New results on path-decompositions and their down-links

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Abstract

In [3] the concept of down-link from a \((K_v, \Gamma)\)-design \(B\) to a \((K_n, \Gamma')\)-design \(B'\) has been introduced. In the present paper the spectrum problems for \(\Gamma' = P_4\) are studied. General results on the existence of path-decompositions and embeddings between path-decompositions playing a fundamental role for the construction of down-links are also presented.

Keywords: \((K_v, \Gamma)\)-design; down-link; embedding.

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1 Introduction

Suppose \(\Gamma \leq K\) to be a subgraph of \(K\). A \((K, \Gamma)\)-design, or \(\Gamma\)-decomposition of \(K\), is a set of graphs isomorphic to \(\Gamma\) whose edges partition the edge set of \(K\). Given a graph \(\Gamma\), the problem of determining the existence of \((K_v, \Gamma)\)-designs, also called \(\Gamma\)-designs of order \(v\), where \(K_v\) is the complete graph on \(v\) vertices, has been extensively studied; see the surveys [4, 5]. In [3] we proposed the following definition.

**Definition 1.1.** Given a \((K, \Gamma)\)-design \(B\) and a \((K', \Gamma')\)-design \(B'\) with \(\Gamma' \leq \Gamma\), a down-link from \(B\) to \(B'\) is a function \(f: B \to B'\) such that \(f(B) \leq B\), for any \(B \in B\).

When such a function \(f\) exists, we say that it is possible to down-link \(B\) to \(B'\).
As seen in [3], down-links are closely related to metamorphoses [8], their generalizations [9] and embeddings [11]. In close analogy to embeddings, we introduced spectrum problems about down-links:

(I) For each admissible \( v \), determine the set \( \mathcal{L}_1(\Gamma)(v) \) of all integers \( n \) such that there exists some \( \Gamma \)-design of order \( v \) down-linked to a \( \Gamma' \)-design of order \( n \).

(II) For each admissible \( v \), determine the set \( \mathcal{L}_2(\Gamma)(v) \) of all integers \( n \) such that every \( \Gamma \)-design of order \( v \) can be down-linked to a \( \Gamma' \)-design of order \( n \).

In [3, Proposition 3.2], we proved that for any \( v \) such that there exists a \( (K_v, \Gamma) \)-design and any \( \Gamma' \leq \Gamma \), the sets \( \mathcal{L}_1(\Gamma)(v) \) and \( \mathcal{L}_2(\Gamma)(v) \) are always non-empty. In the same paper the case \( \Gamma' = \Gamma \) has been investigated in detail.

Here we shall deal with the case \( \Gamma' = P_4 \). In order to get results about down-links to \( P_4 \)-designs, we shall first study path-designs and their embeddings. More precisely, in Section 2 we determine sufficient conditions for the existence of \( P_4 \)-decompositions of any graph \( \Gamma \) and \( P_k \)-decompositions of complete bipartite graphs. In Section 3 applying the results of Section 2 we are able to prove the existence of embeddings and down-links between path-designs. Section 4 is devoted to the cases of cycle systems and path-designs, with general theorems and directed constructions.

Throughout this paper the following standard notations will be used; see also [7]. For any graph \( \Gamma \), write \( V(\Gamma) \) for the set of its vertices and \( E(\Gamma) \) for the set of its edges. If \( \mathcal{B} \) is a collection of graphs, by \( V(\mathcal{B}) \) we will mean the set of the vertices of all its elements. By \( t\Gamma \) we shall denote the disjoint union of \( t \) copies of graphs all isomorphic to \( \Gamma \). As usual, \( P_k = [a_1, \ldots, a_k] \) is the path with \( k-1 \) edges and \( C_k = (a_1, \ldots, a_k) \), \( k \geq 3 \), is the cycle of length \( k \). Also, \( K_{m,n} \) is the complete bipartite graph with parts of size \( m \) and \( n \). When we focus on the actual parts \( X \) and \( Y \), \( K_{X,Y} \) will be written.

## 2 Existence of some path-designs

In this section we present new results on the existence of path decompositions. Recall that a \( (K_n, P_k) \)-design exists if, and only if, \( n(n-1) \equiv 0 \pmod{2(k-1)} \); see [13].

**Proposition 2.1.** Let \( k \) be an even integer. For \( x = k-2, k \) the complete bipartite graph \( K_{k-1,x} \) admits a \( P_k \)-decomposition.

**Proof.** Consider the bipartite graph \( K_{A,I} \) where \( A = \{a_1, \ldots, a_{k-1}\} \) and \( I = \{1, \ldots, x\} \) with \( x = k-2, k \).

Let \( U^t = (1, \ldots, 1) \) be an \( \frac{x}{2} \)-tuple. Set \( P_1^t = (1, \ldots, \frac{x}{2}) \) and for \( i = 1, \ldots, \frac{x}{2} \),
Theorem 2.2. Let $\Gamma$ be a graph with at least two vertices of degree $|V(\Gamma)|-1$. Then $\Gamma$ admits a $P_4$-decomposition if, and only if, $|E(\Gamma)| \equiv 0 \pmod{3}$. If $|E(\Gamma)| \equiv 1, 2 \pmod{3}$, then $\Gamma$ can be partitioned into a $P_4$-decomposition together with one or two (possibly connected) edges, respectively.

Proof. The condition is obviously necessary. For sufficiency, let $\alpha$ and $\beta$ be two vertices of degree $|V(\Gamma)|-1$. Delete $\alpha$ and $\beta$ in $\Gamma$, as to obtain a graph $G$. Let $G'$ be a maximal $P_4$-decomposable subgraph of $G$ and remove from $G$ the edges of $G'$, determining a new graph $G''$. In general, $G''$ is not connected and its connected components are either isolated vertices or stars or cycles of length 3; call $\mathcal{I}$, $\mathcal{S}$ and $\mathcal{C}$ their (possibly empty) sets. Let $\Gamma'$ be the graph obtained removing the edges of $G'$ from $\Gamma$. Clearly, $|E(\Gamma)| \equiv 0 \pmod{3}$ implies $|E(\Gamma')| \equiv 0 \pmod{3}$; thus it remains to show that $E(\Gamma')$ is $P_4$-decomposable. Obviously $\alpha$ and $\beta$ are of degree $|V(\Gamma)|-1$ also in $\Gamma'$. Let $A = \{\alpha, \beta\}$ and consider the following decomposition $\Gamma' = K_A \cup K_{A,I} \cup (C \cup K_{A,V(S)}) \cup (S \cup K_{A,V(S)})$. We begin by providing, separately, $P_4$-decompositions of $K_{A,I}$, $C \cup K_{A,V(S)}$ and $S \cup K_{A,V(S)}$.

i) It is easy to see that for any 3-subset of $\mathcal{I}$, say $H_3$, the graph $K_{A,H_3}$ has a $P_4$-decomposition. Thus, depending on the congruence class modulo 3 of $|\mathcal{I}|$, $K_{A,I}$ can be partitioned into a $P_4$-decomposition together with the following possible remnants.

| Case | $|\mathcal{I}| \equiv i \pmod{3}$ |
|------|----------------|
| (i_1) $|\mathcal{I}| \equiv 0 \pmod{3}$ | the set $\emptyset$ |
| (i_2) $|\mathcal{I}| \equiv 1 \pmod{3}$ | the path $[\alpha, h, \beta]$ with $h \in \mathcal{I}$ |
| (i_3) $|\mathcal{I}| \equiv 2 \pmod{3}$ | the cycle $(h_1, \alpha, h_2, \beta)$ with $h_1, h_2 \in \mathcal{I}$ |

Table 1: Case i.
ii) For any 3-cycle \( C \in \mathcal{C} \), the graph \( C \cup K_{A,V(C)} \) has a \( P_4 \)-decomposition. Thus, \( \mathcal{C} \cup K_{A,V(C)} \) also admits a \( P_4 \)-decomposition.

iii) It is not difficult to see that, for any star \( S_c \in \mathcal{S} \) of center \( c \), the graph \( S_c \cup K_{A,V(S_c)} \) has a partition into a \( P_4 \)-decomposition together with either the path \( [\alpha, c, \beta] \) or the graph \( (\alpha, c, \beta, v) \cup [c, v] \), where \( v \) is any external vertex, depending on whether the number of vertices of \( S_c \) is odd or even.

Let \( S_1 \) (respectively \( S_2 \)) be the set of stars with an odd (even) number of vertices. For any three stars of \( S_1 \) (\( S_2 \)) the remnants give \( P_4 \)-decomposable graphs. So \( S_1 \cup K_{A,V(S_1)} \), as well as \( S_2 \cup K_{A,V(S_2)} \), can be partitioned into a \( P_4 \)-decomposition together with the possible remnants outlined in Tables 2 and 3.

| (iii\_11) | (iii\_12) | (iii\_13) |
|-----------|-----------|-----------|
| \(|S_1| \equiv 0 \pmod{3}\) | \(|S_1| \equiv 1 \pmod{3}\) | \(|S_1| \equiv 2 \pmod{3}\) |
| \(\emptyset\) | the path \( [\alpha, c, \beta] \) where \( c \) is the center of a star | the cycle \( (c_1, \alpha, c_2, \beta) \) where \( c_1, c_2 \) are centers of two stars |

Table 2: Case (iii\_1): \( S_1 \cup K_{A,V(S_1)} \).

| (iii\_21) | (iii\_22) | (iii\_23) |
|-----------|-----------|-----------|
| \(|S_2| \equiv 0 \pmod{3}\) | \(|S_2| \equiv 1 \pmod{3}\) | \(|S_2| \equiv 2 \pmod{3}\) |
| \(\emptyset\) | the graph \( (\alpha, c, \beta, v) \cup [c, v] \) where \( c \) is the center and \( v \) is an external vertex of a star | the graph \( \bigcup_{i=1}^{2} (\alpha, c_i, \beta, v_i) \cup [c_1, v_i] \) where \( c_1, c_2 \) are centers and \( v_1, v_2 \) are external vertices of two stars |

Table 3: Case (iii\_2): \( S_2 \cup K_{A,V(S_2)} \).

The remnants from \( i \), (iii\_1) and (iii\_2) together with the edge \( [\alpha, \beta] \) can be combined in 27 different ways to obtain 27 connected graphs with \( t \) edges. It is a routine to check that we have exactly 9 cases with \( t \equiv i \pmod{3} \), for \( i = 0, 1, 2 \).

In Table 4 we will list in detail the 9 cases with \( t \equiv 0 \pmod{3} \) and, for each of them, in Table 5 we give the corresponding graph.
Table 4: $t \equiv 0 \pmod{3}$.

|   | $i$ | $i_{ii1}$ | $i_{ii2}$ |
|---|---|---|---|
| $a_1$ | $\emptyset$ | $\emptyset$ | $i_{ii22}$ |
| $a_2$ | $\emptyset$ | $i_{i113}$ | $i_{i213}$ |
| $a_3$ | $\emptyset$ | $i_{i112}$ | $\emptyset$ |

|   | $i$ | $i_{ii1}$ | $i_{ii2}$ |
|---|---|---|---|
| $a_4$ | $i_{i2}$ | $\emptyset$ | $\emptyset$ |
| $a_5$ | $i_{i2}$ | $i_{i113}$ | $i_{i222}$ |
| $a_6$ | $i_{i2}$ | $i_{i112}$ | $i_{i233}$ |
| $a_7$ | $i_{i3}$ | $\emptyset$ | $i_{i233}$ |
| $a_8$ | $i_{i3}$ | $i_{i113}$ | $\emptyset$ |
| $a_9$ | $i_{i3}$ | $i_{i112}$ | $i_{i222}$ |

Figure 1: Case $a_1$. Figure 2: Case $a_8$. Figure 3: Cases $a_5$ and $a_9$.

Figure 4: Cases $a_2$, $a_6$ and $a_7$. Figure 5: Cases $a_3$ and $a_4$.

Table 5: Graphs of the remnants plus edge $[\alpha, \beta]$.

It is easy to determine a $P_4$-decomposition of the graphs in Figures 1, 2, 3, 4. In cases $a_3$ and $a_4$ (Figure 5) a $P_4$-decomposition is clearly not possible, thus we proceed back tracking one step in the construction. How to deal with case $a_3$ is explained in Figure 6.

In case $a_4$ we have to distinguish several subcases depending on the size of $\mathcal{I}, \mathcal{C}$ and $\mathcal{S}$. When $|\mathcal{I}| > 1$ see Figure 7. For $|\mathcal{I}| = 1$ and $|\mathcal{C}| \neq 0$, see Figure 8.
Figure 7: If $|\mathcal{I}| > 1$, we recover the two $P_4$’s from 3 vertices of $\mathcal{I}$.

Figure 8: If $|\mathcal{I}| = 1$ and $|\mathcal{C}| \neq 0$ we recover the three $P_4$’s from a $C_3$.

When $|\mathcal{I}| = 1$ and $|\mathcal{C}| = 0$ we have two possibilities. If there is one star of $\mathcal{S}$ with at least two edges, we proceed as explained in Figure 9.

Figure 9: If $|\mathcal{I}| = 1$, $|\mathcal{C}| = 0$, and $\exists S_c \in \mathcal{S}$ with $P_3 \leq S_c$ we recover the two $P_4$’s from 2 radii of $S_c$.

Otherwise, $G''$ consists of an isolated vertex $h$ and a set $\mathcal{P}$ of disjoint $P_2$’s. Since $|E(\Gamma')| \equiv 0 \pmod{3}$, the size of $\mathcal{P}$ is also divisible by 3, let $|\mathcal{P}| = 3p$. It is easy to see that for any 3-subset of $\mathcal{P}$, say $P^3$, the graph $K_{A,P^3}$ has a $P_4$-decomposition. After $p - 1$ steps, the remnant is the graph in Figure 10, which likewise admits a $P_4$-decomposition. This concludes the case $t \equiv 0 \pmod{3}$.

Figure 10: If $|\mathcal{I}| = 1$, $|\mathcal{C}| = 0$ and $S$ is a disjoint union of $P_2$'s we recover the 15 edges from the last but one step.

With similar arguments, when $t \equiv 1, 2 \pmod{3}$ it is possible to find a $P_4$-decomposition of $E(\Gamma)$ leaving as remnants, respectively, one or two edges. \qed
3 Embeddings and down-links to $P_4$-designs

The results presented in the previous section are used to prove the existence of embeddings and down-links to path designs. In particular, we shall focus our attention on $P_4$-decompositions.

**Theorem 3.1.** Any partial $(K_v,P_4)$-design can be embedded into a $(K_n,P_4)$-design for any admissible $n \geq v + 2$.

**Proof.** Let $B$ be a partial $(K_v,P_4)$-design. Let $A$ be a set of vertices disjoint from $V(K_v)$ with $v + |A| \equiv 0,1 \pmod{3}$ and $|A| \geq 2$. Let $\Gamma$ be the graph such that $V(\Gamma) = V(K_v) \cup A$ and $E(\Gamma) = E(K_{v+|A|}) \setminus E(B)$. Since $|A| \geq 2$, by Theorem 3.2 there exists a $(\Gamma,P_4)$-design $B'$ and, clearly, $B \cup B'$ is a $(K_{v+|A|},P_4)$-design.  

**Corollary 3.2.** For any $(K_v,\Gamma)$-design with $P_4 \leq \Gamma$

$$\{n \geq v + 2 \mid n \equiv 0, 1 \pmod{3}\} \subseteq \mathcal{L}_2 \Gamma(v) \subseteq \mathcal{L}_1 \Gamma(v).$$

**Proof.** Let $B$ be a $(K_v,\Gamma)$-design with $P_4 \leq \Gamma$. Choose a $P_4$ in each block of $B$ and call $\mathcal{P}$ the set of such $P_4$’s. Obviously, $\mathcal{P}$ is a partial $P_4$-decomposition of $K_v$. Hence, by Theorem 3.1, $\mathcal{P}$ can be embedded into a $(K_n,P_4)$-design $B'$ for any admissible $n \geq v + 2$. The construction also guarantees the existence of a down-link from $B$ to $B'$.

**Theorem 3.3.** For any even integer $k$, a $P_k$-design of order $n \equiv 0, 1 \pmod{k-1}$ can be embedded into a $P_k$-design of any order $m > n + 1$ with $m \equiv 0, 1 \pmod{k - 1}$.

**Proof.** Let $B$ be a $(K_n,P_k)$-design with $n \equiv 0, 1 \pmod{k-1}$ and let $m = n + s \equiv 0, 1 \pmod{k-1}$. As $K_{n+s} = K_n \cup K_s \cup K_{n,s}$, for the existence of a $(K_m,P_k)$-design embedding $B$ it is enough to find a $P_k$-decomposition of $K_s \cup K_{n,s}$. Since $n,n+s \equiv 0, 1 \pmod{k - 1}$, one of the following cases occurs

- $n = \lambda(k-1), s = \mu(k-1) \Rightarrow K_s \cup K_{n,s} = K_s \cup \lambda \mu K_{k-1,k-1}$
- $n = \lambda(k-1), s = 1+\mu(k-1) \Rightarrow K_s \cup K_{n,s} = K_s \cup K_{\lambda(k-1),1+(\mu-1)(k-1)} = K_s \cup \lambda K_{k-1,k} \cup (\mu - 1)K_{k-1,k-1}$
- $n = 1+\lambda(k-1), s = \mu(k-1) \Rightarrow K_s \cup K_{n,s} = K_s \cup K_{1+\lambda(k-1),\mu(k-1)} = K_s \cup \mu K_{k-1,k} \cup (\lambda - 1)K_{k-1,k-1}$
- $n = 1+\lambda(k-1), s = k-2+\mu(k-1) \Rightarrow K_s \cup K_{n,s} = K_s \cup K_{1+\lambda(k-1),s} = K_{s+1} \cup K_{\lambda(k-1),k-2+\mu(k-1)} = K_{s+1} \cup \lambda K_{k-1,k-2} \cup \lambda \mu K_{k-1,k-1}$

So, to find a $P_k$-decomposition of $K_s \cup K_{n,s}$ it is sufficient to know $P_k$-decompositions of
• $K_s$ and $K_{s+1}$, which exist by [13],
• $K_{k-1,k-1}$, whose existence is proved in [10],
• $K_{k-1,k}$ and $K_{k-1,k-2}$, whose existence follows from Proposition 2.1.

The following corollary is a straightforward consequence of Theorem 3.3.

**Corollary 3.4.** If $n \in \mathcal{L}_i \Gamma(v)$, then

\[
\{ m \geq n + 2 \mid m \equiv 0, 1 \pmod{3} \} \subseteq \mathcal{L}_i \Gamma(v).
\]

**Remark 3.5.** Set $\eta_i = \inf \mathcal{L}_i \Gamma(v)$. By Corollary 3.4, $\mathcal{L}_i \Gamma(v)$ contains all admissible values $m \geq \eta_i$ apart from (possibly) $\eta_i + 1$. Thus to exactly determine the spectra it is enough to compute $\eta_i$ and ascertain if $\eta_i + 1 \in \mathcal{L}_i \Gamma(v)$.

### 4 Cycle systems and path-designs

Here we shall provide some partial results on the existence of down-links from cycle systems and path-designs to $P_4$-designs.

We recall that a $k$-cycle system of order $v$, that is a $(K_v, C_k)$-design, exists if, and only if, $k \leq v$, $v$ is odd and $v(v - 1) \equiv 0 \pmod{2k}$; see [2], [12].

**Theorem 4.1.** For any admissible $v$ and any $k \geq 9$

\[
\left\{ n \geq v - \left\lfloor \frac{k - 9}{4} \right\rfloor \mid n \equiv 0, 1 \pmod{3} \right\} \subseteq \mathcal{L}_2 \Gamma(v) \subseteq \mathcal{L}_1 \Gamma(v).
\]

**Proof.** Let $k \geq 9$ and let $\mathcal{B}$ be a $(K_v, C_k)$-design. Write $t = \left\lfloor \frac{k - 9}{4} \right\rfloor$. Take $t + 2$ distinct vertices $x_1, x_2, \ldots, x_t, y_1, y_2 \in V(K_v)$. Observe that it is possible to extract from each block $C \in \mathcal{B}$ a $P_4$ whose vertices are different from $x_1, x_2, \ldots, x_t, y_1, y_2$, as we are forbidding at most $4(t + 1) + 2 = 4t + 6 = k - 3$ edges from any $k$-cycle. Use these $P_4$’s for the down-link. Let $S$ be the image of the down-link, considered as a subgraph of $K_{v-t} = K_v \setminus \{x_1, \ldots, x_t\}$ and remove the edges of $S$ from $K_{v-t}$ to obtain a new graph $R$. It remains to show that $R$ admits a $P_4$-decomposition. Observe that $|V(R)| = v-t$ and $y_1, y_2$ are two vertices of $R$ of degree $v-t - 1$. To apply Theorem 2.2 we have to distinguish some cases according to the congruence class modulo 3 of $v-t$.

If $v-t \equiv 0 \pmod{3}$, then $|E(R)| \equiv 0 \pmod{3}$ so the existence of a $(R, P_4)$-design is guaranteed by Theorem 2.2. Furthermore, if we add a vertex to $K_{v-t}$ we can apply Theorem 2.2 also to $R' = R \cup K_{1,v-t}$ since $|E(R')| \equiv 0 \pmod{3}$. Hence there exist down-links from $\mathcal{B}$ to $(K_{v-t}, P_4)$-designs and...
to \((K_{v-t+1}, P_4)\)-designs. If \(v-t \equiv 1(\text{mod } 3)\), then \(|E(R)| \equiv 0(\text{mod } 3)\), hence by Theorem 2.2 there exists a \((R, P_4)\)-design. So we determine down-links from \(B\) to \((K_{v-t}, P_4)\)-designs.

Finally, if \(v-t \equiv 2(\text{mod } 3)\), it is sufficient to add either \(u = 1\) or \(u = 2\) vertices to \(K_{v-t}\) and then apply Theorem 2.2 to \(R'' = (K_{v-t} \cup K_u \cup K_{v-t-u}) \setminus S\) in order to down-link \(B\) to \((K_{v-t+1}, P_4)\)-designs or to \((K_{v-t+2}, P_4)\)-designs, respectively. The statement follows from Remark 3.5.

Arguing exactly as in the previous proof it is possible to prove the following result.

**Theorem 4.2.** For any admissible \(v\) and any \(k \geq 12\)

\[
\left\{ n \geq v - \left\lfloor \frac{k - 12}{4} \right\rfloor \mid n \equiv 0, 1 \pmod{3} \right\} \subseteq \mathcal{L}_2 P_k(v) \subseteq \mathcal{L}_1 P_k(v).
\]

### 4.1 Small cases

We shall now investigate in detail the spectrum problems for \(\Gamma = C_4\) and \(\Gamma = P_3\). In order to obtain our results, we shall extensively use the method of gluing of down-links, introduced in [2]. We briefly recall the main idea: a down-link from a \((K_v, \Gamma)\)-design to a \((K_n, \Gamma')\)-design can be constructed as union of down-links between partitions of the domain and the codomain. To give designs suitable for the down-link, we will use difference families; here we recall some preliminaries, for a survey see [1]. Let \(\Gamma\) be a graph. A set \(\mathcal{F}\) of graphs isomorphic to \(\Gamma\) with vertices in \(\mathbb{Z}_v\) is called a \((v, \Gamma, 1)\)-difference family (DF, for short) if the list \(\Delta \mathcal{F}\) of differences from \(\mathcal{F}\), namely the list of all possible differences \(x - y\), where \((x, y)\) is an ordered pair of adjacent vertices of an element of \(\mathcal{F}\), covers \(\mathbb{Z}_v \setminus \{0\}\) exactly once. In [6] it is proved that if \(\mathcal{F} = \{B_1, \ldots, B_t\}\) is a \((v, \Gamma, 1)\)-DF, then the collection of graphs \(B = \{B_i + g \mid B_i \in \mathcal{F}, g \in \mathbb{Z}_v\}\) is a cyclic \((K_v, \Gamma)\)-design.

**Lemma 4.3.** For any \(v \equiv 1, 9 \pmod{24}, v > 1\), there exists a down-link from a \((K_v, C_4)\)-design to a \((K_v, P_4)\)-design. For any \(v \equiv 9, 17 \pmod{24}\) there exists a down-link from a \((K_v, C_4)\)-design to a \((K_{v+1}, P_4)\)-design.

**Proof.** Take \(v = s + 24t \geq 9\), with \(s = 1, 9, 17\), and \(V(K_v) = \mathbb{Z}_v\). Consider the set of 4-cycles

\[
\mathcal{C} = \left\{ C^a = \left(0, a, \frac{v+1}{2}, \frac{v-1}{8} + a \right) \mid a = 1, 2, \ldots, \frac{v-1}{8} \right\}.
\]

It is straightforward to check that

\[
\Delta C^a = \pm \left\{ a, \frac{v+1}{2} - a, \frac{3v+5}{8} - a, \frac{v-1}{8} + a \right\}.
\]
Hence $\Delta C = \mathbb{Z}_v \setminus \{0\}$, so, by [6], the $C^a$ are the $\frac{v-1}{8}$ base blocks of a cyclic $(K_v, C_4)$-design. The development of each base block gives $v$ different 4-cycles, from each of which we extract the edge obtained by developing $[0,a]$. The obtained $P_4$'s will be used to define a down-link in a natural way. The removed edges can be connected to complete the $P_4$-decomposition of $K_v$ as follows: for each triple $\{[0,a+1],[0,a+2],[0,a+3]\}$, for $a \equiv 1 \pmod{3}$ where $a \in \{1,2,\ldots,\frac{v-1}{8}\}$, consider the three developments and connect the edges $\{[i+1,a+1+(i+1)], [i,a+2+i], [i,a+3+i]\}$ obtaining the paths $(i+1,a+i+2,i,a+i+3)$, with $i \in \mathbb{Z}_v$.

If $v \equiv 1 \pmod{24}$, we have the required $P_4$-decomposition.

If $v \equiv 9 \pmod{24}$, we have the required $P_4$-decomposition except for the development of $[0,1]$. The $v$ edges of such a development can be easily connected to give the $v$-cycle $C = (0,1,\ldots,v-1)$, which obviously admits a $P_4$-decomposition. So, for $v \equiv 1,9 \pmod{24}$, there exists a down-link from a $(K_v, C_4)$-design to a $(K_v, P_4)$-design. Under the assumption $v \equiv 9 \pmod{24}$, $n = v + 1$ is also admissible. In this case, add the vertex $\alpha$ to $V(K_v)$ to obtain a $K_{v+1}$ supporting the codomain of the down-link. Actually, the star $S_{[\alpha;V]}$ of center $\alpha$ and external vertices the elements of $V(K_v)$ has been added. Proceed as before till to the last but one step, namely do not decompose the $v$-cycle $C$ obtained by developing $[0,1]$. So it remains to determine a $P_4$-decomposition of the wheel $W = C \cup S_{[\alpha;V]}$.

It is easy to see that $W$ can be decomposed into $3 + 8t$ copies of the graph $W'$ in Figure 11 which evidently admits a $P_4$-decomposition.

![Figure 11: The graph $W'$ as union of two $P_4$'s.](image)

If $v \equiv 17 \pmod{24}$, proceeding as before, we determine the required $P_4$-decomposition except for the two developments, say $d_1$ and $d_2$, of the edges $[0,1]$ and $[0,\frac{v-1}{8}]$. Keeping in mind that we must also add a vertex, say $\alpha$, to the codomain, we have to arrange the edges of $d_1$, $d_2$ and $S_{[\alpha;V]}$. It easy to see that we can obtain the $P_4$'s as $[\alpha,1+i,i,\frac{v-1}{8}+i]$, for $i \in \mathbb{Z}_v$.

So, for $v \equiv 9,17 \pmod{24}$ there exists a down-link from a $(K_v, C_4)$-design to a $(K_{v+1}, P_4)$-design. \qed
Theorem 4.4. For any admissible \( v > 1 \),

\[
\mathcal{L}_1 C_4(v) = \{ n \geq v \mid n \equiv 0, 1 \ (\text{mod} \ 3) \}; \quad (1)
\]

\[
\{ n \geq v + 2 \mid n \equiv 0, 1 \ (\text{mod} \ 3) \} \subseteq \mathcal{L}_2 C_4(v) \subseteq \{ n \geq v \mid n \equiv 0, 1 \ (\text{mod} \ 3) \}. \quad (2)
\]

Proof. Let \( B \) and \( B' \) be, respectively, a \((K_v, C_4)\)-design and a \((K_n, P_4)\)-design. Suppose that \( B \) can be down-linked to \( B' \). Clearly, \( n \geq v \). Hence \( \mathcal{L}_2 C_4(v) \subseteq \mathcal{L}_1 C_4(v) \subseteq \{ n \geq v \mid n \equiv 0, 1 \ (\text{mod} \ 3) \} \).

To prove the reverse inclusion in (1) observe that a \((K_v, C_4)\)-design exists if, and only if, \( v \equiv 1 \ (\text{mod} \ 8) \) and a \((K_n, P_4)\)-design exists if, and only if, \( n \equiv 0, 1 \ (\text{mod} \ 3) \). So it makes sense to look for a down-link from a \((K_v, C_4)\)-design to a \((K_v, P_4)\)-design only for \( v \equiv 1, 9 \ (\text{mod} \ 24) \). Likewise, a down-link from a \((K_v, C_4)\)-design to a \((K_v, P_4)\)-design can exist only if \( v \equiv 9, 17 \ (\text{mod} \ 24) \). The existence of such down-links is proved in Lemma 4.3 The statement of (1) follows from Remark 3.5. The other inclusion in (2) immediately follows from Corollary 3.2.

Theorem 4.5. For any admissible \( v > 1 \),

\[
\mathcal{L}_1 P_5(v) = \{ n \geq v - 1 \mid n \equiv 0, 1(\text{mod} \ 3) \}; \quad (3)
\]

\[
\{ n \geq v + 2 \mid n \equiv 0, 1(\text{mod} \ 3) \} \subseteq \mathcal{L}_2 P_5(v) \subseteq \{ n \geq v \mid n \equiv 0, 1(\text{mod} \ 3) \}. \quad (4)
\]

Proof. The first inclusion in (4) follows from Corollary 3.2. In order to prove the second, it is sufficient to show that for any admissible \( v \) there exists a \((K_v, P_5)\)-design \( B \) wherein no vertices can be deleted. In particular, this is the case if each vertex of \( K_v \) has degree 2 in at least one block of \( B \). First of all note that in a \((K_v, P_5)\)-design there is at most one vertex with degree 1 in each block where it appears. Suppose that there actually exists a \((K_v, P_5)\)-design \( \overline{B} \) with a vertex \( x \) as above. It is easy to see that in \( \overline{B} \) there is at least one block \( P^1 = [x, a, b, c, d] \) such that the vertices \( a, b \) and \( c \) have degree two in at least another block. Let \( P^2 = [x, d, e, f, g] \). By reassembling the edges of \( P^1 \cup P^2 \), it is possible to replace in \( \overline{B} \) these two paths with \( P^3 = [d, x, a, b, c] \), \( P^4 = [c, d, e, f, g] \) if \( c \neq f, g \) or \( P^5 = [a, x, d, c, g], P^6 = [a, b, c, e, d] \) if \( c = f \) or \( P^7 = [c, d, x, a, b], P^8 = [b, c, f, e, d] \) if \( c = g \). Thus we have again a \((K_v, P_5)\)-design. By the assumption on \( a, b, c \) all the vertices of this new design have degree two in at least one block.

Now we consider Relation (3). Let \( B \) and \( B' \) be respectively a \((K_v, P_5)\)-design and a \((K_n, P_4)\)-design. Suppose there exists a down-link \( f : B \rightarrow B' \). Clearly, \( n > v - 2 \). Hence, \( \mathcal{L}_1 P_5(v) \subseteq \{ n \geq v - 1 \mid n \equiv 0, 1(\text{mod} \ 3) \} \).

To show the reverse inclusion in (3) we prove the actual existence of designs providing down-links. Since a \((K_v, P_5)\)-design exists if, and only if, \( v \equiv 0, 1(\text{mod} \ 8) \) and a \((K_n, P_4)\)-design exists if, and only if, \( n \equiv 0, 1(\text{mod} \ 3) \), it makes sense to look for a down-link from a \((K_v, P_5)\)-design to a \((K_{v-1}, P_4)\)-design only if \( v \equiv 1, 8, 16, 17(\text{mod} \ 24) \). For the same reason, it makes sense
to construct a down-link from a \((K_v, P_5)\)-design to a \((K_v, P_4)\)-design only for \(v \equiv 0, 1, 9, 16(\text{mod} \, 24)\). In view of Remark 3.5 in order to complete the proof, we have also to provide a down-link from a \((K_v, P_5)\)-design to a \((K_{v+1}, P_4)\)-design for every \(v \equiv 0, 9(\text{mod} \, 24)\).

To determine the necessary down-links, we analyze a few basic cases and then apply the gluing method. To this end, we will use the following obvious relations in an appropriate way: \(K_{a+b} = K_a \cup K_b \cup K_{a+b}\) and \(K_{a+b,c} = K_{a,c} \cup K_{b,c}\). In particular,

\[
K_{\ell+24t} = K_\ell \cup K_{24t} \cup K_{\ell,24t};
\]

\[
K_{24t} = tK_{24} \cup \left(\frac{t}{2}\right)K_{24,24} = tK_{24} \cup 48\left(\frac{t}{2}\right)K_{3,4};
\]

\[
K_{\ell=rs,24t} = rK_{s,24t} = rtK_{s,24} = 6rtK_{s,4} = 8rtK_{s,3}.
\]

Let us now examine the possible cases.

- \((K_v, P_5) \rightarrow (K_{v-1}, P_4)\)-design with \(v = \ell + 24t > 1, \ell = 1, 8, 16, 17\).

| \(P_5\)-design of order | basic components | \(\rightarrow\) | basic components | \(P_4\)-design of order |
|---------------------------|------------------|----------------|------------------|------------------|
| 1 + 24t \((K_{25}, P_5), (K_{3,4}, P_5)\) | \((K_{24}, P_4), (K_{3,4}, P_4)\) | \(24t\) |
| 8 + 24t \((K_{8}, P_5), (K_{24}, P_5)\) | \((K_{7}, P_4), (K_{24}, P_4)\) | \(7 + 24t\) |
| \(K_{4,3}, P_5\) | \((K_{4,3}, P_4), (K_{3,3}, P_4)\) | |
| 16 + 24t \((K_{16}, P_5), (K_{24}, P_5)\) | \((K_{15}, P_4), (K_{24}, P_4)\) | \(15 + 24t\) |
| \(K_{4,3}, P_5\) | \((K_{4,3}, P_4), (K_{3,3}, P_4)\) | |
| 17 + 24t \((K_{17}, P_5), (K_{24}, P_5)\) | \((K_{16}, P_4), (K_{24}, P_4)\) | \(16 + 24t\) |
| \(K_{4,3}, P_5\) | \((K_{4,3}, P_4), (K_{3,3}, P_4)\) | |

- \((K_v, P_5) \rightarrow (K_{v}, P_4)\)-design with \(v = \ell + 24t > 1, \ell = 0, 1, 9, 16\).

| \(P_5\)-design of order | basic components | \(\rightarrow\) | basic components | \(P_4\)-design of order |
|---------------------------|------------------|----------------|------------------|------------------|
| 24t \((K_{24}, P_5), (K_{3,4}, P_5)\) | \((K_{24}, P_4), (K_{3,4}, P_4)\) | \(24t\) |
| 1 + 24t \((K_{9}, P_5), (K_{16}, P_5)\) | \((K_{9}, P_4), (K_{16}, P_4)\) | \(1 + 24t\) |
| \((K_{24}, P_5), (K_{3,4}, P_5)\) | \((K_{24}, P_4), (K_{3,4}, P_4)\) | |
| 9 + 24t \((K_{9}, P_5), (K_{24}, P_5)\) | \((K_{9}, P_4), (K_{24}, P_4)\) | \(9 + 24t\) |
| \(K_{3,4}, P_5\) | \((K_{3,4}, P_4)\) | |
| 16 + 24t \((K_{16}, P_5), (K_{24}, P_5)\) | \((K_{16}, P_4), (K_{24}, P_4)\) | \(16 + 24t\) |
| \(K_{3,4}, P_5\) | \((K_{3,4}, P_4)\) | |

- \((K_v, P_5) \rightarrow (K_{v+1}, P_4)\)-design with \(v = \ell + 24t > 1, \ell = 0, 9\).

| \(P_5\)-design of order | basic components | \(\rightarrow\) | basic components | \(P_4\)-design of order |
|---------------------------|------------------|----------------|------------------|------------------|
| 24t \((K_{24}, P_5), (K_{3,4}, P_5)\) | \((K_{25}, P_4), (K_{3,4}, P_4)\) | \(1 + 24t\) |
| 9 + 24t \((K_{9}, P_5), (K_{24}, P_5)\) | \((K_{10}, P_4), (K_{24}, P_4)\) | \(10 + 24t\) |
| \((K_{3,4}, P_5), (K_{9,24}, P_5)\) | \((K_{3,4}, P_4), (K_{10,24}, P_4)\) | |
We obtain the image of any $\xi$ by removing the underlined edge. Now, to complete the codomain, we have to add a further vertex to $A$, say $\alpha$, together with all the edges connecting $\alpha$ to the vertices of $B$. Thus, it remains to decompose the graph formed by the removed edges together with the star of center $\alpha$ and external vertices in $B$. Such a $P_4$-decomposition is listed below:

\[
\begin{align*}
[6, a, 12, b, 1] & \quad [1, c, 12, d, 6] & \quad [6, e, 18, f, 1] & \quad [1, g, 12, h, 0] & \quad [12, i, 0, a, 18] \\
[7, a, 13, b, 2] & \quad [2, c, 13, d, 7] & \quad [7, e, 19, f, 2] & \quad [2, g, 13, h, 1] & \quad [13, i, 1, a, 19] \\
[8, a, 14, b, 3] & \quad [3, c, 14, d, 8] & \quad [8, e, 20, f, 3] & \quad [3, g, 14, h, 2] & \quad [14, i, 2, a, 20] \\
[9, a, 15, b, 4] & \quad [4, c, 15, d, 9] & \quad [9, e, 21, f, 4] & \quad [4, g, 15, h, 3] & \quad [15, i, 3, a, 21] \\
[10, a, 16, b, 5] & \quad [5, c, 16, d, 10] & \quad [10, e, 22, f, 5] & \quad [5, g, 16, h, 4] & \quad [16, i, 4, a, 22] \\
[11, a, 17, b, 0] & \quad [0, c, 17, d, 11] & \quad [11, e, 23, f, 0] & \quad [0, g, 17, h, 5] & \quad [17, i, 5, a, 23] \\
[18, b, 6, c, 19] & \quad [19, d, 0, e, 12] & \quad [12, f, 6, g, 19] & \quad [19, h, 6, i, 18] \\
[19, b, 7, c, 20] & \quad [20, d, 1, e, 13] & \quad [13, f, 7, g, 20] & \quad [20, h, 7, i, 19] \\
[20, b, 8, c, 21] & \quad [21, d, 2, e, 14] & \quad [14, f, 8, g, 21] & \quad [21, h, 8, i, 20] \\
[21, b, 9, c, 22] & \quad [22, d, 3, e, 15] & \quad [15, f, 9, g, 22] & \quad [22, h, 9, i, 21] \\
[22, b, 10, c, 23] & \quad [23, d, 4, e, 16] & \quad [16, f, 10, g, 23] & \quad [23, h, 10, i, 22] \\
[23, b, 11, c, 18] & \quad [18, d, 5, e, 17] & \quad [17, f, 11, g, 18] & \quad [18, h, 11, i, 23].
\end{align*}
\]

We obtain the image of any $P_5$ via $\xi$ by removing the underlined edge. Now, to complete the codomain, we have to add a further vertex to $A$, say $\alpha$, together with all the edges connecting $\alpha$ to the vertices of $B$. Thus, it remains to decompose the graph formed by the removed edges together with the star of center $\alpha$ and external vertices in $B$. Such a $P_4$-decomposition is listed below:

\[
\begin{align*}
[6, a, 9, \alpha] & \quad [7, a, 10, \alpha] & \quad [8, a, 11, \alpha] & \quad [1, c, 4, \alpha] & \quad [2, c, 5, \alpha] \\
[3, c, 0, \alpha] & \quad [9, c, 6, \alpha] & \quad [10, e, 7, \alpha] & \quad [11, e, 8, \alpha] & \quad [4, g, 1, \alpha] \\
[5, g, 2, \alpha] & \quad [0, g, 3, \alpha] & \quad [15, i, 12, \alpha] & \quad [16, i, 13, \alpha] & \quad [17, i, 14, \alpha] \\
[22, d, 19, \alpha] & \quad [23, d, 20, \alpha] & \quad [21, d, 18, \alpha] & \quad [12, f, 15, \alpha] & \quad [13, f, 16, \alpha] \\
[14, f, 17, \alpha] & \quad [20, h, 21, \alpha] & \quad [23, h, 22, \alpha] & \quad [20, b, 23, \alpha] & \quad [h, 18, b, 21] & \quad [h, 19, b, 22].
\end{align*}
\]

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