Temporally sparse data assimilation for the generation of
small-scale structures of turbulence

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Abstract

A temporally sparse data assimilation (TSDA) strategy for direct numerical simulation (DNS) of incompressible isotropic turbulence (HIT) is proposed using large assimilation time steps. It is shown that the time step in the TSDA can be relaxed to a much coarser level (1 \sim 2 orders larger) than the existing temporally continuous data assimilation (TCDA) while the accuracy is still maintained. Furthermore, the one-step data assimilation (ODA) is analyzed to explore the mechanism of the TSDA. It is shown that the relaxation effect for errors above the assimilation wavenumber $k_a$ is responsible for the faster error decaying rate of the TSDA than the TCDA. This relaxation effect is assumed to be related to the errors contained in the large scales. These large-scale errors can make the errors in the small scales (large wavenumbers $k > k_a$) decay slower with the TCDA than the TSDA. This mechanism is further confirmed by incorporating different levels of errors in the large scales of the reference velocity field. The advantage of the TSDA is found to grow with the magnitude of the incorporated errors. Thus, it is potentially more beneficial to adopt the large-step assimilation strategy if the reference data contains non-negligible errors. Finally, an outstanding issue raised in previous works regarding the possibility of recovering the dynamics of sub-Kolmogorov scales using DNS data at Kolmogorov scale resolution is also discussed.

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I. INTRODUCTION

Accurate prediction of turbulent flows is crucial in many areas of research and engineering community, including meteorology, aerospace engineering, air pollution control, geosciences and industrial activities [1–3], etc. Among various kinds of flow prediction methods, computational fluid dynamics (CFD) has become a major tool in recent decades due to the significant developments in modern computers and numerical methods [4]. However, due to the strong sensitivity of turbulence to small perturbations in the initial conditions [5], the time horizon for the correct description of a turbulent field is quite limited if CFD is used alone. Indeed, real world measurement inevitably contains errors making it impossible to exactly prescribe the initial conditions for a CFD solver. Meanwhile, CFD simulations always contain numerical errors, which probably further contaminate the CFD-based solution. On the other hand, one may have a time sequence of the observational data, which may be sparse in time and/or space. These observational data also contains some useful information about the true state of the flow field [6]. In this case, improved predictions of the flow can be expected if the CFD simulation and the observational data can be properly merged. Such a strategy belongs to a well established independent subject, namely the data assimilation (DA) of dynamical system [7].

The commonly adopted methodology of DA for fluid mechanics include temporally continuous data assimilation (TCDA) through direct data embedding [1, 2, 8, 9], spatial and Fourier nudging [10, 11], Kalman filtering-based methods [3, 12–18], adjoint-based variational methods [19–24], forward sensitivity method [25], etc. Recently, due to the prosperity of the machine-learning techniques [26–34], artificial neural networks have also become powerful tools for DA [13, 35].

In the direct numerical simulation (DNS), DA is widely exploited in problems including the estimation of initial and boundary conditions [19], the exact flow field reconstruction using large-scale information [2, 8, 9] and data compression [36], etc. In the Reynolds averaged Navier-Stokes (RANS) simulations and large-eddy simulations (LES) [37], DA is mainly used to calibrate the unclosed Reynolds stress or the subgrid-scale (SGS) stress by either directly representing the whole, or part of, the unclosed terms \textit{per se} [20, 35, 38], or by tuning the modeling parameters [12, 39, 40]. Other applications of DA in fluid mechanics can also be found in the areas including the optimization of sensor locations [13, 21, 41],
flow controls [42–44], flow field estimation [14, 15, 22, 23, 45], parameter estimation for wind tunnel wall interference corrections [46, 47], etc.

Of closer relevance to the current study is the exact flow reconstruction of DNS in isotropic turbulence through direct data embedding in the Fourier space [2, 8, 9]. Even though turbulence has strong sensitivity to small perturbations in the initial condition [5], existing works have shown that this sensitivity can be overcome as long as sufficient amount of large-scale structures in a flow field are continuously controlled in time [8, 9]. More precisely, the small-scale information of a flow is completely slaved to the Fourier modes above a critical length scale of wavenumber $k_c$. Slightly different thresholds on $k_c$ have been reported, such as $k_c \eta \approx 0.2$ [8] and $k_c \eta \approx 0.15$ [9], where $\eta$ is the Kolmogorov length scale. Hence, the spatial-temporal information of a flow can be completely recovered with the machine-error level accuracy, as long as the dominant large-scale modes are continuously known and enforced at the same temporal resolution of the reference DNS (i.e. the correct solution).

In the present work, we shall further show that, while the number of necessary large spatial scales for exact flow reconstruction is fixed, the number of time steps for DA can be relaxed to a much coarser level compared to a temporally continuous data assimilation, while the performance is maintained at least on the same level or even slightly better. In this case, the amount of the required data for the assimilation process can be largely reduced. From another point of view, the current treatment can be also viewed as a further data compression in time, in addition to the compression in spatial scales. Nevertheless, it should be noted that the current treatment is essentially different from the the recently proposed data compression scheme [36], which requires the data of the entire spatial domain at large time steps instead of only the large-scale spatial modes.

As noted in the work by Lalescu et al. [9], another motivation of DA is that existing works have shown that length scales smaller than the Kolmogorov length scale $\eta$ can exist due to the spatial intermittency in turbulence [48–52]. Consequently, the DNS at a Kolmogorov scale resolution can be quite inaccurate [50]. A counter argument is that the sub-Kolmogorov scales should be completely recoverable from the DNS of Kolmogorov resolution if the small-scale structures are dominated by the large-scale dynamics. However, Lalescu et al. [9] have also noted that the large-scale data used in the DA process comes from a projection of the solution using a fine grid which is the same as the adopted grid in the DA-based simulation [8, 9]. Hence, whether the sub-Kolmogorov-scale recovery is truly viable is still in question.
since the grid used for the Kolmogorov scale-based DNS is a ‘coarser’ grid relative to the
‘ideal’ grid with which the sub-Kolmogorov scales can be properly resolved. In this regard,
the sub-Kolmogorov-scale recovery problem is also tested and discussed in the current study.

The rest of the paper is organized as follows. The governing equations of incompressible
turbulence and a brief introduction to the DA in the Fourier space are given in Section II,
followed by a detailed investigation on the temporally sparse data assimilation (TSDA) in
Section III, where the TCDA and the TSDA are compared and examined in terms of their
total error and error spectra. In Section IV, the possibility of sub-Kolmogorov-scale recovery
is tested using both the TSDA and the TCDA. Finally, a brief summary of the paper and
comments on future works are given in Section V.

II. GOVERNING EQUATIONS OF INCOMPRESSIBLE TURBULENCE AND A
BRIEF INTRODUCTION OF FOURIER-SPACE DATA ASSIMILATION

The governing equations of incompressible turbulence are presented in the current section.
Afterwards, the data assimilation in Fourier space is briefly introduced with some previous
findings revisited.

A. Governing equations of incompressible turbulence

For incompressible turbulence, the mass and momentum conservation are governed by
the Navier-Stokes equations, namely [53, 54]:

\[
\frac{\partial u_i}{\partial x_i} = 0, \quad (1)
\]

\[
\frac{\partial u_i}{\partial t} + \frac{\partial u_i u_j}{\partial x_j} = -\frac{\partial p}{\partial x_i} + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + F_i, \quad (2)
\]

where \( u_i \) is the velocity component in the \( i \) coordinate direction, \( \nu \) is the kinematic viscosity,
\( p \) is the pressure divided by the constant density, and \( F_i \) is the large-scale forcing applied to
the two lowest wavenumber shells [55, 56]. The summation convention is used throughout
the paper unless otherwise noted. The Kolmogorov length scale \( \eta \) is defined by [53]

\[
\eta = \left( \frac{\nu^3}{\epsilon} \right)^{1/4}, \quad (3)
\]
where $\epsilon$ is the spatially averaged dissipation rate given by $\epsilon = 2\nu\langle S_{ij}S_{ij}\rangle$ with $S_{ij} = \frac{1}{2} (\partial u_i/\partial x_j + \partial u_j/\partial x_i)$ being the strain rate tensor. In the current work, $\langle \cdot \rangle$ invariably stands for the spatial average over the entire physical domain. The Kolmogorov length scale quantifies the size of the smallest eddies in turbulence. Correspondingly, the Kolmogorov time scale is calculated as

$$\tau_\eta = \left(\frac{\nu}{\epsilon}\right)^{1/2}. \quad (4)$$

In addition, the Taylor length scale $\lambda$ is defined by

$$\lambda = \sqrt{\frac{5\nu}{\epsilon}u_{rms}}, \quad (5)$$

with $u_{rms} = \sqrt{\langle u_iu_i \rangle}$ being the root-mean-square (rms) value of velocity magnitude. The Reynolds number based on the Taylor length scale can be calculated as

$$Re_\lambda = \frac{u_{rms}\lambda}{\sqrt{3\nu}}. \quad (6)$$

Finally, the kinetic energy per unit mass is given by

$$\int_0^\infty E(k)dk = \frac{(u_{rms})^2}{2}, \quad (7)$$

where $E(k)$ is the energy spectrum.

### B. The data assimilation in Fourier space

The sensitivity of turbulence to small errors in the initial conditions is well acknowledged in the community of fluid dynamics [5], limiting the predictability of turbulence in real world applications. On the other hand, even though this sensitivity is unavoidable, several works [2, 8, 9] have also shown that, under certain conditions, the perturbation errors can be gradually erased with time if the correct large-scale velocities are continuously supplied to the numerical solution. In other words, the growth of errors due to the aforementioned sensitivity can be gradually suppressed by the dominating effect of large-scales dynamics over the small-scale dynamics in the cost of a continuous time sequence of large-scale data. Consequently, the perturbation errors can be reduced to and maintained at the machine-error level instead of growing out of control. The corresponding data assimilation procedure in Fourier space is briefly introduced in the following.
Assuming that there are two time sequences of numerical solutions of Eqs. (1) and (2), denoted by \( u \) and \( u^{ref} \), computed using the same temporal and spatial resolutions but different initial conditions. Here, we let \( u^{ref} \) denote the true solution and \( u \) be the solution with errors induced by the deviation from the true initial condition (i.e. the initial condition used by \( u^{ref} \)). Previous works have found that the solution of \( u \) would converge to \( u^{ref} \) as long as the velocity modes of \( u \) in Fourier space above a critical length scale are continuously replaced by the corresponding values of \( u^{ref} \) as the numerical solution marches forward [8]. The threshold can be written as \( k_a > k_c \), where \( k_a \) is the assimilation wavenumber below which all the velocity modes of \( u \) are replaced by that of \( u^{ref} \) with time. \( k_c \) is the critical wavenumber necessary for the success of such an assimilation, which depends on the Kolmogorov length scale \( \eta \). By letting \( \hat{u}(k,t) \) denote the Fourier coefficient of \( u \) at wavenumber \( k \) and time \( t \), the assimilation procedure can be written as

\[
\hat{u}(k, t_0 + n\Delta T) = \hat{u}^{ref}(k, t_0 + n\Delta T), \text{ for } k < k_a,
\]

where \( \Delta T \) is the discrete time steps for the data assimilation procedure, \( n = 1, 2, 3 \ldots \) and \( k \) is the magnitude of the wavenumber vector. In the continuous limit, \( \Delta T = \Delta t \), with \( \Delta t \) being the time step of DNS. Different thresholds for \( k_c \) in the continuous limit have been reported, namely \( k_c \eta = 0.2 \) [8] and \( k_c \eta = 0.15 \) [9]. In the current work, we shall show that even though the spatial threshold in terms of the necessary modes of length scales is fixed, the number of assimilation time steps required for the convergence of \( u \) to \( u^{ref} \) can be relaxed to a much coarser level compared to a continuous assimilation.

III. TEMPORALLY CONTINUOUS AND SPARSE DATA ASSIMILATION OF INCOMPRESSIBLE ISOTROPIC TURBULENCE

In the present study, the data assimilation is implemented through direct numerical simulation (DNS) of a forced incompressible isotropic turbulence. The numerical simulations are performed in a cubic box of \( (2\pi)^3 \) with periodic boundary conditions and a uniform grid spacing denoted by \( h_{DNS} \). In this case, the pseudospectral approach is conveniently adopted[54]. Meanwhile, the second-order two-step Adams-Bashforth scheme is used for time marching. The velocity field is forced by prescribing the energy spectrum within the two lowest wavenumber shells[55], and full dealiasing is implemented using the two-thirds
rule [57], with the maximum resolved wavenumber given by \( k_{\max} = N/3 \), where \( N \) is the number of grid points in each spatial direction.

In the current section, three simulation cases for \( \mathbf{u}^{\text{ref}} \) are tested, whose simulation parameters are listed in Table I. As shown in previous work [8], the influences of different choices of resolution parameters \( k_{\max} \eta \) on the results of the data assimilation are quite small. Consequently, following the previous work [8], the resolution parameters are all chosen such that \( k_{\max} \eta \approx 1 \) in the present section. We also note that, due to the CFL condition for numerical stability, the time step \( \Delta t \) is much smaller than the Kolmogorov time scale \( \tau_\eta \). In general, \( \tau_\eta \approx 10 \) to 100 \( \Delta t \). In the current analysis, the ‘erroneous’ field \( \mathbf{u} \) is generated by adding a perturbation to the initial condition, namely

\[
\hat{\mathbf{u}}(\mathbf{k}, t_0) = (1 + \varepsilon)\hat{\mathbf{u}}^{\text{ref}}(\mathbf{k}, t_0),
\]

where \( \varepsilon \) is a small real number. In all the tested cases of the current section, \( \varepsilon = 10^{-2} \) is consistently adopted such that both the decay of error in a successful assimilation and the growth of error in an unsuccessful assimilation can be clearly observed. On the contrary, one may possibly not observe the error growth if initial perturbation was too large, or the error decay if the perturbation was too small (e.g. close to the machine-error level). Otherwise, the way how the initial error is imposed does not affect any of the results based on our test.

To quantify the assimilation error, we define the magnitude of the error vector in the Fourier space as

\[
\delta = \sqrt{\sum_{\mathbf{k}} (\hat{\mathbf{u}} - \hat{\mathbf{u}}^{\text{ref}})^2},
\]

where the dependence on time is implicit. Clearly, if \( \delta \) vanishes as the solution evolves with time, the data assimilation can be deemed as successful.

| Reso. | \( Re_\lambda \) | \( k_{\max} \eta \) | \( \tau_\eta \) | \( \nu \) | \( \Delta t \) | \( \tau_\eta/\Delta t \) |
|-------|------------------|-----------------|-------|-----|--------|----------------|
| 64\(^3\) | 60                | 1.07            | 0.015 | 0.165 | 0.0032 | 52             |
| 128\(^3\) | 105              | 1.02            | 0.006 | 0.093 | 0.0016 | 58             |
| 256\(^3\) | 160              | 1.00            | 0.0024 | 0.057 | 0.0008 | 71             |

TABLE I. Numerical simulation parameters of incompressible isotropic turbulence.
FIG. 1. The evolution of the error magnitude for the TCDA: (a) $N = 64^3$, $Re_\lambda = 60$; (b) $N = 128^3$, $Re_\lambda = 105$; (c) $N = 256^3$, $Re_\lambda = 160$.

A. The spatially continuous and sparse data assimilation

To briefly revisit the assimilation in the continuous limit reported in previous works [8, 9], we plot the temporal evolution of the magnitude of the error vector field in Fig. 1. As previously discussed, the continuous limit is approximated by taking the assimilation time step $\Delta T$ equals to the DNS time step $\Delta t$. Similar to the previous findings [8, 9], the error caused by the difference of the initial condition can be gradually annihilated to machine level (i.e. single precision in the current work), provided that the assimilation wavenumber $k_a$ is above a critical value $k_c$. Meanwhile, an exponential growth or decay is observed in agreement with previous findings. Here the decay (or growth) constant $a$ is defined by

$$\delta(t) = \delta_0 e^{at},$$  \hspace{1cm} (11)

where $\delta_0$ is the initial magnitude of error. In Fig. 1, the values of the exponential constants are also indicated for the steepest decaying curve and the neutral case ($a = 0$).

We also observe in Fig. 1 that, in the $N = 64^3$ case, the exponential behavior is not very concrete for $k_a \eta = 0.15$ and 0.20, presumably because these curves are too close to the neutral state. In the rest of this section, we shall deviate from the continuous limit and investigate the possibility of relaxing assimilation time step from the DNS level to much coarser ones, namely

$$\Delta T = 2^n \Delta t,$$  \hspace{1cm} (12)
where \( m = 1, 2, 3 \ldots \). The evolutions of the magnitude of assimilation error using larger time steps are displayed in Fig. 2. To avoid the inconsistency caused by the initial Euler step used in the two-step Adams-Bashforth scheme, four consecutive (continuous) assimilation steps are performed initially. This should not affect any conclusion since the assimilation can be simply assumed to start from a different error field initially.

In Fig. 2, for practicality, only the cases with \( k_a > k_c \) are considered since the large-step assimilation is not expected to alleviate the restrict on the critical wavenumber \( k_c \) for the temporally continuous assimilation. As can be observed in Fig. 2, these assimilations can still be successful with large assimilation steps even though some oscillatory behaviors are present. Surely, one can expect the data assimilation to be still viable with time steps slightly larger than that in the continuous case. However, something more interesting, and to some extent unexpected, is that in many assimilation cases (e.g. \( \Delta T/\Delta t \leq 2^6 \)) the assimilation error exhibits the same decaying rate as the continuous one, or even decays slightly faster. As such, the amount of data required in the assimilation process can be significantly reduced while the assimilation accuracy is maintained on the same level of the continuous case.

Before further analysis on the assimilation error, we show in Fig. 3 the decay constant of the assimilation error as a function of the assimilation time step \( \Delta T \). These decay constants are obtained from the data presented in Fig. 2 using the least-square linear regression method. In Fig. 3, the assimilation time step is normalized by the Kolmogorov time scale, calculated as \( \tau_\eta = \sqrt{\nu/\epsilon} [53] \), and the exponential constant is scaled by \( (k_a - k_c)\eta/\tau_\eta \).

In Fig. 3a, we observe that, for an assimilation wavenumber that is adequate in the continuous case (i.e. \( k_a\eta > k_c\eta \)), the assimilation may not be successful if a far too large assimilation step is adopted. In principle, there should also be a critical time step for each of the assimilation wavenumbers, represented by the intersections of the curves and the \( a = 0 \) line. Unfortunately, attempts to re-scale these intersections into a single point turn out unsuccessful based on our tests. However, the curves seem to collapse much better below the continuous limit as shown in Fig. 3b, which is a zoom-in view of Fig. 3a in the region close to the continuous limit. This is by no doubt more attractive since it gives a threshold of time scale for the large step assimilation such that its performance is at least the same as, or even slightly better than, the continuous case. Consequently, the amount of the required large-scale data of the ‘true’ field \( u^{ref} \) can be tangibly reduced while the accuracy is maintained at a similar level. Figure 3b shows that as long as the assimilation
FIG. 2. The evolution of the error magnitude for the TSDA: (a) $Re_\lambda = 60, k_a = 6$; (b) $Re_\lambda = 60, k_a = 8$; (c) $Re_\lambda = 105, k_a = 10$; (d) $Re_\lambda = 105, k_a = 15$; (e) $Re_\lambda = 160, k_a = 30$; (f) $Re_\lambda = 160, k_a = 40$.

time step is around $1 \sim 1.5$ times the Kolmogorov time scale, the decaying rate of the error field can be as good as the continuous case.

It should be noted here that, in a typical DNS solution, while the grid space is close to the Kolmogorov length scale $\eta$, the adopted time step is often determined through the CFL condition [55]. In consequence, the time step of DNS is generally one or two orders
FIG. 3. The change of the normalized decay constant with respect to the normalized assimilation time interval: (a) $0 < \Delta T < 20 \tau_\eta$; (b) $0 < \Delta T < 3 \tau_\eta$.

smaller than the Kolmogorov time scale (cf. Table I). In this sense, the current finding gives a relatively consistent threshold regardless of the adopted time step in DNS. Further, it can be seen from Fig. 3b that all the curves merge approximately at the same decay rate in the continuous limit. This is in agreement with the finding by Lalescu et al. [9], who found that the decay rates in the continuous data assimilation follow closely a linear law, namely

$$a \tau_\eta \approx -\beta (k_a - k_c) \eta,$$

where the constant $-\beta \approx -1.35$ represents the intersection of the curves with the vertical axis.

To more directly visualize the performance of the temporally sparse assimilation, we show in Fig. 4 the instantaneous vorticity field at arbitrarily selected $x$-$y$ plane at the end of assimilation ($t \approx 138 \tau_\eta$) for the $Re_\lambda = 105$ case. Here, the vorticity is normalized by its rms value. As can be seen, hardly any similarity in the vorticity field can be recognized between reference field and the one without assimilation due to the uncontrolled growth of the initial error. On the other hand, the temporally continuous data assimilation gives exactly the same vorticity field as the reference field with machine-level errors unrecognizable from the figures. More importantly, the temporally sparse assimilation also recovers the vorticity field correctly even though the time step is two orders coarser than the continuous case, demonstrating its great advantage in data requirement and computational efficiency. In the following, a detailed analysis is performed so as to shed some light on the mechanism behind
FIG. 4. The contour field of normalized vorticity for the TCDA and the TSDA at $t \approx 138 \tau_\eta$.

the large-step assimilation.

B. The mechanism behind the temporally sparse data assimilation

To explore the mechanism behind temporally sparse assimilation, it is natural to consider the one-step data assimilation (ODA), i.e. feeding in the reference data once in the start and observing the evolution of the error thereafter. This numerical experiment of ODA is illustrated in Fig. 5, where the variations of the assimilation error are recorded with a range of assimilation wavenumbers for the cases listed in Table I. Again, four consecutive assimilation steps are performed initially, instead of one, to avoid the inconsistency caused by the initial Euler integration step used in the two-step Adams-Bashforth method.
As can be observed in Fig. 5, the errors invariably experience an initial decaying period, followed by an exponential growth. It should be noted that this decrease of error is entirely due to the initial assimilation step. In other words, a relaxation of the error occurs after the initial supply of the ‘true’ data, making the error continuously decay for some time on the order of the Kolmogorov time scale. This relaxation process eventually ends with the error hitting the minimum as denoted by the circles in the figure. The decay time before the error reaches the bottom is denoted by $t^*$. In this sense, the assimilation does not have to be continuous due to the self-relaxation phenomenon, and it would still be successful unless the time step is too large that error has evolved far into the growing regime.

The Kolmogorov-scale normalized decay time $t^*/\tau_\eta$ of the ODA is plotted in Fig. 6a against the scaled assimilation wavenumber $(k_a - k_c)\eta$. With this scaling, the three curves are close to each other. Meanwhile, the error decay time increases with the assimilation wavenumber but with a subsequent sharp drop to zero once the whole range of wavenumbers are used in the ODA. In Fig. 6b, the variation of the magnitude of the entire error decay, $\delta_0 - \delta_{\text{min}}$, is plotted against the assimilation wavenumber. Here, $\delta_0$ represents the initial error and $\delta_{\text{min}}$ represents the minimum error shown in Fig. 5. Certainly, one would expect larger decay magnitude with more ‘correct’ information being supplied (i.e. large assimilation wavenumber). This is reflected by the initial increase of the decay magnitude with the assimilation wavenumber. However, this trend should not continue with very large assimilation wavenumber due to the following reason: with the increasing of the $k_a$, the errors left for decay at $k > k_a$ also decrease. Therefore, the decay magnitude eventually

FIG. 5. The evolution of the error magnitude for the ODA: (a) $N = 64^3$, $Re_\lambda = 60$; (b) $N = 128^3$, $Re_\lambda = 105$; (c) $N = 256^3$, $Re_\lambda = 160$. 
drops with $k_a$ as shown in the figure. To further understand the evolution of errors, we shall next examine both the initial decay rate and the subsequent growth rate of the error for the ODA.

The initial exponential decay constant $a$ is shown in Fig. 7a, along with the decay constants for the continuous case. As discovered in the previous work [9], the decay constants for continuous assimilation follow closely a linear law as given by Eq. (13), with the data passing through the origin when scaled by $(k_a - k_c) \eta$. The slope constant $-\beta \approx -1.35$ as given in Section IIA. Interestingly, the initial decay constant for the one-step case is very close to the continuous case. The exception is only for the very small assimilation wavenumbers. This should not be surprising since for a moderate level of error, the one-step assimilation is still expected to lower the error for a short period even using a very small assimilation wavenumber. However, this could not occur if the initial error is relatively small, thus any assimilation (being continuous or not) with such assimilation numbers are eventually unsuccessful. A more direct view of the initial decay constant is shown in Fig. 7b, where evolution of errors is shown for both the continuous case and the one-step case. Clearly, the one-step assimilation has the same error decaying rate at the initial stage, but the decay rate gradually decreases with time.

Finally, it is interesting to examine the growing rate of errors after the ODA curves reach the minimum (cf. Fig. 5), as it is related to the instability and chaotic behavior of turbulence. In this consideration, we plot the exponential growth constant against the
Taylor Reynolds number in Fig. 8. Also shown in the figure is the largest Lyapunov exponent curve for incompressible isotropic turbulence reported by Mohan and Fitzsimmons [58], who have found that the largest Lyapunov exponent normalized by the Kolmogorov time scale increases with Taylor Reynolds number with saturation at large Reynolds number as opposed to stay universally constant [59, 60]. As shown in Fig. 8, this trend is also captured by our analysis. It is well known in chaotic theory that a chaotic system would eventually be dominated by the largest Lyapunov exponent [61]. This is also demonstrated by our analysis as we recall that, in Fig. 5, after hitting their minimums, all the curves share a similar slope that depends on the Taylor Reynolds number, reflecting the dominance of the largest Lyapunov exponent.

From the discussions of Fig. 7, it can be now easily understood why the large-time step assimilation can be as good as the continuous case, since there is an initial period when the one-step assimilation has the same decaying rate as the continuous case. However, it is yet unclear regarding the intriguing behavior that the large-step assimilation can have, in some cases, slightly larger decaying rate than the continuous case (cf. Fig. 2 and Fig. 3b). This issue will be examined next.
C. A Further analysis on the error spectrum in data assimilation

To further scrutinize the evolution of errors in the data assimilation process, it is necessary to dissect the assimilation error into each length scale so that the individual contribution from different length scales to the error decaying can be visualized. To this end, we define the error spectrum as [8]

\[
E_\delta(k) = \sum_{k - \frac{1}{2} \leq |k| < k + \frac{1}{2}} \frac{1}{2} [\hat{u}(k, t) - \hat{u}^{ref}(k, t)]^2. \tag{14}
\]

Apparently, the error spectrum should vanish everywhere at all wavenumbers if \( u \) converges to \( u^{ref} \).

In Fig. 9, the error spectrum for the continuous assimilation of the \( Re_\lambda = 105 \) case is illustrated. The results for other cases are very similar and thus not reproduced. In the figure, three assimilation wavenumbers are considered, namely, \( k_a = 5 \) for the unsuccessful assimilation in Figs. 9a and 9b, \( k_a = 8 \) for the neutral case (i.e. marginal assimilation) in Figs. 9c and 9d, and \( k_a = 15 \) for the successful assimilation in Figs. 9e and 9f. The correct energy spectrum \( E(k) \) is also shown in all these figures for comparison. When the assimilation wavenumber is insufficient (cf. Figs. 9a and 9b), the errors invariably grow for the wavenumbers larger than \( k_a \). Meanwhile, the errors at larger wavenumbers grow
faster initially since small scales possess less energy and are thus more sensitive. Also, these scales are somehow less constrained since they are farther away from the assimilation wavenumber. Once the errors in the larger wavenumbers have grown to a certain level, the errors at wavenumbers near the assimilation wavenumber also start to grow due to the accumulation of errors at small scales. The growth of errors finally saturates at the same level of the energy spectrum of the true field, i.e. $E_\delta(k) \approx E(k)$ for $k > k_a$, indicating that $u$ and $u^{ref}$ are almost completely uncorrelated at high wavenumbers.

In the marginal case (cf. Figs. 9c and 9d), there is initially a slight increase of error in the high wavenumber range and a weak decrease of errors for the wavenumbers close to the assimilation wavenumber. But this process soon comes to an end with the distribution of errors among all wavenumbers fixed and maintained at the same level, giving rise to the neutral state. Finally, for a successful data assimilation as shown in Figs. 9e and 9f, the errors at all wavenumbers invariably decrease with time until saturating to the machine-error level with some oscillatory noises. After observing the error spectrum for the continuous data assimilation, we are in a position to re-examine the large-step assimilation and make comparative analysis. This shall be addressed next.

To unravel why the large-step assimilation can have slightly larger decaying rate of errors, it is necessary to look closer into the error evolution at the very early stage of the assimilation. This is shown in Fig. 10, where Fig. 10b is a further zoom-in view of Fig. 10a. Taking $\Delta T = 2^5 \Delta t$ for instance, it is obvious that its initial decaying rate is the same as the continuous case (represented by the $\Delta T = 2^6 \Delta t$ curve), but it gradually decreases with time. However, at each subsequent assimilation step, the error experiences a steep drop which may drag the error below and further below the continuous line such as in the $\Delta T = 2^5 \Delta t$ case. It is precisely this interesting behavior that gives rise to the slightly faster error decaying rate of the large-step case over the continuous case. To further understand this behavior, it is necessary to compare the error spectra of the continuous and one-step data assimilation.

The error spectra of the continuous case and the one-step case are shown in Fig. 11 for all three cases listed in Table I. Here the assimilation wavenumber $k_a = 8$, 15 and 40 for the $Re_\lambda = 60$, 105 and 160 cases, respectively. Also, the normal coordinate scale is used, instead of the widely adopted log-log scale, to more directly visualize the magnitude of the errors from each scale.

From Fig. 11, one immediately observes that the errors at $k > k_a$ of the one-step case
FIG. 9. The evolution of the error spectrum for the TCDA: (a) $Re_\lambda = 60, k_a = 6$; (b) $Re_\lambda = 60, k_a = 8$; (c) $Re_\lambda = 105, k_a = 10$; (d) $Re_\lambda = 105, k_a = 15$; (e) $Re_\lambda = 160, k_a = 30$; (f) $Re_\lambda = 160, k_a = 40$.

decay faster than the continuous case, indicating that, given some free-relaxation time, the errors at small scales can drop faster than the continuous case. One can also observe that, while the errors above the assimilation wavenumber decrease with time, the errors below the assimilation wavenumber increase since no large-scale data is available after the initial supply. On the other hand, the errors at all scales decrease with time in the continuous case since the
assimilation is performed continuously. At the beginning, the advantage of the one-step case above the assimilation wavenumber is roughly in balance with its disadvantage below the assimilation wavenumber compared to the continuous case, such that both the one-step case and continuous case share a similar decaying rate initially. Later, the difference between the one-step and the continuous cases becomes larger as the large-scale errors grow too much for the former.

However, this would not happen in large-step TSDA case if new data in future time steps comes in and annihilates the large-scale deficit for the one-step case while its small scale advantage is maintained. This process is exactly what behind the slightly faster decaying rate for some large-step assimilation cases, and is directly responsible for the sharp drop of total errors as shown in Fig. 10. Certainly, the time step cannot be too large, otherwise the deficit would be too much to overcome by the next supply of data. It is also worth noting that the large-step assimilation only slightly outperforms the continuous case in terms of error dropping rate, since once the error drops, its advantage at larger wavenumbers would also drop. Nevertheless, its much-reduced amount of required data and consequently the higher computational efficiency are definitely meaningful.

At this point, though the mechanism of large-step assimilation is to some extent unraveled, the mathematical or physical reasons regarding why the errors relax faster for large-step assimilation than the continuous one are still not clear. We conjecture that the errors at machine level is potentially a cause to this phenomenon. First, it is understood that, even
though one can deem a particular DNS solution as the ‘correct’ solution, the solution cannot be exactly stored unless with a machine with infinite accuracy. In this sense, the true solution \( u^\text{ref} \) inevitably contains some small errors. In turn, these round-off errors would affect the assimilation process such that the error eventually fluctuates about the machine-error level, instead of going exactly to zero. On the other hand, we also anticipate that these small errors would have more effects on smaller scales (i.e. high wavenumbers) since they possess much less energy, and are thus less stable. Here we also recall the relatively earlier growth of small-scale errors as observed in Fig. 9.

As discussed, the continuous assimilation process in effect introduces more errors (albeit small) than the large-step case. While these ‘errors’ have negligible influences on large scales, they have negative effects on the smaller scales as illustrated in Fig. 11. However, the difference of the final error between the large-step and continuous assimilation is invisible since the machine-error is relatively too small and also contains randomness. To study the effect of errors of large-scale flow fields, we artificially contaminate \( u^\text{ref} \) by incorporating errors much larger than the machine round-off level. The results for the evolution of errors are shown in Fig. 12, where the reference field is replaced by an erroneous field, given as

\[
\hat{u}^e(k,t) = (1 + \varepsilon_e)\hat{u}^\text{ref}(k,t),
\]

where \( \varepsilon_e = 1 \times 10^{-3} \) is deliberately chosen such that \( \varepsilon_e < \varepsilon \) (we recall that \( \varepsilon \) is the perturbation in the initial condition of \( u \)). In this case, it is still reasonable to expect that the magnitude of error \( \delta \) should drop to the order of \( 10^{-3} \) if the assimilation is successful. In Fig. 12, the results for both the TCDA and TSDA with different time steps are included.
Indeed, a drop of errors to the level of $10^{-3}$ occurs for the tested cases. As expected, at the end of the assimilation, the TCDA has an error about 35 percent larger than the TSDA with a properly chosen time step. In the zoom-in view as shown in Fig. 12b, one also clearly recognizes the faster decaying rate of error for the large-step case (see the $\Delta T = 2^5 \Delta t$ case for instance).

A more straightforward visualization of the difference between the performance of the TCDA and the TSDA is given in Fig. 13. Here $\Delta T = 32 \Delta t$ is chosen for the TSDA. As we can see, for both the TCDA and TSDA, the errors at the end of the assimilation linearly increase with the increasing of the large-scale errors. Meanwhile, the TCDA clearly possesses more errors than the TSDA as the errors in the large scales increase. Here, using the least square fitting, the roughly linear behavior of the final assimilation error can be written as

$$\delta_f = c\varepsilon e,$$

where the constant $c = 1.33$ for the TCDA and 1.14 for the TSDA. As a consequence, it is potentially more beneficial to adopt large-step assimilation if the reference data contains non-negligible errors.
IV. INVESTIGATION ON THE POSSIBILITY OF THE SUB-KOLMOGOROV-SCALE RECOVERY USING DATA ASSIMILATION

As reported by several previous works [48–52], flow structures smaller than the Kolmogorov length scales exist due to the spatial intermittency and cannot be resolved by the Kolmogorov scale resolution that is generally adopted in DNS. On the other hand, since the small scales are slaved to the large scales based on the DA experiment, it is tempting to think that the small scales should be recoverable by the DNS of Kolmogorov scale resolution. However, the question is more intriguing than it seems due to the intrinsic issue of the grid resolution for a highly nonlinear problem like the NS system. It should be emphasized that, in the DA scheme of previous sections as well as those in previous works [8, 9], the reference data comes from the same grid resolution as the DA-based numerical solution. In other words, the large-scale data are obtained by projecting the fine solution to a coarser Fourier grid. In this case, to recover the sub-Kolmogorov scales, one would have to use finer grids so as to ‘recover’ them, as if they are computed from the fine grid that is ‘sufficient’ to capture all sub-Kolmogorov scales. With the DNS from a Kolmogorov resolution, however, the reference data is not from a projection of the true fine-grid solution, but from a coarser grid solution instead (i.e. the Kolmogorov resolution). This question is specifically raised in Ref [9].
Indeed, numerical solutions with coarser grid is not expected to give the same result as that with a finer grid for highly nonlinear problems such as the NS equation. This implies that the large-scale data obtained from the DNS of Kolmogorov resolution is not the same as that obtained through a finer grid DNS, and the discrepancy is expected to grow with time. Nevertheless, it is fundamentally interesting to examine what exactly would happen if DA is applied in such cases.

For such a test, we first need a DNS of a sufficient resolution such that the sub-Kolmogorov scales can also be resolved and represented. In the current study, we choose the grid resolution $N = 128^3$ for this ‘correct’ reference simulation with the grid spacing $h_{DNS} = 2\pi/N$ slightly below $\eta/2$. Consequently, at least the scales at $\eta/2$ can be represented, and we assume that to be all the sub-Kolmogorov scales. Meanwhile, we run a second simulation with grid $N = 64^3$ such that $h_{DNS} \approx \eta$, which is taken as a sufficient Kolmogorov-resolution. Further, we also configure a simulation at grid $N = 32^3$ with $h_{DNS} \approx 2\eta$ for later discussion. The details of the numerical parameters are given in Table II. The initial conditions for all three cases are identically taken from the large scales ($k \leq 10$) of a fully-developed turbulent field $u_0$. In addition, the small scales are invariably set to zero. Consequently, we have the following initial conditions for all three cases

$$\hat{u}(k, t_0) = \begin{cases} \hat{u}_0(k), & \text{if } k \leq 10 \\ 0, & \text{if } k > 10 \end{cases}, \quad (17)$$

where $k \leq 10$ is intentionally chosen such that the two-thirds dealiasing rule [57] shall not affect the large scales for any of these adopted grids. With these experiments, it is expected that the sub-Kolmogorov scales should be presented in the $N = 128^3$ case, while certainly not in the $N = 64^3$ case due to the limitation of resolution.

To test the sub-Kolmogorov-scale recovery using DA, a new simulation is run using a $N = 128^3$ grid, with a randomly generated initial condition which is ‘erroneous’ with respect to the reference field. Two reference data sources for the DA process are adopted, namely the $N = 128^3$ and the $N = 64^3$ solutions. The former case corresponds to the usually

| Reso. | 32³ | 64³ | 128³ |
|-------|-----|-----|------|
| $k_{max}\eta$ | 1.11 | 2.27 | 4.48 |

**TABLE II.** Numerical simulation parameters of incompressible isotropic turbulence.
FIG. 14. The evolution of the error magnitude for the TCDA using different data sources and assimilation wavenumbers.

adopted DA procedure, i.e. the large-scale data come from the reference field generated using the same grid resolution, which is known to be viable as long as the large-scale data are sufficient ($k_a > k_c$). In the second case, however, the large-scale reference data are generated using a coarser grid ($N = 64^3$). How such a DA using a coarse-grid solution would behave constitutes the major concern of the current section. Both the TCDA and the TSDA are tested.

The evolution of errors using the TCDA is shown in Fig. 14. As expected, the TCDA-based simulation using the $N = 128^3$ DNS easily converges to the reference field at $k_a = 4$. Indeed, $k_a \eta > 0.2$ requires only $k_a > 2$. In contrast, the DAs using the $N = 64^3$ DNS turn out unsuccessful regardless of the amount of large scales being used. In all the DAs using the $N = 64^3$ data, the errors invariably drop initially but increases with time after reaching a minimum. This is somehow expected since the large-scale information for DA obtained from the coarsened $N = 64^3$ grid is not the ‘correct’ large-scale information which in rigorous sense can only be generated using the same resolution at $N = 128^3$.

As observed in Fig. 14, recovering the sub-Kolmogorov scales using the Kolmogorov DNS is not practical at least for the current isotropic turbulence. Nevertheless, it is interesting to note that the DA errors still decrease for some time initially since large scales contain less error during the initial stage. Thus, the DA would still function for some time even
in the presence of some propagated errors from the small scales as discussed in Section III. This is also reflected in Fig. 14 that the two $k_a = 4$ curves for DAs using the $N = 128^3$ and the $N = 64^3$ DNS initially overlap. However, as the large-scale errors gradually grow in the $N = 64^3$ case, the corresponding DA fails with two curves finally going separated paths. It is also interesting to note the all the curves for DA errors in the failed cases eventually merge into one single curve since the reference data comes from the same resolution. As such, it is not surprising that they all reach the same attractor eventually.

Fig. 15 displays the error evolution for the TSDA, where $k_a = 4$ in Fig. 15a and $k_a = all$ (i.e. all the wavenumbers are used) in Figs. 15b and 15c. Here, it should be noted that more errors are introduced for $k_a = all$ compared to $k_a = 4$. As can be seen, due to the corruption of the large scales with time, the TSDA cannot reconstruct the sub-Kolmogorov scales either. Nevertheless, in Fig. 15b and its zoom-in view Fig. 15c, it is interesting to observe that the TSDA is able to keep the errors below the TCDA level by absorbing data less frequently when $k_a = all$, even though it eventually merges with TCDA due to the growing errors of the reference field. Indeed, if the reference field becomes completely different from the ‘true’ field with time, the data assimilation would eventually fail regardless of whether TCDA or TSDA is adopted. To visualize the corruption of the reference field with time, we present in Fig. 16 the evolution of rms-normalized vorticity field of the large scales generated using different grid resolutions. Here the large scales are obtained using a sharp-spectral filter [53], given by
\[ \hat{u}(k, t) = \begin{cases} \hat{u}(k), & \text{if } k \leq k_\Delta \\ 0, & \text{if } k > k_\Delta \end{cases} \quad (18) \]

where \( k_\Delta = 10 \) is chosen for the consistency with the initial condition given by Eq. (17).

In the figure, the result obtained using the \( N = 32^3 \) grid is also included for comparison. As discussed, we assume that the grid \( N = 128^3 \) is the sufficient resolution that all scales including the sub-Kolmogorov ones can be correctly generated. As the figure shows, the large-scale structures are almost the same at \( t \approx 28\tau_\eta \) for all three grids. When reaching \( t \approx 57\tau_\eta \), however, the results with the \( N = 32^3 \) case becomes very different while the results of the \( N = 64^3 \) and \( 128^3 \) resolutions are still quite alike. Finally at \( t \approx 113\tau_\eta \), both the \( N = 64^3 \) and \( N = 32^3 \) solutions are substantially corrupted with respect to the \( N = 128^3 \) case, due to the coarsening of the grid. With these observations, it is not difficult to understand the failure of DA using numerical solutions from a coarsened resolution: if the large-scale solutions from a coarsened grid deviate substantially from the fine-grid solution, it would be difficult for the DA to be viable based on these ‘corrupted’ large scales.

The foregoing results in this section seem to suggest that DA is not a viable method for recovering the sub-Kolmogorov scales using the Kolmogorov-scale solutions. However, one should also be prudent before questioning the validity of the widely adopted DNS resolution due to the limitations in the isotropic turbulence. We recall that for isotropic turbulence, the commonly adopted periodical boundary condition in effect has very little restrictions on the large scales except for the spatial periodicity. Meanwhile, the forcings on the two largest wavenumbers control only the large-scale energy instead of fluid velocity. In this case, the large scales can freely evolve and have more vulnerability to be affected by small-scale perturbations. In contrast, if a flow is bounded by a solid wall, the flow structures probably do not freely evolve without control (at least in the immediate vicinity of the wall). Such issues shall be pursued in future works.

Indeed, with isotropic turbulence, one concerns more about the flow statistics and the energy cascade instead of the exact spatial-temporal flow structures. As shown in Fig. 17, the energy spectra of all three grid resolutions merge well at the end of the simulation even though their local structures are quite different as shown in Fig. 16.
FIG. 16. The evolution of contour field of normalized vorticity with the same large-scale initial conditions using different grid resolutions.

V. CONCLUSIONS

In the present study, the Fourier-space DA for direct numerical simulation of incompressible HIT is examined in the cases of large assimilation time steps. Compared with the existing TCDA, the proposed TSDA method has significantly reduced the amount of required data while the accuracy is still maintained.

Assimilation time steps $\Delta T$ ranging from $2^1$ to $2^{10}\Delta t$ are tested. In the examined cases, it is shown that for assimilation time step within 1 to 1.5 times the Kolmogorov time scale, the performance of the TSDA is at least the same as the TCDA or even slightly better. Consequently, the amount of required data for the TSDA can be significantly reduced.
FIG. 17. The energy spectrum of the DNS at a fixed Kolmogorov scale using different grid resolutions.

compared to the TCDA.

To explore the mechanism of the TSDA, the ODA is analyzed through both the evolution of total error magnitude and the error spectrum. For ODA, the error is observed to decrease initially followed by a somehow exponential growth after hitting a minimum. The initial error decaying rate of ODA is very close to that of TCDA, but the decaying of error gradually slows down. Meanwhile, the behavior of the exponential growth coefficient for the error after it reaches the minimum is found to be consistent with the behavior of the largest Lyapunov exponent.

A detailed comparison is presented between ODA and TCDA in terms of their error spectra. It is found that the relaxation effect for errors above the assimilation wavenumber $k_a$ is responsible for the faster decaying rate of errors with the TSDA than the TCDA. This relaxation effect is related to the errors contained in the large scales which lead to the errors above $k_a$ decay faster with the TSDA than the TCDA. This is confirmed by artificially incorporating different levels of errors in the reference velocity field. The advantage of the TSDA is found to grow with the increasing of the incorporated errors. Hence, it is potentially more beneficial to adopt the large-step assimilation strategy if the reference data contains non-negligible errors.

Finally, the possibility of recovering sub-Kolmogorov scales with Kolmogorov-scale grids
is investigated using both the TCDA and the TSDA. The results show that it is hard to recover the sub-Kolmogorov scales due to the freedom of large scales of HIT. Further investigations on the sub-Kolmogorov influence with more constrained boundary conditions shall be pursued in future works.

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