The solutions of the $n$-dimensional Bessel diamond operator and the Fourier–Bessel transform of their convolution

HÜSEYIN YILDIRIM*, M ZEKI SARIKAYA and SERMIN ÖZTÜRK

Department of Mathematics, Faculty of Science and Arts, Kocatepe University, Afyon, Turkey
*Corresponding Author.
E-mail: hyildir@aku.edu.tr; sarikaya@aku.edu.tr; ssahin@aku.edu.tr

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Abstract. In this article, the operator $\Box^k_B$ is introduced and named as the Bessel diamond operator iterated $k$ times and is defined by
$$\Box^k_B = [(B_{x_1} + B_{x_2} + \cdots + B_{x_p})^2 - (B_{x_{p+1}} + \cdots + B_{x_{p+q}})^2]^k,$$
where $p + q = n$, $B_{x_i} = \frac{\partial^2}{\partial x_i^2} + \frac{2\nu_i}{x_i} \frac{\partial}{\partial x_i}$, where $2\nu_i = 2\alpha_i + 1$, $\alpha_i > -\frac{1}{2}$ [8], $x_i > 0$, $i = 1, 2, \ldots, n$, $k$ is a non-negative integer and $n$ is the dimension of $\mathbb{R}^+_n$. In this work we study the elementary solution of the Bessel diamond operator and the elementary solution of the operator $\Box^k_B$ is called the Bessel diamond kernel of Riesz. Then, we study the Fourier–Bessel transform of the elementary solution and also the Fourier–Bessel transform of their convolution.

Keywords. Diamond operator; tempered distribution; Fourier–Bessel transform.

1. Introduction
Gelfand and Shilov [2] have first introduced the elementary solution of the $n$-dimensional classical diamond operator. Later, Kananthai [345] has proved the distribution related to the $n$-dimensional ultra-hyperbolic equation, the solutions of $n$-dimensional classical diamond operator and Fourier transformation of the diamond kernel of Marcel Riesz. Furthermore, Kananthai [4] has showed that the solution of the convolution form $u(x) = (-1)^k S_{2k}(x) * R_{2k}(x)$ is a unique elementary solution of the $\Box^k u(x) = \delta$.

In this article, we will define the Bessel ultra-hyperbolic type operator iterated $k$ times with $x \in \mathbb{R}^+_n = \{x: x = (x_1, \ldots, x_n), x_1 > 0, \ldots, x_n > 0\}$,
$$\Box^k_B = (B_{x_1} + B_{x_2} + \cdots + B_{x_p} - B_{x_{p+1}} - \cdots - B_{x_{p+q}})^k, \quad p + q = n.$$ We will show that the generalized function $R_{2k}(x)$ as defined by (10) is the unique elementary solution of the operator $\Box^k_B$, that is $\Box^k_B R_{2k}(x) = \delta$ where $x \in \mathbb{R}^+_n$ and $\delta$ is the Dirac-delta distribution. $S$ is the Schwartz space of any testing functions and $S'$ is a space of tempered distribution.

Furthermore, we will show that the function $E(x)$ as defined by (8) is an elementary solution of the Laplace–Bessel operator.
\[ \Delta_B = \sum_{i=1}^{n} B_{x_i} = \sum_{i=1}^{n} \left( \frac{\partial^2}{\partial x_i^2} + \frac{2v_i}{x_i} \frac{\partial}{\partial x_i} \right), \]

that is, \( \Delta_B E(x) = \delta \) where \( x \in \mathbb{R}^n_+ \).

The operator \( \Box_B^k \) can be expressed as the product of the operators \( \Box_B \) and \( \Delta_B \), that is,

\[ \Box_B^k = \left[ \sum_{i=1}^{p} B_{x_i} \right] - \left( \sum_{i=p+1}^{p+q} B_{x_i} \right) \left[ \sum_{i=1}^{p} B_{x_i} + \sum_{i=p+1}^{p+q} B_{x_i} \right]^k \]

\[ = \Box_B \Delta_B^k. \]

Denoted by \( T^y \) the generalized shift operator acting according to the law [8]

\[ T^y \varphi(x) = C^y \int_0^{\pi} \cdots \int_0^{\pi} \varphi \left( \sqrt{x_1^2 + y_1^2 - 2x_1y_1 \cos \theta_1}, \ldots, \sqrt{x_n^2 + y_n^2 - 2x_ny_n \cos \theta_n} \right) \]

\[ \times \left( \prod_{i=1}^{n} \sin^{2v_i - 1} \theta_i \right) d\theta_1 \cdots d\theta_n, \]

where \( x,y \in \mathbb{R}^+_n, C^y = \prod_{i=1}^{n} \frac{\Gamma(v_i + 1)}{\Gamma(\frac{1}{2} + v_i)}. \) We remark that this shift operator is closely connected with the Bessel differential operator [8].

\[ \frac{d^2U}{dx^2} + \frac{2v}{x} \frac{dU}{dx} = \frac{d^2U}{dy^2} + 2v \frac{dU}{dy}, \]

\[ U(x,0) = f(x), \]

\[ U_y(x,0) = 0. \]

The convolution operator determined by \( T^y \) is as follows:

\[ (f * \varphi)(y) = \int_{\mathbb{R}_n^+} f(y) T^y \varphi(x) \left( \prod_{i=1}^{n} x_i^{2v_i} \right) dy. \]

Convolution (3) is known as a \( B \)-convolution. We note the following properties for the \( B \)-convolution and the generalized shift operator:

(a) \( T^y \cdot 1 = 1 \).
(b) \( T^0 \cdot f(x) = f(x) \).
(c) If \( f(x), g(x) \in C(\mathbb{R}_n^+) \), \( g(x) \) is a bounded function, \( x > 0 \) and

\[ \int_0^\infty |f(x)| \left( \prod_{i=1}^{n} x_i^{2v_i} \right) dx < \infty, \]
then
\[ \int_{\mathbb{R}^n_+} T^y_x f(x) g(y) \left( \prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}^n_+} f(y) T^y_x g(x) \left( \prod_{i=1}^n y_i^{2v_i} \right) dy. \]

(d) From (c), we have the following equality for \( g(x) = 1 \).
\[ \int_{\mathbb{R}^n_+} T^y_x f(x) \left( \prod_{i=1}^n y_i^{2v_i} \right) dy = \int_{\mathbb{R}^n_+} f(y) \left( \prod_{i=1}^n y_i^{2v_i} \right) dy. \]

(e) \((f * g)(x) = (g * f)(x)\).

The Fourier–Bessel transformation and its inverse transformation are defined as follows \([9]\):
\[ (F_B f)(x) = C_v \int_{\mathbb{R}^n_+} f(y) \left( \prod_{i=1}^n j_{v_i - \frac{1}{2}} (x_i, y_i) y_i^{2v_i} \right) dy, \]
\[ (F_B^{-1} f)(x) = (F_B f)(-x), \quad C_v = \left( \prod_{i=1}^n 2^{v_i - \frac{1}{2}} \left( v_i + \frac{1}{2} \right) \right)^{-1}, \]
where \( j_{v_i - (1/2)} (x_i, y_i) \) is the normalized Bessel function which is the eigenfunction of the Bessel differential operator. The following are the equalities for Fourier–Bessel transformation \([6,7]\):
\[ F_B \delta(x) = 1 \]
\[ F_B(f * g)(x) = F_B f(x) \cdot F_B g(x). \] (4)

Now we are finding the solution of the equation
\[ \Delta^{k_0}_B u(x) = \sum_{r=0}^m \Delta^{k_0}_B \delta, \quad \Delta^{k_0}_B \delta = \delta \]

or
\[ \Delta^{k_0}_B \Delta_B^k u(x) = \sum_{r=0}^m \Delta^{k_0}_B \Delta_B^k \delta. \] (5)

In finding the solutions of (5), we use the properties of \( B \)-convolutions for the generalized functions.

**Lemma 1.** There is the following equality for Fourier–Bessel transformation
\[ F_B(|x|^{-\alpha}) = 2^{n+2|v|-2\alpha} \Gamma \left( \frac{n+2}{2} \frac{|v|-\alpha}{2} \right) \Gamma \left( \frac{\alpha}{2} \right)^{-1} \left| x \right|^{\alpha-n-2|v|}, \]
where \(|v| = v_1 + \cdots + v_n\).

The proof of this lemma is given in \([7]\).
Lemma 2. Given the equation $\Delta_B E(x) = \delta$ for $x \in \mathbb{R}_{+}^{n}$, where $\Delta_B$ is the Laplace–Bessel operator defined by (1),

$$E(x) = -S_2(x)$$

is an elementary solution of the operator $\Delta_B$ where

$$S_2(x) = \frac{2^{n+2|\nu| - 4} \Gamma \left(\frac{n+2|\nu| - 2}{2}\right)}{\prod_{i=1}^{n} 2^{\nu_i} \Gamma (\nu_i + \frac{1}{2})} |x|^{2-n-2|\nu|}.$$  

Proof. For the equation $\Delta_B E(x) = \delta$, we have

$$F_B \Delta_B E = F_B \delta.$$  

First, we consider the left side of (7)

$$F_B \Delta_B E(x) = C_v \int_{\mathbb{R}_{+}^{d}} (\Delta_B E(y)) \left(\prod_{i=1}^{n} I_{\nu_i-\frac{1}{2}}(x_i, y_i) y_i^{2\nu_i}\right) dy$$

$$= C_v \int_{\mathbb{R}_{+}^{d}} \left(\sum_{i=1}^{n} \frac{\partial^2 E(y)}{\partial y_i^2}\right) \left(\prod_{i=1}^{n} I_{\nu_i-\frac{1}{2}}(x_i, y_i) y_i^{2\nu_i}\right) dy$$

$$+ C_v \int_{\mathbb{R}_{+}^{d}} \left(\sum_{i=1}^{n} 2\nu_i \frac{\partial E(y)}{\partial y_i}\right) \left(\prod_{i=1}^{n} I_{\nu_i-\frac{1}{2}}(x_i, y_i) y_i^{2\nu_i}\right) dy.$$  

If we apply partial integration twice in the first integral and once in the second integral, then we have

$$F_B \Delta_B E(x) = C_v \int_{\mathbb{R}_{+}^{d}} E(y) \left(\sum_{i=1}^{n} \frac{\partial^2}{\partial y_i^2} \prod_{i=1}^{n} I_{\nu_i-\frac{1}{2}}(x_i, y_i)\right) y_i^{2\nu_i} dy$$

$$+ C_v \int_{\mathbb{R}_{+}^{d}} E(y) \left(\sum_{i=1}^{n} 2\nu_i \frac{\partial}{\partial y_i} \prod_{i=1}^{n} I_{\nu_i-\frac{1}{2}}(x_i, y_i)\right) y_i^{2\nu_i} dy$$

$$= C_v \int_{\mathbb{R}_{+}^{d}} E(y) \left(\sum_{i=1}^{n} B_{\nu_i} \prod_{i=1}^{n} I_{\nu_i-\frac{1}{2}}(x_i, y_i)\right) y_i^{2\nu_i} dy.$$  

Here, if we use the following equality \[8\],

$$\int_{0}^{\infty} E(y) B_{\nu_i} I_{\nu_i-\frac{1}{2}}(x_i, y) y_i^{2\nu_i} dy_i = -x_i^2 \int_{0}^{\infty} E(y) I_{\nu_i-\frac{1}{2}}(x_i, y) y_i^{2\nu_i} dy_i,$$  

then we have

$$F_B \Delta_B E(x) = -(x_1^2 + x_2^2 + \cdots + x_n^2) C_v$$

$$\times \int_{\mathbb{R}_{+}^{d}} E(y) \left(\prod_{i=1}^{n} I_{\nu_i-\frac{1}{2}}(x_i, y_i) y_i^{2\nu_i}\right) dy$$

$$= -|x|^2 F_B E(x).$$
The right side of equality (7) is $F_B \delta = 1$. Then
$$F_B \Delta_B E = - |x|^2 F_B E = 1.$$From Lemma 1 and inverse Fourier–Bessel transformation we obtain
$$E(x) = - \frac{2^{n+2|\nu| - 4} \Gamma \left( \frac{n+2|\nu|-2}{2} \right)}{\prod_{i=1}^{n} 2^{\nu_i - \frac{1}{2}} \Gamma \left( \nu_i + \frac{1}{2} \right)} |x|^{2-n-2|\nu|}. \quad (8)$$That completes the proof. □

**Lemma 3.** Given the equation $\Delta_B^k u(x) = \delta$ for $x \in \mathbb{R}^n_+$, where $\Delta_B^k$ is the Laplace–Bessel operator iterated $k$ times defined by
$$\Delta_B^k = (B_{x_1} + B_{x_2} + \cdots + B_{x_n})^k,$$then $u(x) = (-1)^k S_{2k}(x)$ is an elementary solution of the operator $\Delta_B^k$ where
$$S_{2k}(x) = \frac{2^{n+2|\nu| - 4k} \Gamma \left( \frac{n+2|\nu|-2k}{2} \right)}{\prod_{i=1}^{n} 2^{\nu_i - \frac{1}{2}} \Gamma \left( \nu_i + \frac{1}{2} \right)} |x|^{2k-n-2|\nu|}. \quad (9)$$The proof of Lemma 3 is similar to the proof of Lemma 2.

**Lemma 4.** If $\Box_B^k u(x) = \delta$ for $x \in \Gamma_+ = \{ x \in \mathbb{R}^n_+ ; x_1 > 0, x_2 > 0, \ldots, x_n > 0 \text{ and } V > 0 \}$, where $\Box_B^k$ is the Bessel-ultra hyperbolic operator iterated $k$ times defined by
$$\Box_B^k = (B_{x_1} + B_{x_2} + \cdots + B_{x_n} - B_{x_{p+1}} - \cdots - B_{x_{p+q}})^k, \quad p + q = n,$$then $u(x) = R_{2k}(x)$ is the unique elementary solution of the operator $\Box_B^k$ where
$$R_{2k}(x) = \frac{V \left( \frac{2k-n-2|\nu|}{2} \right)}{K_n(2k)}$$
$$= \frac{(x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^{\left( \frac{2k-n-2|\nu|}{2} \right)}}{K_n(2k)} \quad (10)$$for
$$K_n(2k) = \frac{\pi^{\frac{n+2|\nu|-1}{2}} \Gamma \left( \frac{2+2k-n-2|\nu|}{2} \right) \Gamma \left( \frac{1-2k}{2} \right) \Gamma(2k)}{\Gamma \left( \frac{2+2k-2|\nu|}{2} \right) \Gamma \left( \frac{p-2k}{2} \right)}.$$
The proof of this lemma can be from Lemmas 1–3.

**Lemma 5.** $R_{2k}(x)$ and $S_{2k}(x)$ are homogeneous distributions of order $(2k - n - 2 |\nu|)$.

**Proof.** We need to show that $R_{2k}(x)$ satisfies the Euler equation
$$(2k - n - 2 |\nu|) R_{2k}(x) = \sum_{i=1}^{n} x_i \frac{\partial R_{2k}(x)}{\partial x_i}.$$
Now
\[
\sum_{i=1}^{n} x_i \frac{\partial R_{2k}(x)}{\partial x_i} = \frac{1}{K_n(2k)} \sum_{i=1}^{n} x_i \frac{\partial}{\partial x_i} \left( x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \right) \left( \frac{2k-n-2|v|}{2} \right)
\]
\[
= \frac{(2k-n-2|v|)}{K_n(2k)} \left( x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \right) \left( \frac{2k-n-2|v|}{2} \right)
\]
\[
\times (x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)
\]
\[
= \frac{(2k-n-2|v|)}{K_n(2k)} \left( x_1^2 + x_2^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2 \right) \left( \frac{2k-n-2|v|}{2} \right)
\]
\[
= (2k-n-2|v|) R_{2k}(x).
\]
Hence \( R_{2k}(x) \) is a homogeneous distribution of order \((2k-n-2|v|)\) as required and similarly \( S_{2k}(x) \) is also a homogeneous distribution of order \((2k-n-2|v|)\). \(\square\)

**Lemma 6.** \( R_{2k}(x) \) and \( S_{2k}(x) \) are the tempered distributions.

**Proof.** Choose \( \text{supp} \ R_{2k} = K \subset \Gamma_+ \), where \( K \) is a compact set. Then \( R_{2k} \) is a tempered distribution with compact support and by \(\Pi\), pp. 156–159, \( S_{2k}(x) * R_{2k}(x) \) exists and is a tempered distribution. \(\square\)

**Lemma 7.** (The \( B \)-convolutions of tempered distributions)

\[
S_{2k}(x) * S_{2m}(x) = S_{2k+2m}(x).
\] (11)

**Proof.** From Lemmas 1 and 2 we have

\[
F_B S_{2k}(x) = - |x|^{-2k} \quad \text{and} \quad F_B S_{2m}(x) = - |x|^{-2m}.
\]

Thus from (4) we obtain

\[
F_B (S_{2k}(x) * S_{2m}(x)) = F_B S_{2k}(x) F_B S_{2m}(x)
\]
\[
= |x|^{-2k-2m}
\]
\[
S_{2k}(x) * S_{2m}(x) = F_B^{-1} |x|^{-2k-2m}
\]
\[
= C(v, m, k, n) |x|^{-2k-2m-n-2|v|}
\]
\[
= S_{2k+2m}(x),
\]

where

\[
C(v, m, k, n) = \frac{2^{n+2|v|} \Gamma \left( \frac{n+2|v|-2(m+k)}{2} \right)}{\prod_{i=1}^{n} 2^{v_i+\frac{3}{2}} \Gamma \left( v_i + \frac{1}{2} \right)}.
\]
Now from (6) and (11) with \( m = k = 1 \), we have
\[
E(x) \ast E(x) = (-S_2(x)) \ast (-S_2(x)) = (-1)^2 S_{2+2}(x) = S_4(x).
\]
By induction, we obtain
\[
E(x) \ast E(x) \ast \cdots \ast E(x) = (-1)^k S_{2k}(x). \tag{12}
\]

\[\square\]

**Lemma 8.** Given the equation \( \Delta_k^B u(x) = \delta \), then \( u(x) = (-1)^k S_{2k}(x) \) is an elementary solution of the operator \( \Delta_k^B \) where, \((-1)^k S_{2k}(x)\) is defined by (12).

**Proof.** Now \( \Delta_k^B u(x) = \delta \) can be written in the form
\[
\Delta_k^B \delta \ast u(x) = \delta.
\]

\(B\)-convolving both sides by the function \( E(x) \) defined by (8), we obtain
\[
(E(x) \ast \Delta_k^B \delta) \ast u(x) = E(x) \ast \delta = E(x)
\]
and
\[
(\Delta_k^B E(x) \ast \Delta_k^{k-1} \delta) \ast u(x) = E(x).
\]
Since \( \Delta_k^B E(x) = \delta \) we have
\[
(\delta \ast \Delta_k^{k-1} \delta) \ast u(x) = E(x).
\]
Hence
\[
(\Delta_k^{k-1} \delta) \ast u(x) = E(x).
\]
By keeping on \(B\)-convolving \( E(x) \), \( k - 1 \) times, we obtain
\[
\delta \ast u(x) = E(x) \ast E(x) \ast \cdots \ast E(x). \tag{12}
\]
It follows that
\[
u(x) = (-1)^k S_{2k}(x)
\]
by (12) as required. \[\square\]
Before proving the theorems, we need to define the $B$-convolution of $(-1)^k S_{2k}(x)$ with $R_{2k}(x)$ defined by (10) with $k = 0, 1, 2, \ldots$. Now for the case $2k \geq n + 2 |v|$, we obtain $(-1)^k S_{2k}(x)$ and $R_{2k}(x)$ as analytic functions that are ordinary functions. Thus the $B$-convolution

$$(-1)^k S_{2k}(x) \ast R_{2k}(x)$$

exists. Now for the case $2k < n + 2 |v|$, by Lemma 6 we obtain $(-1)^k S_{2k}(x)$ and $R_{2k}(x)$ as tempered distributions.

Let $K$ be a compact set and $K \subset \Gamma_+$ where $\Gamma_+$ is defined closer to $\Gamma_+$. Choose the support of $R_{2k}(x)$ equal to $K$, then $\text{supp} R_{2k}(x)$ is compact (closed and bounded). So the $B$-convolution

$$(-1)^k S_{2k}(x) \ast R_{2k}(x)$$

exists and is a tempered distribution from Lemma 6.

**Theorem 1.** Given the equation $\bigtriangleup_B^n u(x) = \delta$ for $x \in \mathbb{R}^+_n$, where $\bigtriangleup_B^n$ is a diamond Bessel operator iterated $k$ times defined by (2) then $u(x) = (-1)^k S_{2k}(x) \ast R_{2k}(x)$, defined by (13) and (14), is a unique elementary solution of the operator $\bigtriangleup_B^n$.

**Proof.** Now $\bigtriangleup_B^n u(x) = \delta$ can be written as $\bigtriangleup_B^n u(x) = \bigtriangleup_B^k \Delta_B^k u(x)$ by (2). $\Delta_B^k u(x) = R_{2k}(x)$ is a unique elementary solution of the operator $\bigtriangleup_B^n$ for $n$ odd with $p$ odd and $q$ even, or for $n$ even with $p$ odd and $q$ odd. By the method of $B$-convolution, we have

$$\Delta_B^k \delta \ast u(x) = R_{2k}(x).$$

$B$-convolving both sides by $(-1)^k S_{2k}(x)$, we obtain

$$((-1)^k S_{2k}(x) \ast \Delta_B^k \delta) \ast u(x) = (-1)^k S_{2k}(x) \ast R_{2k}(x)$$

or

$$\Delta_B^k((-1)^k S_{2k}(x)) \ast u(x) = (-1)^k S_{2k}(x) \ast R_{2k}(x).$$

It follows that $u(x) = (-1)^k S_{2k}(x) \ast R_{2k}(x)$ by Lemma 8. That completes the proof. $\square$

**Theorem 2.** For $0 < r < k$,

$$\bigtriangleup_B^r((-1)^k S_{2k}(x) \ast R_{2k}(x)) = (-1)^k S_{2k-2r}(x) \ast R_{2k-2r}(x)$$

and for $k \leq m$

$$\bigtriangleup_B^m((-1)^k S_{2k}(x) \ast R_{2k}(x)) = \bigtriangleup_B^{m-k} \delta.$$

**Proof.** From Theorem 1,

$$\bigtriangleup_B^r((-1)^k S_{2k}(x) \ast R_{2k}(x)) = \delta.$$

Thus

$$\bigtriangleup_B^{k-r} \bigtriangleup_B^r((-1)^k S_{2k}(x) \ast R_{2k}(x)) = \delta.$$
or
\[ \Diamond_h^{k-r} \delta * \Diamond_h^l((-1)^k S_{2k}(x) * R_{2k}(x)) = \delta. \]

$B$-convolving both sides by $(-1)^{k-r} S_{2k-2r}(x) * R_{2k-2r}(x)$, we obtain
\[ \Diamond_h^{k-r}[((-1)^{k-r} S_{2k-2r}(x) * R_{2k-2r}(x)) * \Diamond_h^l((-1)^k S_{2k}(x) * R_{2k}(x))] \]
\[ = [(-1)^{k-r} S_{2k-2r}(x) * R_{2k-2r}(x)] * \delta, \]
or by Theorem 1
\[ \delta * \Diamond_h^l((-1)^k S_{2k}(x) * R_{2k}(x)) = [(-1)^{k-r} S_{2k-2r}(x) * R_{2k-2r}(x)]. \]

It follows that for $0 < r < k$,
\[ \Diamond_h^l((-1)^k S_{2k}(x) * R_{2k}(x)) = (-1)^{k-r} S_{2k-2r}(x) * R_{2k-2r}(x) \]
as required. For $k \leq m$,
\[ \Diamond_h^m((-1)^k S_{2k}(x) * R_{2k}(x)) = \Diamond_h^{m-k} \Diamond_h^k((-1)^k S_{2k}(x) * R_{2k}(x)) \]
\[ = \Diamond_h^{m-k} \delta \]
by Theorem 1. That completes the proofs. \hfill \square

**Theorem 3.** Given the linear differential equation
\[ \Diamond_h^l u(x) = \sum_{r=0}^m c_r \Diamond_h^{r} \delta, \] \hspace{1cm} (15)
where the operator $\Diamond_h^l$ is defined by (2), $n$ is odd with $p$ odd and $q$ even, or $n$ is even with $p$ odd and $q$ odd, $c_r$ is a constant, $\delta$ is the Dirac-delta distribution and $\Diamond_h^{0} \delta = \delta$.

Then the solutions of (15) that depend on the relationship between the values of $k$ and $m$ are as follows:

1. If $m < k$ and $m = 0$, then (15) has the solution $u(x) = c_0 (-1)^k S_{2k}(x) * R_{2k}(x)$, which is an elementary solution of the operator $\Diamond_h^l$ in Theorem 1 and is the ordinary function for $2k < n + 2|v|$.

2. If $0 < m < k$, then the solution of (15) is
\[ u(x) = \sum_{r=1}^m [(-1)^{k-r} S_{2k-2r}(x) * R_{2k-2r}(x)] \]
which is an ordinary function for $2k - 2r \geq n + 2|v|$, and a tempered distribution for $2k - 2r < n + 2|v|$.

3. If $m \geq k$, and suppose $k \leq n + 2|v| \leq M$, then (15) has the solution
\[ u(x) = \sum_{r=k}^M c_r \Diamond_h^{r-k} \]
which is only the singular distribution.
Proof of this theorem can be easily seen from Theorems 1, 2 and [4].

Lemma 9. (The Fourier–Bessel transformation $\diamond^k_B \delta$). Let $\|x\| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$ for $x \in \mathbb{R}^n_+$. Then

$$|F_B \diamond^k_B \delta| \leq C_v \|x\|^{2k}.$$  \hspace{1cm} (16)

That is, $F_B \Delta^k_B \delta$ is bounded and continuous on the space $S'$ of the tempered distribution. Moreover, by the inverse Fourier–Bessel transformation

$$\diamond^k_B \delta = C_v F_B^{-1}[(x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k.$$

**Proof.** From the Fourier–Bessel transform we have

$$F_B \diamond^k_B \delta(x) = C_v \int_{\mathbb{R}^n_+} \diamond^k_B \delta(y) \left( \prod_{i=1}^n I_{\nu - \frac{k}{2}}(x_i, y_i) y_i^{2\nu} \right) \, dy$$

$$= C_v \int_{\mathbb{R}^n_+} \Delta^k_B \delta(y) \left( \prod_{i=1}^n I_{\nu - \frac{k}{2}}(x_i, y_i) y_i^{2\nu} \right) \, dy$$

$$= C_v \int_{\mathbb{R}^n_+} \Delta^k_B \delta(y) \left( \prod_{i=1}^n I_{\nu - \frac{k}{2}}(x_i, y_i) y_i^{2\nu} \right) \, dy,$$

where $g(y) = \Box^k_B \delta(y)$. For $k \in \mathbb{N}$ we have [10]

$$F_B(\Delta^k_B) f = (-1)^k |x|^{2k} F_B f.$$

So we have

$$F_B \diamond^k_B \delta(x) = C_v (-1)^k |x|^{2k} F_B g(x)$$

$$= C_v (-1)^k (x_1^2 + \cdots + x_n^2)^k F_B \Box^k_B \delta(x).$$

The same way we have following equality:

$$F_B \Box^k_B \delta(x) = C_v (-1)^k (x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^k F_B \delta(x).$$

Since $F_B \delta(x) = 1$, we can write

$$F_B \diamond^k_B \delta(x) = C_v (-1)^{2k} (x_1^2 + \cdots + x_n^2)^k$$

$$\times (x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2)^k$$

$$= C_v [(x_1^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k.$$ 

Then there is the following inequality:

$$|F_B \diamond^k_B \delta| = C_v [(x_1^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k$$

$$\leq C_v [(x_1^2 + \cdots + x_p^2)^2]^{2k}$$

$$= C_v \|x\|^{4k}.$$

Therefore $F_B \diamond^k_B \delta$ is bounded and continuous on the space $S'$ of the tempered distribution.
Since $F_B$ is 1-1 transformation from $S'\to \mathbb{R}_n^+$, there is the following equation:

$$\bigotimes B \delta = C_v F_B^{-1} [(x_1^2 + x_2^2 + \cdots + x_n^2) - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2]^k.$$ 

That completes the proof. \qed

**Theorem 4.**

$$F_B[(-1)^k S_{2k}(x) * R_{2k}(x)] \leq C_v \quad \text{for a large } x, \quad \text{where } M \text{ is a constant.}$$

That is, $F_B$ is bounded and continuous on the space $S'$ of the tempered distribution.

**Proof.** By Lemma 8,

$$\bigotimes B[(-1)^k S_{2k}(x) * R_{2k}(x)] = \delta$$

or

$$\bigotimes B \delta \ast [(-1)^k S_{2k}(x) * R_{2k}(x)] = \delta. \quad (17)$$

If we applied the Fourier–Bessel transform on both sides of (17), then we obtain

$$F_B[(\bigotimes B \delta) \ast [(-1)^k S_{2k}(x) * R_{2k}(x)]] = F_B \delta(x)$$

and

$$F_B[(-1)^k S_{2k}(x) * R_{2k}(x)] = C_v.$$ 

By the properties of $B$-convolution

$$C_v \left\langle (\bigotimes B \delta) \ast [(-1)^k S_{2k}(x) * R_{2k}(x)], \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} (x_i, y_i) y_i^{2v_i} \right\rangle = C_v,$$

$$C_v \left\langle [(-1)^k S_{2k}(x) * R_{2k}(x)], \prod_{i=1}^{n} J_{v_i - \frac{1}{2}} (z_i, y_i) y_i^{2v_i} \right\rangle = C_v,$$

$$F_B[(-1)^k S_{2k}(x) * R_{2k}(x)] \frac{1}{C_v} F_B(\bigotimes B \delta) = C_v.$$
By Lemma 9,
\[ F_B[(\mathcal{S}_B^k \mathcal{F}) \ast ([(-1)^k S_{2k}(x) \ast R_{2k}(x))] \times ([x_1^2 + x_2^2 + \cdots + x_p^2] - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2)^k = C_v. \]

It follows that
\[
F_B([-1]^k S_{2k}(x) \ast R_{2k}(x)] = \frac{C_v}{(x_1^2 + x_2^2 + \cdots + x_p^2)^k [\Gamma(\frac{k}{2} + \cdots + \frac{k+m}{2})]^k}.
\]

Now
\[
|F_B([-1]^k S_{2k}(x) \ast R_{2k}(x)]| = \frac{C_v}{(x_1^2 + x_2^2 + \cdots + x_p^2)^k [\Gamma(\frac{k}{2} + \cdots + \frac{k+m}{2})]^k},
\]

where \( x = (x_1, \ldots, x_n) \in \Gamma_+ \) with \( \Gamma_+ \) defined by Lemma 4. Then \((x_1^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2 > 0 \) and for a large \( x_1 \) and a large \( k \), the right-hand side of (18) tends to zero. It follows that it is bounded by a positive constant say \( M \), that is, we obtain (17) as required and also by (17). \( F_B \) is continuous on the space \( S' \) of the tempered distribution.

\[ \square \]

Theorem 5.

\[
F_B[[(-1)^k S_{2k}(x) \ast R_{2k}(x)] \ast [(-1)^m S_{2m}(x) \ast R_{2m}(x)]] = C_v (x_1^2 + x_2^2 + \cdots + x_p^2)^k [\Gamma(\frac{k}{2} + \cdots + \frac{k+m}{2})]^k.
\]

where \( k \) and \( m \) are non-negative integers and \( F_B \) is bounded and continuous on the space \( S' \) of the tempered distribution.

Proof. Since \( S_{2k}(x) \) and \( R_{2k}(x) \) are tempered distribution with compact support, from Lemmas 6 and 7 we have

\[
[(-1)^k S_{2k}(x) \ast R_{2k}(x)] \ast [(-1)^m S_{2m}(x) \ast R_{2m}(x)] = (-1)^{k+m} [S_{2k}(x) \ast S_{2m}(x)] \ast [R_{2k}(x) \ast R_{2m}(x)]
\]

Taking the Fourier transform on both sides and using Theorem 4 we obtain
\[
F_B([-1]^k S_{2k}(x) \ast R_{2k}(x)] \ast [(-1)^m S_{2m}(x) \ast R_{2m}(x)] = \frac{C_v}{(x_1^2 + x_2^2 + \cdots + x_p^2)^k [\Gamma(\frac{k}{2} + \cdots + \frac{k+m}{2})]^k}.
\]
Bessel diamond operator

\[
\begin{align*}
&= \frac{1}{C_v} \left[ (x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2 \right]^k \\
&\quad \times \frac{1}{C_v} \left[ (x_1^2 + x_2^2 + \cdots + x_p^2)^2 - (x_{p+1}^2 + \cdots + x_{p+q}^2)^2 \right]^m \\
&= \frac{1}{C_v} F_B \left[ (-1)^k S_{2k}(x) \ast R_{2k}(x) \right] F_B \left[ (-1)^m S_{2m}(x) \ast R_{2m}(x) \right].
\end{align*}
\]

Since \((-1)^{k+m} S_{2(k+m)}(x) \ast R_{2(k+m)}(x) \in S'\), the space of tempered distribution, and Theorem 4 we obtain that \(F_B\) is bounded and continuous on \(S'\).

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