BIPARTITE CHEBYSHEV POLYNOMIALS
AND ELLIPTIC INTEGRALS EXPRESSIBLE
BY ELEMENTARY FUNCTIONS

KAZUTO ASAI

Center for Mathematical Sciences, University of Aizu,
Aizu-Wakamatsu, Fukushima 965-8580, Japan
e-mail: k-asai@u-aizu.ac.jp
Tel. 0242-37-2644 (Office), 0242-37-2752 (Fax)

Abstract
The article is concerned with polynomials \( g(x) \) whose graphs are “partially packed” between two horizontal tangent lines. We assume that most of the local maximum points of \( g(x) \) are on the first horizontal line, and most of the local minimum points on the second horizontal line, except several “exceptional” maximum or minimum points, that locate above or under two lines, respectively. In addition, the degree of \( g(x) \) is exactly the number of all extremum points +1. Then we call \( g(x) \) a multipartite Chebyshev polynomial associated with the two lines.

Under a certain condition, we show that \( g(x) \) is expressed as a composition of the Chebyshev polynomial and a polynomial defined by the \( x \)-component data of the exceptional extremum points of \( g(x) \) and the intersection points of \( g(x) \) and the two lines. Especially, we study in detail bipartite Chebyshev polynomials, which has only one exceptional point, and treat a connection between such polynomials and elliptic integrals.

1. Introduction

The article is concerned with construction of real polynomials \( g(x) \) whose graphs are partially packed between two horizontal tangent lines. Let \( l_1, l_2 \) be horizontal lines arranged downwards. We consider the case that most of the local maximum points of \( g(x) \) are on \( l_1 \) and most of the local minimum points on \( l_2 \), except that several local maximum or minimum points, which we call the exceptional maximum or minimum points, are located above or under the two lines, respectively. Suppose the degree of \( g(x) \) is exactly the number of all extremum points +1. Then we call \( g(x) \) a multipartite Chebyshev polynomial associated with the two lines. The simplest case is that \( g(x) \) has no exceptional extremum points, then \( g(x) \) is essentially the Chebyshev polynomial.

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of the first kind $T_n(x)$. For basics and several recent results of this polynomial, see [1, 2, 4, 5, 6, 7, 9, 10].

First of all, we consider the above-mentioned “Chebyshev polynomial” case. Let $m$ be a positive real constant. Let the horizontal tangent lines $l_1, l_2$ of $g(x)$ be $y = \pm m$ without loss of generality. Suppose $g(x)$ has positive degree $n$ and no exceptional extremum points. $g(x) - m$ has zeros at every local maximum point of $g(x)$, and also $g(x) + m$ has zeros at every local minimum point of $g(x)$. Additional two zeros of $g(x) - m$ or $g(x) + m$ exist, and we can set them $x = \pm a$ without loss of generality. Hence we have

$$n^2(g^2 - m^2) = (x^2 - a^2)g'^2,$$

and therefore, for $-a \leq x \leq a$,

$$\int \frac{dg}{\sqrt{m^2 - g^2}} = \pm n \int \frac{dx}{\sqrt{a^2 - x^2}}.$$  \hspace{1cm} (2)

Letting $g = m \cos \varphi$, $x = a \cos \theta$, and noting that $|x| \to a$ implies $|g| \to m$, we have

$$-\varphi = \mp n\theta + k\pi. \quad (k \in \mathbb{Z})$$

$$\therefore \quad g = \pm m \cos \left( n \arccos \frac{\varphi}{m} \right) = \pm m T_n \left( \frac{\varphi}{m} \right).$$  \hspace{1cm} (3)

We can treat similarly a multipartite Chebyshev polynomial $g(x)$ of positive degree $n$. Let the horizontal tangent lines $l_1, l_2$ of $g(x)$ be $y = \pm m$. Let $\alpha_1, \ldots, \alpha_r$ be the $x$-components of the intersection (not tangent) points of the graph of $g(x)$ and $l_1$ or $l_2$, and let $\beta_1, \ldots, \beta_\ell$ be the $x$-components of the exceptional extremum points of $g(x)$ outside the two lines. By definition, $r$ is even and $r = 2\ell + 2$. For convenience, we call the data $(\alpha_1, \ldots, \alpha_r; \beta_1, \ldots, \beta_\ell)$ the outside data of $g(x)$, and set $p(x) = (x - \alpha_1) \ldots (x - \alpha_r)$, $q(x) = (x - \beta_1) \ldots (x - \beta_\ell)$. We have

$$n^2(q(x))^2(g^2 - m^2) = p(x)g'^2.$$  \hspace{1cm} (4)

Hence as above, for $x$ such that $p(x) < 0$,

$$\int \frac{dg}{\sqrt{m^2 - g^2}} = \pm n \int \frac{q(x)}{\sqrt{-p(x)}} \frac{dx}{dx}.$$  \hspace{1cm} (5)

In general, the RHS is a hyper-elliptic integral. Also, there is no assurance that $p(x)$ or $q(x)$ has a polynomial solution $g(x)$. Now we assume that for some positive divisor $s$ of $n$, there exists a multipartite Chebyshev polynomial $u(x)$ of degree $s$ that shares the outside data with $g(x)$. Then $u$ satisfies the similar equation as (4):

$$s^2(q(x))^2(u^2 - m^2) = p(x)u'^2.$$  \hspace{1cm} (6)

By (4), (6), noting again that $x \to \alpha_i$ implies $|g|, |u| \to m$, and letting $g = m \cos \varphi$, $u = m \cos \theta$,

$$-\varphi = \int \frac{dg}{\sqrt{m^2 - g^2}} = \pm \frac{n}{s} \int \frac{du}{\sqrt{m^2 - u^2}}$$

$$= \pm \frac{n}{s} \theta + k\pi. \quad (k \in \mathbb{Z})$$

$$\therefore \quad g = \pm m \cos \left( \frac{n}{s} \arccos \frac{\varphi}{m} \right) = \pm m T_n \left( \frac{\varphi}{m} \right).$$  \hspace{1cm} (7)

Equations (4), (6) are necessary conditions for $g$ and $u$, respectively, and they assure that the non-exceptional extremum points of them are located on $l_1, l_2$, but do not
Theorem 1. Let $n$ be a positive integer and $s$ be a positive divisor of $n$. Let $g$ and $u$ be multipartite Chebyshev polynomials of degree $n$ and $s$, respectively, associated with the lines $y = \pm m$, and sharing a common outside data. Then we have $g = \pm mT_{\frac{n}{s}}(\frac{x}{m})$.

2. Bipartite Chebyshev polynomials

A multipartite Chebyshev polynomial $g(x)$ with a unique exceptional extremum point is called a bipartite Chebyshev polynomial. In this case there are four intersection points of $g(x)$ and two tangent lines $y = \pm m$. Suppose $g$ has positive degree $n$ and let the outside data of $g$ be $(\alpha_1, \alpha_2, \alpha_3, \alpha_4; 0)$ ($\alpha_1 > \alpha_2 > 0 > \alpha_3 > \alpha_4$) without loss of generality. If the data satisfies symmetric condition: $\alpha_1 = -\alpha_4$, $\alpha_2 = -\alpha_3$, and $n$ is even, then $g$ is a special case of (7) with quadratic $u$, say, $g = \pm mT_{\frac{n}{s}}(\frac{2x^2 - \alpha_1^2 - \alpha_2^2}{\alpha_1^2 - \alpha_3^2})$.

We now proceed to the case without symmetric condition. In accordance with the outside data, we have

$$n^2x^2(g^2 - m^2) = p(x)g'^2.$$  \hspace{1cm} (8)

Let $s$ be a positive divisor $s$ of $n$ (possibly $s = n$) such that there exists a bipartite Chebyshev polynomial $u$ of degree $s$ sharing a common outside data with $g$, then

$$s^2x^2(u^2 - m^2) = p(x)u'^2.$$  \hspace{1cm} (9)

By Theorem 1, $g$ is represented as $g = \pm mT_{\frac{n}{s}}(\frac{x}{m})$. Thus we study the solution $u$ of equation (9). Since $u'(0) = 0$, we can set

$$u = a_0 + a_2x^2 + a_3x^3 + \cdots + a_xx^s.$$  \hspace{1cm} (10)

Dividing both sides of (9) by $x^2$ and differentiating with respect to $x$, we have

$$2s^2u = 2u''\tilde{p} + u'\tilde{p'},$$  \hspace{1cm} (11)

where $\tilde{p} = p(x)/x^2$. It follows from (11), by setting $\tilde{p} = x^2 + c_1x + c_2 + c_3x^{-1} + c_4x^{-2}$ and comparing the coefficients of $x^k$, that

$$a_k = \frac{1}{2(s^2 - k^2)} \sum_{i=1}^{4} (k + i)(2k + i)c_ia_{k+i} \quad (k = 0, 1, \ldots, s - 1),$$  \hspace{1cm} (12)

where we promise $a_1 = a_{s+1} = a_{s+2} = a_{s+3} = 0$. By (12), the coefficients are determined step by step from $a_s$ to $a_3, a_2, a_0$ in the form:

$$a_k = F_k(c_1, c_2, c_3, c_4)a_s,$$  \hspace{1cm} (13)

where $F_k$ is a polynomial in $c_1, c_2, c_3, c_4$ with rational coefficients for every $k = 0, 2, 3, \ldots, s$. The rest of the condition (9) is the equation (13) for $k = 1$:

$$F_1(c_1, c_2, c_3, c_4) = 0,$$  \hspace{1cm} (14)

and the $x^2$ terms of (9):

$$s^2(a_0^2 - m^2) = 4a_2^2c_4 \iff (s^2F_0^2 - 4c_4F_2^2)a_s^2 = s^2m^2.$$  \hspace{1cm} (15)
Lemma 1. For every $k = 0, 1, \ldots, s$, the polynomial $F_k = F_k(c_1, c_2, c_3, c_4)$ has the following properties:

(i) The coefficient of every term of $F_k$ is positive.
(ii) $F_{s-k}$ consists of the terms of type $c_i c_{i_2} \cdots c_{i_r}$, such that $i_1 + i_2 + \cdots + i_r = k$.
(iii) Every possible term of type $c_i c_{i_2} \cdots c_{i_r}$ with $i_1 + i_2 + \cdots + i_r = k$ appears in $F_{s-k}$ except that $F_0$ does not contain the term $c_1^s$.

Proof. For convenience, write $c_{i_1} c_{i_2} \cdots c_{i_r} = c_{\lambda}$, where $\lambda = (\lambda_1, \ldots, \lambda_r)$ is the weakly decreasing rearrangement of $i_1, \ldots, i_r$. Then $\lambda$ is a partition of $k$ (denoted by $\lambda \vdash k$), $k = i_1 + \cdots + i_r$. Each $\lambda_i$ is called a part of $\lambda$, and $r$ is called the length of $\lambda$ denoted by $\ell(\lambda)$. There is a unique partition of zero, denoted by $\varnothing$, consisting of no parts, having length 0, and we promise that $c_\varnothing = 1$. There are no partitions of negative integers.

By the fact that all coefficients of the linear recurrence relation (12) are positive and by (13), we see (i). Next, we prove (ii) and (iii) by induction on $k$. When $k = 0$, we have $F_s = 1$ which satisfies the proposition. For the induction step, suppose, for every nonnegative integer $j < k$,

$$F_{s-j} = \sum_{\lambda^j; \lambda_1 \leq 4} m_\lambda c_\lambda$$

(16)

with some positive coefficients $m_\lambda$. Then by (12), (13), putting $(s-k+i)(2s-2k+i) = m_i^{(k)}$, we have

$$F_{s-k} = \sum_{i=1}^{4} m_i^{(k)} c_i F_{s-k+i} = \sum_{i=1}^{4} \sum_{\lambda^k-i; \lambda_1 \leq 4} m_i^{(k)} m_\lambda c_i c_\lambda$$

(17)

$$= \sum_{\lambda^k; \lambda_1 \leq 4} m_\lambda c_\lambda.$$

Here, as every partition $\lambda$ of $k$ with $\lambda_1 \leq 4$ is composed of the parts $\lambda_j$ such that $1 \leq \lambda_j \leq 4$, we can reduce the partition $\lambda$ to a partition of $k - i$ by subtracting some part $1 \leq i \leq 4$. Therefore, together with the positivity of the coefficients $m_i^{(k)} m_\lambda$, every coefficient $m_\lambda$ is shown to be positive. Only the case $F_0$, however, is represented as

$$F_0 = \sum_{i=2}^{4} m_i^{(s)} c_i F_i = \sum_{i=2}^{4} \sum_{\lambda^{s-i}; \lambda_1 \leq 4} m_i^{(s)} m_\lambda c_i c_\lambda$$

(18)

because $a_1$ is defined to be 0. From this, it follows that $F_0$ does not contain only the term $c_{(1, \ldots, 1)} = c_1^s$. \qed

Reviewing (17), we have $m_\lambda = \sum_{i=1}^{4} m_i^{(k)} m_{\lambda \ominus(i)}$ for $\lambda \vdash k$, where $\lambda \ominus (i)$ denotes a partition obtained by removing a part of size $i$ from $\lambda$, and set $m_{\lambda \ominus(i)} = 0$ in case that $\lambda$ has no parts of size $i$. Iterating this induction, we can represent the coefficient $m_\lambda$ as follows.
Lemma 2. Let $S(\lambda)$ denote the set of all distinct permutations $i = (i_1, \ldots, i_r)$ of $\lambda$. If $\lambda$ is a partition of the integer less than $s$, then

$$m_\lambda = \sum_{i \in S(\lambda)} \prod_{t=1}^{r} m_{t}^{(i_1 + \cdots + i_r)},$$

(19)

while if $\lambda \vdash s$, $S(\lambda)$ in this formula should be replaced with $S'(\lambda)$, the set of all distinct permutations $(i_1, \ldots, i_r)$ of $\lambda$ where $i_r > 1$.

On the basis of the above-mentioned method, we construct the polynomial $u$ as described in the procedure below:

(i) Choose an arbitrary positive constant $m$, and arbitrary real coefficients $c_2, c_3, c_4$ of $p$ such that $c_2 < 0 < c_4$.

(ii) Represent all coefficients $a_k$ of $u$ as a polynomial in $c_1, \ldots, c_4$ and $a_s$ by using (13), (16) and (19).

(iii) By (14), determine $c_1$ depending on $c_2, c_3, c_4$.

(iv) Determine $a_k$ for all $k = 0, 2, 3, \ldots, s - 1$.

(v) Also determine $a_s$ by (15).

Unfortunately, in the steps (iii) and (v), $c_1$ is possibly an imaginary number, and $a_s$ may be also imaginary, or even does not exist if $s^2 F_0^2 - 4c_4 F_2^2 = 0$. Now we avoid those difficulties by handling the coefficients $c_3, c_4$ ($c_2 < 0$ is fixed). Letting $c_3, c_4 = 0$, equation (19) becomes

$$s^2 x^2 (u^2 - m^2) = x^2 (x^2 + c_1 x + c_2) u^2,$$

(20)

which is clearly reduced to (11), and we have $u = \pm m T_s \left( \frac{x - h}{a} \right)$. But by the continuity of all equations in the above process (i)—(v), we can show that for $c_3$ and $c_4$ sufficiently close to 0, $c_1$ and $a_s$ are determined to be real numbers as follows. Equation (20) has $s - 1$ possible values of $c_1$ for given $c_2$, because the condition (14), i.e., $u'(0) = 0$ obliges one of the extremum points of $u$ to be located on the $y$-axis, which allows $s - 1$ possible graphs of $u = u_1, \ldots, u_{s-1}$. While by Lemma 1, $F_1$ has a leading term $c_1^{s-1}$ with respect to $c_1$, and this has $s - 1$ distinct real roots $c_1$ for $c_3 = c_4 = 0$. Hence, again for $c_3$ and $c_4$ ($c_4 > 0$) sufficiently close to 0, the outside data of $u$ satisfies $c_1 > c_2 > 0 > c_3 > c_4$, and by the continuity of $F_1$, we have $s - 1$ distinct real values of $c_1$, and therefore $s - 1$ different $u$'s (also possible two $\pm a_s$'s), that are obtained graphically from $u_1, \ldots, u_{s-1}$ moving slightly the maximum or minimum point on $y$-axis upward or downward, respectively. Reducing the expression $g = \pm m T_{\frac{u}{s}} \left( \frac{\sqrt{s} - h}{a} \right)$ to $\pm m T_{\frac{s}{u}}(u)$, we have the following.

Theorem 2. For every positive integer $n$ and every positive divisor $s$ of $n$, there exists a bipartite Chebyshev polynomial of degree $n$ of the form $g = \pm m T_{\frac{u}{s}}(u)$ associated with the lines $y = \pm m$, where $u$ is also a bipartite Chebyshev polynomial of degree $s$ associated with the lines $y = \pm 1$ obtained by moving an arbitrary one of the maximum or minimum points of $\pm T_{\frac{u}{s}} \left( \frac{\sqrt{s} - h}{a} \right)$ upward or downward, respectively.
3. Elliptic integrals

It is known that, in general, elliptic integrals are not representable in terms of elementary functions. In this section, however, for a special kind of elliptic integral, we show that there are infinitely many representable cases, which are represented as compositions of the inverse trigonometric/hyperbolic functions and polynomials or compositions of the logarithm and polynomials. For general properties of elliptic integrals and elliptic functions, see e.g. [3, 8].

Focusing again on the integral solution of (9), a special case of the solution (5) (with \( g, n \) replaced by \( u, s \), respectively), for an arbitrary monic real quartic polynomial \( p(x) \),

\[
\int x \sqrt{-p(x)} \, dx = \pm \frac{1}{s} \arccos \frac{u}{m} + C \quad \text{ (in each interval such that } p(x) < 0 \text{ and } -m < u(x) < m) \\
\int x \sqrt{p(x)} \, dx = \pm \frac{1}{s} \arccos \frac{|u|}{m} + C \quad \text{ (in each interval such that } p(x) > 0) .
\]

Hence, if \( u \) is a polynomial (which should be of degree \( s \)), then the elliptic integrals on the LHS are represented as elementary functions. According to Section 2, for every positive integer \( s \), there exists a polynomial solution \( u \) of (9) of degree \( s \), under suitable condition for \( p(x) \), and in this case we obtain a bipartite Chebyshev polynomial \( u \), which satisfies both of (21). For more general \( p(x) \), if we ignore the outside data, and only suppose \( x^2 \notin p(x) \), then the polynomial solution \( u \) of (9) satisfies \( u'(0) = 0 \), and therefore (14) and (15) are satisfied, and conversely, they assure the existence of the polynomial solution.

For the first formula of (21), the domain \( D \) defined by \( p(x) < 0 \) is divided by the condition \( -m < u(x) < m \) into small intervals to avoid the singularities of \( \arccos x \), \( \pm 1 \). If we consider the RHS in \( D \), ignoring the integral constant and \( - \) sign at the beginning, the graph consists of zigzag-like lines packed between the horizontal lines \( y = \frac{\pi}{s} \) and \( x \)-axis, and a small piece of the graph between cusps matches the LHS with the appropriate sign. To complete the formula in \( D \), we should “join” the pieces of the RHS changing the signs and shifting the integral constant by an integral multiple of \( \frac{\pi}{s} \).

For the second formula of (21), we see that \( u(x) > m \) or \( u(x) < -m \) is not necessary to add to the condition \( p(x) > 0 \).

Suppose that \( u \) is a bipartite Chebyshev polynomial of degree \( s \) with the outside data \((\alpha_1, \alpha_2, \alpha_3, \alpha_4; 0)\) (\( \alpha_1 > \alpha_2 > 0 > \alpha_3 > \alpha_4 \)), then \( D \) is composed of the intervals \((\alpha_2, \alpha_1)\), \((\alpha_4, \alpha_3)\). Let the unique exceptional extremum point of \( u(x) \) be the \( k \)-th extremum point counted from the right. From the above consideration of the zigzag-like graph of the first formula, it follows that

\[
\omega_1 = \int_{\alpha_2}^{\alpha_1} \frac{x}{\sqrt{-p(x)}} \, dx = \frac{k}{s} \pi ; \quad \omega_2 = \int_{\alpha_4}^{\alpha_3} \frac{x}{\sqrt{-p(x)}} \, dx = \frac{k-s}{s} \pi ; \quad \omega_1 - \omega_2 = \pi . \quad (22)
\]

Forgetting the connection with multipartite Chebyshev polynomials, we can also deal with the equation similar to (8) for a polynomial \( g \) of positive degree \( n \):

\[
n^2 x^2 (g^2 + m^2) = p(x)g'^2 , \quad (23)
\]
where \( p(x) \) is a monic real quartic polynomial satisfying \( x^2 \mid p(x) \). In the same way as in Section 2, for some positive divisor \( s \) of \( n \), suppose there exists a polynomial \( u \) of degree \( s \) such that

\[
s^2x^2(u^2 + m^2) = p(x)u^2. \tag{24}\]

Letting \( g = m \sinh \varphi \), \( u = m \sinh \theta \) temporarily in the complex domain (to determine the constant of integration), and noting that \( p(x) \to 0 \) implies \( g, u \to \pm mi \), we have

\[
\varphi = \int \frac{dg}{\sqrt{m^2 + g^2}} = \pm \frac{n}{s} \int \frac{du}{\sqrt{m^2 + u^2}} = \pm \frac{n}{s} \theta.
\]

\[
\therefore \quad g = \pm m \sinh \left( \frac{n}{s} \arcsinh \frac{u}{m} \right) \equiv \pm m \tilde{T}_n \left( \frac{u}{m} \right). \tag{25}\]

Here, if \( g, u \to \pm mi \), then \( \varphi \to (\frac{\pi}{2} + k\pi)i \), \( \theta \to (\frac{\pi}{2} + k'\pi)i \) \((k, k' \in \mathbb{Z})\), but this implies that \( n/s \) is odd, which we now assume, and the (real) constant of integration vanishes. One sees that \( \tilde{T}_n(x) = \sinh (n \arcsinh x) \) is monotonously increasing, and a polynomial of degree \( n \) iff \( n \) is an odd positive integer.

Returning to (24), a change from (9) to (24) does not have an effect on the condition (14), while it turns (15) into

\[
s^2(a_0^2 + m^2) = 4a_0^2c_4 \iff (4c_4F_2^2 - s^2F_0^2)a_s^2 = s^2m^2. \tag{26}\]

For convenience, set \( d = s^2F_0^2 - 4c_4F_2^2 \). We see the signature of \( d \) discriminates the existence of \( a_s \) in (15) or (26). On the other hand, the solution of (23) is represented as

\[
\int \frac{x}{\sqrt{p(x)}} \, dx = \pm \frac{1}{s} \arcsinh \frac{u}{m} + C. \tag{27}\]

For the special case \( m = 0 \), one sees that

\[
\int \frac{x}{\sqrt{p(x)}} \, dx = \pm \frac{1}{s} \log |u| + C. \tag{28}\]

**Theorem 3.** Let \( s \) be a positive integer. For a real quartic polynomial \( p(x) = x^4 + c_1x^3 + c_2x^2 + c_3x + c_4 \) such that \( x^2 \mid p(x) \), the elliptic integral

\[
\int \frac{x}{\sqrt{\pm p(x)}} \, dx \tag{29}
\]

is represented as either (21) or (27) or (28) subject to \( d > 0 \) or \( d < 0 \) or \( d = 0 \), respectively, where \( u \) is a polynomial of degree \( s \) determined by (10), (13), (16), (19) and (15) or (26), if and only if (14) is satisfied.

Lastly, we study the set of quartic polynomials \( p(x) \) such that (29) is representable as in Theorem 3, depending on the degree \( s \) of \( u \). For even \( s \), the polynomial \( F_1(c_1, c_2, c_3, c_4) \) has the leading term \( m_{(1,...,1)}c_1^{s-1} \) with respect to \( c_1 \), and thus (as \( s - 1 \) is odd), for arbitrary \( c_2, c_3, c_4 \), we can find the real solution \( c_1 \) to (14). Therefore for given \( c_2, c_3, c_4 \), we have \( p(x) \) suitable for our representation as in Theorem 3. For \( s = 4k - 1 \), \( F_1 \) has the leading term \( m_{(2,...,2)}c_2^{2k-1} \) with respect to \( c_2 \), and therefore for arbitrary \( c_1, c_3, c_4 \), we have \( c_2 \) and \( p(x) \) as desired. For \( s = 8k - 3 \), \( F_1 \) has the leading term \( m_{(4,...,4)}c_4^{2k-1} \) with respect to \( c_4 \), and so for arbitrary \( c_1, c_2, c_3 \), we have \( c_4 \) and \( p(x) \) similarly. The rest case \( s = 8k + 1 \) gives no clear assurance for \( p(x) \).
Theorem 4. If $s$ is not congruent to 1 modulo 8, then for some $i_1, i_2, i_3$ $(1 \leq i_1 < i_2 < i_3 \leq 4)$, given $c_{i_1}, c_{i_2}, c_{i_3}$, we obtain $p(x)$ suitable for representation of \( (29) \) as in Theorem 3 by a polynomial $u$ of degree $s$.

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Center for Mathematical Sciences, University of Aizu, Aizu-Wakamatsu, Fukushima 965-8580, Japan
E-mail address: k-asai@u-aizu.ac.jp