A COHOMOLOGICAL PROPERTY OF SEMI-ABELIAN $p$-GROUPS

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Abstract. We prove a cohomological property for a class of finite $p$-groups introduced earlier by M. Y. Xu, which we call semi-abelian $p$-groups. This result implies that a semi-abelian $p$-group has non-inner automorphisms of order $p$, which settles a longstanding problem for this class. We answer also, independently, an old question of M. Y. Xu about the power structure of semi-abelian $p$-groups.

Dedicated to Professor Ming-Yao Xu for his early work on finite $p$-groups.

1. Introduction

Let $G$ be a finite $p$-group. Following M. Y. Xu (see [14]), we say that $G$ is strongly semi-$p$-abelian, if the following property holds in $G$:

$$(xy^{-1})^{p^n} = 1 \iff x^{p^n} = y^{p^n} \text{ for any positive integer } n.$$ 

For brevity, we shall use the term semi-abelian for such a group. It is easy to see that such a group satisfies the properties:

(i) $\Omega_n(G) = \Omega_{\{n\}}(G)$,

(ii) $|G : G^{p^n}| = |\Omega_n(G)|$, and so $|G : G^{p^n}| \leq |\Omega_n(G)|$.

Hence, semi-abelian $p$-groups share some nice properties with the regular $p$-groups introduced by P. Hall (see [3] for their theory). It is not difficult to show that every regular $p$-group is semi-abelian; however the class of semi-abelian $p$-groups is much larger, and in fact every finite $p$-group can occur as a quotient of a semi-abelian $p$-group (see Section 3).

Let $G$ be a regular $p$-group, and $1 < N < G$ such that $G/N$ is not cyclic. P. Schmid showed in [12], that the Tate cohomology groups $\hat{H}^n(G/N, \mathbb{Z}(N))$ are all non-trivial; where $\mathbb{Z}(N)$ is considered as a $G/N$-module with the action induced by conjugation in $G$. Our first purpose is to show that Schmid’s result holds in a more general context.

**Theorem 1.1.** Let $G$ be a semi-abelian $p$-group, and $1 < N < G$ such that $G/N$ is neither cyclic nor a generalized quaternion group. Then $\hat{H}^n(G/N, \mathbb{Z}(N)) \neq 0$, for all integers $n$.

Let us note that P. Schmid has conjectured that Theorem [13] holds for an arbitrary finite $p$-group $G$ if one takes $N = \Phi(G)$. This conjecture has been refuted by A. Abdollahi in [1]. Although, it is interesting to find other classes of $p$-groups

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which satisfy the conclusion of Theorem 1.1 (see [2] Question 1.2).

The above theorem is intimately related to studying non-inner automorphisms of finite $p$-groups. An abelian normal subgroup $A$ of a group $G$ can be seen as a $G$-module via conjugation, whence we can consider the group of crossed homomorphisms or derivations $\text{Der}(G, A)$. To each derivation $\delta \in \text{Der}(G, A)$, we can associate an endomorphism $\phi_{\delta}$ of $G$, given by $\phi_{\delta}(x) = x\delta(x)$, $x \in G$. This map $\phi$ sends $\text{Der}(G, A)$ into

$$\text{End}_A(G) = \{\theta \in \text{End}(G) | x^{-1}\theta(x) \in A, \text{ for all } x \in G\}$$

and in fact it defines a bijection between the two sets $\text{Der}(G, A)$ and $\text{End}_A(G)$.

If we consider only the set of derivations $\delta : G \to A$, that are trivial on $C_G(A)$ (i.e. $\delta(x) = 1$, for $x \in C_G(A)$), which can be identified to $\text{Der}(G/C_G(A), A)$; then the map $\phi$ induces an isomorphism between $\text{Der}(G/C_G(A), A)$ and the group $\tilde{C}(A)$ of the automorphisms of $G$ acting trivially on $C_G(A)$ and $G/A$.

It is straightforward to see that this isomorphism maps $\text{Ider}(G/C_G(A), A)$ into a group of inner automorphisms lying in $\tilde{C}(A)$, though an inner automorphism lying in $\tilde{C}(A)$ needs not necessarily be induced by an inner derivation; however this case can be avoided by assuming that $C_G(C_G(A)) = A$, as if $\phi_{\delta}(x) = x^g$ for some $g \in G$ and all $x \in G$, then $g \in C_G(C_G(A))$, so $g$ lies in $A$ and $\delta$ is the inner derivation induced by $g^{-1}$. We have established

**Proposition 1.2.** Let $G$ be a group, and $A$ be an abelian normal subgroup of $G$ such that $C_G(C_G(A)) = A$; let $\tilde{C}(A)$ denote the group of the automorphisms of $G$ acting trivially on $C_G(A)$ and $G/A$. Then there is an isomorphism from $\text{Der}(G/C_G(A), A)$ to $\tilde{C}(A)$, which maps $\text{Ider}(G/C_G(A), A)$ exactly to the inner automorphisms lying in $\tilde{C}(A)$. In particular if $\tilde{C}(A) \leq \text{Inn}(G)$, then $\tilde{H}^1(G/C_G(A), A) = 0$.

This well known fact in the literature, which can be found for instance in [3], permits to reduce the problem of existence of non inner automorphisms of some group to a cohomological problem. For instance, this allowed W. Gaschütz to prove that any non simple finite $p$-group has non inner automorphisms of $p$-power order.

It is conjectured by Y. Berkovich that a more refined version of Gaschütz’s result holds, more precisely that a non simple finite $p$-group has non inner automorphisms of order $p$ (see [11] Problem 4.13). While it is not clear that a positive answer to it, has deep implications for our understanding of finite $p$-groups, this problem received a large interest, and its hardness may stimulates further developments of new techniques in finite $p$-group theory. The reader may find more information and the relevant references about this problem in [2].

Our second result settles this problem in the class of semi-abelian $p$-groups.

**Theorem 1.3.** Let $G$ be a semi-abelian finite $p$-group. Then $G$ has a non inner automorphism of order $p$.

Our notation is standard in the litterature. Let $S$ be a group. For a positive integer $n$, we denote by $S^{(p^n)}$ the set of the $p^n$-th powers of all the elements of $S$ and by $S^{p^n}$ the subgroup generated by $S^{(p^n)}$. The subgroup generated by the elements of order dividing $p^n$ is denoted by $\Omega_n(S)$. We denote by $\lambda^n_p(S)$ the terms of the lower $p$-series of $S$ which are defined inductively by :

$$\lambda^n_1(S) = S, \text{ and } \lambda^n_{n+1}(S) = [\lambda^n_n(S), S]\lambda^n_n(S)^p.$$
If $A$ is an $S$-module, then $A_S$ denotes the subgroup of fixed elements in $A$ under the action of $S$.

The remainder of the paper is divided into two sections. In Section 2, we prove Theorem 1.1 and Theorem 1.3; and in Section 3 we answer an old question of M. Y. Xu (see [13, Problem 3]) about the power structure of semi-abelian $p$-groups. This result follows quickly from a result of D. Bubboloni and G. Corsi Tani (see [4]), but it seems not that this link has been noted before.

2. Proofs

Let $Q$ be a finite $p$-group, and $A$ be a $Q$-module of $p$-power order. Recall that $A$ is said to be cohomologically trivial if $\hat{H}^k(S, A) = 0$ for all $S \leq Q$ and all integers $k$.

It is proved by W. Gaschütz and K. Ushida (independently) that $\hat{H}^1(Q, A) = 0$ implies that $\hat{H}^k(S, A) = 0$ for all $S \leq Q$ and all integers $k \geq 1$ (see [8, Lemma 2, §7.5]). This statement can be slightly improved as noted in [9].

**Proposition 2.1.** Let $Q$ be a finite $p$-group, and $A$ be a $Q$-module which is also a finite $p$-group. If $\hat{H}^n(Q, A) = 0$ for some integer $n$, then $A$ is cohomologically trivial.

We shall use Proposition 2.1 to reduce the proof of Theorem 1.1 to the non-vanishing of the Tate cohomology groups in dimension 0, which is easier to handle.

We need also the following result of Schmid (see [12, Proposition 1]).

**Proposition 2.2.** Let $Q$ be a finite $p$-group, and $A \neq 1$ be a $Q$-module which is also a finite $p$-group. If $A$ is cohomologically trivial, then $C_Q(A_K) = K$, for every $K \leq Q$.

We need also to prove the following

**Lemma 2.3.** Under the assumption of Theorem 1.1, set $A = \mathbb{Z}(N)$ and let $S/N$ be a subgroup of exponent $p$ of $G/N$. Then $A^p \leq A_{S/N}$, so that $C_{S/N}(A^p) = S/N$.

**Proof.** Let be $x \in S$ and $a \in A$. We have $x^p \in N$, hence

$$x^p = (x^p)^a = (x^a)^p = (x[a, a])^p.$$ 

As $G$ is semi-abelian, we have $[x, a]^p = 1$. It follows that $(a^{-1} a [a, x])^p = [a, x]^p = 1$, and again since $G$ is semi-abelian, we have

$$a^p = (a[a, x])^p = (a^p)^x.$$ 

This shows that $A^p$ is centralized by every element of $S/N$. $\square$

**Proof of Theorem 1.1.** Assume for a contradiction that $\hat{H}^n(G/N, A) = 0$ for some integer $n$, where $A$ denotes $\mathbb{Z}(N)$. As $G/N$ is not cyclic and different from the generalized quaternion groups $Q_{2^n}$, there is in $G/N$ a subgroup $S/N$ of exponent $p$ and order $\geq p^2$. It follows from Proposition 2.1 that $\hat{H}^n(S/N, A) = 0$, so $A$ is a cohomologically trivial $S/N$-module. Let $K/N \leq S/N$ be a subgroup of order $p$. Proposition 2.1 implies that $\hat{H}^0(K/N, A) = 0$. We have $\hat{H}^0(K/N, A) = A_{K/N}/A^\tau = 0$, where $A^\tau$ is the image of $A$ under the trace homomorphism $\tau : A \to A$ induced by $K/N$. As $K/N$ is cyclic of order $p$, our trace map is given by

$$a^\tau = aa^x \ldots a^{xp-1}$$

for $a \in A$, and any fixed $x \in K - N$.
from which it follows

\[ a^\tau = (ax^{-1})^p x^p. \]

Now as \( G \) is semi-abelian, \( a \in \ker \tau \) if, and only if \( a^p = 1 \); that is \( \ker \tau = \Omega_1(A) \). This implies that \(|A^\tau| = |A^p|\). As \( A_{K/N} = A^\tau \), and \( A^p \leq A_{K/N} \) by Lemma 2.2 we have \( A^p = A_{K/N} \). By Proposition 2.2 \( C_{S/N}(A^p) = C_{S/N}(A_{K/N}) = K/N \), however Lemma 2.2 implies that \( S/N = K/N \), a contradiction.

Before proving Theorem 1.3 we need the following reduction from [5].

**Proposition 2.4.** Let \( G \) be a finite \( p \)-group such that \( C_G(Z(\Phi(G))) \neq \Phi(G) \). Then \( G \) has a non inner automorphism of order \( p \).

Note that M. Ghorasi improved Proposition 2.4 in [7], where he reduced the problem of Berkovich to the \( p \)-groups \( G \) satisfying \( H \leq C_G(H) = \Phi(G) \), where \( H \) is the inverse image of \( \Omega_1(Z(G)/Z(G)) \) in \( G \). A family of examples which satisfy the condition \( C_G(Z(\Phi(G))) = \Phi(G) \) and do not satisfy Ghorashi’s condition can be found in the same paper.

**Proof of Theorem 1.3.** Assume for a contradiction that every automorphism of \( G \) of order \( p \) is inner. Let be \( A = Z(\Phi(G)) \). By Proposition 2.4 we have \( C_G(A) = \Phi(G) \) and so \( C_G(C_G(A)) = A \). If we prove that \( \text{Der}(G/C_G(A), A) = \text{Der}(G/\Phi(G), \Phi(G)) \) has exponent \( p \), then our first assumption together with Proposition 1.2 imply that \( H^1(G/\Phi(G), \Phi(G)) = 0 \), which contradicts Theorem 1.1. So we need only to prove, for any derivation \( \delta \in \text{Der}(G, Z(\Phi(G))) \) which is trivial on \( \Phi(G) \), that \( \delta(x)^p = 1 \), for all \( x \in G \). Indeed

\[ \delta(x^p) = \delta(x)\delta(x)^2 \ldots \delta(x)x^{p-1} = (\delta(x)x^{-1})^p x^p. \]

As \( \delta \) is trivial on \( \Phi(G) \), we have \( \delta(x^p) = (\delta(x)x^{-1})^p x^p = 1 \), and since \( G \) is semi-abelian it follows that \( \delta(x)^p = 1 \). □

3. **Remarks on a particular class of semi-abelian \( p \)-groups**

M. Y. Xu proved in [13], that any finite \( p \)-group \( G \), \( p \) odd, which satisfies \( \Omega_1(\gamma_{p-1}(G)) \leq Z(G) \) is semi-abelian; and he asked if such a group must be power closed, that is every element of \( G^{p^n} \) is a \( p^n \)-th power.

A negative answer to this question will follow from the following important result of D. Bubboloni and G. Corsi Tani (see [4]).

Recall that a \( p \)-central group is a group in which every element of order \( p \) is central. D. Bubboloni and G. Corsi Tani used the term TH-group instead of \( p \)-central group, where TH-group refers to J. G. Thompson, who seems to be the first to observe the importance of \( p \)-central groups (see [10] Hilfssatz III.12.2).

**Theorem 3.1.** Let be \( d \) and \( n \) two positive integers, \( p \) an odd prime, and \( F \) the free group on \( d \) generators. Then \( G_n = F/\lambda_p^n(F) \) is a \( p \)-central \( p \)-group. More precisely we have \( \Omega_1(G_n) = \lambda_p^1(F)/\lambda_p^{n+1}(F) \).

**Corollary 3.2.** A finite \( p \)-group, \( p \) odd, which satisfies \( \Omega_1(\gamma_{p-1}(G)) \leq Z(G) \) needs not be necessarily power closed.
Proof. Obviously a $p$-central $p$-group satisfies $\Omega_1(\gamma_{p-1}(G)) \leq Z(G)$. Assume for a contradiction that the result is false. So the $p$-groups $G_n$ are all power closed. Now every finite $p$-group is a quotient of some $G_n$ (for appropriates $n$ and $d$), and the property of being power closed is inherited by quotients. It follows that every finite $p$-group is power closed, which is a contradiction. □

Let us mention briefly another consequence of Theorem 3.1. It is largely believed that finite $p$-central $p$-groups are dual (in a sense) to powerful $p$-groups. Since the inverse limits of powerful $p$-groups have roughly a uniform structure (see [6, §3]), it is natural to ask if there is a restriction on the structure of a pro-$p$ central $p$-group, that is an inverse limit of finite $p$-central $p$-groups.

Let $F$ be the free group on a finite number of generators. As every normal subgroup of $F$ of $p$-power index contains a subgroup $\lambda^p_n(F)$ for some $n$, it follows that

$$\hat{F}_p \cong \varprojlim F/\lambda^p_n(F)$$

where $\hat{F}_p$ is the pro-$p$ completion of $F$. This shows that the free pro-$p$ group $\hat{F}_p$ is pro-$p$-central, so there is no reasonable restriction on the structure of a (finitely generated) pro-$p$-central $p$-group.

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