AUTOMORPHIC FORMS AND HOLOMORPHIC FUNCTIONS ON THE UPPER HALF-PLANE

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ABSTRACT. We define a set of holomorphic functions in terms of the Hauptmodul of a quotient Riemann surface and prove that these functions are holomorphic on the upper half-plane. It is also shown that these functions are automorphic forms of weight $k$ with respect to a Fuchsian group.

1. Introduction

The group $\text{SL}(2, \mathbb{R})$ is defined by

$$\text{SL}(2, \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, \ ad - bc = 1 \right\}$$

and the group $\text{PSL}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})/\{\pm I_2\}$, where $I_2$ is the $2 \times 2$ identity matrix (see [12, Chapter VII]). Let $\mathbb{H}$ denote the upper half-plane $\{\tau \in \mathbb{C} : \text{Im} \tau > 0\}$. The boundary of $\mathbb{H}$ is $\mathbb{R} \cup \infty$. The group $\text{PSL}(2, \mathbb{R})$ acts on $\mathbb{H}$ as follows:

$$\tau \mapsto \gamma \cdot \tau = \frac{a\tau + b}{c\tau + d}, \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{R}), \ \tau \in \mathbb{H}.$$ 

All transformations of $\text{PSL}(2, \mathbb{R})$ are conformal.

A Fuchsian group is a discrete subgroup of $\text{PSL}(2, \mathbb{R})$, i.e., it is a group of orientation-preserving isometries of $\mathbb{H}$. Study of Fuchsian group is a very interesting topic in many fields of Mathematics. Many mathematicians studied Fuchsian group and various subgroups of Fuchsian group, for example, see [7], [11], [13] and [14]. The Hecke group which is a subgroup of Fuchsian group is studied in [1] and [2] to investigate Ramanujan’s modular equations.

Let $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{PSL}(2, \mathbb{R})$ and let $\text{tr}(\gamma)$ denote the trace of $\gamma$, then the element $\gamma$ is said to be elliptic, parabolic and hyperbolic when $|\text{tr}(\gamma)| < 2$, $|\text{tr}(\gamma)| = 2$ and $|\text{tr}(\gamma)| > 2$, respectively. If $\Gamma \subset \text{PSL}(2, \mathbb{R})$ is a Fuchsian group and $\gamma \in \Gamma$ is an elliptic element, then a point $\tau \in \mathbb{H}$ is called an elliptic point of $\Gamma$ if $\gamma(\tau) = \tau$. Also, for a parabolic element $\sigma \in \Gamma$, a point $x \in \mathbb{R} \cup \{\infty\}$ is called a cusp of $\Gamma$ if $\sigma(x) = x$. If a Fuchsian group $\Gamma$ acts

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on $\mathbb{H}$ properly discontinuously, then we have the quotient Riemann surface $\Gamma \backslash \mathbb{H}$. For a detailed discussion, see [3] and [9].

Let $\mathbb{H}^*$ denote the union of the upper half-plane $\mathbb{H}$ and the set of cusps of a Fuchsian group $\Gamma$. Suppose $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$, $\tau \in \mathbb{H}$ and $f : \mathbb{H} \to \mathbb{C}$ is a holomorphic function. Then the function $f$ is called an automorphic form of weight $k$ with respect to $\Gamma$ if

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

If $k = 0$, then

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = f(\tau)$$

and $f$ is called an automorphic function. When the genus of the quotient Riemann surface $\Gamma \backslash \mathbb{H}^*$ is zero, an automorphic function is called a Hauptmodul. If an automorphic function has no poles, then it is constant according to the consequence of maximum modulus principle. For details, we refer the reader to [5], [6], [8], and [10].

Let $F$ be the fundamental domain for the Fuchsian group $\Gamma$. Let $X$ and $\hat{X}$ denote the quotient Riemann surfaces $\Gamma \backslash \mathbb{H}$ and $\Gamma \backslash \mathbb{H}^*$, respectively. If $F$ is compact, then it has finitely many vertices which are elliptic points and cusps of $\hat{X} = \Gamma \backslash \mathbb{H}^*$. Let $P_1, \ldots, P_r$ be the vertices whose orders are $n_1, n_2, \ldots, n_r$, respectively. If the number of elliptic elements and cusps of $\Gamma$ are $m$ and $l$, respectively, then $m + l = r$. If $g$ is the genus of $\hat{X}$, then we say that $\Gamma$ has signature $(g; n_1, \ldots, n_n)$. For more detailed discussion, reader may consult Section 2.1 of [1], Chapter 4 of [9], and Section 2 of [15]. Let us denote by $A_k$ the space of automorphic forms of weight $k$ with respect to $\Gamma$. The basis for $A_k$ on a Shimura curve $X$ with genus 0 is determined in Theorem 4 of [16]. The following theorem is written according to Theorem 2.23 of [13] to determine the dimension of $A_k$.

**Theorem 1 ([13, Theorem 2.23]).** For a Fuchsian group $\Gamma$ with signature $(g; n_1, \ldots, n_n)$, let $g$ be the genus of the compact quotient Riemann surface $\hat{X} = \Gamma \backslash \mathbb{H}^*$. Then, the dimension, $\dim A_k$, of $A_k$ for an even integer $k$ is given by

$$\dim A_k = \begin{cases} 0 & \text{if } k < 0, \\ 1 & \text{if } k = 0, \\ g & \text{if } k = 2, \\ (g - 1)(k - 1) + \sum_{i=1}^{r} \left\lfloor \frac{k}{2} \left(1 - \frac{1}{n_i}\right) \right\rfloor & \text{if } k \geq 4. \end{cases}$$

In the following section, we present our main results and their proofs.
2. Main Results

Let \((0; n_1, \ldots, n_r)\) be the signature of a Fuchsian group \(\Gamma\), i.e., the genus of the quotient Riemann surface \(\tilde{X} = \Gamma \backslash \mathbb{H}^*\) is 0 and let \(d = \dim A_k\). Then, for an even integer \(k \geq 4\), we have from Theorem 1
\[
d = 1 - k + \sum_{i=1}^{r} \left\lfloor \frac{k}{2} (1 - \frac{1}{n_i}) \right\rfloor.
\]

In the following theorem, we define the functions \(h_j\) for \(j = 0, \ldots, d - 1\) so that the functions are holomorphic on \(\mathbb{H}\). Also, these functions are automorphic forms of weight \(k\) with respect to \(\Gamma\).

**Theorem 2.** Consider the Fuchsian group \(\Gamma\) with signature \((0; n_1, \ldots, n_r)\) and the compact quotient Riemann surface \(\tilde{X} = \Gamma \backslash \mathbb{H}^*\). Let \(\tau_1, \ldots, \tau_r\) be the inequivalent vertices (elliptic points or cusps of \(\tilde{X}\)) of the fundamental domain of \(\Gamma\) of orders \(n_1, \ldots, n_r\), respectively, and let \(w(\tau)\) be a Hauptmodul of \(\tilde{X}\). For an even integer \(k \geq 4\), let
\[
a_i = \left\lfloor \frac{k}{2} \left(1 - \frac{1}{n_i}\right) \right\rfloor \quad \text{and} \quad d = \dim A_k = 1 - k + \sum_{i=1}^{r} a_i.
\]

If \(w(\tau_i) = w_i\) for \(i = 1, \ldots, r\) and the functions
\[
h_j(\tau) = \left(\frac{w'(\tau)}{w(\tau)}\right)^{k/2} (w(\tau))^j \prod_{i=1, w_i\neq \infty}^{r} (w(\tau) - w_i)^{a_i}
\]
for \(j = 0, \ldots, d - 1\) and \(\tau \in \mathbb{H}\), then \(h_j(\tau)\) is holomorphic on \(\mathbb{H}\).

**Proof.** We need to consider the following three cases:

(i) the Hauptmodul \(w(\tau)\) does not have any pole at the points \(\tau_i\) for \(i = 1, \ldots, r\);

(ii) the Hauptmodul \(w(\tau)\) has a pole at one of the points \(\tau_i\) for \(i = 1, \ldots, r\);

(iii) the Hauptmodul \(w(\tau)\) has a pole at another point, say \(\tau = \tau_0\), except the points \(\tau_1, \ldots, \tau_r\).

If a function has a zero of order \(\geq 0\) and has a pole of order \(\leq 0\) at a point, then there is no principal part in the expansion of the function at that point, i.e., the function is holomorphic. Thus, we have to show that \(h_j\) has a zero of order \(\geq 0\) at \(\tau = \tau_i\) for Case (i), \(h_j\) has a pole of order \(\leq 0\) at \(\tau = \tau_i\) for Case (ii) and \(h_j\) has a pole of order \(\leq 0\) at \(\tau = \tau_0\) for Case (iii).

Case (i): If \(w(\tau)\) does not have any pole at \(\tau_i\), then \(w(\tau_i) = w_i \neq \infty\) for \(i = 1, \ldots, r\). Since \(\tau_i\) is a vertex of order \(n_i\), in a neighbourhood of \(\tau = \tau_i\), we have
\[
w(\tau) - w(\tau_i) = b_i (\tau - \tau_i)^{n_i} + O((\tau - \tau_i)^{n_i+1})
\]
or,

\[(2.3) \quad w(\tau) - w_i = (\tau - \tau_i)^{n_i} w^*(\tau),\]

where \(b_i \in \mathbb{C} \setminus \{0\}\), \(w^*(\tau)\) is analytic in a neighbourhood of \(\tau = \tau_i\) and \(w^*(\tau_i) \neq 0\) for \(i = 1, \ldots, r\). Therefore, in a neighbourhood of \(\tau = \tau_i\), one can define a single-valued analytic \(n_i\)-th root of \((w - w_i)\) and this can be done at all points which are equivalent to \(\tau_i\) under the action of the Fuchsian group \(\Gamma\). Since \(w(\tau) - w_i \neq 0\) for \(\tau \neq \tau_i\) and \((w - w_i)\) is analytic on the other part of \(\mathbb{H}\), its \(n_i\)-th root is analytic at each point of the remainder of \(\mathbb{H}\). As \((w(\tau) - w_i)^{n_i}\) is locally analytic and single-valued at each \(\tau \in \mathbb{H}\), so it follows from monodromy theorem that a single-valued and analytic \(n_i\)-th root of \((w - w_i)\) can be defined on the whole \(\mathbb{H}\).

From (2.3), we observe that \((w(\tau) - w_i)\) has a zero of order \(n_i\) at \(\tau = \tau_i\) and \(\prod_{i=1, w_i \neq \infty}^r (w(\tau) - w_i)^{a_i}\) has a zero of order \(n_i a_i = n_i \left[\frac{k}{2} \left(1 - \frac{1}{n_i}\right)\right]\) at \(\tau = \tau_i\). Also, we have from (2.2)

\[(2.4) \quad w'(\tau) = b_i n_i (\tau - \tau_i)^{n_i-1} + O((\tau - \tau_i)^{n_i}).\]

Consequently, at \(\tau = \tau_i\), \((w'(\tau))^{k/2}\) has a zero of order \(\frac{k}{2}(n_i - 1)\). Since

\[\frac{k}{2}(n_i - 1) - n_i \left[\frac{k}{2} \left(1 - \frac{1}{n_i}\right)\right] \geq 0,\]

we conclude from (2.4) that \(h_j\) has a zero of order \(\geq 0\) at \(\tau = \tau_i\). Hence \(h_j\) is holomorphic on \(\mathbb{H}\).

Case (ii): Assume that \(w(\tau)\) has a pole at one of the points \(\tau_i\) for \(i = 1, \ldots, r\). Without loss of generality, suppose \(w(\tau)\) has a pole at \(\tau_1\), i.e., \(w(\tau_1) = w_1 = \infty\). Since \(\tau_1\) is a vertex of order \(n_1\), it follows that

\[w(\tau) = \frac{b_1}{(\tau - \tau_1)^{n_1}} + O((\tau - \tau_1)^{1-n_1}), \quad b_1 \in \mathbb{C} \setminus \{0\}\]

and

\[w'(\tau) = -\frac{b_1 n_1}{(\tau - \tau_1)^{n_1+1}} + O((\tau - \tau_1)^{-n_1}).\]

In this case, from (2.4) we have

\[(2.5) \quad h_j(\tau) = \frac{(w'(\tau))^{k/2}(w(\tau))^j}{\prod_{i=2, w_i \neq \infty}^r (w(\tau) - w_i)^{a_i}}.\]
Now, suppose that \( h_j(\tau) \) defined in (2.5) has a pole of order \( N \) at \( \tau = \tau_1 \). Since \( w(\tau) \) has a pole of order \( n_1 \) at \( \tau = \tau_1 \), \( (w'(\tau))^{k/2} \) has a pole of order \( \frac{k}{2}(n_1 + 1) \) and \( \prod_{i=2, w_i \neq \infty}^r (w(\tau) - w_i)^{a_i} \) has a pole of order \( n_1 \sum_{i=2}^r a_i \) at \( \tau = \tau_1 \). As \( j \) varies from 0 to \( d - 1 \), so the maximum value of \( j \) is \( d - 1 = \sum_{i=1}^r a_i - k \). Hence \( (w(\tau))^j \) has a pole of order at most \( n_1 \sum_{i=1}^r a_i - k \) at \( \tau = \tau_1 \). Therefore, we have

\[
N \leq \frac{k}{2}(n_1 + 1) + n_1 \left( \sum_{i=1}^r a_i - k \right) - n_1 \sum_{i=2}^r a_i
\]

\[
= \frac{k}{2}(n_1 + 1) + n_1 \left( \sum_{i=1}^r \left| \frac{k}{2} \left( 1 - \frac{1}{n_i} \right) \right| - k \right) - n_1 \sum_{i=2}^r \left| \frac{k}{2} \left( 1 - \frac{1}{n_i} \right) \right|
\]

\[
= -\frac{k}{2}(n_1 - 1) + n_1 \left| \frac{k}{2} \left( 1 - \frac{1}{n_1} \right) \right| \leq 0.
\]

Since \( N \leq 0 \), it follows that there is no principal part in the expansion of \( h_j \), i.e., \( h_j \) is holomorphic on \( \mathbb{H} \).

Case (iii): Suppose that \( w(\tau) \) has the value \( \infty \) at the point \( \tau = \tau_0 \) and \( w(\tau_i) \neq \infty \) for \( i = 1, \ldots, r \). Therefore, \( w(\tau) \) has a simple pole at \( \tau_0 \) and we have

\[
(2.6) \quad w(\tau) = \frac{b_0}{(\tau - \tau_0)} + O(1), \quad b_0 \in \mathbb{C} \setminus \{0\}
\]

and

\[
(2.7) \quad w'(\tau) = \frac{b_0}{(\tau - \tau_0)^2} + O(1).
\]

Let \( N_0 \) be the order of the pole of \( h_j \) defined in (2.1) at \( \tau = \tau_0 \). From (2.6) and (2.7), we observe that \( (w'(\tau))^{k/2} \) has a pole of order \( k \), \( \prod_{i=1, w_i \neq \infty}^r (w(\tau) - w_i)^{a_i} \) has a pole of order \( \sum_{i=1}^r a_i \) and \( w^j \) has a pole of order at most \( d - 1 = \sum_{i=1}^r a_i - k \). Therefore, from (2.1), it follows that

\[
N_0 \leq k + \sum_{i=1}^r a_i - k - \sum_{i=1}^r a_i = 0,
\]

which implies that \( h_j \) is holomorphic on \( \mathbb{H} \) in this case also.

\[\Box\]

**Lemma 1.** The functions \( h_j \) for \( j = 0, \ldots, d - 1 \) defined in (2.7) is an automorphic form of weight \( k \) with respect to the Fuchsian group \( \Gamma \).
Proof. For \( j = 0, \ldots, d - 1 \) and \( a_i = \left\lfloor \frac{k}{2} \left(1 - \frac{1}{m_i}\right) \right\rfloor \), we have to show that
\[
h_j \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^k h_j(\tau),
\]
where \( \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma \) and \( \tau \in \mathbb{H} \). Since \( w(\tau) \) is a Hauptmodul of \( \hat{X} \), i.e., \( w(\tau) \) is an automorphic function, thus we have
\[
w \left( \frac{a \tau + b}{c \tau + d} \right) = w(\tau)
\]
and
\[
w' \left( \frac{a \tau + b}{c \tau + d} \right) = (c \tau + d)^{1/2} w'(\tau).
\]
Now,
\[
h_j \left( \frac{a \tau + b}{c \tau + d} \right) = \left( \frac{w' \left( \frac{a \tau + b}{c \tau + d} \right)^{k/2} \left( w \left( \frac{a \tau + b}{c \tau + d} \right) \right)^j}{\prod_{i=1, w_i \neq \infty}^r \left( w \left( \frac{a \tau + b}{c \tau + d} \right) - w_i \right)^{a_i}} \right)\]
\[
= \left( c \tau + d \right)^k \left( w'(\tau) \right)^{k/2} \left( w(\tau) \right)^j \prod_{i=1, w_i \neq \infty}^r \left( w(\tau) - w_i \right)^{a_i}
\]
\[
= (c \tau + d)^k h_j(\tau).
\]
Thus, \( h_j \) is an automorphic form of weight \( k \) with respect to \( \Gamma \).

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