SPECIAL $L$-VALUES AND SELMER GROUPS OF SIEGEL MODULAR FORMS OF GENUS 2

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Abstract. Let $p$ be an odd prime, $N$ a square-free odd positive integer prime to $p$, $\pi$ a $p$-ordinary cohomological irreducible cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ of principal level $N$ and Iwahori level at $p$. Using a $p$-integral version of Rallis inner product formula and modularity theorems for $\text{GSp}_4/\mathbb{Q}$ and $\text{U}_4/\mathbb{Q}$, we establish an identity between the $p$-part of the critical value at 1 of the degree 5 $L$-function of $\pi$ twisted by the non-trivial quadratic Dirichlet character $\xi$ associated to the extension $\mathbb{Q}(\sqrt{-N})/\mathbb{Q}$ and the $p$-part of the Selmer group of the degree 5 Galois representation associated to $\pi$ twisted by $\xi$, under certain conditions on the residual Galois representation.

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1. Introduction

Let $p$ be an odd prime number. In this article we establish an identity between the $p$-part of the special value at 1 of the twist by an odd quadratic character $\xi$ of the standard $L$-function of a $p$-ordinary cuspidal automorphic forms $\varphi$ over $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ and the $p$-part of the Selmer group of the twist by $\xi$ of the $p$-adic standard Galois representation associated to $\varphi$, under certain conditions on the residual Galois representation. The main tool is a $p$-integral Rallis inner product formula, as developed in this article. Denote by $M_{\varphi}$ the rank 4 symplectic motive associated to $\varphi$ and $\rho_{st}: \text{GSp}_4 \to \text{SO}_5$ the standard representation. Then our identity can be seen as supportive evidence for Bloch-Kato Tamagawa Number Conjecture for the motive $\rho_{st}(M_{\varphi}) \otimes \xi$ (cf. [BK90]), except that we consider automorphic periods instead of motivic periods.

To state our result, we first prepare some notations. We fix an odd square-free integer $N > 2$ prime to $p$. Let $\xi$ be the quadratic character associated to the quadratic imaginary extension $E = \mathbb{Q}(\sqrt{-N})/\mathbb{Q}$. Fix an embedding $\iota_p: \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$. Let $\pi = \otimes_{v \leq \infty} \pi_v$ be an irreducible cuspidal automorphic representation of $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ and $\pi' = \pi^\vee$ its contragredient. We assume that $\pi$ is ordinary for $\iota_p$ and unramified outside $Np\infty$, that $\pi_{\infty}$ is in (anti)holomorphic discrete series, that $\pi$ is of principal level $N$ and it has Iwahori level at $p$. For a finite place $\ell$ of $\mathbb{Q}$ such that $\pi_\ell$ is unramified, $\pi_\ell$ has Satake parameter $S_\ell = \text{diag}(\alpha_1, \alpha_2, \alpha_2/\alpha_1, \alpha_1/\alpha_2) \in \text{GSp}_4(\mathbb{C})$. Then the conjugacy class of $\rho_{st}(S_\ell)$ is the same as that of the diagonal matrix $\text{diag}(1, \alpha_1/\alpha_2, \alpha_2/\alpha_1, \alpha_1\alpha_2/\alpha_0, \alpha_0^2/\alpha_1\alpha_2)$. The twisted standard $L$-factor of $\pi_\ell$ by the character $\xi$ is then (for $s \in \mathbb{C}$)

$$L_\ell(s, \text{St}(\pi) \otimes \xi) = (1-\xi(\ell)\ell^{-s})^{-1}(1-\xi(\ell)\ell^{-s}\alpha_1/\alpha_2)^{-1}(1-\xi(\ell)\ell^{-s}\alpha_2/\alpha_1)^{-1}(1-\xi(\ell)\ell^{-s}\alpha_1\alpha_2/\alpha_0^2)^{-1}(1-\xi(\ell)\ell^{-s}\alpha_0^2/\alpha_1\alpha_2)^{-1}.$$  

For any finite set $S$ of places of $\mathbb{Q}$ containing the place $\infty$ and all finite places $\ell$ where $\pi_\ell$ is ramified, we define the partial $L$-function $L_S(s, \text{St}(\pi) \otimes \xi)$ as the product $\prod_{v \not\in S} L_\ell(s, \text{St}(\pi) \otimes \xi)$. Then it is known that $L_S(s, \text{St}(\pi) \otimes \xi)$ converges absolutely for $\text{Re}(s)$ sufficiently large and admits a meromorphic continuation to the whole plane $\mathbb{C}$ with possibly simple poles ([KR90]).

Let $\mathcal{O}$ a sufficiently large extension of $\mathbb{Z}_p$ containing all the eigenvalues of $\pi$ for the Hecke operators. Let $\varpi$ be a uniformizer of $\mathcal{O}$. Fix then $p$-integral ordinary primitive (in terms of Fourier coefficients) factorizable automorphic forms $\varphi \in \pi$ and $\varphi' \in \pi'$ such that their pairing $\langle \varphi, \varphi' \rangle \neq 0$ (cf. Definitions [2.1] and [2.2]). Let $\text{GSO}_6(\mathbb{Q})$ be the similitude orthogonal group over $\mathbb{Q}$ determined by the symmetric matrix diag(2, 2, 2, 2N, 2N, 2N) and $U_4(\mathbb{Q})$ be the unitary group over $E/\mathbb{Q}$ determined by the Hermitian matrix diag(1, 1, 1, 1). We then construct an algebraic $p$-integral theta section $\phi$ in the space of Schwartz-Bruhat functions $S(M_{4 \times 6}(\mathbb{A}_\mathbb{Q}))$ (cf. Section 4). The theta series $\Theta_\phi$ associated to $\phi$ sends the modular form $\varphi \otimes \varphi'$ by theta correspondence to a $p$-integral $\Theta_\phi(\varphi \otimes \varphi')$ on $U_4(\mathbb{A}_\mathbb{Q})$ (via certain exceptional isogeny between GSO$_6$ and $U_4$, see Section 5.1). Now we denote by $\varepsilon(\pi) \in \mathcal{O}$ the congruence number/ideal between $\Theta_\phi(\varphi)$ and other cuspidal automorphic forms on $U_4(\mathbb{A}_\mathbb{Q})$ which are not theta lifts from $\text{GSp}_4(\mathbb{A}_\mathbb{Q})$ (cf. [2.4]). To the automorphic representation $\pi$, one can associate the symplectic Galois representation $\rho_{st}: \Gamma_Q \to \text{GSp}_4(\mathcal{O})$ ([Tay93, Lau05, Wei05]) and we write $\rho_{st}^\text{st} = \rho_{st} \circ \rho_{st}$. Then we can relate the congruence number $\varepsilon(\pi)$ to the Selmer group $\text{Sel}(\mathbb{Q}, \rho_{st}^\text{st} \otimes \xi)$ of the Galois representation $\rho_{st}^\text{st} \otimes \xi$ (in a way similar to [HT16]). Write $\chi(\rho_{st}^\text{st} \otimes \xi)$ for a generator of the Fitting ideal of the Selmer group $\text{Sel}(\mathbb{Q}, \rho_{st}^\text{st} \otimes \xi)$ viewed as an $\mathcal{O}$-module. Then our main result is (see Theorem [6.11] for more details):

**Theorem.** Assume the hypotheses as in [HT16] Theorem 7.3], i.e., $(N, \text{Min}), (\text{RFR}(2))$ and $(\text{BIG}(2))$, then we have the following identities, up to units in $\mathcal{O}$:

$$\frac{L_{Np\infty}(1, \text{St}(\pi) \otimes \xi)\tilde{L}_{Np\infty}(1, \text{St}(\pi) \otimes \xi)}{P_{\pi'}} = \varepsilon(\pi) = \chi(\rho_{st}^\text{st} \otimes \xi).$$
where \( \tilde{L}_{Np^\infty}(1, \text{St}(\pi) \otimes \xi) \) is the product of modified local \( L \)-factors at places dividing \( Np^\infty \) which depends on the local components at places \( Np^\infty \) of \( \varphi, \varphi' \) and \( \phi \), and the Selmer group \( \tilde{L}_{Np^\infty} \) of \( \varphi \) and \( P_{p^\infty} \in \mathbb{R}_{>0} \) is a certain automorphic period of \( \pi \).

**Remark.** Some remarks concerning the theorem are in order:

1. The hypotheses in the above theorem, \((N\text{-Min}), (RFR^{(2)})\) and \((\text{BIG}^{(2)})\) are of Taylor-Wiles type. See Hypothesis \[6.4\] for details.
2. We have explicit formulas for the \( L \)-factors \( \tilde{L}_{\ell}(1, \text{St}(\pi) \otimes \xi) \) for \( \ell \mid Np \). See Section \[4.6\] for the automorphic period \( P_{\varphi^\infty} \), see Lemma \[2.14\]
3. The identity in the above theorem is reminiscent of a conjectural Bloch-Kato formula we recall in the below:

Write \( \text{Tam}(\rho^\text{st}_{\pi} \otimes \xi) \in \mathbb{Z} \) for the Tamagawa number of \( \rho^\text{st}_{\pi} \otimes \xi \) (cf. [FP94, II.5.3.3]) and \( P_{\varphi^\infty} \) for the Deligne period of the motive \( \rho_{\text{st}}(M_{\varphi}) \otimes \xi \). We write \( H^1_f(\mathbb{Q}, \rho^\text{st}_{\pi} \otimes \xi) \) for the Bloch-Kato group of the motive \( \rho_{\text{st}}(M_{\varphi}) \otimes \xi \) using its \( p \)-adic realization \( \rho^\text{st}_{\pi} \otimes \xi \). Then the \( p \)-part of the Bloch-Kato Tamagawa Number Conjecture states as follows

**Conjecture.** We have the following identity, up to units in \( \mathcal{O} \):

\[
\frac{L^N_{p^\infty}(1, \text{St}(\pi) \otimes \xi) \tilde{L}_{Np^\infty}(1, \text{St}(\pi) \otimes \xi)}{P_{\varphi^\infty}} = \chi(H^1_f(\mathbb{Q}, \rho_{\text{st}} \circ \rho_{\pi} \otimes \xi)) \text{Tam}(\rho^\text{st}_{\pi} \otimes \xi).
\]

Here we identify the \( L \)-function of \( \pi \) with that of the associated motive \( \rho_{\text{st}}(M_{\varphi}) \otimes \xi \). The Bloch-Kato Selmer group \( H^1_f(\mathbb{Q}, \rho^\text{st}_{\pi} \otimes \xi) \) is a subgroup of the Greenberg Selmer group \( \text{Sel}(\mathbb{Q}, \rho^\text{st}_{\pi} \otimes \xi)^* \) of finite index (cf. [F190, Theorem 3]). The relation between the automorphic period \( P_{\varphi^\infty} \) and the motivic period \( P_{\varphi^\infty} \) is quite mysterious and we will not touch this topic in this article. There have been many works on the Bloch-Kato conjectures for various motives (cf. [Bro07, BC09, Klo09, Ber15, CGH16]). When one wants to show the non-triviality of the Selmer group, the strategy in common used in these works is that ofRibet. Suppose that \( N_{\pi} \) is a certain motive associated to an automorphic representation \( \pi \) of \( G(\mathbb{A}) \) for some algebraic group \( G/\mathbb{Q} \). Suppose that the associated \( L \)-function \( L(1, N_{\pi}) \neq 0 \) is critical in the sense of Deligne. To construct a non-trivial element in the Selmer group \( H^1_f(N_{\pi}) \) of \( N_{\pi} \), one can try to find some larger algebraic group \( G'/\mathbb{Q} \) such that the \( L \)-group \( G \) of \( G \) maps to \( G' \) (as certain Levi subgroup of a parabolic subgroup of \( \text{GL}^\prime \), for example) such that \( \pi \) lifts to an automorphic representation \( \pi' \) on \( G' \). Suppose that we have an irreducible \( p \)-adic Galois representation associated to \( \pi' \). If \( p \) divides the normalized \( L \)-value \( L(1, N_{\pi}) \), then one can, in favorable situations, construct another automorphic representation \( \pi'' \) of \( G'(\mathbb{A}) \) whose Hecke eigenvalues are congruent to those of \( \pi' \) modulo \( p \). Now the modulo \( p \) Galois representation associated to \( \pi'' \) is reducible and non-semisimple, which gives rise to a non-trivial element in the Selmer group \( H^1_f(N_{\pi}) \) (cf. Introduction of [Klo09]). Previously studied cases include \((G, G') = (\text{GL}_1 \times \text{GL}_1, \text{GL}_2) \) (Ribet), \((\text{GL}_2, \text{GSp}_4) \) (Urban), \((\text{GL}_2, U_{2,2}) \) (Skinner-Urban) and \((\text{Res}_{\mathbb{E}/\mathbb{F}}(\text{GL}_1), U(3)) \) (Bellaiche-Chenevier). Along this strategy, this work can be seen as the case \((G, G') = (\text{GSp}_4, U_4) \). There are two main parts in this article. One part is to construct \( \pi' \) in a \( p \)-integral way: we construct a \( p \)-integral theta series \( \Theta_{\phi} \). Then given a \( p \)-integral cuspidal Siegel modular form \( \varphi \in \pi \), the pairing between \( \varphi \) and \( \Theta_{\phi} \) gives a \( p \)-integral modular form on \( U_4(\mathbb{A}) \). Another part is to relate congruence ideals to Selmer groups. Once we can interpret the theta lift from \( \pi \) to \( \pi' \) in terms of Galois representations, we can use the results similar to those in [HT16] to finish the proof.

As indicated at the end of the last paragraph, we need a \( p \)-integral version of the theta lift from \( \text{GSp}_4 \) to \( \text{GSO}_6 \) (and then transferred to \( U_4 \)). This is the so-called arithmetic Rallis inner product formula. There are already many works on this topic (cf. [BDS12, HK92, Pra06]). The \( p \)-integrality of the theta lift of a single automorphic form is also very useful and there are the works of Kudla...
and Millson (cf. [KM90] and the works cited therein) from an orthogonal group to a symplectic group (see also [Ber14]). We work in the other direction, from a (similitude) symplectic group to a (similitude) orthogonal group. Unlike Kudla and Millson who work with cycles on the symmetric space associated to the orthogonal group and construct theta series using cohomology classes, we construct explicitly local Schwartz-Bruhat functions and show that the associated theta series has $p$-integral Fourier coefficients. Then the pairing between a theta series and a $p$-integral Siegel modular form can be seen as a Serre duality pairing between the coherent cohomology groups $H^0$ and $H^3$ of the Siegel moduli scheme. This strategy is inspired from [HLS05] in which the authors sketched a program to construct theta series using cohomology classes, we construct (similitude) orthogonal group. Unlike Kudla and Millson who work with cycles on the symmetric space (see also [Ber14]). We work in the other direction, from a (similitude) symplectic group to a (similitude) orthogonal group.

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As the reader can see, throughout the article we use heavily the results and ideas on the doubling method developed explicitly and arithmetically in [Lin15b] for the symplectic groups (and [EHL16] for the unitary groups).

Let us give a brief description of the article. Section 2 gives some preliminary notions on Siegel modular forms and congruence ideals that will be used throughout this article. In Section 3 we formulate the theta correspondence for the reductive dual pair $(GSp_4, GSO_6)$. Section 4 is the principal part of the paper. In this section, we give explicit choice of local Schwartz-Bruhat functions in the theta correspondence and the associated Siegel section for the doubling method, and calculate explicitly the non-archimedean local zeta doubling integrals and also show the non-vanishing of local Fourier coefficients. In Section 5, we use known results to write the transfer from automorphic representations of $GSp_4(\mathbb{A})$ to those of $U_4(\mathbb{A})$ in terms of Langlands parameters. This allows us to relate the Galois representations associated to $\pi$ and $\pi'$. In Section 6, we construct an explicit morphism between these two Galois representations and at the end we give the main result of the article.

**Notations.**

1. We fix an odd rational prime $p$. We fix the following isomorphisms and inclusions of fields that are compatible with each other:
   \[ t_p^\infty : \mathbb{C} \simeq \overline{\mathbb{Q}}_p, \quad t_\infty : \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}, \quad t_p = t_p^\infty \circ t_\infty : \overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p. \]

2. We denote by $\mathbb{A} = \mathbb{R} \times \mathbb{A}_{\ell}$ the ring of adeles of $\mathbb{Q}$. For any place $v$ of $\mathbb{Q}$, we write $| \cdot |_v$ for the $v$-adic valuation on $\mathbb{Q}_v$. Any place $v$ of $\mathbb{Q}$ such that $|v| = \ell^{-1}$ and $| \cdot |_\infty$ is the absolute value.

3. We fix a square-free positive integer $N$ prime to $p$. We then write $E = \mathbb{Q}(\sqrt{-N})$. We define a Hecke character $\xi : \mathbb{Q}^\times \backslash \mathbb{A}^\times \to \{ \pm 1 \}$, $x \mapsto (x, -N)$. Here $(x, -N) = \prod_v (x_v, -N)_v$ is the product of local Hilbert symbols for all places $v$ of $\mathbb{Q}$.

4. For each place $v$ of $\mathbb{Q}$, we fix an additive character $e_v : \mathbb{Q}_v \to \mathbb{C}^\times$ as follows: if $v = \infty$, then $e_\infty(x) = \exp(2i\pi x)$; if $v \nmid \infty$, then $e_v(x) = e(-2i\pi \{x\}_v)$ where $\{x\}_v$ is the fractional part of $x$ in $\mathbb{Q}_v$. We then define a character of $\mathbb{A}$, trivial on $\mathbb{Q}$, as the tensor product of all these $e_v$:
   \[ e := \otimes_v e_v : \mathbb{A} \to \mathbb{C}^\times. \]

5. For any algebraic group $G$ defined over $\mathbb{Q}$, we write $[G]$ for the quotient $G(\mathbb{Q}) \backslash G(\mathbb{A})$.

6. Let $R$ be any commutative ring. We write $M_{n\times m}(R)$ to be the set of $R$-valued $n \times m$-matrices. We denote by $I_n$ the identity matrix of size $n \times n$. The Borel subgroup of $GL_n$ consisting of upper triangular matrices is denoted by $B_n$, its maximal torus by $T_n$ and unipotent radical by $N_n$. The transpose of $B_n$ is denoted by $B_n^\circ$, consisting of lower triangular matrices. For any $g \in M_{n\times m}$, we write $g^t$ for its transpose. If $g \in GL_n$, we write $g^{-t}$ for $(g^t)^{-1}$.  

4
In this section we review the notions of Siegel modular variety associated to the symplectic groups \( \text{GSp}_4 \) and \( \text{GSp}_8 \). Many of the materials in this section are simply bookkeeping from the references

(7) We write \( J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix} \). The similitude symplectic group \( G = \text{GSp}_{2n} \) over \( \mathbb{Z} \) is identified with the subgroup of \( \text{GL}_{2n} \) consisting of matrices \( g \) such that \( g^t J_{2n} g = \nu(g) J_{2n} \). Write \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{GSp}_{2n} \) in \( n \times n \)-blocks. We write \( P_G \) to be the subgroup of \( G \) consisting of \( g \) such that \( C = 0 \), \( L_G \) the Levi subgroup of \( P_G \) consisting of those \( g \) such that \( B = 0 \), \( B_G \) the subgroup of \( P_G \) consisting of \( g \) such that \( A \in B_n \), similarly, \( N_G \) consisting of \( g \) such that \( A \in N_n \), \( T_G \) consisting of \( g \) such that \( A \in T_n \) and \( B = 0 \). Let \( Z_G \) be the center of \( G \). We write the Lie algebras of \( G \) and \( T_G \) as \( \mathfrak{g}_G \) and \( \mathfrak{t}_G \) respectively. We fix a \( \mathbb{C} \)-basis for the Lie algebra \( \mathfrak{g}_G(\mathbb{C}) = \mathfrak{gsp}_{2n}(\mathbb{C}) \) as follows

\[
\begin{align*}
\eta^0 &= 1_{2n}, \\
\eta_{i,j} &= E_{i,j} - E_{j+i,n+i+n}, \\
\mu_{i,j} &= E_{i,j} + E_{j+i,n+i}, \\
\mu_{i,j} &= E_{i,n+i}.
\end{align*}
\]

We fix a maximal compact subgroup \( K_{G,\infty} \) of \( G(\mathbb{R}) \) consisting of matrices \( \begin{pmatrix} A & B \\ -\nu B & \nu A \end{pmatrix} \) such that \( A + iB \in U_n(\mathbb{R}) \) and \( \nu = \pm 1 \). We write \( \Gamma(N) \) for the subgroup of \( \text{GSp}_{2n}(\mathbb{Z}) \) consisting of matrices \( g \) such that \( g \equiv 1(\text{mod } N) \) and \( \Gamma = \Gamma(N, p^m) \) the subgroup of \( \Gamma(N) \) consisting of matrices \( g \) such that \( g(\text{mod } p^m) \in N_{\text{sp}_{2n}}(\mathbb{Z}/p^m) \) (\( m > 0 \)). Denote by \( \hat{\Gamma}(N) \), resp., \( \hat{\Gamma} \) the completion of \( \Gamma(N) \), resp., \( \Gamma \) in \( \text{GSp}_{2n}(\mathbb{A}_F) \).

(8) We write \( \rho_G \) for the half sum of the positive roots of \( \mathfrak{gsp}_{2n}(\mathbb{C}) \) with respect to \( T_G \), \( \rho_{L_G} \) for the half sum of the positive roots of the Levi subgroup \( L_G \) with respect to \( T_G \).

(9) We write \( G^1 := \text{Sp}_{2n} \) for the subgroup of \( \text{GSp}_{2n} \) consisting of \( g \) such that \( \nu(g) = 1 \). We write \( B_{G^1}, N_{G^1}, T_{G^1} \) and \( K_{G^1,\infty} \) for the respective intersection of the groups \( B_G, N_G, T_G, K_{G,\infty} \) with \( G^1 \).

(10) For a symmetric matrix \( x \in \text{Sym}_{n \times n} \), we write \( u(x) = \begin{pmatrix} 1_n & x \\ 0 & 1_n \end{pmatrix} \in \text{GSp}_{2n} \). Similarly, for an invertible matrix \( y \in \text{GL}_n \), we write \( m(y) = \begin{pmatrix} y & 0 \\ 0 & y^{-1} \end{pmatrix} \in \text{GSp}_{2n} \). For a finite dimensional \( F \)-vector space \( V \) and any \( y \in \text{GL}_F(V) \), we also write \( m(y) = \begin{pmatrix} y & 0 \\ 0 & y^\vee \end{pmatrix} \) where \( y^\vee \in \text{GL}_F(V^\vee) \) is the dual of \( y \). For any \( g \in \text{GSp}_{2n}(\mathbb{Q}_v) \), the Iwasawa decomposition of \( \text{GSp}_{2n}(\mathbb{Q}_v) \) gives a decomposition \( g = \text{diag}(1, \nu(g)) m(y) u(x) k \) for some \( y, x \) and \( k \) in the standard maximal compact subgroup \( \text{GSp}_{2n}(\mathbb{Z}_v) \) of \( \text{GSp}_{2n}(\mathbb{Q}_v) \). Then we set \( m(g) = \det(a)^{n-1} \nu(g)^{-n(n-1)/2} \).

(11) Let \( \mathbb{H}_n \) denote the Siegel upper half plane, consisting of \( z \in M_{n \times n}(\mathbb{C}) \) such that \( z^t = z \) and \( \text{Im}(z) \) is positive definite. Let \( \text{GSp}_{2n}(\mathbb{R}) \) be the subgroup of \( \text{GSp}_{2n}(\mathbb{R}) \) consisting of \( g \) with \( \nu(g) > 0 \). The group \( \text{GSp}_{2n}^+(\mathbb{R}) \) acts on \( \mathbb{H}_n \) by \( g \cdot z = (Az + B)(Cz + D)^{-1} \). For each \( z \in \mathbb{H}_n \), we define a 1-cocycle on \( \text{GSp}_{2n}(\mathbb{R}) \) as \( \mu(g, z) := Cz + D \).

(12) We fix some sufficiently large extension of \( \mathbb{Q}_p \) containing \( \sqrt{2} \) and roots of unity \( \mu_{N^2C(x)} \) where \( C(x) \) is certain positive number depending on the \( p \)-ordinary automorphic representation (see the beginning of Section 1.5 for its definition). Its ring of integers is denoted by \( \mathcal{O} \) with a fixed uniformizer \( \varpi \).

2. MODULAR FORMS ON \( \text{GSp}_4 \)

In this section we review the notions of Siegel modular variety associated to the symplectic groups \( \text{GSp}_4 \) and \( \text{GSp}_8 \). Many of the materials in this section are simply bookkeeping from the references
given in each subsection and we claim no originality, yet any errors in this section should be the
author’s of this article.

2.1. Siegel modular forms. In this subsection we review nearly holomorphic modular forms on
$G = GSp_{2n}$. We follow closely the treatment in [Liu15a, Liu15b].

2.1.1. Arithmetic modular forms and Hecke operators. A PEL datum is a tuple $\mathcal{P} = (L, \langle \cdot, \cdot \rangle, h)$ where ([Lan12, 1A]):

1. $L$ is a free $\mathbb{Z}$-module of finite rank, and $\langle \cdot, \cdot \rangle : L \times L \to \mathbb{Z}(1) := \text{Ker}(\exp : \mathbb{C} \to \mathbb{C}^\times)$ is a non-
   degenerate symplectic pairing such that for any $b \in \mathbb{Z}$ and $x, y \in L$, we have $\langle bx, y \rangle = \langle x, by \rangle$;
2. $h : \mathbb{C} \to \text{End}_R(L \otimes \mathbb{Z} \mathbb{R})$ is a homomorphism such that for any $z \in \mathbb{C}$ and $x, y \in L \otimes \mathbb{R}$, we have
   $\langle h(z)x, y \rangle = \langle x, h(\overline{z})y \rangle$. Moreover, we require that the new bilinear form $-\langle \cdot, h(i)\cdot \rangle$ on $L$ is
   symmetric and positive definite.

Given such a datum $\mathcal{P}$, suppose that $L \cong \mathbb{Z}^n$, one can then define the similitude symplectic group
$G = Gp$ over $\mathbb{Z}$ associated to the pairing $\langle \cdot, \cdot \rangle$ on $L$ as follows: for any $\mathbb{Z}$-algebra $R$, the $R$-points of
$G$ is given by

$$G(R) = \{(g, \nu) \in \text{GL}_R(L \otimes R) \times (L \otimes R)^\times | \langle gx, gy \rangle = \nu(x, y), \forall x, y \in L \otimes R \}$$

Sometimes we also write an element in $G(R)$ as simply $g' = (g, \nu)$ and refer to $\nu(g')$ as the similitude
factor of $g'$. Fixing a $\mathbb{Z}$-basis of $L$ under which $\langle \cdot, \cdot \rangle$ is of the form $J_{2n}$, one can identify $G$ with
$GSp(2n)$.

We have an algebraic stack $A_{G, \hat{\Gamma}}$ over $\text{Spec}(\mathbb{Z}[1/Np])$ parameterizing the principally polarized
abelian schemes over $\mathbb{Q}$ of dimension $n$, level structure $\hat{\Gamma}$ ([FC90, Chapter I.4.11]). On the other hand,
the complexification $\mathcal{A}_{G, \hat{\Gamma}}(\mathbb{C})$ is isomorphic to the Shimura variety $G^i(\mathbb{Z}) \backslash \mathbb{H}_n \times G(\mathbb{Z})/\hat{\Gamma}$ which
parameterizes the principally polarized abelian varieties over $\mathbb{C}$ of dimension $n$, level structure $\hat{\Gamma}$ ([FC90,
Chapter I.6]). We write $\tilde{A}_{G, \hat{\Gamma}}$ for the toroidal compactification of $A_{G, \hat{\Gamma}}$ with boundary $C = C_{\hat{\Gamma}}$. Let
$\tilde{A}$ be the universal semi-abelian scheme over $\tilde{A}_{G, \hat{\Gamma}}$ with the morphism $p : \tilde{A}_{G, \hat{\Gamma}} \to \tilde{A}_{G, \hat{\Gamma}}$. Then $\tilde{A}_{G, \hat{\Gamma}}$
restricts to the universal abelian scheme $A_{G, \hat{\Gamma}}$ on $A_{G, \hat{\Gamma}}$ and we still write $p : A_{G, \hat{\Gamma}} \to A_{G, \hat{\Gamma}}$.

We write $\omega(\tilde{A}_{G, \hat{\Gamma}}/A_{G, \hat{\Gamma}}) = p_* (\Omega^1_{A_{G, \hat{\Gamma}}/\tilde{A}_{G, \hat{\Gamma}}})$ for the sheaf of invariant differentials, locally free
of rank $n$. The sheaf $\mathcal{H}^1_{\text{dr}}(A_{G, \hat{\Gamma}}/A_{G, \hat{\Gamma}}) = R^1p_* (\Omega^1_{A_{G, \hat{\Gamma}}/\tilde{A}_{G, \hat{\Gamma}}})$ has a canonical extension
$\mathcal{H}^1_{\text{dr}}(A_{G, \hat{\Gamma}}/A_{G, \hat{\Gamma}})^{\text{can}}$ to $\tilde{A}_{G, \hat{\Gamma}}$. One can show that $\mathcal{H}^1(A_{G, \hat{\Gamma}}/A_{G, \hat{\Gamma}})^{\text{can}}$ is locally free of rank $2n$ with the following Hodge filtration

$$0 \to \omega(\tilde{A}_{G, \hat{\Gamma}}/A_{G, \hat{\Gamma}}) \to \mathcal{H}^1_{\text{dr}}(A_{G, \hat{\Gamma}}/A_{G, \hat{\Gamma}})^{\text{can}} \to \text{Lie}(\tilde{A}_{G, \hat{\Gamma}}/A_{G, \hat{\Gamma}}) \to 0.$$
is a corresponding element in $T_H(S)$. We write $\text{Rep}_\mathbb{Q}(P_G)$ for the category of algebraic representations of $P_G$ over $\mathbb{Q}$-vector spaces. With this torsor one can define the functor

$$\mathcal{E}: \text{Rep}_\mathbb{Q}(P_G) \to \text{QCoh}(\tilde{A}_{G,F}), V \mapsto T_H \times^{P_G} V.$$  

The image $\mathcal{E}(V)$ is a locally free sheaf over $\tilde{A}_{G,F}$. In particular, the similitude factor $\nu$ in $G$ defines an element (again denoted by) $\nu$ in $\text{Rep}_\mathbb{Q}(P_G)$ and thus an invertible sheaf $\mathcal{E}(\nu)$ over $\tilde{A}_{G,F}$. We will write $\mathcal{E}(V)(k)$ for $\mathcal{E}(V) \otimes \mathcal{E}(\nu)^{\otimes k}$.

Let $\mathfrak{g}_G$ be the Lie algebra of $G$. Then we write $\text{Rep}_\mathbb{Q}(\mathfrak{g}_G, P_G)$ for the category of $(\mathfrak{g}_G, P_G)$-modules. More precisely, an object in $\text{Rep}_\mathbb{Q}(\mathfrak{g}_G, P_G)$ is an object $W$ in $\text{Rep}_\mathbb{Q}(P_G)$ with an extra action of $\mathfrak{g}_G$ on $W$ such that the restriction of this action to the Lie algebra $\mathfrak{p}_G$ of $P_G$ agrees with the action of $\mathfrak{p}_G$ induced from $P_G$. Moreover, we require that for any $g \in G$, $X \in \mathfrak{g}_G$ and $w \in W$, there is the compatibility $(gX g^{-1})w = (\text{Ad}(g)X)w$. For any $(\mathfrak{g}_G, P_G)$-module $V$, the Gauss-Manin connection $\nabla$ on $\mathcal{H}_{\text{DR}}(\mathcal{A}_{G,F}/\tilde{A}_{G,F})^{\text{can}}$ induces the Gauss-Manin connection on the sheaf ([Liu15a, Proposition 2.2.3])

$$\nabla: \mathcal{E}(V) \to \mathcal{E}(V) \otimes_{\mathcal{O}_{\tilde{A}_{G,F}}} \Omega_{\tilde{A}_{G,F}}(\log C).$$

One can show that this connection induces Hecke-equivariant maps on global sections. We next use $\nabla$ to construct a differential operator. Let $(\rho, W_\rho)$ be a finite dimension algebraic representation of $\text{GL}_n$. We can associate to it a $(\mathfrak{g}_G, P_G)$-module $V_\rho$ as follows ([Liu15a, Section 2.3]): write $\mathbb{Z}$ for the $n \times n$-matrix which is symmetric with entries $Z_{i,j} = Z_{j,i}$ $(1 \leq i, j \leq n)$. As a $\mathbb{Q}$-vector space, we set $V_\rho = W_\rho[\mathbb{Z}]$ to be the space of polynomials in $Z_{i,j}$ with coefficients in $W_\rho$. We define an action of $P_G$ on $V_\rho$ as follows: for any $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in P_G$ and $f(\mathbb{Z})$, we set $\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} f(\mathbb{Z}) := af(a^{-1}b + a^{-1}Zd)$. One can also define an action of $\mathfrak{g}_G$ on $V_\rho$ as in [Liu15b, Section 2.2] and verify that $V_\rho$ becomes a $(\mathfrak{g}_G, P_G)$-module. There is a natural filtration on $V_\rho$ respecting the action of $P_G$ given by the total degree of the polynomials in $V_\rho$: $V_\rho = \bigcup_{r \in \mathbb{N}} V_\rho^r$ where $V_\rho^r = W_\rho[\mathbb{Z}]_{\text{deg} \leq r}$. One can show that $\mathfrak{g}_G V_\rho^r \subset V_\rho^{r+1}$ and the connection $\nabla$ restricts to $\nabla: \mathcal{E}(V_\rho^r) \to \mathcal{E}(V_\rho^{r+1}) \otimes_{\mathcal{O}_{\tilde{A}_{G,F}}} \Omega_{\tilde{A}_{G,F}}(\log C)$. Let $\tau_n: \text{GL}_n \to \text{GL}(n(n+1)/2)$ be the symmetric square representation of $\text{GL}_n$. Then we get the following differential operator ([Liu15b, Section 2.2]):

$$\nabla_\rho: \mathcal{E}(V_\rho^r) \to \mathcal{E}(V_\rho^{r+1}) \otimes_{\mathcal{O}_{\tilde{A}_{G,F}}} \Omega_{\tilde{A}_{G,F}}(\log C) \xrightarrow{K-S} \mathcal{E}(V_\rho \otimes \tau_n)^{r+1}(-1) \to \mathcal{E}(V_\rho \otimes \tau_n)^{r+1}.$$ 

Here the map $K-S$ is the Kodaira-Spencer isomorphism ([Lan12, Propositin 6.9(5)]). Moreover, the composition of the first two maps is Hecke-equivariant.

We next define the $q$-expansion of nearly holomorphic forms on the geometric side. This will give the integral structure on the space of automorphic forms. Let $S_n = (L^+ \otimes \mathbb{Z} L^+)/\langle v \otimes v' = v' \otimes v \rangle$ be the symmetric quotient of $L^+ \otimes \mathbb{Z} L^+$. Write $S_{n,\geq 0}$ for the subset of $S_n$ consisting of elements $(v, v')$ such that $f(v, v') \geq 0$ for any symmetric semi-positive definite bilinear form $f$ on $(L^+ \times L^+) \otimes_{\mathbb{Z}} \mathbb{R}$. Write $\{s_1, s_2, \cdots, s_{n(n+1)/2}\}$ for a $\mathbb{Z}$-basis of $S_n$ lying inside $S_{n,\geq 0}$. We then set the Laurent power series $\mathbb{Z}[[S_{n,\geq 0}]] = \mathbb{Z}[[S_{n,\geq 0}]]/1/s_1 \cdots s_{n(n+1)/2}$. There is a natural embedding $S_{n,\geq 0} \to \mathbb{Z}[[S_{n,\geq 0}]]$ and we denote the image of $\beta \in S_{n,\geq 0}$ by $\beta^\beta$. Now we have a natural map $\mathbb{Z}((e_1^+, \cdots, e_n^+)) \to \mathbb{Z}((e_1^+, \cdots, e_n^+)) \otimes \mathbb{G}_{m/\mathbb{Z}[[S_{n,\geq 0}]]}$, principally polarized by the duality between $\mathbb{Z}(e_1^+, \cdots, e_n^+) \subset \mathbb{Z}(e_1, \cdots, e_n) \otimes \mathbb{G}_{m/\mathbb{Z}[[S_{n,\geq 0}]]}$, given by the symplectic form on $L$. Mumford’s construction ([FC90]) gives an abelian variety $A/\mathbb{Z}[[S_{n,\geq 0}]]$ with a canonical polarization $\lambda_\text{can}$ and a canonical basis $\omega_\text{can} = (\omega_{1,\text{can}}, \cdots, \omega_{n,\text{can}})$ of $\omega(A/\mathbb{Z}[[S_{n,\geq 0}]]))$. We can then define the level structure $\psi_{N,\text{can}}$ and filtration $\text{fil}^{p,m}_{N,\text{can}}$ of this abelian variety $A \times \mathbb{Z}[[S_{n,\geq 0}]]$ by the following
We fix a dominant character \( h \in \mathbb{G}_m \) of the torus \( T_n \) of \( \text{GL}_n \) with respect to \( B_n \) (i.e., \( h \geq 2 \cdots \geq 2 \)). For any \( \rho = \rho_k \) for the irreducible algebraic representation of \( \text{GL}_n \) associated to the character \( k \) (of highest weight \( k \)), which is defined as follows: for any \( \mathbb{Z} \)-algebra \( R \),

\[
W_k((A/B_0/R)/\mathcal{O}_{\mathbb{Z}}) = \{ f \in H^0((\text{GL}_n/R, \mathcal{O}_{\text{GL}_n})| \text{f}(h) = k(h) \text{ for all } h \}.
\]

An element \( g \in \text{GL}_n(R) \) acts on \( f \in W_k((A/B_0/R)/\mathcal{O}_{\mathbb{Z}}) \) by \( (gf)(h) = f(hg) \).

**Definition 2.1.** We fix a dominant character \( k \) of \( T_n \), an integer \( r \geq 0 \), a nearly holomorphic automorphic form of weight \( k \), degree \( r \), level \( \Gamma \) on \( \tilde{A}_{G,\mathbb{F}} \) is a global section of the sheaf \( \mathcal{E}(V^r_k) = \mathcal{E}(V^r_k) \). The holomorphic automorphic form of weight \( k \), degree \( r \) and level \( \Gamma \) on \( \tilde{A}_{G,\mathbb{F}} \) is a global section of the sheaf \( \mathcal{E}(V^0_k) \).

More generally, for any \( \mathbb{Z} \)-algebra \( R \), we write the space of \( R \)-valued, weight \( k \), degree \( r \) and level \( \Gamma \) nearly holomorphic automorphic forms on \( \tilde{A}_{G,\mathbb{F}} \) as \( M_k(\tilde{\Gamma}, R, r) = H^0(\tilde{A}_{G,\mathbb{F}}/R, \mathcal{E}(V^r_k)) \). The subspace of cuspidal forms is \( \mathcal{S}_k(\tilde{\Gamma}, R, r) = H^0(\tilde{A}_{G,\mathbb{F}}/R, \mathcal{E}(V^r_k)/(\mathbb{C})) \).

Similarly we set \( M_k(\tilde{\Gamma}, R) = H^0(\tilde{A}_{G,\mathbb{F}}/R, \mathcal{E}(V^0_k)) \) and \( \mathcal{S}_k(\tilde{\Gamma}, R) = H^0(\tilde{A}_{G,\mathbb{F}}/R, \mathcal{E}(V^0_k)/(\mathbb{C})) \).

The moduli interpretation (à la Katz) of an element \( f \in M_k(\tilde{\Gamma}, R, r) \) as follows: \( f \) assigns to a functorial way to each tuple \( (A/S, \lambda, \psi, \alpha) \) an element \( f(\lambda, \psi, \alpha) \in V^r_k(S) \). Here \( S \) is an \( \mathbb{Z} \)-algebra and \( A/S \) is an abelian scheme over \( S \), \( \lambda \) is a polarization, \( \psi \) is a principal polarization, \( \alpha \) is a basis of \( H^1_{\text{dR}}(A/S) \) respecting the Hodge filtration.

For any algebraic representation \( \rho \), \( M_k(\tilde{\Gamma}, R, r) \) one can consider the top degree coherent cohomology \( H^d(\tilde{A}_{G,\mathbb{F}}/R, \mathcal{E}(V^r_k)) \) where \( d = n(n+1)/2 \). Let \( \rho^\vee \) denote the dual representation of \( \rho \). We have Serre duality

\[
\langle \cdot, \cdot \rangle_{\text{Ser}} : H^0(\tilde{A}_{G,\mathbb{F}}/(\mathbb{C}), \mathcal{E}(V^r_0)) \times H^d(\tilde{A}_{G,\mathbb{F}}/(\mathbb{C}), \mathcal{E}(V^r_0) \otimes \wedge^d \Omega_{\tilde{A}_{G,\mathbb{F}}}) \to \mathbb{C}.
\]
For any dominant weight $\mathbf{k}$ of $T_n$, we write $D^{\mathbf{k}} := (-k_n, -k_{n-1}, \ldots, -k_1) + 2(\rho_G - \rho_{L_G})$ where recall that $\rho_G$, resp., $\rho_{L_G}$ denotes the half sum of positive roots of $G$, resp., the Levi subgroup $L_G$ with respect to $T_G$. Then we have an isomorphism $\mathcal{E}(V^0_{\mathbf{k}}) \otimes \wedge^4 \Lambda_{G, \mathbf{f}} \simeq \mathcal{E}(V^0_{\mathbf{k}})$ [FC90, p.256]. Note that $W_{2(\rho_G - \rho_{L_G})}$ is the 1-dimensional representation of $G$ sending $g$ to $\nu(g)^d$. The Serre duality can be concretely expressed as follows: for any $\phi \in H^0(\mathbf{A}_{G, \mathbf{f}}(\mathbb{C}), \mathcal{E}(W_\rho))$ and $\rho' \in H^d(\mathbf{A}_{G, \mathbf{f}}(\mathbb{C}), \mathcal{E}(W_{\rho'}) \otimes \wedge^4 \Lambda_{G, \mathbf{f}})$, then one has

\begin{equation}
\langle \phi, \phi' \rangle_{\text{Ser}} = \int_{[G]} \phi(g)\phi'(g)\nu(g)^{-d} dg.
\end{equation}

Using the Serre duality, we define

**Definition 2.2.** For any $\mathbb{Z}$-algebra $R$ and any dominant weight $\mathbf{k}$ of $T_n$, we write

\[ \widehat{M}_\mathbf{k}(\mathfrak{g}, R) := H^d(\mathbf{A}_{G, \mathbf{f}, R}, \mathcal{E}(V^0_{\mathbf{k}})) := \text{Hom}_R(H^0(\mathbf{A}_{G, \mathbf{f}, R}, \mathcal{E}(V^0_{\mathbf{k}})), R). \]

\[ \widehat{S}_\mathbf{k}(\mathfrak{g}, R) := H^1(\mathbf{A}_{G, \mathbf{f}, R}, \mathcal{E}(V^0_{\mathbf{k}})) := \text{Hom}_R(H^0(\mathbf{A}_{G, \mathbf{f}, R}, \mathcal{E}(V^0_{\mathbf{k}})), R). \]

These $R$-modules give the integral structures of the top degree (cuspidal) cohomologies ([EHLS16 Section 6.3]).

2.1.2. Automorphic forms. In this subsection, we associate automorphic forms to modular forms defined above. Fix an algebraic representation $(\rho, W_\rho)$ of $\text{GL}_n$. Recall that $\mathbf{A}_{G, \mathbf{f}}(\mathbb{C}) = G^1(\mathbb{Z})\backslash \mathbb{H}_n \times G(\mathbb{Z})/\mathfrak{g}$. By the moduli interpretation mentioned above, to each $(z, k) \in \mathbb{H}_n \times G(\mathbb{Z})$, we can associate an abelian variety $A_{z, k} := \mathbb{C}^n/(\mathbb{Z}^n \oplus z\mathbb{Z}^n)$, with a polarization $\lambda_z$ given by the Hermitian matrix $\text{Im}(z)^{-1}$ for its Riemann form $E_z: \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{R}, (w_1, w_2) \mapsto -i\text{Im}(w_1^t\text{Im}(z)^{-1}w_2)$ and a level structure $\psi_{z, k}: (\mathbb{Z}/N)^n \sim A_{z, k}[N] = ((\mathbb{1}/N)^n \oplus z(\mathbb{1}/N)^n)/(\mathbb{Z}^n \oplus z\mathbb{Z}^n)$ where $k$ acts via its projection to $G(\mathbb{Z}/N)$ ([Pill2 Section 2.1]). For any $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G^1(\mathbb{Z})$, note that $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} z \\ 1 \end{pmatrix} = \begin{pmatrix} z \gamma + D \end{pmatrix}^{-1} \begin{pmatrix} \gamma \gamma^* \\ 1 \end{pmatrix}$, thus the isomorphism $(Cz + D): \mathbb{C}^n \rightarrow \mathbb{C}^n$ sending $z'$ to $(Cz + D)^{-1}$ induces an isomorphism $\phi_\gamma: A_{z,k} \rightarrow A_{\gamma z, \gamma k}$ respecting the polarization. Let $w_1, w_2, \ldots, w_n$ be the coordinates of $\mathbb{C}^n$ and then $dw := \{dw_1, \ldots, dw_n\}$ form a trivialization of $H^1(A_{z,k}(\mathbb{C})$. For any modular form $f \in H^0(\mathbf{A}_{G, \mathbf{f}}(\mathcal{E}(V^0_\rho)))$, we can define a function $F_f: \mathbb{H}_n \times G(\mathbb{Z}) \rightarrow V^0_\rho(\mathbb{C})$ by $F_f(z, k) := f(A_{z,k}, \lambda_z, \psi_{z, k}, dw)$. Then one verifies that for any $\gamma \in G^1(\mathbb{Z})$, $F_f(\gamma z, \gamma k) = \rho(\mu(\gamma, z))F_f(z, k)$ and for any $k' \in \mathfrak{g}$, $F_f(z, k) = F_f(z, kk')$ ([Pill2 Section 2.4] and [Lin16a, Section 2.5]).

By strong approximation for $\text{Sp}(2n)$, we have $G(\mathbb{A}) = G(\mathbb{Q})G^+(\mathbb{R})G(\mathbb{Z})$, so we write each element in $G(\mathbb{A})$ as $g = g_0g_{\infty}g_{\mathfrak{f}}$. Now suppose that there is a real number $m(\rho) \in \mathbb{R}$ such that $\rho_C(\lambda \cdot 1_n) = \lambda^{2m(\rho)}\text{Id}_{W_{\rho}(\mathbb{C})}$ for any $\lambda \in \mathbb{C}^x$. Then to each $f \in H^0(\mathbf{A}_{G, \mathbf{f}}(\mathcal{E}(V^0_\rho)))$, one can associate a function

\[ \tilde{\Phi}(f): G(\mathbb{A}) \rightarrow V^0_\rho(\mathbb{C}), \quad g \mapsto \nu(g_\infty)^{m(\rho)}\rho(\mu(g_\infty, i1_n))^{-1}F_f(g_\infty \cdot i1_n, g_{\mathfrak{f}}). \]

One can then verify that for each $g_0 \in G(\mathbb{Q})$, $a \in Z_G(\mathbb{A})$, $k_n \in K^1_{G, \infty}$ and $k \in \mathfrak{g}$, we have $\tilde{\Phi}(f)(g_0ag(k_n, k)) = \rho(\mu(k_n, i1_n))^{-1}\tilde{\Phi}(f)(g)$. Now for any linear form $w^\vee \in W_{\rho}^\vee$, we define a new function $\Phi(f, w^\vee)$ by $\Phi(f, w^\vee)(g) := w^\vee(\tilde{\Phi}(f)(g)|_{\mathbb{Z}_{\leq 0}}) \in \mathbb{C}$ (recall that $V^0_\rho = W_\rho[I_{\text{deg} \leq 0}]$). So we see that $\Phi(f, w^\vee) \in A(Z_G(\mathbb{A})G(\mathbb{Q})\backslash G(\mathbb{A})/\mathfrak{g})$ is an automorphic form of trivial center character and level $\mathfrak{g}$. Moreover we fix a Hermitian form $\langle \cdot, \cdot \rangle_{\rho}$ on $W_{\rho}^\vee$ invariant under the action of $K^1_{G, \infty}$ (unique up to a scalar), then we can define a Petersson product on $H^0(\mathbf{A}_{G, \mathbf{f}}(\mathcal{E}(V^0_\rho)))$ as follows: for any two cuspidal modular forms $f, f'$, $\langle f, f' \rangle := \int_{\mathbf{A}_{G, \mathbf{f}}(\mathbb{C})} \rho(\text{Im}(z)^{1/2})F_f(z, k), \rho(\text{Im}(z)^{1/2})F_{f'}(z, k)dzdk$. 

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Then one verifies that ([AS] p.195) for any $w^\vee \neq 0$, there is a constant $c > 0$ such that $\langle f, f \rangle = c\int_{Z_G(\A)G(\Q)\backslash G(\A)} |\Phi(f, w^\vee)(g)|^2 dg$ (we will take $c = 1$ in the below).

We next define some differential operators on $A(Z_G(\A)G(\Q)\backslash G(\A)/\hat{T})$. Recall that $h: \C \to \text{End}_\C(L \otimes \R)$ sends $x + iy$ to $\begin{pmatrix} x1_n & y1_n \\ -y1_n & x1_n \end{pmatrix}$. We then let $\C^\times$ act on $G(\R)$ by conjugation composed with $h$, i.e., $(x + iy) \cdot g = h(x + iy)gh(x + iy)^{-1}$. This action induces an action of $\C^\times$ on the complexification $g_{G,\C}$ of the Lie algebra $g_G$ of $G(\R)$. We then write $g_{G,\C}^{a,b}$ for the subspace of $g_{G,\C}$ on which $z \in \C^\times$ acts by the multiplication by the scalar $z^{-a}z^{-b}$. It is easy to verify that we have the following identity:

$$\hat{\Phi}(\cdot, w^\vee) = \Phi(|\cdot|^{a-b}, w^\vee).$$

Moreover, if $f$ is cuspidal, then so is $\Phi(f, w^\vee)$; $f$ is holomorphic, resp., anti-holomorphic, if and only if $g_{G,\C}\Phi(f, w^\vee) = 0$ resp., $g_{G,\C}\Phi(f, w^\vee) = 0$ for all $w^\vee \in W^\vee$.

This shows that the map $\Phi(\cdot, w^\vee)$ is a norm-preserving map from $H^0(\hat{\A}_{G,\hat{F}}, \mathcal{E}(V^\vee_{\rho}))$ to the space $L^2(Z_G(\A)G(\Q)\backslash G(\A))$. In particular, let $(\rho_{\hat{k}}, W_{\hat{k}})$ be the irreducible representation of $GL_n(\A_{\hat{k}})$ of dominant weight $\hat{k}$, then we define the element $\hat{c} \in W_{\hat{k}}$ to be $\hat{c}(f) := f(1_n)$ for any $f \in W_{\hat{k}}(\R)$ and any $\Z$-algebra $R$. We write $A(Z_G(\A)G(\Q)\backslash G(\A)/\hat{T})_{\hat{k}}$ as the $\rho_{\hat{k}}$-isotypic part of $A(Z_G(\A)G(\Q)\backslash G(\A)/\hat{T})$ as $K_{G,\infty}$-representations. Then we have a norm-preserving injective map:

$$\Phi(\cdot, \hat{c}): H^0(\hat{\A}_{G,\hat{F}}, \mathcal{E}(V^\vee_{\hat{k}})) \to A(Z_G(\A)G(\Q)\backslash G(\A)/\hat{T})_{\hat{k}}.$$

This map gives us the correspondence between the automorphic forms of geometric nature and those of adelic nature which is compatible with the differential operators on both sides.

2.1.3. Hecke operators. We define adelic Hecke operators. We fix a prime $\ell$ of $\Q$. Then for any smooth functions $T: G(\Q_\ell) \to \C$ and $f: G(\A) \to \C$, we set

$$(Tf)(g) := \int_{G(\Q_\ell)} T(g')f(gg')dg', \forall g \in G(\A).$$

(here we view $g' \in G(\Q_\ell)$ also as an element in $G(\A)$). Now let $T$ be the characteristic function of the double coset $G(\Q_\ell)MG(\Q_\ell)$ with $M \in G(\Q_\ell)$ and consider two functions $f_1, f_2 \in L^2(Z_G(\A)G(\Q)\backslash G(\A))$, then it is easy to see

Lemma 2.4. We have the following identity:

$$\langle Tf_1, f_2 \rangle = \langle f_1, Tf_2 \rangle.$$

In other words, the operator $T$ is self-adjoint with respect to the Petersson product on $L^2(Z_G(\A)G(\Q)\backslash G(\A))$. 

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For any $\mathbb{Z}$-algebra $A$ and any compact open subgroup $K \subset G(\mathbb{Q}_\ell)$, we write $\mathcal{H}(G(\mathbb{Q}_\ell), K; A)$ for the associative algebra generated over $A$ by characteristic functions $1(KMK)$ of $KMK$ with $M \in G(\mathbb{Q}_\ell)$, where the multiplication is given by convolution $(T_1 T_2)(g) := \int_{G(\mathbb{Q}_\ell)} T_1(x T_2(x^{-1})) dx$. Here the Haar measure on $G(\mathbb{Q}_\ell)$ is the one where $K$ has volume 1. By the Cartan decomposition for $G(\mathbb{Q}_\ell)$, the algebra $\mathcal{H}(G(\mathbb{Q}_\ell), G(\mathbb{Z}_\ell), A)$ is generated by the elements $T^{(n)}_{\ell,i} := 1(G(\mathbb{Z}_\ell)) \text{diag}(\ell \cdot 1_{n-i}, \ell \cdot 1_i, \ell^2 \cdot 1_{n-i}, \ell \cdot 1_i, G(\mathbb{Z}_\ell))$ for $i = 1, \ldots, n$, $T^{(n)}_{\ell,0} := 1(G(\mathbb{Z}_\ell)) \text{diag}(1_n, \ell \cdot 1_n) G(\mathbb{Z}_\ell)$ and $(T^{(n)}_{\ell,0})^{-1}$. Let $T_G$ be the standard torus of $G$ consisting of diagonal elements in $G$. Then we can similarly define an algebra $\mathcal{H}(T_G(\mathbb{Q}_\ell), T_G(\mathbb{Z}_\ell), A)$, $X_0 := 1(T_G(\mathbb{Z}_\ell)) \text{diag}(1_n, \ell \cdot 1_n)$, $X_i := 1(T_G(\mathbb{Z}_\ell)) \text{diag}(1_{n-i}, \ell \cdot 1_i, \ell^{1-i} \cdot 1_{n-i})$ for $i = 1, \ldots, n$, then as $\mathbb{C}$-algebras, $\mathcal{H}(T_G(\mathbb{Q}_\ell), T_G(\mathbb{Z}_\ell), \mathbb{C})$ is isomorphic to $\mathbb{C}[X_0^\pm, \ldots, X_n^\pm]$. For any element $T \in \mathcal{H}(G(\mathbb{Q}_\ell), G(\mathbb{Z}_\ell), \mathbb{C})$, we define its Satake transform $S(T)$ to be an element in $\mathcal{H}(T_G(\mathbb{Q}_\ell), T_G(\mathbb{Z}_\ell), \mathbb{C})$ as follows: $S(T)(g) := |\delta_{BG}(g)|^{1/2} \int_{N_G} T(gn) dn$. Here $\delta_{BG}(\cdot)$ is the modular character of $G$ with respect to the Borel subgroup $B_G$ of $G$, on the diagonal matrices, it is given by: $\delta_{BG}(t_1 \cdot \cdots \cdot t_n) = t_0^{n(n+1)/2} t_2^2 t_4^4 \cdots t_{2n}^{2n}$. Write $W_G$ for the Weyl subgroup of $G$ with respect to the pair $(B_G, T_G)$. Then we have an isomorphism of algebras $S : \mathcal{H}(G(\mathbb{Q}_\ell), G(\mathbb{Z}_\ell), \mathbb{C}) \simeq \mathcal{H}(T_G(\mathbb{Q}_\ell), T_G(\mathbb{Z}_\ell), \mathbb{C})^{W_G}$ [AS Section 3].

We then define some Iwahori Hecke operators. We write $I_{G,\ell}$ for the Iwahori subgroup of $G(\mathbb{Z}_\ell)$ consisting of matrices $g$ which is in $B_G(\mathbb{Z}_\ell)$ modulo $\ell$. Then the dilating Iwahori Hecke algebra $\mathcal{H}^- (G(\mathbb{Q}_\ell), I_{G,\ell}, A)$ is the subalgebra of $\mathcal{H}(G(\mathbb{Q}_\ell), I_{G,\ell}, A)$ (which is no longer commutative) that is generated over $A$ by the elements $U^{(n)}_{\ell,i} := 1(I_{G,\ell}) \text{diag}(\ell \cdot 1_i, 1_{n-i}, \ell \cdot 1_i, \ell^2 \cdot 1_{n-i}) I_{G,\ell}$ for $i = 1, \ldots, n$, $U^{(n)}_{\ell,0} := 1(I_{G,\ell}) \text{diag}(1_n, \ell \cdot 1_n) I_{G,\ell}$ and $(U^{(n)}_{\ell,0})^{-1}$. Similarly, we define $\mathcal{H}^- (T_G(\mathbb{Q}_\ell), T_G(\mathbb{Z}_\ell), A)$ to be the subalgebra of $\mathcal{H}(T_G(\mathbb{Q}_\ell), T_G(\mathbb{Z}_\ell), A)$ generated by the elements $T_G(\mathbb{Z}_\ell) \text{diag}(\ell \cdot 1_i, 1_{n-i}, \ell \cdot 1_i, \ell^2 \cdot 1_{n-i})$ for $i = 1, \ldots, n$, $T_G(\mathbb{Z}_\ell) \text{diag}(1_n, \ell \cdot 1_n)$ and $(T_G(\mathbb{Z}_\ell) \text{diag}(\ell \cdot 1_{2n}))^{-1}$. Then one verifies that the map $S : \mathcal{H}^- (G(\mathbb{Q}_\ell), I_{G,\ell}, A) \to \mathcal{H}^- (T_G(\mathbb{Q}_\ell), T_G(\mathbb{Z}_\ell), A)$ sending generators to the corresponding generators is an isomorphism of algebras ([Cas95 Lemma 4.1.5], [BC00 Proposition 6.4.1]). When $\ell = p$ and we let $U^{(n)}_{p,i}$ act on the automorphic forms, we need to be careful about normalization of these operators. More precisely, for any dominant weight $\ell \in \mathbb{Z}^n$ of $T_n$ and $f \in \mathcal{A}(Z_G(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A}))/k$, we define the action as ([Liu15b (2.5.2)]):

$$(U^{(n)}_{p,i} f)(g) := p^{(k-n_\rho, n_{\text{nc}})\ell} \int_{G(\mathbb{Q}_p)} U^{(n)}_{p,i} (h) f(gh) dh$$

Here $i = (i_0, i_1, \ldots, i_n)$ is an element in $\mathbb{Z}^{n+1}$ such that $i_1 = i_2 = \cdots = i_i = 1$, $i_{i+1} = \cdots = i_n = 0$, $i_0 = 2$ for $i = 1, \ldots, n$ and $0_1 = \cdots = 0_n = 0$, $0_0 = 1$. Moreover $2n_\rho, n_{\text{nc}}$ is the sum of non-compact positive roots of $G$ with respect to $B_G$. More generally we write $C^+_n$ for the subset of $\mathbb{Z}^{n+1}$ consisting of elements $a = (a_0, a_1, \ldots, a_n)$ such that $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$. Then for any $a \in C^+_n$, we write $p^a = \text{diag}(p^{a_1}, \ldots, p^{a_n}, p^{a_{n-1}}, \ldots, p^{a_{a_1}})$. We define $U^{(n)}_{p,a}$ to be the characteristic function $1(I_{G,p}) p^a I_{G,p}$ and let it act on $\mathcal{A}(Z_G(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A}))/k$ by the same formula as above. We write $U_p(f)$ for the subspace of $\mathcal{A}(Z_G(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A}))/k$ generated by $U^{(n)}_{p,a}(f)$ for all $a \in C^+_n$. Then by [Liu15b Proposition 2.5.2], the eigenvalues of these operators $U^{(n)}_{p,a}$ acting on $U_p(f)$ are all $p$-adic integers under the fixed isomorphism $\iota_p : \mathbb{C} \simeq \mathbb{C}_p$. We define $U^{(n)}_p := \prod_{i=1}^n U^{(n)}_{p,i}$. For each nearly-holomorphic modular form $f \in \mathcal{A}(Z_G(\mathbb{A}) G(\mathbb{Q}) \backslash G(\mathbb{A}))/k$, we define an operator $e$ acting on $U_p(f)$ as the limit $e := \lim_{r \to +\infty} U^{(r)}_p$ (the limit is taken with respect to the $p$-adic topology on $U_p(f)$ via $\mathbb{C} \simeq \mathbb{C}_p$). Then we call $e(f)$ the ordinary projection of $f$. It follows from the definition that each $U^{(n)}_{p,a}$ acts on $e(f)$ by $p$-adic units. Moreover, by [Liu15b Proposition 2.5.3], $e(f)$ is in fact a holomorphic modular form.

**Definition 2.5.** Let $R$ be a $\mathbb{Z}$-algebra contained in $\mathbb{C}$. For a prime $\ell$ of $\mathbb{Q}$ such that $K_\ell = G(\mathbb{Q}_\ell)$, we write $T_{K,\ell}(K_\ell, R)$ for the $R$-subalgebra generated by the image of the spherical Hecke algebras...
\( \mathcal{H}(G(\mathbb{Q}_e), G(\mathbb{Z}_e), R) \) in \( \text{End}_C(H^0(\hat{\mathcal{A}}_{G,K}, \mathcal{E}(V_{\mathcal{A}}))) \). Similarly, we write \( T^w_{\mathcal{A}^H}(I_{G,p}, R) \) for the \( R \)-subalgebra generated by the image of the \( U_p \)-operators \( U_p^{(n)} \) in the above endomorphism algebra. Finally we write

\[
T^w_\mathfrak{k}(\hat{\Gamma}, R) = (\bigotimes_{\ell \not| \mathfrak{p}} T^w_{\mathcal{A}_\ell}(K_{\ell}, R)) \bigotimes_{R} T^w_{\mathcal{A}}(I_{G,p}, R).
\]

**2.2. Periods and congruence ideals.** In this subsection we define congruence ideals and some periods of automorphic forms on \([G]\). We follow closely the ideas of \([EHLST16, \text{Section 6}]\).

**Definition 2.6.** Let \( \pi \) be an irreducible cuspidal automorphic representation of \([G]\) with factorization \( \pi = \pi_\infty \otimes \pi_\tau \). We say that \( \pi \) is holomorphic of type \((\mathfrak{k}, \hat{\Gamma})\) if

\[
H^0(\mathfrak{g}^{-1}_G \oplus t_G, K_G, \pi_\infty \otimes W_k) \neq 0 \quad \text{and} \quad \pi_\tau^{\hat{\Gamma}} \neq 0.
\]

Similarly, we say that \( \pi \) is antiholomorphic of type \((\mathfrak{k}, \hat{\Gamma})\) if

\[
H^d(\mathfrak{g}^{-1}_G \oplus t_G, K_G, \pi_\infty \otimes W_k^\tau) \neq 0 \quad \text{and} \quad \pi_\tau^{\hat{\Gamma}} \neq 0.
\]

Let \( \pi \) be a holomorphic automorphic representation of \( G(\mathbb{A}) \) of type \((\mathfrak{k}, \hat{\Gamma})\). Then the action of \( T^w_\mathfrak{k}(\hat{\Gamma}, \mathbb{C}) \) on \( \pi^{\hat{\Gamma}} \) is given by a morphism, denoted by \( \lambda_\pi : T^w_\mathfrak{k}(\hat{\Gamma}, \mathbb{C}) \to \mathbb{C} \). For any subalgebra \( R \subset \mathbb{C} \) containing all the values of \( \lambda_\pi \) in \( \mathbb{C} \), we write

\[
S^w_\mathfrak{k}(\hat{\Gamma}, R)[\lambda_\pi]
\]

for the \( \lambda_\pi \)-isotypic component of \( S^w_\mathfrak{k}(\hat{\Gamma}, R) \). This is equivalent to saying that it is the localization of \( S^w_\mathfrak{k}(\hat{\Gamma}, R) \) at the prime ideal of \( T^w_\mathfrak{k}(\hat{\Gamma}, R) \) given by the kernel of \( \lambda_\pi \). Then one has an embedding of the \( 1 \)-dimensional vector space as \( T^w_\mathfrak{k}(\hat{\Gamma}, R)(\mathbb{C}) \)-modules

\[
\iota_\pi : H^0(\mathfrak{g}^{-1}_G \oplus t_G, K_G, \pi^{\hat{\Gamma}} \otimes \mathbb{C} W_k) \simeq \pi_\tau^{\hat{\Gamma}} \hookrightarrow S^w_\mathfrak{k}(\hat{\Gamma}, \mathbb{C})(\pi).
\]

We write \( m_\pi \) for the maximal ideal of \( T^w_\mathfrak{k}(\hat{\Gamma}, R) \) given by the kernel of the composition \( T^w_\mathfrak{k}(\hat{\Gamma}, R) \xrightarrow{\lambda_\pi} R \to R/m_\pi \) where \( m_\pi \) is the maximal ideal of \( R \). We set

\[
S^w_\mathfrak{k}(\hat{\Gamma}, R)[\pi] := S^w_\mathfrak{k}(\hat{\Gamma}, R)[m_\pi] \cap S^w_\mathfrak{k}(\hat{\Gamma}, R[1/p])[\lambda_\pi].
\]

**Lemma 2.7.** Suppose as above \( R \subset \mathbb{C} \simeq \overline{\mathbb{Q}}_p \) is \( p \)-adically complete. We have

\[
S^w_\mathfrak{k}(\hat{\Gamma}, R[1/p])[\lambda_\pi] = e S^w_\mathfrak{k}(\hat{\Gamma}, R[1/p])[\lambda_\pi].
\]

Moreover, \( \iota_\pi \) induces an isomorphism

\[
\iota_\pi : \pi_\tau^{\hat{\Gamma}} \otimes \pi_\xi^{\hat{\Gamma}} \simeq \pi_\tau^{\hat{\Gamma}} \hookrightarrow S^w_\mathfrak{k}(\hat{\Gamma}, \mathbb{C}).
\]

There is also an integral version: \( \iota_\pi \) identifies \( S^w_\mathfrak{k}(\hat{\Gamma}, R)[\pi] \) with an \( R \)-lattice in \( \pi_\tau^{\hat{\Gamma}} \otimes \pi_\xi^{\hat{\Gamma}} \). It also identifies \( S^w_\mathfrak{k}(\hat{\Gamma}, R)[m_\pi] \) with an \( R \)-lattice in \( \bigoplus_{\pi'} (\pi'^{\tau})^{\text{ord}} \otimes (\pi')^{\hat{\Gamma}} \) where \( \pi' \) runs through all ordinary holomorphic automorphic representations of \([G]\) of type \((\mathfrak{k}, \hat{\Gamma})\) such that \( \lambda_{\pi'} \equiv \lambda_\pi \pmod{m_\pi} \).

Similarly, we can define the following \( T^w_{\mathcal{A}^{\hat{\Gamma}, R}} \)-modules

\[
\widehat{S}^w_\mathfrak{k}(\hat{\Gamma}, R) := \text{Hom}_R(S^w_\mathfrak{k}(\hat{\Gamma}, R), R), \quad \widehat{S}^w_{\text{ord}}(\hat{\Gamma}, R) := \text{Hom}_R(S^w_{\text{ord}}(\hat{\Gamma}, R), R).
\]

We use Serre duality to identify \( \widehat{S}^w_\mathfrak{k}(\hat{\Gamma}, R) \) with

\[
H^d_{\text{w}}(\hat{\Gamma}, R) := \{ \varphi \in H^d(\mathcal{A}_{G,\hat{\Gamma}}(\mathbb{C}), \mathcal{E}(W_k^D)) \mid (S^w_\mathfrak{k}(\hat{\Gamma}, R), \varphi)^{\text{Ser}} \subset R \}. \]
Using the Serre duality, we define $S^\text{ord,⊥}_\mathbb{K}((\hat{\Gamma}, R))$ to be the subspace of $H^d_{\mathbb{K}D}(\hat{\Gamma}, R)$ annihilating $S^\text{ord}_\mathbb{K}(\hat{\Gamma}, R)$. Similarly, one can identify $S^\text{ord}_\mathbb{K}(\hat{\Gamma}, R)$ with

$$H^d_{\mathbb{K}D}(\hat{\Gamma}, R) \{ \varphi \in H^d(\mathbb{A}_G, \pi(W_{\mathbb{K}D})) / S^\text{ord,⊥}_\mathbb{K}(\hat{\Gamma}, R) \langle S^\text{ord}_\mathbb{K}(\hat{\Gamma}, R), \varphi \rangle \}_{\text{Ser}} \subset R \}.$$

Since $\pi$ is holomorphic of type $(\mathbb{K}, \hat{\Gamma})$, its contragredient $\pi^\vee$ is anti-holomorphic of type $(\mathbb{K}, \hat{\Gamma})$. We have the following injection

$$\tilde{\iota}_{\pi^\vee}: H^d(\mathbb{G}_D^{1-1} \oplus \mathfrak{t}_G, K_{G, \infty}, \pi^\vee, \hat{\Gamma} \otimes W_{\mathbb{K}D}) \hookrightarrow (\pi^\vee)\hat{\Gamma} \cong H^d(\mathbb{A}_G, \hat{\Gamma}(\mathbb{C}), \mathcal{E}(W_{\mathbb{K}D})).$$

Similarly, we set

$$H^d_{\mathbb{K}D}(\hat{\Gamma}, R)[\pi^\vee] := H^d_{\mathbb{K}D}(\hat{\Gamma}, R)[\pi^\vee] \bigcap H^d_{\mathbb{K}D}(\hat{\Gamma}, R[1/p])[\lambda_{\pi^\vee}].$$

The dual version of the above lemma is

**Lemma 2.8.** Let $E, \mathbb{E}_R$ be as above. The map $\tilde{\iota}_{\pi^\vee}$ induces an isomorphism

$$\tilde{\iota}_{\pi^\vee}: \pi_{\mathbb{F}S} \cong H^d_{\mathbb{K}D}(\hat{\Gamma}, \mathbb{C})[\pi^\vee].$$

Moreover, $\tilde{\iota}_{\pi^\vee}$ identifies $H^d_{\mathbb{K}D}(\hat{\Gamma}, R)[\pi]$ with an $R$-lattice in $\pi_{\mathbb{F}S}$. It also identifies $H^d_{\mathbb{K}D}(\hat{\Gamma}, R)[\pi^\vee]$ with an $R$-lattice in $\otimes_{\pi^\vee}(\pi')_{\mathbb{F}S}^{\mathbb{F}S}$ where $\pi'$ runs through all ordinary holomorphic automorphic representations of $[G]$ of type $(\mathbb{K}, \hat{\Gamma})$ such that $\lambda_{\pi'} \equiv \lambda_{\pi}(\text{mod } m_R)$.

One can also show that the Serre duality induces perfect $T^\text{ord}_{\mathbb{K}D,R}$-equivariant pairings

$$S^\text{ord}_\mathbb{K}(\hat{\Gamma}, R)[\pi] \bigotimes R H^d_{\mathbb{K}D}(\hat{\Gamma}, R)[\pi^\vee] \to R, \quad S^\text{ord}_\mathbb{K}(\hat{\Gamma}, R)[\pi^\vee] \bigotimes R H^d_{\mathbb{K}D}(\hat{\Gamma}, R)[\pi] \to R.$$

2.2.1. **Congruence ideals.** In this subsection we write $T^\text{ord}$, resp., $T^\text{ord}_{m_s}$ for $eT^\text{ord}_{\mathbb{K}D}(\hat{\Gamma}, R)$, resp., $eT^\text{ord}_{\mathbb{K}D}(\hat{\Gamma}, R)[m_s]$. We make the following Gorenstein hypothesis

**Hypothesis 2.9.** The $R$-algebra $T^\text{ord}_{m_s}$ is Gorenstein and the $T^\text{ord}_{m_s}$-module $S^\text{ord}_\mathbb{K}(\hat{\Gamma}, R)[m_s]$ is a free module.

Note that $T^\text{ord}_{m_s}$ is a reduced $R$-algebra, we have a decomposition of $R$-algebras

$$T^\text{ord}_{m_s}[1/p] := T^\text{ord}_{m_s} \otimes_R R[1/p] = R[1/p] \oplus X$$

where the projection to $R[1/p]$ is induced by the character $\lambda_{\pi}: T^\text{ord}_{m_s} \to R$. We write $1_{\pi}$ for the idempotent element in $T^\text{ord}_{m_s}[1/p]$ corresponding to the projection $T^\text{ord}_{m_s}[1/p] \to R[1/p]$. Let $M$ be a finite $T^\text{ord}$-module flat over $R$ and put $M[1/p] := M \otimes_R R[1/p]$. Define the following $T^\text{ord}$-modules:

$$M^\pi := 1_{\pi} M_{m_s}, \quad M_\pi := M \bigcap 1_{\pi} M[1/p], \quad M^X := (1 - 1_{\pi}) M_{m_s}, \quad M_X := M \bigcap M^X.$$

One can show that there are isomorphisms of $R$-modules

$$\frac{M_{m_s}}{M_\pi + M_X} \simeq \frac{M^\pi}{M_\pi} \simeq \frac{M^X}{M_X} \simeq \frac{M^\pi + M^X}{M_{m_s}}.$$

**Definition 2.10.** The cohomological congruence ideal $c^\text{coh}(M, \pi)$ of $M$ is the annihilator in $R$ of the $R$-module $M^\pi/M_\pi$.

**Lemma 2.11.** Assuming Hypothesis 2.9, one has

$$c^\text{coh}(T^\text{ord}_{m_s}, \pi) = c^\text{coh}(S^\text{ord}_\mathbb{K}(\hat{\Gamma}, R), \pi) = c^\text{coh}(H^d_{\mathbb{K}D}(\hat{\Gamma}, R), \pi).$$

**Proof.** cf. [TU18, Section 2].

**Definition 2.12.** We write $c^\text{coh}(\pi)$ for the ideals in the above lemma.
Remark 2.13. The cohomological congruence ideal defined here is the same as the one given in [EHLS16, after Lemma 6.6.3] for the case of $\text{GSp}_{2n}$. More precisely, write $S_k^\text{ord}(\hat{\Gamma}, R)[\pi]^\perp \subset S_k^\text{ord}(\hat{\Gamma}, R)_{m_\pi}$ for the orthogonal complement of the subspace $S_k^\text{ord}(\hat{\Gamma}, R)[\pi] \subset S_k^\text{ord}(\hat{\Gamma}, R)_{m_\pi}$ under the pairing of Petersson product. Then the ideal $c_\text{coh}(\pi)$ is equal to the ideal of $R$ annihilating the $R$-module $S_k^\text{ord}(\hat{\Gamma}, R)_{m_\pi}/(S_k^\text{ord}(\hat{\Gamma}, R)[\pi] + S_k^\text{ord}(\hat{\Gamma}, R)[\pi]^\perp)$.

The cohomological congruence ideals can be generalized in the following way (cf. [HT16, Section 8.2]). Let $S, S'$ be non-zero reduced algebra over a normal noetherian domain $R$ which are flat of finite type as $R$-modules. Let $f : S \to S'$ be a surjective homomorphism of $R$-algebras. Then we can decompose the total quotient ring $Q(S)$ into a direct product of $R$-algebras $Q(S) = Q(S') \oplus X$. Then the congruence module of the morphism $f$ is defined as $C_0(f) := S/Ker(S \to X) \otimes_{S,f} S'$ and the congruence ideal in $S'$ of $f$ is defined as

$$c(f) = \text{Ann}_{S'}(C_0(f)).$$

The meaning and transfer property of the congruence ideal $c(f)$ are given in [HT16, Proposition 8.3, Lemma 8.5]. Then for the case $\lambda_\pi : T_{m_\pi} \to R$, under Hypothesis 2.9 we have (cf. [TU13, Section 2.1]):

$$c_\text{coh}(\pi) = c(\lambda_\pi).$$

2.2.2. Periods of automorphic representations.

Lemma 2.14. Let $\pi$ be an irreducible holomorphic automorphic representation of $[G]$ of type $(k, \hat{\Gamma})$. The following $R$-modules contained inside $\mathbb{C}$ given by the values of the Petersson products

$$L[\pi] = \langle S_k^\text{ord}(\hat{\Gamma}, R)[\pi], S_k^\text{ord}(\hat{\Gamma}, R)[\pi] \rangle, \quad L_\pi = \langle S_k^\text{ord}(\hat{\Gamma}, R)[\pi], S_k^\text{ord}(\hat{\Gamma}, R)_{m_\pi} \rangle$$

are both of rank 1 over $R$.

Proof. The proof is the same as [Har13, Proposition 2.4.9] which is a simple application of Schur’s lemma. \qed

We fix two positive real numbers as generators of these two $R$-modules $L[\pi]$ and $L_\pi$:

$$P[\pi] = P_G[\pi], \quad P_\pi = P_{G,\pi}.$$  

We have a dual version of Lemma 2.14.

Lemma 2.15. Assume Hypothesis 2.9, then the following $R$-modules

$$\hat{L}[\pi^\vee] = \langle H_k^\text{ord}(\hat{\Gamma}, R)[\pi^\vee], H_k^\text{ord}(\hat{\Gamma}, R)[\pi^\vee] \rangle, \quad \hat{L}_{\pi^\vee} = \langle H_k^\text{ord}(\hat{\Gamma}, R)[\pi^\vee], H_k^\text{ord}(\hat{\Gamma}, R)_{m_{\pi^\vee}} \rangle$$

are both of rank 1 over $R$.

We fix the following positive real numbers as generators of the $R$-module $\hat{L}[\pi^\vee]$, resp., $\hat{L}_{\pi^\vee}$.

$$\hat{P}[\pi^\vee] = \hat{P}_G[\pi^\vee], \quad \hat{P}_{\pi^\vee} = \hat{P}_{G,\pi^\vee}.$$  

These numbers $P[\pi], P_\pi, \hat{P}[\pi^\vee], \hat{P}_{\pi^\vee}$ are called the periods of $\pi$ and $\pi^\vee$. Then we have the following relation of congruence numbers and periods:

Lemma 2.16. we have the following identities, up to units in $R$:

$$c_\text{coh}(\pi)P_\pi = P[\pi], \quad c_\text{coh}(\pi)\hat{P}_{\pi^\vee} = \hat{P}[\pi^\vee], \quad \hat{P}_{\pi^\vee}P_\pi = 1.$$  

Proof. The first identity follows the definitions (cf. [EHLS16, Lemma 6.6.3]) The second one comes from Serre duality. The last one comes from the isomorphism $c_G$ as well as the relation between Serre duality and Petersson products as given by (2.2). \qed
Remark 2.17. In applications to Rallis inner product formula, we will consider antiholomorphic automorphic representations of $[G]$ instead of holomorphic ones. The above results on congruence ideals and periods are still valid in this case up to the exchange of roles of $\pi$ and its contragredient $\pi^\vee$.

2.2.3. Congruence ideals and Petersson products on compact groups. In this subsection we digress to review the relation between congruence ideals and Petersson products. Let $G$ be an algebraic group over $\mathbb{Q}$ compact at $\infty$. Then there is an equivalence between the theory of algebraic modular forms on $G$ and the theory of classical modular forms on $G$. We will use the former theory in this subsection.

Fix a compact open subgroup $U$ of $G(\mathbb{A})$, then $G(\mathbb{Q}) \setminus G(\mathbb{A}) / U$ is a finite set, which we denote by $\bigcup_{i \in I} G(\mathbb{Q}) t_i U$. For any ring $R$, Let $S(U, R)$ be the set of $R$-valued modular forms on $G(\mathbb{A})$ of level $U$, which can be identified with the set of functions $\{f : I \rightarrow R\}$. Suppose $R$ is a subring of $\mathbb{C}$, we define the Petersson inner product on $S(U, R)$ as

$$\langle \cdot, \cdot \rangle : S(U, R) \times S(U, R) \rightarrow R, \quad (f, g) \mapsto \langle f, g \rangle = \sum_i f(t_i) g(t_i^{-1}).$$

This is a perfect pairing, invariant under the action of $G(\mathbb{A})$.

Suppose now that $\mathbb{C}$ is a finite extension of $\mathbb{Z}_p$ with uniformizer $\varpi$ (still viewed as a subring of $\mathbb{C}$). For any automorphic representation $\pi$ of $G(\mathbb{A})$, let $f, f'$ be two $R$-valued modular eigenforms on $G(\mathbb{A})$ in the representation $\pi$ such that $c(f) = f$ and $c(f') = -f'$. Assume that both $f$ and $f'$ are primitive. Let $C$ be an irreducible component of the ordinary Hida-Hecke algebra $\mathcal{T}$ passing through $\pi, \lambda_{\pi} : \mathcal{T} \rightarrow C \xrightarrow{\varpi} R$. Then as in [Hi10] Sections 5.2 and 5.3, we can show that

Lemma 2.18. The Petersson product $\langle f, f' \rangle$ generates the cohomological congruence ideal $\mathfrak{c}(\lambda_{\pi}) \subset R$.

3. Siegel Eisenstein series and theta correspondence

3.1. Vector spaces and groups.

3.1.1. Symplectic groups. Let $V = \mathbb{Q}^4$ be a $\mathbb{Q}$-vector space of dimension 4, equipped with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$. Let $L \subset V$ be a lattice of $V$. Fix a $\mathbb{Z}$-basis of $L$ as follows:

$$B = B_L = B_V = \{e_1^+, e_2^+, e_1^-, e_2^-\}$$

such that $\langle e_i^+, e_j^- \rangle = \delta_{i,j}$ for $i, j = 1, 2$. We define some submodules, resp., subspaces of $L$, resp., $V$ as follows:

$$L^+ = \mathbb{Z}e_1^+ + \mathbb{Z}e_2^+ , \quad L^- = \mathbb{Z}e_1^- + \mathbb{Z}e_2^- , \quad V^\pm = L^\pm \otimes \mathbb{Q}.$$

We then define $L_1 = L_2 = L$, resp., $L_3 = L_4 = L_1 \oplus L_2$ with symplectic form $\langle \cdot, \cdot \rangle_{L_1} = \langle \cdot, \cdot \rangle$, resp., $\langle \cdot, \cdot \rangle_{L_2} = -\langle \cdot, \cdot \rangle, \langle \cdot, \cdot \rangle_{L_3} = \langle \cdot, \cdot \rangle_{L_1} + \langle \cdot, \cdot \rangle_{L_2}$. Then we set $V_i = L_i \otimes \mathbb{Q}$ for $i = 1, \cdots, 4$. We fix the following basis for $V_4$:

$$B_{V_4} = \left\{(e_1^+, 0), (e_2^+, 0), (0, e_1^+), (0, e_2^+), (e_1^-, 0), (e_2^-, 0), (0, e_1^-), (0, e_2^-)\right\}$$

The isometry groups $G_1, G_2, G_4$ for the symplectic spaces $V_1, V_2, V_4$ can be identified under the given basis with the following groups:

$$G_1 = G_2 = G = \left\{g \in \text{GL}_4 | g J_4 g = \nu(g) J_4 \right\}$$

$$G_4 = \left\{g \in \text{GL}_8 | g J_4 g = \nu(g) J_4 \right\}$$

Finally we put $G_3 = \{\text{diag}(g_1, g_2) \in \text{diag}(G_1, G_2) \subset \text{GL}_8 | \nu(g_1) = \nu(g_2)\}$. For each $G_i$ $(1 \leq i \leq 4)$, we define the subgroup

$$G_i^1 = \{g \in G_i | \nu(g) = 1\}.$$
We define an injective morphism $G_1^1 \times G_2^1 \hookrightarrow G_4^1$ as follows:

$$\left( \begin{array}{cc} a_1 & b_1 \\ c_1 & d_1 \end{array} \right), \left( \begin{array}{cc} a_2 & b_2 \\ c_2 & d_2 \end{array} \right) \mapsto \left( \begin{array}{ccc} a_1 & 0 & b_1 \\ 0 & a_2 & 0 & -b_2 \\ c_1 & 0 & d_1 & 0 \\ 0 & -c_2 & 0 & d_2 \end{array} \right).$$

We will also use another maximal polarization of $V_4$ later on which is given as follows:

$$V_4^d = \{(v, v) \in V_4 | v \in V_1\}, \quad V_4,d = \{(v, -v) \in V_4 | v \in V_1\}.$$

Write $P_{V'} \subset G_4$ for the parabolic subgroup stabilizing a subspace $V'$ of $V_4$. Then using the relation between $P_{V''}$ and $P_{V'}$, one verifies easily that the action of $G_1^1 \times G_2^1$ on the flag variety $(P_{V''} \cap G_4^1) \backslash G_4^1$ satisfies the conditions in [GPSR87, p.2]. This will be used in the doubling method in the following.

3.1.2. Orthogonal groups. Let $N \in \mathbb{N}$ be a positive integer prime to $p$. Let $N_1 \in \mathbb{N}$ be a positive integer prime to $Np$ such that the diagonal matrices $\eta_v = \text{diag}(N^2/2, N^2/2, N^2/2, N^2/2, N/N_1, N_1)$ and $\eta_v' = \text{diag}(2N, 2N, 2N, 2N, 2, 2)$ are $\mathbb{Q}$-equivalent. Let $U = \mathbb{Q}^6$ be a $\mathbb{Q}$-vector space of dimension 6 equipped with a non-degenerate symmetric bilinear form $\langle \cdot, \cdot \rangle_U$ such that under a basis $B_U = (u_1, \cdots, u_6)$ of $U$ the symmetric form is the matrix $\eta_u$. We then write $O(U)$, resp. $\text{GSO}(U)$, for the (resp. special similitude) orthogonal group over $\mathbb{Q}$ defined by $\langle \cdot, \cdot \rangle_U$. Under this basis $B_U$, then can by identified with

$$O(U) = \{ h \in \text{GL}_6 | h^t \eta_v h = \eta_v \},$$

$$\text{GSO}(U) = \{ h \in \text{GL}_6 | h^t \eta_u h = \nu(h) \eta_v, \det(h) = \nu(h)^3 \}.$$ 

3.1.3. Tensor products. We then define the tensor products $W = V \otimes \mathbb{Q} U$ and $W_i = V_i \otimes \mathbb{Q} U$ for $1 \leq i \leq 4$. We give these vector spaces symplectic structures as follows: take $W$ as an example, for elements $v \otimes u, v' \otimes u' \in W$, we define a symplectic form $\langle \cdot, \cdot \rangle_W$ on $W$ by

$$\langle v \otimes u, v' \otimes u' \rangle_W = \langle v, v' \rangle_V \times \langle u, u' \rangle_U.$$

The maximal polarizations on $V_i$ induce maximal polarizations on $W_i$ in the obvious way:

$$W_i = (V_i^+ \otimes U) \oplus (V_i^- \otimes U), \quad \forall 1 \leq i \leq 4,$$

$$W_4 = (V_4^d \otimes U) \oplus (V_4,d \otimes U).$$

Similarly, we write $\text{Sp}(W_i)$, resp. $\text{GSp}(W_i)$ for the (resp. similitude) algebraic symplectic groups over $\mathbb{Q}$ defined by the symplectic forms on $W_i$. Note that $\text{Sp}(V_4)$ and $O(U)$ act on $W_4$ by the embedding $\text{Sp}(V_4) \times O(U) \hookrightarrow \text{Sp}(W_4)$. Similarly we have a natural map $\text{GSp}(V_4) \times \text{GSO}(U) \rightarrow \text{GSp}(W_4)$.

Remark 3.1. For the purpose of computations, we will use the following obvious matrix identifications of the above defined various vector spaces:

1. We identify $V_4$ with $M_{8 \times 1}(\mathbb{Q})$: the $k$-th basis element $e_k \in B_{V_4}$ corresponding to the column matrix with 1 at the $k$-th coordinate and 0 elsewhere and extend the map to the whole $V_4$ by $\mathbb{Q}$-linearity. Suppose that $v, v' \in V_4$ correspond to $X, X' \in M_{8 \times 1}(\mathbb{Q})$, then the symplectic form becomes $\langle v, v' \rangle_{V_4} = X^t J_8 X'$. We identify $V_1, V_2, V_4^{\pm}$ with $M_{4 \times 1}(\mathbb{Q})$ in the same way;

2. We identify $U$ with $M_{1 \times 6}(\mathbb{Q})$: the $j$-the basis element $u_j \in B_U$ corresponding to the row matrix with 1 at the $j$-th coordinate and 0 elsewhere. Suppose that $u, u' \in U$ correspond to $Y, Y' \in M_{1 \times 6}(\mathbb{Q})$, then the symmetric form becomes $\langle u, u' \rangle_U = Y \eta_v (Y')^t = \text{tr}(Y' \eta_v Y')$;

3. We then identify $W_4 = V_4 \otimes U$ with $M_{8 \times 6}(\mathbb{Q})$: the $(k, j)$-th basis element $e_k \otimes u_j$ in the basis $B_{W_4} = B_{V_4} \times B_U$ corresponding to the matrix with 1 at the $(k, j)$-the coordinate and 0 elsewhere. Suppose that $w, w' \in W_4$ correspond to $Z, Z' \in M_{8 \times 6}(\mathbb{Q})$, then the symplectic form on $W_4$ becomes $\langle w, w' \rangle_{W_4} = \text{tr}(Z^t J_8 Z' \eta_v)$.
Remark 3.2. We make the following convention on the Haar measures on the groups $G(Q_v)$ and the vector spaces $V(Q_v)$ and the like as follows: (1) the measure on $V_4(Q_\ell)$ is the usual measure $\mu$ with $\mu(V_4(Z_\ell)) = 1$, i.e., the product measure of standard measures on $Q_\ell$ with volume of $Z_\ell$ being 1. Same choices for $V_1(Q_\ell), V_2(Q_\ell)$; (2) the measure on $V_4(\mathbb{R})$ is the measure induced from the standard Lebesgue measure on $M_{8 \times 1}(\mathbb{R})$; (3) the measure $\mu_U$ on $U(Q_\ell)$ is the measure $\mu$ such that $\mu(U(Z_\ell)) = |2^{-4}N^9|_\ell$; (4) the measure $\mu_U$ on $U(\mathbb{R})$ is the measure $2^{-4}N^9\mu_{Leb}$ where $\mu_{Leb}$ is the standard Lebesgue measure on $M_{1 \times 0}(\mathbb{R})$; (5) the measure $\mu_{V_4}(Q_v)$ is the product measure of measures on $V_4(Q_v)$ and $U(Q_v)$; (6) the measure on $Sp(V_i)(Q_\ell)$ ($i = 1, 2, 4$) is the one with volume of $Sp(V_i)(Z_\ell)$ being 1. Similar choice for $O(U)(Q_\ell)$; (7) the measure on $Sp(V_i)(\mathbb{R})$ is taken as follows: $Sp(V_i)(\mathbb{R})$ acts on the Siegel upper half plane $\mathbb{H}_{d(i)}$ with $d(i) = (\dim V_i)/2$ by linear transformation and the subgroup fixing the point $\sqrt{-1} \in \mathbb{H}_{d(i)}$ is a maximal compact subgroup $K_{Sp(V_i)(\mathbb{R})}$ of $Sp(V_i)(\mathbb{R})$. We take the (unique) Haar measure on $K_{Sp(V_i)(\mathbb{R})}$ to be the one with total volume being 1. We take the Haar measure on $\mathbb{H}_{d(i)}$ to be $\det(y)^{-d(i)-1} \prod_{1 \leq i \leq j \leq d(i)} dx_{i,j}dy_{i,j}$. Then the Haar measure on $Sp(V_i)(\mathbb{R})$ is the product of these two measures; (8) the measures on the adelic points of the above mentioned objects are the product measures of the local ones.

3.2. Weil representations. For any maximal polarization $V_4 = V' \oplus V''$, which gives rise to a polarization of $W_4$ as $W_4 = (V' \otimes U) \oplus (V'' \otimes U)$, we will define the local and global Weil representations in this subsection ([Kn96, MW87]). The materials in this subsection are already well-known facts and we will therefore be very brief.

3.2.1. Local Weil representations. We fix a place $v$ of $\mathbb{Q}$ and write $F = Q_v$ for the local field at the place $v$. Let $V$ be a symplectic vector space over $F$ with the symplectic form $\langle \cdot, \cdot \rangle = \langle \cdot, \cdot \rangle_V$. Then the Heisenberg group $H(V)$ associated to $V$ is the set $V \times F$ with the group law

$$(v, t) \cdot (v', t') = (v+v', t+t'+\langle v, v' \rangle/2).$$

The symplectic group $Sp(V)$ defined by the symplectic form $\langle \cdot, \cdot \rangle$ acts on the Heisenberg group by

$$g(v, t) := (gv, t), \quad \forall g \in Sp(V), (v, t) \in H(V).$$

Let $e_F = e_v : F \to \mathbb{C}^\times$ be the exponential character. We can then define a pairing on $W$ by

$$[\cdot, \cdot] : W \times W \to \mathbb{C}^\times, \quad (v, v') \mapsto [v, v'] := e_F(\langle v, v' \rangle).$$

For any subspace $V'$ of $V$, we write $(V')^\perp$ for the subspace of $V$ consisting of vectors $v$ such that $[v, V'] = 1$.

Let $X \subset V$ be a maximal isotropic subspace of $V$ (thus $X^\perp = X$) and we define $H(X)$ to be the Heisenberg group associated to $X$ (with the symplectic form induced from $V$, i.e. the trivial symplectic form). We write $S_X$ to be the set of $C$-valued functions $f : H(V) \to \mathbb{C}$ such that $f(hxh) = e_h(h_X)f(h)$ for any $h_X \in H(X)$ and $h \in H(V)$ and moreover there exists some open subgroup $X'$ of $X$ such that $f(h(x', 0)) = f(h)$ for any $x' \in X'$. Here $e_X((x, t)) := e_F(t)$. For any subspace $V'$ of $V$, we write $S(V')$ for the space of $C$-valued Schwartz-Bruhat functions on $V'$. There is an action of $H(V)$ on $S_X$ given by

$$\rho(h) f(h') = (\rho(h)f)(h') := f(h^{-1}h'), \quad \forall h \in H(V), f(\cdot) \in S_X.$$

For any maximal isotropic subspace $Y$ of $V$ such that $V = X + Y = X \oplus Y$, we have an isomorphism of vector spaces

$$S_X \cong S(Y), \quad f \mapsto \left(y \mapsto f((y, 0)) \right),$$

whose inverse is given by

$$S(Y) \cong S_X, f' \mapsto \left((x + y, t) \mapsto e_F(t - \langle x, y \rangle/2)f'(y) \right).$$
These isomorphisms induce an action of $H(V)$ on $S(Y)$ from that on $S_X$. Explicitly, it is given by
\[
\rho((x + y, t))f'(y') := e_F(t - (x, y) + y/2)f(y + y').
\]
Note that this action depends on the decomposition $V = X + Y$.

Now for any element $g \in \text{Sp}(V)$, there is an obvious isomorphism of vector spaces $A(g) : S_X \to S_{gX}$, $f \mapsto (h \mapsto f(g^{-1}h))$.

One verifies easily that $\rho(h)A(g)f(h') = A(g)\rho(g^{-1}h)f(h'), \forall g \in \text{Sp}(V), f \in S_X$.

For any two maximal isotropic subspaces $X_1, X_2$ of $V$, one can define an intertwining operator
\[
I_{X_1,X_2} : S_{X_1} \xrightarrow{\sim} S_{X_2}, f \mapsto \left(h \mapsto \int_{X_2/X_12}f((x_2, 0)h)dx_2\right),
\]
where $X_{12} = X_1 \cap X_2$. One verifies that $I_{X_1,X_2}(\rho(h)f) = \rho(h)(I_{X_1,X_2}f)$ for $h \in H(W)$. Similarly, for maximal isotropic decompositions $V = X_1 + Y_1 = X_2 + Y_2$, one can use the isomorphisms $S_{X_i} \simeq S(Y_i)$ for $i = 1, 2$ to define intertwining operators $I(Y_1, Y_2) : S(Y_1) \xrightarrow{\sim} S(Y_2)$.

From this one gets the following commutative diagram
\[
\begin{array}{ccc}
S_X & \xrightarrow{A(g)} & S_{gX} \\
\downarrow I_{X,g^{-1}X} & & \downarrow I_{gX,X} \\
S_{g^{-1}X} & \xrightarrow{A(g)} & S_X \\
\end{array}
\]
and we denote their composition by
\[
\Omega_X(g) : S_X \xrightarrow{\sim} S_X.
\]

Using the theorem of Stone and von Neumann (\cite[Theorem 1.1]{Kud96}), one verifies that this is a projective representation of $\text{Sp}(V)$ on $S_X$, i.e. the map $\Omega_X : \text{Sp}(V) \to \text{PGL}_C(S_X)$ is a group homomorphism. Using the isomorphism between $S_X$ and $S(Y)$, one gets a projective representation of $\omega_Y$ of $\text{Sp}(V)$ on $S(Y)$. The projective representation $\omega_Y$ induces a genuine representation $\widetilde{\omega}_Y$ of the metaplectic group $\widetilde{\text{Sp}}(V)$ on $S(Y)$. This irreducible smooth representation is the Schrödinger model of the Weil representation of $\widetilde{\text{Sp}}(V)$ (\cite[p.29]{MVW87}). Here the metaplectic group can be defined as
\[
\widetilde{\text{Sp}}(V) = \{(g, A_g) \in \text{Sp}(V) \times \text{GL}_C(S(Y)) | A_g\rho(h)A_g^{-1} = \rho(gh), \forall h \in H(V)\}
\]
Note that this representation depends on the maximal polarization of $V = X + Y$. Moreover, the metaplectic group $\widetilde{\text{Sp}}(V)$ can also be viewed as the central extension of $\text{Sp}(V)$ by $\mathbb{C}^\times$. The map $\text{Sp}(V) \to \widetilde{\text{Sp}}(V), g \mapsto (g, \omega_Y(g))$ is a section of the natural projection $\text{pr}: \widetilde{\text{Sp}}(V) \to \text{Sp}(V)$. One can then define a 2-cocycle on $\text{Sp}(V)$ valued in $\mathbb{C}^\times$ by
\[
c_Y(g', g) := \omega_Y(g')\omega_Y(g)\omega_Y(g'g^{-1}).
\]
One can use this cocycle to define a group $\text{Sp}(V) \times_{c_Y} \mathbb{C}^\times$ by setting $(g, z) \cdot (g', z') := (gg', zz'c_Y(g, g'))$.

There is a close relation between the metaplectic group $\widetilde{\text{Sp}}(V)$ and the twofold cover $\text{Mp}(V) := \text{Sp}(V) \times_{r_Y} \mu_2$ of $\text{Sp}(V)$. Here $r_Y$ is Rao’s 2-cocycle on $\text{Sp}(V)$ valued in $\mu_2 \subset \mathbb{C}^\times$ (\cite[Chapter I, Theorem 4.5]{Kud96}). To relate these two cocycles, we define a map $\beta : \text{Sp}(V) \to \mathbb{C}^\times$ as in \cite[Chapter I, Theorem 4.5]{Kud96}, then one can show $c_Y(g, g') = \beta(gg')\beta(g)^{-1}\beta(g')^{-1}r_Y(g, g')$. Using this identity, one can define a map $\text{Sp}(V) \times_{r_Y} \mathbb{C}^\times \to \widetilde{\text{Sp}}(V), (g, z) \mapsto (g, z\beta(g)\omega_Y(g))$. 

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which is an isomorphism. Moreover, it is easy to see \( \text{Sp}(V) \times_{\tau_V} \mathbb{C}^\times = \text{Mp}(V) \times_{\mu_2} \mathbb{C}^\times \) where the RHS is the contracted product. Thus \( \tilde{\text{Sp}}(V) \) is isomorphic to \( \text{Mp}(V) \times_{\mu_2} \mathbb{C}^\times \) via the above isomorphism.

Now we restrict ourselves to the special case \( V = W_4 = V_4 \otimes U \) with the polarization \( W_4 = W_4^++W_4^- \). Recall that we have a homomorphism \( \iota: \text{Sp}(V_4) \times O(U) \rightarrow \text{Sp}(W_4) \) with restrictions \( \iota_V := \iota|_{\text{Sp}(V_4) \times 1} \) and \( \iota_U := \iota|_{1 \times O(U)} \). By [Kud96, Chapter II, Corollary 3.3], the morphism \( \iota_V \) lifts uniquely to a morphism of metaplectic groups \( \tilde{\iota}_V: \text{Sp}(V_4) \rightarrow \tilde{\text{Sp}}(W_4) \) whose restriction to the center \( \mathbb{C}^\times \) sends \( z \) to \( z^{\text{dim}(U)} = z^6 \). More precisely, the map is given by

\[
\text{Sp}(V_4) \times_{\tau_V} \mathbb{C}^\times \rightarrow \text{Sp}(W_4) \times_{\tau_W} \mathbb{C}^\times, \quad (g, z) \mapsto (\iota_V(g), z^6 \mu_U(g)).
\]

Here \( \mu_U(g) \) is defined as in [Kud96, Chapter II, Proposition 3.2] such that \( r_{W_4^-}(\iota_V(g), \iota_V(g')) = r_{V_4^-}(g, g')\mu_U(gg')\mu_U(g)^{-1}\mu_U(g')^{-1} \). One can show that this map is a homomorphism and thus gives the homomorphism \( \tilde{\iota}_V \). Moreover, since \( \text{dim}(U) = 6 \) is even, the restriction of \( \tilde{\iota}_V \) to \( \text{Mp}(V_4) = \text{Sp}(V_4) \times_{\tau_V} \mu_2 \) factors through its quotient \( \text{Sp}(V_4) \). Combining these maps, we get the following homomorphism

\[
\text{Sp}(V_4) \rightarrow \tilde{\text{Sp}}(W_4), \quad g \mapsto \left( \iota_V(g), \beta(\iota_V(g))\mu_U(g)\omega_{W_4^-}(\iota_V(g)) \right)
\]

On the other hand, one can show ([Kud96, p.39]) that \( r_{W_4^-} \) is trivial when restricted to \( O(U) \), thus the morphism \( \iota_U \) lifts to a morphism

\[
O(U) \rightarrow \tilde{\text{Sp}}(W_4), \quad h \mapsto \left( \iota_U(h), \omega_{W_4^-}(\iota_U(h)) \right).
\]

The lift is not unique yet we fix this one in this article. In summary, we get a representation \( \omega_{W_4^-} \) of the reductive dual pair \( \text{Sp}(V_4) \times O(U) \) on \( S(W_4^-) \) through the morphism \( \text{Sp}(V_4) \times O(U) \rightarrow \tilde{\text{Sp}}(W_4) \). We next make explicit this representation for future computations. From now on we will only consider the representation \( \omega_{W_4^-}|_{\text{Sp}(V_4) \times O(U)} \) and for any \( g \in \text{Sp}(V_4) \), resp. \( h \in \text{Sp}(U) \), we write \( \omega_{W_4^-}(g) \), resp. \( \omega_{W_4^-}(h) \) instead of \( \omega_{W_4^-}(\iota_V(g)) \), resp. \( \omega_{W_4^-}(\iota_U(h)) \) in the following.

Write elements in \( V_4 \) in the form \( (x, y) \in V_4^+ + V_4^- \). We view \( g \in \text{Sp}(V_4) \) as a matrix of morphisms

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad g \cdot (x, y) = (ax + by, cx + dy);
\]

where \( a \in \text{Hom}(V_4^+, V_4^+), b \in \text{Hom}(V_4^-, V_4^+), c \in \text{Hom}(V_4^+, V_4^-), d \in \text{Hom}(V_4^-, V_4^-) \).

It is easy to see that the stabilizer in \( \text{Sp}(V_4) \) of the decomposition \( V_4^+ + V_4^- \) is the set of matrices \( m(a) \) with \( a \in \text{Aut}(V_4^+) \) and \( a^\vee \in \text{Aut}(V_4^-) \) determined by \( \langle ax, a^\vee y \rangle = \langle x, y \rangle \) for any \( x \in V_4^+, y \in V_4^- \). Any \( g \in \text{Sp}(V_4) \) such that \( g|_{V_4^+} = \text{Id}_{V_4^+} \) is of the form \( u(b) \) with \( \langle bx, x' \rangle = \langle bx', x \rangle \) for any \( x, x' \in V_4^- \). Note that \( \text{Sp}(V_4) \) is generated by elements of the form \( m(a), u(b) \) and \( J_8 \) the longest Weyl element of \( \text{Sp}(V_4) \) exchanging the fixed basis of \( V_4^+ \) and \( V_4^- \). Under the basis \( B_{V_4} \), one can show that \( a^\vee = a^{-1}, b \) is a symmetric matrix in and the longest Weyl element is \( J_8 \). For any element \( w = (w_1, w_2, \ldots, w_8) \in W_4 \simeq U^8 \), we define the Gram matrix as

\[
(w, w) := (\langle w_i, w_j \rangle_U).
\]

For any \( f \in S(W_4^-) \) and \( y \in W_4^- \) ([Kud96, Chapter II, Proposition 4.3]) one has

\[
\omega_{W_4^-} \left( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \right) f(y) = \gamma_{\text{Weil}} \int_{\text{Ker}(c) \setminus W_4^-} \exp \left( \frac{1}{2} \text{tr}(ay, by) - \text{tr}(by, cx) + \frac{1}{2} \text{tr}(cx, dx) \right) f(ay + cx) dx.
\]
Here $\gamma_{\text{Weil}}$ is the Weil index, which is a complex number of absolute value 1. We will discuss Weil index in more detail in Lemma 3.3. In particular, we have

\begin{align}
(3.6) \quad \omega_{W^4_4}(m(a))f(y) &= \xi(\det a)|\det a|^3 f((a'y), \\
(3.7) \quad \omega_{W^4_4}(u(b))f(y) &= e(\text{tr}(by)/2)f(y), \\
(3.8) \quad \omega_{W^4_4}(J_8)f(y) &= \gamma_{\text{Weil}} \int_{W^4_4} f(x) e_F(\langle J_8x, y \rangle)dx, \\
(3.9) \quad \omega_{W^4_4}(h)f(y) &= f(h^{-1}y), \forall h \in O(U),
\end{align}

Remark 3.3. For $\nu = \infty$, since $\xi$ is an odd character, we have $\xi(\det a)|\det a|^3 = \det a^3$.

The intertwining map $I_{W^4_4,W^4_4}$ between $S_{W^4_4}$ and $S_{W^4_4}$ induces an intertwining map between $S(W^4_4)$ and $S(W^4_4)$ for the Weil representations of $Sp(W_4)$. More explicitly ([Li92, p.182]),

\[ \delta : S(W^4_4) \rightarrow S(W^4_4) \]

\[ \phi \mapsto \left( ((x, y), -(x, y)) \mapsto \int_{W^4_1} \phi(u+y, u-y)e(2\langle u, x \rangle_{W^4_4})du \right). \]

Here in the integral we identify $W^4_4$ with $W^1_4$ by sending $((x', y'), (x', y'))$ to $(x', y')$. For any function $f \in S(W^4_1)$, the Fourier inversion theorem gives

\[ \int_{W^4_1} e(\langle y, x' \rangle)dy \int_{W^4_1} f(x)e(\langle x, y \rangle)dx = f(x'). \]

From this we have the inverse of $\delta$ as follows

\[ \delta^{-1} : S(W^4_4) \rightarrow S(W^4_4) \]

\[ \phi' \mapsto \left( ((y, y') \mapsto \int_{W^4_1} \phi'(x/2, (y - y')/2)e(\langle x, (y + y')/2 \rangle)dx \right). \]

3.2.2. Weil representations for similitude groups. We recall briefly how to extend the Weil representation $\omega_{W^4_4}$ of $Sp(V_4) \times O(U)$ to the similitude group $GSp^+(V_4) \times GSO(U)$. Here $GSp^+(V_4)$ is the subgroup of $GSp(V_4)$ given by

\[ GSp^+(V_4) = \{ g \in GSp(V_4)| \nu(g) \in \nu(GSO(U)) \}. \]

The canonical reference is [Rob96]. See also [GT11, Section 2, Similitude theta correspondences].

Suppose $F = \mathbb{Q}_v$ non-archimedean. We consider the subgroup $R_0 = \{ (g, h) \in R| \nu(g)\nu(h) = 1 \}$ of the group $R = GSp^+(V_4) \times GSO(U)$. We define

\[ \omega_{W^4_4}(g, h)f(y) = |\nu(h)|^{-6}\omega_{W^4_4}(g_1, h)f(y) \]

where $g_1 = g\text{diag}(\nu(g)^{-1}, 1) \in Sp(V_4)$. It is easy to see that the scalar element $(\lambda, \lambda^{-1}) \in R_0$ acts as $\omega_{W^4_4}(\lambda, \lambda^{-1})f = \xi(\lambda)^{-4}f$. Now we consider the compactly induced representation of $R$, $\text{Ind}^{R}_{R_0}(\omega_{W^4_4})$. We will still denote this representation by $\omega_{W^4_4}$ ([Rob96, Sections 2,3]).

3.2.3. Global Weil representations. Let $W_4$ be as above, a vector space over $\mathbb{Q}$. We define the global Weil representation $\omega_{W_4^4}$ of $GSp^+(V_4, \mathbb{A}) \times GSO(U, \mathbb{A})$ on $S(W_4^4(\mathbb{A}))$ as the restricted tensor product

\[ \omega_{W_4^4} := \bigotimes_v \omega_{W_4^4 \otimes \mathbb{Q}_v}. \]

3.3. Theta lift.
3.3.1. Local theta lift. Recall $F = \mathbb{Q}_v$. The local theta lift is usually defined on the level of automorphic representations. Let $\pi^+$ be an irreducible admissible representation of $\text{GSp}^+(V_4)$. We define $N(\pi^+) := \bigcap \lambda \text{Ker}(\lambda)$, where $\lambda$ runs through the space $\text{Hom}_{\text{Sp}(V_4)}(\omega_{W_4^-}, \pi^+)$. Then we set $S(\pi^+) := \omega_{W_4^-} / N(\pi^+)$. The space $S(\pi^+)$ is a representation of $\text{GSp}^+(V_4) \times \text{GSO}(U)$. By [MVW87, Chapitre 2, Lemme III.4], there is a smooth representation $\tilde{\Theta}(\pi^+)$ of $\text{GSO}(U)$ unique up to isomorphisms such that $S(\pi^+) \simeq \pi^+ \otimes \tilde{\Theta}(\pi^+)$. Then the Howe duality conjecture says

**Conjecture 3.4.** For any irreducible admissible representation $\pi^+$ of $\text{GSp}^+(V_4)$, either $\tilde{\Theta}(\pi^+)$ vanishes or it is an admissible representation of $\text{GSO}(U)$ of finite length. In the latter case, there exists a unique $\text{GSO}(U)$-invariant submodule $\tilde{\Theta}'(\pi^+)$ of $\tilde{\Theta}(\pi^+)$ such that

$$\Theta(\pi^+) := \tilde{\Theta}(\pi^+) / \tilde{\Theta}'(\pi^+)$$

is an irreducible representation of $\text{GSO}(U)$. If $\tilde{\Theta}(\pi^+) = 0$, we put $\Theta(\pi^+) = 0$. Moreover, for two irreducible admissible representations $\pi_i^+$ and $\pi_j^+$ of $\text{GSO}(U)$, if $\Theta(\pi_i^+)$ and $\Theta(\pi_j^+)$ are both non-zero and isomorphic, then $\pi_i^+$ is isomorphic to $\pi_j^+$ (i.e. the map $\Theta: \pi^+ \mapsto \Theta(\pi^+)$ is injective on the isomorphic classes of irreducible admissible representations of $\text{GSp}^+(V_4)$ with non-zero images by $\tilde{\Theta}$).

In our situation, this conjecture can be proved using the results of [Mor14, GT11b]. We will discuss this in more detail in Section 5.

3.3.2. Global theta lift. To shorten notations, we shall write $(V_4^+_4) = V_4^+ \otimes U$ and the like. For any $\phi \in \mathcal{S}(W_4^-)$, we can define the associated theta series:

$$\Theta_{\phi}(g, h) = \sum_{w \in W_4^+} (\omega_w (g, h) \phi)(w), \forall (g, h) \in \text{GSp}^+(V_4, \mathbb{A}) \times \text{GSO}(U, \mathbb{A})$$

For any cuspidal automorphic form $f \in \mathcal{A}_{\text{cusp}}(\text{GSp}^+(V_4, \mathbb{A}))$, the theta lift $\Theta_{\phi}(f)$ of $f$ to $\text{GO}(U, \mathbb{A})$ is defined as

$$\Theta_{\phi}(f)(h) = \int_{[\text{GSp}^+(V_4)]_u} f(g) \Theta_{\phi}(g, h) dg$$

where $[\text{GSp}^+(V_4)]_u$ is the subset of $[\text{GSp}(V_4)]$ consisting of elements $g$ such that $\nu(g) = \nu(h)$.

Similarly, for any automorphic form $f' \in \mathcal{A}_{\text{cusp}}(\text{GSO}(U)(\mathbb{A}))$, its theta lift $\overline{\Theta}_{\phi}(f')$ to $\text{GSp}^+(V_4, \mathbb{A})$ is defined as

$$\overline{\Theta}_{\phi}(f')(g) = \int_{[\text{GO}(U)]_u} f'(h) \overline{\Theta}_{\phi}(g, h) dh,$$

where $[\text{GO}(U)]_u$ is the subset of $[\text{GO}(U)]$ consisting of elements $h$ such that $\nu(h) = \nu(g)$.

We can extend $\overline{\Theta}_{\phi}(f')$ to the whole $\text{GSp}(V_4, \mathbb{A})$ as follows

$$\overline{\Theta}_{\phi}(f')(g) = \begin{cases} \overline{\Theta}_{\phi}(f')(g) & g \in \text{GSp}(V_4, \mathbb{Q}) \text{GSp}^+(V_4, \mathbb{A}); \\ 0 & \text{otherwise}. \end{cases}$$

The maps $(\Theta_{\phi}, \overline{\Theta}_{\phi})$ are the theta correspondence between $\text{GSp}^+(V_4)$ and $\text{GSO}(U)$.

Similarly, one can define theta correspondences between $\text{GSp}^+(V_i)$ and $\text{GSO}(U)$ for $i = 1, 2$.

On the level of automorphic representations, the theta lift is defined as follows: let $\pi^+$ be an irreducible admissible automorphic representation of $[\text{GSp}^+(V_4)]$. Then the theta lift $\Theta(\pi^+)$ of $\pi$ to $[\text{GSO}(U)]$ is the automorphic representation of $[\text{GSO}(U)]$ on the space

$$\Theta(\pi^+) := \{ \Theta_{\phi}(f) | \forall \phi \in \mathcal{S}(W_4^-), f \in \pi \}$$

Similarly one can define the theta lift of automorphic representations in the other direction.
3.4. **Siegel-Eisenstein series.** We study Siegel-Eisenstein series and define local zeta integrals and Fourier coefficients.

3.4.1. **Siegel-Eisenstein series.** In this subsection, we will work with the group $G = G_4 = \text{GSp}(V_4)$ and its parabolic subgroup $P^d = P_{V_4^d}$ stabilizing the subspace $V_4^d$ of $V_4$, $M^d = M_{P^d}$ the Levi subgroup of $P^d$ preserving the polarization $V_4^d \oplus V_4$, $N^d = N_{P^d}$ the unipotent radical of $P_{V_4^d}$ and the maximal torus $T^d = T_{P^d}$. The modulus character of $P^d$ is defined as

$$\delta_{P^d}: P^d(\mathbb{A}) \to \mathbb{C}^\times, \text{diag}(1, \nu)m(A)u(B) \mapsto |\det A|^{|\nu|^{-10}}.$$

For any complex number $s \in \mathbb{C}$, we define a character $\xi_s$ of $P^d(\mathbb{A})$ as follows

$$\xi_s: P^d(\mathbb{A}) \to \mathbb{C}^\times, \quad p = \text{diag}(1, \nu)m(A)u(B) \mapsto \xi(\det A)\delta_{P^d}(p)^{s/5} = \xi(\det A)|\det A|^{|s|}\nu^{-2s}.$$

We then define the normalized induction as follows

$$\text{Ind}_{P_d(\mathbb{A})}^{G(\mathbb{A})}(\xi_s) = \{ f: G(\mathbb{A}) \to \mathbb{C}, \text{smooth}|f(pg) = \delta_{P^d}(p)^{1/2}\xi_s(p)f(g), \forall p \in P^d(\mathbb{A}), g \in G(\mathbb{A}) \}.$$

We also have the corresponding local version of inductions for each place $v$ of $\mathbb{Q}$, denoted by $\text{Ind}_{P_d(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\xi_s)$). Then one has the restricted tensor product

$$\text{Ind}_{P_d(\mathbb{A})}^{G(\mathbb{A})}(\xi_s) = \bigotimes_v \text{Ind}_{P_d(\mathbb{Q}_v)}^{G(\mathbb{Q}_v)}(\xi_s).$$

For any section $f(s) \in \text{Ind}_{P_d(\mathbb{A})}^{G(\mathbb{A})}(\xi_s)$, we define the Siegel Eisenstein series associated to $f(s)$ as

$$E(g, f(s)) = \sum_{\gamma \in \text{P}^d(\mathbb{Q}) \backslash G(\mathbb{Q})} f(s)(\gamma g).$$

We can do the same thing for another parabolic subgroup $P^+ = P_{V_4^+}$ of $G$ which stabilizes the subspace $V_4^+$ of $V_4$, the Levi subgroup $M^+$ of $P^+$ preserving the polarization $V_4^+ \oplus V_4^-$ of $V_4$ and the unipotent radical $N^+ = N_{P^+}$ of $P^+$ and the maximal torus $T^+ = T_{P^+}$ of $P^+$. We can thus define the normalized induction $\text{Ind}_{P^+(\mathbb{A})}^{G(\mathbb{A})}(\xi_s)$ and Siegel Eisenstein series $E(g, f_{P^+}(s))$ associated to a section $f_{P^+}(s) \in \text{Ind}_{P^+(\mathbb{A})}^{G(\mathbb{A})}(\xi_s)$.

Under the basis $B_{V_4}$, we define $S \in G_4(\mathbb{Q})$ and its inverse as follows

$$S = \begin{pmatrix} 1 & 0 & 0 & 1/2 \\ -1/2 & 0 & 0 & 1/2 \\ 0 & 1 & 1/2 & 0 \\ 0 & -1/2 & 0 & 0 \end{pmatrix}, \quad S^{-1} = \begin{pmatrix} 1/2 & -1/2 & 0 & 0 \\ 0 & 0 & 1/2 & 1/2 \\ 0 & 0 & 1 & -1 \\ 1/2 & 1/2 & 0 & 0 \end{pmatrix}.$$

Then $S$ sends $V_4^+$ to $V_4^d$ (so that $S^{-1}P^dS = P^+, SW_4^+ = W_4^d$ and $SW_4^- = W_4$). Thus we see that for any section $f_{P^+}(s)$ in $\text{Ind}_{P^+(\mathbb{A})}^{G(\mathbb{A})}(\xi_s)$, the function

$$f_{P^+}(s): G(\mathbb{A}) \to \mathbb{C}, \quad g \mapsto f_{P^+}(s)(S^{-1}g)$$

is a section in $\text{Ind}_{P^+(\mathbb{A})}^{G(\mathbb{A})}(\xi_s)$. Moreover, it is easy to see that $E(g, f_{P^+}(s)) = E(g, f_{P^+}(s))$ for any $g \in G(\mathbb{A})$.

Now fix $\phi_i \in S(W_4^-)$ for $i = 1, 2$. Write $\phi^+ = \phi_1 \otimes \phi_2$ for the tensor product which is an element in $S(W_4^-)$. Recall that $G^+ = \{ g \in G|\nu(g) \in \nu(O(U)) \}$ and we put $P^{++} := P^+ \cap G^+$. We then define a map

$$f_{\phi^+}: G^+(\mathbb{A}) \to \mathbb{C}, \quad g \mapsto (\omega_{W_4^-}(g)\phi^+)(0).$$

One can verify that $f_{\phi^+} \in \text{Ind}_{P^{++}(\mathbb{A})}^{G(\mathbb{A})}(\xi_1)$. We can extend $f_{\phi^+}$ to a section $f_{\phi^+}$ in $\text{Ind}_{P^+(\mathbb{A})}^{G(\mathbb{A})}(\xi_1)$ in a unique way as follows: note that $G = P^+G^+$, for any $g = pg^+ \in P^+G^+$, we define $f_{\phi^+}(g) :=$
\(\xi_{1/2}(p)f_{\phi^+}(g^+)\). One can verify that this is well-defined and thus we get a section \(f_{\phi^+} \in \text{Ind}_{F^+}^G(\xi_{1/2})\) from \(\phi^+\). This procedure applies to any other maximal parabolic subgroup other than \(P^+\) and we will extend the sections from \(G^+\) to \(G\) always in this way without further comment.

We define \(\phi^d = \delta(\phi^+) \in \mathcal{S}(W_{4,d}(A))\). Then the function

\[ f_{\phi^d} : G^+(A) \to \mathbb{C}, \quad g \mapsto \omega_{W_{4,d}}(g)\phi^d(0) \]

is in \(\text{Ind}_{F^+}^G(\xi_{1/2})\).

One has the following:

**Lemma 3.5.** The two sections \(f_{\phi^d}\) and \(f_{\phi^+}\) are related by

\[ f_{\phi^d}(g) = f_{\phi^+}(S^{-1}g), \quad \forall g \in G(A). \]

**Proof.** This follows easily from the definition of the Weil representation \(\omega_{W_4^-}\) of \(\text{Sp}(V_4) \times \text{O}(U)\).

Indeed, recall that in the morphism \(\text{Sp}(V_4) \to \widehat{\text{Sp}}(W_4)\), an element \(g\) is sent to the element \((\iota_{V_4}(g), \beta_{W_4^-}(\iota_{V_4}(g))\mu_{U,W_4^-}(g)\omega_{W_4^-}(\iota_{V_4}(g)))\) and by definition the Weil representation of \(\text{Sp}(V_4)\) on the space \(\mathcal{S}(W_4^-)\) is \(\omega_{W_4^-}(g) = \beta_{W_4^-}(\iota_{V_4}(g))\mu_{U,W_4^-}(g)\omega_{W_4^-}(\iota_{V_4}(g))\). Here we add the subscript \(W_4^-\) to indicate the dependence of the functions \(\beta\) and \(\mu\) on the isotropic subspace \(W_4^-\). Similarly, we can define the Weil representation \(\omega_{W_{4,d}}\) of \(\text{Sp}(V_4)\) on the space \(\mathcal{S}(W_{4,d})\), which is given by \(\omega_{W_{4,d}}(g) = \beta_{W_{4,d}}(\iota_{V_4}(g))\mu_{U,W_{4,d}}(g)\omega_{W_{4,d}}(\iota_{V_4}(g))\).

Now by definition,

\[
\begin{align*}
 f_{\phi^d}(g) &= \omega_{W_{4,d}}(g)\phi^d(0) \\
 &= \beta_{W_{4,d}}(\iota_{V_4}(g))\mu_{U,W_{4,d}}(g)\omega_{W_{4,d}}(\iota_{V_4}(g))\phi^d(0) \\
 &= \beta_{W_{4,d}}(\iota_{V_4}(g))\mu_{U,W_{4,d}}(g)A(g) \circ I(W_{4,d}, g^{-1}W_{4,d}) \circ I(W_{4,d}, W_{4,d})\phi^+(0) \\
 &= \beta_{W_{4}}(\iota_{V_4}(S^{-1}g))\mu_{U,W_{4}}(S^{-1}g)A(S) \circ A(S^{-1}g) \circ I(W_{4}^-, (S^{-1}g)^{-1}W_{4}^-)\phi^+(0) \\
 &= \beta_{W_{4}}(\iota_{V_4}(S^{-1}g))\mu_{U,W_{4}}(S^{-1}g)\omega_{W_{4}}(S^{-1}g)\phi^+(0) \\
 &= \beta_{W_{4}}(\iota_{V_4}(S^{-1}g))\mu_{U,W_{4}}(S^{-1}g)\omega_{W_{4}}(S^{-1}g)\phi^+(0) \\
 &= f_{\phi^+}(S^{-1}g).
\end{align*}
\]

From the third line to the fourth line we used the fact that

\[ (3.10) \quad \beta_{W_{4}}(\iota_{V_4}(S^{-1}g))\mu_{U,W_{4}}(S^{-1}g)\text{Id}_{S(W_{4}^-)} = \beta_{W_{4,d}}(\iota_{V_4}(g))\mu_{U,W_{4,d}}(g)I(g^{-1}SW_{4}^-, W_{4}^-) \circ I(SW_{4}^-, g^{-1}SW_{4}^-) \circ I(W_{4}^-, SW_{4}^-) \]

To show this last identity, we need some preliminary results on the composition of intertwining operators defined in 3.3. Use again the notations in Section 3.2, putting \(V = W_4\) over a local field \(F = \mathbb{Q}_v\) and \(X_i\) maximal isotropic subspaces of \(V\). The composition of intertwining operators \(I_{X_2,X_3} \circ I_{X_1,X_2}\) is not exactly \(I_{X_1,X_3}\), but differs from the latter by a scalar in \(\mathbb{C}^\times\). This scalar is given by the Maslov index and Weil character defined below. We write \(\widehat{W}(F)\) to be the Grothendieck group of isometry classes of quadratic forms over \(F\) and define the Witt group \(W(F)\) as \(W(F) = \widehat{W}(F)/\mathbb{Z}H\) where \(H\) is the standard split hyperbolic plane (of dimension 2). Let \((Q,q)\) be a quadratic space over \(F\) with quadratic form \(q\). Consider the pairing \(e_F \circ q : Q \times Q \to \mathbb{C}^\times\). Let \(d\mu_q\) be a measure on \(V\) self-dual with respect to the pairing \(e_F \circ q\). Choose any Schwartz-Bruhat function \(h \in \mathcal{S}(Q)\) such that its Fourier transform is a positive measure and \(h(0) = 1\). Then the Weil character \(\gamma_{\text{Weil}}(q)\) is defined to
be ([Wei64, §14, Théorème 2], [Li08, Proposition 1.2.13])

\[
\gamma_{\text{Weil}}(q) = \lim_{s \to 0} \int_Q h(sx) e_F(q(x,x)/2) d\mu_q(x).
\]

This gives the Weil character of the Witt group

\[
\gamma_{\text{Weil}} : W(F) \to \mathbb{C}^\times, \quad (Q,q) \mapsto \gamma_{\text{Weil}}(q).
\]

For any maximal isotropic subspaces \(X_1, X_2, X_3\) of \(V\), we define \(K := X_1 \oplus X_2 \oplus X_3\) and give it the following quadratic form \(q_K\): for any \(v, w \in K\),

\[
q_K(v, w) = q_K((v_1, v_2, v_3), (w_1, w_2, w_3)) := \frac{1}{2}(v_1, w_2 - w_3, v_2, w_3 - w_1, v_3, w_1 - w_2).
\]

Then the Maslov index \(\tau(X_1, X_2, X_3)\) of \(X_1, X_2, X_3\) is the equivalence class of the quadratic space \((K, q_K)\) in \(W(F)\) ([Li08, Proposition 2.3.3]). Now one can show that ([Li08 Theorem 3.5.1])

\[
I_{X_3, X_1} \circ I_{X_2, X_1} \circ I_{X_1, X_2} = \gamma_{\text{Weil}}(\tau(X_1, X_2, X_3)) \cdot \text{Id}_{S_{X_1}} : S_{X_1} \to S_{X_1}.
\]

Using the fact that \(I_{X_1, X_2} \circ I_{X_2, X_1} = \text{Id} ([Li08 Corollary 3.4.3])\), one sees that

\[
I_{X_2, X_1} \circ I_{X_1, X_2} = \gamma_{\text{Weil}}(\tau(X_1, X_2, X_3)) \cdot I_{X_1, X_3} : S_{X_1} \to S_{X_3}.
\]

Fix a maximal isotropic subspace \(Y\) of \(V\). One can show that for any \(g_1, g_2 \in \text{Sp}(V)\), \(c_\gamma(g_1, g_2) = \gamma_{\text{Weil}}(\tau(Y, g_1 Y, g_1 g_2 Y))\) (cf. [3.3] and [Kud96 Chapter I, Theorem 3.1]).

Now let’s return to our problem and set \(X_1 = W_4^+, X_2 = W_4^d\) and \(X_3 = g^{-1} W_4^d\). One gets

\[
I_{W_4^d, g^{-1} W_4^d} \circ I_{W_4^d, W_4^d} = \gamma_{\text{Weil}}(\tau(W_4^+, W_4^d, g^{-1} W_4^d)) \cdot I_{W_4^+, g^{-1} W_4^d}.
\]

Let’s write \(\gamma(g) = \gamma_{\text{Weil}}(\tau(W_4^+, W_4^d, g^{-1} W_4^d))\).

By the basic properties of the Maslov index \(\tau ([Li08 §3.1 and 4.3.1])\), one has

\[
\gamma_{\text{Weil}}(\tau(W_4^d, S^{-1} W_4^d, g^{-1} W_4^d)) = \gamma_{\text{Weil}}(\tau(W_4^d, S^{-1} W_4^d, g^{-1} W_4^d)).
\]

On the other hand, \(c_{W_4^d, S} = 2\)-cocycle, therefore

\[
c_{W_4^d, S} = c_{W_4^d, S} = c_{W_4^d, S} = c_{W_4^d, S}.
\]

By definition and [Li08 Corollary 3.4.3], one has \(c_{W_4^d, S} = 1 = c_{W_4^d, S}\).

Moreover, by definition one sees that

\[
\beta_{W_4^d}(\nu_4(S^{-1})) = \beta_{W_4^d}(\nu_4(S^{-1})), \quad \mu_{U, W_4^d}(S^{-1}) = \mu_{U, W_4^d}(S^{-1}).
\]

Therefore [3.10] is equivalent to

\[
\beta_{W_4^d}(\nu_4(S^{-1})) \mu_{U, W_4^d}(g S^{-1}) = \beta_{W_4^d}(\nu_4(g)) \mu_{U, W_4^d}(g) c_{W_4^d, S}.
\]

By the morphism \(\text{Sp}(V_4) \to C_p\), this is equivalent to

\[
\beta_{W_4^d}(\nu_4(S^{-1})) \mu_{U, W_4^d}(\nu_4(S^{-1})) = 1.
\]

We can show this in the following way. First we compute the LHS of the above formula for each local case \(F = \mathbb{Q}_v\). Using the Bruhat decomposition for parabolic subgroup \(P_{4,d}\) of \(\text{Sp}(V_4)\), we see that (using the notation as in [Kud96 p.19]) \(j(S^{-1}) = 2\) and \(j(\nu_4(S^{-1})) = 2 \times 6 = 12\). In the same way, \(x(\nu_4(S^{-1})) = x(S^{-1}) = 1 \mod(F^\times)^2\). Thus we see

\[
\beta_{W_4^d}(\nu_4(S^{-1})) = \gamma(\eta)^{-12} = \gamma(-1, \eta)^2 = (-1, -1)_v
\]

where \((\cdot, \cdot)_v\) is the Hilbert symbol in \(F = \mathbb{Q}_v\).
On the other hand, by [Kud96] p.35,
\[ \mu_{U,W,t,d}(S^{-1}) = (\det(U), x(t_{V,S}(S^{-1})))_v \gamma(\det(U), \eta)^{-2} = (\det(U), -1)_v. \]

Taking the product of the above two expressions and using the product formula for Hilbert symbols, one gets the desired identity for \( g \in \sp(V_A)(\A). \) For \( g \in \gsp(V_4)(\A) \backslash \sp(V_4)(\A), \) one can use the definition of the extension of \( f_{\phi^+} \) from \( \sp(V_4)(\A) \) to \( \gsp(V_4)(\A) \) to finish the proof.

For each place \( v \) of \( \Q, \) we define local sections \( \tilde{f}_{\phi^+,v} \in \ind_{pA}^{G_1}(\xi_{1/2}) \) as \( \tilde{f}_{\phi^+,v}(g) := f_{\phi^+,v}(S^{-1}gS). \) Then we put \( \tilde{f}_{\phi^+} := \otimes_v \tilde{f}_{\phi^+,v} \in \ind_{pA}^{G_1}(\xi_{1/2}). \) From the above lemma, we get

**Corollary 3.6.** Let \( f_{\phi^+} \in \ind_{pA}^{G_1}(\xi_{1/2}) \) be the section defined by \( \phi^+ \) as above for \( ? = d, +, \) then we have the following identity

\[ E(g, f_{\phi^+}) = E(g, \tilde{f}_{\phi^+}) = E(g, f_{\phi^+}) \]

**Remark 3.7.** In the following we will use \( f_{\phi^+} \) to compute the Fourier coefficients of the Eisenstein series \( E(\cdot, f_{\phi^+}) \) while we will use \( \tilde{f}_{\phi^+} \) to compute the local zeta integrals given in the next subsection.

### 3.4.2. Zeta integrals.

Let \( \pi = \otimes_v \pi_v \) be an irreducible automorphic representation on \( G_1(\A), \pi^\vee \) its contragredient, which is isomorphic to the complex conjugate \( \overline{\pi}. \) We choose an irreducible \( G_1(\A)-\)constituent \( \pi^1 \) of \( \pi \) that occurs in the space of automorphic forms on \( G_1(\A). \) Similarly for \( \pi^\vee,1 := (\pi^\vee)^1. \)

We assume that \( \pi^1 \) contains the spherical vectors for \( \hat{\Gamma}. \) We will see that the standard \( L \)-function does not depend on the choice \( \pi^1 \) in the decomposition of \( \pi|_{G_1(\A)} \). Let \( S_\pi \) be the places of \( \Q \) dividing \( N. \) We fix non-zero unramified vectors \( \varphi_v,0 \in \pi_v \) and \( \varphi_v,\ 

Similarly for \( \pi^\vee,1:\)
\[ \pi^\vee,1 \simeq \pi^\vee,1 \otimes \pi^\vee,1 \simeq \pi^\vee,1 \otimes (\pi^\vee)^1 \otimes \pi^\vee,1 \otimes \pi^\vee,1. \]

Fix factorizable vectors \( \varphi = \otimes_v \varphi_v \in (\pi^1)^{\hat{\Gamma}} \) and \( \varphi^\vee = \otimes_v \varphi_v^\vee \in (\pi^\vee)^{\hat{\Gamma}}. \) We think of \( \varphi, \) resp. \( \varphi^\vee, \) as an automorphic form on \( G_1(\A), \) resp. \( G_2(\A) \) as follows: for any \( g = g_1g_2 \in G_1(\Q)G_1(\A), \) we set \( \varphi(g) \) to be \( \varphi(g_2), \) while for \( g \in G_1(\A) \backslash G_1(\Q)G_1(\A), \) we set \( \varphi(g) = 0. \) Similarly for \( \varphi^\vee. \)

Moreover, we assume that \( \varphi_v = \varphi_v,0 \) and \( \varphi_v^\vee = \varphi_v^\vee,0 \) for any \( v \notin S_\pi. \)

As for the places \( v|p\infty, \) we will specify \( \varphi_v \) and \( \varphi_v^\vee \) later on.

Let \( Z_1 \) be the center of \( G_1. \) We fix a global Haar measure \( dg \) on \( G_1(\A) \) invariant under left translation of \( G_1(\Q). \) Then we define
\[ \langle \varphi, \varphi^\vee \rangle := \int_{Z_1(\Q)G_1(\A)} \varphi(g)\varphi^\vee(g)dg. \]

This pairing between \( \pi \) and \( \pi^\vee \) decomposes into a product of local pairings \( \langle \cdot, \cdot \rangle_v : \pi_v \otimes \pi_v^\vee \to \C \) that are \( G_1(\Q_v)-\)invariant and \( \langle \varphi_v, \varphi_v^\vee \rangle_v = 1. \)

Now for any section \( f(s) \in \ind_{pA}^{G_1}(\xi_s), \) we define the global zeta integral for \( f(s), \varphi \) and \( \varphi^\vee \) as follows
\[ Z(\varphi, \varphi^\vee, f(s)) := \int_{Z_3(\A)G_3(\Q)G_3(\A)} E((g_1, g_2), f(s))\xi_s^{-1}(\det g_2)\varphi(g_1)\varphi^\vee(g_2)dg_1dg_2. \]

One can show

**Lemma 3.8.** We have
\[ (3.11) \]
\[ Z(\varphi, \varphi^\vee, f(s)) = \int_{G_1(\A)} f(s)((g, 1))\langle \pi(g)\varphi, \varphi^\vee \rangle dg. \]
Proof. The proof is almost the same as [Har93, pp.702-703]. Recall $P_4^d$ is the stabilizer in $G_4$ of $V_4^d$. We put $P_4^{d,1} = P_4^d \cap G_4^1$. Then $P_4^{d,1}\backslash G_4^1 \simeq P_4^d \backslash G_4$. As in [GPSR87], $G_3$ acts by translation on the right on the flag variety $P_4^d \backslash G_4$. The orbits of this action are the same as the orbits of the action $G_1^1 \times G_2^1$ on the flag variety $P_4^{d,1}\backslash G_4^1$. So we see that the orbit $P_4^{d,1} \cdot 1 \cdot G_3$ is the main orbit, as defined in [GPSR87, p.2] and the other orbits are all negligible. We put also $G_3, \gamma(Q) := \gamma_1^{-1}P_4^{d,1}(Q) \gamma$, in particular one has $G_3,1(Q) = G_3^1(Q)$. Therefore we can unfold the global zeta integral in the lemma as follows
\[
Z(\varphi, \varphi^\vee, f(s)) = \int_{Z_3(P) G_3(Q) \backslash G_3} f(s)(\gamma(g_1, g_2)) \xi_s^{-1}(\det g_2) \varphi(g_1) \varphi^\vee(g_2) dg_1 dg_2
\]
\[
= \int_{G_1^1} \sum_{\gamma \in G_1^1} f(s)(\gamma(g_1, g_2)) \xi_s^{-1}(\det g_2) \varphi(g_1) \varphi^\vee(g_2) dg_1 dg_2.
\]

In the last identity we used the fact that only the main orbit $P_4^{d,1}(Q) \cdot 1 \cdot G_3(Q)$ contributes to the integral. Write $G_1^1$ for the image of the map $G_1 \rightarrow G_3$ sending $g$ to $(g, g)$. Then we have an isomorphism
\[
G_3 = \{(g_2, g_2)(g_2^{-1}, 1)| \varphi(g_1) = \nu(g_2)\} = G_1^d G_1^1 \simeq G_1 \times G_1^1.
\]

Note that under the above isomorphism, the subgroup $G_1^1$, resp., $Z_3$ of $G_3$ is sent to the subgroup $1 \times G_1^1$, resp., $1 \times Z_1$ of $G_1 \times G_1^1$. Since $G_1^d \subset P_4^d$, we get
\[
f(s)((g_2, g_2)(g, 1)) \xi_s^{-1}(\det g_2) = f(s)((g, 1)) \xi_s(\det g_2) \xi_s^{-1}(\det g_2) = f(s)((g, 1)).
\]

The above zeta integral becomes
\[
Z(\varphi, \varphi^\vee, f(s)) = \int_{G_1^1} f(s)((g, 1)) \varphi(g_2 g) \varphi^\vee(g_2) dg_2 dg
\]
\[
= \int_{G_1^1} f(s)((g, 1)) dg \int_{G_1^1} \varphi(g_2 g) \varphi^\vee(g_2) dg_2
\]
\[
= \int_{G_1^1} f(s)((g, 1)) \langle \pi(g) \varphi, \varphi^\vee \rangle dg,
\]
which is the desired integral.
\]
3.4.3. Fourier coefficients. Next we define the Fourier coefficients of the Eisenstein series: as above we fix a section \( f(s) \in \text{Ind}^G_{P_4^+}(\xi_s) \).

**Definition 3.10.** For each symmetric matrix \( \beta \in \text{Sym}_{4x4}(\mathbb{Q}) \), the \( \beta \)-th Fourier coefficient of the Eisenstein series \( E(\cdot, f^d(s)) = E(\cdot, f(s)) \) is defined as

\[
E_\beta(g, f(s)) := \int_{[\text{Sym}_{4x4}]} E(u(x)g, f(s))e(-\text{tr} \beta x)dx.
\]

If \( f(s) = \otimes'_v f_v(s) \) is factorizable, then the \( \beta \)-th local Fourier coefficient of \( E(g, f(s)) \) at \( v \) is

\[
(3.13) \quad E_{\beta,v}(g, f(s)) = E_{\beta,v}(g_v, f_v(s)) := \int_{\text{Sym}_{4x4}(\mathbb{A})} f_v(s)(J_gu(x)g_v)e_v(-\text{tr} \beta x_v)d_u x_v.
\]

We call the coset \( P_4^+J_8P_4^+ = P_4^+J_8N_4^+ \) the big cell of \( G_4 \). Here \( N_4^+ \) is the unipotent radical of \( P_4^+ \).

One can show

**Lemma 3.11.** Fix a section \( f(s) \in \text{Ind}^G_{P_4^+}(\xi_s) \) such that there is a finite place \( v_0 \) of \( \mathbb{Q} \) at which in the natural projection \( G_4(\mathbb{A}) \rightarrow G_4(\mathbb{Q}_{v_0}) \), the image of the support of \( f(s) \) in \( G_4(\mathbb{A}) \) is contained in the big cell \( P_4^+(\mathbb{Q}_{v_0})J_8P_4^+(\mathbb{Q}_{v_0}) \). Then for any \( g \in P_4^+(\mathbb{A}) \), one has

\[
E_\beta(g, f(s)) = \int_{\text{Sym}_{4x4}(\mathbb{A})} f(s)(J_gu(x)g)e(-\text{tr} \beta x)dx.
\]

If moreover \( f(s) = \otimes_v f_v(s) \) is factorizable, then

\[
(3.14) \quad E_\beta(g, f(s)) = \prod_v E_{\beta,v}(g_v, f_v(s)).
\]

**Proof.** The proof is essentially contained in [Shi97, Section 18.9]. For any \( g = (g_v)_v \in P_4^+(\mathbb{A}) \), \( f(s)(\gamma g) \neq 0 \) implies that \( \gamma g_{v_0} \in P_4(\mathbb{Q}_{v_0})J_8P_4(\mathbb{Q}_{v_0}) \), thus \( \gamma \in P_4^+(\mathbb{Q})J_8N_4^+(\mathbb{Q}) \). Therefore we have

\[
E(g, f(s)) = \sum_{\gamma \in J_8N_4^+(\mathbb{Q})} f(s)(\gamma g) = \sum_{n \in N_4^+(\mathbb{Q})} f(s)(J_8n g).
\]

From this, we get the \( \beta \)-th Fourier coefficient of \( E(g, f(s)) \) as

\[
E_\beta(g, f(s)) = \int_{[N_4^+]} \sum_{n \in N_4^+(\mathbb{Q})} f(s)(J_8n' g)e(-\text{tr} u(\beta)^t n')dn'
= \int_{N_4^+(\mathbb{A})} f(s)(J_8n' g)e(-\text{tr} u(\beta)^t n')dn',
\]

which finishes the proof of the first part. The second part follows easily from the first part. \qed

3.5. Rallis inner product formula.

3.5.1. Siegel-Weil formula. Now we can state one of the main ingredients of this article: the Siegel-Weil formula. We fix a section \( \phi^+ \in \text{S}(W_4^-(\mathbb{A})) \), \( \Theta_{\phi^+}(g, h) \) the theta series associated to \( \phi^+ \), \( f_{\phi^+} \in \text{Ind}^G_{P_4^+}(\xi_{1/2}) \) the Siegel-Weil section associated to \( \phi^+ \), \( E(g, f_{\phi^+}) \) the Eisenstein series defined by \( f_{\phi^+} \). One can add a complex variable to \( f_{\phi^+} \) as follows:

\[
(3.15) \quad f_{\phi^+}(s)(g) = \omega_{W_4^-}(g)\phi^+(0)|m(g)|^{s-1/2}
\]
Then $f_{\phi^+} = f_{\phi^+}(1/2)$. Moreover one can verify that $f_{\phi^+}(s) \in \text{Ind}_{P_1^+}^{G_4(A)}(\xi_s)$. In the same manner one can define the Eisenstein series $E(g, f_{\phi^+}(s))$ associated to $f_{\phi^+}(s)$. We denote by $[\text{GSO}(U)]_g$ the subset of $[\text{GSO}(U)]$ consisting of elements $h$ such that $\nu(h) = \nu(g)$. One has ([KR88 Main Theorem])

**Theorem 3.12.** Suppose that $\phi^+$ is $K$-finite where $K$ is a maximal compact subgroup of $G_4(A)$. The Eisenstein series $E(g, f_{\phi^+}(s))$ is holomorphic at $s = 1/2$. Moreover for any $g \in G_4^1(A)$,

$$
\int_{[\text{GSO}(U)]_g} \Theta_{\phi^+}(g, h) dh = E(g, f_{\phi^+}).
$$

**Proof.** In [KR88], only the case of isometry groups (i.e., $g \in G_4^1(A)$) is treated. The holomorphicity of the Eisenstein series at $s = 1/2$ follows from the isometry group case. The second part of the theorem also follows from this special case, as follows. For any $g \in G_4^1(A)$, choose any $h_g \in \text{GSO}(U, A)$ such that $\nu(h_g) = \nu(g)$. We then define a new section $\phi_g^+ \in \mathcal{S}(W_4^-(A))$ by $\phi_g^+(x) := \omega_{W_4^-}(h_g) \phi^+(x) = \phi^+(h_g^{-1}x)$ for $x \in W_4^-(A)$. Recall that we write $g_1 := g \text{diag}(\nu(g)^{-1}, 1)$. Then we see that

$$
\int_{[\text{SO}(U)]} \Theta_{\phi^+}(g, hh_g) dh = \int_{[\text{SO}(U)]} dh \sum_{x \in W_4^-(Q)} \omega_{W_4^-}(g, hh_g) \phi^+(x) = |\nu(g)|^{-6} \int_{[\text{SO}(U)]} dh \sum_{x \in W_4^-(Q)} \omega_{W_4^-}(g_1, h) \phi_g^+(x) = |\nu(g)|^{-6} \int_{[\text{SO}(U)]} \Theta_{\phi_g^+}(g_1, h) dh.
$$

This last integral is, by Siegel-Weil formula for isometry groups, equal to $E(g_1, f_{\phi_g^+})$ since $g_1 \in G_4^1(A)$. Moreover, it is easy to see that $E(g_1, f_{\phi_g^+}) = E(g_1, f_{\phi^+})$. Combining with the factor $|\nu(g)|^{-6}$, we see that the Siegel-Weil formula holds for $g \in G_4^1(A)$. \hfill \Box

### 3.5.2. Doubling method.

The definite reference for the doubling method is [GPSR87]. For the case of similitude groups, one can consult [Li90, Har93]. Let $f(s) = \otimes_v f(s)_v \in \text{Ind}_P^{G_4(A)}(s)$ be a factorizable section and $E(g, f(s))$ be the Eisenstein series associated to $f(s)$. Let $\pi$ be a cuspidal automorphic representation of $G(A)$ and $\varphi = \otimes_v \varphi_v$ a factorizable element in $\pi$. Let $\pi^\vee$ be the contragredient of $\pi$ and $\varphi^\vee = \otimes_v \varphi_v^\vee$ a factorizable element in $\pi^\vee$. Recall that $S_\pi$ is a finite set of places of $\mathbb{Q}$ containing $\infty$ and the places where $\pi$ is ramified, or $f(s)_v$ is not a spherical section, or $\varphi_v$ or $\varphi_v^\vee$ is a not spherical vector.

The doubling method says ([Li90 Section 3], [Har93, Section 6.3])

**Theorem 3.13.** We have the following decomposition

$$
Z(\varphi, \varphi^\vee, f(s)) = \prod_{v \in S_\pi} Z_v(\varphi_v, \varphi_v^\vee, f_v(s)) \times L^{S_\pi}(s + 1/2, \text{St}(\pi) \otimes \xi) \times \langle \varphi, \varphi^\vee \rangle.
$$

Here $L^{S_\pi}(s, \text{St}(\pi) \otimes \xi)$ is the partial standard $L$-function of $\pi$ twisted by the character $\xi$.

### 3.5.3. Rallis inner product formula.

The combination of the Siegel-Weil formula and the doubling method gives the Rallis inner product formula, which relates Petersson inner product of an automorphic form to special $L$-value of the automorphic form.

Let $\phi_1 = \otimes_v \phi_{1,v} \in \mathcal{S}(W_1^-(A))$ and $\phi_2 = \otimes_v \phi_{2,v} \in \mathcal{S}(W_2^-(A))$ be two factorizable sections. As above, we write $\phi^+ = \phi_1 \otimes \phi_2 \in \mathcal{S}(W_4^-(A))$ for their tensor product, $f_{\phi^+} \in \text{Ind}_{P_1^+}^{G_4(A)}(\xi_1)$, $\phi^d \in \mathcal{S}(W_4^d(A))$, $f_{\phi^d} \in \text{Ind}_{P_1^+}^{G_4(A)}(\xi_{1/2})$ as above. Let $\pi$ be an anti-holomorphic cuspidal automorphic representation of $G_4(A)$ and $\varphi = \otimes_v \varphi_v \in \pi$ a factorizable element. Similarly, $\varphi^\vee = \otimes_v \varphi_v^\vee \in \pi^\vee$. Then the Rallis inner product formula states
Theorem 3.14. Define the $\mathbb{C}$-bilinear inner product
\[
\langle \Theta_{\phi_1}(\varphi), \Theta_{\phi_2}(\varphi^\vee) \rangle = \int_{[O(U)]} \Theta_{\phi_1}(\varphi)(h)\Theta_{\phi_2}(\varphi^\vee)(h)dh,
\]
then one has
\[
\langle \Theta_{\phi_1}(\varphi), \Theta_{\phi_2}(\varphi^\vee) \rangle = \prod_{v \in S_e} Z_v(\varphi_v, \varphi_v^\vee, f_v(s)) \times L_s^* (s + 1/2, \text{St}(\pi) \otimes \xi) \times \langle \varphi, \varphi^\vee \rangle.
\]

4. Choice of local sections

In this section, we will choose the local sections in the factorizations $\phi_i = \otimes_v \phi_{i,v} \in \mathcal{S}(W^-_i(\mathbb{A}))$ for $i = 1, 2$. Let $\kappa \in \text{Hom}_{\text{cont}}(T^1_2(\mathbb{Z}_p), \overline{\mathbb{Q}}^\times_p)$ be an arithmetic point, i.e., $\kappa$ is a product of an algebraic character $\kappa_{\text{alg}} = k \in \mathbb{Z}^2$ and a finite order character $\kappa_p = (\kappa_1, \kappa_2)$. We say that $\kappa$ is admissible if $k_1 \geq k_2 \geq 3$. Recall that we have four PEL moduli problems $P_i$ ($i = 1, \cdots, 4$) and to each problem we have defined similitude symplectic groups $G = G_1, G_2, G_3$ and $G_4$ and their isometry subgroups $G_i$ ($i = 1, \cdots, 4$).

4.1. Archimedean place. First consider the archimedean place $v = \infty$.

4.1.1. Choice of sections. We write $K_{G_i, \infty}$ the maximal compact subgroup of $G_i(\mathbb{R})$. Note that for an irreducible cuspidal automorphic representation $\pi$ of $G(\mathbb{A})$ whose archimedean component $\pi_\infty$ is a holomorphic discrete series of lowest $K_{G, \infty}$-type of highest admissible weight $\kappa$, the standard $L$-function $L(s, \text{St}(\pi) \otimes \xi)$ is critical at $s = 1$ since $\xi (-1) = -1$ ([BS00 Appendix]).

We define an automorphy factor for any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_4(\mathbb{R})$ as follows: recall that $\mu(g, i) = C_i + D_i$. We set $j(g, i) = \det(\mu(g, i))$, and $j(g, i) = j(g, i) \nu(g)^{-2}$. Using this, we then define a function on $G_4(\mathbb{R})$ as follows:
\[
f^+_\infty(\xi_1/2)(g) := j(g, i)^{-3}.
\]
It is easy to show that $f^+_\infty(\xi_1/2) \in \text{Ind}_{P^+(\mathbb{R})}^{G_4(\mathbb{R})} (\xi_1/2)$ and moreover $f^+_\infty(\xi_1/2)(\lambda \cdot 1_8) = 1$ for any $\lambda \in \mathbb{R}^\times$. We consider a section $\phi^+_\infty \in \mathcal{S}(W^-_4(\mathbb{R}))$ defined as follows: for any $((w_1, w_2)) \in W^-_4(\mathbb{R}) \times W^-_2(\mathbb{R}) = W^-_4(\mathbb{R})$, $\phi^+_{\infty}((w_1, w_2)) = \det(\eta_v)^{\dim V_i} e_\infty (i((w_1, w_2)), \tilde{J}_8((w_1, w_2)))/(w_4/2)$
\]
where $\tilde{J}_8((e_i^\pm, 0)) = \mp(e_i^\pm, 0)$ and $\tilde{J}_8((0, e_i^\pm)) = \mp(0, e_i^\mp)$ for $i = 1, 2$. Under the basis $\mathcal{B}_4 = \mathcal{B}_4 \times \mathcal{B}_U$ of $W_4$, $\tilde{J}_8$ is of the form $J_8 \otimes 1_6$. Let $f^+_{\phi^+_{\infty}} \in \text{Ind}_{P^+(\mathbb{R})}^{G_4(\mathbb{R})} (\xi_1/2)$ be the Siegel section associate to $\phi^+_{\infty}$ via Weil representation. We have the following formula

Lemma 4.1. For any $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_4^1(\mathbb{R})$, the section $f^+_{\phi^+_{\infty}} \phi^+_{\infty}(g) = \det(C_i + D_i)^{-3}$.

Proof. It suffices to prove the lemma for the open dense subset of $G_4^1(\mathbb{R})$ consisting of matrices $g$ with $\det(C) \neq 0$ (using the Bruhat decomposition for $G_4^1(\mathbb{R})$). Moreover, each such matrix $g$ is of the form $m(A')u(B')J_8u(B)$). First, for any $p = m(A')u(B') \in P^+(\mathbb{R})$, $g \in G_4^1(\mathbb{R})$, by Remark 3.3 one has the following $f^+_{\phi^+_{\infty}}(pg) = \omega_{w_4^+}(p)\omega_{w_4^+}(g)\phi^+_{\infty}(0) = \det(A)^3 \phi^+_{\infty}(g)$. Thus it suffices to deal with matrices of the form $g = g_0g_2$ where $g_2 = u(-B')$. Note that $\omega_{W_4^+}(g_2)\phi^+_{\infty}(w) = e_\infty (\langle w, g_2^{-1}w \rangle w_4/2)\phi^+_{\infty}(w)$ for any
$w \in W_4^-(\mathbb{R})$. Thus we get (note Remark 3.2)

$$
\omega_{W_4^+}(g)\phi_{\infty}^+(0) = \int_{W_4^-}(\mathbb{R}) e_\infty(\langle w, g_2^{-1}w \rangle w/2)\phi_{\infty}^+(w) d\mu w_4
$$

$$
= \text{det}(\eta_{W_4^+})^{\dim V_4^+} \int \exp(2i\pi(\langle w, g_2^{-1}w \rangle /2 + i\langle w, g_0w \rangle)) dw.
$$

Now view elements $w$ as matrices in $M_{4\times 6}(\mathbb{R})$, then $\langle w, g_2^{-1}w \rangle + i\langle w, g_0w \rangle = \text{tr}(w^t(D' + i)w\eta_{W_4^-})$. Write $w$ in column vectors $w = (w_1 \ w_2 \ w_3 \ w_4 \ w_5 \ w_6)$, then we have $\text{tr}(w^t(i + D')w\eta_{W_4^-}) = \sum_{j=1}^6(\eta_{W_4^-})_{j,j}w_j(i + D')w_j$. Now a simple calculation of Gaussian type integrals gives

$$
\omega_{W_4^+}(g)\phi_{\infty}^+(0) = \text{det}(\eta_{W_4^+})^{\dim V_4^+} \cdot \text{det}(\eta_{W_4^-})^{-\dim V_4^-} \text{det}(i + D')^{-3} = \text{det}(i + D')^{-3}
$$

which concludes the proof. \hfill \Box

**Corollary 4.2.** We have the identity $f_{\phi_{\infty}^+} = f_{\phi_{\infty}^+}^+$.

We write $D_k$ for the holomorphic discrete series $(\mathfrak{g}_{G_1}, K_{G,\infty})$-module whose lowest $K_{G,\infty}$-type is of highest weight $k$. We then write $D_k(k)$ to be the lowest $K_{G,\infty}$-type in $D_k$. We denote by $D_{k}^+$ the contragredient of $D_k$ and $D_{k}(-k)$ its highest $K_{G,\infty}$-type. Recall in Notation 4(6), we fix a basis $\hat{\mu}_{i,j}$ $(1 \leq i \leq j \leq 4)$ of $\mathfrak{g}_{G_1}^+$. We set $\hat{\mu}_{i,j}^+ := \hat{\mu}_{i,j}$ for $i > j$ and denote by $(\hat{\mu}_{i,j}^+)$ the $4 \times 4$-matrix and in the form of $2 \times 2$-blocks by $\begin{pmatrix} \hat{\mu}_{i,j}^+ & \hat{\mu}_{i,j}^+ \ \hat{\mu}_{i,j}^+ & \hat{\mu}_{i,j}^+ \end{pmatrix}$. Write then $p_{i}^{+1}$ for the subalgebra of $\mathfrak{g}_{G_1}^+$ generated by the elements in $\hat{\mu}_{i}$ for $i = 0, 1, 2$. Denote by $U(\mathfrak{g}_{G_1}^+) \cdot f_{\phi_{\infty}^+}$ the sub-$(\mathfrak{g}_{G_1}^+, K_{G,1,\infty})$-module of the principle discrete series $I_{\nu^2(\mathbb{R})}(1/2, 1)$ generated by $f_{\phi_{\infty}^+}$. For an $n \times n$-matrix $M$, we write $\det_i(M)$ for the determinant of the $i \times i$-minor of $M$ (upper-left $i \times i$-block). Given a dominant weight $k \in \mathbb{Z}^3$, we define the following differential operator

$$
D_k := \det_1 \left( \frac{1}{4i\pi} \hat{\mu}_{i,j}^+ \right)^{k_1-k_2} \det_2 \left( \frac{1}{4i\pi} \hat{\mu}_{i,j}^+ \right)^{k_2-k_3}
$$

and then we put

$$
f_{k}^{+} := D_k f_{\phi_{\infty}^+}.
$$

We then define sections in $S(W_4^-) \mathbb{R})$ as follows:

$$
\phi_{1,\infty}(w_1^-) = e_\infty(i\langle w_1^- , g_0w_1^- \rangle), \quad \phi_{2,\infty}(w_2^-) = e_\infty(i\langle w_2^- , g_0w_2^- \rangle)
$$

(4.1)

$$
\phi_{k,\infty}^+(w_4^-) := D_k^+ \phi_{\infty}^+(w_4^-).
$$

(4.2)

Here the action of the Lie algebra $\mathfrak{g}_{G_1}$ on $S(W_4^-)^- \mathbb{R})$ is induced from the Weil representation of $G_1^+(\mathbb{R})$ on $S(W_4^-)^- \mathbb{R})$. We deduce from Corollary 4.2 that the section $\phi_{k,\infty}^+ = S(W_4^-)^- \mathbb{R})$ gives rise to $f_{k,\infty}^+$.

We next make explicit the action of the differential operator $D_k$ on the space $S(W_4^-)^- \mathbb{R})$. Let $\hat{\mu}_{i,j}^+$ be an entry in $\hat{\mu}_{i,j}^+$ $(i, j = 1, 2)$. Recall that $\mu_{i,j} = E_{i,j} + E_{j,i} + E_{i,j} + E_{j,i}$ and $\hat{\mu}_{i,j}^+ = c\mu_{i,j}^+ c^{-1}$. We write $\mu_{i,j}^+$ in $4 \times 4$-blocks as $\begin{pmatrix} 0 & B_{i,j} \\ 0 & 0 \end{pmatrix}$. For the variable $w_4^- = (w_{i,j}) \in W_4^- \mathbb{R})$ (viewed as a $4 \times 6$-matrix), for each $f \in S(W_4^-)^- \mathbb{R})$, we define $\frac{\partial f}{\partial w_{i,j}}$ to be the $4 \times 6$-matrix $(\frac{Df}{\partial w_{i,j}})$. We define a map from $S(W_4^-)^- \mathbb{R})$ to $M_{4\times 6}(S(W_4^-)^- \mathbb{R}))$ as $Df(w_4^-) = (\sqrt{2\pi}w_4^- \sqrt{\eta} - \sqrt{2\pi}w_4^- \sqrt{\eta}^{-1})f(w_4^-)$ where recall that $\eta = \text{diag}(N^2/2, N^2/2, N^2/2, N^2/2, N/N_1, N_1)$. We define $D^t$ to be the transpose of $D$: $D^t f = (Df)^t$. 30
Lemma 4.3. For any \( f \in S(W_4^-(\mathbb{R})) \) and \( i, j = 1, 2 \), we have

\[
\hat{\mu}_{i,j+6}^+ f(w_4^-) = \frac{-1}{4} \text{tr}(D^4 B_{i,j} \mathbb{D}) f(w_4^-).
\]

Proof. By definition, we have

\[
\hat{\mu}_{i,j+6}^+ = \sqrt{-1} \left( \begin{array}{ccc} -B_{i,j} & 0 & 0 \\ 0 & B_{i,j} & 0 \\ 0 & 0 & 0 \end{array} \right) + \frac{1}{2} \left( \begin{array}{ccc} 0 & B_{i,j} & 0 \\ 0 & 0 & 0 \end{array} \right) + \frac{1}{2} \left( \begin{array}{ccc} 0 & 0 & B_{i,j} \\ 0 & 0 & 0 \end{array} \right).
\]

Then it suffices to treat each term in the sum using the expressions of Weil representation given in \([3,5]\).

Write \( w_4^- = \left( \begin{array}{c} w_1^- \\ w_2^- \end{array} \right) \) with \( w_1^- = \left( \begin{array}{c} w_{1,1}^- \\ w_{1,2}^- \end{array} \right) \in W_1^- (\mathbb{R}) \) and \( w_2^- = \left( \begin{array}{c} w_{2,1}^- \\ w_{2,2}^- \end{array} \right) \in W_2^- (\mathbb{R}) \) for \( k = 1, 2 \) and \( l = 1, \ldots, 6 \). From the above lemma, we get the following

Corollary 4.4. For \( i, j = 1, 2 \), we have

\[
\hat{\mu}_{i,j+6}^+ \phi_\infty^+(w_4^-) = 2i\pi \text{tr}(\phi_\infty^+(w_4^-) \phi_{i,j+6}^-(w_4^-)) = 4i\pi \sum_{l=1}^{6} \left( w_{1,i,l}^+ \sqrt{\eta_{i,l} \phi_{1,\infty}^+(w_1^-)} \right) \left( \phi_{2,\infty}^+(w_2^-) \right).
\]

Define variables \( X_{k;i,l} \) for \( k, i = 1, 2 \) and \( l = 1, \ldots, 6 \), \( Y_{i,j} = \sum_{l=1}^{6} X_{1,i,l} X_{2,j,l} \), \( \det(Y) = (Y_{i,j})_{i,j=1,2,6} \), and \( D^k \phi_\infty^+(w_4^-) = \det(Y) \phi_\infty^+(w_4^-) \). If we set \( X_{k;i,l} = \sqrt{\eta_{i,l} \phi_{i,j}^+(w_i^-)} \), then we have \( D^k \phi_\infty^+(w_4^-) = \phi_\infty^+(w_4^-) \). Develop \( D^k \phi_\infty^+(w_4^-) \) as a sum of monomials on \( X_{k;i,l} \): \( D^k = \sum a(t) X^t \) where \( I = (I_{i,j})_{i,j=1,2} \) with \( I_{i,j} = (I_{i,j_1}, \ldots, I_{i,j_6}) \in (\mathbb{Z}_{\geq 0})^6 \), \( X^t = \prod_{i=1}^{6} \prod_{j=1}^{2} (X_{1,i,j} X_{2,j,l})^{I_{i,j,l}} \) and \( a(t) \in \mathbb{Z} \). We define \( X^t : = \prod_{i=1}^{6} \prod_{j=1}^{2} X_{i,j,l}^t \) for \( t = 1, 2 \) (thus \( X^t = X^t X^2 \)). Write \( \mathfrak{J} = \mathfrak{J} \) for the finite set of \( I \) such that \( a(t) \neq 0 \). For each \( I \in \mathfrak{J} \), we define

\[
\phi_{t,\infty;I}^+(w_4^-) = \sqrt{a_I} X^t \phi_{t,\infty}(w_t^-) \in S(W_4^-(\mathbb{R})), \quad \text{where} \quad t = 1, 2, \quad X_{k;i,l} = \sqrt{\eta_{i,l} \phi_{i,j}^+(w_i^-)};
\]

(4.3)

\[
\phi_{\infty;I}^+(w_1^-, w_2^-) = \phi_{1,\infty;I}^+(w_1^-) \phi_{2,\infty;I}^+(w_2^-) \in S(W_4^-(\mathbb{R})).
\]

(4.4)

By definition, we have \( \sum_{I \in \mathfrak{J}} \phi_{t,\infty;I}^+(w_4^-) = \phi_{\infty;I}^+(w_4^-) \). Now for each \( I \in \mathfrak{J} \), we let

\[
f_{\infty;I}^+ \in \text{Ind}_{\text{P}^{6}(\mathbb{R})}^{G_{4}(\mathbb{R})}(\xi_{1/2})
\]

be the section associated to the section \( \phi_{t,\infty;I}^+ \). Thus \( \sum_{I \in \mathfrak{J}} f_{\infty;I}^+ = f_\infty^+ \).

4.1.2. Fourier coefficients. We next compute the local Fourier coefficients \( E_{\beta,\infty}(g_\infty, f_{\infty;I}^+) \) for each \( I \in \mathfrak{J} \). For any \( \mathbf{z} = x + iy \in \mathbb{H}_4 \), we write \( g_\mathbf{z} = u(\mathbf{x}) m(\sqrt{\mathbf{y}}) \cdot 1_t \in G(\mathbb{A}) \). Recall we write \( \beta = \left( \begin{array}{cc} \beta_1 & \beta_0 \\ \beta_0^t & \beta_2 \end{array} \right) \in \text{Sym}_{4 \times 4}(\mathbb{Q}) \). Then we have \([\text{Shi}82] \) or \([\text{Lin}15b, \text{Section 3.4}]\): \( E_{\beta,\infty}(g_\mathbf{z}, f_{\infty}^+) = \frac{16}{\Gamma_4(3)} \pi^{12} (\det(\beta))^{1/2} (\det(\mathbf{y}))^{3/2} \epsilon_\infty(\text{tr} \beta \mathbf{z}) \)

where \( \Gamma_m(s) = \pi^{m(m-1)/4} \prod_{j=0}^{m-1} \Gamma(m-s-j/2) \). Note that if \( \beta \) is not positive definite, then the local Fourier coefficient \( E_{\beta,\infty}(g_\mathbf{z}, f_{\infty}^+) \) vanishes.

As for \( f_{\infty}^+ \), we have \([\text{Lin}15b, \text{Proposition 4.4.1}]\): \( E_{\beta,\infty}(g_\mathbf{z}, f_{\infty}^+) = D^2(2\beta_0) E_{\beta,\infty}(g_\mathbf{z}, f_{\infty}^+) \).

where \( D^2(2\beta_0) = \det(2\beta_0) \).

We need the following lemma concerning the existence of certain symmetric matrix associated to \( \beta \) equivalent to \( \eta \):
Lemma 4.5. Assume that $\beta \in M_{4 \times 4}(\mathbb{Q})$ is non-singular, then there exist $a, b \in \mathbb{Q}^\times$ such that the matrix $\beta' := \text{diag}(2 \cdot \beta, a, b) \in M_{6 \times 6}(\mathbb{Q})$ is equivalent to $\eta$ (i.e., there exists a matrix $X \in \text{GL}_6(\mathbb{Q})$ such that $X^t \eta X = \beta'$).

Proof. This follows easily from standard results on the theory of quadratic forms over $\mathbb{Q}$ and $\mathbb{Q}_v$.

Note that $\beta'$ is equivalent to $\eta$ over $\mathbb{Q}$ if and only if they are equivalent over each $\mathbb{Q}_v$, and if and only if for each place $v$ of $\mathbb{Q}$, their discriminants are equal $\det(\beta') = \det(\eta) \in \mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2$ and their Hasse invariants are equal $H_v(\beta') = H_v(\eta)$ (Ser73, Chapter IV, Theorems 7 and 9). The first condition gives $ab = \det(\eta) \det(2\beta') \in \mathbb{Q}_v^\times/(\mathbb{Q}_v^\times)^2$, thus $H_v(\beta') = H_v(2\beta')(\det(2\beta), -\det(\eta))_v (a, -\det(\eta) \det(2\beta))_v$ (here $(\cdot, \cdot)_v$ is the Hilbert symbol over $\mathbb{Q}_v^\times$). So it suffices to find an element $a \in \mathbb{Q}^\times$ such that this last identity holds for all $v$. This follows from [Ser73, Chapter III, Theorem 4] and thus we conclude the proof.

In the following, when $\beta$ is non-singular, we fix one such choice of $a, b \in \mathbb{Q}^\times$ and $X \in \text{GL}_6(\mathbb{Q})$ as in the lemma such that $\det(X) = 1$ and denote them by $a_3, b_3$ and $X_3$.

Now we turn to $f_{\infty, z}^+$. For each $\beta \in \text{Sym}_{4 \times 4}(\mathbb{Q})$ definite positive, we write $d\mu_3$ for the induced measure on the compact closed subset $B_3$ of $W_4^- (\mathbb{R})$ consisting of elements $w_4^-$ such that $w_4^- \eta (w_4^-)^t = \beta$. We then denote by $\text{vol}(B_3)$ the volume of $B_3$ under $d\mu_3$. Note that $\text{vol}(B_3) = (\det \beta)^{1/2} \text{vol}(B_1^4)$. For $z = x + iy \in \mathbb{H}_4$, we write $\beta_y = \sqrt{\beta} \sqrt{y}$. We introduce some notations for the next lemma. To a partition $\nu = (\nu_1, \nu_2, \ldots, \nu_l)$ of a positive integer $n$ ($\nu_1 \geq \nu_2 \geq \cdots \geq \nu_l \geq 1$) of length $l(\nu) := l$, we associate an integer $z_\nu := \prod_{i=1}^l \prod_{j=1}^{\nu_i} (5 + 2j - i)$. Moreover we write $h(\nu)$ for the dimension of the irreducible representation over $\mathbb{C}$ of the permutation group $S_{2n}$ of $2n$ elements associated to the partition $\nu$. Then we set $d(\nu) = \text{gcd}_\nu (\frac{2s}{h(\nu)} z_\nu)$ where $\nu$ runs through all the partitions of $n$.

Then we have

Lemma 4.6. For any $I \in \mathcal{J}_z$, the local Fourier coefficient

$$E_{\beta, \infty}(g_z, f_{\phi_{\infty, I}}^+) \in d(k_1 + k_2 - 6)^{-1} a_I e_\infty(\text{tr} \beta z)(\det y)^{3/2}(\det 2\beta \det \eta)^{-1/2} \mathbb{Z} [\sqrt{\beta y}, \sqrt{\eta}^{-1}]$$

Here $\mathbb{Z}[\sqrt{\beta y}]$ is the polynomial ring $\mathbb{Z}[Z]$ on the matrix $Z = (Z_{i,j})_{i,j=1,\ldots,4}$ by identifying $Z$ with $\sqrt{\beta y}$. Similarly $\mathbb{Z}[\sqrt{\eta}^{-1}]$ is the polynomial ring $\mathbb{Z}[Z']$ on the matrix $Z' = (Z'_{i,j})_{i,j,1,\ldots,6}$ by identifying $Z'$ with $\sqrt{\eta}^{-1}$.

Proof. By definition, we have

$$E_{\beta, \infty}(g_z, f_{\phi_{\infty, I}}^+) = e_\infty(\text{tr} \beta x)(\det y)^{3/2} \int_{W_4^- (\mathbb{R})} \phi_{\infty, I}^+(\sqrt{y} w_4^-) d\mu_3$$

$$= e_\infty(\text{tr} \beta x)(\det y)^{3/2} \int_{w_4^- \eta (w_4^-)^t} \phi_{\infty, I}^+(\sqrt{y} w_4^-) d\mu_3$$

$$= a_I e_\infty(\text{tr} \beta x)(\det y)^{3/2} \int_{w_4^- \eta (w_4^-)^t} X^I(\sqrt{y} w_4^-) d\mu_3$$

$$= a_I e_\infty(\text{tr} \beta x)(\det y)^{3/2} \underbrace{X^I(\sqrt{y} w_4^-)}_{=1} d\mu_3$$

Now consider the integral: $\text{vol}^I(Z) = \int_{B_1} X^I(Z w_4^-) d\mu_1(w_4^-)$. We set $N = k_1 + k_2 - 6$. It suffices to show that $d(N) \text{vol}^I(Z) \in \mathbb{Z}[Z, Z']$. We define a map $\psi: O_6(\mathbb{R}) \to B_1$ by writing $M = \begin{pmatrix} M' \\ M'' \end{pmatrix} \in O_6(\mathbb{R})$ with $M' \in M_{4 \times 6}(\mathbb{R})$ and setting $\psi(M) = M'$. By the QR decomposition for rectangular matrices, we see that $\psi$ is surjective. Consider the measure $\mu_{O_6}$, resp. $\mu_{B_1}$, on $O_6(\mathbb{R})$, resp., $B_1$, induced from the Euclidean space $M_{6 \times 6}(\mathbb{R})$, resp., $M_{4 \times 6}(\mathbb{R})$ (inner products both given by $\langle S_1, S_2 \rangle :=$
tr(S_1 S_2^*)). We define the same map \( \psi : M_{6 \times 6}(\mathbb{R}) \to M_{4 \times 6}(\mathbb{R}), w \mapsto w^T \). Then we see that \( \mu_B \) is the induced measure from \( \mu_O \) by \( \psi \). Thus we can write \( \operatorname{vol}^I(Z) = \int_{O_n(\mathbb{R})} X^I(Z \psi(wZ'))d\mu_O(w) \). Develop \( X^I(Z \psi(wZ')) \) as polynomials on \( Z, Z' \): \( X^I(Z \psi(wZ')) = \sum_{S,T} Z^S(Z')^T b_{S,T}(w) \) where \( S \in M_{4 \times 4}(\mathbb{Z}_{\geq 0}) \) and \( T \in M_{6 \times 6}(\mathbb{Z}_{\geq 0}) \). We next show that \( \operatorname{vol}(b_{S,T}) = \int_{O_n(\mathbb{R})} b_{S,T}(w)d\mu_O(w) \) is an integer multiple of \( \frac{1}{d(N)} \int_{O_n(\mathbb{R})} d\mu_O(w) \) for all \((S,T)\). Note that each \( b_{S,T}(w) \) lies in \( \mathbb{Z}[w] \) of total degree \( N \). So it suffices to show that for each monomial \( f(w) \in \mathbb{Z}[w] \) of total degree \( N \) the integral \( \operatorname{vol}(f) = \int_{O_n(\mathbb{R})} f(w)d\mu_O(w) \) is a certain integer multiple of \( d(N)^{-1}\operatorname{vol}(1) \). For this we may apply Weingarten formula given in [CM09]. We write \( \mathcal{M}(2N) \) for the set of pair partitions of the set \( \{1, 2, \cdots, 2N\} \). For a partition \( \nu = (\nu_1, \cdots, \nu_N) \) of \( N \), let \( 2\nu = (2\nu_1, \cdots, 2\nu_N) \) be the partition of \( 2N \). The Weingarten function \( Wg_{O_n}(m,n) \) on \( \mathcal{M}(2N) \times \mathcal{M}(2N) \) is given by \( \frac{2^N N!}{(2N)!} \sum_{\nu} h(2\nu) \omega^\nu(m^{-1}n) \) where \( \nu \) runs through all the partitions of \( N \) such that \( z_\nu \neq 0 \) ([CM09] (3.7)) and \( \omega^\nu(m^{-1}n) \) is a certain integer multiple of \((2^N N!)^{-1}\) since the irreducible characters of the permutation groups always take integer values by the Frobenius formula. By [CM09] Theorem 2.1, \( \operatorname{vol}(f) = \operatorname{vol}(1) \sum_{m,n \in \mathcal{M}(2N)} Wg_{O_n}(m,n) \delta(m, n, f) \) where \( \delta(m,n,f) \) is certain function which is equal to 0 or 1 (note that our measure \( d\mu_O \) is a Haar measure, not necessarily normalized). Thus we conclude that \( \operatorname{vol}(f) \) is a certain integer multiple of \( d(N)^{-1}\operatorname{vol}(1) \).

Corollary 4.7. Supposing \( \beta \) non-singular, then we have
\[
E_{\beta, \infty}(1, f_{\varphi^+, \infty}) \in \text{d}(k_1 + k_2 - 6)^{-1} \omega_{\infty}(\text{tr } i_{\beta}) \text{det}(\eta^{-1})^{1/2} Z \left[ \frac{1}{\sqrt{N}} \right] |X_\beta|.
\]

4.1.3. Local zeta integrals. We show in this subsection the non-vanishing of certain local zeta integrals. We know by [Liu15b Proposition 4.3.1]:

Proposition 4.8. The local zeta integral
\[
Z_\infty(\cdot, \cdot, f^+_{k, \infty}) : D^+_{\infty}(k) \times D'^-_{\infty}(-k) \to \mathbb{C}
\]

is a non-zero pairing. More precisely, let \( \varphi_{k, \infty} \) be a non-zero vector of highest weight in \( D^+_{\infty}(k) \) and \( \varphi'^-_{\infty} \) its dual vector in \( D'^-_{\infty}(-k) \), then
\[
Z_\infty(\varphi_{k, \infty}, \varphi'^-_{\infty}, f^+_{k, \infty}) \neq 0.
\]

The strategy of the proof is to show that (1) \( f^+_{k, \infty}|_{U_{G_1, \infty} \times U_{G_2, \infty}} \) has a non-zero projection to the space \( \sigma_k = D^+_{k}(k) \times D'_-(-k) \); (2) the pairing \( Z_{\infty}(D^+_{k}(k) \times D'^-_{k}(-k)) \) and \( \sigma_k \) is non-zero. In fact, in [Liu15b], the combination of Theorem 4.3.2 and Lemma 4.3.4 shows that \( \sigma_k \) is in fact a direct summand of \( R_{0,6}|_{U_{G_1, \infty} \times U_{G_2, \infty}} \) and \( f^+_{k, \infty} \) actually lies in \( \sigma_k \). Here \( R_{0,2k} \) is as in [Liu15b] p.25, which is the sub-\( G_1^1(\mathbb{R}) \)-representation of the degenerate principal series \( \text{Ind}_{P_k(1)(\mathbb{R})}^G (\xi_{1/2}) \) whose \( g_{G_1^1, \infty} \)-finite part is irreducible and contains the \( K_{G_1^1, \infty} \)-type of scalar weight \( k \) (here we use the case \( k = 3 \)).

Corollary 4.9. For each \( I \in J_k \), the local zeta integral
\[
Z_\infty(\cdot, \cdot, f^+_{\infty, \infty}) : D^+_{k}(k) \times D'^-_{k}(-k) \to \mathbb{C}
\]

is a non-zero pairing.

Proof. It suffices to show that, according to the decomposition \( R_{0,6}|_{U_{G_1 \times G_2}} \simeq \oplus_{a_1 \geq a_2 \geq 3} D_a \boxtimes D_a \) ([Liu15b] (4.3.3)), the vector \( f^+_{\infty, \infty} \) has non-zero component inside \( D^+_{k}(k) \boxtimes D'_-(-k) \). We identify the universal holomorphic discrete series \( M_3 := U(\mathfrak{g}_{G_1, \infty}) \otimes U(\mathfrak{g}_{G_2, \infty}) \) det\(^3 \) with \( R_{0,6} \). On the other hand, \( M_3 \simeq U(\mathfrak{g}_{G_1^1, \infty}) \otimes \mathbb{C} \) det\(^3 \) and when restricted to \( \mathfrak{g}_{G_1^1, \infty} \times \mathfrak{g}_{G_2^1, \infty}, U(\mathfrak{g}_{G_1^1, \infty}) \) is isomorphic to \( U(\mathfrak{g}_{G_1^1, \infty} \times \mathfrak{g}_{G_2^1, \infty}) \otimes \mathbb{C} U(p_0^+) \). We then identify \( U(p_0^{+1}) \) with the polynomial ring \( \mathbb{C}[Y] \) with \( Y = \).
\((Y_{i,j})_{i,j=1,2}\) defined as above. Write \(\mathbb{C}[Y]_r\) for the subspace of \(\mathbb{C}[Y]\) consisting of polynomials of total degree less than or equal to \(r\). Then the action of \(G_1(\mathbb{R}) \times G_2(\mathbb{R})\) (as well as its Lie algebra) on \(U(p_0^{+1})\) translates to an action on \(\mathbb{C}[Y]_r\) and \(\mathbb{C}[Y]\) as \(((g_1,g_2)f)(Y) := f(g_1^*Yg_2)\). We can define a Hermitian product \(\langle \cdot , \cdot \rangle_r\) on \(\mathbb{C}[Y]_r\) as follows: for any monomials \(f(Y) = \prod_{i,j} Y_{i,j}^{F_{i,j}}\) and \(g(Y) = \prod_{k,l} Y_{k,l}^{G_{k,l}}\) in \(\mathbb{C}[Y]_r\), we define

\[
\langle f,g \rangle_r = \frac{1}{r!} \prod_{i,j} \delta_{F_{i,j},G_{3-i,3-j}} \cdot F_{i,j} !.
\]

Then one extends \(\langle \cdot , \cdot \rangle_r\) to the whole \(\mathbb{C}[Y]_r\) by sesquilinearity. One can verify that for any \(g_1 \in G_1(\mathbb{R})\), \(g_2 \in G_2(\mathbb{R})\), we have \(\langle (g_1g_2)f,(g_1g_2)g \rangle_r = \det(g_1)\det(g_2)\langle f,g \rangle_r\). We then extend these products \(\langle \cdot , \cdot \rangle_r\) to the whole \(\mathbb{C}[Y]\) (denoted by \(\langle \cdot , \cdot \rangle\)) by orthogonality and sesquilinearity. Therefore, the decomposition of \(R_{6,6|6_1 \times 6_2}\) is an orthogonal decomposition with respect to this Hermitian product. To show the non-vanishing of the zeta integral, it suffices to show that \(\langle f^{\phi_{\infty,l}}_\ell , f^{\phi^+}_\ell \rangle_{\mathbb{K}_{\infty}} \neq 0\).

For this we need a refined version of the inner product defined above. Recall that \(Y_{i,j} = \sum_{t=1}^6 X_{1:i,t}X_{2:j,t}\). We then write \(\mathbb{C}[\{X_{t,i,l}\}]\) for the polynomial ring on the variables \(X_{t,i,l}\). We define an action of the group \(G_1(\mathbb{R}) \times G_2(\mathbb{R})\) on this polynomial ring as in the above case: write the matrix of variables \(X^{(t)} = \left(\begin{array}{ccc} X_{1:1,t}X_{2:1,t} & X_{1:1,t}X_{2:2,t} \\ X_{1:2,t}X_{2:1,t} & X_{1:2,t}X_{2:2,t} \end{array} \right)\) and we define the action as \(((g_1,g_2)f)(X^{(1)}, \ldots , X^{(6)}) = f(g_1^*X^{(1)}g_2, \ldots , g_1^*X^{(6)}g_2)\). Similarly, one defines an inner product on each \(\mathbb{C}[X^{(t)}]\) exactly as in the case of \(\mathbb{C}[Y]\). Then we extend these inner products on each \(\mathbb{C}[X^{(t)}]\) to their tensor product \(\mathbb{C}[\{X_{t,i,l}\}]\) in the obvious way. Now the natural embedding of inner product spaces \(\mathbb{C}[Y] \to \mathbb{C}[\{X_{t,i,l}\}]\) \(\otimes_{t=1}^6 \mathbb{C}[X^{(t)}]\) sending \(Y_{i,j}\) to \(\sum_{t=1}^6 X_{1:i,t}X_{2:j,t}\) is equivariant for the actions of \(G_1(\mathbb{R}) \times G_2(\mathbb{R})\). We then identify \(f^{\phi^+}_{\phi_{\infty,l}}\) with the polynomial \(X^t = \prod_{t=1}^6 \prod_{i,j=1}^2 (X_{1:i,t}X_{2:j,t})^{\delta_{i,j}}\) in \(\mathbb{C}[\{X_{t,i,l}\}]\) (and identifying \(f^{\phi^+}_{\phi_{\infty,l}}\) with the sum of all the above polynomials with scalar coefficient \(a_I\) for \(I\) running through \(\mathcal{I}_\mathbb{K}\), thus a simple calculation shows that \(\langle f^{\phi_{\infty,l}}_\ell , f^{\phi^+}_\ell \rangle_{\mathbb{K}_{\infty}} \neq 0\).

From the proof of the above corollary, we see that for each \(I \in \mathcal{I}_\mathbb{K}\), \(f^{\phi^+}_{\phi_{\infty,l}}\) lies in some discrete series \(\oplus_{\ell \in \omega_T} D_{\ell}\) where \(\omega_T\) is a finite subset of dominant weights of \(T_2\) depending on \(I\).

### 4.2. Unramified places

Now we consider a finite place \(\ell \mid pN\).

We define the following sections

\[
\phi_{i,\ell}(y_\ell) = 1_{W_{1}^{-}}(z_\ell)(y_\ell), \quad \phi^+_{i,\ell} = \phi_{1,\ell} \otimes \phi^+_{2,\ell} = 1_{W_{+}^{-}}(z_\ell) \in \mathcal{S}(W_{4}^{-}(\mathbb{Q}_\ell)).
\]

This gives rise to a Siegel section \(f_{\phi_{\ell}}^{\phi^+} \in \text{Ind}_{\text{L}_{+},1}(\mathbb{Q}_\ell) \chi_{1/2}\) (note that the \(\ell\)-th component of the Hecke character \(\xi\) is in fact trivial). It is easy to see that \(f_{\phi^+_{\ell}}(G_4(\mathbb{Z}_\ell)) = \{1\}\). We can extend \(f_{\phi^+_{\ell}}\) to a section (still denoted by) \(f_{\phi^+_{\ell}}\) in \(\text{Ind}_{\text{L}_{+},1}(\mathbb{Q}_\ell) \chi_{1/2}\) by setting \(f_{\phi^+_{\ell}}(\text{diag}(1,1,1,1)) = \xi(\nu)|\nu|^{-1/2}\).

Now the local zeta integral

**Theorem 4.10.** We have

\[
Z_\ell(\varphi^+_\ell, \varphi^0; f_{\phi^+_{\ell}}) = \langle \varphi^+_\ell, \varphi^0 \rangle_\ell L_\ell(1, \text{St}(\pi) \otimes \xi).
\]

The local Fourier coefficient is given as (\cite[Theorem 13.6, Proposition 14.9]{Sh})

**Theorem 4.11.** We have

\[
E_{\beta,\ell}(1, f_{\phi^+_{\ell}}) = d_\ell(s, \xi)L_\ell(1, \xi \lambda_\beta)g_{\beta,\ell}(\xi(\ell)\ell^{-3})
\]
where $\lambda_\beta(\ell) = (\frac{\det(2\beta)}{2})$ is the quadratic symbol and $g_{\beta,\ell}(t)$ is element in $\mathbb{Z}[t]$ whose constant term is 1 and is of degree $\leq 8\text{val}_e(\det(2\beta))$.

4.3. Generalized Gauss sums. We will need some results on Gauss sums for the following subsection on the choice of Schwartz-Bruhat functions over places dividing $Np$. For any integers $a, b, c \in \mathbb{Z}$ with $c > 0$, we define the generalized Gauss sum $G(a, b, c)$ as

$$G(a, b, c) = \sum_{j=0}^{c-1} \exp(-2i\pi \frac{aj^2 + bj}{c}).$$

The generalized Gauss sum has the following properties

(1) If $(a, c)|b$, then

$$G(a, b, c) = (a, c)G\left(\frac{a}{(a, c)}, \frac{b}{(a, c)}, \frac{c}{(a, c)}\right);$$

otherwise, $G(a, b, c) = 0$;

(2) We define a function $\varepsilon : 1+2\mathbb{Z} \to \{1, i\}$ with $\varepsilon(m) = 1$ if $m \equiv 1(\text{mod } 4)$ and $= i$ if $m \equiv 3(\text{mod } 4)$. Then assuming $(a, c) = 1$ and $2 \nmid ac$, we have

$$G(a, b, c) = \varepsilon(c)\sqrt{c}\left(\frac{-a}{c}\right)\exp(2i\pi \frac{a_c^{-1}b^2}{c}) \neq 0$$

where $(\frac{a}{c})$ is the Jacobi symbol and $a_c^{-1} \in \mathbb{Z}$ is any integer such that $4a_c^{-1}a \equiv 1(\text{mod } c)$. In particular, if $c = 1$, $G(a, b, c) = 1$.

Using the generalized Gauss sums, we can evaluate some integrals. Let $v = v_{\ell}$ be the $\ell$-adic valuation. For any $S \subset \mathbb{Q}_\ell$ open compact subgroup, we write $v(S)$ to be the integer such that $\ell^{v(S)}\mathbb{Z}_\ell = S$. For any $a, b, r \in \mathbb{Q}_\ell$, we set $a = a\ell^{v(a)}$, $b = b\ell^{v(b)}$ and $r = r\ell^{v(r)}$. Then we consider the following integral

$$\int_{r+S} e_\ell(ax^2 + bx)dx = \int_{r^{\ell^{v(S)}}} e_\ell(a'\ell^{v(a)}x^2 + b'\ell^{v(b)}x)dx$$

To relate the above integral to the generalized Gauss sum, we choose an integer $M \gg 0$ such that $M - v(S) > 0$. Then the above integral becomes

$$\int_{r+S} e_\ell(ax^2 + bx)dx = \ell^{-M} \sum_{j=0}^{\ell^{M-v(S)}-1} e_\ell(a(r + \ell^{v(S)}j)^2 + b(r + \ell^{v(S)}j))$$

$$= \ell^{-M} \sum_{j=0}^{\ell^{M-v(S)}-1} \exp(-2i\pi (a\ell^{2v(S)}j^2 + (2ar\ell^{v(S)} + b\ell^{v(S)}))j + ar^2 + br))$$

$$= \ell^{-M} \exp(-2i\pi (ar^2 + br)) \sum_{j=0}^{\ell^{M-v(S)}-1} \exp(-2i\pi \frac{\ell^{M-v(S)}(a\ell^{2v(S)}j^2 + (2ar\ell^{v(S)} + b\ell^{v(S)}))j}{\ell^{M-v(S)}})$$

$$= \ell^{-M} e_\ell(ar^2 + br)G(a\ell^{M+v(S)}, (2ar + b)\ell^M, \ell^{M-v(S)}).$$

So we see that

**Lemma 4.12.** With the above notations, one has

$$\min(v(aS^2), 0) > v((2ar + b)S) \Leftrightarrow \int_{r+S} e_\ell(ax^2 + bx)dx = 0.$$  

Here we understand $v(aS^2)$ as $v(a) + 2v(S)$. In particular, if $r = 0$ and $v(aS^2) \leq 0$, $v(bS)$, then

$$\int_S e_\ell(ax^2 + bx)dx = \ell^{v(a)/2} e_\ell(-\frac{a}{4a}).$$
We can generalize the above result to higher dimensions. Let \( a \in \text{Sym}_{n \times n}(\mathbb{Q}_\ell) \) be a symmetric matrix (not necessarily non-singular), \( b \in \text{M}_{n \times 1}(\mathbb{Q}_\ell) \) a column matrix. For any element \( r \in \text{M}_{n \times 1}(\mathbb{Q}_\ell) \) and any open compact subgroup \( S \subset \text{M}_{n \times 1}(\mathbb{Q}_\ell) \) of the form \( \ell^{v(S)} \text{M}_{n \times 1}(\mathbb{Z}_\ell) \) for some integer \( v(S) \in \mathbb{Z} \) (the standard open compact subgroups of \( \text{M}_{n \times 1}(\mathbb{Q}_\ell) \)), we want to evaluate the following integral, which is analogous to the above one

\[
\int_{r+S} e_t(x^t ax + b^t x) dx.
\]

First note that \( a \) is \( \mathbb{Z}_\ell \)-equivalent to some diagonal matrix, i.e., there exists some invertible matrix \( D \in \text{GL}_n(\mathbb{Z}_\ell) \) and a diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \in \text{Sym}_{n \times n}(\mathbb{Q}_\ell) \) such that \( a = D^t \Lambda D \) ([Cas82 Chapter 8]). So if we write

\[
r' = Dr = \begin{pmatrix} r'_1 \\ r'_2 \\ \vdots \\ r'_n \end{pmatrix}, \quad S' = DS = \begin{pmatrix} S'_1 \\ S'_2 \\ \vdots \\ S'_n \end{pmatrix}, \quad b' = Db = \begin{pmatrix} b'_1 \\ b'_2 \\ \vdots \\ b'_n \end{pmatrix},
\]

then the integral becomes

\[
\int_{r+S} e_t(x^t ax + b^t x) dx = |\text{det } D|_\ell \int_{r'+S'} e_t(x'^t \Lambda x' + b'^t x') dx' = |\text{det } D|_\ell \prod_{k=1}^n \int_{v'_k+S'_k} e_t(\lambda_k x^2 + b'_k x) dx.
\]

From the above discussion, we see that if there exists some \( k = 1, \ldots, n \) such that \( \min(v(\lambda_k S'_k^2), 0) > v((2\lambda_k r'_k + b'_k) S'_k) \), then \( \int_{r+S} e_t(x^t ax + b^t x) = 0 \). We define \( v(r') = \min_{k=1}^n (v(r'_k)) \), then \( v(r) = v(r') \) and similarly \( v(S) = v(S') \), \( v(b) = v(b') \). Then we have the following lemma

**Lemma 4.13.** Notations as above, one has

\[
\min(v(\lambda_k S'_k^2), 0) > v((2\lambda_k r'_k + b'_k) S'_k) \text{ for some } k \Rightarrow \int_{r+S} e_t(x^t ax + b^t x) dx = 0.
\]

We can again generalize the above results as follows. Let \( a \in \text{Sym}_{n \times n}(\mathbb{Q}_\ell) \) be as above \( \mathbb{Z}_\ell \)-equivalent to \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_n) \) while \( b \in \text{M}_{n \times m}(\mathbb{Q}_\ell) \) be an \( n \times m \)-matrix. Let \( \eta' = \text{diag}(\eta'_1, \ldots, \eta'_m) \) be a diagonal matrix with entries in \( \mathbb{Q}_\ell^* \). Let \( r \in \text{M}_{n \times m}(\mathbb{Q}_\ell) \) and \( S \in \text{M}_{n \times m}(\mathbb{Q}_\ell) \) be an open compact open subgroup. Then we consider the following integral:

\[
\int_{r+S} e_t(\text{tr}(x^t ax \eta' + b' x)) dx.
\]

We write \( r = (r^{(1)}, \ldots, r^{(m)}) \), \( b = (b^{(1)}, \ldots, b^{(m)}) \) and \( x = (x^{(1)}, \ldots, x^{(m)}) \) in column matrices. If we assume \( S \) is of the form \( S = (S^{(1)}, \ldots, S^{(m)}) \), then we can write the above integral as a product \( \prod_{j=1}^m \int_{r^{(j)}+S^{(j)}} e_t((x^{(j)})^t ax^{(j)} \eta_j' + (b^{(j)})^t x^{(j)}) dx^{(j)}. \) As in the above lemma, we can easily give a non-vanishing criterion for this integral: if \( \min(v(\lambda_k (S_k^{(j)}))^2), 0) > v((2\eta'_j \lambda_k r^{(j)}_k + b^{(j)}_k) S_k^{(j)}) \) for some \( j \) and \( k \), then the integral vanishes.

**4.4. Ramified places dividing \( N \).** Now let's consider a finite place \( \ell | N \). We write \( N_\ell \) for \( \ell^{\text{val}_\ell(N)} \). Recall that we have in fact \( N_\ell = \ell \) since \( N \) is square-free. Moreover our symmetric form is \( \eta_\ell = \text{diag}(N^2/2, N^2/2, N^2/2, N^2/2, N/N_1, N_1) \) with \( N_1 \in \mathbb{Z}_{>0} \) prime to \( Np \).
4.4.1. Choice of sections. Recall that we identify \( W_1^- (\mathbb{Q}_\ell) \), \( W_2^- (\mathbb{Q}_\ell) \) with the set of matrices \( M_{2 \times 6} (\mathbb{Q}_\ell) \) under the correspondence \( e_k \otimes u_r \leftrightarrow E_{k,r} \) for \( k = 1, 2 \) and \( r = 1, \ldots, 6 \). We fix the following matrix \( b \) and compact open subgroup \( S \) of \( M_{4 \times 6} (\mathbb{Q}_\ell) \):

\[
 b = \frac{1}{N_\ell} \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}, \quad S = \frac{1}{N_\ell} (\mathbb{Z}_\ell^4, \mathbb{Z}_\ell^4, \mathbb{Z}_\ell^4, \mathbb{Z}_\ell^4, \mathbb{Z}_\ell^4, \mathbb{Z}_\ell^4).
\]

Moreover, we write \( b = (b_{[1]} \ b_{[2]}) \) and \( S = (S_{[1]} \ S_{[2]}) \) in \( 2 \times 6 \)-blocks. We define elements in the following sections:

\[
 \phi_i, \lambda (w_i) = 1_{S_{[i]}}(w_i) e_\ell (\text{tr}(b_{[i]} w_i)) \in S(W_i^- (\mathbb{Q}_\ell)) \text{ where } i = 1, 2.
\]

As above, we set \( \phi^+_\ell = \phi_{1, \ell} \otimes \phi_{2, \ell}, \phi^d_\ell = \delta(\phi^+_\ell) \), which give rise to \( f_{\phi^+_\ell} \in \text{Ind}_{P^+ (\mathbb{Q}_\ell)}^{G_4 (\mathbb{Q}_\ell)} (\xi_1 \xi_2) \) and \( f_{\phi^d_\ell}, \tilde{f}_{\phi^d_\ell} \in \text{Ind}_{P^d (\mathbb{Q}_\ell)}^{G_4 (\mathbb{Q}_\ell)} (\xi_1 \xi_2) \) (as in the case of unramified places, we first define sections in \( \text{Ind}_{P^d (\mathbb{Q}_\ell)}^{G_4 (\mathbb{Q}_\ell)} (\xi_1 \xi_2) \), and then extend them to \( G_4 (\mathbb{Q}_\ell) \) for \( * = +, d \). Then it is easy to see

**Lemma 4.14.** For any \( a \in \text{Sym}_{4 \times 4} (\mathbb{Q}_\ell) \) which is \( \mathbb{Z}_\ell \)-equivalent to the diagonal matrix \( \Lambda = \text{diag}(\lambda_1, \ldots, \lambda_4) \), the generalized Gauss sum \( I = \int_{S} e_\ell (\text{tr}(bx) \Delta) dx \neq 0 \) implies \( v(\lambda_i) \leq -v(N^2) \) for all \( i = 1, \ldots, 4 \). In particular, \( I \neq 0 \) implies that \( \text{det}(a) \neq 0 \).

**Proof.** Indeed, suppose that there is a matrix \( D \in \text{GL}_4 (\mathbb{Z}_\ell) \) such that \( a = D^t \Delta D \). We then write as above \( b' = Db = (b^{(1)}, \ldots, b^{(6)}) \), \( S = DS = (S^{(1)}, \ldots, S^{(6)}) \) and \( x = (x^{(1)}, \ldots, x^{(6)}) \). We define \( I_j = \int_{S^{(j)}} e_\ell (\eta_j (x^{(j)})^t \Delta x^{(j)} + (b^{(j)})^t x^{(j)}) dx \) for \( j = 1, \ldots, 6 \). Then \( I = \prod_{j=1}^6 I_j \). Suppose that \( v(\lambda_k) > -v(N^2) \) for some \( k \). Since \( D \in \text{GL}_4 (\mathbb{Z}_\ell) \), by the definition of \( b \), we see that for the \( k \) above, there exists some \( j = 1, 2, 3, 4 \) such that \( b^{(j)}_k \in \frac{1}{N_\ell} \mathbb{Z}_\ell^x \). Then \( v(b^{(j)}_k S^{(j)}_k) = -v(N^2) < \min(v(\eta_j \lambda_k (S^{(j)}_k)^2), 0) = \min(v(\lambda_k), 0) \) thus by Lemma 4.13 we see that \( I_j = 0 \) and therefore \( I = 0 \). \( \square \)

We write \( \Delta = \begin{pmatrix} 0 & 1_2 \\ 1_2 & 0 \end{pmatrix} \in G_1 (\mathbb{Q}). \) Then it is easy to see that

\[
 S^{-1}(g_1, 1) S = \begin{pmatrix} \frac{1}{2} (g_1 + 1) & \frac{1}{2} (g_1 - 1) \Delta \\ \Delta (g_1 - 1) & \frac{1}{2} \Delta (g_1 + 1) \Delta \end{pmatrix},
\]

Then we have

**Corollary 4.15.** For an element \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in G_4 (\mathbb{Q}_\ell) \), \( f_{\phi^+_\ell} (g) \neq 0 \) implies that \( \text{det}(C^t D) \neq 0 \). For \( g_1 \in G_4^1 (\mathbb{Q}_\ell) \), \( \tilde{f}_{\phi^d_\ell} ((g_1, 1)) \neq 0 \) implies that \( g_1 \in G_4^1 (\mathbb{Z}_\ell) \), \( g_1 \equiv 14 (\text{mod } N^2) \) and \( \text{det}(g_1 - 1) \neq 0 \).

**Proof.** By \( \text{[3.5.]} \) we have that

\[
 f_{\phi^+_\ell} (g) = \gamma_{\text{Weil}} \int_{Ker(C) \setminus W_1^- (\mathbb{Q}_\ell)} e_\ell (\text{tr}(\frac{1}{2} x^t C^t D x + b^t x)) 1_S (C x) dx
\]

\[
 = \gamma_{\text{Weil}} \int_{S} e_\ell (\text{tr}(\frac{1}{2} x^t C^t D x + b^t x)) dx.
\]

By the above lemma, from \( f_{\phi^+_\ell} (g) \neq 0 \) we have \( \text{det}(C^t D) \neq 0 \). We write \( g_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \) in \( 2 \times 2 \)-blocks. Then by definition, \( \tilde{f}_{\phi^d_\ell} ((g_1, 1)) = f_{\phi^+_\ell} (S^{-1}(g_1, 1) S) \). So by the first part of the corollary, if \( \tilde{f}_{\phi^d_\ell} ((g_1, 1)) \neq 0 \), then \( \text{det}((g_1 - 1)(g_1 + 1)) \neq 0 \). Set \( g = S^{-1}(g_1, 1) S \),

\[
\text{37}
\]
since \( \det(C) \neq 0 \), we can write \( g \) as a product

\[
g = u(B')m(A')J_4u(B') = \begin{pmatrix} -B'A' & A' - B'A'B'' \\ -(A')^{-t} & -(A')^{-1}B'' \end{pmatrix}.
\]

Note that \( B'' = \frac{1}{2}(g_1 - 1)^{-1}(g_1 + 1)\Delta \). Therefore \( \int_{\phi} (g) \neq 0 \) if and only if the integral

\[
\int_S \mathbf{e}_t \left( \text{tr}(\frac{1}{2}x^t B'' x\eta + b'x) \right) dx \neq 0.
\]

By the above lemma, this implies that \( (B'')^{-1} \in N^2 \text{Sym}_{4 \times 4}(\mathbb{Z}_\ell) \). Since

\[
g_1 = (1 + (2B''\Delta)^{-1})(1 - (2B''\Delta)^{-1})^{-1}.
\]

We deduce that \( g_1 \in G_1(\mathbb{Z}_\ell) \), \( g_1 \equiv 1_4(\mod N_\ell^2) \) and \( \det(g_1 - 1) \neq 0 \).

4.4.2. Local zeta integral. We next calculate the local zeta integral \( Z_\ell(\varphi, \varphi', \bar{f}_{\bar{\phi}'}^\ell) \). Recall that \( \Gamma(N)_\ell \) is the subgroup of \( G_1(\mathbb{Z}_\ell) \) consisting of matrices \( g \) such that \( g \equiv 1_4(\mod N_\ell) \).

**Proposition 4.16.** Assume that \( \varphi \) is invariant under right translation of \( \Gamma(N)_\ell \). Then the local zeta integral at \( \ell \) is a non-zero rational multiple of \( \langle \varphi, \varphi' \rangle \) (explicitly given in Lemma 4.19):

\[
Z_\ell(\varphi, \varphi', \bar{f}_{\bar{\phi}'}^\ell) \in \mathbb{Q}^x.
\]

**Proof.** By the assumption we have

\[
Z_\ell(\varphi, \varphi', \bar{f}_{\bar{\phi}'}^\ell) = \int_{G_1(\mathbb{Q}_\ell)} \bar{f}_{\bar{\phi}'}^\ell((g, 1)) \langle \pi(g) \varphi, \varphi' \rangle dg = \langle \varphi, \varphi' \rangle \int_{\Gamma(N)_\ell} \bar{f}_{\bar{\phi}'}^\ell((g_1, 1)) dg_1.
\]

So it suffices to treat the integral on the RHS. As in the preceding corollary, we write \( A' = -\Delta(g_1 - 1)^{-t} \) and \( B = (B'')^{-1} = 2((g_1 - 1)^{-1}(g_1 + 1)\Delta)^{-1} \in \text{Sym}_{4 \times 4}(N_\ell \mathbb{Z}_\ell) \) for \( g_1 \in \Gamma(N)_\ell \), i.e. \( g_1 = 2(1 - (2B^{-1}\Delta)^{-1})^{-1} - 1 \). Then by change of variables from \( g_1 \) to \( B \) and taking into account of the Jacobian \( |\det(\partial g_1/\partial B)| = 1 \), we see that

\[
\bar{f}_{\bar{\phi}'}^\ell((g_1, 1)) = |\det(A')|^{3 \xi(\det(A'))} \int_S \mathbf{e}_t \left( \text{tr}(\frac{1}{2}x^t B^{-1} x\eta + b'x) \right) dx
\]

\[
= |\det(B)|^{-3 \xi(\det(B))} \int_S \mathbf{e}_t \left( \text{tr}(\frac{1}{2}x^t B^{-1} x\eta + b'x) \right) dx.
\]

Therefore the integral becomes

\[
\int_{\Gamma(N)_\ell} \bar{f}_{\bar{\phi}'}^\ell((g_1, 1)) dg_1 = \int_{\text{Sym}_{4 \times 4}(\mathbb{Q} \mathbb{Z}_\ell)} |\det(B)|^{-3 \xi(\det(B))} dB \int_S \mathbf{e}_t \left( \text{tr}(\frac{1}{2}x^t B^{-1} x\eta + b'x) \right) dx.
\]

We first treat the inner integral in the above. Without loss of generality we can assume that \( B^{-1} = \text{diag}(B_1^{-1}, \ldots, B_4^{-1}) \). Using the calculation preceding Lemma 4.12 we see that for any \( i = 1, \ldots, 4 \) and \( j = 1, \ldots, 6 \),

\[
\int_{S^{(j)}} \mathbf{e}_t \left( \text{tr}(\frac{1}{2}(x_i^{(j)})^t B_i^{-1} x_i^{(j)} \eta_j + (b_i^{(j)})^t x_i^{(j)}) \right) dx_i^{(j)}
\]

\[
= \ell^{-\frac{1}{2}v_i(B_i) + \frac{1}{2}v_i(\eta_j)} (\mathcal{F}_v(B_i \eta_j))(\frac{-2B_i \eta_j \ell^{-v_i(B_i \eta_j)}/\ell}{v_i(B_i \eta_j)}) \mathbf{e}_t(-\frac{(b_i^{(j)})^2 B_i}{4}).
\]
Note that

\[
\prod_{i,j} \varepsilon(\ell^{v_i(B_i)}) = \prod_{i,j} \varepsilon(\ell^{v_j(B_j)}) = \prod_{i} \varepsilon(\ell^{v_i(B_i)})^2 = \prod_{i} \left(\frac{-1}{\ell}\right)^{v_i(B_i)} = (-1)^{v_i(\det(B))} \cdot \ell^i,
\]

\[
\prod_{i,j} \left(\frac{B_i^j}{\ell}(\eta^j_i)\right)^{v_i(B_i)} = \prod_{i,j} \left(\frac{B_i^j}{\ell}\right)^{v_j(\eta)}(\eta^j_i)\prod_{i,j} (B_i, \eta_j)\ell(-1)^{v_i(B_i)v_j(\eta)} \cdot \ell^i
\]

\[
= (\det(B), \det(\eta))\ell(-1)^{v_i(\det(B))v_j(\det(\eta))} \cdot \ell^i.
\]

So their product gives (for any \(x \in \mathbb{Q}_\ell^x\) we write \(x^o \in \mathbb{Z}_\ell^x\) for \(x \ell^{-v(x)}\))

\[
\int_S e(\text{tr}(\frac{1}{2} x^o B^{-1} x \eta + b^t x))dx = \left(\prod_{i} \ell^{-3v_i(B_i)}\right)(\prod_{j} \ell^{2v_j(\eta)})\prod_{i} e(-\frac{B_i}{\ell^2})
\]

\[
\times \left(\frac{-2}{\ell}\right)^{2\sum_i v_i(B_i) + 4\sum_j v_j(\eta)} \prod_{i,j} \left(\frac{B_i^j}{\ell}(\eta^j_i)\right)^{v_i(B_i) v_j(\eta)}
\]

\[
= |\det(B)|^3|\det(\eta)|^{-2} e(-\text{tr} \frac{B}{\ell^2})(\det(B), \det(\eta))\ell
\]

\[
\times (-1)^{v_i(\det(B))v_j(\det(\eta))} \cdot \ell^i
\]

On the other hand, note that by [Kud96, Chapter II, Proposition 4.3],

\[
\xi(\det(B)) = (\det(B), (-1)^{6x5/2}\det(\eta))\ell
\]

\[
= (\det(B), \det(\eta))\ell(\det(B), -1)\ell
\]

\[
= (\det(B), \det(\eta))\ell(-1)^{v_i(\det(B))} \cdot \ell^i.
\]

Taking into account of the fact \(v_i(\det(\eta)) \equiv 1(\text{mod} 2)\), we get

\[
\int_{\Gamma(N)_{\ell}} \tilde{f}_\ell^\chi((g_1, 1))dg_1 = |\det(\eta)|^{-2} \int_{\text{Sym}_{4x4}(N^2_{\ell}\mathbb{Z}_\ell) \cap \text{GL}(4_{\ell})} e(-\text{tr} \frac{B}{\ell^2})(-1)^{v_i(\det(B))} \cdot \ell^i dB
\]

\[
= \ell^{-20}|\det(\eta)|^{-2} \int_{\text{Sym}_{4x4}(\mathbb{Z}_\ell)} (-1)^{v_i(\det(B))} \cdot \ell^i dB
\]

\((-1)^{v_i(\det(B))} \cdot \ell^i\) is understood to be 0 if \(\det(B) = 0\). We treat this last integral in the following series of lemmas. The result is given in Lemma 4.19.

Recall that we say two matrices \(A, B \in \text{Sym}_{n \times n}(\mathbb{Q}_\ell)\) are \(\mathbb{Z}_\ell\)-equivalent (written as \(A \sim B\)) if there is an invertible matrix \(D \in \text{GL}(n, \mathbb{Z}_\ell)\) such that \(A = DBD^t\). Similarly, we say that \(A, B \in \text{Sym}_{n \times n}(\mathbb{Z}/\ell^N)\) are \(\mathbb{Z}/\ell^N\)-equivalent (written as \(A \sim_N B\)) if there is an invertible matrix \(D \in \text{GL}(n, \mathbb{Z}/\ell^N)\) such that \(A = DBD^t\). We fix a non-square \(z = z_\ell \in \mathbb{Z}_\ell^x\). Moreover, by [Cas82, Chapter 8, Theorem 3.1], any matrix \(A \in \text{Sym}_{n \times n}(\mathbb{Q}_\ell)\) is \(\mathbb{Z}_\ell\)-equivalent to a diagonal matrix \(\Lambda\) of the form \(\Lambda = \text{diag}(\ell^{k_1}\Lambda_1, \ell^{k_2}\Lambda_2, \ldots, \ell^{k_r}\Lambda_r, 0)\) with \(k_1 < k_2 < \cdots < k_r\) and \(\Lambda_i = 1_{n_i}\) or \(\Lambda_i = \text{diag}(1_{n_i-1}, z)\). For any \(A \in \text{Sym}_{n \times n}(\mathbb{Z}_\ell)\), we also write \(A\) for its image in by the canonical projection \(\text{Sym}_{n \times n}(\mathbb{Z}_\ell) \rightarrow \text{Sym}_{n \times n}(\mathbb{Z}/\ell^N)\).

**Lemma 4.17.** Suppose as above \(\Lambda = \text{diag}(\ell^{k_1}\Lambda_1, \ldots, \ell^{k_r}\Lambda_r)\) an invertible matrix with \(k_1 \geq 0\). Then a matrix \(A \in \text{Sym}_{n \times n}(\mathbb{Z}_\ell)\) is \(\mathbb{Z}_\ell\)-equivalent to \(\Lambda\) if and only if \(A\) is \(\mathbb{Z}/\ell^{k_r+1}\)-equivalent to \(\Lambda\).
Proof. It suffices to show that if $A \sim \Lambda$, then for any $A' \in \text{Sym}_{n \times n}(\mathbb{Z}_\ell)$, $A + \ell^{k_r+1}A' \sim \Lambda$. Then it suffices to treat the case $A = \Lambda$. We prove this case by induction on $r$.

For the case $r = 1$, we first assume that $A = 1$. Then for any $A' \in \text{Sym}_{n \times n}(\mathbb{Z}_\ell)$, we define $D(\ell^{k_r+1}A) = \sqrt{1 + \ell^{k_r+1}A} = \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)(\ell^{k_r+1}A)^i$ which is a matrix in $\text{GL}(n, \mathbb{Z}_\ell)$. Then it is clear that $\Lambda + \ell^{k_r+1}A' = D(\ell^{k_r+1}A)\Lambda D(\ell^{k_r+1}A)^t$ thus $\Lambda + \ell^{k_r+1}A' \sim \Lambda$. Next we assume that $\Lambda = \text{diag}(1_{n-1}, z)$. Then for any $A' = \begin{pmatrix} A'_1 & A'_2 \\ (A'_2)^t & A'_3 \end{pmatrix} \in \text{Sym}_{n \times n}(\mathbb{Z}_\ell)$ with $A'_1$ of size $(n-1) \times (n-1)$, we define a matrix $D(\ell^{k_r+1}A) = \begin{pmatrix} D(\ell^{k_r+1}A)'_1 & D(\ell^{k_r+1}A)'_2 \\ D(\ell^{k_r+1}A)'_3 & D(\ell^{k_r+1}A)'_4 \end{pmatrix} \in M_{n \times n}(\mathbb{Q}_\ell)$ with $D(\ell^{k_r+1}A)_1$ of size $(n-1) \times (n-1)$ as follows:

$D(\ell^{k_r+1}A)'_4 := D(z^-1\ell^{k_r+1}A'_3)$,

$D(\ell^{k_r+1}A)'_3 := 0$,

$D(\ell^{k_r+1}A)'_2 := z^-1 D(\ell^{k_r+1}A)'_4^{-1}\ell^{k_r+1}A'_2$,

$D(\ell^{k_r+1}A)'_1 := D(\ell^{k_r+1}A'_1 - (z + \Lambda_z'\ell^{k_r+1}A'_2)(\ell^{k_r+1}A'_2)')$.

Then one verifies that $D(\ell^{k_r+1}A')$ is a matrix in $\text{GL}(n, \mathbb{Z}_\ell)$. Then $\Lambda + \ell^{k_r+1}A' \sim \Lambda$ (note that the argument is independent of the size $n \times n$ of $A'$).

Suppose that we have proved the statement in the beginning for $r - 1 \geq 0$, i.e., for $\Lambda'' = \text{diag}(\ell^{k_2}A_2, \ldots, \ell^{k_r}A_r)$ and any $A \in \text{Sym}_{n \times n}(\mathbb{Z}_\ell)$, there is a matrix $D_{\Lambda''}(\ell^{k_r+1}A) \in \text{GL}(n, \mathbb{Z}_\ell)$ such that $\Lambda'' + \ell^{k_r+1}A = D_{\Lambda''}(\ell^{k_r+1}A)\Lambda'' D_{\Lambda''}(\ell^{k_r+1}A)^t$. We write $\Lambda = \text{diag}(\Lambda', \Lambda'')$ with $\Lambda'$ of size $n_1 \times n_1$. For any $A' = \begin{pmatrix} A'_1 & A'_2 \\ (A'_2)^t & A'_3 \end{pmatrix} \in \text{Sym}_{n \times n}(\mathbb{Z}_\ell)$ with $A'_1$ of size $n_1 \times n_1$. Then we define a matrix $D(\ell^{k_r+1}A') = \begin{pmatrix} D(\ell^{k_r+1}A)'_1 & D(\ell^{k_r+1}A)'_2 \\ D(\ell^{k_r+1}A)'_3 & D(\ell^{k_r+1}A)'_4 \end{pmatrix} \in M_{n \times n}(\mathbb{Q}_\ell)$ with $D(\ell^{k_r+1}A)'_1$ of size $n_1 \times n_1$ as follows:

$D(\ell^{k_r+1}A)'_4 := D_{\Lambda''}(\ell^{k_r+1}A'_3)$,

$D(\ell^{k_r+1}A)'_3 := 0$,

$D(\ell^{k_r+1}A)'_2 := A'_2 D(\ell^{k_r+1}A)'_4^{-1} \ell^{k_r+1}(A'')^{-1}$,

$D(\ell^{k_r+1}A)'_1 := D_{\Lambda''}(\ell^{k_r+1}A'_1 - D(\ell^{k_r+1}A'_2\Lambda'' D(\ell^{k_r+1}A'_2))^t)$.

Then one verifies that $D(\ell^{k_r+1}A') \in \text{GL}(n, \mathbb{Z}_\ell)$ and $\Lambda + \ell^{k_r+1}A' \sim \Lambda(\ell^{k_r+1}A')\Lambda D(\ell^{k_r+1}A)^t$ and therefore $\Lambda + \ell^{k_r+1}A' \sim \Lambda$, which concludes the induction step. \qed

Now for any invertible $\Lambda \in \text{Sym}_{4 \times 4}(\mathbb{Z}_\ell)$, as above, we write $\mathcal{E}(\Lambda)$ for the subset of $\text{Sym}_{4 \times 4}(\mathbb{Z}_\ell)$ consisting of matrices $\mathbb{Z}_\ell$-equivalent to $\Lambda$. By the above lemma, we see that each $\mathcal{E}(\Lambda)$ is an open compact subset of $\text{Sym}_{4 \times 4}(\mathbb{Z}_\ell)$. We also write $\mathcal{Q}$ for the set of all such $\Lambda \in \text{Sym}_{4 \times 4}(\mathbb{Z}_\ell)$. We define $I(\Lambda) = \int_{B \in \mathcal{E}(\Lambda)} \left(-\frac{1}{\ell}\right)v_{\ell}(\det(B)) = \left(-\frac{1}{\ell}\right)v_{\ell}(\det(\Lambda)) \int_{B \in \mathcal{E}(\Lambda)} dB$.

Then we have

$$\int_{\text{Sym}_{4 \times 4}(\mathbb{Z}_\ell)} \left(-\frac{1}{\ell}\right)v_{\ell}(\det(B)) dB = \sum_{\Lambda \in \mathcal{Q}} I(\Lambda).$$

Before going on, we need some results on the order of orthogonal groups over certain finite rings. Let $\Lambda_1 \in \text{GL}(n_1, \mathbb{Z}/\ell)$ be a symmetric matrix. The orthogonal group we consider is $O(\Lambda_1, \mathbb{Z}/\ell^{s+1})$ for any integer $s \geq 0$. We have a canonical projection

$$\text{pr}_s : O(\Lambda_1, \mathbb{Z}/\ell^{s+1}) \to O(\Lambda_1, \mathbb{Z}/\ell^s),$$
Lemma 4.18.\footnote{\cite[Section 3.7]{Wil09}} whose kernel $\text{pr}_s$ can be identified with the subset of $\text{M}_{n_1 \times n_1}(\mathbb{Z}/\ell)$ consisting of matrices $D$ such that $D\Lambda_1 + \Lambda_1 D^t = 0$. So we see that $\#(\text{pr}_s) = \ell^{n_1(n_1-1)/2}$. Therefore by induction on $s$,

$$\#O(\Lambda_1, \mathbb{Z}/\ell^{s+1}) = \ell^{m_1(n_1-1)/2} \#O(\Lambda_1, \mathbb{Z}/\ell).$$

As for the order of the orthogonal group $O(\Lambda_1, \mathbb{Z}/\ell)$, we record results from \cite[Section 3.7]{Wil09}.

1. Assume $\Lambda_1$ is of odd dimension $n_1 = 2m + 1$. Then we know that

$$\#O(\Lambda_1, \mathbb{Z}/\ell) = 2\ell^{m_2}(\ell^2 - 1)(\ell^4 - 1) \cdots (\ell^{2m} - 1).$$

2. Assume that $n_1 = 2m$ is even and $\ell \equiv 1(\text{mod } 4)$ or $n_1 = 2m$ with $m \equiv 0(\text{mod } 2)$ and $\ell \equiv 3(\text{mod } 4)$, then

$$\#O(1_{2m}, \mathbb{Z}/\ell) = 2\ell^{m(m-1)}(\ell^2 - 1)(\ell^4 - 1) \cdots (\ell^{2m-2} - 1)(\ell^m - 1),$$

$$\#O(\text{diag}(1_{2m-1}, z), \mathbb{Z}/\ell) = 2\ell^{m(m-1)}(\ell^2 - 1)(\ell^4 - 1) \cdots (\ell^{2m-2} - 1)(\ell^m + 1).$$

3. Assume that $n_1 = 2m$ is even with $m \equiv 1(\text{mod } 2)$ and $\ell \equiv 3(\text{mod } 4)$, then

$$\#O(1_{2m}, \mathbb{Z}/\ell) = 2\ell^{m(m-1)}(\ell^2 - 1)(\ell^4 - 1) \cdots (\ell^{2m-2} - 1)(\ell^m + 1),$$

$$\#O(\text{diag}(1_{2m-1}, z), \mathbb{Z}/\ell) = 2\ell^{m(m-1)}(\ell^2 - 1)(\ell^4 - 1) \cdots (\ell^{2m-2} - 1)(\ell^m - 1).$$

Now we can evaluate each $I(\Lambda)$. We have

**Lemma 4.18.** Assume $k_1 \geq 0$ in $\Lambda = (\ell^{k_1}\Lambda_1, \ldots, \ell^{k_r}\Lambda_r)$, then the integral is

$$I(\Lambda) = \left(\frac{-1}{\ell}\right)^{\sum_i k_i n_i} \frac{\ell^{6k_r-4}(\ell^4 - 1) \cdots (\ell^4 - 1)}{\prod_i \#O(\Lambda_i, \mathbb{Z}/\ell)}.$$

**Proof.** Note that by the above lemma,

$$\text{vol}(E(\Lambda)) = \ell^{-10(k_r+1)} \#\{B \in \text{Sym}_{4 \times 4}(\mathbb{Z}/\ell^{k_r+1}) | B \sim_{k_r+1} \Lambda\}$$

$$= \ell^{-10(k_r+1)} \frac{\#\text{GL}_4(\mathbb{Z}/\ell^{k_r+1})}{\#\{D \in \text{GL}_4(\mathbb{Z}/\ell^{k_r+1}) | \Lambda = D\Delta D^t\}}$$

$$= \ell^{-10(k_r+1)} \frac{\#\text{GL}_4(\mathbb{Z}/\ell^{k_r+1})}{\#O(\Lambda, \mathbb{Z}/\ell^{k_r+1})}.$$ 

The order of $\text{GL}_4(\mathbb{Z}/\ell^{k_r+1})$ is given by $\ell^{16k_r}(\ell^4 - 3) \cdots (\ell^4 - 1)$ (see for example \cite[Corollary 2.8]{Han06}).

Next we calculate the order of $O(\Lambda, \mathbb{Z}/\ell^{k_r+1})$. For any element $D = (D_{i,j}) \in \text{GL}_4(\mathbb{Z}/\ell^{k_r+1})$ with $D_{i,j} \in \text{M}_{n_i \times n_j}(\mathbb{Z}/\ell^{k_r+1})$, we see that $\hat{\Lambda} = D\Delta D^t = (\Lambda_{i,j})$ with $\Lambda_{i,j} = \ell^{k_i} D_{i,1}^1 \Lambda_1 D_{j,1}^t + \cdots + \ell^{k_r} D_{i,r}^1 \Lambda_r D_{j,r}^t$. If we equate $\hat{\Lambda}$ with $\Lambda$, then $\hat{\Lambda}_{1,1} = \Lambda_{1,1}$ implies that $\ell^{k_1} D_{1,1}^1 \Lambda_1 D_{j,1}^t + \cdots + \ell^{k_r} D_{1,r}^1 \Lambda_r D_{j,r}^t = 0(\text{mod } \ell^{k_r+1})$ with $(*) \in \text{M}_{n_1 \times n_1}(\mathbb{Z}/\ell^{k_r+1})$. For any fixed $(*)$, the number of $D_{1,1}^1$ satisfying the above equation is the same as $\ell^{k_1(n_1+1)/2}$ times the number of $D_{1,1}$ such that $D_{1,1}^1 \Lambda_1 D_{j,1}^t = \Lambda_1$ by the above lemma, which is $\#O(\Lambda_1, \mathbb{Z}/\ell^{k_r+1})$, as given preceding the lemma. Next $\hat{\Lambda}_{2,1} = \Lambda_{2,1} = 0$ implies that $\ell^{k_1} D_{2,1} \Lambda_1 D_{j,1}^t + \ell^{k_2} (*) = 0(\text{mod } \ell^{k_r+1})$ with $(*) \in \text{M}_{n_2 \times n_1}(\mathbb{Z}/\ell^{k_r+1})$. Therefore, $\ell^{k_1} D_{2,1} = 0(\text{mod } \ell^{k_2})$ and $D_{2,1} = -\ell^{k_2-k_1} (*) D_{j,1}^t \Lambda_1^{-1}(\text{mod } \ell^{k_r-k_1})$. Therefore the number of $D_{2,1}$ satisfying the above equation is $\ell^{k_2 n_1 n_2}$. Similarly, for $i = 2, \ldots, r$, one shows that $\hat{\Lambda}_{i,1} = \Lambda_{i,1} = 0$ implies that $\ell^{k_i} D_{i,1} = 0(\text{mod } \ell^{k_2})$ and the number of such $D_{i,1}$ satisfying the above equation is $\ell^{k_k n_1 n_2}$. We can continue the above argument for any $D_{i,j}$ and $D_{j,i}$ with $i > j$, and the number of such $D_{i,j}$ satisfying the corresponding equation is $\ell^{k_k n_i n_j}$ while the number of such $D_{j,i}$ is $\ell^{k_k k_1 n_j}$. For any
Now it suffices to take the sum of \(\ell^{k \cdot n_i(n_i+1)/2} \#O(\Lambda_i, \mathbb{Z}/\ell^{k_i+1})\). So finally we get that
\[
\#O(\Lambda, \mathbb{Z}/\ell^{k_r+1}) = \prod_{i \neq j} \ell^{k_i+k_j \cdot n_i \cdot n_j} \times \prod_i \ell^{k \cdot n_i(n_i+1)/2} \cdot n_i(n_i-1)/2 \#O(\Lambda_i, \mathbb{Z}/\ell).
\]
Combining the above results, we get the formula in the lemma.

Combining the results in Lemmas 4.18 we get the following:

Lemma 4.19.
\[
\int_{\text{Sym}_{4 \times 4}(\mathbb{Z}_\ell)} (\frac{-1}{\ell})_{\nu, \nu}(\text{det}(B)) dB = \frac{((\frac{-1}{\ell}) \ell^5 - 1)(\ell - 1)}{(\ell^5 - \ell)}.
\]

Proof. For any \(\Lambda = \text{diag}((\ell^1 \Lambda_1, \ldots, \ell^{k_r} \Lambda_r)) \in \mathcal{Q}\), we have seen that
\[
I(\Lambda) = \frac{-1}{\ell} \sum \Pi(\Lambda) \cdot \frac{\ell^{4k_r-4-16k_1}}{\#O(\Lambda, \mathbb{Z}/\ell^{k_r+1})} (\ell-1) \cdots (\ell^4-1)
\]
We rewrite \(k_1, \ldots, k_r\) as \(k_1 = \alpha_1, k_2 = \alpha_1 + \alpha_2, \ldots, k_r = \alpha_1 + \cdots + \alpha_r\) with \(\alpha_1 \geq 0, \alpha_i > 0\) for \(i > 1\). Now it suffices to take the sum of \(I(\Lambda)\) over all possible \(r = 1, \ldots, 4, \alpha_1 \geq 0, \ldots, \alpha_r > 0\) and \(\Lambda_1, \ldots, \Lambda_r\). We write \(\alpha = (\ell - 1) \cdots (\ell^4 - 1)\). The results are as follows:

1. \(r = 1\), one has
\[
\sum_{r=1} I(\Lambda) = \frac{\ell^6 \alpha}{(\ell^1 - 1)(\ell^2 - 1)(\ell^4 - 1)};
\]
2. \(r = 2\) and \((n_1, n_2) = (1, 3)\), one has
\[
\sum_{(r, n_1, n_2) = (2, 1, 3)} I(\Lambda) = \frac{\ell^2 \alpha}{(\ell^1 - 1)((\frac{-1}{\ell}) \ell^6 - 1)(\ell^2 - 1)};
\]
3. \(r = 2\) and \((n_1, n_2) = (2, 2)\), one has
\[
\sum_{(r, n_1, n_2) = (2, 2, 2)} I(\Lambda) = \frac{\ell^4 \alpha}{(\ell^1 - 1)(\ell^3 - 1)(\ell^2 - 1)};
\]
4. \(r = 2\) and \((n_1, n_2) = (3, 1)\), one has
\[
\sum_{(r, n_1, n_2) = (2, 3, 1)} I(\Lambda) = \frac{\ell^2 \alpha}{(\ell^1 - 1)((\frac{-1}{\ell}) \ell^6 - 1)(\ell^2 - 1)};
\]
5. \(r = 3\) and \((n_1, n_2, n_3) = (1, 1, 2)\), one has
\[
\sum_{(r, n_1, n_2, n_3) = (3, 1, 1, 2)} I(\Lambda) = \frac{\ell^2 \alpha}{(\ell^1 - 1)((\frac{-1}{\ell}) \ell^6 - 1)(\ell^3 - 1)(\ell^2 - 1)};
\]
6. \(r = 3\) and \((n_1, n_2, n_3) = (1, 2, 1)\), one has
\[
\sum_{(r, n_1, n_2, n_3) = (3, 1, 2, 1)} I(\Lambda) = \frac{\ell^2 \alpha}{(\ell^1 - 1)((\frac{-1}{\ell}) \ell^6 - 1)((\frac{-1}{\ell}) \ell^3 - 1)(\ell^2 - 1)};
\]
7. \(r = 3\) and \((n_1, n_2, n_3) = (2, 1, 1)\), one has
\[
\sum_{(r, n_1, n_2, n_3) = (3, 2, 1, 1)} I(\Lambda) = \frac{\ell^2 \alpha}{(\ell^1 - 1)(\ell^3 - 1)((\frac{-1}{\ell}) \ell^3 - 1)(\ell^2 - 1)};
\]
(8) \( r = 4 \), one has
\[
\sum_{r=4}^{\alpha} I(\Lambda) = \frac{\alpha}{(\ell^{10} - 1)((\frac{1}{\ell})^6 - 1)((\frac{1}{\ell})\ell^{10} - 1)}.
\]
Taking the sum of all these numbers, one gets that if \((\frac{1}{\ell}) = 1\), then \(\sum_{\Lambda \in Q} I(\Lambda) = 1\), and if \((\frac{1}{\ell}) = -1\), then \(\sum_{\Lambda \in Q} I(\Lambda) = (\ell^6 + 1)(\ell - 1)(\ell^6 - 1)^{-1}(\ell + 1)^{-1}\), which is the desired formula in the lemma. □

4.4.3. Local Fourier coefficient. We now turn to the Fourier coefficients

\[
E_{\beta,\ell}(1, f_{\phi^+_t}) = \int_{\text{Sym}_{4 \times 4}(Q_\ell)} f_{\phi^+_t}(J_g u(b)) e_{\ell}(-v(\beta b)) db.
\]

We need to show that \(E_{\beta,\ell}(1, f_{\phi^+_t})\), as a function on \(\beta\), is not identically zero and takes values in \(Q(\zeta_\ell)\). We write

\[
f(b) := f_{\phi^+_t}(J_g u(b)) = \int_{W_4^-(Q_\ell)} e_{\ell}(v(\frac{1}{2} wb w^t \eta U)) \phi^+_t(w) dw.
\]

Note that by Lemma 4.14, the support of \(f(b)\) is contained in the subset of \(b\) such that \(b^{-1} \in N_\ell^2 \text{Sym}_{4 \times 4}(\mathbb{Z}_\ell)\) and by the proof of Proposition 4.16

\[
f(b) = |\det(b)|^{-3} |\det(\eta_U)|^2 (\frac{-1}{\ell})^{v(\det(b))}.
\]

We first show that

Lemma 4.20. \(E_{\beta,\ell}(1, f_{\phi^+_t})\) is continuous on \(\beta \in \text{Sym}_{4 \times 4}(Q_\ell)\) (\(\text{Sym}_{4 \times 4}(Q_\ell)\) is given the \(l\)-adic topology and \(\mathbb{C}\) the Euclidean topology).

Proof. We write \(E_{\beta}\) for \(E_{\beta,\ell}(1, f_{\phi^+_t})\) and \(S_0 = \text{Sym}_{4 \times 4}(\mathbb{Z}_\ell)\) in this proof. For any \(\beta, \beta + \beta_0 \in S_0\),

\[
|E_{\beta} - E_{\beta + \beta_0}|_\infty \leq \int_{b^{-1} \in N_\ell^2 S_0} |\det(b)|^{-3} |1 - e_{\ell}(-v(\beta b))|_\infty db.
\]

Here \(|\cdot|_\infty\) is the usual absolute value in \(\mathbb{C}\). Suppose that \(\beta_0 \in \ell^m S_0\) for \(m \gg 0\), then \(1 - e_{\ell}(-v(\beta_0 b)) = 0\) for any \(b \in \ell^{-m} S_0\). Thus the RHS of the above inequality is equal to

\[
\int_{b^{-1} \in N_\ell S_0, b \notin \ell^{-m} S_0} |\det(b)|^{-3} db.
\]

It is easy to see this last integral has a limit 0 when \(m \to +\infty\), which gives the continuity of \(E_{\beta}\) on \(\beta\).

Corollary 4.21. The Fourier coefficient \(E_{\beta,\ell}(1, f_{\phi^+_t})\) is not identically zero.

Proof. It suffices to show that \(E_{\beta,\ell}(1, f_{\phi^+_t}) \neq 0\) for \(\beta = 0\) by the above lemma. By change of variables from \(b\) to \(b' = b^{-1}\) in \(f(b)\), we see that

\[
E_{0,\ell}(1, f_{\phi^+_t}) = |\det(\eta_U)|^2 \int_{\text{Sym}_{4 \times 4}(\mathbb{Z}_\ell)} e_{\ell}(-v(\beta(b')^{-1})) |\det(b')| (\frac{-1}{\ell})^{v(\det(b'))} db'
\]

\[
= |\det(\eta_U)|^2 N_\ell^{20} \int_{\text{Sym}_{4 \times 4}(\mathbb{Z}_\ell)} |\det(b')| (\frac{-1}{\ell})^{v(\det(b'))} db'.
\]

So it suffices to show that the above integral does not vanish. The calculation of this integral is the same as the one in Lemma 4.19. For any \(\Lambda = \text{diag}(\ell^{k_1} \Lambda_1, \ldots, \ell^{k_r} \Lambda_r) \in Q\), we write

\[
I_0(\Lambda) = \int_{B \sim \Lambda} (\frac{-1}{\ell})^{v(\det(B))} |\det(B)| dB = (\frac{-1}{\ell})^{v(\det(B))} |\det(B)| I(\Lambda).
\]
Then $E_{0,t}(18, f_{\phi_+}) = |\det(\eta_U)|^2 |N_t|^{20} \sum_{\Lambda} I_0(\Lambda)$. Moreover we have (using the same notation as Lemma 4.19):

$$
I(\Lambda) = \frac{\ell^6 \alpha}{(\ell^{14} - 1)(\ell^2 - 1)(\ell^4 - 1)};
$$

$$
I(\Lambda) = \frac{\ell^2 \alpha}{(\ell^{14} - 1)((\frac{1}{\ell}) \ell^9 - 1)(\ell^2 - 1)};
$$

$$
I(\Lambda) = \frac{\ell^4 \alpha}{(\ell^{14} - 1)(\ell^5 - 1)(\ell^2 - 1)^2};
$$

$$
I(\Lambda) = \frac{\ell^2 \alpha}{(\ell^{14} - 1)((\frac{1}{\ell}) \ell^9 - 1)(\ell^2 - 1)};
$$

$$
I(\Lambda) = \frac{\ell^2 \alpha}{(\ell^{14} - 1)((\frac{1}{\ell}) \ell^9 - 1)((\frac{1}{\ell}) \ell^2 - 1)(\ell^2 - 1)};
$$

$$
I(\Lambda) = \frac{\ell^2 \alpha}{(\ell^{14} - 1)((\frac{1}{\ell}) \ell^9 - 1)((\frac{1}{\ell}) \ell^2 - 1)(\ell^2 - 1)};
$$

$$
I(\Lambda) = \frac{\alpha}{(\ell^{14} - 1)((\frac{1}{\ell}) \ell^9 - 1)(\ell^5 - 1)((\frac{1}{\ell}) \ell^2 - 1))}.
$$

Taking the sum of all these numbers, one gets

$$
\sum_{\Lambda \in \mathcal{Q}} I_0(\Lambda) = \frac{(\frac{1}{\ell} \ell^6 - 1))(\ell^3 - 1)(\ell - 1)}{(\ell^7 - 1)(\ell^5 - 1)((\frac{1}{\ell} \ell^2 - 1))} 
\neq 0,
$$

which concludes the proof of the lemma. □

By definition, we have

$$
E_{\beta,t}(18, f_{\phi_+}) = \lim_{n \to +\infty} \int_{t^{-n} \text{Sym}_{4 \times 4}(\mathbb{Z}_t)} e_t(\text{tr}(\frac{1}{2} w^t \eta_U w - \beta) b) db \int_{W_4^-(Q_t)} \phi_+^t(w) dw
$$

$$
= \lim_{n \to +\infty} \ell_0^n \int_{W_4^-(Q_t)} \phi_+^t(w) 1_{t^n \text{Sym}_{4 \times 4}(\mathbb{Z}_t)}(\frac{1}{2} w^t \eta_U w - \beta) dw
$$

$$
= \lim_{n \to +\infty} \ell_0^n \int_S e_t(b^t w) 1_{t^n \text{Sym}_{4 \times 4}(\mathbb{Z}_t)}(\frac{1}{2} w^t \eta_U w - \beta) dw.
$$

It is easy to see that

**Lemma 4.22.** For any positive integer $n > 0$, one has

$$
\int_S e_t(b^t w) 1_{t^n \text{Sym}_{4 \times 4}(\mathbb{Z}_t)}(\frac{1}{2} w^t \eta_U w - \beta) dw = \ell^{-24n} \sum_{w \in S/t^n S} e_t(b^t w) 1_{t^n \text{Sym}_{4 \times 4}(\mathbb{Z}_t)}(\frac{1}{2} w^t \eta_U w - \beta)
$$

We define the local density $D_\beta(n) = \sum_{w \in S/t^n S} e_t(b^t w) 1_{t^n \text{Sym}_{4 \times 4}(\mathbb{Z}_t)}(\frac{1}{2} w^t \eta_U w - \beta)$. Then one has

**Lemma 4.23.** Suppose $\det(\beta) \neq 0$ and $\ell^h \beta^{-1} \in \text{Sym}_{4 \times 4}(\mathbb{Z}_t)$. Then for any $n > h$,

$$
\ell^{-14n} D_\beta(n) = \ell^{-14(n+1)} D_\beta(n + 1).
$$
Proof. The proof is the same as [Kit83, Lemma 5.6.1]. □

Corollary 4.24. Suppose \( \det(\beta) \neq 0 \), then

\[
E_{\beta, \ell}(1_8, f_{\phi^\ell}) \in \mathbb{Z}_p[\zeta]\n\]

Proof. Indeed, the above lemmas show that \( E_{\beta, \ell}(1_8, f_{\phi^\ell}) \) is equal to \( \ell^{-14n}D_\beta(n) \) for \( n \gg 0 \) (depending on \( \beta \) though). It follows from definition that \( D_\beta(n) \in \mathbb{Z}[\zeta] \). Since \( \ell \) is different from \( p \), we see that \( E_{\beta, \ell}(1_8, f_{\phi^\ell}) \) lies in \( \mathbb{Z}_p[\zeta] \). □

Remark 4.25. By Lemma 4.26, we see that the local Fourier coefficient at \( \infty \) vanishes \( E_{\beta, \infty}(g_x, f_{\phi^\infty}) = 0 \) for any \( I \in \mathcal{I}_{\mathbb{Q}} \) when \( \beta \) is singular, thus so does the global Fourier coefficient. So there is no need to consider the range of values of \( E_{\beta, \ell}(1_8, f_{\phi^\ell}) \) for singular \( \beta \) for \( \ell \mid N \).

4.5. Ramified place \( p \). Now consider the case \( \ell = p \).

Recall that we have fixed an admissible character

\[
\kappa = \kappa_{\text{alg}} \times \kappa_{\ell} = (k_0, k_1, k_2) \times (\kappa_0, \kappa_1, \kappa_2): T_{G_1}(\mathbb{Z}_p) \to \mathbb{Q}_p^\times.
\]

For any finite order character \( \chi \) of \( \mathbb{Z}_p^\times \), we write \( p^c(\chi) \) for its conductor and \( C(\chi) = p^{c(\chi)} = p^{\max(1,c'(\chi))} \). We also write \( c(\kappa) = \max(c(\kappa_1), c(\kappa_2)) \). In the following we also write \( c \) for \( c(\kappa) \) when no confusion is possible.

4.5.1. Choice of local sections. We first choose sections in \( \mathcal{S}(W_i^- (\mathbb{Q}_p)) \) for \( i = 1, 2 \).

We write \( \tilde{\kappa}_2 = \kappa_2(\frac{c}{p})^{c'(\kappa_2)} \). We define a Schwartz-Bruhat function \( \alpha_{2, \mathbb{Q}_p} \) on \( \text{Sym}_2(\mathbb{Q}_p) \) whose Fourier transform is given by

\[
\tilde{\alpha}_{2, \mathbb{Q}_p}(\beta_2) = 1_{\text{GL}_2(\mathbb{Z})}(\beta_2) \times (\kappa_0, \kappa_2^{-1})(\det_1(\beta_2)) \times (\kappa_2)(\det_2(\beta_2)).
\]

Similarly, we define also a Schwartz-Bruhat function \( \alpha_{4, \mathbb{Q}_p} \) on \( \text{Sym}_4(\mathbb{Q}_p) \) whose Fourier transform is given by

\[
\tilde{\alpha}_{4, \mathbb{Q}_p}(\beta_2) = 1_{\text{GL}_2(\mathbb{Z})}(\beta_2) \times (\kappa_0, \kappa_2^{-1})(\det_1(\beta_2)) \times (\kappa_2)(\det_2(\beta_2)).
\]

Moreover, we fix \( w_0 \in W_i^-(\mathbb{Z}_p) \) such that its image in \( W_i^-(\mathbb{Z}/p) \) is a matrix of full rank (= 2) and \( w_0 w_U w_0^t = 0 \) (it is easy to see that such \( w_0 \) exists). We fix also some integer \( s \in \mathbb{Z} \) such that \( s + c(\kappa) \leq 0 \) (this gives the freedom in choosing the support of the sections defined below. For application we can take \( s = -c(\kappa) \)). We define \( \phi_{i, \mathbb{Q}_p} \in \mathcal{S}(W_i^- (\mathbb{Q}_p)) \) for \( i = 1, 2 \) as follows:

\[
\phi_{1, \mathbb{Q}_p}(y_1) = 1_{w_0 + C(\chi)W_i^- (\mathbb{Z}_p)}(y_1/p^s),
\]

\[
\phi_{2, \mathbb{Q}_p}(y_2) = 1_{W_i^- (\mathbb{Z}_p)}(y_2/p^s)\tilde{\alpha}_{2, \mathbb{Q}_p}(\frac{1}{2}y_2 w_U w_0^t/2p^s)\tilde{1}_{\text{M}_2(\mathbb{Z}_p)}(y_2 w_U w_0^t/2p^s),
\]

\[
\phi_{2, \mathbb{Q}_p}(y) = \phi_{1, \mathbb{Q}_p}(y_1)\phi_{2, \mathbb{Q}_p}(y_2).
\]

To determine the Siegel section \( f_{\phi_{2, \mathbb{Q}_p}} \) associated to \( \phi_{2, \mathbb{Q}_p} \), we need some results on local densities. For any \( \beta_2 \in \text{Sym}_2(\mathbb{Z}_p) \cap \text{GL}(2, \mathbb{Z}_p) \), we define the following integral (\( c = c(\kappa) \)):

\[
\alpha_{2, \mathbb{Q}_p}(\beta_2) = \int_{W_i^- (\mathbb{Z}_p)} \int_{W_i^- (\mathbb{Z}_p)} 1_{W_i^- (\mathbb{Z}_p)}(y_2)\tilde{1}_{\text{Sym}_2(\mathbb{Z}_p)}(1/2)\tilde{y}_2 w_U w_0^t/2p^s dy_2 dy_1.
\]

Then one can show that

Lemma 4.26. The integral \( \alpha_{2, \mathbb{Q}_p}(\beta_2) \neq 0 \) if and only if \( \det(\beta_2) \det(\eta_U) \) is a square in \( \mathbb{Z}_p^\times \) (i.e.,

\[
\left( \frac{\det(\beta_2)}{p} \right) = \left( \frac{\det(\eta_U)}{p} \right).
\]

Moreover, if \( \left( \frac{\det(\beta_2)}{p} \right) = \left( \frac{\det(\eta_U)}{p} \right) \), we have \( \alpha_{2, \mathbb{Q}_p}(\beta_2) = p^{-23c'(1 - (\frac{1}{p} - \frac{1}{p}))}. \)
Proposition 4.27. \[ \det(A) \]

Otherwise, we have

Proof. We define a function \( 1_0(y) \) on \( W_2^-(\mathbb{Z}/p^c) \) such that \( 1_0(y) = 1 \) if \( y \equiv 0(\text{mod } p^c) \) and \( = 0 \) otherwise. Clearly, we have

\[ \alpha'_{\mathcal{L}, p}(\beta_2) = p^{-24c} \sum_{y_2 \in W_2^-(\mathbb{Z}/p^c)} 1_0(y_2\eta_Uy_2' - \beta_2)1_0(y_2\eta_Uw_0'). \]

One can show that there exists an element \( \tilde{w}_0 \in W_1^-(\mathbb{Z}/p) \) such that \( \tilde{w}_0\eta_Uw_0' \) has its image in \( \text{GL}_2(\mathbb{Z}/p) \) and \( \tilde{w}_0 \) forms the identity matrix \( 1_4 \). Now consider any rank 2-subspace \( U_2(\mathbb{Z}/p) \) of \( \text{GL}_2(\mathbb{Z}/p) \) whose vectors are orthogonal to the row vectors of \( \tilde{w}_0 \). Then the symmetric bilinear form \( \eta_U \) on \( U_2 \) becomes \( \eta_U = \det(\eta_U) = \det(\eta_U)(\text{mod } (\mathbb{Z}/p)^2) \). Now the above sum is equal to

\[ \alpha'_{\mathcal{L}, p}(\beta_2) = p^{-24c} \sum_{y_1 \in (U_2 \otimes V_2^-)(\mathbb{Z}/p^c)} \frac{1}{2}y_2\eta_Uy_2' - \beta_2 = p^{-24c} \sum_{y_1 \in \text{GL}_2(\mathbb{Z}/p^c)} \frac{1}{2}y_2\eta_Uy_2' - \beta_2. \]

We define auxiliary characters \( \kappa_i' = \kappa_i(\frac{a}{p}) \) for \( i = 1, 2 \) and put \( \kappa' = (\kappa'_1, \kappa'_2) \).

Proposition 4.27. Write \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \text{G}_4(\mathbb{Q}_p) \). Then \( f_{\phi^\pm_{\mathcal{L}, p}}(g) \) satisfies: if \( \det(C) \neq 0 \),

\[ f_{\phi^\pm_{\mathcal{L}, p}}(g) = \xi(\det(C)|\det(C)|^{-3p^{-24s-13c}/2}(1 - \frac{1}{p})\frac{1}{p})(\alpha_{\mathcal{L}, p}(p^{2s}C^{-1}D) + \alpha_{\mathcal{L}, p}(p^{2s}C^{-1}D)); \]

otherwise, it is zero.

Proof. We can write \( \phi^\pm_{\mathcal{L}, p}(y) \) as a finite linear combination of characteristic functions \( \sum_i a_i 1_{p^{r_i}+p^{r_i}+\cdots} \) with \( a_i \neq 0 \) and \( r_i \neq r_j \) for \( i \neq j \). Moreover, the image of each \( r_i \) in \( W_i^-(\mathbb{Z}/p) \) is of rank 4. Then by the same reasoning as in Corollary 4.13, we see that the function \( \omega_{W_i^-}(g)\phi^\pm_{\mathcal{L}, p}(0) \) vanishes unless \( \det(CiD) \neq 0 \) and \( \omega_{W_i^-}(J_{8u}(u))\phi^\pm_{\mathcal{L}, p}(0) \) vanishes unless \( D \in p^{-(2s+c)}\text{Sym}_{4 \times 4}(\mathbb{Z}/p) \). By definition of \( f_{\phi^\pm_{\mathcal{L}, p}} \) if \( \det(C) \neq 0 \), we have

\[ f_{\phi^\pm_{\mathcal{L}, p}}(g) = \xi(\det(C)|\det(C)|^{-3p^{-24s-13c}/2}(1 - \frac{1}{p})\frac{1}{p})(\omega_{W_i^-}(u(C^{-1}D))\phi^\pm_{\mathcal{L}, p}(0)). \]
We write \( f(D) = \omega_{W_1}(J_{su}(D))\phi_{\Delta p}^+(0) \). To evaluate \( f(D) \), we first look at its Fourier transform \( \widehat{f}(\beta) \). We have

\[
\widehat{f}(\beta) = \int_{W_1^{-1}(Q_p)} \phi_{\Delta p}^+(y)e_p(\frac{1}{2} \text{tr} y^t D\eta y) dy \int_{p^{-2s+c}\text{Sym}_{4\times 4}(Z_p)} e_p(-\text{tr}(\beta D)) dD
\]

\[
= p^{10(2s+c)} \int_{W_1^{-1}(Q_p)} \phi_{\Delta p}^+(y)p^{2s+c}\text{Sym}_{4\times 4}(Z_p)(\frac{1}{2} y\eta y^t - \beta) dy
\]

\[
= p^{10(2s+c)-24s}\alpha_{2,\Delta p}(p^{-2s}\beta_2)1_{p^{2s}\text{Sym}_{4\times 4}(Z_p)}(p^{-2s}\beta_0)1_{p^{2s}\text{Sym}_{4\times 4}(Z_p)}(p^{-2s}\beta_1)
\]

\[
\times \int_{W_1^{-1}(Z_p)} \int_{\omega_0 + p^{e}W_1^{-1}(Z_p)} 1_{p^{2s}\text{Sym}_{4\times 4}(Z_p)}(\frac{1}{2} y_2\eta y_2 y_2^t - p^{-2s}\beta_2)1_{p^{2s}\text{Sym}_{4\times 4}(Z_p)}(y_2\eta y_2^t y_0^t) dy_1 dy_2
\]

\[
= p^{10(2s+c)-24s}\alpha_{2,\Delta p}(p^{-2s}\beta_2)1_{p^{2s}\text{Sym}_{4\times 4}(Z_p)}(p^{-2s}\beta_0)1_{p^{2s}\text{Sym}_{4\times 4}(Z_p)}(p^{-2s}\beta_1)\alpha_{2,\Delta p}(p^{-2s}\beta_2)
\]

\[
= \frac{p^{10(2s+c)-24s-23c}}{2}(1 - \frac{1}{p})\alpha_{\Delta}(\alpha_{\Delta p}(p^{2s}D) + \alpha_{\Delta p}(p^{2s}D)).
\]

Therefore, taking the inverse Fourier transform, we get

\[
f(D) = \frac{p^{24s-13c}}{2}(1 - \frac{1}{p})\alpha_{\Delta}(\alpha_{\Delta p}(p^{2s}D) + \alpha_{\Delta p}(p^{2s}D)).
\]

This finishes the proof of the lemma.

As for \( f_{\phi_{\Delta}} \in \text{Ind}_{P_d(Q_p)}^{G_d(Q_p)}(\xi) \), we have

**Corollary 4.28.** For any \( g_1 \in G_1^1(Q_p) \), if \( g_1 + 14 \notin p^{-(2s+c)}\text{Sym}_{4\times 4}(Z_p) \), \( f_{\phi_{\Delta}}(\iota(g_1, 1)) = 0 \). Otherwise,

\[
f_{\phi_{\Delta}}(\iota(g_1, 1)) = \frac{p^{24s-13c}}{2}(1 - \frac{1}{p})\alpha_{\Delta}(\alpha_{\Delta p}(p^{2s}g_1 + \frac{1}{2}\Delta) + \alpha_{\Delta p}(p^{2s}g_1 + \frac{1}{2}\Delta)).
\]

Recall that \( \Delta = \begin{pmatrix} 0 & 12 \\ 12 & 0 \end{pmatrix} \).

**Proof.** By definition of \( f_{\phi_{\Delta}} \), if \( \det(\Delta(g_1 - 1)) = 0 \), then \( f_{\phi_{\Delta}}(\iota(g_1, 1)) = 0 \). Otherwise,

\[
f_{\phi_{\Delta}}(\iota(g_1, 1)) = f_{\phi_{\Delta}}(\begin{pmatrix} * & * \\ \Delta(g_1 - 1) & \frac{1}{2}\Delta(g_1 + 1)\Delta \end{pmatrix}).
\]

By the proof of the above lemma, the non-vanishing of \( f_{\phi_{\Delta}}(\iota(g_1, 1)) \) implies that \( \frac{1}{2}(g_1 - 1)^{-1}(g_1 + 1)\Delta \in p^{-(2s+c)}\text{Sym}_{4\times 4}(Z_p) \). This give the first part of the corollary. Now assume that \( g_1 + 1 \in p^{-(2s+c)}\text{Sym}_{4\times 4}(Z_p) \), recall also that \(-2s - c > 0, -2s - 2c \geq 0,\)

\[
\frac{1}{2}(g_1 - 1)^{-1}(g_1 + 1)\Delta = \frac{1}{2}\Delta - \frac{1}{2}(1 - \frac{g_1 + 1}{2})^{-1}\Delta = -(\frac{g_1 + 1}{2} + (\frac{g_1 + 1}{2})^2 + \cdots)\Delta.
\]

It is easy to see that \( \alpha_{\Delta p}(D) = \alpha_{\Delta p}(D + D') \), \( \alpha_{\Delta p}(D) = \alpha_{\Delta p}(D + D') \) for any \( D' \in \text{Sym}_{4\times 4}(Z_p) \).

Now that \( p^{2s}(g_1 + 1)^n \in p^{2s-n(2s+c)}\text{Sym}_{4\times 4}(Z_p) \) and \( 2s - n(2s + c) \geq 0 \) for \( n \geq 2 \). This show that

\[
\alpha_{\Delta p}(p^{2s}g_1 + \frac{1}{2}\Delta) = \alpha_{\Delta p}(p^{2s}g_1 + \frac{1}{2}\Delta)
\]

and similarly for \( \alpha_{\Delta p} \), which proves the lemma.
4.5.2. Local zeta integral. Next we compute the local zeta integral $Z_p(\varphi_p, \varphi'_p, f_{\phi_\xi, p})$. The computation follows closely that of [Liu15b, Section 5]. Let’s first recall the notions of $U_p$ operators and Jacquet modules. For any $\underline{t} = (t_1, t_2) \in \mathbb{Z}^2$, we set $p^\underline{t} = \text{diag}(p^{t_1}, p^{t_2}, p^{-t_1}, p^{-t_2}) \in G_1(\mathbb{Q})$. Let $C^+$ be the set of $\underline{t} \in \mathbb{Z}^2$ such that $t_1 \geq t_2 \geq 0$. Then for any $\underline{t} \in C^+$, we can define the $U_p$-operator $U_{p, \underline{t}}$ acting on $f \in \mathcal{A}(G_1(\mathbb{Q}) \backslash G_1(\mathbb{A}) / \Gamma)_\underline{t}$ as follows:

$$U_{p, \underline{t}} f(g) = p^{\underline{t} + 2 \rho_{\text{Sp}(v), c}} \int_{N_{G_1}(\mathbb{Z}_p)} f(g u p^\underline{t}) du$$

where $\rho_{\text{Sp}(v), c}$ is half the sum of the positive compact roots of $\mathfrak{g}_1$. We then define $U_{p, 1} = U_{p, (1, 0)}$, $U_{p, 2} = U_{p, (1, 1)}$, $U_p = U_{p, 1} U_{p, 2}$. Now for any nearly holomorphic form $\phi$, the limit $\lim_{n \to +\infty} U_{p, n} \phi$ exists and is denoted by $e \phi$. This operator $e$ is in fact a polynomial in $U_p$.

Let $\pi \simeq \otimes_v \pi_v$ be an antiholomorphic irreducible cuspidal automorphic representation of $[G_1]$. Suppose that $\bar{\pi}_\infty \simeq D_k$ is a holomorphic discrete series. We write $e \phi$ for the ordinary projection of any $\phi$ in $\pi$ or $\pi$. It can be shown [Liu15a, Corollary 3.10.3] that $e \pi$ is inside the subspace of antiholomorphic forms in $\pi$. Moreover, if $e \pi$ is non-zero, $\pi_p$ is isomorphic to a composition factor of certain principal series, the projection of $e \pi$ to $\pi_p$ is of dimension 1, the action of the $U_p$ operators on $\bigcap_{\underline{t} \in C^+} U_{p, \underline{t}}(\pi_p)$ is semi-simple (cf. [Liu15b, Section 5.5]). When $e \pi$ is non-zero, there exist $a_1, a_2 \in \mathcal{O}_{p, \pi}^\times$ such that $U_{p, \underline{t}}$ acts on $e \pi$ by the scalar $a_1^{\kappa_1} a_2^{\kappa_2}$ for all $\underline{t} = (t_1, t_2) \in C^+$. Now we set $\kappa_1 = p^{-(k_1 - 1)} a_1$, $\kappa_2 = p^{-(k_2 - 2)} a_2$ and define characters $\theta_1$ and $\theta_2$ of $\mathcal{O}_{p, \pi}^\times$ characterized by

$$\theta_i(x) = \begin{cases} \kappa_i(x) & \text{if } x \in \mathbb{Z}_p^\times \\ \alpha_i & \text{if } x = p. \end{cases}$$

Then by [Liu15b, Section 5.5], $\pi_p$ embeds in the normalized induction $\text{Ind}^{G_1(\mathbb{Q}_p)}_{B_1(\mathbb{Q}_p)}(\theta)$ where $\theta = (\theta_0, \theta_1, \theta_2)$ is a character of $T_{G_1}(\mathbb{Q}_p)$ (we do not precise $\theta_0$). Moreover their ordinary subspaces coincide, both being of dimension 1. Moreover, $(\pi)_p = (\pi')_p$ embeds in $\text{Ind}^{G_1(\mathbb{Q}_p)}_{B_1(\mathbb{Q}_p)}(\theta^{-1})$. We now take a vector $\varphi \in \pi$ such that its ordinary projection $e \varphi = U_{p, \underline{t}} \varphi \neq 0$ for $\underline{t} \gg 0$. Then it is easy to verify that the vector $\tilde{\varphi}_\xi(g) = \langle \varphi_{\text{Sp}(v), \xi} \rangle (\mathcal{E}_{\kappa_1, \xi}, e_{\text{can}})(\cdot, g), e \varphi \rangle$ is also ordinary, therefore is a multiple of $e \varphi$:

$$\tilde{\varphi}_\xi = C_{\kappa, \pi} e \varphi.$$  

We next evaluate this constant $C_{\kappa, \pi} \in \mathbb{C}$. The strategy is to pair the above two automorphic forms $\tilde{\varphi}_\xi$ and $\varphi$ against some other form $\varphi' \in \pi$ and then use some local models to choose some special vectors $\varphi$ and $\varphi'$ to get the value of $C_{\kappa, \pi}$. By definition, we have

$$\langle \tilde{\varphi}_\xi, \varphi' \rangle = \int_{G_1(\mathbb{Q}_p)} \int_{G_1(\mathbb{A})} \int_{[G_1]} f_{\phi_\xi, p}(\epsilon(g', 1)) \varphi(g g' u p^\underline{t}) \varphi'(g) dgdg'du.$$  

We write the local zeta integrals of the above global integral for places $v \nmid p$ as $\tilde{L}_v(1, St(\pi) \otimes \xi) \langle \varphi_v, \varphi'_v \rangle$ and the product of the local L-factors is denoted by $\tilde{L}(1, St(\pi) \otimes \xi) = \prod_{v \nmid p} \tilde{L}_v(1, St(\pi) \otimes \xi)$. Then we have

$$\langle \tilde{\varphi}_\xi, \varphi' \rangle = \tilde{L}(1, St(\pi) \otimes \xi) \int_{G_1(\mathbb{Q}_p)} \int_{G_1(\mathbb{Q}_p)} \int_{[G_1]} f_{\phi_\xi, p}(\epsilon(g', 1)) \varphi(g g' u p^\underline{t}) \varphi'(g) dgdg'du.$$  

From this we see that to evaluate $C_{\kappa, \pi}$, it suffices to evaluate the quotient

$$C_{\kappa, \pi} = \frac{\int_{G_1(\mathbb{Q}_p)} \int_{G_1(\mathbb{Q}_p)} \int_{[G_1]} f_{\phi_\xi, p}(\epsilon(g', 1)) \varphi(g g' u p^\underline{t}) \varphi'(g) dgdg'du}{\int_{G_1(\mathbb{Q}_p)} \int_{[G_1]} \varphi(g u p^\underline{t}) \varphi'(g) dgdg'du}.$$
Now we reduce the above integrals to local situations and use certain local model of $\pi_p$ to do the computation. We take $\varphi' \in \text{Ind}_{B_{G_1}(\mathbb{Q}_p)}^{G_1(\mathbb{Q}_p)}(\theta^{-1})$ supported on $B_{G_1}(\mathbb{Q}_p)J_4 N_{G_1}(\mathbb{Q}_p)$. We then choose some sufficiently small open compact subgroup $K$ of $G_1(\mathbb{Q}_p)$ such that the vectors $\int_{N_{G_1}(\mathbb{Z}_p)} \varphi'(g p^{-u} u^{-1}) d u$ and $\int_{G_1(\mathbb{Q}_p)} \int_{N_{G_1}(\mathbb{Z}_p)} f_{\varphi_1}(u(g', 1)) \varphi'(g p^{-u} u^{-1}(g')^{-1}) d u d g'$ for all $u \gg 0$ are all fixed by translation of $K$.

Then we take $\varphi \in \text{Ind}_{B_{G_1}(\mathbb{Q}_p)}^{G_1(\mathbb{Q}_p)}(\theta)$ supported on $B_{G_1}(\mathbb{Q}_p)K$ and taking value 1 on $K$. The local pairing between $\varphi$ and $\varphi'$ is given by $\langle \varphi, \varphi' \rangle = \int_{G_1(\mathbb{Z}_p)} \varphi(g) \varphi'(g) d g$. So we see that

$$C_{\kappa, \pi} = \frac{\int_{N_{G_1}(\mathbb{Z}_p)} \int_{G_1(\mathbb{Q}_p)} f_{\varphi_1}(u(g', 1)) \varphi(g p^{-u} u^{-1}) \varphi'(g) d g d g'}{\int_{N_{G_1}(\mathbb{Z}_p)} \int_{G_1(\mathbb{Q}_p)} \varphi(g p u^{-1}) \varphi'(g) d g d u}$$

$$= \frac{\int_{N_{G_1}(\mathbb{Z}_p)} \int_{G_1(\mathbb{Q}_p)} \frac{f_{\varphi_1}(u(g', 1)) \varphi(g) \varphi'(g p^{-u} u^{-1}(g')^{-1}) d g d g'}}{\int_{N_{G_1}(\mathbb{Z}_p)} \int_{G_1(\mathbb{Q}_p)} \varphi'(p^{-u} u^{-1}) d g d u}$$

$$= \int_{G_1(\mathbb{Q}_p)} f_{\varphi_1}(u(g', 1)) \varphi'((g')^{-1}) d g'.$$

By Corollary 4.28, we can evaluate the last integral as follows. We denote

$$I(\kappa) = \int_{G_1(\mathbb{Q}_p)} \alpha_{\varphi_1}(-p^{-2s} g + \frac{1}{2} \Delta) \varphi'(g^{-1}) d g, \quad I(\kappa') = \int_{G_1(\mathbb{Q}_p)} \alpha_{\varphi_1}(-p^{-2s} g + \frac{1}{2} \Delta) \varphi'(g^{-1}) d g$$

Note that the support of $\varphi'$ is $B_{G_1}(\mathbb{Q}_p) w_4 N_{G_1}(\mathbb{Q}_p)$, we can write an element $g \in G_1(\mathbb{Q}_p)$ in this set as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = m(A) u(B) J_4 u(B').$$

As such, $\varphi'(g^{-1}) = \theta^{-1}(-A^{-1}) = \theta(c)$. Therefore, we have

$$I(\kappa) = \int_{g+1 \in \mathbb{Z}_p} \alpha_{\varphi_1}(-p^{2s} \left( \begin{array}{cc} b & a + 1 \\ d & c \end{array} \right)) \theta(c) d g$$

$$= p^{-6s} \int_{g+1 \in \mathbb{Z}_p} \left( \begin{array}{cc} -p^{2s} b & 1 \\ d & c \end{array} \right) \alpha_{\varphi_1}(-p^{2s} c) \theta(c) d g$$

$$= p^{16s} \theta^{-1}(-2p^{-2s}) \int_{M_{2 \times 2}(\mathbb{Z}_p)} \alpha_{\varphi_1}(c) \theta(c) d c.$$ (similar formula for $I(\kappa')$) The evaluation of this last integral is given in the next lemma, whose proof follows quite closely [Liu15b, Section 5.7]. For any finite order character $\chi$ of $\mathbb{Z}_p^\times$, we define $\chi^\circ = 1$ if $\chi$ is the trivial character and $\chi^\circ = 0$ otherwise. $G(\chi)$ is the Gauss sum of $\chi$. Recall that $\theta_i(p) = \alpha_i, \theta_i|_{\mathbb{Z}_p^\times} = \kappa_i$ and $\kappa_2 = \kappa_2(p)^c(\kappa_2).

Lemma 4.29. We have

$$(1) \quad \int_{M_{2 \times 2}(\mathbb{Z}_p)} \alpha_{\varphi_1}(c) \theta(c) d c = \left( (1 - \frac{1}{p})(-1)^{\kappa_1^\circ} \frac{1 - \kappa_1^\circ \alpha_1^{-1}}{1 - \kappa_1^\circ \alpha_1 p^{-1}} G(\kappa_1) \alpha_1^{-c(\kappa_1)} \right) \times \left( (1 - \frac{1}{p})(-1)^{\kappa_2^\circ} \frac{1 - \kappa_2^\circ \alpha_2^{-1}}{1 - \kappa_2^\circ \alpha_2 p^{-1}} G(\kappa_2) \alpha_2^{-c(\kappa_2)} \varepsilon(p^{-c(\kappa_2)} p^{-c(\kappa_2)}) \right).$$
\[ (2) \quad \int_{M_{2\times 2}(\mathbb{Q}_p)} \alpha_2 \varphi_p(c)\theta(c)dc = 0. \]

**Proof.** By the support of \( \varphi' \), only those upper triangular matrices \( c \) contribute to the integral in the lemma. Moreover, \( \alpha_2 \varphi_p(c) = \alpha_2 \varphi_p(c') \) where \( c - c' \in \text{Sym}_{2\times 2}(\mathbb{Z}_p) \). Thus we can evaluate the integral by considering only those diagonal matrices \( c = \text{diag}(c_1, c_2) \). First we compute \( \alpha_2 \varphi_p(c) \). Write a matrix \( \left( \begin{array}{ccc} x & y \\ y & z \end{array} \right) \in \text{Sym}_{2\times 2}(\mathbb{Z}_p) \):

\[
\alpha_2 \varphi_p(c) = \int_{\text{Sym}_{2\times 2}(\mathbb{Z}_p)} (\kappa_2^{-1}(x)\bar{\kappa}_2(xz - y^2)e_p(-(c_1x + c_2z))) dxdydz = \int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \kappa_1(x)e_p(-c_1x - c_2y^2x^{-1})dxdy \int_{\mathbb{Z}_p} \bar{\kappa}_2(z)e_p(-c_2z)dz.
\]

First assume that \( \bar{\kappa}_2 \) is a non-trivial character, then \( \int_{\mathbb{Z}_p} \bar{\kappa}_2(z)e_p(-c_2z)dz = p^{-c'(<\hat{\kappa}_2>)}G(\bar{\kappa}_2)\bar{\kappa}_2^{-1}(p^{c'(<\hat{\kappa}_2>)}c_2). \)

Now we have

\[
\alpha_2 \varphi_p(c) = p^{-c'(<\hat{\kappa}_2>)}G(\bar{\kappa}_2)\bar{\kappa}_2^{-1}(p^{c'(<\hat{\kappa}_2>)}c_2)\int_{\mathbb{Z}_p} \kappa_1(x)\left(\frac{x}{p}\right)^{c'(<\hat{\kappa}_2>)}e_p(-c_1x)dx.
\]

We then integrate on the variable \( c_2 \):

\[
\int_{\mathbb{Q}_p} \alpha_2 \varphi_p(\text{diag}(c_1, c_2))\theta_2(c_2)dc_2 = p^{-c'(<\hat{\kappa}_2>/2)}G(\bar{\kappa}_2)\varepsilon(p^{c'(<\hat{\kappa}_2>)}\alpha_2^{-c'(<\hat{\kappa}_2>)}(1 - \frac{1}{p}) \int_{\mathbb{Z}_p} \kappa_1(x)\left(\frac{x}{p}\right)^{c'(<\hat{\kappa}_2>)}e_p(-c_1x)dx.
\]

Note that the product before the above integral is exactly the product in the second big bracket in the lemma for the case \( \bar{\kappa}_2 \) non-trivial.

Next we assume \( \bar{\kappa}_2 = 1 \), then \( \int_{\mathbb{Z}_p} \bar{\kappa}_2(z)e_p(-c_2z)dz = -p^{-1}1_{\mathbb{Z}_p}(c_2) + (1 - p^{-1})1_{\mathbb{Z}_p}(c_2) \). Then the integration on \( c_2 \) gives

\[
\int_{\mathbb{Q}_p} \alpha_2 \varphi_p(\text{diag}(c_1, c_2))\theta_2(c_2)dc_2 = \left((-p^{-1/2})\varepsilon(p)\theta_2^{-1}(p)(1 - \frac{1}{p})\right)\varepsilon(p^{c'(<\hat{\kappa}_2>)}) \int_{\mathbb{Z}_p} \kappa_1(x)\theta_1(x)e_p(-c_1x)dx + \left(1 - \frac{1}{p}\right)^2 \frac{1}{1 - \alpha_2 p^{-1}} \int_{\mathbb{Z}_p} \kappa_1(x)\theta_1(x)e_p(-c_1x)dx.
\]

The term in the first big bracket corresponds to the case \( \kappa_2 = (\frac{\varphi}{p}) \) while the term in the second bracket corresponds to the case \( \kappa_2 = 1 \). Moreover, the product before each integral in each bracket is exactly the formula given in the lemma.

The integration on the variable \( c_1 \) is exactly the same as in the case of \( c_2 \) (in fact simpler than \( c_2 \)). The case of \( \kappa' \) is similar to \( \kappa \). We thus omit the calculation.

In summary we get the following

**Corollary 4.30.** We set \( s = -c = -c(\kappa) \), then for any \( \varphi \in \pi \) and \( \varphi' \in \pi \) such that \( \langle e\varphi, \varphi' \rangle \neq 0 \), we have

\[
\frac{Z_p((e\varphi)_p, \varphi'_p, f_{\text{phi}_p})}{\langle (e\varphi)_p, \varphi'_p \rangle} = (1 - \frac{1}{p})(-1)^{\kappa'} \frac{1 - \kappa_1^\omega \alpha_1^{-1}}{1 - \kappa_1^\omega \alpha_1 p^{-1}} G(\kappa_1) \alpha_1^{-c'(<\kappa_1>)} \times \left(1 - \frac{1}{p}\right)^{-\kappa_2^\omega} \frac{1 - \kappa_2^\omega \alpha_2^{-1}}{1 - \kappa_2^\omega \alpha_2 p^{-1}} G(\bar{\kappa}_2) \alpha_2^{-c'(<\bar{\kappa}_2>)} \varepsilon(p^{c'(<\bar{\kappa}_2>)})p^{-c'(<\bar{\kappa}_2>)}.\]
4.5.3. Local Fourier coefficients. Next we compute the Fourier coefficients \( E_{\beta,p}(1, f_{\phi_{\pm,p}^+}) \). It is easy to see that by Proposition [4.27]

**Lemma 4.31.** The local Fourier coefficient is

\[
E_{\beta,p}(1, f_{\phi_{\pm,p}^+}) = \frac{p^{-4s-13c}}{2} (1 - \left( \frac{-1}{p} \right) \overline{\alpha_{\phi_{\pm,p}}(p^{-2s}\beta)} + \overline{\alpha_{\phi_{-p}^{-1}}(p^{-2s}\beta)}), \quad \forall \beta \in \text{Sym}_{4\times 4}(\mathbb{Q}).
\]

4.6. Summary. Recall that we fix an arithmetic point \( \kappa = (k, \Xi) \in \text{Hom}_{\text{cont}}(T_{G_1}(\mathbb{Z}_p), \mathbb{T}_p^\vee) \) \((k_1 \geq k_2 \geq 3)\). Let \( \pi \) be an anti-holomorphic cuspidal automorphic representation of \( G_1(\mathbb{A}) \) of type \((k, \hat{\Gamma})\). Moreover, we fix \( \varphi = \otimes_v \varphi_v \in \pi \) to be a factorisable automorphic form such that \( \varphi_{\infty} \) lies in the highest \( K_{G_1,\Xi} \)-type of \( \pi_{\infty} \). We fix also a vector \( \varphi^\vee \in \pi^\vee = \mathbb{T} \) such that \( \langle \varphi, \varphi^\vee \rangle \neq 0 \). For each place \( v \) of \( \mathbb{Q} \), we have chosen theta sections \( \phi_{i_0;v}^+ \in \mathcal{S}(W_1^-(\mathbb{Q}_v)) \) \((i = 1, 2)\) as in the following

1. If \( v = \infty \), we fix one \( I \in J_{\kappa_1} \), then for \( X_{k;j,l} = W_{k;j,l}\sqrt{\eta_{1,l}} \) (cf. [4.13]),

\[
\phi_{i_0;\infty;I}(w_{-1}) = \sqrt{\alpha_I} \phi_{i_0;\infty}(w_{-1});
\]

2. If \( \ell \nmid Np \), then (cf. [4.5]):

\[
\phi_{i_0;\ell}(w_{-1}) = 1_{K_{\ell}^-}(w_{-1})
\]

3. If \( \ell | N \), then (cf. [4.13]):

\[
\phi_{i_0;\ell}(w_{-1}) = 1_{S_{10}}(w_{-1}) \phi_{i_0;\ell}(w_{-1})
\]

4. If \( \ell = p \), then (cf. [4.5.1]):

\[
\phi_{1;\ell,p}(y_1) = 1_{w_{1} + C\chi}(w_{1}^p)(y_1/p^s),
\]

\[
\phi_{2;\ell,p}(y_2) = 1_{W_2^-(\mathbb{Q}_v)}(y_2/p^s) \phi_{2;\ell,p}(x_2^p/2y_2^2p^s) \phi_{1;\ell,p}(x_2^p/2y_2^2p^s)
\]

For each place \( v \), we form the tensor product \( \phi_{i_0;v}^+ = \phi_{1;v}^+ \otimes \phi_{2;v}^+ \in \mathcal{S}(W_1^-(\mathbb{Q}_v)) \) and its image under the intertwining operator \( \delta \) is denoted by \( \phi_{\delta;v}^+ \in \mathcal{S}(W_4^d(\mathbb{Q}_v)) \). These sections give rise to Siegel sections \( f_{\phi_{\delta;v}^+} \). The latter then define Siegel Eisenstein series \( E(\cdot, f_{\phi_{\delta;v}^+}) \). We have also studied their local zeta integrals and local Fourier coefficients and showed that they are not identically zero and gave explicit expressions for Fourier coefficient at \( \infty \), zeta integral and Fourier coefficient at \( \ell \nmid Np \) and zeta integral and Fourier coefficient at \( p \). We put the modified local Euler factors as follows (cf. Corollary 4.9, Proposition 4.16, Lemma 4.19 and Corollary 4.30):

\[
L_{\infty}(1, \text{St}(\pi) \otimes \xi) = L_{\infty}(1, \text{St}(\pi) \otimes \xi) = \frac{Z(\varphi_{\kappa,\infty}, \varphi_{\kappa,\infty}, \xi), \phi_{\delta;1,\pi}^+)}{\langle \varphi_{\kappa,\infty}, \varphi_{\kappa,\infty} \rangle};
\]

\[
L_{\ell}(1, \text{St}(\pi) \otimes \xi) = \ell^{-20|\det(\eta_{1,\ell})|/\ell - 2(\frac{1}{\ell}, \ell - 1)(\ell - 1)} \quad \text{for \( \ell | N \)};
\]

\[
L_p(1, \text{St}(\pi) \otimes \xi) = (1 - \left( \frac{-1}{p} \right) \frac{1}{p}) \left[ \frac{1}{p} - \frac{1}{p} \right] \left( \frac{1}{p} - \frac{1}{p} \right) G(\kappa_1) \alpha_{\kappa_1}^{-c(\kappa_1)} \right]
\]

\[
\times \left( \frac{1}{p} - \frac{1}{p} \right) \left( \frac{1}{p} - \frac{1}{p} \right) \left( \frac{1}{p} - \frac{1}{p} \right) G(\kappa_2) \alpha_{\kappa_2}^{-c(\kappa_2)} \right]
\]

Then we write their product as \( L_{\infty}(1, \text{St}(\pi) \otimes \xi) = \prod_{\ell \nmid Np} L_{\ell}(1, \text{St}(\pi) \otimes \xi) \). Recall that we have defined periods \( \hat{P}[\pi] \) and \( P_\pi \) for \( \pi \) in Section 2.2.2. We take \( p \)-integral modular forms \( \varphi, \varphi^\vee \in H^3_{\mathcal{O}_\pi}(\hat{\Gamma}, \mathcal{O}_\pi)[\pi] \) (cf. Section 2.2.2). Then by definition \( \langle \varphi, \varphi^\vee \rangle / \hat{P}[\pi] \in \mathcal{O}_\pi \). We put these results into the following theorem
Theorem 4.32. Let the notations be as above. Assume Hypothesis 2.24, then we have the Rallis inner product formula
\[
\langle \Theta_{\phi_{2,i}}(\varphi), \Theta_{\phi_{2,i}}(\varphi') \rangle_{O(U)} = e^{\text{coh}(\pi)} \frac{\langle \varphi, \varphi' \rangle L_{N_{\text{p},\infty}}(1, \text{St}(\pi) \otimes \xi)L_{N_{\text{p},\infty}}^*(1, \text{St}(\pi) \otimes \xi)}{P_{\pi}}.
\]

4.7. Theta series. We have defined sections \( \phi_{i,\xi} \in \mathcal{S}(W_1^{-}(\mathbb{A})) \) as the restricted tensor product \( \phi_{i,\xi} = \otimes_v \phi_{i,\xi,v} \). In this subsection we study the algebraicity of the Fourier coefficients of the theta series \( \Theta_{\phi_{i,\xi}} \) for \( i = 1, 2 \) and \( \Theta_{\phi_{i,\xi}}^\pm \).

We fix \( h \in O(U)(\mathbb{A}) \). Recall that the Fourier coefficients \( \Theta_{\phi_{i,\xi},\beta}(g, h) \) of the theta series \( \Theta_{\phi_{i,\xi}}^\pm \) is defined as
\[
\Theta_{\phi_{i,\xi},\beta}(g, h) = \int_{[\text{Sym}_{4 \times 4}]} \Theta_{\phi_{i,\xi}}^\pm(u(x)g, h)e(-\text{tr} \beta x)dx
\]
\[
= \sum_{w^- \in W_4^-} \int_{[\text{Sym}_{4 \times 4}]} \omega(u(x)g, h)\phi_{\xi}^\pm(w^-)e(-\text{tr} \beta x)dx.
\]
The second equality comes from the fact that \( \omega(\cdot, h)\phi_{\xi}^\pm(w^-) \) for \( w^- \in W_4^-(\mathbb{Q}) \) is invariant under the action of the rational unipotent matrices \( u(x) \).

Recall that for any element in the Siegel upper half plane \( z = x + iy \in \mathbb{H}_2 \), we have an adelic element \( g_z = u(x)m(\sqrt{y}) \times 1_f \in G_4(\mathbb{A}) \). We then compute \( \Theta_{\phi_{2,\xi},\beta}(g_z, h) \). We have (viewing \( \sqrt{y}, x \) as adelic elements)
\[
\omega(u(x)g_z, h)\phi_{\xi}^\pm(w^-) = e(\text{tr}\left((\sqrt{y}^{-1}w^-)^t(x + x)\sqrt{y}^{-1}w^-\eta_U\right))\omega(1, h)\phi_{\xi}^\pm(\sqrt{y}^{-1}w^-)
\]
\[
= e(\text{tr}\left((\sqrt{y}^{-1}w^-)^t(x\sqrt{y}^{-1}w^-\eta_U\right))e(\text{tr}\left((\sqrt{y}^{-1}w^-)^tx\sqrt{y}^{-1}w^-\right))\omega(1, h)\phi_{\xi}^\pm(\sqrt{y}^{-1}w^-).
\]
Therefore we get (we put \( 1_\beta(x) = 1 \) if \( x = \beta \) and \( 0 \) otherwise)
\[
\int_{[\text{Sym}_{4 \times 4}]} \omega(u(x)g_z, h)\phi_{\xi}^\pm(w^-)e(-\text{tr} \beta x)dx = e(\text{tr}\left((\sqrt{y}^{-1}w^-)^t(x\sqrt{y}^{-1}w^-\eta_U\right))\omega(1, h)\phi_{\xi}^\pm(\sqrt{y}^{-1}w^-)
\]
\[
\times \int_{[\text{Sym}_{4 \times 4}]} e(\text{tr}\left((\sqrt{y}^{-1}w^-)^tx\sqrt{y}^{-1}w^-\eta_U - \beta x\right))dx
\]
\[
= e(\text{tr}(\beta x))\omega(1, h)\phi_{\xi}^\pm(\sqrt{y}^{-1}w^-)1_\beta(\sqrt{y}^{-1}w^- \eta_U(\sqrt{y}^{-1}w^-)^t).
\]
From this, we have
\[
\Theta_{\phi_{2,\xi},\beta}(g_z, h) = e(\text{tr}(\beta x)) \sum_{w^- \in W_4^-} \omega(1, h)\phi_{\xi}^\pm(\sqrt{y}^{-1}w^-)1_\beta(\sqrt{y}^{-1}w^- \eta_U(\sqrt{y}^{-1}w^-)^t).
\]
The condition \( \sqrt{y}^{-1}w^- \eta_U(\sqrt{y}^{-1}w^-)^t = \left((\sqrt{y}^{-1} \times 1_f)w^-\right)\eta_U\left((\sqrt{y}^{-1} \times 1_f)w^-\right)^t = \beta \) means that (at the archimedean place) \( \sqrt{y}^{-1}w^- \eta_U(\sqrt{y}^{-1}w^-)^t = \beta \) and (at the non-archimedean place) \( w^- \eta_U(w^-)^t = \beta \).
This shows that \( \sqrt{y} \beta \sqrt{y} = \beta \) and thus \( \text{tr}(y \beta) = \text{tr}(\beta) \). We then compute each factor \( \omega_v(1, h)\phi_{\xi,\nu}^\pm(\sqrt{y}^{-1}w^-) \) of \( \omega(1, h)\phi_{\xi}^\pm(\sqrt{y}^{-1}w^-) \) for all places \( v \) of \( \mathbb{Q} \).
(1) For $v = \infty$ (cf. 4.3),
\[
\omega_{\infty}(1, h_\infty) \phi^+_{L_\infty}(\sqrt{y}^{-1} w^-) = a_I(X_1 h_\infty)^I(X_2 h_\infty)^I e_\infty(i(\sqrt{y}^{-1} w^-, J_4 \sqrt{y}^{-1} w^-))
= a_I(X_1 h_\infty)^I(X_2 h_\infty)^I e_\infty(i \tr(y));
\]

(2) For $\ell \nmid N_p\infty$,
\[
\omega_{\ell}(1, h_\ell) \phi^+_{L_\ell}(w^-) = 1 w^-(z_\ell)(w^-h_\ell).
\]

(3) For $\ell | N$ (cf. (4.9)),
\[
\omega_{\ell}(1, h_\ell) \phi^+_{L_\ell}(w^-) = \phi^+_{L_\ell}(w^-h_\ell) = 1_{S_{[1]}}(w^-h_\ell)e_\ell(\tr(b_{[1]}^t w^-h_\ell)) \times 1_{S_{[2]}}(w^-h_\ell)e_\ell(\tr(b_{[2]}^t w^-h_\ell)) \in \mathbb{Z}[\mu_{N_\ell^2}].
\]

(4) For $\ell = p$ (cf. 4.5.1),
\[
\omega_{p}(1, h_p) \phi^+_{L_p}(w^-) = \phi^+_{L_p}(w^-h_p) \in \mathbb{Z}[\mu_{N_p^2C(\mathcal{Q})}].
\]

Note that for each fixed $h$, the summation in $\Theta_{\phi^+_{L_p}, \beta}(g_z, h)$ is always a finite sum. Thus

**Proposition 4.33.** The Fourier coefficients $\Theta_{\phi^+_{L_p}, \beta}(g_z, h)e(-\tr(\beta z))$ and $\Theta_{\phi^+_{L_p}, \beta, i}(g_z, h)e(-\tr(\beta_i z_i))$ ($i = 1, 2$) are in the cyclotomic ring $\mathbb{Z}_p[\mu_{N_p^2C(\mathcal{Q})]}]$.

**Definition 4.34.** For $i = 1, 2$, we write $\Theta_{\phi^+_{L_p}, \beta, i}(h)$ for the preimage of $\Theta_{\phi^+_{L_p}, \beta, i}(g_z, h)$ in $H^0(\tilde{\mathbb{A}}_{G_i, \hat{\mathfrak{t}}}; \mathcal{E}(W_i))$ for some algebraic representation $W_i$ of $\text{GL}_2$ by the map $\Phi(\cdot, \varepsilon)$ (cf. (2.3)).

**Remark 4.35.** Though we do not explicate the representation $W_i$, we know at least that it contains a copy of $W_k$ by the non-vanishing of the archimedean local zeta integral (cf. Corollary 4.9).

5. Transfer from $\text{GSp}_4$ to $\text{U}_4$

In this section, we use results of Atobe and Gan [AG16] to deduce the functoriality from the symplectic group $\text{GSp}_4$ to the unitary group $\text{U}_4$.

5.1. **Isogeny from $\text{GSO}_6$ to $\text{U}_4$.** Suppose that $E/\mathbb{Q}$ is a quadratic imaginary extension with $E = \mathbb{Q}(\eta)$ where $\eta^2 = -N$, $H: E^4 \times E^4 \rightarrow E$ is a Hermitian form, positive definite at $\infty$. Suppose also that $H$ is $E$-linear on the first variable and $E$-semilinear on the second variable. Suppose an $E$-basis of $E^4$ is $e_1, e_2, e_3, e_4$. We write $\text{SU}_4, \text{SU}_4, \text{U}_4$ for the (similitude special) unitary algebraic group defined over $\mathbb{Q}$ by the Hermitian form $H$.

5.1.1. **Exceptional morphism.** Here we establish an exceptional isogeny from a similitude special unitary groups $\text{SU}_4$ to a similitude special orthogonal group $\text{GSO}_6$. The construction is taken from [Gar15]. We will define the similitude special orthogonal group $\text{GSO}_6$ later on, which depends on the Hermitian form $H$.

We define a $\text{SL}_4(E)$-invariant $E$-valued symmetric form $\langle \cdot, \cdot \rangle$ on $\wedge^2 E^4$ as follows
\[
\langle x \wedge y, z \wedge w \rangle e_1 \wedge e_2 \wedge e_3 \wedge e_4 = x \wedge y \wedge z \wedge w, \quad (\forall x, y, z, w \in E^4)
\]

We define an $E$-semilinear isomorphism
\[
E^4 \rightarrow (E^4)^* = \text{Hom}_E(E^4, E), \quad x \mapsto (y \mapsto H(y, x)).
\]
This induces a map $\wedge^2 E^4 \rightarrow \wedge^2(E^4)^* \simeq (\wedge^2 E^4)^*$. We also have an $E$-linear isomorphism
\[
\wedge^2 E^4 \rightarrow (\wedge^2 E^4)^*, \quad u \mapsto (v \mapsto \langle v, u \rangle).
\]
Combining these, we get the following $E$-semilinear isomorphism
\[
J: \wedge^2 E^4 \xrightarrow{\langle \cdot, \cdot \rangle} (\wedge^2 E^4)^* \rightarrow \wedge^2(E^4)^* \xrightarrow{\wedge^2 H} \wedge^2 E^4.
\]
Note that $\text{GSU}_4(\mathbb{Q})$, as a subgroup of $\text{SL}_4(\mathbb{E})$, respects $\langle \cdot, \cdot \rangle$. By definition, it respects also $\wedge^2 H$. Thus $J$ commutes with $\text{GSU}_4(\mathbb{Q})$.

Suppose that under the basis $e_1, e_2, e_3, e_4$ of $E^4$, $H$ is of the form $H(e_i, e_j) = a_i^{-1} \delta_{i,j}$, with $0 < a_i \in \mathbb{Q}$ such that $a_1a_2a_3a_4 = 1$. By examining $J(e_i \wedge e_j)$ (evaluating at $e_k \wedge e_l$), we see that $J^2 = 1$. We then pick out the eigenspace $U_J$ of $J$ of eigenvalue $+1$. One can take the following $\mathbb{Q}$-basis of $U_J$:

\[
\begin{align*}
e_1 & \wedge e_2 + a_3 a_4 e_3 \wedge e_4, & \eta e_1 & \wedge e_2 - \eta a_3 a_4 e_3 \wedge e_4, & e_1 & \wedge e_3 - a_2 a_4 e_2 \wedge e_4, \\
\eta e_1 & \wedge e_3 + \eta a_2 a_4 e_2 \wedge e_4, & e_1 & \wedge e_4 + a_2 a_3 e_2 \wedge e_3, & \eta e_1 & \wedge e_4 - \eta a_2 a_3 e_2 \wedge e_3.
\end{align*}
\]

Then the symmetric bilinear form $\langle \cdot, \cdot \rangle|_{U_J}$ is of the form $\text{diag}(2a_3a_4, -2\eta a_3a_4, 2a_2a_4, -2\eta a_2a_4, 2a_2a_3, -2\eta^2 a_2 a_3)$. For application in this article, we can take $a_1 = a_2 = a_3 = a_4 = 1$. Moreover, it is easy to see that $\langle \cdot, \cdot \rangle|_{U_J}$ is $\mathbb{Q}$-equivalent to the quadratic form $\eta_U = \text{diag}(N^2/2, N^2/2, N^2/2, N^2/2, N/N_1, N_1)$ with $0 \neq N_1 \not| N_p$ (using simple properties of Hilbert symbols and density of $\mathbb{Q}$ in $\mathbb{A}$). Therefore, the orthogonal groups $O(\langle \cdot, \cdot \rangle)$ and $O(\eta_U)$ defined by these quadratic forms are isomorphic over $\mathbb{Q}$. We will thus identify these two groups and also their automorphic forms and Hecke algebras without further comment.

5.2. $L$-groups and $L$-parameters. In this subsection we review some facts on the $L$-groups and $L$-parameters of groups that will be used in the sequel.

We define the orthogonal group $O_k(\mathbb{C})$ to be $\{g \in \text{GL}_k(\mathbb{C})| g^t w_k g = w_k\}$ where $w_k$ is the anti-diagonal matrix with $1$ on the anti-diagonal. One defines also the special orthogonal group $\text{SO}_k(\mathbb{C})$, the similitude group $G_0(\mathbb{C}) = \{g \in \text{GL}_k(\mathbb{C})| g^t w_k g = \nu(g) w_k\}$ and the similitude special group $G_0(\mathbb{C}) = \{g \in G_0(\mathbb{C})| \text{det}(g) = \nu(g)^{k/2}\}$ if $k$ is even and $= G_0(\mathbb{C})$ if $k$ is odd.

The $L$-group of the similitude symplectic group $\text{GSp}_4(F)$ is $\text{GSp}_4(\mathbb{C})$. Recall that the standard representation of $\text{GSp}_4(\mathbb{C})$ is given as follows (see [RS07 Appendix 7, pp.286-287]): write $V = (\mathbb{C}^4, \langle \cdot, \cdot \rangle)$ for the symplectic vector space such that under the standard basis $\{e_1, e_2, e_3, e_4\}$ the symplectic form is $J_4$. This gives rise to the similitude isometry group $\text{GSp}_4(\mathbb{C})$. Consider the exterior square $\wedge^2 V$ of $V$ on which $\text{GSp}_4(\mathbb{C})$ by

\[
\rho(g)(v \wedge u) := \nu(g)^{-1}(gv) \wedge (gu)
\]

for $g \in \text{GSp}_4(\mathbb{C})$ and $v \wedge u \in \wedge^2 V$. We define a bilinear form on $\wedge^2 V$ as follows: for any $v \wedge u, v' \wedge u' \in \wedge^2 V$, we set $(v \wedge u, v' \wedge u') := \langle v, v' \rangle \langle u, u' \rangle - \langle v, u' \rangle \langle v', u \rangle$ and then extend to the whole $\wedge^2 V$ by bilinearity. It is clear that this is a symmetric form and

\[
(v \wedge u) \wedge (v' \wedge u') = (v \wedge u, v' \wedge u') \cdot e_1 \wedge e_2 \wedge e_3 \wedge e_4 \in \wedge^4 V.
\]

Therefore an element $g \in \text{GSp}_4(\mathbb{C})$ is sent to an element in $\text{SO}_6(\mathbb{C})$, the isometry group of $(\wedge^2 V, \langle \cdot, \cdot \rangle)$. Consider the basis of $\wedge^2 V$:

\[
\{E_1 = e_1 \wedge e_2, E_2 = e_1 \wedge e_4, E_3 = e_1 \wedge e_3, E_4 = -e_2 \wedge e_4, E_5 = e_2 \wedge e_3, E_6 = e_3 \wedge e_4\}.
\]

It is easy to verify that the symmetric bilinear form of $\wedge^2 V$ under this basis is the anti-diagonal matrix $w_6$. Note that $\text{GSp}_4(\mathbb{C})$ is generated by $J_4, m(a) \text{diag}(1_2, \nu \cdot 1_2)$ for $a \in \text{GL}_2(\mathbb{C})$ and $\nu \in \mathbb{C}^\times$, and $u(b)$ for $b$ a symmetric matrix. Then one can verify case by case that the subspace of $\wedge^2 V$ generated by the vector $E_3 - E_4 = e_1 \wedge e_3 + e_2 \wedge e_4$ is invariant under the action of these elements, thus this subspace is invariant under the action of $\text{GSp}_4(\mathbb{C})$. Now consider its orthogonal complementary subspace $X$ of $\wedge^2 V$ generated by the basis

\[
\{E'_1 = E_1, E'_2 = E_2, E'_3 = (E_3 + E_4)/\sqrt{2}, E'_4 = E_5, E'_5 = E_6\}.
\]

Then the symmetric bilinear form $\langle \cdot, \cdot \rangle|_X$ is $w_5$ under this basis. Therefore, one obtains the standard representation of $\text{GSp}_4(\mathbb{C})$:

\[
\rho_{st} : \text{GSp}_4(\mathbb{C}) \rightarrow \text{SO}_5(\mathbb{C}).
\]

By the definition of the action $\rho$, $\rho_{st}$ factors through $\text{PGSp}_4(\mathbb{C}) = \text{PSp}_4(\mathbb{C})$. As in [RS07, p.287], one verifies that $\rho_{st}$ is surjective. This representation is used in the definition of the standard $L$-functions.
of Siegel modular forms of genus 2. Note that the similitude factor does not play a role in the standard representation of $GSp_4(\mathbb{C})$.

We can modify the action $\rho$ of $GSp_4(\mathbb{C})$ on $\wedge^2 V$ without dividing out the similitude factor:

$$\rho'(g)(v \wedge u) := gv \wedge gu.$$  

In the same manner $GSp_4(\mathbb{C})$ acts on $X$ via $\rho'$. Thus we get a surjective morphism

$$\rho_{st}: GSp_4(\mathbb{C}) \to GSO_5(\mathbb{C})$$

whose kernel is $\{\pm 1\}$. Therefore one identifies $GSp_4(\mathbb{C})$ with the similitude spin group $GSpin_5(\mathbb{C})$. Moreover the images of elements in the standard torus of $GSp_4(\mathbb{C})$:

$$\rho_{st}(\text{diag}(t_1, t_2, t/t_1, t/t_2)) = \text{diag}(t_1 t_2, t t/t_1, t t/t_2, t^2/(t_1 t_2)).$$

This representation will be used in expressing theta lift of unramified representations in terms of Langlands parameters.

The dual group of the similitude special orthogonal group $GSO_6(F)$ is the similitude spin group $GSpin_6(\mathbb{C})$ ([Xu18, p.81]). One can use the above construction to show that $GL_4(\mathbb{C})$ is isomorphic to $GSpin_6(\mathbb{C})$. Indeed, the representation $\rho'$ of $GSp_4(\mathbb{C})$ on $\wedge^2 V$ extends to $GL_4(\mathbb{C})$. This time we get a surjective morphism $\rho_{st}: GL_4(\mathbb{C}) \to GSO_6(\mathbb{C})$, whose kernel is $\{\pm 1\}$ and $\rho_{st}(g) = \det(g)$.

Similarly, under the basis $\{E_1, \cdots, E_6\}$ of $\wedge^2 V$, we have:

$$\rho_{st}(\text{diag}(\lambda_1, \lambda_2, \lambda_3, \lambda_4)) = \text{diag}(\lambda_1 \lambda_2, \lambda_1 \lambda_4, \lambda_1 \lambda_3, \lambda_2 \lambda_4, \lambda_2 \lambda_3, \lambda_3 \lambda_4)$$

and under the basis $\{E'_1, \cdots, E'_5, i(E_3 - E_4)/\sqrt{2}\}$ of $\wedge^2 V$ (note that under this basis, the symmetric bilinear form $(\cdot, \cdot)$ becomes $\text{diag}(J_5, 1)$):

$$\rho_{st}(\text{diag}(t_1, t_2, t/t_1, t/t_2)) = \text{diag}(t_1 t_2, t t/t_1, t t/t_2, t^2/(t_1 t_2), t).$$

The $L$-group of $GSO_6(F) = GSO(U)$ depends on whether $V$ is split over $F$ or not. If $U$ is split over $F$, i.e., $GSO(U) \cong GSO_{3,3}(F)$, then $L GSO(U) = GSpin_6(\mathbb{C}) \times \Gamma_{E/F}$. If $U$ is not split over $F$, i.e., $GSO(U) \cong GSO_{4,2}(F)$, then $L GSO_6(F) = GSO_6(\mathbb{C}) \rtimes \Gamma_{E/F}$ where the Galois group $\Gamma_{E/F} = \{1, c\}$ acts on $GSO_6(\mathbb{C})$ by $c(g)c := J_4 g^{-1} J_4^{-1}$. Note that $c$ fixes each element of $GSp_4(\mathbb{C})$ up to a scalar (the similitude factor). The action of $\Gamma_{E/F}$ on $GSpin_6(\mathbb{C}) \cong GL_4(\mathbb{C})$ descends to $GSO_6(\mathbb{C})$ as follows. We write

$$\epsilon = -\text{diag}(1, 2, 2, 1) \in O_6(\mathbb{C}) \rtimes SO_6(\mathbb{C}).$$

Then we have $\rho_{st}(J_4) = -\epsilon w_6 = -w_6 \epsilon$. Moreover, for any $g \in GL_4(\mathbb{C})$, $\rho_{st}(g^{-1}) = \rho_{st}(g)^{-t}$. Thus

$$\rho_{st}'(c(g)) = (\epsilon w_6) \rho_{st}'(g)^{-t} (\epsilon w_6)^{-1} = \nu(p_{st}(g)) \epsilon \rho_{st}'(g) \epsilon^{-1} = \det(g) \epsilon \rho_{st}'(g) \epsilon.$$

In other words,

**Lemma 5.1.** The element $c \in \Gamma_{E/F}$ acts on $GSO_6(\mathbb{C})$ by conjugation by $\epsilon$ up to the similitude factor. The conjugate action of $\epsilon$ on $GSO_6(\mathbb{C})$ lifts to the action $c$ on $GL_4(\mathbb{C})$ up to the determinant factor.

To have a uniform expression for the cases split and non-split, we write in both cases the $L$-group of $GSO_6(F)$ as

$$L GSO_6(F) = GSpin_6(\mathbb{C}) \rtimes \Gamma_{E/F}.$$  

It should be kept in mind that when $U$ is split over $F$, this is a direct product.

**Remark 5.2.** (1) We discuss the relation between $O_6(\mathbb{C})$ and $SO_6(\mathbb{C})$ and the related similitude groups as well as (similitude) spin/pin groups, which will be useful later on when we express the theta lift of unramified representations in terms of Langlands parameters. We define a morphism $\mu: O_6(\mathbb{C}) \to \{0, 1\}$ where $\mu(g) = 0$ if $\det(g) = 1$ and $\mu(g) = 1$ otherwise. Then we can define an isomorphism

$$O_6(\mathbb{C}) \cong SO_6(\mathbb{C}) \rtimes \{1, \epsilon\}, \quad g \mapsto (\epsilon g \mu(g), \epsilon \nu(g)).$$
Here $\epsilon$ acts on $\text{SO}_6(\mathbb{C})$ by conjugation. One can extend $\mu$ to $\text{GO}_6(\mathbb{C})$, the Pin group $\text{Pin}_6^+(\mathbb{C})$ and the similitude Pin group $\text{GPin}_6^+(\mathbb{C})$ as follows: if $\det(g) = \nu(g)^3$, then $\mu(g) = 0$, otherwise $\mu(g) = 1$. Using this $\mu$, one obtains similar isomorphisms

$$\text{GO}_6(\mathbb{C}) \simeq \text{GSO}_6(\mathbb{C}) \rtimes \{1, \epsilon\}, \quad \text{Pin}_6^+(\mathbb{C}) \simeq \text{Spin}_6(\mathbb{C}) \rtimes \{1, \bar{\epsilon}\},$$

$$\text{GPin}_6^+(\mathbb{C}) \simeq \text{GSpin}_6(\mathbb{C}) \rtimes \{1, \bar{\epsilon}\} \simeq \text{GL}_4(\mathbb{C}) \rtimes \Gamma_{E/F}.$$

Here $\bar{\epsilon} \in \text{Pin}_6^+(\mathbb{C})$ is any element that is mapped to $\epsilon$ under the natural projection.

(2) We define the adjoint representations $\rho_{ad}$ of $\text{GSp}_4(\mathbb{C}) \rtimes \Gamma_{E/F}$ and $\text{GL}_4(\mathbb{C}) \rtimes \Gamma_{E/F}$ as follows. We define a symmetric bilinear form on $\text{M}_{4 \times 4}(\mathbb{C})$ by $(X, Y) := \text{tr}(XY)$. The action of $\text{GL}_4(\mathbb{C})$ on $\text{M}_{4 \times 4}(\mathbb{C})$ is defined to be $\rho_{ad}(g)(X) = g \cdot X := gXg^{-1}$. The action of $c \in \Gamma_{E/F}$ is $\rho_{ad}(c)(X) = -J_4X^tJ_4^{-1}$. It is clear that the action of $\text{GL}_4(\mathbb{C}) \rtimes \Gamma_{E/F}$ preserves the bilinear form. There is a subspace of $\text{M}_{4 \times 4}(\mathbb{C})$ of dimension 5 defined to be $\tilde{V} = \{X \in \text{M}_{4 \times 4}(\mathbb{C}) | J_4X^tJ_4^{-1} = X, \text{tr}(X) = 0\}$. It is easy to verify that $\tilde{V}$ is invariant under $\text{GSp}_4(\mathbb{C}) \rtimes \Gamma_{E/F}$. Moreover, this representation $\rho_{ad}$ of $\text{GSp}_4(\mathbb{C})$ is isomorphic to the standard representation $\rho_{st}: \text{GSp}_4(\mathbb{C}) \to \text{SO}_5(\mathbb{C})$. One sees by definition that $\rho_{ad}(c)(X) = -X$ for $X \in \tilde{V}$. This is compatible with the conjugate action of $\epsilon$ on $\tilde{V}'$ of the Lie algebra $\mathfrak{so}_6(\mathbb{C}) = \{X \in \text{M}_{6 \times 6}(\mathbb{C}) | X^t\mathfrak{w}_6 = -\mathfrak{w}_6X\}$ defined by $\tilde{V}' = \{X | \rho_{st}(J_4)X^t\rho_{st}(J_4)^{-1} = X\}$. Indeed, let’s write $w = \rho_{st}(J_4)w = w^t$. Then for any $X \in \tilde{V}'$, $\epsilon X = \mathfrak{w}_6w X = \mathfrak{w}_6(wX)^t = \mathfrak{w}_6X^t = -X\mathfrak{w}_6w = -X\epsilon$. Note that $c$, resp., $\epsilon$ acts on the 1-dimensional subspace $\mathbb{C} \cdot 1_4 \subset \text{M}_{4 \times 4}(\mathbb{C})$, resp., $\mathbb{C} \cdot 1_6 \subset \mathfrak{gso}_6(\mathbb{C})$ as $\rho_{ad}(c)(1_4) = 1_4$, $\rho_{st}(\epsilon)(1_6) = 1_6$.

The dual group of $\text{U}_4(F)$ is $\text{GL}_4(\mathbb{C}) \simeq \text{GSpin}_6(\mathbb{C})$. The $L$-group of $\text{U}_4(F)$ depends also on whether $V$ is split over $F$ or not. If $U$ is split over $F$, then $E \simeq F \oplus F$. In this case, $\text{U}_4(F) \simeq \text{GL}_4(F)$ (also an isomorphism for both groups as algebraic groups), thus the $L$-group of $\text{U}_4(F)$ is just $\text{U}_4^+(F) = \text{GL}_4(\mathbb{C}) \rtimes \Gamma_{E/F}$. If $U$ is not split over $F$, then $E/F$ is a quadratic field extension. In this case, $\text{U}_4(F) = \text{GL}_4(\mathbb{C}) \rtimes \Gamma_{E/F}$ where $c$ acts on $\text{GL}_4(\mathbb{C})$ by $c(g)c := J_4g^{-t}J_4^{-1}$ (the same as in the case of $\text{GSO}_6(F)$). As in the case of $\text{GSO}_6(F)$, we write the $L$-group of $\text{U}_4(F)$ as $\text{GL}_4(\mathbb{C}) \rtimes \Gamma_{E/F}$ for both cases where the semi-direct product is understood to be direct product when $U$ is split over $F$.

From the above discussion, we have the following identification of $L$-groups:

$$\text{L} \text{U}_4(F) \simeq \text{L} \text{GSO}_6(F) \simeq \text{GSpin}_6(\mathbb{C}) \rtimes \Gamma_{E/F} \simeq \text{GL}_4(\mathbb{C}) \rtimes \Gamma_{E/F}.$$ 

### 5.3. Local theta lift from $\text{GSp}^+(V)$ to $\text{GSO}(U)$.

Let $F$ be the local field $\mathbb{Q}_p$. Now we consider the subgroup $\text{GSp}^+(V)$ of $\text{GSp}(V)$ over $F$ consisting of elements $g$ such that $\nu(g) \in \nu(\text{GSO}(U))$.

One has the following ([GT11 Proposition 2.3])

**Proposition 5.3.** Suppose that $\pi$ is a supercuspidal representation of $\text{GSp}^+(V)$, then $\Theta(\pi)$ is either zero or is an irreducible representation of $\text{GSO}(U)$. Moreover, if $\pi'$ is another supercuspidal representation of $\text{GSp}^+(V)$ such that $\Theta(\pi') \neq 0$, then $\pi = \pi'$.

#### 5.3.1. Unramified local theta lift from $\text{GSp}^+_4$ to $\text{GSO}_4$, non-archimedean case.

Suppose $F$ is non-archimedean and that $-N$ is a non-square in $F$, then $U$ is $F$-equivalent to the quadratic space $F(\sqrt{-N}) \oplus \mathbb{H}^2$ where $\mathbb{H}$ is the split hyperbolic plane. Thus $\text{GSO}(U) \simeq \text{GSO}_4$.

The results for the theta lift can be found in [Mor14, Theorem 6.21]. Concerning the theta lift of unramified representations, we can proceed as follows. We define a map $\tilde{\mu}: W_F \to \{0, 1\}$ by $\tilde{\mu}(g) = 0$ if $\xi(g) = 1$ and $\tilde{\mu}(g) = 1$ otherwise. We then define a morphism of $L$-groups

$$\iota: \text{L} \text{GSp}_4(F) = \text{GSpin}_5(\mathbb{C}) \rtimes \Gamma_{E/F} \to \text{L} \text{GSO}_6(F) = \text{GSpin}_6(\mathbb{C}) \rtimes \Gamma_{E/F},$$

$$(g, \sigma) \mapsto (\text{diag}(g, \nu(g)))_{\mathbb{R}}, \sigma)$$

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where the subscript $\mathfrak{B}'$ means that we consider the element $\text{diag}(g\xi(\sigma), 1)$ under the basis $\mathfrak{B}' = \{E_1, \cdots, E_5, (E_3 - E_4)/\sqrt{2}\}$ of $\wedge^2 \mathbb{V}$. For $\xi(\sigma) = -1$, it is easy to see that under the basis $\mathfrak{B} = \{E_1, \cdots, E_6\}$ of $\wedge^2 \mathbb{V}$, the element $\text{diag}(\xi(\sigma)1_5, 1)_{\mathfrak{B}'}$ becomes $\text{diag}(-1_2, -J_2, -1_2)_{\mathfrak{B}} = \epsilon$.

One verifies that:

**Lemma 5.4.** The map $\iota$ is a morphism of groups.

*Proof.* It suffices to show that $\tilde{\iota}$ commutes with each element in the image of $\text{GSpin}_5^0(\mathbb{C})$ in $\text{GPin}_6^+(\mathbb{C})$. By Lemma 5.1, $\tilde{\iota}$ acts by conjugation on $\text{GSpin}_4(\mathbb{C}) = \text{GL}_4(\mathbb{C})$ via the action of $c$. Now for any $g \in \text{GSp}_4(\mathbb{C})$, we have $c(g) = J_4g^{-t}J_4 = g$. We conclude that $\tilde{\iota}$ commutes with each element of $\text{GSpin}_5(\mathbb{C})$ viewed as subgroup of $\text{GPin}_6^+(\mathbb{C})$. □

We can interpret the above morphism in terms of classical groups: for $\tilde{\mu}(\sigma) = 1$, i.e., $\xi(\sigma) = -1$, we have (be careful about the definition of the $E_i$’s):

$$(\rho_{st})^{-1}(\text{diag}(-1_2, -J_2, -1_2)_{\mathfrak{B}}\tilde{\iota}(\sigma)) = (\rho_{st}')^{-1}(-1_6) = i \cdot 1_4 \in \text{GL}_4(\mathbb{C}).$$

We write $A$ for this element. Then the above morphism becomes

$$\iota: \text{L} \text{GSp}_4(F) = \text{GSp}_4(\mathbb{C}) \times \Gamma_{E/F} \to \text{L} \text{GSO}_6(F) = \text{GL}_4(\mathbb{C}) \times \Gamma_{E/F}, \quad (h, \sigma) \mapsto (hA\tilde{\iota}(\sigma), \sigma).$$

Note that the morphism $\iota$ is independent of whether $U$ is split over $F$ or not.

Now we have ([Mor14, Corollary 6.23]):

**Theorem 5.5.** Let $\pi = \pi(s)$ be an unramified, resp., twisted Steinberg representation of $\text{GSp}_4(F)$ corresponding to the semi-simple class $s \in \text{GSp}_4(\mathbb{C}) \times \Gamma_{E/F}$ and $\pi_+(s)$ an irreducible constituent of $\pi|_G$. Then $\Theta^*(\pi(s)) := \Theta(\pi_+(s))$ is the unramified, resp., twisted Steinberg representation $\text{GSO}_6(F)$ corresponding to the semi-simple class $\iota(s) \in \text{GL}_4(\mathbb{C}) \times \Gamma_{E/F}$. More precisely, suppose that $\pi(s)$ is the unique irreducible unramified submodule of $\text{Ind}_B(\chi_1, \chi_2, \chi)$ and set $t_1 = \chi_1(\omega_F)$, $t_2 = \chi_2(\omega_F)$, $t = \chi(\omega_F)$, the Satake parameter of $\pi$

$$s = (\text{diag}(t, t_1t, t_1t_2, t_2), \sigma) \in \text{GSp}_4(\mathbb{C}) \times \Gamma_{E/F},$$

then the Satake parameter of $\Theta^*(\pi(s))$ is

$$\iota(s) = s \cdot (A\tilde{\iota}(\sigma), 1) = \begin{cases} 
(i \cdot \text{diag}(t, t_1t, t_1t_2, t_2), \sigma) & \text{if } \xi(\sigma) = -1, \\
(\text{diag}(t, t_1t, t_1t_2, t_2), \sigma) & \text{otherwise}.
\end{cases}$$

In terms of $L$-factors, if $\pi$ is unramified, we have

$$L(s, \Theta^*(\pi), \text{st}) = L(s, \text{St}(\pi) \otimes \xi)(1 - \ell^{-s})^{-1}.$$
5.3.3. Archimedean theta lift from $\text{GSp}_4^+$ to $\text{GO}_6$. For the case $F = \mathbb{R}$, the group $\text{GO}(U)$ is just $\text{GO}_6 = \text{GO}_6(\mathbb{R})$. Moreover, it is easy to see $\nu(\text{GO}_6(\mathbb{R})) = \mathbb{R}^\times$ and thus $\text{GSp}_4^+ = \text{GSp}_4(\mathbb{R})$. The results for this theta lift can be found in [Pau05]. For the completeness of the article and the convenience of the reader, we reproduce the relevant results of [Pau05]. See also [Mor14] Section 7.

We first recall the classification of discrete series representations of $\text{Sp}_4(\mathbb{R})$. We have the characters $\epsilon_1, \epsilon_2$ of $T_{\text{Sp}_4}$. Then the set of roots of $\text{Sp}_4$ is $\Sigma = \{ \pm \epsilon_1 + \epsilon_2, \pm 2\epsilon_1, \pm 2\epsilon_2 \}$. A set of positive compact roots is $\Sigma^+_c = \{ \epsilon_1 - \epsilon_2 \}$. Then positive root systems containing $\Sigma^+_c$ are the following ones:

- $\Sigma^+_1 = \{ \epsilon_1 - \epsilon_2, 2\epsilon_1, \epsilon_1 + \epsilon_2, 2\epsilon_2 \}$
- $\Sigma^+_2 = \{ \epsilon_1 - \epsilon_2, -2\epsilon_1, -\epsilon_1 - \epsilon_2, -2\epsilon_2 \}$
- $\Sigma^+_3 = \{ \epsilon_1 - \epsilon_2, 2\epsilon_1, \epsilon_1 + \epsilon_2, -2\epsilon_2 \}$
- $\Sigma^+_4 = \{ \epsilon_1 - \epsilon_2, 2\epsilon_1, -\epsilon_1 - \epsilon_2, -2\epsilon_2 \}$

The corresponding sets of dominant weights are $(i = 1, 2, 3, 4)$ (recall that $\langle e_j, e_k \rangle = 2\delta_{j,k}$)

$$X_i = \{ (\lambda_1, \lambda_2) \in \mathbb{Z}^2 | \langle \lambda_1 \epsilon_1 + \lambda_2 \epsilon_2, \alpha \rangle > 0, \forall \alpha \in \Sigma^+_i \}.$$ 

The Harish-Chandra parameter of the discrete series representations of $\text{Sp}_4(\mathbb{R})$ is then the union $\bigcup_i X_i$. Moreover, $X_1$ parameterizes holomorphic discrete series representations, $X_2$ parameterizes anti-holomorphic discrete series representations while $X_3 \bigcup X_4$ parameterizes generic discrete series representations.

We then recall the classification of discrete series representations of $\text{SO}_6(\mathbb{R})$. We have the characters $\epsilon'_1, \epsilon'_2, \epsilon'_3$ of $T_{\text{SO}_6}$. The set of roots of $\text{SO}_6$ is $\Sigma' = \{ \pm \epsilon_i + \epsilon_j | 1 \leq i < j \leq 3 \}$. A set of positive compact roots is $\Sigma'^+_c = \{ \epsilon'_i + \epsilon'_j | 1 \leq i < j \leq 3 \}$. Then the positive root system compatible containing $\Sigma'^+_c$ is $\Sigma'^+_c = \Sigma'^+_c$. The corresponding set of dominant weights are $X' = \{ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3 | \langle \lambda_1 \epsilon'_1 + \lambda_2 \epsilon'_2 + \lambda_3 \epsilon'_3, \alpha \rangle > 0, \forall \alpha \in \Sigma'^+_c \} = \{ (\lambda_1, \lambda_2, \lambda_3) \in \mathbb{Z}^3 | \lambda_1 > \lambda_2 > |\lambda_3| \}.$

If $\pi$ is a discrete series of $\text{Sp}_4(\mathbb{R})$ in $\Sigma^+_i$, resp. $\Sigma^+_j$ of weight $(\lambda_1, \lambda_2)$ (such that $\lambda_1 > \lambda_2 > 0$, resp. $0 > \lambda_1 > \lambda_2$), then by [Pau05] Theorem 15], the theta lift $\Theta(\pi)$ to $\text{SO}_6(\mathbb{R})$ is non-zero of weight $(\lambda_1, \lambda_2, 0)$. If $\pi$ is in $\Sigma'^+_3$ of weight $(\lambda_1, \lambda_2)$ (such that $\lambda_1 > |\lambda_2|$ and $\lambda_2 < 0$), then again by [Pau05] Theorem 15], the theta lift $\Theta(\pi)$ to $\text{SO}_6(\mathbb{R})$ vanishes. Similarly, if $\pi$ is in $\Sigma'^+_4$, the theta lift $\Theta(\pi)$ vanishes too. From this we conclude that the theta lift of a (anti-)holomorphic discrete series of $\text{Sp}_4(\mathbb{R})$ to $\text{SO}_6(\mathbb{R})$ is non-zero while the theta lift of generic discrete series vanishes. The theta lift for the corresponding similitude groups can be similarly discussed and the result is the same: the theta lift of (anti-)holomorphic discrete series of $\text{GSp}_4(\mathbb{R})$ to $\text{GSO}_6(\mathbb{R})$ is non-zero while theta lift of generic discrete series vanishes.

5.4. Transfer between $\text{GSp}_4(F)$ and $\text{Sp}_4(F)$, $U_4(F)$ and $\text{SU}_4(F)$. In this subsection, we review some results on the transfer of representations between $\text{GSp}_4(F)$ and $\text{Sp}_4(F)$, $U_4(F)$ and $\text{SU}_4(F)$, globally and locally ([LS86] Section 3]. In order to have a uniform treatment, we write $G$ to be $\text{Sp}_4(F)$ (for $F$ local field), $\text{Sp}_4(\mathbb{A}_F)$ (for $F$ global field), $\text{SU}_4(F)$ or $\text{SU}_4(\mathbb{A}_F)$, resp., $\tilde{G}$ to be $\text{GSp}_4(F)$, $\text{GSp}_4(\mathbb{A}_F)$, $\text{U}_4(F)$ or $\text{U}_4(\mathbb{A}_F)$).

5.4.1. $L$-packets. Note that in any case, $G$ is a normal subgroup of $\tilde{G}$. For any irreducible admissible representation $\pi$ of $G$ and any element $g \in \tilde{G}$, we define a new representation $\pi^g$ of $G$ as follows: $\pi^g(h) := \pi(ghg^{-1})$ for any $h \in G$. We say that two irreducible admissible representations $\pi$ and $\pi'$ of $G$ are equivalent if $\pi' \simeq \pi^g$ for some $g \in \tilde{G}$. We denote by $\mathcal{L}(G)$ the quotient set of the set of irreducible admissible representation of $G$ by this equivalence relation and each element in $\mathcal{L}(G)$ is an $L$-packet of $G$. The usual equivalence relation on representations of $\tilde{G}$ defines the $L$-packets of $\tilde{G}$. We write the set of $L$-packets of $\tilde{G}$ as $\mathcal{L}(\tilde{G})$. We also write $\mathcal{E} = (\tilde{G}/G)^\vee$ for the set of characters of the quotient $\tilde{G}/G$. Moreover, we say two irreducible admissible representations $\pi$ and $\pi'$ of $\tilde{G}$ $\mathcal{E}$-equivalent if $\pi' \simeq \pi \otimes \chi$ for some $\chi \in \mathcal{E}$. We then write $\mathcal{E}(\tilde{G})$ the set of $\mathcal{E}$-equivalent classes of irreducible admissible representations of $\tilde{G}$. Thus there is a canonical projection $L(\tilde{G}) \to \mathcal{E}(\tilde{G})$. 

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One can show the following

**Lemma 5.7.** For an irreducible admissible representation \( \tilde{\pi} \) of \( \tilde{G} \), its restriction to \( G \) depends only on its class in \( \mathscr{E}(\tilde{G}) \). Moreover, \( \tilde{\pi}|_G \) is a direct sum of irreducible admissible representations of \( G \).

The proof is the same as for [LS86, Lemma 3.2].

**Proof.** The first point follows from the definition of \( \mathscr{E}(\tilde{G}) \).

For the second point, first assume that \( F \) is local. Let \( \tilde{Z} \) be the center of \( \tilde{G} \). Note that \( \tilde{Z}G/\tilde{G} \) is compact and abelian (as connected reductive algebraic groups, we have an \( F \)-isogeny \( ZG \to \tilde{G} \to 1 \)). By [Sil79, Theorem], one knows that \( \tilde{\pi}|_{\tilde{Z}G} \) is a direct sum of irreducible admissible representations of \( \tilde{Z}G \). Since \( \tilde{Z} \) acts on each irreducible factor in \( \tilde{\pi}|_{\tilde{Z}G} \) by scalars, we see that \( \tilde{\pi}|_G \) is also a direct sum of irreducible admissible representations of \( G \). For \( F \) global, we can decompose \( \tilde{\pi} \) into local components and apply the above result. \( \square \)

**Remark 5.8.** For \( F \) local, if \( \tilde{\pi} \) is an unramified representation of \( \tilde{G} \), then its restriction \( \tilde{\pi}|_G \) has only one irreducible admissible submodule which is unramified.

### 5.4.2. The transfer

First we assume \( F \) local. For an irreducible admissible representation \( \tilde{\pi} \) of \( \tilde{G} \), fix \( \pi \) an irreducible submodule of the restriction \( \tilde{\pi}|_G \). Then \( \tilde{\pi}|_G \) is a direct sum of representations of the form \( \pi^g \) for some \( g \in \tilde{G} \). Now we can define the following map

\[ R: \mathcal{E}(\tilde{G}) \to \mathcal{L}(G), \tilde{\pi} \mapsto \pi. \]

By the above discussion, \( R \) is well-defined and one can show by Frobenius reciprocity that \( R \) is bijective.

Now assume \( F \) global. We say an \( L \)-packet of \( G \) is cuspidal if some element in this \( L \)-packet is a cuspidal representation. We write \( \mathcal{L}_0(G) \) for the subset of \( \mathcal{L}(G) \) consisting of cuspidal \( L \)-packets. Similarly we define cuspidal equivalence classes of in \( \mathcal{E}(\tilde{G}) \) and denote by \( \mathcal{E}_0(\tilde{G}) \) the subset of \( \mathcal{E}(\tilde{G}) \) consisting of cuspidal classes. Then one can show that the restriction of \( R \) to \( \mathcal{E}_0(\tilde{G}) \) is again bijective ([Sil79, Lemma 1]):

\[ R: \mathcal{E}_0(\tilde{G}) \to \mathcal{L}_0(G). \]

As above, we write \( \tilde{Z} \) for the center of \( \tilde{G} \) and \( Z = G \cap \tilde{Z} \). Fix a unitary Hecke character \( \tilde{\chi} \) of \( \tilde{Z} \) and \( \chi = \tilde{\chi}|_Z \). We write \( \rho_\chi \) for the representation of \( G \) by right translation on the space of \( \mathbb{C} \)-valued cuspidal square-integrable functions \( L^2_0([G], \chi) \) on which \( \tilde{Z} \) acts by the character \( \chi \). One can extend \( \rho_\chi \) to a representation \( \rho_\chi' \) on the same space of the group \( G' := \tilde{Z}G(F)G \) as follows: for any \( z\gamma g \in \tilde{Z}G(F)G \) and \( f \in L^2_0([G], \chi) \), we set \( \rho_\chi'(z\gamma g)f(x) := \tilde{\chi}(z)f(\gamma^{-1}x\gamma g) \). In the same manner as \( \rho_\chi \), one can define the representation \( \rho_\tilde{\chi} \) of \( \tilde{G} \) on the space \( L^2_0([\tilde{G}], \tilde{\chi}) \) of square-integrable functions by right translation such that \( \tilde{Z} \) acts on by the character \( \tilde{\chi} \). Then one has

\[ \rho_\tilde{\chi} = \text{Ind}_G^{\tilde{G}}(\rho_\chi'). \]

Moreover, \( \rho_\chi' \) is a direct sum of irreducible admissible representations of \( G' \) (with multiplicity one) and each irreducible cuspidal representation \( \tilde{\pi} \) of \( \tilde{G} \) of central character \( \tilde{\chi} \) occurs (with multiplicity one) in \( \text{Ind}_G^{\tilde{G}}(\pi') \) for some submodule \( \pi' \) of \( \rho_\chi' \). As above, for any such representation \( \pi' \) of \( G' \), let \( \pi \) be an irreducible submodule of \( \pi'|_G \). Then \( \pi'|_G \) is a direct sum of \( \pi^g \) for some \( g \in \tilde{G} \). Then we see that \( \pi \) and \( R(\tilde{\pi}) \) are in the same \( L \)-packet of \( G \).

Now we restrict ourselves to automorphic representations. Denote by \( \mathcal{E}\mathcal{A}(\tilde{G}), \mathcal{E}\mathcal{A}_0(\tilde{G}) \), the subset of \( \mathcal{E}(\tilde{G}) \), resp., \( \mathcal{E}_0(\tilde{G}) \), consisting of irreducible, resp., irreducible cuspidal, automorphic representations of \( \tilde{G} \). We denote also by \( \mathcal{L}\mathcal{A}(G), \mathcal{L}\mathcal{A}_0(G), \mathcal{L}\mathcal{A}_0(G) \), the subset of \( \mathcal{L}(G) \) consisting of
irreducible, resp., irreducible cuspidal automorphic representations of $G$. Using [Lang79, Proposition 2], one has

**Lemma 5.9.** The map $R$ defined above induces bijections

$$R: \mathcal{E}\mathcal{A}(\widetilde{G}) \rightarrow \mathcal{L}\mathcal{A}(G), \mathcal{E}\mathcal{A}_0(\widetilde{G}) \rightarrow \mathcal{L}\mathcal{A}_0(G).$$

In summary, let $\tilde{\chi}$ be a Hecke character of $\tilde{Z}$ and $\chi = \tilde{\chi}|_Z$, then one has

**Corollary 5.10.** Any irreducible admissible (resp., cuspidal) automorphic representation $\pi$ of $G$ of central character $\chi$ can be extended to an irreducible admissible (resp., cuspidal) automorphic representation $\bar{\pi}$ of $\tilde{G}$ of central character $\bar{\chi}$.

**Remark 5.11.** Since one can always extend a Hecke character $\chi$ of $Z$ to a character $\bar{\chi}$ of $\tilde{Z}$, the above corollary shows that we can always extend an irreducible admissible (resp., cuspidal) automorphic representation of $G$ of central character $\chi$ to an irreducible admissible (resp., cuspidal) automorphic representation $\bar{\pi}$ of $\tilde{G}$. Moreover, for any irreducible cuspidal automorphic representation $\bar{\pi}$ of $\tilde{G}$, all the irreducible submodules of $\bar{\pi}|_G$ have the same unramified local components and thus the same partial $L$-function.

For an automorphic form $f$ on $G(\mathbb{A}_F)$, we can extend it to an automorphic form $\tilde{f}$ on $\tilde{G}(\mathbb{A}_F)$ as follows: for $g = g_1g_2$ with $g_1 \in \tilde{G}(F)$ and $g_2 \in G(\mathbb{A}_F)$, we set $\tilde{f}(g) = f(g_2)$, otherwise, $\tilde{f}(g) = 0$.

**5.5. Theta lift from $GSp_4$ to $U_4$.** In this subsection, we use the previous results on the theta lift of the reductive dual pair $(GSp^+(V), GSO(U))$ to deduce the transfer/theta lift from $GSp_4$ to $U_4$.

Let $\pi = \otimes_v \pi_v$ be a cohomological cuspidal irreducible automorphic representation of $GSp_4(\mathbb{A})$ of Iwahori level $Np^m$ and of trivial central character. Let $f \in \pi$ be an automorphic form which is $p$-integral. Write $f_+$ its restriction to $Sp_4(\mathbb{A})$. The theta lift $\Theta_\phi(f_+)$ to $SO_6(\mathbb{A})$ can be viewed as an automorphic form on $SU_4(\mathbb{A})$ and then extended by zero to an automorphic form on $U_4(\mathbb{A})$ (see below Remark 5.11), which we denote by $\Theta^\prime_\phi(f)$. Then write $\Theta^\prime_\phi(f) = \{\Theta^\prime_\phi(f)|f \in \pi, \phi \in \mathcal{S}(W_4^-(\mathbb{A}))\}$ for the transfer of the representation $\pi$ on $GSp_4(\mathbb{A})$ to $U_4(\mathbb{A})$. In terms of automorphic representations, write $\pi^+$ for an irreducible submodule of $\pi_\pm$. Then $\pi^+ = \bigoplus_{i \in I}(\pi^+)^{\gamma_i}$ and all these $(\pi^+)^{\gamma_i}$ are in the same $L$-packet of $Sp_4(\mathbb{A})$. Then $\Theta^\prime_\phi(f) = \sum_i \Theta((\pi^+)^{\gamma_i})$ with $g_i \in GSp_4(\mathbb{A})$ as representations of $U_4(\mathbb{A})$ (Lemma 5.9 and Corollary 5.10).

To see the relation of the Langlands parameters of unramified components of $\pi$ and $\Theta^\prime_\phi(f)$, we proceed as follows: for any unramified local component $\pi_v$ on $GSp_4(Q_v)$ of $\pi$, let $\pi^+_v$ be the unique irreducible admissible unramified submodule of $(\pi_v)_+$, the restriction of $\pi_v$ to $GSp_4^+(Q_v)$. Suppose that $\pi^+_v$ corresponds to the semi-simple class $s(\pi_v) \in GSp_4(Q_v) \rtimes \Gamma_{E_v/Q_v}$, then for any irreducible submodule $\Theta^\prime_\phi((\pi^+)^{\gamma_i})$ of $\Theta^\prime_\phi(f)$, its unramified local component $(\Theta^\prime_\phi((\pi^+)^{\gamma_i}))_v$ corresponds to the semi-simple class $\langle s(\pi_v) \rangle \in GL_4(C) \rtimes \Gamma_{E_v/Q_v}$ (Theorems 5.3 and 5.6). The same result holds for the Steinberg components of $\pi$ and $\Theta^\prime_\phi(f)$ by the same theorems.

We summarize the discussion into the following:

**Theorem 5.12.** Let $\pi$ be a cohomological cuspidal irreducible automorphic representation of $GSp_4(\mathbb{A})$ of Iwahori level $Np^m$. If the automorphic representation $\Theta^\prime_\phi(f)$ of $U_4(\mathbb{A})$ is not zero, let the representations $\rho_\pi: \Gamma_Q \rightarrow GSp_4(C)$, resp., $\rho_{\Theta^\prime_\phi(f)}: \Gamma_Q \rightarrow GL_4(C) \rtimes \Gamma_{E/Q}$, be the Galois representations associated to $\pi$, resp., $\Theta^\prime_\phi(f)$. Then $\rho_{\Theta^\prime_\phi(f)}(\sigma) = \rho_\pi(\sigma)\sigma$), where $\overline{\sigma}$ is the image of $\sigma$ under the projection $\Gamma_Q \rightarrow \Gamma_{E/Q}$. The adjoint representations of $\rho_\pi$ and $\rho_{\Theta^\prime_\phi(f)}$ are related by

$$\text{ad}(\rho_{\Theta^\prime_\phi(f)}(\sigma))|_V = \text{ad}(\rho_\pi)|_V \times \xi.$$

Here $V$ is the subspace of $M_{4 \times 4}(C)$ defined in Remark 5.12.
6. Selmer groups, congruence ideals and L-values

In this section, we first establish morphisms between the Hecke algebras and universal deformation rings on GSp\(_4\) and U\(_4\). This permits us to relate Selmer groups to congruence ideals and finally using theta correspondence we get an identity of the characteristic element of the Selmer group and the special L-value.

6.1. Hecke algebras and Galois representations. In this subsection we will establish homomorphisms of Hecke algebras and Galois representations on the groups GSp\(_4\) and U\(_4\) using the theta lift in the preceding section.

6.1.1. Case: GSp\(_4\). Recall that we have defined Hecke operators on \(G = \text{GSp}_4\): for any \(\mathbb{Z}\)-algebra \(R\), the spherical Hecke operator \(\mathcal{H}(G(\mathbb{Q}_\ell), G(\mathbb{Z}_\ell), R)\), the dilating Iwahori operator \(\mathcal{H}^-(G(\mathbb{Q}_\ell), I_{G,p}, R)\). Then the abstract global Hecke algebra of \(G(\mathbb{Q})\) is the restricted tensor product:

\[
H^s(R) = \left( \bigotimes_{\ell|N_p} \mathcal{H}(G(\mathbb{Q}_\ell), G(\mathbb{Z}_\ell), R) \bigotimes \mathcal{H}^-(G(\mathbb{Q}_\ell), I_{G,p}, R) \right).
\]

Moreover, we define the Hecke polynomial as

\[
P_t(X) = T_t(1)X^4 - (T_t(2) + T_{t,0}(2) \ell^2 - \ell^2 T_{t,2}(2)(\ell^2) + T_{t,0}(2) T_{t,1}(2)(\ell^2)) X^2 - \ell^3 T_{t,0}(2) T_{t,1}(2)(\ell^2) + \ell^6 T_{t,2}(2)(\ell^2)^2.
\]

We have also defined the space of cuspidal automorphic forms \(S^s_k(\mathbb{A})\) on GSp\(_4\)(\(\mathbb{A}\)) of weight \(k\), with coefficients in \(R\) and invariant under \(\widehat{\Gamma}\). To avoid possible confusion of notations with the case of \(U_4\), we add a superscript ‘s’ to this space: \(S^s_k(\widehat{\Gamma}, R)\). Then \(H^s(R)\) acts on the space \(S^s_k(\widehat{\Gamma}, R)\) and the image of \(H^s(R)\) in \(\text{End}_R(S^s_k(\widehat{\Gamma}, R))\) is denoted by \(T^s_k(\widehat{\Gamma}, R)\). Suppose moreover that \(R\) is \(p\)-adically complete, then as in Section 2, we have the idempotent element \(e \in T^s_k(\widehat{\Gamma}, R)\) and also the ordinary parts \(T^s_{k,0}(\widehat{\Gamma}, R) := eT^s_k(\widehat{\Gamma}, R)\) and \(S^s_{k,0}(\widehat{\Gamma}, R) := eS^s_k(\widehat{\Gamma}, R)\). Finally we write \(S^s_{k,0}(\widehat{\Gamma}^\infty, R)\) for the space of ordinary \(p\)-adic modular forms on \(\text{GSp}_4(\mathbb{Q})\), which is the projective limit of \(S^s_{k,0}(\widehat{\Gamma}(N, p^n), R)\) for \(m \in \mathbb{Z}_{>0}\) (in [Pill12], Théorème 2.1(2)), it is denoted by \(\psi_{\text{cusp}^s}\). This is a finite free module over \(\Lambda^s\) where \(\Lambda^s\) is the Iwasawa weight algebra given by

\[
\Lambda^s := \mathcal{O}[[T(1 + p\mathbb{Z}_p)]] \simeq \mathcal{O}[[X_1, X_2, X_3]],
\]

\[
\text{diag}(1_{2-i}, (1 + p) \cdot 1_i, (1 + p)^2 \cdot 1_{2-i}, (1 + p) \cdot 1_i) \mapsto 1 + X_{i+1} \ (i = 0, 1, 2)
\]

We then define the big ordinary Hecke algebra \(T^s_{k,0}(\widehat{\Gamma}^\infty, \mathcal{O})\) as the image of \(H^s \otimes_{\mathbb{Z}} \Lambda^s\) in the endomorphism algebra \(\text{End}_\mathcal{O}(S^s_{k,0}(\widehat{\Gamma}^\infty, \mathcal{O}))\) (in [Pill12] below Théorème 6.1, it is denoted by \(\overline{T}\)).

Next we recall several properties of the Galois representation associated to an automorphic representation of GSp\(_4\)(\(\mathbb{A}\)). Let \(\pi = \otimes_v \pi_v\) be a cohomological cuspidal irreducible automorphic representation of GSp\(_4\)(\(\mathbb{A}\)) of weight \(k = (k_0, k_1, k_2)\) with \(k_1 \geq k_2\). Moreover, assume that \(\pi\) is not CAP, \(\pi_\infty\) is a discrete series holomorphic representations of GSp\(_4\)(\(\mathbb{R}\)) and \(\pi_\ell\) is unramified outside \(Np\) and Steinberg representation at \(p\) by [RS07] Tables 1 and 15], \(\pi_p\) has a unique vector fixed by the Iwahori subgroup \(I_{p,1}\) of GSp\(_4\)(\(\mathbb{Z}_p\)). Then there is a \(p\)-adic Galois representation \(\rho_\pi : \Gamma_{\mathbb{Q}} \to \text{GSp}_4(\mathcal{O})\) associated to \(\pi\) ([Lay93, GT05, Jor12, Wei05]). Then for any local component \(\pi_\ell\), we have an isomorphism of semi-simplified Weil-Deligne representations

\[
(1.1) \quad \text{WD}(\rho_\pi|_{D_\ell})^{ss} \simeq \iota(\text{rec}(\pi_\ell \otimes | \cdot |^{-3/2})^{ss}).
\]

Here rec is the local Langlands correspondence for GSp\(_4\), \(D_\ell\) is the decomposition group at \(\ell\), and \(\iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_p\) is the isomorphism fixed since the beginning ([Jor12, Theorem A]). On the other hand, we
can relate the Frobenius to the Hecke polynomial as follows ([Wei05, Theorem I]): let $X$ be a variable and $\text{Fr}_\ell$ the Frobenius element at $\ell$, if $\ell \nmid N_p$, then
\[
\det(X - \text{Fr}_\ell, \rho_\pi|_{D_p}) = P_\ell(X).
\]
Moreover, $\rho_{\pi,p} := \rho_\pi|_{D_p}$ is crystalline ([Fal89]). We write $\rho_{\pi,p}^{\text{cr}} = (\rho_{\pi,p} \otimes B_{\text{cr}})^{D_p}$ where $B_{\text{cr}}$ is Fontaine’s $p$-adic period ring and the absolute Frobenius element $\text{Fr}_p$ acts on $\rho_{\pi,p}^{\text{cr}}$. Then we have ([Urb05 Théorème I])
\[
\det(X - \text{Fr}_p', \rho_{\pi,p}^{\text{cr}}) = P_p(X).
\]
Suppose that $\pi$ is ordinary at $p$ (cf. [Pil12] below Théorème 3.3, i.e., $\pi_p$ is spherical and the entries in $\lambda_\pi((\text{diag}(v_1u_1, v_2u_1, u_1u_1, v_1u_2)) \in (\mathbb{Q}_p^\times)^4/W_G$ have $p$-adic valuations $\{0, k_2 + 1, k_1 + 2, k_1 + k_2 + 3\}$). We write these numbers as $\alpha_i$ $(i = 1, \ldots, 4)$ with $p$-adic valuations $0, k_2 + 1, k_1 + 2, k_1 + k_2 + 3$. Write $\chi_p: D_p \to \mathbb{Z}_p^\times$ for the $p$-adic cyclotomic character and for any $a \in \mathbb{Z}_p^\times$, $\text{ur}(a)$ denotes the unramified character of $D_p$ such that $\text{ur}(a)(\text{Fr}_p) = a$. If we assume that $\rho_\pi$ is irreducible, then $\rho_{\pi,p}$ is conjugate to an upper triangular representation
\[
(6.2) \quad \rho_{\pi,p} \simeq \begin{pmatrix}
\chi_1 & * & * & * \\
0 & \chi_2 & * & * \\
0 & 0 & \chi_3 & * \\
0 & 0 & 0 & \chi_4
\end{pmatrix}
\]
whose diagonal entries are characters defined as $\chi_1 := \text{ur}(\alpha_1)$, $\chi_2 := \text{ur}(\alpha_2p^{-k_2-1})\chi_p^{-k_2-1}$, $\chi_3 := \text{ur}(\alpha_3p^{-k_1-2})\chi_p^{-k_1-2}$ and $\chi_4 := \text{ur}(\alpha_4p^{-k_1-k_2-3})\chi_p^{-k_1-k_2-3}$ ([Urb05 Corollaire 1]). We write $\overline{\pi}_p: \Gamma_Q \to \text{GSp}_4(\mathbb{F})$ to be the residual Galois representation of $\rho_\pi$. Then there is a filtration $(\text{Fil}_i)_{i=0}^4$ on $\overline{\pi}_p$, such that $\text{Fil}_i/\text{Fil}_{i-1} = \chi_i$. We define a unipotent matrix (corresponding to the matrix $\varepsilon$ in [Pil12 Section 3.4.1])
\[
\varepsilon = \begin{pmatrix}
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0
\end{pmatrix}
\]
Then for $\ell|N$, the restriction $\overline{\pi}_|_{I_\ell}$ to the inertia subgroup $I_\ell$ of $\overline{\rho}_\pi$ has image in the unipotent subgroup $\text{exp}(\mathbb{F}\varepsilon) \subset \text{GSp}_4(\mathbb{F})$ by the local-global compatibility of Galois representations associated to Siegel automorphic forms ([Sor10 Theorems A and B], [Jor12 Theorem A]) and the classification of Iwahori-spherical representations ([RS07 Table 15]).

We define deformation functors and their universal deformation rings for $\overline{\pi}_p$. We write $\text{CNL}_A$ for the category of complete noetherian local $\mathcal{O}$-algebras $A$ with maximal ideal $\mathfrak{m}_A$ such that $A/\mathfrak{m}_A = \mathbb{F}$. Similarly we write $\text{AL}_A$ for the subcategory of $\text{CNL}_A$ of (complete) artinian local $\mathcal{O}$-algebras. We define deformation functors as follows
\[
\mathcal{D}_\pi, \mathcal{D}_{\overline{\pi}'}: \text{AL}_A \to \text{Sets}
\]
where $\mathcal{D}_\pi(A)$ is the set of equivalence classes of liftings $\rho_A: \Gamma_Q \to \text{GSp}_4(A)$ of $\overline{\pi}_p$ such that
1. $\rho_A$ is unramified outside $N_p$;
2. for each $\ell|N$, up to a conjugation, the image $\rho_A(I_\ell)$ of the inertia subgroup is contained in the unipotent subgroup $\text{exp}(A\varepsilon)$;
3. there is a filtration $(\text{Fil}_i^A)_{i=0}^4$ on $\rho_A$ lifting $(\text{Fil}_i)$ and stable under $D_p$;
4. the character of $D_p$ on $\text{Fil}_i^A/\text{Fil}_{i-1}^A$ is unramified.

For $\mathcal{D}_{\overline{\pi}''}(A)$, we require moreover as in [Pil12 Section 5.6] that in (3) above the character $\chi_i'$ of $D_p$ via $\rho_A$ on $\text{Fil}_i^A/\text{Fil}_{i-1}^A$ is of the form $\chi_i'(\sigma) = \text{Art}(\sigma)^{k''_i}$ lifting $\chi_i$ for $\sigma \in I_p$ and $\text{Art}(\sigma) \in \mathbb{Z}_p^\times$ by the local Artin map (clearly $\mathcal{D}_{\overline{\pi}''}(A)$ is nonempty only if $k''_i \equiv (0, -k_2 - 1, k_1 - 2, -k_1 - k_2 - 3)(\text{mod } p - 1)$,

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which we will assume in the following). Two liftings \( \rho_A \) and \( \rho'_A \) are equivalent if there is a matrix \( g \in 1 + m_A \cdot \mathfrak{sp}_4(A) \) such that \( \rho_A = g \rho'_A g^{-1} \). We have the following hypotheses

**Hypothesis 6.1.**

(1) (N-Min) For each \( \ell \mid N \), the restriction to inertia group \( \mathfrak{T}_\pi \mid \mathfrak{I}_\ell \) contains a regular unipotent element;

(2) (RFR(2)) The images of \( \alpha_1, \alpha_2 p^{-k_2-1}, \alpha_3 p^{-k_1-2} \) and \( \alpha_4 p^{-k_1-k_2-3} \) in \( \mathcal{O} / \mathfrak{w} \) are mutually distinct;

(3) (BIG(2)) The image of the residual Galois representation \( \overline{\mathfrak{T}_\pi (\Gamma_Q)} \) contains \( \text{Sym}^3(\text{SL}_2(\mathbb{F}_p)) \).

Then by [Til06] Proposition 2.1, the functors \( \mathcal{D}_\pi \), resp., \( \mathcal{D}_{\mathfrak{K}''} \) is pro-representable, say by the couple \( (R^s_\pi, R^s_{\mathfrak{K}''}) \), resp., \( (R^s_{\mathfrak{K}''}, R^s_{\mathfrak{K}'}) \), is an object in \( \text{CNL}_\mathcal{O} \) and \( \rho^s \), resp., \( \rho^s_{\mathfrak{K}''} \), is a Galois representation \( \Gamma_Q \to \text{GSp}_4(R^s_\pi) \), resp., \( \Gamma_Q \to \text{GSp}_4(R^s_{\mathfrak{K}'}) \). We write \( m_\pi \) for the maximal ideal of \( \mathbb{T}^{s, \text{ord}}(\hat{\Gamma}_\infty, \mathcal{O}) \) associated to the residual representation \( \overline{\mathfrak{T}_\pi} \) and set

\[
\mathbb{T}_\pi^s := \mathbb{T}_\pi^{s, \text{ord}}(\hat{\Gamma}_\infty, \mathcal{O})_{m_\pi}, \quad S_\pi^s := S_\pi^{s, \text{ord}}(\hat{\Gamma}_\infty, \mathcal{O}_p / \mathbb{Z}_p)_{m_\pi},
\]

\[
\mathbb{T}_{\mathfrak{K}''}^s := \mathbb{T}_{\mathfrak{K}''}^{s, \text{ord}}(\hat{\Gamma}, \mathcal{O}), \quad S_{\mathfrak{K}''}^s := S_{\mathfrak{K}''}^{s, \text{ord}}(\hat{\Gamma}, \mathcal{O}_p / \mathbb{Z}_p).
\]

Using pseudo-representations, one can construct a Galois representation \( \rho_{\mathbb{T}_4} : \Gamma_Q \to \text{GSp}(\mathbb{T}_\pi^s) \) such that it is a lift of \( \overline{\rho}^s \) (see [Pil12] Section 6.6). Thus by the definition of \( R^s_\pi \), we have a morphism \( R^s_\pi \to \mathbb{T}_\pi^s \) of \( \Lambda^s \)-algebras (the \( \Lambda^s \)-algebra structure on \( R^s \) is given as in [Pil12] Section 5.5]). The method of Taylor-Wiles gives ([Pil12] Théorème 7.1):

**Theorem 6.2.** Assume Hypothesis \([\ell, 1]\) then the morphism \( R^s_\pi \to \mathbb{T}_\pi^s \) is an isomorphism and \( \mathbb{T}_\pi^s \) is finite flat complete intersection over \( \Lambda^s \). The Hecke module \( S^s_\pi \) is a finite free \( \mathbb{T}_\pi^s \)-module. Moreover, the specialization map \( \mathbb{T}_\pi^s \otimes_{\Lambda^s} \mathcal{O} \to \mathbb{T}_{\mathfrak{K}''}^s \) is an isomorphism. In particular, we have an isomorphism \( R^s_{\mathfrak{K}''} \cong \mathbb{T}_{\mathfrak{K}''}^s \).

**Proof.** All the parts in the theorem are proved except the part that \( S^s_\pi \) is finite free over \( \mathbb{T}_\pi^s \). In [Pil12] Théorème 7.1, it is shown that the linear dual \( \text{Hom}_\mathcal{O}(S^s_\pi, \mathcal{O}) \) is finite free over \( \mathbb{T}_\pi^s \). Now that \( \mathbb{T}_\pi^s \) is complete intersection, thus is Gorenstein, which implies that \( S^s_\pi \) is finite free over \( \mathbb{T}_\pi^s \). \( \square \)

**Remark 6.3.** From this theorem, we see that Hypothesis \([2, 9]\) is satisfied.

6.1.2. Case \( U_4 \). In the case of unitary groups, we follow closely [Ge10] Section 2. As in [Ge10], we are only interested in defining Hecke operators at finite places \( \ell \) of \( \mathbb{Q} \) that split in \( E / \mathbb{Q} \). Since in this case \( U_4(\mathbb{Q}_\ell) \) is isomorphic to \( \text{GL}_4(\mathbb{Q}_\ell) \), thus we are essentially defining Hecke operators of \( \text{GL}_4(\mathbb{Q}_\ell) \), as we will do in the following. Suppose that \( \ell \) splits as \( \mathfrak{f} \subseteq E \). Then we fix an isomorphism

\[
i_\ell : U_4(\mathbb{Z}_\ell) \cong \text{GL}_4(\mathbb{O}_E, 1) \cong \text{GL}_4(\mathbb{Z}_\ell).
\]

We can use this isomorphism \( i_\ell \) to carry over the Hecke theory of \( \text{GL}_4 \) to \( U_4 \). We write \( G' = \text{GL}_4(\mathbb{Q}_\ell) \) and the maximal compact open subgroup \( K' = \text{GL}_4(\mathbb{Z}_\ell) \). We define the spherical Hecke algebra \( H^u_\ell \) of \( G' \) as the \( \mathbb{Z} \)-algebra of finite \( \mathbb{Q} \)-linear combinations of double cosets \( K' \gamma K' \) for \( \gamma \in G' \). We have several distinguished Hecke operators in \( H^u_\ell \) as follows ([Ge10] p.10])

\[
T^u_\ell(\ell) = K' \text{diag}(\ell \cdot 1_4, 1_{4-i}) K'
\]

\((i = 1, 2, 3, 4)\).

As \( \mathbb{Q} \)-algebras, \( H^u_\ell \cong \mathbb{Q}[T^u_1(\ell), \ldots, T^u_4(\ell), T^u_4(\ell)^{-1}] \). Let \( T_{G'} \) be the standard torus of \( G' \) and define \( H^u_\ell(T_{G'}) \) to be the \( \mathbb{Q} \)-algebra of finite \( \mathbb{Q} \)-linear combinations of (double) cosets of \( T_{G'}(\mathbb{Z}_\ell) \)\( \gamma \) for \( \gamma \in T_{G'} \). Then the Satake isomorphism identifies \( H^u_\ell \) with the subalgebra of \( H^u_\ell(T_{G'}) \) consisting of elements invariant under the Weyl group \( W_{G'} \) of \( T_{G'} \) in \( G' \). We have an isomorphism of \( H^u_\ell(T_{G'}) \) with the polynomial algebra \( \mathbb{Q}[w_1^{\pm 1}, \ldots, w_4^{\pm 1}] \) by sending \( T_{G'}(\mathbb{Z}_\ell) \)\( \text{diag}(\ell a_1, \ldots, \ell a_4) \) to \( w_1^{a_1} \cdots w_4^{a_4} \). One can identify each morphism of \( \mathbb{C} \)-algebras \( \chi : H^u_\ell \otimes \mathbb{C} \to \mathbb{C} \) to the element \( \chi(w_1 \cdots w_4) \) in \( (\mathbb{C}^\times)^4 / W_{G'} \). This is the Satake parameter of \( \chi \). Since each unramified representation \( \pi' \) of \( G' \) corresponds to a \( \lambda' \), such a representation \( \pi' \) has its Satake parameter \( \lambda'(w_1 \cdots w_4) \). Moreover, \( \pi' \) can be embedded into
a principal series representation $\text{Ind}_{B_{G'}}^G (\xi_1, \cdots, \xi_4)$ where $\xi_1, \cdots, \xi_4$ are unramified characters of $\mathbb{Q}_\ell^*$. Then we have $\chi(w_1 \cdots w_4) = (\xi'_1, \cdots, \xi'_4)(\ell \cdot 1_4) \in (\mathbb{C}^*)^4/W_{G'}$. The Hecke polynomial $P^u_{\ell}(X)$ at $\ell$ is defined to be

$$P^u_{\ell}(X) := X^4 - T_{11}^u(\ell) X^3 + \ell T_{12}^u(\ell) X^2 - \ell^2 T_{13}^u(\ell) X + \ell^3 T_{14}^u(\ell).$$

We define the Iwahori Hecke algebra $H^u_{p,\text{Iw}}$ of $G' = \text{GL}_4(\mathbb{Q}_p)$ for $\ell = p$ as follows. For any integers $c \geq b \geq 0$, we write $Iw_{p,c}^{b,c}$ to be the subgroup of $K'$ such that

$$Iw_{p,c}^{b,c} \equiv \left( \begin{array}{cccc} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{array} \right) \pmod{p^b}, \quad Iw_{p,c}^{b,c} \equiv \left( \begin{array}{cccc} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{array} \right) \pmod{p^c}.$$

We fix a dominant algebraic character $\tilde{k} = (k_1, \cdots, k_4)$ of $T_{G'}$ such that $k_1 \geq k_2 \geq k_3 \geq k_4$. Let $w_{G'}$ be the longest element in $W_{G'}$. Then we define the following Hecke operators ([Ge10, p.10])

$$U_{k,i}^{b,c} = (w_{G'} \tilde{k})(\text{diag}(p \cdot 1_i, 1_{4-i}))^{-1} Iw_{p,c}^{b,c} \text{diag}(p \cdot 1_i, 1_{4-i}) Iw_{p,c}^{b,c}, \quad (i = 1, 2, 3, 4).$$

We define also the diamond operators: for each $u \in T'(\mathbb{Z}_p)$, we write $\langle u \rangle = Iw_{p,c}^{b,c} u Iw_{p,c}^{b,c}$. Then $H^u_{p,k}$ is the $\mathbb{Z}$-algebra generated by these $U_{k,i}^{b,c}$ and $\langle u \rangle$.

We then define the global Hecke algebra $H^u_{p,k}$ of $U_4(\mathbb{Q})$ Iwahori at $p$ of character $\tilde{k}$ to be the restricted tensor product

$$H^u_{p,k} = \bigotimes_{\ell \neq p, \text{split in } E} H_{p,k}^\ell \bigotimes H_{p,k}^{u,b,c}.$$

We briefly recall the notion of automorphic forms on $U_4(\mathbb{A})$ as in [Ge10, Section 2.2]. As above, let $\tilde{k}$ be a dominant character of $T_{G'}$ and we write $\tilde{M}_{\tilde{k}}$ for the algebraic representation $\text{Ind}_{B_{G'}}^G (w_{G'} \tilde{k})/\mathbb{Z}_p$ and $M_{\tilde{k}} := \tilde{M}_{\tilde{k}}(\mathbb{Z}_p)$. For any $\mathbb{Z}_p$-module $R$, we then write $S^u_{\tilde{k}}(R)$ for the space of functions ([Ge10, Definition 2.2.4])

$$f : U_4(\mathbb{Q}) \otimes U_4(\mathbb{A}_\infty) \to M_{\tilde{k}} \otimes_{\mathbb{Z}_p} R$$

such that there is a compact open subgroup $\tilde{K}$ of $U_4(\mathbb{A}_\infty \times U_4(\mathbb{Z}_p)$ and for any $u \in \tilde{K}$ and $g \in U_4(\mathbb{A}_\infty)$, we have $u_p(f(gu)) = f(g)$. The group $U_4(\mathbb{A}_\infty \times U_4(\mathbb{Z}_p)$ acts on $S^u_{\tilde{k}}(R)$ by $(g : f)(g') := g_p(f(g'g))$. For any subgroup $\tilde{K}$ of $U_4(\mathbb{A}_\infty \times U_4(\mathbb{Z}_p)$, we then write $S^u_{\tilde{k}}(R)$ for the $\tilde{K}$-invariant submodule of $S^u_{\tilde{k}}(R)$. For any compact open subgroup $\tilde{K}$ of $U_4(\mathbb{A}_\infty)$ such that $U_\ell = U_4(\mathbb{Z}_\ell)$ for any $\ell$ split in $E$ (including the the case $p$), we set $\tilde{K}^{b,c} = \tilde{K}^p \times Iw_{p,c}^{b,c}$. If $R$ is a $\mathbb{Z}_p$-algebra, we let the above defined global Hecke algebra $H^u_{\tilde{k}}^{b,c}$ act on $S^u_{\tilde{k}}(\tilde{K}^{b,c}, R)$ and write the $R$-subalgebra of $\text{End}_R(S^u_{\tilde{k}}(\tilde{K}^{b,c}, R))$ generated by the image of $H^u_{\tilde{k}}^{b,c}$ as $\mathbb{T}^u_{\tilde{k}}(\tilde{K}^{b,c}, R)$. If $R$ is only a $\mathbb{Z}_p$-module, we replace $R$-subalgebra by $\mathbb{Z}_p$-subalgebra. In the Hecke algebra $\mathbb{T}^u_{\tilde{k}}(\tilde{K}^{b,c}, R)$, we consider the idempotent element $e := \lim_{n \to \infty} (\prod_{i=1}^4 U_{k,i}^{b,c})^n$ and we define the ordinary Hecke algebra and ordinary automorphic forms as ([Ge10, Definition 2.4.1])

$$\mathbb{T}^u_{\tilde{k}}(\tilde{K}^{b,c}, R) = e \mathbb{T}^u_{\tilde{k}}(\tilde{K}^{b,c}, R), \quad S^u_{\tilde{k}}(\tilde{K}^{b,c}, R) = e S^u_{\tilde{k}}(\tilde{K}^{b,c}, R).$$

Finally we define the big ordinary Hecke algebra ([Ge10, Definition 2.4.5])

$$\mathbb{T}^u_{\tilde{k}}(\tilde{K}^{\infty}, \mathcal{O}) = \lim_{\ell > 0} \mathbb{T}^u_{\tilde{k}}(\tilde{K}^{c,c}, \mathcal{O}), \quad S^u_{\tilde{k}}(\tilde{K}^{\infty}, \mathbb{Q}_p/\mathbb{Z}_p) = \lim_{\ell > 0} S^u_{\tilde{k}}(\tilde{K}^{c,c}, \mathbb{Q}_p/\mathbb{Z}_p).$$
Note that \( \mathbb{T}^{u, \text{ord}}(K, \mathcal{O}) \) is a finite faithful algebra over \( \Lambda^u \) where
\[
\Lambda^u = \mathcal{O}[\{ T_i(1 + pZ_p) \}] \simeq \mathcal{O}[\{ Y_1, Y_2, Y_3, Y_4 \}],
\]
\[
\text{diag}(1_{i-1}, 1 + p, 1_{4-i}) \mapsto 1 + Y_i, \quad (i = 1, 2, 3, 4).
\]
is the Iwasawa weight algebra (\cite[Corollary 2.5.4]{Ge10}).

We next recall some results on the \( p \)-adic Galois representations associated to automorphic representations of \( U_4(\mathbb{A}) \). Let \( \pi' = \otimes'_v \pi'_v \) be a (cuspidal) irreducible automorphic representation of \( U_4(\mathbb{A}) \) of dominant weight \( k' = (k'_1, k'_2, k'_3, k'_4) \), unramified outside \( Np \), ordinary at \( p \) (i.e., \( \pi'_p \) is \( \text{Iw}^{0.1} \)-spherical and \( \pi' \subset \mathcal{S}^{u, \text{ord}}(K^{0.1}, \mathcal{O}) \)). We define a group scheme over \( \mathbb{Z} \):
\[
\mathcal{G}_4 = (\text{GL}_4 \times \text{GL}_4) \rtimes \{ 1, j \}
\]
where \( j(g, \mu)^{-1} = (\mu g^{-t}, \mu) \) for \( (g, \mu) \in \text{GL}_4 \times \text{GL}_4 \). Then there exists a Galois representation \( \rho_{\pi'} : \Gamma \to \mathcal{G}_4(\mathcal{O}) \) associate to \( \pi' \) (\cite[Section 2.7]{Ge10}) with certain properties introduced below. For any finite place \( \ell \nmid Np \) of \( \mathbb{Q} \) split as \( \mathbb{F} \) in \( E \), we have an isomorphism of semi-simplified Weil-Deligne representations (\cite[Proposition 2.7.2(1)]{Ge10})
\[
(6.3) \quad \text{WD}(\rho_{\pi'}|D) \simeq \iota(\text{rec}(\pi'_i \otimes \cdot | -^{3/2})^{ss}).
\]
Here \( \text{rec} \) is the local Langlands correspondence for \( \text{GL}_4 \) and \( \iota : \mathbb{C} \simeq \overline{\mathbb{Q}}_p \). Moreover, suppose \( p \) splits as \( pp' \) in \( E \) and the Hecke operators \( U^{0.1}_{k'_i,i} \) acts on the space \( \pi' \) by scalars \( u_{k'_i,i} \in \mathcal{O}^\times \) \( (i = 1, 2, 3, 4) \), then by \cite[Corollary 2.7.8]{Ge10}, if \( k'_i \) is regular (i.e., \( k_1 > k_4 \)), we have that \( \rho_{\pi',p} := \rho_{\pi}|D_p \) is crystalline and is conjugate to an upper triangular representation
\[
(6.4) \quad \rho_{\pi',p} \simeq \begin{pmatrix}
\chi'_1 & * & * & * \\
0 & \chi'_2 & * & * \\
0 & 0 & \chi'_3 & * \\
0 & 0 & 0 & \chi'_4
\end{pmatrix}
\]
whose diagonal entries are characters defined as \( \chi'_i = \text{ur}(u_{k'_i,i}/u_{k'_i,-1})^{-k'_{5-i-1}} \) (we set \( u_{k'_0,0} = 1 \)). On the side of automorphic representations, \( \pi_p \) is unramified and therefore (\cite[Lemma 2.7.5]{Ge10}): 
\[
\det(X - \text{Fr}_p', \rho_{\pi',p}^{\text{cryst}}) = \prod_{i=1}^4 (X - p^{i-1+k'_{5-i-1}} u_{k'_i,i}/u_{k'_i,-1}).
\]
Here \( \text{Fr}_p' \) and \( \rho_{\pi',p}^{\text{cryst}} \) are defined as in the case of \( \text{GSp}_4 \).

Next we define deformation functors and universal deformation rings of the residual representation \( \overline{\rho}_{\pi'} \) as follows (\cite[Section 2.3]{HT16}):
\[
\mathcal{D}_{\pi', k'} : \text{AL}_{\mathcal{O}} \to \text{Sets}
\]
where \( \mathcal{D}_{\pi'}(A) \) is the set of equivalence classes of liftings \( \rho_A : \Gamma \to \mathcal{G}_4(A) \) such that
\begin{enumerate}
\item \( \rho_A \) is unramified outside \( Np \);
\item \( \text{the projection of } \rho_A|D_p \text{ to the factor } \text{GL}_4(A) \) is conjugate by an element \( g \in 1 + m_A M_{4 \times 4}(A) \) to an upper triangular representation whose diagonal entries are characters \( \overline{\chi}_i \rho^{-1}_A \) such that \( \overline{\chi}_i \) lifts the character \( \text{ur}(u_{k'_i,i}/u_{k'_i,-1})^{-k'_{5-i-1}} \).
\end{enumerate}
For \( \mathcal{D}_{\pi', k'}(A) \), we require moreover that in (2) above \( \overline{\chi}_i \) is of the form \( \overline{\chi}_i(\sigma) = \text{Art}(\sigma)^{k'_{5-i}} \) for \( \sigma \in I_p \) and \( \text{Art}(\sigma) \in \mathbb{Z} \) by the local Artin map (of course \( \mathcal{D}_{\pi', k'}(A) \) is non-empty only if \( k' = k'_i \text{ (mod } p-1) \)). Two liftings \( \rho_A \) and \( \rho'_A \) are equivalent if there is an element \( g \in \text{Ker}(\text{GL}_4(A) \to \text{GL}_4(\mathbb{F})) \) such that \( \rho'_A \simeq g \rho_A g^{-1} \). If we assume that the diagonal entries \( \chi'_i \) in \( \overline{\rho}_{\pi',p} \) are mutually distinct either on the
Frobenius element $\text{Fr}_p$ or the inertia subgroup $I_p$ of $D_p$, then by [HT16, Lemma 2.6], $D_{\pi'}$ is pro-representable by $(\rho_\pi', R_{\pi'})$ where $R_{\pi'}$ is in CNL and $\rho_\pi': \Gamma_Q \to \mathcal{G}_4 (R_{\pi'})$. We then write $m_\pi'$ for the maximal ideal of $T_{k'}^u (K^\infty, \mathcal{O})$ associated to the residual representation $\overline{\pi}_\pi'$ and then set
\[
T_{k'}^u := T_{k'}^{u, \text{ord}} (K^\infty, \mathcal{O})_{m_\pi'}, \quad S_{k'}^u := S_{k'}^{u, \text{ord}} (K^\infty, \mathcal{O}/pZ_p)_{m_\pi'}.
\]

Then there is a Galois representation $\rho_{T_{k'}^u}: \Gamma_Q \to \mathcal{G}_4 (T_{k'}^u)$ lifting $\overline{\pi}_\pi'$ and thus we have a morphism $R_{\pi'}^u \to T_{k'}^u$ of $\Lambda^u$-algebras. The method of Taylor-Wiles gives ([HT16, Section 2.7]):

**Theorem 6.4.** Assume Hypothesis 6.1, then the morphism $R_{\pi'}^u \to T_{k'}^u$ is an isomorphism and $T_{k'}^u$ is a finite flat complete intersection over $\Lambda^u$. The Pontryagin dual $(S_{k'}^u)^\vee$ is a finite free $T_{k'}^u$-module. Moreover, the specialization map $T_{k'}^u \otimes_{\Lambda^u, k'} \mathcal{O} \to T_{k'}^u$ is an isomorphism for any dominant weight $k'' \equiv k' (\text{mod} \ p - 1)$. In particular, we have an isomorphism $R_{\pi'}^u \simeq T_{k'}^u$.

### 6.2. Morphism of the transfer.
Let $\pi$ be an irreducible automorphic representation of $\text{GSp}_4 (\mathbb{A})$ of type $(\Lambda, \hat{\Gamma})$ (holomorphic or antiholomorphic). We write $\pi' = \Theta (\pi)$ for the theta lift of $\pi$ to $U_4 (\mathbb{A})$. Suppose that $\Theta (\pi) \neq 0$. In this subsection we define morphisms $\Lambda^u \to \Lambda^s$, $T_{k'}^u \to T_{k'}^s$ and $R_{\pi'}^u \to R_{\pi'}^s$.

#### 6.2.1. Morphism of Iwasawa weight algebras.
We define a map of tori $T^s$ of $\text{GSp}_4 (\mathbb{Q}_p)$ and $T^u$ of $\text{GL}_4 (\mathbb{Q}_p)$ as follows (natural inclusion):
\[
T^s \to T^u, \quad \text{diag} (t_0, t_0t_1, t_0t_2, t_0t_1t_2) \mapsto \text{diag} (t_0, t_0t_1, t_0t_2, t_0t_1t_2).
\]

This induces a morphism of Iwasawa algebras
\[
\Lambda^u = \mathcal{O} [[T^u (1 + pZ_p)]] \to \Lambda^s = \mathcal{O} [[T^s (1 + pZ_p)]].
\]

#### 6.2.2. Morphism of Hecke algebras.
We define a map of Hecke algebras $T_{k'}^u \to T_{k'}^s$ as follows: for any prime $\ell \nmid N$ split as $\mathbb{F}_p$ in $E$, the morphism of local Hecke algebras $(T_{k'}^u)_{(\ell)} \to (T_{k'}^s)_{(\ell)}$ is given by [Ra82, Remark 4.4(A)] (in the notation of loc.cit., for the case $i = 3, n = 2$). Note that in [Ra82], only the case $\pi_\ell$ unramified is treated. For the case of $\pi_\ell$ Steinberg, the same result follows using Theorem 5.5 and the Satake isomorphism of Iwahori Hecke algebras. Moreover, for the diamond operators $(u)$, we send the image $\text{Im} (\mathcal{O} [[T^u (Z_p)]] \to \mathcal{O} [[T^s (Z_p)]]$ as induced by the map $T^s (Z_p) \to T^u (Z_p)$ defined above. This finishes the definition of the map $T_{k'}^u \to T_{k'}^s$. Similarly we define $T_{k'}^u \to T_{k'}^s$.

#### 6.2.3. Morphism of Galois representations.
Write $\Gamma_{E/Q} = \{1, c\}$. Define a group scheme $\mathcal{G}'_4$ over $\mathbb{Z}$ by $(\text{GL}_4 \times \text{GL}_1) \rtimes \{1, c\}$ where $c(g, \mu)^{-1} := (\mu J_4 g^{-1} J_4^{-1}, \mu)$. It is easy to verify that the following map is an isomorphism
\[
\mathcal{G}'_4 \to \mathcal{G}_4, \quad (g, \mu, 1) \mapsto (g, \mu, 1), \quad (g, \mu, c) \mapsto (gJ_4, -\mu, j).
\]

Moreover, the following map is a morphism of groups:
\[
\text{GSp}_4 \times \Gamma_{E/Q} \to \mathcal{G}'_4, \quad (g, x) \mapsto (g, \nu(g), x).
\]

Let $R$ be a topological ring. Then for any continuous Galois representation $\rho: \Gamma_Q \to \text{GSp}_4 (R)$, we define
\[
\Theta (\rho): \Gamma_Q \to \mathcal{G}'_4 (R), \quad \sigma \mapsto (\rho (\sigma), \nu (\rho (\sigma)), \overline{\sigma})
\]
where $\overline{\sigma}$ is the projection image of $\sigma$ in $\Gamma_{E/Q}$. Composing with the isomorphism $\mathcal{G}'_4 \simeq \mathcal{G}_4$, we define the following (cf. [CHT08, Lemma 2.1.2])
\[
\Theta (\rho): \Gamma_Q \xrightarrow{\Theta (\rho)} \mathcal{G}'_4 (R) \xrightarrow{\sim} \mathcal{G}_4 (R).
\]
Thus $\Theta(\rho)$ and $\Theta'(\rho)$ are isomorphic as representations of $\Gamma_{E/\mathbb{Q}}$.

We can relate the adjoint representations of $\rho$ and $\Theta(\rho)$ as follows:

**Lemma 6.5.** Suppose that $\sqrt{2} \in \mathbb{R}^\times$. We have the following isomorphisms of representations of $\Gamma_{\mathbb{Q}}$:  
\[ \text{ad}(\Theta(\rho)) \simeq \text{ad}(\Theta'(\rho)) \simeq \text{ad}(\rho) \oplus (\rho_{st} \circ \rho \otimes \xi) \]

**Proof.** The first isomorphism is clear. For the second, let $G_1(R)$ act by adjoint action on $\mathfrak{sl}_4(R)$. Note that $c$ acts on $\mathfrak{sl}_4(R)$ by $c(X) := -J_4XJ_4^{-1}$ while $\text{GL}_4(R)$ acts trivially on $\mathfrak{sl}_4(R)$. As in Remark 5.2, we have the following decomposition $\mathfrak{sl}_4(R) = V_1 \oplus V_2$ as representations of $\Gamma_{\mathbb{Q}}$, where  
\[ V_1 = \{ X \in \mathfrak{sl}_4(R) | J_4XJ_4^{-1} = -X \}, \quad V_2 = \{ X \in \mathfrak{sl}_4(R) | J_4XJ_4^{-1} = X \}. \]

Note that $V_1 = \mathfrak{sp}_4(R)$, thus $V_1$ is isomorphic to $\text{ad}(\rho)$ as representations of $\Gamma_{\mathbb{Q}}$. On the other hand, by definition, $c$ acts on $V_2$ by $-1$, so we see that the action of $\Gamma_{\mathbb{Q}}$ on $V_2$ via the adjoint action of $\Theta'(\rho)$ is isomorphic to the action via $\rho_{st} \circ \rho \otimes \xi$ (in the definition of $\rho_{st}$, the element $\sqrt{2}$ is used, that is why we suppose $\sqrt{2} \in \mathbb{R}^\times$). \qed

**Remark 6.6.** Note the difference of our decomposition given in the lemma and the one given in [HT16, Before Theorem 7.3]. This difference comes from the fact that we use the theta correspondence between $\text{GSp}_4$ and $\text{GSO}_6$ while [HT16] uses the theta correspondence between $\text{GSp}_4$ and $\text{GSO}_{3,3}$. More explicitly, in [HT16], a Galois representation $\rho: \Gamma_{\mathbb{Q}} \to \text{GSp}_4(R)$ is taken to a Galois representation $\Gamma_{\mathbb{Q}} \to G_4(R)$ by sending $g \in \text{GSp}_4(R)$ to $(g, \nu(g), 1) \in G_4(R)$.

6.2.4. *Morphism of universal deformation rings.* We apply the map $\Theta$ to the continuous $p$-adic Galois representation $\rho_{\pi}: \Gamma_{\mathbb{Q}} \to \text{GSp}_4(O)$.

**Lemma 6.7.** The representation $\Theta(\rho_{\pi})$ is a lift of $\overline{\rho}_{\pi'}$.

**Proof.** For any finite place $\ell \nmid N$, $\pi_{\ell}$ is unramified or Steinberg. By the correspondence of Satake parameters of local theta correspondence (Theorem 5.12), the relations of Weil-Deligne representations and Galois representations 6.1 and 6.3, we see that $\Theta(\rho_{\pi})|_{D_{\ell}}$ is equal to $\rho_{\pi'}|_{D_{\ell}}$ and thus lifts $\overline{\rho}_{\pi'}|_{D_{\ell}}$. Moreover, it is clear that $\Theta(\rho_{\pi})$ is unramified outside $N_p$. \qed

The same argument shows that $\Theta(\rho_{\pi})$ lifts $\overline{\rho}_{\pi'}$, thus by the universal property of $R_{\pi'}^u$, there is a natural map of $\mathcal{O}$-algebras:

\[ (6.6) \quad R_{\pi'}^u \to R_{\pi}^u, \text{ similarly, } R_{\pi''}^u \to R_{\pi'}^u. \]

We have the following:

**Lemma 6.8.** The morphisms defined above are all surjective and compatible with each other:

\[ \Lambda^u \longrightarrow T_{\pi'}^u \longrightarrow R_{\pi'}^u \longrightarrow R_{\pi''}^u \]

\[ \Lambda^s \longrightarrow T_{\pi}^s \longrightarrow R_{\pi}^s \longrightarrow R_{\pi'}^s \]

**Proof.** The commutativity of the left square follows from the definition of the map $T_{\pi'}^u \to T_{\pi}^s$. The commutativity of the right square is clear. For the middle square, it suffices to look at their respective Galois representations: for each $\ell \nmid N_p$ split in $E$, under the morphism $T_{\pi'}^u \to T_{\pi}^s$, the unramified Galois representation $\rho_{\pi'}|_{D_{\ell}}$ is mapped to $\Theta(\rho_{\pi'})|_{D_{\ell}}$ by [Ral82, Section 7]. Similarly, by the definition of diamond operators $\{u\}$, the ordinary Galois representation $\rho_{T_{\pi}^s}|_{D_p}$ is mapped to $\Theta(\rho_{T_{\pi}^s})|_{D_p}$. This shows the compatibility of $T_{\pi'}^u \to T_{\pi}^s$ with $R_{\pi'}^u \to R_{\pi}^s$.

We next show that $\Lambda^u \to \Lambda^s$ is surjective. Since $\mathcal{O}[[1 + p\mathbb{Z}_p]]$ is the projective limit of the group rings $\mathcal{O}[[1 + p\mathbb{Z}_p]]/\mathcal{O}[[1 + p^{n+1}\mathbb{Z}_p]] \simeq \mathcal{O}[\mathbb{Z}/p^n\mathbb{Z}]_n$, it suffices to show that the inclusion of the finite
group \( T^u_n := T^u(1 + p\mathbb{Z}_p)/T^u(1 + p^{n+1}\mathbb{Z}_p) \) in \( T^u_n := T^u(1 + p\mathbb{Z}_p)/T^u(1 + p^{n+1}\mathbb{Z}_p) \) induces a surjective morphism of group rings \( \mathcal{O}[T^u_n] \to \mathcal{O}[T^u_n] \). This is clear since \( T^u_n \) is a direct summand of \( T^u_n \).

Finally we show that \( R^u_n \to R^u_n \) is surjective, which also gives the surjectivity of the other two vertical arrows. The argument is similar to [Zha18 Lemma 4.10]. For this it suffices to show that for any object \( A \) in \( \text{AL}_{\mathcal{O}} \), for any two lifts \( \rho_A \) and \( \rho_A' \) of \( \pi_A \), if \( \Theta(\rho_A) \) and \( \Theta(\rho_A') \) are equivalent (conjugate by an element in \( \text{GL}_4(A) := \text{Ker}(\text{GL}_4(A) \to \text{GL}_4(F)) \)), then \( \rho_A \) and \( \rho_A' \) are equivalent (conjugate by an element in \( \text{GSp}_4(A) := \text{Ker}(\text{GSp}_4(A) \to \text{GSp}_4(F)) \)). Suppose that there is some \( g \in \text{GL}_4(A) \) such that \( \Theta'(\rho_A) = g\Theta(\rho_A)g^{-1} \). We write \( \rho_{A,p} = \rho_{A|D_p} \) and \( \rho'_{A,p} = \rho'_{A|D_p} \). By definition there are elements \( h, h' \in \text{GSp}_4(A) \) such that the representations of \( D_p, \tilde{\rho} := h^{-1}\rho_{A,p}h \) and \( \tilde{\rho}' := (h')^{-1}\rho'_{A,p}h' \) are upper triangular. Recall that the diagonal entries of \( \tilde{\rho} \), resp., \( \tilde{\rho}' \), are mutually distinct characters, so there is some element \( \sigma_0 \in D_p \) such that the diagonal elements of \( \tilde{\rho}(\sigma_0) \), resp., \( \tilde{\rho}'(\sigma_0) \), are mutually distinct. We can choose \( h \), resp., \( h' \) such that \( \tilde{\rho}(\sigma_0) \), resp., \( \tilde{\rho}'(\sigma_0) \), are diagonal matrices with mutually distinct diagonal entries and are equal to each other. Since \( \rho_{A,p} = g^{-1}\rho_{A,p}g \) (note that \( p \) splits in \( E/Q \)), we conclude that \( h' = gh \). This gives \( g = h'h^{-1} \in \text{GSp}_4(A) \cap \text{GL}_4(A) = \text{GSp}_4(A) \).

### 6.3. Selmer groups and congruence ideals

In this subsection we define Selmer groups for the Galois representations \( \rho_{\mathfrak{p}^n} \) with \( ? = s, u \) following the treatment in [HT16, Section 1].

Let \( R \) be an object in \( \text{Cln}_{\mathcal{O}} \) and \( \rho: \Gamma_Q \to G(R) \) a continuous Galois representation, unramified outside \( Np \), ordinary at \( p \). Here \( G = \text{GSp}_4, \mathcal{G}_4 \) or a classical subgroup of \( \text{GL}_n \). We say that \( \Gamma_Q \) acts on the free \( R \)-module \( M \) with \( M = R^4 \), \( R^n \) according to \( G = \text{GSp}_4, \mathcal{G}_4 \) or a subgroup of \( \text{GL}_n \). By the ordinarity assumption, there is a filtration (Fil\(_0(M)\)) on \( M \) stable under \( \rho_{D_p} \) (or \( \rho_{D_p} \) if \( G = \mathcal{G}_4 \)). Assume moreover that \( \rho_{D_p} \) acts by \( \text{ur}(a_i)\chi_{\mathfrak{p}_i}^{-i} \) on \( \text{Fil}_0(M)/\text{Fil}_{-1}(M) \). We write \( M^r = \text{Hom}_{\mathcal{O}}(M, \mathcal{O}[1/p]/\mathcal{O}) \) for the Pontryagin dual of \( M \). We define a subspace \( L_p \) of \( H^1(D_p, M \otimes_R R^r) \) as the image of

\[
\text{Ker}(H^1(D_p, \text{Fil}_0(M) \otimes_R R^r) \to H^1(I_p, \text{Fil}_0(M)/\text{Fil}_{-1} \otimes_R R^r)).
\]

Then we define the discrete Selmer group of \( \rho \) as

\[
\text{Sel}(\rho) = \text{Ker}\left(H^1(\mathbb{Q}, M \otimes_R R^r) \to \prod_{\ell \neq p} H^1(I_\ell, M \otimes_R R^r) \times H^1(D_p, M/\text{Fil}_0(M) \otimes_R R^r)/L_p\right).
\]

Now let \( \rho: \Gamma_Q \to \text{GSp}_4(R) \) be a continuous Galois representation as above. Then the following Galois representations also satisfies the above assumptions, ad \( \rho: \Gamma_Q \to \text{GL}_R(\mathfrak{sp}_4(R)), \Theta(\rho): \Gamma_Q \to \mathcal{G}_4(R) \), ad \( \Theta(\rho): \Gamma_Q \to \text{GL}_R(\mathfrak{s}_4(R)), \rho_{st} \circ \rho: \Gamma_Q \to \text{SO}_5(R) \). Moreover, since \( \text{ad}(\Theta(\rho)) \simeq \text{ad}(\rho \circ (\rho_{st} \circ \rho \otimes \xi)) \), we have the following decomposition:

\[
\text{Sel}(\text{ad}(\Theta(\rho))) \simeq \text{Sel}(\text{ad}(\rho)) \bigoplus \text{Sel}(\rho_{st} \circ \rho \otimes \xi).
\]

Now we consider the morphism defined above \( \vartheta: T^u_k \to T^u_k \) where \( k \) is the weight of \( \pi \). The Hida family of \( \pi \) in \( T^u_k \) also gives a surjective map of \( \mathcal{O} \)-algebras \( \vartheta_{\pi}: T^u_k \to \mathcal{O} \). Note that both \( T^u_k \) and \( T^u_k \) are complete intersections over \( \mathcal{O} \) by the \( R = T \) theorems, so we can apply [HT16 Corollary 8.6] to conclude that

\[
\mathfrak{c}(\vartheta_{\pi} \circ \vartheta) = \mathfrak{c}(\vartheta_{\pi}) \vartheta_{\pi}(\mathfrak{c}(\vartheta)).
\]

Recall that for any \( \mathcal{O} \)-module of finite type \( M, M \) is isomorphic to a unique \( \mathcal{O} \)-module of the form \( \mathcal{O}^r \times \prod_{i=1}^j \mathcal{O}/\varpi^r_i \) for some \( r, j, r_i \geq 0 \). Then the Fitting ideal \( \text{Fit}(M) \) of \( M \) is 0 if \( r > 0 \) and \( \prod_{i=1}^j \varpi^r_i \mathcal{O} \) otherwise. By [HT16 Proposition 3.4], the Fitting ideal \( \text{Fit}(\text{Sel}(\text{ad}(\Theta(\rho)))) \), resp., \( \text{Fit}(\text{Sel}(\text{ad}(\rho_{st} \circ \rho \otimes \xi))) \) is equal to the congruence ideal \( \mathfrak{c}(\vartheta_{\pi} \circ \vartheta) \), resp., \( \mathfrak{c}(\vartheta_{\pi}) \). Therefore we conclude

**Proposition 6.9.** Assume Hypothesis 6.1, then the Fitting ideal \( \text{Fit}(\text{Sel}(\xi \circ \rho_{st} \circ \rho \otimes \xi)) \) is equal to the congruence ideal \( \vartheta_{\pi}(\mathfrak{c}(\vartheta)) \).
6.4. Conclusion. Recall that we fix a $p$-ordinary antiholomorphic cohomological cuspidal irreducible automorphic representation $\pi$ of $[G_1]$, of trivial central character, of type $(\mathbb{K}, \Gamma)$ with $k_1 \geq k_2 \geq 3$ ($\pi$ is anti-holomorphic on $G_1$ while it is holomorphic on $G_2$). We assume that the Hida family $\mathbb{T}^n_\pi$ in $\mathbb{T}^n$ passing through $\pi$ is of weight $\mathfrak{w} = (\mathfrak{w}_1, \mathfrak{w}_2) \in \text{Hom}(T_{G_1}(\mathbb{Z}_p), \overline{\mathbb{Q}}_p^+)$, and the $U_p$-operators $U_{p, \xi}$ act on the ordinary subspace $e\pi$ by the scalar $a_\xi^1a_\xi^2$ for all $\xi \in C^\times$. Moreover, in Section 4, we have constructed sections $\phi_{i, k} \in S(W_i(\mathbb{A}))$ for $i = 1, 2$.

We show the theta lift $\Theta_{\phi_1, \xi}(\varphi)$ is a $p$-integral primitive modular form on $U_4(\mathbb{A})$ up to a certain power of $p$ if $\varphi \in \pi$ is a $p$-integral primitive form. In Proposition 4.33, we have shown that the Fourier coefficients of the theta series $\Theta_{\phi_1}^\varphi$ are all in $\mathcal{O}$, thus for each $h \in U_4(\mathbb{A})$, $\Theta_{\phi_1}^\varphi(\cdot, h)|\nu(\cdot)|^3 \in H^0(\mathbb{A}_{G_1, \hat{\Gamma}/\hat{\mathcal{O}}, \mathcal{E}(V)})$ for some algebraic representation $W_i$ of $\text{GL}_2/\mathbb{Q}$ by the map $\Phi(\cdot, \xi)$ (cf. Definition 4.34). Thus for any anti-holomorphic $p$-integral cuspidal Siegel modular form $f_i \in H^3(G_i, \hat{\Gamma}, \mathcal{O})$, the Serre duality shows that for each fixed $h \in U_4(\mathbb{A})$, we have

$$
\Theta_{\phi_1, \xi}^\varphi(f_i)(h) = \langle \Theta_{\phi_1, \xi}(\cdot, h)|\nu(\cdot)|^3, f_i(\cdot) \rangle_{\text{Ser}} = \int_{[G_i]} f_i(g_i)\Theta_{\phi_1, \xi}(g_i, h)dg_i \in \mathcal{O}
$$

(we extend $\Theta_{\phi_1, \xi}(f_i)$ by zero from $SU_4(\mathbb{A})$ to $U_4(\mathbb{A})$, cf. Section 5.5). Therefore, we can choose some $\nu(f_i) \in \mathbb{Z}_{\geq 0}$ (which is clearly unique) such that $p^{-\nu(f_i)}\Theta_{\phi_1, \xi}^\varphi(f_i)$ is primitive. We define new Schwartz functions $\tilde{\phi}_{i, k} = \otimes_i \tilde{\phi}_{i, k, v}$ whose local components are given by

$$
\tilde{\phi}_{i, k, \infty} = p^{-\nu(f_i)}\phi_{i, k, \infty}, \quad \tilde{\phi}_{i, k, \ell} = \phi_{i, k, \ell}, \forall \ell.
$$

Thus $\Theta_{\phi_1, \xi}^\varphi(f_i) = p^{-\nu(f_i)}\Theta_{\phi_1, \xi}^\varphi(f_i)$ is a $p$-integral primitive modular form on $U_4(\mathbb{A})$. Now we take $f_1$ and $f_2$ to be cuspidal ordinary $p$-integral Siegel modular forms $\varphi_1 \in H^3_{\text{ord}}(G_1, \mathbb{A}, \mathcal{O})[\pi]$ and $\varphi_2 \in H^3_{\text{ord}}(G_2, \mathbb{A}, \mathcal{O})[\pi]$ such that $\langle \varphi_1, \varphi_2 \rangle$ is equal to the period $\hat{P}[\pi]$ (cf. Lemma 2.16). We have integers $\nu(\varphi_1)$ and $\nu(\varphi_2)$ as above and also $\tilde{\phi}_{i, k} = p^{-\nu(\varphi_i)}\phi_{i, k}$ whose local components are given as above. As such we have two $p$-integral primitive modular forms $\Theta_{\phi_1, \xi}^\varphi(\varphi_i)$ on $U_4(\mathbb{A})$. Recall from Theorem 4.32 we have defined the modified local Euler factor $L^*_{\pi}(1, \text{St}(\pi) \otimes \xi)$ for $v|Np\infty$ which depends on $\phi_{i, k, v}$.

Now we put

$$
\tilde{L}_{\infty}(1, \text{St}(\pi) \otimes \xi) = p^{-\nu(\varphi_1) - \nu(\varphi_2)}L^*(1, \text{St}(\pi) \otimes \xi), \quad \tilde{L}_{\ell}(1, \text{St}(\pi) \otimes \xi) = L^*(1, \text{St}(\pi) \otimes \xi), \forall |\ell|Np.
$$

These are the local Euler factors given by the new Schwartz functions $\tilde{\phi}_{i, k, v}$ for $v|Np\infty$. Taking into account Remark 6.33 (which implies $c(\varphi_\pi) = c_{\text{coh}}(\pi)$), the Rallis inner product formula now reads

**Theorem 6.10.** Let the notations be as above, then we have the following identity up to a unit in $\mathcal{O}$:

$$
\langle \Theta_{\phi_1, \xi}^\varphi(\varphi_1), \Theta_{\phi_2, \xi}^\varphi(\varphi_2) \rangle_{U_4} = c(\varphi_\pi)L^{Np\infty}(1, \text{St}(\pi) \otimes \xi)\tilde{L}_{Np\infty}(1, \text{St}(\pi) \otimes \xi)P_{\pi^\infty}.
$$

By the same argument as in [Ber14, Lemma 23], one can show that the theta series $\Theta_{\phi_1, \xi}^\varphi$ are both eigenforms for the Hecke algebra $\mathbb{T}^n_\mathbb{A}$ (we first show that they are eigenforms for the Hecke algebra of the orthogonal group $O(U)$ and then use the exceptional isogeny between the orthogonal group and unitary group to identify their respective Hecke algebras). Therefore we see that the Siegel modular forms $\varphi_i$ give rise to $\mathbb{T}^n_\mathbb{A}$-eigenforms $\Theta_{\phi_1, \xi}^\varphi(\varphi_i)$ by Lemma 2.18 the Petersson product $\langle \Theta_{\phi_1, \xi}^\varphi(\varphi_1), \Theta_{\phi_2, \xi}^\varphi(\varphi_2) \rangle_{U_4}$ generates the congruence ideal $c(\varphi_\pi \circ \varphi_\pi)$. Write $\chi(\text{Sel}(\xi \otimes \rho_{st} \circ \rho_p))$ for a generator of the Fitting ideal $\text{Fit}(\text{Sel}(\xi \otimes \rho_{st} \circ \rho_p))$. Combining the above theorem with Proposition 6.9, we get the following (for the local factors $L^{Np\infty}_{\pi}(1, \text{St}(\pi) \otimes \xi)$, see Theorem 4.32):
Theorem 6.11. Let the notations be as in the last theorem. We make the assumptions (1) Hypothesis 6.1; (2) the pairing $\langle \varphi_1, \varphi_2 \rangle = \hat{P}[\pi] \neq 0$ and $\varphi_1, \varphi_2$ are $p$-ordinary. Then we have the following identity up to units in $\mathcal{O}$:

$$
\frac{L^{Np,\infty}(1, \text{St}(\pi) \otimes \xi) \tilde{L}_{Np,\infty}(1, \text{St}(\pi) \otimes \xi)}{P_{\pi^\vee}} = \chi(\text{Sel}(\xi \otimes \rho_{st} \circ \rho_{\pi})).
$$

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