Essential spectra of weighted composition operators induced by elliptic automorphisms

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Abstract
The spectrum of a weighted composition operator $C_{\psi,\varphi}$ that is induced by an automorphism has been investigated for over 50 years. However, many results are got only under the condition that the weight function $\psi$ is continuous up to the boundary. In this paper we study the spectra and essential spectra of $C_{\psi,\varphi}$ on weighted Bergman spaces when $\varphi$ is an elliptic automorphism, without the assumption that $\psi$ is continuous up to the boundary.

Keywords Weighted Composition operator · Essential spectrum · Elliptic automorphism · Weighted Bergman space · Maximal ideal space

Mathematics Subject Classification Primary 47B33; Secondary 46E15 · 47B48

1 Introduction

Let $D$ be the open unit disk in complex plane $\mathbb{C}$, and let $H(D)$ denote the set of all analytic functions on $D$. Given an analytic self-map $\varphi$ of $D$ and a function $\psi \in H(D)$, one can define a linear weighted composition operator $C_{\psi,\varphi}$ on $H(D)$ by

$C_{\psi,\varphi}f = \psi \cdot f \circ \varphi$

for all $f \in H(D)$. By taking $\psi = 1$ one obtains the composition operator $C_{\varphi}$. And putting $\varphi$ be the identity map on $D$, gives the analytic multiplication $M_\psi$. 
The spectral properties of bounded weighted composition operators on kinds of analytic function spaces have been actively investigated over the past 50 years. In 1978, Kamowitz [11] determined the spectrum of $C_{\psi,\varphi}$ on the disk algebra $A(D) = H(D) \cap C(\overline{D})$, when $\varphi$ is an automorphism of $D$. In the past 10 years, several papers carried Kamowitz’s project to different Banach spaces of analytic functions, such as Hardy space and Bergman space, see [4,7,9]. However, the results of these papers have a fatal shortcoming: most of the results in [4,7,9] are got under the condition that the weight function $\psi$ of the operator $C_{\psi,\varphi}$ belongs to $A(D)$, that is, $\psi$ is continuous up to the boundary of $D$. This condition is quite unnatural when we consider the property of $C_{\psi,\varphi}$ on a function space other than $A(D)$.

When $\psi$ is not continuous up to the boundary of $D$, the spectral properties of $C_{\psi,\varphi}$ can be much more complicated, and new methods are needed to overcome the difficulties. Quite recently, Kitover and Orhon in [13] gave some descriptions about the spectrum of $C_{\psi,\varphi}$ on variety kinds of function spaces when $\varphi$ is a rotation, without requiring that $\psi \in A(D)$.

In this paper we investigate the spectrum and essential spectrum of weighted composition operator $C_{\psi,\varphi}$, when $\varphi$ is an elliptic automorphism, on weighted Bergman spaces without the assumption that $\psi \in A(D)$. It’s an open question posed in [13] (Problem 7.13 therein).

For $\alpha > -1$ and $p \geq 1$, the weighted Bergman space $A^p_\alpha$ is a Banach space consisting of all analytic functions on $D$ such that

$$\|f\|^p = \int_D |f(z)|^p dA_\alpha(z) < \infty,$$

where the measure

$$dA_\alpha(z) = \frac{1 + \alpha}{\pi} (1 - |z|^2)^\alpha dA(z)$$

is normalized on $D$. Here $\|f\|$ is the norm of $f$ in $A^p_\alpha$. An elliptic automorphism is an biholomorphic map from $D$ to $D$ that has a unique fixed point in $D$. Note that if $\varphi$ is an elliptic automorphism, then the necessary and sufficient condition for $C_{\psi,\varphi}$ to be bounded on $A^p_\alpha$ is that $\psi$ is a bounded analytic function on $D$.

Our main results are in Sects. 5 and 6. In Sect. 5, we give a complete description of the spectrum and essential spectrum of $C_{\psi,\varphi}$ on weighted Bergman spaces when it is invertible. The spectrum and essential spectrum of a non-invertible weighted composition operator $C_{\psi,\varphi}$ is discussed in Sect. 6. We will show that the essential spectrum of $C_{\psi,\varphi}$ depends much on the location of the zeros of $\psi$ in the maximal ideal space of bounded analytic functions on $D$.

The main results of this paper are Corollaries 5.4, 6.3, and Theorem 6.4.

2 Preliminaries

In this section we will recall some basic facts about the maximal ideal space of bounded analytic functions on $D$, which is a key tool in our discussion throughout this paper. Let $H^\infty$ denote the space of bounded analytic functions defined on $D$. Equipped with the supremum norm $\| \cdot \|_\infty$, $H^\infty$ is a Banach algebra.

For a Banach algebra $A$, the maximal ideal space of $A$ is the collection of all non-zero multiplicative linear functionals on $A$, equipped with the weak * topology induced from the dual space of $A$. We use $\mathcal{M}_\infty$ to denote maximal ideal space of $H^\infty$. It is easy to check that $\mathcal{M}_\infty$ is a compact Hausdorff space. The unit disk $D$ can be embedded homeomorphically into $\mathcal{M}_\infty$, in a way that each point in $D$ is regarded as an evaluation functional.

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The Gelfand transform of a function \( f \in H^\infty \), denoted by \( \hat{f} \), is defined as
\[
\hat{f}(m) = m(f), \quad \forall m \in \mathcal{M}_\infty.
\]
Obviously, \( \hat{f} \) is a continuous extension of \( f \) from \( D \) to \( \mathcal{M}_\infty \). Thus, \( H^\infty \) can be seen as a subalgebra of \( C(\mathcal{M}_\infty) \).

Let \( L^\infty(\partial D) \) be the space of \( L^\infty \) functions on the unit circle \( \partial D \). Then it is also a Banach algebra. We use \( \mathcal{M}_{L^\infty} \) to denote the maximal ideal space of \( L^\infty(\partial D) \). It is well known that \( \mathcal{M}_{L^\infty} \) is exactly the Shilov boundary of \( \mathcal{M}_\infty \), that is, \( \mathcal{M}_{L^\infty} \) is the smallest closed set contained in \( \mathcal{M}_\infty \) satisfying that
\[
\|f\|_\infty = \sup_{m \in \mathcal{M}_{L^\infty}} |\hat{f}(m)|
\]
for all \( f \in H^\infty \).

The Gelfand transform of a function \( f \in L^\infty(\partial D) \), also denoted by \( \hat{f} \), is defined as
\[
\hat{f}(m) = m(f), \quad \forall m \in \mathcal{M}_{L^\infty}.
\]
Similar with the case in \( H^\infty \), for each \( f \in L^\infty(\partial D) \), \( \hat{f} \) is a continuous function on \( \mathcal{M}_{L^\infty} \). In the rest part of this paper, we shall always identify a function \( f \) with its Gelfand transform \( \hat{f} \) whenever \( f \) belongs to either \( H^\infty \) or \( L^\infty(\partial D) \).

Since \( H^\infty \) is a logmodular algebra, see Lemma 4.3 in Sect. 4, according to Theorem 4.2 in [5] there exists an unique positive Borel measure \( \mu_0 \) on \( \mathcal{M}_{L^\infty} \) such that
\[
\mu_0(\mathcal{M}_{L^\infty}) = 1
\]
and
\[
f(0) = \int_{\mathcal{M}_{L^\infty}} f d\mu_0
\]
for all \( f \in H^\infty \). \( \mu_0 \) is called the representing measure for the point 0. Moreover, for any \( w \in D \) and \( f \in H^\infty \), we have
\[
f(w) = \int_{\mathcal{M}_{L^\infty}} f P_w d\mu_0,
\]
(2.1)
where \( P_w(e^{i\theta}) = \frac{1-|w|^2}{|e^{i\theta} - w|^2} \) is the Poisson kernel for the point \( w \). In fact, by the Riesz representation theorem, \( \mu_0 \) is determined by the linear functional
\[
g \mapsto \frac{1}{2\pi} \int_0^{2\pi} g(e^{i\theta}) d\theta
\]
for \( g \in L^\infty(\partial D) \).

Suppose \( f \in H^\infty \) is an inner function, then \( f \) is unimodular in \( L^\infty(\partial D) \). Therefore one have \( |f(m)| = 1 \) for all \( m \in \mathcal{M}_{L^\infty} \). As a consequence, if \( \psi \) is a function in \( H^\infty \), then \( |\psi(m)| = |\psi(m)| \) for all \( m \in \mathcal{M}_{L^\infty} \). This fact will be used repeatedly in this paper.

For more information about the maximal ideal space of \( H^\infty \), one can refer to Chapter X in [5]. And some recent results on the topology structure of the maximal ideal space of \( H^\infty \) can be found in [1].

3 Notations

The notations introduced in this section will be used throughout this paper.
We use \( \mathbb{N} \) to denote the set of all nonnegative integers, and \( \mathbb{Z} \) to denote the set of all integers.

\((H^\infty)^{-1}\) denote the collection of all invertible functions in \( H^\infty \). That is,

\[(H^\infty)^{-1} = \{f : f \in H^\infty, 1/f \in H^\infty\}.

Let \( \phi \) be an analytic self-map of the unit disc \( D \). Then by \( \phi_n \) we mean the \( n \)-th iteration of the map \( \phi \), here \( n \in \mathbb{N} \). Set \( \phi_0(z) = z \) be the identity. If \( \phi \) is an automorphism, then we denote by \( \phi_{-1} \) the inverse of \( \phi \), and \( \phi_{-n} \) is the \( n \)-th iteration of \( \phi_{-1} \) for \( n \in \mathbb{N} \).

Suppose \( \psi \in H^\infty \). For a given analytic self-map \( \phi \) of \( D \), define

\[\psi(n) = \prod_{j=0}^{n-1} \psi \circ \phi_j\]

for \( n \in \mathbb{N} \). Set \( \psi(0)(z) = 1 \). Define \( \rho_{\psi,\phi} = \lim_{n \to \infty} \|\psi(n)\|^{1/n}_\infty \).

For a weighted composition operator \( C_{\psi,\phi} \), we denote its spectrum on \( A_a^\infty \) by \( \sigma(C_{\psi,\phi}) \).

Also the following notations are used to denote the different parts of the spectrum of \( C_{\psi,\phi} \):

\[\sigma_e(C_{\psi,\phi}) = \{\lambda \in \mathbb{C} : C_{\psi,\phi} - \lambda \text{ is not Fredholm}\}\]

is the essential spectrum;

\[\sigma_{ap}(C_{\psi,\phi}) = \{\lambda \in \mathbb{C} : C_{\psi,\phi} - \lambda \text{ is not bounded from below}\}\]

is the approximate point spectrum;

\[\sigma_p(C_{\psi,\phi}) = \{\lambda \in \mathbb{C} : C_{\psi,\phi} - \lambda \text{ is not an injection}\}\]

is the set of eigenvalues.

Each analytic self-map \( \phi \) of \( D \) has a natural extension, also denoted by \( \phi \) in this paper, as a self-map of \( \mathcal{M}_\infty \), which is defined as follow:

\[\phi(m)(f) = m(f \circ \phi), \quad \forall f \in H^\infty, \forall m \in \mathcal{M}_\infty.\]

When \( \phi(z) = e^{i\theta}z \) is a rotation, then for any \( m \in \mathcal{M}_\infty \) we will use \( e^{i\theta}m \) to denote \( \phi(m) \). This is an abuse of notation, but it will cause no ambiguity in this paper.

Define \( e^{i\theta}E = \{e^{i\theta}x : x \in E\} \), here \( E \) is a subset of either \( \mathcal{M}_\infty \) or \( \partial D \).

### 4 Spectral radius

Let \( \phi \) be an elliptic automorphism of \( D \). The order of \( \phi \) is the smallest positive integer \( n \) such that \( \varphi_n(z) = z \) is the identity on \( D \). If such \( n \) does not exist, then we say \( \phi \) is of order \( \infty \).

When the order of \( \phi \) is finite, say \( n_0 \in \mathbb{N} \), then \( C_{\psi,\phi} = M_{\psi(\phi)}^{n_0} \) is an analytic multiplication. So in this case, it is not difficult to figure out the spectrum of \( C_{\psi,\phi} \). For a complete discussion, see [4] for example.

In this section we will investigate the spectral radius of \( C_{\psi,\phi} \) where \( \phi \) is an elliptic automorphism of order \( \infty \). Assume that \( a \in D \) is the fixed point of \( \phi \). Let \( \varphi_a(z) = \frac{z - a}{1 - \overline{a}z} \) be the involution automorphism that exchanges 0 and \( a \). Then \( \tau = \varphi_a \circ \phi \circ \varphi_a \) is an automorphism fixing the point 0, hence a rotation. A simple calculation shows that

\[C_{\varphi_a}^{-1}C_{\psi,\phi}C_{\varphi_a} = C_{\psi \circ \varphi_a \circ \tau}. \quad (4.1)\]

So \( C_{\psi,\phi} \) is similar to the weighted composition operator \( C_{\psi \circ \varphi_a \circ \tau} \), which means that they have the same spectrum. Note that a rotation is of order \( \infty \) if and only if it is an irrational one.

Therefore in the rest part of this paper, we shall just handle the case when \( \phi \) is an irrational rotation.

It is easy to check that

\[C_{\psi,\phi}^n = C_{\psi(n),\psi_n}.\]
Since $C_{\psi,\varphi}$ is unitary on $A^0_{\alpha}$ when $\varphi$ is a rotation, the spectral radius of $C_{\psi,\varphi}$ is
\[
\rho_{\psi,\varphi} = \lim_{n \to \infty} \| (\psi(n))^{1/n} \|.
\]

The following theorem, known as the Birkhoff’s Ergodic Theorem, is a useful tool in our discussion.

**Theorem 4.1** Suppose $(X, \mathcal{F}, \mu)$ is a probability space. Let $T$ be a surjective map from $X$ onto itself such that for any $A \in \mathcal{F}$, we have $T^{-1}A \in \mathcal{F}$ and $\mu(T^{-1}A) = \mu(A)$. If $T$ is ergodic, then for any $f \in L^1(X, \mu)$ we have
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} f(T^k x) = \int_X f \, d\mu
\]
for $\mu$-almost every $x \in X$.

Here by saying a map $T$ is ergodic we mean that $T^{-1}A = A$ implies $\mu(A)$ is 0 or 1 for all $A \in \mathcal{F}$. For more information about this theorem, one can turn to [3].

The unit circle $\partial D$, along with the Lebesgue measure $d\theta/2\pi$, is a probability space. If $\varphi$ is a rotation, then $\varphi$ can be seen as a surjective map from $\partial D$ onto itself. It is well known that the rotation $\varphi$ is ergodic on $\partial D$ if and only if it is an irrational one.

Suppose $\psi \in H^\infty$ is not identically zero, then $\log |\psi|$ belongs to $L^1(\partial D, d\theta/2\pi)$, see Theorem 2.7.1 in [14]. So by Ergodic Theorem, if $\varphi$ is an irrational rotation, then
\[
\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} \log |\psi(\varphi_k(\zeta))| = \frac{1}{2\pi} \int_0^{2\pi} \log |\psi(e^{i\theta})| d\theta
\]
for almost every $\zeta \in \partial D$. Or equivalently we have
\[
\lim_{n \to \infty} |\psi(n)(\zeta)|^{1/n} = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log |\psi(e^{i\theta})| d\theta \right)
\]
for almost every $\zeta \in \partial D$. Therefore, if we write $v$ for the outer part of the function $\psi$, then the spectral radius of $C_{\psi,\varphi}$ is no less than
\[
\exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log |\psi(e^{i\theta})| d\theta \right) = |v(0)|.
\]

For the general case, if $\varphi$ is an elliptic automorphism with fixed point $a \in D \setminus \{0\}$ of order $\infty$, then using (4.1) one can conclude that the spectral radius of $C_{\psi,\varphi}$ is no less than $|v(a)|$, where $v$ is the outer part of $\psi$. Thus we have proved the following proposition.

**Proposition 4.2** Suppose $\psi$ is an elliptic automorphism of order $\infty$ and $\psi \in H^\infty$. Then the spectral radius of $C_{\psi,\varphi}$ on $A^0_{\alpha}$ is $\rho_{\psi,\varphi}$.

If we take $v$ to be the outer part of function $\psi$, and assume that $a \in D$ is the fixed point of $\varphi$, then $\rho_{\psi,\varphi} \geq |v(a)|$.

It has been shown in [4,7] that if $\psi \in A(D)$, then the spectral radius of $C_{\psi,\varphi}$ is exactly $|v(a)|$. However, one can never expect $\rho_{\psi,\varphi} = |v(a)|$ holds for all the cases, see the following Example 4.5.

Recall that $H^\infty$ is a logmodular subalgebra of $L^\infty(\partial D)$, which means that the set
\[
\{ \log |f| : f \in (H^\infty)^{-1} \}
\]
is dense in $L^\infty(\partial D)$. In fact, we have the following lemma, which is Theorem 4.5 in [5].
Lemma 4.3 Every real-valued function $g(t)$ in $L^\infty(\partial D)$ has the form $\log |f(t)|$, where $f \in (H^\infty)^{-1}$.

Remark 4.4 Suppose $g(t)$ is a real-valued function in $L^1(\partial D)$ that is bounded above, then a similar discussion shows that $g(t)$ is of the form $\log |f(t)|$ for some $f \in H^\infty$. See page 53 of [8].

Example 4.5 According to Rudin [15], a Borel set $U$ in $\partial D$ is called permanently positive if

$$\frac{d\theta}{2\pi} \left( \bigcap_{j=1}^{n} \zeta_j U \right) > 0$$

for any $n \in \mathbb{N}$ and any choice of $\{\zeta_j\}_{j=1}^{n}$ on $\partial D$. Now suppose $U$ is a permanently positive set with $\frac{d\theta}{2\pi} (U) < 1$, and $\chi_U \subset L^\infty(\partial D)$ is the characteristic function for $U$. Define $\widehat{U} = \{ m \in \mathcal{M}_{L^\infty} : \chi_U(m) = 1 \}$, then $\widehat{U}$ is a clopen subset of $\mathcal{M}_{L^\infty}$. For any finite set $\{\zeta_1, \zeta_2, ..., \zeta_n\}$ in $\partial D$, $\bigcap_{j=1}^{n} \zeta_j U$ has positive Lebesgue measure, so the set

$$\bigcap_{\zeta \in \partial D} \zeta \widehat{U} \neq \emptyset.$$

Take $m \in \bigcap_{\zeta \in \partial D} \zeta \widehat{U}$, which means that $\zeta m \in \widehat{U}$ for all $\zeta \in \partial D$. Now if we take a function $\psi \in (H^\infty)^{-1}$ such that $\log |\psi|$ equals to $\chi_U$ in $L^\infty(\partial D)$, then $|\psi(\zeta m)| = e$ for all $\zeta \in \partial D$.

So for any irrational rotation $\varphi(z) = \eta z$, we have

$$\rho_{\psi, \varphi} \geq \lim_{n \to \infty} \prod_{j=0}^{n-1} |\psi(\eta^j m)|^{1/n} = e.$$

On the other hand, we know $\rho_{\psi, \varphi} \leq \|\psi\|_\infty = e$. Therefore the spectral radius of $C_{\psi, \varphi}$ on $A_\varphi^0$ is $\rho_{\psi, \varphi} = e$, which is larger than $|\psi(0)| = e\frac{d\theta}{2\pi} (U)$.

In [12] Kitover proves that for any irrational rotation $\varphi$ and $\psi \in H^\infty$,

$$\rho_{\psi, \varphi} = \max_{\mu \in M_{\varphi}} \exp \left( \int_{\mathcal{M}_{L^\infty}} \log |\psi| d\mu \right),$$

where $M_{\varphi}$ is the set of all $\psi$-invariant regular probability Borel measures on $\mathcal{M}_{L^\infty}$. However, this is not a explicit description of $\rho_{\psi, \varphi}$. In some special cases, $\rho_{\psi, \varphi}$ can have a more clear expression. The next lemma is Theorem 4.2 in [13].

Lemma 4.6 Suppose $\varphi$ is an irrational rotation and $f$ is an upper semi-continuous function on $\partial \mathbb{D}$, then

$$\lim_{n \to \infty} \sup_{\zeta \in \partial D} \left( \frac{1}{n} \sum_{j=0}^{n-1} f \circ \varphi_j (\zeta) \right) = \frac{1}{2\pi} \int_0^{2\pi} f(e^{i\theta}) d\theta.$$
Let \( v \in H^\infty \) be an outer function. Then according to Remark 4.4, we can define \( v^* \) to be the outer function such that

\[
|v^*(e^{i\theta})| = \limsup_{z \to e^{i\theta}} |v(z)|
\]
a.e. on \( \partial D \). Then \( v^* \) is unique up to multiplication by a constant of modulus one. Similarly, if \( g(e^{i\theta}) = \liminf_{z \to e^{i\theta}} \log |v(z)| \) is in \( L^\infty(\partial D) \), we can define \( v_* \) to be the outer function such that

\[
|v_*(e^{i\theta})| = \liminf_{z \to e^{i\theta}} |v(z)|
\]
a.e. on \( \partial D \). Otherwise, we shall just set \( v_* = 0 \). Denote \( v^{**} = ((v^*)_*)^* \). Now we will improve Proposition 4.2 as follows.

**Proposition 4.7** Suppose \( \phi \) is an elliptic automorphism of order \( \infty \) with fixed point \( a \in D \) and \( \psi \in H^\infty \). Let \( v \) be the outer part of \( \psi \). Then

\[
\max \{ |v(a)|, |v_{**}(a)| \} \leq \rho_{\psi,\phi} \leq |v^*(a)|.
\]

**Proof** We will just proof the situation where \( a = 0 \), or equivalently, \( \phi \) is an irrational rotation. Then the general case follows directly from (4.1).

By the definition of \( v^* \), \( \log |v^*| \) equals to an upper semi-continuous function a.e. on \( \partial D \). Moreover, since the inner part of \( \psi \) take a value of modulus one at each point in \( M_{L\infty} \), we have \( |\psi(m)| \leq |v^*(m)| \) for all \( m \in M_{L\infty} \). Therefore by Lemma 4.6 we have

\[
\rho_{\psi,\phi} \leq \rho_{v^*,\psi} = \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log |v^*(e^{i\theta})| d\theta \right) = |v^*(0)|.
\]

Now let us proof that \( |v_{**}(0)| \leq \rho_{\psi,\phi} \). Fix any \( \epsilon > 0 \). By the definition of \( v_* \), for almost every \( \xi \in \partial D \) there exists an open arc \( I_\xi \) center at \( \xi \) such that

\[
\log |(v_*)^*(\xi')| > \log |v_{**}(\xi)| - \epsilon, \quad \forall \xi' \in I_\xi.
\]

As a consequence, the set

\[
E_\xi = \{ \xi' \in I_\xi : \log |v_*(\xi')| > \log |v_{**}(\xi)| - \epsilon \}
\]

is dense in \( I_\xi \). Moreover, since \( |v_*| \) equals to a lower semi-continuous function on \( \partial D \), we may assume \( E_\xi \) to be open after modifying the values of \( v_* \) on a set of Lebesgue measure zero. So for any finite set of points \( \{ \xi_j \}_{j=1}^n \) on \( \partial D \), we have

\[
d\theta/2\pi \left( \bigcap_{j=1}^n \overline{E}_\xi \right) > 0.
\]

(4.2)

Let

\[
\widehat{E}_\xi = \{ m \in M_{L\infty} : \log |v_*(m)| \geq \log |v_{**}(\xi)| - \epsilon \}.
\]

Then (4.2) implies that \( \bigcap_{j=1}^n \overline{E}_\xi \) is not empty. Since each \( \widehat{E}_\xi \) is closed in \( M_{L\infty} \), by the finite intersection property we can know that

\[
\bigcap_{\xi \in \partial D} \overline{\widehat{E}_\xi} \neq \emptyset.
\]
Take $m_0 \in \cap_{\zeta \in \partial D} \overline{\xi E}_{\zeta}$, then for all $\theta \in [0, 2\pi)$ we have
\[
\log |v(e^{i\theta} m_0)| \geq \log |v_*(e^{i\theta} m_0)| \geq \log |v_*(e^{i\theta})| - \epsilon.
\]
Hence by Ergodic Theorem, we have
\[
\rho_\psi,\varphi \geq \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log |v(e^{i\theta} m_0)| \right) \\
\geq \exp \left( \frac{1}{2\pi} \int_0^{2\pi} \log |v_*(e^{i\theta})| - \epsilon \right) \\
= |v_*(0)|e^{-\epsilon}.
\]
Letting $\epsilon \to 0$, we have $\rho_\psi,\varphi \geq |v_*(0)|$. \hfill \Box

**Example 4.8** Let $G$ be an open dense subset of $\partial D$ with $\frac{d\theta}{d\tau}(G) < 1$. And $\chi_G \subset L^\infty(\partial D)$ is the characteristic function for $G$. We now take a function $\psi \in (H^\infty)^{-1}$ such that $\log |\psi|$ equals to $\chi_G$ in $L^\infty(\partial D)$, and consider the spectrum of $C_\psi,\varphi$, where $\varphi$ is an irrational rotation, on $A^p_\alpha$.

Firstly, note that the open dense $G$ is clearly permanently positive. So Example 4.5 shows directly that $\rho_\psi,\varphi = e$.

But here let’s discuss this example by using Proposition 4.7 other than Example 4.5. Since $\psi$ is outer in this case, it is easy to see that $\psi^* = \psi_*$ is $e$. So by Proposition 4.7, the spectral radius of $C_{\psi,\varphi}$ is $\rho_{\psi,\varphi} = e$. On the other hand, a simple calculation shows that $C_{\psi,\varphi}^{-1} = C_{\psi,\varphi^{-1}}$ where $\psi^{-1} = \frac{1}{\psi(0)}\psi_{\varphi^{-1}}$. Since $G$ is open, one has $\psi^* = \tilde{\psi}$. So the spectral radius of $C_{\psi,\varphi}^{-1}$ is $\rho_{\tilde{\psi},\varphi} = \frac{1}{\psi(0)} = e^{-\frac{d\theta}{d\tau}(G)}$.

According to the discussion in Sect. 4, one in fact has
\[
\sigma(C_{\psi,\varphi}) = \sigma_e(C_{\psi,\varphi}) = \{ \lambda \in \mathbb{C} : e^{\frac{d\theta}{d\tau}(G)} \leq |\lambda| \leq e \}.
\]
This example can also be found in [13], but it can actually be derived from the proof of Theorem 1.1 in [10].

**Remark 4.9** Another useful model for Ergodic Theorem involved in our paper is $M_{L^\infty}$ with measure $\mu_0$. It is easy to see that each irrational rotation $\varphi$ is ergodic on $M_{L^\infty}$ with respect to the measure $\mu_0$, see [10]. So by Ergodic Theorem, for any $\psi \in H^\infty$, the following equation
\[
\lim_{n \to \infty} |\psi^{(n)}(m)|^{1/n} = \exp \left( \int_{M_{L^\infty}} \log |\psi| d\mu_0 \right) = |v(0)|
\]
holds for $\mu_0$-almost every $m \in M_{L^\infty}$. Here $v$ is the outer part of $\psi$.

## 5 Spectra of invertible operators

In this section, we will give a complete description of the spectrum and essential spectrum of invertible operator $C_{\psi,\varphi}$ when $\varphi$ is an elliptic automorphism of order $\infty$. Note that when $\varphi$ is an elliptic automorphism, $C_{\psi,\varphi}$ is invertible if and only if $\psi \in (H^\infty)^{-1}$, i.e., $\psi$ has no zero in $M_{L^\infty}$.

The proof of next lemma is the same as the proof of Proposition 7.11 in [13].

**Lemma 5.1** Suppose $\varphi$ is an irrational rotation and $\psi \in H^\infty$. Let $\sigma_e(C_{\psi,\varphi})$ be the essential spectrum of $C_{\psi,\varphi}$ on $A^p_\alpha$. If $\lambda \in \sigma_e(C_{\psi,\varphi})$, then for any $\theta \in \mathbb{R}$ we have $e^{i\theta}\lambda \in \sigma_e(C_{\psi,\varphi})$. 

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The next lemma is crucial for our discussion in this section. The proof is based on Remark 4.9.

**Lemma 5.2** Suppose \( \varphi \) is an irrational rotation and \( \psi \in H^\infty \). Let \( v \) be the outer part of \( \psi \). If \( \rho_{\psi, \varphi} = \lim_{n \to \infty} \| \psi(n) \|_\infty^{1/n} \) is greater than \( |v(0)| \), then for any \( \epsilon > 0 \) and \( n \in \mathbb{N} \), there exist \( m_0 \in \mathcal{M}_{L^\infty} \) and \( n' > n \) such that

\[
|\psi(n)(m_0)|^{1/n} > \rho_{\psi, \varphi} - \epsilon
\]

and

\[
|\psi(n') \circ \varphi_{-n'}(m_0)|^{1/n'} < |v(0)| + \epsilon.
\]

**Proof** According to Remark 4.9, the equation

\[
\lim_{k \to \infty} \left| \prod_{j=1}^{k} \psi \circ \varphi_{-j}(m) \right|^{1/k} = |v(0)|
\]

holds for \( \mu_0 \)-almost every \( m \) in \( \mathcal{M}_{L^\infty} \). By the definition of \( \rho_{\psi, \varphi} \), for any \( \epsilon > 0 \) and \( n \in \mathbb{N} \), the set

\[
J = \{ m \in \mathcal{M}_{L^\infty} : |\psi(n)(m)| > (\rho_{\psi, \varphi} - \epsilon)^n \}
\]

is a non-empty open set in \( \mathcal{M}_{L^\infty} \). Moreover, since \( \| \psi(n) \|_\infty \geq \rho_{\psi, \varphi}^n \), by (2.1) we have \( \mu_0(J) > 0 \). Therefore we can pick a point \( m_0 \in J \) such that (5.1) holds for \( m_0 \). Thus for \( n' \in \mathbb{N} \) large enough, we have

\[
|\psi(n') \circ \varphi_{-n'}(m_0)|^{1/n'} = \left| \prod_{j=1}^{n'} \psi \circ \varphi_{-j}(m_0) \right|^{1/n'} < |v(0)| + \epsilon.
\]

So \( m_0 \) is the point we want. \( \square \)

We now prove our main theorem in this section, with the help of Lemma 5.2.

**Theorem 5.3** Suppose \( \varphi \) is an irrational rotation and \( \psi \in H^\infty \). Let \( v \) be the outer part of \( \psi \). If \( \rho_{\psi, \varphi} \) is greater than \( |v(0)| \), then the set

\[
\{ \lambda \in \mathbb{C} : |v(0)| \leq |\lambda| \leq \rho_{\psi, \varphi} \}
\]

is contained in the essential spectrum of \( C_{\psi, \varphi} \) on \( A_\alpha^p \).

**Proof** First fix an arbitrary positive number \( \lambda \) in the interior of the target set, and take \( q_1, q_2 > 0 \) such that \( |v(0)| < q_1 \lambda < q_2 < \rho_{\psi, \varphi} \).

We now claim that for each \( k \in \mathbb{N} \), we can find \( n_k > 2k \) and \( g_k \in A_\alpha^p \) such that

\[
\| \psi(k) \cdot g_k \| > q_2^k \| g_k \|
\]

and

\[
\| \psi(n_k) \cdot g_k \circ \varphi_{n_k-k} \| < q_1^{n_k-k} \| \psi(k)g_k \|.
\]

In fact, by Lemma 5.2, for any fixed \( k \in \mathbb{N} \) and any \( \epsilon > 0 \) there exist \( m \in \mathcal{M}_{L^\infty} \) and \( n_k > 2k \) such that

\[
|\psi(k)(m)|^{1/k} = l_1 > \rho_{\psi, \varphi} - \epsilon
\]
and
\[ |\psi(n_k-k) \circ \varphi_{k-n_k}(m)|^{\frac{1}{n_k-k}} = l_2 < |v(0)| + \epsilon. \]

Let
\[ h_k = (1 + \epsilon) \cdot \min \left\{ \left| \frac{\psi(k)}{I^k_1} \right|, \frac{l_2^{n_k-k}}{\left| \psi(n_k-k) \circ \varphi_{k-n_k} \right|} \right\}, \]
then \( h_k \) is a continuous function on \( \mathcal{M}_{L^\infty} \), and \( h_k(m) = 1 + \epsilon \). Moreover, since
\[ \int_{\mathcal{M}_{L^\infty}} \log h_k d\mu_0 > -\infty, \]
by Remark 4.4 we can take outer function \( \tilde{h}_k \in H^\infty \subset A^p \) such that 
\[ \|\tilde{h}_k\|_{\infty} \geq |\tilde{h}_k(m)| = h_k(m) = 1 + \epsilon > 1. \]

Now let \( E_k = \{z \in D : |\tilde{g}_k| \geq 1\} \). Notice that
\[ E_k \subset \left\{ z : |\psi(k)(z)| \geq \frac{l_1^k}{1+\epsilon} \right\} \cap \left\{ z : |\psi(n_k-k) \circ \varphi_{k-n_k}(z)| \leq (1 + \epsilon)l_2^{n_k-k} \right\}. \]

So by the Lebesgue Dominated Convergence Theorem, for \( N \in \mathbb{N} \) large enough we have
\[ \left\| \psi(k) \cdot \tilde{g}_k^N \right\| \geq \left( \int_{E_k} |\psi(k) \tilde{g}_k^N| ^p dA_\alpha \right)^{1/p} \]
\[ \geq \frac{l_1^k}{1+\epsilon} \left( \int_{E_k} |\tilde{g}_k|^p dA_\alpha \right)^{1/p} \]
\[ \geq \frac{1-\epsilon}{1+\epsilon} \cdot (\rho_{\psi,\varphi} - \epsilon) \left\| \tilde{g}_k^N \right\|, \]
and
\[ \left\| \psi(n_k-k) \circ \tilde{g}_k^N \circ \varphi_{n_k-k} \right\| = \left\| \psi(n_k-k) \circ \varphi_{k-n_k} \cdot \psi(k) \cdot \tilde{g}_k^N \right\| \]
\[ \leq \frac{1}{1-\epsilon} \left( \int_{E_k} \left| \psi(n_k-k) \circ \varphi_{n_k-k} \cdot \psi(k) \cdot \tilde{g}_k^N \right|^p dA_\alpha \right)^{1/p} \]
\[ \leq \frac{1+\epsilon}{1-\epsilon} \cdot l_2^{n_k-k} \left( \int_{E_k} |\psi(k) \tilde{g}_k^N|^p dA_\alpha \right)^{1/p} \]
\[ \leq \frac{1+\epsilon}{1-\epsilon} \cdot (|v(0)| + \epsilon)^{n_k-k} \left\| \psi(k) \tilde{g}_k^N \right\|. \]

By taking \( \epsilon > 0 \) sufficiently small and taking \( g_k = \tilde{g}_k^N \), we have (5.2) and (5.3) hold.

Let
\[ f_k = \sum_{j=0}^{n_k-1} \lambda^{k-j} \psi(j) \cdot g_k \circ \varphi_{j-k}. \]

Then
\[ (C_{\psi,\varphi} - \lambda) f_k = \frac{\psi(n_k) \cdot g_k \circ \varphi_{n_k-k}}{\lambda^{n_k-k}} - \lambda^k g_k \circ \varphi_{-k}. \]
So

\[
\| (C_{\psi, \varphi} - \lambda) f_k \| \leq \frac{\| \psi(n_k) \cdot g_k \circ \varphi_{n_k - k} \|}{\lambda^{n_k - k}} + \lambda^k \| g_k \circ \varphi_{-k} \|
\]

\[
\leq \left( \frac{q_1}{\lambda} \right)^{n_k - k} \| \psi(k) g_k \| + \left( \frac{\lambda}{q_2} \right)^k \| \psi(k) g_k \|. 
\]

(5.4)

Here the last inequality follows from (5.2) and (5.3).

Let \( \lambda_{k,s} = \frac{\lambda e^{2\pi i nk}}{ns} \) for \( s = 1, 2, 3, ..., n_k \). Replacing \( \lambda \) in the definition of each function \( f_k \) by \( \{ \lambda_{k,s} \}_{s=1}^{n_k} \), we get \( n_k \) functions, denoted by \( \{ f_{k,s} \}_{s=1}^{n_k} \) respectively. It is easy to check that

\[
\sum_{s=1}^{n_k} \lambda_{k,s} f_{k,s} = n_k \cdot \psi(k) \cdot g_k,
\]

so there exists \( s_k \in \{ 1, 2, 3, ..., n_k \} \) such that \( \lambda \| f_{k,s_k} \| \geq \| \psi(k) g_k \| \). However, (5.4) implies that

\[
\lim_{k \to \infty} \frac{\| (C_{\psi, \varphi} - \lambda_{k,s_k}) f_{k,s_k} \|}{\| \psi(k) g_k \|} = 0.
\]

By passing to a subsequence, we may assume that the sequence \( \{ \lambda_{k,s_k} \}_{k=1}^{\infty} \) converges to a point \( \lambda_0 \). Then

\[
\lim_{k \to \infty} \frac{\| (C_{\psi, \varphi} - \lambda_0) f_{k,s_k} \|}{\| f_{k,s_k} \|} = \lim_{k \to \infty} \frac{\| (C_{\psi, \varphi} - \lambda_{k,s_k}) f_{k,s_k} \|}{\| f_{k,s_k} \|}
\]

\[
\leq \lambda \cdot \lim_{k \to \infty} \frac{\| (C_{\psi, \varphi} - \lambda_{k,s_k}) f_{k,s_k} \|}{\| \psi(k) g_k \|}
\]

\[
= 0.
\]

This means that the operator \( C_{\psi, \varphi} \) is not bounded from below on \( A_p^\alpha \). Now we want to show that \( \lambda_0 \) is not an eigenvalue of \( C_{\psi, \varphi} \). If this is true, then \( \lambda_0 \) belongs to \( \sigma_{ap}(C_{\psi, \varphi}) \setminus \sigma_p(C_{\psi, \varphi}) \subset \sigma_e(C_{\psi, \varphi}) \), and by Lemma 5.1 we can get our conclusion.

To this end, let’s assume that there exists \( f' \in A_p^\alpha \setminus \{ 0 \} \) such that \( C_{\psi, \varphi} f = \lambda_0 f \). Suppose \( f = \sum_{j=K}^{\infty} a_j z^j \) where \( a_K \neq 0 \). Then

\[
\psi \cdot \sum_{j=K}^{\infty} a_j z^j = \sum_{j=K}^{\infty} \lambda_0 a_j z^j.
\]

Taking \( K \)-th derivative at zero to both sides of this equation we can see that \( |\lambda_0| = |\psi(0)| \).

But this is impossible since

\[
|\lambda_0| = \lambda > |v(0)| \geq |\psi(0)|.
\]

\[\square\]

Now we can give our final result in this section as follows.

**Corollary 5.4** Suppose \( \psi \in (H^\infty)^{-1} \) and \( \varphi \) is an elliptic automorphism of order \( \infty \). Then on \( A_p^\alpha \) one has

\[
\sigma(C_{\psi, \varphi}) = \sigma_e(C_{\psi, \varphi}) = \{ \lambda \in \mathbb{C} : \rho_{\frac{1}{\varphi \cdot \varphi}}^{-1} \leq |\lambda| \leq \rho_{\psi \cdot \varphi} \}. 
\]
Proof If $\rho_{\psi,\varphi}^{-1} = \rho_{\psi,\varphi}$, then the result follows directly from Lemma 5.1.

If $\rho_{\psi,\varphi}^{-1} < \rho_{\psi,\varphi}$, then the result is a combination of Theorem 5.3 and the fact that $C_{\psi,\varphi}$ is invertible with $C_{\psi,\varphi}^{-1} = C_{\psi}^{-1} \circ \varphi^{-1}$.

6 Spectra of non-invertible operators

In this section, we will discuss the (essential) spectrum of $C_{\psi,\varphi}$ when it is not invertible. Since $\varphi$ is an automorphism, if $C_{\psi,\varphi}$ is not invertible, then $\psi$ must have zeros in $\mathcal{M}_\infty$. It turns out that the essential spectrum of $C_{\psi,\varphi}$ depends much on the location of the zeros of $\psi$.

First let us treat the case when $\psi$ has zeros in $\mathcal{M}_L \infty$. This is equivalent to the condition that the outer part of $\psi$ does not belong to $(H^\infty)^{-1}$. The following two results, Lemma 6.1 and Theorem 6.2, are parallel to Lemma 5.2 and Theorem 5.3 respectively.

Lemma 6.1 Suppose $\psi \in H^\infty$ and $\varphi$ is an irrational rotation. Let $v$ be the outer part of $\psi$ and

$$r_{\psi,\varphi} = \lim_{k \to \infty} \inf_{m \in \mathcal{M}_{L \infty}} |\psi(k)(m)|^{1/k}.$$ 

If $r_{\psi,\varphi}$ is less than $|v(0)|$, then for any $\epsilon > 0$ and $n \in \mathbb{N}$, there exist $m_0 \in \mathcal{M}_{L \infty}$ and $n' > n$ such that

$$|\psi(n')(m_0)|^{1/n'} > |v(0)| - \epsilon$$

and

$$|\psi(n) \circ \varphi^{-n}(m_0)|^{1/n} < r_{\psi,\varphi} + \epsilon.$$ 

Proof The proof is similar with the proof of Lemma 5.2.

In fact, according to Remark 4.9, the Ergodic Theorem shows that the equation

$$\lim_{k \to \infty} |\psi(k)(m)|^{1/k} = |v(0)|$$

holds for $\mu_0$-almost every $m$ in $\mathcal{M}_{L \infty}$. For any $n \in \mathbb{N}$ and $\epsilon > 0$, let

$$\mathcal{J} = \{m \in \mathcal{M}_{L \infty} : |\psi(m)| < (r_{\psi,\varphi} + \epsilon)^n\}.$$ 

Then $\mathcal{J}$ is non-empty. Moreover, since

$$\log |v(m)(w)| = \int_{\mathcal{M}_{L \infty}} \log |\psi(m)| \cdot P_w d\mu_0$$

for $w \in D$, where $P_w \in L^\infty(\partial D)$ is the Poisson kernel for the point $w$, the fact

$$\inf_{w \in D} |v(m)(w)|^{1/n} = \inf_{m \in \mathcal{M}_{L \infty}} |\psi(m)|^{1/n} \leq r_{\psi,\varphi}$$

implies that $\mu_0(\mathcal{J}) > 0$. Therefore we can take $m_1 \in \mathcal{J}$ such that (6.1) holds for $\psi_n(m_1)$. Then take $n'$ large enough, and $m_0 = \varphi^{-n}(m_1)$ is the point we want. □

Note that in the previous lemma, if the function $\psi$ has zeros in $\mathcal{M}_{L \infty}$, then $r_{\psi,\varphi} = 0$. Otherwise, if $\psi$ has no zero in $\mathcal{M}_{L \infty}$, then its outer part $v$ belongs to $(H^\infty)^{-1}$, so we have $r_{\psi,\varphi} = \rho_{\psi}^{-1}$.

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Theorem 6.2 Suppose $\psi \in H^\infty$ and $\varphi$ is an irrational rotation. Let $v$ be the outer part of $\psi$ and
\[ r_{\psi, \varphi} = \lim_{k \to \infty} \inf_{m \in M_{L^\infty}} |\psi(k)(m)|^{1/k}. \]
If $r_{\psi, \varphi}$ is less than $|v(0)|$, then
\[ \{ \lambda \in \mathbb{C} : r_{\psi, \varphi} \leq |\lambda| \leq |v(0)| \} \]
is contained in the essential spectrum of $C_{\psi, \varphi}$.

Proof Fix an arbitrary positive number $\lambda$ such that $r_{\psi, \varphi} < \lambda < |v(0)|$ and $\lambda \neq |\psi(0)|$. Take $q_1, q_2 > 0$ satisfying that $r_{\psi, \varphi} < q_1 < \lambda < q_2 < |v(0)|$. Then for each $k \in \mathbb{N}$, by Lemma 6.1 and using the same method as in the proof of Theorem 5.3, one can find $n_k > k$ and $g_k \in A^p_\alpha$ such that
\[ \| \psi(n_k)g_k \| > q_2^{n_k} \| g_k \| \]
and
\[ \| \psi(n_k+k) \cdot g_k \circ \varphi_k \| < q_1^k \| \psi(n_k)g_k \|. \]
Now by repeating the proof of Theorem 5.3 we can see that there exists $\lambda_0$ such that $|\lambda_0| = \lambda$ and $C_{\psi, \varphi} - \lambda_0$ is not bounded from below. In the last part of the proof of Theorem 5.3 we have shown that each possible eigenvalue of $C_{\psi, \varphi}$ must have the same modulus with $\psi(0)$. Since $\lambda \neq |\psi(0)|$, $\lambda_0$ can not be a eigenvalue of $C_{\psi, \varphi}$, hence $C_{\psi, \varphi} - \lambda_0$ has no closed range. This means that $\lambda_0 \in \sigma_e(C_{\psi, \varphi})$. Finally, by Lemma 5.1 and the fact that $\sigma_e(C_{\psi, \varphi})$ is closed, we get our conclusion.

\[ \square \]

Corollary 6.3 Suppose $\psi \in H^\infty$ and $\varphi$ is an elliptic automorphism of order $\infty$. If $\psi$ has zeros in $M_{L^\infty}$, then on $A^p_\alpha$ one has
\[ \sigma_e(C_{\psi, \varphi}) = \{ \lambda \in \mathbb{C} : |\lambda| \leq \rho_{\psi, \varphi} \}. \]

Proof This corollary is a combination of Theorem 5.3 and Theorem 6.2 as soon as one notices that $r_{\psi, \varphi} = 0$ in this situation.

The previous corollary gives a complete description of the spectrum and essential spectrum of $C_{\psi, \varphi}$ when $\psi$ has zeros in $M_{L^\infty}$, or equivalently, the outer part of $\psi$ is not bounded from below on $D$.

On the other hand, if all the zeros of $\psi$ lie in $D$, then the outer part of $\psi$ now belongs to $(H^\infty)^{-1}$, and the inner part of $\psi$ is a finite Blaschke product. The next Theorem shows that in this case $\sigma_e(C_{\psi, \varphi})$ no longer coincides with $\sigma(C_{\psi, \varphi})$.

Theorem 6.4 Suppose $\psi \in H^\infty$ and $\varphi$ is an elliptic automorphism of order $\infty$. If $\psi$ has no zero in $M_{\infty} \setminus D$ but has zeros in $D$, then on $A^p_\alpha$ one has
\[ \sigma_e(C_{\psi, \varphi}) = \{ \lambda \in \mathbb{C} : \rho_{\psi, \varphi}^{-1} \leq |\lambda| \leq \rho_{\psi, \varphi} \} \]
and
\[ \sigma(C_{\psi, \varphi}) = \{ \lambda \in \mathbb{C} : |\lambda| \leq \rho_{\psi, \varphi} \}. \]
Proof For any fixed $\lambda$ with $|\lambda| < \rho_{\frac{1}{1+,\psi}}^{-1}$, we will show that $C_{\psi, \varphi} - \lambda$ is bounded from below.

Take $q > 0$ such that $|\lambda| < q < \rho_{\frac{1}{1+,\psi}}^{-1}$. Assume that $\psi = \tau \cdot v$ where $\tau$ and $v$ are the inner and outer part of $\psi$ respectively. Since $\psi$ has no zero in $M_{\infty} \setminus D$, $\tau$ is a finite Blaschke product and $v \in (H^\infty)^{-1}$. By the definition of $\rho_{\frac{1}{1+,\psi}}$, for $n_0 \in \mathbb{N}$ large enough we have

$$q^{n_0} < \inf_{m \in M_{\infty}} |\psi(m)| = \inf_{z \in D} |v(z)|.$$

For any $\epsilon > 0$ there exists $R \in (0, 1)$ such that $|\tau(z)| > 1 - \epsilon$ whenever $R < |z| < 1$. Let $R' = R^{+1}$. For any fixed $f \in A_\sigma^p$, write $\|f\|^p = I_1 + I_2$, where

$$I_1 = \frac{1 + \alpha}{\pi} \int_0^R dr \int_0^{2\pi} |f(re^{i\theta})|^p (1 - r^2)^\alpha r d\theta$$

and

$$I_2 = \frac{1 + \alpha}{\pi} \int_R^{R'} dr \int_0^{2\pi} |f(re^{i\theta})|^p (1 - r^2)^\alpha r d\theta.$$

Then

$$I_2 \geq \frac{1 + \alpha}{\pi} \int_R^{R'} dr \int_0^{2\pi} |f(re^{i\theta})|^p (1 - r^2)^\alpha r d\theta$$

$$\geq \frac{1 + \alpha}{\pi} (1 - R'^2)^\alpha R \int_R^{R'} dr \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

$$\geq \frac{1 + \alpha}{\pi} (1 - R'^2)^\alpha (R' - R) \int_0^R dr \int_0^{2\pi} |f(re^{i\theta})|^p d\theta$$

$$\geq (1 - R'^2)^\alpha (R' - R) I_1.$$

So $I_2 \geq \frac{C}{1 + \epsilon} \|f\|^p$, where $C = (1 - R'^2)^\alpha (R' - R)$. Therefore, for $n > n_0$ we have

$$\left\| C^n_{\psi, \varphi} f \right\|^p \geq q^{np} \cdot \left\| (\tau(n) \cdot f \circ \varphi_n \right\|^p$$

$$\geq q^{np} \cdot \frac{1 + \alpha}{\pi} \int_0^1 dr \int_0^{2\pi} |\tau(n)(re^{i\theta}) f \circ \varphi_n(re^{i\theta})|^p (1 - r^2)^\alpha r d\theta$$

$$\geq q^{np} (1 - \epsilon)^{np} I_2$$

$$\geq q^{np} (1 - \epsilon)^{np} \frac{C}{1 + \epsilon} \|f\|^p.$$

By taking $\epsilon > 0$ sufficiently small and $n \in \mathbb{N}$ large enough, we can make $q^{np} (1 - \epsilon)^{np} \frac{C}{1 + \epsilon}$ greater than $|\lambda|^{np}$. This means that $C_{\psi, \varphi} - \lambda$ is bounded from below.

Since $\tau$ is a finite Blaschke product, the codimension of the range of $C_{\psi, \varphi}$ is finite, Hence $C_{\psi, \varphi}$ is Fredholm. By the stability of the index of Fredholm operators, we can infer that $C_{\psi, \varphi} - \lambda$ is always a Fredholm operator whose index is not zero whenever $|\lambda| < \rho_{\frac{1}{1+,\psi}}^{-1}$. Thus we have proved that

$$\{ \lambda \in \mathbb{C} : |\lambda| < \rho_{\frac{1}{1+,\psi}}^{-1} \} \subset \sigma(C_{\psi, \varphi}) \setminus \sigma_e(C_{\psi, \varphi}).$$

On the other hand, Theorem 5.3 and Theorem 6.2 show that

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{λ ∈ C : ρ_{\tilde{\psi},\varphi}^{-1} ≤ |λ| ≤ ρ_{\psi,\varphi}} ⊂ σ_e(C_{\psi,\varphi}).

Finally there remains the case where ψ has zeros in M_∞ \ D but has no zero in M_{L,∞}. We shall list this case as an open question here.

**Problem 6.5** Suppose ψ ∈ H^∞ and φ is an irrational rotation. Does one have

σ_e(C_{\psi,\varphi}) = \{λ ∈ C : |λ| ≤ ρ_{\psi,\varphi}\}

on A^p_α whenever ψ has zeros in M_∞ \ D but has no zero in M_{L,∞}.

Note that this will happen only when the outer part of ψ is in (H^∞)^{-1} and the inner part of ψ is not a finite Blaschke product. Theorem 5.3 and Theorem 6.2 are still available in this situation, so we have

{λ ∈ C : ρ_{\tilde{\psi},\varphi}^{-1} ≤ |λ| ≤ ρ_{\psi,\varphi}} ⊂ σ_e(C_{\psi,\varphi}).

Moreover, since C_{\psi,\varphi} is not Fredholm in this case, we have 0 ∈ σ_e(C_{\psi,\varphi}). In fact, when the outer part of ψ is in (H^∞)^{-1}, it is known that C_{\psi,\varphi} has closed range if and only if the inner part of ψ is a finite product of interpolating Blaschke products, if and only if ψ does not vanish on any trivial Gleason part of M_∞. See [2,6]. So if all the zeros of ψ lie in non-trivial Gleason parts, then some neighbourhood of 0 is contained in σ_e(C_{\psi,\varphi}) \ σ_{ap}(C_{\psi,\varphi}); otherwise, if ψ take zeros on some trivial Gleason parts, then 0 ∈ σ_e(C_{\psi,\varphi}) \ σ_{ap}(C_{\psi,\varphi}). Therefore, the structure of the (essential) spectrum of C_{\psi,\varphi} in these situations can be expected to rely much on the location of the zeros of ψ as well.

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