UNIFORM LOCAL EXISTENCE FOR INHOMOGENEOUS ROTATING FLUID EQUATIONS

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Abstract. We investigate the equations of anisotropic incompressible viscous fluids in \( \mathbb{R}^3 \), rotating around an inhomogeneous vector \( B(t, x_1, x_2) \). We prove the global existence of strong solutions in suitable anisotropic Sobolev spaces for small initial data, as well as uniform local existence result with respect to the Rossby number in the same functional spaces under the additional assumption that \( B = B(t, x_1) \) or \( B = B(t, x_2) \). We also obtain the propagation of the isotropic Sobolev regularity using a new refined product law.

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1. Introduction

The aim of this work is to investigate the well-posedness theory of inhomogeneous rotating fluids in the framework of Sobolev spaces. We consider the equations governing incompressible, viscous fluids in $\mathbb{R}^3$, rotating around an inhomogeneous vector $B(t, x_h) = (b_1(t, x_h), b_2(t, x_h), b_3(t, x_h))$ where $x_h$ stands for the horizontal variables (that is $x = (x_h, x_3)$). This is a generalization of the usual rotating fluid model, where $B = e_3 := (0, 0, 1)$.

More precisely, we are interested in the following system

\[
\begin{align*}
\partial_t u^\varepsilon + (u^\varepsilon \cdot \nabla) u^\varepsilon - \nu_h \Delta_h u^\varepsilon - \nu_v \partial^2_{x_3} u^\varepsilon + \frac{1}{\varepsilon} \left( u^\varepsilon \times B \right) + \nabla p^\varepsilon &= 0 \quad \text{in} \quad \mathbb{R}^3 \\
\text{div} \, u^\varepsilon &= 0 \quad \text{in} \quad \mathbb{R}^3 \\
u^\varepsilon(0, x) &= u_0(x)
\end{align*}
\]

where $u$ is the velocity field and $p$ is the pressure. The constants $\nu_h > 0$, $\nu_v \geq 0$ and $\varepsilon > 0$ represent respectively the horizontal viscosity, the vertical viscosity and the Rossby number. We have written $\Delta_h = \partial_1^2 + \partial_2^2$ and $(u \cdot \nabla) u = \sum_{j=1}^3 u_j \partial_j u$ where $u = (u_1, u_2, u_3)$ and $\partial_j$ represents the partial derivative in the direction $x_j$. We will also write $\nabla_h = (\partial_1, \partial_2)$ and $\text{div}_h u = \partial_1 u_1 + \partial_2 u_2$. Throughout this article, $B$ will be a smooth function with bounded derivatives. Additional assumptions on $B$ will be made later.

Before going any further, let us recall some well-known facts about the constant case $B = e_3$. There is a large literature dealing with these equations in the constant case. The first results for systems with large skew-symmetric perturbations are contained in [23], [21], [22]. In [1], A. Babin, A. Mahalov and B. Nicolaenko studied the incompressible rotating Euler and Navier-Stokes equations on the torus. Using the method introduced by S. Schochet (see [30] and [31]), I. Gallagher studied in [11] and [12] this problem in its abstract hyperbolic form. In the case of the incompressible rotating Navier-Stokes equations on the torus, it is shown (see [1] and [16]) that the solutions exist globally in time and converge to a solution of some diffusion equation. The case of rotating fluids evolving between two horizontal plates with Dirichlet boundary conditions was studied by E. Grenier and N. Masmoudi [17] (see also the paper of N. Masmoudi [25] and the recent work of J.-Y. Chemin, B. Desjardin, I. Gallagher and E. Grenier [8]). Motivated by this case, J.-Y. Chemin and al. studied the incompressible fluids with anisotropic viscosity in the whole space in [7] and [8]. They obtained local existence of the solution in the anisotropic Sobolev spaces $H^{0,1/2+\eta}(\mathbb{R}^3)$ and the global existence for data which are small compared to the horizontal viscosity. They also proved the global existence of the solution for anisotropic rotating fluids. The key of their proof is an anisotropic version of Strichartz estimates. In [27], global existence of the solution for rotating fluids with vanishing vertical viscosity was shown in the periodic case. Finally, we also refer to the related works [2], [4], [9] and [15].

In both constant and non constant cases, if $\nu_v > 0$, it is easy to construct global weak solutions. This is simply due to the fact that the singular perturbation is a linear skew-symmetric operator. The behavior of these solutions (as $\varepsilon \to 0$) has been studied by I. Gallagher and L. Saint-Raymond in a recent paper [13]. It is proved in [13] that the weak solutions $u^\varepsilon$ converge to the solution of a heat equation...
when the vector field $B$ is non stationary. The proofs of [13] are based on weak compactness arguments. For a detailed analysis of the rotating geophysical fluids we refer to [14].

In this paper, we would like to investigate the question of existence of strong solutions for $(IRF^\varepsilon)$. Our aim is to prove the existence and uniqueness of a solution on a uniform time interval, or the global existence and uniqueness for small initial data. To do so we need an energy estimate in Sobolev spaces as in [10]. Unfortunately one sees quickly that this is not an easy matter just by taking horizontal derivatives of the equation. But since $B$ does not depend on the third variable, all vertical derivatives are allowed in the energy estimate. Only horizontal derivatives create extra terms. So the idea is to start by taking the initial data in an anisotropic-type Sobolev space. Such spaces have been introduced by Iftimie in [18] and are very useful for anisotropic problems (see also [7], [19, 20], [26, 27]). Using the skew-symmetry of the Coriolis operator in the anisotropic Sobolev space $H^{0,s}$ (see Definition 2.2 below) and some product laws we shall prove the global existence and uniqueness for small data in $H^{0,s}$ for $s > 1/2$.

The next step consists in proving uniform local existence and uniqueness for large data in the same anisotropic Sobolev space. For technical reasons, we are not able to prove such result in the general case but only under the assumption that the rotation vector depends on one space variable. Let us remark that this assumption on $B$ is consistent with some models of geophysical flows (see [14]).

Once these steps are achieved, we return to the propagation of the isotropic Sobolev regularity. This is a delicate problem due to the lack of vertical viscosity. The major difficulty is to estimate the term $\langle (u_\times B(x_h)), u_\varepsilon \rangle_{H^{0,s}}$ in a good way. In the third section we obtain a new refined product law (see Lemma 3.1) which enables us to propagate the $H^s$ regularity as claimed in Theorems 1.3, 1.4 and 1.5.

1.1. Statement of the results. Our first result is the global existence of solutions of $(IRF^\varepsilon)$ in suitable anisotropic Sobolev spaces when the initial data are small enough.

**Theorem 1.1.** Assume that $B = B(t,x_1,x_2)$. Let $s > 1/2$ be a real number and $u_0$ be a divergence-free vector field in $H^{0,s}(\mathbb{R}^3)$. Assume that $\|u_0\|_{H^{0,s}} \leq c\nu_h$ where $c$ is small enough. Then, there exists a unique global solution $u_\varepsilon$ of $(IRF^\varepsilon)$ such that

$$u_\varepsilon \in C_b(\mathbb{R}^+; H^{0,s}) \quad \text{and} \quad \nabla_h u_\varepsilon \in L^2(\mathbb{R}^+; H^{0,s}).$$

Moreover, for any $t \geq 0$,

$$\|u_\varepsilon(t)\|_{H^{0,s}}^2 + 2\nu_h \int_0^t \|\nabla_h u_\varepsilon(\tau)\|_{H^{0,s}}^2 \, d\tau \leq \|u_0\|_{H^{0,s}}^2.$$

The proof of this theorem involves no real difficulties if one takes into account the earlier results of [26]. It is based on the following simple fact

$$\langle u \times B(t,x_h), u \rangle_{H^{0,s}} = 0.$$

The next theorem gives the local existence, uniformly in $\varepsilon$, in the space $H^{0,s}$ when the rotation vector only depends on one space variable, say $x_1$. 
Theorem 1.2. Assume that \( B = B(t, x_1) \). Let \( s > 1/2 \) be a real number and \( u_0 \in H^{0,s}(\mathbb{R}^3) \) be a divergence-free vector field. Then, there exists a positive time \( T \) such that, for all \( \varepsilon > 0 \), the system \( (IRF^\varepsilon) \) has a unique solution \( u^\varepsilon \) such that

\[
 u^\varepsilon \in C([0, T], H^{0,s}) \quad \text{and} \quad \nabla_h u^\varepsilon \in L^2([0, T], H^{0,s}).
\]

The assumption on \( B \) made in this theorem seems to be physical. Indeed, in the literature, \( B \) usually depends only on one variable, for example \( B = (1 + \beta x_2) e_3 \), where \( x_2 \) is the latitude variable and \( \beta \) is a parameter. Let us explain briefly the mathematical reasons for the assumption made on \( B \). Generally speaking, we can prove local existence of the solution as long as we can prove that the solution of the linear part of the equation stays uniformly small in a short time. In the general case \( (B = B(t, x_1, x_2)) \), we can prove only that the solution of the linear corresponding equation is bounded, but we cannot prove that the solution is small for short time. Contrary to this situation, if \( B = B(t, x_1) \) or \( B = B(t, x_2) \) we can prove that the solution of the linear problem is small in uniformly small time and so we can prove the uniformly local existence result.

Theorem 1.2 is not completely satisfactory, since the initial data belongs to an anisotropic Sobolev space which is adapted to the equation. A natural question to ask is then the following: is it possible to propagate the isotropic Sobolev regularity?

Let us first consider the particular case \( B \equiv 0 \), that is, the case of the anisotropic Navier-Stokes equations.

\[
(\text{NS}_h) \quad \begin{cases} 
\partial_t u + u \cdot \nabla u - \nu_h \Delta_h u + \nabla p = 0 \\
\text{div} \ u = 0 \\
u(0, x) = u_0(x)
\end{cases}
\]

This system was studied in [7], where the local existence for arbitrary initial data and global existence for small initial data in \( H^{0,s}(\mathbb{R}^3) \) for \( s > 1/2 \) and uniqueness for \( s > 3/2 \) are proved. In [20], the author filled the gap between existence and uniqueness and proved that uniqueness holds when existence does; that is for \( s > 1/2 \). In the third section we will prove the following theorem which can be seen as a propagation of the isotropic Sobolev regularity in time.

Theorem 1.3. Let \( s > 1/2 \) be a real number, and let \( u_0 \in H^s(\mathbb{R}^3) \) be a divergence-free vector field. There exists a positive time \( T \) and a unique solution \( u \) of \( (\text{NS}_h) \) defined on \([0, T] \times \mathbb{R}^3 \) such that

\[
 u \in C([0, T]; H^s) \quad \text{and} \quad \nabla_h u \in L^2([0, T]; H^s).
\]

If the maximal time \( T^* \) of existence is finite, then

\[
\lim_{t \to T^*} \int_0^t \|\nabla_h u(\tau)\|_{L^\infty(L^2_h)}^2 (1 + \|u(\tau)\|_{L^\infty(L^2_h)})^2 d\tau = +\infty.
\]

Furthermore, there exists a positive constant \( c \) such that if \( \|u_0\|_{H^s} \leq c \nu_h \), then the solution is global in time.

The fact that the lack of vertical viscosity prevents us from gaining vertical regularity is the main difficulty in the proof of Theorem 1.3. The main tool to overcome
this serious difficulty is a new refined product law stated in Section 3 (see Lemma 3.1) and proved in Section 6.

From this theorem we derive the following two results. The first one concerns the propagation of the isotropic Sobolev regularity for \((\text{IRF}^\epsilon)\). The second one concerns the uniform local existence in the isotropic Sobolev space.

**Theorem 1.4.** Let \( s > 1/2 \) be a real number. There exists \( c > 0 \) small enough such that, for any divergence-free vector field \( u_0 \in H^s(\mathbb{R}^3) \) with \( \|u_0\|_{H^s} \leq c
u h \), and for any \( \epsilon > 0 \), the system \((\text{IRF}^\epsilon)\) has a unique global solution \( u^\epsilon \) such that
\[
u^\epsilon \in \mathcal{C}(\mathbb{R}^+; H^s(\mathbb{R}^3)) \cap L^\infty_{\text{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^3)) \quad \text{and} \quad \nabla_h u^\epsilon \in L^2_{\text{loc}}(\mathbb{R}^+; H^s(\mathbb{R}^3)).
\]

**Theorem 1.5.** Assume that \( B = B(t, x_1) \). Let \( s > 1/2 \) be a real number, and let \( u_0 \in H^s(\mathbb{R}^3) \) be a divergence-free vector field. Then, there exists a positive time \( T \) such that, for all \( \epsilon > 0 \), the system \((\text{IRF}^\epsilon)\) has a unique solution \( u^\epsilon \) satisfying
\[
u^\epsilon \in \mathcal{C}([0, T]; H^s) \quad \text{and} \quad \nabla_h u^\epsilon \in L^2([0, T]; H^s).
\]

**Remark 1.1.** We point out that all the results stated above still holds in the periodic case. The proofs are similar except for some technical differences (see [28]).

The structure of the paper is as follows: Section 2 contains some notations and technical results which will be used in the whole paper. The third section deals with the propagation of the isotropic Sobolev regularity for the anisotropic Navier-Stokes equations. Section four is devoted to the proofs of global existence results for small initial data. In the fifth section, we give the proofs of the uniform local existence theorems. Section six deals with the proof of the main product law stated in the third section (see Lemma 3.1). Finally, the appendix is devoted to the proofs of some technical lemmas and product laws, which are more or less contained in earlier papers. For the sake of completeness we give their proofs here.

## 2. Notations and technical lemmas

In view of the anisotropy of the problem, we shall have to use functions spaces that take into account this anisotropy. The main tools will be the energy estimate and anisotropic Sobolev spaces. Such spaces have been introduced by Iftimie in [18] and used by several authors (see for instance [26, 27]-[7]-[19, 20]).

The definition of these spaces requires an anisotropic dyadic decomposition in the Fourier spaces. Let us first recall the isotropic Littlewood-Paley theory. We refer to [6] for a precise decomposition of the Fourier space.

### 2.1. Isotropic Littlewood-Paley theory

The main idea is to localize in frequencies. The interest of this theory consists in the fact that it allows us to define, at least formally, the product of two distributions as para-products and remainder.

Consider two smooth, compactly supported functions \( \varphi \) and \( \chi \), with support respectively in a fixed ring of \( \mathbb{R} \) far from the origin, and in a fixed ball containing the origin and such that
\[
\forall \ s \in \mathbb{R}\setminus\{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j} s) = 1 \quad \text{and} \quad \forall \ s \in \mathbb{R}, \quad \chi(s) + \sum_{j \in \mathbb{N}} \varphi(2^{-j} s) = 1
\]
Let us note that there exists an integer $N_0$ such that if $|j-k| \geq N_0$, then $\text{supp} \varphi(2^{-j}) \cap \varphi(2^{-k}) = \emptyset$.

Next, we define the following operators of localization in Fourier space:

$$
\text{if } j \geq 0, \quad \mathcal{F}(\Delta_j u)(\xi) = \varphi(2^{-j}|\xi|) \mathcal{F}(u)(\xi)
$$

$$
\mathcal{F}(\Delta_{-1} u)(\xi) = \chi(|\xi|) \mathcal{F}(u)(\xi)
$$

$$
\Delta_j u = 0 \text{ for } j < -1
$$

where

$$
\mathcal{F}(u)(\xi) = \hat{u}(\xi) = \int_{\mathbb{R}^3} e^{-ix \cdot \xi} u(x) \, dx, \quad \xi = (\xi_1, \xi_2, \xi_3) := (\xi_h, \xi_3) \in \mathbb{R}^3
$$

is the Fourier transform of the function $u$.

We recall also the definition of the anisotropic Lebesgue spaces:

**Definition 2.1.** We denote by $L^p_h(L^r_v)$ the space $L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}; L^r(\mathbb{R}_{x_3}))$ endowed with the norm

$$
\|f\|_{L^p_h(L^r_v)} = \left\|\|f(x,h,\cdot)\|_{L^r_v(x_3)}\right\|_{L^p_h(x_1, x_2)}.
$$

In the same way, we define $L^r_v(L^p_h)$ to be the space $L^r(\mathbb{R}_{x_3}; L^p(\mathbb{R}_{x_1} \times \mathbb{R}_{x_2}))$ endowed with the norm

$$
\|f\|_{L^r_v(L^p_h)} = \left\|\|f(\cdot, x_3)\|_{L^p_h(x_1, x_2)}\right\|_{L^r_v(x_3)}.
$$

The operators $\Delta_j$ satisfy the following property:

**Lemma 2.1.** The operator $\Delta_j$ is uniformly bounded in $L^p_h L^r_v$ for all $1 \leq q \leq \infty$ and $1 \leq r \leq \infty$.

**Lemma 2.2.** There exists a positive constant $C$ such that, for every $a \in S'(\mathbb{R}^3)$ with $\nabla a \in L^\infty_v(L^1_h)$ and $b \in L^p_v(L^1_h)$

$$
\left\|[\Delta_j; a]b\right\|_{L^p_h L^r_v} \leq C 2^{-j} \|\nabla a\|_{L^\infty_v L^1_h} \|b\|_{L^p_v L^1_h}
$$

where $1/r = 1/s + 1/t$, $1 \leq p \leq \infty$ and $[\Delta_j; a]b = \Delta_j(ab) - a \Delta_j b$.

**Lemma 2.3.** There exists a positive constant $C$ such that, for every $a \in S'(\mathbb{R}^3)$, for $j \geq 0$, $1 \leq p \leq \infty$ and $1 \leq r \leq \infty$, we have

$$
\|\Delta_j a\|_{L^p_h L^r_v} \leq C 2^{-j} \|\Delta_j \nabla a\|_{L^p_h L^r_v}.
$$

The proofs of Lemmas 2.1, 2.2 and 2.3 are essentially contained in [26] (see also [5] for a more general setting). For the sake of completeness we give the proofs in the Appendix.

We recall now Bony’s decomposition [3]. It is well known that the dyadic decomposition is useful for defining the product of two distributions. Formally, we can
write, for any two distributions \( u \) and \( v \)
\[
u = \sum_q \Delta_q u \quad \text{and} \quad v = \sum_q \Delta_q v \quad \text{and} \quad uv = \sum_{q,q'} \Delta_q u \Delta_{q'} v.
\]
We split this sum into three terms
\[
uvw = Tu v + Tv u + R(u, v),
\]
where we have denoted
\[
Tu v = \sum_{q' \leq q-2} \Delta_{q'} u \Delta_q v = \sum_q S_{q-1} u \Delta_q v
\]
\[
R(u, v) = \sum_q \sum_{j \in \{0, \pm 1\}} \Delta_q u \Delta_{q+j} v.
\]
In the first term, the frequencies of \( u \) are smaller than those of \( v \). In the second term, the frequencies of \( v \) are smaller than those of \( u \). In the third term, the frequencies of \( u \) and \( v \) are comparable. The first two sums are called the para-products and the third sum is the remaining term.

We have the following quasi-orthogonality properties
\[
\Delta_q \left(S_{q'-1} u \Delta_{q'} v\right) = 0 \quad \text{if} \quad |q - q'| \geq N_0
\]
\[
\Delta_q \left(S_{q'+1} u \Delta_{q'} v\right) = 0 \quad \text{if} \quad q' \leq q - N_0.
\]
Note that \( u \) is in the isotropic Sobolev space \( H^s(\mathbb{R}^3) \) if and only if
\[
\left( \sum_q 2^{qs} \|\Delta_q u\|_{L^2}^2 \right)^{1/2} < \infty,
\]
with equivalence of the norms.

2.2. The anisotropic case. In the frame of anisotropic Lebesgue spaces, the Hölder and Young inequalities reads.

**Proposition 2.1.** We have the following inequalities:
\[
\|fg\|_{L_r^r(L_h^p)} \leq \|f\|_{L_{r'}^r(L_h^{p'})} \|g\|_{L_{r''}^r(L_h^{p''})}
\]
where \( 1/r = 1/r' + 1/r'' \) and \( 1/p = 1/p' + 1/p'' \).
\[
\|f * g\|_{L_r^r(L_h^p)} \leq \|f\|_{L_{r'}^r(L_h^{p'})} \|g\|_{L_{r''}^r(L_h^{p''})}
\]
where \( 1 + 1/r = 1/r' + 1/r'' \) and \( 1 + 1/p = 1/p' + 1/p'' \).

We recall the definition of anisotropic Sobolev spaces and their properties.

**Definition 2.2.** Let \( s \) and \( s' \) be two real numbers. The anisotropic Sobolev space \( H^{s,s'}(\mathbb{R}^3) \) is the space of tempered distributions \( u \) such that \( \hat{u} \) belongs to \( L_{loc}^1(\mathbb{R}^3) \) and
\[
\|u\|_{H^{s,s'}}^2 := \int_{\mathbb{R}^3} \left(1 + |\xi_1|^2\right)^s \left(1 + |\xi_3|^2\right)^{s'} |\hat{u}(\xi)|^2 d\xi < \infty.
\]
It is obvious that the space $H^{s,s'}(\mathbb{R}^3)$ endowed with the norm $||.||_{H^{s,s'}}$ is a Hilbert space.

As in the isotropic case, we have the following interpolation property (see [18]).

**Proposition 2.2.** Let $s, t, s', t' \in \mathbb{R}$ and $0 \leq \alpha \leq 1$. If $u$ is in $H^{s,s'}(\mathbb{R}^3) \cap H^{t,t'}(\mathbb{R}^3)$, then $u$ belongs to $H^{s+1-\alpha t, s'+(1-\alpha)t'}(\mathbb{R}^3)$ and

$$||u||_{H^{s+1-\alpha t, s'+(1-\alpha)t'}} \leq ||u||_{H^{s,s'}}^{\alpha} ||u||_{H^{t,t'}}^{1-\alpha}.$$

The multiplicative properties of the anisotropic Sobolev spaces have been studied by several authors [7, 18, 19, 29]. The following result is proved in [19].

**Theorem 2.1.** Let $s, t < 1$, $s+t > 0$, and $s', t' < \frac{1}{2}$, $s'+t' > 0$. There exists a constant $C > 0$ such that, for any $u \in H^{s,s'}(\mathbb{R}^3)$ and $v \in H^{t,t'}(\mathbb{R}^3)$, the product $uv$ belongs to $H^{s+t-1, s'+t'-1/2}(\mathbb{R}^3)$ and

$$||uv||_{H^{s+t-1, s'+t'-1/2}} \leq C||u||_{H^{s,s'}} ||v||_{H^{t,t'}}.$$

This theorem is not sufficient for our purpose since the regularity we need is greater than $1/2$ in the vertical direction. The following theorem, proved in [20], deals with this difficulty.

**Theorem 2.2.** Let $s, t < 1$, $s+t > 0$, and $s' > \frac{1}{2}$, $t' \leq s'$, $s'+t' > 0$. If $u \in H^{s,s'}(\mathbb{R}^3)$ and $v \in H^{t,t'}(\mathbb{R}^3)$, then $uv \in H^{s+t-1,t'}(\mathbb{R}^3)$ and there exists a constant $C > 0$ such that, for any $u \in H^{s,s'}(\mathbb{R}^3)$, $v \in H^{t,t'}(\mathbb{R}^3)$, we have

$$||uv||_{H^{s+t-1,t'}(\mathbb{R}^3)} \leq C||u||_{H^{s,s'}(\mathbb{R}^3)} ||v||_{H^{t,t'}(\mathbb{R}^3)}.$$

The proofs of the above theorems and those of some product laws, which will be stated below, essentially use the anisotropic Littlewood-Paley theory. Let us briefly recall this theory (for more details, we refer to [3, 26, 27]).

The basic idea of this theory is to localize in vertical frequencies. Let us introduce the operators of localization in vertical frequencies in the following way:

$$\text{if } j \geq 0, \quad \Delta_j^V u = \mathcal{F}^{-1} \left( \varphi(2^{-j} |\xi_3|) \mathcal{F} u(\xi) \right)$$

$$\Delta_{-1}^V u = \mathcal{F}^{-1} \left( \chi(|\xi_3|) \mathcal{F} u(\xi) \right)$$

$$\Delta_j^V u = 0 \quad \text{for } j < -1.$$

We define also the operator

$$S_q^V u = \sum_{q' \leq q-1} \Delta_{q'}^V u.$$

The interest of this decomposition consists in the fact that any vertical derivative of a function localized in vertical frequencies of size $2^q$ acts like the multiplication by $2^q$. The following lemma, which is often referred to as Bernstein Lemma, will be very useful in what follows.
Lemma 2.4. Let $u$ be a function such that $\operatorname{supp} F^v u \subset \mathbb{R}^2 \times 2^k C$, where $C$ is a dyadic ring. Let $p \geq 1$ and $r \geq r' \geq 1$ be real numbers. Then, there exists a constant $C > 0$ such that

\begin{align}
(2.1) \quad & 2^{qk} C^{-k} \|u\|_{L^p_h(L^r_v)} \leq \|\partial_{x^3} u\|_{L^p_h(L^r_v)} \leq 2^{qk} C^k \|u\|_{L^p_h(L^r_v)}, \\
(2.2) \quad & 2^{qk} C^{-k} \|u\|_{L^p_h(L^r_v)} \leq \|\partial_{x^3} u\|_{L^p_h(L^r_v)} \leq 2^{qk} C^k \|u\|_{L^p_h(L^r_v)}, \\
(2.3) \quad & \|u\|_{L^p_h(L^r_v)} \leq C 2^{q(k - \frac{1}{2})} \|u\|_{L^p_h(L^r_v')}, \\
(2.4) \quad & \|u\|_{L^p_h(L^r_v)} \leq C 2^{q(k - \frac{1}{2})} \|u\|_{L^p_h(L^r_v')}.
\end{align}

We give now the Bony’s decomposition in the vertical variable (see [3], [7], [18]). Formally, we can write, for two distributions $u$ and $v$

\[
u = \sum_{q} \Delta_q^v u \quad ; \quad v = \sum_{q} \Delta_q^v v \quad \text{and} \quad uv = \sum_{q,q'} \Delta_q^v u \Delta_{q'}^v v.
\]

We split this sum into three terms. In the first term, the vertical frequencies of $u$ are smaller than those of $v$ while in the second term we have that the vertical frequencies of $v$ are smaller than those of $u$. The first two sums are called the para-products. The third term is called the remaining term and is such that the vertical frequencies of $u$ and $v$ are comparable. We shall denote:

\[
T_{u,v} = \sum_{q' \leq q - 2} \Delta_q^v u \Delta_{q'}^v v = \sum_{q} S_{q-1}^v u \Delta_q^v v \\
R(u,v) = \sum_{q} \sum_{j \in \{0,\pm 1\}} \Delta_q^v u \Delta_{q+j}^v v.
\]

We have the following quasi-orthogonality properties

\[
\Delta_q^v (S_{q-1}^v u \Delta_q^v v) = 0 \quad \text{if} \quad |q - q'| \geq N_0 \\
\Delta_q^v (S_{q+1}^v u \Delta_q^v v) = 0 \quad \text{if} \quad q' \leq q - N_0.
\]

Note that $u$ is in the anisotropic Sobolev space $H^{s,s}_0(\mathbb{R}^3)$ if and only if

\[
\left( \sum_{q} 2^{2qs} \|\Delta_q^v u\|_{L^2}^2 \right)^{1/2} < \infty,
\]

with equivalence of the norms.

3. Propagation of regularity for the anisotropic Navier-Stokes system

In this section we will prove Theorem 1.3. Let us first remark that this theorem is only a result about the propagation of the isotropic Sobolev regularity of the initial data in the case of anisotropic Navier-Stokes equations. The existence of the solution is known in the larger space $H^{0,s}$ with $s > 1/2$ ([7]) and in the critical Besov space $B^{0,1/2}$ ([26]). We shall use the energy estimate in $H^s$ to prove that the solution
constructed in \cite{7} preserves the isotropic Sobolev regularity. The main difficulty is the control of the term $\langle (u, \nabla)u, u \rangle_{H^s(\mathbb{R}^3)}$ without using vertical derivatives of $u$. The following lemma is the key argument in the proof.

**Lemma 3.1.** Let $s > 1/2$ be a real number. There exists a positive constant $C$ such that, for any divergence-free vector field $u \in H^s(\mathbb{R}^3)$, we have

$$
\left| \langle u, \nabla u, u \rangle_{H^s} \right| \leq C \left( \| u \|_{L^\infty_x(L^2_t)}^{1/2} \| \nabla u \|_{L^\infty_x(L^2_t)}^{1/2} \| u \|_{H^s}^{1/2} + \| \nabla u \|_{L^\infty_x(L^2_t)} \| u \|_{H^s} \right).
$$

The proof of this lemma will be given in Section 6. Let us go back to the proof of Theorem 1.3.

**Proof of Theorem 1.3.**

The energy estimate in $H^s(\mathbb{R}^3)$ yields

$$
\frac{d}{dt} \| u(t) \|_{H^s}^2 + 2\nu_h \| \nabla_h u(t) \|_{H^s}^2 \leq 2 \left| \langle u, \nabla u, u \rangle_{H^s} \right|.
$$

Using Lemma 3.1, we obtain

$$
\left| \langle u, \nabla u, u \rangle_{H^s} \right| \leq C \left( \| u \|_{L^\infty_x(L^2_t)}^{1/2} \| \nabla u \|_{L^\infty_x(L^2_t)}^{1/2} \| u \|_{H^s}^{1/2} + \| \nabla u \|_{L^\infty_x(L^2_t)} \| u \|_{H^s} \right).
$$

The Gronwall’s Lemma gives

$$
\| u(t) \|_{H^s}^2 \leq \| u_0 \|_{H^s}^2 \exp \left( \int_0^t \left( C\nu_h^{-3}(\nu_h^2 + \| u(\tau) \|_{L^\infty_x(L^2_t)}^2) \| \nabla_h u(\tau) \|_{L^\infty_x(L^2_t)}^2 \right) d\tau \right).
$$

This implies that, if

$$
\int_0^t \left( C\nu_h^{-3}(\nu_h^2 + \| u(\tau) \|_{L^\infty_x(L^2_t)}^2) \| \nabla_h u(\tau) \|_{L^\infty_x(L^2_t)}^2 \right) d\tau < +\infty,
$$

for all $0 \leq t \leq T$, then $T^* \geq T$, where $T^*$ is the lifespan of the solution in $H^s$. In particular, if $\| u_0 \|_{H^s} \leq c\nu_h$ where $c$ is small enough, then the anisotropic norm of $u_0$ is small enough ($\| u_0 \|_{H^{0,s}} \leq c\nu_h$) and by the results proved in \cite{7}, there exists a unique global in time solution in the space $H^{0,s}$. In addition

$$
\forall \ t \geq 0, \ \| u(t) \|_{H^{0,s}}^2 + \nu_h \int_0^t \| \nabla_h u(\tau) \|_{H^{0,s}}^2 d\tau \leq \| u_0 \|_{H^{0,s}}^2.
$$

Using the Sobolev embedding $H^{0,s} \hookrightarrow L^\infty_x(L^2_t)$ for $s > 1/2$, we get

$$
\forall \ t \geq 0, \ \| u(t) \|_{L^\infty_x(L^2_t)}^2 + \nu_h \int_0^t \| \nabla_h u(\tau) \|_{L^\infty_x(L^2_t)}^2 d\tau \leq \| u_0 \|_{H^{0,s}}^2.
$$

Therefore, the solution exists globally in $H^s$ and it is small compared with the horizontal viscosity. This ends the proof of Theorem 1.3. \hfill \blacksquare
4. Global existence for small initial data

In this section, we shall prove the global existence of solutions of \((IRF^c)\) in \(H^{0,s}(\mathbb{R}^3)\) and \(H^s(\mathbb{R}^3)\), when the initial data are small enough. It should be noted here that the 2D Navier-Stokes equations are globally well-posed in \(L^2\), and that the 3D case can be dealt with by splitting the horizontal and vertical variables and using Sobolev embedding \(H^{0,s} \hookrightarrow L_0^\infty(\mathbb{R}^3)\).

The main tool in the proof of Theorem 4.1. is the following product law (see [7] and [20] for the critical case).

**Lemma 4.1.** Let \(s > 1/2\) be a real number. There exists a positive constant \(C_s\) such that, for any \(u, v \in H^{0,s}(\mathbb{R}^3)\), \(u\) being divergence-free, we have

\[
\|u, \nabla v, v\|_{H^{0,s}} \leq C_s \left( \|u\|_{H^{0,s}}^{1/2} \|\nabla_h u\|_{H^{0,s}}^{1/2} \|v\|_{H^{0,s}}^{1/2} \|\nabla_h v\|_{H^{0,s}}^{3/2} + \|v\|_{H^{0,s}} \|\nabla_h v\|_{H^{0,s}} \|\nabla_h u\|_{H^{0,s}} \right).
\]

In particular

\[
\|u, \nabla u, u\|_{H^{0,s}} \leq C_s \|u\|_{H^{0,s}} \|\nabla_h u\|_{H^{0,s}}^2.
\]

Note that, in such inequalities, the following simple fact is fundamental

\[
\text{div} u = 0 \implies \partial_3 u_3 = -\text{div}_h u_h.
\]

**Proof of Theorem 4.1.**

Without loss of generality, one can assume that \(\nu_v = 0\). For simplicity, we omit the dependance on \(\varepsilon\) in \(u^\varepsilon\). Therefore, \(u\) satisfies

\[
\partial_t u + (u, \nabla) u - \nu_h \Delta_h u + \frac{1}{\varepsilon} (u \times B(t, x_h)) = -\nabla p.
\]

Taking the scalar product of (4.5) with \(u\) in \(H^{0,s}\), and using the fact that

\[
\langle u \times B(t, x_h), u \rangle_{H^{0,s}} = 0,
\]

we infer

\[
\frac{d}{dt} \|u(t)\|_{H^{0,s}}^2 + 2\nu_h \|\nabla_h u(t)\|_{H^{0,s}}^2 \leq 2 \left| \langle u, \nabla u, u \rangle_{H^{0,s}} \right|.
\]

By Lemma 4.1, (4.6) becomes

\[
\frac{d}{dt} \|u(t)\|^2_{H^{0,s}} + 2\nu_h \|\nabla_h u(t)\|^2_{H^{0,s}} \leq 2C_s \|u\|_{H^{0,s}} \|\nabla_h u\|_{H^{0,s}}^2.
\]

Define \(T^*\) by

\[
T^* = \sup \left\{ T > 0 \mid \forall 0 \leq t \leq T, \|u(t)\|_{H^{0,s}} \leq 2c\nu_h < \frac{\nu_h}{2C_s} \right\}.
\]

Using (4.7), we get

\[
\forall 0 \leq t < T^*, \|u(t)\|^2_{H^{0,s}} + \nu_h \int_0^t \|\nabla_h u(\tau)\|^2_{H^{0,s}} d\tau \leq \|u_0\|^2_{H^{0,s}} \leq (c\nu_h)^2.
\]

Thus

\[
\forall 0 \leq t < T^*, \|u(t)\|_{H^{0,s}} \leq c\nu_h < 2c\nu_h,
\]

and so

\[
T^* = +\infty.
\]
Moreover, we have the following energy estimate

\[ \forall \ t \geq 0, \quad \|u(t)\|_{H^{0,s}}^2 + \nu_h \int_0^t \|\nabla_h u(\tau)\|_{H^{0,s}}^2 \, d\tau \leq \|u_0\|_{H^{0,s}}^2. \]

Theorem 1.1 is then completely proved.

We now come to the proof of Theorem 1.4 about global existence for small initial data in the isotropic Sobolev space $H^s$. Here, the product law stated in Lemma 3.1 play a crucial role.

**Proof of Theorem 1.4.**

The energy estimate in $H^s(\mathbb{R}^3)$ implies

\[ \frac{d}{dt} \|u(t)\|_{H^s}^2 + 2\nu_h \|\nabla_h u(t)\|_{H^s}^2 \leq 2 \langle (u, \nabla) u, u \rangle_{H^s} + \frac{2}{\varepsilon} \langle u \times B, u \rangle_{H^s}. \]

The assumption on $B$ yields

\[ \left| \langle u \times B, u \rangle_{H^s} \right| \leq C \|u\|_{H^s}^2. \]

Using Lemma 3.1 and the injection $H^{0,s} \hookrightarrow L^\infty_v(L^2_h)$, we infer

\[ \langle (u, \nabla) u, u \rangle_{H^s} \leq C \left[ \|u\|_{H^{0,s}}^{1/2} \|\nabla_h u\|_{H^{0,s}}^{1/2} \|u\|_{H^s}^{1/2} \|\nabla_h u\|_{H^s}^{3/2} + \|\nabla_h u\|_{H^{0,s}} \|u\|_{H^s} \|\nabla_h u\|_{H^s} \right] \]

\[ \leq \nu_h/2 \|\nabla_h u\|_{H^s}^2 + C \nu_h^{-3} (\nu_h^2 + \|u\|_{H^{0,s}}^2) \|\nabla_h u\|_{H^{0,s}}^2 \|u\|_{H^s}^2. \]

The Gronwall’s Lemma leads to

\[ \|u(t)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 \exp \left( \int_0^t \left( \frac{C}{\varepsilon} + C \nu_h^{-3} (\nu_h^2 + \|u(\tau)\|_{H^{0,s}}^2) \|\nabla_h u(\tau)\|_{H^{0,s}}^2 \right) \, d\tau \right). \]

The fact that $u_0$ is small in $H^{0,s}$ together with Theorem 1.1 implies that $u$ is global in time. Moreover

\[ \|u(t)\|_{H^{0,s}}^2 + \nu_h \int_0^t \|\nabla_h u(\tau)\|_{H^{0,s}}^2 \, d\tau \leq \|u_0\|_{H^{0,s}}^2, \quad \text{for any } \ t \geq 0. \]

It follows that

\[ \|u(t)\|_{H^s}^2 \leq \|u_0\|_{H^s}^2 \exp \left( C \nu_h^{-3} (\nu_h^2 + \|u_0\|_{H^{0,s}}^2) \|u_0\|_{H^{0,s}}^2 \right) \exp \left( \frac{C t}{\varepsilon} \right). \]

This completes the proof of Theorem 1.4.

**5. Uniform local existence**

In this section we wish to investigate the uniform local existence in both anisotropic and isotropic cases. We prove Theorem 1.2 and Theorem 1.3 stated in the introduction. As remarked before, we shall assume that $B$ only depends on one space variable since we are unable to recover similar results in general case. Before we come to the details of the proofs, let us first explain rapidly the scheme and make some clarifying comments.

We split the initial data $u_0$ in a small part in $H^{0,s}$ and a smooth one. First, we solve the linear system with smooth initial data to obtain a global and bounded
solution \( v^\varepsilon \). Then, we consider the perturbed system satisfied by \( w^\varepsilon := u^\varepsilon - v^\varepsilon \) and with small initial data. We have to prove that \( w^\varepsilon \) exists on a uniform time interval. The strategy here is to use energy estimate in \( H^{0,s} \) and a Gronwall’s lemma in order to prove that \( w^\varepsilon \) remains small on a uniform time interval. The main difficulty comes from the term
\[
\langle (v^\varepsilon, \nabla) v^\varepsilon, w^\varepsilon \rangle_{H^{0,s}},
\]
but using the fact that two frequency directions of \( v^\varepsilon \) are bounded, one can prove
\[
\left| \langle (v^\varepsilon, \nabla) v^\varepsilon, w^\varepsilon \rangle_{H^{0,s}} \right| \leq C \|v^\varepsilon\|^3 \|\nabla_h v^\varepsilon\| L^2 + \frac{\nu_h}{10} \|\nabla_h w^\varepsilon\|^2_{H^{0,s}}.
\]
This inequality, together with the fact that
\[
\int_0^t |v^\varepsilon(\tau)|^3 \|\nabla_h v^\varepsilon(\tau)\| L^2 d\tau \lesssim t^{1/2} \|u_0\|^4_{L^2},
\]
allows us to carry out the proofs.

We now come to the details.

**Proof of Theorem 1.2.**

As mentioned before, the main argument consists in splitting the initial data in a small part in \( H^{0,s} \) and a regular one. To do this, we set \( S_N u = \mathcal{F}^{-1}(\chi(2^{-N}|\xi|) \mathcal{F} u) \) and we take \( N \) sufficiently large (depending on \( u_0 \) and \( \nu_h \)) such that \( \|(I - S_N) u_0\|_{H^{0,s}} \leq c \nu_h \), where \( c > 0 \) is small enough. Then, \( u_0 = S_N u_0 + (I - S_N) u_0 \), with \( S_N u_0 \in H^\infty(\mathbb{R}^3) \). We point out that throughout the rest of this section, \( C \) will denotes a positive constant which may depends on \( N, \nu_h, s \) but not on \( \varepsilon \).

First, we consider the following linear system:
\[
\begin{aligned}
\partial_t v^\varepsilon_N &= -\nu_h \Delta_h v^\varepsilon_N + \frac{1}{\varepsilon} (v^\varepsilon_N \times B(t, x_1)) = -\nabla p^\varepsilon_N \\
\text{div} \, v^\varepsilon_N &= 0 \\
v^\varepsilon_N|_{t=0} &= S_N u_0.
\end{aligned}
\]

Clearly (5.8) has a unique global solution
\[
v^\varepsilon_N \in C(\mathbb{R}_+; H^{0,s}) \quad \text{with} \quad \nabla_h v^\varepsilon_N \in L^2(\mathbb{R}_+; H^{0,s}).
\]

Since \( B \) only depends on \( t \) and \( x_1 \), we obtain \( S_N^{x_2,x_3}(u \times B(t, x_1)) = S_N^{x_2,x_3} u \times B(t, x_1) \), where \( S_N^{x_2,x_3} \) is the operator of localization in \( \xi_2 \) and \( \xi_3 \) direction defined by \( S_N^{x_2,x_3} u = \mathcal{F}^{-1} (\tilde{\chi}(2^{-N}|(\xi_2, \xi_3)|) \mathcal{F} u(\xi)) \), \( \tilde{\chi} \) being a smooth compactly supported function whose value is 1 on the support of \( \chi \). By uniqueness and the fact that \( S_N^{x_2,x_3} S_N u_0 = S_N u_0 \), we get \( v^\varepsilon_N = S_N^{x_2,x_3} v^\varepsilon_N \). This implies the important fact that \( v^\varepsilon_N \) is a regular function with respect to the \( x_2 \) and \( x_3 \) variables.

Next, we consider the perturbed system:
\[
\begin{aligned}
\partial_t w^\varepsilon_N + (w^\varepsilon_N + v^\varepsilon_N) \nabla(w^\varepsilon_N + v^\varepsilon_N) - \nu_h \Delta_h w^\varepsilon_N + \frac{1}{\varepsilon} (w^\varepsilon_N \times B(t, x_1)) &= -\nabla q^\varepsilon_N \\
\text{div} \, w^\varepsilon_N &= 0 \\
w^\varepsilon_N|_{t=0} &= (I - S_N) u_0.
\end{aligned}
\]
We shall now be interested in proving that \( w_N^\varepsilon \) exists on a uniform time interval. The energy estimate in \( H^{0,s} \) yields

\[
\frac{1}{2} \frac{d}{dt} \left\| w_N^\varepsilon(t) \right\|_{H^{0,s}}^2 + \nu_h \| \nabla_h w_N^\varepsilon(t) \|_{H^{0,s}}^2 \leq \left| \langle w_N^\varepsilon \nabla w_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} \right| \\
+ \left| \langle v_N^\varepsilon \nabla w_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} \right| + \left| \langle w_N^\varepsilon \nabla v_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} \right| \\
+ \left| \langle v_N^\varepsilon \nabla v_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} \right|
\]

For the first two terms in the right hand side, we apply Lemma 4.1 to obtain

\[
\left| \langle w_N^\varepsilon \nabla w_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} \right| \leq C_s \| w_N^\varepsilon \|_{H^{0,s}} \| \nabla_h w_N^\varepsilon \|_{H^{0,s}}^2,
\]

and

\[
\left| \langle v_N^\varepsilon \nabla w_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} \right| \leq C_s \| v_N^\varepsilon \|_{H^{0,s}}^{1/2} \| \nabla_h v_N^\varepsilon \|_{H^{0,s}}^{1/2} \| w_N^\varepsilon \|_{H^{0,s}}^{1/2} \| \nabla_h w_N^\varepsilon \|_{H^{0,s}}^{3/2} \\
+ \| w_N^\varepsilon \|_{H^{0,s}} \| \nabla_h w_N^\varepsilon \|_{H^{0,s}} \| \nabla_h v_N^\varepsilon \|_{H^{0,s}} \leq C_s \nu_h^{-3} (\nu_h^2 + \| v_N^\varepsilon \|_{H^{0,s}}^2) \| \nabla_h v_N^\varepsilon \|_{H^{0,s}}^2 \| w_N^\varepsilon \|_{H^{0,s}}^2 + \frac{\nu_h}{10} \| \nabla_h w_N^\varepsilon \|_{H^{0,s}}^2.
\]

For the third term, we write

\[
\langle w_N^\varepsilon \nabla v_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} = \langle w_{N,h}^\varepsilon \nabla_h v_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} + \langle w_{N,3}^\varepsilon \partial_3 v_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}}.
\]

The anisotropic product law given in Theorem 2.2 implies

\[
\left| \langle w_{N,h}^\varepsilon \nabla_h v_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} \right| \leq C_s \| \nabla_h v_N^\varepsilon \|_{H^{0,s}} \| w_N^\varepsilon \|_{H^{0,s}} \| \nabla_h w_N^\varepsilon \|_{H^{0,s}} \\
\leq C_s \nu_h^{-1} \| \nabla_h v_N^\varepsilon \|_{H^{0,s}}^2 \| w_N^\varepsilon \|_{H^{0,s}}^2 + \frac{\nu_h}{10} \| \nabla_h w_N^\varepsilon \|_{H^{0,s}}^2.
\]

We also have

\[
\left| \langle w_{N,3}^\varepsilon \partial_3 v_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} \right| \leq C_s \| \partial_3 v_N^\varepsilon \|_{H^{0,s}} \| w_N^\varepsilon \|_{H^{0,s}} \| \nabla_h w_N^\varepsilon \|_{H^{0,s}}.
\]

Since \( v_N^\varepsilon \) is localized in the \( \xi_3 \) direction, we can apply Lemma 2.4 to obtain

\[
\| \partial_3 v_N^\varepsilon \|_{H^{0,s}} \leq C 2^N \| v_N^\varepsilon \|_{H^{0,s}}.
\]

Finally

\[
\left| \langle w_{N,3}^\varepsilon \partial_3 v_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} \right| \leq C_s 2^N \| v_N^\varepsilon \|_{H^{0,s}} \| w_N^\varepsilon \|_{H^{0,s}} \| \nabla_h w_N^\varepsilon \|_{H^{0,s}} \\
\leq C_s 2^N \nu_h^{-1} \| v_N^\varepsilon \|_{H^{0,s}}^2 \| w_N^\varepsilon \|_{H^{0,s}}^2 + \frac{\nu_h}{10} \| \nabla_h w_N^\varepsilon \|_{H^{0,s}}^2.
\]

What remains to estimate, therefore, is the term \( \langle v_N^\varepsilon \nabla v_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} \). Note that (and this will be clear in the rest of the proof) this term is the only one for which we need that \( B \) only depends on \( t \) and \( x_1 \).

Since the vector field \( v_N^\varepsilon \) is divergence-free, we can write

\[
\langle v_N^\varepsilon \nabla v_N^\varepsilon, w_N^\varepsilon \rangle_{H^{0,s}} = (\text{div} \langle v_N^\varepsilon \otimes v_N^\varepsilon \rangle, w_N^\varepsilon)_{H^{0,s}} = \langle v_N^\varepsilon \otimes v_N^\varepsilon, \nabla_h w_N^\varepsilon \rangle_{H^{0,s}} + \langle \partial_3 (v_N^\varepsilon \otimes v_N^\varepsilon), w_N^\varepsilon \rangle_{H^{0,s}}.
\]

We have the estimate
Similarly, one has
\[ \left\| v_N^{\varepsilon} \right\|_{L^2_1(L^2_{2,:3})} \leq C \left\| v_N^{\varepsilon} \right\|_{L^2(L^2_{2,:3})} \]
By the Gagliardo-Nirenberg inequality, we get
\[ \left\| v_N^{\varepsilon} \right\|_{L^2_1(L^2_{2,:3})} \leq C \left\| v_N^{\varepsilon} \right\|_{L^2(L^2_{2,:3})} \]
Hence
\[ \left\| v_N^{\varepsilon} \right\|_{L^2_1(L^2_{2,:3})} \leq C \left\| v_N^{\varepsilon} \right\|_{L^2_2(L^2_{2,:3})} \leq C \left\| v_N^{\varepsilon} \right\|_{L^2} \left\| \partial_x v_N^{\varepsilon} \right\|_{L^2}. \]
Similarly, one has
\[ \left\| \partial_t (v_N^{\varepsilon} \otimes v_N^{\varepsilon}), w_N^{\varepsilon} \right\|_{H^{0,s}} \leq C 2^{N+1} \left\| v_N^{\varepsilon} \right\|_{L^2} \left\| \partial_t v_N^{\varepsilon} \right\|_{L^2} \left\| w_N^{\varepsilon} \right\|_{H^{0,s}} \]
To summarize, we combine above estimates to obtain
\[ \frac{d}{dt} \left\| w_N^{\varepsilon}(t) \right\|_{H^{0,s}}^2 + \frac{\nu_h}{10} \left\| \nabla_h w_N^{\varepsilon}(t) \right\|_{H^{0,s}}^2 \leq \left\| w_N^{\varepsilon}(t) \right\|_{H^{0,s}} \left\| \nabla_h w_N^{\varepsilon}(t) \right\|_{H^{0,s}}^2 \]
Since \( w_N^{\varepsilon}(0) \) is small in \( H^{0,s} \), we can define the time \( T_{\varepsilon,N} > 0 \) by
\[ T_{\varepsilon,N} = \sup \left\{ t > 0; \quad \forall 0 \leq t' \leq t, \quad \left\| w_N^{\varepsilon}(t') \right\|_{H^{0,s}} \leq 2c \nu_h \} \]
On the time interval \([0, T_{\varepsilon,N}]\), we have
\[ \frac{d}{dt} \left\| w_N^{\varepsilon}(t) \right\|_{H^{0,s}}^2 + \frac{\nu_h}{10} \left\| \nabla_h w_N^{\varepsilon}(t) \right\|_{H^{0,s}}^2 \leq C \left\| w_N^{\varepsilon}(t) \right\|_{L^2} \left\| \nabla_h v_N^{\varepsilon}(t) \right\|_{L^2} \]
Using Gronwall’s Lemma, energy estimate and the fact that
\[ \int_0^t \left\| \nabla_h v_N^{\varepsilon}(\tau) \right\|_{L^2} d\tau \leq \frac{t}{\left( \nu_h \right)^{1/2}} \left\| u_0 \right\|_{L^2}, \]
we infer
\[ \left\| w_N^{\varepsilon}(t) \right\|_{H^{0,s}} \leq \left( (4\nu_h)^2 + \frac{t}{2} C \left\| u_0 \right\|_{L^2} \right) \exp \left( t C \left( 1 + \left\| u_0 \right\|_{H^{0,s}}^2 + 1 + \left\| u_0 \right\|_{H^{0,s}}^2 \right) \right). \]
It follows that \( T_{\varepsilon,N} \geq T \) where \( T > 0 \) is given by
\[(cv_h)^2 + T^2 C \|u_0\|_{L^2}^4 \exp \left( T C (\|u_0\|_{H^{0,s}}^2 + (1 + \|u_0\|_{H^{0,s}}^2)\|u_0\|_{L^2}^2) \right) \leq \left( \frac{3}{2} cv_h \right)^2.\]

Thus we have proved that \(w^\varepsilon_s\) exists on a uniform time interval and the conclusion follows. This being said we have only to add the remark that the time \(T\) depends on the distribution of the frequencies of the initial data and not only on the size of the data. \(\blacksquare\)

**Proof of Theorem 1.3**

Let us now take \(u_0 \in H^s\) with \(s > 1/2\). By Theorem 1.2, there exists a positive time \(T\) independent of \(\varepsilon > 0\) and a unique solution \(u^\varepsilon \in C([0, T], H^{0,s})\) of the system \((IRF^\varepsilon)\), with \(\nabla_h u^\varepsilon \in L^2([0, T], H^{0,s})\). Since \(H^{0,s}\) is continuously embedded in \(L^\infty_c(L^2_h)\), we deduce that the norms \(\|u^\varepsilon(t)\|_{L^\infty_c(L^2_h)}\) and \(\int_0^t \|\nabla_h u^\varepsilon(\tau)\|_{L^\infty_c(L^2_h)}^2 d\tau\) are bounded on the time interval \([0, T]\). By Theorem 1.3, we obtain that the lifespan \(T^\varepsilon\) of the solution \(u^\varepsilon\) in the space \(H^s\) satisfies the lower bound \(T^\varepsilon > T\). This means that the solution exists on a uniform time interval in the \(H^{0,s}\) space as well as in the \(H^s\) space. \(\blacksquare\)

6. **Proof of Lemma 3.1**

We begin with two key estimates based on the divergence-free condition.

**Proposition 6.1.** There exists a positive constant \(C\), such that for every divergence-free vector field \(u = (u_1, u_2, u_3)\),

\[(6.10) \quad \|\nabla u_3\|_{H^s(\mathbb{R}^3)} \leq C \|\nabla_h u\|_{H^s(\mathbb{R}^3)}\]

\[(6.11) \quad \|\Delta_q u\|_{L^2_{-h} L^1_h} \leq 2^{-qs} c_q \|u\|_{H^s(\mathbb{R}^3)}^{1/2} \|\nabla_h u\|_{H^s(\mathbb{R}^3)}^{1/2}.\]

Hereafter, \((b_q)\) and \((c_q)\) stands for generic positive sequences (which could depend on \(t\)) such that

\[\sum_q b_q \leq 1\] and \[\sum_q c_q^2 \leq 1.\]

**Proof of Proposition 6.1.**

The divergence-free condition implies that

\[\nabla u_3 = (\nabla_h u_3, -div_h u^h),\]

from which the estimate \((6.10)\) directly follows.

To prove \((6.11)\), we write

\[
\|\Delta_q u\|_{L^2_{-h} L^1_h} \leq \|\Delta_q u\|_{L^2(\mathbb{R}^3)}^{1/2} \|\Delta_q \nabla_h u\|_{L^2(\mathbb{R}^3)}^{1/2}
\]
\[
\leq \left(2^{-\frac{qs}{2}} c_q^{1/2} \|u\|_{H^s(\mathbb{R}^3)}^{1/2}\right) \cdot \left(2^{-\frac{qs}{2}} c_q^{1/2} \|\nabla_h u\|_{H^s(\mathbb{R}^3)}^{1/2}\right)
\]
\[
\leq 2^{-qs} c_q \|u\|_{H^s(\mathbb{R}^3)}^{1/2} \|\nabla_h u\|_{H^s(\mathbb{R}^3)}^{1/2}.\]

\(\blacksquare\)
Now, we return to the proof of Lemma 3.1. Using the fact that
\[ \langle f, g \rangle_{H^s(\mathbb{R}^3)} \approx \sum_{q \geq -1} 2^{2qs} \langle \Delta_q f, \Delta_q g \rangle_{L^2(\mathbb{R}^3)}, \]
we only need to prove that
\[ \left| \langle \Delta_q (u \cdot \nabla u), \Delta_q u \rangle_{L^2(\mathbb{R}^3)} \right| \leq C 2^{-2qs} c_q \left( \| u \|_{L^{\infty}_v L^2_h}^{1/2} \| \nabla_h u \|_{L^2_v L^2_h}^{1/2} \| u \|_{H^s}^{1/2} \| \nabla_h u \|_{H^s}^{3/2} \right) \]
\[ + \| \nabla_h u \|_{L^\infty_v L^2_h} \| u \|_{H^s} \| \nabla_h u \|_{H^s} \].

We shall split the term \( \langle \Delta_q (u \cdot \nabla u), \Delta_q u \rangle_{L^2(\mathbb{R}^3)} \) in the following way
\[ \int_{\mathbb{R}^3} \Delta_q (u \cdot \nabla u) \Delta_q u \, dx = T^h_q + T^v_q \]

where
\[ T^h_q = \int_{\mathbb{R}^3} \Delta_q (u_h \cdot \nabla_h u) \Delta_q u \, dx \]
\[ T^v_q = \int_{\mathbb{R}^3} \Delta_q (u_3 \partial_3 u) \Delta_q u \, dx. \]

Let us start with the study of the horizontal terms. Using the para-differential decomposition of Bony, we can write
\[ T^h_q = T^{h,q}_1 + T^{h,q}_2 + R^{h,q}, \]

where
\[ T^{h,q}_1 = \sum_{|q' - q| < N_0} \int_{\mathbb{R}^3} \Delta_q (S_{q'-1} u_h \cdot \nabla_h \Delta_{q'} u) \Delta_q u \, dx \]
\[ T^{h,q}_2 = \sum_{|q' - q| < N_0} \int_{\mathbb{R}^3} \Delta_q (\Delta_{q'} u_h \nabla_h S_{q'-1} u) \Delta_q u \, dx \]
\[ R^{h,q} = \sum_{q' > q - N_0, \ i \in \{0, \pm 1\}} \int_{\mathbb{R}^3} \Delta_q (\Delta_{q'} u_h \cdot \nabla_h \Delta_{q'+i} u) \Delta_q u \, dx. \]

**Estimate of \( T^{h,q}_1 \)**

By the Hölder inequality, we infer
\[ |T^{h,q}_1| \leq \sum_{|q' - q| < N_0} \| \Delta_q (S_{q'-1} u_h \cdot \nabla_h \Delta_{q'} u) \|_{L^2_v L^{4/3}_h} \| \Delta_q u \|_{L^2_v L^4_h}. \]

Proposition 5.1 gives
\[ \| \Delta_q u \|_{L^2_v L^4_h} \leq 2^{-qs} c_q \| u \|_{H^s(\mathbb{R}^3)}^{1/2} \| \nabla_h u \|_{H^s(\mathbb{R}^3)}^{1/2}. \]
Using Lemma 2.1 and Hölder inequality, we get
\[
\|\Delta_q(S_{q'-1}u_h, \nabla_h \Delta_{q'}u)\|_{L^4_h L^4_h} \leq C \|S_{q'-1}u_h, \nabla_h \Delta_{q'}u\|_{L^2_h L^{4/3}_h} \\
\leq C \|S_{q'-1}u_h\|_{L^\infty L^4_h} \|\nabla_h \Delta_{q'}u\|_{L^2_h L^2_h} \\
\leq C 2^{-q's} c_{q'} \|u\|_{L^\infty L^4_h} \|\nabla_h u\|_{H^s(\mathbb{R}^3)} \\
\leq C 2^{-q's} c_{q'} \|u\|^{1/2}_{L^\infty L^2_h} \|\nabla_h u\|^{1/2}_{L^\infty L^2_h} \|\nabla_h u\|^{3/2}_{H^s(\mathbb{R}^3)}.
\]

Finally
\[
|I_h(q)| \leq 2^{-2q's} b_q \|u\|^{1/2}_{L^\infty L^2_h} \|\nabla_h u\|^{1/2}_{L^\infty L^2_h} \|u\|^{1/2}_{H^s(\mathbb{R}^3)} \|\nabla_h u\|^3_{H^s(\mathbb{R}^3)}.
\]

**Estimate of \(T_{h,q}'\)**

Arguing as in the estimate of the term \(I_h(q)\), we get
\[
|T_{h,q}'| \leq \sum_{\{q-q'|<N_0\}} \|\Delta_q(\Delta_{q'}u_h, \nabla_h S_{q'-1}u)\|_{L^4_h L^4_h} \|\Delta_q u\|_{L^2_h L^4_h} \\
\leq C \sum_{\{q-q'|<N_0\}} \|\Delta_q u\|_{L^2_h L^4_h} \|\nabla_h S_{q'-1}u\|_{L^\infty L^2_h} \|\Delta_q u\|_{L^2_h L^4_h} \\
\leq 2^{-2q's} b_q \|\nabla_h u\|_{L^\infty L^2_h} \|u\| \|\nabla_h u\|_{H^s}.
\]

**Estimate of \(R_{h,q}^c\)**

Using the Hölder inequality, we get
\[
|R_{h,q}^c| \leq \|\Delta_q u\|_{L^2_h L^4_h} \sum_{q' \geq q - 4} \sum_{i \in \{0, \pm 1\}} \|\Delta_q(\Delta_{q'}u_h, \nabla_h \Delta_{q' + i}u)\|_{L^2_h L^4_h}.
\]

Proposition 6.1 implies that
\[
\|\Delta_q u\|_{L^2_h L^4_h} \leq 2^{-q's} c_q \|u\|^{1/2}_{H^s(\mathbb{R}^3)} \|\nabla_h u\|^{1/2}_{H^s(\mathbb{R}^3)}.
\]

Moreover, we have
\[
\|\Delta_q(\Delta_{q'}u_h, \nabla_h \Delta_{q' + i}u)\|_{L^2_h L^4_h} \leq C \|\Delta_q u_h, \nabla_h \Delta_{q' + i}u\|_{L^2_h L^4_h} \\
\leq C \|\Delta_q u_h\|_{L^\infty L^4_h} \|\nabla_h \Delta_{q' + i}u\|_{L^2(\mathbb{R}^3)} \\
\leq 2^{-q's} c_{q'} \|\nabla_h u\|_{H^s(\mathbb{R}^3)} \|u\|^{1/2}_{L^\infty L^2_h} \|\nabla_h u\|^{1/2}_{L^\infty L^2_h}.
\]

Finally, we obtain
\[
|R_{h,q}^c| \leq 2^{-2q's} b_q \|u\|^{1/2}_{L^\infty L^2_h} \|\nabla_h u\|^{1/2}_{L^\infty L^2_h} \|u\|^{1/2}_{H^s} \|\nabla_h u\|^{3/2}_{H^s}.
\]

**Estimate of \(I_q^v\)**

The vertical term is more delicate since the vertical viscosity vanishes, and hence we loose the regularity in the vertical direction. To avoid this difficulty, we proceed in the following way. We write the Bony’s decomposition for \(\Delta_q(u_3 \partial_3 u)\) in two para-product terms and in the rest term. Using some commutators, we decompose now the para-product term \(\Delta_q T_{u_3} \partial_3 u\) as following
\[
\Delta_q(S_{q'-1}u_3 \partial_3 \Delta_{q'}u) = S_{q'-1}u_3 \partial_3 \Delta_q \Delta_{q'}u + (S_{q'-1}u_3 - S_{q'-1}u_3) \partial_3 \Delta_q \Delta_{q'}u + [\Delta_q, S_{q'-1}u_3] \partial_3 \Delta_{q'}u.
\]
Summing in $q'$, we obtain finally the following decomposition
\[ T^v_q = T^{v,q}_1 + T^{v,q}_2 + T^{v,q}_3 + T^{v,q}_4 + R^{v,q}, \]
where
\[ T^{v,q}_1 = \int_{\mathbb{R}^3} (S_{q-1} u_3) \partial_3 \Delta_q u \cdot \Delta_q u \, dx \]
\[ T^{v,q}_2 = \sum_{|q'-q| < N_0} \int_{\mathbb{R}^3} [\Delta_q, S_{q'-1} u_3] \partial_3 \Delta_q u \cdot \Delta_q u \, dx \]
\[ T^{v,q}_3 = \sum_{|q'-q| < N_0} \int_{\mathbb{R}^3} (S_{q'-1} u_3 - S_{q-1} u_3) \partial_3 \Delta_q u \cdot \Delta_q u \, dx \]
\[ T^{v,q}_4 = \sum_{|q'-q| < N_0} \int_{\mathbb{R}^3} \Delta_q (\Delta_q u_3 \partial_3 S_{q'-1} u) \cdot \Delta_q u \, dx \]
\[ R^{v,q} = \sum_{q' > q - N_0} \int_{\mathbb{R}^3} \Delta_q (\Delta_q u_3 \partial_3 S_{q''} u) \cdot \Delta_q u \, dx. \]

We shall now estimate each of the above terms.

- **Estimate of $T^{v,q}_1$**
  Integrating by parts yields
  \[ T^{v,q}_1 = -\frac{1}{2} \int_{\mathbb{R}^3} (S_{q-1} \partial_3 u_3) \Delta_q u \cdot \Delta_q u \, dx. \]
  Using Proposition 6.1 we get
  \[ |T^{v,q}_1| \leq C \left\| \nabla_h u \right\|_{L^\infty L_h^2} \left\| \Delta_q u \right\|_{L^2 L_h^4}^2 \leq 2^{-2q} b_q \left\| \nabla_h u \right\|_{L^\infty L_h^2} \left\| u \right\|_{H^s} \left\| \nabla_h u \right\|_{H^s}. \]

- **Estimate of $T^{v,q}_2$**
  By Hölder inequality, we have
  \[ |T^{v,q}_2| \leq \left\| [\Delta_q, S_{q'-1} u_3] \partial_3 \Delta_q u \right\|_{L^2 L_h^4} \left\| \Delta_q u \right\|_{L^2 L_h^4}. \]
  Using Lemma 2.2, Proposition 6.1 and the fact that $\Delta_q u$ is localized in vertical frequencies, we infer
  \[ |T^{v,q}_2| \leq C 2^{-q} \left\| S_{q'-1} \nabla u_3 \right\|_{L^\infty L_h^2} \left\| \partial_3 \Delta_q u \right\|_{L^2 L_h^4} \left\| \Delta_q u \right\|_{L^2 L_h^4} \leq C 2^{q'} \left\| \nabla_h u_3, \partial_3 u_3 \right\|_{L^\infty L_h^2} \left\| \Delta_q u \right\|_{L^2 L_h^4} \left\| \Delta_q u \right\|_{L^2 L_h^4} \leq C \left\| \nabla_h u \right\|_{L^\infty L_h^2} \left\| \Delta_q u \right\|_{L^2 L_h^4} \left\| \Delta_q u \right\|_{L^2 L_h^4} \leq C 2^{-2q} b_q \left\| \nabla_h u \right\|_{L^\infty L_h^2} \left\| u \right\|_{H^s} \left\| \nabla_h u \right\|_{H^s}. \]

- **Estimate of $T^{v,q}_3$**
Since $S_{q' - 1} u_3 - S_{q - 1} u_3$ is localized in frequency in a ring of size $2^q$, we get
\[
|T_{3}^{v,q}| \leq \sum_{q' \sim q} \|S_{q' - 1} u_3 - S_{q - 1} u_3\|_{L^\infty_h L^2_v} \|\partial_3 \Delta_q \Delta_{q'} u\|_{L^2_h L^4_v} \|\Delta_q u\|_{L^2_v L^4_h}
\]
\[
\leq \sum_{q' \sim q} 2^{-q} \|(S_{q' - 1} - S_{q - 1}) \nabla u_3\|_{L^\infty_h L^2_v} \|\partial_3 \Delta_q \Delta_{q'} u\|_{L^2_h L^4_v} \|\Delta_q u\|_{L^2_v L^4_h}
\]

Using Proposition [6.1] we obtain
\[
|T_{3}^{v,q}| \leq C 2^{-2q} b_q \|\nabla_h u\|_{L^\infty_h L^2_v} \|u\|_{H^s} \|\nabla_h u\|_{H^s}.
\]

**Estimate of $T_{4}^{v,q}$**

As in the above estimates, we have
\[
|T_{4}^{v,q}| \leq C \sum_{|q - q'| < N_0} \|\Delta_q (\Delta_{q'} u_3 \partial_3 S_{q' - 1} u)\|_{L^2_v L^4_h} \|\Delta_q u\|_{L^2_v L^4_h}
\]
\[
\leq C \sum_{|q - q'| < N_0} \|\Delta_q u_3 \partial_3 S_{q' - 1} u\|_{L^2_v L^4_h} \|\Delta_q u\|_{L^2_v L^4_h}
\]
\[
\leq C \sum_{|q - q'| < N_0} \|\Delta_q u_3\|_{L^2_v L^2_h} \|\partial_3 S_{q' - 1} u\|_{L^\infty_h L^2_v} \|\Delta_q u\|_{L^2_v L^4_h}.
\]

Proposition [6.1] implies that
\[
\|\Delta_{q'} u_3\|_{L^2} \leq C 2^{-q'} \|\Delta_q \nabla u_3\|_{L^2} \leq C 2^{-q} 2^{-q} c_q \|\nabla_h u\|_{H^s},
\]
and
\[
\|\Delta_q u\|_{L^2_v L^4_h} \leq 2^{-q} c_q \|u\|_{H^s} \|\nabla_h u\|_{H^s}^{1/2}.
\]

In addition
\[
\|\partial_3 S_{q' - 1} u\|_{L^\infty_h L^4_v} \leq C 2^{q'} \|u\|_{L^\infty_h L^4_v}
\]
\[
\leq C 2^{q'} \|u\|_{L^\infty_h L^2_v}^{1/2} \|\nabla_h u\|_{L^\infty_h L^2_v}^{1/2}.
\]

Hence
\[
|T_{4}^{v,q}| \leq 2^{-2q} b_q \|u\|_{L^\infty_h L^2_v}^{1/2} \|\nabla_h u\|_{L^\infty_h L^2_v}^{1/2} \|u\|_{H^s}^{1/2} \|\nabla_h u\|_{H^s}^{3/2}.
\]

**Estimate of $R_{v,q}$**

We easily see that
\[
|R_{v,q}| \leq C \sum_{q' > q - N_0} \|\Delta_q (\Delta_{q'} u_3 \partial_3 \Delta_{q''} u)\|_{L^2_v L^4_h} \|\Delta_q u\|_{L^2_v L^4_h}
\]
\[
\leq C \sum_{q' > q - N_0} \|\Delta_{q'} u_3 \partial_3 \Delta_{q''} u\|_{L^2_v L^4_h} \|\Delta_q u\|_{L^2_v L^4_h}
\]
\[
\leq \sum_{q' > q - N_0} \|\Delta_{q'} u_3\|_{L^2} \|\partial_3 \Delta_{q''} u\|_{L^\infty_h L^4_v} \|\Delta_q u\|_{L^2_v L^4_h}.
\]
Using Proposition 6.1 and Lemma 2.4, we infer
\[ \|\Delta_q u_3\|_{L^2} \leq 2^{-q'}\|\Delta_q \nabla u_3\|_{L^2} \]
\[ \leq 2^{-q'}\|\Delta_q \nabla h_3\|_{L^2} \]
\[ \leq 2^{-q'}2^{-q' \alpha_c}(5.1) \|\nabla h\|_{H^s}, \]
\[ \|\partial_3 \Delta_q u\|_{L^6 L^4_\Xi} \leq 2^{q''}\|u\|_{L^6 L^4_\Xi} \]
\[ \leq 2^{q''}\|u\|_{L^6 L^4_\Xi} \|\nabla h\|_{L^6 L^4_\Xi}, \]
and
\[ \|\Delta_q u\|_{L^6 L^4_\Xi} \leq 2^{-q' \alpha_c}\|u\|_{H^s} \|\nabla h\|_{H^s}. \]
Finally
\[ |R^{u}\| \leq 2^{-2q' \alpha_c}b_q \|u\|_{L^6 L^4_\Xi} \|\nabla h\|_{L^6 L^4_\Xi} \|\nabla h\|_{H^s}^{3/2}. \]
The proof of Lemma 3.1 is completed.

7. Appendix

This section is devoted to the proofs of some technical lemmas stated in the second section. Note that the proofs of such results are contained in several papers (see for instance [5, 26, 27]). Here we give them for the convenience of the reader.

Proof of Bernstein Lemma 2.3
Let ̃ be a ring such that ̃ ⊂ ̃ and φ be a smooth compactly supported function in ̃, whose value is 1 near ̃. Since supp ̃ u ⊂ 2φ̃, we infer
\[ \mathcal{F} \Delta_q u(\xi) = \phi(2^{-q} \xi) \mathcal{F} \Delta_q u(\xi) = \sum_{i=1,3} \phi_i(2^{-q} \xi) \xi_i \partial_{\xi_i} \Delta_q u(\xi), \]
where φ(ξ) = \sum_{i=1,3} \phi_i(ξ) is such that ξ_i ≠ 0 on the support of φ_i. If we set ψ_i(ξ) = \phi_i(ξ)/ξ_i, then ψ_i ∈ C^∞(R^3), has the support in ̃ and
\[ \mathcal{F} \Delta_q u(\xi) = 2^{-q} \sum_{i=1,3} \psi_i(2^{-q} \xi) \partial_{\xi_i} \Delta_q u(\xi). \]
We denote by h the rapidly decaying function such that \mathcal{F} h_i = ψ_i. Therefore
\[ \Delta_q u(x) = 2^{-q} h_i(2^q \xi) \partial_{\xi_i} \Delta_q u(x). \]
From this equality, we easily deduce that
\[ \|\Delta_q u\|_{L^6 L^4_\Xi} \leq C 2^{-q} \sum_{i=1,3} \|\Delta_q u\|_{L^1} \|\partial_{\xi_i} \Delta_q u\|_{L^6 L^4_\Xi} \]
and so
\[ \|\Delta_q u\|_{L^6 L^4_\Xi} \leq C 2^{-q} \|\nabla\Delta_q u\|_{L^6 L^4_\Xi}. \]
Proof of Lemma 2.1.
We introduce the function \( h_j(x) = 2^{3j} h(2^j x) \), where \( h \in \mathcal{S}(\mathbb{R}^3) \) is such that \( \mathcal{F}(h) = \varphi \in C_0^\infty(\mathbb{R}^3) \). Let

\[
u_j(x) := \Delta_j u(x) = \int_{\mathbb{R}^3} h_j(x - y) u(y) dy
\]

\[
= \int_{\mathbb{R}^3} h_j(x_h - y_h, x_v - y_v) u(y_h, y_v) dy_h dy_v
\]

\[
= \int_\mathbb{R} \left( \int_{\mathbb{R}^2} h_j(x_h - y_h, x_v - y_v) u(y_h, y_v) dy_h \right) dy_v
\]

\[
= \int_\mathbb{R} \left( h_j(\cdot, x_v - y_v) \ast_{x_h} u(\cdot, y_v) \right)(x_h) dy_v.
\]

Then

\[
\| u_j(\cdot, x_v) \|_{L_h^r} \leq \int_\mathbb{R} \| h_j(\cdot, x_v - y_v) \ast_{x_h} u(\cdot, y_v) \|_{L_h^r} dy_v.
\]

By the Young inequality, we get

\[
\| h_j(\cdot, x_v - y_v) \ast_{x_h} u(\cdot, y_v) \|_{L_h^r} \leq \| h_j(\cdot, x_v - y_v) \|_{L_h^r} \| u(\cdot, y_v) \|_{L_h^r}
\]

and so

\[
\| u_j(\cdot, x_v) \|_{L_h^r} \leq \int_\mathbb{R} \| h_j(\cdot, x_v - y_v) \|_{L_h^r} \| u(\cdot, y_v) \|_{L_h^r} dy_v
\]

\[
= \left( \| h_j(\cdot, \cdot) \|_{L_h^r} \ast_{x_v} \| u(\cdot, \cdot) \|_{L_h^r} \right)(x_v).
\]

Now, we take the \( L_v^p \) norm in the above inequality to obtain

\[
\left\| \| u_j(\cdot, x_v) \|_{L_h^r} \right\|_{L_v^p} \leq \left\| \| h_j(\cdot, \cdot) \|_{L_h^r} \right\|_{L_v^p} \left\| \| u(\cdot, \cdot) \|_{L_h^r} \right\|_{L_v^p}.
\]

Finally

\[
\| \Delta_j u \|_{L_v^p L_h^r} \leq \| h \|_{L^1(\mathbb{R}^3)} \| u \|_{L_v^p L_h^r}.
\]

\( \blacksquare \)

Proof of Lemma 2.2.
We will prove here, that, for all \( p, r, s, t \geq 1 \) such that \( \frac{1}{r} = \frac{1}{s} + \frac{1}{t} \), we have

\[
\| [\Delta_q; a] b \|_{L_v^p L_h^r} \leq C 2^{-q} \| \nabla a \|_{L_v^\infty L_h^s} \| b \|_{L_v^p L_h^t}.
\]

We begin by writing the commutator in the following form
$$[\Delta_q; a]b(x) = 2^{3q} \int_{\mathbb{R}^3} h(2^q y)(a(x - y) - a(x))b(x - y)dy$$

$$= -2^{3q} \int_{\mathbb{R}^3} h(2^q y) \int_0^1 y\nabla a(x - \tau y)d\tau b(x - y)dy$$

$$= -2^{3q} \int_{\mathbb{R}^3 \times [0,1]} \sum_{i=1,3} h_i(2^q y)\partial_i a(x - \tau y)b(x - y)d\tau dy$$

$$= 2^{-q}2^{3q} \int_{\mathbb{R}^3 \times [0,1]} \sum_{i=1,3} h_i(2^q y)\cdot (\partial_i a)(x - \tau y)b(x - y)d\tau dy,$$

where $h_i(z) = h(z)z_i$. We take the $L^r_h$ norm in the horizontal variable and we apply

the Hölder inequality to obtain

$$\|\langle \Delta_q; a\rangle b(\cdot, x_v)\|_{L^r_h} \leq 2^{-q} \int_{\mathbb{R}^3 \times [0,1]} \sum_{i=1,3} 2^{3q} |h_i(2^q y)|\|\partial_i a(\cdot - \tau y_h, x_v - \tau y_v)b(\cdot - y_h, x_v - y_v)\|_{L^r_h} d\tau dy$$

$$\leq 2^{-q} \int_{\mathbb{R}^3 \times [0,1]} \sum_{i=1,3} 2^{3q} |h_i(2^q y)|\|\partial_i a(\cdot - \tau y_h, x_v - \tau y_v)\|_{L^r_h} \|b(\cdot - y_h, x_v - y_v)\|_{L^{1r}_x} d\tau dy$$

$$= 2^{-q} \int_{\mathbb{R}^3 \times [0,1]} \sum_{i=1,3} 2^{3q} |h_i(2^q y)|\|\partial_i a(\cdot, x_v - \tau y_v)\|_{L^r_h} \|b(\cdot, x_v - y_v)\|_{L^1_h} d\tau dy.$$

We take now the $L^p_v$ norm in the vertical variable and we obtain

$$\|\langle \Delta_q; a\rangle b\|_{L^p_v L^r_h} \leq 2^{-q} \int_{\mathbb{R}^3 \times [0,1]} \sum_{i=1,3} 2^{3q} |h_i(2^q y)|\|\partial_i a(\cdot, x_v - \tau y_v)\|_{L^p_v} \|b(\cdot, x_v - y_v)\|_{L^r_h} d\tau dy$$

$$\leq 2^{-q} \int_{\mathbb{R}^3 \times [0,1]} \sum_{i=1,3} 2^{3q} |h_i(2^q y)|\sup_{x_v} \|\partial_i a(\cdot, x_v - \tau y_v)\|_{L^p_v} \|b(\cdot, x_v - y_v)\|_{L^{p_r}_x} d\tau dy$$

$$= 2^{-q} \int_{\mathbb{R}^3 \times [0,1]} \sum_{i=1,3} 2^{3q} |h_i(2^q y)|\|\partial_a\|_{L^p_v(L^r_h)} \|b\|_{L^p_v(L^r_h)} \|h_i\|_{L^1(\mathbb{R}^3)}.$$

This leads to

$$\|\langle \Delta_q; a\rangle b\|_{L^p_v(L^r_h)} \leq C2^{-q} \|\nabla a\|_{L^p_v(L^r_h)} \|b\|_{L^p_v L^r_h}.$$

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References

[1] A. Babin, A. Mahalov and B. Nicolaenko, Regularity and integrability of 3D Euler and Navier-Stokes equations for rotating fluids, Asymptotic Analysis, 15, p. 103-150, 1997.
[2] A. Babin, A. Mahalov and B. Nicolaenko, Global Regularity of 3D Rotating Navier-Stokes Equations for Resonant Domains, Indiana University Mathematics Journal, Vol. 48, No. 3, p. 1133-1176, 1999.
[3] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Annales scientifiques de l’École Normale Supérieure, 14, p. 209-246, 1981.
[4] D. Bresch, D. Gérad-Varet and E. Grenier, Derivation of the Planetary Geostrophic Equations, Arch. Rat. Mech. Anal. 182, No 2, p. 387-413, 2006.
[5] J.-Y. Chemin, Localization in Fourier space and Navier-Stokes system, Preprint.
[6] J.-Y. Chemin, Fluides parfaits incompressibles, Astérisque 230, 1995.
[7] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Fluids with anisotropic viscosity, M2AN. Math. Numer. Anal., 34, No. 2, p. 315-335, 2000.
[8] J.-Y. Chemin, B. Desjardins, I. Gallagher and E. Grenier, Ekman boundary layers in rotating fluids, ESAIM Contrôle optimal et Calcul des Variations, Special Tribute issue to Jacques-Louis Lions, 8, p. 441-466, 2002.
[9] C. Cheverry, Propagation of oscillations in Real Vanishing Viscosity Limit, Commun. Math. Phys. 247, p. 655-695, 2004.
[10] H. Fujita and T. Kato, On the Navier-Stokes initial value problem I, Archiv for Rational Mechanic and Analysis, 16, p. 269-315, 1964.
[11] I. Gallagher, Applications of Schochet’s Methods to Parabolic Equation, Journal de Mathématiques Pures et Appliquées, 77, p. 989-1054, 1998.
[12] I. Gallagher, Asymptotics of the Solutions of Hyperbolic Equations With a Skew-Symmetric Perturbation, Journal of Differential Equations, 150, p. 363-384, 1998.
[13] I. Gallagher and L. Saint-Raymond, Weak convergence results for inhomogeneous rotating fluid equations, Journal d’Analyse Mathématique, Vol. 99, p. 1-34, 2006.
[14] I. Gallagher and L. Saint-Raymond, On the influence of the Earth’s rotation on geophysical flows, Handbook of Mathematical Fluid Dynamics Vol 4, p. 201-329.
[15] I. Gallagher and Laure Saint-Raymond, On pressureless gases driven by a strong inhomogeneous magnetic field, SIAM Journal for Mathematical Analysis, 36, no. 4, p. 1159-1176, 2005.
[16] E. Grenier, Oscillatory Perturbations of the Navier-Stokes Equations, Journal de Mathématiques Pures et Appliquées, 76, p. 477-498, 1997.
[17] E. Grenier and N. Masmoudi, Ekman layers of rotating fluids, the case of well prepared initial data, Comm. Partial Differential Equations 22, no. 5-6, p. 953-975, 1997.
[18] D. Iftimie, Resolution of the Navier-Stokes equations in anisotropic spaces, Revista Matemática Iberoamericana, 15, no. 1, p. 1-36, 1999.
[19] D. Iftimie, The 3D Navier-Stokes equations seen as a perturbation of the 2D Navier-Stokes equations, Bulletin de la Société Mathématique de France, 127, p. 473-517, 1999.
[20] D. Iftimie, A Uniqueness result for the Navier-Stokes Equations with vanishing vertical viscosity, SIAM J. Math. Anal. Vol. 33, No. 6, p. 1483-1493, 2002.
[21] J. Joly, G. Métivier and J. Rauch, Generic rigourous Asymptotic Expansions for Weakly Nonlinear Multidimensional Oscillatory Waves, Duke Mathematical Journal, 70, p. 373-404, 1993.
[22] J. Joly, G. Métivier and J. Rauch, Coherent and Focusing Multidimensional Oscillatory Nonlinear Geometric optics, Annales Scientifiques de L’ENS, 28, p. 51-113, 1995.
[23] S. Klainerman and A. Majda, Singular Limits of Quasilinear Hyperbolic system with Large Parameters, and the Incompressible Limit of Compressible Fluids, Communications on pure and applied Mathematics, 34, p. 481-524, 1981.
[24] J. Leray, Sur le mouvement d’un liquide visqueux remplissant l’espace, Acta Math, 63, p. 193-248, 1934.
[25] N. Masmoudi, *Ekman layers of rotating fluids: the case of general initial data*, Comm. Pure Appl. Math. 53, no. 4, p. 432-483, 2000.

[26] M. Paicu, *Équation anisotrope de Navier-Stokes dans des espaces critiques*, Rev. Mat. Iberoamericana, 21, no. 1, p. 179–235, 2005.

[27] M. Paicu, *Étude asymptotique pour les fluides anisotropes en rotation rapide dans le cas périodique*, Journal des Mathématiques Pures et Appliquées, 83, p. 163-242, 2004.

[28] M. Paicu, *Équation périodique de Navier-Stokes sans viscosité dans une direction*, Comm. Partial Differential Equations, 30, no. 7-9, p. 1107–1140, 2005.

[29] M. Sablé-Tourgeron, *Régularité microlocale pour des problèmes aux limites non linéaires*, Ann. Inst. Fourier, 36, p. 39-82, 1986.

[30] S. Schochet, *The Compressible Euler Equations in a Bounded Domain: Existence of Solutions and the Incompressible Limit*, Comm. Math. Physics, 104, p. 49-75, 1986.

[31] S. Schochet, *Fast Singular Limits of Hyperbolic PDEs*, Journal of Differential Equations, 114, p. 476-512, 1994.

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