Dielectric response of Anderson and pseudogapped insulators

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1. Introduction

An Anderson insulator [1–3] with a relatively long localization length $\xi$ is a paradigmatic model for studies of different physical effects due to disorder present in many poor conductors. In particular, it was recently shown that one specific class of quantum phase transitions between superconducting and insulating states (superconductor-insulator transitions, SIT) can be understood in terms of a relatively weak Cooper attraction between Anderson-localized electrons [4]. In such materials, superconductivity was shown to develop in two stages as temperature decreases: first, localized electrons are progressively bound into localized pairs (forming a pseudogapped insulator [5]), and then at lower temperatures macroscopic coherence between such pairs is established, leading to superconductivity.

However, the theory developed in [4, 5] is somewhat limited since it neglects long-range Coulomb interaction between electron pairs (the importance of Coulomb repulsion is known from a more standard approach to SIT based upon a model of a weak-link Josephson array where the SIT is driven by a competition between the Josephson coupling energy and the Coulomb charging energy [5]). The first step in filling this omission is studying the dielectric response of disordered Anderson insulators in the regime of a large localization length (where the resulting dielectric constant may become large). Knowledge of the dielectric constant is also important for studying high-frequency response of superconducting resonators made of strongly disordered superconductors like InO$_x$, TiN and NbN, where pseudogap-related phenomena were detected [7].

In order to specify what we understand by a 'large' localization length, let us briefly review the length and energy scales involved in the problem. If we denote the electron density $n$, the average inter-electron distance is $n^{-1/d}$ (in $d$ dimensions), and we can define the Fermi kinetic energy (the energy scale associated with the inter-particle distance) as

$$\varepsilon_F = \nu^{-1} n,$$

where $\nu$ is the $d$-dimensional density of states at the Fermi level.
We will work in the localized phase with the localization length $\xi$ (the decay rate of the average logarithm of the square of an eigenstate: $-\ln|\langle \psi(x) \rangle^2/|\psi(0)\rangle^2| \sim x/\xi$, assuming the maximum of the amplitude $|\psi(x)|$ at $x = 0$), which defines the typical level spacing in the localization volume

$$\delta_\xi = (\nu d^6)^{-1}.$$  

Our large localization length assumption means that $\delta_\xi \ll \varepsilon_\text{e}$ (or, equivalently, the localization length is much larger than the inter-particle distance, $\xi \gg n^{-1/d}$).

In the discussion of the dielectric constant and dielectric screening, one more important scale is the bare Coulomb interaction strength,

$$\varepsilon_C \approx e^2 n^{1/d}.$$  

Note that this is not the actual energy of Coulomb interactions, since this definition includes neither dielectric screening by the localized conductance electrons (discussed in the present paper), nor possible screening by electrons from the valence band (which can be strong in some materials with large DoS in the valence band close to the Fermi level). In typical metals, $\varepsilon_C \sim \varepsilon_\text{e}$, and we may assume a similar relation here.

Finally, in a pseudogapped insulator, one more energy scale appears: the pseudogap $\Delta_p$. It is assumed to be much smaller than $\delta_\xi$.

The issue of dielectric response of an Anderson insulator with a long localization length is not a new one, but is far from settled. Pertinent results belong to two major groups, according to the treatment of Coulomb electron–electron interaction. The first well-known approach [8–11] is to consider polarizability of non-interacting electrons, with an idea to add Coulomb interaction self-consistently at the later stage. This way one comes to the scaling dependence for the dielectric constant $\varepsilon \propto \varepsilon^2 \xi^2$, with some undefined numerical prefactor of the order of unity (another result within the same approach belongs to Efetov [12], who has studied Anderson transition at high dimensions $d \gg 1$ by means of some effective–medium theory developed on the basis of the Bethe–lattice solution of the localization problem; his theory predicts $\chi \propto \xi$).

An alternative approach was initiated by McMillan [13] and then continued in a number of later works [14–16]; for recent developments along similar lines see [17, 18]. Here, the general idea is to account for both disorder and Coulomb correlations simultaneously, as both of them are equally important near real metal–insulator transition in 3D. Then dielectric permeability is found to obey scaling relation $\varepsilon \propto \xi^{2-z}$, where $z < 3$ is a dynamic critical exponent. In particular, [15, 17] provide arguments in favor of $z \approx 2$ near 3D Anderson transition.

We will follow the first of the approaches described above, i.e., we start from the calculation of the polarizability $\chi$ in the model of non-interacting electrons, with the attitude to find an accurate numerical coefficient in the $\varepsilon(\xi)$ dependence. Later on, we discuss the effects of electron–electron interactions. The rest of the paper is organized as described below.

In section 2 we study the polarizability $\chi$ in the model of non-interacting electrons under the assumption $\xi \gg n^{-1/d}$. This problem was solved exactly in the one-dimensional case [19, 20], with the conclusion that $\chi \propto \xi^2$. In higher dimensions $d > 1$, no exact results are available. Our results for $d = 2, 3$, based on a combination of analytical and numerical methods, confirm scaling $\chi \propto \xi^2$, similarly to the one-dimensional case. As a check of the method, for $d = 1$, our computations reproduce the exact result [20].

Then in section 3 we use our results on polarizability to study the dielectric response of interacting systems. Here, we consider quasi-one-dimensional, quasi-two-dimensional and isotropic three-dimensional cases and additionally require $\varepsilon_C \gtrsim \delta_\xi^{3/2} \varepsilon_\text{e}^{1/3}$ (this condition is automatically satisfied if $\xi \gg n^{-1/d}$ and $\varepsilon_C \sim \varepsilon_\text{e}$). Under this condition, the polarizability $\chi$ is large, thus providing an efficient dielectric screening of Coulomb repulsion between electrons. We approximate the dielectric constant $\varepsilon$ in the usual ‘mean-field’ way as

$$\varepsilon = 4\pi \chi \pm 1.$$  

This approximation neglects two potentially important effects: (a) mesoscopic fluctuations of dielectric screening and (b) interaction-induced electron–electron correlations.

The effects of type (a) may be, in principle, taken into account at the RPA level by a proper disorder averaging. We discuss it in more detail in section 3.1 and estimate that these effects should be small in quasi-1D and quasi-2D cases, but may become relevant in the isotropic 3D case.

The effects of type (b) go beyond RPA: they renormalize the polarization response $\chi$ itself and are related to the Coulomb-gap effect (modification of the density of states around the Fermi level due to interactions). These effects are considered in section 3.3 after a short analysis of the relation to low-frequency conductivity in section 3.2. We find that, due to dielectric screening, the Coulomb-gap is renormalized to a numerically small fraction of $\delta_\xi$, which suggests that these effects do not have large impact on the static dielectric response.

Finally, in section 3.4, we apply our results to the theory of pseudogapped insulators and estimate the effect of the pseudogap on the dielectric constant $\varepsilon$. Since the pseudogap suppresses the low-frequency part of the
electromagnetic response, it correspondingly reduces $\epsilon$. In the regime, where the pseudogap is much smaller than $\delta_\xi$, this reduction is small, but we propose to observe it as a function of an externally applied magnetic field, where it may show up as a linear term in the dependence of $\epsilon$ on the magnetic field.

We conclude the paper with a discussion of the results in section 4.

2. Static polarizability of the Anderson-localized electron gas

In this section, we show, using both analytical and numerical means, that the static polarizability $\chi$ of disordered non-interacting fermions scales proportionally to the square of the localization length $\xi$ in a wide range of parameters, and compute the proportionality coefficient.

The zero-temperature polarizability of a non-interacting electron gas in the presence of a random potential can be obtained via a perturbation theory with respect to a uniform electric field:

$$\chi_{\alpha\beta} = \frac{e^2}{V} \sum_{\epsilon_m > \epsilon_F} \langle n | x_{\alpha}|m\rangle\langle m | x_{\beta}|n\rangle + \langle n | x_{\alpha}|m\rangle \langle m | x_{\beta}|n\rangle,$$

where $|m\rangle$ and $|n\rangle$ are exact single-electron eigenfunctions with energies $\epsilon_m$ and $\epsilon_n$, $\epsilon_F$ is the Fermi level, $V$ is the volume of the system, $x_{\alpha}$ are the coordinate components, and $e$ is the electron charge. Equation (5) is exact, applies in any dimension, and holds for any specific realization of the random potential. By averaging over disorder (assuming that the disorder average is isotropic) and taking the limit $V \to \infty$, we find

$$\chi = -e^2 \iint_{\epsilon_1 < \epsilon_2 < \epsilon_2} \nu(\epsilon_1)\nu(\epsilon_2) \int d^dx S(\epsilon_1, \epsilon_2, x) x_1^2,
$$

where the disorder-averaged response function is defined as

$$S(\epsilon_1 = \epsilon_2, x) = \frac{1}{\nu(\epsilon_1)\nu(\epsilon_2)} \sum_{n,m} \delta(\epsilon_n - \epsilon_1)\delta(\epsilon_m - \epsilon_2) \psi_n^*(x)\psi_m(x)\psi_n(0)\psi_m(0),$$

Here $(\ldots)_{\text{dis}}$ denotes the disorder average and $\nu(\epsilon)$ is the $d$-dimensional density of states (here and below we consider spinless fermions).

Now we assume that the integrals are limited to a small region around the Fermi energy, so that the density of states can be approximated by a constant ($\nu(\epsilon) = \nu$) and $S(\epsilon_1, \epsilon_2, x)$ only depends on the difference $\omega = |\epsilon_1 - \epsilon_2|$. The characteristic energy scale for $S(\epsilon_1, \epsilon_2, x)$ is $\delta_\xi$, therefore the above assumption is equivalent to $\delta_\xi \ll \epsilon_{\text{e}},$ where $\epsilon_{\text{e}}$ is the characteristic energy scale of the density of states variations defined in equation (1). This condition is satisfied for large localization lengths, $\xi \gg n^{-1/d}$ (the localization length is larger than the inter-particle distance).

Under this assumption, we can simplify equation (6) as

$$\chi = -e^2 \nu^2 \int_0^\infty d\omega \int d^dx S(\omega, x) x_1^2.$$

The function $S(\omega, x)$ was studied in the one-dimensional case in [21] and was further conjectured in the quasi-one-dimensional case and in higher dimensions in [22, 23]. We introduce its integral over energy $F(x)$ which can further be related to correlations of a single-wave-function using the orthogonality of eigenstates:

$$F(x) = -2\nu \int_0^\infty d\omega \ S(\omega, x = 0) = \frac{1}{\nu} \left( \sum_n \delta(\epsilon_n - \epsilon) |\psi_n(x)|^2 |\psi_n(0)|^2 \right)_{\text{dis}}.$$

Thus defined quantity $F(x)$ has the dimensionality $[\text{length}]^{-d}$ and obeys the normalization condition

$$\int d^dx F(x) = 1.$$

This allows us to express $\chi$ in terms of single-wave-function statistics:

$$\chi = e^2 \nu^2 \int d^dx F(x) x_1^2.$$

The above derivation only uses the assumption that the localization length is sufficiently large so that the polarizability is determined by a vicinity of the Fermi surface. The result should be valid for any type of impurities (without the single-parameter-scaling assumption) and in any dimension.

On general grounds, we expect that $F(x)$ is a function exponentially decaying at a length scale of order $\xi$, and therefore

$$\chi = \alpha \ e^2 \nu \xi^2,$$
where

\[ \alpha = \frac{1}{2d} \int d^d x \, F(x) \left( \frac{x}{\xi} \right)^2 \]  

(13)

is a numerical coefficient depending on the dimension \( d \). In principle, this coefficient may also depend on the type of disorder and disorder strength. In the limit of white-noise disorder, localization is described (except in the pure 1D case) by a nonlinear supersymmetric sigma model [12, 24], and we may expect a universal form of the function \( F(x) \) (more precisely, its non-oscillating part rescaled in the units of the localization length \( \xi \)).

In fact, in the one-dimensional (and quasi-one-dimensional) case, the function \( F(x) \) in the white-noise limit is known analytically [12, 22, 25–29]:

\[ F(x) = 2\pi^2 \xi \frac{\partial^2}{\partial x^2} \int_0^\infty k \, dk \, \frac{\tanh \frac{\pi k}{\cosh \frac{\pi k}{\xi}}} {\cos^2 \frac{\pi k}{\xi}}. \]  

(14)

On substituting (14) into (13), we integrate by parts twice to obtain

\[ \alpha = 4\pi^2 \int_0^\infty dk \, \frac{k}{k^2 + 1/4} \, \frac{\tanh \frac{\pi k}{\cosh \frac{\pi k}{\xi}}} {\cos^2 \frac{\pi k}{\xi}} = 4\zeta(3) = 4.808... \]  

(15)

Abrikosov and Ryzhkin in [20] present an equivalent formula (below their equation (2.95)); an apparent difference by a factor of 2 from our result is due to the spin degeneracy accounted for in [20].

We have further studied numerically the function \( F(x) \) and the coefficient \( \alpha \) for two models of disorder at different strengths of disorder levels in one, two, and three-dimensions. Namely, we considered the so-called Anderson model: the tight-binding model on the square lattice (linear chain or cubic lattice, depending on dimension) with a random on-site disorder (see the appendix for details). The distribution of disorder was taken either uniform-box or Gaussian. We have analyzed a variety of levels of disorder strength at different levels of \( \xi \) and have found that \( F(x) \) is nearly universal when rescaled in length units of \( \xi \). Our results for \( F(x) \) (multiplied by a suitable power of \( x \)) are shown in figure 1 and the corresponding values of \( \alpha \) are reported in table 1 and in figure 2.

In one-dimension, the numerical results for \( \alpha \) agree with the analytical value (approximately 4.8). In two and three-dimensions, the value of \( \alpha \) is nearly universal (3.2 in two-dimensions and 2.8 in three-dimensions) for the studied range of parameters. A slight trend to the increase of \( \alpha \) with increasing \( \xi \) in 3D may be due to either real or an artifact of our numerical methods. In the former case, the extrapolated value of \( \alpha \) at the localization transition \( \xi \to \infty \) is about 3.1.

Note that \( F(x) \) has qualitatively very similar behavior in all the three dimensionalities. In all the three cases, the main contribution to the integral (11) comes from large distances, of the order of 5–10 localization lengths. At such large distances, the function \( F(x) \) is determined not by typical localized states, but by relatively rare Mott hybridization events [23, 32]. As a consequence, \( F(x) \) decays not at the length scale \( \xi \), but at a larger one (at the length scale of \( 4\xi \) in one-dimension in the white-noise limit; we are not aware of similar exact results in higher dimensions).

3. Effects of interaction

3.1. Dielectric constant

The calculation of the previous section was for a single-particle response to an external field. It only assumed that the localization length is much larger than the inter-particle spacing, \( \xi \gg n^{-1/d} \). In a physical material, one is instead interested in the dielectric constant \( \varepsilon \) describing the dielectric screening of the external field by polarized localized states and reducing the Coulomb interaction between electrons. In three-dimensions, the result (12) may be rewritten as

\[ \chi = \alpha \varepsilon_C \delta_\xi^{-2/3} \varepsilon_\omega^{-1/3}, \]  

(16)

using the energy scales (1)–(3) defined in the Introduction. Under the assumption \( \delta_\xi \ll \varepsilon_\omega \sim \varepsilon_C \), we find \( \chi \approx 1 \), and then the dielectric constant (4) is even larger by an extra order of magnitude.

However, the standard formula (4) implies the uniformity of the actual electric field inside in the sample, which is not obviously granted near Anderson transition. This problem exists already within the standard RPA approach: the screened Coulomb potential \( V(\mathbf{r}, \mathbf{r'}) \) for a given realization of disorder is determined (within the RPA) by the equation

\[ (4\pi \varepsilon_0)^{-1} \nabla^2 V(\mathbf{r}, \mathbf{r'}) + \int d^d \eta \Pi(\mathbf{r}, \mathbf{r}) V(\mathbf{r}, \mathbf{r'}) = \delta(\mathbf{r} - \mathbf{r'}), \]  

(17)

where \( \Pi(\mathbf{r}, \mathbf{r'}) = \lim_{\omega \to 0} \Pi(\mathbf{r}, \mathbf{r'}; \omega) \) is the polarization function of the free Fermi gas with random potential. Then, the dielectric constant \( \varepsilon \) is determined by the long-scale asymptotics of the averaged potential.
The procedure we used in section 2 to calculate the polarizability $\chi$ was equivalent to averaging the polarization function, $\Pi(r, \tau; \omega) \to \Pi(|r - \tau|; \omega)$, instead of averaging the resulting potential $V(r, r')$. Physically, the problem with such a preliminary averaging is the it neglects the fact that the microscopic electric field is inhomogeneous in a disordered sample. This effect is expected to be important near 3D Anderson transition, due to strong fractality of electron wave functions with energies near the mobility edge, see [33] and references therein.

In low-dimensional systems with long localization length, wave-function fractality is absent ($d = 1$), or rather weak ($d = 2$). Thus averaging over disorder, already at the level of the polarization function, should not lead to any considerable error. Therefore our results (12), (13) imply direct consequences for the macroscopic

$$V(r, r') \approx \frac{e^2}{\varepsilon |r - r'|}.$$ (18)

Figure 1. The function $F(x) x^{d-1} / \xi$ versus the scaled coordinate $x / \xi$ calculated for the cases listed in table 1. The three panels show the one-, two-, and three-dimensional results, respectively. The function $F(x)$ is multiplied by the appropriate power of $x$, in order to reflect the contribution to the integral, equation (11). In order to reduce statistical fluctuations, the points represents averages over bins of size $0.2x / \xi$. In one-dimension, the black solid line shows the analytical result (14).
dielectric constant in quasi-1D materials (conducting 1D wires with moderate disorder in a poorly conducting matrix) and quasi-2D (layered) materials with relatively large Drude sheet conductance \( \sigma \gg 1 \) and very poor interlayer tunneling. In both these cases, the material becomes insulating at low temperature, and we provide expressions for bulk dielectric constants, assuming the direction of the electric field to be parallel to conducting wires or planes:

\[
\varepsilon_1 = 16\pi \zeta (3)e^2n_z\nu_1\xi^2 \quad \text{(quasi-1D)} \tag{19}
\]

\[
\varepsilon_2 = 4\pi\alpha_2 e^2\nu_2 c^2 \xi^2 \quad \text{(quasi-2D)} \tag{20}
\]

where \( n_z \) is the two-dimensional density of wires in the quasi-1D sample, \( \alpha_2 \approx 3.2 \), and \( d \) is the interlayer distance in a quasi-2D material (note that equation (19) is known since [20]). Note that the materials we discuss

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**Table 1.** The values of \( \alpha \) calculated using equation (13). The model is the Anderson model on the square lattice with on-site random potential distributed on each site independently with either Gauss or uniform-box distribution. The disorder strength \( W \) is the standard deviation for the Gauss distribution and the width for the box distribution (see the appendix for details), in the units of the hopping amplitude. The Fermi energy \( \varepsilon_F \) is the energy at which the wave functions are considered, in the units of the hopping amplitude, measured from the middle of the band. The localization length \( \xi \) is calculated independently using the transfer-matrix method of [30,31]. The energy \( \delta_\xi \) is calculated according to equation (2). The error bars for the last digit are indicated in brackets. The first line contains the analytical result (15).

| Dimension | Disorder | \( W \) | \( \varepsilon_F \) | \( \xi \) | \( \nu \) | \( \delta_\xi \) | \( \alpha \) |
|-----------|----------|--------|----------------|--------|--------|----------------|--------|
| 1         | White-noise | ---   | ---            | ---    | ---    | ---            | 4.808  |
| 1         | Gauss    | 0.288 | 1.0            | 36.1(2) | 0.184(5) | 0.151(5)       | 4.76(9)  |
| 1         | Gauss    | 0.288 | 1.4            | 24.7(1) | 0.222(8) | 0.182(7)       | 4.75(9)  |
| 1         | Gauss    | 0.288 | 1.8            | 10.1(5) | 0.334(8) | 0.296(9)       | 4.9(1)  |
| 1         | Box      | 1.3   | 0.0            | 30.6(2) | 0.161(3) | 0.203(5)       | 4.86(9)  |
| 1         | Box      | 2.5   | 0.0            | 8.2(4)  | 0.160(4) | 0.76(2)        | 4.84(9)  |
| 1         | Box      | 3.5   | 0.0            | 4.1(2)  | 0.159(4) | 1.51(5)        | 4.82(9)  |
| 2         | Gauss    | 2.0   | 3.0            | 8.0(1)  | 0.084(2) | 0.185(8)       | 3.3(2)  |
| 2         | Gauss    | 2.0   | 3.5            | 5.23(6) | 0.078(1) | 0.47(2)        | 3.2(2)  |
| 2         | Gauss    | 2.0   | 4.0            | 3.33(3) | 0.065(2) | 1.38(6)        | 3.2(2)  |
| 2         | Box      | 8.0   | 0.0            | 5.50(7) | 0.105(1) | 0.32(1)        | 3.1(2)  |
| 2         | Box      | 9.0   | 0.0            | 3.65(4) | 0.095(1) | 0.79(3)        | 3.2(2)  |
| 2         | Box      | 10.0  | 0.0            | 2.66(3) | 0.088(1) | 1.61(5)        | 3.1(2)  |
| 3         | Gauss    | 4.0   | 7.0            | 3.35(7) | 0.032(3) | 0.86(2)        | 2.9(2)  |
| 3         | Gauss    | 4.0   | 7.5            | 2.05(4) | 0.028(1) | 4.1(2)         | 2.8(2)  |
| 3         | Gauss    | 4.0   | 8.0            | 1.35(2) | 0.023(1) | 17.6(9)        | 2.6(2)  |
| 3         | Box      | 21.0  | 0.0            | 2.98(5) | 0.045(3) | 0.85(5)        | 2.9(2)  |
| 3         | Box      | 23.0  | 0.0            | 1.89(4) | 0.041(3) | 3.6(2)         | 2.8(2)  |
| 3         | Box      | 25.0  | 0.0            | 1.44(2) | 0.038(3) | 8.7(4)         | 2.5(2)  |

**Figure 2.** Values of the proportionality coefficient \( \alpha \) in equation (12) in two and three-dimensions as a function of the inverse localization length.
here are considered to be quasi-1D or quasi-2D with regard to their electronic properties only; in particular, typical interchain (interplane) length scales \( (n^{-1/2}, d) \) are only slightly larger than the interatomic distances along highly conductive directions. Therefore, large values of 1D or 2D polarizability also imply large values of \( \epsilon_{1,2} \) as presented in equations (19), (20).

In a bulk (isotropic 3D) system close to localization threshold, the expression

\[
    \epsilon = 4\pi\alpha_3 e^2\xi^2 \quad \alpha_3 \approx 3
\]

may serve just as an order of magnitude estimate until a solution to the random RPA equation (1) with a more accurate disorder averaging is available. We will further use equation (21) to estimate possible effects of Coulomb interaction which go beyond RPA.

3.2. Low-frequency conductivity and Kramers–Kronig relation

Effects of interactions between electrons with the characteristic energy scale below \( \delta_\xi \) may be estimated from the frequency-dependent conductivity via the Kramers–Kronig relation. At frequencies below \( \delta_\xi \), the conductivity is expected to obey the Mott–Berezinsky law,

\[
    \sigma(\omega) \approx \sum_d \left( \frac{e^2}{\hbar} \right)^{2-d} \xi^{2-d} \left( \frac{\omega}{\delta_\xi} \right)^{\frac{1}{d-1}} \ln^{d-1} \left( \frac{\omega}{\delta_\xi} \right),
\]

where \( \delta_\xi = c_d \xi \) and \( c_d \) is some number of the order of unity. In the 1D case both numerical coefficients are known: \( \Sigma_1 = 4\pi \) (for spinless fermions), and \( c_1 = 1/\pi \), see [20].

Now we can use the Kramers–Kronig relation

\[
    \chi = \frac{2}{\pi} \int_0^\infty \frac{d\omega}{\omega^2} \sigma(\omega),
\]

in order to estimate the dominant contribution to \( \chi \) and compare it with our results obtained in the previous section. Namely, we approximate equation (23) as

\[
    \chi \approx \frac{2}{\pi} \int_0^{\delta_\xi} \frac{d\omega}{\omega^2} \sigma(\omega) \approx \frac{2(d + 1)!}{\pi} \sum_d c_d e^2 \xi^2, \quad \text{assuming that the dominant contribution to } \chi \text{ comes from resonant Mott pairs with } \omega < \delta_\xi, \text{ see equation (22).}
\]

This is in agreement with the shape of the function \( F(\lambda) \) shown in figure 1, where the region of \( \lambda ~ 5–10 \xi \) provides the main contribution to the integral (13).

Using the Kramers–Kronig relation (24) and our results for \( \chi \) in section 2, we can relate the coefficients \( \Sigma_d \) and \( c_d \) in equation (22) (for spinless fermions) as

\[
    \alpha_d \approx \frac{2(d + 1)!}{\pi} \Sigma_d c_d, \quad \text{up to a numerical coefficient of order one. For } d = 1 \text{ the above relation can be tested by means of exact results: it is accurate within a } 6\% \text{ error.}
\]

3.3. Effects of Coulomb repulsion

Our calculations of \( \chi \) do not take into account Coulomb interactions between localized electrons. Such interactions, due to their long-range nature, modify the density of states close to the Fermi energy (the so-called Coulomb–gap) and the low-frequency dynamics. In three-dimensions, the magnitude of the Coulomb-gap is estimated as [34]

\[
    \Delta_C \approx \frac{e^3\rho^{1/2}}{\epsilon^{3/2}}.
\]

In our case, the large dielectric constant \( \epsilon \) is produced by the same localized electrons, and, using (21), we estimate

\[
    \Delta_C \approx \frac{\delta_\xi}{(4\pi\alpha_3)^{1/2}} \approx \frac{\delta_\xi}{250}.
\]

In other words, the Coulomb–gap scale is only by a numerical factor smaller than the localization energy scale \( \delta_\xi \). Our estimate of this smallness (27) should be taken with care, since both \( \delta_\xi \) and the Coulomb-gap are only estimated up to a numerical coefficient of order one. If it turns out that \( \Delta_C \) is of the same magnitude as \( \delta_\xi \), our calculations would need to be revised by self-consistently taking into account the effect of Mott-hybridized states on the electrostatic potential.

An attempt to take into account the effect of the Coulomb repulsion on the Mott conductivity (22) was done in [35]. Even though their results are sometimes used in the discussion of conductivity in general-type Anderson insulators (or electron glasses) [36], they were originally derived for doped semiconductors with strongly
localized electronic states. As we show below, in our case (large \( \xi \) and, consequently, large dielectric constant \( \epsilon \), see section 3.1), the applicability regime of the results of \cite{35} is small or vanishes.

Indeed, according to \cite{35}, the Coulomb interaction between electrons should be taken into account by replacing one of the \( \omega \) factors in equation (22) by \( \omega + \epsilon^2/\tau_c \epsilon \), where \( \tau_c = 2 \xi \ln(\delta \xi/\omega) \) is the Mott hybridization length. At low enough \( \omega \) the additional term \( \epsilon^2/\tau_c \epsilon \) becomes dominant, which changes frequency-dependence of conductivity crucially, in comparison with equation (22), \( \sigma(\omega) \propto \omega \ln(\delta \xi/\omega) \). Clearly, this dependence cannot be valid down to arbitrary low \( \omega \), otherwise the integral (23) would diverge. Indeed it is explained in \cite{35} that applicability of this result is limited by the condition

\[
\frac{\epsilon^2}{\tau_c \epsilon} \gg \Delta_c.
\]

(28)

otherwise the Coulomb-gap effect needs to be solved self-consistently.

Thus, the result of \cite{35} contains two opposite conditions on the relevant frequencies. The first limitation is \( \epsilon^2/\tau_c \epsilon \gg \omega \) (otherwise the effect of the Coulomb correction is small). This condition translates, in our case of large \( \xi \), into

\[
\omega \lesssim \frac{\delta \xi}{8\pi \alpha} \approx 0.01 \delta \xi.
\]

(29)

The other condition is the actual applicability limitation (28), which translates into

\[
\omega \gg \epsilon \delta \xi \exp(-\sqrt{4\pi \alpha}) \approx 0.05 \delta \xi.
\]

(30)

As we see, those conditions are both formulated in terms of a numerically small fraction of \( \delta \xi \) of similar magnitude. Therefore, in our situation of large \( \xi \), the range of applicability of the results of \cite{35} practically vanishes (the above discussion has no bearing on the original application of results of \cite{35} to the strongly localized regime with short \( \xi \)).

Note that the condition (30) also sets the limit for the applicability of the Mott–Berezinsky formula (22).

### 3.4. Effects of phonon-mediated attraction: Anderson insulators with a pseudogap

Based on the Kramers–Kronig relation, we can analyze the dielectric response of an Anderson insulator with a local phonon-mediated attraction \cite{4,5,37,38}. Such an attraction leads to a 'pseudogap' in the single-particle density of states \cite{39} (similar to the 'parity gap' in ultra-small metallic grains \cite{40}). The energy scale of the pseudogap \( \Delta_p \) is given by the typical interaction energy of a localized state. We assume that \( \Delta_p \ll \delta \xi \), but not too small, in order to neglect the effects of Coulomb interaction (in three-dimensions, we estimate this condition to be \( \Delta_p \gg 0.05 \delta \xi \) from equation (30)).

The influence of the pseudogap upon frequency-dependent Mott conductivity of an insulator was studied by two of the present authors in \cite{41}. It was found that asymptotic form (22) remains valid also below \( \Delta_p \), but with a strongly reduced prefactor \( \Sigma_0 \) (by a factor more than 10). We can therefore estimate the effect of the pseudogap on the dielectric susceptibility (proportional to \( \chi \)) by simply excluding the low-frequency part \( \omega < \Delta_p \) from the Kramers–Kronig integral (23).

This leads to the following result for the reduction of the dielectric response due to the pseudogap:

\[
-\frac{d \ln \epsilon}{d \Delta_p} \approx \frac{8 \sigma(\omega)}{\epsilon^2} \omega^2 \bigg|_{\omega=\Delta_p} \approx \frac{1}{\delta \xi} \left( \ln \frac{\delta \xi}{\Delta_p} \right)^{d+1}
\]

(31)

(up to a numerical coefficient of order one).

This effect of the pseudogap on the dielectric response may be convenient to observe by applying an external magnetic field. A magnetic field \( B \) suppresses the parity gap via Zeeman effect, \( \Delta_p(B) = \Delta_p - g_{\text{eff}} \mu_B B \).

(32)

The effective Lande factor \( g_{\text{eff}} \) is material-dependent: it may vary from its bare value 2 in the absence of spin-orbit coupling to small values of order \( \tau_\omega \Delta_p / \hbar \ll 1 \), if strong spin–orbit scattering is present (\( \tau_\omega \) is the spin-order scattering time) \cite{42,43}. Combining equations (31) and (32) we find a positive linear effect of small magnetic fields upon \( \epsilon \):

\[
\frac{d \ln \epsilon}{dB} \approx g_{\text{eff}} \frac{\mu_B}{\delta \xi} \left( \ln \frac{\delta \xi}{\Delta_p(B)} \right)^{d+1}.
\]

(33)

Another effect of the magnetic field on the dielectric constant \( \epsilon \) is via modification of the localization length \( \xi \) due to orbital effect. However, the corresponding \( \xi(B) \) dependence is quadratic at weak magnetic fields, \( \delta \xi(B)/\xi \sim (B_0^2/\hbar \delta \xi)^2 \), therefore the pseudogap-related contribution (33) dominates at low fields.

The results (31) and (33) should also be applicable for two-dimensional films, but the applicability condition at low \( \Delta_p \) (replacing equation (30)) requires further study.
4. Conclusions and discussion

The main result of the paper is equation (12) for the polarizability of a non-interacting Anderson insulator with a large localization length. This result was derived by a combination of analytic and numerical methods in dimensionality \( d = 1, 2, 3 \), and the coefficient \( \alpha \) was computed in all the three cases (in one-dimension, our computation agrees with the exact analytical result). The main contribution to polarizability comes from relatively large distances (of order 5–10 localization lengths), which implies an important contribution from relatively rare Mott-hybridized eigenstates [23, 32]. As a by-product of this study, we found that the rescaled correlation function \( F(x) \propto d^{1/3} \) is of nearly universal shape in all dimensions \( d = 1, 2, 3 \), see figure 1.

It follows from our main result that, in the case of a large localization length, the polarizability is also large. For quasi-1D and quasi-2D materials it means also a very large dielectric constant, see equations (19), (20). Therefore, Coulomb interactions between electrons in these materials are strongly reduced, and all the energy scales arising from Coulomb interactions are renormalized down to a numerically small fraction of \( \delta \zeta \). Note however that the actual relevant energy scale is also a certain fraction of \( \delta \zeta \) (we denote it \( \delta \zeta \) in section 3.2). Thus, the role of Coulomb effects may be larger than expected. However, since all the relevant energy scales are numerical fractions of \( \delta \zeta \), we expect that the renormalization of the dielectric response due to Coulomb-gap effects is also only by a numerical factor of order one, and therefore our results (19), (20) give correct order of magnitude estimates.

For isotropic 3D materials with large but finite localization length, similar conclusions should be taken with care, as explained in section 3.1. Indeed, the numerical results [17] obtained within Hartree–Fock scheme indicate a crucial role of Coulomb interaction in the pinning of the mobility edge to the Fermi level in such a system. Similar conclusions about the importance of Coulomb interactions come out of the renormalization-group study [18]. If such a situation—with pinned mobility edge—is realized, then Coulomb interaction is strongly relevant while our approach is not adequate. One should note, however, that the value of \( z \) for the 3D metal–insulator transition with Coulomb interaction is still unknown. Although numerical data [17] suggest \( z \approx 2 \), it is not evident that the size of the system used in these computations is sufficient to make any definite conclusion, since, according to our results (see figure 1), the major contribution to polarizability comes from large spatial scales \( r \sim 5–10\xi \).

In some disordered materials a large dielectric constant \( \varepsilon \gg 1 \) is already provided by the contribution of valence band electrons with energies separated from the Fermi level by some non-zero but relatively small gap \( E_{0} \ll E_{F} \). Then Coulomb interaction between electrons in the conduction band (subject to Anderson localization due to strong disorder) is reduced accordingly. In such a case, two different scenarios may be realized: (a) unpinned mobility edge with ‘non-interacting scaling’ \( \varepsilon \propto \xi^{\zeta} \), that is realized at moderately large \( \xi \ll \xi_{1} \) (for some crossover scale \( \xi_{1} \)), and (b) at further increase of \( \xi \) beyond \( \xi_{1} \), a pinning of the mobility edge to the Fermi level occurs, which is accompanied by a crossover to the ‘interacting scaling’ similar to the one described in [17, 18].

We also note the difference between our case and the findings of [35] (derived in the regime of small localization lengths) where Coulomb interactions modify the frequency-dependent conductivity drastically in a wide range of frequencies.

One of the applications of our results is for Anderson insulators with a phonon-mediated pseudogap. In such materials, the dielectric constant is slightly reduced due to the pseudogap and, together with the pseudogap, may be tuned by an external magnetic field. We predict a linear dependence of this correction on the magnetic field and estimate the magnitude of this effect.

The major remaining challenge in these topics is the simultaneous account for wave-function fractality and Coulomb-gap effects. This is a many-body problem, which goes beyond the methods of the present paper.

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Appendix. Details of numerical calculations

For numerical calculations of section 2, we sampled single-particle eigenstates in the tight-binding model on the cubic lattice (in \( d \) dimensions) with the random on-site potential. The Hamiltonian has the usual form:
\[ H = -\sum_{\langle ij \rangle} a_i^{\dagger} a_j + \sum_i V_i a_i^{\dagger} a_i, \]  

where the first term describes nearest-neighbor hopping and the second term the random potential \((a_i^{\dagger} a_i)\) are the creation and annihilation operators. The hopping amplitude is taken to be one. The on-site disorder \(V_i\) is uncorrelated on different sites and is drawn from the normal distribution centered at zero with the standard deviation \(W\) for ‘Gaussian’ models and from the uniform distribution between \(-W/2\) and \(W/2\) for ‘Box’ models.

For calculating \(P(x)\) and \(\alpha\), we used systems of cubic geometry with periodic boundary conditions. The linear system sizes were equal to 2000 for one-dimensional case, 750 for two-dimensions and 100 for three-dimensions. The data for each system were obtained by averaging over about 1500 eigenstates in many disorder realizations in 2D and 3D and over \(3.2 \times 10^6\) eigenstates in 1D.

The localization length \(\xi\) was computed independently using the transfer-matrix method of [30, 31]. Note that the definition of the localization length usually employed while using this method is two times larger than the one we used here.

References

[1] Anderson P W 1958 Phys. Rev. 109 1492
[2] Lee P A and Ramakrishnan T V 1985 Rev. Mod. Phys. 57 287
[3] Kramer B and MacKinnon A 1993 Rep. Prog. Phys. 56 1469
[4] Feigel’man M V, Ioffe I B, Kravtsov V E and Yuzbashyan E A 2007 Phys. Rev. Lett. 98 027001
[5] Feigelman M V, Ioffe I B, Kravtsov V E and Cuevas E 2010 Ann. Phys. 325 1390
[6] van der Zant H S J, Elion W J, Geerligs L and Mooij J E 1996 Phys. Rev. B 54 10081
[7] Fazio R and van der Zant H 2001 Phys. Rep. 355 235
[8] de Visser P J, Baselmans J A, Yates S C, Diener P, Endo A and Klapwijk T M 2012 Appl. Phys. Lett. 100 162601
[9] Efetov K B 1983 Phys. Rev. B 28 3436
[10] Efetov K B 1997 Introduction to the Theory of Disordered Systems (New York: Cambridge University Press)
[11] Efetov K B 1991 Phys. Rev. B 44 2739
[12] Baranovski D I, Shklovskii B I and Efros A L 1984 Zh. Eksp. Teor. Fiz. 87 1793 [1984 Sov. Phys.—JETP 60 1031]
[13] Finkel’stein A M 1990 Sov. Sci. Rev. A 14 1
[14] Amini M, Kravtsov V E and Muller M 2014 New J. Phys. 16 015022
[15] Burmistrov I S, Gornyi I V and Mirlin A D 2014 Phys. Rev. B 89 035430
[16] Berezinsky V L 1973 Zh. Eksp. Teor. Fiz. 65 1251 [1974 Sov. Phys.—JETP38 620]
[17] Abrikosov A A and Ryzhkin I A 1978 Adv. Phys. 27 147
[18] Gor’kov L P, Dorokhov O N and Prigara F V 1983 Zh. Eksp. Teor. Fiz. 85 1470 [1983 Sov. Phys.—JETP 58 852]
[19] Ivanov D A, Ostrovsky P M and Skvortsov M A 2009 Phys. Rev. B 79 265108
[20] Ivanov D A, Skvortsov M A, Ostrovsky P M and Pominov Ya V 2012 Phys. Rev. B 85 035109
[21] Efetov K B 1983 Adv. Phys. 32 53
[22] Gogolin A A 1976 Zh. Eksp. Teor. Fiz. 71 1912 [1976 Sov. Phys.—JETP 44 1003]
[23] Lifshits I M and Gredeskul S A and Pastur L A 1988 Introduction to the Theory of Disordered Systems (New York: Wiley)
[24] Kolokolov I V 1995 Physica D 86 134
[25] Mirlin A D 1997 J. Math. Phys. 38 1888
[26] Efetov K B and Larkin A I 1983 Zh. Eksp. Teor. Fiz. 85 764 [1983 Sov. Phys.—JETP 58 444]
[27] MacKinnon A and Kramer B 1983 Z. Phys. B 53 1
[28] Slevin K and Ohtsuki T 2014 New J. Phys. 16 015012
[29] Mott N F 1968 Phil. Mag. 17 1259
[30] Mott N F 1970 Phil. Mag. 22 227
[31] Efros A I and Kravtsov V E 2007 Phys. Rev. B 76 235119
[32] Efros A I and Shklovskii B I 1975 J. Phys. C: Solid State Phys. 8 L49
[33] Shklovskii B I and Efros A I 1981 Zh. Eksp. Teor. Fiz. 81 406 [1981 Sov. Phys.—JETP 54 218]
[34] Efetov K B and Larkin A I 1997 Phys. Rev. Lett. 78 1374
[35] Ivanov D A and Feigel’man M V 2017 Phys. Rev. B 95 045147
[36] Anderson P W 1958 Phys. Rev. B 76 235119
[37] Efetov K B 1983 Phys. Rev. B 28 3436
[38] Ivanov D A and Feigel’man M V 2017 Phys. Rev. B 95 045147
[39] Anderson P W 1958 Phys. Rev. B 76 235119