm^4$ unit are $1700, 624, 82, 52, 217, 2, 306$ and $1138$ for the eight $H_Q$’s respectively. Thus, out of the eight $SU(3)$ algebras, two of them generate oblate shape and the remaining six prolate shape. Out of these six, one of them generates very small quadrupole moments and $B(E2)$ values. All these SM results for the energies of the ground $K = 0$ band, $Q(J)$’s and $B(E2)$’s are also well reproduced (within 5% difference) by DSM using a single intrinsic state as in the $(sdg)$ examples presented in [21]. The HF $sp$ spectrum for the $(sdg)^{6p}$ system is essentially same as the one shown in Figure 1 except for a scale factor for the $sp$ energies and the single intrinsic state employed in DSM calculations corresponds to 2 protons each in the lowest two $k = 1/2$ $sp$ levels and the lowest $k = 3/2$ $sp$ level shown in Figure 1. Let us add that the lowest intrinsic state gives the intrinsic (or mass) quadrupole moments (in units of $b^2$) to be $60, 35, 15, -9, -21, 3, 23$ and $48$ for $(\alpha) = (-, -, -), (-, -, +), (-, +, -), (-, +, +), (+, +, +), (+, +, -), (+, -, +), (+, -, -)$ respectively; note that the quadrupole operator is $Q^2_{\eta}(-, -, -; p)$. Now, we will consider the larger space example of $(sdgi)^{6p,6n}$ where SM calculations are not feasible and DSM gives the results.

4.3 Results for $(sdgi)^{6p,6n}$ system

Carrying out DSM calculations for the $(sdgi)^{(6p,6n)T=0}$ system using the eight $H_Q$’s, it is found that all of them generate the same HF $sp$ spectrum as shown in Figure 1. The lowest intrinsic state shown in the figure gives the intrinsic quadrupole moments (in units of $b^2$) to be $120, 70, 34, -16, -43, 8, 43, 94$ for $(\alpha) = (-, -, -), (-, -, +), (-, +, -), (-, +, +), (+, +, +), (+, +, -), (+, -, +), (+, -, -)$ respectively. Here and in the next subsection, used are proton and neutron quadrupole moment operators $Q^2_{\eta}(-, -, -; p)$ and $Q^2_{\eta}(-, -, -; n)$. The intrinsic quadrupole moments show that, again as in the $sdgi$IBM, in SM also two of the eight $SU(3)$ algebras generate oblate shape and rest of the six generate prolate shape. Out of these six, one of them generates very small quadrupole moment. Going further, after angular momentum projection from the lowest intrinsic state shown in the figure, the ground state energy and the excited state energies of the yrast levels are found to be within 1% of the exact $SU(3)$ results (DSM generates yrast states with $S = 0$ and $J = L$). Note that the $SU(3)$ irrep generating the ground $K = 0$ band is $(60,0)$ giving $E(J = L) =$
−3780 + 0.75J(J + 1) for the yrast 0\(^+\), 2\(^+\), 4\(^+\), \ldots levels. Turning to \(Q(J)\) and \(B(E2)\)’s, in the calculations used are \(e_{eff}^p = 1.5e\), \(e_{eff}^n = 0.5e\) and \(b^2 = 4.644 \text{ fm}^2\).

Here, equation (10) applies for the states from \(H_Q^{(--,-)}\); note that \((\lambda_p, \mu_p) = (30, 0)\) and \((\lambda_n, \mu_n) = (30, 0)\). For \(\alpha = (-, -, -)\), the DSM results shown in Table 2 agree with the \(SU(3)\) formulas to within 3\%; However, the results from the other seven \(H_Q\)’s are quite different as in the previous \((sdgi)^{6p}\) example. Again, it is seen from Table 2 that the results for \(Q(J)\) and \(B(E2)\)’s from \(H_Q^{(--,-)}\) and \(H_Q^{(+,-,-)}\) are strong and the \(B(E2)\)’s from \(H_Q^{(+,+,-)}\) are much smaller in magnitude. Moreover, six of them generate prolate shape and two of them oblate shape as in the previous examples. Thus, the results in Tables 1 and 2 give the generic result that out of the eight \(SU(3)\) algebras in the \((sdgi)\) space, six will give prolate and two oblate shape and in addition, one of them [(+, +, −)] gives very small quadrupole moments. For further elucidating the difference between the eight \(SU(3)\) algebras, we show in Table 3 the \(sp\) wave functions for the lowest \(sp\) state in Figure 1 (this is same for protons and neutrons). With \(sp \ j\) values taking 1/2 to 13/2, the structure of the \(sp\) wave function is

\[
|k_r(\alpha)\rangle = \sum_j \left( -1 \right)^{\phi_j(\alpha)} C_{k_r}^{j} |jk_r\rangle , \quad \text{with} \quad C_{k_r}^{j} \geq 0. \quad (11)
\]
generate the same HF sp spectrum as shown in Figure 2. The lowest intrinsic state shown in the figure generates the ground \( k \) out using the eight \( H \). The most important point here is that for a given \( k \), the \( | k \rangle \) for the eight \((1/2)\) for the eight \((1/2)\) \((12p_6n)\) system of 6 protons and 6 neutrons (with \( T = 0 \) in \( \eta = 6 \) shell). Results are given for the eight \((-1/4) Q^2(\alpha) \cdot Q^2(\alpha) \) Hamiltonians. In the table, \( Q \) denotes \( Q(J) \) and \( B(E2) \) denotes \( B(E2; J \rightarrow J - 2) \).

\[
\begin{array}{c|cccccc}
\alpha & | \alpha \rangle & \langle \alpha | & j & \langle \alpha | & j & \langle \alpha | & j \\
\hline
(-, -, -) & Q & -159 & -202 & -222 & -234 & -241 & \langle \alpha | & j \\
B(E2) & 6482 & 9236 & 10123 & 10523 & 10711 & & \langle \alpha | & j \\
(-, -, +) & Q & -95 & -121 & -133 & -141 & -146 & & \langle \alpha | & j \\
B(E2) & 2309 & 3263 & 3525 & 3587 & 3550 & & \langle \alpha | & j \\
(-, +, -) & Q & -39 & -51 & -57 & -62 & -67 & & \langle \alpha | & j \\
B(E2) & 391 & 575 & 667 & 750 & 840 & & \langle \alpha | & j \\
(-, +, +) & Q & 25 & 31 & 35 & 38 & 40 & & \langle \alpha | & j \\
B(E2) & 161 & 225 & 238 & 235 & 223 & & \langle \alpha | & j \\
(+, +, -) & Q & 56 & 72 & 78 & 82 & 84 & & \langle \alpha | & j \\
B(E2) & 822 & 1172 & 1286 & 1340 & 1368 & & \langle \alpha | & j \\
(+, +, +) & Q & 27 & 36 & 42 & 48 & 52 & & \langle \alpha | & j \\
B(E2) & 1029 & 1435 & 1515 & 1490 & 1409 & & \langle \alpha | & j \\
(+, -, -) & Q & -127 & -161 & -175 & -183 & -186 & & \langle \alpha | & j \\
B(E2) & 4164 & 5907 & 6426 & 6609 & 6633 & & \langle \alpha | & j \\
\end{array}
\]

Table 2. Deformed shell model results for quadrupole moments \( Q(J) \) (in \( e\,fm^2 \) unit) and \( B(E2; J \rightarrow J - 2) \) values (in \( e^2\,fm^4 \) unit) for the ground \( K = 0^+ \) band members for a system of 6 protons and 6 neutrons (with \( T = 0 \) in \( \eta = 6 \) shell). Results are given for the eight \((-1/4) Q^2(\alpha) \cdot Q^2(\alpha) \) Hamiltonians in \((sdgi)\) space. The \( C_{1/21}^j \) for the seven sp states (in the order given in the table) are 0.366, 0.430, 0.527, 0.380, 0.425, 0.191 and 0.207 respectively.

\[
\begin{array}{c|c|cccccc}
|k_r\rangle & \alpha & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 \\
\hline
1/2_1 & (-, -, -) & - & - & + & + & - & - & - & + \\
& (-, -, +) & - & - & + & + & - & + & - \\
& (-, +, -) & - & - & - & + & + & - & + \\
& (-, +, +) & - & - & - & + & + & - & + \\
& (+, -, -) & + & + & + & + & + & + & + \\
& (+, -, +) & + & + & + & + & + & + & + \\
& (+, +, -) & + & + & + & + & + & + & + \\
& (+, +, +) & + & + & + & + & + & + & + \\
\end{array}
\]

Table 3. Phases \((-1)^{\phi_3(\alpha)}\) in equation (11) for the lowest Hartree–Fock sp state \(|1/2_i\rangle \) (see Fig. 1) for the eight \((-1/4) Q^2(\alpha) \cdot Q^2(\alpha) \) Hamiltonians in \((sdgi)\) space. The \( C_{1/2_1}^j \) for the seven \( j \) states (in the order given in the table) are 0.366, 0.430, 0.527, 0.380, 0.425, 0.191 and 0.207 respectively.

Most important point here is that for a given \( k_r \), the \( C_{k_r}^j \) do not depend on \( \alpha \). Thus, the \( sp \) wave functions differ only in \((-1)^{\phi_3}\). Now, we will consider the \((sdgi)^{12p,6n}\) example.

### 4.4 Results for \((sdgi)^{12p,6n}\) system

In our third example, DSM calculations for the \((sdgi)^{(12p,6n)}_{T=3}\) system are carried out using the eight \( H_Q \)'s. As in the previous examples, it is found that all of them generate the same HF sp spectrum as shown in Figure 2. The lowest intrinsic state shown in the figure generates the ground \( K = 0 \) band and its \( SU(3) \) structure (same
Fig. 2. Hartree–Fock sp spectrum for the lowest intrinsic state for the \((sdg)^{12p,6n}\) system generated by the eight \(H_Q\) operators. In the figure, the symbol \(\bigodot\) denotes protons and \(\times\) denotes neutrons. The spectrum is same for all the eight Hamiltonians although the sp wave functions are different. See text for further details.

for all eight \(H_Q\)'s is

\[
\left| (sdg)^{12p}(48,0), (sdg)^{6n}(30,0) : (78,0)K = 0, L; S = 0; J = L; T = 3 \right|. 
\]

For the lowest intrinsic state, the intrinsic quadrupole moments (in units of \(b^2\)) are 156, 71, –16, –51, 35, 34, 121 for \((\alpha) = (-, -, -), (-, +, -), (-, +, +), (+, +, +), (+, +, -), (+, -), (-, - +), (-, +, -)\) respectively. The intrinsic quadrupole moments show that, in this SM example also two of the eight \(SU(3)\) algebras generate oblate shape and rest of the six generate prolate shape. Out of the six prolate examples, one of them generates very small quadrupole moments. Going further, after angular momentum projection from the lowest intrinsic state shown in the figure, the ground state energy and the excited state energies of the yrast levels are found to be within 1% of the exact \(SU(3)\) result \(E(J = L) = -6318 + 0.75J(J + 1)\) for the yrast \(0^+, 2^+, 4^+, \ldots\) levels. Turning to \(Q(J)\) and \(B(E2)'s\), in the calculations used is \(T^{E2}\) in equation (9) with \(e_{eff}^p = 1.5e, e_{eff}^n = 0.5e\) and \(b^2 = 4.61 \text{fm}^2\). The DSM results are shown in Table 4. It is easy to see, by applying equation (10), that the DSM results for \(\alpha = (-, -, -)\) agree with the exact \(SU(3)\) formulas. However, the results from the other seven \(H_Q''s\) are quite different as in the previous \((sdg)\) example. Again, it is seen from Table 4 that the results for \(Q(J)\) and \(B(E2)'s\) from \(H_Q^{(-,-,-)}\) and \(H_Q^{(+,-,-)}\) are strong and the \(B(E2)'s\) from \(H_Q^{(-,+,+)}\) are much smaller in magnitude. Moreover, six of them generate prolate shape and two of them oblate shape as in the previous examples. Thus, the numerics in Tables 1, 2 and 4 give the generic result that out of the eight \(SU(3)\) algebras in the \((sdg)\) space, six will give prolate and two oblate shape and in addition, one of them gives small quadrupole moments. This need to be coupled with the fact that all of them generate the same spectrum.
Table 4. Deformed shell model results for quadrupole moments $Q(J)$ (in $e\,fm^2$ unit) and $B(E2; J \rightarrow J - 2)$ values (in $e^2\,fm^4$ unit) for the ground $K = 0^+$ band members for a system of 12 protons and 6 neutrons (with $T = 3$) in $\eta = 6$ shell. Results are given for the eight $(-1/4)Q^2(\alpha) \cdot Q^2(\alpha)$ Hamiltonians. In the table, $Q$ denotes $Q(J)$ and $B(E2)$ denotes $B(E2; J \rightarrow J - 2)$.

| $\alpha$       | 2   | 4   | 6   | 8   | 10  |
|----------------|-----|-----|-----|-----|-----|
| $(-,-,-)$      | $Q$ | $-234$ | $-297$ | $-327$ | $-344$ | $-355$ |
| $B(E2)$        | $13288$ | $18952$ | $20815$ | $21699$ | $22167$ |
| $(-,-,+)$      | $Q$ | $-92$ | $-118$ | $-130$ | $-138$ | $-144$ |
| $B(E2)$        | $2065$ | $2953$ | $3256$ | $3414$ | $3514$ |
| $(-,+,-)$      | $Q$ | $-120$ | $-152$ | $-167$ | $-176$ | $-181$ |
| $B(E2)$        | $3501$ | $4988$ | $5467$ | $5682$ | $5782$ |
| $(-,+,+)$      | $Q$ | $22$ | $27$ | $30$ | $31$ | $31$ |
| $B(E2)$        | $114$ | $161$ | $176$ | $182$ | $184$ |
| $(+,+,+)$      | $Q$ | $76$ | $96$ | $104$ | $109$ | $110$ |
| $B(E2)$        | $1395$ | $1988$ | $2180$ | $2267$ | $2309$ |
| $(+,+,-)$      | $Q$ | $-66$ | $-84$ | $-93$ | $-98$ | $-101$ |
| $B(E2)$        | $1055$ | $1501$ | $1642$ | $1703$ | $1727$ |
| $(+,-,+)$      | $Q$ | $-38$ | $-49$ | $-55$ | $-60$ | $-64$ |
| $B(E2)$        | $352$ | $504$ | $559$ | $591$ | $615$ |
| $(+,,-,-)$     | $Q$ | $-180$ | $-229$ | $-252$ | $-266$ | $-276$ |
| $B(E2)$        | $7846$ | $11191$ | $12290$ | $12812$ | $13087$ |

5 Conclusions and future outlook

Multiple $SU(3)$ algebras appearing in both the shell model and the interacting boson model opened a new paradigm in the applications of $SU(3)$ symmetry in nuclei. The $sdgi$IBM and SM examples presented in Sections 2–4 show that the eight $SU(3)$ algebras in the $(sdgi)$ space of IBM and SM exhibit quite different properties with regard to the quadrupole collectivity as brought out by the quadrupole moments $Q(J)$ and $B(E2)$’s in the ground $K = 0$ bands. Six of them generate prolate shape, two oblate shape and in the six prolate, one of them generates small quadrupole moments. However, they all generate the same rotational spectra. Thus, with multiple $SU(3)$ algebras it is possible to have rotational spectra with strong quadrupole collectivity and also rotational spectra with weak quadrupole collectivity. In addition, some of them give prolate and other oblate shapes. All these conclusions are consistent with the earlier $(sdg)$ space results [21] and therefore establish that these are generic results valid both in shell model and interacting boson model spaces.

All the results above offer criterion to find experimental examples for multiple $SU(3)$ algebras in nuclei and which of the $SU(3)$ algebras is good for a given nucleus and they may change with proton and/or neutron number variation. This analysis of experimental data will be considered in a future publication. Another approach to establish the role of multiple $SU(3)$ algebras in nuclei is to study possible transition from one realization to another simulating a phase transition between different shapes, let us say a prolate to oblate transition, or about some kind of shape coexistence. Results of such phase transitions in $sdI$IBM are well known [24]. We are investigating phase transitions in $sdgIBM$ and $sdgiIBM$ with $H = -Q \cdot Q$ where the quadrupole operator $Q$ is same as the one given in equation (1) with the difference that $\alpha_{\ell,\ell+2}$’s are treated as free parameters varying from $-1$ to $+1$. Results of this work will be available soon. We will now list a number of other investigations that need to be
carried out in future for deeper understanding and applications of multiple $SU(3)$
algebras in nuclei.

Further understanding of multiple $SU(3)$ algebras in $sdgiIBM$ spaces will follow
by analyzing the structure of the low-lying $\gamma$ and $\beta$ bands. It is also of interest to
analyze multiple $SU(3)$ algebras in IBM-2 in $(sdg)$ and $(sdgi)$ spaces. In some of
these IBM studies, large $N$ results given in [26,32] will be useful. Note that in IBM-2,
it is possible to consider multiple $SU(3)$ algebras in proton bosons space and neutron
boson space separately, leading to a much larger class of $SU(3)$ algebras. Turning to
SM spaces, SM and DSM analysis of systems with lowest $SU(3)$ irrep of the type $(\lambda\mu)$
with $\mu \neq 0$ will be important. Here, $SU(3) \supset SO(3)$ integrity basis operators that
are 3- and 4-body are needed as demonstrated for example in [3]. Also, as in IBM-
2, in the shell model studies of heavy nuclei with protons and neutrons in different
oscillator shells, there will be multiple $SU(3)$ algebras in the SM space for protons
and in the SM space for neutrons separately. Combining these proton and neutron
$SU(3)$ algebras will again lead to a much larger class of multiple $SU(3)$ algebras
in SM. Let us add that, multiple $SU(3)$ algebras discussed here combined with the
multiple pairing algebras in SM and IBM spaces [20,33] will generate multiple pairing
plus quadrupole-quadrupole $(P + Q \cdot Q)$ Hamiltonians. These are expected to give
new insights into the structures generated by $(P + Q \cdot Q)$ Hamiltonians in nuclei; see
[34] for new interest in the studies using $(P + Q \cdot Q)$ Hamiltonians.

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**Author contribution statement**

All authors contributed equally to the work in the manuscript.

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**Appendix A:**

Methods for obtaining the spe and TBME for the quadrupole-quadrupole interaction
operator $Q^2(\alpha) \cdot Q^2(\alpha)$ (for all phase choices $\alpha$) are well known [30]. In order to
derive the formulas for the spe and TBME, we begin with equation (8) and drop the
factor ‘2’. Also, we do not show $\alpha$ in $Q^2_q(\alpha)$ when there is no confusion. For a many
particle system,

$$Q \cdot Q = \sum_{i=1}^{m} Q(i) \cdot Q(i) + 2 \sum_{i<k=1}^{m} Q(i) \cdot Q(k)$$

(A.1)

where $i$ and $k$ are particle indices and $m$ is number of particles. The first sum generates
spe and the second term TBME. Given the shell model sp $(n\ell j)$-orbits (note that the
oscillator shell number $\eta = 2n + \ell$), matrix elements of $Q(1) \cdot Q(2)$ in the two-particle
antisymmetric states (called a.s.m.) can be written in terms of the matrix elements
in the two-particle non-antisymmetric states (called n.a.s.m.) as,

\[ \langle (j_a j_b)JT | Q(1) \cdot Q(2) | (j_c j_d)JT \rangle_{n.a.s.m.} = \frac{1}{\sqrt{(1 + \delta_{ab})(1 + \delta_{cd})}} \times \left[ \langle (j_a j_b)JT | Q(1) \cdot Q(2) | (j_c j_d)JT \rangle_{n.a.s.m.} \right. \\
\left. + (-1)^{J+T-j_c-j_d} \langle (j_a j_b)JT | Q(1) \cdot Q(2) | (j_d j_c)JT \rangle_{n.a.s.m.} \right]. \]

(A.2)

Using angular momentum algebra it is easy to recognize that,

\[ \langle (j_a j_b)JT | Q(1) \cdot Q(2) | (j_c j_d)JT \rangle_{n.a.s.m.} = (-1)^{j_b+j_c} \left\{ \frac{j_a - j_b - J}{j_d - j_c} \right\} \times \langle j_a || Q || j_c \rangle\langle j_b || Q || j_d \rangle. \]

(A.3)

The reduced matrix elements \( \langle || Q || \rangle \) are given by,

\[ \langle \eta, \ell_f, j_f || Q^2(\alpha) || \eta, \ell_i, j_i \rangle = (-1)^{\ell_f + \frac{j_f}{2} + j_i + 2} \times \sqrt{5(2j_i + 1)(2j_f + 1)} \left\{ \frac{\ell_f}{j_i} \frac{j_f}{\ell_i} \frac{1}{2} \right\} t_{\ell_f, \ell_i}(\alpha). \]

(A.4)

Combining equations (A.3) and (A.4) with equation (A.2) and equation (A.1) will give the TBME of the \( Q^2(\alpha) \cdot Q^2(\alpha) \) operator. The spe \( \epsilon_{\ell j}^\alpha \) of the \( Q^2(\alpha) \cdot Q^2(\alpha) \) are simply given by

\[ \epsilon_{\ell j}^\alpha = \frac{5}{2\ell + 1} \sum_{\ell'} |t_{\ell\ell'}(\alpha)|^2. \]

(A.5)

An important property is,

\[ -\frac{1}{4} Q^2(\alpha) \cdot Q^2(\alpha) = C_2(SU^\alpha(3)) + \frac{3}{4} L \cdot L. \]

(A.6)

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