ON THE VANISHING OF THE LOWER $K$-THEORY OF THE HOLOMORPH OF A FREE GROUP ON TWO GENERATORS

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Abstract. We show that the holomorph of the free group on two generators satisfies the Farrell–Jones Fibered Isomorphism Conjecture. As a consequence, we show that the lower $K$-theory of the above group vanishes.

1. Introduction

Certain obstructions that appear in problems of topological rigidity of manifolds are elements of algebraic $K$-groups, specially lower $K$-groups. For this reason, the calculation of the lower $K$-groups has implications in geometric topology.

The main modern tool for calculating lower $K$-groups (and other geometrically interesting obstruction groups) is the Farrell–Jones Fibered Isomorphism Conjecture. The Conjecture provides an inductive method for calculating the obstruction groups of a group from those of certain subgroups. More specifically, if a group satisfies the Fibered Isomorphism Conjecture for a specific theory, then the obstruction groups can be calculated from the obstruction groups of the virtually cyclic subgroups. The last class of subgroups consists the finite subgroups and groups that are virtually infinite cyclic. The virtually infinite cyclic groups are of two types:

- Groups $V$ that surject onto the infinite cyclic group $\mathbb{Z}$ with finite kernel, i.e. $V = H \times \mathbb{Z}$ with $H$ finite.
- Groups $W$ that surject onto the infinite dihedral group $D_\infty$ with finite kernel, i.e. $W = A \ast_B C$ with $B$ finite and $[A : B] = [C : B] = 2$.

Obstruction groups that can be calculated this way are pseudoisotopy groups, $K$-groups of group rings, $K$-groups of $C^*$-algebras, $L^{-\infty}$-groups.

The fundamental work of Farrell–Jones ([8]) deals with the Fibered Isomorphism Conjecture for the pseudoisotopy spectrum. It should be noted that if a group satisfies the Fibered Isomorphism Conjecture for pseudoisotopies, then it satisfies the Isomorphism Conjecture for the lower $K$-groups. The reason for this is that the lower homotopy groups of the pseudoisotopy spectrum and the $K$-theory spectrum are isomorphic. For $K$-groups there is a refinement of the Conjecture that was given in [6]. They showed that finite and virtually infinite cyclic subgroups of the first type suffice in detecting the $K$-theory of the group.

Our main interest is in computing the lower $K$-groups of the holomorph of $F_2$, the free group on two generators. For a group $G$, the holomorph of $G$ is the universal split extension of $G$. Thus, it fits into a split exact sequence:

$$1 \to G \to \text{Hol}(G) \to \text{Aut}(G) \to 1.$$ 

The main result of the paper is the following:
Theorem (Main Theorem). The group $\text{Hol}(F_2)$ satisfies the Fibered Isomorphism Conjecture for pseudoisotopies. Furthermore, if $\Gamma < \text{Hol}(F_2)$ then

$$\text{Wh}(\Gamma) = \tilde{K}_0(\mathbb{Z}\Gamma) = K_i(\mathbb{Z}\Gamma) = 0, \quad \text{for } i \leq -1.$$ 

Notice that $\text{Hol}(F_2)$ is equipped with a sequence of surjections:

$$\text{Hol}(F_2) \to \text{Aut}(F_2) \to \text{GL}_2(\mathbb{Z}).$$

The second surjection is induced by sending an automorphism of $F_2$ to an automorphism of its abelianization $\mathbb{Z}^2$. Notice that the kernel of both surjections is isomorphic to $F_2$. To show that $\text{Hol}(F_2)$ satisfies the Fibered Isomorphism Conjecture, we use the fact that every automorphism of $F_2$ is geometric, i.e. it can be realized by a diffeomorphism on a surface with boundary. That allows us to show first that $\text{Aut}(F_2)$ satisfies the Conjecture and using the same fact again that $\text{Hol}(F_2)$ does also.

The group $\text{GL}_2(\mathbb{Z})$ splits as an amalgamated free product of finite dihedral groups:

$$\text{GL}_2(\mathbb{Z}) = D_4 *_{D_2} D_6.$$ 

Thus both $\text{Aut}(F_2)$ and $\text{Hol}(F_2)$ split as amalgamated free products. Using this fact and the properties of elements of finite order in $\text{Aut}(F_2)$ given in [10, 11], we determine the finite and the virtually infinite cyclic subgroups of $\text{Aut}(F_2)$ and $\text{Hol}(F_2)$, up to isomorphism. The list of groups is short and their lower $K$-theory vanishes. The Main Theorem follows from this observation.

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2. Preliminaries

Let $G$ be a discrete group. By a class of subgroups of $G$ we mean a collection of subgroups of $G$ that is closed under taking subgroups and conjugates. In our application we consider the following classes of subgroups:

- $\text{Fin}$, the class of finite subgroups of $G$.
- $\text{FBC}$ for the class of finite by cyclic subgroups. Those are subgroups $H < G$ such that
  $$1 \to A \to H \to C \to 1$$
  where $C$ is cyclic (finite or infinite) group and $A$ is finite. Notice that $\text{FBC}$ contains $\text{Fin}$ and subgroups $H = A \rtimes \mathbb{Z}$, when $C$ is the infinite cyclic group.
- $\text{VC}$ for the class of virtually cyclic subgroups of $G$. For a such a subgroup $H$ either $H \in \text{Fin}$, $H \in \text{FBC}$, or $H = A *_{B} C$, where $A$, $B$, $C$ are finite and $|A : B| = |C : B| = 2$.
- $\text{ALL}$ for the class of all subgroups of $G$.

Let $C$ be a class of subgroups of $G$. The classifying space for $C$, $E_C G$, is a $G$-CW-complex such that the isotropy groups of the actions are in $C$ and, for each $H \in C$, the fixed point set of $H$ is contractible (for more details [5, 13]).

Remark 2.1. For $\text{Aut}(F_2)$ the classifying space for finite groups is the outer space ([10, 11]).
The Fibered Isomorphism Conjecture (FIC) was stated by Farrell–Jones ([8]). For the groups that hold, it provides an inductive method for computing obstruction groups in geometric topology (for a review see [14]). If $G$ satisfies the FIC then the natural map

$$H^G_n(E_{VC}G; \mathbb{KZ}^{-\infty}) \to H^G_n(E_{ALL}G; \mathbb{KZ}^{-\infty}) = K_n(\mathbb{Z}G)$$

is an isomorphism for $n \leq 1$. Notice that the left hand side of the isomorphism can be computed from the virtually cyclic subgroups of $G$.

In general, there are “forgetful maps”

$$H^G_n(E_{Fin}G; \mathbb{KZ}^{-\infty}) \to H^G_n(E_{FBC}G; \mathbb{KZ}^{-\infty}) \to H^G_n(E_{VC}G; \mathbb{KZ}^{-\infty})$$

The difference between the class $\text{Fin}$ and the class $\text{VC}$ is that the second class can be captured by the Waldhausen and Bass–Farrell Nil-groups of the infinite virtually cyclic subgroups. In [4] and [6], it was shown that the second map is an isomorphism. Essentially, the authors proved that the Waldhausen’s Nil-groups that appear in the $K$-theory of virtually infinite cyclic subgroups can be detected by the Bass–Farrell Nil-groups that appear in the $FBC$ class.

The FIC is known to hold for certain classes of groups. One class of interest for this paper is the class of strongly poly-free groups. A group $\Gamma$ is called strongly poly-free if there is a filtration:

$$\Gamma = \Gamma_0 \supseteq \Gamma_1 \supseteq \cdots \supseteq \Gamma_n = \{1\}$$

such that:

1. $\Gamma_i$ is normal in $\Gamma$ for each $i$.
2. $\Gamma_i/\Gamma_{i+1}$ is finitely generated free for all $0 \leq i \leq n - 1$.
3. For each $\gamma \in \Gamma$ there is a compact surface $S$ and a diffeomorphism $f : S \to S$ such that the induced homomorphism $f_* : \pi_1(S)$ is equal to $c_\gamma$ in $\text{Out}(\pi_1(S))$, where $c_\gamma$ is the action of $\gamma$ on $\Gamma_i/\Gamma_{i+1}$ by conjugation and $\pi_1(S)$ is identified with $\Gamma_i/\Gamma_{i+1}$ via a suitable isomorphism.

In [2] and [9] it was shown that a finite extension of a strongly poly-free group satisfies the FIC.

Remark 2.2. Let $\Gamma$ be a group that satisfies (1) and (2) above. We assume that $\Gamma_i/\Gamma_{i+1} \cong F_2$. Then $G$ is strongly poly-free. For this, let $T^2$ be the torus and $p = (1, 1) \in T^2$. Then $\pi_1(T^2 \setminus \{p\}, x) = F_2$. In this case,

$$\text{Out}(F_2) = \text{Aut}(F_2)/\text{Inn}(F_2) = \text{GL}_2(\mathbb{Z})$$

where $\text{Aut}(F_2)$ denotes the automorphism group of $F_2$. Let $c_\gamma$ be an induced homomorphism as in (3) above. Then the image of $c_\gamma$ to $\text{Out}(F_2)$ can be represented by a diffeomorphism $f$ of $T^2$ that fixes $p$. After an isotopy starting at the identity on $T^2$, we can assume that $f$ fixes a small open disk $D$ around $p$. Then $f$ induces a diffeomorphism on the compact surface $f : T^2 \setminus D \to T^2 \setminus D,$

that fixes the boundary. Thus $f_* = c_\gamma$ in $\text{Out}(F_2)$.

Start with an exact sequence of groups.

$$1 \to A \to B \to C \to 1.$$ 

In the Appendix of [8], it was shown the FIC holds for $B$ if:

- It holds for $C$. 

• For each virtually cyclic subgroup $V$ of $C$, it holds for $r^{-1}(V)$.

Using this result we show the following

**Proposition 2.3.** Let

$$1 \to F_2 \to G \xrightarrow{r} H \to 1$$

be an exact sequence. If the FIC holds for $H$, then it holds for $G$.

**Proof.** Using the result in [8], it is enough to show that the FIC holds for $r^{-1}(V)$, where $V$ is a virtually cyclic subgroup of $H$.

If $V$ is finite, then $r^{-1}(V)$ is a finite extension of $F_2$ and $r^{-1}(V)$ is a finite extension of a free group. The result follows from Remark 2.2.

If $V$ is infinite, then $V$ contains an infinite cyclic normal subgroup $W$ of finite index. Then $r^{-1}(W)$ is a normal subgroup of $r^{-1}(V)$ and fits into an exact sequence:

$$1 \to F_2 \to r^{-1}(W) \to W \to 1.$$ 

Then there is a filtration $r^{-1}(W) > F_2 > \{1\}$, with the first quotient being an infinite cyclic group. Obviously, every homomorphism of $Z$ is realized by a diffeomorphism of $S^1 \times [0,1]$. Using Remark 2.2 we see that $r^{-1}(W)$ is strongly poly-free. Therefore, $r^{-1}(V)$ is a finite extension of a strongly poly-free group. By [9], it satisfies the FIC, completing the proof of the proposition. 

Let Hol($F_2$) denote the holomorph of $F_2$, namely, the universal split extension of $F_2$:

$$1 \to F_2 \to \text{Hol}(F_2) \xrightarrow{\theta} \text{Aut}(F_2) \to 1.$$ 

Notice that there is an exact sequence

$$1 \to \text{Inn}(F_2) \to \text{Aut}(F_2) \xrightarrow{\theta} \text{GL}_2(Z) \to 1$$

that is induced by mapping the automorphisms of $F_2$ to the automorphisms of its abelianization. That induces an exact sequence:

$$1 \to F_2 \to \text{Aut}(F_2) \xrightarrow{\theta} \text{GL}_2(Z) \to 1.$$ 

**Proposition 2.4.** The FIC holds for $\text{Aut}(F_2)$ and $\text{Hol}(F_2)$.

**Proof.** The group $\text{GL}_2(Z)$ contains a subgroup of finite index that is isomorphic to $F_2$. In fact, the following short exact sequence is known to hold, as a result of the standard action that $\text{GL}_2(Z)$ admits on the upper half plane:

$$1 \to F_2 \to \text{GL}_2(Z) \to D_{12} \to 1$$

(see for example [7]). Thus the FIC holds for $\text{GL}_2(Z)$. Now $\text{Aut}(F_2)$ fits into an exact sequence:

$$1 \to F_2 \to \text{Aut}(F_2) \to \text{GL}_2(Z) \to 1.$$ 

By Proposition 2.3 the FIC holds for $\text{Aut}(F_2)$. Also, $\text{Hol}(F_2)$ fits into an exact sequence:

$$1 \to F_2 \to \text{Hol}(F_2) \to \text{Aut}(F_2) \to 1.$$ 

By Proposition 2.3 again, the FIC holds for $\text{Hol}(F_2)$. 

□
3. Infinite Finite-by-Cyclic Subgroups of $\text{Hol}(F_2)$

Since there is an exact sequence

$$1 \to \text{Inn}(F_2) \to \text{Aut}(F_2) \xrightarrow{p} \text{GL}_2(\mathbb{Z}) \to 1$$

and $F_2 = \langle a, b \rangle$ is torsion free, every finite subgroup of $\text{Aut}(F_2)$ maps isomorphically to a finite subgroup of $\text{GL}_2(\mathbb{Z})$. On the other hand, $\text{GL}_2(\mathbb{Z})$ admits a decomposition as an amalgamated free product of the form

$$\text{GL}_2(\mathbb{Z}) = D_4 \ast_D D_6$$

where $D_2$, $D_4$ and $D_6$ are dihedral groups of orders 4, 8 and 12 respectively. Hence, any finite subgroup of $\text{GL}_2(\mathbb{Z})$ is a subgroup of a conjugate of either $D_2$ or $D_4$ or $D_6$ and hence, so is every finite subgroup of $\text{Aut}(F_2)$.

Now a presentation for $\text{Aut}(F_2)$ is given by

$$(p, x, y, \tau_a, \tau_b) \mapsto (p^2 = (px)^2 = 1, (py)^2 = \tau_b, x^2 = y^3 \tau_a^{-1} \tau_b, p^{-1} \tau_a p = x^{-1} \tau_a x = y^{-1} \tau_a y = \tau_b, p^{-1} \tau_b = \tau_a, x^{-1} \tau_a x = \tau_a^{-1}, y^{-1} \tau_b y = \tau_a^{-1} \tau_b)$$

where $\tau_a, \tau_b$ are the inner automorphisms of $F_2$ corresponding to $a, b$ respectively (see for example [15]). Moreover, a presentation for $\text{GL}_2(\mathbb{Z})$ is given by

$$\text{GL}_2(\mathbb{Z}) = \langle P, X, Y \mid X^4 = P^2 = (PX)^2 = (PY)^2 = 1, X^2 = Y^3 \rangle$$

and $G$ maps onto $\text{GL}_2(\mathbb{Z})$ by $p \mapsto P, x \mapsto X, y \mapsto Y, \tau_a, \tau_b \mapsto 1$.

As shown in [16], if $g$ is an element of finite order in $\text{Aut}(F_2)$, then $g$ is conjugate in $\text{Aut}(F_2)$, to one of the following elements $p, px, px\tau_a, x^2, y^2 \tau_a^{-1}$ or $x$ with orders 2, 2, 2, 2, 3 or 4 respectively. This fact implies that $\text{Aut}(F_2)$ cannot contain finite subgroups isomorphic to $D_6$. Moreover, any element of $\text{Aut}(F_2)$ can be written uniquely in the form $p^r u(x, y) x^{2s} u(\tau_a, \tau_b)$ where $r, s \in \{0, 1\}$, $u(\tau_a, \tau_b)$ is a reduced word in $\text{Inn}(F_2)$ and $u(x, y)$ is a reduced word where $x, y, y^{-1}$ are the only powers of $x, y$ appearing (see [16, 15]).

Also, due to the decomposition [11] of $\text{GL}_2(\mathbb{Z})$, $\text{Aut}(F_2)$ is also an amalgamated free product of the form

$$(2) \quad \text{Aut}(F_2) = B \ast_D C$$

where $B, C$ and $D$ fit into the following short exact sequences

$$1 \to \text{Inn}(F_2) \to B \to D_4 \to 1$$
$$1 \to \text{Inn}(F_2) \to C \to D_6 \to 1$$
$$1 \to \text{Inn}(F_2) \to D \to D_2 \to 1.$$

Moreover, since every one of $B, C$ and $D$ are free-by-finite groups they admit an action on a tree with finite quotient graph and finite vertex and edge stabilizers (as a corollary of the Almost Stability Theorem of Dicks and Dunwoody [7]). In fact, they are also amalgamated free products of the form

$$B = D_4 \ast_{\mathbb{Z}/2\mathbb{Z}} D_2 = \langle x, p \rangle \ast_{\langle px \rangle} \langle px, x^2 \tau_b \rangle$$
$$C = D_3 \ast_{\mathbb{Z}/2\mathbb{Z}} D_2 = \langle y^2 \tau_b^{-1} \tau_a, p \rangle \ast_{\langle p \rangle} \langle p, x^2 \rangle$$
$$D = D_2 \ast_{\langle \mathbb{Z}/2\mathbb{Z} \rangle} \langle p, x^2 \rangle \ast_{\langle x^2 \tau_b \rangle} \langle x \rangle.$$

Once again, the elements of finite order are $p, px, x^2 \tau_b, px^2 \tau_b, x^2, px^2, y^2 \tau_b^{-1} \tau_a$ and $x$.

To be in accordance with Meskin, we see that $x^2 = y^2 \tau_b^{-1} \tau_a x^2 = y^3 \tau_b^{-1} \tau_a (y^2 \tau_b^{-1} \tau_a) \tau_a^{-1} \tau_b y^{-3} = y^2 \tau_b^{-1}, \tau_a^{-1} x^{-1} (px^3 \tau_b) x \tau_a = px \tau_a, y (x^2 \tau_b) y^{-1} = x^2$ and that $x^{-1} (px^2) x = p$.
By definition, $G = \text{Hol}(F_2)$ is the universal split extension of Aut($F_2$) and thus it fits to the split exact sequence

$$1 \to F_2 \to \text{Hol}(F_2) \to \text{Aut}(F_2) \to 1.$$ 

So $\text{Hol}(F_2) = F_2 \rtimes \text{Aut}(F_2)$. Hence, the above presentation for Aut($F_2$) provides us with a presentation for Hol($F_2$). Namely,

$$\text{Hol}(F_2) = \langle p, x, y, \tau_a, \tau_b, a, b \mid x^4 = p^2 = (px)^2 = 1, (py)^2 = \tau_b, x^2 = y^3\tau_b^{-1}\tau_a, p^{-1}\tau_ap = x^{-1}\tau_ax = y^{-1}\tau_ay = \tau_b, p^{-1}\tau_bp = \tau_a, x^{-1}\tau_bx = \tau_a^{-1}, y^{-1}\tau_by = \tau_a^{-1}\tau_b, \rangle$$

Moreover, the decomposition (2) of Aut($F_2$) provides an amalgamated free product decomposition for Hol($F_2$):

$$\text{Hol}(F_2) = (F_2 \rtimes B) *_{F_2 \rtimes D} (F_2 \rtimes C)$$

and based on (4) we have

$$F_2 \rtimes B = (F_2 \rtimes D_4) *_{F_2 \rtimes \mathbb{Z}/2\mathbb{Z}} (F_2 \rtimes D_2)$$

$$F_2 \rtimes C = (F_2 \rtimes D_3) *_{F_2 \rtimes \mathbb{Z}/2\mathbb{Z}} (F_2 \rtimes D_2)$$

$$F_2 \rtimes D = (F_2 \rtimes D_2) *_{(F_2 \rtimes \mathbb{Z}/2\mathbb{Z})}$$

Based again on the Almost Stability Theorem, we see that every vertex group in the above graphs of groups is a free-by-finite group so, it also admits a decomposition as a graph of groups with finite vertex groups. An analysis, based on the presentations and also on the fact that the action of $x, y, p$ on $a, b$ is the same as that on $\tau_a, \tau_b$, would give us the following: In $F_2 \rtimes B$,

$$F_2 \rtimes D_4 = D_4 *_{\mathbb{Z}/2\mathbb{Z}} D_2 = \langle x, p \rangle *_{(px)} \langle px, x^2b \rangle$$

$$F_2 \rtimes D_2 = D_2 *_{\mathbb{Z}/2\mathbb{Z}} D_2 = \langle px, a^2\tau_b \rangle *_{(px)} \langle px, a^2\tau_b^{-1} \rangle = \langle px, a^2\tau_b \rangle *_{\mathbb{Z}/2\mathbb{Z}} \langle px, a^2\tau_b^{-1} \rangle = \langle px \rangle *_{\mathbb{Z}/2\mathbb{Z}} \langle px, a^2\tau_b \rangle *_{\mathbb{Z}/2\mathbb{Z}} \langle px, a^2\tau_b^{-1} \rangle$$

In $F_2 \rtimes C$,

$$F_2 \rtimes D_3 = D_3 *_{\mathbb{Z}/2\mathbb{Z}} D_3 = \langle y^2\tau_b^{-1}, p \rangle *_{(p)} \langle y^2\tau_b^{-1}, a, p \rangle$$

$$F_2 \rtimes D_2 = D_2 *_{\mathbb{Z}/2\mathbb{Z}} \langle p, x^2 \rangle = \langle p, x^2 \rangle *_{\mathbb{Z}/2\mathbb{Z}} \langle p, x^2 \rangle$$

$$F_2 \rtimes \mathbb{Z}/2\mathbb{Z} = \langle p, x^2 \rangle *_{\mathbb{Z}/2\mathbb{Z}} \langle p, x^2 \rangle = \langle p, x^2 \rangle *_{\mathbb{Z}/2\mathbb{Z}} \langle p, x^2 \rangle$$

Finally, in $F_2 \rtimes D$,

$$F_2 \rtimes D_2 = D_2 *_{\mathbb{Z}/2\mathbb{Z}} \langle p, x^2 \rangle = \langle p, x^2 \rangle *_{\mathbb{Z}/2\mathbb{Z}} \langle p, x^2 \rangle$$

In the above, $S*_t$ denotes the HNN-extension with base group $S$ and stable letter $t$.

For example, $F_2 \rtimes (px, a^2\tau_b)$ has a presentation of the form

$$\langle \xi_1, \xi_2, a, b \mid \xi_1^2 = 1, [\xi_1, \xi_2] = 1, \xi_1a\xi_1 = a^{-1}, \xi_1b\xi_1 = b, \xi_2a\xi_2 = b^{-1}a^{-1}b, \xi_2b\xi_2 = b^{-1} \rangle$$

where $\xi_1 = px$ and $\xi_2 = x^2\tau_b$. By setting $\xi_2 = b\xi_2$ and eliminating $b$, we get

$$\langle \xi_1, \xi_2, a, \xi_2 \mid \xi_1^2 = 1, [\xi_1, \xi_2] = 1, \xi_1a\xi_1 = a^{-1}, \xi_2a\xi_2 = a^{-1} \rangle$$

Now by setting $\xi_3 = \xi_1a$ and eliminating $a$ we get

$$\langle \xi_1, \xi_2, \xi_3, \xi_2 \mid \xi_1^2 = 1, [\xi_1, \xi_2] = 1, [\xi_2, \xi_3] = [\xi_1, \xi_2] = 1 \rangle$$
which is the desired decomposition.

So now we can prove the following result which generalizes the result in [16] on the elements of finite order in \( \text{Aut}(F_2) \).

**Lemma 3.1.** An element of finite order in \( \text{Hol}(F_2) \) is conjugate to exactly one of \( p, px, pxτ, pxτa, pxτa, a^2, x^2b, y^2τ_b, y^2τ_b^{-1}, a \) and \( x \) with orders 2, 3, 4.

**Proof.** Given the above decomposition, every element of finite order is a conjugate of an element of a vertex group. So it suffices to observe the following:

\[
\begin{align*}
x^2(px^3τ_b)x^2 &= pxτa, \quad x^2(px^3b)x^2 = pxτa, \quad x^2(px^2)x^{-1} = p, \\
y(x^2τ_b)y^{-1} &= x^2, \quad b^{-1}(x^2τ_b^{-1}b) = x^2τ_b^{-1}b, \quad xτ_a^{-1}τ_b^{-1}(x^2τ_b^{-1}a)τ_b^{-1}τ_a^{-1} = x^2τ_b \\
yb^{-1}xτ_a^{-1}τ_b^{-1}(x^2τ_b)τ_b^{-1}τ_a^{-1}by^{-1} &= x^2b.
\end{align*}
\]

Notice also that \( x^2b \) is no longer conjugate to \( x^2 \) since the relation \( x^2 = y^3τ_b^{-1}τ_a \) has no equivalent for \( a \) and \( b \) due to the semidirect product structure of \( G \).

From the fact that \( \text{Hol}(F_2) = \langle a, b \rangle \rtimes \text{Aut}(F_2) \) we have that every element \( W \) of \( \text{Hol}(F_2) \) can be written uniquely in the form

\[
W = V z(a, b)
\]

where \( V \in \text{Aut}(F_2) \) and \( z(a, b) \) is a word in the free group \( \langle a, b \rangle \). So, the normal form for the elements of \( \text{Aut}(F_2) \) implies the existence of a normal form for the elements of \( \text{Hol}(F_2) \):

\[
W = p^ru(x, y)x^{2s}w(τ_a, τ_b)z(a, b)
\]

where \( u(t_a, t_b) \) is a reduced word in the free group \( \langle t_a, t_b \rangle \), \( u(x, y) \) is a reduced word where \( x, y, y^{-1} \) are the only powers of \( x, y \) appearing and \( r, s \in \{0, 1\} \). Moreover, every vertex group in the decomposition (1) has also a normal form. More specifically, every element in \( F_2 \rtimes B \) can be written uniquely in the form \( p^r x^s x^{2s} w(τ_a, τ_b) z(a, b) \) where \( r, n, s \in \{0, 1\} \) and every element in \( F_2 \rtimes C \) can be written uniquely in the form \( p^r y^s x^s w(τ_a, τ_b) z(a, b) \) where \( r, s \in \{0, 1\} \) and \( n \in \{0, 1, -1\} \) and \( w(τ_a, τ_b) \) is a reduced word in \( \langle τ_a, τ_b \rangle \) and \( z(a, b) \) is a reduced word in \( \langle a, b \rangle \).

Notice that one can define a natural epimorphism

\[
\text{Hol}(F_2) \to \text{GL}_2(\mathbb{Z})
\]

with kernel \( \langle a, b \rangle \rtimes \langle τ_a, τ_b \rangle \). In fact, \( \text{Hol}(F_2) \) fits into the following short exact sequence

\[
1 \to F_2 \rtimes F_2 \to \text{Hol}(F_2) \to \text{GL}_2(\mathbb{Z}) \to 1
\]

although such a sequence does not split.

We are searching for subgroups of \( G = \text{Hol}(F_2) \) which are isomorphic to \( A \rtimes \mathbb{Z} \) where \( A \) is a finite subgroup of \( G \). In our argument we shall make extensive use of the following well known result of Bass-Serre theory [17]. Let \( M \) be a group that acts on its standard tree \( T \) and \( m \in M \) such that \( m \) stabilizes two distinct vertices of \( T \). Then \( m \) stabilizes the (unique reduced) path that connects the two vertices. In particular, \( m \) is an element of every edge stabilizer of every edge that constitute the path that connects the two vertices.

In fact we shall show the following:

**Proposition 3.2.** The only finite-by-cyclic subgroups of \( G \) are \( \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z} \).
Lemma 3.1. Hence, a relation of the form $Dg \langle \langle g \rangle \rangle$ is conjugate to the other, i.e. there are no $F$'s be an element of $D$ unless $g = 1$, and by an element $x$ to the relation $gpg^{-1}$. The above relation implies the existence of the following relation in $\text{GL}(Z)$:

$$P'U(X,Y)X^{2s}PX^{2s}U^{-1}(X,Y)P'' = P$$

Proof. Claim 1. The only subgroups isomorphic to $A \times Z$ with $A$ finite cyclic are isomorphic to $Z/2Z \times Z$.

One can easily check that $\langle px, b \rangle \cong Z/2Z \times Z$ and so $Z/2Z \times Z$ is a subgroup group of $G$.

Now the only elements of order three in $\text{Hol}(F_2)$ are conjugates of $y^2 \tau^{-1}_b$ or $y^2 \tau^1_b$. Assume that there is a subgroup of $G$ isomorphic to $Z/3Z \times Z$. Then, conjugating if necessary, we may assume that there is an element of infinite order in $G$, say $g$, such that $g^{-1}(y^2 \tau^1_b a^s)g = (y^2 \tau^1_b a^s)^{±1}$ with $s \in \{0, 1\}$. Based on the decomposition 4, we see that the above relation implies that $y^2 \tau^1_b a^s$ stabilizes both vertices $F_2 \times C$ and $g^{-1}(F_2 \times C)$ and hence the path that connects them. So it belongs to the edge stabilizer $F_2 \times D$, unless $g \in F_2 \times C$. But if $g \in F_2 \times D$ we have a contradiction since by decomposition 4, $F_2 \times D$ cannot contain elements of order 3. Now if $g \in F_2 \times C$, then based again on the decomposition 4 of $F_2 \times C$, we have that $g$ stabilizes both $F_2 \times D$ and $g^{-1}(F_2 \times D)$ and so it belongs to $F_2 \times Z/2Z$, a further contradiction, unless again $g \in F_2 \times D$.

Finally, by the decomposition of $F_2 \times D_3 = D_3 \ast_{Z/2Z} D_3$ we have again that $g$ has to be an element of either of the two $D_3$ vertices and hence an element of finite order.

We shall show now that $G$ cannot contain subgroups isomorphic to $Z/4Z \times Z$. Assume that $G$ contains such a subgroup, say $A$. Then, $A$ is generated by a conjugate of $x$, since the conjugacy class of $x$ is the only class of elements of order 4, and by an element $g$ of $G$. Using conjugation if necessary, we may assume the element of order 4 in $A$ is $x$. Now let $g \in G$ such that $\langle x, y \rangle \cong Z/4Z \times Z$. Then $g^{-1}xg = x^{±1}$. Let $G$ act to the tree that corresponds to the decomposition 4. Then, due to the above relation, $x$ stabilizes both $F_2 \times B$ and $g^{-1}(F_2 \times B)$ and so it stabilizes the path between the two vertices. Hence, $x \in F_2 \times D$ a contradiction, unless $g \in F_2 \times B$.

Moreover, using the decomposition 4 we see that $g$ can only be an element of $F_2 \times D_4$ and using the fact that $F_2 \times D_4 = D_4 \ast_{Z/2Z} D_2$ it can only be an element of $D_4$ and so is of finite order. This completes the proof of claim 1.

Claim 2. There are no subgroups of $G$ of the form $D_2 \times Z$ or of the form $D_3 \times Z$.

Up to conjugacy, the possible $D_2$ in $G$ are $\langle x^2, p \rangle$, $\langle px, x^2 \rangle$, $\langle px, x^2b \rangle$, $\langle px, x^2\tau_b \rangle$, $\langle px, x^2\tau_b^{-1} \rangle$ and $\langle px, x^2\tau_b \rangle$, $\langle px, x^2\tau_b^{-1} \rangle$. Now notice that all last five, $\langle px, x^2 \rangle$, $\langle px, x^2b \rangle$, $\langle px, x^2\tau_b \rangle$, $\langle px, x^2\tau_b^{-1} \rangle$ and $\langle px, x^2\tau_b \rangle$, $\langle px, x^2\tau_b^{-1} \rangle$ appear only once in the graph of groups decomposition 4 of $\text{Hol}(F_2)$, as vertex groups. Moreover, in all five, none of the generators is conjugate to the other, i.e. there are no $g \in G$ such that $gpxg^{-1} = x^2$ or $gpxg^{-1} = x^2b$ or $gpxg^{-1} = x^2\tau_b$ or $gpxg^{-1} = x^2\tau_b^{-1}$ or $gpxg^{-1} = x^2\tau_b$ or $gpxg^{-1} = x^2\tau_b^{-1}$ by Lemma 4.1. Hence, a relation of the form $gD_2g^{-1} = D_2$ implies (repeating again the argument of claim 1) that $g$ is an element of finite order. So the only possibility for a semidirect product $D_2 \times Z$ lies with $\langle x^2, p \rangle$.

So assume that there is an element $g \in G$ such that $\langle g, p, x^2 \rangle = D_2 \times Z$. Then, since $p$ and $x^2$ are not conjugates and $px^2$ is conjugate to $p$, the action of $g$ is either $gpg^{-1} = p$ and $gx^2g^{-1} = x^2$, or $gpg^{-1} = px^2$ and $gx^2g^{-1} = x^2$. Let us concentrate to the relation $gpg^{-1} = p$. Given the normal form of $g = p' u(x, y)x^{2s}w(\tau_a, \tau_b)z(a, b)$ we have that

$$p' u(x, y)x^{2s}w(\tau_a, \tau_b)z(a, b)pz^{-1}(a, b)w^{-1}(\tau_a, \tau_b)x^{-2s}u^{-1}(x, y)p^{-r} = p.$$
which is equivalent to
\[ UPU^{-1} = P. \]

By the normal form for the elements of GL$_2$($\mathbb{Z}$), we have that $U$ is of the form $U = XY^{e_1} \ldots XY^{e_k}$ with $e_i \in \{\pm 1\}$. So the word $UPU^{-1}$ becomes
\[
XY^{e_1} \ldots XY^{e_k} PY^{-e_1} X^{-1} \ldots Y^{-e_1} X^{-1} = \\
XY^{e_1} \ldots XY^{e_k} Y^{e_1} X \ldots Y^{e_1} X : P = \\
X^2 \cdot XY^{e_1} \ldots XY^{e_k-1} X \cdot Y^{-1} \cdot XY^{e_k-1} \ldots Y^{e_1} X \cdot P \text{ if } e_k = 1 \\
X^2 \cdot XY^{e_1} \ldots XY^{e_k-1} X \cdot Y \cdot XY^{e_k-1} \ldots Y^{e_1} X \cdot P \text{ if } e_k = -1 \\
XY^{e_1} \ldots XY^{-e_1} X \ldots Y^{e_1} XP \text{ if } e_k = 0 \text{ and } e_{k-1} = \mp 1.
\]

In all cases, the relation $UPU^{-1} = P$ is impossible, since, after deletions of $P$, the remaining word is reduced as written so is never trivial, unless $U = 1$. Hence $u = 1$ and so the only possible $g = p^r x^{2s} w(\tau_a, \tau_b) z(a, b)$. Then the relation $g pg^{-1} = p$ gives
\[
x^{2s} w(\tau_a, \tau_b) z(a, b) p z^{-1}(a, b) w^{-1}(\tau_a, \tau_b) x^{2s} = p
\]
i.e. $w(\tau_b, \tau_a) w^{-1}(\tau_a, \tau_b) = 1$. One can easily see that if $w(\tau_a, \tau_b)$ is a reduced word in $\text{Inn}(F_2) \cong F_2$ then the word $w(\tau_a, \tau_b)$ is reduced and cyclically reduced as written. Hence, a relation $w(\tau_b, \tau_a) w^{-1}(\tau_a, \tau_b) = 1$ is impossible unless $w = 1$. That implies $z(b, a) z^{-1}(a, b) = 1$ and again $z = 1$. Then $g = p^r x^{2s}$ which has finite order for all possible $r, s$.

Let us now examine the possibility $g pg^{-1} = px^2$. This implies that
\[
p^r u(x, y) x^{2s} w(\tau_a, \tau_b) z(a, b) p z^{-1}(a, b) w^{-1}(\tau_a, \tau_b) x^{-2s} u^{-1}(x, y) p^{-r} = px^2.
\]

Projection to GL$_2$($\mathbb{Z}$) gives
\[
P^r U(X, Y) X^{2s} P X^{2s} U^{-1}(X, Y) P^r = PX^2
\]
which is equivalent to
\[
UPU^{-1} = PX^2.
\]

Performing the same analysis as above for $UPU^{-1}$ we get that the only possibility is $U = X$, hence $u = x$. Then $g = p^r x^{2s} w(\tau_a, \tau_b) z(a, b)$ and so the relation $p^r x^{2s} w(\tau_a, \tau_b) z(a, b) p z^{-1}(a, b) w^{-1}(\tau_a, \tau_b) x^{2s} x^{-1} p^r = px^2$ implies again that $w(\tau_b, \tau_a) w^{-1}(\tau_a, \tau_b) = 1$ which possible if and only if $w = 1$ and so $z(b, a) z^{-1}(a, b) = 1$ which is possible if and only if $z = 1$.

Finally, one can easily check that existence of $g \in G$ such that $g D_3 g^{-1} = D_3$ can only occur for $g$ of finite order (using again the previous arboreal argument), so we have that no subgroup of the form $D_3 \rtimes \mathbb{Z}$.

The only case left is subgroups isomorphic to $D_4 \rtimes \mathbb{Z}$. It is easy to see that then it will contain subgroups isomorphic to $D_2 \rtimes \mathbb{Z}$, which is impossible. \hfill $\Box$

4. **Vanishing of lower $K$-theory of Hol($F_2$)**

We will prove the main result of the paper. For a group $G$, we write
\[
\text{Wh}_q(G) = \begin{cases} 
    \text{Wh}(G), & \text{if } q = 1, \\
    K_0(\mathbb{Z}G), & \text{if } q = 0, \\
    K_q(\mathbb{Z}G), & \text{if } q < 0.
\end{cases}
\]

**Theorem 4.1.** Let $\Gamma < \text{Hol}(F_2)$. Then for all $q \leq 1$, $\text{Wh}_q(\Gamma) = 0$. 
Proof. We will show the theorem for $G = \text{Hol}(F_2)$. The proof for $\text{Aut}(F_2)$ is similar. By Proposition 2.4, $G$ satisfies the FIC. Let $\Gamma < \text{Hol}(F_2)$. Then by [8], $\Gamma$ also satisfies the FIC. Thus the maps
\[ H^q_G(E_{FBC} \Gamma; \mathbb{K} \mathbb{Z}^{-\infty}) \to \text{Wh}_q(\mathbb{Z}) \eta, \quad q \leq 1, \]
are isomorphisms. There is a spectral sequence that computes the left hand side of such an isomorphism:
\[ E^2_{i,j} = H^i_G(E_{FBC} \Gamma; \text{Wh}_j(V)) \Rightarrow \text{Wh}_{i+j}(\Gamma), \]
where $V$ is in $FBC$. Now, by the decomposition of $\text{Hol}(F_2)$ and Proposition 3.2:

1. If $V$ is finite, $V$ will isomorphic to one of the following groups: $\mathbb{Z}/2\mathbb{Z}$, $\mathbb{Z}/3\mathbb{Z}$, $\mathbb{Z}/4\mathbb{Z}$, $D_2$, $D_4$. But in this case from the lists in [11] and [12]
   \[ \text{Wh}_q(V) = 0, \quad \text{for } q \leq 1. \]

2. If $V$ is infinite, then $V = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}$. Using the Bass–Heller–Swan Formula and the calculations of the Nil-groups in [3], we have that:
   \[ \text{Wh}_q(V) = 0, \quad \text{for } q \leq 1. \]
Thus $\text{Wh}_q(\Gamma) = 0$ for all $q \leq 1$. □

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