DUALITY FOR LOCAL FIELDS AND SHEAVES ON THE CATEGORY OF FIELDS

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Abstract. Duality for complete discrete valuation fields with perfect residue field with coefficients in (possibly p-torsion) finite flat group schemes was obtained by Béguéri, Bester and Kato. In this paper, we give another formulation and proof of this result. We use the category of fields and a Grothendieck topology on it. This simplifies the formulation and proof and reduces the duality to classical results on Galois cohomology. A key point is that the resulting site correctly captures extension groups between algebraic groups.

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1. Introduction

1.1. **Aim of the paper.** Let $K$ be a complete discrete valuation field with algebraically closed (more generally, perfect) residue field $k$ of characteristic $p > 0$. Duality for $K$ with coefficients in finite flat group schemes (with torsion not necessarily prime to $p$) was obtained by Bégueri ([Bég81]), Bester ([Bes78]) and Kato (unpublished, but announcements in a much more general setting can be found in [Kat86], [Kat91, §3.3]). This is a generalization of Serre’s local class field theory ([Ser61]) in the style of local Tate duality. A summary of the work of Bégueri and Bester can be found in Milne’s book [Mil80, III, §4, §10]. The hard part of this duality is how to give a nice geometric structure for cohomology groups.

In this paper, we give another formulation and proof of this result. Our method is simple and straightforward, requiring only classical results on Galois cohomology of such discrete valuation fields written for example in [Ser79] or [Ser02] together with Serre’s local class field theory, once we define a certain Grothendieck site as a set-up for the duality and establish its basic site-theoretic properties. The underlying category of the site we use is the category of fields (possibly transcendental) over $k$, in contrast to other usual sites whose underlying categories are categories of rings or schemes. An easy fact is that the cohomology theory of our site is simple: it is essentially Galois cohomology. A difficult fact is that extensions between sheaves on our site are rich: they correctly capture extensions between algebraic groups.

In [Suz20], we apply our formulation and results to Grothendieck’s conjecture on the special fibers of abelian varieties over $K$ that he made in SGA7.

1.2. **Main results.** Now we formulate here a relatively simple part of the duality. Let $k$ be a perfect field of characteristic $p > 0$. We say that a $k$-algebra is rational if it is a finite direct product of the perfect closures of finitely generated fields over $k$. We denote the category of rational $k$-algebras by $k_{\text{rat}}$. This is essentially the category of (perfect) fields. Give it the étale topology and denote the resulting site by $\text{Spec} k_{\text{rat}}$. We call this site the rational étale site of $k$.

Let $K$ be a complete discrete valuation field with ring of integers $\mathcal{O}_K$ and residue field $k$. We denote by $\mathcal{O}_K(k') = W(k') \hat{\otimes}_{W(k)} \mathcal{O}_K$ and $K(k') = \mathcal{O}_K(k') \otimes_{\mathcal{O}_K} K$.

Note that if $k'$ has only one direct factor, then $K(k')$ is the complete discrete valuation field obtained from $K$ by extending its residue field from $k$ to $k'$. The sheaves of invertible elements of $\mathcal{O}_K$ and $K$ are denoted by $U_K$ and $K^\times$, respectively.

We define a category $K_{\text{et}}/k_{\text{rat}}$ as follows. An object is a pair $(L, k_L)$, where $k_L \in k_{\text{rat}}$ and $L$ is an étale $K(k_L)$-algebra. A morphism $(L, k_L) \to (L', k_{L'})$ consists of a $k$-algebra homomorphism $k_L \to k_{L'}$ and a ring homomorphism $L \to L'$ such
Theorem A. Assume that $K$ has mixed characteristic.

1. For any torsion étale group scheme $A$ over $K$ ($p$-torsion being allowed), we have $H^i(K_{et}, A) = 0$ for $i \neq 0, 1$.

2. The Kummer sequence gives an isomorphism

$$H^1(K_{et}, \mathbb{Q}/\mathbb{Z}(1)) = \mathbb{K}^\times \otimes_{\mathbb{Z}} \mathbb{Q}/\mathbb{Z}$$

in $A(k_{et}^{\text{rat}})$. By composing this with the valuation map $\mathbb{K}^\times \to \mathbb{Z}$, we obtain a morphism

$$H^1(K_{et}, \mathbb{Q}/\mathbb{Z}(1)) \to \mathbb{Q}/\mathbb{Z},$$

which we call the trace map.

3. Let $A$ be a finite étale group scheme over $K$. Consider the pairing

$$\mathbf{R}\Gamma(K_{et}, A^{\text{CD}}) \times \mathbf{R}\Gamma(K_{et}, A) \to \mathbb{Q}/\mathbb{Z}[-1]$$

in $D(k_{et}^{\text{rat}})$ given by the cup product and the trace map. The induced morphism

$$\mathbf{R}\Gamma(K_{et}, A^{\text{CD}}) \to \mathbf{R}\text{Hom}_{k_{et}^{\text{rat}}}^{\text{CD}}(\mathbf{R}\Gamma(K_{et}, A), \mathbb{Q}/\mathbb{Z}[-1])$$

is an isomorphism.
For example, we have $H^2(K_{et}, \mathbb{Q}/\mathbb{Z}(1)) = 0$. This is equivalent to the classical vanishing result of $H^2(K(k')_{et}, \mathbb{Q}/\mathbb{Z}(1))$ is the Brauer group of the complete discrete valuation field $K(k')$ with any algebraically closed residue field $k'$ (over $k$). Here it is essential to use the category $k_{rat}^{et}$ of fields. If $k'$ is replaced by a more general $k$-algebra $R$, then the cohomology of a similarly defined ring $K(R)$ is at least not classical.

The proof of Assertion (3) requires the following theorem to understand extension groups over $\text{Spec} k_{rat}^{et}$. Note that the quotient $U_K/(U_K)^n$ by $n$-th power elements for any $n \geq 1$ is represented by a quasi-algebraic group of units over $k$ studied by Serre (Ser61). Recall that a quasi-algebraic group is the perfection (inverse limit along Frobenii) of an algebraic group (Ser60). Let $\text{Alg}/k$ be the category of commutative affine quasi-algebraic groups over $k$ and $\text{Ext}^n_{\text{Alg}/k}$ the $n$-th Ext functor for $\text{Alg}/k$.

**Theorem B.** For any $A, B \in \text{Alg}/k$ and any $n \geq 0$, we have

$$\text{Ext}^n_{k_{rat}^{et}}(A, B) = \text{Ext}^n_{\text{Alg}/k}(A, B).$$

It is important for this theorem that the category $k_{rat}^{et}$ contains the generic points of $A, B \in \text{Alg}/k$. With this theorem, assuming $k$ algebraically closed, which we may, the essential part of the spectral sequence associated with the morphism in (3) is the sequence

$$0 \to \text{Ext}^1_{\text{Alg}/k}(U_K/(U_K)^n, \mathbb{Z}/n\mathbb{Z}) \to H^1(K_{et}, \mathbb{Z}/n\mathbb{Z})$$

$$\to \text{Hom}_{\text{Alg}/k}(\mathbb{Z}/n\mathbb{Z}(1)(K), \mathbb{Z}/n\mathbb{Z}) \to 0,$$

where $\mathbb{Z}/n\mathbb{Z}(1)(K)$ is the finite étale group over $k$ given by the kernel of multiplication by $n$ on $K^\times$. We can show that this sequence agrees with the exact sequence given by Serre’s local class field theory. This proves the exactness of our sequence and hence Assertion (3).

This paper does not contain essentially new duality results. Trying to give such results is not the purpose of this paper. Our purpose is to give a clear exposition of the use of sheaves on the category of fields in the duality theories of Bégueri, Bester and Kato. We hope that similar techniques can be applied to other situations to reduce cohomology of schemes to cohomology of fields.

When writing this paper, the author was informed that Pépin (Pép14) found a similar formulation to ours, independently at almost the same time. He views cohomology of $K$ as a functor on the category of fields over $k$, which is a key common feature between his and our formulations. This might explain how the method of the category of fields may naturally arise.

1.3. **Organization.** The details of the duality are explained in Section 2 except for the proof of Theorem 13. The proof of Theorem A finishes at Section 2.7. We treat the following more general setting. Using fppf cohomology of $K$, the equal and mixed characteristic cases are treated together. Since the group $U_K$ is an infinite-dimensional proalgebraic group, we treat proalgebraic groups in order to give a transparent argument. This option is actually necessary in the equal characteristic case, since even the group $U_K/(U_K)^p$ in that case is infinite-dimensional and $\text{Ext}^n_{k_{rat}^{et}}(U_K/(U_K)^p, \mathbb{Z}/p\mathbb{Z})$ does not agree with the extension group for the category $\text{PAlg}/k$ of proalgebraic groups. This leads us to define a larger site than $\text{Spec} k_{rat}^{et}$, which we call the ind-rational étale site of $k$, denoted by $\text{Spec} k_{\text{indrat}}^{et}$. Its
underlying category $k^{\text{indrat}}$ consists of ind-rational $k$-algebras, which are defined as filtered unions of rational $k$-algebras. The corresponding generalization of Theorem [2] is Theorem [2.1.5] where $A$ is allowed to be proalgebraic. Assuming Theorem [2.1.5] the proof of our duality theorem goes mostly in the manner outlined above. Unfortunately, however, the option we take raises an additional problem. We have to compute $H^2(K(k')_{\text{et}}, G_m)$ with $k'$ ind-rational that may have infinitely many direct factors. In this case, the ring $K(k')$ is no longer a finite product of fields, not even a direct limit of finite products of fields. In Section [2.3] we compute this type of cohomology by approximating it with cohomology of complete discrete valuation subfields.

We do not redo all results of Bégueri, Bester and Kato. We do, however, include a duality for varieties over $K$ (assuming $K$ has mixed characteristic) with possibly $p$-torsion coefficients in Section [2.8]. This is a simple combination of the duality for $K$ and the Poincaré duality for varieties over an algebraic closure of $K$, which does not seem to have been written down elsewhere. We also restate Theorem [A] as a Verdier-type duality for $K$ and discuss a possible relation with the Albanese property of $K^\times$ ([CC94]) in Section [2.9]. When writing this paper, the author realized that some categories closely related to our site have already been studied by Rovinsky [Rov05] and Jannsen-Rovinsky [JR10] in their study of motives. We explain these relations in Section [4].

The proof of Theorem [2.1.5], thereby Theorem [B], occupies the entire part of Section [3]. We outline the proof. First note that if $A$ is an algebraic group over $k$ with generic point $\xi_A$, then the group operation map

$$\xi_A \times_k \xi_A \to A$$

is faithfully flat. In other words, any point of $A$ can be written as the sum of two generic points. A key observation is to regard this well-known fact as saying that $A$ is covered by fields. Now let $\text{Spec } k^{\text{perf}}$ be the category of perfect $k$-algebras (having invertible Frobenius) endowed with the étale topology. Breen’s results [Bre70] and [Bre81] tell us that

$$\text{Ext}^n_{k^{\text{perf}}}(A, B) = \text{Ext}^n_{\text{PAlg}/k}(A, B).$$

It does not seem possible to directly compare extensions over $\text{Spec } k^{\text{perf}}_\text{et}$ and $\text{Spec } k^{\text{indrat}}_\text{et}$. The étale topology is too coarse to treat the above morphism $\xi_A \times_k \xi_A \to A$ as a covering. Instead of trying a direct comparison, we define another site, the perfect pro-fppf site, denoted by $\text{Spec } k^{\text{perf}}_\text{proppf}$. Its underlying category is the category of perfect $k$-algebras, where a covering of a perfect affine $k$-scheme $X$ is a jointly surjective finite family of filtered inverse limits of perfect flat affine $X$-schemes of finite presentation. This is a flat analog of Scholze’s pro-étale site [Sch13]. Using the pro-fppf topology, we can include the faithfully flat morphism $\xi_A \times_k \xi_A \to A$ above as a covering, even though it is not of finite presentation or pro-étale. There is no difference between cohomology with respect to $\text{Spec } k^{\text{perf}}_\text{proppf}$ and $\text{Spec } k^{\text{indrat}}_\text{et}$ with coefficients in a quasi-algebraic group $B$ since $B$ is the perfection of a smooth algebraic group. (There seems to be no known analogous comparison result for the fpqc cohomology instead of pro-fppf for the third cohomology $H^3(R, G_m)$ or higher.) The comparison of extensions over $\text{Spec } k^{\text{perf}}_\text{proppf}$ and $\text{Spec } k^{\text{indrat}}_\text{et}$ is complicated due to

\[1\text{See } [\text{DG70}, \text{II, §5, Lemma 1.2}] \text{ for example, applying the limit argument given in } [\text{DG70}, \text{III, §3, Lemma 7.1}].\]
the fact that the category $k^{\text{rat}}$ of fields do not have all finite fiber sums and the pullback functor for the natural continuous functor $\text{Spec} k^{\text{perf}} \to \text{Spec} k^{\text{indrat}}$ is not exact. To overcome this, we follow Breen’s method ([Bre78]) to write extension groups of $A$ in terms of spectral sequences whose $E_2$-terms are given by cohomology groups of products of $A$. More precisely, we use Mac Lane’s resolution defined as the bar construction for the cubical construction ([ML57]), and apply it to the left variable $A$ of the functor $\text{Ext}^n(A, B)$. A key property of the cubical construction is that it is an additive functor up to a very explicit and simple chain homotopy called the splitting homotopy ([ML57 §5, Lemme 2]). This homotopy and variants of the covering $\xi_A \times_k \xi_A \to A$ allow us to replace the cohomology groups appearing in the $E_2$-terms by cohomology groups of fields. This is the hardest part of this paper.

Here are the logical connections of the sections:

\[
\begin{array}{c}
\text{2.1} \longrightarrow \text{2.2} \\
\downarrow \\
\text{4} \leftarrow \text{5} \longrightarrow \text{2.3, 2.7} \longrightarrow \text{2.8, 2.9}
\end{array}
\]

Section 2.2 does not substitute Section 3. It proves only the case $n = 0, 1$ of Theorem B which, though, gives some ideas about the general case. Section 2.5 can be skipped if one is only interested in the mixed characteristic case. Hence the quickest way to Theorem A may be 2.1–2.7 with 2.5 skipped, interpreting the ind-rational étale site as the rational étale site.

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Notation. We fix a perfect field $k$ of characteristic $p > 0$. A perfect field over $k$ is said to be finitely generated if it is the perfection (direct limit along Frobenii) of a finitely generated field over $k$. The same convention is applied to morphisms of perfect $k$-algebras or $k$-schemes being finite type, finite presentation etc. A perfect $k$-scheme of finite type is also said to be quasi-algebraic following Serre’s terminology [Ser60]. The perfections of $G_a$, $G_m$ and $A^n_k$ over $k$ are denoted by the same symbols $G_a$, $G_m$ and $A^n_k$ by abuse of notation. A point of a scheme is a point of its underlying set unless otherwise noted, usually identified with the Spec of its residue field. If $X$ is a perfect (hence reduced) $k$-scheme of finite type, then its generic point, denoted by $\xi_X$, means the disjoint union of the generic points of its irreducible components. For a set $X$, we denote by $\mathbb{Z}[X]$ the free abelian group generated by $X$. We denote by $\text{Set}$, $\text{Ab}$, $\text{GrAb}$, $\text{DGAb}$ the categories of sets, abelian groups, graded abelian groups, differential graded abelian groups, respectively. Set theoretic issues are omitted for simplicity as the main results hold independent of the choice of universes. All groups (except for Galois groups) are assumed to be
two sufficient conditions for a continuous map prevents us from using the Grothendieck spectral sequence for example. There are will have to use continuous maps without exact pullbacks in Sections 3 and 4. The called $S$ \cite{1.3 (5)}; or $f$ maps in this section satisfy one of these conditions, hence morphism $s$ of sites.

For an object $X$ of $S$, the category $S/X$ of objects of $S$ over $X$ is equipped with the induced topology \cite{1.3}, which is the localization of $S$ at $X$ \cite{1.3, §5}. For $F \in S(S)$, we denote by $\mathbb{Z}[F]$ the sheafification of the presheaf $X \mapsto \mathbb{Z}[F(X)]$. By a continuous map $f : S' \to S$ between sites $S'$ and $S$ we mean a continuous functor from the underlying category of $S$ to that of $S'$, which means that the right composition (or the pushforward $f_*$) sends sheaves on $S'$ to sheaves on $S$. By a morphism $f : S' \to S$ of sites we mean a continuous map whose pullback functor $S(S) \to S(S')$ is exact. For an abelian category $A$, we denote by $\text{Ext}^i_A$ the $i$-th Ext functor for $A$. The bounded, bounded below, bounded above and unbounded derived categories of $A$ are denoted by $D^b(A)$, $D^+(A)$, $D^-(A)$ and $D(A)$, respectively. If $A = \text{Ab}(S)$, we also write $D^+(S) = D^+(\text{Ab}(S))$ for $* = b, +, -$ and (blank), and $\text{Ext}^*_S = \text{Ext}^*_A$. For sites such as Spec $k^{\text{rat}}_n$, we also use $\text{Ext}^*_S(k^{\text{rat}}_n)$ etc. omitting Spec from the notation. When a continuous map $f : S' \to S$ of subcanonical sites is obtained by the identity functor on the underlying categories, we simply write $f_*F = F$ for a representable sheaf $F$.

2. Duality for local fields with perfect residue field

2.1. The ind-rational étale site. Before introducing the ind-rational étale site, we first need to fix our terminology on Grothendieck sites \cite{1.3}, which we will use throughout the paper, and recall some basic facts. All sites we use are (fortunately) defined by covering families or pretopologies \cite{1.3, II, Définition 1.1] except for the dominant topology DM$_k$ that we study in Sections 4.2 and 4.3. As in Notation, a continuous map $f : S' \to S$ of sites (called a continuous functor in \cite{1.3]), except for those in Sections 4.2 and 4.3, satisfy this extra condition. The pushforward functor of such a continuous map (between sites defined by pretopologies) sends acyclic sheaves to acyclic sheaves and hence induces Leray spectral sequences \cite{1.1.3.2.4}. We say that $A \in \text{Ab}(S)$ is acyclic if $H^n(X, A) = 0$ for any $X$ in $S$ and any $n \geq 1$. This is called $S$-acyclic in \cite{1.3, V, Définition 4.1] and flask in \cite{1.1.3.2.4}.

When $f^*$ is exact, we say that $f$ is a morphism of sites \cite{1.1.3.4.9]. We will have to use continuous maps without exact pullbacks in Sections 3 and 4. The pushforward functor of such a functor does not send injectives to injectives, which prevents us from using the Grothendieck spectral sequence for example. There are two sufficient conditions for a continuous map $f : S' \to S$ to be a morphism of sites: $S$ has all finite limits and $f$ commutes with finite limits \cite{1.1.3.1.3 (5)}; or $f$ admits a left adjoint \cite{1.1.3.2.5]. All continuous maps in this section satisfy one of these conditions, hence morphisms of sites.

commutative. We denote by $\text{Alg}/k$ the category of (commutative) affine quasi-algebraic groups over $k$ in the sense of Serre \cite{Ser60}, that is, group objects in the category of affine quasi-algebraic schemes over $k$. Its procategory $\text{PAlg}/k$ is the category of affine proalgebraic groups over $k$. We denote by $L\text{Alg}/k$ the category of the perfections of smooth group schemes over $k$. For a Grothendieck site $S$ and a category $C$, we denote by $C(S)$ the category of sheaves on $S$ with values in $C$. For an object $X$ of $S$, the category $S/X$ of objects of $S$ over $X$ is equipped with the induced topology \cite{AGV72a, III, §3], which is the localization of $S$ at $X$ \cite{AGV72a, III, §5]. For $F \in S(S)$, we denote by $\mathbb{Z}[F]$ the sheafification of the presheaf $X \mapsto \mathbb{Z}[F(X)]$. By a continuous map $f : S' \to S$ between sites $S'$ and $S$ we mean a continuous functor from the underlying category of $S$ to that of $S'$, which means that the right composition (or the pushforward $f_*$) sends sheaves on $S'$ to sheaves on $S$. By a morphism $f : S' \to S$ of sites we mean a continuous map whose pullback functor $S(S) \to S(S')$ is exact. For an abelian category $A$, we denote by $\text{Ext}^i_A$ the $i$-th Ext functor for $A$. The bounded, bounded below, bounded above and unbounded derived categories of $A$ are denoted by $D^b(A)$, $D^+(A)$, $D^-(A)$ and $D(A)$, respectively. If $A = \text{Ab}(S)$, we also write $D^+(S) = D^+(\text{Ab}(S))$ for $* = b, +, -$ and (blank), and $\text{Ext}^*_S = \text{Ext}^*_A$. For sites such as Spec $k^{\text{rat}}_n$, we also use $\text{Ext}^*_S(k^{\text{rat}}_n)$ etc. omitting Spec from the notation. When a continuous map $f : S' \to S$ of subcanonical sites is obtained by the identity functor on the underlying categories, we simply write $f_*F = F$ for a representable sheaf $F$.

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2.1. The ind-rational étale site. Before introducing the ind-rational étale site, we first need to fix our terminology on Grothendieck sites \cite{AGV72a, III}, which we will use throughout the paper, and recall some basic facts. All sites we use are (fortunately) defined by covering families or pretopologies \cite{AGV72a, II, Définition 1.1] except for the dominant topology DM$_k$ that we study in Sections 4.2 and 4.3. As in Notation, a continuous map $f : S' \to S$ of sites (called a continuous functor in \cite{AGV72a, III, Définition 1.1]) is a functor from $S$ to $S'$ such that the pushforward $f_*$ sends sheaves on $S'$ to sheaves on $S$. In general, this is stronger than saying that $f$ sends coverings in $S'$ to coverings in $S'$. But these are equivalent if $f(Y \times_X Z) = f(Y) \times_{f(X)} f(Z)$ in $S'$ for any morphism $Y \to X$ in $S$ appearing in a covering family and any morphism $Z \to X$ in $S$. (See \cite{AGV72a, III, Proposition 1.6] for these two facts.) All continuous maps in this paper, except for those in Sections 4.2 and 4.3, satisfy this extra condition. The pushforward functor of such a continuous map (between sites defined by pretopologies) sends acyclic sheaves to acyclic sheaves and hence induces Leray spectral sequences \cite{Art62, §2.4]. We say that $A \in \text{Ab}(S)$ is acyclic if $H^n(X, A) = 0$ for any $X$ in $S$ and any $n \geq 1$. This is called $S$-acyclic in \cite{AGV72a, V, Définition 4.1] and flask in \cite{Art62, §2.4].

When $f^*$ is exact, we say that $f$ is a morphism of sites \cite{AGV72a, IV, §4.9]. We will have to use continuous maps without exact pullbacks in Sections 3 and 4. The pushforward functor of such a functor does not send injectives to injectives, which prevents us from using the Grothendieck spectral sequence for example. There are two sufficient conditions for a continuous map $f : S' \to S$ to be a morphism of sites: $S$ has all finite limits and $f$ commutes with finite limits \cite{AGV72a, III, Proposition 1.3 (5)}; or $f$ admits a left adjoint \cite{AGV72a, III, Proposition 2.5]. All continuous maps in this section satisfy one of these conditions, hence morphisms of sites.
Now we define key notions.

**Definition 2.1.1.** We say that a perfect $k$-algebra $k'$ is rational if it is a finite direct product of finitely generated perfect fields over $k$, and ind-rational if it is a filtered union of rational $k$-subalgebras. The rational (resp. ind-rational) $k$-algebras form a full subcategory of the category of perfect $k$-algebras, which we denote by $k^{\text{rat}}$ (resp. $k^{\text{indrat}}$).

Note that $k^{\text{indrat}}$ is naturally equivalent to the ind-category of $k^{\text{rat}}$ and $k^{\text{rat}}$ consists of the compact objects in $k^{\text{indrat}}$. Rational $k$-algebras appear as the rings of rational functions on perfect $k$-schemes of finite type. Examples of ind-rational $k$-algebras include a perfect field over $k$ formed as a direct product of finitely generated perfect fields over $k$. An algebra $k$ is rational (resp. strict-rational) if it is ind-rational. A homomorphism in $k^{\text{indrat}}$ is flat. A homomorphism in $k^{\text{indrat}}$ is faithfully flat if and only if it is injective.

**Proposition 2.1.2.** Any $k$-algebra étale over a rational (resp. ind-rational) $k$-algebra is rational (resp. ind-rational).

**Proof.** Let $k_2$ be a $k$-algebra étale over a perfect $k$-algebra $k_1$. If $k_1$ is rational, clearly so is $k_2$. If $k_1$ is ind-rational, then we can write $k_2 = k_2' \otimes k_1'$ with $k_2'$ a rational $k$-subalgebra of $k_1$ and $k_1'$ étale over $k_1$. Write $k_1$ as a filtered union of rational $k$-subalgebras $k_{1,\lambda}$ containing $k_1'$. Then $k_2$ is the filtered union of the rational $k$-subalgebras $k_2' \otimes k_{1,\lambda}$, hence ind-rational. □

The proposition above leads to the following definition.

**Definition 2.1.3.** We define the rational étale site $\text{Spec}k^{\text{rat}}_\text{et}$ (resp. ind-rational étale site $\text{Spec}k^{\text{indrat}}_\text{et}$) to be the category $k^{\text{rat}}$ (resp. $k^{\text{indrat}}$) endowed with the étale topology.

Note that usual tensor products of rings do not always give fiber sums for $k^{\text{rat}}$ or $k^{\text{indrat}}$, and not all fiber sums in $k^{\text{rat}}$ or $k^{\text{indrat}}$ exist. Therefore a continuous map to the site $\text{Spec}k^{\text{indrat}}_\text{et}$ does not always have an exact pullback functor (an example is given in Proposition 3.2.3). In this section, we will always need to check that the continuous maps we have are indeed morphisms of sites.

Some care is needed for localizations (see Notation) of $\text{Spec}k^{\text{rat}}_\text{et}$ and $\text{Spec}k^{\text{indrat}}_\text{et}$. If $k' \in k^{\text{indrat}}_\text{et}$ is a field, then a rational $k'$-algebra is ind-rational over $k$ since it is a finite product of field extensions of $k$. Hence an ind-rational $k'$-algebra is ind-rational over $k$. But a $k'$-algebra ind-rational over $k$ is not necessarily ind-rational over $k'$. Hence the category $k^{\text{indrat}}/k'$ of objects over $k'$ in $k^{\text{indrat}}$ can strictly contain $k^{\text{indrat}}_\text{et}$ and the localization $\text{Spec}k^{\text{indrat}}_\text{et}/k'$ can be different from $\text{Spec}k^{\text{indrat}}_\text{et}$. But if $k' \in k^{\text{rat}}$ is a field (that is, if $k'$ is a finitely generated perfect field over $k$), then $k^{\text{indrat}}/k' = k^{\text{indrat}}_\text{et}$ and $k^{\text{rat}}/k' = k^{\text{rat}}_\text{et}$. Therefore for any $k' \in k^{\text{rat}}$, we can define

$$k^{\text{indrat}}_\text{et} := k^{\text{indrat}}/k', \quad k^{\text{rat}}_\text{et} := k^{\text{rat}}/k', \quad k^{\text{indrat}} := \text{Spec}k^{\text{indrat}}_\text{et}/k', \quad k^{\text{rat}} := \text{Spec}k^{\text{rat}}_\text{et}/k'.$$

2An example is given as follows. Let $k''_n$ be the perfection of $k(x_1, \ldots, x_n)$ and $k''_n = (k')^n \times k'_{n+1}$ with the natural $k''_n$-algebra structure. Consider the inclusion $k''_n \hookrightarrow k''_{n+1}$ and the $k''_n$-algebra homomorphism $\phi''_n \hookrightarrow k''_{n+1}$ given by $(f_1, \ldots, f_{n+1}) \mapsto (f_1, \ldots, f_{n+1}, \phi(f_{n+1}))$, where $\phi(f(x_1, \ldots, x_{n+1})) = f(x_1, \ldots, x_n, f_{n+1})$. Let $k' = \bigcup k''_n$ and consider the $k'$-algebra $k'' = \bigcup k''_n$ ind-rational over $k$. Then it can be shown that $k''$ is not ind-rational over $k'$. 


without ambiguity. At any rate, the localization \( \text{Spec} \, k^{\text{indrat}}_{\text{et}}/k' \) for any \( k' \in k^{\text{indrat}} \) is the category of \( k' \)-algebras ind-rational over \( k \) endowed with the étale topology.

Also if \( k' \) is an algebraic extension field of \( k \), then \( k^{\text{indrat}}_{\text{et}}/k' = k^{\text{indrat}} \). To see this, let \( k'' = \bigcup k''_\lambda \) with \( k''_\lambda \in k^{\text{rat}} \) and suppose \( k'' \) has a structure of a \( k' \)-algebra. Then \( k''_\lambda \otimes_k k' \in k^{\text{rat}} \) since \( k' \) is algebraic (hence separable) over \( k \). Let \( k''_\lambda \in k^{\text{rat}} \) be the image of \( k''_\lambda \otimes_k k' \) in \( k'' \). Then \( k'' = \bigcup k''_\lambda \in k^{\text{rat}} \). In particular, we have

\[
\mathfrak{C}^{\text{indrat}}_{\text{et}} = k^{\text{indrat}}_{\text{et}}/k, \quad \text{Spec} \mathfrak{C}^{\text{indrat}}_{\text{et}} = \text{Spec} \, k^{\text{indrat}}_{\text{et}}/\mathfrak{C},
\]

where \( \mathfrak{C} \) is an algebraic closure of \( k \).

The cohomology theory of the site \( \text{Spec} \, k^{\text{indrat}}_{\text{et}} \) is essentially Galois cohomology, as follows.

**Proposition 2.1.4.** Let \( k' \in k^{\text{indrat}} \). Let \( f : \text{Spec} \, k^{\text{indrat}}_{\text{et}}/k' \rightarrow \text{Spec} k^{\text{indrat}}_{\text{et}} \) be the morphism defined by the identity. Then \( f_* \) is exact. We have

\[
R\Gamma(k^{\text{indrat}}_{\text{et}}/k', A) = R\Gamma(k^{\text{indrat}}_{\text{et}}, f_* A)
\]

for any \( A \in \text{Ab}(k^{\text{indrat}}_{\text{et}}/k') \), where the left-hand side is the cohomology of the site \( \text{Spec} \, k^{\text{indrat}}_{\text{et}}/k' \) (at the final object \( k' \)) with coefficients in \( A \).

**Proof.** First note that \( f \) is a morphism of sites since the underlying category of the target site \( \text{Spec} \, k^{\text{indrat}}_{\text{et}} \) has all finite limits and \( f \) commutes with finite limits. The exactness of \( f_* \) is obvious. Hence the Grothendieck spectral sequence yields the result. \( \square \)

For the rest of the paper, we will denote the object \( R\Gamma(k^{\text{indrat}}_{\text{et}}/k', A) \) appearing in the proposition simply by \( R\Gamma(k^{\text{indrat}}_{\text{et}}, A) \). For any sheaf \( A \in \text{Ab}(k^{\text{indrat}}_{\text{et}}) \), the cohomology of \( k' \in k^{\text{indrat}} \) with coefficients in \( A \) (that is, the derived functor of \( \Gamma(k', \cdot) : \text{Ab}(k^{\text{indrat}}_{\text{et}}) \rightarrow \text{Ab} \)) is given by the cohomology of the site \( \text{Spec} \, k^{\text{indrat}}_{\text{et}}/k' \) with coefficients in the restriction \( A|_{k'} \in \text{Ab}(k^{\text{indrat}}_{\text{et}}/k') \) by the relation between cohomology and localization [AGV72], V, §2.2, 1st paragraph; [AGV72a], IV, §5.1, 1st paragraph]. Hence the above is enough for describing cohomology of any \( k' \in k^{\text{indrat}} \).

As in Notation, we denote by \( \text{Alg}/k \) and \( \text{PAlg}/k \) the categories of affine quasi-algebraic and affine proalgebraic groups over \( k \), respectively. We also denote by \( \text{LAlg}/k \) the category of the perfections of smooth group schemes over \( k \), which contains \( \text{Alg}/k \) and \( \mathbb{Z} \) for example.\(^3\) Recall from [Ser60] §3.6, Proposition 13] that the Hom group and the \( \text{Ext}^n \) groups between the perfections of algebraic groups \( A, B \) are the direct limits of those between \( A, B \) along Frobenii. We frequently apply results on algebraic groups to quasi-algebraic groups by passing to limits along Frobenii, without giving the detailed procedures. By evaluation, we have natural functors from any of the categories \( \text{Alg}/k, \text{PAlg}/k \) and \( \text{LAlg}/k \) to \( \text{Ab}(k^{\text{rat}}) \)

\(^3\)The “L” stands for “locally”. A related fact that will not be used later is that if a perfect group scheme \( G \) over \( k \) is covered by quasi-algebraic open affine subschemes, then it is the perfection of a smooth group scheme. Use the fact that the underlying topological space of \( G \) is locally noetherian and hence the disjoint union of open connected components. The identity component \( G_0 \) of \( G \) is quasi-compact and separated due to the group structure. Hence \( G_0 \) is the perfection of a smooth algebraic group \( G_0' \) by [Ser60] §1.4, Proposition 10]. If \( N \) is the kernel of the natural morphism \( G_0 \rightarrow G_0' \), then \( G/N \) is a smooth group scheme since \( \pi_0(G) \) is étale. Then \( G \) is the perfection of \( G/N \).
and $\text{Ab}(k_{\text{indrat}})$. The functors $\text{Alg}/k \to \text{Ab}(k_{\text{et}}) \hookrightarrow \text{Ab}(k_{\text{indrat}})$ are exact and hence induce homomorphisms

$$\text{Ext}_{\text{Alg}/k}^n(A, B) \to \text{Ext}_{k_{\text{et}}}^n(A, B) \to \text{Ext}_{k_{\text{indrat}}}^n(A, B)$$

for any $A, B \in \text{Alg}/k$ and $n \geq 0$. Hence for $A = \varprojlim A_\lambda \in \text{PAlg}/k$ with $A_\lambda \in \text{Alg}/k$ and $B \in \text{Alg}/k$, we have a homomorphism

$$\text{Ext}_{\text{PAlg}/k}^n(A, B) = \lim_{\lambda} \text{Ext}_{\text{Alg}/k}^n(A_\lambda, B) \to \lim_{\lambda} \text{Ext}_{k_{\text{et}}}^n(A_\lambda, B) \to \text{Ext}_{k_{\text{indrat}}}^n(A, B)$$

(see [Ser60] §3.4, Proposition 7) for the first isomorphism. We will prove Theorem [B] and the following generalization in Section [B].

**Theorem 2.1.5.** Let $A \in \text{PAlg}/k, B \in \text{LAlg}/k$ and $k' = \bigcup k'_\nu \in k_{\text{indrat}}$ with $k'_\nu \in k_{\text{rat}}$. Then for any $n \geq 0$, we have

$$\text{Ext}_{k_{\text{indrat}}/k'}^n(A, B) = \lim_{\nu} \text{Ext}_{(k'_\nu)_{\text{indrat}}}^n(A, B).$$

If $k'$ is a field, then this is further isomorphic to $\text{Ext}_{k_{\text{indrat}}}^n(A, B)$. If $B \in \text{Alg}/k$, then

$$\text{Ext}_{k_{\text{et}}}^n(A, B) = \text{Ext}_{\text{PAlg}/k}^n(A, B),$$

$$\text{Ext}_{k_{\text{et}}}^n(A, \mathbb{Q}) = \text{Hom}_{k_{\text{et}}}^n(A, \mathbb{Z}) = 0,$$

$$\text{Ext}_{k_{\text{et}}}^{n+1}(A, \mathbb{Z}) = \lim_{m} \text{Ext}_{\text{PAlg}/k}^n(A, \mathbb{Z}/m\mathbb{Z}).$$

If $A \in \text{Alg}/k$, then the isomorphisms in the last three lines also hold with $k_{\text{et}}$ replaced by $k_{\text{et}}$ and $\text{PAlg}/k$ by $\text{Alg}/k$.

In particular, the functors $\text{Alg}/k \to \text{Ab}(k_{\text{et}})$ and $\text{PAlg}/k \to \text{Ab}(k_{\text{et}})$ are fully faithful. Using this theorem, we will interpret Serre’s local class field theory as a duality for local fields with coefficients in $\mathbb{G}_m$ and extensions by $\mathbb{Z}$. (We denote the perfection of $\mathbb{G}_m$ simply by $\mathbb{G}_m$, as in Notation.) The result is Theorem [2.6.1]. We will then deduce Theorem [A] by trivializing finite (multiplicative) $A$ and embedding it into a product of copies of $\mathbb{G}_m$.

**Remark 2.1.6.** Theorem [2.1.5] holds also for commutative quasi-algebraic groups $A$ and $B$ not necessarily affine, for example, abelian varieties. See Remark [3.8.4]. Affine groups as stated above are sufficient for the purpose of duality for local fields with coefficients in finite group schemes.

### 2.2. Comparison of Ext in low degrees: birational groups.

In this subsection, we give a proof of the following important part of Theorem [B] (or [2.1.5]):

**Proposition 2.2.1.** Let $A, B \in \text{Alg}/k$. Then we have

$$\text{Ext}_{\text{Alg}/k}^n(A, B) \xrightarrow{\sim} \text{Ext}_{k_{\text{et}}}^n(A, B)$$

for $n = 0, 1$.

This part can be treated by means of *generic translations* and *birational group laws* in the sense of Weil ([Wei55]; see also [Ser88] V, §1.5]). By evaluating an element of $\text{Ext}_{k_{\text{et}}}^n(A, B) \in \text{Alg}$. At the generic point of $A$, we can obtain a homomorphism of birational groups if $n = 0$ and a rational symmetric factor system ([Ser88] VII, §1.4) if $n = 1$, which are regularizable and come from $\text{Ext}_{\text{Alg}/k}^n(A, B)$ by means of generic translations. This proof is illustrative and helpful to understand the general
case \( n \geq 2 \), where we will use Mac Lane’s resolution and apply a similar but more involved regularization (or generification) technique for it instead of Weil’s results. The cases \( n = 0, 1 \) are important since the groups \( \text{Ext}^n_{\text{PAlg}/k}(A, B) \) for \( n \geq 2 \) (resp. \( n \geq 3 \)) with \( k \) algebraically closed are mostly (resp. always) zero \((\text{Ser}60, \S 10)\). Our explanation of the duality, however, does need the same vanishing for \( \text{Ext}^1_{k_{\text{et}}}^{\prime}(A, B) \).

**Lemma 2.2.2.** Proposition [2.2.1] is true for \( n = 0 \).

**Proof.** For the injectivity, let \( \varphi : A \to B \) in \( \text{Alg}/k \) become zero in \( \text{Ab}(k_{\text{et}}^{\prime \prime}) \). Let \( \xi_A \) be the generic point of \( A \) (which is defined to be the disjoint union of the generic points of all irreducible components of \( A \)). Then \( \varphi(\xi_A) \in B(\xi_A) \) is zero. This means that \( \varphi \) is generically zero. A generic translation then shows that \( \varphi \) is everywhere zero.

For the surjectivity, let \( \varphi \in \text{Hom}_{k^{\prime \prime}}(A, B) \). Consider the element \( \varphi(\xi_A) \in B(\xi_A) \). This corresponds to a rational map \( \tilde{\varphi} : A \dashrightarrow B \). Consider the following commutative diagram in \( \text{Set}(k_{\text{et}}^{\prime \prime}) \):

\[
\begin{array}{ccc}
A \times_k A & \xrightarrow{\varphi \times \varphi} & B \times_k B \\
\downarrow & & \downarrow \\
A & \xrightarrow{\varphi} & B
\end{array}
\]

The vertical arrows are given by the group operations. Applying this commutativity for the element \( \xi_{A \times A} \in A \times A \), we obtain an equality \( \tilde{\varphi}(x_1 + x_2) = \tilde{\varphi}(x_1) + \tilde{\varphi}(x_2) \) as rational maps \( A \times A \dashrightarrow B \). This and a generic translation imply that the rational map \( \tilde{\varphi} : A \dashrightarrow B \) is everywhere regular and is a homomorphism of algebraic groups \((\text{Ser}SS, \S 1.5, \text{Lemma } 6)\).

It remains to show that \( \tilde{\varphi} = \varphi \) in \( \text{Hom}_{k^{\prime \prime}}(A, B) \). Let \( k' \in k^{\prime \prime} \) and \( a \in A(k') \). We want to decompose \( a \) into the sum of two generic points. Let \( A_{k'} = \text{Spec} k' \times_k A \) and \( \text{Spec} k'' \) the generic point of \( A_{k'} \). The natural projection \( A_{k'} \to A \) gives a point \( x \in \xi_A(k'') \). We have \( \varphi(x) = \tilde{\varphi}(x) \) by definition of \( \tilde{\varphi} \). If we show that \( x - a \in \xi_A(k'') \), then \( \varphi(a - x) = \varphi(a) \), so

\[
\varphi(a) = \varphi(a - x) + \varphi(x) = \varphi(a - x) + \tilde{\varphi}(x) = \tilde{\varphi}(a),
\]

which proves \( \varphi = \tilde{\varphi} \). To show that \( a + x \in \xi_A(k'') \), note that the correspondence \( y \mapsto a - y \) gives an automorphism of the scheme \( A_{k'} \). Hence \( x \in \xi_A(k'') \) implies that \( a - x \in \xi_A(k'') \). This completes the proof of \( \text{Hom}_{\text{Alg}/k}(A, B) = \text{Hom}_{k^{\prime \prime}}(A, B) \). \( \square \)

In particular, the exact functor \( \text{Alg}/k \to \text{Ab}(k_{\text{et}}^{\prime \prime}) \) is fully faithful.

**Lemma 2.2.3.** For the case \( n = 1 \) of Proposition [2.2.1], it is enough to show the surjectivity of the map, and we may assume that \( A \) and \( B \) are connected and \( H^1(k_{\text{et}}, B) = 0 \) for all \( k' \in k^{\prime \prime} \).

**Proof.** If \( 0 \to B \to C \to A \to 0 \) is an extension in \( \text{Alg}/k \) that admits a splitting \( s : A \to C \) in \( \text{Ab}(k_{\text{et}}^{\prime \prime}) \), then \( s \in \text{Hom}_{\text{Alg}/k}(A, C) \) by the \( n = 0 \) case, hence the extension splits also in \( \text{Alg}/k \). This shows the injectivity.

For the second half of the lemma, let \( k'' \) be a finite Galois extension of \( k \). We first show that if the statement of the \( n = 1 \) case of the proposition is true for \( A \times_k k'' : B \times_k k'' \), that is,

\[
\text{Ext}^1_{\text{Alg}/k''}(A, B) = \text{Ext}^1_{k_{\text{et}}^{\prime \prime}}(A, B),
\]

then
then it is true for $A, B$. For any $i \geq 0$, let $k'' \otimes_{i+1}$ be the tensor product of $i+1$ copies of $k''$ over $k$ and $B^i$ the Weil restriction $\text{Res}_{k''/i+1/k} B \in \text{Alg}/k$. The usual formulas for coboundary maps of augmented Čech complexes define a complex $0 \to B \to B^0 \to B^1 \to \cdots$ in $\text{Alg}/k$ (see the proof of \cite{Mil80} III, Theorem 3.9). On $\kappa$-points, it is the augmented Čech complex of $k'' \otimes_k \kappa/k$ with coefficients in $B$, which is exact since $\kappa$ has trivial cohomology. Hence the complex $B^\bullet = \{B^i\}_{i \geq 0}$ is a resolution of $B$ in $\text{Alg}/k$. Let $F^1_j = \text{Ext}^1_{\text{Alg}/k}(A, \cdot)$ and $F^2_j = \text{Ext}^1_{k''}(A, \cdot)$ for $j \geq 0$. For $n = 1, 2$ and $j \geq 0$, denote the $i$-th cohomology of the complex $F^1_n(B^\bullet) = \{0 \to F^1_n(B^0) \to F^1_n(B^1) \to \cdots\}$ by $H^i F^1_n(B^\bullet)$. Since $\{F^1_j\}_{j \geq 0}$ is a $\delta$-functor for $n = 1, 2$, a simple diagram chase shows that we have a four term exact sequence

$$0 \to H^1 F^0_n(B^\bullet) \to F^1_n(B) \to H^0 F^1_n(B^\bullet) \to H^2 F^0_n(B^\bullet)$$

for $n = 1, 2$. The functor $\text{Alg}/k \to \text{Ab}(k''_{\text{rat}})$ induces a morphism of sequences from the above sequence for $n = 1$ to that for $n = 2$. Thus $F^1_1(B^\bullet) \to F^1_2(B^\bullet)$ is an isomorphism for $i \geq 0$ by Lemma \cite{TakshiSu21} For each $i \geq 0$, the $k$-algebra $k'' \otimes_{i+1}$ is a finite product of copies of $k''$ and the group $B^1$ is a finite product of copies of $\text{Res}_{k''/k} B$. We have

$$F^1_1(\text{Res}_{k''/k} B) = \text{Ext}^1_{\text{Alg}/k''}(A, B),$$

$$F^2_2(\text{Res}_{k''/k} B) = \text{Ext}^1_{k''}(A, B).$$

Therefore if the statement is true for $A \times_k k''$, $B \times_k k''$, then $F^1_1(B^\bullet) \to F^1_2(B^\bullet)$ is an isomorphism for $i \geq 0$ and hence $F^1_1(B) \to F^1_2(B)$ is an isomorphism, which is the statement for $A, B$.

Now assume that the statement of the $n = 1$ case of the proposition is true for connected $A, B \in \text{Alg}/k$ with $H^1(k''_{\text{rat}}, B) = 0$ for $k' \in k''_{\text{rat}}$. We show that the same statement is true for arbitrary $A, B \in \text{Alg}/k$.

Case $A, B$ connected: We can take a finite Galois extension $k''$ such that $B \times_k k''$ is a successive extension of copies of $G_a$ and $G_m$. In particular, $H^1(k''_{\text{et}}, B) = 0$ for $k' \in k''_{\text{rat}}$, and the statement is true for $A \times_k k''$, $B \times_k k''$. So is for $A, B$ by the above Galois descent.

Case $A$ étale and $B$ arbitrary: We may assume $A = \mathbb{Z}/m\mathbb{Z}$ for some $m \geq 1$ by again extending $k$. The long exact sequence for $\text{Ext}^1_{k''_{\text{rat}}} (\cdot, B)$ associated with the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/m\mathbb{Z} \to 0$ yields an exact sequence

$$0 \to B(k)/mB(k) \to \text{Ext}^1_{k''_{\text{rat}}}(\mathbb{Z}/m\mathbb{Z}, B) \to H^1(k''_{\text{et}}, B)[m] \to 0,$$

where $[m]$ denotes the $m$-torsion part. The group $\text{Ext}^1_{\text{Alg}/k}(\mathbb{Z}/m\mathbb{Z}, B)$ also fits in the middle of this sequence. Hence we have $\text{Ext}^1_{k''_{\text{rat}}}(\mathbb{Z}/m\mathbb{Z}, B) = \text{Ext}^1_{\text{Alg}/k}(\mathbb{Z}/m\mathbb{Z}, B)$.

Case $A$ arbitrary and $B$ connected: We denote the identity component of $A$ by $A_0$. Let $0 \to B \to C \to A_0 \to 0$ be an extension in $\text{Ab}(k''_{\text{rat}})$. Let $0 \to B \to C' \to A_0 \to 0$ be its pullback by $A_0 \to A$. It defines an element of $\text{Ext}^1_{k''_{\text{rat}}}(A_0, B)$. The connected case above implies that $C' \in \text{Alg}/k$. We have an exact sequence $0 \to C' \to C \to \pi_0(A) \to 0$, or an element of $\text{Ext}^1_{k''_{\text{rat}}}(\pi_0(A), C')$. Since $\pi_0(A)$ is étale, the previous case implies that $C \in \text{Alg}/k$. Then the sequence $0 \to B \to C \to A \to 0$ is an exact sequence in $\text{Alg}/k$ since $\text{Alg}/k \to \text{Ab}(k''_{\text{rat}})$ is fully faithful.

Case $A$ arbitrary and $B$ étale: This reduces to the previous case by embedding $B$ into a connected affine quasi-algebraic group and the five lemma.
Case $A, B$ arbitrary: We denote the identity component of $B$ by $B_0$. Let $0 \to B \to C \to A \to 0$ be an extension in $\text{Ab}(k_{\text{et}}')$. Let $0 \to \pi_0(B) \to C' \to A \to 0$ be its pushout by $B \to \pi_0(B)$. It defines an element of $\text{Ext}^1_{k_{\text{et}}'}(A, \pi_0(B))$. The previous case implies that $C' \in \text{Alg}/k$. We have an exact sequence $0 \to B_0 \to C \to C' \to 0$, or an element of $\text{Ext}^1_{k_{\text{et}}'}(C', B_0)$. Since $B_0$ is connected, the previous case implies that $C \in \text{Alg}/k$. Then the sequence $0 \to B \to C \to A \to 0$ is an exact sequence in $\text{Alg}/k$. □

The following finishes the proof of Proposition 2.2.4.

**Lemma 2.2.4.** Proposition 2.2.1 is true for $n = 1$ if $A$ and $B$ are connected and $H^1(k_{\text{et}}', B) = 0$ for all $k' \in k_{\text{rat}}$.

**Proof.** Let $0 \to B \to C \to A \to 0$ be an extension in $\text{Ab}(k_{\text{et}}')$. By pulling back by the inclusion $\xi_A \hookrightarrow A$, we have an element of $H^1((\xi_A)_{\text{et}}, B)$, which is trivial by assumption. Hence we have a section $s: \xi_A \rightarrow C$ (in $\text{Set}(k_{\text{et}}')$) to the projection $C \rightarrow A$. Consider the associated factor system

$$f: \xi_{A \times A} \rightarrow B,$$

$$(x_1, x_2) \mapsto s(x_1) + s(x_2) - s(x_1 + x_2).$$

We have equalities

$$f(x_2, x_3) - f(x_1 + x_2, x_3) + f(x_1, x_2 + x_3) - f(x_1, x_2) = 0,$$

$$f(x_1, x_2) = f(x_2, x_1)$$

as morphisms $\xi_{A \times A} \rightarrow B$ and $\xi_{A \times A} \rightarrow B$. This means that $f$ as a rational map $A \times A \dashrightarrow B$ is a rational symmetric factor system ([Ser78 VII, §1.4]). Hence $f$ defines a birational group, which comes from a true quasi-algebraic group $D$ by Weil’s theorem, which fits in an extension $0 \to B \to D \to A \to 0$ of quasi-algebraic groups ([Ser78 VII, §1.4]). By construction, we have a rational section $A \dashrightarrow D$ whose associated rational factor system agrees with $f$. Consider the Baer difference of the two extensions

$$[0 \to B \to E \to A \to 0] := [0 \to B \to C \to A \to 0] - [0 \to B \to D \to A \to 0]$$

in $\text{Ab}(k_{\text{et}}')$. This admits a section $t: \xi_A \rightarrow E$ (in $\text{Set}(k_{\text{et}}')$) whose associated factor system is $f - f = 0$. This means the equality

$$t(x_1 + x_2) = t(x_1) + t(x_2) \text{ in } E(k')$$

for $(x_1, x_2) \in \xi_{A^2}(k')$.

We want to extend $t: \xi_A \rightarrow E$ to a homomorphism $\tilde{t}: A \rightarrow E$ in $\text{Ab}(k_{\text{et}}')$ in a unique way. We need two sublemmas.

**Sublemma 2.2.5.**

(1) $t(x_1 + \cdots + x_m) = t(x_1) + \cdots + t(x_m)$ in $E(k')$ for $m \geq 2$ and $x_1, \ldots, x_m \in \xi_A(k')$ with $x_1 + \cdots + x_m \in \xi_A(k')$.

(2) $t(-y) = -t(y)$ in $E(k')$ for $y \in \xi_A(k')$.

**Proof of Sublemma 2.2.5.** (1) We prove this only for $m = 2$ as the general case is similar. As before, let $A_{k'} = \text{Spec} k' \times_k A$ and $k''$ the function field of $A_{k'}$. The natural projection $A_{k'} \rightarrow A$ gives a point $x \in \xi_A(k'')$. The morphism $x_2: \text{Spec} k' \rightarrow A$ corresponding to the point $x_2 \in \xi_A(k')$ is flat. Taking the fiber product of this morphism with $A$, we have a flat morphism $A_{k'} \rightarrow A^2$, hence a morphism
Ext comes from the inclusion Proof of Sublemma 2.2.6. We have to show that and hence the element \( t \) is equivalent. Since \( D \) is exact since \( x \in \xi_A(k') \) implies \( (x_1, x_2 + x) \in \xi_A(k'') \). Therefore
\[
t(x_1 + x_2) + t(x) = t(x_1) + t(x_2 + x) = t(x_1) + t(x_2) + t(x),
\]
and hence the element \( t(x_1) + t(x_2) - t(x_1 + x_2) \) of \( B(k') \) becomes zero in \( B(k'') \). Since \( B(k') \to B(k'') \) is injective, we have \( t(x_1) + t(x_2) = t(x_1 + x_2) \) in \( E(k') \).

\[ \text{2} \quad \text{Using (1) for } m = 3, \text{ we have } t(y + t(-y) + t(y) = t(y), \text{ so } t(-y) = -t(y). \]

**Sublemma 2.2.6.** For any field extension \( k''/k' \) in \( k^{\text{rat}} \), the sequence
\[
0 \to E(k') \to E(k'') \to E(\text{Frac}(k'' \otimes_{k'} k''))
\]
is exact, where \( \text{Frac} \) denotes the total quotient ring, and the first homomorphism comes from the inclusion \( k' \hookrightarrow k'' \) and the second the difference of the two homomorphisms coming from the inclusions \( k'' \rightrightarrows \text{Frac}(k'' \otimes_{k'} k'') \) into the two factors.\(^3\)

**Proof of Sublemma 2.2.7.** Consider the similar sequences
\[
0 \to A(k') \to A(k'') \to A(\text{Frac}(k'' \otimes_{k'} k'')),
0 \to B(k') \to B(k'') \to B(\text{Frac}(k'' \otimes_{k'} k'')).
\]
These are exact, since \( A \) and \( B \) are quasi-algebraic groups, and a rational map \( g \) from a variety to a quasi-algebraic group (which in particular is separated) such that \( g(x_1) = g(x_2) \) for independent generic points \( (x_1, x_2) \) is constant everywhere. Also the sequence
\[
0 \to B(k') \to E(k') \to A(k') \to 0
\]
is exact since \( H^1(k'_{et}, B) = 0 \). These yield the required exactness by a diagram chase.

Now let \( a \in A(k') \). As above, let \( A_{k'} = \text{Spec } k' \times_k A \) and \( k'' \) the function field of \( A_{k'} \). The natural projection \( A_{k'} \to A \) gives a point \( x \in \xi_A(k'') \), which satisfies \( a - x \in \xi_A(k'') \). We define
\[
t(a) = t(a - x) + t(x) \in E(k'').
\]
We show that \( t(a) \in E(k') \). The two inclusions \( k'' \rightrightarrows \text{Frac}(k'' \otimes_{k'} k'') \) and the point \( x \in \xi_A(k'') \) defines two points \( x_1, x_2 \in \xi_A(\text{Frac}(k'' \otimes_{k'} k'')) \). In view of Sublemma 2.2.6, we have to show that \( t(a - x_1) + t(x_1) = t(a - x_2) + t(x_2) \). This follows from (1) and (2) of Sublemma 2.2.6. We show that \( t(a + b) = t(a) + t(b) \) for \( a, b \in A(k') \). This is equivalent to the equality \( t(a + b - x) = t(a - x) + t(b - x) + t(x) \) in \( E(k'') \), which follows from (1) of Sublemma 2.2.6. The homomorphism \( A \to E \) thus obtained is the only possible extension of \( t \).

Therefore the extension \( 0 \to B \to E \to A \to 0 \) in \( \text{Ab}(k'_{et}) \) is trivial. Hence the two extensions \( 0 \to B \to C \to A \to 0 \) and \( 0 \to B \to D \to A \to 0 \) are equivalent. Since \( D \) is a quasi-algebraic group, this proves that \( \text{Ext}^1_{\text{Alg}/k}(A, B) = \text{Ext}^1_{k_{et}}(A, B) \).

\(^3\)This sublemma says that \( E \) is a sheaf for the rational flat topology defined in Section 4.2. See Proposition 4.2.1. The argument below shows that \( A \) and \( B \) are sheaves for the rational flat topology. Hence so is \( E \).
Remark 2.2.7. The above proof works for an arbitrary commutative quasi-algebraic group $A$ and affine commutative quasi-algebraic group $B$. To allow a non-affine quasi-algebraic group $B$, we need the following modification. An element of $\text{Ext}_{\mathcal{O}_K}^{1}(A, B)$ defines a non-zero element $x$ of $H^1((\xi_A)_{\text{et}}, B)$ in general. This element is primitive in the sense of [Ser87 VII, §3.14], that is, $s^*x = \text{proj}^1_*x + \text{proj}^2_*x$ in $H^1((\xi_A)_{\text{et}}, B)$, where $s, \text{proj}^1, \text{proj}^2: \xi_A \to \xi_A$ are the group operation, first and second projections, respectively. There is a homomorphism from the subgroup of primitive classes of $H^1((\xi_A)_{\text{et}}, B)$ to a certain subquotient of $H^0((\xi_A)_{\text{et}}, B) \oplus H^0((\xi_A)_{\text{et}}, B)$, whose kernel consists of the classes coming from $\text{Ext}_{\mathcal{O}_K}^{1}(A, B)$. This homomorphism comes from Eilenberg-Mac Lane’s abelian complex ([Bre59]). From such an element $x$, we can again define a birational group just as in the proof of [Ser87 VII, §3.15, Theorem 5].

In the next section, we will use the cubical construction and Mac Lane’s resolution instead. With this, we can treat arbitrary commutative quasi-algebraic groups $A, B$ and all higher Ext groups.

2.3. The relative fppf site of a local field. Let $K$ be a complete discrete valuation field with ring of integers $\mathcal{O}_K$ and perfect residue field $k$ of characteristic $p > 0$. We denote by $W$ the affine ring scheme of Witt vectors of infinite length. As in Introduction, we define sheaves of rings on the site $\text{Spec} \mathcal{O}_K^{\text{indrat}}$ by assigning to each $k' \in \mathcal{O}_K^{\text{indrat}}$,

$$\mathcal{O}_K(k') = W(k') \otimes_W \mathcal{O}_K, \quad K(k') = \mathcal{O}_K(k') \otimes_{\mathcal{O}_K} K.$$ 

We define a category $K/\mathcal{O}_K^{\text{indrat}}$ as follows. An object is a pair $(S, k_S)$, where $k_S \in \mathcal{O}_K^{\text{indrat}}$ and $S$ is a $K(k_S)$-algebra of finite presentation. A morphism $(S, k_S) \to (S', k_{S'})$ consists of a $k$-algebra homomorphism $k_S \to k_{S'}$ and a ring homomorphism $S \to S'$ such that the diagram

$$\begin{CD}
K(k_S) @>>> K(k_{S'}) \\
@VVV @VVV \\
S @>>> S'
\end{CD}$$

commutes. The composite of two morphisms is defined in the obvious way. We say that a morphism $(S, k_S) \to (S', k_{S'})$ is flat/étale if $S \to S'$ is flat and $k_S \to k_{S'}$ is étale.

Proposition 2.3.1. Let $(S, k_S) \to (S', k_{S'})$, $(S, k_S) \to (S'', k_{S''})$ be two morphisms in $K/\mathcal{O}_K^{\text{indrat}}$. Assume that the first one is flat/étale. Then we have $K(k_{S'}) \otimes_{K(k_S)} K(k_{S''}) = K(k_{S'} \otimes_{k_S} k_{S''})$. The pair $(S' \otimes_S S'', k_{S'} \otimes_{k_S} k_{S''})$ is the fiber sum of $(S', k_{S'})$ and $(S'', k_{S''})$ over $(S, k_S)$ in the category $K/\mathcal{O}_K^{\text{indrat}}$, and is flat/étale over $(S'', k_{S''})$.

Proof. For the equality $K(k_{S'}) \otimes_{K(k_S)} K(k_{S''}) = K(k_{S'} \otimes_{k_S} k_{S''})$, note that we can write $k_{S'} = k_{1}' \otimes_{k_1} k_S$ with $k_1$ a rational $k$-subalgebra of $k_S$ and $k_{1}'$ étale over $k_1$. Writing $k_1$ as a finite product of fields $k_{1,i}$, we know that $k_1 \to k_{1,i}'$ is a finite product of finite free étale homomorphisms $k_{1,i} \to k_{1,i}'$, where $k_{1,i}'$ can be zero. Hence $k_S \to k_{S'}$ can be written as a finite product of finite free étale homomorphisms $k_{S,i} \to k_{S',i}$. Therefore $W(k_S) \to W(k_{S'})$ can be written as the product of the finite free étale homomorphisms $W(k_{S,i}) \to W(k_{S',i})$. Hence the equality $K(k_{S'}) \otimes_{K(k_S)} K(k_{S''}) = K(k_{S'} \otimes_{k_S} k_{S''})$ follows. The rest is obvious. $\square$
Definition 2.3.2. We define the relative fppt site of $K$ over $k$, $\text{Spec} \ K_{\text{fppt}}/k^{\text{indrat}}$, to be the category $K/k^{\text{indrat}}$ endowed with the topology whose covering families over an object $(S,k_S) \in K_{\text{fppt}}/k^{\text{indrat}}$ are finite families $\{(S_i,k_{S_i})\}$ each flat/étale over $(S,k_S)$ with $\prod_i S_i$ faithfully flat over $S$.

Note that in this definition, $\prod_i S_i$ is not required to be faithfully flat over $k_S$.

Proposition 2.3.3. Let $k' \to k''$ be étale in $k^{\text{indrat}}$, $(S,k'') \in K/k^{\text{indrat}}$ and $A \in \text{Ab}(K_{\text{fppt}}/k^{\text{indrat}})$. Then the natural morphism $(S,k') \to (S,k'')$ in $K/k^{\text{indrat}}$ induces an isomorphism $A(S,k') \xrightarrow{\sim} A(S,k'')$.

Proof. The object $(S,k'')$ covers $(S,k')$. The sheaf condition says that the sequence

$$0 \to A(S,k') \to A(S,k'') \to A(S,k'' \otimes_{k'} k'')$$

is exact. Since $k''$ is étale over $k'$, the diagonal homomorphism $k'' \otimes_{k'} k'' \to k''$ is étale. Hence $(S,k'')$ covers $(S,k'' \otimes_{k'} k'')$. In particular, the homomorphism $A(S,k'' \otimes_{k'} k'') \to A(S,k'')$ is injective. This and the above sequence imply the result. \qed

A little closer observation shows that a representable presheaf $F = (S,k_S)$ on $\text{Spec} K_{\text{fppt}}/k^{\text{indrat}}$ is a sheaf if and only if $k_S = k$. In particular, an affine scheme $\text{Spec} S$ finitely presented over $K$ can be regarded as a sheaf on $\text{Spec} K_{\text{fppt}}/k^{\text{indrat}}$ by identifying it with the sheaf $(S,k)$. By patching, any locally finitely presented scheme over $K$ can be regarded as a sheaf on $\text{Spec} K_{\text{fppt}}/k^{\text{indrat}}$. The sheafification of $F = (S,k_S)$ sends an object $(S',k_{S'})$ to the filtered direct limit of $F(S',k')$, where the limit is indexed by pairs $k'$ (an étale $k_{S'}$-algebra) and $K(k') \to S'$ (a $K(k_{S'})$-algebra homomorphism). When we write $\mathbb{Z}[F]$, we mean the sheafification of the presheaf of free abelian groups generated by $F$.

The cohomology theory of the site $\text{Spec} K_{\text{fppt}}/k^{\text{indrat}}$ is essentially fppt cohomology of $K$, as follows.

Proposition 2.3.4. Let $(S,k_S) \in K/k^{\text{indrat}}$. Let $\text{Spec} S_{\text{fppt}}$ be the fppt site of the ring $S$ (on the category of finitely presented $S$-algebras). Let $(\text{Spec} K_{\text{fppt}}/k^{\text{indrat}})/(S,k_S)$ be the localization of $\text{Spec} K_{\text{fppt}}/k^{\text{indrat}}$ at the object $(S,k_S)$. Finally let

$$f : (\text{Spec} K_{\text{fppt}}/k^{\text{indrat}})/(S,k_S) \to \text{Spec} S_{\text{fppt}}$$

be the morphism of sites defined by sending a finitely presented $S$-algebra $S'$ to the object $(S',k_S)$ of $K/k^{\text{indrat}}$. Then $f_*$ is exact. We have

$$R\Gamma((S,k_S)_{\text{fppt}}, A) = R\Gamma(S_{\text{fppt}}, f_*A)$$

for any sheaf $A$ of abelian groups on the target $(\text{Spec} K_{\text{fppt}}/k^{\text{indrat}})/(S,k_S)$, where the left-hand side is the cohomology of this site (at the final object $(S,k_S)$) with coefficients in $A$.

Proof. First $f$ is a morphism of sites since the underlying category of the target $\text{Spec} S_{\text{fppt}}$ has all finite limits and the functor $S' \to (S',k_S)$ commutes with these limits (or tensor products). We show the exactness of $f_*$. Let $\varphi : A \to B$ be a surjection of sheaves on $\text{Spec} K_{\text{fppt}}/k^{\text{indrat}})/(S,k_S)$, $S'$ a finitely presented $S$-algebra and $b \in (f_*B)(S') = B(S',k_S)$. The surjectivity says that there exists a cover $(S'',k_{S''})$ of $(S',k_S)$ and an element $a \in A(S'',k_{S''})$ such that $\varphi(a) = b$ in $B(S'',k_{S''})$. By Proposition 2.3.3, we may assume that $k_{S''} = k_S$. Then $a \in (f_*A)(S'')$ and $\varphi(a) = b \in (f_*B)(S'')$. Therefore $f_*A \to f_*B$ is surjective. Hence $f_*$
is exact. The equality of cohomology complexes then follows from the Grothendieck spectral sequence.

The above is enough for describing the cohomology theory of Spec $K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}}$; see the paragraph after Proposition 2.1.4.

2.4. The structure morphism of a local field and the cup product pairing.

**Proposition 2.4.1.** The functor $k^{\text{indrat}} \to K/k^{\text{indrat}}$ defined by

$$k' \mapsto (K(k'), k')$$

sends étale coverings to étale (hence fppf) coverings and has a right adjoint given by

$$(S, k_S) \mapsto k_S.$$  

**Proof.** If $k''/k'$ is an étale covering in $k^{\text{indrat}}$, then $K(k'') \otimes_{K(k')} K(k') = K(k'' \otimes_{k'} k'')$ and $K(k'')/K(k')$ is an étale covering, as we saw in Proposition 2.3.1 and its proof. The adjointness is obvious. □

**Definition 2.4.2.** The functor $k' \mapsto (K(k'), k')$ above defines a morphisms of sites

$$\pi: \text{Spec } K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}} \to \text{Spec } k_{\text{et}}^{\text{indrat}}.$$  

We call this the (fppf) structure morphism of $K$ over $k$. We denote

$$\Gamma(K_{\text{fppf}}, \cdot) = \pi_*, \quad H^i(K_{\text{fppf}}, \cdot) = R^i\pi_*,$$

$$R\Gamma(K_{\text{fppf}}, \cdot) = R\pi_*: D(K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}}) \to D(k_{\text{et}}^{\text{indrat}})$$

Note that the pullback $\pi^*$ sends a sheaf $A$ to the sheafification of the presheaf $(S, k_S) \mapsto A(k_S)$ by the above proposition. This is exact, so $\pi$ is indeed a morphism of sites. A remark is that by forgetting $k'$ from $(K(k'), k')$, we have a continuous map from the fppf site of $K$-algebras to the site Spec $K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}}$. But neither of it or its composite with our morphisms $\pi$ has an obvious reason to be a morphism of sites. Therefore we only use $\pi: \text{Spec } K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}} \to \text{Spec } k_{\text{et}}^{\text{indrat}}$. For the four operation formalism (about pushforward, pullback, Hom and tensor product) for an arbitrary morphism of sites and unbounded derived categories, see [KS06] Chapter 18.

In a concrete term, we have $\Gamma(K_{\text{fppf}}, A)(k') = A(K(k'), k')$ for any $k' \in k^{\text{indrat}}$ and $A \in \text{Ab}(K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}})$, and the sheaf $H^i(K_{\text{fppf}}, A)$ on Spec $k_{\text{et}}^{\text{indrat}}$ is the étale sheafification of the presheaf

$$k' \mapsto H^i(K(k')_{\text{fppf}}, f_*A),$$

where $(f_*A)(S) = A(S, k')$ as in Proposition 2.3.3. If $A$ is a locally algebraic group scheme over $K$ viewed as a sheaf on Spec $K_{\text{fppf}}/k_{\text{et}}^{\text{indrat}}$ (see the paragraph after Proposition 2.3.3), then $A(S, k') = A(S)$, hence $f_*A = A$. Recall that if $k' \in k_{\text{et}}$, then $K(k')$ is a finite direct product of complete discrete valuation fields with perfect residue field. Therefore $R\Gamma(K_{\text{fppf}}, A)$, at least when restricted to the site Spec $k_{\text{et}}$, is essentially a classical object when $A$ is a locally algebraic group scheme over $K$. In the next subsection, we will need to understand cohomology of $K(k')$ when $k'$ is ind-rational but not necessarily rational.

To simplify the notation, we will write

$$\Gamma(\cdot) = \pi_*, \quad H^i(\cdot) = R^i\pi_*: \text{Ab}(K) \to \text{Ab}(k),$$

$$R\Gamma(\cdot) = R\pi_*: D(K) \to D(k)$$
when there is no confusion. Let $\text{Hom}$ (resp. $\text{Ext}^i$) denote the sheaf-Hom (resp. the $i$-th sheaf-Ext) and $R\text{Hom}$ its derived version. We want to define a morphism

$$R\Gamma(Z) \to R\text{Hom}_k(K^\times, Z)$$

of duality with coefficients in $G_m$. We need two propositions.

**Proposition 2.4.3.** There exists a canonical morphism

1. $R\Gamma R\text{Hom}_k(A, B) \to R\text{Hom}_k(R\Gamma(A), R\Gamma(B))$

in $D(k)$ for $A, B \in D(K)$, called the morphism of functoriality of $R\Gamma$. If a sequence

2. $0 \to B \to C_n \to \cdots \to C_1 \to A \to 0$

in $\text{Ab}(K)$ is exact with $H^i(A) = H^i(B) = H^i(C_j) = 0$ for any $i \geq 1$ and any $j$, then the sequence

3. $0 \to \Gamma(B) \to \Gamma(C_n) \to \cdots \to \Gamma(C_1) \to \Gamma(A) \to 0$

is exact, and the extension class of $1$ maps to that of $2$ via the induced map

4. $\text{Ext}^n_k(A, B) \to \text{Ext}^n_k(\Gamma(A), \Gamma(B))$.

**Proof.** The adjunction between $\Gamma^*$ and $R\pi_*$ induces morphisms

$$R\Gamma R\text{Hom}_k(A, B) \to R\Gamma R\text{Hom}_k(\Gamma^* R\pi(A), B) \to R\text{Hom}_k(R\Gamma(A), R\pi(B)).$$

See [KS05, Theorem 18.6.9] for this setting of a morphism of sites, sheaf-Hom and unbounded derived categories. This construction shows that (4) is the homomorphism

$$\text{Hom}_{D(K)}(A, B[n]) \to \text{Hom}_{D(k)}(\Gamma(A), \Gamma(B[n])).$$

The extension class of (2) corresponds to the morphism

$$A \xrightarrow{\sim} [B \to C_n \to \cdots \to C_1] \to B[n],$$

where the $B$ in the complex in the middle is placed at degree $-n$. The assumptions show that

$$R\Gamma[B \to C_n \to \cdots \to C_1] = [\Gamma(B) \to \Gamma(C_n) \to \cdots \to \Gamma(C_1)].$$

Hence we obtain the corresponding morphism

$$\Gamma(A) \xrightarrow{\sim} [\Gamma(B) \to \Gamma(C_n) \to \cdots \to \Gamma(C_1)] \to \Gamma(B[n]).$$

$\square$

Note that the morphism of functoriality of $R\Gamma$ is equivalent to the cup-product pairing

$$R\Gamma(A) \otimes^L K R\Gamma(C) \to R\Gamma(A \otimes^L K C)$$

(where $\otimes^L K$ and $\otimes^L K$ denote the derived tensor products over $k$ and $K$, respectively) by the derived tensor-hom adjunction [KS05, Theorem 18.6.4 (vi)] via the change of variables $R\text{Hom}_k(A, B) \to C$ and $A \otimes^L K C \to B$. We prefer $\text{Ext}$ groups rather than Tor for the treatment of proalgebraic groups.

Before the next proposition, we recall the functorial valuation map from [SY12, §4.1]. (The definition of this map does not depend on the choice of the topology, fpqc or étale.) For $k' \in k^{\text{indrat}}$ and $m \in \text{Spec} k'$, the ring $K(k'/m)$ is a complete discrete valuation field extending $k$. We denote by $v_m$ the composite of the natural surjection $K(k')^\times \to K(k'/m)^\times$ and the normalized valuation $K(k'/m)^\times \to Z$. For $\alpha \in K(k')^\times$, the map $m \mapsto v_m(\alpha)$ is a locally constant $Z$-valued function on the underlying topological space of $\text{Spec} k'$ ([SY12, Proposition 4.1.1]). This defines a
morphism of sheaves $K^\times \rightarrow \mathbb{Z}$ in $\text{Ab}(k^{\text{indrat}})$, which we call the \textit{valuation map} for $K^\times$. Denote $U_K = O_K^\times$ as in Introduction. The sequence

$$0 \rightarrow U_K \rightarrow K^\times \rightarrow \mathbb{Z} \rightarrow 0$$

in $\text{Ab}(k^{\text{indrat}})$ is split exact ([SY12 Proposition 4.1.2]).

**Proposition 2.4.4.** We have

$$R\Gamma(G_m) = \Gamma(G_m) = K^\times.$$  

We call the composite of this isomorphism and the valuation map $K^\times \rightarrow \mathbb{Z}$ (see above) the \textit{trace map}.

We will prove this proposition in the next subsection. Note that the sheaf $\mathcal{H}^1(G_m)$ on $\text{Spec} k^{\text{indrat}}$ is the étale sheafification of the presheaf

$$k' \in k^{\text{indrat}} \mapsto H^1(K(k')^{\text{fppf}}, G_m) = H^1(K(k')^{\text{et}}, G_m),$$

where we used the fact that the fppf cohomology with coefficients in a smooth group scheme agrees with the étale cohomology ([Mil80 III, Remark 3.11 (b)]). When restricted to $\text{Spec} k^{\text{et}}$, its vanishing for $i \geq 1$ follows from classical results on Galois cohomology of complete discrete valuation fields with algebraically closed residue field: for each field $k' \in k^{\text{rat}}$ and $i \geq 1$, we have

$$H^i(K(k')^{\text{et}}, G_m) = H^i(K(k')^{\text{ur}}, G_m) = 0$$

by [Ser79 V, §4, Proposition 7, and X, §7, Proposition 11], where $K(k')^{\text{ur}}$ is the maximal unramified extension of the complete discrete valuation field $K(k')$ and $K(\overline{k})$ its completion.

Using Propositions 2.4.3 and 2.4.4, we have a desired morphism

$$R\Gamma(\mathbb{Z}) \rightarrow R\Gamma R\text{Hom}_K(G_m, G_m) \rightarrow R\text{Hom}_k(K^\times, K^\times) \rightarrow R\text{Hom}_k(K^\times, \mathbb{Z}).$$

**Remark 2.4.5.** The site $\text{Spec} K^{\text{fppf}}/k^{\text{indrat}}$ has the following relationship with oriented products of sites defined by Laumon [Lau83 §3.1.3]. Consider the continuous map $\pi_0$: $\text{Spec} K^{\text{fppf}} \rightarrow \text{Spec} k^{\text{indrat}}$ defined by the functor $k' \mapsto K(k')$ (here we allow all $K$-algebras in the underlying category of the site $\text{Spec} K^{\text{fppf}}$). We denote the empty site by $\emptyset$. Then we have two continuous maps $\text{Spec} K^{\text{fppf}} \rightarrow \text{Spec} k^{\text{indrat}} \leftarrow \emptyset$.

Their oriented product $\text{Spec} K^{\text{fppf}} \times_{\text{Spec} k^{\text{indrat}}} \emptyset$ can be defined as in loc. cit., even though $\pi_0$ is not a morphism of sites, and it agrees with our site $\text{Spec} K^{\text{fppf}}/k^{\text{indrat}}$.

The projection $\text{Spec} K^{\text{fppf}} \rightarrow_{\text{Spec} k^{\text{indrat}}} \emptyset$ agrees with our morphism $\pi$ of sites.

More generally, let $S, S'$ be sites defined by pretopologies. Let $\pi_0: S' \rightarrow S$ be a continuous map whose underlying functor (also denoted by $\pi_0$) sends coverings to coverings such that $\pi_0(Y \times_X Z) = \pi_0(Y) \times_{\pi_0(X)} \pi_0(Z)$ if $Y \rightarrow X$ in $S$ appears in a covering family. Then we can define the relative site $S'/S$ in the same way as above and it agrees with $S' \times_S \emptyset$. The morphism $\pi: S'/S \rightarrow S$ of sites is similarly defined and agrees with the projection $S' \times_S \emptyset \rightarrow S$. The effect of multiplying $\emptyset$ is to force a continuous map to have an exact pullback functor. Assume moreover that if $Y \rightarrow X$ is a morphism in $S$ that appears in a covering family, then the diagonal morphism $Y \rightarrow Y \times_X Y$ also appears in a covering family. Then the cohomology theory of $S'/S$ is essentially that of $S'$ in the sense of Proposition 2.3.4.
2.5. Cohomology of local fields with ind-rational base. To prove Proposition 2.4.4, we first need to understand the étale site of $K(k')$ when $k' \in \kappa^{\text{indrat}}$. We define subsheaves $O_{K}^{0} \subset O_{K}$ and $K^{0} \subset K$ by

$$O_{K}^{0}(k') = \bigcup_{\lambda} O_{K}(k'_{\lambda}), \quad K^{0}(k') = \bigcup_{\lambda} K(k'_{\lambda}).$$

for each $k' = \bigcup_{\lambda} k'_{\lambda} \in \kappa^{\text{indrat}}$ with $k'_{\lambda} \in \kappa^{\text{rat}}$. The strategy is to compare $K(k')_{\text{et}}$ with $K^{0}(k')_{\text{et}}$, the latter of which is described by the étale sites of complete discrete valuation fields. The argument in this comparison goes basically in the same line as the proof of Krasner’s lemma using Hensel’s lemma. Additional complications come from the underlying topological space of $\text{Spec} k'$, which is a profinite space. We treat this topology and the topology coming from the valuation simultaneously.

Then we will be reduced to considering the cohomology of the pushforward of $G_{\text{an}}$ from $K(k')_{\text{et}}$ to $K^{0}(k')_{\text{et}}$. The computation of this is essentially classical. Up to a notational preparation, we only need to recall the fact that for a finite extension $L/K$, any element of $K^{\infty}$ becomes a norm in $L^{\infty}$ after a finite unramified extension (see [Ser79, V, §3]).

We need notation and several lemmas to prove Proposition 2.4.4. We fix $k' \in \kappa^{\text{indrat}}$. If $m$ is a maximal ideal of $k'$, then the kernel $K(m)$ of the natural surjection $K(k') \to K(k'/m)$ is a maximal ideal of $K(k')$. Conversely, we have:

**Lemma 2.5.1.** Any maximal ideal of $K(k')$ is of the form $K(m)$ for some maximal ideal $m$ of $k'$.

**Proof.** Let $n$ be a maximal ideal of $K(k')$. Let $m \subset k'$ be the ideal given by the image of the ideal $O_{K}(k') \cap n$ via the natural surjection $O_{K}(k') \to k'$.

We show that for an element $a \in k'$ to be in $m$, it is necessary and sufficient that $\omega(a) \in n$, where $\omega$ is the Teichmüller lift. Clearly this is sufficient. For necessity, let $\alpha = O_{K}(k') \cap n$ map to $a \in m$. Take a prime element $\pi \in O_{K}$ and write $\alpha = \omega(a) + \pi \beta$, where $\beta \in O_{K}(k')$. Since $k' \in \kappa^{\text{indrat}}$, we can write $a = u\pi$ for an idempotent $e \in k'$ and a unit $u \in k^{\infty}$. Then $\omega(e)\alpha = \omega(e)\omega(u) + \pi \beta$. Since $\omega(u) + \pi \beta \in O_{K}(k')^{\infty}$ and $\alpha \in n$, we have $\omega(e) \in n$. Hence $\omega(a) \in n$.

This characterization shows that $m \subset k'$ is a prime ideal. It has to be maximal since $k' \in \kappa^{\text{indrat}}$. To finish the proof, it is enough to show that $n \subset K(m)$. Let $\alpha = \sum_{i \geq n} \omega(a_{i})\pi^{i} \in n$ be any element ($n \in \mathbb{Z}$). Since $\pi \in K^{\infty}$, we have $\pi^{-n}\alpha \in O_{K}(k') \cap n$. Hence $a_{n} \in m$ by definition of $m$. Hence $\omega(a_{n})\pi^{n} \in n$ and $\alpha - \omega(a_{n})\pi^{n} = \sum_{i \geq n+1} \omega(a_{i})\pi^{i} \in O_{K}(k') \cap n$. Inductively, we have $a_{i} \in m$ for all $i$. Hence $\alpha \in K(m)$. Therefore $n \subset K(m)$.

Hence the maximal spectrum of $K(k')$ is in bijection with $\text{Spec} k'$. (The whole prime spectrum is much different. Results from [Arn73], in the equal characteristic case, show that $O_{K}(k') \cong k'[[T]]$ and $K(k') \cong k'[[T]][1/T]$ have infinite Krull dimensions if $k'$ has infinitely many direct factors.)

**Lemma 2.5.2.** Let $m \subset k'$ be maximal. A neighborhood base of the maximal ideal $K(m)$ in $\text{Spec} K(k')$ is given by the family $\text{Spec} K(k')[1/\omega(a)] = \text{Spec} K(k' / \langle a \rangle)$ of open sets, where $a \in k' \setminus m$. In particular, any Zariski covering of $\text{Spec} K(k')$ can be refined by a disjoint Zariski covering.

**Proof.** Let $\alpha = \sum_{i \geq n} \omega(a_{i})\pi^{i} \in K(k') \setminus K(m)$ be any element ($\pi$ a prime), so $a_{m} \notin m$ for some $m \geq n$. We may assume that $a_{n}, \ldots, a_{m-1} \in m$. Let $e_{\leq m} \in k'$
be the idempotent generating the ideal \((a_n, \ldots, a_{m-1})\) of \(k'\). Write \(a_m = u_m e_m\), where \(e_m\) is an idempotent of \(k'\) and \(u_m\) is a unit of \(k'\). Then

\[
\omega \left( (1 - e_{<m}) e_m \right) = \omega \left( (1 - e_{<m}) e_m \right) \left( \omega(u_m) \pi^m + \sum_{i \geq m+1} \omega(a_i) \pi^i \right).
\]

The term in the large brackets is a unit in \(K(k')\). Hence

\[
\text{Spec } K(k')[1/\omega((1 - e_{<m}) e_m)] \subset \text{Spec } K(k')[1/\alpha].
\]

This proves the lemma since \((1 - e_{<m}) e_m \notin m\).

By a similar argument, we know that if an element of \(K(k')\) becomes a unit in \(K(k'/m)\) (resp. lies in \(O_{K(k'/m)}\)), then it is a unit in \(K(k'[1/\alpha])\) (resp. lies in \(O_{K(k'[1/\alpha])}\)) for some \(\alpha \in k' \setminus m\).

**Lemma 2.5.3.** Any étale covering of \(K(k')\) can be refined by a covering coming from an étale covering of \(K^0(k')\). More precisely, let \(S\) be a faithfully flat étale \(K(k')\)-algebra. Then there exist a faithfully flat étale \(K^0(k')\)-algebra \(L^0\) and a \(K(k')\)-algebra homomorphism \(S \to L^0 \otimes_{K^0(k')} K(k')\).

**Proof.** We may assume that \(S\) is a standard étale \(K(k')\)-algebra by Lemma 2.5.2. This means that \(S = K(k'[x]/g(x))/(f(x))\) for some polynomials \(f(x)\) and \(g(x)\) such that \(f(x)\) is monic and \(f'(x) \in S^x\). Fix a maximal ideal \(m \subset k'\). By the same lemma, it is enough to show the existence of such \(L^0\) after localizing \(k'\) by an element not in \(m\). We denote by \(f_m(x)\) the image of \(f(x)\) in \(K(k'[m])[x]\). Since \(S/SS(K(m)) \neq 0\), there exists a simple root \(\alpha\) (in a separable closure of \(K(k'/m))\) of the polynomial \(f_m(x)\) such that \(g_m(\alpha) \neq 0\). Let \(\alpha_1, \alpha_2, \ldots\) be the other roots of \(f_m(x)\). We can take an algebraic element \(x\) over \(K^0(k'/m)\) arbitrarily close to \(\alpha\) since \(K^0(k'/m)\) is the completion of \(K^0(k'/m)\). We choose such \(x\) so that: it is separable over \(K^0(k'/m)\); \(f_m'(x) \neq 0\); and there exists an element \(\gamma \in K^x\) such that \(|\alpha - \beta| < |\gamma| \leq |\alpha_i - \beta|\) for all \(i\) (\(|\cdot|\) denotes an absolute value). Let \(h(y) \in K^0(k'/m)[y]\) be the monic minimal polynomial of \(\beta\). Take a rational \(k\)-subalgebra \(k'' \subset k'\) such that \(h(y) \in K(k''/k''\cap m)[y]\). We may assume that \(k'' \cap m = 0\) and so \(k''\) is a field by localizing \(k'\) by an element not in \(m\). We define \(L = K(k''\cap m)/h(y) \cong K(k'')[\beta]\). This is a finite separable extension of the complete discrete valuation field \(K(k'')\) linearly disjoint from \(K(k'/m)\). We define \(L^0 = K(k'')[y]/h(y)\). This is a faithfully flat étale \(K^0(k')\)-algebra. Since \(f_m'(x) \neq 0\), we may assume that \(f'(y)\) is a unit in \(L^0 \otimes_{K^0(k')} K(k')\) by a similar localization (apply the argument right before this lemma to the norm \(N_{L/K(k'')} f'(y) \in K(k')\)). Consider the polynomial 

\[
(\gamma f_m'(x))^{-1} f_m(\gamma x + \beta) + \text{coefficients in the complete discrete valuation field } K(k'/m)[\beta] \cong L \otimes_{K(k')} K(k'/m).\]

By looking at the Newton polygon and by the choice of \(\beta\), we know that this polynomial has integral coefficients, the constant term has positive valuation, and the coefficient for \(x = 1\). Hence we can localize \(k'\) by an element not in \(m\) so that \((\gamma f'(y))^{-1} f(\gamma x + y)\) is in \(O_L \otimes_{O_{K(k'')}} O_K(k')\), the constant term in \(x\) is in \(p_L \otimes_{O_{K(k'')}} O_K(k')\), and the coefficient for \(x = 1\) is \(1\), where \(O_L\) is the ring of integers of \(L\) and \(p_L\) its maximal ideal. The pair \((O_L \otimes_{O_{K(k'')}} O_K(k'), p_L \otimes_{O_{K(k'')}} O_K(k'))\) is Henselian as the pairs \((W(k'), (p))\) and \((k'[T],[T])\) are so. Therefore the polynomial \((\gamma f'(y))^{-1} f(\gamma x + y)\) of \(x\) has a unique root in \(O_L \otimes_{O_{K(k'')}} O_K(k')\) whose image in \(k_L \otimes_{k''} k'\) is zero, where \(k_L\) is the residue field of \(O_L\). Write this root as \(x = (z - y)/\gamma\), so \(z \in L \otimes_{K(k'')} K(k')\) is a root.
of the polynomial \( f(x) \). Sending \( x \) to \( z \), we have a \( K(k') \)-algebra homomorphism \( K(k'[x]/(f(x)) \to L \otimes_{K(k')} K(k') = L^0 \otimes_{K(k')} K(k') \). The image of \( z \) in the field \( L \otimes_{K(k')} K(k'/m) = K(k'/m)[\alpha] \) is \( \alpha \) by uniqueness. Hence the image of \( g(z) \) in the same field is \( g_m(\alpha) \neq 0 \). Therefore we can localize \( k' \) by an element not in \( m \) so that \( g(z) \) is a unit in \( L \otimes_{K(k')} K(k') \). Thus we get a \( K(k') \)-algebra homomorphism \( S \to L^0 \otimes_{K(k')} K(k') \), as required.

**Lemma 2.5.4.** The tensor product \( (\cdot) \otimes_{K^0(k')} K(k') \) induces a fully faithful embedding from the category of étale \( K^0(k') \)-algebras into the category of étale \( K(k') \)-algebras.

**Proof.** Let \( f : \text{Spec } K(k') \to \text{Spec } K^0(k') \) be the natural morphism. Let \( L_1^0 \) and \( L_2^0 \) be étale \( K^0(k') \)-algebras. It is enough to show that the natural sheaf morphism

\[
\text{Hom}_{K(k')} (L_1^0, L_2^0) \to f, \text{Hom}_{K(k')} (L_1^0 \otimes_{K(k')} K(k'), L_2^0 \otimes_{K(k')} K(k'))
\]

in \( \text{Set} (K^0(k')_{et}) \) is an isomorphism. We may check this étale locally on \( K^0(k') \). Note that \( K^0(k') \) is a filtered union of finite products of fields. Hence by taking an étale cover of \( K^0(k') \), we may assume that \( L_1^0 \) and \( L_2^0 \) are direct factors of \( K^0(k') \). Since \( k' \in k^\text{indrat} \), we know that \( L_i^0 = K^0(k'_i) \) for some direct factor \( k'_i \) of \( k' \) for each \( i = 1, 2 \). The morphism above becomes

\[
\text{Hom}_{K(k')} (K^0(k'_1), K^0(k'_2)) \to f, \text{Hom}_{K(k')} (K(k'_1), K(k'_2)).
\]

The both sides are a point if \( \text{Spec } k'_2 \subset \text{Spec } k'_1 \) and empty otherwise. \( \square \)

**Lemma 2.5.5.** Let \( f : \text{Spec } K(k') \to \text{Spec } K^0(k') \) be the natural morphism. Then the pushforward \( f_* : \text{Ab}(K(k')_{et}) \to \text{Ab}(K^0(k')_{et}) \) is exact.

**Proof.** This is a formal consequence of Lemmas 2.5.3 and 2.5.4. \( \square \)

Therefore we have \( H^i(K(k')_{et}, G_m) = H^i(K^0(k')_{et}, f_* G_m) \) for all \( i \). To compute this, we recall Tate cohomology sheaves from \( [SY12, \S 3.4] \) and give a variant of the vanishing result \( [SY12, \text{Proposition } 4.2] \). Let \( L/K \) be a finite Galois extension with Galois group \( G \). Let \( \mathcal{O}_L \) be the ring of integers of \( L \). Define sheaves \( \mathbf{L}_k, \mathbf{U}_{L,k} \) on \( \text{Spec } k_{et} \) by

\[
k'' \in k_{et} \mapsto L \otimes_K K(k''), \quad k'' \in k_{et}^{\text{indrat}} \mapsto (\mathcal{O}_L \otimes_{\mathcal{O}_K} \mathcal{O}_K(k''))^\times,
\]

respectively. The finite group \( G \) acts on them as morphisms of sheaves. Define a complex

\[
\cdots \to \prod_{\sigma_1, \sigma_2 \in G} L_k^\times \to \prod_{\sigma_1 \in G} L_k^\times \to L_k^\times \xrightarrow{\mathbf{N}} L_k^\times \to \prod_{\sigma_1 \in G} L_k^\times \to \prod_{\sigma_1, \sigma_2 \in G} L_k^\times \to \cdots
\]

in \( \text{Ab}(k_{et}^{\text{indrat}}) \) using the differentials for inhomogeneous chains and cochains (\[ \text{Set}_{79}, \S 3 \) and \( \S 4 \)) and the norm map \( N \). For each \( i \in \mathbb{Z} \), define the Tate cohomology sheaf \( \hat{H}^i(G, L_k^\times) \) to be the \( i \)-th cohomology of this complex. This is the étale sheafification of the presheaf \( k'' \mapsto \hat{H}^i(G, L_k^\times(k'')) \) of Tate cohomology groups. We will show below that the sheaves \( \hat{H}^i(G, L_k^\times) \) on \( \text{Spec } k_{et}^{\text{indrat}} \) are all zero. The difference from \( [SY12, \text{Proposition } 4.2] \) is the choice of the site. To kill a cocycle in \( \hat{H}^i(G, L_k^\times(k'')) \) with \( k'' \in k_{et}^{\text{indrat}} \), we only need to extend \( k'' \) to its finite extension.

**Lemma 2.5.6.** We have \( \hat{H}^i(G, L_k^\times) = 0 \) for all \( i \in \mathbb{Z} \).
Proof. For any \(k'' \in k^{\text{indrat}}\), any Zariski locally free sheaf on \(\text{Spec} K(k'')\) of constant rank is free by Lemma 2.5.2. Hence \(\text{Pic}(K(k'')) = 0\) and

\[
\hat{H}^1(G, L_k^x(k'')) = \text{Ker}(\text{Pic}(K(k''))) \rightarrow \text{Pic}(L_k(k'')) = 0,
\]

so \(\hat{H}^1(G, L_k^x) = 0\). Therefore it is enough to show that \(\hat{H}^0(G, L_k^x) = 0\) by [Ser79] IX, §5, Theorem 8 (to apply this theorem on group cohomology groups to our situation of cohomology sheaves, take an arbitrary injective object \(I\) in \(\text{Ab}(k^{\text{indrat}})\) and consider a corresponding statement for \(\hat{H}^i(G, \text{Hom}_{\text{indrat}}(L_k^x, I))\)). It is enough to show that the norm map \(N: U_{L,k} \rightarrow U_K\) is surjective in \(\text{Ab}(k^{\text{indrat}})\).

Let \(k' \in k^{\text{indrat}}\) and \(x \in U_K(k')\). The map \(N\) is a surjective homomorphism of proalgebraic groups over \(k\) ([Ser61] §2.1; [Ser79] §3, Corollary to Proposition 1). The norm computation given in [Ser79] V, §3 shows that almost all subquotients of the filtration \(\{\text{Ker}(N) \cap U^n_{L,k}\}\) of \(\text{Ker}(N)\) are connected unipotent, where \(U^n_{L,k}\) is the group of \(n\)-th principal units. Assume that the subquotients are connected unipotent for \(n \geq n_0\). Consider the cartesian diagram of surjections of proalgebraic groups

\[
\begin{array}{ccc}
U_{L,k}/(\text{Ker}(N) \cap U_{L,k}^{n_0}) & \longrightarrow & U_K \\
\downarrow & & \downarrow \\
U_{L,k}/U_{L,k}^{n_0} & \longrightarrow & U_K/NU_{L,k}^{n_0}.
\end{array}
\]

The lower left term and hence the lower right term are quasi-algebraic. Hence the image of the \(k'\)-point \(x\) in the lower right term lifts to a \(k''\)-point of the lower left term for some faithfully flat étale \(k'\)-algebra \(k''\). Since the diagram is cartesian, the \(k'\)-point \(x\) of the upper right term itself lifts to a \(k''\)-point \(x_0\) of the upper left term. Consider the cartesian diagram of surjections of proalgebraic groups

\[
\begin{array}{ccc}
U_{L,k}/(\text{Ker}(N) \cap U_{L,k}^{n_0+1}) & \longrightarrow & U_{L,k}/(\text{Ker}(N) \cap U_{L,k}^{n_0}) \\
\downarrow & & \downarrow \\
U_{L,k}/U_{L,k}^{n_0+1} & \longrightarrow & U_{L,k}/(U_{L,k}^{n_0+1} \cap \text{Ker}(N) \cap U_{L,k}^{n_0}).
\end{array}
\]

The kernel \(G\) of the lower horizontal morphism is connected unipotent quasi-algebraic by assumption. The image of the \(k''\)-point \(x_0\) in the lower right term defines an element of \(H^1(k''_{et}, G)\). The group \(H^1(k''_{et}, G)\) is trivial since it is isomorphic to coherent cohomology by [Mil80] III, Remark 3.8 and the affine scheme \(\text{Spec} k''\) has trivial coherent cohomology. This implies that \(H^1(k''_{et}, G) = 0\). Hence the image of the \(k''\)-point \(x_0\) in the lower right term lifts a \(k''\)-point of the lower left term (with no need to extend \(k''\)). Since the diagram is cartesian, the \(k''\)-point \(x_0\) of the upper right term itself lifts to a \(k''\)-point \(x_1\) of the upper left term. Iteratively applying this argument, we obtain a \(k''\)-point of

\[
\lim_{n \geq n_0} U_{L,k}/(\text{Ker}(N) \cap U_{L,k}^n) = U_{L,k}
\]

whose norm is \(x \in U_K(k')\). This shows the surjectivity of \(N: U_{L,k} \rightarrow U_K\) in \(\text{Ab}(k^{\text{indrat}})\).

\[\square\]

Proof of Proposition 2.4.4. By Lemma 2.5.3 we have

\[
H^i(K(k')_{et}, G_m) = H^i(K^0(k')_{et}, f_* G_m),
\]

where \(f: K(k') \rightarrow K\) is the base change.
Proof. \( \text{of } R \mathcal{K}(b) \). Conversely, this isomorphism for any \( 1.17(a) \). Hence the homomorphism (5) is the direct limit of
\[ R^i(K^0(k')_{et}, f_*G_m) = \lim_{\lambda, L} H^i(\text{Gal}(L/K(k'_\lambda)), (L \otimes_{K(k'_\lambda)} K(k'))^\times), \]
where the \( L \) runs through the finite Galois extensions of \( K(k'_\lambda) \). The coefficient group on the right-hand side can be written as \( L_{k'_\lambda}^\times (k') \) in the notation above. Therefore for \( i \geq 1 \), its element is killed after an étale faithfully flat extension of \( k' \) by Lemma 2.5.6. Hence \( H^i(G_m) = 0 \) for \( i \geq 1 \). This proves the proposition. \( \square \)

The following will be needed in the next subsection to reduce the computation of \( R \Gamma(Z) \) to that of cohomology of complete discrete valuation fields.

**Proposition 2.5.7.** Let \( k' \in k^{\text{idrat}}. \) Let \( A \) be a constant sheaf of abelian groups. Then \( R^i(K(k')_{et}, A) = R^i(K^0(k')_{et}, A) \).

**Proof.** By Lemma 2.5.4, this amounts to saying that \( f_*A = A \) in \( \text{Ab}(K^0(k')_{et}) \), where \( f: \text{Spec } K(k') \to \text{Spec } K^0(k') \) is the natural morphism. Let \( L^0 \) be any étale \( K^0(k') \)-algebra and set \( L = L^0 \otimes_{K^0(k')} K(k') \). We need to show that \( A(L^0) = A(L) \). It is enough to show that the inclusion \( L^0 \hookrightarrow L \) induces a bijection on the set of idempotents. The set of idempotents of \( L^0 \), \( L \) can be identified with
\[ \text{Hom}_{K^0(k')}(K^0(k') \times K^0(k'), L^0), \text{Hom}_{K(k')}(K(k') \times K(k'), L), \]
respectively. They are in bijection by Lemma 2.5.4. \( \square \)

2.6. **Duality with coefficients in \( G_m \).** The following states the duality for \( K \) with coefficients in \( G_m \).

**Theorem 2.6.1.** The morphism
\[ R \Gamma(Z) \to R \text{Hom}_k(K^\times, Z) \]
in \( D(k) \) defined at the end of Section 2.4 is an isomorphism.

We prove this in this subsection. This implies, by taking \( R \Gamma(k'_{et}, \cdot) \) of the both sides for any \( k' \in k^{\text{idrat}} \), an isomorphism
\[ R \Gamma(K(k')_{et}, Z) \to R \text{Hom}_{k^{\text{idrat}}/k'}(K^\times, Z) \]
in \( D(Ab) \), where we again used the comparison between fppf and étale cohomology with coefficients in a smooth group scheme ([Mil80 III, Remark 3.11 (b)]). Conversely, this isomorphism for any \( k' \in k^{\text{idrat}} \) implies the theorem. Let \( k' = \bigcup \lambda k'_\lambda \) with \( k'_\lambda \in k^{\text{idrat}} \). The left-hand side of (3) is the direct limit of \( R \Gamma(K(k'_\lambda)_{et}, Z) \) in \( \lambda \) by Proposition 2.5.7 and [Mil80 III, Lemma 1.16, Remark 1.17 (a)]. For the right-hand side, note that \( K^\times \cong Z \times U_K \), where \( U_K = O_K^\times \). The sheaf \( U_K \) is represented by the proalgebraic group of units of \( K \) studied by Serre [Ser61], which is affine. Hence Theorem 2.4 shows that \( R \text{Hom}_{k^{\text{idrat}}/k'}(U_K, Z) \) is the direct limit of \( R \text{Hom}_{(k'_\lambda)^{\text{idrat}}/k'}(U_K, Z) \) in \( \lambda \). On the other hand, Proposition 2.4 shows that \( R \text{Hom}_{k^{\text{idrat}}/k'}(Z, Z) = R^i(k'_{et}, Z) \), which is the direct limit of \( R \text{Hom}_{(k'_\lambda)^{\text{idrat}}/k'}(Z, Z) = R^i(k'_\lambda_{et}, Z) \) in \( \lambda \) by [Mil80 III, Lemma 1.16, Remark 1.17 (a)]. Hence the homomorphism (3) is the direct limit of
\[ R \Gamma(K(k'_\lambda)_{et}, Z) \to R \text{Hom}_{(k'_\lambda)^{\text{idrat}}/k'}(K^\times, Z). \]
We want to show that this is an isomorphism for any \( \lambda \). Replacing \( K \) with \( K(k'_\lambda) \), we only need to consider the case \( k'_\lambda = k \):

\[
R\Gamma(K_{et}, Z) \to R\operatorname{Hom}_{k_{et}^{\text{indrat}}}(K^\times, Z),
\]

Compare this morphism with the morphism for the case \( k' = \overline{k} \):

\[
R\Gamma(\overline{K}(\mathcal{O})_{et}, Z) \to R\operatorname{Hom}_{\overline{k}_{et}^{\text{indrat}}}(\overline{K^\times}, Z) = R\operatorname{Hom}_{k_{et}^{\text{indrat}}}(K^\times, Z),
\]

where we used the fact \( \text{Spec} k_{et}^{\text{indrat}} = \text{Spec} k_{et}^{\text{indrat}}/\overline{k} \) observed in Section 2.1. By applying \( R\Gamma(k_{et}, \cdot) \) to the latter, we may assume that \( k = \overline{k} \). What we have to show is hence

\[
H^i(K, Z) \cong \operatorname{Ext}^i_k(K^\times, Z)
\]

for algebraically closed \( k \). We first treat the part \( i \neq 2 \).

**Proposition 2.6.2.** We have

\[
\Gamma(K, Z) = \operatorname{Hom}_k(K^\times, Z) = Z,
\]

\[
H^i(K, Z) = \operatorname{Ext}^i_k(K^\times, Z) = 0 \quad \text{for} \quad i \neq 0, 2.
\]

**Proof:** We compute \( \operatorname{Ext}^i_k(U_K, Z) \) for \( i \neq 2 \). The case \( i = 0 \) is done in Theorem 2.1.5 If \( i \neq 0 \), then we have

\[
\operatorname{Ext}^i_k(U_K, Z) = \lim_{n} \operatorname{Ext}^{i-1}_{k_{\text{PAlg}}}(U_K, Z/nZ).
\]

by the same theorem. By [Ser60] §5.4, Corollary to Proposition 7, this group is the Pontryagin dual of the \( (i-1) \)-st homotopy group \( \pi_{i-1}(U_K) \) of the proalgebraic group \( U_K \) in the sense of Serre. We have \( \pi_{i-1}(U_K) = 0 \) if \( i - 1 = 0 \) by the connectedness of \( U_K \), and if \( i - 1 \geq 2 \) by [Ser60] §10, Theorem 2]. This finishes the proof of the statements for \( \operatorname{Ext}^i_k(K^\times, Z) \).

The groups \( H^i(K, Z) \) are Galois cohomology groups of the complete discrete valuation field \( K \) with algebraically closed residue field. Such a field has cohomological dimension 1 by [Ser02] II, §3.3]. The result then follows. \( \square \)

Therefore it is enough to show that the homomorphism

\[
H^2(K, Z) \to \operatorname{Ext}^2_k(K^\times, Z)
\]

is an isomorphism. Note that the left-hand side is equal to \( H^1(K, Q/Z) = \operatorname{Ext}^1_k(Z, Z) \) and the right-hand side \( \operatorname{Ext}^1_k(K^\times, Q/Z) = \lim_n \operatorname{Ext}^1_k(K^\times, Z/nZ) \) as shown in the proof of the above proposition.

**Proposition 2.6.3.** The above homomorphism

\[
H^1(K, Q/Z) \to \operatorname{Ext}^1_k(K^\times, Q/Z)
\]

sends a finite cyclic extension \( L \) of \( K \) of degree \( n \) with Galois group \( G = \langle \sigma \rangle \) to the extension class

\[
0 \to G^{\sigma \mapsto \sigma^2} \to L^\times / I_GU_L \to N_G^{\mathbb{K}^\times} \to 0,
\]

where \( L \) is defined from \( L \) in the same way as we defined \( K, I_G = Z[G](\sigma - 1) \) the augmentation ideal of the integral group ring \( Z[G] \), \( \pi_L \) a prime element of \( \mathcal{O}_L \), and \( N_G \) the norm map.
Proof. The extension $L/K$ as an element of $\text{Ext}^2_K(\mathbb{Z}, \mathbb{Z})$ is represented by

$$0 \to \mathbb{Z} \xrightarrow{N_G} \mathbb{Z}[G] \xrightarrow{\sigma^{-1}} \mathbb{Z}[G] \xrightarrow{\sigma^{-1}} \mathbb{Z} \to 0,$$

Its image in $\text{Ext}^2_K(G_m, G_m)$ is

$$0 \to G_m \xrightarrow{\text{incl}} \text{Res}_{L/K} G_m \xrightarrow{\sigma^{-1}} \text{Res}_{L/K} G_m \xrightarrow{N_G} G_m \to 0,$$

where $\text{Res}_{L/K}$ is the Weil restriction functor. We have

$$\text{R}\Gamma(K_{\text{fppf}}, \text{Res}_{L/K} G_m) = \text{R}\Gamma(L_{\text{fppf}}, G_m) = L \times.$$

Hence we can use the second half of Proposition 2.4.3. Applying $\Gamma$, we have an exact sequence

$$0 \to K^\times \xrightarrow{\text{incl}} L^\times \xrightarrow{\sigma^{-1}} L^\times \xrightarrow{N_G} K^\times \to 0$$

in $\text{Ab}(k)$. Pushing out this extension by the valuation map $K^\times \to \mathbb{Z}$ from the left term, we find that the image in $\text{Ext}^2_k(K^\times, \mathbb{Z})$ is given by

$$0 \to \mathbb{Z} \xrightarrow{1-\pi_K} L^\times / U_K \xrightarrow{\sigma^{-1}} L^\times \xrightarrow{N_G} K^\times \to 0$$

where $\pi_K$ is a prime element of $\mathcal{O}_K$. We have a natural quotient map from this extension to the extension

$$0 \to \mathbb{Z} \xrightarrow{n \pi_K} \mathbb{Z} \xrightarrow{1-\sigma(\pi_L)/\pi_L} L^\times / I_G U_L \xrightarrow{N_G} K^\times \to 0.$$

This extension as an element of the subgroup $\text{Ext}^1_k(K^\times, \mathbb{Z}/n)$ of $\text{Ext}^2_k(K^\times, \mathbb{Z})$ is

$$0 \to G \xrightarrow{\sigma^{-1}} \mathbb{Z} \xrightarrow{1-\sigma(\pi_L)/\pi_L} L^\times / I_G U_L \xrightarrow{N_G} K^\times \to 0,$$

as required. $\square$

Proof of Theorem 2.6.1. We have

$$\text{Ext}^1_{\text{reg}}(K^\times, \mathbb{Q}/\mathbb{Z}) = \lim_n \text{Ext}^1_{\text{PAlg}/k}(U_K, \mathbb{Z}/n\mathbb{Z})$$

as seen in the proof of Proposition 2.6.2. Under this isomorphism, the proposition above says that our homomorphism, or its Pontryagin dual

$$\pi_1(U_K) \to \text{Gal}(K^{ab}/K),$$

(ab denotes the maximal abelian extension) agrees with the reciprocity isomorphism of Serre’s local class field theory [Ser61]. This proves the theorem. $\square$

A corollary is that

$$H^1(K, \mathbb{Q}/\mathbb{Z}) = \text{Ext}^1_k(K^\times, \mathbb{Q}/\mathbb{Z}).$$

When $k$ is not necessarily algebraically closed, this recovers the main theorem of [SY12].
2.7. **Duality with coefficients in a finite flat group scheme.** Let $A$ be a finite flat group scheme over $K$. Assume that $A$ does not have connected unipotent part (which is always the case if $K$ has mixed characteristic). As in the previous subsection, we obtain a morphism

\[ R\Gamma(A^{CD}) \to R\Gamma R Hom_K(A, G_m) \]
\[ \to R Hom_k(R\Gamma(A), \mathbb{Z}) = R Hom_k(R\Gamma(A), \mathbb{Q}/\mathbb{Z}[-1]), \]

where the last equality is from the exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ and that $A$ is finite.

**Theorem 2.7.1.** The morphism

\[ R\Gamma(A^{CD}) \to R Hom_k(R\Gamma(A), \mathbb{Q}/\mathbb{Z}[-1]) \]

defined above is an isomorphism.

**Lemma 2.7.2.** The theorem is true for $A = \mu_l$, where $l$ is a prime (possibly $l = p$).

**Proof.** Consider the Kummer sequence $0 \to \mu_l \to G_m \to G_m \to 0$ and its Cartier dual $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/l\mathbb{Z} \to 0$. They yield a commutative diagram

\[
\begin{array}{ccc}
R\Gamma(\mathbb{Z}) & \longrightarrow & R\Gamma(\mathbb{Z}) \\
\downarrow & & \downarrow \\
R Hom_k(R\Gamma(G_m), \mathbb{Z}) & \longrightarrow & R Hom_k(R\Gamma(G_m), \mathbb{Z}) \\
\end{array}
\]

whose two rows are distinguished triangles. The first and second vertical morphisms are isomorphisms by Theorem 2.6.1. Hence so is the third. \qed

The following finishes the proof of the theorem in the mixed characteristic case.

**Lemma 2.7.3.** The theorem is true for a multiplicative $A$.

**Proof.** We may assume that $A$ has $l$-power order for a prime $l$ (possibly $l = p$). We reduce the lemma to the previously treated case $A = \mu_l$. Let $L$ be a finite Galois extension of $K$ such that the étale group $A^{CD}$ becomes constant over $L$. Let $M$ be the intermediate field of $L/K$ that corresponds to an $l$-Sylow subgroup of $Gal(L/K)$. Then the $l$-power torsion finite abelian group $A^{CD}(L)$ is equipped with the action of the $l$-group $Gal(L/M)$. Therefore, over $M$, the group $A^{CD}$ (resp. $A$) has a filtration whose successive subquotients are all isomorphic to $\mathbb{Z}/l\mathbb{Z}$ (resp. $\mu_l$). Hence by the above lemma, we have an isomorphism

\[ R\Gamma(M, A^{CD}) \to R Hom_k(R\Gamma(M, A), \mathbb{Q}/\mathbb{Z}[-1]) \]

in $D(k')$, where $k'$ is the residue field of $M$. We have a commutative diagram

\[
\begin{array}{ccc}
Spec M_{fppt}/k'_{et}^{indrat} & \longrightarrow & Spec k'_{et}^{indrat} \\
\downarrow & & \downarrow \\
Spec K_{fppt}/k'_{et}^{indrat} & \longrightarrow & Spec k'_{et}^{indrat} \\
\end{array}
\]

of morphisms of sites, where the left vertical morphism is defined by the functor $(S, k_S) \mapsto (S \otimes_K M, k_S \otimes_k k')$. Hence by applying the Weil restriction functor $Res_{k'/k}$ for the both sides of \((\text{6})\) and using the duality for the finite étale morphism $Spec k' \to Spec k$ [Mil80 V, Proposition 1.13], we have an isomorphism

\[ R\Gamma(K, Res_{M/K} A^{CD}) \to R Hom_k(R\Gamma(K, Res_{M/K} A), \mathbb{Q}/\mathbb{Z}[-1]) \]
in $D(k)$. The inclusion $A \hookrightarrow \text{Res}_{M/K} A$ followed by the norm map $\text{Res}_{M/K} A \to A$ is given by multiplication by $[M : K]$. This is an isomorphism since $A$ is $l$-power torsion and $M/K$ is an extension of degree prime to $l$. Hence $A$ is a canonical direct summand of $\text{Res}_{M/K} A$. A similar relation holds for $A^{\text{CD}}$. The above isomorphism induces an isomorphism on these direct summands. This proves the lemma. □

Note that the distinguished triangle $R\Gamma(\mu_l) \to R\Gamma(G_m) \to R\Gamma(G_m)$ used in the above proof and Proposition 2.3.4 show that
\[
H^i(\mu_l) = 0 \quad \text{for} \quad i \neq 0, 1,
\]
\[
H^1(\mu_l) = K^\times/(K^\times)^l.
\]

**Lemma 2.7.4.** The theorem is true for $A = \mathbb{Z}/p\mathbb{Z}$ when $K$ has equal characteristic.

**Proof.** By the previous lemma, the morphism in the theorem this case can be written as
\[
R\Gamma(\mu_p) \to R\text{Hom}_k(R\text{Hom}_k(R\Gamma(\mu_p), \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}),
\]
which agrees with the canonical evaluation morphism. With the remark just above, we know that $R\Gamma(\mu_p)$ is acyclic outside degree $1$ and
\[
H^1(\mu_p) = K^\times/(K^\times)^p = \mathbb{Z}/p\mathbb{Z} \times U_K^1/(U_K^1)^p = \mathbb{Z}/p\mathbb{Z} \times G_a^N,
\]
where the last isomorphism is given by the Artin-Hasse exponential map (See [Ser88 V, §3.16, Proposition 9]). We need to show that the evaluation morphism
\[
G_a^N \to R\text{Hom}_k(R\text{Hom}_k(G_a^N, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z})
\]
is an isomorphism. By the Breen-Serre duality on perfect unipotent groups ([Ser60 8.4, Remarque], [Mil06 III, Theorem 0.14]) and Theorem 2.1.5, the evaluation morphism
\[
G_a \to R\text{Hom}_k(R\text{Hom}_k(G_a, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}),
\]
is an isomorphism. We have
\[
R\text{Hom}_k(G_a^N, \mathbb{Q}/\mathbb{Z}) = R\text{Hom}_k(G_a, \mathbb{Q}/\mathbb{Z})^{\otimes N},
\]
since
\[
\text{Ext}_k^i(G_a^N, \mathbb{Q}/\mathbb{Z}) = \lim_{m} \text{Ext}_k^i(G_a^N, \mathbb{Z}/m\mathbb{Z}) = \lim_{m} \text{Ext}_k^i(G_a, \mathbb{Z}/m\mathbb{Z})^{\otimes N} = \text{Ext}_k^i(G_a, \mathbb{Q}/\mathbb{Z})^{\otimes N}
\]
for any $i$ by Theorem 2.1.5 ([Ser60] §3.4, Proposition 7) and the argument in the paragraph right after Theorem 2.6.1. Hence we have
\[
R\text{Hom}_k(R\text{Hom}_k(G_a^N, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = R\text{Hom}_k(R\text{Hom}_k(G_a, \mathbb{Q}/\mathbb{Z})^{\otimes N}, \mathbb{Q}/\mathbb{Z}) = R \prod_{n \in \mathbb{N}} R\text{Hom}_k(R\text{Hom}_k(G_a, \mathbb{Q}/\mathbb{Z}), \mathbb{Q}/\mathbb{Z}) = R \prod_{n \in \mathbb{N}} G_a,
\]
5The key here is that $\text{Ext}_k^i(G_a, \mathbb{Q}/\mathbb{Z}) = 0$ for $i \neq 1$ and $\text{Ext}_k^1(G_a, \mathbb{Q}/\mathbb{Z}) = k$ via the Artin-Schreier isogeny, so $R\text{Hom}_k(G_a, \mathbb{Q}/\mathbb{Z}) = G_a[-1]$. See [Ser60] §8.
where $R \prod_{n \in \mathbb{N}}$ denotes the derived functor of the direct product functor \((\mathbf{Roo06})\) for $\text{Ab}(k^{\text{indrat}})$. Hence it is enough to show that $R^i \prod_{n \in \mathbb{N}} G_a = 0$ for $i \geq 1$. This follows from \((\mathbf{Roo06}, \text{Proposition 1.6})\) and that $H^i(k^{\text{et}}_\text{indrat}, G_a) = 0$ for any $k' \in k^{\text{indrat}}$.

**Lemma 2.7.5.** The theorem is true for an étale unipotent $A$.

*Proof.* We can reduce this to the previously treated case $A = \mathbb{Z}/p\mathbb{Z}$ by a similar method used in the proof of Lemma 2.7.3. \qed

**Proof of Theorem 2.7.1.** A general $A$ with no infinitesimal unipotent part is an extension of an étale unipotent one by a multiplicative one. This finishes the proof of the theorem. \qed

Theorem 2.7.1 implies

$$R\Gamma(A^{\text{CD}}) = R\text{Hom}_k(R\Gamma(A), \mathbb{Q}/\mathbb{Z}[-1])$$

by taking $R\Gamma(k, \cdot)$ of the both sides.

From the proof, it follows that $H^i(A) = 0$ for $i \geq 2$. The sheaf $\Gamma(A) \in \text{Ab}(k^{\text{indrat}})$ is a finite étale group. The sheaf $H^1(A)$ is an extension of a finite étale group by a connected ind-pro-unipotent group (that is, a filtered direct limit of connected unipotent proalgebraic groups).

If $K$ has mixed characteristic, this ind-pro-unipotent part is a (finite-dimensional) unipotent group. Hence the theorem remains valid if we use the site $\text{Spec } k^{\text{rat}}$ instead of $\text{Spec } k^{\text{indrat}}$ by Theorem (2.1.5). Also we can use the étale topology for $K$ instead of the fppf topology since $A$ over $K$ is étale and the étale cohomology with coefficients in $A$ agrees with the fppf cohomology. This means that we can use $\text{Spec } K^{\text{et}}/k^{\text{rat}}$ instead of $\text{Spec } K^{\text{fppf}}/k^{\text{indrat}}$. The argument of the paragraph before Lemma (2.7.4) shows that $H^1(\mathbb{Z}/n\mathbb{Z}(1)) = K^{\times} \otimes \mathbb{Z}/n\mathbb{Z}$ for any $n \geq 1$. Hence we have $H^1(\mathbb{Q}/\mathbb{Z}(1)) = K^{\times} \otimes \mathbb{Q}/\mathbb{Z}$. In this manner, Theorem 2.7.1 for mixed characteristic $K$ becomes Theorem A in Introduction.

**Remark 2.7.6.**

1. There are some generalizations of Theorem 2.7.1 essentially covered in Bégouri’s and Bester’s papers. Here we merely indicate the formulation in our setting without proof.

   (a) Theorem 2.7.1 holds even if $A$ is a general finite flat group scheme over $K$. It is reduced to the case $A = \alpha_p$. This case can be treated by the same argument as \[AM76\] or \[Bes78\]. Namely, by pushing forward $A$ from the relative fppf site $\text{Spec } K^{\text{fppf}}/k^{\text{indrat}}$ to the relative étale site $\text{Spec } K^{\text{et}}/k^{\text{indrat}}$, and using the dlog map and the Cartier operator, our duality is reduced to the duality for coherent coefficients over $K$.

   (b) We can also formulate and prove a similar duality for the ring $\mathcal{O}_K$ of integers. Namely, we can define a similar site $\text{Spec } \mathcal{O}_K^{\text{fppf}}/\mathcal{O}_k^{\text{indrat}}$, a morphism $\pi_\ast : (\text{Spec } \mathcal{O}_K^{\text{fppf}}/\mathcal{O}_k^{\text{indrat}}) \to \text{Spec } \mathcal{O}_K^{\text{indrat}}$ and fppf cohomology $R\Gamma_x$ with support on the closed point $x = \text{Spec } k$ (\[Bes78 \S 2.2\], \[Mil06 III, \S 0]). Again by classical results on étale cohomology of $K$ and $\mathcal{O}_K$, we have $R\Gamma_x((\mathcal{O}_K^{\text{fppf}}, G_m) = \mathbb{Z}[-1]$, and the morphisms

   $$R\Gamma_x((\mathcal{O}_K^{\text{fppf}}, Z) \to R\text{Hom}_k(R\Gamma((\mathcal{O}_K^{\text{fppf}}, G_m), Z[-1]),$$

   $$R\Gamma_x((\mathcal{O}_K^{\text{fppf}}, A^{\text{CD}}) \to R\text{Hom}_k(R\Gamma((\mathcal{O}_K^{\text{fppf}}, A), \mathbb{Q}/\mathbb{Z}[-2]))$$
with finite flat $A$ are isomorphisms in $D(k)$.

(2) If we avoid Theorem 2.1.5 (hence avoid the entire part of Section 3) and use Proposition 2.2.1 instead, then what we can prove for mixed characteristic $K$ is an apparently weaker statement

$$R\Gamma(A^{CD}) = \tau_{<1}R\text{Hom}_{k^{rat}}(R\Gamma(A), \mathbb{Q}/\mathbb{Z}[-1]),$$

where $\tau$ denotes the truncation functor.

(3) If one feels strange about the shift, one can define the compact support cohomology $R\Gamma_c(K, A) = R\pi_* A$ of $K$ to be $R\Gamma(K, A)[-1]$. Then Theorem 2.7.4 may be written as

$$R\Gamma(K, A^{CD}[2]) = R\text{Hom}_{k^{indrat}}(R\Gamma_c(K, A), \mathbb{Q}/\mathbb{Z}).$$

Note that if $K$ has mixed characteristic, then $A^{CD}[2] = A^{PP}(1)[2]$, where $PD = \text{Hom}_K(\cdot, \mathbb{Q}/\mathbb{Z})$ denotes the Pontryagin dual. The complex $R\Gamma(K, A)$ is concentrated in degrees 0, 1 and $R\Gamma_c(K, A)$ in degrees 1, 2. Hence our duality takes the same form as the Poincaré duality for $l$-adic cohomology of an affine curve over a closed field, just as if $\text{Spec } K$ were a punctured disc over $\text{Spec } k$ via the structure morphism $\pi$.

Similarly, the duality for the cohomology of $\mathcal{O}_K$ above takes the same form as the duality for an open disc by setting $R\Gamma_c(\mathcal{O}_K, \cdot) = R\Gamma_c(\mathcal{O}_K, \cdot)$.

We can show that $R\Gamma(\mathcal{O}_K, j_i \cdot) = 0$, where $j: \text{Spec } \mathcal{O}_K \rightarrow \text{Spec } K$ is the open immersion, and the localization exact sequence induces an isomorphism $R\Gamma_c(K, j_i \cdot) \sim R\Gamma_c(\mathcal{O}_K, j_i \cdot)$. This justifies our definition of $R\Gamma_c(K, \cdot)$.

2.8. Duality for a variety over a local field. In this subsection, we assume that the complete discrete valuation field $K$ has mixed characteristic. As in Introduction, we use the site $\text{Spec } k^{rat}$ instead of $\text{Spec } k^{indrat}$ and $\text{Spec } K^{rat}/k^{rat}$ instead of $\text{Spec } K^{ppf}/k^{indrat}$. We denote the étale structure morphism by

$$\pi_{K/k}: \text{Spec } K^{rat}/k^{rat} \rightarrow \text{Spec } k^{rat}.$$
where \((f, A)(Y') = A(Y', k_Y)\) for étale \(Y\)-schemes \(Y'\). For \((Y, k_Y) \in X_{\text{et}}/k_{\text{rat}}\), if \(k_Y\) is a field, then we denote by \(\pi_0(Y) = \text{Spec} L_Y\) the finite étale \(K(k_Y)\)-scheme of connected components of \(Y\) (\cite{DG} I, \S 4, Definition 6.6). If \(k_Y\) is not a field but a product of fields \(\prod k_{Y,i}\), then we set \(\pi_0(Y) = \text{Spec} L_Y = \bigcup \pi_0(Y_i)\) where \(Y_i\) is the fiber over \(K(k_{Y,i})\). The functor

\[
K_{\text{et}}/k_{\text{rat}} \rightarrow X_{\text{et}}/k_{\text{rat}}, \quad (L, k_L) \mapsto (X \times_K L, k_L)
\]

admits a left adjoint given by \((Y, k_Y) \mapsto (L_Y, k_Y)\). Hence it defines a morphism of sites

\[
\pi_{X/K} : X_{\text{et}}/k_{\text{et}} \rightarrow \text{Spec} K_{\text{et}}/k_{\text{et}}.
\]

The composite \(\pi_{K/k} \circ \pi_{X/K}\) is denoted by \(\pi_{X/k}\):

\[
\pi_{X/k} : X_{\text{et}}/k_{\text{et}} \xrightarrow{\pi_{X/K}} \text{Spec} K_{\text{et}}/k_{\text{et}} \xrightarrow{\pi_{K/k}} \text{Spec} k_{\text{et}}.
\]

We denote

\[
\Gamma(X_{\text{et}}, \cdot) = \pi_{X/k,*}, \quad R\Gamma(X_{\text{et}}, \cdot) = R\pi_{X/k,*}
\]

(where we denote the pushforward \(\pi_{X/k,*}\) by \(\pi_{X/k,*}\) for brevity). For \(d \in \mathbb{Z}\) and a torsion sheaf \(A \in \text{Ab}(X_{\text{et}}/k_{\text{et}})\) or \(A \in \text{Ab}(K_{\text{et}}/k_{\text{et}})\), we denote the \(d\)-th Tate twist of \(A\) by \(A(d)\). We have \(R\pi_{X/K,*}(A(d)) = (R\pi_{X/K,*}(A))(d)\) since \(\mathbb{Z}/n\mathbb{Z}(d)\) for any \(n \geq 1\) is étale locally constant on \(K\). Similarly Tate twists commute with \(R\text{Hom}_X\) or \(R\text{Hom}_K\).

Now assume that \(X\) is proper, smooth and geometrically connected over \(K\). Let \(\mathbf{X} = X \times_K \mathbf{K}\). We are going to regard the Poincaré duality for the variety \(\mathbf{X}\) over \(\mathbf{K}\) as duality for \(\pi_{X/K}\) and combine it with the duality for \(\pi_{K/k}\) established in Theorem 2.7.1 to obtain a duality for \(\pi_{X/k}\). Let \(d = \dim(X)\). A part of the Poincaré duality for \(\mathbf{X}\) gives the isomorphism

\[
\tau_{\geq 2d} R\Gamma(\mathbf{X}_{\text{et}}, \mathbf{Q}/\mathbf{Z}(d)) = \mathbf{Q}/\mathbf{Z}[\cdot]^{-2d},
\]

where \(\tau\) denotes the truncation functor. The same is true when \(X\) over \(K\) is replaced by \(X \times_K K(k')\) over \(K(k')\) for any field \(k' \in k_{\text{rat}}\). Hence

\[
\tau_{\geq 2d} R\pi_{X/K,*} \mathbf{Q}/\mathbf{Z}(d) = \mathbf{Q}/\mathbf{Z}[\cdot]^{-2d},
\]

and we have a morphism

\[
R\Gamma(X_{\text{et}}, \mathbf{Q}/\mathbf{Z}(d + 1)) = R\Gamma(K_{\text{et}}, R\pi_{X/K,*} \mathbf{Q}/\mathbf{Z}(d + 1))
\]

\[
\rightarrow R\Gamma(K_{\text{et}}, \mathbf{Q}/\mathbf{Z}(1))[\cdot]^{-2d}
\]

\[
\rightarrow \mathbf{H}^1(K_{\text{et}}, \mathbf{Q}/\mathbf{Z}(1))[\cdot]^{-2d - 1}
\]

\[
\rightarrow \mathbf{Q}/\mathbf{Z}[\cdot]^{-2d - 1}
\]

using the trace map (Theorem A). Let \(A\) be a constructible sheaf on \(X_{\text{et}}\). We denote by \(A^{\text{PD}} = R\text{Hom}_X(A, \mathbf{Q}/\mathbf{Z})\) the Pointryagin dual of \(A\). We have a morphism

\[
R\Gamma(X_{\text{et}}, A^{\text{PD}}(d + 1))
\]

\[
\rightarrow R\text{Hom}_k(\mathbf{R}\Gamma(X_{\text{et}}, A), \mathbf{R}\Gamma(X_{\text{et}}, \mathbf{Q}/\mathbf{Z}(d + 1)))
\]

\[
\rightarrow R\text{Hom}_k(\mathbf{R}\Gamma(X_{\text{et}}, A), \mathbf{Q}/\mathbf{Z}[\cdot]^{-2d - 1}).
\]

**Theorem 2.8.1.** The above defined morphism

\[
R\Gamma(X_{\text{et}}, A^{\text{PD}}(d + 1)) \rightarrow R\text{Hom}_k(\mathbf{R}\Gamma(X_{\text{et}}, A), \mathbf{Q}/\mathbf{Z}[\cdot]^{-2d - 1})
\]

is an isomorphism.
Proof. By the Poincaré duality for the variety $\mathcal{X}$ over $\overline{K}$, we have

$$R\Gamma(\mathcal{X}_{et}, A^{PD}(d)) = R\Gamma(\mathcal{X}_{et}, A)^{PD}[-2d].$$

The same is true when $X$ over $K$ is replaced by $X \times_K K(k')$ over $K(k')$ for any field $k' \in k_{rat}$. Hence

$$R\pi_{X/K,*}(A^{PD})(d) = (R\pi_{X/K,*}A)^{PD}[-2d].$$

The complexes $R\pi_{X/K,*}A$, $R\pi_{X/K,*}(A^{PD})$ are bounded complexes in $D(K)$ whose cohomology at each degree is a finite étale group scheme over $K$ by the constructibility of $A$ and the proper base change theorem. Therefore we can apply Theorem 2.7.1 to get

$$\begin{align*}
R\Gamma(X_{et}, A^{PD}(d + 1)) &= R\Gamma(K_{et}, (R\pi_{X/K,*}A)^{PD}(1))[-2d] \\
&= R\text{Hom}_k(R\Gamma(K_{et}, R\pi_{X/K,*}A), Q/Z)[-2d - 1] \\
&= R\text{Hom}_k(R\Gamma(X_{et}, A), Q/Z)[-2d - 1],
\end{align*}$$

as desired. \qed

Remark 2.8.2. As in Remark 2.7.1 [3], one might want to define the compact support cohomology $R\Gamma_c(X_{et}, A) = R\pi_{X/K,*}A$ to be $R\Gamma(X_{et}, A)[-1] = R\Gamma_c(K, R\pi_{X/K,*}A)$ or $R\pi_{X/K,*} = R\pi_{K/k!} \circ R\pi_{X/K,*}$. Then the above theorem may be written as

$$R\Gamma(X_{et}, A^{PD}(d + 1)(2d + 2)) = R\text{Hom}_k(R\Gamma_c(X_{et}, A), Q/Z),$$

as if $X$ were a $(d + 1)$-dimensional smooth variety over $k$.

2.9. Duality as adjunction. In this subsection, we make no assumption on the characteristic of $K$ and return to the fppf structure morphism $\pi : \text{Spec } K_{fpf}/k_{\text{indrat}} \to \text{Spec } k_{\text{et}}$. Recall that we have denoted $R\pi_* = R\Gamma$. As in Remarks 2.7.1 [3] and 2.8.2 we set

$$R\pi_! = R\pi_* [-1].$$

We denote by $\otimes_k^L$, $\otimes_K^L$ the derived tensor products for $D(k)$, $D(K)$, respectively. Let $B \in \text{Ab}(k)$ and $C \in \text{Ab}(K)$. By the paragraph after Proposition 2.4.3 and adjunction, we have a natural morphism and a cup product pairing

$$B \otimes_k^L R\pi_* C \to R\pi_* \pi^* B \otimes_k^L R\pi_* C \to R\pi_!(\pi^* B \otimes_K^L C).$$

With a shift, we have a morphism

$$B \otimes_k^L R\pi C \to R\pi!(\pi^* B \otimes_K^L C) \quad (7)$$

in $D(k)$.

Proposition 2.9.1. Assume that $B$ is a commutative finite étale group scheme over $k$. The morphism (7) is an isomorphism.

Proof. We may check the statement étale locally on $k$. Therefore we may assume that $B$ is a constant group. Hence we may replace $B$ by $\mathbb{Z}$. This case is obvious. \qed

We continue assuming $B$ to be a finite étale group scheme over $k$. We define

$$\pi^1 B = \pi^* B \otimes_K^L \mathbb{G}_m[1].$$

An explicit description is given as follows. If $K$ has mixed characteristic, we have $\pi^* B \otimes_K^L \mathbb{G}_m \cong \pi^* B \otimes_K^L \mathbb{Q}/\mathbb{Z}(1)$ since the quotient of $\mathbb{G}_m$ by its torsion subgroup
Q/Z(1) is uniquely divisible. The exact sequence 0 → Z → Q → Q/Z → 0 induces an isomorphism π∗B ⊗_K Q/Z(1) ∼= (π∗B)(1)[1]. Thus π∗B ∼= (π∗B)(1)[2] (again, compare this with Remarks 2.7.6 [3] and 2.8.2). For general K, let Tor^K_i be the n-th left derived functor of ⊗_K. Then we have π∗B ∼= Tor^K_i(π∗B, G_m)[2], or in other words, Tor^K_i(π∗B, G_m) = 0 for n ≠ 1, and Tor^K_i(π∗B, G_m) is a finite multiplicative group scheme over K. To see this, we may assume that B is constant and moreover B = Z/mZ for m ≥ 1 since we may check the statement étale locally on k making an unramified extension of K. The exact sequence 0 → Z → Z → Z/nZ → 0 shows Tor^K_i(Z/mZ, G_m) = 0 for n ≠ 1 and Tor^K_i(Z/mZ, G_m) = μ_m. Hence the statement follows.

The above proposition and the trace map Rπ∗G_m[1] = RΓ(G_m) → Z yield a morphism

Rπ∗π∗B ∼= B ⊗_K Rπ∗G_m[1] → B.

Let A ∈ Ab(K_{pfp}/k_{ind}). We have a morphism

Rπ∗τ≤−1RHom_K(A, π∗B) → Rπ∗RHom_K(A, π^1B) → RHom_k(Rπ∗A, Rπ∗π∗B) → RHom_k(Rπ∗A, B).

The truncation τ≤−1 is needed to ignore extensions of degree ≥ 2 as sheaves over positive characteristic K that do not come from extensions as commutative group schemes (see also [Bre69]). This is unnecessary if K has characteristic zero.

Proposition 2.9.2. Assume that A (resp. B) is a commutative finite flat (resp. étale) group scheme over K (resp. k) without connected unipotent part. Then the morphism

(9) Rπ∗τ≤−1RHom_K(A, π∗B) → RHom_k(Rπ∗A, B)

in D(k) defined above is an isomorphism.

Proof. Take a resolution 0 → B_1 → B_2 → B → 0 of B by étale group schemes B_1, B_2 over k whose groups of geometric points are finite free abelian groups. We have a distinguished triangle

RHom_K(A, π∗B_1 ⊗_K G_m[1]) → RHom_K(A, π∗B_2 ⊗_K G_m[1]) → RHom_K(A, π∗B_1).

We have Ext^K_i(A, G_m) = 0 by Cartier duality ([Mil80, III, Lemma 4.17]). Hence the sheaves Ext^K_i(A, π∗B_1 ⊗_K G_m) = 0 for i = 1, 2 are locally zero and hence zero. Hence we have a distinguished triangle

Hom_K(A, π∗B_1 ⊗_K G_m)[1] → Hom_K(A, π∗B_1 ⊗_K G_m)[1] → τ≤−1RHom_K(A, π∗B_1).

Therefore, what we need to show is that the morphism

Rπ∗Hom_K(A, π∗B_1 ⊗_K G_m)[1] → RHom_k(Rπ∗A, B_i)

is an isomorphism for i = 1, 2. This is reduced to the case B_i = Z for i = 1, 2 by extending the base k to its finite extension. The statement is then Theorem 2.7.1.

Remark 2.9.3.

(1) Propositions 2.9.1 and 2.9.2 are true for any étale group scheme B, using the same definition (8) of π∗B. The second proposition needs the fact that the functors RHom_K(A, ·), RHom_k(Rπ∗A, ·) with A a finite flat group scheme over K commute with filtered direct limits. See Lemma 3.8.2.
In fact, the definition (8) of $π^! B$ makes sense for any sheaf $B ∈ \text{Ab}(k_{\text{indrat}})$. But Proposition 2.9.1 seems false in general. An interesting observation in this direction is the following. Let $A = \mathbf{G}_m$ and $B ∈ \text{Alg}/k$. About the left-hand side of (9), we have a natural morphism
\[ Rπ_∗π^! B[1] → Rπ_∗R\text{Hom}_K(\mathbf{G}_m, π^! B ⊗_k \mathbf{G}_m[1]) = Rπ_∗R\text{Hom}_K(\mathbf{G}_m, π^! B), \]
and about the right, an isomorphism
\[ R\text{Hom}_k(Rπ_! \mathbf{G}_m, B) = R\text{Hom}_k(\mathbf{K}^\times, B)[1]. \]
We do not seem to have a morphism
\[ Rπ_∗π^* B \to R\text{Hom}_k(\mathbf{K}^\times, B) \]
between them. Even taking $H^0$ and evaluating the both sheaves by $k$, we still do not seem to have a morphism
\[ (π^* B)(K) → \text{Hom}_k(\mathbf{K}^\times, B). \]
A remark is that this hypothetical morphism is taking its shape quite similar to the Albanese property of $\mathbf{K}^\times$ (CC94) in the equal characteristic case. We recall this property here.

Assume that $K$ has equal characteristic. Let $k_{\text{sch}}$ be the category of (not necessarily perfect) $k$-algebras. The functor $K$ makes sense also on $k_{\text{sch}}$ by the same definition: it sends a $k$-algebra $R$ to $(R \otimes_k \mathcal{O}_K) \otimes_{\mathcal{O}_K} K$. Since $K$ is a $k$-algebra, we have a morphism $π_0: \text{Spec } K → \text{Spec } k$ of schemes. Let $B$ be a commutative smooth algebraic group over $k$ and $π_0^* B = B \times_k K$ the base extension of $B$ from $k$ to $K$. Then there exists an isomorphism
\[ (π_0^* B)(K) ≅ \text{Hom}_{k_{\text{sch}}}(\mathbf{K}^\times, B). \]
This is the Albanese property of $\mathbf{K}^\times$ (CC94).

We do not know whether or not there exists a way to modify our formulation and Proposition 2.9.2 on duality so that it contains the Albanese property of $\mathbf{K}^\times$ as the special case where $A = \mathbf{G}_m$ and $B ∈ \text{Alg}/k$.

3. Extensions of algebraic groups as sheaves on the rational étale site

In this section, we prove Theorem 2.7.5. This and the next sections do not involve complete discrete valuation fields.

3.1. The pro-fppf site. Some general results on perfect schemes in modern language can be found in [BS17] §3. We first make some remarks about the fppf topology on the category $k_{\text{perf}}^\text{perf}$ of perfect $k$-algebras. As in Notation, we say that a homomorphism $R → S$ in $k_{\text{perf}}^\text{perf}$ is finitely presented if it can be written as the perfection of a $k$-algebra homomorphism $R → S_0$ of finite presentation. For a perfect $k$-algebra $R$, we say that a perfect $R$-algebra $S$ is flat of finite presentation if it is the perfection of a flat $R$-algebra of finite presentation. (It is not clear if this is stronger than saying that $S$ is flat and the perfection of an $R$-algebra of finite presentation).

Note that if $R$ is the perfection of a $k$-algebra $R_0$ and $S$ is a perfect $R$-algebra flat of finite presentation in the above sense, then there exists an $R_0$-algebra $S_0$ flat of finite presentation (in the usual sense) whose perfection is isomorphic to $S$ as an $R$-algebra. To see this, write $S$ as the perfection of a flat $R$-algebra $S'_0$ of
finite presentation. Let \( R_0 \to R_0^{(1)} \to R_0^{(2)} \to \cdots \) be the Frobenius morphisms over \( k \), where \( R_0^{(n)} \) is the ring \( R_0 \) with \( k \)-algebra structure given by \( a \cdot b := a^p b \), \( a \in k, b \in R_0 \). There is an \( R_0^{(n)} \)-algebra \( S_0'' \) of finite presentation for some \( n \) such that \( R \otimes_{R_0^{(n)}} S_0'' \cong S_0' \) as \( R \)-algebras. The flatness of \( S_0' \) over \( R \) implies that we can take \( S_0'' \) to be flat over \( R_0^{(n)} \) (for some larger \( n \)) by the permanence property of flatness under passage to limits ([Gro66, Corollary 11.2.6.1]). Then \( S_0''^{(-n)} \) is an \( R_0 \)-algebra flat of finite presentation and we have \( R \otimes_{R_0} S_0''^{(-n)} \cong S_0''^{(-n)} \) as \( R^{(-n)} \)-algebras. Hence \( S \) is the perfection of the flat \( R_0 \)-algebra \( S_0 = S_0''^{(-n)} \) of finite presentation.

The permanence property of flatness under passage to limits used above has the following variant for perfect flat algebras of finite presentation: if \( \{ R_\lambda \to S_\lambda \} \) is a filtered direct system of finitely presented homomorphisms between perfect \( k \)-algebras such that \( S_\mu = S_\lambda \otimes_{R_\lambda} R_\mu \) for any \( \mu \geq \lambda \) and if its direct limit \( R \to S \) is a perfect \( k \)-algebra homomorphism flat of finite presentation, then so is \( R_\mu \to S_\mu \) for some \( \mu \). To see this, take a flat \( R \)-algebra \( S_0 \) of finite presentation in the usual sense whose perfection is \( S \). For some \( \lambda \), there exists a flat \( R_\lambda \)-algebra \( S_0 \lambda \) of finite presentation in the usual sense such that \( S_0 \cong S_0 \lambda \otimes_{R_\lambda} R \) by [Gro66, Corollary 11.2.6.1]. Set \( S_0 n = S_0 \lambda \otimes_{R_\lambda} R \) for \( \mu \geq \lambda \). Take an \( R_\lambda \)-algebra \( S_1 \lambda \) of finite presentation in the usual sense whose perfection is \( S_\lambda \) and set \( S_1 = S_1 \lambda \otimes_{R_\lambda} R \) and \( S_1 \mu = S_1 \lambda \otimes_{R_\lambda} R_\mu \) for \( \mu \geq \lambda \). Since the perfections of \( S_0 \) and \( S_1 \) are both isomorphic to \( S \) as \( R \)-algebras, there exist \( R \)-algebra homomorphisms \( \varphi : S_0 \to S_1^{(n)} \) and \( \psi : S_1 \to S_0^{(m)} \) for some \( n, m \geq 0 \) such that \( \psi \circ \varphi \) and \( \varphi \circ \psi \) are \( p^{n+m} \)-th power Frobenius homomorphisms. For some \( \mu \geq \lambda \), the homomorphisms \( \varphi \) and \( \psi \) come (via base change \( \otimes_{R_\mu} R \)) from some \( R_\mu \)-algebra homomorphisms \( \varphi_\mu : S_0 \mu \to S_1^{(n)} \mu \) and \( \psi_\mu : S_1 \mu \to S_0^{(m)} \mu \) such that \( \psi_\mu \circ \varphi_\mu \) and \( \varphi_\mu \circ \psi_\mu \) are \( p^{n+m} \)-th power Frobenius homomorphisms by [Gro66, Theorem 8.8.2 (i)]. Hence the perfections of \( S_0 \mu \) and \( S_1 \mu \) are isomorphic to each other as \( R_\mu \)-algebras. Therefore \( S_\mu \) is a perfect flat \( R_\mu \)-algebra of finite presentation.

Note also that if \( R \to S \) and \( S \to T \) are flat homomorphisms of finite presentation between perfect \( k \)-algebras, then so is the composite \( R \to T \). To see this, write \( S \) as the perfection of a flat \( R \)-algebra \( S_0 \) of finite presentation. By what we saw in the second last paragraph, \( T \) is the perfection of an \( S_0 \)-algebra \( T_0 \) flat of finite presentation. Hence \( T \) is the perfection of the flat \( R \)-algebra \( T_0 \) of finite presentation.

Therefore we can define the perfect fpf site \( \text{Spec} k_{\text{perf}} \) to be the category \( k_{\text{perf}} \) of perfect \( k \)-algebras endowed with the topology whose covering families over \( R \in k_{\text{perf}} \) are finite families \( \{ S_i \} \) of perfect flat \( R \)-algebras of finite presentation with \( \prod_i S_i \) faithfully flat over \( R \). The perfection functor from the category of all \( k \)-algebras to \( k_{\text{perf}} \) induces a morphism of sites \( \text{Spec} k_{\text{perf}} \to \text{Spec} k_{\text{ppf}} \) to the usual fpf site of \( k \), whose pushforward functor is exact. Hence it induces an isomorphism on cohomology theory. For a perfect \( k \)-algebra \( R \), the category of objects over \( R \) in \( k_{\text{perf}} \) is nothing but the category of perfect \( R \)-algebras, in contrast to the case of the category of ind-rational \( k \)-algebras \( k_{\text{indrat}} \). Hence we can write the localization of \( \text{Spec} k_{\text{perf}} \) at \( R \) by \( \text{Spec} k_{\text{perf}} / R = \text{Spec} R_{\text{perf}} \) without ambiguity.

To define the perfect pro-\( \text{ppf} \) topology, we need the following class of homomorphisms.
Definition 3.1.1. Let $R$ be a perfect $k$-algebra. We say that a perfect $R$-algebra $S$ is flat of ind-finite presentation if there exists a filtered direct system $\{R_\lambda\}$ of perfect $R$-algebras such that each $R_\lambda$ is flat of finite presentation over $R$ and $S$ is isomorphic to $\lim_\rightarrow R_\lambda$ as an $R$-algebra. In this case, we also say that $\text{Spec } S \to \text{Spec } R$ is flat of profinite presentation. If moreover $S$ is faithfully flat over $R$, we say that $S$ is faithfully flat of ind-finite presentation over $R$ and $\text{Spec } S$ is faithfully flat of profinite presentation over $\text{Spec } R$.

Note that the transition homomorphisms $R_\lambda \to R_\mu$ are not required to be flat. An example of a flat homomorphism of ind-finite presentation is a localization $S/k$. Note that not every flat homomorphism is of ind-finite presentation.

The permanence property of flatness under passage to limits used above with a standard limit argument shows the following.

Proposition 3.1.2. If $R \to S$ and $S \to T$ are flat homomorphisms of ind-finite presentation between perfect $k$-algebras, then so is the composite $R \to T$.

Proof. Write $S = \lim_\rightarrow S_\lambda$ with $S_\lambda$ perfect flat of finite presentation over $R$ and $T = \lim_\rightarrow T_\mu$ with $T_\mu$ perfect flat of finite presentation over $S$. By [Swa98, Lemma 1.5] (plus perfection), it is enough to show that any $R$-algebra homomorphism $T' \to T$ from a perfect $R$-algebra $T'$ of finite presentation factors through a perfect $R$-algebra $T''$ flat of finite presentation.

The $R$-algebra homomorphism $T' \to T$ factors through $T_\mu$ for some $\mu$. For some $\lambda$, there is a perfect $S_\lambda$-algebra $T_{\lambda \mu}$ of finite presentation such that $S \otimes_{S_\lambda} T_{\lambda \mu} \cong T_\mu$ as $S$-algebras. For $\lambda' \geq \lambda$, let $T_{\lambda' \mu} = S_{\lambda'} \otimes_{S_\lambda} T_{\lambda \mu}$. Then $T_{\lambda' \mu}$ is a perfect $S_{\lambda'}$-algebra flat of finite presentation for some $\lambda' \geq \lambda$ by the perfect-algebras variant of the permanence property of flatness under passage to limits shown above. The $R$-algebra homomorphism $T' \to T_{\lambda' \mu}$ factors through $T_{\lambda'' \mu}$ for some $\lambda'' \geq \lambda'$. Therefore the $R$-algebra homomorphism $T' \to T$ factors through the perfect $R$-algebra $T_{\lambda'' \mu}$ flat of finite presentation.

Clearly the base extension of a flat homomorphism $R \to S$ of ind-finite presentation by arbitrary homomorphism $R \to R'$ is flat of ind-finite presentation. Hence we can make the following definition.

Definition 3.1.3. We define the perfect pro-fppf site $\text{Spec } k_{\text{perf}}^{\text{pro-fppf}}$ to be the category $k_{\text{perf}}^{\text{pro-fppf}}$ of perfect $k$-algebras endowed with the topology whose covering families over $R \in k_{\text{perf}}^{\text{pro-fppf}}$ are finite families $\{S_i\}$ of perfect flat $R$-algebras of ind-finite presentation with $\prod_\lambda S_i$, faithfully flat over $R$. For a perfect affine scheme $X$, the localization $\text{Spec } k_{\text{perf}}^{\text{pro-fppf}}/X$ of $\text{Spec } k_{\text{perf}}^{\text{pro-fppf}}$ at $X$ (see Notation in Introduction) is also denoted by $X_{\text{pro-fppf}}$.

Remark 3.1.4.

(1) In the same way, we can define pro-smooth morphisms and the perfect pro-smooth site using [Gro67, Proposition 17.7.8] instead. The rest of this section also works when we use the perfect pro-smooth site instead of pro-fppf.

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6Stacks Project, http://stacks.math.columbia.edu, Tag 0ATE.
If we want to allow proper varieties to the underlying category of the pro-fppf site, we can use the category of perfect quasi-compact quasi-separated schemes over $k$ instead of affine ones, with the help of the absolute noetherian approximation.

3.2. Basic relations with the ind-rational étale site. We have continuous maps of sites

$$
\Spec k_{\text{prof}}^{\text{perf}} \to \Spec k_{\text{fppf}}^{\text{perf}} \to \Spec k_{\text{et}}^{\text{perf}} \to \Spec k_{\text{et}}^{\text{indrat}}
$$
defined by the identity. The first two are morphisms of sites (that is, have exact pullback functors), but the third is not (see Proposition 3.2.3 below for an example).

**Definition 3.2.1.** Let $X = \Spec R$ be a perfect affine $k$-scheme. Write $R = \bigcup \lambda R_\lambda$ as a filtered union of perfect $k$-subalgebras $R_\lambda$ of finite type.

1. A profinite set of points of $X$ is a subsystem $x = \varprojlim \lambda x_\lambda \subset X = \varprojlim \lambda \Spec R_\lambda$ consisting of finite subsets $x_\lambda$ of $\Spec R_\lambda$.

2. Assume that all the transition homomorphisms $R_\lambda \to R_\mu$ are flat. Then we have $\text{Frac } R = \varinjlim \lambda \text{Frac } R_\lambda \in k^{\text{indrat}}$, where $\text{Frac }$ denotes the total quotient ring. We call the profinite set of points $\xi_X = \Spec \text{Frac } R$ the generic point of $X$.

These notions are independent of the presentation $R = \bigcup \lambda R_\lambda$ (with flat transition homomorphisms in the second definition).

Note that a profinite set of points is the Spec of an ind-rational $k$-algebra. The set of all profinite sets of points of a given $X$ is a directed set by inclusion. Any morphism from a perfect $k$-scheme $\Spec k'$ with $k' \in k^{\text{indrat}}$ to $X$ factors through a uniquely determined profinite set of points of $X$. Hence we have the following.

**Proposition 3.2.2.** A perfect affine $k$-scheme $X$ regarded as a sheaf on $\Spec k_{\text{et}}^{\text{indrat}}$ is the filtered directed union of the profinite sets of points of $X$. If $X$ is of finite type over $k$, then $X$ regarded as a sheaf on $\Spec k_{\text{et}}^{\text{indrat}}$ is the disjoint union of points of $X$.

If $f : Y \to X$ is a morphism in $k^{\text{prof}}$ and $y$ a profinite set of points of $Y$, then $f(y)$ is a profinite set of points of $X$. Under the assumption made in (2) of Definition 3.2.1 the natural inclusion $\xi_X \hookrightarrow X$ is flat of profinite presentation and dominant. We can speak of the generic point of a proalgebraic group $A \in \text{PAAlg}/k$ since $A$ satisfies the assumption in (2). Any $k$-algebra homomorphism from an ind-rational $k$-algebra to a perfect $k$-algebra is flat of ind-finite presentation, which is faithfully flat if it is injective.

We should be careful when using the continuous map to $\Spec k_{\text{et}}^{\text{indrat}}$ due to the following.

**Proposition 3.2.3.** Let $h : \Spec k_{\text{prof}}^{\text{perf}} \to \Spec k_{\text{et}}^{\text{indrat}}$ be the continuous map defined by the identity. Let $h^{\text{set}} : \text{Set}(k_{\text{et}}^{\text{indrat}}) \to \text{Set}(k_{\text{prof}}^{\text{perf}})$ be the pullback functor for sheaves of sets. Let $\mathbb{A}^1$ be the affine line over $k$. Then the natural morphism

$$
h^{\text{set}}(\mathbb{A}^1 \times_k \mathbb{A}^1) \to h^{\text{set}}\mathbb{A}^1 \times_k h^{\text{set}}\mathbb{A}^1
$$
is not an isomorphism. In particular, $h^{\text{set}}$ is not exact and $h$ is not a morphism of sites.
Proof. By Proposition 3.2.2, we have
\[ \mathbb{A}^1 \times_k \mathbb{A}^1 = \bigsqcup_{z \in \mathbb{A}^2} z, \quad \mathbb{A}^1 = \bigsqcup_{x \in \mathbb{A}^1} x \]
in \( \text{Set}(k_{\text{indrat}}) \), where the disjoint unions are over points of the underlying set of the schemes. Hence
\[ h^* \text{set}(\mathbb{A}^1 \times_k \mathbb{A}^1) = \bigsqcup_{z \in \mathbb{A}^2} z, \quad h^* \text{set}\mathbb{A}^1 = \bigsqcup_{x \in \mathbb{A}^1} x \times_k y \]
in \( \text{Set}(k_{\text{perf}}^{\text{profppf}}) \). The natural morphism between them is clearly not an isomorphism. \( \square \)

By the same reason, the continuous maps from \( \text{Spec} k_{\text{perf}}^{\text{profppf}} \) or \( \text{Spec} k_{\text{et}}^{\text{perf}} \) to \( \text{Spec} k_{\text{et}}^{\text{indrat}} \) are not morphisms of sites.

3.3. Pro-fppf, fppf and étale cohomology.

Proposition 3.3.1. Let \( f : \text{Spec} k_{\text{profppf}}^{\text{perf}} \rightarrow \text{Spec} k_{\text{ppf}}^{\text{perf}} \) be the morphism defined by the identity. Let \( B \in \text{Ab}(k_{\text{profppf}}^{\text{perf}}) \). Assume that \( B \) as a functor on \( k_{\text{perf}}^{\text{profppf}} \) commutes with filtered direct limits. Then we have
\[ R\Gamma(k_{\text{profppf}}^{\text{perf}}, B) = R\Gamma(k_{\text{ppf}}^{\text{perf}}, f_* B) \]
for any \( R \in k_{\text{perf}}^{\text{perf}} \). In other words, we have \( R^j f_* B = 0 \) for any \( j \geq 1 \).

We need a lemma. Let \( B \) and \( R \) as in the proposition. We denote by \( H^i(R_{\text{profppf}}^{\text{perf}}, B) \) the pro-fppf Čech cohomology and by \( H^i_{\text{profppf}}(B) \) the presheaf \( S \mapsto H^i(S_{\text{profppf}}^{\text{perf}}, B) \). The same notation is applied to the fppf cohomology. Note that Čech cohomology is defined for any presheaf. Hence both the pro-fppf Čech cohomology and the fppf Čech cohomology with coefficients in \( H^j_{\text{ppf}}(f_* B) \) make sense.

Lemma 3.3.2. Let \( B \) and \( R \) as in the proposition. We have
\[ \check{H}^i(R_{\text{profppf}}^{\text{perf}}, H^j_{\text{ppf}}(f_* B)) = \check{H}^i(R_{\text{ppf}}^{\text{perf}}, H^j_{\text{ppf}}(f_* B)) \]
for any \( i, j \geq 0 \).

Proof. Let \( S \in k_{\text{perf}}^{\text{perf}} \) be faithfully flat of ind-finite presentation over \( R \). By definition, we can write \( S = \lim_{\rightarrow \lambda} S_{\lambda} \) by a filtered directed system of faithfully flat perfect \( R \)-algebras \( S_{\lambda} \) of finite presentation. It is enough to show that
\[ \check{H}^i(S/R, H^j_{\text{ppf}}(f_* B)) = \lim_{\rightarrow \lambda} \check{H}^i(S_{\lambda}/R, H^j_{\text{ppf}}(f_* B)). \]

We have \( S_{\lambda}^{\otimes R(i+1)} = \lim_{\rightarrow \lambda} S_{\lambda}^{\otimes R(i+1)} \). Since \( B \) commutes with filtered direct limits, we have
\[ H^j((S_{\lambda}^{\otimes R(i+1)})_{\text{ppf}}, f_* B) = \lim_{\rightarrow \lambda} H^j((S_{\lambda}^{\otimes R(i+1)})_{\text{ppf}}, f_* B) \]
for \( j \geq 0 \) (argue inductively using the Čech-to-derived functor spectral sequence). Therefore the Čech complex of \( S/R \) with coefficients in \( H^j_{\text{ppf}}(f_* B) \) is the direct limit of the Čech complex of \( S_{\lambda}/R \) with coefficients in \( H^j_{\text{ppf}}(f_* B) \). The result follows by taking cohomology. \( \square \)
Proof of Proposition 3.3.1. We need to prove that the $j'$-th cohomology groups of the both sides are isomorphic for all $j' \geq 0$. We prove this by induction. The case $j' = 0$ is obvious. Fix $j' \geq 1$. Assume the equality of the $j$-th cohomology for $j = 0, 1, \ldots, j' - 1$. Consider the Čech-to-derived functor spectral sequences

$$E^{ij}_{2, \text{profppf}} = \hat{H}^i(R\text{profppf}, H^j_{\text{profppf}}(B)) \implies H^{i+j}(R\text{profppf}, B),$$

$$E^{ij}_{2, \text{fppf}} = \hat{H}^i(R\text{fppf}, H^j_{\text{fppf}}(f_*B)) \implies H^{i+j}(R\text{fppf}, f_*B).$$

The induction hypothesis implies that $H^j_{\text{profppf}}(B) = H^j_{\text{fppf}}(f_*B)$ for $j = 0, 1, \ldots, j' - 1$. Therefore we have $E^{ij}_{\text{et}, \text{profppf}} = E^{ij}_{\text{et}, \text{fppf}}$ for $i + j = j'$, $j = 0, 1, \ldots, j' - 1$ by the above lemma. On the other hand, we have $E^{ij}_{2, \text{profppf}} = E^{ij}_{2, \text{fppf}} = 0$ by [Mil80, III, Proposition 2.9]. Hence the case $j = j'$ follows. \hfill \Box

Objects of $\text{Alg}/k$ and, more generally, $\text{LAlg}/k$ are perfections of smooth group schemes and hence commutes with filtered direct limits as functors on $k^\text{perf}$. Since the fppf cohomology with coefficients in a smooth group scheme agrees with the étale cohomology ([Mil80, III, Remark 3.11 (b)]), we have the following.

Corollary 3.3.3. Let $B \in \text{LAlg}/k$. Then we have

$$R\Gamma(R\text{profppf}, B) = R\Gamma(R\text{et}, B)$$

for any $R \in k^\text{perf}$. In other words, we have $Rf_*B = 0$ for any $j \geq 1$, where $f: \text{Spec} R^{\text{profppf}} \to \text{Spec} R^{\text{et}}$ is the morphism defined by the identity.

3.4. Review of Mac Lane’s resolution. We review Mac Lane’s resolution [ML57]. We review only the part necessary for our constructions and proofs. In particular, the base ring is taken to be $\mathbb{Z}$. We denote by $\text{Ab}$, $\text{GrAb}$, $\text{DGAb}$ the categories of abelian groups, graded abelian groups, differential graded abelian groups, respectively. For a set $X$, we denote by $\mathbb{Z}[X]$ the free abelian group generated by $X$.

Let $A$ be an abelian group. Let $C_n = \{0, 1\}^n$ for $n \geq 0$. Define a graded abelian group $Q'(A) = \bigoplus_{n \geq 0} Q'_n(A)$ by

$$Q'_n(A) = \mathbb{Z}[\text{Map}(C_n, A)] = \mathbb{Z}[A^{2^n}].$$

There are certain differential maps $\partial: Q'_n(A) \to Q'_{n-1}(A)$ coming from the combinatorics of the vertices $C_n$ of the $n$-cubes and the group structure of $A$. The explicit definition is not needed for our purpose, but see [ML57 §4] especially for formulas in low degrees. The assignment $A \mapsto Q'(A)$ defines a (non-additive) functor $Q'$: $\text{Ab} \to \text{DGAb}$. There is a certain differential graded subgroup $N_A$ of $Q'(A)$ generated by the so-called “norms”. The cubical construction $Q(A)$ is the normalization defined as the differential graded abelian group $Q(A) = Q'(A)/N_A$. The assignment $A \mapsto Q(A)$ defines a (non-additive) functor $Q$: $\text{Ab} \to \text{DGAb}$. For each $n \geq 0$, the degree $n$ part $Q_n(A)$ is a direct summand of the abelian group $Q'_n(A)$ and a splitting $Q'_n(A) \cong Q_n(A) \oplus (N_A)_n$ can be taken functorially ([Pr96 §5], [JP91 Proposition 2.6]).

Mac Lane’s resolution $M(A)$ of $A$ can be written functorially as a graded abelian group as

$$M(A) = Q(A) \otimes_{\mathbb{Z}} \mathcal{B},$$

where $\mathcal{B}$ is a certain graded abelian group $\mathcal{B}(0, Q(\mathbb{Z}), \eta_Q)$ in Mac Lane’s notation [ML57 §7, Remarque 1]. The group $\mathcal{B}$ does not depend on $A$ and has homogeneous
parts all free abelian groups. In particular, each homogeneous part of \( M(A) \) is a direct summand of a direct sum of groups of the form \( \mathbb{Z}[A^m] \) for various \( m \geq 0 \). There are certain differential maps \( \partial: M_n(A) \to M_{n-1}(A) \) (which is not the tensor product of the differentials for \( Q(A) \) and \( B \)). See [ML57 §7] for formulas in low degrees. The assignment \( A \mapsto M(A) \) defines a (non-additive) functor \( M\colon \text{Ab} \to \text{DGAb} \).

(In the language of two-sided bar constructions, the group \( B \) is the unnormalized bar construction \( B(\mathbb{Z}, Q(\mathbb{Z}), \mathbb{Z}) \) for the augmented differential graded ring \( Q(\mathbb{Z}) \) and the group \( M(A) \) is \( B(Q(A), Q(\mathbb{Z}), \mathbb{Z}) \).) Mac Lane’s theorem is that \( M(A) \) is a resolution of \( A \), namely

\[
H_n(M(A)) = 0 \quad \text{for} \quad n > 0, \\
H_0(M(A)) = A \quad \text{(functorial isomorphism)}.
\]

For later use, we define \( M'(A) = Q'(A) \otimes_\mathbb{Z} B \in \text{GrAb} \). Each homogeneous part of \( M(A) \) is a direct sum of groups of the form \( \mathbb{Z}[A^m] \) for various \( m \geq 0 \). Since \( Q(A) \) is a functorial direct summand of \( Q'(A) \) in \( \text{GrAb} \), we know that \( M(A) \) is a functorial direct summand of \( M'(A) \) in \( \text{GrAb} \).

In the proof of our theorem, we will need splitting homotopy \( V \) for \( Q(A) \) and \( M(A) \) [ML57 §5 and §8, respectively] with respect to additive projections, so we recall it here. Assume that \( A \) is the direct sum of two abelian groups \( A_0 \) and \( A_1 \). Let \( p_0, p_1: A \to A \) be the corresponding projections. These maps induce endomorphisms \( p_0, p_1: Q(A) \to Q(A) \) and \( p_0, p_1: M(A) \to M(A) \) of differential graded abelian groups. Note that \( p_0 + p_1 \neq \text{id} \) on \( Q(A) \) or \( M(A) \) in general since \( Q \) and \( M \) are not additive functors. Define an endomorphism \( V \) on the graded abelian group \( Q(A) \) of degree +1 by

\[
(Vt)(\epsilon, e) = p_\epsilon(t(e))
\]

for \( t: C_n \to A, e \in C_n \) and \( \epsilon = 0, 1 \). Then

\[
-\partial V - V \partial = \text{id} - p_0 - p_1 \quad \text{on} \quad Q(A).
\]

The endomorphism \( V \) induces the endomorphism \( V = V \otimes \text{id} \) on the graded abelian group \( M(A) = Q(A) \otimes_\mathbb{Z} B \) of degree +1. Then

\[
-\partial V - V \partial = \text{id} - p_0 - p_1 \quad \text{on} \quad M(A).
\]

We can do these constructions with arbitrary pairs of homomorphisms generalizing projections in the following way (see also [LM51 Theorem 11.2]). Let \( A \) and \( B \) be abelian groups and \( p_0, p_1: A \to B \) any homomorphisms. Set \( p = p_0 + p_1: A \to B \). The product homomorphism \( (p_0, p_1): A \to B^2 \) and the functoriality of \( Q \) induce a morphism \( Q(A) \to Q(B^2) \) in \( \text{DGAb} \). As above, we have the splitting homotopy \( Q(B^2) \to Q(B) \) with respect to the natural two projections on \( B^2 \). The summation map \( B^2 \to B \) and the functoriality of \( Q \) induce a morphism \( Q(B^2) \to Q(B) \) in \( \text{DGAb} \). Composing these three morphisms in this order, we obtain a homomorphism \( Q(A) \to Q(B) \) of graded abelian groups of degree +1, which we again call the splitting homotopy (with respect to \( p_0 \) and \( p_1 \)) and denote by the same symbol \( V \). A similar construction gives a homomorphism \( V: M(A) \to M(B) \) of graded abelian groups of degree +1. The splitting homotopy conditions above yield equalities

\[
-\partial V - V \partial = p - p_0 - p_1: Q(A) \to Q(B), \\
-\partial V - V \partial = p - p_0 - p_1: M(A) \to M(B).
\]
Remark 3.4.1. In [ML57] §5 and §8, the splitting homotopy $V$ was defined also for non-additive projections. The non-additivity makes the behavior of $V$ much more complicated. In this paper, we use $V$ only for additive projections.

3.5. Pullback of Mac Lane’s resolution from the category of fields. Let $\mathcal{A}$ be either $\text{Ab}$, $\text{GrAb}$ or $\text{DGAb}$ and $F : \text{Ab} \to \mathcal{A}$ a (non-additive) functor, for example $F = \mathbb{Z}[\cdot]$, $Q', Q, M', M$, where $\mathbb{Z}[\cdot] : A \to \mathbb{Z}[A]$ is the free abelian group functor as above. For a sheaf $A$ of abelian groups on a Grothendieck site $S$, we define $F(A)$ to be the sheafification of the presheaf $X \mapsto F(A(X))$. Then $M(A)$ is a resolution of the sheaf $A$ since sheafification is exact. We have

$$Q'_n(A) = \mathbb{Z}[A^{2^n}] \in \text{Ab}(S), \quad M'(A) = Q'(A) \otimes_{\mathbb{Z}} \mathcal{B} \in \text{GrAb}(S),$$

where $\mathcal{B}$ is regarded as a constant sheaf. The quotient morphisms $Q'_n(A) \to Q_n(A)$ and $M'_n(A) \to M_n(A)$ in $\text{Ab}(S)$ admit sections.

Let $k' \in k^{\text{indrat}}$. Let $h : \text{Spec} k^{\text{perf}}_{\text{profppf}} \to \text{Spec} k^{\text{indrat}}_{\text{et}} / k'$ be the continuous map defined by the identity. For $A' \in \text{Ab}(k^{\text{perf}}_{\text{profppf}})$, we consider

$$h^*F(h_*A') \in \mathcal{A}(k^{\text{perf}}_{\text{profppf}}).$$

We first define a morphism $h^*F(h_*A') \to F(A')$ in $\mathcal{A}(k^{\text{perf}}_{\text{profppf}})$ using adjunction. Let $h^{-1}$ be the pullback functor for presheaves with values in $\mathcal{A}$. Let $\text{sh}_{\text{profppf}}^{\text{perf}}$ (resp. $\text{sh}_{\text{et}}^{\text{indrat}}$) be the sheafification functor with respect to the perfect pro-lf topology (resp. the ind-rational étale topology). Let $F^{\text{pre}}(A')$ be the presheaf $X \mapsto F(A'(X))$. Then we have natural isomorphisms and an adjunction morphism

$$h^*F(h_*A') = \text{sh}_{\text{profppf}}^{\text{perf}} h^{-1} \text{sh}_{\text{et}}^{\text{indrat}} F^{\text{pre}}(h_*A')$$

$$= \text{sh}_{\text{profppf}}^{\text{perf}} h^{-1} h_* F^{\text{pre}}(A')$$

$$\to \text{sh}_{\text{profppf}}^{\text{perf}} F^{\text{pre}}(A') = F(A').$$

If $A \in \text{PAlg}/k$, we denote $A' = A \times_k k'$ and write $h_* A' = A'$.

**Proposition 3.5.1.** Let $k' \in k^{\text{indrat}}$. Let $F = Q$ or $M$. For any $A \in \text{PAlg}/k$, the above defined morphism

$$h^*F(A') \to F(A') \quad \text{in} \quad \text{DGAb}(k^{\text{perf}}_{\text{profppf}})$$

with $A' = A \times_k k'$ is a quasi-isomorphism. In particular, we have

$$H_n(h^*M(A')) = 0 \quad \text{for} \quad n > 0 \quad \text{and}$$

$$H_0(h^*M(A')) = A'.$$

This is the hardest proposition in this paper. Its proof occupies the next subsection. Before the proof, we need to write $h^*F(A')$ more explicitly. When $F = \mathbb{Z}[\cdot]$, we have $h^*\mathbb{Z}[A'] = \mathbb{Z}[h^*\text{set} A']$, where $h^*\text{set}$ denotes the pullback for sheaves of sets. By Proposition 3.2.2, we have

$$h^*\text{set} A' = \bigcup_{x \in A'} x, \quad h^*\mathbb{Z}[A'] = \mathbb{Z}[h^*\text{set} A'] = \bigcup_{x \in A'} \mathbb{Z}[x],$$

where $x$ runs through all profinite sets of points of $A'$. The $x$ are Spec’s of objects of $k^{\text{indrat}} / k'$. The morphism $h^*\mathbb{Z}[A'] \to \mathbb{Z}[A']$ is induced from the natural inclusion.
$h^\text{set}A' \subset A'$, or $x \subset A'$. Also we have

$$h^*Q_n(A') = \bigcup_{x \subset A'} \mathbb{Z}[x] \subset \mathbb{Z}[A'^{\text{et}}] = Q_n(A'),$$

$$h^*M'(A') = h^*Q'(A') \otimes_{\mathbb{Z}} \mathcal{B}.$$

Therefore, roughly speaking, Proposition 3.5.1 says that $A \in \text{PAlg}/k$ can completely be resolved and described by its field-valued points and group operation.

**Remark 3.5.2.** The special case $k' = k$ of Proposition 3.5.1 does not imply the general case by the following reason. The tensor product $\langle \cdot \rangle \otimes_k k'$ does not define a functor $k^\text{indrat} \to k^\text{indrat}/k'$, so the localization morphism $\text{Set}(k^\text{indrat}/k') \to \text{Set}(k^\text{indrat})$ of topoi (whose pullback functor is the restriction functor; [AGV 72a IV, §5.2]) does not come from a morphism $\text{Spec} k^\text{indrat}/k' \to \text{Spec} k^\text{indrat}$. Moreover, the diagram

$$\begin{array}{ccc}
\text{Set}(k^\text{perf}_\text{proppf}) & \leftarrow & \text{Set}(k^\text{indrat}/k') \\
\uparrow & & \uparrow \\
\text{Set}(k^\text{perf}_\text{proppf}) & \leftarrow & \text{Set}(k^\text{indrat})
\end{array}$$

of pullback functors is not commutative. For example, a representable sheaf $X = \text{Spec} k'' \in k^\text{indrat}$ in the right-lower corner maps to $\lim_{x} x$ (where $x$ runs through all profinite sets of points of $X \times_k k' = \text{Spec}(k'' \otimes_k k')$) through the right-upper corner and to $X \times_k k'$ through the left-lower corner. Therefore arguments about $\text{Spec} k^\text{perf}_\text{proppf} \to \text{Spec} k^\text{indrat}$ do not naturally restrict to $h \colon \text{Spec} k^\text{perf}_\text{proppf} \to \text{Spec} k^\text{indrat}/k'$.

### 3.6. Proof of the acyclicity of the pullback of Mac Lane’s resolution.

The idea of proof of Proposition 3.5.1 is the following. Let $X \in k^\text{perf}$ and $a \in A'(X)$. We need to modify $a$ so as to put it into $h^\text{set}A'$, which is the union of the profinite sets of points of $A'$. By “points” in the definition of profinite sets of points, we actually mean generic points of closed subschemes. The $X$-valued point $a$ is not generic to a closed subscheme of $A'$ in general. We can write $a = (a - \tilde{a}) + \tilde{a}$ by any other $\tilde{a} \in A'(X)$. Since $\xi_{A'} \times_k \xi_{A'} \to A'$ is pro-fppf, by suitably extending $X$ to its pro-fppf cover $Z$, we can choose $\tilde{a} \in A'(Z)$ so that each of $\tilde{a}$ and $a - \tilde{a}$ is generic to a closed subscheme of $A'$. Since $F$ is a non-additive functor, the decomposition of this type $a = (a - \tilde{a}) + \tilde{a}$ in $A'$ cannot directly be translated into a similar decomposition in $F(A')$. This difference is managed by the splitting homotopy $V$ recalled in Section 3.1. We choose $Z$ and $\tilde{a}$ carefully so that the assignment $a \mapsto \tilde{a}$ is a homomorphism, which is the requirement for $V$ to behave simply. The technical core is Lemma 3.6.3 below.

We fix $A \in \text{PAlg}/k$ and $k' \in k^\text{indrat}$, so that $A' = A \times_k k'$. We will need the following two lemmas to create sufficiently many pro-fppf covers.

**Lemma 3.6.1.** Let $Y \to X$, $Y' \to X$ be morphisms in $k^\text{perf}$ and let $Y' = X' \times_X Y$. Assume that $X' \to X$ is flat. If $Y \to X$ is dominant, so is $Y' \to X'$.

**Proof.** Write $X = \text{Spec} R$, $Y = \text{Spec} S$, $X' = \text{Spec} R'$ and $Y' = \text{Spec} S'$. Note that the homomorphism $R \to S$ between perfect (hence reduced) rings being dominant is equivalent that it is injective. The definition of flatness then says that $R' \to S'$ is injective. □
Lemma 3.6.2. Let $Z_i \to Y \to X$ be morphisms in $k^{\text{perf}}$, $i = 1, \ldots, n$, and let $Z = Z_1 \times_X \cdots \times_X Z_n$. Assume that $Y/X$ is faithfully flat of profinite presentation. Assume also the following conditions for each $i$:

- $Z_i \to Y$ is flat of profinite presentation,
- the morphism $(Z_i)_x \to Y_x$ on the fiber over any point $x \in X$ is dominant.

Then $Z$ also satisfies these two conditions. In particular, $Z/X$ is faithfully flat of profinite presentation.

Proof. This follows from the previous lemma.

Let $L$ be a finitely generated abelian group regarded as a constant sheaf over $k$. To simplify the notation in the next lemma, we denote the sheaf-Hom by $\text{Hom}_{\text{profppf}}(L, A) \in \text{Ab}(k^{\text{perf}}_{\text{profppf}})$ by $[L, A]$. We have $[L, A](X) = \text{Hom}(L, A(X))$ for any $X \in k^{\text{perf}}$. We have $[Z, A] = A$. Take an exact sequence $0 \to Z^n \to Z^m \to L \to 0$. The left exactness of sheaf-Hom yields an exact sequence $0 \to [L, A] \to A^n \to A^m$. Hence $[L, A] \in \text{PAlg}/k$. If $a \in L$, we define by $\tilde{a}$ the homomorphism $[L, A] \to A$ given by evaluation at $a$.

Lemma 3.6.3. Let $X \in k^{\text{perf}}$, $L$ a finitely generated subgroup of $A'(X)$ and $Y = X \times_k [L, A] = X \times_k [L, A]'$. For any element $a \in L$, there exists $Z \in k^{\text{perf}}$ and a $k'$-morphism $Z \to Y$ satisfying the two conditions in Lemma 3.6.2 such that the natural images $\tilde{a}, a - \tilde{a} \in A'(Z)$ are contained in the subset $(h^{\text{set}} A')(Z)$. If $a \in (h^{\text{set}} A')(X)$, then we can take $Z$ so that $(a - \tilde{a}, \tilde{a}) \in (h^{\text{set}} A'(Z))$. 

Proof. Consider the following commutative diagram with a cartesian square:

\[
\begin{array}{ccc}
L, A' & \xrightarrow{\tilde{a}} & \text{Im}(\tilde{a})' \\
\downarrow \text{incl} & & \downarrow \text{incl} \\
[L, A]' & \xrightarrow{\tilde{a}} & \text{Im}(\tilde{a})' & \xrightarrow{\text{incl}} & A',
\end{array}
\]

where $\text{Im}(\tilde{a})' = \text{Im}(\tilde{a}) \times_k k'$ and $\xi_{\text{Im}(\tilde{a})'}$ its generic point (Definition 3.2.1). Note that $\text{Im}(\tilde{a}) \in \text{PAlg}/k$ and hence $\text{Im}(\tilde{a})'$ satisfies the assumption of Definition 3.2.1 (2). The bottom arrow in the square is faithfully flat of profinite presentation since it is a surjection of proalgebraic groups base-changed to $k'$. The right arrow is dominant flat of profinite presentation. Hence the left arrow is dominant flat of profinite presentation by Lemma 3.6.1. We define $Z_1 = X \times_k \tilde{a}^{-1}(\xi_{\text{Im}(\tilde{a})})$. Then the natural morphism $Z_1 \to Y$ satisfies the two conditions in Lemma 3.6.2 by Lemma 3.6.3. The natural image $\tilde{a} \in A'(Z_1)$ is a morphism $Z_1 \to A'$ that factors through $\tilde{a}^{-1}(\xi_{\text{Im}(\tilde{a})'}) \subset h^{\text{set}} A'$. Hence $\tilde{a} \in (h^{\text{set}} A')(Z_1)$.

The natural inclusion $L \subset A'(X)$ defines a morphism $\iota : X \to [L, A]'$. We have an automorphism of the $X$-scheme $Y = X \times_k [L, A]'$ given by $(x, \varphi) \mapsto (x, \iota(x) - \varphi)$. The composite of this with the morphism $\tilde{a} : Y \to A'$ is $a - \tilde{a}$. We define $Z_2 \to Y$ to be the inverse image of the morphism $Z_1 \to Y$ by this $X$-automorphism of $Y$. Then we have $a - \tilde{a} \in (h^{\text{set}} A')(Z_2)$ and $Z_2$ satisfies the two conditions in Lemma 3.6.2. We define $Z = Z_1 \times_X Z_2$. Then we have $\tilde{a}, a - \tilde{a} \in (h^{\text{set}} A')(Z)$ and $Z$ satisfies the two conditions in Lemma 3.6.2.

Next assume that $a \in (h^{\text{set}} A')(X)$. Consider the automorphism $(b, c) \mapsto (b + c, c)$ of the group $A^2$, which maps $(a - \tilde{a}, \tilde{a})$ to $(a, \tilde{a})$. Hence it is enough to show that we can take $Z$ so that $(a, \tilde{a}) \in (h^{\text{set}} A'(Z))$. We identify $a : X \to h^{\text{set}} A'$ with its image, which is an object of $k^{\text{indrat}}/k'$, so that we have a faithfully flat
morphism $a: X \to a$ of profinite presentation. Consider the following commutative square:

\[
\begin{array}{ccc}
(a, \tilde{a})^{-1}(\xi_{a \times_k \text{Im}(\tilde{a})'}) & \xrightarrow{(a, \tilde{a})} & \xi_{a \times_k \text{Im}(\tilde{a})'} \\
\downarrow \text{incl} & & \downarrow \text{incl} \\
X \times_{k'} [L, A'] & \xrightarrow{(a, \tilde{a})} & a \times_{k'} \text{Im}(\tilde{a})' \xrightarrow{\text{incl}} A'^2
\end{array}
\]

Again, the generic point of $a \times_{k'} \text{Im}(\tilde{a})' = a \times_k \text{Im}(\tilde{a})$ is well-defined. We define $Z = (a, \tilde{a})^{-1}(\xi_{a \times_k \text{Im}(\tilde{a})'})$. Then $(a, \tilde{a}) \in (h^\text{set}(A'^2))(Z)$ by the same argument as above. The square in the above diagram can be split into the following two cartesian squares:

\[
\begin{array}{ccc}
Z & \xrightarrow{(\text{id}, (a, \tilde{a}))} & X \times a \xi_{a \times_k \text{Im}(\tilde{a})'} \\
\downarrow \text{incl} & & \downarrow \text{proj}_1 \times \text{proj}_3 \text{incl} \\
Y & \xrightarrow{(\text{id}, \tilde{a})} & X \times_{k'} \text{Im}(\tilde{a})' \xrightarrow{(a, \text{id})} a \times_{k'} \text{Im}(\tilde{a})'
\end{array}
\]

The bottom two arrows are faithfully flat of profinite presentation. The third vertical arrow is dominant flat of profinite presentation. By pulling back the left square by a point of $X$ and using Lemma 3.6.1, we see that the morphism $Z \to Y$ satisfies the two conditions of Lemma 3.6.2.

Let $X, L \subset A'(X), Y = X \times_{k'} [L, A']$ be as in the lemma. Define two homomorphisms

\[
p_0, p_1: L \to A'(X) \oplus A'([L, A']), \quad p_0(a) = a - \tilde{a}, \quad p_1(a) = \tilde{a}.
\]

The composites of them with the natural map $A'(X) \oplus A'([L, A']) \to A'(X \times_{k'} [L, A']) = A'(Y)$ are denoted by the same letters $p_0, p_1$. Their sum is the inclusion $L \subset A'(X) \subset A'(Y)$. Let $F: \text{Ab} \to \text{A}$ be one of the functors $Z[\_], Q', Q, M'$ or $M$. By functoriality, we have homomorphisms

\[
(p_0, p_1): F(L) \to F(A'(Y)^2) = F^{\text{pre}}(A'(Y)) \to F(A'(Y)),
\]

\[
(p_0, p_1): F(L) \to F(A'(Y)^2) = F^{\text{pre}}(A'^2)(Y) \to F(A'^2)(Y).
\]

Define

\[
T: F(L) \to F(A'(Y)), \quad Tt = p_0(t) + p_1(t),
\]

\[
V: F(L) \to F(A'^2)(Y), \quad Vt = (p_0(t), p_1(t)).
\]

Note that $T \neq \text{incl}$ since $F$ is non-additive. For $F = Q$ or $M$, let $F' = Q'$ or $M'$, respectively. For each $n \geq 0$, the identification $\mathbb{Z}[(A'(Y)^2)^{2^n}] \cong \mathbb{Z}[(A'(Y)^2)^{2^{n+1}}]$ defines identifications $F'_n(A'(Y)^2) \cong F_{n+1}(A'(Y))$ and $F'_n(A'^2) \cong F_{n+1}(A')$, where $F'_n$ is the degree $n$ part of $F'$. We have a commutative diagram

\[
\begin{array}{ccc}
F'_n(L) & \xrightarrow{V} & F'_n(A'(Y)^2) \cong F_{n+1}(A'(Y)) \\
\downarrow & & \downarrow \\
F_n(L) & \xrightarrow{V} & F_{n+1}(A')(Y),
\end{array}
\]

where: the upper $V$ is defined above; the lower $V$ is the splitting homotopy $F_n(L) \to F_{n+1}(A'(Y))$ recalled in Section 3.3; followed by the natural homomorphism $F_{n+1}(A'(Y)) \to F_{n+1}(A')(Y)$; and the vertical morphisms are the natural
quotient maps. In this sense, the above $V$ is compatible with the splitting homotopy. By the splitting homotopy condition in Section 3.4, we have

$$\text{Proof.} \quad \text{We only need to show this for} \quad t \in F(L) \quad \text{with the composite} \quad Z \to Y \to X \text{faithfully flat of profinite presentation such that the natural image } Tt \in F(A')(Z) \text{ is contained in the subgroup } (h^*F(A'))(Z). \quad \text{If } t \in (h^*F(A'))(X), \text{ then we can take } Z \text{ so that } Vt \in (h^*F(A^Z))(Z).$$

**Lemma 3.6.4.** In the setting of the previous lemma and for $F = \mathbb{Z}[\ ]$, $Q'$, $Q$, $M'$ or $M$, let $t \in F(L)$. Then there exists $Z \in k'^{\text{perf}}$ and a $k'$-morphism $Z \to Y$ with the composite $Z \to Y \to X$ faithfully flat of profinite presentation such that the natural image $Tt \in F(A')(Z)$ is contained in the subgroup $(h^*F(A'))(Z)$. If $t \in (h^*F(A'))(X)$, then we can take $Z$ so that $Vt \in (h^*F(A^Z))(Z)$.

**Proof.** We only need to show this for $F = \mathbb{Z}[\ ]$ in view of the structures of homogeneous parts of $Q', Q, M', M$ recalled in Section 3.3. Write $t = \sum_{i=1}^{n} m_i(a_i)$, $m_i \in \mathbb{Z}$, $a_i \in L$, where $(a_i)$ is the image of $a_i$ in $\mathbb{Z}[L]$. For any $i$, take $Z_i$ corresponding to $a_i$ in $\mathbb{Z}[L]$. Then $T(a_i) \in Z[h^*A'](Z_i) = (h^*Z[A'])(Z_i)$. Let $Z = Z_1 \times_Y \cdots \times_Y Z_n$. Then $Z/X$ is faithfully flat of profinite presentation by Lemma 3.6.2, and we have $Tt = \sum m_iT(a_i) \in (h^*Z[A'])(Z)$. The statement for $V$ is similar. \qed

**Proof of Proposition 3.5.1.** Let $F = Q$ or $M$. We want to show that

$$H_n(h^*F(A')) \simeq H_n(F(A')) \text{ in } \text{Ab}(k'^{\text{perf}}_{\text{proff}})$$

for all $n \geq 0$. We want to show that the inverse is given by $T$ (with pro-fppf locally defined chain homotopy to the identity given by $V$). To give a short and rigorous proof, we avoid doing this directly but as follows.

We first treat the injectivity. Let $F' = Q'$ or $M'$ if $F = Q$ or $M$, respectively. Let $t \in (h^*F_n(A'))(X)$ be an element such that $t = \partial s$ for some $s \in F_{n+1}(A')(X))$. We are ignoring the difference between $F'_n(A'(X)) = F_{n}^{\text{preff}}(A')(X)$ and $F'_n(A')(X)$ since an element of the latter comes from an element of the former after taking a pro-fppf cover of $X$ and it is enough to argue pro-fppf locally on $X$.) Since $F'_n(A') \to F_n(A')$ is a split surjection, we can take a lift $t' \in (h^*F'_n(A'))(X)$ of $t$. Take a finitely generated subgroup $L$ of $A'(X)$ large enough so that $t' \in F'_n(L)$ and $s \in F_{n+1}(L)$. Then by the above lemma, there exists $Z \in k'^{\text{perf}}$ faithfully flat of profinite presentation over $X$ such that $Vt' \in (h^*F'_n(A^Z))(Z)$ and $Ts \in (h^*F_{n+1}(A'))(Z)$. Hence $Vt \in (h^*F_{n+1}(A'))(Z)$. A simple computation using the splitting homotopy condition (10) shows that $t = \partial(Ts - Vt) \in (h^*F_n(A'))(Z)$. This shows the injectivity.

Next we show the surjectivity. Let $t \in F_n(A'(X))$ be an element with $\partial t = 0$. Take a finitely generated subgroup $L$ of $A'(X)$ large enough so that $t \in F_n(L)$. Then by the above lemma, there exists $Z \in k'^{\text{perf}}$ faithfully flat of profinite presentation over $X$ such that $Tt \in (h^*F_n(A'))(Z)$. Then

$$t = Tt - \partial Vt - V\partial t = Tt - \partial Vt.$$

This shows the surjectivity. \qed

**Remark 3.6.5.**

(1) Proposition 3.5.1 is not true if we use the fppf topology instead of the pro-fppf topology. Namely, let $h_0 : \text{Spec } k'^{\text{fppf}} \to \text{Spec } k'^{\text{inff}}$ be the continuous map defined by the identity. Then we can show that $H_1(h_0^*Q(G_a)) \neq 0$, even though $H_1(Q(G_a))$, which is the first stable homology (see below), vanishes. This difference comes from the fact that the inclusion $\xi_{\text{im} (\tilde{h})} : \infty 

\rightarrow \to \kappa$.
sends acyclic sheaves to acyclic sheaves. Then the pullback functor \( k' \) sends acyclic sheaves to acyclic sheaves (see the first paragraph of Section 2.1).

We first show that \( k' \) admits a left derived functor. We know that \( k' \) sends acyclic sheaves to acyclic sheaves (see the first paragraph of Section 2.1).

**Proposition 3.7.1.** Let \( A \in \text{PAlg}/k, B \in \text{LAlg}/k \) and \( k' \in \text{h}^{\text{indrat}} \). Then we have
\[
R\text{Hom}_{\text{k}^{\text{indrat}}/k}(A, B) = R\text{Hom}_{\text{k}^{\text{rat}}}(A, B).
\]
If \( k' \) is a field, this is further isomorphic to \( R\text{Hom}_{\text{k}^{\text{rat}}}(A, B) \). If \( A \in \text{Alg}/k \) and \( k' = k \), then the same equality is true with \( \text{k}^{\text{indrat}} \) replaced by \( \text{k}^{\text{rat}} \).

We prove this below. Throughout this subsection, let \( k' \in \text{k}^{\text{indrat}} \) and let \( h: \text{Spec}\, k^{\text{proppf}} \rightarrow \text{Spec}\, k^{\text{indrat}}/k' \) be the continuous map defined by the identity. We first show that \( h^* \) admits a left derived functor. We know that \( h^* \) sends acyclic sheaves to acyclic sheaves (see the first paragraph of Section 2.1).

**Lemma 3.7.2.** Let \( h: S' \rightarrow S \) be a continuous map of sites. Assume that \( h^* \) sends acyclic sheaves to acyclic sheaves. Then the pullback functor \( h^*: \text{Ab}(S) \rightarrow \text{Ab}(S') \) admits a left derived functor \( Lh^*: D(S) \rightarrow D(S') \), which is left adjoint to \( Rh^*: D(S') \rightarrow D(S) \). We have \( L_n h^* \mathbb{Z}[X] = 0 \) for any object \( X \) of \( S \) and \( n \geq 1 \).

**Proof.** By [KS06] Theorem 14.4.5, it is enough to show the existence of an \( h^* \)-projective full subcategory of \( \text{Ab}(S) \) in the sense of [KS06] Definition 13.3.4 that contains sheaves of the form \( \mathbb{Z}[X] \). Consider the following condition for a sheaf \( P \in \text{Ab}(S) \):
\[
\text{Ext}^n_{\mathbb{Z}}(P, I) = 0
\]
for any acyclic sheaf \( I \in \text{Ab}(S) \) and \( n \geq 1 \). The sheaves satisfying this condition form a full subcategory of \( \text{Ab}(S) \), which contains sheaves of the form \( \mathbb{Z}[X] \). We want to show that this full subcategory is \( h^* \)-projective. We check the conditions of the dual of [KS06] Corollary 13.3.8 (which is a standard criterion for the existence of derived functors).

First, every object of \( \text{Ab}(S) \) is a quotient of a direct sum of objects of the form \( P = \mathbb{Z}[X] \). Next, if \( 0 \rightarrow P \rightarrow Q \rightarrow R \rightarrow 0 \) is an exact sequence in \( \text{Ab}(S) \) with \( Q \) and \( R \) satisfying (11), then \( P \) also satisfies the same condition. Moreover, the homomorphism \( \text{Hom}_{S'}(h^*Q, I') \rightarrow \text{Hom}_{S'}(h^*P, I') \) for any acyclic sheaf \( I' \in \text{Ab}(S') \) is identified with \( \text{Hom}_{S}(Q, h_! I') \rightarrow \text{Hom}_{S}(P, h_! I') \), which is surjective by (11). Taking \( I' \) to be an injective sheaf containing \( h^*P \), we know that \( 0 \rightarrow h^*P \rightarrow h^*Q \rightarrow h^*R \rightarrow 0 \) is exact. Hence we have verified the required conditions. \[\square\]

The purpose of the previous two subsections was to prove the following proposition. This proposition is highly non-trivial since \( h^{\text{sect}}: \text{Set}(k^{\text{indrat}}/k') \rightarrow \text{Set}(k^{\text{proppf}}/k') \) is not exact (Proposition 3.2.3) and we do not know whether \( h^*: \text{Ab}(k^{\text{indrat}}/k') \rightarrow \text{Ab}(k^{\text{proppf}}/k') \) is exact or not. It is crucial to use Mac Lane’s resolution here.

**Proposition 3.7.3.** We have \( Lh^* A' = A \) for any \( A \in \text{PAlg}/k \), where \( A' = A \times_k k' \).
Proof. Let \( M(A') \) be Mac Lane’s resolution of \( A' \) in \( \text{Ab}(k^{\text{indrat}}/k) \). By Proposition 3.5.1, we have \( h^* M(A') = A' \). For each \( n \geq 0 \), the sheaf \( h^* M_n(A') \) is a direct summand of a filtered union of a direct sum of sheaves of the form \( H[Z[\text{Spec } k''/k] \) for \( k'' \in k^{\text{indrat}}/k' \). Hence \( Lh^* M(A') = h^* M(A') = A' \).

**Proposition 3.7.4.** Let \( f : \text{Spec } k^{\text{perf}}_{\text{proppf}} \to \text{Spec } k^{\text{perf}}_{\text{et}} \) and \( g : \text{Spec } k^{\text{perf}}_{\text{et}} \to \text{Spec } k^{\text{indrat}}_{\text{et}}/k' \) be the continuous maps defined by the identities. Let \( A \in \text{PAlg}/k \) and \( B \in \text{Ab}(k^{\text{perf}}_{\text{proppf}}) \). Then we have

\[
\text{RHom}_{k^{\text{perf}}_{\text{proppf}}} (A, B) = \text{RHom}_{k^{\text{perf}}_{\text{et}}} (A, Rf_* B) = \text{RHom}_{k^{\text{indrat}}_{\text{et}}/k'} (A, g_* Rf_* B).
\]

(Here we are using the same symbol \( A' = A \times_k k' \).)

Proof. The first equality is clear since \( f \) is a morphism of sites. For the second, note that \( g_* Rf_* = Rh_* \) since \( g_* \) is exact. The previous lemma shows that

\[
\text{RHom}_{k^{\text{perf}}_{\text{proppf}}} (A', B) = \text{RHom}_{k^{\text{perf}}_{\text{proppf}}} (Lh^* A', B) = \text{RHom}_{k^{\text{indrat}}_{\text{et}}/k'} (A', Rh_* B).
\]

**Proof of Proposition 3.7.4.** Let \( f : \text{Spec } k^{\text{perf}}_{\text{proppf}} \to \text{Spec } k^{\text{perf}}_{\text{et}} \) be the morphism defined by the identity. Let \( A \in \text{PAlg}/k \) and \( B \in \text{LAlg}/k \). We have \( Rf_* B = B \) by Corollary 3.3.3. The proposition above then gives the first statement of the proposition.

For the second statement, if \( k' \) is a field, then the first statement with \( k \) replaced by \( k' \) implies that

\[
\text{RHom}_{k^{\text{indrat}}_{\text{et}}} (A, B) = \text{RHom}_{k^{\text{perf}}_{\text{et}}} (A, B).
\]

This gives the second statement.

For the third, assume that \( A \in \text{Alg}/k \). It is enough to show that

\[
\text{RHom}_{k^{\text{indrat}}_{\text{et}}} (A, B) = \text{RHom}_{k^{\text{et}}} (A, B).
\]

Let \( \alpha : \text{Spec } k^{\text{indrat}}_{\text{et}} \to \text{Spec } k^{\text{et}}_{\text{et}} \) be the morphism of sites defined by the identity. Clearly \( \alpha_* \) is exact and \( \alpha_* \alpha^* = \text{id} \). Note that \( A \) is the disjoint union of its points \( x \in A \) as sheaves of sets and these points are in \( k^{\text{et}} \). Hence \( \alpha^* A = A \) as sheaves of abelian groups. This implies the above equality.

3.8. **Algebraic groups as sheaves on the perfect étale site.** The following proposition finishes the proof of Theorem 2.1.3 and hence Theorem B.

**Proposition 3.8.1.** Let \( A \in \text{PAlg}/k \), \( B \in \text{LAlg}/k \) and \( k' = \bigcup \nu, k'_\nu \in k^{\text{indrat}} \) with \( k'_\nu \in k^{\text{rat}} \). Then for any \( n \geq 0 \), we have

\[
\text{Ext}^n_{k^{\text{perf}}_{\text{et}}} (A, B) = \lim_{\nu} \text{Ext}^n_{k^{\text{et}}_\nu} (A, B).
\]

If \( B \in \text{Alg}/k \), then

\[
\begin{align*}
\text{Ext}^n_{k^{\text{perf}}_{\text{et}}} (A, B) &= \text{Ext}^n_{\text{PAlg}/k} (A, B), \\
\text{Ext}^n_{k^{\text{perf}}_{\text{et}}} (A, Q) &= \text{Hom}_{k^{\text{perf}}_{\text{et}}} (A, Q) = 0, \\
\text{Ext}^{n+1}_{k^{\text{perf}}_{\text{et}}} (A, Z) &= \text{Ext}^n_{k^{\text{et}}_\nu} (A, Q/Z) = \lim_{m} \text{Ext}^n_{\text{PAlg}/k} (A, Z/mZ),
\end{align*}
\]

This proposition is a consequence of results of Breen [Bre70], [Bre81]. We make this clear below. We first need a lemma on limit arguments.
Lemma 3.8.2. Let $A = \lim_{\lambda} A_\lambda \in \text{PAlg}/k$ with $A_\lambda \in \text{Alg}/k$. Let $\{R_\nu\}$ be a filtered direct system in $k^{\text{perf}}$ with limit $R$.

(1) If $B \in \text{LAlg}/k$, then we have

$$\text{Ext}^i_{(R_\nu^{\text{perf}})}(A, B) = \lim_{\lambda, \nu} \text{Ext}^i_{(R_\nu^{\text{perf}})}(A_\lambda, B)$$

for all $i \geq 0$.

(2) The functor $\text{Ext}^i_{(R_\nu^{\text{perf}})}(A, \cdot)$ on $\text{Ab}(R_\nu^{\text{perf}})$ commutes with filtered direct limits.

We prove this lemma by reducing it to the corresponding facts for cohomology:

$$H^i(A \times_k R, B) = \lim_{\lambda, \nu} H^i(A_\lambda \times_k R_\nu, B)$$

and the functor $H^i(A \times_k R, \cdot)$ on $\text{Ab}(R_\nu^{\text{perf}})$ commutes with filtered direct limits. The first fact is given in [Mil80] III, Lemma 1.16, Remark 1.17 (a)] and second in [Mil80] III, Remark 3.6 (d)]. It is important here that $A \times_k R$ is quasi-compact and $B$ is locally of finite presentation. We do this reduction by describing Ext groups by cohomology groups using the cubical construction and its homology. (Mac Lane’s resolution does not behave well with respect to limits since the graded abelian group $B$ in Section 3.4 is not finite rank in each degree.)

Proof. It is enough to show the two statements in the case the system $\{R_\nu\}$ is constant (that is, $R_\nu = R$ for all $\nu$). Indeed, let $f: \text{Spec} R_{\nu}^{\text{perf}} \to \text{Spec} k^{\text{perf}}$ and $f_\nu: \text{Spec}(R_\nu)^{\text{perf}} \to \text{Spec} k^{\text{perf}}$ be the natural morphisms. Then for $B \in \text{LAlg}/k$ and $j \geq 0$, the sheaf $R^j f_* B \in \text{Ab}(k^{\text{perf}})$ is the sheafification of the presheaf $R' \mapsto H^j(R' \otimes_k R, B)$ and the sheaf $\lim_{\nu} R^j f_* B$ is the sheafification of the presheaf $R' \mapsto \lim_{\nu} H^j(R' \otimes_k R_\nu, B)$. These sheaves are isomorphic by [Mil80] III, Lemma 1.16, Remark 1.17 (a)]. Therefore we have

$$\lim_{\nu} \text{Ext}^i_{(R_\nu^{\text{perf}})}(A, R^j f_* B) = \text{Ext}^i_{(R_\nu^{\text{perf}})}(A, R^j f_* B)$$

for any $i, j \geq 0$ if we use (2). Since the pushforward functors $f_{\nu,*}$ and $f_*$ send injectives to injectives, we have Grothendieck spectral sequences

$$E_2^{ij} = \lim_{\nu} \text{Ext}^i_{(R_\nu^{\text{perf}})}(A, R^j f_{\nu,*} B) \Longrightarrow \lim_{\nu} \text{Ext}^{i+j}_{(R_\nu^{\text{perf}})}(A, B),$$

$$E_2^{ij} = \text{Ext}^i_{(R_\nu^{\text{perf}})}(A, R^j f_* B) \Longrightarrow \lim_{\nu} \text{Ext}^{i+j}_{(R_\nu^{\text{perf}})}(A, B).$$

The isomorphisms between the $E_2$-terms induce isomorphisms between the $E_\infty$-terms since the spectral sequences come from the commutative diagram

$$\lim_{\nu} \text{Hom}_{(R_\nu^{\text{perf}})}(A, f_{\nu,*} J) \rightarrow \lim_{\nu} \text{Hom}_{(R_\nu^{\text{perf}})}(A, J)$$

$$\text{Hom}_{(R_\nu^{\text{perf}})}(A, f_* J) \rightarrow \text{Hom}_{(R_\nu^{\text{perf}})}(A, J)$$

on the level of complexes, where $J$ is an injective resolution of $B$ in $\text{Ab}(L^{\text{perf}})$. (Here $J$ is pulled back to $R_\nu$ and $R$, which remains injective since the pullback functors $f_{\nu,*}$ and $f_*$ send injectives to injectives by [Mil80] III, Lemma 1.11].) Thus

$$\lim_{\nu} \text{Ext}^i_{(R_\nu^{\text{perf}})}(A, B) = \text{Ext}^i_{(R^{\text{perf}})}(A, B).$$
Therefore if we have shown
\[ \lim_{\lambda} \Ext^{i}_{(R_{\lambda})_{\text{et}}} (A_{\lambda}, B) = \Ext^{i}_{(R_{\lambda})_{\text{et}}} (A, B) \]
for each fixed \( \nu \), the general case of (1) follows.

(1) when \( R_{\nu} = R \) for all \( \nu \): We need several reduction steps. First, it is enough to show the statement with the étale topology replaced by the pro-fppf topology:
\[ \lim_{\lambda} \Ext^{i}_{R_{\text{et}}^\text{perf}} (A_{\lambda}, B) = \Ext^{i}_{R_{\text{et}}^\text{perf}} (A, B) \]
for \( B \in \text{LAlg}/k \) and \( i \geq 0 \). Indeed, let \( f : \Spec R_{\text{et}}^\text{perf} \to \Spec R_{\text{et}}^\text{perf} \) be the morphism defined by the identity. Then we have \( Rf_{\text{et}} B = B \) by Corollary 3.3.3. Hence
\[ R \hom_{R_{\text{et}}^\text{perf}} (A, B) = R \hom_{R_{\text{et}}^\text{perf}} (A, Rf_{\text{et}} B) = R \hom_{R_{\text{et}}^\text{perf}} (A, B). \]
The same is true with \( A \) replaced by \( A_{\lambda} \). Hence
\[ \Ext^{i}_{R_{\text{et}}^\text{perf}} (A, B) = \Ext^{i}_{R_{\text{et}}^\text{perf}} (A, B), \quad \Ext^{i}_{R_{\text{et}}^\text{perf}} (A_{\lambda}, B) = \Ext^{i}_{R_{\text{et}}^\text{perf}} (A_{\lambda}, B). \]
We will prove a slightly more general statement:
\[ \lim_{\lambda} \Ext^{i}_{R_{\text{et}}^\text{perf}} (A_{\lambda}, B) = \Ext^{i}_{R_{\text{et}}^\text{perf}} (A, B) \]
for any sheaf \( B \in \text{Ab}(R_{\text{et}}^\text{perf}) \) that commutes with filtered direct limits as a functor on \( R^\text{perf} \). For this form of the statement, it is actually enough to treat the case \( R = k \). Indeed, let \( f : \Spec R_{\text{et}}^\text{perf} \to \Spec k_{\text{et}}^\text{perf} \) be the natural morphism. As before, we have Grothendieck spectral sequences
\[ E^{ij}_{2} = \lim_{\lambda} \Ext^{i}_{R_{\text{et}}^\text{perf}} (A_{\lambda}, R^{j} f_{*} B) = \lim_{\lambda} \Ext^{i+j}_{R_{\text{et}}^\text{perf}} (A_{\lambda}, B), \]
and a morphism between them compatible with the \( E_{\infty} \)-terms. By Proposition 3.3.3, the sheaf \( R^{j} f_{*} B \in \text{Ab}(k^\text{perf}) \) is the pro-fppf sheafification of the presheaf \( R' \to H^{j}((R' \otimes_{k} R)_{\text{et}}, B) \). The pro-fppf sheaf appearing here commutes with filtered direct limits as a functor on \( k^{\text{perf}} \) by [MASH III, Lemma 1.16, Remark 1.17 (d)]. Hence the fpf sheafification of this presheaf is already a pro-fppf sheaf, and \( R^{j} f_{*} B \) commutes with filtered direct limits as a functor on \( k^{\text{perf}} \). Hence we may replace \( R \) by \( k \) and \( B \) by \( R^{j} f_{*} B \). Thus we are reduced to showing that
\[ \lim_{\lambda} \Ext^{i}_{R_{\text{et}}^\text{perf}} (A_{\lambda}, B) = \Ext^{i}_{R_{\text{et}}^\text{perf}} (A, B) \]
for any \( B \in \text{Ab}(k_{\text{et}}^\text{perf}) \) that commutes with filtered direct limits as a functor on \( k^{\text{perf}} \).

Note that the functor \( \text{PAlg}/k \to \text{Ab}(k_{\text{et}}^\text{perf}) \) is exact since a surjection in \( \text{PAlg}/k \) is a faithfully flat morphism of profinite presentation and hence a surjection in \( \text{Ab}(k_{\text{et}}^\text{perf}) \). For each \( \lambda \), let \( A_{\lambda}', A_{\lambda}'', A_{\lambda}'''' \) be the kernel, cokernel, image of \( A \to A_{\lambda} \). If we know that
\[ \lim_{\lambda} \Ext^{i}_{R_{\text{et}}^\text{perf}} (A_{\lambda}', B) = \lim_{\lambda} \Ext^{i}_{R_{\text{et}}^\text{perf}} (A_{\lambda}'', B) = 0 \]
for any $i \geq 0$, then
\[
\text{Ext}^i_{\text{perf}}(A, B) = \lim_{\lambda} \text{Ext}^i_{\text{perf}}(A^\lambda, B) = \lim_{\lambda} \text{Ext}^i_{\text{perf}}(A, B)
\]
for any $i \geq 0$. Therefore it is enough to show the following statement: if \{A_\lambda\}_\lambda is a filtered inverse system of proalgebraic groups over $k$ with $\lim A_\lambda = 0$ and $B \in \text{Ab}(k_{\text{profppf}})$ commutes with filtered direct limits, then
\[
\lim_{\lambda} \text{Ext}^i_{\text{perf}}(A_\lambda, B) = 0
\]
for any $i \geq 0$.

We prove this statement. We denote by $K^+(k_{\text{profppf}})$ and $K^+(\text{Ab})$ the homotopy categories of bounded below complexes in $\text{Ab}(k_{\text{perfppf}})$ and $\text{Ab}$, respectively. For a filtered direct system \{D_\lambda\} of bounded above complexes in $\text{Ab}(k_{\text{perfppf}})$, consider the triangulated functor
\[
K^+(k_{\text{perfppf}}) \to K^+(\text{Ab}), \quad C \mapsto \lim_{\lambda} \text{Hom}_{k_{\text{perfppf}}}(D_\lambda, C),
\]
where we denote by $\text{Hom}_{k_{\text{perfppf}}}(D_\lambda, C)$ the total complex of the Hom double complex and take its term-wise direct limit in $\lambda$. This admits a right derived functor
\[
D^+(k_{\text{perfppf}}) \to D^+(\text{Ab}), \quad C \mapsto \lim_{\lambda} R\text{Hom}_{k_{\text{perfppf}}}(D_\lambda, C)
\]
since $\text{Ab}(k_{\text{perfppf}})$ has enough injectives and by [KSO96, Proposition 13.2.3]. For any $i$, its $i$-th cohomology is $\lim_{\lambda} H^i R\text{Hom}_{k_{\text{perfppf}}}(D_\lambda, C)$. For $n \geq 0$, let
\[
X^n = \lim_{\lambda} R\text{Hom}_{k_{\text{perfppf}}}(H_nQ(A_\lambda), B),
\]
\[
Y^n = \lim_{\lambda} R\text{Hom}_{k_{\text{perfppf}}}(\tau_{\leq -n}Q(A_\lambda), B),
\]
where $\tau$ denotes truncation (in cohomological grading). We have a diagram
\[
\begin{array}{cccccc}
X^0 & \xrightarrow{X^1[-1]} & X^2[-2] & \xrightarrow{X^3[-3]} & \cdots \\
\downarrow & & \downarrow & & \downarrow \\
Y^0 & \xrightarrow{Y^1} & Y^2 & \xrightarrow{Y^3} & \cdots
\end{array}
\]
in $D^+(\text{Ab})$, where each L shape triangle $X^n[-n] \to Y^n \to Y^{n+1}$ is distinguished. The homology groups $H_n(Q(A))$ are the stable homology of $A$ (Remark 3.6.3). We use the following fact ([Bre70, Theorem 3]): $H_0(Q(A)) = A$, and for each $n \geq 1$, the group $H_n(Q(A))$ is a finite direct sum of the kernel or the cokernel of multiplication by $l$ on $A$ for various primes $l \geq 2$. (This is a finite direct sum for each fixed $n$ since by the definition of admissible sequences [Bre70, (1.18)], there are finitely many $l$-admissible sequences $(a_1, a_2, \ldots)$ of degree $n$ with $a_i \equiv 0 \mod 2l - 2$ for any prime $l$ and there are no such if $2l - 2 > n$.) In particular, we have
\[
X^0 = \lim_{\lambda} R\text{Hom}_{k_{\text{perfppf}}}(A_\lambda, B),
\]
of which we want to show the vanishing.
We first show that \( Y^0 = 0 \). We have a hyperext spectral sequence
\[
E_{i,j}^1 = \lim_{\lambda} \text{Ext}^j_{k_{\text{et}}} (Q_i(A_\lambda), B) \Longrightarrow H^{i+j} Y^0.
\]
Recall from Section 5.1 that for each \( i \), the \( i \)-th term \( Q_i(A_\lambda) \) is a direct summand of the sheaf \( \mathbb{Z}[A_\lambda^2] \). Therefore we can write the Ext groups in the above spectral sequences in terms of pro-fppf cohomology groups of schemes of the form \( A_\lambda^2 \). Since \( \lim_{\lambda} A_\lambda = 0 \), we have
\[
\lim_{\lambda} H^{i,j} (A_\lambda^2, B) = H^{i,j} (0, B)
\]
by the remark before the proof (more precisely, by the pro-fppf version of [Mil80, III, Remark 1.17 (d)]). Hence we have \( E_{i,j}^{1} = \text{Ext}^j_{k_{\text{et}}} (Q_i(0), B) \). Note that the zero map \( A \to 0 \) induces a morphism on their spectral sequences of the above type compatible with the filtrations on the \( E_\infty \)-terms, since it comes from the morphism
\[
\lim_{\lambda} \text{Hom}_{k_{\text{et}}} (Q(A_\lambda), J) \to \text{Hom}_{k_{\text{et}}} (Q(0), J)
\]
of double complexes, where \( J \) is an injective resolution of \( B \). Hence
\[
Y^0 = R \text{Hom}_{k_{\text{et}}} (Q(0), B) = 0
\]
since \( Q(0) \) has zero homology.

Now we show that \( X^0 = 0 \). We prove that \( \tau_{\leq n} X^0 = 0 \) by induction on \( n \geq 0 \). The base case \( n = 0 \) follows from \( Y^0 = 0 \). Assume that \( \tau_{\leq n} X^0 = 0 \) (for all \( \{A_\lambda\} \) satisfying the assumption). Then
\[
\tau_{\leq n} \lim_{\lambda} R \text{Hom}_{k_{\text{et}}} (A_\lambda^2, B) = 0,
\]
where \( A_\lambda^2 \in \text{PAlg}/k \) is the kernel or cokernel of multiplication by any positive integer on \( A_\lambda \). Hence
\[
\tau_{\leq n+1} (X^1[-1]) = \tau_{\leq n+1} (X^2[-2]) = \cdots = 0
\]
by the structure of the homology of \( Q \). Note that \( Y^n \) is concentrated in degrees \( \geq n \). Hence by the above diagram and \( Y^0 = 0 \), we have
\[
\tau_{\leq n+1} X^0 = \tau_{\leq n} X^1 = \tau_{\leq n} Y^2 = \cdots = \tau_{\leq n} Y^{n+1} = 0.
\]

\[2\] We may assume that \( R = k \). Indeed, let \( f : \text{Spec} R_{et}^{\text{perf}} \to \text{Spec} k_{et}^{\text{perf}} \) be the natural morphism. Then for any filtered direct system \( \{C_\mu\} \) in \( \text{Ab}(R_{et}^{\text{perf}}) \) with direct limit \( C \), \( j \geq 0 \) and \( R' \in k^{\text{perf}} \), we have
\[
\lim_{\mu} H^j (R' \otimes_k R, C_\mu) = H^j (R' \otimes_k R, C)
\]
by [Mil80, III, Remark 3.6 (d)]. Hence \( \lim_{\mu} R^j f_* C_\mu = R^j f_* C \). We have Grothendieck spectral sequences
\[
E_{i,j}^2 = \lim_{\mu} \text{Ext}^j_{k_{et}} (A, R^j f_* C_\mu) \Longrightarrow \lim_{\mu} \text{Ext}^{i+j}_{k_{et}} (A, C_\mu),
\]
\[
E_{i,j}^2 = \text{Ext}^j_{k_{et}} (A, R^j f_* C) \Longrightarrow \text{Ext}^{i+j}_{k_{et}} (A, C).
\]
There is a morphism from the first spectral sequence to the second compatible with the \( E_\infty \)-terms by the following reason. By [KS06, Corollary 9.6.6], there exists a
functorial choice of embeddings $D \hookrightarrow \Psi(D)$ for arbitrary $D \in \text{Ab}(\text{R}_{\text{et}}^{\text{perf}})$ with $\Psi(D)$ injective. Let $\Psi^0(D) = \Psi(D)$, $\Psi^1(D) = \Psi(\Psi(D)/D)$ and so on. Then we have a functorial injective resolution $\Psi^\bullet(D)$ of $D$. We have a commutative diagram

$$
\begin{array}{c}
\lim_{\mu} \text{Hom}_{k_{\text{et}}} (A, f \ast \Psi^\bullet(C_{\mu})) \\
\downarrow \\
\text{Hom}_{k_{\text{et}}} (A, f \ast \Psi^\bullet(C))
\end{array}
\begin{array}{c}
\lim_{\mu} \text{Hom}_{R_{\text{et}}^{\text{perf}}} (A, \Psi^\bullet(C)) \\
\downarrow \\
\text{Hom}_{R_{\text{et}}^{\text{perf}}} (A, \Psi^\bullet(C))
\end{array}
$$

of complexes in $\text{Ab}$. This induces a desired morphism of spectral sequences. Replacing $R$ by $k$ and $C_{\mu}$ by $R^j f_\ast C_{\mu}$, we are reduced to the case $R = k$.

Now assume $R = k$. Let $M$ be a directed set viewed as a filtered category. Consider the category $\text{Ab}(k_{\text{et}}^{\text{perf}})(M)$ of functors from $M$ to $\text{Ab}(k_{\text{et}}^{\text{perf}})$ (that is, direct systems in $\text{Ab}(k_{\text{et}}^{\text{perf}})$ indexed by $M$). This is an abelian category with enough injectives ([Mil80, III, Remark 3.6 (d)]). The additive bifunctor

$$\text{Ab}(k_{\text{et}}^{\text{perf}})^{\text{op}} \times \text{Ab}(k_{\text{et}}^{\text{perf}})(M) \to \text{Ab}, \quad (D, \{C_{\mu}\}) \mapsto \lim_{\mu} \text{Hom}_{k_{\text{et}}^{\text{perf}}}(D, C_{\mu})$$

(op means the opposite category) extends to a triangulated bifunctor

$$K^-(k_{\text{et}}^{\text{perf}})^{\text{op}} \times K^+ (\text{Ab}(k_{\text{et}}^{\text{perf}})(M)) \to K^+ (\text{Ab})$$

by [KS08] Proposition 11.6.4 (i)]. Evaluating the left variable at any bounded above complex $D$ in $\text{Ab}(k_{\text{et}}^{\text{perf}})$, we have a triangulated functor

$$K^+ (\text{Ab}(k_{\text{et}}^{\text{perf}})(M)) \to K^+.$$

This admits a right derived functor

$$D^+ (\text{Ab}(k_{\text{et}}^{\text{perf}})(M)) \to D^+(\text{Ab}), \quad \{C_{\mu}\} \mapsto \lim_{\mu} R \text{Hom}_{k_{\text{et}}^{\text{perf}}}(D, C_{\mu})$$

by [KS08] Proposition 13.2.3], whose $i$-th cohomology is $\lim_{\mu} H^i R \text{Hom}_{k_{\text{et}}^{\text{perf}}}(D, C_{\mu})$. We have a morphism

$$\lim_{\mu} R \text{Hom}_{k_{\text{et}}^{\text{perf}}}(D, C_{\mu}) \to R \text{Hom}_{k_{\text{et}}^{\text{perf}}}(D, \lim_{\mu} C_{\mu})$$

by universality. We consider the cubical construction $Q(A)$ for $A$ viewed as an object of $\text{DGA}(k_{\text{et}}^{\text{perf}})$, which is the étale (rather than pro-fppf) sheafification of the presheaf $R \mapsto Q(A(R))$. Let $\{C_{\mu}\} \in \text{Ab}(k_{\text{et}}^{\text{perf}})(M)$ with $\lim_{\mu} C_{\mu} = 0$ and set

$$X^j = \lim_{\mu} R \text{Hom}_{k_{\text{et}}^{\text{perf}}}(H_j Q(A), C_{\mu}),$$

$$Y^j = \lim_{\mu} R \text{Hom}_{k_{\text{et}}^{\text{perf}}}(\tau_{\leq -j} Q(A), C_{\mu}).$$

We know $Y_0 = 0$ from

$$\lim_{\mu} H^3(A^{2'}, C_{\mu}) = H^3(A^{2'}, 0) = 0$$

by the same argument as the proof of the previous assertion. For a fixed $n \geq 0$, assume $\tau_{\leq n} X^0 = 0$ for any $A \in \text{PAlg}/k$. Then

$$\tau_{\leq n} \lim_{\mu} R \text{Hom}_{k_{\text{et}}^{\text{perf}}}(A[l], C_{\mu}) = 0.$$
for any \( l \geq 1 \), where \( A[l] \in \text{PAlg}/k \) is the \( l \)-torsion part. This and \( \tau_{\leq n} X^0 = 0 \) imply

\[
\tau_{\leq n} \lim_{\mu} R \text{Hom}_{k_{\text{prof}}}^m(A/\text{et} l, C_\mu) = 0,
\]

where \( A/\text{et} l \in \text{Ab}(k_{\text{et}}^{\text{perf}}) \) is the cokernel of multiplication by \( l \) on \( A \) in \( \text{Ab}(k_{\text{et}}^{\text{perf}}) \) (which might not be in \( \text{PAlg}/k \) since \( \text{PAlg}/k \rightarrow \text{Ab}(k_{\text{et}}^{\text{perf}}) \) is only left exact). From these, we can deduce \( \tau_{\leq n+1} X^0 = 0 \) by the same argument as the proof of the previous assertion. Induction then finishes the proof. \( \square \)

With this lemma, the statements of the proposition are reduced to showing the following parts

\[
\begin{align*}
\text{Ext}_k^n(A, B) &= \text{Ext}_k^n(A, B), \\
\text{Ext}_k^n(A, Q) &= 0
\end{align*}
\]

(12) for \( A, B \in \text{Alg}/k \). (Note that \( \text{Hom}_{k_{\text{et}}}^n(A, Q) = 0 \) implies that its subgroup \( \text{Hom}_{k_{\text{et}}}^n(A, Z) \) is zero.)

Next we will reduce these statements to the case that \( k \) is algebraically closed.

Let \( A \in \text{Alg}/k \) and \( B \in \text{LAlg}/k \). Assertion (1) of the above lemma implies that

\[
\text{Ext}_k^j(A, B) = \lim_{k'/k} \text{Ext}_k^j(A, B),
\]

where \( k' \) runs through all finite Galois subextensions of \( \bar{k}/k \). Therefore the Hochschild-Serre spectral sequences for finite Galois extensions give an isomorphism

\[
(13) \quad R\Gamma(\text{Gal}(\bar{k}/k), R \text{Hom}_{k_{\text{et}}}^n(A, B)) = R \text{Hom}_{k_{\text{et}}}^n(A, B)
\]

in \( D(\text{Ab}) \). Hence the second line of (12) is reduced to the case \( \bar{k} = k \).

For the first line, assume \( B \in \text{Alg}/k \). Then [Mil70] \( \text{§1, Proposition} \) (or its proof) shows that the functor \( \text{Hom}_{\text{PAlg}/k}^n(\cdot, B) \) takes projectives in \( \text{PAlg}/k \) to \( \Gamma(\text{Gal}(\bar{k}/k), \cdot) \)-acyclics. Hence we have an isomorphism

\[
(14) \quad R\Gamma(\text{Gal}(\bar{k}/k), R \text{Hom}_{\text{PAlg}/k}^n(A, B)) = R \text{Hom}_{\text{PAlg}/k}^n(A, B)
\]

in \( D(\text{Ab}) \). The functor \( \text{PAlg}/k \rightarrow \text{Ab}(k_{\text{profppf}}) \) is exact as we saw in the proof of the previous lemma. This induces morphisms

\[
\begin{align*}
R \text{Hom}_{\text{PAlg}/k}^n(A, B) &\rightarrow R \text{Hom}_{k_{\text{profppf}}}^n(A, B), \\
R \text{Hom}_{\text{PAlg}/k}^n(A, B) &\rightarrow R \text{Hom}_{k_{\text{profppf}}}^n(A, B)
\end{align*}
\]

in \( D(\text{Ab}) \), \( D(\text{Gal}(\bar{k}/k)) \) (= the derived category of discrete \( \text{Gal}(\bar{k}/k) \)-modules), respectively. The right-hand sides are isomorphic to \( R \text{Hom}_{k_{\text{et}}}^n(A, B), R \text{Hom}_{k_{\text{et}}}^n(A, B) \), respectively, by Proposition [3.7.4] and Corollary [3.3.3]. Hence we have morphisms

\[
\begin{align*}
R \text{Hom}_{\text{PAlg}/k}^n(A, B) &\rightarrow R \text{Hom}_{k_{\text{et}}}^n(A, B), \\
R \text{Hom}_{\text{PAlg}/k}^n(A, B) &\rightarrow R \text{Hom}_{k_{\text{et}}}^n(A, B)
\end{align*}
\]

in \( D(\text{Ab}) \), \( D(\text{Gal}(\bar{k}/k)) \), respectively. Applying \( R\Gamma(\text{Gal}(\bar{k}/k), \cdot) \) to the second morphism results the first by (13) and (14). Therefore if we show that

\[
R \text{Hom}_{\text{PAlg}/k}^n(A, B) = R \text{Hom}_{k_{\text{et}}}^n(A, B),
\]

then

\[
R \text{Hom}_{\text{PAlg}/k}^n(A, B) = R \text{Hom}_{k_{\text{et}}}^n(A, B).
\]
Noting that $\operatorname{Ext}^n_{\text{Alg}/k}(A, B) = \operatorname{Ext}^n_{\text{Alg}/k}(A, B)$, we are reduced to the case $\bar{k} = k$.

We show (12) in the case $\bar{k} = k$.

**Lemma 3.8.3.** Let $A, B \in \text{Alg}/k$ be connected. Then the group $\operatorname{Ext}^i_{k^{\text{perf}}}(A, B)$ consists of $p$-power-torsion elements for any $i \geq 2$. In particular, we have $\operatorname{Ext}^i_{k^{\text{perf}}}(G_m, G_m) = 0$ for all $i \geq 2$.

**Proof.** Let $A_0, B_0$ be commutative affine algebraic groups over $k$ whose perfections are $A, B$, respectively. Let $k^{\text{sch}}$ be the category of affine $k$-schemes and $\text{Spec} k^{\text{sch}}$ the étale site on it. By [Bre81] Lemmas 1.1-1.3 and their proofs, we have

$$\operatorname{Ext}^i_{k^{\text{perf}}} (A, B) = \operatorname{Ext}^i_{k^{\text{sch}}} (A_0, \lim_{n} B_0^{(-n)}) ,$$

where $B_0 \to B_0^{(-1)} \to B_0^{(-2)} \to \cdots$ are the Frobenius morphisms over $k$. Assertion (2) of Lemma 3.8.2 is also true for $\text{Spec} k^{\text{sch}}$ by exactly the same proof. Hence we have

$$\operatorname{Ext}^i_{k^{\text{perf}}} (A, B) = \lim_{n} \operatorname{Ext}^i_{k^{\text{sch}}} (A_0, B_0^{(-n)}) .$$

We want to show that $\operatorname{Ext}^i_{k^{\text{sch}}} (A_0, B_0^{(-n)})$ consists of $p$-power-torsion elements in the category of $k^{\text{sch}}$ schemes and $\text{Spec} k^{\text{sch}}$ the é tale site on it. By [Bre70] §10, 6th paragraph (p.43), we know that $\operatorname{Ext}^i_{k^{\text{perf}}} (G_m, G_m)$ consists of $p$-power-torsion elements in the category of $k^{\text{perf}}$ schemes. (As the proof of loc. cit. shows, the meaning of an abelian group being “at most $p$-torsion” in the terminology of [Bre70] is that it consists of $p$-power-torsion elements.) This proves the desired property of $\operatorname{Ext}^i_{k^{\text{perf}}} (A, B)$. On the other hand, the $p$-th power map on the group $G_m$ over $k^{\text{perf}}$ is an isomorphism. Hence $\operatorname{Ext}^i_{k^{\text{perf}}} (G_m, G_m) = 0$ for all $i \geq 2$. \hfill $\square$

**Proof of Proposition 3.8.1** Let $A, B \in \text{Alg}/k$. We first prove

$$\operatorname{Ext}^n_{k^{\text{perf}}} (A, B) = \operatorname{Ext}^n_{\text{Alg}/k} (A, B)$$

for $A, B \in \text{Alg}/k$ and $n \geq 0$. The case $n = 0, 1$ is classical (apply the direct limit in Frobenii to [Oor66] Corollary 17.5 and argue as in the proof of the previous lemma). We may assume that each of $A$ and $B$ is $G_a$, $G_m$, or finite. The case $A$ is finite is easy. The case $B$ is finite is reduced to the case $B$ is connected affine by embedding a finite $B$ into a connected affine group. There are no higher extensions between $G_a$ (killed by $p$) and $G_m$ (having invertible $p$-th power map). For the case $A = B = G_m$ and $n \geq 2$, the right-hand side is zero since the category of quasi-algebraic groups of multiplicative type is equivalent to the category of finitely generated $\mathbb{Z}[1/p]$-modules [Ser60] §7.2, Proposition 1, 2. The left-hand side is also zero by Lemma 3.8.3 For the case $A = B = G_a$ and $n \geq 2$, the right-hand side is zero by [Ser60] §8.6, Corollaries 4 and 5 to Proposition 6]. The left-hand side is also zero by [Bre81] Corollary 1.7].
Next we prove $\text{Ext}^n_{\text{et}}(A, \mathbb{Q}) = 0$ for $n \geq 0$. This is trivial if $A$ is finite. Hence we may assume that $A$ is connected. Let $M(A)$ be Mac Lane’s resolution of $A$ in $\text{Ab}(k'_\text{et})$. Consider the hyperext spectral sequence

$$E_1^{ij} = \text{Ext}^j_{k'_\text{et}}(M_i(A), \mathbb{Q}) \implies \text{Ext}^{i+j}_{k'_\text{et}}(A, \mathbb{Q}).$$

Each term of $M(A)$ is a direct summand of a direct sum of sheaves of the form $\mathbb{Z}[A^n]$ for various $n$. Hence $\text{Ext}^j_{k'_\text{et}}(M_i(A), \mathbb{Q})$ is a direct factor of a direct product of abelian groups of the form $H^j(A^n, \mathbb{Q})$. If $j \geq 1$, this is zero since the étale cohomology of a normal scheme with coefficients in $\mathbb{Q}$ vanishes in non-zero degree by [Den88 §2.1]. If $j = 0$, this is $\mathbb{Q} = H^0(0, \mathbb{Q})$ since $A$ is assumed to be connected. Therefore

$$\text{Ext}^n_{k'_\text{et}}(A, \mathbb{Q}) = \text{Ext}^n_{k'_\text{et}}(0, \mathbb{Q}) = 0.$$

\[\square\]

Remark 3.8.4. Results of this section can be extended for abelian varieties instead of affine proalgebraic groups. This can be done as follows. Let $\text{Alg}'/k$ be the category of (not necessarily affine) quasi-algebraic groups over $k$. Consider its pro-category $\text{PAlg}'/k$ (which is Serre’s category of proalgebraic groups over $k$ [Ser60 §2.6, Proposition 12]). Let $\text{PAlg}'/k$ be the full subcategory of $\text{PAlg}'/k$ consisting of extensions of perfections of abelian varieties by affine proalgebraic groups. This category contains projective envelopes of perfections of abelian varieties by $\text{PAlg}'/k$ [Ser60 §9.2, Proposition 4]. It follows that any object $G \in \text{PAlg}'/k$ has a projective resolution $G_\bullet$ in $\text{PAlg}'/k$ such that $G_n \in \text{PAlg}'/k$ (hence projective also in $\text{PAlg}'/k$) for any $n$. We use $\text{PAlg}'/k$ instead of the category $\text{PAlg}/k$ of affine proalgebraic groups.

Let $k'^{\text{perf}}$ be the category of quasi-compact quasi-separated perfect schemes over $k$. We use $k^{\text{perf}}$ instead of the category $k^{\text{perf}}$ of perfect $k$-algebras. The category $k^{\text{perf}}$ contains the perfection $A$ of an abelian variety over $k$; its generic point $\xi_A$ and the inclusion morphism $\xi_A \hookrightarrow A$. A morphism $Y \to X$ in $k'^{\text{perf}}$ is said to be flat of finite presentation (in the perfect sense) if it can be written as the perfection of a $k$-scheme morphism $Y_0 \to X$ flat of finite presentation in the usual sense. A morphism $Y \to X$ in $k'^{\text{perf}}$ is said to be flat of profinite presentation if it can be written as the inverse limit of a filtered inverse system $\{Y_\lambda \to X\}$ of morphisms in $k'^{\text{perf}}$ flat of finite presentation such that the transition morphisms $Y_\mu \to Y_\lambda$ are affine. A finite family $\{Y_i \to X\}$ of morphisms in $k'^{\text{perf}}$ is called an fppf (resp. pro-fpfp) covering if each $Y_i \to X$ is a flat morphism of finite (resp. profinite) presentation and $\bigsqcup Y_i \to X$ is surjective. This defines an fppf (resp. pro-fpfp) site $\text{Spec} k'^{\text{perf}}$ (resp. $\text{Spec} k'^{\text{profpf}}$) on the category $k'^{\text{perf}}$. The étale site $\text{Spec} k'_\text{et}$ on $k'^{\text{perf}}$ can be defined in the usual way, noting that an étale scheme over a perfect $k$-scheme is again perfect. The identity functor defines a morphism of sites $\text{Spec} k'^{\text{perf}} \to \text{Spec} k'_\text{et}$, which induces an equivalence on the topos.

Then all the definitions, statements and proofs in this section before Lemma 3.8.3 can be generalized with $\text{PAlg}/k$ replaced by $\text{PAlg}'/k$ and $k^{\text{perf}}$ replaced by $k'^{\text{perf}}$. For Lemma 3.8.3 we need to prove the following generalization: the group $\text{Ext}^i_{k'_\text{et}}(A, B)$ consists of $p$-power-torsion elements for any $i \geq 2$ and any connected $A, B \in \text{Alg}'/k$. To prove this, by the same argument as the original proof, it is enough to show that $\text{Ext}^i_{k'_\text{et}}(A_0, B_0)$ consists of $p$-power-torsion elements for any
we explain a relation between the site \( \text{DM}^a_k \) introduced in \cite{Rov05}. Their purpose was to study motives. In this section, actions by the field automorphism group of a universal domain over \( k \) \cite{JR10}, Jannsen and Rovinsky defined and used the site \( \text{DM}^a_k \) to change.

The statement to the case where \( A \) are abelian varieties. Then using either one of the above two sets of sequences reduces gives the desired result in this case. Finally, let \( A \) be the perfection of a semi-abelian variety and \( B \) argument using the exact sequences on the third terms of these sequences is an isomorphism for any \( n \geq 3 \). Hence we can inductively prove that the morphism of these sequences is an isomorphism if \( n \) isomorphism for any \( n \). The case \( n = 0,1 \) is again classical (this time using the not-necessarily-affine version \cite[Remark after Corollary 17.5]{Oor66}). We have \( \text{Ext}^n_{\text{Alg}'_k/k}(A,B) = 0 \) for \( n \geq 3 \) by \cite[§10.1, Theorem 1]{Ser60}. As noted after the proof of the cited theorem, the group \( \text{Ext}^2_{\text{Alg}'_k/k}(A,B) \) is zero if \( A \) and \( B \) are elementary in the sense of \cite[§3.2, Definition 1]{Ser60} except the case \( A = \mathbb{G}_a \) and \( B = \mathbb{Z}/p\mathbb{Z} \), where \( \text{Ext}^2_{\text{Alg}'_k}(\mathbb{G}_a, \mathbb{Z}/p\mathbb{Z}) \) is killed by \( p \). By dévissage, this implies that \( \text{Ext}^2_{\text{Alg}'_k/k}(A,B) \) is torsion for any \( A,B \in \text{Alg}'_k \). By the same argument as the proof of the original Lemma \ref{lem:exact-sequence} we may assume that \( A \) and \( B \) are \( \mathbb{G}_a, \mathbb{G}_m \) or perfections of abelian varieties. First, let \( A \) be connected affine and \( B \) the perfection of a semi-abelian variety. Then for any \( n,m \geq 1 \), the Kummer sequence \( 0 \to B[m] \to B \to 0 \) induces exact sequences

\[
0 \to \text{Ext}^{n-1}_{\text{Alg}'_k/k}(A,B)/m \to \text{Ext}^n_{\text{Alg}'_k/k}(A,B[m]) \to \text{Ext}^n_{\text{Alg}'_k/k}(A,B)[m] \to 0,
\]

\[
0 \to \text{Ext}^{n-1}_{k_{\text{perf}}}(A,B)/m \to \text{Ext}^n_{k_{\text{perf}}}(A,B[m]) \to \text{Ext}^n_{k_{\text{perf}}}(A,B)[m] \to 0,
\]

where \( /m \) denotes the cokernel of multiplication by \( m \). We have a natural morphism from the first sequence to the second. As we saw, the morphism on the first terms of these sequences is an isomorphism if \( n \leq 2 \). The group \( B[m] \) is finite and hence affine. Hence the affine case implies that the morphism on the second terms is an isomorphism for any \( n \). The group \( \text{Ext}^n_{k_{\text{perf}}}(A,B) \) is torsion for \( n \geq 2 \) by Lemma \ref{lem:exact-sequence} generalized right above. Hence we can inductively prove that the morphism on the third terms of these sequences is an isomorphism for any \( n \). Second, let \( A \) be the perfection of a semi-abelian variety and \( B \) connected affine. Then the same argument using the exact sequences

\[
0 \to \text{Ext}^{n-1}_{\text{Alg}'_k/k}(A,B)/m \to \text{Ext}^n_{\text{Alg}'_k/k}(A,m) \to \text{Ext}^n_{\text{Alg}'_k/k}(A,B)[m] \to 0,
\]

\[
0 \to \text{Ext}^{n-1}_{k_{\text{perf}}}(A,B)/m \to \text{Ext}^n_{k_{\text{perf}}}(A,m) \to \text{Ext}^n_{k_{\text{perf}}}(A,B)[m] \to 0
\]

gives the desired result in this case. Finally, let \( A \) and \( B \) be perfections of semi-abelian varieties. Then using either one of the above two sets of sequences reduces the statement to the case where \( A \) or \( B \) is affine. This finishes the case of arbitrary \( A,B \in \text{Alg}'_k \). The final paragraph of the proof of Proposition \ref{prop:exact-sequence} needs no change.

4. Comparison with Jannsen-Rovinsky’s dominant topology

Let \( \text{DM}_k \) be the category of the perfections of smooth \( k \)-morphisms of smooth affine \( k \)-schemes with the topology where a cover is a dominant morphism. In \cite{JR10}, Jannsen and Rovinsky defined and used the site \( \text{DM}_k \) (without taking perfections or affineness). Its topos is equivalent to the category of sets with smooth actions by the field automorphism group of a universal domain over \( k \), which Rovinsky introduced in \cite{Rov05}. Their purpose was to study motives. In this section, we explain a relation between the site \( \text{DM}_k \) and our sites \( k_{\text{perf}}^\text{rat}, \text{Spec} k_{\text{perf}}^\text{profppf} \).
We hope our sites and Theorem 4.3 are useful for the study of motives and the site $\text{DM}_k$.

Note that the scheme-theoretic fiber product $Y \times_X Z$ for $X, Y, Z \in \text{DM}_k$ (which is always in $\text{DM}_k$) gives the fiber product in the category $\text{DM}_k$ when $Y \to X$ or $Z \to X$ is étale, but not always. Morphisms in $\text{DM}_k$ are restricted to be smooth. A little more precise definition of the topology of $\text{DM}_k$ is that a sieve on an object $X \in \text{DM}_k$ is a covering if it contains a (finite) family of morphisms $X_i \to X$ such that $\bigsqcup X_i \to X$ is dominant. A presheaf $F$ on $\text{DM}_k$ is a sheaf if and only if it sends disjoint unions to direct products and the sequence $F(X) \to F(Y) \cong F(Y \times_X Y)$ is exact for any dominant morphism $Y \to X$ in $\text{DM}_k$.

4.1. The Zariski and étale cases. Let $\text{DM}_{k,\text{et}}$ (resp. $\text{DM}_{k,\text{zar}}$) be the same category as $\text{DM}_k$ with the topology where a cover is a dominant étale morphism (resp. a dominant open immersion). Let $\text{Spec}_{\text{rat}}$ be the category $k_{\text{rat}}$ with the topology where a cover is the identity map (so a presheaf $F$ is a sheaf if and only if $F(k_1 \times \cdots \times k_n) = F(k_1) \times \cdots \times F(k_n)$ for fields $k_1, \ldots, k_n \in k_{\text{rat}}$). Note that for a perfect scheme $X$ essentially of finite type over $k$, its generic point $\xi_X$ of $X$ (Definition 3.2.1) is the Spec of a rational $k$-algebra.

**Proposition 4.1.1.** The functor taking an object of $\text{DM}_k$ to its generic point defines morphisms of sites

\[ f: \text{Spec}_{\text{rat}} \to \text{DM}_{k,\text{zar}}, \quad g: \text{Spec}_{\text{et}} \to \text{DM}_{k,\text{et}}, \]

which induce equivalences on the associated toponoi.

**Proof.** First we treat $f$. A presheaf $F$ on $\text{DM}_{k,\text{zar}}$ is a sheaf if and only if $F(X \cup Y) = F(X) \times F(Y)$ for $X, Y \in \text{DM}_{k,\text{zar}}$ and $F(X) \cong F(Y)$ for a dominant open immersion $Y \to X$ in $\text{DM}_{k,\text{zar}}$. The pullback $f^* F$ for $F \in \text{Set}(\text{DM}_{k,\text{zar}})$ is given by $(f^* F)(x) = F(X)$ for any $x \in k_{\text{rat}}$, where $X \in \text{DM}_{k,\text{zar}}$ is any object with $\xi_X = x$. Obviously $f_* f^* = \text{id}$ and $f^* f_* = \text{id}$. Hence $f$ induces an equivalence on the toponoi.

Hence for $g$, we only need to show that $f^*$ maps the subcategory $\text{Set}(\text{DM}_{k,\text{et}})$ to $\text{Set}(k_{\text{et}})$. Let $F \in \text{Set}(\text{DM}_{k,\text{et}})$. Let $y \to x$ be a faithfully flat étale morphism in $k_{\text{rat}}$. We want to show that the sequence

\[ f^* F(x) \to f^* F(y) \cong f^* F(y \times_x y) \]

is exact. Take a dominant étale morphism $Y \to X$ in $\text{DM}_{k,\text{et}}$ whose associated morphism $\xi_Y \to \xi_X$ gives $y \to x$. Then $\xi_Y \times_X Y = y \times_x y$. Since $F \in \text{Set}(\text{DM}_{k,\text{et}})$, the sequence

\[ F(X) \to F(Y) \cong F(Y \times_X Y) \]

is exact. This sequence is identical to the above sequence. \qed

**Remark 4.1.2.** The composite $\text{Spec}_{\text{et}} \to \text{Spec}_{\text{rat}} \to \text{DM}_{k,\text{et}}$ is defined by the functor sending $X \in \text{DM}_k$ to its generic point $\xi_X$. This functor is different from the identity functor $X \to X$. The identity functor does not define a continuous map $\text{Spec}_{\text{et}} \to \text{DM}_{k,\text{et}}$.

4.2. The flat case. We define a site $\text{Spec}_{\text{rat}}$ and compare it with $\text{DM}_k$. Let $k_{\text{rat}}$ be the full subcategory of the category of perfect $k$-schemes consisting of (not necessarily finite) disjoint unions of the Spec’s of finitely generated perfect fields over $k$. The fiber product of any objects $y, z$ over any object $x$ in $k_{\text{rat}}$ exists. It is given by the disjoint union of the points of the underlying set of the usual fiber
product $y \times_z z$ as a scheme. We denote it by $y x z$. We endow the category $\tilde{k}^\text{rat}$ with the topology where a covering is a faithfully flat morphism. The resulting site is denoted by $\text{Spec } \tilde{k}^{\text{rat}}$. We call this the rational flat site of $k$. We denote $y \times_z z := \xi_{y x z}$.

**Proposition 4.2.1.** In order for a presheaf $F$ on $\text{Spec } \tilde{k}^\text{rat}$ to be a sheaf, it is necessary and sufficient that the following two conditions be satisfied:

1. For any fields $k_\lambda \in k^\text{rat}$ and $x_\lambda = \text{Spec } k_\lambda$, we have $F(\bigcup x_\lambda) = \prod F(x_\lambda)$.
2. For any field extension $k''/k'$ in $k^\text{rat}$ and $x' = \text{Spec } k'$, $y = \text{Spec } k''$, the sequence

$$F(x) \to F(y) \xrightarrow{p_1} F(y \times_z^\text{gen} y)$$

is exact, where $p_1$ and $p_2$ are given by the first and the second projections $y \times_z^\text{gen} y \xrightarrow{\pi_2} y$.

**Proof.** Clearly these conditions are sufficient. For necessity, let $F \in \text{Set}(\tilde{k}^\text{rat})$. The first condition is clear. For the second, let $s \in F(y)$ satisfy $p_1(s) = p_2(s)$ in $F(y \times_z^\text{gen} y)$. We need to show that for any connected component $z \subset y x z$ (or in other words, a point of $y x z$), we have $p_1(s) = p_2(s) = F(z)$. Since $F \in \text{Set}(\tilde{k}^\text{rat})$, the projection $z \times_z^\text{gen} y \to z$ induces an injection $F(z) \to F(z \times_z^\text{gen} y)$. Hence it is enough to show that the elements $p_1(s)$ and $p_2(s)$ map to the same element of $F(z \times_z^\text{gen} y)$. The third projection $p_3: z \times_z^\text{gen} y \to y$ sends $s \in F(y)$ to another element $p_3(s) \in F(z \times_z^\text{gen} y)$. We show that $p_1(s) = p_3(s)$ and $p_2(s) = p_3(s)$ in $F(z \times_z^\text{gen} y)$. The morphism $(p_1, p_3): z \times_z y \to y x x y$ is flat. Hence this induces a morphism $(p_1, p_3): z \times_z^\text{gen} y \to y \times_y^\text{gen} y$. Since $p_1(s) = p_2(s)$ in $F(y \times_z^\text{gen} y)$ by assumption, we have $p_1(s) = p_3(s)$ in $F(z \times_z^\text{gen} y)$. Similarly we have $p_2(s) = p_3(s)$ in $F(z \times_z^\text{gen} y)$. Hence $p_1(s) = p_2(s)$ in $F(z \times_z^\text{gen} y)$. \qed

**Proposition 4.2.2.** The functor taking an object of $\text{DM}_k$ to its generic point defines a morphism of sites

$$h: \text{Spec } \tilde{k}^\text{rat} \to \text{DM}_k,$$

which induces an equivalence on the associated topos.

**Proof.** It is enough to show that $f_* (= h_*)$ and $f^*$ used for the Zariski sites send sheaves to sheaves. If $F \in \text{Set}(\tilde{k}^\text{rat})$ and $Y \to X$ dominant in $\text{DM}_k$, then $\xi_Y \to \xi_X$ is faithfully flat in $k^\text{rat}$. Therefore Proposition [3.2.1] implies that the sequence $F(\xi_X) \to F(\xi_Y) \xrightarrow{\theta} F(\xi_Y \times_X Y)$ is exact. Hence $f^* F \in \text{Set}(\text{DM}_k)$. Conversely, let $F \in \text{Set}(\text{DM}_k)$, $k''/k'$ a field extension in $k^\text{rat}$, $x = \text{Spec } k'$ and $y = \text{Spec } k''$. Take a dominant morphism $Y \to X$ in $\text{DM}_k$ whose associated morphism $\xi_Y \to \xi_X$ gives $y \to x$. Then $\xi_Y \times_X Y = y \times_y^\text{gen} y$. The sheaf condition for $F \in \text{Set}(\text{DM}_k)$ says that the sequence $F(X) \to F(Y) \xrightarrow{\theta} F(Y \times X X)$ is exact. This sequence is identical to the sequence $(f^* F)(x) \to (f^* F)(y) \xrightarrow{\theta} (f^* F)(y \times_y^\text{gen} y)$. Hence $f^* F \in \text{Set}(\tilde{k}^\text{rat})$ by Proposition [3.2.1]. \qed

### 4.3. The dominant topology and the pro-fppf topology

Next we relate $\text{Spec } \tilde{k}^\text{rat}$ and $\text{DM}_k$ to a variant of $\text{Spec } k^\text{perf}$. Let $k^\text{perf}$ be the full subcategory of $k^\text{perf}$ consisting of the perfects of $k$-algebras essentially of finite type. Let $\text{Spec } k^\text{perf}$ be the site obtained by restricting $\text{Spec } k^\text{perf}$ to $k^\text{perf}$. (In the category $k^\text{perf}$, every flat morphism is of profinite presentation, and the pro-fppf
and fpqc topologies coincide, but we do not need this fact.) We need a generic variant of the Čech complex.

**Proposition 4.3.3.** Let $k'/k$ be a field extension in $k^{\text{rat}}$ and $x = \text{Spec } k'$, $y = \text{Spec } k''$. Let $C^\text{gen}(y/x)$ be the complex

\[ \cdots \to \mathbb{Z}[y \times x \times y] \to \mathbb{Z}[y \times x] \to \mathbb{Z}[y] \]

(\$\mathbb{Z}[y]\$ placed in degree 0) in $\text{Ab}(k'_{\text{proflf}})$ with differential given by the usual formula for Čech cohomology. Give it the augmentation $C^\text{gen}(y/x) \to \mathbb{Z}[x]$ and denote the resulting complex by $C^\text{gen}_0(y/x)$ (so $\mathbb{Z}[x]$ is placed in degree 1). Then $C^\text{gen}_0(y/x)$ is acyclic.

We need a lemma. To simplify the notation, we write $y^n_x := y \times x \cdots \times y$, and $y^n_x \times x \cdots \times y$. Note that $y^n_x = y^n_x \times x$.

**Lemma 4.3.2.** Let $y/x$ as above. Let $X \in k_{\text{perf}}$, $Y = y \times_k X$ and $n \geq 0$.

1. Let $a \in y^n_x \text{gen}(X)$ and $Ua = (\text{id}, a) \in (y \times x y^n_x \text{gen})(Y)$ the natural extension of $a$. Then there exists $Z \in k^{\text{perf}}$ and a morphism $Z \to Y$ satisfying the two conditions of Lemma 3.6.3 such that the natural image $Ua \in (y \times_x y^n_x \text{gen})(Z)$ is contained in the subset $y^{n+1}_x \text{gen}(Z)$.

2. Let $t \in \mathbb{Z}[y^n_x \text{gen}(X)]$ and $Ut \in \mathbb{Z}[y \times x y^n_x \text{gen}(Y)]$ the natural extension of $t$. Then there exists $Z \in k^{\text{perf}}$ and a morphism $Z \to Y$ with the composite $Z \to Y \to X$ faithfully flat of profinite presentation such that the natural image $Ut \in \mathbb{Z}[y \times x y^n_x \text{gen}(Z)]$ is contained in the subset $\mathbb{Z}[y^{n+1}_x \text{gen}(Z)]$.

**Proof.** (1) Define $Z$ by the following cartesian diagram:

\[
\begin{array}{ccc}
Z & \to & Y \\
\downarrow & & \downarrow \\
\left(\begin{array}{c}
y^{n+1}_x \text{gen} \\
\end{array}\right) & \to & \left(\begin{array}{c}
y^n_x \text{gen} \\
\end{array}\right)
\end{array}
\]

Then we can check the requirements for $Z$ in the same way as the proof of Lemma 3.6.3.

(2) We can deduce this from (1) in the same way we deduced Lemma 3.6.4 from Lemma 3.6.3.

**Proof of Proposition 4.3.4.** We use the same notation as the lemma. Let $n \geq 0$ and $a \in y^n_x \text{gen}(X)$ (which in particular gives an element of $x(X)$, or a morphism $X \to x$). Write $a = (a_0, \ldots, a_{n-1}) \in y^n_x (X)$, with $a_i \in y(X)$. Define $Ua = (c, a_0, \ldots, a_{n-1}) \in y^{n+1}_x (Y)$, where $c$ corresponds to the identity map $y \to y$. Then we have $\partial Ua + U \partial a = a$ in $\mathbb{Z}[y^n_x \text{gen}(Z)]$, where $\partial$ is the boundary map for the Čech complex. By linearity, we have $\partial U t + U \partial t = t$ in $\mathbb{Z}[y^n_x \text{gen}(Z)]$ for any $t \in \mathbb{Z}[y^n_x \text{gen}(X)]$. Therefore if $t \in \mathbb{Z}[y^n_x \text{gen}(X)]$ is a cycle, it is a boundary of a chain in $\mathbb{Z}[y^{n+1}_x \text{gen}(Z)]$. Since $Z$ is faithfully flat of profinite presentation over $X$, this proves the result.

**Proposition 4.3.3.**

1. The identity functor defines a continuous map

\[ f : \text{Spec } k_{\text{proflf}}^{\text{perf}} \to \text{Spec } k^{\text{rat}}_0. \]

2. The functor $f_*$ is exact.
(3) We have $f_*f^* = \text{id}$.
(4) The functor $f^* : \text{Set}(\mathbb{k}_{\text{fl}}^{\text{rat}}) \to \text{Set}(\mathbb{k}_{\text{proppf}}^{\text{perf}})$ is fully faithful.
(5) The functor $f_* : \text{Ab}(k_{\text{proppf}}^{\text{perf}}) \to \text{Ab}(k_{\text{fl}}^{\text{rat}})$ sends acyclic sheaves (see the first paragraph of Section 2.7) to acyclic sheaves.
(6) We have

$$R\Gamma(\mathbb{k}_{\text{fl}}^{\text{rat}}, f_* A) = R\Gamma(X, f_* A) = R\Gamma(\mathbb{k}_{\text{proppf}}^{\text{perf}}, A)$$

for any $A \in \text{Ab}(k_{\text{proppf}}^{\text{perf}})$, $k' \in k_{\text{fl}}^{\text{rat}}$ and $X \in \text{DM}_k$ with $\xi_X = k'$.
(7) Let $g : \text{Spec} \mathbb{k}_{\text{fl}}^{\text{rat}} \to \text{Spec} \mathbb{k}_{\text{et}}^{\text{rat}}$ be the morphism defined by the identity and $B \in \text{Alg}/k$. Then we have $Rg_* B = B$.
(8) Theorem [7] also holds when replacing $\text{Spec} \mathbb{k}_{\text{et}}^{\text{rat}}$ with $\text{Spec} \mathbb{k}_{\text{fl}}^{\text{rat}}$ or $\text{DM}_k$.

Actually we need to extend $k_{\text{perf}}$ to allow infinite disjoint unions. Alternatively, we could define the site $\text{Spec} \mathbb{k}_{\text{fl}}^{\text{rat}}$ to be the category of finitely generated perfect fields over $k$ endowed with the topology whose covering sieves are all non-empty sieves. These modifications do not change the associated topoi, so we ignore this issue.

**Proof.** [1] We need to show that $f_*$ sends sheaves to sheaves. Let $F \in \text{Set}(k_{\text{proppf}}^{\text{perf}})$. Let $k''/k'$ be a field extension in $k_{\text{rat}}$ and $x = \text{Spec} k'$, $y = \text{Spec} k''$. By Proposition 4.2.1 it is enough to show that the sequence $F(x) \to F(y) \to F(y \times^x y)$ is exact. Let $s \in F(y)$ be such that $p_1(s) = p_2(s)$ in $F(y \times^x y)$, where $p_i$ is the $i$-th projection. To show $s \in F(x)$, it is enough to show that $p_1(s) = p_2(s)$ in $F(y \times^x y)$ since $F \in \text{Set}(k_{\text{proppf}}^{\text{perf}})$. Let $X = y \times^x y$, $Y = y \times y \times_x y$, $Z_1 = (y \times^x y) \times_x y \to Y$. Let $Z_2 \to Y$ be obtained from $Z_1 \to Y$ by flipping the last two factors $y \times^x y$ of $Y$. Let $Z = Z_1 \times_y Z_2$. Let $p_0, p_1, p_2 : Y \times^x y \times^x y \to y$ be the projections. In $F(Z)$, we have $p_0(s) = p_1(s)$. In $F(Z_2)$, we have $p_0(s) = p_2(s)$. Hence in $F(Z)$, we have $p_1(s) = p_0(s) = p_2(s)$. Hence it is enough to show that $Z/X$ is faithfully flat of finite presentation. We check the two conditions of Lemma 3.6.2 for $Z_1$ and $Z_2$. It is enough to check them for $Z_1$ only. The flatness of $Z_1 \to Y$ is obvious. Take a point $x_0$ of $X = y \times^x y$. Then we have a commutative diagram

$$\begin{array}{ccc}
  y \times y \times x_0 & \longrightarrow & y \times x_0 = Y_{x_0} \\
  \downarrow & & \downarrow \\
  (y \times^{x_0} y) \times_x y = Z_1 & \longrightarrow & y \times y \times_x y = Y.
\end{array}$$

Hence the fiber $(Z_1)_{x_0}$ contains the generic point of $Y_{x_0}$. Hence the morphism $(Z_1)_{x_0} \to Y_{x_0}$ is dominant. This completes the proof of [1].

[2] It is enough to show that for any field $k' \in k_{\text{rat}}$ and a covering $X \to x$ in $\text{Spec} k_{\text{proppf}}^{\text{perf}}$ with $x = \text{Spec} k'$, there exists a covering $y \to x$ in $\text{Spec} k_{\text{fl}}^{\text{rat}}$ and an $x$-morphism $y \to X$. We can take such a $y$ to be any point of $X \neq \emptyset$.

[3] The argument done right above shows that $f_*$ commutes with sheafification. Let $f^{-1}$ be the pullback for presheaves of sets. Clearly $f_* f^{-1} = \text{id}$. Combining these two facts, we have $f_* f^* = f_* f^{-1} = \text{id}$.

[4] This follows from [2].

[5] First note that the complex of the form [15] in Proposition 3.5.1 remains isomorphic to $\mathbb{Z}[x]$ even if it is considered in $\text{Ab}(k_{\text{fl}}^{\text{rat}})$ by [2] and [3]. We denote this complex in $\text{Ab}(k_{\text{fl}}^{\text{rat}})$ by the same symbol $\mathbb{C}[\mathbb{Z}][y/x]$. 


Let $x = \text{Spec } k'$ and $I \in \text{Ab}(k'_{\text{prof}})$ acyclic. We want to show that $H^i(x_{\text{fl}}^\text{rat}, f_*I) = 0$ for $j \geq 1$. What we just saw above gives a spectral sequence

$$E_2^{ij} = \lim_{y/x} H^i H^j((y_{x_{\text{gen}}}^\text{rat})_{\text{fl}}, f_*I) \Rightarrow H^{i+j}(x_{\text{fl}}^\text{rat}, f_*I)$$

where $y = \text{Spec } k''$ and $k''$ runs over all field extensions of $k'$, and the $H^i$ above is the $i$-th cohomology of the complex $\text{Ext}^j_{k'_{\text{fl}}}(\tilde{C}_{\text{gen}}(y/x), f_*I)$ of term-wise Ext groups. We have $E_2^{0j} = 0$ for $j > 0$ since it is the $\lim_{y/x}$ of the kernel of $H^j(y_{\text{fl}}^\text{rat}, f_*I) \rightarrow H^j((y \times y_{\text{gen}}^\text{rat})_{\text{fl}}, f_*I)$.

For $E_2^{i0}$ for $i > 0$, note that it is the $i$-th cohomology of the complex $\text{Hom}_{k'_{\text{fl}}}(\tilde{C}_{\text{gen}}(y/x), f_*I) = \text{Hom}_{k_{\text{perf}}}((\tilde{C}_{\text{gen}}(y/x), I)$.

Since $I$ is an acyclic sheaf, we have

$$\text{Ext}^i_{k'_{\text{prof}}}(Z[x], I) = H^i(\widetilde{\mathcal{O}}_{k'_{\text{prof}}}, I) = 0$$

for any $z \in k_{\text{rat}}$ and $i > 0$. Hence the isomorphism $\tilde{C}_{\text{gen}}(y/x) = Z[z]$ (Proposition 1.3.1) implies the acyclicity of $\text{Hom}_{k_{\text{perf}}}(\tilde{C}_{\text{gen}}(y/x), I)$ in positive degrees, so $E_2^{i0} = 0$ for $i > 0$. Hence the above spectral sequence implies the result by induction on $j$.

(b) The property (5) is precisely what we need to apply the Grothendieck spectral sequence for $R^j\Gamma$ (that is, the Leray spectral sequence). With (2), the result follows.

(7) This follows from (6) and Corollary 3.3.3.

(8) This follows from (7).

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