The red-shift effect and radiation decay on black hole spacetimes

Mihalis Dafermos\textsuperscript{*} \hspace{5mm} Igor Rodnianski\textsuperscript{†}

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Abstract

We consider solutions to the linear wave equation $\Box g \phi = 0$ on a (maximally extended) Schwarzschild spacetime. We assume only that the solution decays suitably at spatial infinity on a complete Cauchy hypersurface $\Sigma$. (In particular, the support of $\phi$ may contain the bifurcate event horizon.) It is shown that the energy flux of the solution through arbitrary achronal subsets of the black hole exterior region is bounded by $C(v_+^2 + u_+^2)$, where $v$ and $u$ denote the infimum of the Eddington-Finkelstein advanced and retarded time of the subsets, and $v_+$ denotes $\max\{1, v\}$, etc. (This applies in particular to subsets of the event horizon or null infinity.) It is also shown that $\phi$ satisfies the pointwise decay estimate $|\phi| \leq Cv_+^{-1}$ in the entire exterior region, and the estimate $|r\phi| \leq C_\mu (1 + |u|)^{-\frac{1}{2}}$ in the region $\{r \geq \hat{R}\} \cap J^+(\Sigma)$, for any $\hat{R} > 2M$. The estimates near the event horizon exploit an integral energy identity normalized to local observers. This estimate can be thought to quantify the celebrated red-shift effect. The results in particular give an independent proof of the classical result $|\phi| \leq C$ of Kay and Wald without recourse to the discrete isometries of spacetime.

1 Introduction

The concept of a black hole is a central one in general relativity: Spacetime is said to contain a black hole when it admits a complete null infinity whose past has a regular future boundary. This boundary is called the event horizon and the black hole itself is defined to be its future; the past of null infinity is known as the black hole exterior.

The simplest solutions of the Einstein vacuum equations of general relativity,

$$R_{\mu\nu} = 0$$

\textsuperscript{*}University of Cambridge, Department of Pure Mathematics and Mathematical Statistics, Wilberforce Road, Cambridge CB3 0WB United Kingdom

\textsuperscript{†}Princeton University, Department of Mathematics, Fine Hall, Washington Road, Princeton, NJ 08544 United States
containing black holes, the one-parameter Schwarzschild family of solutions, were written down in local coordinates \cite{19} in 1916, but only correctly understood as describing spacetimes with black holes in the sense above, around 1960. Until that time, there were many arguments in the physics literature (e.g. \cite{10}) purporting to show that such solutions would be pathological and unstable. It was only when Kruskal demonstrated \cite{15} that the event horizon could be covered by regular coordinates that its true geometric character became clear, and the problem of stability could be given a sensible and well-defined formulation.

The Schwarzschild family turns out to be a sub-family of the two-parameter Kerr family which describe stationary rotating black holes. In its proper rigorous formulation, the problem of nonlinear stability of the Kerr family is one of the major open problems in general relativity.\footnote{In particular, it is conjectured that perturbations of Schwarzschild initial data should evolve into a spacetime with complete null infinity whose past “suitably” approaches a nearby Kerr exterior.}

At the heuristic level, however, considerable progress has been made in the last 40 years towards an understanding of the issues involved. In particular, a very influential role was played by the work of R. Price \cite{18} in 1972, who discovered a heuristic mechanism allowing for the decay of scalar field linear perturbations on the Schwarzschild exterior. The mechanism depends on the following fact, known as the red-shift effect, which had been understood previously in the context of geometric optics. Given two observers $A$ and $B$, depicted below\footnote{What is depicted is the Penrose diagram of a subset of Schwarzschild. See Section \ref{diagrams}. The reader unfamiliar with these diagrams can refer to \cite{8}.},

\begin{center}
\includegraphics[width=0.5\textwidth]{schwarzschild_diagram.png}
\end{center}

then if $A$ emits a signal at a constant rate with respect to his own proper time, the frequency of the signal as received by $B$ is infinitely shifted to the red as $B$’s proper time goes to infinity.

For spherically symmetric solutions of the coupled Einstein-scalar field system, and more generally, the Einstein-Maxwell-scalar field system, the heuristic picture put forth by Price is now a theorem \cite{8}, and the red-shift effect described above plays a central role in the proof. Moreover, one of the results of \cite{8}, namely the decay rate

\begin{equation}
|\phi| + |\partial_v \phi| \leq Cv_+^{-3+\epsilon}
\end{equation}

along the event-horizon\footnote{Here, $v$ is a naturally defined Eddington-Finkelstein-like advanced time coordinate.}, has important implications for the causal structure of the interior of the black hole: In \cite{7}, it is shown using \cite{2} that, in the charged
case, the spacetimes of [8] do not generically terminate in everywhere spacelike singularities as originally widely thought, but rather, their future boundary has a null component across which the spacetime metric can be continuously extended. In particular, this implies that the $C^0$-inextendibility formulation [4] of strong cosmic censorship is false for the system considered.

For the Einstein vacuum equations [1] in the absence of symmetry assumptions, results analogous to [8, 7] seem out of reach at present. Clearly, a first step towards attacking geometric non-linear stability questions is proper understanding of the linear theory in an appropriate geometric setting. This will be the subject of the present paper. Our main result is the following:

**Theorem 1.1.** Let $\phi$ be a sufficiently regular solution of the wave equation

$$\square_g \phi = 0$$

on the (maximally extended) Schwarzschild spacetime $(\mathcal{M}, g)$, decaying suitably at spatial infinity on an arbitrary complete asymptotically flat Cauchy surface $\Sigma$. Fix retarded and advanced Eddington-Finkelstein coordinates $u$ and $v$ on one of the exterior regions. For any achronal hypersurface $S$ in the closure of this region, let $F(S)$ denote the flux of the energy through $S$, where energy is here measured with respect to the timelike Killing vector field. Let $v_+ = \max\{v, 1\}$, $u_+ = \max\{u, 1\}$, and $v_+(S) = \max\{\inf S v, 1\}$, $u_+(S) = \max\{\inf S u, 1\}$. We have

$$F(S) \leq C((v_+(S))^{-2} + (u_+(S))^{-2}).$$

(4)

(We also allow $S$ to be a subset of null infinity, interpreted in the obvious limiting sense.) In addition, we have the pointwise decay rates

$$|\phi| \leq Cv_+^{-1} \quad \text{in} \quad J^-(I^+) \cap J^+(I^-),$$

$$|r\phi| \leq C\hat{R}(1 + |u|)^{-\frac{1}{2}} \quad \text{in} \quad \{r \geq \hat{R} > 2M\} \cap J^+(\Sigma).$$

(5)

In the spherically symmetric case, the above result follows from a very special case of [8]. (See also [16].) Decay for $\phi$, without however a rate, was first proven in the thesis of Twainy [21]. The uniform boundedness of $\phi$ is a classical result of Kay and Wald [13]. For the more general Kerr family, even uniform boundedness remains an open problem (see however [20]).

We should also note that, independently of us, a variant of the problem considered here is being studied by [3].

A statement of Theorem 1.1 in local coordinates will be given later. This, in particular, will explain the dependence on $C$, and the minimum regularity assumptions necessary. The reader wishing to penetrate deeper into this problem, however, is strongly encouraged to learn the language necessary for the above geometric formulation. For neither the correct conditions on initial data, nor the desired statement of decay, have particularly natural formulations when stated with respect to Regge-Wheeler coordinates.\(^4\) The motivation for consider-

\(^4\)In particular, it is not correct to restrict to “compactly” supported initial data in Regge-Wheeler coordinates, because $r^* = -\infty$ corresponds to points in the actual spacetime. These issues are well known in the relativity community, and the reader should consult the nice discussion in [13].
ering decay as stated in (4) is that it is this formulation that has direct relevance both in the astrophysical regime, as well as for the fate of observers who enter the black hole region. For (4) applied to \( S = I^+ \) gives decay rates for the energy radiated to infinity (this is what is astrophysically observable), while applied to \( S = H^+ \), it gives decay rates for the energy thrown into the black hole (this is what concerns the observer entering the black hole).

It is interesting to note that the decay rates (4)–(5) are sufficiently fast so as to suggest that the picture established in [7] may remain valid in the absence of symmetry assumptions, in particular, the existence of a marginally trapped tube which becomes achronal and terminates at \( i^+ \), and a weak null singular boundary component to spacetime, emanating from \( i^+ \), across which the spacetime can still be continuously extended.

The techniques of this paper are guided by the principle that they should be relevant for non-linear stability problems. For such problems, in the absence of symmetry, energy-type estimates have proven the most robust [5]. For Lagrangian theories like the homogeneous wave equation, such estimates naturally arise by contracting suitable vector fields \( V^\alpha \) with the energy-momentum tensor \( T_{\alpha\beta} \), to produce a one form \( P_\alpha \). (See [6] for a general discussion.) The divergence theorem relates the spacetime integral of \( \nabla^\alpha P_\alpha \) with suitable boundary terms. The method can be used to estimate the spacetime integral from the boundary terms, but also the future boundary terms from the past boundary and the spacetime integral. As we shall see below, both implications will be used here.

Let us recall the situation for the wave equation (3) on Minkowski space. The technique described here was introduced by Morawetz, see [17] and Klainerman [14]. By applying the method of the previous paragraph to the Killing vector field \( \frac{\partial}{\partial t} \), one obtains the usual energy conservation. By applying the method to the conformally Killing “Morawetz” vector field

\[
K = v^2 \frac{\partial}{\partial v} + u^2 \frac{\partial}{\partial u},
\]

in the region \( \{1 \leq t \leq t_1\} \), one obtains an identity relating a spacetime integral in this region to boundary integrals of weighted energy densities on \( \{t = 1\} \) and \( \{t = t_1\} \). The spacetime integral can be completely removed by a second application of the divergence theorem, which yields additional positive quantities on the boundary hypersurfaces. In particular, the weights are sufficient to derive decay of the form (4), assuming that the initial boundary integral is bounded. Pointwise decay estimates of the form (5) can then be obtained by Sobolev inequalities, after commuting the equation with angular momentum operators \( \Omega \).

Turning to equation (3) on the Schwarzschild exterior, we have again a time-like Killing vector field \( \frac{\partial}{\partial t} \), and thus one immediately obtains conservation of

\footnote{For even the weakest results of [7], one still requires the analogue of \( \phi \leq C v^{-\frac{1}{2}} e^{-\epsilon} \) along the event horizon, for an \( \epsilon > 0 \).}
the associated energy. Applying as before the above method to the vector field $K$ in the region $\{1 \leq t \leq t_i\}$, we again obtain boundary hypersurface integrals with a sign, and strong weights, in particular controlling $t_i^{-2}$ times the energy density on $\{t = t_i\} \cap \{v \geq t_i\} \cap \{u \geq t_i\}$.

On the other hand, even after an additional integration by parts, the spacetime term arising from $\nabla^\alpha P_\alpha$ no longer vanishes. In the regions $r \leq r_0$ and $r \geq R$, for certain constants $r_0$, $R$, however, this spacetime term has a good sign, and can be ignored. On the other hand, in the region $r_0 \leq r \leq R$, it turns out the spacetime integral arising from $K$ can be controlled by $t$ times the spacetime integrals–summed–arising from vector fields $X_\ell$ of the form

$$X_\ell = f_\ell \left( \frac{\partial}{\partial v} - \frac{\partial}{\partial u} \right),$$

for a carefully chosen function $f_\ell$, applied to each spherical harmonic $\phi_\ell$, (see also [2] in connection with vector fields $X$). In fact, for this bound, the vector fields $X_\ell$ must also be applied to angular derivatives of $\phi$. This leads to loss of derivatives in the argument. The boundary hypersurface integrals arising from $X_\ell$ are controlled in turn by the total energy. Putting the information together from these two vector fields immediately yields energy decay (4), but with power $-1$ in place of $-2$.

The boundary integrals arising from the sum of the $X_\ell$ identities, when suitably localized to the future and past boundaries of a dyadic characteristic rectangle $\{t_0 \leq t \leq 1.1t_0\} \cap \{r_0 \leq r \leq R\}$, can in fact be bounded by $t_0^{-2}$ times the boundary integrals arising from $K$ on constant $t = t_0$ and $t = 1.1t_0$ hypersurfaces. Using this fact, we can iterate the procedure, to obtain that the boundary integrals arising from $K$ are in fact bounded, and thus that (4) holds as stated.

The above methods do not give good control near the horizon $H^+$. For this we need another estimate, which has no analogue in Minkowski space. This estimate arises from applying a vector field of the form

$$Y \sim \frac{1}{1 - \mu \partial u} \frac{\partial}{\partial u}$$

---

6 defined with respect to suitably normalized Eddington-Finkelstein advanced and retarded coordinates $v$ and $u$.

7 The necessity of taking angular derivatives is related to the presence of the so-called photosphere at $r = 3M$.

8 This is not surprising, as $K$ is normalized to $I^+$. 5
in the characteristic rectangle depicted. The \( v \) length of this rectangle is chosen to be of the order of \( t_i \). Note that \( Y \) extends regularly to the horizon \( \mathcal{H}^+ \). The boundary terms arising from \( Y \) are related to the energy that would be observed by a local observer crossing the event horizon.\(^9\) The associated spacetime integral contains a term with a good sign, and terms that can be controlled by the sum of the spacetime integrals arising from the \( X_\ell \). As a first step, one can show using the energy identities for \( \frac{\partial}{\partial t} \) and \( X_\ell \), and the nature of the spacetime integral arising from \( Y \), that the boundary terms arising from \( Y \) are uniformly bounded. Then one can go back and, using a pigeonhole argument, extract from the term in the spacetime integral with a good sign a constant-\( v \) slice of the rectangle such that the integral of the energy density as measured by a local observer is bounded by \( t_i^{-1} \). Finally, one applies again the energy identity of \( Y \) in a subrectangle to obtain that the integral of the energy density measured by a local observer is bounded by \( t_i^{-1} \) on the segment \( v = t_i \) depicted. One repeats the procedure to obtain decay of \( t_i^{-2} \).

Note that the procedure above for extracting decay for energy as measured by a local observer by means of a pigeonhole argument applied to the spacetime integral arising from the energy identity for \( Y \) is the analytic manifestation in our technique of the redshift effect described previously. Here, the estimate is to be compared with the geometric optics argument of the first diagram, but where the observers \( A \) and \( B \) both cross the event horizon, with \( B \) at advanced time \( t_i^{-1} \) later than \( A \).

As in Minkowski space, one obtains pointwise estimates \(^5\) from the \( L^2 \) estimates via the Sobolev inequality, after commuting the equation with angular momentum operators. One should note however, that the necessary \( L^2 \) estimates near the horizon for the angular derivatives arise from the decay rate of \( t_i^{-2} \) for the energy flux measured by \( Y \). The boundary terms of \( K \) and \( \frac{\partial}{\partial t} \) do not contain angular derivatives on \( \mathcal{H}^+ \cup \mathcal{H}^- \).

Our use of the vector field \( Y \) is inspired by estimates in \(^8\). In that paper, one could continue the iteration further to obtain better decay rates than \(^4\). \(^6\). Here on the other hand, the weights on the boundary term arising from the Morawetz identity give an upper bound to the amount of decay that can be extracted. It would be interesting to explore whether one can surpass this barrier using additional techniques.

A final interesting aspect of our argument is that it does not require inverting the Laplacian on initial data or appealing to the discrete isometries of Schwarzschild. (In particular, the results here yield an independent proof of the classical uniform boundedness theorem \(^13\) of Kay and Wald.) It may be useful for non-linear applications to avoid techniques so heavily dependent on the exact staticity. In this sense, the argument given here is perhaps more robust.

\(^9\)More precisely, exactly the part of the energy not seen by \( \frac{\partial}{\partial t} \).
2 Schwarzschild

Let \((M, g)\) denote the maximally extended Schwarzschild spacetime with parameter \(M > 0\). This manifold is spherically symmetric, i.e. the group \(SO(3)\) acts by isometry. We recall briefly the usual description of the global causal structure of \(M\), via its so-called Penrose diagram. (We refer the reader to standard references, for instance [12].)

For spherically symmetric spacetimes, recall that Penrose diagrams are just the image of global bounded null coordinate systems on the Lorentzian quotient \(Q = M / SO(3)\) viewed as maps in the obvious way from \(Q \to \mathbb{R}^{1+1}\). In the case of Schwarzschild, the Penrose diagram is depicted below:

The curve \(S\) depicts the projection to \(Q\) of a particular choice of complete Cauchy surface \(\Sigma \subset M\). We will call the sets \(J^-(I_+^A) \cap J^+(I^-_A)\) and \(J^-(I^-_B) \cap J^+(I^-_B)\) the exterior regions.\(^{10}\) We call \(H^+_A\) the future event horizon corresponding to the end \(I^+_A\) and \(H^+_B\) the future event horizon corresponding to \(I^+_B\). We could also consider \(H^-_B\) as the past event horizon corresponding to \(I^-_A\), and thus may denote it alternatively by \(H^-_A\).

Defining a function \(r : Q \to \mathbb{R}\)

\[
r(q) = \sqrt{\text{Area}(q)/4\pi},
\]

the two exteriors are each covered by a coordinate system \((r, t)\) so that the metric \(g\) may be written:

\[
- \left( 1 - \frac{2M}{r} \right) dt^2 + \left( 1 - \frac{2M}{r} \right)^{-1} dr^2 + r^2 d\sigma^2_S,
\]

where \(d\sigma^2_S\) denotes the standard metric on the unit sphere. The vector field \(\frac{\partial}{\partial t}\) is clearly timelike Killing. These coordinates break down on \(H^+_A \cup H^+_B\). The Schwarzschild solution was originally understood as the spacetime described by the expression (7). As explained in the introduction, the realization that this was actually just one of the exterior regions of a larger spacetime, took a surprisingly long time.\(^{11}\)

\(^{10}\)For an explanation of the notation \(J^-(I^+)\), etc., and more about Penrose diagrams, see the appendix of [8].

\(^{11}\)Note that the expression \(\frac{\partial}{\partial t}\) also describes the metric in the black hole interior \(J^+(H^+_A) \cap J^-(H^-_B) \setminus (H^+_B \cup H^+_A)\). Here \(\frac{\partial}{\partial t}\) is spacelike Killing.
The sphere $\mathcal{H}_B^+ \cap \mathcal{H}_A^+$ is known as the bifurcation sphere of the event horizon. If we chose $S$ to contain $\mathcal{H}_B^+ \cap \mathcal{H}_A^+$, then

$$S' = S \cap J^-(\mathcal{I}_A^+) \cap J^+(\mathcal{I}_A^-)$$

is a Cauchy surface for the exterior region $J^-(\mathcal{I}_A^+) \cap J^+(\mathcal{I}_A^-)$. In particular, solutions of the linear wave equations on $J^-(\mathcal{I}_A^+) \cap J^+(\mathcal{I}_A^-)$ are determined by their data on $S'$.

The above means that if one is only interested in the behaviour of solutions to (3) in $J^- (\mathcal{I}_A^+) \cap J^+ (\mathcal{I}_A^-)$, one can study the problem in any coordinate system defined globally in this region, in particular, so-called Schwarzschild coordinates $(r, t)$. For convenience, we shall in fact use a null coordinate system $(u, v)$, which “sends” $\mathcal{H}_A^+$ to $u = \infty$ and $\mathcal{H}_B^+$ to $v = -\infty$. These coordinates, so-called Eddington-Finkelstein retarded and advanced coordinates, will be described in the next section.

The reader should not think, however, that imposing such a coordinate system removes the geometry from this problem. For one must not forget that the correct assumptions on $\phi$ are those expressible geometrically, not those that happen to look natural in the chosen coordinate system. When written in local coordinates, these assumptions would be difficult to motivate without knowing the origin of the problem. Moreover, the same can be said for the form of many of the techniques used here. In particular, the form of the vector field $Y$ of Section 9 is best understood by passing to a new regular coordinate system on $\mathcal{H}^+$. In the coordinate system chosen here, $Y$ appears asymptotically singular.

With these warnings in place, we turn to a description of two related coordinate systems covering $J^- (\mathcal{I}_A^+) \cap J^+ (\mathcal{I}_A^-)$.

### 3 Eddington-Finkelstein and Regge-Wheeler coordinates

Let $(r, t)$ denote the Schwarzschild coordinates of $\mathcal{M}$. Define first the so-called Regge-Wheeler tortoise coordinate $r^*$ by

$$r^* = r + 2M \log(r-2M) - 3M - 2M \log M,$$

and define retarded and advanced Eddington-Finkelstein coordinates $u$ and $v$, respectively, by

$$t = v + u$$

and

$$r^* = v - u.$$

---

12 For instance, one should not restrict to $\phi$ of compact support on $S'$, for this would mean that $\phi$ necessarily vanishes at $\mathcal{H}_B^+ \cap \mathcal{H}_A^+$, an actual sphere in the spacetime. See also the remarks in the Introduction.

13 Coordinates $(r^*, t)$ are together known as Regge-Wheeler coordinates. We have centred $r^*$ so as for $r^* = 0$ to correspond to $r = 3M$. This is the so-called photosphere.
These coordinates turn out to be null: Setting \( \mu = \frac{2M}{r} \), the metric has the form
\[-4(1 - \mu)du dv + r^2 d\sigma^2.\]

We shall move freely between the two coordinate systems \((r^*, t)\) and \((u, v)\) in this paper. Note that in either, \( J^- (\mathcal{I}_A^+) \cap J^+ (\mathcal{I}_A^-) \) is covered by \((-\infty, \infty) \times (-\infty, \infty)\). By appropriately rescaling \( u \) and \( v \) to have finite range, one can construct coordinates which are in fact regular on \( \mathcal{H}^+ \) and \( \mathcal{H}^- \). By a slight abuse of language, one can parametrise the future and past event horizons in our present \((u, v)\) coordinate systems as \( \mathcal{H}^+ = \{(u, v) \mid v \in (-\infty, \infty) \} \) and \( \mathcal{H}^- = \{(u, -\infty) \mid u \in (-\infty, \infty) \}. \)

Finally, we collect various formulas for future reference:
\[
\mu = \frac{2M}{r},
\]
\[
g_{uv} = (g^{uv})^{-1} = -2(1 - \mu),
\]
\[
\partial_v r = (1 - \mu), \quad \partial_u r = -(1 - \mu)
\]
\[
dt = dv + du, \quad dr^* = dv - du,
\]
\[
\frac{\partial}{\partial t} = \frac{1}{2} \left( \frac{\partial}{\partial v} + \frac{\partial}{\partial u} \right),
\]
\[
\frac{\partial}{\partial r^*} = \frac{1}{2} \left( \frac{\partial}{\partial v} - \frac{\partial}{\partial u} \right),
\]
\[
dVol_M = r^2 (1 - \mu) dv \, du \, d\sigma^2,
\]
\[
dVol_{t=\text{const}} = r^2 \sqrt{1 - \mu} dr^* \, d\sigma^2.
\]

\[
\Box \psi = \nabla^\alpha \nabla_\alpha \psi = -(1 - \mu)^{-1} \left( \partial^2_t \psi - r^{-2} \partial_{r^*} (r^2 \partial_{r^*} \psi) \right) + \nabla^A \nabla_A \psi.
\]

Here \( \nabla \) denotes the induced covariant derivative on the group orbit spheres.

4 The class of solutions

Let \( S' \) denote the surface \( \{ t = 1 \} \) say in the exterior, and let \( S \) be a Cauchy surface for \( \mathcal{M} \) such that \( S \cap J^- (\mathcal{I}_A^+) \cap J^+ (\mathcal{I}_A^-) = S' \). Let \( N \) denote the future-directed unit normal to \( S \).

We proceed to describe the solutions \( \phi : \mathcal{M} \to \mathbb{R} \) of the wave equation
\[
\Box \phi = 0 \quad \text{(8)}
\]
on \( \mathcal{M} \), which we shall consider in this paper. Given \( \phi, N \phi \) on \( S \), define the quantities
\[
E_0 = \sum_{i=0}^{2} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} r^2 (1 - \mu)^{-\frac{1}{2}} \left( (\partial_i (r^i \nabla^i \phi))^2 + (\partial_{r^*} (r^i \nabla^i \phi))^2 \right. \\
\left. +(1 - \mu) |\nabla r^i \nabla^i \phi|^2 \right) (1, r^*, \sigma_{S^2}) dr^* \, d\sigma^2, \quad \text{(9)}
\]
\begin{equation}
\bar{E}_1 = \sum_{i=0}^{2} \int_{-\infty}^{\infty} \int_{S^2} r^2 \left( u^2 (\partial_u (r^i \nabla^i \phi))^2 + v^2 (\partial_v (r^i \nabla^i \phi))^2 \right) + (1 - \mu) (u^2 + v^2) |\nabla (r^i \nabla^i \phi)|^2 \right) (1, r^*, \omega) \cdot d\sigma_{S^2} 
\end{equation}

\begin{equation}
\bar{E}_2 = \sum_{i=0}^{4} \int_{-\infty}^{\infty} \int_{S^2} r^2 \left( u^2 (\partial_u (r^i \nabla^i \phi))^2 + v^2 (\partial_v (r^i \nabla^i \phi))^2 \right) + (1 - \mu) (u^2 + v^2) |\nabla (r^i \nabla^i \phi)|^2 \right) (1, r^*, \omega) \cdot d\sigma_{S^2} 
\end{equation}

Our weakest result, namely, the uniform boundedness, requires the boundedness of $\bar{E}_0$. Our energy decay result will require the boundedness of $\bar{E}_1$ and our full pointwise decay results will require the boundedness of $\bar{E}_2$.

We note that the boundedness of $\bar{E}_0$ follows from the statement that $\phi$ is $C^3$, $N \phi$ is $C^2$ on $S$, and that $\phi$ and decays suitably at spacelike infinity, for instance, if $\phi$ vanishes identically in a neighborhood of $i^0$. Similarly, the boundedness of $\bar{E}_1$ follows for suitably decaying $C^4 \phi$ and $C^3 N \phi$, and finally the boundedness of $\bar{E}_2$ follows from $C^6 \phi$, etc. There is no assumption of the vanishing of $\phi$ on $\mathcal{H}^+ \cap \mathcal{H}^-$.

We will state a coordinate version of Theorem 1.1 in Section 7. As this theorem will refer to the energy flux defined by the Killing vector field $\partial/\partial t$, we will first need some general results regarding conservation laws. These will be given in the next two sections.

5 Conservation laws

As discussed in the introduction, the results of this paper will rely on estimates of energy-type. Such estimates arise naturally in view of the Lagrangian structure of the theory. We review briefly here.
In general coordinates, the energy-momentum tensor for $\phi$ is given by

$$T_{\alpha\beta} = \partial_\alpha \phi \partial_\beta \phi - \frac{1}{2} g_{\alpha\beta} g^{\gamma\delta} \partial_\gamma \phi \partial_\delta \phi.$$ 

This is divergence free, i.e. we have

$$\nabla^\alpha T_{\alpha\beta} = 0. \quad (12)$$

For the null coordinates we have defined, we compute the components

$$T_{uu} = (\partial_u \phi)^2, \quad T_{vv} = (\partial_v \phi)^2,$$

$$T_{uv} = -\frac{1}{2} g_{uv} |\nabla \phi|^2 = (1 - \mu) |\nabla \phi|^2.$$ 

Let $V^\alpha$ denote an arbitrary vector field. Let $n^\alpha$ denote the normal to a hypersurface$^{14}$. Let $\pi^{\alpha\beta}_V$ denote the deformation tensor of $V$, i.e.,

$$\pi^{\alpha\beta}_V = \frac{1}{2} (\nabla^\alpha V^\beta + \nabla^\beta V^\alpha). \quad (13)$$

We will denote in what follows the tensor $\pi^{\alpha\beta}_V$ just by $\pi^{\alpha\beta}$. In local coordinates we have the following expression:

$$T_{\alpha\beta} \pi^{\alpha\beta} = \frac{1}{4(1 - \mu)} \left( (\partial_u \phi)^2 \partial_v (V_v (1 - \mu)^{-1}) + (\partial_v \phi)^2 \partial_u (V_u (1 - \mu)^{-1}) \right)$$

$$+ |\nabla \phi|^2 (\partial_u V_v + \partial_v V_u) - \frac{1}{2r} (V_u - V_v) (|\nabla \phi|^2 - \phi^\alpha \phi_\alpha).$$

Let $S$ be a region bounded to the future and past by two hypersurfaces $\Sigma_1$ and $\Sigma_0$, respectively. Let $P^\alpha = g^{\alpha\beta} T_{\beta\delta} X^\delta$. The divergence theorem together with (12) and (13) gives

$$\int_S T_{\alpha\beta} \pi^{\alpha\beta} dVol_S = \int_{\Sigma_1} g_{\alpha\beta} P^\alpha n^\beta dVol_{\Sigma_1} - \int_{\Sigma_0} g_{\alpha\beta} P^\alpha n^\beta dVol_{\Sigma_0}. \quad (14)$$

6 Conservation of energy

Let us apply the above to the vector field $\frac{\partial}{\partial t}$. As this is Killing, by (12) and (13), the associated vector field $P$ is divergence free, so there is no space-time

$^{14}$Note the convention in Lorentzian geometry
term. The (negative of the) boundary terms on constant-\(t\)-hypersurfaces are:

\[
E_\phi(t_i) = \int_{-\infty}^{\infty} \int_{S^2} \frac{1}{\sqrt{1-\mu}} \left( \frac{1}{4} T_{vv} + \frac{1}{4} T_{uu} + \frac{1}{2} T_{uv} \right) \cdot (r^*, t_i, \omega) \cdot r^2 \sqrt{1-\mu} \, dr^* \, d\sigma_{S^2}
\]

\[
= \int_{-\infty}^{\infty} \int_{S^2} \frac{1}{4\sqrt{1-\mu}} ((\partial_v \phi)^2 + (\partial_u \phi)^2 + 2(1-\mu)|\nabla \phi|^2) \cdot r^2 \sqrt{1-\mu} \, dr^* \, d\sigma_{S^2}
\]

\[
= \int_{-\infty}^{\infty} \int_{S^2} \frac{1}{2\sqrt{1-\mu}} ((\partial_t \phi)^2 + (\partial_r \phi)^2 + (1-\mu)|\nabla \phi|^2) \cdot r^2 \sqrt{1-\mu} \, dr^* \, d\sigma_{S^2}
\]

on constant-\(u\) hypersurfaces are given by:

\[
F_{\phi}(\{u\} \times [v_1, v_2]) = \int_{v_1}^{v_2} \int_{S^2} \frac{1}{4(1-\mu)} (T_{uu} + T_{uv}) \cdot r^2 (1-\mu) \, d\sigma_{S^2}
\]

\[
= \int_{v_1}^{v_2} \int_{S^2} \frac{1}{4(1-\mu)} ((\partial_v \phi)^2 + (1-\mu)|\nabla \phi|^2) \cdot r^2 (1-\mu) \, d\sigma_{S^2}
\]

and on constant-\(v\) hypersurfaces are given by:

\[
F_{\phi}([u_1, u_2] \times \{v\}) = \int_{u_1}^{u_2} \int_{S^2} \frac{1}{4(1-\mu)} (T_{uu} + T_{uv}) \cdot r^2 (1-\mu) \, d\sigma_{S^2}
\]

\[
= \int_{u_1}^{u_2} \int_{S^2} \frac{1}{4(1-\mu)} ((\partial_u \phi)^2 + (1-\mu)|\nabla \phi|^2) \cdot r^2 (1-\mu) \, d\sigma_{S^2}.
\]

The identity (14) applied to the above shows that

\[
E_\phi(t_0) = E_\phi(1) \doteq E_\phi
\]

and that the fluxes satisfy

\[
F_{\phi}([u_1, u_2] \times \{v\}) \leq E_\phi,
\]

\[
F_{\phi}([u] \times [v_1, v_2]) \leq E_\phi.
\]

In particular, we can define a function \(\varphi_\phi\) on \(Q \cap J^- (T^+ + J^+ (I^-))\) by

\[
\varphi_\phi(u, v) \doteq F_{\phi}([u, \infty] \times \{v\}) + F_{\phi}(\{\infty\} \times [v, \infty]).
\]

(15)

It is clear that the bound

\[
\varphi_\phi \leq 2E_\phi,
\]

(16)

follows immediately.
7 Coordinate version of the main theorem

**Theorem 7.1.** Let $\phi_0(r^*, \omega), \phi_1(r^*, \omega)$ be functions such that the quantity $\bar{E}_0$ of (9) is bounded\(^{15}\), and let $\phi$ be the unique solution of (3) on $J^-(I^+) \cap J^+(I^-)$ with $\phi(r^*, 1, \omega) = \phi_0(r^*, \omega), \partial_t \phi(r^*, 1, \omega) = \phi_1(r^*, \omega)$. Let $\varpi_\phi$ be as defined in (15). Then there exists a universal constant $C$ such that

$$|\phi| \leq C \bar{E}_0 r^{-\frac{1}{2}}. \quad (17)$$

If the quantity $\bar{E}_1$ of (10) is bounded, then

$$\varpi_\phi \leq C \bar{E}_1 (v_+^{-2} + u_+^{-2}). \quad (18)$$

If the quantity $\bar{E}_2$ of (11) is bounded, then

$$|\phi| \leq C \bar{E}_2 v_+^{-1} \quad (19)$$

for all $(u, v, \omega)$, while

$$|r \phi| \leq C R \bar{E}_2 (1 + |u|)^{-\frac{1}{2}} \quad (20)$$

for all $r \geq R > 2M, t \geq 1$.

In view of our previous remarks, it is clear that Theorem 7.1 implies Theorem 1.1.

**Remark 7.1.** The number of derivatives required in the definitions of $\bar{E}_1, \bar{E}_2$ can be reduced by a slight refinement of the analysis in Section 8. We shall not pursue this here.

8 The vector fields $X_\ell$

In this section we shall define, for each spherical harmonic $\phi_\ell$, a vector field $X_\ell$ by

$$X_\ell = -\frac{1}{2} f_\ell \frac{\partial}{\partial u} + \frac{1}{2} f_\ell \frac{\partial}{\partial v} = f_\ell \frac{\partial}{\partial r^*}, \quad (21)$$

for some function $f_\ell = f_\ell(r^*)$ to be determined later. We shall show that the function $f_\ell$ can be chosen so as for the spacetime term corresponding to $X_\ell$ to be positive, and so as for the boundary terms arising to be controlled by the usual (conserved) $\frac{\partial}{\partial r^*}$-energy, after application of a Hardy inequality. (See Proposition 11.2 from the next section.) These $X_\ell$-energy identities can then be summed so as to yield an identity relating a positive spacetime integral and boundary terms controlled by the usual $\frac{\partial}{\partial r^*}$-energy. (See Section 8.4.)

\(^{15}\)In the obvious sense, i.e. when $\phi$ is replaced by $\phi_0$ and $\partial_t \phi$ is replaced by $\phi_1$ when evaluating (9).
8.1 Identities

We first collect various identities for vector fields of the form (21) for a function \( f \). For functions of \( r^* \), let \( ' \) here denote \( \frac{d}{dr^*} \). The spacetime integral given by the left hand side of (14) in a region \( R = \{ t_0 \leq t \leq t_1 \} \) is:

\[
\hat{I}_X^\phi (R) = \int_R T_{\alpha \beta} \pi_{\alpha \beta} dVol
\]

\[
= \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \int_{S^2} \frac{f' (\partial_r \phi)^2}{1 - \mu} + \frac{1}{2} |\nabla \phi|^2 \left( \frac{2}{r} - \frac{3}{r} \right) f \\
- \frac{1}{4} \left( 2f' + 4 \frac{1 - \mu}{r} f \right) \phi^2 \phi_\alpha \cdot r^2 (1 - \mu) dt \, dr^* \, d\sigma_{S^2}
\]

while the boundary terms are given on a constant \( t \)-hypersurfaces by:

\[
\hat{E}_X^\phi (t_i) = \int_{-\infty}^{t_i} \int_{-\infty}^{\infty} \int_{S^2} \frac{f' (\partial_t \phi)^2}{1 - \mu} \partial_t \phi \, \phi (t_i, r^*, \omega) r^2 \sqrt{1 - \mu} \, dr^* \, d\sigma_{S^2}.
\]

We have the identity

\[
\hat{I}_X^\phi (R) = \hat{E}_X^\phi (t_1) - \hat{E}_X^\phi (t_0).
\]

(23)

To produce an identity with terms for which we can control the signs, we wish to replace the spacetime integral \( \hat{I}_X^\phi \) with a new integral obtained by applying Green’s theorem to its last term. Defining

\[
I_\phi^X (R) = \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \int_{S^2} \left( \frac{f'}{1 - \mu} (\partial_r \phi)^2 + |\nabla \phi|^2 \left( \frac{\mu'}{2 (1 - \mu)} + \frac{1 - \mu}{r} \right) f \\
- \frac{1}{4} \left( \nabla^2 \left( f' + 2 \frac{1 - \mu}{r} f \right) \right) \phi^2 \right) r^2 (1 - \mu) dt \, dr^* \, d\sigma_{S^2},
\]

(24)

we have by Green’s theorem that

\[
I_\phi^X (R) = \hat{I}_\phi^X (R) + \int_{-\infty}^{\infty} \int_{S^2} \frac{1}{2} \left( f' + 2 \frac{1 - \mu}{r} \right) \partial_t \phi \, \phi (t_1, r^*, \omega) r^2 \, dr^* \, d\sigma_{S^2}
\]

\[
- \int_{-\infty}^{\infty} \int_{S^2} \frac{1}{2} \left( f' + 2 \frac{1 - \mu}{r} \right) \partial_t \phi \, \phi (t_0, r^*, \omega) r^2 \, dr^* \, d\sigma_{S^2},
\]

and thus we have the identity

\[
I_\phi^X (R) = E_\phi^X (t_1) - E_\phi^X (t_0)
\]

where

\[
E_\phi^X (t_i) = \hat{E}_\phi^X (t_i) + \int_{-\infty}^{t_i} \int_{S^2} \frac{1}{2} \left( f' + 2 \frac{1 - \mu}{r} \right) \partial_t \phi \, \phi (t_i, r^*, \omega) r^2 \sqrt{1 - \mu} \, dr^* \, d\sigma_{S^2}.
\]

(25)
Finally, let us compute the expression with the d’Alambertian on the right hand side of (24). Since, for a function \( \psi \) dependent only on \( r \), we have \( \Box \psi = (1 - \mu)^{-1} \psi'' + \frac{2}{r} \psi' \), it follows that

\[
\Box \left( f' + 2 \frac{1 - \mu}{r} f \right) = \frac{1}{1 - \mu} f''' + \frac{2}{r} f'' + \frac{2}{r} f''
\]

\[
+ \frac{2}{1 - \mu} \left( \left( \frac{1 - \mu}{r} \right)' - 2 f' \frac{(1 - \mu)^2}{r^2} - 2 f' \frac{r}{r} \right)
\]

\[
+ \frac{4}{r^2} (1 - \mu) f' - \frac{4}{r^3} (1 - \mu)^2 f - \frac{4}{r^2} \mu' f.
\]

In view of the relations

\[
\left( \frac{1 - \mu}{r} \right)' = \frac{(1 - \mu)^2}{r^2} - \frac{\mu'}{r}
\]

and

\[
\left( \frac{1 - \mu}{r} \right)'' = \frac{2(1 - \mu)^3}{r^3} + \frac{3(1 - \mu)\mu'}{r^2} - \frac{\mu''}{r},
\]

we obtain the expression

\[
\Box \left( f' + 2 \frac{1 - \mu}{r} f \right) = \frac{1}{1 - \mu} f''' + \frac{4}{r} f'' - \frac{4\mu'}{r(1 - \mu)} f''
\]

\[
+ \frac{2}{r} \left( \frac{\mu'(1 - \mu)}{r} - \mu'' \right) f.
\]  

(26)

8.2 The choice of \( f_\ell \) for \( \ell \geq 1 \)

In this section, we shall choose \( f_\ell \) for the higher spherical harmonics \( \ell \geq 1 \).

Our goal is to make the spacetime integral \( I^{X_\ell}_\phi (R) \) positive, and the boundary terms \( E^{X_\ell}_\phi \) controllable by the energy \( E_\phi \). Thus, one would like the function \( f_\ell \) to be bounded, and, in view of the first term in (24), \( f_\ell' \) should be positive. For this to be the case, however, \( f_\ell''' \) must become positive in a neighborhood of the horizon. Examining, however, (24) and (26), one sees that this term enters in \( I^{X_\ell}_\phi \) with the wrong sign, and, what is more, it is multiplied by \( (1 - \mu)^{-1} \), and thus seems to dominate the only term with the correct sign to cancel it, namely \( f_\ell'' \).

To generate a term which can indeed cancel this “bad term”, we must borrow from the term in (24) with \( \partial_r \cdot \phi \). This is possible as follows: For any \( C^1 \) function \( \beta = \beta(r^*) \) decaying to 0 as \( r^* \to \pm \infty \), we note the following identity:

\[
\int_{-\infty}^{\infty} \frac{f'}{1 - \mu} (\partial_r \cdot \phi)^2 r^2 (1 - \mu) dr^* = \int_{-\infty}^{\infty} \frac{f'}{1 - \mu} (\partial_r \cdot \phi + \beta \phi)^2 r^2 (1 - \mu) dr^* + \int_{-\infty}^{\infty} \phi^2 \left( - \frac{f'}{1 - \mu} \beta^2 + \frac{f''}{1 - \mu} \beta + \frac{f'}{1 - \mu} \beta' + \frac{2}{r} f' \beta \right) r^2 (1 - \mu) dr^*.
\]
Let us set $M = 1$ (i.e. $\mu = \frac{2}{3}$) and, for a sufficiently large constant $\alpha$ to be determined below, let $x$ denote the coordinate

$$x = r^* - \alpha - 1.$$

Define

$$\beta = \frac{1 - \mu}{r} - \frac{x}{\alpha^2 + x^2}, \quad \beta' = -\frac{(1 - \mu)^2}{r^2} - \frac{\mu'}{r} - \frac{1}{1 + x^2} + \frac{2x^2}{(\alpha^2 + x^2)^2}$$

so that

$$\beta' - \beta^2 + \frac{2}{r}(1 - \mu)\beta = \frac{\mu'}{r} - \frac{\alpha^2}{(\alpha^2 + x^2)^2}.$$

We may thus write

$$I_{\phi \ell}^X(R) = \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \int_{S^2} \left( \frac{f_{\ell}}{1 - \mu} \left( \partial_{r^*} \phi_\ell + \left( \frac{1 - \mu}{r} - \frac{x}{\alpha^2 + x^2} \right) \phi_\ell \right)^2 + \frac{2 - 3\mu}{2r} f_{\ell} |\nabla \phi_\ell|^2 ight. \\
- \frac{1}{4} \frac{1}{1 - \mu} \left( f_{\ell}'' + \frac{4f_{\ell}'}{\alpha^2 + x^2} + \frac{4\alpha^2 f_\ell'}{(\alpha^2 + x^2)^2} \right) \phi_\ell^2 \\
- \frac{\mu f_{\ell}}{2r^3} (4\mu - 3) \phi_\ell^2) r^2 (1 - \mu) dt dr^* d\sigma_{S^2}. \quad (27)$$

For each spherical harmonic number $\ell$, in view of the relation

$$\int_{S^2} |\nabla \phi_\ell|^2 r^2 d\sigma_{S^2} = \int_{S^2} (\ell(\ell + 1)) r^2 (\phi_\ell)^2 d\sigma_{S^2},$$

we may rewrite

$$I_{\phi \ell}^X(R) = \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \int_{S^2} \left( \frac{f_{\ell}'}{1 - \mu} \left( \partial_{r^*} \phi_\ell + \left( \frac{1 - \mu}{r} - \frac{x}{\alpha^2 + x^2} \right) \phi_\ell \right)^2 \\
- \frac{1}{4} \frac{1}{1 - \mu} \left( f_{\ell}'' + \frac{4f_{\ell}'}{\alpha^2 + x^2} + \frac{4\alpha^2 f_\ell'}{(\alpha^2 + x^2)^2} \right) \phi_\ell^2 \\
+ \left( \ell(\ell + 1) + \frac{3 - \mu}{2r^3} + \frac{3 - 4\mu}{r^4} \right) f_{\ell} \phi_\ell^2 \right) r^2 (1 - \mu) dt dr^* d\sigma_{S^2}. \quad (27)$$

The expression multiplying $f_{\ell} \phi_\ell^2$ in the last term above vanishes at a unique value of $r^*$. Denote this by $-\gamma_{\ell}$. Note that

$$-4/3 \leq r^*(8/3) \leq -\gamma_{\ell} \leq r^*(3) = 0.$$

We may now define $f_{\ell} = f_{\ell}(r^*)$ by setting

$$f_{\ell}(\gamma_{\ell})) = 0,$$
\[(f_\ell)' = (\alpha^2 + x^2)^{-1}.
\]

Let us drop the \(\ell\) in what follows. We have

\[f'' = -\frac{2x}{(\alpha^2 + x^2)^2},\]

\[f''' = \frac{8x^2}{(\alpha^2 + x^2)^3} - \frac{2}{(\alpha^2 + x^2)^2}.
\]

Denoting by

\[F = -\frac{1}{4(1 - \mu)} \left( f''' + \frac{4f''x}{\alpha^2 + x^2} + \frac{4f'\alpha^2}{(\alpha^2 + x^2)^2} \right),
\]

we compute

\[F = \frac{1}{2(1 - \mu)} \frac{x^2 - \alpha^2}{(\alpha^2 + x^2)^3}.
\]

Note that \(F\) is nonnegative for \(x \leq -\alpha\), and \(x \geq \alpha\).

We would like to show that for a sufficiently large \(\alpha\), independent of \(\ell\), we can dominate the second term in (27) pointwise by the last term, i.e. we want to show

\[-F \leq \left( \ell(\ell + 1) \frac{2 - 3\mu}{2r^3} + \frac{3 - 4\mu}{r^4} \right) f. \quad (28)\]

In view of the sign of \(F\), it suffices to consider \(-\alpha < x < \alpha\).

For \(-\alpha < x < \alpha\) then, we estimate

\[f = \int_{-\alpha - 1 - \gamma}^{x} f' \geq \frac{x + \alpha + 1}{2\alpha^2 + 2\alpha + 1}.
\]

To show (28), it suffices then to show

\[\left( \ell(\ell + 1) \frac{2 - 3\mu}{2r^3} + \frac{3 - 4\mu}{r^4} \right) \frac{x + \alpha + 1}{2\alpha^2 + 2\alpha + 1} \geq \frac{1}{2(1 - \mu)} \frac{\alpha^2 - x^2}{(\alpha^2 + x^2)^3}. \quad (29)\]

Consider first the region \(-\alpha < x \leq -2\alpha/3\). For \(\alpha\) sufficiently large, we have

\[\frac{1}{2(1 - \mu)} \frac{\alpha^2 - x^2}{(\alpha^2 + x^2)^3} \leq \frac{3\sqrt{2}}{2} \frac{\alpha + x}{(\frac{13}{9}\alpha^2)^{5/2}}.
\]

On the other hand, since for \(x \leq -2\alpha/3\) we have, say \(r \leq 5\alpha/12\), and since

\[r - 3 = \int_{0}^{r^*} (1 - \mu) \geq \frac{1}{3} r^*,
\]

we have \((2 - 3\mu)(r^*) \geq 1/5\) for \(r^* \geq 1\). Then for \(\ell \geq 1\),

\[\ell(\ell + 1) \frac{2 - 3\mu}{2r^3} \frac{x + \alpha + 1}{2\alpha^2 + 2\alpha + 1} \geq \frac{x + \alpha}{\alpha^5}.
\]
for sufficiently large $\alpha$. This gives (29) for the region considered.

Consider now the region $-2\alpha/3 \leq x < \alpha$. Note that as $\alpha \to \infty$, we have $r \sim x + \alpha$, $\mu \sim 0$ in this region. Thus, we have

$$\ell(\ell + 1) \frac{2 - 3\mu x + \alpha + 1 + \gamma}{2\alpha^2 + 2\alpha + 1} \sim \frac{\ell(\ell + 1)}{2} \frac{x + \alpha}{(x + \alpha)^3 \alpha^2},$$

while

$$\frac{1}{2(1 - \mu)} \frac{\alpha^2 - x^2}{(\alpha^2 + x^2)^3} \sim \frac{1}{2} \frac{(\alpha - x)(\alpha + x)}{(\alpha^2 + x^2)^3}.$$

To show (29) for sufficiently large choice of $\alpha$, it suffices then to show the bound

$$\frac{(\alpha - x)(x + \alpha)^3 \alpha^2}{2(\alpha^2 + x^2)^3} < 1. \tag{30}$$

For $x < 0$, (30) is immediate from

$$\frac{(\alpha - x)\alpha^5}{2(\alpha^2 + x^2)^3} \leq \frac{\sqrt{2(\alpha^2 + x^2)}\alpha^5}{2(\alpha^2 + x^2)^3} \leq 2^{-\frac{3}{2}}.$$

For $x \geq 0$, we have on the one hand

$$\frac{(\alpha - x)(x + \alpha)^3 \alpha^2}{2(\alpha^2 + x^2)^3} \leq \frac{(x + \alpha)^3 \alpha^3}{2(\alpha^2 + x^2)^3} \leq \frac{1}{2} \left( \frac{\alpha x + \alpha^2}{x^2 + \alpha^2} \right)^3.$$

On the other hand,

$$\alpha x + \alpha^2 \leq \frac{q}{2} x^2 + \left( 1 + \frac{1}{2q} \right) \alpha^2.$$

Set

$$q = 2 + \frac{1}{q}, \quad q^2 - 2q - 1 = 0.$$

Then $q = 1 + \sqrt{2}$ and

$$\alpha x + \alpha^2 \leq \frac{1}{2} (1 + \sqrt{2})(\alpha^2 + x^2).$$

The bound (30) then follows from the inequality

$$((1/2)(1 + \sqrt{2}))^3 < 2.$$

### 8.3 The case $\ell = 0$

For $\ell = 0$, we have the identity

$$2T_{\alpha\beta} \pi^{\alpha\beta} = \frac{f'_0}{(1 - \mu)} ((\partial_t \phi_0)^2 + (\partial_r \cdot \phi_0)^2) + \frac{2}{r} ((\partial_t \phi_0)^2 - (\partial_r \cdot \phi_0)^2).$$
Applying (23) with the sharp cut-off function \( f_0 = \chi(-\infty, R^*) \), so that \( f'_0 = -\delta(r^* - R^*) \), we obtain

\[
\int_{t_0}^{t_1} \int_{\mathbb{S}^2} \frac{1}{1 - \mu} \left( \left( \partial_t \phi_0 \right)^2 + \left( \partial_r \phi_0 \right)^2 \right) (t, R^*, \omega) r^2 (1 - \mu) d\sigma dt
\]

\[
+ \int_{t_0}^{t_1} \int_{-\infty}^{R^*} \frac{2}{r} \left( \left( \partial_r \phi_0 \right)^2 - \left( \partial_t \phi_0 \right)^2 \right) r^2 (1 - \mu) d\sigma dr^* dt \leq 4E_{\phi_0}.
\]

Let us denote

\[
F(r^*) = \int_{t_0}^{t_1} \int_{-\infty}^{r^*} \frac{2}{r} \left( \partial_t \phi_0 \right)^2 r^2 (1 - \mu) d\sigma dr^* d\rho dt.
\]

Then, in particular,

\[
\frac{r}{2(1 - \mu)} F'(r^*) \leq F(r^*) + 4E_{\phi_0}.
\]

Hence,

\[
\left( e^{-\int_{-\infty}^{r^*} 2r^{-1}(1 - \mu) F(r^*)} \right)' \leq 4e^{-\int_{-\infty}^{r^*} 2r^{-1}(1 - \mu) 2r^{-1}(1 - \mu) E_{\phi_0}^\prime}.
\]

Note that \( \int_{-\infty}^{r^*} 2r^{-1}(1 - \mu) dr^* = \log \mu^{-2} \). Also observe that \( F(-\infty) = 0 \). Therefore, integrating we obtain

\[
\mu^2 F(r^*) \leq 4E_{\phi_0} \int_{2M}^{r^*} \mu^2 d\bar{r} \leq 4E_{\phi_0},
\]

and thus,

\[
F(r^*) \leq 4\mu^{-2} E_{\phi_0}.
\]

It now follows that

\[
\int_{t_0}^{t_1} \int_{\mathbb{S}^2} \frac{1}{1 - \mu} \left( \left( \partial_t \phi \right)^2 + \left( \partial_r \phi \right)^2 \right) (t, r^*, \omega) r^2 (1 - \mu) d\sigma dt \leq 4\mu^{-2} E_{\phi_0}.
\]

Multiplying by \( \mu^2 \) and averaging over \( r^* \) we obtain

\[
\tilde{I}_{\phi_0}^{X_0} \equiv \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \frac{4\mu^2}{(1 - \mu)(1 + |r^*|)^{1+}} \left( \left( \partial_t \phi \right)^2 + \left( \partial_r \phi \right)^2 \right) r^2 (1 - \mu) d\sigma dr^* dt \leq (4+) E_{\phi_0}.
\]

(31)
8.4 The identity, summed

We introduce the notation

\[ E^{X_{\geq 1}}_{\phi} = \sum_{\ell \geq 1} E^{X_{\ell}}_{\phi}, \]

and

\[ I^{X_{\geq 1}}_{\phi} = \sum_{\ell \geq 1} I^{X_{\ell}}_{\phi}. \]

Our final energy identity for the totality of vector fields \( X_{\ell}, \ell \geq 1, \) can be summarized by the statement

\[ 0 \leq I^{X_{\geq 1}}_{\phi}(R) = E^{X_{\geq 1}}_{\phi}(t_1) - E^{X_{\geq 1}}_{\phi}(t_0). \] (32)

Defining

\[ I^{X}_{\phi} = I^{X_0}_{\phi} + I^{X_{\geq 1}}_{\phi}, \]

\[ E^{X}_{\phi} = (2+)E^{X_0}_{\phi} + E^{X_{\geq 1}}_{\phi}, \]

we have the inequality

\[ I^{X}_{\phi}(R) \leq |E^{X}_{\phi}(t_1)| + |E^{X}_{\phi}(t_0)|. \]

9 The vector field \( Y \)

In this section, we shall introduce a vector field which will give good control on the solution near the event horizon \( H^+. \) Although the computations are done in Eddington-Finkelstein coordinates, the reader is encouraged to compare them with computations in a null coordinate system which is regular on the event horizon.

We set \( Y \) to be the vector field

\[ Y = -\frac{1}{2} f \frac{\partial}{\partial u} - \frac{1}{2} \tilde{f} \frac{\partial}{\partial v}. \]

where

\[ f = \frac{\alpha(r^*)}{1 - \mu}, \quad \tilde{f} = \beta(r^*), \]

for functions \( \alpha \) and \( \beta \) do be defined later. Below \( ' \) will denote \( \frac{d}{dr}. \) We have

\[ T_{\gamma\delta\pi\pi} = - \frac{(\partial_u \phi)^2}{4(1 - \mu)^2} \left( \frac{\alpha \mu}{r} - \alpha' \right) - \frac{(\partial_v \phi)^2}{4(1 - \mu)} \beta' \]

\[ - \frac{1}{4} |\nabla \phi|^2 \left( \frac{\alpha'}{1 - \mu} - \frac{(\beta(1 - \mu))'}{1 - \mu} \right) + \frac{1}{2r} \left( \frac{\alpha}{1 - \mu} - \beta \right) \partial_u \phi \partial_v \phi. \]
Integrating in the characteristic rectangle \( \tilde{R}' = [u_1, u_2] \times [v_1, v_2] \), we obtain
\[
F^Y_{\phi}(\{u_2\} \times [v_1, v_2]) + F^Y_{\phi}([u_1, \infty] \times \{v_2\}) = I^Y_{\phi}(\tilde{R}') + F^Y_{\phi}(u_1 \times [v_1, v_2]) + F^Y_{\phi}([u_1, \infty] \times \{v_1\})
\]
(33)

where
\[
F^Y_{\phi} (\{u\} \times [v_1, v_2]) \doteq \int_{v_1}^{v_2} \int_{S^2} (\alpha |\nabla \phi|^2 + \beta (\partial_v \phi)^2)(\infty, v) r^2 dv d\sigma_{S^2},
\]
\[
F^Y_{\phi} ([u_1, u_2]) \doteq \int_{u_1}^{\infty} \int_{S^2} \left( \frac{\alpha}{1 - \mu} (\partial_u \phi)^2 + (1 - \mu) \beta |\nabla \phi|^2 \right) r^2 du d\sigma_{S^2},
\]
\[
I^Y_{\phi}(\tilde{R}') \doteq \int_{v_1}^{v_2} \int_{u_1}^{u_2} \int_{S^2} \left( - \left( \frac{\partial_u \phi}{(1 - \mu)} (\frac{\alpha \mu}{r} - \alpha') + (\partial_v \phi)^2 \beta' + |\nabla \phi|^2 (\alpha' - (\beta(1 - \mu))' \right) \right.
\]
\[
+ \frac{2}{r} (\alpha - \beta(1 - \mu)) \partial_u \phi \partial_v \phi \right) r^2 du dv d\sigma_{S^2}.
\]

We define \( \alpha, \beta \) as follows. Let \( r_0 > 2M \) be a constant sufficiently close to \( 2M \), to be determined by various restrictions that follow. Set \( \alpha = 1 \) and \( \beta = 0 \) on the event horizon. Furthermore, require that \( \alpha, \beta \) both be non-negative functions supported in the region \( r \leq 1.2r_0 \), with
\[
\alpha'(r^*) \sim C(1 + |r^*|)^{-1}, \quad \beta(r^*) \sim C(1 + |r^*|)^{-1}, \quad \forall r \leq r_0
\]
for some constant \( C > 0 \).

We will always apply the above identity in \([u_1, \infty] \times [v_1, v_2]\). We have then that
\[
F^Y_{\phi}(\{\infty\} \times [v_1, v_2]) = \int_{v_1}^{v_2} \int_{S^2} |\nabla \phi|^2(\infty, v) r^2 dv d\sigma_{S^2}.
\]

Let us set
\[
I^Y_{\phi}(\tilde{R}') \doteq \int_{v_1}^{v_2} \int_{u_1}^{\infty} \int_{S^2} \left( \frac{(\partial_u \phi)^2}{(1 - \mu)} \left( \frac{\alpha \mu}{r} - \alpha' \right) + (\partial_v \phi)^2 \beta' + |\nabla \phi|^2 (\alpha' - (\beta(1 - \mu))' \right) \right)
\]
\[
+ \frac{2}{r} (\alpha - \beta(1 - \mu)) \partial_u \phi \partial_v \phi \right) r^2 du dv d\sigma_{S^2}.
\]

The quantity \( r_0 \) will be chosen sufficiently small so that all terms in the above integrand for \( I^Y_{\phi} \) are non-negative in the region \( r \leq r_0 \), and so that moreover
\[
\left( \frac{\alpha \mu}{r} - \alpha' \right) \geq \frac{1}{2r} (\alpha - \beta(1 - \mu))^2, \quad (34)
\]
in $r \leq r_0$. In addition, we will require of $r_0$ that $1.2r_0 < 3M$ (see Proposition 11.5 and 51).

Defining

$$\hat{I}^Y(\tilde{R}') \doteq \int_{v_1}^{v_2} \int_{u_1}^{u_2} \int_{\Sigma^2} \frac{2}{r}(\alpha - \beta(1 - \mu)) \partial_u \phi \partial_v \phi r^2 \, du \, dv \, d\sigma_{\Sigma^2},$$

we may rewrite 33 as

$$F^Y(\{\infty\} \times [v_1, v_2]) + F^Y([u_1, \infty] \times \{v_2\}) + \hat{I}^Y(\tilde{R}')$$
$$= \hat{I}^Y(\tilde{R'}) + F^Y(u_1 \times [v_1, v_2]) + F^Y([u_1, \infty] \times \{v_1\}).$$

In our argument (see Proposition 11.5), the terms $\hat{I}^Y$ and $\tilde{I}^Y$ in the region $r \geq r_0$ will be controlled by $I^X$. The relation of $F^Y$ with the integrand of $\hat{I}^Y$ (as exploited in Proposition 11.6) can be thought of as the manifestation of the red-shift effect as measured by two local observers. It is to be compared with the fact that, in equation (51) of [8] for $\partial_v \phi$, the factor in front of $\phi$ on the right hand side is bounded above and below away from 0 uniformly in the region $r \leq r_0$.

10 The vector field $K$

In this section, we shall define the Morawetz vector field $K$.

We set $K$ to be the vector field

$$K = -\frac{1}{2} u^2 \frac{\partial}{\partial u} - \frac{1}{2} v^2 \frac{\partial}{\partial v}.$$

We have

$$K^u = -\frac{1}{2} u^2, \quad K^v = -\frac{1}{2} v^2, \quad K_u = (1 - \mu)u^2, \quad K_v = (1 - \mu)v^2.$$

We compute

$$T_{\alpha\beta} \pi^{\alpha\beta} = t|\nabla \phi|^2 \left( \frac{1}{2} + \frac{r^*}{4r} - \frac{r^*(1 - \mu)}{2r} \right) + tr^* \frac{(1 - \mu)}{4r} \Box \phi^2.$$

Let $\mathcal{R} = \{t_0 \leq t \leq t_1\}$. The identity 14 gives

$$\hat{I}^K_\phi(\mathcal{R}) = \hat{E}^K_\phi(t_0) - \hat{E}^K_\phi(t_1),$$

where

$$\hat{I}^K_\phi(\mathcal{R}) = \int_\mathcal{R} T_{\alpha\beta} \pi^{\alpha\beta} \, dV ol_{\Sigma}$$
$$= \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \int_{\Sigma^2} \left( t|\nabla \phi|^2 \left( \frac{1}{2} + \frac{r^*}{4r} - \frac{r^*(1 - \mu)}{2r} \right) + \frac{tr^* (1 - \mu)}{4r} \Box \phi^2 \right) \cdot r^2 (1 - \mu) d\sigma_{\Sigma^2} \, dr^* \, dt$$

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and
\[
\dot{E}_\phi^K(t_i) = \int_{\{t=t_i\}} g_{\alpha\beta} n^\alpha P^\beta dVol_{\{t=t_i\}}
\]
\[
= \int_{-\infty}^{\infty} \int_{S^2} \frac{1}{2\sqrt{1-\mu}} (K^\nu T_{uv} + K^u T_{uv} + K^u T_{uv} + K^\nu T_{uv})(r^*, t_i, \omega)
\]
\[
\cdot r^2 \sqrt{1-\mu} d\sigma_{S^2}
\]
\[
= \int_{-\infty}^{\infty} \int_{S^2} \frac{1}{4\sqrt{1-\mu}} (u^2 T_{uv} + v^2 T_{uv} + (u^2 + v^2) T_{uv})
\]
\[
\cdot r^2 \sqrt{1-\mu} d\sigma_{S^2}
\]
\[
= \int_{-\infty}^{\infty} \int_{S^2} \frac{1}{4\sqrt{1-\mu}} (u^2 (\partial_u \phi)^2 + v^2 (\partial_v \phi)^2 + (1-\mu)(u^2 + v^2) \|\nabla \phi\|^2)
\]
\[
\cdot r^2 \sqrt{1-\mu} d\sigma_{S^2}.
\]

In view of the identity
\[
\int_{\mathcal{R}} \psi \Box (\phi^2) dVol_{\mathcal{R}} - \int_{\mathcal{R}} \Box (\phi^2) dVol_{\mathcal{R}}
\]
\[
= \int_{\{t=t_1\}} (2 g^{\alpha\beta} \phi \nabla_\alpha \phi n_\beta \psi - g^{\alpha\beta} \nabla_\alpha \psi n_\beta \phi^2) dVol_{\{t=t_1\}}
\]
\[
- \int_{\{t=t_0\}} (2 g^{\alpha\beta} \phi \nabla_\alpha \phi n_\beta \psi - g^{\alpha\beta} \nabla_\alpha \psi n_\beta \phi^2) dVol_{\{t=t_0\}},
\]
applied to
\[
\psi = \frac{tr^*(1-\mu)}{4r},
\]
we obtain the identity
\[
I^K_\phi(\mathcal{R}) = E^K_\phi(t_1) - E^K_\phi(t_0)
\]
where
\[
I^K_\phi(\mathcal{R}) = \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \int_{S^2} \left( t \|\nabla \phi\|^2 \left( \frac{1}{2} + \frac{\mu r^*}{4r} - \frac{r^*(1-\mu)}{2r} \right) \right)
\]
\[
\cdot r^2 \sqrt{1-\mu} dt^* d\sigma_{S^2}
\]
\[
+ \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \int_{S^2} \frac{1}{4\mu r^2 |\phi|^2} \left( \frac{r^* (4\mu-3)}{r} \right) \cdot r^2 (1-\mu) dt^* d\sigma_{S^2}
\]
and
\[
E^K_\phi(t_i) = \dot{E}_\phi^K(t_i) + \int_{\{t=t_i\}} \frac{1}{2\sqrt{1-\mu}} \left( \frac{tr^*(1-\mu)}{2r} \phi \partial_t \phi - \phi^2 \frac{r^*(1-\mu)}{4r} \right) \cdot r^2 \sqrt{1-\mu} d\sigma_{S^2}.
\]

For the analogue of the following argument in Minkowski space, see [14]. Let $S = v \partial_v + u \partial_u$ and $\mathcal{S} = v \partial_v - u \partial_u$. Then
\[
v^2 (\partial_v \phi)^2 + u^2 (\partial_u \phi)^2 = \frac{1}{2} \left( (S \phi)^2 + (\mathcal{S} \phi)^2 \right),
\]
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\[ S = 2(t \partial_t + r^* \partial_r), \quad \mathcal{S} = 2(t \partial_{r^*} + r^* \partial_t), \]
\[ 2t \partial_t \phi = S \phi - 2r^* \partial_r \phi, \quad 2t \partial_{r^*} \phi = \frac{t}{r} \mathcal{S} - 2 \frac{t^2}{r^*} \partial_r \phi. \]

Thus, by integration by parts, we obtain\(^{16}\):
\[
\int_{-\infty}^{\infty} 2t \frac{(1 - \mu)r^*}{r} \partial_t \phi \partial_r^{2} \, dr^* = \int_{-\infty}^{\infty} \left( r^2 \frac{(1 - \mu)r^*}{r} S \phi \phi + \partial_r \left( (1 - \mu) r(r^*)^2 \right) \phi^2 \right) \, dr^* \\
= \int_{-\infty}^{\infty} (1 - \mu)r^2 \left( \frac{r^*}{r} S \phi + \left( \frac{(r^*)^2}{r^2} + 2 \frac{r^*}{r} \right) \phi^2 \right) \, dr^*.
\]

It now follows that
\[
\int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \left( \frac{1}{4}(1 - \mu)(S \phi)^2 + 2t \frac{(1 - \mu)r^*}{r} \partial_t \phi \phi - \frac{(1 - \mu)r^*}{r} S \phi \phi \right) r^2 \, dr^* d\sigma_{\mathbb{S}^2} \\
= \int_{-\infty}^{\infty} (1 - \mu)r^2 \left( \frac{1}{2} S \phi + \frac{r^*}{r} \phi \right)^2 \, dr^* d\sigma_{\mathbb{S}^2}.
\]

We may thus write
\[
E^K_{\phi}(t_i) = \frac{1}{4} \int_{-\infty}^{\infty} \int_{\mathbb{S}^2} \frac{1}{\sqrt{1 - \mu}} \left( \frac{1}{4}(1 + \mu)(S \phi)^2 + \frac{1}{2}(\mathcal{S} \phi)^2 + (1 - \mu)(u^2 + v^2)|\nabla \phi|^2 \right) \\
+ (1 - \mu) \left( \frac{1}{2} S \phi + \frac{r^*}{r} \phi \right)^2 \, dr^* d\sigma_{\mathbb{S}^2}. \quad (39)
\]

In particular, \(E^K_{\phi}(t_i)\) is nonnegative.

### 11 Comparison estimates

Let us introduce one final spacetime integral quantity: For \(X\) a region denote
\[
I_{\phi}(X) = \int_X \left( |\partial_r \phi|^2 + |\partial_t \phi|^2 + (1 - \mu)|\nabla \phi|^2 \right) \, dVol.
\]

We have the following

**Proposition 11.1.** If \(X\) is a rectangle defined by
\[
X = \{ t_0 \leq t \leq t_1 \} \cap \{ 2M < r_0 \leq r \leq \hat{R} \},
\]
then
\[
I_{\phi}(X) \leq C(I_{\phi}^X(X) + I_{\phi_0}^X(X)). \quad (40)
\]

Moreover, for either \(\hat{R} < 3M\) or \(r_0 > 3M\),
\[
I_{\phi}(X) \leq CI_{\phi}^X(X). \quad (41)
\]

\(^{16}\)Note that there is no problem near the bifurcate sphere in the integration by parts argument due to the presence of the exponentially decaying \((1 - \mu)\) factor.
Proof. From the definition of the functions \( f_\ell \), we have that for \( \ell \geq 1 \),
\[
\int_X \frac{\phi^2_\ell}{(1 - \mu)(1 + (r^*)^2)} \, \text{dVol} \leq C I^X_{\phi_\ell}(X), \tag{42}
\]
and thus, summing over \( \ell \geq 1 \), we obtain
\[
\int_X \frac{\phi^2_{\ell \geq 1}}{(1 - \mu)(1 + (r^*)^2)} \, \text{dVol} \leq C I^X_{\phi_{\ell \geq 1}}(X). \tag{43}
\]
Now using this bound, we can apply \( \text{(24)} \) again with the function
\[
f(r^*) = \int_0^{r^*} \frac{d\rho}{1 + (\rho)^2}. \tag{44}
\]
Note that this choice ensures the non-negativity of the space-time integral containing the angular derivatives. We obtain
\[
\int_X |\partial_{r^*} \phi_{\ell \geq 1}|^2 \, \text{dVol} + \int_X f(r^*) \frac{(2 - 3\mu)|\nabla \phi_{\ell \geq 1}|^2}{2r} \, \text{dVol} \leq C I^X_{\phi_{\ell \geq 1}}(X). \tag{45}
\]
Applying \( \text{(22)} \), using the representation
\[
\Box (\phi^2_{\ell \geq 1}) = \frac{2}{1 - \mu} (- (\partial_t \phi_{\ell \geq 1})^2 + (\partial_{r^*} \phi_{\ell \geq 1})^2 + (1 - \mu)|\nabla \phi_{\ell \geq 1}|^2) \tag{46}
\]
and the function \( \hat{f}(r^*) = f(r^*) \cdot (2 - 3\mu)/(2r)^{-1} \), we obtain from \( \text{(43)} \) and \( \text{(45)} \) the bound
\[
\int_X f(r^*) (2 - 3\mu)(2r)^{-1} |\partial_{r^*} \phi_{\ell \geq 1}|^2 \, \text{dVol} \leq C I^X_{\phi_{\ell \geq 1}}(X). \tag{47}
\]
On the other hand, by \( \text{(31)} \), we clearly have
\[
\int_X \frac{|\partial_{r^*} \phi_0|^2 \mu^2}{(1 - \mu)(1 + (r^*)^2)^{1+}} \, \text{dVol} + \int_X \frac{|\partial_t \phi_0|^2 \mu^2}{(1 - \mu)(1 + (r^*)^2)^{1+}} \, \text{dVol} \leq C I^X_{\phi_0}(X). \tag{48}
\]
Estimates \( \text{(48)} \), \( \text{(47)} \) and \( \text{(45)} \) immediately yield
\[
\int_X f(r^*) (2 - 3\mu) |\partial_{r^*} \phi|^2 \mu^2 \, \text{dVol} + \int_X \frac{|\partial_t \phi|^2 \mu^2}{2r(1 + (r^*)^2)^{1+}} \, \text{dVol} + \int_X f(r^*) (2 - 3\mu)|\nabla \phi|^2 \, \text{dVol} \leq C I^X_{\phi}. \tag{49}
\]
This clearly gives \( \text{(41)} \).

Applying our argument again to the solution \( \phi_\omega = \Omega \phi \) of the wave equation
\[\Box \phi_\omega = 0, \tag{41}\]
where \( \Omega \) is an angular momentum operator, in view of the fact that this solution has vanishing zeroth spherical harmonic, we obtain as in \( \text{(38)} \)
\[
\int_X \frac{r^2|\nabla \phi|^2}{(1 - \mu)(1 + (r^*)^2)^2} \, \text{dVol} \leq C I^X_{\phi_\omega}(X). \tag{50}
\]
Here $I_{\phi}$ actually denotes the sum of $I$ applied to $\Omega \phi$ where $\Omega$ range over an appropriate basis of angular momentum operators. Then, using (46) and (48), and arguing as before we also obtain that

$$\int \frac{|\partial_t \phi|^2 \mu^2}{(1 - \mu)(1 + (r^*)^2)^2} dVol \leq C \left( I^{X} (\mathcal{X}) + I^{X}_\phi (\mathcal{X}) \right). \quad (51)$$

The estimates (49), (50) and (51) together give (40).

**Proposition 11.2.** Consider the hypersurface $\{t = t_i\}$. We have

$$|E^{t_i}_\phi| \leq C E^{t_i}_\phi.$$  

**Proof.** In view of the Cauchy-Schwarz inequality applied to the boundary terms defined by (25), it suffices to establish the Hardy inequality

$$\int_{\mathbb{S}^2} \int_{\mathbb{S}^2} (1 + |r^*|)^{-2} |\phi|^2 r^2 dr^* d\sigma \leq C \int_{\mathbb{S}^2} \int_{\mathbb{S}^2} |\partial_r \phi|^2 r^2 dr^* d\sigma.$$ \quad (52)

Define the function $f(r^*)$ by solving the equation

$$f' = \left( \frac{r}{1 + |r^*|} \right)^2$$

with the boundary condition $f(-\infty) = 0$, i.e.,

$$f(r^*) = \int_{-\infty}^{r^*} \left( \frac{r(\rho)}{1 + |\rho|} \right)^2 d\rho.$$

Here $'$ denotes $\frac{d}{dr^*}$. Clearly,

$$f(r^*) \sim |r^*|^{-1}, \quad (\text{for } r^* \to -\infty) \quad \text{and} \quad f(r^*) \sim r^*, \quad (\text{for } r^* \to +\infty).$$

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Now,}
\[
\int_{-\infty}^{\infty} \int_{S^2} (1 + |r^*|)^{-2} |\phi|^2 r^2 dr^* = \int_{-\infty}^{\infty} \int_{S^2} f'(r^*)|\phi|^2 dr^*
\]
\[
= -2 \int_{-\infty}^{\infty} \int_{S^2} f(r^*) \partial_r \phi \phi dr^*
\]
\[
\leq 2 \left( \int_{-\infty}^{\infty} \int_{S^2} \left( \frac{(f(r^*))^2}{f'(r^*)} |\partial_r \phi|^2 dr^* \right)^{1/2} \right)
\[
\cdot \left( \int_{-\infty}^{\infty} \int_{S^2} f'(r^*) |\phi|^2 dr^* \right)^{1/2}
\]
\[
= 2 \left( \int_{-\infty}^{\infty} \int_{S^2} \left( \frac{(1 + |r^*|) f(r^*)^2}{r^2} |\partial_r \phi|^2 dr^* \right)^{1/2} \right)
\[
\cdot \left( \int_{-\infty}^{\infty} \int_{S^2} (1 + |r^*|)^{-2} |\phi|^2 r^2 dr^* \right)^{1/2}
\]

Inequality (52) follows immediately from
\[
\frac{((1 + |r^*) f(r^*))^2}{r^2} \leq C r^2.
\]

We will denote by $E_{\phi,\omega}$ the total energy associated with the solutions of the wave equation $\phi_\omega = \Omega \phi$, obtained by applying a basis angular momentum operators $\Omega$ to $\phi$, and summing. We will also set $E_{\phi,\phi,\omega} = E_\phi + E_{\phi,\omega}$ etc.

Proposition 11.3. Consider the rectangle $\mathcal{X}$ of Proposition 11.1. We have
\[
I_\phi(\mathcal{X}) \leq CE_{\phi,\phi,\omega}.
\]
If $\mathcal{X} \cap ([t_0, t_1] \times \{r = 3M\}) = 0$ then
\[
I_\phi(\mathcal{X}) \leq CE_\phi.
\]

Proof. This follows immediately from the previous two Propositions. 

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Proposition 11.4. Consider the rectangle $\tilde{R} = [u_1, \infty] \times [v_1, v_2]$, and the shaded triangle $R$ depicted.

We have

$$F_Y^\phi ([u_1, \infty] \times \{v_1, v_2\}) \leq C(\omega_\phi(u_1, v_2) - \omega_\phi(u_1, v_1)),$$

$$\omega_\phi(\infty, v_2) - \omega_\phi(\hat{u}(v_2), v_2) \leq CF_\phi^Y ([u_1, \infty] \times \{v_2\}),$$

$$\int_{\hat{u}(v_2) \mathbb{S}^2} \frac{(\partial_\phi^2)^2}{1 - \mu} d\sigma_{\mathbb{S}^2} du \leq CF_\phi^Y ([u_1, \infty] \times \{v_2\})$$

$$\int_{v_1}^\infty \int_{v_2}^{\hat{v}(u)} |\nabla_\phi|^2 d\sigma_{\mathbb{S}^2} dv \leq F_\phi^Y ([u_1, \hat{v}(u)]), \quad \forall u \geq u_1$$

where $\hat{u}(v_2)$ and $\hat{v}(u)$ are defined so that $r(\hat{u}(v_2), v_2) = r_0$ and $r(u, \hat{v}(u)) = r_0$ respectively.

Proof. This follows immediately from the definition of $Y$ in Section 9.

The next proposition shows that in the identity (36) generated by vector field $Y$, the space-time terms without a sign can be controlled with the help of $I^X$.

Proposition 11.5. With $\tilde{R}$, $R$ as above, we have

$$F_Y^\phi ([v_1, v_2]) + F_Y^\phi ([u_1, \infty] \times \{v_2\}) + \frac{1}{2} \hat{I}_\phi^Y (\tilde{R})$$

$$\leq C \left( I_\phi^X (R) + F_\phi^Y ([u_1, \infty] \times \{v_1\}) + \omega_\phi(u_1, v_2) - \omega_\phi(u_1, v_1) \right).$$

Proof. By Proposition 11.4, we have that

$$F_Y^\phi ([u_1, v_2]) \leq C(\omega_\phi(u_1, v_2) - \omega_\phi(u_1, v_1)).$$

From (36), it would suffice then to show that

$$\hat{I}_\phi^Y (\tilde{R}) \leq CI_\phi^X (R) + \frac{1}{2} \hat{I}_\phi^Y (\tilde{R}).$$

(53)

Let us decompose:

$$\hat{I}_\phi^Y (\tilde{R}) = \hat{I}_\phi^Y (\tilde{R} \setminus R) + \hat{I}_\phi^Y (R).$$
By an application of Cauchy-Schwarz and (34), (35), we have in the region $\tilde{R} \setminus R$ that
\[
\frac{2}{r} (\alpha - \beta)(1 - \mu) \partial_u \phi \partial_v \phi \leq \frac{1}{2} \left( \frac{(\alpha - \beta)(1 - \mu)}{2r(1 - \mu)} (\partial_u \phi)^2 + 8r^{-1}(1 - \mu)(\partial_v \phi)^2 \right) \\
\leq \frac{1}{2} \left( \frac{(\partial_u \phi)^2}{(1 - \mu)} \left( \frac{\alpha \mu}{r} - \alpha' \right) + \beta' (\partial_v \phi)^2 \right),
\]
so integrating, we obtain
\[
\hat{I}_Y \phi \left( \tilde{R} \setminus R \right) \leq \frac{1}{2} \hat{I}_Y \phi (\tilde{R} \setminus R).
\]
On the other hand, by Proposition 11.1 and the condition $1.2r_0 < 3M$, we easily see that
\[
\hat{I}_Y \phi (R) \leq C \left( I_X \phi (R) \right).
\]
Since
\[
0 \leq \hat{I}_Y \phi \left( \tilde{R} \setminus R \right) = \hat{I}_Y \phi (\tilde{R}') - \hat{I}_Y \phi (R),
\]
and
\[
|\hat{I}_Y \phi (R)| \leq C \left( I_X \phi (R) \right),
\]
(53) follows immediately.

The next proposition can be thought of as a pigeonhole argument, which will allow us to pick values of $v$ where the boundary terms generated by the vector field $Y$ gain additional decay.

**Proposition 11.6.** With $\tilde{R}$ as above, we have
\[
\inf_{v_1 \leq v \leq v_2} F_Y ([u_1, \infty] \times \{v\}) \leq C (v_2 - v_1)^{-1} \hat{I}_Y \phi (R) + C(\varphi(u_1, v) - \varphi(u_2, v)).
\]

**Proof.** This follows immediately from the fact that $\int_a^b f \geq (b - a) \inf f$, and the inequality
\[
F_Y \phi ([u_1, \infty] \times \{v\}) \leq C \int_{u_1}^{\infty} \int_{\mathbb{S}^2} \left( \frac{(\partial_u \phi)^2}{(1 - \mu)} \left( \frac{\alpha \mu}{r} - \alpha' \right) + (\partial_v \phi)^2 \beta' + |\nabla \phi|^2 (\alpha' - (\beta(1 - \mu))) \right) r^2 dud\sigma
\]
\[
+ \varphi(u_1, v) - \varphi(u_2, v).
\]

The next proposition estimates the positive part of $I^K$ (i.e. the terms with the wrong sign) by $I(\mathcal{A}')$, with a certain loss:
Proposition 11.7. Let \( \tilde{R}' \) and \( X \) denote regions \( \tilde{R}' = \{ t_0 \leq t \leq t_1 \}, X = \tilde{R}' \cap \{ r_0 \leq r \leq R \} \) depicted below:

![Diagram of regions](image)

where \( R \) is sufficiently large. We have

\[
I^K_\phi (\tilde{R}') \leq t_1 I_\phi (X).
\]

Proof. Recall that

\[
I^K_\phi (\tilde{R}') = \int_{t_0}^{t_1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left( \frac{t}{2} |\nabla \omega \phi|^2 \left( 1 + \frac{3\mu - 2}{2r} r^* \right) + \frac{r}{4} \left( \frac{2\mu}{r^2} + \left( \frac{4\mu - 3}{r^3} \right) r^* \right) \phi^2 \right) r^2 (1 - \mu) \, dt \, dr^* \, d\sigma_{\mathbb{S}^2}.
\]

We have required of \( r_0 \) that

\[
r_0^* < \max \left( -\frac{2r_0}{4\mu_0 - 3}, -\frac{2r_0}{3\mu_0 - 2} \right) < 0. \tag{54}
\]

We require of \( R \) that \( R \) is sufficiently large, to be determined below, and, in particular, that \( R > r_2 \) where \( r_2 \) is the infimum of all \( r \) satisfying, for all \( \bar{r} \geq r \), the inequality

\[
\bar{r}^* > \min \left( -\frac{2\bar{r}}{4\bar{\mu} - 3}, -\frac{2\bar{r}}{3\bar{\mu} - 2} \right) > 0. \tag{55}
\]

Note that \( r_2 < \infty \). The choices \( r_0 \), \( r_2 \) ensure that in the regions \( \{ r \leq r_0 \} \), \( \{ r \geq R \} \), the integrand of \( I^K_\phi \) is nonpositive. Thus, we have

\[
I^K_\phi (\tilde{R}') = I^K_\phi (\tilde{R}' \setminus X) + I^K_\phi (X) \leq I^K_\phi (X).
\]

On the other hand, one sees easily that

\[
\int_X \frac{t}{2} |\nabla \phi|^2 \left( 1 + \frac{3\mu - 2}{2r} r^* \right) \, dVol \leq C t_1 I_\phi (X).
\]

It will then suffice to prove the inequality

\[
\int_X \frac{t}{4} \left( \frac{2\mu}{r^2} + \left( \frac{4\mu - 3}{r^3} \right) r^* \right) \phi^2 \, dVol \leq -c I^K_\phi (\tilde{R}' \setminus X) + C t_1 I_\phi (X).
\]
for some sufficiently small $\epsilon > 0$. The function

$$H(r^*) = \left(\frac{2\mu}{r^2} + \frac{(4\mu - 3)\mu}{r^3} r^*ight)$$

is positive only in a subset of the region of $2M < r_0 < r_1 \leq r \leq r_2 < R$. Moreover, $H(r^*)$ behaves like $r^{-3}$ as $r \to +\infty$. The desired result will then follow from the following inequality:

$$\int_{r_1}^{r_2} \phi^2 (1-\mu) d\sigma_{S^2} dr^* \leq C \int_{r_1}^{r_2} (\phi')^2 r^2 (1-\mu) d\sigma_{\mathcal{S}^2} dr^* + c \int_{r_2}^{r_3} r^{-3} \phi^2 (1-\mu) d\sigma_{S^2} dr^*$$

for some small constant $c > 0$ and some large $r_3 \leq R$. This in turn easily follows from rescaling the following one dimensional estimate:

**Lemma 11.1.** For any $c > 0$ there exist sufficiently large constants $C$ and $A$ such that

$$\int_{1}^{2} \phi^2(x) dx \leq C \int_{1}^{A} (\phi')^2 dx + 2c \int_{1}^{A} x^{-1} \phi^2 dx.$$

**Proof.** We write

$$\phi^2(x) - \phi^2(y) = 2 \int_{x}^{y} \phi' \phi dz \leq 2 \left( \int_{x}^{y} z \phi'^2 dz \right)^{\frac{1}{2}} \left( \int_{x}^{y} z^{-1} \phi'^2 dz \right)^{\frac{1}{2}}.$$

Thus

$$\int_{1}^{2} \phi^2(x) dx \leq c^{-2} \int_{1}^{y} z \phi'^2 dz + c \int_{1}^{y} z^{-1} \phi'^2 dz + \phi^2(y).$$

Dividing by $y$ and integrating in $y$ in the region $[2, A]$ with $A = e^{1/c}$, we obtain

$$(c^{-1} - \log 2) \int_{1}^{2} \phi^2(x) dx \leq c^{-3} \int_{1}^{A} z \phi'^2 dz + c \int_{1}^{A} z^{-1} \phi'^2 dz + \int_{2}^{A} y^{-1} \phi^2(y) dy.$$

The desired conclusion follows immediately with $C \sim Ac^{-2}$. 

**Proposition 11.8.** For the region $\mathcal{R}'$ as above, we have

$$E^K_{\phi} (t_1) \leq C(I^K_{\phi} (\mathcal{R}') + E^K_{\phi} (t_0)).$$

**Proof.** This follows immediately from (37). 

**Proposition 11.9.** On a constant $t_1$ hypersurface, we have the bounds

$$\int_{-\infty}^{\infty} \int_{S^2} \frac{1}{\sqrt{1-\mu}} \left( u^2 (\partial_u \phi)^2 + v^2 (\partial_v \phi)^2 \right)$$

$$+ (1-\mu)(u^2 + v^2) |\nabla \phi|^2 \cdot r^2 \sqrt{1-\mu} dr^* d\sigma_{S^2} \leq E^K_{\phi} (t_1),$$

(56)
\[
\int_{-\infty}^{\infty} \int_{S^2} \sqrt{1-\mu} r^2 \cdot r^2 \sqrt{1-\mu} \, dr \, d\sigma_{S^2} \leq E^K_{\phi}(t_1), \tag{57}
\]
\[
\int_{-\infty}^{\infty} \int_{S^2} \sqrt{1-\mu} r^2 \cdot r^2 \sqrt{1-\mu} \, dr \, d\sigma_{S^2} \leq CE^K_{\phi}(t_1). \tag{58}
\]

**Proof.** The bounds (56) and (57) follow immediately from (39).

For (58), write
\[
\int_{2t}^{(1-\mu) r^*} \partial_t \phi \, dr^* = \int \left( \frac{r^2 (1-\mu) t}{r^2} S\phi + t^2 \partial_r r^2 \phi^2 \right) \]
\[
= \int (1-\mu) r^2 \left( \frac{t}{r} S\phi + \frac{t^2}{r^2} \phi^2 \right) \]
\[
= \int (1-\mu) r^2 \left( \frac{1}{2} S\phi + \frac{t}{r} \phi \right)^2 - \frac{1}{4} \int (1-\mu) r^2 (S\phi)^2, \]
and use (39).

We now compare the boundary terms \(E^K_{\phi}(t_i)\) generated by \(K\) with the flux of \(\frac{\partial}{\partial t}\)-energy.

**Proposition 11.10.** With \(\tilde{R}\) as before, let \((r^*_1, t_1), (\tilde{r}^*_1, t_1)\), be so as for \(t_1 - \tilde{r}^*_1 \geq 1, t_1 + r^*_1 \geq 1\). Then
\[
\varpi_{\phi}(\tilde{r}^*_1, t_1) - \varpi_{\phi}(r^*_1, t_1) \leq (t_1 - \tilde{r}^*_1)^{-2} E^K_{\phi}(t_1) + (t_1 + r^*_1)^{-2} E^K_{\phi}(t_1).
\]

**Proof.** This follows immediately from (56) and the geometry of the region considered.

**Proposition 11.11.** With \(X\) as above, suppose
\[
t_1 \leq 1.1 t_0, \quad |r^*(r_0)| + |r^*(R)| \leq 1.1 t_0. \tag{59}
\]
Then we have
\[
I^X_{\phi}(X) \leq C t_0^{-2} E^K_{\phi}(t_0).
\]

**Proof.** Let \(\chi\) be a smooth cut-off function equal to one on the interval \([-1, 1]\) and vanishing outside the interval \([-1.5, 1.5]\). Define \(\psi\) to be a solution of the wave equation \(\Box_g \psi = 0\) with initial conditions
\[
\psi(t_0, r^*) = \chi(2r^*/t_0) \phi(t_0, r^*), \quad \partial_t \psi(t_0, r^*) = \chi(2r^*/t_0) \partial_t \phi(t_0, r^*).
\]
We claim that
\[
I^X_{\phi}(X) = I^X_{\psi}(X).
\]
This follows from the choices  and the fact that initial data for \( \psi(t_0, \cdot) \) coincides with those of \( \phi(t_0, \cdot) \) for the values of \( -0.5t_0 \leq r^* \leq 0.5t_0 \).

We now apply Proposition 11.3 to the function \( \psi \) to obtain

\[
I_X(\tilde{R}') \leq CE\psi(t_0).
\]

It remains to show that

\[
E\psi(t_0) \leq C t_0^{-2} E\phi(t_0).
\]

From the definition of the energy \( E\psi \) and the properties of the cut-off function \( \chi \) we see immediately that

\[
E\psi(t_0) \leq \varpi\phi(-0.75t_0, t_0) - \varpi\phi(0.75t_0, t_0) + C t_0^{-2} \int_{-0.75t_0}^{0.75t_0} \int_S |\phi|^2 r^2 \sigma dr^*.
\]

In view of Proposition 11.10 it suffices to show that

\[
\int_{-0.75t_0}^{0.75t_0} \int_S |\phi|^2 r^2 \sigma dr^* \leq CE\phi(t_0).
\]

We will rely on the one-dimensional inequality

\[
\int_{-a}^{a} |f(x)|^2 dx \leq Ca^2 \left( \int_{-a}^{a} |\partial_x f(x)|^2 dx + \int_{-1}^{1} |f(x)|^2 dx \right).
\]

Applying this to the function \( r\phi \), and then integrating over \( S^2 \), we obtain

\[
\int_{-0.75t_0}^{0.75t_0} \int_S |\phi|^2 r^2 \sigma dr^* \leq C t_0^2 \left( \int_{-0.75t_0}^{0.75t_0} \left( |\partial_r \phi|^2 + \frac{(1 - \mu)^2}{r^2} |\phi|^2 \right) r^2 \sigma dr^* \right) + \int_{-1}^{1} \int_{-\frac{1}{2}}^{\frac{1}{2}} |\phi|^2 r^2 dr^* \leq CE\phi(t_0),
\]

where the last inequality follows from  and .
12 Local observers’ uniform energy boundedness

For any $v_2 \geq 1$ construct a rectangle $\tilde{R}$ and triangle $\mathcal{R}$ as in the figure of Proposition 11.4 with $v_1 = 1$. Let

$$H_\phi = \int_{-\infty}^{\infty} \int_{S^2} r^2(1 - \mu)^{-\frac{1}{2}} ((\partial_t \phi)^2 + (\partial_r \phi)^2 + (1 - \mu)|\nabla \phi|^2) (1, r^*, \sigma_{S^2}) dr^* d\sigma_{S^2}.$$  

Similar to $E_{\phi, \omega}, E_{\phi, \phi, \omega}$ etc. we define $H_{\phi, \omega}, H_{\phi, \phi, \omega}$ etc. Note that $H_{\phi, \phi, \phi, \omega} \leq \bar{E}_0$. By Cauchy stability\footnote{To see this, one must actually change coordinates.}, we have that

$$F_{\phi}^Y ([u_1, \infty] \times \{1\}) \leq CH_{\phi}. $$

On the other hand, by Proposition 11.3 we have the uniform estimate

$$I_{\phi}^X (\mathcal{R}) \leq CE_{\phi}$$

and thus, by Proposition 11.5 we have

$$I_{\phi}^X (\tilde{R} \setminus \tilde{S}’ \cup \tilde{S}’') \leq C(E_{\phi} + H_{\phi}),$$  

(60)

$$F_{\phi}^Y ([u_1, \infty] \times \{v_2\}) \leq C(E_{\phi} + H_{\phi}),$$  

(61)

$$F_{\phi}^Y ([\infty] \times [1, v_2]) \leq C(E_{\phi} + H_{\phi}).$$

Note: Using the above bounds, the classical result of Kay and Wald can be reproven without exploiting discrete isometries of Schwarzschild. See Section 13.3.

13 Proof of Theorem 7.1

Set $t_0 = 1$, $t_{i+1} = 1.1t_i$, and let $u_i, v_i$ be defined by the relations $r(u_i, v_i) = r_0$, $u_i + v_i = t_i$. Define the sets

$$\tilde{R}_i = [u_i, \infty] \times [v_i, v_{i+1}],$$

$$\tilde{R}_i' = \{t_i \leq t \leq t_{i+1}\},$$

$$\tilde{S}_i' = \bigcup_{j=0}^{i} \tilde{R}_j',$$

$$\mathcal{X}_i = \tilde{R}_i' \cap \{r_0 \leq r \leq R\},$$

$$\mathcal{Y}_i = \bigcup_{j=0}^{i} \mathcal{X}_i.$$
13.1 Energy decay

Applying first Proposition 11.3 to $\tilde{Y}_i$, we obtain

$$I_\phi^X(\tilde{Y}_i) \leq CE_\phi,$$

$$I_{\phi,\omega}^X(\tilde{Y}_i) \leq CE_{\phi,\omega}.$$

Applying now Propositions 11.7 and 11.8 with $\tilde{R}' = \tilde{S}'_i$, $\mathcal{X}' = \mathcal{Y}_i$, we obtain

$$E^K_\phi(t_i) \leq Ct_iE_{\phi,\phi,\omega} + CE^K_\phi(1),$$

and similarly

$$E^K_\phi(t_i) \leq Ct_iE_{\phi,\phi,\omega} + CE^K_\phi(1).$$

Applying now Proposition 11.11 to $X_i$, we obtain

$$I_\phi^X(X_i) \leq Ct_i^{-1}E_{\phi,\phi,\omega} + Ct_i^{-2}E^K_\phi(1),$$

and also

$$I_{\phi,\omega}^X(X_i) \leq Ct_i^{-1}E_{\phi,\phi,\omega} + Ct_i^{-2}E^K_\phi(1).$$

Applying Proposition 11.12 and once again Proposition 11.7 but now with $\tilde{R}' = \tilde{R}'_i$, $\mathcal{X}' = \mathcal{X}_i$, we obtain

$$I_\phi^K(\tilde{R}'_i) \leq CE_{\phi,\phi,\omega} + Ct_i^{-1}E^K_\phi(1).$$

Summing over $i$, one obtains

$$I_\phi^K(Y_i) = \sum_{j=0}^{i} I_\phi^K(X_i) \leq C(\log t_i)E_{\phi,\phi,\omega} + CE^K_\phi(1).$$

Proposition 11.8 then gives

$$E^K_\phi(t_i) \leq C(\log t_i)E_{\phi,\phi,\omega} + CE^K_\phi(1),$$

and similarly,

$$E^K_{\phi,\omega}(t_i) \leq C(\log t_i)E_{\phi,\phi,\phi,\omega} + CE^K_{\phi,\phi,\omega}(1).$$

One repeats the procedure one final time to remove the log term:

$$E^K_\phi(t_i) \leq C(E_{\phi,\phi,\phi,\omega} + E^K_{\phi,\phi,\phi,\omega}(1)) = CE_{1}. \quad (62)$$

Note one also obtains then from Proposition 11.11 that

$$I_\phi^X(\mathcal{X}_i) \leq CE_{1}t_i^{-2}. \quad (63)$$

Proposition 11.10 62 and energy conservation now immediately imply 18.
13.2 Local observers’ energy decay

Refer to the following diagramme:

\[
\begin{align*}
\hat{R}_i & \quad t = t_i \\
\hat{R}_i & \quad v = v_{i+1} \\
\hat{R}_i & \quad u = u_{i+1} \\
\hat{R}_i & \quad t = t_{i+1}
\end{align*}
\]

From (60), we have already shown

\[
\hat{P}_\phi^Y(R_i) \leq C(E_\phi + H_\phi).
\]

Apply Proposition 11.6 together with (18) to obtain the existence of a \( \tilde{v}_i \) such that

\[
F^Y_\phi ([u_1, \infty] \times \{v\}) \leq C(E_\phi + H_\phi)(v_{i+1} - v_i)^{-1} \leq C(E_\phi + H_\phi)t_i^{-1}. \quad (64)
\]

Now let \( \tilde{u}_i \) be defined by \( r(\tilde{u}_i, \tilde{v}_i) = r_0 \), and construct rectangle \( \tilde{R}_i \) and triangle \( \hat{R}_i \), as depicted:

\[
\begin{align*}
\tilde{R}_i & \quad t = t_i \\
\tilde{R}_i & \quad v = v_{i+1} \\
\tilde{R}_i & \quad u = u_{i+1} \\
\tilde{R}_i & \quad t = t_{i+1}
\end{align*}
\]

By Proposition 11.5 and (64), we have that

\[
\varpi_\phi(\tilde{u}_i, \tilde{v}_i) - \varpi_\phi(\infty, \tilde{v}_i) \leq F^Y_\phi ([u_1, \infty] \times \{v_i\}) \leq C(E_\phi + H_\phi)t_i^{-1}.
\]

On the other hand, by (63) and the fact that \( u \sim v \sim t_i \) on \( \hat{R}_i \), we have

\[
\varpi_\phi(\tilde{u}_i, v_{i+1}) - \varpi_\phi(\tilde{u}_{i+1}, \tilde{v}_i)) \leq C\bar{E}_2t_i^{-2}.
\]

Finally, from (63), we have the bound

\[
I^X_\phi(\hat{R}_i) \leq C\bar{E}_1t_i^{-2}.
\]

Thus, we obtain from Proposition 11.5 and the inequality \( E_\phi + H_\phi \leq \bar{E}_0 \), the bound

\[
F^Y_\phi ([u_1, \infty] \times \{v_{i+1}\}) \leq C(\bar{E}_1 + \bar{E}_0)t_i^{-1}.
\]

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We now repeat the procedure to obtain

\[ F_Y^Y([u_i, \infty] \times \{v_i\}) \leq C(\tilde{E}_1 + \tilde{E}_0) v_i^{-2}. \]

Further application of Proposition 11.5 now yields

\[ F_Y^Y([u, \infty] \times \{v\}) \leq C(\tilde{E}_1 + \tilde{E}_0) v^{-2}, \tag{65} \]

\[ F_Y^Y(\{u\} \times [v, \tilde{v}]) \leq C(\tilde{E}_1 + \tilde{E}_0) v^{-2}. \tag{66} \]

Note that, by Proposition 11.4, this implies in particular, that if \((u, \tilde{v})\) is such that \(r(u, \tilde{v}) \leq r_0\) and if \((\tilde{u}, v)\) is such that \(r(\tilde{u}, v) \leq r_0\) then

\[ \int_{\tilde{u}}^{\tilde{u}} \int_{S^2} |\nabla \phi|^2 d\sigma_{S^2} dv \leq C(\tilde{E}_1 + \tilde{E}_0) v^{-2}, \tag{67} \]

\[ \int_{\tilde{u}}^{\tilde{u}} \int_{S^2} \frac{(\partial_{\rho} \phi)^2}{1 - \mu} d\sigma_{S^2} dv \leq C(\tilde{E}_1 + \tilde{E}_0) v^{-2}. \tag{68} \]

Recall that these \(L^2\) estimates are not available in this region from the usual energy estimate.

More generally one easily shows the following

**Theorem 13.1.** Let \(S\) be an achronal hypersurface in the closure of the exterior. Then \(F_Y^Y(\mathcal{S}) \leq C(\tilde{E}_1 + \tilde{E}_0)(v_+(\mathcal{S}))^{-2}\), where \(v_+(\mathcal{S})\) is as in the statement of Theorem 7.2 and \(F_Y^Y(\mathcal{S})\) denotes the flux of \(T_{\alpha\beta}Y^\alpha\) through \(\mathcal{S}\).

The above theorem applies in particular to subsets of the event horizon \(\mathcal{H}^+\).

### 13.3 Uniform boundedness of \(\phi\)

We now reprove the classical result of Kay and Wald stated as estimate 11 of Theorem 7.1. First consider the region \(r \geq r_0\). We have

\[ |\phi(t, r^*, \omega)| \leq \int_{r^*}^{\infty} |\partial_{r^*} \phi(t, \rho, \omega)| d\rho \leq r^{-\frac{1}{2}} \left( \int_{r^*}^{\infty} |\partial_{r^*} \phi(t, \rho, \omega)|^2 r^2(\rho) d\rho \right)^{\frac{1}{2}}. \]

By the Sobolev embedding on \(S^2\),

\[ \int_{r^*}^{\infty} |\partial_{r^*} \phi(t, \rho, \omega)|^2 r^2(\rho) d\rho \leq C \int_{r^*}^{\infty} (|\partial_{r^*} \phi(t, \rho, \omega)|^2 + |\partial_{r^*} \Omega \phi(t, \rho, \omega)|^2 \]

\[ + |\partial_{r^*} \Omega^2 \phi(t, \rho, \omega)|^2) r^2(\rho) d\sigma_{S^2} d\rho \]

\[ \leq CE_{\phi, \phi, \phi, \phi, \omega}. \]

Thus, for \(r \geq r_0\)

\[ |\phi|^2 \leq Cr^{-1} E_{\phi, \phi, \phi, \phi, \omega}. \tag{69} \]
In the region \( r \leq r_0 \) we define \( \tilde{u}(v) \) so that \( r(\tilde{u}(v), v) = r_0 \). Then

\[
|\phi(u, v, \omega)| \leq |\phi(\tilde{u}(v), v, \omega)| + \int_{\tilde{u}(v)}^{u} |\partial_u \phi(u', v, \omega)| \, du' \\
\leq |\phi(\tilde{u}(v), v, \omega)| + c \left( \int_{\tilde{u}(v)}^{u} \left( \frac{(\partial_u \phi(u', v, \omega))^2}{1 - \mu} \right) \, du' \right)^{\frac{1}{2}}.
\]

Since \( r(\tilde{u}(v), v) = r_0 \), by (69) we have

\[
|\phi(\tilde{u}(v), v, \omega)|^2 \leq C r_0^{-1} E_{\phi, \phi, \phi}.
\]

On the other hand, by the Sobolev embedding, we have

\[
\int_{\tilde{u}(v)}^{\infty} \frac{(\partial_u \phi(u', v, \omega))^2}{1 - \mu} \, du' \leq C \int_{\tilde{u}(v)}^{\infty} \left( (\partial_u \phi(u', v, \omega))^2 + (\partial_u \Omega \phi(u', v, \omega))^2 \right) \, dv \\
+ (\partial_u \Omega^2 \phi(u', v, \omega))^2 \, d\sigma_{S^2} \, du' \\
\leq C (E_{\phi, \phi, \phi} + H_{\phi, \phi, \phi}),
\]

where the last inequality follows from Proposition 11.4 and 61. Thus, for \( r \leq r_0 \),

\[
|\phi|^2 \leq C (E_{\phi, \phi, \phi} + H_{\phi, \phi, \phi}).
\]

Combining the estimates for \( r \geq r_0 \) and \( r \leq r_0 \) and observing that \( \tilde{E}_0 = E_{\phi, \phi, \phi} + H_{\phi, \phi, \phi} \) we obtain the desired result.

### 13.4 Pointwise decay for \( \phi \)

#### 13.4.1 \( |\phi| \leq C v_{+}^{-1} \) decay near \( H^+ \)

For a fixed large \( \hat{R} \) consider the region

\[
\mathcal{U} = \{(u, v) \in \{ r \leq \hat{R} \} : |\phi(\tilde{u}, \tilde{v}, \omega)| < \hat{C} \tilde{v}^{-1}, \forall \tilde{v}, \tilde{v} \leq v, \ r(\tilde{u}, \tilde{v}) \leq \hat{R}\},
\]

for a \( \hat{C} \) to be chosen later. \( \mathcal{U} \) is clearly open. Moreover, for sufficiently large \( \hat{C} \), \( \mathcal{U} \supset \{ r \leq \hat{R} \} \cap \{ v \leq 2 \} \). We will show that \( \mathcal{U} \) is closed, and thus, coincides with \( r \leq \hat{R} \). For this, clearly it suffices to show

\[
|\phi| \leq (\hat{C}/2) v_{+}^{-1}
\]

in \( \mathcal{U} \cap \{ v \geq 2 \} \).

Let \( (\tilde{u}, \tilde{v}) \in \mathcal{U} \cap \{ v \geq 2 \} \), and consider the null segment \( \{ \tilde{u} \} \times [\tilde{v} - 1, \tilde{v}] \). Note that this null segment is contained in \( r \leq \hat{R} \). We have a Sobolev inequality

\[
|\phi|^2(u, v, \omega) \leq K \int_{\tilde{v} - 1}^{\tilde{v}} \int_{S^2} \left( |\phi|^2 + |\partial_{\tilde{v}} \phi|^2 + |\nabla \phi|^2 + |\nabla \partial_{\tilde{v}} \phi|^2 \right) + \left( |\nabla^2 \phi|^2 + |\nabla^2 \partial_{\tilde{v}} \phi|^2 \right) \, d\sigma_{S^2} \, dv,
\]

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in \( r \leq \hat{R} \). Our energy decay estimate (18) and our local observers’ energy decay estimate (66), in particular (67), imply that, after commuting the equation twice with angular momentum operators \( \Omega \),

\[
\int \int_{\bar{v} - 1} \left( |\partial_v \phi|^2 + |\nabla \phi|^2 + |\nabla \partial_v \phi|^2 + |\nabla \nabla \phi|^2 + |\nabla \nabla \partial_v \phi|^2 \right) d\sigma_{\mathbb{S}^2} dv \leq \bar{E}_2 \bar{v}^{-2},
\]

in \( r \leq \hat{R} \). Choosing \( \hat{C} \) sufficiently large, it remains to show, say that

\[
\int \int_{\bar{v} - 1} |\phi|^2 (u, v, \omega) d\sigma_{\mathbb{S}^2} dv \leq \left( \frac{\hat{C}^2}{8K} \right) \bar{v}^{-2}.
\]

Let \( u_0(v) \) be defined by \( r^*(u_0(v), v) = v - \check{v} + \hat{R}^* + 1 \), in particular \( u_0(v) + v = \check{v} - \hat{R}^* - 1 = t \) is independent of \( v \). We have

\[
\int \int_{\bar{v} - 1} |\phi|^2 (u, v, \omega) d\sigma_{\mathbb{S}^2} dv = \int \int_{\bar{v} - 1} |\phi|^2 (u_0, v, \omega) d\sigma_{\mathbb{S}^2} dv + 2 \int \int_{v_1 u_0} \int \partial_v \phi \cdot d\sigma_{\mathbb{S}^2} d\sigma_{\mathbb{S}^2} dv
\]

\[
\leq \int \int_{\bar{v} - 1} |\phi|^2 (u_0, v) d\sigma_{\mathbb{S}^2} dv + \int \int_{\bar{v} - 1} \int (\partial_u \phi)^2 \frac{1}{1 - \mu} d\sigma_{\mathbb{S}^2} d\sigma_{\mathbb{S}^2} dv + \int \int_{\bar{v} - 1} (1 - \mu) \phi^2 d\sigma_{\mathbb{S}^2} d\sigma_{\mathbb{S}^2} dv.
\]

From the bounds (58), (62), we have

\[
\int \int_{\bar{v} - 1} |\phi|^2 (u_0(v), v) d\sigma_{\mathbb{S}^2} dv \leq \bar{C} E_1 \check{v}^{-2}.
\]

On the other hand, from our local observers’ energy estimate (68), we have

\[
\int \int_{u_0} (\partial_u \phi)^2 \frac{1}{1 - \mu} d\sigma_{\mathbb{S}^2} du \leq C (\bar{E}_1 + \bar{E}_0) \check{v}^{-2}.
\]

Thus, choosing \( \hat{C} \) sufficiently large, we are left with showing, say, the bound

\[
\int \int_{\bar{v} - 1} (1 - \mu) \phi^2 d\sigma_{\mathbb{S}^2} d\sigma_{\mathbb{S}^2} dv \leq \left( \frac{\hat{C}^2}{16K} \right) \check{v}^{-2}.
\]

We have the inclusion

\[
[u_0(v), \infty) \times [\check{v} - 1, \check{v}] \subset \{(t, r^*) : \check{v} - \hat{R}^* - 1 \leq t < \infty, \ \check{v} - t - 1 \leq r^* \leq \check{v} - t\}.
\]
Thus,

$$\int_{\tilde{v}^{-1} u_0(v)}^{\tilde{v}} \int_{\tilde{v}^{-1} - 1}^{\tilde{v}^{-1} - t - 1} (1 - \mu) |\phi|^2 d\sigma_{S^2} du' dv \leq C \int_{\tilde{v}^{-1} - 1}^{\tilde{v}^{-1} - t - 1} \int_{\tilde{v}^{-1} - t - 1}^{\tilde{v}^{-1}} (1 - \mu) |\phi|^2 d\sigma_{S^2} dr^* dt.$$  

We may write

$$\int_{\tilde{v}^{-1} - 1}^{\tilde{v}^{-1} - t - 1} \int_{\tilde{v}^{-1} - t - 1}^{\tilde{v}^{-1}} (1 - \mu) |\phi|^2 d\sigma_{S^2} dr^* dt = \int_{\tilde{v}^{-1} - 1}^{\tilde{v}^{-1} - t - 1} \int_{\tilde{v}^{-1} - t - 1}^{\tilde{v}^{-1}} (1 - \mu) |\phi|^2 d\sigma_{S^2} dr^* dt$$

$$+ \int_{\tilde{v}^{-1} - 1}^{\tilde{v}^{-1} - t - 1} \int_{\tilde{v}^{-1} - t - 1}^{\tilde{v}^{-1}} (1 - \mu) |\phi|^2 d\sigma_{S^2} dr^* dt.$$  

The first integral can be estimated with the help of (58), (62).

$$\int_{\tilde{v}^{-1} - 1 + A}^{\tilde{v}^{-1} - t + A} \int_{\tilde{v}^{-1} - 1}^{\tilde{v}^{-1} - t - 1} (1 - \mu) |\phi|^2 d\sigma_{S^2} dr^* dt \leq A(1 + |\tilde{v} - \tilde{R}^* - 1|)^{-2} C \sup_{t \in [\tilde{v}^{-1} - 1 + A, \tilde{v}^{-1} - t - 1 + A]} E^K_1(t)$$

$$\leq A C \tilde{E}_1(\tilde{R}^*)^2 \tilde{v}^{-2}. \quad (72)$$

For the second integral we use the fact that, for $A$ large enough, the region of integration above is contained in $\mathcal{U}$, as $r^*(u, v) \leq \tilde{R}^* - A + 1$ and $\tilde{v} - 1 \leq v \leq \tilde{v}$, so we have

$$|\phi| \leq \tilde{C} v^{-1}.$$  

On the other hand, we have $(1 - \mu) \leq h e^{r^*}$ in this region. Thus

$$\int_{\tilde{v}^{-1} - 1 + A}^{\tilde{v}^{-1} - t + A} \int_{\tilde{v}^{-1} - 1}^{\tilde{v}^{-1} - t - 1} (1 - \mu) |\phi|^2 d\sigma_{S^2} dr^* dt \leq \tilde{C}^2 \tilde{v}^{-2} \int_{-\infty}^{\tilde{A} + \tilde{R}^* + 1} e^{-A + \tilde{R}^* + 1} dr^* \leq h \tilde{C}^2 e^{-A + \tilde{R}^* + 1} \tilde{v}^{-2}.$$  

Choosing $A$ to be sufficiently large, say $A = 10 + \log h + \tilde{R}^*$, and $\tilde{C}$ large as before, and satisfying in addition, say

$$\tilde{C} \geq 32 AK(\tilde{R}^*)^2 (\tilde{E}_2 + \tilde{E}_1 + \tilde{E}_0),$$

we obtain (74), as desired. Note that $\tilde{E}_2$ dominates $\tilde{E}_1$ and $\tilde{E}_0$.

### 13.4.2 Decay in $r \geq \tilde{R}$

We turn to $r \geq \tilde{R}$. First consider the region $\{r \geq \tilde{R}\} \cap \{u \geq 1\}$. By the Sobolev inequality,

$$r^2 |\phi(u, v, \omega)|^2 \leq C \int_{S^2} |\phi|^2 r^2 d\sigma_{S^2} + C \int_{S^2} (|r \nabla \phi|^2 + |r^2 \nabla \phi|^2) r^2 d\sigma_{S^2}.$$  

40
For any $\hat{R} \leq \hat{r} \leq \hat{R} + 1$, for $k = 0, 1, 2$, and any $r$ we have
\[
\int_{S^2} |\Omega^k \phi|^2 r^2(t, r, \omega)d\sigma_{S^2} \leq \int_{S^2} |\Omega^k \phi|^2 r^2(t, \tilde{r}^*, \omega)d\sigma_{S^2} + C \int_{\tilde{r}^*}^{r^*} \int_{S^2} \left( |\partial_r \Omega^k \phi| \right) |\Omega^k \phi| + r^{-1} |\Omega^k \phi|^2 r^2 dr d\sigma_{S^2}.
\]
Applying a pigeonhole argument in $\tilde{r}$ and (58), (62) with $\Omega^k \phi$, we obtain that $\tilde{r}$ can be chosen so as for
\[
\int_{S^2} |\Omega^k \phi|^2 r^2(t, \tilde{r}^*, \omega)d\sigma_{S^2} \leq \bar{E}_2 t^{-2}.
\]
On the other hand, from (58), (62), and (62) applied to $\Omega^k \phi$, we have
\[
\int_{r^*_0}^{r^*} \int_{S^2} |\Omega^k \phi|^2 r^2 dr d\sigma_{S^2} \leq \bar{E}_2 r t^{-2},
\]
\[
\int_{r^*_0}^{r^*} \int_{S^2} |\partial_r \Omega^k \phi| |\Omega^k \phi| r^2 dr d\sigma_{S^2} \leq \bar{E}_2 r t^{-1} u^{-1}.
\]
Therefore, since in the region $u \geq 1$ we have that (say) $t \geq r/2$, it follows that
\[
|\phi| \leq \bar{E}_2 (r t u)^{-\frac{1}{2}} \leq 2\bar{E}_2 r^{-1} u^{-\frac{1}{2}}.
\]
On the other hand, since $\max(r, u) \geq v/2$ we also obtain
\[
|\phi| \leq \bar{E}_2 (1 + |v|)^{-1}.
\]
The region $\{r \geq r_0\} \cap \{u \leq 1\} \cap \{t \geq 1\}$ can be treated similarly by integrating out to spatial infinity using the estimates
\[
\int_{r^*_0}^{\infty} \int_{S^2} |\partial_r \Omega^k \phi| |\Omega^k \phi| r^2 dr d\sigma_{S^2} \leq \bar{E}_2 |u|^{-1},
\]
\[
\int_{r^*_0}^{\infty} \int_{S^2} |\Omega^k \phi|^2 r dr d\sigma_{S^2} \leq \bar{E}_2 r^{-1}.
\]
In this region, one obtains
\[
|\phi| \leq C r^{-1} (1 + |u|)^{-\frac{1}{2}}, \quad |\phi| \leq \bar{E}_2 v^{-1} (1 + |u|)^{-\frac{1}{2}}.
\]
This completes the proof.
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