The Weak Field Limit of Higher Order Gravity

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2004 - 2007
To my lovely family:
My grandmothers Angela and Giuseppina
My grandfathers Antonio and Arturo
My parents Giuseppina and Pasquale
My brother Antonio
and ... She ... Georgia
## Contents

**The Weak Field Limit of Higher Order Gravity**

**Introduction: History and Motivations**

- 0.1 General considerations on Gravitational theories
- 0.2 Issues from Cosmology
- 0.3 The Weak Field Limit of Higher Order Gravity
- 0.4 Plan of thesis

**1 Extended theories of gravity: a review**

- 1.1 What a good theory of Gravity has to do: General Relativity and its extensions
- 1.2 The Extended Theories of Gravity: $F(R, \Box R, \ldots, \Box^k R, \phi)$
- 1.3 Conformal transformations
- 1.4 The Palatini Approach and the Intrinsic Conformal Structure
- 1.5 The general $f$ theory
- 1.6 The field equations for the $R_{\alpha\beta}R^{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ invariants
- 1.7 Generalities on spherical symmetry

**2 Exact and perturbative solutions in General Relativity**

- 2.1 The Schwarzschild, Schwarzschild — de Sitter and Reissner — Nordstrom solutions: the Birkhoff theorem in General Relativity
- 2.2 Perturbations of the Schwarzschild solution: The Eddington parameters $\beta$ and $\gamma$
- 2.3 General remarks on the Newtonian and the post-Newtonian approximation of Einstein equation
- 2.4 General remarks on the post-Minkowskian approximation of Einstein equation

**3 Spherical symmetry in $f$ — gravity**

- 3.1 The Ricci curvature scalar in spherical symmetry
### CONTENTS

3.2 Solutions with constant curvature scalar ........................................ 55  
3.3 Solutions with curvature scalar function of $r$ ................................. 60  
3.4 Perturbing the spherically symmetric solutions ................................. 61

4 The Noether Symmetries of $f$ − gravity ........................................ 69  
4.1 The point-like $f$ Lagrangian in spherical symmetry ........................... 69  
4.2 The Noether Symmetry Approach ................................................... 75  
4.3 The Noether Approach for $f$ − gravity in spherical symmetry ............ 79  
4.4 Perspectives of Noether symmetries approach ................................. 84

5 $f$ − gravity and scalar-tensor gravity: affinities and differences .......... 87  
5.1 PPN − parameters in Scalar − Tensor and Fourth Order Gravity ........ 87  
5.2 Comparing with experimental measurements .................................. 90  
5.3 Newtonian limit of $f$ − gravity by O’Hanlon theory analogy ............... 93  
5.4 Differences of a generic scalar − tensor theory in the Jordan and Einstein frames .................................................. 100

6 The Newtonian limit of Fourth Order Gravity theory .......................... 103  
6.1 The Newtonian limit of $f$ − gravity in spherically symmetric background ........ 103  
6.1.1 Newtonian and post − Newtonian limit in the harmonic gauge ........ 110  
6.2 The Newtonian limit of quadratic gravity ....................................... 110  
6.3 Green functions for systems with spherical symmetry ........................ 115  
6.4 Solutions using the Green function ................................................. 120  
6.4.1 Particular solutions ..................................................................... 121  
6.4.2 The general solution by Green function $G^A(x - x')$ .................... 123  
6.4.3 Further solutions by the Green functions $G^B(x - x')$ and $G^C(x - x')$ .... 127  
6.5 Post − Newtonian scheme of $f$ − gravity ...................................... 129

7 The post-Minkowskian approximation in $f$ − gravity: Gravitational Waves in higher order gravity ......................................................... 133  
7.1 The Post − Minkowskian approximation in spherically symmetric solution ........ 133  
7.2 The post − Minkowskian limit of $f$ − gravity toward gravitational waves .... 136  
7.3 Strong gravitational waves in a general $f$ − gravity ........................... 141  
7.4 Energy-momentum tensor of $f$ − gravity ....................................... 142

8 Discussions and conclusions .............................................................. 149

List of papers ......................................................................................... 153
Acknowledges
The Weak Field Limit of Higher Order Gravity

The Higher Order Theories of Gravity - $f(R, R_{\alpha\beta}R^{\alpha\beta})$ - theory, where $R$ is the Ricci scalar, $R_{\alpha\beta}$ is the Ricci tensor and $f$ is any analytic function - have recently attracted a lot of interest as alternative candidates to explain the observed cosmic acceleration, the flatness of the rotation curves of spiral galaxies and other relevant astrophysical phenomena. It is a crucial point testing these alternative theories in the so called weak field and newtonian limit of a $f(R, R_{\alpha\beta}R^{\alpha\beta})$ - theory. With this ”perturbation technique” it is possible to find spherically symmetric solutions and compare them with the ones of General Relativity. On both approaches we found a modification of General Relativity: the behaviour of gravitational potential presents a modification Yukawa - like in the newtonian case and a massive propagation in the weak field case. When the modification of the theory is removed (i.e. $f(R, R_{\alpha\beta}R^{\alpha\beta}) = R$, Hilbert - Einstein lagrangian) we find the usual outcomes of General Relativity. Also the Noether symmetries technique has been investigated to find some time independent spherically symmetric solutions.

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Introduction: History and Motivations

0.1 General considerations on Gravitational theories

General Relativity (GR) is a comprehensive theory of spacetime, gravity and matter. Its formulation implies that space and time are not “absolute” entities, as in Classical Mechanics, but dynamical quantities strictly related to the distribution of matter and energy. As a consequence, this approach gave rise to a new conception of the Universe itself which, for the first time, was considered as a dynamical system. In other words, Cosmology has been enclosed in the realm of Science and not only of Philosophy, as before the Einstein work. On the other hand, the possibility of a scientific investigation of the Universe has led to the formulation of the Standard Cosmological Model \[1\] which, quite nicely, has matched with observations.

Despite of these results, the study of possible modifications of Einstein’s theory of gravitation has a long history which reaches back to the early 1920s \[2, 3, 4, 5, 6, 7\]. While the proposed early amendments of Einstein’s theory were aimed toward the unification of gravity with the other interactions of physics, like electromagnetism and whether GR is the only fundamental theory capable of explaining the gravitational interaction, the recent interest in such modifications comes from cosmological observations (for a comprehensive review, see \[8\]). Such issues come, essentially, from Cosmology and Quantum Field Theory. In the first case, the presence of the Big Bang singularity, the flatness and horizon problems \[9\] led to the statement that Cosmological Standard Model, based on GR and Standard Model of Particle Physics, is inadequate to describe the Universe at extreme regimes. These observations usually lead to the introduction of additional ad-hoc concepts like dark energy/matter if interpreted within Einstein’s theory. On the other hand, the emergence of such stopgaps could be interpreted as a first signal of a breakdown of GR at astrophysical and cosmological scales \[10, 11\], and led to the proposal of several alternative modifications of the underlying gravity theory (see \[12\] for a review). Besides from Quantum Field Theory point view, GR is a classical theory which does not work as a fundamental theory, when one wants to achieve a full quantum description of spacetime (and then of gravity).
While it is very natural to extend Einstein’s gravity to theories with additional geometric degrees of freedom, (see for example [13, 14, 15] for general surveys on this subject as well as [16] for a list of works in a cosmological context), recent attempts focused on the old idea of modifying the gravitational Lagrangian in a purely metric framework, leading to higher order field equations. Such an approach is the so-called Extended Theories of Gravity (ETG) which have become a sort of paradigm in the study of gravitational interaction. They are based on corrections and enlargements of the Einstein theory. The paradigm consists, essentially, in adding higher order curvature invariants and minimally or non-minimally coupled scalar fields into dynamics which come out from the effective action of quantum gravity [17].

The idea to extend Einstein’s theory of gravitation is fruitful and economic also with respect to several attempts which try to solve problems by adding new and, most of times, unjustified ingredients in order to give a self-consistent picture of dynamics. The today observed accelerated expansion of the Hubble flow and the missing matter of astrophysical large scale structures, are primarily enclosed in these considerations. Both the issues could be solved changing the gravitational sector, i.e. the l.h.s. of field equations. The philosophy is alternative to add new cosmic fluids (new components in the r.h.s. of field equations) which should give rise to clustered structures (dark matter) or to accelerated dynamics (dark energy) thanks to exotic equations of state. In particular, relaxing the hypothesis that gravitational Lagrangian has to be a linear function of the Ricci curvature scalar $R$, like in the Hilbert-Einstein formulation, one can take into account an effective action where the gravitational Lagrangian includes other scalar invariants.

In summary, the general features of ETGs are that the Einstein field equations result to be modified in two senses: i) geometry can be non-minimally coupled to some scalar field, and / or ii) higher than second order derivative terms in the metric come out. In the former case, we generically deal with scalar-tensor theories of gravity; in the latter, we deal with higher order theories. However combinations of non-minimally coupled and higher-order terms can emerge as contributions in effective Lagrangians. In this case, we deal with higher-order-scalar-tensor theories of gravity.

Due to the increased complexity of the field equations in this framework, the main amount of works dealt with some formally equivalent theories, in which a reduction of the order of the field equations was achieved by considering the metric and the connection as independent fields [18, 19, 20, 21, 22]. In addition, many authors exploited the formal relationship to scalar-tensor theories to make some statements about the weak field regime, which was already worked out for scalar-tensor theories more than ten years ago [23].

Other motivations to modify GR come from the issue of a full recovering of the Mach principle which leads to assume a varying gravitational coupling. The principle states that the local inertial
frame is determined by some average of the motion of distant astronomical objects [24]. This fact implies that the gravitational coupling can be scale-dependent and related to some scalar field. As a consequence, the concept of “inertia” and the Equivalence Principle have to be revised. For example, the Brans-Dicke theory [25] is a serious attempt to define an alternative theory to the Einstein gravity: it takes into account a variable Newton gravitational coupling, whose dynamics is governed by a scalar field non-minimally coupled to the geometry. In such a way, Mach’s principle is better implemented [25, 26, 27].

As already mentioned, corrections to the gravitational Lagrangian, leading to higher order field equations, were already studied by several authors [3, 6, 7] soon after the GR was proposed. Developments in the 1960s and 1970s [28, 29, 30, 31, 32], partially motivated by the quantization schemes proposed at that time, made clear that theories containing only a $R^2$ term in the Lagrangian were not viable with respect to their weak field behavior. Buchdahl, in 1962 [28] rejected pure $R^2$ theories because of the non-existence of asymptotically flat solutions.

Another concern which comes with generic higher order gravity (HOG) theories is linked to the initial value problem. It is unclear if the prolongation of standard methods can be used in order to tackle this problem in every theory. Hence it is doubtful that the Cauchy problem could be properly addressed in the near future, for example within $1/R$ theories, if one takes into account the results already obtained in fourth order theories stemming from a quadratic Lagrangian [33, 34].

Starting from the Hilbert-Einstein Lagrangian

$$\mathcal{L}_{GR} = \sqrt{-g}R$$

the following terms

\[
\begin{align*}
\mathcal{L}_1 &= \sqrt{-g}R^2 \\
\mathcal{L}_2 &= \sqrt{-g}R_{\alpha\beta}R^{\alpha\beta} \\
\mathcal{L}_3 &= \sqrt{-g}R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}
\end{align*}
\]

and combinations of them, represent a first obvious choices for an extended gravity theory with improved dynamics with respect to GR. Since the variational derivative of $\mathcal{L}_3$ can be linearly expressed [5, 35] via the variational derivatives of $\mathcal{L}_1$ and $\mathcal{L}_2$, one can omit $\mathcal{L}_3$ in the final Lagrangian of a HOG without loss of generality.
Introduction: Historical motivations

Besides, every unification scheme as Superstrings, Supergravity or Grand Unified Theories, takes into account effective actions where non-minimal couplings to the geometry or higher order terms in the curvature invariants are present. Such contributions are due to one-loop or higher loop corrections in the high curvature regimes near the full (not yet available) quantum gravity regime [17]. Specifically, this scheme was adopted in order to deal with the quantization on curved spacetimes and the result was that the interactions among quantum scalar fields and background geometry or the gravitational self-interactions yield corrective terms in the Hilbert-Einstein Lagrangian [36]. Moreover, it has been realized that such corrective terms are inescapable in order to obtain the effective action of quantum gravity at scales closed to the Planck one [37]. All these approaches are not the “full quantum gravity” but are needed as working schemes toward it.

In summary, higher order terms in curvature invariants (such as $R^2$, $R_{\alpha\beta}R^{\alpha\beta}$, $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$, $R_{\Box}R$, or $R_{\Box^k}R$) or non-minimally coupled terms between scalar fields and geometry (such as $\phi^2 R$) have to be added to the effective Lagrangian of gravitational field when quantum corrections are considered. For instance, one can notice that such terms occur in the effective Lagrangian of strings or in Kaluza-Klein theories, when the mechanism of dimensional reduction is used [38].

On the other hand, from a conceptual viewpoint, there are no a priori reason to restrict the gravitational Lagrangian to a linear function of the Ricci scalar $R$, minimally coupled with matter [19]. More precisely, higher order terms appear always as contributions of order two in the field equations. For example, a term like $R^2$ gives fourth order equations [39], $R_{\Box}R$ gives sixth order equations [40] [41], $R_{\Box^2}R$ gives eighth order equations [42] and so on. By a conformal transformation, any 2nd order derivative term corresponds to a scalar field [43]: for example, fourth order gravity gives Einstein plus one scalar field, sixth order gravity gives Einstein plus two scalar fields and so on [40] [43]. Furthermore, the idea that there are no “exact” laws of physics could be taken into serious account: in such a case, the effective Lagrangians of physical interactions are “stochastic” functions. This feature means that the local gauge invariances (i.e. conservation laws) are well approximated only in the low energy limit and the fundamental physical constants can vary [44].

0.2 Issues from Cosmology

Beside fundamental physics motivations, these theories have acquired a huge interest in Cosmology due to the fact that they “naturally” exhibit inflationary behaviors able to overcome the shortcomings of Cosmological Standard Model (based on GR). The related cosmological models seem

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2 The dynamics of such scalar fields is usually given by the corresponding Klein-Gordon equation, which is second order.
realistic and capable of matching with the Cosmic Microwave Background Radiation (CMBR) observations [34, 45, 46]. Furthermore, it is possible to show that, via conformal transformations, the higher order and non-minimally coupled terms always correspond to the Einstein gravity plus one or more than one minimally coupled scalar fields [33, 40, 47, 48, 49].

Furthermore, it is possible to show that the \( f(R) \)-gravity (\( f \)-gravity) is equivalent not only to a scalar-tensor one but also to the Einstein theory plus an ideal fluid [50]. This feature results very interesting if we want to obtain multiple inflationary events since an early stage could select “very” large-scale structures (clusters of galaxies today), while a late stage could select “small” large-scale structures (galaxies today) [41]. The philosophy is that each inflationary era is related to the dynamics of a scalar field. Finally, these extended schemes could naturally solve the problem of “graceful exit” bypassing the shortcomings of former inflationary models [46, 51].

In recent years, the efforts to give a physical explanation to the today observed cosmic acceleration [52, 53, 54, 55] have attracted a good amount of interest in \( f \)-gravity, considered as a viable mechanism to explain the cosmic acceleration by extending the geometric sector of field equations [56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74]. There are several physical and mathematical motivations to enlarge GR by these theories. For comprehensive reviews, see [75, 76, 77].

Specifically, cosmological models coming from \( f \)-gravity were firstly introduced by Starobinsky [45] in the early 80’ies to build up a feasible inflationary model where geometric degrees of freedom had the role of the scalar field ruling the inflation and the structure formation.

In addition to the revision of Standard Cosmology at early epochs (leading to the Inflation), a new approach is necessary also at late epochs. ETGs could play a fundamental role also in this context. In fact, the increasing bulk of data that have been accumulated in the last few years have paved the way to the emergence of a new cosmological model usually referred to as the *Cosmological Concordance Model* (\( \Lambda \) Cold Dark Matter: \( \Lambda \)CDM).

The Hubble diagram of Type Ia Supernovae (hereafter SNeIa), measured by both the Supernova Cosmology Project [54, 78] and the High-z Team [52, 79] up to redshift \( z \sim 1 \), the luminosity distance of Ia Type Supernovae [52, 53, 54, 55], the Large Scale Structure [72] and the anisotropy of CMBR [73, 74] are the evidence that the Universe is undergoing a phase of accelerated expansion. On the other hand, balloon born experiments, such as BOOMERanG [80] and MAXIMA [81], determined the location of the first and second peak in the anisotropy spectrum of the CMBR strongly pointing out that the geometry of the Universe is spatially flat. If combined with constraints coming from galaxy clusters on the matter density parameter \( \Omega_M \), these data indicate that the Universe is dominated by a non-clustered fluid with negative pressure, generically dubbed *dark energy*, which is able to drive the accelerated expansion. This picture has been further strength-
ened by the recent precise measurements of the CMBR spectrum, due to the WMAP experiment \cite{73, 74, 82}, and by the extension of the SNeIa Hubble diagram to redshifts higher than 1 \cite{52}. From this amount of data, the widely accepted \( \Lambda \)CDM is a spatially flat Universe, dominated by cold dark matter (CDM (\( \sim 0.25 \div 0.3\% \)) which should explain the clustered structures) and dark energy (\( \Lambda \sim 0.65 \div 0.7\% \)), in the form of an “effective” cosmological constant, giving rise to the accelerated behavior.

After these observational evidences, an overwhelming flood of papers has appeared: they present a great variety of models trying to explain this phenomenon. In any case, the simplest explanation is claiming for the well known cosmological constant \( \Lambda \) \cite{83}. Although it is the best fit to most of the available astrophysical data \cite{73, 74, 82}, the \( \Lambda \)CDM model fails in explaining why the inferred value of \( \Lambda \) is so tiny (120 orders of magnitude lower than the value of quantum gravity vacuum state!) if compared with the typical vacuum energy values predicted by particle physics and why its energy density is today comparable to the matter density (the so called *coincidence problem*).

Although the cosmological constant \cite{84, 85, 86} remains the most relevant candidate to interpret the accelerated behavior, several proposals have been suggested in the last few years: quintessence models, where the cosmic acceleration is generated by means of a scalar field, in a way similar to the early time inflation \cite{45}, acting at large scales and recent epochs \cite{87, 88}; models based on exotic fluids like the Chaplygin-gas \cite{89, 90, 91}, or non-perfect fluids \cite{92}); phantom fields, based on scalar fields with anomalous signature in the kinetic term \cite{93, 94, 95, 96}, higher dimensional scenarios (braneworld) \cite{97, 98, 99, 100}. These results can be achieved in metric and Palatini approaches \cite{20, 57, 64, 65, 66, 68, 66, 69, 70, 101}. In addition, reversing the problem, one can reconstruct the form of the gravity Lagrangian by observational data of cosmological relevance through a ”back scattering” procedure. All these facts suggest that the function \( f \) should be more general than the linear Hilbert-Einstein one implying that HOG could be a suitable approach to solve GR shortcomings without introducing mysterious ingredients as dark energy and dark matter (see e.g. \cite{102, 103}).

Actually, all of these models, are based on the peculiar characteristic of introducing new sources into the cosmological dynamics, while it would be preferable to develop scenarios consistent with observations without invoking further parameters or components non-testable (up to now) at a fundamental level.

Moreover, it is not clear where this scalar field originates from, thus leaving a great uncertainty on the choice of the scalar field potential. The subtle and elusive nature of dark energy has led many authors to look for completely different scenarios able to give a quintessential behavior without the need of exotic components. To this aim, it is worth stressing that the acceleration of
the Universe only claims for a negative pressure dominant component, but does not tell anything
about the nature and the number of cosmic fluids filling the Universe.

Actually, there is still a different way to face the problem of cosmic acceleration. As stressed
in [104], it is possible that the observed acceleration is not the manifestation of another ingredient
in the cosmic pie, but rather the first signal of a breakdown of our understanding of the laws of
gravitation (in the infra-red limit).

From this point of view, it is thus tempting to modify the Friedmann equations to see whether
it is possible to fit the astrophysical data with models comprising only the standard matter. Interest-
ing examples of this kind are the Cardassian expansion [105] and the Dvali-Gabadaze-Porrati
gravity [97]. Moving in this same framework, it is possible to find alternative schemes where a
quintessential behavior is obtained by taking into account effective models coming from funda-
mental physics giving rise to generalized or HOG actions [56, 60, 59, 20] (for a comprehensive
review see [76]).

For instance, a cosmological constant term may be recovered as a consequence of a non-
vanishing torsion field thus leading to a model which is consistent with both SNeIa Hubble diagram
and Sunyaev-Zel’dovich data coming from clusters of galaxies [106]. SNeIa data could also be
efficiently fitted including higher-order curvature invariants in the gravity Lagrangian [58, 107,
108, 109]. It is worth noticing that these alternative models provide naturally a cosmological
component with negative pressure whose origin is related to the geometry of the Universe thus
overcoming the problems linked to the physical significance of the scalar field.

It is evident, from this short overview, the large number of cosmological models which are
viable candidates to explain the observed accelerated expansion. This abundance of models is,
from one hand, the signal of the fact that we have a limited number of cosmological tests to
discriminate among rival theories, and, from the other hand, that a urgent degeneracy problem has
to be faced. To this aim, it is useful to remark that both the SNeIa Hubble diagram and the angular
size-redshift relation of compact radio sources [110, 111] are distance based methods to probe
cosmological models so then systematic errors and biases could be iterated. From this point of
view, it is interesting to search for tests based on time-dependent observables.

For example, one can take into account the lookback time to distant objects since this quan-
tity can discriminate among different cosmological models. The lookback time is observationally
estimated as the difference between the present day age of the Universe and the age of a given
object at redshift \( z \). Such an estimate is possible if the object is a galaxy observed in more than one
photometric band since its color is determined by its age as a consequence of stellar evolution. It
is thus possible to get an estimate of the galaxy age by measuring its magnitude in different bands
and then using stellar evolutionary codes to choose the model that reproduces the observed colors
The resort to modified gravity theories, which extend in some way the GR, allows to pursue this different approach (no further unknown sources) giving rise to suitable cosmological models where a late time accelerated expansion naturally arises.

The idea that the Einstein gravity should be extended or corrected at large scales (infrared limit) or at high energies (ultraviolet limit) is suggested by several theoretical and observational issues. Quantum field theories in curved spacetimes, as well as the low energy limit of string theory, both imply semi-classical effective Lagrangians containing higher-order curvature invariants or scalar-tensor terms. In addition, GR has been tested only at solar system scales while it shows several shortcomings if checked at higher energies or larger scales.

Of course modifying the gravitational action asks for several fundamental challenges. These models can exhibit instabilities [112, 113, 114] or ghost-like behaviors [32], while, on the other side, they should be matched with the low energy limit observations and experiments (solar system tests). Despite of all these issues, in the last years, several interesting results have been achieved in the framework of f-gravity at cosmological, galactic and solar system scales.

For example, models based on generic functions of the Ricci scalar \(R\) show cosmological solution with late time accelerating dynamics [20, 59, 60, 61, 62, 63], in addition, it has been shown that some of them could agree with CMBR observational prescriptions [115, 116], nevertheless this matter is still argument of debate [117, 118]. For a review of the models and their cosmological applications see, e.g., [76, 118].

Moreover, considering \(f\)-gravity in the low energy limit, it is possible to obtain corrected gravitational potentials capable of explaining the flat rotation curves of spiral galaxies without considering huge amounts of dark matter [10, 119, 120, 121, 122] and, furthermore, this seems the only self-consistent way to reproduce the universal rotation curve of spiral galaxies [123]. On the other hand, several anomalies in Solar System experiments could be framed and addressed in this picture [124, 125].

Summarizing, almost 95% of matter-energy content of the universe is unknown in the framework of Standard Cosmological Model while we can experimentally probe only gravity and ordinary (baryonic and radiation) matter. Considering another point of view, anomalous acceleration (Solar System), dark matter (galaxies and galaxy clusters), dark energy (cosmology) could be nothing else but the indications that shortcomings are present in GR and gravity is an interaction depending on the scale. The assumption of a linear Lagrangian density in the Ricci scalar \(R\) for the Hilbert-Einstein action could be too simple to describe gravity at any scale and more general approaches should be pursued to match observations. Among these schemes, several motivations suggest to generalize GR by considering gravitational actions where generic functions of curvature
invariants are present.

0.3 The Weak Field Limit of Higher Order Gravity

It is well known that GR is the cornerstone theory among several attempts proposed to describe gravity. It represents an elegant approach giving several phenomenological predictions, but its validity, in the Newtonian limit regime, is experimentally probed only at Solar System scales. However, also at these scales, some conundrums come out as an apparent, anomalous, long-range acceleration revealed from the data analysis of Pioneer 10/11, Galileo, and Ulysses spacecrafts. Such a feature is difficult to be framed in the standard scheme of GR and its low energy limit [126, 127], while it could be framed in a general theoretical scheme by taking corrections to the Newtonian potential into account [128]. Furthermore, at galactic distances, huge bulks of dark matter are needed to provide realistic models matching with observations. In this case, retaining GR and its low energy limit, implies the introduction of an actually unknown ingredient. We face a similar situation even at larger scales: clusters of galaxies are gravitationally stable and bounded only if large amounts of dark matter are supposed in their potential wells.

Taking into account the weak field limit approximation, ETGs are expected to reproduce GR [129]. This fact is matter of debate since several relativistic theories do not reproduce exactly the Einstein results in the Newtonian approximation but, in some sense, generalize them. As it was firstly noticed by Stelle [32], a $R^2$-theory gives rise to Yukawa-like corrections in the Newtonian potential. Such a feature could have interesting physical consequences. For example, some authors claim to explain the flat rotation curves of galaxies by using such terms [130]. Others [131] have shown that a conformal theory of gravity is nothing else but a HOG model containing such terms in the Newtonian limit.

In general, any relativistic theory of gravitation yields corrections to the Newton potential (see for example [132]) which, in the post-Newtonian formalism, could be a test for the same theory [129]. Furthermore the newborn gravitational lensing astronomy [133] is giving rise to additional tests of gravity over small, large, and very large scales which soon will provide direct measurements for the variation of the Newton coupling [134], the potential of galaxies, clusters of galaxies and several other features of self-gravitating systems. Such data could be, very likely, capable of confirming or ruling out the physical consistency of GR or of any ETG.

In recent papers, some authors have confronted this kind of theories even with the Post Parameterized Newtonian (PPN) prescriptions in metric and Palatini approaches. The results seem controversial since in some cases [135, 136] it is argued that GR is always valid at Solar System
scales and there is no room for other theories; nevertheless, some other studies [137, 138] find that recent experiments as Cassini and Lunar Laser Ranging allow the possibility that ETGs could be seriously taken into account. In particular, it is possible to define PPN-parameters in term of $f$-gravity functions and several classes of fourth order theories result compatible with experiments in Solar System [137].

In a recent paper [139], spherically symmetric solutions for $f$-gravity in vacuum have been found considering relations among functions defining the spherical metric or imposing a constant Ricci curvature scalar. The authors have been able to reconstruct, at the end, the form of some $f$-theories, discussing their physical relevance. In [140], the same authors have discussed static spherically symmetric perfect fluid solutions for $f$-gravity in metric formalism. They showed that a given matter distribution is not capable of determining the functional form of $f$.

The discussion about the short scale behavior of HOG has been quite vivacious in the last years since GR shows is best predictions just at the Solar System level. As matter of fact, measurements coming from weak field limit tests like the bending of light, the perihelion shift of planets, frame dragging experiments represent inescapable tests for whatever theory of gravity. Actually, in our opinion, there are sufficient theoretical predictions to state that HOG can be compatible with Newtonian and post-Newtonian prescriptions [A]. In other papers [137], it has been that this result can be achieved by means of the analogy of $f$-models with scalar-tensor gravity.

Nevertheless, up to now, the discussion on the weak field limit of $f$-theories is far to be definitive and there are several papers claiming for opposite results [125, 138, 142, 143, 144, 145, 148, 149, 150], or stating that no progress has been reached in the last years due to the several common misconceptions in the various theories of gravity [141].

In the last few years, several authors have dealt with this matter with contrasting conclusions, in particular with respect to the PPN limit [135, 137, 138, 143, 151]. On the other hand, the investigation of spherically symmetric solutions for such kind of models has been developed in several papers [139, 140, 142, 152]. Such an analysis deserves particular attention since it can allow to draw interesting conclusions on the effective modification of the gravitational potential induced by HOG at low energies and, in addition, it could shed new light on the PPN limit of such theories at least in a preliminary way. For example, theories like $f = R + \frac{\mu}{R}$, which fairly address the cosmic acceleration issue [56, 57, 58], suffer a ill-defined PPN-limit since a theory containing terms like $R^{-1}$ is singular in $R = 0$ and does not admit any Minkowski limit and then any other background solution which is Ricci flat. On the other hand, affirming that the unique $f$-gravity spherically symmetric ”static” solution, corresponding to a realistic mass source and matching the present cosmic background at infinity has the PPN parameter $\gamma = 1/2$, in conflict with experiments which give $\gamma \simeq 1$ could be misleading since by assuming, for example, $f = R^{1+\epsilon}$ with $\epsilon \to 0$ has
to give results compatible with GR (i.e. $\gamma \sim 1$).

As a matter of fact, defining a PPN-limit in such a case is a quite delicate issue, since in order to develop an analytical study of the deviation from the Newtonian approximation requires that the spacetime should be, at least, asymptotically Minkowski. Finally, approaching the problem considering a curvature constant metric, as in the case of the Schwarzschild-de Sitter solution, could induce to flawed conclusions. As a consequence, understanding the properties of HOG with respect to spherically symmetric solutions in the weak field limit turns out to be a key issue from several points of view.

This is only one example about the debate on the weak limit field limit: authors approached the weak limit issue following different schemes and developing different parameterizations which, in some cases, turn out to be not necessarily correct.

The purpose of this thesis (also referring to the published papers [C], [D]) is to take part to the debate, building up a rigorous formalism which deals with the formal definition of weak field and small velocities limit applied to HOG gravity. In a series of papers [C], [D], [E], the aim is to pursue a systematic discussion involving: i) the Newtonian limit of $f$-gravity, ii) spherically symmetric solutions toward the weak field limit of $f$-gravity; and, finally, iii) considering general HOG theories where also invariants as $R_{\mu\nu}R^{\mu\nu}$ or $R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$ are considered. Besides the Birkhoff theorem is not a general result in HOG models [C] also if it holds for several interesting classes of these theories as discussed, for example, in [153] [154].

The analysis is based on the metric approach, developed in the Jordan frame, assuming that the observations are performed in it, without resorting to any conformal transformation as done in several cases [135]. This point of view is adopted in order to avoid dangerous variable changes which could compromise the correct physical interpretation of the results.

A relevant aspect of HOG theories, in the post-Minkowskian limit is the propagation of the gravitational fields. It turns out that wave signals can be characterized with both tensorial and scalar mode [F]. This issue represents a quite striking difference between GR-like models and extended gravity models since in the standard Einstein scheme only tensorial degrees of freedom are allowed. As matter of facts, the gravitational wave limit of these models can represent an interesting framework to study, in order to discuss the physical observable footprints which discriminate between GR and HOG experimental predictions.

*In this PhD thesis, we are going to analyze and discuss, in a general way and without specifying a priori the form of the Lagrangian, the relation between the spherical symmetry and the weak field limit, pointing out the differences and the relations with respect to the post-Newtonian and the post-Minkowskian limits of $f$-gravity. Our aim is to develop a systematic approach considering the theoretical prescriptions to obtain a correct weak field limit in order to point out the analogies*
Introduction: Historical motivations

and the differences with respect to GR. A fundamental issue is to recover the asymptotically flat solution in absence of gravity and the well-known results related to the specific case $f = R$, i.e. GR. Only in this situation a correct comparison between GR and any ETG is possible from an experimental and a theoretical viewpoint. For example,

In literature, there are several definitions and several claims in this direction but clear statements and discussion on these approaches urge in order to find out definite results to be tested by experiments [141].

0.4 Plan of thesis

The layout of the PhD thesis is organized as follows. In the first chapter we report a general review of ETGs and the fundamental aspects of GR. In particular we display all fundamental tools: Einstein Equation, Bianchi Identity, Conformal transformations, Metric and Palatini formalism, ETG theories (Scalar-tensor, HOG theories and so on), Coordinates system transformations and the relations between them (for example standard, isotropic coordinates etc).

In the second chapter some "exact" spherically symmetric solution of GR is shown (Schwarzschild, Schwarzschild-de Sitter, Reissner-Nordstrom solution). On the other hand, we show the technicality of development of field equation [C] with respect to Newtonian and Post-Newtonian approach: in such case we also introduce the Eddington parameters. Finally we perform the post-Minkowskian limit: the gravitational waves. The developments are computed in generic coordinates systems and in the gauge harmonic.

The third chapter is devoted to some general remarks on spherical symmetry in $f$-gravity [D]. In particular, the expression of the Ricci scalar and the general form of metric components are derived in spherical symmetry discussing how recovering the correct Minkowski flat limit. We discuss the spherically symmetric background solutions with constant scalar curvature considering, in particular Schwarzschild-like and Schwarzschild-de Sitter-like solutions with constant curvature; we discuss also the cases in which the spherical symmetry is present also for the Ricci scalar depending on the radial coordinate $r$. This is an interesting situation, not present in GR. In fact, as it is well known, in the Einstein theory, the Birkhoff theorem states that a spherically symmetric solution is always stationary and static [157] and the Ricci scalar is constant. In $f$ the situation is more general and then the Ricci scalar, in principle, can evolve with radial and time coordinates. Finally, the last part of chapter is devoted to the study of a perturbation approach starting from a spherically symmetric background considering the general case in which the Ricci scalar is $R = R(r)$. The motivation is due to the fact that, in GR, the Schwarzschild solution and the weak
field limit coincide under suitable conditions.

In the fourth one we want to seek for a general method to find out spherically symmetric solutions in $f$-gravity and, eventually, in generic ETG [B]. Asking for a certain symmetry of the metric, we would like to investigate if such a symmetry holds for a generic theory of gravity. In particular for the $f$-theories. Specifically, we want to apply the Noether Symmetry Approach [26] in order to search for spherically symmetric solutions in generic $f$-theories of gravity. This means that we consider the spherical symmetry for the metric as a Noether symmetry and search for $f$ Lagrangians compatible with it.

In the fifth chapter, we follow a different approach. Starting from the definitions of PPN-parameters in term of a generic analytic function $f$ and its derivatives, we deduce a class HOG theories, compatible with data, by means of an inverse procedure which allows to compare PPN-conditions with data [A]. As a matter of fact, it is possible to show that a third order polynomial, in the Ricci scalar, is compatible with observational constraints on PPN-parameters. The degree of deviation from GR depends on the experimental estimate of PPN-parameters. The second part of chapter is dedicated to very strong debate about the analogy or not between $f$- and Scalar-tensor gravity [H]. In fact for some authors the Newtonian limit of $f$-gravity is equivalent to the one of Brans-Dicke gravity with $\omega_{BD} = 0$, so that the PPN parameters of these models turn out to be ill defined. We don’t agree with this claim. We show that this is indeed not true. We discuss that HOG models are dynamically equivalent to a O’Hanlon Lagrangian which is a special case of Brans-Dicke theory characterized by a self-interaction potential and that, in the low energy and small velocity limit, this will imply a non-standard behaviour. This result turns out to be completely different from the one of a pure Brans-Dicke model and in particular suggests that it is completely misleading to consider the PPN parameters of this theory with $\omega_{BD} = 0$ in order to characterize the homologous quantities of $f$-gravity.

In the sixth one we analyze the Newtonian limit of HOG theory. We are going to focus exclusively on the weak field limit within the metric approach [C], [E], [G]. At this point we remind the readers that it was already shown in [158] that different variational procedures do not lead to equivalent results in the case of quadratic order Lagrangians, casting a shadow on several newer works in which this equivalence was implicitly assumed. In the first part we use the development shown in the second chapter for a generic analytic function $f$ and find the solution in the vacuum. For the sake of completeness we will treat the problem also by imposing the harmonic gauge on the field equations. Besides, we show that the Birkhoff theorem is not a general result for $f$-gravity since time-dependent evolution for spherically symmetric solutions can be achieved depending on the order of perturbations. In the second part we find again the Newtonian limit but for a so-called quadratic gravity lagrangian with the Green function method. We find the internal and external
potential generated by an extended spherically symmetric matter source. In the last part we outline the general approach to find the expression of metric tensor at fourth order perturbative for a generic $f$-theory.

In the seventh chapter we develop a formal description of the gravitational waves propagation in HOG models focusing on the scalar degrees of freedom and the characteristic of such scalar candidate in the gravity sector of gauge bosons $[F]$. As in the previously chapter we performed the Newtonian limit in vacuum with a spherically symmetric solution in standard coordinates, now we repeat the development but in the post-Minkowskian limit. In addition we discuss, in such a framework, the definition of the energy-momentum tensor of gravity which is a fundamental quantity in order to calculate the gravitational time delay in Pulsar timing. Some considerations on the differences between GR and HOG in the post-Minkowskian limit are addressed.

Finally in the last chapter we report the discussions and conclusions.
Chapter 1

Extended theories of gravity: a review

1.1 What a good theory of Gravity has to do: General Relativity and its extensions

From a phenomenological point of view, there are some minimal requirements that any relativistic theory of gravity has to match. First of all, it has to explain the astrophysical observations (e.g. the orbits of planets, the potential of self-gravitating structures).

This means that it has to reproduce the Newtonian dynamics in the weak-energy limit. Besides, it has to pass the classical Solar System tests which are all experimentally well founded [129].

As second step, it should reproduce galactic dynamics considering the observed baryonic constituents (e.g. luminous components as stars, sub-luminous components as planets, dust and gas), radiation and Newtonian potential which is, by assumption, extrapolated to galactic scales.

Thirdly, it should address the problem of large scale structure (e.g. clustering of galaxies) and finally cosmological dynamics, which means to reproduce, in a self-consistent way, the cosmological parameters as the expansion rate, the Hubble constant, the density parameter and so on. Observations and experiments, essentially, probe the standard baryonic matter, the radiation and an attractive overall interaction, acting at all scales and depending on distance: the gravity.

The simplest theory which try to satisfies the above requirements was formulated by Albert Einstein in the years 1915 - 1916 [159] and it is known as the Theory of General Relativity. It is firstly based on the assumption that space and time have to be entangled into a single spacetime structure, which, in the limit of no gravitational forces, has to reproduce the Minkowski spacetime structure. Einstein profitted also of ideas earlier put forward by Riemann, who stated that the Universe should be a curved manifold and that its curvature should be established on the basis of astronomical observations [160].
In other words, the distribution of matter has to influence point by point the local curvature of the spacetime structure. The theory, eventually formulated by Einstein in 1915, was strongly based on three assumptions that the physics of gravitation has to satisfy.

The "Principle of Relativity", which states that amounts to require all frames to be good frames for Physics, so that no preferred inertial frame should be chosen a priori (if any exist).

The "Principle of Equivalence", that amounts to require inertial effects to be locally indistinguishable from gravitational effects (in a sense, the equivalence between the inertial and the gravitational mass).

The "Principle of General Covariance", that requires field equations to be "generally covariant" (today, we would better say to be invariant under the action of the group of all spacetime diffeomorphisms) [161].

And - on the top of these three principles - the requirement that causality has to be preserved (the "Principle of Causality", i.e. that each point of spacetime should admit a universally valid notion of past, present and future).

Let us also recall that the older Newtonian theory of spacetime and gravitation - that Einstein wanted to reproduce at least in the limit of small gravitational forces (what is called today the "post-Newtonian approximation") - required space and time to be absolute entities, particles moving in a preferred inertial frame following curved trajectories, the curvature of which (i.e., the acceleration) had to be determined as a function of the sources (i.e., the "forces").

On these bases, Einstein was led to postulate that the gravitational forces have to be expressed by the curvature of a metric tensor field \( ds^2 = g_{\alpha\beta} dx^\alpha dx^\beta \) on a four-dimensional spacetime manifold, having the same signature of Minkowski metric, i.e., the so-called "Lorentzian signature", herewith assumed to be \((+,-,-,-)\). He also postulated that spacetime is curved in itself and that its curvature is locally determined by the distribution of the sources, i.e. - being spacetime a continuum - by the four-dimensional generalization of what in Continuum Mechanics is called the "matter stress-energy tensor", i.e. a rank-two (symmetric) tensor \( T_{\mu\nu} \).

Once a metric \( g_{\mu\nu} \) is given, the inverse \( g^{\mu\nu} \) satisfies the condition

\[
g^{\mu\alpha} g_{\alpha\beta} = \delta^\mu_\nu. \tag{1.1}
\]

Its curvature is expressed by the Riemann tensor (curvature)

\[
R^\alpha_{\mu\beta\nu} = \Gamma^\alpha_{\mu\beta,\nu} - \Gamma^\alpha_{\mu\nu,\beta} + \Gamma^\sigma_{\mu\nu} \Gamma^\alpha_{\sigma\beta} - \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\mu\beta}. \tag{1.2}
\]
where the comas are partial derivatives. The $\Gamma^\alpha_{\mu\nu}$ are the Christoffel symbols given by

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} g^{\alpha\sigma} (g_{\mu\sigma,\nu} + g_{\nu\sigma,\mu} - g_{\mu\nu,\sigma}),$$  \hspace{1cm} (1.3)$$

if the Levi-Civita connection is assumed. The contraction of the Riemann tensor \(^{(1.2)}\)

$$R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} = \Gamma^\sigma_{\mu\sigma,\nu} - \Gamma^\sigma_{\mu\sigma,\nu} + \Gamma^\sigma_{\mu\nu,\sigma},$$  \hspace{1cm} (1.4)$$
is the \textit{Ricci tensor} and the scalar

$$R = g^{\tau\sigma} R_{\tau\sigma} = R^\sigma_{\sigma} = g^{\tau\xi} \Gamma^\sigma_{\tau\xi,\sigma} - g^{\tau\xi} \Gamma^\sigma_{\tau\sigma,\xi} + g^{\tau\xi} \Gamma^\sigma_{\tau\xi,\sigma} - g^{\tau\xi} \Gamma^\sigma_{\tau\sigma,\xi} - g^{\tau\xi} \Gamma^\sigma_{\tau\xi,\sigma} - g^{\tau\xi} \Gamma^\sigma_{\tau\sigma,\xi},$$  \hspace{1cm} (1.5)$$
is called the \textit{scalar curvature} of $g_{\mu\nu}$. The Riemann tensor \(^{(1.2)}\) satisfies the so-called \textit{Bianchi identities}:

$$\begin{cases}
R_{\alpha\mu\beta\nu} + R_{\alpha\mu\delta\beta} + R_{\alpha\nu\delta;\beta} = 0 \\
R_{\alpha\mu\beta\nu;\alpha} + R_{\mu\beta;\nu} - R_{\mu\nu;\beta} = 0 \\
2R_{\alpha\beta;\alpha} - R_{\alpha\beta} = 0 \\
2R_{\alpha\beta} - \Box R = 0
\end{cases}$$  \hspace{1cm} (1.6)$$

where the covariant derivative is $A^{\alpha\beta...\delta}_{;\mu} = \nabla_\mu A^{\alpha\beta...\delta} = A^{\alpha\beta...\delta}_{;\mu} + \Gamma^\alpha_{\sigma\mu} A^{\beta...\delta}_{;\sigma} + \Gamma^\beta_{\sigma\mu} A^{\alpha...\delta}_{;\sigma} + \Gamma^\delta_{\sigma\mu} A^{\alpha\beta...\sigma} + \nabla_\alpha \nabla_\beta \Box R = \frac{\partial}{\partial \sigma} \sqrt{-g} \frac{\partial}{\partial \sigma}$ is the d’Alembert operator with respect to the metric $g_{\mu\nu}$.

Einstein was led to postulate the following equations for the dynamics of gravitational forces

$$R_{\mu\nu} = \mathcal{X} T_{\mu\nu}$$  \hspace{1cm} (1.7)$$

where $\mathcal{X} = 8\pi G$ is a coupling constant (we will use the convention $c = 1$). These equations turned out to be physically and mathematically unsatisfactory.
As Hilbert pointed out [161], they have not a variational origin, i.e. there was no Lagrangian able to reproduce them exactly (this is slightly wrong, but this remark is unessential here). Einstein replied that he knew that the equations were physically unsatisfactory, since they were contrasting with the continuity equation of any reasonable kind of matter. Assuming that matter is given as a perfect fluid, that is

\[ T_{\mu\nu} = (p + \rho)u_\mu u_\nu - pg_{\mu\nu} \]  

where \( u_\mu u_\nu \) define a comoving observer, \( p \) is the pressure and \( \rho \) the density of the fluid, then the continuity equation requires \( T_{\mu\nu} \) to be covariantly constant, i.e. to satisfy the conservation law

\[ T^{\mu\sigma \cdot \sigma} = 0 . \]  

In fact, it is not true that \( R^{\mu\sigma \cdot \sigma} \) vanishes (unless \( R = 0 \)). Einstein and Hilbert reached independently the conclusion that the wrong field equations (1.7) had to be replaced by the correct ones

\[ G_{\mu\nu} = \mathcal{X} T_{\mu\nu} \]  

where

\[ G_{\mu\nu} = R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} \]  

that is currently called the "Einstein tensor" of \( g_{\mu\nu} \). These equations are both variational and satisfy the conservation laws (1.9) since the following relation holds

\[ G^{\mu\sigma \cdot \sigma} = 0 , \]  

as a byproduct of the so-called Bianchi identities that the curvature tensor of \( g_{\mu\nu} \) has to satisfy [1, 162].

The Lagrangian that allows to obtain the field equations (1.10) is the sum of a matter Lagra-
1.1 What a good theory of Gravity has to do: General Relativity and its extensions

The action of GR is

$$A = \int d^4x \sqrt{-g}(R + \mathcal{L}_m).$$ \hspace{1cm} (1.14)

From the action principle, we get the field equations (1.10) by the variation:

$$\delta A = \delta \int d^4x \sqrt{-g}(R + \mathcal{L}_m) = \int d^4x \sqrt{-g} \left[R_{\mu\nu} - \frac{R}{2} g_{\mu\nu} + \mathcal{X} T_{\mu\nu}\right] \delta g_{\mu\nu} + \int d^4x \sqrt{-g} g_{\mu\nu} \delta R_{\mu\nu} = 0,$$ \hspace{1cm} (1.15)

where $T_{\mu\nu}$ is energy momentum tensor of matter:

$$T_{\mu\nu} = -\frac{1}{\sqrt{-g}} \frac{\delta (\sqrt{-g} \mathcal{L}_m)}{\delta g^{\mu\nu}}.$$ \hspace{1cm} (1.16)

The last term in (1.15) is a 4-divergence

$$\int d^4x \sqrt{-g} g^{\mu\nu} \delta R_{\mu\nu} = \int d^4x \sqrt{-g} \left[(-\delta g^{\mu\nu})_{\mu\nu} - \Box (g^{\mu\nu} \delta g_{\mu\nu})\right]$$ \hspace{1cm} (1.17)

then we can neglect it and we get the field equation (1.10). For the variational calculus (1.15) we used the following relations
Chapter 1 Extended theories of gravity: a review

\[
\begin{align*}
\delta \sqrt{-g} &= -\frac{1}{2} \sqrt{-g} g_{\alpha\beta} \delta g^{\alpha\beta} \\
\delta R &= R_{\alpha\beta} \delta g^{\alpha\beta} + g^{\alpha\beta} \delta R_{\alpha\beta} \\
\delta R_{\alpha\beta} &= \frac{1}{2} (\delta g^\rho_{\alpha;\beta\rho} + \delta g^\rho_{\beta;\alpha\rho} - \Box \delta g_{\alpha\beta} - g^{\rho\sigma} \delta g_{\rho\sigma;\alpha\beta})
\end{align*}
\]

The choice of Hilbert and Einstein was completely arbitrary (as it became clear a few years later), but it was certainly the simplest one both from the mathematical and the physical viewpoint. As it was later clarified by Levi-Civita in 1919, curvature is not a ”purely metric notion” but, rather, a notion related to the ”linear connection” to which ”parallel transport” and ”covariant derivation” refer [163].

In a sense, this is the precursor idea of what, in the sequel, would be called a ”gauge theoretical framework” [164], after the pioneering work by Cartan in 1925 [165]. But at the time of Einstein, only metric concepts were at hands and his solution was the only viable.

It was later clarified that the three principles of relativity, equivalence and covariance, together with causality, just require that the spacetime structure has to be determined by either one or both of two fields, a Lorentzian metric \(g\) and a linear connection \(\Gamma\), assumed to be torsionless for the sake of simplicity.

The metric \(g\) fixes the causal structure of spacetime (the light cones) as well as its metric relations (clocks and rods); the connection \(\Gamma\) fixes the free-fall, i.e. the locally inertial observers. They have, of course, to satisfy a number of compatibility relations which amount to require that photons follow the null geodesics of \(\Gamma\), so that \(\Gamma\) and \(g\) can be independent, \textit{a priori}, but constrained, \textit{a posteriori}, by some physical restrictions. These, however, do not impose that \(\Gamma\) has necessarily to be the Levi Civita connection of \(g\) [166].

This justifies - at least on a purely theoretical basis - the fact that one can envisage the so-called ”alternative theories of gravitation”, that we prefer to call ”Extended Theories of Gravitation” since their starting points are exactly those considered by Einstein and Hilbert: theories in which gravitation is described by either a metric (the so-called ”purely metric theories”), or by a linear connection (the so-called ”purely affine theories”) or by both fields (the so-called ”metric-affine theories”, also known as ”first order formalism theories”). In these theories, the Lagrangian is a scalar density of the curvature invariants constructed out of both \(g\) and \(\Gamma\).

The choice (1.13) is by no means unique and it turns out that the Hilbert-Einstein Lagrangian is in fact the only choice that produces an invariant that is linear in second derivatives of the metric.
A Lagrangian that, unfortunately, is rather singular from the Hamiltonian viewpoint, in much than same way as Lagrangians, linear in canonical momenta, are rather singular in Classical Mechanics (see e.g. [167]).

A number of attempts to generalize GR (and unify it to Electromagnetism) along these lines were followed by Einstein himself and many others (Eddington, Weyl, Schrodinger, just to quote the main contributors; see, e.g., [168]) but they were eventually given up in the fifties of XX Century, mainly because of a number of difficulties related to the definitely more complicated structure of a non-linear theory (where by "non-linear" we mean here a theory that is based on non-linear invariants of the curvature tensor), and also because of the new understanding of physics that is currently based on four fundamental forces and requires the more general "gauge framework" to be adopted (see [169]).

Still a number of sporadic investigations about "alternative theories" continued even after 1960 (see [129] and refs. quoted therein for a short history). The search of a coherent quantum theory of gravitation or the belief that gravity has to be considered as a sort of low-energy limit of string theories (see, e.g., [170]) - something that we are not willing to enter here in detail - has more or less recently revitalized the idea that there is no reason to follow the simple prescription of Einstein and Hilbert and to assume that gravity should be classically governed by a Lagrangian linear in the curvature.

Further curvature invariants or non-linear functions of them should be also considered, especially in view of the fact that they have to be included in both the semi-classical expansion of a quantum Lagrangian or in the low-energy limit of a string Lagrangian.

Moreover, it is clear from the recent astrophysical observations and from the current cosmological hypotheses that Einstein equations are no longer a good test for gravitation at Solar System, galactic, extra-galactic and cosmic scale, unless one does not admit that the matter side of Eqs.(1.10) contains some kind of exotic matter-energy which is the "dark matter" and "dark energy" side of the Universe.

The idea which we propose here is much simpler. Instead of changing the matter side of Einstein Equations (1.10) in order to fit the "missing matter-energy" content of the currently observed Universe (up to the 95% of the total amount!), by adding any sort of inexplicable and strangely behaving matter and energy, we claim that it is simpler and more convenient to change the gravitational side of the equations, admitting corrections coming from non-linearities in the Lagrangian. However, this is nothing else but a matter of taste and, since it is possible, such an approach should be explored. Of course, provided that the Lagrangian can be conveniently tuned up (i.e., chosen in a huge family of allowed Lagrangians) on the basis of its best fit with all possible observational tests, at all scales (solar, galactic, extragalactic and cosmic).
Something that - in spite of some commonly accepted but disguised opinion - can and should be done before rejecting a priori a non-linear theory of gravitation (based on a non-singular Lagrangian) and insisting that the Universe has to be necessarily described by a rather singular gravitational Lagrangian (one that does not allow a coherent perturbation theory from a good Hamiltonian viewpoint) accompanied by matter that does not follow the behavior that standard baryonic matter, probed in our laboratories, usually satisfies.

### 1.2 The Extended Theories of Gravity: \( F(R, \Box R, ..., \Box^k R, \phi) \)

With the above considerations in mind, let us start with a general class of higher-order-scalar-tensor theories in four dimensions given by the action

\[
A = \int d^4x \sqrt{-g} \left[ F(R, \Box R, \Box^2 R, ..., \Box^k R, \phi) + \omega(\phi) \phi, \phi^\alpha + X L_m \right], \tag{1.19}
\]

where \( F \) is an unspecified function of curvature invariants and of a scalar field \( \phi \). The term \( L_m \), as above, is the minimally coupled ordinary matter contribution and \( \omega(\phi) \) is a generic function of the scalar field \( \phi \). For example its values could be \( \omega(\phi) = \pm 1, 0 \) fixing the nature and the dynamics of the scalar field which can be a standard scalar field, a phantom field or a field without dynamics (see [112, 171] for details).

In the metric approach, the field equations are obtained by varying (1.19) with respect to \( g_{\mu\nu} \). We get

\[
\hat{F} G_{\mu\nu} - \frac{1}{2} g_{\mu\nu} (\hat{F} - \hat{\Box} R) - \hat{\Box}_{\mu\nu} + g_{\mu\nu} \Box \hat{F} + g_{\mu\nu} \left[ (\Box^{i-1} R)^{\alpha \beta} \frac{\partial F}{\partial \Box^i R} \right]_{;\alpha} +
\]

\[
- \frac{1}{2} \sum_{i=1}^{k} \sum_{j=1}^{i} \left[ g_{\mu\nu} (\Box^{j-i})^{\alpha \beta} \left( \Box^{i-j} \frac{\partial F}{\partial \Box^i R} \right)_{;\alpha} + (\Box^{j-i})_{,\mu} \left( \Box^{i-j} \frac{\partial F}{\partial \Box^i R} \right)_{;\mu} \right]_{;\alpha} +
\]

\[
- \frac{\omega(\phi)}{2} \phi, \phi^\alpha g_{\mu\nu} + \omega(\phi) \phi, \phi, \phi, \phi = X T_{\mu\nu} \tag{1.20}
\]

where \( G_{\mu\nu} \) is the above Einstein tensor (1.11) and
1.2 The Extended Theories of Gravity: $F(R, \Box R, ..., \Box^k R, \phi)$

$$\dot{F} = \sum_{j=0}^{n} \Box^j \frac{\partial F}{\partial \Box^j R}.$$  \hfill (1.21)

The differential Equations (1.20) are of order $(2k + 4)$. The (eventual) contribution of a potential $V(\phi)$ is contained in the definition of $F$. By varying with respect to the scalar field $\phi$, we obtain the Klein - Gordon equation

$$\Box \phi = \frac{1}{2} \frac{\delta \ln \omega(\phi)}{\delta \phi} \phi_{,\alpha} \phi^{,\alpha} + \frac{1}{2 \omega(\phi)} \frac{\delta F(R, \Box R, \Box^2 R, ..., \Box^k R, \phi)}{\delta \phi}.$$  \hfill (1.22)

Several approaches can be used to deal with such equations. For example, as we said, by a conformal transformation, it is possible to reduce an ETG to a (multi) scalar - tensor theory of gravity [23, 40, 48, 132, 172].

The simplest extension of GR is achieved assuming

$$F = f(R), \quad \omega(\phi) = 0,$$  \hfill (1.23)

in the action (1.19); $f$ is an arbitrary (analytic) function of the Ricci curvature scalar $R$. Then

$$A^f = \int d^4 x \sqrt{-g} \left[ f + \chi L_m \right]$$  \hfill (1.24)

where the standard Hilbert-Einstein action is, of course, recovered for $f = R$. Varying the (1.24) with respect to $g_{\mu\nu}$, we get the field equations

$$H_{\mu\nu} = f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} - f'_{,\mu\nu} + g_{\mu\nu} \Box f' = \chi T_{\mu\nu},$$  \hfill (1.25)

which are fourth-order equations due to the terms $f'_{,\mu\nu}$ and $\Box f'$; the prime indicates the derivative with respect to $R$. The trace of (1.25) is

$$H = g^{\alpha\beta} H_{\alpha\beta} = 3 \Box f' + f' R - 2 f = \chi T.$$  \hfill (1.26)
The peculiar behavior of \( f = R \) is due to the particular form of the Lagrangian itself which, even though it is a second order Lagrangian, can be non-covariantly rewritten as the sum of a first order Lagrangian plus a pure divergence term. The Hilbert-Einstein Lagrangian can be in fact recast as follows:

\[
L_{HE} = \sqrt{-g}R = \sqrt{-g}g^{\alpha\beta}(\Gamma^\rho_{\alpha\sigma}\Gamma^\sigma_{\rho\beta} - \Gamma^\rho_{\rho\sigma}\Gamma^\sigma_{\alpha\beta}) + \nabla_\sigma(\sqrt{-g}g^{\alpha\beta}u^\sigma_{\alpha\beta});
\]  

(1.27)

\( \Gamma \) is the Levi-Civita connection of \( g \) and \( u^\sigma_{\alpha\beta} \) is a quantity constructed out with the variation of \( \Gamma \) [1, 162]. Since \( u^\sigma_{\alpha\beta} \) is not a tensor, the above expression is not covariant; however a standard procedure has been studied to recast covariance in the first order theories [173]. This clearly shows that the field equations should consequently be second order and the Hilbert-Einstein Lagrangian is thus degenerate.

From the action (1.19), it is possible to obtain another interesting case by choosing

\[
F = F(\phi)R + V(\phi),
\]  

(1.28)

where \( V(\phi) \) and \( F(\phi) \) are generic functions describing respectively the potential and the coupling of a scalar field \( \phi \). In this case, we get

\[
A^{ST} = \int d^4x \sqrt{-g}[F(\phi)R + \omega(\phi)\frac{\partial\phi}{\partial x}\frac{\partial\phi}{\partial x} + V(\phi) + XL_m].
\]  

(1.29)

The Brans-Dicke theory of gravity is a particular case of the action (1.29) in which we have \( V(\phi) = 0 \) and \( \omega(\phi) = -\frac{\omega_{BD}}{\phi} \). In fact we have

\[
A^{BD} = \int d^4x \sqrt{-g} \left[ \phi R - \omega_{BD} \frac{\partial \phi}{\partial x} \frac{\partial \phi}{\partial x} + XL_m \right].
\]  

(1.30)

The variation of (1.29) with respect to \( g_{\mu\nu} \) and \( \phi \) gives the second-order field equations.
Let us now introduce conformal transformations to show that any higher-order or scalar-tensor theory, in absence of ordinary matter, e.g. a perfect fluid, is conformally equivalent to an Einstein theory plus minimally coupled scalar fields. If standard matter is present, conformal transformations allow to transfer non-minimal coupling to the matter component \[175\]. The conformal transformation on the metric \( g_{\mu\nu} \) is

\[
\tilde{g}_{\mu\nu} = A(x^\lambda)g_{\mu\nu}
\]

with \( A(x^\lambda) > 0 \). \( A \) is the conformal factor. Obviously the transformation rule for the contravariant metric tensor is \( \tilde{g}^{\mu\nu} = A^{-1}g^{\mu\nu} \). The various mathematical quantities in the so-called Einstein frame (EF) (quantities referred to \( \tilde{g}_{\mu\nu} \)) are linked to the ones in the so-called Jordan Frame (JF) (quantities referred to \( g_{\mu\nu} \)) as follows:

The third equation in (1.31) is the trace of field equation for \( g_{\mu\nu} \) and the last one is a combination of the trace and of the one for \( \phi \). This last equation is equivalent to the Bianchi contracted identity \[174\]. Standard fluid matter can be treated as above.
where \( \phi \equiv \ln A^{1/2} \). But we can have also the inverse relations

\[
\begin{align*}
\Gamma^\alpha_{\mu\nu} &= \Gamma^\alpha_{\mu\nu} + \phi_{,\mu} \delta^\alpha_\nu - \phi_{,\nu} \delta^\alpha_\mu + \phi^{,\alpha} g_{\mu\nu} \\
\tilde{R}^\alpha_{\mu\beta\nu} &= R^\alpha_{\mu\beta\nu} + \delta^\alpha_\beta (\phi_{,\mu} + \phi_{,\nu} + g_{\mu\nu} \phi^{,\sigma} \phi_{,\sigma}) - \delta^\alpha_\nu (\phi_{,\mu} + \phi_{,\nu} + g_{\mu\nu} \phi^{,\sigma} \phi_{,\sigma}) + \\
&\quad + g_{\mu\beta} (\phi^{,\alpha}_\beta - \phi^{,\alpha}_\beta) + g_{\mu\beta} (\phi^{,\alpha}_\nu - \phi^{,\alpha}_\nu) \\
\tilde{R}_{\mu\nu} &= R_{\mu\nu} - 2 \phi_{,\mu} + 2 \phi_{,\nu} - g_{\mu\nu} \Box \phi - 2 g_{\mu\nu} \tilde{\phi}_{,\sigma} \phi^{,\sigma} \\
\tilde{G}_{\mu\nu} &= G_{\mu\nu} - 2 \phi_{,\mu} + 2 \phi_{,\nu} + 2 g_{\mu\nu} \Box \phi + g_{\mu\nu} \tilde{\phi}_{,\sigma} \phi^{,\sigma} \\
\end{align*}
\]

where \( \Box \) and \( \tilde{\Box} \) are the d’Alembert operators with respect to the metric \( g_{\mu\nu} \) and \( \tilde{g}_{\mu\nu} \). The transformation between the operators is \( \tilde{\Box} = e^{2\phi} \Box - 2 \phi^{,\nu} \partial_\nu \).

Under these transformations, the action in (1.29) can be reformulated as follows
\[ A_{EF}^{ST} = \int d^4x \sqrt{-\tilde{g}} \left[ \Lambda \tilde{R} + \Omega(\varphi)\varphi_\alpha^\alpha + W(\varphi) + \mathcal{X}\tilde{L}_m \right]. \tag{1.35} \]

in which \( \tilde{R} \) is the Ricci scalar relative to the metric \( \tilde{g} \) and \( \Lambda \) is a generic constant. The relations between the quantities in two frames are

\[
\begin{align*}
\Omega(\varphi)d\varphi^2 &= \Lambda \left[ \frac{\omega(\phi)}{F(\phi)} - \frac{3}{2} \left( \frac{d \ln F(\phi)}{d \phi} \right)^2 \right] d\phi^2 \\
W(\varphi) &= \frac{\Lambda^2}{F(\phi(\varphi))^2} V(\phi(\varphi)) \\
\tilde{L}_m &= \frac{\Lambda^2}{F(\phi(\varphi))^2} L_m \left( \frac{\Lambda \tilde{g}_{\mu\nu}}{F(\phi(\varphi))} \right) \\
F(\phi)A(x^\lambda)^{-1} &= \Lambda
\end{align*}
\tag{1.36}
\]

The field equations for the new fields \( \tilde{g}_{\mu\nu} \) and \( \varphi \) are

\[
\begin{align*}
\Lambda \tilde{G}_{\mu\nu} - \frac{1}{2} W(\varphi) \tilde{g}_{\mu\nu} + \Omega(\varphi) \left[ \varphi_{;\mu} \varphi_{;\nu} - \frac{1}{2} \varphi_{;\alpha} \varphi^{;\alpha} \tilde{g}_{\mu\nu} \right] &= \mathcal{X} \tilde{T}_{\mu\nu} \\
2\Omega(\varphi) \tilde{\square} \varphi - \Omega,_{\varphi}(\varphi) \varphi_\alpha^\alpha - W_{,\varphi}(\varphi) &= \mathcal{X} \tilde{L}_{m,\varphi} \\
\tilde{R} &= -\frac{1}{\Lambda} \left( \mathcal{X} \tilde{T}^\varphi + 2W(\varphi) + \Omega(\varphi) \tilde{g}_{\sigma\tau} \varphi;_{;\sigma} \varphi;_{;\tau} \right)
\end{align*}
\tag{1.37}
\]

Therefore, every non-minimally coupled scalar-tensor theory, in absence of ordinary matter, e.g. perfect fluid, is conformally equivalent to an Einstein theory, being the conformal transformation and the potential suitably defined by (1.36). The converse is also true: for a given \( F(\phi) \), such that is valid the relations (1.36), we can transform a standard Einstein theory into a non-minimally coupled scalar-tensor theory. This means that, in principle, if we are able to solve the field equations in the framework of the Einstein theory in presence of a scalar field with a given potential, we should be able to get the solutions for the scalar-tensor theories, assigned by the coupling \( F(\phi) \), via the conformal transformation (1.32) with the constraints given by (1.36). Following the standard terminology, the “Einstein frame” is the framework of the Einstein theory with the minimal
coupling and the “Jordan frame” is the framework of the non-minimally coupled theory [175].

This procedure can be extended to more general theories. If the theory is assumed to be higher than fourth order, we may have Lagrangian densities of the form [40, 158],

\[ L = L(R, □R, ..., □^k R). \]

Every □ operator introduces two further terms of derivation into the field equations. For example, a theory like

\[ L = \sqrt{-g} R □R, \]

is a sixth-order theory and the above approach can be pursued by considering a conformal factor of the form

\[ A = \left| \frac{\partial (R □R)}{\partial R} + □ \frac{\partial (R □R)}{\partial □R} \right|. \]

In general, increasing two orders of derivation in the field equations (i.e. for every term □R), corresponds to adding a scalar field in the conformally transformed frame [40]. A sixth-order theory can be reduced to an Einstein theory with two minimally coupled scalar fields; a 2n-order theory can be, in principle, reduced to an Einstein theory plus (n – 1) scalar fields. On the other hand, these considerations can be directly generalized to higher - order - scalar - tensor theories in any number of dimensions as shown in [47].

The analogy between scalar-tensor gravity and HOG, although mathematically straightforward, requires a careful physical analysis. Recasting fourth - order gravity as a scalar - tensor theory, often the following steps, in terms of a generic scalar field \( \phi \), are considered

\[ f + L_m \rightarrow f'(\phi)R + f(\phi) - f'(\phi)\phi + L_m \rightarrow f'(\phi)R + V(\phi) + L_m, \]

where, by analogy, \( \phi \rightarrow R \) and the ”potential” is \( V(\phi) = f(\phi) - f'(\phi)\phi \). Clearly the kinetic term is not present so that (1.41) is usually referred as a scalar-tensor description of \( f \) - gravity where \( \omega(\phi) = 0 \). This is the so-called O’Hanlon Lagrangian [33]:
1.3 Conformal transformations

| $\mathcal{L}_{HOG}$  | $\mathcal{L}_{ST}$  | $\mathcal{L}_{HE} + \varphi$ |
|---------------------|--------------------|-----------------------------|
| $\downarrow$        | $\downarrow$       | $\downarrow$                |
| HOG Eqs.            | ST Eqs.            | Einstein Eqs. + $\varphi$   |
| $\uparrow$          | $\uparrow$         | $\uparrow$                  |
| HOG Solutions       | ST Solutions       | Einstein Solutions          |

Table 1.1: Summary of the three approaches: Scalar-Tensor (ST), Einstein +$\varphi$, and $f$ and their relations at Lagrangians, field equations and solutions levels. The solutions are in the Einstein frame for the minimally coupled case while they are in Jordan frame for $f$ and ST - gravity. Clearly, $f$ and ST theories can be rigorously compared only recasting them in the Einstein frame.

\[
\mathcal{A}^{OH} = \int d^4x \sqrt{-g} \left[ \phi R + V(\phi) + \mathcal{L}_m \right]. \tag{1.42}
\]

As concluding remarks, we can say that conformal transformations work at three levels: 

i) on the Lagrangian of the given theory;

ii) on the field equations;

iii) on the solutions. The table 1.1 summarizes the situation for HOG, non-minimally coupled scalar-tensor theories (ST) and standard Hilbert-Einstein (HE) theory. Clearly, direct and inverse transformations correlate all the steps of the table but no absolute criterion, at this point of the discussion, is able to select which is the “physical” framework since, at least from a mathematical point of view, all the frames are equivalent [175].

However, the typical Brans-Dicke action is the (1.30) where no scalar field potential is present and $\omega_{BD}$ is a constant, while the O’Hanlon Lagrangian (1.42) has a potential but has no kinetic term. The most general situation is in (1.29) where we have non-minimal coupling, kinetic term, and scalar field potential. This means that fourth-order gravity and scalar tensor gravity can be “compared” only by means of conformal transformations where kinetic and potential terms are preserved. In particular, it is misleading to state that PPN - limit of HOG is not working since these models provide $\omega_{BD} = 0$ and this is in contrast with observations [135][136].

Scalar-tensor theories and $f$-theories can be rigorously compared, after conformal transformations, in the Einstein frame where both kinetic and potential terms are present.
1.4 The Palatini Approach and the Intrinsic Conformal Structure

As we said, the Palatini approach, considering $g$ and $\Gamma$ as independent fields, is “intrinsically” bi-metric and capable of disentangling the geodesic structure from the chronological structure of a given manifold. Starting from these considerations, conformal transformations assume a fundamental role in defining the affine connection which is merely “Levi - Civita” only for the Hilbert-Einstein theory.

In this section, we work out examples showing how conformal transformations assume a fundamental physical role in relation to the Palatini approach in ETGs [176].

Let us start from the case of fourth-order gravity where Palatini variational principle is straightforward in showing the differences with Hilbert-Einstein variational principle, involving only metric. Besides, cosmological applications of $f$-gravity have shown the relevance of Palatini formalism, giving physically interesting results with singularity - free solutions [20, 64, 65, 66, 67, 68, 69]. This last nice feature is not present in the standard metric approach.

An important remark is in order at this point. The Ricci scalar entering in $f$ is $R \equiv R(g, \Gamma) = g^{\alpha\beta}R_{\alpha\beta}(\Gamma)$ that is a generalized Ricci scalar and $R_{\mu\nu}(\Gamma)$ is the Ricci tensor of a torsion-less connection $\Gamma$, which, a priori, has no relations with the metric $g$ of spacetime. The gravitational part of the Lagrangian is controlled by a given real analytical function of one real variable $f$, while $\sqrt{-g}$ denotes a related scalar density of weight 1. Field equations, deriving from the Palatini variational principle are:

$$\begin{align*}
&f'R_{(\mu\nu)}(\Gamma) - \frac{1}{2}f g_{\mu\nu} = \mathcal{X} T_{\mu\nu} \\
&\nabla_{\alpha}^{\Gamma}(\sqrt{-g} f'g^{\mu\nu}) = 0
\end{align*}$$

where $\nabla^{\Gamma}$ is the covariant derivative with respect to $\Gamma$. We shall use the standard notation denoting by $R_{(\mu\nu)}$ the symmetric part of $R_{\mu\nu}$, i.e. $R_{(\mu\nu)} \equiv \frac{1}{2}(R_{\mu\nu} + R_{\nu\mu})$.

In order to get the first one of (1.43), one has to additionally assume that $\mathcal{L}_{m}$ is functionally independent of $\Gamma$; however it may contain metric covariant derivatives $\nabla_{\alpha}^g$ of fields. This means that the matter stress-energy tensor $T_{\mu\nu} = T_{\mu\nu}(g, \Psi)$ depends on the metric $g$ and some matter fields denoted here by $\Psi$, together with their derivatives (covariant derivatives with respect to the Levi - Civita connection of $g$). From the second one of (1.43) one sees that $\sqrt{-g} f'g^{\mu\nu}$ is a symmetric twice contravariant tensor density of weight 1. As previously discussed in [176,177], this naturally
leads to define a new metric \( h_{\mu\nu} \), such that the following relation holds:

\[
\sqrt{-g} f' g^{\mu\nu} = \sqrt{-h} h^{\mu\nu}.
\]  
(1.44)

This \textit{ansatz} is suitably made in order to impose \( \Gamma \) to be the Levi-Civita connection of \( h \) and the only restriction is that \( \sqrt{-g} f' g^{\mu\nu} \) should be non-degenerate. In the case of Hilbert-Einstein Lagrangian, it is \( f' = 1 \) and the statement is trivial.

Eq. (1.44) imposes that the two metrics \( h \) and \( g \) are conformally equivalent. The corresponding conformal factor can be easily found to be \( f' \) (in dim \( \mathcal{M} = 4 \)) and the conformal transformation results to be ruled by:

\[
h_{\mu\nu} = f' g_{\mu\nu}
\]  
(1.45)

Therefore, as it is well known, Eq. (1.43) implies that \( \Gamma = \Gamma_{LC}(h) \) and \( R_{(\mu\nu)}(\Gamma) = R_{\mu\nu}(h) \equiv R_{\mu\nu} \). Field equations can be supplemented by the scalar-valued equation obtained by taking the trace of (1.43)

\[
f' R - 2f = \lambda T
\]  
(1.46)

which controls solutions of (1.43).

We shall refer to this scalar-valued equation as \textit{the structural equation} of the spacetime. In the vacuum case (and spacetimes filled with radiation, such that \( T = 0 \)) this scalar-valued equation admits constant solutions, which are different from zero only if one add a cosmological constant. This means that the universality of Einstein field equations holds \( [177] \), corresponding to a theory with cosmological constant \( [83] \).

In the case of interaction with matter fields, the structural equation (1.45), if explicitly solvable, provides an expression of \( R = R(T) \) and consequently both \( f \) and \( f' \) can be expressed in terms of \( T \). The matter content of spacetime thus rules the bi-metric structure of spacetime and, consequently, both the geodesic and metric structures which are intrinsically different. This behavior generalizes the vacuum case and corresponds to the case of a time-varying cosmological constant. In other words, due to these features, conformal transformations, which allow to pass from a metric structure to another one, acquire an intrinsic physical meaning since “select” metric and geodesic structures which, for a given ETG, in principle, \textit{do not} coincide.
Let us now try to extend the above formalism to the case of non-minimally coupled scalar-tensor theories. The effort is to understand if and how the bi-metric structure of spacetime behaves in this cases and which could be its geometric and physical interpretation.

We start by considering scalar-tensor theories in the Palatini formalism, calling $A_1$ the action functional. After, we take into account the case of decoupled non-minimal interaction between a scalar-tensor theory and a $f$-theory, calling $A_2$ this action functional. We finally consider the case of non-minimal-coupled interaction between the scalar field $\phi$ and the gravitational fields $(g, \Gamma)$, calling $A_3$ the corresponding action functional. Particularly significant is, in this case, the limit of low curvature $R$. This resembles the physical relevant case of present values of curvatures of the Universe and it is important for cosmological applications.

The action (1.29) for scalar-tensor gravity can be generalized, in order to better develop the Palatini approach, as:

$$A_1 = \int d^4x\sqrt{-g}[F(\phi)R + \omega(\phi) \nabla_\mu \phi \nabla^\mu \phi + V(\phi) + \mathcal{X} \mathcal{L}_m(\Psi, \nabla_\Psi)]$$

(1.47)

As above, the values of $\omega(\phi) = \pm 1$ selects between standard scalar field theories and quintessence (phantom) field theories. The relative “signature” can be selected by conformal transformations. Field equations for the gravitational part of the action are, respectively for the metric $g$ and the connection $\Gamma$:

$$\begin{cases}
F(\phi)[R_{\mu\nu} - \frac{R}{2} g_{\mu\nu}] = \mathcal{X} T^\mu_\nu + \frac{1}{2} \omega(\phi) \nabla_\mu \phi \nabla^\mu \phi g_{\mu\nu} + \frac{1}{2} V(\phi) g_{\mu\nu} \\
\nabla_\alpha(\sqrt{-g} F(\phi) g^{\mu\nu}) = 0
\end{cases}$$

(1.48)

$R_{\mu\nu}$ is the same defined in (1.43). For matter fields we have the following field equations:

$$\begin{cases}
2\omega(\phi) \Box \phi + \omega_\phi(\phi) \nabla_\alpha \phi \nabla^\alpha \phi + V_\phi(\phi) + F_\phi(\phi) R = 0 \\
\frac{\delta \mathcal{L}_m}{\delta \Psi} = 0
\end{cases}$$

(1.49)

In this case, the structural equation of spacetime implies that:
1.4 The Palatini Approach and the Intrinsic Conformal Structure

\[ R = -\mathcal{X}T + 2\omega(\phi) \frac{\partial}{\partial \phi} \partial^\alpha \phi \partial_\alpha \phi + 2V(\phi) \]  

(1.50)

which expresses the value of the Ricci scalar curvature in terms of the traces of the stress-energy tensors of standard matter and scalar field (we have to require \( F(\phi) \neq 0 \)). The bi-metric structure of spacetime is thus defined by the ansatz:

\[ \sqrt{-g} F(\phi) g_{\mu\nu} = \sqrt{-h} h_{\mu\nu} \]  

(1.51)

such that \( g \) and \( h \) result to be conformally related

\[ h_{\mu\nu} = F(\phi) g_{\mu\nu} \]  

(1.52)

The conformal factor is exactly the interaction factor. From (1.50), it follows that in the vacuum case \( T = 0 \) and \( \omega(\phi) \partial_\alpha \phi \partial^\alpha \phi + V(\phi) = 0 \): this theory is equivalent to the standard Einstein one without matter. On the other hand, for \( F(\phi) = F_0 \) we recover the Einstein theory plus a minimally coupled scalar field: this means that the Palatini approach intrinsically gives rise to the conformal structure (1.52) of the theory which is trivial in the Einstein, minimally coupled case.

As a further step, let us generalize the previous results considering the case of a non-minimal coupling in the framework of \( f \)-theories. The action functional can be written as:

\[ \mathcal{A}_2 = \int d^4x \sqrt{-g} [F(\phi)f(R) + \omega(\phi) \partial_\alpha \phi \partial^\alpha \phi + V(\phi) + 2\mathcal{X}\mathcal{L}_m(\Psi, \partial_\mu \Psi)] \]  

(1.53)

where \( f \) is, as usual, any analytical function of the Ricci scalar \( R \). Field equations (in the Palatini formalism) for the gravitational part of the action are:

\[
\begin{align*}
F(\phi)[f'R_{\mu\nu} &- \frac{f'}{2} g_{\mu\nu}] = \mathcal{X}T_{\mu\nu} + \frac{1}{2} \omega(\phi) \partial_\alpha \phi \partial^\alpha \phi g_{\mu\nu} + \frac{1}{2} V(\phi) g_{\mu\nu} \\
\partial_\alpha (\sqrt{-g} F(\phi) f' g^{\mu\nu} ) &= 0 
\end{align*}
\]  

(1.54)

For scalar and matter fields we have, otherwise, the following field equations:
\[ 2\omega(\phi) \Box \phi + \omega_{,\phi}(\phi) \nabla_\alpha \phi \nabla^\alpha \phi + V_{,\phi}(\phi) + F_{,\phi}(\phi)f(R) = 0 \]  
\[ \delta \mathcal{L}_m = 0 \]  
where the non-minimal interaction term enters into the modified Klein-Gordon equations. In this case the structural equation of spacetime implies that:

\[ f' R - 2f = \frac{\chi T + 2\omega(\phi) \nabla_\alpha \phi \nabla^\alpha \phi + 2V(\phi)}{F(\phi)} \]

We remark again that this equation, if solved, expresses the value of the Ricci scalar curvature in terms of traces of the stress-energy tensors of standard matter and scalar field (we have to require again that \( F(\phi) \neq 0 \)). The bi-metric structure of spacetime is thus defined by the ansatz:

\[ \sqrt{-g}F(\phi)f'g^{\mu \nu} = \sqrt{-h}h^{\mu \nu} \]

such that \( g \) and \( h \) result to be conformally related by:

\[ h_{\mu \nu} = F(\phi)f'g_{\mu \nu} \]

Once the structural equation is solved, the conformal factor depends on the values of the matter fields \((\phi, \Psi)\) or, more precisely, on the traces of the stress-energy tensors and the value of \( \phi \). From equation (1.56), it follows that in the vacuum case, i.e. both \( T = 0 \) and \( \omega(\phi) \nabla_\alpha \phi \nabla^\alpha \phi + V(\phi) = 0 \), the universality of Einstein field equations still holds as in the case of minimally interacting \( f \)-theories [177]. The validity of this property is related to the decoupling of the scalar field and the gravitational field.

Let us finally consider the case where the gravitational Lagrangian is a general function of \( \phi \) and \( R \). The action functional can thus be written as:

\[ \mathcal{A}_3 = \int d^4x \sqrt{-g}[K(\phi, R) + \omega(\phi) \nabla_\alpha \phi \nabla^\alpha \phi + V(\phi) + \chi \mathcal{L}_m(\Psi, \nabla \Psi)] \]
1.4 The Palatini Approach and the Intrinsic Conformal Structure

Field equations for the gravitational part of the action are:

\[
\left\{ \frac{\partial K(\phi, R)}{\partial R} R_{\mu\nu} - \frac{K(\phi, R)}{2} g_{\mu\nu} = X T_{\mu\nu} + \frac{1}{2} \omega(\phi) g^{\alpha} \nabla_\alpha \phi \nabla^\alpha \phi g_{\mu\nu} + \frac{1}{2} V(\phi) g_{\mu\nu} \right\} 
\]

(1.60)

For matter fields, we have:

\[
\left\{ 2 \omega(\phi) \Box \phi + \omega,\phi(\phi) \nabla_\alpha \phi \nabla^\alpha \phi + V,\phi(\phi) + \frac{\partial K(\phi, R)}{\partial \phi} = 0 \right\} 
\]

(1.61)

The structural equation of spacetime can be expressed as:

\[
\nabla^\alpha \left[ \sqrt{-g} \left( \frac{\partial K(\phi, R)}{\partial R} \right) g^{\mu\nu} \right] = 0
\]

The structural equation of spacetime can be expressed as:

\[
\frac{\partial K(\phi, R)}{\partial R} R - 2K(\phi, R) = X T + 2\omega(\phi) \nabla_\alpha \phi \nabla^\alpha \phi + 2V(\phi)
\]

(1.62)

This equation, if solved, expresses again the form of the Ricci scalar curvature in terms of traces of the stress-energy tensors of matter and scalar field (we have to impose regularity conditions and, for example, \(K(\phi, R) \neq 0\)). The bi-metric structure of spacetime is thus defined by the ansatz:

\[
\sqrt{-g} \frac{\partial K(\phi, R)}{\partial R} g^{\mu\nu} = \sqrt{-h} h^{\mu\nu}
\]

(1.63)

such that \(g\) and \(h\) result to be conformally related by

\[
h_{\mu\nu} = \frac{\partial K(\phi, R)}{\partial R} g_{\mu\nu}.
\]

(1.64)

Again, once the structural equation is solved, the conformal factor depends just on the values of the matter fields and (the trace of) their stress energy tensors. In other words, the evolution, the definition of the conformal factor and the bi-metric structure is ruled by the values of traces of the stress-energy tensors and by the value of the scalar field \(\phi\). In this case, the universality of Einstein field equations does not hold anymore in general. This is evident from (1.62) where the
strong coupling between $R$ and $\phi$ avoids the possibility, also in the vacuum case, to achieve simple constant solutions.

We consider, furthermore, the case of small values of $R$, corresponding to small curvature spacetimes. This limit represents, as a good approximation, the present epoch of the observed Universe under suitably regularity conditions. A Taylor expansion of the analytical function $K(\phi, R)$ can be performed:

$$K(\phi, R) = K_0(\phi) + K_1(\phi) R + o(R^2)$$  \hspace{1cm} (1.65)

where only the first leading term in $R$ is considered and we have defined:

$$\begin{align*}
K_0(\phi) &= K(\phi, R)_{R=0} \\
K_1(\phi) &= \left( \frac{\partial K(\phi, R)}{\partial R} \right)_{R=0}
\end{align*}$$  \hspace{1cm} (1.66)

Substituting this expression in (1.62) and (1.64) we get (neglecting higher order approximations in $R$) the structural equation and the bi-metric structure in this particular case. From the structural equation, we get:

$$R = -T + 2\omega(\phi) \nabla_\alpha \phi \nabla^\alpha \phi + 2V(\phi) + 2K_0(\phi)$$  \hspace{1cm} (1.67)

such that the value of the Ricci scalar is always determined, in this first order approximation, in terms of $\omega(\phi) \nabla_\alpha \phi \nabla^\alpha \phi + V(\phi)$, $T$ e $\phi$. The bi-metric structure is, otherwise, simply defined by means of the first term of the Taylor expansion, which is

$$h_{\mu\nu} = K_1(\phi) g_{\mu\nu}.$$  \hspace{1cm} (1.68)

It reproduces, as expected, the scalar-tensor case (1.52). In other words, scalar-tensor theories can be recovered in a first order approximation of a general theory where gravity and non-minimal couplings are any (compare (1.67) with (1.50)). This fact agrees with the above considerations where Lagrangians of physical interactions can be considered as stochastic functions with local
gauge invariance properties [44].

Finally we have to say that there are also bi-metric theories which cannot be conformally related (see for example the summary of alternative theories given in [129]) and torsion field should be taken into account, if one wants to consider the most general viewpoint [13, 178]. We will not take into account these general theories in this review.

After this short review of ETGs in metric and Palatini approach, we are going to face some remarkable applications to cosmology and astrophysics. In particular, we deal with the straightforward generalization of GR, the \( f \)-gravity, showing that, in principle, no further ingredient, a part a generalized gravity, could be necessary to address issues as missing matter (dark matter) and cosmic acceleration (dark energy). However what we are going to consider here are nothing else but toy models which are not able to fit the whole expansion history, the structure growth law and the CMB anisotropy and polarization. These issues require more detailed theories which, up to now, are not available but what we are discussing could be a useful working paradigm as soon as refined experimental tests to probe such theories will be proposed and pursued. In particular, we will outline an independent test, based on the stochastic background of gravitational waves, which could be extremely useful to discriminate between ETGs and GR or among the ETGs themselves. In this latter case, the data delivered from ground-based interferometers, like VIRGO and LIGO, or the forthcoming space interferometer LISA, could be of extreme relevance in such a discrimination.

Finally, we do not take into account the well known inflationary models based on ETGs (e.g. [45]) since we want to show that also the last cosmological epochs, directly related to the so called Precision Cosmology, can be framed in such a new ”economic” scheme.

\section{The general \( f \) – theory}

Let \( f \) be an analytic function of Ricci scalar \( R \). We can formulate a HOG starting from the action principle (1.24). By varying the action (1.24) and by using the properties (1.18) we get the field equations:
\[ \delta A = \delta \int d^4x \sqrt{-g} [f + \mathcal{X} \mathcal{L}_m] = \]
\[ = \int d^4x \sqrt{-g} \left( f' R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - \mathcal{X} T_{\mu\nu} \right) \delta g^{\mu\nu} + g_{\mu\nu} f' \delta R^{\mu\nu} \]
\[ = \int d^4x \sqrt{-g} \left( f' R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - \mathcal{X} T_{\mu\nu} \right) \delta g^{\mu\nu} + \]
\[ + f' \left[ - (\delta g^{\mu\nu})_{;\mu\nu} - \Box (g^{\mu\nu} \delta g_{\mu\nu}) \right] \sim \]
\[ \sim \int d^4x \sqrt{-g} \left( f' R_{\mu\nu} - \frac{f}{2} g_{\mu\nu} - \mathcal{X} T_{\mu\nu} - f' \delta g_{\mu\nu} + g_{\mu\nu} \Box f' \right) \delta g^{\mu\nu} = \]
\[ = \int d^4x \sqrt{-g} (H_{\mu\nu} - \mathcal{X} T_{\mu\nu}) \delta g^{\mu\nu} = 0 \] (1.69)

where the symbol \( \sim \) means that we neglected a pure divergence; then we obtain the field equation (1.25). Eq. (1.25) satisfies the condition \( H^{\alpha\mu} ;_{\alpha} = \mathcal{X} T^{\alpha\mu} ;_{\alpha} = 0 \). In fact it is easy to check that

\[ H^{\alpha\mu} ;_{\alpha} = f'_\alpha R^{\alpha\mu} + f' R^{\alpha\mu} ;_{\alpha} - \frac{1}{2} f^{t\mu} - f'^{\alpha\mu} + f'^{\alpha} ;_{\alpha} = \]
\[ f'' R^{\alpha\mu} R ;_{\alpha} - f'^{t\alpha} R^{\alpha} ;_{\alpha} = \]
\[ f'' R^{\alpha\mu} R ;_{\alpha} - f'^{t\alpha} R^{\alpha} ;_{\alpha} = \]
\[ f'^{t} R^{\alpha\mu} R ;_{\alpha} = 0 ; \] (1.70)

where we used the properties \( G^{\alpha\mu} ;_{\alpha} = 0 \) and \( [\nabla^\mu, \nabla^\alpha] f'^{t\alpha} = -f'^{t\alpha} R^{\mu} ;_{\alpha} \). If we develop the covariant derivatives in (1.25) and in (1.26) we obtain the complete expression for a generic \( f \)-theory

\[ \begin{cases} 
H_{\mu\nu} = f' R_{\mu\nu} - \frac{1}{2} f g_{\mu\nu} + \mathcal{H}_{\mu\nu} = \mathcal{X} T_{\mu\nu} \\
H = f' R - 2f + \mathcal{H} = \mathcal{X} T 
\end{cases} \] (1.71)

where the two quantities \( \mathcal{H}_{\mu\nu} \) and \( \mathcal{H} \) read:
1.6 The field equations for the $R_{\alpha\beta}R^{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ — invariants

The technicality is ever the same one. We start from the action principle for a Lagrangian densities $\sqrt{-g} R_{\alpha\beta}R^{\alpha\beta}$ and $\sqrt{-g} R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}$ and we get their field equations:

\[
\mathcal{H}_{\mu\nu} = -f''\left\{ R_{\mu\nu} - \Gamma_{\mu\nu}^{\sigma} R_{\sigma} - g_{\mu\nu} \left[ \left( g^{\sigma\tau} \ln \sqrt{-g_{\sigma\tau}} \right) R_{\sigma\tau} + g^{\sigma\tau} R_{\sigma\tau} \right] \right\} - f''' \left( R_{\mu} R_{\nu} - g_{\mu\nu} g^{\sigma\tau} R_{\sigma\tau} R_{\sigma\tau} \right)
\]

\[
\mathcal{H} = 3f'' \left[ \left( g^{\sigma\tau} R_{\sigma\tau} + g^{\sigma\tau} \ln \sqrt{-g_{\sigma\tau}} \right) R_{\sigma\tau} + g^{\sigma\tau} R_{\sigma\tau} \right] + 3f''' g^{\sigma\tau} R_{\sigma\tau} R_{\sigma\tau}
\]

$\Gamma_{\mu\nu}^{\sigma}$ are the standard Christoffel’s symbols defined by (1.3). We conclude, then, this paragraph having shown the most general expression of field equations of $f$-gravity in metric formalism.
\[\delta A = \delta \int d^4x \sqrt{-g} [R_{\alpha \beta} R^{\alpha \beta} + X \mathcal{L}_m] = \]
\[= \delta \int d^4x \sqrt{-g} [R_{\alpha \beta} g^{\alpha \rho} g^{\beta \sigma} R_{\rho \sigma} + X \mathcal{L}_m] = \]
\[= \int d^4x \sqrt{-g} \left( 2 R^{\alpha \beta} R_{\alpha \beta} - \frac{R_{\alpha \beta} R^{\alpha \beta}}{2} g_{\mu \nu} - X T_{\mu \nu} \right) \delta g^{\mu \nu} + \]
\[+ 2 R^\mu_\nu \delta R^\mu_\nu \right] = \]
\[= \int d^4x \sqrt{-g} \left( 2 R^{\alpha \beta} R_{\alpha \beta} - \frac{R_{\alpha \beta} R^{\alpha \beta}}{2} g_{\mu \nu} - X T_{\mu \nu} \right) \delta g^{\mu \nu} + \]
\[+ R^\mu_\nu (2 g^{\rho \sigma} \delta g_{\rho (\mu \nu) \sigma} - \Box \delta g_{\mu \nu} - g^{\rho \sigma} \delta g_{\rho \mu \nu}) \right) \sim \]
\[\sim \int d^4x \sqrt{-g} \left( 2 R^{\alpha \beta} R_{\alpha \beta} - \frac{R_{\alpha \beta} R^{\alpha \beta}}{2} g_{\mu \nu} - X T_{\mu \nu} \right) \delta g^{\mu \nu} + \]
\[+ \Box R_{\mu \nu} \delta g^{\mu \nu} + R^\sigma_{\mu \nu \sigma} g_{\mu \nu} \delta g^{\mu \nu} \right] = \]
\[= \int d^4x \sqrt{-g} \left( 2 R^{\alpha \beta} R_{\alpha \beta} - \frac{R_{\alpha \beta} R^{\alpha \beta}}{2} g_{\mu \nu} - 2 R^\sigma_{(\mu \nu) \sigma} + \right. \]
\[+ \Box R_{\mu \nu} + g_{\mu \nu} R^\sigma_{(\mu \nu) \sigma} - X T_{\mu \nu} \left. \right) \delta g^{\mu \nu} = 0. \quad (1.73) \]

Then, the field equations are

\[H^{Ric}_{\mu \nu} = 2 R^{\alpha \beta} R_{\alpha \beta} - \frac{R_{\alpha \beta} R^{\alpha \beta}}{2} g_{\mu \nu} - 2 R^\sigma_{(\mu \nu) \sigma} + \Box R_{\mu \nu} + g_{\mu \nu} R^\sigma_{(\mu \nu) \sigma} = X T_{\mu \nu} \quad (1.74) \]

and the trace is

\[H^Ric = 2 \Box R = X T, \quad (1.75) \]

where we used the Bianchi identity contracted (1.6).

Let us calculate the field equations for the \( R_{\alpha \beta \gamma \delta} R^{\alpha \beta \gamma \delta} \) - invariant:
1.6 The field equations for the $R_{\alpha\beta} R^{\alpha\beta}$ and $R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}$ — invariants

\[ \delta A = \delta \int d^4 x \sqrt{-g} \left[ R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + \mathcal{L}_m \right] = \]
\[ = \delta \int d^4 x \sqrt{-g} \left[ R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta} + \mathcal{L}_m \right] = \]
\[ = \int d^4 x \sqrt{-g} \left[ \left( 4 R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma} - \frac{R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}}{2} g_{\mu\nu} - \mathcal{L}_{\mu\nu} \right) \delta g^{\mu\nu} + + 2 R^{\alpha\beta\gamma\delta} \delta R_{\alpha\beta\gamma\delta} \right] = \]
\[ = \int d^4 x \sqrt{-g} \left[ \left( 4 R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma} - \frac{R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}}{2} g_{\mu\nu} - \mathcal{L}_{\mu\nu} \right) \delta g^{\mu\nu} + + R^{\alpha\beta\gamma\delta} \left( \delta g_{\alpha\beta;\gamma} + \delta g_{\alpha\delta;\beta} - \delta g_{\beta\delta;\alpha} - \delta g_{\alpha\beta;\gamma} - \delta g_{\alpha\gamma;\beta} + \delta g_{\beta\gamma;\alpha} \right) \right] \sim \]
\[ \sim \int d^4 x \sqrt{-g} \left[ 2 R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma} - \frac{R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}}{2} g_{\mu\nu} - 4 R_{\mu \nu \alpha\beta} - \mathcal{L}_{\mu\nu} \right] \delta g^{\mu\nu} = 0 \quad (1.76) \]

We used the expressions

\[ \begin{cases} 
\delta R_{\alpha\beta\gamma\delta} = \delta (g_{\alpha\sigma} R^{\sigma}_{\beta\gamma\delta}) = R^{\sigma}_{\beta\gamma\delta} \delta g_{\alpha\sigma} + g_{\alpha\sigma} \delta R^{\sigma}_{\beta\gamma\delta} \\
\delta R^{\sigma}_{\beta\gamma\delta} = \frac{1}{2} \left( \delta g_{\alpha\beta;\gamma} + \delta g_{\alpha\delta;\beta} - \delta g_{\beta\delta;\alpha} - \delta g_{\alpha\beta;\gamma} - \delta g_{\alpha\gamma;\beta} + \delta g_{\beta\gamma;\alpha} \right) \end{cases} \quad (1.77) \]

Then, the field equations, from (1.76), are

\[ H^{Rie}_{\mu\nu} = 2 R_{\mu\alpha\beta\gamma} R^{\alpha\beta\gamma} - \frac{R_{\alpha\beta\gamma\delta} R^{\alpha\beta\gamma\delta}}{2} g_{\mu\nu} - 4 R_{\mu \nu \alpha\beta} = \mathcal{L}_{\mu\nu} \quad (1.78) \]

and the trace is

\[ H^{Rie} = -4 R_{\gamma \alpha\beta} = \mathcal{L}. \quad (1.79) \]
1.7 Generalities on spherical symmetry

Since we are interested in understanding the modifications of predictions of GR when one considers a concentration of matter in the space, it is fundamental requiring particular properties of metric $g_{\mu\nu}$. The first step, and also the easiest, we will study, in the next chapters, the gravitational potential generated by spherically symmetric matter distribution (point-like and not) and the choice of mathematical form of metric becomes very important. Starting from the matter spherically symmetric we expect also the metric has the same symmetries.

We conclude this chapter showing the principal relations between some coordinates systems we will use in this PhD thesis.

The most general spherically symmetric metric\footnote{The metric is spherically symmetric if it depends only on $\mathbf{x}$ and $d\mathbf{x}$ only through the rotational invariants $d\mathbf{x}^2$, $\mathbf{x} \cdot d\mathbf{x}$ and $\mathbf{x}^2$.} can be written as follows:

$$ds^2 = g_1(t, |\mathbf{x}|) \, dt^2 + g_2(t, |\mathbf{x}|) \, dt \cdot d\mathbf{x} + g_3(t, |\mathbf{x}|)(\mathbf{x} \cdot d\mathbf{x})^2 + g_4(t, |\mathbf{x}|)d|\mathbf{x}|^2$$  \hspace{1cm} (1.80)

where $g_i$ are functions of the distance $|\mathbf{x}|$ and of the time $t$. The set of coordinates is $x^\mu = (t, x^1, x^2, x^3)$. The scalar product is defined as usual form: $\mathbf{x} \cdot d\mathbf{x} = x^1 dx^1 + x^2 dx^2 + x^3 dx^3$. By spherically symmetric form of (1.80) it is convenient to replace $\mathbf{x}$ with spherical polar coordinates $r, \theta, \phi$ defined as usual by

$$x^1 = r \sin \theta \cos \phi, \quad x^2 = r \sin \theta \sin \phi, \quad x^3 = r \cos \phi.$$  \hspace{1cm} (1.81)

The proper time interval (1.80) then becomes

$$ds^2 = g_1(t, r) \, dt^2 + r g_2(t, r) \, dt \, dr + r^2 g_3(t, r) \, dr^2 + g_4(t, r)(dr^2 + r^2 d\Omega),$$  \hspace{1cm} (1.82)

where $d\Omega = d\theta^2 + \sin^2 \theta d\phi^2$ is the solid angle. We are free to reset our clocks by defining the time coordinate

$$t = t' + \zeta(t', r),$$  \hspace{1cm} (1.83)
with \( \zeta(t', r) \) an arbitrary function of \( t' \) and \( r \). This allows us to eliminate the off-diagonal element \( g_{tr} \) in the metric (1.82) by setting

\[
\frac{d\zeta(t', r)}{dr} = -\frac{rg_2(t', r)}{2g_1(t', r)},
\]

the metric (1.82) becomes

\[
ds^2 = g_1(t', r) \left[ 1 + \frac{d\zeta(t', r)}{dt'} \right]^2 dt'^2 + \left[ r^2 g_3(t', r) - \frac{r^2 g_2(t', r)^2}{4g_1(t', r)} + g_4(t', r) \right] dr^2 +
\]

\[
+ g_4(t', r) r^2 d\Omega,
\]

where if we introduce a new metric coefficients \( g_{tt}(t', r) \), \( g_{rr}(t', r) \) and \( g_{\Omega\Omega}(t', r) \) we can recast the (1.85) as follows

\[
ds^2 = g_{tt}(t', r) dt'^2 - g_{rr}(t', r) dr'^2 - g_{\Omega\Omega}(t', r) d\Omega;
\]

if we introduce a new radial coordinate \( (r') \) by considering a further transformation

\[
r' = (\text{const}) e^\int dr' \sqrt{\frac{g_{rr}(t', r)}{g_{\Omega\Omega}(t', r)}}
\]

it is possible to recast Eq. (1.86) into the isotropic form (isotropic coordinates)

\[
ds^2 = g_{tt}(t', r') dt'^2 - g_{ij}(t', r') dx^i dx^j;
\]

and then it is possible also to choose \( g_{\Omega\Omega}(t', r) = r'^2 \) (this condition allows us to obtain the standard definition of the circumference with radius \( r'' \)) and to have the metric (1.86) in the standard form (standard coordinates)

\[
ds^2 = g_{tt}(t', r'') dt'^2 - g_{rr}(t', r'') dr'^2 - r'^2 d\Omega.
\]
Obviously the functions $g_{tt}(t', r'')$ and $g_{rr}(t', r'')$ are not the same of (1.86). If we suppose $g_{ij}(t', r') = Y(t', r')\delta_{ij}$ we note that it is possible pass from (1.88) to (1.89) by the coordinate transformations:

$$r' = r'(r'') = (\text{const})e^{\int dr'' \sqrt{\tilde{g}^{rr}}}.$$  

(1.90)

We can, then, affirm that the expressions (1.86), (1.88) and (1.89) are equivalent to the metric (1.80) and we can consider them without loss of generality as the most general definitions of a spherically symmetric metric compatible with a pseudo-Riemannian manifold without torsion. The choice of the form of the metric is only a practical issue. With this hypothesis, by inserting these metrics into the field equations (1.25), one obtains:

$$H_{\mu\nu} = f'R_{\mu\nu} - \frac{1}{2}fg_{\mu\nu} - f''\left\{ R_{\mu\nu} - \Gamma^t_{\mu\nu}R_t - \Gamma^r_{\mu\nu}R_r - g_{\mu\nu}\left[ \left( g_{tt}R_{tt} + g_{rr}R_{rr} \right) + \Gamma^t_{tt}R_{tt} + \Gamma^t_{rt}R_{rt} + \Gamma^r_{rr}R_{rr} \right] - 3f'\left[ R_{rt}R_{rr} - g_{\mu\nu}\left( g_{tt}R_{tt}^2 + g_{rr}R_{rr}^2 \right) \right] \right\}$$

(1.91)

$$H = f'R - 2f + 3f''\left[ g_{tt}R_{tt} + g_{rr}R_{rr} \right] + 3f'\left[ g_{tt}R_{tt}^2 + g_{rr}R_{rr}^2 \right]$$

Eqs. (1.91) are the starting-point for the next chapter. All our studies in the next chapters are referred ever to field equations (1.91), except the second part of sixth chapter where we have to insert in the field equations also the contribution of $R_{\alpha\beta}R^{\alpha\beta}$-invariant (1.74).

We conclude having shown the most general spherically symmetric metric tensor for our aim and before starting from third chapter with a systematic study of $f$-gravity we want to stop, in the second chapter, to consider the principal spherically symmetric solutions in GR. Some of these solutions will be the starting-point to find, with perturbative methods, the corrections induced by $f$-theory.
Chapter 2

Exact and perturbative solutions in General Relativity

In this chapter we show, starting from knowledge of outcomes of GR, the mathematical tools needed for aims of present thesis. First of all we present the particular spherically symmetric solutions in GR and consequent Birkhoff theorem (§ 2.1). In §2.3 we show the technicality of development of field equations with respect to Newtonian and Post-Newtonian approach [C]. Finally, in §2.4 we perform the post-Minkowskian limit: the gravitational waves. The developments are computed in generic coordinates systems and in the gauge harmonic.

The $f$-gravity theory, from mathematical point-view, is more complicated than GR. Giving the exact solutions of Eqs. (1.25) is vary hard challenge. Nevertheless, known the basic solutions of GR, we can try to find new solutions by requiring the Newtonian and post-Newtonian limit approach. This approach is very useful when we consider the astrophysical problems or the study of planet motion in the Solar System. An another field of comparison between GR and $f$-gravity is possible in the post-Minkowskian regime. In this case we can study the propagation of gravitational field induced by $f$-gravity.

We dedicate, then, this chapter to understanding the outcomes of GR and to showing all mathematical tools needed for the next chapters.
2.1 The Schwarzschild, Schwarzschild – de Sitter and Reissner – Nordstrom solutions: the Birkhoff theorem in General Relativity

We can rewrite the metric (1.89) as follows

\[ ds^2 = e^{\nu(t,r)} dt^2 - e^{\mu(t,r)} dr^2 - r^2 d\Omega , \]  

(2.1)

where we recalled the radial coordinate. The only nonvanishing components of metric tensor \( g_{\mu\nu} \) are

\[ g_{tt} = e^{\nu(t,r)} , \quad g_{rr} = -e^{\mu(t,r)} , \quad g_{\theta\theta} = -r^2 , \quad g_{\phi\phi} = -r^2 \sin^2 \theta \]  

(2.2)

with functions \( \mu(t, r) \) and \( \nu(t, r) \) that are to be determined by solving the field equations in GR (1.10). Since \( g_{\mu\nu} \) is diagonal, it is easy to write down all the nonvanishing components of its inverse:

\[ g^{tt} = e^{-\nu(t,r)} , \quad g^{rr} = -e^{-\mu(t,r)} , \quad g^{\theta\theta} = -r^{-2} , \quad g^{\phi\phi} = -r^{-2} \sin^{-2} \theta . \]  

(2.3)

Furthermore, the determinant of the metric tensor is

\[ g = -e^{\mu(t,r)+\nu(t,r)} r^4 \sin^2 \theta \]  

(2.4)

so the invariant volume element is

\[ \sqrt{-g} \ dr \ d\theta \ d\phi = r^2 e^{-\mu(t,r)+\nu(t,r)/2} \sin \theta \ dr \ d\theta \ d\phi . \]  

(2.5)

The only nonvanishing components of symbols Christoffel (1.3) are
2.1 The Schwarzschild, Schwarzschild − de Sitter and Reissner − Nordstrom solutions: the Birkhoff theorem in General Relativity

\[
\begin{align*}
\Gamma_{tt}^t &= \frac{\dot{\nu}(t,r)}{2}, & \Gamma_{rr}^r &= \frac{\rho'(t,r)}{2}, & \Gamma_{tr}^r &= \frac{\nu'(t,r)}{2}e^{\nu(t,r) - \mu(t,r)}, \\
\Gamma_{rr}^t &= \frac{\mu'(t,r)}{2}e^{\mu(t,r) - \nu(t,r)}, & \Gamma_{rr}^t &= \frac{\nu'(t,r)}{2}, & \Gamma_{rr}^r &= \frac{e^{\nu(t,r)}}{2}, \\
\Gamma_{\theta\theta}^r &= -r e^{-\mu(t,r)}, & \Gamma_{r\theta}^\theta &= \Gamma_{r\phi}^\phi = \frac{1}{r}, & \Gamma_{\phi\phi}^r &= -r e^{-\mu(t,r)} \sin^2 \theta, \\
\Gamma_{\phi\phi}^\theta &= -\sin \theta \cos \theta, & \Gamma_{\phi\phi}^\phi &= \cot \theta,
\end{align*}
\]

(2.6)

and the field equations (1.10) become

\[
\begin{align*}
\frac{1}{r^2} - e^{-\mu(t,r)} \left[ \frac{1}{r^2} - \frac{\mu'(t,r)}{r} \right] &= \mathcal{X} T^t_t \\
\frac{\dot{\mu}(t,r)}{r}e^{-\mu(t,r)} &= \mathcal{X} T^r_t \\
\frac{1}{r^2} - e^{-\mu(t,r)} \left[ \frac{\nu'(t,r)}{r} + \frac{1}{r^2} \right] &= \mathcal{X} T^r_r \\
\frac{e^{-\nu(t,r)}}{2} \left[ \dot{\mu}(t,r) + \frac{\mu^2(t,r)}{2} - \frac{\mu(t,r)\nu(t,r)}{2} \right] + \\
-\frac{e^{-\mu(t,r)}}{2} \left[ \nu''(t,r) + \frac{\nu^2(t,r)}{2} \right. &+ \left. \frac{\nu'(t,r) - \mu'(t,r)}{r} - \frac{\nu'(t,r)\mu'(t,r)}{2} \right] &= \mathcal{X} T^\theta_\theta = \mathcal{X} T^\phi_\phi 
\end{align*}
\]

(2.7)

and if we suppose a tensor of matter like \( T_{\mu\nu} = \rho u_\mu u_\nu \) with \( \rho = M \delta(x) \) the density of matter time-independent we obtain the so-called Schwarzschild solution in standard coordinates:

\[
ds^2 = \left[ 1 - \frac{r_g}{r} \right] dt^2 - \frac{dr^2}{1 - \frac{r_g}{r}} - r^2 d\Omega
\]

(2.8)

where \( r_g = 2GM \) is the so called Schwarzschild radius.

Metric (2.8) determines completely the gravitational field in the vacuum generated by a spherically matter density distribution. Furthermore the Schwarzschild solution is valid also when we consider a moving source with a spherical distribution. The spatial metric is determined by expression of spatial distance element
\[ dl^2 = \frac{dr^{''2}}{1 - \frac{r_g}{r^{''}}} + r^{''2} d\Omega. \] 

(2.9)

We have to note that, while the length of circumference with "radius" \( r^{''} \) is the usual one \( 2\pi r^{''} \), the distance between two points on the same radius is given by the integral

\[
\int_{r^{''}_1}^{r^{''}_2} \frac{dr^{''}}{\sqrt{1 - \frac{r_g}{r^{''}}} > r^{''}_2 - r^{''}_1;}
\]

(2.10)

then the space is curved. Besides we note that \( g_{tt} \leq 1 \), then, by the relation between the time coordinate \( t \) and the proper time \( \tau \) (\( d\tau = \sqrt{g_{tt}} dt \)), we get the condition

\[ d\tau \leq dt. \] 

(2.11)

At infinity, the time coordinate coincides with physical time. We can state that when we are at a finite distance from the the mass, there is a slowdown of the time with respect to the time measured at infinity.

In presence of matter the situation is the following. In fact from the first equation in the (2.7), when \( r \to 0 \), \( \mu(t, r) \) has to vanish as \( r^2 \); otherwise \( T^t_t \) could have a singular point in the origin. By integrating formally the equation with the condition \( \mu(t, r)|_{r=0} = 0 \), one get

\[
\mu(t, r) = - \ln \left[ 1 - \frac{\mathcal{X}}{r} \int_{0}^{r} T^t_t \hat{r}^2 d\hat{r} \right].
\]

(2.12)

It is easy to demonstrate also in the matter with spherical symmetry that the proprieties (2.10), (2.11) and \( \mu(t, r) + \nu(t, r) \leq 0 \) are verified [162]. If the gravitational field is created by spherical body with "radius" \( \xi \), we have \( T^t_t = 0 \) outside the body \( r > \xi \) and we can write

\[
\mu_{\xi}(t, r) = - \ln \left[ 1 - \frac{\mathcal{X}}{r} \int_{0}^{\xi} T^t_t \hat{r}^2 d\hat{r} \right]
\]

(2.13)

and obtain the analogous expression of (2.8) in the matter:
2.1 The Schwarzschild, Schwarzschild – de Sitter and Reissner – Nordstrom solutions: the Birkhoff theorem in General Relativity

\[ ds^2 = \left[ 1 - \frac{r_g(r'')}{r''} \right] dt^2 - \frac{dr''^2}{1 - \frac{r_g(r'')}{r''}} - r''^2 d\Omega, \]  

(2.14)

where we introduced the Schwarzschild radius linked to the quantity of matter included in the sphere with radius \( r'' \):

\[ r_g(r'') = \mathcal{X} \int_0^{r''} T^t_t \hat{r}^2 d\hat{r}; \]  

(2.15)

obviously when the distance is bigger than the radius of the body, the metric (2.14) is equal to (2.8).

If we consider the transformation (1.90), which in the case of Schwarzschild solution is

\[ r' = \frac{2r'' - r_g + 2\sqrt{r''^2 - r_g r''}}{4}, \]  

(2.16)

it is possible to obtain the Schwarzschild solution (2.8) in isotropic coordinates:

\[ ds^2 = \left[ \frac{1 - \frac{r_g}{r''}}{1 + \frac{r_g}{4r''}} \right]^2 dt^2 - \left[ 1 + \frac{r_g}{4r''} \right]^4 (dr'^2 + r'^2 d\Omega). \]  

(2.17)

In both cases, the solutions (2.8) and (2.14) agree with the trace equation of Einstein equation: \( R = -\mathcal{X} T \). Since in the vacuum the trace of matter tensor is vanishing (except the origin, in which the trace is proportional to \( \delta(x) \)) we can state that the Schwarzschild solution is "Ricci flat": \( R = 0 \).

If we add in the Hilbert - Einstein lagrangian (1.13) a term like \( -2\sqrt{-g} \Lambda \) with \( \Lambda \) a generic constant the field equations (1.10) are modified as follows

\[ G_{\mu\nu} + \Lambda g_{\mu\nu} = \mathcal{X} T_{\mu\nu}, \]  

(2.18)

and if we consider a point-like source, we find the Schwarzschild - de Sitter solution
\[ ds^2 = \left[ 1 - \frac{r_g}{r'^n} + \frac{\Lambda}{3} r''^2 \right] dt^2 - \frac{dr'^2}{1 - \frac{r_g}{r'} + \frac{\Lambda}{3} r''^2} - r''^2 d\Omega. \] 

(2.19)

In this case the trace of (2.18) is

\[ R = 4\Lambda - \mathcal{X} \mathcal{T} \] 

(2.20)

from which we note that this solution does not admit solution in the vacuum, since also in absence of ordinary matter \( T_{\mu\nu} = 0 \) we have a nonvanishing scalar curvature. The contribution is given by cosmological constant \( \Lambda \). It is also possible in this case to find the analogous of (2.14).

Finally let us consider as source a radial and static electric field \( \mathbf{E} = \frac{Q}{|x|} / |x|^3 \). We know that the Lagrangian of electromagnetic field is \( -\frac{1}{4\pi} F_{\alpha\beta} F^{\alpha\beta} \) where \( F_{\alpha\beta} \) is the electromagnetic tensor. Then, the Hilbert - Einstein lagrangian is

\[ \mathcal{L}_{HE} = \sqrt{-g} \left( R - \frac{1}{4\pi} F_{\alpha\beta} F^{\alpha\beta} \right), \] 

(2.21)

and the Einstein equation (1.10) becomes

\[ G_{\mu\nu} = -\frac{1}{8\pi} \left( g_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta} - 4F_{\mu\alpha} F^{\alpha}_{\nu} \right). \] 

(2.22)

The solution for a spherically symmetric system is the Reissner - Nordstrom solution:

\[ ds^2 = \left[ 1 - \frac{r_g}{r'^n} + \frac{Q^2}{r'^2} \right] dt^2 - \frac{dr'^2}{1 - \frac{r_g}{r'} + \frac{Q^2}{r'^2}} - r'^2 d\Omega. \] 

(2.23)

In all the above cases shown the Birkhoff theorem holds: The metric tensor generated in vacuum by a matter density distribution with a spherical symmetry is time-independent. Also a time-dependent source with a spherical symmetry produces a static metric. The curvature of spacetime in the matter, a distance \( r \) from the origin, is proportional only to the matter inside the sphere of radius \( r \). This conclusion is compatible with the Gauss theorem of classical mechanics.

One of the goal of the present thesis is to develop similar considerations in the case of \( f \)-gravity.
2.2 Perturbations of the Schwarzschild solution: The Eddington parameters $\beta$ and $\gamma$

The Schwarzschild solution (2.17) is a mathematically exact solution and is true everywhere. But in some cases the physical conditions could permit a "reduction" of them. In fact the Schwarzschild radius $r_g$ is a scale-length induced by theory. Then we could be at radial distance $r' \ll 1$ for which we have $r_g / r' \ll 1$ and the (2.17) becomes

$$ds^2 \simeq \left[ 1 - \frac{r_g}{r'} + \frac{1}{2} \left( \frac{r_g}{r'} \right)^2 + \ldots \right] dt^2 - \left[ 1 + \frac{r_g}{r'} + \ldots \right] \left[ dr'^2 + r'^2 d\Omega \right].$$  \hspace{1cm} (2.24)

Since we are interested to investigate the deviations, induced by $f$-gravity, from behavior (2.24) it is useful to introduce the method taking into account such deviations with respect to GR. A standard approach is the Parameterized-Post-Newtonian (PPN) expansion of the Schwarzschild metric (2.17). Eddington parameterized deviations with respect to GR, considering a Taylor series in term of $r_g / r'$ assuming that in Solar System, the limit $r_g / r' \ll 1$ holds [129]. The resulting metric is

$$ds^2 \simeq \left[ 1 - \frac{r_g}{r'} + \frac{\beta}{2} \left( \frac{r_g}{r'} \right)^2 + \ldots \right] dt^2 - \left[ 1 + \gamma \frac{r_g}{r'} + \ldots \right] \left[ dr'^2 + r'^2 d\Omega \right],$$  \hspace{1cm} (2.25)

where $\alpha$, $\beta$ and $\gamma$ are unknown dimensionless parameters (Eddington parameters) which parameterize deviations with respect to GR. The reason to carry out this expansion up to the order $(r_g / r')^2$ in $g_{tt}$ and only to the order $(r_g / r')$ in $g_{ij}$ is that, in applications to celestial mechanics, $g_{ij}$ always appears multiplied by an extra factor $v^2 \sim M / r'$. It is evident that the standard GR solution for a spherically symmetric gravitational system in vacuum, is obtained for $\alpha = \beta = \gamma = 1$ giving again the "perturbed" Schwarzschild solution (2.24). Actually, the parameter $\alpha$ can be settled to the unity due to the mass definition of the system itself [129]. As a consequence, the expanded metric (2.25) can be recast in the form:

$$ds^2 \simeq \left[ 1 - \frac{r_g}{r''} + \frac{\beta - \gamma}{2} \left( \frac{r_g}{r''} \right)^2 + \ldots \right] dt^2 - \left[ 1 + \gamma \frac{r_g}{r''} + \ldots \right] \left[ dr''^2 + r''^2 d\Omega \right],$$  \hspace{1cm} (2.26)

where we have restored the standard spherical coordinates by means of the transformation $r'' = \ldots$
Chapter 2 Exact and perturbative solutions in General Relativity

The two parameters $\beta$, $\gamma$ have a physical interpretation. The parameter $\gamma$ measures the amount of curvature of space generated by a body of mass $M$ at radius $r'$. In fact, the spatial components of the Riemann curvature tensor are, at post-Newtonian order,

$$R_{ijkl} = \frac{3}{2} \gamma \frac{r_g}{r^2} N_{ijkl}$$

independently of the gauge choice, where $N_{ijkl}$ represents the geometric tensor properties (e.g. symmetries of the Riemann tensor and so on). On the other side, the parameter $\beta$ measures the amount of non-linearity ($\sim (r_g/r')^2$) in the $g_{tt}$ component of the metric. However, this statement is valid only in the standard post-Newtonian gauge.

2.3 General remarks on the Newtonian and the post — Newtonian approximation of Einstein equation

At this point, it is worth discussing some general issues on the Newtonian and post-Newtonian limits. Basically there are some general features one has to take into account when approaching these limits, whatever the underlying theory of gravitation is. In fact here we are not interested in entering the theoretical discussion on how to formulate a mathematically well founded Newtonian limit (and post-Newtonian) of general relativistic field theories, nevertheless we point the interested reader to [15, 179, 180, 181, 182, 183, 184]. In this section, we provide the explicit form of the various quantities needed to compute the approximations in the field equations in GR theory and any metric theory of gravity. We only mention that there is also been a discussion on alternative ways to define the Newtonian and Post - Newtonian limit in higher-order theories in the recent literature, see for example [185]. In this work, the Newtonian and Post - Newtonian limit is identified with the maximally symmetric solution, which is not necessarily Minkowski spacetime in $f$ - theories which could be singular.

If one consider a system of gravitationally interacting particles of mass $\bar{M}$, the kinetic energy $\frac{1}{2} \bar{M} \bar{v}^2$ will be, roughly, of the same order of magnitude as the typical potential energy $U = G \bar{M}^2 / \bar{r}$, with $\bar{M}$, $\bar{r}$, and $\bar{v}$ the typical average values of masses, separations, and velocities of these particles. As a consequence:

$$\bar{v}^2 \sim \frac{G \bar{M}}{\bar{r}} ,$$
(for instance, a test particle in a circular orbit of radius $r$ about a central mass $M$ will have velocity $v$ given in Newtonian mechanics by the exact formula $v^2 = GM/r$.)

The post-Newtonian approximation can be described as a method for obtaining the motion of the system to higher approximations than the first order (approximation which coincides with the Newtonian mechanics) with respect to the quantities $\bar{G} \bar{M}/\bar{r}$ and $\bar{v}^2$ assumed small with respect to the squared light speed. This approximation is sometimes referred to as an expansion in inverse powers of the light speed.

The typical values of the Newtonian gravitational potential $\Phi$ are nowhere larger (in modulus) than $10^{-5}$ in the Solar System (in geometrized units, $\Phi$ is dimensionless). On the other hand, planetary velocities satisfy the condition $\bar{v}^2 \lesssim -\Phi$, while the matter pressure $p$ experienced inside the Sun and the planets is generally smaller than the matter gravitational energy density $-\rho \Phi$, in other words $\lbrack p/\rho \lesssim -\Phi \rbrack$. Furthermore one must consider that even other forms of energy in the Solar System (compressional energy, radiation, thermal energy, etc.) have small intensities and the specific energy density $\Pi$ (the ratio of the energy density to the rest-mass density) is related to $U$ by $\Pi \lesssim U$ ($\Pi$ is $\sim 10^{-5}$ in the Sun and $\sim 10^{-9}$ in the Earth\[129\]). As matter of fact, one can consider that these quantities, as function of the velocity, give second order contributions:

\[ -\Phi \sim v^2 \sim p/\rho \sim \Pi \sim O(2). \]  

Therefore, the velocity $v$ gives $O(1)$ terms in the velocity expansions, $U^2$ is of order $O(4)$, $Uv$ of $O(3)$, $U\Pi$ is of $O(4)$, and so on. Considering these approximations, one has

\[ \frac{\partial}{\partial t} \sim v \cdot \nabla, \]  

and

\[ \frac{|\partial/\partial t|}{|\nabla|} \sim O(1). \]  

Now, particles move along geodesics:

\[ ^1 \text{Typical values of } p/\rho \text{ are } \sim 10^{-5} \text{ in the Sun and } \sim 10^{-10} \text{ in the Earth\[129\].} \]
\[ \frac{d^2 x^\mu}{ds^2} + \Gamma^\mu_{\sigma\tau} \frac{dx^\sigma}{ds} \frac{dx^\tau}{ds} = 0 , \quad (2.32) \]

which can be written in details as

\[ \frac{d^2 x^i}{dt^2} = -\Gamma^i_{tt} - 2\Gamma^i_{tm} \frac{dx^m}{dt} - \Gamma^i_{mn} \frac{dx^m}{dt} \frac{dx^n}{dt} + \left[ \Gamma^t_{tt} + 2\Gamma^t_{tm} \frac{dx^m}{dt} + 2\Gamma^t_{mn} \frac{dx^m}{dt} \frac{dx^n}{dt} \right] \frac{dx^i}{dt} . \quad (2.33) \]

In the Newtonian approximation, that is vanishingly small velocities and only first-order terms in the difference between \( g_{\mu\nu} \) and the Minkowski metric \( \eta_{\mu\nu} \), one obtains that the particle motion equations reduce to the standard result :

\[ \frac{d^2 x^i}{dt^2} \simeq -\Gamma^i_{tt} \simeq -\frac{1}{2} \frac{\partial g_{tt}}{\partial x^i} . \quad (2.34) \]

The quantity \( 1 - g_{tt} \) is of order \( G\bar{M}/\bar{r} \), so that the Newtonian approximation gives \( \frac{d^2 x^i}{dt^2} \) to the order \( G\bar{M}/\bar{r}^2 \), that is, to the order \( \bar{v}^2/\bar{r} \). As a consequence if we would like to search for the post-Newtonian approximation, we need to compute \( \frac{d^2 x^i}{dt^2} \) to the order \( \bar{v}^4/\bar{r} \). Due to the Equivalence Principle and the differentiability of spacetime manifold, we expect that it should be possible to find out a coordinate system in which the metric tensor is nearly equal to the Minkowski one \( \eta_{\mu\nu} \), the correction being expandable in powers of \( G\bar{M}/\bar{r} \sim \bar{v}^2 \). In other words one has to consider the metric developed as follows :

\[ \begin{align*}
g_{tt}(t, \mathbf{x}) & \simeq 1 + g^{(2)}_{tt}(t, \mathbf{x}) + g^{(4)}_{tt}(t, \mathbf{x}) + O(6) \\
g_{ti}(t, \mathbf{x}) & \simeq g^{(3)}_{ti}(t, \mathbf{x}) + O(5) \\
g_{ij}(t, \mathbf{x}) & \simeq -\delta_{ij} + g^{(2)}_{ij}(t, \mathbf{x}) + O(4)
\end{align*} \quad (2.35) \]

where \( \delta_{ij} \) is the Kronecker delta, and for the controvariant form of \( g_{\mu\nu} \), one has
2.3 General remarks on the Newtonian and the post–Newtonian approximation of Einstein equation

\[
\begin{align*}
  g^{tt}(t, x) &\simeq 1 + g^{(2)tt}(t, x) + g^{(4)tt}(t, x) + O(6) \\
g^{ti}(t, x) &\simeq g^{(3)ti}(t, x) + O(5) \\
g^{ij}(t, x) &\simeq -\delta_{ij} + g^{(2)ij}(t, x) + O(4)
\end{align*}
\] (2.36)

The inverse of the metric tensor (2.35) is defined by (1.1). The relations among the higher than first order terms turn out to be

\[
\begin{align*}
  g^{(2)tt}(t, x) &= -g^{(2) tt}(t, x) \\
g^{(4) tt}(t, x) &= g^{(2) tt}(t, x)^2 - g^{(4) tt}(t, x) \\
g^{(3) ti} &= g^{(3) ti} \\
g^{(2) ij}(t, x) &= -g^{(2) ij}(t, x)
\end{align*}
\] (2.37)

In evaluating $\Gamma^\mu_{\alpha\beta}$ we must take into account that the scale of distance and time, in our systems, are respectively set by $\bar{r}$ and $\bar{r}/\bar{v}$, thus the space and time derivatives should be regarded as being of order

\[
\frac{\partial}{\partial x^i} \sim \frac{1}{\bar{r}}, \quad \frac{\partial}{\partial t} \sim \frac{\bar{v}}{\bar{r}}.
\] (2.38)

Using the above approximations (2.35), (2.36) and (2.37) we have, from the definition (1.3),
\[
\left\{
\begin{aligned}
\Gamma^{(3)}_{tt} &= \frac{1}{2} g_{tt,t} \\
\Gamma^{(2)}_{jk} &= \frac{1}{2} \left( g_{jk,i} - g_{ij,k} - g_{ik,j} \right) \\
\Gamma^{(3)}_{ij} &= \frac{1}{2} \left( g_{ii,j} + g_{jj,i} - g_{ij,t} \right) \\
\Gamma^{(4)}_{tj} &= \frac{1}{2} \left( g_{ji,j} + g_{ij,2} - 2 g_{ij,t} \right) \\
\Gamma^{(4)}_{ti} &= \frac{1}{2} \left( g_{it,i} + g_{2i,t} - 2 g_{it,t} \right)
\end{aligned}
\right.
\] (2.39)

The Ricci tensor components (1.4) are

\[
\left\{
\begin{aligned}
R^{(2)}_{tt} &= \frac{1}{2} g_{tt,mm} \\
R^{(4)}_{tt} &= \frac{1}{2} g_{tt,mm} + \frac{1}{2} g_{mm,ng_{tt,n}} + \frac{1}{2} g_{mn,ng_{tt,mn}} + \frac{1}{2} g_{mm,tt} - \frac{1}{4} g_{mm,ng_{tt,mn}} - \frac{1}{4} g_{mn,ng_{tt,n}} - g_{tt,nn} \\
R^{(3)}_{ti} &= \frac{1}{2} g_{ti,mm} - \frac{1}{2} g_{mm,mt} - \frac{1}{2} g_{mt,mi} + \frac{1}{2} g_{mm,ti} \\
R^{(2)}_{ij} &= \frac{1}{2} g_{ij,mm} - \frac{1}{2} g_{mm,mj} - \frac{1}{2} g_{jm,mi} - \frac{1}{2} g_{tt,ij} + \frac{1}{2} g_{mm,ij}
\end{aligned}
\right.
\] (2.40)

and the Ricci scalar (1.5) is
2.3 General remarks on the Newtonian and the post – Newtonian approximation of Einstein equation

\[
\begin{align*}
R^{(2)} &= R^{(2)}_{tt} - R^{(2)}_{mm} = g^{(2)}_{tt,mm} - g^{(2)}_{mm,mm} + g^{(2)}_{mm,mm} \\
R^{(4)} &= R^{(4)}_{tt} - g^{(2)}_{tt} R^{(2)}_{tt} - g^{(2)}_{mm} R^{(2)}_{mn} = \\
&= \frac{1}{2} g^{(4)}_{tt,mm} + \frac{1}{2} g^{(2)}_{mn,mm} g^{(2)}_{tt,n} + \frac{1}{2} g^{(2)}_{mn,mm} g^{(2)}_{tt,m} - \frac{1}{4} g^{(2)}_{tt,tt} g^{(2)}_{mm,mm} + \\
&\quad - g^{(2)}_{nt,lm} g^{(2)}_{mm,tt} - g^{(2)}_{nt,lm} g^{(2)}_{mm,tt} - \frac{1}{2} g^{(2)}_{mm,tt} \left( g^{(2)}_{mm,tt} - g^{(2)}_{mm,tt} \right) \\
\end{align*}
\]

(2.41)

The Einstein tensor components (1.11) are

\[
\begin{align*}
G^{(2)}_{tt} &= R^{(2)}_{tt} - \frac{1}{2} R^{(2)} = \frac{1}{2} g^{(2)}_{mm,nn} + \frac{1}{2} g^{(2)}_{mm,nn} \\
G^{(4)}_{tt} &= R^{(4)}_{tt} - \frac{1}{2} R^{(4)} - \frac{1}{2} g^{(2)}_{tt} R^{(2)} = \ldots \\
G^{(3)}_{ti} &= R^{(3)}_{ti} = \frac{1}{2} g^{(3)}_{ti,mm} + \frac{1}{2} g^{(2)}_{mm,ti} \\
G^{(2)}_{ij} &= R^{(2)}_{ij} + \frac{\delta_{ij}}{2} R^{(2)} = \frac{1}{2} g^{(2)}_{ij,mm} - \frac{1}{2} g^{(2)}_{jm,mi} - \frac{1}{2} g^{(2)}_{jm,mi} + \frac{1}{2} g^{(2)}_{mm,ij} + \\
&\quad + \frac{\delta_{ij}}{2} \left[ g^{(2)}_{tt,mm} - g^{(2)}_{mm,mm} + g^{(2)}_{mm,mm} \right] \\
\end{align*}
\]

(2.42)

By assuming the harmonic gauge

\[ g^{\rho\sigma} \Gamma^\mu_{\rho\sigma} = 0 \]

(2.43)

it is possible to simplify the components of Ricci tensor (2.40). In fact for \( \mu = 0 \) one has

---

\(^2\)The gauge transformation is \( \hat{h}_{\mu\nu} = h_{\mu\nu} - \zeta_{\mu,\nu} - \zeta_{\nu,\mu} \) when we perform a coordinate transformation as \( x'\mu = x^\mu + \zeta^\mu \) with \( O(\zeta^2) \ll 1 \). To obtain our gauge and the validity of field equation for both perturbation \( h_{\mu\nu} \) and \( \hat{h}_{\mu\nu} \) the \( \zeta_{\mu} \) have satisfy the harmonic condition \( \Box \zeta_{\mu} = 0 \).
\[ 2g^{\sigma\tau}\Gamma^t_{\sigma\tau} \approx g^{(2)}_{tt,t} - 2g^{(3)}_{tm,m} + g^{(2)}_{mm,t} = 0, \quad (2.44) \]

and for \( \mu = i \)

\[ 2g^{\sigma\tau}\Gamma^i_{\sigma\tau} \approx g^{(2)}_{tt,i} + 2g^{(2)}_{mi,m} - g^{(2)}_{mm,i} = 0. \quad (2.45) \]

Differentiating Eq.(2.44) with respect to \( t, x^j \) and (2.45) and with respect to \( t \), one obtains

\[ g^{(2)}_{tt,tt} - 2g^{(3)}_{tm,mt} + g^{(2)}_{mm,tt} = 0, \quad (2.46) \]

\[ g^{(2)}_{tt,tj} - 2g^{(3)}_{mt,jm} + g^{(2)}_{mm,tj} = 0, \quad (2.47) \]

\[ g^{(2)}_{tt,ti} + 2g^{(2)}_{mi,tm} - g^{(2)}_{mm,ti} = 0. \quad (2.48) \]

On the other side, combining Eq.(2.47) and Eq.(2.48), we get

\[ g^{(2)}_{mm,ti} - g^{(2)}_{mi,tm} - g^{(2)}_{mt,mi} = 0. \quad (2.49) \]

Finally, differentiating Eq.(2.45) with respect to \( x^j \), one has:

\[ g^{(2)}_{it,ij} + 2g^{(2)}_{mi,jm} - g^{(2)}_{mm,ij} = 0 \]

and redefining indexes as \( j \rightarrow i, i \rightarrow j \) since these are mute indexes, we get

\[ g^{(2)}_{tt,ij} + 2g^{(2)}_{mj,im} - g^{(2)}_{mm,ij} = 0. \quad (2.51) \]
2.3 General remarks on the Newtonian and the post – Newtonian approximation of Einstein equation

Combining Eq. (2.50) and Eq. (2.51), we obtain

\[ g^{(2)}_{tt,ij} + g^{(2)}_{mi,jm} + g^{(2)}_{mj,im} - g^{(2)}_{mm,ij} = 0. \] (2.52)

Relations (2.46), (2.49), (2.52) guarantee us to rewrite Eqs. (2.40) as

\[
\begin{align*}
R^{(2)}_{tt}|_{HG} &= \frac{1}{2} \triangle g^{(2)}_{tt} \\
R^{(4)}_{tt}|_{HG} &= \frac{1}{2} \triangle g^{(4)}_{tt} + \frac{1}{2} g^{(2)}_{mn} g^{(2)}_{tt,mn} - \frac{1}{2} g^{(2)}_{tt,tt} - \frac{1}{2} \nabla g^{(2)}_{tt} |^2 \\
R^{(3)}_{ti}|_{HG} &= \frac{1}{2} \triangle g^{(3)}_{ti} \\
R^{(2)}_{ij}|_{HG} &= \frac{1}{2} \triangle g^{(2)}_{ij}
\end{align*}
\] (2.53)

and Eqs. (2.41) becomes

\[
\begin{align*}
R^{(2)}|_{HG} &= \frac{1}{2} \triangle g^{(2)}_{tt} - \frac{1}{2} \triangle g^{(2)}_{mm} \\
R^{(4)}|_{HG} &= \frac{1}{2} \triangle g^{(4)}_{tt} + \frac{1}{2} g^{(2)}_{mn} g^{(2)}_{tt,mn} - \frac{1}{2} g^{(2)}_{tt,tt} - \frac{1}{2} \nabla g^{(2)}_{tt} |^2 - \frac{1}{2} g^{(2)}_{tt} \triangle g^{(2)}_{tt} - \frac{1}{2} g^{(2)}_{mn} \triangle g^{(2)}_{mn}
\end{align*}
\] (2.54)

where \( \nabla \) and \( \triangle \) are, respectively, the gradient and the Laplacian in flat space. The Einstein tensor components (1.11) in the harmonic gauge are

\[
\begin{align*}
G^{(2)}_{tt}|_{HG} &= \frac{1}{4} \triangle g^{(2)}_{tt} + \frac{1}{4} \triangle g^{(2)}_{mm} \\
G^{(4)}_{tt}|_{HG} &= \ldots \\
G^{(3)}_{ti}|_{HG} &= \frac{1}{2} \triangle g^{(3)}_{ti} \\
G^{(2)}_{ij}|_{HG} &= \frac{1}{2} \triangle g^{(2)}_{ij} + \frac{\delta_{ij}}{4} \left[ \triangle g^{(2)}_{tt} - \triangle g^{(2)}_{mm} \right]
\end{align*}
\] (2.55)

On the matter side, i.e. right-hand side of the field equations (1.10), we start with the general definition of the energy-momentum tensor of a perfect fluid
\[ T_{\alpha\beta} = (\rho + \Pi \rho + p) u_\alpha u_\beta - p g_{\alpha\beta}. \]  
(2.56)

Following the procedure outlined in [155], we derive the explicit form of the energy-momentum as follows

\[
\begin{align*}
T_{tt} &= \rho + \rho(v^2 - 2U + \Pi) + \rho \left[ v^2 \left( \frac{\rho}{\rho} + v^2 + 2V + \Pi \right) + \sigma - 2\Pi U \right] \\
T_{ti} &= -\rho v^i + \rho \left[ -v^i \left( \frac{\rho}{\rho} + 2V + v^2 + \Pi \right) + h_{ti} \right] \\
T_{ij} &= \rho v^i v^j + p \delta_{ij} + \rho \left[ v^i v^j \left( \Pi + \frac{\rho}{\rho} + 4V + v^2 + 2U \right) - 2v^c \delta_{c(i} h_{0j)} + 2\frac{p}{\rho} V \delta_{ij} \right]
\end{align*}
\]  
(2.57)

We are now ready to make use of Einstein field equations (1.10), which we assume in the form

\[ R_{\mu\nu} = \kappa \left[ T_{\mu\nu} - \frac{T}{2} g_{\mu\nu} \right]. \]  
(2.58)

From their interpretation as the energy density, momentum density and momentum flux, then, we have \( T_{tt}, T_{ti} \) and \( T_{ij} \) at various order

\[
\begin{align*}
T_{tt} &= T_{tt}^{(0)} + T_{tt}^{(2)} + O(4) \\
T_{ti} &= T_{ti}^{(1)} + O(3) \\
T_{ij} &= T_{ij}^{(2)} + O(4)
\end{align*}
\]  
(2.59)

where \( T_{\mu\nu}^{(N)} \) denotes the term in \( T_{\mu\nu} \) of order \( \tilde{M}/\tilde{r}^3 \tilde{v}^N \). In particular \( T_{tt}^{(0)} \) is the density of restmass, while \( T_{tt}^{(2)} \) is the non-relativistic part of the energy density. What we need is

\[ S_{\mu\nu} = T_{\mu\nu} - \frac{T}{2} g_{\mu\nu}. \]  
(2.60)
2.3 General remarks on the Newtonian and the post-Newtonian approximation of Einstein equation

But $G\bar{M}/\bar{r}$ is of order $v^2$, so (2.35) and (2.59) give

\[
S_{tt} = S_{tt}^{(0)} + S_{tt}^{(2)} + O(6)
\]
\[
S_{ti} = S_{ti}^{(1)} + O(3)
\]
\[
S_{ij} = S_{ij}^{(0)} + O(2)
\]

where $S_{\mu\nu}^{(N)}$ denotes the term in $S_{\mu\nu}$ of order $\bar{M}/\bar{r}^3 \bar{v}^N$. In particular

\[
S_{tt}^{(0)} = \frac{1}{2} T_{tt}^{(0)}
\]
\[
S_{tt}^{(2)} = \frac{1}{2} T_{tt}^{(2)} + \frac{1}{2} T_{mm}^{(2)}
\]
\[
S_{ti}^{(1)} = T_{ti}^{(1)}
\]
\[
S_{ij}^{(0)} = \frac{1}{2} \delta_{ij} T_{tt}^{(0)}
\]

Using the (2.53) and (2.61) in the field equation (2.58) we find that the field equations in harmonic coordinates are indeed consistent with the expansions we are using, and give

\[
R_{tt}^{(2)} = \mathcal{X} S_{tt}^{(0)}
\]
\[
R_{tt}^{(4)} = \mathcal{X} S_{tt}^{(2)}
\]
\[
R_{ti}^{(3)} = \mathcal{X} S_{ti}^{(0)}
\]
\[
R_{ij}^{(2)} = \mathcal{X} S_{ij}^{(0)}
\]

and in particular
\[
\begin{aligned}
\triangle g_{tt}^{(2)} &= \mathcal{X} T_{tt}^{(0)} \\
\triangle g_{tt}^{(4)} &= \mathcal{X} \left[ T_{tt}^{(2)} + T_{mm}^{(2)} \right] - g_{mm}^{(2)} g_{tt,mm} + g_{tt,tt}^{(2)} + \left| \nabla g_{tt}^{(2)} \right|^2 \\
\triangle g_{t_i}^{(3)} &= 2 \mathcal{X} T_{t_i}^{(1)} \\
\triangle g_{i_j}^{(2)} &= \mathcal{X} \delta_{i_j} T_{tt}^{(0)}
\end{aligned}
\]

(2.64)

From the first one of (2.64), we find, as expected, the Newtonian mechanics:

\[
g_{tt}^{(2)} = -\frac{\mathcal{X}}{4\pi} \int d^3x' \frac{T_{tt}^{(0)}(x')}{|x - x'|} = -2G \int d^3x' \frac{T_{tt}^{(0)}(x')}{|x - x'|} = 2\Phi(x)
\]

(2.65)

where \(\Phi(x)\) is the gravitational potential which, in the case of point-like source with mass \(M\), is

\[
\Phi(x) = -\frac{GM}{|x|}.
\]

(2.66)

From the third and fourth equations of (2.64) we find that

\[
\begin{aligned}
g_{t_i}^{(3)} &= -\frac{\mathcal{X}}{2\pi} \int d^3x' \frac{T_{t_i}^{(1)}(x')}{|x - x'|} = Z_i(x) \\
g_{i_j}^{(2)} &= -\frac{\mathcal{X}}{4\pi} \delta_{i_j} \int d^3x' \frac{T_{i_j}^{(0)}(x')}{|x - x'|} = 2\delta_{i_j} \Phi(x)
\end{aligned}
\]

(2.67)

The second equation of (2.64) can be rewritten as follows

\[
\triangle \left[ g_{tt}^{(4)} - 2\Phi^2 \right] = \mathcal{X} \left[ T_{tt}^{(2)} + T_{mm}^{(2)} \right] - 8\Phi \triangle \Phi + 2\Phi_{tt}
\]

(2.68)

and the solution for \(g_{tt}^{(4)}\) is
2.3 General remarks on the Newtonian and the post–Newtonian approximation of Einstein equation

\[
g^{(4)}_{tt} = 2\Phi^2 - \frac{\mathcal{V}}{4\pi} \int d^3x \frac{T^{(2)}_{tt}(x') + T^{(2)}_{mm}(x')}{|x - x'|} + \frac{\mathcal{V}}{\pi} \int d^3x' \frac{\Phi(x') \Delta x' \Phi(x')}{|x - x'|} - \frac{1}{2\pi} \int d^3x' \frac{\Phi_{tt}(x')}{|x - x'|} = 2\Theta(x). \tag{2.69}
\]

By using the equations at second order we obtain the final expression for the correction at fourth order in the time-time component of the metric:

\[
\Theta(x) = \Phi(x)^2 - \frac{\mathcal{V}}{8\pi} \int d^3x \frac{T^{(2)}_{tt}(x') + T^{(2)}_{mm}(x')}{|x - x'|} + \frac{\mathcal{V}}{\pi} \int d^3x' \frac{\Phi(x') T^{(0)}_{tt}(x')}{|x - x'|} - \frac{1}{4\pi} \partial_{tt} \int d^3x' \frac{\Phi(x')}{|x - x'|}. \tag{2.70}
\]

We can rewrite the metric expression \((2.35)\) as follows

\[
g_{\mu\nu} \sim \left( 1 + 2\Phi + 2\Theta \sqrt{\mathcal{Z}^T \mathcal{Z} - \delta_{ij} (1 - 2\Phi)} \right) \tag{2.71}
\]

Finally the Lagrangian of a particle in presence of a gravitational field can be expressed as proportional to the invariant distance \(d\sqrt{s}/2\), thus we have:

\[
L = \left( g^{\rho\sigma} \frac{dx^\rho}{dt} \frac{dx^\sigma}{dt} \right)^{1/2} = \left( g_{tt} + 2g_{tm}v^m + g_{mn}v^m v^n \right)^{1/2} = \left( 1 + g^{(2)}_{tt} + g^{(4)}_{tt} + 2g^{(3)}_{tm}v^m - v^2 + g^{(2)}_{mn}v^m v^n \right)^{1/2}, \tag{2.72}
\]

which, to the \(O(2)\) order, reduces to the classic Newtonian Lagrangian of a test particle \(L_{\text{New}} = \left( 1 + 2\Phi - v^2 \right)^{1/2}\), where \(v^m = \frac{dx^m}{dt}\) and \(|v|^2 = v^m v^m\). As matter of fact, post-Newtonian physics has to involve higher than \(O(2)\) order terms in the Lagrangian. In fact we obtain

\[
L \sim 1 + \left[ \Phi - \frac{1}{2}v^2 \right] + \frac{3}{4} \left[ \Theta + Z_m v^m + \Phi v^2 \right]. \tag{2.73}
\]

An important remark concerns the odd-order perturbation terms \(O(1)\) or \(O(3)\). Since, these
terms contain odd powers of velocity $v$ or of time derivatives, they are related to the energy dissipation or absorption by the system. Nevertheless, the mass-energy conservation prevents the energy and mass losses and, as a consequence, prevents, in the Newtonian limit, terms of $O(1)$ and $O(3)$ orders in the Lagrangian. If one takes into account contributions higher than $O(4)$ order, different theories give different predictions. GR, for example, due to the conservation of post-Newtonian energy, forbids terms of $O(5)$ order; on the other hand, terms of $O(7)$ order can appear and are related to the energy lost by means of the gravitational radiation.

2.4 General remarks on the post—Minkowskian approximation of Einstein equation

We suppose the metric to be close to the Minkowski metric $\eta_{\mu\nu}$:

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$$  \hspace{1cm} (2.74)

with $h_{\mu\nu}$ small quantities ($O(h)^2 \ll 1$). To first order in $h$, the Christoffel symbols (1.3) are

$$\Gamma^\alpha_{\mu\nu} = \frac{1}{2} \eta^{\alpha\sigma}(h_{\mu\sigma,\nu} + h_{\nu\sigma,\mu} - h_{\mu\nu,\sigma}).$$  \hspace{1cm} (2.75)

As long as we restrict ourselves to first order in $h$, we must raise and lower all indices using $\eta_{\mu\nu}$, not $g_{\mu\nu}$; that is

$$\eta^{\sigma\tau}h_{\sigma\tau} = h^\sigma_\sigma = h, \quad \eta^{\sigma\tau}\frac{\partial}{\partial x^\tau} = \frac{\partial}{\partial x^\tau}, \quad \text{etc.}$$  \hspace{1cm} (2.76)

With this assumptions, the Ricci tensor and scalar (1.4) - (1.5) are then

$$\begin{cases} R^{(1)}_{\mu\nu} &= h^\sigma_{(\mu,\nu)\sigma} - \frac{1}{2} \square \eta h_{\mu\nu} - \frac{1}{2} h_{,\mu\nu} \\ R^{(1)} &= h_{\sigma\tau}^{\sigma\tau} - \square \eta h \end{cases}$$  \hspace{1cm} (2.77)

where $\nabla^\alpha \nabla_\alpha \sim \sigma^{\sigma} = \square \eta$ is the d’Alembertian operator in the flat space. The field equation (1.10) becomes
2.4 General remarks on the post – Minkowskian approximation of Einstein equation

\[ G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2} R^{(1)} \eta_{\mu\nu} = \mathcal{X} T^{(0)}_{\mu\nu} \]  

where \( T_{\mu\nu} \) is fixed at zero-order in (2.78) since in this perturbation scheme the first order on Minkowski space has to be connected with the zero order of the standard matter energy momentum tensor. Eqs. (2.78) in terms of \( h_{\mu\nu} \) are

\[ h^\sigma_{(\mu,\nu)} - \frac{1}{2} \Box \eta h_{\mu\nu} - \frac{1}{2} h_{,\mu\nu} - \frac{1}{2} h \tau^{\sigma\tau} - \square_\eta h_{\eta\mu\nu} = \mathcal{X} T^{(0)}_{\mu\nu}. \]  

(2.79)

Since \( T_{\mu\nu} \) is taken to the lowest order in \( h_{\mu\nu} \), so it is independent of \( h_{\mu\nu} \), it has to satisfies the ordinary conservation conditions:

\[ T^{\alpha\mu}_{,\alpha} = 0. \]  

(2.80)

Note that it is this form of the conservation law that is needed for the consistency of (2.79), because (2.80) implies

\[ G^{(1)}_{\mu\sigma} = 0 \]  

(2.81)

whereas the linearized Ricci tensor satisfies Bianchi identities (1.6) of the form

\[ R^{(1)}_{\sigma\mu} - \frac{1}{2} \left[ h^{\alpha\beta},_{\alpha\beta} - \Box_\eta h \right]^{,\mu} = \frac{1}{2} R^{(1)}_{,\mu}. \]  

(2.82)

By choosing the transformation \( \tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{h}{2} \eta_{\mu\nu} \) and the gauge condition \( \tilde{h}^{\mu\nu}_{,\mu} = 0 \) (harmonic gauge (2.43)) one obtains that field equations read

\[ \square \tilde{h}_{\mu\nu} = -2 \mathcal{X} T^{(0)}_{\mu\nu}. \]  

(2.83)

\(^3\)In this perturbation scheme the first order on Minkowski space has to be connected with the zero order of the standard matter energy momentum tensor. This formalism descends from the theoretical setting of Newtonian mechanics which requires the appropriate scheme of approximation and coincides with a gravity theory analyzed at the first order of perturbations in the curved spacetime metric.
One solution is the *retarded potential*

\[
\bar{h}_{\mu\nu}(t, x) = 4G \int d^3 x' \frac{T_{\mu\nu}(x', t - |x - x'|)}{|x - x'|}
\]

or in terms of perturbation \( h_{\mu\nu} \)

\[
h_{\mu\nu}(t, x) = 4G \int d^3 x' \frac{S_{\mu\nu}(x', t - |x - x'|)}{|x - x'|}.
\]

The propagation of \( h_{\mu\nu} \) is possible with a particle massless.
Chapter 3

Spherical symmetry in $f$ – gravity

Spherical symmetry in $f$-gravity is discussed in details considering also the relations with the weak field limit $[D]$. Exact solutions are obtained for constant Ricci curvature scalar and for Ricci scalar depending on the radial coordinate. In particular, we discuss how to obtain results which can be consistently compared with General Relativity giving the well known post-Newtonian and post-Minkowskian limits. Furthermore, we implement a perturbation approach to obtain solutions up to the first order starting from spherically symmetric backgrounds. Exact solutions are given for several classes of $f$-theories in both $R = \text{constant}$ and $R = R(r)$.

3.1 The Ricci curvature scalar in spherical symmetry

Starting by the definition of Ricci scalar $[1.5]$ and imposing the spherical symmetry $[1.89]$, the Ricci scalar in terms of the gravitational potentials $(g_{tt}$ and $g_{rr})$ reads:

$$R(t, r) = \frac{1}{2r^2 g_{tt}^2 g_{rr}^2} \left\{ g_{rr} \left[ g_{tt} \ddot{g}_{rr} - \dot{g}_{tt}^2 \right] r^2 + g_{tt} \left[ r \left( \dot{g}_{tt}^2 - \dot{g}_{tt} \dot{g}_{rr} \right) + 2g_{rr} \left( 2 \dot{g}_{tt} + r g_{tt}^\prime - r \ddot{g}_{rr} \right) \right] \ight. - 4g_{tt} \left[ \dot{g}_{rr}^2 - g_{rr} + r g_{rr}^\prime \right] \right\} \quad (3.1)$$

where, for the sake of brevity, we have discarded the explicit dependence in $g_{tt}(t, r)$ and $g_{rr}(t, r)$ and the prime indicates the derivative with respect to $r$ while the dot with respect to $t$. If the metric $[1.89]$ is time-independent, i.e. $g_{tt}(t, r) = a(r)$, $g_{rr}(t, r) = b(r)$, the (3.1) assumes the simpler form:
\[ R(r) = \frac{1}{2r^2a(r)^2b(r)^2} \left\{ a(r) \left[ 2b(r) \left( 2a'(r) + ra''(r) \right) - ra'(r)b'(r) \right] - b(r)a'(r)^2r^2 \right. \\
-4a(r)^2 \left( b(r)^2 - b(r) + rb'(r) \right) \left\} \right. \\
\] (3.2)

where the radial dependence of the gravitational potentials is now explicitly shown. This expression can be seen as a constraint for the functions \( a(r) \) and \( b(r) \) once a specific form of Ricci scalar is given. In particular, it reduces to a Bernoulli equation of index two with respect to the metric potential \( b(r) \):

\[ b'(r) + \left\{ \frac{r^2a'(r)^2 - 4a(r)^2 - 2ra(r)[2a(r)' + ra(r)'']}{ra(r)[4a(r) + ra'(r)]} \right\} b(r) \\
+ \left\{ \frac{2a(r)}{r} \left[ \frac{2 + r^2R(r)}{4a(r) + ra'(r)} \right] \right\} b(r)^2 = b'(r) + h(r)b(r) + l(r)b(r)^2 = 0 . \] (3.3)

A general solution of (3.3) is:

\[ b(r) = \frac{\exp\left[ -\int dr h(r) \right]}{K + \int dr l(r) \exp\left[ -\int dr h(r) \right]} , \] (3.4)

where \( K \) is an integration constant while \( h(r) \) and \( l(r) \) are the two functions which, respectively, define the coefficients of the quadratic and the linear term with respect to \( b(r) \), as in the standard definition of the Bernoulli equation \[156\]. Looking at the equation, we can notice that it is possible to have \( l(r) = 0 \) which implies to find out solutions with a Ricci scalar scaling as \( -\frac{2}{r^2} \) in term of the radial coordinate. On the other side, it is not possible to have \( h(r) = 0 \) since otherwise we will get imaginary solutions. A particular consideration deserves the limit \( r \to \infty \). In order to achieve a gravitational potential \( b(r) \) with the correct Minkowski limit, both \( h(r) \) and \( l(r) \) have to go to zero provided that the quantity \( r^2R(r) \) turns out to be constant: this result implies \( b'(r) = 0 \), and, finally, also the metric potential \( b(r) \) has a correct Minkowski limit. In general, if we ask for the asymptotic flatness of the metric as a feature of the theory, the Ricci scalar has to evolve at infinity as \( r^{-n} \) with \( n \geq 2 \). Formally it has to be:

\[ \lim_{r \to \infty} r^2R(r) = r^{-n} \] (3.5)
with \( n \in \mathbb{N} \). Any other behavior of the Ricci scalar could affect the requirement of the correct asymptotic flatness. This result can be easily deduced from (3.3). In fact, let us consider the simplest spherically symmetric case:

\[
\begin{align*}
\begin{array}{c}
\mathrm{d}s^2 = a(r) \mathrm{d}t^2 - \frac{\mathrm{d}r^2}{a(r)} - r^2 \mathrm{d}\Omega.
\end{array}
\end{align*}
\]

The Bernoulli equation (3.3) is easy to integrate and the most general metric potential \( a(r) \), compatible with the Ricci scalar constraint (3.2), is:

\[
a(r) = 1 + \frac{k_1}{r} + \frac{k_2}{r^2} + \frac{1}{r^2} \int \left[ \int r^2 R(r) \mathrm{d}r \right] \mathrm{d}r
\]

where \( k_1 \) and \( k_2 \) are integration constants. Actually one gets the standard result \( a(r) = 1 \) (Minkowski) for \( r \to \infty \) only if the condition (3.5) is satisfied, otherwise we get a diverging gravitational potential.

### 3.2 Solutions with constant curvature scalar

Let us assume a scalar curvature constant \((R = R_0)\). The field equations (1.25) and (1.26) reduce to:

\[
\begin{align*}
\begin{array}{c}
f'(R_0) R_{\mu\nu} - \frac{1}{2} f(R_0) g_{\mu\nu} = \mathcal{X} T_{\mu\nu} \\
f'(R_0) R_0 - 2 f(R_0) = \mathcal{X} T
\end{array}
\end{align*}
\]

Such equations can be arranged as:

\[
\begin{align*}
\begin{array}{c}
R_{\mu\nu} + \lambda g_{\mu\nu} = q \mathcal{X} T_{\mu\nu} \\
R_0 = q \mathcal{X} T - 4 \lambda
\end{array}
\end{align*}
\]

where \( \lambda = -\frac{f(R_0)}{2 f'(R_0)} \) and \( q^{-1} = f'(R_0) \). Since we are analyzing the weak field limit of HOG, it is reasonable to consider Lagrangians which work as the Hilbert-Einstein one when \( R \to 0 \) (this
even means that we can suitably put the cosmological constant to zero), that is:

$$\lim_{R \to 0} f \sim R.$$  \hfill (3.10)

In such a case, the trace equation of (3.9) indicates that in the vacuum case ($T_{\mu \nu} = 0$), one obtains a solutions with constant curvature $R = R_0$.

Let us now suppose that the above theory, for small curvature values, evolves to a constant as $\lim_{R \to 0} f = f_0$. Even in this case, considering only the trace equation of (3.8), some interesting features emerge. In fact if we consider the expression for $f$:

$$f = f_0 + f_1 R + F(R).$$  \hfill (3.11)

where $f_0$ and $f_1$ are a coupling constants, while $F(R)$ is a generic analytic function of $R$ satisfying the condition

$$\lim_{R \to 0} R^{-2} F(R) = \text{constant},$$  \hfill (3.12)

it is evident that no zero-curvature solutions are obtainable since:

$$F'(R_0) R_0 - 2F(R_0) - f_1 R_0 - 2f_0 = \mathcal{X} T.$$  \hfill (3.13)

Furthermore, in this case, even in absence of matter, there are no Ricci flat solution of the field equations since the higher order derivative terms give curvature constant solutions corresponding to a sort of effective cosmological constant. Of course, this is not the case for GR. In fact in the standard Einstein theory, constant curvature solutions different from zero are in order only in presence of matter because of the proportionality of the Ricci scalar to the trace of energy-momentum tensor of matter. Besides, one can get a similar situation in presence of a cosmological constant put by hands into the dynamics.

In other words, the difference between GR and HOG is that the Schwarzschild-de Sitter solution is not necessarily given by a $\Lambda$-term while the effect of an “effective” cosmological constant can be achieved by the higher order derivative contributions. This result has been extensively investigated in several recent papers as, for example, [140]. For a discussion, see also [44].
Let us consider now the search for a general solution of (1.25) and (1.26) considering a spherically symmetric metric as (1.89). By substituting the metric into the field equations, one obtains that the \( \{t, r\} \) - component of (1.25) gives
\[
\dot{g}_{rr}(t, r) \frac{g_{rr}(t, r)}{g_{rr}(t, r)} = 0.
\]
This means that \( g_{rr}(t, r) \) has to be time independent and we can write
\[
g_{rr}(t, r) = b(r).
\]
On the other hand, by the \( \{\theta, \theta\} \) - component of (1.25), one gets the relation
\[
g''_{tt}(t, r) \frac{g_{tt}(t, r)}{g_{tt}(t, r)} = \zeta(r)
\]
with \( \zeta(r) \) a given time independent function:
\[
g_{tt}(t, r) = \tilde{a}(t) \exp \left[ \int \zeta(r) dr \right] = \tilde{a}(t) \frac{b(r)}{r^2} \exp \left[ 2 \int dr \frac{1 - r^2(\lambda + qXp)b(r)}{r} \right],
\]
where \( \lambda \) and \( q \) are defined as above and \( p \) is the pressure of a perfect fluid being the stress-energy tensor of matter
\[
T_{\mu\nu} = (\rho + p)u_\mu u_\nu - pg_{\mu\nu};
\]
\( \rho \) is the energy density and \( u^\mu = dx^\mu / ds \) is the 4-velocity. Therefore, the function \( g_{tt}(t, r) \) has to be a separable function of time and radial coordinates, i.e. \( g_{tt}(t, r) = \tilde{a}(t)a(r) \). As a matter of facts, the metric (1.89) becomes
\[
ds^2 = \tilde{a}(t)a(r)dt^2 - b(r)dr^2 - r^2d\Omega
\]
which can be recast as
\[
ds^2 = a(r)d\tilde{t}^2 - b(r)dr^2 - r^2d\Omega,
\]
by a suitable time - redefinition \( d\tilde{t} = \sqrt{\tilde{a}(t)}dt \). Nevertheless we redefine the time \( \tilde{t} \) as \( t \).

This exact result states that whenever one has a constant scalar curvature spacetime, any spherically symmetric background has to be necessarily static. In other words, this means that the Birkhoff theorem holds for the \( f \) - theories with constant curvature as it has to be (see [157]). It has to be noticed that such a result is in striking contrast with what has been argued elsewhere about the fact that Birkhoff theorem does not hold, in general, for HOG theories.

A remark is in order at this point. We have obtained this result considering a constant Ricci scalar spacetime and deducing some conditions on the form of gravitational potentials. Nevertheless one can even reverse the problem. It is possible to argue that whenever the gravitational
potential $g_{tt}(t, r)$ is described by a separable functions and $g_{rr}(t, r)$ is time-independent, by the definition of the Ricci scalar, one gets that $R = R_0$ and at the same time the final solutions of the field equations will be static if the spherical symmetry is invoked.

Actually, for a complete analysis of the problem, one should take into account the remaining field equations descending from (1.25) and (1.26) which have to be solved by taking even into account the definition of the Ricci scalar (3.2). In summary, we have to solve the system:

\[
\begin{align*}
R_{tt} + \lambda a(r) - q\mathcal{X}[\rho + p(1 - a(r))] &= 0 \\
R_{rr} - \lambda b(r) - q\mathcal{X} p b(r) &= 0 \\
R_0 - q\mathcal{X}(\rho - 3p) + 4\lambda &= 0 \\
R(a(r), b(r)) &= R_0
\end{align*}
\]

A general solution of the above set of equations is achieved for $p = -\rho$ and reads:

\[
ds^2 = \left(1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - \lambda r^2}{3}\right)dt^2 - \frac{dr^2}{1 + \frac{k_1}{r} + \frac{q\mathcal{X}\rho - \lambda r^2}{3}} - r^2d\Omega.
\]

In other words, any $f$-theory in the case of constant curvature scalar ($R = R_0$) exhibits solutions with cosmological constant as the solution Schwarzschild - de Sitter (2.19) if we consider the relation $\Lambda = q\mathcal{X}\rho - \lambda = \frac{2\mathcal{X}\rho + f(R_0)}{2f'(R_0)}$. This is one of the reason why the dark energy issue can be addressed using these theory [20, 56, 57, 58, 59, 60, 61, 62, 63]. This fact is well known using the FRW metric [44].

If we neglect the cosmological constant $f_0$ and $f_1$ is set to zero in the (3.11), we obtain a new class of theories which, in the limit $R \rightarrow 0$, does not reproduce GR (from (3.12), we have $\lim_{R \rightarrow 0} f \sim R^2$). In such a case analyzing the whole set of (1.25) and (1.26), one can observe that both zero and constant $\neq 0$ curvature solutions are possible. In particular, if $R = R_0 = 0$ the field equations are solved for every form of gravitational potentials entering the spherically symmetric background (3.17), provided that the Bernoulli equation (3.3), relating these functions, is fulfilled for the particular case $R(r) = 0$. The solutions are thus defined by the relation.
Table 3.1: Examples of $f$-models admitting constant and zero scalar curvature solutions. In the right hand side, the field equations are given for each model. The power $n, m$ are natural numbers while $\xi_i$ are generic real constants.
Chapter 3  Spherical symmetry in \( f - \) gravity

3.3 Solutions with curvature scalar function of \( r \)

Up to now we have discussed the behavior of \( f - \)gravity seeking for spherically symmetric solutions \((1.89)\) with constant scalar curvature. This situation is well known in GR and give rise to the Schwarzschild solution \((R = 0)\) and the Schwarzschild-de Sitter solution \((R = R_0 \neq 0)\). The problem can be generalized in \( f - \) gravity investigating considering the Ricci scalar as an arbitrary function of the radial coordinate \( r \).

This approach is interesting since, in general, HOG theories are supposed to admit such kind of solutions and several examples have been found in literature \([10, 32, 119, 139, 140]\). Here we want to face the problem from general point of view.

If we choose the Ricci scalar \( R \) as a generic function of the radial coordinate \((R = R(r))\), it is possible to show that also in this case the solution of the field equations \((1.25) \text{ and } (1.26)\) is time independent \((if T_{\mu
u} = 0)\). In other words, the Birkhoff theorem has to hold. The crucial point of the approach is to study the off-diagonal \( \{t, r\} \) component of \((1.25)\) as well as in the case of GR. This equation, for a generic \( f \) reads:

\[
\frac{d}{dr} \left( r^2 f' \right) \dot{g}_{rr}(t, r) = 0, \tag{3.21}
\]

and two possibilities are in order. Firstly, we can choose \( \dot{g}_{tt}(t, r) \neq 0 \). This choice implies that \( f' \sim r^{-2} \). If this is the case, the remaining field equation turn out to be not fulfilled and it can be easily recognized that the dynamical system encounters a mathematical incompatibility.

The only possible solution is given by \( \dot{g}_{rr}(t, r) = 0 \) and then the gravitational potential has to be \( g_{rr}(t, r) = b(r) \). Considering also the \( \{\theta, \theta\} \) - equation of \((1.25)\) one can determine that the gravitational potential \( g_{tt}(t, r) \) can be factorized with respect to the time, so that we get solutions of the type \((3.16)\) which can be recast in the stationary spherically symmetric form \((3.17)\) after a suitable coordinate transformation.

As a matter of fact, even the more general radial dependent case admit time-independent solutions. From the trace equation and the \( \{\theta, \theta\} \) - component, we deduce a relation which links \( a(r) \) and \( b(r) \):

\[
a(r) = \frac{b(r) e^{\int \frac{Rf'}{R^2 f''} dr}}{r^4 R^2 f''}, \tag{3.22}
\]

(with \( f'' \neq 0 \) and one which relates \( b(r) \) and \( f \) (see also \([139, 140]\) for a similar result):
3.4 Perturbing the spherically symmetric solutions

\[ b(r) = \frac{6 \left[ f'(rR'f'')' - rR'^2 f'' \right]}{rf(rR'f'' - 4f')} + 2f'(rR(f' - rR'f') - 3R'f''). \] (3.23)

As above, three further equations has to be satisfied to completely solve the system (respectively the \{t, t\} and \{r, r\} components of the field equations and the Ricci scalar constraint) while the only unknown functions are \( f \) and the Ricci scalar \( R(r) \).

If we now consider a fourth order model of the form \( f = R + \mathcal{F}(R) \), with \( \mathcal{F}(R) \ll R \) we are capable of satisfying the whole set of equations up to third order in \( \mathcal{F} \). In particular, we can solve the whole set of equations: the relations (3.22) and (3.23) will give the general solution depending only on the forms of \( \mathcal{F} \) and \( R = R(r) \), that is:

\[
\begin{align*}
  a(r) &= \frac{b(r)e^{-\frac{3}{2} \int \frac{[R + (2\mathcal{F} - R\mathcal{F}')b(r)]}{rR^2 \mathcal{F}''} dr}}{r^4R^2 \mathcal{F}''^2} \\
  b(r) &= -\frac{3(rR' \mathcal{F}'')'}{rR}.
\end{align*}
\] (3.24)

Once the radial dependence of the scalar curvature is obtained, (3.24) allow to write down the solution of the field equations and the gravitational potential, related to the function \( a(r) \), can be deduced. Furthermore one can check the physical relevance of such a potential by means of astrophysical data, see for example the analysis in [152].

3.4 Perturbing the spherically symmetric solutions

The search for solutions in \( f \) - gravity, in the case of Ricci scalar dependent on the radial coordinate, can be faced by means of a perturbation approach. There are several perturbation techniques by which HOG can be investigated in the weak field limit. A general approach is starting from analytical \( f \) - theories assuming that the background model slightly deviates from the Einstein GR (this means to consider \( f = R + \mathcal{F}(R) \) where \( \mathcal{F}(R) \ll R \) as above). Another approach can be developed starting from the background metric considered as the 0th - order solution. Both these approaches assume the weak field limit of a given HOG theory as a correction to GR, supposing that zero order approximation should yield the standard lore.

Both these methods can provide interesting results on the astrophysical scales where spherically symmetric solutions characterized by small values of the scalar curvature, can be taken into account.
In the following, we will consider the first approach assuming that the background metric matches, at zero order, the GR solutions.

In general, searching for solutions by a perturbation technique means to perturb the metric

\[ g_{\mu\nu} = g^{(0)}_{\mu\nu} + g^{(1)}_{\mu\nu}. \]  

This implies that the field equations (1.25) and (1.26) split, up to first order, in two levels. Besides, a perturbation on the metric acts also on the Ricci scalar \( R \) (1.5):

\[ R \sim R^{(0)} + R^{(1)}, \]  

and then we can Taylor expand the analytic \( f \) about the background value of \( R \), i.e.:

\[
\begin{align*}
f &= \sum_n \frac{f^{(n)}(R^{(0)})}{n!} \left[ R - R^{(0)} \right]^n = \sum_n \frac{f^{(n)}(0)}{n!} R^{(1)} R^{(0)} = f^{(0)} + f^{(0)} R^{(1)} + \frac{f^{(0)}}{2} R^{(1)^2} \\
&\quad + \frac{f^{(0)}}{6} R^{(1)^3} + \frac{f^{(0)}}{24} R^{(1)^4} + \ldots \\
\frac{df}{dR} &= \sum_n \frac{f^{(n+1)}(R^{(0)})}{n!} \left[ R - R^{(0)} \right]^n = f^{(0)} + f^{(0)} R^{(1)} + \frac{f^{(0)}}{2} R^{(1)^2} + \frac{f^{(0)}}{6} R^{(1)^3} \\
&\quad = df^{(1)}/dR \\
f'' &= \sum_n \frac{f^{(n+2)}(R^{(0)})}{n!} \left[ R - R^{(0)} \right]^n = f^{(0)} + f^{(0)} R^{(1)} + \frac{f^{(0)}}{2} R^{(1)^2} + \frac{f^{(0)}}{6} R^{(1)^3} \\
&\quad = df^{(1)}/dR \\
f''' &= \sum_n \frac{f^{(n+3)}(R^{(0)})}{n!} \left[ R - R^{(0)} \right]^n = f^{(0)} + f^{(0)} R^{(1)} = df^{(2)}/dR \\
f'''' &= \sum_n \frac{f^{(n+4)}(R^{(0)})}{n!} \left[ R - R^{(0)} \right]^n = f^{(0)} + f^{(0)} R^{(1)} = df^{(3)}/dR \\
\end{align*}
\]

However the above condition \( F(R) \ll R \) has to imply the validity of the linear approximation \( \frac{f''(R^{(0)})(R^{(1)})}{f(R^{(0)})} \ll 1 \). This is demonstrated by assuming \( f' = 1 + F' \) and \( f'' = F'' \). Immediately we obtain that the condition is fulfilled for
3.4 Perturbing the spherically symmetric solutions

\[ \frac{\mathcal{F}''(R(0)) R^{(1)}}{1 + \mathcal{F}'(R(0))} \ll 1. \]  

(3.28)

For example, given a Lagrangian of the form \( f = R + \frac{R_0}{R} \), the \( (3.28) \) means

\[ \frac{2R_0 R^{(1)}}{R(0)^2 - R_0} \ll 1, \]  

(3.29)

while, for \( f = R + \alpha R^2 \), the \( (3.28) \) means

\[ \frac{2\alpha R^{(1)}}{1 + 2\alpha R^{(0)}} \ll 1. \]  

(3.30)

This means that the validity of the approximation strictly depends on the form of the models and the value of the parameters, in the previous case \( R_0 \) and \( \alpha \). For the considerations below, we will assume that it holds. A detailed discussion for the Palatini formalism is in [142, 186].

The zero order field equations read:

\[ f^{(0)} R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} f^{(0)} + \mathcal{H}_{\mu\nu}^{(0)} = \mathcal{X} T_{\mu\nu}^{(0)} \]  

(3.31)

where

\[ \mathcal{H}_{\mu\nu}^{(0)} = -f''(0) \left\{ R_{\mu\nu}^{(0)} - \Gamma^{(0)}_{\mu\nu} R^{(0)} - g_{\mu\nu}^{(0)} \left( g^{(0)}_{\rho\sigma} R_{\rho\sigma}^{(0)} + g^{(0)}_{\rho\sigma} R_{\rho\sigma}^{(0)} \right) + g^{(0)}_{\rho\sigma} \ln \sqrt{-g} R_{\rho\sigma}^{(0)} \right\} - f''''(0) \left\{ R_{\mu\nu}^{(0)} - g_{\mu\nu}^{(0)} g^{(0)}_{\rho\sigma} R_{\rho\sigma}^{(0)} R_{\rho\sigma}^{(0)} \right\}. \]  

(3.32)

At first order one has:

\[ f^{(0)} \left\{ R_{\mu\nu}^{(1)} - \frac{1}{2} g_{\mu\nu}^{(0)} R^{(1)} \right\} + f''(0) R_{\mu\nu}^{(1)} R_{\rho\sigma}^{(0)} - \frac{1}{2} f^{(0)} g_{\mu\nu}^{(1)} + \mathcal{H}_{\mu\nu}^{(1)} = \mathcal{X} T_{\mu\nu}^{(1)} \]  

(3.33)

with
Chapter 3  Spherical symmetry in $f$ gravity

\[ \mathcal{H}_{\mu\nu}^{(1)} = - f''(0) \left\{ R_{\mu\nu}^{(1)} - \Gamma_{\mu\nu}^{(0)\rho} R_{\rho}^{(1)} - \Gamma_{\mu\rho}^{(1)\nu} R_{\nu}^{(0)} - g_{\mu\nu}^{(0)} \left[ g_{\rho\sigma}^{(0)\rho} R_{\rho}^{(1)} + g_{\rho\sigma}^{(1)\rho} R_{\rho}^{(0)} + g_{\rho\sigma}^{(1)\rho} \ln \sqrt{-g_{\rho\sigma}^{(0)}} R_{\rho}^{(1)} + g_{\rho\sigma}^{(1)\rho} \ln \sqrt{-g_{\rho\sigma}^{(0)}} R_{\rho}^{(0)} + \right. \right. \]

\[ + \left. \left. g_{\rho\sigma}^{(0)\rho} \ln \sqrt{-g_{\rho\sigma}^{(0)}} R_{\rho}^{(0)} \right] \right\} - f''(0) \left\{ R_{\mu\nu}^{(0)} R_{\rho}^{(1)} + R_{\mu\nu}^{(1)} R_{\rho}^{(0)} - g_{\mu\nu}^{(0)} g_{\rho\sigma}^{(0)\rho} \left( R_{\rho\sigma}^{(1)} \right) + \right. \]

\[ + \left. R_{\mu\nu}^{(1)} \left( R_{\rho\sigma}^{(0)} \right) - g_{\mu\nu}^{(0)} \left( R_{\rho\sigma}^{(1)} \right) - g_{\mu\nu}^{(0)} g_{\rho\sigma}^{(0)\rho} \left( R_{\rho\sigma}^{(0)} \right) \right\} - f'''(0) \left\{ R_{\mu\nu}^{(1)} + \right. \]

\[ - \Gamma_{\mu\nu}^{(0)\rho} R_{\rho}^{(1)} - g_{\mu\nu}^{(0)} \left( R_{\rho\sigma}^{(1)} \right) - g_{\mu\nu}^{(0)} g_{\rho\sigma}^{(0)\rho} \left( R_{\rho\sigma}^{(0)} \right) + g_{\rho\sigma}^{(0)\rho} \ln \sqrt{-g_{\rho\sigma}^{(0)}} R_{\rho}^{(1)} \right\} + \]

\[ - f^{IV}(0) \left\{ R_{\mu\nu}^{(1)} - g_{\mu\nu}^{(0)} g_{\rho\sigma}^{(0)\rho} R_{\rho\sigma}^{(0)} \right\}. \]  

(3.34)

A part the analyticity, no hypothesis has been invoked on the form of $f$. As a matter of fact, $f$ can be completely general. At this level, to solve the problem, it is required the zero order solution of (3.31) which, in general, could not be a GR solution. This problem can be overcome assuming the same order of perturbation on the $f$, that is:

\[ f = R + \mathcal{F}(R), \]  

(3.35)

where $\mathcal{F}$ is a generical function of the Ricci scalar as above. Then we have

\[ f = R^{(0)} + R^{(1)} + \mathcal{F}^{(0)} , \quad f' = 1 + \mathcal{F}'^{(0)} , \quad f'' = \mathcal{F}''^{(0)} , \quad f''' = \mathcal{F}'''^{(0)} , \]  

(3.36)

and the (3.31) reduce to the form

\[ R_{\mu\nu}^{(0)} - \frac{1}{2} R^{(0)} g_{\mu\nu}^{(0)} = G_{\mu\nu}^{(0)} = \mathcal{X} T_{\mu\nu}^{(0)} . \]  

(3.37)
On the other hand, the (3.33) reduce to

$$R_{\mu\nu}^{(1)} - \frac{1}{2}g_{\mu\nu}^{(0)}R^{(1)} - \frac{1}{2}g_{\mu\nu}^{(1)}R^{(0)} - \frac{1}{2}g_{\mu\nu}^{(0)}\mathcal{F}^{(1)} + \mathcal{F}^{(0)}R_{\mu\nu}^{(0)} + \mathcal{H}_{\mu\nu}^{(1)} = \mathcal{X}^T_{\mu\nu}$$  \hspace{1cm} (3.38)

where

$$\mathcal{H}_{\mu\nu}^{(1)} = -\mathcal{F}^{m(0)}\left\{R_{\mu\nu}^{(0)} - \frac{1}{2}g_{\mu\nu}^{(0)}g^{(0)}_{\sigma\tau}R^{(0)}_{s,\sigma}R^{(0)}_{s,\tau}\right\} - \mathcal{F}^{m(0)}\left\{R_{\mu\nu}^{(0)} - \Gamma_{\mu\nu}^{(0)}R_{s,s}^{(0)}\right\}$$

$$-g_{\mu\nu}^{(0)}\left(g^{(0)}_{\sigma,\tau}R^{(0)}_{s,\sigma} + g^{(0)}_{\sigma,\tau}R^{(0)}_{s,\tau} + g^{(0)}_{\sigma,\tau}\ln\sqrt{-g_{s,\tau}^{(0)}R_{s,\tau}^{(0)}}\right)\right\}. \hspace{1cm} (3.39)$$

The new system of field equations is evidently simpler than the starting one and once the zero order solution is obtained, the solutions at the first order correction can be easily achieved. In Tables 3.2 and 3.3, a list of solutions, obtained with this perturbation method, is given considering different classes of $f$-models.

Some remarks on these solutions are in order at this point. In the case of $f$ models which are evidently corrections to the Hilbert - Einstein Lagrangian as $\Lambda + R + \epsilon R \ln R$ and $R + \epsilon R^n$, with $\epsilon << 1$, one obtains exact solutions for the gravitational potentials $a(r)$ and $b(r)$ related by $a(r) = b(r)^{-1}$. The first order expansion is straightforward as in GR. If the functions $a(r)$ and $b(r)$ are not related, for $f = \Lambda + R + \epsilon R \ln R$, the first order system is directly solved without any prescription on the perturbation functions $x(r)$ and $y(r)$. This is not the case for $f = R + \epsilon R^n$ since, for this model, one can obtain an explicit constraint on the perturbation function implying the possibility to deduce the form of the gravitational potential $\Phi(x)$ from $a(r) = 1 + 2 \Phi(x)$. In such a case, no corrections are found with respect to the standard Newtonian potential. The theories $f = R^n$ and $f = \frac{R}{(R_0 + R)}$ show similar behaviors. The case $f = R^2$ is peculiar and it has to be dealt independently.
Chapter 3  Spherical symmetry in $f$ – gravity

$f$ - theory: $\Lambda + R + \epsilon R \ln R$

spherical potentials: $a(r) = b(r)^{-1} = 1 + \frac{k}{r} - \frac{\Lambda r^2}{6} + \delta x(r)$

solutions: $x(r) = \frac{k_2}{r} + \frac{c\Lambda \ln(-2\Lambda - 1)r^2}{6\delta}$

first order metric: $a(r) = 1 - \frac{\Lambda r^2}{6} + \delta x(r), \quad b(r) = \frac{1}{1 - \frac{\Lambda r^2}{6}} + \delta y(r)$

\[
x(r) = (\Lambda r^2 - 6) \left\{ k_1 + \int \frac{dr}{36r\delta(\Lambda r^2 - 6)} \right\} \left\{ 4\delta(2\Lambda^2 r^4 - 15 r^2 + 18)y(r) + r\{36r\epsilon\Lambda[\ln(-2\Lambda) - 1] \right. \\
\left. + \delta(\Lambda r^2 - 6)^2 y'(r) \right\}
\]

solutions: $y(r) = \frac{k_2 \delta - 6r^3\epsilon\Lambda[\ln(-2\Lambda) - 1]}{r\delta(r^2\Lambda - 6)^2}$

$f$ - theory: $R + \epsilon R^n$

spherical potentials: $a(r) = b(r)^{-1} = 1 + \frac{k}{r} + \delta x(r)$

solutions: $x(r) = \frac{k_2}{r}$

first order metric: $a(r) = 1 + \delta \frac{x(r)}{r}, \quad b(r) = 1 + \delta \frac{y(r)}{r}$

solutions: $x(r) = k_1 + k_2 r, \quad y(r) = k_3$

$f$ - theory: $R/(R_0 + R)$

first order metric: $a(r) = 1 + \delta \frac{x(r)}{r}, \quad b(r) = 1 + \delta \frac{y(r)}{r}$

solutions:

\[
x(r) = -\frac{4e^{R_1/2r}}{R_0} \left\{ k_1 - 2\sqrt{6e^{R_1/2r}R_0^{1/2}} k_2 + k_3 r \right\}
\]

\[
y(r) = -\frac{4e^{R_1/2r}}{36^{1/2} \sqrt{6} R_0^{1/2}} \left\{ k_1 - \frac{R_1^{1/2} e^{R_1/2r}}{R_0^{1/2}} (\sqrt{6} - R_0^{1/2}) k_2 \right\}
\]

Table 3.2: A list of exact solutions obtained via the perturbation approach for several classes of $f$ - theories; $k_i$ are integration constants; the potentials $a(r)$ and $b(r)$ are defined by the metric (3.17).
3.4 Perturbing the spherically symmetric solutions

\[ \text{\textbf{f - theory:}} \]

spherical potentials:

\[ a(r) = b(r)^{-1} = 1 + \frac{k_1}{r} + \frac{R_0 r^2}{12} + \delta x(r) \]

\[ R^n \]

\[ n = 2, \ R_0 \neq 0 \text{ and } x(r) = \frac{3k_2 - k_3}{3} + \frac{k_3 r^2}{12} + \frac{k_4}{r} \int dr r^2 \left\{ \int dr \exp \left[ \frac{R_0 r^2 \ln(r - r_0)}{8 + 3R_0 r_0^2} \right] \right\} \]

solutions:

with \( r_0 \) satisfying the condition

\[ 6k_1 + 8r_0 + R_0 r_0^3 = 0 \]

\[ n \geq 2, \ \text{System solved only with } R_0 = 0 \]

and no prescriptions on \( x(r) \)

first order metric:

\[ a(r) = 1 + \delta \frac{x(r)}{r} , \ b(r) = 1 + \delta \frac{y(r)}{r} \]

\[ n = 2 \]

\[ y(r) = -\frac{R_0 r^3}{6} - \frac{x(r)}{2} + \frac{1}{2} r x'(r) + k_1, \]

\[ R(r) = \delta R_0 \]

solutions:

\[ n \neq 2 \]

\[ y(r) = -\frac{1}{2} \int dr r^2 R(r) - \frac{x(r)}{2} + \frac{1}{2} r x'(r) + k_1 \]

with \( R(r) \) whatever

first order metric:

\[ a(r) = 1 - \frac{r_2}{r} + \delta x(r) , \ b(r) = \frac{1}{1 - \frac{r_2}{r}} + \delta y(r) \]

\[ n = 2 \]

\[ y(r) = \frac{r k_1}{3r_2^2 - 2r_2 r + 4r^2} + \frac{r^2 k_3}{3(3r_2^2 - 2r_2 r + 4r^2)} \]

\[ + \frac{r_2 x r x'(r) + 2(r_2 r^3 - r^3) x'(r)}{(3r_2 - 4r)(r_2 - 4r^2)} \]

solutions:

\[ n \neq 2 \]

whatever functions \( x(r) \), \( y(r) \) and \( R(r) \)

\[ \]
Chapter 4

The Noether Symmetries of $f$ – gravity

We search for spherically symmetric solutions of $f$-theories of gravity via the Noether Symmetry Approach \[B\]. A general formalism in the metric framework is developed considering a point-like $f$-Lagrangian where spherical symmetry is required. New exact solutions are given.

4.1 The point-like $f$ Lagrangian in spherical symmetry

As hinted in the plan of thesis, the aim of this chapter is to work out an approach to obtain spherically symmetric solutions in HOG by means of Noether Symmetries. In order to develop this approach, we need to deduce a point-like Lagrangian from the action (1.24). Such a Lagrangian can be obtained by imposing the spherical symmetry in the field action (1.24). As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number. The technique is based on the choice of a suitable Lagrange multiplier defined by assuming the Ricci scalar, argument of the function $f$ in spherical symmetry. Elsewhere, this approach has been successfully used for the FRW metric with the purpose to find out cosmological solutions \[26, 174, 187, 188\].

In general, a spherically symmetric spacetime can be described assuming that the metric (1.86) is time independent:

$$ds^2 = A(r)dt'^2 - B(r)dr^2 - M(r)d\Omega,$$

(4.1)

where $g_{tt}(t', r) \equiv A(r)$, $g_{rr}(t', r) \equiv B(r)$ and $g_{\Omega\Omega}(t', r) \equiv M(r)$. Obviously the conditions $M(r) = r^2$ and $B(r) = A^{-1}(r)$ are requested to obtain the standard Schwarzschild case of GR.
Our goal is to reduce the field action (1.24) to a form with a finite degrees of freedom, that is the canonical action of the form

$$\mathcal{A} = \int dr \mathcal{L}(A, A', B, B', M, M', R, R')$$ (4.2)

where the Ricci scalar $R$ and the potentials $A$, $B$, $M$ are the set of independent variables defining the configuration space. Prime indicates now the derivative with respect to the radial coordinate $r$.

In order to achieve the point-like Lagrangian in this set of coordinates, we write, in the vacuum, the (1.24) as

$$\mathcal{A} = \int d^4x \sqrt{-g} \left[ f(R) - \lambda(R - \bar{R}) \right] ,$$ (4.3)

where $\lambda$ is the Lagrangian multiplier and $\bar{R}$ is the Ricci scalar (1.5) expressed in terms of the metric (4.1):

$$\bar{R} = \frac{A''}{AB} + 2\frac{M''}{BM} + \frac{A'M'}{ABM} - \frac{A'^2}{2A^2B} - \frac{M'^2}{2BM^2} - \frac{A'B'}{2AB^2} - \frac{BM'}{B^2M} - \frac{2}{M} ,$$ (4.4)

which can be recast in the more compact form

$$\bar{R} = R^* + \frac{A''}{AB} + 2\frac{M''}{BM} ,$$ (4.5)

where $R^*$ collects first order derivative terms. The Lagrange multiplier $\lambda$ is obtained by varying the action (4.3) with respect to $R$. One gets $\lambda = f_R[1]$. By expressing the determinant $g$ and $\bar{R}$ in terms of $A$, $B$ and $M$, we have, from (4.3),

$$\mathcal{A} = \int dr A^{1/2}B^{1/2}M \left[ f - f_R \left( R - R^* - \frac{A''}{AB} - 2\frac{M''}{BM} \right) \right] = \int dr \left\{ A^{1/2}B^{1/2}M \left[ f - f_R (R - R^*) \right] - \left( \frac{f_R M}{A^{1/2}B^{1/2}} \right) A' - 2 \left( \frac{A^{1/2}}{B^{1/2}f_R} \right) M' \right\} .$$ (4.6)

The two lines differs for a divergence term which we discard integrating by parts. Therefore, the

---

1 In this chapter the derivatives of $f$ will be indicated like $d^n f/dR^n = f_R....R$. 

The point-like Lagrangian in spherical symmetry

which is canonical since only the configuration variables and their first order derivatives with respect to \( r \) are present. Eq. (4.7) can be recast in a more compact form introducing the matrix formalism:

\[
\mathcal{L} = q^\prime \cdot \hat{T} q' + V
\]  

(4.8)

where \( q = (A, B, M, R) \) and \( q' = (A', B', M', R') \) are the generalized positions and velocities associated to \( \mathcal{L} \). The index “\( \prime \)” indicates the transposed column vector. The kinetic tensor is given by \( \hat{T}_{ij} = \frac{\partial^2 \mathcal{L}}{\partial q'_i \partial q'_j} \). \( V = V(q) \) is the potential depending only on the configuration variables. The Euler-Lagrange equations read

\[
\frac{d}{dr} \nabla_{q'} \mathcal{L} - \nabla_q \mathcal{L} = 2 \frac{d}{dr} \left( \hat{T} q' \right) - \nabla_q V - q' \left( \nabla_q \hat{T} \right) q' = 2 \hat{T} q'' + 2 \left( q' \cdot \nabla_q \hat{T} \right) q' - \nabla_q V - q' \left( \nabla_q \hat{T} \right) q' = 0
\]  

(4.9)

which furnish the equations of motion in term of \( A, B, M \) and \( R \), respectively. The field equation for \( R \) corresponds to the constraint among the configuration coordinates. It is worth noting that the Hessian determinant of (4.7), \( \left| \frac{\partial^2 \mathcal{L}}{\partial q'_i \partial q'_j} \right| \), is zero. This result clearly depends on the absence of the generalized velocity \( B' \) into the point-like Lagrangian. As matter of fact, using a point-like Lagrangian approach implies that the metric variable \( B \) does not contributes to dynamics, but the equation of motion for \( B \) has to be considered as a further constraint equation.

Beside the Euler-Lagrange equations (4.9), one has to take into account the energy \( E_{\mathcal{L}} \):

\[
E_{\mathcal{L}} = q' \cdot \nabla_{q'} \mathcal{L} - \mathcal{L}
\]  

(4.10)
which can be easily recognized to be coincident with the Euler-Lagrange equation for the component $B$ of the generalized position $q$. Then the Lagrangian (4.7) has three degrees of freedom and not four, as we would expect “a priori”.

Now, since the motion equation describing the evolution of the metric potential $B$ does not depend on its derivative, it can be explicitly solved in term of $B$ as a function of other coordinates:

$$B = \frac{2M^2f_{RR}A'R' + 2Mf_RA'M' + 4AMf_{RR}M'R' + Af_RM^2}{2AM[(2 + MR)f_R - Mf]}.$$ (4.11)

By inserting the (4.11) into the Lagrangian (4.7), we obtain a non-vanishing Hessian matrix removing the singular dynamics. The new Lagrangian reads:

$$\mathcal{L}^* = \mathbf{L}^{1/2}$$ (4.12)

with

$$\mathbf{L} = q'^t \mathbf{L} q' = \frac{[(2 + MR)f_R - fM]}{M} \times [2M^2f_{RR}A'R' + 2MM'(f_RA' + 2Af_{RR}R') + Af_RM^2].$$ (4.13)

Since $\frac{\partial \mathbf{L}}{\partial r} = 0$, $\mathbf{L}$ is canonical ($\mathbf{L}$ is the quadratic form of generalized velocities, $A'$, $M'$ and $R'$ and then coincides with the Hamiltonian), so that we can consider $\mathbf{L}$ as the new Lagrangian with three degrees of freedom. The crucial point of such a replacement is that the Hessian determinant is now non-vanishing, being:

$$\left| \frac{\partial^2 \mathbf{L}}{\partial q_i \partial q'_j} \right| = 3AM[(2 + MR)f_R - Mf]^3f_Rf_{RR}^2.$$ (4.14)

Obviously, we are supposing that $(2 + MR)f_R - Mf \neq 0$, otherwise the above definitions of $B$, (4.11), and $\mathbf{L}$, (4.13), lose of significance, besides we assume $f_{RR} \neq 0$ to admit a wide class of HOG models. The case $f = R$ requires a different investigation. In fact, considering the

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2 Lowering the dimension of configuration space through the substitution (4.11) does not affect the dynamics, since $B$ is a non-evolving quantity. In fact, introducing the (4.11) directly into the dynamical equations given by (4.7), they coincide with those derived by (4.13).

---
GR point-like Lagrangian needs a further lowering of degrees of freedom of the system and the previous results cannot be straightforwardly considered. From (4.7), we get:

\[ L_{GR} = -\frac{A^{1/2}}{2MB^{1/2}}M'^2 - \frac{1}{A^{1/2}B^{1/2}}A'M' - 2A^{1/2}B^{1/2}, \]

where Euler-Lagrange equations provide the standard equations of GR for Schwarzschild metric. It is easy to see the absence of the generalized velocity \( B' \) in (4.15). Again, the Hessian determinant is zero. Nevertheless, considering, as above, the constraint (4.11) for \( B \), it is possible to obtain a Lagrangian with a non-vanishing Hessian. In particular one has:

\[ B_{GR} = \frac{M'^2}{4M} + \frac{A'M'}{2A}, \]

\[ L^*_{GR} = L_{GR}^{1/2} = \left( \frac{M'(2MA' + AM')}{M} \right)^{1/2}, \]

and then the Hessian determinant is

\[ \left| \frac{\partial^2 L_{GR}}{\partial q_i' \partial q_j'} \right| = -1, \]

which is nothing else but a non-vanishing sub-matrix of the \( f \) Hessian matrix.

Considering the Euler-Lagrange equations coming from (4.16) and (4.17), one obtains the vacuum solutions of GR (2.8), that is:

\[ A = k_4 - \frac{k_3}{r + k_1}, \quad B = \frac{k_2k_4}{A}, \quad M = k_2(r + k_1)^2. \]

In particular, the standard form of Schwarzschild solution is obtained for \( k_1 = 0, k_2 = 1, k_3 = r_g \) and \( k_4 = 1 \).

A formal summary of the field equations descending from the point-like Lagrangian and their relation with respect to the ones of the standard approach is given in Tab. (??).

The explicit form of field equations (1.25) - (1.26) in the vacuum with the metric (4.1) are
Chapter 4  The Noether Symmetries of \( f \) – gravity

| Field equations approach | Point-like Lagrangian approach |
|--------------------------|-------------------------------|
| \( \delta \int d^4x \sqrt{-g} f = 0 \) | \( \delta \int dr \mathcal{L} = 0 \) |
| \( H_{\mu\nu} = \partial_\mu \left( \frac{\partial \sqrt{-g} f}{\partial g_{\mu\nu}} \right) - \frac{\partial (\sqrt{-g} f)}{\partial g^\nu} = 0 \) | \( \frac{d}{dr} \nabla_q \mathcal{L} - \nabla_q \mathcal{L} = 0 \) |
| \( H = g^\alpha\beta H_{\alpha\beta} = 0 \) | \( E_\mathcal{L} = q' \cdot \nabla_q \mathcal{L} - \mathcal{L} \) |
| \( H_{tt} = 0 \) | \( \frac{d}{dr} \frac{\partial \mathcal{L}}{\partial A^r} - \frac{\partial \mathcal{L}}{\partial A} = 0 \) |
| \( H_{rr} = 0 \) | \( \frac{d}{dr} \frac{\partial \mathcal{L}}{\partial M^r} - \frac{\partial \mathcal{L}}{\partial M} = 0 \) |
| \( H_{\theta\theta} = \csc^2 \theta H_{\phi\phi} = 0 \) | A combination of the above equations |

Table 4.1: The field-equations approach and the point-like Lagrangian approach differ since the symmetry, in our case the spherical one, can be imposed whether in the field equations, after standard variation with respect to the metric, or directly into the Lagrangian, which becomes point-like. The energy \( E_\mathcal{L} \) corresponds to the \( tt \) - component of \( H_{\mu\nu} \). The absence of \( B' \) in the Lagrangian implies the proportionality between the constraint equation for \( B \) and the energy function \( E_\mathcal{L} \). As a consequence, the number of independent equations is three (as the number of unknown functions). Finally it is obvious the correspondence between \( \theta\theta \) component and field equation for \( M \).

\[
H_{tt} = 2A^2B^2Mf + \{BMA^2 - A[2BA'M' + M(2BA'' - A'B')]\}f_R + \\
+(-2A^2MB'R' + 4A^2BM'R' + 4A^2BM'R')f_{RR} + \\
+4A^2BM'R^2f_{RRR} = 0, \quad (4.20)
\]

\[
H_{rr} = 2A^2B^2M^2f + (BM^2A'^2 + AM^2A'B' + 2A^2MB'M' + 2A^2BM' + \\
-2ABM^2A'' - 4A^2BM''')f_R + (2ABM^2A'R' + \\
+4A^2BM'M'R')f_{RR} = 0, \quad (4.21)
\]
4.2 The Noether Symmetry Approach

In order to find out solutions for the Lagrangian (4.13), we can search for symmetries related to cyclic variables and then reduce dynamics. This approach allows, in principle, to select $f$-gravity models compatible with spherical symmetry. As a general remark, the Noether Theorem states that conserved quantities are related to the existence of cyclic variables into dynamics [167, 189, 190]. Let us give a summary of the approach for finite dimensional dynamical systems.

Let $\mathcal{L}(q^i, \dot{q}^i)$ be a canonical, non-degenerate point-like Lagrangian where

$$\frac{\partial \mathcal{L}}{\partial \dot{\lambda}} = 0 \quad \det H_{ij} \equiv \det \left| \frac{\partial^2 \mathcal{L}}{\partial \dot{q}^i \partial \dot{q}^j} \right| \neq 0,$$

with $H_{ij}$ as above, the Hessian matrix related to $\mathcal{L}$. The dot indicates derivatives with respect to the affine parameter $\lambda$ which, ordinarily, corresponds to time $t$. In our case, it is the radial coordinate $r$. In standard problems of analytical mechanics, $\mathcal{L}$ is in the form

$$\mathcal{L} = T(q, \dot{q}) - V(q),$$

where $T$ and $V$ are the "kinetic" and "potential energy" respectively. $T$ is a positive definite

$$H_{\theta\theta} = 2AB^2 M f + (4AB^2 - BA'M' + AB'M' - 2ABM'')f_R +$$
$$+ (2BMA'R' - 2AMB'R' + 2ABM'R' + 4ABMR'')f_{RR} +$$
$$+ 4ABMR^2 f_{RRR} = 0,$$  (4.22)

$$H_{\phi\phi} = \sin^2 \theta H_{\theta\theta} = 0.$$  (4.23)

$$H = 4AB^2 M f - 2AB^2 M R f_R + 3(BMA'R' - AMB'R' +$$
$$+ 2ABM'R' + 2ABMR'')f_{RR} + 6ABMR^2 f_{RRR} = 0$$  (4.24)
quadratic form in $\dot{q}$. The energy function associated with $\mathcal{L}$ is

$$E_\mathcal{L} \equiv \frac{\partial \mathcal{L}}{\partial q^i} \dot{q}^i - \mathcal{L}, \quad (4.27)$$

which is the total energy $T + V$. It has to be noted that $E_\mathcal{L}$ is, in any case, a constant of motion. In this formalism, we are going to consider only transformations which are point-transformations. Any invertible and smooth transformation of the "positions" $Q^i = Q^i(q)$ induces a transformation of the "velocities" such that

$$\dot{Q}^i(q) = \frac{\partial Q^i}{\partial q^j} \dot{q}^j; \quad (4.28)$$

the matrix $\mathcal{J} = ||\partial Q^i/\partial q^j||$ is the Jacobian of the transformation on the positions, and it is assumed to be nonzero. The Jacobian $\tilde{\mathcal{J}}$ of the "induced" transformation is easily derived and $\mathcal{J} \neq 0 \rightarrow \tilde{\mathcal{J}} \neq 0$. Usually, this condition is not satisfied in the whole space but only in the neighbor of a point. It is local transformation. If one extends the transformation to the maximal submanifold such that $\mathcal{J} \neq 0$, it is possible to get troubles for the whole manifold due to possible different topologies [190].

A point transformation $Q^i = Q^i(q)$ can depend on one (or more than one) parameter. Let us assume that a point transformation depends on a parameter $\varepsilon$, i.e. $Q^i = Q^i(q, \varepsilon)$, and that it gives rise to a one-parameter Lie group. For infinitesimal values of $\varepsilon$, the transformation is then generated by a vector field: for instance, as well known, $\partial/\partial x$ represents a translation along the $x$ axis, $x(\partial/\partial y) - y(\partial/\partial x)$ is a rotation around the $z$ axis and so on. In general, an infinitesimal point transformation is represented by a generic vector field on $Q$

$$X = \alpha^i(q) \frac{\partial}{\partial q^i}. \quad (4.29)$$

The induced transformation (4.28) is then represented by

$$X^c = \alpha^i(q) \frac{\partial}{\partial q^i} + \left[ \frac{d}{d\lambda} \alpha^i(q) \right] \frac{\partial}{\partial q^i}. \quad (4.30)$$

$X^c$ is called the "complete lift" of $X$ [190]. A function $f(q, \dot{q})$ is invariant under the transformation
\( L_{X^c} f \) if

\[
L_{X^c} f = \alpha^i(q) \frac{\partial f}{\partial q^i} + \left[ \frac{d}{d\lambda} \alpha^i(q) \right] \frac{\partial f}{\partial q^i} = 0 ,
\]  

(4.31)

where \( L_{X^c} f \) is the Lie derivative of \( f \). In particular, if \( L_{X^c} \mathcal{L} = 0 \), \( X^c \) is said to be a symmetry for the dynamics derived by \( \mathcal{L} \).

In order to see how Noether’s theorem and cyclic variables are related, let us consider a Lagrangian \( \mathcal{L} \) and its Euler-Lagrange equations

\[
\frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} = 0 .
\]  

(4.32)

Let us consider also the vector field (4.30). Contracting (4.32) with the \( \alpha^i \)’s gives

\[
\alpha^i \left( \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} - \frac{\partial \mathcal{L}}{\partial q^i} \right) = 0 .
\]  

(4.33)

Being

\[
\alpha^i \frac{d}{d\lambda} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{d}{d\lambda} \left( \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) - \left( \frac{d\alpha^i}{d\lambda} \right) \frac{\partial \mathcal{L}}{\partial q^i} ,
\]  

(4.34)

from (4.33), we obtain

\[
\frac{d}{d\lambda} \left( \alpha^i \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \right) = L_{X^c} \mathcal{L} .
\]  

(4.35)

The immediate consequence is the Noether Theorem\(^3\):

If \( L_{X^c} \mathcal{L} = 0 \), then the function

\[
\Sigma_0 = \alpha^i \frac{\partial \mathcal{L}}{\partial q^i} ,
\]  

(4.36)

is a constant of motion.

\(^3\)In the following, with abuse of notation, we shall write \( X \) instead of \( X^c \), whenever no confusion is possible.
Remark. Eq. (4.36) can be expressed independently of coordinates as a contraction of $X$ by a Cartan one-form

$$\theta_L = \frac{\partial L}{\partial \dot{q}^i}dq^i.$$  \hfill (4.37)

For a generic vector field $Y = y^i\partial/\partial x^i$, and one-form $\beta = \beta_i dx^i$, we have, by definition, $i_Y\beta = y^i\beta_i$. Thus Eq. (4.36) can be written as

$$i_X\theta_L = \Sigma_0.$$  \hfill (4.38)

By a point-transformation, the vector field $X$ becomes

$$\tilde{X} = i_X\theta_L = \partial \frac{\partial L}{\partial Q^k} \partial \dot{Q}^k + \left[ \frac{d}{d\lambda} (i_x dQ^k) \right] \frac{\partial}{\partial \dot{Q}^k}.$$  \hfill (4.39)

We see that $\tilde{X}'$ is still the lift of a vector field defined on the ”space of positions”. If $X$ is a symmetry and we choose a point transformation such that

$$i_XdQ^1 = 1; \quad i_XdQ^i = 0 \quad i \neq 1,$$  \hfill (4.40)

we get

$$\tilde{X} = \frac{\partial}{\partial Q^1}; \quad \frac{\partial L}{\partial Q^1} = 0.$$  \hfill (4.41)

Thus $Q^1$ is a cyclic coordinate and the dynamics can be reduced \textit{[167, 189]}.

Remarks:

1. The change of coordinates defined by (4.40) is not unique. Usually a clever choice is very important.

2. In general, the solution of (4.40) is not well defined on the whole space. It is \textit{local} in the sense explained above.
3. It is possible that more than one $X$ is found, say for instance $X_1, X_2$. If they commute, i.e. $[X_1, X_2] = 0$, then it is possible to obtain two cyclic coordinates by solving the system

$$i_{X_1}dQ^1 = 1; \quad i_{X_2}dQ^2 = 1; \quad i_{X_1}dQ^i = 0; \quad i \neq 1; \quad i_{X_2}dQ^i = 0; \quad i \neq 2.$$ \hspace{1cm} (4.42)

The transformed fields will be $\partial/\partial Q^1, \partial/\partial Q^2$. If they do not commute, this procedure is clearly not applicable, since commutation relations are preserved by diffeomorphisms. Let us note that $X_3 = [X_1, X_2]$ is also a symmetry, indeed, being $L_{X_3}L = L_{X_1}L_{X_2}L - L_{X_2}L_{X_1}L = 0$. If $X_3$ is independent of $X_1, X_2$, we can go on until the vector fields close the Lie algebra. The usual way to treat this situation is to make a Legendre transformation, going to the Hamiltonian formalism and to a Lie algebra of Poisson brackets. If we look for a reduction with cyclic coordinates, this procedure is possible in the following way:

- we arbitrarily choose one of the symmetries, or a linear combination of them, and get new coordinates as above. After the reduction, we get a new Lagrangian $\tilde{L}(Q)$;
- we search again for symmetries in this new space, make a new reduction and so on until possible;
- if the search fails, we try again with another of the existing symmetries.

Let us now assume that $L$ is of the form (4.26). As $X$ is of the form (4.30), $L_XL$ will be a homogeneous polynomial of second degree in the velocities plus a inhomogeneous term in the $q^i$. Since such a polynomial has to be identically zero, each coefficient must be independently zero. If $n$ is the dimension of the configuration space, we get $1 + n(n + 1)/2$ partial differential equations (PDE). The system is overdetermined, therefore, if any solution exists, it will be expressed in terms of integration constants instead of boundary conditions. It is also obvious that an overall constant factor in the Lie vector $X$ is irrelevant. In other words, the Noether Symmetry Approach can be used to select functions which assign the models and, as we shall see below, such functions (and then the models) can be physically relevant. This fact justifies the method at least \textit{a posteriori}.

### 4.3 The Noether Approach for $f$–gravity in spherical symmetry

Since the above considerations, if one assumes the spherical symmetry, the role of the \textit{affine parameter} is played by the coordinate radius $r$. In this case, the configuration space is given by
\( Q = \{A, M, R\} \) and the tangent space by \( TQ = \{A, A', M, M', R, R'\} \). On the other hand, according to the Noether theorem, the existence of a symmetry for dynamics described by the Lagrangian (4.13) implies a constant of motion. Let us apply the Lie derivative to the (4.13), we have:

\[
L_X L = \alpha \cdot \nabla_q L + \alpha' \cdot \nabla_q' L = q'^t \left[ \alpha \cdot \nabla_q \hat{L} + 2 \left( \nabla_q \alpha \right)^t \hat{L} \right] q',
\]

which vanish if the functions \( \alpha \) satisfy the following system

\[
\alpha \cdot \nabla_q \hat{L} + 2(\nabla_q \alpha)' \hat{L} = 0 \quad \rightarrow \quad \alpha_i \frac{\partial \hat{L}_{km}}{\partial q_i} + 2 \frac{\partial \alpha_i}{\partial q_k} \hat{L}_{im} = 0.
\]

The system (4.44) assumes the following explicit form

\[\text{From now on, } q \text{ indicates the vector } (A, M, R).\]
4.3 The Noether Approach for $f$ gravity in spherical symmetry

\[
\begin{align*}
\Upsilon \left( \frac{\partial \alpha_3}{\partial A} f_R + M \frac{\partial \alpha_3}{\partial A} f_{RR} \right) &= 0 \\
\frac{2}{M} \left[ (2 + MR) \alpha_3 f_{RR} - \frac{2 \alpha_2}{M} f_R \right] f_R + \\
&\quad + \Upsilon \left[ \left( \frac{\alpha_1}{M} + 2 \frac{\partial \alpha_1}{\partial M} + 2 A \frac{\partial \alpha_2}{\partial M} \right) f_R + A \left( \frac{\alpha_3}{M} + 4 \frac{\partial \alpha_3}{\partial M} \right) f_{RR} \right] = 0 \\
\Upsilon \left( M \frac{\partial \alpha_1}{\partial R} + 2 A \frac{\partial \alpha_2}{\partial R} \right) f_{RR} &= 0 \\
\alpha_2 (f - R f_R) f_R - \Upsilon \left[ (\alpha_3 + M \frac{\partial \alpha_1}{\partial M} + 2 A \frac{\partial \alpha_2}{\partial A}) f_{RR} + \\
&\quad + \left( \frac{\partial \alpha_2}{\partial M} + \frac{\partial \alpha_3}{\partial A} + \frac{A}{M} \frac{\partial \alpha_3}{\partial A} \right) f_R \right] = 0 \tag{4.45}
\end{align*}
\]

where $\Upsilon = (2 + MR) f_R - M f$. Solving the system (4.45) means to find out the functions $\alpha_i$ which assign the Noether vector. However the system (4.44) implicitly depends on the form of $f$ and then, by solving it, we get also $f$ theories compatible with spherical symmetry. On the other hand, by choosing the $f$ form, we can explicitly solve (4.44). As an example, one finds that the system (4.44) is satisfied if we chose

\[
f = f_0 R^s \quad \alpha = (\alpha_1, \alpha_2, \alpha_3) = \left( (3 - 2s) k A, -k M, k R \right) \tag{4.46}
\]
with $s$ a real number, $k$ a real number, an integration constant and $f_0$ a dimensional coupling constant. This means that for any $f = R^s$ exists, at least, a Noether symmetry and a related constant of motion $\Sigma_0$:

$$
\Sigma_0 = \alpha \cdot \nabla q' L = 2skMR^{2s-3}[2s + (s-1)MR][(s-2)RA' - (2s^2 - 3s + 1)AR'].
$$

A physical interpretation of $\Sigma_0$ is possible if one gives an interpretation of this quantity in GR. In such a case, with $s = 1$, the above procedure has to be applied to the Lagrangian (4.17). We obtain the solution

$$
\alpha_{GR} = (-kA, kM).
$$

The functions $A$ and $M$ give the Schwarzschild solution (4.19), and then the constant of motion acquires the form

$$
\Sigma_0 = r_g.
$$

In other words, in the case of Einstein gravity, the Noether symmetry gives as a conserved quantity the Schwarzschild radius or the mass of the gravitating system. Another solution can be find out for $R = R_0$ where $R_0$ is a constant. In this case, the field equations (1.25) reduce to

$$
R_{\mu\nu} + k_0 g_{\mu\nu} = 0,
$$

where $k_0 = -\frac{1}{2}f(R_0)/f_R(R_0)$. The general solution is

$$
A(r) = \frac{1}{B(r)} = 1 + \frac{k_0}{r} + \frac{R_0}{12} r^2, \quad M = r^2
$$

with the special case

---

The dimensions are given by $R^{3-s}$ in term of the Ricci scalar. For the sake of simplicity we will put $f_0 = 1$ in the forthcoming discussion.
4.3 The Noether Approach for $f -$ gravity in spherical symmetry

\[ A(r) = \frac{1}{B(r)} = 1 + \frac{k_0}{r}, \quad M = r^2, \quad R = 0. \quad (4.52) \]

The solution (4.51) is the well known Schwarzschild-de Sitter one which is a solution in most of modified gravity theories. It evades the Solar System constraints due to the smallness of the effective cosmological constant. However, other spherically symmetric solutions, different from this, are more significant for Solar System tests. In the general case $f = R^s$, the Lagrangian (4.13) becomes

\[ L = \frac{sR^{2s-3}[2s + (s - 1)MR]}{M} \times [2(s - 1)M^2A'R' + 2MRmA' + 4(s - 1)AMR'R' + ARM'^2], \quad (4.53) \]

and the expression (4.11) for $B$ is

\[ B = \frac{s[2(s - 1)M^2A'R' + 2MRmA' + 4(s - 1)AMR'R' + ARM'^2]}{2AMR[2s + (s - 1)MR]} \quad (4.54) \]

As it can be easily checked, GR is recovered when $s = 1$. Using the constant of motion (4.47), we solve in term of $A$ and obtain

\[ A = R^{\frac{2s-3}{s-2}} \left\{ k_1 + \Sigma_0 \int \frac{R^{\frac{4s-9s+5}{s-2}}}{2ks(s - 2)M[2s + (s - 1)MR]} \, dr \right\} \quad (4.55) \]

for $s \neq 2$, with $k_1$ an integration constant. For $s = 2$, one finds

\[ A = \frac{-\Sigma_0}{12kr^2(4 + r^2R)RR'}. \quad (4.56) \]

These relations allow to find out general solutions for the field equations giving the dependence of the Ricci scalar on the radial coordinate $r$. For example, a solution is found for

\[ s = 5/4, \quad M = r^2, \quad R = 5r^{-2}, \quad (4.57) \]
obtaining the spherically symmetric metric

\[ ds^2 = \frac{1}{\sqrt{5}}(k_2 + k_1 r)dt^2 - \frac{1}{2} \left( \frac{1}{1 + \frac{k_2}{k_1 r}} \right) dr^2 - r^2 d\Omega, \]

with \( k_2 = \frac{32\Sigma_0}{225k} \). It is worth noting that such exact solution is in the range of \( s \) values ruled out by Solar System observations, as pointed out in [145, 146, 147].

### 4.4 Perspectives of Noether symmetries approach

In this chapter, we have discussed a general method to find out exact solutions in Extended Theories of Gravity when a spherically symmetric background is taken into account. In particular, we have searched for exact spherically symmetric solutions in \( f \)-gravity by asking for the existence of Noether symmetries. We have developed a general formalism and given some examples of exact solutions. The procedure consists in: \( i \) considering the point-like \( f \) Lagrangian where spherical symmetry has been imposed; \( ii \) deriving the Euler-Lagrange equations; \( iii \) searching for a Noether vector field; \( iv \) reducing dynamics and then integrating the equations of motion using conserved quantities. Viceversa, the approach allows also to select families of \( f \) models where a particular symmetry (in this case the spherical one) is present. As examples, we discussed power law models and models with constant Ricci curvature scalar. However, the above method can be further generalized. If a symmetry exists, the Noether Approach allows, as discussed in §4.2, transformations of variables where the cyclic ones are evident. This fact allows to reduce dynamics and then to get more easily exact solutions. For example, since we know that \( f = R^s \) -gravity admit a conserved quantity, a coordinate transformation can be induced by the Noether symmetry. We ask for the coordinate transformation:

\[
\mathbf{L} = \mathbf{L}(q, q') = \mathbf{L}(A, M, R, A', M', R') \rightarrow \tilde{\mathbf{L}} = \tilde{\mathbf{L}}(\tilde{M}, \tilde{R}, \tilde{A}', \tilde{M}', \tilde{R}'),
\]

for the Lagrangian (4.13), where the Noether symmetry, and then the conserved quantity, corresponds to the cyclic variable \( \tilde{A} \). If more than one symmetry exists, one can find more than one cyclic variables. In our case, if three Noether symmetries exist, we can transform the Lagrangian \( \mathbf{L} \) in a Lagrangian with three cyclic coordinates, that is \( \tilde{A} = \tilde{A}(q), \tilde{M} = \tilde{M}(q) \) and \( \tilde{R} = \tilde{R}(q) \) which are function of the old ones. These new functions have to satisfy the following system
4.4 Perspectives of Noether symmetries approach

\[
\begin{aligned}
(3 - 2s)A \frac{\partial \bar{A}}{\partial A} - M \frac{\partial \bar{A}}{\partial M} + R \frac{\partial \bar{A}}{\partial R} &= 1, \\
(3 - 2s)A \frac{\partial \bar{q}_i}{\partial A} - M \frac{\partial \bar{q}_i}{\partial M} + R \frac{\partial \bar{q}_i}{\partial R} &= 0,
\end{aligned}
\]

with \( i = 2, 3 \) (we have put \( k = 1 \)). A solution of (4.60) is given by the set (for \( s \neq 3/2 \))

\[
\begin{aligned}
\bar{A} &= \frac{\ln A}{(3 - 2s)} + F_A(A^{\frac{\eta_A}{3 - 2s}} M^{\eta_A}, A^{\frac{\xi_A}{3 - 2s}} M^{\xi_A}) \\
\bar{q}_i &= F_i(A^{\frac{\eta_i}{3 - 2s}} M^{\eta_i}, A^{\frac{\xi_i}{3 - 2s}} M^{\xi_i})
\end{aligned}
\]

and if \( s = 3/2 \)

\[
\begin{aligned}
\bar{A} &= -\ln M + F_A(A) G_A(MR) \\
\bar{q}_i &= F_i(A) G_i(MR)
\end{aligned}
\]

where \( F_A, F_i, G_A \) and \( G_i \) are arbitrary functions and \( \eta_A, \eta_i, \xi_A \) and \( \xi_i \) integration constants.

These considerations show that the Noether Symmetries Approach can be applied to large classes of gravity theories. Up to now the Noether symmetries Approach has been worked out in the case of FRW - metric. In this chapter, we have concentrated our attention to the development of the general formalism in the case of spherically symmetric spacetimes. Therefore the fact that, even in the case of a spherical symmetry, it is possible to achieve exact solutions seems to suggest that this technique can represent a paradigmatic approach to work out exact solutions in any theory of gravity. At this stage, the systematic search for exact solution is well beyond the aim of this thesis. A more comprehensive analysis in this sense will be the argument of forthcoming studies. The results presented in this chapter point out that it does not hold in general for the specific \( f \) theories considered. However, the above technique could be a good approach to select suitable classes of theories where such a theorem holds.
Chapter 5

$f -$ gravity and scalar-tensor gravity: affinities and differences

In the last years a very strong debate has been pursued about the Newtonian limit of HOG models. According to some authors the Newtonian limit of $f$ - gravity is equivalent to the one of Brans-Dicke gravity with $\omega_{BD} = 0$, so that the PPN parameters of these models turn out to be ill defined. In this chapter we show that this is indeed not true. We discuss that HOG models are dynamically equivalent to a O’Hanlon Lagrangian which is a special case of Scalar-tensor theory characterized by a self-interaction potential and that, in the low energy and small velocity limit, this will imply a non-standard behaviour $[H]$. This result turns out to be completely different from the one of a pure Brans-Dicke model and in particular suggests that it is completely misleading to consider the PPN parameters of this theory with $\omega_{BD} = 0$ in order to characterize the homologous quantities of $f$-gravity.

By using the definition of the PPN-parameters $\gamma$ and $\beta$ $[2.25]$ in term of $f$-theories, we show that a family of third-order polynomial theories, in the Ricci scalar $R$, turns out to be compatible with the PPN - limit and the deviation from GR, theoretically predicted, can agree with experimental data $[A]$.

5.1 PPN – parameters in Scalar – Tensor and Fourth Order Gravity

If one takes into account a more general theory of gravity, the calculation of the PPN - limit can be performed following a well defined pipeline shown in the §2.3 which straightforwardly generalizes the standard GR. A significant development in this sense has been pursued by Damour and Esposito.
- Farese [23, 191, 192, 193] which have approached to the calculation of the PPN-limit of scalar - tensor gravity by means of a conformal transformation (see §1.3) to the standard Einstein frame. This scheme provides several interesting results up to obtain an intrinsic definition of $\gamma, \beta$ in term of the non - minimal coupling function $F(\phi)$. The analogy between scalar - tensor gravity and higher order theories of gravity has been widely investigated [33, 43, 48].

Starting from this analogy, the PPN results for scalar - tensor gravity can be extended to HOG [137]. In fact, identifying $\phi \rightarrow R$ [48], it is possible to extend the definition of the scalar-tensor PPN - parameters [23, 194] to the case of HOG:

$$
\gamma - 1 = -\frac{f''}{f' + 2f''}, \quad \beta - 1 = \frac{1}{4} \left( \frac{f' \cdot f''}{2f' + 3f''} \right) \frac{d\gamma}{dR}.
$$

In [137], these definitions have been confronted with the observational upper limits on $\gamma$ and $\beta$ coming from Mercury Perihelion Shift [195] and Very Long Baseline Interferometry [196]. Actually, it is possible to show that data and theoretical predictions from (5.1) agree in the limits of experimental measures for several classes of fourth order theories. Such a result tells us that extended theories of gravity are not ruled out from Solar System experiments but a more careful analysis of theories against experimental limits has to be performed. A possible procedure could be to link the analytic form of a generic fourth order theory with experimental data. In fact, the matching between data and theoretical predictions, found in [137], holds provided some restrictions for the model parameters but gives no general constraints on the theory. In general, the function $f$ could contain an infinite number of parameters (i.e. it can be conceived as an infinite power series [43]) while, on the contrary, the number of useful relations is finite (in our case we have only two relations). An attempt to deduce the form of the gravity Lagrangian can be to consider the relations (5.1) as differential equations for $f$, so that, taking into account the experimental results, one could constrain, in principle, the model parameters by the measured values of $\gamma$ and $\beta$. This hypothesis is reasonable if the derivatives of $f$ function are smoothly evolving with the Ricci scalar. Formally, one can consider the r.h.s. of the definitions (5.1) as differential relations which have to be matched with values of PPN - parameters. In other words, one has to solve the equations (5.1) where $\gamma$ and $\beta$ are two parameters. Based on such an assumption, on can try to derive the largest class of $f$ - theories compatible with experimental data. In fact, by the integration of (5.1), one obtains a solution parameterized by $\beta$ and $\gamma$ which have to be confronted with the experimental quantities $\beta_{exp}$ and $\gamma_{exp}$.

Assuming $f' + 2f'' \neq 0$ and defining $A = \left| \frac{1-\gamma}{2\gamma-1} \right|$ we obtain from (5.1) a differential equation for $f$:
5.1 PPN – parameters in Scalar – Tensor and Fourth Order Gravity

\[ f'''' = A f'. \]  

(5.2)

The general solution of such an equation is a third order polynomial \( f = a R^3 + b R^2 + c R + d \) whose coefficients have to satisfy the conditions: \( a = b = c = 0 \) and \( d \neq 0 \) (trivial solution) or \( a = \frac{4}{12} \) and \( b = \pm \frac{\sqrt{2}}{2} \), with \( c, d \neq 0 \). Thus, the general solution for the non-trivial case, in natural units, reads

\[
 f = \frac{1}{12} \left| \frac{1-\gamma}{2\gamma-1} \right| R^3 \pm \frac{\sqrt{2}}{2} \sqrt{\frac{1-\gamma}{2\gamma-1}} R^2 + c R + d. \]  

(5.3)

It is evident that the integration constants \( c \) and \( d \) have to be compatible with GR prescriptions and, eventually, with the presence of a cosmological constant. Indeed, when \( \gamma \to 1 \), which implies \( f \to c R + d \), the GR - limit is recovered. As a consequence the values of these constants remain fixed (\( c = 1 \) and \( d = \Lambda \), where \( \Lambda \) is the cosmological constant). Therefore, the fourth order theory provided by (5.3) becomes

\[
 f_\pm = \frac{1}{12} \left| \frac{1-\gamma}{2\gamma-1} \right| R^3 \pm \frac{1}{2} \sqrt{\frac{1-\gamma}{2\gamma-1}} R^2 + R + \Lambda, \]  

(5.4)

where we have formally displayed the two branch form of the solution depending on the sign of the coefficient entering the second order term. Since the constants \( a, b, c, d \) of the general solution satisfy the relation \( 3 a c - b^2 = 0 \), one can easily verify that it gives:

\[
 \left. \frac{d\gamma}{dR} \right|_{f_\pm} = - \left. \frac{d}{dR} \frac{f''''}{f'' + 2f''''} \right|_{f_\pm} = 0, \]  

(5.5)

where the subscript \( f_\pm \) refers the calculation to the solution (5.4). This result, compared with the second differential equation (5.1), implies \( 4(\beta - 1) = 0 \), which means the compatibility of the solution even with this second relation.
5.2 Comparing with experimental measurements

Up to now we have discussed a family of fourth order theories (5.4) parameterized by the PPN - quantity $\gamma$; on the other hand, for this class of Lagrangians, the parameter $\beta$ is compatible with GR value being unity.

| Experiment                        | Constraint                        |
|-----------------------------------|------------------------------------|
| Mercury Perihelion Shift          | $|2\gamma - \beta - 1| < 3 \times 10^{-3}$ |
| Lunar Laser Ranging               | $4\beta - 3 = -(0.7 \pm 1) \times 10^{-3}$ |
| Very Long Baseline Interf.       | $|\gamma - 1| = 4 \times 10^{-4}$ |
| Cassini Spacecraft                | $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$ |

Table 5.1: A schematic resume of recent experimental constraints on the PPN - parameters. They are the perihelion shift of Mercury [195], the Lunar Laser Ranging [197], the upper limit coming from the Very Long Baseline Interferometry [196] and the results obtained by the estimate of the Cassini spacecraft delay into the radio waves transmission near the Solar conjunction [198].

Now, the further step directly characterizes such a class of theories by means of the experimental estimates of $\gamma$. In particular, by fixing $\gamma$ to its observational estimate $\gamma_{exp}$, we will obtain the weight of the coefficients relative to each of the non-linear terms in the Ricci scalar of the Lagrangian (5.4). In such a way, since GR predictions require exactly $\gamma_{exp} = \beta_{exp} = 1$, in the case of HOG, one could to take into account small deviations from this values as inferred from experiments. Some plots can contribute to the discussion of this argument. In figure 5.1 the Lagrangian (5.4) is plotted. It is parameterized for several values of $\gamma$ compatible with the experimental bounds coming from the Mercury perihelion shift (see Table 1 and [195]). The function is plotted in the range $R \geq 0$. Since the property $f_+ (R) = -f_- (-R)$ holds for the function (5.4), one can easily recover the shape of the plot in the negative region. As it is reasonable, the deviation from GR becomes remarkable when scalar curvature is large.

In order to display the differences between the theory (5.4) and Hilbert-Einstein one, the ratio $f/R$ is plotted in figure 5.2. Again it is evident that the two Lagrangians differ significantly for great values of the curvature scalar. It is worth noting that the formal difference between the PPN - inspired Lagrangian and the GR expression can be related to the physical meaning of the parameter $\gamma$ which is the deviation from the Schwarzschild - like solution. It measures the spatial curvature of the region which one is investigating, then the deviation from the local flatness can be due to the influence of higher order contributions in Ricci scalar. On the other hand, one can reverse the argument and notice that if such a deviation is measured, it can be recast in the framework of HOG, and in particular its “amount” indicates the deviation from GR. Furthermore, it is worth considering that, in the expression (5.4), the modulus of the coefficients in $\gamma$ (i.e. the strength of
5.2 Comparing with experimental measurements

Figure 5.1: Plot of the two branch solution provided in (5.4). The $f_+$ (dotted line) branch family is up to GR solution (straight line), while the one indicated with $f_-$ (dotted-dashed line) remains below this line. The different plots for each family refer to different values of $\gamma$ fulfilling the condition $|\gamma - 1| \leq 10^{-4}$ and increased by step of $10^{-5}$.

The term) decreases by increasing the degree of $R$. In particular, the highest values of cubic and squared terms in $R$ are, respectively, of order $10^{-4}$ and $10^{-2}$ (see figure 5.3), then GR remains a viable theory at short distances (i.e. Solar System) and low curvature regimes.

A remark is in order at this point. The class of theories which we have discussed is a third order function of the Ricci scalar $R$ parameterized by the experimental values of the PPN parameter $\gamma$. In principle, any analytic $f$ can be compared with the Lagrangian (5.4) provided suitable values of the coefficients. However, more general results can be achieved relaxing the condition $\beta = 1$ which is an intrinsic feature for (5.4) (see for example [137]). These considerations suggest to take into account, as physical theories, functions of the Ricci scalar which slightly deviates from GR, i.e. $f = f_0 R^{(1+\epsilon)}$ with $\epsilon$ a small parameter which indicates how much the theory deviates from

Figure 5.2: The ratio $f/R$. It is shown the deviation of the HOG from GR considering the PPN - limit. Dotted and dotted-dashed lines refer to the $f_+$ and $f_-$ branches plotted with respect to several values of $\gamma$ (the step in this case is $2.5 \times 10^{-5}$).
In fact, supposing $\epsilon$ sufficiently small, it is possible to approximate this expression

$$|R|^{1+\epsilon} \simeq |R| \left(1 + \epsilon \ln |R| + \frac{\epsilon^2 \ln^2 |R|}{2} + \ldots \right).$$

This relation can be easily confronted with the solution (5.4) since, also in this case, the corrections have very small “strength”.

We can conclude this paragraph having shown how a polynomial Lagrangian in the Ricci scalar $R$, compatible with the PPN - limit, can be recovered in the framework of HOG. The approach is based on the formulation of the PPN - limit of such gravity models developed in analogy with scalar-tensor gravity [137]. In particular, considering the local relations defining the PPN fourth order parameters as differential expressions, one obtains a third-order polynomial in the Ricci scalar which is parameterized by the PPN - quantity $\gamma$ and compatible with the limit $\beta = 1$. The order of deviation from the linearity in $R$ is induced by the deviations of $\gamma$ from the GR expectation value $\gamma = 1$. Actually, the PPN parameter $\gamma$ may represent the key parameter to discriminate among relativistic theories of gravity. In particular, this quantity should be significatively tested at Solar System scales by forthcoming experiments like LATOR [199]. From a physical point of view, any analytic function of $R$, by means of its Taylor expansion, can be compared with (5.4). Therefore, a theory like $f = f_0 R^{(1+\epsilon)}$, indicating small deviations from standard GR, is in agreement with the proposed approach, so, in principle, the experimental $\gamma$ could indicate the value of the parameter $\epsilon$. In conclusion, one can reasonably state that generic fourth-order gravity models could be viable candidate theories even in the PPN - limit. In other words, due to the presented results, they cannot be a priori excluded at Solar System scales.
5.3 Newtonian limit of $f$– gravity by O’Hanlon theory analogy

Recently several authors claimed that HOGs and among these, in particular, HOG models are characterized by an ill defined behaviour in the Newtonian regime. In particular, in a series of papers \cite{135, 143, 144, 149, 200, 201} it is addressed that Post-Newtonian corrections of the gravitational potential violate experimental constraints since these quantities can be recovered by a direct analogy with Brans-Dicke gravity \cite{25} simply supposing the Brans - Dicke characteristic parameter $\omega_{BD}$ vanishing. Actually despite the calculation of the Newtonian and the post-Newtonian limit of $f$ - theory, performed in a rigorous manner, have showed that this is not the case \cite{137, 151, 185, 202}, it remains to clarify why the analogy with Brans - Dicke gravity seems to fail its predictions. The issue is easily overcame once the correct analogy between $f$-gravity and the Brans-Dicke model is taken into account.

In literature, it is suggested that HOG models can be rewritten in term of a scalar-field Lagrangian non minimally coupled with gravity but without any kinetic term by fact implying $\omega_{BD} = 0$ (as shown in the chapter 1). Actually, the simplest case of scalar - tensor gravity models has been introduced some decades ago by Brans and Dicke in order to give a general mechanism capable of explaining the inertial forces by means of a background gravitational interaction. The explicit expression of such gravitational action is (1.30), while the general action of $f$-gravity is (1.24). As said above, $f$-gravity can be recast as a scalar-tensor theory by introducing a suitable scalar field $\phi$ which nonminimally couples with the gravity sector. It is important to remark that such an analogy holds in a formalism in which the scalar field displays no kinetic term but is characterized by means of a self-interaction potential which determines the whole dynamics (O’Hanlon Lagrangian) \cite{33}. We can resume the actions as follow

\[
\begin{align*}
\mathcal{A}_{fF}^f &= \int d^4x \sqrt{-g} \left[ f + \mathcal{L}_m \right] \\
\mathcal{A}_{fF}^{BD} &= \int d^4x \sqrt{-g} \left[ \phi R - \omega_{BD} \frac{\phi \phi^{\alpha}}{\phi} + \mathcal{L}_m \right] \\
\mathcal{A}_{fF}^{OH} &= \int d^4x \sqrt{-g} \left[ \phi R + V(\phi) + \mathcal{L}_m \right]
\end{align*}
\]

(5.7)

This consideration, therefore, implies that the scalar field Lagrangian equivalent to the purely
geometrical $f$-gravity turns out to be quite different with respect to the ordinary Brans - Dicke definition (1.30). This point represents a crucial aspect of our analysis. In fact, as we afterwards will see, such a difference will imply completely different results in the Newtonian limit of the two models and, consequently, the impossibility of extend predictions from the PPN approximation of Brans-Dicke models to $f$-gravity. Considering natural units, the O’Hanlon Lagrangian [203] is the third of (5.7) or (1.42). The field equations are obtained by varying Eq. (1.42) with respect to both $g_{\mu\nu}$ and $\phi$ which now represent the dynamical variables (the same field equations are given setting $\omega(\phi) = 0$ and $F(\phi) = \phi$ in (1.31)). Thus, one obtains

\[
\begin{align*}
\phi G_{\mu\nu} - \frac{1}{2} V(\phi) g_{\mu\nu} - \phi_{;\mu\nu} + g_{\mu\nu} \Box \phi &= \mathcal{X} T_{\mu\nu} \\
R + \frac{dV(\phi)}{d\phi} &= 0 \\
\Box \phi + \frac{1}{3} \phi \frac{dV(\phi)}{d\phi} - \frac{2}{3} V(\phi) &= \frac{\mathcal{X}}{3} T
\end{align*}
\]

(5.8)

where the second line of (5.8) the field equation for $\phi$. While the third equation is a combination of the trace of the first one and of the second one. The two schemes can be mapped one into the other considering the following equivalences

\[
\begin{align*}
\phi &= f' \\
V(\phi) &= f - f'R \\
\phi \frac{dV(\phi)}{d\phi} - 2V(\phi) &= f'R - 2f
\end{align*}
\]

(5.9)

where we are the Jacobian matrix of the transformation $\phi \leftrightarrow f'$ is non-vanishing. Henceforth we can consider instead of (1.25) - (1.26) a new set of field equations determined by the equivalence of $f$-gravity with the O’Hanlon approach:

\[
\begin{align*}
\phi R_{\mu\nu} + \frac{1}{6} \left( V(\phi) + \phi \frac{dV(\phi)}{d\phi} \right) g_{\mu\nu} - \phi_{;\mu\nu} &= \mathcal{X} \Sigma_{\mu\nu} \\
\Box \phi + \frac{1}{3} \left( \phi \frac{dV(\phi)}{d\phi} - 2V(\phi) \right) &= \frac{\mathcal{X}}{3} T
\end{align*}
\]

(5.10)
where $\Sigma_{\mu\nu} = T_{\mu\nu} - \frac{1}{3} T g_{\mu\nu}$.

Let us, now, calculate the Newtonian limit of Eqs. (5.10). Considering the perturbations of the metric tensor $g_{\mu\nu}$ and of the scalar field $\phi$ with respect to a background value, we search for solution at the O(2) - order in terms of the metric entries and of the scalar field itself (see § 2.3):

$$g_{\mu\nu} \sim \begin{pmatrix} 1 + g^{(2)}_{tt} & \vec{0}^T \\ \vec{0} & -\delta_{ij} + g^{(2)}_{ij} \end{pmatrix}$$  \hspace{1cm} (5.11)

$$\phi \sim \phi^{(0)} + \phi^{(2)}$$  \hspace{1cm} (5.12)

where we neglected the vectorial component in the metric. The differential operators turn out to be approximated as

$$\Box \approx \partial^2_t - \Delta \quad \text{and} \quad \nabla_\mu \nabla_\nu \approx \partial^2_{\mu\nu}.$$  \hspace{1cm} (5.13)

Actually in order to simplify calculations we can exploit the intrinsic gauge freedom intrinsic in the metric definition. In particular, we choose the harmonic gauge (2.43) and the expressions of Ricci tensor components given by Eqs. (2.53). In relation with the adopted approximation we coherently develop the self-interaction potential at second order. In particular, the quantities in (5.10) read:

$$\begin{align*}
\frac{\phi V(\phi)}{d\phi} + \phi \frac{dV(\phi)}{d\phi} & \simeq V(\phi^{(0)}) + \phi^{(0)} \frac{dV(\phi^{(0)})}{d\phi} + \left[ \phi^{(0)} \frac{d^2V(\phi^{(0)})}{d\phi^2} + 2 \frac{dV(\phi^{(0)})}{d\phi} \right] \phi^{(2)} \\
\phi \frac{dV(\phi)}{d\phi} - 2V(\phi) & \simeq \phi^{(0)} \frac{dV(\phi^{(0)})}{d\phi} - 2V(\phi^{(0)}) + \left[ \phi^{(0)} \frac{d^2V(\phi^{(0)})}{d\phi^2} - \frac{dV(\phi^{(0)})}{d\phi} \right] \phi^{(2)}
\end{align*}$$  \hspace{1cm} (5.14)

The field equations (5.10), solved at 0-th order of approximation, provide the two solutions

$$V(\phi^{(0)}) = 0 \quad \text{and} \quad \frac{dV(\phi^{(0)})}{d\phi} = 0$$  \hspace{1cm} (5.15)

which fix the 0-th order terms in the development of the self-interaction potential; therefore we
have

\[
\begin{align*}
V(\phi) + \phi \frac{dV(\phi)}{d\phi} &\simeq \phi(0) \frac{d^2V(\phi(0))}{d\phi^2} \phi(2) \mp 3\lambda^2 \phi(2) \\
\phi \frac{dV(\phi)}{d\phi} - 2V(\phi) &\simeq \phi(0) \frac{d^2V(\phi(0))}{d\phi^2} \phi(2) \mp 3\lambda^2 \phi(2) 
\end{align*}
\] (5.16)

where constant factors \( \phi(0) \frac{d^2V(\phi(0))}{d\phi^2} \) have been condensed within the quantity \( 3\lambda^2 \). Such a constant can be easily interpreted as a mass term as will become clearer in the following. Now, taking into account the above simplifications, we can rewrite field equations (5.10) at the at O(2) - order in the form:

\[
\begin{align*}
\nabla^2 g^{(2)}_{tt} &= \frac{2\lambda}{\phi(0)} \Sigma_{tt}^{(0)} - \lambda^2 \frac{\phi^{(2)}}{\phi(0)}, \\
\nabla^2 g^{(2)}_{ij} &= \frac{2\lambda}{\phi(0)} \Sigma_{ij}^{(0)} + \lambda^2 \frac{\phi^{(2)}}{\phi(0)} \delta_{ij} + 2 \frac{\phi^{(2)}}{\phi(0)} \\
\nabla^2 \phi^{(2)} - \lambda^2 \phi^{(2)} &= -\frac{\lambda}{3} T^{(0)}. 
\end{align*}
\] (5.17-5.19)

The scalar field solution can be easily obtained from the (5.19) as :

\[
\phi(x) = \phi(0) + \frac{\lambda}{3} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{T^{(0)}(k)e^{ikx}}{k^2 + \lambda^2} 
\] (5.20)

while for \( g^{(2)}_{tt} \) and \( g^{(2)}_{ij} \) we have

\[
g^{(2)}_{tt}(x) = -\frac{\lambda}{2\pi \phi(0)} \int d^3x' \frac{\Sigma_{tt}^{(0)}(x')}{|x - x'|} + \frac{\lambda^2}{4\pi \phi(0)} \int d^3x' \frac{\phi^{(2)}(x')}{|x - x'|},
\] (5.21)

1The factor 3 is introduced to simplify an analogous factor present in the field equations (5.10).
5.3 Newtonian limit of $f$-gravity by O’Hanlon theory analogy

$$g^{(2)}_{ij}(x) = -\frac{\chi}{2\pi \phi(0)} \int d^3x' \Sigma^{(0)}_{ij}(x') - \frac{\lambda^2 \delta_{ij}}{4\pi \phi(0)} \int d^3x' \phi^{(2)}(x') |x - x'|^2 + 2 \phi(0) \left[ \frac{x_i x_j}{x^2} \phi^{(2)}(x) + \left( \delta_{ij} - \frac{3 x_i x_j}{x^2} \right) \frac{1}{|x|^3} \int_0^{|x|} d|x'| |x'|^2 \phi^{(2)}(x') \right]. \quad (5.22)$$

The above three solutions represent a completely general result. In particular adopting the transformation rules (5.9), one can straightforwardly obtain the solutions in the pure $f$-scheme.

Let us analyze the above results with a simple example. We can consider a HOG Lagrangian of the form $f = a_1 R + a_2 R^2$ so that the “dummy” scalar field reads $\phi = a_1 + 2a_2 R$. The self-interaction potential turns out to be $V(\phi) = -\frac{(\phi - a_1)^2}{4a_2}$ satisfying the conditions $V(a_1) = 0$ and $V'(a_1) = 0$. In relation with the definition of the scalar field, we can opportunistically identify $a_1$ with a constant value $\phi(0) = a_1$. Furthermore, the scalar field “mass” can be expressed in terms of the Lagrangian parameters as $\lambda^2 = \frac{1}{3} \phi(0)^2 \frac{\delta^2 V(\phi(0))}{\delta \phi^2} = -\frac{a_1}{6a_2}$. Since the Ricci scalar at the second order reads

$$R \simeq R^{(2)} = \frac{\phi^{(2)}}{2a_2} = \frac{\chi}{6a_2} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{\tilde{T}^{(0)}(k) e^{i k \cdot x}}{k^2 + \lambda^2}, \quad (5.23)$$

if we consider a point-like mass $M$, the energy-momentum tensor components become respectively $T_{tt} = \rho$, $T \sim \rho$ while $\rho = M \delta(x)$, therefore we obtain

$$R^{(2)} = \frac{GM}{3\pi^2 a_2} \int d^3k \frac{e^{i k \cdot x}}{k^2 + \lambda^2} = -\sqrt{\frac{\pi}{2 \alpha_1}} \frac{r_g \lambda^2}{a_1} e^{-\lambda|x|}. \quad (5.24)$$

The immediate consequence is that the solution for the scalar field $\phi$ at second order is

$$\phi^{(2)} = 2a_2 R^{(2)} = \sqrt{\frac{\pi r_g}{2 \alpha_1}} \frac{e^{-\lambda|x|}}{|x|}. \quad (5.25)$$

while the complete scalar field solution up to the second order of perturbation is given by

$$\phi = a_1 + \sqrt{\frac{\pi r_g}{2 \alpha_1}} \frac{e^{-\lambda|x|}}{|x|}. \quad (5.26)$$

Once the behavior of the scalar field has been obtained up to the second order of perturbation, in
the same way, one can deduce the expressions for \( g_{tt}^{(2)} \) and \( g_{ij}^{(2)} \), where \( \Sigma_{tt}^{(0)} = \frac{2}{3} \rho \) and \( \Sigma_{ij}^{(0)} = \frac{1}{3} \rho \delta_{ij} = \frac{1}{2} \Sigma_{tt}^{(0)} \delta_{ij} \). As matter of fact the metric solutions at the second order of perturbation are

\[
\begin{align*}
  g_{tt} &= 1 - \frac{2}{3a} \left( \frac{r_0}{|x|} - \sqrt{\frac{\pi}{2} \frac{1}{3a}} e^{-\lambda |x|} \right), \\
  g_{ij} &= -\left\{ 1 + \frac{1}{3a} \left( \frac{r_0}{|x|} - \sqrt{\frac{\pi}{2} \frac{1}{3a}} \left[ \left( \frac{1}{|x|} - \frac{2}{\lambda^2 |x|^2} - \frac{2}{\lambda^3 |x|^3} \right) e^{-\lambda |x|} - \frac{2}{\lambda^2 |x|^2} \right] \right) \right\} \delta_{ij} \\
  &+ \frac{(2\pi)^{1/2} r_0}{3a} \left[ \left( \frac{1}{|x|} + \frac{3}{\lambda^2 |x|^2} + \frac{3}{\lambda^3 |x|^3} \right) e^{-\lambda |x|} - \frac{3}{\lambda^2 |x|^2} \right] \frac{x_i x_j}{|x|^4}
\end{align*}
\]

This quantity, which is directly related to the gravitational potential, shows that the gravitational potential of the O’Hanlon Lagrangian is non-Newtonian like. Such a behavior prevents from obtaining a natural definition of the PPN parameters as corrections to the Newtonian potential. As matter of fact since it is indeed not true that a generic \( f \)-gravity model corresponds to a Brans-Dicke model with \( \omega_{BD} = 0 \) coherently to its Post-Newtonian approximation. In particular it turns out to be wrong considering the PPN parameter \( \gamma = \frac{1+\omega_{BD}}{2+\omega_{BD}} \) (see, for example, [129]) of Brans-Dicke gravity and evaluating this at \( \omega_{BD} = 0 \) so that one has \( \gamma = 1/2 \) as suggested in [135, 143, 149, 200].

Differently, because of the presence of the self-interaction potential \( V(\phi) \), in the O’Hanlon Lagrangian, a Yukawa like correction appears in the Newtonian limit appears. Such a correction in a completely different way even at the post-Newtonian limit. As matter of fact, one obtains a completely different gravitational potential with respect to the ordinary Newtonian one and as matter of fact the fourth order corrections in term of the \( v/c \) ratio (Newtonian level), have to be evaluated in a complete new general way. In other words, considering a Brans-Dicke Lagrangian and an O’Hanlon one, despite their similar structure, will imply completely different predictions in the weak field and small velocity limits. Such a result represents a significant argument against the claim that HOG models can be ruled out only on the bases of the analogy with Brans-Dicke PPN parameters.

An important consideration is in order now. The definition of PPN-parameters \( \gamma \) and \( \beta \), in the GR realm, is intended as a correction to the Newtonian-like behaviour of the gravitational potentials (2.25). In particular, the PPN parameter \( \gamma \) is related to the second order correction of to the gravitational potential while \( \beta \) is linked with the fourth order level of perturbation. Actually, if we consider the limit \( f \to R \), from Eqs. (5.27), we have
5.3 Newtonian limit of $f$– gravity by O’Hanlon theory analogy

\[ g_{tt} = 1 - \frac{2}{3a} \frac{r_a}{|x|} \]
\[ g_{ij} = - \left( 1 + \frac{1}{3a} \frac{r_a}{|x|} \right) \delta_{ij} \]

(5.28)

Since $a$ is an arbitrary constant, in order to match the Newtonian gravitational potential of GR, we should fix $a = 2/3$. This assumption implies

\[ g_{tt} = 1 - \frac{r_a}{|x|} \]
\[ g_{ij} = - \left( 1 + \frac{r_a}{2|x|} \right) \delta_{ij} \]

(5.29)

which suggest that the PPN parameter $\gamma$, in this limit, results $1/2$ which is in striking contrast with GR predictions ($\gamma \sim 1$). Such a result is in reality not surprising. In fact, the GR limit of the O’Hanlon Lagrangian requires $\phi \sim \text{const}$ and $V(\phi) \to 0$ but such approximations induce mathematical inconsistencies in the field equations of $f$-gravity once these have been obtained by a general O’Hanlon Lagrangian. In reality this is a general issue of O’Hanlon Lagrangian. In fact it can be demonstrated that the field equations (5.10) do not reduce to the standard GR ones (for $V(\phi) \to 0$ and $\phi \sim \text{const}$) since we have:

\[ R_{\mu\nu} = \frac{\lambda}{a_1} \Sigma_{\mu\nu} \]
\[ 0 = \frac{\lambda}{3} T \]

(5.30)

But $\Sigma_{\mu\nu}$ components read $\Sigma_{tt} = \frac{2}{3} \rho$ and $\Sigma_{ij} = \frac{1}{2} \rho \delta_{ij} = \frac{1}{2} \Sigma_{tt} \delta_{ij}$ in place of $S_{tt} = \frac{1}{2} \rho$ and $S_{ij} = \frac{1}{2} \rho \delta_{ij} = S_{tt} \delta_{ij}$ as usual, while the GR field equations are the (2.58). Such a pathology is in order even when the GR limit is performed from a pure Brans - Dicke Lagrangian. In such a case, in order to match the Hilbert - Einstein Lagrangian, one needs $\phi \sim \text{const}$ and $\omega_{BD} = 0$, the immediate consequence is that the PPN parameter $\gamma$ turns out to be $1/2$, while it is well known that Brans - Dicke model fulfils low energy limit prescriptions in the limit $\omega \to \infty$. Even in this case, the problem, with respect to the GR prediction, is that the GR limit of the model introduces inconsistencies in the field equations. In other words, it is not possible to impose the
same transformation which leads the Brans-Dicke theory into GR at the Lagrangian level on the solutions obtained by solving the field equations descending from the general Lagrangian. The relevant aspect of this discussion is that considering a \( f \) model, in analogy with the O’Hanlon Lagrangian and supposing that the self-interaction potential is negligible, introduces a pathological behaviour of the dynamical solutions and induces to obtain a PPN parameter \( \gamma = 1/2 \). This is what happens when an effective approximation scheme is introduced in the field equations in order to calculate the weak field limit of HOG by means of Brans-Dicke model. Such a result seems, from another point of view, to enforce the claim that HOG models have to be carefully investigated in this limit and their analogy with scalar-tensor gravity should be opportunely considered.

5.4 Differences of a generic scalar — tensor theory in the Jordan and Einstein frames

up to now Since along the chapter we have discussed the weak field and small velocity limit of HOG models in term of Brans-Dicke like Lagrangian remaining in the Jordan frame. There we show what are the predictions of the weak field and small velocity limit when a conformal transformation (1.32) is applied on the O’Hanlon Lagrangian. In other words, we discuss HOG models conformally transformed in the Einstein frame. The generic scalar-tensor action \( A_{ST}^{EF} \) in the Jordan frame (1.29) is linked to the generic action \( A_{ST}^{EF} \) in the Einstein frame (1.35) via the transformations (1.36) between the quantities in the two frames. In the case of the O’Hanlon theory in the Jordan frame, (1.42), i.e. \( F(\phi) = \phi \) and \( \omega(\phi) = 0 \), the action (1.35) in the Einstein frame is simplified and the transformation between the two scalar fields reads

\[
\Omega(\phi) d\phi^2 = -\frac{3\Lambda}{2} \frac{d\phi^2}{\phi^2}. \tag{5.31}
\]

If, now, we suppose \( \Omega(\phi) = -\Omega_0 < 0 \) we have

\[
\phi = k e^{\pm Y \phi}, \tag{5.32}
\]

where \( Y = \sqrt{\frac{2\Omega_0}{3\Lambda}} \) and \( k \) is an integration constant. We obtain the transformed of (1.42) in the Einstein frame is
5.4 Differences of a generic scalar – tensor theory in the Jordan and Einstein frames

\[ \mathcal{A}_{EF}^{OH} = \int d^4x \sqrt{-\tilde{g}} \left[ \Lambda \tilde{R} - \Omega_0 \varphi ; \alpha \varphi^{\alpha} + \frac{\Lambda^2}{k^2} e^{\mp 2Y \varphi} V(k e^{\pm Y \varphi}) + \frac{\mathcal{A}}{k} e^{\mp 2Y \varphi} \mathcal{L}_m \left( \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho \sigma} \right) \right] . \]  

(5.33)

The field equations on the other side are

\[
\begin{cases}
\Lambda \tilde{G}_{\mu \nu} - \frac{\Lambda^2}{2k^2} e^{\mp 2Y \varphi} V(k e^{\pm Y \varphi}) \tilde{g}_{\mu \nu} - \Omega_0 \left( \varphi_{; \mu} \varphi_{; \nu} - \frac{1}{2} \varphi_{; \alpha} \varphi^{\alpha} \tilde{g}_{\mu \nu} \right) = \mathcal{A} \tilde{T}_{\mu \nu} \\
2 \Omega_0 \Box \varphi + \frac{\Lambda^2}{k^2} e^{\mp 2Y \varphi} \left( \frac{\mathcal{V}}{k^2} (k e^{\pm Y \varphi}) \mp 2 Y V(k e^{\pm Y \varphi}) \right) + \mathcal{A} \tilde{L}_{m, \varphi} = 0 \end{cases}
\]  

(5.34)

where the matter tensor, which now coupled with the scalar field \( \varphi \), in the Einstein frame reads

\[
\tilde{T}_{\mu \nu} = - \frac{1}{\sqrt{-\tilde{g}}} \left( \delta \left( \sqrt{-\tilde{g}} \tilde{L}_m \right) \right) = \frac{\Lambda^2}{2k^2} e^{\mp 2Y \varphi} \left[ \mathcal{L}_m \left( \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho \sigma} \right) \tilde{g}_{\mu \nu} - 2 \delta \left( \delta \tilde{g}_{\mu \nu} \mathcal{L}_m \left( \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho \sigma} \right) \right) \right] ,
\]  

(5.35)

and

\[
\tilde{L}_{m, \varphi} = \mp \frac{\Lambda^2 Y}{k^2} e^{\mp 2Y \varphi} \left[ 2 \mathcal{L}_m \left( \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho \sigma} \right) + \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho \sigma} \delta \mathcal{L}_m \left( \frac{\Lambda}{k} e^{\mp Y \varphi} \tilde{g}_{\rho \sigma} \right) \right] .
\]  

(5.36)

Actually, in order to calculate the weak field and small velocity limit of the model in the Einstein frame, we can develop the two scalar fields at the second order \( \varphi \sim \varphi^{(0)} + \varphi^{(2)} \) and \( \varphi \sim \varphi^{(0)} + \varphi^{(2)} \) with respect to a background value. This choice gives the relations:

\[
\begin{cases}
\varphi^{(0)} = \pm Y^{-1} \ln \frac{\phi^{(0)}}{k} \\
\varphi^{(2)} = \pm Y^{-1} \frac{\phi^{(2)}}{\phi^{(0)}}
\end{cases}
\]  

(5.37)
Let us consider the conformal transformation \( \tilde{g}_{\mu\nu} = \frac{\phi}{\Lambda} g_{\mu\nu} \) \( (1.32) \). From this relation and considering the \( (5.32) \) one obtains, if \( \phi^{(0)} = \Lambda \), that

\[
\begin{align*}
\tilde{g}^{(2)}_{tt} & = g^{(2)}_{tt} + \frac{\phi^{(2)}}{\phi^{(0)}} \\
\tilde{g}^{(2)}_{ij} & = g^{(2)}_{ij} - \frac{\phi^{(2)}}{\phi^{(0)}} \delta_{ij}
\end{align*}
\]  

(5.38)

As matter of fact, since \( g^{(2)}_{tt} = 2\Phi^{JF} \), \( g^{(2)}_{ij} = 2\Psi^{JF} \delta_{ij} \) and \( \tilde{g}^{(2)}_{tt} = 2\Phi^{EF} \), \( \tilde{g}^{(2)}_{ij} = 2\Psi^{EF} \delta_{ij} \) from \( (5.38) \) it descends a relevant relation which links the gravitational potentials of Jordan and Einstein frame:

\[
\begin{align*}
\Phi^{EF} & = \Phi^{JF} + \frac{\phi^{(2)}}{2\phi^{(0)}} = \Phi^{JF} \pm \frac{Y}{2} \phi^{(2)} \\
\Psi^{EF} & = \Psi^{JF} - \frac{\phi^{(2)}}{2\phi^{(0)}} = \Psi^{JF} \mp \frac{Y}{2} \phi^{(2)}
\end{align*}
\]  

(5.39)

If we introduce the variations of two potentials: \( \Delta \Phi = \Phi^{JF} - \Phi^{EF} \) and \( \Delta \Psi = \Psi^{JF} - \Psi^{EF} \) we obtain the most relevant result of this paragraph:

\[
\Delta \Phi = - \Delta \Psi = - \frac{\phi^{(2)}}{2\phi^{(0)}} = \mp \frac{Y}{2} \phi^{(2)} \propto a_2 \propto f''(0) .
\]  

(5.40)

From the above expressions, one can notice that there is an evident difference between the behaviour of the two gravitational potentials in the two frames. Such a result suggests that, at the Newtonian level, it is possible to discriminate between the two mathematical frame thus one can deduce what is the true physical one. In particular, once, the gravitational potential is calculated in the Jordan frame and the dynamical evolution of \( \phi \) is taken into account at the suitable perturbation level, these can be substituted in the first of \( (5.39) \) so that to obtain its Einstein frame evolution. The final step is that the two potentials have to be matched with experimental data in order to investigate what is the true physical solution. A similar result has been provided in a recent paper \[204\].
Chapter 6

The Newtonian limit of Fourth Order Gravity theory

The Newtonian limit of HOG is worked out discussing its viability with respect to the standard results of GR. We exclusively investigate the limit in the metric approach, refraining from exploiting the formal equivalence of higher-order theories by considering the analogy with specific scalar-tensor theories, i.e., we work in the Jordan frame in order to avoid possible misleading interpretations of the results. Considering the Taylor expansion of a generic $f$-gravity, it is possible to obtain general solutions in terms of the metric coefficients up to the third order of approximation. Furthermore, we show that the Birkhoff theorem is not a general result for $f$-gravity since time-dependent evolution for spherically symmetric solutions can be achieved depending on the order of perturbations $[C]$, $[G]$ . Furthermore we provide explicit solutions for several different types of Lagrangians containing powers of the Ricci scalar as well as combinations of the other curvature invariants $[E]$. In particular, we develop the Green function method for fourth-order theories in order to find out solutions. Finally, the consistency of the results with respect to GR is discussed. In particular, the solution relative to the $g_{tt}$ component gives a gravitational potential always corrected with respect to the Newtonian one of the linear theory $f = R$.

6.1 The Newtonian limit of $f$ – gravity in spherically symmetric background

Exploiting the formalism of Newtonian and post-Newtonian approximation described in paragraph (2.3), we can develop a systematic analysis in the limits of weak field and small velocities for the $f$-gravity. We are going to assume, as background, a spherically symmetric spacetime and we are
going to investigate the vacuum case. Considering the metric (1.89), we have, for a given $g_{\mu\nu}$:

$$
\begin{align*}
    g_{tt}(t, r) &\simeq 1 + g^{(2)}_{tt}(t, r) + g^{(4)}_{tt}(t, r) \\
    g_{rr}(t, r) &\simeq -1 + g^{(2)}_{rr}(t, r) \\
    g_{\theta\theta}(t, r) &= -r^2 \\
    g_{\phi\phi}(t, r) &= -r^2 \sin^2 \theta
\end{align*}
$$

while considering Eqs. (2.36):

$$
\begin{align*}
    g^{tt} &\simeq 1 - g^{(2)}_{tt} + [g^{(2)}_{tt}, g^{(4)}_{tt}] \\
    g^{rr} &\simeq -1 - g^{(2)}_{rr}
\end{align*}
$$

The determinant reads

$$
g \simeq r^4 \sin^2 \theta \{ -1 + [g^{(2)}_{rr} - g^{(2)}_{tt}] + [g^{(2)}_{tt} g^{(2)}_{rr} - g^{(4)}_{tt} g^{(4)}_{rr}] \}. \tag{6.3}
$$

The Christoffel symbols (2.39) are

$$
\begin{align*}
    \Gamma^{(3)}_{tt} &= \frac{g^{(2)}_{tt}}{2} \\
    \Gamma^{(2)}_{tt} + \Gamma^{(4)}_{tt} &= \frac{g^{(2)}_{tt}}{2} + \frac{g^{(2)}_{tt} - g^{(4)}_{tt}}{2} \\
    \Gamma^{(3)}_{tr} &= \frac{g^{(2)}_{tr}}{2} \\
    \Gamma^{(2)}_{tr} + \Gamma^{(4)}_{tr} &= \frac{g^{(2)}_{tr}}{2} + \frac{g^{(2)}_{tr} - g^{(4)}_{tr}}{2} \\
    \Gamma^{(3)}_{rr} &= \frac{g^{(2)}_{rr}}{2} \\
    \Gamma^{(2)}_{rr} + \Gamma^{(4)}_{rr} &= -\frac{g^{(2)}_{rr}}{2} - \frac{g^{(2)}_{rr} g^{(2)}_{rr}}{2} \\
    \Gamma^{\phi\phi}_{\theta\theta} &= \sin^2 \theta \Gamma^{\phi\phi}_{\theta\theta} \\
    \Gamma^{(0)}_{\theta\theta} + \Gamma^{(2)}_{\theta\theta} + \Gamma^{(4)}_{\theta\theta} &= -r - r g^{(2)}_{rr} - r g^{(2)}_{rr}
\end{align*}
$$

while the Ricci tensor component, (2.40), are
\[ R_{tt}^{(2)} = \frac{rg_{tt,rr}^{(2)} + 2g_{tt,r}^{(2)}}{2r} \]

\[ R_{rr}^{(4)} = -\frac{rg_{tt,r}^{(2)} + 4g_{tt}^{(4)} + 4g_{r,r}^{(2)} + 2g_{rr}^{(2)} + 2g_{tt,rr}^{(2)} + 2rg_{tt,rr}^{(2)} + 2rg_{rr,rr}^{(2)}}{4r} \]

\[ R_{θθ}^{(2)} = -\frac{g_{rr,t}^{(2)}}{r} \]

\[ R_{φφ}^{(2)} = \sin^2 \theta R_{θθ}^{(2)} \]

and, finally, the Ricci scalar expression is

\[ R^{(2)} = \frac{2g_{rr}^{(2)} + 2g_{r,r}^{(2)} + 2g_{rr,rr}^{(2)}}{r^2} \]

\[ R^{(4)} = \frac{1}{2r^2} \left[ 4g_{rr}^{(2)} + 2rg_{rr}^{(2)} + 4g_{tt}^{(2)} + 4g_{r,r}^{(2)} + 2g_{rr}^{(2)} \right] + r \left\{ -rg_{tt,r}^{(2)} + 4g_{tt}^{(4)} + \right. \]

\[ +rg_{tt,r}^{(2)}g_{rr}^{(2)} - 2g_{tt}^{(2)} \left[ 2g_{tt,r}^{(2)} + rg_{rr}^{(2)} \right] + 2rg_{tt,rr}^{(2)} + 2rg_{rr,rr}^{(2)} \right\} \]

In order to derive the Newtonian and post-Newtonian approximation for a generic function \( f \), one should specify the \( f - \) Lagrangian into the field equations (1.25). This is a crucial point because once a certain Lagrangian is chosen, one will obtain a particular approximation referred to such a choice. This means to lose any general prescription and to obtain corrections to the Newtonian potential, \( Φ(x) \), which refer "univocally" to the considered \( f - \) function. Alternatively, one can restrict to analytic \( f - \) functions expandable with respect to a certain value \( R = R_0 = \) constant or, at least, its non-analytic part, if exists at all, goes to zero faster than \( R^n \), with \( n \geq 2 \) at \( R \rightarrow 0 \). In general, such theories are physically interesting and allow to recover the GR results and the correct boundary and asymptotic conditions. Then we assume
Chapter 6  The Newtonian limit of Fourth Order Gravity theory

\[ f = \sum_{n} \frac{f^{n}(R_0)}{n!} (R - R_0)^n \simeq f_0 + f_1 R + f_2 R^2 + f_3 R^3 + \ldots, \quad (6.7) \]

One has to note that the development (6.7), also if similar to (3.27), is very different from the one in the Chapter 3: in fact, \( R^{(0)} \) is a general space-time function linked to the background metric \( g_{\mu \nu}^{(0)} \) (3.25), while here \( R_0 \) is a constant value of scalar curvature. Besides the coefficients \( f_0, f_1, f_2, f_3 \) are not proportional, respectively, to zero-th, first, second, third coefficient of Taylor development of \( f \). In fact, we have

\[
\begin{align*}
  f_0 &= f(R_0) - R_0 f'(R_0) + \frac{1}{2} R_0^2 f''(R_0) - \frac{1}{6} R_0^3 f'''(R_0) \\
  f_1 &= f'(R_0) - R_0 f''(R_0) + \frac{1}{2} R_0^2 f'''(R_0) \\
  f_2 &= \frac{1}{2} f''(R_0) - \frac{1}{2} R_0 f'''(R_0) \\
  f_3 &= \frac{1}{6} f'''(R_0)
\end{align*}
\]

(6.8)

If we consider a flat background, then \( R_0 = 0 \) and the coefficients \( f_0, f_1, f_2, f_3 \) are the terms of Taylor series. But if we are finding the solutions at Newtonian and (possibility) post-newtonian level we have to consider a vanishing background scalar curvature. It is possible to obtain the Newtonian and post-Newtonian approximation of \( f \) - gravity considering such an expansion (6.7) into the field equations (1.25) and expanding the system up to the orders \( O(0), O(2), O(3) \) and \( O(4) \). This approach provides general results and specific (analytic) Lagrangians are selected by the coefficients \( f_i \) in (6.7). Developing the equations in the case of vanishing matter, i.e. \( T_{\mu \nu} = 0 \), we have

\[
\begin{align*}
  H_{\mu \nu}^{(0)} &= 0, \quad H^{(0)} = 0 \\
  H_{\mu \nu}^{(2)} &= 0, \quad H^{(2)} = 0 \\
  H_{\mu \nu}^{(3)} &= 0, \quad H^{(3)} = 0 \\
  H_{\mu \nu}^{(4)} &= 0, \quad H^{(4)} = 0
\end{align*}
\]

(6.9)
and, in particular, from the O(0) order approximation, one obtains

\[ f_0 = 0 \]  

(6.10)

which trivially follows from the above assumption that the space-time is asymptotically Minkowskian (asymptotically flat background). This result suggests a first consideration. If the Lagrangian is developable around a vanishing value of the Ricci scalar \( R_0 = 0 \) the relation (6.10) will imply that the cosmological constant contribution has to be zero whatever is the \( f \)-gravity theory.

If we now consider the O(2) - order approximation, the equations system (6.9), in the vacuum case, results to be

\[
\begin{align*}
  f_1 r R^{(2)} - 2 f_1 g^{(2)}_{tt,r} + 8 f_2 R^{(2)}_{,r} - f_1 r g^{(2)}_{tt,rr} + 4 f_2 r R^{(2)}_{,r} &= 0 \\
  f_1 r R^{(2)} - 2 f_1 g^{(2)}_{rr,r} + 8 f_2 R^{(2)}_{,r} - f_1 r g^{(2)}_{tt,rr} &= 0 \\
  2 f_1 g^{(2)}_{rr} - r [ f_1 r R^{(2)} - f_1 g^{(2)}_{tt,r} - f_1 g^{(2)}_{rr,r} + 4 f_2 R^{(2)}_{,r} + 4 f_2 r R^{(2)}_{,rr} ] &= 0 \\
  f_1 r R^{(2)} + 6 f_2 [ 2 R^{(2)}_{,r} + r R^{(2)}_{,rr} ] &= 0 \\
  2 g^{(2)}_{rr} + r [ 2 g^{(2)}_{tt,r} - r R^{(2)} + 2 g^{(2)}_{rr,r} + r g^{(2)}_{tt,rr} ] &= 0
\end{align*}
\]

(6.11)

The last equation of the system (6.11) is the definition of Ricci scalar (1.5) at O(2) - order. The trace equation (the fourth line in the (6.11)), in particular, provides a differential equation with respect to the Ricci scalar which allows to solve, if sign\([ f_1 ] = - \)sign\([ f_2 ]\), the system (6.11) at O(2) - order:

\[
\begin{align*}
  g^{(2)}_{tt} &= \delta_0 - \frac{\delta_1}{f_1 r} - \frac{\delta_2(t)}{3 \lambda} \frac{e^{-\lambda r}}{\lambda r} + \frac{\delta_3(t)}{6 \lambda^2} \frac{e^{\lambda r}}{\lambda r} \\
  g^{(2)}_{rr} &= - \frac{\delta_1}{f_1 r} - \frac{\delta_2(t)}{3 \lambda} \frac{\lambda r + 1}{\lambda r} e^{-\lambda r} + \frac{\delta_3(t)}{6 \lambda^2} \frac{\lambda r - 1}{\lambda r} e^{\lambda r} \\
  R^{(2)} &= \delta_2(t) \frac{e^{-\lambda r}}{r} + \frac{\delta_3(t)}{2 \lambda} \frac{e^{\lambda r}}{r}
\end{align*}
\]

(6.12)

where
\[ \lambda \equiv \sqrt{-\frac{f_1}{6f_2}}, \quad (6.13) \]
with the dimension of length\(^{-1}\). Let us notice that the integration constant \(\delta_0\) has to be dimensionless, \(\delta_1\) has the dimension of length, while the time-dependent functions \(\delta_2\) and \(\delta_3\), respectively, have the dimensions of length\(^{-1}\) and length\(^{-2}\). The functions \(\delta_i(t)\) \((i = 2, 3)\) are completely arbitrary since the differential equation system (6.23) contains only spatial derivatives. Besides, the integration constant \(\delta_0\) can be set to zero, as in the theory of the potential, since it represents an unessential additive quantity. When we consider the limit \(f \to R\), in the case of a point-like source of mass \(M\), we recover the perturbed version of standard Schwarzschild solution (2.8) at O(2)-order with \(\delta_1 = r_g\). In order to match at infinity the Minkowskian prescription of the metric, we discard the Yukawa growing mode present in (6.12), we have:

\[
\begin{align*}
\left\{ 
\begin{array}{l}
\quad ds^2 = \left[ 1 - \frac{r_g}{f_1 r} + \frac{\delta_2(t) e^{-\lambda r}}{3\lambda} \right] dt^2 - \left[ 1 + \frac{r_g}{f_1 r} + \frac{\delta_2(t) e^{\lambda r + 1}}{3\lambda} \right] dr^2 - r^2 d\Omega \\
\quad R = \frac{\delta_2(t) e^{-\lambda r}}{r}
\end{array}
\right.
\end{align*}
\quad (6.14)
\]

At this point one can provide the solution in terms of the gravitational potential. In such a case, we have an explicit Newtonian-like term into the definition, according to previous results obtained with less rigorous methods \([32, 132]\). The first of (6.12) provides the second order solution in terms of the metric expansion (see the definition (6.1)), but, this term coincides with the gravitational potential at the Newtonian order (2.65). In particular, since we have \(g_{tt} = 1 + 2\Phi\), the gravitational potential of a HOG theory, analytic in the Ricci scalar \(R\), is

\[ \Phi = -\frac{GM}{f_1 r} + \frac{\delta_2(t) e^{-\lambda r}}{6\lambda} \quad (6.15) \]

As first remark, one has to notice that the structure of the potential (6.15), for a given \(f\)-theory, is determined by the parameter \(\lambda\), (6.13), which depends on the first and the second derivative of \(f\), once developed around a vanishing value of Ricci scalar. The potential (6.15) is coherent with respect to the results in \([132, 172]\), obtained for higher order Lagrangians as \(f = R + \sum_{k=0}^{p} a_k R \Box^k R\). In this last case, it is demonstrated that the number of Yukawa corrections to the gravitational potential was strictly related to the order of the theory.
From (6.14) one can notice that the Newtonian limit of any analytic $f$-theory is related only to the first and second term of the Taylor expansion (6.7) of the given theory. *In other words, the gravitational potential is always characterized by two Yukawa corrections and only the first two terms of the Taylor expansion of a generic $f$ Lagrangian turn out to be relevant. This is indeed a general result.*

Let us now consider the system (6.9) at third order contributions. The first important issue is that, at this order, one has to consider even the off-diagonal equation:

$$f_1 g_{r r, t}^{(2)} + 2 f_2 r R_{r t}^{(2)} = 0,$$  \hspace{1cm} (6.16)

which relates the time derivative of the Ricci scalar to the time derivative of $g_{r r}^{(2)}$. From this relation, it is possible to draw a relevant consideration. One can deduce that, if the Ricci scalar depends on time, so it is the same for the metric components and even the gravitational potential turns out to be time-dependent. This result agrees with the analysis provided in the Chapter 3 where a complete description of the weak field limit of HOG has been provided in terms of the dynamical evolution of the Ricci scalar. Moreover, it was demonstrated that if one supposes the time independent Ricci scalar (3.21), static spherically symmetric solutions result of the form (3.24).

Eq. (6.16) confirms this result and provides the formal theoretical explanation of such a behavior. In particular, together with Eqs. (6.14), it suggests that if one considers the problem at a lower level of approximation (i.e. the second order) the background spacetime metric can have static solutions according to the Birkhoff theorem; this is no more verified when the problem is faced with approximations of higher order. In other words, the debated issue to prove the validity of the Birkhoff theorem in higher order theories of gravity, finds here its physical answer. In the Chapter 3 and here, the validity of this theorem is demonstrated for $f$-theories only when the Ricci scalar is time independent or, in addition, when the solutions of Eqs. (1.25) are investigated up to the second order of approximation in the metric coefficients (6.1). *Therefore, the Birkhoff theorem is not a general result for HOG but, on the other hand, in the limit of small velocities and weak fields (which is enough to deal with the Solar System gravitational experiments), one can assume that the gravitational potential is effectively time independent according to (6.14) and (6.15).*

The above results fix a fundamental difference between GR and HOG theories. While, in GR, a spherically symmetric solution represents a stationary and static configuration difficult to be related to a cosmological background evolution, this is no more true in the case of HOG. In the latter case, a spherically symmetric background can have time-dependent evolution together with the radial dependence. In this sense, a relation between a spherical solution and the cosmological Hubble...
flow can be easily achieved.

From the system (6.9), one can notice that the general solution is characterized only by the first three orders of the $f$ expansion (6.7). Such a result is in agreement with the $f$ reconstruction which can be induced by the post-Newtonian parameters adopting a scalar-tensor analogy as discussed in Chapter 5.

### 6.1.1 Newtonian and post-Newtonian limit in the harmonic gauge

Up to now, the discussion has been developed without any gauge choice. In order to overcome the difficulties related to the nonlinearities of calculations, we can work considering some gauge choice obtaining less general solutions for the metric entries. If we consider the gauge (2.43) we can use the Ricci tensor components (2.53) and the Ricci scalar expression (2.54). The gauge choice does not affect the Christoffel symbols. Thus, by solving the field equations (1.25) one obtains

\[
\begin{align*}
  g_{tt\mid HG}(t, r) &= 1 + k_2^{(2)} + k_0^{(4)} - \frac{k_1^{(2)} + 2k_1^{(2)}k_3^{(2)} + k_2^{(4)}}{r} + \frac{k_3^{(2)} - 2k_2^{(2)}k_3^{(2)}}{r^2} - 2k_1^{(2)} k_4^{(2)} \log r \\
  g_{rr\mid HG}(t, r) &= -1 + k_4^{(2)} - \frac{k_2^{(2)}}{r}
\end{align*}
\]

(6.17)

where the constants $k_i^{(2)}$ are relative to approximation level $O(2)$, while $k_i^{(4)}$ to $O(4)$. The Ricci scalar is zero both at $O(2)$ and at $O(4)$ approximation orders.

Eqs. (6.17) suggest some interesting remarks. It is easy to check that the GR prescriptions are immediately recovered for a particularly choice of integration constants. The $g_{rr}$ component displays only the second order term, as required by a GR-like behavior, while the $g_{tt}$ component shows also the fourth order corrections which determine the second post-Newtonian parameter $\beta$ [129]. It has to be stressed here that a full post-Newtonian formalism requires to take into account matter in the system (6.9): the presence of matter links the second and fourth order contributions in the metric coefficients [129].

### 6.2 The Newtonian limit of quadratic gravity

Since terms resulting from $R^n$ with $n \geq 3$ do not contribute in the Newtonian limit, as seen previously, the most general choice for the Lagrangian is
6.2 The Newtonian limit of quadratic gravity

\[ f = a_1 R + a_2 R^2 + a_3 R_{\alpha\beta} R^{\alpha\beta} \] (6.18)

where \( a_1, a_2, a_3 \) are constants. We have to note that the field equations (1.25) (with \( f = R^2 \)), (1.74) and (1.78) satisfy, in four dimensions, the condition

\[ H_{\mu\nu} R^{2\mu\nu} - 4 H_{\mu\nu}^{\text{Ric}} + H_{\mu\nu}^{\text{Rie}} = 0, \] (6.19)

then only two of the three expressions are independent. Such a quantity is related to the Gauss-Bonnet topological invariant. We can consider Eq. (6.18) as the most general quadratic theory of gravity. The field equations of (6.18) are a linear combination of (1.10), (1.25) and (1.74), that is

\[ a_1 G_{\mu\nu} + a_2 H_{\mu\nu}^{R^2} + a_3 H_{\mu\nu}^{\text{Ric}} = \mathcal{X} T_{\mu\nu}. \] (6.20)

If we introduce the generalization of the gravitational potentials in the isotropic metric (1.88) by the quantities \( \Phi \) and \( \Psi \) linked to \( g^{(2)}_{tt} \) and \( g^{(2)}_{ij} \), we can investigate the solution of field equations (6.20) in the Newtonian limit:

\[ ds^2 = \left[ 1 + 2\Phi \right] dt^2 - \left[ 1 - 2\Psi \right] \delta_{ij} dx^i dx^j. \] (6.21)

Up to the Newtonian order the left-hand side of the field equations (1.10), (1.25), (1.74) and (1.78) are

1. for the GR - theory:

\[
\begin{align*}
G_{tt} &\sim G^{(2)}_{tt} = 2 \Delta \Psi \\
G_{ij} &\sim G^{(2)}_{ij} = \Delta (\Phi - \Psi) \delta_{ij} - (\Phi - \Psi)_{,ij}
\end{align*}
\] (6.22)

2. for \( R^2 \) - theory:

Note that \(|a_2| = |a_3| = \text{length}^2\) and \(|a_1| = \text{length}^0\)
\[
\begin{align*}
H^{R2}_{tt} & \sim H^{R2(2)}_{tt} = 4\Delta^2(\Phi - 2\Psi) \\
H^{R2}_{ij} & \sim H^{R2(2)}_{ij} = 4\left[\Delta^2(2\Psi - \Phi)\delta_{ij} + (\Delta \Phi - 2\Delta \Psi),_{ij}\right]
\end{align*}
\] (6.23)

3. for \(R_{\alpha\beta}R^{\alpha\beta}\) - theory:

\[
\begin{align*}
H^{Ric}_{tt} & \sim H^{Ric(2)}_{tt} = 2\Delta^2(\Phi - \Psi) \\
H^{Ric}_{ij} & \sim H^{Ric(2)}_{ij} = \Delta^2(3\Psi - \Phi)\delta_{ij} + (\Delta \Phi - 3\Delta \Psi),_{ij}
\end{align*}
\] (6.24)

4. for \(R_{\alpha\beta\gamma\delta}R^{\alpha\beta\gamma\delta}\) - theory:

\[
\begin{align*}
H^{Rie}_{tt} & \sim H^{Rie(2)}_{tt} = 4\Delta^2\Phi \\
H^{Rie}_{ij} & \sim H^{Rie(2)}_{ij} = 4\left[\Delta^2\Psi\delta_{ij} - (\Delta \Psi),_{ij}\right]
\end{align*}
\] (6.25)

If we take into account the results (6.22), (6.23) and (6.24) for the geometric side and the results (2.57) for the matter side, the explicit form of the field equations (6.20) up to the Newtonian order is

\[
\begin{align*}
2a_1\Delta \Psi - 2(4a_2 + a_3)\Delta^2\Psi + 2(2a_2 + a_3)\Delta^2\Phi &= \mathcal{X}\rho, \\
\Delta \left[a_1(\Psi - \Phi) + (4a_2 + a_3)\Delta \Phi - (8a_2 + 3a_3)\Delta \Psi\right]\delta_{ij} \\
- \left[a_1(\Psi - \Phi) + (4a_2 + a_3)\Delta \Phi - (8a_2 + 3a_3)\Delta \Psi\right],_{ij} &= 0
\end{align*}
\] (6.26)

By introducing two new auxiliary functions (\(\tilde{\Phi}\) and \(\tilde{\Psi}\)), the equations (6.26) become
6.2 The Newtonian limit of quadratic gravity

\[
\begin{align*}
\frac{2a_2}{3a_2+a_3} \Delta^2 \tilde{\Psi} &= \frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)} \Delta^2 \tilde{\Phi} - \frac{4a_2+a_3}{2a_2+a_3} \Delta \tilde{\Phi} - \frac{a_1}{2a_2+a_3} \Delta \tilde{\Psi} = \mathcal{X} \rho \\
\Delta \left[ \Phi + \Delta \tilde{\Psi} \right] \delta_{ij} - \left[ \tilde{\Phi} + \Delta \tilde{\Psi} \right]_{,ij} &= 0
\end{align*}
\]

(6.27)

where \( \tilde{\Phi} \) and \( \tilde{\Psi} \) are linked to \( \Phi \) and \( \Psi \) via

\[
\begin{align*}
\Phi &= -\frac{(8a_2+3a_3)\tilde{\Phi}+a_1 \tilde{\Psi}}{2a_1(2a_2+a_3)} \\
\Psi &= -\frac{(4a_2+a_3)\tilde{\Phi}+a_1 \tilde{\Psi}}{2a_1(2a_2+a_3)}
\end{align*}
\]

(6.28)

Obviously we must require \( a_1(2a_2 + a_3) \neq 0 \), which is the determinant of the transformations (6.28). Let us introduce the new function \( \Xi \) defined as follows:

\[
\Xi := \tilde{\Phi} + \Delta \tilde{\Psi}.
\]

(6.29)

At this point, we can use the new function \( \Xi \) to uncouple the system (6.26). With the choice \( \tilde{\Phi} = \Xi - \Delta \tilde{\Psi} \), it is possible to rewrite equations (6.26) as follows

\[
\begin{align*}
\frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)} \Delta^2 \tilde{\Psi} + \frac{6a_2+a_3}{2a_2+a_3} \Delta^2 \tilde{\Phi} - \frac{a_1}{2a_2+a_3} \Delta \tilde{\Phi} &= \mathcal{X} \rho + \tau, \\
\Delta \Xi \delta_{ij} - \Xi_{,ij} &= 0
\end{align*}
\]

(6.30)

where \( \tau \equiv \frac{4a_2+a_3}{2a_2+a_3} \Delta \Xi + \frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)} \Delta^2 \Xi \). We are interested in the solution of (6.27) in terms of the Green function \( G(x, x') \) defined by

\[
\tilde{\Psi}(x) = Y \int d^3x' G(x, x') \sigma(x'),
\]

(6.31)

where
\[
\sigma(x) \doteq \lambda \rho(x) + \tau(x),
\]
and \(Y\) being a constant, which we introduce for dimensional reasons. Then Eqs. (6.26) are equivalent to

\[
\begin{align*}
\left\{ \begin{array}{l}
\frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)} \triangle^3 G(x, x') + \frac{6a_2+a_3}{2a_2+a_3} \triangle^2 G(x, x') - \frac{a_1}{2a_2+a_3} \triangle G(x, x') = Y^{-1} \delta(x - x')
\end{array} \right.
\end{align*}
\]

\[\triangle \Xi(x) \delta_{ij} - \Xi(x),_{ij} = 0\]

where \(\delta(x - x')\) is the 3-dimensional Dirac \(\delta\)-function. The general solution of (6.27) for \(\Phi(x)\) and \(\Psi(x)\), in terms of the Green function \(G(x, x')\) and the function \(\Xi(x)\), are

\[
\begin{align*}
\Phi(x) &= Y \left( \frac{8a_2+3a_3}{2a_1(2a_2+a_3)} \right) \int d^3x' G(x, x') \left[ \lambda \rho(x') \right. \\
&\quad + \frac{4a_2+a_3}{2a_2+a_3} \triangle x' \Xi(x') + \frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)} \triangle^2 x' \Xi(x') \\
&\quad - \frac{8a_2+3a_3}{2a_1(2a_2+a_3)} \Xi(x)
\end{align*}
\]

\[\Psi(x) = Y \left( \frac{4a_2+a_3}{2a_1(2a_2+a_3)} \right) \int d^3x' G(x, x') \left[ \lambda \rho(x') \right. \\
&\quad + \frac{4a_2+a_3}{2a_2+a_3} \triangle x' \Xi(x') + \frac{2a_3(3a_2+a_3)}{a_1(2a_2+a_3)} \triangle^2 x' \Xi(x') \\
&\quad - \frac{4a_2+a_3}{2a_1(2a_2+a_3)} \Xi(x)
\]

Eqs. (6.27) represent a coupled set of fourth order differential equations. The total number of integration constant is eight. With the substitution (6.29), it has been possible to decouple the set of equations, but now the differential order is changed. The total differential order is the same, indeed we have one equation of sixth order and another equation of second order, while previously we had two equations of fourth order. The number of integration constants is conserved. We can conclude that, with our approach, also introducing the new quantities \(\tilde{\Phi}, \tilde{\Psi}\) does not contradict the paradigm of a metric theory of HOG. The price is that now the r.h.s. of \(tt\)-component of field equation has been modified: there is an additional matter term \(\tau\) coming from the \(ij\)-component. (see the redefinition of the matter density (6.32)). In Table 6.1 we show particular cases of Eqs. (6.27) for different choices of coupling constants of the theory with vanishing the determinant of...
6.3 Green functions for systems with spherical symmetry

We are interested in the solutions of field Eqs. (6.27) at Newtonian order by using the method of Green functions and assuming a system with spherical symmetry: \( G(x, x') = G(|x - x'|) \). Let us introduce the radial coordinate \( r = |x - x'| \); with this choice, the first equation of (6.33) for \( r \neq 0 \) becomes

\[
2a_3(3a_2 + a_3) \Delta^3 G(x, x') + (6a_2 + a_3) \Delta^2 G(x, x') - a_1^2 \Delta r G(x, x') = 0, \tag{6.35}
\]

where \( \Delta_r = r^{-2} \partial_r (r^{-2} \partial_r) \) is the radial component of the Laplacian in polar coordinates. The solution of (6.35) is:

\[
G(r) = K_1 - \frac{1}{r} \left[ K_2 + \frac{a_3}{a_1} \left( K_3 e^{-\sqrt{\frac{a_2}{a_3}} r} + K_4 e^{\sqrt{\frac{a_2}{a_3}} r} \right) \right.
- \left. 2(3a_2 + a_3) \left( K_5 e^{-\sqrt{\frac{a_2}{3(a_2 + a_3)}} r} + K_6 e^{\sqrt{\frac{a_2}{3(a_2 + a_3)}} r} \right) \right], \tag{6.36}
\]

where \( K_i \) are constants. We note that, if \( a_2 = a_3 = 0 \), the Green function of the Newtonian mechanics is found. It is the same of the Electromagnetism. The integration constants \( K_i \) have to be fixed by imposing the boundary conditions at infinity and in the origin. In fact Eqs. (6.36) is a solution of (6.35) and not of the first equation in (6.33). A physically acceptable solution has to satisfy the condition \( G(x, x') \to 0 \) if \( |x - x'| \to \infty \), then the constants \( K_1, K_4, K_6 \) in (6.36) have to vanish. To obtain the conditions on the constants \( K_2, K_3, K_5 \) we consider the Fourier transformation of \( G(x, x') \), that is

\[
\mathcal{G}(x, x') = \tilde{G}(k) = \int \frac{d^3 k}{(2\pi)^{3/2}} \tilde{G}(k) e^{ik \cdot (x - x')} \tag{6.37}
\]

Eq. (6.33), in terms of Fourier transform, becomes
Table 6.1: Explicit form of the field equations for different choices of the coupling constants for which the determinant of the transformations (6.28) vanishes.

| Case | Choices of constants | Corresponding field equations |
|------|----------------------|-------------------------------|
| A    | $a_2 = 0$ $a_3 = 0$ | $\nabla^2 \Psi = \frac{X}{2a_1} \rho,$  
$\nabla \left[ \Phi(x) + \frac{G}{a_1} \int d^3x' \frac{\rho(x')}{|x-x'|} \right] \delta_{ij}$  
$- \left[ \Phi(x) + \frac{G}{a_1} \int d^3x' \frac{\rho(x')}{|x-x'|} \right]_{ij} = 0$ |
| B    | $a_1 = 0$ $a_3 = 0$ | $\nabla^2 (2\Psi - \Phi) = -\frac{X}{4a_2} \rho,$  
$\nabla \left[ (2\Psi - \Phi) \right] \delta_{ij} - \left[ (2\Psi - \Phi) \right]_{ij} = 0$ |
| C    | $a_1 = 0$ $a_2 = 0$ | $\nabla^2 (2\Psi - \Phi) = \frac{X}{2a_1} \rho,$  
$\nabla \left[ (2\Psi - \Phi) \right] \delta_{ij} - \left[ (2\Psi - \Phi) \right]_{ij} = 0$ |
| D    | $a_3 = -2a_2$ | $2a_2 \nabla^2 \Psi - a_1 \nabla \Psi = -\frac{X}{2} \rho,$  
$\nabla^2 \left[ a_1 \Phi(x) - 2a_2 \nabla \Phi(x) + G \int d^3x' \frac{\rho(x')}{|x-x'|} \right] \delta_{ij}$  
$- \left[ a_1 \Phi(x) - 2a_2 \nabla \Phi(x) + G \int d^3x' \frac{\rho(x')}{|x-x'|} \right]_{ij} = 0$ |
| E    | $a_1 = 0$ $a_3 = -4a_2$ | $\nabla^2 \Phi = -\frac{X}{4a_2} \rho,$  
$\nabla \left[ (2\Psi - \Phi) \right] \delta_{ij} - \left[ (2\Psi - \Phi) \right]_{ij} = 0$ |
| F    | $a_1 = 0$ $a_3 = -2a_2$ | $\nabla^2 \Phi = -\frac{X}{4a_2} \rho,$  
$\nabla \left[ (2\Psi - \Phi) \right] \delta_{ij} - \left[ (2\Psi - \Phi) \right]_{ij} = 0$ |
| G    | $a_1 = 0$ $a_3 = -\frac{8a_2}{3}$ | $\nabla^2 (2\Psi + \Phi) = -\frac{3X}{4a_2} \rho,$  
$\nabla \left[ (2\Psi + \Phi) \right] \delta_{ij} - \left[ (2\Psi + \Phi) \right]_{ij} = 0$ |
6.3 Green functions for systems with spherical symmetry

\[ \int \frac{d^3k}{(2\pi)^{3/2}} e^{i k \cdot (x - x')} \left\{ \hat{G}(k) \left[ -\frac{2a_3(3a_2 + a_3)}{a_1(2a_2 + a_3)} k^6 + \frac{6a_2 + a_3}{2a_2 + a_3} k^4 + \frac{a_1}{2a_2 + a_3} k^2 \right] - Y^{-1} \right\} = 0. \] (6.38)

The Green function can be expressed as follows:

\[ \hat{G}(x - x') = -Y^{-1} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{d|k|}{2a_3(3a_2 + a_3)} |k|^2 \frac{|x - x'|}{|k^2 - \frac{a_1}{a_3}|} \frac{e^{i k \cdot (x - x')}}{k^2 + \frac{a_1}{2(3a_2 + a_3)}}. \] (6.39)

Since we are investigating systems with spherical symmetry, it is better to introduce polar coordinates in the \( k \)-space. Eq. (6.39) becomes

\[ \hat{G}(x - x') = -Y^{-1} \int_0^\infty \frac{d|k|}{2a_3(3a_2 + a_3)} |k|^2 \frac{|x - x'|}{|k^2 - \frac{a_1}{a_3}|} \frac{e^{i k \cdot (x - x')}}{k^2 + \frac{a_1}{2(3a_2 + a_3)}}. \] (6.40)

The analytic expression of \( \hat{G}(x - x') \) depends on the nature of the poles of \( |k| \) and on the values of the arbitrary constants \( a_1, a_2, a_3 \). If we define two new quantities \( \lambda_1, \lambda_2 \in \mathbb{R} \):

\[ \lambda_1^2 = -\frac{a_1}{a_3}, \quad \lambda_2^2 = \frac{a_1}{2(3a_2 + a_3)}, \] (6.41)

we obtain a particular expression of (6.39):

\[ \hat{G}(x - x') = -\sqrt{\frac{\pi}{18 |x - x'|}} Y^{-1} \left[ \frac{\lambda_2^2 - \lambda_1^2}{\lambda_1^2 \lambda_2^2} \frac{e^{-\lambda_1 |x - x'|}}{\lambda_1^2} + \frac{e^{-\lambda_2 |x - x'|}}{\lambda_2^2} \right]. \] (6.42)

This Green function corresponds to the one in (6.36) but now we have also the conditions in the origin. Obviously, we have three possibilities to introduce \( \lambda_1 \) and \( \lambda_2 \). In Table 6.2, we provide the complete set of Green functions \( \hat{G}(x - x') \), depending on the choices of the coefficients \( a_2 \) and \( a_3 \).
(with a fixed sign of $a_1$).

| Case | Choices of constants | Green function |
|------|----------------------|----------------|
| A    | $a_3 < 0$            | $G^A(x - x') = \sqrt{\frac{\pi}{18}} \frac{Y^{-1}}{||x-x'||} \left[ \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 \lambda_2^2} + e^{\frac{\lambda_1 |x+x'|}{\lambda_1^2}} - \frac{e^{\lambda_2 |x-x'|}}{\lambda_2^2} \right]$ |
|      | $3a_2 + a_3 > 0$     |                |
| B    | $a_3 > 0$            | $G^B(x - x') = \sqrt{\frac{\pi}{18}} \frac{Y^{-1}}{||x-x'||} \left[ \frac{\lambda_1^2 - \lambda_2^2}{\lambda_1^2 \lambda_2^2} - \frac{\cos(\lambda_1 |x-x'|)}{\lambda_1^2} + \frac{\cos(\lambda_2 |x-x'|)}{\lambda_2^2} \right]$ |
|      | $3a_2 + a_3 < 0$     |                |
| C    | $a_3 < 0$            | $G^C(x - x') = \sqrt{\frac{\pi}{18}} \frac{Y^{-1}}{||x-x'||} \left[ \frac{\lambda_1^2 + \lambda_2^2}{\lambda_1^2 \lambda_2^2} + e^{\frac{\lambda_1 |x-x'|}{\lambda_1^2}} + \frac{\cos(\lambda_2 |x-x'|)}{\lambda_2^2} \right]$ |
|      | $3a_2 + a_3 < 0$     |                |

**Table 6.2:** The complete set of Green functions for equations (6.39). It is possible to have a further choice for the scale lengths which turns out to be dependent on the two known length scales. In fact, if we perform the substitution $\lambda_1 \leftrightarrow \lambda_2$, we obtain a fourth choice. In addition, for a correct Newtonian component, we assumed $a_1 > 0$. In fact when $a_2 = a_3 = 0$ the field equations (6.26) give us the Newtonian theory if $a_1 = 1$.

When one considers a point-like source, $\rho \propto \delta(x)$, and by setting $\Xi(x) = 0$, the potentials (6.34) are proportional to $G(x - x')$. Without losing of generality, we have:

$$\Phi(x) \sim \frac{1}{|x|} + \frac{e^{-\lambda_1 |x|}}{|x|} + \frac{e^{-\lambda_2 |x|}}{|x|}, \quad (6.43)$$

an analogous behavior is obtained for the potential $\Psi(x)$. We note that, in the vacuum case, we found Yukawa-like corrections to Newtonian mechanics but with two scale lengths related to the quadratic corrections in the Lagrangian (6.18). See also the above expressions (6.41). This behavior is strictly linked to the sixth order of (6.33), which depends on the coupled form of the system of equations (6.26). In fact if we consider the Fourier transform of the potentials $\Phi$ and $\Psi$:

$$\Phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{\Phi}(k) e^{ik \cdot x}, \quad \Psi(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \hat{\Psi}(k) e^{ik \cdot x}, \quad (6.44)$$

the solutions are
6.3 Green functions for systems with spherical symmetry

\[
\begin{align*}
\Phi(x) &= -\frac{X}{2} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{[a_1 + (8a_2 + 3a_3)k^2] \hat{\rho}(k)e^{ik\cdot x}}{k^2(a_1 - a_3k^2)[a_1 + 2(3a_2 + a_3)k^2]} \\
\Psi(x) &= -\frac{X}{2} \int \frac{d^3k}{(2\pi)^{3/2}} \frac{[a_1 + (4a_2 + a_3)k^2] \hat{\rho}(k)e^{ik\cdot x}}{k^2(a_1 - a_3k^2)[a_1 + 2(3a_2 + a_3)k^2]}
\end{align*}
\] (6.45)

where \( \hat{\rho}(k) \) is the Fourier transform of the matter density. We can see that the solutions have the same poles as (6.39). Finally, if \( \hat{\rho}(k) = \frac{M}{(2\pi)^{3/2}} \) (the Fourier transform of a point-like source) the solutions (6.45) are similar to (6.43). In fact, if we suppose that \( a_3 \neq 0 \) and \( 3a_2 + a_3 \neq 0 \), the solutions (6.45) are

\[
\begin{align*}
\Phi(x) &= -\frac{GM}{a_1|x|} \left( 1 - \frac{4}{3}e^{-\lambda_1|x|} + \frac{1}{3}e^{-\lambda_2|x|} \right) \\
\Psi(x) &= -\frac{GM}{a_1|x|} \left( 1 - \frac{2}{3}e^{-\lambda_1|x|} - \frac{1}{3}e^{-\lambda_2|x|} \right)
\end{align*}
\] (6.46)

Then the metric (6.21) becomes

\[
ds^2 = \left[ 1 - \frac{r_g}{a_1|x|} \left( 1 - \frac{4}{3}e^{-\lambda_1|x|} + \frac{1}{3}e^{-\lambda_2|x|} \right) \right] dt^2 + \\
- \left[ 1 + \frac{r_g}{a_1|x|} \left( 1 - \frac{2}{3}e^{-\lambda_1|x|} - \frac{1}{3}e^{-\lambda_2|x|} \right) \right] \delta_{ij}dx^idx^j
\] (6.47)

It is interesting to note that, if \( a_3 = 0 \) (\( \lambda_1 \to \infty \)), we have the missing of a scale length (a pole is missed) with only a Yukawa-like term as for the Electrodynamics. The Green function, in this case, is:

\[
\tilde{G}(k)_{a_3=0} = \frac{2a_2Y^{-1}}{6a_2k^4 + a_1k^2},
\] (6.48)

and the Lagrangian becomes: \( f = a_1R + a_2R^2 \). Since at the level of the Newtonian limit, as discussed, the powers of Ricci scalar higher than two do not contribute, we can conclude that (6.48) is the Green function for any \( f \)-gravity at Newtonian order, if \( f \) is an analytic function of the Ricci scalar. We found the same situation in the Newtonian limit of \( f \) - theory in standard
coordinates (§ 6.1). In fact if we consider the analogy

\[ a_1 = f_1, \quad a_2 = f_2 \]  

(6.49)

and if \( \text{sign}[a_2] = -\text{sign}[a_1] \) we found the same characteristic scale - length: \( \lambda = \lambda_2 \). Finally the presence of the pole is achieved considering a particular choice of the constants in the theory, e.g. \( a_3 = -2a_2 \). In Table [6.1](Case D), we provide the field equations for this choice and the relative Green function is:

\[ \tilde{G}_{(2a_2 \nabla^4 - a_1 \nabla^2)}(k) \propto \frac{1}{2a_2 k^4 + a_1 k^2} . \]  

(6.50)

The spatial behavior of (6.48) - (6.50) is the same but the coefficients are different since the theories are different. In conclusion we need the Green function for the differential operator \( \Delta^2 \). The only possible physical choice for the squared Laplacian is:

\[ \tilde{G}_{(\Delta^2)}(x - x') \propto \frac{1}{|x - x'|} , \]  

(6.51)

since the other choice is proportional to \( |x - x'| \) and cannot to be accepted. Considering the last possibility, we will end up with a force law increasing with distance [31]. Summary, we have shown the general approach to find out solutions of the field equations by using the Green functions. In particular, the vacuum solutions with point-like source have been used to find out directly the potentials, however it remains the most important issue to find out solutions when we consider systems with extended matter distribution.

### 6.4 Solutions using the Green function

Unlike the § 6.1 we are going to find solutions with Green functions method. Before to investigate the general solution of Eqs. (6.26) we want discuss, in the first subparagraph, all cases shown in the Table [6.1](Case D). While in the next subsections we will analyze the solution in presence of matter using the Green functions shown in Table [6.2].
6.4 Solutions using the Green function

6.4.1 Particular solutions

In Table 6.3 we provide solutions, in terms of the Green function of the corresponding differential operator, of the field equations shown in Table 6.1. Case A corresponds to the Newtonian theory and the arbitrary constant $a_1$ can be absorbed in the definition of matter Lagrangian. The solutions are:

\[ A\Phi(x) = A\Psi(x) = -G \int d^3x' \frac{\rho(x')}{|x - x'|}. \] (6.52)

For Case D, instead, we have the field equations of a sort of modified electrodynamic-like representation. The solution can be expressed as follows:

\[ D\Phi(x) = D\Psi(x) = -G \int d^3x' \left[ 1 - e^{-\sqrt{\frac{\pi}{2a_2}}|x - x'|} \right] \rho(x'). \] (6.53)

The solutions make sense only if $a_1/a_2 > 0$, then we can introduce a new scale-length. A particular expression of (6.53), for a fixed matter density $\rho(x)$, will be found in a more general context in the next section. Nevertheless these two cases are the only ones which exhibit the Newtonian limit (obviously the first one!), while for the remaining cases there are serious problems with divergences and incompatibilities. In fact, Case B presents an incompatibility between the solution obtained from the $t\bar{t}$ - component and the one from the $ij$ - component. The incompatibility can be removed if we consider, as the Green function for the differential operator $\nabla^4$, the trivial solution: $\mathcal{G}(\Delta^2)|_B = \text{const}$. With this choice, the arbitrary integration constant $\Phi_0$ can be interpreted as $-GM$. However another problem remains: namely the divergence at the origin. The interpretation of the constant $\Phi_0$ as a total mass and not as a generic integral $\int d^3x' \rho(x')$ does not avoid the singularity. We can conclude, then, the solution

\[ 2_B \Psi(x) - B \Phi(x) = -\frac{GM}{|x - x'|}. \] (6.54)

holds only in vacuum. Before continuing our analysis of the various cases, the term $\int d^3x' \mathcal{G}(\Delta^2)(x - x')\rho(x')$ has to be discussed for the choice (6.51). A generic field equation with $\Delta^2$ (from Table 6.1) is
we conclude that the only consistent possibility is to set $\rho(x) = 0$. In the remaining cases, we can only consider vacuum solutions.

Table 6.3: Here we provide the solutions of the field equations in Table 6.1. The solutions are found by setting $\Xi = 0$ in the $ij \cdot$ component of the field equation (6.33). The solutions are displayed in terms of the Green functions. $\Phi_0$ is a generic integration constant.
6.4 Solutions using the Green function

6.4.2 The general solution by Green function $G^A(x - x')$

In this section, we explicitly determine the gravitational potential in the inner and in the outer region of a spherically symmetric matter distribution. The first consequence of the extended gravity theories which we are considering is the non-validity of the Gauss theorem. In fact, in the Newtonian limit of GR, the equation for the gravitational potential, generated by a point-like source

$$\nabla_x G_{\text{New.mech.}}(x - x') = -\frac{4\pi\delta(x - x')}{|x - x'|^3}$$

(6.56)

is not satisfied by the new Green functions developed above. If we consider the flux of gravitational field $g_{\text{New.mech.}}$ defined as

$$g_{\text{New.mech.}} = -\frac{GM(x - x')}{|x - x'|^3} = -GM\nabla_x G_{\text{New.mech.}}(x - x'),$$

(6.57)

we obtain, as standard, the Gauss theorem:

$$\int d\Sigma \ g_{\text{New.mech.}} \cdot \hat{n} \propto M,$$

(6.58)

where $\Sigma$ is a generic two-dimensional surface and $\hat{n}$ its surface normal vector. The flux of field $g_{\text{New.mech.}}$ on the surface $\Sigma$ is proportional to the matter content $M$, inside the surface independently of the particular shape of surface (Gauss theorem, or Newton theorem for the gravitational field [205]). On the other hand, if we consider the flux defined by the new Green function, its value is not proportional to the enclosed mass but depends on the particular choice of the surface:

$$\int d\Sigma \ g_{\text{New.mech.}} \cdot \hat{n} \propto M_{\Sigma},$$

(6.59)

Hence $M_{\Sigma}$ is a mass-function depending on the surface $\Sigma$. Then we have to find the solution inside/outside the matter distribution by evaluating the quantity

$$\int d^3x G^A(x - x')\rho(x'),$$

(6.60)
and by imposing the boundary condition on the separation surface. By considering solutions (6.34) with the Green function \( G^A(x - x') \) from Table 6.2 and by assuming \( \Xi(x) = 0 \), we have

\[
\begin{align*}
A \Phi(x) &= Y \mathcal{X} \frac{(8a_2 + 3a_3) \Delta x - a_1}{2a_1(2a_2 + a_3)} \int d^3x' G^A(x, x') \rho(x') \\
&= (\mu_1 + \mu_2 \Delta x) \int d^3x' G^A(x - x') \rho(x') \\
A \Psi(x) &= Y \mathcal{X} \frac{(4a_2 + a_3) \Delta x - a_1}{2a_1(2a_2 + a_3)} \int d^3x' G^A(x, x') \rho(x') \\
&= (\mu_1 + \mu_3 \Delta x) \int d^3x' G^A(x - x') \rho(x')
\end{align*}
\]  

(6.61)

where \( \mu_1 := -\frac{4\pi Y}{2a_2 + a_3} = -12\pi Y \frac{\lambda_2^2 \lambda_3^2}{\lambda_1^2 - \lambda_2^2}, \mu_2 := \frac{4\pi Y (8a_2 + 3a_3)}{a_1(2a_2 + a_3)} = \frac{4\pi Y(4a_2 + a_3)}{a_1(2a_2 + a_3)}, \mu_3 := \frac{4\pi Y (4a_2 + a_3)}{a_1(2a_2 + a_3)} = \frac{4\pi Y (2\lambda_2^2 + \lambda_3^2)}{a_1(\lambda_1^2 - \lambda_2^2)}. \) We have to note that our working hypothesis, \( \Xi(x) = 0 \), is not particular, since when we considered the Hilbert-Einstein Lagrangian in §6.2 to give the Newtonian solution, we imposed an analogous condition. For the potential \( \Phi(x) \), one obtains:

\[
A \Phi(x) = (\mu_1 + \mu_2 \Delta x) \int d^3x' \rho(x') \left[ \sum_{i=0}^{2} G_i^A e^{-\lambda_i |x - x'|} \right]
\]

\[
= \sum_{i=0}^{2} G_i^A (\mu_1 + \mu_2 \Delta x) T^{A, \lambda_i}(x) = \sum_{i=0}^{2} G_i^A \Phi^{A, \lambda_i}(x),
\]

(6.62)

where

\[
T^{A, \lambda_i}(x) \doteq \int d^3x' \rho(x') e^{-\lambda_i |x - x'|} |x - x'|,
\]

(6.63)

\[
\Phi^{A, \lambda_i}(x) \doteq (\mu_1 + \mu_2 \Delta x) T^{A, \lambda_i}(x),
\]

(6.64)

and \( G_i^A \) are the coefficients. Here the numbers \( \lambda_i \) assume the above values 0, \( \lambda_1 \), \( \lambda_2 \). Supposing a matter density \( \rho(x) = \rho(|x|) \) and denoting the radius of the sphere with total mass \( M \) by \( \xi \), we have
\[ T^{A, \lambda_i}(x) = G \int_{0}^{\xi} d|x'|||x'||^{2} \rho(|x'|) \int_{0}^{2\pi} d\phi' \]
\[ \times \int_{0}^{\pi} d\theta' \sin \theta' \frac{e^{-\lambda_i \sqrt{|x|^2 + |x'|^2 - 2|x||x'| \cos \alpha}}}{\sqrt{|x|^2 + |x'|^2 - 2|x||x'| \cos \alpha}}, \quad (6.65) \]

where \( \cos \alpha = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi') \) and \( \alpha \) is the angle between two vectors \( x, x' \). In the spherically symmetric case, we can choose \( \theta = 0 \) without losing generality. The \((6.65)\) becomes

\[ T^{A, \lambda_i}(x) = \frac{2\pi G}{\lambda_i |x|} \int_{0}^{\xi} d|x'|||x'||^{2} \rho(|x'|) \left[ e^{-\lambda_i |x|} - e^{-\lambda_i (|x| + |x'|)} \right]. \quad (6.66) \]

For a constant radial profile of density, \( \rho(|x|) = \frac{3M}{4\pi \xi^3} \), we have:

\[ T^{A, \lambda_i}(x) = \begin{cases} 
\frac{3GM}{\lambda_i \xi^{4}} \left[ 1 - (1 + \lambda_i \xi)e^{-\lambda_i \xi \sinh(\lambda_i |x|) / \lambda_i |x|} \right] & |x| < \xi \\
\frac{3GM}{\lambda_i \xi^{4}} \left[ \lambda_i \xi \cosh(\lambda_i \xi) - \sinh(\lambda_i \xi) \right] e^{-\lambda_i |x| / \lambda_i |x|} & |x| > \xi,
\end{cases} \quad (6.67) \]

in the inner and in the outer region respectively. The limit

\[ \lim_{\lambda_i \to 0} T^{A, \lambda_i}(x) = \begin{cases} 
\frac{3GM}{2\xi^2} - \frac{GM}{2\xi^2} |x|^2 & |x| < \xi \\
\frac{GM}{|x|} & |x| > \xi,
\end{cases} \quad (6.68) \]

gives us the internal and the external Newtonian behavior. The internal and the external potential for a given \( \lambda_i \) is

\[ \Phi^{A, \lambda_i}_{\text{in}}(x) = \frac{3GM}{\lambda_i \xi^{4}} \left[ \mu_1 - e^{-\lambda_i \xi} (1 + \lambda_i \xi)(\mu_1 + \lambda_i^2 \mu_2) \frac{\sinh(\lambda_i |x|)}{\lambda_i |x|} \right] \]
\[ \Phi^{A, \lambda_i}_{\text{out}}(x) = \frac{3GM}{\lambda_i \xi^{4}} \left[ \lambda_i \xi \cosh(\lambda_i \xi) - \sinh(\lambda_i \xi) \right] (\mu_1 + \lambda_i^2 \mu_2) e^{-\lambda_i |x| / \lambda_i |x|} \]

\[ (6.69) \]
The boundary condition on the surface $|x| = \xi$ is

$$
A\Phi_{in}(\xi) - A\Phi_{out}(\xi) = -\frac{3GM}{\xi^3} \mu_2 \sum_{i=0}^{3} G^A_i,
$$

(6.70)

but the last relation is identically vanishing (see Table 6.2). The internal and external potential is given by:

$$
\begin{align*}
A\Phi_{in}(x) &= -\frac{\sqrt{2} \pi^{3/2} GM}{a_1} \left[ \frac{\lambda_1^2 (2+3\lambda_2^2 \xi^2) - 8\lambda_2^2}{\lambda_1^2 \lambda_2^2} \right] - |x|^2 + 8e^{-\lambda_1 \xi} (1 + \lambda_1 \xi) \frac{\sinh(\lambda_1 |x|)}{\lambda_1 |x|} \\
&\quad - 2e^{-\lambda_2 \xi} (1 + \lambda_2 \xi) \frac{\sinh(\lambda_2 |x|)}{\lambda_2 |x|} \\
A\Phi_{out}(x) &= -\frac{2\sqrt{2} \pi^{3/2} GM}{a_1} |x| + \frac{8\sqrt{2} \pi^{3/2} GM}{\lambda_1^2 \lambda_2^2} \left[ \lambda_1 \xi \cosh(\lambda_1 \xi) - \sinh(\lambda_1 \xi) \right] e^{-\lambda_1 |x|} \\
&\quad - 2\sqrt{2} \pi^{3/2} \frac{GM}{\lambda_2^2 \xi} \left[ \lambda_2 \xi \cosh(\lambda_2 \xi) - \sinh(\lambda_2 \xi) \right] e^{-\lambda_2 |x|} \\
\end{align*}
$$

(6.71)

The relations (6.71) give the solutions for the gravitational potential $\Phi$ inside and outside the constant spherically symmetric matter distribution. A similar relation is found for $\Psi$ by substituting $\mu_2 \to \mu_3$. We note that the corrections to the Newtonian terms are ruled by $G^A(x - x')$. If we perform a Taylor expansion for $\lambda|x| \ll 1$, we have:

$$
\frac{\sinh(\lambda |x|)}{\lambda |x|} \simeq \text{constant} + |x|^2 + \ldots.
$$

(6.72)

For fixed value of the distance $|x|$, the external potential $\Phi_{out}^A(x)$ depends on the value of the radius $\xi$, then we have that the Gauss theorem does not work. In this case, the potential depends on the total mass and on the distribution of matter in the space. In particular, if the matter distribution takes a bigger volume, the potential $|\Phi_{out}^A(x)|$ increases and viceversa. We can write

$$
\lim_{\xi \to \infty} \frac{\lambda \xi \cosh(\lambda \xi) - \sinh(\lambda \xi)}{\lambda^3 \xi^3} = \infty;
$$

(6.73)
obviously the limit of $\xi$ has to be interpreted up to the maximal value where the generic position $|x|$ in the space is fixed. If we consider the limit $\xi \to 0$ (the point-like source limit), we obtain

$$\lim_{\xi \to 0} 3 \frac{\lambda \xi \cosh(\lambda \xi) - \sinh(\lambda \xi)}{\lambda^3 \xi^3} = 1.$$ \hspace{1cm} (6.74)

For $A\Phi_{\text{out}}(x)$, we have

$$\lim_{\xi \to 0} A\Phi_{\text{out}}(x) = -\frac{2\sqrt{2}}{a_1} \frac{\pi^{3/2} GM}{|x|} + \frac{8\sqrt{2}}{3a_1} \frac{\pi^{3/2} GM e^{-\lambda_1 |x|}}{|x|}$$

$$-\frac{2\sqrt{2}}{3a_1} \frac{\pi^{3/2} GM e^{-\lambda_1 |x|}}{|x|}.$$ \hspace{1cm} (6.75)

The last expression is compatible with the discussion in §6.3.

### 6.4.3 Further solutions by the Green functions $G^B(x - x')$ and $G^C(x - x')$

For the sake of completeness, let us derive solutions for the other Green functions. Starting from Case B in Table 6.2, we have:

$$T^{B, \lambda_i}(x) = \begin{cases} 
\frac{3GM}{\lambda_i \xi^3} \left\{ -1 + \left[ \cos(\lambda_i \xi) + \lambda_i \xi \sin(\lambda_i \xi) \right] \frac{\sin(\lambda_i |x|)}{\lambda_i |x|} \right\} & |x| < \xi \\
\frac{3GM}{\lambda_i \xi^3} \left[ \sin(\lambda_i \xi) - \lambda_i \xi \cos(\lambda_i \xi) \right] \frac{\cos(\lambda_i |x|)}{\lambda_i |x|} & |x| > \xi 
\end{cases}$$ \hspace{1cm} (6.76)

in the inner and outer region. Also in this case, if we consider the limit of $\lambda_i \to 0$, one obtains the Newtonian limit (6.68). The internal and external potential for given $\lambda_i$ is

$$\begin{align*}
\Phi_{\text{in}}^{B,\lambda_i}(x) &= \frac{3GM}{\lambda_i \xi^3} \left\{ -\mu_1 + (\mu_1 - \lambda_i^2 \mu_2) [\cos(\lambda_i \xi) + \lambda_i \xi \sin(\lambda_i \xi)] \frac{\sin(\lambda_i |x|)}{\lambda_i |x|} \right\} \\
\Phi_{\text{out}}^{B,\lambda_i}(x) &= \frac{3GM}{\lambda_i \xi^3} (\mu_1 - \lambda_i^2 \mu_2) [\sin(\lambda_i \xi) - \lambda_i \xi \cos(\lambda_i \xi)] \frac{\cos(\lambda_i |x|)}{\lambda_i |x|}
\end{align*}$$ \hspace{1cm} (6.77)

The boundary condition on the surface $|x| = \xi$ is
Chapter 6  The Newtonian limit of Fourth Order Gravity theory

\[ B \Phi_{\text{in}}(\xi) - B \Phi_{\text{out}}(\xi) = -\frac{3GM}{\xi^3} \mu_2 \sum_{i=0}^{3} G_i B = 0 \]  \hfill (6.78)

(see Table 6.2). The internal and external potential are given by

\[
B \Phi_{\text{in}}(x) = -\frac{\sqrt{2} \pi^{3/2} GM}{a_1} \left\{ \frac{\lambda_1^2 (3 \lambda_1^2 \xi^2 - 2) + 8 \lambda_2^2}{\lambda_1^2 \lambda_2^2} - |x|^2 - \frac{8}{\lambda_1^2} [\cos(\lambda_1 \xi) + \lambda_1 \xi \sin(\lambda_1 \xi)] \right\}
\]

\[
+ \frac{\lambda_3^2}{\lambda_2^2} [\cos(\lambda_2 \xi) + \lambda_2 \xi \sin(\lambda_2 \xi)] \frac{\sin(\lambda_1 |x|)}{\lambda_1 |x|}
\]

\[
B \Phi_{\text{out}}(x) = -\frac{2 \sqrt{2} \pi^{3/2} GM}{a_1} |x| + \frac{8 \sqrt{2} \pi^{3/2} GM}{a_1^2 \lambda_1^2 |x|^3} [\sin(\lambda_1 \xi) - \lambda_1 \xi \cos(\lambda_1 |x|)] \frac{\cos(\lambda_1 |x|)}{|x|} +
\]

\[
-\frac{2 \sqrt{2} \pi^{3/2} GM}{a_1^2 \lambda_2^2 |x|^3} [\sin(\lambda_2 \xi) - \lambda_2 \xi \cos(\lambda_2 |x|)] \frac{\cos(\lambda_2 |x|)}{|x|}
\]

\hfill (6.79)

The above considerations hold also for the first of (6.79). The correction term to the Newtonian potential in the external solution, second line of (6.79), can be interpreted as the Fourier transform of the matter density. In fact, we have:

\[
\int \frac{d^3 x'}{(2 \pi)^{3/2}} \rho(x') e^{-i k \cdot x'} = \frac{3M}{(2 \pi)^{2/3}} \frac{\sin(|k| \xi) - |k| \xi \cos(|k| \xi)}{|k|^3 \xi^3},
\]

and in the limit

\[
\lim_{\xi \to 0} \int \frac{d^3 x'}{(2 \pi)^{3/2}} \rho(x') e^{-i k \cdot x'} = \frac{M}{(2 \pi)^{2/3}},
\]

we obtain again the external solution for point-like source as limit of external solution (6.79):

\[
\lim_{\xi \to 0} B \Phi_{\text{out}}(|x|) = -\frac{2 \sqrt{2} \pi^{3/2} GM}{a_1 |x|^3} + \frac{4 \sqrt{2} \pi^{3/2} GM \cos(\lambda_1 |x|)}{3 a_1 |x|}
\]

\[
-\frac{\sqrt{2} \pi^{3/2} GM \cos(\lambda_2 |x|)}{3 a_1 |x|}.
\]

\hfill (6.82)

Finally for Case C in Table 6.2, we have
\[ C \Phi_{\text{in}}(x) = -\frac{\sqrt{2}}{a_1} \frac{\pi^{3/2}}{\xi^4} \left\{ \frac{\lambda_1^2 (3 \lambda_2^2 \xi^2 - 8 \lambda_2)}{\lambda_1^2 \lambda_2^2} - |x|^2 + \frac{8}{\lambda_1^2} e^{-\lambda_1 \xi} (1 + \lambda_1 \xi) \frac{\sinh(\lambda_1 |x|)}{\lambda_1 |x|} \right. \\
+ \frac{2}{\lambda_2} [\cos(\lambda_2 \xi) + \lambda_2 \xi \sin(\lambda_2 |x|) \sin(\lambda_2 |x|)] \left\} \right. \]
\[ C \Phi_{\text{out}}(x) = -\frac{2 \sqrt{2}}{a_1} \frac{\pi^{3/2}}{|x|} + \frac{8 \sqrt{2}}{a_1} \frac{\pi^{3/2}}{\lambda_1^2 \xi^4} (\lambda_1 \xi \cosh(\lambda_1 |x|) - \sinh(\lambda_1 |x|)) \frac{e^{-\lambda_1 |x|}}{|x|} \]
\[ -\frac{2 \sqrt{2}}{a_1} \frac{\pi^{3/2}}{\lambda_2^2 \xi^4} [\sin(\lambda_2 \xi) - \lambda_2 \xi \cos(\lambda_2 |x|) \cos(\lambda_2 |x|)] \frac{\cos(\lambda_2 |x|)}{|x|}. \]

The limit of point-like source is valid also in this last case:

\[ \lim_{\xi \to 0} C \Phi_{\text{out}}(x) = -\frac{2 \sqrt{2}}{a_1} \frac{\pi^{3/2}}{|x|} + \frac{8 \sqrt{2}}{3 a_1} \frac{\pi^{3/2}}{\lambda_2^2 \xi^4} \frac{GM e^{-\lambda_1 |x|}}{|x|} \]
\[ -\frac{2 \sqrt{2}}{3 a_1} \frac{\pi^{3/2}}{\lambda_2^2 \xi^4} \frac{GM \cos(\lambda_2 |x|)}{|x|}. \]

These results means that for suitable distance scales, the Gauss theorem is recovered and the theory agrees with the standard Newtonian limit of GR.

### 6.5 Post-Newtonian scheme of $f - gravity$

In this last paragraph of sixth Chapter we want to trace a methodological approach to "perturbed" Eqs. (1.25) when we consider $f \sim f_1 R + f_2 R^2 + f_3 R^3 + \ldots$ (see the Taylor expanse (6.7)) and the metric tensor completed at post-Newtonian order (2.35). Obviously the matter tensor is Eq. (2.57). Eqs. (1.25) - (1.26) at O(2) - order become

\[ H_{tt}^{(2)} = f_1 R_{tt}^{(2)} - \frac{f_2}{2} R^{(2)} - 2 f_2 \nabla R^{(2)} = \mathcal{X} T_{tt}^{(0)} \]
\[ H_{ij}^{(2)} = f_1 R_{ij}^{(2)} + \left[ \frac{f_2}{2} R^{(2)} + 2 f_2 \nabla R^{(2)} \right] \delta_{ij} - 2 f_2 R_{ij}^{(2)} = \mathcal{X} T_{ij}^{(0)} \]
\[ H^{(2)} = -6 f_2 \nabla R^{(2)} - f_1 R^{(2)} = \mathcal{X} T^{(0)} \]
at O(3) - order

\[ H^{(3)}_{\alpha\beta} = f_1 R^{(3)}_{\alpha\beta} - 2 f_2 R^{(2)}_{\alpha\beta} = \mathcal{X} T^{(1)}_{\alpha\beta} \] (6.86)

and by remember the expressions for the Christoffel symbols (2.39) and \( \ln \sqrt{-g} \sim \frac{1}{2} [g^{(2)}_{tt} - g^{(2)}_{mm}] + \ldots \) finally, at O(4) - order,

\[
\begin{cases}
H^{(4)}_{tt} = f_1 R^{(4)}_{tt} + 2 f_2 R^{(2)}_{tt} - \frac{f_1}{2} R^{(4)} - \frac{f_1}{2} g^{(2)}_{tt} R^{(2)} - \frac{f_2}{2} R^{(2)}^2 \\
-2 f_2 \left[ g^{(2)}_{mn,m} R^{(2)}_{,n} + \triangle R^{(4)} + g^{(2)}_{mn} R^{(2)}_{,mn} - \frac{1}{2} \nabla g^{(2)}_{mn} \cdot \nabla R^{(2)} \right] \\
-6 f_3 \left[ |\nabla R^{(2)}|^2 + R^{(2)} \triangle R^{(2)} \right] = \mathcal{X} T^{(2)}_{tt}
\end{cases}
\] (6.87)

\[ H^{(4)} = -6 f_2 \triangle R^{(4)} - f_1 R^{(4)} - 18 f_3 \left[ |\nabla R^{(2)}|^2 + R^{(2)} \triangle R^{(2)} \right] \\
+6 f_2 \left[ R^{(2)}_{tt} - g^{(2)}_{mn} R^{(2)}_{,mn} - \frac{1}{2} \nabla (g^{(2)}_{tt} - g^{(2)}_{mm}) \cdot \nabla R^{(2)} - g^{(2)}_{mn} R^{(2)}_{,mn} \right] = \mathcal{X} T^{(2)}
\]

The solution for the Ricci scalar \( R^{(2)} \), in the last line of \( (6.85) \) is similar to \( (5.24) \). In fact we have

\[ R^{(2)}(x) = \mathcal{X} \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{\bar{T}^{(0)}(k) e^{ik \cdot x}}{f_1} = - \frac{8\pi G \lambda^2}{f_1} \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{\bar{T}^{(0)}(k) e^{ik \cdot x}}{k^2 + \lambda^2} = \\
- \frac{2G}{f_1} \int d^3 x' T^{(0)}(x') e^{-\lambda |x - x'|} \] (6.88)

where, again, \( \lambda \) is one defined previously \( (6.13) \) or \( (6.41) \). Considering a generalization of the metric \( (2.71) \)

\[ g_{\mu\nu} = \begin{pmatrix}
1 + 2\Phi + 2\Theta & \bar{Z}^T \\
\bar{Z} & -\delta_{ij} + g^{(2)}_{ij}
\end{pmatrix} \] (6.89)
and requiring the harmonic gauge (2.43) the solution for the metric tensor is found. Now, from the expression of time-time component of Ricci tensor at Newtonian level (2.53) we obtain the modified newtonian potential as solution of the first line of (6.85)

\[ \Phi(x) = -\frac{2G}{f_1} \int d^3x' \frac{T_{tt}(x')}{|x - x'|} - \frac{1}{8\pi} \int d^3x' \frac{R^{(2)}(x')}{|x - x'|} - \frac{1}{3\lambda^2} R^{(2)}(x). \] (6.90)

We can check immediately that when \( f \to R \) we find \( \Phi(x) \to -G \int d^3x' \frac{\rho(x')}{|x - x'|} \). This outcome is a generalization of solution (6.15). From the (6.86) we find the "vectorial" solution

\[ Z_i = -\frac{4G}{f_1} \int d^3x' \frac{T_{ti}(x')}{|x - x'|} + \frac{1}{6\pi\lambda^2} \int d^3x' \frac{R_{ti}(x')}{|x - x'|}, \] (6.91)

and from second line of (6.85) the spatial tensorial solution

\[ g_{ij}^{(2)} = -\frac{4G}{f_1} \int d^3x' \frac{T_{ij}(x')}{|x - x'|} + \frac{\delta_{ij}}{4\pi} \int d^3x' \frac{R^{(2)}(x')}{|x - x'|} + \frac{2\delta_{ij}}{3\lambda^2} R^{(2)}(x) \]

\[ - \frac{2}{3\lambda^2} \left[ x_i x_j R^{(2)}(x) + \left( \delta_{ij} - \frac{3x_i x_j}{x^2} \right) \frac{1}{|x|^3} \int_0^{|x|} d|x'| |x'|^2 R^{(2)}(x') \right]. \] (6.92)

We can affirm that it is possible to have solution non-Ricci-flat in vacuum: HOG mimics a matter source.

From the fourth order of field equation, we note also the Ricci scalar \( R^{(4)} \) propagates with the same \( \lambda \) (the second line of (6.87)) and the solutions at second order originates a supplementary matter source in r.h.s. of (1.25):

\[ \frac{1}{\lambda^2} \triangle R^{(4)} - R^{(4)} = \frac{18f_3}{f_1} T_a + \frac{1}{\lambda^2} T_b + \frac{x}{f_1} T^{(2)} \] (6.93)

where the functions \( T_a, T_b \) are known. The Ricci scalar at fourth order is
\[ R^{(4)}(x) = -\frac{\lambda}{4\pi f_1} \int d^3x'T^{(2)}(x') \frac{e^{-\lambda|x-x'|}}{|x-x'|} - \frac{18f_3}{4\pi f_1} \int d^3x'T^{(4)}(x') \frac{e^{-\lambda|x-x'|}}{|x-x'|} - \frac{1}{4\pi \lambda^2} \int d^3x'T^{(4)}(x') \frac{e^{-\lambda|x-x'|}}{|x-x'|} \] (6.94)

Also in this case we can have a non-vanishing curvature in absence of matter. At last the \( tt \)-component at fourth order can be reformulated as follows

\[ \Delta \Theta = \frac{\lambda}{f_1} T^{(2)}_{tt} + \text{contributions from previously order} \] (6.95)

then it is possible to find a general solution for \( tt \)-component at fourth order of metric tensor.

With this last paragraph we wanted to resume a methodological approach to Post-Newtonian limit of \( f \)-gravity, if \( f \) is an analytical function of Ricci scalar. The development of \( f \) is performed in \( R = 0 \) and the Ricci tensor components are expressed in the harmonic gauge.
Chapter 7

The post-Minkowskian approximation in $f$ – gravity: Gravitational Waves in higher order gravity

In this chapter, we develop the post-Minkowskian limit of HOG theories [F]. It is well known that when dealing with GR such an approach provides massless spin-two waves as propagating degree of freedom of the gravitational field while ETGs imply other additional propagating modes in the gravity spectra. We show that a general analytic HOG model, together with a standard massless graviton, is characterized by a massive scalar particle with a finite-distance interaction. We briefly discuss how such massive gravitational mode can have relevant consequences both on cosmological and small scales distances affecting the stochastic background of gravitational waves and representing a valid alternative to Dark Matter on galactic scales. Furthermore we develop an analytic definition of the energy-momentum tensor of the gravitational field in such a scheme. Such a tensor represents a basic quantity in order to calculate the gravitational time delay in Pulsar timing.

7.1 The Post – Minkowskian approximation in spherically symmetric solution

In Chapter 6, we have found the spherically symmetric solution of $f$-gravity in the Newtonian limit with the metric (1.89). Here we want to discuss a different limit of these theories, pursued when the small velocity assumption is relaxed and only the weak field approximation is retained. This
situation is related to the Minkowski limit of the underlying gravity theory as well as the discussion of the Chapter 6 was related to the Newtonian one. In order to develop such an analysis, we can reasonably resort to the metric (1.89), considering the gravitational potentials $g_{tt}(t, r)$ and $g_{rr}(t, r)$ in the suitable form

$$
\begin{array}{l}
g_{tt}(t, r) = 1 + g_{tt}^{(1)}(t, r) \\
g_{rr}(t, r) = 1 + g_{rr}^{(1)}(t, r)
\end{array}
$$

(7.1)

with $g_{tt}^{(1)}, g_{rr}^{(1)} \ll 1$. Let us now perturb the field equations (1.25), with respect to approach (3.31) - (3.33), considering, again, the Taylor expansion (6.7) for a generic $f$ - theory. For the vacuum case ($T_{\mu\nu} = 0$), at the first order with respect to $g_{tt}^{(1)}$ e $g_{rr}^{(1)}$, it is

$$
\begin{array}{l}
f_0 = 0 \\
f_1 \left\{ R^{(1)}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R^{(1)} \right\} + \mathcal{H}^{(1)}_{\mu\nu} = 0
\end{array}
$$

(7.2)

where

$$
\mathcal{H}^{(1)}_{\mu\nu} = -f_2 \left\{ R^{(1)}_{\mu\nu} - R^{(0)}_{\mu\nu} R^{(1)}_{\rho\sigma} - g_{\mu\nu} \left[ g^{(0)\rho\sigma} R^{(1)}_{\rho\sigma} + g^{(0)\rho\sigma} R^{(1)}_{\rho\sigma} + g^{(0)\rho\sigma} \ln \sqrt{-g_{\rho\sigma}} R^{(1)}_{\rho\sigma} \right] \right\}.
$$

(7.3)

In this approximation, the Ricci scalar turns out to be zero while the derivatives, in the previous relations, are calculated at $R = 0$. Again we find $f$-Lagrangians without the cosmological contribution as in (6.10).

Let us now consider the limit for large $r$, i.e. we study the problem far from the source of the gravitational field. In such a case the (7.2) become
7.1 The Post – Minkowskian approximation in spherically symmetric solution

\[
\frac{\partial^2 g_{tt}^{(1)}}{\partial t^2} - \frac{\partial^2 g_{rr}^{(1)}}{\partial r^2} = 0
\]

\[
f_1 \left[ g_{tt}^{(1)} - g_{rr}^{(1)} \right] - 8 f_2 \left[ \frac{\partial^2 g_{tt}^{(1)}}{\partial t^2} + \frac{\partial^2 g_{rr}^{(1)}}{\partial r^2} - 2 \frac{\partial^2 g_{rr}^{(1)}}{\partial t^2} \right] = \text{any function of time } t
\]

The above equations are two coupled wave equations in terms of the two functions \( g_{tt}^{(1)} \) and \( g_{rr}^{(1)} \). Therefore, we can ask for a wave-like solutions for the gravitational potentials

\[
\begin{align*}
  g_{tt}^{(1)} &= \int \frac{d\omega dk}{2\pi} \tilde{g}_{tt}^{(1)}(\omega, k) e^{i(\omega t - kr)} \\
  g_{rr}^{(1)} &= \int \frac{d\omega dk}{2\pi} \tilde{g}_{rr}^{(1)}(\omega, k) e^{i(\omega t - kr)}
\end{align*}
\]

and substituting these into the above equations. We find the condition

\[
\begin{align*}
  \tilde{g}_{tt}^{(1)}(\omega, k) &= \tilde{g}_{rr}^{(1)}(\omega, k) & \omega &= \pm k \\
  \tilde{g}_{tt}^{(1)}(\omega, k) &= \left[ 1 + \frac{3\lambda^2}{4k^2} \right] \tilde{g}_{rr}^{(1)}(\omega, k) & \omega &= \pm \sqrt{k^2 + \frac{3\lambda^2}{4}}
\end{align*}
\]

where \( \lambda \) is defined in the (6.13). In particular, for \( f_1 = 0 \) or \( f_2 = 0 \) one obtains solutions with a dispersion relation \( \omega = \pm k \). In other words, for \( f_i \neq 0 \) (\( i = 1, 2 \)), that is in the case of non-linear \( f \), the above dispersion relation suggests that massive modes are in order. In particular the mass of the graviton is \( m_{grav} = \frac{\sqrt{3}}{2} \lambda \) and, coherently, it is obtained for a modified real gravitational potential. As matter of fact, a gravitational potential deviating from the Newtonian regime induces a massive degree of freedom into the particle spectrum of the gravity sector with interesting perspective for the detection and the production of gravitational waves [206]. It has to be remarked that the presence of massive gravitons in the wave spectrum of HOG is a well known result since the paper of [32]. Nevertheless it is our opinion that this issue has been always considered under a negative perspective and has been not sufficiently investigated.

In the post-Minkowskian approximation, as expected, the gravitational field propagates by means of wave-like solutions. This result suggests that investigating the gravitational waves behavior of HOG can represent an interesting issue where a new phenomenology (massive gravitons) has to be seriously taken into account. Besides, such massive degrees of freedom could be a realistic
7.2 The post–Minkowskian limit of $f$–gravity toward gravitational waves

Let us formally develop, in this section, the post-Minkowskian limit of HOG models. Such investigation completes the analysis of the weak field regime of $f$-gravity and it has to be considered in this present work thesis. The post-Minkowskian limit of whatever gravity theory arises when the regime of small field is considered without any prescription in term of the propagation velocity of the field. This case has to be clearly distinguished with respect to the Newtonian limit which, differently, requires both the small velocity and the weak field approximations. Often in literature such a distinction is not clearly remarked and several cases of pathological analysis can be accounted. The post-Minkowskian limit of GR naturally furnishes massless waves as the propagating behavior of gravity in this regime. We can now develop an analogous study (see paragraph 2.4) considering in place of the Hilbert-Einstein Lagrangian a general function of the Ricci scalar. Actually, in order to perform a post-Minkowskian development of field equations one has to implement the field equations (1.25) with a small perturbation on the Minkowski background $\eta_{\mu\nu}$ (2.74). It is reasonable to assume that the $f$-Lagrangian is an analytic expression in term of the Ricci scalar (6.7) (i.e. Taylor expandable around the Ricci scalar value $R = R_0 = 0$). In such a case field equations (1.25), at the first order of approximation in term of the perturbation become:

$$f_1 \left[ R^{(1)}_{\mu\nu} - \frac{R^{(1)}}{2} \eta_{\mu\nu} \right] - 2f_2 \left[ R^{(1)}_{\mu\nu} - \eta_{\mu\nu} \Box \eta R^{(1)} \right] = \mathcal{X} T^{(0)}_{\mu\nu}.$$  \hspace{1cm} (7.8)

From zero-order of (1.25) one gets again $f(0) = 0$ while $T_{\mu\nu}$ is fixed at zero-order in (7.8) as in the paragraph 2.4. The explicit expressions of Ricci tensor and scalar are the same of (2.77). The (7.8) can be rewritten in a more suitable form by introducing the constant $\lambda$ (6.13):

$$h^\rho_{(\mu,\nu)\sigma} - \frac{1}{2} \Box \eta h_{\mu\nu} - \frac{1}{2} h_{\mu\nu} - \frac{1}{2} (h_{\sigma\tau} \sigma\tau - \Box \eta h) \eta_{\mu\nu} +$$

$$+ \frac{1}{3 \lambda^2} (\partial^2_{\mu\nu} - \eta_{\mu\nu} \Box \eta) (h_{\sigma\tau} \sigma\tau - \Box \eta h) = \frac{\mathcal{X}}{f_1} T^{(0)}_{\mu\nu}.$$  \hspace{1cm} (7.9)
and by choosing the harmonic gauge (2.43): \( \tilde{h}_{\mu\nu} = h_{\mu\nu} - \frac{\mathbf{k}}{2} \eta_{\mu\nu} \) with the condition \( \tilde{h}^{\mu\nu,\mu} = 0 \), one obtains that field equations and the trace equation respectively read

\[
\begin{align*}
\square_{\eta} \tilde{h}_{\mu\nu} + \frac{1}{3\lambda^2} (\eta_{\mu\nu} \square_{\eta} - \partial^2_{\mu\nu}) \square_{\eta} \tilde{h} &= - \frac{2\chi}{f_1} T^{(0)}_{\mu\nu} \\
\square_{\eta} \tilde{h} + \frac{1}{\lambda^2} \partial^2_{\eta} \tilde{h} &= - \frac{2\chi}{f_1} T^{(0)}
\end{align*}
\]

(7.10)

In order to deduce the analytic solutions of (7.10), we can now adopt a dual space (momentum space) description, this approach can simplify the equations system and, above all, allows to directly observe what are the physical properties of our problem. In such a scheme we have:

\[
\begin{align*}
k^2 \tilde{h}_{\mu\nu}(k) + \frac{1}{3\lambda^2} (k_{\mu} k_{\nu} - k^2 \eta_{\mu\nu}) k^2 \tilde{h}(k) &= \frac{2\chi}{f_1} T^{(0)}_{\mu\nu}(k) \\
k^2 \tilde{h}(k)(1 - \frac{k^2}{\lambda^2}) &= \frac{2\chi}{f_1} T^{(0)}(k)
\end{align*}
\]

(7.11)

where

\[
\begin{align*}
\tilde{h}_{\mu\nu}(k) &= \int \frac{d^4 x}{(2\pi)^4} \tilde{h}_{\mu\nu}(x) e^{-ikx} \\
T^{(0)}_{\mu\nu}(k) &= \int \frac{d^4 x}{(2\pi)^4} T^{(0)}_{\mu\nu}(x) e^{-ikx}
\end{align*}
\]

(7.12)

are the Fourier transforms of the perturbation \( \tilde{h}_{\mu\nu}(x) \) and of the matter tensor \( T^{(0)}_{\mu\nu} \). We have defined, as usual, \( k x = \omega t - \mathbf{k} \cdot \mathbf{x} \) and \( k^2 = \omega^2 - k^2 \). On the other side \( \tilde{h}(k) \) and \( T^{(0)}(k) \) are the traces of \( \tilde{h}_{\mu\nu}(k) \) and \( T^{(0)}_{\mu\nu}(k) \). In the momentum space one can easily recognize the solutions of (7.11); the expression for \( \tilde{h}_{\mu\nu}(k) \) turns out to be

\[
\tilde{h}_{\mu\nu}(k) = \frac{2\chi T^{(0)}_{\mu\nu}(k)}{f_1} - \frac{2\chi}{3f_1} k_{\mu} k_{\nu} - \frac{2\chi}{f_1} k^2 T^{(0)}(k),
\]

(7.13)

which fulfills the condition \( \tilde{h}^{\mu\nu,\mu} = 0 \) (\( \tilde{h}^{\mu\nu}(k) k_{\mu} = 0 \)). The true perturbation variable \( h_{\mu\nu}(k) \) can be obtained inverting the relation with the tilded variables, in particular inserting the matter functions \( S^{(0)}_{\mu\nu}(k) = T^{(0)}_{\mu\nu}(k) - \frac{1}{2} \eta_{\mu\nu} T^{(0)}(k) \) and \( S^{(0)}(k) = \eta^{\mu\nu} S^{(0)}_{\mu\nu}(k) \), one obtains:
which represents a wavelike solution, in the momentum space, with a massless and a massive contribute since the pole in the denominator of the second term, whose mass is directly related with the pole itself. The explicit wavelike solution can be obtained returning the the configuration space inverting the Fourier transform of $h_{\mu\nu}$.

Let us remark that field equations (1.25), for a generical $f$ - Lagrangian, can be rewritten isolating the Einstein tensor in the l.h.s. as usual for Curvature Quintessence [56, 57, 58]. In such a case higher than second order differential contributes, in term of the metric tensor, are considered in the r.h.s. as a source component of the space-time dynamics as well as the energy momentum tensor of ordinary matter does:

$$G_{\mu\nu} = T_{\mu\nu}^{(m)} + T_{\mu\nu}^{(curv)},$$

(7.15)

where

$$\left\{ \begin{array}{l}
T_{\mu\nu}^{(m)} = \frac{\chi T_{\mu\nu}}{f'} \\
T_{\mu\nu}^{(curv)} = \frac{1}{2}g_{\mu\nu} - \frac{f'}{f'^2} R + \frac{f' - g_{\mu\nu} \Box f'}{f'}
\end{array} \right.$$  

(7.16)

Actually if we consider the perturbed metric (2.74) and develop the Einstein tensor up to the first order of perturbation we have

$$G_{\mu\nu} \sim G_{\mu\nu}^{(1)} = h^{\sigma}_{(\mu,\nu)} = \frac{1}{2} \Box h_{\mu\nu} - \frac{1}{2} h_{,\mu\nu} - \frac{1}{2} (h_{\sigma\tau,^{\sigma\tau} - \Box h})\eta_{\mu\nu}$$

(7.17)

while the curvature tensor will give the other contributes

$$T_{\mu\nu}^{(curv)} \sim \frac{1}{3\lambda^2} (\eta_{\mu\nu} \Box h - \partial_{\mu\nu}^2) (h_{\sigma\tau,^{\sigma\tau} - \Box h})$$

(7.18)

This expression easily allow to recognize that, in the dual space of momentum, such a quantity
will be responsible of the pole-like term which implies the introduction of a massive degree of freedom into the particle spectrum of gravity. In fact, inserting these two expressions into the the field equations (7.15) and considering the (2.77) we obtain the solution:

\[ \Box \eta h_{\mu\nu}(x) = -\frac{2\chi}{f_1} \left[ S_{\mu\nu}^{(0)}(x) + \Sigma^\lambda_{\mu\nu}(x) \right] \quad (7.19) \]

where \( \Sigma^\lambda_{\mu\nu}(x) \) is related with the Curvature tensor and is defined as

\[ \Sigma^\lambda_{\mu\nu}(x) = -\frac{1}{6} \int \frac{d^4k}{(2\pi)^2} \frac{k^2 \eta_{\mu\nu} + 2k_\mu k_\nu}{k^2 - \lambda^2} S^{(0)}(k) \, e^{ikx}. \quad (7.20) \]

The general solution for the metric perturbation \( h_{\mu\nu}(x) \), when equation are given as in (7.15), can be rewritten as

\[ h_{\mu\nu}(x) = \frac{2\chi}{f_1} \int \frac{d^4k}{(2\pi)^2} S_{\mu\nu}^{(0)}(k) \, e^{ikx} - \frac{\chi}{3f_1} \int \frac{d^4k}{(2\pi)^2} \frac{k^2 \eta_{\mu\nu} + 2k_\mu k_\nu}{k^2(k^2 - \lambda^2)} S^{(0)}(k) \, e^{ikx}, \quad (7.21) \]

which displays in the second term a pole whose properties can be easily evaluated in vacuum. In fact, in such a case (i.e. \( T_{\mu\nu} = 0 \)), the (7.10) becomes

\[ \begin{cases} 
  k^2[\tilde{h}_{\mu\nu}(k) + \frac{1}{\lambda^2}(k_\mu k_\nu - k^2 \eta_{\mu\nu})\tilde{h}(k)] = 0 \\
  k^2\tilde{h}(k)(1 - \frac{k^2}{\lambda^2}) = 0 
\end{cases} \quad (7.22) \]

showing that allowed solutions are of two types along the relations:

\[ \begin{cases} 
  \omega = \pm|k| \\
  h_{\mu\nu}(x) = \frac{d^4k}{(2\pi)^2} h_{\mu\nu}(k) \, e^{ikx} \quad \text{with} \quad h(k) = 0 
\end{cases} \quad (7.23) \]

and
Chapter 7  The post-Minkowskian approximation in \( f \) gravity: Gravitational Waves in higher order gravity

\[
\omega = \pm \sqrt{k^2 + \lambda^2}
\]

\[
h_{\mu\nu}(x) = -\int \frac{d^4k}{(2\pi)^4} \left[ \frac{\lambda^2 \eta_{\mu\nu} + 2k_{\mu}k_{\nu}}{6\lambda^4} \right] h(k) \ e^{ikx} \quad \text{with} \quad h(k) \neq 0
\]  

(7.24)

It is evident, that the first solution represents a massless graviton according with standard prescriptions of GR while the second one gives a massive degree of freedom with \( m^2 = \lambda^2 \). In this sense we can furtherly rewrite the (7.10) introducing \( \phi \equiv \Box \tilde{h} \) so that the general system can be rearranged in the following way

\[
\left\{ 
\begin{array}{l}
\Box \eta \tilde{h}_{\mu\nu} = -\frac{2\chi}{f_1} T^{(0)}_{\mu\nu} + \left[ \frac{\partial_{\mu\nu} - \eta_{\mu\nu} \Box}{3m^2} \right] \phi \\
(\Box \eta + m^2) \phi = -\frac{2\chi}{f_1} m^2 T^{(0)}
\end{array}
\right.
\]

(7.25)

which suggests that the higher order contributes act in the post-Minkowskian limit as a massive scalar field whose mass depends on the degree of deviation \( \left( f', f'' \right) \), calculated on the background unperturbed metric, of the initial \( f \) - Lagrangian with respect to the standard Hilbert-Einstein expression.

It is important to remark that the peculiarity of a massive contribute in the wave spectrum of HOG is strictly related with the peculiar behavior of the trace equation with respect to GR. In fact in the case of HOG theories the trace equation establishes a constraint for the Ricci scalar under the form of a dynamical equation. This relation allows a more complex evolution of the system since the Ricci scalar is not univocally fixed by the trace equation as in GR. In fact, while in the framework of GR vacuum solutions imply \( R = 0 \) (i.e. it holds the Poisson equation in the Newtonian Limit), in the HOG models \( R \) can assume a generical dynamical evolution according with (1.26), which assumes the zero value only under certain hypotheses on the nonlinear Lagrangian. This behaviour, as widely displayed in chapter 6, implies, as natural consequence, a modify of the Poisson equation in the Newtonian limit to a form which allows a modified gravitational potential in such a regime. This characteristic is directly related with the massive degree of freedom obtained in the post-Minkowskian limit of these theories. In other words, such peculiarity is a different representation, at a different scale (or energy range), of the same effect. One can easily notice that the characteristic length of the modified gravitational potential enters in the wave solution exactly as the mass parameter \( m^2 \equiv -\frac{f_1}{6f_2} \) of the additive component in the gravitational wave spectrum.
7.3 Strong gravitational waves in a general $f -$ gravity

We are interesting to study the field equation for a small perturbation $h_{\mu\nu}$ on the background metric $g_{\mu\nu}^{(0)} (O(h)^2 \ll 1)$. Where $g_{\mu\nu}^{(0)}$ is a solution of GR with Ricci scalar $R = R^{(0)} = 0$ (this solution is in the vacuum). Then the relativistic invariant is described as

$$ds^2 = g_{\sigma\tau} dx^\sigma dx^\tau = (g_{\sigma\tau}^{(0)} + h_{\sigma\tau}) dx^\sigma dx^\tau$$  \hspace{1cm} (7.26)

Obviously the lowering and rising of the index have been made with the metric background $g_{\mu\nu}^{(0)}$. The field equation (1.25) at zero order, if we consider the development shown in the paragraph [3.4], is

$$f^\prime(0) R_{\mu\nu}^{(0)} - \frac{1}{2} g_{\mu\nu}^{(0)} f^{(0)} = \mathcal{X} T_{\mu\nu}^{(0)}$$  \hspace{1cm} (7.27)

from the which the trace equation states $f^{(0)} = -\frac{\mathcal{X}}{2} T$. But in the vacuum the trace is vanishing and we have to impose the condition $f^{(0)} = 0$ (we neglect the cosmological contribute). Then the field equation (7.27) becomes

$$R_{\mu\nu}^{(0)} = 0$$  \hspace{1cm} (7.28)

and the metric $g_{\mu\nu}^{(0)}$ is solution for the field equation (1.25) at zero order in the vacuum (Schwarzschild solution). At the first order we have

$$f^\prime(0) \left\{ R_{\mu\nu}^{(1)} - \frac{1}{2} g_{\mu\nu}^{(0)} R^{(1)} \right\} - f^{\prime\prime(0)} \left\{ R_{;\mu\nu}^{(1)} - g_{\mu\nu}^{(0)} \square g^{(0)} R^{(1)} \right\} = \mathcal{X} T_{\mu\nu}^{(1)}$$  \hspace{1cm} (7.29)

where the derivatives covariant have been calculated with respect to metric $g_{\mu\nu}^{(0)}$. Now the expressions for Ricci tensor and scalar are

$$\left\{ \begin{array}{l}
R_{\mu\nu}^{(1)} = h_{(\mu;\nu)}^\sigma - \frac{1}{2} \square g^{(0)} h_{\mu\nu} - \frac{1}{2} h_{;\mu\nu} \\
R^{(1)} = h_{\sigma\tau ;\sigma\tau} - \square g^{(0)} h
\end{array} \right.$$  \hspace{1cm} (7.30)
and the general field equation perturbed (7.29) is

\[ 2h_{(\mu\nu)\sigma} - \Box g^{(0)} h_{\mu\nu} - h_{(\mu\nu)\sigma} - g^{(0)}_{\mu\nu} \left( h_{\sigma\tau}^{\sigma\tau} - \Box g^{(0)} h \right) + \]
\[ + \frac{2}{3\lambda^2} \left( \nabla_{\mu} \nabla_{\nu} - g^{(0)}_{\mu\nu} \Box g^{(0)} \right) \left( h_{\sigma\tau}^{\sigma\tau} - \Box g^{(0)} h \right) = \frac{X}{f^{(0)}} T^{(1)}_{\mu\nu} \]  
(7.31)

where \( \tilde{\lambda} = -\frac{f''(0)}{3f'''} \). If \( f \to R \) one has \( f'' \to 0, \tilde{\lambda} \to \infty \) and the (7.31) becomes

\[ G^{(1)}_{\mu\nu} = R^{(1)}_{\mu\nu} - \frac{1}{2} g^{(0)}_{\mu\nu} R^{(1)} = \lambda T^{(1)}_{\mu\nu} \]  
(7.32)

i.e. the first order for Einstein equation in GR with the same conditions for the metric \( g^{(0)}_{\mu\nu} \). The trace of (7.31) is

\[ \Box g^{(0)} \left( h_{\sigma\tau}^{\sigma\tau} - \Box g^{(0)} h \right) + \tilde{\lambda}^2 \left( h_{\sigma\tau}^{\sigma\tau} - \Box g^{(0)} h \right) = -\frac{\tilde{\lambda}^2 X}{f^{(0)}} T^{(1)} \]  
(7.33)

By using the harmonic gauge condition (2.43) from the equations (7.31) and (7.33) one has

\[ \begin{cases} 2\tilde{h}_{(\mu\nu)\sigma} - \Box g^{(0)} \tilde{h}_{\mu\nu} + \frac{1}{\lambda^2} \left( \nabla_{\mu} \nabla_{\nu} - g^{(0)}_{\mu\nu} \Box g^{(0)} \right) \Box g^{(0)} \tilde{h} = \frac{X}{f^{(0)}} T^{(1)}_{\mu\nu} \\
(\Box g^{(0)} + \lambda^2) \Box g^{(0)} \tilde{h} = -\frac{X}{f^{(0)}} \lambda^2 T^{(1)} \end{cases} \]  
(7.34)

7.4 Energy-momentum tensor of \( f - gravity \)

In order to detect gravitational waves the construction of a number of sensitive detectors for gravitational waves (GWs) is underway today. At the moment there are several laser interferometers already built like the VIRGO detector (Italy), the GEO 600 detector (Germany), the two LIGO detectors (United States), the TAMA 300 detector (Japan) and many bar detectors are currently in operation too. Since very soon there will be a huge amount of experimental data the results of these detectors will have a fundamental impact on astrophysics and gravitation physics. GW detectors will be of fundamental importance in order to probe GR and, above all, to check every
alternative theory of gravitation \cite{208, 209, 210}. A possible target of these experiments is the so called stochastic background of gravitational waves \cite{220, 221, 211, 212, 213, 214} which can be related with the inflationary scenario settled in the early universe evolution. Actually there is another very challenging test dealing with gravitational waves phenomenology: the gravitational time delay in Pulsar timing. This experiment is one of the most important evidence of GR validity, since allows to verify the correction to the orbital period of pulsars as predicted by Einstein gravity theory \cite{215}. Therefore, this experiment represents an unescapable test in order to check a whatever viable gravity theory. An analytic calculation of this problem has been performed in the case of the Brans - Dicke theory with positive results since there are not significant constraint on the Brans - Dicke parameter $\omega$ \cite{216, 217}. Actually, in order to calculate what is the physical effect of HOG model on a pulsar system one has calculate the energy-momentum tensor of the gravitational field. This quantity will characterize the energy loss due to the gravitational irradiation. Although the procedure to calculate the energy-momentum tensor of the gravitational field in GR is often debated, one can extend the formalism developed for a generical field theory and obtain this quantity varying functionally on the gravity Lagrangian in term of the Lagrange operator obtaining a so called pseudo-tensor whose properties does not completely fulfils diphieomorphisms invariance\footnote{This quantity is typically referred as the Landau-Lifshitz energy-momentum tensor, nevertheless other kinds of energy-momentum tensor of the gravitational field can be defined}. Such calculation need to be extended when dealing with an HOG model since field equations are of order higher than two.

In standard field theory, given a generical Lagrangian $L = L(g_{\mu \nu}, g_{\mu \nu,\rho}, g_{\mu \nu,\rho \sigma})$ depending even on accelerations, field equations are obtained considering a variational principle which considers all the explicit functions. Thus, in the case of a HOG Lagrangian which depends on the metric and its derivatives up to the second order one has

\[
\delta \int d^4x \sqrt{-g} f = \delta \int d^4x L(g_{\mu \nu}, g_{\mu \nu,\rho}, g_{\mu \nu,\rho \sigma}) \sim \\
\int d^4x \left( \frac{\partial L}{\partial g_{\rho \sigma}} - \partial_\lambda \frac{\partial L}{\partial g_{\rho \sigma,\lambda}} + \partial_\xi^2 \frac{\partial L}{\partial g_{\rho \sigma,\lambda \xi}} \right) \delta g_{\rho \sigma} = \int d^4x \sqrt{-g} H^{\rho \sigma} \delta_{\rho \sigma} = 0 \tag{7.35}\]

where $\sim$ means we neglected a pure divergence. Then we can set :

\[
\int d^4x \partial_\lambda \left[ \left( \frac{\partial L}{\partial g_{\rho \sigma,\lambda}} - \partial_\xi \frac{\partial L}{\partial g_{\rho \sigma,\lambda \xi}} \right) \delta g_{\rho \sigma} + \frac{\partial L}{\partial g_{\rho \sigma,\lambda \xi}} \delta g_{\rho \sigma,\xi} \right] \to 0 . \tag{7.36}\]
As matter of facts, one can write generalized Euler-Lagrange equations for this framework:

\[ H^{\rho\sigma} = \frac{1}{\sqrt{-g}} \left[ \frac{\partial L}{\partial g_{\rho\sigma}} - \partial_{\lambda} \frac{\partial L}{\partial g_{\rho\sigma,\lambda}} + \partial_{\lambda} \partial_{\lambda} \frac{\partial L}{\partial g_{\rho\sigma,\lambda\xi}} \right] = 0, \tag{7.37} \]

which coincide with the field equations (1.25) in the vacuum. Actually, even in the case of such general model it is possible to define an energy momentum tensor of the field, in particular \( t^\lambda_\alpha \) turns out to be defined as follows:

\[ t^\lambda_\alpha = \frac{1}{\sqrt{-g}} \left[ \left( \frac{\partial L}{\partial g_{\rho\sigma,\lambda}} - \partial_\xi \frac{\partial L}{\partial g_{\rho\sigma,\lambda\xi}} \right) g_{\rho\sigma,\alpha} + \frac{\partial L}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\xi\alpha} - \delta^\lambda_\alpha L \right]. \tag{7.38} \]

The (7.38) quantity together the energy-momentum tensor of matter \( T_{\mu\nu} \) satisfies a conservation according with standard requirements. In fact since in presence of matter \( H_{\mu\nu} = \mathcal{X} T_{\mu\nu} \), one has

\[ (\sqrt{-g} t^\lambda_\alpha)_\lambda = -\sqrt{-g} H^{\rho\sigma} g_{\rho\sigma,\alpha} = -\mathcal{X} \sqrt{-g} T^{\rho\sigma} g_{\rho\sigma,\alpha} = -2\mathcal{X} (\sqrt{-g} T^\lambda_\alpha)_\lambda \tag{7.39} \]

and as a consequence

\[ [\sqrt{-g}(t^\lambda_\alpha + 2\mathcal{X} T^\lambda_\alpha)]_\lambda = 0 \tag{7.40} \]

which demonstrates the conservation law. We can now write down the expression of the energy-momentum tensor \( t^\lambda_\alpha \) of the gravitational field in term of the \( f \)-gravity action and the respective derivatives, in such a way to have a completely general expression:

\[ t^\lambda_\alpha = f' \left\{ \left[ \frac{\partial R}{\partial g_{\rho\sigma,\lambda}} - \frac{1}{\sqrt{-g}} \partial_\xi \left( \sqrt{-g} \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}} \right) \right] g_{\rho\sigma,\alpha} + \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\xi\alpha} \right\} - f'' R_{\xi\lambda} \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\alpha} - \delta^\lambda_\alpha f. \tag{7.41} \]

Let us notice that while in GR \( t^\lambda_\alpha \) is non-covariant quantity, the relative generalization in HOG models turns out to satisfy the covariance prescription behaving as an ordinary tensor. One can easily verify that such an expression reduces to the usual definition of the Landau-Lifshitz
energy-momentum tensor of GR when $f = R$.

$$t^\lambda_{\alpha|_{GR}} = \frac{1}{\sqrt{-g}} \left( \frac{\partial \mathcal{L}_{GR}}{\partial g_{\rho\sigma,\lambda}} g_{\rho\sigma,\alpha} - \delta^\lambda_\alpha \mathcal{L}_{GR} \right)$$  \hspace{1cm} (7.42)$$

where the GR Lagrangian has been considered in its effective form, i.e. the symmetric part of the Ricci tensor, which effectively characterizes the variation principle leading to the motion equations

$$\mathcal{L}_{GR} = \sqrt{-g} g^{\mu\nu} \left( \Gamma^\rho_{\mu\sigma} \Gamma^\sigma_{\rho\nu} - \Gamma^\sigma_{\mu\nu} \Gamma^\rho_{\sigma\rho} \right).$$  \hspace{1cm} [218]$$

It is important to stress that GR definition of the energy-momentum tensor and HOG definition are quite different. These discrepancies are due to the presence, in the second case, of higher than second order differential term in the gravity action, which cannot be discarded by means of a boundary integration as it is done in GR. We have remarked above that the effective Lagrangian of GR turns out to be the symmetric part of the Ricci scalar since the second order terms present in the definition of $R$ can be discarded by means of integration by part.

A generic analytic $f$ - Lagrangian is characterized from the dynamical point of view only by the first two terms of its Taylor expansion once the perturbation is implemented at the linear level, i.e. $f \sim f'(0) R + \mathcal{F}(R)$, where the function $\mathcal{F}$ satisfies the condition: $\lim_{R \to 0} \mathcal{F} = R^2$. As a consequence one can rewrite the explicit expression of (7.41) as:

$$t^\lambda_{\alpha} = f'(0) t^\lambda_{\alpha|_{GR}} + \mathcal{F}' \left\{ \left[ \frac{\partial R}{\partial g_{\rho\sigma,\lambda}} - \frac{1}{\sqrt{-g}} \partial_\xi \left( \sqrt{-g} \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}} \right) \right] g_{\rho\sigma,\alpha} + \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\alpha\xi} \right\} - \mathcal{F}'' R_{,\xi} \frac{\partial R}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\alpha} - \delta^\lambda_\alpha \mathcal{F}$$  \hspace{1cm} (7.43)$$

Let us recall the general expression of the Ricci scalar (1.5) splitting its linear ($R^*$) and quadratic ($\bar{R}$) dependence once a perturbed metric is considered

$$R = R^* + \bar{R}$$  \hspace{1cm} (7.44)$$

(notice that $\mathcal{L}_{GR} = -\sqrt{-g} \bar{R}$). Actually, in the case of GR $t^\lambda_{\alpha|_{GR}}$ the Landau - Lifshitz tensor shows as a first non vanishing term a $h^2$ contribute. A similar result can be obtained in the case of HOG models. In fact considering the expression (7.43) one obtains that at the lower expansion order $t^\lambda_{\alpha}$ reads:
Chapter 7 The post-Minkowskian approximation in $f −$ gravity: Gravitational Waves in higher order gravity

\[ t^\lambda_\alpha \sim t^\lambda_\alpha |_{h^2} = f'(0) t^\lambda_\alpha |_{GR} + f''(0) R^* \left[ -\partial_\xi \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\alpha} + \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\xi\alpha} \right] \]

\[ -f''(0) R^*_\xi \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\alpha} - \frac{1}{2} f''(0) \delta^\lambda_\alpha R^* = \]

\[ = f'(0) t^\lambda_\alpha |_{GR} + f''(0) \left[ R^* \left( \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\alpha} - \frac{1}{2} R^* \delta^\lambda_\alpha \right) \right] \]

\[ -\partial_\xi \left( R^* \frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\alpha} \right) \right]. \quad (7.45) \]

Now, since for a perturbed metric (2.74) $R^* \sim R^{(1)}$, where $R^{(1)}$ is defined as in (2.77), one has

\[ \left\{ \begin{array}{l}
\frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} \sim \frac{\partial R^{(1)}}{\partial h_{\rho\sigma,\lambda\xi}} = \eta^\rho_\lambda \eta^\sigma_\xi - \eta^\lambda_\xi \eta^\rho_\sigma \\
\frac{\partial R^*}{\partial g_{\rho\sigma,\lambda\xi}} g_{\rho\sigma,\alpha} \sim \tilde{h}^\lambda_\alpha \eta^\xi_\alpha - \delta^\lambda_\alpha
\end{array} \right. \]

(7.46)

and the first significant term in (7.45) is of second order in the perturbation. We can now write down the expression of the energy - momentum tensor explicitly in term of the perturbation $h$:

\[ t^\lambda_\alpha \sim f'(0) t^\lambda_\alpha |_{GR} + f''(0) \{ (h^\rho_\sigma \eta^\rho_\sigma - \Box h) [h^\lambda_\alpha \eta^\xi_\alpha - h^\lambda_\alpha - \frac{1}{2} \delta^\lambda_\alpha (h^\rho_\sigma \eta^\rho_\sigma - \Box h)] \]

\[ -h^\rho_\sigma \eta^\rho_\sigma \tilde{h}^\lambda_\alpha \eta^\xi_\alpha + h^\rho_\sigma \eta^\rho_\sigma \lambda h_\alpha + h^\lambda_\alpha \eta^\xi_\alpha \Box h_\xi - \frac{1}{4} (\Box \tilde{h}) \delta^\lambda_\alpha \} , \quad (7.47) \]

in term of the tilded perturbation $\tilde{h}_{\mu\nu}$ the new contribution reads:

\[ t^\lambda_\alpha |_f = \frac{1}{2} \left[ \frac{1}{2} \tilde{h}^\lambda_\alpha \Box \tilde{h} - \frac{1}{2} \tilde{h}_\alpha \Box \tilde{h}^\lambda - \tilde{h}^\lambda_\sigma \alpha \Box \tilde{h}^\sigma - \frac{1}{4} (\Box \tilde{h}) \delta^\lambda_\alpha \right] . \quad (7.48) \]

As matter of facts the energy-momentum tensor of the gravitational field, which expresses the energy transport of this field during its propagation, can have a natural generalization in the case of HOG models. We have adopted in our case the Landau-Lifshitz definition, however some other approaches are in order as outlined in [219]. The general definition of $t^\lambda_\alpha$ obtained above consists
of a sum considering the GR contribute plus a term which takes into account corrections induced by the higher differential order of $f$ - theories:

$$t^\lambda_\alpha = f'(0) t^\lambda_\alpha|_{GR} + f''(0) t^\lambda_\alpha|_{f},$$

(7.49)

and again when $f = R$ we obtains $t^\lambda_\alpha = t^\lambda_\alpha|_{GR}$ as already discussed.

Quantities obtained along this section represent the basic elements in order to develop an analytic calculation of the gravitational time delay in the pulsar timing in the framework of HOG models. Nevertheless such analysis is beyond the purposes of the current study and will be argument of a forthcoming investigation.
Chapter 8

Discussions and conclusions

ETGs are good candidates to solve several shortcomings of modern astrophysics and cosmology since they seem, in a natural way, to address the problem of cosmological dynamics without introducing unknown forms of dark matter and dark energy (see e.g. [10, 56]). Nevertheless, a "final" alternative theory solving all the issues has not been found out up to now and the debate on modifying gravitational sector or adding new (dark) ingredients is still open. Beside this general remark related to the paradigm (extending gravity and/or adding new components), there is the methodological issue to "recover" the standard and well-tested results of GR in the framework of these enlarged schemes. The recover of a self-consistent Newtonian limit (or a weak field limit) is the test bed of any theory which pretends to enlarge or correct the GR. In fact GR has been consistently tested in physical situations implying, essentially, spherical symmetry and weak field limit [129].

One of the fundamental and obvious issue that any theory of gravity should satisfy is the fact that, in absence of gravitational field or very far from a given distribution of sources, the spacetime has to be asymptotically flat (Minkowski). Any alternative or modified gravitational theory (beside the diffeomorphism invariance and the general covariance) should address these physical requirements to be consistently compared with GR. This is a crucial point which several times is not considered when people is constructing the weak field limit of alternative theories of gravity.

In our opinion, such a task has to be pursued in the natural frame of the theory otherwise the results could be misleading. Specifically, we have to develop the limit in the Jordan frame without conformal transformations to the Einstein frame since such transformations could alter the interpretation of the results.

In this PhD thesis, we have considered the Taylor expansion of a generic $f$-theory, obtaining general solutions in term of the metric coefficients up to the second order of approximation when matter is neglected. In particular, the solution relative to the $g_{tt}$ metric component gives the grav-
Chapter 8 Discussions and conclusions

Gravitational potential which is corrected with respect to the Newtonian one of $f = R$. The general gravitational potential is given by a Yukawa-like terms, combined with the Newtonian potential, which is effectively achieved at small distances. Besides also starting from the standard corrections to the Hilbert - Einstein Action (the well-known quadratic ones), but now the matter is present, we have faced, in same systematic way, the problem to find out solutions. The solutions are found using the Green function method and we have derived several solutions where the Newton potential is corrected by combinations Yukawa-like terms. We have classified the results considering $i$) the parameters in the Lagrangian, $ii$) the field equations and $iii$) the resulting potential. In relation to the sign of the characteristic coefficients entering the $g_{tt}$ component, one can obtain real or complex solutions. In both cases, the resulting gravitational potential has physical meanings. A discussion on the non-validity of the Gauss theorem has been given. Furthermore, for spherically symmetric distributions of matter, we discussed the inner and the outer solutions. Furthermore, it has been shown that the Birkhoff theorem is not a general result for $f$ - gravity. This is a fundamental difference between GR and HOG. While in GR a spherically symmetric solution is static, here time-dependent evolution can be achieved depending on the order of perturbations [C, E, G].

From other hand it is possible also to calculate Newtonian limit of such theories with a redefinition of the degrees of freedom by some scalar field leading to the so called O’Hanlon Lagrangian [203]. In fact, considering this latter approach, we get a scalar-tensor theory with vanishing kinetic term and a potential term linked to $f$-theory. Also in this case we found a Yukawa-like correction to classic Newtonian potential. Nevertheless when we turn off the modification of Hilbert-Einstein Lagrangian we do not obtain the right Newtonian potential. In fact only in this limit $f \rightarrow R$ it has sense speaking about the Eddington parameter $\gamma$ and its value is $1/2$ and not $\gamma \sim 1$ as observed. The origin of inconsistency is in the not-well defined field equation when $f \rightarrow R$. In fact this problem in present also in Brans-Dicke theory and only by requiring $\omega_{BD} \rightarrow \infty$ we obtain the GR [H].

We have discussed the differences between the post-Newtonian and the post - Minkowskian limit in $f$ - gravity. The main result of such an investigation is the presence of massive degrees of freedom in the spectrum of gravitational waves which are strictly related to the modifications occurring into the gravitational potential. This occurrence could constitute an interesting opportunity for the detection and investigation of gravitational waves. To do this it needs to generalize the energy-momentum tensor for a generic $f$-gravity. In the last chapter we tried to find a new expression for a HOG [F].

Starting from Tensor-multi-scalar theory of gravity [23] we can show how a polynomial Lagrangian in the Ricci scalar $R$, compatible with the PPN-limit, can be recovered in the framework of HOG. The approach is based on the formulation of the PPN-limit of such gravity models de-
developed in analogy with scalar-tensor gravity [137]. In particular, considering the local relations defining the PPN fourth order parameters as differential expressions, one obtains a third-order polynomial in the Ricci scalar which is parameterized by the PPN-quantity $\gamma$ and compatible with the limit $\beta = 1$. The order of deviation from the linearity in $R$ is induced by the deviations of $\gamma$ from the GR expectation value $\gamma = 1$. Actually, the PPN parameter $\gamma$ may represent the key parameter to discriminate among relativistic theories of gravity [A].

Besides we investigated also the viability to find spherical solutions is in $f$-theories with an perturbation methodic analysis with respect to standard results of GR when we consider the limits $r \to \infty$ and $f \to R$. Essentially, spherical solutions can be classified, with respect to the Ricci curvature scalar $R$, as $R = 0$, $R = R_0 \neq 0$, and $R = R(r)$, where $R_0$ is a constant and $R(r)$ is a function of the radial coordinate $r$. In order to achieve exact spherical solutions, a crucial role is played by the relations existing between the metric potentials and between them and the Ricci curvature scalar. In particular, the relations between the metric potentials and the Ricci scalar can be used as a constraint: this gives a Bernoulli equation. Solving it, in principle, spherically symmetric solutions can be obtained for any analytic $f$ function, both for constant curvature scalar and for curvature scalar depending on $r$. Such spherically symmetric solutions can be used as background to test how generic $f$ theories of gravity deviate from GR. Particularly interesting are theories that imply $f \to R$ in the weak field limit. In such cases, the experimental comparison is straightforward and also experimental results, evading GR constraints, can be framed in a self-consistent picture [128]. Finally, we have constructed a perturbation approach in which we search for spherical solutions at the 0th-order and then we search for solutions at the first order. The scheme is iterative and could be, in principle, extended to any order in perturbations. The crucial request is to take into account $f$ - theories which are Taylor expandable about some value $R = R_0$ of the curvature scalar. A important remark is in order at this point. Considering interior and exterior solutions, the junction conditions are related to the integration constants of the problem and strictly depend on the source (e.g. the form of $T_{\mu\nu}$). We have not considered this aspect here since we have, essentially, searched for vacuum solutions. However, such a problem has to be carefully faced in order to deal with physically consistent solutions. For example, the Schwarzschild solution $R = 0$, which is one of the exterior solutions which we have considered, always satisfies the junction conditions with physically interesting interior metric. This is not the case for several spherically symmetric solutions which could give rise to unphysical junction conditions and not be in agreement with Newtons law of gravitation, also asymptotically. In these cases, such solutions have to be discarded [D].

We have discussed a general method to find out exact solutions in ETGs when a spherically symmetric background is taken into account. In particular, we have searched for exact spherically
symmetric solutions in $f$-gravity by asking for the existence of Noether symmetries. We have developed a general formalism and given some examples of exact solutions. The procedure consists in: 

1) considering the point-like $f$ Lagrangian where spherical symmetry has been imposed; 
2) deriving the Euler-Lagrange equations; 
3) searching for a Noether vector field; 
4) reducing dynamics and then integrating the equations of motion using conserved quantities. 

Viceversa, the approach allows also to select families of $f$ models where a particular symmetry (in this case the spherical one) is present. As examples, we discussed power law models and models with constant Ricci curvature scalar. However, the above method can be further generalized. If a symmetry exists, the Noether Approach allows transformations of variables where the cyclic ones are evident. This fact allows to reduce dynamics and then to get more easily exact solutions. These considerations show that the Noether Symmetries Approach can be applied to large classes of gravity theories. Up to now the Noether symmetries Approach has been worked out in the case of FRW metric. In this PhD work, we have concentrated our attention to the development of the general formalism in the case of spherically symmetric spacetimes. Therefore the fact that, even in the case of a spherical symmetry, it is possible to achieve exact solutions seems to suggest that this technique can represent a paradigmatic approach to work out exact solutions in any theory of gravity. A more comprehensive analysis in this sense will be the argument of forthcoming studies [B].
List of papers

A Fourth-order gravity and experimental constraints on Eddington parameters - S. Capozziello, A. Stabile, A. Troisi, Modern Physics Letters A 21, 2291 (2006);

B Spherically symmetric solutions in $f(R)$-gravity via Noether symmetry approach - S. Capozziello, A. Stabile, A. Troisi, Classical and Quantum Gravity 24, 2153 (2007);

C Newtonian limit of $f(R)$-gravity - S. Capozziello, A. Stabile, A. Troisi, Physical Review D 76, 104019 (2007);

D Spherical symmetry in $f(R)$-gravity - S. Capozziello, A. Stabile, A. Troisi, Classical and Quantum Gravity 25, 085004 (2008);

E The Newtonian limit of metric gravity theories with quadratic Lagrangians - S. Capozziello, A. Stabile, Submitted to Classical and Quantum Gravity;

F The post Minkowskian limit of $f(R)$-gravity - S. Capozziello, A. Stabile, A. Troisi, (in preparation);

G A general solution in the Newtonian limit of $f(R)$-gravity - S. Capozziello, A. Stabile, A. Troisi, Submitted to Modern Physics Letters A;

H Comparing scalar-tensor gravity and $f(R)$-gravity in the Newtonian limit - Capozziello S. Capozziello, A. Stabile, A. Troisi, (in preparation).
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