LARGE CONFORMAL DEFORMATIONS
AND SPACETIME STRUCTURE

GREGORY PELTS

Department of Physics
The Rockefeller University
1230 York Avenue
New York, NY 10021-6399

Abstract

We demonstrate in detail how the space of two-dimensional quantum field theories can be parametrized by off-shell closed string states. The dynamic equation corresponding to the condition of conformal invariance includes an infinite number of higher order terms, and we give an explicit procedure for their calculation. The symmetries corresponding to equivalence relations of CFT are described. In this framework we show how to perform nonperturbative analysis in the low-energy limit and prove that it corresponds to the Brans-Dicke theory of gravity interacting with a skew symmetric tensor field.

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1 Introduction

Classical closed string states are believed to be associated with quantum conformal field theories in two dimensions (CFT), which are usually defined as theories of the single string moving in some nontrivial spacetime background. The condition of anomaly cancellation leads to the so-called $\beta$-function equation on the background fields. The main advantage of this approach is its more or less explicit connection to spacetime geometry, and the main drawback is that it usually focuses only on massless fields. Treatment of massive fields is problematic and, therefore, characterization of dynamical degrees of freedom is obscure. Symmetries also are not explicit because classically equivalent CFT may correspond to inequivalent quantum theories. Approaches \cite{1-5} based on the operator formalism \cite{6} encounter problems dealing with ambiguity of the vertex operator commutator due to contact singularity of their $T$-product. As we will see, this ambiguity is principal as it actually makes the string theory nonlinear.

2 Vertex operators

We will consider CFT as a family of amplitudes $\langle 0 \rangle_\Sigma$ assigned to Riemann surfaces $\Sigma$ and obeying sewing property (see details in \cite{7}). It is similar to the Segal’s definition \cite{8}. In this formalism vertex operators $\Psi(z_0)$ inserted at the point $z_0$ can be defined as a family of states $\langle \Psi(z_0) \rangle_D \in \mathcal{H}^{\partial D}$ associated with disc like environments $D$ of $z_0$ obeying the condition

$$\langle \Psi(z_0) \rangle_D^2 = \text{Sp}_{\partial D} \langle 0 \rangle_D^2 \otimes \langle \Psi(z_0) \rangle_D \quad (D_1 \subset D_2).$$

Here $\text{Sp}_\Gamma$ denotes contraction of amplitudes corresponding to sewing of Riemann surfaces along the contour $\Gamma$. The $T$-product of such vertex operators can be defines as

$$\langle \Psi_0(z_0) \cdots \Psi_n(z_n) \rangle_\Sigma = \text{Sp}_{\partial \Sigma_{\text{ext}}} \bigotimes_{i=0}^n \langle \Psi_i(z_i) \rangle_{D_i} \otimes \langle 0 \rangle_{\Sigma_{\text{ext}}}.$$  \hspace{1cm} (2.1)

Here $D_i$ are nonintersecting environments of $z_i$ and $\Sigma_{\text{ext}}$ is their complement in $\Sigma$.

The Virasoro algebra does not have a bounded natural representation in $\mathcal{H}^{\partial D}$. The conformal transformations deform the boundary of the disk and,
therefore, corresponding to them linear operators are not automorphisms. However, we can define such a representation in the space of vertex operators, which is independent of the position of the boundary.

3 Infinitesimal Deformation of CFT

Infinitesimal deformations of amplitudes can be parametrized by (1,1)-primary fields

\[ \langle 0 \rangle_{\Sigma} \rightarrow \langle 0 \rangle_{\Sigma} + \delta \langle 0 \rangle_{\Sigma}, \quad \delta \langle 0 \rangle_{\Sigma} = \frac{1}{\pi} \int_{\Sigma} \langle \Psi(z) \rangle_{\Sigma} d^2z. \]

Formally we can parametrize vertex operators of the deformed theory by vertex operators of the initial theory as

\[ \langle \Upsilon \rangle_{D} \rightarrow \langle \Upsilon \rangle_{D} + \delta \langle \Upsilon \rangle_{D}, \quad \delta \langle \Upsilon(z_0) \rangle_{D} = \frac{1}{\pi} \int_{D} \langle \Psi(z) \Upsilon(z_0) \rangle_{D} d^2z. \]

However, in general, the integral here may be divergent because of the contact singularity of the \( T \)-product. The simple cutoff regularization

\[ \delta_R \langle \Upsilon(z_0) \rangle_{\Sigma} = \frac{1}{\pi} \int_{\Sigma \setminus D_{z_0,R}} \langle \Psi(z) \Upsilon(z_0) \rangle_{\Sigma} d^2z, \quad D_{z_0,R} = \{ z \in \Sigma, |z - z_0| \leq R \} \]

will violate sewing properties for small disks. Instead we will make an average of such regularizations over infinitely small cutoff radiiuses

\[ \delta \langle \Upsilon(z_0) \rangle_{\Sigma} = \int_{0}^{\infty} \delta_r \langle \Upsilon(z_0) \rangle_{\Sigma} d\mu(r). \] (3.2)

Here \( d\mu \) is a generalized measure in \( \mathbb{R}_+ \) having support in 0 and integrable in a product with all the functions having a finite degree singularity at \( r = 0 \). Such a measure exists and is fully described by the function

\[ \Lambda(\alpha) = \int_{0}^{\infty} r^{2\alpha} \mu(r) \, dr \quad (\alpha \in \mathbb{R}), \]

satisfying

\[ \Lambda(0) = 1, \quad \Lambda(\alpha) = 0 \quad (\alpha > A) \]

for some positive \( A \).
4 Residue-like operations

We will call local linear operators from the space of functions on $\Sigma^{N+1}$ having diagonal singularities to the space of functions on $\Sigma$ residue-like operators of rank $N$ and denote them as

$$G(z_0) = R_{z_N = \ldots = z_0} F(z_1, \ldots, z_0) \quad \text{or} \quad G(z_0) = R_{z_N = \ldots = z_0} F(z_1, \ldots, z_0).$$

Using $T$-product (2.1) we can define representation of residue-like operations by multilinear products in the space of vertex operator functions:

$$\{ \Upsilon_i \}_{i=1}^N \rightarrow \Upsilon = R_{z_N = \ldots = z_0} \Upsilon_0 \cdots \Upsilon_N,$$

$$\langle \Upsilon(z_0) \rangle_\Sigma \overset{\text{def}}{=} R_{z_N = \ldots = z_0} \langle \Upsilon_0 \cdots \Upsilon_N \rangle_\Sigma.$$

The deformation (3.2) induce the deformation of this representation

$$\delta R_{z_n = \ldots = z_0} \Upsilon_0(z_0) \cdots \Upsilon_n(z_n) = R_{z_n = \ldots = z_0} \Upsilon_0(z_0) \cdots \Upsilon_n(z_n) \Psi(z). \quad (4.3)$$

Here $R_{z_n = \ldots = z_0}$ is a next rank residue-like operation defined as

$$R_{z_n = \ldots = z_0} F = R_{z_n = \ldots = z_0} \frac{1}{\pi} \int_\Sigma F d^2 z - \frac{1}{\pi} \int_\Sigma R_{z_n = \ldots = z_0} F d^2 z. \quad (4.4)$$

It is, indeed, a residue-like operation, because the right part of (4.4) does not depend on the area of integration as far as it includes $z_0$. We will call this operation a successor of $R_{z_n = \ldots = z_0}$. A successor of antiholomorphic derivative can be shown to be

$$\partial_{\bar{z}} R_{z_n = \ldots = z_0} F = - \text{Res}_{z = z_0} F.$$

Here $\text{Res}_{z = z_0}$ is a generalized residue operation defined for nonholomorphic functions as

$$\text{Res}_{z = z_0} \frac{(z - z_0)^k}{|z - z_0|^{2\alpha}} = \frac{\Lambda(k - \alpha)}{k!(k + 1)!} \delta_{k,-1} \quad (\alpha \in \mathbb{R}, k \in \mathbb{Z}).$$

5 Finite Deformations

Let deformed amplitudes be defined by the formula

$$\langle 0 \rangle^\Psi_\Sigma = \left\langle \exp \frac{1}{\pi} \int_\Sigma \Psi d^2 z \right\rangle_\Sigma.$$
Here $\Psi$ is some vertex operator function (not necessarily primary), and the contact divergences are regularized by the method \((3.2)\). Then the sewing property is automatically satisfied, and only the condition of conformal invariance remains to be implemented. Hereafter we mark all the deformed objects with the superscript symbol of the vertex operator function parameterizing the deformation. For parametrization of deformed vertex operators we will use the formula

$$\langle \Upsilon(z_0) \rangle^\Psi = \langle \Upsilon(z_0) \exp \frac{1}{\pi} \int_{\Sigma} \Psi(z) d^2z \rangle^\Sigma.$$

The energy-momentum tensors for the family of deformed theories parametrize by scaled vertex operator function $\tau \Psi$ ($\tau \in \mathbb{R}$) can be shown to obey the differential equation

$$\frac{\partial}{\partial \tau} T_{zz}^{\tau \Psi}(z) = B_{z_1=z}^{\tau \Psi} \Psi(z_1) T_{zz}^{\tau \Psi}(z) + A_{z_1=\bar{z}}^{\tau \Psi} \Psi(z_1) T_{zz}^{\tau \Psi}(z) + \Psi(z). \quad (5.5)$$

Here $A_{z_1=z}, B_{z_1=\bar{z}}$ are residue-like operations satisfying

$$\partial_z A_{z_1=z} + \partial_{\bar{z}} B_{z_1=\bar{z}} = \text{Res}_{z_1=z} + \text{Res}_{\bar{z}=z_1}.$$ 

Differentiating \((5.5)\) with respect to $\tau$ and applying \((4.3)\) we can recurrently calculate all the higher derivatives of the energy-momentum tensor and then substitute them to the Taylor expansion. Thus we have a perturbative formula for $T^{\Psi}$ with higher order terms expressed through residue-like operations $\text{Res}_{z_1=z_0}, A_{z_1=z_0}, B_{z_1=\bar{z}_0}$ and their successors \((4.4)\). Details on calculation of these operations can be found in [7]. The transformation of energy-momentum tensor

$$T_{zz}^{\Psi, \Phi} = T_{zz}^{\Psi} - \partial_z^{\Psi} \Phi_z, \quad T_{z\bar{z}}^{\Psi, \phi} = T_{z\bar{z}}^{\Psi} - \partial_z^{\Psi} \Phi_{\bar{z}}, \quad T_{\bar{z}z}^{\Psi, \phi} = T_{\bar{z}z}^{\Psi} + \partial_{\bar{z}}^{\Psi} \Phi_z, \quad T_{\bar{z}\bar{z}}^{\Psi, \phi} = T_{\bar{z}\bar{z}}^{\Psi} + \partial_{\bar{z}}^{\Psi} \Phi_{\bar{z}}$$

does not affect translation operators $L_{-1}, \overline{L}_{-1}$. If the theory is conformally symmetrical, there exists $\Phi_z, \Phi_{\bar{z}}$ trivializing the contradiagonal components of $T^{\Psi, \Phi}$. Then the diagonal components of $T^{\Psi, \Phi}$, will be (anti)holomorphic. They can be used for calculation of the deformed Virasoro representation.
6 Symmetries

The transformations of \( \Psi \)

\[
\delta \xi \Psi(z) = \partial_z \xi(z) + \text{Res}_{z \rightarrow z_1} \xi(z_1) \Psi(z) + z \leftrightarrow \bar{z} + O(\Psi^2) \tag{6.6}
\]

can be shown not to affect equivalence classes of theories. They are parameterized by the pair of vertex operator functions \( \xi = (\xi_z, \xi_{\bar{z}}) \). The corresponding covariant transformation of vertex operators are

\[
\hat{\xi} \Upsilon(z_0) = \text{Res}_{z \rightarrow z_0} \xi(z) \Upsilon(z_0) + \frac{1}{2} \text{Res}_{z \rightarrow z_1} \xi(z_1) \Upsilon(z_0) + z \leftrightarrow \bar{z} + O(\Psi^2).
\]

The symmetry transformation of \( \Phi \) is defined by the requirement for the energy-momentum tensor \( T^{\psi, \phi} \) to transform covariantly. Note that the commutator of such symmetries depend on \( \Psi \) and regularization parameters. Some symmetries related to global spacetime transformations where also described in [9,10]

7 Linear approximation

In the linear approximations the energy-momentum tensor and symmetries can be shown to be

\[
T^{\psi, \phi}_{zz} = \mathcal{O} \Psi - L_{-1} \Phi_z, \quad T^{\psi, \phi}_{\bar{z}\bar{z}} = \overline{\mathcal{O}} \Psi - L_{-1} \Phi_{\bar{z}}
\]

\[
T^{\psi, \phi}_{z\bar{z}} = T_{z\bar{z}} + L_{-1} \Phi_z, \quad T^{\psi, \phi}_{\bar{z}z} = T_{\bar{z}z} + L_{-1} \Phi_{\bar{z}}
\]

\[
\delta \xi \Psi = -\bar{L}_{1}\xi_z - L_{1}\xi_{\bar{z}}, \quad \delta \xi \Phi_z = \mathcal{O} \xi_z, \quad \delta \xi \Phi_{\bar{z}} = \overline{\mathcal{O}} \xi_{\bar{z}}.
\]

Here

\[
\mathcal{O} = 1 + \sum_{j=0}^{\infty} \frac{(L_{-1})^j L_j}{(j + 1)!}, \quad \overline{\mathcal{O}} = 1 + \sum_{j=0}^{\infty} \frac{(\bar{L}_{-1})^j \bar{L}_j}{(j + 1)!}.
\]

Then the equations

\[
T^{\psi, \phi}_{z\bar{z}} = T^{\psi, \phi}_{\bar{z}z} = 0
\]

are satisfied if \( \Phi \) is trivial and \( \Psi \) is a primary field. It corresponds to the deformations of the Virasoro operators

\[
L_k = L_k + \frac{1}{2\pi i} \oint_{\Gamma} \Psi(z-z_0)^k dz, \quad \bar{L}_k = \bar{L}_k + \frac{1}{2\pi i} \oint_{\Gamma} \Psi(\bar{z}-\bar{z}_0)^k d\bar{z},
\]
which is equivalent to the deformations proposed in [1]. Some deformations corresponding to nonprimary fields were first found in [11,12] in the low-energy limit. This relaxation of the equations of motion is compensated by the symmetries and does not create additional physical degrees of freedom.

8 Low-energy limit

Let us consider deformations corresponding to the vertex operator function \( \Psi = H_{\nu\mu} \partial X^{\nu} \bar{\partial} X^{\mu}(z') \), where \( H_{\nu\mu} \) is a Hermitian matrix with slowly-varying coefficients. Then the coordinate vertex operators can be shown to have the obey

\[
\langle X^{\nu}(z') X^{\mu}(z) \rangle_{\Psi} \approx -2 \langle :g^{\nu\mu}(z) : \rangle_{\Sigma} \ln |z' - z|,
\]

\[
\bar{\partial} \Psi \frac{\partial}{\partial \Psi} : \psi : \approx : \partial_{\psi}^{\nu} d \omega_{\eta \eta' \nu} \partial \Psi X^{\nu} \bar{\partial} \Psi X^{\mu} : .
\]

The contravariant metric \( g^{\nu\mu} \) and skew symmetric tensor \( \omega_{\nu\mu} \) here are equal to

\[
g = U(1) U^T(1), \quad \omega = \Im \left( \int_{0}^{1} (U^{-1})^T \frac{\partial}{\partial \tau} U^{-1} \ d\tau \right),
\]

where

\[
U(\tau) = \cosh \left( \tau \sqrt{HH^T} \right) + H^T \frac{\sinh \left( \tau \sqrt{HH^T} \right)}{\sqrt{HH^T}}.
\]

Local spacetime symmetries

\[
\delta g_{\nu\mu} = \varepsilon_{\nu;\mu} + \varepsilon_{\mu;\nu}, \quad \delta \omega_{\nu\mu} = \varepsilon^{\eta}_{\nu} \omega_{\eta \mu} + \varepsilon^{\nu}_{\mu} \omega_{\eta \eta} + \varepsilon^{\nu \eta}_{\mu} \omega_{\eta \mu} + d\xi_{\nu \mu}
\]

are a particular case of more general symmetries (6.6) with the following choice of parameters:

\[
\xi_z = : \left( \varepsilon_{\nu} + i \xi_{\nu} \right) \partial X^{\nu} : , \quad \xi_{\bar{z}} = : \left( \varepsilon_{\mu} - i \xi_{\mu} \right) \bar{\partial} X^{\mu} : .
\]

The contradiagonal components of the energy-momentum tensor can be shown to be

\[
T_{\bar{z}z}^{\Psi} \approx T_{z\bar{z}}^{\Psi} \approx -\frac{1}{2} : \left( R_{\nu\mu} - d \omega_{\nu \sigma} d \omega_{\mu \rho} + i d \omega_{\eta \eta' \nu \mu} \right) \partial \Psi X^{\nu} \bar{\partial} \Psi X^{\mu} : .
\]
Putting here $\Phi_\phi = \partial_\phi \phi$, $\Phi_\bar{\phi} = \partial_\bar{\phi} \phi$: we will come to the following equations of motion:

$$R_{\nu\mu} = d\omega_\nu^{\sigma\rho} d\omega_{\mu\sigma\rho} + 2\phi_{,\nu\mu}, \quad d\omega_{\eta_{\mu\nu}} = 2\phi^{,\eta} d\omega_{\eta_{\mu\nu}}.$$

As a consequence of this equations it can be shown that

$$2\Box \phi + 4\phi^{,\nu} \phi_{,\nu} = m^2 - \frac{2}{3} d\omega_\nu^{\sigma\rho} d\omega_{\nu\sigma\rho}.$$

The parameter $m$ here is the topological constant of the theory which can be interpreted as a dilaton mass It is responsible for deformation of central charge

$$c = D + \frac{1}{2}m^2.$$

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