Weyl’s law for the eigenvalues of the Neumann–Poincaré operators in three dimensions: Willmore energy and surface geometry

Yoshihisa Miyanishi *

Abstract

We deduce eigenvalue asymptotics of the Neumann–Poincaré operators in three dimensions. The region \( \Omega \) is \( C^{2,\alpha} \) \((\alpha > 0)\) bounded in \( \mathbb{R}^3 \) and the Neumann–Poincaré operator \( \mathcal{K}_{\partial \Omega} : L^2(\partial \Omega) \to L^2(\partial \Omega) \) is defined by

\[
\mathcal{K}_{\partial \Omega}[\psi](x) := \frac{1}{4\pi} \int_{\partial \Omega} \frac{\langle y - x, n(y) \rangle}{|x - y|^3} \psi(y) \, dS_y
\]

where \( dS_y \) is the surface element and \( n(y) \) is the outer normal vector on \( \partial \Omega \). Then the ordering eigenvalues of the Neumann–Poincaré operator \( \lambda_j(\mathcal{K}_{\partial \Omega}) \) satisfy

\[
|\lambda_j(\mathcal{K}_{\partial \Omega})| \sim \left\{ \frac{3W(\partial \Omega) - 2\pi \chi(\partial \Omega)}{128\pi} \right\}^{1/2} j^{-1/2} \quad \text{as } j \to \infty.
\]

Here \( W(\partial \Omega) \) and \( \chi(\partial \Omega) \) denote, respectively, the Willmore energy and the Euler characteristic of the boundary surface \( \partial \Omega \). This formula is the so-called Weyl’s law for eigenvalue problems of Neumann–Poincaré operators.

1 Introduction and Results

The Neumann–Poincaré (abbreviated by NP) operator is a boundary integral operator which appears naturally when solving classical boundary value problems using layer potentials. Its study (for the Laplace operator) goes back to C. Neumann [41] and H. Poincaré [45] as the name of the operator suggests. If the boundary of the domain, on which the NP operator is defined, is \( C^{1,\alpha} \) smooth, then the NP operator is compact. Thus the Fredholm integral equation, which appears when solving Dirichlet or Neumann problems, can be solved using the Fredholm index theory [22]. If the domain has corners, the NP operator is not any more a compact operator, but a singular integral operator. The solvability of the corresponding integral equation was established in [58].

Regarding spectral properties of the NP operator, it is proved in [42] that the NP operator can be realized as a self-adjoint operator by introducing a new inner product on the \( H^{-1/2} \)-space (see also [29]), and so the NP spectrum consists of continuous spectrum and discrete spectrum (and possibly the limit points of discrete spectrum). If the domain has corners, the corresponding NP operator may exhibit a continuous spectrum (as well as eigenvalues). For recent development in this direction we refer to [28, 30, 43, 44]. If the domain has the smooth boundary, then the spectrum consists of eigenvalues converging to 0. We refer to [9, 40] for progress on the convergence rate of NP eigenvalues in two dimensions. However the satisfactory answers of decay rates in three dimensions were less-known even for smooth cases since it uses the smoothness of the kernel of

*Center for Mathematical Modeling and Data Science, Osaka University, Osaka 560-8531, Japan. Email: miyanishi@sigmath.es.osaka-u.ac.jp.
NP operators in two dimensions \[20\]. With this in mind, the purpose of this paper is to prove the so-called “Weyl law” which is the asymptotic behavior of NP eigenvalues in three dimensions.

To state the result in a precise manner, let \(\Omega\) be a \(C^{1,\alpha}\) bounded region in \(\mathbb{R}^3\). The NP operator \(K_{\partial\Omega} : L^2(\partial\Omega) \to L^2(\partial\Omega)\) is defined by

\[
K_{\partial\Omega}[\psi](x) := \frac{1}{4\pi} \int_{\partial\Omega} \frac{(y - x, n(y))}{|x - y|^3} \psi(y) \, dS_y
\]  

(1.1)

where \(dS_y\) is the surface element and \(n(y)\) is the outer normal vector on \(\partial\Omega\). As described above, we know that \(K_{\partial\Omega}\) is a compact operator on \(L^2(\partial\Omega)\) and its eigenvalues consist of at most countable numbers, with 0 the only possible limit point. It is also known that the eigenvalues of the NP operator lie in the interval \((-1/2, 1/2]\) and the eigenvalue 1/2 corresponds to constant eigenfunctions. We denote the set of NP eigenvalues counting multiplicities by

\[
\sigma(K_{\partial\Omega}) = \{ \lambda_j(K_{\partial\Omega}) \mid \frac{1}{2} = |\lambda_0(K_{\partial\Omega})| > |\lambda_1(K_{\partial\Omega})| \geq |\lambda_2(K_{\partial\Omega})| \geq \cdots \geq 0 \}. \tag{1.2}
\]

Here our main purpose is to deduce the asymptotic behavior of NP eigenvalues by using the basic ingredients of surface geometry. To do this, we also define the Willmore energy \(W(\partial\Omega)\) by

\[
W(\partial\Omega) := \int_{\partial\Omega} H^2(x) \, dS_x
\]  

(1.3)

where \(H(x)\) is the mean curvature of the surface. Then we have:

**Theorem 1.1.** Let \(\Omega\) be a \(C^{2,\alpha}\) bounded region with \(\alpha > 0\). Then

\[
|\lambda_j(K_{\partial\Omega})| \sim \left\{ \frac{3W(\partial\Omega) - 2\pi \chi(\partial\Omega)}{128\pi} \right\}^{1/2} j^{-1/2} \quad \text{as } j \to \infty. \tag{1.4}
\]

Here \(W(\partial\Omega)\) and \(\chi(\partial\Omega)\) denote, respectively, the Willmore energy and the Euler characteristic of the surface \(\partial\Omega\).

Thus the NP operator has infinite rank \[52\] and the decay rate of NP eigenvalues is \(j^{-1/2}\) for \(C^{2,\alpha}\) regions. Furthermore, the integral (1.3) is specially interesting because it has the remarkable property of being invariant under Möbius transformations of \(\mathbb{R}^3\) \[14, 59\]. Thus we find that the asymptotic behavior of NP eigenvalues is also Möbius invariant since the Euler characteristic is topologically invariant. We will present some further facts and applications later (See section 5).

To clarify the meaning of Theorem 1.1 let us consider the case \(\partial\Omega = S^2\). It is proved by Poincaré \[45\] that the NP eigenvalues on a two-dimensional sphere are \(\frac{1}{2(2k+1)}\) for \(k = 0, 1, 2 \ldots\) and their multiplicities are \(2k + 1\) (see also \[10\]). So we may enumerate them as

\[
\begin{align*}
\lambda_1 & = \frac{1}{6}, \quad \lambda_2 = \frac{1}{10}, \quad \lambda_3 = \frac{1}{10}, \\
\lambda_4 & = \frac{1}{5}, \quad \lambda_5 = \frac{1}{5}, \quad \lambda_6 = \frac{1}{10}, \quad \lambda_7 = \frac{1}{10}, \\
\lambda_8 & = \frac{1}{2(2k+1)}, \quad \lambda_9 = \frac{1}{2(2k+1)}, \quad \lambda_{10} = \frac{1}{2(2k+1)}, \\
& \quad \vdots
\end{align*}
\]

It easily follows that the \(j = k^2\)th eigenvalue satisfies

\[
|\lambda_j(K_{S^2})| = \frac{1}{2(2k + 1)} \sim \frac{1}{4} j^{-1/2}.
\]

In contrast, one can verify these asymptotics from Theorem 1.1

\[
|\lambda_j(K_{S^2})| \sim \left\{ \frac{3W(S^2) - 2\pi \chi(S^2)}{128\pi} \right\}^{1/2} j^{-1/2} = \frac{1}{4} j^{-1/2}
\]

since \(W(S^2) = 4\pi\) and \(\chi(S^2) = 2\). This calculation, of course, is consistent with the asymptotic of the explicit eigenvalues.

It is worth comparing the decay rates for three dimensional NP eigenvalues obtained here with those for two dimensional NP eigenvalues. For the two dimensional cases, it is well known that
the eigenvalues of the integral operator $K_{\partial \Omega}$ are symmetric with respect to the origin \[15, 51]. The only exception is the eigenvalue $1/2$ corresponding to constant eigenfunctions. NP eigenvalues are invariant under Möbius transformations \[50]. One of the main distinguished features is that the decay rates deeply depend on the smoothness of the boundary. Indeed, we \[9, 40\] proved the decay rate which depends on the smoothness of the $C^k$ smooth boundary $\partial \Omega$, that is, for any $\tau > -k + 3/2$,

$$|\lambda_j^\pm (K_{\partial \Omega})| = o(j^\tau) \quad \text{as} \quad j \to \infty,$$

where $o$ means the small order. Moreover for the analytic boundary, we have the exponential decay rate:

$$|\lambda_j^\pm (K_{\partial \Omega})| \leq C e^{-j\epsilon} \quad \text{as} \quad j \to \infty,$$

for any $j$. Here $\epsilon$ is the modified Grauert radius of $\partial \Omega$ (See \[9\] for the precise statement).

We also remark that $K_{\partial \Omega}$ is not self-adjoint on $L^2(\partial \Omega)$. This difficulty can be circumvented by restating from singular values into NP eigenvalues. To do this, this paper is organized as follows: In the next section we introduce the notations of surface geometry and state the relationships among singular- and eigenvalues using Ky-Fan theorem and the triangular representation of NP operators. In section 3 we provide the approximate pseudo-differential operators for NP operators. Then the relationships given in section 2 yield the Weyl law of NP eigenvalues in section 4. The applications are provided in section 5. This paper ends with some discussions and a brief conclusion.

**Acknowledgement:** I am grateful to Prof. H. Kang and Prof. K. Ando for useful discussions on the early stages of this work. I would also like to thank the members of Inha university for their hospitality during my visit, when the main results of this paper were obtained.

## 2 Preliminaries and Notations

As preliminaries, we shall mention the notations of surface geometry and some results of Schatten class used in this paper.

### 2.1 Surface geometry

Let $M$ be a two-dimensional surface without boundary and $\mathbf{r} = \mathbf{r}(s, t)$ be a regular parametrization of a surface in $\mathbb{R}^3$, where $\mathbf{r}$ is a smooth (at least $C^2$) vector-valued function of two variables. It is common to denote the partial derivatives of $\mathbf{r}$ with respect to $s$ and $t$ by $\mathbf{r}_s$ and $\mathbf{r}_t$. The first fundamental form is the inner product on the tangent space of a surface in three-dimensional Euclidean space which is induced canonically from the dot product of $\mathbb{R}^3$. We denote the first fundamental form the Roman numeral $I$:

$$I = Eds^2 + 2Fdsdt + Gdt^2.$$  \hspace{1cm} (2.1)

Let $\mathbf{r}(s, t)$ be a parametric surface. Then the inner product of two tangent vectors is

$$I(a\mathbf{r}_s + b\mathbf{r}_t, c\mathbf{r}_s + d\mathbf{r}_t)$$

$$= ac(\mathbf{r}_s \cdot \mathbf{r}_s) + (ad + bc)(\mathbf{r}_s \cdot \mathbf{r}_t) + bd(\mathbf{r}_t \cdot \mathbf{r}_t)$$

$$= Eac + F(ad + bc) + Gbd,$$

Thus

$$E = \mathbf{r}_s \cdot \mathbf{r}_s, \quad F = \mathbf{r}_s \cdot \mathbf{r}_t, \quad G = \mathbf{r}_t \cdot \mathbf{r}_t$$  \hspace{1cm} (2.2)

and hereafter we often write $E = g_{11}, \quad F = g_{12} = g_{21}, \quad G = g_{22}$.

The second fundamental form of a general parametric surface is defined as follows: Regularity of the parametrization means that $\mathbf{r}_s$ and $\mathbf{r}_t$ are linearly independent for any $(s, t)$ in the domain of $\mathbf{r}$, and hence span the tangent plane to $M$ at each point. Equivalently, the cross product $\mathbf{r}_s \times \mathbf{r}_t$
is a nonzero vector normal to the surface. The parametrization thus defines a field of unit normal vectors $\mathbf{n}$:

$$\mathbf{n} = \frac{\mathbf{r}_s \times \mathbf{r}_t}{|\mathbf{r}_s \times \mathbf{r}_t|}. $$

Then the second fundamental form is usually written as

$$\mathbf{II} = L ds^2 + 2M ds dt + N dt^2. \quad (2.3)$$

Here the coefficients $L, M, N$ at a given point in the parametric $st$-plane are given by the projections of the second partial derivatives of $\mathbf{r}$ at that point onto the normal line to $M$ and can be computed with the aid of the dot product as follows:

$$L = \mathbf{r}_{ss} \cdot \mathbf{n}, \quad M = \mathbf{r}_{st} \cdot \mathbf{n}, \quad N = \mathbf{r}_{tt} \cdot \mathbf{n}.$$ 

Under these notations, we can denote the Gaussian curvature $K$ and the mean curvature $H$ as

$$K := \det A = \frac{LN - M^2}{EG - F^2}, \quad H := \frac{1}{2} \text{tr} A$$

where $A = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} L & M \\ M & N \end{pmatrix}$ is the Weingarten matrix.

We also recall the conformal parameters used in this paper: The parameter $(s, t)$ is said to be isothermal or conformal if the first fundamental form is written as

$$I = e^{2\sigma} (ds^2 + dt^2) \quad (i.e. \ E = G = e^{2\sigma}, \ F = 0) \quad (2.4)$$

where $\sigma := \sigma(s, t)$ is a $C^2$ function in $(s, t)$. It is emphasized that one can always take such coordinates without loss of regularity \cite{19}. For the isothermal parameters, we find

$$K = \frac{LN - M^2}{E^2}, \quad H = \frac{L + N}{2E}$$

and the Gauss-Bonnet formula

$$\int_M K \, dS = 2\pi \chi(M)$$

holds true even for $C^2$ oriented compact surfaces \cite{47}.

### 2.2 Schatten class

Let $K$ be a compact operator in separable Hilbert space $H$. We denote the singular values $\{s_j(K)\}$ as the family of eigenvalues of $(K^*K)^{1/2}$. Since the singular values are non-negative, we always assume the singular values are non-increasing:

$$\sigma_{\text{sing}}(K) = \{ s_j(K) \mid s_1(K) \geq s_2(K) \geq s_3(K) \geq \cdots \}. \quad (2.5)$$

Then the Schatten $p$-norm of $K$ is defined by

$$\|K\|_{S^p} := \text{tr}(K^*K)^{p/2} = \sum_{j=1}^{\infty} |s_j(K)|^p$$

and the $p$-th Schatten-class operator is a bounded linear operator on a Hilbert space with finite Schatten $p$-norm. Especially for $p = 2$, 2nd Schatten-class operator is so-called Hilbert-Schmidt class operator. Here we can show the convergence rate of singular values of the $p$-th Schatten class operators:

**Lemma 2.1.** If $K$ is in $p$-th Schatten class, then ordered singular values satisfy

$$s_j(K) = O(j^{-1/p}).$$
Proof.
\[ \|K\|_{L_p}^p = \sum_{j=1}^{\infty} |s_j(K)|^p < \infty. \]

Thus, for all \( \epsilon > 0 \), there exists \( N \in \mathbb{N} \) such that
\[ (n - N)|s_n(K)|^p \leq \sum_{N+1}^{n} |s_j(K)|^p < \epsilon, \]
and hence,
\[ n|s_n(K)|^p < 2\epsilon \quad \text{for all } n > 2N. \]

Accordingly, \( s_j(K) = o(j^{-1/p}) \) as desired.

The class of operators \( K \), which satisfy \( s_j(K) = o(j^{-1/p}) \), is called weakly \( p \)-Schauder class [54]. Thus if we know the class of operators, then the upper bounds of decay rates are obtained. We also have the precise relation between singular- and eigenvalues, which is convenient to derive NP eigenvalues from singular values:

**Proposition 2.2.** Let \( K \) be a compact operator. Assume the following (1)-(3):

1. \( K - K^* \) is in Hilbert-Schmidt class.
2. Eigenvalues of \( K \) consist of real values.
3. \( s_j(K) \sim Cj^{-1/2} \) as \( j \to \infty \).

Then \( |\lambda_j(K)| \sim s_j(K) \sim Cj^{-1/2} \) as \( j \to \infty \).

We call the operator \( K \), satisfying (1)-(3), "almost self-adjoint operator".

**Proof.** As it is well known [25], for any compact operator \( K \), there are a compact normal operator \( D \) and a compact quasinilpotent operator \( V \), such that
\[ K = D + V \quad \text{and} \quad \sigma(D) = \sigma(K). \]

Since the spectrum \( \sigma(D) = \sigma(K) \) is real, \( D \) is a compact self-adjoint operator and
\[ K^* = D + V^*. \]

Thus \( K - K^* = V - V^* \) is in Hilbert-Schmidt class. The compact quasinilpotent operator \( V \) with Hilbert-Schmidt imaginary part \( \Im(V) = \frac{V - V^*}{2i} \) is also in Hilbert-Schmidt class (See e.g. [23] Lemma 6.5.1 and [24]).

From Ky-Fan theorem (see e.g. [20] and references therein) and Lemma 2.1, the Hilbert-Schmidt operator \( V \) is considered as a small perturbation of \( K \). Thus
\[ |\lambda_j(K)| = |\lambda_j(D)| = s_j(D) \sim s_j(K) \sim Cj^{-1/2} \quad \text{as } j \to \infty. \]

3 NP operators as pseudo-differential operators

We study the asymptotics of singular numbers of \( \Psi \)DO’s which correspond to compact operators of the form \( (1.1) \). To consider an integral operator as a \( \Psi \)DO is technically convenient also because the asymptotics can be expressed directly in terms of symbols. The starting point for us is to construct the approximate \( \Psi \)DO of the NP operator \( K_{\partial \Omega} \) modulo Hilbert-Schmidt operators. This is done by using local coordinates.
3.1 NP operators and approximations

Let \( x_0 \in \partial \Omega \) and choose open neighborhoods \( U_j \) \((j = 1, 2, 3)\) of \( x_0 \) in \( \partial \Omega \) so that \( U_1 \cup U_2 \subset U_3 \) and \( U_3 \) has a local parametrization as in section 2.1. Let \( \varphi_j \) \((j = 1, 2)\) be a smooth functions such that \( \text{supp}\ \varphi_1 \subset U_1 \) and \( \text{supp}\ \varphi_2 \subset U_2 \). Such a situation allows us to taking the local coordinates \((x(s, t)), y(s, t), z(s, t)\) and the surface element is given by \( d\sigma(s, t) = |(x_s, y_s, z_s) \times (x_t, y_t, z_t)| ds \wedge dt \). Thus we obtain

\[
\varphi_2 K_{\partial \Omega}[\varphi_1 f](x) := \varphi_2(x) \int_{\mathbb{R}^2} \left[ \frac{(x(s_1, t_1) - x(s_2, t_2))(y_s z_t - z_s y_t)(s_1, t_1)}{4\pi|x - y|^3} \right. \\
+ \left. \frac{(y(s_1, t_1) - y(s_2, t_2))(z_s x_t - x_s z_t)(s_1, t_1)}{4\pi|x - y|^3} \right. \\
+ \left. \frac{(z(s_1, t_1) - z(s_2, t_2))(x_s y_t - y_s x_t)(s_1, t_1)}{4\pi|x - y|^3} \right] \varphi_1(y) f(y) d s_1 \wedge d t_1.
\]

Remark that \( dx = x_s ds + x_t dt \), \( dy = y_s ds + y_t dt \) and \( dz = z_s ds + z_t dt \) on local charts, the metric of the surface is denoted as

\[
dx^2 + dy^2 + dz^2 = (x_s^2 + y_s^2 + z_s^2) ds^2 + 2(x_s x_t + y_s y_t + z_s z_t) ds dt + (x_t^2 + y_t^2 + z_t^2) dt^2 = Eds^2 + 2F ds dt + G dt^2.
\]

Letting \( y' = (s_1, t_1) \) and \( x' = (s_2, t_2) \), we also find that

\[
|\mathbf{x} - \mathbf{y}'|^2 \\
= (x(s_1, t_1) - x(s_2, t_2))^2 + (y(s_1, t_1) - y(s_2, t_2))^2 + (z(s_1, t_1) - z(s_2, t_2))^2 \\
= (E(x'))(s_1 - s_2)^2 + 2F(x')(s_1 - s_2)(t_1 - t_2) + G(x')(t_1 - t_2)^2 + O(|x' - y'|),
\]

and so

\[
\frac{1}{|\mathbf{x} - \mathbf{y}'|^3} = \frac{1}{(E(x')(s_1 - s_2)^2 + 2F(x')(s_1 - s_2)(t_1 - t_2) + G(x')(t_1 - t_2)^2)^{3/2}}(1 + O(|x' - y'|)) \\
= K(x', x' - y') + E_1(x', y').
\]

Here

\[
K(x', x' - y') = \frac{1}{(E(x')(s_1 - s_2)^2 + 2F(x')(s_1 - s_2)(t_1 - t_2) + G(x')(t_1 - t_2)^2)^{3/2}}
\]

and

\[
|E_1(x', y')| \lesssim |(s_1, t_1) - (s_2, t_2)|^{-2}.
\]
Similarly the numerator of the integral kernel $\varphi_2 K_{\partial\Omega}[\varphi_1 f](\mathbf{x})$ can be denoted as

\[
\begin{aligned}
&(x(s_1, t_1) - x(s_2, t_2))(y_s z_t - z_s y_t)(s_1, t_1) + (y(s_1, t_1) - y(s_2, t_2))(z_s x_t - x_s z_t)(s_1, t_1) \\
&+ (z(s_1, t_1) - z(s_2, t_2))(x_s y_t - y_s x_t)(s_1, t_1) \\
= &\left(x_t(t_1 - t_2) + \frac{1}{2} (s_1 - s_2)^2 x_{ss} + x_{st}(s_1 - s_2)(t_1 - t_2) + \frac{1}{2} x_{tt}(t_1 - t_2)^2\right)(y_s z_t - z_s y_t)(s_1, t_1) \\
&+ (y_s(s_1 - s_2) + y_t(t_1 - t_2) + \frac{1}{2} (s_1 - s_2)^2 y_{ss} + y_{st}(s_1 - s_2)(t_1 - t_2) + \frac{1}{2} y_{tt}(t_1 - t_2)^2)(z_s x_t - x_s z_t)(s_1, t_1) \\
&+ (z_s(s_1 - s_2) + z_t(t_1 - t_2) + \frac{1}{2} (s_1 - s_2)^2 z_{ss} + z_{st}(s_1 - s_2)(t_1 - t_2) + \frac{1}{2} z_{tt}(t_1 - t_2)^2)(x_s y_t - y_s x_t)(s_1, t_1) \\
&+ O(|x' - y'|^{2+\alpha}) \\
= &\frac{1}{2} (s_1 - s_2)^2 \{x_{ss}(y_s z_t - z_s y_t) + y_{ss}(z_s x_t - x_s z_t) + z_{ss}(x_s y_t - y_s x_t)\}(s_1, t_1) \\
&+ (s_1 - s_2)(t_1 - t_2) \{x_{st}(y_s z_t - z_s y_t) + y_{st}(z_s x_t - x_s z_t) + z_{st}(x_s y_t - y_s x_t)\}(s_1, t_1) \\
&+ \frac{1}{2} (t_1 - t_2)^2 \{x_{tt}(y_s z_t - z_s y_t) + y_{tt}(z_s x_t - x_s z_t) + z_{tt}(x_s y_t - y_s x_t)\}(s_1, t_1) \\
&+ O(|x' - y'|^{2+\alpha}) \\
= &\frac{1}{2} \left(L'(y')(s_1 - s_2)^2 + 2M'(y')(s_1 - s_2)(t_1 - t_2) + N'(y')(t_1 - t_2)^2\right)\{(x_s y_s, z_s) \times (x_t y_t, z_t)\}(s_1, t_1) \\
&+ O(|x' - y'|^{2+\alpha}) \\
= &\frac{1}{2} \left(L'(x')(s_1 - s_2)^2 + 2M'(x')(s_1 - s_2)(t_1 - t_2) + N'(x')(t_1 - t_2)^2\right)\{(x_s y_s, z_s) \times (x_t y_t, z_t)\}(s_1, t_1) \\
&+ O(|x' - y'|^{2+\alpha}).
\end{aligned}
\]

Here we used the second fundamental form:

\[
\mathcal{I} = L ds^2 + 2M ds dt + N dt^2
\]

and

\[
L(x') - L(y') = O(|x' - y'|^\alpha), \quad M(x') - M(y') = O(|x' - y'|^\alpha), \quad N(x') - N(y') = O(|x' - y'|^\alpha).
\]

Summarizing the dominator and the numerator of the kernel,

\[
\begin{align*}
\varphi_2 K_{\partial\Omega}[\varphi_1 f](\mathbf{x}) &= \frac{1}{8\pi} \varphi_2(\mathbf{x}) \int_{\mathbb{R}^2} \left\{ (L(s_1 - s_2)^2 + 2M(s_1 - s_2)(t_1 - t_2) + N(t_1 - t_2)^2) \\
&\quad \times (x_s y_s, z_s) \times (x_t y_t, z_t) \right\} \varphi_1(y)f(y)d\sigma(s, t) + \mathbf{H}[f](\mathbf{x}) \\
&= \frac{1}{8\pi} \varphi_2(\mathbf{x}) \int_{\mathbb{R}^2} \left\{ (L(s_1 - s_2)^2 + 2M(s_1 - s_2)(t_1 - t_2) + N(t_1 - t_2)^2) \\
&\quad \times K(x', x' - y')(x_s y_s, z_s) \times (x_t y_t, z_t) + E_2(x', y') \right\} \varphi_1(y)f(y)d\sigma(s, t) + \mathbf{H}[f](\mathbf{x}) \\
&= \frac{1}{8\pi} \varphi_2(\mathbf{x}) \int_{\mathbb{R}^2} \left\{ (L(s_1 - s_2)^2 + 2M(s_1 - s_2)(t_1 - t_2) + N(t_1 - t_2)^2) \\
&\quad \times K(x', x' - y') \right\} \varphi_1(y)f(y)d\sigma(s, t) + \mathbf{H}[f](\mathbf{x})
\end{align*}
\]

where $\mathbf{H}[f](\mathbf{x}) = \frac{1}{8\pi} \varphi_2(\mathbf{x}) \int_{\mathbb{R}^2} E_2(x', y') \varphi_1(y)f(y)d\sigma(s, t) + \mathbf{H}[f](\mathbf{x})$ and $E_2(x', y') = O(|x' - y'|^{1-\alpha})$.

It follows that $E_2(x', y')$ is in $L^2_{\text{loc}}(\mathbb{R}^4)$ and $\mathbf{H}$ is in Hilbert-Schmidt class by Mercer's theorem (See e.g. [17, 54]).
3.2 Symbols of the approximate NP operators

From the calculations in the previous subsection, the NP operator on local charts is denoted as
\[
\varphi_2 K_{\partial} [\varphi_1 f] (x) \equiv \varphi_2 P [\varphi_1 f] (x) \\
:= \frac{1}{8\pi} \varphi_2 (x) \int_{\mathbb{R}^2} \left[ (L(s_1 - s_2)^2 + 2M(s_1 - s_2)(t_1 - t_2) + N(t_1 - t_2)^2 \right] K(x',x' - y') \varphi_1 (y) f (y) d\sigma (s,t)
\]
where the notation \( \equiv \) stands for modulo Hilbert-Schmidt operators. Let us denote \( P [f] (x) \) as the pseudo-differential operator (See e.g. [53, 56] for the details). The classical \( \Psi DO \) is defined as
\[
\text{Op}(\sigma) [f] (x) = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \sigma(x,\xi) e^{i(x-y) \cdot \xi} f(y) dx dy
\]
where \( \sigma(x,\xi) \) is called the symbol of \( \text{Op}(\sigma) \). Write \((x,y)\) instead of \((s,t)\) as usual. Our purpose is to get the \( \Psi DO \) representation of the operator \( P_{jk}^\beta \):
\[
P_{jk}^\beta [f] (x') := \frac{1}{8\pi} \int_{\mathbb{R}^2} (x_j - y_j)(x_k - y_k) K(x',x' - y') f(y) d\sigma
\]
where
\[
K(x',x' - y') = \frac{1}{[g_{11}(x')(x_1 - y_1)^2 + 2g_{12}(x')(x_1 - y_1)(x_2 - y_2) + g_{22}(x')(x_2 - y_2)^2]^{3/2}}.
\]
We can calculate the principal symbol of \( P_{jk}^\beta \) with the aid of the surface Riesz transforms \( R_k^\beta \):
\[
R_k^\beta [f] (x') = \frac{1}{2\pi} \int_{\mathbb{R}^2} (x_k - y_k) K(x',x' - y') f(y) dy'.
\]
Here \( R_k^\beta \) is, in fact, a homogeneous pseudo-differential operator [8]. Let us recall the symbol of \( R_k^\beta \) for the reader’s convenience:

**Lemma 3.1.** For \( f \in C_0^\infty (\mathbb{R}^2) \)
\[
R_k^\beta [f] (x') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{-i}{\sqrt{\det (g_{kl}(x))}} \sum_l g^{kl}(x') \xi_l \frac{\sum_k g^{kl}(x') \xi_k}{\sum_k g^{kl}(x') \xi_k} e^{i(x'-y) \cdot \xi} f(y) dy' d\xi.
\]

**Proof.** The matrix (tensor) \( G(x) = (g_{ij}(x)) \) is symmetric and one can diagonalize via orthogonal matrices:
\[
P^{-1}(x) G(x) P(x) = \begin{pmatrix} \alpha^2(x) & 0 \\ 0 & \beta^2(x) \end{pmatrix}
\]
and
\[
\begin{pmatrix} \alpha^{-1}(x) & 0 \\ 0 & \beta^{-1}(x) \end{pmatrix} P^{-1}(x) G(x) P(x) \begin{pmatrix} \alpha^{-1}(x) & 0 \\ 0 & \beta^{-1}(x) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\]
Since \( \alpha(x) \beta(x) = \sqrt{\det (g_{kl}(x))} \), putting \( z_k = x_k - y_k \) \((k = 1,2)\) and
\[
z = \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = P(x) \begin{pmatrix} \alpha^{-1}(x) & 0 \\ 0 & \beta^{-1}(x) \end{pmatrix} \begin{pmatrix} \bar{z}_1 \\ \bar{z}_2 \end{pmatrix} = \tilde{P}(x) \tilde{z},
\]
(3.2) becomes
\[
R_k^\beta [f] (x') = \frac{1}{2\pi} \int_{\mathbb{R}^2} \frac{[\tilde{P}(x') \tilde{z}]_k}{|\tilde{z}|^3} f(y) \frac{d\tilde{z}}{\alpha(x') \beta(x')} = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \frac{-i}{\sqrt{\det (g_{kl}(x))}} \sum_k g^{kl}(x') \xi_l \frac{\sum_k g^{kl}(x') \xi_k}{\sum_k g^{kl}(x') \xi_k} e^{i\xi \cdot (x'-y)} f(y) dy' d\xi.
\]
Lemma 3.2. For $f \in C_0^\infty(\mathbb{R}^2)$, we get the symbol of $P^g_{jk}$ as homogeneous pseudo-differential operators:

$$P^g_{jk}[f](x') = \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \left[ \frac{(-1)^{i-j} \hat{\xi}_j \hat{\xi}_k}{4 \det(g_{ij}(x')) \left\{ \sum_{j,k} g^{jk}(x') \xi_j \xi_k \right\}^3} \right] e^{i(x'-y') \cdot \xi} f(y) dy' d\xi \quad (3.6)$$

Here $\hat{\xi}_1 = \xi_2$ and $\hat{\xi}_2 = \xi_1$.

Proof. From Lemma 3.1 and using the integration by parts as oscillatory integrals [53], we have for all $f \in C_0^\infty(\mathbb{R}^2)$,

$$P^g_{jk}[f](x') = \frac{1}{4} \mathcal{P}^g_{jk} \left[ \sqrt{\det(g(x'))} (x_j - y_j) f(x') \right]$$

$$= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \left[ \frac{1}{4 \sqrt{\sum_{k,l} g^{kl}(x') \xi_k \xi_l}} \frac{1}{i} \frac{\partial}{\partial \xi_j} e^{i(x'-y') \cdot \xi} f(y) dy' d\xi \right]$$

$$= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \left[ \frac{1}{4 \sqrt{\sum_{k,l} g^{kl}(x') \xi_k \xi_l}} \frac{\partial}{\partial \xi_j} \left[ \frac{\sum_{k,l} g^{kl}(x') \xi_k}{\sum_{j,k} g^{jk}(x') \xi_j} \right] e^{i(x'-y') \cdot \xi} f(y) dy' d\xi \right]$$

$$= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \left[ \frac{1}{4 \sqrt{\sum_{k,l} g^{kl}(x') \xi_k \xi_l}} \left( \sum_{k,l} g^{kl}(x') \xi_k \xi_l \right) e^{i(x'-y') \cdot \xi} f(y) dy' d\xi \right]$$

$$= \frac{1}{(2\pi)^2} \int_{\mathbb{R}^4} \left[ \frac{1}{4 \sqrt{\sum_{k,l} g^{kl}(x') \xi_k \xi_l}} \left( \sum_{j,k} g^{jk}(x') \xi_j \xi_k \right) \right] e^{i(x'-y') \cdot \xi} f(y) dy' d\xi$$

as desired. \(\square\)

Hence the principal symbol of $P^g_{jk}$ is a (strictly) homogeneous symbol of order $-1$ and the summation immediately yields the principal symbol of $P$:

Lemma 3.3. Let $\partial \Omega$ be a bounded $C^2, \alpha$ surface. Then

$$P \equiv Op \left( \frac{L(x') \xi_2^2 - 2M(x') \xi_1 \xi_2 + N(x') \xi_1^2}{4 \det(g_{ij}) \left\{ \sum_{j,k} g^{jk}(x') \xi_j \xi_k \right\}^3} \right) \text{ modulo Hilbert-Schmidt operators.}$$

We remark that the above $\Psi DO$ is defined even for $f \in L^2(\partial \Omega)$ since the localizations of $\Psi DO$ on local coordinates coincide with the sum of (3.1).

9
4 Weyl’s law of compact pseudo-differential operators and NP operators

Let us introduce Weyl’s law of singular values of the pseudo-differential operators with $C^\alpha$ smooth in x-variable. M. S. Birman and M. Z. Solomyak [13] showed the asymptotics under weak smoothness hypothesis both in the x- and $\xi$-variable (See [20] [27] [39] for recent progress). In our situation, we employ the results as the asymptotics of singular values of $-1$ homogeneous $\Psi$DO in two dimensions.

**Theorem 4.1** (27 Theorem 2.1 and Theorem 2.5). On a closed manifold $M$ of dimension 2, let $P$ be defined in local coordinates from symbols $p(x,\xi)$ that are homogeneous in $\xi$ of degree $-1$. Assume that the symbols restricted to $\xi \in S^{m-1} = \{|\xi| = 1\}$ are in $C(S^{m-1}, C^\epsilon)$ for some $\epsilon$. Then

$$s_j(P) \sim C(\partial \Omega)^{1/2} j^{-1/2} \quad j \to \infty.$$ 

Here

$$C(\partial \Omega) = \frac{1}{8\pi^2} \int_{\partial \Omega} |\sigma_0(x,\xi)|^2 dx d\xi,$$

(4.1)

$S^* M$ denotes the cosphere bundle and $\sigma_0$ is the principal symbol of $Op(\sigma(x,\xi))$.

From Theorem 4.1 immediately we have the Weyl’s law of singular values:

**Lemma 4.2.** Let $\Omega$ be a $C^{2,\alpha}$ bounded region. Then

$$s_j(\mathcal{K}_{\partial \Omega}) \sim \left( \frac{3W(\partial \Omega) - 2\pi \chi(\partial \Omega)}{128\pi} \right) \frac{1}{j} \quad j \to \infty,$$

(4.2)

where $W(\partial \Omega) = \int_{\partial \Omega} H^2 dS$ and $\chi(\partial \Omega)$ denotes, respectively, the Willmore energy and the Euler characteristic of the surface $\partial \Omega$.

**Proof.** In the preceding section, we proved that $\mathcal{K}_{\partial \Omega}$ is a $\Psi$DO modulo Hilbert-Schmidt class:

$$\mathcal{K}_{\partial \Omega} = Op \left( \left( \frac{L(x)\xi_2^2 - 2M(x)\xi_1\xi_2 + N(x)\xi_1^2}{4\det(g_{ij}(x)} \right) \frac{1}{\sqrt{\sum_j g_{jk}(x)\xi_j\xi_k}} \right) \quad \text{modulo Hilbert-Schmidt operator } H.$$ 

(4.3)

From Ky-Fan theorem [20], $H$ is considered as the small perturbation of the $\Psi$DO. Thus $s_j(\mathcal{K}_{\partial \Omega})$ also satisfies

$$s_j(\mathcal{K}_{\partial \Omega}) \sim C(\partial \Omega) j^{-1/2} \quad j \to \infty.$$ 

To calculate the positive constant $C(\partial \Omega)$ in Theorem 4.1, we take the isothermal charts introduced in section 2.1. The surface element is given by $dS_x = E(x) dx$ and

$$C(\partial \Omega)^2 = \frac{1}{8\pi^2} \int_{\partial \Omega} \int_{S^1} \left[ \frac{L(x)\cos^2 \theta - 2M(x)\cos \theta \sin \theta + N(x)\sin^2 \theta}{4E^2(x)E^{-3/2}(x)} \right]^2 d\xi dx$$

$$= \frac{1}{128\pi^2} \int_{\partial \Omega} \int_{S^1} \left[ \frac{L(x)\cos^2 \theta - 2M(x)\cos \theta \sin \theta + N(x)\sin^2 \theta}{E(x)E^{-3/2}(x)} \right] d\xi dx$$

$$= \frac{1}{128\pi^2} \int_{\partial \Omega} \int_{S^1} \left[ \frac{\frac{3\pi}{4} L^2(x) + \frac{3\pi}{4} N^2(x) + \pi M^2(x) + \frac{\pi}{2} L(x)N(x)}{E(x)} \right] dx$$

$$= \frac{1}{128\pi^2} \int_{\partial \Omega} \int_{S^1} \left[ \frac{\frac{3\pi}{4} L^2(x) + \frac{3\pi}{4} N^2(x) + \pi (L(x)N(x) - E^2(x)K(x)) + \frac{\pi}{2} L(x)N(x)}{E(x)} \right] dx$$

$$= \frac{3}{512\pi} \int_{\partial \Omega} \left[ \left( \frac{L(x) + N(x)}{E(x)} \right)^2 - \frac{4}{3} K(x) \right] E(x) dx$$

$$= \frac{3}{512\pi} \int_{\partial \Omega} 4H^2(x) dx - \frac{1}{64} \chi(\partial \Omega)$$

$$= \frac{3}{128\pi} [W(\partial \Omega) - 2\pi \chi(\partial \Omega)].$$
Thus we have a Weyl's type formula for singular values of NP operators.

**Proof of Theorem 1.1.** From Proposition 2.2, we need to prove only the almost self-adjointness of \( K_{\partial \Omega} \), that is,

\[
K_{\partial \Omega}^* - K_{\partial \Omega}
\]

is a Hilbert-Schimidt operator. This fact follows from that the approximate \( \Psi DO's \) of \( K_{\partial \Omega}^* \) and \( K_{\partial \Omega} \) have just the same symbol from the similar calculations in section 3.

---

## 5 Applications

Theorem 1.1 states that the NP operators in three dimensions have infinite rank. Regarding the rank of NP operators, Khavinson-Putinar-Shapiro [32] propose a question:

“The disk is the only planar domain for which the NP operator has finite rank. It is not known whether there are such domains in higher dimensions.”

Our answer to this question is summarized as the following:

**Corollary 5.1 (Finite-rank problem).** Let \( \Omega \) be a bounded \( C^{2,\alpha} \) \( (\alpha > 0) \) region in \( \mathbb{R}^n \) \((n = 2, 3)\). If the NP operator has finite rank, then

\[
n = 2 \text{ and } \partial \Omega = S^1.
\]

Thus the finite rank NP operator is rank one.

Some results are also obtained from the known facts of the Willmore energy: The Willmore energy [13] is known as the best thought of as a measure of ‘roundness’, it is not hard to prove

\[
W(\partial \Omega) \geq 4\pi,
\]

with equality if and only if \( \partial \Omega \) is an round sphere. For higher genus cases, F. C. Marques and A. Neves [35] proved the celebrated “Willmore conjecture”

\[
W(\partial \Omega) \geq 2\pi^2.
\]

The equality is achieved by the torus of revolution whose generating circle has radius 1 and center at distance \( \sqrt{2} \) from the axis of revolution:

\[
T^2_{\text{Clifford}} := \{(\sqrt{2} + \cos u) \cos v, (\sqrt{2} + \cos u) \sin v, \sin u) \in \mathbb{R}^3 \mid (u, v) \in \mathbb{R}^2\}
\]

As results, we obtain the spectral geometry nature of NP eigenvalues:

**Corollary 5.2.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded region of class \( C^{2,\alpha} \). Then

\[
|\lambda_j(K_{\partial \Omega})| \gtrsim \frac{1}{4} j^{-1/2}.
\]

The minimum asymptotic is achieved if and only if \( \partial \Omega = S^2 \). Especially if \( \sigma_p(K_{\partial \Omega}) = \sigma_p(K_{S^2}) \) then \( \partial \Omega = S^2 \).

**Corollary 5.3.** Let \( \Omega \subset \mathbb{R}^3 \) be a bounded region of class \( C^{2,\alpha} \) with genus \( g(\partial \Omega) \geq 1 \). Then

\[
|\lambda_j(K_{\partial \Omega})| \gtrsim \frac{\sqrt{3}\pi}{8} j^{-1/2}.
\]

Especially if \( \sigma_p(\partial \Omega) = \sigma_p(T^2_{\text{Clifford}}) \) then \( \partial \Omega \cong T^2_{\text{Clifford}} \). Here \( \cong \) means modulo Möbius transforms.

Any other properties of the Willmore energy can also be interpreted as the asymptotics of NP eigenvalues. For instance, Langevin and Rosenberg [34] showed that any knotted embedding of a torus in \( \mathbb{R}^3 \) was bounded below by \( 8\pi \), namely,

\[
W(\partial \Omega) \geq 8\pi.
\]

In the terminology of NP eigenvalues, we have
Corollary 5.4. Let $\Omega \subset \mathbb{R}^3$ be a bounded region of class $C^{2,\alpha}$ and $\partial \Omega$ be a knot torus. Then

$$|\lambda_j(K_{\partial \Omega})| \gtrsim \frac{\sqrt{3}}{4} j^{-1/2}.$$ 

As the last application, let us consider plasmonic eigenvalues (See e.g. [26] and references therein). A real number $\epsilon$ is called a plasmonic eigenvalue if the following problem admits a solution $u$ in the space $H^1(\mathbb{R}^3)$:

$$
\begin{cases}
\Delta u = 0 & \text{in } \mathbb{R}^3 \setminus \partial \Omega, \\
u|_-= u|_+ & \text{on } \partial \Omega, \\
\epsilon \partial_n u|_- = -\partial_n u|_+ & \text{on } \partial \Omega.
\end{cases}
$$

(5.1)

where the subscript $\pm$ on the left-hand side respectively denotes the limit (to $\partial \Omega$) from the outside and inside of $\Omega$. The well-known relation [10] among the plasmonic eigenvalue $\epsilon$ and the NP eigenvalue $\lambda$ gives

$$|\epsilon_j - 1| = \frac{2\lambda_j}{\lambda_j - 1/2} \sim |4\lambda_j| \sim \left\{ \frac{3W(\partial \Omega) - 2\pi \chi(\partial \Omega)}{8\pi} \right\}^{1/2} j^{-1/2}.$$ 

(5.2)

Hence the plasmonic eigenvalues consist of the sequence with 1 as the limit, and the (R.H.S.) of (5.2) gives its converging rate.

6 Conclusion and discussions

We discussed about the Weyl's law of NP eigenvalues. It depends on the Willmore energy and the Euler characteristics. However Theorem 1.1 holds only for $C^{2,\alpha}$ smooth surfaces, while $K_{\partial \Omega}$ is compact for $C^{1,\alpha}$ surfaces. This fact indicates that the Weyl's law on surfaces having only $C^{1,\alpha}$ smoothness is probably changed to be the interpolation between $C^{0,1}$ and $C^{2,\alpha}$.

Moreover we don’t know the asymptotics of signed NP eigenvalues:

$$\lambda_1^+ > \lambda_2^+ > \cdots > 0 > \cdots > \lambda_2^- > \lambda_1^-.$$ 

(6.1)

When we denote the signed Browder-Gårding density [11] as

$$C_{\pm}(\partial \Omega) = \frac{1}{8\pi^2} \int_{S^* \partial \Omega} \frac{L(x')\xi_2^2 - 2M(x')\xi_1\xi_2 + N(x')\xi_1^2}{4 \det(g_{ij}) \left( \sqrt{\sum_{j,k} g^{jk}(x')\xi_j \xi_k} \right)^3} \pm dx d\xi$$

(6.2)

where the subscript $\pm$ denotes the positive and negative part respectively, we believe that

$$\lambda_j^+ \sim \pm C_{\pm}^{1/2}(\partial \Omega) j^{-1/2} \text{ as } j \to \infty.$$ 

If so, for the case of ellipsoids, $C_{-}(\partial \Omega) = 0$ and negative NP eigenvalues decay faster than $j^{-1/2}$ [3]. For a torus, $C_{-}(\partial \Omega) > 0$ and infinitely many negative NP eigenvalues exist.

We hope that the truth or falsehood of these problems will be established in the recent future.

References

[1] L. V. Ahlfors, Remarks on the Neumann-Poincaré integral equation, Pacific. J. Math., 2 (1952), 271–280.

[2] J. F. Ahner, Some spectral properties of an integral operator in potential theroy, Proc. Edinburgh Math. Soc., 29 (1986), 405–411.

[3] J. F. Ahner, On the eigenvalues of the electrostatic integral operator II, J. Math. Anal. Appl., 181 (1994), 328–334.
[4] J. F. Ahner and R. F. Arenstorf, *On the eigenvalues of the electrostatic integral operator*, J. Math. Anal. Appl., **117** (1986), 187–197.

[5] H. Ammari, G. Ciaolo, H. Kang, H. Lee and G. W. Milton. Spectral theory of a Neumann-Poincaré-type operator and analysis of cloaking due to anomalous localized resonance, *Arch. Ration. Mech. An.* **208** (2013), 667–692.

[6] H. Ammari, H. Kang, and H. Lee, *Layer potential techniques in spectral analysis*, Mathematical Surveys and Monographs, **153** American Math. Soc., Providence RI, (2009).

[7] K. Ando and H. Kang, Analysis of plasmon resonance on smooth domains using spectral properties of the Neumann–Poincaré operator, *J. Math. Anal. Appl.* **435**(1) (2016), 162–178.

[8] K. Ando, H. Kang and Y. Miyaniishi, Elastic Neumann-Poincaré operators on three-dimensional smooth domains: Polynomial compactness and spectral structure, *Int. Math. Res. Notices*, rmx258, (2017).

[9] K. Ando, H. Kang, Y. Miyaniishi, Exponential decay estimates of the eigenvalues for the Neumann–Poincaré operator on analytic boundaries in two dimensions, *To appear in J. Integral Equations*, arXiv:1606.01483.

[10] K. Ando, H. Kang, Y. Miyaniishi and E. Ushikoshi, The first Hadamard variation of Neumann–Poincaré eigenvalues, *To appear in Proc. Amer. Math. Soc.*, arXiv:1805.02414.

[11] A. S. Andreev, Asymptotics of the spectrum of compact pseudodifferential operators in a Euclidean domain, *Mat. Sbornik* **137** (1988), 203–223; Math. USSR Sbornik **65** (1990).

[12] R. Bhatia, Some inequalities for norm ideals, *Comm. Math. Phys.*, **111** (1987), 33–39.

[13] M. S. Birman and M. Z. Solomyak, Asymptotic behavior of the spectrum of pseudodifferential operators with anisotropically homogeneous symbols, *Vestnik Leningrad Univ.* **13** (1977), 13–21; English translation in Vestin. Leningr. Univ. Math. **10**, 237–247.

[14] W. Blaschke, *Vorlesungen Über Differentialgeometrie III*, Berlin: Springer (1929)

[15] J. Blumenfeld and W. Mayer, Über poincaré fundamental funktionen, *Sitz. Wien. Akad. Wiss., Math.-Nat. Klasse* **122**, Abt. Ia (1914), 2011–2047.

[16] E. Bonnetier and F. Triki, On the spectrum of Poincaré variational problem for two close-to-touching inclusions in 2D, *Arch. Ration. Mech. An.* **209** (2013), 541–567.

[17] R. Courant and D. Hilbert, *Methods of Mathematical Physics I*, Wiley-Interscience, (1953).

[18] R. R. Coifman and Y. Meyer, Au dulà des opérateurs pseudodifférentiels, *Asterisque* **57** (1978), 1–185.

[19] D. Deturck and L. Kazdan, Some regularity theorems in Riemannian geometry, *Ann. Sci. Éc. Norm. Supér.* (4), **14** (3) (1981), 249–260.

[20] M. Dostanić, A theorem of Ky-Fan type, *Matematički Vesnik*, **47** (1995), 7–10.

[21] J. Delgado and M. Ruzhansky, Schatten classes on compact manifolds: Kernel conditions, *J. Funct. Anal.*, **267** (2014), 772–798.

[22] E. I. Fredholm, Sur une classe d’équations fonctionnelles, *Acta Mathematica*, **27** (1903), 365–390.

[23] M. I. Gil’, Lower bounds for eigenvalues of Schatten-Von Neumann operators, *J. Inequal. Pure Appl. Math.*, **8**(3) (2007), Art. 66.

[24] M. I. Gil’, *Operator functions and localization of spectra*, Lectures Notes in Mathematics, vol.1830, Springer-Verlag, Berlin, 2003.
[25] I. C. Gohberg and M. G. Krein, Introduction to the Theory of Linear Nonselfadjoint Operators, Trans. Math., Monographs, 18: (1969) Amer. Math. Soc., Providence, R.I.

[26] D. Grieser, The plasmonic eigenvalue problem, Rev. Math. Phys. 26 (2014), 1450005.

[27] G. Grubb, Spectral asymptotics for nonsmooth singular green operators, Comm. P. D. E., 39: (2014), 530–573.

[28] J. Helsing, H. Kang and M. Lim, Classification of spectra of the Neumann–Poincaré operator on planar domains with corners by resonance, Ann. I. H. Poincare, AN, to appear., arXiv:1603.03522, 2016.

[29] H. Kang, K. Kim, H. Lee, J. Shin and S. Yu, Spectral properties of the Neumann–Poincaré operator and uniformity of estimates for the conductivity equation with complex coefficients, J. London Math. Soc. (2) 93 (2016), 519–546.

[30] H. Kang, M. Lim and S. Yu, Spectral resolution of the Neumann-Poincaré operator on intersecting disks and analysis of plasmon resonance, arXiv:1501.02952, 2015.

[31] O. D. Kellogg, Foundations of Potential Theory, Dover, New York, 1953.

[32] D. Khavinson, M. Putinar, and H.S. Shapiro, Poincaré's variational problem in potential theory. Arch. Ration. Mech. Anal. 185(1) (2007), 143–184.

[33] H. KÖNIG, Eigenvalue distribution of compact operators. Operator theory: Advances and Applications, 16. Birkhäuser Verlag, Basel, (1986).

[34] R. Langevin and H. Rosenberg, On curvature integrals and knots. Topology, 15:405416, (1976).

[35] F. C. Marques and A. Neves, Min-Max theory and the Willmore conjecture, Anal. Math. 179 (2014), 683–782.

[36] E. Martensen, A spectral property of the electrostatic integral operator, J. Math. Anal. Appl., 238 (1999), 551–557.

[37] C. A. McCarthy, $C_p$, Israel J. Math., 5 (1967), 249–271.

[38] I. D. Mayergoyz, D. R. Fredkin and Z. Zhang, Electrostatic (plasmon) resonances in nanoparticles, Phys. Rev. B. 72 (2005), 155412.

[39] G.W. Milton and N.-A.P. Nicorovici, On the cloaking effects associated with anomalous localized resonance, Proc. R. Soc. A 462 (2006), 3027–3059.

[40] Y. Miyaniishi and T. Suzuki, Eigenvalues and eigenfunctions of double layer potentials, Trans. Amer. Math. 369 (2017), 8037–8059

[41] C. Neumann, Über die Methode des arithmetischen Mittels, Erste und zweite Abhandlung, Leipzig 1887/88, in Abh. d. Kgl. Sächs Ges. d. Wiss., IX and XIII.

[42] J. Plemeij, Potentialtheoretische Untersuchungen, Preisschriften der Füristlich Jablonowskischen Gesellschaft zu Leipzig, Teubner-Verlag, Leipzig, (1911).

[43] K. Perfekt and M. Putinar, Spectral bounds for the Neumann–Poincaré operator on planar domains with corners, J. Anal. Math. 124 (2014), 39–57.

[44] K. Perfekt and M. Putinar, The essential spectrum of the Neumann–Poincaré operator on a domain with corners, arXiv:1601.03183v2.

[45] H. Poincaré, La méthode de Neumann et le problème de Dirichlet, Acta Math. 20 (1897), 59–152.
[46] M. Reed and B. Simon, Methods of Modern Mathematical Physics, I. Functional Analysis, Academic Press, New York, (1972).

[47] Y. G. Reshetnyak, Two-Dimensional Manifolds of Bounded Curvature, in Geometry IV. Nonregular Riemannian Geometry, English Translation by E. Primrose, Springer Encyclopedia of Mathematical Sciences, 70 (1993), 3–163.

[48] S. Ritter, The spectrum of the electrostatic integral operator for an ellipsoid, in “Inverse Scattering and Potential Problems in Mathematical Physics,” (R.F.Kleinman, R.Kress, and E.Marstensen, Eds.), Lang, Frankfurt/Bern, (1995), 157–167.

[49] G. Rozenblum and G. Tashchiyan, Eigenvalue asymptotics for potential type operators on Lipschitz surfaces, Russ. J. Math. Phys. 13 (3), (2006), 326–339.

[50] M. Schiffer, The Fredholm eigenvalues of plane domains, Pacific J. Math. 7 (1957), 1187–1225.

[51] M. Schiffer, Fredholm eigenvalues and conformal mapping, Autovalori e autosoluzioni, C.I.M.E. Summer Schools 27, Springer (2011), 203–234.

[52] H. S. Shapiro, The Schwarz function and its generalization to higher dimensions, University of Arkansas Lecture Notes in the Mathematical Sciences, 9. A Wiley Interscience Publication. John Wiley and Sons, Inc., New York, (1992).

[53] M. A. Shubin, Pseudodifferential Operators and Spectral Theory (Second Edition), Springer, (2001).

[54] B. Simon, Trace ideals and their applications, 2nd ed., Amer. Math. Soc. (2005).

[55] O. Steinbach and W. L. Wendland, On C.Neumann’s method for second-order elliptic systems in domains with non-smooth boundaries, J. Math. Anal. Appl.262 (2001), 733–748.

[56] M. E. Taylor, Tools for PDE: Pseudodifferential Operators, Paradifferential Operators, and Layer Potentials, Mathematical Surveys and Monographs, 81 American Math. Soc., Providence RI, (2000).

[57] F. G. Tricomi, Integral Equations, Wiley, New York (1957).

[58] G.C. Verchota, Layer potentials and boundary value problems for Laplace’s equation in Lipschitz domains, J. Funct. Anal. 59 (1984), 572–611.

[59] J. H. White, A global invariant of conformal mappings in space, Proc. Amer. Math. Soc. 38 (1973), 162–164.

[60] K. Yosida, Functional analysis, Sixth edition, Springer-Verlag, Berlin-New York, 1980.