AdaptSPEC-X: Covariate-Dependent Spectral Modeling of Multiple Nonstationary Time Series

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ABSTRACT
We present the AdaptSPEC-X method for the joint analysis of a panel of possibly nonstationary time series. The approach is Bayesian and uses a covariate-dependent infinite mixture model to incorporate multiple time series, with mixture components parameterized by a time-varying mean and log spectrum. The mixture components are based on AdaptSPEC, a nonparametric model which adaptively divides the time series into an unknown number of segments and estimates the local log spectra by smoothing splines. AdaptSPEC-X extends AdaptSPEC in three ways. First, through the infinite mixture, it applies to multiple time series linked by covariates. Second, it can handle missing values, a common feature of time series which can cause difficulties for nonparametric spectral methods. Third, it allows for a time-varying mean. Through these extensions, AdaptSPEC-X can estimate time-varying means and spectra at observed and unobserved covariate values, allowing for predictive inference. Estimation is performed by Markov chain Monte Carlo (MCMC) methods, combining data augmentation, reversible jump, and Riemann manifold Hamiltonian Monte Carlo techniques. We evaluate the methodology using simulated data, and describe applications to Australian rainfall data and measles incidence in the United States. Software implementing the method proposed in this article is available in the R package BayesSpec. Supplementary files for this article are available online.

1. Introduction
When the available data are multiple time series thought to be realizations of nonstationary random processes, estimation of their time-varying mean and spectrum offers insight into the behavior of the processes, including whether and how they have changed over time. For example, an analysis of the spatial distribution of the frequency domain characteristics of time series of rainfall for sites spanning a wide spatial field can quantify the cyclical variability of the underlying process, while allowing for nonstationarity can suggest ways in which the climate has changed over the observation period.

Joint modeling of multiple time series with similar or identical local spectra improves estimation of the spectra by borrowing strength, and this is the motivation behind the approach taken in this article. This is particularly necessary when the time series are nonstationary and the data are historical, because the prospect of the spectrum changing implies that future observations may not help to estimate the local spectrum in earlier time periods (Priestley 1965; Dahlhaus 1997). This is reflected, for example, in theoretical frameworks for estimating evolutionary spectra, such as those of Priestley (1965), Dahlhaus (1997), and Dahlhaus (2012). In those frameworks, the relevant asymptotics are based on infill, requiring new observations of the process at finer time intervals; such infill is not possible for historical time series. Joint modeling of multiple time series with similar or identical local spectra is one way to ameliorate this problem, as estimates for different time series can borrow strength from each other (the same approach is also useful for stationary time series; see, e.g., Diggle and al Wasel 1997). If additional information is available, such as covariates, then it should be incorporated into the model to improve estimation.

This article addresses this problem by presenting methodology for analyzing a panel of possibly nonstationary time series using a covariate-dependent infinite mixture model, with mixture components parameterized by their time-varying mean and spectrum. The mixture components are based on AdaptSPEC (Rosen, Wood, and Stoffer 2012), which partitions a (centered) time series into an unknown but finite number of segments, estimating the spectral density within each segment by smoothing splines. As part of the proposed method, AdaptSPEC is extended to handle missing values, a common feature of time series which can cause difficulties for nonparametric spectral methods. A second extension is the incorporation of a time-varying mean, which avoids having to de-mean (center) the time series as a preliminary step. The covariates, which are assumed to be time-independent, are incorporated via the mixture using the logistic stick breaking process (LSBP) of Rigon and Durante (2021), where the log odds for each “stick break” are modeled using a thin plate spline Gaussian process (GP) over the covariates. The model is formulated in a Bayesian framework, where Markov

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chain Monte Carlo (MCMC) methods are used for parameter estimation and to deal with missing values. Specifically, as in AdaptSPEC, reversible jump MCMC (RJMCMC) is used to estimate the mixture component parameters, while the LSBP parameters are estimated via the Pólya-Gamma based latent variable expansion of Rigon and Durante (2021) (see also Polson, Scott, and Windle 2013, for the original latent variable expansion in the finite mixture case). The model and sampling scheme are capable of handling large panels, such as that of the measles application which has nearly 200,000 observations. In addition to estimating time-varying spectra for each time series in the panel, the covariate-dependent mixture structure allows inference about the underlying process at unobserved covariate values, enabling predictive inference. For instance, in this work, we use longitude and latitude as covariates when modeling Australian rainfall data, and are able to infer the predictive time-varying spectrum of the rainfall process at unobserved locations.

Many methods have been proposed for the spectral analysis of time series. As the focus of this article is multiple time series, we provide background in the form of an overview of methods for nonstationary single time series. We then review methods for multiple time series, stationary or otherwise. This excludes methods for multivariate time series, and we refer readers to Li and Krafty (2019) for a review of the past and recent work in this active research area.

Approaches to spectral estimation for a single nonstationary time series include fitting parametric time series models with time-varying parameters (Kitagawa and Gersch 1996; Dahlhaus 1997; West, Prado, and Krystal 1999; Yang, Holan, and Wikle 2016), smoothing the log periodogram (Ombao et al. 2001; Guo et al. 2003; Qin and Wang 2008), dividing the time series into locally stationary segments (Adak 1998; Davis, Lee, and Rodriguez-Yam 2006; Rosen, Stoffer, and Wood 2009; Rosen, Wood, and Stoffer 2012), using short time Fourier transforms (Yang and Zhou 2020), and through the use of wavelets (Nason, Sachs, and Kroisandt 2000). For a recent and extensive overview of methods for single nonstationary time series, see Yang, Holan, and Wikle (2016). Most directly relevant to this article, Rosen, Stoffer, and Wood (2009) estimated the log of the spectral density using a Bayesian mixture of splines. The time series is partitioned into small sections, and it is assumed that the log spectral density within each partition is given by a mixture of smoothing splines. The mixture weights are assumed to be time-varying. Rosen, Wood, and Stoffer (2012) introduced the AdaptSPEC method, which avoids the fixed partitions of Rosen, Stoffer, and Wood (2009). AdaptSPEC partitions the time series into one or more variable length segments in an adaptive manner, modeling the log spectral density within each segment via a smoothing spline. This results in better estimates than those obtained from the method of Rosen, Stoffer, and Wood (2009). Furthermore, by averaging over the possible locations of the partition points, the method can accommodate both slowly and abruptly varying processes, as well as identify stationary processes. AdaptSPEC forms the basis of our proposed model for multiple time series.

For multiple stationary time series, the seminal work of Diggle and al Wasel (1997) established a frequentist framework for replicated stationary time series in which the log spectral density for each replicate varies randomly around a baseline population density. The variation is captured through a mixed-effects model as arising from both random and deterministic (i.e., covariate-dependent) causes, and is estimated through a probability model over the log periodograms of the replicates. Freyermuth et al. (2010) extend this method using tree-structured wavelets to improve the estimation of both the population and the replicate log spectra (see also Chau and von Sachs 2016, for wavelet estimation for spectra of replicated time series). Krafty, Hall, and Guo (2011) constructed a covariate-dependent model for multiple stationary time series in which the log spectrum has a mixed effects representation, where the effects are functions over the frequency domain.

Iannaccone and Coles (2001) described a Bayesian extension of the work of Diggle and al Wasel (1997), in which the population spectral density is modeled using a nonparametric structure as in Carter and Kohn (1997), and MCMC is used for estimation. Macaro and Prado (2014) proposed a Bayesian model for multiple stationary time series with a covariate-dependent spectral density composed as a sum of spectral densities corresponding to different levels of two or more factors and, following Choudhuri, Ghosal, and Roy (2004), model the spectral density associated with these factors via Bernstein-Dirichlet priors. Krafty et al. (2017) presented a Bayesian model for stationary multivariate time series based on the work of Rosen and Stoffer (2007), where multiple (multivariate) time series from different subjects are available, and subjects have an associated single covariate. Cadonna, Kottas, and Prado’s (2019) model multiple stationary time series using a Bayesian hierarchical model. The log-periodogram of a single stationary time series is modeled as a mixture of Gaussian distributions where the mixture weights and mean functions are frequency-dependent. The hierarchical model for multiple time series is constructed by setting the mean functions to be common to all time series while letting the weights vary between time series.

For multiple nonstationary time series (Qin, Guo, and Litt 2009) present a frequentist method for estimating the covariate-dependent time-varying spectral density of a collection of time series with covariates. The covariates are allowed to vary with time, and the local log spectral density at each time within each series is a linear combination of time-frequency surfaces, where the coefficients on the combination are the covariates. This has similarities to the approach for stationary time series used by Diggle and al Wasel (1997), but random variation in the replicates is not accounted for. Estimation is performed in two stages. First, the time period of the study is split into blocks, and the local log periodogram is calculated within each block and for each replicate. The covariate-dependent time-varying spectral density is then estimated using an approximate functional model over the local log periodograms. Also frequentist, Fiecas and Ombao (2016) performed spectral estimation for replicated nonstationary multivariate time series, where the spectrum changes slowly between replications. The application is to multiple time series collected from the same subject over time. The estimation technique, which has similarities to that of Qin, Guo, and Litt (2009), first splits the time series into blocks and computes local periodogram matrices, and then smooths over the blocks to estimate the spectrum. Bruce et al. (2018) presented a method for multiple nonstationary time series with a single covariate that is referred to as conditional adaptive
Bayesian spectrum analysis (CABS), which adaptively partitions both time- and covariate-space, modeling the spectrum within each partition by smoothing splines (as in Rosen, Wood, and Stoffer 2012).

Augmenting this literature, we present methodology, referred to as AdaptSPEC-X, which combines four features: multiple time series, nonstationarity in both mean and spectrum, multiple covariates, and missing data. We demonstrate the method on simulated data, and show how it can be used to estimate the mean and spectra in two application areas: Australian rainfall data, and measles incidence in the United States. Software implementing AdaptSPEC-X is available in the R package BayesSpec.¹

The AdaptSPEC-X methodology constitutes an advance of the existing state-of-the-art in several respects. A near-universal feature in the literature on spectral estimation is to assume the mean is known or fixed over time (see, e.g., Diggle and al Wasel 1997), whereas AdaptSPEC-X allows for the mean to be unknown and to change over time. A fixed-mean assumption is appropriate in many settings, but not universally so. In particular, the assumption would be inappropriate in the applications presented in the article. In many cases, particularly those with highly variable short time series, there may also be considerable uncertainty regarding the mean, so assuming it is known would lead to overconfident and potentially different inference about other aspects of the time series. Another advance is AdaptSPEC-X’s treatment of missing values, which are accommodated via the MCMC scheme; while most parametric models have little trouble with missing data, they are typically not handled by nonparametric spectral methods. These first two advances are in fact extensions of AdaptSPEC (Rosen, Wood, and Stoffer 2012), nonparametric spectral methods. The other comparable method to AdaptSPEC-X is that of Qin, Guo, and Litt (2009), which also allows for multiple time-varying covariates; the converse is true for the ability to handle time-varying means and missing values. A further difference is that AdaptSPEC-X is fully Bayesian, and, through its use of thin-plate splines, incorporates covariates in a more general fashion than that provided by the additive model used by Qin, Guo, and Litt (2009).

The article proceeds as follows. Section 2 describes AdaptSPEC, the method for single nonstationary time series forming the basis for the analysis of multiple nonstationary time series. AdaptSPEC-X, a covariate-dependent infinite mixture model, is presented in Section 3. Section 4 outlines the MCMC scheme used to estimate the model parameters. Section 5 presents simulation studies that examine the performance of the method. Section 6 describes the application areas. Australian rainfall is analyzed in Section 6.1, and measles incidence in the United States is discussed in Section 6.2. The article is accompanied by a set of Appendices in the supplementary material. Appendix A provides details of the conditional distributions necessary for the sampling scheme, and Appendix B expands on the covariance structure used to derive the conditional distribution of the missing values.

2. Model for Single Time Series

Rosen, Wood, and Stoffer (2012) presented the AdaptSPEC method for modeling single nonstationary time series which we summarize in this section. Let \( x = (x_1, \ldots, x_n)' \) be a time series of length \( n \). Assume for ease of notation that \( n \) is even, and suppose initially that \( x \) is a realization from a stationary process \( \{X_t\} \) with constant mean \( \mu \) and a bounded positive spectral density \( f(\omega) \) for \( \omega \in (-\frac{1}{2}, \frac{1}{2}) \). Whittle (1957) shows that, for large \( n \), the likelihood of \( x \) can be approximated as

\[
p(x | \mu, f) = \frac{1}{(2\pi)^{n/2}} \prod_{k=1}^{n} f(\omega_k)^{1/2} \exp \left\{ -\frac{1}{2} \sum_{k=1}^{n} I_k \right\},
\]

where \( \omega_k = \frac{k-1}{n} \) for \( k = 1, \ldots, n \) are the Fourier frequencies, \( I_k = |d_k|^2 \) is the periodogram at \( \omega_k \), and

\[
d_k = \frac{1}{\sqrt{n}} \sum_{t=1}^{n} (x_t - \mu) e^{-2\pi i \omega(t-1)}
\]

is the discrete Fourier transform (DFT) at \( \omega_k \), with \( i = \sqrt{-1} \). Rosen, Wood, and Stoffer (2012) followed Wahba (1990) by expressing \( \log f \) as

\[
\log f(\omega) = a_0 + h(\omega),
\]

and placing a smoothing spline prior on \( h(\omega) \). Due to the evenness of \( f(\omega) \) and the periodogram, \( h(\omega) \) is modeled on the domain \( \omega \in [0, 0.5] \), corresponding to the first \( \frac{\omega}{2} + 1 \) Fourier frequencies. This prior is expressed via a linear combination of \( f \) basis functions, where \( J < \frac{\omega}{2} + 1 \) is chosen to balance prior flexibility and computational resources. See Appendix A.1 in the supplementary material and Rosen, Wood, and Stoffer (2012) for details.

Next we allow the underlying process \( \{X_t\} \) to be nonstationary. Let a time series consist of a number of segments, \( m \), and let \( \xi_{s,m} \) be the end of the \( s \)th segment, \( s = 1, \ldots, m \), where \( \xi_{0,m} = 0 \)

¹Available from GitHub at https://github.com/mbertolacci/BayesSpec/. As of publication, the version of the package on CRAN does not contain AdaptSPEC-X.
and $\xi_{m,m} = n$. Then we assume that $\{X_t\}$ is piecewise stationary, with

$$X_t = \sum_{s=1}^{m} X_t^s \delta_{s,m}(t),$$

where the processes $\{X_t\}$ are independent and stationary with means $\mu_{s,m}$, spectral densities $f_{s,m}(\omega)$, and $\delta_{s,m}(t) = 1$ if $t \in (\xi_{s-1,m}, \xi_{s,m}]$. Consider a realization $x$ from (4). Rosen, Wood, and Stoffer (2012) approximated the likelihood of $x$ by

$$g(x | \Theta) = \prod_{s=1}^{m} p(x_{ts} | \mu_{s,m}, f_{s,m}),$$

where $\Theta = \{m, \xi_{m}, \mu_{m}, f_{1,m}, \ldots, f_{m,m}\}$, $\xi_{m} = (\xi_{1,m}, \ldots, \xi_{m,m})$, $\mu_{m} = (\mu_{1,m}, \ldots, \mu_{m,m})$, $x_{s,m} = \{x_t : \delta_{s,m}(t) = 1\}$ are the data for the $s$th segment, and $p(x_{ts} | \mu_{s,m}, f_{s,m})$ is the Whittle likelihood (1). The values of $m$, $\xi_{m}$, and $f_{s,m}$ for $s = 1, \ldots, m$ are considered unknown and are assigned priors. For $f_{s,m}$, the prior in (3) is used, while $m$ is given the discrete uniform prior between 1 and $M$ (see Rosen, Wood, and Stoffer 2012, for the details of the prior on $\xi_{m}$). Rosen, Wood, and Stoffer (2012) considered $\mu_{s,m}$ to be known and equal to zero, but this work considers it unknown and assigns to $\mu_{s,m}$ a uniform prior with support $\mu_- < \mu_{s,m} < \mu_+$. A minimum segment length $h_{\text{min}}$ is set to ensure that there are sufficient time periods within each segment so that the Whittle likelihood approximation is appropriate.

Rosen, Wood, and Stoffer (2012) described a reversible jump Markov chain Monte Carlo algorithm (Green 1995) that samples from the posterior distribution of this model that allows AdaptSPEC to handle both abruptly and slowly varying nonstationary time series, as well as identify whether a time series is stationary. AdaptSPEC forms the basis of our spectral estimation technique for multiple time series.

3. Model for Multiple Time Series

In this section we extend the model of Section 2 to multiple time series. Suppose now that the stochastic process $\{X_t\}$ has associated covariates $u = (u_1, \ldots, u_p)'$. We model $\{X_t\}$ with a covariate-dependent mixture structure

$$\{X_t\} \sim \sum_{h=1}^{H} \pi_h(u) g_h(\{X_t\} | \Theta_h),$$

where the mixture component distributions $g_h$ are instances of AdaptSPEC (Equation (5)) with parameters $\Theta_h = \{m^h, \xi^h, \mu_{m^h}^{h}, f_{1,m^h}^{h}, \ldots, f_{m,m^h}^{h}\}$, $2 \leq H \leq \infty$, and the mixture weights $\pi_h(\cdot)$ satisfy $0 \leq \pi_h(\cdot) \leq 1$ and $\sum_{h=1}^{H} \pi_h(u) = 1$. Equation (6) implies that $\{X_t\}$’s distribution is determined by its covariates $u$, which, importantly, do not vary with time. The purpose of the mixture structure in Equation (6) is to induce covariate-dependence in a flexible, semi-parametric manner, and we do not use this structure to perform inference about clustering or the number of clusters among multiple time series.

Let $\{x_1, \ldots, x_N\}$ be a finite collection of $N$ time series, of length $n$ each, where each time series $x_j = (x_{1,j}, \ldots, x_{n,j})'$ has covariates $u_j = (u_{1,j}, \ldots, u_{p,j})'$ for $j = 1, \ldots, N$. Assuming independence conditional on $\pi_h(\cdot)$ and $\Theta_h$, it follows from Equation (6) that the joint distribution of the collection is

$$p(x_1, \ldots, x_N) = \prod_{j=1}^{N} \sum_{h=1}^{H} \pi_h(u_j) g_h(x_j | \Theta_h).$$

3.1. Model for Mixture Weights

For the mixture weights $\pi_h(u)$ in Equation (6), we use the LSBP of Rigon and Durante (2021), according to which $\pi_h(u)$ is given by

$$\pi_h(u) = v_h(u) \prod_{h'=1}^{h-1} (1 - v_{h'}(u)),$$

where logit $v_h(u) = w_h(u)$, so that $w_h(u)$ are the covariate-dependent log odds. This prior allows for $2 \leq H \leq \infty$ mixture components, with $v_1(u) = 1$ when $H = \infty$.

As in Rigon and Durante (2021), this construction can be interpreted via sequential (continuation-ratio) logits (Agresti 2018). Let $z_j \in \{1, 2, \ldots, H\}$ be a latent indicator such that $(x_j | z_j = h) \sim g_h(x_j | \Theta_h)$. Then the LSBP can be represented in a generative manner as a sequence of decisions, where $p(z_j = 1) = v_1(u_j) = (1 + \exp(-w_1(u_j)))^{-1}$, $p(z_j = 2 | z_j > 1) = v_2(u_j) = (1 + \exp(-w_2(u_j)))^{-1}$, and so on, such that in general, $p(z_j = h | z_j > h - 1) = (1 + \exp(-w_h(u_j)))^{-1}$.

The model in Equations (7) and (8) has a similar structure to the classic mixture of experts model (Jacobs et al. 1991), in which the weights of a finite mixture depend on covariates through multinomial logits. Our motivation for choosing the LSBP (Equation (8)) over multinomial logits is to obviate the choice of the number of mixture components. The model in Equations (7) and (8) is in principle an infinite mixture and so the question of the number of components becomes irrelevant. In practice, however, it is common to truncate the infinite representation at a suitably high but finite $K$ (Ishwaran and James 2001). The LSBP is analogous to the probit stick-breaking process (Chung and Dunson 2009), where a probit link function is used in place of the logit.

3.2. Model for Log Odds

We model the log odds by a GP prior $w_h(u) \sim \text{GP}(\beta_0, \beta^2 \Sigma_w)$, where $\beta_0$ is an intercept, $\beta_h = (\beta_{1,h}, \ldots, \beta_{p,h})'$ is a vector of regression coefficients, $\tau_h^2$ is a smoothing parameter, and $\Sigma_w(u, u')$ is the covariance kernel constructed via the reproducing kernel Hilbert space defined by a $P$-dimensional thin-plate GP prior (see Wood 2013). For a finite collection of $N$ time series $\{x_1, \ldots, x_N\}$, with associated covariates $\{u_1, \ldots, u_N\}$, the log odds vector, $w_h = (w_h(u_1), \ldots, w_h(u_N))'$, has a multivariate normal distribution

$$w_h \sim N(\beta_0 1_N + \beta_0 1_N, \Sigma_w),$$

where $1_N$ is an $n \times 1$ vectors of ones, $U = (u_1, \ldots, u_N)'$ is an $N \times P$ matrix, and $\Sigma_w$ is an $N \times N$ matrix whose $(j,k)$th entry is equal to $\Omega(u_j, u_k)$. To facilitate the posterior sampling scheme in Section 4, we transform the problem via a basis expansion
Let $\Sigma_w = QDQ'$ be the eigenvalue decomposition of $\Sigma_w$, where $Q$ is an $N \times N$ orthogonal matrix whose columns are the eigenvectors of $\Sigma_w$, and $D$ is a diagonal matrix containing the eigenvalues of $\Sigma_w$. Define $U^t = (1_N, U, QD^{1/2})$ by columnwise concatenation, let $\beta^{GP}_{h}$ be an $N \times 1$ vector, and $\beta^t_h = (\beta_{0h}, \beta'_h, \beta^{GP}_{h})'$. The first column of $U^t$ is a vector of ones, the next $P$ columns of $U^t$ (equal to $U$) are the original covariates, while the last $N$ columns (equal to $QD^{1/2}$) are basis functions, with associated coefficients $\beta^{GP}_{h}$. For computational convenience and parsimony, we truncate the basis expansion to the first $B < N$ basis functions, so that $U^t$ is $N \times (P + B + 1)$ and $\beta^t_h$ is $(P + B + 1) \times 1$. Equation (9) now takes the form

$$
\begin{align*}
\beta^{GP}_{h} &= U^t \beta^t_h, \\
\beta^t_h &\sim \mathcal{N}(0_B, \tau^t_h I_B),
\end{align*}
$$

(10)

where $0_B$ is an $B \times 1$ vector of zeros, and $I_B$ is the $B \times B$ identity matrix. This basis expansion, combined with the interpretation via sequential logits given in the previous section, facilitate the development of the posterior sampling scheme presented in Section 4.

The expression for $v_h(u_j)$ becomes logit $v_h(u_j) = u_j^t \beta^t_h$, where $u_j^t$ is the $j$th row of $U^t$. The prior placed on $(\beta_{0h}, \beta'_h, \beta^{GP}_{h})' \sim \mathcal{N}(\mu_{\beta}, \Sigma_{\beta})$, where $\mu_{\beta}$ is a $(P + 1) \times 1$ vector and $\Sigma_{\beta}$ is a $(P + 1) \times (P + 1)$ covariance matrix, so that

$$
\beta^t_h \sim \mathcal{N}\left(\begin{pmatrix} \mu_{\beta} \\ 0 \end{pmatrix}, \begin{pmatrix} \Sigma_{\beta} & 0 \\ 0 & \tau^t_h I_B \end{pmatrix}\right).
$$

Finally, to complete the model specification, we assign $t_h$ a half-$t$ distribution (Gelman 2006) with density

$$
p(t_h) \propto \left(1 + \frac{1}{v_t} \left(\frac{t_h}{A_t}\right)^2\right)^{-\frac{(v_t+1)}{2}}, \quad t_h > 0,
$$

where $A_t$ and $v_t$ are scale and degrees of freedom parameters, respectively. As described in Appendix A in the supplementary material, the half-$t$ distribution can be expressed as a scale mixture of inverse Gamma distributions, which simplifies the sampling of $t_h$. For the results in this article, we set $\mu_{\beta} = 0$, $\Sigma_{\beta} = 100I_{P+1}$, $v_t = 3$ and $A_t = 10$.

Figure 1 displays a graphical summary of the model in Equations (5)–(10), showing the dependence between the data, covariates, parameters, and hyperparameters.

### Figure 1

A graphical representation of AdaptSPEC-X. The bottom row lists the main hyperparameters.

#### 3.3. Missing Values

The model can accommodate missing values by exploiting the fact that the Whittle likelihood describes a multivariate normal distribution (Whittle 1953). Under the assumption that the missing data mechanism is ignorable (see Rubin 1976, for minimal sufficient conditions under which this is true), we show below that this property of the Whittle likelihood implies that the conditional distribution of the missing values is also multivariate normal, and we use this fact to accommodate missing values by integrating them out as part of the MCMC scheme in Section 4. Ignorability of the missing data mechanism is sufficient for inference on the model parameters, but further assumptions, such as that the data are missing completely at random (MCAR; see Rubin 1976), are required if inference is made on the missing values themselves.

For ease of exposition, in this section we first return to the case where $x$ is a single stationary time series with spectral density $f$, then later describe how this is extended to multiple nonstationary time series. Define the $n \times n$ matrix $V$ with entries $V_{ik} = \frac{1}{\sqrt{n}} \exp(-(2\pi i(t-i)\omega_k))$ for $t = 1, \ldots, n$ and $k = 0, \ldots, n - 1$. It then follows that $V(x - \mu) = (d_1, \ldots, d_n)'$, where $d_k$ is given in (2). Noting that $V$ is a unitary matrix, (1) may be rewritten as

$$
p(x \mid \mu, f) = \frac{1}{(2\pi)^{n/2} |R|^{1/2}} \exp \left\{ -\frac{1}{2} (x - \mu)' V R V^* (x - \mu) \right\},
$$

where $R = \text{diag}(r)$, $r = (1/f(\omega_1), \ldots, 1/f(\omega_n))'$, and $V^*$ is the conjugate transpose of $V$. The precision matrix $\Lambda = V R V^*$ is symmetric and circulant, and is thus defined by its first column, with entries $\Lambda_{1t} = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{f(\omega_k)} e^{-2\pi i(t-i)\omega_k}$ (see Appendix B in the supplementary material for a derivation). Suppose some values of $x$ are missing, and write $x = (x_{\text{mis}}, x_{\text{obs}})'$, where $x_{\text{mis}}$ and $x_{\text{obs}}$ are the missing and observed values, respectively. From standard multivariate normal conditioning results and under the assumption of ignorability,

$$
(x_{\text{mis}} \mid x_{\text{obs}}, \mu, f) \sim \mathcal{N}(\mu_{\text{mis}}|\text{obs}, \Lambda_{\text{mis}}^{-1}|\text{obs}),
$$

(11)

where

$$
\mu_{\text{mis}}|\text{obs} = \mu - \Lambda_{\text{mis},\text{mis}}^{-1} \Lambda_{\text{mis},\text{obs}}(x_{\text{obs}} - \mu), \quad \Lambda_{\text{mis}} = \Lambda_{\text{mis},\text{mis}},
$$

(12)

and the quantities in Equation (12) can be obtained from expressing $\Lambda$ as

$$
\Lambda = \begin{bmatrix} \Lambda_{\text{mis}\text{mis}} & \Lambda_{\text{mis},\text{obs}} \\ \Lambda_{\text{mis},\text{obs}} & \Lambda_{\text{obs},\text{obs}} \end{bmatrix} = \begin{bmatrix} V_{\text{mis}} R_{2\times1} V_{\text{mis}}^* & V_{\text{mis}} R_{2\times1} V_{\text{obs}}^* \\ V_{\text{obs}} R_{2\times1} V_{\text{mis}}^* & V_{\text{obs}} R_{2\times1} V_{\text{obs}}^* \end{bmatrix},
$$

(13)

in which $V_{\text{mis}}$ and $V_{\text{obs}}$ are matrices made up of the rows of $V$ corresponding to the missing and observed times, respectively. The quantities in Equation (13) can be computed efficiently by the fast Fourier transform. When simulating from Equation (11), the most computationally intensive step is the inversion, $\Lambda_{\text{mis},\text{mis}}^{-1}$, in Equation (12). In a general framework for spectral estimation with stationary time series, Guinness (2019) described computationally efficient methods for missing data that could be used to simulate from Equation (11). For this article we compute $\Lambda_{\text{mis},\text{mis}}^{-1}$ the usual way, i.e., via its Cholesky decomposition.
Missing values may be accommodated in AdaptSPEC by partitioning the data in each segment, $x_{mis}$, into missing and observed times as above, and sampling from (11) for each segment as part of the MCMC scheme. As shown in the next section, this is extended to the full AdaptSPEC-X model by conditioning on the latent indicators $z$.

The Whittle likelihood is used in AdaptSPEC-X in two ways: first, as a nonparametric technique based on its asymptotic properties (see Section 2 and Rosen, Wood, and Stoffer 2012), and second, in the assumption that the missing values follow a multivariate normal distribution. We justify the latter assumption in several ways. First, Guinness (2019), who makes a similar assumption regarding missing values, found through both theory and numerical experiments that the assumption did not have a deleterious effect on spectral estimation. Second, it is hard to avoid making some assumption about the distribution of the missing values, and multivariate normality is arguably the most parsimonious choice, as it requires only the first two central moments to be specified (the minimum necessary to have mean and spectrum consistent with the observed data). Finally, as shown above and in Guinness (2019), the assumption is computationally convenient.

4. Sampling Scheme

Define $z = (z_1, \ldots, z_N)'$, $x^{all} = (x_1', \ldots, x_N')'$, $\beta^\dagger = \{\beta_1', \ldots, \beta_{H-1}'\}$, $\Theta = \{\Theta_1', \ldots, \Theta_H\}$, and $\tau = (\tau_1, \ldots, \tau_{H-1})'$. Let $x^{all} = (x_{mis}^{all}, x_{obs}^{all})$ be the decomposition of $x^{all}$ into missing and observed times, respectively. Values produced in steps 2, 3, 4, and 5 of the MCMC sampling scheme below are indicated by the superscript $[l + 0.5]$; and are then used in a label swapping move in Step 6 to produce the $(l + 1)$th iteration. As described below, this move improves convergence of the following sampling scheme.

Step 1. $(x_{mis}^{all}[l+1], \beta_1[l], \Theta[l], x_{obs}^{all})$ as per Section 3.3.

Step 2. $(\Theta_h[l+0.5], x_{mis}^{all}[l+0.5], z[l], x_{obs}^{all})$ for each $h = 1, \ldots, H$. This step is potentially transdimensional, and uses the reversible-jump MCMC scheme of Rosen, Wood, and Stoffer (2012), with two modifications. The first one samples the segment means, $\mu_{mis}^h$, as described in Appendix A.1.1 in the supplementary material. The second modification incorporates a Riemann manifold Hamiltonian Monte Carlo (RMHMC for short, see Girolami and Calderhead 2011) step to accelerate convergence, as described in Appendix A.1.2.

Step 3. $(z[l+0.5], \beta_1[l], \Theta[l+0.5], x_{mis}^{all}[l+1], x_{obs}^{all})$ for each $h = 1, \ldots, H$. This uses the Polya-Gamma data augmentation scheme developed by Polson, Scott, and Windle (2013), as applied to the LSBP by Rigon and Durante (2021).

Step 4. $(\beta_h[l+0.5], \tau[l+0.5], z[l+0.5])$ for each $h = 1, \ldots, H$. This step is potentially transdimensional, and uses the reversible-jump MCMC scheme of Rosen, Wood, and Stoffer (2012).

Step 5. $(\tau_h[l+0.5], \beta_h[l+0.5])$ for each $h = 1, \ldots, H$.

Step 6. $(\Theta_h[l+1], z[l+1], \beta_1[l+1], \tau[l+1], x_{mis}^{all}[l+1], x_{obs}^{all})$ using a label swapping step, described below.

The details of Steps 2–5 are presented in Appendix A in the supplementary material, while Step 1, for $x_{mis}$, is described in Section 3.3. For Step 6, we adapt a label swapping move from Hastie, Liverani, and Richardson (2015), who find that it improves convergence in the context of MCMC samplers for Dirichlet process mixture models. The label swapping step is composed of the following substeps:

Step 6a. Pick uniformly at random components $h_1, h_2 \in \{1, \ldots, H\}, h_1 < h_2$, to swap.

Step 6b. Construct proposal component indicators $z_{\text{swap}}^{h_1}$ and $\Theta_{\text{swap}}$ such that

$$z_{\text{swap}}^{h_1} = \begin{cases} h_2 & \text{if } z_{l+0.5} = h_1, \\ h_1 & \text{if } z_{l+0.5} = h_2, \\ z_j & \text{otherwise,} \end{cases}$$

$$\Theta_{\text{swap}} = \begin{cases} \Theta_{h_1} & \text{if } h = h_1, \\ \Theta_{h_2} & \text{if } h = h_2, \\ \Theta_j & \text{otherwise.} \end{cases}$$

Step 6c. Construct proposal $\tau_{\text{swap}}$ by setting

$$\tau_{\text{swap}} = \begin{cases} \tau_{h_1} & \text{if } h = h_1, \\ \tau_{h_2} & \text{if } h = h_2, \\ \tau_j & \text{otherwise,} \end{cases}$$

and sample proposal $\beta_{\text{swap}}^{h_1}, \beta_{\text{swap}}^{h_2}$ from

$$q(\beta_{\text{swap}} | z_{l+0.5}, \tau_{l+0.5}) \sim N(\mu_{\text{mode}}^{h_1}, \Sigma_{\text{mode}}^{h_1}),$$

where $\mu_{\text{mode}}^{h_1}$ and $\Sigma_{\text{mode}}^{h_1}$ are the mode and the negative inverse of the Hessian, respectively, of $\log p(\beta_{\text{swap}} | z_{l+0.5}, \tau_{l+0.5})$.

Step 6d. Accept the swap with probability equal to the Metropolis-Hastings ratio

$$\min \left\{ 1, \frac{p(\beta_{\text{swap}}^{h_1}, \Theta_{\text{swap}}, \tau_{\text{swap}}, z_{\text{swap}}^{h_1} | x_{\text{all}}^{l+1}, x_{\text{obs}}^{l+1}) q(\beta_{\text{swap}}^{h_1} | z_{\text{swap}}^{h_1}, \tau_{\text{swap}})}{p(\beta_{\text{swap}}^{h_2}, \Theta_{\text{swap}}, \tau_{\text{swap}}, z_{\text{swap}}^{h_2} | x_{\text{all}}^{l+1}, x_{\text{obs}}^{l+1}) q(\beta_{\text{swap}}^{h_2} | z_{\text{swap}}^{h_2}, \tau_{\text{swap}})} \right\}.$$
5. Simulation Studies

This section reports on simulation results to study the proposed method, as well as to compare it with another method in the literature. Section 5.1 focuses on the advantages of AdaptSPEC-X for joint modeling of multiple nonstationary time series, comparing it to separate applications of AdaptSPEC to multiple time series. To this end, multiple time series with time varying means and spectra at both observed and unobserved covariates are generated repeatedly. This simulation demonstrates AdaptSPEC-X’s ability to provide out-of-sample predictions, and unobserved covariate values. This study also illustrates AdaptSPEC-X’s ability to estimate time varying means and spectra at both observed and unobserved covariates. Our simulation results demonstrate superior performance of AdaptSPEC-X in terms of mean squared error.

5.1. Joint Versus Separate Modeling

Let \( U = (u_1, \ldots, u_{100}) \) be a \( 100 \times 2 \) design matrix corresponding to \( N = 100 \) subjects, each with two covariates, where the \( u_j, j = 1, \ldots, N, \) are sampled uniformly from \([0,1] \times [0,1]\). Each \( u_j \) is mapped deterministically to a region \( r_j \in \{1, 2, 3, 4\} \), according to the plot shown in Figure 2(a), which also includes the locations of the 100 sampled points, denoted by crosses. Four locations are chosen as example time series, marked in green circles, and labeled D1 through to D4 (corresponding to \( r_j = 1 \) through \( r_j = 4 \), respectively). Four more locations are marked in red diamonds labeled T1 through to T4 (again for \( r_j = 1 \) to 4). These values of \( u \) have no corresponding time series and are used as test points to evaluate the predictive inferences. The four different regions correspond to four different data-generating processes. Each time series \( x_{jt} \), within region \( r_j, j = 1, \ldots, 100, \) is a realization of length \( n = 256 \) from

\[
(x_{jt} - \mu_{r_j,t}) = \phi_{r_j,1,t}(x_{jt-1} - \mu_{r_j,t-1}) + \phi_{r_j,2,t}(x_{jt-2} - \mu_{r_j,t-2}) + \epsilon_{jt},
\]

where \( \epsilon_{jt} \sim N(0,1) \), and the values of \( \mu_{r_j,t} \) and \( \phi_{r_j,p,t} \) are given in the following table:

| \( r_j \) | \( \mu_{r_j,t} \) | \( \phi_{r_j,1,t} \) | \( \phi_{r_j,2,t} \) | \( \mu_{r_j,t} \) | \( \phi_{r_j,1,t} \) | \( \phi_{r_j,2,t} \) |
|---|---|---|---|---|---|---|
| 1 | -1.5 | 1.5 | -0.75 | 2 | -0.8 | 0 |
| 2 | 1 | -0.8 | 0 | -1 | -0.8 | 0 |
| 3 | 0 | 1.5 | -0.75 | 0 | 1.5 | -0.75 |
| 4 | 1 | 0.2 | 0 | 1 | 1.5 | -0.75 |

Thus, time series with \( r_j = 1 \) have two segments with different means and different spectra, those with \( r_j = 2 \) have two segments with different means but with the same spectra, time series with \( r_j = 3 \) have only one stationary segment, and those with \( r_j = 4 \) have two segments with the same mean but with different spectra. In each time series, 10% of the times are set as missing. Figure 2(b) displays example realizations from Process (14) for \( r_j = 1, 2, 3 \) and 4, showing the time series values, underlying time-varying mean, and the times at which values are missing.

We sample 100 replicates from Process (14), and to each fit AdaptSPEC-X using the MCMC sampling scheme of Section 4. We run 50,000 iterations of the MCMC scheme, where the first...
10,000 are discarded as burn-in. Each mixture component has $M = 4$, as the maximum number of segments, $s_{\min} = 40$, as the minimum segment length, $J = 25$, as the number of basis functions for the smoothing spline prior on the log spectra (see Section 6.1, where we discuss how the selection of $J$ relates to that of $s_{\min}$), and $(\mu_{-\}, \mu_{+}) = (-10, 10)$, as the support of the prior on $\mu^{h}_{s,m}$. The LSBP is truncated at $H = 25$ components, and has $B = 10$ basis functions. The MCMC scheme for each replicate takes around 1.5 hours to run when using two cores on a computer with an Intel Core i9-9900K CPU with a clock speed of 3.60 GHz.

To assess the quality of the estimated time-varying mean, we define the mean squared error (MSE) for the mean as

$$\text{MSE}_{\text{mean}}(u) = \frac{1}{n} \sum_{t=1}^{n} \left[ \hat{\mu}(t, u) - \mu(t, u) \right]^2,$$

where $\hat{\mu}(t, u)$ is the estimate of $\mu(t, u)$, the true time-varying mean at covariates $u$. Similarly, we define the MSE for the spectrum as

$$\text{MSE}_{\text{spec}}(u) = \frac{1}{n} \sum_{t=1}^{n} \sum_{k=1}^{k_{\max}} \left[ \log f_i(t, \frac{k-1}{2k_{\max}}, u) - \log f_i(t, \frac{k-1}{2k_{\max}}, u) \right]^2,$$

where $k_{\max} = 128$, and $\log f_i(t, \omega, u)$ is the estimate of $\log f(t, \omega, u)$, the true time-varying log spectral density at location $u$.

Figure 3 presents boxplots of $\text{MSE}_{\text{mean}}$ (top) and $\text{MSE}_{\text{spec}}$ (bottom), at each observed location, D1–D4, and unobserved test location, T1–T4, from left to right, respectively. We now fit the AdaptSPEC model (Rosen, Wood, and Stoffer 2012), to each replicate of D1–D4 individually, and the green boxplots in Figure 3 show $\text{MSE}_{\text{mean}}$ and $\text{MSE}_{\text{spec}}$ for those fits.

The median $\text{MSE}_{\text{mean}}$ of AdaptSPEC-X is substantially smaller than that of the individual AdaptSPEC fits at D1, D3, and D4, and its interquartile range is narrower. Similarly, the median $\text{MSE}_{\text{spec}}$ of AdaptSPEC-X is smaller than that of individual AdaptSPEC fits at D1, D3, and D4. This demonstrates the benefits of applying a joint model to multiple time series, in that the ability to group them together via the covariates can lead to improved estimation performance. The exception to this is the performance at D2, for which AdaptSPEC-X does worse, and we discuss possible reasons for this below. For the test points T1–T4, the predictive performance of AdaptSPEC-X as measured by $\text{MSE}_{\text{mean}}$ and $\text{MSE}_{\text{spec}}$ is comparable to the observed points D1–D4. The test points are unobserved, so individual fits of AdaptSPEC to the observed series do not allow prediction of their time-varying mean and spectra. This is another benefit of jointly modeling the data by including covariates.

AdaptSPEC-X’s estimates of the time-varying mean and spectrum corresponding to the median MSE values are shown in Figures 4 and 5, respectively. These qualitatively match the $\text{MSE}_{\text{mean}}$ and $\text{MSE}_{\text{spec}}$ scores, in that the estimates for points other than D2 and T2 are visually very close to the truth, while for D2 and T2 some differences are visible.

There are several explanations for the worse performance for D2 and T2 relative to the other points. One reason is that the spatial model has relatively little information to identify the existence of a region: these points belong to region two, which is represented by only eight time series (in contrast to regions one, three, and four, which have 41, 18, and 33 members, respectively), and which has the smallest spatial area in the study. Another reason is that the process mean for region two (equal to 1 in the first half and $-1$ in the second half) is similar to that of the surrounding region four (constant mean of 1), making it harder to distinguish between the regions. For both reasons, it is not surprising to see worse performance for T2 and D2, which corresponds to good model behavior in the sense that it represents genuine model uncertainty. This is seen in Figure 4, where the $\hat{\mu}(t, u)$ for T2 and D2 in the second half of the time series is shrunk toward that of region four, the enclosing region. It is also worth noting that the median $\text{MSE}_{\text{mean}}$ and $\text{MSE}_{\text{spec}}$ for T2 and D2 are small relative to the scales of their mean and log spectrum, respectively. This can also be seen qualitatively by the similarity between the true and estimated mean and spectra in Figures 4 and 5.
5.2. Comparison of AdaptSPEC-X and CABS

CABS is a method for estimating the time-varying spectra of a panel of time series, \( \{x_j : j = 1, \ldots, N\} \), for \( x_j = (x_{1,j}, \ldots, x_{n,j})' \), where each time series is associated with a single covariate value \( u_j \). It extends AdaptSPEC to multiple time series by partitioning the covariate in the same way that AdaptSPEC partitions the time series, such that the spectral density is constant as a function of the covariate within a “rectangle” in time and covariate space. Unlike CABS, AdaptSPEC-X is able to handle an arbitrary number of covariates, because there is no need to adaptively divide the covariate space. AdaptSPEC-X is richer in features than CABS, but for the purposes of comparison, we evaluate AdaptSPEC-X and CABS in a setting that is applicable to both methods. To this end, we reproduce the simulation study presented by Bruce et al. (2018), which examined the ability of CABS to estimate the time-varying spectra of a collection of time series drawn from known processes. The simulation study considers two cases: one in which time series have abrupt changes in their spectra, and another where changes occur slowly. In both cases, there are \( N \) time series of length \( n \) each, with a single covariate, taking the values \( u_j = (j - 1)/(N - 1), j = 1, \ldots, N \), that in turn determine the spectra of the corresponding time series. The \( j \)th abruptly varying time series is simulated from a piecewise AR(1) processes, \( \{x_{t,j} : t = 1, \ldots, n\} \), for which

\[
x_{t,j} = \begin{cases} 
-\phi_j x_{t-1,j} + \epsilon_{t,j}, & 1 \leq t \leq n/2 \\
\phi_j x_{t-1,j} + \epsilon_{t,j}, & n/2 < t \leq n 
\end{cases}
\]

where \( \phi_j = \begin{cases} 
-0.5 + t/n, & 0 \leq u_j \leq 0.5 \\
-0.9 + 9t/(5n), & 0.5 < u_j \leq 1 
\end{cases} \) (17)

and \( \epsilon_{t,j} \sim \text{i.i.d.} \ N(0,1) \). The \( j \)th slowly varying time series is simulated from a time-varying AR(1) process with

\[
x_{t,j} = \phi_{t,j} x_{t-1,j} + \epsilon_{t,j}, \text{ where}
\]

\[
\phi_{t,j} = \begin{cases} 
-0.5 + t/n, & 0 \leq u_j \leq 0.5 \\
-0.9 + 9t/(5n), & 0.5 < u_j \leq 1 
\end{cases}
\]

and \( \epsilon_{t,j} \sim \text{i.i.d.} \ N(0,1) \).
Bruce et al. (2018) applied CABS to 100 replicates from processes (17) and (18) over the nine \((N,n)\) combinations, where \(N \in \{20,40,80\}\) and \(n \in \{1000,2000,4000\}\), and evaluated its ability to estimate the time-varying log spectra of the time series. Here we repeat this evaluation using AdaptSPEC-X. To facilitate comparison, for quantities that are common to CABS and AdaptSPEC-X, we use the same values: \(m_{\text{min}} = 50\) (minimum segment length); \(M = 10\) (the maximum number of segments); and \(J = 7\) (the number of basis functions for the smoothing spline on the log spectrum; \(B\) in the notation of Bruce et al. (2018)). The number of segments in covariate space in CABS is set equal to eight. In AdaptSPEC-X, we set the number of mixture components to \(H = 10\), and the number of basis functions in covariate space to \(B = 10\). Bruce et al. (2018) run MCMC chains for 5000 iterations with a warm-up of 1000 iterations; we run AdaptSPEC-X for 10,000 iterations with a warm-up of 1000 iterations. For AdaptSPEC-X, running the MCMC scheme took between 5 min (for \(N = 20, n = 1000\)) and 30 min (for \(N = 80, n = 4000\)) using one core of a desktop with an Intel Core i9-9900K CPU with a clock speed of 3.60 GHz.

To evaluate the performance of CABS at estimating the true time-varying log spectrum, Bruce et al. (2018) used the mean-squared error (MSE), calculated according to Equation (16). For a particular combination \((n, N)\), and underlying process (abrupt or slow), a mean MSE was calculated across the \(N\) time series within each of the 100 replicates. The mean and standard deviation of the mean MSE across the replicates was then calculated. Table 1 presents the mean MSE (and standard deviation of the mean MSE across the replicates) for estimating the log spectra of the time series within each of the 100 replicates. The mean and spectrum of Australian rainfall. As part of a report on climate change tendered by several Australian government agencies, Cai et al. (2007) found that, since 1950, the Australian north has seen increased annual rainfall, while the southeast and southwest have experienced the opposite. The causes of these trends have been the subject of study and debate (see, among others, Hope, Drosdowsky, and Nicholls 2006; Unmehof et al. 2009; Pook, Risbey, and McIntosh 2012; Risbey, McIntosh, and Pook 2013). Apart from trends in overall rainfall, several authors have reported relative increases in heavy rainfall events, indicating changes in the variability of rainfall (Cai et al. 2007; Gallant et al. 2013). We contribute to this literature by using AdaptSPEC-X to analyze the time-varying mean and spectrum of Australian rainfall from sites dispersed over a wide spatial field, addressing simultaneously the question of whether changes have occurred in rainfall levels and rainfall variability.

We use data from Bertolacci et al. (2019), who studied the climatology of Australian daily rainfall using measurements from 17,606 sites across the continent. In particular, we use 151 of these sites characterized by having long and nearly contiguous rainfall records, the locations of which are displayed in Figure 6(a). These sites are among those identified by Lavery, Kariko, and Nicholls (1992) as having high-quality records suitable for monitoring and assessing climate change. The raw time series are daily, and observations are typically made at 9 a.m. local time, recording the total rainfall in millimeters (mm) for the previous 24 hr. Aggregation to monthly data is performed

| \(n\) | \(N\) | CABS | AdaptSPEC-X | Ratio | CABS | AdaptSPEC-X | Ratio |
|---|---|---|---|---|---|---|---|
| 1000 | 20 | 0.1190 (0.0580) | 0.0339 (0.0113) | 3.5 | 0.0353 (0.0052) | 0.0245 (0.0027) | 1.4 |
| 1000 | 40 | 0.1130 (0.0598) | 0.0266 (0.0071) | 4.3 | 0.0260 (0.0031) | 0.0184 (0.0021) | 1.4 |
| 1000 | 80 | 0.1079 (0.0557) | 0.0201 (0.0027) | 5.4 | 0.0206 (0.0026) | 0.0135 (0.0015) | 1.5 |
| 2000 | 20 | 0.0748 (0.0252) | 0.0316 (0.0084) | 2.4 | 0.0249 (0.0035) | 0.0180 (0.0017) | 1.4 |
| 2000 | 40 | 0.0753 (0.0274) | 0.0242 (0.0039) | 3.1 | 0.0185 (0.0017) | 0.0130 (0.0010) | 1.4 |
| 2000 | 80 | 0.0702 (0.0230) | 0.0198 (0.0039) | 3.5 | 0.0148 (0.0013) | 0.0101 (0.0009) | 1.5 |
| 4000 | 20 | 0.0617 (0.0121) | 0.0200 (0.0076) | 3.1 | 0.0177 (0.0019) | 0.0135 (0.0010) | 1.3 |
| 4000 | 40 | 0.0610 (0.0133) | 0.0249 (0.0064) | 2.5 | 0.0137 (0.0012) | 0.0101 (0.0007) | 1.4 |
| 4000 | 80 | 0.0607 (0.0135) | 0.0192 (0.0032) | 3.2 | 0.0110 (0.0009) | 0.0082 (0.0007) | 1.3 |

NOTES: The ratio of the mean MSE for CABS to the mean MSE for AdaptSPEC-X is also presented. The values for CABS are taken from Bruce et al. (2018).
by calculating the average daily rainfall for the month. To avoid artifacts, we consider as missing any month with fewer than fifteen days of measurements available (i.e., not missing). We choose to aggregate the data to monthly intervals in order to focus the analysis on climatic changes to the spectrum, which occur on time scales much longer than a month. A secondary benefit is that doing so requires much fewer basis functions for the smoothing splines on the log spectrum. This is due to the fact that the dominant feature of the data is the seasonal cycle at a period of 365.25 days for the daily data, and such a low frequency spike can only be accommodated by a high value of $J$.

The resulting time series span the 1078 months from September, 1914 to June, 2004 (inclusive), for a total of 162,778 observations, of which 4095 are missing. The smallest possible measurement is 0 mm, corresponding to no rainfall for the month; this is true for 9933 months. Time series for four example sites are displayed in Figure 6(b), and their locations are marked on inset
maps (also marked in green in Figure 6(a)). The four time series span a wide range of average rainfall levels from around 1 mm at site 47053 to 4 mm at site 14042. They also exhibit varying levels of seasonality, where sites 10525 and 14042 have highly seasonal rainfall, while rainfall at sites 47,053 and 69,018 is less seasonal. All maps show strong positive seasonality, where sites 10525 and 14042 have highly seasonal rainfall. The four times series Figure 7. 

Estimated time-varying means $\mu(t, u)$ for four monthly rainfall sites (first two rows), and four locations without observations (last two rows).

The estimated time varying means $\hat{\mu}(t, u)$ are shown in Figure 7, where the first two rows display the estimates for the four example sites, and the last two rows present predictive estimates for the four test locations which do not have observations. The corresponding estimated time varying log spectra $\log f(t, \omega, u)$ are shown in Figure 8. Figures 7 and 8 show that the model has accommodated considerable variation between sites, with mean monthly rainfall ranging from 1 mm at site 47053 to 3.5 mm at 14042, and with spectra similarly varied. The dominant period at all sites is 12 months, corresponding to the Earth’s orbit around the sun. There is also considerable power at lower frequencies, indicating long-term dependence. The predictive means and spectra at unobserved sites match qualitative expectations. For example, among the unobserved locations, Test Point 2 has the heaviest mean rainfall and the most power in its spectrum, reflecting its location in the tropics.

Two major droughts occurred during the study period: the World War II drought of 1937–1945, and the Millennium drought that started in 1996 and was still ongoing by the end of the study period in 2004 (Ummenhofer et al. 2009). Table 2 presents estimated posterior probabilities of changes in the mean $\mu(t, u)$ or variance $\sigma^2(t, u) = \int_0^{365.25} f(t, \omega, u) d\omega$ around these times. Specifically, it shows that $\hat{P}(\mu_{1940}^0 < \mu_{1950}^0) > 0.9$ and $\hat{P}(\sigma_{1940}^0 < \sigma_{1950}^0) > 0.9$ at all four sites. However, $\hat{P}(\mu_{1940}^0 < \mu_{1930}^0) < 0.7$ and $\hat{P}(\sigma_{1940}^0 < \sigma_{1930}^0) < 0.57$ except for site 69018 for which these probabilities are greater than 0.8. The Millennium drought is associated with drops in $\mu(t, u)$, $\sigma^2(t, u)$ or both at sites 14,042, 47,053, and 69,018 as can be seen from the estimated probabilities at these sites: $\hat{P}(\mu_{1990}^0 < \mu_{1990}^0) > 0.9$ and $\hat{P}(\sigma_{1990}^0 < \sigma_{1990}^0) > 0.93$. Site 10525 in the southwest of the continent does not exhibit a drop with probability greater than 0.9, consistent with the fact that the drought principally affected southeastern Australia (Ummenhofer et al. 2009).

Cai et al. (2007) reported large trends in rainfall since 1950. For southeast Australia, they report reductions in annual rainfall corresponding to 10–15 mm/decade for the site 47,053 and 50 mm/decade for 69,018. Consistent with this, Table 2 shows that, for these sites, AdaptSPEC-X estimates that both $\mu(t, u)$ and $\sigma^2(t, u)$ declined between January, 1950 and January, 2004 with probability greater than 0.94. The estimated drop in $\mu(t, u)$ for site 47,053 corresponds to a reduction of 1–16 mm/decade$^3$ (10th–90th percentile), consistent with Cai et al.’s (2007) esti-

\[ \text{3Calculated as } 10 \times 365.25 \times \text{difference in daily average / (2004} - 1950) \]
Figure 8. Estimated time-varying spectra \( \log \tilde{f}(t, \omega, u) \) for four monthly rainfall sites (first two rows) and four locations without observations (last two rows). The color indicates the log power at the corresponding time and frequency. The \( \omega \)-axis is on a square-root scale. The axis on the right-hand side displays the period \((1/\omega)\).

Table 2. Estimated posterior probabilities \( \hat{p}(\cdot | x) \) of various events for monthly rainfall at the four example sites.

| Event                  | \( \hat{p}(\cdot | x) \) | Site [u]     |
|------------------------|--------------------------|--------------|
| **WW2 drought**        |                          | 10525        |
| \( \mu(1940-01, u) < \mu(1930-01, u) \) | 0.540                    | 0.442        |
| \( \sigma^2(1940-01, u) < \sigma^2(1930-01, u) \) | 0.574                    | 0.391        |
| \( \mu(1940-01, u) < \mu(1950-01, u) \) | 0.917                    | 0.980        |
| \( \sigma^2(1940-01, u) < \sigma^2(1950-01, u) \) | 0.997                    | 0.996        |
| **Millenium drought** |                          | 14042        |
| \( \mu(2004-01, u) < \mu(1990-01, u) \) | 0.878                    | 0.901        |
| \( \sigma^2(2004-01, u) < \sigma^2(1990-01, u) \) | 0.747                    | 0.930        |
| **Long term**          |                          | 47053        |
| \( \mu(2004-01, u) < \mu(1950-01, u) \) | 0.820                    | 0.906        |
| \( \sigma^2(2004-01, u) < \sigma^2(1950-01, u) \) | 0.961                    | 0.955        |

NOTES: The events correspond to the WW2 drought (first four rows), the Millenium drought (second two rows), and long-term change (last two rows). For each event and site \( u \), the table presents probabilities that the mean \( \mu(t, u) \) and the variance \( \sigma^2(t, u) \) changed before or after the event.
mate. However, for site 69,018, the estimated reduction is 6–42mm/decade, substantially less than 50mm/decade. For the site 14,042 in the tropical north, Cai et al. (2007) estimated increases of around 40mm/decade since 1950, while the estimates of Table 2 indicate a decline over the same period. Finally, Cai et al. reported a decline of 20–30mm/decade in the region of southwest Australia containing the site 10,525, but Table 2 does not indicate a significant change in \( \mu(t, u) \) at this location (though the variance does appear to have declined). In contrast to Cai et al. and Gallant et al. (2013), the reduction in \( \sigma^2(t, u) \) since 1950 at all locations suggests that rainfall variability has declined. This could have resulted from the use of different definitions of variability, for example, counts of extreme events as in Cai et al. (2007), versus our use of change in variance.

6.2. Measles Incidence in the United States

Measles is a highly contagious disease that causes fever, cough, runny nose, and a rash (Moreno 2018). Complications of measles can include pneumonia, deafness, or death (Moreno 2018). Prior to the licensing of a vaccine in 1963, the incidence rate of measles averaged 318 cases per 100,000 population per year, with outbreaks occurring annually or every other year (van Panhuis et al. 2013). After vaccine licensure, incidence declined dramatically, and endemic measles transmission was declared eliminated from the United States in 2000 (Katz and Hinman 2004). As part of a study on the impact of vaccination for a variety of contagious diseases, van Panhuis et al. (2013) presented a unique dataset by digitizing weekly surveillance reports from the United States of several nationally notifiable diseases, including measles, and have made these data available online at the Project Tycho website. In this section, we analyze Project Tycho’s measles data using AdaptSPEC-X.

The data comprise weekly time series of measles incidence for each state, reporting the weekly (where the week starts on a Sunday) incidence rate per 100,000 population. In this work, we use time series from the continental United States (that is, excluding Hawaii and Alaska), plus the District of Columbia. These span the 3914 weeks from week one of 1928 (which we write as 1928-01) to week one of 2003 (2003-01). Across all 49 time series, there are 191,786 observations. Of these, 50,067 (26%) are missing, and 30,439 (16%) have incidence equal to zero. The time series are shown in Figure 9, where each panel presents the series for a state, and the incidence axis is on a square-root scale (note however that the spectral analysis performed later is applied to the untransformed data). The layout of the panels roughly matches the geographic distribution of the states. The most striking aspect of these plots is the dramatic decline in both the level and volatility of incidence starting in 1963, the year of vaccine licensure.

We set \( u_j = (\text{lon}_j, \text{lat}_j) \), the longitude and latitude of the centroid of each state, and fit AdaptSPEC-X to the measles time series. Each mixture component has \( t_{\text{min}} = 208 \) (4 years) as the minimum segment length. This was chosen to ensure four

\[\text{https://www.tycho.pitt.edu/}\]
observations of the annual cycle in each segment. The maximum number of segments was set as $M_{\text{max}} = 18$, the maximum allowed by the combination of $t_{\text{min}}$ and the number of weeks in the data. The support of the prior on $\mu_{\text{s,m}}$ was set to $(\mu_- , \mu_+) = (0, 20)$; the lower bound represents the known positivity of the incidence rates (and is particularly helpful to constrain $\mu_{\text{s,m}}$ in periods with very small counts), while the upper bound is twice as large as the posterior mean of this parameter. As in the rainfall application, preliminary fits of AdaptSPEC to individual measles time series indicated that the estimated log spectra did not change much for $J > 60$, and so we set $J = 60$ in this application as well. The LSBP is truncated at $H = 10$ components, as we found higher values did not change the results. Finally, the thin-plate GP prior for the LSBP has $B = 20$ basis functions, which captures more than 95% of the variation implied by the prior. The MCMC scheme took 3 hr to run on a computer using four cores of an Intel Core i9-9900K CPU with a clock speed of 3.60 GHz. The resulting estimated time-varying means and spectra are shown in Figures 10 and 11, respectively.

As in Figure 9, the post-vaccine drop in the mean and power of incidence is the most obvious feature. This drop occurs in steps, starting with a dramatic drop following the licensing of the Edmonston vaccine in 1963, stalling around 1970, then dropping again around 1980. This corresponds to the waxing and waning of government funding and effort targeted at measles elimination, which culminated in an intensified elimination drive (Hinman, Brandling-Bennett, and Nieburg 1979; Atkinson, Orenstein, and Krugman 1992). An outbreak during the early 1990s is visible as an increase in power in Figure 11; this outbreak received much attention and resulted in changes to the immunization schedule for children (Atkinson, Orenstein, and Krugman 1992).

In the pre-vaccine period, the spectra in Figure 11 have peaks around frequencies 1/52 and 0, indicating annual seasonality and long-term dependence, respectively. After the introduction of the vaccine, the annual peak disappears. Grenfell, Bjornstad, and Kappey (2001) identify a biennial cycle in similar measles data for the United Kingdom, but this does not appear to be a feature of the U.S. data. Figures 10 and 11 also indicate changes in the mean and spectrum during the pre-vaccine years, where all states exhibit periods of increased mean incidence and power centered around 1940 and 1955. This concords with van Panhuis et al. (2013), who noted that incidence rates had variable patterns in the pre-vaccine time period, speculating that these may have been due to sanitation, hygiene, or demographic factors.

Because of the extreme nonstationarity introduced by the vaccine, the time-varying spectra in Figure 11 span such a wide range of powers that the spectra for all states look almost identical. This is not the case: a deeper inspection, for example by magnifying Figure 11, reveals that the spectra highlights the existence of geographic heterogeneity between states, where higher power is more typical of the west and north, compared to the south and east.

Since the elimination of endemic measles in the United States in 2000, there have been a number of outbreaks associated with
individuals “importing” measles by acquiring the disease while outside the United States and spreading it upon their return (Parker et al. 2006; CDC 2019). Phadke et al. (2016) associated several of these outbreaks with individuals unvaccinated for nonmedical reasons, which they term as vaccine refusal. Some authors have even declared that a resurgence of measles has occurred (Lynfield and Daum 2014). Using the AdaptSPEC-X fits, we tested whether the mean \( \mu(t, u) \) or variance \( \sigma^2(t, u) \) of measles increased from 1995-01, a few years after the big outbreak in the early 1990s, to 2003-01, the last time period in the data. We find no evidence of increase in \( \mu(t, u) \) in any state, which can be seen from the fact that the highest posterior probability of an increase equals 0.68 in Oklahoma. As for \( \sigma^2(t, u) \), the posterior probabilities of an increase in all states range between 0.81 and 0.87, which we consider to be weak evidence of change. Unfortunately, because the data end in 2003, it is not possible to assess changes to the mean or spectrum of measles incidence in more recent years.

7. Discussion

This article has presented AdaptSPEC-X, a Bayesian method for analyzing a panel of possibly nonstationary time series using a covariate-dependent infinite mixture model, with mixture components parameterized by their time-varying mean and spectrum. AdaptSPEC-X extends AdaptSPEC to accommodate multiple time series, each with its own covariate values. Specifically, the covariates, which are assumed to be time-independent, are incorporated via the mixture weights using the logistic stick breaking process. The mixture components are based on AdaptSPEC, which handles a single nonstationary time series. In particular, it partitions a time series into an unknown but finite number of segments, and estimates the spectral density within each segment by smoothing splines. New features which have been added to the AdaptSPEC components include estimation of time-varying means and handling of missing observations. The model and sampling scheme can accommodate large panels, such as that of the measles application. In addition to estimating time-varying spectra for each time series in the panel, AdaptSPEC-X allows inference about the underlying process at unobserved covariate values, enabling predictive inference. Efficient software implementing AdaptSPEC-X is available in the R package BayesSpec.

In Section 3.2, the log odds of the LSBP, which determine the mixture weights, are modeled using a thin-plate GP prior. While this prior is flexible, it is also smooth and stationary. This property may be inappropriate in settings where changes in the mean or spectrum of the individual time series occur abruptly over the covariate space. An extension to a nonstationary prior...
for the log odds, or a piecewise prior as in Bruce et al. (2018), may be of interest in these cases.

AdaptSPEC (and therefore AdaptSPEC-X) relies on Whittle’s approximation to the likelihood (Equation (1)). The Whittle likelihood is asymptotically correct for both Gaussian and non-Gaussian time series (Hannan 1973), but is known to be inefficient for small sample sizes (Contreras-Cristán, Gutiérrez-Peña, and Walker 2006). Several methods exist in the literature to ameliorate this problem (see, e.g., Sykulska et al. 2019), and these methods might produce useful extensions to AdaptSPEC for settings with short time series or small segment lengths (i.e., small \( T_{\text{min}} \)). Another feature of AdaptSPEC that is inherited by AdaptSPEC-X is the use of a smoothing splines prior to model the log spectrum within a segment. While this is a very flexible model, it may not be optimal in all settings. For instance, if the rainfall application were repeated at the daily scale, the smoothing splines would require a very large number of basis functions to accommodate the low-frequency spike at 1/365.25 days. Future work into alternative prior structures would be valuable for this and similar settings.

Neither AdaptSPEC nor AdaptSPEC-X account explicitly for measurement error. For i.i.d. Gaussian measurement error, this should not cause a problem, as the added variance would appear as an added constant (that is, white noise) in the spectrum. On the other hand, if the measurement error is changing over time, this may be estimated as false nonstationary, in the sense that the underlying process is not actually changing. Adding an explicit layer for measurement error to AdaptSPEC-X would be a useful extension for these cases. More specialized forms of measurement error may be useful in other settings. For example, in the measles application of Section 6.2, the data are incidence rates

Reproducibility

Data and code reproducing the figures and tables in this manuscript are available online at https://github.com/mbertolacci/adaptspecx.

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Supplementary Material

The article is accompanied by a set of Appendices in the supplementary material. Appendix A provides details of the conditional distributions necessary for the sampling scheme, and Appendix B expands on the covariance structure used to derive the conditional distribution of the missing values.

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