Global potential function on complete special holonomy manifolds

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Abstract

In this article, we introduce and study the notion of a complete special holonomy manifold which given by a global potential function. We establish some vanishing theorems on the $L^2$ harmonic forms under some growth assumptions on the global potential function.

Keywords. $L^2$ harmonic form; $G_2$-manifold; $Spin(7)$-manifold; global potential function

1 Introduction

Let $X$ be a $C^\infty$-manifold equipped with a differential form $\omega$. This form is called parallel if $\omega$ is preserved by the Levi-Civita connection: $\nabla \omega = 0$. This identity gives a powerful restriction on the holonomy group $\text{Hol}(X)$. In Kähler geometry the parallel forms are the Kähler form and its powers. The algebraic geometers obtained many results of topological and geometric on studying the corresponding algebraic structure. In $G_2$- or $Spin(7)$-manifold the parallel form is the $G_2$- or $Spin(7)$-structure.

In [25], the author had generalized some of these results on Kähler manifolds to other manifolds with a parallel form, especially the parallel $G_2$-manifolds. The results which obtained on [25] can be summarized as Kähler identities for $G_2$-manifolds.

The theory of $G_2$-manifolds is one of the places where mathematics and physics interact most strongly [16, 18]. In string theory, $G_2$-manifolds are expected to play the same role as Calabi-Yau manifolds in the usual A- and B-models of type-II string theories. There are many results on the constructed of $G_2$-manifolds [11, 14, 15, 17]. Hitchin constructed a geometry flow [10] which physicists called Hitchin’s flow, it turned out to be extremely important in string physics.

The study of $L^2$ harmonic forms on a complete special holonomy manifold is a very interesting and important subject; it also has numerous applications in the field of Mathematical Physics, See for example [9]. In Kähler geometry (holonomy $U(n)$) the parallel forms are the Kähler form $\omega$. 

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and its powers. Studying the corresponding algebraic structures, the algebraic geometers amassed an amazing wealth of topological and geometric information. There are many vanishing results on Kähler geometry. The first general result in the non-compact case is due to Donnelly-Fefferman [6]. If $X$ is a strongly pseudoconvex domain in $\mathbb{C}^n$, they showed in [6] that $\mathcal{H}^{p,q}_{(2)}(X)$, $p + q \neq n$, if $\omega$ is the Bergman metric. In [7], Gromov introduced the notion of Kähler hyperbolicity and established the vanishing of $\mathcal{H}^{p,q}_{(2)}(X)$, outside the middle dimension, for any $(X, \omega)$ which is Kähler hyperbolic and which covers a compact manifold. In [5, 13], the authors proved that $\mathcal{H}^{p,q}_{(2)}(X) = 0$, $p + q \neq n$, if $\omega = d\alpha$ with $||\alpha||_{L^\infty(X)}$ growing slower than the Riemannian distance associated to $\omega$. If we assume that $\omega$ is given by a global potential function, i.e., there is a $\lambda \in \mathcal{C}^2(X)$ such that

$$\omega = i\partial\bar{\partial}\lambda = \frac{1}{2}dd^C\lambda,$$

where $d^C := [L_\omega, d^*] = -i(\partial - \bar{\partial})$. In [19, 20], McNeal proved two vanishing on $\mathcal{H}^{p,q}_{(2)}(X)$ when $p + q \neq n$, under some growth assumptions on the global potential function $f$.

For the case of complete $G_2$- (or $\text{Spin}(7)$-)manifold $X$, it well-known that $\mathcal{H}^{i}_{(2)}(X) = 0$, $i = 0, 1$, since $X$ is Ricci-flat. The author in [11] proved that $\mathcal{H}^2_{(2)}(X) = 0$ if the structure from $\omega = d\alpha$ with $||\alpha||_{L^\infty(X)}$ growing slower than the Riemannian distance associated to the metric $g_\omega$ induced by $\omega$.

Inspired by Kähler geometry, we define some complete manifolds $X$ equipped with a parallel differential form $\omega$, See Definition 4.1. It given by a global potential function $f$, i.e., there is a $f \in \mathcal{C}^2(X)$ such that

$$\omega = (-1)^C dd_C\omega = dd^*(f\omega),$$

where $d_C$ is the twisted de Rham operator of $(X, \omega)$, See [25] Definition 2.3. The main purpose of this article is to prove some vanishing results on $\mathcal{H}^{p,q}_{(2)}(X)$, under some growth assumptions on the global potential function $f$. The crucial condition is pointwise comparison $df$ and $f$ itself, See Definition 4.6.

**Example 1.1.** There are some trivial examples of $G_2$- and $\text{Spin}(7)$-manifolds satisfy the growth conditions required.

(i) Let $(X, \omega, \Omega)$ be a nearly Kähler 6-fold, see [23, 24]. There is a $(3, 0)$-form $\Omega$ with $|\Omega| = 1$, and

$$d\omega = 3\lambda \text{Re}\Omega, \quad dIm\Omega = -2\lambda\omega^2,$$

where $\lambda$ is a nonzero real constant. For simply, we choose $\lambda = 1$. Denote by $C(X)$ the Riemannian cone of $(X, g)$. The Riemannian cone $(C(X), dr^2 + g)$ is a $G_2$-manifold with torsion-free $G_2$-structure $\phi$ defined by

$$\phi := r^2\omega \wedge dr + r^3\text{Re}\Omega.$$

We denote $f = \frac{1}{6}r^2$, thus $df = \frac{1}{3}rdr$. In a direct calculate,

$$L_{df}\phi = di_{df}\phi = d\left(\frac{1}{3}r^3\omega\right) = \phi.$$
Therefore the Riemannian cone $C(X)$ given by a global potential $\frac{1}{6}r^2$.

(ii) Let $(X, \phi)$ be a nearly parallel $G_2$-manifold. See [12]. There is a 3-form $\phi$ with $|\phi|^2 = 7$ such that
\[ d\phi = 4 \ast \phi. \]

Then the Riemannian cone $(C(X), dr^2 + g)$ is a $Spin(7)$-manifold with $Spin(7)$-structure $\Phi$ defined by
\[ \Phi := r^3 dr \wedge \phi + r^4 \ast \phi. \]

We denote $f = \frac{1}{8}r^2$, thus $df = \frac{1}{4}rdr$. In a direct calculate,
\[ L_{df}\Phi = d_{df}\Phi = d\left( \frac{1}{4}r^4\phi \right) = \Phi. \]

Therefore the Riemannian cone $C(X)$ given by a global potential $\frac{1}{8}r^2$.

Suppose that the $G_2$ (or $Spin(7)$-) structure form is given by a global potential function $f$. In this article, we will prove two vanishing theorem on $H^k(X)$ under some growth assumptions on the function $f$.

(I) Suppose also that $f$ is dominated by a constant times $f$, then $H^2(X) = 0$.

**Theorem 1.2.** Suppose $(X, \omega)$ is a complete $G_2$- (or $Spin(7)$-) manifold given by a global potential function $\omega = (-1)^6ddCf$ for some $f \in \Lambda^0(X)$, $f \geq 1$. Suppose, also that (1) $f$ dominates $df$ and, (2) $f$ is an exhaustion function on $X$. Then
\[ H^2(X) = 0. \]

(II) Suppose also that the constant $B$ as in Definition 4.6 is small enough, we obtain a lower bound on $(\Delta u, u)$ for $\Lambda^k(X)$, $k = 0, 1, 2$.

**Theorem 1.3.** Suppose $(X, \omega)$ is a complete $G_2$- (or $Spin(7)$-) manifold given by a global potential $\omega := (-1)^6ddCf$ for some $f \in \Lambda^0(X)$, $f \geq 1$. Suppose that $f$ dominates $df$. Then there is a positive constant $\delta \in (0, 1)$ with following significance. If $B \leq \delta$, there exist constants $m, M$ depending only on universal constants and the constants $A, B$ such that
\[ m \int_X \frac{1}{f + M} |u|^2 \leq (\|du\|^2 + \|d^*u\|^2), \forall u \in \Lambda^2_0(X), \] \[ (1.1) \] In particular, $H^2(X) = 0$.

As we derive estimates our article, there will be many constants which appear. Sometimes we will take care to bound the size of these constants, but we will also use the following notation whenever the value of the constants are unimportant. We write $\alpha \lesssim \beta$ to mean that $\alpha \leq C\beta$ for some positive constant $C$ independent of certain parameters on which $\alpha$ and $\beta$ depend. The parameters on which $C$ is independent will be clear or specified at each occurrence. We also use $\beta \lesssim \alpha$ and $\alpha \approx \beta$ analogously.
2 Preliminaries

2.1 $L^2$-harmonic forms

We recall some basic on $L^2$ harmonic forms [3, 4]. Let $M$ be a smooth manifold of dimension $n$, let $\Lambda^k(M)$ and $\Lambda^k_0(M)$ denote the smooth $k$-forms on $M$ and the smooth $k$-forms with compact support on $M$, respectively. We assume now that $M$ is endowed with a Riemannian metric $g$. Let $\langle \cdot, \cdot \rangle$ denote the pointwise inner product on $\Lambda^k(M)$ given by $g$. The global inner product is defined

$$(\alpha, \beta) = \int_M \langle \alpha, \beta \rangle dVol_g.$$ 

We also write $|\alpha|^2 = \langle \alpha, \alpha \rangle$, $\|\alpha\|^2 = \int_M |\alpha|^2 dVol_g$, and let

$$\Lambda^k_0(M) = \{ \alpha \in \Lambda^k(M) : \|\alpha\|^2 < \infty \}.$$ 

The operator of exterior differentiation is $d : \Lambda^k_0(M) \to \Lambda^k_0(M)$ and it satisfies $d^2 = 0$; its formal adjoint is $d^* : \Lambda^{k+1}_0(M) \to \Lambda^k_0(M)$; we have

$$\forall \alpha \in \Lambda^k_0(M), \forall \beta \in \Lambda^{k+1}_0(M), \int_M \langle d\alpha, \beta \rangle = \int_M \langle \alpha, d^* \beta \rangle.$$ 

We consider the space of $L^2$ closed forms

$$Z^k_2(M) = \{ \alpha \in \Lambda^k_0(M) : d\alpha = 0 \},$$ 

where it is understood that the equation $d\alpha = 0$ holds weakly that is to say

$$\forall \beta \in \Lambda^k_0(M), \langle \alpha, d^* \beta \rangle = 0.$$ 

That is we have

$$Z^k_2(M) = (d^*(\Lambda^{k+1}(M)))^\perp.$$ 

We can also define

$$\mathcal{H}^k_{(2)}(M) = (d^*(\Lambda^{k+1}(M)))^\perp \cap (d^*(\Lambda^{k-1}(M)))^\perp = Z^k_2(M) \cap \{ \alpha \in \Lambda^k_0(M) : d^* \alpha = 0 \} = \{ \alpha \in \Lambda^k_0(M) : d\alpha = d^* \alpha = 0 \}.$$ 

Because the operator $d + d^*$ is elliptic, we have by elliptic regularity: $\mathcal{H}^k_{(2)}(M) \subset \Lambda^k(M)$. The space $\Lambda^k_0(M)$ has the following of Hodge-de Rham-Kodaira orthogonal decomposition

$$\Lambda^k_0(M) = \mathcal{H}^k_{(2)}(M) \oplus \overline{d(\Lambda^{k-1}_0(M))} \oplus \overline{d^*(\Lambda^{k+1}_0(M))},$$ 

where the closure is taken with respect to the $L^2$ topology.
2.2 Estimates on Ricci-flat manifolds

Let $M$ be an oriented smooth Riemannian manifold of dimension $n$. In a local orthonormal frame $\{x_i\}_{i=1}^n$, the metric $ds^2 = g_{ij}dx^i dx^j$, the Laplace operator
\[
\Delta = -\frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (\sqrt{g} g^{ij} \frac{\partial}{\partial x^i}) = (d + d^*)^2,
\]
where $g^{ij} = (g^{-1})_{ij}$, $g = \det(g_{ij})$.

**Theorem 2.1.** ([21] Theorem 1.2) Let $M$ be a complete Ricci-flat Riemannian manifold. If $u \geq 0$ satisfies
\[
\Delta u \leq 0.
\]
Then
\[
\sup_{B(R)} u^2 \leq \frac{C_1}{Vol B(R)} \int_{B(R)} u^2,
\]
where $C_1$ is a positive constant.

**Proposition 2.2.** If the function $f$ over a complete Ricci-flat manifold $M$ satisfies
\[
\Delta u = C,
\]
where $C$ is a constant. Then
\[
\sup_{B(R)} |du|^2 \leq \frac{C_2}{Vol B(R)} \left( \frac{1}{R^2} \int_{B(2R)} |u|^2 + \int_{B(2R)} |u| \right),
\]
where $C_2$ is a positive constant.

**Proof.** For any $f \in C^\infty(X)$, we have the pointwise identity
\[
\frac{1}{2} d^* d|f|^2 = -|\nabla f|^2 + \langle \nabla^* \nabla f, f \rangle \text{ on } X.
\]
For any $\alpha \in \Lambda^1(X)$, the Weitzenböck formula gives
\[
\nabla^* \nabla(df) = (d^* d + dd^*) df = d(d^* df).
\]
We take $f = u$ in above identity, thus $\nabla^* \nabla (du) = 0$. Therefore,
\[
\Delta |du|^2 = -2|\nabla (du)|^2 \leq 0.
\]
Following Theorem 2.1 we have
\[
\sup_{B(R)} |du|^2 \leq \frac{C_1}{Vol B(R)} \int_{B(R)} |du|^2.
\]
We choose a cut-off function $\psi$ over $M$ such that $|\nabla \psi| \leq \frac{c}{R}$ and
\[
\psi = \begin{cases} 
1 & x \in B(R), \\
0 & x \in X \setminus B(2R),
\end{cases}
\]
We then have
\[
(\Delta u, \psi^2 u) = (\nabla u, 2(\psi \nabla \psi) u) + (\nabla u, \psi^2 \nabla u),
\]
\[
\int_{B(2R)} \psi^2 |\nabla u|^2 \leq -2 \int_{B(2R)} \psi u \nabla \psi \cdot \nabla u + C \int_{B(2R)} \psi^2 u
\leq \frac{1}{2} \int_{B(2R)} \psi^2 |\nabla u|^2 + 2 \int_{B(2R)} u^2 |\nabla \psi|^2 + C \int_{B(2R)} \psi^2 u.
\]
Therefore, we obtain that
\[
\int_{B(R)} |\nabla u|^2 \leq \int_{B(2R)} \psi^2 |\nabla u|^2
\leq 4 \int_{B(2R)} u^2 |\nabla \psi|^2 + 2C \int_{B(2R)} \psi^2 u
\leq \frac{4c^2}{R^2} \int_{B(2R)} |u|^2 + 2|C| \int_{B(2R)} |u|.
\]
The inequality (2.1) follows the inequalities (2.2)–(2.3) and the fact $|du| \leq |\nabla u|$.

**Corollary 2.3.** If $(X, \omega)$ is a complete $G_2$- (or $\text{Spin}(7)$-) manifold given by a global potential function $f$. Suppose also that $f$ satisfies
\[
|f|(x) \leq A + B \rho^2(x),
\]
where $A, B$ are positive constant, $\rho(x) = \text{dist}(x_0, x)$, $x_0$ is a fix point on $X$. Then we have
\[
\sup_{B(\frac{4}{5})} |df|^2 \leq C_0(1 + \frac{1}{R} + R),
\]
where $C_0 = C_0(A, B)$ is a positive constants.

## 3 Riemannian manifolds with a parallel differential form

### 3.1 The structure operator and the twisted differential

In this section, we recall some notations and definitions on differential geometry [25]. Let $X$ be a $C^\infty$-manifold. Given an odd or even from $\alpha \in \Lambda^*(X)$, we denote by $\tilde{\alpha}$ its parity, which is equal to 0 for even forms, and 1 for odd forms. An operator $f \in \text{End}(\Lambda^*(X))$ preserving parity is called even, and one exchanging odd and even forms is odd, $\tilde{f}$ is equal to 0 for even forms and 1 for odd ones.
Given a $C^\infty$-linear map $\Lambda^1(X) \xrightarrow{p} \Lambda^{\text{odd}}(X)$ or $\Lambda^1(X) \xrightarrow{p} \Lambda^{\text{even}}(X)$, $p$ can be uniquely extended to a $C^\infty$-linear derivation $\rho$ on $\Lambda^*(X)$, using the rule

$$
\rho|_{\Lambda^0(X)} = 0, \quad \rho|_{\Lambda^1(X)} = p, \quad \rho(\alpha \wedge \beta) = \rho(\alpha) \wedge \beta + (-1)^{\tilde{\alpha} \tilde{\beta}} \rho(\beta).
$$

Verbitsky gave a definition of the structure operator of $(X, \omega)$ [25, Definition 2.1].

**Definition 3.1.** Let $X$ be a Riemannian manifold equipped with a parallel differential $k$-form $\omega$. Consider an operator $C : \Lambda^1(X) \to \Lambda^{k-1}(X)$ mapping $\alpha \in \Lambda^1(X)$ to $*(\omega \wedge \alpha)$. The corresponding differentiation

$$
C : \Lambda^*(X) \to \Lambda^{*(k-2)}(X)
$$

is called the structure operator of $(X, \omega)$. The parity of $C$ is equal to that of $\omega$.

**Lemma 3.2.** Let $X$ be a Riemannian manifold equipped with a parallel differential $k$-form $\omega$, and $L_\omega$ the operator $\alpha \mapsto \alpha \wedge \omega$. Then

$$
d_C := L_\omega d^* - (-1)^\tilde{C} d^* L_\omega = \{L_\omega, d^*\},
$$

where $d_C$ is the supercommutator $\{d, C\} := dC - (-1)^\tilde{C} Cd$.

We recall some Generalized Kähler identities which proved by Verbitsky [25, Proposition 2.5].

**Proposition 3.3.** Let $X$ be a Riemannian manifold equipped with a parallel differential $k$-form $\omega$, $d_C$ the twisted de Rham operator constructed above, and $d_C^*$ its Hermitian adjoint. Then:

(i) The following supercommutators vanish:

$$
\{d, d_C\} = 0, \quad \{d, d_C^*\} = 0, \quad \{d^*, d_C\} = 0, \quad \{d^*, d_C^*\} = 0.
$$

(ii) The Laplacian $\Delta = \{d, d^*\}$ commutes with $L_\omega : \alpha \mapsto \alpha \wedge \omega$ and it adjoint operator, denoted as $\Lambda_\omega : \Lambda^i(X) \to \Lambda^{i-k}(X)$.

**Corollary 3.4.** (25 Corollary 2.9) Let $(X, \omega)$ be a Riemannian manifold equipped with a parallel differential $k$-form $\omega$, and $\alpha$ a harmonic form on $X$. Then $\alpha \wedge \omega$ is harmonic.

### 3.2 $G_2$-manifolds

We begin with a crash course in $G_2$-geometry, touching upon the basic concepts and facts relevant for this article. For a more thorough and comprehensive discussion we refer to Joyce’s book [15].

Let $V$ be a 7-dimensional vector space equipped with a non-degenerate 3-form $\phi$. Here by non-degenerate we mean that for each non-zero vector $v \in V$ the 2-form $i_v \phi$ on the quotient is $V/\langle v \rangle$ is symplectic. Then $V$ carries a unique inner product $g$ and orientation such that

$$
i_{v_1} \phi \wedge i_{v_2} \phi \wedge \phi = 6g(v_1, v_2) d\text{vol}, \forall v_1 \in V.
$$
An appropriate choice of basis identifies $\phi$ with the model
\[
\phi_0 = dx_{123} + dx_{145} + dx_{167} + dx_{246} - dx_{257} - dx_{347} - dx_{356},
\]
where $dx^{ijk} = dx^i \wedge dx^j \wedge dx^k$ and $\{x_1, \ldots, x_7\}$ are standard coordinates on $\mathbb{R}^7$. The stabiliser of $\phi_0$ in $GL(\mathbb{R}^7)$ is known to be isomorphic to the exceptional Lie group $G_2$.

**Definition 3.5.** A $G_2$-manifold is a 7-manifold $X$ equipped with a torsion-free $G_2$-structure $\phi$, that is
\[
\nabla_{g_\phi} \phi = 0,
\]
where $g_\phi$ is the metric induced by $\phi$.

Under the action of $G_2$, the space $\Lambda^2(X)$ splits into irreducible representations, as follows:
\[
\Lambda^2(X) = \Lambda^2_7(X) \oplus \Lambda^2_{14}(X),
\]
where $\Lambda^i_j$ is an irreducible $G_2$-representation of dimension $j$. These summands can be characterized as follows:
\[
\begin{align*}
\Lambda^2_7(X) &= \{ \alpha \in \Lambda^2(X) \mid *(\alpha \wedge \phi) = 2\alpha \} = \{ *(u \wedge *\phi) : u \in \Lambda^1(X) \}, \\
\Lambda^2_{14}(X) &= \{ \alpha \in \Lambda^2(X) \mid *(\alpha \wedge \phi) = -\alpha \} = \{ \alpha \in \Lambda^2(X) \mid \alpha \wedge *\phi = 0 \}.
\end{align*}
\]
We will show that the map $L_\phi : \Lambda^p \rightarrow \Lambda^{p+2}$ on the complete $G_2$-manifold is injective for $p = 0, 1, 2$.

**Lemma 3.6.** Let $(X, \phi)$ be a complete $G_2$-manifold. Then any $\alpha \in \Lambda^k(X)$, $k = 0, 1, 2$, satisfies the inequalities
\[
\|\alpha\|_{L^2(X)} \approx \|\alpha \wedge \phi\|_{L^2(X)}.
\]

**Proof.** Let $\alpha, \beta \in \Lambda^0(X)$, we observe that:
\[
(\alpha \wedge \phi) \wedge *(\beta \wedge \phi) = 7\alpha \beta \ast 1.
\]
We take $\beta = \alpha$, then
\[
\|\alpha\|_{L^2(X)}^2 = \frac{1}{7} \|\alpha \wedge \phi\|_{L^2(X)}^2.
\]
Let $\alpha, \beta \in \Lambda^1(X)$, we also observe that:
\[
*(\alpha \wedge \phi) \wedge (\beta \wedge \phi) = 4 \ast \alpha \wedge \beta,
\]
where we use the identity $* (\alpha \wedge \phi) \wedge \phi = -4 \ast \alpha$, See [2]. We take $\beta = \alpha$, then
\[
\|\alpha\|_{L^2(X)}^2 = \frac{1}{4} \|\alpha \wedge \phi\|_{L^2(X)}^2.
\]
Let $\alpha \in \Lambda^2(X)$, we can write $\alpha = \alpha^7 + \alpha^{14}$, then $\alpha \wedge \phi = 2 \ast \alpha^7 - \ast \alpha^{14}$. Hence
\[
\|\alpha \wedge \phi\|_{L^2(X)}^2 = 4 \|\alpha^7\|_{L^2(X)}^2 + \|\alpha^{14}\|_{L^2(X)}^2 \approx \|\alpha\|_{L^2(X)}^2.
\]
3.3 $Spin(7)$-manifolds

In this section we approach $Spin(7)$-geometry by thinking of the 4-form $\Phi$, and not the metric, as the defining structure.

**Definition 3.7.** A 4-form $\Phi$ on an 8-dimensional vector space $W$ is called admissible if there exists a basis of $W$ in which it is identified with the 4-form $\Phi_0$ on $\mathbb{R}^8$ defined by

$$\Phi_0 = dx^{1234} + dx^{1256} + dx^{1357} - dx^{1458} - dx^{1467} - dx^{2358} - dx^{2367} - dx^{2457} + dx^{2468} + dx^{3456} + dx^{3478} + dx^{5678},$$

where $dx^{ijkl} = dx^i \wedge dx^j \wedge dx^k \wedge dx^l$ and $\{x_1, \ldots, x_8\}$ are standard coordinates on $\mathbb{R}^8$. The space of admissible forms on $W$ is denoted by $\mathcal{A}(W)$.

**Definition 3.8.** A $Spin(7)$-structure on an 8-dimensional manifold $X$ is an admissible 4-form $\Phi \in \Gamma((TX)) \subset \Lambda^4(X)$. An 8-manifold together with a $Spin(7)$-structure is called an almost $Spin(7)$-manifold.

It follows that each almost $Spin(7)$-manifold is canonically equipped with a metric $g_\Phi$ and an orientation.

**Definition 3.9.** A $Spin(7)$-manifold is an 8-manifold $X$ equipped with a torsion-free $Spin(7)$-structure $\Phi$, that is

$$\nabla_{g_\Phi} \Phi = 0.$$ 

Under the action of $Spin(7)$, the space $\Lambda^2(X)$ splits into irreducible representations, as follows:

$$\Lambda^2(X) = \Lambda_7^2(X) \oplus \Lambda_{21}^2(X).$$

These summands can be characterized as follows:

$$\Lambda_7^2(X) = \{ \alpha \in \Lambda^2(X) \mid *(\alpha \wedge \Phi) = 3\alpha \},$$

$$\Lambda_{21}^2(X) = \{ \alpha \in \Lambda^2(X) \mid *(\alpha \wedge \Phi) = -\alpha \}.$$ 

We will also show that the map $L_\Phi : \Lambda^p \to \Lambda^{p+4}$ on the complete $Spin(7)$-manifold is injective for $p = 0, 1, 2$.

**Lemma 3.10.** Let $(X, \Phi)$ be a complete $Spin(7)$-manifold. Then any $\alpha \in \Lambda^k(X)$, $k = 0, 1, 2$, satisfies the inequalities

$$\|\alpha\|_{L^2(X)} \approx \|\alpha \wedge \Phi\|_{L^2(X)}.$$

**Proof.** Let $\alpha, \beta \in \Lambda^0(X)$, we observe that:

$$(\alpha \wedge \Phi) \wedge *(\beta \wedge \Phi) = 14\alpha \beta \ast 1,$$

then

$$\|\alpha\|^2_{L^2(X)} = \frac{1}{14} \|\alpha \wedge \Phi\|^2_{L^2(X)}.$$
Let \( \alpha, \beta \in \Lambda^1(X) \), we also observe that:

\[
* (\alpha \wedge \Phi) \wedge (\beta \wedge \Phi) = 4 * \alpha \wedge \beta,
\]

where we use the identity \( * (\alpha \wedge \Phi) \wedge \Phi = 4 \alpha \). We take \( \beta = \alpha \), then

\[
\|\alpha\|_{L^2(X)}^2 = \frac{1}{4} \|\alpha \wedge \Phi\|_{L^2(X)}^2.
\]

Let \( \alpha \in \Lambda^2(X) \), we write \( \alpha = \alpha^7 + \alpha^{21} \), then \( \alpha \wedge \Phi = 3 \alpha^7 - \alpha^{21} \). Hence

\[
\|\alpha \wedge \Phi\|_{L^2(X)}^2 = 9 \|\alpha^7\|_{L^2(X)}^2 + \|\alpha^{21}\|_{L^2(X)}^2 \approx \|\alpha\|_{L^2(X)}^2.
\]

\[\square\]

4 Vanishing theorems

4.1 A global potential function

**Definition 4.1.** Let \((X, \omega)\) be a complete manifold equipped with a non-zero parallel differential form \( \omega \). We call \((X, \omega)\) is a complete manifold given by a global potential, i.e., there is a \( f \in C^2(X) \) such that

\[
\omega = (-1)^g d d^c f.
\]

If we denote \( L_{df} \) by the Lie derivative of the vector field (dual of the 1-form \( df \), we also denote by \( df \)). Then

\[
\omega := L_{df} \omega.
\]

**Remark 4.2.** To defined \( \phi \)-plurisubharmonic function on a calibrated manifold \((X, \phi)\) where \( \text{deg}(\phi) = p \). Harvey and Lawson [8] introduced a second order differential operator \( H^\phi : C^\infty(X) \to \Lambda^p(X) \), the \( \phi \)-Hessian given by

\[
H^\phi (f) = \lambda_\phi (\text{Hess} f),
\]

where \( \text{Hess} f \) is the Riemannian Hessian of \( f \) and \( \lambda_\phi : \text{End}(TX) \to \Lambda^p(X) \) is the bundle map given by \( \lambda_\phi A = D_{A^*}(\phi) \) where \( D_{A^*} : \Lambda^p T^* X \to \Lambda^p T^* X \) is the natural extension of \( A^* : T^* X \to T^* X \) as a derivation.

When the calibration \( \phi \) is parallel there is a natural factorization

\[
H^\phi = d d^\phi,
\]

where \( d \) is the de Rahm differential and \( d^\phi : C^\infty(X) \to \Lambda^{p-1}(X) \) is given by

\[
d^\phi f = i_{\nabla f} \phi.
\]
Proposition 4.3. If \((X, \omega)\) is a complete \(G_2\)- (or \(\text{Spin}(7)\)-) manifold given by a global potential function \(f\). Then \(f\) satisfies
\[
d^*df = C,
\]
where \(C = -\frac{7}{3}\) for \(G_2\)-manifold and \(C = -\frac{7}{2}\) for \(\text{Spin}(7)\)-manifold.

Proof. By the hypothesis of \(G_2\)-manifold, the \(G_2\)-structure form \(\phi\) satisfies
\[
7 = *(\phi \wedge* \phi) = *(dd_f \phi \wedge* \phi) = -* (d^*(f \phi) \wedge* \phi)
\]
\[
= *(d * (df \wedge* \phi) \wedge* \phi)) = *d(* (df \wedge* \phi) \wedge* \phi)
\]
\[
= *d * (3df) = -3d^* df.
\]
Here we use the identity \(*(*(\alpha \wedge* \phi) \wedge* \phi) = 3\alpha \) for \(\alpha \in \Lambda^1(X)\), See [2] (3.5).

By the hypothesis of \(\text{Spin}(7)\)-manifold, the \(\text{Spin}(7)\)-structure form \(\Phi\) satisfies
\[
14 = *(\Phi \wedge* \Phi) = *(dd_f \phi \wedge* \phi) = -* (d^*(f \phi) \wedge* \phi)
\]
\[
= *(d * (df \wedge* \Phi) \wedge* \Phi)) = *d(* (df \wedge* \Phi) \wedge* \Phi)
\]
\[
= *d * (4df) = -4d^* df.
\]
Here we use the identity \(*(*(\alpha \wedge* \Phi) \wedge* \Phi) = 4\alpha \) for \(\alpha \in \Lambda^1(X)\). Thus, we complete this proof.

Let us recall some concepts introduced by Cao-Xavier in [5]. A differential form \(\alpha\) on a complete non-compact Riemannian manifold \((M, g)\) is called \(d\)-(sublinear) if there exist a differential form \(\beta\) and a number \(c > 0\) such that
\[
|\alpha(x)|_g \leq c, \quad |\beta(x)|_g \leq c(1 + \rho(x, x_0)),
\]
where \(\rho(x, x_0)\) stands for the Riemannian distance between \(x\) and a base point \(x_0\) with respect to \(g\).

Jost and Zuö’s theorem stated that if a complete Kähler manifold \(X\) with a \(d\)-(linear) Kähler form \(\omega\), then the only \(L^2\)-harmonic forms lie in the middle dimension. In [5], Cao-Xavier also obtained the same result of Jost-Zuo by another way. We extend the idea of Cao-Xavier’s to the case of Riemannian manifold equipped with a parallel differential form. We then have

Theorem 4.4. ([17] Theorem 2.9) Let \((X, \omega)\) be a Riemannian manifold equipped with a parallel differential \(k\)-form \(\omega\). Suppose also that \(\omega\) is \(d\)-(linear). Then for any \(\alpha \in \mathcal{H}^p_{(2)}(X)\), we have

By the estimates Ricci-flat manifold, we then have

Corollary 4.5. Suppose that \((X, \omega)\) is a complete \(G_2\)- (or \(\text{Spin}(7)\)-) manifold given by a global potential function \(f\). Suppose also that \(f\) satisfies
\[
|f(x)| \leq A + B \rho^2(x),
\]
where \(A, B\) are positive constant, \(\rho(x) = \text{dist}(x_0, x)\), \(x_0\) is a fix point on \(X\). Then
\[
\mathcal{H}^2_{(2)}(X) = 0
\]

Proof. Following Corollary 4.3 and Theorem 4.4, it follows that \(\omega \wedge \alpha = 0\) for any \(\alpha \in \mathcal{H}^k_{(2)}(X) = 0\).
Therefore, combining Lemma 3.6, 3.10, we have \(\alpha = 0\) for \(k = 0, 1, 2\).
4.2 The $L^2$ estimates

In this section, we will prove two vanishing theorem on $\mathcal{H}^k_{(2)}(X)$, Theorem 1.3, Theorem 1.2, along with some related results.

**Definition 4.6.** Let $g \in C^2(X)$ be a function on $X$, $g \geq 1$. $g$ dominates its gradient, or $g$ dominates $dg$, if there exist constant $A \geq 0$ and $B \geq 0$ such that

$$|dg|^2(x) \leq A + Bg(x), \quad \forall x \in X.$$  \hfill (4.1)

**Proposition 4.7.** Let $X$ be a complete Riemannian manifold, $\dim X = n$. Suppose that there is a function $f \in \Lambda^0(X)$, $f \geq 1$ such that

$$\Delta f = C, \quad |df|^2 \leq A + Bf, \quad B < |C|,$$

where $A, B$ are positive constant and $C$ is non-zero constant. Then

$$m \int_X \frac{1}{f + M} |u|^2 \leq \|du\|^2, \quad \forall u \in \Lambda^0(X),$$  \hfill (4.2)

where $M, m$ are positive constants depending on $A, B$. Furthermore, if $X$ is Ricci-flat, then inequality (4.2) is also correct in $\Lambda^1_0(X)$.

**Proof.** If $\lambda$ is smooth function on $X$, we have an inequality

$$\|du + ud\lambda\|^2 = \|du\|^2 + \|ud\lambda\|^2 + (du^2, d\lambda) \geq 0.$$ 

Thus

$$-(u^2, d^*d\lambda) \leq \|du\|^2 + \|ud\lambda\|^2.$$  \hfill (4.3)

Suppose now that $f$ dominates $df$. Replacing $f$ by $\tilde{f} = tf + 1$, $t > 0$ and small, we may assume

(i) $\tilde{f} \geq 1, \quad x \in X$

(ii) $|df|^2 \leq B\tilde{f}, \quad x \in X$,

where $B$ in (ii) above is the constant appearing in Definition 4.6. Fix a $t$ such that (i) and (ii) hold. For notational convenience, we will continue to denote $\tilde{f}$ as just $f$, but unravel this abuse of notation at the end of the proof.

For $\varepsilon > 0$ to be determined, let $\lambda = -\varepsilon \sigma \log f$, where $\sigma = \text{sign}(C)$. Note that

$$-d^*d\lambda = \sigma \frac{\varepsilon d^*df}{f^2} - \sigma \frac{\varepsilon \ast (df \wedge df)}{f^2}.$$ 

It follows that

$$-(u^2, d^*d\lambda) \geq \int_X \varepsilon \frac{(C - B)}{f} |u|^2.$$  \hfill (4.4)

Note also that

$$|d\lambda|^2 = \frac{\varepsilon^2}{f^2} |df|^2 \leq \varepsilon^2 \frac{B}{f}.$$  \hfill (4.5)
Substituting (4.4)-(4.5) into (4.3), we obtain
\[ \int_X \frac{\varepsilon(|C| - B) - \varepsilon B^2}{f} |u|^2 \leq \|du\|^2. \] (4.6)

As \(|C| - B > 0\), choose \(\varepsilon\) so that \(|C| - B - \varepsilon B = \kappa > 0\). It follow from (4.6) that (4.3) holds with \(\tilde{f}\) in place of \(f\) when \(M = 0\) and \(m = \kappa \varepsilon\). Recalling that \(\tilde{f} = tf + 1\), it follows that (4.6) holds for \(f\) with \(m = \kappa \varepsilon\) and \(M = \frac{1}{t}\), which completes the proof.

If \(X\) is Ricci-flat, the Weitzenböck formula gives
\[ \|du\|^2 + \|d^* u\|^2 = \|\nabla u\|^2, \forall u \in \Lambda^1_0(X). \]
Following the Kato inequality \(|\nabla|u|| \leq |\nabla u|\) and (4.2), we have
\[ m \int_X \frac{1}{f + M} |u|^2 \leq \|\nabla|u\||^2 \leq \|\nabla u\|^2 \leq \|du\|^2 + \|d^* u\|^2. \]
We complete this proof.

If \(f\) is not bounded on \(X\), Proposition 4.7 does not imply
\[ \|du\|^2 \geq L\|u\|^2, \forall u \in \Lambda^0_0(X), \] (4.7)
for a positive constant \(L\). The next proposition gives two situations where we can obtain (4.7) with a reasonable estimate on \(L\).

**Proposition 4.8.** Let \(X\) be a complete Riemannian manifold, \(\dim X = n\). Suppose that there is a function \(f \in \Lambda^0(X)\), satisfying (1) \(\Delta f = C, C \neq 0\), and (2) \(|df|^2 \leq A\), i.e., \(B = 0\) in Definition 4.6. Then (4.7) holds with \(L = \frac{C^2}{2A}\).

**Proof.** Let \(\lambda = \varepsilon \sigma f\), where \(\sigma = \text{sign}(C)\). Following (4.3), we obtain that
\[ C \varepsilon \|u\|^2 \leq \|du\|^2 + A \varepsilon^2 \|u\|^2. \]
The constant \(L\) follows directly from above inequality by setting \(\varepsilon = \frac{C}{2A}\).

We shall first prove the stronger (both in terms of conclusions and hypotheses) of the two vanishing results.

**Lemma 4.9.** Let \((X, \omega)\) be a complete \(G_2\)- (or \(\text{Spin}(7)\)-) manifold. If \(u \in \Lambda^2(X)\), we denote \(u = u_1 + u_2\), where \(u_i \in \Lambda^2_1(X)\), then \(\Delta u_i \in \Lambda^2_1(X)\). Furthermore, we have identity
\[ \langle \Delta u, u \rangle = \langle \Delta u_1, u_1 \rangle + \langle \Delta u_2, u_2 \rangle. \]

**Proof.** Let \(u_i \in \Lambda^2_1(X)\), i.e., \(u_i \wedge \omega = c_i \ast u_i\), where \(c_i\) is constant, See Section 3.2 and 3.3. Following Proposition 3.3 the Laplacian \(\Delta = \{d, d^*\}\) commutes with \(L_\omega\). Thus \(\Delta u_i \wedge \omega = \Delta(u_i \wedge \omega) = \Delta \ast c_i u_i = \ast c_i \Delta u_i\), i.e., \(\Delta u_i \in \Lambda^2_1(X)\).
Proof of Theorem 7.3 Over a complete $G_2$- (or $\text{Spin}(7)$-) manifold, $u \in \Lambda^2(X)$ is decomposed into $u = u_1 + u_2$, where $u_1 \in \Lambda^2_7(X)$, $u_2 \in \Lambda^2_{14}(X)$. Hence, we have identities $\ast(u_i \wedge \omega) = c_i \omega$, where $c_1, c_2$ are constant.

For notational convenience, we will continue to denote $f$ as in the proof of Proposition 4.7. We denote $u_i = (-1)^{\tilde{C}} u_i f^{ \frac{-1}{2} }$. Since $u_i$ has compact support, an integration by parts gives

\begin{equation}
(u_i \wedge \omega, d(u_i \wedge \omega)) = (d^\ast(u_i \wedge \omega), u_i \wedge dCf).
\end{equation}

(4.8)

Since $\omega = (-1)^{\tilde{C}} ddCf$, we have

\begin{equation}
(d(u_i \wedge dCf)) = du_i \wedge dCf + u_i \wedge \omega.
\end{equation}

(4.9)

We now substitute (4.9) into (4.8), it gives that

\begin{equation}
(u_i \wedge \omega, u_i \wedge \omega) = - (u_i \wedge \omega, du_i \wedge dCf) - (d^\ast(u_i \wedge \omega), u_i \wedge dCf).
\end{equation}

(4.10)

For the term coming from on the right-hand side of (4.10), the Cauchy-Schwarz inequality implies

\begin{equation}
I \lesssim \int_X \left| u_i \wedge du_i \wedge dCf \right| \lesssim \int_X f^{-1}|u_i||du_i||df| + \int_X f^{-2}|u_i||dCf|^2 \lesssim \int_X |du_i|^2 + \int_X f^{-2}|u_i|^2|df|^2 \lesssim \int_X |du_i|^2 + B \int_X f^{-1}|u_i|^2,
\end{equation}

(4.11)

for constants independent on $k$ and $A, B$ as in Definition 4.6.

For the term coming from on the left-hand side of (4.10), the dominated convergence theorem implies

\begin{equation}
\lim_{k \to \infty} (u_i \wedge \omega, u_i \wedge \omega) = c_i^2 \int_X \frac{u_i^2}{f}.
\end{equation}

(4.12)

Substituting (4.11)–(4.12) into (4.10), it follows that

\begin{equation}
\int_X \frac{u_i^2}{f} \leq C\|du_i\|^2 + CB \int_X \frac{u_i^2}{f}
\end{equation}

(4.13)

where $C$ is a positive constant independent of $k$ and $A, B$. Provide $CB \leq \frac{1}{2}$, rearrangement gives

\begin{align*}
\int_X \frac{u_i^2}{f} &\leq 2\left( \int_X \frac{u_1^2}{f} + \int_X \frac{u_2^2}{f} \right) \\
&\leq 4C(\|du_1\|^2 + \|du_2\|^2) \\
&\leq 4C(\|du\|^2 + \|d^\ast u\|^2)
\end{align*}

where we use the Lemma 4.9.

\qed
The inequalities (1.1) on differential forms has an important application in the following problem: The $L^2$-existence theorem and $L^2$-estimate of the Cartan-De Rham equation

$$dv = u$$

where $u \in L^2(\Lambda^k(X))$ is given $(k + 1)$-form satisfying

$$du = 0.$$

**Proposition 4.10.** Assume the hypothesis on Theorem 1.3. Suppose that $f$ dominates $df$ and the constant $B$ in Definition 4.6 is small enough. Then for any $u \in \Lambda^k(X)$ with $k = 0, 1, 2$ such that (i) $du = 0$ and (ii) $fu \in \Lambda^k(2)$ there exist a solution to $dv = u$ which satisfies the estimate

$$\|v\|^2 \leq C \int_X |u|^2 \cdot (f + M),$$

where the positive constant $C$ depends only on $A, B$.

**Proof.** Our proof here use McNeal’s argument in [19] for $\bar{\partial}$-equation. Let $N = \{\alpha \in \Lambda^k(2) : d\alpha = 0\}$ and $S = \{d^*\beta : \beta \in \Lambda^k_0 \cap N\}$. On $S$ consider the linear functional

$$d^*\beta \rightarrow (\beta, u).$$

Using (1.1), we obtain

$$|\langle \beta, u \rangle| = |\langle \frac{1}{\sqrt{f + M}} \beta, \sqrt{f + Mu} \rangle|$$

$$\leq \left( \int_X \frac{1}{f + M} |\beta|^2 \right)^{1/2} \cdot \left( \int_X (f + M)|u|^2 \right)^{1/2}$$

$$\lesssim \|d^*\beta\| \left( \int_X (f + M)|u|^2 \right)^{1/2}.$$  \hfill (4.14)

Thus the functional is bounded on $S$. However we also have $\langle \beta, u \rangle = 0$ if $\beta \in S^\perp$ since $du = 0$, so (4.14) actually holds for all $\beta \in \Lambda^k_0(X)$. Since $\Lambda^k_0(X)$ is dense in $\text{Dom}(d^*) := \{u \in \Lambda^k(2) : d^*u \in \Lambda^{k-1}_0(2) \}$ in the norm $\|u\|^2 + \|d^*u\|^2$, (4.14) holds for all $\beta \in \text{Dom}(d^*)$. The Hahn-Banach theorem extends the function to all of $\Lambda^k(2)$ and then the Riesz representation theorem gives a $v \in \Lambda^{k-1}(X)$ such that

$$\langle d^*\beta, v \rangle = (\beta, u), \forall \beta \in \text{Dom}(d^*).$$

This is equivalent to $dv = u$, and

$$\|v\| \lesssim \left( \int_X |u|^2 \cdot (f + M) \right)^{1/2},$$

which is the claimed norm estimate. \qed
The second main result of this section is a weaker vanishing theorem for $H^k_1(X)$, one without an estimate from below on the Dirichlet form of $\Delta$. Recall that a function $f$ is an exhaustion function on $X$, i.e.,
\[ X_k = \{ x \in X : f(x) < k \} \subset X, \forall k \in \mathbb{R}. \]

**Proposition 4.11.** Let $(X, \omega)$ be a complete manifold equipped with a non-zero parallel differential form $\omega$. Suppose that $\omega$ given by a global potential function $\omega = (-1)^C d_C f$ for some $f \in \Lambda^0(X)$, $f \geq 1$. Suppose, also that $f$ dominates $df$ and, (2) $f$ is an exhaustion function on $X$. Then for any $h \in H^p_1(X)$, we have
\[ \omega \wedge h = 0. \]

**Proof.** Let $\chi : \mathbb{R} \to \mathbb{R}$ be smooth, $0 \leq \chi \leq 1$ with
\[ \chi(x) = \begin{cases} 
1 & x \geq 1, \\
0 & x \leq 0,
\end{cases} \]
and define, for $k \in \mathbb{N}^+$,
\[ \psi_k(x) = \chi(k - f(x)). \]

Note that $supp \psi_k \subset X_k$ and $\psi_k \equiv 1$ on $X_{k-1}$.

Suppose $k \leq 3$ and let $h \in H^k_1(X)$. By Corollary 3.4, $\omega \wedge h \in H^{k+p}_1(X)$ and so it implies that $\omega \wedge h$ is co-closed. Let $h = (-1)^C d_C f \wedge h$. Since $\psi_k \cdot h$ has compact support, an integration by parts gives
\[ (\omega \wedge h, d(\psi_k \cdot h)) = (d^*(\omega \wedge h), \psi_k \cdot h) = 0. \]
(4.15)

Since $\omega = (-1)^C d_C f$,
\[ d(\psi_k \cdot h) = \psi_k'(k - f) \cdot df \wedge d_C f \wedge h + \psi_k \cdot \omega \wedge h. \]
(4.16)

We now substitute (4.16) into (4.15) and consider the two terms coming from the right-hand side of (4.16) separately. For the first term, the Cauchy-Schwarz inequality and the fact that $\omega$ is bounded in the $\langle \cdot \rangle$ inner product imply
\[ |(\omega \wedge h, \psi_k' \cdot df \wedge d_C f \wedge h)| \lesssim \int_{X_k \setminus X_{k-1}} |df \wedge d_C f| |h|^2 \]
\[ \lesssim \int_{X_k \setminus X_{k-1}} (A + Bf) |h|^2 \]
(4.17)
\[ \lesssim (A + Bk) \int_{X_k \setminus X_{k-1}} |h|^2, \]
for constants independent of $k$ and $A, B$ as in Definition 4.6. The second inequality follows from our hypothesis on $df$. 
The assumption that $h \in H^k(2)(X)$ implies that there exists a subsequence \( \{k_l\} \) such that
\[
k_l \int_{X \setminus X_{k-1}} |h|^2 \to 0 \text{ as } l \to \infty.
\]
(4.18)
Otherwise, for some $c > 0$,
\[
\int_X |h|^2 = \sum_{k=1}^{\infty} \int_{X \setminus X_{k-1}} |h|^2 \geq c \sum_{k=1}^{\infty} \frac{1}{k} = \infty,
\]
a contradiction.

For the term coming from the second term on the right-hand side for (4.16), the dominated convergence theorem implies
\[
\lim_{k \to \infty} (\omega \wedge h, \psi_k \cdot \omega \wedge h) = \|\omega \wedge h\|^2.
\]
(4.19)
Substituting (4.17)–(4.19) into (4.15), it follows that $\omega \wedge h = 0$. $\square$

**Proof of Theorem 1.2** Let $\alpha$ be a $L^2$ $k$-form on $X$. By Proposition 4.11, we have $\omega \wedge \alpha = 0$. Combining Lemma 3.6, 3.10 it gives $\alpha = 0$ for $k = 0, 1, 2$. $\square$

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**References**

[1] R. Bryant, Metrics with exceptional holonomy. *Ann. of Math.* **126**(2), (1987) 525–576.

[2] R. Bryant, Some remarks on $G_2$-structures. *Proceedings of Gökova Geometry-Topology Conference 2005*, 75–109.

[3] G. Carron, $L^2$ harmonic forms on non-compact Riemannian manifolds. *Surveys in analysis and operator theory* (Canberra, 2001), 49–59, Proc. Centre Math. Appl. Austral. Nat. Univ., 40, Austral. Nat. Univ., Canberra, 2002.

[4] G. Carron, $L^2$-cohomology of manifolds with flat ends. *Geom. Funct. Anal. GAFA*, **13**(2), (2003) 366–395.

[5] J. G. Cao, X. Frederico, Kähler parabolicity and the Euler number of compact manifolds of non-positive sectional curvature. *Math. Ann.* **319**, (2001) 483–491.

[6] H. Donnelly, C. Fefferman, $L^2$ cohomology and index theorem for the Bergman metric. *Ann. Math.* **118**, (1983) 593–618.

[7] M. Gromov, Kähler hyperbolicity and $L_2$-Hodge theory. *J. Diff. Geom.* **33**, (1991) 263–292.

[8] F. R. Harvey, H. B. Lawson, An introduction to potential theory in calibrated geometry. *Amer. J. Math.* **131**(4), (2009) 893–944.
[9] N. J. Hitchin, $L^2$ cohomology of hyper-Kähler quotients. *Comm. Math. Phys.* **211**, (2000) 153–165.

[10] N. J. Hitchin, The geometry of three-forms in six and seven dimensions. *J. Diff. Geom.* **55**(3), (2003) 547–576.

[11] T. Huang, $L^2$ harmonic forms on complete special holonomy manifolds. *Ann. Glob. Anal. Geom.* (2019). Doi: 10.1007/s10455-019-09654-z

[12] S. Ivanov, Connections with torsion, parallel spinors and geometry of $Spin(7)$ manifolds. *Math. Res. Lett.* **11**, (2004) 171–186.

[13] J. Jost, K. Zuo, Vanishing theorems for $L^2$-cohomology on infinite coverings of compact Kähler manifolds and applications in algebraic geometry. *Comm. Anal. Geom.* **8**, (2000) 1–30.

[14] D. Joyce, Compact Riemannian 7-manifolds with holonomy $G_2$, I,II. *J. Diff. Geom.* **43**(2), (1996) 291–328. 329–375.

[15] D. Joyce, Compact manifolds with special holonomy, in: Oxford Mathematical Monographs, Oxford University Press, Oxford, 2000.

[16] S. Karigiannis, N. C. Leung, Hodge theory for $G_2$-manifolds: intermediate Jacobians and Abel-Jacobi maps. *Proc. Lond. Math. Soc.* **99**(3), (2009) 297325.

[17] A. Kovalev, Twisted connected sums and special Riemannian holonomy. *J. Reine Angew. Math.* **565**, (2003) 125–160.

[18] J. H. Lee, N. C. Leung, Geometric structures on $G_2$ and $Spin(7)$-manifolds. *Adv. Theor. Math. Phys.* **13**(1), (2009) 131.

[19] J. D. McNeal, $L^2$ harmonic forms on some complete Kähler manifolds. *Math. Ann.* **323**, (2002) 319–349.

[20] J. D. McNeal, A vanishing theorem for $L^2$ cohomology on complete manifolds. *J. Korean. Math. Soc.* **40**(4), (2003) 747–756.

[21] P. Li, R. Schoen, $L^p$ and mean value properties of subharmonic functions on Riemannian manifolds. *Acta Math.* **153**, (1984) 279–301.

[22] R. Schoen, S. T. Yau, Lectures on Differential Geometry. International Press, Cambridge, MA (1994)

[23] M. Verbitsky, An intrinsic volume functional on almost complex 6-manifolds and nearly Kähler geometry. *Pacific J. Math.* **235**(2), (2008) 323–344.

[24] M. Verbitsky, Hodge theory on nearly Kähler manifolds. *Geom. Topo.* **15**, (2011) 2111–2133.

[25] M. Verbitsky, Manifolds with parallel differential forms and Kähler identities for $G_2$-manifolds. *J. Geom. Phys.* **61**(6), (2011) 1001–1016.

[26] R. Wells, Differential Analysis on Complex Manifolds, (Third ed.) Graduate Texts in Math. 65, Springer, New York, Berlin, (2008)