From Generalized Gauss Bounds to Distributionally Robust Fault Detection With Unimodality Information

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Abstract—The need for exact distributions in probabilistic fault detection design is hardly fulfilled. The recent moment-based distributionally robust fault detection (DRFD) design secures robustness against inexact distributions but suffers from overpessimism. To address this issue, we develop a new DRFD design scheme by using unimodality, a ubiquitous property of real-life distributions. To evaluate worst-case false alarm rates, a new generalized Gauss bound is first attained, which is less conservative than known Chebyshev bounds that underpin moment-based DRFD. This also yields analytical solutions to DRFD design problems, which are suboptimal but provably less conservative than known ones disregarding unimodality. A tightened Gauss bound is further derived by assuming bounded uncertainty, based on which convex programming approximation of DRFD problems is developed. Results on physical system data elucidate that the proposed DRFD design can reduce conservatism of moment-based ones by using unimodality information, and attaining a better robustness-sensitivity trade-off than prevalent data-centric design with moderate sample sizes.

Index Terms—Fault detection, optimization, uncertain systems, unimodality.

I. INTRODUCTION

With the rapidly growing complexity of modern technical systems, requirements of operational safety and reliability in an uncertain environment are becoming more critical. This has stimulated the tremendous development and successful applications of fault detection, diagnosis, and fault-tolerant control techniques over the past few decades [1], [2]. From a unified viewpoint, residual generation and residual evaluation are centerpieces of fault detection design [1]. The former aims to construct an indicator that can sensitively unveil the occurrence of anomalous events in dynamical systems, while the latter decides whether an alarm shall be raised via some detection logic.

The ubiquity of uncertainties raises significant challenges for fault detection design. In an uncertain environment, the false alarm rate (FAR) and fault detection rate (FDR) are indices of immediate interest for performance evaluation under fault-free and faulty conditions. Since FAR and FDR are essentially probabilities, it is rational to formulate fault detection design problems probabilistically [3], where a detailed description of underlying distribution shall be available. Current endeavors in this vein widely hinge on statistical inference under the Gaussian assumption, e.g., the generalized likelihood ratio test (GLRT), whereby the $\chi^2$-distribution has been used for thresholding; however, in practice the Gaussian assumption itself may be unjustifiable and, thus, vulnerable. Once the true distribution deviates from the assumed normality, a significant degradation in detection performance could occur. Specifically, a high FAR can raise the “alarm flood” issue, eventually leading to mistrust of the alarm system and fatal vulnerability to abnormal events [4].

Along an alternative route is the set-membership technique, which accounts for all admissible uncertainty realizations within a norm-bound set (e.g., zonotope) and then optimizes the worst-case performance [5], [6]. Despite its distribution-free nature, the ensuing fault detection design may be over-pessimistic due to the absence of necessary statistical information.

The abovementioned limitations are being recognized and addressed by distributionally robust optimization (DRO), an emerging roadmap in operations research community [7], [8], [9], [10]. As an intermediate to aforesaid two mainstreams, DRO shows wider applicability in practical situations where only partial stochastic information is available. The crux is to construct an ambiguity set as a family of admissible distributions sharing some common properties, such as the moments and support, based on which an optimal worst-case performance is pursued. In this way, the resultant decision can hedge against the ambiguity in probability distributions of unknowns. Moreover, for a large class of DRO problems the convexity of problems can be recaptured, which ensures computational tractability [8].
Such a new uncertainty characterization has also been popularized in systems and control, see, e.g., [11], [12], [13]. Recently, distributionally robust fault detection (DRFD) has been extensively investigated [14], [15], [16], where robust integrated design of residual generator and alarm threshold are obtained, reliably enforcing constraints on FAR, FDR, and other indices irrespective of imprecisely known distributions [15], [16], [17].

The robustness level of DRFD design relies on the ambiguity set, which is mostly constructed based on the mean and covariance in prior work [14], [15], [16]. Such a description caters to the broad interest in using the first two moments to characterize a distribution; however, the induced design always shows over pessimism, as can be evidenced from a recent work [15] where the true FAR tends to be excessively lower than the tolerance but compromises the sensitivity against faults. In fact, the moment-based ambiguity set encompasses an excessively large class of probability distributions, among which the worst-case one is always pathologically discrete, as documented by [7] and [18]. In real-world systems, however, it is unlikely that uncertainties, especially disturbances, are governed by discrete distributions with only few atoms. Thereby, the robustness against such implausibility is deemed as a main cause for over conservatism.

Thus, this article is oriented toward an improved DRFD design scheme that alleviates the conservatism and strikes a sensible tradeoff between FAR and FDR. The idea is to incorporate unimodality information of distributions, alongside moment and support information, into the ambiguity set. This effectively rules out unrealistic discrete distributions and enables utilizing a broader spectrum of “ambiguous” information as compared to the previous work [15]. Notably, the unimodality assumption bears a meaningful implication that smaller deviations are always likely than larger ones, a common property possessed by a plethora of distributions in probability theory. More importantly, unimodality can be evidenced in many practical situations, e.g., from histograms or scatter plots, which also justifies the extensive usage of Gaussian distributions as approximations. As such, departing from usual methods based on either Gaussian or norm-bounded assumptions, the unimodality-induced ambiguity set yields a “coarse” yet practically sound description to uncertainty governed by unknown distributions. This yields an integrated design problem with various types of prior knowledge judiciously exploited, which maximizes the overall fault detectability while robustly regulating false alarms under inexact or even varied distributions.

We remark that unimodality has been used for uncertainty description in diverse fields including DRO [19], [20], control theory [21], [22], power system operations [23], [24], [25], and statistics [26], [27], [28]. Despite these efforts, resolving the DRFD design problem under unimodality information remains a considerable challenge, which arises mostly from evaluating the worst-case FAR (WCFAR) over all admissible distributions within the ambiguity set. Indeed, evaluating the FAR is simply a quantification problem of tail probability that a random vector deviates from its mean in a multidimensional setting. Thus, given the mean and covariance, the WCFAR is nothing but an extension of the classic univariate Chebyshev bound [14]. With unimodality additionally assumed, the WCFAR can be also viewed as generalizing the Gauss bound. Unfortunately, multivariate unimodal Gauss bounds developed in [19] and [28] are not applicable to fault detection design. This is because quadratic evaluation functions are typically used in fault detection, and thus, the tail probability outside an ellipsoidal region is of major interest. However, to the best of authors knowledge, until now there have been no results regarding the multivariate Guass bound of an ellipsoidal acceptance region, while only the polyhedral case was addressed in previous contributions [19], [28].

The main contributions of this article consist in two aspects, namely two new generalized Gauss bounds useful for less conservative evaluations of FAR, and the resulting DRFD design using moment, unimodality, and support information simultaneously. Given moment, unimodality and unbounded support information, a new multivariate Gauss bound is first developed in closed form (Theorem 2), which generalizes and strictly improves over the known Chebyshev bound [15], [29], [30]. When the support is known to be bounded, we develop a tightened Gauss bound by solving a tractable convex program (Theorem 4), and it is always no higher than the previous bound provided a tuning parameter is suitably chosen. The derived bounds lay the groundwork for tractably solving DRFD design problems under unimodality information (Theorems 6–8), which yields DRFD design with safe but less conservative guarantee of FAR. In particular, analytical expressions of feasible solutions to DRFD design problems are obtained under unbounded support information. Despite suboptimality, the resulting design enables faithful regulation of FAR and yields strictly better detectability than the known DRFD scheme [15] that disregards unimodality information. The effectiveness of the developed DRFD designs is eventually illustrated on data collected from a physical three-tank apparatus, which elucidates the benefit of utilizing unimodality as new ambiguity information in enhancing sensitivity while safely regulating false alarms. Meanwhile, as compared to prevalent data-centric design schemes, such as the Wasserstein distance-based DRFD design [9], [31] and scenario approach [3], [32], our scheme can still robustly safeguard FAR with a desirable sensitivity-robustness tradeoff without abundant data samples.

The rest of this article organized as follows. Section II revisits preliminary knowledge of residual generation, unimodality, and Chebyshev/Gauss bounds in probability theory. In Section III, the main results of this article are presented, while in Section IV case studies are reported. Finally, Section V concludes this article.

Notation: Given a signal \( \{\xi(k)\} \) and an integer \( s > 0 \), the concatenated vector is defined as \( \xi_s(k) = [\xi(k)^\top \xi(k-1)^\top \cdots \xi(k-s)^\top]^\top \). For a matrix \( A \), its null space and Moore–Penrose inverse are denoted by \( \ker(A) \) and \( A^\dagger \), respectively, and for a symmetric \( A \) its positive semidefiniteness is indicated by \( A \succeq 0 \). \( I_n \) denotes the identity matrix of size \( n \times n \), and \( 0_{n \times m} \) indicates \( n \times m \) matrix of all zeros. The line segment connecting two points \( x, y \in \mathbb{R}^n \) is denoted by \( [x, y] \subseteq \mathbb{R}^n \). \( \delta(\xi) \) denotes the Dirac point distribution at \( \xi \).
II. PRELIMINARIES AND PROBLEM FORMULATION

A. Residual Generation and Evaluation

Let us consider the following linear discrete-time (LTI) stochastic system:

\[
\begin{cases}
x(k+1) = Ax(k) + Bu(k) + B_d d(k) + B_f f(k) \\
y(k) = Cx(k) + Du(k) + D_d d(k) + D_f f(k)
\end{cases}
\]  \hspace{1cm} (1)

where \( x \in \mathbb{R}^{n_x}, y \in \mathbb{R}^{n_y}, u \in \mathbb{R}^{n_u}, d \in \mathbb{R}^{n_d}, \) and \( f \in \mathbb{R}^{n_f} \) stand for the process state, measured output, control input, unknown stochastic disturbance, and faults, respectively. System matrices in \( (1) \) have conformable dimensions and \( (C, A) \) is assumed to be observable. Meanwhile, \( d(k) \) is assumed to be random with an unknown probability distribution. To construct a residual generator, a unified way is to adopt the stable kernel representation (SKR) \( K(z) \) based on so-called analytical redundancy [33]. In the absence of faults and disturbances, viz. \( f(k) = 0 \) and \( d(k) = 0 \), one obtains SKR of the LTI system \( (1) \)

\[
K(z) \begin{bmatrix} u(z) \\ y(z) \end{bmatrix} = 0
\]  \hspace{1cm} (2)

where \( z \) denotes the time-shift operator. Thus, a residual signal \( r \) can be generated as an information carrier sensitive to anomalies

\[
r(z) = K(z) \begin{bmatrix} u(z) \\ y(z) \end{bmatrix} \in \mathbb{R}^{n_r}.
\]  \hspace{1cm} (3)

Parallel to the residual generator \( (3) \) is its design form, which delineates the law governing the dynamics of residuals

\[
r(k) = P(Wd_s(k) + Vf_s(k))
\]  \hspace{1cm} (4)

where \( \{W, V\} \) and \( P \) are coefficient matrices and the design matrix that correspond to the residual generator \( K(z) \), and \( s > n_x \) is the given order of concatenated vectors [33]. Typical examples of \( (3) \) and \( (4) \) include the parity space method [34], observer-based method [35], and subspace identification [33], which give rise to different parameterizations of \( \{W, V\} \). Note that the realization of residual generator \( K(z) \) is fully specified by the choice of \( P \) in virtue of \( (4) \); see [33] for more details on the implementation of \( K(z) \) given \( P \). In order to judiciously design \( P \), a variety of classical methods are available, e.g., the unified solution [36]

\[
P^* = \arg \min_P \frac{\|PW\|}{\|PV\|}
\]

which balances between sensitivity to fault \( f \) and robustness against unknown disturbances \( d(k) \). Given \( P \), the realization of residual generator \( (3) \) can be determined and \( J(r) = \|r\|^2 \) is then adopted for residual evaluation. Once \( J(r) \) goes beyond a decision threshold \( J_{th} > 0 \), an alarm is declared to signify an ongoing anomalous situation. Under routine fault-free conditions \( (f = 0) \), alarms are induced by additive uncertainty, as denoted by \( \xi := ds(k) \in \mathbb{R}^n, n = n_d(s + 1) \). The alarm frequency, namely FAR, is an important performance index in residual generation and evaluation.

**Definition 1 (FAR):** Given the threshold \( J_{th} \), the FAR of the residual generator is defined as

\[
\text{FAR} = \mathbb{P}_\xi \{ \|r\|^2 > J_{th} \}
\]

\[
= \mathbb{P}_\xi \{ \|PW\xi\|^2 > J_{th} \}. \hspace{1cm} (5)
\]

The computation of FAR critically resorts to full knowledge about the true distribution \( \mathbb{P}_\xi \). In the fault detection literature, it is widely assumed that \( \mathbb{P}_\xi \) is Gaussian. Consequently, \( J_{th} \) can be calibrated based on \( \chi^2 \)-distribution under a given confidence level \( \varepsilon \), e.g., 0.05, which stands for the highest frequency of false alarms tolerable in engineering practice. It can be decided, for example, by the maximal labor cost affordable in routine operations for effective alarm removal.

B. Distributionally Robust Integrated Design Perspective

The usual assumption that \( \mathbb{P}_\xi \) is completely known is rather restrictive. In this work, we take a distributionally robust design perspective by assuming that \( \xi \) follows an unknown distribution \( \mathbb{P}_\xi \) that is enclosed by an ambiguity set \( \mathcal{D} \). To construct \( \mathcal{D} \), some partial statistical information, such as moment, support, and unimodality information can be integratedly leveraged. In the literature, a prevalent formulation of \( \mathcal{D} \) is developed based on the first two moments and support.

**Definition 2 (Moment-based ambiguity set, [7]):** Given the support \( \Xi \subseteq \mathbb{R}^n \), the estimated mean \( \mu_0 \) and covariance \( S_0 \), the moment-based ambiguity set is defined as

\[
\mathcal{D}(\gamma_1, \gamma_2, \Xi) = \left\{ \mathbb{P}(d\xi) \mid \left\{ \begin{array}{l}
\mathbb{P}\{\xi \in \Xi\} = 1 \\
\mathbb{E}_\mathbb{P}\{\xi\} - \mu_0 \leq \gamma_1 \\
(\mathbb{E}_\mathbb{P}\{\xi\} - \mu_0)^\top S_0^{-1}(\mathbb{E}_\mathbb{P}\{\xi\} - \mu_0) \leq \gamma_2 S_0 \\
\mathbb{E}_\mathbb{P}\{(\xi - \mu_0)(\xi - \mu_0)^\top\} \leq \gamma_2 S_0
\end{array} \right\} \right\}.
\]

Intuitively, \( \mathcal{D}(\gamma_1, \gamma_2, \Xi) \) consists of admissible distributions whose mean and covariance are close to \( \mu_0 \) and \( S_0 \). In this way, the true distribution may still be captured even in the presence of estimation errors. The estimation error of \( \mu_0 \) can be accounted for by an ellipsoid centered at \( \mu_0 \) whose size can be adjusted by \( \gamma_1 > 0 \). Meanwhile, the estimation error of \( S_0 \) is addressed by \( \gamma_2 \geq \max\{\gamma_1, 1\} \) where the second-order moment is upper bounded in a semidefinite sense. \( \gamma_1 \) and \( \gamma_2 \) can be effectively tuned using the bootstrap strategy [31].

When \( J(r) = \|r\|^2 \) is used for residual evaluation, it is customary to perform centering onto \( \xi \) to obtain zero-mean residuals, which better distinguishes nominal variations from anomalies [33]. Thus, it is assumed throughout that \( \mu_0 = 0 \), which causes no loss of generality. Without knowing \( \mathbb{P}_\xi \) exactly, it is meaningful to define the WCFAR within an ambiguity set \( \mathcal{D} \).

**Definition 3 (WCFAR):** Given an ambiguity set \( \mathcal{D} \), the WCFAR of a residual generator is defined as

\[
\text{WCFAR} = \sup_{\mathbb{P}_\xi \in \mathcal{D}} \mathbb{P}_\xi \{ \|r\|^2 > J_{th} \}
\]

\[
= \sup_{\mathbb{P}_\xi \in \mathcal{D}} \mathbb{P}_\xi \{ \|PW\xi\|^2 > J_{th} \}. \hspace{1cm} (6)
\]
By constraining the WCFAR, the DRFD design problem is formally cast as a distributionally robust chance constrained program [15]

$$\max_P \rho(P)$$

s.t. $\sup_{P \in \mathcal{D}} \mathbb{P}_P \{\|PW\xi\|^2 > 1\} \leq \varepsilon$ (DRFD)

where $\rho(\cdot)$ is an overall fault detectability metric to be maximized. The threshold $J_{th} = 1$ is trivially adopted because otherwise one could always attain infinitely many tuples $\{P, J_{th}\}$ with identical detection performance. $\varepsilon$ is a prescribed upper-bound of WCFAR. Solving (DRFD), one attains an integrated design $P$ of a residual generator, which not only maximizes detectability but also safely ensures that the WCFAR over all distributions in $\mathcal{D}$ does not exceed the tolerance. This circumvents the usual two-step procedure where $P$ is first designed and $J_{th}$ is then calibrated. In other words, our design phase itself enables a clear control mechanism of FAR, so that alarm overloading and safety hazard can be reliably circumvented under fault-free conditions. In this way, one strikes a sensible tradeoff between robustness against disturbance and sensitivity against faults. Viable choices of $\rho(\cdot)$ include the Frobenius norm metric [32]

$$\rho_1(P) = \|PV\|^2 = \text{Tr}\{V^\top P^\top PV\}$$ (7)

and the pseudodeterminant metric [15]:

$$\rho_2(P) = \log \text{pdet}(V^\top P^\top PV) = \log \det(\Lambda^\top U_1^\top P^\top PU_1\Lambda)$$ (8)

where $V = U_1\Lambda U_2^\top$ is the compact singular value decomposition with $\Lambda \in \mathbb{R}^{m_1 \times m_1}$ being diagonal and invertible. In a nutshell, the usage of $\rho_1(P)$ and $\rho_2(P)$ amounts to maximizing, respectively, the sum of eigenvalues and the product of positive eigenvalues of $V^\top P^\top PV$. More generally, a weighted combination of two metrics is also possible.

**Remark 1:** Note that the inexactness of distribution of additive disturbance $\xi = d_\alpha(k)$ can be addressed by solving problem (DRFD). Indeed, it allows for broader usage even when knowledge about $d_\alpha(k)$ itself is unavailable (e.g., unmeasurable disturbance) but a running residual signal $v(k)$ has been constructed, whose dynamics are governed by $v(k) = W_\alpha d_\alpha(k) + V_\alpha f_\alpha(k)$. In this case, we can regard the residual under routine fault-free conditions as additive uncertainty, viz. $\xi = v_0(k) := W_\alpha d_\alpha(k)$, whose distribution is unknown but fault-free data samples are available to build $\mathcal{D}$. It then follows that $v(k) = \xi(k) + V_\alpha f_\alpha(k)$ and the goal is to identify a design matrix $P$ to "refine" the present residual generator as

$$r(k) = P v(k) = P[v_0(k) + V_\alpha f_\alpha(k)].$$ (9)

In this case, our formulation (DRFD) still applies with $W = I$ and $V$ replaced by $V_\alpha$ in the definition of $\rho(\cdot)$.

**Remark 2:** A large body of work postulate exact moment matching, which uses $\mathcal{D}(0, 1, \mathbb{R}^n)$ and replaces "$\leq\"$ with "$=\"$" therein. In fact, in the context of DRFD, it suffices to consider $\mathcal{D}(\gamma_1, \gamma_2, \Xi)$ with a semidefinite constraint because in (DRFD) the worst-case distribution tends to be maximally spread out, which attains the upper bound on the covariance. Thus, the results developed under moment ambiguity apply straightforwardly to the setup of exact moment matching. Similar arguments have been made in [7] and [37].

As a primary modeling tool of ambiguous uncertainty, the moment-based set $\mathcal{D}(\gamma_1, \gamma_2, \Xi)$ is known to be over conservative. It was shown in [7] that the worst-case distribution tends to be "impulsive," thereby deviating from reality. In the context of fault detection, this renders the realistic FAR much lower than the tolerance while unnecessarily sacrificing fault detectability [15]. To alleviate such over conservatism, incorporating structural properties, such as unimodality and monotonicity has been suggested in the operations research community, see, e.g., [19], [28]. As a minimal structural property, unimodality is not only ubiquitous in real-life situations but also inherited by numerous distributions in probability theory.

In this work, we aim to address the design problem (DRFD) by assuming that the true distribution of uncertainty $\xi$ in system (1) is known to be unimodal, thereby utilizing an increased amount of ambiguous information as compared to [15]. Conceptually, unimodality of a distribution indicates that larger deviations are less likely than smaller ones, which is a realistic assumption on uncertain disturbance and yields a new type of partial information. In the univariate case $\xi \in \mathbb{R}$, unimodality asserts the existence of a mode $m$ where the probability density culminates. In the multivariate case, the $\alpha$-unimodality offers a straightforward generalization.

**Definition 4 ($\alpha$-unimodality, [38]):** For any $\alpha > 0$, a multivariate distribution $\mathcal{P}$ is $\alpha$-unimodal about 0 if $t^\alpha \mathcal{P}(S(t))$ is nondecreasing in $t \in (0, \infty)$ for every Borel set $S \subset \mathcal{B}(\mathbb{R}^n)$.

The $\alpha$-unimodality regulates the minimal decreasing rate of density along rays emitted from the mode. In this sense, $\alpha$ can be interpreted as characterizing the “degree of unimodality.” For example, $n$-dimensional normal distributions and uniform distributions are $n$-unimodal [19]. Denoting by $\mathcal{P}_\alpha$, the set of all $\alpha$-unimodal distributions with zero mode, we adopt the following strengthened ambiguity set to describe the unknown distribution of $\xi$ in (1), which encodes moment and support information together with unimodality

$$\mathcal{D}_\alpha(\gamma_1, \gamma_2, \Xi) \triangleq \mathcal{D}(\gamma_1, \gamma_2, \Xi) \cap \mathcal{P}_\alpha.$$ (10)

It is known that for any finite $\alpha$ the Dirac distribution $\delta_\alpha$ with $\alpha = 0$ does not belong to $\mathcal{P}_\alpha$ [19]. This sheds light on the capability of $\mathcal{D}_\alpha$ in eliminating unrealistic discrete distributions. As a result, $\mathcal{D}_\alpha(\gamma_1, \gamma_2, \Xi)$ has a smaller “volume” than $\mathcal{D}(\gamma_1, \gamma_2, \Xi)$ for any finite $\alpha$. This eventually helps us to derive a solution to (DRFD) that is less conservative than [15]. For clarity, we may refer to $\mathcal{D}_\alpha$ with its dependence on $\gamma_1, \gamma_2, \Xi$ dropped when no confusion is made.

**Remark 3:** Setting $\alpha = n$, the generic notion of star-unimodality is recovered [39], which requires the density function to be nonincreasing along any ray emanating from the mode. Intuitively, the density function is “bell-shaped” in a high-dimensional space. Thus, it is practically meaningful to specify $\alpha = n$ when disturbance at a higher energy level is known to be less likely. Moreover, $\mathcal{P}_\alpha$ enjoys the nesting property $\mathcal{P}_\alpha \subset \mathcal{P}_{\alpha_2}$ if $\alpha_1 < \alpha_2 \leq \infty$. As $\alpha$ tends to infinity, the restriction on
unimodality vanishes. Henceforth, a smaller $\alpha$ helps reducing the conservatism of $D_\alpha$. To verify the $\alpha(< n)$-unimodality of a density $P_\xi(d\xi)$ from data, a viable approach in the light of [20] is to sample a number of "directions" $\zeta_0$ on the $n$-dimensional sphere and then inspect the unimodality of scaled univariate densities $P_{\alpha,\zeta_0}(dt) = t^{v-\eta}P_\xi(td\zeta_0)$ by projection onto each $\zeta_0$.

C. Optimal Inequalities in Probability Theory

Due to the consideration of unimodality information, the solution algorithm in [15] is no longer applicable to our design problem (DRFD). Next we focus on the WCFAR in (DRFD) and show that it is germane to optimal inequalities in probability theory. By definition, the FAR in (5) is essentially a tail probability of unfortunate events. In the univariate case $\xi \in \mathbb{R}$ with mean and variance given, the worst-case tail probability in (DRFD) can be evaluated using the Chebyshev inequality, a fundamental result in probability theory.

Theorem 1 (Chebyshev inequality): For a random variable $\xi \in \mathbb{R}$ with mean $\mu$ and variance $\sigma^2$, it always holds that

$$\Pr \{ |\xi - \mu| \geq \kappa \sigma \} \leq \min \left\{ \frac{1}{\kappa^2}, 1 \right\}, \quad \kappa > 0. \quad (11)$$

Due to its distribution-free nature, the Chebyshev inequality constitutes the foundation of a variety of probabilistic methods, such as the minimax probability machines and DRFD design [14], [16], [40]. Note that the Chebyshev bound is tight due to the existence of an extremal distribution

$$\left(1 - \frac{1}{\kappa^2}\right) \delta_{\mu}(\cdot) + \frac{1}{2\kappa^2} \delta_{\mu+\delta}(\cdot) + \frac{1}{2\kappa^2} \delta_{\mu-\delta}(\cdot) \quad (12)$$

which makes (11) an equality. Such a discrete distribution is germane to the interplay between the Chebyshev bound and DRO problems. That is, the r.h.s. of (11) can be interpreted as the optimal value of the following worst-case probability problem with the moment-based ambiguity set [19]:

$$\sup_{P \in D'} \Pr \{ |\xi - \mu| \geq \kappa \sigma \} = \min \left\{ \frac{1}{\kappa^2}, 1 \right\} \quad (13)$$

where $D'$ encloses univariate distributions sharing the same mean $\mu$ and variance $\sigma^2$

$$D' = \{ P(d\xi) \mid \mathbb{E}_P \{ \xi \} = \mu, \mathbb{E}_P \{ (\xi - \mu)^2 \} = \sigma^2 \} \quad (14)$$

More importantly, the optimal solution to (13) is given by the discrete distribution (12) in the worst-case. The over conservatism of the Chebyshev inequality can be alleviated by considering unimodal distributions solely. This yields the Gauss inequality [41], which is interpretable as the worst-case probability problem (13) with a more informed $\alpha$-unimodal ambiguity set

$$D'_\alpha \triangleq D_\alpha \cap \{ P(d\xi) \mid \Pr \{ \xi \} \text{ is unimodal about } \mu \}$$

in lieu of $D'$.

There is a vast literature on multivariate generalizations of the Chebyshev inequality [26], [27], [29], [30]. Meanwhile, multivariate extensions of the univariate Gauss inequality have been investigated as well [19], [28], [42]. All of them concentrate on the highest risk of a random vector residing outside a polytope. A notable result is [19, Lemma 4], which assumes the polytope having the form $\{ \xi \mid |\xi_i - \mu_i| \leq \kappa \sigma, i = 1, \ldots, n \}$ and yields an exact improvement factor

$$c_\alpha = \left( \frac{2}{\alpha + 2} \right)^{2/\alpha} \quad (15)$$

III. MAIN RESULTS

A. New Multivariate Generalizations of Gauss Inequality

Insofar as the probability outside an ellipsoid is concerned in the design problem (DRFD), known multivariate $\alpha$-unimodal Gauss bounds are no longer applicable. To evaluate the WCFAR in the constraint of (DRFD), new multivariate extensions of $\alpha$-unimodal Gauss bounds are first derived in this section. Before proceeding, we shall recall a useful fact that the family of $\alpha$-unimodal distributions can be reparameterized explicitly based on radial $\alpha$-unimodal distributions as extremal ones.

Definition 5 (Radial $\alpha$-unimodal distributions, [19]): For any $\alpha > 0$ and $x \in \mathbb{R}^n$, $\delta_{[0,x]}^\alpha(\cdot)$ stands for the radial distribution supported on the line segment $[0, x] \subset \mathbb{R}^n$ with the property $\delta_{[0,x]}^\alpha(\cdot)([0, \lambda x]) = \lambda^\alpha, \forall \lambda \in [0, 1]$.

The $\alpha$-unimodality of radial distributions $\delta_{[0,x]}^\alpha(\cdot)$ is indeed easy to verify. Moreover, they are extremal distributions in $\mathcal{P}_\alpha$ [19], which are not representable as strict convex combinations of two distinct distributions in $\mathcal{P}_\alpha$. Thus, invoking the Choquet theory [43], the family of $\alpha$-unimodal distributions can be explicitly reparameterized by “mixing” extremal ones.

Lemma 1 (Choquet representation of unimodal distributions): For every distribution $P \in \mathcal{P}_\alpha$ supported on $\Xi$, there exists a unique distribution $Q$ supported on $\Xi$ such that

$$P(d\xi) = \int_{\Xi} \delta_{[0,w]}^\alpha(d\xi)Q(dw). \quad (16)$$

Proof: The proof of [39, Th. 3.5] applies with minor modifications to the present setup and is thus omitted. ■

Lemma 1 asserts that every $\alpha$-unimodal distribution supported on $\Xi$ is expressible as a mixture of radial distributions $\delta_{[0,x]}^\alpha(\cdot)$, $w \in \Xi$, with $Q$ being the mixture distribution. This allows us to recast the worst-case probability problem explicitly in the following supporting lemma.

Lemma 2: Given an ellipsoidal confidence region $\mathcal{E} = \{ \xi \in \mathbb{R}^n \mid \xi^\top M \xi \leq 1 \}$ with $M \succeq 0$, the worst-case probability outside $\mathcal{E}$, i.e.,

$$\sup_{P \in D_{\alpha}^\gamma(\gamma_1, \gamma_2, \Xi)} \Pr \{ \xi \notin \mathcal{E} \}$$

is equal to the optimal value of the following semi-infinite optimization problem:

$$\min_{Q, q_0, \gamma_0} \gamma_2 \text{Tr} \{ QS_0^2 \} + q_0$$

s.t. $\xi^\top Q \xi + 2\xi^\top q + q_0 \geq L_\alpha(\xi), \forall \xi \in \Xi$ \quad (17)

$$Q \succeq 0 \quad Q \succeq 0$$

where

$$L_\alpha(\xi) = \max \left\{ 1 - \| M^{1/2} \xi \|^2 - \frac{\alpha}{\alpha + 2}, 0 \right\}, \quad S_0^\alpha = \frac{\alpha + 2}{\alpha} S_0.$$
Proof: We draw ideas from [28, Th. 1]. Thanks to Lemma 1, it suffices to optimize over the unstructured mixture distribution \( Q \) instead of \( \mathbb{P} \). Using the reparameterization (16) yields

\[
\sup_{\mathbb{P} \in \mathcal{P}_n(\gamma_1, \gamma_2, \Xi)} \mathbb{P} \{ \xi^T M \xi > 1 \} = \sup_{\mathbb{P} \in \mathcal{P}_n(\gamma_1, \gamma_2, \Xi)} \int_{\Xi} \mathbb{P} (\xi) \mathbb{P} (d\xi) = \sup_{\mathbb{P} \in \mathcal{P}_n(\gamma_1, \gamma_2, \Xi)} \int_{\Xi} \mathbb{P} (\xi) \mathbb{P} (d\xi) \]

where the last equality stems from the fact that \( \delta_{[0, w]}(\xi) \) is supported on the line segment \([0, w]\) only and thus it suffices to integrate over \([0, w]\) using a scalar \( \lambda \). Then, the objective amounts to

\[
\sup_{\mathbb{P} \in \mathcal{P}_n(\gamma_1, \gamma_2, \Xi)} \int_{\Xi} \mathbb{P} (\xi) \mathbb{P} (d\xi) = \sup_{\mathbb{P} \in \mathcal{P}_n(\gamma_1, \gamma_2, \Xi)} \int_{\Xi} \mathbb{P} (\xi) \mathbb{P} (d\xi) = \sup_{\mathbb{P} \in \mathcal{P}_n(\gamma_1, \gamma_2, \Xi)} \int_{\Xi} \mathbb{P} (\xi) \mathbb{P} (d\xi) = \sup_{\mathbb{P} \in \mathcal{P}_n(\gamma_1, \gamma_2, \Xi)} \int_{\Xi} \mathbb{P} (\xi) \mathbb{P} (d\xi)
\]

By similar arguments, two moment constraints can be translated into

\[
\begin{align*}
\int_{\Xi} w^T Q (d\xi) &= \frac{S_0}{\int_0^1 \lambda^2 \delta_{[0, w]}(\xi) (w d\xi)} = \frac{\alpha + 2}{\alpha} S_0 \triangleq S_0^{\alpha} \\
\int_{\Xi} w Q (d\xi) &= \frac{\mu_0}{\int_0^1 \lambda \delta_{[0, w]}(\xi) (w d\xi)} = \frac{\alpha + 1}{\alpha} \mu_0 \triangleq \mu_0^{\alpha}.
\end{align*}
\]

In this way, assessing the WCFAR can be recast as the following worst-case expectation problem with the unstructured distribution \( Q \) being the decision variable, which resides in the generic moment-based set

\[
\sup_{\mathbb{Q}} \mathbb{E}_\mathbb{Q} \{ L_\alpha (w) \} = \sup_{\mathbb{Q} \in \mathcal{D}(S_0^{\alpha}, \mu_0^{\alpha}, \Xi)} \mathbb{E}_\mathbb{Q} \{ L_\alpha (w) \}
\]

Its full equivalence with problem (17) immediately follows from the duality argument in [7, Lemma 1], which completes the proof.

Resolving problem (17) with unbounded support \( \Xi = \mathbb{R}^n \), we arrive at a new multivariate \( \alpha \)-unimodal Gauss bound in closed form, which not only complements [19, Lemma 4] but also paves the way for subsequent DRFD design.

Theorem 2 (Generalized \( \alpha \)-unimodal Gauss bound): Consider \( \mathcal{D} = \mathcal{D}_n(\gamma_1, \gamma_2, \mathbb{R}^n) \) that includes all \( \alpha \)-unimodal distributions on unbounded support \( \Xi = \mathbb{R}^n \) subject to moment constraints. The induced worst-case probability of the event \( \xi \notin \mathcal{E} \) is upper bounded by

\[
\sup_{\mathbb{P} \in \mathcal{P}_n(\gamma_1, \gamma_2, \mathbb{R}^n)} \mathbb{P} \{ \xi \notin \mathcal{E} \} \leq \begin{cases} 
 c_\alpha \gamma_2 \mathbb{Tr} \{ MS_0 \}, & \text{if } c_\alpha \gamma_2 \mathbb{Tr} \{ MS_0 \} \leq \frac{\alpha}{\alpha + 2} \\
 1 - \frac{1}{(\gamma_2 \mathbb{Tr} \{ MS_0^+ \})^{n/2}}, & \text{otherwise}
\end{cases} \quad (18)
\]

The proof is relegated to the Appendix due to its complexity. Next we dwell on the effect of introducing \( \alpha \)-unimodality. In the limiting case \( \alpha \to \infty \), where \( \alpha \)-unimodality vanishes, the following established result is recalled.

Theorem 3 (Generalized Chebyshev bound, [15], [29], [30]): Consider all distributions \( \mathbb{P} \in \mathcal{D}(\gamma_1, \gamma_2, \mathbb{R}^n) \) that have unbounded support and are subject to moment constraints only. The worst-case probability of the event \( \xi \notin \mathcal{E} \) is given by

\[
\sup_{\mathbb{P} \in \mathcal{D}(\gamma_1, \gamma_2, \mathbb{R}^n)} \mathbb{P} \{ \xi \notin \mathcal{E} \} = \min \{ \gamma_2 \mathbb{Tr} \{ MS_0 \}, 1 \} \quad (19)
\]

Theorem 3 offers an equality, whereas under \( \alpha \)-unimodality assumption one merely attains via Theorem 2 an upper bound that is not necessarily sharp. One may wonder whether restricting the search within unimodal distributions leads to a more conservative quantification of tail probability. Interestingly, it turns out that introducing unimodality always helps us to reduce conservatism of evaluating WCFAR even if the tightness of (18) is currently unknown.

Corollary 1: For any finite \( \alpha \), the generalized Gauss bound (18) is always lower than its Chebyshev counterpart (19).

Proof: It follows from \( c_\alpha < 1 \) that \( c_\alpha \gamma_2 \mathbb{Tr} \{ MS_0 \} < \gamma_2 \mathbb{Tr} \{ MS_0 \} \). Then, two cases are distinguished. When \( c_\alpha \gamma_2 \mathbb{Tr} \{ MS_0 \} \leq \frac{\alpha}{\alpha + 2} < 1 \), one obtains

\[
c_\alpha \gamma_2 \mathbb{Tr} \{ MS_0 \} < \min \{ \gamma_2 \mathbb{Tr} \{ MS_0 \}, 1 \}.
\]

As for the case \( c_\alpha \gamma_2 \mathbb{Tr} \{ MS_0 \} > \frac{\alpha}{\alpha + 2} \), we have

\[
1 - \frac{1}{(\gamma_2 \mathbb{Tr} \{ MS_0^+ \})^{n/2}} < \min \{ c_\alpha \gamma_2 \mathbb{Tr} \{ MS_0 \}, 1 \}
\]

which yields the result.

Remark 4: When \( \alpha \to \infty \), one can verify that the upper bound in (18) becomes the generalized Chebyshev bound (19), which is known to be tight. Thus, we conjecture that the generalized \( \alpha \)-unimodal Gauss bound (18) is tight as well, which will be deferred to further investigation.

Remark 5: The proposed generalized \( \alpha \)-unimodal Gauss bound (18) improves upon its Chebyshev counterpart (19) by a factor of \( c_\alpha \) when \( \gamma_2 \mathbb{Tr} \{ MS_0 \} \) is suitably small. This coincides with the gain within known multivariate formulations of \( \alpha \)-unimodal Gauss bound [19], [42]. In Fig. 1, we depict the value of \( c_\alpha \) under varied \( \alpha \), where for a moderately valued \( \alpha \) (e.g., between 5 and 50), the improvement factor ranges from 0.6 to 0.8. When \( \alpha \to \infty \), the value of \( c_\alpha \) tends to one, and thus the generalized Chebyshev bound (19) is recovered.

In practice, a stochastic disturbance within a finite time period typically has bounded energy. This justifies the prevalence of
set-membership regime in model-based filtering and fault diagnosis [5], [6], [44], [45], where uncertainty is confined to a compact set. Such knowledge can be further utilized to enrich the information within ambiguity sets by endowing \( D_{\alpha} \) with a bounded support \( \Xi_b \subset \mathbb{R}^n \). Through a confluence of moment, unimodality, and bounded support information, the resulting ambiguity set \( D_{\alpha}(\gamma_1, \gamma_2, \Xi_b) \) offers a "hybrid" description to uncertainty \( \xi \), which can be understood as "interpolating" between the Gaussianity assumption and norm-bounded description. On the one hand, unimodality as well as mean-covariance information underlying Gaussian distributions is preserved. On the other hand, bounded support information is taken into account.

All information is seamlessly synthesized by \( D_{\alpha}(\gamma_1, \gamma_2, \Xi_b) \) to develop a strengthened \( \alpha \)-unimodal Gauss bound, which admits a tractable convex programming approximation.

**Theorem 4: (Generalized \( \alpha \)-unimodal Gauss bound for bounded uncertainty)** Suppose the support of distributions is described as the intersection of ellipsoids

\[
\Xi_b = \{ \xi | (\xi - a_j)^\top \Theta_j (\xi - a_j) \leq 1, j \in \{1, N_e\} \}. 
\]

Then, the worst-case probability of the event \( \xi \notin \mathcal{E} \) under \( D_{\alpha}(\gamma_1, \gamma_2, \Xi_b) \) is no higher than the optimal value of the following semidefinite program (SDP):

\[
\begin{align*}
\min_{Q,q,\eta,\alpha,\beta} & \; \gamma_{2} \text{Tr} \{ QS_0^\top \} + q_0 \\
\text{s.t.} & \; \eta \geq 0, \; \beta_j \geq 0, \; \tilde{\beta}_j \geq 0, \; j = 1, \ldots, N_e \\
& \begin{bmatrix} Q - \eta M & q \\ q^\top & q_0 - 1 \end{bmatrix} + \sum_{j=1}^{N_e} \beta_j \Phi_j \geq 0 \\
& \frac{1}{\tau_0^\alpha} \begin{bmatrix} 0 & 0 & 0 \\ 0 & \alpha + 1 & -\alpha \tau_0^\alpha \\ 0 & \alpha \tau_0^\alpha & 0 \end{bmatrix} \geq 0 \\
& \begin{bmatrix} Q & q \\ q^\top & q_0 \end{bmatrix} + \sum_{j=1}^{N_e} \tilde{\beta}_j \tilde{\Phi}_j \geq 0 \\
& Q \succeq 0
\end{align*}
\]

where

\[
\Phi_j = \begin{bmatrix} \tilde{\Phi}_j & 0 \\ 0 & 0 \end{bmatrix}, \; \tau_0 > 0
\]

\[
\tilde{\Phi}_j = \begin{bmatrix} \Theta_j & -\Theta_j a_j \\ -a_j^\top \Theta_j & a_j^\top \Theta_j a_j - 1 \end{bmatrix}
\]

**Proof:** By virtue of Lemma 2, the worst-case probability problem can be recast as

\[
\begin{align*}
\min_{Q,q,\eta,\alpha,\beta} & \; \gamma_{2} \text{Tr} \{ QS_0^\top \} + q_0 \\
\text{s.t.} & \; \xi^\top Q \xi + 2\xi^\top q + q_0 \geq 1 - ||M^2 \xi||^{-\alpha} \quad \forall \xi \in \Xi_b \\
& \xi^\top Q \xi + 2\xi^\top q + q_0 \geq 0 \quad \forall \xi \in \Xi_b \\
& Q \succeq 0
\end{align*}
\]

We first tackle the semi-infinite constraints \( \xi^\top Q \xi + 2\xi^\top q + q_0 \geq 1 - ||M^2 \xi||^{-\alpha} \), \( \forall \xi \in \Xi_b \), which are implied by the following constraint by the same linearization technique in Theorem 2 (see Appendix):

\[
\xi^\top Q \xi + 2\xi^\top q + q_0 + (\alpha + 1)\tau_0^{-\alpha} - 1
\]

\[
\geq \alpha \tau_0^{-\alpha - 1} ||M^2 \xi|| \quad \forall \xi \in \Xi_b
\]

for an arbitrary \( \tau_0 > 0 \). This turns out to be equivalent to the following semi-infinite constraint without conic terms

\[
\xi^\top Q \xi + 2\xi^\top q + q_0 + (\alpha + 1)\tau_0^{-\alpha} - \alpha \tau_0^{-\alpha - 1} t \geq 1 - 0,
\]

\[
\forall (\xi,t) \in \left\{ (\xi, t) \left| \left( \xi - a_j \right)^\top \Theta_j (\xi - a_j) \leq 1, j \in \{1, N_e\} \right. \right\}
\]

By invoking the S-procedure [46], a sufficient condition for the above semi-infinite constraint to hold is the existence of Lagrangian multipliers \( \{ \beta_j \geq 0, j = 1, \ldots, N_e \} \) and \( \eta \geq 0 \) such that \( \forall (\xi,t) \in \mathbb{R}^{n+1} \)

\[
\xi^\top Q \xi + 2\xi^\top q + q_0 + (\alpha + 1)\tau_0^{-\alpha} - \alpha \tau_0^{-\alpha - 1} t
\]

\[
+ \eta t^2 - \eta \xi^\top M \xi + \sum_{j=1}^{N_e} \beta_j (\xi - a_j)^\top \Theta_j (\xi - a_j) \geq 0
\]

which amounts to the linear matrix inequality (LMI) (21c).

By similar arguments, the semi-infinite constraints \( \xi^\top Q \xi + 2\xi^\top q + q_0 \geq 0, \forall \xi \in \Xi_b \) are implied by the existence of \( \{ \tilde{\beta}_j \geq 0, j = 1, \ldots, N_e \} \) assuring (21d).

**Remark 6:** It is not restrictive to express \( \Xi_b \) as an intersection of ellipsoids. Indeed, one can show along a similar route that the case of polytopic support \( \Xi_b = \{ \xi | G \xi \leq h \} \) is also tractable by invoking the S-procedure.

Recall that in Theorem 4, we have derived an upper bound of worst-case probabilities, whose tightness is unwarranted. Thus, it is doubtful whether the exploitation of support information does help to refine the \( \alpha \)-unimodal Gauss bound (18) and leads to less conservative evaluation of WCFAR. Said another way, a possible yet unwanted outcome is that the bound in Theorem 4 is even higher than that in Theorem 2. Next we show that this
case can be effectively circumvented by judiciously choosing the tuning parameter \( \tau_0 > 0 \).

**Theorem 5:** Given
\[
\tau_0 = \max \left\{ \frac{1}{\sqrt{c_0}}, \sqrt{\gamma_2 \text{Tr}\{MS^0\}} \right\}
\]
the optimal value of the associated problem (21) is always lower than or equal to the generalized \( \alpha \)-unimodal Gauss bound (18) under \( \Xi = R^n \).

**Proof:** We first consider the case where \( \tau_0 = 1/\sqrt{c_0} > \sqrt{\gamma_2 \text{Tr}\{MS^0\}} \). From the proof of Theorem 2, the generalized \( \alpha \)-unimodal Gauss bound \( c_\alpha \gamma_2 \text{Tr}\{MS^0\} \) in (18) coincides with the optimal value of problem (28) with \( \tau_0 = 1/\sqrt{c_0} \). Denote by \( \{Q^*, q^*, \tilde{q}_0, \eta^*\} \) the associated optimal solution to (28). It immediately follows that \( \{Q^*, q^*, \tilde{q}_0, \eta, \beta_1 = 0, \beta_0 = 0\} \) is feasible for (21). Henceforth, (21) is less constrained than (28) with the same objective, indicating that the optimal value of (21) is no lower than that of (28). The case of \( \tau_0 = \sqrt{\gamma_2 \text{Tr}\{MS^0\}} \geq 1/\sqrt{c_0} \) can be treated similarly, from which the claim follows.

**B. DRFD Tradeoff Design Scheme**

Based on established \( \alpha \)-unimodal Gauss bounds, we are now ready to address the robust design problem (DRFD) under the ambit set \( \mathcal{D}_\alpha \). We first investigate the case of unbounded uncertainty \( \xi \in R^n \). By inserting the generalized Gauss bound (18) into the design problem (DRFD), feasible closed-form solutions can be derived under different detectability metrics \( \rho_\alpha (\cdot) \). Even though there is no guarantee of optimality, these design solutions are of significance since they strictly improve upon known DRFD designs without considering unimodality.

**Theorem 6:** A feasible solution to problem (DRFD) with the Frobenius norm metric \( \rho_1 (\cdot) \) under \( \mathcal{D} = \mathcal{D}_\alpha (\gamma_1, \gamma_2, R^n) \) is given by
\[
P^*_1 = \begin{cases} \frac{\sqrt{\omega_1}}{\gamma_2 c_0}, \frac{p^1_1}{\sqrt{\gamma_1 p_1}}, 0 < \varepsilon \leq \frac{\alpha}{\alpha + 2}, \\
\frac{\omega_1}{\gamma_1 (\alpha + 2) + \gamma_2 p_1}, \frac{p^1_1}{\sqrt{\gamma_1 p_1}}, \frac{\alpha}{\alpha + 2} < \varepsilon \leq 1 \end{cases}
\]
where \( \{\omega_1, p_1\} \) are the largest eigenvalue and the associated eigenvector of the following generalized eigen-decomposition problem:
\[
VV^T p = \omega W^T S_0 W^T p.
\]

**Proof:** We first consider the case \( 0 < \varepsilon \leq \alpha / (\alpha + 2) \). Noting that the right-hand side of (18) is strictly increasing in \( \gamma_2 \text{Tr}\{MS^0\} \) and plugging \( M = W^T P^* PW \) into (18), the constraint on the WCFAR in (DRFD) is implied by the inequality
\[
c_\alpha \gamma_2 \text{Tr}\{W^T P^* PW S_0\} \leq \varepsilon.
\]
Thus, solving the following problem always yields a feasible solution to (DRFD) with metric \( \rho_1 (\cdot) \)
\[
\max_{P^*} \text{Tr}\{V^T P^* PV\}
\]
s.t. \( c_\alpha \gamma_2 \text{Tr}\{W^T P^* PW S_0\} \leq \varepsilon \).

One of its optimal solutions is given by the first case in (23) according to [15, Th. 2]. Next we embark on the case \( \alpha / (\alpha + 2) < \varepsilon \leq 1 \). The constraint on WCFAR is implied by
\[
1 - \frac{1}{(\gamma_2 \text{Tr}\{MS^0\})^{\alpha/2}} \leq \varepsilon
\]
which amounts to
\[
\gamma_2 \text{Tr}\{W^T P^* PW S_0\} \leq \frac{1}{(1 - \varepsilon)^{2/\alpha}}.
\]
By the same token, a feasible solution to the design problem is attained as the second case in (23), from which the claim follows.

**Theorem 7:** A feasible solution to problem (DRFD) with the pseudodeterminant metric \( \rho_2 (\cdot) \) under \( \mathcal{D} = \mathcal{D}_\alpha (\gamma_1, \gamma_2, R^n) \) is given by
\[
P^*_2 = \begin{cases} \frac{\varepsilon}{m_f \gamma_2 c_0}, P^{\text{GLRT}}, 0 < \varepsilon \leq \frac{\alpha}{\alpha + 2} \\
\frac{1}{m_f \gamma_2 (1 - \varepsilon)^{2/\alpha}}, P^{\text{GLRT}}, \frac{\alpha}{\alpha + 2} < \varepsilon \leq 1 \end{cases}
\]
where
\[
P^{\text{GLRT}} = S_0^{-1/2} V (V^T S_0^{-1} V)^{-1} V^T S_0^{-1}, S_0 = W S_0 W^T
\]
is celebrated GLRT design for fault detection [47, 48].

**Proof:** We follow the same outline as Theorem 6 and take the case \( 0 < \varepsilon \leq \alpha / (\alpha + 2) \) as an example. In this case, the following problem acts as a conservative approximation to (DRFD) with metric \( \rho_2 (\cdot) \)
\[
\max_{P^*} \log \det \left( \Lambda^* U^T J^T P^* P U J \right)
\]
s.t. \( c_\alpha \gamma_2 \text{Tr}\{W^T P^* PW S_0\} \leq \varepsilon \)
which admits a closed-form optimal solution in terms of the first case in (24) due to [15, Th. 3]. The case \( \alpha / (\alpha + 2) < \varepsilon \leq 1 \) can be treated similarly.

Some remarks on Theorems 6 and 7 are made in order.
and collapsed into the Dirac distribution $\delta_0$. In this case, it is expected that an “infinitely large” $P_0^\ast$ is attained. This can also be inspected from (23) and (24) since for $\varepsilon > 0$, $\frac{\alpha}{\alpha+2} < \varepsilon \leq 1$ always holds, resulting in

$$\lim_{\alpha \to 0} \frac{1}{(1-\varepsilon)^{2/\alpha}} = \infty.$$ 

**Remark 9:** The GLRT design $P_{GLRT}$ itself is developed based on the Gaussian assumption, making $J(r)$ follow a $\chi^2$-distribution with $m_J$ degrees of freedom [47], [48]. Under confidence level $\varepsilon \in (0,1)$, a theoretical threshold for $J(r)$ is given by $\chi^2_{m_J,1-\varepsilon}$, which enjoys the strong concentration property [49]

$$\chi^2_{m_J,1-\varepsilon} \sim O(\log(1/\varepsilon)), \varepsilon \to 0.$$ 

Under distributional ambiguity, however, such a property no longer remains since Theorem 7 implies that for a sufficiently small $\varepsilon$, a safe threshold for $P_{GLRT}$ is

$$m_J\gamma_2c_a/\varepsilon \sim O(1/\varepsilon)$$

as $\varepsilon \to 0$, which is the price we have to pay for being distributionally robust.

We further show that under bounded uncertainty $\xi \in \Xi_b$, the design problem (DRFD) with $D = D_0(\gamma_1, \gamma_2, \Xi_b)$ can be approximated as a tractable convex program by embedding the SDP formulation (21) in (DRFD). To do so, we need to define a positive semidefinite matrix $\tilde{P} = P^\top P \succeq 0$ as the optimization variable. Then, the design problem (DRFD) becomes

$$\max_{\tilde{P} \succeq 0} \rho(\tilde{P}^{1/2})$$

s.t. $\sup_{\xi \in \Xi} \{\xi^\top W^\top \tilde{P}W\xi > 1\} \leq \varepsilon $(DRFD')

**Theorem 8:** Solving the following problem always yields a suboptimal solution to the design problem (DRFD') with $D = D_0(\gamma_1, \gamma_2, \Xi_b)$:

$$\max_{\tilde{P},Q,q,q_0,\eta,\beta,M} \rho(\tilde{P}^{1/2})$$

s.t. $\gamma_2 \text{Tr}\{QS_q^0\} + q_0 \leq \varepsilon \eta$

$$\eta \geq 0, \beta_j \geq 0, \tilde{\beta}_j \geq 0, j = 1, \ldots, N_e$$

$$\begin{bmatrix}
    Q - W^\top \tilde{P}W & q & 0 \\
    q^\top & q_0 - \eta & 0 \\
    0 & 0 & 1
\end{bmatrix} + \sum_{j=1}^{N_e} \beta_j \Phi_j \geq 0 \quad (25)$$

$$\begin{bmatrix}
    \eta \\
    \alpha + 1 \\
    \frac{\alpha}{\alpha + 2}
\end{bmatrix} \geq 0$$

$$\begin{bmatrix}
    Q \\
    q^\top \\
    q_0
\end{bmatrix} + \sum_{j=1}^{N_e} \tilde{\beta}_j \Phi_j \geq 0, Q \succeq 0, \tilde{P} \succeq 0$$

where $\tau_0 > 0$.

**Proof:** In virtue of Theorem 4, we arrive at the following inner approximation to the design problem (DRFD'):

$$\min_{\tilde{P},Q,q,q_0,\eta,\beta,M} \rho(\tilde{P}^{1/2})$$

s.t. $\gamma_2 \text{Tr}\{QS_q^0\} + q_0 \leq \varepsilon$

Constraints (21b) - (21e)

$$M = W^\top \tilde{P}W, \tilde{P} \succeq 0.$$ 

Because $\{\tilde{P},M\}$ have to be optimized in conjunction with $\eta$, the constraint (21c) now becomes a bilinear matrix inequality. By defining new variables $Q := Q/\eta, q := q/\eta, q_0 := q_0/\eta, \beta_j := \beta_j/\eta, \tilde{\beta}_j := \tilde{\beta}_j/\eta, \eta := 1/\eta$, bilinear terms can be eliminated, giving rise to the reformulation (25).

**Remark 10:** Because both metrics $\rho_1(P^{1/2})$ and $\rho_2(P^{1/2})$ are concave in $\tilde{P}$ [50], the design problem (25) is always a convex program amenable to off-the-shelf solvers.

**Remark 11:** As an approximation to (DRFD), the pessimism underlying Theorem 8 is closely related to the value of $\tau_0 > 0$. It can be readily deduced that by choosing $\tau_0$ based on (22), the resulting design matrix $\tilde{P}$ has detectability that is at least as good as previous $P^\ast$ in Theorems 6 and 7, while safely controlling FAR. This essentially shows the value of injecting boundedness information in enhancing the performance of the proposed DRFD scheme. In order for better performance, gridding of $\tau_0$ can be efficiently carried out around (22) and the best solution can be selected from all candidates. Given the optimal solution $\tilde{P}^\ast$ to (25), the projection matrix $P^\ast$ can be built from the Cholesky decomposition $P^\ast = LL^\top$.

**C. Safe Thresholding for Change Detection**

Serendipitously, the developed generalized Gauss bounds also provide a recipe for safe alarm thresholding in versatile change detection tasks across different thematic fields of systems and control, where anomaly detectors in quadratic forms $\text{Index}(\xi) = \xi^\top M \xi$ play a central role. Some notable instances are described below.

1) **Attack detection in CPS:** Very often, CPS is modeled as a discrete-time linear time-invariant system. Using a state estimator, one defines residuals as output errors, based on which a quadratic failure detector is commonly constructed [14], [51].

2) **Control performance monitoring (CPM):** To detect performance degradation, the celebrated Harris index in minimum variance control and quadratic cost in linear quadratic Gaussian control are widely adopted as CPM indices, see e.g., [52], [53].

3) **Multivariate statistical process monitoring:** Popular monitoring indices, such as the Hotelling’s $T^2$ and the squared prediction error can be constructed with multivariate analysis methods including principal component analysis and slow feature analysis [54], [55].

Given an index $\text{Index}(\xi)$, its realistic FAR is dependent on the alarm threshold $J_{th}$, which is generically calibrated under the Gaussian assumption on $P_0$. A common concern in abovementioned thematic fields is that the Gaussianity itself...
could be unjustified in engineering practice, and thus robustness against ambiguity and/or variations in probability distributions is desired. In this case, \( D_\alpha \) acts as an appealing choice to delineate unknown distribution of \( \xi \). As a consequence, generalized Gauss bounds in Theorems 2 and 4 yield practically useful alternatives to generic \( \chi^2 \)-quantiles, which help to calibrate \( J_{th} \) robustly. Here, we only discuss the case of \( \alpha \)-unimodal set \( D_\alpha(\gamma_1, \gamma_2, \Xi_b) \) with bounded support, by making use of Theorem 4. The proof is omitted since it requires no new ideas.

**Theorem 9:** Given a quadratic anomaly detector \( \text{Index}(\xi) = \xi^T M \xi \) where the distribution of \( \xi \) is included in the ambiguity set \( D = D_\alpha(\gamma_1, \gamma_2, \Xi_3) \) and \( \Xi_3 \) is given in (20). An alarm threshold \( J_{th} \) rendering FAR no higher than the tolerance \( \varepsilon \in (0, 1) \) can be obtained by solving the following SDP:

\[
\min_{Q,q,q_0,\eta,\beta,J_{th}} J_{th}
\text{ s.t. } \gamma_2 \text{Tr}(Q S^2_{b}) + q_0 \leq \varepsilon J_{th}
\eta \geq 0, \beta_j \geq 0, \tilde{\beta}_j \geq 0, j = 1, \ldots, N_e
\begin{bmatrix}
Q - \eta M & q & 0 \\
q^T & q_0 - J_{th} & 0 \\
0 & 0 & \eta
\end{bmatrix} + \sum_{j=1}^{N_e} \beta_j \Phi_j
\begin{bmatrix}
0 & 0 & 0 \\
0 & \alpha + 1 & \frac{\alpha}{\tau_0} \\
0 & \frac{\alpha}{\tau_0} & 0
\end{bmatrix} \succeq 0
\]

\[
\begin{bmatrix}
Q & q \\
q^T & q_0
\end{bmatrix} + \sum_{j=1}^{N_e} \beta_j \tilde{\Phi}_j \geq 0, \ Q \succeq 0
\]

where \( \tau_0 > 0 \) is a tuning parameter.

### IV. Case Studies on a Physical System

Next, we perform case studies using realistic data collected from an experimental three-tank system. The primary goal is to comprehensively investigate the detection performance of DRFD designs proposed in Theorems 6–8, under assumptions of unbounded/bounded uncertainty and different choices of detectability metric \( \rho(\cdot) \). A pictorial description of this apparatus is given in Fig. 2, where three tanks are interconnected via two pipelines. Water is fed into Tanks 1 and 2 by two pumps, whose flow-rates act as system inputs \( u(k) \in \mathbb{R}^2 \). System states \( x(k) = [x_1(k) \ x_2(k) \ x_3(k)]^T \) are the levels of three tanks, among which \( y(k) = [x_1(k) \ x_3(k)]^T \) constitute the outputs. By means of physical knowledge, ordinary differential equations can be established; see [56] for more details. Through model linearization around the operating point with a sampling interval \( \Delta t = 5 \) s, coefficient matrices in state-space equations are derived as follows:

\[
A = \begin{bmatrix}
0.8945 & 0.0048 & 0.1005 \\
0.0048 & 0.8500 & 0.8081 \\
0.1005 & 0.0801 & 0.8164
\end{bmatrix}, \quad B = \begin{bmatrix}
0.0317 & 0.0001 \\
0.0001 & 0.0309 \\
0.0018 & 0.0014
\end{bmatrix}
\]

Then, the parity space approach [34] is adopted to construct a deadbeat filter such that a residual signal \( v(k) \in \mathbb{R}^9 \) with \( s = 6 \) is generated, whose dynamics are governed by \( v(k) = W_v d_s(k) + V_v f_s(k) \).

Following Remark 1, we seek to obtain a “calibrated” residual \( r(k) = P v(k) \) with distributional robustness, by optimizing \( P \) while regarding the fault-free realization \( v_0(k) = W_v d_s(k) \) as uncertainty \( \xi \) that follows an unknown distribution. \( N = 200 \) “training” data samples of \( v_0(k) \) are collected in routine fault-free operations, based on which statistical information of \( \xi \) such as covariance and support can be estimated. In particular, the support \( \Xi_b \) is determined as a hyper-rectangular \( \Xi_b = \{ \xi | \xi_i \leq 1.2 \times \xi_i^{\text{max}}, \forall i = 1, \ldots, 9 \} \) that reliably covers all samples and is also representable as (20), where \( \xi_i^{\text{max}} \) is the element-wise maximum of \( |\xi_i| \) on empirical samples. Meanwhile, unimodality can be inspected from scatter plots of \( v_0(k) \); hence, we choose \( \alpha = n = 9 \) trivially, which corresponds to star-unimodality, to encode such minimal structural information. With these information, various ambiguity sets can be constructed, and for a particular metric \( \rho_i(\cdot) \), \( i = 1, 2 \), the following DRFD designs are developed.

1) **DRi-U:** Known design by solving (DRFD) under \( D(\gamma_1, \gamma_2, \mathbb{R}^n) \) [15, Ths. 2 and 3].

2) **DRi-U_c:** Proposed design based on suboptimal solutions to (DRFD) under \( D_\alpha(\gamma_1, \gamma_2, \mathbb{R}^n) \) in Theorems 6 and 7, which are strictly less conservative than DRi-U (see Remark 7).

3) **DRi-B:** Known design by solving (DRFD) under \( D(\gamma_1, \gamma_2, \Xi_b) \) with a bounded support [15, Th. 4].

4) **DRi-B_c:** Proposed design based on suboptimal solutions to (DRFD) under \( D_\alpha(\gamma_1, \gamma_2, \Xi_b) \) with a bounded support.
in Theorem 8. With $\tau_0$ calibrated by gridding, it is provably no more conservative than $\text{DR}_1-U_\alpha$ (see Remark 11).

For comprehensiveness of comparisons, the standard GLRT-based anomaly detection design [47], [48] is implemented, which imposes Gaussian assumption upon $\xi$. Meanwhile, we also investigate the performance of recently prevailing data-centric probabilistic designs that utilize empirical data distributions for uncertainty description.

1) $\text{DR}_i-W$: The Wasserstein distance-based DRFD design [9], [31]. The ambiguity set in (DRFD) is characterized by the Wasserstein distance to the empirical data distribution, i.e.,

$$
\mathcal{D} = \{ \mathbb{P}(d\xi) | \Delta(\mathbb{P}, \hat{\mathbb{P}}_N) \leq \theta \}
$$

where $\Delta(\cdot, \cdot)$ is the Wasserstein distance between two distributions [57], $\mathbb{P}_N$ is the empirical distribution defined by $N$ samples $\{\xi[k], k = 1, \ldots, N\}$, $\theta > 0$ is the radius of Wasserstein ball. The robustness level critically relies on the choice of $\theta$, which is generically determined via cross-validation [31].

2) $\text{DR}_i-S$: The scenario program (SP)-based fault detection design [3], [32], which solves the following sample-based problem as a surrogate:

$$
\max_P \rho_i(P) \quad \text{s.t.} \quad \|PW_{\xi[k]}\|^2 \leq 1, \ k = 1, \ldots, N.
$$

Given sufficient samples, the induced solution $P^*_N$, which is inherently random, enables satisfaction of the chance constraint $\mathbb{P}_\xi(\|PW_{\xi}\|^2 > 1) \leq \varepsilon$ with high probability [58]. Because such a relation holds in a distribution-free manner, it also manifests distributional robustness.

All induced convex programs are solved using YALMIP interface in MATLAB equipped with the mosek solver. We first elucidate the effect of introducing $\alpha$-unimodality in reducing conservatism of robust design. The problem (DRFD) is resolved under various choices of ambiguity sets and protection levels $\varepsilon$. The resulting optimal values of (DRFD) are displayed in Fig. 3. It can be seen that by introducing unimodality information, higher values of detectability metrics are attained, indicating a smaller feasible region and thus a reduction of pessimism. Under a relatively large $\varepsilon$, the effect of assuming unimodality tends to outweigh the usage of bounded support information. When $\varepsilon$ is small, $\text{DR}_i-B$ outdoes $\text{DR}_i-U$ because $\text{DR}_i-B$ tends to classic set-membership robust design, while the worst-case distribution in $\text{DR}_i-U$ has masses outside the support and is, thus, unrealistic. By synthesizing all information, the best detectability can always be attained by $\text{DR}_i-B_\alpha$.

Next, we implement different fault detection methods under a prescribed tolerance $\varepsilon = 0.1$ and verify their detection performance using another test dataset collected from the apparatus. It consists of both normal and faulty samples, where the first 200 normal samples are collected under healthy conditions, based on which residuals $r(k)$ can be generated and FAR can be evaluated

$$
\text{FAR} = \frac{\text{Number of alarms declared}}{\text{Number of normal samples}} \times 100%.
$$

Meanwhile, a fault of leakage in Tank 3 is introduced from the 201st sample till the end, based on which FDR can be evaluated as

$$
\text{FDR} = \frac{\text{Number of alarms declared}}{\text{Number of faulty samples}} \times 100%.
$$

The detection results of all relevant methods are depicted in Fig. 4, and their performance indices are summarized in Table I. The classic GLRT design scheme yields an FAR of 45%, which results in massive nuisance alarms. This highlights the fragility of assuming Gaussianity and the necessity of introducing distributional robustness into fault detection design. Next, we focus on the proposed unimodality-based DRFD designs ($\text{DR}_i-U_\alpha$ and $\text{DR}_i-B_\alpha$) and their counterparts disregarding unimodality ($\text{DR}_i-U$ and $\text{DR}_i-B$). For $\rho_1(\cdot)$, all the four detectors successfully keep FARs lower than the tolerance, showcasing the guaranteed robustness. On the other hand, the detectability of both $\text{DR}_1-U$ and $\text{DR}_1-B$ is improved by their $\alpha$-unimodal counterparts proposed in this article, where a desirable tradeoff is achieved by $\text{DR}_1-B_\alpha$ with an FDR of 87%. In general, the DRFD designs induced by $\rho_2(\cdot)$ appear to be more conservative. In $\text{DR}_2-U$, $\text{DR}_2-U_\alpha$, and $\text{DR}_2-B$, the fault remains largely undetected, while the exploitation of support and unimodality information in $\text{DR}_2-B_\alpha$ helps better showcase the fault and reduce the pessimism. These comparisons demonstrate that, the proposed DRFD design enables usage of moment, bounded support, and unimodality information simultaneously, which has

\[2\] In our study, we use the same 200 data samples to define $\hat{\mathbb{P}}_N$.  

---

Fig. 3. Optimal values of problem (DRFD) under different ambiguity sets and detectability metrics $\rho(\cdot)$.  

\[\text{FAR} = \frac{\text{Number of alarms declared}}{\text{Number of normal samples}} \times 100%.
\]

\[\text{FDR} = \frac{\text{Number of alarms declared}}{\text{Number of faulty samples}} \times 100%.
\]
TABLE I
FAULT DETECTION PERFORMANCE OF DIFFERENT METHODS (%)

|        | GLRT | DR1-U | DR1-Uα | DR1-B | DR1-Bα | DR1-W | DR1-S | DR2-U | DR2-Uα | DR2-B | DR2-Bα | DR2-W | DR2-S |
|--------|------|-------|--------|-------|--------|-------|-------|-------|--------|-------|--------|-------|-------|
| FAR    | 45.00| 3.50  | 6.00   | 6.00  | 7.50   | 27.50 | 53.00 | 0.00  | 0.00   | 0.00  | 25.50  | 49.00 |
| FDR    | 98.99| 24.50 | 46.00  | 76.50 | 87.00  | 98.49 | 98.49 | 1.50  | 4.50   | 21.50 | 46.00  | 98.49 | 99.50 |

Fig. 4. Detection of leakage in Tank 3 using different algorithms. Red dashed lines indicate detection thresholds.

Fig. 5. Cross-validation results of DR1-W under different Wasserstein radii.

Reduced conservatism in controlling FAR and better detectability than existing methods [15] that makes no use of unimodality.

Finally, we make a contrast among the proposed method, the Wasserstein distance-based design (DR1-W) [9], [31] as well as the scenario-based probabilistic design (DR1-S) [3], [32] by using 200 normal training samples. For DR1-W, the radius parameter is determined as $\theta = 0.0007$ according to five-fold cross-validation. In Fig. 5, FARs under different radii in a cross-validation setting are reported, where a significant decline can be observed on $[10^{-4}, 10^{-3}]$. Due to the lack of explicit interpretation, however, one has to perform cross-validation across a broad range of $\theta$. This is much involved especially when a meticulous calibration of $\theta$ is required. What is more, the out-of-sample performance of DR1-W is unsatisfactory as realizations of FAR are beyond the tolerance, as shown in Table I and Fig. 4. This could be primarily due to the curse of dimensionality of the Wasserstein distance [59], i.e., an inadequate sample size in high-dimensional setup, as well as potential complex time-varying characteristics of realistic data.

As for the scenario approach [3], [32], the over conservatism is even more severe. This is because there is no trade-off parameter for regulating pessimism in (SP), and sufficient samples are required to ensure a high confidence of constraint satisfaction, well-known as the following probabilistic guarantee [58, Proposition 1]:

$$P\{P_\xi\{\|P_N W_\xi\|^2 > 1\} \leq \varepsilon\} \geq 1 - \beta,$$

where the outer randomness arises from sampling $N$ points and $\beta$ is the confidence level. $n_p$ denotes the number of free decision variables in (SP). For example, given $\varepsilon = 0.10$ and $\beta = 10^{-5}$ ("practical certainty"), the minimal number of independent samples needed is $N \geq \frac{2}{\varepsilon}(\log \frac{1}{\beta} + n_p) \approx \frac{2}{0.10}(\log \frac{1}{10^{-5}} + \frac{n(n+1)}{2}) \approx 1131$, which turns out to be demanding. By contrast, the proposed DRFD design still exhibits guaranteed robustness under a moderate sample size, thanks to a judicious usage of ambiguous information such as moment, support, and unimodality.

V. CONCLUDING REMARKS

In this article, a new distributionally robust design scheme is developed to maximize fault detectability and regulate false alarm ratio (FAR) reduction, which is of primary importance in practical applications. The key idea was to exploit ambiguous information such as moment, support, and unimodality to design fault detection systems that are distributionally robust to data uncertainty. The proposed method, which we refer to as distributionally robust fault detection (DRFD), was shown to exhibit significant advantages over existing methods, particularly in terms of fault detectability and reduced conservatism in controlling FAR. The effectiveness of the proposed method was demonstrated through numerical simulations and real data experiments, showcasing its potential for practical implementation in a variety of applications.

---

3The number $n_p$ of free variables in $P$ is $n(n+1)/2$. 

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alarms without requiring precise distribution knowledge. By further integrating $\alpha$-unimodality information, the conservatism of previous moment-based DRFD schemes can be reliably alleviated. To tackle the constraint on WCFAR, we first establish a new multivariate $\alpha$-unimodal Gauss bound on the tail probability outside an ellipsoid, which strictly improves upon its Chebyshev counterpart with the improvement factor explicitly given. Based on this, suboptimal solutions to DRFD problems are derived in closed form. Then, we develop a tightened $\alpha$-unimodal Gauss bound by further injecting support information, which enables us to tackle the DRFD design problem approximately by solving a convex program and further alleviate the design conservatism. The developed multivariate Gauss bounds are also applicable to general change detection tasks across different areas in systems and control. The efficacy of the new fault detection design scheme in reducing conservatism is illustrated using data collected from an experimental three-tank apparatus. It is also shown that compared to recent data-centric design such as Wasserstein distance-based methods and scenario approach, our method manifests a more stable performance with moderate sample sizes. A future direction is to investigate the usage of recent data-driven dynamic ambiguity sets [13] in DRFD design.

**APPENDIX**

**Proof of Theorem 2:** We first derive a convex approximation of (17). Define $h(\tau) = -\tau^{-\alpha}$, which is a concave in $\tau > 0$. Then, given $\tau_0 := \|M^\frac{1}{2}\xi_0\|$ and $\tau := \|M^\frac{1}{2}\xi\| > 0$, $h(\tau)$ is majorized by its first-order Taylor approximation

$$h(\tau) \leq h(\tau_0) + (\tau - \tau_0) h'(\tau_0) = -\tau_0^{-\alpha} + \alpha(\tau - \tau_0)$$

which amounts to

$$-\|M^\frac{1}{2}\xi\|^{-\alpha} \leq -(\alpha + 1)\tau_0^{-\alpha} + \alpha\tau^{-\alpha-1}\|M^\frac{1}{2}\xi\|.$$ 

One then obtains an inner approximation to (17)

$$\min_{Q,q,q_0} \gamma_2 \text{Tr} \{QS_0^\alpha\} + q_0$$

(26a) with

$$\begin{cases}
Q \succeq 0, \quad q'q_0 \geq 0
\end{cases}$$

(26b)

where

$$g(\xi, t) \triangleq \xi'Q\xi + 2\xi' q + q_0 - 1 + (\alpha + 1)\tau_0^{-\alpha} - \alpha\tau^{-\alpha-1} t + \eta t^2$$

$$-\eta\xi'M\xi.$$ 

Note that (27) can be rewritten as an LMI. Consequently, problem (26) equals to the following SDP:

$$\min_{Q,q,q_0,\eta} \gamma_2 \text{Tr} \{QS_0^\alpha\} + q_0$$

(28a) with

$$\begin{cases}
Q \succeq 0, \quad \eta \geq 0
\end{cases}$$

(28b)

$$\begin{pmatrix}
Q - \eta M & q \\
q' & q_0 - 1
\end{pmatrix} \succeq 0,$$

$$+ \frac{1}{\tau_0^2}
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \alpha + 1 & \frac{\alpha}{\tau_0} & 0 \\
0 & \frac{\alpha}{\tau_0} & 0 & 0
\end{pmatrix} \succeq 0$$

(28c)

where (27) is recast as (28c). Since $\tau_0 \geq 1$, it holds that $\alpha\tau_0^{-\alpha-1}/2 > 0$, and with the aim of securing the positive semidefiniteness of (28c), it must be that $\eta > 0$ and $q_0 + (\alpha + 1)\tau_0^{-\alpha} - 1 > 0$. As a result, according to the Schur’s complement, the LMI (28c) can be recast as

$$Q - \eta M \succeq 0$$

(29a)

$$Q - \eta M \geq \frac{q'q_0}{q_0 + (\alpha + 1)\tau_0^{-\alpha} - 1}$$

(29b)

$$\eta [q_0 + (\alpha + 1)\tau_0^{-\alpha} - 1] \geq \frac{\alpha^2}{4} \tau_0^{-2\alpha-2}$$

(29c)

$$\eta > 0, \quad q_0 + (\alpha + 1)\tau_0^{-\alpha} - 1 > 0$$

(29d)

where (29a) is implied by (29b) and, thus, can be omitted. Next we distinguish between three cases.

Case 1: $(\alpha + 1)\tau_0^{-\alpha} \leq 1$. In this case, $q_0 > 1 - (\alpha + 1)\tau_0^{-\alpha} \geq 0$. It follows from the Schur’s complement that the LMI (28b) reduces to $Q \succeq q'q_0$. Thus, problem (28) becomes

$$\min_{Q,q,q_0,\eta} \gamma_2 \text{Tr} \{QS_0^\alpha\} + q_0$$

(27) with

$$\begin{cases}
Q \succeq q'q_0, \quad Q - \eta M \geq \frac{q'q_0}{q_0 + (\alpha + 1)\tau_0^{-\alpha} - 1} \\
\eta [q_0 + (\alpha + 1)\tau_0^{-\alpha} - 1] \geq \frac{\alpha^2}{4} \tau_0^{-2\alpha-2} \\
q_0 + (\alpha + 1)\tau_0^{-\alpha} - 1 > 0
\end{cases}$$

$$q_0 + (\alpha + 1)\tau_0^{-\alpha} - 1 > 0.$$
which amounts to
\[
\begin{align*}
\min_{q_0, q_r} & \quad r + q_0 \\
\text{s.t.} & \quad r \geq \frac{\gamma_2 q_r^T S_0 q_r}{q_0} + \frac{\gamma_2 q_r^T S_0 q_r}{q_0} + (\alpha + 1) \tau_0^{-\alpha} - 1 + \tau_0^{-2\alpha-2} \\
& \quad \eta [q_0 + (\alpha + 1) \tau_0^{-\alpha} - 1] \geq \frac{\alpha^2}{4} \tau_0^{-2\alpha-2} \\
& \quad g_0 + (\alpha + 1) \tau_0^{-\alpha} - 1 > 0 \\
\end{align*}
\]
Clearly, it holds that \(q^* = 0, r^* = \gamma_2 \eta \text{Tr}\{MS_0^\alpha\}\), which yields
\[
\begin{align*}
\min_{q_0, q_r} & \quad \gamma_2 \eta \text{Tr}\{MS_0^\alpha\} + q_0 \\
\text{s.t.} & \quad \eta [q_0 + (\alpha + 1) \tau_0^{-\alpha} - 1] \geq \frac{\alpha^2}{4} \tau_0^{-2\alpha-2} \\
& \quad q_0 + (\alpha + 1) \tau_0^{-\alpha} - 1 > 0 \\
& \quad g_0 + (\alpha + 1) \tau_0^{-\alpha} - 1 > 0 \\
\end{align*}
\]
\[
= 4[q_0 + (\alpha + 1) \tau_0^{-\alpha} - 1] + q_0 \\
\text{s.t.} & \quad q_0 + (\alpha + 1) \tau_0^{-\alpha} - 1 > 0 \\
& \quad g_0 + (\alpha + 1) \tau_0^{-\alpha} - 1 > 0 \\
= \alpha\tau_0^{-\alpha-1} \sqrt{\gamma_2 \text{Tr}\{MS_0^\alpha\} - (\alpha + 1) \tau_0^{-\alpha} + 1} \\
& \quad \eta \text{Tr}\{MS_0^\alpha\} + q_0 \\
\end{align*}
\]

where
\[
g_0^* = \frac{\alpha}{2} \tau_0^{-\alpha-1} \sqrt{\gamma_2 \text{Tr}\{MS_0^\alpha\}} + 1 - (\alpha + 1) \tau_0^{-\alpha} > 1 - (\alpha + 1) \tau_0^{-\alpha}
\]
is attainable. Case 2: \(1 < (\alpha + 1) \tau_0^{-\alpha} \leq 1 + \frac{\alpha}{\sqrt{2}} \tau_0^{-\alpha-1} \sqrt{\gamma_2 \text{Tr}\{QS_0\}^\alpha} \). If \(q_0 = 0\), (28b) boils down to \(Q \succeq 0\) with \(q^* = 0\), and consequently the optimal value of (28) is obtained as
\[
f_2(\tau_0) \triangleq \frac{\alpha^2 \tau_0^{-2\alpha-2} \gamma_2 \text{Tr}\{MS_0^\alpha\}}{4[(\alpha + 1) \tau_0^{-\alpha} - 1]}.
\]
If \(q_0 > 0\) otherwise, proceeding analogously to Case 1, one obtains the optimal value as \(f_1(\tau_0)\) in (30). Because the value of (31) is lower than that of (30), one concludes that the optimal value is \(f_1(\tau_0)\) and thus Case 2 can be combined with Case 1 as a single case. Case 3: \((\alpha + 1) \tau_0^{-\alpha} > 1 + \frac{\alpha}{\sqrt{2}} \tau_0^{-\alpha-1} \sqrt{\gamma_2 \text{Tr}\{QS_0\}^\alpha}\). It is an easy exercise to verify that \(g_0^* = 0\), and thus, the optimal value is \(f_2(\tau_0)\). Summarizing abovementioned cases yields the optimal value of the relaxed problem (28)
\[
\sup_{\mathcal{P}_E} \mathbb{P}_E \{\xi^T M \xi > 1\}
\]
\[
\leq \begin{cases} f_1(\tau_0), & \text{if } (\alpha + 1) \tau_0^{-\alpha} \leq 1 + \frac{\alpha}{\sqrt{2}} \tau_0^{-\alpha-1} \sqrt{\gamma_2 \text{Tr}\{MS_0^\alpha\}} \ \\
 f_2(\tau_0), & \text{otherwise.}
\end{cases}
\]
Next, we seek the best approximation among all choices of \(\tau_0 > 0\). For convenience, we define \(z = 1/\tau_0 \in (0, 1]\), \(g_1(z) = f_1(\tau_0), g_2(z) = f_2(\tau_0)\), and \(g(z) = 1 + \frac{\alpha}{\sqrt{2}} \tau_0^{-\alpha-1} \sqrt{\gamma_2 \text{Tr}\{MS_0^\alpha\}} z^{\alpha+1} - (\alpha + 1) z^\alpha\). In this way, (32) becomes
\[
\sup_{\mathcal{P}_E} \mathbb{P}_E \{\xi^T M \xi > 1\} \leq h(z) \triangleq \begin{cases} g_1(z), & \text{if } g(z) \geq 0 \ \\
 g_2(z), & \text{if } g(z) < 0
\end{cases}
\]
where \(h(z)\) is continuous. Thus, it suffices to resolve \(\min_{z \in (0,1]} h(z)\). By setting \(g_1(z) = 0\) \((i = 1, 2)\), one obtains a unique stationary point of \(g_1(z)\) on \(\mathbb{R}_+^+\)
\[
\frac{1}{\sqrt{\gamma_2 \text{Tr}\{MS_0^\alpha\}}}, \quad z_*^1 = \sqrt{c_0}.
\]
Note that \(0 < z_*^1 < 1\) always holds. Meanwhile, \(g_i(z)\) is first decreasing on \((0, z_*^i)\) and then increasing. As for \(g(z)\), there is also a unique stationary point \(z^* = 2/\sqrt{\gamma_2 \text{Tr}\{MS_0^\alpha\}} > 0\). Next, the following cases are distinguished.

Case 1: \(g(z^*) \geq 0\), which equals to \(\gamma_2 \text{Tr}\{MS_0^\alpha\} \geq 4\). In this case, \(g(z) \geq 0\) always holds for \(z \in (0, 1]\), and the minimizer \(z_*^1 \leq 1/2\) of \(g_1(z)\) is always attainable. Thus
\[
\min_{z \in (0,1]} h(z) = g_1(z_*^1) = 1 - \frac{1}{(\gamma_2 \text{Tr}\{MS_0^\alpha\})^{\alpha/2}}.
\]
Case 2: \(g(z^*) < 0\) and \(g(z_*^2) \geq 0\), which amounts to
\[
\frac{1}{c_0} \leq \gamma_2 \text{Tr}\{MS_0^\alpha\} < 4
\]
further indicating \(0 < z_*^1 \leq z_*^2 < 1\). Moreover, it can be verified that \(g(1) < 0\). Thus, \(g(z)\) has a unique root \(\hat{z} \in (z_*^2, 1)\), and one obtains
\[
h(z) = \begin{cases} g_1(z), & \text{if } z \in (0, \hat{z}] \ \\
 g_2(z), & \text{if } z \in (\hat{z}, 1]
\end{cases}
\]
Note that \(\min_{z \in (0,1]} g_1(z) = g_1(z_*^1)\), which is \(g_1(z) \geq g_1(z_*^1)\). It immediately follows that \(\min_{z \in (0,1]} h(z) = g_1(z_*^1)\).

Case 3: \(g(z^*) < 0\) and \(g(z_*^2) < 0\). In this case, it holds that
\[
\gamma_2 \text{Tr}\{MS_0^\alpha\} < \frac{1}{c_0}
\]
which implies \(g(z_*^1) < 0\) and \(z_*^1 < z_*^2\). Because there always exists \(\hat{z} \in (0, z_*^2)\) such that \(g(\hat{z}) = 0\), \(h(z)\) can be expressed as (33), and we have \(\min_{z \in (0,1]} g_1(z) = g_2(\hat{z}) \geq g_2(z_*^2)\) and \(\min_{z \in (0,1]} g_2(z) = g_2(z_*^2)\). This gives rise to
\[
\min_{z \in (0,1]} h(z) = g_2(z_*^2) = c_0 \gamma_2 \text{Tr}\{MS_0^\alpha\}.
\]
Summarizing abovementioned cases yields (18).

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