ON THE SPECTRALITY AND SPECTRAL EXPANSION OF THE NON-SELF-ADJOINT MATHIEU-HILL OPERATOR IN $L_2(\mathbb{R})$

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Abstract. In this paper we investigate the non-self-adjoint operator $H$ generated in $L_2(\mathbb{R})$ by the Mathieu-Hill equation with a complex-valued potential. We find a necessary and sufficient conditions on the potential for which $H$ has no spectral singularity at infinity and it is an asymptotically spectral operator. Moreover, we give a detailed classification, stated in term of the potential, for the form of the spectral decomposition of the operator $H$ by investigating the essential spectral singularities.

1. Introduction. Let $L(q)$ be the Hill operator generated in $L_2(\mathbb{R})$ by the expression

$$l(y) = -y'' + qy, \quad (1)$$

where $q$ is a complex-valued summable function on $[0,1]$ and $q(x+1) = q(x)$ a.e.. It is well-known that (see [3, 8, 9]) the spectrum $S(L(q))$ of the operator $L(q)$ is the union of the spectra $S(L_t(q))$ of the operators $L_t(q)$ for $t \in (-\pi,\pi]$, where $L_t(q)$ is the operator generated in $L_2[0,1]$ by (1) and the boundary conditions

$$y(1) = e^{it}y(0), \quad y'(1) = e^{it}y'(0). \quad (2)$$

The spectrum of $L_t(q)$ for $t \in \mathbb{C}$ consist of the eigenvalues that are the roots of

$$F(\lambda) = 2\cos t, \quad (3)$$

where $F(\lambda) = \varphi'(1,\lambda) + \theta(1,\lambda)$, $\varphi$ and $\theta$ are the solutions of the equation $l(y) = \lambda y$ satisfying the initial conditions $\theta(0,\lambda) = \varphi'(0,\lambda) = 1$ and $\theta'(0,\lambda) = \varphi(0,\lambda) = 0$.

The operators $L_t(q)$ and $L(q)$ are denoted by $H_t(a,b)$ and $H(a,b)$ respectively when

$$q(x) = ae^{-i2\pi x} + be^{i2\pi x}, \quad (4)$$

where $a$ and $b$ are the complex numbers. In this paper we consider the spectrality and spectral expansion of the non-self-adjoint Mathieu-Hill operator $H(a,b)$ defined in $L_2(\mathbb{R})$. For this aim, first, in Section 3 we obtain the uniform with respect to $t$ in some neighborhood of 0 and $\pi$ asymptotic formulas for the eigenvalues of the operators $L_t(q)$ and $H_t(a,b)$. These formulas are the preliminary investigations and have an auxiliary nature. Then, in Section 4 using these asymptotic formulas, we find a necessary and sufficient condition, stated in term of potential (4), for the asymptotic spectrality of the operator $H(a,b)$. Finally, in Section 5 we classify in detail the form of the spectral expansion of $H(a,b)$ in term of $a$ and $b$. 

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Gesztesy and Tkachenko [6] proved two versions of a criterion for the Hill operator $L(q)$ with $q \in L^2[0, 1]$ to be a spectral operator of scalar type, in sense of Danford, one analytic and one geometric. The analytic version was stated in term of the solutions of Hill’s equation. The geometric version of the criterion uses algebraic and geometric properties of the spectra of periodic/antiperiodic and Dirichlet boundary value problems.

The problem of describing explicitly, for which potentials $q$ the Hill operators $L(q)$ are spectral operators appeared to have been open for about 60 years. In paper [13] we found the explicit conditions on the potential $q$ such that $L(q)$ is an asymptotically spectral operator. In this paper we find a criterion for asymptotic spectrality of $H(a, b)$ stated in term of $a$ and $b$. Note that these investigations show that the set of potentials $q$ for which $L(q)$ is spectral is a small subset of the periodic functions and it is very hard to describe explicitly the required subset. Moreover, the papers [17, 18] and this paper show that the investigation of the spectrality is ineffective for the construction of the spectral expansion for $L(q)$. For this in [17, 18] we introduced a new notions essential spectral singularity (ESS) and ESS at infinity and proved that they determine the form of the spectral expansion for $L(q)$. In this paper investigating the ESS and ESS at infinity for $H(a, b)$ we classify the form of its spectral expansion in term of $a$ and $b$.

To describe more precisely the main results of this paper let us introduce some notations and definitions of the needed notions. The spectrum of $L_t(q)$ consist of the eigenvalues. In [16] we proved that the eigenvalues $\lambda_n(t)$ of $L_t$ can be numbered (counting the multiplicity) by elements of $\mathbb{Z}$ such that, for each $n$ the function $\lambda_n(\cdot)$ is continuous on $(-\pi, \pi]$ and $|\lambda_{\pm n}(t)| \to \infty$ as $n \to \infty$. The spectrum of $L(q)$ is the union of the continuous curves $\Gamma_n = \{\lambda_n(t) : t \in (-\pi, \pi]\}$ for $n \in \mathbb{Z}$. Let $\Psi_{n,t}$ be the normalized eigenfunction corresponding to the simple eigenvalue $\lambda_n(t)$ and $\Psi^*_{n,t}$ be the normalized eigenfunction of $(L_t(q))^*$ corresponding to $\overline{\lambda_n(t)}$. It is well-known that (see p. 39 of [10]) if $\lambda_n(t)$ is a simple eigenvalue of $L_t$, then the spectral projection $e(\lambda_n(t))$ defined by contour integration of the resolvent of $L_t(q)$ over the closed contour containing only the eigenvalue $\lambda_n(t)$, has the form

$$e(\lambda_n(t))f = \frac{1}{d_n(t)}(f, \Psi^*_{n,t})\Psi_{n,t},$$

where

$$d_n(t) = \langle \Psi_{n,t}, \Psi^*_{n,t} \rangle, \quad ||e(\lambda_n(t))|| = |d_n(t)|^{-1},$$

and $(\cdot, \cdot)$ is the inner product in $L^2[0, 1]$. Note that in this paper the number $d_n(t)$ is defined only for the simple eigenvalues $\lambda_n(t)$. If $\lambda_n(t)$ is a simple eigenvalue then the normalized eigenfunctions $\Psi_{n,t}$ and $\Psi^*_{n,t}$ are determined uniquely up to constant of modulus 1. Therefore $|d_n(t)|$ is uniquely defined and it is the norm of the projection $e(\lambda_n(t))$. Note also that $\lambda_n(t)$ is a simple eigenvalue if $F'(\lambda_n(t)) \neq 0$ and the roots of the equation $F'(\lambda) = 0$ is a discrete set, since $F'$ is an entire function. Thus $\lambda_n(t)$ is a simple eigenvalue for $t \in (-\pi, \pi] \setminus A_n$, where $A_n$ is at most a finite set. Moreover $|d_n(t)|$ is continuous at $t$ if $\lambda_n(t)$ is a simple eigenvalue. Therefore in this paper we prefer the following definitions stated in term of $d_n(t)$. McGarvey [8] proved that $L(q)$ is a spectral operator if and only if there exists $c_1 > 0$ such that $||e(\lambda_n(t))|| < c_1$ for $n \in \mathbb{Z}$ and for almost all $t \in (-\pi, \pi]$. It can be stated in terms of $d_n(t)$ and $A_n$ as follows.
Definition 1. We say that \( L(q) \) is a spectral operator if there exists \( c_1 > 0 \) such that
\[
|d_n(t)|^{-1} < c_1 \tag{7}
\]
for all \( n \in \mathbb{Z} \) and \( t \in ((-\pi, \pi) \backslash A_n) \).

Note that here and in subsequent relations we denote by \( c_i \) for \( i = 1, 2, \ldots \) the positive constants whose exact values are inessential. Similarly, we use the following definition.

Definition 2. We say that \( L(q) \) is an asymptotically spectral operator if there exists \( N > 0 \) such that (7) holds for all \( |n| > N \) and \( t \in ((-\pi, \pi) \backslash A_n) \).

As was noted in the paper [16], the spectral singularity of the operator \( L(q) \) are the points \( \lambda \in \sigma(L(q)) \) for which the projections \( e(\lambda_n(t)) \) corresponding to the simple eigenvalues \( \lambda_n(t) \) lying in some neighborhood of \( \lambda \) are not uniformly bounded. Therefore we have the following definitions for the spectral singularities in term of \( d_n \).

Definition 3. A point \( \lambda \in \sigma(L(q)) \) is said to be a spectral singularity of \( L(q) \) if there exist \( n \in \mathbb{Z} \) and sequence \( \{t_k\} \subset ((-\pi, \pi) \backslash A_n) \) such that \( \lambda_n(t_k) \to \lambda \) and \( |d_n(t_k)| \to 0 \) as \( k \to \infty \). We say that the operator \( L(q) \) has a spectral singularity at infinity if there exist sequences \( \{n_k\} \subset \mathbb{Z} \) and \( \{t_k\} \subset ((-\pi, \pi) \backslash A_{n_k}) \) such that \( |n_k| \to \infty \) and \( |d_{n_k}(t_k)| \to 0 \) as \( k \to \infty \).

It is clear that the operator \( L(q) \) has no the spectral singularity at infinity if and only if it is asymptotically spectral operator. Now let us list the main results.

Theorem 1 (Main result for spectrality). The operator \( H(a,b) \) has no spectral singularity at infinity and is an asymptotically spectral operator if and only if \( a = b \) and
\[
\inf_{q,p \in \mathbb{N}} \{|q\alpha - (2p - 1)|\} \neq 0, \tag{8}
\]
where \( \alpha = \pi^{-1} \text{arg}(ab), \mathbb{N} = \{1, 2, \ldots\} \).

This main result of Section 4 implies the following

Corollary 1. Let \( ab \in \mathbb{R} \). Then \( H(a,b) \) is a spectral operator if and only it is self adjoint.

These results show that the theory of spectral operator is ineffective for the study of the spectral expansion for the non-self-adjoint operator \( H(a,b) \) too. It was proven in [5] that in the self-adjoint case the spectral expansion of \( L(q) \) has the following elegant form
\[
f = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{(-\pi, \pi]} a_n(t) \Psi_{n,t} dt, \tag{9}
\]
where
\[
a_n(t) = \frac{1}{d_n(t)} \left( \int_{\mathbb{R}} f(x) \overline{\Psi_{n,t}^*(x)} dx \right). \tag{10}
\]
In the non-self-adjoint case to obtain the spectral expansion, we need to consider the integrability of \( a_n(t)\Psi_{n,t} \) with respect to \( t \) over \((-\pi, \pi]\) which is connected with the integrability of \( \frac{1}{d_n} \). Therefore in [17] we introduced the following notions.

Definition 4. A number \( \lambda_0 \in \sigma(L) \) is said to be an essential spectral singularity (ESS) of \( L \) if there exist \( t_0 \in (-\pi, \pi] \) and \( n \in \mathbb{Z} \) such that \( \lambda_0 = \lambda_n(t_0) \) and \( \frac{1}{d_n} \) is not integrable over \((t_0 - \delta, t_0 + \delta)\) for all \( \delta > 0 \).
It is clear that $\lambda_0 = \lambda_n(t_0)$ is ESS if and only if there exists sequence of closed intervals $I(s)$ approaching $t_0$ such that $\lambda_n(t)$ for $t \in I(s)$ are the simple eigenvalue and
\[
\lim_{s \to \infty} \int_{I(s)} |d_n(t)|^{-1} dt = \infty. \quad (11)
\]

It the similar way in [17] we defined ESS at infinity.

**Definition 5.** We say that the operator $L(q)$ has ESS at infinity if there exist sequence of integers $n_s$ and sequence of closed subsets $I(s)$ of $(-\pi, \pi]$ such that $\lambda_{n_s}(t)$ for $t \in I(s)$ are the simple eigenvalues and
\[
\lim_{s \to \infty} \int_{I(s)} |d_{n_s}(t)|^{-1} dt = \infty. \quad (12)
\]

Note that it follows from the above definitions that the boundlessness of $|d_n(\cdot)|^{-1}$ is the characterization of the spectral singularities and the considerations of the spectral singularities play only the crucial role for the investigations of the spectrality of $L(q)$. On the other hand, the periodic differential operators, in general, is not a spectral operator. Therefore to construct the spectral expansion for the operator $L$, in the general case, in [17, 18] we introduced the new concepts ESS which connected with the nonintegrability of $|d_n(\cdot)|^{-1}$ and proved that the spectral expansion has the elegant form (9) if and only if $L(q)$ has no ESS and ESS at infinity. In Section 5 investigating the ESS and ESS at infinity for $H(a, b)$ we obtained the following main results for its spectral expansion.

**Theorem 2.** If $0 < |ab| < 16/9$, then $H(a, b)$ has no ESS and ESS at infinity and its spectral expansion has the elegant form (9).

For the largest subclass of the potentials (4) we prove the following criterion

**Theorem 3.** The non-self-adjoint operator $H(a, b)$ has no ESS at infinity, has at most finite number of ESS and its spectral expansion has the asymptotically elegant form
\[
f(x) = \frac{1}{2\pi} \left( \sum_{n \in \mathbb{S}} a_n(t)\Psi_{n,t}(x)dt + \sum_{n \in \mathbb{Z}\setminus\mathbb{S}} \int_{(-\pi,\pi]} a_n(t)\Psi_{n,t}(x)dt \right) \quad (13)
\]
if and only if $ab \neq 0$, where $\mathbb{S}$ is at most a finite set and is the set of the indices $n$ for which $\Gamma_n$ contains at least one ESS.

For the remaining potentials we prove the following criterion

**Theorem 4.** The operator $H(a, b)$ has ESS at infinity and infinitely many ESS and its spectral expansion has the following form if and only if either $a = 0$ or $b = 0$
\[
f(x) = f_0(x) + f_\pi(x) + \frac{1}{2\pi} \sum_{n \in \mathbb{Z}} \int_{B(h)} a_n(t)\Psi_{n,t}(x)dt, \quad (14)
\]
where
\[
2\pi f_0(x) = \int_{[-h, h]} a_0(t)\Psi_{0,t}(x)dt + \sum_{n=1}^{\infty} \int_{[-h, h]} (a_{-n}(t)\Psi_{{-n},t}(x) + a_n(t)\Psi_{{n},t}(x)) dt, \quad (15)
\]
\[
2\pi f_\pi(x) = \sum_{n=0}^{\infty} \int_{[-\pi, \pi+h]} (a_n(t)\Psi_{{n},t}(x) + a_{-(n+1)}(t)\Psi_{{-(n+1)},t}(x)) dt, \quad (16)
\]
B(h) = [h, \pi - h] \cup [\pi + h, 2\pi - h], \ 0 < h < \frac{1}{157}.

Note that if the conditions requested for \( H(a, b) \) in Theorem 3 do not hold then either \( a = 0 \) or \( b = 0 \), that is, the conditions requested for \( H(a, b) \) in Theorem 4 hold. It means that all cases of the potential (4) are investigated in Theorem 3 and Theorem 4. In Theorem 2 some subcase of Theorem 3 is studied.

2. Preliminary facts. In this section we present some results of [12, 13, 2] which are used in this paper.

Theorem 5. (Theorem 2 of [12]). The eigenvalues \( \lambda_n(t) \) and eigenfunctions \( \Psi_{n,t} \)
of the operators \( L_t(q) \) for \( t \neq 0, \pi \), satisfy the following asymptotic formulas

\[
\lambda_n(t) = (2\pi n + t)^2 + O(n^{-1} \ln |n|), \quad \Psi_{n,t}(x) = e^{i(2\pi n t + x)} + O(n^{-1}).
\]

(17)

for \( |n| \to \infty \). For any fixed number \( \rho \in (0, \pi/2) \), these asymptotic formulas are uniform with respect to \( t \) in \( [\rho, \pi - \rho] \). Moreover, there exists a positive number \( N(\rho) \), independent of \( t \), such that the eigenvalues \( \lambda_n(t) \) for \( t \in [\rho, \pi - \rho] \) and \( |n| > N(\rho) \) are simple.

Note that, the formula \( f(n, t) = O(h(n)) \) is said to be uniform with respect to \( t \) in a set \( I \) if there exist positive constants \( M \) and \( N \), independent of \( t \), such that \( |f(n, t)| < M h(n) \) for all \( t \in I \) and \( |n| \geq N \). We use Remark 2.1 and lot of formulas of [13] that are listed in Remark 1 and as formulas (20)-(36).

Remark 1. In Remark 2.1 of [13] we proved that here exists a positive integer \( N(0) \) such that the disk \( U(n, t, \rho) =: \{ \lambda \in \mathbb{C} : |\lambda - (2\pi n + t)^2| \leq 15\pi n \rho \} \) for \( t \in [0, \rho] \), where \( 15\pi \rho < 1 \), and \( n > N(0) \) contains two eigenvalues (counting with multiplicities) denoted by \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) and these eigenvalues can be chosen as a continuous function of \( t \) on the interval \( [0, \rho] \). Similarly, there exists a positive integer \( N(\pi) \) such that the disk \( U(n, t, \rho) \) for \( t \in [\pi - \rho, \pi] \) and \( n > N(\pi) \) contains two eigenvalues (counting with multiplicities) denoted again by \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) that are continuous function of \( t \) on the interval \( [\pi - \rho, \pi] \).

Thus for \( n > N =: \max \{ N(\rho), N(0), N(\pi) \} \), the eigenvalues \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) are continuous on \( [0, \rho] \cup [\pi - \rho, \pi] \) and for \( n > N \) the eigenvalue \( \lambda_n(t) \), defined by (17), is continuous on \( [\rho, \pi - \rho] \). Moreover, \( \lambda_{n,1} \) and \( \lambda_{n,2} \) can be chosen so that

\[
\lambda_{n,1}(\rho) = \lambda_{n,1}(0), \quad \lambda_{n,2}(\rho) = \lambda_{n,2}(0),
\]

(18)

\[
\lambda_{n,1}(\pi - \rho) = \lambda_n(\pi - \rho), \quad \lambda_{n,2}(\pi - \rho) = \lambda_{n+1}(\pi - \rho).
\]

(19)

Let us redenote \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) by \( \lambda_{n}(t) \) and \( \lambda_{n}(t) \) respectively for \( n > N \) and \( t \in [0, \rho] \). Similarly, redenote \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) by \( \lambda_{n}(t) \) and \( \lambda_{n-1}(t) \) respectively for \( n > N \) and \( t \in [\pi - \rho, \pi] \). Defining \( \lambda_{n}(-t) = \lambda_{n}(t) \) we obtain continuous function \( \lambda_{n} \) on \( (-\pi, \pi] \). In this paper we use both notations: \( \lambda_{n}(t) \) and \( \lambda_{n,j}(t) \).

One can readily see that

\[
|\lambda - (2\pi n - k)^2| > |k| |2n - k|, \quad \forall \lambda \in U(n, t, \rho)
\]

(20)

for \( k \neq 0, 2n \) and \( t \in [0, \rho] \), where \( |n| > N \).

In [13] to obtain the uniform, with respect to \( t \in [0, \rho] \), asymptotic formulas for the eigenvalues \( \lambda_{n,j}(t) \) we used (20) and the iteration of the formula

\[
(\lambda_{n,j}(t) - (2\pi n + t)^2)(\Psi_{n,j,t} e^{i(2\pi n t + x)}) = (q\Psi_{n,j,t}, e^{i(2\pi n t + x)}),
\]

(21)
where Ψ_{n,j,t} is any normalized eigenfunction corresponding to \(\lambda_{n,j}(t)\). Iterating (21) infinite times we get the following formula

\[
(\lambda_{n,j}(t) - (2\pi n + t)^2 - A(\lambda_{n,j}(t), t))u_{n,j}(t) = (q_{2n} + B(\lambda_{n,j}(t), t))v_{n,j}(t),
\]

where \(u_{n,j}(t) = (\Psi_{n,j,t} e^{i(2\pi n + t)x}), v_{n,j}(t) = (\Psi_{n,j,t} e^{i(-2\pi n + t)x}), q_n = (q, e^{2\pi n x}), \)

\[
A(\lambda, t) = \sum_{k=1}^{\infty} a_k(\lambda, t), \quad B(\lambda, t) = \sum_{k=1}^{\infty} b_k(\lambda, t),
\]

\[
a_k(\lambda, t) = \sum_{n_1, n_2, ..., n_k} q_{n_1-n_2-...-n_k} \prod_{s=1}^{k} q_{n_s} (\lambda - (2\pi(n - n_1 - ... - n_s) + t)^2)^{-1},
\]

\[
b_k(\lambda, t) = \sum_{n_1, n_2, ..., n_k} q_{2n_1-n_2-...-n_k} \prod_{s=1}^{k} q_{n_s} (\lambda - (2\pi(n - n_1 - ... - n_s) + t)^2)^{-1}
\]

for \(\lambda \in U(n, t, \rho)\) (see (37) of [13]).

Similarly, we obtained the formula

\[
(\lambda_{n,j}(t) - (-2\pi n + t)^2 - A'(\lambda_{n,j}(t), t))v_{n,j}(t) = (q_{-2n} + B'(\lambda_{n,j}(t), t))u_{n,j}(t),
\]

where

\[
A'(\lambda, t) = \sum_{k=1}^{\infty} a'_k(\lambda, t), \quad B'(\lambda, t) = \sum_{k=1}^{\infty} b'_k(\lambda, t),
\]

\[
a'_k(\lambda, t) = \sum_{n_1, n_2, ..., n_k} q_{n_1-n_2-...-n_k} \prod_{s=1}^{k} q_{n_s} (\lambda - (2\pi(n + n_1 + ... + n_s) - t)^2)^{-1},
\]

\[
b'_k(\lambda, t) = \sum_{n_1, n_2, ..., n_k} q_{-2n_1-n_2-...-n_k} \prod_{s=1}^{k} q_{n_s} (\lambda - (2\pi(n + n_1 + ... + n_s) - t)^2)^{-1}
\]

for \(\lambda \in U(n, t, \rho)\) (see (38) of [13]).

The sums in (24), (25) and (28), (29) are taken under conditions \(n_1 + n_2 + ... + n_s \neq 0, 2n\) and \(n_1 + n_2 + ... + n_s \neq 0, -2n\) respectively, where \(s = 1, 2, ...\).

Besides, it was proved [13] that the equalities

\[
a_k(\lambda, t), b_k(\lambda, t), a'_k(\lambda, t), b'_k(\lambda, t) = O \left((n^{-1} \ln |n|)^k\right)
\]

hold uniformly for \(t \in [0, \rho]\) and \(\lambda \in U(n, t, \rho)\) (see (34) and (36) of [13]), and derivatives of these functions with respect to \(\lambda\) are \(O(n^{-k-1})\) (see the proof of Lemma 2.5) which imply that the functions \(A(\lambda, t), A'(\lambda, t), B(\lambda, t)\) and \(B'(\lambda, t)\) are analytic on \(U(n, t, \rho)\). Moreover, there exists a constant \(K\) such that

\[
|A(\lambda, t)| < Kn^{-1}, \quad |A'(\lambda, t)| < Kn^{-1}, \quad |B(\lambda, t)| < Kn^{-1}, \quad |B'(\lambda, t)| < Kn^{-1},
\]

\[
|A(\lambda, t) - A(\mu, t)| < Kn^{-2} |\lambda - \mu|, \quad |A'(\lambda, t) - A'(\mu, t)| < Kn^{-2} |\lambda - \mu|, \quad |B(\lambda, t) - B(\mu, t)| < Kn^{-2} |\lambda - \mu|, \quad |B'(\lambda, t) - B'(\mu, t)| < Kn^{-2} |\lambda - \mu|,
\]

\[
|C(\lambda, t)| < tKn^{-1}, \quad |C(\lambda, t) - C(\mu, t)| < tKn^{-1} |\lambda - \mu|,
\]

for all \(n > N, t \in [0, \rho]\) and \(\lambda, \mu \in U(n, t, \rho)\), where \(N\) and \(U(n, t, \rho)\) are defined in Remark 1, and

\[
C(\lambda, t) = \frac{1}{2}(A(\lambda, t) - A'(\lambda, t))
\]

(see Lemma 2.3 and Lemma 2.5 of [13]).
In this paper we use also the following, uniform with respect to \( t \in [0, \rho] \), equalities from [13] (see (26)-(28) of [13]) for the normalized eigenfunction \( \Psi_{n,j,t} \):

\[
\Psi_{n,j,t}(x) = u_{n,j}(t)e^{i(2\pi n+t)x} + v_{n,j}(t)e^{i(-2\pi n+t)x} + h_{n,j,t}(x),
\]

(35)

\( (h_{n,j,t}, e^{i(2\pi n+t)x}) = 0, \)  \( \|h_{n,j,t}\| = O(n^{-1}), \)  \( |u_{n,j}(t)|^2 + |v_{n,j}(t)|^2 = 1 + O(n^{-2}), \)

(36)

Here we also use formula (55) of [2] about estimations of \( B(\lambda, 0) \) and \( B'(\lambda, 0) \) as follows:

*Let the potential has the form (4), \( \lambda = (2\pi n)^2 + z \), where \( |z| < 1 \), and

\[
p_{n_1,n_2,\ldots,n_k}(\lambda, 0) = q_{2n-n_1-n_2-\ldots-n_k} \prod_{s=1}^{k} q_{n_s} (\lambda - (2\pi(n - n_1 - \ldots - n_k))^2)^{-1}
\]

(37)

be summands of \( b_k(\lambda, t) \) for \( t = 0 \) (see (14)). Then using (55) of [2] with \( q \geq 2 \) and the estimation

\[
\sum_{q \geq 2} (n+q)^2 \left( \frac{|ab|}{n^2} \right)^q = O(n^{-2})
\]

of [2] (see the estimation after formula (55) of [2]) and taking into account that if \( k \) changes from \( 2n+3 \) to \( \infty \), then the number of steps \(-2 \) (that is, in our notations the number of indices \( n_1, n_2, \ldots, n_k \) of (37) that are equal to 1) changes from 2 to \( \infty \), we obtain

\[
\sum_{k=2n+3}^{\infty} \sum_{n_1, n_2, \ldots, n_k} |p_{n_1,n_2,\ldots,n_k}(\lambda, 0)| = b_{2n-1}(\lambda, 0)O(n^{-2}).
\]

(38)

3. **On the operators** \( L_t(q) \) and \( H_t(a, b) \). One can readily see from (22), (26), (31) and Remark 1 that

\[
\lambda_{n,j}(t) \in (d^{-}(r(n), t)) \cup d^{+}(r(n), t) \subset U(n, t, \rho),
\]

(39)

for all \( n > N, t \in [0, \rho] \), where \( r(n) = \max\{|q_{2n}|, |q_{-2n}|\} + 2Kn^{-1} \) and \( d^{\pm}(r(n), t) \) is the disk with center \( \pm 2\pi n + t^2 \) and radius \( r(n) \). Indeed if \( |u_{n,j}(t)| \geq |v_{n,j}(t)| \) then using (22), if \( |v_{n,j}(t)| > |u_{n,j}(t)| \), then using (26) and (31) we get (39).

**Theorem 6.** A number \( \lambda \in U(n, t, \rho) \) is an eigenvalue of \( L_t(q) \) for \( t \in [0, \rho] \) and \( n > N \), where \( U(n, t, \rho) \) and \( N \) are defined in Remark 1, if and only if

\[
(\lambda - (2\pi n + t)^2) - A(\lambda, t)))(\lambda - (2\pi n - t)^2) - A'(\lambda, t) = (q_{2n} + B(\lambda, t))q_{-2n} + B'(\lambda, t).
\]

(40)

Moreover \( \lambda \in U(n, t, \rho) \) is a double eigenvalue of \( L_t \) if and only if it is a double root of (40).

**Proof.** If \( u_{n,j}(t) = 0 \), then by (36) we have \( v_{n,j}(t) \neq 0 \). Therefore, (22) and (26) imply that

\[
q_{2n} + B(\lambda_{n,j}(t), t) = 0, \ \lambda_{n,j}(t) - (-2\pi n + t)^2 - A'(\lambda_{n,j}(t), t) = 0,
\]

that is, the right-hand side and the left-hand side of (40) vanish when \( \lambda \) is replaced by \( \lambda_{n,j}(t) \). Hence \( \lambda_{n,j}(t) \) satisfies (40). In the same way we prove that if \( v_{n,j}(t) = 0 \) then \( \lambda_{n,j}(t) \) is a root of (40). It remains to consider the case \( u_{n,j}(t)v_{n,j}(t) \neq 0 \). In this case multiplying (22) and (26) side by side and canceling \( u_{n,j}(t)v_{n,j}(t) \) we get an equality obtained from (40) by replacing \( \lambda \) with \( \lambda_{n,j}(t) \). Thus, in any case \( \lambda_{n,j}(t) \) is a root of (40).
Now we prove that the roots of (40) lying in \( U(n, t, \rho) \) are the eigenvalues of \( L_i(q) \). Let \( F(\lambda, t) \) be the left-hand side minus the right-hand side of (40). Using (31) one can easily verify that the inequality

\[
|F(\lambda, t) - G(\lambda, t)| < |G(\lambda, t)|,
\]

where \( G(\lambda, t) = (\lambda - (2\pi n + t)^2)(\lambda - (2\pi n - t)^2) \), holds for all \( \lambda \) from the boundary of \( U(n, t, \rho) \). Since the function \( (\lambda - (2\pi n + t)^2)(\lambda - (2\pi n - t)^2) \) has two roots in the set \( U(n, t, \rho) \), by the Rouche’s theorem from (41) we obtain that \( F(\lambda, t) \) has two roots in the same set. Thus \( L_i(q) \) has two eigenvalue (counting with multiplicities) lying in \( U(n, t, \rho) \) (see Remark 1) that are the roots of (40). On the other hand, (40) has precisely two roots (counting with multiplicities) in \( U(n, t, \rho) \). Therefore \( \lambda \in U(n, t, \rho) \) is an eigenvalue of \( L_i(q) \) if and only if (40) holds.

If \( \lambda \in U(n, t, \rho) \) is a double eigenvalue of \( L_i(q) \), then by Remark 1 \( L_i(q) \) has no other eigenvalues in \( U(n, t, \rho) \) and hence (40) has no other roots. This implies that \( \lambda \) is a double root of (40). By the same argument one can prove that if \( \lambda \) is a double root of (40) then it is a double eigenvalue of \( L_i(q) \)

One can readily verify that equation (40) can be written in the form

\[
(\lambda - (2\pi n + t)^2 - \frac{1}{2}(A + A') + 4\pi nt)^2 = D,
\]

where

\[
D(\lambda, t) = (4\pi nt)^2 + q_2nq_{-2n} + 8\pi ntC + C^2 + q_2nB' + q_{-2n}B + BB'
\]

and, for brevity, we denote \( C(\lambda, t), B(\lambda, t), A(\lambda, t) \) etc. by \( C, B, A \) etc. It is clear that \( \lambda \) is a root of (42) if and only if it satisfies at least one of the equations

\[
\lambda - (2\pi n + t)^2 - \frac{1}{2}(A(\lambda, t) + A'(\lambda, t)) + 4\pi nt = -\sqrt{D(\lambda, t)}
\]

and

\[
\lambda - (2\pi n + t)^2 - \frac{1}{2}(A(\lambda, t) + A'(\lambda, t)) + 4\pi nt = \sqrt{D(\lambda, t)},
\]

where

\[
\sqrt{D} = \sqrt{|D|}e^{(\arg D)/2}, \quad -\pi < \arg D \leq \pi.
\]

**Remark 2.** It is clear from the construction of \( D(\lambda, t) \) that this function is continuous with respect to \( (\lambda, t) \) for \( t \in [0, \rho] \) and \( \lambda \in U(n, t, \rho) \). Moreover, by Remark 1 the eigenvalues \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) continuously depend on \( t \in [0, \rho] \). Therefore \( D(\lambda_{n,j}(t), t) \) for \( n > N \) and \( j = 1, 2 \) is a continuous functions of \( t \in [0, \rho] \). By (43), (34), (23), (27) and (30) we have

\[
D(\lambda_{n,j}(\rho), \rho) = (4\pi nt)^2 + o(1), \quad A(\lambda_{n,j}(\rho), \rho) + A'(\lambda_{n,j}(\rho), \rho) = o(1)
\]

as \( n \to \infty \). Therefore by (18) and Theorem 2 of [12] the eigenvalues \( \lambda_{n,1}(\rho) \) and \( \lambda_{n,2}(\rho) \) are simple, \( \lambda_{n,1}(\rho) \), satisfies (44) and \( \lambda_{n,2}(\rho) \) satisfies (45). If \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) are simple for \( t \in [t_0, \rho] \), where \( 0 \leq t_0 \leq \rho \), then these functions are analytic function on \( [t_0, \rho] \) and \( \lambda_{n,1}(t) \neq \lambda_{n,2}(t) \) for all \( t \in [t_0, \rho] \).

**Theorem 7.** Suppose that \( \sqrt{D(\lambda_{n,j}(t), t)} \) continuously depends on \( t \) at \( [t_0, \rho] \) and

\[
D(\lambda_{n,j}(t), t) \neq 0, \quad \forall t \in [t_0, \rho]
\]

for \( n > N \) and \( j = 1, 2 \), where \( \rho \) and \( N \) are defined in Remark 1 and \( \sqrt{D} \) is defined in (46) and \( 0 \leq t_0 \leq \rho \). Then for \( t \in [t_0, \rho] \) the eigenvalues \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \)
defined in Remark 1 are simple, \( \lambda_{n,1}(t) \) satisfies (44) and \( \lambda_{n,2}(t) \), satisfies (45). That is
\[
\lambda_{n,j}(t) = (2\pi n + t)^2 + \frac{1}{2}(A(\lambda_{n,j}(t), t) + A'(\lambda_{n,j}(t))) - 4\pi nt + (-1)^j \sqrt{D(\lambda_{n,j}(t))}
\]
for \( t \in [t_0, \rho] \), \( n > N \) and \( j = 1, 2 \).

Proof. By Remark 2, the eigenvalues \( \lambda_{n,1}(\rho) \) and \( \lambda_{n,2}(\rho) \) are simple, \( \lambda_{n,1}(\rho) \) satisfies (44) and \( \lambda_{n,2}(\rho) \) satisfies (45). Let us prove that \( \lambda_{n,1}(t) \) satisfies (44) for all \( t \in [t_0, \rho] \). Suppose to the contrary that this claim is not true. Then there exists \( t \in [t_0, \rho) \) and the sequences \( p_n \to t \) and \( q_n \to t \), where one of them may be a constant sequence, such that \( \lambda_{n,1}(p_n) \) and \( \lambda_{n,1}(q_n) \) satisfy (44) and (45) respectively.

Using the continuity of \( \sqrt{(D(\lambda_{n,j}(t), t))} \), we conclude that \( \lambda_{n,1}(t) \) satisfies both (44) and (45). However, it is possible only if \( D(\lambda_{n,1}(t), t) = 0 \) which contradicts (47).

Hence \( \lambda_{n,1}(t) \) satisfies (44) for all \( t \in [t_0, \rho] \). In the same way we prove that \( \lambda_{n,2}(t) \) satisfies (45) for all \( t \in [t_0, \rho] \). If \( \lambda_{n,1}(t) = \lambda_{n,2}(t) \) for some value of \( t \in [t_0, \rho] \), that is if \( \lambda_{n,j}(t) \) is a double eigenvalue then it satisfies both (44) and (45) which again contradicts (47).

Now we study the operator \( H_t \). Note that we consider only the case \( t \in [0, \rho) \) due to the following reason. The case \( t \in [\rho, \pi - \rho] \) was considered in [12]. The case \( t \in [\pi - \rho, \pi] \) is similar to the case \( t \in [0, \rho] \) and we explain it in Remark 3. Besides, the eigenvalues of \( H_{-t} \) coincides with the eigenvalues of \( H_t \).

When the potential \( q \) has the form (4) then
\[
q_{-1} = a, \quad q_1 = b, \quad q_n = 0, \quad \forall n \neq \pm 1
\]
and hence the formulas (22), (26), (42) and (43) have the form
\[
(\lambda_{n,j}(t) - (2\pi n + t)^2 - A(\lambda_{n,j}(t), t))u_{n,j}(t) = B(\lambda_{n,j}(t), t)v_{n,j}(t),
\]
\[
(\lambda_{n,j}(t) - (-2\pi n + t)^2 - A'(\lambda_{n,j}(t), t))v_{n,j}(t) = B'(\lambda_{n,j}(t), t)u_{n,j}(t),
\]
\[
(\lambda - (2\pi n + t)^2 - \frac{1}{2}(A(\lambda, t) + A'(\lambda, t)) + 4\pi nt)^2 = D(\lambda, t),
\]
\[
D(\lambda, t) = (4\pi nt + C(\lambda, t))^2 + B(\lambda, t)B'(\lambda, t).
\]
Moreover, by Theorem 5, \( \lambda \in U(n, t, \rho) \) is a double eigenvalue of \( H_t \) if and only if it satisfies (52) and the equation
\[
2(\lambda - (2\pi n + t)^2 - \frac{1}{2}(A + A') + 4\pi nt)^2(1 - \frac{1}{2} \frac{\partial}{\partial \lambda} (A + A')) = \frac{\partial}{\partial \lambda} (D(\lambda, t)).
\]
By (39) and (49) \( \lambda_{n,j}(t) \in (d^- (2Kn^{-1}, t) \cup d^+ (2Kn^{-1}, t)) \subseteq U(n, t, \rho) \). Therefore the formula
\[
\lambda_{n,j}(t) = (2\pi n)^2 + O(n^{-1})
\]
holds uniformly, with respect to \( t \in [0, n^{-2}] \), for \( j = 1, 2 \), i.e., there exist positive constants \( M \) and \( N \) such that \( |(\lambda_{n,j}(t) - (2\pi n)^2) < Mn^{-1} | for n \geq N \) and \( t \in [0, n^{-2}] \).

Let us consider the functions taking part in (50)-(52). From (49) we see that the indices in formulas (24), (25) for the case (4) satisfy the conditions
\[
\{n_1, n_2, ..., n_k\} \subset \{-1, 1\}, \quad n_1 + n_2 + ... + n_k \neq 0, 2n,
\]
\[
\{n_1, n_2, ..., n_k, 2n - n_1 - n_2 - ... - n_k\} \subset \{-1, 1\}, \quad n_1 + n_2 + ... + n_k \neq 0, 2n
\]
for \( s = 1, 2, ..., k \) respectively. Hence, by (49) \( q_{-n_1-n_2-...-n_k} = 0 \) if \( k \) is an even number. Therefore, by (24) and (28)
\[
a_{2m}(\lambda, t) = 0, \quad a_{2m}'(\lambda, t) = 0, \quad \forall m = 1, 2, ...
\]
Since the indices \( n_1, n_2, ..., n_k \) take two values (see (56)) the number of the summands in the right-hand side of (24) is not more than \( 2^k \). Clearly, these summands for \( k = 2m - 1 \) have the form
\[
a_k(\lambda, n_1, n_2, ..., n_k, t) = (ab)^m \prod_{s=1,2,...,k}^\lambda \left( (2\pi(n - n_1 - n_2 - ... - n_s) + t)^2 \right)^{-1}
\]
(see (24), (49) and (56)). Therefore, we have
\[
a_{2m-1}(\lambda_{n,j}(t), t) = (4ab)^m O(n^{-2m+1}).
\]
(59)

If \( t \in [0, n^{-2}] \), then one can readily see that
\[
a_1(\lambda_{n,j}, t) = \frac{(2\pi n)^2 + O(n^{-1}) - (2\pi(n - 1))^2}{2\pi(2\pi(2n - 1) - 2\pi(2\pi(2n + 1)) + O \left( \frac{1}{n!} \right) = O \left( \frac{1}{n^2} \right)}.
\]

The same estimations for \( a'_{2m-1}(\lambda_{n,j}(t), t) \) and \( a'_1(\lambda_{n,j}(t), t) \) hold respectively. Thus, by (23), (27), (30) and (58), we have
\[
A(\lambda_{n,j}(t), t) = O(n^{-2}), \quad A'(\lambda_{n,j}(t), t) = O(n^{-2}), \quad \forall t \in [0, n^{-2}].
\]
(60)

Now we study the functions \( B(\lambda, t) \) and \( B'(\lambda, t) \) (see (23), (25) and (27), (29)).

First let us consider \( b_{2n-1}(\lambda, t) \). If \( k = 2n - 1 \), then by (57) \( n_1 = n_2 = ... = n_{2k-1} = 1 \). Using this and (49) in (25) for \( k = 2n - 1 \), we obtain
\[
b_{2n-1}(\lambda, t) = b^2 \prod_{s=1}^{2n-1} (\lambda - (2\pi(n - s) + t)^2)^{-1}.
\]
(61)

If \( k < 2n - 1 \) or \( k = 2m \), then, by (49), \( q_{2n-n_1-n_2-...-n_k} = 0 \) and by (25)
\[
b_k(\lambda, t) = 0.
\]
(62)

In the same way, from (29) we obtain
\[
b'_{2n-1}(\lambda, t) = ab^2 \prod_{s=1}^{2n-1} (\lambda - (2\pi(n - s) - t)^2)^{-1}, \quad b'_k(\lambda_{n,j}(t), t) = 0
\]
(63)

for \( k < 2n - 1 \) or \( k = 2m \).

Now, (30), (62) and (63) imply that the equalities
\[
B(\lambda, t) = O (n^{-5}), \quad B'(\lambda, t) = O (n^{-5})
\]
(64)

hold uniformly for \( t \in [0, \rho] \) and \( \lambda \in U(n, t, \rho) \). From (50) and (51) (if \( |v_{n,j}(t)| \geq |v_{n,j}(t)| \) then use (50) and if \( |v_{n,j}(t)| > |u_{n,j}(t)| \) then use (51)) by using (60) and (64) we obtain that the formula
\[
\lambda_{n,j}(t) = (2\pi n)^2 + O(n^{-2})
\]
(65)

holds uniformly, with respect to \( t \in [0, n^{-3}] \), for \( j = 1, 2 \).

More detail estimations of \( B \) and \( B' \) are given in the following lemma, where we use the following notation. We say that \( a_n \) is of order of \( b_n \) and write \( a_n \sim b_n \) if \( a_n = O(b_n) \) and \( b_n = O(a_n) \) as \( n \to \infty \).

**Lemma 1.** If \( q \) has the form (4), then the formulas
\[
B(\lambda, t) = \beta_n \left( 1 + O(n^{-2}) \right), \quad B'(\lambda, t) = \alpha_n \left( 1 + O(n^{-2}) \right),
\]
(66)
\[
\frac{\partial}{\partial \lambda} (B'(\lambda, t) B(\lambda, t)) \sim \alpha_n \beta_n n^{-1} \ln |n|,
\]
(67)
where \( \beta_n = b^{2n} \left( (2\pi)^{2n-1}(2n-1)! \right)^{-2} \), \( \alpha_n = a^{2n} \left( (2\pi)^{2n-1}(2n-1)! \right)^{-2} \) hold uniformly for 
\[ t \in [0, n^{-3}], \; \lambda = (2\pi n)^2 + O(n^{-2}). \tag{68} \]

Proof. Using (61) and (63) by direct calculations we get

\[ b_{2n-1}(2\pi n, 0) = \beta_n, \; b_{2n-1}'(2\pi n, 0) = \alpha_n. \tag{69} \]

If \( 1 \leq s \leq 2n-1 \) then for any \((\lambda, t)\) satisfying (68) there exists \( \lambda_1 = (2\pi n)^2 + O(n^{-2}) \) and \( \lambda_2 = (2\pi n)^2 + O(n^{-2}) \) such that

\[ | \lambda_1 - (2\pi(n-s))^2 | < | \lambda - (2\pi(n-s) + t)^2 | < | \lambda_2 - (2\pi(n-s))^2 |. \tag{70} \]

Therefore from (61) we obtain that

\[ |b_{2n-1}(\lambda_1, 0)| < |b_{2n-1}(\lambda, t)| < |b_{2n-1}(\lambda_2, 0)|. \tag{71} \]

On the other hand, differentiating (61) with respect to \( \lambda \), we conclude that

\[ \frac{\partial}{\partial \lambda} b_{2n-1}(2\pi n, 0) = b_{2n-1}(2\pi n, 0) \sum_{s=1}^{2n-1} \frac{1 + O(n^{-1})}{s(2n-s)}. \tag{72} \]

Now taking into account that the last summation is of order \( n^{-1} \ln |n| \) and using (69), we get

\[ \frac{\partial}{\partial \lambda} b_{2n-1}(2\pi n, 0) \sim \beta_n n^{-1} \ln |n|. \tag{73} \]

Arguing as above one can easily see that the \( m \)-th derivative, where \( m = 2, 3, \ldots. \) of \( b_{2n-1}(\lambda, 0) \) is \( O(\beta_n) \). Hence using the Taylor series of \( b_{2n-1}(\lambda, 0) \) for \( \lambda = (2\pi n)^2 + O(n^{-2}) \) about \( (2\pi n)^2 \), we obtain

\[ b_{2n-1}(\lambda, 0) = \beta_n (1 + O(n^{-2})), \forall i = 1, 2. \]

This with (71) yields

\[ b_{2n-1}(\lambda, t) = \beta_n (1 + O(n^{-2})) \tag{74} \]

for all \((\lambda, t)\) satisfying (68). In the same way, we get

\[ \frac{\partial}{\partial \lambda} b_{2n-1}'(2\pi n, 0) \sim \alpha_n \left( \frac{\ln n}{n} \right), \; b_{2n-1}'(\lambda, t) = \alpha_n (1 + O(n^{-2})). \tag{75} \]

Now let us consider \( b_{2n+1}(\lambda, t) \). By (57) the indices \( n_1, n_2, \ldots, n_{2n+1} \) taking part in \( b_{2n+1}(\lambda, t) \) are 1 except one, say \( n_{s+1} = -1 \), where \( s = 2, 3, \ldots, 2n-1 \). Moreover, if \( n_{s+1} = -1 \), then \( n_1 + n_2 + \ldots + n_{s+1} = n_1 + n_2 + \ldots + n_{s-1} = s-1 \) and

\[ n_1 + n_2 + \ldots + n_{s+2} = n_1 + n_2 + \ldots + n_s = s. \] Therefore, by (25), \( b_{2n+1}(\lambda, t) \) for

\[ \lambda = (2\pi n)^2 + O(n^{-1}), \; t \in [0, n^{-3}] \tag{76} \]

has the form

\[ b_{2n-1}(\lambda, t) \sum_{s=2}^{2n-1} \frac{ab}{(2\pi n)^2 - (2\pi(n-s+1))^2 + O(n^{-1})} \left( \frac{(2\pi n)^2}{(2\pi(n-s))^2} + O(n^{-1}) \right). \]

One can easily see that the last sum is \( O(n^{-2}) \). Thus we have

\[ b_{2n+1}(\lambda, t) = b_{2n-1}(\lambda, t) O(n^{-2}) = \beta_n O(n^{-2}) \tag{77} \]

for all \((\lambda, t)\) satisfying (76).

Now let us estimate \( b_k(\lambda, t) \) for \( k > 2n+1 \). Since the sums in (25) are taken under conditions (57), we conclude that \( 1 \leq n_1 + n_2 + \cdots + n_s \leq 2n-1 \). Using this
instead of $1 \leq s \leq 2n - 1$ and repeating the proof of (71) we obtain that for any $(\lambda, t)$ satisfying (76) there exists

$$\lambda_3 = (2\pi n)^2 + O(n^{-1}) \quad \& \quad \lambda_4 = (2\pi n)^2 + O(n^{-1})$$

such that

$$|p_{n_1,n_2,\ldots,n_k}(\lambda_3,0)| < |p_{n_1,n_2,\ldots,n_k}(\lambda,t)| < |p_{n_1,n_2,\ldots,n_k}(\lambda_4,0)|, \forall k < 2n - 1,$$

where $p_{n_1,n_2,\ldots,n_k}(\lambda,0)$ is defined in (37). This with (38) and (77) implies that

$$\sum_{k=2n+1}^{\infty} |b_k(\lambda,t)| = \beta_n O(n^{-2})$$

(78)

for all $(\lambda, t)$ satisfying (76). In the same way, we obtain

$$\sum_{k=2n+1}^{\infty} |b_k'(\lambda,t)| = \alpha_n O(n^{-2}).$$

(79)

Thus (66) follows from (74), (75), (78) and (79).

Now we prove (67). It follows from (78), (79) and the Cauchy’s inequality that

$$\frac{\partial}{\partial \lambda} \left( \sum_{k=2n+1}^{\infty} b_k(\lambda,t) \right) = \beta_n O(n^{-1}), \quad \frac{\partial}{\partial \lambda} \left( \sum_{k=2n+1}^{\infty} b_k'(\lambda,t) \right) = \alpha_n O(n^{-1}).$$

(80)

Therefore (67) follows from (73) and (75).

From Lemma 1 it easily follows the following statement.

**Theorem 8.** If $\lambda_{n,j}(t)$ for $t \in [0, \rho]$ is a multiple eigenvalue of $H_t$, then

$$(4\pi n)^2 = -\beta_n \alpha_n (1 + O(n^{-2})).$$

(81)

**Proof.** If $\lambda_{n,1}(t) = \lambda_{n,2}(t) = : \lambda_n(t)$ is a multiple eigenvalue, then as it is noted in the above, it satisfies (52) and (54) from which we obtain

$$4D(\lambda_n(t), t) \left( 1 - \frac{1}{2} \frac{\partial}{\partial \lambda} (A(\lambda_n(t),t) + A'(\lambda_n(t),t)) \right)^2 = \left( \frac{\partial}{\partial \lambda} D(\lambda_n(t),t) \right)^2.$$  

(82)

By (32) and (34) we have

$$\frac{\partial}{\partial \lambda} (A(\lambda_n(t),t) + A'(\lambda_n(t),t)) = O(n^{-2}),$$

(83)

$$(4\pi nt + C(\lambda_n(t),t))^2 = (4\pi nt)^2(1 + O(n^{-2})),$$

(84)

$$\frac{\partial}{\partial \lambda} (4\pi nt + C(\lambda_n(t),t))^2 = (4\pi nt)^2(1 + O(n^{-2}))O(n^{-3})$$

(85)

for $t \in [0, \rho]$. On the other hand, it follows from (64) and (33) that

$$B(\lambda_n(t),t)B'(\lambda_n(t),t) = O(n^{-10}), \quad \frac{\partial}{\partial \lambda} (B'(\lambda_n(t),t)B(\lambda_n(t),t)) = O(n^{-7}).$$

(86)

Therefore from (53) and (84)-(86) we obtain

$$D((\lambda_n(t),t)) = (4\pi nt)^2(1 + O(n^{-2})) + O(n^{-10})$$

(87)

and

$$\frac{\partial}{\partial \lambda} (D(\lambda_n(t),t)) = (4\pi nt)^2(1 + O(n^{-2}))O(n^{-3}) + O(n^{-7}).$$

(88)

Using the equalities (83), (87) and (88) in (82) we get

$$4(4\pi nt)^2(1 + O(n^{-2})) = (4\pi nt)^2O(n^{-4}) + O(n^{-8}).$$

(89)
Hence, we have $t \in [0, n^{-3}]$. Then by (65), $t$ and $\lambda := \lambda_n(t)$ satisfy (68) and by Lemma 1

$$B(\lambda_n(t), t) = \beta_n \left(1 + O(n^{-2})\right), \quad B'(\lambda_n(t), t) = \alpha_n \left(1 + O(n^{-2})\right),$$

(90)

$$\frac{\partial}{\partial \lambda}(B'(\lambda_n(t), t)B(\lambda_n(t), t)) \sim \alpha_n \beta_n n^{-1} \ln |n|.$$ (91)

Therefore by (53), (84) and (85) we have

$$D(\lambda_n(t), t) = (4\pi nt)^2 \left(1 + O(n^{-2})\right) + \beta_n \alpha_n \left(1 + O(n^{-2})\right)$$

(92)

and

$$\frac{\partial}{\partial \lambda}(D(\lambda_n(t), t)) = (4\pi nt)^2 \left(1 + O(n^{-2})\right)O(n^{-3}) + O(\alpha_n \beta_n n^{-1} \ln |n|).$$ (93)

Now using (83), (92) and (93) in (82) we obtain

$$(4\pi nt)^2 \left(1 + O(n^{-2})\right) + \beta_n \alpha_n \left(1 + O(n^{-2})\right) = (4\pi nt)^2 O(n^{-4}) + (O(\alpha_n \beta_n n^{-1} \ln |n|))^2$$

which implies (81)

Note that in (92) the terms $O(n^{-2})$ don't depend on $t$, i.e., there exists $c > 0$ such that

$$|O(n^{-2})| < cn^{-2}$$ (94)

for all $t \in [0, n^{-3}]$. Henceforward, for brevity of notation, $1 + O(n^{-2})$ is denoted by $[1]$.

Now we are ready to prove the main result of this section by using Theorems 6 and 7.

**Theorem 9.** Let $S$ be the set of integer $n > N$ such that

$$-\pi + 3cn^{-2} \leq \arg(\beta_n \alpha_n) \leq \pi - 3cn^{-2}$$ (95)

and $\{t_n : n > N\}$ be a sequence defined as follows: $t_n = 0$ if $n \in S$ and

$$(4\pi nt_n)^2 (1 - cn^{-2}) = -(1 + cn^{-2} + n^{-3}) \Re(\beta_n \alpha_n)$$ (96)

if $n \notin S$, where $c$ is defined in (94). Then the eigenvalues $\lambda_{n,1}(t)$ and $\lambda_{n,2}(t)$ defined in Remark 1 are simple and satisfy (48) for $t \in [t_n, \rho]$.

**Proof.** Let $n \notin S$. It follows from (53), (64) and (84) that if $t \geq n^{-3}$ then

$$\Re D(\lambda_{n,j}(t)) > 0.$$ (97)

If $t \in [0, n^{-3}]$ then we have formula (92). Since the terms $O(n^{-2})$ in (92) satisfy (94) we have the following estimate for the real part of the first term in the right-hand side of (92):

$$\Re((4\pi nt)^2 [1]) > (4\pi nt)^2 (1 - cn^{-2}) \geq (4\pi nt_n)^2 (1 - cn^{-2})$$ (98)

for $t \in [t_n, n^{-3}]$. On the other hand if $n \notin S$ then by the definition of $S$ (95) does not hold, which implies that

$$\beta_n \alpha_n = -|\beta_n \alpha_n| e^{i\theta}, \quad |\theta| < 3cn^{-2}, \quad \Im(\beta_n \alpha_n) = O(n^{-2}) \Re(\beta_n \alpha_n).$$ (99)

Using this and (94), we obtain the following estimate for the real part of the second term in the right-hand side of (92)

$$|\Re(\beta_n \alpha_n [1])| < (1 + cn^{-2} + n^{-3}) |\Re(\beta_n \alpha_n)|.$$
and by Remark 2 it continuously depends on \( t \). Therefore the proof follows from
Theorem 6.

Now consider the case \( n \in \mathbb{S} \). By (94) we have

\[
-cn^{-2} - n^{-3} < \arg(1) < cn^{-2} + n^{-3}.
\]

Using (99) and (94) we obtain

\[
-\pi + 2cn^{-2} - n^{-3} < \arg(\beta_n\alpha_n[1]) < -\pi - 2cn^{-2} + n^{-3},
\]

\[
-cn^{-2} - n^{-3} < \arg((4\pi nt)^2[1]) < cn^{-2} + n^{-3}
\]

and the acute angle between the vectors \((4\pi nt)^2[1]\) and \(\beta_n\alpha_n[1]\) is less than \( \pi \).

Therefore by the parallelogram law of vector addition we have

\[
-\pi < \arg(D(\lambda_{n,j}(t))) < \pi, \quad D(\lambda_{n,j}(t)) \neq 0
\]

for \( t \in [0, \rho] \). Thus the proof again follows from Theorem 6.

\[ \square \]

**Corollary 2.** If the relation

\[
\inf_{q,p \in \mathbb{N}} \{ |2q\alpha - (2p - 1)\} \neq 0
\]

holds, then there exists \( c_2 \in (0, 1) \) and \( c_3 \in (0, 1) \) such that for all \( n > N \) the relations

\[
-\pi + c_2 < \arg(\alpha_n\beta_n) < \pi - c_2,
\]

\[
|\text{Im}(\alpha_n\beta_n)| > c_3|\text{Re}(\alpha_n\beta_n)|
\]

hold and the eigenvalues \( \lambda_{n,1}(t) \) and \( \lambda_{n,2}(t) \) are simple and satisfy (48) for \( t \in [0, \rho] \),
where \( \lambda_{n,1}(t) \), \( \lambda_{n,2}(t) \) and \( N \) are defined in Remark 1.

**Proof.** By (100), there exists \( c_2 \in (0, 1) \) such that

\[
-\pi + c_2 < \arg((ab)^{2n}) < \pi - c_2
\]

for all \( n \in \mathbb{N} \). Hence by the definition of \( \beta_n \) and \( \alpha_n \) (see Lemma 1) (101) and hence (102) holds. Moreover, (101) implies that (95) holds. Therefore the proof follows from Theorem 8.

\[ \square \]

**Remark 3.** Let \( \bar{A}, \bar{B}, \bar{A}', \bar{B}' \) and \( \bar{C} \) be the functions obtained from \( A, B, A', B' \) and \( C \) by replacing \( a_k, a'_k, b_k, b'_k \) with \( \bar{a}_k, \bar{a}'_k, \bar{b}_k, \bar{b}'_k \), where \( \bar{a}_k, \bar{a}'_k, \bar{b}_k, \bar{b}'_k \) differ from \( a_k, a'_k, b_k, b'_k \) respectively, in the following sense. The sums in the expressions for \( \bar{a}_k, \bar{a}'_k, \bar{b}_k, \bar{b}'_k \) are taken under condition \( n_1 + n_2 + ... + n_s \neq 0, \pm(2n + 1) \) instead of the condition \( n_1 + n_2 + ... + n_s \neq 0, \pm 2n \) for \( s = 1, 2, ..., k \). In \( \bar{b}_k, \bar{b}'_k \) the multiplicant \( q_{\pm 2n-n_1-n_2-...-n_k} \) of \( b_k, b'_k \) is replaced by \( q_{\pm 2(n+1)-n_1-n_2-...-n_k} \). To consider the case \( t \in [\pi - \rho, \pi] \) instead of (22), (26) we use

\[
(\lambda_{n,j}(t) - (2\pi n + t)^2 - \bar{A}(\lambda_{n,j}(t)))u_{n,j}(t) = (q_{2n+1} + \bar{B}(\lambda_{n,j}(t)))v_{n,j}(t),
\]

\[
(\lambda_{n,j}(t) - (-2\pi(n + 1) + t)^2 - \bar{A}'(\lambda_{n,j}(t)))v_{n,j}(t) = (q_{-2n-1} + \bar{B}'(\lambda_{n,j}(t)))u_{n,j}(t)
\]

and repeat the investigations of the case \( t \in [0, \rho] \). Note that instead of (20) for \( k \neq 0, 2n \) using the same inequality for \( k \neq 0, 2n + 1 \) and \( t \in [\pi - \rho, \pi] \) from (21) we obtain the last equalities instead of (22) and (26).

In the case \( t \in [\pi - \rho, \pi] \) instead of (48) we obtain

\[
\lambda_{n,j}(t) = (2\pi n + t)^2 - 2\pi(2n + 1)(t - \pi) + \frac{1}{2}(\bar{A}' + \bar{A}) + (-1)^j \sqrt{D(\lambda_{n,j}(t))},
\]

(103)
where $\tilde{D} = \left(2\pi(2n+1)(t-\pi)+\tilde{\zeta}\right)^2 + \tilde{B} \tilde{B}'$. Similarly, instead of (66), (81), (96) and (100) we obtain respectively the following relations

$$\tilde{B}(\lambda, t) = \tilde{\beta}_n \left(1 + O(n^{-2})\right), \quad \tilde{B}'(\lambda, t) = \tilde{\alpha}_n \left(1 + O(n^{-2})\right),$$

$$(2\pi(2n+1)(t-\pi))^2 = -\tilde{\beta}_n \tilde{\alpha}_n \left(1 + O(n^{-2})\right),$$

$$(2\pi(2n+1)(\tilde{t}_n - \pi))^2 = -\left(1 + cn^{-2} + n^{-3}\right) \operatorname{Re}(\tilde{\beta}_n \tilde{\alpha}_n)$$

$$\inf_{q,p \in \mathbb{N}} \left\{ \left| (2q + 1)\alpha - (2p - 1) \right| \right\} \neq 0,$$  

(104)

where $\tilde{\beta}_n = b^{2n+1} \left(2\pi(2n)(2n)!\right)^{-2}$, $\tilde{\alpha}_n = a^{2n+1} \left(2\pi(2n)(2n)!\right)^{-2}$, $\tilde{t}_n \in [\pi - \rho, \pi]$ and Theorems 7 and 8 and Corollary 2 continue to hold under the corresponding replacements.

As we noted in Section 2 (see Theorem 2 of [12] and Remark 1) the large eigenvalues of $H_t$ for $t \in [\rho, \pi - \rho]$ consist of the simple eigenvalues $\lambda_n(t)$ for $|n| > N$ satisfying the, uniform with respect to $t$ in $[\rho, \pi - \rho]$, asymptotic formula (17). Thus by Theorem 8 and by the just noted similar investigation, the eigenvalues $\lambda_{n,j}(t)$ for $n > N$, $j = 1, 2$ and $t \in ([\tilde{t}_n, \rho] \cup [\pi - \rho, \tilde{t}_n])$ and the eigenvalues $\lambda_{n}(t)$ for $t \in [\rho, \pi - \rho]$ and $|n| > N$ are simple. These eigenvalues satisfy (48), (17) and (103) for $t \in [\tilde{t}_n, \rho]$, $t \in [\rho, \pi - \rho]$ and $t \in [\pi - \rho, \tilde{t}_n]$ respectively. Finally, note that (100) and (104) hold if and only if (8) holds.

4. On the spectrality of $H(a,b)$. In this section we find necessary and sufficient condition on $a$ and $b$ for the asymptotic spectrality of the operator $H(a,b)$, that is, we prove Theorem 1 formulated in the introduction. To prove this main result of this section we first prove the following two statements which easily follows from the results of Section 3.

**Theorem 10.** If (8) holds, then there exists $N$ such that for $|n| > N$ the component $\Gamma_n$ of the spectrum $S(H)$ of the operator $H$ is a separated simple analytic arc with the end points $\lambda_n(0)$ and $\lambda_n(\pi)$. These components do not contain spectral singularities. In other words, the number of the spectral singularities of $H$ is finite.

**Proof.** As we noted in the end of Remark 3 if (8) holds, then (100) and (104) hold too. Therefore by Corollary 2, Theorem 2 of [13], and Remark 3 the eigenvalues $\lambda_n(t)$ for $|n| > N$ and $t \in [0, \pi]$ are simple. Therefore for $|n| > N$ the component $\Gamma_n$ of the spectrum of the operator $H$ is a separated simple analytic arc with the end points $\lambda_n(0)$ and $\lambda_n(\pi)$. It is well-known that the spectral singularities of $H$ are contained in the set of multiple eigenvalues of $H_t$ (see Proposition 2 of [17]). Hence, $\Gamma_n$ for $|n| > N$ does not contain the spectral singularities. On the other hand, the multiple eigenvalues are the zeros of the entire function $\frac{dF(\lambda)}{d\lambda}$, where $F(\lambda)$ is defined in (3). Since the entire function has a finite number of roots on the bounded sets the number of the spectral singularities of $H(a,b)$ is finite.

It was noted in [2] that (see page 539 of [2]) if $|a| \neq |b|$, then the results of [6] and [2] show that $H(a,b)$ is not a spectral operator. Since our aim is to prove the necessary and sufficient condition for asymptotic spectrality and the fact that $H(a,b)$ is not a spectral operator does not imply that it is not asymptotic spectral operator, here we prove the following fact which easily follows from the formulas of Section 3.


Proposition 1. If \(|a \neq b|\), then the operator \(H(a,b)\) has the spectral singularity at infinity and hence is not an asymptotically spectral operator.

Proof. Suppose, without loss of generality, that \(|a| < |b|\). By Theorem 7 large periodic eigenvalues \(\lambda_n(0)\) are simple. Due to (49), formulas (22), (26) and (36) for \(t = 0\) have the forms

\[
\begin{align*}
(\lambda_n(0) - (2\pi n)^2 - A(\lambda_n(0), 0))u_n &= B(\lambda_n(0), 0)v_n, \quad (105) \\
(\lambda_n(0) - (2\pi n)^2 - A'(\lambda_n(0), 0))v_n &= B'(\lambda_n(0), 0))u_n, \quad (106)
\end{align*}
\]

where \(u_n = (\Psi_{n,0}, e^{i2\pi nx}), v_n = (\Psi_{n,0}, e^{-i2\pi nx})\) and \(\lambda_n(0)\) is denoted by \(\lambda_n(0)\). By (66), \(B(\lambda_n(0), 0)\) and \(B'(\lambda_n(0), 0))\) are nonzero numbers. Moreover, by Lemma 3 of [11] we have \(A(\lambda_n(0), 0)) = A'(\lambda_n(0), 0))\). Therefore equalities (105)-(107) imply that

\[
(\lambda_n(0) - (2\pi n)^2 - A(\lambda_n(0), 0)) = (\lambda_n(0) - (2\pi n)^2 - A(\lambda_n(0), 0)) \neq 0, \quad u_n \neq 0 \text{ and } v_n \neq 0.
\]

Thus, dividing (105) and (106) side by side and using (66) we get

\[
\frac{u_n^2(0)}{v_n^2(0)} = \frac{B(\lambda_n(0), 0))}{B'(\lambda_n(0), 0))} = O\left(\frac{|a|^n}{|b|^n}\right) = O(n^{-2}). \quad (108)
\]

Using this equality and (35) we obtain

\[
u_n = v_n O(n^{-1}) = O(n^{-1}), \quad \Psi_{n,0}(x) = c_4 e^{-i2\pi nx} + O(n^{-1}),
\]

where \(|c_4| = 1\) and \(\Psi_{n,0}(x)\) is the normalized eigenfunction corresponding to \(\lambda_n(0)\). Replacing \(a\) and \(b\) by \(\bar{a}\) and \(\bar{b}\) respectively, in the same way we obtain

\[
\Psi_{n,0}^*(x) = c_5 e^{i2\pi nx} + O(n^{-1}), \quad |c_5| = 1.
\]

Thus \((\Psi_{n,0}, \Psi_{n,0}^*(x)) \to 0\) as \(n \to \infty\) and hence the proof follows from Definitions 2 and 3.

Thus the last theorem shows that if \(|a \neq b|\), then \(H(a,b)\) is not an asymptotically spectral operator and for this in is enough to consider the case \(t = 0\). However the inverse statement is not true. Moreover, by (6) and Definition 2, to find the condition for asymptotic spectrality we need to consider \(\lambda_n(t) = (\Psi_{n,t}, \Psi_{n,t}^*(x))\) and get an estimation (7) for large \(n\) and for all values of \(t \in (-\pi, \pi]\), when \(\lambda_n(t)\) is a simple eigenvalue. The proof of (7) for \(t \in [\rho, \pi - \rho]\) follows from (17). Now we estimate \(|\lambda_n(t)|^{-1}\) for \(t \in [0, \rho]\), which is the main difficulty of this section. Especially, it is very hard to estimate it when \((4\pi n t)^2\) lies in the neighborhood of \(-\beta_n\alpha_n\), since in this case \(\lambda_n(t)\) may became a multiple eigenvalue due to Theorem 7. The estimation for \(t \in [\pi - \rho, \pi]\) is similar.

Remark 4. Henceforward, for brevity of notation and according to Remark 1 instead of \(\lambda_{n,t}(t), u_{n,t}(t)\) and \(v_{n,t}(t)\) we use the notation \(\lambda_n(t), u_n(t)\) and \(v_n(t)\) and consider the case \(n > N\). The case \(n < -N\) is similar. Moreover, we redenote the numbers \(C(\lambda_n(t), t), D(\lambda_n(t), t), B(\lambda_n(t), t), B'(\lambda_n(t), t)\) by \(C(\lambda_n(t)), D(\lambda_n(t)), B(\lambda_n(t)), B'(\lambda_n(t))\). By (84), Lemma 1 and (53) the equalities

\[
4\pi n t + C(\lambda_n(t)) = (4\pi n t)[1], \quad B(\lambda_n(t)) = \beta_n[1], \quad B'(\lambda_n(t)) = \alpha_n[1]
\]

hold uniformly for \(t \in [0, n^{-3}]\), where \([1] = 1 + O(n^{-2})\). By Theorem 8, \(\lambda_n(t)\) satisfies (48) for \(j = 2\) and \(t \in [t_n, \rho]\). If \(t \in [0, t_n]\), then \(\lambda_n(t)\) satisfies either (44) or (45).
Using formula (48) for \( j = 2 \) in (50) and (51) and taking into account the notations and arguments of Remark 4 we obtain

\[
E_-(\lambda_n(t))u_n(t) = \beta_n v_n(t)[1],
\]

\[
E_+(\lambda_n(t))v_n(t) = \alpha_n u_n(t)[1],
\]

where \([1] = 1 + O(n^{-2}),\)

\[
E_{\pm}(\lambda_n(t)) = s(t)\sqrt{(4\pi nt)^2[1] + \alpha_n\beta_n[1] \pm 4\pi nt[1]},
\]

\[4\pi nt[1] = C(\lambda_n(t)) + 4\pi nt,\]

\(s(t)\) is \(-1\) or \(1\) if \(\lambda_n(t)\) satisfies (44) or (45) respectively.

Since the boundary condition (2) is self-adjoint we have \((H_t(q))^* = H_t(q)\). Therefore, all formulas and theorems obtained for \(H_t\) are true for \(H^*_t\) if we replace \(a\) and \(b\) by \(\overline{a}\) and \(\overline{b}\) respectively. For instance, (35) and (36) hold for the operator \(H^*_t\) and hence we have

\[
\Psi_{n,t}^*(x) = u_n^*(t)e^{i(2\pi n t)x} + v_n^*(t)e^{-i(2\pi n t)x} + h_{n,t}^*(x),
\]

\[(h_{n,t}^*, e^{i(2\pi n t)x}) = 0, \quad \|h_{n,t}^*\| = O(n^{-1}), \quad |u_n^*(t)|^2 + |v_n^*(t)|^2 = 1 + O(n^{-2}).\]

Similarly the formulas (109) and (110) for the operator \(H^*_t\) have the form

\[
E_-(\lambda_n(t))u_n^*(t) = \alpha_n v_n^*(t)[1], \quad E_+(\lambda_n(t))v_n^*(t) = \beta_n u_n^*(t)[1].
\]

For \(n \geq N\) it follows from (35), (36), (112) and (113) that

\[
(\Psi_{n,t}, \Psi_{n,t}^*) = u_n(t)\overline{u_n^*(t)} + v_n(t)\overline{v_n^*(t)} + O(n^{-1}).
\]

By (6) and Definition 2 to study the asymptotic spectrality we need to consider the expression \((\Psi_{n,t}, \Psi_{n,t}^*)\). First let us note the following simple statement.

**Proposition 2.** The equality \((\Psi_{n,t}, \Psi_{n,t}^*) = 1 + O(n^{-1})\) holds uniformly for \(t \in [n^{-3}, \rho]\).

**Proof.** If \(t \in [n^{-3}, \rho]\), then by (84), (53) and (64) the coefficient of \(v_n(t)\) in (110) is greater than \(n\) times the coefficient of \(u_n(t)\). Therefore from (35) and (36) we get

\[
\Psi_{n,t}(x) = e^{i(2\pi n t)x} + O(n^{-1}).
\]

Instead of (110), (35) and (36) using (114), (112) and (113) in the same way we obtain that \(\Psi_{n,t}^*\) satisfies the same formula. These formulas imply the proof of the proposition. \(\Box\)

By (115) to estimate \((\Psi_{n,t}, \Psi_{n,t}^*)\) we need to consider \(u_n(t)\overline{u_n^*(t)} + v_n(t)\overline{v_n^*(t)}\). Using (110), (114) and the obvious equality

\[
E_+(\lambda_n(t))E_-(\lambda_n(t)) = \alpha_n\beta_n[1]
\]

we obtain

\[
v_n(t)\overline{v_n^*(t)}(E_+(\lambda_n(t)))^2 = u_n(t)\overline{u_n^*(t)}(E_+(\lambda_n(t))\alpha_n\beta_n[1] \quad \text{and}
\]

\[
\frac{v_n(t)\overline{v_n^*(t)}}{u_n(t)\overline{u_n^*(t)}} = \frac{\alpha_n\beta_n[1]}{(E_+(\lambda_n(t)))^2} = \frac{E_-(\lambda_n(t))[1]}{E_+(\lambda_n(t))}.
\]

This with definition of \(E_{\pm}(\lambda_n(t))\) implies that

\[
u_n(t)\overline{u_n^*(t)} + v_n(t)\overline{v_n^*(t)} = u_n(t)\overline{u_n^*(t)}F_+(\lambda_n(t)),
\]

where

\[
F_+(\lambda_n(t)) = \frac{G(\lambda_n(t))}{E_+(\lambda_n(t))}, \quad G(\lambda_n(t)) = O(n^{-2})E_-(\lambda_n(t)) + 2s(t)\sqrt{(4\pi nt)^2[1] + \alpha_n\beta_n[1]}.
\]

Therefore in the following lemma we investigate \(F_+(\lambda_n(t))\) and \(u_n(t)\overline{u_n^*(t)}\).
Remark 5. By Proposition 2 we need to estimate \((\Psi_{n,t}, \Psi_{n,t}^*)\) for \(t \in [0, n^{-3}]\). For this we divide the last interval into three subintervals \(I_1, I_2\) and \(I_3\), where \(I_k\) for \(k = 1, 2, 3\) are respectively the set of all \(t \in [0, n^{-3}]\) such that \(4\pi nt\) belongs to the sets \([0, \frac{1}{2} \varepsilon_n]\), \((\frac{1}{2} \varepsilon_n, \frac{3}{2} \varepsilon_n)\) and \([\frac{3}{2} \varepsilon_n, 4\pi n^{-2}]\), where \(\varepsilon_n = \sqrt{|\alpha_n\beta_n|}\). It follows from (96) that \(I_3 \subset [t_n, n^{-3}]\). Therefore \(s(t) = 1\) if \(t \in I_3\).

Lemma 2. (a) The relation

\[ F_+ (\lambda_n(t)) \sim 1 \]  

holds uniformly for \(t \in (I_1 \cup I_3)\). If \(|a| = |b|\), then there exists \(c_0 > 0\) such that

\[ |u_n(t)\overline{u_n}(t)| > c_0 \]  

for all \(t \in (I_1 \cup I_3)\).

(b) If (8) holds then the relation (119) hold uniformly for \(t \in I_2\). If \(|a| = |b|\), then (120) holds for all \(t \in I_2\).

Proof. (a) If \(t \in I_1\) then we have \(4\pi nt \leq \frac{1}{2} \varepsilon_n\). It implies that

\[ G(\lambda_n(t)) \sim \varepsilon_n, \ E_+(\lambda_n(t)) \sim \varepsilon_n \]  

and hence (119) holds. If \(t \in I_3\), then \(s(t) = 1\) (see Remark 5) and by (46) we have

\[ G(\lambda_n(t)) \sim 4\pi nt, \ E_+(\lambda_n(t)) \sim 4\pi nt. \]  

Therefore (119) holds. Now suppose that \(|a| = |b|\). If \(t \in I_1\), then using (121) and taking into account that \(\alpha_n \sim \beta_n\) when \(|a| = |b|\), we obtain \(E_+(\lambda_n(t)) \sim \alpha_n\). Therefore from (110) and (36) we obtain \(u_n(t) \sim v_n(t) \sim 1\). In the same way from (114) and (113) we get \(u_n^*(t) \sim v_n^*(t) \sim 1\). These relations imply (120). If \(t \in I_3\) and \(|a| = |b|\), then using (46) we see that \(|E_+(\lambda_n(t))| > |\alpha_n|\). Therefore using (110) and (36) we obtain \(|u_n(t)| > 2/3\). Similarly \(|u_n^*(t)| > 2/3\). Thus (120) holds.

(b) If (8) holds then we have inequality (102). Using it and the relation \(t \in I_2\) we obtain

\[ \text{Im}((4\pi nt)^2[1 + \alpha_n^2\beta_n]) \sim \alpha_n \beta_n \sim (4\pi nt)^2, \ E_+(\lambda_n(t)) \sim G(\lambda_n(t)) \sim \varepsilon_n. \]

It implies (119). If \(|a| = |b|\) and \(t \in I_2\), then \(|\alpha_n| = |\beta_n|\) and \(4\pi nt \sim \alpha_n\). Therefore one can easily verify that

\[ E_+(\lambda_n(t)) \sim \alpha_n. \]

Using it and arguing as in the case \(t \in I_1\) we get the proof of (120).

Now we prove the main result (extended version of Theorem 1) of this section.

Theorem 11. (a) The operator \(H\) has no the spectral singularity at infinity and is an asymptotically spectral operator if and only if \(|a| = |b|\) and (8) holds.

(b) Let \(|a| = |b|\). If \(\alpha\) is a rational number, that is, \(\alpha = \frac{m}{q}\) where \(m\) and \(q\) are irreducible integers and \(\alpha\) is defined in (8), then the operator \(H\) has no the spectral singularity at infinity and is an asymptotically spectral operator if and only if \(m\) is an even integer. If \(\alpha\) is an irrational number, then \(H\) has the spectral singularity at infinity and is not an asymptotically spectral operator if and only if there exists a sequence of pairs \(\{(q_k, p_k)\} \subset \mathbb{N}^2\) such that

\[ |\alpha - (2p_k - 1)(q_k)^{-1}| = o \left((q_k)^{-1}\right), \]

where \(2p_k - 1\) and \(q_k\) are irreducible integers.
Proof. It is clear that (b) follows from (a). First we prove the sufficiency of (a). For this assuming that \(| a | = | b |\) and (8) holds, we prove (7) for large \(n\). If \(t \in [n^{-3}, \rho]\), then by Proposition 2, (7) holds. Using (115), (118), and Lemma 2 we get (7) in the case \(t \in [0, n^{-3}]\). Hence (7) for \(t \in [0, \rho]\) is proved. In the same way, by using Remark 3, we prove (7) for \(t \in [\pi - \rho, \pi]\). If \(t \in [\rho, \pi - \rho]\), then (7) follows from (17). Thus (7) folds for \(t \in [0, \pi]\) and \(n > N\). In the same way we prove it for \(t \in (-\pi, 0)\) and \(n < -N\).

It remains to prove the necessity of (a). Suppose that \(H(a, b)\) is an asymptotically spectral operator. Then by Proposition 1, \(| a | = | b |\). Now we prove that (8) holds. Suppose to the contrary that (8) does not hold. Then there exists a sequence of pairs \(\{(q_k, p_k)\}\) such that \(q_k a - (2p_k - 1) \to 0\). First suppose that the sequence \(\{q_k\}\) contains infinite many of even number. Then one can easily verify that there exists a sequence \(\{n_k\}\) satisfying

\[
\text{Im}((ab)^{2n_k}) = o((ab)^{2n_k}), \quad \lim_{k \to \infty} \text{sgn}(\text{Re}((ab)^{2n_k})) = -1.
\]

By Theorem 8, for the sequence \(\{t_{n_k}\}\) defined by (96) and now, for simplicity, redenoted by \(\{t_k\}\) the eigenvalues \(\lambda_{n_k,j}(t_k)\) are simple and the following relations hold

\[
(4\pi n_k t_k)^2 = -\text{Re}(\beta_{n_k} \alpha_{n_k})(1 + o(1)) = -(\beta_{n_k} \alpha_{n_k})(1 + o(1)),
\]

\[
(4\pi n_k t_k)^2 + (\beta_{n_k} \alpha_{n_k}) = o(\beta_{n_k}^2), \quad 4\pi n_k t_k \sim \beta_{n_k} \sim \alpha_{n_k}.
\]

Therefore we have \(G(\lambda_{n_k}(t_k)) = o(\beta_{n_k})\). It with (124) implies that \(F_{+}(\lambda_{n_k}(t_k)) = o(1)\) as \(k \to \infty\). Thus \(|d_{n_k}(t_k)| \to 0\) as \(k \to \infty\), due to (115) and (118). In the same way we prove it when \(\{q_k\}\) contains infinite number of odd number. It contradicts to the assumption that \(H(a, b)\) is an asymptotically spectral operator, due to Definition 2.

Now using Theorem 1, that is, Theorem 10 (a) we prove Corollary 1 (see introduction).

The proof of Corollary 1. Since any self-adjoint operator is spectral, we need to prove that if \(H(a, b)\) is a spectral operator and \(ab\) is real, then (4) is a real potential. By Definitions 1 and 2 the spectral operator is also asymptotically spectral operator. Thus \(ab\) is real and by Theorem 1, (8) holds. If \(ab < 0\) then \(\alpha - 1 = 0\), where \(\alpha = \pi^{-1} \arg(ab)\), which contradicts (8). Hence we have \(ab > 0\). On the other hand, Proposition 1 implies that \(| a | = | b |\). From the last two relations we obtain \(b = \pi\). It means that (4) is a real potential and \(H(a, b)\) is a self-adjoint operator.

5. On the spectral expansion of \(H(a, b)\). Now we consider the forms of the spectral expansion of \(H(a, b)\). For this as is noted in the introduction we need to investigate in detail the ESS and ESS at infinity for \(H(a, b)\). Besides, we use the following results of the papers [14, 17, 18] formulated as summary.

Summary 1. (a) The spectral expansion has the elegant form (9) if and only if \(L(q)\) has no ESS and ESS at infinity (see page 7 of [18]).

(b) If \(L(q)\) has no ESS at infinity, then the number of ESS is at most finite and the spectral expansion has the asymptotically elegant form (13) (see Theorem 3.13 of [18]).

(c) ESS of \(L(q)\) is a multiple 2-periodic eigenvalue. Note that the eigenvalues of \(L_0(q)\) and \(L_\pi(q)\) is called as 2-periodic eigenvalues. If the geometric multiplicity of the multiple 2-periodic eigenvalue is 1, then it is ESS (see Proposition 4 of [17]).
(d) If $0 < |ab| < 16/9$ then all 2-periodic eigenvalues of $H(a, b)$ are simple (see Theorems 13 and 15 of [14]).

(e) If $\Lambda = \lambda_n(t_0)$ is a multiple eigenvalue, then the sum of the expressions $a_k(t)\Psi_{k,t}(x)$ for $k \in \{s \in \mathbb{Z} : \lambda_s(t_0) = \Lambda\}$ is integrable in some neighborhood of $t_0$. If $\Lambda$ is an ESS then at least two of these expressions are nonintegrable (see Remark 2 of [17]).

As we noted at the end of introduction, in this section we consider the spectral expansion of $H(a, b)$ for all potential of the form (4) by dividing it into two complementary cases:

Case 1: $ab \neq 0$ and Case 2: either $a = 0$ or $b = 0$.

First we consider Case 1 and prove that in this case the operator $H(a, b)$ has no ESS at infinity. Therefore by Summary 1(b) the number of ESS is at most finite and the spectral expansion has the asymptotically elegant form (13). For this, due to Definition 5, we need to study the existence and the behavior of

$$
\int_{(-\pi, \pi]} |d_n(t)|^{-1} dt
$$

for large $n$. Note that the estimations that was done in Section 4 for $|d_n(t)|^{-1}$ are not enough for the estimations of (125). Here we need more sharp and complicated estimations due to the followings. In Section 4 some estimations for $(\Psi_{n,t}, \Psi_{n,t}^*)$ were done under assumption $|a| = |b|$ (first condition) and the estimations for $t \in I_2$, where a multiple eigenvalues may appear, were done under condition (8) (second condition), while in this section the estimations are done for all cases of the potential (4). Moreover in Section 4 we considered only boundlessness of $|d_n(t)|^{-1}$, while here we consider its integrability and investigate the limit of (125) as $n \to \infty$.

In this section to estimate $(\Psi_{n,t}, \Psi_{n,t}^*)$ we also use (118). However, if the first condition does not hold, say if $|a| < |b|$, then the multiplicand $u_n(t)\Sigma_n^*(t)$ and hence the left-hand side of (118) is $O(|a|^n |b|^{-n})$. Therefore (115) is ineffective for the estimation of $(\Psi_{n,t}, \Psi_{n,t}^*)$. For this first of all we consider the Bloch functions in detail which was done in Theorem 11. Moreover, in Section 4 we have used essentially the second condition (8) to estimate $(\Psi_{n,t}, \Psi_{n,t}^*)$ for $t \in I_2$. To do this estimation without condition (8) we develop a new approach at the end in this section. Besides we use the first statements of Lemma 2(a) and Lemma 2(b) which hold without assumption $|a| = |b|$. Finally, note that in the proof of Lemma 2(b) we proved that if (102) holds then (119) holds uniformly for $t \in I_2$.

**Theorem 12.** If $|a| < |b|$ then

$$(\Psi_{n,t}, \Psi_{n,t}^*) = u_n(t)\Sigma_n^*[1] + v_n(t)\Sigma_n^*[1] + O(n^{-1} a^n b^{-n}) .$$

**Proof.** First we write the function $h_{n,2,t}$ defined in (35) as sum of $f_n$ and $g_n$ defined by

$$
f_n(x) = \sum_{0 < |k| \leq s} \left( (\Psi_{n,t}, e^{i(2\pi(n+k)+t)x}) e^{i(2\pi(n+k)+t)x} + (\Psi_{n,t}, e^{i(2\pi(-n-k)+t)x}) e^{i(2\pi(-n-k)+t)x} \right)
$$

and

$$
g_n(x) = \sum_{k; |k| > s} (\Psi_{n,t}, e^{i(2\pi k+t)x}) e^{i(2\pi k+t)x} .$$
where $s$ is a positive integer of order $n/\ln n$ such that
\[ n^{-s} = O\left(n^{-1}a^n b^{-n} \left(|a| + |b| \right)^{-s}\right). \] (127)

If $|k| > s$ then iterating the formula
\[ (\Psi_{n,t}, e^{i(2\pi k + t)x}) = \frac{\alpha(\Psi_{n,t}, e^{i(2\pi (k+1) + t)x}) + b(\Psi_{n,t}, e^{i(2\pi (k-1) + t)x})}{\lambda_n(t) - (2\pi k + t)^2} \] (128)
s times we obtain
\[ (\Psi_{n,t}, e^{i(2\pi k + t)x}) = \sum_{n_1, n_2, \ldots, n_s} \prod_{j=1, 2, \ldots, s} q_{n_j} q_{n_j} \cdots q_{n_s} \left(\frac{1}{\lambda_n(t) - (2\pi (n + k + t))^2}\right) \]
where $n_j$ is either 1 or $-1$ and $q_{-1} = a$, $q_1 = b$. Therefore using (20) and (127) we obtain
\[ |(\Psi_{n,t}, e^{i(2\pi k + t)x})| \leq \sum_{j=1, 2, \ldots, s} \left(\frac{|a| + |b|}{n^2 - (n + k - j)^2}\right) = O \left(\frac{a^n}{kn^s}\right) = O \left(\frac{a^n}{knb^n}\right). \]

If $0 < |k| \leq s$ then we iterate the formula
\[ (\Psi_{n,t}, e^{i(2\pi (n + k) + t)x}) = \frac{\alpha(\Psi_{n,t}, e^{i(2\pi ((n+k) + t)x}) + b(\Psi_{n,t}, e^{i(2\pi ((n+k-1) + t)x})}{\lambda_n(t) - (2\pi (n + k + t))^2} \]
obtained from (128) by replacing $k$ with $n + k$ as follows. After each iteration we isolate the term containing $(\Psi_{n,t}(x), e^{i(2\pi n_k + t)x})$ and iterate the other terms. Continuing this procedure $s$ times and estimating as above we obtain
\[ (\Psi_{n,t}, e^{i(2\pi (n + k) + t)x}) = u_n(t) O(n^{-2}) + O \left(n^{-1}k^{-1}a^n b^{-n}\right). \]

In the same way we obtain
\[ (\Psi_{n,t}, e^{i(2\pi (n - k) + t)x}) = v_n(t) O(n^{-2}) + O \left(n^{-1}k^{-1}a^n b^{-n}\right). \]
The same estimations hold for the eigenfunction $\Psi_{n,t}$. These estimations imply (126).

Now to estimate $|d_n(t)|$ we use (126) and the following formula
\[ u_n \overline{u}_n[1] + v_n \overline{v}_n[1] = [1] u_n \overline{u}_n F_+(\lambda_n(t)), \] (129)
where $F_+(\lambda_n(t))$ is defined in (118) and the proof of (129) can be obtained by repeating the proofs of (118). In Lemma 2 the expression $u_n \overline{u}_n$ is estimated under condition $|a| = |b|$. Now we estimate it if this condition does not holds and without loss of generality assume that $|a| < |b|$. First we estimate $u_n$.

**Lemma 3.** If $|a| < |b|$ and $t \in [0, \rho]$, then
\[ u_n(t) = 1 + O(n^{-1}). \] (130)

**Proof.** First study the case $4\pi nt \in [0, 2\varepsilon_n]$, where $\varepsilon_n$ is defined in Remark 5. Then by (111) $|E-(\lambda_n(t))| < 5\varepsilon_n$. Therefore, using (109) and then taking into account that
\[ |\alpha_n| |\beta_n|^{-1} = |a^{2n}| |b^{2n}|^{-1} = O \left(n^{-2}\right) \] (131)
(see Lemma 1) we obtain
\[ |v_n(t)| \leq \frac{|E-(\lambda_n(t))|}{|\beta_n[1]|} |u_n(t)| \leq \frac{6\sqrt{|\alpha_n\beta_n|}}{|\beta_n|} |u_n(t)| = O \left(\frac{1}{n}\right). \]
Now consider the case $4\pi nt > 2\varepsilon_n$. Then by Theorem 8 we have $t > \tau_n$ and in formula \((111)\) we should take $s(t) = 1$. Hence using \((111)\) and \((46)\) one can conclude that $|E_+(\lambda_n(t))| > \varepsilon_n$. Therefore from \((110)\) and \((131)\) it follows that

$$|v_n(t)| = \frac{|\alpha_n|}{|E_+(\lambda_n(t))|} |u_n(t)[1]| \leq \frac{\sqrt{\alpha_n}}{\beta_n} |u_n(t)[1]| = O\left(\frac{1}{n}\right).$$

These estimations for $|v_n(t)|$ together with \((36)\) imply \((130)\).

Now using the last lemma and the formulas \((129)\) and \((126)\) we estimate $d_n(t)$.

\begin{lemma}
Let $I_4$ and $I_5$ be respectively the set of all $t \in I_3$ such that $4\pi nt$ belongs to the intervals $\left[\frac{4}{5} \varepsilon_n, |\beta_n|\right]$ and $|\beta_n|$, $4\pi n^{-2}$, where $I_5$ is defined in Remark 5. Let $|a| < |b|$. 
\begin{enumerate}
  \item If $t \in I_5$, then $|d_n(t)| \sim 1$.
  \item If $t \in (I_1 \cup I_4)$, then there exists $c_7 > 0$ such that
    $$|d_n(t)| \geq c_7 |a^n b^{-n}|.$$ \hfill \(\text{(132)}\)
  \item If $t \in I_2$ and \((102)\) holds, then \((132)\) is satisfied.
\end{enumerate}
\end{lemma}

\textbf{Proof.} \(\text{(a)}\) If $t \in I_5$, then it follows from \((111)\) and \((131)\) that $|E_+(\lambda_n(t))| > \frac{\sqrt{3}}{2} |\beta_n|$. Using it and \((114)\) we obtain

$$|v_n^*(t)| = |\beta_n| |E_+(\lambda_n(t))|^{-1} |u_n^*(t)[1]| \leq |u_n^*(t)|.
$$

It with \((113)\) implies that $|u_n^*(t)| > 2/3$. Therefore we get the proof of \((a)\) by using \((126)\), \((129)\), and \((130)\) and taking into account that \((119)\) holds for $|a| \neq |b|$ too.

\(\text{(b)}\) Using \((111)\) and the definitions of $I_1$ and $I_4$ one can easily see that

$$E_+(\lambda_n(t)) \geq c_8 \varepsilon_n$$ \hfill \(\text{(133)}\)

for $t \in (I_1 \cup I_4)$. It with \((114)\) and \((131)\) implies that

$$|u_n^*(t)| \geq c_8 \left| v_n^* \frac{\sqrt{\alpha_n \beta_n}}{\beta_n[1]} \right| \geq c_8 \left| v_n^* \frac{\sqrt{\alpha_n}}{\sqrt{\beta_n[1]}} \right| \geq c_8 \left| v_n^* \frac{a^n}{b^n} \right|.
$$

Therefore using \((133)\) we obtain $|u_n^*(t)| \geq c_9 |a^n b^{-n}|$. Now \((132)\) follows from \((126)\), \((129)\), \((130)\) and \((119)\).

\(\text{(c)}\) If \((102)\) holds, then \((133)\) holds for $t \in I_2$ too. Using it and repeating the proof of \(\text{(b)}\) we get the proof of \(\text{(c)}\).

The obtained estimations are enough to prove the main results if \((102)\) holds. Now we estimate $d_n(t)$ for $t \in I_2$ when \((102)\) doesn’t hold. This case is the most complicated case. In this case to estimate $(d_k(t))^{-1}$ we use the following formula

$$\frac{-1}{d_k(t)} = \frac{\| \Phi(\cdot, \lambda_n(t)) \| \| \Phi_{-1}(\cdot, \lambda_n(t)) \|}{\varphi(1, \lambda_n(t)) F'(\lambda_n(t))} = \frac{\| G(\cdot, \lambda_n(t)) \| \| G_{-1}(\cdot, \lambda_n(t)) \|}{\theta'(1, \lambda_n(t)) F'(\lambda_n(t))} \quad \text{(134)}$$

(see \((29)\) and \((32)\) of \([17]\)) where

$$\Phi_2(x, \lambda) = \varphi \theta(x, \lambda) + (e^{it} - \theta) \varphi(x, \lambda), \quad G_2(x, \lambda) = \varphi' \varphi(x, \lambda) + (e^{it} - \varphi') \theta(x, \lambda),$$

$\varphi = \varphi_1(1, \lambda), \varphi' = \varphi'(1, \lambda), \theta = \theta(1, \lambda), \theta' = \theta'(1, \lambda), \varphi(x, \lambda)$ and $\theta(x, \lambda)$ are defined in \((3)\). Using \((3)\) and taking into account the Wronskian equality $\theta \varphi - \varphi' \theta = 1$ we obtain $(e^{it} - \theta)(e^{-it} - \theta) = -\varphi \theta'$. Therefore at least one of the following inequality holds

$$|e^{it} - \theta| \leq 1, \quad \left| \frac{e^{-it} - \theta}{\varphi} \right| \leq 1, \quad \left| \frac{e^{it} - \theta}{\theta'} \right| \leq 1, \quad \left| \frac{e^{-it} - \theta}{\theta'} \right| \leq 1.$$
Without loss of generality we suppose that the first equality holds. Then we have
\[
\frac{\|\Phi_{t}(\cdot, \lambda_{n}(t))\|}{\|\varphi(1, \lambda_{n}(t))\|} \leq \|\theta(\cdot, \lambda_{n}(t))\| + \|\varphi(\cdot, \lambda_{n}(t))\|.
\]
It with (65) and the following asymptotic formulas (see page 63 of [3])
\[
\theta(x, \lambda) = \cos \mu x + \frac{\sin \mu x}{2\mu} Q(x) + \frac{\cos \mu x}{4\mu^2} (q(x) - q(0)) - \frac{1}{2} Q^2(x) + O(\mu^{-3}),
\]
\[
\varphi(x, \lambda) = \frac{\sin \mu x}{\mu} - \frac{\cos \mu x}{2\mu^2} Q(x) + \frac{\sin \mu x}{4\mu^3} (q(x) + q(0)) - \frac{1}{2} Q^2(x) + O(\mu^{-4}),
\]
where \( \mu = \sqrt{x}, |\text{Im} \mu| < 3 \), \( Q(x) = \int_0^x q(x)dx \) implies the following inequalities
\[
\frac{\|\Phi_{t}(\cdot, \lambda_{n}(t))\|}{\|\varphi(1, \lambda_{n}(t))\|} < c_{10}, \quad \|\Phi_{-t}(\cdot, \lambda_{n}(t))\| < c_{10} n^{-4}
\]
for \( t \in I_2 \). Therefore using the substitution \( \lambda = \mu^2 \), \( f(\mu) = F(\lambda), F'(\lambda) = f'(\mu) \frac{1}{2\mu} \), we get
\[
\frac{1}{d_{n}(t)} \leq \frac{c_{11} n^{-3}}{|f'(\mu_n(t))|}, \tag{135}
\]
where \( f'(\mu) = \frac{df}{d\mu}, \mu_n(t) = \sqrt{\lambda_n(t)} \in \{z \in \mathbb{C} : |z - 2\pi n| < 2\} = \Omega(2\pi n, 2) \).

Now using the well-known asymptotic formula for the Hill discriminant \( f(\mu) = F(\lambda) \):
\[
f(\mu) = 2 \cos \mu + R(\mu), \quad |R(\mu)| < c_{12} |\mu|^3
\]
(see page 64 of [3]) and the Cauchy’s integral formula
\[
f^{(k)}(\mu_n(t)) = \frac{k!}{2\pi i} \int_{\gamma} \frac{f(\xi)}{(\xi - \mu_n(t))^{k+1}} d\xi,
\]
where \( R(\mu) \) is an entire function and \( \mu \in \Omega(2\pi n, 2), \gamma = \{z \in \mathbb{C} : |z - \mu_n(t)| = 1\} \) and \( \mu_n(t) \in \Omega(2\pi n, 1) \) we estimate \( |f'_{\mu_n(t)}| \). Using (136) and (137) for \( k = 1 \) we obtain
\[
f'_{\mu_n(t)} = -2 \sin \mu + R'(\mu), \quad |R'(\mu)| < c_{13} |\mu|^3.
\]
By (138) and Rouche’s theorem the equation \( f'(\mu) = 0 \) has a unique root \( \mu_n \) in \( c_{14} n^{-3} \) neighborhood on \( 2\pi n \) for large \( n \). Moreover, using (136) and (3) we see that \( \mu_n = \mu_n(t_0), |t_0| < c_{15} n^{-3/2} \) and \( (\mu_n(t_0))^2 \) is a multiple eigenvalue of the operator \( H_{t_0}(a, b) \) satisfying \( |\mu_n(t_0) - 2\pi n| < c_{14} n^{-3} \). It with (65) gives
\[
|\mu_n(t) - \mu_n(t_0)| < c_{16} n^{-3}
\]
for \( t \in I_2 \). Since the proof of (81) is unchanged if \([0, \rho]\) is replaced by \( \{t| t < c_{15} n^{-3/2}\} \) we have
\[
(4\pi n t_0)^2 = -\beta_n \alpha_n (1 + O(n^{-2})).
\]
\[ \tag{140} \]

**Remark 6.** As is noted in above we consider the case when (102) does not hold say for \( c_3 = 1/9 \). Then \(-\beta_n \alpha_n = u_n + i v_n, |u_n| \geq 9 |v_n| \). Since \( \mu_n(t_0) = \mu_n(t_0) \), without using of generality it can be assumed that \( \text{Re} t_0 \geq 0 \). These arguments with (140) imply that \( t_0 = u_0 + i v_0, u_0 > 8 |v_0| \). Using it, (140) and definition of \( I_2 \) we obtain the following relations which we use in the proof of the main results
\[
|t_0| = (4\pi n)^{-1} \varepsilon_n (1 + O(n^{-2})), \quad \frac{3}{4} |t_0| \leq u_0 \leq |t_0|, \quad |t - t_0| < |t_0|
\]
for all \( t \in I_2 \), where \( \varepsilon_n \) is defined in Remark 5.
Lemma 5. If \( t \in I_2 \), then
\[
\frac{1}{|d_k(t)|} \leq \frac{c_{11}n^{-3}}{|\sin t_0|\sqrt{|t-t_0|}}.
\] (142)

Proof. By formulas (136) and (137) for \( k = 2 \) we have
\[
f''(\mu) = -2\cos\mu + R''(\mu), \quad |R''(\mu)| < c_{17} |\mu|^{-3}
\] (143)
for \( |\mu - \mu_n(t_0)| < c_{18}n^{-3} \). Using the Taylor’s theorem for \( f'(\mu) \) and taking into account that \( f'(\mu_n(t_0)) = 0 \) we get
\[
f'(\mu_n(t)) = \int_{\mu_n(t)}^{\mu_n(t_0)} f''(\mu_n(t))(\mu_n(t) - \mu_n(t_0)) + f_2(\mu_n(t))(\mu_n(t) - \mu_n(t_0))^2,
\]
where \( |f_2(\mu_n(t))| < c_{19} |\mu_n(t)|^{-1} \) (see pages 125 and 126 of [1]). It with (143), (139) and (65) imply that
\[
|f'(\mu_n(t))| > |\mu_n(t) - \mu_n(t_0)|.
\] (144)
Similarly, using the Taylor’s theorem for \( f(\mu) \) and \( \cos t \) and taking into account that \( f(\mu_n(t_0)) = 2\cos t_0, f'(\mu_n(t_0)) = 0, f''(\mu_n(t)) = -2 + O(n^{-1}) \) we obtain
\[
f(\mu_n(t)) = 2\cos t_0 - (\mu_n(t) - \mu_n(t_0))^2(1 + o(1)),
\]
\[
2\cos t - 2\cos t_0 - 2(\sin t_0) (t - t_0) - (t - t_0)^2 (1 + o(1)).
\]
These equalities with \( f(\mu_n(t)) = 2\cos t \) and (141) imply that
\[
|\mu_n(t) - \mu_n(t_0)| > |\sin t_0|\sqrt{|t-t_0|}.
\]
It with (135) and (144) implies (142). \( \square \)

Theorem 13. If \( ab \neq 0 \), then the operator \( H(a,b) \) has no ESS at infinity.

Proof. Using Lemma 4 and the definitions of \( I_1, I_4, I_5, \varepsilon_n \) and \( \beta_n \) one can verify that
\[
\int_{[0,n^{-3}] \setminus I_2} |d_n(t)|^{-1} \, dt = O(n^{-3}).
\] (145)
On the other hand, by Proposition 2 the integral of \( |d_n(t)|^{-1} \) over \( [n^{-3}, \rho] \) is less than \( c_{20} \). If (102) holds then by Lemma 3(c) in (145) the integral of \( |d_n(t)|^{-1} \) over \( [0,n^{-3}] \setminus I_2 \) can be replaced by the integral over \( [0,n^{-3}] \). If (102) does not hold, then using (142), (141) and the obvious relations \( I_2 \subset [0,2|t_0|] \) and \( |t-u_0| \leq |t-t_0| \) we obtain
\[
\int_{I_2} |d_n(t)|^{-1} \, dt \leq \int_{[0,2|t_0|]} \frac{c_{17}n^{-3}}{|\sin t_0|\sqrt{|t-u_0|}} \, dt = O(n^{-3}).
\]
Thus the integral of \( |d_n(t)|^{-1} \) over \( [0,\rho] \) is less than \( c_{21} \). Similarly, integral of \( |d_n(t)|^{-1} \) over \( [\pi - \rho, \pi] \) is less than \( c_{21} \). These inequalities with (17) imply that the integral of \( |d_n(t)|^{-1} \) over \( [0,\pi] \) is less than \( c_{22} \). Since \( \lambda_n(-t) = \lambda_n(t) \) (see Remark 1) it follows from (134) that \( |d_n(-t)|^{-1} = |d_n(t)|^{-1} \). Therefore the integral of \( |d_n(t)|^{-1} \) over \( (-\pi, \pi] \) is less than \( 2c_{22} \) and hence by Definition 5 the operator \( H(a,b) \) has no ESS at infinity. \( \square \)
The proofs of Theorems 2, 3 and 4. The proofs of Theorems 2 and 3 follow from Theorem 12 and Summary 1. Now we prove Theorem 4. It is well-known that (see [4], [7] and [15]) if either \( a = 0 \) or \( b = 0 \), then \( \lambda_n(0) = (2\pi n)^2 \) for \( n \in \mathbb{Z} \setminus \{0\} \) and \( \lambda_n(\pi) = (2\pi n + \pi)^2 \) for \( n \in \mathbb{Z} \) are the double 2-periodic eigenvalues. Moreover in [7] it was proven that the geometric multiplicities of these eigenvalues is 1. Thus \( k \) is nonintegrable. Besides it readily follows from Definitions 4 and 5 that if \( L \) holds, where

\[
\mathcal{I} = \int_{h,h} f(t) dt,
\]

then the sum of two expressions \( a_k(t)\Psi_{k,t}(x) \) and \( a_{-k}(t)\Psi_{-k,t}(x) \) corresponding to the ESS \( \lambda_k(0) \) is integrable on \([-h,h]\), while both of them is nonintegrable. Besides \( a_0(t)\Psi_{0,t} \) is integrable since \( \lambda_0(0) \) is a simple eigenvalue and hence is not an ESS. Therefore we have

\[
\int_{-h,h} \sum_{|n| \leq N(h)} a_n(t)\Psi_{n,t} dt = \int_{-h,h} a_0(t)\Psi_{0,t} dt + \sum_{n=1}^{N(h)} \int_{-h,h} (a_n(t)\Psi_{n,t} + a_n(t)\Psi_{n,t}) dt.
\]

Using it in (146) we get (15). In the same way from the last equality for \( f_\pi \) we obtain (16).

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