Research Article

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**Approximation properties of Kantorovich type \( q \)-Balázs-Szabados operators**

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**Abstract:** In this paper, we introduce a new kind of \( q \)-Balázs-Szabados-Kantorovich operators called \( q \)-BSK operators. We give a weighted statistical approximation theorem and the rate of convergence of the \( q \)-BSK operators. Also, we investigate the local approximation results. Further, we give some comparisons associated with the convergence of \( q \)-BSK operators.

**Keywords:** Balázs-Szabados operators, \( q \)-calculus, rate of convergence, Peetre’s K-functional

**MSC:** 41A25, 41A35, 41A36

**1 Introduction and auxiliary results**

Approximation theory is an important area of research. Recently, several interesting studies have been conducted (see [1–7]). The statistical approximation properties of the some operators have also been recently investigated by several authors. For example, in [8] Meyer-König and Zeller operators based on \( q \)-integers; in [9] \( q \)-analogues of Bernstein-Kantrovich operators; in [10] \( q \)-Bleimann, Butzer and Hahn operators; in [11] \( q \)-Baskokov-Kantrovich operators; in [12] Kantrovich type \( q \)-Bernstein operators; in [13] \( q \)-Stancu-Beta operators; in [14] Kantorovich-type \( q \)-Bernstein-Stancu operators were defined and their statistical approximation properties were investigated.

Firstly, we recall some basic definitions used in \( q \)-calculus. Details can be found in [15–17]. For any non-negative integer \( r \), the \( q \)-integer of the number \( r \) is defined by

\[
[r]_q = \begin{cases} 
\frac{1-q^r}{1-q} & \text{if } q \neq 1; \\
\frac{1}{r} & \text{if } q = 1,
\end{cases}
\]

where \( q \) is a fixed positive real number. The \( q \)-factorial is defined by

\[
[r]_q! = \begin{cases} 
[1]_q [2]_q \cdots [r]_q & \text{if } r = 1, 2, \ldots; \\
1 & \text{if } r = 0.
\end{cases}
\]

For integers \( n, r \) with \( 0 \leq r \leq n \), the \( q \)-binomial coefficients are defined by

\[
\binom{n}{r}_q = \frac{[n]_q!}{[r]_q! [n-r]_q!}.
\]

For fixed \( 0 < q < 1 \), we denote the \( q \)-derivative of a function \( f \) with respect to \( x \)

\[
D_q f(x) = \begin{cases} 
\frac{f(qx)-f(x)}{(q-1)x} & \text{if } x \neq 0; \\
f'(0) & \text{if } x = 0,
\end{cases}
\]

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and we get

$$\lim_{q \to 1} D_q [f(x)] = f'(x).$$

The definite $q$-integral is defined by

$$\int_0^b f(t) \, dq_t = (1 - q) b \sum_{j=0}^{\infty} f(q^j b) q^j, \quad 0 < q < 1, b > 0,$$

and

$$\int_a^b f(t) \, dq_t = \int_0^b f(t) \, dq_t - \int_0^a f(t) \, dq_t, \quad 0 < a < b.$$

Bernstein-type rational functions are defined by Balázs [18]. Balázs and Szabados modified and studied the approximation properties of these operators [19]. The $q$-analogue of the Balázs-Szabados operators is defined by Doğru [20] as follows

$$R_n (f; q, x) = \frac{1}{n!} \sum_{j=0}^{\infty} q^{j(1-1/2)} \left( \frac{[n]_q}{b_n} \right) \left[ \begin{array}{c} n \\ j \end{array} \right]_q (a_n x)^j,$$  \hspace{1cm} (1.1)

where $x \in [0, \infty)$, $a_n = [n]_q^{\beta-1}$ and $b_n = [n]_q^\beta$ for all $n \in \mathbb{N}$ and $0 < \beta \leq \frac{3}{2}$. Also, Doğru gave the following equalities

$$R_n (e_0; q, x) = 1, \hspace{1cm} (1.2)$$

$$R_n (e_1; q, x) = \frac{x}{1 + a_n x}, \hspace{1cm} (1.3)$$

$$R_n (e_2; q, x) = \frac{[n-1]_q}{[n]_q} \frac{q^2 x^2}{(1 + a_n x)(1 + a_n qx)} + \frac{x}{b_n (1 + a_n x)}, \hspace{1cm} (1.4)$$

In (1.4), using the equality $[n]_q = q [n - 1]_q + 1$, we get

$$R_n (e_2; q, x) = \frac{1 - \frac{a_n}{b_n}}{(1 + a_n x)(1 + a_n qx)} + \frac{x}{b_n (1 + a_n x)}. \hspace{1cm} (1.5)$$

We will use (1.5) instead of (1.4) throughout this paper.

In [21], a kind of real and complex $q$-Balázs-Szabados-Kantorovich operators were defined, and it was given an upper estimate on compact disks.

Now, we give the following new kinds of $q$-Balázs-Szabados-Kantorovich operators:

**Definition 1.** A new kind of $q$-Balázs-Szabados-Kantorovich operators is defined as follows:

$$\tilde{R}_n (f; q, x) = \frac{b_n}{n \prod_{s=0}^{n-1} (1 + q^s a_n x)} \sum_{j=0}^{\infty} q^{j(1-1/2)} \left[ \begin{array}{c} n \\ j \end{array} \right]_q \left( \frac{a_n x}{b_n} \right)^j \int_0^b f(t) \, dq_t,$$  \hspace{1cm} (1.6)

where $f$ is a nondecreasing and continuous function on $[0, \infty)$, $a_n = [n]_q^{\beta-1}$ and $b_n = [n]_q^\beta$ for all $n \in \mathbb{N}$, $q \in (0, 1)$ and $0 < \beta \leq \frac{3}{2}$.

$q$-Balázs-Szabados-Kantorovich operators can be called as $q$-BSK operators for convenience. Since $f$ is nondecreasing and from the definition of $q$-integral, $q$-BSK operator is a positive operator. And also, $q$-BSK operator is linear, so $q$-BSK operator is a linear and positive operator.

We have the following lemma for the $q$-BSK operators:
Lemma 1. The following equalities hold for the $q$-BSK operators

\begin{align*}
\tilde{R}_n (e_0; q, x) &= 1, \\
\tilde{R}_n (e_1; q, x) &= \frac{2qx}{[2]_q (1 + a_n x)} + \frac{1}{[2]_q b_n}, \\
\tilde{R}_n (e_2; q, x) &= \frac{1 - \frac{a_n}{b_n}}{3} 3 q^3 x^2 + \frac{3q [2]_q x}{[3]_q b_n (1 + a_n x)} + \frac{1}{[3]_q b_n^2}.
\end{align*}

Proof. Using (1.2), (1.3), (1.5) and the equality $[n]_q = q [n-1]_q + 1$, we have

\begin{equation}
\int \frac{d_0 t}{\psi_{10}^n} = \frac{1}{b_n}, \tag{1.10}
\end{equation}

\begin{equation}
\int \frac{td_0 t}{\psi_{10}^n} = \frac{1}{[2]_q b_n^2} (2q [j]_q + 1), \tag{1.11}
\end{equation}

\begin{equation}
\int \frac{t^2 d_0 t}{\psi_{10}^n} = \frac{1}{[3]_q b_n^2} (3q [j]_q^2 + 3q [j]_q + 1). \tag{1.12}
\end{equation}

Using (1.2) and (1.10), we get

\begin{equation*}
\tilde{R}_n (e_0; q, x) = 1.
\end{equation*}

Similarly, using (1.3) and (1.11), we obtain

\begin{equation*}
\tilde{R}_n (e_1; q, x) = \frac{1}{[2]_q b_n} R_n (e_1; q, x) + \frac{2q}{[2]_q} R_n (e_0; q, x)
\end{equation*}

\begin{equation*}
= \frac{2qx}{1 + a_n x} + \frac{1}{[2]_q b_n}.
\end{equation*}

Finally, from (1.2), (1.3), (1.5) and (1.12), we find that

\begin{equation*}
\tilde{R}_n (e_2; q, x) = \frac{3q^2}{[3]_q} R_n (e_2; q, x) + \frac{3q}{[3]_q b_n} R_n (e_1; q, x) + \frac{1}{[3]_q b_n^2} R_n (e_0; q, x)
\end{equation*}

\begin{equation*}
= \frac{1 - \frac{a_n}{b_n}}{3} 3 q^3 x^2 + \frac{3q [2]_q x}{[3]_q b_n (1 + a_n x)} + \frac{1}{[3]_q b_n^2}.
\end{equation*}

Lemma 2. The following equalities hold for the $q$-BSK operators

\begin{align*}
\tilde{R}_n ((e_1 - x); q, x) &= - \frac{a_n x^2}{1 + a_n x} + \left( \frac{1 - q}{[2]_q (1 + a_n x)} + \frac{1}{[2]_q b_n} \right), \\
\tilde{R}_n ((e_2 - x^2); q, x) &= - \frac{a_n^2 q x^4 + a_n [2]_q x^3}{(1 + a_n x) (1 + a_n x)} + \frac{3q [3]_q - 3q^3 \frac{a_n}{b_n}}{[3]_q (1 + a_n x) (1 + a_n x)} + \frac{3q [2]_q x}{[3]_q b_n (1 + a_n x)} + \frac{1}{[3]_q b_n^2}.
\end{align*}
\[ R_n \left( (e_1 - x)^2 ; q, x \right) = \frac{a_n^2 q x^4}{(1 + a_n x)(1 + a_n qx)} + \frac{a_n (1 - q) (2q + [2]_q) x^3}{[2]_q (1 + a_n x)(1 + a_n qx)} - \frac{(1 - q) 3q^2 x^2}{[3]_q (1 + a_n x)(1 + a_n qx)} \]

\[ + \frac{1}{[2]_q [3]_q (1 + a_n x)(1 + a_n qx)} \cdot 3q [2]_q x + \frac{2x}{[2]_q b_n} + \frac{1}{[3]_q b_n^2}. \quad (1.15) \]

**Proof.** Using Lemma 2, after a simple calculation, the proof can be obtained easily, so we omit it. \(\square\)

## 2 Weighted statistical approximation properties

The concept of the statistical convergence was introduced by H. Fast [22]. We recall some definitions about the statistical convergence. The density of a set \( K \subset \mathbb{N} \) is defined by

\[ \delta \{ k \leq n : k \in K \} \]

The natural density, \( \delta \), of a set \( K \subset \mathbb{N} \) is defined by

\[ \lim_{n \to \infty} \frac{1}{n} \left| \{ k \leq n : k \in K \} \right|, \]

provided the limits exist [23]. A sequence \( x = (x_k) \) is called statistically convergent to a number \( L \) if

\[ \delta \{ k : |x_k - L| \geq \varepsilon \} = 0 \]

for every \( \varepsilon > 0 \) and it is denoted as \( st - \lim x_k = L \). Any convergent sequence is statistically convergent but not conversely (see [20]). A real function \( \rho \) is called a weight function if it is continuous on \( \mathbb{R} \) and \( \lim_{|x| \to \infty} \rho (x) = \infty \), \( \rho (x) \geq 1 \) for all \( x \in \mathbb{R} \).

Let denote by \( B_{\rho_0} (\mathbb{R}_+ \) the weighted space of the real valued functions \( f \) defined on \( \mathbb{R}_+ \) satisfying \( |f(x)| \leq Mf \rho_0 (x) \) for all \( x \in \mathbb{R}_+ \), where \( \mathbb{R}_+ = [0, \infty) \), \( \rho_0 (x) = 1 + x^2 \) and \( Mf \) is a constant depending on the function \( f \). We also denote with \( C_{\rho_0} (\mathbb{R}_+) := \{ f \in B_{\rho_0} (\mathbb{R}_+) : f \text{ continuous on } \mathbb{R}_+ \} \) the weighted subspace of \( B_{\rho_0} (\mathbb{R}_+) \).

\( B_{\rho_0} (\mathbb{R}_+) \) and \( C_{\rho_0} (\mathbb{R}_+) \) are Banach spaces with \( \|f\|_{\rho_0} = \sup_{x \in \mathbb{R}_+} \frac{|f(x)|}{\rho_0(x)} \).

Let \( q = (q_n) \) a sequence satisfying

\[ st - \lim_n q_n = 1 \quad \text{and} \quad st - \lim_n q_n^2 = c, \quad (0 \leq c < 1). \quad (2.1) \]

Now we are ready to prove the following convergence theorems for the \( q \)-BSK operators:

**Theorem 1.** Let \( q = (q_n) \) be a sequence satisfying the conditions given as in (2.1). If \( f \) is a nondecreasing function in \( C_{\rho_0} (\mathbb{R}_+) \), then it holds for the \( q \)-BSK operators

\[ st - \lim_n \| \hat{R}_n (f; q_n, \cdot) - f \|_{\rho_0} = 0. \]

**Proof.** From (1.7) in Lemma 1, it is clear that

\[ st - \lim_n \| \hat{R}_n (e_0; q_n, \cdot) - e_0 \|_{\rho_0} = 0. \tag{2.2} \]

Using (1.13) in Lemma 2, we get

\[ \| \hat{R}_n (e_1; q_n, \cdot) - e_1 \|_{\rho_0} \leq a_n \| e_2 \|_{\rho_0} + \frac{1 - q_n}{[2]_q q_n} \| e_1 \|_{\rho_0} + \frac{1}{[3]_q q_n b_n}. \]

For a given \( \varepsilon > 0 \), let us define the following sets:

\[ A := \left\{ k : \| \hat{R}_k (e_1; q_k, \cdot) - e_1 \|_{\rho_0} \geq \varepsilon \right\}, \]

\[ A_1 := \left\{ k : a_k \geq \frac{\varepsilon}{3} \right\}, \]

\[ A_2 := \left\{ k : \zeta_k \geq \frac{\varepsilon}{3} \right\}, \]

\[ A_3 := \left\{ k : \gamma_k \geq \frac{\varepsilon}{3} \right\}. \]
with \( a_k = a_k \| e_2 \|_{\rho_0} \), \( \zeta_k = \frac{1-q_k}{[1+q_k]} \), and \( \gamma_k = \frac{1}{[q_k]^{1/4}} \). Since \( A \subseteq A_1 \cup A_2 \cup A_3 \), we get

\[
\delta \left\{ k \leq n : \| \tilde{R}_n (e_1; q_n, \cdot) - e_1 \|_{\rho_0} \geq \varepsilon \right\} \leq \delta \left\{ k \leq n : a_k \geq \frac{\varepsilon}{3} \right\} + \delta \left\{ k \leq n : \zeta_k \geq \frac{\varepsilon}{3} \right\} + \delta \left\{ k \leq n : \gamma_k \geq \frac{\varepsilon}{3} \right\}.
\]

Under the conditions given in (2.1), it is clear that

\[
st - \lim_n \alpha_n = s - \lim_n \zeta_n = s - \lim_n \gamma_n = 0,
\]

which implies

\[
st - \lim_n \| \tilde{R}_n (e_1; q_n, \cdot) - e_1 \|_{\rho_0} = 0
\]
Using (1.14) in Lemma 2, we can write

\[
\| \tilde{R}_n (e_2; q_n, \cdot) - e_2 \|_{\rho_0} \leq a_k^2 q_n \| e_2 \|_{\rho_0} + a_n [2]_{q_0} \| e_3 \|_{\rho_0} + \frac{3 q_n^3}{\| e_2 \|_{\rho_0} + \frac{3 q_n}{[2]_{q_0}} \| e_1 \|_{\rho_0}} + \frac{1}{B_n [3]_{q_0}} \| e_0 \|_{\rho_0}.
\]

Again for a given \( \varepsilon > 0 \), let us define the following sets:

\[
B : = \left\{ k : \| \tilde{R}_k (e_2; q_k, \cdot) - e_2 \|_{\rho_0} \geq \varepsilon \right\},
B_1 : = \left\{ k : v_k \geq \frac{\varepsilon}{5} \right\}, \quad B_2 : = \left\{ k : \varphi_k \geq \frac{\varepsilon}{5} \right\},
B_3 : = \left\{ k : \eta_k \geq \frac{\varepsilon}{5} \right\}, \quad B_4 : = \left\{ k : \mu_k \geq \frac{\varepsilon}{5} \right\},
B_5 : = \left\{ k : \sigma_k \geq \frac{\varepsilon}{5} \right\}
\]

by choosing \( v_k = a_k^2 q_k \| e_2 \|_{\rho_0} \), \( \varphi_k = a_n [2]_{q_0} \| e_3 \|_{\rho_0} \), \( \eta_k = \frac{3 q_n [3]_{q_0} - 3 q_n^3}{[3]_{q_0}} \| e_2 \|_{\rho_0} \), \( \mu_k = \frac{3 q_n}{[2]_{q_0}} \| e_1 \|_{\rho_0} \) and \( \sigma_k = \frac{1}{[q_k]^{1/4}} \| e_0 \|_{\rho_0} \). Since \( B \subseteq B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_5 \), we have

\[
\delta \left\{ k \leq n : \| \tilde{R}_k (e_2; q_k, \cdot) - e_2 \|_{\rho_0} \geq \varepsilon \right\} \leq \delta \left\{ k \leq n : v_k \geq \frac{\varepsilon}{5} \right\} + \delta \left\{ k \leq n : \varphi_k \geq \frac{\varepsilon}{5} \right\} + \delta \left\{ k \leq n : \eta_k \geq \frac{\varepsilon}{5} \right\} + \delta \left\{ k \leq n : \mu_k \geq \frac{\varepsilon}{5} \right\} + \delta \left\{ k \leq n : \sigma_k \geq \frac{\varepsilon}{5} \right\}.
\]

Under the conditions given in (2.1), we have

\[
st - \lim_n v_n = s - \lim_n \varphi_n = 0,
\]

\[
st - \lim_n \eta_n = 0,
\]

\[
st - \lim n \mu_n = st - \lim_n \sigma_n = 0.
\]

From (2.4), we obtain

\[
st - \lim_n \| \tilde{R}_n (e_2; q_n, \cdot) - e_2 \|_{\rho_0} = 0.
\]

Since

\[
\| \tilde{R}_n (f; q_n, \cdot) - f \|_{\rho_0} \leq \| \tilde{R}_n (e_0; q_n, \cdot) - e_0 \|_{\rho_0} + \| \tilde{R}_n (e_1; q_n, \cdot) - e_1 \|_{\rho_0} + \| \tilde{R}_n (e_2; q_n, \cdot) - e_2 \|_{\rho_0},
\]

from (2.2), (2.3) and (2.5), the proof of theorem is completed.

In this part, we give the rates of convergence of the \( q \)-BSK operators by means of the weighted modulus of smoothness.
The weighted modulus of smoothness for the functions $f$ in $B_{p_0}(\mathbb{R}_+)$ is defined as
\[
\Omega_{p_0}(f; \delta) = \sup_{x \in \mathbb{R}_+, |h| < \delta} \frac{|f(x + h) - f(x)|}{1 + (x + h)^2}
\] (2.6)
for $\delta > 0$. It is clear that for each $f$ in $B_{p_0}(\mathbb{R}_+)$, $\Omega_{p_0}(f; \cdot)$ is well-defined and satisfies the following properties (see [24])
\[
\begin{align*}
\Omega_{p_0}(f; \lambda \delta) & \leq (\lambda + 1) \Omega_{p_0}(f; \delta), \quad \delta > 0, \\
\Omega_{p_0}(f; n \delta) & \leq n \Omega_{p_0}(f; \delta), \quad \delta > 0, \quad n \in \mathbb{N}, \\
\Omega_{p_0}(f; \delta) & \leq 2 \|f\|_{p_0}, \quad \delta > 0, \quad f \in B_{p_0}(\mathbb{R}_+), \\
\lim_{\delta \to 0^+} \Omega_{p_0}(f; \delta) & = 0,
\end{align*}
\] (2.7)

We give the following rate of convergence for the $q$-BSK operators.

**Theorem 2.** Let $q = (q_n)$ be a sequence satisfying the conditions given as in (2.1). For all nondecreasing functions $f$ in $B_{p_0}(\mathbb{R}_+)$, we have
\[
|\bar{R}_n(f; q_n, x) - f| \leq 2 \sqrt{\bar{R}_n\left(\Omega_{p_0}^2\left(\kappa_x^2(t); q_n, x\right)\right) \Omega_{p_0}(f; \mu_n(x))},
\]
for $x \in \mathbb{R}_+, \delta > 0, n \in \mathbb{N}$, where $\kappa_x(t) := 1 + (x + |t - x|)^2$ for $t \geq 0$ and $\mu_n(x) = \left(\bar{R}_n\left((\theta_1 - x)^2; q_n, x\right)\right)^{1/2}$.

**Proof.** Let $n \in \mathbb{N}$ and $f \in B_{p_0}(\mathbb{R}_+)$. From (2.6) and (2.7), we can write
\[
|f(t) - f(x)| \leq \left(1 + (x + |t - x|)^2\right)^{1/2} \Omega_{p_0}(f; \delta),
\]
\[
= \kappa_x(t) \left(1 + \frac{1}{\delta} |t - x|\right) \Omega_{p_0}(f; \delta).
\] (2.8)

From the linearity and positivity of the $q$-BSK operators and using Cauchy-Schwarz inequality, we obtain
\[
|\bar{R}_n(f; q_n, \cdot) - f| \leq \bar{R}_n(f(t) - f(x); q_n, x)
\leq \left\{\bar{R}_n(\kappa_x(t); q_n, x) \frac{1}{2} \bar{R}_n(|t - x|; q_n, x)\right\} \Omega_{p_0}(f; \delta)
\leq \sqrt{\bar{R}_n\left(\kappa_x^2(t); q_n, x\right) \left(1 + \frac{1}{\delta} \sqrt{\bar{R}_n\left((t - x)^2; q_n, x\right)}\right) \Omega_{p_0}(f; \delta)}.
\]
Finally, choosing $\delta = \mu_n(x)$, the proof is completed. \qed

### 3 Local approximation

Let $C_B(\mathbb{R}_+)$ be the space of all real valued continuous bounded functions defined on $\mathbb{R}_+$. The norm on the space $C_B(\mathbb{R}_+)$ is the supremum norm $\|f\| = \sup \{|f(x)| : x \in \mathbb{R}_+\}$. Also, Peetre’s $K$-functional is defined
\[
K_2(f, \delta) = \inf_{g \in W^2} \left\{\|f - g\| + \delta \|g^\prime\|\right\},
\]
where $W^2 = \{g \in C_B(\mathbb{R}_+) : g', g'' \in C_B(\mathbb{R}_+)\}$.

By [25] (p.117), there exists a positive constant $C > 0$ such that
\[
K_2(f, \delta) \leq C \omega_2\left(f, \delta^{1/2}\right), \quad \delta > 0,
\]
where
\[
\omega_2\left(f, \delta^{1/2}\right) = \sup \{|f(x + 2h) - 2f(x + h) + f(x)| : x \in \mathbb{R}_+, 0 < |h| < \delta^{1/2}\}
\]
is the second order modulus of continuity of functions $f$ in $C_B[0, \infty)$. Further, the usual modulus of continuity is defined by
\[
\omega\left(f, \delta^{1/2}\right) = \sup \{|f(x + h) - f(x)| : x \in \mathbb{R}_+, \quad 0 < |h| < \delta^{1/2}\}.
\]
Now, we give local results for the $q$-BSK operators.
Theorem 3. Let \( q = (q_n) \) be a sequence satisfying the conditions given as in (2.1) and \( f \in C_B(\mathbb{R}+) \). Then for all \( n \in \mathbb{N} \), there exists a positive constant \( C > 0 \) such that

\[
|\tilde{R}_n (f; q_n, x) - f(x)| \leq C \omega_2 \left( f, \sqrt{\delta_n(x)} \right) + \omega(f, \alpha_n(x)),
\]

where \( \delta_n(x) = \tilde{R}_n ((t-x)^2; q_n, x) + (\tilde{R}_n ((t-x); q_n, x))^2, \alpha_n(x) = |\tilde{R}_n ((t-x); q_n, x)|, \) where \( \tilde{R}_n ((t-x); q_n, x) \) and \( \tilde{R}_n ((t-x)^2; q_n, x) \) are given as in and (1.13) and (1.15).

Proof. For \( x \in \mathbb{R}^+ \), we introduce the auxiliary operator as follows:

\[
\tilde{R}_n (f; q_n, x) = \tilde{R}_n (f; q_n, x) + f(x) - f(\xi_n(x)),
\]

where \( \xi_n(x) = x - \frac{q_n}{1+q_n} + \frac{(1-q_n)x}{(2\ln(1+q_n)x)} + \frac{1}{(2\ln(q_n))^2} \). Using (3.1), we have

\[
\tilde{R}_n ((t-x); q_n, x) = \tilde{R}_n ((t-x); q_n, x) + f(x) - f(\xi_n(x)) = 0.
\]

Let \( x \in \mathbb{R}^+ \) and \( g \in W^2 \). Using the Taylor formula

\[
g(t) = g(x) + g'(x)(t-x) + \int_x^t (t-u)g''(u)du.
\]

Applying \( \tilde{R}_n \) to both sides of the above equation, we obtain

\[
\tilde{R}_n (g(t); q_n, x) - g(x) = \tilde{R}_n (t-x)g'(x); q_n, x) + \tilde{R}_n \left( \int_x^t (t-u)g''(u)du; q_n, x \right)
\]

\[
= g'(x)\tilde{R}_n ((t-x); q_n, x)
\]

\[
+ \tilde{R}_n \left( \int_x^t (t-u)g''(u)du; q_n, x \right) - \int_x^t (\xi_n(x) - u)g''(u)du
\]

\[
= \tilde{R}_n \left( \int_x^t (t-u)g''(u)du; q_n, x \right) - \int_x^t (\xi_n(x) - u)g''(u)du.
\]

On the other hand,

\[
\int_x^t (t-u)g''(u)du \leq \int_x^t |t-u|g''(u)du \leq \|g''\| \int_x^t |t-u|du \leq \|g''\| (t-u)^2
\]

and

\[
\int_x^t (\xi_n(x) - u)g''(u)du \leq \|g''\| (\xi_n(x) - x)^2 = \|g''\| (\tilde{R}_n ((t-x); q_n, x))^2,
\]

which implies

\[
|\tilde{R}_n (g; q_n, x) - g(x)| \leq |\tilde{R}_n \left( \int_x^t (t-u)g''(u)du; q_n, x \right) + \int_x^t (\xi_n(x) - u)g''(u)du|
\]

\[
\leq \|g''\| \left\{ \tilde{R}_n ((t-x)^2; q_n, x) + (\tilde{R}_n ((t-x); q_n, x))^2 \right\}
\]

\[
\leq \|g''\| \delta_n(x).
\]
Considering $\tilde{R}_n (1; q_n, x) = 1$, we have also

$$\left| \tilde{R}_n (f; q_n, x) \right| \leq \left| \tilde{R}_n (f; q_n, x) \right| + | f(\xi_n(x)) | \leq \tilde{R}_n (f; q_n, x) + 2 \| f \| \leq 3 \| f \| .$$

Therefore

$$\left| \tilde{R}_n (f; q_n, x) - f(x) \right| \leq \left| \tilde{R}_n ((f - g); q_n, x) - (f - g)(x) \right| + | f(\xi_n(x)) - f(x) | + \left| \tilde{R}_n (g; q_n, x) - g(x) \right| \leq 4 \| f - g \| + \omega (f; \alpha_n(x)) + \| g'' \| \delta_n(x) .$$

Taking infimum on the right hand side over all $g \in W^2$, we obtain

$$\left| \tilde{R}_n (f; q_n, x) - f(x) \right| \leq 4K_2 (f, \delta_n(x)) + \omega (f; \alpha_n(x)) .$$

By the inequality $K_2 (f, \delta) \leq C \omega_2 \left( f, \delta^{1/2} \right)$ for $\delta > 0$, we obtain the desired result.

Let $E$ be any subset of $\mathbb{R}_+$ and $\alpha \in (0, 1)$. Then $\text{Lip}_M (E, \alpha)$ denotes the space of functions $f$ in $C_B (\mathbb{R}_+)$ satisfying the condition

$$| f (t) - f (x) | \leq M_f (| t - x |^\alpha), \quad t \in \bar{E} \text{ and } x \in [0, \infty),$$

where $M_f$ is a constant depending on $f$ and $\bar{E}$ denotes the closure of $E$ in $\mathbb{R}_+$.

**Theorem 4.** Let $q = (q_n)$ be a sequence satisfying the conditions given as in (2.1) and the function $f$ in $C_B [0, \infty) \cap \text{Lip}_M (E, \alpha)$, $\alpha \in (0, 1)$ and $E$ be any bounded subset of $\mathbb{R}_+$. Then for each $x \in \mathbb{R}_+$, we have

$$\left| \tilde{R}_n (f; q_n, x) - f(x) \right| \leq M_f \left\{ (\mu_n(x))^\alpha + 2 (d(x, E))^\alpha \right\} ,$$

where $\mu_n (x) = \left( \tilde{R}_n \left( (t-x)^2; q_n, x \right) \right)^{1/2}$ and $M_f$ is a constant depending on $f$, and $d(x, E)$ is a distance between points $x$ and $E$, that is $d(x, E) = \inf \{|t-x|: t \in E\}$.

**Proof.** Let $E$ be the closure of the set $E$. Then there exists an $x_0 \in E$ such that $| x - x_0 | = d(x, E)$, where $x \in \mathbb{R}_+$. Thus we can write

$$\left| \tilde{R}_n (f; q_n, x) - f(x) \right| \leq \tilde{R}_n \left( (f(t) - f(x_0)); q_n, x \right) + \tilde{R}_n \left( |f(x_0) - f(x)|; q_n, x \right) \leq M_f \left( \tilde{R}_n \left( (t-x)^\alpha; q_n, x \right) + |x-x_0|^{\alpha} \right) \leq M_f \left( \tilde{R}_n \left( (t-x)^\alpha; q_n, x \right) + 2 |x-x_0|^{\alpha} \right) .$$

Using the H"older inequality with $p = \frac{2}{\alpha}$ and $q = \frac{2}{2-\alpha}$, we obtain

$$\left| \tilde{R}_n (f; q_n, x) - f(x) \right| \leq M_f \left\{ \tilde{R}_n \left( (t-x)^{\alpha p}; q_n, x \right) \right\}^{1/p} \left\{ \tilde{R}_n \left( 1; q_n, x \right) \right\}^{1/q} + 2 (d(x, E))^\alpha \leq M_f \left( (\mu_n(x))^{\alpha} + 2 (d(x, E))^\alpha \right) ,$$

which completes the proof.

**Remark 1.** Let $q = (q_n)$ be a sequence satisfying the conditions given as in (2.1), then

$$st - \lim_n a_n = st - \lim_n \frac{1}{b_n} = 0,$$

so the results in weighted space give us the weighted approximation degree of the $q$-BSK operators to $f$, and also the results of local approximation give us the approximation degree of the $q$-BSK operators to $f$.

We give an illustrative example which shows the rate of convergence of the operators $q$-BSK to certain functions in the following example:

**Example 1.** In case of $\beta = 0.5$, the convergence of the $q$-BSK operators to $f(x) = x + \sin(3x)$ is illustrated in Figure 1 and Figure 2 according to increasing values of $n$ and $q$, respectively.

The figures clearly show that, for increasing values of $n$ and $q$, the degree of approximation becomes better.
Figure 1: The convergence of the operators $\tilde{R}_n(f; 0.95, x)$ to $f(x)$ for increasing values of $n$.

Figure 2: The convergence of the operators $\tilde{R}_{50}(f; q, x)$ to $f(x)$ for increasing values of $q$. 
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