K-stability, Futaki invariants and cscK metrics on orbifold resolutions

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Abstract

In this paper we compute the Futaki invariant of adiabatic Kähler classes on resolutions of Kähler orbifolds with isolated singularities. Combined with previous existence results of extremal metrics by Arezzo-Lena-Mazzieri, this gives a number of new existence and non-existence results for cscK metrics.

1 Introduction

In this paper we address the question of existence (and non-existence) of constant scalar curvature Kähler metrics (cscK from now on) in adiabatic Kähler classes on resolutions of compact cscK orbifolds with isolated singularities.

Form a purely conceptual point of view the basic existence result for extremal Kähler metrics proved in [2] can be reinterpreted in the following form in the spirit of Szekelyhidi’s work on blow ups of smooth points [16, 17]:

Theorem 1.1. Let $M$ be a Kähler orbifold of dimension $m$ with finite singular set $S \subset M$, and let $\pi: M' \rightarrow M$ be a resolution of singularities with local model $\pi_p: X_p \rightarrow C^m/\Gamma_p$ at each $p \in S$. Assume that $M$ admits a Kähler metric $\omega$ with constant scalar curvature and that each $X_p$ admits a scalar flat ALE Kähler metric $\eta_p$.

Then there exists $\varepsilon_0 > 0$ such that for all $\varepsilon < \varepsilon_0$, the following are equivalent

1. $M'$ has a Kcsc metric in the class $\pi^*\omega + \sum_{p \in S} \varepsilon[\eta_p]$ 

2. $(M', \pi^*\omega + \sum_{p \in S} \varepsilon[\eta_p])$ is K-stable.

$1 \implies 2$ is proved in [17] Proposition 38, while $2 \implies 1$ is a simple consequence of [2] Theorem 1.1.

While this result settles the celebrated Tian-Yau-Donaldson Conjecture [7, 8] for these type of manifolds and classes, because of the known difficulty in checking $K$-stability for a polarized manifold, it remains of great interest to have some effective method to give some geometric conditions on $S$ which guarantee the existence of cscK metrics. This is the primary aim of this paper.

In [1] and [2] partial results have been obtained in this direction by performing a careful analysis of the PDE, very much in the spirit of the analogue results of Arezzo-Pacard [3, 4] for
blow ups of smooth points. This approach produces a variety of sufficient conditions for the existence of a cscK metric, all of which follow from the more general results of the present work.

The main result of this paper, Theorem 2.3, is the computation of the Futaki invariant of adiabatic Kähler classes on resolution of singularities in terms of corresponding objects on the base orbifold and the geometry of $X_p$. In fact, the nonuniqueness of the resolution one decides to consider, prevents from using the algebraic techniques already employed in the analogue situation for the blow ups of smooth points by many authors (Stoppa [15], [6], Odaka [14] and Szekelyhidi [16]).

What the PDE analysis showed is that a critical difference in the behaviour of this problem comes from the ADM mass of the local model of the resolution. While it has been long known how to relate this number to the behaviour at infinity of ALE metrics, only very recently Hein-LeBrun [10] have discovered some very elegant interpretation of this quantity in purely cohomological terms for scalar flat metrics. Objects coming into the computation of the Futaki invariant are different from theirs, yet their work has been a key source of inspiration to bypass the problem of non-uniqueness of the resolution.

A number of consequences follows from our Theorem 2.3 both getting a new proof of the results of [1] and [2], but more importantly of new existence and various nonexistence results, which are discussed in details in Section 4 below. Using this approach, we can distinguish three different situations (it is worth recalling for the convenience of the reader that an ALE manifold is allowed to have zero ADM mass without being isometric to the flat Euclidean space, as pointed out in [12]):

1. For all $p \in S$, the local model $X_p$ has a scalar flat metric with zero ADM mass;

2. There exist $p, q \in S$ such that the local model $X_p$ has a scalar flat metric with zero ADM mass, the local model $X_q$ has a scalar flat metric with non-zero ADM mass, and the adiabatic classes have the same scales of volumes of exceptional divisors;

3. There exist $p, q \in S$ such that local model $X_p$ has a scalar flat metric with zero ADM mass, the local model $X_q$ has a scalar flat metric with non-zero ADM mass, and the adiabatic classes have different scales of volumes of exceptional divisors.

In each of these cases we give a sufficient condition which generalizes the ones found in [1] and [2] in terms of the position of the singular points to be resolved, which we prove to be essentially also necessary in Theorem 4.1. This is done in Theorems 4.3, 4.4 and 4.5 respectively.

2 The Futaki invariant of an orbifold resolution

Let $(M, \omega)$ be a compact Kähler orbifold of complex dimension $m$ with finite singular set $S \subset M$. This means that $M$ is a compact Hausdorff topological space endowed with a structure of $n$-dimensional complex manifold on the subset $M \setminus S$ such that for each singular point $p \in S$ there exist the following data:

- a neighborhood $U_p$ intersecting $S$ just at $p$,
- a non-trivial finite subgroup $\Gamma_p \subset U(m)$,
- a homeomorphism between $U_p$ and a ball $B(r)/\Gamma_p \subset \mathbb{C}^m/\Gamma_p$ which restricts to a biholomorphism between $U_p \setminus \{p\}$ and the punctured ball $B'(r)/\Gamma_p$,
Moreover, $\omega$ restricts to a Kähler form of a genuine Kähler metric on $M \setminus S$, and the restriction of $\omega$ to $U_p$ lifts to a $\Gamma_p$-invariant Kähler form on the ball $B(r) \subset \mathbb{C}^m$.

The quotient $\mathbb{C}^m/\Gamma_p$ is called the local model for the singularity at $p$. We stress that different singular points may have different local models. Finally note that, in principle, the radius $r$ may depend on $p$, but taking the minimum as $p$ varies on $S$ we can suppose that $r$ is indeed independent of the point.

2.1 Resolution of orbifold singularities

Let $p \in S$ be a singular point of $M$, and let

$$\pi_p : X_p \to \mathbb{C}^m/\Gamma_p$$

be a resolution of singularities of the local model at $p$. In view of our applications, we will always assume that $X_p$ admits a Kähler metric. By definition, $\pi_p$ is a proper birational morphism from a $m$-dimensional complex manifold $X_p$ to $\mathbb{C}^m/\Gamma_p$ which restricts to a biholomorphism on the complement of $\pi_p^{-1}(0)$. It follows by definition that $\pi_p^{-1}(0)$ is a union of compact complex submanifolds of $X$. The biholomorphism between $U_p \setminus \{p\}$ and the punctured ball $B'(r)/\Gamma_p$ existing by definition of complex orbifold, and the fact that $\pi_p$ is a biholomorphism on a neighborhood of $\partial B(r)/\Gamma_p$ also allow to replace each neighborhood $U_p \subset M$ with the resolved ball $\pi_p^{-1}(B(r)/\Gamma_p)$ and obtain a complex manifold $M'$ and a resolution of singularities

$$\pi : M' \to M.$$

This map collects all maps $\pi_p$ as $p$ varies in $S$. More precisely, $\pi$ acts on $\pi_p^{-1}(B(r)/\Gamma_p)$ as the composition of $\pi_p$ together with the homeomorphism from $B(r)/\Gamma_p$ to $U_p$. Moreover $\pi$ is the identity on the complement of $\pi^{-1}(U)$, where $U$ is the union of all $U_p$ as $p$ varies in $S$. In particular $\pi$ turns out to be a biholomorphism when restricted to the complement of $\pi^{-1}(S)$.

2.2 A Kähler metric on $M'$

In this subsection, we construct a Kähler metric on $M'$ which is, in some respect, a deformation of the Kähler metric $\omega$ on $M$.

For any $p \in S$, let $\eta_p$ be a Kähler metric on the model resolution $X_p$ of the form

$$\eta_p = \xi_p + dd^c \varphi_p,$$

where $\xi_p$ is a $(1,1)$-form supported in $\pi_p^{-1}(B(r)/\Gamma_p)$, and $\varphi_p$ is a smooth function. Since we constructed $M'$ by replacing each singular ball $U_p$ with the resolved ball $\pi_p^{-1}(B(r)/\Gamma_p)$, we can think of each $\xi_p$ as a $(1,1)$-form on $M'$. Thus, for all real $\varepsilon$, we can consider the following $(1,1)$-form on $M'$

$$\omega_\varepsilon = \pi^* \omega + \varepsilon \sum_{p \in S} \xi_p.$$

Note that $\omega_\varepsilon$ defines a Kähler metric on $M'$ for all $\varepsilon > 0$ sufficiently small. This can be seen by considering the restriction of $\omega$ on $\pi^{-1}(U)$ and on $M' \setminus \pi^{-1}(U)$. The latter is positive since there the map $\pi$ restricts to a biholomorphism and $\xi$ vanishes. Thus it remains to check that for $\varepsilon$ sufficiently small $\omega_\varepsilon$ is positive around the resolution $\pi^{-1}(U_p)$ of any singular point $p \in S$. Over $\pi^{-1}(U_p)$ the form $\omega_\varepsilon$ restricts to $\pi_p^* \omega + \varepsilon \xi_p$ (here, for ease of notation, we wrote $\omega$ instead of the pullback of $\omega|_{U_p}$ to the ball $B(r)/\Gamma_p$). Since $\omega$ comes from a $\Gamma_p$-invariant form on $B(r)$,
we can suppose $\omega = dd^c h_p$ for some smooth function $h_p$ on the ball $B(r)/\Gamma_p$. As a consequence, on $\pi^{-1}(U_p)$ we have

$$\omega_\varepsilon = dd^c (\pi^{-1}_p h_p - \varepsilon \varphi_p) + \varepsilon \eta_p.$$ 

Note that $\eta_p$ is positive in any neighborhood of the exceptional set $\pi^{-1}(p)$. On the other hand, for $\varepsilon$ sufficiently small the function $\pi^{-1}_p h_p - \varepsilon \varphi_p$ is plurisubharmonic on the complement of the exceptional set. This shows that $\omega_\varepsilon$ is positive on any $\pi^{-1}(U_p)$ provided $\varepsilon$ is sufficiently small, as claimed.

2.3 Pushing down vector fields

In this section we show that any holomorphic vector field on $M'$ induces a holomorphic vector field on $M$. If the vector field on $M'$ is also Hamiltonian, the induced vector field on $M$ is Hamiltonian too.

**Lemma 2.1.** Any holomorphic vector field $V$ on $M'$ descends to a holomorphic vector field $\pi_\ast V$ on $M$ which vanishes at all points of $S$.

**Proof.** Since $\pi$ is a biholomorphism on the complement of $\pi^{-1}(S)$, pushing down the restriction of $V$ to that set defines a vector field $V'$ on $M \setminus S$. Given $p \in S$, the restriction of $V'$ to $U_p \setminus \{p\}$ lifts to a $\Gamma_p$-invariant vector field on the punctured ball $B'(r)$ of $\mathbb{C}^m$. By Hartog’s theorem such a vector field extends to a holomorphic vector field on the whole ball $B(r)$. Of course such a vector field is $\Gamma_p$-invariant, and so it gives a holomorphic vector field on $U_p$ which is equal to $V'$ on $U_p \setminus \{p\}$. Therefore one ends up with a holomorphic vector field $\pi_\ast V$ on $M$.

It remains to show that $\pi_\ast V$ vanishes at any $p \in S$. To this end, note that the fiber $\pi^{-1}(p)$ is a union of compact complex submanifolds. Therefore $V$ must be tangent to it, and consequently $\pi_\ast V$ must tend to zero as approaching to $p$. By continuity we can then conclude that $\pi_\ast V$ actually vanishes at $p$. \qed

**Lemma 2.2.** If $V$ is a holomorphic vector field on $M'$ and is Hamiltonian with respect to $\omega_\varepsilon$, then $\pi_\ast V$ is Hamiltonian with respect to $\omega$ on $M$. Moreover, if $u_\varepsilon$ and $u$ are Hamiltonian potentials for $V$ and $\pi_\ast V$ respectively, then one has

$$u_\varepsilon = \pi_\ast u + \varepsilon \sum_{p \in S} u_p + c(\varepsilon),$$

where $u_p$ is a smooth function supported in $\pi^{-1}(U_p)$ satisfying $du_p = i_V \xi_p$, and $c(\varepsilon)$ is a constant.

**Proof.** On the complement of $\pi^{-1}(U)$ the Kähler form $\omega_\varepsilon$ is equal to $\omega$ and the vector field $V$ is equal to $\pi_\ast V$. Therefore the Hamiltonian potential $u_\varepsilon$ of $V$ restricts to a function on $M' \setminus \pi^{-1}(U)$, say $u$, which does not depend on $\varepsilon$ and is a Hamiltonian potential for $\pi_\ast V$ with respect to $\omega$ on $M \setminus U$.

Now let $p \in S$. Identifying $\pi^{-1}(U_p)$ with the resolved ball $\pi^{-1}_p (B(r)/\Gamma_p)$ in the local model $X_p$, one can write

$$\omega_\varepsilon = \pi^{-1}_p dd^c h_p + \varepsilon \xi_p,$$

where $h_p$ is a Kähler potential for $\omega$ in the ball $B(r)/\Gamma_p$. Since, by hypothesis, $V$ is Hamiltonian with potential $u_\varepsilon$, one has

$$du_\varepsilon = i_V dd^c \pi^{-1}_p h_p + \varepsilon i_V \xi_p.$$

On the other hand, $V$ is holomorphic, therefore Cartan formula yields

$$du_\varepsilon = d^c V(\pi^{-1}_p h_p) + \frac{1}{4} dJV(\pi^{-1}_p h_p) + \varepsilon i_V \xi_p.$$
whence it follows that
\[ i_V \xi_p = du_p - \frac{1}{\varepsilon} dF(V(\pi_p^* h_p)), \]
for some smooth function \( u_p \). Note that the last summand does not depend on \( \xi_p \), but it is forced to vanish where \( \xi_p \) does. Therefore, by arbitrariness of the metric \( \eta_p \) we started with, it must vanish everywhere. As a consequence the support of \( du_p \) is contained in the support of \( \xi_p \).

In particular, up to adding a suitable constant, we can suppose that \( u_p \) is supported in \( \pi^{-1}(U_p) \). Thus \( u_p \) has all the properties stated above. Moreover we proved that on \( \pi^{-1}(U_p) \) it holds
\[ u_\varepsilon = \frac{1}{4} JV(\pi_p^* h_p) + \varepsilon u_p + c(p, \varepsilon), \]
where \( c(p, \varepsilon) \) is a constant.

Since the support of \( \xi_p \) is compactly contained in the resolved ball \( \pi_p^{-1}(B(r)/\Gamma_p) \), on a neighborhood of the boundary \( \pi^{-1}(\partial U_p) \) it holds
\[ u_\varepsilon = \pi_p^* \left( \frac{1}{4} (J\pi_* V)(h_p) + c(p, \varepsilon) \right). \]

Therefore \( u \) extends to \( \frac{1}{4} (J\pi_* V)(h_p) + c(p, \varepsilon) \) on \( U_p \). Finally note that it holds \( du = i_{\pi_* V} \omega \), that is \( u \) is a Hamiltonian potential for \( \pi_* V \). Since Hamiltonian potentials are defined just up to an additive constant, equation (1) then follows. \( \square \)

### 2.4 The Futaki invariant

Given a holomorphic vector field \( V \) on the resolution \( M' \), and supposing that \( V \) is Hamiltonian with respect to \( \omega_\varepsilon \) with potential \( u_\varepsilon \), one can form the Futaki invariant
\[ \text{Fut}(V, \omega_\varepsilon) = \int_{M'} (u_\varepsilon - \bar{u}_\varepsilon) \rho_\varepsilon \wedge \omega_\varepsilon^{m-1} \]
where \( \rho_\varepsilon \) is the Ricci form of \( \omega_\varepsilon \), and \( \bar{u}_\varepsilon = \int u_\varepsilon \omega_\varepsilon^m / \int \omega_\varepsilon^m \) is the mean value of \( u_\varepsilon \) with respect to \( \omega_\varepsilon \).

On the other hand, thanks to Lemmata 2.1 and 2.2, \( V \) descends to a holomorphic vector field \( \pi_* V \) on \( M \) which is Hamiltonian with respect to \( \omega \) with potential, say, \( u \). Thus one can also consider the Futaki invariant
\[ \text{Fut}(\pi_* V, \omega) = \int_M (u - \bar{u}) \rho \wedge \omega^{m-1} \]
where \( \rho \) is the Ricci form of \( \omega \), and \( \bar{u} = \int u \omega^m / \int \omega^m \). The Futaki invariants \( \text{Fut}(V, \omega_\varepsilon) \) and \( \text{Fut}(\pi_* V, \omega) \) are relate by the following

**Theorem 2.3.** As \( \varepsilon \to 0 \) one has
\[
\text{Fut}(V, \omega_\varepsilon) = \text{Fut}(\pi_* V, \omega) + \varepsilon^{-1} \sum_{p \in S} (u(p) - \bar{u}) \int_{X_p} \frac{\rho_p \wedge \xi_p^{m-1}}{(m-1)!} \]
\[
- \varepsilon \sum_{p \in S} (g(u(p) - \bar{u}) + \Delta u(p)) \int_{X_p} \frac{\xi_p^m}{m!} + O(\varepsilon^m) \tag{2}
\]
where \( \rho_p \) is the Ricci form of the chosen ALE Kähler metric \( \eta_p \) on the model resolution \( X_p \), and \( g = m \int \rho \wedge \omega^{m-1} / \int \omega^m \) is the mean scalar curvature of \( \omega \).
Proof. The Futaki invariant of the vector field $V$ can be written as

$$\text{Fut}(V, \omega_\varepsilon) = \int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge \frac{(\omega_\varepsilon + u_\varepsilon)^m}{m!} - u_\varepsilon \int_{M'} \rho_\varepsilon \wedge \omega_\varepsilon^{m-1} \wedge (m-1)!,$$

where $\Delta_\varepsilon$ denotes the Laplacian of the Kähler metric on $M'$ associated to $\omega_\varepsilon$. The first integral and the average of $u_\varepsilon$ perhaps could be calculated using equivariant cohomology theory. However one can avoid that theory and prove the statement by means of more elementary arguments. In order to understand the formula above, note that $\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon$ and $\omega_\varepsilon + u_\varepsilon$ are non-homogeneous differential forms on $M'$. Their wedge product is the usual one, and so the integrand is a sum of even degree differential forms. The integral of such a form is, by definition, the integral of its $2m$-degree component. Now consider the differential operator

$$d_V = d - i_V$$

acting on differential forms on $M'$. The fact that $u_\varepsilon$ is a Hamiltonian potential for $V$ with respect to $\omega_\varepsilon$ can be stated as

$$d_V(\omega_\varepsilon + u_\varepsilon) = 0.$$

In other words, $\omega_\varepsilon + u_\varepsilon \in \Omega^*(M)$ is a $d_V$-closed differential form. After expanding

$$\omega_\varepsilon + u_\varepsilon = \pi^*(\omega + u) + \varepsilon \sum_{p \in S} \xi_p + u_p,$$

one sees that $\pi^*(\omega + u)$ and $\xi_p + u_p$ are $d_V$-closed as well. Finally, by a standard local calculation one can check that

$$d_V(\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) = 0.$$

Now pick a singular point $p \in S$ and let $\pi^{-1}(p)$ be the exceptional divisor over $p$. The proof of the statement of the Theorem rests essentially on the following

Claim. Any $d_V$-closed differential form $\alpha \in \Omega^*(M')$ which restricts to the zero form on the exceptional divisor $\pi^{-1}(p)$ satisfies

$$\int_{M'} \alpha \wedge (\xi_p + u_p) = 0.$$

In order to prove the claim note that by Poincaré-Lelong equation one can find a path $F_t$ of smooth functions on $M'$, with $t > 0$, such that $\xi_p + d\bar{\partial}F_t$ weakly converges to a current supported on $\pi^{-1}(p)$ as $t \to 0$. A moment’s thought should show that $\xi_p + u_p + d\bar{\partial}F_t$ also converges to the same current. On the other hand, note that by Stokes’ Theorem the integral of any $d_V$-closed form on $M'$ vanishes. As a consequence one has

$$\int_{M'} \alpha \wedge (\xi_p + u_p) = \int_{M'} \alpha \wedge (\xi_p + u_p + d\bar{\partial}F_t) \to 0$$

as $t \to 0$ thanks to the hypothesis that $\alpha$ restricts to the zero form on $\pi^{-1}(p)$.

Now we can proceed with the proof of the Theorem. We need to expand (3) in powers of $\varepsilon$. First of all consider the average $u_\varepsilon = \int u_\varepsilon \omega_\varepsilon^m / \int \omega_\varepsilon^m$. Note that

$$(m+1) \int_{M'} u_\varepsilon \omega_\varepsilon^m = \int_{M'} (\omega_\varepsilon + u_\varepsilon)^{m+1}. \quad (5)$$
Substituting (4) yields
\[
\int_{M'} (ω_ε + u_ε)^{m+1} = \int_M (ω + u)^{m+1} + \varepsilon \sum_{p\in S} \pi^*(ω + u)^{m+1-ℓ} \wedge (ξ_p + u_p)^{ℓ} \\
+ \varepsilon^{m+1} \sum_{p\in S} \int_{M'} (ξ_p + u_p)^{m+1}. \tag{6}
\]

Focus on the middle summands of the right hand side. Note that for all 1 ≤ ℓ ≤ m and p ∈ S the differential form
\[
α = π^*(ω + u)^{m+1-ℓ} \wedge (ξ_p + u_p)^{ℓ-1} - u(p)^{m+1-ℓ}(ξ_p + u_p)^{ℓ-1}
\]
satisfies the hypotheses of the Claim, whence it follows
\[
\int_{M'} π^*(ω + u)^{m+1-ℓ} \wedge (ξ_p + u_p)^{ℓ} = u(p)^{m+1-ℓ} \int_{M'} (ξ_p + u_p)^{ℓ}.
\]
Since ℓ runs from 1 to m, the right hand side integral vanishes unless ℓ = m, in which case it reduces to the integral of ξ_p^m. Therefore, substituting in (6) and (5) we get the expansion
\[
\int_{M'} u_ε ω_ε^m = \int_M u ω^m + \varepsilon^m \sum_{p\in S} u(p) \int_{M'} ξ_p^m + O(ε^{m+1}). \tag{7}
\]

In order to get the expansion of u_ε we need to divide the expression above by the total volume of ω_ε. Recalling that ω_ε = π^*ω + ε Σ ξ_p, one has
\[
\int_{M'} ω_ε^m = \int_M ω^m + \varepsilon^m \sum_{p\in S} ξ_p^m. \tag{8}
\]
Since ξ_p is supported on π^{-1}(U_p) and π^*ω is exact on there, Stokes’ Theorem yields
\[
\int_{M'} ω_ε^m = \int_M ω^m + \varepsilon^m \sum_{p\in S} \int_{X_p} ξ_p^m.
\]
Note that we replaced the integral of ξ_p over M' with the integral over the model resolution X_p. This is possible since the support of ξ_p is contained in π^{-1}(U_p), which in turn we have identified with a neighborhood of the exceptional divisor of X_p. Dividing (7) by (8) finally gives
\[
u_ε = u + ε^m \sum_{p\in S} (u(p) - u) \frac{\int_{X_p} ξ_p^m}{\int_M ω^m} + O(ε^{m+1}). \tag{9}
\]

Now we pass to consider the total scalar curvature of ω_ε. Arguing as above, after substituting ω_ε = π^*ω + ε Σ ξ_p and applying Stokes’ Theorem, one gets
\[
\int_{M'} ρ_ε \wedge ω_ε^{m-1} = \int_{M'} ρ_ε \wedge π^*ω^{m-1} + ε^{m-1} \sum_{p\in S} \int_{M'} ρ_ε \wedge ξ_p^{m-1}. \tag{10}
\]
Consider the two summands separately. Adding and subtracting π^*ρ in the first summand yields
\[
\int_{M'} ρ_ε \wedge π^*ω^{m-1} = \int_{M'} π^*ρ \wedge π^*ω^{m-1} + \int_{M'} (ρ_ε - π^*ρ) \wedge π^*ω^{m-1}.
\]
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The first summand reduces to the integral of $\rho \wedge \omega^{m-1}$ over $M$, and the second summand vanishes once again by Stokes’ Theorem. Indeed $\omega_c = \omega$ on the complement of $U$. As a consequence $\rho_c - \pi^* \rho$ vanishes on that set. On the other hand $\pi^* \omega$ is exact on any connected component of $U$. Therefore

$$\int_{M'} \rho_c \wedge \pi^* \omega^{m-1} = \int_M \rho \wedge \omega^{m-1}. \quad (11)$$

Now consider the second summand of the right hand side of (10). Since the support of $\rho_c$ is contained in $\pi^{-1}(U)$, we identified with a neighborhood of the exceptional divisor of the model resolution $X_p$, we can consider the Ricci form $\rho_p$ of the chosen ALE Kähler metric $\eta_p$ on $X_p$ and thought of $\rho_p \wedge \xi_p^{m-1}$ as a differential form on $M'$. After noting that $\rho_c - \rho_p = d\epsilon \log(\eta_p^m/\omega_c^m)$, Stokes’ Theorem yields

$$\int_{M'} (\rho_c - \rho_p) \wedge \xi_p^{m-1} = 0.$$

On the other hand, we can also consider $\rho_p \wedge \xi_p^{m-1}$ as a differential form on the local resolution $X_p$. As a consequence we can rewrite equation above in the form

$$\int_{M'} \rho_c \wedge \xi_p^{m-1} = \int_{X_p} \rho_p \wedge \xi_p^{m-1}. \quad (12)$$

Therefore substituting (12) together with (11) into (10) yields

$$\int_{M'} \rho_c \wedge \omega_c^{m-1} = \int_M \rho \wedge \omega^{m-1} + \epsilon^{m-1} \sum_{p \in S} \int_{X_p} \rho_p \wedge \xi_p^{m-1}. \quad (13)$$

Thanks to (9) and (13), the second summand of the right hand side of (3) expands as

$$u \int_{M'} \rho_c \wedge \omega_c^{m-1} = u \int_M \rho \wedge \omega^{m-1} + \epsilon^{m-1} \sum_{p \in S} \int_{X_p} \rho_p \wedge \xi_p^{m-1}$$

$$+ \epsilon^m \sum_{p \in S} s(u(p) - u) \int_{X_p} \xi_p^m / m! + O(\epsilon^{m+1}). \quad (14)$$

Finally it remains to consider the first summand of (3). Substituting (14) yields

$$\int_{M'} (\rho_c - \Delta_c u_c) \wedge (\omega_c + u_c)^m / m! = \int_{M'} (\rho_c - \Delta_c u_c) \wedge \pi^*(\omega + u)^m / m!$$

$$+ \sum_{\ell=1}^m \epsilon^\ell \sum_{p \in S} \int_{M'} (\rho_c - \Delta_c u_c) \wedge \pi^*(\omega + u)^{m-\ell} / (m-\ell)! \wedge (\xi_p + u_p)^{\ell-1} / \ell!. \quad (15)$$

Focus on the summands of the second line. Fix $p \in S$ and suppose $0 < \ell < m$, so that the differential form

$$\alpha = (\rho_c - \Delta_c u_c) \wedge \pi^*(\omega + u)^{m-\ell} / (m-\ell)! \wedge (\xi_p + u_p)^{\ell-1} / \ell!$$

satisfies the hypotheses of the Claim above, whence it follows that

$$\int_{M'} (\rho_c - \Delta_c u_c) \wedge \pi^*(\omega + u)^{m-\ell} / (m-\ell)! \wedge (\xi_p + u_p)^{\ell-1} / \ell! = u(p)^{m-\ell} / (m-\ell)! \int_{M'} (\rho_c - \Delta_c u_c) \wedge (\xi_p + u_p)^{\ell} / \ell!.$$
If \( \ell < m - 1 \), the integrand differential form of the right hand side has no component of degree \( 2m \), therefore the integral is zero. On the other hand, for \( \ell = m - 1 \), equation above together with (12) give
\[
\int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge \pi^* (\omega + u) \wedge \frac{(\xi_p + u_p)^{m-1}}{(m-1)!} = u(p) \int_X \rho_p \wedge \xi_p^{m-1}.
\]
By discussion above, (15) reduces to
\[
\int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge \frac{(\omega_\varepsilon + u_\varepsilon)^m}{m!} = \int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge \frac{\pi^* (\omega + u)^m}{m!}
+ \varepsilon^{m-1} \sum_{p \in S} u(p) \int_X \rho_p \wedge \xi_p^{m-1} (m-1)! + \varepsilon^m \sum_{p \in S} \int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge \frac{(\xi_p + u_p)^m}{m!}.
\] (16)
In order to treat the first summand of the right hand side, note that the difference \( \rho_\varepsilon - \Delta_\varepsilon u_\varepsilon - \pi^* (\rho - \Delta u) \) is a \( dV \)-closed differential form on \( M' \) which is compactly supported in the union of all \( \pi^{-1}(U_p) \) as \( p \) varies in \( S \). The proof of the Claim above works also replacing the form \( \xi_p + u_p \) with \( \rho_\varepsilon - \Delta_\varepsilon u_\varepsilon - \pi^* (\rho - \Delta u) \). As a consequence, since \( \pi^* (\omega + u)^m \) restricts to a constant function on any exceptional divisor, one then has
\[
\int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge \frac{\pi^* (\omega + u)^m}{m!} = \int_{M'} \pi^* (\rho - \Delta u) \wedge \frac{\pi^* (\omega + u)^m}{m!}.
\]
Therefore (16) simplifies to
\[
\int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge \frac{(\omega_\varepsilon + u_\varepsilon)^m}{m!} = \int_{M'} (\rho - \Delta u) \wedge \frac{(\omega + u)^m}{m!} + \varepsilon^{m-1} \sum_{p \in S} u(p) \int_X \rho_p \wedge \xi_p^{m-1} (m-1)! + \varepsilon^m \sum_{p \in S} \int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge \frac{(\xi_p + u_p)^m}{m!}.
\] (17)
Finally consider the last summand, which can be rewritten in the form
\[
\int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge \frac{(\xi_p + u_p)^m}{m!} = \int_{M'} \pi^* (\rho - \Delta u) \wedge \frac{(\xi_p + u_p)^m}{m!}
+ \int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon - \pi^* (\rho - \Delta u)) \wedge \frac{(\xi_p + u_p)^m}{m!}.
\] (18)
The first summand of the right hand side can be trated once again by the Claim above and the Stokes’ Theorem. Indeed the differential form \( \alpha = (\pi^* (\rho - \Delta u) + \Delta u(p)) \wedge (\xi_p + u_p)^{m-1} \) on \( M' \) satisfies the hypothesis of the Claim, whence arguing as above yields
\[
\int_{M'} \pi^* (\rho - \Delta u) \wedge \frac{(\xi_p + u_p)^m}{m!} = -\Delta u(p) \int_X \frac{\xi_p^m}{m!}.
\]
On the other hand, the second summand of (18) is \( O(\varepsilon) \), as follows by (4). As a consequence, (18) reduces to \( \int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge (\xi_p + u_p)^m = -\Delta u(p) \int_X \frac{\xi_p^m}{m!} + O(\varepsilon) \). Substituting this into (17) yields
\[
\int_{M'} (\rho_\varepsilon - \Delta_\varepsilon u_\varepsilon) \wedge \frac{(\omega_\varepsilon + u_\varepsilon)^m}{m!} = \int_M (\rho - \Delta u) \wedge \frac{(\omega + u)^m}{m!}
+ \varepsilon^{m-1} \sum_{p \in S} u(p) \int_X \frac{\rho_p \wedge \xi_p^{m-1}}{(m-1)!} - \varepsilon^m \sum_{p \in S} \Delta u(p) \int_X \frac{\xi_p^m}{m!} + O(\varepsilon^{m+1}).
\] (19)
Finally the thesis follows by plugging this and (14) into (3).
3 ALE resolutions

In this section we introduce Kähler metrics on local models having behavior at infinity suitable for applications in next section.

Let $\Gamma$ be a finite subgroup of the unitary group $U(m)$ and suppose that $\Gamma$ acts freely on the complement of $0 \in \mathbb{C}^m$. The quotient $(\mathbb{C}^m \setminus \{0\})/\Gamma$ is therefore a complex manifold and the Euclidean metric on $\mathbb{C}^m$ descend to a Kähler metric $\eta_0$ on the quotient. Such a Kähler metric serves as a model at infinity for ALE resolutions.

By ALE resolution (of $\mathbb{C}^m/\Gamma$) we mean a non-compact complex manifold $X$ equipped with a complete Kähler metric $\eta$ satisfying the following requirements:

- There exists a finite subgroup $\Gamma \subset U(m)$ acting freely on $\mathbb{C}^m \setminus \{0\}$ and a proper birational morphism $\pi: X \to \mathbb{C}^m/\Gamma$ which restricts to a biholomorphism of $X \setminus \pi^{-1}(0)$ onto $(\mathbb{C}^m \setminus \{0\})/\Gamma$.
- The metric $\eta$ approximates smoothly the model metric $\eta_0$ at infinity. More specifically, for all integers $k \geq 0$ one has
  \[
  \nabla^k(\pi^*\eta - \eta_0) = O(|z|^{2-2m-k}) \quad \text{as } |z| \to \infty,
  \]
  where $\nabla$ denotes the Euclidean connection.

In particular, an ALE resolution turns out to be an ALE Kähler manifold subject to a couple of more restricting requirements. Firstly, here we assume that $\pi$ is a biholomorphism at all smooth points of the quotient $\mathbb{C}^m/\Gamma$ (hence the name resolution), whereas an ALE Kähler manifold (with one end) is just required to contain a compact subset $K$ whose complement is biholomorphic to the complement of a ball centered at the origin in $\mathbb{C}^m/\Gamma$. Secondly, ALE Kähler metrics are commonly allowed to have quite permissive fall-off order at infinity. However, in second point above the order $2 - 2m$ is chosen in accordance with the well-known decay of scalar flat Kähler metrics on $\mathbb{C}^m/\Gamma$ [3, Lemma 7.2], being metrics of that kind our main interest for applications.

On ALE resolutions one can develop Hodge theory as for compact Kähler manifolds [11 pp. 182-186]. The upshot is that any cohomology class in $H^{1,1}(X)$ can be represented by a closed compactly supported $(1,1)$-form on $X$ [11 Theorem 8.4.3]. Moreover one can suppose that

\[
\eta = \xi + dd^c\varphi
\]

for some $(1,1)$-form $\xi$ compactly supported around the exceptional locus $\pi^{-1}(0)$ and some smooth real function $\varphi$ on $X$ [11 Theorem 8.4.4]. Clearly $\xi$ and $\varphi$ are not uniquely defined. In particular $\varphi$ can be added by a function in the kernel of $dd^c$ operator. However we rule out this indeterminacy by requiring that

$\varphi - |z|^2/4 = O(|z|^{4-2m})$

for large $z$. Therefore by (21) the derivative $\nabla^k(\varphi - |z|^2/4)$ must be $O(|z|^{4-2m-k})$ for all $k \geq 0$.

Finally, here we recall an elementary result that will be useful in the following.

\textbf{Lemma 3.1.} Let $B(R) \subset \mathbb{C}^m$ be the ball centered at zero of radius $R$. One has

\[
\int_{\partial B(R)} d^c|z|^2 \wedge (dd^c|z|^2)^{m-1} = (4\pi)^m R^{2m}.
\]
Proposition 3.2. For \( k \) for all integer curvature of some subsets of \( X \) formula for the volume and the total scalar curvature of \( \pi \) so that the point of definition of ALE resolution. Moreover, note that the second summand of \( 4 \) whence the thesis follows by \( \int_{B(R)} \Omega_E = \text{vol}(S^{2m-1}) \int_0^R t^{2m-1} dt = \frac{2\pi^m}{1(m)2^m} R^{2m} \).

3.1 Asymptotic formulae for volume and total scalar curvature

Let \( m > 1 \) be an integer and consider the real function \( f \) on \((0, +\infty)\) defined by

\[
f(t) = \frac{1}{4} t + \frac{e - t^{2-m}}{2 - m} + ct^{1-m}
\]

for some real constants \( e, c \). In this section we consider an \( m \)-dimensional ALE resolution whose Kähler metric is of the form \( \eta = \xi + dd^c \varphi \) with \( \xi \) compactly supported around \( \pi^{-1}(0) \) and \( \varphi \) satisfying

\[
\nabla^k \varphi = \nabla^k f(|z|^2) + O(|z|^{2m-k}) \quad \text{as} \quad |z| \to +\infty
\]

for all integer \( k \geq 0 \). This ensures that \( \eta \) satisfies the fall-off requirement (20) of the third point of definition of ALE resolution. Moreover, note that the second summand of \( f \) is chosen so that \( f \) depends smoothly (in fact analytically) on the dimension \( m \) and for \( m = 2 \) one has \( f(t) = t/4 - e \log(t) + ct^{-1} \).

We shall compute, at least up to some controlled error, the volume and the total scalar curvature of some subsets of \( X \) with respect to \( \eta \). More specifically we shall give an asymptotic formula for the volume and the total scalar curvature of \( \pi^{-1}(B(R)/\Gamma) \) for large \( R \).

Proposition 3.2. For \( R \to +\infty \) one has

\[
\int_{\pi^{-1}(B(R)/\Gamma)} \eta^m = \frac{\pi^m}{m! |\Gamma|} R^{2m} - \frac{4\pi^m e}{(m - 1)! |\Gamma|} R^2 - \frac{4\pi^m (c - e^2 R^{1-2m})}{(m - 2)! |\Gamma|} + \int_X \xi^m + O(R^{-1})
\]

(Note that the term of order \( R^{1-2m} \) is not infinitesimal just in dimension \( m = 2 \)).

Proof. Since \( \eta = \xi + dd^c \varphi \), the volume form of \( \eta \) is given by

\[
\eta^m = \xi^m + d \left( \sum_{\ell=1}^m \binom{m}{\ell} \xi^{m-\ell} \wedge dd^c \varphi \wedge (dd^c \varphi)^{\ell-1} \right).
\]

Recall that \( \xi \) is supported in \( K \), which in turns is compactly contained in \( \pi^{-1}(B(R)/\Gamma) \). Therefore, integrating formula above and applying Stokes’ theorem gives

\[
\int_{\pi^{-1}(B(R)/\Gamma)} \eta^m = \int_X \xi^m + \int_{\partial B(R)/\Gamma} dd^c \varphi \wedge (dd^c \varphi)^{m-1}.
\]

Equation (23) yields \( dd^c \varphi = df + O(|z|^{-2m-1}) \) and \( dd^c \varphi = df + O(|z|^{-2m-2}) \), whence by easy calculations one gets

\[
d^c \varphi \wedge (dd^c \varphi)^{m-1} = (f')^m d^c |z|^2 \wedge (dd^c |z|^2)^{m-1} + O(|z|^{-2m-1}).
\]
Integrating over $\partial B(R)/\Gamma$ and applying lemma 3.1 gives
\[
\int_{\partial B(R)/\Gamma} d^c \varphi \wedge (dd^c \varphi)^{m-1} = \left(4\pi m R^{2m} f'(R^2)^m \right) + O(R^{-1}). \tag{25}
\]
By (22) one calculates $f'(t)^m = 4^{-m} \left(1 - 4m e^{t/m} - 4m(1-2m) + O(t^{1-2m})\right)$, whence the thesis follows after substituting in (25) and the result in (24). \hfill \Box

We now aim to determine an asymptotic formula for the total scalar curvature of $\pi^{-1}(B(R)/\Gamma)$ as $R$ grows. Our main interest is in Corollary 3.4, which also follows by [10, Theorem C] and the classical fact (see for example [12]) that $e$ is, up to a positive normalization constant depending on the dimension $m$ and the order of $\Gamma$, the ADM mass of the ALE metric associated with $\eta$. Nevertheless we include a complete, direct proof for the reader convenience.

**Proposition 3.3.** Let $s$ be the scalar curvature of $\eta$. For $R \to +\infty$ one has
\[
\int_{\pi^{-1}(B(R)/\Gamma)} s \eta^m = m \rho \wedge \eta^{m-1}. \tag{26}
\]
Proof. The Ricci form $\rho$ of $\eta$ satisfies $s \eta^m = m \rho \wedge \eta^{m-1}$. Thanks to (21) one has
\[
\eta^{m-1} = \xi^{m-1} + d \left( \sum_{\ell=1}^{m-1} \binom{m-1}{\ell} \xi^{m-\ell} \wedge d^c \varphi \wedge (dd^c \varphi)^{\ell-1} \right).
\]
Since $\xi$ is supported in $K$, which in turns is compactly contained in $\pi^{-1}(B(R)/\Gamma)$, integrating $m \rho \wedge \eta^{m-1}$ and applying Stokes’ theorem gives
\[
\int_{\pi^{-1}(B(R)/\Gamma)} s \eta^m = \int_K m \rho \wedge \xi^{m-1} + \int_{\partial B(R)/\Gamma} m \rho \wedge d^c \varphi \wedge (dd^c \varphi)^{m-2}. \tag{26}
\]
The fact that $\xi$ is supported in $K$, also implies that $\eta = dd^c \varphi$ in a neighborhood $U$ of $\partial B(R)/\Gamma$. On the other hand the two-form $\tilde{\eta} = dd^c f(|z|^2)$ defines a Kähler metric on $U$ at least if $R$ is sufficiently large. Of course large $R$ has to be $R$ depends on the value of the constants $c$ and $e$. However, we are interested in large $R$ asymptotic, therefore we can suppose that $\tilde{\eta}$ is Kähler with no loss of generality. Let $\tilde{\rho}$ be the Ricci form of $\tilde{\eta}$, so that on $U$ one has
\[
\rho = \tilde{\rho} - dd^c \log((dd^c \varphi)^m / (dd^c f)^m).
\]
By (23) derivatives of $\varphi$ equals derivatives of $f(|z|^2)$ up to a controlled error. In particular one has
\[
dd^c \log((dd^c \varphi)^m / (dd^c f)^m) = O(|z|^{-2m-2}).
\]
On the other hand, since $\tilde{\eta} = f' dd^c |z|^2 + f'' d|z|^2 \wedge d^c |z|^2$ one can readily calculate
\[
\tilde{\rho} = -dd^c \log \left[ (f')^{m-1} \left( f' + |z|^2 f'' \right) \right].
\]
The second summand of (26) is then given by
\[
\int_{\partial B(R)/\Gamma} m \rho \wedge d^c \varphi \wedge (dd^c \varphi)^{m-2} = \int_{\partial B(R)/\Gamma} md^c \log \left[ (f')^{m-1} \left( f' + |z|^2 f'' \right) \right] \wedge (dd^c \varphi)^{m-1} + O(R^{-4}).
\]
Again by (23) one has $dd^c\varphi = dd^c f + O(|z|^{-2m-2})$ so that

$$\int_{\partial B(R)/\Gamma} m\rho \wedge d^c\varphi \wedge (dd^c\varphi)^{m-2} = \int_{\partial B(R)/\Gamma} m\rho \wedge d^c \log [(f')^{m-1} (f' + |z|^2 f'')] \wedge (dd^c f)^{m-1} + O(R^{-2}). \quad (27)$$

Since $f$ depends only on $|z|^2$, one has

$$d^c \log [(f')^{m-1} (f' + |z|^2 f'')] \wedge (dd^c f)^{m-1} = h(|z|^2) d^c |z|^2 \wedge (dd^c |z|^2)^{m-1} \quad (28)$$

where $h = [(f')^{m-1} (f' + t f'')] / (f' + t f'')$. By definition of $f$ and elementary computations one gets

$$f'(t)^{m-1} = 4^{1-m} (1 - 4(m-1)ct^{1-m} - 4(m-1)^2ct^{-m}) + O(t^{2-2m})$$

where the error term vanishes in dimension $m = 2$. Moreover one calculates

$$f'(t) + tf''(t) = \frac{1}{4} + (m-2)ct^{1-m} + (m-1)^2ct^{-m}, \quad (29)$$

so that

$$[f'(t)^{m-1}(f'(t) + t f'')(t)]' = 4^{1-m}(1 - 4ct^{1-m}) + O(t^{1-2m}). \quad (30)$$

Dividing (30) by (29) one finally gets the following expansion

$$h(t) = 4^{2-m}(m-1)ct^{-m} + O(t^{1-2m}).$$

Therefore substituting (29) into (27) and applying lemma 3.1 yields

$$\int_{\partial B(R)/\Gamma} m\rho \wedge d^c\varphi \wedge (dd^c\varphi)^{m-2} = \frac{m(m-1)(4\pi)^m e}{|\Gamma|} + O(R^{-2}).$$

whence the thesis follows after substituting in (26).

A straightforward consequence of Proposition 3.3 is that the scalar curvature is integrable on $X$. More specifically it holds the following

**Corollary 3.4 ([10] Theorem C]).** The total scalar curvature of $\eta$ is given by

$$\int_X s^{m-1} \frac{\eta^m}{m!} = \int_X \rho \wedge \xi^{m-1} / (m-1)! + \frac{16\pi^m e}{(m-2)!|\Gamma|}.\]
\[ \eta_p = \xi_p + dd^c \varphi_p \] representing the fixed Kähler class on the model resolution \( X_p \) of the singular point \( p \). In order to get an explicit expression, write \( \varphi_p \) as

\[ \varphi_p = \frac{1}{4} |z|^2 + e_p \frac{1 - |z|^{4-2m}}{2 - m} + c_p |z|^{2-2m} + O(|z|^{-2m}), \]

for some real constants \( e_p \) and \( c_p \) as \( |z| \to +\infty \). Thanks to Corollary 3.4 then one has

\[ \int_{X_p} \frac{\rho_p \wedge c_p^{-1}}{(m-1)!} = \int_{X_p} \frac{\eta_p^m}{m!} - \frac{16\pi^m c_p}{(m-2)\|\Gamma_p\|}, \]

where \( s_p \) is the scalar curvature of \( \eta_p \). Moreover, Proposition 3.2 yields

\[ \int_{X_p} \frac{c_p^m}{m!} = \frac{4\pi^m c_p}{(m-2)\|\Gamma_p\|} + \lim_{R \to +\infty} \int_{R^{-1}(B(R)/\Gamma_p)} \frac{\eta_p^m}{m!} - \frac{\pi^m}{m\|\Gamma_p\|} \left( R^{2m} - 4me_p R^2 + 8m(m-1)c_p^2 R^{4-2m} \right). \quad (31) \]

Note that formula above reduces drastically whenever \( \eta_p \) is Ricci-flat. In this case, the volume form of \( \eta_p \) equals the euclidean volume form [1, Proposition 2.4] and moreover \( e_p = 0 \), hence

\[ \int_{X_p} \frac{c_p^m}{m!} = \frac{4\pi^m c_p}{(m-2)\|\Gamma_p\|}. \quad (32) \]

### 4 Constant scalar curvature Kähler resolutions

In this section we prove the existence and non-existence results for constant scalar curvature Kähler metrics on resolutions of Kähler orbifolds under suitable hypotheses.

The set-up is similar to that of the previous sections. In particular, we assume that \( \pi : M' \to M \) is a resolution of a \( m \)-dimensional orbifold \( M \) having finite singular set \( S \subset M \). More precisely, around each singular point \( p \in S \), the map \( \pi \) is equal to a resolution \( \pi_p : X_p \to C^n / \Gamma_p \) restricted to some neighborhood of the exceptional set \( \pi_p^{-1}(0) \). Moreover we assume given a Kähler metric \( \omega \) on \( M \) with constant scalar curvature \( s \) and we denote by \( \mu : M \to \mathbb{g}^* \) a moment map for the action of the group of holomorphic Hamiltonian diffeomorphisms of \((M, \omega)\), normalized so that \( \int_M \mu \omega^n = 0 \). We also assume given for each singular point \( p \in S \) a scalar-flat ALE metric \( \eta_p \) on the model resolution \( X_p \) of the form \( \eta_p = \xi_p + dd^c \varphi_p \), with \( \xi_p \) compactly supported around the exceptional set of \( \pi_p \).

Note that by Corollary 3.4 and Proposition 3.3 the scalar-flatness hypothesis of \( \eta_p \) affects the expansion of the ALE Kähler potential \( \varphi_p \) for large \( z \). More precisely, it gives a cohomological formula for the ADM mass of \( \eta_p \). In particular one has

\[ \varphi_p = \frac{1}{4} |z|^2 + e_p \frac{1 - |z|^{4-2m}}{2 - m} + c_p |z|^{2-2m} + O(|z|^{-2m}) \]

with ADM mass \( e_p = \frac{1}{\|\Gamma_p\|} \int_{X_p} \rho_p \wedge c_p^{-1} \). Finally, consider for each \( p \in S \) the positive constant \( a_p = \frac{1}{\|\Gamma_p\|} \int_{X_p} \xi_p^m \). Note that \( a_p \) is related to \( c_p \) and \( e_p \) by formula (31) and that, by discussion after that formula, one has \( a_p = \frac{e_p}{4\|\Gamma_p\|} \) whenever \( \eta_p \) is Ricci-flat.

Recalling that the Futaki invariant is an obstruction for the existence of constant scalar curvature Kähler metrics, the asymptotic formula for the Futaki invariant of Theorem 2.3 and discussion above on the ADM mass readily give the following non-existence result
Theorem 4.1. If one of the following conditions holds
\[ \sum_{p \in S} \frac{e_p}{|\Gamma_p|} \mu(p) \neq 0, \]
\[ \sum_{p \in S} a_p(s\mu + \Delta\mu)(p) \neq 0, \]
then for all \( \varepsilon > 0 \) sufficiently small the Kähler class \( \pi^*[\omega] + \varepsilon \sum_{p \in S} [\xi_p] \in H^{1,1}(M') \) contains no constant scalar curvature Kähler metrics.

The starting point for our existence results is the following theorem on existence of extremal Kähler metrics on resolutions [2]:

Theorem 4.2. With the notation above, assume that \( \omega \) is an extremal Kähler metric on the orbifold \( M \). Then for all \( \varepsilon > 0 \) sufficiently small there is an extremal Kähler metric on \( M' \) which in the Kähler class \( \pi^*[\omega] + \varepsilon \sum_{p \in S} [\xi_p] \in H^{1,1}(M') \).

At this point we are in position to state and prove our main existence results.

Theorem 4.3. Let \( e_p \) be the ADM mass of \( \eta_p \), and let \( Q \subset S \) be the subset of singular points with non-zero ADM mass. If \( Q \) is non-empty and
\[ \sum_{q \in Q} \frac{e_q}{|\Gamma_q|} \mu(q) = 0 \quad \text{and} \quad \text{span}\{\mu(q) \mid q \in Q\} = \mathfrak{g}^* \]
then, after identifying each \( \xi_p \) with a \((1,1)\)-form on \( M' \) as above, for all \( \varepsilon > 0 \) sufficiently small there exists \( \lambda_q(\varepsilon) > 0 \) and a constant scalar curvature Kähler metric \( \omega'_\varepsilon \) such that
\[ [\omega'_\varepsilon] = \pi^*[\omega] + \sum_{q \in Q} \lambda_q(\varepsilon)[\xi_q] + \varepsilon \sum_{p \in S \setminus Q} [\xi_p] \in H^{1,1}(M') \]  \hspace{1cm} (33)
and \( \lambda_q(\varepsilon) \sim \varepsilon \) as \( \varepsilon \) tends to 0.

An analytic proof of this result has been given in [2]. We skip the new proof being similar and simpler to the one of the next result.

Theorem 4.4. Suppose that \( e_p = 0 \) for all \( p \in S \). If
\[ \sum_{p \in S} a_p(s\mu + \Delta\mu)(p) = 0 \quad \text{and} \quad \text{span}\{(s\mu + \Delta\mu)(p) \mid p \in S\} = \mathfrak{g}^* \]  \hspace{1cm} (34)
then, after identifying each \( \xi_p \) with a \((1,1)\)-form on \( M' \) as above, for all \( \varepsilon > 0 \) sufficiently small there exists \( \lambda_p(\varepsilon) > 0 \) and a constant scalar curvature Kähler metric \( \omega'_\varepsilon \) such that
\[ [\omega'_\varepsilon] = \pi^*[\omega] + \sum_{p \in S} \lambda_p(\varepsilon)[\xi_p] \in H^{1,1}(M') \]  \hspace{1cm} (35)
and \( \lambda_p(\varepsilon) \sim \varepsilon \) as \( \varepsilon \) tends to 0.

This result extends an analogue one in [1], proved under the additional assumption of Ricci-flatness of the local resolutions, in which case \( a_p = \frac{c_p}{8\pi g_1} \), as remarked above.
Proof. For each $p \in S$ and for all real $t_p$ such that $|t_p| < 1$, take $\varepsilon > 0$ sufficiently small and consider on $M'$ the Kähler metric

$$\omega_{t,\varepsilon} = \pi^* \omega + \varepsilon \sum_{p \in S} (1 + t_p)^{1/m} \xi_p.$$  

Note that this metric is invariant with respect to any holomorphic vector field with zeroes $V$ on $M'$ since $\pi^* \omega$ and each $\xi_p$ are. Moreover note that any such vector field is Hamiltonian with respect to $\omega_{t,\varepsilon}$. In particular, by theorem [2,3] one has

$$\text{Fut}(V, \omega_{t,\varepsilon}) = -\frac{16 \pi^m \varepsilon^m}{(m-2)!} \sum_{p \in S} (1 + t_p) a_p (s(u(p) - u) + \Delta u(p)) + O(\varepsilon^{m+1}),$$  

(36)

where we used the hypothesis that $\omega$ has constant scalar curvature (hence vanishing Futaki invariant), for each $p \in S$ the model metric $\eta_p$ is Ricci flat (hence $\rho_p = 0$), and the scalar curvature $s$ of $\omega$ is constant. On the other hand, note that by general theory of Futaki invariant, $\text{Fut}(V, \omega_{t,\varepsilon})$ depends polynomially on the cohomology class of $\omega_{t,\varepsilon}$, hence on $\varepsilon$ and $(1 + t_p)^{1/m}$. Finally, observe that the normalized Hamiltonian potential $u - u$ of $\pi_* \omega$ is equal to $\langle \mu, V \rangle$. Therefore, letting

$$F(t, \varepsilon)(V) = -\frac{(m-2)!}{16 \pi^m \varepsilon^m} \text{Fut}(V, \omega_{t,\varepsilon})$$

defines a smooth function $F$ on $(-1,1)^{|S|} \times \mathbb{R}$ with values in $g^*$, being $|S|$ the cardinality of the singular set $S$. By (36), for small $\varepsilon$ one has

$$F(t, \varepsilon) = \sum_{p \in S} (1 + t_p) a_p (s\mu + \Delta\mu)(p) + O(\varepsilon).$$

Therefore, hypotheses (33) ensure that $F(0,0) = 0$, and the Jacobian $\partial F/\partial t$ at the point $(0,0)$ has rank equal to $\dim g$. As a consequence, by implicit function theorem one can find $\varepsilon_0 > 0$ and a smooth function $t$ on $(-\varepsilon_0, \varepsilon_0)$ with values to $(-1,1)^{|S|}$ such that $F(t(\varepsilon), \varepsilon) = 0$, and $t(0) = 0$. Clearly there are $|S| - \dim g$ free parameters in doing this, but we don’t need this extra flexibility for our purposes. By discussion above, for all holomorphic vector field with zeroes on $M'$ then one has $\text{Fut}(V, \omega_{t(\varepsilon),\varepsilon}) = 0$ if $0 < \varepsilon < \varepsilon_0$. Therefore, letting $\lambda_p(\varepsilon) = \varepsilon (1 + t(\varepsilon)p)^{1/m}$ yields a family of Kähler classes

$$\pi^* \omega + \sum_{p \in S} \lambda_p(\varepsilon) |\xi_p| \in H^{1,1}(M')$$  

(37)

with vanishing Futaki invariant and approaching the class $|\pi^* \omega|$ as $\varepsilon \to 0$.

By the elementary remark that extremal Kähler metrics with vanishing Futaki invariant have constant scalar curvature [5, 9], in order to get the thesis we are now reduced to show that classes as in (37) contain an extremal Kähler metric, at least when $\varepsilon$ is sufficiently small. This follows by Theorem 4.2 and openness of the extremal cone [13]. Indeed, by a standard perturbation argument these two theorems imply that under our hypotheses there is a small open ball $H^{1,1}(M')$ centered at $|\pi^* \omega|$ whose intersection $A$ with the Kähler cone of $M'$ is constituted by extremal Kähler classes, i.e. Kähler classes representable by extremal Kähler metrics. Since classes as in (37) are contained in $A$ for $\varepsilon$ sufficiently small, it follows that all these classes contain extremal Kähler metrics.

Comparing formulae for Kähler classes (33) and (35) somehow suggests that one can still get cscK metrics in adiabatic classes of $M'$ in cases not covered by theorems above. This can be
done by choosing different scaling volumes with respect to $\varepsilon$ to exceptional divisors, according they project via $\pi$ to a singular point with zero or non-zero ADM mass. More precisely one has the following result which is not covered by previous works.

**Theorem 4.5.** Let $Q \subset S$ be the subset constituted by those singular points of $M$ which have non-zero ADM mass and let $P$ its complement. If $P$ and $Q$ are both not empty and

$$\sum_{q \in Q} \frac{e_q}{\Gamma_q} \mu(q) + \sum_{p \in P} a_p(s\mu + \Delta\mu)(p) = 0 \quad \text{and} \quad \text{span}\{\mu(q), (s\mu + \Delta\mu)(p) | q \in Q, p \in P\} = \mathfrak{g}^*$$

then, after identifying each $\xi_p$ with a $(1,1)$-form on $M'$ as above, for all $\varepsilon > 0$ sufficiently small there exist $\lambda_p(\varepsilon) > 0$ and a constant scalar curvature Kähler metric $\omega'_{\varepsilon}$ such that

$$[\omega'_{\varepsilon}] = \pi^*[\omega] + \sum_{q \in Q} \lambda_q(\varepsilon)[\xi_q] + \sum_{p \in P} \lambda_p(\varepsilon)[\xi_p] \in H^{1,1}(M').$$

Moreover, as $\varepsilon$ tends to 0, one has $\lambda_q(\varepsilon) \sim \varepsilon$ for all $q \in Q$ and $\lambda_p(\varepsilon) \sim \varepsilon^\frac{m-1}{m}$ for all $p \in P$.

**Proof.** The statement follows exactly from the same line of arguments as in the proof of theorem 4.4 once one starts with the Kähler metric on $M'$ defined by

$$\omega_{t,\varepsilon} = \pi^*\omega + \varepsilon \sum_{q \in Q} (1 + t_q)\frac{1}{\Gamma_q} \xi_q + \varepsilon^\frac{m-1}{m} \sum_{p \in P} (1 + t_p)^\frac{1}{m} \xi_p$$

with $t_p, t_q \in \mathbb{R}$ such that $|t_p|, |t_q| < 1$, and $\varepsilon > 0$ sufficiently small. \qed

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