The new structures of stochastic solutions for the nonlinear Schrödinger’s equations

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Abstract
The nonlinear Schrödinger’s equations (NLSEs) is a famous model used to investigate the propagation of optical solitons via nonlinear optical fibers. We applied the unified solver method in order to extract some new stochastic solutions for three types of NLSEs forced by multiplicative noise in Itô sense. The acquired solutions describe the propagation of solitons in nonlinear optical fibers. We exhibit the influence of presence of noise term on the solution for the NLSEs. The theoretical analysis and presented solutions illustrate that the proposed solver is powerful and efficient. Finally, the wave amplitudes may be controlled by the effects performance of physical parameters of the NLSEs in the presence of noise term in Itô sense. Finally, we present He’s frequency formulation.

Keywords
Closed-form solutions, stochastic Schrödinger equations, optical solitons, symbolic computations

AMS subject classifications: 35A08, 35A22, 35C08, 60H10, 60H15, 35Q55, and 35Q60

Introduction
The nonlinear wave phenomena plays a fundamental role in different fields of natural sciences, like nonlinear optics, superfluid, high-energy physics, biology, nuclear physics, gravitation, engineering, solid state physics, and so on.¹⁻⁶ Noise (randomness) is of great importance in many phenomena, thus it has become important to involve random effects when explaining different physical phenomena in chemical engineering, physics, economy, digital simulation, robotics control, networked systems, and many others.⁷,⁸ The nonlinear partial differential equations (NPDEs) that consider time-dependent randomness are called stochastic NPDEs. The nonlinear wave phenomena exist for solutions of deterministic and stochastic NPDEs.⁹⁻¹¹ Various studies focused on the nonlinear wave in NPDEs and their applications.¹²⁻¹⁹

Consider nonlinear partial differential equation for \( \mathcal{H}(x,t) \) in the existence of noise term in Itô sense given by

\[
\Gamma(\mathcal{H}, \mathcal{H}_x, \mathcal{H}_t, \mathcal{H}_{xx}, \mathcal{H}_{xt}, \mathcal{H}_{tt}, \ldots) = 0.
\]

Utilizing wave transformation:

\[
\mathcal{H}(x,t) = \mathcal{H}(\xi), \quad \xi = x - vt,
\]

to equation (1.1) where

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\[
\mathcal{H}_t = \frac{d\mathcal{H}}{d\xi} \frac{d\xi}{dx} = \mathcal{H}', \quad \mathcal{H}_\xi = \frac{d\mathcal{H}}{d\xi} \frac{d\xi}{dt} = -v\mathcal{H}', \quad \mathcal{H}_{\xi\xi} = \frac{d\mathcal{H}_\xi}{d\xi} \frac{d\xi}{dx} = \mathcal{H}''', \quad \mathcal{H}_{\xi\xi\xi} = \frac{d\mathcal{H}_{\xi\xi}}{d\xi} \frac{d\xi}{dt} = v^2 \mathcal{H}'''
\]

and the prime corresponds to the differentiation with respect to \(\xi\). \(\xi\) is the wave transformation and \(v\) is the velocity speed of the traveling wave. After putting equation (1.2) into equation (1.1) and simplification, the nonlinear partial differential equation (1.1) converted to the following ODE:

\[
\Lambda(\mathcal{H}, \mathcal{H}', \mathcal{H}'', \mathcal{H}''', \ldots) = 0.
\]

Various models in natural sciences presented in form of equation (1.1), converted to the following duffing ODE

\[
C_1 \mathcal{H}'' + C_2 \mathcal{H}^3 + C_3 \mathcal{H} = 0,
\]

where \(C_1\), \(C_2\), and \(C_3\) are certain constants depending on the constants of the main problem and the velocity speed of the wave transformations. In ref. 21, we introduced a unified solver technique to solve equation (1.1) in deterministic case in a completely unified way. The effect of stochastic terms of the NPDEs is so important to explain many vital phenomena in many fields of real life problems, such as fluid mechanics, biology, engineering, chemical engineering, fluid dynamics, solid state physics, signal processing. In this work, we developed this solver to solve new families of equations in presence of noise in Itô sense. This solver gives closed form of solutions in explicit way. Indeed, this solver will be used as a box solver for physicists, engineers and mathematicians.

The nonlinear Schrödinger equations (NLSEs) mainly describe the dynamics of optical soliton promulgation in optical fibers, electromagnetic wave propagation, deep water, plasma physics, super conductivity, quantum electronics, magnetostatic spin waves, optoelectronics, and photonics. Many real circumstances of media have given various forms of the NLSEs, such as cubic nonlinear term; cubic-quintic-septimal nonlinear terms; derivative nonlinear term and other more. Lebowitz et al. investigated the statistical mechanics of a complex field whose dynamics is determined by NLSE. Such fields characterize in suitable idealizations, Langmuir waves in a plasma and a propagating laser field in a nonlinear medium. Recently, the NLSEs have acquired significant attention and have been a point of discussion of various studies, like auxiliary equation approach; sine-Gordon expansion approach; \((\xi)^n\) expansion approach; RB sub-ODE approach; direct algebraic approach; unified solver approach; modified Kudraysov approach; and sine–cosine approach. In this work, we consider different types of NLSEs in the presence of noise term in Itô sense. Namely, we introduce new solutions for these equations. These solutions are applicable in explaining various complex phenomena in new physics.

This article is organized as follows. The closed-form structures introduces the closed-form solutions for a wide range of NPDEs. Nonlinear Schrödinger’s equations forced by multiplicative noise in Itô sense presents some new stochastic solutions for the NLSE forced by multiplicative noise in Itô sense. Results and discussion gives the explanation for the acquired solutions for the stochastic NLSEs. We also introduce He’s frequency formulation for the duffing equation. Finally, conclusion is provided in Conclusions.

**The closed-form structures**

We give closed-form solutions for the following equation:

\[
C_1 \mathcal{H}'' + C_2 \mathcal{H}^3 + C_3 \mathcal{H} = 0.
\]

In view of the unified solver technique introduced in ref. 21, the solutions for equation (2.1) are:

**Rational solutions: (at \(C_3 = 0\))**

\[
\mathcal{H}_{1,2}(x,t) = \left( \mp \frac{\sqrt{|C_3|}}{2C_1} (\xi + \varsigma) \right)^{-1},
\]

where \(\varsigma\) is an arbitrary constant.

**Trigonometric solutions: (at \(\frac{C_3}{C_1} < 0\))**

\[
\mathcal{H}_{3,4}(x,t) = \pm \sqrt{\frac{C_3}{C_1}} \tan \left( \sqrt{\frac{C_3}{2C_1}} (\xi + \varsigma) \right),
\]
\[ H_{5,6}(x,t) = \pm \sqrt{\frac{C_3}{C_2}} \cot\left( \sqrt{\frac{C_1}{2C_1}} (\xi + \zeta) \right). \] (2.4)

**Hyperbolic solutions: (at \( \frac{C_3}{C_1} > 0 \))**

\[ H_{7,8}(x,t) = \pm \sqrt{\frac{-C_3}{C_2}} \tanh\left( \sqrt{\frac{C_3}{2C_1}} (\xi + \zeta) \right), \] (2.5)

\[ H_{9,10}(x,t) = \pm \sqrt{\frac{-C_3}{C_2}} \coth\left( \sqrt{\frac{C_3}{2C_1}} (\xi + \zeta) \right). \] (2.6)

**Nonlinear Schrödinger’s equations forced by multiplicative noise in Itô sense**

The proposed closed-form of solutions is tested through a range of applications. Namely, we applied this solver to different forms of the NLSEs in presence of noise term in Itô sense.

**NLS**

We first consider the NLSE given by:

\[ i\psi_t + \psi_{xx} + 2\gamma |\psi|^2 \psi + \sigma \psi \beta_t = 0, \] (3.1)

\( \gamma \in \mathbb{R} - \{0\} \), \( \psi(x, t) \) is a complex valued function, whereas \( \sigma \) is the noise strength. The noise \( \beta_t \) is the time derivative of the Brownian motion \( \beta(t) \).

Using wave transformation\(^{34}\)

\[ \psi(x,t) = \Psi(\xi)e^{i(px + rt + \sigma \beta(t))}, \quad \xi = x + vt, \] (3.2)

\( p, r, \) and \( v \) are constants, yields

\[ \Psi'' + 2\gamma \Psi^3 - (p^2 + r) \Psi = 0, \quad v = -2p. \] (3.3)

The solutions for equation (3.1) are:

**Rational solutions:**

\[ \Psi_{1,2}(x,t) = \left( \mp \sqrt[3]{i\gamma} (\xi + \zeta) \right)^{-1}. \] (3.4)

Consequently, the stochastic solutions of the equation (3.1) are

\[ \psi_{1,2}(x,t) = \left( \mp \sqrt[3]{i\gamma} (\xi + \zeta) \right)^{-1} e^{i(px + rt + \sigma \beta(t))}. \] (3.5)

**Trigonometric solutions:**

\[ \Psi_{3,4}(x,t) = \pm \sqrt{-\frac{(p^2 + r)}{2\gamma}} \tan\left( \sqrt{\frac{p^2 + r}{2}} (x + vt + \zeta) \right) \] (3.6)

and

\[ \Psi_{5,6}(x,t) = \pm \sqrt{-\frac{(p^2 + r)}{2\gamma}} \cot\left( \sqrt{\frac{p^2 + r}{2}} (x + vt + \zeta) \right). \] (3.7)

Consequently, the stochastic solutions of the equation (3.1) are

\[ \psi_{3,4}(x,t) = \pm \sqrt{-\frac{(p^2 + r)}{2\gamma}} \tan\left( \sqrt{\frac{p^2 + r}{2}} (x + vt + \zeta) \right) e^{i(px + rt + \sigma \beta(t))}. \] (3.8)
and

\[ \psi_{5,6}(x,t) = \pm \sqrt{-\frac{(p^2 + r)}{2\gamma}} \cot \left( \sqrt{\frac{p^2 + r}{2}} (x + vt + \varsigma) \right) e^{i(p^2 + r + \sigma \beta t)}. \]  

(3.9)

**Hyperbolic solutions:**

\[ \psi_{7,8}(x,t) = \pm \sqrt{\frac{p^2 + r}{2\gamma}} \tanh \left( \sqrt{\frac{-(p^2 + r)}{2}} (x + vt + \varsigma) \right) \]  

(3.10)

and

\[ \psi_{9,10}(x,t) = \pm \sqrt{\frac{p^2 + r}{2\gamma}} \coth \left( \sqrt{\frac{-(p^2 + r)}{2}} (x + vt + \varsigma) \right). \]  

(3.11)

Consequently, the stochastic solutions of the equation (3.1) are

\[ \psi_{7,8}(x,t) = \pm \sqrt{\frac{p^2 + r}{2\gamma}} \tanh \left( \sqrt{\frac{-(p^2 + r)}{2}} (x + vt + \varsigma) \right) e^{i(p^2 + r + \sigma \beta t)}. \]  

(3.12)

and

\[ \psi_{9,10}(x,t) = \pm \sqrt{\frac{p^2 + r}{2\gamma}} \coth \left( \sqrt{\frac{-(p^2 + r)}{2}} (x + vt + \varsigma) \right) e^{i(p^2 + r + \sigma \beta t)}. \]  

(3.13)

**NLS**

We second consider the NLSE given by:

\[ i\psi_t + \psi_{xx} - 2\gamma |\psi|^2 \psi + \sigma \psi \beta_t = 0. \]  

(3.14)

Using the traveling wave transformation

\[ \psi(x,t) = \Psi(\zeta)e^{i(p^2 + r + \sigma \beta t)}, \quad \zeta = x + vt, \]  

(3.15)

where \( p, r, \) and \( v \) are constants and \( \sigma \) is the noise strength, yields

\[ \Psi'' - 2\gamma \Psi^3 - (p^2 + r) \Psi = 0, \quad v = -2p. \]  

(3.16)

The solutions for equation (3.16) are:

**Rational solutions:**

\[ \Psi_{1,2}(x,t) = \left( \mp \sqrt{\gamma}(\zeta + \varsigma) \right)^{-1}. \]  

(3.17)

Consequently, the stochastic solutions of the equation (3.1) are

\[ \psi_{1,2}(x,t) = \left( \mp \sqrt{\gamma}(\zeta + \varsigma) \right)^{-1} e^{i(p^2 + r + \sigma \beta t)}. \]  

(3.18)

**Trigonometric solutions:**

\[ \Psi_{3,4}(x,t) = \pm \sqrt{\frac{p^2 + r}{2\gamma}} \tan \left( \sqrt{\frac{p^2 + r}{2}} (x + vt + \varsigma) \right) \]  

(3.19)

and

\[ \Psi_{5,6}(x,t) = \pm \sqrt{\frac{p^2 + r}{2\gamma}} \cot \left( \sqrt{\frac{p^2 + r}{2}} (x + vt + \varsigma) \right). \]  

(3.20)
Consequently, the stochastic solutions of the equation (3.1) are

$$\psi_{3,4}(x,t) = \pm \sqrt{\frac{p^2 + r}{2\gamma}} \tan \left( \sqrt{\frac{p^2 + r}{2}} (x + \nu t + \zeta) \right) e^{i(px + rt + \sigma \beta t)}$$ (3.21)

and

$$\psi_{5,6}(x,t) = \pm \sqrt{\frac{p^2 + r}{2\gamma}} \cot \left( \sqrt{\frac{p^2 + r}{2}} (x + \nu t + \zeta) \right) e^{i(px + rt + \sigma \beta t)}.$$ (3.22)

**Hyperbolic solutions:**

The solutions of equation (3.16) are

$$\Psi_{7,8}(x,t) = \pm \sqrt{-\frac{(p^2 + r)}{2\gamma}} \tanh \left( \sqrt{-\frac{(p^2 + r)}{2}} (x + \nu t + \zeta) \right)$$ (3.23)

and

$$\Psi_{9,10}(x,t) = \pm \sqrt{-\frac{(p^2 + r)}{2\gamma}} \coth \left( \sqrt{-\frac{(p^2 + r)}{2}} (x + \nu t + \zeta) \right).$$ (3.24)

Consequently, the stochastic solutions of the equation (3.1) are

$$\psi_{7,8}(x,t) = \pm \sqrt{-\frac{(p^2 + r)}{2\gamma}} \tanh \left( \sqrt{-\frac{(p^2 + r)}{2}} (x + \nu t + \zeta) \right) e^{i(px + rt + \sigma \beta t)}$$ (3.25)

and

$$\psi_{9,10}(x,t) = \pm \sqrt{-\frac{(p^2 + r)}{2\gamma}} \coth \left( \sqrt{-\frac{(p^2 + r)}{2}} (x + \nu t + \zeta) \right) e^{i(px + rt + \sigma \beta t)}.$$ (3.26)

**The complex cubic NLSE with δ-potential**

The complex cubic NLSE with repulsive δ-potential forced by multiplicative noise in Itô sense is

$$i\psi_t + \frac{1}{2} \psi_{xx} - \alpha \delta \psi - \rho |\psi|^2 \psi + \sigma \psi \beta_t = 0,$$ (3.27)

where $\alpha, \delta, \rho \in \mathbb{R} - \{0\}; \delta$ is Dirac measure at the origin. Equation (3.27) explains the resonant nonlinear propagation of light via optical wave guides with localized defects. The delta potential is “repulsive” at $\alpha > 0$, whereas “attractive” at $\alpha < 0$. Segata et al. considered the scattering and long time behavior to the solutions of equation (3.27). Goodman et al. studied the stability of soliton of equation (3.27). Holmer and Zworski investigated the behavior for flow through equation (3.27).

Utilizing wave transformation

$$\psi(x,t) = \Psi(\xi) e^{i(px + rt + \sigma \beta t)}; \quad \xi = \mu(x - \nu t),$$ (3.28)

yields $w = \rho$, from imaginary part, while real part yields

$$\mu^2 \Psi'' - 2\rho \Psi' - \left( p^2 + 2(r + \alpha \delta) \right) \Psi = 0.$$ (3.29)

The solutions for equation (3.29) are:

**Rational solutions:**

$$\Psi_{1,2}(x,t) = \left( \mp \frac{\sqrt{\rho}}{\mu} (\mu(x - \nu t) + \zeta) \right)^{-1}.$$ (3.30)
Consequently, the stochastic solutions of the equation (3.27) are

\[
\psi_{1,2}(x,t) = \left( \mp \frac{\sqrt{\nu}}{\mu} \left( \mu(x - pt) + \zeta \right) \right)^{-1} e^{i(px + rt + \kappa \sigma^2(t))}.
\]  

(3.31)

**Trigonometric solutions:**

\[
\Psi_{3,4}(x,t) = \pm \sqrt{\frac{p^2 + 2(r + a\delta)}{2\rho}} \tan \left( \frac{\sqrt{p^2 + 2(r + a\delta)}}{\sqrt{2\mu}} \left( \mu(x - pt) + \zeta \right) \right).
\]  

(3.32)

and

\[
\Psi_{5,6}(x,t) = \pm \sqrt{\frac{p^2 + 2(r + a\delta)}{2\rho}} \cot \left( \frac{\sqrt{p^2 + 2(r + a\delta)}}{\sqrt{2\mu}} \left( \mu(x - pt) + \zeta \right) \right).
\]  

(3.33)

Consequently, the stochastic solutions of the equation (3.27) are

\[
\psi_{3,4}(x,t) = \pm \sqrt{\frac{p^2 + 2(r + a\delta)}{2\rho}} \tan \left( \frac{\sqrt{p^2 + 2(r + a\delta)}}{\sqrt{2\mu}} \left( \mu(x - pt) + \zeta \right) \right) e^{i(px + rt + \kappa \sigma^2(t))}
\]  

(3.34)

and

\[
\psi_{5,6}(x,t) = \pm \sqrt{\frac{p^2 + 2(r + a\delta)}{2\rho}} \cot \left( \frac{\sqrt{p^2 + 2(r + a\delta)}}{\sqrt{2\mu}} \left( \mu(x - pt) + \zeta \right) \right) e^{i(px + rt + \kappa \sigma^2(t))}.
\]  

(3.35)

**Hyperbolic solutions:**

\[
\Psi_{7,8}(x,t) = \pm \sqrt{\frac{-p^2 + 2(r + a\delta)}{2\rho}} \tanh \left( \frac{\sqrt{-p^2 + 2(r + a\delta)}}{\sqrt{2\mu}} \left( (x - pt) + \zeta \right) \right)
\]  

(3.36)

and

\[
\Psi_{9,10}(x,t) = \pm \sqrt{\frac{-p^2 + 2(r + a\delta)}{2\rho}} \coth \left( \frac{\sqrt{-p^2 + 2(r + a\delta)}}{\sqrt{2\mu}} \left( (x - pt) + \zeta \right) \right).
\]  

(3.37)

Consequently, the stochastic solutions of the equation (3.27) are

\[
\psi_{7,8}(x,t) = \pm \sqrt{\frac{-p^2 + 2(r + a\delta)}{2\rho}} \tanh \left( \frac{\sqrt{-p^2 + 2(r + a\delta)}}{\sqrt{2\mu}} \left( (x - pt) + \zeta \right) \right) e^{i(px + rt + \kappa \sigma^2(t))}
\]  

(3.38)

and

\[
\psi_{9,10}(x,t) = \pm \sqrt{\frac{-p^2 + 2(r + a\delta)}{2\rho}} \coth \left( \frac{\sqrt{-p^2 + 2(r + a\delta)}}{\sqrt{2\mu}} \left( (x - pt) + \zeta \right) \right) e^{i(px + rt + \kappa \sigma^2(t))}.
\]  

(3.39)

**Remark 3.1.**

1. The proposed solver in this work can be applied for the large classes of nonlinear stochastic partial differential equations (NSPDEs).
2. The presented solver can be easily extended for solving nonlinear stochastic fractional partial differential equations (NSFPDEs).38–40
3. The proposed solver avoids complex and tedious calculations and presents vital solutions in an explicit way.

**Remark 3.2.** Even the presented solver can be utilized to solve all classes of NPDEs reduced to form (2.1), it is fail to solve other classes of NPDEs, which considered a disadvantage of the proposed approach.
Results and discussion

To illustrate the propagations of optical profiles in appropriate fiber environments modes, it is indispensable to solve NLSEs. We present the solutions for different forms of NLSEs forced by multiplicative noise in Itô sense. The closed-form structure of waves has been efficiently introduced to construct various new solutions involving rational, trigonometric, hyperbolic functions in the explicit form. These solutions represented the nonlinear wave propagation in optical fibers communications. These optical features emerge from rational, trigonometric, and hyperbolic function properties along with complex structures. In this respect, from physical and mathematical points of views, the presented results play a crucial role in explaining wave propagation of NLSEs arising in optical fibers. For example, Kerr law nonlinearity of NLSE emanates when a light wave in an optical fiber faces nonlinear responses from nonharmonic motion of electrons bound in molecules, caused by an external electric field. Indeed, the behavior of acquired solutions are soliton, periodic, explosive, rough or dissipative, is an indication for the physical parameters in the NLSEs. For example, the behavior of wave varies from compressive to rarefactive at critical points and stability regions varies to unstable regions at certain values of wave number called critical values. The shock and periodic solutions produced in stability regions. The instability regions are represented in presence of waves increase extremely like huge waves. The presented solutions realize very significant fact for investigation the qualitative interpretations for various phenomena in our nature. It was expected that the acquired profiles can be interpret the fundamentals of Bloch, capillary profiles, plasma physics, nonlinear optics, spatiotemporal pattern, modeling of deep water, femtosecond pulse, and switching techniques. Moreover, the presented solver is a straightforward sturdy and efficient.

The amplitude (strength) of a wave is its height, which is, half the distance from trough to crest. Amplitude can be measured for sound wave traveling through air, water waves, or for any other kind of wave through a liquid or a gas. Even waves traveling through a solid have an amplitude, as in waves shaking the earth due to an earthquake. The amplitude of a quantum wave is also called the probability amplitude. The output of the key equation of quantum mechanics, the NLSE, is probability densities. In sequel we present the effect of the noise term on the amplitude of the waves. Also, it was spotted that our new forms are concerning to the real observed physics that depends on the nonlinearity coefficients.

The effect of $\gamma$

One of the fundamental aims of this work is to clarify the effect of $\gamma$ on the properties of the wave modes. Figure 1 depicts the profile of solution (3.12) for distinct values of $\gamma$. In Figure 1 the variation of solutions with $x$ has been plotted. By increasing $\gamma$ the amplitude of optical soliton solution (3.12) is decreased. Also, no any shift or direction reverse in this amplitude.

He’s frequency formulation

For the sake of completeness of this article, we present He’s frequency formulation. We noted that there are so many types of partial differential equation (1.1) transferred to the ODE (1.5). Equation (1.5) is equal to autonomous planar dynamical system

$$\mathcal{H}' = z', z' = -\frac{C_2}{C_1} H^3 - \frac{C_3}{C_1} H.$$  

The above system of equations $\mathcal{H}' = f_1(\mathcal{H}, z), z' = f_2(\mathcal{H}, z)$ is Hamiltonian system if exists $H(\mathcal{H}, z)$ such that:

$$f_1 = \frac{\partial H}{\partial \mathcal{H}}, f_2 = \frac{\partial H}{\partial z}.$$  

A necessary and sufficient condition for planar system (4.1) to be a Hamiltonian system is

$$\frac{\partial f_1}{\partial \mathcal{H}} + \frac{\partial f_2}{\partial z} = 0.$$  

Our system (4.1) is a Hamiltonian system with Hamiltonian function

$$H = \frac{z^2}{2} + \frac{C_2}{4C_1} \mathcal{H}^4 + \frac{C_3}{2C_1} \mathcal{H}^2.$$  

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Equation (1.5) looks like the equation

\[
\mathcal{H}'' = \frac{\partial V(\mathcal{H})}{\partial \mathcal{H}} \\
= -\frac{C_2}{C_1} \mathcal{H}^3 - \frac{C_3}{C_1} \mathcal{H} \tag{4.5}
\]

which looks like the equation of motion of a classical particle moving in a potential well \( V(\mathcal{H}) \), which is often referred to a pseudopotential or sagdeev potential and is obtained by

\[
V(\mathcal{H}) = -\int F(\mathcal{H})d\mathcal{H} \\
= \int \left( \frac{C_2}{C_1} \mathcal{H}^3 - \frac{C_3}{C_1} \mathcal{H} \right) d\mathcal{H} \tag{4.6}
\]

\[
= \frac{C_2}{4C_1} \mathcal{H}^4 + \frac{C_3}{2C_1} \mathcal{H}^2.
\]

Thus integrating equation (1.4) with respect to \( \mathcal{H} \), gives:

**Figure 1.** Variations of solution (3.12) for \( \gamma = 2, 4, 6, 8 \).
\[
\frac{1}{2} \dot{\mathcal{H}}^2 + \frac{C_2}{4C_1} \mathcal{H}^4 + \frac{C_3}{2C_1} \mathcal{H}^2 = \text{const},
\]  

(4.7)

which represent the energy integral (Jacobi’s integral), that means that our system is a Hamiltonian system. The importance of knowing that our system is Hamiltonian is the fact that we can essentially draw the phase portrait without solving the system where for Hamiltonian system \(H\) is constant along every solutions curve. If the system is not Hamiltonian there is neither solitary nor periodic wave solutions as there does not exist any closed trajectory in the phase plane.

Also He’s frequency formulation\(^{54}\) is accessible and extremely simple. Equation (4.5) can be written in the form:

\[
\mathcal{H}'' + f(\mathcal{H}) = 0; f(\mathcal{H}) = -\frac{\mathcal{F}(\mathcal{H})}{\mathcal{H}}.
\]  

(4.8)

Assuming that \(\mathcal{H}(0) = A, \mathcal{H}'(0) = 0\) and for nonlinear oscillators, it requires that \(f(\mathcal{H})/\mathcal{H} > 0\). He’s frequency formulation predicts the square of the frequency in the form

\[
\omega^2 = f'(\mathcal{H})|_{\mathcal{H} = \mathcal{H}^*},
\]  

(4.9)

where \(\mathcal{H}^* = \frac{A^2}{2}\).

**Conclusions**

The current work involves three families of stochastic NLSEs forced by multiplicative noise in Itô sense. We implemented the closed-form wave structures, as an efficient technique for extracting solutions of stochastic NLSEs. We successfully obtained optical solitons for the proposed three equations. Special values of the physical parameters were used to draw the profiles pictures of acquired soliton solutions. Our work shows that the proposed closed-form structures is effective in dealings with NPDEs to construct various classes of new stochastic solutions. Finally, the proposed solver can also be applied to further models arising in natural science.

**Authors’ contributions**

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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