Revisiting the Hubble Constant, Spatial Curvature, and Cosmography with Strongly Lensed Quasar and Hubble Parameter Observations

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Abstract

It is well known that time delays due to strong lensing offer the opportunity of a one-step measurement of the Hubble constant $H_0$ that is independent of the cosmic distance ladder. In this paper, we go further and propose a cosmological model-independent approach to simultaneously determine the Hubble constant and cosmic curvature with measurements of the time delay due to strong lensing, without any prior assumptions regarding the content of the universe. The data we use comprise the recent compilation of six well studied strongly lensed quasars, while the cosmic chronometer data are utilized to reconstruct distances via cosmographic parameters. In the framework of third-order Taylor expansion and $(2, 1)$ order Padé approximation for cosmographic analysis, our results provide model-independent estimations of the Hubble constant $H_0 = 72.24^{+2.52}_{-2.22}$ km s$^{-1}$ Mpc$^{-1}$ and $H_0 = 72.45^{+2.02}_{-1.93}$ km s$^{-1}$ Mpc$^{-1}$, which are well consistent with that derived from the local distance ladder by the SH0ES collaboration. The measured cosmic curvature $\Omega_k = 0.062^{+0.017}_{-0.078}$ and $\Omega_k = 0.069^{+0.117}_{-0.103}$ shows that zero spatial curvature is supported by the current observations of time delays due to strong lensing and cosmic chronometers. Imposing the prior of spatial flatness leads to more precise (at 1.6% level) determinations of the Hubble constant $H_0 = 70.47^{+1.15}_{-1.13}$ km s$^{-1}$ Mpc$^{-1}$ and $H_0 = 71.66^{+1.37}_{-1.52}$ km s$^{-1}$ Mpc$^{-1}$, values located between the results from Planck and the SH0ES collaboration. If a prior of local (SH0ES) $H_0$ measurement is adopted, the constraint on curvature parameter can be further improved to $\Omega_k = 0.123^{+0.046}_{-0.040}$ and $\Omega_k = 0.101^{+0.072}_{-0.071}$, supporting no significant deviation from a flat universe. Finally, we also discuss the effectiveness of the Padé approximation in reconstructing the cosmic expansion history for redshifts up to $z \sim 2.3$, considering its better performance in the Bayes information criterion.

Unified Astronomy Thesaurus concepts: Strong gravitational lensing (1643); Cosmological parameters (339)

1. Introduction

In modern cosmology, the spatially flat model with cosmological constant plus cold dark matter ($\Lambda$CDM) has withstood tests against most of the observational evidence (Riess et al. 2007; Cao et al. 2011, 2012, 2015; Cao & Zhu 2014; Planck Collaboration et al. 2016; Alam et al. 2017). However, the standard cosmological model suffers the well-known fine-tuning problem and coincidence problem (Weinberg 1989, 2000; Dalal et al. 2001; Chen et al. 2010; Cao et al. 2011; Qi et al. 2018). In particular, there is a $4.4\sigma$ tension between the Hubble constant ($H_0$) inferred within $\Lambda$CDM from data on the cosmic microwave background (CMB) anisotropy (temperature and polarization) (Planck Collaboration et al. 2020) and its value measured through the Cepheid-calibrated distance ladder by the Supernova H0 for the Equation of State collaboration (SH0ES) (Riess et al. 2019). This inconsistency may be caused by unknown systematic errors in astrophysical observations or may reveal new physics significantly different from $\Lambda$CDM (Freeman 2017). Recently, some works suggested that the $H_0$ tension could possibly be caused by the inconsistency of spatial curvature between the early universe and late universe (Di Valentino et al. 2020, 2021; Handley 2021). More specifically, combining the Planck temperature and polarization power spectra data, the work showed that a closed universe with $\Omega_k = -0.044^{+0.018}_{-0.015}$ was supported, as a consequence of higher lensing amplitude (Planck Collaboration et al. 2020) supported by the data. However, a combination of the Planck lensing data and observations of redshift baryon acoustic oscillations (BAOs) leads to the preference of a flat universe with curvature parameter precisely constrained within $\Omega_k = 0.0007 \pm 0.0019$ (Di Valentino et al. 2020). It should be emphasized that both methods invoke a particular cosmological model—the non-flat $\Lambda$CDM. To better understand the inconsistency between the Hubble constant and the curvature of the universe inferred from local measurements and Planck observations, it is necessary to seek other methods that could provide cosmological-model independent measurements of both $H_0$ and $\Omega_k$.

Strong gravitational lensing (SGL) systems, which have already demonstrated their ability to constrain dark matter density (Cao et al. 2011; Cao & Zhu 2012), dark energy equation of state (Cao et al. 2012; Liu et al. 2019), and the velocity dispersion function of early-type galaxies (Ma et al. 2019; Geng et al. 2021), provide a valuable opportunity for model-independent constraints on both $H_0$ and $\Omega_k$. First, from the measurement of time delay between multiple images from SGL, one can measure the Hubble constant effectively and directly. For details, practical approaches, and results of the Hubble constant measurements using the SGL time delay one can refer to Liao et al. (2019, 2020), Lyu et al. (2020), and Taubenberger et al. (2019). Second, combining the time-delay measurements with other SGL observations and using the well-
known distance sum rule, one can simultaneously determine the spatial curvature and the Hubble constant without adopting any particular cosmological model. Recently, the work by Collett et al. (2019) first presented such a methodology to simultaneously constrain the spatial curvature and the Hubble constant with Type Ia supernova observations and SGL time-delay measurement. In further research, Wei & Melia (2020) extended this approach by considering the nonlinear relation between observations of the ultraviolet and X-ray fluxes of high-redshift quasars. However, the SGL time-delay measurements applied to cosmology require three angular diameter distances. Hence, observations using angular diameter distances are more intuitive and efficient in this context, and following this path Qi et al. (2021) further determined $H_0$ and $\Omega_k$ by combining SGL time delays with compact structure in radio quasars acting as standard rulers, and obtained the stringent results $H_0 = 78.3 \pm 2.9$ km s$^{-1}$ Mpc$^{-1}$ and $\Omega_k = 0.49 \pm 0.24$. However, one should emphasize here that whether using quasars as standard rulers or supernovae as standard candles, the problem of relative distance calibration is unavoidable. It should be done without invoking any particular background cosmology model in order not to fall into a circularity. One general approach often used by researchers is to take a polynomial expansion of the cosmological distance in redshift (or a certain function of redshift such as $\log(1+z)$). Such a reconstruction procedure often involves questions about the physical meaning of polynomial coefficients.

In this paper, we will determine the curvature parameter and the Hubble constant simultaneously using the direct measurements of Hubble expansion rates provided by the proposed differential ages of passively evolving galaxies (also known as cosmic chronometers; Jimenez & Loeb 2002), focusing on six SGL time-delay measurements released by the $H_0$ Lenses in COSMOGRAIL’s Wellspring (H0LiCOW) collaboration (Wong et al. 2020). For the purpose of distance reconstruction, instead of the polynomial expansion, we will use a method named cosmography (Collett et al. 2019; Wei & Melia 2020; Qi et al. 2021). Examples of using a cosmographic approach can be found in, e.g., Risaliti & Lusso (2015, 2019), Visser (2015), Dunsby et al. (2016), and Capozziello et al. (2019). The advantage of cosmography consists in minimal prior assumptions about the universe (only its homogeneity and isotropy) and in using the expansion of the Hubble function around the present time, which is much more physical than polynomial expansion of distance. This paper is organized as follows. In Section 2, we present the methodology and observational data. The results and discussion are given in Section 3. Finally, we summarize our findings in Section 4.

## 2. Methodology and Observations

For a general homogeneous and isotropic universe, the FLRW metric reads

$$ds^2 = c^2dt^2 - \frac{a(t)^2}{1-kr^2}dr^2 - a(t)^2r^2d\Omega^2,$$

where $a(t)$ is the scale factor, $c$ is the speed of light, and $k$ is dimensionless curvature taking one of three values $\{-1, 0, 1\}$. The cosmic curvature parameter $\Omega_k$ is related to $k$ and the Hubble constant $H_0$ as $\Omega_k = -c^2k/a_0^2H_0^2$. Having in mind an SGL system consisting of three elements—the source, the lens, and the observer—let us introduce dimensionless comoving distances $d_l \equiv d(0,z_l)$, $d_s \equiv d(0,z_s)$, and $d_b \equiv d(z_b,z)$ where the cosmological length scale $c/H_0$ is factored out. As a specific example, the dimensionless comoving distance between the lensing galaxy (at redshift $z_l$) and the background source (at redshift $z_s$) is given by

$$\frac{d(z_l,z_s)}{d(z_l)} = \frac{H_0}{c}(1 + z_s)D_A(z_l,z_s),$$

$$= \frac{1}{\sqrt{|\Omega_k|}} \left( \frac{\int_{z_l}^{z_s} dz'/E(z')}{\int_{z_l}^{z_s} dz'/E(z')} \right), \quad (2)$$

where

$$f(x) = \begin{cases} \sin(x), & \Omega_k < 0, \\ x, & \Omega_k = 0, \\ \sin(x), & \Omega_k > 0. \end{cases} \quad (3)$$

$D_A(z_l,z_s)$ represents the angular diameter distance from $z_l$ to $z_s$, which is given by $D_A(z_l,z_s) = \frac{c}{1+z_l} \int_{z_l}^{z_s} dz' H(z')$, and the expansion rate at redshift $z$ is $H(z) = H_0 E(z)$. In the framework of the FLRW metric, one can use the well-known distance sum rule (DSR) to relate these three distances in a non-flat universe (Räsänen et al. 2015):

$$d_{ls} = d_s \sqrt{1 + \Omega_k d_l^2} - d_l \sqrt{1 + \Omega_k d_s^2}.$$

Note that in terms of dimensionless comoving distances the DSR will reduce to an additive relation $d_{ls} = d_s + d_l$ in the flat universe ($\Omega_k = 0$). The original idea of cosmological application of the DSR in general, and with respect to gravitational lensing data in particular, can be traced back to Räsänen et al. (2015). Moreover, one can rearrange Equation (4) as

$$\frac{dd_{ls}}{dd_{l}} = \frac{1}{\sqrt{1/d_s^2 + \Omega_k} - \sqrt{1/d_l^2 + \Omega_k}}, \quad (5)$$

which provides a model-independent test for the Copernican principle (Qi et al. 2019; Cao et al. 2019; Zhang et al. 2022) and the viscosity of dark matter (Cao et al. 2021, 2022), with different measurements of cosmic curvature and dark matter viscosity at different redshifts. Expression (5) multiplied by $1 + z_l$ and with dimensionality of length recovered is known as the time delay distance $D_{\Delta t}$.

**Time-delay distances from lensed quasars.** In strong lensing systems with quasars acting as background sources, the time difference (time delay) between two images of the source, which could be measured from variable light curves of active galactic nuclei, depends on the time-delay distance $D_{\Delta t}$ and the geometry of the universe as well as the gravitational potential of the lensing galaxy (Perlick 1990a, 1990b):
requires deep and wide-field imaging of the area around the lens system, \( \theta_i \) and \( \theta_j \) are angular positions of images \( i \) and \( j \) in the lens plane (Treu & Marshall 2016). The two-dimensional lensing potentials at the image positions, \( \psi(\theta_i) \) and \( \psi(\theta_j) \), and the unlensed source position \( \beta \) can be determined by the mass model of the system (Rathna Kumar et al. 2015). The time-delay distance \( D_{\Delta t} \) is a combination of three angular diameter distances expressed as

\[
D_{\Delta t} = (1 + z_i) \frac{D_\Delta^A D_j^A}{D_k^A} = \frac{c}{H_0} \frac{d_i d_j}{d_k},
\]

where the superscript (A) denotes angular diameter distance. One can clearly see that the distance ratios \( d_i/d_k \) can be assessed from the observations of time delay in SGL systems. Meanwhile, if the other two dimensionless distances \( d_i \) and \( d_j \) can be obtained from observations, the measurement of \( \Omega_k \) and \( H_0 \) could be directly obtained with time-delay measurements of \( \Delta t_{ij} \) and a well-reconstructed lens potential difference of \( \Delta \phi_{ij} \) between two images. See Liao (2019), Harvey (2020), Ding et al. (2021), Sonnenfeld (2021), Liao (2021), and Bag et al. (2022) for more recent works on the extensive applications of SGL time delays.

In the milestone paper of Wong et al. (2020), the sample of six lensed quasars was jointly used to measure \( H_0 \) by the H0LiCOW collaboration under six dark energy cosmological models. The six lenses with measured time delays are B1608 +656 (Suyu et al. 2010; Jee et al. 2019), RXJ1131–1231 (Suyu et al. 2013, 2014; Chen et al. 2019), HE 0435–1223 (Chen et al. 2019; Wong et al. 2017), SDSS 1206+4332 (Birrer et al. 2019), WFI2033−4723 (Rusu et al. 2020), and PG 1115+080 (Chen et al. 2019) and their sources cover the redshift range \( 0.654 < z_s < 1.789 \). The time-delay distance and the luminosity distance for lensed quasar systems are summarized in Table 2 of Wong et al. (2020). For the earliest studied lens, B1608 +656, its \( D_{\Delta t} \) measurement was represented by a skewed log-normal distribution. The \( D_{\Delta t} \) of other lenses were given in the form of Monte Carlo Markov chains (MCMCs) in a series of H0LiCOW papers. The posterior distributions of distances are available on the H0LiCOW website. The future Legacy Survey of Space and Time (LSST) of the Vera C. Rubin Observatory, should discover a considerable number of lensed quasar systems whose time delays could be measured (Liao et al. 2015). The project named “Time Delay Challenge” was proposed to test the measurability of SGL time delays with a blind analysis of mock LSST data (Dobler et al. 2015; Liao et al. 2015). After testing many different algorithms, the results showed that time-delay measurements of the precision and accuracy needed for time-delay cosmography would indeed be possible (Ding et al. 2018; Millon et al. 2020a, 2020b; Shajib et al. 2022).

**Cosmographic approach to reconstruct the distance.** In order to determine the curvature parameter and the Hubble constant, two dimensionless distances \( d_i \) and \( d_j \) are required. Other ingredients are directly measurable. In this work, we will reconstruct distances corresponding to redshifts \( z_i \) and \( z_j \) by using the cosmographic approach to the data regarding the cosmic expansion history available from cosmic chronometers. Cosmographic techniques allow one to study the evolution of the universe in a model-independent way through kinematic variables. The procedure starts by introducing the series of cosmographic functions defined by subsequent time derivatives of the scale factor (Weinberg 1972, 2008; Chiba & Nakamura 1998; Alam et al. 2003; Visser 2004). The first four such functions are known as the Hubble function, deceleration, jerk, and snap:

\[
H \equiv \frac{\dot{a}}{a}, \quad q \equiv -\frac{\ddot{a}}{a H^2}, \quad j \equiv \frac{\dddot{a}}{a H^3}, \quad s \equiv \frac{\ddddot{a}}{a H^4}.
\]

where the dots represent cosmic time derivatives and \( a^{(n)} \) stands for the \( n \)th time derivative of the scale factor. With the above preparation, one can Taylor-expand the Hubble parameter:

\[
H(z) = H_0 + \frac{dH}{dz} \bigg|_{z=0} z + \frac{1}{2!} \frac{d^2H}{dz^2} \bigg|_{z=0} z^2 + \frac{1}{3!} \frac{d^3H}{dz^3} \bigg|_{z=0} z^3 + \cdots.
\]

and express the coefficients of the expansion in terms of cosmographic functions:

\[
\mathcal{H}_i^2 = 1, \quad \mathcal{H}_i^4 = 1 + q_0, \quad \mathcal{H}_i^6 = \frac{1}{2} (j_0 - q_0^2), \quad \mathcal{H}_i^8 = \frac{1}{6} (-3j_0 - 4q_0 j_0 + 3q_0^2 + 3q_0^3 - s_0).
\]

Here, the subscript “0” indicates the parameters at the present epoch (\( z = 0 \)). In particular, \( q_0 < 0 \) indicates that the universe is decelerating, and \( q_0 > 0 \) corresponds to an accelerating universe. A positive sign of \( j_0 \) means a transition time between the deceleration and acceleration.

Furthermore, from (9) and (10) one is able to build the expansion of dimensionless comoving distance \( d(z) \),

\[
d(z) = \sum_{i=1} D_i^j z^i,
\]

with

\[
D_0^1 = 1, \quad D_2^2 = \frac{1}{2} (1 + q_0), \quad D_4^3 = \frac{1}{6} (2 - j_0 + 4q_0 + 3q_0^2), \quad D_6^4 = -\frac{1}{24} (6 - 9j_0 + 18q_0 - 10j_0 q_0
\]

\[
+ 27q_0^2 + 15q_0^3 - s_0).
\]

Using the above expressions, we can reconstruct the theoretical dimensionless comoving distance for the given strong lensing system at lens and source with the Taylor series expansion.

It has to be admitted that our work involves data reaching high redshifts—strong lensing systems up to redshift \( z_s = 1.789 \) and cosmic chronometers up to redshift \( z = 2.3 \). Thus one might seriously worry about the convergence of the cosmographic Taylor series. One possible approach could be to consider auxiliary variables extending the convergence radii of

5 http://www.h0licow.org
standard Taylor expansions, such as $y = \arctan(z)$ and $y = 1 - 1/(1 + z)$ (Cattoën & Visser 2007; Aviles et al. 2012), which leads to better convergence of these new expansions in the higher redshift range. However, such an approach is in a sense artificial—new variables lose their physical meaning. On the other hand, Gruber & Luongo (2014) proposed the use of Padé approximants (PAs) for cosmographic analysis, which is much better than Taylor series expansion, as the convergence radius of PAs is larger than that of Taylor series expansion.

The $(n, m)$ order PA of a generic function $f(z)$ is its representation by a rational function

\[ P_{n,m}(z) = \frac{\sum_{i=0}^{n} a_i z^i}{1 + \sum_{j=0}^{m} b_j z^j}, \]

which agrees with $f(z)$ and all its derivatives up to $(n + m)$th order at $z = 0$. As compared with the Taylor expansion, PAs are known to have much better convergence, to better approximate singular points, and to reduce uncertainty propagation. A comprehensive discussion on these issues can be found in Capozziello et al. (2020). In particular, it has been argued that the most stable order of the Padé series is $(2, 1)$. Therefore, we choose $P_{2,1}$ to approximate the Hubble parameter $H(z)$ in this work:

\[ H_{2,1}(z) = \frac{P_0 + P_1 z + P_2 z^2}{1 + Q_1 z}, \]

where $P_0$, $P_1$, $P_2$, and $Q_1$ are PA parameters. They also have no obvious physical meaning, but fortunately one can always relate them to the physically relevant kinematic quantities $q_0$, $j_0$, and $s_0$, and explicit expressions of this kind can be found in Capozziello & Ruchika (2019). In the framework of PAs, the expansion of dimensionless comoving distance $d(z)$ at lensing redshift $z_l$ and source redshift $z_s$ also can be reconstructed by using this technique (see, e.g., Capozziello et al. 2020).
For the current data regarding the Hubble parameter, we turn to the newest compilation covering the redshift range $0.01 < z < 2.3$ (Jimenez et al. 2003; Moresco et al. 2012, 2016; Moresco 2015), which consists of 31 cosmic chronometer $H(z)$ data (denoted as CC) and 10 BAO $H(z)$ data. However, one should be aware that the BAO method relies slightly on cosmological models, i.e., the measurements based on the identification of BAOs and the Alcock–Paczynski distortion from galaxy clustering depend on how “standard rulers” evolve with redshift (Blake et al. 2012). On the other hand, the approach of cosmic chronometers makes use of differential ages of passively evolving galaxies and thus is independent of any specific cosmological model. Hence the resulting $H(z)$ measurements are cosmological-model-independent (Liu et al. 2020a, 2021a). Therefore, in our work we consider only the cosmic chronometer data to reconstruct the $H(z)$ function and then derive dimensionless comoving distances ($d_3$ and $d_t$) within the redshift range $0 < z < 2.3$, which covers the redshift of six SGL systems very well.

$H(z)$ data from cosmic chronometers are sensitive to cosmographic parameters $p = \{H_0, q_0, j_0\}$, and the corresponding likelihood function $L_{CC}$ is defined according to the formula

$$L_{CC} = \prod_{i=1}^{31} \frac{1}{\sqrt{2\pi\sigma_i^2}} \times \exp\left\{-\frac{[H_i^{\text{th}}(z_i; p) - H_i^{\text{obs}}]^2}{2\sigma_i^2}\right\},$$ (15)

where $H_i^{\text{obs}}$ is the $i$th Hubble parameter measured by cosmic chronometers, with $\sigma_i$ representing its uncertainty, and the $H_i^{\text{th}}$ is the theoretical Hubble parameter expressed in terms of cosmographic parameters; see Equations (9) and (14). It should be emphasized that convergence and goodness of fit decrease when higher-order terms are considered. In this paper we analyze independently the cosmographic expansion at the third order with different technologies, i.e., Taylor series expansion denoted as $T_3$ and (2, 1) Padé series expansion denoted as $P_{2,1}$.

In the case of SGL systems, the time delay distance $D_{\Delta t}$ is the observable for which the likelihood was defined. For the lens B1608+656 the likelihood was given as a skewed log-normal distribution

$$L_{D_{\Delta t}} = \frac{1}{\sqrt{2\pi(x - \lambda_D)\sigma_D}} \exp\left\{-\frac{(\ln(x - \lambda_D) - \mu_D)^2}{2\sigma_D^2}\right\},$$ (16)

with the parameters $\lambda_D = 4000.0$, $\sigma_D = 0.22824$, and $\mu_D = 7.0531$, where $x = D_{\Delta t}/(1 \text{ Mpc})$. For the other five lenses, the posterior distributions of $D_{\Delta t}$ were released in the form of MCMCs by the H0LiCOW team. A kernel density estimator was used to compute the likelihood $L_{D_{\Delta t}}$ from the chains. Finally, the log-likelihood of the time-delay distances $L_{D_{\Delta t}}$ of six lenses and the log-likelihood of the Hubble parameter, being independent, were used jointly to compute the posterior distributions of the cosmological parameters ($H_0, \Omega_k$) and cosmography parameters ($q_0, j_0$). Hence, the log-likelihood sampled by using the Python MCMC module EMCEE (Foreman-Mackey et al. 2013) is given by

$$\ln L = \ln(L_{CC}) + \ln(L_{D_{\Delta t}}).$$ (17)

Concerning the priors on parameters, we employed uniform priors on $H_0$ in the range $[0, 150]$ km s$^{-1}$ Mpc$^{-1}$, $\Omega_k$ in the range $[-1, 1]$, $q_0$ in the range $[-2, 0.5]$, and $j_0$ in the range $[-5, 5]$ to make sure that they are either physically motivated or sufficiently conservative.

### 3. Results and Discussion

First, we consider the Hubble parameter using the third order of Taylor expansion in cosmography. The 1D marginalized probability distributions and 2D regions with $1\sigma$ and $2\sigma$ confidence levels for the parameters ($H_0, \Omega_k, q_0, j_0$) constrained by the six time-delay measurements and cosmic chronometer data, are displayed in Figure 1. The median values plus the distances to the 16th and 84th percentiles for the four parameters are $H_0 = 72.24^{+2.72}_{-2.28}$ km s$^{-1}$ Mpc$^{-1}$, $\Omega_k = 0.062^{+0.107}_{-0.078}$, $q_0 = -0.645^{+0.126}_{-0.119}$ and $j_0 = 0.901^{+0.241}_{-0.225}$. Our results support the flat universe at $1\sigma$ confidence level, and the central value of $H_0$ is in agreement with the result inferred from the H0LiCOW collaboration, which demonstrates the consistency and validity of our method. This clear result from available lensed quasars is also consistent with recent analysis of distance ratios in strong lensing systems (Xia et al. 2017; Qi et al. 2019; Liu et al. 2020, Liu et al. 2020b, 2021b; Zhou & Li 2020), which is a technique complementary to SGL time delays. In order to clearly demonstrate and compare the constraining power of different probes on the Hubble constant, spatial curvature, and cosmography, individual constraint results will also be presented in the following analysis. Specially, the fitting results of $H_0, q_0$, and $j_0$ from Hubble parameter observations are shown in Figure 2, in the framework of cosmography using third-order Taylor series expansion. The best-fit values of the three parameters with $1\sigma$ uncertainties are $H_0 = 67.78^{+3.09}_{-3.35}$ km s$^{-1}$ Mpc$^{-1}$, $q_0 = -0.437^{+0.176}_{-0.158}$ and $j_0 = 0.564^{+0.256}_{-0.224}$. Our results suggest that larger and more accurate sample of strongly lensed quasars can be a valuable complementary probe to simultaneously determine the Hubble constant, cosmic curvature, and...
cosmography. This tendency could be seen from more stringent constraints from the combination of the two probes in Figure 1.

If we assume a flat universe, one can clearly see from Table 1 and Figure 3 that $H_0 = 70.47^{+1.14}_{-1.13}$ km s$^{-1}$ Mpc$^{-1}$, which falls between the SH0ES and Planck CMB results. The detailed numerical results from analysis of joint time delays + cosmic chronometers are displayed in rows 1 and 2 of Table 1 for a non-flat and a flat universe, respectively. In particular, the cosmographic coefficients do not differ much between these two cases. The deceleration parameters $q_0 = -0.645^{+0.126}_{-0.124}$ and $q_0 = -0.560^{+0.063}_{-0.061}$ corresponding to the non-flat and flat cases, respectively, are mutually consistent. Their values indicate an accelerating expansion of the universe. To better understand the meaning of these parameters, let us consider a flat ΛCDM model. Then, one can relate cosmographic parameters to the physical quantity $\Omega_{m0}$ (i.e., the matter density parameter at present time) according to $q_0 = -1 + (3/2)\Omega_{m0}$ and $j_0 = 1$. It is easy to check that our results are compatible with a flat ΛCDM model with the current matter density parameter $\Omega_{m0} = 0.311$ obtained from Planck CMB observations within the 1σ confidence level. Capozziello et al. (2020) have discussed the role of spatial curvature in the cosmographic context. Degenoracy among coefficients ($q_0$, $j_0$) and spatial curvature appears due to the fact that all cosmographic parameters are related to the Hubble function $H(z)$. In order to highlight the influence of spatial curvature on cosmographic parameters, in another scenario we fixed the value of the Hubble constant as $H_0 = 73.24$ km s$^{-1}$ Mpc$^{-1}$. We adopted the Gaussian kernel density estimator to compute the probability density function of the spatial curvature parameter $\Omega_k$ in the right panel of Figure 3. Full numerical results regarding the cosmographic parameters are given in Table 1. In this case, our findings support an open universe at 2.6σ confidence level. Moreover, our results also confirm the conclusions of Capozziello et al. (2020).

The results for the (2, 1) order Padé approximation used for cosmographic analysis are also reported in Table 1 and displayed graphically in Figure 4. Comparing the obtained value of $H_0 = 72.45^{+1.95}_{-2.02}$ km s$^{-1}$ Mpc$^{-1}$ with $H_0 = 72.24^{+2.73}_{-2.52}$ km s$^{-1}$ Mpc$^{-1}$ inferred from third-order Taylor expansion, we see very little effect on the results coming from different reconstruction methods. Most likely this is because the inferred value of $H_0$ is dominated by the contribution of very precise data regarding time delays due to gravitational lensing. Regarding the cosmic curvature parameter, our results show that there is almost no difference between the curvature parameters obtained by the two reconstruction methods, which once again proves the effectiveness of our methods and techniques. However, the PA method yields the smaller value of deceleration and the higher value of jerk parameter: $q_0 = -0.823^{+0.115}_{-0.218}$ and $j_0 = 1.161^{+0.237}_{-0.204}$.

Table 1
Summary of the Constraints on the Hubble Constant $H_0$, Spatial Curvature $\Omega_k$, and Cosmographic Parameters $q_0$ and $j_0$ in the Framework of Third-order Taylor Series and (2, 1) Order Padé Expansions in Cosmography by Using Time Delay + Cosmic Chronometer Observations

| Parametric Order | $H_0$ (km s$^{-1}$ Mpc$^{-1}$) | $\Omega_k$ | $q_0$ | $j_0$ | $\chi^2_{\nu}$ | BIC |
|------------------|-----------------|-----------|------|------|-------------|-----|
| Taylor $T_3$     | 72.24$^{+2.33}_{-2.52}$ | 0.062$^{+0.117}_{-0.078}$ | -0.645$^{+0.126}_{-0.124}$ | 0.901$^{+0.241}_{-0.225}$ | 1.43 | 65.98 |
| Taylor $T_3$ (fixed $\Omega_k$) | 70.47$^{+1.14}_{-1.13}$ | --- | -0.560$^{+0.063}_{-0.061}$ | 0.746$^{+0.112}_{-0.114}$ | 1.43 | 62.30 |
| Taylor $T_3$ (fixed $H_0$) | --- | 0.123$^{+0.060}_{-0.046}$ | -0.716$^{+0.051}_{-0.051}$ | 1.024$^{+0.127}_{-0.127}$ | 1.44 | 62.75 |
| Padé $P_{2,1}$ | 72.45$^{+1.95}_{-2.02}$ | 0.069$^{+0.116}_{-0.103}$ | -0.822$^{+0.015}_{-0.218}$ | 1.161$^{+0.237}_{-0.204}$ | 1.39 | 64.19 |
| Padé $P_{2,1}$ (fixed $\Omega_k$) | 71.66$^{+1.15}_{-1.17}$ | --- | -0.795$^{+0.100}_{-0.081}$ | 1.128$^{+0.122}_{-0.129}$ | 1.42 | 62.18 |
| Padé $P_{2,1}$ (fixed $H_0$) | --- | 0.101$^{+0.090}_{-0.072}$ | -0.847$^{+0.056}_{-0.072}$ | 1.179$^{+0.110}_{-0.110}$ | 1.38 | 60.58 |
Figure 4. The 1D marginalized probability distributions and 2D regions with 1σ and 2σ contours corresponding to these parameters ($H_0$, $\Omega_k$, $q_0$, $j_0$) in the (2, 1) order Padé expansion, constrained by the six time-delay measurements and cosmic chronometer data.

Figure 5. The same as Figure 3, but using the (2, 1) order Padé expansion.
respectively. This result is likely due to strong degeneracy between the cosmographic parameters themselves. Therefore, we expect to seek other types of astronomical data samples that might break the degeneracy.

Assuming zero spatial curvature, we obtain $H_0 = 71.66^{+1.15}_{-1.37}$ km s$^{-1}$ Mpc$^{-1}$ (see the posterior probability density function in left panel of Figure 5; representing a precision of 2.2%), $q_0 = -0.795^{+1.010}_{-0.081}$, and $j_0 = 1.128^{+0.122}_{-0.279}$, which also fall between the SH0ES and Planck CMB results. Similarly to third-order Taylor expansion, we fix the value of the Hubble constant as $H_0 = 73.24$ km s$^{-1}$ Mpc$^{-1}$ to explore the spatial curvature; the curvature parameter of the posterior probability density function under this assumption is shown in Figure 5. An open universe is also supported at 1.3σ confidence level. The differences between PA-derived cosmological parameters and those obtained with the Taylor expansion method are consistent within 1σ uncertainties.

In order to quantify the relative performance of Taylor expansion and PA, we use the Bayes information criterion (BIC), which is a well-known tool of model selection theory (Burnham & Anderson 2002). It is defined as

$$\text{BIC} = -2 \ln L_{\text{max}} + k \ln N,$$

where $L_{\text{max}}$ is the maximum value of the likelihood, $N$ denotes the number of data points, and $k$ represents the number of free parameters. The BIC values for Taylor and Padé expansions are listed in Table 1, together with reduced chi-square $\chi^2_{\text{dof}}$, i.e., $\chi^2_{\text{dof}} = \chi^2_{\text{max}}/36$ with 36 degrees of freedom ($31 + 6 - 1$). According to these results, we conclude that Padé expansion is slightly preferred over the Taylor expansion with the BIC difference $\Delta \text{BIC} = 0.79$. Evidence for a mild difference between the support given to competing models by the data starts with the difference $\Delta \text{BIC} \geq 2$ and can be regarded as strong with $\Delta \text{BIC} \geq 6$ (Schwarz 1978). Hence, we conclude that adopting different extension techniques has only a minimal impact on our results. On the other hand, one can reconstruct the cosmic expansion history with the best-fit values of the cosmographic parameters $q_0$, $j_0$, and Hubble constant $H_0$. In Figure 6, we show the cosmic expansion history based on third-order Taylor cosmography, which is well consistent with the cosmic chronometer observations. When bigger samples of observational data are available in the future, our approach will yield a much more accurate determination of the Hubble constant and cosmic curvature, together with precise reconstruction of the cosmic expansion history over a wide redshift range.

### 4. Conclusion

The Hubble constant and spatial curvature of the universe are still the most important cosmological parameters in modern cosmology. In this work, we used a model-independent approach to determine $H_0$ and $\Omega_k$ simultaneously by analyzing cosmic chronometer data in combination with time-delay measurements of six well studied SGL systems. Unlike many recent applications of various cosmological probes based on the polynomial expansion, our method used cosmography to reconstruct distances from cosmic chronometer observations. Beyond a very general basic assumption of homogeneity and isotropy of the universe (independent of the details of any cosmological model), the advantage of cosmography is that parameters such as $H_0$, $q_0$, $j_0$, and $\Omega_k$ can be directly obtained from the data without any need to calibrate the intercept (nuisance parameter) as in the case of standard candles or standard rulers.

First, in the framework of third-order Taylor expansion, our results showed that the mean value of $H_0$ is in agreement with the value inferred by the H0LiCOW collaboration, and a flat universe is supported within the 1σ confidence level based on current observational data. This is consistent with the previous results given in Collett et al. (2019), Wei & Melia (2020), and Qi et al. (2021), in which the authors respectively used the Pantheon Type Ia supernova, the nonlinear relation between the ultraviolet and X-ray fluxes in quasars, and the angular size versus redshift relation from compact structures in radio quasars combined with time-delay measurements of SGL systems to obtain $H_0$ and $\Omega_k$. However, their methods usually required the reconstruction of cosmological distances, phenomenologically modeled by some polynomials. This approach often involves problems regarding the physical meaning of polynomial coefficients and, what is more important, the calibration of a “nuisance parameter.” Calibration introduces additional systematic uncertainties, hence one may claim that our results regarding $H_0$ and $\Omega_k$ are more accurate than these previous assessments. The deceleration parameter turned out to be $q_0 = -0.645^{+0.126}_{-0.124}$, which indicates an accelerating expansion of the universe. Our results are compatible, within the 1σ confidence level, with a flat $\Lambda$CDM model having the current matter density parameter $\Omega_m = 0.311$ as suggested from Planck CMB observations.

Assuming a flat universe, we obtained $H_0 = 70.47^{+1.14}_{-1.15}$ km s$^{-1}$ Mpc$^{-1}$, which falls between the results from SH0ES and Planck CMB data. In order to highlight the influence of spatial curvature on cosmographic parameters, we fixed the value of the Hubble constant as $H_0 = 73.24$ km s$^{-1}$ Mpc$^{-1}$. Under this assumption, our findings supported an open universe at 2.6σ confidence level. This means that there is a positive correlation between the Hubble constant and the spatial curvature parameter. Moreover, our results also confirm the conclusions formulated in Capozziello et al. (2020). It should be stressed, however, that our work offers a new approach to constrain both the spatial curvature and cosmographic parameters. Although the degeneracy between them.

![Figure 6. Reconstructed Hubble expansion history by using third-order Taylor series and (2, 1) order Padé expansions.](imageURL)
still exists, our method alleviates it to some extent in comparison with cosmography alone. Considering that redshift coverage of the data we used extends to $z \sim 2.3$, the issue of convergence regarding the Taylor series cosmographic expansion becomes important. Therefore we also used the technique of Padé approximants, in particular the (2, 1) PA known to perform the best in cosmography (Capozziello et al. 2020). As discussed above, the results regarding $H_0$ and cosmic curvature were similar to those using the Taylor expansion, while deceleration and jerk parameters turned to be noticeably different. The BIC criterion used to quantify the performance of two competing reconstruction techniques yielded a weak preference of PA over Taylor expansion. This result, however, is inconclusive.

As a final remark, there are many potential ways to improve our method. For instance, current and future surveys such as the Dark Energy Survey (Treu et al. 2018), the Hyper SuprimeCam Survey (More et al. 2017), and the LSST (Oguri & Marshall 2010; Collett 2015) will bring us hundreds of thousands of lensed quasars in the most optimistic discovery scenario. Even if only some small fraction of them have precise measurements of time delays between multiple images, the resulting statistics will outshine current catalogs. With high-quality auxiliary observations, one can use high-cadence, high-resolution, and multiple-filter imaging of the resolved lensed images to derive an accurate determination of the Fermat potential, which will increase the precision of time-delay distance by an order of magnitude. At the same time, we also expect that following surveys such as the WiggleZ Dark Energy Survey (Drinkwater et al. 2010) will bring us hundreds of thousands of lensed quasars in the most optimistic discovery scenario. With high-cadence, high-resolution, and multiple-filter imaging of the resolved lensed images, the spatial curvature, with future surveys of SGL systems, can be measured to expectations.

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