Hurwitz spaces of triple coverings of elliptic curves and moduli spaces of abelian threefolds

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Dedicated to the memory of Fabio Bardelli

Abstract

We prove that the moduli spaces $A_3(D)$ of polarized abelian threefolds with polarizations of types $D = (1,1,2), (1,2,2), (1,1,3)$ or $(1,3,3)$ are unirational. The result is based on the study of families of simple coverings of elliptic curves of degree 2 or 3 and on the study of the corresponding period mappings associated with holomorphic differentials with trace 0. In particular we prove the unirationality of the Hurwitz space $H_{3,A}(Y)$ which parameterizes simply branched triple coverings of an elliptic curve $Y$ with determinants of the Tschirnhausen modules isomorphic to $A^{-1}$.

Introduction

The problem of calculating the Kodaira dimension of the moduli spaces $A_g(d_1,d_2,\ldots,d_g)$ which parameterize abelian varieties of dimension $g$ with polarizations of type $D = (d_1,d_2,\ldots,d_g)$, $d_i|d_{i+1}$ has been a topic of intensive study in the last 20 years. The case of principal polarizations is almost settled. Due to the work of Clemens, Donagi, Tai, Freitag and Mumford it is known that the moduli space $A_g = A_g(1,\ldots,1)$ is unirational if $g \leq 5$ and is of general type if $g \geq 7$. The Kodaira dimension of $A_6$ is unknown. Much work has been devoted to the modular varieties $A_2(1,d)$. Due to the work of Birkenhake, Lange, van Straten, Horrocks and Mumford, Manolache and Schreyer, O’Grady, Gross and Popescu it is known that $A_2(1,d)$ is unirational if $2 \leq d \leq 11$. The interested reader may find a detailed discussion on these and related results in [GP]. Gritsenko proved in [Gri] that $A_2(1,d)$ is not uniruled if $d \geq 13$ and $d \neq 14, 15, 16, 18, 20, 24, 30, 36$. Sankaran proved in [Sa] that $A_2(1,d)$ is of general type if $d$ is a prime number $\geq 173$.

Much less is known about the modular varieties of non-principally polarized abelian varieties of dimension $\geq 3$. Tai proved in [Ta] that $A_g(D)$ is of general type if $g \geq 16$ and $D$...
is an arbitrary polarization type or if \( g \geq 8 \) for certain \( D \). The only result about unirationality of such modular varieties known to the author is due to Bardelli, Ciliberto and Verra who proved in [BCV] that \( A_d(1, 2, 2, 2) \) is unirational. In the recent paper [BL3] Birkenhake and Lange proved that \( A_g(D) \equiv A_g(\hat{D}) \) where \( D = (d_1, d_2, \ldots, d_g) \), \( \hat{D} = (\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_g) \) with \( \hat{d}_i = \frac{d_i d_{i+1}}{d_{i+1} - d_i} \). As a consequence \( A_4(1, 1, 1, 2) \) is unirational. Whether \( A_d(1, 1, 2, 2) \) is unirational seems to be unknown.

In the present paper we consider three-dimensional abelian varieties with polarizations of exponent 2 or 3. We prove that the following modular varieties are unirational: \( A_3(1, 1, 2) \), \( A_3(1, 2, 2) \), \( A_3(1, 1, 3) \) and \( A_3(1, 3, 3) \). By the result of Birkenhake and Lange cited above it suffices to verify the unirationality only of \( A_3(1, 1, 2) \) and \( A_3(1, 1, 3) \). The idea of the proof is the following. We observe that given a covering \( \pi : X \to Y \) of degree \( d = 2 \) or 3, with \( g(X) = 4 \), \( g(Y) = 1 \), the associated Prym variety \( P = \text{Ker}(Nm_\pi : J(X) \to J(Y)) \) has polarization of type \((1, 1, d)\). We consider a smooth elliptic fibration \( Y \to Z \subset \mathbb{P}^1 \) with a section obtained from a general pencil of cubic curves in \( \mathbb{P}^2 \). We construct an open subset \( T \subset \mathbb{P}(\mathbb{H}) \) where \( \mathbb{H} \) is a certain vector bundle over \( Z \) and a family of simple degree \( d \) coverings \( p : \mathcal{X} \to \mathcal{Y}_T = \mathcal{Y} \times_Z T \). With this family one associates the Prym mapping \( \Phi : T \to A_3(1, 1, d) \). We prove that the differential of the Prym mapping \( d\Phi \) is generically surjective and this fact implies the unirationality of \( A_3(1, 1, d) \). Both the construction of \( T \) and the generic surjectivity of \( d\Phi \) are easier when \( d = 2 \). The case \( d = 3 \) requires the use of Prym varieties in a generalized sense, associated with triple coverings. Here we use Miranda’s result which relates triple coverings with rank 2 vector bundles on the base [M1] and Atiyah’s results about vector bundles over elliptic curves [A4]. Defining the Prym mapping and calculating its differential is a necessary work that we do in Section 3 and Section 4 for families of coverings of arbitrary degree. Using the results of the present paper and following the same pattern we prove in [K3] the unirationality of \( A_3(1, 1, 4) \). The modular variety \( A_3(1, 1, 5) \) may be studied in a similar manner which we intend to address elsewhere.

Here is an outline of the content of the paper by sections. Section 1 contains two lemmas which connect coverings of elliptic curves with abelian varieties with polarization of type \((1, \ldots, 1, d)\). In Section 2 we study the Hurwitz space \( \mathcal{H}_{d,n}(Y) \) which parameterizes degree \( d \) coverings \( \pi : X \to Y \) of an elliptic curve \( Y \) simply branched in \( n \geq 2 \) points. The direct summand \( E^\vee \) in the decomposition \( \pi_\ast \mathcal{O}_X \cong \mathcal{O}_Y \oplus E^\vee \) is called the Tschirnhausen module of the covering. Given \( A \in \text{Pic}^n/2Y \) we denote by \( \mathcal{H}_{d,A}(Y) \) the subset of \( \mathcal{H}_{d,n}(Y) \) parameterizing coverings with \( \text{det} E \cong A \). We define and give some simple properties of \( \mathcal{H}_{d,n}(Y) \) and \( \mathcal{H}_{d,A}(Y) \) in \((2.1) - (2.6)\). In \((2.7)\) we focus on triple coverings of elliptic curves and make a dimension count of the number of parameters on which depend triple coverings with a Tschirnhausen module of a given type. In order to make a conclusion about the type of the Tschirnhausen module of a general triple covering in Proposition \((2.9)\) we need the technical result of Lemma \((2.8)\). We then prove in Theorem \((2.10)\) that \( \mathcal{H}_{3,A}(Y) \) is rational if \( \text{deg} A \) is odd and is unirational if \( \text{deg} A \) is even. This result is of independent interest and is analogous to the classically known unirationality of the moduli space of trigonal curves. The way we prove the unirationality of \( \mathcal{H}_{3,n}(Y) \) suggests the construction in Proposition \((2.14)\) of the family of coverings \( p : \mathcal{X} \to \mathcal{Y}_T \) over a rational base \( T \) of dimension 6 which we discussed
in the previous paragraph.

Large part of Section 3 is an overview of polarized Hodge structures of weight one and their variations. Some simple facts we need are usually included as particular cases of more general theorems and also usually only unimodular polarizations are considered. It seems to us appropriate to include in the paper some of the material we use. With every covering \( \pi : X \to Y \) of smooth projective curves one can associate two dual abelian varieties, the Prym variety \( P = \text{Ker}(Nm_\pi : J(X) \to J(Y))^0 \), and its dual \( \hat{P} = \text{Pic}^0 X/\pi^* \text{Pic}^0 Y \) polarized naturally by dual polarizations. Given a family of coverings over a smooth base \( T \) one obtains respectively two morphisms \( \Phi : T \to A(D) \) and \( \hat{\Phi} : T \to A(\hat{D}) \) into moduli spaces of abelian varieties. The morphisms \( \Phi \) and \( \hat{\Phi} \) are constructed by means of variations of Hodge structures of weight one. In Proposition 3.21 we give a multiplicative formula for the differential of \( \Phi \).

Section 4 is devoted to the local study of the Prym mapping. In the first part of the section we consider an arbitrary simple branched covering of smooth, projective curves \( \pi : X \to Y \) with \( g(Y) \geq 1 \) and its minimal versal deformation over the base \( N \times H \). Here \( N \) is the base of a minimal versal deformation of \( Y \) and \( H \) is a product of small disks centered at the branch points of \( \pi \). Considering the spaces of holomorphic differentials with trace 0 one obtains a polarized variation of Hodge structures of weight 1 and a corresponding period mapping \( \tilde{\Phi} : N \times H \to D \) where \( D \) is a period domain biholomorphically equivalent to a Siegel upper half space. In (4.1) – (4.9) we work out various details necessary for obtaining in Proposition 4.9 a formula for the differential \( d\tilde{\Phi} \). In the remaining part of the section we restrict ourselves to the case of elliptic \( Y \). In (4.10) – (4.17) we obtain a geometric criterion in terms of the cover \( X \) in order that the kernel of the differential \( d\tilde{\Phi}(s_0) \), evaluated at the reference point corresponding to \( \pi : X \to Y \), has dimension 1 (the minimal possible). The check of this criterion for double coverings of elliptic curves is easy and is done in Proposition 4.18. The remaining part of the section is devoted to the proof that the criterion is valid for general triple covers of genus 4, a fact needed for the proof of the unirationality of \( A_3(1,1,3) \). We first verify it for a union of two elliptic curves which intersect transversally in 3 points, and then obtain the result for general triple covers of genus 4 by smoothing.

In Section 5 we prove our main results. In Theorem 5.1 we give an alternative proof of a result of Birkenhake and Lange that \( A_2(1,2) \) and \( A_2(1,3) \) are unirational [BL1]. Using the same argument and applying the results of Gritsenko and Sankaran cited above we prove in Theorem 5.2 that if \( d \geq 13 \) and \( d \neq 14, 15, 16, 18, 20, 24, 30, 36 \) and if \( A \in \text{Pic}^2 Y \) is a fixed invertible sheaf then every connected component of the Hurwitz space \( \mathcal{H}_{d,A}(Y) \) corresponding to coverings for which \( \text{Ker} (\pi^* : J(Y) \to J(X)) = 0 \) is not uniruled. Moreover every connected component of \( \mathcal{H}_{d,A}(Y) \) is of general type if \( d \) is prime and \( d \geq 173 \). Finally in Theorem 5.3 we prove that \( A_3(1,1,2) \) and \( A_3(1,1,3) \) are unirational.

An argument of Section 4 uses that degenerating coverings of smooth projective curves into a covering of reduced curves the limit of the trace mapping of holomorphic differentials equals the trace mapping of regular differentials. As discussed in [Li] p.7 such a statement does not seem to follow from relative duality. We give a proof of the statement we need in Appendix A.
In Appendix B we prove the openness of the stability, the semistability and the regular polystability conditions for families of vector bundles over a family of elliptic curves. These are well-known facts that we use but for which we could not find a reference.

**Notation and conventions.** We use the term morphism only in the category of schemes. When working with complex analytic spaces we use the term holomorphic mapping. Unless otherwise specified we make distinction between locally free sheaves and vector bundles and we denote differently their projectivizations. If $E$ is a locally free sheaf of $Y$ and if $E$ is the corresponding vector bundle, i.e. $E \cong O_Y(E)$, then $P(E) := \text{Proj}(S(E)) \cong \mathbb{P}(E^*)$. A morphism (or holomorphic mapping) $\pi : X \to Y$ is called covering if it is finite, surjective and flat. If $X$ and $Y$ are smooth, then finiteness and surjectivity imply flatness (see e.g. [Mat] p.179 and [Fi] p.158). A covering of irreducible projective curves $\pi : X \to Y$ of degree $d$ is called simple if $X$ and $Y$ are smooth and for each $y \in Y$ one has $d-1 \leq \# \pi^{-1}(y) \leq d$. All schemes are assumed separated of finite type over the algebraically closed base field. Unless otherwise specified, or clear from the context, curve means integral scheme of dimension one. Unless otherwise specified we assume the base field $k = \mathbb{C}$.

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## 1 Prym varieties of coverings of elliptic curves

Let $\pi : X \to Y$ be a covering of smooth, projective curves of degree $d \geq 2$, suppose $g(Y) \geq 1$. Let $P = \text{Ker}(Nm_{\pi} : J(X) \to J(Y))^0$ be the Prym variety of the covering. Let $\Theta$ be the canonical polarization of $J(X)$ and let $\Theta_P$ be its restriction on $P$. We give a proof of the following fact stated in [BNR] Remark 2.7.

**Lemma 1.1.** The following three conditions are equivalent: $\pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ is surjective; $\pi^* : J(Y) \to J(X)$ is injective; $\text{Ker}(Nm_{\pi})$ is connected. Suppose these conditions hold and let $P = \text{Ker}(Nm_{\pi})$. Then the polarization $\Theta_P$ is of type $(1, \ldots, 1, d, \ldots, d)$ where the $d$’s are repeated $g(Y)$ times.

**Proof.** To prove the first statement we may look at these abelian varieties as real tori: $J(X) \cong H_1(X, \mathbb{R})/H_1(X, \mathbb{Z})$, $\text{Pic}^0X \cong H^1(X, \mathbb{R})/H^1(X, \mathbb{Z})$ and similar isomorphisms hold for $Y$. Then $\text{Ker}(Nm_{\pi})/\text{Ker}(Nm_{\pi})^0 \cong H_1(Y, \mathbb{Z})/\pi_*H_1(X, \mathbb{Z})$ The dual finite group $\subset H^1(Y, \mathbb{R})/H^1(Y, \mathbb{Z})$ equals $\text{Ker}(\pi^* : \text{Pic}^0Y \to \text{Pic}^0X)$. Let us prove now the second statement. Let $J(X) = V/\Lambda$. Let $B = \pi^*J(Y)$. We have $J(X) = B + P$. Let $V = V_B \oplus V_P$ be the
interested in simple ramified coverings. Let \( R \) be the Riemann form of the polarization \( \Theta \), \( E|_\Lambda = (\cdot, \cdot)_X \), let \( E_B \) and \( E_P \) be the restrictions on \( V_B \) and \( V_P \) respectively. The sublattices \( \Lambda_B \) and \( \Lambda_P \) are primitive and it is a standard fact that \( \Lambda_B^*/\Lambda_B \cong \Lambda_P^*/\Lambda_P \) where the duals are taken with respect to \( E_B, E_P \). By hypothesis \( \pi^* : J(Y) \to B \) is an isomorphism and by the projection formula \( (\pi^*\lambda, \pi^*\mu)_X = d(\lambda, \mu)_X \). Thus \( E_B|_{\Lambda_B} \) is \( d \) times a unimodular form, so \( \Lambda_B^*/\Lambda_B \cong \Lambda_P^*/\Lambda_P \cong (\mathbb{Z}/d\mathbb{Z})^{2g(Y)} \).

**Lemma 1.2.** Let \( \pi : X \to Y \) be a covering of smooth, projective curves of degree \( d \), let \( g(Y) = 1 \) and let \( g(X) \geq 2 \). Let \( P = \text{Ker}(Nm_\pi : J(X) \to J(Y))^0 \) and \( \Theta_P = \Theta|_P \) be as above. Let \( d_2 = |H_1(Y, \mathbb{Z}) : \pi_*H_1(X, \mathbb{Z})| = |\text{Ker} \pi^* : J(Y) \to J(X)| \). Then \( \pi \) may be decomposed as \( X \xrightarrow{\pi_1} \tilde{Y} \xrightarrow{\pi_2} Y \) where \( \pi_2 \) is an isogeny of degree \( d_2 \) and \( \text{deg} \pi_1 = d_1 = \frac{d}{d_2} \). The type of the polarization \( \Theta_P \) is \((1, \ldots, 1, d_1)\). In particular if \( d \) is prime then \( \text{Ker}(Nm_\pi : J(X) \to J(Y)) \) is connected and the type of \( \Theta_P \) is \((1, \ldots, 1, d)\).

**Proof.** The decomposition \( X \xrightarrow{\pi_1} \tilde{Y} \xrightarrow{\pi_2} Y \) is clear and furthermore \( \pi_{1*} : H_1(X, \mathbb{Z}) \to H_1(\tilde{Y}, \mathbb{Z}) \) is surjective. The statements of the lemma follow thus from Lemma 1.1.

Let \( Y \) be an elliptic curve. Let \( X \) be a smooth, irreducible, projective curve which is a cover \( \pi : X \to Y \) of degree \( d \) simply branched at \( B \subset Y \). By Hurwitz' formula \( g = g(X) = \#B + 1 \). Let \( Nm_\pi : J(X) \to J(Y) \) be the induced map of the Jacobians. Then \( P = \text{Ker}(Nm_\pi)^0 \) is an abelian variety of dimension \( g - 1 = \#B - \frac{1}{2} \). A composition of \( \pi \) with an automorphism of \( Y \) does not change \( P \subset J(X) \). Counting parameters we see that there are two cases in which one might obtain a generic abelian \((g - 1)\)-fold with polarization of type \((1, \ldots, 1, d)\) by this construction.

**Case A.** \( \dim P = 2 \). Here \( \dim \mathcal{A}_2(1, d) = 3 \). One obtains the same number of moduli by fixing \( Y \), varying 4 branch points of simple \( d \)-sheeted coverings and subtracting 1 for the action of \( \text{Aut}(Y) \).

**Case B.** \( \dim P = 3 \). Here \( \dim \mathcal{A}_3(1, 1, d) = 6 \). This number of moduli is obtained by varying \( Y \), varying the 6 branch points of simple \( d \)-sheeted coverings and subtracting 1 for the action of \( \text{Aut}(Y) \).

The reader is referred to Corollary 5.4 for results about the representation of general abelian surfaces and general abelian threefolds (with appropriate polarizations) as Prym varieties of coverings of elliptic curves.

### 2 Hurwitz spaces of triple coverings of elliptic curves

Let \( \pi : X \to Y \) be a covering of an elliptic curve \( Y \) of degree \( d \). We say that \( \pi' : X' \to Y \) is equivalent to \( \pi \) if there is an isomorphism \( f : X \to X' \) such that \( \pi = \pi' \circ f \). We are mainly interested in simple ramified coverings. Let \( R \subset X \) be the ramification locus and \( B \subset Y \) be the discriminant locus bijective to \( R \). We have \( \#B = n = 2e \).
Lemma 2.1. Suppose \( d \geq 2 \). Let \( n = 2e \geq 2 \) and let \( B \subset Y \) be a subset of \( n \) points. Then there exists a simple covering \( \pi : X \to Y \) of degree \( d \) branched in \( B \) with monodromy group \( S_d \).

**Proof.** Let \( B = \{b_1, \ldots, b_n\} \) and let \( y_0 \in Y - B \). The fundamental group of \( Y - B \) is isomorphic to\( \pi_1(Y - B, y_0) \cong \langle \alpha_1, \ldots, \alpha_n, \gamma, \delta \mid \alpha_1 \cdots \alpha_n \gamma \delta^{-1} \delta^{-1} = 1 \rangle \) where \( \alpha_i \) is a simple loop around \( b_i \). Consider the homomorphism \( m : \pi_1(Y - B, y_0) \to S_d \) defined by:

\[
m(\alpha_1) = (12), \ldots, m(\alpha_n) = (12), m(\gamma) = (12 \ldots d), m(\delta) = (12 \ldots d)^{-1}.
\]

Since \( (12) \) and \( (12 \ldots d) \) generate \( S_d \) by Riemann’s existence theorem \( m \) is the monodromy homomorphism of a connected simple covering \( \pi : X \to Y \) of degree \( d \) branched at \( B \). \( \square \)

2.2. The set of equivalence classes of simple coverings of \( Y \) of degree \( d \) branched at \( n \geq 2 \) points is parameterized by the Hurwitz space \( H_{d,n}(Y) \) (see e.g. [M]). The Hurwitz space is an étale cover of \( Y^{(n)} - \Delta \), where \( \Delta \) is the codimension one subvariety consisting of nonsimple divisors of degree \( n \). So it is smooth equidimensional of dimension \( n \). We denote by \( p : \mathcal{X} \to Y \times H_{d,n}(Y) \) the universal family of simple \( d \)-sheeted coverings branched in \( n \) points. It has the following properties:

- \( \mathcal{X} \) is smooth;
- \( p : \mathcal{X} \to Y \times H_{d,n}(Y) \) is a covering of degree \( d \) and the composition \( \pi_2 \circ p : \mathcal{X} \to H_{d,n}(Y) \) is a smooth, proper morphism with connected fibers;
- for every simple \( d \)-sheeted covering \( \pi : X \to Y \) branched in \( n \) points there is a unique \( s \in H_{d,n}(Y) \) such that \( p_s : \mathcal{X}_s \to Y \times \{s\} \) is equivalent to \( \pi : X \to Y \).

The connection between triple coverings and rank 2 bundles is due to R. Miranda [M]. We recall some basic facts taken from [CE]. Let \( \pi : X \to Y \) be a covering of degree \( d \) of smooth, projective curves. The Tschirnhausen module of the covering is the quotient sheaf \( E^\vee \) defined by the exact sequence

\[
0 \longrightarrow \mathcal{O}_Y \xrightarrow{\pi^*} \pi_* \mathcal{O}_X \longrightarrow E^\vee \longrightarrow 0.
\]

One has \( E^\vee \cong \ker(\text{Tr}_x : \pi_* \mathcal{O}_X \to \mathcal{O}_Y) \) and this is a locally free sheaf of rank \( d - 1 \). There is a canonical embedding \( i : X \to \mathbf{P}(E) \) and \( i^* \mathcal{O}_{\mathbf{P}(E)}(1) \cong \omega_{X/Y} \cong \omega_X \otimes (\pi^* \omega_Y)^{-1} \) (see [CE] p.448).

Lemma 2.3. Let \( R \) be the ramification divisor of \( \pi : X \to Y \). Then \( \deg R = 2 \deg E \).
Proof. One has $\chi(\mathcal{O}_X) = \chi(\pi_*\mathcal{O}_X) = \chi(\mathcal{O}_Y) \oplus \chi(E^\vee)$. By Riemann-Roch $\chi(E^\vee) = \deg E^\vee + (d-1)(1-g(Y))$. Therefore

$$1 - g(X) = \deg E^\vee + d(1-g(Y)).$$

The equality $2 \deg E = \deg R$ follows thus from Hurwitz’ formula. \hfill \Box

2.4. Let $X \to Y \times \mathcal{H}_{d,n}(Y)$ be the universal family of simple coverings of degree $d \geq 2$ branched in $n = 2e$ points. Let $E^\vee$ be the corresponding Tschirnhausen module. Let $A = \det \mathcal{E}$. By Lemma 2.3 for every $z \in \mathcal{H}_{d,n}(Y)$ one has $\deg(A_z) = e$. We obtain a morphism $h : \mathcal{H}_{d,n}(Y) \to \text{Pic}^eY$, $h(z) = A_z$. Let $A \in \text{Pic}^eY$. We let $\mathcal{H}_{d,A}(Y) = h^{-1}(A)$.

Lemma 2.5. Let $d$ and $e$ be integers such that $d \geq 2$, $e \geq 1$. Let $n = 2e$ and let $A \in \text{Pic}^eY$. The following properties hold.

i. The morphism $h : \mathcal{H}_{d,n}(Y) \to \text{Pic}^eY$ is surjective.

ii. If $A' \in \text{Pic}^eY$ then $\mathcal{H}_{d,A}(Y) \cong \mathcal{H}_{d,A'}(Y)$.

iii. $\mathcal{H}_{d,A}(Y)$ is smooth, equidimensional of dimension $n-1$.

Proof. (ii). Let $A \in \text{Pic}^eY$ and let $t_\alpha : Y \to Y$ be the corresponding translation. If $E^\vee$ is the Tschirnhausen module of $\pi : X \to Y$ then $t_\alpha^*E^\vee$ is the Tschirnhausen module of $t_\alpha \circ \pi : X \to Y$. So $A = \det E$ and if $A' = t_\alpha^*A$ then the mapping $[\pi : X \to Y] \mapsto [t_\alpha \circ \pi : X \to Y]$ yields an isomorphism between $\mathcal{H}_{d,A}(Y)$ and $\mathcal{H}_{d,A'}(Y)$.

(i) and (iii). Given an element $[\pi : X \to Y] \in \mathcal{H}_{d,n}(Y)$ the translation $[t_\alpha \circ \pi : X \to Y]$ with $\alpha \in \text{Pic}^eY$ belongs to the same connected component of $\mathcal{H}_{d,n}(Y)$. Thus the morphism $h : \mathcal{H}_{d,n}(Y) \to \text{Pic}^eY$ has the property that its restriction on every connected component of $\mathcal{H}_{d,n}(Y)$ is surjective, so Part (i) holds. Hence every sufficiently general fiber $\mathcal{H}_{d,A}(Y)$ is equidimensional of dimension $n-1$ and is furthermore smooth \[III.10.7\]. Using (ii) we conclude that every fiber $\mathcal{H}_{d,A}(Y)$ has these properties. \hfill \Box

2.6. If $\pi : X \to Y$ is a triple covering of smooth projective curves with Tschirnhausen module isomorphic to $E^\vee$ then $\mathbf{P}(E)$ is a ruled surface, $\varphi : \mathbf{P}(E) \to Y$, and $i(X)$ is a divisor of the linear system $|\mathcal{O}_{\mathbf{P}(E)}(3) \otimes \varphi^*(\det E)^{-1}|$. Hence $\pi : X \to Y$ is uniquely determined by $E$ and by an element $\langle \eta \rangle \in \mathbb{P}H^0(Y,S^3E \otimes (\det E)^{-1})$. Two equivalent coverings $\pi : X \to Y$ and $\pi' : X' \to Y$ yield isomorphic Tschirnhausen modules such that $\langle \eta \rangle$ is transformed into $\langle \eta' \rangle$ under the isomorphism. Conversely, given a rank 2 locally free sheaf $E$ on $Y$ with associated ruled surface $\varphi : \mathbf{P}(E) \to Y$ any nonsingular, irreducible divisor $X \in |\mathcal{O}_{\mathbf{P}(E)}(3) \otimes \varphi^*(\det E)^{-1}|$ determines a triple covering $\pi : X \to Y$ with Tschirnhausen module isomorphic to $E^\vee$. The group $\text{PGL}_Y(E)$ acts faithfully on the set of reduced divisors of $|\mathcal{O}_{\mathbf{P}(E)}(3) \otimes \varphi^*(\det E)^{-1}|$ and the orbit $\text{PGL}_Y(E) \cdot X$ corresponds to the equivalence class $[X \to Y]

2.7. We now consider simple triple branched coverings $\pi : X \to Y$ where $Y$ is elliptic. The number of branch points $n = \#B = \#R = 2 \deg E$, $n \geq 2$. Let $\mathcal{H}_{3,n}(Y)$ be the corresponding Hurwitz space. We wish to determine the types of the Tschirnhausen modules of general
coverings. We need to bound above the number of moduli of \([X \to Y] \in \mathcal{H}_{3,n}(Y)\) with certain types of Tschirnhausen modules (Cases 1 – 5 considered below). We do not need to address the question whether such coverings with smooth, irreducible \(X\) actually exist. Let us first consider the case of decomposable Tschirnhausen modules. Let \(\text{deg} E = e, \ n = 2e, \ E = Lu \oplus Mv\) where \(u \in \Gamma(Y, E \otimes L^{-1}), \ v \in \Gamma(Y, E \otimes M^{-1})\). Let \(a = \text{deg} L, \ b = \text{deg} M, \ e = a + b\). we may assume \(a \leq b\). We have

\[
S^3E \otimes (\det E)^{-1} = L^2M^{-1}v^3 + Lu^2v + Muv^2 + M^2L^{-1}v^3.
\]

Hence a global section \(\eta \in H^0(Y, S^3E \otimes (\det E)^{-1})\) may be decomposed as \(\eta = \alpha u^3 + \beta u^2v + \gamma uv^2 + \delta v^3\). In order that \(E^\vee\) is the Tschirnhausen module of the irreducible covering \(\pi : X \to Y\) determined by \(\eta\) it is necessary that \(\alpha \neq 0\) and \(\delta \neq 0\) (see [At] p.1145 or [CDC] p.266). Hence \(L^2 \geq M\) and \(M^2 \geq L\). If \(a \leq 0\) then \(b \leq 2a \leq 0\) which is absurd. Hence \(a \geq 1, \ b \geq 1, \ 2a \geq b, \) and \(2b > a\) since \(b \geq a\). We see that if \(n = 2\) and if \(X\) is irreducible then the Tschirnhausen module is indecomposable. The splitting (1) yields

\[
h^0(Y, S^3E \otimes (\det E)^{-1}) = (2a - b + \epsilon) + a + b + 2b - a = 2e + \epsilon
\]

where \(\epsilon = 1\) if \(L^2 \cong M\) and \(\epsilon = 0\) otherwise. We have

\[
\dim \text{Aut}_Y (E) = h^0(Y, \text{End}(E)) = 2 + h^0(Y, LM^{-1}) + h^0(Y, ML^{-1})
\]

**Case 1.** \(a < b, \ 2a > b\). Here \(L\) and \(M\) may vary independently, so

\[
\# \text{ moduli} \leq 2 + h^0(Y, S^3E \otimes (\det E)^{-1}) - h^0(Y, \text{End}(E)) = 2 + 2e - 2 - (b - a) = n - (b - a)
\]

**Case 2.** \(a = b, \ L \not\cong M\). This case is possible only if \(n \equiv 0 \pmod{4}\) since \(n = 2(a + b)\). Here

\[
\# \text{ moduli} \leq 2 + 2e - 2 = n.
\]

**Case 3.** \(a = b, \ L \cong M\). Here \(\# \text{ moduli} \leq 1 + 2e - 4 = n - 3\).

**Case 4.** \(2a = b\). Here one has two subcases: \(L^2 \not\cong M\) and \(L^2 \cong M\). In both subcases

\[
\# \text{ moduli} \leq 2 + 2e - 2 - (b - a) = n - (b - a).
\]

Recall from [At] p.432 that on an elliptic curve for every \(r \geq 1\) up to isomorphism there is a unique indecomposable locally free sheaf \(F_r\) of rank \(r\) and degree 0 with \(h^0(Y, F_r) \neq 0\). Furthermore any indecomposable locally free sheaf \(E\) of degree \(e\) and rank \(r\) such that \(r|e\) is isomorphic to \(L \otimes F\) where \(L\) is an invertible sheaf of degree \(\frac{e}{r}\).

**Case 5.** The Tschirnhausen module of \(\pi : X \to Y\) is indecomposable of even degree. Let \(n = 2e, \ e\) is even. Let us fix an indecomposable \(E\) with \(\text{deg} E = e\). Let \(E \cong L \otimes F_2\). We have by [At] p.438 that

\[
S^3E \otimes (\det E)^{-1} \cong L^3 \otimes S^3F_2 \otimes L^{-2} \cong L \otimes F_4.
\]

Hence by [At] p.430

\[
h^0(S^3E \otimes (\det E)^{-1}) = h^0(L \otimes F_4) = \text{deg}(L \otimes F_4) = 4\frac{e}{2} = n.
\]

One has \(\text{End}_Y (E) \cong F_2 \otimes F_2' \cong F_2 \otimes F_2 \cong F_1 \oplus F_3\) by [At] pp.433,437, hence \(h^0(\text{End}_Y (E)) = 2\). Varying \(E\) (i.e. \(L \in \text{Pic}^{e/2}Y\)) we obtain \(\# \text{ moduli} \leq 1 + n - 2 = n - 1\).
We need a technical result related to [CE] Theorem 3.6. Although we need it only in the case of families of elliptic curves we state and prove it for arbitrary dimensions and arbitrary algebraically closed base field \( k \) of characteristic 0. Let \( Y \) be a smooth integral scheme over \( k \). We recall from [CE] Definition 3.3 that given a rank 2 locally free sheaf \( E \) on \( Y \) an element \( \eta \in H^0(Y, S^3E \otimes (\det E)^{-1}) \) is called of right codimension in every \( y \in Y \) if \( \eta(y) \in (S^3E \otimes (\det E)^{-1}) \otimes_{\mathcal{O}_Y} k(y) \) is nonzero for every \( y \in Y \). Every such \( \eta \) determines a Gorenstein triple covering \( X_\eta \to Y \), \( X_\eta \subset \mathbb{P}(E) \) with Tschirnhausen module isomorphic to \( E^\vee \). [

Lemma 2.8. Let \( q : \mathcal{Y} \to Z \) be a smooth proper morphism with connected fibers, where \( Z \) is smooth. Let \( \mathcal{E} \) be a locally free sheaf of rank 2 on \( \mathcal{Y} \) such that \( h^0(\mathcal{Y}_z, S^3\mathcal{E}_z \otimes (\det \mathcal{E}_z)^{-1}) \) is independent of \( z \in Z \) and is \( \neq 0 \). Consider the locally free sheaf \( \mathcal{H} = q_* (S^3\mathcal{E} \otimes (\det \mathcal{E})^{-1}) \) on \( Z \). Let \( f : \mathbb{H} \to Z \) be the associated vector bundle with fibers \( \mathbb{H}_z = H^0(\mathcal{Y}_z, S^3\mathcal{E}_z \otimes (\det \mathcal{E}_z)^{-1}) \). Then the subset \( \mathbb{H}_0 \subset \mathbb{H} \) consisting of \( \eta \) which satisfy the following three conditions is Zariski open in \( \mathbb{H} \).

\( \mathcal{H}_0 \neq \emptyset \). Consider the base change \( \mathcal{Y}' = \mathcal{Y} \times_Z \mathbb{H}_0 \) and let \( \mathcal{E}' = \pi_1^* \mathcal{E} \). Then there is a smooth subscheme \( X \subset \mathbb{P}(\mathcal{E}') \) and a triple covering \( p : X \to \mathcal{Y}' \) such that for every \( \eta \in \mathbb{H}_0 \) with \( f(\eta) = z \) the fiber \( X_\eta \to \mathcal{Y}_\eta' \) is equivalent to \( X_\eta \to \mathcal{Y}_\eta \).

**Proof.** The statement is local with respect to \( Z \) so we may assume \( Z \) is irreducible. If \( \mathbb{H}_0 \) is empty there is nothing to prove. Suppose \( \mathbb{H}_0 \neq \emptyset \).

**Step 1.** Let \( \mathbb{H}' \) be the set of \( \eta \in \mathbb{H} \) for which (a) holds. We claim \( \mathbb{H}' \) is Zariski open in \( \mathbb{H} \). Consider the incidence correspondence \( \Gamma \subset \mathbb{P}(\mathcal{E}) \times_Z \mathbb{H} \) defined as follows.

\[ \Gamma = \{(x, \eta) | \eta(x) = 0 \text{ where } x \in \mathbb{P}(\mathcal{E})_y, \eta \in \mathbb{H}_z, y \in \mathcal{Y}_z \}. \]

Consider the projection \( \varepsilon : \Gamma \to \mathcal{Y} \times_Z \mathbb{H}, \varepsilon(x, \eta) = (y, \eta) \). An element \( \eta \in \mathbb{H}_z \) fails to be of right codimension in \( y \in \mathcal{Y}_z \) if and only if \( \eta(x) = 0 \) for \( \forall x \in \mathbb{P}(\mathcal{E})_y \). Equivalently \( (y, \eta) \in \Sigma \) where \( \Sigma \subset \mathcal{Y} \times_Z \mathbb{H} \) is the subset of points for which \( \dim \varepsilon^{-1}(y, \eta) \geq 1 \). Hence \( \Sigma \) is closed in \( \mathcal{Y} \times_Z \mathbb{H} \). Since properness is preserved under base change the projection of \( \Sigma \) in \( \mathbb{H} \) is closed. This proves \( \mathbb{H}' \) is open in \( \mathbb{H} \).

**Step 2.** Let \( \mathbb{H}'' \subset \mathbb{H}' \) be the set of \( \eta \in \mathbb{H} \) for which both (a) and (b) hold. We claim \( \mathbb{H}'' \) is open in \( \mathbb{H}' \). There is a commutative diagram

\[
\begin{array}{ccc}
S^3E \otimes (\det E)^{-1} & \xrightarrow{\pi_1} & Y \\
\downarrow & & \downarrow \\
\mathcal{Y} \times_Z \mathbb{H} & \xrightarrow{\mathcal{Y}} & \mathcal{Y} \\
\end{array}
\]
where $N'(y, \eta) = \eta(y)$. Let $E_{\mathbb{H}} = \pi^*E$. Since $\pi^*(S^3E \otimes (\det E)^{-1}) \cong S^3E_{\mathbb{H}} \otimes (\det E_{\mathbb{H}})^{-1}$ we obtain a tautological section $N \in H^0(Y \times Z \mathbb{H}, S^3E_{\mathbb{H}} \otimes (\det E_{\mathbb{H}})^{-1})$. Restricting to $Y \times Z \mathbb{H}'$ we obtain a section which is of right codimension for every $(y, \eta) \in Y \times Z \mathbb{H}'$. By $[CE]$ Theorem 3.4 one obtains a closed subscheme $X' \subset P(E_{\mathbb{H}''}) = (P(E_{\mathbb{H}'})$ and a Gorenstein triple covering $p : X' \to Y \times Z \mathbb{H}'$ whose Tschirnhausen module is isomorphic to $E_{\mathbb{H}''}$. Let $f : X' \to \mathbb{H}'$ be the projection morphism. Let $f(x) = \eta$. The point $x$ is nonsingular in the fiber $X'_{\eta}$ if and only if

(i) $x$ is a nonsingular point of $X'$, (ii) $T_xf : T_xX' \to T_\eta\mathbb{H}'$ is surjective (see e.g. $[AK]$ pp.131,145). Since $f$ is proper $f(Sing X')$ is closed in $\mathbb{H}'$. Let $\mathbb{H}'_1 = \mathbb{H}' - f(Sing X')$. The set of $x \in f^{-1}(\mathbb{H}'_1)$ where the rank of $T_xf$ is not maximal is closed in $f^{-1}(\mathbb{H}'_1)$, so again by properness its image in $\mathbb{H}'_1$ is closed. The complement of the latter in $\mathbb{H}'_1$ is the set $U$ consisting of $\eta \in \mathbb{H}'$ for which $X'_{\eta}$ is smooth. Considering the Stein factorization we see that all fibers of $f^{-1}(U) \to U$ have the same number of irreducible components. Since by assumption $\mathbb{H}_0 \neq \emptyset$ and $Z$ is irreducible we conclude that all fibers $X'_{\eta}$, $\eta \in U$ are irreducible. So the set $\mathbb{H}''$ consisting of $\eta \in \mathbb{H}$ for which both (a) and (b) hold equals $U$ and is therefore open in $\mathbb{H}'$.

Step 3. We claim $\mathbb{H}_0$ is open in $\mathbb{H}''$. Let $X'' \to Y \times Z \mathbb{H}''$ be the restriction of $X' \to Y \times Z \mathbb{H}'$. Let $B \subset Y \times Z \mathbb{H}''$ be the discriminant subscheme (see e.g. $[AK]$ pp.123-124). We apply to the projection $B \to \mathbb{H}''$ the same argument as that in Step 2 to conclude that $\mathbb{H}_0$ is open in $\mathbb{H}''$. This proves $\mathbb{H}_0$ is open in $\mathbb{H}$. If $\mathbb{H}_0 \neq \emptyset$ let $X' \to Y' = Y \times Z \mathbb{H}_0$ be the restriction of $X' \to Y \times Z \mathbb{H}'$ from Step 2. The last claim of the lemma follows from the functoriality of Miranda’s construction.

Proposition 2.9. Let $n = 2e \geq 2$. There is a Zariski open dense subset of the Hurwitz space $U \subset H_{3,n}(Y)$ such that for every $[X \to Y] \in U$ one has

1. if $e \equiv 1(mod 2)$ the Tschirnhausen module $E^\vee$ is indecomposable of degree $-e$.

2. if $e \equiv 0(mod 2)$ the Tschirnhausen module $E^\vee$ is isomorphic to $L^{-1} \oplus M^{-1}$ where $\deg L = \deg M = \frac{e}{2}$ and $L \not\cong M$.

Proof. We need to prove that in each of the cases 1, 3 and 5 from (2.7) either there is no triple covering $\pi : X \to Y$ with Tschirnhausen module of that type or if it exists the set of equivalence classes of such coverings is contained in a closed subscheme of $H_{3,n}(Y)$ of codimension $\geq 1$. We apply Lemma 2.8 with $Y = Y \times Z$, $q = \pi_2$ where $Z$ and $E$ are constructed in the various cases as follows. In Case 1 we let $Z = \text{Pic}^aY \times \text{Pic}^bY$ and if $L \to Y \times \text{Pic}^aY$ and $M \to Y \times \text{Pic}^bY$ are the Poincaré invertible sheaves we let $E = \pi_{12}^*L \oplus \pi_{13}^*M$. Here $\pi_{12}$ and $\pi_{13}$ are the corresponding projections of $Y \times Z = Y \times \text{Pic}^aY \times \text{Pic}^bY$. In Case 3 we let $Z = \text{Pic}^aY$ and $E = L \oplus L$. Case 4 splits into two subcases. When $L^2 \not\cong M$ we repeat the construction of Case 1 letting $Z \subset \text{Pic}^cY \times \text{Pic}^bY$ be the open subset $Z = \{(u, v) \mid L_u \not\cong M_v\}$. In the subcase $L^2 \cong M$ we let $Z = \text{Pic}^cY$ and $E = L \oplus L^2$. In Case 5 we let $Z = \text{Pic}^c/2Y$ and let $E = (\pi_1^*F_2) \otimes L$. Suppose there is a triple covering of one of the considered types. Let $\mathbb{H}_0$ and $X \to Y \times \mathbb{H}_0$ be as in Lemma 2.8. By the
universal property of the Hurwitz space there is an associated morphism $f : \mathbb{H}_0 \to \mathcal{H}_{3,n}(Y)$. The closure $\overline{f(\mathbb{H}_0)}$ is a closed subscheme of codimension $\geq 1$ according to the calculations of the number of moduli in the various cases from (2.7).

Theorem 2.10. Let $Y$ be an elliptic curve, let $n$ be a pair integer $n = 2e \geq 2$. Let $A \in \text{Pic}^eY$. If $d = 2$ or $3$ the Hurwitz spaces $\mathcal{H}_{d,n}(Y)$ and $\mathcal{H}_{d,A}(Y)$ are irreducible. The variety $\mathcal{H}_{d,A}(Y)$ is rational if $d = 2$ or if $d = 3$ and $e \equiv 1(\text{mod } 2)$ and is unirational if $d = 3$ and $e \equiv 0(\text{mod } 2)$.

Proof. According to Lemma 2.5 $h : \mathcal{H}_{d,n}(Y) \to \text{Pic}^eY$ is surjective. It has fibers $\mathcal{H}_{d,A}(Y)$, $A \in \text{Pic}^eY$ which are isomorphic to each other. Hence it suffices to prove the statements for $\mathcal{H}_{d,A}(Y)$.

If $d = 2$ what is claimed is obvious, since a simple double covering $\pi : X \to Y$ such that $\pi_*\mathcal{O}_X \cong \mathcal{O}_Y \oplus A^{-1}$ is uniquely determined by an element $\langle \eta \rangle \in \mathbb{P} H^0(Y, A^2)$ such that $\text{div}(\eta)$ is a simple divisor (see e.g. [Wa]). If $\Delta \subset \mathbb{P} H^0(Y, A^2)$ is the closed subset of nonsimple divisors one obtains an isomorphism $f : \mathbb{P} H^0(Y, A^2) - \Delta \to \mathcal{H}_{2,A}(Y)$.

If $d = 3$ we have two cases.

Case 1. $e \equiv 1(\text{mod } 2)$. According to Atiyah’s results [At] up to isomorphism there is a unique indecomposable rank 2 locally free sheaf $E$ on $Y$ with $\text{deg } E \cong A$. By Proposition 2.9 and Lemma 2.5 there exist simple triple coverings with Tschirnhausen module isomorphic to $E'$. Applying Lemma 2.5 with $Z = \{\ast\}$, $\mathcal{E} = E$ one obtains a covering $\mathcal{X} \to Y \times \mathbb{H}_0$, where $\mathbb{H}_0$ is Zariski open nonempty subset of $\mathbb{H} = H^0(Y, S^3E \otimes (\text{det } E)^{-1})$ which moreover is invariant with respect to the action of $\mathbb{C}^*$. Using the universal property of the Hurwitz space $\mathcal{H}_{3,n}(Y)$ one obtains a morphism $f : \mathbb{P} \mathbb{H}_0 \to \mathcal{H}_{3,A}(Y)$. This morphism is dominant by Proposition 2.9. It is injective since $h^0(Y, \text{End}(E)) = 1$ (cf. [At] Lemma 22 and (2.6)). Hence $\mathcal{H}_{3,A}(Y)$ is a rational variety.

Case 2. $e \equiv 0(\text{mod } 2)$. Let $e = 2a$. Let $\sigma : \text{Pic}^aY \to \text{Pic}^aY$ be the involution $L \mapsto A \otimes L^{-1}$ and let $\mu : \text{Pic}^aY \to \mathbb{P}^1$ be the quotient map. Let $Z \subset \mathbb{P}^1$ be the complement of the branch locus of $\mu$. Consider the double covering $1 \times \mu : Y \times \text{Pic}^aY \to Y \times \mathbb{P}^1$. Let $\mathcal{L}$ be the Poincaré invertible sheaf on $Y \times \text{Pic}^aY$ and let $\mathcal{E} = (1 \times \mu)_*\mathcal{L}|_{Y \times Z}$. By construction for every $z \in Z$ one has $\mathcal{E}_z \cong \mathcal{L}_z \oplus (A \otimes \mathcal{L}_z^{-1})$. We apply Lemma 2.5 with $Y = Y \times Z$, $q = \pi_2$ and using Proposition 2.9 and Lemma 2.5 we conclude that $\mathbb{H}_0$ is nonempty. Here $\mathbb{H}$ is a vector bundle over $Z$ and its Zariski open subset $\mathbb{H}_0$ is $\mathbb{C}^*$-invariant. The family of triple coverings $\mathcal{X} \to Y \times \mathbb{H}_0$ yields a morphism $f : \mathbb{P} \mathbb{H}_0 \to \mathcal{H}_{3,A}(Y)$ which is dominant by Proposition 2.9. Hence $\mathcal{H}_{3,A}(Y)$ is irreducible and unirational.

Remark 2.11. Graber, Harris and Starr proved in [GHS] the irreducibility of the space $\mathcal{H}_{d,n}^{a}(Y)$ parameterizing simple coverings with monodromy group $S_d$ for any $Y$ of positive genus when $n \geq 2d$. This result implies the irreducibility of $H_{3,n}(Y)$ stated in the above theorem when $n \geq 6$.

The proof of the theorem ($d = 3$, Case 1) shows the following result.
Corollary 2.12. Let $A \in \text{Pic}^e Y$, $e \equiv 1 \text{ (mod 2)}$, $e \geq 1$, and let $E$ be an indecomposable rank 2 locally free sheaf on $Y$ with $\det E \cong A$. Then a Zariski open nonempty subset of the Hurwitz space $H_{3,A}(Y)$ consists of equivalence classes of coverings $[X \to Y]$ with Tschirnhausen module isomorphic to $E^\vee$.

2.13. So far in this section we considered families of triple coverings of a fixed elliptic curve. We now want to vary also the elliptic curve. We need only the case of triple coverings simply branched at 6 points and this is the case we work out. We consider a sufficiently general pencil of cubic curves in $\mathbb{P}^2$. Blowing up the nine base points and discarding the singular fibers we obtain a smooth family $q : Y \to Z$, $Z \subset \mathbb{P}^1$ of elliptic curves with 9 sections. Let us choose one of the sections and call it $D$. We construct a rank 2 locally free sheaf on $Y$ as in [Ha] Ch.V Ex.2.11.6. Namely, the extensions

$$0 \to \mathcal{O}_Y \to \mathcal{F} \to \mathcal{O}_Y(D) \to 0$$

are parameterized by $H^1(Y, \mathcal{O}_Y(-D)) = H^0(Z, R^1q_*\mathcal{O}_Y(-D))$ by Leray’s spectral sequence. The sheaf $R^1q_*\mathcal{O}_Y(-D)$ is locally free of rank one by Grauert’s theorem. Replacing $Z$ by a smaller affine set we may assume $R^1q_*\mathcal{O}_Y(-D)$ is trivial. A trivializing section yields an extension (2) with the property that $\mathcal{F}_z$ is indecomposable over $Y_z$ for every $z \in Z$. Let $\mathcal{E} = \mathcal{F} \otimes \mathcal{O}_Y(D)$. By semicontinuity, replacing $Z$ by a smaller open set, we may assume $h^0(Y_z, S^3\mathcal{E}_z \otimes (\det \mathcal{E}_z)^{-1})$ is independent of $z$. We may now apply Lemma 2.8. If $\mathcal{H} = q_*(S^3\mathcal{E} \otimes (\det \mathcal{E})^{-1})$, if $f : \mathbb{H} \to Z$ is the corresponding vector bundle and $\mathbb{H}_0 \subset \mathbb{H}$ the open subset that satisfies the three conditions of Lemma 2.8 then $\mathbb{H}_0 \neq \emptyset$ by Corollary 2.12. According to that lemma letting $\mathbb{P}(\mathbb{H}_0) = T$ we obtain a family of triple coverings

$$\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
T & \longrightarrow & Z
\end{array}$$

Letting $Y_T = Y \times_Z T$ one obtains a family of coverings over $T$:

$$\begin{array}{ccc}
X & \longrightarrow & Y_T \\
\downarrow & & \downarrow \\
T & \longrightarrow & Y_T
\end{array}$$

In the case of double coverings one has an analogous and simpler construction. Here one lets $\mathcal{A} = \mathcal{O}_Y(3D)$, $\mathcal{H} = q_*\mathcal{A}^2$ and $T = \mathbb{P}(\mathbb{H}_0)$. One obtains a family as above with $\deg p = 2$.

Proposition 2.14. Let $d = 2$ or 3. The constructed family of coverings of degree $d$ has the following properties.

a. Every sufficiently general elliptic curve is isomorphic to a fiber of $Y \to Z$. 

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b. Let $z \in \mathbb{Z}$. The fibers $\mathcal{X}_\eta \to \mathcal{Y}_z$ with $\eta \in T$, $f(\eta) = z$ correspond to a Zariski open nonempty subset of the Hurwitz space $\mathcal{H}_{d,A}(\mathcal{Y}_z)$ with $A = \mathcal{O}_{\mathcal{Y}_z}(3D_z)$.

c. $T$ is a rational variety of dimension 6.

**Proof.** Part (b) follows from Corollary 2.122 and Lemma 2.8. The other parts are clear.

\[ \square \]

### 3 Families of coverings and variations of Hodge structures

We start by recalling some well known facts and fixing notation (cf. [Gr], [GS], [Ke]).

#### 3.1. A polarized Hodge structure of weight one is given by the following data.

i. A free abelian group $M$ of rank $2g$.

ii. A complex subspace $U \subset M_C = M \otimes \mathbb{C}$ such that $M_C = U \oplus \overline{U}$.

iii. An integer valued, skew-symmetric, nondegenerate form $Q : M \times M \to \mathbb{Z}$ whose $\mathbb{C}$-bilinear extension $Q_C$ satisfies the Riemann relations

\[
\begin{align*}
(c) & \quad Q_C(U, U) = 0, \\
& \quad iQ_C(u, \overline{u}) > 0 \text{ for } \forall u \in U, u \neq 0
\end{align*}
\]

One defines the real Weyl operator $C$ so that it has eigenvalue $i$ on $U$ and $-i$ on $\overline{U}$. Given such data one may define a polarized abelian variety $P(U) = \overline{U}/\pi^{0,1}(M)$. The polarization on $P(U)$ is defined by the Hermitian form $H'$ on $\overline{U}$ such that $Im(H') = E' := -Q_{\mathbb{R}}$. Let us check a part of the last statement. If $J = -C$ is the complex structure on $M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$ defined by $J(\varphi + \overline{\varphi}) = -i\varphi + i\overline{\varphi}$, then $\pi^{0,1} : (M_{\mathbb{R}}, J) \to \overline{U}$ is a $\mathbb{C}$-linear isomorphism. We want to verify that $E'(J\omega, \omega) > 0$ for every nonzero $\omega \in M_{\mathbb{R}}$. Indeed, if $\omega = \varphi + \overline{\varphi}$ then

\[
- Q_{\mathbb{R}}(J\omega, \omega) = -Q_C(-i\varphi + i\overline{\varphi}, \varphi + \overline{\varphi}) = -(-iQ_C(\varphi, \overline{\varphi}) + iQ_C(\overline{\varphi}, \varphi)) = 2iQ_C(\varphi, \overline{\varphi}) > 0
\]

**Example 3.2.** Let $X$ be a compact Riemann surface of genus $g$. Let $M = H^1(X, \mathbb{Z}), U = H^{1,0}(X), M_C = H^1(X, \mathbb{C}) = H^{1,0}(X) \oplus H^{0,1}(X) = U \oplus \overline{U}$ be the Hodge decomposition, $Q(\varphi, \psi) = \int_X \varphi \wedge \psi$. Then $P(U) = H^{0,1}(X)/\pi^{0,1}H^1(X, \mathbb{Z}) \cong Pic^0(X)$ is polarized by an hermitian form $H'$ with $E' = Im(H')$ where $E'(\varphi, \psi) = -\int_X \varphi \wedge \psi$.

#### 3.3. Consider a polarized Hodge structure as in (3.1). Let $\Lambda = M^* = Hom_\mathbb{Z}(M, \mathbb{Z})$. Let $V = (U^\perp) ^* \cong U^*, \overline{V} = U^\perp \cong \overline{U}^*$. The transposed Weyl operator $t^*C$ has eigenvalues $i, -i$ on $V$ and $\overline{V}$ respectively. Consider the corresponding splitting $\Lambda_C = V \oplus \overline{V}$. The complex torus $A(U) = V/\pi_V(\Lambda)$ is dual to $P(U)$. Indeed, define a complex structure on $\Lambda_{\mathbb{R}}$ by $I = i^*C$, i.e. $I(v + \overline{v}) = iv - i\overline{v}$. Then the $\mathbb{C}$-linear isomorphism $\pi_V : (\Lambda_{\mathbb{R}}, I) \to V$ induces
an isomorphism of complex tori \( \mathbb{A}_\mathbb{R}/\Lambda \to V/\pi_V(\Lambda) = A(U) \). As we saw above we have \( (M_\mathbb{R}, J)/M \cong P(U) \). The \( \mathbb{R} \)-extension of the perfect pairing \( \langle , \rangle : \Lambda \times M \to \mathbb{Z} \) satisfies \( \langle I v, J \varphi \rangle = \langle v, \varphi \rangle \). Thus \( A(U) \) is dual to \( P(U) \). Let \( \omega_1, \ldots, \omega_g \) be a \( \mathbb{C} \)-basis of \( U \) and let \( \gamma_1, \ldots, \gamma_{2g} \) a \( \mathbb{Z} \)-basis of \( \Lambda \). Let \( \{ \gamma_\rho \}_\rho \) be the dual basis of \( M \) and let \( \Pi = (\pi_{\alpha \rho}) \) be the \( g \times 2g \) period matrix with entries \( \pi_{\alpha \rho} = \langle \gamma_\rho, \omega_\alpha \rangle \). Then \( \omega_\alpha = \sum_\rho \pi_{\alpha \rho} \gamma_\rho^* \). In matrix form this can be written as \( t(\omega_1, \ldots, \omega_g) = \Pi^t(\gamma_1^*, \ldots, \gamma_{2g}^*) \). The matrix \( \Pi \) is the period matrix of the torus \( A(U) \). In Example 3.2 we have \( \Lambda = H_1(X, \mathbb{Z}), V \cong H^{1,0}(X)^*, A(U) = V/\pi_V(\Lambda) \) is the Jacobi variety \( J(X) \) of \( X \).

3.4. The material of this paragraph is related to [BL2], [BL3]. Let \( \Lambda \) be a lattice of rank 2g. Let \( E \) be a nondegenerate, integer valued, skew-symmetric form on \( \Lambda \). Let \( \phi : \Lambda_\mathbb{R} \to \Lambda_\mathbb{R}^* \) be the associated isomorphism \( v \mapsto E(v, -) \). We define a skew-symmetric form \( E^* \) on \( \Lambda_\mathbb{R}^* \) by the equality \( E = \phi^* E^* \), i.e. \( E(v, w) = E^*(\phi(v), \phi(w)) \). We need to multiply \( E^* \) by an integer in order to make it integer-valued on \( \Lambda^* \). Let \( A = (a_{\rho \sigma}) \) be the matrix of \( E \) in a basis \( \gamma_1, \ldots, \gamma_{2g} \) of \( \Lambda \). Let \( \{ \gamma_\rho \}_\rho \) be the dual basis of \( \Lambda^* \), \( \langle \gamma_\rho, \gamma_\sigma^* \rangle = \delta_{\rho \sigma} \). Since \( \phi(\rho \sigma) = \sum_\sigma a_{\rho \sigma} \gamma_\sigma^* \) one has that the matrix of \( E^* \) in the basis \( \{ \gamma_\rho \}_\rho \) satisfies \( A = AA^* t A \). Hence \( A^* = -A^{-1} \).

Let \( \lambda_1, \ldots, \lambda_{2g} \) be a simplectic basis of \( \Lambda \), thus \( A = (\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}) \) where \( D = diag(d_1, \ldots, d_g) \) and let \( d_1 | d_{i+1} \) and \( e = d_i + d_{i+1} \) then \( A^* = (\begin{smallmatrix} 0 & D \\ -D & 0 \end{smallmatrix}) \) and hence \( eA^* \) is with integer entries. We see that \( \hat{E} = eE^* \) is an integer valued skew-symmetric form on \( \Lambda^* \) with elementary divisors \( (d_1, \ldots, d_g) \), \( d_i = \frac{d_1}{d_{i-1} + d_i} \). If \( A = (a_{\rho \sigma}) \) is the matrix of \( E \) in the basis \( \{ \gamma_\rho \}_\rho \) of \( \Lambda \) and if \( A^{-1} = (b_{\sigma \tau}) \), then \( eA^{-1} = (-e b_{\sigma \tau}) \) is the matrix of \( \hat{E} \) in the dual basis \( \gamma_\sigma^* \) of \( \Lambda^* \). Let \( \hat{\phi} : \Lambda_\mathbb{R}^* \to \Lambda_\mathbb{R} \) be the isomorphism associated with \( \hat{E} \), \( \omega \mapsto \hat{E}(\omega, -) \). One has \( \hat{\phi}(\gamma_\sigma^*) = \sum_\tau e b_{\sigma \tau} \gamma_\tau \), so

\[
(5) \quad \hat{\phi} \circ \phi = -e \mathbf{1}, \quad \phi \circ \hat{\phi} = -e \mathbf{1}.
\]

The equality \( \phi^* \hat{E} = eE \) and [5] imply \( \hat{\phi} E = e \hat{E} \), so \( \hat{E}^* = \frac{1}{e} \hat{E} \) and since the maximal elementary divisor of \( \hat{E} \) is \( \frac{\frac{\hat{E}}{\hat{E}}}{\frac{\hat{E}}{\hat{E}} \hat{E}} \), one obtains \( (\hat{E})^* = \frac{\hat{E}}{\hat{E}} \). Thus if \( d_1 = 1 \) one has \( (\hat{E})^* = E, d_g = e = \hat{d}_g \) and the following relations hold

\[
(6) \quad \hat{\phi}^* \hat{E} = eE, \quad \hat{\phi}^* E = e \hat{E}
\]

Let \( P = \mathbb{C}^g/\Lambda \) be an abelian variety with polarization \( E = Im H \). It is convenient to consider \( \mathbb{C}^g \) as the real space \( \Lambda_\mathbb{R} \) and the multiplication by \( i \) as a linear operator \( I : \Lambda_\mathbb{R} \to \Lambda_\mathbb{R} \) with \( I^2 = -1 \). The Riemann conditions for \( E \) are:

(i) \( E(\Lambda, \Lambda) \subset \mathbb{Z} \), (ii) \( E(Iv, Iv) = E(v, w) \), (iii) \( E(Iv, v) > 0 \) for all \( v \neq 0 \).

Let \( \hat{P} = \Lambda_\mathbb{R}^* / \Lambda^* \) be the dual complex torus. Here the complex structure on \( \Lambda_\mathbb{R}^* \) is defined by \( J = -tI \) so that \( \langle Iv, J\omega \rangle = \langle v, \omega \rangle \). The mapping \( v \mapsto E(v, -) \) yields a \( \mathbb{C} \)-isomorphism \( \phi : (\Lambda_\mathbb{R}, I) \sim \hat{P}(\Lambda_\mathbb{R}, J) \) and an isogeny \( \varphi : P \to \hat{P} \) such that \( \varphi_\ast = \phi \). If the polarization is defined by an invertible sheaf \( L \) on \( P \) then \( \varphi = \varphi_L \) where \( \varphi(x) = T_x^* L \otimes L^{-1} \in \text{Pic}^0 P = \hat{P} \) (see [Ke] p.7). Now the above construction yields a polarization \( \hat{E} \) on \( \hat{P} \) such that the polarization mapping \( \hat{\phi} : \hat{P} \to P \) satisfies \( \hat{\phi} \circ \varphi = -e \mathbf{1}_P, \quad \varphi \circ \hat{\phi} = -e \mathbf{1}_P \).
We summarize the arguments of (3.1) – (3.5) in the following statement.

Let
\[
\lambda_j = \sum_{i=1}^{g} Z_{ij} \frac{1}{d_i} \lambda_{g+i}, \quad j = 1, \ldots, g
\]

One has \( \phi(\lambda_j) = d_j \lambda_{g+j}^* \), \( \phi(\lambda_{g+j}) = -d_i \lambda_i^* \), so applying \( \phi \) to \( \Lambda \) and dividing by \( d_j \) one obtains

\[
\lambda_{g+j}^* = \sum_{i=1}^{g} \frac{1}{d_j} Z_{ij} (-\lambda_i^*) = \sum_{i=1}^{g} e \frac{1}{d_i} Z_{ij} \frac{1}{d_j} \left( \frac{e}{d_i} \right)^{-1} (-\lambda_i^*) = \sum_{i=1}^{g} Z_{ij} \left( \frac{e}{d_i} \right)^{-1} (-\lambda_i^*)
\]

where \( Z' = eD^{-1}ZD^{-1} \). Now, let \( \hat{\lambda}_i = \lambda_{g-i+1}^*, \hat{\lambda}_{g+i} = -\lambda_{g-i+1}^*, d_i = \frac{e}{d_{g-i+1}}, \hat{Z}_{ij} = Z'_{g-i+1,g-j+1} \). Then \( \{ \hat{\lambda}_i \} \) is a simplectic basis of \( \Lambda^* \) for \( \hat{E} \) with elementary divisors \( \{ d_1, d_2, \ldots, d_g \} \) and the corresponding normalized period matrix of \( \hat{P} \) is \( (\hat{Z}, \hat{D}) \) where \( \hat{D} = \text{diag}(d_1, \ldots, d_g) \), and

\[
(8) \quad \hat{Z} = S(eD^{-1}ZD^{-1})S \quad \text{with} \quad S = \begin{pmatrix}
0 & \ldots & 0 & 1 \\
0 & \ldots & 1 & 0 \\
\cdots & \cdots & \cdots & \cdots \\
1 & \ldots & 0 & 0
\end{pmatrix}
\]

3.5. Let \( M, M_C = U \oplus \overline{U}, Q : M \times M \rightarrow \mathbb{Z} \) be a polarized Hodge structure of weight one. Let \( \Lambda = M^* \), let \( \psi : M_R \rightarrow \Lambda_\mathbb{R} \) be the linear isomorphism \( \psi(m) = Q(m, -) \) and let \( Q : \Lambda \times \Lambda \rightarrow \mathbb{Z} \) be the dual form as defined in (3.4). Since \( Q_R(C_1, C_2) = Q_R(\omega_1, \omega_2) \) one obtains a \( \mathbb{C} \)-linear isomorphism \( \psi : (M_R, C) \rightarrow (\Lambda_\mathbb{R}, -^t C) \). With respect to \( -^t C \) one has the splitting \( \Lambda_C = W \oplus \overline{W} \) with eigenvalues \( i \) and \( -i \) respectively, where \( W = V = U^\perp \) and \( \overline{W} = V = (\overline{U})^\perp \) (cf. (3.3)). Furthermore \( (\Lambda, \Lambda_C = W \oplus \overline{W}, Q) \) is a polarized Hodge structure of weight one. We call it the dual polarized Hodge structure of \( (M, M_C = U \oplus \overline{U}, Q) \). One has \( P(W) = A(U), A(W) = P(U) \).

We summarize the arguments of (3.1) – (3.5) in the following statement.

**Proposition 3.6.** Let \( M, M_C = U \oplus \overline{U}, Q : M \times M \rightarrow \mathbb{Z} \) be a polarized Hodge structure of weight one. Then one may associate to it a pair of dual abelian varieties \( A(U) = U^*/\pi_{U^*}(\Lambda), P(U) = U/\pi^{0,1}(M) \) with polarizations \( E, \hat{E} \) which have types \( (d_1, d_2, \ldots, d_g) \), resp. \( (\hat{d}_1, \hat{d}_2, \ldots, \hat{d}_g) \) where \( \hat{d}_i = d_g/d_{g-i+1} \). Furthermore \( \hat{E} = -\frac{1}{c}Q \) where \( c \) is the first elementary divisor of \( Q \) and \( E = -\hat{Q} \). In appropriate simplectic bases the normalized period matrices of \( A(U) \) and \( P(U) \) are respectively \( \Pi = (Z, D), \hat{\Pi} = (\hat{Z}, \hat{D}) \) where \( \hat{Z} = S(d_dD^{-1}ZD^{-1})S, D = \text{diag}(d_1, \ldots, d_g), \hat{D} = \text{diag}(\hat{d}_1, \ldots, \hat{d}_g) \) and \( S \) is the matrix defined in (8). The dual polarized Hodge structure is defined by \( \Lambda = M^*, \Lambda_C = W \oplus \overline{W}, Q, \) where \( W = U^\perp, \overline{W} = (\overline{U})^\perp \) and one has \( A(U) = P(W), P(U) = A(W) \).
Example 3.7. Consider $A(U) \cong J(X)$ from (3.2) and (3.3). Let $E : \Lambda \times \Lambda \to \mathbb{Z}$ be the skew-symmetric form $E(\gamma, \delta) = -(\gamma, \delta)_X$. Then $\phi : \Lambda \to \Lambda^* = M$ equals $\phi = -D_X : H_1(X, \mathbb{Z}) \to H^1(X, \mathbb{Z})$, where $D_X$ is the Poincaré isomorphism. From the property $(\gamma, \delta)_X = \int_X D_X(\gamma) \wedge D_X(\delta)$ (cf. [GH] Ch.0) one concludes $E = E'$ where $E'$ was defined in Example 3.2. Hence $-(\gamma, \delta)_X = E = (E')^\perp$ is the canonical principal polarization of $J(X)$. The period $g \times g$ matrix $Z$ as defined in (3.3) is the same as the classical one. In fact if $\{A_i, B_j\}_{i,j=1}^g$ is a standard system of cycles on $X$, $(A_i, B_j)_X = \delta_{ij}$, then $(\lambda_i = B_i, \lambda_{g+j} = A_j)$ is a symplectic basis for $E$. Thus if $\omega_1, \ldots, \omega_g$ is a normalized basis of differentials, $\int_{A_i} \omega_i = \delta_{ij}$ then $Z_{ij} = \int_{B_j} \omega_i$.

The polarized Hodge structure dual to the one of Example 3.2 is $H_1(X, \mathbb{Z}), H_1(X, \mathbb{C}) = W \oplus \overline{W}, \hat{Q}$, where $W = H^{1,0}(X) \perp, \overline{W} = H^{0,1}(X) \perp, \hat{Q}(\gamma, \delta) = (\gamma, \delta)_X$.

3.8. Let $\pi : X \to Y$ be a covering of smooth projective curves, $g(Y) \geq 1$. Then $\pi^* : H^1(Y, \mathbb{Z}) \to H^1(X, \mathbb{Z})$ and $\pi^* = \pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ induce morphisms of the corresponding Hodge structures. We let $M = H^1(X, \mathbb{Z})/H^1(X, \mathbb{Z}) \cap \pi^* H^1(Y, \mathbb{R})$ with a dual lattice $\Lambda = Ker(\pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z}))$. The Hodge structures $M_\mathbb{C} = U \oplus \overline{U}$ and $\Lambda_\mathbb{C} = W \oplus \overline{W}$ are defined respectively by $U = H^{1,0}(X)/\pi^* H^{1,0}(Y), W = H^{1,0}(X) \perp \cap \Lambda_\mathbb{C}$.

The corresponding pair of dual abelian varieties is

$$A(U) = Ker(Nm_\pi : J(X) \to J(Y))^0, \quad P(U) = Pic^0(X)/\pi^* Pic^0(Y).$$

Consider the restriction of $-(\gamma, \delta)_X$ on $\Lambda$. It is a nondegenerate form as evident from the orthogonal decomposition $H_1(X, \mathbb{R}) = Ker \pi_* \oplus \pi^* H_1(Y, \mathbb{R})$. Dividing it by its smallest elementary divisor we obtain a polarization $\hat{E} : \Lambda \times \Lambda \to \mathbb{Z}$ on $A(U)$. The dual form $\hat{E} : M \times M \to \mathbb{Z}$ is a polarization on $P(U)$, both $E$ and $\hat{E}$ have first elementary divisor 1, have the same exponent and their types are related as in Proposition 3.6. The Hodge structures $M_\mathbb{C} = U \oplus \overline{U}$ and $\Lambda_\mathbb{C} = W \oplus \overline{W}$ are polarized respectively by $Q = -\hat{E}$ and $\hat{Q} = -E$.

Example 3.9. (i) Let $\pi : X \to Y$ be a covering of smooth, projective curves of prime degree $d$, let $g(X) \geq 3, g(Y) = 1$. Then by Lemma 1.2, $P = Ker(Nm_\pi)$ is connected and the polarization $\hat{E}$ induced from $J(X)$ has type $(1, \ldots, 1, d)$. hence the dual abelian variety $\hat{P} = Pic^0(X)/\pi^* Pic^0(Y)$ has dual polarization $\hat{E}$ of type $(1, d, \ldots, d)$.

(ii) Let $\pi : X \to Y$ be a double covering of smooth, projective curves where $g(X) = 7, g(Y) = 3$ as in [RCV]. Then $Ker Nm_\pi$ is connected and equals the Prym variety $P$ with induced polarization $E$ of type $(1, 2, 2, 2)$. Thus the dual variety $\hat{P} = Pic^0X/\pi^* Pic^0Y$ has dual polarization $\hat{E}$ of type $(1, 1, 1, 2)$.

3.10. We want to adapt the arguments in [GH] pp.576,577 to the case of arbitrary polarizations. Let $M, M_\mathbb{C} = U \oplus \overline{U}, Q : M \times M \to \mathbb{Z}$ be a polarized Hodge structure. Let $\Lambda = M^*, \Lambda_\mathbb{C} = W \oplus \overline{W}, \hat{Q}$ be the dual Hodge structure and let $-\hat{Q} = E : \Lambda \times \Lambda \to \mathbb{Z}$ and $\hat{E} : M \times M \to \mathbb{Z}$ be the associated skew-symmetric forms as defined in Proposition 3.6. Let $\{\omega_\nu\}$ be a basis of $U$, let $\{\lambda_\nu\}$ be a symplectic basis of $\Lambda$ with respect to $E$ and let $\Pi$ be the corresponding period matrix

$$(9) \quad \Pi'(\omega_1, \ldots, \omega_g) = \Pi'(\lambda^*_1, \ldots, \lambda^*_g).$$
The matrix of $\hat{E}$ in $\{\lambda^*_\rho\}$ is $(\hat{E}(\lambda^*_\rho, \lambda^*_\rho)) = (0_{-D^{-1}} \ 1_{-D^{-1}})$ where $D = \text{diag}(d_1, \ldots, d_g)$. Since $Q = -c\hat{E}$, $c \in \mathbb{N}$, the Riemann relations of (3.11) may be written as (a) $(\hat{E}(\omega_{\alpha}, \omega_{\beta})) = 0$, (b) $i(\hat{E}(\omega_{\alpha}, \omega_{\beta})) < 0$ or equivalently in matrix form for $\Pi = (\Pi_1, \Pi_2)$

$$
\begin{pmatrix}
(P_1 & P_2)
\begin{pmatrix}
0 & D^{-1} \\
-D^{-1} & 0
\end{pmatrix}
\begin{pmatrix}
(P_1 & P_2)
\begin{pmatrix}
0 & D^{-1} \\
-D^{-1} & 0
\end{pmatrix}
\begin{pmatrix}
< 0.
\end{pmatrix}
\end{pmatrix}
$$

In this form the relations are the same as in [LB] p.77. Changing the basis of $U$ by $t(\phi_1, \ldots, \phi_g) = A^t(\omega_1, \ldots, \omega_g)$ and the simplectic basis of $\Lambda$ by $(\mu_1, \ldots, \mu_2g) = (\lambda_1, \ldots, \lambda_{2g})$ one obtains an equivalent period matrix $\Pi' \sim \Pi$, $P' = A\Pi R$. The same argument as in [Gr] p.577 shows that det $\Pi_1 \neq 0$, det $\Pi_2 \neq 0$, so every $\Pi$ is equivalent to $D\Pi_2^{-1}(\Pi_1, \Pi_2) = (ZD)$ with $Z \in \mathcal{S}_g$ as follows from Riemann’s relations. The matrix $R \in M_{2g}(\mathbb{Z})$ satisfies the equality $tR(a_{-D} b_{-D}) = (c_{-D} d_{-D})$. If we let $R = t(c_{-D} d_{-D})$, then $(c_{-D} d_{-D}) \in G_D = Sp^D(\mathbb{Z})$, the group defined in [LB] p.219. Multiplying on the right a normalized period matrix $(ZD)$ by $R = t(c_{-D} d_{-D})$ and then normalizing one obtains $(Z'D)$ where $Z' = (aZ + bD)(D^{-1}cZ + D^{-1}dD)^{-1}$. This is the left action of $G_D$ on $\mathcal{S}_g$ defined in [LB] p.219. The quotient $G_D/\mathcal{S}_g$ is the moduli space $A_D$ for polarized abelian varieties of type $D$ (ibid). In conclusion one obtains a correspondence

$$
(10) \quad (M, M_C = U \oplus \overline{U}, Q) \quad \mapsto \quad Z(\text{mod } G_D) \in A_D.
$$

Using the dual polarized Hodge structure $(\Lambda, \Lambda_C = W \oplus \overline{W}, \overline{Q})$ one obtains similarly another correspondence

$$
(11) \quad (M, M_C = U \oplus \overline{U}, Q) \quad \mapsto \quad \hat{Z}(\text{mod } G_D) \in A_D.
$$

Comparing with Proposition 3.6 we see that (10) and (11) associate to a polarized Hodge structure of weight one respectively the isomorphism classes $[A(U)] \in A_D$ and $[P(U)] \in A_D$.

3.11. Let $T$ be a connected complex manifold. A polarized variation of Hodge structure of weight one (VHS) over $T$ is given by the following data.

i. A flat bundle of rank $2g$ lattices $\mathbb{M} \to T$.

ii. A holomorphic rank $g$ subbundle $F \subset M_C$ such that $F \oplus \overline{F} = M_C$.

iii. A flat skew-symmetric form $Q : \mathbb{M} \times \mathbb{M} \to \mathbb{Z}$ which satisfies fiberwise the Riemann relations of (3.11).

Given a VHS one may consider the dual VHS $L = \text{Hom}_Z(M, Z)$, $L_C = G \oplus \overline{G}$, where $G = (F)^{\perp}$, $G = (F)^{\perp}$ and the polarization $\hat{Q}$ is obtained from $Q$ as described in the beginning of 3.11. Dividing by an appropriate negative integer $-c$ one obtains flat, integer valued, skew-symmetric forms $E = \hat{Q}$, $\hat{D} = \frac{-1}{c}Q$ such that for each $s \in T$ the forms $E_s$, $\hat{D}_s$ are respectively polarizations of types $D = (1, d_2, \ldots, d_g)$, $\hat{D} = (1, \hat{d}_2, \ldots, \hat{d}_g)$ of the associated complex tori $A(F_s)$, $P(F_s)$ as in Proposition 3.6. Let $S \subset T$ be an open set in the Hausdorff
topology over which \( M \) and \( \mathbb{F} \) are trivial. Choosing a frame \( \lambda_1, \ldots, \lambda_{2g} \) of \( \mathcal{L}|_S \), a frame \( \omega_1, \ldots, \omega_g \) of \( \mathcal{F}|_S \) and normalizing the associated period matrices as in (3.10) one obtains a holomorphic mapping \( \Phi : S \to \mathfrak{H}_g \). Passing to the quotient \( \Gamma_D/\mathfrak{H}_g = A_D \) one obtains a holomorphic mapping \( \Phi_S : S \to A_D \). Covering \( T \) with such open sets \( T = \cup S_i \) and gluing \( \Phi_{S_i} \) as evident from (3.10) one obtains the period mapping \( \Phi : T \to A_D \). One has \( \Phi(s) = [A(\mathbb{F}_s)] \). The same argument applied to the dual VHS yields a dual period mapping \( \hat{\Phi} : T \to A^\ast_D \) such that \( \hat{\Phi}(s) = [P(\mathbb{F}_s)] \). This proves part of the following statement.

**Proposition 3.12.** Let \( T \) be a connected complex manifold and let \( \mathcal{M} = \mathbb{F} \oplus \mathbb{F}_L \), \( Q : \mathcal{M} \times \mathcal{M} \to \mathbb{Z} \) be a polarized variation of Hodge structures of weight one. Let \( D \) and \( D^\ast \) be the dual polarization types as defined in (3.11). Then one can define period mappings \( \Phi : T \to A_D \) and \( \Phi^\ast : T \to A_D^\ast \) which transform \( s \in T \) respectively into the isomorphism classes of polarized abelian varieties \( \Phi(s) = [A(\mathbb{F}_s)] \), \( \Phi^\ast(s) = [P(\mathbb{F}_s)] \). If \( T \) is algebraic, so are the period mappings \( \Phi \) and \( \Phi^\ast \). The mapping \( \Phi^\ast \) is dominant if and only if \( \Phi \) is dominant. \( \square \)

**Proof.** That \( \Phi \) and \( \Phi^\ast \) are algebraic if \( T \) is algebraic follows from Borel’s extension theorem [Bo] Theorem 3.10. The last statement follows from comparing the differentials of \( \Phi \) and \( \Phi^\ast \) by means of (\ref{eq:diff}).

**3.13.** We now consider a family of coverings of curves and associate to it two dual VHS. Suppose we are given a commutative diagram of holomorphic mappings

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{p} & \mathcal{Y} \\
\downarrow f & & \downarrow q \\
T & & \\
\end{array}
\]

where \( T \) is a connected complex manifold, \( f \) and \( q \) are smooth, proper of relative dimension one and \( p \) is surjective. Then we have the standard VHS associated with \( f \) and \( q \): \( H_X = R^1f_*\mathbb{Z}, \ F_X \subset H_X \otimes \mathbb{C} \) where \( \mathcal{O}_T(F_X) = f_*\Omega^1_{X/T} \) and similarly for \( q \). One has a morphism of VHS \( p^* : (H_Y, F_Y) \to (H_X, F_X) \). Define \( M, F \subset M_C \) as follows (cf. (3.3))

\[
\mathcal{M} = R^1f_*\mathbb{Z}/R^1f_*\mathbb{Z} \cap p^*R^1q_*\mathbb{R}, \quad \mathcal{O}_T(F) = f_*\Omega^1_{X/T}/p^*q_*\Omega^1_{Y/T}.
\]

We obtain a dual VHS letting \( L = M^\ast \), \( G = (F)^\perp \). Let \( \hat{Q} : R^1f_*\mathbb{Z} \times R^1f_*\mathbb{Z} \to R^2f_*\mathbb{Z} = \mathbb{Z} \) be the cup-product which is an unimodular polarization of \( (H_Y, F_X) \) (cf. Example 3.2). According to (3.5) we let \( \hat{Q} : (H^\ast_X \times H^\ast_X) \to \mathbb{Z} \) be the dual polarization. Since \( M \) is a quotient of \( H_X \) the dual \( L = M^\ast \) may be embedded in \( H^\ast_X \). Restricting \( \hat{Q} \) on \( L \) and dividing the obtained flat skew-symmetric form by its least elementary divisor one obtains a polarization \( \hat{Q} : L \times L \to \mathbb{Z} \). Its dual \( Q : M \times M \to \mathbb{Z} \) polarizes the VHS \( (\mathcal{M}, F) \). The flat forms \( E = -\hat{Q} \) and \( \hat{E} = -Q \) polarize fiberwise respectively the associated complex tori \( Ker(Nm_p^\ast) \) and \( Pic^0X^\ast/p^\ast_sPic^0Y^\ast \) for \( \forall s \in T \). Applying Proposition 3.12 we obtain the following result.
**Proposition 3.14.** Let \( p : \mathcal{X} \to \mathcal{Y} \) be a covering of smooth \( T \)-curves over a smooth connected algebraic base \( T \) (cf. [12]). Fix \( o \in T \) and suppose the restriction of the intersection form \((\ , \ )_{\mathcal{X}_o} \) on \( \text{Ker}((p_o)_* : H_1(\mathcal{X}_o, \mathbb{Z}) \to H_1(\mathcal{Y}_o, \mathbb{Z})) \) has elementary divisors \((m, md_2, \ldots, md_g), \ d_i|d_{i+1}\). Let \( \mathcal{D} = (1, d_2, \ldots, d_g) \), \( \hat{\mathcal{D}} = (1, \hat{d}_2, \ldots, \hat{d}_g) \) where \( \hat{d}_i = d_g/d_{g-i+1} \). Then the period mappings \( \Phi : T \to \mathcal{A}_D \) and \( \hat{\Phi} : T \to \mathcal{A}_{\hat{D}} \) given by \( \Phi(s) = [\text{Ker}(Nm_{p_s})^0] \) and \( \hat{\Phi}(s) = [\text{Pic}^0 \mathcal{X}_s/p_s^* \text{Pic}^0 \mathcal{Y}_s] \) are algebraic morphisms. The morphism \( \Phi \) is dominant if and only if the morphism \( \hat{\Phi} \) is dominant. If this is the case and if \( T \) is unirational, then both \( \mathcal{A}_D \) and \( \mathcal{A}_{\hat{D}} \) are unirational varieties.

**Remark 3.15.** A recent result of Birkenhake and Lange [BL3] shows that \( \mathcal{A}_D \) and \( \mathcal{A}_{\hat{D}} \) are in fact isomorphic to each other.

**Example 3.16.** In [RCV] p.124 it is proved that \( \mathcal{A}_4(1, 2, 2, 2) \) is unirational considering the Prym mapping for a family as in [12] where \( T \) is a Zariski open set in \( |\mathcal{O}_{\mathbb{P}^2}(4)(-p_1 - p_2 - p_3 - p_4)| \), where \( p_1, \ldots, p_4 \) are general points in \( \mathbb{P}^2 \), \( q : \mathcal{Y} \to T \) is the corresponding family of plane quartics and \( p : \mathcal{X} \to \mathcal{Y} \) is a suitable double covering. From Proposition 3.14 it follows that every general abelian variety of dimension four with polarization of the type \((1, 1, 1, 2)\) is isomorphic to \( \text{Pic}^0 \mathcal{X}_s/p_s^* \text{Pic}^0 \mathcal{Y}_s \) for some \( s \in T \) and \( \mathcal{A}_4(1, 1, 1, 2) \) is unirational. The latter follows of course from the result of Birkenhake and Lange cited above.

**Question 3.17.** Is it true that the moduli space \( \mathcal{A}_4(1, 1, 2, 2) \) is unirational?

**3.18.** We now wish to give a formula for the differential of the period mapping \( \Phi \) of the VHS considered in Proposition 3.14. Let us first consider the general set-up of polarized VHS of weight one. Since the problem is local replacing \( T \) by a smaller open set \( S \) we may restrict ourselves to the case where \( M = M \times S \) is constant and the holomorphic subbundle \( \mathcal{F} \subset M \) is trivial. Let us fix a basis \( \lambda_1^*, \ldots, \lambda_g^* \) of \( M \) and a frame \( \omega_1, \ldots, \omega_g \) of \( \mathcal{F} \). Transposing \( \Phi \) we may write

\[
(\omega_1(s), \ldots, \omega_g(s)) = (\lambda_1^*, \ldots, \lambda_g^*)^{t} \Pi(s).
\]

By the first Riemann relation \((3.1\text{iii})\) every \( g \)-plane \( \mathcal{F}(s) \subset M \) is isotropic with respect to \( Q. \) Let us denote by \( \hat{\mathcal{D}} \subset Gr(g, M) \) the simplectic grassmanian of isotropic \( g \)-planes and let \( \mathcal{D} \subset \hat{\mathcal{D}} \) be the open subset of those \( g \)-planes satisfying the second Riemann relation \([GS] \) p.54. One obtains a holomorphic mapping \( \hat{\Phi} : S \to \mathcal{D} \subset \hat{\mathcal{D}} \), \( \hat{\Phi}(s) = \mathcal{F}(s) \). Considering \( g \)-planes is equivalent to taking quotient modulo the equivalence relation \( ^t \Pi \sim ^t \Pi ^t A, \quad A \in GL(g, \mathbb{C}) \). Thus \( \hat{\Phi} \) is a coordinate free description of the mapping \( \hat{\Phi}_S : S \to \mathfrak{g}_g \) considered in \([5.11]\).

Let \( \mathcal{U} \to \hat{\mathcal{D}} \) be the tautological vector bundle. The cotangent bundle \((T \hat{\mathcal{D}})^t\) is isomorphic to \( Sym^2 \mathcal{U} \). Fiberwise this isomorphism is explicitly described as follows. If \( z = [U \subset M] \in \hat{\mathcal{D}} \) the vector \( \phi \in \text{Hom}(U, M/C/U) = T_z Gr(g, M) \) belongs to the tangent space to \( \hat{\mathcal{D}} \) if and only if \( Q(\phi(u), v) + Q(u, \phi(v)) = 0 \) for all \( u, v \in U \). Since \( Q \) is skew-symmetric this is equivalent to saying that the bilinear form \( q_\phi(u, v) = Q(\phi(u), v) \) is symmetric. Considering the trilinear form

\[
T \hat{\mathcal{D}} \times \mathcal{U} \times \mathcal{U} \to \mathbb{C}, \quad (\phi, u, v) \mapsto Q(\phi(u), v)
\]
yields the isomorphisms

\[ T\tilde{D} \xrightarrow{\sim} Sym^2\mathbb{U}^*, \quad \phi \mapsto q_\phi, \quad q_\phi(u, v) = Q(\phi(u), v), \]
\[ Sym^2\mathbb{U} \xrightarrow{\sim} (T\tilde{D})^*, \quad \langle \phi, u \otimes v \rangle = Q(\phi(u), v). \]

Consider the period mapping \( \tilde{\Phi} : S \to \tilde{D} \). Let \( s \in S \). The differential of \( \tilde{\Phi} \) at \( s \in S \) yields a trilinear form

\[ T_sS \times \mathbb{F}(s) \times \mathbb{F}(s) \to \mathbb{C}, \quad \left( \frac{\partial}{\partial \tau}, \omega_1, \omega_2 \right) = Q(d\tilde{\Phi} \left( \frac{\partial}{\partial \tau} \right)(\omega_1, \omega_2). \]

From (14) one obtains a formula for \( ^t d\tilde{\Phi}(s) \):

\[ ^t d\tilde{\Phi}(s) : Sym^2\mathbb{F}(s) \to (T_sS)^*, \quad \left( \frac{\partial}{\partial \tau}, ^t d\tilde{\Phi}(\omega_1 \cdot \omega_2) \right) = Q(d\tilde{\Phi} \left( \frac{\partial}{\partial \tau} \right)(\omega_1, \omega_2), \]

where \( \cdot \) denotes the product in the symmetric algebra induced from \( \otimes \).

3.19. Let us apply the above to the VHS associated with a surjective holomorphic mapping \( p : X \to Y \) of smooth families of curves over \( T \) as in (3.13). Let \( s_0 \in T \) and let \( X = X_{s_0}, \quad Y = Y_{s_0}, \quad \pi = p_{s_0} \). We may replace \( T \) by a smaller neighborhood \( S \) of \( s_0 \) such that \( f : X \to S \) and \( q : Y \to S \) are \( C^\infty \)-trivial fibrations and all bundles occurring in the constructions of (3.13) are trivial (in the corresponding category). Abusing notation let us denote by \( \pi_* \) both the homomorphism \( \pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z}) \) and the Gysin homomorphism \( D_Y \circ \pi_* \circ D_X^1 : H^1(X, \mathbb{Z}) \to H^1(Y, \mathbb{Z}) \), where \( D_X, D_Y \) are the Poincaré isomorphisms. Let \( (\quad)_- \) denote the kernel of \( \pi_* \) on the corresponding object. The Gysin homomorphism preserves the Hodge type and coincides with the trace map \( Tr_{p*} \) on \( H^{1,0}(X_s) = H^0(X_s, \Omega^1_{X_s}) \). We thus obtain a VHS as follows: \( M' = H_1(X, \mathbb{C})^- \), \( M' = M' \times S, \quad \mathbb{F}^- = \mathbb{M}'c_1 \cap \mathbb{F}_X = Ker(\pi_* : \mathbb{F}_X \to \mathbb{F}_Y) \). Notice that for the Weyl operators one has \( D_X \circ (-c_{X_s}) = c_{X_s} \circ D_X \), thus \( D_X(H_1(X, \mathbb{Z})^-) = H^1(X, \mathbb{Z})^- \), \( D_X(\mathbb{G}) = \mathbb{F}^- \), \( D_X(\mathbb{G}) = \mathbb{F}^- \) (cf. (3.13)). Let \( Q^- \) be the restriction of \( Q \) on \( \mathbb{M}' \). Thus \( Q^- \) is the constant skew-symmetric form induced from

\[ Q^- (\omega_1, \omega_2) = \int_X \omega_1 \wedge \omega_2, \quad \text{where} \quad \omega_1, \omega_2 \in H^1(X, \mathbb{Z})^- \].

The canonical homomorphism \( j : H^1(X, \mathbb{Z})^- \to H^1(X, \mathbb{Z})/\pi^*H^1(Y, \mathbb{Z}) \) of lattices of equal rank has kernel 0 and induces a morphism of VHS \( (\mathbb{M}', \mathbb{F}') \to (\mathbb{M}, \mathbb{F}) \). We wish to compare the polarization \( Q \) on \( (\mathbb{M}, \mathbb{F}) \) as defined in (3.13) with \( Q^- \) from (17).

**Lemma 3.20.** Suppose the restriction of the intersection form \( (\quad, \quad)_X \) on \( H_1(X, \mathbb{Z})^- \) has elementary divisors \( (m, md_2, \ldots, md_g) \), \( d_id_{i+1} \). Then \( j^*Q = md_gQ^- \)

**Proof.** Using the notation of (3.8) and (3.13) we have \( \Lambda = H_1(X, \mathbb{Z})^- \), \( (\quad, \quad)_X|_\Lambda = -mE \). The homomorphism \( \varphi : \Lambda \to \Lambda^* = M \) is given by \( \langle \delta, \varphi(\lambda) \rangle = E(\lambda, \delta) \) and one has

\[ \varphi^*(Q) = \varphi^*(-\hat{E}) = \varphi^*(-d_gE^*) = -d_gE = \frac{d_g}{m}(\quad, \quad)_X|_\Lambda. \]

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On the other hand the Poincaré isomorphism $D_X : H_1(X, \mathbb{Z}) \to H^1(X, \mathbb{Z})$ transforms $\Lambda = H_1(X, \mathbb{Z})^{-}$ onto $H^1(X, \mathbb{Z})^{-}$ and one has $D_X^*(\lambda, \delta) = (\lambda, \delta)|_X$. Consider the composition mapping

$$\phi' : \Lambda \longrightarrow H_1(X, \mathbb{Z}) \xrightarrow{D_X} H^1(X, \mathbb{Z}) \xrightarrow{j} H^1(X, \mathbb{Z})/\pi^*H^1(Y, \mathbb{Z}).$$

One has for every $\lambda, \delta \in \Lambda$

$$\langle \delta, \phi'(\lambda) \rangle = \langle \delta, D_X(\lambda) \rangle = (\lambda, \delta) = -mE(\lambda, \delta) = \langle \delta, -m\phi(\lambda) \rangle.$$

Hence $\phi' = -m\phi$ and $(\phi')^*(\lambda) = m^2\phi^*(\lambda) = \delta_{\phi}(\lambda)$. Comparing the equality $D_X^* \circ j^*(\lambda) = (\phi')^*(\lambda) = \delta_{\phi}'(\lambda)$ with $D_X^*(\lambda, \delta) = (\lambda, \delta)|_X$ we conclude that $j^*(\lambda) = \delta_{\phi}'(\lambda)$.

We conclude from the lemma that via the isomorphism $j : M' \sim M_C$ the mapping $\Phi : S \to \mathbb{D} \subset \mathbb{H}$ associated with the polarized VHS $(\mathbb{M}, \mathbb{F}, \mathbb{Q})$ may be identified with the mapping associated with $(\mathbb{M'}, \mathbb{F}', \mathbb{Q}')$. Via this identification $\mathbb{D}$ is the simplectic grassmannian of $g$-planes in $H^1(X, \mathbb{C})^{-}$ isotropic with respect to $Q_C(\omega_1, \omega_2) = \int_X \omega_1 \wedge \omega_2$ and $\Phi(s) = H^0(X, \Omega_X^{-})$.

**Proposition 3.21.** Let $p : \mathcal{X} \to \mathcal{Y}$ be a surjective holomorphic mapping of smooth families of curves over $T$ as in (12). Let $\mathbb{M}, \mathbb{F}, \mathbb{Q}$ be the polarized variation of Hodge structures defined in (3.13). Let $s_0 \in T$, $X = X_{s_0}$, $Y = Y_{s_0}$, $\pi = p_{s_0}$ and let $H^0(X, \Omega_X^{-}) = \text{Ker}(Tr_{s_0} : H^0(X, \Omega_X) \to H^0(Y, \Omega_Y))$. Let $S$ be a small neighborhood of $s_0$ as in (3.18) and let $\Phi : S \to \mathbb{D} \subset \mathbb{H}$ be the local period mapping. Let $p : T_{s_0}S \to H^1(X, TX)$ be the Kodaira-Spencer mapping. Then the differential of $\Phi$ at $s_0$ is given by the following formula, where $(\ , \ )$ is the Serre duality pairing (in the next formula $\omega_1 \cdot \omega_2$ denotes the product in the symmetric algebra, while $\omega_1 \omega_2$ is a quadratic differential).

$$\partial d\Phi(s_0) : \text{Sym}^2 H^0(X, \Omega_X^{-}) \longrightarrow (T_{s_0}S)^*, \quad \langle \frac{\partial}{\partial t}, \partial d\Phi(\omega_1 \cdot \omega_2) \rangle = (\rho(\frac{\partial}{\partial t}), \omega_1 \omega_2)$$

**Proof.** Let $\frac{\partial}{\partial t} \in T_{s_0}S$ and let $Z \subset S$ be a smooth, complex analytic curve tangent to $T_{s_0}S$ at $s_0$. The restricted family $\mathcal{X}_Z \to Z$ yields the standard VHS $H^{1,0}(\mathcal{X}_u) \subset H^1(X, \mathbb{C})$, $u \in Z$. By [G] p.816 the differential of the period mapping of this VHS transforms $\frac{\partial}{\partial t}$ into an element of $H^{1,0}(\mathcal{X}_u, H^{0,1}(X))$ given by the cup-product $\omega \mapsto \rho(\frac{\partial}{\partial t}) \circ \omega$ (i.e. $\cup$ composed with contraction). From the splitting

$$H^{1,0}(\mathcal{X}_u) = H^{1,0}(\mathcal{X}_u) \oplus \pi^*H^{1,0}(\mathcal{Y}_u) \subset H^1(X, \mathbb{C})^{-} \oplus \pi^*H^1(Y, \mathbb{C})$$

one concludes that $d\Phi(\frac{\partial}{\partial t}) = \phi \in Hom(H^{1,0}(X), H^{0,1}(X)^{-})$ is given as well by the cup product: $\phi(\omega) = \rho(\frac{\partial}{\partial t}) \circ \omega$. Applying (10) we have that for $\forall \omega_1, \omega_2 \in H^0(X, \omega_X)$

$$\langle \frac{\partial}{\partial t}, \partial d\Phi(\omega_1 \cdot \omega_2) \rangle = Q^{-}(\phi(\omega_1), \omega_2) = \int_X \left(\rho(\frac{\partial}{\partial t}) \circ \omega_1\right) \wedge \omega_2.$$
We may replace in this integral \( \rho(\frac{\partial}{\partial \tau}) \circ \omega_1 \) by \( v^{0,1} \wedge \omega_1 \), where \( v^{0,1} \in H^0(X, \mathcal{T}_X) \) is a Dolbeault representative of \( \rho(\frac{\partial}{\partial \tau}) \). The new integrand \((v^{0,1} \wedge \omega_1) \wedge \omega_2 \) is a \((1,1)\) form which is given locally by

\[
((a \frac{\partial}{\partial z} \otimes d\bar{z}) \wedge b dz) \wedge c dz = ab d\bar{z} \wedge c dz = abc d\bar{z} \wedge dz.
\]

Locally \( \omega_1 \wedge \omega_2 = bc(dz)^2 \), so \((v^{0,1} \wedge \omega_1) \wedge \omega_2 \) equals \( v^{0,1} \wedge (\omega_1 \wedge \omega_2) \). Integrating this \((1,1)\) form we obtain by the definition of Serre’s duality the required formula for \( 'd\Phi \).

\[\square\]

## 4 Local study of the Prym mapping

4.1. Let \( Y \) be a smooth, projective curve of genus \( \geq 0 \) and let \( \pi: X \to Y \) be a simple covering of degree \( d \geq 2 \) branched in \( n \) points \( B = \{b_1, \ldots, b_n\} \). Let \( y_0 \in Y - B \) and let \( \Delta_i \) be a small open disk centered at \( b_i, \quad i = 1, \ldots, n \). Fixing the monodromy \( m: \pi_1(Y - \bigcup_{i=1}^n \Delta_i, y_0) \to S_d \) and varying the branch points in \( \Delta_i \) one obtains a family of \( d \)-sheeted coverings \((\Psi, f): X \to Y \times H \) where \( H = \Delta_1 \times \cdots \times \Delta_n \). Let \( w_i \) be local coordinates of \( Y \) in \( \Delta_i \) satisfying \( w_i(b_i) = 0 \).

We define coordinates \( t = (t_1, \ldots, t_n) \) in \( H \) by \( t_i(y_1, \ldots, y_i, \ldots, y_n) = w_i(y_i) \). At points of ramification which project to \( \Delta_i \) the mapping \( \Psi \) is given by \( w_i = \Psi_i(z_i, t) = z_i^2 + t_i \) and \( f(z_i, t) = t \).

4.2. Suppose first \( g(Y) = 1 \). Choose a point \( c \notin \{b_1, \ldots, b_n, y_0\} \) and a small disk \( D \) centered at \( c \). Let \( \pi^{-1}(D) = D_1 \cup \ldots \cup D_d \) be a disjoint union of disks biholomorphically equivalent to \( D \). Let \( x_i = \pi^{-1}(c) \cap D_i \). Let \( v \) be a local coordinate in \( D \) with \( v(c) = 0 \). Let \( u_i = \pi^*(v)|_{D_i} \) be the corresponding local coordinates in \( D_i \). We consider a Schiffer variation of \( Y \) (we follow here \[\text{ACGH}\] Vol. II). Namely paste together \( A \times \Delta := \{\zeta: |\zeta| \leq \epsilon\} \times \{s: |s| < r\} \) with \((Y - \{v: |v| < \frac{\epsilon}{2}\}) \times \Delta\) by means of \( \zeta = v + \frac{u}{v} \). Here \( 0 < \epsilon \ll 1 \) and \( r \) depends on \( \epsilon \). One obtains a family \( q: \mathcal{Y} \to N = \Delta \) with \( \mathcal{Y}_0 \cong \mathcal{Y} \). Performing the corresponding pastings by means of

\[
\zeta_i = u_i + \frac{s}{u_i} \quad \text{at all} \quad D_i
\]

and varying the branch points as in the preceding paragraph one obtains a smooth proper mapping \( f: X \to N \times H \) and a holomorphic mapping \( p: X \to \mathcal{Y} \) which fit into a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{p} & \mathcal{Y} \\
\downarrow f & & \downarrow q \\
N \times H & \xrightarrow{\pi_1} & N
\end{array}
\]

If \( g(Y) \geq 2 \) one chooses \( m = 3g(Y) - 3 \) general points \( c_1, \ldots, c_m \) on \( Y \) which impose independent conditions on \( H^0(Y, \omega_Y^2) \), one repeats the same pasting procedure simultaneously at \( c_i, \quad i = 1, \ldots, m \) and obtains a family \( q: \mathcal{Y} \to N = \Delta^m \) with \( \mathcal{Y}_o \cong \mathcal{Y} \). In both cases,
$g(Y) = 1$ or $g(Y) \geq 2$, $q : \mathcal{Y} \to N = \Delta^m$ is a minimal versal deformation of $Y$ since the Kodaira-Spencer mapping $\rho : T_oN \to H^1(Y, T_Y)$ is an isomorphism [KS]. Similarly to the above one obtains a holomorphic mapping $p : \mathcal{X} \to \mathcal{Y}$ and a smooth, proper mapping $f : \mathcal{X} \to N \times H$ which fit into a commutative diagram as in (19). The constructed family of coverings is versal for deformations of $\pi : X \to Y$ as shows the following proposition.

**Proposition 4.3.** Let $\pi : X \to Y$ be a covering of a smooth, projective curve of genus $g(Y) \geq 1$ simply branched in $B \subset Y$. Let

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{p'} & \mathcal{Y}' \\
\downarrow{f'} & & \downarrow{q'} \\
T & \xrightarrow{h} & Z
\end{array}
$$

be a commutative diagram of holomorphic mappings of complex manifolds, where $f'$ and $q'$ are proper, smooth and surjective, $p'$ is finite and surjective, and there is a point $s_0 \in T$ such that the covering $p'_{s_0} : \mathcal{X}'_{s_0} \to \mathcal{Y}'_{h(s_0)}$ is isomorphic to $\pi : X \to Y$. Then there exist neighborhoods $S$ and $V$ with $s_0 \in S \subset T$, $h(s_0) \in V \subset Z$, $h(S) \subset V$ and holomorphic mappings $\mu$ and $\nu$ which fit into the commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{h} & V \\
\downarrow{\mu} & & \downarrow{\nu} \\
N \times H & \xrightarrow{\pi_1} & N
\end{array}
$$

such that the restricted family of coverings $p' : \mathcal{X}'_S \to \mathcal{Y}'_V$ is the pull-back via $\mu, \nu$ of the covering (19).

**Proof.** As mentioned above the existence of $V \ni h(s_0)$ and a holomorphic mapping $\nu : V \to N$ such that $\mathcal{Y}'_V \cong \mathcal{Y} \times_N V$ follows from Kodaira-Spencer's theorem of completeness [KS]. We may replace $T$ by $h^{-1}(V)$ and $f' : \mathcal{X}' \to T$ by the corresponding restriction. We then obtain a deformation into the family $q : \mathcal{Y} \to N$ as defined in [Ho2] § 5:

$$
\begin{array}{ccc}
\mathcal{X}' & \xrightarrow{f'} & \mathcal{Y} \\
\downarrow{\nu \circ h} & & \downarrow{q} \\
T & \xrightarrow{\nu \circ h} & N
\end{array}
$$

An easy calculation which uses [Ho2] Lemma 5.1 shows that the characteristic map $\tau : T_{(s_0, B)}N \times H \to D_\mathcal{X}/\mathcal{Y}$ of the deformation (19) is an isomorphism. Therefore by Horikawa's theorem of completeness [Ho2] Theorem 5.2 there is a neighborhood $S$ of $s_0 \in T$ and a holomorphic mapping $\mu : S \to N \times H$ such that the restriction of (20) on $S$ is isomorphic to the pull-back of (19) by $\mu$. This proves the proposition. \qed

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Let \( Y \) be a curve of genus \( g \geq 0 \) and let \( \pi : X \to Y \) be a simple covering of degree \( d \geq 2 \) with \( n \geq 2 \) branch points. Let \( C \to M \), \( C_0 \cong X \) be a minimal versal deformation of \( X \). Let \( (\Psi, f) : \mathcal{X} \to Y \times H \) be the (local Hurwitz) family considered in [4.1]. Shrinking \( H \) if necessary, there is a holomorphic mapping \( h : H \to M \) such that \( f : \mathcal{X} \to H \) is the pull-back of \( C \to M \) via \( h \).

**Proposition 4.4.** Let \( \pi : X \to Y \) be as above, let \( B = \{b_1, \ldots, b_n\} \subset Y \) be the branch locus and let \( R = \{a_1, \ldots, a_n\} \subset X \), \( \pi(a_i) = b_i \) be the ramification locus. Let \( w_i, w_i(b_i) = 0 \) be local coordinates of \( Y \) at \( b_i, i = 1, \ldots, n \) and let \( t = (t_1, \ldots, t_n) \) be the corresponding coordinates of \( H \) as defined in [4.1]. Then the transpose of the differential of \( h \)

\[
(22) \quad (dh)_o : H^0(X, \omega_X^2) \cong (T_0M^*) \to (T_BH)^* \cong \bigoplus_{i=1}^n C(d_t)_o
\]

is given by the formula

\[
(21) \quad \varphi \mapsto 2\pi \sqrt{-1} \sum_{i=1}^n \text{Res}_{a_i} \frac{\varphi}{\pi^* dw_i} (dt_i)_o
\]

**Proof.** We are in the situation considered by Horikawa in [Ho1]. At the ramification points of \((\Psi, f) : \mathcal{X} \to Y \times H\) the mapping \( \Psi : \mathcal{X} \to Y \) is given by (cf. [4.1]) \( w_i = \Psi_i(z_i, t) = z_i^2 + t_i \). Consider as in [Ho1] the sequence

\[
0 \to T_X \xrightarrow{dz} \pi^* T_Y \to \mathcal{T} \to 0.
\]

Here \( \mathcal{T} \) is a skyscraper sheaf (non-canonically) isomorphic to \( \bigoplus_{i=1}^n C_{a_i} \). Then Horikawa’s characteristic mapping \( \tau : T_BH \to H^0(X, \mathcal{T}) \) is given by

\[
\tau(\frac{\partial}{\partial t_i}) = \frac{\partial}{\partial w_i} \text{ mod } d\pi(X)_{a_i} \in C_{a_i} \subset \bigoplus_{k=1}^n C_{a_k}.
\]

Consider the exact cohomology sequence

\[
0 \to H^0(X, T_X) \to H^0(X, \pi^* T_Y) \to H^0(X, \mathcal{T}) \xrightarrow{\delta} H^1(X, T_X).
\]

The Kodaira-Spencer mapping equals the composition \( \rho = \delta \circ \tau \) [Ho1] p.376. In order to prove (21) it suffices to verify that for every \( \varphi \in H^0(X, \omega_X^2) \) and every \( i = 1, \ldots, n \) for the Serre duality pairing one has

\[
(22) \quad (\rho(\frac{\partial}{\partial t_i}) \cdot \varphi) = 2\pi \sqrt{-1} \text{Res}_{a_i} \left( \frac{\varphi}{\pi^* dw_i} \right).
\]

For every \( i = 1, \ldots, n \) let \( U_i \) be a small disc containing \( a_i \) with local coordinate \( z_i, z_i(a_i) = 0 \), such that \( \pi(U_i) \subset \Delta_i \) and such that \( \pi \) is locally given by \( w_i = z_i^2 \). Let \( U_0 = X - R \). Consider the Stein covering \( \mathcal{U} = \{U_0, U_1, \ldots, U_n\} \). By the definition of \( \tau(\frac{\partial}{\partial t_i}) \) it is immediate that \( \rho(\frac{\partial}{\partial t_i}) = \delta(\tau(\frac{\partial}{\partial t_i})) \) is given by a 1-cochain \( \{\xi_{\alpha\beta}\} \in C^1(\mathcal{U}, T_X) \), such that \( \xi_{i0} = \frac{1}{2z_i} \frac{\partial}{\partial z_i} = -\xi_{0i} \) while the other \( \xi_{\alpha\beta} = 0 \). Let \( \varphi = f_i(z_i)(dz_i)^2 \) in \( U_i \). Then \( \xi_{i0} \cdot \varphi = \frac{f_i(z_i)}{2z_i} dz_i = \frac{\varphi}{\pi^* dw_i} \). A calculation similar to the one in [ACGH] pp.14,15 shows (22). \( \square \)
Corollary 4.5. Let the hypothesis be as in Proposition 4.4. Then the annihilator of \(dh(T_BH)\) in \((T_oM)^*\) equals \(H^0(X,\omega_X^2(-R))\)

Remark 4.6. When \(Y = \mathbb{P}^1\) this result was proved by Donagi and Green by a different argument [DS] Appendix.

Let us consider now the case of a simple branched covering \(\pi : X \to Y\) of degree \(d\) where \(g(Y) \geq 1\). Let \(q : Y \to N\) be the minimal versal deformation of \(Y\) constructed by Schiffer variations at non-branched points of \(Y\) (cf. [4,2]) and let us consider the deformation [19]. Let \(C \to M,\ C_o \cong X\) be a minimal versal deformation of \(X\). Shrinking \(N\) and \(H\) if necessary, there exists a holomorphic mapping \(h : N \times H \to M\) such that \(f : X \to N \times H\) is the pull-back of \(C \to M\) via \(h\).

Proposition 4.7. Let \(\pi : X \to Y\) be as above, let \(B = \{b_1, \ldots, b_n\} \subset Y\) be the branch locus and let \(R = \{a_1, \ldots, a_n\} \subset X,\ \pi(a_i) = b_i\) be the ramification locus. Let \(m = 1\) if \(g(Y) = 1\) and let \(m = 3g(Y) - 3\) if \(g(Y) \geq 2\). Let \(c_1, \ldots, c_m\) be general points in \(Y\) and let \(\pi^{-1}(c_i) = \{x_{i1}, \ldots, x_{id}\}\). Let \(v_i\) be local coordinates at \(c_i,\ i = 1, \ldots, m\) and let \(w_j\) be local coordinates at \(b_j,\ j = 1, \ldots, n\). Then the transpose of the differential of \(h\)
\[
^t dh : H^0(X,\omega_X^2) \longrightarrow (T_{(o,B)}N \times M)^* = (\bigoplus_{i=1}^m \mathbb{C}(d s_i)_o) \oplus (\bigoplus_{j=1}^n \mathbb{C}(d t_j)_o)
\]
is determined by the following formulas

\[
\langle \frac{\partial}{\partial s_i}, \ ^tdh(\varphi) \rangle = 2\pi \sqrt{-1} \sum_{k=1}^d \frac{\varphi}{\pi^*(dv_i)^2}(x_{ik}) \tag{23}
\]

\[
\langle \frac{\partial}{\partial t_j}, \ ^tdh(\varphi) \rangle = 2\pi \sqrt{-1} \text{Res}_{o,j} \frac{\varphi}{\pi^*dw_j} \tag{24}
\]

Proof. The second formula was proved in Proposition 4.4. We give the proof of the first one in the case \(g(Y) = 1\). The case \(g(Y) \geq 2\) is similar. Using the notation of [14,2] we consider the Stein covering of \(X,\ X = D_0 \cup D_1 \cup \ldots \cup D_d\), where \(D_d = X - \pi^{-1}(a)\). The Kodaira-Spencer class \([\theta] = \rho(\frac{\partial}{\partial s_i}) \in H^1(X,\mathcal{T}_X)\) of the deformation of \(X\) defined by [18], the branch points remaining fixed, is given by the 1-cocycle \(\theta_{i0} = -\theta_{0i} = \frac{1}{u_i} \frac{\partial}{\partial u_i} \in H^0(D_i \cap D_0, \mathcal{T}_X)\) for \(i \geq 1\) and \(\theta_{ij} = 0\) for \(i, j \geq 1\). Let \(\varphi \in H^0(X,\omega_X^2)\) and let \(\varphi_i = f_i(du_i)^2\) in \(D_i\). Then \(\theta_{i0} \cdot \varphi = \frac{f_i(u_i)}{u_i} du_i\). Thus \(\text{Res}_{x_i}(\theta_{i0} \cdot \varphi) = f_i(0) = \frac{\varphi}{\pi^*(dv_i)}(x_i)\). Using a calculation similar to the one in [ACGH] pp.14,15 one obtains for the Serre duality pairing

\[
([\theta], \varphi) = 2\pi \sqrt{-1} \sum_{k=1}^d \frac{\varphi}{\pi^*(dv)^2}(x_k)
\]

\[\square\]

4.8. Let \(\pi : X \to Y\) be a simple branched covering of degree \(d \geq 2\) where \(Y\) is a smooth projective curve of genus \(g(Y) \geq 1\). Consider the deformation [19] of the covering \(\pi\) as
described at the end of (1.2). Let \( S = N \times H \) and let \( q' : \mathcal{Y}' \to S \) be the pullback of the family \( q : \mathcal{Y} \to N \) via \( \pi_1 : S \to N \). We obtain a commutative triangle as in (1.2):

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\psi'} & \mathcal{Y}' \\
\downarrow f & & \downarrow q' \\
S & & \\
\end{array}
\]

By (3.13) one associates to it a polarized VHS \((\mathcal{M}, F, Q)\). Shrinking \( N \) and \( H \) if necessary one may define a lifting of the period mapping \( \Phi : S \to \mathcal{D} \subset \check{\mathcal{D}} \) as in (3.18) and furthermore this mapping may be identified with the local period mapping associated with the polarized VHS \((\mathcal{M}', F', Q^-)\) as in (3.19) and Lemma 3.20. Let \( s_0 = (o, B) \) be the reference point of \( S = N \times H \).

**Proposition 4.9.** Using the notation of Proposition 4.7 one has the following formula for the differential of \( \Phi \) at \( s_0 \):

\[
^t d\Phi(s_0) : \text{Sym}^2 H^0(X, \omega_X)^- \longrightarrow (\oplus_{i=1}^m C(ds_i)_o) \oplus (\oplus_{j=1}^n C(dt_j)_o).
\]

For every \( \omega_1, \omega_2 \in H^0(X, \omega_X)^- \) it holds

\[
\langle \frac{\partial}{\partial s_i}, (^t d\Phi)(\omega_1 \cdot \omega_2) \rangle = 2\pi \sqrt{-1} \sum_{k=1}^d \frac{\omega_1 \omega_2}{\pi^*(dv_k)^2} (x_{ik})
\]

(25)

\[
\langle \frac{\partial}{\partial t_j}, (^t d\Phi)(\omega_1 \cdot \omega_2) \rangle = 2\pi \sqrt{-1} \text{Res}_{s_0} \frac{\omega_1 \omega_2}{\pi^*dw_j}
\]

(26)

**Proof.** One applies Proposition 3.21 with \( \frac{\partial}{\partial s_i} = \frac{\partial}{\partial s_i} \) or \( \frac{\partial}{\partial t_j} \). That \( (\rho(\frac{\partial}{\partial t}), \omega_1 \omega_2) \) equals the right-hand side of (25) and (26) in the respective cases follows from the proofs of Proposition 4.7 and Proposition 4.4. \( \square \)

4.10. We now restrict ourselves to the case of coverings of elliptic curves \( \pi : X \to Y \) with \( g = g(X) \geq 3 \) and first we consider deformations of \( \pi : X \to Y \) with fixed \( Y \). Consider the canonical map \( \phi_K : X \to \mathbb{P}^{g-1} = |\omega_X|_* \). The space \( H^0(\omega_X)^- = \{ \omega \mid Tr_\pi(\omega) = 0 \} \) is a hyperplane in \( H^0(\omega_X) \) since \( g(Y) = 1 \). It defines a point \( q^- \in \mathbb{P}^{g-1} \) with the property that the differentials \( \omega \in H^0(\omega_X)^- \) define hyperplanes in \( \mathbb{P}^{g-1} \) containing \( q^- \). Let \( \alpha \in H^0(Y, \omega_Y) \), \( \alpha \neq 0 \). Let \( H_\alpha \subset \mathbb{P}^{g-1} \) be the hyperplane in \( \mathbb{P}^{g-1} \) which corresponds to \( \pi^*\alpha \in H^0(\omega_X) \). Clearly \( \text{div}(\pi^*\alpha) = R = \phi_K^*(H_\alpha) \).

**Lemma 4.11.** Let \( \pi : X \to Y \) be a covering of an elliptic curve of degree \( d \geq 2 \). Then \( q^- \notin H_\alpha \)

**Proof.** \( Tr_\pi(\pi^*\alpha) = d\alpha \neq 0 \), hence by definition \( q^- \notin H_\alpha \). \( \square \)

Let \( \pi : X \to Y \) be a simple covering of an elliptic curve of degree \( d \geq 2 \) branched at \( B \subset Y \) where \( g = g(X) \geq 3 \). Consider the deformation \( (\Psi, f) : \mathcal{X} \to Y \times H \) defined in (1.1). Let
\[ \Phi : H \rightarrow \mathbb{D} \cong \mathfrak{H}_{g-1} \] be the period mapping corresponding to the variation \( H^{1,0}(\mathcal{X}_s)^{-}_s \subset H^1(X, \mathbb{C})^{-}_s \), \( s \in H \) (cf. \( \text{[6.19]} \)). Clearly the mapping \( \Phi \) is invariant with respect to the action of the germ of \( \text{Aut}(Y) \) at \( id \). So \( \dim \ker d\Phi_Y(s_0) \geq 1 \) where \( s_0 = B \in H \).

**Proposition 4.12.** Let \( \pi : X \rightarrow Y \) be a simple covering of an elliptic curve. Suppose \( X \) is not hyperelliptic and \( g(X) \geq 3 \). Then \( \dim \ker d\tilde{\Phi}_Y(s_0) = 1 \). In particular if \( g(X) = 3 \), then \( d\tilde{\Phi}_Y(s_0) \) is an epimorphism, and if \( g(X) = 4 \) the image \( d\tilde{\Phi}_Y(TBH) \) is a hypersurface in \( T_H \mathbb{D} \) where \( H = \Phi(s_0) \).

**Proof.** We identify \( X \) with its image \( \phi_K(X) \subset \mathbb{P}^{g-1} \). Then \( H_\alpha \cap X = R \) where \( R \) is the ramification divisor of \( \pi : X \rightarrow Y \) and the span \( \langle R \rangle = H_\alpha \). By \( \text{[26]} \) the annihilator of \( d\tilde{\Phi}_Y(TBH) \) equals the subspace of \( S^2 H^0(\omega_X)^{-}_s \) consisting of elements which vanish in \( R \). Identifying \( H^0(\omega_X) \) with \( H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1)) \) we consider the restriction map \( r : S^2 H^0(\omega_X)^{-}_s \rightarrow H^0(H_\alpha, \mathcal{O}_{H_\alpha}(2)) \). This is an isomorphism since both spaces have the same dimension and the kernel is 0. Indeed any element of the kernel corresponds to a reducible quadric \( H' \cup H_\alpha \) with center \( H' \cap H_\alpha \). The quadrics of \( |S^2 H^0(\omega_X)^{-}_s| \) are singular and contain \( q^{-}_1 \) in their center. By Lemma \( \text{[11]} \) \( q^{-}_1 \notin H_\alpha \), thus no quadric of the type \( H' \cup H_\alpha \) could belong to \( |S^2 H^0(\omega_X)^{-}_s| \). We conclude that \( \ker d\tilde{\Phi}_Y(s_0) \cong H^0(H_\alpha, \mathcal{O}_{H_\alpha}(2) \otimes J_R) \) where \( J_R \) is the ideal sheaf of \( R \subset H_\alpha \).

Let \( g(X) = 3 \). Then \( X \subset \mathbb{P}^2 \) is a quartic and \( H_\alpha \) is a line intersecting \( X \) in 4 distinct points. Here \( H^0(H_\alpha, \mathcal{O}_{H_\alpha}(2) \otimes J_R) \cong H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) = 0 \) so \( d\tilde{\Phi}_Y(s_0) \) is epimorphic with kernel of dimension 1 since \( \dim H = 4 \), \( \dim \mathbb{D} = 3 \).

Suppose \( g(X) \geq 4 \). The restriction mapping

\[
H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2) \otimes J_X) \rightarrow H^0(H_\alpha, \mathcal{O}_{H_\alpha}(2) \otimes J_R)
\]

is an isomorphism (see the proof of \( \text{[15]} \) Lemma 2.10). Therefore \( h^0(H_\alpha, \mathcal{O}_{H_\alpha}(2) \otimes J_R) = \frac{(g-2)(g-3)}{2} \). We conclude that

\[
\dim d\tilde{\Phi}_Y(TBH) \quad = \quad \frac{(g-1)g}{2} - \frac{(g-2)(g-3)}{2} \quad = \quad 2g - 3.
\]

By Hurwitz’ formula \( 2g - 2 = n + d(2g(Y) - 2) = n = \dim H \). Hence \( \dim \ker d\tilde{\Phi}_Y(s_0) = 1 \).

**Remark 4.13.** If \( X \) is hyperelliptic, then similar calculation shows that \( \dim d\tilde{\Phi}_Y(TBH) = g - 1 \). Hence \( \dim \ker d\tilde{\Phi}_Y(s_0) = g - 1 \geq 2 \) for \( g \geq 3 \). The next proposition shows that if \( n \geq 4 \) and if \( [X \rightarrow Y] \in \mathcal{H}_{d,n}(Y) \) is sufficiently general, then the curve \( X \) is not hyperelliptic.

**Proposition 4.14.** Let \( d \geq 2, n = 2e \geq 4 \). Then the locus of points \( [X \rightarrow Y] \in \mathcal{H}_{d,n}(Y) \) with hyperelliptic \( X \) is contained in a closed subset of \( \mathcal{H}_{d,n}(Y) \) of codimension \( \geq \frac{n}{2} - 1 \)

**Proof.** Suppose \( \pi : X \rightarrow Y \) is a covering of degree \( d \) simply branched in \( n \) points such that \( X \) is hyperelliptic. Let \( \mu : X \rightarrow \mathbb{P}^1 \) be the corresponding covering of degree 2. Consider \( \nu = (\mu, \pi) : X \rightarrow \mathbb{P}^1 \times Y \). Let \( \nu(X) = X' \). The morphism \( \nu : X \rightarrow X' \) is birational. Indeed,
since \( \mu = \pi_1 \circ \nu \), the other possibility would be deg \( \nu = 2 \), \( X' \cong \mathbb{P}^1 \). This is however absurd, since \( \pi_2 : X' \to Y \) is an epimorphism. The curve \( X' \) belongs to \( |\pi_1^* \mathcal{O}_{\mathbb{P}^1}(d) \otimes \pi_2^* L| \) for some invertible sheaf \( L \in \text{Pic}^2 Y \). Let \( \varphi = \varphi_x : Y \to \mathbb{P}^1 \) and let \( C = (1 \times \varphi)(X') \subset \mathbb{P}^1 \times \mathbb{P}^1 \). The curve \( C \) is irreducible and belongs to \( |\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(d, 1)| \), hence it is isomorphic to \( \mathbb{P}^1 \). We obtain that the covering \( \pi : X \to Y \) fits into a commutative diagram

\[
\begin{array}{ccc}
X & \xrightarrow{\nu} & X' \quad \xrightarrow{\varphi} \quad Y \\
\downarrow{\mu} & & \downarrow{\varphi} \\
C & \xrightarrow{\psi} & \mathbb{P}^1
\end{array}
\]

where \( \psi : C \to \mathbb{P}^1 \) is a morphism of degree \( d \). Conversely, given \( \varphi : Y \to \mathbb{P}^1 \) of degree 2 and \( \psi : C \to \mathbb{P}^1 \) of degree \( d \) one defines \( X' = C \times_{\mathbb{P}^1} Y \). Normalizing \( X' \) one obtains \( \pi : X \to Y \) such that \( X \) is hyperelliptic. Composing in (27) with an automorphism \( \tau : \mathbb{P}^1 \to \mathbb{P}^1 \) does not change the equivalence class of \( \pi : X \to Y \). Thus we may suppose the discriminant locus of \( \varphi \) is a fixed set \( B_\varphi = \{0, 1, \lambda, \infty\} \) where \( \lambda \) is determined by the \( j \)-invariant of \( Y \) up to a finite number of choices [14] Ch.IV 8.4. If \( c \in \mathbb{P}^1 - B_\varphi \) is a branch point of \( \psi \), then the two points \( \varphi^{-1}(c) \) are branch points of \( \pi \). One concludes firstly that all branch points of \( \psi \) contained in \( \mathbb{P}^1 - B_\varphi \) are simple and secondly \( \#(B_\psi - B_\varphi) \leq \frac{d}{2} \). Families of coverings \( \pi : X \to Y \) that fit into the diagram (27) are constructed by: (a) considering Hurwitz spaces of coverings \( \psi : C \to \mathbb{P}^1 \) where deg \( \psi = d \), \( C \cong \mathbb{P}^1 \), and the branch points of \( \psi \) belonging to \( \mathbb{P}^1 - B_\psi \) are simple, (b) composing the morphisms in the diagram (27) with an automorphism of \( Y \). The dimensions of the possible Hurwitz spaces of (a) are \( \leq \frac{d}{2} \). Hence the equivalence classes of coverings \( [X \to Y] \) with hyperelliptic \( X \) are contained in a closed subset of \( \mathcal{H}_{d, n}(Y) \) of dimension \( \leq \frac{d}{2} + 1 \). \( \square \)

The next lemma follows from the proof of Proposition [14]. Abusing notation we will not distinguish between \( S^m H^0(X, \omega_X) \) and \( H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(m)) \). We denote by \( S^2 H^0(X, \omega_X)^{-}(\mathcal{O}_X) \) the space of elements of \( S^2 H^0(X, \omega_X)^{-} \) which vanish on the ramification locus \( R \).

**Lemma 4.15.** Let the assumptions be as in Proposition 4.12. Let \( g(X) \geq 4 \). Then for every \( F \in H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2) \otimes J_X) \) there is a unique element \( F^- \in S^2 H^0(X, \omega_X)^{-}(\mathcal{O}_X) \) such that

\[
F^- = \omega \cdot \pi^* \alpha + F
\]

for some \( \omega \in H^0(X, \omega_X) \). The mapping \( F \mapsto F^- \) yields a linear isomorphism between \( H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2) \otimes J_X) \) and \( S^2 H^0(X, \omega_X)^{-}(\mathcal{O}_X) \).

We now consider the deformation [19] of \( \pi : X \to Y \). Let \( \tilde{\Phi} : N \times H \to \mathbb{D} \) be the period mapping defined in [14]. We want to calculate \( \text{Ker} \; d\tilde{\Phi}(s_0) \) where \( s_0 = (o, B) \). Since \( \tilde{\Phi}|_{\{0\} \times H} = \tilde{\Phi}_Y \) this kernel has dimension \( \geq 1 \).

**Proposition 4.16.** Let \( \pi : X \to Y \) be a simple covering of an elliptic curve. Suppose \( X \) is not hyperelliptic and \( g(X) \geq 4 \). Then \( \dim \text{Ker} \; d\tilde{\Phi}(s_0) = 1 \) if and only if the point \( q^- \in \mathbb{P}^{g-1} \) which corresponds to \( H^0(\omega_X)^{-} \) does not belong to the intersection of quadrics which contain \( \phi_K(X) \).
Proof. Identify $X$ with $\phi_K(X) \subset \mathbb{P}^{g-1}$. Let $c \in Y$ and let $v$ be a local coordinate as in (4.2). Let $\pi^{-1}(c) = \{x_1, \ldots, x_d\}$. Let $m : S^2 H^0(X, \omega_X) \to H^0(X, \omega_X^2)$ be the multiplication map. By Proposition 4.12 and Proposition 4.9 one has $\dim \ker d\Phi(s_0) = 1$ if and only if the linear functional $\lambda$ on $S^2 H^0(X, \omega_X)^{-}(R)$ defined by

$$\lambda(F^{-}) = \sum_{k=1}^{d} \frac{m(F^{-})}{\pi^*(dv)^2}(x_k)$$

is not identically zero. For every $F^{-}$ we have by Lemma 4.15 the representation $F^{-} = \omega \cdot \pi^* \alpha + F$ where $F \in H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(2) \otimes J_X)$. It is clear that

$$\lambda(F^{-}) = \frac{\text{Tr}_\pi(\omega)}{\partial v}(c) \frac{\alpha}{\partial v}(c).$$

Since $\text{Tr}_\pi(\omega) = \text{const} \alpha$ we conclude that $\lambda(F^{-}) = 0$ if and only if $\text{Tr}_\pi(\omega) = 0$, i.e. iff $\omega \in H^0(X, \omega_X)^{-}$, or equivalently iff the linear form of $H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1))$ corresponding to $\omega$ vanishes in $q^{-}$.

Suppose there is a quadric $Q \subset \mathbb{P}^{g-1}$ given by an equation $F = 0$ such that $Q \supset X$ and $q^{-} \notin Q$. Let $F^{-} = \omega \cdot \pi^* \alpha + F$ be the element in $S^2 H^0(X, \omega_X)^{-}(-R)$ corresponding to $F$ as in Lemma 4.14. Since $q^{-}$ belongs to the center of the quadric $F^{-} = 0$ the form $\omega$ cannot belong to $H^0(X, \omega_X)^-$, thus $\lambda(F^{-}) \neq 0$.

Conversely, let $\lambda(F^{-}) \neq 0$ for some $F^{-} \in S^2 H^0(\omega_X)^{-}(-R)$. Let $F^{-} = \omega \cdot \pi^* \alpha + F$ be the representation of Lemma 4.15. By Lemma 4.14 and the argument above the linear forms of $H^0(\mathbb{P}^{g-1}, \mathcal{O}_{\mathbb{P}^{g-1}}(1))$ which correspond to $\omega$ and $\pi^* \alpha$ do not vanish in $q^{-}$. Since $q^{-}$ belongs to the center of $F^{-} = 0$ we conclude that $F$ does not vanish in $q^{-}$. \qed

Corollary 4.17. Let $X \subset \mathbb{P}^{3}$ be a canonical curve which is a simple covering of an elliptic curve $\pi : X \to Y$. Let $Q$ be the unique quadric in $\mathbb{P}^{3}$ which contains $X$ and let $q^{-} \in \mathbb{P}^{3}$ be the point corresponding to the hyperplane of holomorphic differentials with trace zero. Let $\Phi : N \times H \to \mathbb{D} \cong S_3$ be the period mapping defined in (4.8). Then the differential $d\Phi(s_0)$ at the point corresponding to $\pi : X \to Y$ is surjective if and only if $q^{-} \notin Q$.

Proposition 4.18. Let $\pi : X \to Y$ be a covering of an elliptic curve of degree $d = 2$. Suppose $X$ is not hyperelliptic and $g(X) \geq 4$. Let $\Phi : N \times H \to \mathbb{D} \cong S_{g-1}$ be the mapping of (4.8) obtained by the periods of the Prym differentials. Then $\dim \ker d\Phi(s_0) = 1$. In particular if $g(X) = 4$ then $d\Phi(s_0)$ is surjective.

Proof. $X$ is a bi-elliptic curve and it is well-known that $\phi_K(X)$ lies on a normal elliptic cone $C$ of degree $g - 1$ (cf. [ACGH], p.269). The vertex of the cone is exactly the point $q^{-}$ defined in (4.10). One has $q^{-} \notin X$, the projection from the vertex maps $X$ into $Y \subset \mathbb{P}^{g-2}$ and this projection coincides with $\pi$. Every quadric $Q$ which contains $X \cup \{q^{-}\}$ should contain $C$ as well since the generators of $C$ are secants of $X$. By Enriques-Babbage theorem $\cap_{Q \supset X} Q$ equals either $X$ or a surface of degree $g - 2$. Since $\deg C = g - 1$ we conclude there is a quadric $Q$ which contains $X$ and does not contain $q^{-}$ so that Proposition 4.16 may be applied. \qed
4.19. We now consider the deformation (19) in the case of triple covers of genus 4 and we wish to prove that the differential of the period mapping \( \Phi : N \times H \rightarrow \mathbb{D} \cong \mathcal{S}_3 \) is surjective for \([X \rightarrow Y]\) belonging to a dense open subset of \(\mathcal{H}_{3,6}(Y)\). Applying Corollary 1.17 and using the irreducibility of the Hurwitz spaces \(\mathcal{H}_{3,6}(Y)\) proved in Theorem 2.10 it suffices to construct a single \(\pi : X \rightarrow Y\) with non-hyperelliptic \(X\) such that \(q^- \notin Q\). We do not know explicit examples of triple covers of genus 4 which satisfy this condition. For example our calculations show that in the case of cyclic triple covers one has in fact \(q^- \in Q\). We resolve the problem by considering singular curves and smoothing.

Let \(Y\) be a fixed elliptic curve and let \(B' = \{b_1, b_2, b_3\} \subset Y\). Let \(C_1 = Y\), let \(p_1 : C_1 \rightarrow Y\) be the identity mapping and let \(x_i = p_1^{-1}(b_i)\). Let \(p_2 : C_2 \rightarrow Y\) be an isogeny of degree 2. Let \(p_2^{-1}(b_i) = \{y_i, y_i'\}\). We consider the following triple covering of \(Y\):

\[
X' = C_1 \cup C_2/\{x_i \sim y_i\}_{i=1}^3, \quad \pi' = p_1 \cup p_2 : X' \rightarrow Y.
\]

The regular differentials of \(X'\) are of the form \(\omega = (\omega_1, \omega_2)\) where \(\omega_1 \in H^0(C_1, \Omega C_1(\sum x_i))\), \(\omega_2 \in H^0(C_2, \Omega C_2(\sum y_i))\) and \(\text{Res}_{x_i} \omega_1 + \text{Res}_{y_i} \omega_2 = 0\) for \(i = 1, 2, 3\). In order to describe the canonical image \(\phi_K(X')\) we proceed as follows. We have embeddings \(\phi_1 = \phi_1|_{\sum x_i} : C_1 \hookrightarrow \mathbb{P}^2\) and \(\phi_2 = \phi_2|_{\sum y_i} : C_2 \hookrightarrow \mathbb{P}^2\) such that \(\sum x_i\) and \(\sum y_i\) are respectively pull-backs of the lines \(\ell_1\) and \(\ell_2\). There is a unique projective linear mapping of \(\ell_1\) into \(\ell_2\) that transforms \(x_i\) into \(y_i\). Identifying \(\ell_1\) with \(\ell_2\) along this transformation we obtain a reducible quadric

\[
Q = \mathbb{P}^2 \cup \mathbb{P}^2/\ell_1 \sim \ell_2 = H_1 \cup H_2 \subset \mathbb{P}^3.
\]

By dimension count one easily verifies that

\[
\phi_1(C_1) \cup \phi_2(C_2) = Q \cap F
\]

where \(F\) is an irreducible cubic surface in \(\mathbb{P}^3\). Since the degrees of both sides in (30) are equal to 6 one obtains that \(\phi_1(C_1) \cup \phi_2(C_2)\) is a canonical curve of arithmetic genus 4. This shows

**Claim.** The canonical map \(\phi_K : X' \rightarrow \mathbb{P}^3\) is a regular embedding and \(\phi_K(X') = Q \cap F\) where \(Q\) is a reducible quadric \(H_1 \cup H_2\) and \(F\) is an irreducible cubic surface.

**Proposition 4.20.** Let \(H^0(X', \omega_{X'})\) be the hyperplane of regular differentials with trace 0 and let \(q^- \in [\omega_{X'}]^* = \mathbb{P}^3\) be the corresponding point. Then \(q^- \notin H_1 \cup H_2 = Q\).

**Proof.** Let \(A = \{y_1, y_2, y_3\}\), let \(\sigma : C_2 \rightarrow C_2\) be the involution interchanging the branches of \(p_2 : C_2 \rightarrow Y\) and let \(G = \{\text{id}, \sigma\}\). The mapping \(p_2^* \circ \text{Tr}_{p_2}\) transforms injectively \(H^0(C_2, \Omega C_2(A))\) into \(H^0(C_2, \Omega C_2(A + \sigma A))\). Since these spaces as well as \(H^0(Y, \Omega_Y(B'))\) all have dimension 3 we conclude that

\[
\text{Tr}_{p_2} : H^0(C_2, \Omega C_2(A)) \xrightarrow{\sim} H^0(Y, \Omega_Y(B'))
\]

is an isomorphism. Let \((\omega_1, \omega_2) \in H^0(X', \omega_{X'})\) be a regular differential with trace 0, \(\text{Tr}_{X'}(\omega_1, \omega_2) = \text{Tr}_{p_1}(\omega_1) + \text{Tr}_{p_2}(\omega_2) = 0\). Then \(\omega = \text{Tr}_{p_1}(\omega_1) \in H^0(Y, \Omega_Y(B'))\) and \(\text{Tr}_{p_2}(\omega_2) = 0\).
−ω. Conversely by the isomorphism \(31\) the regular differentials with trace 0 are the pairs \((ω_1, ω_2)\) that satisfy the latter property. The fact that the restrictions \((ω_1, ω_2) \mapsto ω_1\) and \((ω_1, ω_2) \mapsto ω_2\) are isomorphisms of \(H^0(X', ω_{X'}) \cong H^0(P^3, O_{P^3}(1) \otimes J_p)\) with respectively \(H^0(C_1, Ω_{C_1}(∑x_i)) \cong H^0(H_1, O_{H_1}(1))\) and \(H^0(C_2, Ω_{C_2}(∑y_i)) \cong H^0(H_2, O_{H_2}(1))\) implies that \(q^- \notin H_1 \cup H_2\).

**Proposition 4.21.** Let \(π' : X' \to Y\) be the singular triple covering of \(4.14\) with sufficiently general branch points \(\{b_1, b_2, b_3\} \subset Y\). Then there exists a finite, flat, surjective morphism \(p = (Ψ, f) : X \to Y \times T\) where \(X\) and \(T\) are smooth and irreducible varieties such that the following properties hold: the morphism \(f : X \to T\) is proper and flat; the scheme-theoretic fibers of \(f\) are reduced and have dimension 1 and arithmetic genus 4; for every sufficiently general \(s ∈ T\) the covering \(p_s : X_s \to Y \times \{s\}\) is simple; there is a point \(s_0 ∈ T\) such that \(p_{s_0} : X_{s_0} \to Y\) is equivalent to \(π' : X' \to Y\).

**Proof.** Consider the curve \(X_0 = C_1 \sqcup C_2\) and the covering \(π_0 = p_1 \sqcup p_2 : X_0 \to Y\). One has \((p_2)_*O_{C_2} = O_Y \oplus η\) where \(η = Ker(Tr_{p_2} : (p_2)_*O_{C_2} → O_Y)\) is an invertible sheaf of \(Y\) with \(η^{⊗2} \cong O_Y\). The Tschirnhausen module of \(π_0 : X_0 \to Y\) is the quotient sheaf \(E_0^\vee\) defined by

\[
0 \to O_Y \xrightarrow{π_0^\#} (π_0)_*O_{X_0} \to E_0^\vee \to 0.
\]

It is isomorphic to \(Ker(Tr_{π_0} : (π_0)_*O_{X_0} → O_Y))\). There is a canonical embedding \(X_0 \hookrightarrow P(E_0)\) such that \(i^*O_{P(E_0)}(1) \cong ω_{X_0/Y} \cong ω_{X_0} \otimes π_0^*(ω_Y)^{-1}\) (cf. \[CE\] p.448). We claim \(E_0^\vee \cong O_Y \oplus η\). Indeed, \((π_0)_*O_{X_0} \cong O_Y \oplus (O_Y \oplus η)\). The trace map is given on local sections by \(Tr_{π_0}(a_1, a_2 + b) = a_1 + 2a_2\). Thus the homomorphism \(O_Y \oplus η \to Ker Tr_{π_0}\) given by \((a, b) \mapsto (-2a, a + b)\) is an isomorphism. Let \(W_0 = P(E_0) = P(O_Y \oplus η)\). This is an elliptically ruled surface of invariant \(e_0 = 0\) and with two sectional curves \(Y_0\) and \(Y_∞\) with minimal self-intersection \(−e_0 = 0\). Clearly \(C_2 \neq Y_0\) and \(C_2 \neq Y_∞\), thus one may choose \(b_1\) in such a way that if \(π_0^{-1}(b_1) = \{x_1, y_1, y_1'\}\) with \(x_1 ∈ C_1, y_1, y_1' ∈ C_2\) none of the points \(y_1\) or \(y_1'\) belongs to \(Y_0 ∪ Y_∞\). Performing an elementary transformation with center at the point \(y_1\), which consists of blowing-up \(y_1'\) and blowing-down the strict transform of the fiber of \(W_0 \to Y\) passing through \(x_1\) and \(y_1\) we obtain a new ruled surface \(W_1 \to Y\) with invariant \(e_1 = −1\) \[SE\] p.210 and an embedding of \(X_1 = X_0/x_1 \sim y_1\) in \(W_1\). Let \(π_1 : X_1 → Y\) be the covering induced by \(π_0\). Choose a second non-branch point \(b_2 \in Y, b_2 \neq b_1\) and let \(π_1^{-1}(b_2) = \{x_2, y_2, y_2'\}\) with \(x_2 ∈ C_1, y_2, y_2' ∈ C_2\). There is a sectional curve with minimal self-intersection \(−e_1 = 1\) passing through \(y_2'\) \[HA\] Ex.V.2.7. Performing an elementary transformation with center in \(y_2'\) we obtain a new ruled surface \(W_2 \to Y\) with invariant \(e_2 = 0\) \[SE\] p.210 and an embedding of \(X_2 = X_1/x_2 \sim y_2\) in \(W_2\). We claim that \(W_2\) may have only a finite number of sectional curves with minimal self-intersection \(−e_2 = 0\). Indeed, otherwise there would be an \(∞\) family of such curves and if \(z_2 ∈ W_2\) is the image of the blown-down curve of \(W_1\) there would be a sectional curve with minimal self-intersection passing through \(z_2\). Performing an elementary transformation of \(W_2\) with center in \(z_2\) we would obtain a ruled surface with invariant \(e = 1\). This surface is however isomorphic to \(W_1\) with invariant \(e_1 = −1\) which is absurd. Let \(π_2 : X_2 = X_0/\{x_i \sim y_i\}_{i=1}^2 → Y\) be the mapping induced by \(π_0\). Since \(C_2 ⊂ X_2\)
cannot be a sectional curve we may choose $b_3$ in such a way that if $\pi_2^{-1}(b_3) = \{x_3, y_3, y_3'\}$ with $x_3 \in C_1, y_3, y_3' \in C_2$ then none of $y_3$ or $y_3'$ belongs to the union of the sectional curves with minimal self-intersection. Performing an elementary transformation of $W_3$ with center in $y_3'$ we obtain $W = W_3$ with invariant $\varepsilon = -1$ and an embedding of $X' = X_0/\{x_i \sim y_i\}_{i=1}^3$ in $W$. Let $Y_0$ be a sectional curve with minimal self-intersection $Y_0^2 = -\varepsilon = 1$ and let $F$ be a fiber of $W \to Y$. We have numerical equivalence $X' \equiv 3Y_0 + bF$ since $X' \cdot F = \deg(\pi' : X' \to Y) = 3$.

For the canonical class $K_W$ of $W$ one has $K_W \equiv -2Y_0 + F \ [\text{[Ha] Cor.V.2.11.}]$. Hence $2p_a(X') - 2 = 6 = X' \cdot (X' + K_W) = (3Y_0 + bF)(Y_0 + (b + 1)F) = 3 + 3(b + 1) + b = 6 + 4b$. Therefore $b = 0$. We now apply a criterion of very ampleness due to Biancofiore and Livorni [BLi] p.183. Namely, $b > 3e + 2 = -1$, hence $O_W(X')$ is a very ample invertible sheaf. The required family in the proposition is constructed as follows. Let $\mathbb{P}^n = |O_W(X')|^\vee$ and let $\varphi : W \to \mathbb{P}^n$ be the closed embedding. We let $Z = \tilde{P}^n$, $X' \subset W \times Z$ be the closed subset $X' = \{(x, \gamma) | \varphi(x) \in H_3\}$. It is clear that $X'$ is smooth and projective. The projection $\pi_1 : X' \to W$ has fibers $\cong \mathbb{P}^{n-1}$, so $X'$ is irreducible. The projection $W \to Y$ induces $p' : X' \to Y \times Z$ which is a finite surjective morphism to a smooth variety, so it is flat. Let $f' = \pi_2 : X' \to Z$. This is a flat morphism since it is a composition of two flat morphisms: $p' : X' \to Y \times Z$ and $Y \times Z \to Z$. Let $T = \{s \in Z | X'_s \text{ is reduced}\}$. We prove below $T$ is open in $Z$. Assuming this we let $X = (p')^{-1}(Y \times T)$ and $p : X \to Y \times T$ be the restriction of $p'$ on $X$.

For proving the openness of $T$ we use that a locally noetherian scheme is reduced if and only if it satisfies the conditions $R_0$ and $S_1$ [AK] p.132. Every $X'_s$ is a hyperplane sections of the smooth surface $W$, so it is Cohen-Macaulay and satisfies the conditions $S_k$ for all $k$. The condition $R_0$ means the scheme is smooth at every generic point. Let $R' \subset X'$ be the closed subset where the differential of $f'$ has not a maximal rank. Since $f'$ is proper $B = f'(R')$ is closed in $Z$ and $B \neq Z$ by Bertini’s theorem. Let $B_1 = \{s \in B | \dim(f'|_{R'})^{-1}(s) \geq 1\}$. This is a closed subset of $B$ and the condition $R_0$ holds for $s \in Z$ iff $s \in Z - B_1$. Thus $T = Z - B_1$ is open.

It remains to prove that for sufficiently general $s \in T$ the covering $p_s : X_s \to Y$ is simple. The ruled surface $W$ has invariant $\varepsilon = -1$, so $W = \mathbb{P}(F)$ where $F$ is a normalized indecomposable locally free sheaf [Ha] Ch.VI § 2. If $E$ is the dual of the Tschirnhausen module of $\pi' : X' \to Y$, then by [Mi] p.1150 one has $E \cong F \otimes L$ for a certain invertible sheaf $L$ on $Y$, hence $E$ is indecomposable of degree $3$ (see the proof of Lemma 2.3). Let $A = \det E$. By Proposition 2.9 and Lemma 2.5 every sufficiently general simple covering $[X \to Y] \in \mathcal{H}_{3,A}(Y)$ has Tschirnhausen module isomorphic to $E^\vee$ and $X \in O_{\mathbb{P}(E)}(3) \otimes (\pi'(\det E)^{-1})$. The curve $X'$ belongs to the same linear system. Since simpleness is an open condition we obtain that for sufficiently general $s \in T$ the covering $p_s : X_s \to Y$ is simple.

\textbf{Proposition 4.22.} Let $Y$ be an elliptic curve and let $A \in \text{Pic}^3 Y$. Then every sufficiently general simple triple covering $\pi : X \to Y$ having a Tschirnhausen module $E^\vee$ with $\det E \cong A$ satisfies the condition of Corollary 4.17. $X$ is not hyperelliptic and $q^- \notin Q$.

\textbf{Proof.} Composing $\pi$ by an automorphism of $Y$ does not change neither $q^- \in \mathbb{P}^3$ nor the quadric $Q \supset \phi_K(X)$. By the argument of Lemma 2.5 it suffices to prove the claim
only for one line bundle in $\text{Pic}^3(Y)$. Let $E'$ be the Tschirnhausen module of the reducible covering $\pi' : X' \to Y$ as at the end of the preceding proof. Let $A = \det E$. By the proofs of Theorem 2.10 (cf. Corollary 2.12) and Proposition 4.21 a Zariski open nonempty subset of $\mathcal{H}_{3,A}(Y)$ consists of equivalence classes of coverings $[X \to Y]$ such that $X \in |\mathcal{O}_W(X')|$

We claim that for every $C \in |\mathcal{O}_W(X')|$ the dualizing sheaf $\omega_C$ is spanned, i.e. the canonical homomorphism $H^0(C, \omega_C) \otimes \mathcal{O}_C \to \omega_C$ is surjective. Indeed, using the notation of the proof of Proposition 4.21 one has $K_W + X' \equiv Y_0 + F$. Hence $\mathcal{O}_W(K_W + X')$ is base-point-free by [BL1] p.183. Thus by the adjunction formula $\omega_C \cong \mathcal{O}_W(K_W + X') \otimes \mathcal{O}_C$ is spanned. Let us consider the family $f : \mathcal{X} \to T$ constructed in Proposition 4.21. Let $\omega_{X/T} := \Omega^n \otimes f^*(\Omega^n_T)^{-1}$. By flatness $f_*\omega_{X/T}$ is locally free of rank 4. By the above claim the canonical morphism $f^*(f_*\omega_{X/T}) \to \omega_{X/T}$ is an epimorphism. This yields a well-defined relative canonical map

$$f \quad \varphi \quad P(f_*\omega_{X/T}) = \mathbb{P}$$

The trace mapping $\text{Tr}_p : p_*(\Omega^n_a) \to \Omega^n_{X \times T}$ is an epimorphism since for every $\Phi \in \Gamma(V, \Omega^n_{X \times T})$ one has $\text{Tr}_p(\frac{1}{\text{deg}p^*}\Phi) = \Phi$. This yields an epimorphism $\text{Tr}_r : f_*\omega_{X/T} \to H^0(Y, \omega_Y) \otimes \mathcal{O}_T = \mathcal{O}_T$, hence a section $\mu$ of $g : \mathbb{P} \to T$. According to Proposition 4.22 the restriction of $\text{Tr}_r$ at the fiber over any $s \in T$ equals $\text{Tr}_{p_s} : H^0(X_s, \omega_{X_s}) \to H^0(Y, \omega_Y)$. Hence $\mu(s)$ is the point $q^−$ for $p_s : X_s \to Y$ as defined in (4.10).

Consider the homomorphism $\varphi^* : g_*\mathcal{O}_P(2) \to f_*\omega_{X/T}^{\otimes 2}$. If $s_0 \in T$ is the point which corresponds to the reducible covering $\pi' : X' = C_1 \cup C_2 \to Y$, then $\varphi^* \otimes k(s_0) : S^2H^0(X_{s_0}, \omega_{X_{s_0}}) \to H^0(X_{s_0}, \omega_{X_{s_0}}^{\otimes 2})$ is surjective since $\phi_K(X_{s_0})$ is contained in a unique quadric. Hence replacing $T$ by an affine neighborhood of $s_0$ if necessary we may assume that for every $s \in T$ the fiber homomorphism $\varphi^* \otimes k(s)$ is surjective with one dimensional kernel. In particular if $X_s$ is smooth it is non-hyperelliptic. We obtain a relative quadric $Q \subset \mathbb{P}$ which contains $\varphi(\mathcal{X})$. The set $V = \{s \in T | \mu(s) \in Q(s)\}$ is closed in $T$ and $s_0 \notin V$. The proposition is proved. □

5 Unirationality results

We first give an alternative proof of a known result [BL1].

**Theorem 5.1.** The moduli spaces of polarized abelian surfaces $\mathcal{A}_2(1,2)$ and $\mathcal{A}_2(1,3)$ are unirational.

**Proof.** Let $d = 2$ or $3$. We fix an elliptic curve $Y$ and let $T$ be the Hurwitz space $\mathcal{H}_{d,4}(Y)$. We consider a commutative diagram as in (12)

$$\mathcal{X} \quad p \quad Y \times T$$

$$f \quad \quad \quad q = \pi_2$$

33
where $p$ is the universal family of simple coverings of degree $d$ branched in 4 points. According to Theorem 2.10 $T$ is irreducible. Applying Lemma 4.12 and Proposition 3.14 we obtain a period mapping $\Phi : T \to A_2(1, d)$. We claim $\Phi$ is dominant. Let $s_0 \in T$ and let $H \subset T$ be a polydisk centered in $s_0$ as in (4.1). Then we have a lifting of the period mapping and a commutative diagram

$$
\begin{array}{ccc}
H & \xrightarrow{\Phi} & \tilde{H_2} \\
\downarrow & & \downarrow \\
T & \xrightarrow{\Phi} & A_2(1, 3)
\end{array}
$$

By Propositions 4.14 and (4.12) if $s_0$ is chosen general enough the differential $d\Phi(s_0)$ is surjective. Hence by the implicit function theorem a neighborhood of $\Phi(s_0)$ is contained in $\tilde{\Phi}(H)$. Therefore $\Phi$ is dominant. Let us choose $A \in \text{Pic}^2 Y$. Then the images by $\Phi$ of $H_{d,A}(Y)$ and $H_{d,A}(Y)$ are the same (cf. Proposition 3.14 and the proof of Lemma 2.5). Thus the restriction of $\Phi : H_{d,A}(Y) \to A_2(1, d)$ is dominant as well. By Theorem 2.10 $H_{d,A}(Y)$ is unirational. Therefore $A_2(1, d)$ is unirational.

Gritsenko proved in [Gri] that if $\tilde{A}_2(1, d)$ is a nonsingular, projective model of $A_2(1, d)$, then the geometric genus $p_g(A_2(1, d)) \geq 1$ when $d \geq 13$ and $d \neq 14, 15, 16, 18, 20, 24, 30, 36$. Sankaran proved in [Sa] that if $d$ is prime and $d \geq 173$ then $\tilde{A}_2(1, d)$ is of general type. If $g(Y) = 1$ none of the Hurwitz spaces $H_{d,n}(Y)$ is uniruled because of the epimorphism $h : H_{d,n}(Y) \to \text{Pic}^{n/2} Y$ (cf. (2.4)). We denote by $H_{d,n}^0(Y)$ the subset of $H_{d,n}(Y)$ whose points correspond to coverings $\pi : X \to Y$ with the property that $\pi_* : H_1(X, \mathbb{Z}) \to H_1(Y, \mathbb{Z})$ is surjective, or equivalently that $\pi^* : J(Y) \to J(X)$ is injective. This property is preserved under deformation, so $H_{d,n}^0(Y)$ is a union of connected components of $H_{d,n}(Y)$. We denote by $H_{d,A}^0(Y)$ the intersection $H_{d,A}^0(Y) \cap H_{d,A}(Y)$. We notice that $H_{d,n}^0(Y)$ and $H_{d,A}^0(Y)$ are non-empty for every $n \geq 2$ and every $A \in \text{Pic}^{n/2} Y$ as follows from Lemma 2.5. Lemma 4.1 and Lemma 2.5. In a direction opposite to the one of Theorem 5.1 we have the following result.

**Theorem 5.2.** Let $Y$ be an elliptic curve and let $A \in \text{Pic}^2 Y$. Let $d \geq 13$ and let $d \neq 14, 15, 16, 18, 20, 24, 30, 36$. Then every connected component of $H_{d,A}^0(Y)$ has the property that each of its nonsingular, projective models has geometric genus $\geq 1$. In particular none of the connected components of $H_{d,A}^0(Y)$ is uniruled. If furthermore $d$ is prime and $d \geq 173$, then every projective nonsingular model of any connected component of $H_{d,A}(Y)$ is of general type.

**Proof.** Using Lemma 4.1 and repeating the argument of Theorem 5.1 we obtain a morphism $\Phi : H_{d,A}^0(Y) \to A_2(1, d)$ whose restriction on every connected component is dominant. The theorem follows thus from the results of Gritsenko and Sankaran cited above.

**Theorem 5.3.** The moduli spaces of polarized abelian threefolds $A_3(1, 1, 2)$, $A_3(1, 2, 2)$, $A_3(1, 1, 3)$ and $A_3(1, 3, 3)$ are unirational.
Proof. Let $d = 2$ or 3. We consider the family of coverings from (2.13) and Proposition 2.14. According to Lemma 1.2 and Proposition 3.14 the period mapping $\Phi : T \to A_3(1, 1, d)$ defined by $\Phi(s) = \text{Ker}(Nm_{p_s})$ is an algebraic morphism. We wish to prove $\Phi$ is dominant. Let $s_0 \in T$. Let us denote by $\pi : X \to Y$ the covering corresponding to $s_0$. By the definition of $\Phi$ (cf. (3.11)) one may choose a neighborhood $S$ of $s_0$ (in the Hausdorff topology) such that $\Phi|_S$ may be lifted to a holomorphic mapping $\tilde{\Phi}' : S \to A_3(1, 1, 3)$.

Consider the restriction of the family (4) on $S$. Applying Proposition 4.3 to the deformation (3) we conclude that, shrinking $S$ if necessary, there is a holomorphic mapping $\mu : S \to N \times H$ such that the family of coverings over $S$ is the pull-back of the family induced from (19). Considering the period mappings one obtains a commutative diagram

$$
\begin{array}{ccc}
S & \xrightarrow{\tilde{\Phi}'} & \tilde{\Phi}_3 \\
\downarrow i & & \downarrow \\
T & \xrightarrow{\Phi} & A_3(1, 1, 3)
\end{array}
$$

By Proposition 4.22 if $s_0 \in T$ is chosen general enough the differential $d\tilde{\Phi}(s_0)$ is epimorphic. Hence by the implicit function theorem a neighborhood of $\tilde{\Phi}(s_0)$ is contained in $\tilde{\Phi}(N \times H)$. Consider the Tschirnhausen module $E^\nu$ of the covering $X \to Y \times H (N \times H)$ induced from (19). The set of $u \in N \times H$ such that $E_u$ is stable is open in $N \times H$ (cf. Proposition B.4). For vector bundles of rank 2 and degree 3 over an elliptic curve being stable and being indecomposable are equivalent conditions [Tu] p.20. Hence by Proposition 2.9 if one chooses $s_0 \in T$ general enough a neighborhood of $\mu(s_0)$ in $N \times H$ corresponds to triple coverings with indecomposable Tschirnhausen modules of degree $-3$. Given a triple covering of an elliptic curve, composing the covering by a translation results in translation of the determinant of the Tschirnhausen module (cf. the proof of Lemma 2.5) while the kernel of the norm map of the Jacobians remains the same. This shows that $\tilde{\Phi}(S)$ contains the image by $\tilde{\Phi}$ of a certain neighborhood of $\mu(s_0)$ in $N \times H$. Therefore a neighborhood of $\tilde{\Phi}'(s_0)$ in $\tilde{\Phi}_3$ is contained in $\tilde{\Phi}'(S)$. This shows that $\Phi : T \to A_3(1, 1, d)$ is dominant.

By construction $T$ is rational. Therefore $A_3(1, 1, d)$ is unirational. The unirationality of $A_3(1, d, d)$ then follows either from the result of Birkenhake and Lange [BL3] or from the weaker statement in Proposition 3.12. 

The proofs of Theorem 5.1, Theorem 5.2 and Theorem 5.3 together with Proposition 3.14 yield the following corollary.

**Corollary 5.4.** Let $Y$ be an elliptic curve. Every sufficiently general abelian surface with polarization of type $(1, d)$ is isomorphic to the Prym variety of a simple degree $d$ covering of
For every sufficiently general abelian threefold $A$ with polarization of type $(1,1,d)$, $d=2$ or 3 there exists an elliptic curve $Y$ (depending on $A$) such that $A$ is isomorphic to the Prym variety of a simple covering $\pi : X \to Y$ of degree $d$ branched in 6 points. For every sufficiently general abelian threefold $B$ with polarization of type $(1,d,d)$, $d=2$ or 3 there exists an elliptic curve $Y$ and a simple covering $\pi : X \to Y$ of degree $d$ branched in 6 points such that $B$ is isomorphic to $\text{Pic}^0 X/\pi^* \text{Pic}^0 Y$.

A Traces of differential forms

A.1. In this appendix we assume the base field $k$ is algebraically closed of characteristic 0. Let $X$ and $Y$ be smooth, irreducible varieties of dimension $n$, let $p : X \to Y$ be a finite, surjective morphism. One defines the trace mapping $\text{Tr}_p : \Omega^n(X) \to \Omega^n(Y)$ for rational differential forms $n$-forms as follows. Let $Y_0 \subset Y$ be an open subset such that if $X_0 = p^{-1}(Y_0)$ the restriction $p : X_0 \to Y_0$ is étale. If $v_1, \ldots, v_n$ are local parameters at some point $y \in Y_0$, then $\text{Tr}_p(a dp^* v_1 \wedge \cdots \wedge dp^* v_n) := \text{Tr}_p(a dv_1 \wedge \cdots \wedge dv_n)$. It is well-known that the trace mapping transforms regular differentials of $X$ into regular differentials of $Y$ (cf. [Li] Example 2.1.2). More generally one may define in the same way $\text{Tr}_p : \Omega^n(X) \to \Omega^n(Y)$ for every finite, surjective morphism $p : X \to Y$ between reduced, equidimensional schemes of dimension $n$ where $\Omega^n(X)$ and $\Omega^n(Y)$ are the sheaves of rational $n$-forms.

Proposition A.2. Suppose we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{p} & \mathcal{Y} \\
\downarrow f & & \downarrow g \\
T & & \\
\end{array}
\]

where $\mathcal{X}, \mathcal{Y}$ and $T$ are smooth and irreducible, $T$ is affine, $\dim \mathcal{X} = \dim \mathcal{Y} = n$, $p$ is finite and surjective, the morphisms $f$ and $g$ are proper of relative dimension 1 with reduced fibers. Suppose the discriminant subscheme of $p$ does not contain fibers of $g$. Then for every $s \in T$ one has canonical isomorphisms given by Poincaré residues $(\Omega^n_X)_s \cong \omega_{\mathcal{X}_s}$, $(\Omega^n_Y)_s \cong \omega_{\mathcal{Y}_s}$ and the following commutative diagram holds

\[
\begin{array}{ccc}
H^0(\mathcal{X}, \Omega^n_X) \otimes k(s) & \xrightarrow{\text{Tr}_p \otimes k(s)} & H^0(\mathcal{Y}, \Omega^n_Y) \otimes k(s) \\
\cong & & \cong \\
H^0(\mathcal{X}_s, \omega_{\mathcal{X}_s}) & \xrightarrow{\text{Tr}_{p_s}} & H^0(\mathcal{Y}_s, \omega_{\mathcal{Y}_s}) \\
\end{array}
\]

In particular for every $s \in T$ the trace mapping $\text{Tr}_{p_s}$ transforms the regular differentials of $\mathcal{X}_s$ into regular differentials of $\mathcal{Y}_s$.

Proof. Let $t_1, \ldots, t_{n-1}$ be local parameters of $T$ at $s$. If $x \in \mathcal{X}_s$ and if $u_1, \ldots, u_n$ are local parameters of $\mathcal{X}$ at $x$ then a generator of $(\omega_{\mathcal{X}_s})_x$ is given by the Poincaré residue
Applying $E$ one fixes one indecomposable vector bundle if and only if it is a direct sum of indecomposable vector bundles of the same slope. A vector

if and only if $E \sim E_r \oplus \cdots \oplus E_n$ where $E_i$ is stable for every $i = 1, \ldots, n$. Let $r(E) = r$, $d(E) = dh$, where $(r, d) = 1$. $E$ is called regular if $h^0(\text{End } E) = h$ (cf. [FMW]).

**Definition B.1.** Let $E$ be a semistable vector bundle over an elliptic curve. $E$ is called polystable if $E \cong E_1 \oplus \cdots \oplus E_n$, where $E_i$ is stable for every $i = 1, \ldots, n$. Let $r(E) = rh$, $d(E) = dh$, where $(r, d) = 1$. $E$ is called regular if $h^0(\text{End } E) = h$ (cf. [FMW]).

**Lemma B.2.** Let $E$ be a semistable vector bundle over an elliptic curve of rank $rh$ and degree $dh$, where $(r, d) = 1$. Then $h^0(\text{End } E) \geq h$. The vector bundle $E$ is regular, polystable if and only if $E \cong E_1 \oplus \cdots \oplus E_h$, where every $E_i$ is indecomposable of rank $r$ and degree $d$ and $E_i \not\cong E_j$ for $\forall i \neq j$. 

In this appendix we will make the customary identification between vector bundles and locally free sheaves. In the next proposition we prove a result which should be well-known, but we could not find an appropriate reference. The openness of the stability and the semistability conditions for families of vector bundles over a fixed curve of genus $\geq 2$ are treated in [Pa] p.459.
Proposition B.4. Let $E \cong E_1 \oplus \cdots \oplus E_n$ where every $E_i$ is indecomposable. We have $\mu(E_i) = \mu(E) = d/r$, so $r(E_i) = rh_i$, $d(E_i) = dh_i$. By \cite{At} Lemma 24 and the proof of Lemma 23 one has for every $i = 1, \ldots, n$ that $E_i \cong E_i' \otimes F_{h_i}$, where $E_i'$ is indecomposable of rank $r$ and degree $d$ and furthermore $h^0(\End E_i) = h_i$. Thus $h^0(\End E) \geq \sum h_i = h$. The vector bundle $E$ is polystable if and only if $h_i = 1$ for every $i$ and therefore $n = h$. It is regular polystable if moreover $h^0(E_i \otimes E_j^\vee) = 0$, or equivalently $E_i \not\cong E_j$ for $\forall i \neq j$. \hfill $\square$

Lemma B.3. Let $E$ be a vector bundle over an elliptic curve. Then $E$ is not stable (resp. not semistable) if and only if there exists a stable vector bundle $F$ of rank $< r(E)$ and slope $\mu(F) \geq \mu(E)$ (resp. $\mu(F) > \mu(E)$) and a nonzero homomorphism of $F$ into $E$. $E$ is semistable, but is not regular polystable if and only if there exists an indecomposable vector bundle $G$ with $\mu(G) = \mu(E)$ and $r(G) \leq r(E)$ such that $\dim \Hom(G, E) \geq 2$.

Proof. The statement about non-stability resp. non-semistability is proved in \cite{NS} Proposition 4.6 for vector bundles over smooth, projective curves of arbitrary genus $\geq 1$. Let $E$ be semistable and let $E \cong E_1 \oplus \cdots \oplus E_n$ be a direct sum of indecomposable vector bundles, which by the semistability have slopes equal to $\mu(E)$. Suppose $E$ is not regular polystable. Then either one of $E_i$ has $(r(E_i), d(E_i)) > 1$ and in this case we let $G = E_i$, or $E_j \cong E_k$ for some pair $j \neq k$ and in this case we let $G = E_j$. Then clearly $\dim \Hom(G, E) \geq 2$. Conversely, suppose $E$ is regular polystable. Let $r(E) = rh$, $d(E) = dh$ where $(r, d) = 1$ and let $E \cong E_1 \oplus \cdots \oplus E_h$ be the decomposition of Lemma B.2. If $G$ is indecomposable with $\mu(G) = d/r$, then $G \cong G' \otimes F_t$, where $G'$ is indecomposable of rank $r$ and degree $d$ \cite{At} Lemma 26. For every $i$ we have $G' \otimes E_i \cong (G')^\vee \otimes E_i \otimes F_t^\vee \cong (\sum_{j=1}^{r^2} L_j) \otimes F_t$ where $L_j$ are line bundles of degree 0 (cf. \cite{At} pp.434,437,439). Only one of $L_j$ might be isomorphic to the trivial line bundle and this happens if and only if $G' \cong E_i$. By \cite{At} Theorem 5 we conclude $\dim \Hom(G, E) = h^0(G^\vee \otimes E) \leq 1$. \hfill $\square$

Proposition B.4. Let $q : \mathcal{Y} \to B$ be a smooth family of elliptic curves. Here in the algebraic setting $B$ is a scheme and $q$ is a smooth morphism, in the complex analytic setting $B$ is an analytic space and $q$ is a smooth holomorphic mapping. Let $\mathcal{E}$ be a vector bundle over $\mathcal{Y}$. Then the sets $B_q = \{ b \in B | \mathcal{E}_b \text{ is stable over } \mathcal{Y}_b \}$, $B_{s} = \{ b \in B | \mathcal{E}_b \text{ is semistable over } \mathcal{Y}_b \}$, $B_{s\text{ss}} = \{ b \in B | \mathcal{E}_b \text{ is regular semistable over } \mathcal{Y}_b \}$ and $B_{s\text{ps}} = \{ b \in B | \mathcal{E}_b \text{ is regular polystable over } \mathcal{Y}_b \}$ are all open in $B$.

Proof. Replacing $B$ by $B_{\text{red}}$ we may assume $B$ is reduced. The statement is local so we may further assume $B$ is connected and is either affine scheme (in the algebraic setting) or is a Stein space (in the complex analytic setting). Furthermore it is obvious it suffices to prove the statement of the proposition for elliptic fibrations which have a section $\sigma : B \to \mathcal{Y}$. We need a lemma.

Lemma B.5 (Atiyah). Let $r, d$ be a pair of integers, $r \geq 1$. Then there exists a vector bundle $E(r, d)$ over $\mathcal{Y}$ such that for every $b \in B$ the fiber $E(r, d)_b$ is indecomposable of rank $r$ and degree $d$ over $\mathcal{Y}_b$ and if $d = 0$ it holds $h^0(\mathcal{Y}_b, E(r, d)_b) = 1$. 

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**Proof.** We only indicate the modifications one needs to make in order to apply the arguments of [Al]. Let $D = \sigma(B)$. Tensoring by powers of $\mathcal{L}(D)$ we see it suffices to construct $E(r, d)$ for $0 \leq d < r$. One proceeds by induction on $r$. If $r = 1$ one lets $E(1, 0) = \mathcal{O}_Y$. Suppose $E(\ell, d)$ is constructed for all $\ell < r$. Consider $E(r - d, d)$. By [Al] Lemma 15 one has $h^1(\mathcal{Y}_b, E(r - d, d)\mathcal{Y}) = h$ where $h = \max\{1, d\}$. By Grauert’s theorem the sheaf $G = R^1_qE(r - d, d)^\vee$ is locally free. The extensions of $q^*G$ by $E(r - d, d)^\vee$ are classified by $\text{Ext}^1(q^*G, E(r - d, d)^\vee) \cong H^1(\mathcal{Y}, q^*G^\vee \otimes E(r - d, d)^\vee) \cong H^0(B, R^1q_*(q^*G^\vee \otimes E(r - d, d)^\vee)) \cong \text{Hom}_B(G, R^1q_*(E(r - d, d)^\vee))$. To the identity homomorphism it corresponds an extension

$$0 \to E(r - d, d)^\vee \to F \to \mathcal{G} \to 0.$$ 

One lets $E(r, d) = F^\vee$.

We continue the proof of Proposition [B.4]. Let us consider $\mathcal{Y} \times_B \mathcal{Y}$. Let $\Delta \subset \mathcal{Y} \times_B \mathcal{Y}$ be the diagonal. Let $\mathcal{E}(r, d) = p_1^*E(r, d) \otimes L(\Delta - p_2^*D)$ where $p_i : \mathcal{Y} \times_B \mathcal{Y} \to \mathcal{Y}$ are the two projections. Looking at $p_2 : \mathcal{Y} \times_B \mathcal{Y} \to \mathcal{Y}$ as a base change of the fibration $q : \mathcal{Y} \to B$ we have that $\mathcal{E}(r, d)$ is a Poincaré vector bundle for the elliptic fibration $q : \mathcal{Y} \to B$. Namely, every fiber $\mathcal{E}(r, d)_z$, $z \in \mathcal{Y}$ is an indecomposable vector bundle over $\mathcal{Y}_b$, where $q(z) = b$, and for every indecomposable vector bundle $E$ of rank $r$ and degree $d$ over $\mathcal{Y}_b$ there exists a $z \in \mathcal{Y}_b$ such that $E \cong \mathcal{E}(r, d)_z$ (cf. [Al] Theorem 10).

Let $\mu = d(E_b)/r(E_b)$. Let us fix integers $r \geq 1$ and $d \in \mathbb{Z}$ such that $(r, d) = 1$ and $d/r \geq \mu$ (resp. $> \mu$). Let $A \subset B$ be the subset of $b \in B$ such that there exists a stable vector bundle $F$ over $\mathcal{Y}_b$ of rank $r$ and degree $d$ and a nonzero homomorphism $F \to \mathcal{E}_b$. We claim $A$ is closed in $B$. Indeed, consider $\mathcal{H} = \mathcal{E}(r, d)^\vee \otimes p_1^*E$ over $\mathcal{Y} \times_B \mathcal{Y}$. From the upper semi-continuity theorem for cohomology applied to $\mathcal{H}$ and $p_2 : \mathcal{Y} \times_B \mathcal{Y} \to \mathcal{Y}$ it follows that the set $S = \{z \in \mathcal{Y} | h^0(\mathcal{Y}_b, \mathcal{E}(r, d)^\vee \otimes \mathcal{E}_b) \geq 1 \text{ for } b = q(z)\}$ is closed in $\mathcal{Y}$. Since $q : \mathcal{Y} \to B$ is proper and $A = q(S)$ we conclude $A$ is closed in $B$.

If a vector bundle $E$ over a smooth, projective curve $X$ is not stable (resp. not semistable) there exists a proper stable vector subbundle $F$ of $E$ such that $\mu(F) \geq \mu(E)$ (resp. $\mu(F) > \mu(E)$) (cf. [NS] Proposition 4.5). Furthermore the slopes of such vector bundles are bounded from above by a constant. We recall how one obtains a majorant for $\mu(F)$ (see e.g. [Br] p.82). Let $L$ be a line bundle on $X$ of positive degree and let $E^\vee \otimes L^m$, $m \gg 0$ be generated by global sections. Then $E$ is isomorphic to a vector subbundle of the semistable bundle $(L^m)^{\oplus N}$, so for each vector subbundle $F \subset E$ one has $\mu(F) \leq m\deg(L)$.

If $q : \mathcal{Y} \to B$ is an algebraic family then there exists an $m \gg 0$ such that $q^*q_*\mathcal{E}\mathcal{Y} \otimes \mathcal{L}(mD) \to \mathcal{E}^\vee \otimes \mathcal{L}(mD)$ is surjective. If $\mathcal{E}_b$ is not stable (resp. not semistable) then as we saw in the preceding paragraph there exists a proper, stable vector subbundle $F \subset \mathcal{E}_b$ such that $\mu = \mu(\mathcal{E}_b) \leq \mu(F) \leq m$ (resp. $\mu = \mu(\mathcal{E}_b) < \mu(F) \leq m$). The rank and degree of such vector subbundles belong to a finite set of integers. So, from what we proved above and from Lemma [B.3] the set $B - B_3$ (resp. $B - B_{ss}$) consisting of $b \in B$ such that $\mathcal{E}_b$ is not stable (resp. not semistable) is a finite union of closed subsets of $B$ and hence it is closed in $B$.

In the complex analytic setting a small change of the above argument is necessary. It suffices to check that $U \cap (B - B_3)$ (resp. $U \cap (B - B_{ss})$) is closed for every relatively compact open subset $U \subset B$. For each such $U$ there exists an $m \gg 0$ such that $q^*(q_*\mathcal{E}\mathcal{Y} \otimes \mathcal{L}(mD)|_U)$ →
$E^\vee \otimes \mathcal{L}(mD)|_{q^{-1}U}$ is surjective. Repeating the argument of the preceding paragraph we see that both $B - B_s$ and $B - B_{ss}$ are closed in $B$.

That $B_{ss}$ is open in $B_{ss}$ follows from Lemma [B.2] and the upper semi-continuity theorem for cohomology.

Let $r(E) = rh$, $d(E) = dh$ where $(r, d) = 1$. Let $1 \leq \ell \leq h$. Consider the vector bundle $\mathcal{H} = E(\ell, d\ell)^\vee \otimes p^*_z E$ over $Y \times_B Y$. From the upper semi-continuity theorem for cohomology applied to $\mathcal{H}$ and $p_2 : Y \times_B Y \to Y$ it follows that the set $S_\ell = \{z \in Y | h^0(Y_b, E(\ell, d\ell)^\vee \otimes E_b) \geq 2 \text{ for } b = q(z)\}$ is closed in $Y$. Since $q : Y \to B$ is proper the image $A_\ell = q(S_\ell)$ is closed in $B$. According to Lemma [B.3] the set $B_{rps}$ is the complement in $B_{ss}$ of $\bigcup_{\ell=1}^h A_\ell$. Therefore $B_{rps}$ is open in $B$. □

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