The Gell-Mann-Okubo and Coleman-Glashow relations for octet and decuplet baryons
in the $SU_q(3)$ quantum algebra

Antonio E Cárcamo Hernández

Departamento de Física, Universidad Nacional de Colombia, Bogotá, Colombia

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The $q$-deformed Clebsch-Gordan coefficients corresponding to the $\{3\} \times \{21\}$ reduction of the $SU_q(3)$ quantum algebra are computed. From these results and using the quantum Clebsch-Gordan coefficients for the $\{21\} \times \{21\}$ reduction found by Z.Q.Ma, the $q$-deformed Gell-Mann-Okubo mass relations for octet and decuplet baryons are determined by generalizing the procedure used for the $SU(3)$ algebra. We also determine the Coleman-Glashow relations for octet and decuplet baryons in the $SU_q(3)$ algebra. Finally, by using the experimental particle masses of the octet and decuplet baryons, two values of the $q$-parameter are found and adjusted for the predicted expressions of the masses (one for the Gell-Mann-Okubo mass relations and the other for the Coleman-Glashow relations) and a possible physical interpretation is given.

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I. INTRODUCTION

The $q$-deformed algebras extend the domain of classical group theory and constitute a new and growing field of mathematics with vast potential for applications in physics. The quantum algebra $SU_q(3)$, as a generalization of the $SU(3)$ algebra, has been studied by several authors with several applications in particle physics, conformal field theory, statistical mechanics, quantum optics, condensed matter, molecular, atomic and nuclear spectroscopy. In these applications either an existing model is identified with a quantum algebraic structure or a standard model is deformed to show a underlying quantum algebraic structure which reveals new features.

Applications of quantum algebras in particle physics have been explored in several works. In hadronic phenomenology $q$-deformed mass relations between particle families in the octet and decuplet baryons have been determined from the computation of the mass operator’s expectation value. In these works, the mass operator has been defined in terms of generators of $SU_q(4)$ and $SU_q(5)$ and its expectation value has been computed from the determination of their matrix elements.

Our aim in this article is to determine the $q$-deformed Gell-Mann-Okubo and Coleman-Glashow relations for octet and decuplet baryons in the $SU_q(3)$ quantum algebra. First, we develop the $SU_q(3)$ quantum algebra and use it to compute the $q$-deformed Clebsch-Gordan coefficients...
coefficients corresponding to the \{3\} \times \{21\} reduction. Second, to derive the \(q\)-deformed Gell-Mann-Okubo mass relations for octet and decuplet baryons in the \(SU_q(3)\) quantum algebra we generalize the traditional procedure for the \(SU(3)\) algebra [11] and use the previous results together with the quantum Clebsch-Gordan coefficients corresponding to the \{21\} \times \{21\} reduction [7]. After that, we obtain the \(q\)-deformed Coleman-Glashow relations for octet and decuplet baryons by following the same procedure used for the \(SU(3)\) algebra [2]. Finally, by using the experimental particle masses of the octet and decuplet baryons, two values of the \(q\)-parameter are found and adjusted for the Gell-Mann-Okubo and Coleman-Glashow mass relations and a possible physical interpretation is given.

II. THE \(SU_q(3)\) QUANTUM ALGEBRA

The quantum algebra \(SU_q(3)\) is generated by the operators \(\tilde{T}, \tilde{h}_1, \tilde{h}_2, \tilde{X}_t^+, \tilde{X}_t^-\), which satisfy the following commutation relations [10]:

\[
[\tilde{h}_i, \tilde{h}_j] = 0, \quad [\tilde{h}_i, \tilde{X}_t^+] = \pm a_{ij} \tilde{X}_t^+ \quad (1)
\]

\[
[\tilde{X}_t^+, \tilde{X}_t^-] = \delta_{ij} \tilde{h}_q \quad i, j = 1, 2 \quad (2)
\]

together with:

\[
\left( \tilde{X}_t^+ \right)^2 \tilde{X}_t^+ + \tilde{X}_t^- \left( \tilde{X}_t^- \right)^2 - [2_q]_q \tilde{X}_t^+ \tilde{X}_t^+ \tilde{X}_t^- = 0 \quad i \neq j \quad (3)
\]

where \(a_{ij}\) is the Cartan matrix given by:

\[
a_{ij} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \quad (4)
\]

Additional generators \(\tilde{X}_3^\pm\) are introduced by [8]:

\[
\tilde{X}_3^+ = q^{1/2} \left[ \tilde{X}_1^+, \tilde{X}_2^+ \right]_{q^{-1}}, \quad \tilde{X}_3^- = -q^{-1/2} \left[ \tilde{X}_2^-, \tilde{X}_1^- \right]_q \quad (5)
\]

where:

\[
\tilde{X}_1^\pm = \tilde{T}_\pm, \quad \tilde{X}_2^\pm = \tilde{U}_\pm, \quad \tilde{h}_1 = 2\tilde{T}_3, \quad (6)
\]

\[
\tilde{h}_2 = -\tilde{T}_3 + \frac{3}{2} \tilde{Y}, \quad [\tilde{A}, \tilde{B}]_q = \tilde{A}\tilde{B} - q\tilde{B}\tilde{A} \quad (7)
\]

From the previous expressions, we obtain:

\[
[\tilde{X}_3^+, \tilde{X}_3^-] = -\frac{q^{\tilde{h}_3} - q^{-\tilde{h}_3}}{q - q^{-1}} = -\left[ \tilde{h}_3 \right]_q, \quad (8)
\]

\[
[\tilde{h}_1, \tilde{X}_3^\pm] = \pm \tilde{X}_3^\pm \quad \text{With} \quad \tilde{h}_3 = \tilde{h}_1 + \tilde{h}_2 = \tilde{T}_3 + \frac{3}{2} \tilde{Y} \quad (9)
\]

According to the standard coproduct definition at \(SU_q(3)\), the following expressions are obtained [2]:

\[
\Delta \tilde{T}_\pm = \tilde{T}_\pm \otimes q^{\tilde{T}_3} + q^{-\tilde{T}_3} \otimes \tilde{T}_\pm, \quad (10)
\]

\[
\Delta \tilde{U}_\pm = \tilde{U}_\pm \otimes q^{(3\tilde{Y} - 2\tilde{T}_3)/4} + q^{-(3\tilde{Y} - 2\tilde{T}_3)/4} \otimes \tilde{U}_\pm \quad (11)
\]

Hence, the coproduct of two \(SU_q(3)\) irreps is defined as [2]:

\[
\tilde{T}_\pm \left( \psi_\eta^{(\lambda_1, \lambda_2)} \psi_\beta^{(\mu_1, \mu_2)} \right) = \left( q^{\tilde{T}_3} \psi_\eta^{(\lambda_1, \lambda_2)} \right) \left( \tilde{T}_\pm \psi_\beta^{(\mu_1, \mu_2)} \right) + \left( \tilde{T}_\pm \psi_\eta^{(\lambda_1, \lambda_2)} \right) \left( q^{\tilde{T}_3} \psi_\beta^{(\mu_1, \mu_2)} \right)
\]

\[
\tilde{U}_\pm \left( \psi_\eta^{(\lambda_1, \lambda_2)} \psi_\beta^{(\mu_1, \mu_2)} \right) = \left( \tilde{U}_\pm \psi_\eta^{(\lambda_1, \lambda_2)} \right) \left( q^{(3\tilde{Y} - 2\tilde{T}_3)/4} \psi_\beta^{(\mu_1, \mu_2)} \right) + \left( q^{-(3\tilde{Y} - 2\tilde{T}_3)/4} \psi_\eta^{(\lambda_1, \lambda_2)} \right) \left( \tilde{U}_\pm \psi_\beta^{(\mu_1, \mu_2)} \right)
\]

where \(\psi_\alpha^{(\nu)}\), with \(\alpha = 1, 2, \cdots \dim \{\nu\}\), is the state with eigenvalues \(t_\alpha, t_{3\alpha}\) and \(y_\alpha\), which belongs to the \(SU_q(3)\) representation \(\{\nu\} = \{\nu_1, \nu_2\} \) [s]

[s] In this article, we consider \(q \in R\) following the Quesne’s prescription.
III. THE $q$-DEFORMED GELL-MANN-OKUBO MASS RELATIONS FOR OCTET AND DECUPLET BARYONS

In order to get expressions of the masses corresponding to the octet and decuplet baryons, we introduce the following mass operator:

$$\hat{M} = \hat{M}_S + \hat{M}_T$$

(12)

where $\hat{M}_S$ is a $SU_q(3)$ invariant and $\hat{M}_T$ an isospin scalar.

By computing the expectation value of the mass operator in a $|\psi^{(\lambda)}_\alpha\rangle$ state belonging to a $\{\lambda\}$ representation, we obtain that the mass of the particle corresponding to this state is given by:

$$m(\{\lambda\}, \{\mu\}) = \langle\psi^{(\lambda)}_\alpha|\hat{M}_S|\psi^{(\lambda)}_\alpha\rangle + \langle\psi^{(\lambda)}_\alpha|\hat{M}_T|\psi^{(\lambda)}_\alpha\rangle$$

(13)

with:

$$|\psi^{(\lambda)}_\alpha\rangle = |\{\lambda\}, \{\mu\}\rangle = |\{\lambda\}, y, t, t_z\rangle$$

$$\{\mu\} = \{y, t, t_z\}$$

(14)

where in order to simplify the notation we have taken $y = y_\alpha$, $t = t_\alpha$ and $t_z = t_{3\alpha}$ being $y$, $t$ and $t_z$ the hypercharge, the isospin and the $z$ isospin component respectively.

As $\hat{M}_S$ is a $SU_q(3)$ scalar, its expectation value is the same for all members of the multiplet corresponding to a $\{\lambda\}$ representation. Therefore, according to the previous, we obtain:

$$\langle\psi^{(\lambda)}_\alpha|\hat{M}_S|\psi^{(\lambda)}_\alpha\rangle = m_S(\{\lambda\})$$

(15)

We also note that the $\hat{M}_T$ operator can be written as an expansion of irreducible tensor operators of $SU_q(3)$, according to:

$$\hat{M}_T = \sum_{\mu\nu} \hat{T}_\mu^{(\nu)}$$

(16)

Then, by taking into account that $[\hat{M}_T, \hat{Y}] = [\hat{M}_T, \hat{T}] = 0$, from the previous expression we obtain that the irreducible tensor operators $\hat{T}_\mu^{(\lambda)}$ should satisfy the following conditions:

$$[\hat{T}_\mu^{(\nu)}, \hat{Y}] = 0, \quad [\hat{T}_\mu^{(\nu)}, \hat{T}] = 0$$

(17)

On the other hand, according to the Wigner-Eckart theorem for the quantum group $SU_q(3)$, we have that the matrix element of the irreducible tensor operator $\hat{T}_\mu^{(\lambda_2)}$ is given by [12]:

$$\langle\{\lambda_3\}, y_3, t_3, t_{3z}|\hat{T}_\mu^{(\lambda_2)}|\{\lambda_1\}, y_1, t_1, t_{1z}\rangle = \sum_{\gamma} \left(\begin{array}{ccc} \lambda_1 & \lambda_2 & \lambda_{3\gamma} \\ \mu_1 & \mu_2 & \mu_3 \end{array}\right)_q \langle\{\lambda_3\}|\hat{T}^{(\lambda_2)}|\{\lambda_1\}\rangle_\gamma$$

(18)

where the reduced matrix element $\langle\{\lambda_3\}|\hat{T}^{(\lambda_2)}|\{\lambda_1\}\rangle_\gamma$ only depends on the represen-
tations involved and the sum is performed over \( \gamma \), being \( \gamma \) the index which labels the copies of the \( \{ \lambda_3 \} \) representation in the \( \{ \lambda_1 \} \otimes \{ \lambda_2 \} \) reduction. We also have that the coupling factor in the expression \( (18) \) corresponds to the \( q \)-deformed Clebsch-Gordan Coefficient of \( SU_q(3) \) [12].

Moreover, in the \( |\psi_\alpha^{(\lambda)}\rangle = |\{ \lambda \}, y, t, t_z \rangle \) basis, we have:

\[
\langle \{ \lambda \}, y, t, t_z | y^t t_z^t | \hat{T}_\mu^{(\nu)} | \{ \lambda \} \rangle = \delta_\mu^0 \delta_\nu^0 \delta_{t_z}^t \delta_{t}^y \langle \{ \lambda \} | \hat{T}_\mu^{(\nu)} | \{ \lambda \} \rangle
\]

(19)

Therefore, by comparing the previous result with the corresponding to the \( q \)-deformed Wigner-Eckart theorem, we obtain that \( \mu \) should be \( (y t t_z) = (000) \).

Then, according to the previous we have that the \( \hat{M}_T \) operator is given by:

\[
\hat{M}_T = \sum_\nu \hat{T}_{000}^{(\nu)}
\]

(20)

where the sum is performed over all physically allowed \( SU_q(3) \) representations. Hence, we have \( \{ \nu \} = \{ 0 \}, \{ 21 \}, \{ 42 \}, \ldots \).

Therefore, the particle mass corresponding to a \( |\psi_\alpha^{(\lambda)}\rangle = |\{ \lambda \}, y, t, t_z \rangle \) state of a multiplet belonging to a \( \{ \lambda \} \) representation of \( SU_q(3) \) is given by:

\[
m (\{ \lambda \}, y, t, t_z) = m_S (\{ \lambda \}) + \sum_\nu \langle \{ \lambda \}, y, t, t_z | \hat{T}_{000}^{(\nu)} | \{ \lambda \}, y, t, t_z \rangle
\]

(21)

It is important to point out that the \( \langle \psi_\alpha^{(\lambda)} | \hat{T}_{000}^{(0)} | \psi_\alpha^{(\lambda)} \rangle \) term remains absorbed in the \( m_S (\{ \lambda \}) \) term, which is a \( SU_q(3) \) scalar. We also have that the dominant contribution comes from the \( \hat{T}_{000}^{(21)} \) component of the octet tensor operator. Hence, according to the previous we obtain [2]:

\[
m (\{ \lambda \}, y, t, t_z) = m_S (\{ \lambda \}) + \langle \{ \lambda \}, y, t, t_z | \hat{T}_{000}^{(21)} | \{ \lambda \}, y, t, t_z \rangle
\]

(22)

On the other hand, it is well known that the product of the \( SU_q(3) \) representations \( \{ 21 \} \) and \( \{ 21 \} \) is given by:

\[
\{ 21 \} \times \{ 21 \} = \{ 42 \} + \{ 3 \} + \{ 3^2 \} + \{ 21 \}_S + \{ 21 \}_A + \{ 0 \}
\]

(23)

where \( \{ 42 \}, \{ 3 \}, \{ 3^2 \}, \{ 21 \} \), and \( \{ 0 \} \) are representations with dimensions equal to 27, 10, 10, 8, and 1, respectively. Moreover, \( \{ 21 \}_S \) and \( \{ 21 \}_A \) are symmetric and antisymmetric representations with the same transformation properties under the quantum group \( SU_q(3) \).

We also have that the \( \{ 21 \} \) representation corresponding to the octet baryon exhibits the following decomposition under the subgroup \( U(1)_Y \times SU_q(2)_T \):

\[
\{ 21 \} \downarrow \{ 3 \} \times \{ 1 \} + \{ 0 \} \times \{ 2 \} + \{ 0 \} \times \{ 0 \} + \{ \overline{3} \} \times \{ 1 \}
\]

(24)

where the products \( \{ 3 \} \times \{ 1 \}, \{ 0 \} \times \{ 2 \}, \{ 0 \} \times \{ 0 \}, \) and \( \{ \overline{3} \} \times \{ 1 \} \) represent 2, 3, 1, and 2 states corresponding to the \( N, \Sigma, \Lambda, \) and \( \Xi \) particles of the octet baryon [13].

Then, by applying the \( q \)-deformed Wigner-Eckart theorem to the second term of expression \( (22) \) taking into account that in the \( \{ 21 \} \times \{ 21 \} \) reduction, the \( \{ 3 \} \times \{ 1 \}, \{ 0 \} \times \{ 2 \}, \{ 0 \} \times \{ 0 \}, \) and \( \{ \overline{3} \} \times \{ 1 \} \) states of the first \( \{ 21 \} \) representation are coupled to the \( \{ 0 \} \times \{ 0 \} \) state of the second one \( \{ 21 \} \), where the coupling coefficients are...
the $q$-deformed isoscalar factors, we obtain the following expressions corresponding to the masses of the particle families in the octet baryon

\[
m_N = m_S \{21\} + \frac{\{21\} \{21\} \parallel \{21\}_S}{1} \langle \{21\}\|\{21\}\rangle_q + \frac{\{21\} \{21\} \parallel \{21\}_A}{1} \langle \{21\}\|\{21\}\rangle_A \\
m_S = m_S \{21\} + \frac{\{21\} \{21\} \parallel \{21\}_S}{10} \langle \{21\}\|\{21\}\rangle_q + \frac{\{21\} \{21\} \parallel \{21\}_A}{10} \langle \{21\}\|\{21\}\rangle_A \\
m_A = m_S \{21\} + \frac{\{21\} \{21\} \parallel \{21\}_S}{00} \langle \{21\}\|\{21\}\rangle_q + \frac{\{21\} \{21\} \parallel \{21\}_A}{00} \langle \{21\}\|\{21\}\rangle_A \\
m_2 = m_S \{21\} + \frac{\{21\} \{21\} \parallel \{21\}_S}{1 - \frac{1}{2}} \langle \{21\}\|\{21\}\rangle_q + \frac{\{21\} \{21\} \parallel \{21\}_A}{1 - \frac{1}{2}} \langle \{21\}\|\{21\}\rangle_A
\]

(25)

(26)

(27)

(28)

We also have that according to Racah’s factorization lemma, the $q$-deformed isoscalar factors for the $SU_q(3)$ quantum algebra are given by [14]:

\[
\begin{pmatrix}
\lambda_1 & \lambda_2 & \lambda_3 \\
\mu_1 & \mu_2 & \mu_3 \\
t_1 y_1 & t_2 y_2 & t y
\end{pmatrix}_q = \frac{\lambda_1 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3} C_{1t_1 t_2 t_t}^{(\lambda_1 \times (\lambda_2 \times \lambda_3))} q C_{t_1 t_2 t_t}^{(\lambda_1 \times (\lambda_2 \times \lambda_3))} C_{t_1 t_2 t_t}^{(\lambda_1 \times (\lambda_2 \times \lambda_3))}
\]

(29)

where $q C_{t_1 t_2 t_t}^{(\lambda_1 \times \lambda_2)}$ is the $q$-deformed Clebsch-Gordan coefficient for $SU_q(2)$. For this case, we have [14]:

\[
q C_{\frac{1}{2} \frac{1}{2}}^{(\frac{1}{2})} = q C_{\frac{1}{2} \frac{1}{2}}^{(1)} = q C_{\frac{1}{2} \frac{1}{2}}^{(0)} = q C_{\frac{1}{2} \frac{1}{2}}^{(0)} = 1
\]

(30)

Then, for the octet baryon we get:

\[
\begin{pmatrix}
\{21\} \{21\} \parallel \{21\}_S, A \\
t y & 00 & ty
\end{pmatrix}_q = q C_{(\alpha 5) (\alpha)}^{(21) \times (21)} C_{(\alpha 5) (\alpha)}^{(21) \times (21)} C_{(\alpha 5) (\alpha)}^{(21) \times (21)}
\]

(31)

Therefore, by combining expressions (25)-(28) and using the quantum Clebsch-Gordan coefficients corresponding to the $\{21\} \times \{21\}$ reduction obtained in [2], the following $q$-deformed mass relation for octet baryons is obtained [2]:
\[ [3]_q \sqrt{\frac{[3]_q + [2]_q}{[4]_q + 1}} m_A + m_o (C(q) - E(q) - D(q)) = m_N \left\{ q^{-2}E(q) + \frac{q^2}{2} \left( A(q) + \frac{\left[ \frac{3}{2} \right]_q}{[3]_q \left[ \frac{3}{2} \right]_q} \sqrt{[5]_q} B(q) \right) \right\} + m_e \left\{ q^2E(q) + \frac{q^3}{2} \left( A(q) - \frac{\left[ \frac{3}{2} \right]_q}{[3]_q \left[ \frac{3}{2} \right]_q} \sqrt{[5]_q} B(q) \right) \right\} \] (32)

where \( A(q), B(q), C(q) \) and \( D(q) \) are functions of the \( q \)-real parameter given by:

\[
A(q) = 1 - q^{-1} + q^{-2} + q^{-3} - q^{-4} + q^{-5}, \quad B(q) = 1 + q^{-1} + q^{-2} - q^{-3} - q^{-4} - q^{-5}, \quad E(q) = \sqrt{\frac{[3]_q + [2]_q}{[4]_q + 1}} \]

\[
C(q) = \frac{q^{5/2} \left( [2]_q + 1 \right) (q^{3/2} + q^{-3/2})}{2 \left[ [3]_q \right]} A(q), \quad D(q) = \frac{q^{5/2} \left( [2]_q - 1 \right) (q^{3/2} - q^{-3/2}) \left[ \frac{3}{2} \right]_q}{2 \left( [3]_q \right) \left[ \frac{3}{2} \right]_q} B(q) \] (33)

By replacing the average multiplet masses in the expression and performing a program in C that finds the roots of this expression, we obtain [1]:

\[ q_1 = 0.9870 \pm 0.0002 \] (34)

Throughout this article, the errors of the \( q \)-parameters were obtained by applying the formula:

\[ \Delta q = \sqrt{\sum_{i=1}^{N} \left( \frac{\partial q}{\partial m_i} \right)^2 \left( \Delta m_i \right)^2} \] (35)

\[
\frac{\partial q}{\partial m_A} = \frac{-F(q)}{m_A \frac{dF(q)}{dq} + m_o \frac{dG(q)}{dq} - m_N \frac{dH(q)}{dq} - m_e \frac{dI(q)}{dq}}, \quad \frac{\partial q}{\partial m_N} = \frac{H(q)}{m_A \frac{dF(q)}{dq} + m_o \frac{dG(q)}{dq} - m_N \frac{dH(q)}{dq} - m_e \frac{dI(q)}{dq}} \]

\[
\frac{\partial q}{\partial m_o} = \frac{-G(q)}{m_A \frac{dF(q)}{dq} + m_o \frac{dG(q)}{dq} - m_N \frac{dH(q)}{dq} - m_e \frac{dI(q)}{dq}}, \quad \frac{\partial q}{\partial m_e} = \frac{I(q)}{m_A \frac{dF(q)}{dq} + m_o \frac{dG(q)}{dq} - m_N \frac{dH(q)}{dq} - m_e \frac{dI(q)}{dq}} \]

[1] In this article we choose between the roots of the mass relations which satisfy \( 0 \leq q \leq 1 \).
With the functions $F(q)$, $G(q)$, $H(q)$ and $I(q)$ given by:

$$F(q) = [3]_q \sqrt{[3]_q [2]_q}$$

$$G(q) = C(q) - E(q) - D(q)$$

$$H(q) = q^{-2} E(q) + \frac{q^2}{2} \left( A(q) + \left( \frac{[\frac{3}{2}]_q}{[\frac{3}{2}]_q [\frac{1}{2}]_q [\frac{2}{2}]_q} \right)^2 \sqrt{[5]_q} B(q) \right)$$

$$I(q) = q^2 E(q) + \frac{q^3}{2} \left( A(q) - \left( \frac{[\frac{3}{2}]_q}{[\frac{3}{2}]_q [\frac{1}{2}]_q [\frac{2}{2}]_q} \right)^2 \sqrt{[5]_q} B(q) \right)$$

To determine the $q$-deformed mass relations for the decuplet baryon, we follow the same procedure used for the octet baryon, taking into account that when the $q$-deformed Wigner Eckart theorem is applied to the second term of expression (22), the $\{3\} \times \{3\}$, $\{0\} \times \{2\}$, $\{3\} \times \{1\}$, and $\{0\} \times \{0\}$ states of the $\{3\}$ representation (which represent 4, 3, 2, and 1 states respectively, corresponding to the $\Delta$, $\Sigma$, $\Xi$, and $\Omega$ particles of the decuplet baryon) are coupled to the $\{0\} \times \{0\}$ state of the $\{21\}$ representation, where the coupling coefficients are the $q$-deformed isoscalar factors. Hence, according to the previous, we obtain the following expressions corresponding to the masses of the particle families in the decuplet baryon [2]:

$$m_\Delta = m_S (\{3\}) + \left( \begin{array}{ccc} \{3\} & \{21\} & \{3\} \\ \frac{3}{2} & 1 & 0 \\ 0 & 0 & \frac{1}{2} \end{array} \right)_q \langle \{3\} || \hat{T}^{(21)} || \{3\} \rangle$$

$$m_\Sigma = m_S (\{3\}) + \left( \begin{array}{ccc} \{3\} & \{21\} & \{3\} \\ 10 & 0 & 0 \\ 0 & 0 & 10 \end{array} \right)_q \langle \{3\} || \hat{T}^{(21)} || \{3\} \rangle$$

$$m_\Xi = m_S (\{3\}) + \left( \begin{array}{ccc} \{3\} & \{21\} & \{3\} \\ \frac{1}{2} & -1 & 0 \\ 0 & 0 & \frac{1}{2} \end{array} \right)_q \langle \{3\} || \hat{T}^{(21)} || \{3\} \rangle$$

$$m_\Omega = m_S (\{3\}) + \left( \begin{array}{ccc} \{3\} & \{21\} & \{3\} \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{array} \right)_q \langle \{3\} || \hat{T}^{(21)} || \{3\} \rangle$$

Where for this case, we have [1, 2].

Then, for the decuplet baryon we have:

$$q C_{\frac{3}{2}\frac{3}{2}}^{\frac{3}{2}\frac{3}{2}} = q C_{\frac{3}{2}\frac{3}{2}}^{\frac{3}{2}\frac{3}{2}} = q C_{\frac{3}{2}\frac{3}{2}}^{\frac{3}{2}\frac{3}{2}} = 1$$

(40)
Therefore, by combining expressions (36)-(39) and using the $q$-deformed Clebsch-Gordan coefficients corresponding to the $\{3\} \times \{21\}$ reduction given in the Appendix, the following $q$-deformed mass relations for decuplet baryons is obtained [2]:

\[ m_{\Sigma^*} - m_{\Omega} = \frac{1}{2q} (m_\Delta - m_{\Xi^*}) \]
\[ m_{\Xi^*} - m_{\Omega} = \frac{q^3}{1+q} (m_{\Sigma^*} - m_{\Omega}) \]
\[ m_{\Omega} - m_{\Xi^*} = \frac{q^3}{1+q-q^3} (m_{\Xi^*} - m_{\Sigma^*}) \]

where the $q$-deformation parameter, according to the expression [22] is given by:

\[ q = \frac{m_\Delta - m_{\Xi^*} \pm \sqrt{(m_\Delta - m_{\Xi^*})^2 - 4 (m_{\Sigma^*} - m_{\Omega}) (m_{\Xi^*} + m_{\Xi^*} - m_{\Omega} - m_\Delta)}}{2 (m_{\Sigma^*} + m_{\Xi^*} - m_{\Omega} - m_\Delta)} \]

As in the case of the octet baryons, we replace the average multiplet masses obtaining the following values for the $q$-parameter:

\[ q_2 = 0.917 \pm 0.012 \quad q_3 = 0.986 \pm 0.003 \]
\[ q_4 = 0.985 \pm 0.002 \]

With the aim to determine a unique $q$-parameter for the $q$-deformed Gell-Mann-Okubo mass relations for octet and decuplet baryons, we perform a $\chi^2$ adjustment given by the fitted function:

\[ \chi^2 = \sum_{i=1}^{4} \frac{(q_i - q)^2}{q^2} \]

By minimizing the $\chi^2$ function, we obtain that the fitted $q$-parameter is given by:

\[ q = 0.970 \pm 0.003 \]

Besides that, by comparing the generalization of the Gell-Mann-Okubo mass formula for pseudoscalar meson with its $q$-deformed version, the following relation is obtained [5]:

\[ \frac{f_K^2}{f_\pi} = \frac{[2]_q}{2 \left( [2]_q - [3]_q \right)} \]

where $f_K$ and $f_\pi$ are the decay constants for the $K$ and $\pi$ mesons, respectively.

Moreover, the ratio $\frac{f_K}{f_\pi}$ can be expressed in terms of the Cabbibo angle as follows [5]:

\[ \tan^2 \theta_C = \frac{m_{\pi}^2}{m_K^2} \frac{f_K}{f_\pi} \]

From expressions [49] and [50], a connection between the $q$-deformation parameter and the Cabbibo angle is observed [5].

On the other hand, as we have chosen before that $0 \leq q \leq 1$, we can introduce a new $\tau$ parameter according to:
TABLE I: Error percentages of the Gell-Mann-Okubo mass relations for the $SU_q(3)$ and $SU(3)$ algebras.

| Mass Relation | $q$ | $\frac{RHS-LHS}{RHS} \times 100$ |
|---------------|-----|---------------------------------|
| (32)          | 0.970 | 0.33                           |
| (32)          | 1.000 | 0.58                           |
| (42)          | 0.970 | 2.99                           |
| (42)          | 1.000 | 4.49                           |
| (43)          | 0.970 | 4.26                           |
| (43)          | 1.000 | 3.40                           |
| (44)          | 0.970 | 7.95                           |
| (44)          | 1.000 | 8.23                           |

$q = \cos \tau \qquad (51)$

Then, from expressions (48) and (51) we get:

$$\tau = (0.246 \pm 0.012) \text{ rad} \qquad (52)$$

As the Cabbibo angle is equal to $\theta_C = (0.226 \pm 0.002) \text{ rad}$, the expression (52) implies:

$$\tau = (1.085 \pm 0.063) \theta_C \qquad (53)$$

Hence, according to the previous result, it is possible to interpretate the $q$-deformation parameter with the cosine of the Cabbibo angle.

By replacing the $q$-deformation parameter obtained by the $\chi^2$ fitting in the Gell-Mann-Okubo mass relations and comparing with the $q = 1$ case, we obtain Table II.

In Table I we can see that for the $q$-deformed Gell-Mann-Okubo mass relations for the octet and decuplet baryons, a better agreement with the experimental masses than the predicted by the $SU(3)$ algebra is obtained, except for expression (43) corresponding to the decuplet baryon, where the difference between the error percentages is only 0.86%.

IV. THE $q$-DEFORMED COLEMAN-GLASHOW RELATIONS FOR OCTET AND DECUPLET BARYONS

The center of the octet baryon is degenerate and so the $U$ eigenfunctions differ from isospin eigenfunctions. The $U = 1$ and $U = 0$ eigenfunctions are:

$$|\Sigma^0_U\rangle = \sqrt{\frac{3}{2}_q}|\Lambda^0\rangle - \frac{1}{2}_q|\Sigma^0\rangle \qquad (54)$$

$$|\Lambda^0_U\rangle = \sqrt{\frac{3}{2}_q}|\Lambda^0\rangle + \frac{1}{2}_q|\Sigma^0\rangle \qquad (55)$$

For this case, we consider the following expression for the mass operator:

$$\tilde{M} = \tilde{M}_S + \tilde{M}_T + \tilde{M}_U \qquad (56)$$

where the operators $\tilde{M}_S$, $\tilde{M}_T$ and $\tilde{M}_U$ are $SU_q(3)$ scalar, isospin scalar and $U$ spin scalar, respectively.

The mass operators $\tilde{M}_T$ and $\tilde{M}_U$ have matrix elements related by:

$$m_T(n) = m_T(p), \quad m_T(\Sigma^+) = m_T(\Sigma^0), \quad m_U(n) = m_U(\Sigma^0_U), \quad m_U(p) = m_U(\Sigma^+_U)$$

$$m_T(\Sigma^-) = m_T(\Sigma^0), \quad m_T(\Xi^-) = m_T(\Xi^0), \quad m_U(\Sigma^-) = m_U(\Xi^-), \quad m_U(\Xi^0) = m_U(\Sigma^0_U) \qquad (57)$$
Moreover, by taking into account that:

\[ m_U(\Sigma_U^0) = \langle \Sigma_U^0 | \hat{M}_U | \Sigma_U^0 \rangle \quad \langle \Sigma_U^0 | \hat{M}_U | \Lambda_U^0 \rangle = 0 \]

\[ m(\Sigma^0\Lambda^0) = \langle \Sigma^0 | \hat{M}_U | \Lambda^0 \rangle = \langle \Sigma^0 | \hat{M}_U | \Lambda^0 \rangle = m_U(\Sigma^0\Lambda^0) \]

and using the expressions corresponding to the \(|\Sigma_U^0\rangle\), \(|\Lambda_U^0\rangle\) states, we get:

\[ m_U(\Sigma_U^0) = \frac{[3]_q m_U(\Lambda^0) + m_U(\Sigma^0) - 2\sqrt{[3]_q} m_U(\Sigma^0\Lambda^0)}{[2]_q^2} \quad (58) \]

Then, from the expressions (58), we find:

\[ m_U(\Sigma_U^0) - m_U(\Sigma^0) = -\frac{[4]_q}{[2]_q \sqrt{[3]_q}} m_U(\Sigma^0\Lambda^0) \quad (59) \]

Therefore, by replacing (59) in expressions (56) and (60), the following relations are obtained:

\[ m(n) - m(p) + m(\Sigma^+) - m(\Sigma^0) = \frac{[3]_q}{[2]_q^2} (m_U(\Lambda^0) - m_U(\Sigma^0)) - \frac{2\sqrt{[3]_q}}{[2]_q^2} m_U(\Sigma^0\Lambda^0) \quad (60) \]

\[ m(\Xi^-) - m(\Xi^0) + m(\Sigma^0) - m(\Sigma^-) = \frac{2\sqrt{[3]_q}}{[2]_q^2} m_U(\Sigma^0\Lambda^0) + \frac{[3]_q}{[2]_q^2} (m_U(\Sigma^0) - m_U(\Lambda^0)) \quad (61) \]

From the linear combinations of the previous relations we find:

\[ m_{\Xi^-} - m_{\Xi^0} = m_{\Sigma^-} - m_{\Sigma^0} + m_p - m_n \quad (62) \]

Hence, the Coleman-Glashow relation for the octet baryon is independent of \(q\).

On the other hand, it is known that when the electromagnetic interactions are neglected, we have that the mass operator for hadron multiplets is given by:

\[ \hat{M} = \hat{M}_S + \hat{M}_T \quad (65) \]

where \(\hat{M}_S\) is a \(SU_q(3)\) invariant whereas \(\hat{M}_T\) is an operator corresponding to the \(SU_q(3)\) symmetry breaking.

We also have that under \(SU_q(2)_U \times U(1)_Q\) the \(\hat{M}_T\) operator in expression (65) transforms as the sum of a \(U\) vector spin and a \(U\) scalar \(\Xi\). Applying the \(SU_q(2)\)
Wigner-Eckart theorem to this subgroup, we have:

\[ M(U, U_3) = A + C(U, q) q^{-U_3} U_3 \]  \hspace{1cm} (66)

where \( C(U, q) \) is given by:

\[ C(U, q) = \frac{q^{U+\frac{1}{2}}}{|2U + 1|} \langle \alpha U | \mathcal{M}_U | \alpha U \rangle \]  \hspace{1cm} (67)

From expressions (66) and (67) we get:

\[ m(U, U_3) - m(U, U_3 - 1) = C(U, q) q^{-U_3 + 1} \]

\[ + C(U, q) q^{-U_3} (1 - q) U_3 \]  \hspace{1cm} (68)

By applying the previous formula to the octet baryon, we find:

\[ m(n) - m(\Sigma^0_U) = C(1, q) q^{-1} \]

\[ m(\Sigma^0_U) - m(\Xi^0) = C(1, q) q \]

Then, the following relation holds:

\[ q \left( m_n - m_{\Sigma^0_U} \right) = q^{-1} \left( m_{\Sigma^0_U} - m_{\Xi^0} \right) \]  \hspace{1cm} (69)

Hence, by using (63), (69), and (69) we obtain a \( q \) deformed mass relation for the particles in the octet baryon:

\[ q^2 m_n + m_p + q^{-2} m_{\Xi^0} + m_{\Sigma^-} = [3] q \left( m_{\Lambda^0} + m_{\Sigma^+} + m_{\Sigma^-} - m_{\Sigma^0} \right) \]  \hspace{1cm} (70)

with the \( q \) deformation parameter given by:

\[ q = \pm \sqrt{s + \sqrt{s^2 - 4 (m_n - m_{\Lambda^0})(m_{\Xi^0} - m_{\Lambda^0})}} \]  \hspace{1cm} (71)

where the \( s \) parameter has been introduced according to:

\[ s = m_{\Sigma^+} + m_{\Sigma^-} + m_{\Lambda^0} - m_p - m_{\Xi^-} - m_{\Sigma^0} \]  \hspace{1cm} (72)

By replacing the experimental octet particle masses in expression (61), we obtain:

\[ q_1 = 0.965 \pm 0.001 \]  \hspace{1cm} (73)

On the other hand, when formula (68) is applied to the decuplet baryon, we get the following expressions:

\[ m_{\Delta^-} - m_{\Sigma^-} = \frac{1}{2} C \left( \frac{3}{2}, q \right) q^{-3/2} (3 - q) \]

\[ m_{\Sigma^0} - m_{\Xi^-} = \frac{1}{2} C \left( \frac{3}{2}, q \right) q^{-1/2} (1 + q) \]

\[ m_{\Xi^-} - m_{\Omega^-} = \frac{1}{2} C \left( \frac{3}{2}, q \right) q^{1/2} (3q - 1) \]

\[ m_{\Delta^0} - m_{\Sigma^0} = C(1, q) q^{-1} \]

\[ m_{\Sigma^0} - m_{\Xi^0} = C(1, q) q \]

Hence, for the decuplet baryons we obtain:

\[ \frac{q^{3/2} (m_{\Delta^-} - m_{\Sigma^-})}{3 - q^{-1}} = \frac{q^{1/2} (m_{\Sigma^0} - m_{\Xi^-})}{1 + q} \]

\[ = \frac{q^{-1/2} (m_{\Xi^-} - m_{\Omega^-})}{3q - 1} \]  \hspace{1cm} (74)

\[ q (m_{\Delta^0} - m_{\Sigma^0}) = q^{-1} (m_{\Sigma^0} - m_{\Xi^0}) \]  \hspace{1cm} (75)

Then, from expressions (64) and (64) and using the experimental decuplet particle masses we obtain:

\[ q_2 = 0.976 \pm 0.007 \]

\[ q_3 = 0.965 \pm 0.004 \]

\[ q_4 = 0.970 \pm 0.003 \]

\[ q_5 = 0.988 \pm 0.008 \]  \hspace{1cm} (76)
TABLE II: Error percentages of the Coleman-Glashow relations for the $SU_q(3)$ and $SU(3)$ algebras.

| Mass Relation | $q$ | $\frac{|RHS - LHS|}{RHS}$% |
|---------------|-----|----------------------------|
| (70)          | 0.973 | 0.14                      |
| (70)          | 1.000 | 0.60                      |
| (74)          | 0.973 | 1.29                      |
| (74)          | 1.000 | 1.42                      |
| (74)          | 0.973 | 1.27                      |
| (74)          | 1.000 | 7.03                      |
| (74)          | 0.973 | 1.03                      |
| (74)          | 1.000 | 9.72                      |
| (74)          | 0.973 | 2.97                      |
| (74)          | 1.000 | 2.51                      |

Up to this point we have obtained five different values for the $q$-deformation parameter from the $q$-deformed Coleman-Glashow relations for octet and decuplet baryons. In order to obtain a unique $q$-parameter we perform a $\chi^2$ adjustment obtaining:

$$q = 0.973 \pm 0.002$$  \hspace{1cm} (78)

For this case, we have obtained that the $\tau$ parameter is given by:

$$\tau = (0.233 \pm 0.010) \text{ rad} = (1.030 \pm 0.053) \theta_C$$  \hspace{1cm} (79)

which implies that the $q$-deformation parameter can be interpreted with the cosine of the Cabbibo angle.

By replacing the $q$-deformation parameter obtained by the $\chi^2$ adjustment in the Coleman-Glashow relations and comparing with the predicted by the $SU(3)$ algebra, we obtain the Table II. From Table II, a better agreement of the $q$-deformed Coleman-Glashow relations with the experiment than the predicted by the $SU(3)$ algebra is obtained in all cases except in the last (relation (74)), where the difference between the error percentages is only 0.46%.

V. CONCLUSIONS

The quantum group $SU_q(3)$ provides a good tool to solve problems in particle physics, especially when one needs to describe the mass splitting for particles from isomultiplets within octet and decuplet baryons.

The $q$-deformed Clebsch-Gordan coefficients corresponding to the $\{3\} \times \{21\}$ reduction of the $SU_q(3)$ algebra were computed. The $q$-deformed mass relations for octet and decuplet baryons have been explicitly obtained from the quantum Clebsch-Gordan coefficients corresponding to the $\{21\} \times \{21\}$ and $\{3\} \times \{21\}$ reductions. The Coleman-Glashow relations for octet and decuplet baryons have been found in the $SU_q(3)$ quantum algebra. From the Coleman-Glashow relation for the octet baryon, a $q$-deformed mass relation between its particles has been obtained.

By performing an adjustment of the $q$-deformation parameter in the $q$-deformed Gell-Mann-Okubo and Coleman-Glashow relations for octet and decuplet baryons, we obtain that the corresponding values for this parameter are $q = 0.970 \pm 0.003$ and $q = 0.973 \pm 0.002$, respectively. These values are directly connected with the cosine of the Cabbibo angle. That is, a unique relation between the $q$-deformation parameter and the Cabbibo angle has been found, which differs with the results given in [5] in the fact that in this reference two different relations $q = e^{2i\theta_C}$ and $q = e^{i\theta_C}$, between these parameters have been obtained for the octet and decuplet baryons, respectively, exhibiting error percentages of approximately $0.07\%$ and $0.53\%$ when these relations are replaced in the $q$ deformed masses expressions.
In spite of the fact that the error percentages of the $q$-deformed mass relations obtained in [5] are lower than those obtained in this article, it is important to point out that we have shown that when two approximately equivalent values of the $q$-deformation parameter are used, the $q$-deformed Gell-Mann-Okubo and Coleman-Glashow relations for octet and decuplet baryons exhibit a very good agreement with the experimental results, in most cases better than the predicted by the $SU(3)$ algebra. The error percentages of the $q$-deformed Gell-Mann-Okubo and Coleman-Glashow relations are lower than 7.95% and 2.97%, respectively.

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APPENDIX A: QUANTUM CLEBSCH-GORDAN COEFFICIENTS FOR THE $\{3\} \times \{21\}$ REDUCTION

The product of the $SU_q(3)$ representations $\{3\}$ and $\{21\}$ is given by:

$$\{3\} \times \{21\} = \{51\} + \{42\} + \{3\} + \{21\} \quad (A1)$$

where $\{51\}$, $\{42\}$, $\{3\}$, and $\{21\}$ are representations with dimensions equal to 35, 27, 10, and 8 respectively.

These representations are shown in the Figures 1-4.

Through a straightforward calculation in which the angular momentum addition rules are taken into account together with the requirements of orthogonality, the $q$-deformed Clebsch-Gordan coefficients corresponding to
the \{3\} \times \{21\} reduction are determined and shown in the Table[III] In this case, we have that \( \chi_i, i = 1, 2 \cdots 8 \) and \( \xi_j, j = 1, 2 \cdots 10 \) are the wave functions which describe the octet and decuplet baryons states corresponding to the \{21\} and \{3\} representations.
| (a) $\xi_1 \chi_1$ | (b) $\xi_4 \chi_6$ | (c) $\xi_1 \chi_3$ | (d) $\xi_4 \chi_8$ | (e) $\xi_1 \chi_7$ | (f) $\xi_1 \chi_8$ |
|------------------|------------------|------------------|------------------|------------------|------------------|
| (a) $\psi_1^{(51)}$ | (b) $\psi_5^{(51)}$ | (c) $\psi_6^{(51)}$ | (d) $\psi_{15}^{(51)}$ | (e) $\psi_{34}^{(51)}$ | (f) $\psi_{35}^{(51)}$ |

**TABLE III: The $q$-deformed Clebsch-Gordan coefficients for the $\{3\} \times \{2\}$ reduction.**

| (a) $\xi_2 \chi_1$ | (b) $\xi_3 \chi_2$ | (d) $\xi_3 \chi_3$ | (e) $\xi_8 \chi_7$ | (g) $\xi_6 \chi_6$ | (c) $\xi_3 \chi_4$ |
|------------------|------------------|------------------|------------------|------------------|------------------|
| (a) $\psi_2^{(51)}$ | (b) $\psi_4^{(51)}$ | (c) $\psi_3^{(42)}$ | (d) $\psi_{10}^{(51)}$ | (e) $\psi_{10}^{(42)}$ | (f) $\psi_{24}^{(51)}$ |
| (g) $\psi_{23}^{(51)}$ | (h) $\psi_{18}^{(51)}$ | (i) $\psi_{27}^{(42)}$ | (j) $\psi_{38}^{(51)}$ | (k) $\psi_{19}^{(42)}$ | (l) $\psi_{24}^{(42)}$ |

| (a) $\xi_1 \chi_4$ | (b) $\xi_5 \chi_6$ | (c) $\xi_7 \chi_8$ | (d) $\xi_1 \chi_7$ | (e) $\xi_1 \chi_6$ | (f) $\xi_1 \chi_6$ |
|------------------|------------------|------------------|------------------|------------------|------------------|
| (a) $\psi_{31}^{(51)}$ | (b) $\psi_{12}^{(51)}$ | (c) $\psi_{26}^{(42)}$ | (d) $\psi_{10}^{(51)}$ | (e) $\psi_{19}^{(51)}$ | (f) $\psi_{20}^{(51)}$ |

| (a) $\xi_1 \chi_5$ | (b) $\xi_1 \chi_5$ | (c) $\xi_1 \chi_5$ | (d) $\xi_1 \chi_5$ | (e) $\xi_1 \chi_5$ | (f) $\xi_1 \chi_5$ |
|------------------|------------------|------------------|------------------|------------------|------------------|
| (a) $\xi_1 \chi_8$ | (b) $\xi_1 \chi_8$ | (c) $\xi_1 \chi_8$ | (d) $\xi_1 \chi_8$ | (e) $\xi_1 \chi_8$ | (f) $\xi_1 \chi_8$ |

| (a) $\xi_1 \chi_7$ | (b) $\xi_1 \chi_7$ | (c) $\xi_1 \chi_7$ | (d) $\xi_1 \chi_7$ | (e) $\xi_1 \chi_7$ | (f) $\xi_1 \chi_7$ |
|------------------|------------------|------------------|------------------|------------------|------------------|
| (a) $\xi_1 \chi_4$ | (b) $\xi_1 \chi_4$ | (c) $\xi_1 \chi_4$ | (d) $\xi_1 \chi_4$ | (e) $\xi_1 \chi_4$ | (f) $\xi_1 \chi_4$ |
TABLE III: (Continued.)

|       | \( \psi_{10}^{(51)} \) | \( \psi_{20}^{(42)} \) | \( \psi_{21}^{(42)} \) | \( \psi_{3}^{(1)} \) | \( \psi_{4}^{(21)} \) | \( \psi_{5}^{(21)} \) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| (a)   | \( \xi_{10}\chi_{1} \) | \( \xi_{0}\chi_{1} \) | \( q^3 \sqrt{[\frac{3}{4}]_q} \) | 0                | \( \sqrt{[\frac{3}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |\( \sqrt{[\frac{2}{4}]_q} \) |
|       | \( \xi_{9}\chi_{3} \) | \( \xi_{7}\chi_{3} \) | \( -q^{-1} \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |\( -q^{-2} \) |
|       | \( \xi_{5}\chi_{5} \) | \( \xi_{5}\chi_{6} \) | \( -q^{-2} \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |\( -q^{-2} \) |
| (a)   | \( \xi_{8}\chi_{2} \) | \( \xi_{8}\chi_{8} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |
|       | \( \xi_{8}\chi_{4} \) | \( \xi_{6}\chi_{4} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |
|       | \( \xi_{6}\chi_{5} \) | \( \xi_{6}\chi_{5} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |
|       | \( \xi_{8}\chi_{6} \) | \( \xi_{7}\chi_{3} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |
|       | \( \xi_{9}\chi_{6} \) | \( \xi_{9}\chi_{6} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |
|       | \( \xi_{9}\chi_{5} \) | \( \xi_{9}\chi_{5} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |
|       | \( \xi_{8}\chi_{4} \) | \( \xi_{8}\chi_{4} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |

TABLE III: (Continued.)

|       | \( \psi_{28}^{(51)} \) | \( \psi_{22}^{(42)} \) | \( \psi_{23}^{(42)} \) | \( \psi_{3}^{(1)} \) | \( \psi_{4}^{(21)} \) | \( \psi_{5}^{(21)} \) |
|-------|-----------------|-----------------|-----------------|-----------------|-----------------|-----------------|
| \( \xi_{10}\chi_{2} \) | \( q^3 \sqrt{[\frac{3}{4}]_q} \) | 0                | \( \sqrt{[\frac{3}{4}]_q} \) | \( \sqrt{[\frac{3}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |\( \sqrt{[\frac{2}{4}]_q} \) |
| \( \xi_{8}\chi_{6} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |\( \sqrt{[\frac{2}{4}]_q} \) |
| \( \xi_{7}\chi_{7} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |\( \sqrt{[\frac{2}{4}]_q} \) |
| \( \xi_{9}\chi_{8} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |\( \sqrt{[\frac{2}{4}]_q} \) |
| \( \xi_{9}\chi_{5} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |\( \sqrt{[\frac{2}{4}]_q} \) |
| \( \xi_{9}\chi_{4} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) | \( \sqrt{[\frac{2}{4}]_q} \) |\( \sqrt{[\frac{2}{4}]_q} \) |
| \(\xi_1 \chi_4\) | (a) \(\psi_{18}^{(51)}\) | (b) \(\psi_{18}^{(51)}\) | (a) \(\psi^{(51)}\) | (b) \(\psi^{(51)}\) | (a) \(\psi_{18}^{(42)}\) | (b) \(\psi^{(42)}\) | (a) \(\psi_1^{(3)}\) | (b) \(\psi_1^{(3)}\) | (a) \(\psi_{13}^{(3)}\) | (b) \(\psi_{13}^{(3)}\) |
|---|---|---|---|---|---|---|---|---|---|---|
| (a) \(\xi_1 \chi_5\) | (a) \(\xi_1 \chi_6\) | (a) \(\xi_2 \chi_3\) | (b) \(\xi_2 \chi_3\) | (b) \(\xi_3 \chi_1\) | (b) \(\xi_3 \chi_1\) | (a) \(\xi_2 \chi_8\) | (b) \(\xi_2 \chi_8\) | (b) \(\xi_3 \chi_1\) | (b) \(\xi_3 \chi_1\) |

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| \(\xi_1 \chi_4\) | (a) \(\psi_{18}^{(51)}\) | (b) \(\psi_{18}^{(51)}\) | (a) \(\psi^{(51)}\) | (b) \(\psi^{(51)}\) | (a) \(\psi_{18}^{(42)}\) | (b) \(\psi^{(42)}\) | (a) \(\psi_1^{(3)}\) | (b) \(\psi_1^{(3)}\) | (a) \(\psi_{13}^{(3)}\) | (b) \(\psi_{13}^{(3)}\) |
|---|---|---|---|---|---|---|---|---|---|---|
| (a) \(\xi_1 \chi_5\) | (a) \(\xi_1 \chi_6\) | (a) \(\xi_2 \chi_3\) | (b) \(\xi_2 \chi_3\) | (b) \(\xi_3 \chi_1\) | (b) \(\xi_3 \chi_1\) | (a) \(\xi_2 \chi_8\) | (b) \(\xi_2 \chi_8\) | (b) \(\xi_3 \chi_1\) | (b) \(\xi_3 \chi_1\) |

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| \(\xi_1 \chi_4\) | (a) \(\psi_{18}^{(51)}\) | (b) \(\psi_{18}^{(51)}\) | (a) \(\psi^{(51)}\) | (b) \(\psi^{(51)}\) | (a) \(\psi_{18}^{(42)}\) | (b) \(\psi^{(42)}\) | (a) \(\psi_1^{(3)}\) | (b) \(\psi_1^{(3)}\) | (a) \(\psi_{13}^{(3)}\) | (b) \(\psi_{13}^{(3)}\) |
|---|---|---|---|---|---|---|---|---|---|---|
| (a) \(\xi_1 \chi_5\) | (a) \(\xi_1 \chi_6\) | (a) \(\xi_2 \chi_3\) | (b) \(\xi_2 \chi_3\) | (b) \(\xi_3 \chi_1\) | (b) \(\xi_3 \chi_1\) | (a) \(\xi_2 \chi_8\) | (b) \(\xi_2 \chi_8\) | (b) \(\xi_3 \chi_1\) | (b) \(\xi_3 \chi_1\) |

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| \(\xi_1 \chi_4\) | (a) \(\psi_{18}^{(51)}\) | (b) \(\psi_{18}^{(51)}\) | (a) \(\psi^{(51)}\) | (b) \(\psi^{(51)}\) | (a) \(\psi_{18}^{(42)}\) | (b) \(\psi^{(42)}\) | (a) \(\psi_1^{(3)}\) | (b) \(\psi_1^{(3)}\) | (a) \(\psi_{13}^{(3)}\) | (b) \(\psi_{13}^{(3)}\) |
|---|---|---|---|---|---|---|---|---|---|---|
| (a) \(\xi_1 \chi_5\) | (a) \(\xi_1 \chi_6\) | (a) \(\xi_2 \chi_3\) | (b) \(\xi_2 \chi_3\) | (b) \(\xi_3 \chi_1\) | (b) \(\xi_3 \chi_1\) | (a) \(\xi_2 \chi_8\) | (b) \(\xi_2 \chi_8\) | (b) \(\xi_3 \chi_1\) | (b) \(\xi_3 \chi_1\) |

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| \(\xi_1 \chi_4\) | (a) \(\psi_{18}^{(51)}\) | (b) \(\psi_{18}^{(51)}\) | (a) \(\psi^{(51)}\) | (b) \(\psi^{(51)}\) | (a) \(\psi_{18}^{(42)}\) | (b) \(\psi^{(42)}\) | (a) \(\psi_1^{(3)}\) | (b) \(\psi_1^{(3)}\) | (a) \(\psi_{13}^{(3)}\) | (b) \(\psi_{13}^{(3)}\) |
|---|---|---|---|---|---|---|---|---|---|---|
| (a) \(\xi_1 \chi_5\) | (a) \(\xi_1 \chi_6\) | (a) \(\xi_2 \chi_3\) | (b) \(\xi_2 \chi_3\) | (b) \(\xi_3 \chi_1\) | (b) \(\xi_3 \chi_1\) | (a) \(\xi_2 \chi_8\) | (b) \(\xi_2 \chi_8\) | (b) \(\xi_3 \chi_1\) | (b) \(\xi_3 \chi_1\) |
| $\xi_1 \chi_3$ | $\psi_9^{(51)}$ | $\psi_{10}^{(51)}$ | $\psi_5^{(42)}$ | $\psi_9^{(42)}$ | $\psi_7^{(3)}$ | $\psi_1^{(21)}$ |
|---------------|----------------|----------------|---------------|---------------|---------------|---------------|
| $q^2 \sqrt{[2]_{q}[4]_{q}[3]_{q}[3]_{q}}$ | $-q^{-3}[2]_{q}$ | $q^{-1}\sqrt{[2]_{q}[4]_{q}[3]_{q}[3]_{q}}$ | $q^{-4}\sqrt{[2]_{q}[4]_{q}[3]_{q}[3]_{q}}$ | $\sqrt{[2]_{q}}$ | $\sqrt{[2]_{q}}$ | $\sqrt{[2]_{q}}$ |
| $\xi_6 \chi_1$ | 0 | $\sqrt{[2]_{q}[4]_{q}[5]_{q}}$ | $\sqrt{[2]_{q}[4]_{q}[5]_{q}}$ | $-\sqrt{[2]_{q}[4]_{q}[5]_{q}}$ | $\sqrt{[2]_{q}[4]_{q}[5]_{q}}$ | $\sqrt{[2]_{q}[4]_{q}[5]_{q}}$ |
| $\xi_2 \chi_4$ | $\frac{[2]_{q}\sqrt{[2]_{q}[4]_{q}[5]_{q}}}{[4]_{q}[6]_{q}}$ | $q^{-1/2}[2]_{q}-q^{-5}$ | $q^{-1/2}[2]_{q}-q^{-4}$ | $\frac{[2]_{q}}{[2]_{q}}$ | $\sqrt{[2]_{q}[4]_{q}[5]_{q}}$ | $\frac{[2]_{q}}{[2]_{q}}$ |
| $\xi_5 \chi_2$ | 0 | $\sqrt{[3]_{q}[5]_{q}}$ | $\sqrt{[3]_{q}[5]_{q}}$ | $\sqrt{[3]_{q}[5]_{q}}$ | $\sqrt{[3]_{q}[5]_{q}}$ | $\sqrt{[3]_{q}[5]_{q}}$ |
| $\xi_1 \chi_6$ | $\frac{[2]_{q}\sqrt{[2]_{q}[4]_{q}[5]_{q}}}{[4]_{q}[6]_{q}}$ | $-q^{-3}$ | $\sqrt{[2]_{q}[4]_{q}[5]_{q}}$ | $\sqrt{[2]_{q}[4]_{q}[5]_{q}}$ | $\sqrt{[2]_{q}[4]_{q}[5]_{q}}$ | $\sqrt{[2]_{q}[4]_{q}[5]_{q}}$ |
| $\xi_2 \chi_5$ | 0 | $\sqrt{[3]_{q}[5]_{q}}$ | $\sqrt{[3]_{q}[5]_{q}}$ | $\sqrt{[3]_{q}[5]_{q}}$ | $\sqrt{[3]_{q}[5]_{q}}$ | $\sqrt{[3]_{q}[5]_{q}}$ |
| $\psi_9^{(51)}$ | $\psi_{10}^{(51)}$ | $\psi_5^{(42)}$ | $\psi_9^{(42)}$ | $\psi_7^{(3)}$ | $\psi_1^{(21)}$ |
| $\xi_7 \chi_5$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | 0 | 0 | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ |
| $\xi_4 \chi_7$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ |
| $\xi_9 \chi_2$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | 0 | 0 | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ |
| $\xi_3 \chi_8$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ |
| $\xi_6 \chi_6$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ |
| $\xi_7 \chi_4$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ |
| $\psi_9^{(51)}$ | $\psi_{10}^{(51)}$ | $\psi_5^{(42)}$ | $\psi_9^{(42)}$ | $\psi_7^{(3)}$ | $\psi_1^{(21)}$ |
| $\xi_2 \chi_6$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ |
| $\xi_6 \chi_2$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ |
| $\xi_3 \chi_4$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ | $\frac{[2]_{q}[4]_{q}[6]_{q}}{[4]_{q}[6]_{q}}$ |
\begin{table}
\centering
\begin{tabular}{c c c c c c c}
\hline
\(\xi_{7\chi_1}\) & 0 & \(q^2 \sqrt{\frac{[5]}{[4][6]}}\) & \(\frac{1}{\sqrt{[2][3][5]}}\) & \(\sqrt{\frac{[2][4]}{[3][6]}}\) & \(\sqrt{\frac{[2]}{[5][6]}}\) & \(\sqrt{\frac{[2]}{[5][6]}}\) \\
\hline
\(\xi_{4\chi_3}\) & \(q^3 \sqrt{\frac{[2]}{[4][5][6]}}\) & \(-q^{-2} \sqrt{\frac{[3]}{[2][5][6]}}\) & \(\frac{1}{\sqrt{[2][3][6]}}\) & \(q^{-3} \sqrt{\frac{[2]}{[3][6]}}\) & \(q \sqrt{\frac{[3]}{[2][6]}}\) & \(\sqrt{\frac{[5]}{[2][6]}}\) \\
\hline
\(\xi_{3\chi_5}\) & \(0\) & \(q^{-3} \sqrt{\frac{[3]}{[2][5][6]}}\) & \(\frac{3}{\sqrt{[2][5][6]}}\) & \(0\) & \(\sqrt{\frac{3}{[2][5][6]}}\) & \(0\) \\
\end{tabular}
\end{table}

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