1 Abstract

The index formula is a local statement made on global and local data; for this reason we introduce local Alexander - Spanier cohomology, local periodic cyclic homology, local Chern character and local $T^*_r$-theory. Index theory should be done: Case 1: for arbitrary rings, Case 2: for rings of functions over topological manifolds. Case 1 produces general index theorems, as for example, over pseudo-manifolds. Case 2 gives a general treatment of classical and non-commutative index theorems. All existing index theorems belong to the second category. The tools of the theory would contain: local $T^*_r$-theory, local periodic cyclic homology, local Chern character. These tools are extended to non-commutative topology. The index
formula has three stages: **Stage I** is done in $T_i^{loc}$-theory, **Stage II** is done in the local periodic cyclic homology and **Stage III** involves products of distributions, or restriction to the diagonal. For each stage there corresponds a **topological index** and an **analytical index**. The construction of $T_*$-theory involves the $T$-completion. It involves also the need to work with **half integers**; this should have important consequences.

La formule de l’indice est une formule locale faite sur les données globales et locales; pour cette raison, nous introduisons local Alexander - Spanier co-homologie, homologie cyclique périodique locale, caractère de Chern locale et la théorie local $T_*$. La théorie de l’indice doit être fait: **Case 1**: pour anneaux arbitraires, **Case 2**: pour anneaux de fonctions sur les variétés topologiques. **Cas 1** est le cas des théorèmes de l’indice général, comme par exemple, par exemple pseudo-viariétés. **Cas 2** donne un traitement général des théorèmes d’index classiques et de la géométrie non-commutatives. Tous les théorèmes d’index existants appartiennent à la deuxième catégorie. Le outils de la théorie contiendra: local $T_*$- théorie, homologie cyclique périodique locale, caractère de Chern locale. Ces outils sont tendus topologie non-commutative. La formule de l’indice a ha trois tapes: **Etape I** est fait dans $T_i^{loc}$- théorie, **Etape II** est fait dans l’homologie cyclique périodique local et **Etape III** implique des produits de distributions, ou la restriction la diagonale. Pour chaque tape correspond un **index topologique** et une **index analytique**. La construction de $T_*$- théorie utilise le $T$-completion. Elle implique également la nécessité de travailler avec **demi entiers**; cela devrait avoir des conséquences importantes.

**Part I**

**Local algebraic structures.**

2 Localised rings.

**Definition 1** Localised rings.

*Let $A$ be an unital associative ring. The ring $A$ is called localised ring provided it is endowed with an additional structure satisfying the axioms (1) - (3) below.*
Axiom 1. The underlying space $\mathcal{A}$ has a decreasing filtration by sub-spaces $\{\mathcal{A}_\mu\}_{\mu \in \mathbb{N}} \subset \mathcal{A}$.

Axiom 2. $C.1 \subset \mathcal{A}_\mu$, for any $\mu \in \mathbb{N}$

Axiom 3. For any $\mu, \mu' \in \mathbb{N}^+$, $\mathcal{A}_\mu \cdot \mathcal{A}_{\mu'} \subset \mathcal{A}_{Mn(\mu, \mu')-1}$, $(\mathcal{A}_0 \cdot \mathcal{A}_0 \subset \mathcal{A}_0)$.

**Remark 2** A ring $\mathcal{A}$ could have different localisations.

### 3 Local Alexander - Spanier co-homology.

**Definition 3** Let $\mathcal{A} = \mathcal{A}_\mu$ be a localised unital ring and $G$ an Abelian group. Define

$$C^{p}_{AS}(\mathcal{A}_\mu, G) = \{ \sum_{i} g_i a^i_0 \otimes a^i_1 \otimes \cdots \otimes a^i_p, \ g_i \in G, \ a_i \in \mathcal{A}_\mu \}_{p=0,1,\ldots,\infty}. \tag{1}$$

The boundary map $d$ is defined as in the classical definition of Alexander - Spanier co-homology

$$d : C^{p}_{AS}(\mathcal{A}_\mu, G) \rightarrow C^{p+1}_{AS}(\mathcal{A}_\mu, G) \tag{2}$$

$$d(\sum_{i} g_i a^i_0 \otimes a^i_1 \otimes \cdots \otimes a^i_p) = \sum_{i} g_i a_0 \otimes a^i_1 \otimes \cdots \otimes a^i_p + \cdots + (-1)^{p+1} a^i_0 \otimes a^i_1 \otimes \cdots \otimes a^i_p \otimes 1. \tag{3}$$

The local Alexander - Spanier co-homology is

$$H^{loc,p}_{AS}(\mathcal{A}) = \text{ProjLim}_\mu H_p(C^{*}_{AS}(\mathcal{A}_\mu, G), d). \tag{6}$$

### 4 Local periodic cyclic homology. Long exact sequence.

We assume that $\mathcal{A}_\mu$ is a localised unital ring.

The operators, see [9], $T$, $N$, $B$, $I$, $S$ pass to localised rings. Therefore local cyclic homology, local periodic cyclic homology may be extended to localised rings.

**Theorem 4** The local Hochschild and cyclic homology are well defined for a localised ring.
One has the exact sequence (analogue of Connes’ exact sequence, see Connes [9])

\[ \ldots \xrightarrow{B} H^\text{loc}_p(A) \xrightarrow{I} H^\text{loc,}\lambda_p(A) \xrightarrow{S} H^\text{loc}_{p-2}(A) \xrightarrow{B} H^\text{loc}_{p-1}(A) \xrightarrow{S} \ldots \]  

(7)

**Theorem 5** For \( A = C^\infty(M) \), one has

\[ H^\text{loc}_p(C^\infty(M)) = H^\text{loc}_p(C^\infty(M)). \]  

(8)

The local bi-complex \((b, B)(A_\mu)\) is well defined for localised rings too. The general term of the \((b, B)(A_\mu)\) bi-complex is

\[ C_{p,q} = \otimes^{p-q+1} A_\mu, \quad q \leq p \]  

(9)

\[ b : C_{p,q} \longrightarrow C_{p,q-1} \]  

(10)

\[ B : C_{p,q} \longrightarrow C_{p+1,q}. \]  

(11)

**Definition 6** The homology of the bi-complex \((b, B)(A_\mu)\) has two components: the even, resp. odd, component corresponding to the \( p - q = \text{even number} \), resp. \( p - q = \text{odd number} \).

The local periodic cyclic homology of the localised ring \( A \) is

\[ H^\text{loc,per,\lambda}_{ev,odd}(A) := \lim_{\text{Proj}} H_{ev,odd}(A_\mu). \]  

(12)

## 5 Local periodic Chern character

In this section we localise the periodic even/odd Chern character. All operations involved in the construction of the cyclic homology may be localised. Here we use the bi-complex \((b, B)(A_\mu)\) only onto non degenerate elements, i.e. onto the range of the idempotent \( \Pi \).

**Definition 7** local periodic cyclic Chern character.

Let \( A_\mu \) be a localised ring.

1. For any idempotent \( e \in M_n(A_\mu) \) define

\[ Ch_{ev,A_\mu}(e) = \text{Tr} e + \sum_{p=1}^{\infty} (-1)^p \frac{(2p)!}{p!} (e - \frac{1}{2})(de)^{2p}. \]  

(13)

\( Ch_{ev,A_\mu}(e) \) is an even cycle in the \((b, B)(A_\mu)\) cyclic complex. Its homology class is the periodic cyclic Connes Chern character of \( e \).
Let $e \in T^0_{\text{loc}}(A)$; define

$$Ch_{ev}(e) = \text{ProjLim}_\mu Ch_{ev,A_\mu}(e) \in H^0_{\text{loc,per,}\lambda}(A).$$  \hspace{1cm} (14)

2. For any invertible $u \in \mathbb{M}_n(A_\mu)$, define

$$Ch_{odd}(u) = \sum_p (-1)^{2p+1}(2p+1)! \text{Tr}(u^{-1}du)(du^{-1}du)\ldots(du^{-1}du).$$  \hspace{1cm} (15)

Let $u \in T^1_{\text{loc}}(A)$; define

$$Ch_{odd}(u) = \text{ProjLim}_\mu Ch_{odd,A_\mu}(u) \in H^1_{\text{loc,per,}\lambda}(A).$$  \hspace{1cm} (16)

In non-commutative topology, discussed later in this article, we will present a definition of the Chern character of idempotents.

6 **Local index theorem**

In this section we consider an elliptic pseudo-differential operator $A$. We are going to define the Chern character of differences of idempotents; the skew-symmetrisation is not necessary. For more details about this topic see Teleman [25], [29].

**Lemma 8** Let $R(A) := P - e$, where $P$ and $e$ are idempotents. Then $R(A)$ satisfies the identity

$$R(A)^2 = R(A) - (e.R(A) + R(A).e).$$  \hspace{1cm} (17)

**Definition 9** Let $R(A) := P - e$, where $P$ and $e$ are idempotents. Let $\Lambda = \mathbb{C} + e.\mathbb{C}$. Then,

$$C_q(R(A)) := \text{Tr} \otimes^q_\Lambda R(A)$$  \hspace{1cm} (18)

is the Chern character of $R(A)$. For the definition of $R(A)$ see §23, Definition 57.

**Lemma 10** $C_q(R(A))$ is a cycle in the complex $\{\otimes^q_\Lambda \mathbb{M}_N(A)\}_*$, such that

$$\lambda.a_0 \otimes_\Lambda a_1 \otimes_\Lambda \ldots \otimes_\Lambda a_k = a_0 \otimes_\Lambda a_1 \otimes_\Lambda \ldots \otimes_\Lambda a_k.\lambda$$  \hspace{1cm} (19)

for any $\lambda \in \Lambda$. 

Proof. One has
\[
\begin{align*}
    b' \ \ & C_q (R(A)) = Tr \left[ (R(A) - (e.R(A) + R(A).e)) \otimes \Lambda \cdots \otimes \Lambda R(A) \ldots \right] \quad (20) \\
    & -R(A) \otimes \Lambda R(A) \otimes \Lambda \cdots \otimes \Lambda (R(A) - (e.R(A) + R(A).e)) \left] = (21) \\
    & Tr \left[ -(e.R(A) + R(A).e)) \otimes \Lambda \cdots \otimes \Lambda R(A) \ldots \right] \quad (22) \\
    & -R(A) \otimes \Lambda R(A) \otimes \Lambda \cdots \otimes \Lambda (- (e.R(A) + R(A).e) \right] = (23) \\
    & -Tr \left[ e.R(A) \otimes \Lambda \cdots \otimes \Lambda R(A) - R(A) \otimes \Lambda \cdots \otimes \Lambda R(A).e \right] = 0. \quad (24)
\end{align*}
\]

\(C_q (R(A))\) is a cyclic cycle. It represents a cyclic homology class in \(H^*_\Lambda (\otimes^{2q+1} (A_\mu)).\)

The ring \(\Lambda\) is separable. Indeed the mapping \(\mu : \Lambda \otimes \Lambda^{\text{op}} \rightarrow \Lambda, \quad \mu(1 \otimes 1) = 1, \ \mu(1 \otimes e) = \mu(e \otimes 1) = e, \ \mu(e \otimes e) = e\)

has the \(\Lambda\)-bimodule splitting \(s : \Lambda \rightarrow \Lambda \otimes \Lambda^{\text{op}}, \quad s(1) = e \otimes e + (1 - e) \otimes (1 - e), \ s(e) = e \otimes e.\)

Theorem 1.2.13 \([16]\) states that the homology of the Hochschild complex is isomorphic to the homology of the complex \(C^S(A)\). Using the long exact sequence in homology associated to the complex \(H^*_\Lambda (\otimes^{2q+1} (A_\mu))\) and the Connes exact sequence, we find that the cyclic homology of \(H^*_\Lambda (\otimes^{2q+1} (A_\mu))\) is isomorphic to the cyclic homology of the algebra \(A\). Here we have used the localisation of the algebra \(M_N(A)\) defined by the supports of the operators. \(\blacksquare\) It remains to solve the problem of producing the \textit{whole index class}. This could be obtained by the formula \(Ch_{ev} R(A) - Ch_{ev} e_{II},\) \(25\)
where \(R(A)\) is the the operator associated to the signature operator.

Part II

Prospects in Index Theory

7 Local algebraic \(T_i\) theory.

This section deals with the program presented in \([28]\).
For any associative ring \( A \) we define the commutative groups \( T_i(A) \), \( i = 0, 1 \). We introduce the notion of localised rings, see Definition 1, \( \mathcal{A} = \{ A_\mu \} \), given by a linear filtration of the algebra \( A \) and we associate the commutative groups \( T_i^{\text{loc}}(A) \). Although we define solely \( T_i^{\text{loc}}(A) \) for \( i = 0, 1 \), we expect our construction could be extended in higher degrees.

We stress that our construction of \( T_i(A) \) and \( T_i^{\text{loc}}(A) \), uses exclusively matrices. The projective modules are totally avoided. The role of equivalence of projective modules, used in the classical construction of the algebraic \( K \)-theory, is played by conjugation.

The commutative group \( T_0(A) \) is by definition the Grothendieck completion of the space of idempotent matrices factored through the equivalence relations: -1) stabilisation \( \sim_s \), -2) conjugation \( \sim_c \), and -3) for localised groups, \( T_0^{\text{loc}}(A) \), projective limit with respect to the filtration, denoted \( \sim_p \). The non-localised group \( T_0(A) \) coincides with the classical algebraic \( K_0 \)-theory group.

The groups \( T_1(A) \) are by definition the quotient of \( \text{GL}(A) \) through the equivalence relation generated by -1) stabilisation \( \sim_s \), -2) conjugation \( \sim_c \) and -3) \( \sim_{O(A)} \), where \( O(A) \) is the sub-group generated by elements of the form \( u \oplus u^{-1} \), for any \( u \in \text{GL}(A) \). For any \( u_1, u_2 \in \text{GL}(A) / (\sim_s \cup \sim_c) \), one defines \( u_1 \sim_{O(A)} u_2 \) if there exist \( \xi_1, \xi_2 \in O(A) \) such that \( u_1 + \xi_1 = u_2 + \xi_2 \). The third relation is a particular case of a new completion procedure which we call \( T \)-completion. The operation

\[
\text{GL}(A) / (\sim_s \cup \sim_c) \longrightarrow \text{GL}(A) / (\sim_s \cup \sim_c \cup \sim_{O(A)})
\]

transforms the commutative semi-group \( \text{GL}(A) / (\sim_s \cup \sim_c) \) in the commutative group \( \text{GL}(A) / (\sim_s \cup \sim_c \cup \sim_{O(A)}) \).

The groups \( T_1^{\text{loc}}(A) \) follow the same construction as that of \( T_1(A) \), provided the supports of the elements belong to \( A_\mu \). Our definition of \( T_1(A) \) and \( T_1^{\text{loc}}(A) \) does not use the commutator sub-group \( \{ \text{GL}(A), \text{GL}(A) \} \) nor elementary matrices in its construction.

We define short exact sequences of localised rings and we get the corresponding open six terms exact sequence (Theorem).

We stress that one has to take the tensor product of the expected six terms exact sequence by \( \mathbb{Z}[\frac{1}{2}] \) in order to get the open six terms exact sequence. We expect the factor \( \mathbb{Z}[\frac{1}{2}] \) to have important implications, among them, Pontrjagin classes, existence of a generator of
the $K$-homology fundamental class and Kirby-Siebenmann obstruction class.

Our work shows that the basic relations which define $T_1$ and $T_1^{\text{loc}}$ reside in the additive sub-group generated by elements of the form $u \oplus u^{-1}$, $u \in \text{GL}(\mathcal{A})$, rather than in the multiplicative commutator sub-group $[\text{GL}(\mathcal{A}), \text{GL}(\mathcal{A})]$.

Even for trivially filtered algebras, $\mathcal{A} = \{\mathcal{A}_\mu\}$, for all $\mu \in \mathbb{N}$, the groups $T_1^{\text{loc}}(\mathcal{A})$ provides more information than the classical group $T_1(\mathcal{A})$. For the computation of the groups $T_i^{\text{loc}}(\mathbb{C})$ see [30].

8 Motivation

To motivate the next considerations we have in mind the index formula applied onto the algebra of integral operators and pseudo-differential operators. The index formula is a global statement whose ingredients may be computed by local data. Our leading idea is to localise $K$-theory and periodic cyclic homology along the lines of the Alexander-Spanier co-homology in such a way that the new tools operate naturally with it, see [28].

This article goes along the lines of [28]. We define localised rings $\mathcal{A}$ and we define and their local $T$-theory, $T_i^{\text{loc}}(\mathcal{A})$, $i = 0, 1$. Keeping in mind that both the topological and analytic indices of an elliptic operator are stable under cutting the operators about the diagonal, leads us naturally to the notion of localised rings. Based on this notion we pass to the problem of finding a natural $K$-theory able to control these local entities. These are the local algebraic $T$-theory groups, $T_i^{\text{loc}}(\mathcal{A})$, $i = 0, 1$ and local cyclic homology.

For the construction of $K_i$-theory groups, in the pure algebraic context, the reader may consult the following books [5], [10], [18], [26], in the Banach algebras or $C^*$-algebras category, [21], [12].

Our construction of $T_0^{\text{loc}}$ uses exclusively idempotent matrices. The reasons why we chose to avoid projective modules are: -i) projective modules, in comparison with idempotent matrices, contain more ambiguity and -ii) matrices are more suitable for controlling the ring filtration data and are more prone to make calculations. No reference to projective modules is used in our constructions.

Regarding our construction of $T_1^{\text{loc}}$ we recall that the classical algebraic $K$-theory group $K_1(\mathcal{A})$ of the algebra $\mathcal{A}$, see [2], [6], [5],
is by definition the Whitehead group

$$K_1(\mathcal{A}) := \text{GL}(\mathcal{A})/[\text{GL}(\mathcal{A}), \text{GL}(\mathcal{A})],$$

(26)

where $[\text{GL}(\mathcal{A}), \text{GL}(\mathcal{A})]$ is the commutator normal sub-group of the group of invertible matrices $\text{GL}(\mathcal{A})$. Our definition of local $T$-theory groups needs to keep track of the number of multiplications performed inside the algebra $\mathcal{A}$. In order for our constructions to hold it is necessary to involve a bounded number of multiplications. It is important to state that, in general, the number of multiplications needed to generate the whole commutator sub-group is un-bounded. It is known that the commutator sub-group is also generated by the elementary matrices. This is the reason why our definition of $T^\text{loc}_1(\mathcal{A})$ avoids entirely factorising $\text{GL}(\mathcal{A})$ through the commutator sub-group or the sub-group generated by elementary matrices.

$T^\text{loc}_0(\mathcal{A})$ is by definition the Grothendieck completion of the semi-group of idempotent matrices in $\mathbb{M}_n(\mathcal{A}_\mu)$ modulo three equivalence relations: -i) stabilisation $\sim_s$, -ii) local conjugation $\sim_c$ by invertible elements $u \in \text{GL}_n(\mathcal{A}_\mu)$ and -iii) projective limits with respect to $\mu \in \mathbb{N}$.

To understand the relationship between our definition of the group $T^\text{loc}_1$ and $K_1$, recall that the commutator sub-group $[\text{GL}(\mathcal{A}, \text{GL}(\mathcal{A})]$ is given by arbitrary products of multiplicative commutators

$$[A, B] := ABA^{-1}B^{-1}, \quad \text{for any } A, B \in \text{GL}_n(\mathcal{A}).$$

On the other side, supposing that $A$ and $B$ are conjugated, i.e. $A = UBU^{-1}$, we have

$$A = UBU^{-1} = UBU^{-1}B^{-1}B = [U, B]B.$$

This shows that if $A$ and $B$ are conjugated, they differ, multiplicatively, by a commutator. To complete this remark, we say that $A$ and $B$ are locally conjugated provided $A, B$ and $U$ belong to some $\text{GL}_n(\mathcal{A}_\mu)$; here, $\mathcal{A}_\mu$ denote the terms of the filtration of $\mathcal{A}$.

It is important to note that in the particular case $\mathcal{A} = \mathbb{C}$ the quotient of $\text{GL}(\mathbb{M}(\mathbb{C}))$ through the commutator sub-group $\text{GL}(\mathbb{M}(\mathbb{C})), \text{GL}(\mathbb{M}(\mathbb{C}))$ gives much less information than $T^\text{loc}_1(\mathbb{M}(\mathbb{C}))$. 

9
8.1 Factorisation of $u \otimes u^{-1}$

**Proposition 11** In the algebra of matrices one has the identity

$$
\begin{pmatrix}
u & 0 \\
0 & u^{-1}
\end{pmatrix} =
\begin{pmatrix}1 & u \\
0 & 1
\end{pmatrix}
\begin{pmatrix}1 & 0 \\
-u^{-1} & 1
\end{pmatrix}
\begin{pmatrix}1 & u \\
0 & 1
\end{pmatrix}
\begin{pmatrix}0 & -1 \\
1 & 0
\end{pmatrix}.
\tag{27}
$$

This formula will play an important role in the construction of $T_\ast$-theory.

9 Algebraic $T_i$ and $T_{i}^{\text{loc}}$-theory.

9.1 Generalities and Notation.

Let $\mathcal{A}$ be a ring, with or without unit. If the unit will be needed, the unit will be adjoined.

**Definition 12** Given the ring $\mathcal{A}$ we denote by $\mathcal{M}_n(\mathcal{A})$ the space of $n \times n$ matrices with entries in $\mathcal{A}$. $\mathcal{M}_n(\mathcal{A})$ is a bi-lateral $\mathcal{A}$-module.

Let $\text{Idemp}_n \subset \mathcal{M}_n(\mathcal{A})$ be the subset of idempotents $p$ ($p^2 = p$) of size $n$ with entries in $\mathcal{A}$.

Suppose $\mathcal{A}$ has an unit. We denote by $\text{GL}_n(\mathcal{A})$ the sub-space of matrices $M$ of size $n$, with entries in $\mathcal{A}$ which are invertible, i.e. there exists the matrix $M^{-1} \in \mathcal{M}_n(\mathcal{A})$ such that $MM^{-1} = M^{-1}M = 1$. $\text{GL}_n(\mathcal{A})$ is a non-commutative group under multiplication.

**Definition 13** The inclusions

- $\text{Idemp}_n(\mathcal{A}) \longrightarrow \text{Idemp}_{n+1}(\mathcal{A})$

$$
p \mapsto p' = \begin{pmatrix} p & 0 \\
0 & 0
\end{pmatrix}
\tag{28}
$$

- $\text{GL}_n(\mathcal{A}) \longrightarrow \text{GL}_{n+1}(\mathcal{A})$

$$
u \mapsto u' = \begin{pmatrix} u & 0 \\
0 & 1
\end{pmatrix}
\tag{29}
$$

are stabilisations.

Stabilisations of idempotents -i) and invertible matrices -ii) define two direct systems with respect to $n \in \mathbb{N}$.

- $\text{iii)}$ If $p$ or $p'$ are idempotents and one of them is an iterated stabilisation of the other we write $p \sim_\ast p'$.

If $u$ or $u'$ are invertible matrices and one of them is an iterated stabilisation of the other we write $u \sim_\ast u'$. 

10
10 Localised rings

We recall the definition of localised rings given before.

**Definition 14** Let \(A\) be an unital associative ring. The ring \(A\) is called localised ring provided it is endowed with an additional structure satisfying the axioms (1) - (4) below.

Axiom 1. The underlying space \(A\) has a decreasing filtration by sub-spaces \(\{A_{\mu}\}_{\mu \in \mathbb{N}} \subset A\).

Axiom 2. \(C.1 \subset A_{\mu}\), for any \(\mu \in \mathbb{N}\)

Axiom 3. For any \(\mu, \mu' \in \mathbb{N}^+\), \(A_{\mu} \cdot A_{\mu'} \subset A_{\min(\mu, \mu')} - 1\), \((A_0 \cdot A_0 \subset A_0)\).

**Definition 15** Homomorphisms of localised rings. Induced homomorphism.

- i) A homomorphism from the localised ring \(A = \{A_{\mu}\}_{\mu \in \mathbb{N}}\) to the localised ring \(B = \{B_{\mu}\}_{\mu \in \mathbb{N}}\) is an ring homomorphism \(\phi : A \rightarrow B\) such that \(\phi : A_{\mu} \rightarrow B_{\mu}\), for any \(\mu \in \mathbb{N}\).

- ii) Let \(f : A \rightarrow B\) be a localised ring homomorphism.

Let \(f_* : \mathbb{M}_n(A_{\mu}) \rightarrow \mathbb{M}_n(B_{\mu})\) be the induced homomorphism which replaces any component \(a_{ij}\) of the matrix \(M\) with the component \(f(a_{ij})\) of the matrix \(f_*(M)\).

**Remark 16** The notion of localised ring differs from the notion of \(m\)-algebras defined by Cuntz \([22]\), in many respects. The sub-spaces \(A_n\) are not required to be algebras, or even more, topological algebras. In the Cuntz’ definition of localised Banach algebras, the projective limit of these sub-algebras might be the zero Banach algebra. However, even in these cases, the corresponding local \(T\)-theory could be not trivial.

**Remark 17** In this section any localisation of the algebra of bounded operators on the Hilbert space \(H := L_2(M \times M)\) derives from a decreasing filtration \(\{U_{\mu}\}\), towards the diagonal, of the space of bounded operators on \(M \times M\). In such a case, if \(\mu' > \mu\), one has the close inclusions

\[
L_2(U_{\mu'}) \subset L_2(U_{\mu})
\]

and any internal authomorphism of the algebra \(\mathbb{B}(U_{\mu'})\) is an internal authomorphism of the algebra \(\mathbb{B}(U_{\mu})\) (any auto-morphism of the algebra \(L_2(U_{\mu})\) will be extended by the identity on the complement of the Hilbert space \(L_2(U_{\mu'})\)).
Remark 18 The immediate application of this theory regards pseudo-differential operators. The pseudo-differential operators on a compact smooth manifold form a localised (Banach) algebra. The filtration is defined in terms of the support of the operators; the bigger the filtration order is, the smaller the supports of the operators towards the diagonal are.

11 Local Mayer-Vietoris Diagrams.

In this section we adapt Milnor’s [5] description of the first two algebraic $K$-theory groups to the case of localised rings.

Let $\Lambda, \Lambda_1, \Lambda_2$ and $\Lambda'$ be rings with unit 1 and let

$$\Lambda \xrightarrow{i_1} \Lambda_1 \xrightarrow{j_1} \Lambda'$$

$$\Lambda_2 \xrightarrow{j_2} \Lambda'$$

be a commutative diagram of ring homomorphisms. All ring homomorphisms $f$ are assumed to satisfy $f(1) = 1$. Any module in this paper is a left module.

We assume the diagram satisfies the three conditions below.

**Hypothesis 1.** All rings and ring homomorphisms are localised, see Definition 1.

**Hypothesis 2.** $\Lambda$ is a local product of $\Lambda_1$ and $\Lambda_2$, i.e. for any pair of elements $\lambda_1 \in \Lambda_{1,\mu}$ and $\lambda_2 \in \Lambda_{2,\mu}$ such that $j_1(\lambda_{1,\mu}) = j_2(\lambda_{2,\mu}) = \lambda' \in \Lambda'$, there exists only one element $\lambda_n \in \Lambda_\mu$ such that $i_1(\lambda_\mu) = \lambda_{1,\mu}$ and $i_2(\lambda_\mu) = \lambda_{2,\mu}$.

The ring structure in $\Lambda$ is defined by

$$(\lambda_1, \lambda_2) + (\lambda_1', \lambda_2') := (\lambda_1 + \lambda_1', \lambda_2 + \lambda_2'), \ (\lambda_1, \lambda_2).(\lambda_1', \lambda_2') := (\lambda_1.\lambda_1', \lambda_2.\lambda_2'),$$

i.e. the ring operations in $\Lambda$ are performed component-wise.

**Hypothesis 3.** At least one of the homomorphisms $j_1$ and $j_2$ is surjective.

**Remark 19** -i) Any matrix $M \in M_n(\Lambda)$ consists of a pair of matrices $(M_1, M_2) \in M_n(\Lambda_1) \times M_n(\Lambda_2)$ subject to the condition $j_{1,*}M_1 = j_{2,*}M_2$. Any matrix $M \in M_n(\Lambda)$ is called double matrix.
-ii) if \((M_1, M_2), (N_1, N_2)\) are double matrices, (resp. belong to 
\(\mathbb{M}(\Lambda_1) \times \mathbb{M}(\Lambda_2)\)) and \((\lambda_1, \lambda_2) \in \Lambda\), (resp. \((\lambda_1, \lambda_2) \in \Lambda_1 \times \Lambda_2\)) then relations (21.6) induce onto the space of double matrices, (resp. the space \(\mathbb{M}(\Lambda_1) \times \mathbb{M}(\Lambda_2)\)) the following relations

\[
(\lambda_1, \lambda_2) \, (M_1, M_2) = (\lambda_1 \, M_1, \lambda_2 \, M_2)
\]

\[
(M_1, M_2) + (N_1, N_2) = (M_1 + N_1, M_2 + N_2)
\]

\[
(M_1, M_2) \cdot (N_1, N_2) = (M_1 \cdot N_1, M_2 \cdot N_2)
\]

**Definition 20** A commutative diagram satisfying Hypotheses 1, 2, 3. will be called local Mayer-Vietoris diagram.

**Standing Hypothesis.** In the remaining part of this chapter we assume that the ring \(\mathcal{A}\) is localised. We assume also that \(\Lambda_1 = \Lambda_2 = \mathcal{A}\) and that \(J \subset \mathcal{A}\) is a bi-lateral ideal. Define \(\Lambda := \{(\lambda_1, \lambda_2) \in \mathcal{A} \oplus \mathcal{A}, \text{ such that } \lambda_1 - \lambda_2 \in J\}\). Denote \(i_\alpha(\lambda_1, \lambda_2) := \lambda_\alpha, \alpha = 1, 2\). Denote also \(\Lambda' := \mathcal{A}/J\) and \(j_\alpha(\lambda_1, \lambda_2) := \lambda_\alpha \mod. J\). One has the ring short exact sequence

\[
0 \to \Lambda \xrightarrow{(i_1, i_2)} \Lambda_1 \oplus \Lambda_2 \xrightarrow{j_1 - j_2} \Lambda' \to 0.
\]

We assume that the above scheme is a localised Mayer - Vietoris diagramme.

### 12 Preparing the definition of \(T_{0}^{\text{loc}}(\mathcal{A})\) and \(T_{1}^{\text{loc}}(\mathcal{A})\).

**Definition 21** We assume the ring \(\mathcal{A}\) is localised.

We consider the space of matrices with entries in \(\mathcal{A}_\mu\) and we denote it by \(\mathbb{M}_n(\mathcal{A}_\mu)\).

Let \(\text{Idemp}_n(\mathcal{A}_\mu)\) denote the space of idempotent matrices of size \(n\) with entries in \(\mathcal{A}_\mu\).

Let \(\text{GL}_n(\mathcal{A}_\mu)\) denote the space of invertible matrices \(M\) of size \(n\) with the property that the entries of both \(M\) and \(M^{-1}\) belong to \(\mathcal{A}_\mu\).

Let \(\text{Idemp}(\mathcal{A}_\mu) := \text{inj lim}_n \text{Idemp}_n(\mathcal{A}_\mu)\).

Let \(\text{GL}(\mathcal{A}_\mu) := \text{inj lim}_n \text{GL}_n(\mathcal{A}_\mu)\).

**Definition 22** -i) Two matrices \(s, t \in \mathbb{M}_n(\mathcal{A})\) are called conjugated and we write \(s \sim_c t\), provided they are related through an inner auto-morphism, i.e. there exists \(u, u^{-1} \in \text{GL}_n(\mathcal{A})\) such that \(s = utu^{-1}\).
-ii) Two matrices \( s, t \in M_n(A) \) will be called locally conjugated and we write \( s \sim_{lc} t \) provided they are related through a local inner auto-morphism defined by \( u, u^{-1} \in GL_n(A) \) such that \( s = utu^{-1} \).

In particular,

-ii.1) two idempotents \( p, q \in \text{Idemp}_n(A) \) are locally conjugated and we write \( p \sim_{lc} q \) provided there exists \( u, u^{-1} \in GL_n(A) \) such that \( q = u p u^{-1} \) and

-ii.2) two invertible matrices \( s, t \in GL_n(A) \) are locally conjugated and we write \( s \sim_{lc} t \) provided there exists \( u, u^{-1} \in GL_n(A) \) such that \( s = utu^{-1} \).

Sometimes, if it is clear from the context, \( \sim_{lc} \) will be simply denoted by \( \sim_c \).

**Proposition 23** -i) \( \text{Idemp}_n(A) \) and \( GL_n(A) \) are semigroups with respect to the direct sum

\[
A + B := \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix}
\]  

(ii) The operations of stabilisation, conjugation and local conjugation of idempotents, resp. invertible matrices, commute.

The spaces \( \text{Idemp}_n(A), GL_n(A) \) are compatible with stabilisations.

The direct sum addition of idempotents, resp. invertibles, is compatible with the local conjugation equivalence relation. Indeed, if \( s_1, s_2 \) are conjugated through an inner auto-morphism defined by the element \( u_1 (s_1 \sim_{lc} s_2) \) and \( t_1, t_2 \) are conjugated through an inner auto-morphism defined by the element \( u_2 (t_1 \sim_{lc} t_2) \), then \( (s_1 + t_1) \sim_{lc} (s_2 + t_2) \) are conjugated through the inner auto-morphism \( u_1 \oplus u_2 \).

With this observation, the associativity of the addition is now immediate.

These show that \( \text{Idemp}(A) / \sim_{lc}, \text{GL}(A) / \sim_l \) is an associative semi-group.

**Proposition 24** The semi-groups \( \text{Idemp}(A) / \sim_c, \text{GL}(A) / \sim_{lc} \) are commutative.

**Proof.** The result follows from the following identity valid for any
two matrices \(A, B \in M_n(\mathcal{A}_\mu)\)

\[
\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^{-1},
\]

which tells that \((A + B) \sim_c (B + A)\), resp. \((A + B) \sim_{lc} (B + A)\).

For more information about the relationship between the classical algebraic \(K\)-theory and the local \(T\)-theory see §.

13 Definition of \(T_0(\mathcal{A}_\mu)\) and \(T_0^{loc}(\mathcal{A})\).

We suppose the stabilisation is involved without being specified.

**Definition 25** Suppose \(\mathcal{A}\) is a localised unital associative ring. We define

- \(T_0(\mathcal{A}_\mu) = G\ (\text{Idemp}_n(\mathcal{A}_\mu) / \sim_c).\) \hspace{1cm} (35)

- \(T_0(\mathcal{A}) = \text{ProjLim}_{\mu \in \mathbb{N}} T_0(\mathcal{A}_\mu).\) \hspace{1cm} (36)

where \(G\) is Grothendieck completion.

Any local inner automorphism induces the identity on \(T_0^{loc}(\mathcal{A})\).

**Proposition 26** For any unital associative algebra \(\mathcal{A}\), trivially localised \((\mathcal{A}_\mu = \mathcal{A})\), \(T_0(\mathcal{A}) = K_0(\mathcal{A})\).

**Proof.** See Rosenberg [18] Lemma 1.2.1. ■

14 \(T_1(\mathcal{A}_\mu)\) and \(T_1^{loc}(\mathcal{A})\).

As in the previous sub-section, we assume the stabilisation is involved without being specified.

The equivalence class of the invertible element \(u \in \mathbb{GL}(\mathcal{A})\) modulo conjugation, \([u]_{\sim_c}\), will be called the abstract Jordan canonical form of \(u\). The group \(T_1(\mathcal{A}_\mu)\) we are going to define preserves much of the information provided by the abstract Jordan form. The classical definition of \(K_1\) extracts a minimal part of the abstract Jordan form. As the addition in the semi-group \(\mathbb{GL}(\mathcal{A})\) is given by the direct sum and the Jordan canonical form \(J(u)\) (in the classical case of the algebra \(\mathbb{GL}_n(\mathbb{R})\)) \(J\) behaves additively \((J(u + v) = J(u) + J(V)\)
modulo permutations of the Jordan blocks), given an arbitrary element \( u \in \text{GL}(A) \), it is not reasonable to expect existence of an element \( \tilde{u} \) such that \([u + \tilde{u}]_{\sim_c} = [1_{2n}]_{\sim_c}\). Given that we want \( T_1(A_\mu) \) to be a group, we introduce the group structure (opposite elements) forcibly. In the case of the classical \( K_1 \)-theory, the class of the element \( u^{-1} \) represents the opposite class, \(-[u] \in K_1(A)\). In our case, the opposite elements will be introduced by means of the \( T \)-completion technique, to be explained next.

15 **T-completion**

**Proposition 27** Let \( S \) be an additive commutative semi-group with zero element 0. Let \( I : S \rightarrow S \) be an additive involution, \((I^2 = Id)\), such that \( I(0) = 0 \).

Define the equivalence relation \( \sim_\sigma \) in \( S \): \( u \sim_\sigma v \) iff these exist two elements \( u_0, u_1 \in S \), such that the elements \( u, v, \xi_0 = u_0 + I(u_0) \) and \( \xi_1 = u_1 + I(u_1) \), satisfy

\[
    u + \xi_0 = v + \xi_1.
\]

(i) The equivalence relation \( \sim_\sigma \) is compatible with the addition in \( S \).

(ii) \( S/\sim_\sigma \) is a group.

(iii) Let \( [u] \) denote the \( \sim_\sigma \)-equivalence class of \( u \). Then

\[
    -[u] = [I(u)].
\]

**Definition 28** Define

\[
    O(u) := u + I(u) \quad \text{for any } u \in S
\]

\[
    O(S) := \{ O(u) \mid u \in S \}
\]

**Proof.**

(i) Obvious.

(ii) \( \sim \) is clearly reflexive and symmetric. If \( u_1 + \xi_1 = u_2 + \xi_2 \) and \( u_2 + \xi_3 = u_3 + \xi_4 \), \( u_1, u_2, u_3, u_4 \in O(S) \), then \( u_1 + (\xi_1 + \xi_3) = u_2 + (\xi_2 + \xi_3) = u_3 + (\xi_2 + \xi_4) \), which shows that \( \sim \) is transitive. Therefore \( \sim \) is an equivalence relation.

(iii) One the other side, for any element \( u \in S \), one has \([u] = [u + 0] = [u] + [0] = [0] + [u]\), which shows that the class of \([0] \) is a the zero element of \( S/\sim_\sigma \).
We have also \([u] + [I(u)] = [O(u)] = [0 + O(u)] = [0]\) because \(0 \sim (0 + O(u))\); therefore, \(-[u] = [I(u)]\) exists. From this we get further \([O(u)] = [u + I(u)] = [u] + [I(u)] = 0\). 

16 Definition of \(T_1(A_\mu)\) and \(T^\text{loc}_1(A)\)

The construction of the groups \(T_1(A_\mu)\) and \(T^\text{loc}_1(A)\) involves the \(T\)-completion \([15]\). We take

- \(S = \mathbb{GL}(A_\mu) \sim_c\)
- the involution \(I : \mathbb{GL}(A_\mu) \sim_{cl} \mathbb{GL}(A_\mu) \sim_{cl}\) is given by \(I(u) = u^{-1}\)
- the identity element \(I \in \mathbb{GL}(A_\mu)/ \sim_c\) becomes the zero element of \(S\).

Here \(\sim_l\) indicates that the invertible elements and their conjugation occurs locally. Note that if \(u_1 \sim_{cl} u_2\), then \(u_1^{-1} \sim_{cl} u_2^{-1}\).

The next proposition summarises the properties of \(O(A_\mu)\)

**Proposition 29**

- \(\text{i})\) The space \(O(A_\mu)\) is a commutative group; the zero element is the class if the identity
- \(\text{ii})\) the mapping

\[
O : \mathbb{GL}(A_\mu) \longrightarrow O(A_\mu)
\]

is additive and commutes with local conjugation

\[
O(u_1 + u_2) = O(u_1) + O(u_2)
\]

\[
O(\lambda u \lambda^{-1}) = \lambda O(u) \lambda^{-1}, \quad \lambda \in \mathbb{GL}(A_\mu),
\]

\(i.e.\) if \(u_1 \sim_{st} u_2\) then \(O(u_1) \sim_{st} O(u_2)\)

- \(\text{iii})\)

\[
O(u^{-1}) \sim_c O(u).
\]

- \(\text{iv})\)

\[
O(u_1 u_2) \neq O(u_1)O(u_2),
\]

**Definition 30**

\(T(A_\mu) := T - \text{completion of } \mathbb{GL}(A_\mu)/ \sim_{lc}\) \hspace{1cm} (46)

and

\(T^\text{loc}(A) := \text{proj lim}_{\mu \in \mathbb{N}} T_1(A_\mu)\) \hspace{1cm} (47)
Proposition 31

-i) $T_1(A_\mu)$ and $T^{loc}_1(A)$ are commutative groups.

-ii) The inverse element of $[u]$ is $-[u] = [u^{-1}]$.

Proof. -i) and -ii) follow from the properties of the T-completion, see Proposition 21.5 §15.

For the computation of the local algebraic $T$-theory, $i = 0, 1$, of the algebra of complex numbers $\mathbb{C}$ see Teleman [30].

For the computation of the local cyclic homology of the algebra of Hilbert-Schmidt operators on simplicial spaces see [27].

For the local index theorem see [29], [25].

17 Induced homomorphisms.

Definition 32 Let $f : A \rightarrow B$ be a localised ring homomorphism. Then the ring homomorphism $f$ induces homomorphisms

\[ f_* : T_0(A_\mu) \rightarrow T_0(B_\mu), \quad f_* : T^{loc}_0(A) \rightarrow T^{loc}_0(B) \]

and

\[ f_* : T_0(A_\mu) \rightarrow T_0(B_\mu), \quad f_* : T^{loc}_1(A) \rightarrow T^{loc}_1(B) \]

18 Constructing idempotents and invertible matrices over $\Lambda_\mu$.

We come back to the situation presented in the section 11. Here we produce local idempotents and invertibles of the ring $\Lambda$. These results will be used in the proof of the six terms exact sequence Theorem 7.

18.1 Constructing idempotents over $\Lambda_\mu$.

Theorem 33 Let $p_1 \in \mathbb{Idemp}_n(\Lambda_{1,\mu})$ be an idempotent matrix with entries in $\Lambda_{1,\mu}$, resp. $p_2 \in \mathbb{Idemp}_n(\Lambda_{2,\mu})$, such that

\[ j_{1*}(p_1) = u j_{2*}(p_2) u^{-1}, \]

where $u \in \mathbb{GL}_n(\Lambda'_\mu)$ is an invertible matrix.
-i) Then there exists an idempotent double matrix \( p \in \mathbb{Idemp}(\Lambda_\mu) \) such that
\[
i_{1*}(p) = p_1 \oplus 0_n \quad \text{and} \quad i_{2*}(p) = \tilde{p}
\] (51)
where the idempotent \( \tilde{p}_2 \in \mathbb{M}_{2n}(\Lambda_{2,\mu}) \) is conjugated to \( p_2 \oplus 0_n \) through an invertible matrix \( \tilde{U}(u) \in \mathbb{GL}_{2n}(\Lambda_{2,\mu}) \), that is
\[
\tilde{p}_2 = \tilde{U}(u)(p_2 \oplus 0_n)\tilde{U}(u)^{-1}
\] (52)
\[
j_{2*,}(\tilde{p}_2) = (u j_{1*}(p_1) u^{-1}) \oplus 0_n = j_{1*}(p_1 \oplus 0_n)
\] (53)

-ii) The corresponding double matrix idempotent is denoted \( p = (p_1, p_2, \tilde{U}(u)). \)

**Definition 34** Let \( \tilde{U}(u) \) be the lifting (27) of \( u \otimes u^{-1} \) in \( \Lambda_{2,\mu} \).

Denote by \( p = (p_1, p_2, \tilde{U}(u)) \) the idempotent over \( \Lambda_\mu \) produced by Theorem 33.

Condition (50) says that \([j_{1*}p_1] = [j_{2*}p_2] \in T_0(\Lambda'_\mu). \) Part -i) says that the pair \(([p_1 \oplus 1_n], [\tilde{p}_2]) \in T_0(\Lambda_1) \oplus T_0(\Lambda_2) \) belongs to the image of \((i_{1*}, i_{2*}). \)

**Proof.** To prove this theorem we will operate onto objects related to \( \Lambda_{2,\mu}. \)

**Lemma 35** Let \( p_1 = (a_{ij}) \in \mathbb{Idemp}_n(\Lambda_{1,\mu}) \) and \( p_2 = (b_{ij}) \in \mathbb{Idemp}_n(\Lambda_{2,\mu}) \) be idempotents.

Suppose the idempotents \( j_{1*}(p_1), \ j_{2*}(p_2) \) are conjugated through an inner automorphism defined by \( u \in \mathbb{GL}_n(\Lambda'_\mu), \) i.e.
\[
j_{1*}(p_1) = u \ j_{2*}(p_2) \ u^{-1}.
\] (54)
Assume, additionally, that the invertible element \( u \) lifts to an invertible element \( \tilde{u} \in \mathbb{GL}_n(\Lambda_{2,\mu}) \) (i.e. \( j_{2*}\tilde{u} = u \)).

Then \( p = (p_1, p'_2, u) \in \mathbb{Idemp}_n(\Lambda_\mu) \) is an idempotent given by the double matrix
\[
p = ((a_{ij}, c_{ij})),
\] (55)
where
\[
(a_{ij}) = p_1 \in \mathbb{Idemp}_n(\Lambda_{1,\mu}) \quad \text{and} \quad p'_2 = (c_{ij}) := \tilde{u} \ p_2 \ \tilde{u}^{-1} \in \mathbb{Idemp}_n(\Lambda_{2,\mu}).
\] (56)

remark that in this lemma the size of the double matrix \( p \) does not change.
Proof of Lemma 35. We use Remark 19-ii). It is clear that the matrix $p$ given by (55) is an idempotent. In fact, to evaluate $p^2$ amounts to compute separately the square of the first and second component matrices of the matrix $p$, i.e. the squares of $(a_{ij})$ and $(c_{ij})$. These are

$$(a_{ij})^2 = (a_{ij}) \quad \text{and} \quad (c_{ij})^2 = (\bar{u} p_2 \bar{u}^{-1})^2 = \bar{u} p_2^2 \bar{u}^{-1} = \bar{u} p_2 \bar{u}^{-1} = (c_{ij}).$$

(57)

It remains to verify that $p \in \mathbb{M}_n(\Lambda_{\mu})$, i.e. $j_{1*}(a_{ij}) = j_{2*}(c_{ij})$. This follows from (40) combined with (54)

$$j_{1*}(p_1) = u j_{2*}(p_2) u^{-1} = j_{2*}(\bar{u} p_2 \bar{u}^{-1}) = j_{2*}(\tilde{p}_2).$$

(58)

This ends the proof of Lemma 35.

Lemma 36 Let $p_1 = (a_{ij}) \in \mathbb{Idemp}_n(\Lambda_{1,\mu})$ and $p_2 = (b_{ij}) \in \mathbb{Idemp}_n(\Lambda_{2,\mu})$ be idempotents.

Suppose the idempotents $j_{1*}(p_1)$, $j_{2*}(p_2)$ are conjugated through an inner automorphism defined by $u \in \mathbb{GL}_n(\Lambda'_{\mu})$, i.e.

$$j_{1*}(p_1) = u j_{2*}(p_2) u^{-1}. $$

(59)

Then

- i) $j_{1*}(p_1 \oplus 0_n)$ and $j_{2*}(p_2 \oplus 0_n)$ are conjugated by $U := u \oplus u^{-1} \in \mathbb{GL}_2n(\Lambda'_\mu)$, i.e.

$$j_{1*}(p_1) \oplus 0_n = j_{1*}(p_1 \oplus 0_n) = U j_{2*}(p_2 \oplus 0_n) U^{-1} = (u j_{2*}(p_2) u^{-1}) \oplus 0_n.$$

(60)

- ii) Supposing that $j_2$ is surjective, the invertible matrix $U$ lifts to an invertible matrix $\tilde{U} \in \mathbb{M}_2n(\Lambda_{2,\mu})$. Let

$$\tilde{p}_2 := \tilde{U} (p_2 \oplus 0_n) \tilde{U}^{-1}. $$

(61)

Then

$$j_{1*}(p_1 \oplus 0_n) = (u j_{2*}(p_2) u^{-1}) \oplus 0_n = j_{2*}(\tilde{p}_2);$$

(62)

i.e. the matrices $p_1 \oplus 0_n$, $\tilde{p}_2$ form a double matrix idempotent in $\mathbb{Idemp}_{2n}(\Lambda'_{\mu})$, denoted $p := (p_1, p_2, u) \in \mathbb{Idemp}_{2n}(\Lambda_{\mu})$ and

$$(i_1\ast, i_2\ast) p = (p_1, \tilde{p}_2)$$

(63)

- iii) The pair of idempotents $p_1 \in \mathbb{Idemp}_n(\Lambda_{1,\mu})$, $p_2 \in \mathbb{Idemp}_n(\Lambda_{2,\mu})$ is stably equivalent to the pair of idempotents $p_1 \oplus 0_n$, $p_2 \oplus 0_n$ and $p_2 \oplus 0_n \sim_1 \tilde{p}_2$. In other words

$$([p_1], [p_2]) = ([p_1], [\tilde{p}_2]) \in T_0(\Lambda_{1,\mu}) \oplus T_0(\Lambda_{2,\mu}).$$

(64)
Note that in this lemma, in comparison with the preceding Lemma 35, the size of the desired idempotent doubles; otherwise, the important modifications still occur onto matrices associated with \( \Lambda_{2,\mu} \).

**Proof of Lemma 36.** Part -i) is clear.

The proof of -ii) uses \( \mathcal{O}_{2n}(u) \), where \( u \in \mathbb{G}L_n(\mathcal{A}_\mu) \). Recall that \( \mathcal{O}_{2n}(u) \) may be written as a product of elementary matrices and a scalar matrix \([27]\)

\[
U := \mathcal{O}_{2n}(u) = \begin{pmatrix} u & 0 \\ 0 & u^{-1} \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -u^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \in M_{2n}(\Lambda_{2,\mu}).
\]

(65)

Having in mind the localisation of the ring \( \mathcal{A} \), it is important to note that the entries of the formula (65) depend on \( u \) and \( u^{-1} \) only.

The proof will be complete after we will have shown that the invertible matrix \( U \) has an invertible lifting \( \tilde{U} \in \Lambda_{2,\mu} \). This follows from the properties of elementary matrices (valid for matrices and block matrices) to be discussed next.

**Definition 37** Elementary Matrices.

A matrix \( E_{ij}(a) \in GL_n(\mathcal{A}_\mu) \) having all entries equal to zero, except for the diagonal entries equal to 1 and just one \((i, j)\)-entry \( a \in \mathcal{A}_\mu \), \( 0 \leq i \neq j \leq n \) is called elementary matrix with entry in \( \mathcal{A}_\mu \).

The space of elementary matrices with entries in \( \mathcal{A}_\mu \) is by definition

\[
E_n(\mathcal{A}_\mu) := \{ E_{ij}(a) \mid 1 \leq i \neq j \leq n, a \in \mathcal{A}_\mu \}. \tag{66}
\]

Let \( E_n(\mathcal{A}) \) the sub-group generated by all elementary matrices and

\[
E(\mathcal{A}) := \text{inj lim}_{n \in \mathbb{N}} E_n(\mathcal{A}). \tag{67}
\]

**Lemma 38** Begin The elementary matrices satisfy

\[
E_{i,j}(a).E_{i,j}(b) = E_{i,j}(a + b) \tag{68}
\]

\[
E_{i,j}(a)^{-1} = E_{i,j}(-a), \tag{69}
\]

therefore \( E_{i,j}(a) \in \mathbb{G}L(\mathcal{A}_\mu) \). Any elementary matrix is a commutator

\[
E_{ij}(A) = [E_{ik}(A), E_{kj}(1)], \quad \text{for any } i, j, k \text{ distinct indices}. \tag{70}
\]
We come back to the proof of Lemma 36 -ii). As $j_2$ is surjective, each of the entries of factors of the RHS of (65) has a lifting in $\mathbf{M}_{2n} (\Lambda_2, \mu)$; each of the elementary matrix factor lifts as an invertible elementary matrix. The last factor lifts as it is. Therefore, $U$ has an invertible lifting $\tilde{U} \in \mathbb{G}L_{2n} (\Lambda_2, \mu)$. Lemma 35 completes the proof of Lemma 36 -ii).

Lemma 36 -iii) follows from the definition of $T_{0}^{loc}(\Lambda)$. This completes the proof of Lemma 21.2.

Theorem 33 follows from Lemma 34 combined with Lemma 35.

Theorem 33 refers to the construction and description of idempotents over $\Lambda_\mu$. We need to extend Theorem 33 to elements of $T_{0}^{loc}(\mathcal{A})$, i.e. to formal differences of local idempotents.

**Lemma 39** (compare \cite{5} Lemma 1.1)

Let $p_1, p_2, q_1, q_2 \in \text{Idemp}_n(\mathcal{A}_\mu)$ be idempotents and let $[ \ ]$ denote their $T_{0}(\mathcal{A}_\mu)$ class. Suppose

$$[p_1] - [p_2] = [q_1] - [q_2] \in T_{0}(\mathcal{A}_\mu).$$  \hspace{1cm} (71)

Then $p_1 + q_2$ and $p_2 + q_1$ are locally, stably isomorphic, $p_1 + q_2 \sim_{ls} p_2 + q_1$.

**Proof.** The stabilisation and Grothendieck completion imply that there exists an idempotent $s \in \text{Idemp}_m(\mathcal{A}_\mu)$ such that the idempotents

$$p_1 + q_2 + s, \quad p_2 + q_1 + s$$

are locally, stably isomorphic. We assume that the idempotent $s$ is already sufficiently stabilised. This means there exists an invertible matrix $u \in \mathbb{G}L_{2n+m}(\mathcal{A}_\mu)$ such that

$$p_1 + q_2 + s = u \left( p_2 + q_1 + s \right) u^{-1}.$$

We add to both sides of this equality the idempotent $1_m - s$ and we extend $u$ to be the identity on the last summand. We get

$$p_1 + q_2 + s + (1 - s), = \ u \left( p_2 + q_1 + s + (1 - s) \right) u^{-1}.$$

From this we get further

$$p_1 + q_2 + 1_{2m}, = \ u \left( p_2 + q_1 + 1_{2m} \right) u^{-1},$$
that is, the idempotents $p_1 + q_2$, $p_2 + q_1$ are locally, stably isomorphic

$$p_1 + q_2 \sim_{sl} p_2 + q_1.$$ 

Lemma 40 Let $p, q \in I\text{demp}_n(A_\mu)$ be idempotents. Suppose

$$[p] - [1_n] = [q] - [1_n] \in T_0(A_\mu). \quad (72)$$

Then

- i) $p$ and $q$ are locally, stably isomorphic, $p \sim_{ls} q$.
- ii) there exists an $N \in \mathbb{N}$ and an $u \in GL_{n+N}(A_\mu)$ such that

$$p + 1_N = u q u^{-1} + 1_N = u (q + 1_N) u^{-1}. \quad (73)$$

Proof. -i) Lemma 39 says that the idempotents $p + 1_n$, $q + 1_n$ are locally, stably isomorphic. This means that the idempotents $p$ and $q$ are locally, stably isomorphic. Part -ii) tells precisely this. ■

Theorem 41 Let $p_{ij}$ be idempotents

$$[p_1] = [p_{11}] - [p_{12}] \in T_0(\Lambda_{1,\mu})$$

$$[p_2] = [p_{21}] - [p_{22}] \in T_0(\Lambda_{2,\mu})$$

with the property that

$$j_1 \ast [p_1] = j_2 \ast [p_2] \in T_0(\Lambda_{\mu}').$$

Then there exists $[p] = [p_{01}] - [p_{02}] \in T_0(\Lambda_{\mu})$ with the property that

$$i_1 \ast [p] = [p_1] \text{ and } i_2 \ast [p] = [p_2].$$

Proof. We may describe the two $T$-theory classes differently

$$[p_1] = [p_{11}] - [p_{12}] = [p_{11} + (1 - p_{12})] - [p_{12} + (1 - p_{12})] = [p'_{12}] - [1_n] \in T_0(\Lambda_{1,\mu})$$

and

$$[p_2] = [p_{21}] - [p_{22}] = [p_{21} + (1 - p_{22})] - [p_{22} + (1 - p_{22})] = [p'_{22}] - [1_n] \in T_0(\Lambda_{2,\mu}).$$

Then

$$j_1 \ast [p_1] = j_1 \ast ([p'_{12}] - [1_n]) = (j_1 \ast [p'_{12}]) - [1_n]$$
and 
\[ j_2^*[p_2] = j_2^*([p'_{22}] - [1_n]) = (j_2^*[p'_{22}]) - [1_n]. \]

The hypothesis says that
\[ (j_1^*[p'_{12}]) - [1_n] = (j_2^*[p'_{22}]) - [1_n]. \]

Lemma 39 says that the idempotents \( j_1^*[p'_{12}], j_2^*[p'_{22}] \) are locally, stably isomorphic. Now we are in the position to use Theorem 21.1. Let \( u \in GL(\Lambda_2(A_\mu)) \); consider the conjugation
\[ j_1^*(p'_{12}) = u j_2^*(p'_{22}) u^{-1}. \quad (74) \]

Theorem 21.1 provides the idempotent
\[ p = (j_1^*(p'_{12}), (j_2^*(p'_{2}), u). \]

The desired idempotents are
\[ p_{10} = p = (j_1^*(p'_{12}), (j_2^*(p'_{2}), u) \in Idemp(\Lambda_{\mu}) \]
\[ p_{20} = 1_N \in Idemp(\Lambda_{\mu}). \]

\[ \blacksquare \]

18.2 Constructing invertibles over \( \Lambda_{\mu} \).

**Theorem 42** Let \( s_1 \in GL_n(\Lambda_{1,\mu}), s_2 \in GL_n(\Lambda_{2,\mu}), \) be invertible matrices with entries in \( \Lambda_{1,\mu}, \) resp. \( \Lambda_{2,\mu}, \) such that
\[ j_1^*(u_1) = u j_2^*(u_2) u^{-1}, \quad (75) \]

where \( u \in GL_n(\Lambda'_{\mu}). \)

-\( i) \) Then there exists an invertible matrix \( s \in GL_{2n}(\Lambda_{\mu}) \) such that
\[ i_1^*(s) = s_1 \oplus 1_n \quad \text{and} \quad i_2^*(s) = \tilde{s}_2 \quad (76) \]

where the invertible matrix \( \tilde{s}_2 \in GL_{2n}(\Lambda_{2,\mu}) \) is conjugated to \( s_2 \oplus 1_n \) through an inner auto-morphism defined by the invertible matrix \( \tilde{U} \in GL_{2n}(\Lambda_{2,\mu}), \) that is
\[ \tilde{s}_2 = \tilde{U}(s_2 \oplus 1_n)\tilde{U}^{-1} \quad (77) \]
\[ j_2^* \tilde{s}_2 = (u j_2^*(s_2) u^{-1}) \oplus 1_n = j_1^*(s_1 \oplus 1_n) \quad (78) \]

-\( ii) \) The corresponding invertible double matrix \( s \) is denoted \( s = (s_1, s_2, \tilde{U}). \)
The proof of Theorem 42 goes along the same lines as the proof of Theorem 33. The proof of Theorem 33 is based on the following facts:

- a) operations with double matrices respect Remark 19 - ii), lifting of the invertible element \( U = O_{2n}(u) := u ⊕ u^{-1} ∈ GL_{2n}(\Lambda_{2,µ}) \) by means of the factorisation of \( U \) by elementary matrices, see (27), and
- c) the fact that any inner automorphism keeps invariant any zero vector sub-space.

To prove Theorem 42 we use the same arguments -a), -b), -c) with the following changes:

- idempotents are replaced by invertible elements and for -c) we use the fact that the inner auto-morphisms transform the mapping \( 1_n \) into itself. This ends the proof of the theorem.

The next theorem is the analogue of Theorem 41 in the \( T_1(A_µ) \) case.

**Theorem 43**

Suppose \( j_1 \) and \( j_2 \) are epi-morphisms.

Let \( [s_1] ∈ T_1^{loc}(Λ_{1,µ}) \) and \( [s_2] ∈ T_1^{loc}(Λ_{2,µ}) \) be such that

\[
  j_{1*}[s_1] = j_{2*}[s_2] ∈ T_1^{loc}(Λ'_{µ}).
\]

(79)

Then there exists \([s] ∈ T_1^{loc}(Λ_µ)\) such that

\[
  i_{1*}[s] = [s_1] ∈ T_1^{loc}(Λ_{1,µ}) \quad \text{and} \quad i_{2*}[s] = [s_2] ∈ T_1^{loc}(Λ_{2,µ}).
\]

(80)

**Proof.**

The definition of \( T_1^{loc}(Λ'_{µ}) \) involves an ambiguity belonging to the sub-module \( O(Λ'_{µ}) \). We assume the elements \( s_1 \) and \( s_1 \) are sufficiently stabilised. Equality (79) tells there exist two elements \( ξ_1, ξ_2 ∈ O_{2n}(Λ'_{µ}) \) such that the invertible matrices \( j_{1*}(s_1) + ξ_1 \), \( j_{2*}(s_2) + ξ_2 \) are locally conjugated by means of a matrix \( u ∈ GL_{2n}(Λ') \)

\[
  j_{1*}(s_1) + ξ_1 = u (j_{2*}(s_2) + ξ_2) u^{-1}.
\]

(81)

The problem of finding the element \( s \) will be split in two separate problems

1. find \( ξ_1 ∈ O(Λ_{1,µ}) \), resp. \( ξ_2 ∈ O(Λ_{2,µ}) \), lifts of the elements \( ξ_1 \), resp. \( ξ_2 \),

2. apply Theorem 43 with \( s_1 \), resp. \( s_2 \), replaced by \( s_1 + ξ_1 \), resp. \( s_2 + ξ_2 \), and the invertible element \( u \).

1. The first lift is obtained in two steps:
   - i) we find invertible lifts \( \tilde{ξ}_1 ∈ Λ_{1,µ} \), resp. \( \tilde{ξ}_2 ∈ Λ_{2,µ} \), of the elements \( ξ_1 \), resp. \( ξ_2 \), and we verify that -ii) such lifts belong to \( O_{2n}(Λ_{i,µ}) \), \( i = 1, 2 \).
As the lifts of the elements $\xi_i$ belong to $O_{2n}(\Lambda_{i,\mu})$, the lifted elements will not change the corresponding $T$-completion classes.

18.3 Lifting of $O(\Lambda'_{\mu})$

We illustrate the procedure on the element $\xi_1$; for $\xi_2$ we use the same procedure.

The element $\xi_1$ has the form

$$\xi_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_{-1} \end{pmatrix} = \alpha_1 \oplus \alpha_{-1}^{-1} \in O_{2n}(\Lambda'_{\mu}). \quad (82)$$

We use the factorisation (27) to decompose of $\xi_1$

$$\xi_1 = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_{-1} \end{pmatrix} = \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\alpha_{-1}^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & \alpha_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \quad (83)$$

The homo-morphism $j_1$ being an epi-morphism, there exists $\tilde{\alpha}_1$, resp. $\tilde{\beta}_1 \in \Lambda_{1,\mu}$ such that $j_1(\tilde{\alpha}_1) = \alpha_1$, resp. $j_1(\tilde{\beta}_1) = \alpha_{-1}^{-1}$. The lifted element is

$$\tilde{\xi}_1 = \begin{pmatrix} 1 & \tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -\tilde{\beta}_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & \tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad (84)$$

$$j_1(\tilde{\xi}_1) = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha_{-1}^{-1} \end{pmatrix} = \xi_1.$$ 

We use formula (27) and property (69) of elementary matrices to find the inverse of $\xi_1$

$$\tilde{\xi}_1^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\tilde{\alpha}_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}. \quad (85)$$

We have

$$j_1(\tilde{\xi}_1^{-1}) = \begin{pmatrix} \alpha_1 \\ 0 \end{pmatrix} \begin{pmatrix} 0 & \alpha_{-1}^{-1} \end{pmatrix}^{-1} \sim_{el} \xi_1^{-1}$$

because $j_1$ is a unital ring homo-morphism.

The lifted element is

$$\tilde{\xi}_1 = \tilde{\xi}_1 \oplus \tilde{\xi}_1^{-1} \in M_{4n}(\Lambda_{1,\mu}).$$
18.4 Back to the proof of Theorem 43

The elements $s_1 + \tilde{\xi}_1 \in \mathbb{GL}_{4n}(A_{1,\mu})$, $s_2 + \tilde{\xi}_2 \in \mathbb{GL}_{4n}(A_{2,\mu})$ and the invertible element $u$ satisfy the relation

$$j_{1,*}(s_1 + \tilde{\xi}_1) = u (s_2 + \tilde{\xi}_2) u^{-1}. \quad (86)$$

Theorem 42 follows from Theorem 43. ■

19 $K_1(A)$ vs. $T_1(A)$

The following identities are well known and used as building blocks of $K$-theory, see [2], [5], [10], [12], [18], [21], [26].

We summarise some basic facts from the classical algebraic $K_1$-theory and compare them with $T_1$.

Theorem 44 -i) Whitehead group $K_1(A)$ is

$$K_1(A) := \text{inj lim}_{n \in \mathbb{N}} \mathbb{GL}_n(A)/[\mathbb{GL}_n(A), \mathbb{GL}_n(A)] \quad (87)$$

The group structure in $K_1(A)$ is given by matrix multiplication and direct sum addition

$$[A] + [B] := [A.B]. \quad (88)$$

$T_1(A)$ is the set of Jordan canonical forms of matrices over $A$ modulo $O(A)$. The sum in $T_1(A)$ is the direct sum.

-ii) Any commutator is stably isomorphic to a product of elements of the form $O_{2n}(A)$. More specifically, for any $A, B \in \mathbb{GL}(A)$

$$\begin{pmatrix} ABA^{-1}B^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & B^{-1} \end{pmatrix} \begin{pmatrix} (BA)^{-1} & 0 \\ 0 & BA \end{pmatrix}. \quad (89)$$

-iii) If $A \in \mathbb{GL}_n(A)$, then (see [27])

$$O_{2n}(A) := \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} = \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -A^{-1} & 1 \end{pmatrix} \begin{pmatrix} 1 & A \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}. \quad (90)$$

-iv) Any elementary matrix is a commutator

$$E_{ij}(A) = [E_{ik}(A), E_{kj}(1)], \text{ for any } i, j, k \text{ distinct indices.} \quad (91)$$
-iv) For any \( A, B \in \mathbb{GL}_n(A_\mu) \), \( A + B \) is stably equivalent to \( AB \) and \( BA \) modulo (multiplicatively) elements of the form \( O_{2n}(A) \)

\[
\begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} AB & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} = \begin{pmatrix} B^{-1} & 0 \\ 0 & B \end{pmatrix} \begin{pmatrix} BA & 0 \\ 0 & 1 \end{pmatrix}.
\] (92)

-v) For any \( A, B \in \mathbb{GL}_n(A_\mu) \) one has the identity

\[
\begin{pmatrix} ABA^{-1} & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A^{-1} & 0 \\ 0 & A \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & 0 \\ 0 & A^{-1} \end{pmatrix}^{-1}.
\] (93)

Theorem 45 (see [2], [6], [1], [5], [10], [18])

-ii) \( [\mathbb{GL}(A), \mathbb{GL}_n(A)] = \mathbb{E}_n(A) \)

-iii.1) \( A + B = [AB] = [BA] \in K_1(A) \)

-iii.2) \( A + B = [B] + [A] \in T_{1\text{loc}}^1(A) \). Therefore \( T_{1\text{loc}}^1(A) \) is an Abelian group.

-iv) \( [O_{2n}(A)] = [1_n] = 0 \) in \( K_1(A) \) and \( T_{1\text{loc}}^1(A) \)

-v) \(-[A] = [A^{-1}] \) in \( K_1(A) \) and \( T_1(A)^{\text{loc}} \).

-vi) \( [ABA^{-1}] = [B] \in K_1(A) \).

Proof. -i) Relation (70) says that any commutator is a product of matrices of type \( O_n(A) \). Formula (27) says that any matrix of type \( O_n(A) \) is a product of elementary matrices and a scalar matrix \( 27 \). This proves -i).

-ii) Follows from the definition of \( K_1(A) \).

-iii.1) and -iii.2). See definitions.

-iv) For any invertible element \( x_0 \) one has

\[
(x_0 + (\xi + \xi^{-1})) \sim_\sigma (x_0 + (\xi + \xi^{-1})) + (\xi + \xi^{-1})
\] (94)

along with the fact that \( T_1 \) is a group.

-v) Relation for \( K_1 \) follows from \( AA^{-1} = 1 \), which is the zero element in \( K_1 \).

For \( T_1 \) the relation follows from -iv).

-vi) \( (ABA^{-1}).B^{-1} \) is a commutator; it is the zero element in \( K_1 \).

On the other side \( [B^{-1}] = [B] \in K_1 \). □

The next remark explains the specific parts of the construction of \( T_{1\text{loc}}^1 \) which make our construction different from the classical one.
Remark 46 -i) In our construction the factorisation of the elements of $\GL(A)$ through the commutator sub-group (or the subgroup generated by elementary matrices) is avoided because the number of products needed to generate these sub-groups might be unbounded. Having in mind the algebra of integral operators, or pseudo-differential operators, one realises that any product increases the size of the support; an un-bounded number of products does not allow a support control.

For this reason, in the whole chapter we don’t use more than three products of elements in the algebra. Products are replaced by sums and direct sums. The increase of the size of matrices replaces in our construction the need to perform multiple products. In our definition of $T_1^{\text{loc}}(A)$, where products could not be avoided, the corresponding increase in the size of the supports is absorbed by the projective limit.

-ii) Our construction uses the elements $O(A)$ additively and not multiplicatively. In the classical construction of $K_1(A)$, the elements of $O(A)$ are used multiplicatively. Unfortunately, to generate the commutator sub-group it is necessary to perform an unbounded number of products; in our construction an un-bounded number of products is not allowed.

-iii) The construction of $T_1^{\text{loc}}(A)$ uses the factorisation of $\GL(A)$ through the smaller (than the commutator) sub-group of inner automorphisms. The class of an invertible element $u$ modulo inner automorphisms, which is nothing but the abstract Jordan form $J(u)$ of $u$, contains more information than the class of the invertible element modulo the commutator sub-group. Both, in the classical $K_1(A)$ and $T_1(A)$ the elements of $O_n(A)$ represent the zero element.

If $A$ were the algebra of complex matrices and $u \in M_n(\mathbb{C})$, then $J(u)$ could be identified with the Jordan canonical form of the matrix $u$. The Jordan canonical form of the matrix $u \oplus u^{-1}$ is precisely $J(u) \cup J(u^{-1})$ modulo permutations of the Jordan blocks. It is clear that $u \oplus u^{-1}$ could never be conjugated to the the identity element unless $u = 1_n$. In general, $u \oplus u^{-1}$ could not be conjugated to $1_n \oplus 1_n$, unless $u \sim_s 1_n$.

For the computation of the groups $T_i^{\text{loc}}(\mathbb{C})$, where $\mathbb{C}$ is the algebra of complex numbers endowed with the trivial filtration, see [30] and §22.

The elements $O(A)$ represent the zero element in $T_1$.

For the definition of $T_1^{\text{loc}}(A)$ we find it natural to consider the
quotient space of $\mathbb{GL}(A)$ modulo the equivalence relation $\sim_{st}$. This factorisation uses the additive sub-group $O(A)$. This factorisation decrees that the Jordan canonical forms of the elements $u$ and $u^{-1}$ are opposite one to each other in $T_{1}^{loc}$. The additive group generated by elements $O(A)$ is contained in the commutator sub-group; this property insures the fact that there exists a natural epi-morphism from $T_{1}^{loc}(A)$ to $K_{1}(A)$.

-iv) The factorisation through the sub-group $O(A)$ does not appear to kill much information. A partial argument in support of this is the fact that for any $u, v \in \mathbb{GL}_{n}(A)$, $(u \oplus u^{-1}) \sim_{l} (v \oplus v^{-1})$ if and only if $(u \oplus 1_{n}) \sim_{l} (v \oplus 1_{n})$.

-v) Additionally, the projective limit (Alexander-Spanier type construction, made possible by the filtration $A_{\mu}$ of the algebra), makes the algebraic $T_{i}^{loc}$-theory richer than the classical $K_{i}$-theory, $i = 0, 1$.

Theorem 47 -i) There is a canonical epi-morphism

$$\Pi : T_{1}^{loc}(A) \longrightarrow K_{1}(A)$$

$$Ker \Pi = [\mathbb{GL}(A), \mathbb{GL}(A)] / Inner(A).$$

20 Connecting homo-morphism $\partial : T_{1}^{loc}(A) \longrightarrow T_{0}^{loc}(A) \otimes \mathbb{Z}[\frac{1}{2}]$

In this section we assume that the diagram satisfies Hypotheses 1, 2, 3. In this section we define the connecting homomorphism $\partial : T_{1}^{loc} \longrightarrow T_{0}^{loc} \otimes \mathbb{Z}[\frac{1}{2}]$.

The reader will notice that the construction of $\partial$ involves from the very beginning idempotents.

Definition 48 Let $[u] \in T_{1}(A_{\mu})$. Recall that the elements of $T_{1}(A_{\mu})$ are equivalence classes of invertible matrices modulo $\sim_{O(A_{\mu})}$; we may assume that $u \in \mathbb{GL}_{n}(A_{\mu})$.

Define the connecting homomorphism $\partial : T_{1}(A_{\mu}) \longrightarrow T_{0}(A_{\mu}) \otimes \mathbb{Z}[\frac{1}{2}]$

$$\partial[u] = [p(1_{n}, 1_{n}, \tilde{U}(u)) - [\Lambda^{n}(A_{\mu})],$$

where $\tilde{U}(u)$ is obtained through the decomposition of $u \oplus u^{-1}$ as in (84), and lifted in $\Lambda_{1,\mu}$, as in (84).
Proposition 49 Suppose the homo-morphisms \( j_1, j_2 \) are epi-morphisms.

Then the connecting homomorphism is well defined.

We have to show that \( \partial \) is compatible with the equivalence relation \( \sim_{\mathcal{O}_n(A)} \). For, suppose that \( u_1 + \xi_1 = u_0 + \xi_0 \), where \( u_0, u_1 \in \mathcal{G}\mathcal{L}_n(A_{\mu}) \) and \( \xi_0, \xi_1 \in \mathcal{O}_n(A_{\mu}) \). We have to prove that \( \partial \mathcal{O}(u_1) = \partial \mathcal{O}(u_2) = \text{trivial idempotent} \).

We proved in §18.3 that any invertible element \( \xi \) belonging to \( \mathcal{O}_n(\Lambda'_{\mu}) \) lifts to an invertible elements \( \tilde{\xi}_1, \tilde{\xi}_2 \), belonging to \( \Lambda_{\mu,1}, \Lambda_{\mu,2} \). In our specific case, the element \( 1_n(\Lambda_{1,\mu}), \Lambda_{\mu,2} \), is conjugated through the lifted elements \( \tilde{\xi}_i, i = 1, 2 \).

The conjugation inside \( \Lambda_{\mu,i} \) preserves the \( T_{1}(\Lambda_{\mu,i}) \) classes. Recall that the idempotents of \( \Lambda_{\mu} \) consist of pairs of idempotents \( (p_1, p_2) \in \Lambda_{\mu,1} \oplus \Lambda_{\mu,2} \) such that \( i_1 p_1 = i_2 p_2 \). The idempotents \( p_1 = \xi_1 1_n \xi_1^{-1}, \ p_2 = \xi_2 1_n \xi_2^{-1} \) satisfy this condition. The proposition is proven.

21 Six terms exact sequence.

Theorem 50 The following sequences are exact

\[ T_{0}^{\text{loc}}(\Lambda) \xrightarrow{(i_1,i_2)} T_{0}^{\text{loc}}(\Lambda_1) \oplus T_{0}^{\text{loc}}(\Lambda_2) \xrightarrow{j_{1*}-j_{2*}} T_{0}^{\text{loc}}(\Lambda') \] (98)

\[ T_{1}^{\text{loc}}(\Lambda) \xrightarrow{(i_1,i_2)} T_{1}^{\text{loc}}(\Lambda_1) \oplus T_{1}^{\text{loc}}(\Lambda_2) \xrightarrow{j_{1*}-j_{2*}} T_{1}^{\text{loc}}(\Lambda') \] (99)

\[ T_{0}^{\text{loc}}(\Lambda) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{(i_1,i_2)} (T_{0}^{\text{loc}}(\Lambda_1) \otimes \mathbb{Z}[\frac{1}{2}]) \oplus (T_{0}^{\text{loc}}(\Lambda_2) \otimes \mathbb{Z}[\frac{1}{2}]). \] (100)

Therefore, the following six terms sequence is exact

\[ T_{1}^{\text{loc}}(\Lambda) \xrightarrow{(i_1,i_2)} T_{1}^{\text{loc}}(\Lambda_1) \oplus T_{1}^{\text{loc}}(\Lambda_2) \xrightarrow{j_{1*}-j_{2*}} T_{1}^{\text{loc}}(\Lambda') \xrightarrow{\partial} \] (102)

\[ T_{0}^{\text{loc}}(\Lambda) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{(i_1,i_2)} T_{0}^{\text{loc}}(\Lambda_1) \otimes \mathbb{Z}[\frac{1}{2}] \oplus T_{0}^{\text{loc}}(\Lambda_2) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{j_{1*}-j_{2*}} T_{0}^{\text{loc}}(\Lambda') \otimes \mathbb{Z}[\frac{1}{2}] \] (103)
Proof.

- i)
  i.1.) \( \text{Im}(i_{1*}, i_{2*}) \subset \text{Ker}(j_{1*} - j_{2*}) \).
  Easy to verify.
  
  i.2.) \( \text{Ker}(j_{1*} - j_{1*}) \subset \text{Im}(i_{1*}, i_{2*}) \)
  We are going to verify it.

Lemma 51 Let \( p \in \text{Idemp}_n(A_{\mu}) \).

Then
\[
p \oplus (1 - p) \sim_{ls} 1_n. \tag{104}
\]

Proof. The proof is based on the following identity
\[
\begin{pmatrix}
  p & 0 \\
  0 & 1 - p
\end{pmatrix}
= \begin{pmatrix}
  p & 1 - p \\
  1 - p & p
\end{pmatrix}
\begin{pmatrix}
  1_n & 0 \\
  0 & 1_n
\end{pmatrix}
\begin{pmatrix}
  p & 1 - p \\
  1 - p & p
\end{pmatrix}
\tag{105}
\]
along with the observation that
\[
\begin{pmatrix}
  p & 1 - p \\
  1 - p & p
\end{pmatrix}^2
= \begin{pmatrix}
  p^2 + (1 - p)^2 & p(1 - p) + (1 - p)p \\
  (1 - p)p + p(1 - p) & p^2 + (1 - p)^2
\end{pmatrix}
= \begin{pmatrix}
  1_n & 0 \\
  0 & 1_n
\end{pmatrix}
\tag{106}
\]
which shows that
\[
\begin{pmatrix}
  p & 1 - p \\
  1 - p & p
\end{pmatrix}^{-1}
= \begin{pmatrix}
  p & 1 - p \\
  1 - p & p
\end{pmatrix}, \tag{107}
\]
which entitles us to say that the RHS of (107) is an inner automorphism. ■

For any idempotent \( p \) of size \( n \) we convine to write \( \bar{p} := 1_n - p \).
Let \( ([p_1] - [p_2], [q_1] - [q_2]) \in T^\text{loc}_0(\Lambda_{1,\mu}) \oplus T^\text{loc}_0(\Lambda_{2,\mu}) \) be such that
\[
0 = j_{1*}([p_1] - [p_2]) - j_{2*}([q_1] - [q_2]),
\]
where \( p_1, p_2, q_1, q_2 \) are idempotents. The pair of \( T \)-theory classes may be re-written
\[
0 = j_{1*}([p_1 + \bar{p}_2] - [p_2 + \bar{p}_2]) - j_{2*}([q_1 + \bar{q}_2] - [q_2 + \bar{q}_2])
\]
or
\[
0 = j_{1*}([p_1 + \bar{p}_2] - [1_s]) - j_{2*}([q_1 + \bar{q}_2] - [1_s]).
\]
By adding all sides a trivial idempotent of sufficiently large size, we may assume that \( r = s \) is large. This relation may be re-written

\[
0 = j_{1,*}([p_1 + \bar{p}_2] - j_{2,*}([q_1 + \bar{q}_2]).
\]

This means that there exists an idempotent \( \xi \in \text{Idemp}_N(\Lambda'_\mu) \) such that the idempotents

\[
(j_{1,*}([p_1 + \bar{p}_2] + \xi)) - (j_{2,*}([q_1 + \bar{q}_2] + \xi)
\]

are isomorphic. We add further the idempotent \( \bar{\xi} := 1_N - \xi \) to get isomorphic idempotents

\[
(j_{1,*}([p_1 + \bar{p}_2] + \xi + \bar{\xi}), \ (j_{2,*}([q_1 + \bar{q}_2] + \xi + \bar{\xi})) \in \text{Idemp}_N(\Lambda'_\mu)
\]

( \( \bar{N} \) being the size of these idempotents) or

\[
(j_{1,*}([p_1 + \bar{p}_2] + 1_N), \ (j_{2,*}([q_1 + \bar{q}_2] + 1_N).
\]

This means there exists \( u \in \text{GL}_{\bar{N}}(\Lambda') \) which conjugates these two idempotents.

Theorem 41 says that there exists the idempotent \( p \in \text{Idemp}_{2\bar{N}}(\Lambda_\mu) \) such that

\[
i_{1,*}p = (p_1 + \bar{p}_2 + 1_N) \oplus 1_N
\]

and

\[
i_{2,*}p = U ((q_1 + \bar{q}_2 + 1_N) \oplus 1_N) U^{-1}
\]

where

\[
U \in \text{GL}_{2\bar{N}}(\Lambda'_\mu).
\]

This means the image of the \( T \)-theory class of \( p - 1_N \in \Lambda_\mu \) through the pair of homomorphisms \( (i_{1,*}, i_{2,*}) \) is

\[
(i_{1,*}, i_{2,*})[p - 1_N] = \]

\[
= ( (p_1 + \bar{p}_2 + 1_N) \oplus 1_N - 1_N, \ U ((q_1 + \bar{q}_2 + 1_N) \oplus 1_N) U^{-1} - 1_N ) = \]

\[
= ( (p_1 + \bar{p}_2 + 1_N) \oplus 1_N - 1_N, \ U ((q_1 + \bar{q}_2 + 1_N) \oplus 1_N - 1_N) U^{-1} = \]

\[
= ([p_1 - p_2], [q_1 - q_2],
\]

which completes the proof of the part i).
-ii)  
  -ii.1.) \( \text{Im}(i_1, i_2) \subset \text{Ker}(j_{1,*} - j_{2,*}) \). 
  If \( u = (u_1, u_2) \in \mathcal{G}(\Lambda_\mu) \) belongs to \( \text{Im}(i_1, i_2) \), then according to the definition of \( \Lambda_\mu \), \( u \) may be chosen so that \( j_1(u_1) = j_2(u_2) \) and hence \( [u] \in \text{Ker}(j_{1,*} - j_{2,*}) \). 
  -ii.2.) \( \text{Ker}(j_{1,*} - j_{2,*}) \subset \text{Im}(i_1, i_2) \). 
  Let \( u = (u_1, u_2) \in \text{Ker}(j_{1,*} - j_{2,*}) \). This means there exist two elements \( \xi_1, \xi_2 \in \mathcal{O}(\Lambda_\mu) \) so that \( j_1(u_1) + \xi_1 = j_2(u_2) + \xi_2 \). The elements \( j_1(u_1), j_2(u_2) \) lift to \( u_1 \in \Lambda_{1,\mu} \), resp. \( u_2 \in \Lambda_{2,\mu} \). On the other side, see \[ (18.3) \] the elements \( \xi_1 \), resp. \( \xi_2 \), lift to elements \( \tilde{\xi}_1 \in \mathcal{O}(\Lambda_{1,\mu}) \), resp. \( \tilde{\xi}_2 \in \mathcal{O}(\Lambda_{2,\mu}) \). This means 
  \[ u_1 + \tilde{\xi}_1 \in \Lambda_{1,\mu} \text{ and } u_1 + \tilde{\xi}_2 \in \Lambda_{2,\mu} \] 
  are so that 
  \[ j_1(u_1 + \tilde{\xi}_1) = j_2(u_2 + \tilde{\xi}_2). \]
  Therefore, 
  \[ [(u_1 + \tilde{\xi}_1), (u_2 + \tilde{\xi}_2)] = [(u_1, u_2)] \in T_1(\Lambda_{1,\mu}) \oplus T_1(\Lambda_{2,\mu}), \]
  which proves the statement.

-iii)  
  -iii.1.) \( \text{Im}(j_{1,*} - j_{2,*}) \subset \text{Ker}\partial \).
  It is sufficient to prove \( \partial \circ j_{1,*} = 0 \). Consider \( u \in \Lambda_{1,\mu} \). We have to compute \( \partial \circ j_1(u) \). The result is independent of the lift of \( (j_1(u)) \otimes (j_1(u))^{-1} \). We may use the lift \( u \otimes u^{-1} \). We get then
  \[ \partial \circ j_1(u) = (u \otimes u^{-1})(1_n \oplus 1_n)(u \otimes u^{-1})^{-1} = 1_n \oplus 1_n. \quad (108) \]
  -iii.2.) \( \text{Ker}\partial \subset \text{Im}(j_{1,*} - j_{2,*}) \). Let \( u \in \Lambda'_\mu \) be such that
  \[ \partial u = [(1_n, 1_n, U(u))] - [\mathcal{A}_{\mu}] = 0. \]
  We have to prove that there exist \( v_1 \in \Lambda_{1,\mu} \), \( v_2 \in \Lambda_{2,\mu} \) such that \([u] = [j_{1,*}v_1 - j_{2,*}v_2] \).
  The hypothesis says
  \[ [(1_n, 1_n, U(u))] = [\mathcal{A}_\mu]. \]
  By adding an idempotent \( q \) and its complementary to both sides of this equation, we get
  \[ (1_{m+n}, 1_{m+n}, U(u \oplus 1_m)) \sim_c \mathcal{A}_{\mu}^{m\cdot n}. \]
We may transfer the conjugation onto the second term of this equation to get
\[ \partial(u) = \Lambda_{\mu}^{m+n}, \]
which proves the statement.

-iii.3.) \( \text{Im } \partial \subset \text{Ker } (i_1, i_2)_* \).

Let \( u \in T_1(\Lambda'_{\mu}) \) and \( p = \partial u \in T_0(\Lambda_{\mu}) \). We have to prove that \((i_1, i_2)_*(p) = 0\).

We have
\[ [i_1 \circ \partial u] = [i_1 \left( 1_n \oplus 0_n, u(1_n \oplus 0_n)u^{-1}\right) - \Lambda^n(\mathcal{A}_{\mu})] = (109) \]
\[ [1_n \oplus 0_n] - [\Lambda^n(\mathcal{A}_{\mu})] = 0 \]

On the other side
\[ [i_2 \circ \partial u] = [i_2 \left( 1_n \oplus 0_n, u(1_n \oplus 0_n)u^{-1}\right) - \Lambda^n(\mathcal{A}_{\mu})] = (110) \]
\[ [u(1_n \oplus 0_n)u^{-1}] - [\Lambda^n(\mathcal{A}_{\mu})] = 0. \]

-iii.4.) \( \text{Ker}(i_1, i_2)_* \subset \text{Im}(\partial \otimes \mathbb{Z}[\frac{1}{2}]). \)

Let \( p \in T_0(\Lambda_{\mu}) \) and \((i_1, i_2)_*(p) = 0\). We have to prove that there exists \( u \in T_1(\Lambda'_{\mu}) \) such that \( p = \partial u \in T_0(\Lambda_{\mu}). \)

That is, we look for an invertible matrix \( u \in M_n(\Lambda'_{\mu}) \) such that
\[ 0 = (i_1, i_2)_*[p] = (i_1, i_2)_*[\left( 1_n \oplus 0_n, u(1_n \oplus 0_n)u^{-1}\right)] - [\Lambda^n_{\mu}] = (111) \]
\[ ([1_n \oplus 0_n], [u(1_n \oplus 0_n)u^{-1}]) - [\Lambda^n_{\mu}]. \]

We stabilise the idempotents \( 1_n \oplus 0_n, \Lambda^n_{\mu}. \) The conjugation transforms trivial idempotents in trivial idempotents. The stabilised idempotents \( 1_{n+m} \oplus 0_{n+m}, \Lambda^n_{\mu+m} \) still satisfy the condition
\[ p_{1,*}(1_{n+m} \oplus 0_{n+m}) = 0, \quad p_{2,*}(\Lambda^n_{\mu})^{n+m} = 0. \]

We choose an isomorphism \( \tilde{u} \) between the trivial idempotents \( 1_{n+m} \oplus 0_{n+m}, \Lambda^n_{\mu+m}. \) We reduce these idempotents, modulo the ideal \( J; \) we obtain idempotents in \( \Lambda'(\mathcal{A}_{\mu}). \) Let \( u \) be the restriction of \( \tilde{u} \) modulo \( J. \) The element \( u \) is the element we are looking for.

If we started with the element \( p \otimes 1/2^m \in T_0(\Lambda_{\mu}) \otimes \mathbb{Z}[\frac{1}{2}], \) the corresponding element would be \( u \otimes 1/2^m. \) ■
22 Relative $T$- groups: $T_i(\mathcal{A}_\mu, J)$.

**Definition 52** Let $\mathcal{A}$ be a localised ring and $\mathcal{J}$ be a localised bi-lateral ideal in $\mathcal{A}$. Here we make reference to the Mayer - Vietoris Diagram [11]: we take
\[ \Lambda_\mu = J_\mu, \quad \Lambda_{1,\mu} = \Lambda_{2,\mu} = \mathcal{A}_\mu \] \[ \Lambda_\mu = \{ (\lambda_1, \lambda_2) \mid \lambda_1 \in \Lambda_{1,\mu}, \lambda_2 \in \Lambda_{2,\mu}, \lambda_1 - \lambda_2 \in J_\mu \}. \]
and
\[ \Lambda'_\mu = \mathcal{A}_\mu / J_\mu \]
The above structure will be called $(\mathcal{A}, J)$ localised ideal. Define, for $i = 1, 2$
\[ j_i : \Lambda_\mu \longrightarrow \mathcal{A}/J, \quad j_i (\lambda_1, \lambda_2) := \lambda_i \mod J_\mu. \]

**Definition 53**
\[ T_0^{loc}(\mathcal{A}, J) := \text{Ker} ( T_0^{loc}(\Lambda_\mu) \xrightarrow{j_2^*} T_0^{loc}(\mathcal{A}_\mu/J_\mu) ) \]
\[ T_1^{loc}(\mathcal{A}, J) := \text{Ker} ( T_1^{loc}(\Lambda_\mu) \xrightarrow{j_2^*} T_1^{loc}(\mathcal{A}_\mu/J_\mu) ) \]

**Theorem 54** -i) One has the exact sequence
\[ 0 \longrightarrow J_\mu \xrightarrow{\iota} \mathcal{A}_\mu \xrightarrow{\pi} \mathcal{A}_\mu / J_\mu \longrightarrow 0, \]
where $\iota$ is the inclusion and $\pi$ is the canonical projection.
-ii) The exact sequence [7] becomes
\[ T_1^{loc}(\mathcal{A}, J) \xrightarrow{\iota} T_1^{loc}(\mathcal{A}) \xrightarrow{\pi} T_1^{loc}(\mathcal{A}/J) \xrightarrow{\partial} \]
\[ T_0^{loc}(\mathcal{A}, J) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{i^*} T_0^{loc}(\mathcal{A}) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{\pi^*} T_0^{loc}(\mathcal{A}/J) \otimes \mathbb{Z}[\frac{1}{2}]. \]

**Proof.** We leave the check to the reader. ■

23 Connecting homo-morphism - second form

In this section we define Mayer - Vietoris diagramms associated with elliptic operators between two different Hilbert spaces and we exhibit the corresponding connecting homomorphism.
To motivate the next definition, let $D : H_0 \rightarrow H_1$ be an (integral or pseudo-differential) elliptic operator. Let $\sigma(D) := D \mod{\text{compact operators}}$ be its symbol.

We consider the ring

$$\Lambda_1 = \Lambda := \text{Hom}_{\text{vect}}(H_0 \oplus H_1, H_0 \oplus H_1)$$

(121)

and

$$J := \text{Compact operators on } H_0 \oplus H_1$$

(122)

**Definition 55** Consider the following structure.

Let $\Lambda_1 = \Lambda_2$ be a ring. Let $J \subset \Lambda_i, i = 1, 2$, be a bi-lateral ideal.

Introduce the rings

$$\Lambda := \{(\lambda_1, \lambda_2) \mid \lambda_1 \in \Lambda_1, \lambda_2 \in \Lambda_2, \lambda_1 - \lambda_2 \in J\}$$

(123)

$$\Lambda' := \Lambda_1/J = \Lambda_2/J.$$ 

(124)

The ring structure in $\Lambda$ is

$$(\lambda_1, \lambda_2) + (\lambda'_1, \lambda'_2) := (\lambda_1 + \lambda'_1, \lambda_2 + \lambda'_2)$$

$$(\lambda_1, \lambda_2) \cdot (\lambda'_1, \lambda'_2) := (\lambda_1 \cdot \lambda'_1, \lambda_2 \cdot \lambda'_2)$$

Define $i_i, i = 1, 2$,

$$i_i : \Lambda \rightarrow \Lambda_1 \oplus \Lambda_2, \quad i_i (\lambda_1, \lambda_2) := \lambda_i.$$ 

(125)

and

$$j_i : \Lambda_1 \oplus \Lambda_2 \rightarrow \Lambda', \quad j_i (\lambda_1, \lambda_2) := \lambda_i/J.$$ 

(126)

**Proposition 56** \{\Lambda, \Lambda_1, \Lambda_2, \Lambda', i_1, i_2, j_1, j_2\} is a Mayer-Vietoris diagram, see [111].

If $\Lambda_i, J$ are localised rings, then this is a localised Mayer-Vietoris diagram.

To this localised Mayer-Vietoris diagram there corresponds a six term exact sequence, see Theorem [7].

Next, we are going to exhibit the corresponding connecting homomorphism $\partial : T^{\text{loc}}_1(\Lambda') \rightarrow T^{\text{loc}}_0(\Lambda)$.

**Definition 57** Let $[u] \in T^{\text{loc}}_1(\Lambda')$, where $u \in \text{GL}_n(\Lambda'_\mu)$ for some $n$ and $\mu$.

Let $A$, resp. $B$, $\in \mathbb{M}_n(\Lambda_1, \mu)$ be liftings of $u$, resp. $u^{-1}$, in $\mathbb{M}_n(\Lambda_1, \mu)$.

Such liftings exist because the canonical mapping $j_1$ is surjective.
With A and B one associates

\[ S_0 = 1 - BA \in M_n(\Lambda_{1,\mu-1}), \quad S_1 = 1 - AB \in M_n(\Lambda_{1,\mu-1}). \]

The matrices \( S_0, S_1 \) satisfy

\[ j_1(S_0) = j_1(S_1) = 0. \quad (127) \]

With these matrices one associates the invertible matrix (ref. [15])

\[ L = \begin{pmatrix} S_0 & -(1 + S_0)B \\ A & S_1 \end{pmatrix} \in \text{GL}_{2n}(\Lambda_{1,\mu-2}); \quad (128) \]

the inverse of the matrix \( L \) is

\[ L^{-1} = \begin{pmatrix} S_0 & (1 + S_0)B \\ -A & S_1 \end{pmatrix} \in \text{GL}_{2n}(\Lambda_{1,\mu-2}). \quad (129) \]

Let \( e_1, e_2 \) be the idempotents

\[ e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \Lambda_{1,\mu-2}, \quad e_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \Lambda_{2,\mu-2}. \quad (130) \]

The invertible matrix \( L \) is used to produce the idempotent

\[ P := Le_1L^{-1} = \begin{pmatrix} S_0^2 & S_0(1 + S_0)B \\ S_1A & 1 - S_1^2 \end{pmatrix} \in \text{Idemp}_{2n}(\Lambda_{1,\mu-2}). \quad (131) \]

with

\[ j_1(P) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]

The idempotent \( P \) is used to construct the double matrix idempotent \( P_U \)

\[ P_U := \begin{pmatrix} (S_0^2, 0) & (S_0(1 + S_0)B, 0) \\ (S_1A, 0) & (1 - S_1^2, 1) \end{pmatrix} \in \text{Idemp}_{2n}(\Lambda_{1,\mu-2}) \oplus \text{Idemp}_{2n}(\Lambda_{2,\mu-2}). \quad (132) \]

with

\[ \pi(P_U) = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \in \text{Idemp}(\Lambda_{2,\mu}). \]

The idempotent \( P_U \) satisfies \((j_{1,*} - j_{2,*})(P_U) = 0\). Therefore, \( P_U \in M_{2n}(\Lambda_{\mu-2}). \)
Definition 58  Connecting homomorphism - second form. (for K-theory see [15]).

For any \([U] \in T_{1}^{\text{loc}}(\Lambda')\) we define the connecting homomorphism
\(\partial_{II} : T_{1}^{\text{loc}}(\Lambda') \to T_{0}^{\text{loc}}(\Lambda)\) by
\[
\partial_{II}[U] := [P_{U}] - [\begin{pmatrix} e_{2} & e_{2} \end{pmatrix}] \in T_{0}^{\text{loc}}(\Lambda).
\] (133)

In this case it is easy to check that \(\partial\) is compatible with the equivalence relation \(\sim_{U}\). Indeed, any element \(\xi = u \oplus u^{-1} \in \Lambda'_{\mu}\) has an invertible lifting in \(\Lambda_{1,\mu}\), see §18.3. For this reason, both \(A\) and \(B\) may be chosen to be inverse one to each other. Therefore, \(S_{0} = 0\) and \(S_{1} = 0\) and hence
\[
P = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \partial[U] = 0.
\]

23.1 Connecting homomorphism and stabilisation

In this sub-section we show how \(\partial\) depends on stabilisations. In the proof of the exactness, Theorem 7 -iii), we will need to know how the connecting homomorphism, Definition 23, behaves with respect to stabilisations. For this purpose we consider a more general situation than that considered in the previous section §23.

Let \(U \in \mathbb{GL}_{m+n}(\Lambda')\) and
\[
e_{(0,n)} = \begin{pmatrix} 0 & 0 \\ 0 & 1_{n} \end{pmatrix} \in M_{m+n}(\Lambda')
\] (134)
be such that the diagram
\[
\begin{array}{ccc}
\Lambda'_{m+n} & \xrightarrow{U} & \Lambda'_{m+n} \\
\downarrow^{e_{(0,n)}} & & \downarrow^{e_{(0,n)}} \\
\Lambda'_{m+n} & \xrightarrow{U} & \Lambda'_{m+n}
\end{array}
\] (135)
is commutative. This condition will be needed to show that \(P_{U} \in \mathbb{Idemp}_{m+n}(\Lambda)\), formula ()

The case discussed in §23 corresponds in this subsection to \(m = 0\).

We proceed as in §23. Let \(A, B \in \mathbb{GL}_{m+n}(\Lambda_{1,\mu})\) be liftings of \(U\), resp. \(U^{-1}\), in \(M_{m+n,\mu}(\Lambda_{1})\). Such liftings exist because we assume \(j_{1}\) is surjective.
With $A$ and $B$ one associates $S_0 = 1 - BA \in \text{GL}_{m+n}(\Lambda_{1,\mu})$ and $S_1 = 1 - AB \in \text{GL}_{m+n}(\Lambda_{1,\mu})$. The matrices $S_0, S_1$ satisfy
\begin{equation}
 j_{1,*}(S_0) = j_{1,*}(S_1) = 0.
\end{equation}
(136)
With these matrices one associates the invertible matrix
\begin{equation}
 L = \begin{pmatrix} S_0 & -(1 + S_0)B \\ A & S_1 \end{pmatrix} \in \text{GL}_{2(m+n)}(\Lambda_{1,\mu-2});
\end{equation}
(137)
the inverse of the matrix $L$ is
\begin{equation}
 L^{-1} = \begin{pmatrix} S_0 & (1 + S_0)B \\ -A & S_1 \end{pmatrix} \in \text{GL}_{2(m+n)}(\Lambda_{1,\mu-2}).
\end{equation}
(138)
Let $e_1$ be the idempotent
\begin{equation}
 e_1 = \begin{pmatrix} e_{(0,n)} & 0 \\ 0 & 0 \end{pmatrix} \in \text{GL}_{2(m+n)}(\Lambda_{1,\mu}).
\end{equation}
(139)
and
\begin{equation}
 e_2 = \begin{pmatrix} 0 & 0 \\ 0 & e_{(0,n)} \end{pmatrix} \in \text{GL}_{2(m+n)}(\Lambda_{1,\mu}).
\end{equation}
(140)
The invertible matrix $L$ is used to produce the idempotent
\begin{equation}
 P_U := Le_1L^{-1} = \begin{pmatrix} S_0 e_{(0,n)} & S_0 e_{(0,n)} (1 + S_0)B \\ A e_{(0,n)} & A e_{(0,n)} (1 + S_0)B \end{pmatrix} \in \text{Idemp}_{2(m+n)}(\Lambda_{1,\mu-2}).
\end{equation}
(141)

**Definition 59**
\begin{equation}
 R(U) := P_U - e_2.
\end{equation}
(142)
The idempotent $P_U$ is used to construct the double-matrix idempotent
\begin{equation}
 P_U := \begin{pmatrix} (S_0 e_{(0,n)} & 0) & (S_0 e_{(0,n)} (1 + S_0)B, 0) \\ (A e_{(0,n)} & 0) & (A e_{(0,n)} (1 - S_0)B, e_{(0,n)}) \end{pmatrix}
\end{equation}
\begin{equation}
 \in \text{Idemp}_{2(m+n)}(\Lambda_{1,\mu-2} \oplus \Lambda_{2,\mu-2}).
\end{equation}
(143)
(144)
The matrix $P_U$ is an idempotent in $\text{M}_{2(m+n)}(\Lambda_{\mu-2})$. We may verify directly that $(j_{1,*} - j_{2,*})p_U = 0$. Indeed, $j_{1,*}$ is a ring homomorphism and $j_{1,*}(S_0) = j_{1,*}(S_1) = 0$; finally, the hypothesis (135) gives
\begin{equation}
 23. j_1(A e_{(0,n)} (1 - S_0)B = U e_{(0,n)} U^{-1} = e_{(0,n)} = j_2(e_{(0,n)}).
\end{equation}
(145)
Therefore $P_U \in M_{2n}(\Lambda_{\mu-2})$. The fact that the matrix $P_U$ is an idempotent in $M_{2(m+n)}(\Lambda_{\mu-2})$ follows from the fact that $P$ and $e_2$ are idempotents along with the discussion above.

**Definition 60** Connecting homomorphism - third form.

We suppose the assumptions and constructions of §23 above are in place. For any $[U] \in T^1_{1}(\Lambda')$ one defines the connecting homomorphism $\partial : T^1_{1}(\Lambda') \longrightarrow T^0_{1}(\Lambda)$ by

$$\partial[U] := [P_U] - [(e_2, e_2)] =$$

$$= \begin{pmatrix} (0, 0) & (0, 0) \\ (0, 0) & (Ae_{(0,n)}B, e_{(0,n)}) \end{pmatrix} - [(e_2, e_2)] \in T^0_{1}(\Lambda). \quad (147)$$

It remains to follow up how $P_U$ depends of the choice of the lifts $A$ and $B$ of $U$ and $U^{-1}$. A different choice of $A$ and $B$ has the effect of modifying the matrix $L$. We will show that if $A'$ and $B'$ are two such different lifts and $L'$ is the corresponding matrix, then

$$L' = \tilde{L} L, \quad \text{with} \quad \tilde{L} \in GL_{2(m+n)}(\Lambda) \quad (148)$$

and hence the corresponding idempotents $P_U := Le_1L^{-1}$, $P'_U := L'e_1L'^{-1}$ are conjugate.

To better organise the computation, we change the liftings one at the time.

We begin with $A$. Let $\tilde{A} = A + T$ with $j_1(T) = 0$. Let $\tilde{S}_0 = 1 - B\tilde{A}$, $\tilde{S}_1 = 1 - \tilde{A}B$, $\tilde{L}$, $\tilde{P}$ and $\tilde{P}_U$ be the corresponding elements. A direct computation gives

$$\tilde{L}L^{-1} = \begin{pmatrix} 1 - BT & -BTB \\ T & 1 + TB \end{pmatrix} \quad (149)$$

or

$$\tilde{L} = \begin{pmatrix} 1 - BT & -BTB \\ T & 1 + TB \end{pmatrix} L. \quad (150)$$

We know that $\tilde{L}$ and $L$ are invertible matrices; therefore the RHS of (150) is an invertible matrix.
The corresponding idempotent $\tilde{P}$ is

$$\tilde{P} = \tilde{L}e_1\tilde{L}^{-1} = \begin{pmatrix} 1 - BT & -BTB \\ T & 1 + TB \end{pmatrix} L e_1 L^{-1} \begin{pmatrix} 1 - BT & -BTB \\ T & 1 + TB \end{pmatrix}^{-1} = $$

(151)

$$= \begin{pmatrix} 1 - BT & -BTB \\ T & 1 + TB \end{pmatrix} P \begin{pmatrix} 1 - BT & -BTB \\ T & 1 + TB \end{pmatrix}^{-1}$$

(152)

and furthermore

$$\tilde{P}_U = \left(\begin{pmatrix} 1 - BT & -BTB \\ T & 1 + TB \end{pmatrix}, 1\right) P U \left(\begin{pmatrix} 1 - BT & -BTB \\ T & 1 + TB \end{pmatrix}, 1\right)^{-1}.$$  

(153)

Therefore, $[\tilde{P}_U] = [P_U] \in T_{0}^{\text{loc}}(\Lambda)$.

It remains to see what happens if $A$ remains unchanged and the lifting $B$ is changed. Let $\tilde{B} = B + H$, with $j_1(H) = 0$. Let $\tilde{L}$, $\tilde{P}$ and $\tilde{P}_U$ be the corresponding matrices. A direct computation gives

$$\tilde{L}L^{-1} = \begin{pmatrix} 1 + \Delta_{11} & \Delta_{12} \\ \Delta_{21} & 1 + \Delta_{22} \end{pmatrix} \in \mathcal{M}_{2n}(\Lambda_{\mu - 4})$$

(154)

where

$$\Delta_{11} = HA - HAHA - BAH A$$
$$\Delta_{12} = -2H + HAB + AH + BAH - BAHAB - HAHAB$$
$$\Delta_{21} = AHA$$
$$\Delta_{22} = -AH + AHA.$$

The RHS of (154) is a product of invertible matrices; therefore, it is an invertible matrix. Proceeding as above we get

$$\tilde{P}_U = \left(\begin{pmatrix} 1 + \Delta_{11} & \Delta_{12} \\ \Delta_{21} & 1 + \Delta_{22} \end{pmatrix}, 1\right) P_U \left(\begin{pmatrix} 1 + \Delta_{11} & \Delta_{12} \\ \Delta_{21} & 1 + \Delta_{22} \end{pmatrix}, 1\right)^{-1}.$$  

(155)

This completes the discussion about the choice of the liftings $A$ and $B$. 

42
Part III
Topological index and analytical index. Reformulation of index theory.

24 Level I: Index theory at the $\mathbb{T}_{\mu}^{\text{loc}}$-theory level.

Definition 61 Let $\mathcal{J} \subset \mathcal{A}$ be a localised bi-lateral ideal of the unital ring $\mathcal{A}$. There corresponds the short ring exact sequence

$$0 \to \mathcal{J}_\mu \to \mathcal{A}_\mu \xrightarrow{\pi} \mathcal{A}_\mu / \mathcal{J}_\mu \to 0.$$  \hfill (156)

Definition 62 The ring $\mathcal{A}_\mu / \mathcal{J}_\mu$ is the analogue of the Calkin ring. If $U \in \mathcal{A}_\mu$ then $\pi(U) \in \mathcal{A} / \mathcal{J}$ is called the symbol of $U$.

Consider the 6-term exact sequence in $\mathbb{T}_{\mu}^{\text{loc}}$-theory \hfill (119)

$$\mathbb{T}_{\mu}^{\text{loc}}(\mathcal{A}, \mathcal{J}) \xrightarrow{i_\ast} \mathbb{T}_{1}^{\text{loc}}(\mathcal{A}) \xrightarrow{\pi_\ast} \mathbb{T}_{1}^{\text{loc}}(\mathcal{A} / \mathcal{J}) \xrightarrow{\partial} \mathbb{T}_{0}^{\text{loc}}(\mathcal{A}) \xrightarrow{i_\ast} \mathbb{T}_{0}^{\text{loc}}(\mathcal{A} / \mathcal{J}) \xrightarrow{\pi_\ast} \mathbb{T}_{0}^{\text{loc}}(\mathcal{A} / \mathcal{J}) \otimes \mathbb{Z}[\frac{1}{2}].$$  \hfill (157)

$$\mathbb{T}_{0}^{\text{loc}}(\mathcal{A}, \mathcal{J}) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{i_\ast} \mathbb{T}_{0}^{\text{loc}}(\mathcal{A}) \otimes \mathbb{Z}[\frac{1}{2}] \xrightarrow{\pi_\ast} \mathbb{T}_{0}^{\text{loc}}(\mathcal{A} / \mathcal{J}) \otimes \mathbb{Z}[\frac{1}{2}].$$  \hfill (158)

1. Topological index of $u$ is

$$\text{Top}^T \text{Index} \ (u) := \delta[u] \in \mathbb{T}_{0}^{\text{loc}}(\mathcal{A}) \otimes \mathbb{Z}[\frac{1}{2}],$$  \hfill (159)

where $[u] \in \mathbb{T}_{1}^{\text{loc}}(\mathcal{A} / \mathcal{J})$.

2. Analytical Index of $u$ is

$$\text{An}^T \text{Index} \ (u) := \text{R} (\delta[u]) = \delta_{11}[u] \in \mathbb{T}_{0}^{\text{loc}}(\mathcal{A}) \otimes \mathbb{Z}[\frac{1}{2}]$$  \hfill (160)

where $\delta_{11}$ is the second definition of boundary map, see Definition 23.
Case 1.

**Problem 63** Define significant classes of extensions (156)

\[ \Delta(\mathcal{A}, \mathcal{J}) := \text{Top}^T \text{Index} (u) - \text{An}^T \text{Index}(u) \]  

(161)
can be computed.

Case 2.

**Conjecture 64** -1) Let \( M^{2l} \) be a closed compact quasi-conformal manifold. Let \( \Omega^*(M) \) be the algebra of differential forms on \( M^{2l} \). Let \( H \) be the Hilbert space of \( L_2 \)-forms of degree \( l \). Let be the exact sequence associated to the short exact sequence

\[ 0 \rightarrow \mathcal{L}^{(1,\infty)} \xrightarrow{i} \Psi \text{Diff} \xrightarrow{\pi} \Psi \text{Diff}/\mathcal{L}^{1,\infty} \rightarrow 0. \]

For any \( u \in \mathbb{GL}_N(\Psi \text{Diff}/\mathcal{L}^{1,\infty}) \) one has

\[ (\text{Top}^T \text{Index}) (u) \otimes \mathbb{Q} = (\text{An}^T \text{Index}) (u) \otimes \mathbb{Q}. \]

(162)

-2) Let \( A \) be an elliptic pseudo-differential operator on \( M \). Let \( u = \pi(A) \in \mathbb{GL}_N(A/\mathcal{J}) \) be the image of \( A \) in the quotient space. Then

\[ \text{An}^T \text{Index}(u) \otimes \mathbb{Q} = \text{Top}^T \text{Index}(u) \otimes \mathbb{Q} = \]

\[ = \frac{1}{(2\pi i)^q (2q)!} (-1)^{\dim M} \, \text{Ch}(u) \cap [T^*M]. \]

(163) \hspace{1cm} (164)

\( \text{Ch}(u) \) in the formula (163) is the periodic cyclic homology of \( u \).

-3) Let \( A \) be a pseudo-differential elliptic operator on the quasi-conformal manifold \( M \). Let \( u \) be its classical symbol. Produce the residue operator \( \mathbf{R}(u) \) (\( K_1 \) is replaced by \( T_1^{loc} \)). Let \( f \in C_{\text{AS}}^q(M) \) be an Alexander - Spanier co-cycle; let \( [f] \) be its co-homology class. Let \( \tau(T^*(M)) \) be the Todd class of \( M \).

Then

\[ \text{An}^T \text{Index}(u) \otimes \mathbb{Q} [f] = \text{Top}^T \text{Index}(u) \otimes \mathbb{Q} [f] = \]

\[ = \frac{1}{(2\pi i)^q (2q)!} (-1)^{\dim M} \, \text{Ch}(u) \cup \tau(T^*(M)) \cap [T^*M]. \]

(165) \hspace{1cm} (166)
25 Level II: Index Theory in local periodic cyclic homology.

Definition 65 Let \( \mathcal{A}_\mu \) be a localised ring. Consider the exact sequence \([156]\). Let \( u \in T^1_{\text{loc}}(\mathcal{A}_\mu/\mathcal{J}_\mu) \).

1. Topological index of \( u \) is
\[
\text{Top}^{\text{Ch}}\text{Index} (u) := Ch[\delta[u]] \in C^{\text{loc,per,}\lambda}(\mathcal{A}),
\]  
(167)

2. Analytical Index of \( u \) is
\[
\text{An}^{\text{Ch}}\text{Index} (u) := Ch_{ev}(\delta [u]) = Ch_{ev}(\delta_{II} [u]) \in H^{0}_{\text{loc,per,}\lambda}(\mathcal{A}),
\]  
(168)

see Definition \([23]\).

The topological index could be defined in a different way. One could consider the connecting homomorphism \( \delta_{\lambda} \) in the local cyclic periodic homology instead of the connecting homomorphism in the \( T^1_{\text{loc}} \) exact sequence. This leads to the topological index
\[
\text{Top}^{\text{Ch}_{\lambda}}\text{Index} (u) := Ch\delta_{\lambda}[u] \in H^{ev}_{\text{loc,per,}\lambda}(\mathcal{A}),
\]  
(169)

Case 1

Problem 66 Define significant classes of extensions \([156]\) for which the difference
\[
\Delta^{\text{Ch}}(\mathcal{A}, \mathcal{J}) := \text{Top}^{\text{Ch}}\text{Index} (u) - \text{An}^{\text{Ch}}\text{Index} (u)
\]  
(170)
can be computed.

Case II.

Conjecture 67 Let \( M \) be a quasi-conformal closed manifold. Let \( u \in \mathcal{GL}_N(\mathcal{A}_\mu/\mathcal{J}_\mu) \), where \( \mathcal{A}_\mu \) is the algebra of pseudo-differential operators on \( M \) and \( \mathcal{J} = L^{1,\infty}_{\text{loc}} \), localised by the support of operators about the diagonal. Then
\[
\text{Top}^{\text{Ch}}\text{Index}(u) = \text{An}^{\text{Ch}}\text{Index}(u).
\]  
(171)
26  Level III: Index theory restricted at the diagonal.

This situation applies only when the local periodic cyclic homology has a limit to the diagonal. It depends on the regularity of the structure. The classical index theorems belong to this class.

Part IV
Noncommutative Topology

Abstract

We intend to produce a theory which generalises topological spaces; we call it non-commutative topology. In non-commutative differential geometry the basic homology theory is the periodic cyclic homology, based on the bi-complex \((b, B)\). In non-commutative topology this structure will be replaced by the bi-complex \((\tilde{b}, d)\); the boundary \(b\) is called modified Hochschild boundary. These ideas combine A. Connes' work [17] with ideas of the articles by Teleman N. and Teleman K. [3], [4], [25], [27], [28], [29], [30].

27 Modified Hochschild homology

The Alexander - Spanier co-homology uses solely the topology of the space; it does not require any kind of analytical regularity. We use the Alexander - Spanier construction and the definition of local periodic cyclic co/homology as the departure point for non-commutative topology. Recall we extended the Alexander - Spanier co-homology to arbitrary localised rings, see §21.

As an application we compute the local modified periodic cyclic homology of the topological algebra of smooth functions, Theorem , of the Banach agebra of continuous functions, Theorem , and of the algebra of arbitrary functions, Theorem ??, on a smooth manifold. These results are significant because it is known that the Hochschild and (periodic) cyclic homology of Banach algebras are either trivial or not interesting, see Connes [9], [17], [15]. Entire cyclic cohomology, due to Connes [17] (15, 14) gives a different solution to the problem of defining the cyclic homology of Banach algebras. The chains of the entire cyclic cohomology are elements of the infinite product \((b, B)\) which satisfy a certain bi-degree asymptotic growth condition. Connes constructed a Chern character of \(\theta\)-summable Fredholm modules with values in the entire cyclic cohomology, see Connes [11].
28 The idempotent $\Pi$.

Recall the operator $\sigma$ was introduced in §2.4.10 It is well defined on the whole complex $C_\ast(A)$

$$db + bd = 1 - \sigma,$$  \hfill (172)

where $d$ is the non-localised Alexander - Spanier co-boundary and $b$ is the Hochschild boundary.

The main properties of $\sigma$ are

1. $\sigma$ commutes both with $d$ and $b$.
2. $\sigma$ is a chain homomorphism both in the Alexander-Spanier and in the Hochschild complex. Hence, the range of the operator $\sigma$, and its powers, are sub-complexes both in the Alexander-Spanier and Hochschild complexes.
3. The range of the homomorphism $\Pi$ consists of non-degenerate chains.
4. $\sigma$ is homotopic to the identity. Therefore, the inclusions of these sub-complexes into the Alexander-Spanier, resp. Hochschild, complexes induce isomorphisms between their homologies.

The operator $\Pi(k)$ is defined by the formula

$$\Pi(k) := (1 - bd)\sigma(k).$$  \hfill (173)

The operator $\Pi(k)$ has the properties

1. $\Pi(k)$ is an idempotent. Let $\{\tilde{C}_\ast(A)\}$ be its range. Hence

$$ (1 - bd)\sigma(k) = 1 \text{ on } \tilde{C}_k(A)$$  \hfill (174)

2. $\Pi(k)$ commutes with $d, b$, and hence with $\sigma$. Therefore
3. $\{\tilde{C}_\ast(A)\}$ is a $b$ and $d$ sub-complex
4. The mapping $\Pi(k)$ induces isomorphisms between the $b$, resp. $d$, homologies.

The sub-complex $\{(1 - \Pi(k))C_k(A)\}_{k \in \mathbb{Z}}$ is acyclic.

29 Modified Hochschild homology.

On the range of the idempotent $\Pi$, i.e. $\{\tilde{C}_\ast(A)\}$, one has the identity

$$\sigma^k = 1 + bd\sigma^k.$$  \hfill (175)

From this formula we get

**Proposition 68** On the complex $\{\tilde{C}_\ast(A)\}$ one has the identity

$$0 = \sum_{k=1}^{n+1} (-1)^{k-1} C_{n+1}^k (bd)^k + \sum_{k=1}^{n} (-1)^{k-1} C_n^k (db)^k.$$  \hfill (176)
Proof. The relation (176) is obtained from the formula
\[ 1 = (1 - bd)\sigma^n. \]  
by making the substitution \( \sigma = 1 - (db + bd) \). For more details see [24].

Formula (176) suggests to introduce on the spaces \( C_*(A) \) the operators \( \tilde{b}, \tilde{d} \).

**Definition 69** The operators \( \tilde{b}_n \) and \( \tilde{d}_n \) acting on \( C_n(A) \) are defined by the formulae
\[ \tilde{b}_n := b \sum_{k=1}^{n} (-1)^{k-1} C^k_n(db)^{k-1} \]  
\[ \tilde{d}_n := d \sum_{k=1}^{n} (-1)^{k-1} C^k_{n+1}(bd)^{k-1}. \]

**Proposition 70** The operators \( \tilde{b} \) and \( \tilde{d} \) anti-commute in \( \tilde{C}(A) \); the same relation holds for the operators \( b \) and \( \tilde{d}_n \)
\[ 0 = \tilde{b}_{n+1}d_n + d_{n-1}\tilde{b}_n \]  
\[ 0 = \tilde{d}_{n-1}b_n + b_{n+1}\tilde{d}_n. \]

**Definition 71** The homology of the complex \( \{C_*(A, \tilde{b})\}_* \) is called modified Hochschild homology.

The expression of the operator \( \tilde{b}_n \) contains the factor \( b \) both to the left and to the right.

**Definition 72** -1) \( \tilde{b} \tilde{b} = 0 \) and hence \( \tilde{b} \) is a boundary operator.

-2) The complex \( \{C_*(A, \tilde{b})\}_* \) is called modified Hochschild complex. It makes sense even for non-localised algebras.

The homology of the modified Hochschild complex is called modified Hochschild homology.

-3) We assume \( A \) to be a localised ring; the Alexander - Spanier and the Hochschild complex are localised. On the space of normalised chains \( (\tilde{b}, \tilde{d}) \) introduces a bi-complex structure. The boundary \( \tilde{b} \) defined on the space of normalised chains is called local modified Hochschild homology.

The homology of the bi-complex \( (\tilde{b}, \tilde{d}) \) is called local modified periodic cyclic homology; it is \( \mathbb{Z}_2 \)-graded.

The definition of the local modified periodic cyclic homology is analogues to the definition of periodic cyclic homology, comp. A. Connes [17], see also J. - L. Loday [16] Sect. 5.1.7., pag.159.

More specifically, given the localised ring \( A \), one has
\[ (\tilde{b}, \tilde{d})^{\lambda, per}_*(A) = \{C_{p,q}(A)\}_{(p,q) \in \mathbb{Z} \times \mathbb{Z}}, \]
where \( C_{p,q}(A) := \otimes^{p-q}A \), for \( q \leq p \). The boundary maps are
\[ \tilde{b} : C_{p,q} \rightarrow C_{p,q-1} \]
and
\[ d : C_{p,q} \rightarrow C_{p+1,q}; \]
Proposition 73 For any ring \( A \), the homology of the modified Hochschild complex contains the Hochschild homology.

Proof. As said before, the operator \( \tilde{b} \) contains the factor \( b \) both to the left as well as to the right. For this reason any Hochschild cycle is a \( \tilde{b} \) cycle; for the same reason any \( \tilde{b} \) boundary is a \( b \) boundary. ■

29.1 Local modified periodic cyclic homology of the algebra of smooth functions.

In this subsection we consider the Fréchet topological algebra of smooth functions over a compact smooth manifold \( M \). We know that the Hochschild boundary is well defined on germs at the diagonals and that the Hochschild homology depends only on the quotient complex, see Teleman [19].

The first two terms of the spectral sequence associated to the first filtration (with respect to \( d \) and then with respect to \( \tilde{b} \)) of the \((\tilde{b}, d)\) bi-complex are

\[
E^1_{p,q} = H_p(C_\ast,q,d) \cong H_{dR}^{q-p}(M)
\]

and

\[
E^2_{p,q} = H_q(E^1_{p,*}, \tilde{b}) \cong H_{dR}^{q-p}(M).
\]

In fact, the \( E^1 \)-term is the Alexander - Spanier co-homology of the complex of smooth chains. The Alexander - Spanier chain complex contains the sub-complex \( \sigma_\ast C_\ast(A) \). We know that this complex is co-homologous with the original Alexander - Spanier complex. For this reason, for the computation of \( E^2 \), we may represent any element of \( E^1 \) by an element of the this sub-complex. The term \( E^2 = b \)-homology is a quotient group of a sub-group of \( E^1 \); from this we get

\[
\dim \{ \tilde{b} \text{-homology of a quotient group of a sub-group of } E^1 \} = \dim E^2 \leq \dim E^1.
\] (183)

The term \( E^1 \), being the Alexander - Spanier co-homology of \( M \), is isomorphic to the periodic \( b \)-homology of the algebra \( A \).

\[
\dim E^1 = \dim \{ \text{b-homology of } C^\infty(M) \}
\] (184)

On the other side, the \( \tilde{b} \)-homology contains the \( b \)-homology, Proposition [178]

\[
\dim (b \text{- homology}) \leq \dim(\tilde{b} \text{- homology}).
\] (185)

From the equations (183), (184) and (185) we get

\[
E^2 \cong H_{dR}(M).
\] (186)

We have proved the following result.

Theorem 74 The local modified periodic cyclic homology of the algebra of smooth functions on a compact smooth manifold is isomorphic to the \( \mathbb{Z}_2 \)-graded de Rham co-homology of the manifold.
30 Characteristic classes of idempotents.

Definition 75 Let $A$ be a localised ring which contains the rational numbers. Let $e \in \mathcal{M}_N(A)$ be an idempotent.

The Chern character of $e$ is the even local periodic cyclic homology class defined by

$$\text{Ch}(e) := e + \frac{(-1)^1}{1!} e \cdot (de)^2 + \frac{(-1)^2}{2!} e \cdot (de)^4 + \cdots + \frac{(-1)^n}{n!} e \cdot (de)^{2n} + \cdots$$  \hspace{1cm} (187)

Remark 76 It is important to notice that the Chern character defined above does not use the trace.

Theorem 77 Let $A$ be a localised ring. Suppose the ring $\mathbb{K}$ contains $\mathbb{Q}$. Let $e \in \mathcal{M}_N(A)$ be an idempotent.

Then the RHS of the equation (187) is a cycle in the bi-complex $(\tilde{b}, d)$.

Proof. We compute $\tilde{b} e (d e)^{2n}$. We have

$$\tilde{b}_n(e (d e)^{2n}) = b \sum_{k=1}^{2n} (-1)^{k-1} C_{2n}^k (d b)^{k-1} =$$  \hspace{1cm} (188)

$$b C_{2n}^1 + b \sum_{k=2}^{n} (-1)^{k-1} C_{2n}^k (d b)^{k-1} =$$  \hspace{1cm} (189)

$$\tilde{b}_{2n}(e (d e)^{2n}) = 2n e (d e)^{2n-1} + b \sum_{k=2}^{2n} (-1)^{k-1} C_{2n}^k (d b)^{k-2}(d e)^{2n-1} =$$  \hspace{1cm} (190)

$$2n e (d e)^{2n-1} + b \sum_{k=2}^{2n} (-1)^{k-1} C_{2n}^k (d b)^{k-2}(d e)^{2n-1} =$$  \hspace{1cm} (191)

$$2n e (d e)^{2n-1} + b \sum_{k=2}^{2n} (-1)^{k-1} C_{2n}^k (d b)^{k-2}(d e)^{2n-1} =$$  \hspace{1cm} (192)

$$2n e (d e)^{2n-1} + b \sum_{k=2}^{2n} (-1)^{k-1} C_{2n}^k (d b)^{k-2}(d e)^{2n-1} =$$  \hspace{1cm} (193)

$$2n e (d e)^{2n-1} + b \sum_{k=2}^{2n} (-1)^{k-1} C_{2n}^k (d b)^{k-2}(d e)^{2n-1} =$$  \hspace{1cm} (194)

$$2n e (d e)^{2n-1} + \sum_{k=2}^{2n} (-1)^{k-1} C_{2n}^k (d b)^{k-2}(d e)^{2n-1} =$$  \hspace{1cm} (195)

Lemma 78 -1)

$$2n + \sum_{k=2}^{2n} (-1)^{k-1} C_{2n}^k 2^{k-1} = 0,$$  \hspace{1cm} (196)

-2)

$$\sum_{k=2}^{2n} (-1)^{k} C_{2n}^k 2^{k-2} = n.$$  \hspace{1cm} (197)
Proof. -1) 
\[ 2n + \sum_{k=2}^{2n} (-1)^{k-1} C_{2n}^k 2^{k-1} = 2n - \frac{1}{2} \sum_{k=2}^{2n} (-1)^k C_{2n}^k 2^k = \] 
\[ 2n - \frac{1}{2} [(1 - 2)^{2n} - (1 - 2.2n)] = 2n - \frac{1}{2}[1 - 1 + 4n] = 0. \] 

-2) 
\[ \sum_{k=2}^{2n} (-1)^k C_{2n}^k 2^{k-2} = \frac{1}{4} \sum_{k=2}^{2n} (-1)^k C_{2n}^k 2^k = \frac{1}{4} [(1 - 2)^{2n} - (1 - 2n.2) ] = n. \] 

\[ \text{From Lemma 78 we get} \] 
\[ \tilde{b}_{2n} ( e(de)^{2n} ) = n (de)^{2n-1}. \] 

\[ \text{On the other side} \] 
\[ d ( e (de)^{2n-2} ) = (de)^{2n-1}. \] 

These prove the theorem.

**Definition 79** Let \( \Delta \subset \tilde{C}_* (A) \) be the \( \mathbb{K} \) sub-module generated by all chains 
\[ e(de)^n, \ (de)^n, b (e(de)^n), b (de)^n, B (e(de)^n), B (de)^n, d (e(de)^n), d (de)^n \] 
where \( n \in \mathbb{N} \) and \( e \in M_*(A) \) is an arbitrary idempotent.

\( \Delta \) is called characteristic sub-complex.

**Proposition 80** \( \Delta \) is a sub-complex both in the \( (b,B) \) and \( (\tilde{b},d) \) complexes.

Although not all relations below are used in this chapter, we provide them for the benefit of the reader.

\[ B(a_0a_1 \ldots da_n) = \sum_{j=n}^{j=n} (-1)^{jn} da_j \ldots da_n da_0 \ldots da_{j-1}. \] 

On the same space the formulas hold
\[ B^2 = 0, \ bB + Bb = 0. \] 

The following relations hold on normalised local/non-local chains. Although not all of them are necessary in what follows, we mention them for the benefit of the reader, see J. M. Garcia - Bondia, H. Figueroa, J. C. Varilly [20], Pg. 447

\[ b( e(de)^{2n} ) = e(de)^{2n-1}, \ b( e(de)^{2n-1} ) = 0 \] 
\[ b (de)^{2n} = (2e - 1)(de)^{2n-1}, \ b (de)^{2n-1} = 0 \] 
\[ B( e(de)^{2n} ) = (2n + 1)(de)^{2n+1}, \ B( (de)^{2n} ) = 0 \] 
\[ B (e(de)^{2n-1} ) = 0, \ B (de)^{2n-1} = 0, \] 
\[ d( e(de)^{n} ) = (de)^{n+1}, \ d (de)^n = 0. \]
The following relation is a property of the idempotent $\Pi$:

\[ \Pi_{p+1}(Bx) = d\Pi_p(x). \] (211)

**Theorem 81** [187] Let $M$ be a compact smooth manifold and $e$ an idempotent of the matrix algebra over $A = C^\infty(M)$.

Then the Chern character $\text{Ch}(e)$ defined by the formula (187) coincides, up to a scalar factor, with the entire cyclic homology Chern character of $e$.

**Proof.** The relations

\[ \tilde{b}(e(de)^{2n-1}) = 0 \] (212)
\[ d(e(de)^{2n-1}) = e(de)^{2n} \] (213)

show that

\[ e(de)^{2n} = \frac{1}{2}(e - \frac{1}{2})(de)^{2n} - (\tilde{b} + d)\frac{1}{2}(e(de)^{2n-1}). \] (214)

This means the classes $e(de)^{2n}$, $\frac{1}{2}(e - \frac{1}{2})(de)^{2n}$ are co-homologous in the local and non-local complex $\sigma(C_\ast(A))$. The non-local class $\frac{1}{2}(e - \frac{1}{2})(de)^{2n}$ represents the Connes - Chern character of the idempotent $e$ in the non-local entire cyclic homology, see Getzler, Senes [13] and [20], Pg. 447.

**Theorem 82** The Chern character defines a homomorphism

\[ \text{Ch} : T_0(A) \longrightarrow H^\text{per,loc}_\ast(A) \] (215)

**Proof.** The Chern character is compatible with all equivalences which define $T_0(A)$: stabilisation and conjugation.

### 31 Rational Pontrjagin classes of topological manifolds

#### 31.1 Existence of direct connections on topological manifolds.

We intend to construct a direct connection $A$ on $M$. Recall that a direct connection, see [23], on $M$ consists of a set of isomorphisms $A(x,y)$ where, in general

\[ A(x,y) = \text{isomorphism from a neighbourhood of } y \text{ to a neighbourhood of } x. \] (216)

with the property

\[ A(x,x) = \text{Identity}. \] (217)

The connection $A$ will be constructed by induction using a handlebody decomposition of $M$, see [8] 7.1. Pg. 319.

The second condition on direct connections assures that the inductive construction of the connection $A$ may be performed without meeting homotopic obstructions in $\pi_n(TOP)$. 

52
31.2 The characteristic class

We consider the chain
\[ \Psi_{2n}(x_0, \ldots, x_{2n}) = N A(x_{i_0}, x_{i_{2n}}) \circ dA(x_{i_{2n}}, x_{i_{2n-1}}) \cdots \circ dA(x_{i_0}, x_{i_{2n}}), \]
(218)
where \( N \) is the cyclic symmetrisation with respect to the variables \( x \) and
\[ dA(y, x) = 1 \otimes A(y, x) - A(y, x) \otimes 1. \]
(219)

Given that \( A(x, x) = \text{Identity} \), the chain \( \Psi_{2n} \) is a cycle in the \( b' \)-complex. Therefore,
1. it is a \( b' \)-cycle
2. it is \( T \)-invariant.

\( \Psi_{2n} \) is a cycle in the cyclic homology of the algebra \( A \).

References

[1] Swan R., Vector bundles and projective modules. Trans. Amer. Math. Soc. 105, 264 - 277, 1962
[2] Whitehead G. W.: Generalised homology theories. Trans. Amer. Math. Soc., 102, pp. 227 - 283, 1962.
[3] Teleman N.: A geometrical definition of some Andre Weil forms which can be associated with an infinitesimal connection, (Roumanian). St. Cerc. Math. Tom. 18, No. 5, pp. 753-762, Bucarest, 1966.
[4] Teleman K.: Sur le charact'ere de Chern dun fibre complexe differentiable, Rev. Roumaine Math. Pures Appl. 12, pp. 725-731, 1967.
[5] Milnor J.: Introduction to Algebraic \( K \)-Theory. Annals of Math. Studies Vol. 72, 1971
[6] Bass H.: Introduction to some Methods of Algebraic \( K \)-theory. AMS, CBMS, 1974
[7] Milnor J.: Characteristic Classes, Annals of Mathematics Studies Nr. 76, Princeton, 1974
[8] Kirby R., Siebenmann L.: Foundational Essays on Topological Manifolds, Smoothings and Triangulations. Annals of Mathematical Studies 88, Princeton Univ. Press, 1977
[9] Connes A.: Noncommutative differential Geometry, Publ. Math. IHES 62, pp.257 - 360, 1985
[10] Karoubi M., Homologie cyclique et \( K \)-Théorie. Astérisque No. 149, 147 pp., 1987
[11] Connes A.: Entire cyclic cohomology of Banach algebras and characters of \( \theta \)-summable Fredholm modules, K-Theory 1, 519-548, 1998
[12] Blackadar B.: \( K \)-Theory for Operator Algebras, Second Ed, Cambridge University Press, 1998
[13] Getzler E., Szenes A.: On the Chern character of a theta-summable Fredholm module. J. Func. Anal. 84, 343-357, 1989.

[14] Connes A., Gromov M., Moscovici H.: Conjectures de Novikov et fibrés presque plats, C. R. Acad. Sci. Paris Sér. A-B 310, 273-277, 1990

[15] Connes A., Moscovici H.: Cyclic Cohomology, The Novikov Conjecture and Hyperbolic Groups, Topology Vol. 29, pp. 345-388, 1990.

[16] Loday J.-L.: Cyclic Homology, Grundlehren in mathematischen Wissenschaften 301, Springer Verlag, Berlin Heidelberg, 1992.

[17] Connes A.: Noncommutative Geometry, Academic Press, 1994

[18] Rosenberg J.: Algebraic K-Theory and its Applications. Graduate Texts Nr. 147, Springer, Berlin, 1994.

[19] Teleman N.: Microlocalisation de l’Homologie de Hochschild, Compt. Rend. Acad. Scie. Paris, Vol. 326, 1261-1264, 1998.

[20] Bondia Jose’ M. Garcia - Bondia, Figueroa Hector, Varilly Joseph C. Elements of Noncommutative Geometry, Birkhauser Advanced Texts, 2000

[21] Rordam M., Lansen F., Lautsen N.: An Introduction to K-Theory for C*-Algebras. London Math. Soc. Student Texts Nr. 49, Cambridge, 2000

[22] Cuntz J.: Cyclic Theory, Bivariant K-theory and the bivariant Chern-Connes character, Operator Algebras in Noncommutative Geometry II, Encyclopedia of Mathematical Sciences, Vol. 121, Pg. 1 - 71, Springer Verlag, 2004

[23] Teleman N.: Direct Connections and Chern Character. Proceedings of the International Conference in Honour of Jean-Paul Brasselet, Luminy, World Scientific, May 2005.

[24] Teleman N.: Modified Hochschild and Periodic Cyclic Homology. Central European Journal of Mathematics, “C*-Algebras and Elliptic Theory II”, Trends in Mathematics, 251-265, Birkhauser, 2008

[25] Teleman N.: Local^3 Index Theorem. arXiv: 1109.6095v1, math.KT, 28 Sep. 2011.

[26] Weibel C.: K-theory, 2012.

[27] Teleman N.: Local Hochschild Homology of Hilbert-Schmidt Operators on Simplicial Spaces. arXiv hal-00707040, Version 1, 11 June 2012

[28] Teleman N.: Local Algebraic K-Theory, 26 lug 2013 - arXiv.org.math [arXiv:1307.7014] 2013

[29] Teleman N.: The Local Index Theorem, HAL-00825083, arXiv 1305.5329, 22 May 2013.

[30] Teleman N.: \(K^i_{loc}(\mathbb{C}), i = 0, 1\) 10 set 2013 - [arXiv:1309.2421v1 [math.KT]] 10 Sep 2013.