Systematic Study of Theories with Quantum Modified Moduli

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Abstract

We begin the process of classifying all supersymmetric theories with quantum modified moduli. We determine all theories based on a single $SU$ or $Sp$ gauge group with quantum modified moduli. By flowing among theories we have calculated the precise modifications to the algebraic constraints that determine the moduli at the quantum level. We find a class of theories, those with a classical constraint that is covariant but not invariant under global symmetries, that have a singular modification to the moduli, which consists of a new branch.

11.30.Pb, 11.15.-q, 11.15.Tk
I. INTRODUCTION

Supersymmetric gauge theories with matter fields generally have a large degeneracy of inequivalent vacua. The space of vacua, or ‘moduli space’, can be readily determined at the classical level. After quantization the problem of determining the moduli space is more difficult because asymptotically free theories can be strongly coupled. Seiberg has studied the phase structure of SUSY QCD with $N_c$ colors and $N_f$ flavors [1]. It was known that the theory has runaway vacua [2] for $N_f < N_c$. Seiberg argued that for $N_f = N_c$ the moduli space (of vacua) is modified by quantum effects while for $N_f = N_c + 1$ the theory displays confinement without chiral symmetry breaking. For $N_f > N_c + 1$ he found dual descriptions in which the magnetic dual coupling is weak when the electric one is strong and vice versa.

Following Seiberg many have studied a number of specific SUSY gauge theories. Intriligator and Pouliot repeated Seiberg’s analysis for $Sp(N_c)$ gauge groups with $2N_f$ flavors (matter fields in the fundamental representation) [3]. Pouliot [4] and Trivedi and Poppitz [5] studied $SU(N)$ theories with an antisymmetric tensor field and Pouliot and Strassler [6] with a symmetric tensor. Many other examples and references can be found in reviews [7].

In those investigations the emphasis was on finding the behavior of a particular theory, or class of theories. Csaki, Schmaltz and Skiba took a different approach [8]. They attempted to find all theories that display a particular effect. To this end they define “s-confinement” as a generalization of confinement without chiral symmetry breaking as obtained by Seiberg for SUSY QCD with $N_f = N_c + 1$. They proceed to find all SUSY gauge theories based on a simple gauge group that display s-confinement.

In this paper we take a similar track. We begin a study of all SUSY gauge theories with a quantum modified moduli space. We determine all theories based on a simple $SU(N)$ or $Sp(N)$ gauge group with a quantum modified moduli space. We have not attempted to study exceptional or orthogonal gauge groups. Theories with a modified moduli space are of interest per se. The quantum modification is poorly understood, inferred only from consistency conditions. These theories can be used to fabricate models of dynamical SUSY breaking [12].

One may describe the classical moduli space in terms of gauge invariant composite operators. The moduli is the space of values these operators may take, modulo algebraic constraints. At the quantum level the description of the moduli space is still in terms of these operators. The modification of the moduli space is to be found in a modified algebraic constraint. It is therefore useful to know, a priori, how many constraints one must have (given a choice of composite operators to describe the moduli). We derive a simple formula for the dimension of the moduli space which then gives us the number of required constraints.

The constraint specifying the moduli may be either invariant or covariant under the non-anomalous global symmetries of the theory. In the former case the quantum moduli space differs from the classical in that the origin has been smoothly excised. But when the constraint is covariant the origin remains in the quantum moduli space. In order for the ‘t Hooft anomaly condition to be satisfied at the origin, one mode must be excluded, and this can be implemented in two distinct ways. The constraint can be used to express one mode in terms of the rest, and therefore this mode does not contribute to the anomaly. Alternatively one can implement the constraint with a Lagrange multiplier in a superpotential. In this way
we find that one of the constrained modes, classically massless, becomes massive. Integrating out this mode leaves unconstrained the rest of the modes. The only quantum effect has been to pick which mode to eliminate by the constraint, save for an interesting subtlety. Going to infinity in moduli space along a particular direction we find a new branch of moduli space. On this branch the global $U(1)_R$ symmetry is spontaneously broken. A similar situation has been found for theories with branches in a Coulomb phase [9–11], but with the obvious distinction that in the theories we consider there is no local symmetry on the branch.

The methods we use are similar to those of Csaki et al. They used a condition on a certain sum of indices of the representations for the particle and gauge contents. This condition significantly reduced the number of all possible theories. The number of theories was further reduced by studying the flow to other theories with a phase structure incompatible with s-confinement. The remaining theories were checked one by one to be in the s-confining phase.

An index condition can also be used to classify theories with a quantum modified moduli space. With the help of a generalized flow, we not only check the phase structure of our potentially interesting theories, but also use it to determine how the gauge invariant operators and the constraints flow from one theory to another. In this way we can determine the quantum modified constraints explicitly. One could also use our generalized flow to determine explicitly the precise form of the constraints in the s-confining theories considered by Csaki et al.

The paper is organized as follows. In Sect. II we classify theories according to whether the algebraic constraint specifying the moduli is invariant or covariant under global symmetries and discuss the correspondingly different structure of the moduli space. In Sect. III we review the index condition for the s-confining theories and for theories with quantum modified moduli. We also explain there the additional conditions from the flow of the theories and give some examples of gauge invariant operator flow. Our formula for the number of constraints is explained in Sect. IV. The methods introduced are then put to work in an explicit example in Sect. V. The results for the $SU(N)$ and $Sp(N)$ theories are presented in Sect. VI. We list all the theories obeying the index condition along with their phase structure. For the theories not yet discussed in the literature, we write down the gauge invariant operators and the exact constraint. We come to a conclusion in Sect. VII.

In the appendix we list all the gauge invariant operators with their precise index structure. This is important because there is no unique choice of operators. Another choice will generally change the precise form of the constraint.

It may appear that the qualitative results obtained here do not require a precise determination of the form of the constraints. However, care must be exercised in not choosing redundant operators. That is, some of the operators used to described the classical moduli space may not be independent even though it may appear so a priori. We have found that deriving quantitatively precise constraints guards against such errors. Moreover, we believe it will be of general use to both model builders and field theorists to have a complete tabulation of the precise constraints. We have undertaken the task here.

II. THEORIES WITH QUANTUM MODIFIED MODULI

The theories with quantum modified moduli (QMM) generalize Seiberg’s SUSY QCD with $N_c = N_f$. QMM theories are confining. The moduli space is described by a set of
composite gauge invariant operators. A generic feature of QMM theories is that the dimension of the vacuum is smaller than the number of independent gauge invariant ‘composite’ operators. Both classically and quantum mechanically the moduli space is specified by algebraic constraints among the composite operators. In theories with QMM the quantum and classical constraints differ.

Returning to our prototype, supersymmetric QCD with $N_c = N_f \equiv N$, we recall that the moduli is described by a matrix valued composite $M_{ij}$ transforming as $(N, \bar{N})$ under the global symmetry group $SU(N) \times SU(N)$, and two composites $B$ and $\tilde{B}$ transforming as singlets. The classical moduli is the space of these composites, modulo the constraint $\det(M) - B\tilde{B} = 0$. At the quantum level the origin is excised from the moduli space; the QMM is described by the modified constraint $\det(M) - B\tilde{B} = \Lambda^{2N}$. Notice that the constraints remain invariant under the global symmetry group.

In this example the origin of moduli space was taken out. This is not generic for quantum modified theories. When the classical constraint $F(\phi_i) = 0$ for composites $\phi_i$ is covariant but not invariant under the global symmetries, the quantum modification cannot be simply replacing the right hand side by a non-vanishing constant. This would break the global symmetries. Instead, the right hand side will turn out to be of the form $\Lambda^p \phi_k$, where $\phi_k$ is a composite with the right transformation properties under the global symmetry group and the power $p$ is governed by dimensional analysis.

Theories in which the constraint is covariant (c-QMM’s) have different physics than those with invariant constraints (i-QMM’s). In c-QMM’s the particle corresponding to the composite $\phi_k$ that appears in the quantum modification becomes massive. The description of the moduli space should not include $\phi_k$. In contrast, in i-QMM’s one is free to solve the constraint for any one composite in terms of the others.

To see this introduce a lagrange multiplier chiral superfield $\lambda$ and use it to enforce the constraint by means of a superpotential

$$W = \lambda (F(\phi_i) - \Lambda^p).$$

for i-QMM’s and

$$W = \lambda (F(\phi_i) - \Lambda^p \phi_k).$$

for c-QMM’s. In Eq. (2.1) the lagrange multiplier simply enforces the constraint $F(\phi_i) = \Lambda^p$. But in Eq. (2.2) the lagrange multiplier plays a dynamical physical role: it pairs up with the composite $\phi_k$ into massive states. Integrating out these massive states leaves a theory with $\phi_k$ (and $\lambda$) excluded and a vanishing superpotential.

We believe this to be the correct realization of the constraint for the case of theories with c-QMM. The obvious alternative is to apply the constraint $F(\phi_i) = \Lambda^p \phi_k$ directly to the description of the moduli. There are physical distinctions between these two approaches, as seen in the paragraph below. Our believe in the lagrange multiplier method can be supported by the following argument. As discussed below, s-confining theories flow into theories with i-QMM’s. These have constraints implemented by lagrange multipliers which can be identified with modes of the parent s-confining theories.

Thus the field $\lambda$ must be considered a dynamical field. And this leads to a surprising modification of the c-QMM. There is a branch parametrized by $\lambda$ itself, with the other fields determined from
\[
\lambda \frac{\partial F}{\partial \phi_i} = 0 \quad (i \neq k) \tag{2.3}
\]
\[
\lambda \left( \frac{\partial F}{\partial \phi_k} - \Lambda^p \right) = 0 \tag{2.4}
\]
\[
F(\phi_i) - \Lambda^p \phi_k = 0 \tag{2.5}
\]

To solve these for arbitrary \( \lambda \) generally requires that one of the \( \phi_i \) tend to infinity as some of the others approach the origin. This may seem bizarre, but we know of no reason why such solutions should be excluded. On this moduli subspace the \( U(1)_R \) symmetry is broken. In addition, if \( \phi_k \) carries any other non-anomalous global symmetry then \( \lambda \) must carry the opposite charge and this symmetry is also broken on this branch. The two real scalar components of \( \lambda \) can be understood as the corresponding goldstone bosons.

III. INDEX CONDITIONS AND FLOWS

A. The Index Condition

1. The Index Condition for s-confining theories

Csaki, Schmaltz and Skiba introduced “smooth confinement without chiral symmetry breaking and with a non-vanishing confining superpotential”, or “s-confinement” for short, as a generalization of SUSY QCD with \( N_f = N_c + 1 \). It is defined as follows. An s-confining theory must admit a description in terms of gauge invariant composite operators everywhere on the moduli space. The infrared effective theory must have a smooth superpotential, ie, polynomial in the gauge invariant operators. The origin of the classical moduli space must also be a vacuum of the quantum moduli space. The definition excludes theories which admit a Coulomb phase somewhere on the moduli space and theories which have boundaries in the moduli space between distinct Higgs and confinement phases. Consequently the t’Hooft anomalies should match between the short and long distance descriptions everywhere in the moduli space, and this was found to be true by explicit computation.

To explain the index condition for s-confining theories we need to introduce some notation. Consider a supersymmetric theory with gauge group \( G \) and \( N \) chiral matter multiplets, \( Q_1, \ldots, Q_N \). In the absence of a superpotential there are \( N \) global \( U(1) \) symmetries, one for each matter field, corresponding to separate flavor number. There is also a \( U(1) \) \( R \)-symmetry. All these symmetries are broken at the quantum level by anomalies, but one may combine the \( U(1) \) \( R \)-symmetry with each of the global flavor numbers to form \( N \) conserved \( R \)-symmetries, \( U(1)_{R_1}, \ldots, U(1)_{R_N} \) with the following charge assignments:

| \( U(1)_{R_1} \) | \( U(1)_{R_2} \) | \( \cdots \) | \( U(1)_{R_N} \) |
|---|---|---|---|
| \( Q_1 \) | \( a_1 \) | 0 | 0 |
| \( Q_2 \) | 0 | \( a_2 \) | 0 |
| \( \cdots \) | \( \cdots \) | \( \cdots \) | \( \cdots \) |
| \( Q_N \) | 0 | 0 | \( a_N \). |
The $R$-charges $a_i$ are fixed by requiring the vanishing of the gauge anomaly. Denoting by $\mu_G$ and $\mu_i$ the indices of the adjoint and of the representation of $Q_i$, normalized to unity for the fundamental representation, one finds:

$$a_i = \frac{\left( \sum_{j=1}^{N} \mu_j - \mu_G \right)}{\mu_i}.$$

Now, $s$-confining theories must admit a smooth superpotential. It must carry 2 units of every one of the $R$ charges. Since only the $i$-th field carries $R_i$ charge, it must enter the superpotential as $Q_i^{2/a_i}$. The superpotential must be a combination of terms of the form

$$\Lambda^3 \prod_{i=1}^{N} (Q_i/\Lambda)^{2\mu_i/(\sum_{j=1}^{N} \mu_j - \mu_G)}.$$

$\Lambda$, a dynamical mass scale, is introduced by dimensional analysis. If there is at least one chiral superfield transforming as the fundamental (or antifundamental) of the gauge group, which is always the case in Csaki et al, then the smoothness of the superpotential requires $\sum_{j=1}^{N} \mu_j - \mu_G = 1$ or 2. Csaki et al argue that, in fact, only the second solution is available. For $Sp(N)$ theories this can be seen from Witten’s anomaly, which requires an even number of fundamentals. The index condition for $s$-confinement for theories with at least one fundamental is therefore

$$\sum_{j=1}^{N} \mu_j - \mu_G = 2.$$

If the theory has no matter fields transforming as the fundamental (or antifundamental) representation the index condition is relaxed: $\sum_{j=1}^{N} \mu_j - \mu_G$ or $(\sum_{j=1}^{N} \mu_j - \mu_G)/2$ must be a common divisor of all the $\mu_i$.

2. The Index Condition for Theories with Quantum Modified Moduli

The classical constraint between the composites $\phi_i$ is a non trivial polynomial,

$$\sum_{n=1}^{m} \left( \prod_{i=1}^{k_n} \phi_i \right)_n = 0.$$

The quantum modification generically is of the form

$$\sum_{n=1}^{m} \left( \prod_{i=1}^{k_n} \phi_i \right)_n = \prod_{i=1}^{r} \phi_i \Lambda^p.$$

Notice that we have allowed for a product of composites on the right hand side. In all the cases we study we find, however, at most one composite on the right hand side. The exact form of the left hand side, $\sum_{n=1}^{m} \left( \prod_{i=1}^{k_n} \phi_i \right)_n$, is determined by the classical limit.

The index condition now follows from requiring that the constraint be covariant under global $U(1)$ R-symmetries. As in our review of $s$-confining theories we introduce an anomaly.
free $U(1)$ R-symmetry for each chiral superfield. Because the left and right sides of the constraint in Eq. (3.1) have different number of composites, at least one of the R-charges must vanish. For this we must have the index condition \[ \sum_{i=1}^{n} \mu_i - \mu_G = 0. \] (3.2)

In an alternative derivation of the index condition we adopt the point of view that $\Lambda_{b_0}$ is a background chiral superfield. Now consider the $R$ symmetry with all the $R$ charges of the chiral superfields set to vanish. The assigned R-charge of $\Lambda_{b_0}$ is given by the anomaly, \[ Q_R(\Lambda_{b_0}) = \sum_{i=1}^{n} \mu_i - \mu_G. \]

The left side of our constraint, however, has an R-charge of zero. Therefore, $\Lambda_{b_0}$ has an R-charge of zero and we have again Eq. (3.2).

To find all QMM theories one must begin by classifying all theories that satisfy Eq. (3.2). Since the fundamental representation has $\mu_{\text{fund}} = 1$, adding a pair of chiral superfields, one in the fundamental and one in the antifundamental representations, to a QMM theory gives a theory with $\sum_{i=1}^{n} \mu_i - \mu_G = 2$. These are candidates for s-confinement and were classified by Csaki et al. Therefore all theories satisfying the index condition (3.2) can be obtained from the list of s-confinement candidates of Csaki et al by removing a fundamental and an antifundamental. Clearly removing a pair ensures that all the gauge anomalies remain absent. Section VI contains tables listing the complete set of QMM candidates based on $SU$ and $Sp$ gauge groups.

B. The Flow

1. The Flow of the Theories

The index condition gives only a necessary condition. To find out if a candidate theory actually has a QMM, one must make some other investigations. As the next step to sort out all QMM theories we consider points in the classical moduli space where the gauge group of our candidate theory is broken. The gauge fields which correspond to the broken generators acquire a mass proportional to the vacuum expectation value of the Higgs field. These massive gauge superfields pair up with chiral superfields which become massive through the Higgs mechanism as well. Together they form a massive supermultiplet. We integrate out these heavy degrees of freedom. The new theory, which is an effective theory of the original ‘UV’ theory, should be in a phase consistent with the UV theory being in a quantum modified phase. We refer to this as ‘the flow’ of the UV theory to an effective theory. If the theory flows to a theory in a Coulomb phase we say that the theory has a Coulomb branch, not a QMM. By studying the flow we can, therefore, rule out quite a few theories which fulfill the index condition.

It is useful to tabulate the manner in which theories may flow. Below we list the gauge groups together with their particle content. The latter is contained in square brackets and is represented by the Young tableaux of the corresponding representation, with a possible
multiplier when there are more than one field for that representation. We don’t list any gauge singlets that may remain in the effective theory. These are not all the possible flow diagrams. They were, however, sufficient for our classification work.

\[
SU(N) [ N(\blacksquare + \blacksquare) ] \rightarrow SU(N - 1) [ (N - 1)(\blacksquare + \blacksquare) ] \tag{3.3}
\]

\[
SU(N) [ \blacksquare + (N - 1)\blacksquare + 3\blacksquare ] \rightarrow SU(N - 1) [ \blacksquare + (N - 2)\blacksquare + 3\blacksquare ]
\]

\[
\downarrow
Sp(N) [ (N + 2)\blacksquare ] \rightarrow Sp(N - 2) [ (N)\blacksquare ]
\]

\[
SU(N) [ \blacksquare + \blacksquare + 2(\blacksquare + \blacksquare) ] \rightarrow SU(N - 1) [ \blacksquare + \blacksquare + 2(\blacksquare + \blacksquare) ]
\]

\[
\downarrow
Sp(2N) [ \blacksquare + 4\blacksquare ]
\]

\[
SU(N) [ \text{Adj} ] \rightarrow \text{Coulomb branch} \tag{3.6}
\]

\[
SU(6) [ \blacksquare + 3(\blacksquare + \blacksquare) ] \rightarrow SU(3) \times SU(3) [ 3((1,\blacksquare) + (\blacksquare,1) + (1,\blacksquare) + (\blacksquare,1)) ]
\]

\[
\downarrow
SU(5) [ 2\blacksquare + 1\blacksquare + 3\blacksquare ] \rightarrow Sp(4) [ \blacksquare + 4\blacksquare ]
\]

\[
\downarrow
SU(4) [ 2\blacksquare + 2(\blacksquare + \blacksquare) ] \quad \text{(This is special case of } SU(N) [ \blacksquare + \blacksquare + 2(\blacksquare + \blacksquare) ].)
\]

\[
SU(4) [ 3\blacksquare + 1(\blacksquare + \blacksquare) ] \rightarrow Sp(4) [ 2\blacksquare + 2\blacksquare ] \rightarrow (SU(2) \times SU(2)) [ (\blacksquare,1) + (1,\blacksquare) + (\blacksquare,\blacksquare) ]
\]

\[
\downarrow
SU(3) [ 3(\blacksquare + \blacksquare) ]
\]

\[
SU(4) [ 4\blacksquare ] \rightarrow Sp(4) [ 3\blacksquare ] \rightarrow \text{Coulomb branch} \tag{3.9}
\]

\[
SU(5) [ 2\blacksquare + \blacksquare + 1\blacksquare ] \rightarrow Sp(4) [ 2\blacksquare + 2\blacksquare ] \rightarrow (SU(2) \times SU(2)) [ (\blacksquare,1) + (1,\blacksquare) + (\blacksquare,\blacksquare) ] \tag{3.10}
\]

\[
SU(6) [ 2\blacksquare ] \rightarrow SU(3) \times SU(3) [ ((\blacksquare,\blacksquare) + (\blacksquare,\blacksquare)) ] \rightarrow \text{Coulomb branch} \tag{3.11}
\]

\[
SU(7) [ \blacksquare + 4\blacksquare + 2\blacksquare ] \rightarrow SU(3) \times SU(3) [ (3(1,\blacksquare) + (\blacksquare,1)) + (\blacksquare,\blacksquare) ]
\]

\[
\downarrow
SU(6) [ \blacksquare + \blacksquare + 2\blacksquare ] \rightarrow SU(3) \times SU(3) [ (3(1,\blacksquare) + (\blacksquare,1)) + (\blacksquare,\blacksquare) ]
\]

\[
\downarrow
Sp(6) [ \blacksquare + 3\blacksquare ] \rightarrow Sp(4) [ 2\blacksquare + 2\blacksquare ] \rightarrow (SU(2) \times SU(2)) [ (\blacksquare,1) + (1,\blacksquare) + (\blacksquare,\blacksquare) ]
\]

\[
\downarrow
SU(3) [ 3(\blacksquare + \blacksquare) ]
\]

\[
Sp(2N) [ \blacksquare = \text{Adj} ] \rightarrow \text{Coulomb branch.} \tag{3.13}
\]
2. Flow of Operators

The flow is useful in determining the quantum modified constraints precisely. For example given a classical constraint, one can immediately write a putative quantum modified constraints as in Eqs. (2.1) or (2.2). The precise coefficient is then determined by flowing to a theory with exactly known constraint, such as SUSY QCD with $N_f = N_c$.

This works because the flow maps not just theories but also specific operators between the UV and effective theories. This is very useful in the determination of the classical constraints too. Given a classical constraint in a UV theory one can generally determine the constraint in any of its effective theories by following the flow. It is not so obvious that one may infer the constraint of a UV theory if the constraints of its effective theories are known. In fact, in practice one finds that for many cases one needs only the constraints of one of the effective theories. We found this reverse flow procedure of central importance in our investigations of the more complicated theories.

For example, one can start from the known theory $SU(3)[3(\square + \bar{\square})]$ and map the gauge invariant operators up to $SU(4)[3\square + 1(\square + \bar{\square})]$. Then mapping down to $Sp(4)[2\square + 2\bar{\square}]$ is possible. One can then determine how the operators are mapped from one theory to the next. For example,

$$SU(4)[Q\bar{Q}] \rightarrow Sp(4)[Q_1Q_2]$$

where $Q\bar{Q}$ and $Q_1Q_2$ represent composites of the $SU(4)$ and $Sp(4)$ theories, respectively (if the notation is not self-evident, it will be clarified in Section [VI]). Thus one can find all the constraints of these $SU(4)$ and $Sp(4)$ theories. The constraints are determined explicitly, that is, all the numerical coefficients are fixed. One can obtain all the results for the remaining theories with similar sequences of reverse and forward flows.

Mapping the gauge invariant operators from a smaller theory to a bigger theory may be problematic. The Higgs mechanism may map some of the gauge invariant operators to singlets in the effective theory or may even render some of the gauge invariant operators in the UV theory equal to zero. This implies that we would break our theory to a place of the moduli space which has a dimension strictly smaller by more than one. The zero operators or the singlet operators would not show up and all the terms in which they do show up would not be in the constraint if we map the constraint from the effective theory to the UV theory. This happened only twice in our analysis. When this happened, however, we could always determine the constraint by requiring covariance under global symmetries.

Other useful examples of operator mapping are presented in the sequences below, in which $Q$ and $\bar{Q}$ always stand for a fundamental and anti-fundamental ($\square$ and $\bar{\square}$), and $A$ and $B$ for a two and three index antisymmetric tensors ($\square$ and $\bar{\square}$), respectively. The composite operators are denoted by their components, and a subscript “anti” is included when only the antisymmetric part is included (the precise description of the operators can be found in the appendix). The mappings are

$$SU(7)[B^3\bar{Q}^3Q_{anti}] \rightarrow SU(6)[(B^2A\bar{Q}^2)_{anti}] \rightarrow Sp(6)[B\bar{Q}^3] \rightarrow Sp(4)[Q_1Q_2]$$

for the theory flow in (3.12), and

$$SU(5)[A^2\bar{A}\bar{Q}^2] \rightarrow Sp(4)[Q_1Q_2]$$

for the flow in (3.10).
IV. DIMENSION OF THE MODULI SPACE

We derive formulas for the dimension of the classical moduli space. These formulas give a relation between the number of gauge invariant operators and the number of constraints. Because the SUSY lagrangian is invariant under the complexified gauge group $G_c$, the moduli space $M_0$ is equal to

$$M_0 = F \parallel G_c.$$ 

$F$ is the space of all constant field configurations if there is no superpotential. It is the space of all extrema of the superpotential if there is a superpotential. The equivalence relation between two elements $\Phi$ and $\Phi_0$ of the same $G_c$ orbit is of the generalized form

$$\lim_{g_i} \Phi = \Phi_0$$

with $g_i \in G_c$.

$F \parallel G_c$ can be described as an algebraic variety of all gauge invariant holomorphic polynomials [13]. Therefore the dimension of the vacuum is

$$\dim \text{vacuum} = N_{\text{Ops}} - N_{\text{Con}}$$

where $N_{\text{Ops}}$ and $N_{\text{Con}}$ are the number of independent gauge invariant operators and constraints, respectively. But there is a natural map $\pi$ between $F$ and $M_0$. This map induces a map between the tangent spaces of $F$ at the generic point $\phi \in F$ and the tangent space at the point $\pi(\phi)$. This map is a surjective homomorphism if $\phi$ is a point on the moduli space which breaks the gauge group totally. The kernel is $G_c \phi$ [14]. That $\phi$ is on the moduli space implies of course that the D-flat condition is fulfilled. It follows directly that:

$$\dim \text{vacuum} = \dim F - \dim G_c.$$ (4.2)

$s$-confining theories and quantum modified theories have no superpotential and they have always points on the moduli space where the gauge group is completely broken.

The two formulas for the dimension of the moduli space allow us to calculate easily the difference between the number of gauge invariant operators and constraints. As an example consider $SU(5)$ with $(2\Box + \Box + 3\Box)$ (example 4.1.3 in Ref. [8]). There are 18 gauge invariant operators and the dimension of the moduli space, as given by Eq. (4.2) is 16, so that there must be two constraints. The constraints are easily obtained by integrating out $(Q_2Q_4)$ and $(A^2Q_1Q_2Q_4)$ from the superpotential of the corresponding $s$-confining theory, $SU(5)$ with $(2\Box + 2\Box + 4\Box)$ (here we are using the notation of Ref. [8]). Alternatively one can use the operator flow between $SU(5)$ with $(2\Box + \Box + 3\Box)$ and $SU(4)$ with $(2\Box + 2\Box + 2\Box)$ to map the two constraints for the $SU(4)$ theory to the constraints of the $SU(5)$ theory. One obtains two constraints, one is quantum modified and the other is not. The corresponding superpotential is:

$$W = \lambda[(A^3Q)^2(Q\overline{Q}) + (A^3\overline{Q})(A^2Q)(A\overline{Q})^2] - \Lambda^{10} + \mu[(A^3\overline{Q})^a(A\overline{Q}^2)_{jk}^b \epsilon^{jk} \epsilon_{ab}].$$
V. AN EXAMPLE

Before giving our results we present an example in which we apply all the tools presented above. Consider an $SU(4)$ gauge theory with 3 antisymmetric tensors $A_{i\beta}$, a fundamental $Q_{\alpha}$ and an antifundamental $\bar{Q}^\alpha$. Since $\mu_G = 8$, $\mu_A = 2$ and $\mu_Q = \mu_{\bar{Q}} = 1$ we see that the index condition, $\sum_{i=1}^{n} \mu_i - \mu_G = 0$, is satisfied. Adding an additional $Q$, $\bar{Q}$, gives a theory with $\sum_{i=1}^{n} \mu_i - \mu_G = 2$ which is not s-confining.

According to Eq. (3.8) the dimension of the classical moduli space is $3 \times 6 + 2 \times 4 - 15 = 11$. To determine the number of constraints we need a choice of composites. Consider the obvious choice

\begin{equation}
(AA_{\text{sym}})^{ij} = A_{\alpha\beta}^i A_{\gamma\delta}^j \epsilon^{\alpha\beta\gamma\delta}
\end{equation}

\begin{equation}
(AAQ\bar{Q})^{ij} = A_{\alpha\beta}^i A_{\gamma\delta}^j Q_{\eta} \bar{Q}^\alpha \epsilon^{\beta\gamma\delta\eta}
\end{equation}

It would seem that these 15 operators are sufficient to characterize the 11 dimensional moduli space if four constraints are imposed. However subspaces of the moduli characterized by $A = 0$ with arbitrary $Q = \bar{Q}^\dagger$ are not properly parametrized by these composites. We see that we need in addition

\begin{equation}
(Q\bar{Q}) = Q_{\alpha} \bar{Q}^\alpha
\end{equation}

This set of operators is not independent. One can verify that the part of $(AAQ\bar{Q})$ symmetric under $i \leftrightarrow j$ is proportional to $(QQ)(AA_{\text{sym}})$. We do not consider this a constraint, for the relation involves $(AAQ\bar{Q}_{\text{sym}})$ linearly: one should simply exclude this operator.

What operators might we need, in addition to $(QQ)$, $(AA_{\text{sym}})$ and $(AAQ\bar{Q}_{\text{anti}})$, to describe the moduli? To answer this we flow to $SU(3)$, along directions of non-vanishing $Q = \bar{Q}^\dagger$, as in Eq. (3.8). This theory is the familiar example analyzed by Seiberg and has a classical constraint $\det(M) - BB = 0$ involving baryons. However none of the operators above flow to these baryons. To remedy this we include in our list

\begin{equation}
(AAAQQ) = 1/6(Q^\alpha A_{\alpha\beta}^i Q^\gamma A_{\gamma\delta}^j A_{\eta}^k \epsilon^{\beta\gamma\delta\eta} \epsilon_{ijk})
\end{equation}

\begin{equation}
(AAAQQ) = 1/6(A_{\alpha\beta}^i A_{\gamma\delta}^j A_{\eta}^k Q_{\eta} Q_{\lambda} \epsilon^{\alpha\beta\gamma\lambda} \epsilon_{ijk})
\end{equation}

which flow to $B$ and $\bar{B}$.

The set of operators $(QQ)$, $(AA_{\text{sym}})$, $(AAQ\bar{Q}_{\text{anti}})$, $(AAQQ)$ and $(AAAQ)$ is what we list in Sect. VIA. With $N_{\text{ops}} = 12$ Eq. (4.1) implies we need one constraint. The constraint must flow to $\det(M) - BB = 0$ in $SU(3)$. Now, $(QQ)(AA_{\text{sym}}) + (AAQ\bar{Q}_{\text{anti}})$ flows to $M$ and $(AAQQ)$ and $(AAQQ)$ flow to $B$ and $\bar{B}$. It follows that the classical constraint must be of the form $\det[(QQ)(AA_{\text{sym}}) + (AAQ\bar{Q}_{\text{anti}})] - (AAQQ)(AAAQ) = 0$, or by expanding the determinant and keeping track of numerical constants

\begin{equation}
1/6(AA_{\text{sym}})^3(QQ)^2 + 4(AA_{\text{sym}})(AAQ\bar{Q}_{\text{anti}})^2 + 64(AAAQQ)(AAAQ) = 0
\end{equation}

where

\begin{equation}
(AA_{\text{sym}})^3 = (AA_{\text{sym}})^{ij}(AA_{\text{sym}})^{kl}(AA_{\text{sym}})^{mn} \epsilon_{ikm} \epsilon_{jln}
\end{equation}

\begin{equation}
(AAQ\bar{Q}_{\text{anti}})^2(AA_{\text{sym}}) = (AAQQ)^{ij}(AAQQ)^{kl}(AA)^{mn} \epsilon_{ijm} \epsilon_{kln}.
\end{equation}
It is now a simple exercise to verify this constraint (with the help of symbolic manipulator programs).

To explore the quantum moduli we note, as above, that the 't Hooft anomaly matching conditions are satisfied everywhere except at the origin which must therefore be excluded by modifying the classical constraint. The theory has a non-anomalous global $U(1)$ symmetry under which the fields $A$, $Q$ and $\bar{Q}$ transform with charges $1$, $-3$ and $-3$, respectively. The left hand side of the constraint in Eq. (5.4) transforms non-trivially, with charge $-6$. This is an example of a c-QMM. The composite $(Q\bar{Q})$ has charge $-6$. It is straightforward to check that the constraint

$$1/6(AA_{sym})^3(Q\bar{Q})^2 + 4(AA_{sym})(AAQ\bar{Q}_{anti})^2 + 64(AAAQ\bar{Q})(AAAQQ) = \Lambda^8(Q\bar{Q})$$

(5.5)

flows to the corresponding $SU(3)$ constraint.

This c-QMM constraint does not exclude the origin of moduli space where the 't Hooft anomaly condition is not satisfied. However, if the constraint is implemented by a lagrange multiplier, $\lambda$, via a superpotential

$$W = \lambda[1/6(AA_{sym})^3(Q\bar{Q})^2 + 4(AA_{sym})(AAQ\bar{Q}_{anti})^2 + 64(AAAQ\bar{Q})(AAAQQ) - \Lambda^8(Q\bar{Q})],$$

and $\lambda$ is interpreted as a dynamical field, both $\lambda$ and $Q\bar{Q}$ become massive. This removes one composite from the spectrum and leaves the others unconstrained, and the 't Hooft anomaly matching conditions are satisfied.

This superpotential exhibits a new, purely quantum mechanical, branch of the moduli space. Consider directions on the moduli given by the scalings $(AA_{sym}) \sim \epsilon^{-1}$, $(Q\bar{Q}) \sim \epsilon^3$, $(AAQ\bar{Q}_{anti}) \sim \epsilon^{1+x}$ (any $x > 0$) and $(AAAQQ) = (AAAQQ) = 0$. These are in the moduli only if $\lambda = 0$, but in the limit $\epsilon \rightarrow 0$ the moduli includes the branch $\lambda \neq 0$. Since $\lambda$ carries 2 units of $U(1)_R$, the symmetry is spontaneously broken on this branch. Although $(AA_{sym}) \rightarrow \infty$, there remains at least an unbroken $SU(2)$ gauge group, which is strongly coupled in the neighborhood of this branch. This suggests the interpretation of $\lambda$ as a glueball superfield, and $\lambda \neq 0$ as gaugino condensation.

VI. ALL QUANTUM MODIFIED THEORIES

This section contains our results. In tables I and II we list all gauge and Witten anomaly free theories that satisfy the index constraint (3.2) for $SU$ and $Sp$ gauge groups, respectively. We give the gauge group in the first column, the matter content in the second and state whether the theory has a QMM or a Coulomb branch in the last column.

For all theories which are derived from an s-confining theory by taking out a fundamental and antifundamental, the superpotential is easily determined by integrating out a fundamental and an antifundamental. This was done by Csaki et al and we do no reproduce their results here. For the rest of the theories, those which only follow from theories with $\sum_{j=1}^{N} \mu_j - \mu_G = 2$ that are not s-confining, we use the flow to determine the classical constraint.

1 However, we do not agree with some of their results. For details see section [V].
Next, in separate sub-sections, we give the precise results for those theories which cannot be obtained from an s-confining theory by integrating out a fundamental-antifundamental pair. In each case we give a table. The upper part of each table lists the chiral superfields, the representation they belong to under the gauge group and finally their global symmetry properties. The second part of the table shows the analogous information for the composite operators. The composite operators are labeled by their component fields. There is often more than one way to construct an invariant operator from the given component fields. The precise construction used is specified in the appendix. In some tables there is a third part which introduces shorthand notation convenient for giving the constraint. The explicit constraint is then given.

A. The Quantum Modified $SU(N)$ Theories

| $SU(N)$ | $N(\square + \square)$ | $SU(N)$ | $\square + (N-1)\square + 3\square$ | $SU(N)$ | $\square + \square + 2(\square + \square)$ | $SU(N)$ | $Adj$ | Coulomb branch |
|---------|-------------------------|---------|---------------------------------|---------|---------------------------------|---------|------|----------------|
| $SU(4)$ | $3\square + 1(\square + \square)$ | $SU(4)$ | $4\square$ | $SU(4)$ | 3 $\square + 1(\square + \square)$ | $SU(4)$ | 4 | c-quantum modified |
| $SU(5)$ | $2\square + 1\square + 3\square$ | $SU(5)$ | $2\square + 2\square + 1\square$ | $SU(5)$ | 2 $\square + 1\square + 3\square$ | $SU(5)$ | 2 | c-quantum modified |
| $SU(6)$ | $2\square + 4\square$ | $SU(6)$ | $\square + 3(\square + \square)$ | $SU(6)$ | $\square + \square + 2\square$ | $SU(6)$ | 2 | i-quantum modified |
| $SU(6)$ | $\square + 3(\square + \square)$ | $SU(6)$ | $\square + \square + 2\square$ | $SU(6)$ | $3\square + 1\square$ | $SU(6)$ | $2\square$ | Coulomb branch |
| $SU(7)$ | $\square + 3\square + 1\square$ | $SU(7)$ | $\square + 3\square + 1\square$ | $SU(7)$ | $\square + 3\square + 1\square$ | $SU(7)$ | $3\square + 1\square$ | c-quantum modified |

TABLE I. These are all $SU$ theories satisfying $\sum_j \mu_j - \mu_G = 0$ and free of gauge anomalies. We list the gauge group and the field content of the theories in the first and second column. In the third column, we indicate whether the theory has a quantum modified moduli space or a Coulomb branch. The prefix “i” indicates an invariant quantum modification and the prefix “c” a covariant quantum modification.
1. SU(4) with $3\Box + (\Box + \Box)$

|        | SU(4) | SU(3) | U(1)$_A$ | U(1)$_B$ | U(1)$_R$ |
|--------|-------|-------|----------|----------|----------|
| $A$    | $\Box$ | 0     | 1        | 0        |          |
| $Q$    | $\Box$ | 1     | 1        | -3       | 0        |
| $\bar{Q}$ | $\Box$ | 1     | -1       | -3       | 0        |
| $QQ$   | 1     | 1     | 0        | -6       | 0        |
| $AA_{sym}$ | 1     | $\Box$ | 0        | 2        | 0        |
| $AAQ\bar{Q}_{anti}$ | 1     | $\Box$ | 0        | -4       | 0        |
| $AAAQQ$ | 1     | 1     | -2       | -6       | 0        |
| $AAA\bar{Q}$ | 1     | 1     | 2        | -6       | 0        |

The constraint is:

$$\frac{1}{6}(AA_{sym})^3(Q\bar{Q})^2 + 4(AA_{sym})(AAQ\bar{Q}_{anti})^2 + 64(AAA\bar{Q})(AAAQQ) = \Lambda^8(Q\bar{Q})$$

2. SU(5) with $2\Box + \Box + \Box$

|        | SU(5) | SU(2) | U(1)$_A$ | U(1)$_B$ | U(1)$_R$ |
|--------|-------|-------|----------|----------|----------|
| $A$    | $\Box$ | $\Box$ | 1        | 0        | 0        |
| $\bar{A}$ | $\Box$ | $\Box$ | 1        | -2       | 1        |
| $Q$    | $\Box$ | $\Box$ | 1        | 0        | -3       |
| $AA$   | 1     | $\Box$ | -1       | 1        | 0        |
| $A^2\bar{A}^2$ | 1     | $\Box$ | -2       | 2        | 0        |
| $\bar{A}^2\bar{Q}$ | 1     | $\Box$ | 1        | -4       | -1       |
| $A^3\bar{Q}$ | 1     | $\Box$ | 3        | -3       | 0        |
| $A^4\bar{A}\bar{Q}$ | 1     | $\Box$ | 2        | -2       | 0        |
| $A^2\bar{A}\bar{Q}^2$ | 1     | $\Box$ | 1        | 0        | -5       |

The constraint is:

$$(2^{10} f_1 + 2^9 f_3 + 2^7 f_4)A^2\bar{A}\bar{Q}^2 + (5 f_5^2 + 2^2 f_6 - 2^7 f_2 - 2^6 f_7)\bar{A}^2\bar{Q} = \Lambda^8(A^2\bar{A}\bar{Q}^2)$$
The constraint is:

\[-12(6S_6 + S_1S_3)f_1 + 18f_2 - 27S_3f_3 - 648S_4f_4 - 16(18S_4 + S_2S_3)S_5^2 + 48(12S_6 - S_1S_3)S_4S_1 + 96S_2S_6^2 = \Lambda^2S_5\]

4. SU(7) with $3 + 3 + 3$

| SU(7) | SU(3) | U(1)$_A$ | U(1)$_B$ | U(1)$_R$ |
|--------|-------|----------|----------|----------|
| $B$    |       | 1 0 1 0 0 |          |          |
| $Q$    |       | 1 -3 -10 0|          |          |
| $\bar{Q}$ |     | 1 0 0 0 0 |          |          |
| $f_1 = [(B^2\bar{Q})(B^3\bar{Q})]^\text{flavorsym}$ | 1 | 1 |          |          |
| $f_2 = [(B^2\bar{Q})^2(B^3\bar{Q})]^\text{flavorsym}$ | 1 | 1 |          |          |
| $f_3 = [(B^3\bar{Q})^2]^\text{flavorsym}$ | 1 | 1 |          |          |
| $f_4 = [(B^2\bar{Q})^2]^\text{flavorsym}$ | 1 | 1 |          |          |
The constraint is:

\[\begin{align*}
7f_1(B^4Q^2) + 6f_2(B^7) - 288f_3(B^7)(B\bar{Q}^3) + 1008f_4 + 12(B^7)(B^3\bar{Q}^3 Q_{anti})^2 - 72(B^4Q^2)(B^7)(B\bar{Q}^3)^2 &= \Lambda^{14}(B^3\bar{Q}^3 Q_{anti})
\end{align*}\]

**B. The Quantum Modified \(Sp(N)\) Theories**

| \(Sp(2N)\)         | \(2N + 2\) | i-quantum modified |
|---------------------|-------------|---------------------|
| \(Sp(2N)\)         | + 4        | i-quantum modified  |
| \(Sp(2N)\)         |             | Coulomb branch      |

| \(Sp(4)\)         | 2           | c-quantum modified  |
|---------------------|-------------|---------------------|
| \(Sp(4)\)         | + 2        | Coulomb branch      |
| \(Sp(6)\)         | 2           | Coulomb branch      |
| \(Sp(6)\)         | + 3        | c-quantum modified  |

**TABLE II.** These are all \(Sp\) theories satisfying \(\sum_j \mu_j - \mu_G = 0\) and the Witten anomaly condition. We list the gauge group and the field content of the theories in the first and second column. In the third column, we indicate which theories are quantum modified. The prefix “i” indicates an invariant quantum modification and the prefix “c” a covariant quantum modification.

1. \(Sp(4)\) with \(2 \square + \square\)

|          | \(Sp(4)\) | \(SU(2)_A\) | \(SU(2)_Q\) | \(U(1)_A\) | \(U(1)_R\) |
|----------|-----------|-------------|-------------|-------------|-------------|
| \(A\)    | \(\square\) | \(\square\) | 1           | 1           | 0           |
| \(Q\)    | \(\square\) | 1           | \(\square\) | -2          | 0           |
| \(Q_1Q_2\)| 1         | 1           | 1           | -4          | 0           |
| \(AA_{sym}\)| 1        | \(\square\)| 1           | 2           | 0           |
| \(AQ_1Q_2\)| 1        | \(\square\)| 1           | -3          | 0           |
| \(AAQ_iQ_j\)| 1        | 1           | \(\square\)| -2          | 0           |

The constraint is:

\[\begin{align*}
(AA_{sym})^2(Q_1Q_2)^2 - 4((AA_{sym})(AQ_1Q_2)^2) - 16(AAQ_iQ_j)^2 &= \Lambda^6(Q_1Q_2)
\end{align*}\]
2. Sp(6) with $\begin{bmatrix} 3 \end{bmatrix} + 3 \begin{bmatrix} \end{bmatrix}$

| $Sp(6)$ | $SU(3)$ | $U(1)_A$ | $U(1)_R$ |
|---------|----------|-----------|-----------|
| $B$     | $\begin{bmatrix} \end{bmatrix}$ | 1 | 3 | 0 |
| $Q$     | $\begin{bmatrix} \end{bmatrix}$ | $\begin{bmatrix} \end{bmatrix}$ | $-5$ | 0 |
| $QQ$    | 1 | $\begin{bmatrix} \end{bmatrix}$ | $-10$ | 0 |
| $BQ^3$  | 1 | 1 | $-12$ | 0 |
| $B^2Q^2_{\text{sym}}$ | 1 | $\begin{bmatrix} \end{bmatrix}$ | $-4$ | 0 |
| $B^4$   | 1 | 1 | 12 | 0 |
| $B^3Q^3$ | 1 | 1 | $-6$ | 0 |

The constraint is:

$$1728(B^3Q^3)^2 - 8(BQ^3)^2(B^4) + 12(B^2Q^2_{\text{sym}})^3 + 3(B^4)(B^2Q^2_{\text{sym}})(QQ)^2 = \Lambda^8(BQ^3)$$

VII. CONCLUSIONS

Adding a fundamental and an antifundamental matter multiplet to a theory with a quantum modified moduli space one obtains a theory satisfying the index condition $\sum_{i=1}^{n} \mu_i - \mu_G = 2$. If this theory is s-confining, the algebraic constraint defining the moduli is invariant under all global symmetries, and the invariant quantum modified moduli (i-QMM) is given by $F(\phi_i) = \Lambda^p$. However, if the resulting theory is not s-confining the constraint of the original theory is only covariant under global symmetries. This gives a covariant quantum modified moduli (c-QMM), characterized by $F(\phi_i) = \Lambda^p\phi_k$.

Theories with i-QMM are by now commonplace. Less familiar are theories with c-QMM. In these theories we believe one must take the lagrange multiplier enforcing the constraint via a superpotential seriously as a dynamical degree of freedom. But it follows immediately that the c-QMM has branches, absent at the classical level, for which global $U(1)_R$ symmetry is broken.

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APPENDIX A: GAUGE INVARIANT OPERATORS IN DETAIL

1. SU(N) Theories

a. $SU(4)$ with $\begin{bmatrix} 3 \end{bmatrix} + (\begin{bmatrix} \end{bmatrix} + \begin{bmatrix} \end{bmatrix})$

The gauge invariant operators:
\[QQ = Q_\alpha Q^\alpha\]  
\[AA_{\text{sym}} = 1/2(A^i_{\alpha\beta}A^j_{\gamma\delta} \epsilon^{\alpha\beta\gamma\delta} + A^j_{\alpha\beta}A^i_{\gamma\delta} \epsilon^{\alpha\beta\gamma\delta})\]  
\[AAQ\bar{Q}_{\text{anti}} = A^i_{\alpha\beta}A^j_{\gamma\delta}Q_\eta Q^\alpha \epsilon^{\gamma\delta\eta}\]  
\[AAAQ\bar{Q} = 1/6(Q_\alpha A^i_{\alpha\beta}A^j_{\gamma\delta}A^k_{\mu\nu} \epsilon^{\alpha\beta\gamma\delta\mu\nu} \epsilon_{ijk})\]  
\[AAAQQ = 1/6(A^i_{\alpha\beta}A^j_{\gamma\delta}A^k_{\mu\nu}Q_\alpha Q_\eta \epsilon^{\alpha\beta\gamma\delta\mu\nu} \epsilon_{ijk})\]

The gauge and flavor invariant operators are:

\[(AA_{\text{sym}})^3 = (AA_{\text{sym}})^{ij}(AA_{\text{sym}})^{kl}(AA_{\text{sym}})^{ml} \epsilon_{ikm} \epsilon_{jln}\]  
\[(AAQ\bar{Q}_{\text{anti}})^2 = (AAQ\bar{Q})^{[ij]}(AAQ\bar{Q})^{[kl]}(AA)^{mn} \epsilon_{ijm} \epsilon_{kln}\]

b. \(SU(5)\) with \(\Box + \Box + \Box\)

The gauge invariant operators are:

\[A\bar{A} = (A_{\alpha\beta}\bar{A}_{\alpha\beta})^i\]  
\[A^2\bar{A}^2 = (A_{\alpha\beta}\bar{A}_{\alpha\beta})^{ij}\]  
\[\bar{A}^2\bar{Q} = \bar{A}_{\alpha\beta}A_{\alpha\beta}Q^\alpha \epsilon_{\alpha\beta} \epsilon_{\gamma\delta}\]  
\[A^3\bar{Q} = 2/3(A^i_{\alpha\beta}A^j_{\gamma\delta}A^k_{\mu\nu} Q^\lambda \epsilon^{\alpha\beta\gamma\delta\mu\nu} \epsilon_{ijk})\]  
\[A^4\bar{A}\bar{Q} = (A^i_{\alpha\beta}A^j_{\gamma\delta}A^k_{\mu\nu} Q^\lambda A^l_{\mu\nu} \epsilon^{\beta\delta\mu\nu})\]  
\[A^2\bar{A}^2\bar{Q} = (A_{\alpha\beta}\bar{A}_{\alpha\beta})^{ij}(A_{\alpha\beta}^{kl} \epsilon_{ikm} \epsilon_{jln})\]

The flavor and gauge invariant operators are:

\[f_1 = (A^4\bar{A}\bar{Q})^{ij}(A^2\bar{A}^2)^{kl} \epsilon_{ikm} \epsilon_{jln}\]  
\[f_2 = (A^4\bar{A}\bar{Q})^{ij}(A^4\bar{A}\bar{Q})^{kl} \epsilon_{ikm} \epsilon_{jln}\]  
\[f_3 = (A^4\bar{A}\bar{Q})^{ij}(AA)^{kl} \epsilon_{ikm} \epsilon_{jln}\]  
\[f_4 = (A^2\bar{A}^2)^{ij}(AA)^{kl} \epsilon_{ikm} \epsilon_{jln}\]  
\[f_5 = (AA)^{ij}(A^3\bar{Q})^{kl} \epsilon_{ikm} \epsilon_{jln}\]  
\[f_6 = (A^2\bar{A}^2)^{ij}(A^3\bar{Q})^{kl} \epsilon_{ikm} \epsilon_{jln}\]  
\[f_7 = (A^4\bar{A}\bar{Q})^{ij}(AA)^{kl} \epsilon_{ikm} \epsilon_{jln}\]

c. \(SU(6)\) with \(\Box + \Box + 2\Box\)

With

\[Adj^\delta_T = B_{\alpha\beta\gamma}B_{\mu\nu\tau} \epsilon^{\alpha\beta\gamma\mu\nu\tau}\]

it follows:
\[ A\bar{Q}^2 = 1/2(A_{\alpha\beta}\bar{Q}^i_{\alpha}\bar{Q}^j_{\beta} - A_{\alpha\beta}\bar{Q}^j_{\alpha}\bar{Q}^i_{\beta}) \]  
(A22)

\[ A^3 = A_{\alpha\beta}A_{\gamma\delta}A_{\nu\tau}\epsilon^{\alpha\beta\gamma\delta\nu\tau} \]  
(A23)

\[ B^4 = \text{Adj}_\alpha^2\text{Adj}_\beta^3 \]  
(A24)

\[ (B^4A^3) = \text{Adj}_\alpha^2A_{\beta\gamma}A_{\nu\rho}A_{\tau\sigma}\epsilon^{\alpha\beta\gamma\delta\nu\rho\tau\sigma} \]  
(A25)

\[ (BA^2\bar{Q}) = (B_{\alpha\beta\gamma}\bar{Q}^i_{\alpha}\bar{Q}^j_{\beta}\bar{Q}^k_{\gamma})A_{\mu\nu}\epsilon^{\alpha\beta\gamma\delta\mu\nu\tau\sigma} \]  
(A26)

\[ (B^2A^2\bar{Q})_{\text{sym}} = (B_{\alpha\beta\gamma}\bar{Q}^i_{\alpha}\bar{Q}^j_{\beta}\bar{Q}^k_{\gamma})A_{\delta\epsilon}\epsilon^{\alpha\beta\gamma\delta\mu\nu\tau\sigma}[ij] \]  
(A27)

\[ (B^2A^2\bar{Q})_{\text{anti}} = (B_{\alpha\beta\gamma}\bar{Q}^i_{\alpha}\bar{Q}^j_{\beta}\bar{Q}^k_{\gamma})A_{\delta\epsilon}\epsilon^{\alpha\beta\gamma\delta\mu\nu\tau\sigma}[ij] \]  
(A28)

\[ (B^3A^2\bar{Q}) = \text{Adj}_\alpha^2A_{\beta\gamma\delta}B_{\mu\nu\tau}\bar{Q}^i_{\sigma}\epsilon^{\alpha\beta\gamma\delta\mu\nu\tau\sigma}\epsilon^{\gamma\delta\mu\nu\tau\sigma} \]  
(A29)

\[ (B^4A^2\bar{Q})_{\text{anti}} = (\text{Adj}_\alpha^2B_{\beta\gamma\delta\tau}Q^i_{\sigma}B_{\gamma\delta\epsilon\nu}\epsilon^{\alpha\beta\gamma\delta\mu\nu\tau\sigma}\epsilon^{\gamma\delta\mu\nu\tau\sigma})[ij] \]  
(A30)

The flavor and gauge invariant operators are:

\[ f_1 = (BA^2\bar{Q})^i(BA^2\bar{Q})^j\epsilon_{ij} \]  
(A31)

\[ f_2 = ((B^2A^2\bar{Q})_{\text{sym}})^{ij}(B^3A^2\bar{Q})^k(B^3A^2\bar{Q})^l\epsilon_{ik}\epsilon_{jl} \]  
(A32)

\[ f_3 = ((B^2A^2\bar{Q})_{\text{sym}})^{ij}(BA^2\bar{Q})^k(BA^2\bar{Q})^l\epsilon_{ik}\epsilon_{jl} \]  
(A33)

\[ f_3 = 1/2(((B^2A^2\bar{Q})_{\text{sym}})^{ij}((B^2A^2\bar{Q})_{\text{sym}})^{kl}\epsilon_{ik}\epsilon_{jl}) \]  
(A34)

\[ d.\ SU(7)\ with\ \boxed{\bigcirc + \bigcirc + 3\bigcirc} \]

With

\[ \bar{B}^{\mu\nu\rho\tau} = B_{\alpha\beta\gamma\epsilon}^{\epsilon\beta\gamma\mu\nu\tau} \]  
(A35)

\[ H^{\alpha}_{\tau\mu\nu} = \bar{B}^{\alpha\beta\gamma\delta}B_{\gamma\delta\tau\beta\mu\nu} \]  
(A36)

it follows:

\[ Q\bar{Q} = (Q^i_{\alpha}\bar{Q}^i_{\alpha}) \]  
(A37)

\[ B\bar{Q}^3 = (B_{\alpha\beta\gamma}\bar{Q}^i_{\alpha}\bar{Q}^j_{\beta}\bar{Q}^k_{\gamma})[ijk] \]  
(A38)

\[ B^3\bar{Q}^2 = (B^{\mu\nu\rho\tau}B_{\mu\nu\gamma}\bar{Q}^i_{\gamma}\bar{Q}^j_{\rho}\epsilon_{\alpha\beta\gamma\mu\nu\tau\rho\sigma})[ij] \]  
(A39)

\[ B^3Q^3_{\text{anti}} = (B^{\mu\nu\rho\tau}B_{\mu\nu\gamma}\bar{Q}^i_{\gamma}\bar{Q}^j_{\rho}\epsilon_{\alpha\beta\gamma\mu\nu\tau\rho\sigma})[ijk] \]  
(A40)

\[ B^4\bar{Q}^2 = \bar{B}^{\mu\nu\rho\tau}Q_{\tau}B_{\mu\nu\gamma}\bar{B}^{\alpha\beta\gamma\delta}B_{\alpha\beta\rho} \]  
(A41)

\[ B^5\bar{Q}^2Q = (\bar{B}^{\alpha\beta\gamma\delta}B_{\alpha\beta\rho}\bar{Q}^i_{\rho}/B_{\mu\nu\gamma}\bar{Q}^j_{\nu}/B_{\gamma\delta\epsilon\nu}\epsilon_{\tau\mu\nu\gamma\delta\epsilon\nu\tau})[ij] \]  
(A42)

\[ B^7 = H^{\alpha}_{\tau\mu\nu}H^{\beta}_{\rho\sigma\lambda}\bar{B}_{\alpha\beta\gamma\epsilon}^{\epsilon\gamma\mu\nu\rho\sigma\lambda} \]  
(A43)

The flavor and gauge invariant operators are:

\[ f_1 = (B^3\bar{Q}^2)^{ij}(B^3\bar{Q}^2)^{kl}(B^3\bar{Q}^2)^{mn}\epsilon_{ijk}\epsilon_{jln} \]  
(A44)

\[ f_2 = (B^3\bar{Q}^2)^{ij}(B^3\bar{Q}^2)^{kl}(QQ)^n(Q\bar{Q})^n\epsilon_{ikm}\epsilon_{jln} \]  
(A45)

\[ f_3 = 1/4((B^5\bar{Q}^2Q)^{ij}(QQ)^{k}\epsilon_{ijk}) \]  
(A46)

\[ f_4 = 1/16((B^5\bar{Q}^2Q)^{ij}(B^5\bar{Q}^2Q)^{kl}(B^3\bar{Q}^2)^{mn}\epsilon_{ijk}\epsilon_{klm}) \]  
(A47)
2. The $Sp(2N)$ Theories

We recall that there is an invariant tensor, $J^{\alpha\beta} = (1_{N \times N} \otimes i\sigma^2)^{\alpha\beta}$. In the following examples indices have been freely raised using $J^{\alpha\beta}$.

a. $Sp(4)$ with $2 \begin{bmatrix} \Box & \Box \end{bmatrix}$

The gauge invariant operators are:

\[ Q_1 Q_2 = Q_{1\alpha} Q_{2}^{\alpha} \]  \hspace{1cm} (A48)
\[ AA_{\text{sym}} = (A_{\alpha\beta} A_{\gamma\delta} \epsilon^{\alpha\beta\gamma\delta})^{\{a,b\}} \]  \hspace{1cm} (A49)
\[ AQ_1 Q_2 = (A_{\alpha\beta} Q_{1}^{\alpha} Q_{2}^{\beta})^{b} \]  \hspace{1cm} (A50)
\[ AAQ_i Q_j = (A_{\alpha\beta} Q_{\gamma}^{\beta} A_{\gamma\delta} Q_{\tau}^{\gamma} \epsilon^{\alpha\gamma\delta\tau})^{[ab]}{\{ij\}} \]  \hspace{1cm} (A51)

The gauge and flavor invariant operators are:

\[ AA_{\text{sym}}^2 = 1/2(AA_{\text{sym}}^{ab} AA_{\text{sym}}^{cd} \epsilon_{ac} \epsilon_{bd}) \]  \hspace{1cm} (A52)
\[ (AA_{\text{sym}})(AQ_1 Q_2)^2 = AA_{\text{sym}}^{ab}(AQ_1 Q_2)^{c}(AQ_1 Q_2)^{d} \epsilon_{ac} \epsilon_{bd} \]  \hspace{1cm} (A53)
\[ (AAQ_i Q_j)^2 = 1/2((AAQ_i Q_j)^{ab}(AAQ_i Q_j)^{cd} \epsilon_{ac} \epsilon_{bd}) \]  \hspace{1cm} (A54)

b. $Sp(6)$ with $\begin{bmatrix} \Box & \Box \end{bmatrix} + 3 \Box$

With

\[ B_{\alpha\beta\gamma} = B_{\mu\nu\tau} \epsilon^{\alpha\beta\gamma\mu\nu\tau} \]  \hspace{1cm} (A55)

it follows:

\[ QQ = (Q_{\alpha} Q^{\alpha})^{[ij]} \]  \hspace{1cm} (A56)
\[ BQ^3 = (B_{\alpha\beta\gamma} Q^{\alpha} Q^{\beta} Q^{\gamma})^{[ijk]} \]  \hspace{1cm} (A57)
\[ B^2 Q^{2}_{\text{sym}} = (B_{\alpha\beta\gamma} B_{\alpha\beta\gamma} Q_{\gamma})^{\{ij\}} \]  \hspace{1cm} (A58)
\[ B^4 = B_{\alpha\beta\gamma} B_{\mu\nu\tau} B_{\alpha\beta\gamma} B_{\mu\nu\tau} \]  \hspace{1cm} (A59)
\[ B^3 Q^3 = (B_{\alpha\beta\gamma} B_{\alpha\beta\gamma} Q_{\gamma} B_{\gamma\mu} Q^{\mu} Q^{\nu})^{[ijk]} \]  \hspace{1cm} (A60)

The gauge and flavor invariant operators are:

\[ (B^2 Q_{\text{sym}}^{2})^3 = (B^2 Q_{\text{sym}}^{2})^{ij}(B^2 Q_{\text{sym}}^{2})^{kl}(B^2 Q_{\text{sym}}^{2})^{mn} \epsilon_{ikm} \epsilon_{jln} \]  \hspace{1cm} (A61)
\[ (B^2 Q_{\text{sym}}^{2})(QQ)^2 = (B^2 Q_{\text{sym}}^{2})^{ij}(QQ)^{kl}(QQ)^{mn} \epsilon_{ikl} \epsilon_{jmn} \]  \hspace{1cm} (A62)
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