A New Low-Redundancy Restricted Array with Reduced Mutual Coupling

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Abstract

In array signal processing, a fundamental problem is to design a sensor array with high degrees of freedom (DOFs) (or low-redundancy) and reduced mutual coupling, which are the main features to improve the performance of direction-of-arrival (DOA) estimation. A sensor array is called low-redundancy if the ratio \( R = N(N-1)/(2L) \) is approaching the Leech’s bound \( 1.217 \leq R_{opt} \leq 1.332 \), where \( N \) is the physical sensor number and \( L \) is the maximum inter-sensor spacing of the sensor array. Up to now, the best known infinite classes of low-redundancy arrays (LRAs) satisfy \( R \leq 1.5 \). The objective of this paper is to propose a new LRA configuration with \( R < 1.5 \) for any \( N \geq 18 \). The new array preserves all the good properties of existing sparse arrays, namely, the different co-array is hole-free, and the sensor locations are uniquely determined by a closed-form expression as a function of the number of sensors. Notably, compared to existing sparse arrays, the proposed array can significantly reduce the effects of mutual coupling between sensors, by decreasing the number of sensor pairs with minimum inter-spacing to the lowest number 1. Numerical simulations are conducted to verify the superiority of the new array over other known sparse arrays.

Index Terms

Sparse arrays, degrees of freedom (DOFs), mutual coupling, low-redundancy arrays, direction-of-arrival (DOA) estimation.

I. INTRODUCTION

Array signal processing is a fundamental technology used in various applications such as radar, sonar, navigation, wireless communications, electronic surveillance and radio astronomy [13], [37]. Key benefits of using sensor arrays include spatial selectivity and the capability to mitigate interference and improve signal quality. The most commonly used sensor arrays are conventional uniform linear arrays (ULAs), where the inter-element spacing is constant and is no more than half wavelength to avoid spatial aliasing. However, the ULAs have some shortcomings: For an \( N \)-sensor ULA, the traditional subspace-based methods [34], [38] can resolve only up to \( N \) sources. To increase this number, additional sensors are required, thus leading to a high complexity that may be impractical or infeasible. In practice, electromagnetic characteristics cause mutual coupling between sensors, making the sensor response interfere with each other [2], [37]. This has an adverse effect on the estimation of parameters (e.g., DOA). Generally, the smaller the inter-spacing between sensors is, the larger the mutual coupling becomes. Thus ULAs suffer from severe mutual coupling effects between array sensors. Although many methods aim to remove the effect of mutual coupling from the received data by using proper mutual coupling models [2], [37], they are usually computationally expensive, and sensitive to model mismatch.

To overcome these problems caused by traditional ULAs, nonuniform linear arrays (NLAs) (also referred as sparse arrays) were introduced [37]. Consider an \( N \)-sensor NLA with sensors located at \( n_d \), where \( n_i \) belongs to an integer set

\[ S = \{s_1, s_2, \ldots, s_N\}, \]

and \( d \) is the unit inter-element spacing, usually equal to a half wavelength \( \lambda/2 \). The difference co-array (DCA) of the NLA is defined as the array which has sensors located at positions given by the set \( \mathcal{D} = \{s_i - s_j : i,j = 1,2,\ldots,N\} \), and the cardinality of \( \mathcal{D} \) is called the degrees of freedom (DOF) of the NLA. The DOF directly decides the distinct values of the cross correlation terms in the covariance matrix of the signal received by an antenna array, which can be used to detect uncorrelated sources. Each such technique, like co-array MUSIC, actually amounts to using a maximal central ULA segment, whose cardinality is called uniform DOF (denoted as uDOF), of its resulting DCA instead of the original array to perform DOA estimation. If the uniform DOF is \( U \), then the number of uncorrelated sources that can be detected by using co-array MUSIC is \( (U-1)/2 \) [29], [40]. Note that there are totally \( N^2 \) elements in \( \mathcal{D} \), although some locations maybe repeated. There is a possibility that we can get \( O(N^2) \) uniform DOF using only \( N \) physical sensors, which results in that up to \( O(N^2) \) uncorrelated sources can be identified [12]. In fact, many works have been done for this purpose [29], [43]. This implies a significant increased number of detected sensors compared to traditional ULAs. Moreover, mutual coupling effects may also be reduced due to the larger inter-sensor spacing in sparse arrays [3]. For simplicity, we will use \( S \) and \( \mathcal{D} \) to denote a sparse array and its corresponding DCA respectively, in the rest of this paper.

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A sparse array is called restricted (or hole-free) if its DCA is a ULA, i.e., \( D = [-L, L] \), where \( L \) denotes the maximal inter-sensor spacing. For example, \( \{0, 1, 4, 6\} \) is a 4-sensor restricted array since its DCA \( D = [-6, 6] \). Relatively, a sparse array is called general if its DCA is not an ULA. The array \( \{0, 1, 4\} \) is such an example since the spacing 2 is missing in its DCA. In this paper, we consider only the restricted problem. Unless stated otherwise, the term “array” will always refer to a restricted array. About the research on general arrays, interested readers can refer to [11], [12], [13], [40] for details.

Without loss of generality, let \( S = \{s_1 = 0, s_2, \ldots, s_N = L\} \) be an \( N \)-sensor restricted array. The redundancy ratio \( R \) of \( S \) is quantitatively defined as the number of pairs of sensors divided by \( L: R = (N(\!-\!1)\!/(2L)) \) [25]. Obviously \( R \geq 1 \). Bracewell [5] proved that arrays with \( R = 1 \) (“zero-redundancy” arrays) only exist for \( N \leq 4 \). In 1956, Leech [22] demonstrated that for minimum redundancy arrays (MRA) (achieving the lowest \( R \)): \( 1.217 \leq R_{\text{opt}} \leq 1.332 \) for \( N \to \infty \), and provided some optimal solutions for \( N \leq 11 \). For larger value of \( N \), the optimal design of such arrays is not easy and in most cases, they are restricted to computer simulations or complicated algorithms for sensor placement [14], [25]. Based on the difficulty of MRAs, several early attempts have been made to construct large low-redundancy linear arrays (LRA) (approaching Leech’s bound).

The problem of finding LRAs has been first investigated as a pure number-theoretic issue of “restricted difference bases” [11], [22], [26], [44], [45], which is a set of integers \( 0 = a_1 < a_2 < \ldots < a_N = n \) such that for every integer \( v, 1 \leq v \leq n \), there are two terms \( a_i, a_j \) with \( a_i - a_j = v \). By observing some results in [22], Wichmann [44] proposed the first class of LRAs with \( R \leq 1.5 \) for any \( N = 4 \) (mod 6). Later in 1966, Bracewell constructed LRAs with \( R \approx 2 \) for any \( N \) by combining an ULAs, an additional sensor and a sparse ULA [5]. Ishiguro [14] found that there are apparent regular patterns in the configurations of optimal LRAs for large \( N \), i.e., the largest spacing between sensors (called base of the pattern) repeated many times at the central of the array, and constructed large LRAs from small MRAs, but the \( R \) of resulting arrays increased monotonically with the number of elements and LRAs with prime \( N \) can not be constructed by Ishiguro’s method. Wichmann’s results [44] are of such pattern where the base is congruent to 3 mod 4. In 1993, Linebarger et al. [17] presented two similar patterns of LRAs with \( R \leq 1.5 \) for any \( N \), dependent on the base congruent to 0 and 1 mod 4, respectively. Especially, the pattern for bases congruent to 0 mod 4 only contains two sensor pairs whose inter-spacing is \( \lambda/2 \), i.e., \( \omega(1) = 2 \). Here \( \omega(a) \) is the weight function denoting the number of sensor pairs whose inter-element spacing is \( a\lambda/2 \) (see Definition 1). Thus the mutual coupling is greatly reduced compared with other known patterns of LRAs. By changing the base number of Wichmann’s array from \( 2r + 1 \) to any \( 2r + k \) (\( 1 \leq k \leq 6 \)), a \((4r + 3)\)-Type array (i.e., the base is congruent to 3 mod 4) can exist for any \( N \), which has the largest uniform DOFs up to now. With the help of powerful modern computers, Camps et al. [6] proposed a fast technique for the direct synthesis of large LRAs and summarized main LRAs configurations for \( N \leq 37 \) which are the best and most extensive results known up to 2001. Inspired by the patterns found by Ishiguro [14], Dong et al. [7] summarized a general structure of large LRAs, and proposed an efficient method using the general structure to guide an ant-colony-optimization-based search for a rapid exploration of optimal LRAs. Using this method, they presented some new array patterns with \( R \leq 1.5 \) for any \( N \) dependent on how the base of the array reduces mod 4, superior to the results given by Camps [6].

After 2010, several LRAs with \( R \approx 2 \) are successively obtained, inspired by the nested array (NA) proposed in [29]. The NA is obtained by combining two or more ULAs with increased inter-element spacing, and can provide at most \( N_2^2 + N - 1/3 \) uniform DOFs with \( N \) physical sensors for the two-level case. The authors in [29] also proposed a novel spatial smoothing based approach to DOA estimation, and a new approach to beamforming based on a nonlinear preprocessing, which can effectively utilize the DOFs offered by any restricted arrays. In [46], an improved nested array (INA) was proposed via increasing the inter-spacing element spacing of the outer ULA and adding an additional sensor. The INA can provide at most \( N_2^2 + 2N - 3 \) uniform DOFs, which is higher compared with the NA. Note that the NA and INA all contain a dense ULA in the physical array, which result in significantly higher mutual coupling effects between elements [9]. In [19], the (second-order) super nested array (SNA) was introduced by rearranging the dense ULA part of a NA in such a way that the difference co-array remains unchanged, but mutual coupling effect can be significantly reduced since the optimal solutions for SNA configuration have \( \omega(1) = 1 \). In [20], a generalization of SNAs, referred to as Qth-order SNA (\( Q > 2 \)), was introduced to further reduce the mutual coupling effects by reducing \( \omega(2) \) to only about half that of the second-order SNAs. All the SNA configurations have the same uniform DOFs as the NAs. In [24], augmented nested array (ANA) was proposed by splitting the dense ULA of a NA into two or four parts, which can be relocated at the two sides of the sparse ULA of a NA. Depending on how the splitting takes place, four different ANAs, i.e., ANAI-1, ANAI-2, ANAII-1 and ANAII-2, are derived. Although these ANAs provide a higher number of uniform DOFs compared with the existing NA and SNA configurations, they also have some disadvantages: ANAI-1 and ANAI-2 still contain a dense ULA; ANAI-1 and ANAI-2 have to satisfy complicated conditions in order to obtain hole-free DCAs. In [48], a new array configuration based on the maximum inter-element spacing constraint (MISC) criterion was introduced which consists of three sparse ULAs plus two separate sensors that are appropriately placed. The MISC array can provide a higher number of uniform DOFs at most \( N_2^2 + 3N - 8.5 \) and also reduce the mutual coupling since \( \omega(1) = 1 \). Thus, it is the best LRAs with \( R \approx 2 \) up to now, not only from the uniform DOFs, but also for the mutual coupling.

In the design of LRAs, the sensor locations should be described for any \( N \) using simple rules or closed forms for practical applications. To have a better comparison of such known LRAs, we list three Tables: Table 1 summarizes known LRAs with...
$R \approx 2$; Table III summarizes known LRAs with $R \leq 1.5$, and Table III summarizes known LRAs with $\omega(1) \leq 2$. All the LRAs are represented in terms of the inter-element spacing set, i.e., $D = \{d_1, d_2, \ldots, d_{N-1}\}$ with $d_i = s_{i+1} - s_i$ for any $1 \leq i \leq N - 1$.

**TABLE I: A SUMMARY OF LRAS WITH REDUNDANCY $R \approx 2$.**

| Array Structure | uDOFs | Weight Injection |
|-----------------|-------|-----------------|
| Braccioff (1996) | $\{ (m-1, (m+1)^{m-1} \}, \text{ if } N = 2m$ | $\frac{2}{\omega(1)} + 1$, if $N = 2m$ |
| | $\{ (m, (m+2), (m+1)^{m-1} \}, \text{ if } N = 2m + 1$ | $\frac{2}{\omega(1)} + \frac{1}{\omega(2)}$, if $N = 2m + 1$ |
| NA (2010) | $\{ (1, m, (m+1)^{m-1} \}, \text{ if } N \leq 2m \}$ | $\frac{2}{\omega(1)} + 1$, if $N \leq 2m$ |
| | $\{ (1, m, (m+1)^{m-1} \}, \text{ if } N \geq 2m \}$ | $\frac{2}{\omega(1)} + \frac{1}{\omega(2)}$, if $N \geq 2m$ |
| INA (2016) | $\{ (1, m, (m+1)^{m-1}, m \}, \text{ if } N = 2m \}$ | $\frac{2}{\omega(1)} + 2N - 3$, if $N = 2m$ |
| | $\{ (1, m, (m+1)^{m-1}, m \}, \text{ if } N = 2m + 1 \}$ | $\frac{2}{\omega(1)} + 2N - 2$, if $N = 2m + 1$ |
| SNA (2016) | $\{ (1, 2, \frac{m-1}{3}, \frac{m+1}{3}, \frac{2m-1}{3}, (m+1)^{m-2}, m, 1 \}, \text{ if } N = 2m \}$ | $\frac{2}{\omega(1)} + 1$, if $N = 2m$ |
| | $\{ (1, 2, \frac{m-1}{3}, \frac{m+1}{3}, \frac{2m-1}{3}, (m+1)^{m-2}, m, 1 \}, \text{ if } N = 2m + 1 \}$ | $\frac{2}{\omega(1)} + \frac{1}{\omega(2)}$, if $N = 2m + 1$ |
| ANM (2017) | $\{ (1, 2, \frac{m-1}{3}, \frac{m+1}{3}, \frac{2m-1}{3}, (m+1)^{m-1}, 1, 2^{m-4}, 1 \}, \text{ if } N \equiv 0 \mod 4 \}$ | $\frac{2}{\omega(1)} + 2N - 5$, if $N \equiv 0 \mod 4$ |
| | $\{ (1, 2, \frac{m-1}{3}, \frac{m+1}{3}, \frac{2m-1}{3}, (m+1)^{m-1}, 1, 2^{m-4}, 1 \}, \text{ if } N \equiv 1 \mod 4 \}$ | $\frac{2}{\omega(1)} + 2N - 4$, if $N \equiv 1 \mod 4$ |
| SIC (2019) | $\{ (1, P - 3, P-N, 2, \frac{3m-1}{3}, \frac{2m-1}{3}, \frac{m+1}{3}, 1 \}, \text{ if } N \equiv 4 \mod 5 \}$ | $\frac{2}{\omega(1)} + 3N - 8$, if $N \equiv 4 \mod 5$ |
| | $\{ (1, P - 3, P-N, 2, \frac{3m-1}{3}, \frac{2m-1}{3}, \frac{m+1}{3}, 1 \}, \text{ if } N \equiv 2 \mod 5 \}$ | $\frac{2}{\omega(1)} + 3N - 9$, if $N \equiv 2 \mod 5$ |
| | $\{ (1, P - 3, P-N, 2, \frac{3m-1}{3}, \frac{2m-1}{3}, \frac{m+1}{3}, 1 \}, \text{ if } N \equiv 0 \mod 5 \}$ | $\frac{2}{\omega(1)} + 3N - 10$, if $N \equiv 0 \mod 5$ |

**TABLE II: A SUMMARY OF LRAS WITH REDUNDANCY $R \leq 1.5$.**

| Array Structure | N | Weight Function |
|-----------------|---|-----------------|
| Wetlaq (1982) | $\{ 1, r = 1, k = 1, t = 1 \}$ | $2^{N/2} - 2^{N/3} - 2^{N/4} - 2^{N/5}$ |
| | $\{ 1, r = 1, k = 2, t = 1 \}$ | $2^{N/2} - 2^{N/3} - 2^{N/4} - 2^{N/5}$ |
| (4r-3)-Type (1982) | $\{ 1, r = 1, k = 2, t = 1 \}$ | $2^{N/2} - 2^{N/3} - 2^{N/4} - 2^{N/5}$ |
| (4r-1)-Type (1982) | $\{ 1, r = 1, k = 2, t = 1 \}$ | $2^{N/2} - 2^{N/3} - 2^{N/4} - 2^{N/5}$ |
| (4r)-Type (1982) | $\{ 1, r = 1, k = 2, t = 1 \}$ | $2^{N/2} - 2^{N/3} - 2^{N/4} - 2^{N/5}$ |
| (4r+3)-Type (1982) | $\{ 1, r = 1, k = 2, t = 1 \}$ | $2^{N/2} - 2^{N/3} - 2^{N/4} - 2^{N/5}$ |
| (4r+1)-Type (1982) | $\{ 1, r = 1, k = 2, t = 1 \}$ | $2^{N/2} - 2^{N/3} - 2^{N/4} - 2^{N/5}$ |
| (4r-3)-Type (1982) | $\{ 1, r = 1, k = 2, t = 1 \}$ | $2^{N/2} - 2^{N/3} - 2^{N/4} - 2^{N/5}$ |
| (4r)-Type (1982) | $\{ 1, r = 1, k = 2, t = 1 \}$ | $2^{N/2} - 2^{N/3} - 2^{N/4} - 2^{N/5}$ |
| (4r+1)-Type (1982) | $\{ 1, r = 1, k = 2, t = 1 \}$ | $2^{N/2} - 2^{N/3} - 2^{N/4} - 2^{N/5}$ |

Seen from the summary above, the MISC array is the best array configuration with $R \approx 2$. For the LRAs with $R \leq 1.5$, the (4r-3)-Type array is the best array configuration in terms of uniform DOFs; while in terms of mutual coupling, the (4r)-Type array is the best array configuration. In this paper, we introduce a new array configuration which also satisfy $R \approx 1.5$, but has reduced mutual coupling, i.e., $\omega(1) = 1$. The new array can be seen as a modification of the (4r)-Type array in [17], and still satisfies that the base (the largest $d_i$ in $D$) is congruent to 0 mod 4. Generally, the new array configuration consists of five sparse ULAs plus four separate sensors which are appropriately placed. For a given number of sensors, the sensor locations of the new array are uniquely determined by a closed-form expression and the number of achievable uniform DOFs is analytically presented as well. Numerical results demonstrate its superiority in comparison to different existing sparse array configurations.

The rest of this paper is organized as follows. Some necessary preliminaries are introduced in Section III. In Section IV, we present the new array configuration, include its design rules and array structure, the number of uniform DOFs, and the weight functions. Numerical results are provided in Section V to demonstrate the superiority of the proposed new array. Finally, conclusions are given in Section VI.
A. Difference Co-array Signal Model

Consider an $N$-sensor NLA whose sensor positions are given by \( S = \{s_1, s_2, \ldots, s_N\} \), with difference co-array \( D \). Assume that \( K \) far-field, uncorrelated narrowband signals impinge on the array from distinct directions \( \{\theta_1, \theta_2, \ldots, \theta_K\} \) with powers \( \{\sigma_1^2, \sigma_2^2, \ldots, \sigma_K^2\} \). The signal received by the array at time \( t \) is modeled as

\[
x(t) = \sum_{k=1}^{K} a_k(t) s_k(t) + n(t) = As(t) + n(t),
\]

where \( s(t) = [s_1(t), s_2(t), \ldots, s_K(t)]^T \) is the signal waveform vector, \( A = [a_1(t), a_2(t), \ldots, a_K(t)] \) is the \( N \times K \) array manifold matrix, and \( a_k(t) = [e^{j2\pi s\theta_k}, e^{j2\pi s\theta_k}, \ldots, e^{j2\pi s\theta_k}]^T \) is the steering vector of the array corresponding to the \( k \)-th signal with \( \theta_k = \sin\theta_k/2 \) denoting the normalized direct-of-arrival (DOA) satisfying \(-1/2 \leq \theta_k \leq 1/2\). The noise \( n(t) \) is assumed to be temporally and spatially white, and uncorrelated from the sources. The \( \theta_k \) is considered to be fixed but unknown, and need to be estimated by the signal model (i.e., DOA estimation).

The theoretical covariance matrix of \( x(t) \) can be expressed as

\[
R_{xx} = E[x(t)x^H(t)] = A \text{diag}(\sigma_1^2, \ldots, \sigma_K^2)A^H + \sigma_n^2 I_N
\]

where \( \sigma_n^2 \) represents the noise variance. Since the entries in \( a(\theta_k)A^H(\theta_k) \) are of the form \( e^{j2\pi \delta \theta_k d} \) for \( d \in D \), it enables us to reshape \( 2 \) into an autocorrelation vector \( z \) as in \[18], \[29] \n
\[
z = \sum_{k=1}^{K} \sigma_k^2 b(\theta_k) + \sigma_n^2 e_0
\]

where \( b(\tilde{\theta}_k) = [e^{j2\pi \delta \tilde{\theta}_k d}]_{d \in D} \), \( B = [b(\tilde{\theta}_1), b(\tilde{\theta}_2), \ldots, b(\tilde{\theta}_K)] \), \( p = [\sigma_1^2, \ldots, \sigma_K^2] \) and \( \langle e_0 \rangle_d = \delta_{d,0} \) for \( d \in D \). Here \( \delta_{p,q} \) is the Kronecker delta. Compared \[11] with \[3 \], the vector \( z \) can be viewed as the received data from a coherent source signal vector \( p \) with a single snapshot, and \( \sigma_n^2 e_0 \) becomes a deterministic noise term. Hence, the original model in \[1 \] in the physical array domain \( S \), is converted into another model \[3 \] in the difference co-array domain \( D \), and the DOA estimation can be applied to the data in \[3 \] instead of \[1 \]. Each such technique, like co-array MUSIC \[29], \[30], actually amount to using a subvector \( z_{ul} \) of \( z \) to perform DOA estimation, where \( U = [-L_u, L_u] \) is the maximal central ULA segment of \( D \), and the number of uncorrelated sources that can be identified is \( |U| - 1/2 \).

B. Mutual Coupling

The received signal vector \[1 \] assumes that the sensors do not interfere with each other. In practical application, however, the mutual coupling effect between the elements with small separation cannot be neglected. After incorporating the mutual coupling effect, \[1 \] can be rewritten as

\[
x(t) = CA s(t) + n(t),
\]
where $C$ is the $N \times N$ mutual coupling matrix. Thus the coupling-free model (1) can be regarded as a special case of (4), where $C$ is an identity matrix.

Generally, the expression for $C$ is rather complicated [19], [24]. In the ULA configuration, $C$ can be approximated by a B-banded symmetric Toeplitz matrix as follows [8], [16], [36], [39], [47]:

$$
(C)_{n_i, n_j} = \begin{cases} 
  c_{|n_i - n_j|}, & \text{if } |n_i - n_j| \leq B, \\
  0, & \text{otherwise}, 
\end{cases}
$$

(5)

where $n_i, n_j \in S$ and $c_0, c_1, \ldots, c_B$ are coupling coefficients satisfying $|c_0| = 1 > |c_1| > |c_2| > \cdots > |c_B|$. It is assumed that the magnitudes of their sensor separations, i.e., $|c_k/c_\ell| = \ell/k$ for $k, \ell > 0$ [3]. To evaluate the mutual coupling effect, the weight function and coupling leakage are usually used.

**Definition 1** (Weight Function): The weight function $\omega(d)$ of an array $A$ is defined as the number of sensor pairs that lead to co-array index $d$. Namely,

$$
\omega(d) = |\{(n_i, n_j) \in S^2 : n_i - n_j = d\}|, \ d \in D.
$$

(6)

**Definition 2** (Coupling Leakage): For a given number of sensors, the coupling leakage is defined as the energy ratio [19]:

$$
L_c = \frac{||C - \text{diag}(C)||_F}{||C||_F}
$$

(7)

where $|| \cdot ||_F$ is the energy of all the off-diagonal components, which characterizes the level of mutual coupling. A small value of $L_c$ implies that the mutual coupling is less significant.

### III. The New Array Configuration

In this section, we will develop a new array configuration which also satisfies $R < 1.5$. The new array can be see a modification of the $(4r)$-type array in [17], since its base is congruent to $0 \mod 4$. Compared with the $(4r+3)$-Type array and the $(4r)$-type array in [17], although our new array has a less uniform DOFs for fixed number of sensors, but it has reduced mutual coupling because of smaller value of the crucial weights $\omega(1) = 1$ and $\omega(2) \approx N/3$. Compared with the MISC array, our new arrays possess a higher uniform DOFs and less mutual coupling effect by reducing $\omega(2)$ to about two-thirds of that of MISC arrays. The numerical simulations in the next section also demonstrate the superiority of our new array over other existing LRAs.

#### A. New Array structure

Dong et al [7] summarized a common general structure of large LRAs in terms of the inter-element spacing set as follows:

$$
D = \{a_1, a_2, \ldots, a_s, c^\ell, b_1, b_2, \ldots, b_s\}
$$

(8)

where $c^\ell$ denotes the base (i.e., the largest spacing) repeats $\ell$ times, $\ell = 2, 3, \cdots$ [6]. The difference value of $\ell$ would affect the redundancy of the array while still keeping all spacings; $s_1$ and $s_2$ denote the lengths of the sub-sequences relating to the base $c$ by

$$
s_1 + s_2 = c - 1.
$$

(9)

To ensure obtaining LRAs, they also restricted $s_1, s_2$ in the general structure by

$$
|s_1 - s_2| \leq \begin{cases} 
  2, & r \text{ is odd}; \\
  3, & r \text{ is even}. 
\end{cases}
$$

(10)

By applying the ant-colony-optimization (ACO) procedure to (8) with constraints (9) and (10), they obtain four types of LRAs with $R \leq 1.5$ for any given number of sensors in nearly zero computer time (See Table II). Their LRAs are classified dependent on how that base of the array remainder mod 4, and include the $(4r+3)$-Type array and $(4r)$-Type array in [17] as special cases.

In this paper, we slightly modified the constraints (9) of the general structure (8) of LRAs to be

$$
s_1 + s_2 = c,
$$

(11)

and obtain a new array configuration as follow:

$$
D_{\text{New}} = \{1, 2^{r-2}, 3, (2r - 1)^r, (4r)^{2r+k-1}, (2r + 1)^{r-1}, 2, 2r - 1, 2^{r-1}\},
$$

(12)

where $i^m$ correspond to $m$ repetitions of the inter-sensor spacing $i$. The sensor number of the new array is $N = 6r + k$, $-3 \leq k \leq 2$. Since MRAs have been known for $N \leq 17$ through computer search [42], thus in this paper we let $N \geq 18$, i.e., $r \geq 3$.,
The sensor position set corresponding to $D_{New}$ is expressed as

$$S_{New} = \{0, 1, 3, \ldots, 2r - 3, 2r - 1, 6r - 2, \ldots, 2r^2 + r, 2r^2 + 5r, 2r^2 + 9r, \ldots, 10r^2 + (4k - 3)r, 10r^2 + (4k - 1)r + 1, \ldots, 12r^2 + (4k - 4)r - 1, 12r^2 + (4k - 4)r + 1, 12r^2 + (4k - 2)r, 12r^2 + (4k - 2)r + 2, \ldots, 12r^2 + 4kr - 2\}. \quad (13)$$

It is clear that the sensor positions can be represented as a function of $r$ and $k$, which are uniquely determined by $N$. Therefore, the new array have close-from expressions for the sensor position with respect to an arbitrary number of sensors.

Note that the new array yields a hole-free difference co-array, as formally described in Lemma 1.

**Lemma 1:** The difference co-arrays of the new arrays (13) are hole-free ULAs, i.e., $D_{New} = [-L, L]$ with $L = 12r^2 + 4rk - 2$.

**Proof.** In terms of $S_{New}$, the maximum distance between sensors is $L = 12r^2 + 4rk - 2$. Since $D_{New}$ is symmetric about 0, we only need to prove that its positive part $D_{New}^+$ contains the consecutive set $F = \{1, 2, 3, \ldots, L\}$.

Based on $S_{New} = \{s_1, \ldots, s_n\}$, we may construct the $N - 1$ positive difference sets as:

- $D_1 = \{s_2 - s_1, s_3 - s_1, \ldots, s_N - s_1\}$,
- $D_2 = \{s_3 - s_2, s_4 - s_2, \ldots, s_N - s_2\}$,
- $\ldots$,
- $D_{N-2} = \{s_{N-1} - s_{N-2}, s_N - s_{N-2}\}$,
- $D_{N-1} = \{s_N - s_{N-1}\}$.

Let $D_0 = \bigcup_{i=1}^{N-1} D_i$ and $F = \bigcup_{i=0}^{3r+k-1} F_i$, where $F_i$ is the consecutive lags of $F$ defined as:

- $F_i = \{4ri + 1, 4ri + 2, \ldots, 4r(i + 1)\}$, for $0 \leq i \leq 3r + k - 2$,
- $F_{3r+k-1} = \{4r(3r + k - 1) + 1, 4r(3r + k - 1) + 2, \ldots, 4r(3r + k) - 2\}$.

The proof of $F \subseteq D_0$ can be carried out by finding the consecutive lags $F_i$ from some subsets of $D_0$. We will do this case by case:

- $F_0 \subset D_1 \cup \ldots \cup D_r \cup D_{2r+1} \cup D_{4r+k} \cup D_{5r+k-2} \cup D_{5r+k-1} \cup D_{5r+k+1}$;
- when $1 \leq i \leq \lceil \frac{r}{2} \rceil - 2$,
  - $F_i \subset D_1 \cup \ldots \cup D_r \cup D_{2r+1-2i} \cup \ldots \cup D_{2r+1} \cup D_{4r+k-i} \cup \ldots \cup D_{4r+k+[\frac{i-1}{2}]} \cup \ldots \cup D_{5r+k+2i}$;
- when $i = \lceil \frac{r}{2} \rceil - 1$,
  - $F_i \subset D_1 \cup \ldots \cup D_r \cup D_{2r+1+[\frac{i-1}{2}]} \cup \ldots \cup D_{2r+1} \cup D_{4r+k-i} \cup \ldots \cup D_{4r+k+[\frac{i-1}{2}]-i}$;
- when $\lceil \frac{r}{2} \rceil \leq i \leq 2r + k - 2$,
  - $F_i \subset D_1 \cup \ldots \cup D_{2r+1} \cup F_{4r+k-i} \cup \ldots \cup D_{4r+k+[\frac{r-i}{2}]}$;
- when $2r + k - 1 \leq i \leq 2r + k - 2 + \lceil \frac{r+1}{2} \rceil$,
  - $F_i \subset D_1 \cup \ldots \cup D_{4r+k-i+[\frac{r+1}{2}]}$;
- when $i = 2r + k - 2 + \lceil \frac{r+1}{2} \rceil + k'$ for $1 \leq k' \leq \lceil \frac{r}{2} \rceil$,
  - $F_i \subset D_1 \cup \ldots \cup D_{2(r-k')+[\frac{r+1}{2}]-[\frac{r-1}{2}]}$.

Therefore, the difference co-array of our new array is a hole-free ULA, i.e., $D_{New} = [-L, L]$ with $L = 12r^2 + 4rk - 2$. □

Now we use an example to illustrate the procedure of proof in Lemma 1.

**Example 1:** Let $N = 18$, i.e., $r = 3, k = 0$. The structure of our new array with 18-sensor is

$$\{0, 1, 3, 6, 11, 16, 21, 33, 45, 57, 69, 81, 88, 95, 97, 102, 104, 106\}.$$
Thus we obtain 17 positive difference sets as follows:

\[
\begin{align*}
D_1 &= \{1, 3, 6, 11, 16, 21, 33, 45, 57, 69, 81, 88, 95, 97, 102, 104, 106\}, \\
D_2 &= \{2, 5, 10, 15, 20, 32, 44, 56, 68, 80, 87, 94, 96, 101, 103, 105\}, \\
D_3 &= \{3, 8, 13, 18, 30, 42, 54, 66, 78, 85, 92, 94, 99, 101, 103\}, \\
D_4 &= \{5, 10, 15, 27, 39, 51, 63, 75, 82, 89, 91, 96, 98, 100\}, \\
D_5 &= \{5, 10, 22, 34, 46, 58, 70, 77, 84, 86, 91, 93, 95\}, \\
D_6 &= \{5, 17, 29, 41, 53, 65, 72, 79, 81, 86, 88, 90\}, \\
D_7 &= \{12, 24, 36, 48, 60, 67, 74, 76, 81, 83, 85\}, \\
D_8 &= \{12, 24, 36, 48, 55, 62, 64, 69, 71, 73\}, \\
D_9 &= \{12, 24, 36, 43, 50, 52, 57, 59, 61\}, \\
D_{10} &= \{12, 24, 31, 38, 40, 45, 47, 49\}, \\
D_{11} &= \{12, 19, 26, 28, 33, 35, 37\}, \\
D_{12} &= \{7, 14, 16, 21, 23, 25\}, \\
D_{13} &= \{7, 9, 14, 16, 18\}, \\
D_{14} &= \{2, 7, 9, 11\}, \\
D_{15} &= \{5, 7, 9\}, \\
D_{16} &= \{2, 4\}, \\
D_{17} &= \{2\}.
\end{align*}
\]

We want to prove that \(D_{\text{New}}^+ = [0, 106]\). Define \(F_i = \{12i + 1, 12i + 2, \ldots, 12(i + 1)\}\) \((0 \leq i \leq 7)\) and \(F_8 = \{97, 98, \ldots, 106\}\).

It is easy to obtain

\[
\begin{align*}
\{1, 2, \ldots, 12\} &\subset D_1 \cup D_2 \cup D_3 \cup D_7 \cup D_{12} \cup D_{13} \cup D_{16}, \\
\{13, 14, \ldots, 24\} &\subset D_1 \cup D_2 \cup D_3 \cup D_5 \cup D_6 \cup D_7 \cup D_{11} \cup D_{12}, \\
\{25, 26, \ldots, 36\} &\subset D_1 \cup D_2 \cup \ldots \cup D_7 \cup D_{10} \cup D_{11} \cup D_{12}, \\
\{37, 38, \ldots, 48\} &\subset D_1 \cup D_2 \cup \ldots \cup D_7 \cup D_9 \cup D_{10} \cup D_{11}, \\
\{49, 50, \ldots, 60\} &\subset D_1 \cup D_2 \cup \ldots \cup D_7 \cup D_8 \cup D_9, \\
\{61, 62, \ldots, 72\} &\subset D_1 \cup D_2 \cup \ldots \cup D_7 \cup D_8 \cup D_9, \\
\{73, 74, \ldots, 84\} &\subset D_1 \cup D_2 \cup \ldots \cup D_7 \cup D_8, \\
\{85, 86, \ldots, 96\} &\subset D_1 \cup D_2 \cup \ldots \cup D_6, \\
\{97, 98, \ldots, 106\} &\subset D_1 \cup D_2 \cup \ldots \cup D_4.
\end{align*}
\]

which coincide with the process of proof in Lemma \(\text{I}\) Thus we obtain that the new array is hole-free.

B. Uniform DOFs and Weight Functions

From Lemma \(\text{I}\) we see that the difference co-array of the \(N\)-sensor new array is given by a consecutive set between \(-(12r^2 + 4rk - 2)\) and \(12r^2 + 4rk - 2\). Thus we obtain the following result.

Theorem 1: Let \(N = 6r + k\) with \(r \geq 3\) and \(-3 \leq k \leq 2\). The uniform DOFs for the \(N\)-sensor new array \(\text{[13]}\) is:

\[u\text{DOFs}_{\text{New}} = \frac{2N^2}{3} - \frac{2k^2}{3} - 3\]

for any sensor number \(N \geq 18\). Thus its redundancy ratio is: \(R_{\text{New}} < 1.5\), and \(R_{\text{New}} \approx 1.5\) when \(N \rightarrow \infty\).

Proof. From \(\text{[13]}\) and Lemma \(\text{I}\) we know that

\[u\text{DOFs}_{\text{New}} = 2L + 1 = 24r^2 + 8rk + 3 = \frac{2N^2}{3} - \frac{2k^2}{3} - 3.\quad (14)\]

Thus the redundancy ratio is

\[R = \frac{N(N - 1)}{2L} = \frac{36r^2 + 6r(2k - 1) + k^2 - k}{24r^2 + 8rk - 4} = \frac{3}{2} - \frac{6r - k^2 + k - 6}{24r^2 + 8rk - 4} < 1.5.\quad (15)\]

The last inequality is obtained by \(N \geq 18\), i.e., \(r = 3, 0 \leq k \leq 2\) and \(r \geq 4, -3 \leq k \leq 2\). \(\Box\)

Obviously, the new array has closed-from expressions for the achievable number of uniform DOFs with respect to the number of sensors by Theorem \(\text{I}\).
It is known that the weight functions at small separations are more important for mutual coupling effects [24]. In particular, the first three weight functions, \( \omega(1), \omega(2) \) and \( \omega(3) \), have a major impact one the mutual coupling of an array, and \( \omega(1) \) provides the greatest impact [19]. Therefore, in the following, we derive the expressions for the first three weight functions of the new array to evaluate the mutual coupling effects.

**Theorem 2:** Let \( N = 6r + k \) with \( r \geq 3 \) and \( -3 \leq k \leq 2 \). For the \( N \)-sensor new array (13), its weight function \( \omega(m) \) at \( m = 1, 2, 3 \) is

\[
\omega(1) = 1, \quad \omega(2) = 2r - 2, \quad \omega(3) = 3.
\]

**Proof.** This can be easily obtained by the structure (12) of the new array. \( \Box \)

For comparison, the uniform DOFs and weight functions for MISC arrays in [48], (4\( r \))-Type Arrays in [17] and (4\( r \)+3)-Type Arrays in [17] also given by

**MISC Arrays:**

\[
uDOF_{MISC} = \begin{cases} 
\frac{N^2}{2} + 3N - 8.5, & \text{if } N \equiv 1 \pmod{4}, \\
\frac{N^2}{2} + 3N - 9, & \text{if } N \equiv 0 \pmod{2}, \\
\frac{N^2}{2} + 3N - 10.5, & \text{if } N \equiv 3 \pmod{4}.
\end{cases}
\]

\[
\omega_{MISC}(1) = 1, \quad \omega_{MISC}(2) = 2 \left\lfloor \frac{N}{4} \right\rfloor - 2, \quad \omega_{MISC}(3) = \begin{cases} 
1, & \text{if } N \neq 9, \\
2, & \text{if } N = 9.
\end{cases}
\]

**(4\( r \))-Type Arrays:**

\[
uDOF_{(4r)} = \frac{2N^2}{3} - \frac{2k^2}{3} + 1.
\]

\[
\omega_{(4r)}(1) = 2, \quad \omega_{(4r)}(2) = 2r - 1, \quad \omega_{(4r)}(3) = 2.
\]

**([4r]+3)-Type Arrays:**

\[
uDOF_{(4r+3)} = \frac{2N^2}{3} + \frac{2N}{3} - \frac{2k^2 - 16k + 33}{3}.
\]

\[
\omega_{(4r+3)}(1) = 2r, \quad \omega_{(4r+3)}(2) = 2r - 1, \quad \omega_{(4r+3)}(3) = 2r - 2.
\]

Based on (14), (17), (19) and (21), the (4\( r \)+3)-Type array seems to be the best array since it owns the maximum uniform DOFs, but this is not always true. The estimation performance depends heavily on the mutual coupling coefficients. If the mutual coupling is negligible, these performance is governed by the uniform DOFs, implying

\[
uDOF_{New}, \quad \omega_{New}, \quad \text{uniform DOFs}, \quad \text{uniform DOFs}, \quad \text{uniform DOFs}, \quad \text{uniform DOFs}.
\]

In this section, numerical examples will be provided to illustrate the superiority of the proposed new arrays over the existing sparse arrays in terms of weight functions, mutual coupling matrices and DOA estimation performance. Seven types of LFRs will be used for comparison: nested array (NA) [29], second-order super nested array (SNA) [19], ANAI-2 [24], MISC array, the (4\( r \)-Type array in 1993 [17], the (4\( r \))-Type array in 1993 [17], and our new array. Other array configurations are not considered for comparison, since they either have less uniform DOFs, or larger mutual coupling. Co-array MUSIC algorithm is used to execute DOA estimation. Moreover, we assume that all incident sources have equal power and the number of sources is
known. To evaluate the DOA estimation performance of the sparse arrays, the root-mean-square error (RMSE) of the estimated normalized DOAs is shown as:

\[
\text{RMSE} = \sqrt{\frac{1}{QK} \sum_{q=1}^{Q} \sum_{k=1}^{K} (\hat{\theta}_k - \theta_k)^2}
\]

where \(Q\) is the number of independent trials, and \(\hat{\theta}_k\) is the estimate of \(\theta_k\) for the \(i\)th trial. Similar to [19] and [20], in what follows, we focus on the uniform DOFs, rather than the array aperture, to investigate the overall estimation performance.

### A. Weight Functions and Mutual Coupling Matrices

In this subsection, we compare the weight functions and the mutual coupling matrices of seven array configurations: NA, SNA, ANAI-2, MISC, the \((4r+3)\)-Type arrays, the \((4r)\)-Type arrays, and our proposed new array defined in (12). For all these arrays, three different cases are considered where the number of sensors is 18, 23, and 36, respectively. Moreover, the sensor position sets for these arrays are given in Table IV. We also adapt the mutual coupling model in [19], which is characterized by \(c_1 = 0.3e^{\pi j / 3}, B = 100\) and \(c_2 = e^{-j(\pi - \theta_l)/l} / l\) for \(2 \leq l \leq B\).

**TABLE IV: SENSOR POSITION SUMMARY OF 18, 23, 36 ELEMENTS LRAs**

| Array Type | Sensor Position |
|------------|-----------------|
| 18-element | NA [0,1,2,3,4,5,6,7,8,9,11,21,23,25,29,39,49,59,69,79,89] |
|            | SNA [0,2,4,6,8,10,11,13,15,17,19,21,23,25,29,39,49,59,69,79,89,99] |
|            | ANAI-2 [0,1,3,5,7,9,11,21,23,25,29,39,49,59,69,79,99,90,92,94,96] |
|            | MISC [0,1,8,18,28,38,48,58,68,78,88,98,99,90,92,94,96,100,102] |
| (4r+3) - Type | [0,1,2,8,14,20,31,42,53,64,75,86,97,102,107,110,111,112] |
| (4r) - Type | [0,1,3,7,14,21,33,45,57,69,81,93,98,103,104,106,108] |
| New        | [0,1,3,6,11,16,21,23,25,27,59,61,83,95,97,102,104,106] |
| 23-element | NA [0,1,2,3,4,5,6,7,8,9,10,11,23,25,29,49,59,69,79,89] |
|            | SNA [0,2,4,6,8,10,11,13,15,17,19,21,23,25,29,39,49,59,69,79,89,99] |
|            | ANAI-2 [0,1,3,5,7,9,11,21,23,25,29,39,49,59,69,79,99,101,103] |
|            | MISC (4r+3) - Type [0,1,2,8,14,20,31,42,53,64,75,86,97,102,107,110,111,112] |
|            | MISC (4r) - Type [0,1,3,7,14,21,33,45,57,69,81,93,98,103,104,106,108] |
| New        | [0,1,3,6,11,16,21,23,25,27,59,61,83,95,97,102,104,106] |
| 36-element | NA [0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,37,56,75,94,113,122,131,150,169,188,207,226,245,264,283,302,321,340] |
|            | SNA [0,2,4,6,8,10,11,13,15,17,19,21,23,25,29,39,49,59,69,79,89,99] |
|            | ANAI-2 [0,1,3,5,7,9,11,21,23,25,29,39,49,59,69,79,99,101,103] |
|            | MISC [0,1,2,8,14,20,31,42,53,64,75,86,97,102,107,110,111,112] |
| (4r+3) - Type | [0,1,2,8,14,20,31,42,53,64,75,86,97,102,107,110,111,112] |
| (4r) - Type | [0,1,3,7,14,21,33,45,57,69,81,93,98,103,104,106,108] |
| New        | [0,1,3,6,11,16,21,23,25,27,59,61,83,95,97,102,104,106] |

Table IV provides a summary of the weight function and the mutual coupling leakage for the seven LRAs, where only the weight functions \(\omega(1), \omega(2)\) and \(\omega(3)\) are shown, since the small separations have major impact on the mutual coupling of the array. It is shown that the nested array exhibits the largest, and the \((4r+3)\)-Type array exhibits the second largest, weight functions and mutual coupling leakages, due to the dense ULA in their configurations. Thus they will suffer from severe mutual coupling effect. The rest five LRAs have relatively small weight functions and mutual coupling leakages, due to the dense ULA in their configurations. Thus they will suffer from severe mutual coupling effect. The rest five LRAs have relatively small weight functions and mutual coupling leakage because their small inter-sensor separations. Especially, our new array attains the smallest weight functions and mutual coupling leakage, since it has the least \(\omega(1)\) and \(\omega(2)\), compared with other LRAs, which implies that our array is the best to resist mutual coupling effect.

Fig. 1 and Fig. 2 give visual representations of the weight functions, or the magnitudes of the mutual coupling matrices for the six LRAs, respectively. In Fig. 1 the heights of these line segments represent the sizes of weight functions \(\omega(l)\) with \(-20 \leq l \leq 20\), which is symmetrical about \(l = 0\). It is easy to see that our new array has the lowest line segments at \(l = 1, 2\), compared with other LRAs. In Fig. 2 the color of blocks represents the energy in the corresponding entry, where the less color implies less energy. Thus we want the light-colored blocks to be as little as possible. From Fig. 2 we can still demonstrate the superiority of our new array.

### B. DOA Estimation in the Absence of Mutual Coupling

In this subsection, we compare the DOA estimation performance in the absence of mutual coupling, among \((4r+3)\)-Type arrays, \((4r)\)-Type arrays, NAs, ANAI-2, MISC and the proposed New arrays. The same number of 23 sensors is used for all array configurations.
Fig. 1: The weight functions for six kinds of 23-element LRAs. (a) NA. (b) ANAI-2. (c) MISC. (d) (4r+3)-Type. (e) (4r)-Type. (f) New.

Fig. 2: The magnitudes of the mutual coupling matrices for six kinds of 23-element LRAs. (a) NA. (b) ANAI-2. (c) MISC. (d) (4r+3)-Type. (e) (4r)-Type. (f) New.
TABLE V: THE WEIGHT FUNCTIONS AND MUTUAL COUPLING LEAKAGE FOR SEVEN KINDS OF LRAs

| Array | 18 sensors | (4r + 3)-Type array | (4r)-Type array | NA | SNA | ANAI-2 | MISC | New |
|-------|------------|---------------------|----------------|----|-----|--------|------|-----|
|       | N₁ = 9, N₂ = 9 |                     |                |    |     |        |      |     |
| w₁(1) | 4           | 2                   | 9              | 1  | 2   | 1      | 1    |     |
| w₁(2) | 2           | 5                   | 8              | 8  | 7   | 6      | 4    |     |
| w₁(3) | 1           | 2                   | 7              | 1  | 2   | 1      | 2    |     |
| L₁    | 0.2209      | 0.1993              | 0.3364         | 0.1979 | 0.2121 | 0.1802 | 0.1659 |
|       | N₁ = 11, N₂ = 12 |                   |                |    |     |        |      |     |
| w₁(1) | 6           | 2                   | 11             | 1  | 2   | 1      | 1    |     |
| w₁(2) | 4           | 7                   | 10             | 10 | 9   | 8      | 6    |     |
| w₁(3) | 2           | 2                   | 9              | 1  | 2   | 1      | 2    |     |
| L₁    | 0.2383      | 0.1902              | 0.3322         | 0.1910 | 0.2021 | 0.1762 | 0.1612 |
|       | N₁ = 18, N₂ = 18 |                   |                |    |     |        |      |     |
| w₁(1) | 10          | 2                   | 18             | 2  | 2   | 1      | 1    |     |
| w₁(2) | 8           | 11                  | 17             | 15 | 16  | 16     | 10   |     |
| w₁(3) | 6           | 2                   | 16             | 4  | 2   | 1      | 2    |     |
| L₁    | 0.2499      | 0.1737              | 0.3440         | 0.1988 | 0.1967 | 0.1856 | 0.1524 |

1) MUSIC Spectra: Fig. 3 shows the MUSIC spectra for six kinds of 23-element arrays when K = 95 sources are uniformly distributed at \( \theta_k = -0.3 + 0.6(k - 1)/(K - 1) \), 1 \( \leq k \leq K \). The SNR is fixed at 0 dB and the number of snapshots is set as \( T = 1000 \). As shown in Fig. 3, the nested array and ANAI-2 fail to identify 95 sources due to the limitation in the number of uniform DOFs. The MISC array has false peak, i.e., there is a big error in DOA estimation, while the remaining arrays can resolve 95 true peaks. Furthermore, the (4r + 3)-Type array, the (4r)-Type array and the proposed new array exhibit higher peaks than other arrays. Therefore, they can provide higher DOA estimation resolution than other arrays in the absence of mutual coupling.

2) RMSE Performance: The next simulations consider the RMSE performance versus the input SNR, the number of snapshots, and the number of sources. The fixed parameter setting is SNR = 0 dB, \( T = 1000 \) snapshots, and \( K = 35 \) sources. The sources are uniformly located at \( \theta_k = -0.4 + 0.8(k - 1)/(K - 1) \), 1 \( \leq k \leq K \).

Fig. 4, Fig. 5, and Fig. 6 show the RMSE of the normalized DOA estimates versus the SNRs, the number of snapshots and the number of sources, respectively. In the absence of mutual coupling, all the simulations are totally determined by the uniform DOFs. The higher the uniform DOF is, the lower the curve becomes. Observe from the three figures, the flow of these curves completely coincide with the size of their uniform DOFs.

C. DOA Estimation in the Presence of Mutual Coupling

In practice, the influence of mutual coupling on DOA estimation can not be ignored. Thus in this subsection, we compare the DOA estimation performance of six LRAs: NA, SNA, MISC array, (4r + 3)-Type array, (4r)-Type array and our proposed...
Fig. 4: RMSE of normalized DOA estimates versus the SNR when $K = 35$ sources are located at $\theta_k = -0.4 + 0.8(k-1)/34, 1 \leq k \leq 35$. Each simulated point is averaged from 1000 trials.

Fig. 5: RMSE of normalized DOA estimates versus the number of snapshots when $K = 35$ sources are located at $\theta_k = -0.4 + 0.8(k-1)/34, 1 \leq k \leq 35$. Each simulated point is averaged from 1000 trials.

Fig. 6: RMSE of normalized DOA estimates versus the number of sources when $K$ sources are located at $\theta_k = -0.4 + 0.8(k-1)/(K - 1), 1 \leq k \leq K$. Each simulated point is averaged from 1000 trials.
new array, in the presence of mutual coupling. The same number of 23 sensors is used for all arrays, whose configurations are the same as those in Section IV-A.

1) **MUSIC Spectra**: Fig. 7 shows the MUSIC spectra for six kinds of 23-element array when \( K = 40 \) sources are uniformly located at \( \hat{\theta}_k = -0.45 + 0.9(k - 1)/39, 1 \leq k \leq 40 \). The SNR is fixed at 0 dB and the number of snapshots is set as \( T = 1000 \). Observe from Fig. 7 only the \((4r)\)-Type array and our new array is capable of detecting all 40 sources clearly, while the other arrays (with false peaks or missing peaks) are not. Especially, our new array has higher peaks than the \((4r)\)-Type array, indicating the higher resolution. Since the numbers of uniform DOFs of these arrays are all higher than 40, our new array is more effective than the remaining arrays against strong mutual coupling.

Fig. 7: The MUSIC spectra for six kinds of 23-element arrays when \( K = 40 \) sources are located at \( \hat{\theta}_k = -0.45 + 0.9(k - 1)/39, 1 \leq k \leq 40 \). (a) NA. (b) SNA. (c) MISC. (d) \((4r + 3)\)-Type. (e) \((4r)\)-Type. (f) New.

2) **RMSE Performance**: The simulations in this part focus on the RMSE performance versus the input SNR, the number of snapshots, and the modulus of coupling coefficient \( c_1 \), respectively. The mutual coupling model (5) is characterized by \( c_1 = 0.3e^{j\pi/3}, B = 100, \) and \( c_\ell = c_1e^{-j(\ell-1)\pi/8}/\ell \) (except the case where \( |c_1| \) varies). The fixed parameter setting is SNR = 0 dB, \( T = 1000 \) snapshots, and \( K = 35 \) sources (except the case where \( K \) varies). The source are located at \( \hat{\theta}_k = -0.45 + 0.9(k - 1)/(K - 1), 1 \leq k \leq K \).

Fig. 8 shows the RMSE of the normalized DOA estimation versus the SNRs. Observe from this figure, we can see that our new array yields the lowest RMSE over the entire SNR range, thus it exhibits better performance than all the other arrays. This indicates that our new array outperform the other arrays against mutual coupling effects, and is least sensitive to the mutual effects. We should note that although the \((4r + 3)\)-type array has the largest uniform DOFs, its DOA estimation is relatively poor when mutual coupling is considered, because of the dense ULA in its configuration.

Fig. 9 illustrates the RMSE of the normalized DOA estimation versus the number of snapshots. It is observed that, as the number of snapshots increases, the RMSE of all arrays are reduced rapidly until \( T \) reaches about 900, except for NA and \((4r + 3)\)-Type array. This is because the both arrays have higher coupling leakage compared to other arrays. Especially, our new array has the lowest RMSE than other arrays over the entire snapshots range.

Fig. 10 depicts the RMSE of the normalized DOA estimates versus the number of sources. When \( K \) is small, the \((4r + 3)\)-Type array has the minimum RMSE, but its RMSE curve increases rapidly as \( K \) exceeds 20, which implies that the angle measurement accuracy decreases rapidly. The nest array also presents the worst RMSE than the other arrays in most of the range, which reflects the sensitivity to the mutual coupling. In contrast, the performance of our new array is more stable and begin to evidently deteriorate only when \( K \) is more than 35. Additionally, our new array will outperform the other arrays when \( K \) exceeds 25, except only in very few cases.

Fig. 11 illustrates the RMSE of the normalized DOA estimation versus \( |c_1| \). For any array geometry, along with the increase of \( |c_1| \), the corresponding RMSE increases. That is because a higher value of \( |c_1| \) introduces more severe mutual coupling effect. When \( |c_1| \) is less than 0.7, our new array yields the best performance while the nested array achieves the worst performance, which is because the estimation accuracy is severely affected by uniform DOFs and mutual coupling effects together. When \( c_1 \) is greater than 0.7, the SNA offers smaller RMSE results than other test arrays, which indicates that the SNA outperforms other arrays when the mutual coupling is very severe.
Fig. 8: RMSE of normalized DOA estimates versus the SNR when \( K = 35 \) sources are located at \( \theta_k = -0.4 + 0.8(k-1)/34, 1 \leq k \leq 35 \). Each simulated point is averaged from 1000 trials.

Fig. 9: RMSE of normalized DOA estimates versus the number of snapshots when \( K = 35 \) sources are located at \( \theta_k = -0.4 + 0.8(k-1)/34, 1 \leq k \leq 35 \). Each simulated point is averaged from 1000 trials.

Fig. 10: RMSE of normalized DOA estimates versus the number of sources when \( K \) sources are located at \( \theta_k = -0.4 + 0.8(k-1)/(K-1), 1 \leq k \leq K \). Each simulated averaged from 1000 trials.
Fig. 11: RMSE of normalized DOA estimates versus snapshots when \( K = 35 \) sources are located at \( \theta_k = -0.4 + 0.8(k - 1)/34, 1 \leq k \leq 35 \). Each simulated point is averaged from 1000 trials.

V. Conclusions

In 2010, Dong et al. [7] summarized a common general structure of large LRAs, and obtain several types of LRAs with \( R \leq 1.5 \) for any given number of sensors by applying the aunt-colony-optimization procedure. Almost all the known classes of LRAs coincide with this structure. In this paper, we slightly modified this structure, and obtained a new class of LRAs with \( R < 1.5 \) for any \( N \geq 18 \). The new array preserves all the properties of existing sparse arrays, namely, the different co-array is hole-free, and the sensor locations are uniquely determined by a closed-form expression as a function of the number of sensors. More importantly, compared to existing sparse arrays with \( R \leq 1.5 \), our new array can significantly reduce the effects of mutual coupling between sensors, by decreasing the number of sensor pairs with minimum inter-spacing \( \omega(1) \) to the lowest number 1, while this can not be done for the existing LRA configurations with \( R \leq 1.5 \). Furthermore, by comparing with the best array configuration with \( R = 2 \) for a fixed number of sensors, our new array not only has a higher uniform DOFs, but also has a reduced mutual coupling effect, since it has a smaller number of sensor pairs with the second-smallest inter-spacing \( \omega(2) \) while the number \( \omega(1) \) remains unchanged. Numerical results verified that our new array is superior to the existing LRAs in terms of weight functions, mutual coupling matrices as well as DOA estimation performance.

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