On holomorphic sector of higher-spin theory

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Abstract
Recent investigation of the locality problem for higher-spin fields led to a vertex reconstruction procedure that involved elements of contraction of the original Vasiliev interaction algebra. Inspired by these results we propose the Vasiliev-like generating equations for the holomorphic higher-spin interactions in four dimensions based on the observed contracted algebra. We specify the functional class that admits evolution on the proposed equations and brings in a systematic procedure of extracting all-order holomorphic vertices. A simple consequence of the proposed equations is the space-time locality of the gauge field sector. We also show that vertices come with a remarkable shift symmetry.
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1 Introduction

1.1 Higher-spin locality problem

A notoriously difficult open problem is the structure of (non)-local higher-spin (HS) gauge interactions \([1]-[4]\). The question is whether the theory is local and if not what a missing locality should be replaced with? A certain kind of non-locality is expected due to a HS symmetry that mixes fields of arbitrary large spins via higher space-time derivatives \([5], [6], [7]\). This results in that while a free theory carries no more than two derivatives, the cubic interaction \(V_{s_1,s_2,s_3}\) involves higher order ones. Their number grows with spin being bounded from above by \(\#\partial = s_1 + s_2 + s_3\) \([8]\). Since the HS spectrum is unbounded in \(d > 3\) this implies the theory is non-local beyond free level. A cubic example illustrates however that this type of non-locality is under full control once interaction is restricted to fixed three spins. The corresponding vertex contains finitely many derivatives and therefore is local. Such vertices can be treated using standard field theory tools. In particular, they can be recovered from a free CFT \([9]\) by inverting the Witten diagrams of HS/O\((N)\) duality \([10], [11], [12], [13]\) testing the Klebanov-Polyakov conjecture \([10]\) at this order along the way.

A natural question is whether a quartic HS vertex \(V_{s_1,s_2,s_3,s_4}\) is local or not. The holographic reconstruction of scalar self-interaction \(0−0−0−0\) from the \(O(N)\) four-point correlation function carried out in \([14]\) has sowed doubts \([15]\) on the locality at this order as the final result appeared to involve infinitely many space-time derivatives even for fixed spins \(s_i = 1\). Although the non-local result of \([14]\) can not be treated as a solid prove of irremovable non-locality, it raises a big concern on the existence of a HS theory in the form of a local field theory. Apart from the assumption of the holographic duality at the \(4pt\) – level itself, a weak spot of \([15]\) from the CFT side is an infinite series of single trace conformal blocks which singularity is not fully understood (see \([16]\) for analysis of this issue). On the \(AdS\) side the non-commutativity of the \(AdS\) derivatives \([D, D] \sim R^{-2}\) may affect the reconstruction procedure as well. It should be also stressed that the inherent field redefinition ambiguity that may change a local form of the vertex into a non-local one is very hard to take into account within the holographic reconstruction. This is why it is important to investigate the locality issue using the \(AdS/CFT\) independent tools.

1.2 Unfolded approach

The language of differential forms that take values in the HS algebra is at the core of the unfolding formalism for higher spins \([17]\). Its classical differential equations of motion \([17]\) are of first order, while gauge invariance is inbuilt through a formal consistency. The price one pays for an apparent simplicity is a necessity of an infinite number of auxiliary fields. For the purely symmetric bosonic fields in \(d\) dimensions the spectrum of these fields is governed by the Eastwood-Vasiliev HS algebra \([19], [20]\). These are the two-row traceless \(o(d−1, 2)\) Young diagrams. However, despite the full nonlinear equations of motion for these gauge fields are known in a closed form \([20]\), their analysis is not yet well elaborated beyond free level.

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1 The holographic reconstruction of \([14]\) should be considered more like a signal of a potential problem rather than a derivation of the vertex. Indeed, the appearance of infinitely many derivatives affects the Lagrangian analysis and calls for a precise definition of its functional class. For example, by allowing \(\Box\) in the Lagrangian one no longer can set apart bulk and boundary integral contributions.

2 See \([18]\) for an application of the unfolded dynamics at quantum level.
While the HS locality problem can be addressed in any space-time dimension, a particular instance of $d = 4$ has a great advantage as compared to general $d$. It is in this case that the HS algebra admits a very simple realization in terms of the two-component spinors [1] reducing it to the enveloping of $Y_A = (y_\alpha, \bar{y}_\dot{\alpha})$, $\alpha, \dot{\alpha} = 1, 2$ modded by
\[
[y_\alpha, y_\beta]_* = 2i\epsilon_{\alpha\beta}, \quad [y_\alpha, \bar{y}_\dot{\beta}]_* = 0, \quad [\bar{y}_\dot{\alpha}, \bar{y}_\dot{\beta}]_* = 2i\epsilon_{\dot{\alpha}\dot{\beta}},
\] (1.1)

where star product $\star$ can be chosen to be the Moyal one
\[
f \star g = f(y, \bar{y})e^{i\epsilon_{\alpha\beta} \partial_\alpha \bar{\partial}_\beta} g(y, \bar{y})
\] (1.2)
and $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ is the $\text{sp}(2)$ invariant form (same for $\epsilon_{\dot{\alpha}\dot{\beta}}$). Note that the star product is given by a tensor product of two pieces – the holomorphic that acts on $y$ and the anti-holomorphic one acting on $\bar{y}$
\[
f \star g = f(y, \bar{y})e^{i\epsilon_{\alpha\beta} \partial_\alpha \bar{\partial}_\beta} g(y, \bar{y}).
\] (1.3)

The $sl(2, \mathbb{C})$ dictionary implies that the two-row $o(3, 2) = sp(4, \mathbb{R})$ Young diagrams are mere symmetric multispinors in dotted and undotted indices and therefore can be packed into generating functions as formal polynomials of $y$ and $\bar{y}$. This way one introduces a space-time one-form $\omega(y, \bar{y})$ and a zero-form $C(y, \bar{y})$. The unfolded HS equations are
\[
d^1 x \omega + \omega \star \omega = \Upsilon(\omega, \omega, C) + \Upsilon(\omega, \omega, C, C) + \ldots
\] (1.4)
\[
d^2 x C + [\omega, C]_* = \Upsilon(\omega, C, C) + \Upsilon(\omega, C, C, C) + \ldots
\] (1.5)

Let us specify some important properties of these equations

- The space-time derivative appears in the form of de-Rahm differential in (1.4)-(1.5). Particularly, no manifest derivatives are there in $\Upsilon$’s. It does not of course imply that the space-time vertices contain no derivatives, rather it says that the first derivative of certain fields is expressed in terms of other fields on-shell. This is a manifestation of the fact that $\omega(y, \bar{y})$ and $C(y, \bar{y})$ along with the physical fields contain plenty of the auxiliary ones that get expressed via the former on (1.4)-(1.5). Therefore, any vertex $\Upsilon$ accumulates space-time derivatives when expressed in terms of physical fields.

- Vertices $\Upsilon$ can be found order by order in $C$ by inspecting the integrability requirement $d^2 x = 0$. The resulting relations stem from the HS algebra action present on the l.h.s of (1.4)-(1.5). This procedure is naturally defined up to a field redefinition $\omega \rightarrow \omega + f_\omega(\omega, C, \ldots, C)$ and $C \rightarrow C + f_C(C, \ldots, C)$. This ambiguity is at the core of the locality problem. A systematic way of extracting $\Upsilon$’s is given by the all order Vasiliev equations [21]. Their remarkable feature is that being field redefinition independent they allow for an all order analysis of a functional class corresponding to a one or another field frame. This makes the Vasiliev approach very suitable for the locality analysis.

- The vacuum equations correspond to $\Upsilon = 0$ which are satisfied by a nontrivial $AdS_4$ background connection $\omega_0$ or any other HS flat connection. The linearized equations correspond to the vanishing of all $\Upsilon(\omega, C, \ldots, C) = 0$ and a nontrivial $\Upsilon(\omega, \omega, C)$. We note

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3The commutator on the l.h.s of (1.5) is in the twisted-adjoint representation of HS algebra rather than in the adjoint as (1.5) suggests. This is usually achieved by introducing outer Klein operators $k$ and $\bar{k}$ within the field dependence [1] that implement the twisted automorphism. We omit those for brevity.
that the linearized vertex contains some information on the cubic interactions (quartic in Lagrangian counting). This happens because the HS algebra action on the l.h.s of (1.4)-(1.5) already stores some details of quadratic interaction. This makes the unfolded deformation procedure crucially different from the standard Noether one which can be trivialized by non-local field redefinitions. In the unfolded case there are no field redefinitions that trivialize cubic coupling on a general HS vacuum because this coupling partly comes with the free one.

• Vertices of (1.4)-(1.5) depend on a free constant complex parameter \( \eta \), which unless \( \eta = 1 \) or \( \eta = i \) breaks parity of HS interactions. From the boundary side \( \eta \) arguably interpolates between free (critical) boson and free (critical) fermion via the 3d Chern-Simons interaction manifesting the so-called 3d bosonization [13], [22]. \( \Upsilon \)'s depend on \( \eta \) in the following fashion

\[
\Upsilon(\omega, \omega, C, \ldots, C) = \sum_{k=0}^{N} \eta^k \bar{\eta}^{N-k} \Upsilon_k(\omega, \omega, C, \ldots, C),
\]

(1.6)

\[
\Upsilon(\omega, C, \ldots, C) = \sum_{k=0}^{N-1} \eta^k \bar{\eta}^{N-k-1} \Upsilon_k(\omega, C, \ldots, C),
\]

(1.7)

where \( \Upsilon_k \) are \( \eta \)-independent.

• Since \( \eta \) and \( \bar{\eta} \) appear as formally independent variables in any of consistency relations for \( (1.4)-(1.5) \) one can set, say, \( \bar{\eta} = 0 \). The resulting system remains consistent. This way one reduces HS equations down to the holomorphic sector, originally called the self-dual [21] and sometimes chiral [23]. As it follows from the early analysis by Metsaev [24], [25] based on the light-cone approach (see e.g., [26] for a recent account) this sector is fixed by a cubic approximation receiving no higher order corrections on the Minkowski background. One does not expect a similar behavior within the covariant approach on the AdS or general HS background though.

### 1.3 Unfolded view of (non)-locality

As it was stressed, the unfolded approach somehow obscures the derivative structure of interaction in terms of the auxiliary fields. To get at the derivative map let us have a closer look at the generating fields \( \omega \) and \( C \) from (1.4)-(1.5). For a detailed analysis we refer the reader to [27], [28]. A spin \( s \) field is singled out by the following conditions

\[
\left( y^a \frac{\partial}{\partial y^a} + \bar{y}^{\dot{a}} \frac{\partial}{\partial \bar{y}^{\dot{a}}} \right) \omega = 2(s - 1) \omega \, , \tag{1.8}
\]

\[
\left( y^a \frac{\partial}{\partial y^a} - \bar{y}^{\dot{a}} \frac{\partial}{\partial \bar{y}^{\dot{a}}} \right) C = \pm 2s \, C \, , \tag{1.9}
\]

where \( \omega \) contains gauge degrees of freedom, while \( C \) accounts for the gauge invariant combinations (spin \( s \) analogs of Maxwell’s tensor). Note, that as follows from (1.8), fields \( \omega_{\alpha_1 \ldots \alpha_m \beta_1 \ldots \beta_n} \)

\footnote{Such a reduction ruins the reality condition implying that the (anti)holomorphic sector is essentially complex.}
describe spin $s$ iff $m + n = 2(s - 1)$, which says that both $m$ and $n$ are bounded by the value of spin. This gives one a spin increasing but a finite set of fields for a given gauge field. In particular, the Fronsdal spin $s$ component is stored in $\omega_{\alpha_1...\alpha(s-1),\dot{\beta}_1...\dot{\beta}(s-1)}$, while the rest fields are auxiliary. The approximate derivative map that relates these to the Fronsdal ones at free level is as follows

$$
\omega[m],[n] \sim \left( \frac{\partial}{\partial x} \right)^{|m-s+1|} \omega[|s-1],[s-1]|, \quad (1.10)
$$

where by $[m]$ we denote a group of $m$ indices and the spinor version of $x_a \sim x_a\alpha\dot{\alpha}$. Things are different with $C$ as from (1.9) it follows that a spin $s$ component $C_{\alpha_1...\alpha_m,\dot{\beta}_1...\dot{\beta}_n}$ satisfies $|m-n| = 2s$ and therefore there are infinitely many components for a given spin. The physical one is purely (anti)-holomorphic $C_{\alpha_1...\alpha_{2s}}$ and $C_{\dot{\beta}_1...\dot{\beta}_{2s}}$. Again, one can estimate now how the auxiliary components are expressed via the physical ones

$$
C[m],[n] \sim \left( \frac{\partial}{\partial x} \right)^{\min(m,n)} C[2s]. \quad (1.11)
$$

1.3.1 Spin locality and ultra-locality

It is clear now where a potential non-local obstruction may come from. Whenever one encounters infinitely many pairs of contracted dotted and undotted indices in $C$'s (mind to keep spins fixed by (1.9)) the corresponding contribution is non-local by argument (1.11). An example of such a non-locality is easy to devise

$$
\sum_n a_n C_{\alpha_1...\alpha_{2s_1}\gamma_1...\gamma_n,\dot{\gamma}_1...\dot{\gamma}_n} C_{\dot{\beta}_1...\dot{\beta}_{2s_2}\gamma_1...\gamma_n,\dot{\gamma}_1...\dot{\gamma}_n}, \quad (1.12)
$$

provided coefficients $a_n$ above have infinitely many nonzero values. Here we have a non-local interactions of spin $s_1$ and spin $s_2$.

One can define the notion of spin locality [29], [27] for the vertices from (1.4)-(1.5). We call $\Upsilon$ a spin local if being restricted to fixed spins it contains no more than a finite amount of contractions between different $C$'s. By a single contraction we assume a contraction of a one pair of dotted and undotted indices, e.g. $f_{\alpha\beta}\gamma^{\alpha\beta}$.

Few comments are now in order. While one may expect that the notion of spin locality is equivalent to the space-time locality, this may not be necessarily the case. As is carefully analyzed in [27] the two notions are equivalent if the number of physical fields is finite. The equivalence was not shown otherwise (see however [28] where this problem has been recently detailed). Another comment is that there is no need to bother of contractions between two $\omega$'s or between $\omega$ and $C$ since once restricted to a given spin, $\omega_s$ becomes polynomial thanks to (1.8) and therefore such contractions are always spin local.

A very important concept is the spin ultra-locality [30] which is known to persist at lower orders in $\Upsilon(\omega,\omega,\bullet)$ from (1.4). $\Upsilon(\omega^{t_1},\omega^{t_2},\bullet)$ is called a spin ultra-local if the number of different index contractions is bounded for all spins in its $C$ part (for fixed spins $t_{1,2}$). In other words, the vertex dependence on field $C^{s_1}$ must be organized via its physical (primary) component or the auxiliary (descendant) ones which depth can not grow with spin $s_i$. In practice this implies that the structure of such a vertex in its $C$ dependence is roughly the following

$$
\Upsilon(\omega^{t_1},\omega^{t_2},C^{s_1},\ldots,C^{s_n}) \sim \Phi(y, \partial y_i, \bar{y}, \partial \bar{y}) C^{s_1}(y_1, \bar{y}) \ldots C^{s_n}(y_n, \bar{y}) \bigg|_{y_i=0}, \quad (1.13)
$$
where $\Phi$ depends polynomially on $y$ and $\partial_y$, such that its degree is independent of $s_i$. Note that the dependence on $y$ (or similarly $\bar{y}$) drops off from $C$’s in the ultra-local case. So, while a spin-local expression cannot contain infinitely many contractions for fixed spins, in the spin ultra-local case the number of such contractions in addition is bounded for any spins that enter $C$’s. An example of an ultra-local ‘vertex’ comes already from the free equations, where it has the form of the so-called central on-mass-shell theorem \[17, 31\]

\[
\Upsilon(\omega_0, \omega_0, C) \sim \eta H^{\alpha\beta} \frac{\partial^2}{\partial \bar{y}^\alpha \partial \bar{y}^\beta} C(0, \bar{y}) + \bar{\eta} H^{\alpha\beta} \frac{\partial^2}{\partial y^\alpha \partial y^\beta} C(y, 0),
\]

(1.14)

where $H$ is the background AdS two-form.

Let us also note that the meaning of ultra-locality within the Frönsdal formulation is not that transparent as the corresponding ultra-local expressions when expressed in terms of physical fields are just local and contain spin dependent derivatives. It is the rate of an $s$-dependence that allows one distinguishing local expressions from ultra-local ones.

1.3.2 Star product and (non)-locality

The global HS symmetry naturally introduces a certain non-locality by mixing an infinite tower of gauge fields. Therefore, it is not surprising that star product (1.2) can be a source of non-local terms. For example,

\[
C^{s_1} \star C^{s_2} = C^{s_1}(y, \bar{y}) e^{i \vec{\partial} \cdot \vec{\partial} + i \vec{\bar{\partial}} \cdot \vec{\bar{\partial}}} C^{s_2}(y, \bar{y})
\]

(1.15)

is spin non-local for projection on any $s$ and for any fixed $s_1$ and $s_2$, because it inevitably contains infinitely many contractions of two types of indices. If however one leaves only half of the star product nontrivial, say the one that acts on $\bar{y}$, then a similar expression

\[
C^{s_1} \bar{\star} C^{s_2} = C^{s_1}(y, \bar{y}) e^{i \vec{\bar{\partial}} \cdot \vec{\bar{\partial}}} C^{s_2}(y, \bar{y})
\]

(1.16)

is perfectly spin-local for it contains infinitely many contractions of one type (dotted) only. We then note that the fate of locality heavily relies on the structure of the underlying HS algebra. If we are to keep the first relation in (1.1) nontrivial only by setting $[\bar{y}_\alpha, \bar{y}_\beta] = 0$, the resulting HS algebra delivers no apparent non-localities\(^5\).

Let us look at the effect that star product (1.2) produces on (non)-locality of (1.4)-(1.5). The left hand side has the quadratic vertices governed by the HS symmetry. These are $\omega \star \omega$ and $[\omega, C]_\star$. Both are spin-local since $C$ appears once at best. The vertices on the right come from a HS symmetry deformation, which emerges from the consistency constraint $d_x^2 = 0$. For example, one easily extracts the following condition\(^6\) at leading order by applying $d_x$ to (1.4) and (1.5)

\[
\Upsilon_1(\omega, \omega, C) \star C = \Upsilon_0(\omega, \omega \star C, C) - \Upsilon_0(\omega \star \omega, C, C) + \omega \star \Upsilon_0(\omega, C, C),
\]

(1.17)

where $\Upsilon_1$ and $\Upsilon_0$ come from 1-form (1.4) and 0-form (1.5) sectors with all $\omega$’s at mostly left position in $\Upsilon_{0,1}$. From (1.17) it is clear that $\Upsilon_0(\omega, C, C)$ can not be local if the left hand side

\(^5\)The space-time algebra made of billinears in $y$’s is no longer the AdS one in this case. Note, that it does not reduce to the Minkowski either.

\(^6\)It is convenient to assume extra Chan-Paton color indices on $\omega$ and $C$. It allows one considering different orderings of $\omega$’s and $C$’s separately.
is non-local. The latter having a star product of two $C$’s is non-local unless $\Upsilon_1(\omega, \omega, C)$ brings no dependence on $y$ or $\bar{y}$ in the argument of $C$. This happens if $\Upsilon_1(\omega, \omega, C)$ is ultra-local. From this simple analysis one concludes that the star product places very stringent constraints for the vertices to be potentially local. Particularly, some vertices have to be ultra-local to support locality. An important lesson here is that even local field redefinitions at a given leading order may result in non-local vertices at the next-to-leading one. Indeed, $\Upsilon(\omega, \omega, C)$ having only one $C$ is always local. It is its ultra-locality however that gives $\Upsilon(\omega, C, C)$ a local chance. The nature of this phenomenon is directly related to an infinite spectrum of HS fields and has no analog in the case of a finite spectrum. In particular, this makes the canonical form of (1.14) crucially important for the cubic HS interactions to be local.

1.4 Present state of HS (non)-locality

There are several approaches devoted to the HS interaction problem in the literature (see e.g. [4] for a recent pedagogical review and references therein). Some, like in [32], [33] give up on locality from the very beginning assuming that once it breaks down sooner or later in perturbations there is no need in hanging it on. This point of view is supported in some sense by the holographic quartic analysis [14] that unlike the cubic one [9] points out at non-locality. The holographic approach pronounces HS theory non-local beyond cubic order. It remains not clear so far how to define the theory in the bulk without involving a conjectural holographic dual one on this way. The matching of bulk and boundary pictures works fine while local though [34], [35], [36], [37] (see also [38] for a very recent analysis at the cubic order). At any rate, one has to accept that the holographic HS reconstruction is paused until the quartic vertex is settled up. An attempt to qualify the holographic non-locality arising at this order is given in [16]. Another feasible option proposed in [39], [28] is the proper boundary dual for the HS theory might be the conformal HS theory rather than a vector model. If that is the case then the holographic reconstruction should be revisited.

The approach based on the Vasiliev equations on the other hand gives access to all order vertices in their unfolded form (1.4)-(1.5). What it does not tell is which field frame one should pick for the interactions to be local or properly non-local if the former is not possible. Being independent of any particular field frame, the Vasiliev equations allow for all order control of the functional class and therefore gives a tool for a systematic analysis of (non)-locality. In a series of papers [40]-[46], [27]-[30], the locality problem was analyzed at first few interaction orders and some all order statements were obtained. The results of these papers can be summarized as follows.

Vertices that appear in (1.4) are shown to be local and explicitly found\footnote{Note that the result of [40] invalidates the argument of [47] which states that if the Vasiliev quadratic vertex is chosen to be non-local via the standard homotopy resolution, there is no regular improvement that can bring it into a local form.} at least up to $\Upsilon(\omega, \omega, C, C)$, while those from (1.5) are local at least up to $\Upsilon(\omega, C, C, C)$ in the holomorphic sector ($\bar{\eta} = 0$, see (1.7)). An explicit form of (1.5) is known for $\Upsilon(\omega, C, C)$, [44] and for a fragment of $\Upsilon(\omega, C, C, C)$, [46]. Note that the locality is proven to quite a high order with some of the vertices from quintic interactions in the Lagrange counting. Still, neither of those cover the complete quartic vertex as the fate of HS quartic interaction remains indefinite. Particularly, the $\eta\bar{\eta}$ – part of $\Upsilon(\omega, C, C, C)$ that completes quartic $V_{0,s_1,s_2,s_3}$ is not yet analyzed.
In obtaining these results a number of important observations and conjectures have been made. Let us specify those playing an important role in our investigation.

**Functional class** In the sequel we deal with the Vasiliev generating equations for (1.4)-(1.5). The master fields entering these equations belong to a certain space which is large enough to encompass both the local and (non)-local field frames. Therefore, one has to specify it exactly in order to provide a local frame if exists. While *a priori* it is not clear how to do it, whatever this class is it should be closed on the operations of the generating equations. This problem was first analyzed in [48], where a certain class was proposed. The would be local class if exists should be a subclass of this one as it was specified further in [27].

**Pfaffian locality theorem** The degree of non-locality can be estimated by the important theorem [29] which states, that a perturbation theory prescribed by the specific resolution operators reduces generic non-locality as measured by rank of a certain Pfaffian. Particularly, this result alone is sufficient to prove locality of vertex Υ(ω, C, C). Although the result of [29] is a crude estimate of the exponential behavior of the non-local contractions and does not answer whether the theory is local or not, it sets the stage for a deeper analysis into HS interactions that meet the Pfaffian locality theorem conditions.

**Star-product re-ordering limit** In constructing the resolutions that respect the aforementioned functional class, one arrives at the homotopy operators that can be reinterpreted in terms of conventional homotopies for the star-product re-ordered Vasiliev equations followed by a certain limit of the re-ordering parameter $\beta \to -\infty$, [44]. Specifically, the limit in question represents a contraction of the large star-product algebra that changes some commutation relations. This is a very important result that shows that the (non)-locality is much affected by the type of the large algebra. In different context a recent investigation on the ordering (in)dependence can be found in [49].

**Holomorphic sector** The analysis of the HS vertex structure in the holomorphic sector can be carried out independently from the rest. Moreover, its status from the locality perspective is crucial for the locality of the whole theory. By now it is known that the holomorphic leading $\Upsilon(\omega, \omega, C)$ and next-to-leading $\Upsilon(\omega, \omega, C, C)$ are ultra-local, [44]. In [29] an important conjecture called the $z$ – dominance lemma was proposed, which if true allows to state a vertex locality without its manifest calculation. The conjecture was proven at $CC$ – order and verified by a brute force calculation of $\Upsilon(\omega, C, C, C)$ in [46]. This technical condition if satisfied serves as a useful guiding principle in probing locality. In [50] we specify conditions for the $z$ – dominance lemma and prove its validity. The all order spin locality conjecture of the holomorphic sector has been put forward in [29].

### 1.5 Structure of the paper

The paper is organized as follows. In section 2 we formulate our goals and summarize the main results. Section 3 is devoted to the Vasiliev formulation of the HS generating equations.

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8The fact of equivalence upon re-ordering of the standard homotopy with the properly shifted one was first noted for $\beta = 1$ in [43].
In §3.1 we provide a brief reminder of the original ideas, in section §3.2 we modify the Vasiliev equations by introducing the $\beta$–ordering freedom, in section §3.3 their $\beta \to -\infty$ contraction is investigated. In section §3.4 we specify the proper functional class and in section §3.4.1 analyze star products within that class. Section §4 contains generating equations for the HS holomorphic interactions. Its perturbation theory is given in section §4.1 with the lower order examples of interaction vertices in section §5. The locality is proven in section §5.1 while an observation of a certain shift symmetry is given in §5.2. We conclude in section §6. The paper is supplemented with two appendices. In Appendix A we derive our functional class. Appendix B provides with a proof of the central projector identity (4.9).

2 Goals and main results

Our primary interest is the holomorphic sector. One would like to understand whether the ultra-locality extends beyond $\Upsilon(\omega, \omega, C, C)$, as well as whether the spin locality extends beyond $\Upsilon(\omega, C, C, C)$? In attacking this problem we specify the functional class for the Vasiliev master fields. The proper functions are chosen to be invariant under the aforementioned $\beta$–re-ordering. This is a crucial observation based on our experience of the earlier analysis of a few interaction orders [44]. Such defined class appears to be a subclass of the one proposed in [27], but unlike the latter it is not respected by the original Vasiliev star product. Nevertheless it turns out to be respected by a star product that emerges in the re-ordering limit (contraction).

So, our goal is to come up with the Vasiliev-like generating equations for the specified functional class. Should that be possible one no longer needs quite a complicated homotopy resolutions from [44], which are designed to bring fields to the proper class after star-product action. Indeed, as shown in [44] the complicated homotopy operators reduce to the standard contracting homotopy upon the re-ordering followed by a contraction.

To reach this goal we scrutinize the $\beta$–ordering freedom of the Vasiliev generating system and investigate the locality limit $\beta \to -\infty$ at the level of equations of motion. Even though the naive limit does exist, the resulting equations appear to make no sense beyond lower orders as we show. At higher orders the new star product brings infinities. This result is exceptionally interesting for on the one hand we know that some higher order vertices were indeed effectively calculated [44] using the limiting star-product, but on the other, this limit could not be taken at the level of the Vasiliev equations directly. That suggests the existence of the different Vasiliev type equations based on a new algebra.

We show that such equations do exist and indeed differ from those obtained via a naive limit $\beta \to -\infty$. Equations (4.12)-(4.15) is the central result of our work. In particular they do not contain a zero-form module $B$ which is usually responsible for vertices $\Upsilon(\omega, C, \ldots, C)$. In our case these missing vertices come up automatically in a way to complete consistency. In addition to the standard Vasiliev case, the mechanism that makes the whole system consistent rests on the existence of a unique element called $\Lambda_0$ in the space of $dz$ – one-forms. Along with the usual Klein two-form the two are the main building blocks of the generating system. We observe a remarkable projector identity (4.9) that involves $\Lambda_0$. The identity is responsible for the consistency of the whole system and is a major observation of the present paper. The appearance of $\Lambda_0$ is something that makes difference between our system and the standard Vasiliev equations.

A simple consequence of the obtained generating equations is the all order ultra-locality of
Υ(ω, ω, C, . . . , C) and spin locality of Υ(ω, C, . . . , C) as we also show. Thus we give a proof of
the locality conjecture of [29]. Another important result is a shift symmetry of the holomorphic
vertices that holds to all orders. An investigation of this symmetry was inspired by the structure
lemma of [29] that underlies the Pfaffian locality theorem and prescribes certain homotopy shifts
in perturbation theory. We show that precisely these shifts generate a symmetry of HS vertices.
This observation indicates a relation of the observed symmetry with the HS locality.

In connection with our work let us point out a recent paper [51] where the all order locality
conjecture of the holomorphic sector was also put forward. While the result of [51] is as
well based on the limiting star product of [44], unlike our approach the authors engaged an
accessory assumption called the ‘duality map’ that they checked at lower orders. The duality
map is supposed to help reaching out Υ(ω, ω, C, . . . , C) from Υ(ω, ω, C, . . . , C). Lack of manifest
consistency however forces the authors examining the resulting vertices against consistency.
This way [51] explicitly checks few vertices up to Υ(ω, ω, C, C) reproducing some of the earlier
results [30], [44] and conjectures the higher order ones. Our approach is free from any outside
assumptions on vertices and rests on the formal all order consistency.

3 Vasiliev equations

3.1 A brief reminder

Here we would like to recall the basic elements behind the Vasiliev equations. We are being
somewhat sketchy there. Our exposition is slightly different from the original [21] (for reviews
see [1], [2], [53]) for a reason that will be clear soon.

The idea of the generating system for the unfolded HS equations (1.4)-(1.5) is to extend
the dependence of fields ω and C onto a larger space that includes variables Z_A = (z_α, \bar{z}_\dot{\alpha})
and together with Y’s form what can be referred to as the large algebra. Introducing

\begin{align*}
W(Z, Y) &= \omega + F_\omega(\omega, C; Y, Z) + F_\omega(\omega, C, C; Y, Z) + \ldots, \\
B(Z, Y) &= C + F_C(C, C; Y, Z) + F_C(C, C, C; Y, Z) + \ldots,
\end{align*}

and setting

\begin{align*}
d_x W + W \star W &= 0, \\
& \quad (3.3) \\
d_x B + [W, B]_* = 0, \\
& \quad (3.4)
\end{align*}

where ω and C do not depend on Z, while F_ω and F_C are supposed to encode the would be
HS vertices on the right of (1.4)-(1.5). Such an extension requires star product \star to act not
only on Y variables but on Z as well and as such should extend (1.2) preserving associativity.
System (3.3)-(3.4) is obviously consistent under d_x^2 = 0. Therefore, in order to have a form of
(1.4)-(1.5), the dependence on Z must identically vanish d_x((3.3), (3.4)) \equiv 0. The requirement
of course imposes a problem of the Z – dependence of functions (3.3), (3.4). This problem was
solved by Vasiliev [21] as he proposed the following evolution equations along Z by introducing
an auxiliary field Λ

\begin{align*}
\Lambda(Z, Y) &= \theta^a \Lambda_a + \bar{\theta}^{\dot{a}} \bar{\Lambda}_{\dot{a}}, \\
& \quad (3.5) \\
d_x W + \{W, \Lambda\}_* + d_x \Lambda = 0, \\
& \quad (3.6)
\end{align*}

\footnote{The original Vasiliev equations have a slightly different form with field S instead of Λ. The two are related
by a shift S = \Lambda + i/2 \theta^a Z_a for the original commutation relations of Z’s}
where we denote
\[ dz^\alpha \equiv \theta^\alpha , \quad d\bar{z}^{\dot{\alpha}} \equiv \bar{\theta}^{\dot{\alpha}} . \] (3.7)

Now by acting with \( dz \) one easily verifies that (3.3) and (3.4) are free from the \( z \)-dependence. Eqs. (3.5)-(3.6) determine otherwise unspecified functions \( F_\omega \) and \( F_C \) in terms of a yet unknown connection \( \Lambda \). The missing equation for \( \Lambda \) can be guessed by checking consistency of (3.5)-(3.6), \( d_z^2 = 0 \), that fulfills by setting
\[ d_z \Lambda + \Lambda \star \Lambda = \eta B \star \gamma + \bar{\eta} B \star \bar{\gamma} , \] (3.8)

where \( \gamma \) and \( \bar{\gamma} \) are the so called Klein two-forms, which we specify in what follows and \( \eta \) is a free parity breaking parameter. The reason why the Klein two-forms \( \gamma \) and \( \bar{\gamma} \) appear in (3.8) is due to the twisting in commutators (3.4) and (3.6) that we sloppily assume. Eqs. (3.3)-(3.4), (3.5)-(3.6) and (3.8) are called the Vasiliev equations.

One notes that from the point of view of a mere consistency and a formal form of (1.4)-(1.5), the Vasiliev equations allow for any star operation \( \star \) so long it is associative, admits a well-defined Klein two-form \( \gamma \) and leads to a regular product of master fields. We are going to introduce this freedom following [44] as a one parameter re-ordering of the original star product.

### 3.2 \( \beta \)-reordered form. Holomorphic sector

From now on we restrict ourselves to the holomorphic sector, that is we set \( \bar{\eta} = 0 \). The HS generating system then amounts to
\[ d_z W + W \star W = 0 , \] (3.9)
\[ d_z B + W \star B - B \star \pi(W) = 0 , \] (3.10)
\[ d_z W + \{ W, \Lambda \} + d_z \Lambda = 0 , \] (3.11)
\[ d_z B + \Lambda \star B - B \star \pi(\Lambda) = 0 , \] (3.12)
\[ d_z \Lambda + \Lambda \star \Lambda = B \star \gamma , \] (3.13)

where \( \star \) is the standard Vasiliev star product. Since we are in the holomorphic sector one needs no \( \bar{z} \) variable from \( Z = (z, \bar{z}) \)
\[ (f \star g)(z, Y) = \frac{1}{(2\pi)^2} \int d^2 u d^2 v f(z + u; y + u; \bar{y}) \star g(z - v; y + v; \bar{y}) \exp(iu_\alpha v^\alpha) \] (3.14)

and \( \bar{\star} \) is the leftover star product acting on \( \bar{y} \) (1.3). We are going to systematically omit \( \bar{\star} \) as well as the \( \bar{y} \)-dependence itself from now onwards.

We use the manifest form of the twisting now
\[ \pi(f(y, z)) = f(-y, -z) \] (3.15)

instead of the usual outer Klein operators, which we do not introduce in our definitions. The two-form \( \gamma \) is given by
\[ \gamma = \frac{1}{2} e^{iz_\alpha y^\alpha} \theta_\beta \theta_{\bar{\beta}} . \] (3.16)

In what follows it is convenient to introduce the following notation that we will use to simplify expressions. For two commuting spinors \( a_\alpha \) and \( b_\beta \) let us define their contraction as follows
\[ a_\alpha b^\alpha \equiv ab = -ba . \] (3.17)
Now, following [44] one can replace the original star product (3.14) with the \( \beta \) - reordered one that arises in the locality analysis

\[
f \star_{\beta} g = \int \frac{d^2u d^2v' d^2v' d^2u'}{(2\pi)^4} f(z+u', y+u) g(z-(1-\beta)v-v', y+v+(1-\beta)v') \exp(iu_\alpha v^\alpha + iu'_\alpha v'^\alpha). \tag{3.18}
\]

The commutation relations are

\[
y \star_{\beta} = y + i \frac{\partial}{\partial y} - i(1-\beta) \frac{\partial}{\partial z}, \quad \star_{\beta} y = y - i \frac{\partial}{\partial y} - i(1-\beta) \frac{\partial}{\partial z}, \tag{3.19}
\]

\[
z \star_{\beta} = z - i \frac{\partial}{\partial z} + i(1-\beta) \frac{\partial}{\partial y}, \quad \star_{\beta} z = z + i(1-\beta) \frac{\partial}{\partial y} + i \frac{\partial}{\partial z}. \tag{3.20}
\]

Note that the product reduces to (1.2) for \( z \) - independent functions and to (3.14) for \( \beta = 0 \). One can also observe that

\[
f(y) \star_{\beta} g(z,y) = f(y) \star g(z,y), \quad g(z,y) \star_{\beta} f(y) = g(z,y) \star f(y), \tag{3.21}
\]

which makes \( \star_{\beta} \) indistinguishable from the original Vasiliev star product at lower orders, where no product of two \( z \) - dependent functions is yet present. Parameter \( \beta \) interpolates between different orderings. So, the Weyl ordering corresponds to \( \beta = 1 \), while \( \beta = 2 \) corresponds to the anti-Vasiliev one. An operator that maps symbols in the original ordering (3.14) to the re-ordered one is

\[
O_{\beta} f(z,y) = \int \frac{d^2u d^2v}{(2\pi)^2} f(z+v, y+\beta u) \exp(iu_\alpha v^\alpha), \tag{3.22}
\]

\[
O^{-1}_{\beta} f(z,y) = O_{-\beta} f(z,y) = \int \frac{d^2u d^2v}{(2\pi)^2} f(z+v, y-\beta u) \exp(iu_\alpha v^\alpha). \tag{3.23}
\]

Using it one easily calculates Klein two-form \( \gamma_{\beta} \) corresponding to ordering (3.18) that appears in (3.13), [44]

\[
\gamma_{\beta} = \frac{1}{2} O_{\beta}(\exp(izy)) \theta_\alpha \theta'^\alpha = \frac{1}{2} \frac{1}{(1-\beta)^2} \exp \left( \frac{i}{1-\beta} zy \right) \theta_\alpha \theta'^\alpha. \tag{3.24}
\]

Note, that \( \gamma \) remains invariant if the re-ordering is supplemented by the proper rescaling of \( z \). It makes it convenient introducing

\[
O_{\beta} f(z,y) := O_{\beta} f(z,y) \bigg|_{z \to (1-\beta)z} \tag{3.25}
\]

leading to\[\text{10}\]

\[
O_{\beta} \gamma = \gamma, \quad \forall \beta \tag{3.26}
\]

Remarkably, as far as the lower orders are concerned, the \( \beta \) - re-ordered Vasiliev equations result in vertices that do not depend on \( \beta \) at all. These are \( \Upsilon(\omega, \omega, C) \) and \( \Upsilon(\omega, C, C) \). This fact was noted in [44] and earlier for the case of the central on-mass-shell theorem (1.14) calculated for \( \beta = 1 \) in [33]. In particular, these vertices are insensitive of \( \beta \to -\infty \). Moreover,

\[\text{10} \text{Note that rescaling } z \to (1-\beta)z \text{ implies } dz = \theta \to (1-\beta)\theta.\]
the perturbative master fields $\Lambda^{(1)}$, $W^{(1)}$ and $B^{(2)}$ do not depend on $\beta$, upon trivial rescaling $z \to (1 - \beta)z$ either. This suggests that at least at lower orders the functional class of the Vasiliev master fields is $\beta$-ordering independent (modulo trivial rescaling of $z$) functions

$$
\mathcal{O}_\beta(W) = W, \\
\mathcal{O}_\beta(B) = B, \\
\mathcal{O}_\beta(\Lambda) = \Lambda.
$$

(3.27)

We may emphasize that from the point of view of HS consistency and lower order interaction, star product (3.18) is not any worse or better than the original one (3.14). What can make a difference is the functional class that should correspond to the perturbatively well-defined equations (1.4)-(1.5). This class is of course sensitive to a particular star product.

### 3.3 Limit $\beta \to -\infty$

While the lower order vertices are insensitive to a particular $\beta$-ordering, things change at the level of $\Upsilon(\omega, \omega, C, C)$ and higher, where parameter $\beta$ essentially survives and it is only at $\beta \to -\infty$ that one recovers an (ultra-)local result. There are different ways of thinking of what goes on at higher orders. From the point of view of functional class (3.27) (yet to be specified), star-product (3.14) does not respect its invariance as soon as both multipliers are $z$-dependent. To keep fields within the class amounts to introducing the special contracting homotopy operators for solving for the master field $z$-dependence that somehow undo the effect of star-product (3.14). Technically this procedure assumes a limit $\beta \to -\infty$, which enters the homotopy operators. Given the effect of $\beta$ is equivalent to the re-ordering and rescaling, $\beta \to -\infty$ looks like a certain contraction of the original star product (3.14). This suggests that the limiting star-product respects functional class (3.27) and might be implemented at the level of the Vasiliev equations.

Let us have a detailed look at whether such a limit really makes sense at the level of equations of motion. Naively, $\beta \to -\infty$ can be straightforwardly taken from (3.18). To carry it out one should keep in mind to rescale properly (see also [44])

$$
z \to (1 - \beta)z, \quad \theta = dz \to (1 - \beta)\theta.
$$

(3.28)

This amounts to set the multiplied functions to be $f = f(\frac{1}{1-\beta}, y)$. The result for the limiting star product $\beta \to -\infty$ gets immediately available then,

$$
f \ast g = \int \frac{d^2u'd^2v'd^2u d^2v}{(2\pi)^4} f(z + u', y + u) g(z + v, y + v + v') \exp(iu_\alpha v^\alpha + iu'_\alpha v'^\alpha). 
$$

(3.29)

The obtained $\ast$-product is naturally associative and gives the following rules

$$
y \ast = y + i \frac{\partial}{\partial y} - i \frac{\partial}{\partial z}, \quad z \ast = z + i \frac{\partial}{\partial y}, 
$$

(3.30)

$$
\ast y = y - i \frac{\partial}{\partial y} - i \frac{\partial}{\partial z}, \quad \ast z = z + i \frac{\partial}{\partial y}.
$$

(3.31)

Some important properties of $\ast$ are in order. Unlike the original product (3.14), oscillators $z$ commute in the limiting case

$$
[z_\alpha, z_\beta]_\ast = 0.
$$

(3.32)
This means that limit $\beta \to -\infty$ is really a contraction rather than just a re-ordering. Another feature is whenever one of the multiplier is $z$-independent, (3.29) acts exactly as (3.14), e.g.,

$$f(y) \ast g(z, y) = f(y) \ast g(z, y) , \quad g(z, y) \ast f(y) = g(z, y) \ast f(y)$$  \hfill (3.33)

going in line with a more general statement of $\beta$-independence of lower order vertices.

Eqs. (3.9)-(3.13) formally survive limit $\beta \to -\infty$ and preserve the same form with the natural replacement $\ast \to \cdot$. In taking the limit the form of the Vasiliev equations in terms of field $\Lambda$ rather than $S$ (see footnote 9) was important. The naive limit just implemented is not harmless though. The room for a potential problem is easy to isolate. Unlike the original Klein operator, the one corresponding to (3.29) has certain ill-defined properties. For example,

$$e^{izy} \cdot e^{izy} = \infty = \delta^2(0).$$  \hfill (3.34)

The worrisome divergency calls for a thorough analysis of the functional class that survives under (3.29).

### 3.4 Classes of functions

In order to proceed with the functional class of the contracted Vasiliev equations analysis let us sort out their form. There are two space-time equations

$$d_x W + W \ast W = 0 , \quad (3.35)$$

$$d_x B + W \ast B - B \ast \pi(W) = 0 , \quad (3.36)$$

two equations that determine field $z$-dependence

$$d_z W + \{W, \Lambda\} \ast + d_x \Lambda = 0 , \quad (3.37)$$

$$d_z B + \Lambda \ast B - B \ast \pi(\Lambda) = 0 \quad (3.38)$$

and an equation for the auxiliary field $\Lambda$

$$d_z \Lambda + \Lambda \ast \Lambda = B \ast \gamma , \quad \gamma = \frac{1}{2} e^{izy} \theta \theta . \quad (3.39)$$

To identify the required class of functions one notes that fields entering the equations are $\theta$-graded. $W$ and $B$ are zero-forms in $\theta$, while $\Lambda$ is a one-form and $\gamma$ is a two-form. The class of functions we look for $C^r = \{\phi(z, y; \theta)\}$ can be labelled by the $\theta$-degree $r = 0, 1, 2$ for zero-, one- and two-forms, correspondingly. We require $C^r$ to be closed under $\ast$ and respect $\theta$-grading

$$C^{r_1} \ast C^{r_2} \to C^{r_1 + r_2} , \quad (3.40)$$

$$d_z C^r \to C^{r+1} . \quad (3.41)$$

Note, that since $\theta \theta \theta = 0$ this implies $C^3 = \emptyset$. In addition to (3.40)-(3.41), our definition of the class includes invariance (3.27) under re-ordering (3.25), i.e.,

$$C^r = \{\phi(z, y; \theta) : \quad O_\beta(\phi) = \phi , \quad \forall \phi_{1,2} : \quad O_\beta(\phi_1 \ast \phi_2) = \phi_1 \ast \phi_2\} . \quad (3.42)$$
While we do not have a clear insight into condition (3.27), we may note that it holds experimentally at a few available interaction orders, as we have stressed earlier. So, our strategy is to look at it as the all order exact. Condition (3.42) may happen to be in tension with (3.40), but it turns out that the all three are consistent if well-defined with star product (3.29), unlike (3.14). The details of derivation of $C_r$ are given in the Appendix A. The final result is a subclass of the one proposed earlier in [27]. It includes functions of the following form

$$\phi(z, y; \theta) = \int_0^1 \frac{d\tau}{\tau} f\left( tz + v, (1 - \tau)(y + u); \frac{\tau}{1 - \tau} \right) e^{i\tau z_0 y^\alpha \cdot v + iu_\alpha v^\alpha},$$

(3.43)

where $f$ is such that integration over $\tau$ makes sense and is otherwise arbitrary. A convenient way of looking at (3.43) is using the generating functions. Taking $f \sim \exp(iyA + izB + i\varepsilon A \cdot B)$ with sources $A$ and $B$ and conveniently setting $\varepsilon = 1$ for $r = 0$ and $\varepsilon = 0$ for $r = 1, 2$ in order to control over $\tau -$ poles one then finds by explicit integration over $u$ and $v$ in (3.43)

$$\theta^0: \quad \int_0^1 d\tau \frac{1 - \tau}{\tau} e^{i\tau z_0 y^\alpha + i(1 - \tau)A^\alpha y_\alpha + i\tau B^\alpha z_\alpha - i\tau A^\alpha B_\alpha},$$

(3.44)

$$\theta^1: \quad \int_0^1 d\tau e^{i\tau z_0 y^\alpha + i(1 - \tau)A^\alpha y_\alpha + i\tau B^\alpha z_\alpha + i(1 - \tau)A^\alpha B_\alpha},$$

(3.45)

$$\theta^2: \quad \int_0^1 d\tau \frac{\tau}{1 - \tau} e^{i\tau z_0 y^\alpha + i(1 - \tau)A^\alpha y_\alpha + i\tau B^\alpha z_\alpha + i(1 - \tau)A^\alpha B_\alpha}.$$  

(3.46)

Now, any function $\phi$ from (3.43) can be viewed as being originated from (3.44)-(3.46) by means of the decomposition in sources $A, B$. For example, at $r = 0$ the generating function should be at least linear in $B$ in order to be well-defined as an integral over $\tau$. Similarly, it should be at least linear in $A$ for $r = 2$.

Let us note a persistent factor $\exp(i\tau z y)$ as well as a specific $\tau -$ dependence of $z, y$ and $\theta$ that class (3.43) has. These are characteristic features of the functional class of [48], [27] too. What makes (3.43) different is a specific 'tail behavior' of the $AB -$ contractions in (3.44)-(3.46). It is these contractions that are necessary for (3.43) to be $O_\beta -$ invariant, (3.25). Moreover, the original star product (3.14) does not respect the invariance. In other words, $\ast -$ product of two functions from (3.43) does not remain in the class in general.

A convenient way to operate with (3.44)-(3.46) is to note that these can be rewritten in a somewhat factorized form

$$e^{i\tau z_0 y^\alpha + i(1 - \tau)A^\alpha y_\alpha + i\tau B^\alpha z_\alpha - i\tau A^\alpha B_\alpha} = e^{iA^\alpha y_\alpha} \circ e^{i\tau z_0 (y + B)^\alpha},$$

(3.47)

where

$$f(y) \circ g(z, y) = \int \frac{dudv}{(2\pi)^2} f(y + u)g(z - v, y) \exp(iu_\alpha v^\alpha)$$  

(3.48)

is a certain product. Note that the left multiplier in (3.48) is $z -$ independent. We are going to use the factorized representation (3.47) throughout the paper in what follows.

Let us look closer at (3.43). As HS equations (1.4)-(1.5) are formulated in terms of $z -$ independent functions the natural question is whether (3.43) contains functions of variable $y$ only. The answer to this question is affirmative within the $r = 0$ class.
\[ \phi(z, y) = \int_0^1 d\tau f^\alpha(y) \circ (1 - \tau)z_\alpha e^{i\tau y + q} \equiv -e^{izy} \partial_\alpha \int_0^1 d\tau \left( f^\alpha(x)e^{-izx} \right), \] (3.49)

where \( f^\alpha(y) \) is yet an arbitrary function and \( q \) is an arbitrary spinor (it can be zero, for example) and

\[ x = (1 - \tau)y - \tau q. \] (3.50)

Choosing now \( f_\alpha = -(y + q)^\alpha f(y) \) we can easily see that

\[ \phi(z, y) = f(y), \] (3.51)

i.e., \( \phi \) is \( z \)-independent. Indeed,

\[ -e^{izy} \partial_\alpha \int_0^1 d\tau f^\alpha(x)e^{-izx} = \]

\[ = e^{izy} \int_0^1 d\tau \left( 2(1 - \tau) -(1 - \tau)^2 \partial \overline{\partial} f(x) e^{-izx} \right) = -e^{izy} \int_0^1 d\tau \partial_\tau \left( (1 - \tau)^2 f(x)e^{-izx} \right) = f(y), \] (3.52)

where we integrated by parts over \( \tau \) in the last line. There are many ways to represent one and the same \( z \)-independent function. For example, up to a coefficient it can be represented as follows

\[ f(y) \sim \int_0^1 d\tau (y + q)^\alpha(n) f(y) \circ (1 - \tau)\tau^{n-1}z_\alpha \ldots z_\alpha e^{i\tau y + q} \] (3.53)

as can be checked by the multiple partial integration. Therefore, any \( z \)-independent \( \theta \)-zero-form belongs to \( C^0 \).

**Klein operator**  Another important object that manifests in (3.39) is Klein operator \( \kappa = e^{izy} \). Notably, there is no such function within \( C^0 \). Instead, it resides in sector \( C^2 \) just as it appears in (3.39) as a two-form \( \gamma \). To find it consider the following element from \( C^2 \)

\[ \kappa := \int_0^1 d\tau \frac{\tau}{1 - \tau} e^{\alpha\beta} \partial A^\alpha \partial B^\beta e^{ixz_\alpha y^\alpha + i(1 - \tau)A^\alpha y_\alpha + iB^\alpha z_\alpha + i(1 - \tau)A^\alpha B_\alpha} \bigg|_{A=B=0}. \] (3.54)

One then finds by partial integration,

\[ \kappa = \int_0^1 d\tau (2i\tau y e^{i\tau z y} = e^{izy}. \] (3.55)

### 3.4.1 Star products

Class (3.40)-(3.42) seemingly contains all the required elements to operate on (3.35)-(3.39). The important question however is whether star product (3.29) provides a meaningful outcome. We noted already that the replacing of the original product (3.14) with (3.29) may not be a harmless endeavor, (3.34). So, let us check out if the left hand side of (3.40) really makes sense. The good news is

\[ (C^0 \star C^0, \ C^0 \star C^1, \ C^1 \star C^0) \text{ - well-defined} \] (3.56)
This can be checked straightforwardly by using (3.44) and (3.45) as one finds that the $\tau$ integrations result in analytic expressions in that case. The problem blows up for other products

$$(C^1 \ast C^1, \ C^0 \ast C^2, \ C^2 \ast C^0) - \text{ill-defined},$$  

(3.57)

unless $C^0$ is $z$–independent in which case $C^0 \ast C^2$ and $C^2 \ast C^0$ exist and coincide with those calculated using the Vasiliev star product (3.33).

As an instructive example let us look at the product of $f \in C^0$ given by

$$f_\alpha = \int_0^1 d\tau (1 - \tau) z_\alpha e^{i\tau z y},$$  

(3.58)

with the Klein. From (3.29) one has

$$f(z, y) \ast e^{i\tau y} = \int \frac{dudv}{(2\pi)^2} e^{i\tau a(y+v) + iuv\alpha} f(v, u)$$  

(3.59)

and thus substituting (3.58) we find

$$f_\alpha \ast e^{i\tau y} = e^{i\tau y} \int_0^1 d\tau \frac{1}{1 - \tau} \int dv \delta(v) v_\alpha = \infty \cdot 0.$$

(3.60)

Uncertainty (3.60) illustrates a problem with star product (3.29) for the case of $C^0 \ast C^2$ and $C^2 \ast C^0$. It does not exist unless $d_z C^0 = 0$. In other words, the product makes sense for $z$–independent functions $f(y) \in C^0$ only.

Similarly, one can look at $C^1 \ast C^1$. Choosing, for example,

$$\Lambda_\alpha = \int_0^1 d\tau \tau z_\alpha e^{i\tau z(y+p)},$$

(3.61)

where $p$ is an arbitrary spinor parameter, we analogously arrive at

$$\Lambda_\alpha \ast \Lambda^\alpha = \infty \cdot 0.$$  

(3.62)

This example illustrates a general phenomenon that star product (3.29) is ill-defined for certain products that however inevitably present perturbatively within HS generating equations (3.35)-(3.39). More precisely, star product (3.29) is consistent with most of the generating equations (3.35)-(3.38), because these equations contain products of type (3.56). However, it does not allow one to determine evolution of field $\Lambda$ from (3.39) beyond free level. At first order $\Lambda^{(1)}$ can be determined from (3.39) because the right hand side contains $C(y) \ast \gamma$ which is well defined being a product of a $z$–independent function by a two-form.

A conclusion here is eqs. (3.35)-(3.39) cannot be taken as the generating ones for higher-spin system (1.4)-(1.5) beyond lower orders. Precisely, they reproduce $\Upsilon(\omega, \omega, C)$ and $\Upsilon(\omega, C, C)$ and then stubborn in $0 \cdot \infty$ at higher orders. Note the stark contrast with the original Vasiliev product (3.14), which has no obstruction in formal $\Upsilon$ reconstruction at any orders. One arrives at a curious case. The analysis of [44] plainly indicates the emergence of algebra (3.29) via the special homotopy resolutions as soon as one insists on higher order locality. Yet, a naive re-ordering followed by a contraction $\beta \rightarrow -\infty$ at the level of the Vasiliev equations that leads to (3.29) seemingly makes no sense in (3.39). The situation gets even more mysterious if noted that vertex $\Upsilon_{C\omega\omega C}$ residing in $W_1 \ast W_1$ part of (3.33) was effectively calculated using $\ast$ in place of $\ast$ in [44].

A possible resolution is that the Vasiliev equations may admit a consistent modification that keeps space-time equations (3.33) and (3.5) intact, while featuring a different condition for $\Lambda$ in the case of product (3.29).
4 Generating equations for the holomorphic sector

Let us start over with the derivation of the Vasiliev type generating equations. We keep the original idea that (1.4) can be reached by setting

\[ d_x W + W \ast W = 0, \quad (4.1) \]
\[ d_z W + \{ W, \Lambda \} \ast + d_x \Lambda = 0, \quad (4.2) \]

where we replace Vasiliev product (3.14) with (3.29). We also constrain ourselves to the specific functional class (3.43)

\[ W \in C^0, \quad \Lambda \in C^1, \quad (4.3) \]

which makes all operations within (4.1) and (4.2) well-defined. Recall, condition (4.2) guarantees that (4.1) is \( z \)-independent as can be checked by applying \( d_z \) to (4.1) and making use of (4.2). In so doing one never encounters undefined operations (3.57). Similarly, applying \( d_x \) to (4.2) one finds no further constraints or ill-defined structures either. A pinnacle of the problem is of course the \( d_z \)-consistency of (4.2) that normally leads to the introduction of the zero-form module \( B \), (3.4) and eventually to (3.39). In our case, however, (3.39) makes no sense as it contains structures \( C^1 \ast C^1 \) and \( C^0 \ast C^2 \). To check whether (4.2) may have solutions or not, we apply \( d_z \) to see that there are none unless

\[ d_z \{ W, \Lambda \} \ast = d_x d_z \Lambda. \quad (4.4) \]

The important comment is while both parts of (4.4) are well-defined on classes (3.43), neither \([d_z W, \Lambda] \ast\) nor \([W, d_z \Lambda] \ast\) being products of type (3.57) exists, unless \( W \) is in cohomology of \( d_z \), i.e., \( z \)-independent, (3.33). It is this feature that prevents one from re-expressing \( d_z W \) from (4.2) for substitution into (4.1). Equivalently, one can not use the Leibniz rule on the left hand side of (4.4). Specifically, if one takes e.g. \( d_z W \) from (4.2) and plug it into \([d_z W, \Lambda] \ast\) there would be the terms containing products of two \( \Lambda \)'s. These however are ill-defined being of the type \( C^1 \ast C^1 \). Let us stress once again that the original expression \( d_z \{ W, \Lambda \} \ast\) is perfectly alright. Indeed, once \( \{ W, \Lambda \} \ast\) is well defined so is \( d_z \{ W, \Lambda \} \ast\).

To proceed further we need an equation that determines evolution of \( \Lambda \) along \( z \). Typically of the Vasiliev-like approach, such an evolution is determined in terms of \( \Lambda \) itself and in terms of some new zero-form field \( B(z, y) \in C^0 \). As a matter of principle all possible contributions to a \( d_z \Lambda \)-equation can contain the two-form \( \Lambda \ast \Lambda \) and a two-form composed of \( B \). The latter being a zero-form should be accompanied with a certain two-form \( \Gamma \in C^2 \). The problem here is that once \( d_z \Lambda \in C^2 \), it can not be expressed via undefined \( C^1 \ast C^1 \). This excludes the \( \Lambda \ast \Lambda \) contribution. Similarly, \( C^2 \ast C^0 \) and \( C^0 \ast C^2 \) are both ill-defined unless the \( C^0 \)-multiplier is \( z \)-independent implying that \( B \) can not depend on \( z \), \( B := C(y) \). Therefore, the remaining option is a product of \( C \) by \( \Gamma \in C^2 \). At free level such dependence is given by

\[ d_z \Lambda = C \ast \gamma, \quad (4.5) \]

where \( C(y) \) can depend on \( y \) only and \( \Gamma = \gamma \). Let us stress that any \( z \)-dependent corrections of the form (3.44) to field \( C \) will lead to a meaningless result. This implies that (4.5) should be

\[ \text{As an illustration, one can think of } d_z (C^0 \ast C^1) = d_z C^0 \ast C^1 + C^0 \ast d_z C^1 \text{ as of a decomposition of a convergent integral into a sum of two divergent ones.} \]
taken as either all-order exact or the higher order corrections modify\footnote{We thank the anonymous Referee for pointing out this option to us.} \( \gamma \to \gamma^{\text{int}} = \gamma + O(C) \) (modulo field redefinition \( \gamma^{\text{int}} \to f(C) \ast \gamma \)). While the latter alternative cannot be \textit{a priori} excluded, we will see that (4.5) can indeed be taken as the exact one upon specifying its solution. So, our strategy is to postulate (4.5) in what follows. Plugging then (4.5) into (4.4) one arrives at

\[
d_x (\gamma - \gamma) = d_x \{W, \Lambda\} \ast ,
\]

which leads to a yet another consistency check with respect to \( d_x^2 = 0 \). As the left hand side is \( d_x \)– exact, the right one should satisfy \( d_x d_x \{W, \Lambda\} = 0 \), which is indeed the case in view of (4.1) and (4.2). As a result, eqs. (4.1) and (4.2) are consistent provided (4.6) is satisfied. The latter equation being interpreted as (1.5) (modulo field redefinition \( \gamma \) to \( f(C) \ast \gamma \)) places a stringent constraint on the right hand side of (4.6). Namely, once \( d_x C \ast \gamma \) is a product of \( z \)– independent function \( d_x C(y|x) \) by Klein two-form \( \gamma \), the same dependence of type \( f(y) \ast \gamma \) should be on the right hand side of (4.6) too. Specifically, comparing (4.6) with (1.5), we have

\[
d_x C \ast \gamma = (\Upsilon(\omega, C) + \Upsilon(\omega, C, C) + \Upsilon(\omega, C, C, C) + \ldots) \ast \gamma, \tag{4.7}
\]

with \( \Upsilon \)'s being \( z \)– independent by definition (1.5) and where

\[
\Upsilon(\omega, C, \ldots, C) \ast \gamma = d_x \{W^{(n-1)}, \Lambda\} \ast \tag{4.8}
\]

and \( n \) – is the order of perturbative expansion in \( C \). If this is not the case, then (4.6) can not have the form of (4.7).

Let us note that at this stage our consistency analysis is general and applicable to any associative star product and any functional class, provided expressions make sense. In particular, one could have written down the same system with the original star-product (3.14). Not surprisingly, (4.6) acquires a wrong dependence on \( z \) that is \( C \) can not be \( z \)– independent in this case, such that it loses interpretation in terms of (1.5) beyond free level unless higher order corrections of \( \Lambda \) are taken into account. Therefore, it can not describe higher-spin dynamics with (4.5). Things change radically with star product (3.29).

**Projector identity**  A crucial observation is – for any function \( f \in C^0 \) the following projector identity takes place

\[
d_z (f \ast \Lambda_0) = F_R(y) \ast \gamma, \quad d_z (\Lambda_0 \ast f) = F_L(y) \ast \gamma, \tag{4.9}
\]

where \( \Lambda_0 \) is the following solution of (4.5)

\[
\Lambda_0 = \theta \alpha \int_0^1 d\tau \tau \zeta_\alpha C(-\tau z)e^{izy}, \tag{4.10}
\]

\( f \) is given by (3.43) at \( \theta = 0 \) and \( F_{R,L} \) are the following \( z \)– independent functions

\[
F_L := (C \ast y f(-z, -y)) \bigg|_{z=y}, \quad F_R := (f(-z, y) \ast y C) \bigg|_{z=y}, \tag{4.11}
\]

where $*_y$ is the standard Weyl star-product (1.2) that acts on variables $y$ and ignores $z$. Eq. (4.9) says that the $z$ dependence of its left hand side collapses into the Klein two form no matter what function $f(z, y) \in C^0$ is. This result is not at all obvious and can be seen after the detailed calculation of (4.9) that boils down to a total derivative in one of the two integrals over $\tau$’s followed by a proper integration variable change. The projector identity plays a central role in the HS interpretation of eq. (4.6) and we prove it in Appendix B. That $f \in C^0$ and $\Lambda_0$ is (4.10) is crucially important. In particular, (4.9) is not going to hold for the $z$– dependent $f$ if one to replace $\Lambda_0$ with $\Lambda_0 + d_z \xi$, where $\xi \in C^0$ is an arbitrary function. It should be noted also that (4.11) are well-defined for a well-defined $f \in C^0$. Therefore, vertices $\Upsilon$ in (4.8) are indeed $z$– independent, while the consistency requirement (4.6) for (4.2) to admit solutions is fulfilled. Eventually, the complete system that generates the holomorphic higher-spin interactions reads

$$d_x W + W * W = 0,$$

$$d_z W + \{W, \Lambda_0\}*_z + d_x \Lambda_0 = 0,$$

$$d_z \Lambda_0 = C * \gamma ,$$

$$d_x C * \gamma = d_z \{W, \Lambda_0\}*_z,$$

where $\Lambda_0$ is given by (4.10). Let us summarize its main features.

- The system makes sense as the higher-spin generating if $W \in C^0$ only, which will be shown to take place at least in perturbations. Outside that class either some star products are ill-defined or $C$ can not be $z$ – independent. Most spectacular is a peculiar $z$ – dependence via Klein form $\gamma$ that shows up on the right hand side (4.15) for any $W \in C^0$. This makes $\Lambda_0$ from (4.10) a unique object playing a distinguished role in the whole construction. Recall also that $C^0$ contains all $z$ – independent functions.

- As different from the standard Vasiliev system (3.9)-(3.13), where (1.5) is reached via the $B$ - module, eq. (4.14) is so restrictive that constrains the zero-form vertices in (4.15) leaving no room for any $C$– corrections typically stored in $B$. This happens because the right hand side of (4.14) is the only meaningful $C^2$ – expression that one can write down modulo field redefinition $C \rightarrow f(C)$, provided the Klein two-form $\gamma$ is kept field independent.

Another comment is while the $z$ – independence of vertices (1.5) follows from the $z$ – independence of (3.10) within the Vasiliev framework, in our case we should set $C$ that appears in (4.14) and (4.15) to be $z$ – independent in the first place. This choice might have been inconsistent with the actual dynamics governed by the equations, but turns out to be perfectly fine with it due to (4.9).

- The local gauge symmetry of (4.12)-(4.15) generated by $\epsilon \in C^0$ is easy to identify

$$\delta \epsilon \Lambda_0 = d_z \epsilon + [\Lambda_0, \epsilon]_*, \quad (4.16)$$

$$\delta \epsilon W = d_x \epsilon + [W, \epsilon]_*, \quad (4.17)$$

$$\gamma * \delta \epsilon C = d_z [\epsilon, \Lambda_0]_*.$$

13The case is somewhat analogous to the early formulation of HS equations [52], where retrospectively the auxiliary connection $\Lambda$ was suitably fixed. We thank M.A. Vasiliev for the related discussion.
Note, however, the presence of $\Lambda_0$, (4.10) in the gauge transformations. While formally the invariance holds for any $\Lambda$ satisfying (4.14), it is only for $\Lambda_0$ that $\delta \epsilon C$ from (4.18) remains $z$–independent due to (4.9).

• The natural and simplest vacuum of (4.12)-(4.15) is the AdS space-time

$$W_0 = -\frac{i}{4}(\omega^{\alpha\beta} y_{\alpha} y_{\beta} + \bar{\omega}^{\dot{\alpha}\dot{\beta}} \bar{y}_{\dot{\alpha}} \bar{y}_{\dot{\beta}} + 2e^{\alpha\beta} y_{\alpha} \bar{y}_{\beta})$$

and

$$C_0 = 0,$$

where $\omega$, $\bar{\omega}$ and $e$ are the AdS connection fields. As expected, the Minkowski space is not a solution of the theory. However, one can consider a contraction of the original star-product algebra (1.1)

$$\bar{y} \rightarrow \rho^{-1} \bar{y}, \quad \rho \rightarrow 0$$

that results in

$$[y_{\alpha}, y_{\beta}]_* = 2i\epsilon_{\alpha\beta}, \quad [y_{\alpha}, \bar{y}_{\beta}]_* = 0, \quad [\bar{y}_{\dot{\alpha}}, \bar{y}_{\dot{\beta}}]_* = 0.$$ (4.22)

The contraction is well-defined on eqs. (4.12)-(4.15) and leads to a specific 'flat' limit with the vacuum of the form

$$W'_0 = -\frac{i}{4}(\omega^{\alpha\beta} y_{\alpha} y_{\beta} + e^{\alpha\beta} y_{\alpha} \bar{y}_{\beta}).$$ (4.23)

Note, this vacuum can not be regarded as the Minkowski one since it lacks the anti-chiral part of the Lorentz connection $\bar{\omega}^{\dot{\alpha}\dot{\beta}}$ (see also footnote 5). Put it differently, one can choose a coordinate system on the Minkowski space-time with $\bar{\omega}^{\dot{\alpha}\dot{\beta}} = 0$ which is consistent with (4.23). In this case, however, the manifest Lorentz covariance appears to be lost.

• Finally, since $W$, $\Lambda_0$ and $C$ depend on variable $\bar{y}$, which we systematically omit, one should remember of implicit $\bar{x}$ – product present in (4.12)-(4.15).

4.1 Perturbation theory

The higher-spin vertices in (1.4)-(1.5) can be determined from (4.12) and (4.15) order by order

$$\Upsilon(\omega, \omega, C, \ldots, C) = -\sum_{i=1}^{n} d_x W^{(i)} - \sum_{i+j=n} W^{(i)} * W^{(j)},$$

and

$$\Upsilon(\omega, C, \ldots, C) * \gamma = d_z \{W^{(n-1)}, \Lambda_0\}_*.$$ (4.25)

Note that $d_x W^{(i)}$ contributes to the $n^{th}$-order (4.24) for all $1 \leq i \leq n$. To calculate the vertices one should solve for $W^{(i)}$ using (4.13). So, at $0^{th}$-order in $C$ we have

$$d_z W^{(0)} = 0 \Rightarrow W^{(0)} = \omega(y) \in C^0.$$ (4.26)

Note that the solution belongs to $C^0$ as required. In fact, one can prove that any order solution of (4.13) belongs to $C^0$ once $\Lambda_0 \in C^1$. Indeed, let $W^{(n-1)} \in C^0$, then from (3.40) it follows that $\{W^{(n-1)}, \Lambda_0\}_* \in C^1$ and therefore

$$d_z W^{(n)} = \theta^a X_\alpha, \quad X \in C^1.$$ (4.27)
where $X$ comes from the $(n - 1)$-order solution and by construction belongs to $C^1$

$$X_\alpha = \int_0^1 d\tau \int \frac{du \, dv}{(2\pi)^2} f_\alpha^{(n-1)}(\tau z + v, (1 - \tau)(y + u)) e^{ir\zeta + iuv}. \quad (4.28)$$

Solving for $W^{(n)}$ using the standard homotopy

$$\Delta_0 X := z^\alpha \frac{\partial}{\partial \theta^\alpha} \int_0^1 d\tau \frac{1}{\tau} X(\tau z, y; \tau \theta) \quad (4.29)$$

and carry out a simple integration variable change one arrives at

$$W^{(n)} = \int_{[0,1]^2} d\tau d\sigma \int \frac{du \, dv}{(2\pi)^2} \frac{1 - \tau}{\tau} \tau z^\alpha f_\alpha^{(n-1)}(\tau z + v, \sigma(1 - \tau)(y + u)) e^{ir\zeta + iuv} + \phi^{(n)}(y). \quad (4.30)$$

The result is by definition (3.43) belongs to $C^0$. Indeed, the first term in (4.30) is manifestly in $C^0$ and since the freedom in the homogeneous solution $\phi$ is any $z$–independent function which too belongs to $C^0$, we conclude that any perturbative $W^{(n)} \in C^0$. Although not necessary, a convenient choice is to set $\phi^{(n)} = 0$.

5 Vertices

In what follows it is convenient to use the notation from [30]. We switch to the Taylor form of our fields by introducing

$$C(y, \bar{y}) \equiv e^{-iy^\alpha \gamma_\alpha} C(y', \bar{y}) \Big|_{y' = 0}, \quad p_\alpha = -i \frac{\partial}{\partial y'^\alpha}, \quad (5.1)$$

$$\omega(y, \bar{y}) \equiv e^{-iy^\alpha \gamma_\alpha} \omega(y', \bar{y}) \Big|_{y' = 0}, \quad t_\alpha = -i \frac{\partial}{\partial y'^\alpha}, \quad (5.2)$$

where we intentionally tagged the same operator with either $p$ or $t$ to distinguish its action on $C$ from its action on $\omega$. In this terms any vertex that system (4.12)-(4.15) produces has the following schematic form

$$\Upsilon_1 = \Phi^{[\delta_1, \delta_2]}(y; t_1, t_2, p_1) (C \star \ldots \star \omega \star \ldots \star C) (y'_1, \bar{y}) \Big|_{y'_1 = 0}, \quad (5.3)$$

$$\Upsilon_0 = \Phi^{[\delta]}(y; t, p_j) (C \star \ldots \star \omega \bar{C} \star \ldots \omega \ldots \bar{C}) (y'_j, \bar{y}) \Big|_{y'_j = 0}, \quad (5.4)$$

where $\Upsilon_1$ and $\Upsilon_0$ contribute to (1.4) and (1.5) correspondingly. All $C$’s on the right are ordered as they stand from left to right with one or two ordered $\omega$ – ‘impurities’ that may appear in any place of the string. Each $p$ acts on the corresponding $C$ and each $t$ acts on its $\omega$ and $\# I = \# i + 2$, $\# J = \# j + 1$. The structure of vertices is therefore totally encoded by functions $\Phi$’s, which however depend on the place of $\omega$’s in the $C$ – string. This place we denote by $[\delta_1, \delta_2]$, $\delta_2 > \delta_1$ in (5.3) with two $\omega$’s and by $[\delta]$ in (5.4) with a single $\omega$.

To simplify the subsequent star product calculation we will use

$$\Lambda_0 \rightarrow \theta^\alpha \int_0^1 d\tau \tau z_\alpha e^{ir\zeta(y + p)} \quad (5.5)$$
in place of $\Lambda_0$, (4.10) assuming that (5.3) acts on the corresponding $C$ in accordance with (5.1) and similarly

$$\omega(y) \rightarrow e^{it_1 y}$$  (5.6)

in place of $\omega(y)$, (5.2). The vertices are not difficult to calculate at a given perturbation order using (4.29) for solving for $W$ in (4.13). In so doing it is convenient to set $\phi = 0$ in (4.30). This way one finds

$$W^{(n)} = -\Delta_0 \left( \{ W^{(n-1)}, \Lambda_0 \} \right).$$  (5.7)

Note that contribution $d_x \Lambda_0$ trivially vanishes and $W^{(n)}$ equals zero at $z = 0$. It makes it convenient finding the $z$-independent vertices in (4.24) by setting $z = 0$ in each of the two terms. Since $d_x W^{(n)}_{z=0} = 0$ one finds

$$\Upsilon(\omega, \omega, C, \ldots, C) = -\left( \sum_{i+j=n} W^{(i)} \ast W^{(j)} \right) \bigg|_{z=0}.$$  (5.8)

As an illustration, consider the lower order examples. Using (4.26) and (5.7) we have at the first order

$$W^{(1)}_{\omega C} = -i^\alpha \int \left[ 0, 1 \right]^2 d\tau d\rho e^{i\rho ty + i(1-\rho)p t} \circ (1-\tau) z_\alpha e^{i\tau z(y + p + t)}, \quad (5.9)$$

$$W^{(1)}_{C\omega} = -i^\alpha \int \left[ 0, 1 \right]^2 d\tau d\rho e^{i\rho ty + i(1-\rho)p t} \circ (1-\tau) z_\alpha e^{i\tau z(y + p - t)}. \quad (5.10)$$

Recall again that the above expressions generate $W^{(1)}$ by acting on $\hat{\omega} \hat{C}$ and $C \hat{\omega}$ in accordance with (5.1), (5.2). Substituting it to (5.8) we obtain up to terms quadratic in $C$

$$\Upsilon(\omega, \omega, C) = \left( \Phi^{[1,2]}(\omega \hat{\omega} \omega \hat{C} C) + \Phi^{[1,3]}(\omega \hat{\omega} C \hat{\omega} \omega) + \Phi^{[2,3]}(C \hat{\omega} \hat{\omega} \hat{\omega}) \right) \bigg|_{y_1' = y_2' = y_3' = 0}; \quad (5.11)$$

where

$$\Phi^{[1,2]} = -\left( e^{it_1 y_1} \ast W^{(1)}_{\omega C} \right) \bigg|_{z=0} \quad (5.12)$$

$$\Phi^{[1,3]} = -\left( e^{it_1 y_1} \ast W^{(1)}_{C\omega} + W^{(1)}_{\omega C} \ast e^{it_2 y} \right) \bigg|_{z=0} \quad (5.13)$$

$$\Phi^{[2,3]} = -\left( W^{(1)}_{C\omega} \ast e^{it_2 y} \right) \bigg|_{z=0} \quad (5.14)$$

or explicitly,

$$\Phi^{[1,2]} = t_2 t_1 \int \left[ 0, 1 \right]^2 d\tau d\rho (1 - \tau) e^{i(1-\tau)(t_1 + \rho t_2)y - i(\tau t_1 + (1-\rho) t_2 + \rho t_2)p + i(\tau (1-\tau)\rho)t_2 t_1}, \quad (5.15)$$

$$\Phi^{[1,3]} = t_1 t_2 \int \left[ 0, 1 \right]^2 d\tau d\rho (1 - \tau) e^{i(1-\tau)(t_2 + \rho t_1)y - i(\tau t_2 + (1-\rho) t_1 + \rho t_1)p + i((1-\tau)\rho - \tau)t_2 t_1 +$$

$$+ t_2 t_1 \int \left[ 0, 1 \right]^2 d\tau d\rho (1 - \tau) e^{i(1-\tau)(t_1 + \rho t_2)y - i(\tau t_1 + (1-\rho) t_2 + \rho t_2)p + i((1-\tau)\rho - \tau)t_2 t_1}, \quad (5.16)$$

$$\Phi^{[2,3]} = t_1 t_2 \int \left[ 0, 1 \right]^2 d\tau d\rho (1 - \tau) e^{i(1-\tau)(t_2 + \rho t_1)y - i(\tau t_2 + (1-\rho) t_1 + \rho t_1)p + i(\tau (1-\tau)\rho)t_2 t_1}. \quad (5.17)$$
Eqs. (5.15)-(5.17) naturally reproduce a generalization of the central on-mass-shell theorem for an arbitrary HS background obtained in [30]. Their exponential parts carry neither \( pp \) nor \( yp \) contractions. Therefore, these vertices are ultra-local.

The lowest order form of (1.5) corresponding to the vanishing of its right hand side comes from (4.25) as follows. Plugging (4.26) into (4.25) one gets

\[
d_x C \gamma = d_x (\omega \Lambda_0 + \Lambda_0 \omega) = -\omega \gamma + \gamma \omega , \tag{5.18}
\]

where we used (4.14) and the Leibniz rule, which can be applied since \( \omega \) is \( z \)-independent. Noting then that

\[
\gamma \omega = \pi(\omega) \gamma \tag{5.19}
\]

where \( \pi \) was defined in (3.15), we have at this order

\[
d_x C + \omega C - C \pi(\omega) = 0 . \tag{5.20}
\]

To arrive further at the leading order one substitutes (5.9), (5.10) into (4.15) and use (4.9) to find

\[
\Upsilon(\omega, C, C) = (\Phi[1](\omega \Lambda_0 C) + \Phi[2](\Lambda_0 \omega C) + \Phi[3](\Lambda_0 C \omega)) \bigg|_{y_1 = y_2 = y_3 = 0} \tag{5.21}
\]

with

\[
\Phi[1] \gamma = d_x \left( W^{(1)}_{\omega C} \right) , \tag{5.22}
\]

\[
\Phi[2] \gamma = d_x \left( W^{(1)}_{\Lambda_0 C} + \Lambda_0 W^{(1)}_{\omega C} \right) , \tag{5.23}
\]

\[
\Phi[3] \gamma = d_x \left( \Lambda_0 W^{(1)}_{\omega C} \right). \tag{5.24}
\]

Explicit expressions can be found using (B.6), (B.7) and (B.8) with the final result being

\[
\Phi[1] = ty \int_{[0,1]^2} d\rho d\sigma \sigma e^{i(\sigma \rho \rho_2 + (1-\sigma) \rho_1) t + i(1-\sigma) \rho_1 + (1-\sigma + \sigma \rho) t} y , \tag{5.25}
\]

\[
\Phi[2] = ty \int_{[0,1]^2} d\rho d\sigma \sigma e^{i(\sigma \rho \rho_2 + (1-\sigma) \rho_1) t + i(1-\sigma) \rho_1 - (1-\sigma - \sigma \rho) t} y , \tag{5.26a}
\]

\[
+ yt \int_{[0,1]^2} d\rho d\sigma \sigma e^{i(\sigma \rho \rho_1 + (1-\sigma) \rho_2) t + i(1-\sigma) \rho_1 - (1-\sigma - \sigma \rho) t} y , \tag{5.26b}
\]

\[
\Phi[3] = yt \int_{[0,1]^2} d\rho d\sigma \sigma e^{i(\sigma \rho \rho_1 + (1-\sigma) \rho_2) t + i(1-\sigma) \rho_1 + (1-\sigma) \rho_2 - (1-\sigma + \sigma \rho) t} y . \tag{5.27}
\]

These vertices coincide with those found earlier in [30] modulo the change of the integration over \( \sigma \) and \( \rho \) to the integration over a two-dimensional simplex. Note there are no \( p_1 p_2 \) contractions within the exponentials meaning that the result is spin-local. Similarly one can go on to higher orders in this fashion.

### 5.1 Locality

While it is not difficult to come up with expressions for vertices \( \Upsilon \) reproduced via the standard homotopy (4.29), some notable features are accessible without the detailed calculation. It is
easy to see that any order vertex from (1.4) is ultra-local. This can be reached in two steps. First, we note that (4.29) applied for solving \( W^{(n)} \) leads to the following schematic result

\[
W^{(n)} \sim \sum_{\delta} \int D\rho \, t^{a_1} \ldots t^{a_n} e^{i\rho_{\tau}^{a_n y_n + it^a P_{\alpha}}} \circ \int_0^1 d\tau \frac{1-\tau}{\tau} \tau z_{\alpha_1} \ldots \tau z_{\alpha_n} e^{i\tau z_\alpha (y-P^{(\alpha)})},
\]  

(5.28)

where we recall that (5.28) acts on the string of \( n \) \( C \)'s with one impurity \( \omega \) at certain place \( \delta \). The sum is taken over all possible positions of \( \omega \). Here \( \int D\rho \) denotes all repeated integrations that show up in the process of applying (4.29) except for the single one over \( \tau \). \( P_z \) and \( P_t \) are the linear combinations with \( \rho \) – dependent coefficients of different \( p \)'s

\[
P_t = a_1(\rho)p_1 + a_2(\rho)p_2 + \ldots + a_n(\rho)p_n, \quad P_z = b_1(\rho)p_1 + b_2(\rho)p_2 + \ldots + b_n(\rho)p_n.
\]

(5.29)

While \( P_t \) and \( P_z \) depend on \( \delta \), we deliberately ignore any particular order of \( \omega \) and \( C \) since it is not important for our conclusion of locality. Note that (5.28) is free from nonlocalities as the only contraction that appears after \( \circ \) – computation (3.47) is between \( \omega \) and \( C \) but never between two \( C \)'s. Moreover, since there are no contractions between \( p \)'s and \( y \) in the left hand exponential of (5.28), the whole expression (5.28) is ultra-local having no occurrence of \( y \) in \( C \)'s. Now, the vertex from (1.4) is given by (5.8) which contains star products of \( W \)'s from different orders. These star products remain ultra-local as follows from (A.11). Indeed, taking some \( W^{(i)} \) from the \( i^{th} \) – order

\[
W^{(i)} \sim \int D\rho^{(i)} \left( t \cdot \frac{\partial}{\partial P_z^{(i)}} \right)^i \left( e^{i\rho_{\tau}^{(i)} y_n + it^a P_{\alpha}^{(i)}} \circ \int_0^1 d\tau \frac{1-\tau}{\tau} e^{i\tau z_\alpha (y-P^{(\alpha)})} \right),
\]

(5.30)

one finds

\[
W^{(i)} \ast W^{(j)} \sim \int D\rho^{(i)} D\rho^{(j)} \frac{d\sigma}{(1-\sigma)} \left( t_1 \cdot \frac{\partial}{\partial P_z^{(i)}} \right)^i \left( t_2 \cdot \frac{\partial}{\partial P_z^{(j)}} \right)^j \times \int \frac{1-\tau}{\tau} e^{i\tau z_\alpha (y-P^{(\alpha)})},
\]

(5.31)

where

\[
P_z^{(i,j)} = \sigma(P_z^{(i)} - \rho_y^{(j)})t + (1-\sigma)(P_z^{(j)} + \rho_y^{(j)})t.
\]

(5.32)

The above star product brings no \( y \cdot p \) terms into the exponential leaving all \( C \)'s \( y \) – independent in each contribution of (5.31). Therefore the final result (5.8) is ultra-local. This proves ultra-locality of (1.4) in the holomorphic sector.

Similarly, the holomorphic part of the 0-form vertices (1.5) can be extracted from (4.25). Taking (5.28) and substituting it into (4.25) up to an irrelevant order of \( \omega \) in the string we obtain

\[
d_z(W^{(n-1)} \ast \Lambda_0) = \Phi_L(y) \ast \gamma,
\]

(5.33)

where

\[
\Phi_L(y) = \int D\rho \int_0^1 d\sigma \frac{1-\sigma}{\sigma} \sigma^{n-1} (y^\beta t_\beta)^{n-1} \exp \left\{ iy^\alpha (-\sigma P_z^{(n-1)} + (1-\sigma)\rho_y^{(n-1)}t + (1-\sigma)p_n)^\alpha + it^\alpha P_{\alpha}^{(n-1)} + i(\sigma P_z^{(n-1)} - \rho_y^{(n-1)}(1-\sigma)p_n)^\alpha t_\alpha \right\}.
\]

(5.34)
Analogously with $d_z(\Lambda_0 \ast W^{(n-1)}) = \Phi_R(y) \ast \gamma$. Note that (5.34) contains $y \cdot p$ contraction within the exponential. Therefore, the corresponding vertex is not ultra-local. However, since it has no $p \cdot p$ contractions it is still spin local. This makes the holomorphic part of vertices in (1.3) spin-local.

5.2 Shift symmetry

Now we want to specify the dependence of $P_t$ and $P_z$ on $\rho$ variables. This dependence has the following remarkable properties

$$P_t(p_1 + a, p_2 + a, \ldots, p_n + a) = P_t(p_1, p_2, \ldots, p_n) + (1 - \rho_y) a, \quad (5.35)$$

$$P_z(p_1 + a, p_2 + a, \ldots, p_n + a, t) = P_z(p_1, p_2, \ldots, p_n, t) - a. \quad (5.36)$$

Here $a$ is an arbitrary spinor. One can prove that the following properties indeed take place by induction.

It is easy to see that the base of induction, namely $W^{(1)}$, respects such property. For example,

$$W^{(1)}_{\omega C} = \int_{\sigma} d\rho \ e^{i\rho y \cdot t_\alpha + i(1 - \rho) p_\alpha t_\alpha} t_\beta \ o \ \int_{\theta} d\tau \ \frac{1 - \tau}{\tau} \tau z_\beta e^{i\tau z_\alpha(y + p)\alpha}. \quad (5.37)$$

Similarly with $W^{(1)}_{\omega C}$. We already know that at any order of perturbation theory $W^{(n)}$ acquires the form (5.28). Assuming that properties (5.35), (5.36) are satisfied for the $n$-th order we are going to show that they are satisfied for the $(n + 1)$ order. Straightforward computation yields

$$\Delta_0(W^{(n)} \ast \Lambda_0) = \int D\rho \ \rho \ \int_\sigma d\sigma \ \frac{1 - \sigma}{\sigma} \sigma^n \ \int_\theta d\tau \ \frac{1 - \tau}{\tau} \tau z_\beta \ \cdot \ o \ \int_{\theta} d\tau \ \frac{1 - \tau}{\tau} \tau z_\beta \ \cdot \ o \ \int_\theta d\tau \ \tau z_{\beta + 1} e^{i\tau z_\alpha(y - \sigma P_z + (1 - \sigma) p_{n+1})\alpha}. \quad (5.38)$$

From the above expressions one immediately sees that provided properties (5.35) and (5.36) are satisfied for order $n$ they are satisfied for order $n + 1$.

Such a symmetry of the master field $W$ induces the corresponding symmetric properties on the vertices. To compute the zero-form vertices one uses (4.25). Straightforward computation yields

$$d_z(W^{(n)} \ast \Lambda) = \frac{1}{2} \theta^2 \ \int D\rho \ \int_\sigma d\sigma \ \frac{1 - \sigma}{\sigma} \sigma^n (z_\beta t_\beta)^n \ exp \ \left\{ i z_\alpha (y - \sigma P_z + (1 - \sigma) p_{n+1})^\alpha + \ i t_\alpha P_t + i (\sigma P_z - (1 - \sigma) p_{n+1})^\alpha \ rho_{\gamma}^\alpha \right\}. \quad (5.39)$$

To extract vertices for the zero-form sector one has to rewrite this expression in the form

$$d_z(W^{(n)} \ast \Lambda_0) = \int D\rho \ \int_\sigma d\sigma \ \frac{1 - \sigma}{\sigma} \sigma^n (y \cdot t_\beta)^n \ exp \ \left\{ i y^\alpha (-\sigma P_z + (1 - \sigma) p_{n+1})^\alpha + \ i t_\alpha P_t + i (\sigma P_z - (1 - \sigma) p_{n+1})^\alpha \ rho_{\gamma}^\alpha \right\} \ast \gamma. \quad (5.40)$$
From the last expression one derives the following property of the vertex

$$\Phi^{[\delta]}(y; t, p_i + a) = e^{i(t + y)^a} \Phi^{[\delta]}(y; t, p_i) \quad (5.41)$$

Now consider vertices in the one-form sector. Contributions to these vertices come from various products $W^{(n)} \ast W^{(n')}$. Straightforward computation yields

$$\begin{align*}
(W^{(n)} \ast W^{(n')})|_{z=0} &= \left( \int D\rho \, e^{i\rho \rho_y y^a t^a} P_t t^{\beta_1} \ldots t^{\beta_n} \int_0^1 d\tau \, \frac{1 - \tau}{\tau} z_{\beta_1} \ldots z_{\beta_n} \, e^{i\tau z_0 (y - P_z)^a} \ast \\
&\ast \int D\rho' \, e^{i\rho' y^a t'^a} P_{t'} t'^{\sigma_1} \ldots t'^{\sigma_{n'}} \int_0^1 d\tau' \, \frac{1 - \tau'}{\tau'} z_{\sigma_1} \ldots z_{\sigma_{n'}} \, e^{i\tau' z_0 (y - P'_z)^a} \right)|_{z=0} = \\
&= \int D\rho \int D\rho' \int_0^1 d\sigma \frac{1}{(1 - \sigma)} \int_0^1 d\tau \frac{1 - \tau}{\tau} \, (\tau \sigma \rho_y)^n (1 - \sigma) \rho_y \tau^{n'} (t^{\alpha} t'^{\alpha'})^{n+n'} \times \\
&\times \exp \left\{ i(1 - \tau) y^\alpha (\rho_y t + \rho'_y t')^\alpha - i \rho_y \rho_y' t^\alpha t'^{\alpha'} + i \rho_y P_t + i t^\alpha P_{t'} \\
&+ i \tau (\rho_y t + \rho'_y t') + (1 - \sigma) P_z - (1 - \sigma) \rho_y t (\rho_y t + \rho'_y t')^\alpha \right\} \quad (5.42)
\end{align*}$$

Using (5.35) and (5.36) one can derive the following property of the vertices for one-forms

$$\Phi^{[\delta_1, \delta_2]}(y - a; t_1, t_2, p_i + a) = e^{i(t_1 + t_2)^a} \Phi^{[\delta_1, \delta_2]}(y; t_1, t_2, p_i) \quad (5.43)$$

Taking $a = \nu(t + y)$ in (5.41) and $a = \chi(t_1 + t_2)$ in (5.43), where $\nu$ and $\chi$ are arbitrary numbers one notes that

$$\Phi^{[\delta]}(y; t, p_i + \nu(t + y)) = \Phi^{[\delta]}(y; t, p_i) \quad (5.44)$$
$$\Phi^{[\delta_1, \delta_2]}(y - \chi(t_1 + t_2); t_1, t_2, p_i + \chi(t_1 + t_2)) = \Phi^{[\delta_1, \delta_2]}(y; t_1, t_2, p_i) \quad (5.45)$$

The differential version of symmetries (5.41) and (5.43) reads

$$\left( t + y - i \sum_j \frac{\partial}{\partial p_j} \right) \Phi^{[\delta]}(y; t, p_i) = 0 \quad (5.46)$$
$$\left( t_1 + t_2 + i \frac{\partial}{\partial y} - i \sum_j \frac{\partial}{\partial p_j} \right) \Phi^{[\delta_1, \delta_2]}(y; t_1, t_2, p_i) = 0 \quad (5.47)$$

A few comments on the shift symmetry are in order. Its action both in (5.41) and (5.43) is somewhat ‘off-shell’ in a sense that it remains valid for the integrands of eqs. (5.40), (5.42). Such a behavior is reminiscent of the structure lemma from [29] which controls parameters of the homotopy operators used in the perturbation theory and underlies the so called Pfaffian locality theorem. While this lemma relies heavily on the Vasiliev star product (3.14) and besides that is not about any symmetries at all, we showed that similar shift of parameters $p_i$ (accompanied by the shift $y \to y - a$ in the sector of 1-form (5.43)) lead to the exact symmetry of vertices based on star product (3.29). Another comment is as shown in [50], by postulating symmetry (5.41) one can prove the $Z$ dominance conjecture from [29]. This implies that the observed symmetry (5.41) and (5.43) is strongly intertwined with HS locality.

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14 The homotopy shifts of the structure lemma [29] include sign flips which are not present in our case. This difference is artificial and is related to the presence of outer Klein operator $k$ within the formalism of [29] which induces the aforementioned sign alteration.
6 Conclusion

The main result of our paper is the generating system (4.12)-(4.15) for order by order corrections of higher-spin interactions in the holomorphic sector. The obtained equations are of the Vasiliev type in a sense of being based on the zero-curvature condition of a certain large algebra containing HS algebra as a subalgebra. The original Vasiliev algebra is a square of the HS one. It is constructed by introducing the noncommutative variables $z$, while in our case the auxiliary $z$'s commute as the new algebra can be viewed as a contraction of the Vasiliev one. The obtained equations allow us to prove the all order locality of the holomorphic HS interactions as well as to derive a remarkable shift symmetry (5.41), (5.43) of the interaction vertices.

The appearance of a different algebra that underlies (4.12)-(4.15) has been identified in [44] while studying the locality properties of the holomorphic interaction. It was noted there that the homotopy operators introduced for solving the Vasiliev equations which result in the local interactions can be re-interpreted as a one-parameter $\beta$ – re-ordering of the original star product followed by a contraction $\beta \to -\infty$. This procedure effectively leads to a different large algebra in place of the original Vasiliev one and unambiguously prescribes the ordering of its generating elements via star product formula (3.29).

Quite puzzling however is even though some HS vertices were calculated using (3.29) in [44], the naive replacement of the Vasiliev algebra with the contracted one used in this paper at the level of the Vasiliev equations makes no sense beyond lower orders as we explain in section 3.4.1. The reason is certain star products present in the Vasiliev system get ill-defined with star product (3.29) unlike those with the original one (3.14). All that suggests that on star product (3.29) there exists a consistent Vasiliev type system that differs from the original one in some constraints. To check if it is really so, we used the following important observation of [44], [43]. Namely, master fields of the Vasiliev equations that exhibit (spin)-local vertices are invariant under $\beta$ – re-ordering (3.27). Taking it as a definition of the proper functional class along with a natural requirement of being closed on the operations of the generating equations we come up with the following results

- The relevant functions are those given by (3.42). They have a natural grading with respect to degree $dz \equiv \theta$ being zero-, one- and two-forms. Each sector contains important building blocks of HS dynamics. The dynamical fields appear as $d_z$ – cohomologies from $C^0$, while the interaction vertices are generated via a special 1-form $\Lambda_0 \in C^1$ and the Klein 2-form $\gamma \in C^2$.

- Star product (3.14) does not respect this class unlike (3.29), which does, provided $\theta$ – rank of the product is less than two, (3.50), (3.57). Otherwise the product is ill-defined unless one of the multiplier is $z$ – independent. That explains why (3.29) makes no sense on the Vasiliev equations at higher orders yet being applicable at the lower ones.

In constructing our generating equations the latter fact turned out to be of a crucial importance. Unlike the standard Vasiliev formulation, it prevents from higher order corrections to the auxiliary connection $\Lambda$ (4.14) responsible for the consistency. This leaves no room for the zero-form module $B$ that naturally appears in the Vasiliev case, bringing instead a very unusual constraint (4.15). This constraint is seemingly in tension with the HS interpretation of field $C$ as $z$ – independent by definition. Remarkably, (4.15) turned out to be fully consistent with its
z – independence thanks to the curious projector identity (4.9), which rests on our functional class and a very special element from the algebra (4.10). The identity is somehow responsible for projecting away the z – dependence. This observation being a major result of our work plays a central role for the consistency and allows one reproducing (1.4)-(1.5) from the system. On a side note it would be very interesting to understand what makes $\Lambda_0$ so special from the algebraic point of view.

Once the system is written down and is shown to be consistent, we briefly analyze what kind of HS vertices it delivers. Solving our equations using the standard contracting homotopy one is able to see that all holomorphic vertices from (1.4) are ultra-local. Following analysis from [27] this implies the all order space-time locality. Similarly, we find the vertices from (1.5) to be spin-local. These results prove the locality conjecture of [29] in the holomorphic sector and extend the recent analysis [30], [44], [45], [46] to all orders.

Another interesting observation is a shift symmetry of HS vertices (5.41), (5.43). The investigation of that kind of a symmetry was motivated by a remarkable result of [29], where the so called Pfaffian locality theorem was proven that allows one reducing the degree of non-locality using a class of shifted homotopies. Its base is the so called structure lemma that prescribes parametric shifts in the perturbative homotopy operators. The shifts are designed to keep track of the non-local pp contractions within the exponential part of the vertices. We were able to show that this observation has an analog in the form of the exact shift symmetry of our local vertices. Let us also note that in [50] it is shown that the symmetry is a sufficient ingredient for the proof of the $Z –$ dominance conjecture of [29]. It would be interesting to see its CFT dual realization.

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Appendix A. Deriving functional class

Let us prove that (3.43) satisfies the class closure requirements

$$\mathcal{C}^r = \{ \phi(z, y; \theta) : \mathcal{O}_\beta(\phi) = \phi, \forall \beta \},$$

(A.1)

$$\mathcal{C}^{r_1} \star \mathcal{C}^{r_2} \to \mathcal{C}^{r_1 + r_2}, \ r_1 + r_2 < 2$$

(A.2)

$$d_z \mathcal{C}^r \to \mathcal{C}^{r+1},$$

(A.3)

where $\mathcal{O}_\beta$ is given by (3.25) and $r$ counts rank of $\theta –$ degree. Consider condition (A.1). The invariance it places amounts to an integral equation

$$\int \frac{dudv}{(2\pi)^2} \phi((1 - \beta)z + v, y + \beta u; (1 - \beta)\theta) \exp(iuv) = \phi(z, y; \theta).$$

(A.4)
Let us start with zero-forms \( r = 0 \). To solve it we propose the following ansatz

\[
\phi = \int_0^1 d\tau \rho(\tau) \int \frac{du^2 dv^2}{(2\pi)^2} f(\tau z + v, (1 - \tau)(y + u)) e^{i\tau z + iuv},
\]

(A.5)

where \( \rho(\tau) \) is unknown function. Applying \( \mathcal{O}_\beta \) to (A.5) and after some algebra one finds

\[
\mathcal{O}_\beta(\phi) = \int_0^1 d\tau \frac{\rho(\tau)}{(1 - \beta \tau)^2} \int \frac{du^2 dv^2}{(2\pi)^2} f\left(\frac{\tau(1 - \beta)z + v, (1 - \tau)(y + u)}{1 - \beta \tau}\right) e^{i\tau z + iuv}.\]

(A.6)

Making use of the integration variable change

\[
\tau \rightarrow \frac{\tau}{1 - \beta + \beta \tau} \in [0, 1]
\]

(A.7)

one arrives at the same expression as in (A.5) provided the following functional equation on \( \rho \) satisfied

\[
\rho\left(\frac{\tau}{1 - \beta + \beta \tau}\right) = (1 - \beta) \rho(\tau),
\]

(A.8)

which solution up to an arbitrary factor is

\[
\rho(\tau) = \frac{1 - \tau}{\tau}.
\]

(A.9)

It is easy to see that with (A.8) one has \( \mathbb{C}^0 \ast \mathbb{C}^0 \rightarrow \mathbb{C}^0 \). A convenient way seeing this is by using generating functions (3.44) and representation (3.47). Taking

\[
\phi = \int_0^1 d\tau e^{iyA} \circ \frac{1 - \tau}{\tau} e^{i\tau z(y+B)},
\]

(A.10)

one ends up with

\[
\phi_1 \ast \phi_2 = \int_{[0,1]^2} d\tau d\sigma \frac{1}{\sigma(1 - \sigma)} \left(e^{iyA_1} \ast e^{iyA_2}\right) \circ \frac{1 - \tau}{\tau} e^{i\tau z(y+B_{1,2})} \in \mathbb{C}^0,
\]

(A.11)

where

\[
B_{1,2} = \sigma(B_1 + A_2) + (1 - \sigma)(B_2 - A_1).
\]

(A.12)

Recall that (A.10) is considered as the generating function with respect to sources \( A \) and \( B \). Among different integrals it provides we pick only those for which \( \tau - \) pole cancels out (see discussion after (3.44)-(3.46)). In this case 'poles' at \( \sigma = 0, \sigma = 1 \) and \( \tau = 0 \) in (A.11) are fictious just as well. Applying now \( \mathcal{D}_z \) to (A.5) we find

\[
\mathcal{D}_z \phi = \theta \int_0^1 \int \frac{du^2 dv^2}{(2\pi)^2} d\tau(1 - \tau)(y + u)f(\tau z + v, (1 - \tau)(y + u)) e^{i\tau z + iuv},
\]

(A.13)

which belongs to \( \mathbb{C}^1 \) from (3.43) for \( r = 1 \). Thus,

\[
\mathcal{D}_z \mathbb{C}^0 \rightarrow \mathbb{C}^1.
\]

(A.14)

It is straightforward to see that such defined \( \mathbb{C}^1 \) enjoys

\[
\mathcal{O}_\beta(\mathbb{C}^1) = \text{inv}.
\]

(A.15)
Again, using (3.29) one makes sure that
\[ C^0 \ast C^1 \rightarrow C^1, \quad C^1 \ast C^0 \rightarrow C^1. \tag{A.16} \]

Product \( C^1 \ast C^1 \) however is generally ill-defined as it gains \( \tau \) – pole even for the well-defined multipliers which leads to divergency, see e.g., (3.62). Finally, one readily checks that \( d_z C^1 \) results in \( r = 2 \) class from (3.43) which is also \( O_\beta \) – invariant. Being a two-form, its only product which is not identically zero is the one with functions from \( C^0 \). Generally, this product does not exist for \( z \) – dependent \( C^0 \) either, see e.g., (3.60).

Appendix B. Projector identity

Here we sketch the proof of (4.9). We need to calculate \( d_z (f \ast \Lambda_0) \) and \( d_z (\Lambda_0 \ast f) \), where \( f \in C^0 \) and \( \Lambda_0 \) is given in (4.10). A convenient way of doing this is to represent \( \Lambda_0 \) as
\[ \Lambda_0 \rightarrow \theta^\alpha \int_0^1 d\tau \tau z_\alpha e^{i \tau z (y+p)} \quad p = -i \partial, \tag{B.1} \]

where this formula should be understood as a generating one for (4.10) with the help of translation operator \( p \) that acts on \( C \). Similarly, we can take \( f \in C^0, (3.43) \)
\[ f(z, y) = \int_0^1 d\tau \frac{1 - \tau}{\tau} \int \frac{du dv}{(2\pi)^2} \psi(\tau z + v, (1 - \tau)(y + u)) e^{i \tau z y + iuv} \tag{B.2} \]

using its Taylor representation for
\[ \psi(z, y) \rightarrow e^{iyA+izB}, \quad B = i \partial_1, \quad A = i \partial_2, \tag{B.3} \]
\[ \psi(z, y) \equiv e^{iyA+izB} \psi(0, 0), \tag{B.4} \]

where \( \partial_1, 2 \) act on the first and the second argument of \( \psi \) correspondingly. Doing \( uv \) integral in (B.2) with \( \psi \) from (B.3) and using symbol \( \circ \) from (3.48) we have
\[ f \rightarrow \int_0^1 d\tau \frac{1 - \tau}{\tau} e^{iyA+izB} \circ e^{i \tau z (y+B)}, \tag{B.5} \]

where we recall that we consider regular functions \( f(z, y) \) only for which \( \tau \) – pole cancels out (see below (3.44)-(3.46)). Now, it is not difficult to come up with the following generating expressions
\[ \Lambda_0 \ast f \rightarrow \theta^\alpha \int_{[0,1]^2} d\tau d\sigma \frac{\sigma}{1 - \sigma} e^{iyA+izB} \circ \tau z_\alpha e^{i \tau z (y+\sigma(p+A)+(1-\sigma)B)}, \tag{B.6} \]
\[ f \ast \Lambda_0 \rightarrow \theta^\alpha \int_{[0,1]^2} d\tau d\sigma \frac{\sigma}{1 - \sigma} e^{iyA+izB} \circ \tau z_\alpha e^{i \tau z (y+\sigma(p-A)+(1-\sigma)B)}, \tag{B.7} \]

which are obtained directly by using (3.29) along with an appropriate change of variable \( \tau \). The last step is to check that by applying \( d_z \) to (B.6) and (B.7) the result amounts to \( F_\pm(y) \ast \gamma \), (4.11). To this end we observe the following identity
\[ d_z \int_0^1 d\tau \phi(y) \circ \tau \theta^\alpha z_\alpha e^{i \tau z (y+q)} = \phi(-q) e^{-iyq} \ast \gamma, \tag{B.8} \]
which holds for any $\phi(y)$ and $q$. Eq. (B.8) is not hard to prove by using the two-component Schouten identity
\[
\theta^\alpha \theta^\beta = \frac{1}{2} \theta^\gamma \theta^\gamma \epsilon^{\alpha\beta},
\] (B.9)
which allows one to get to the following expression upon carrying out $\circ$ – product
\[
d_z \int_0^1 d\tau \phi(y) \circ \tau \theta^\alpha z_\alpha e^{i\tau z(y+q)} = \frac{1}{2} \theta^\gamma \epsilon^{\alpha\gamma} \int_0^1 d\tau \partial_\gamma \left( \tau^2 \phi(x) e^{-izx} \right), \quad x = (1-\tau)y-\tau q. \quad (B.10)
\]
The integral over $\tau$ is given by a total derivative which reduces the integration to the value at $\tau = 1$ thus proving (B.8). It remains to plug (B.6) and (B.7) into (B.8) and fold up the generating functions (B.1) and (B.5) back to obtain (4.11). In doing so we observe that substitution of (B.7) into (B.8) upon changing $\sigma = 1 - \tau$ can be rewritten in the following form
\[
d_z (\Lambda_0 \ast f) = \left. \left( \int_0^1 d\tau \frac{1-\tau}{\tau} e^{i(1-\tau)(-y-p)A+iz(-y-p+B)+(1-\tau)BA} \right) \right|_{z=-y} \ast \gamma. \quad (B.11)
\]
Now we see that due to (B.5) the integral above is
\[
\int_0^1 d\tau \frac{1-\tau}{\tau} e^{i(1-\tau)(-y-p)A+iz(-y-p+B)+(1-\tau)BA} = \int_0^1 d\tau \frac{1-\tau}{\tau} e^{i(-y-p)A+iBA} \circ e^{iz(-y-p+B)} \rightarrow f(z, -y - p). \quad (B.12)
\]
Finally, one is left to note that with prescription $C(y) \rightarrow e^{-iyp}$ and using $y$ – star product (1.2) the result indeed amounts to (4.11)
\[
f(-z, -y - p)|_{z=y} e^{-iyp} \rightarrow (C(y) \ast_y f(-z, -y))|_{z=y}. \quad (B.14)
\]
The case of $d_z (f \ast \Lambda_0)$ is reached analogously.

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