Degenerate 0-Schur Algebras and Nil-Temperley-Lieb Algebras

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Abstract

In Jensen and Su (J. Pure Appl. Algebra 219(2), 277–307 2014) constructed 0-Schur algebras, using double flag varieties. The construction leads to a presentation of 0-Schur algebras using quivers with relations and the quiver presentation naturally gives rise to a new class of algebras, which are introduced and studied in this paper. That is, these algebras are defined on the quivers of 0-Schur algebras with relations modified from the defining relations of 0-Schur algebras by a tuple of parameters $t$. In particular, when all the entries of $t$ are 1, we recover 0-Schur algebras. When all the entries of $t$ are zero, we obtain a class of basic algebras, which we call the degenerate 0-Schur algebras. We prove that the degenerate algebras are both associated graded algebras and quotients of 0-Schur algebras. Moreover, we give a geometric interpretation of the degenerate algebras using double flag varieties, in the same spirit as Jensen and Su (J. Pure Appl. Algebra 219(2), 277–307 2014), and show how the centralizer algebras are related to nil-Hecke and nil-Temperley-Lieb algebras.

Keywords 0-Schur algebras · Quivers · Nil-Hecke algebras · Nil-Temperley-Lieb algebras · Double flag varieties

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1 Introduction

It is well known that the classical Schur algebras are specialisations of $q$-Schur algebras (see [5] and [6]) at $q = 1$. Analogously, 0-Schur algebras are specialisations of $q$-Schur algebras at $q = 0$. The 0-Schur algebras have been studied by Donkin [6, §2.2] in terms of 0-Hecke algebras of symmetric groups, by Krob and Thibon [13] in connection with noncommutative symmetric functions, and by Deng and Yang on their presentations and representation types [3, 4].

A new approach towards 0-Schur algebras was investigated by Jensen and Su [11], by considering their monoid structure. Inspired by Beilinson, Lusztig and MacPherson's geometric construction of $q$-Schur algebras [1] and Reineke's work on a monoid structure of Hall algebras [16], Su defined a generic multiplication in the positive part of $q$-Schur algebras [19]. The generic multiplication was then generalised by Jensen and Su [11] to give a global geometric construction of 0-Schur algebras. This geometric construction produces a monoid structure, simplifies the multiplication and provides a new approach to studying the structure of 0-Schur algebras. In [12] we gave a construction of indecomposable projective modules and studied homomorphism spaces between projective modules.

The nature of 0-Schur algebras exposed in [11] leads to several interesting related algebras. First of all, we can modify the generating relations of 0-Schur algebras, relying on multiple parameters $t$. In particular, when all the parameters are 1, we recover 0-Schur algebras and when all parameters are 0, we obtain a class of basic algebras. We prove that similarly to 0-Schur algebras, these newly defined algebras have reduced paths as basis (see [12]), which are in one-to-one correspondence with $\text{GL}(V)$-orbits in double flag varieties or certain sets of integral matrices. So they have the same dimension as the corresponding 0-Schur algebras and they are proved to be degenerations of 0-Schur algebras. We further show that they are isomorphic to quotients of 0-Schur algebras by naturally defined ideals and to the associated graded algebras of 0-Schur algebras, which notably have a natural structure of filtered algebras. We also investigate relations between their centralizer algebras, Nil-Hecke and Nil-Temperley-Lieb algebras.

The remainder of this paper is organised as follows. In Section 2, we give a brief background on $q$-Schur and 0-Schur algebras and discuss how to consider 0-Schur algebras as filtered algebras. In Section 3, we prove some preliminary results on a family of idempotent ideals. In Section 4, we construct a class of algebras $D_t(n, r)$ using quivers and modified relations of 0-Schur algebras and prove an isomorphism theorem between different $D_t(n, r)$. In Section 5, we first construct the associated graded algebras $DS_0(n, r)$ of 0-Schur algebras and give them a geometric interpretation. We then show that $D_t(n, r)$ for $t = 0$ is a degenerate 0-Schur algebra and prove our main result that the three algebras, $DS_0(n, r)$, $D_0(n, r)$ and the quotient of $S_0(n, n + r)$ modulo a natural idempotent ideal are isomorphic. In Section 6, we discuss relations between centralizer algebras of the degenerate algebras, nil-Hecke and nil-Temperley-Lieb algebras.

2 Background on the Algebra $S_0(n, r)$

2.1 The Algebra $S_q(n, r)$

We first recall the construction of quantised Schur algebras given by Beilinson-Lusztig-MacPherson [1]. Let $k$ be a field and let $V$ be a $k$-vector space of dimension $r$. Let $\mathcal{F} = \mathcal{F}(V)$ be the set of $n$-step flags

$$0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V.$$
The natural action of $\text{GL}(V)$ on the vector space $V$ induces a diagonal action of $\text{GL}(V)$ on $\mathcal{F} \times \mathcal{F}$ defined by

$$g(f, f') = (gf, gf'),$$

where $g \in \text{GL}(V)$ and $f, f' \in \mathcal{F}$. Denote the orbit of $(f, f')$ by $[f, f']$.

Let $\Xi(n, r)$ be the set of $n \times n$-matrices $A = (a_{ij})_{i,j}$ with $a_{ij}$ nonnegative integers and $\sum_{1 \leq i,j \leq n} a_{ij} = r$. Then there is a bijection from $\mathcal{F} \times \mathcal{F}/\text{GL}(V)$ to $\Xi(n, r)$ sending the orbit of $(f, f')$ to $A = (a_{ij})_{i,j}$ with

$$a_{ij} = \dim \frac{V_i \cap V_j'}{V_{i-1} \cap V_j' + V_i \cap V_{j-1}'}$$

for $1 \leq i, j \leq n$, (1)

where $f = (0 = V_0 \subseteq V_1 \subseteq V_2 \subseteq \cdots \subseteq V_n = V)$ and $f' = (0 = V'_0 \subseteq V'_1 \subseteq V'_2 \subseteq \cdots \subseteq V'_n = V)$. Denote by $e_A$ the orbit in $\mathcal{F} \times \mathcal{F}$ corresponding to a matrix $A$. Consider the diagram

$$\begin{array}{ccc}
\mathcal{F} \times \mathcal{F} \times \mathcal{F} & \xrightarrow{\Delta} & (\mathcal{F} \times \mathcal{F}) \times (\mathcal{F} \times \mathcal{F}) \\
\pi \downarrow & & \downarrow (\pi \times \pi) \\
\mathcal{F} \times \mathcal{F} & & \\
\end{array}$$

where $\pi(f, g, h) = (f, h)$ and $\Delta(f, g, h) = ((f, g), (g, h))$.

Following [1, Prop. 1.2], for any given $A, B, C \in \Xi(n, r)$, there is a polynomial $g_{A,B,C}(q) \in \mathbb{Z}[q]$ such that for all finite fields $k$,

$$g_{A,B,C}(k) = \frac{|\pi^{-1}(e_C) \cap \Delta^{-1}(e_A \times e_B)|}{|e_C|},$$

where $|X|$ denotes the cardinality of a set $X$.

Following a remark by Du [7], the $q$-Schur algebra studied by Dipper and James in [5] (see also [6]) can now be defined as follows.

**Definition 2.1** [1] The quantised Schur algebra $S_q(n, r)$ is the free $\mathbb{Z}[q]$-module with basis $\{e_A \mid A \in \Xi(n, r)\}$ and multiplication given by

$$e_A \cdot e_B = \sum_{C \in \Xi(n, r)} g_{A,B,C}(q)e_C,$$

for all $A, B \in \Xi(n, r)$.

For an $n \times n$-matrix $A = (a_{ij})$, define the row and column vectors of $A$ by

$$\text{ro}(A) = (\sum_{j=1}^{n} a_{1j}, \ldots, \sum_{j=1}^{n} a_{nj}) \quad \text{and} \quad \text{co}(A) = (\sum_{i=1}^{n} a_{i1}, \ldots, \sum_{i=1}^{n} a_{in}).$$

Note that if $g_{A,B,C}(q) \neq 0$, then

$$\text{ro}(A) = \text{ro}(C), \quad \text{co}(A) = \text{co}(B) \quad \text{and} \quad \text{co}(B) = \text{co}(C).$$

**2.2 Definition of $S_0(n, r)$**

In [11], Jensen and Su defined a generic multiplication of orbits in $\mathcal{F} \times \mathcal{F} = \mathcal{F}(V) \times \mathcal{F}(V)$ by

$$e_A \cdot e_B = \begin{cases} e_C & \text{if } \Delta^{-1}(e_A \times e_B) \neq \emptyset, \\ 0 & \text{otherwise}, \end{cases}$$

(3)
where $V$ is defined over an algebraically closed field $k$ and $e_C$ is the unique open orbit in $\pi^{-1}(e_A \times e_B)$. This defines an associative $\mathbb{Z}$-algebra $G(n, r)$ with basis $\mathbb{Z}(n, r)$ and in fact $G(n, r)$ is isomorphic to the 0-Schur algebra (Theorem 7.2.1 in [11]).

$$S_0(n, r) = S_q(n, r) \otimes_{\mathbb{Z}[q]} \mathbb{Z},$$

where $\mathbb{Z}$ is viewed as the $\mathbb{Z}[q]$-module $\mathbb{Z}[q]/(q)$. As the multiplication in $G(n, r)$ is much simplified (e.g. the multiplication of two orbits is either 0 or again an orbit), in the rest of this paper, we will take $G(n, r)$ as the 0-Schur algebra $S_0(n, r)$.

Remark 2.2 We will see below (Lemma 2.3, 2.4 and 2.7) that the multiplication can be described combinatorially, independent of the choice of $k$.

Let $\Lambda_1(n, r)$ be the set of compositions of $r$ into $n$ parts. For each $\lambda = (\lambda_1, \ldots, \lambda_n) \in \Lambda_1(n, r)$, let $\text{diag}(\lambda)$ denote the diagonal matrix $\text{diag}(\lambda_1, \ldots, \lambda_n)$ and write $k_\lambda = e_{\text{diag}(\lambda)}$.

By definition, for each $A \in \Xi_1(n, r)$,

$$k_\lambda \cdot e_A = \begin{cases} e_A, & \text{if } \lambda = \text{ro}(A); \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad e_A \cdot k_\lambda = \begin{cases} e_A, & \text{if } \lambda = \text{co}(A); \\ 0, & \text{otherwise}. \end{cases}$$

Thus, $\sum_{\lambda \in \Lambda_1(n, r)} k_\lambda$ is the identity of $S_0(n, r)$.

Let $E_{ij}$ be the elementary $n \times n$ matrix with a single nonzero entry 1 at $(i, j)$. Denote by $e_{i, \lambda}$ (resp. $f_{j, \lambda}$) the basis element of $S_0(n, r)$ corresponding to the matrix that has column vector $\lambda$ and the only nonzero off diagonal entry is 1 at $(i, i+1)$ (resp. $(j+1, j)$).

### 2.3 The Fundamental Multiplication Rules

Note that $S_0(n, r)$ is generated by $e_{i, \lambda}$, $f_{i, \lambda}$ and $k_\lambda$, where $1 \leq i \leq n - 1$ and $\lambda \in \Lambda(n, r)$ (see Lemma 6.9 in [11]). Let

$$e_i = \sum_{\lambda \in \Lambda(n, r)} e_{i, \lambda} \quad \text{and} \quad f_i = \sum_{\lambda \in \Lambda(n, r)} f_{i, \lambda}.$$ 

Note that for any given orbit $e_A$, by the definition of the multiplication,

$$e_i e_A = e_{i, \text{co}(A)} e_A.$$

This says that only one term remains in the product and the same holds for $e_A e_i$, $e_A f_i$ and $f_i e_A$. The following are the fundamental multiplication rules in $S_0(n, r)$, which describe the action of generators on basis elements.

**Lemma 2.3** [11, Lemma 6.11] Let $e_A \in S_0(n, r)$ with $\text{ro}(A) = \lambda$.

1. If $\lambda_i + 1 > 0$, then $e_i e_A = e_X$, where $X = A + E_{i,p} - E_{i+1,p}$ with $p = \max\{j \mid a_{i+1,j} > 0\}$.
2. If $\lambda_i > 0$, then $f_i e_A = e_Y$, where $Y = A - E_{i,p} + E_{i+1,p}$ with $p = \min\{j \mid a_{i,j} > 0\}$.

Symmetrically, there are the following formulas.

**Lemma 2.4** [12, Lemma 2.2] Let $e_A \in S_0(n, r)$ with $\text{co}(A) = \mu$.

1. If $\mu_{i+1} > 0$, then $e_A f_i = e_X$, where $X = A + E_{p,i} - E_{p,i+1}$ with $p = \max\{j \mid a_{j,i+1} > 0\}$.
2. If $\mu_{i} > 0$, then $e_A e_i = e_Y$, where $Y = A - E_{p,i} + E_{p,i+1}$ with $p = \min\{j \mid a_{j,i} > 0\}$.
2.4 Presenting $S_0(n,r)$ by Quiver with Relations

Let

$$\alpha_i = (0, \ldots, 0, 1, -1, 0, \ldots, 0) \in \mathbb{Z}^n,$$

where the only nonzero entries 1 and $-1$ are at the $i$th and $(i+1)$th positions, respectively. We define a quiver $\Sigma(n,r)$ with vertices corresponding to $\lambda \in \Lambda(n,r)$ and arrows

$$k_{\lambda + \alpha_i} \xrightarrow{f_{i,\lambda}} k_{\lambda},$$

and let $J$ be the ideal of the path algebra $\mathbb{Z}[\Sigma(n,r)]$ generated by the binomial relations

$$P_{ij,\lambda} = k_\mu P_{ij} k_\lambda,$$

$$N_{ij,\lambda} = k_\mu N_{ij} k_\lambda,$$

$$C_{ij,\lambda} = k_{\lambda + \alpha_i - \alpha_j} C_{ij} k_\lambda,$$

where

$$P_{ij} = \begin{cases} e_i^2 e_j - e_i e_j e_i & \text{for } i = j - 1, \\ -e_i e_j e_i + e_j e_i^2 & \text{for } i = j + 1, \end{cases}$$

and $\mu = \begin{cases} \lambda + 2\alpha_i + \alpha_j & \text{if } i = j \pm 1 \\ \lambda + \alpha_i + \alpha_j & \text{otherwise}; \end{cases}$

$$N_{ij} = \begin{cases} -f_i f_j f_i + f_j f_i^2 & \text{for } i = j - 1, \\ f_i^2 f_j - f_i f_j f_i & \text{for } i = j + 1, \end{cases}$$

and $\mu = \begin{cases} \lambda - 2\alpha_i - \alpha_j & \text{if } i = j \pm 1 \\ \lambda - \alpha_i - \alpha_j & \text{otherwise}; \end{cases}$

and

$$C_{ij} = e_i f_j - f_j e_i - \delta_{ij} \left( \sum_{\mu: \mu_{i+1} = 0} k_\mu - \sum_{\mu: \mu_i = 0} k_\mu \right).$$

Note that the ideal $J$ is not admissible, as the relations $C_{ij}$ involve idempotents.

The quotient algebra $\mathbb{Z}[\Sigma(n,r)]/J$ is isomorphic to $S_0(n,r)$, by an isomorphism that maps the arrows $e_{i,\lambda}, f_{i,\lambda}$ and vertices $k_\lambda$ to the corresponding elements in $S_0(n,r)$ (Theorem 7.1.2 in [11]). The relations $P_{ij,\lambda}, N_{ij,\lambda}$ are called the Serre relations of $S_0(n,r)$. The relations $C_{ij,\lambda}$ are commutative relations if $i \neq j$ or $\lambda_i \lambda_{i+1} = 0$, idempotent relations if exactly one of $\lambda_i, \lambda_{i+1}$ is zero, and empty relations otherwise.

2.5 Bases of Reduced Paths and $S_0(n,r)$ as a Filtered Algebra

We say that a path in $\Sigma(n,r)$ is reduced if it is not equal to a path of shorter length in $\Sigma(n,r)$ modulo $J$. Equivalence classes of reduced paths form a multiplicative basis for $S_0(n,r)$, which coincides with the basis $e_A, A \in \Sigma(n,r)$. There are in general many paths equal to $e_A$, modulo $J$, but by the fundamental multiplication rules and the presentation of $S_0(n,r)$, the numbers of occurrences of $e_i$ and $f_i$ in any reduced path are

$$E_i(e_A) = \sum_{l \leq i < m} a_{l,m}$$

and $F_i(e_A) = \sum_{m \leq i < l} a_{l,m},$ respectively.

**Lemma 2.5** For any $1 \leq i \leq n - 1$, we have

$$E_i(e_A) + E_i(e_B) - E_i(e_A \cdot e_B) = F_i(e_A) + F_i(e_B) - F_i(e_A \cdot e_B) \geq 0.$$
Proof The generating relations $P_{ij}$ and $N_{ij}$ do not change the length of a path. The length of a path either remains unchanged or decreases by 2, when we apply the relation $C_{ii,\lambda}$. It decreases if and only if $\lambda_i = 0$ or $\lambda_{i+1} = 0$ (but not both), as in these cases the relation $C_{ii,\lambda}$ becomes

$$k_\lambda f_i e_i k_\lambda = k_\lambda$$

if $\lambda_i = 0$ and $\lambda_{i+1} \neq 0$;

or

$$k_\lambda e_i f_i k_\lambda = k_\lambda$$

if $\lambda_{i+1} = 0$ and $\lambda_i \neq 0$.

In this situation, the numbers of the occurrences of $e_i$ and $f_i$ both decrease by 1. Therefore

$$E_i(e_A) + E_i(e_B) - E_i(e_A \cdot e_B) = F_i(e_A) + F_i(e_B) - F_i(e_A \cdot e_B) \geq 0,$$

as stated.

We define vectors $E(e_A)$ and $F(e_A)$ by

$$E(e_A) = (E_1(e_A), E_2(e_A), \ldots, E_{n-1}(e_A))$$

and

$$F(e_A) = (F_1(e_A), F_2(e_A), \ldots, F_{n-1}(e_A)).$$

Comparing tuples $(E(e_A), F(e_A))$ component-wise gives the following.

Corollary 2.6 $S_0(n, r)$ is a filtered algebra with degree function $e_A \mapsto (E(e_A), F(e_A))$.

For $j \geq i$, let

$$e(i, j) = e_i \cdot e_{i+1} \cdot \ldots \cdot e_j$$

and

$$f(j, i) = f_j \cdot \ldots \cdot f_{i+1} \cdot f_i.$$

We give two explicit descriptions of reduced paths equal to a basis element $e_A$, analogous to the monomial and PBW bases of $S_q(n, r)$, respectively.

Lemma 2.7 Let $e_A \in S_0(n, r)$ with $A = (a_{i,j})_{i,j}$. Then we can write $e_A$ as reduced paths as follows.

1. $e_A = (\prod_{s=n-1}^1 \prod_{l=1}^s e_l \sum_{1 \leq p \leq f} a_{p,s+1}) \cdot (\prod_{s=1}^{n-1} \prod_{l=n-1}^s f_l \sum_{1 \leq p \leq n} a_{p,s}) \cdot k_{co(A)}$;

2. $e_A = (\prod_{j=n}^1 \prod_{i=j-1}^1 e(i, j-1)^{a_{i,j}}) \cdot (\prod_{j=1}^{n-1} \prod_{i=n}^{j+1} f(i-1, j)^{a_{i,j}}) \cdot k_{co(A)}$.

Proof The formulas follow by repeatedly applying the fundamental multiplication rules in Lemma 2.3. In both (1) and (2), the numbers of occurrences of $e_i$ and $f_i$ are $E_i(e_A)$ and $F_i(e_A)$. So the paths are reduced.

We explain by an example on how to achieve the above formulas.

Example 2.8 Let $A = (a_{i,j})$, where

$$A = \begin{pmatrix}
0 & 1 & 2 \\
3 & 0 & 4 \\
5 & 6 & 0
\end{pmatrix}$$

and $B = \begin{pmatrix}
8 & 0 & 0 \\
0 & 7 & 0 \\
0 & 0 & 6
\end{pmatrix}$. Note that $e_B = k_{co(A)}$. Using the fundamental multiplication rules, we have the following.

$$(1) \quad B = \begin{pmatrix}
8 & 0 & 0 \\
0 & 7 & 0 \\
0 & 0 & 6
\end{pmatrix} = f_2^6 \leftarrow f_2^5 \leftarrow f_2^4 \leftarrow f_2^3 \leftarrow f_2^2 \leftarrow f_2^1 \leftarrow \begin{pmatrix}
0 & 0 & 0 \\
3 & 1 & 0 \\
5 & 6 & 6
\end{pmatrix} = e_1 \leftarrow e_0$$

$$(2) \quad e_1 \leftarrow e_0 \leftarrow e_1 \leftarrow e_0 \leftarrow e_1 \leftarrow \begin{pmatrix}
0 & 1 & 2 \\
3 & 0 & 4 \\
5 & 6 & 0
\end{pmatrix} = A.$$
That is,
\[ e_A = e_1^2 \cdot e_2^6 \cdot e_1 \cdot f_2^5 \cdot f_1^8 \cdot f_2^6 \cdot e_B \]
\[ = e_1^{a_{13}} \cdot e_2^{(a_{13} + a_{23})} \cdot e_1^{a_{12}} \cdot f_2^{a_{31}} \cdot f_1^{(a_{21} + a_{31})} \cdot f_2^{a_{32}} \cdot e_B. \]

(2) \[ B = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{f_2^5} \begin{pmatrix} 8 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \xrightarrow{(f_2 f_1)^5} \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 6 & 6 \end{pmatrix} \xrightarrow{f_1^3} \begin{pmatrix} 0 & 1 & 2 \\ 0 & 1 & 2 \\ 5 & 6 & 4 \end{pmatrix} \xrightarrow{(e_1 e_2)^2} \begin{pmatrix} 0 & 1 & 2 \\ 3 & 0 & 4 \\ 5 & 6 & 0 \end{pmatrix} = A. \]

That is,
\[ e_A = e_2^4 \cdot (e_1 e_2)^2 \cdot e_1 \cdot f_1^3 \cdot (f_2 f_1)^5 \cdot f_2^6 \cdot e_B \]
\[ = e(2, 2)^4 \cdot e(1, 2)^2 \cdot e(1, 1) \cdot f(1, 1)^3 \cdot f(2, 2)^6 \cdot e_B \]
\[ = e(2, 2)^{a_{23}} \cdot e(1, 2)^{a_{13}} \cdot e(1, 1)^{a_{12}} \cdot f(2, 2)^{a_{21}} \cdot f(2, 1)^{a_{31}} \cdot f(2, 2)^{a_{32}} \cdot e_B. \]

In both cases, the lower triangular parts/upper triangular parts are created column-wise. The main differences are, for instance in the lower triangular parts, in (2) each step creates an entry \( a_{ij} \) at a time, starting from the lowest entry and then moving upwards, while in (1) it goes downwards and in each step we apply the maximal times of \( f_i \) so that afterwards we have an exactly right entry in row \( i \).

## 3 Idempotents and Idempotent Ideals of \( S_0(n, r) \)

The set of compositions \( \Lambda(n, r) \) (i.e. the vertices in the quiver \( \Sigma(n, r) \)) can be drawn on an \((n - 1)\)-simplex, where compositions with zero entries lie on the boundary. We call an idempotent \( k_\lambda \) boundary if \( \lambda \) lies on the boundary of the simplex, and interior otherwise. In this section, we are interested in the ideal generated by all boundary idempotents. We obtain a dimension formula for the quotient algebra, which turns out to be useful when we consider degenerations of 0-Schur algebras in Section 5.

Let \( I_i(n, r) \) be the ideal generated by the idempotents \( k_\lambda \) with the number of nonzero entries in \( \lambda \) less than or equal to \( i \). That is, \( I_i(n, r) \) is generated by idempotents corresponding to \((i - 1)\)-faces of the simplex. There are in general several \((i - 1)\)-faces in \( \Sigma(n, r) \), but as the following lemma shows, any one of them contains enough idempotents to generate the whole of \( I_i(n, r) \).

**Lemma 3.1** The ideal \( I_i(n, r) \) is generated by all idempotents \( k_\lambda \) lying in any chosen \((i - 1)\)-face in \( \Sigma(n, r) \).

**Proof** For any \( \lambda = (\ldots, \lambda_{i-1}, 0, \lambda_{i+1}, \ldots) \in \Lambda(n, r) \),
\[ k_\lambda = f_i^{\lambda_{i+1}} k_\mu e_i^{\lambda_{i+1}}, \]
where \( \mu = (\ldots, \lambda_{i-1}, \lambda_{i+1}, 0, \ldots) \) with the other entries equal to those of \( \lambda \). Therefore \( k_\lambda \) is contained in the ideal generated by \( k_\mu \). Similarly, \( k_\mu \) is contained in the ideal generated by \( k_\lambda \), and so these two ideals are equal. This shows that \( I_i(n, r) \) is generated by all idempotents \( k_\lambda \) lying in any chosen \((i - 1)\)-face in \( \Sigma(n, r) \). \( \square \)
Denote by $I(n, r)$ the ideal of $S_0(n, r)$ generated by all the boundary idempotents. If $r \geq n$, then $I(n, r) = I_{n-1}(n, r)$. If $r \lt n$, then $I(n, r) = S_0(n, r)$.

**Lemma 3.2** The ideal $I(n, r)$ has a basis consisting of $e_A$, where $A \in \mathcal{Z}(n, r)$ is a matrix with at least one diagonal entry equal to 0.

**Proof** Denote by $S$ the subspace of $S_0(n, r)$ spanned by $e_A$ with at least one diagonal entry of $A$ equal to 0. We first show that $S$ is an ideal. It is enough to prove that for any generator $x$ and any $eD \in S$, both $xeD$ and $eDx$ are contained in $S$. We only prove that $xeD \in S$ for $x = e_j$ and $D = (d_{ij})_{ij}$ with $d_{ii} = 0$, as the other case can be done similarly. Let $e_X = e_j e_D$ and $X = (x_{ij})_{ij}$. By Lemma 2.3, multiplying $e_j$ with $e_D$ from the left only affects two entries in the $(j+1)$th-row and $j$th-row in $D$, respectively. We have either

$$x_{ii} = d_{ii} = 0 \text{ or } x_{ii} = 1.$$  

The latter case occurs only if $d_{i+1,i+1} = 0$ (and $j = i + 1$), but then

$$x_{i+1,j+1} = 0$$  

as well. In either case, $xeD = e_X \in S$, as required. So $S$ is an ideal in $S_0(n, r)$.

As $I(n, r)$ is generated by idempotents $k_\lambda$ with $\lambda_i = 0$ for some $i$, we have $k_\lambda \in S$ and therefore $I(n, r) \subseteq S$, since $S$ is an ideal.

It remains to show that $S \subseteq I(n, r)$. Suppose that $e_A \in S$ has the diagonal entry $a_{ii} = 0$, for some $1 \leq i \leq n$. We will show that $e_A \in I(n, r)$. Let $B$ be the matrix with

$$b_{ss} = \begin{cases} \sum_{s \leq l \leq i} a_{ls} & \text{if } s < i, \\ 0 & \text{if } s = i, \\ \sum_{i \leq l \leq s} a_{ls} & \text{if } s > i \end{cases}$$

on the diagonal and

$$b_{st} = \begin{cases} 0 & \text{if } i \geq s > t, \\ 0 & \text{if } i \leq s < t, \\ a_{st} & \text{otherwise} \end{cases}$$

off the diagonal. The $i$’th row in $B$ is zero, and so we have $e_B = k_\lambda e_B \in I(n, r)$, where $\lambda = \text{ro}(B)$. To complete the proof we will construct elements $x$ and $y$ such that $xye_B = e_A$, which implies that $e_A \in I(n, r)$. In fact, $y$ is a product of generators $e_j$ for $j > i$ and $x$ is a product of $f_j$ for $j \leq i$. Explicitly, for instance, we have

$$y = \prod_{s=n-1}^{i} \prod_{l=i}^{s} e_i^{\sum_{p \leq l} a_{p,s+l+1}} = (e_i^{a_{i,n}}_i e_i^{a_{i,n+1}}_i \cdots e_n^{a_{n-1,n}}_n \cdots e_i^{a_{i,i+1}}_i)$$

and a similar formula for $x$. Multiplying $e_B$ with $x$ and $y$ produces the correct entries at $(s, t)$ in the two zero regions $i \geq s > t$ and $i \leq s < t$ from the diagonal entries in $B$. So

$$xye_B = e_A,$$

as required. \qed

We give an example to illustrate the construction in the proof above.
Example 3.3 Let \( A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} \), where the diagonal entry \( a_{33} = 0 \). Then \( B = \begin{pmatrix} 3 & 1 & 2 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 \end{pmatrix} \), \( x = f_2^2 f_1^2 f_2 \) and \( y = e_3 \). We have \( e_A = x y e_B \).

Lemma 3.2 shows that \( I(n,r) \) is a summand of \( S_0(n,r) \) as \( \mathbb{Z} \)-modules, and so \( S_0(n,r)/I(n,r) \) is a free \( \mathbb{Z} \)-module. We have the following formulas which will be used Section 5.

Corollary 3.4

1. \( \text{rank } I(n,r) = \sum_{s=1}^{n} \binom{n}{s} \left( \frac{n^2 + r - n - 1}{r + s - n} \right). \)

2. \( \text{rank } S_0(n,r)/I(n,r) = \left( \frac{n^2 + r - n - 1}{r - n} \right). \)

Note that in part (1), each term in the sum counts the matrices with exactly \( s \) zero diagonal entries.

4 Modified Algebras of \( S_0(n,r) \)

In the remainder of this paper, we will work with algebras defined over a field \( F \). Note that the 0-Schur algebra \( S_0(n,r) \) is a free \( \mathbb{Z} \)-module. When we extend the ground ring \( \mathbb{Z} \) to a field \( F \),

\[ S_0(n,r) \otimes_{\mathbb{Z}} F, \]

the results in Sections 2 and 3 still hold, where replace rank by dimension in Corollary 3.4. By abuse of notation, we continue to denote the \( F \)-algebra by \( S_0(n,r) \).

Modifying the generating relations in Section 2.4, we construct a family of modified algebras \( D_t(n,r) \) of \( S_0(n,r) \) and prove when two modified algebras are isomorphic. Further properties will be discussed in the next section, in relation to degenerations of 0-Schur algebras and their geometric interpretation.

Let \( B(n,r) \subseteq \Lambda(n,r) \) be the set of elements corresponding to boundary idempotents. That is, this is the set of \( \lambda \) such that there is some \( \lambda_i = 0 \). Let

\[ t = (t_i)_{1 \leq i \leq n-1} \]

with each entry of \( t \) in \( F \). Denote by \( J(t) \) the ideal of \( F \Sigma(n,r) \) generated by \( P_{ij,\lambda} \), \( N_{ij,\lambda} \) and \( C_{ij,\lambda}(t) \), where \( P_{ij,\lambda} \) and \( N_{ij,\lambda} \) are defined as in Section 2.4 and

\[ C_{ij,\lambda}(t) = k_{\lambda+\alpha_i-\alpha_j} C_{ij}(t) k_{\lambda}, \tag{5} \]

where

\[ C_{ij}(t) = e_i f_j - f_j e_i - \delta_{ij} \cdot t_i \cdot \left( \sum_{\mu : \mu_i+1 = 0} k_{\mu} - \sum_{\mu : \mu_i = 0} k_{\mu} \right). \]

Note that if \( \lambda_i = \lambda_{i+1} = 0 \), then \( C_{ij,\lambda}(t) \) is an empty relation.

Definition 4.1 Let \( D_t(n,r) := F \Sigma(n,r)/J(t) \).
Remark 4.2 Note that for $i \neq j$, $C_{ij}(t) = e_i f_j - f_j e_i$, which gives the following commutative diagram in $D_\omega(n, r)$

$$
\begin{array}{cccc}
  k_\lambda & \xrightarrow{e_i} & k_\lambda + \alpha_i \\
  f_j & \downarrow & \downarrow f_j \\
  k_\lambda - \alpha_j & \xleftarrow{e_i} & k_\lambda + \alpha_i - \alpha_j.
\end{array}
$$

When $i = j$ and either $\lambda_i \neq 0$ or $\lambda_{i+1} \neq 0$, we have the following three cases

$$
C_{ii,\lambda}(t) = \begin{cases} 
(e_i f_i - f_i e_i)k_\lambda, & \text{if } \lambda_i \lambda_{i+1} \neq 0; \\
(e_i f_i - t_i k_\lambda)k_\lambda, & \text{if } \lambda_{i+1} = 0; \\
(t_i k_\lambda - f_i e_i)k_\lambda, & \text{if } \lambda_i = 0.
\end{cases}
$$

For the second and the third case, $\lambda$ is boundary and the relations are not admissible if $t_i \neq 0$.

When all the $t_i = 1$, we recover the 0-Schur algebra $S_0(n, r)$. When all the $t_i = 0$, $D_\omega(n, r)$ is a basic algebra, which we call the degenerate 0-Schur algebra.

Define the map

$$
\Phi_{t, s} : D_\omega(n, r) \longrightarrow D_\omega(n, r)
$$
given by

$$(k_\lambda \mapsto k_\lambda, f_{i,\lambda} \mapsto f_{i,\lambda}, e_{i,\lambda} \mapsto \frac{t_i}{s_i} e_{i,\lambda}),$$

if $t_i s_i \neq 0$ and $e_{i,\lambda} \mapsto e_{i,\lambda}$ otherwise.

For $\underline{a} = (a_1, a_2, \ldots, a_{n-1}) \in \mathbb{F}^{n-1}$, we write $\underline{t} = \underline{a} \underline{s}$ if $t_i = a_i s_i$ for all $1 \leq i \leq n - 1$.

We have the following proposition.

**Proposition 4.3** The map $\Phi_{t, s}$ defined above is an isomorphism of algebras if and only if the sets $\{i \mid t_i \neq 0\}$ and $\{i \mid s_i \neq 0\}$ coincide.

**Proof** If the sets $\{i \mid t_i \neq 0\}$ and $\{i \mid s_i \neq 0\}$ coincide, it can be easily verified that the maps $\Phi_{t, s}$ and $\Phi_{s, t}$ are well-defined and that they are mutually inverse. Therefore $\Phi_{t, s}$ is an isomorphism.

On the other hand, if the two sets do not coincide, then either $\Phi_{t, s}$ has a non-trivial kernel or is not a homomorphism of algebras. So it is not an isomorphism.

**Corollary 4.4** Suppose that $s_i = 0$ if $t_i = 0$ and $s_i = 1$ if $t_i \neq 0$. Then

$$
D_\omega(n, r) \cong D_\omega(n, r).
$$

In particular, if $t_i \neq 0$ for all $1 \leq i \leq n - 1$, then

$$
D_\omega(n, r) \cong S_0(n, r).
$$
5 The Degenerations $D_{t}S_{0}(n, r)$ of $S_{0}(n, r)$

5.1 A New Algebra Defined on Double Flag Varieties

Let $D_{t}S_{0}(n, r)$ be the $\mathbb{F}$-space with basis $\{ e_{A} \mid A \in \Xi(n, r) \}$. Define a multiplication on $D_{t}S_{0}(n, r)$ as follows,

$$e_{A} \ast_{t} e_{B} = \frac{t}{E(e_{A})+E(e_{B})-E(e_{A}e_{B})} e_{A} \cdot e_{B},$$

where $\frac{1}{E(e_{X})}$ is the product of $\frac{1}{t_{i}E(e_{X})}$ and we use the convention that $t_{i}^{0} = 1$ when $t_{i} = 0$. Following Lemma 3.2, let $I(n, r) \subseteq D_{t}S_{0}(n, r)$ be the subspace with basis $e_{A}$, where $A$ has a zero on the diagonal.

**Lemma 5.1** $D_{t}S_{0}(n, r)$ with the product $\ast_{t}$ is an associative $\mathbb{F}$-algebra. Moreover, $I(n, r)$ is a two-sided ideal.

**Proof** We only need to show that the product $\ast_{t}$ is associative. We have

$$(e_{A} \ast_{t} e_{B}) \ast_{t} e_{C} = \frac{1}{E(e_{A})+E(e_{B})-E(e_{A}e_{B})} (e_{A}e_{B}) \ast_{t} e_{C} = \frac{1}{E(e_{A})+E(e_{B})+E(e_{C})-E(e_{A}e_{B}e_{C})} e_{A}e_{B}e_{C}$$

Similarly,

$$e_{A} \ast_{t} (e_{B} \ast_{t} e_{C}) = \frac{1}{E(e_{A})+E(e_{B})+E(e_{C})-E(e_{A}e_{B}e_{C})} e_{A}e_{B}e_{C}$$

and so the product is associative.

As $e_{A} \ast_{t} e_{B}$ and the multiplication $e_{A}e_{B}$ in $S_{0}(n, r)$ only differ by a scalar, that $I(n, r)$ is a two sided ideal in $D_{t}S_{0}(n, r)$ follows from Lemma 3.2.

For the special case where $t_{i} = 0$ for all $i$, we write $\ast$ for the product $\ast_{t}$, and we write $DS_{0}(n, r)$ for $D_{0}S_{0}(n, r)$.

**Example 5.2** Let $A = \begin{pmatrix} 2 & 2 \\ 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ and $C = \begin{pmatrix} 3 & 1 \\ 1 & 0 \end{pmatrix}$. Then

$$e_{A}e_{B} = e_{C}, \text{ but } e_{A} \ast e_{B} = 0.$$

By Lemma 3.2, the $e_{C}$ in Example 5.2 is contained in the ideal $I(2, 5)$. This indicates that there should be a link between $DS_{0}(n, r)$ and the quotient algebra $S_{0}(n, r)/I(n, r)$. We will explore the link and their relation to $D_{0}(n, r)$ in the remaining part of this section.

**Lemma 5.3** The algebra $D_{t}S_{0}(n, r)$ is generated by $e_{i, \lambda}$, $f_{i, \lambda}$ and $k_{\lambda}$, where $\lambda \in \Lambda(n, r)$ and $1 \leq i \leq n - 1$.

**Proof** Recall that Lemma 2.7 expresses each basis element $e_{A}$ in $S_{0}(n, r)$ as reduced paths. Observe that the corresponding products in $D_{t}S_{0}(n, r)$ determined by the reduced paths are not zero, as the products satisfy

$$E(e_{A}) + E(e_{B}) = E(e_{A} \cdot e_{B}).$$

So the lemma follows.
Recall the algebra

\[ D_t(n, r) = \mathbb{F} \Sigma(n, r)/\mathcal{J}(t) \]

defined in Section 4. Up to isomorphism (see Corollary 4.4), the difference between \( D_t(n, r) \) and \( S_0(n, r) \) is that some of the defining relations of \( S_0(n, r) \) involving idempotents are replaced by zero relations. Recall also for \( j \geq i \),

\[ e(i, j) = e_i \cdot e_{i+1} \cdot \ldots \cdot e_j. \]

**Lemma 5.4** Let \( i < j \) and \( a, b \geq 0 \) such that \( j - i > a + b \). Then

\[ e(i, j) \cdot e(i + a, j - b) = e(i + a, j - b) \cdot e(i, j) \]

in \( D_t(n, r) \).

**Proof** It suffices to prove that \( e(i, j) \cdot e_l = e_l \cdot e(i, j) \) for all \( l = i, \ldots, j \). In fact, due to the commutativity relation \( e_m e_{m'} = e_{m'} e_m \) when \(|m - m'| > 1\), we need only prove \( e(i, j) \cdot e_l = e_l \cdot e(i, j) \) for \((i, j) = (l - 1, l), (l - 1, l + 1), (l, l + 1)\). All the three cases follow from the relations \( \mathcal{P}_{mm'} \).

**Proposition 5.5** \( \dim_{\mathbb{F}} D_t(n, r) = \dim_{\mathbb{F}} S_0(n, r) \).

**Proof** Observe that in both algebras, a complete list of representatives of nonzero paths is a basis. In particular, the paths of the form in Lemma 2.7 (2), which are all reduced paths, form a basis for \( S_0(n, r) \).

As the relations \( \mathcal{P}_{ij} \) and \( \mathcal{N}_{ij} \) are the same in both algebra, if two nonzero paths are equal in \( D_t(n, r) \), they are equal in \( S_0(n, r) \). Further, any reduced path that is nonzero in \( S_0(n, r) \) is also nonzero in \( D_t(n, r) \). Indeed, take a nonzero reduced path \( \rho = \ldots e_{i,\lambda} f_{j,\mu} \ldots \) in \( \Sigma(n, r) \). Now suppose that \( 0 = \rho \in D_t(n, r) \). This implies that using relations \( \mathcal{P}_{ij}, \mathcal{N}_{ij} \),

\[ \rho = \ldots e_{i,\lambda} - a_i f_{i,\lambda} k_{i,\lambda} \cdots \text{ or } \ldots k_{i,\lambda} f_{i,\lambda} + a_i e_{i,\lambda} \cdots \quad (\dagger) \]

with \( \lambda \) at the boundary. So the new expression \((\dagger)\) also holds in \( S_0(n, r) \). By the relations \( \mathcal{C}_{ij,\lambda} \),

\[ e_{i,\lambda} - a_i + a_{i+1} f_{i,\lambda} k_{\lambda} = k_{i,\lambda} f_{i,\lambda} + a_{i+1} e_{i,\lambda} = k_{\lambda}, \]

which contradicts the minimality of the number of arrows in \( \rho \). So \( \rho \) is a non-zero path in \( D_t(n, r) \). Therefore

\[ \dim_{\mathbb{F}} D_t(n, r) \geq \dim_{\mathbb{F}} S_0(n, r). \quad (7) \]

Next we claim that any reduced path \( \rho \) in \( D_t(n, r) \) is equal to a path of the form (2) in Lemma 2.7 and thus by the inequality \((7)\), the dimensions of the two algebras are the same. We proceed with the proof of the claim by induction on the length of \( \rho \). When the length of \( \rho \) is at most 1, then it already has the required form. Assume that the length is larger than 1, and that \( \rho = \rho \cdot k_{\lambda} \). We have

\[ \rho = \rho' e_i k_{\lambda} \text{ or } \rho' f_i k_{\lambda} \]

where \( \rho' \) is a path of length one less than \( \rho \). By the induction hypothesis, we may write \( \rho' \) of the form (2) in Lemma 2.7,

\[ \rho' = E F k_{\lambda'} \]

where \( E = e(i_1, j_1) \cdot \ldots \cdot e(i_1, j_1) \) and \( F = f(i_1', j_1') \cdot \ldots \cdot f(i_1', j_1') \). We first consider the case \( \rho = \rho' e_i k_{\lambda} \). Since \( \rho' \) is reduced, we have

\[ \rho = E F e_i k_{\lambda} = E e_i F k_{\lambda}. \]
If \( j_m \neq i - 1 \) for all \( m \), the relations \( P_{ij} \) give us the required form
\[
E e_i F k_\lambda = e(i_s, j_s) \cdot \ldots \cdot e(i, j_i) \cdot e(i, i) \cdot \ldots \cdot e(i_1, j_1) F k_\lambda,
\]
where \( l \) is the smallest integer such that \( j_l > i \).

Let \( m \) be the smallest integer with \( j_m = i - 1 \). We have
\[
E e_i = e(i_s, j_s) \cdot \ldots \cdot e(i_m, j_m + 1) \cdot \ldots \cdot e(i_1, j_1).
\]

By repeatedly applying Lemma 5.4 and relations \( P_{ij} \), we can move \( e(i_m, j_m + 1) \) to the appropriate position.

The case \( \rho = \rho' f i k_\lambda \) is similar, and so we skip the details. In either case, the path \( \rho \) is equal to a path of the form (2) in Lemma 2.7, as claimed.

We can now prove the first main result of this paper.

**Theorem 5.6** There is an isomorphism of \( F \)-algebras
\[
D_t(n, r) \cong D_t S_0(n, r),
\]
where \( k_\lambda \mapsto k_\lambda, e_{i,\lambda} \mapsto e_{i,\lambda} \) and \( f_{i,\lambda} \mapsto f_{i,\lambda} \).

*Proof* As the defining relations of \( D_t(n, r) \) are satisfied in \( D_t S_0(n, r) \), the map is well-defined. By Lemma 5.3, the map is an epimorphism. Note that by definition,
\[
\dim D_t S_0(n, r) = \dim S_0(n, r)
\]
and thus by Proposition 5.5,
\[
\dim D_t S_0(n, r) = \dim D_t(n, r).
\]
Thus the map is an isomorphism, as claimed.

Our second main result realises the algebra \( D S_0(n, r) \) as a quotient of the algebra \( D_t S_0(n, r) \).

**Theorem 5.7** There is an isomorphism of \( F \)-algebras
\[
D_t S_0(n, r+n)/I(n, r+n) \cong D S_0(n, r).
\]

*Proof* We embed \( \Sigma(n, r) \) into the interior of \( \Sigma(n, r+n) \) via \( \phi : k_\lambda \mapsto k_\mu \) and embed the arrows correspondingly, where \( \mu = (\lambda_1 + 1, \lambda_2 + 1, \ldots, \lambda_n + 1) \in \Lambda(n, r+n) \).

By abuse of notation, we also denote the ideal of \( D_t(n, r+n) \) generated by the boundary idempotents by \( I(n, r) \). Observe that the relations \( \phi(P_{ij,\lambda}), \phi(N_{ij,\lambda}) \) and \( \phi(C_{ij,\lambda}(0)) \) are zero in \( D_t(n, r+n)/I_t(n, r+n) \). So we have a surjective map
\[
D_0(n, r) \rightarrow D_t(n, r+n)/I(n, r+n).
\]
Further, following the isomorphism in Theorem 5.6, which maps the ideal \( I(n+r) \) of \( D_t(n, r+n) \) into the ideal \( I(n, r+n) \) of \( D_t S_0(n, r+n) \), we have a surjective map
\[
D_t(n, r+n)/I(n, r+n) \rightarrow D_t S_0(n, r+n)/I(n, r+n).
\]
By Corollary 3.4,
\[
\dim S_0(n, r+n)/I(n, r+n) = \left( \frac{n^2 + (n+r) - n - 1}{n + r - n} \right) = \left( \frac{n^2 + r - 1}{r} \right),
\]

\[\square\]
which is the dimension of $S_0(n, r)$. Therefore,
\[
\dim D_L S_0(n, r + n)/I(n, r + n) = \dim S_0(n, r + n)/I(n, r + n) \\
= \dim S_0(n, r) \\
= \dim DS_0(n, r).
\]
Applying the isomorphism in Theorem 5.6 for $t = 0$, we have surjective maps
\[
DS_0(n, r) \twoheadrightarrow D_L (n, r + n)/I(n, r + n) \twoheadrightarrow DS_0(n, r + n)/I(n, r + n)
\]
with the dimensions of two end terms the same. Therefore we have the isomorphism as claimed.

The construction of the isomorphism in the previous theorem gives us the following inverse system of algebra surjections (similar to the analogous inverse system for $q$-Schur algebras when $q$ is invertible [1]).

**Corollary 5.8** There is a sequence of surjective algebra homomorphisms
\[
DS_0(n, r) \leftarrow DS_0(n, r + n) \leftarrow \cdots \leftarrow DS_0(n, r + kn) \leftarrow \cdots
\]

**Proof** If $t_i = 0$ for all $i$, then the previous theorem gives us the surjective composition
\[
DS_0(n, r + kn) \rightarrow DS_0(n, r + kn)/I(n, r + kn) \cong DS_0(n, r + (k - 1)n)
\]
for any $k \geq 0$. 

**Remark 5.9** (1) Assume that $t$ and $s$ are given with the property that $s_i = 0$ if $t_i = 0$. Theorem 5.6 and Corollary 4.4 imply that $D_L S_0(n, r)$ is a degeneration of $D_L S_0(n, r)$. In particular, $D_L S_0(n, r)$ is a degeneration of $S_0(n, r)$ and $DS_0(n, r)$ is a degeneration of $D_L S_0(n, r)$ for all $L$.

(2) These degenerations can be understood as passing from a filtered algebra to its associated graded algebra. In particular, $DS_0(n, r)$ is the graded algebra associated to the filtered algebra structure on $S_0(n, r)$ given in Corollary 2.6.

We end this section by giving a more geometric interpretation of the multiplication in $DS_0(n, r)$ using double flag varieties. Suppose that
\[
e_A = [f, g], \ e_B = [g, h], \text{ and } e_C = [f', h']
\]
with
\[
e_A \cdot e_B = e_C
\]
in $S_0(n, r)$. Recall that $V$ is an $r$-dimensional vector space defined over a field $k$. We view an $n$-step flag in $V$ as a representation of a linear quiver $A_n$, where vertex 1 is a source and vertex $n$ is a sink. Associated to such a representation $f$, we denote the dimension vector by $\dim f$.

**Lemma 5.10** $\dim f' \cap h' \geq \dim f \cap g + \dim g \cap h - \dim g$. Consequently,
\[
\dim f' \cap h' = \dim f \cap g + \dim g \cap h - \dim g
\]
if and only if
\[
E(e_A) + E(e_B) = E(e_A \cdot e_B).
\]
Proof Note that 
\[ E(e_A) = \dim f - \dim f \cap g, \quad E(e_B) = \dim g - \dim h \cap g, \quad E(e_Ae_B) = \dim f' - \dim f' \cap h' \]
and \( f \cong f' \) as representations. Now the lemma follows from Lemma 2.5.

Therefore the product \( \star \) can be defined as below.

**Lemma 5.11**
\[
[f, g] \star [g, h] = \begin{cases} 
[f', h'] & \text{if } \dim f' \cap h' = \dim f \cap g + \dim g \cap h - \dim g, \\
0 & \text{otherwise}
\end{cases}
\]

6 Centralizer Algebras of \( D_\xi(n, r) \)

In this section, we discuss relations between the 0-Hecke algebra \( H_0(r) \), the nil-Hecke algebra \( NH_0(r) \), the nil-Temperley-Lieb algebra \( NTL(r) \) and the degenerate 0-Schur algebra \( DS_0(r, r) \). Throughout this section, we let \( \alpha = (1, \ldots, 1) \) be the composition in \( \Lambda(r, r) \) with all the entries equal to 1.

6.1 The Algebras \( H_0(r), NH_0(r) \) and \( DS_0(r, r) \)

We first recall some definitions and key facts. We then show that the associated graded algebras of \( H_0(r) \) and \( NH_0(r) \), with the filtration induced by the degree function defined in Section 2.5, are isomorphic to a subalgebra of \( k_\alpha DS_0(r, r)k_\alpha \).

**Definition 6.1** [2, 8] The 0-Hecke algebra, denoted by \( H_0(r) \), is the \( \mathbb{F} \)-algebra generated by \( T_1, T_2, \ldots, T_{r-1} \) with defining relations
\[
\begin{align*}
T_i^2 &= T_i, & & \text{for } 1 \leq i \leq r - 1; \\
T_iT_j &= T_jT_i, & & \text{for } 1 \leq i, j \leq r - 1 \text{ with } |i - j| > 1; \\
T_iT_{i+1}T_i &= T_{i+1}T iT_{i+1}, & & \text{for } 1 \leq i \leq r - 2.
\end{align*}
\]

**Theorem 6.2** [11, Theorem 10.4] The algebras \( H_0(r) \) and \( k_\alpha S_0(r, r)k_\alpha \) are isomorphic via the map \( T_i \mapsto k_\alpha f_i e_i k_\alpha \).

**Definition 6.3** [10, 17] The nil-Hecke algebra, denoted by \( NH_0(r) \), is the unital \( \mathbb{F} \)-algebra generated by \( T_1, T_2, \ldots, T_{r-1} \) with defining relations
\[
\begin{align*}
T_i^2 &= 0, & & \text{for } 1 \leq i \leq r - 1; \\
T_iT_j &= T_jT_i, & & \text{for } 1 \leq i, j \leq r - 1 \text{ with } |i - j| > 1; \\
T_iT_{i+1}T_i &= T_{i+1}T iT_{i+1}, & & \text{for } 1 \leq i \leq r - 2.
\end{align*}
\]

Assume that \( w = s_{i_1} \cdots s_{i_t} = s_{j_1} \cdots s_{j_t} \) are reduced expressions in the symmetric group \( S_r \) on \( r \) letters, where \( s_i \) is the transposition \( (i, i + 1) \). As reduced expressions of \( w \) can be obtained from each other by using the braid relations only (see [14, 20]), we have
\[
T_{i_1} \cdots T_{i_t} = T_{j_1} \cdots T_{j_t}
\]
in both \( H_0(r) \) and \( NH_0(r) \), and thus the element
\[
T_w = T_{i_1} \cdots T_{i_t}
\]
is well-defined in both algebras. One can also deduce that \( \{ T_w \mid w \in S_r \} \) is a basis for both algebras. Further,

\[
\text{in } H_0(r), \quad T_i T_w = \begin{cases} 
T_{i(w)}, & \text{if } \ell(s_i w) = \ell(w) + 1; \\
T_w, & \text{otherwise},
\end{cases}
\]

(10)

and

\[
\text{in } N H_0(r), \quad T_{w_1} T_{w_2} = \begin{cases} 
T_{w_1 w_2}, & \text{if } \ell(w_1 w_2) = \ell(w_1) + \ell(w_2); \\
0, & \text{otherwise},
\end{cases}
\]

(11)

where \( \ell : W \to \mathbb{N} \cup \{ 0 \} \) is the length function of elements in \( S_r \).

By Theorem 6.2, the filtration of the 0-Schur algebra \( S_0(r, r) \) discussed in Section 2.5 induces a filtration on \( H_0(r) \). When the length equations in Eqs. 10 or 11 hold, the multiplications of \( T_w \) and \( T_{w'} \) in \( H_0(r) \) and \( N H_0(r) \) are the same, so we also have a filtration on \( N H_0(r) \). Further, together with the fact that \( DS_0(r, r) \) is the associated graded algebra of \( S_0(r, r) \), this implies the following.

**Proposition 6.4** The associated graded algebras of \( H_0(r) \) and \( N H_0(r) \) are isomorphic to the algebra \( k_\alpha DS_0(r, r)k_\alpha \).

Define a degree function \( d \) for a \( \mathbb{Z} \)-grading on \( k_\alpha DS_0(r, r)k_\alpha \) given by

\[
d(w) := d(T_w) = \sum_i E_i(e_A),
\]

where \( A \) is the permutation matrix corresponding to \( w \). Then via the isomorphism in Theorem 6.2, we can reformulate the multiplication in \( k_\alpha DS_0(r, r)k_\alpha \) as follows

\[
T_{w_1} T_{w_2} = \begin{cases} 
T_{w_1 w_2}, & \text{if } d(w_1 w_2) = d(w_1) + d(w_2); \\
0, & \text{otherwise},
\end{cases}
\]

(12)

Note that in \( k_\alpha DS_0(r, r)k_\alpha \), some \( T_w \) can not be expressed as a product in the \( T_i \).

**Example 6.5** Consider the product \( T_1 T_2 T_1 \), which has length 3 and so is non-zero in the nil-Hecke algebra. The corresponding element \( T_{s_1 s_2 s_2} \) has degree 2, therefore the product \( T_1 T_2 T_1 \) is zero in \( k_\alpha DS_0(r, r)k_\alpha \).

By inspecting the entries above the diagonal in the matrix \( A \), we can reinterpret \( d(e_A) \) in terms of permutations as

\[
d(w) = \sum_{w(j) < j} (j - w(j)).
\]

It is shown in [15], that any permutation can be written as a product of transpositions, such that the degrees add up. In other words, the corresponding product in \( k_\alpha DS_0(r, r)k_\alpha \) is non-zero. Hence the set of elements \( T(i,j) \) corresponding to transpositions \( (i, j) \) generates the algebra \( k_\alpha DS_0(r, r)k_\alpha \).

### 6.2 Nil-Temperley-Lieb Algebras

The nil-Temperley-Lieb algebra \( NTL(r) \) is the quotient algebra of \( N H_0(r) \) modulo the ideal generated by \( T_i T_{i+1} T_i \) for \( 1 \leq i \leq r - 2 \) (see e.g. [9]).

**Lemma 6.6** The algebra \( NTL(r) \) has a basis consisting of \( T_w \), where \( w \) does not contain a subword of the form \( s_i s_j s_i \) in any of its reduced expressions, for any \( i \) and \( j \) with \( |i - j| = 1 \).
Proof The lemma follows from the fact that \( \{ T_w \mid w \in S_r \} \) is a basis of \( NH_0(r) \) and the definition of \( NTL(r) \).

Note that the dimension of \( NTL(r) \) is the Catalan number \( \frac{1}{r+1} \binom{2r}{r} \) and there is a useful combinatorial parametrisation of the elements in a basis of \( NTL(r) \), using Dyck words. Pictorially, Dyck words can be described using peak pictures in a triangle with \( r \) dots on each edge (cf [18]). For instance, when \( r = 3 \), the five peak pictures are as follows.

\begin{align*}
&\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
&\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
&\begin{array}{c}
\vdots \\
\vdots \\
\vdots \\
\end{array} \\
\end{align*}

Let

\[ x_i = k_\alpha f_i e_i k_\alpha \in k_\alpha DS_0(n, r) k_\alpha. \]

Denote by \( DS(r, \alpha) \) the subalgebra of \( k_\alpha DS_0(r, r) k_\alpha \) generated by \( x_i \) for \( 1 \leq i \leq r - 1 \). When \( r = 3 \), the algebra \( DS(r, \alpha) \) is five dimensional, the orbit basis elements are determined by the following matrices.

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix},
\begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix}.
\]

Similar to \( T_w \),

\[ x_w = x_{i_1} \ldots x_{i_t} \]

is well-defined, where \( s_{i_1} \ldots s_{i_t} \) is a reduced expression of \( w \). By direct computation following the fundamental multiplication rules and the definition of \( DS_0(n, r) \), we have the following lemma.

Lemma 6.7 The elements \( x_i \) for \( 1 \leq i \leq r - 1 \) satisfy the generating relations of \( NTL(r) \). Consequently, there is an epimorphism

\[ NTL(r) \longrightarrow DS(r, \alpha), \quad T_i \mapsto x_i, \quad \forall i. \]

Any matrix that determines a nonzero orbit in \( DS(r, \alpha) \) is a permutation matrix. We call a nonzero entry that is below the diagonal a peak entry. For a peak entry \((i, j)\), which implies that \( j < i \), we call the entries \((j, j)\) and \((i, i)\) the feet of the peak. For the five matrices above, which determine the orbit basis elements in \( DS(r, \alpha) \), we connect the peaks, feet and diagonal 1-entry, using zig-zag lines as follows.

\[
\begin{pmatrix}
1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix}
\]
In this way, we obtain a well defined one-to-one correspondence between the basis elements of \( NTL(3) \) and \( DS(3, \alpha) \). In fact, such a one-to-one correspondence exists for all \( r \).

**Theorem 6.8** The two algebras \( NTL(r) \) and \( DS(r, \alpha) \) are isomorphic.

**Proof** First note the nonzero orbits \( e_A \) in \( DS(r, \alpha) \) form a basis and each orbit \( e_A \) is uniquely determined by the matrix \( A \). By Lemma 6.7, there is a surjective homomorphism from \( NTL(r) \) to \( DS(r, \alpha) \). So it suffices to show that there exists a nonzero element generated by the \( x_i \)'s such that the corresponding matrix produces the peak picture.

First identify the triangle, in which we draw the peak pictures, with the lower triangular part of an \( r \times r \) matrix. The peaks give us peak entries, \((i_1, j_1), \ldots, (i_s, j_s)\) where for any \( l \),

\[
j_l < i_l < i_{l+1} \text{ and } j_l < j_{l+1}.
\]

Let

\[
x(l) = x_{i_l-1} \ldots x_{j_l+1}x_{j_l} \text{ and } x = x^{(1)} \ldots x^{(s)}.
\]

By the fundamental multiplication rules, the multiplication of \( x_{i_l+1} \) with \( x_i \ldots x_{j_l}k_{\alpha} \) moves a 1 on or above the diagonal further up in the same column and a 1 on or below the diagonal further down in the same column and thus

\[
E(x_{i_l+1}) + E(x_i \ldots x_{j_l}k_{\alpha}) = E(x_{i_l+1}x_i \ldots x_{j_l}k_{\alpha}).
\]

That is, the equality in the definition of \( \star \) holds. Therefore,

\[
x^{(s)}k_{\alpha} \neq 0.
\]

Similarly,

\[
x^{(l)}(x^{(l+1)} \ldots x^{(s)}k_{\alpha}) \neq 0
\]

and when compared with \( x^{(l+1)} \ldots x^{(s)}k_{\alpha} \), it has a new peak at \((i_l, j_l)\). Thus we have found a nonzero element \( x = xk_{\alpha} \) such that the corresponding matrix gives us the required peak picture.

**Example 6.9** In this example, we demonstrate the process of obtaining the matrix

\[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\]

which as indicated has two peaks, using the construction in the proof of the theorem. The two peak entries are (2, 1) and (3, 2). By definition,

\[
x^{(2)} = x_2 \text{ and } x^{(1)} = x_1.
\]

Then \( x^{(2)}k_{\alpha} \) and \( x^{(1)}x^{(2)}k_{\alpha} \) are the orbit basis elements corresponding to the matrices, respectively,

\[
\begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{pmatrix} \quad \text{and} \quad \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix}.
\]

We have the following special version of Lemma 5.11.
Lemma 6.10 Use the same notation Lemma 5.11. Assume that all the flags are isomorphic to $\oplus_{i=1}^n F_i$ and $[f, g] = x_i$ for some $i$. Then the following is true in $D\Sigma(r, \alpha)$

$$[f, g] \star [g, h] = \begin{cases} 
[f', h'] & \text{if } \dim f' \cap h' = \dim g \cap h - 1, \\
0 & \text{otherwise}.
\end{cases}$$

Proof Note that as a representation, $g/(f \cap g)$ is isomorphic to the simple representation of the linear quiver $A_n$ at vertex $i$, since $[f, g] = x_i$. Now the lemma follows from Lemma 5.10 and 5.11.

Remark 6.11 In the light of Lemma 6.10, Theorem 6.8 gives a geometric realisation of nil-Temperley-Lieb algebras, via double flag varieties.

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References

1. Beilinson, A.A., Lusztig, G., MacPherson, R.: A geometric setting for the quantum deformation of $GL_n$. Duke Math. J. 61(2), 655–677 (1990)
2. Deng, B., Du, J., Parshall, B., Wang, J.: Finite Dimensional Algebras and Quantum Groups. Mathematical Surveys and Monographs, vol. 150. American Mathematical Society, Providence (2008). xxvi+759 pp
3. Deng, B., Yang, G.: On 0-Schur algebras. J. Pure Appl. Alg. 216, 1253–1267 (2012)
4. Deng, B., Yang, G.: Representation type of 0-Hecke algebras. Sci. China Math. 54, 411–420 (2011)
5. Dipper, R., James, G.: The $q$-Schur algebra. Proc. London. Math. Soc. (3) 59(1), 23–50 (1989)
6. Donkin, S.: The $q$-Schur Algebra, London Mathematical Society Lecture Note Series, vol. 253. Cambridge University Press, Cambridge (1998). x+179 pp
7. Du, J.: A note on quantised Weyl reciprocity at roots of unity. Algebra Colloq. 2(4), 363–372 (1995)
8. Duchamp, G., Hivert, F., Thibon, J.: Noncommutative symmetric functions. VI. Free quasi-symmetric functions and related algebras. Internat. J. Algebra Comput. 12(5), 671–717 (2002)
9. Fomin, S., Greene, C.: Noncommutative Schur functions and their applications. Selected papers in honor of Adriano Garsia (Taormina, 1994). Discrete Math. 193(1–3), 179–200 (1998)
10. Fomin, S., Stanley, R.P.: Schubert polynomials and the nilCoxeter algebra. Adv. Math. 103, 196–207 (1994)
11. Jensen, B.T., Su, X.: A geometric realisation of 0-Schur and 0-Hecke algebras. J. Pure Appl. Algebra 219(2), 277–307 (2014)
12. Jensen, B.T., Su, X., Yang, G.: Projective modules of 0-Schur algebras. J. Algebra 454, 181–205 (2016)
13. Krob, D., Thibon, J.Y.: Noncommutative symmetric functions. IV. Quantum linear groups and Hecke algebras at $q = 0$. J. Algebraic Combin. 6, 339–376 (1997)
14. Matsumoto, H.: Generateurs et relations des groupes de Weyl generalises. C. R. Acad. Sci. Paris 258, 3419–3422 (1964)
15. Petersen, T.K., Tenner, B.: The depth of a permutation. J. Comb. 6, 145–178 (2015)
16. Reineke, M.: Generic extensions and multiplicative bases of quantum groups at $q = 0$. Represent. Theory 5, 147–163 (2001)
17. Rouquier, R.: Quiver Hecke algebras and 2-Lie algebras. Algebra Colloq. 19(2), 359–410 (2012)
18. Stanley, R.P.: Enumerative Combinatorics. Cambridge Studies in Advanced Mathematics, vol. 2, p. 62. Cambridge University Press, Cambridge (1999)
19. Su, X.: A generic multiplication in quantised Schur algebras. Quart. J. Math. 61, 497–510 (2010)
20. Tits, J.: Le probleme des mots dans les groupes de Coxeter. 1969 Symposia Mathematica (IN- DAM, Rome, 1967/68), vol. 1, pp. 175–185. Academic Press, London (1969).

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