LECTURES ON TWO-LOOP SUPERSTRINGS

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Abstract

In these lectures, recent progress on multiloop superstring perturbation theory is reviewed. A construction from first principles is given for an unambiguous and slice-independent two-loop superstring measure on moduli space for even spin structure. A consistent choice of moduli, invariant under local worldsheet supersymmetry is made in terms of the super-period matrix. A variety of subtle new contributions arising from a careful gauge fixing procedure are taken into account.

The superstring measure is computed explicitly in terms of genus two theta-functions and reveals the importance of a new modular object of weight 6. For given even spin structure, the measure exhibits a behavior under degenerations of the worldsheet that is consistent with physical principles. The measure allows for a unique modular covariant GSO projection. Under this GSO projection, the cosmological constant, the 1-, 2- and 3-point functions of massless supergravitons vanish pointwise on moduli space. A certain disconnected part of the 4-point function is shown to be given by a convergent integral on moduli space. A general consistent formula is given for the two-loop cosmological constant in compactifications with central charge $c = 15$ and with $\mathcal{N} = 1$ worldsheet supersymmetry. Finally, some comments are made on possible extensions of this work to higher loop order.

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1 Introduction

In these Lectures, a review is presented of recent advances [1, 2, 3, 4] made in the conceptual understanding and concrete calculation of two loop Type II and Heterotic superstring amplitudes. To tree and one loop level, the basic scattering amplitudes for the Type II and Heterotic superstrings had been calculated in the foundational papers [5] and [6], where these string theories were first constructed. To higher loop level, however, a reliable and concrete formulation of even the simplest amplitudes (such as the zero point function or cosmological constant) had remained unavailable until these works.

The key complication at higher loop level, in the Ramond-Neveu-Schwarz (RNS) formulation of superstring theory, is the emergence from the gauge fixing procedure of odd-Grassmann-valued supermoduli [7, 8]. Odd supermoduli are absent at tree level and at one-loop level with even spin structures. There is an odd supermodulus to one-loop level with odd spin structure, but its role is merely that of a bookkeeping device and is dealt with easily [9]. A graphical representation of the 4-point function to tree, one and two loop levels is given in Fig.1.

A proposal based on worldsheet conformal field theory, BRST invariance and the picture changing operator was made in [7], in which it was proposed to summarize the effects of odd supermoduli in terms of the picture changing operator on a purely bosonic worldsheet specified by bosonic moduli only. Although this approach is both natural and appealing, a concrete calculation [10] has demonstrated that to two loop level, results are obtained that depend on the gauge slices chosen, and are thus inconsistent. To remedy this situation, a general procedure for generating correction terms and restoring slice independence, based on Cech cohomology, was developed in [11]. A construction from first principles remained, however, out of reach and this scheme was not applied in practice.

![Figure 1: String 4-point function to (a) tree-level, (b) one-loop level and (c) two-loop level.](image-url)
Considerable efforts were made by many authors to overcome the obstacles identified in [10] in terms of alternative prescriptions. These drew from a variety of fundamental principles, such as modular invariance [12], the lightcone gauge [13], the global geometry of Teichmüller space [14], the unitary gauge [15, 16, 17], the operator formalism [18, 19], group theoretic methods [20], factorization [21], and algebraic geometry [22]. The basic problem, however, that gauge-fixing required a local gauge slice and that any consistent prescription must be independent of the choice of such slice, remained unsolved. In fact, this state of affairs raised the undesirable possibility that the definition of higher loop superstring amplitudes could be inherently ambiguous [23, 24] and that it may be necessary to consider other options, such as the Fischler-Susskind mechanism [25].

In [9] and [26], we had proposed that the difficulties encountered in the earlier prescriptions were the result of improper gauge-fixing procedures which did not respect worldsheet local supersymmetry. As a point of departure, a superspace formulation of the worldsheet [27, 28] was used. Superholomorphic anomalies, generalizing those found for the bosonic string in [29], had already been identified and their cancellation in the critical superstring had already revealed the key role that would be played by supermoduli space [30]. Finally, a first principles gauge fixing of the superstring amplitudes and chiral splitting were carried out in [9, 26], thereby producing a consistent formulation of superstring amplitudes as integrals over supermoduli space. In particular, this formulation was used in [31] to show the perturbative unitarity of the superstring amplitudes.

Supermoduli space for higher genus surfaces is a delicate and complicated object, and the ultimate goal in superstring perturbation theory is to integrate out the odd supermoduli and achieve a formulation in terms of integrals over bosonic moduli only. It may be inferred from the work of [10] that the problems with the BRST picture changing operator approach arise from an elimination of the odd supermoduli that is inconsistent with local worldsheet supersymmetry. To take proper account of this supersymmetry, we had outlined in [9, 32] a new gauge-fixing procedure based on projecting supergeometries onto their super period matrices instead of onto their underlying bosonic geometries. Unlike the projection to the bosonic geometries, the projection to the super period matrix is invariant under local worldsheet supersymmetry. In the recent papers [1, 2, 3, 4], it is shown that this approach, applied to two loop level, solves all the problems encountered with the previous prescriptions.
1.1 Summary of results

- A gauge-fixed formula $d\mu[\delta](\Omega)$ for the contribution to the superstring measure of each even spin structure $\delta$, which is independent of the choice of gauge slice was constructed from first principles in [2]. The ambiguities plaguing the earlier prescriptions have now disappeared, as was shown in [2, 3].

- In [4], the chiral measure $d\mu[\delta](\Omega)$ is evaluated explicitly in terms of the genus two $\vartheta$-constants, and a new modular object, $\Xi_6[\delta](\Omega)$ emerges naturally from the construction. For each $\delta$, $d\mu[\delta](\Omega)$ transforms covariantly under modular transformations. There is a unique assignment of relative phases $\eta_\delta$ so that $\sum_\delta \eta_\delta d\mu[\delta](\Omega)$ is a modular form, and hence a unique way of implementing the Gliozzi-Scherk-Olive (GSO) projection [33].

- The superstring measure, when summed over all $\delta$, and therefore also the cosmological constant, vanishes point by point on moduli space [4]. In establishing this property, use is made of a 2-loop generalization of the Jacobi identity which, remarkably, is not a consequence of the genus 2 Riemann identities only. Instead, it is equivalent to the identity, special to genus 2, that any modular form of weight 8 must be proportional to the square of the unique modular form of weight 4.

- Similarly, the 1-, 2- and 3-point functions for the scattering of the supergraviton multiplet vanish by a variety of novel identities. This result was announced in [1]; a detailed proof will appear in [34].

- The 4-point function may be evaluated explicitly in terms of $\vartheta$-functions and modular forms. For a certain disconnected part of the 4-point function, explicit formulas are presented here; they are manifestly finite, in the regime of purely imaginary Mandelstam variables. As is well known [35], the other regimes are accessible only after proper analytic continuation in the external momenta. The connected part and the full 4-point function will appear in a forthcoming publication [36].

- Finally, we provide a simple slice independent formula for the even spin structure superstring measure and cosmological constant for general compactifications with matter central charge $c = 15$ and $\mathcal{N} = 1$ worldsheet supersymmetry. This formula was announced in [1]; it will be applied to the case of $\mathbb{Z}_2$ orbifolds in a forthcoming publication [37].
2 The Ramond-Neveu-Schwarz Formulation

In the RNS formulation [38] of superstring theory, the fundamental string degrees of freedom are the bosonic position $x^\mu$ and the fermionic counterpart $\psi^\mu$. Both are fields on the worldsheet $\Sigma$ and transform as vectors under the Lorentz transformations of flat Minkowski space-time. The formulation also appeals to the worldsheet metric $g_{mn}$ and gravitino $\chi_m$ fields, which are non-dynamical. The starting point for the formulation of scattering amplitudes is the worldsheet supergravity action of [39], given by

$$I_m = \frac{1}{4\pi} \int_{\Sigma} d^2z \sqrt{g} \left( \frac{1}{2} g^{mn} \partial_m x^\mu \partial_n x^\mu + \psi^\mu \gamma^m \partial_m \psi^\mu 
- \psi^\mu \gamma^n \gamma^m \chi_n \partial_m x^\mu - \frac{1}{4} \psi^\mu \gamma^n \gamma^m \chi_n (\chi_m \psi^\mu) \right)$$

(2.1)

The action is constructed so as to be invariant under diffeomorphisms, local $N = 1$ supersymmetry, Weyl and super Weyl transformations of the worldsheet. In view of the key role played by local supersymmetry, it is convenient to reformulate the action in terms of a matter superfield $X^\mu$ and a supergeometry specified by a local frame $E_M^a$ and local $U(1)$ connection superfield $\Omega_M$ [27, 28, 30, 9]. For a brief summary see Appendix B of [2]. The relation between component and superfields is, ($A$ and $F^\mu$ are auxiliary fields)

$$X^\mu \equiv x^\mu + \theta \psi^\mu + \bar{\theta} \bar{\psi}^\mu + i \theta \bar{\theta} F^\mu$$

$$E_m^a \equiv e^a_m + \theta \gamma^a \chi_m - \frac{i}{2} \theta \bar{\theta} A e^a_m$$

(2.2)

In terms of these superfields, the worldsheet action (2.1) takes the simple form,

$$I_m = \frac{1}{4\pi} \int_{\Sigma} d^2z \ E \ D_+ X^\mu D_- X^\mu \quad E \equiv \text{sdet} E_M^A$$

(2.3)

where $D_\pm$ are supercovariant derivatives, whose precise form may be found in Appendix B of [2] but will not be needed here.

The starting point for the scattering amplitudes is the Polyakov formulation of string perturbation theory [40], in which a summation is performed over all surfaces (including their topologies, specified by the number of handles $h$) and all fields on the surface,

$$A_\mathcal{O} = \sum_{h=0}^{\infty} \int \frac{D(E\Omega)}{\text{Vol}(\text{Symm})} \int DX^\mu \ \mathcal{O} \ e^{-I_m}$$

(2.4)

The operator $\mathcal{O}$ stands for the insertion of any set of physical state vertex operators, whose construction in superspace was given in [41]. In the critical dimension, 10, the quantum string is invariant under the full set of classical symmetries,

$$\text{Symm} = \text{sDiff}(\Sigma) \times \text{sWeyl}(\Sigma) \times \text{sU}(1)(\Sigma)$$

(2.5)

which must be factored out. Finally, $\delta(T)$ indicates that the torsion constraints of the $\mathcal{N} = 1$ supergeometry are to be enforced.
3 Reliable Superspace Gauge Fixing

As was shown in [9], a reliable gauge fixing procedure may be derived from first principles by reducing the integral over all supergeometries to a finite-dimensional integral over the quotient of all supergeometries by all the local symmetries of (2.5). This quotient is referred to as supermoduli space; its dimensions are as follows,

\[ sM_h \equiv \{ E_M^A, \Omega_M + \text{torsion constraints} \} / \text{sDiff} \times \text{sWeyl} \times \text{sU}(1) \]

\[
\dim(sM_h) = \begin{cases} 
(0|0) & h = 0 \\
(1|0) & h = 1 \\
(3h - 3|2h - 2) & h \geq 2
\end{cases} \tag{3.1}
\]

The subscripts \( e \) and \( o \) refer to the cases of even and odd spin structures respectively.

Being a quotient, supermoduli space does not admit a canonical parametrization, and one is led to choosing a local slice \( S \) of the same dimension as \( sM_h \), and transverse to the orbits of the symmetry group \( \text{sDiff}(\Sigma) \times \text{sWeyl}(\Sigma) \times \text{sU}(1)(\Sigma) \). Specializing to \( h \geq 2 \), we parametrize \( S \) by \( m_A = (m^a|\zeta^a), a = 1, \cdots, 3h - 3 \text{ even and } \alpha = 1, \cdots, 2h - 2 \text{ odd super-moduli.} \)

A gauge fixed formulation in terms of a slice representing supermoduli space \( sM_h \) was proposed in [10] and derived from first principles in [9]. It involves ghost superfields \( B \) and \( C \) (as well as their complex conjugates) which are related to the customary ghost fields \( b \) and \( c \) and superghost fields \( \beta \) and \( \gamma \) by

\[
B \equiv \beta + \theta b + \text{auxiliary fields} \\
C \equiv c + \theta \gamma + \text{auxiliary fields} \tag{3.2}
\]

The gauge fixed expression for the amplitudes is given by

\[
A_\mathcal{O} = \int_{sM} |dm^A|^2 \int D(XBC) \prod_A \delta(\langle H_A|B \rangle) |^2 \mathcal{O} e^{-I} \tag{3.3}
\]

The combined matter and ghost action is given by

\[
I \equiv \frac{1}{2\pi} \int_\Sigma d^2z E \left( \frac{1}{2} \mathcal{D}_+ X^\mu \mathcal{D}_- X_\mu + B \mathcal{D}_- C + \bar{B} \mathcal{D}_+ \bar{C} \right) \tag{3.4}
\]

The super-Beltrami differential, defined by

\[
(H_A)_-^z \equiv (-)^{A(M+1)} E_{-M} \frac{\partial E_M^z}{\partial m^A} = \bar{\theta}(\mu_A - \theta \chi_A) \bigg|_{\text{WZ}} \tag{3.5}
\]

represents the tangent vectors to the slice \( S \).
4 Chiral Splitting

The formulation of the amplitudes given in (3.3) is such that left moving (or holomorphic) and right moving (or anti-holomorphic) degrees of freedom are related to one another by complex conjugation. This result emerges naturally when the starting point of string theory is in terms of a summation over actual surfaces with Euclidean signature worldsheet metrics. In some deep sense, the original theory was rather in terms of a Minkowskian worldsheet where the fermions of left and right chiralities are independent of one another. This independence is a crucial ingredient in the very definition of both the Type II and Heterotic string theories, since left and right chiralities are assigned independent spin structures and a GSO projection must be carried out independently on left and right chirality degrees of freedom.

To recover the independence of left and right chiralities on a worldsheet with an Euclidean signature metric, a process of splitting the chiralities from one another must be applied. A glance at the worldsheet action (2.1) immediately reveals that this process appears to have some basic obstructions; the quartic term in fermions couples left and right chiralities to one another and the zero mode of the scalar field $x^\mu$ cannot be split. Similar obstructions appear when vertex operators are included. Nonetheless, the splitting is possible within each conformal block, labeled here by the internal loop momenta, and the chiralities may be identified with holomorphic and anti-holomorphic dependence on supermoduli [30, 9, 26].

The chiral splitting procedure may be summarized in terms of a set of effective rules [26, 31], which we now spell out. Chiral splitting may be achieved for each conformal block, which is labelled uniquely by a set of $h$ independent internal loop momenta $p_I^\mu$, $I = 1, \cdots, h$. It will be convenient to choose a basis for the first homology of the surface in terms of canonical $A_I$ and $B_I$, $I = 1, \cdots, h$ cycles, such as depicted in Fig.2 for genus 2.

![Figure 2: Choice of Canonical homology basis for genus 2](image-url)
The independent internal loop momenta $p_I^\mu$, $I = 1, \cdots, h$ may then be viewed as the momenta traversing the cycles $A_I$. The following effective prescription for the scalar superfield correlation functions emerges,

$$\langle \prod_{i=1}^N V_i(k_i, \epsilon_i) \rangle_{X^\nu} = \int dp_I^\mu \left| \left\langle \prod_{i=1}^N V_i^{\text{chi}}(k_i, \epsilon_i; p_I^\mu) \right\rangle_+ \right|^2$$

(4.1)

Here, $\langle \cdots \rangle_+$ denotes the fact that the effective rules for the contractions of the vertex operators $V_i^{\text{chi}}(k_i, \epsilon_i; p_I^\mu)$ are used, as given in Table 1.

| Internal Loop momenta | Original | Effective Chiral |
|----------------------|----------|-----------------|
| Bosons               | $x^\mu(z)$ | $x^\mu_I(z)$    |
| Fermions             | $\psi^\mu_+(z)$ | $\psi^\mu_+(z)$ |
| $x$-propagator       | $\langle x^\mu(z)x^\nu(w) \rangle$ | $-\delta^{\nu\mu}\ln E(z, w)$ |
| $\psi_+$-propagator  | $\langle \psi^\mu_+(z)\psi^\nu_+(w) \rangle$ | $-\delta^{\nu\mu}S_8(z, w)$ |
| Covariant Derivatives| $D_+$    | $\partial_\theta + \theta \partial_z$ |

Table 1: Effective Rules for Chiral Splitting

In this table, $E(z, w)$ is the prime form, and $S_8(z, w)$ is the Szegö kernel. The point of the effective rules is that they only involve meromorphic notions, unlike the $x$-propagator $\langle x^\mu(z)x^\nu(w) \rangle$ which is given by the scalar Green’s function $\delta^{\mu\nu}G(z, w)$. The superghost correlation functions are manifestly split. We obtain the following formula,

$$A_\mathcal{O}[\delta] = \int|\prod_A dm_A|^2 \int dp_I^\mu e^{i\pi p_I^\mu \hat{\Omega}_I p_I^\mu} A_\mathcal{O}[\delta]$$

(4.2)

where $A_\mathcal{O}[\delta]$ is the following effective chiral correlator

$$A_\mathcal{O}[\delta] = \left\langle \prod_A \delta(\langle H_A|B\rangle) \mathcal{O}_+ \exp \left\{ \int_{\Sigma} \frac{dz}{2\pi} \chi_z^+ S(z) \right\} \right\rangle_+$$

(4.3)

and $S(z)$ is the total supercurrent

$$S(z) = -\frac{1}{2} \psi_+^\mu \partial_z x_+^\mu + \frac{1}{2} b \gamma - \frac{3}{2} \beta \partial_z c - (\partial_z \beta)c,$$

(4.4)

Here, $\hat{\Omega}_I$ is the super period matrix, defined by [9, 26] for any genus. For genus 2, its expression simplifies considerably and is given by

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int_{\Sigma} d^2z \int_{\Sigma} d^2w \omega_I(z) \chi_z^+ S_8(z, w) \chi_w^+ \omega_J(w)$$

(4.5)
Here $\Omega_{IJ}$ is the period matrix corresponding to the complex structure of the metric $g_{mn}$. The $\omega_I(z)$ span a basis of holomorphic Abelian differentials dual to the $A_I$-cycles, so that

$$\oint_{A_I} \omega_J = \delta_{IJ} \quad \oint_{B_I} \omega_J = \Omega_{IJ} \quad (4.6)$$

The period matrix may also be obtained in an intrinsic way from the superholomorphic 1/2 forms $\hat{\omega}_I$, which are the super analogs of the ordinary holomorphic Abelian 1-forms $\omega_I$. Given again the choice of canonical homology cycles as depicted in Fig.2, the $\hat{\omega}_I$ may be canonically normalized on $A_I$-cycles and yield the super period matrix when integrated over $B_I$ cycles,

$$\mathcal{D} \cdot \hat{\omega}_I = 0 \quad \oint_{A_I} \hat{\omega}_J = \delta_{IJ} \quad \oint_{B_I} \hat{\omega}_J = \hat{\Omega}_{IJ} \quad (4.7)$$
in complete analogy with the ordinary Abelian differentials.

The chirally split expression (4.2) and (4.3) is our first significant departure [9, 26] from the proposals of other authors in the late 1980’s, in that it is the super period matrix $\hat{\Omega}_{IJ}$ which appears as covariance of the internal loop momenta $p_{\mu}^I$, and not the ordinary period matrix $\Omega_{IJ}$. Therefore, a correct chiral splitting points to the super period matrix $\hat{\Omega}_{IJ}$ as the proper locally supersymmetric moduli for gauge-fixing.

The actual amplitudes for the Type II and Heterotic superstrings are obtained by assembling the contributions from left and right movers endowed with the same period matrix and internal momenta, but with independent spin structures (or winding sectors for the bosonic formulation of the right moving part of the heterotic string). The correct amplitudes are then given by

$$A_{IIO} = \int dp_I^\mu \sum_{\delta, \bar{\delta}} \int_{s_{M_h}} |dm^A|^2 | \exp\{i\pi p_I^\mu \hat{\Omega}_{IJ} p_J^\mu\} | \mathcal{A}_{\Omega}[\delta](\hat{\Omega}) \mathcal{A}_{\Omega}^*[\bar{\delta}](\hat{\Omega}^*)$$

$$A_{HO} = \int dp_I^\mu \sum_{\delta, \bar{\delta}} \int_{s_{M_h}} dm^A \int_{M_{\bar{m}}} dm^A | \exp\{i\pi p_I^\mu \hat{\Omega}_{IJ} p_J^\mu\} | \mathcal{A}_{\bar{\Omega}}[\delta](\hat{\Omega}) \mathcal{B}_{\bar{\Omega}}(\Omega^*) \quad (4.8)$$

where $B_{\Omega}$ stands for the chiral half of the 26-dimensional heterotic string, compactified on a self-dual even lattice, suitable for the heterotic string. Here, $M_{\bar{m}}$ stands for the bosonic moduli space of Riemann surfaces at genus $h$. Note that the period matrix $\Omega_{IJ}$ characterizing the right moving heterotic sector is set equal to the superperiod matrix $\hat{\Omega}_{IJ}$ on the left moving superstring sector. The phases $\eta_{\delta, \bar{\delta}}$ and $\bar{\eta}_\delta$ are chosen so as to be consistent with modular invariance and are present to enforce the GSO projection independently on left and right movers, if so desired.
5 Integrating out Odd Supermoduli

As a consistent formulation of the scattering amplitudes as integrals over supermoduli space is thus available, the question now becomes as to whether and how a consistent formulation as integrals on moduli space may be obtained. In other words, how the odd supermoduli can be integrated out. We shall restrict to addressing this question for genus 2 and even spin structures, where a rigorous treatment is now available. Comments on higher genus will be made in the last section of this paper at the level of conjecture.

5.1 Naive derivation of the BRST picture changing Ansatz

It was shown in [10] that the formulation of superstring amplitudes of [7] may be recovered from the one in terms of supermoduli space provided certain assumptions are made on the choices of gauge slice. Of course, given that the work of [10] also demonstrates that the Ansatz of [7] leads to slice dependent formulas, it must be that the assumptions made in the derivation of this Ansatz are incorrect. It is very instructive to see where they fail.

The key assumption made by [10] in their rederivation of [7] by integrating over the odd supermoduli is that the metric and gravitino slices are chosen so that

$$g_{mn}(m^a) \chi = \sum_{\alpha=1}^{2} \zeta^\alpha \chi_\alpha(m^a) \quad (5.1)$$

The interpretation of these formulas is that the bosonic moduli are associated with the metric $g_{mn}$, independently of the odd supermoduli $\zeta^\alpha$. The functions $\chi_\alpha$ characterize the slice chosen for the odd supermoduli, and are taken to be of pointlike support at insertion points $z_\alpha$. This choice leads directly to the Ansatz of [7] in terms of BRST invariance and the picture changing operator $Y(z_\alpha)$,

$$\left\langle \mathcal{O} \prod_{a=1}^{3h-3} \langle \mu_a | b \prod_{\alpha=1}^{2h-2} Y(z_\alpha) \right\rangle \prod_{a=1}^{3h-3} dm^a \quad (5.2)$$

If all the assumptions had held correct, this Ansatz ought to have produced amplitudes that are independent of the insertion points $z_\alpha$ of the picture changing operators. An explicit calculation to two loop level in [10] has shown, however, that there is residual dependence on these points.

This situation spells disaster. If an analogy were sought with the quantization of Yang-Mills theory, the present situation would be as if the perturbative evaluation of a gauge invariant correlation function of gauge invariant operators in $\xi$-gauge were to yield a result that is not independent of $\xi$. In Yang-Mills theory, it is clear that this situation signals a faulty gauge fixing procedure. So it does for superstring theory.
5.2 The key role of local supersymmetry

Odd supermoduli may be viewed as fibers over even supermoduli and the operation of integrating out the odd supermoduli may be viewed as a projection along the fibers of supermoduli space onto its even base. The assumptions made in the previous subsection are equivalent to the following projection,

\[
(g_{mn}, \chi_m) \sim (g'_{mn}, \chi'_m) \quad \text{under SUSY}
\]

\[
g_{mn} \sim g'_{mn} \quad \text{under Diff × Weyl} \quad (5.3)
\]

The interpretation of this diagram is as follows. The projection onto the even moduli amounts to omitting \(\chi\), as the moduli \(m^a\) are functions of only the metric and not \(\chi\), cfr (5.1). The local supersymmetry transformation of the metric is given by

\[
\delta g_{mn} = 2\xi^+ \chi_{(m}^+ e_n)_{\bar{z}} \quad (5.4)
\]

This variation produces a change in the moduli \(m^a\) defined above, so that the moduli \(m^a\) are defined in a manner that is not invariant under local supersymmetry. Therefore, the projection (5.3) itself is inconsistent with local supersymmetry and it stands to reason that its use will lead to ambiguities.

A consistent projection can only be obtained when the even supermoduli \(m^a\) are defined in a manner invariant under the action of local supersymmetry. Therefore, moduli should be viewed as defined by another metric \(\hat{g}_{mn}\). The action of local supersymmetry transformations on this metric \(\hat{g}_{mn}\) must descend to an action of diffeomorphisms and Weyl transformations only, without the admixture of variations in moduli. Schematically, this type of consistent supersymmetric projection may be represented as follows,

\[
(g_{mn}, \chi_m) \sim (g'_{mn}, \chi'_m) \quad \text{under SUSY}
\]

\[
\hat{g}_{mn}(m^a) \sim \hat{g}'_{mn}(m^a) \quad \text{under Diff × Weyl} \quad (5.5)
\]

The fundamental guarantee that this projection exists and has the desired properties rests on the fact that the super period matrix \(\hat{\Omega}_{IJ}\), introduced earlier, is invariant under local supersymmetry. The form of \(\hat{g}_{mn}\) will be needed at intermediate stages of the calculation, but when all parts are assembled, the chiral measure will involve only the conformal class of \(\hat{\Omega}_{IJ}\), which is uniquely determined by \(\hat{\Omega}_{IJ}\).

Henceforth, we shall use this consistent projection (5.5) for genus 2; comments on higher genus will be deferred to the last section.
6 Construction of the Chiral Measure

It will be helpful to spell out the key ingredients in the construction the chiral measure.

- We make use of supersymmetric supermoduli \( m^A = (\hat{\Omega}_{IJ}, \zeta^\alpha) \).

- All quantities, calculated originally for the metric \( g_{mn} \) with complex structure \( \Omega_{IJ} \) are re-expressed in terms of the super period matrix \( \hat{\Omega}_{IJ} \). In correlation functions, this change is achieved via the insertion of the stress tensor,

\[
\Omega_{IJ} \rightarrow \hat{\Omega}_{IJ} \quad \begin{cases} 
\frac{g}{\partial \bar{z}} & \rightarrow \hat{g} = g + \hat{\mu} \\
\hat{\partial}_{\bar{z}} & \rightarrow \hat{\partial}_{\bar{z}} = \partial_{\bar{z}} + \hat{\mu} \partial_{\bar{z}} \\
\langle \cdots \rangle(g) = \langle \cdots \rangle(\hat{g}) + \int \hat{\mu} \langle T \cdots \rangle(\hat{g}) 
\end{cases}
\]  

The Beltrami differential \( \hat{\mu} \) is associated with the deformation of complex structure \( \Omega_{IJ} \) to \( \hat{\Omega}_{IJ} \). In view of the relation between the two period matrices, \( \hat{\mu} \) is determined by

\[
\int_\Sigma \hat{\mu} \omega_I \omega_J = \frac{1}{8\pi} \int_\Sigma d^2z \int_\Sigma d^2w \omega_I(z) \chi_{\bar{z}}^+ S_8(z, w) \chi_{\bar{w}}^+ \omega_J(w)
\]

Although ultimately only the conformal class of \( \hat{\mu} \) will enter, calculations at intermediate stages of the chiral measure will appeal to the actual metric \( \hat{g} \). The final cancellation of all dependence on the choice of metric \( \hat{g} \) within its conformal class serves both as a check of the consistency of the approach, and of the actual calculations.

- Superholomorphic forms on supermoduli space project to holomorphic forms on moduli space (with complex structure \( \hat{\Omega}_{IJ} \)) plus exact differentials. The simplest example is provided by the superholomorphic \( 1/2 \) forms, which obey \( \partial_- \hat{\omega}_I = 0 \) and project as follows,

\[
\hat{\omega}_I(\Omega, \zeta) = \theta \omega_I(\hat{\Omega}) + \partial_+ \Lambda_I
\]

The nature of this projection for general forms is of great interest for the study of scattering amplitudes; it will be discussed in detail in the forthcoming papers [34, 36].

- The fact that the super period matrix is unchanged under variations of the odd supermoduli \( \zeta \) implies that none of the components of the super Beltrami differential introduced in (3.5) will vanish,

\[
\delta_\zeta \hat{\Omega}_{IJ} = 0 \quad \Rightarrow \quad \begin{cases} 
H_A = \tilde{\theta}(\mu_A - \theta \chi_A) \\
\mu_A \neq 0 \quad \& \quad \chi_A \neq 0 
\end{cases}
\]

This situation is contrasted with that of the choice (5.1) where, for example, \( \mu_\alpha = 0 \).

- Dual to the super Beltrami differential are superholomorphic \( 3/2 \) forms of odd type \( \Phi_{IJ} \) and of even type \( \Phi_\alpha \). Of key importance will be the explicit formula for

\[
\Phi_{IJ} = -\frac{i}{2} \left( \hat{\omega}_i \partial_+ \hat{\omega}_J + \hat{\omega}_J \partial_+ \hat{\omega}_i \right)
\]

which is normalized to satisfy \( \langle H_a | \Phi_{IJ} \rangle = \delta_{a,IJ} \) and \( \langle H_a | \Phi_{IJ} \rangle = 0 \).
7 Calculation of the Chiral Measure

Before starting the computation of the chiral measure, using the ingredients developed in the preceding sections, one more obstacle must be overcome.

7.1 Change of basis of super Beltrami differentials

First, a change of basis is to be performed on the super Beltrami differentials. This is needed because our use of supersymmetric bosonic moduli forces all components of $H_A$ to be non-vanishing, as indicated in (6.4). Without it, the product of $\delta(\langle H_A|B\rangle)$ factors would produce an exceedingly complicated and untractable form for the correlation functions.

Under the assumption that the vertex operators occurring in $\mathcal{O}$ are independent of the ghost superfield $B$ (as is the case for all NS vertex operators [41]), the pairing of $H_A$ is effectively with a superholomorphic $B$ field. Therefore, we may change basis from $H_A$ to new super Beltrami differentials $H^*_A$, chosen for maximal simplicity to be

\begin{align}
H^*_a &= \bar{\theta} \delta(z, p_a) & a &= 1, 2, 3 \\
H^*_\alpha &= \bar{\theta} \theta \delta(z, q_\alpha) & \alpha &= 1, 2
\end{align}

Denoting an arbitrary complete set of linearly independent even and odd superholomorphic $3/2$ forms by $\Phi_C$, we have

\begin{equation}
\prod_A \delta(\langle H_A|B\rangle) = \frac{\text{sdet}(H_A|\Phi_C)}{\text{sdet}(H^*_A|\Phi^*_C)} \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha))
\end{equation}

Clearly, this formula is independent of the choice of $\Phi_C$. Its considerable advantage is that all correlation functions are now with respect to standard insertions and all the complication inherent in $H_A$ has been relegated to a single multiplicative factor.

Two natural choices of basis emerge for $\Phi_C$. The first, denoted simply by $\Phi_C$, is dual to $H_A$, while the second, denoted by $\Phi^*_C$, is dual to $H^*_A$,

\begin{equation}
\langle H_A|\Phi_C \rangle = \langle H^*_A|\Phi^*_C \rangle = \delta_{AC}
\end{equation}

The explicit form of the basis vectors $\Phi^*_C$ is known and may be found in [2], Appendix B. The explicit form for $\Phi_C$ is only known for half of its components. Indeed, it was already established in (6.5) that $\Phi_{IJ}$ provides the odd components of $\Phi_C$, namely for the even indices $c = \{IJ\}$. The even partners $\Phi^*_c$, however, have no such canonical expression. On general grounds, they may be expressed as a linear combination,

\begin{equation}
\Phi^*_c(z) = \Phi^*_c(z) C^c{}_{\gamma} + \Phi_{IJ}(z) D^{IJ}{}_{\gamma}
\end{equation}

for some $z$-independent, but moduli dependent, matrices $C$ and $D$. By pairing with $H_\alpha$ and using the fact that $\langle H_\alpha|\Phi_{IJ} \rangle = 0$, we readily have $\det C \times \det \langle H_\alpha|\Phi^*_c \rangle = 1$. Taking all
factors into account,

\[
\frac{\text{sdet}(H_A|\Phi_C)}{\text{sdet}(H_A^*|\Phi_C)} = \frac{1}{\det \Phi_{IJ}(p_a) \times \det(H_A|\Phi^*)}
\]  \hspace{1cm} (7.5)

The components \(\Phi_{IJ}(p_a)\) are explicitly known. The components \(\mu_\alpha\) and \(\chi_\alpha\) of \(H_\alpha = \bar{\theta}(\mu_\alpha - \theta \chi_\alpha)\) are known in the following manner. The objects \(\chi_\alpha\) represent the choice of worldsheet gravitini slice, and should be viewed as input into the gauge fixing process (which, in the end is to be independent of the choice of \(\chi_\alpha\)). The object \(\mu_\alpha\) may be shown [2] to be given by \(\mu_\alpha = \partial \hat{\mu}/\partial \zeta^\alpha\). All ingredients in the gauge fixed formula are thus known explicitly, and we have the following formula for the chiral measure,

\[
\mathcal{A}[\delta] = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)) \rangle}{\det \Phi_{IJ}(p_a) \det(H_\alpha|\Phi^*)} \left\{ 1 + \frac{1}{2\pi} \int \hat{\mu}(T) - \frac{1}{8\pi^2} \int \int \chi \chi S S \right\}
\]  \hspace{1cm} (7.6)

As a consistency check, it was demonstrated in [2] that this expression is indeed invariant under local supersymmetry on the worldsheet, as is expected on general grounds.

7.2 The calculation in components

In order to achieve workable formulas, the above expression is henceforth considered for a gravitino slice supported at two arbitrary generic points \(x_1\) and \(x_2\),

\[
\chi_\alpha(z) = \delta(z, x_\alpha)
\]  \hspace{1cm} (7.7)

The chiral measure may then be expressed entirely in terms of quantities that are meromorphic on the worldsheet. These include the prime form \(E(z, w)\), the Szegő kernel \(S_\delta(z, w)\), the \(b-c\) ghost Green function \(G_2(z, w)\) (which is defined to vanish when \(z = p_1, p_2, p_3\) in view of the \(b\)-insertions at \(p_a\)) and the superghost Green function \(G_{3/2}(z, w)\) (which is defined to vanish when \(z = q_1, q_2\) in view of the \(\delta(\beta)\)-insertions at \(q_\alpha\)). They also include a number of holomorphic differentials; \(\psi^*_\alpha(z)\) and \(\bar{\psi}_\alpha(z)\) are holomorphic 3/2 forms normalized so that \(\psi^*_\alpha(q_\beta) = \bar{\psi}_\alpha(x_\beta) = \delta_{\alpha\beta}\), and the quantity \(\varpi_a(z, w)\) provides a one-to-one map between holomorphic 2 forms in one variable and holomorphic forms of two variables of weight 1 each. It obeys the normalization \(\varpi_a(p_b, p_b) = \delta_{ab}\).

The chiral measure is given as follows,

\[
\mathcal{A}[\delta] = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)) \rangle}{\det \omega_{IJ}(p_a) \cdot \det \psi^*_\beta(x_\alpha)} \left\{ 1 + \frac{\zeta^1 \zeta^2}{16\pi^2} \sum_{i=1}^6 \chi_i \right\}
\]  \hspace{1cm} (7.8)
with the following expressions for $X_i$,

\[
\begin{align*}
X_1 &= -10S_\delta(x_1, x_2)\partial_{x_1}\partial_{x_2} \ln E(x_1, x_2) \\
&\quad -3\partial_{x_2}G_2(x_1, x_2)G_{3/2}(x_2, x_1) - 2G_2(x_1, x_2)\partial_{x_2}G_{3/2}(x_2, x_1) - (1 \leftrightarrow 2) \\
X_2 &= S_\delta(x_1, x_2)\omega_I(x_1)\omega_J(x_2)\partial_I\partial_J \ln \left( \frac{\vartheta[\delta](0)^5\vartheta(p_1 + p_2 + p_3 - 3\Delta)}{\vartheta[\delta](q_1 + q_2 - 2\Delta)} \right) \\
X_3 &= 2S_\delta(x_1, x_2)\sum_a \varpi_a(x_1, x_2)[B_2(p_a) + B_{3/2}(p_a)] \\
X_4 &= 2S_\delta(x_1, x_2)\sum_a \partial_{p_a}\partial_{x_1} \ln E(p_a, x_1)\varpi_a(p_a, x_2) - (1 \leftrightarrow 2) \quad (7.9) \\
X_5 &= \sum_a S_\delta(p_a, x_1)\partial_{p_a}S_\delta(p_a, x_2)\varpi_a(x_1, x_2) - (1 \leftrightarrow 2) \\
X_6 &= 3\partial_{x_2}G_2(x_1, x_2)G_{3/2}(x_2, x_1) + 2f_{3/2}(x_1)G_2(x_1, x_2)\partial\tilde{\psi}_1(x_2) - (1 \leftrightarrow 2) \\
&\quad + 2G_{3/2}(x_2, x_1)G_2(x_1, x_2)\partial\tilde{\psi}_2(x_2) + \partial_{x_2}G_2(x_2, x_1)\partial\tilde{\psi}_1(x_2) - (1 \leftrightarrow 2)
\end{align*}
\]

where we have used the following notations,

\[
\begin{align*}
&f_n(w) = \omega_I(w)\partial_I\ln \vartheta[\delta](D_n) + \partial_w\ln(\prod_i \sigma(w)E(w, z_i)) \\
&B_2(w) = -27T_1(w) + \frac{1}{2}f_2(w)^2 - \frac{3}{2}\partial_wf_2(w) - 2\sum_a \partial_{p_a}\partial_w \ln E(p_a, w)\varpi_a(p_a, w) \\
&B_{3/2}(w) = 12T_1(w) - \frac{1}{2}f_{3/2}(w)^2 + \partial_wf_{3/2}(w) + \frac{3}{2}\partial_{x_1}G_2(w, x_1) + \frac{3}{2}\partial_{x_2}G_2(w, x_2) \\
&\quad - \frac{3}{2}\partial_wG_{3/2}(x_1, w)\tilde{\psi}_1(w) - \frac{3}{2}\partial_{a}G_{3/2}(x_2, w)\tilde{\psi}_2(w) - \frac{1}{2}G_{3/2}(x_1, w)\partial\tilde{\psi}_1(w) \\
&\quad - \frac{1}{2}G_{3/2}(x_2, w)\partial\tilde{\psi}_2(w) + G_2(w, x_1)\partial\tilde{\psi}_1(x_1) + G_2(w, x_2)\partial\tilde{\psi}_2(x_2) \quad (7.10)
\end{align*}
\]

### 7.3 Fundamental Consistency Check

The above expression for the chiral measure is a sum of terms that are manifestly well-defined scalar meromorphic functions of $x_\alpha, q_a$ and $p_a$. By inspection of any possible singularities when a given point approaches any of the remaining 6 points, it may be shown directly [3] that the above result is actually holomorphic in each point, and thus independent of all 7 points $x_\alpha, q_a$ and $p_a$. This important result checks that our approach and calculations are indeed consistent.
7.4 Corrections to picture changing operators

In view of the independence on the points \(x_{\alpha}, q_{\alpha}, p\), it is very interesting to rein-
vestigate the problems that emerged in the old approach of [7] and [10]. In particular,
in the old approach, the product of the picture changing operators was singular. In our
approach, no such singularity can emerge. Their cancellation proceeds as follows. The
only contribution in (7.9) common with [7] and [10] arises from \(\mathcal{X}_1\) which contains

\[
\mathcal{X}_1 \sim \langle S(x_1)\delta(\beta(q_1)) S(x_2)\delta(\beta(q_2)) b(p_1)b(p_2)b(p_3) \rangle
\]  

(7.11)

Formally, the picture changing operator is defined by \(Y(q) \sim S(q)\delta(\beta(q))\) in the limit
\(x_{\alpha} \rightarrow q_{\alpha}\). However, this limit on the terms in \(\mathcal{X}_1\) alone,

\[
G_{3/2}(x_1, x_2) \rightarrow \begin{cases} 
0 & \text{as } x_1 \rightarrow q_1 \\
\infty & \text{as } x_2 \rightarrow q_2
\end{cases}
\]  

(7.12)

is ill-defined and singular. An identical and opposite singularity arises from the finite-
dimensional determinants summarized in \(\mathcal{X}_6\), however, and the combination of both con-
tributions is regular. Thus, a correct definition of the picture changing operators must
include the appropriate contributions of associated finite-dimensional determinants.

7.5 The choice of a convenient gauge

- Given the independence of the chiral measure on all points, we may set \(x_{\alpha} = q_{\alpha}\).

- Furthermore, terms \(\mathcal{X}_2, \mathcal{X}_3, \mathcal{X}_4\) are proportional to \(S_\delta(x_1, x_2) = S_\delta(q_1, q_2)\) and will
  vanish upon choosing the split gauge \(S_\delta(q_1, q_2) = 0\). This gauge is also natural since
  it implies \(\hat{\Omega}_{IJ} = \Omega_{IJ}\).

- Finally, it is advantageous to choose the points \(p_a\) to be the three zeros of a holo-
morphic 3/2 form \(\psi_A(z)\). This choice yields a particularly useful form for the \(b - c\)
  Green function \(G_2\) in terms of \(\psi_A\) and the Szego kernel,

\[
G_2(z, w) = S_\delta(z, w)\psi_A(z)/\psi_A(w)
\]  

(7.13)

so that in split gauge we have \(G_2(q_1, q_2) = 0\). Combining all contributions, we now
have \(\mathcal{X}_1 + \mathcal{X}_6 = \mathcal{X}_2 = \mathcal{X}_3 = \mathcal{X}_4 = 0\), and only \(\mathcal{X}_5 \neq 0\) remains.

The evaluation of the single remaining term \(\mathcal{X}_5\) is quite involved and will not be repro-
duced here; it may be found in [4]. The final result of the calculation will be discussed in
the next section.
8 Explicit Formulas in terms of $\vartheta$-constants

The fundamental result of [4] is a concise formula for the chiral superstring measure in terms of $\vartheta$-constants and modular forms. Henceforth, only the superperiod matrix will appear, which we shall now denote by $\Omega_{IJ}$ to simplify notation. The measure is,

$$d\mu[\delta](\Omega) = \frac{\Xi_6[\delta](\Omega)}{16\pi^6} \frac{\vartheta[\delta]^4(0, \Omega)}{\Psi_{10}(\Omega)} \, d^3\Omega_{IJ}$$

(8.1)

It remains to explain the various ingredients in this formula.

- On a genus 2 surface, there are 16 independent spin structures, which may be labelled by half integer characteristics,

$$\kappa = (\kappa'|\kappa'') \quad \kappa', \kappa'' \in (0, \frac{1}{2})^2$$

(8.2)

Here the two components $\kappa'_I$ of $\kappa'$ refer to the spin structure assignments along the homology cycles $A_I$, while the components $\kappa''_I$ refer to those on the cycles $B_I$. One distinguishes even and odd spin structures according to whether $4\kappa' \cdot \kappa''$ is even or odd. We have,

$$\begin{cases}
\kappa & \text{even/odd iff } 4\kappa' \cdot \kappa'' \text{ even/odd} \\
\delta & 10 \text{ even spin structures} \\
\nu & 6 \text{ odd spin structures} \\
\delta & = \nu_{i_1} + \nu_{i_2} + \nu_{i_3} = \nu_{i_4} + \nu_{i_5} + \nu_{i_6}
\end{cases}$$

(8.3)

The last lines states the fact, specific to genus 2, that every even spin structure may be written (in exactly two different ways) as the sum of three distinct odd spin structures. A compact notation will be used for the pairing,

$$\langle \kappa|\rho \rangle \equiv \exp\{4\pi i (\kappa' \cdot \rho'' - \rho' \cdot \kappa'')\}$$

(8.4)

which by construction takes on the values $\pm 1$.

- The $\vartheta$-functions with characteristic $\kappa$ are defined by

$$\vartheta[\kappa](v, \Omega) \equiv \sum_{n - \kappa' \in \mathbb{Z}^2} \exp\{i\pi n'\Omega n + 2\pi in'(v + \kappa'')\}$$

(8.5)

and are manifestly holomorphic in all arguments. They are even/odd functions of $v \in \mathbb{C}^2$ according to whether $\kappa$ is an even/odd spin structure.

- We shall make heavy use of modular forms for genus 2. These were classified long ago in [42]. Using the $\vartheta$-constants $\vartheta[\delta](0, \Omega)$, one may readily produce an infinite series of
modular forms, for \( k = 1, 2, 3, \ldots \)

\[
\Psi_{4k}(\Omega) \equiv \sum_\delta \vartheta[\delta]^{8k}(0, \Omega) \tag{8.6}
\]

These forms are not all independent. Instead, they form a polynomial ring with a finite number of generators [42]. We shall need in particular the form of weight 10 given by,

\[
\Psi_{10}(\Omega) \equiv \prod_{\delta \text{ even}} \vartheta[\delta]^2(0, \Omega) \tag{8.7}
\]

Finally, the evaluation introduces a new modular quantity, \( \Xi_6[\delta](\Omega) \) which may be defined as follows. Let the even spin structure \( \delta \) be decomposed as the sum of three distinct odd spin structures \( \delta = \nu_1 + \nu_2 + \nu_3 = \nu_4 + \nu_5 + \nu_6 \). We then have,

\[
\Xi_6[\delta](\Omega) \equiv \sum_{1 \leq i < j \leq 3} \langle \nu_i | \nu_j \rangle \prod_{k=4,5,6} \vartheta[\nu_i + \nu_j + \nu_k]^4(0, \Omega) \tag{8.8}
\]

Alternatively, \( \Xi_6[\delta] \) may be expressed via even spin structures only,

\[
\Xi_6[\delta](\Omega) = \sum_{[\delta, \delta_1, \delta_2, \delta_3]} \left( -\frac{1}{2} \prod_{i=1}^{3} \langle \delta | \delta_i \rangle \vartheta[\delta_i]^4(0, \Omega) \right) \tag{8.9}
\]

To specify the sum over the even spin structures \( \delta_i, i = 1, 2, 3 \), we introduce the symbol \( e \) as well as the following nomenclature [42],

\[
e(\delta, \epsilon, \eta) \equiv \langle \delta | \epsilon \rangle \langle \epsilon | \eta \rangle \langle \eta | \delta \rangle = \begin{cases} +1 & \text{syzygous triple} \\ -1 & \text{asyzygous triple} \end{cases} \tag{8.10}
\]

The sum over quartets \( [\delta, \delta_1, \delta_2, \delta_3] \) is defined to be such that any of its 4 distinct triplets is asyzygous.

\[
\begin{align*}
  e(\delta_1, \delta_2, \delta_3) &= e(\delta, \delta_1, \delta_2) = -1 \\
  e(\delta, \delta_2, \delta_3) &= e(\delta, \delta_3, \delta_1) = -1 \tag{8.11}
\end{align*}
\]

Note that while the definition of \( \Xi_6[\delta] \) given in (8.8) is restricted to genus 2, the form given in (8.9) makes sense for any genus, and may be viewed as a definition of \( \Xi_6[\delta] \) in arbitrary genus. Also note that \( \Xi_6[\delta] \) is not a modular form, since it has an explicit dependence on the spin structure \( \delta \). There does exist a modular form \( \Psi_6 \) of weight 6, obtained by summing products of three \( \vartheta^4 \) over all 60 asyzygous (or 60 syzygous) triplets in the following formula,

\[
\Psi_6(\Omega) \equiv \sum_{e(\delta_1, \delta_2, \delta_3)=-1} \pm \vartheta[\delta_1]^4(0, \Omega) \vartheta[\delta_2]^4(0, \Omega) \vartheta[\delta_3]^4(0, \Omega) \tag{8.12}
\]

But this modular form is not the same object as \( \Xi_6[\delta] \).
Modular transformations are defined to leave the canonical intersection matrix invariant, and thus form the group $Sp(4, \mathbb{Z})$,
\[
\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix}^t = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(4, \mathbb{Z}) \tag{9.1}
\]

The action on spin structure is given by \[\tilde{\kappa}' = (D - C - B A) \kappa' \kappa'' + \frac{1}{2} \text{diag} \left(CD^T \ A B^T \right), \tag{9.2}\]

Here, $\text{diag}(M)$ of an $n \times n$ matrix $M$ is an $1 \times n$ column vector whose entries are the diagonal entries of $M$. On the period matrix, the transformation acts by \[\tilde{\Omega} = (A \Omega + B)(C \Omega + D)^{-1} \tag{9.3}\]
while on the Jacobi $\vartheta$-functions, the action is given by \[\vartheta[\tilde{\kappa}](\{(C \Omega + D)^{-1}\} v, \tilde{\Omega}) = \epsilon(\kappa, M) \det(C \Omega + D)^{\frac{1}{2}} e^{i \pi v^T (C \Omega + D)^{-1} C v} \vartheta[\kappa](v, \Omega) \tag{9.4}\]
The phase factor $\epsilon(\kappa, M)$ depends upon both $\kappa$ and the modular transformation $M$ and obeys $\epsilon(\kappa, M)^8 = 1$. As a result, we obtain the following transformation laws
\[
\begin{align*}
d^3 \tilde{\Omega}_{IJ} &= \det(C \Omega + D)^{-3} d^3 \Omega_{IJ} \\
\vartheta[\tilde{\delta}](0, \tilde{\Omega}) &= \epsilon^4 \det(C \Omega + D)^2 \vartheta[\delta](0, \Omega) \\
\Xi_6[\tilde{\delta}](\tilde{\Omega}) &= \epsilon^4 \det(C \Omega + D)^6 \Xi_6[\delta](\Omega) \\
\Psi_{10}(\tilde{\Omega}) &= \det(C \Omega + D)^{10} \Psi_{10}(\Omega) \tag{9.5}
\end{align*}
\]
Note that $\Xi_6[\delta](\Omega)$ does not transform as a modular form.

The modular transformation law of the chiral measure may be readily deduced,
\[d\mu[\tilde{\delta}](\tilde{\Omega}) = \det(C \Omega + D)^{-5} d\mu[\delta](\Omega) \tag{9.6}\]
The weight $-5$ is related to the critical dimension, 10, as may be seen most easily after the integration over internal momenta has been carried out. The resulting factor of $\det \text{Im} \Omega$ has the following modular transformation law,
\[d\mu[\tilde{\delta}](\tilde{\Omega}) = \det(C \Omega + D)^{-5} d\mu[\delta](\Omega) \tag{9.6}\]
The weight $-5$ is related to the critical dimension, 10, as may be seen most easily after the integration over internal momenta has been carried out. The resulting factor of $\det \text{Im} \Omega$ has the following modular transformation law,
\[d\mu[\tilde{\delta}](\tilde{\Omega}) = \det(C \Omega + D)^{-5} d\mu[\delta](\Omega) \tag{9.6}\]
Therefore, the full measure combining left and right movers is modular covariant,
\[\left(\det \text{Im} \tilde{\Omega}\right)^{-5} d\mu[\tilde{\delta}](\tilde{\Omega}) \times d\mu[\delta](\Omega) = \left(\det \text{Im} \Omega\right)^{-5} d\mu[\delta](\Omega) \times d\mu[\delta](\Omega) \tag{9.8}\]
The phase factor in (9.6) is 1 for every spin structure, so that all GSO phases, consistent with modular invariance, must be equal and may be set to 1. Notice that this phase assignment is unique.
10 Behavior under Degenerations

A key consistency check on the calculation of any superstring measure or scattering amplitude is that it must obey the proper factorizations onto physical states when the string worldsheet degenerates. The general structure of such factorizations is known on general physical grounds. This check was carried out in [4]. There are two inequivalent cases, according to whether the degeneration separates the surface into two disconnected parts or not. We make the choice of canonical homology cycles of Fig.2, and use the following parametrization of the period matrices in this homology basis,

\[ \Omega = \begin{pmatrix} \tau_1 & \tau \\ \tau & \tau_2 \end{pmatrix} \]  \hspace{1cm} (10.1)

Clearly, the separating degeneration corresponds to the limit \( \tau \to 0 \) as \( \tau_1 \) and \( \tau_2 \) are kept fixed, while the non-separating degeneration may be taken to correspond to the limit \( q \equiv \exp\{i\pi \tau_2\} \to 0 \) as \( \tau_1 \) and \( \tau \) are kept fixed. The two cases are illustrated in Fig.3.

Figure 3: Degenerations of a genus 2 surface: (a) separating, (b) non-separating.
10.1 Separating Degeneration: $\tau \to 0$, $\tau_{1,2}$ fixed

In this limit, we distinguish between two cases. First is the case of 9 out of the 10 even spin structures $\delta$ for which the genus 1 spin structures $\mu_1$ and $\mu_2$ on the genus 1 connected components are both even; this is the NS-NS case. Second is the case of the single even spin structure for which the genus 1 spin structures are both odd and equal to $\nu_0$; this is the R-R case. The asymptotic behavior in both cases may be worked out using the limit of the $\vartheta$-function, which may be expressed in the following way,

$$
\vartheta \left[ \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right] (0, \Omega) = \sum_{p=0}^{\infty} \frac{(2\tau)^{2p}}{(2p)!} \partial_{\tau_1}^p \vartheta_1[\mu_1](0, \tau_1) \partial_{\tau_2}^p \vartheta_1[\mu_2](0, \tau_2)
$$

$$
\vartheta \left[ \begin{array}{c} \nu_0 \\ \nu_0 \end{array} \right] (0, \Omega) = \frac{1}{4\pi i} \sum_{p=0}^{\infty} \frac{(2\tau)^{2p+1}}{(2p+1)!} \partial_{\tau_1}^p \vartheta_1'[\nu_0](0, \tau_1) \partial_{\tau_2}^p \vartheta_1'[\nu_0](0, \tau_2)
$$

(10.2)

Here, $\vartheta_1$ are genus 1 $\vartheta$-functions. The limits of $\Psi_{10}(\Omega)$ and $\Xi_6[\delta](\Omega)$ are given by

$$
\Psi_{10}(\Omega) = -(2\pi \tau)^2 \cdot 2^{12} \cdot \eta(\tau_1)^{24} \eta(\tau_2)^{24} + \mathcal{O}(\tau^4)
$$

$$
\Xi_6 \left[ \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right] (\Omega) = -2^8 \cdot \langle \mu_1 | \nu_0 \rangle \langle \mu_2 | \nu_0 \rangle \eta(\tau_1)^{12} \eta(\tau_2)^{12} + \mathcal{O}(\tau^2)
$$

$$
\Xi_6 \left[ \begin{array}{c} \nu_0 \\ \nu_0 \end{array} \right] (\Omega) = -3 \cdot 2^8 \cdot \eta(\tau_1)^{12} \eta(\tau_2)^{12} + \mathcal{O}(\tau^2)
$$

(10.3)

Combining all contributions, the limit of the measure is found to be

$$
\text{NS} - \text{NS} \quad d\mu \left[ \begin{array}{c} \mu_1 \\ \mu_2 \end{array} \right] = \frac{d^3\tau}{\tau^2} \prod_{i=1,2} \frac{\langle \mu_i | \nu_0 \rangle \eta[\mu_i]^4(\tau_i)}{32\pi^4 \eta(\tau_i)^{12}} + \mathcal{O}(\tau^0)
$$

$$
\text{R} - \text{R} \quad d\mu \left[ \begin{array}{c} \nu_0 \\ \nu_0 \end{array} \right] = \frac{3\tau^2 d^3\tau}{2^6\pi^4} + \mathcal{O}(\tau^4)
$$

(10.4)

The NS-NS case reproduces the correct 1-loop factors, including the GSO phases appropriate for 1-loop amplitudes. Notice that these phases emerged from the limit of $\Xi_6[\delta]$. The $\tau^{-2}$ prefactor indicates the presence of the tachyon intermediate state, which is indeed expected in the NS-NS sector. Upon combining this limiting measure with the right moving part, and including the effects of the internal momenta, a massless pole will also be present. Partial GSO summation in one or the other 1-loop component will cancel both the tachyon and the massless singularities, as is expected. The R-R case has neither tachyon nor massless intermediate singularities, as indeed is expected.
10.2 Non-Separating Degeneration : $q \to 0, \tau_1, \tau$ fixed

The case of non-separating degenerations is analogous, and we only quote the results here; detailed derivations may be found in [4]. We have

$$
\begin{align*}
    d\mu_{\left[ \mu_i \right]}^{00} &= + V_i(\tau, \tau_1) \frac{d^3 \tau}{q} + \mathcal{O}(q^0) \\
    d\mu_{\left[ \mu_i \right]}^{0\frac{1}{2}} &= - V_i(\tau, \tau_1) \frac{d^3 \tau}{q} + \mathcal{O}(q^0) \\
    d\mu_{\left[ \mu_i \right]}^{\frac{1}{2}0} &= \mathcal{O}(q^0) \\
    d\mu_{\left[ \nu_0 \right]}^{\nu_0} &= \mathcal{O}(q^0)
\end{align*}
$$

(10.5)

In the first three lines, $\mu_i$ stands for any of the three even genus 1 spin structures on handle 1, and $V_i(\tau, \tau_1)$ stands for the tachyon 2-point function on the degenerate genus 1 surface. The $q^{-1}$ singularity corresponds to the tachyon traversing the homology cycle $A_2$ when the spin structure in handle 2 is either $[00]$ or $[0 \ 1/2]$, which corresponds to NS boundary conditions. This is as expected. Note that a partial summation over spin structures in the NS sector alone eliminates the tachyon singularity, again as expected. On the other hand, no tachyon appears when the spin structures $[1/2 \ 0]$ and $[1/2 \ 1/2]$ correspond to R boundary conditions, again as expected.

To conclude, the measure passes all immediate degeneration checks carried out in this section, confirming its validity.
11 Vanishing of the Cosmological Constant

General arguments have been given long ago that space-time supersymmetry guarantees the vanishing of the cosmological constant in superstring theory considered in flat space-time [43]. This vanishing is just one example of a whole array of non-renormalization results in supersymmetric Yang-Mills theory and string theory (for recent reviews see [44, 45, 46]). As discussed in the Introduction, attempts had been made by several authors [10, 12, 15, 24] to derive this vanishing from a first principles calculation, but, as explained earlier, progress was halted by the ambiguities that were believed to plague superstring perturbation theory. Now that we have a consistent formula for the measure available, the vanishing of the cosmological constant may be derived from a first principles calculation in superstring theory.

The two-loop contribution to the cosmological constant for both the Type II and Heterotic superstrings are the most immediate quantities that may be evaluated once the superstring chiral measure is known. They are given by

\[
\Lambda_{II} = \int_{\mathcal{M}_2} \frac{|d^3\Omega|^2}{(\det \text{Im}\Omega)^5} \times \frac{\Upsilon_8(\Omega)\overline{\Upsilon_8(\Omega)}}{2^{8\pi^2}|\Psi_{10}(\Omega)|^2}
\]

\[
\Lambda_H = \int_{\mathcal{M}_2} \frac{|d^3\Omega|^2}{(\det \text{Im}\Omega)^5} \times \frac{\Upsilon_8(\Omega)\overline{\Psi_8(\Omega)}}{2^{8\pi^2}|\Psi_{10}(\Omega)|^2}
\]

We have used the fact that the chiral measure for the 26-dimensional bosonic string, partially compactified on the Cartan tori of \(E_8 \times E_8\) or \(\text{Spin}(32)/\mathbb{Z}_2\) is proportional to \(\Psi_8(\Omega)/\Psi_{10}(\Omega)\), as was established in [4].

The remaining ingredient is the chiral measure for the left moving superstring, which is obtained by summing over all even spin structures as

\[
\Upsilon_8(\Omega) \equiv \sum_\delta \Xi_6[\delta](\Omega) \vartheta[\delta]^4(0, \Omega)
\]

Here, we have made use of the unique GSO phase factor assignment that follows from requiring modular invariance.

We shall now show that \(\Upsilon_8 = 0\), so that both cosmological constants vanish \(\Lambda_{II} = \Lambda_H = 0\). First, \(\Upsilon_8\) is a modular form of weight 8, by its very construction. Next, we know from the asymptotics derived in (10.2) and (10.3) that \(\Upsilon_8 \to 0\) in the limit of separating degeneration. By the general classification of genus 2 modular forms of [42], any modular form of weight 8 must be proportional to \(\Psi_4(\Omega)^2\), so that we must have \(\Upsilon_8(\Omega) = c\Psi_4(\Omega)^2\) for some (moduli independent) constant \(c\). But \(\Psi_4(\Omega)\) tends to a non-zero value in the limit of separating degeneration. Therefore, the constant \(c\) and thus \(\Upsilon_8(\Omega)\) must vanish. Note that this relation amongst \(\vartheta\)-constants implied by the vanishing of \(\Upsilon_8\) does not result from the Riemann relations alone and is equivalent instead to the relation \(\psi_4(\Omega)^2 = 4\Psi_8(\Omega)\).
12 Scattering Amplitudes

The vertex operators for the scattering of $N$ massless bosons are given by

$$\prod_{i=1}^{N} V(k_i, \epsilon_i) = \prod_{i=1}^{N} \int_{\Sigma} d^2 z_i \ E(z_i) \ e^{i k_i \cdot z_i} \ D_+ X_{\mu} D_- X_{\bar{\mu}} e^{i k_i \cdot X}(z_i)$$

As in the case of the measure, the superstring scattering amplitudes require a GSO summation over spin structures of the conformal blocks of $\langle \prod_{i=1}^{N} V(k_i, \epsilon_i) \rangle_X$ in the $X^\mu$ superconformal field theory.

12.1 Vanishing of the 1-, 2- and 3-point functions

On general grounds, the vanishing of the 1-, 2- and 3-point functions is expected from space-time supersymmetry [43] and, using our measure, may be shown from first principles. The 0-, 1-, 2- and 3-point functions in both the Type II and the heterotic strings are then found to vanish pointwise on moduli space and without the appearance of boundary terms, in view of the following new identities,

$$\sum_\delta \Xi_6[\delta](\Omega) \vartheta[\delta](0, \Omega) S_6(z_1, z_2)^2 = 0$$
$$\sum_\delta \Xi_6[\delta](\Omega) \vartheta[\delta](0, \Omega) S_6(z_1, z_2) S_6(z_3, z_5) = 0$$

which were proven in [4] using the Fay trisecant identity [47, 48]. A full discussion of the calculations and proofs involved will be presented in a forthcoming paper [34].

12.2 The 4-point function

The 4-point function receives contributions from two distinct parts. The first arises from the connected part of the correlators

$$\langle S(z)S(w) \prod_{i=1}^{4} V(k_i, \epsilon_i)^{\text{chi}} \rangle_{\text{conn}} \quad \text{and} \quad \langle T(z) \prod_{i=1}^{4} V(k_i, \epsilon_i)^{\text{chi}} \rangle_{\text{conn}}$$

(12.2)

The second arises from the disconnected part

$$\langle S(z)S(w) \prod_{i=1}^{4} V(k_i, \epsilon_i)^{\text{chi}} \rangle \quad \text{and} \quad \langle T(z) \prod_{i=1}^{4} V(k_i, \epsilon_i)^{\text{chi}} \rangle$$

(12.3)

of these correlators and combines with the gauge fixing determinants into a contribution proportional to the measure $d\mu[\delta](\Omega)$. The connected part is more complicated and requires an independent treatment to appear in a later publication [36].
The disconnected part (for example for the Type II superstrings) is given by

\[
\langle \prod_{i=1}^{4} V(\epsilon_i, k_i) \rangle = g_s^2 \delta(k) \int_{\mathcal{M}_2} \frac{|d\Omega|^2}{(\det \text{Im}\Omega)^{\frac{5}{2}}} \prod_{i=1}^{4} d^2 z_i |\mathcal{F}|^2 \exp\left(-\sum_{i<j} k_i \cdot k_j G(z_i, z_j)\right)
\]  

(12.4)

Here, \(g_s\) is the string coupling, the scalar Green’s function is given by

\[
G(z, w) = -\log|E(z, w)|^2 + 2\pi \text{Im} \int_z^w \omega_j (\text{Im} \Omega)^{-1} \text{Im} \int_z^w \omega_j
\]

while \(k\) is the total momentum, and \(\mathcal{F}\) is a holomorphic 1-form in each \(z_i\), given by

\[
\mathcal{F} = C_S \mathcal{S}(1234) + \sum_{(i,j,k) = \text{perm}(2,3,4)} C_T(i|j,k) \mathcal{T}(i|j,k)
\]  

(12.5)

The combinations \(C_S\) and \(C_T\) are kinematical factors, which depend only on the polarization vectors \(\epsilon_i\) and the external momenta \(k_i\) through the gauge invariant combinations \(f_i^{\mu\nu} \equiv \epsilon_i^\mu k_i^\nu - \epsilon_i^\nu k_i^\mu\) and are given by

\[
C_S = f_1^{\mu\nu} f_2^{\nu\sigma} f_3^{\sigma\rho} f_4^{\rho\mu} + f_1^{\mu\nu} f_2^{\nu\sigma} f_3^{\sigma\rho} f_4^{\rho\mu} + f_1^{\mu\nu} f_2^{\nu\sigma} f_3^{\sigma\rho} f_4^{\rho\mu}
\]

(12.6)

\[
C_T(i|j,k) = f_i^{\mu\nu} f_j^{\nu\sigma} f_k^{\sigma\rho} f_l^{\rho\mu} - f_i^{\mu\nu} f_j^{\nu\sigma} f_k^{\sigma\rho} f_l^{\rho\mu} + 2f_i^{\mu\nu} f_j^{\nu\sigma} f_k^{\sigma\rho} f_l^{\mu\rho} - 2f_i^{\mu\nu} f_j^{\nu\sigma} f_l^{\sigma\rho} f_k^{\rho\mu}
\]

The kinematical combination \(C_S\) coincides with the unique kinematical invariant of the NS 4-point function encountered at tree and 1-loop level, which is often expressed in terms of the rank 8 tensor \(t\) (see [5, 49]), \(C_S = -8t_{\kappa_1\lambda_1\kappa_2\lambda_2\kappa_3\lambda_3\kappa_4\lambda_4} f_1^{\kappa_1\lambda_1} f_2^{\kappa_2\lambda_2} f_3^{\kappa_3\lambda_3} f_4^{\kappa_4\lambda_4}\). Finally, the forms \(\mathcal{S}\) and \(\mathcal{T}\) are given by

\[
\mathcal{S}(1234) = -\frac{1}{192\pi^6 \Psi_{10} \omega_1(z_1)\omega_2(z_2)\omega_3(z_3)\omega_4(z_4)} \sum_{\delta} \Xi_6[\delta] \delta[\delta]^3 \partial_1 \partial_2 \partial_3 \partial_4 \vartheta[\delta](\Omega)
\]

\[
\mathcal{T}(ij|kl) = -\frac{1}{8\pi^2} \omega_1(z_1)\omega_2(z_2)\omega_3(z_3)\omega_4(z_4)
\]  

(12.7)

The \(\delta\)-sum for the \(\mathcal{T}\)-term was carried out explicitly, and no \(\Psi_{10}\) appears in its contribution. \(\mathcal{S}\) and \(C_S\) are totally symmetric, while \(\mathcal{T}\) and \(C_T\) are odd under the interchange of \(i \leftrightarrow j\) or \(k \leftrightarrow l\). As a result, the \(\mathcal{T}\)-term is novel at 2 loops and could not exist at 1 loop.

12.3 Finiteness of the disconnected part

The disconnected part of the 4-point function for massless bosons, calculated above, is finite. This is the case at least when the Mandelstam variables \(k_i \cdot k_j\) are purely imaginary. As is now well known [35], finiteness for general \(k_i \cdot k_j\) cannot be read off directly, but has to be established by analytic continuation.
To show convergence, we recall that the modular form $\Psi_{10}(\Omega)$ vanishes of second order along the divisor of separating nodes. This corresponds to the propagation of a tachyon, and was responsible for the divergence in the bosonic string [29]. Here, the modular tensor

$$\sum_\delta \Xi_0[\delta](\Omega)\vartheta^3[\delta](\Omega)\partial_I\partial_J\partial_K\partial_L\vartheta[\delta](\Omega)$$

also vanishes of second order along the divisor of separating nodes, rendering the superstring amplitude finite. We illustrate this cancellation mechanism in a typical case, where, say, two of the indices $I, J, K, L$ are with respect to the variable $\zeta_1$ in $\vartheta[\delta](\zeta, \Omega)$, and the other two are with respect to the variable $\zeta_2$. In this case, the asymptotics are given by

$$\partial_I\partial_J\partial_K\partial_L\vartheta^{[\mu_1}_{\mu_2]}(0, \Omega) = -2^4\pi^2\vartheta_{\tau_1}\vartheta_{\mu_1}[\mu_2](0, \tau_1)\vartheta_{\mu_2}[\mu_2](0, \tau_2) + O(\tau^2)$$

$$\partial_I\partial_J\partial_K\partial_L\vartheta^{[\nu_0}_{\nu_0]}(0, \Omega) = 2^5\pi^3\tau\partial_{\tau_1}\eta^3(\tau_1)\partial_{\tau_2}\eta^3(\tau_2) + O(\tau^3)$$

It follows from (10.2) that the asymptotics of the $\vartheta$-constants themselves are given by

$$\vartheta^{[\mu_1}_{\mu_2]}(0, \Omega) = \vartheta[\mu_1][0, \tau_1]\vartheta[\mu_2][0, \tau_2] + O(\tau^2)$$

$$\vartheta^{[\nu_0}_{\nu_0]}(0, \Omega) = \frac{1}{2\pi i}\tau\partial_{\tau_1}\vartheta[\nu_0][0, \tau_1]\partial_{\tau_2}\vartheta[\nu_0][0, \tau_2] + O(\tau^3)$$

Combining these factors in the disconnected part of the 4-point function, we see that the contribution of the last spin structure in (12.10) is of order $O(\tau^3)$, and can be ignored. The sum over the remaining even spin structures $\delta$ produces, up to $O(\tau^2)$

$$\sum_\delta \Xi_0[\delta](\Omega)\vartheta^3[\delta](\Omega)\partial_I\partial_J\partial_K\partial_L\vartheta[\delta](\Omega) = 2^{12}\pi^2\eta^{12}(\tau_1)\eta^{12}(\tau_2)$$

$$\times \sum_{\mu_1}\langle \mu_1|\nu_0]\vartheta^3_{\mu_1}[0, \tau_1]\partial_{\tau_1}\vartheta[\mu_1][0, \tau_1] \sum_{\mu_2}\langle \mu_2|\nu_0]\vartheta^3_{\mu_2}[0, \tau_2]\partial_{\tau_2}\vartheta[\mu_2][0, \tau_2]$$

This vanishes, in view of the derivative of the Jacobi identity. Next, we discuss the case of non-separating degenerations. As explained earlier in section §10.2, the degenerating parameter is then $q \to 0$. The asymptotics of $\Psi_{10}(\Omega)$ are given by

$$\Psi_{10}(\Omega) = -2^{12}q^2\eta^{18}(\tau_1)\vartheta^2[\nu_0](\tau, \tau_1) + O(q^3)$$

so the finiteness will result from the vanishing to order $O(q^2)$ of (12.8). To see that this is indeed the case, we note that for all even genus 2 spin structures $\delta$ whose component along cycle 2 produces an $\mathcal{R}$ sector along the degenerating $B_2$ cycle, both the terms $\vartheta[\delta](0, \Omega)$ and $\partial_I\partial_J\partial_K\partial_L\vartheta[\delta](0, \Omega)$ vanish to order $O(q^{1/2})$, while $\Xi_0[\delta](\Omega)$ vanishes to order $O(q)$. Thus the contributions of all such spin structures are of order $O(q^2)$, and can be ignored in the
proof of finiteness. The remaining genus 2 even spin structures $\delta$ correspond to the NS sector along the $B_2$ cycle and are of the form

$$\delta_{\text{NS}} = \left[ \frac{\mu}{(0|\kappa''')} \right]$$

(12.12)

The asymptotics of $\vartheta[\delta_{\text{NS}}]$ and $\partial_I \partial_J \partial_K \vartheta[\delta_{\text{NS}}](0, \Omega)$ are, up to order $O(q^2)$, invariant under the interchange $\kappa'' = 0 \leftrightarrow \kappa'' = 1/2$. On the other hand, under the same interchange, the asymptotics of $\Xi_6[\delta_{\text{NS}}]$ are odd, again up to order $O(q^2)$. It follows that the contribution to (12.8) of the spin structures $\delta_{\text{NS}}$ vanishes up to the order $O(q^2)$. We observe that, as expected on physical grounds, the cancellation mechanism here does not require the one-loop Jacobi identity.

### 12.4 The Supergravity Limit

In the low energy limit, the exponential factor of the scalar Green’s function in (12.4) tends to 1. It is instructive to identify the kinematical factors that emerge from the integration over the 4 vertex insertion points $z_i$ of $|\mathcal{F}|^2$ in (12.4) in the Type II superstrings (analogous expressions may be derived for the heterotic strings). The first contribution is from the product $C_S \bar{C}_S$, and yields the well-known $ttR^4$ term of four Riemann tensors contracted with two copies of the rank 8 tensor $t$ as obtained in [49]. As argued in the preceding paragraph, this contribution is given by a convergent integral. The second contribution is from the products $C_S \bar{C}_T$; it vanishes in view of the complete symmetry in the points $z_i$ in $S$ and the antisymmetry in two pairs of points in $T$. The third contribution is from the product $C_T \bar{C}_T$ for which the $z_i$ integrals may be carried out using the Riemann bilinear relations. The resulting kinematical factors is again a quadrilinear in the Riemann tensor and is proportional to

$$C_T \bar{C}_T \rightarrow +(R_{\alpha\beta\mu\nu} R^{\alpha\beta\mu\nu})^2 - R_{\alpha\beta\mu\nu} R^\gamma\delta\mu
u R^{\alpha\beta\rho\sigma} R_{\gamma\delta\rho\sigma}$$

$$+ 4R_{\alpha\beta\mu\nu} R^{\gamma\delta\mu\nu} R^\beta_{\gamma\rho\sigma} R^\delta_{\alpha\rho\sigma} - 4R_{\alpha\beta\mu\nu} R_{\delta\alpha\mu\nu} R_{\beta\gamma\rho\sigma} R^\gamma\delta\rho\sigma$$

$$+ 4R_{\alpha\beta\mu\nu} R_{\beta\gamma\nu\rho} R^{\gamma\delta\rho\sigma} R_{\delta\alpha\sigma\mu} - 4R_{\alpha\beta\mu\nu} R_{\beta\gamma\nu\rho} R^{\gamma\delta\rho\sigma} R_{\delta\alpha\sigma\mu}$$

(12.13)

While it is possible that this term, which arose from the disconnected contributions in (12.3), will be cancelled by similar contributions arising from the connected contributions in (12.2), the above contribution to the low energy effective action has at least one remarkable property: the integral over moduli space becomes simply the volume of moduli space with respect to the $Sp(4, \mathbb{Z})$ invariant volume form $|d^3\Omega|^2 (\det \text{Im}\Omega)^{-3}$. We note that the problem of loop corrections in Type II superstrings and their contribution to low energy effective actions has witnessed a resurgence of interest recently (see for example [50, 51, 52]).
13 Compactification & The Cosmological Constant

One of the most important paradoxes in theoretical physics is the extraordinary smallness of the cosmological constant as compared to typical particle physics scales. In theories with global space-time supersymmetry, such as super Yang-Mills theories, general arguments show that the cosmological constant must be comparable to the scale at which supersymmetry is broken. In theories with local space-time supersymmetry, such as supergravity and superstring theory, the situation is different and it is possible to have vanishing cosmological constant despite the fact that supersymmetry is broken [53].

13.1 Motivation from Orbifold Compactifications

Recently, this alternative has been investigated in a number of papers [54, 55] within the context of orbifold compactifications. The models of [54] are constructed in a manner such that the cosmological constant vanishes to 1-loop level, despite the fact that supersymmetry is broken; it was proposed that the cosmological constant should vanish also to higher orders. Lacking reliable formulas for multiloop amplitudes, however, it was impossible to check these claims with concrete calculations. We shall now revisit these issues.

In orbifold compactifications with non-Abelian orbifold groups, qualitatively novel effects emerge starting at two loop level. The reason for this is as follows. The orbifolding of one of the 10-dimensional flat space-time superstring theories is carried out by making identifications of the string fields $x^\mu$ and $\psi^\mu$ under an orbifold group $G$, which is a subgroup of the Poincaré group $ISO(1,9)$,

\[ \rho : \pi_1(\Sigma) \rightarrow G \]

\[
\begin{align*}
x(\gamma z) &= \rho_x(\gamma) \ x(z) \\
\psi(\gamma z) &= \rho_\psi(\gamma) \ \psi(z)
\end{align*}
\]  

(13.1)

as the point $z$ is taken around a homotopy cycle $\gamma \in \pi_1(\Sigma)$ on $\Sigma$. The maps $\rho_x(\gamma)$ and $\rho_\psi(\gamma)$ are representations of $\pi_1(\Sigma)$ in $G$. The representations $\rho_x$ and $\rho_\psi$ will in general be different because $x$ may be translated as well as rotated, while $\psi$ may only be rotated.

Some of the most interesting cases are when $G$ is non-Abelian [54]. To tree and one loop levels, however, $\pi_1(\Sigma)$ is Abelian and therefore runs through an Abelian subgroup of $G$. Thus, one-loop contributions to the cosmological constant in non-Abelian orbifold models can capture only part of the orbifold phenomena. To two loop level and higher, $\pi_1(\Sigma)$ is always non-Abelian and the non-Abelian effects of the orbifolding process will start to play a role.
13.2 The Cosmological constant for general compactifications

The calculations of the cosmological constant presented earlier for flat space-time may be extended to the case where some of the space-time directions are compactified to form a total space-time manifold $C$, under the following mild assumptions \[1, 37\].

- The compactification only modifies the matter conformal field theory, leaving the superghost part unchanged;
- The compactification respects $\mathcal{N} = 1$ local worldsheet supersymmetry, so that the super-Virasoro algebra with matter central charge $c = 15$ is preserved.

Chiral splitting must be carried out with some care, as the superconformal families will be labeled no longer only by the internal loop momenta $p_\mu^I$ of flat, characteristic of flat space-time. Instead, superconformal families will be labeled by $f$, which may include internal loop momenta as well as twist sectors and any other quantum numbers specifying the superconformal families. The spin structure $\delta$ will not be included in $f$ as this label also enters into the ghost superconformal field theory. We shall denote by $Z_M[\delta](\Omega)$ the partition function for flat Minkowski space-time (omitting the Gaussian factor involving the internal momenta), and by $Z_C[\delta](\Omega, f)$ the partition function for compactification onto the manifold $C$, associated with superconformal family $f$.

Under these assumptions, the superstring measure is independent of any choices of gauge slice, and a simple expression was derived in \[1\] for split gauge $S_\delta(q_1, q_2) = 0$,

$$d\mu_C[\delta](\Omega, f) = \frac{Z_C[\delta](\Omega, f)}{Z_M[\delta](\Omega)} \left\{ \frac{\Xi_6[\delta] \vartheta[\delta]^4}{4\Psi_{10}}(\Omega) - Z\langle S_C(q_1)S_C(q_2)c(\Omega, f) \rangle \right\} \frac{d^3\Omega}{4\pi^6} \quad (13.2)$$

where

$$Z = \frac{\langle \prod_a b(p_a) \prod_\alpha \delta(\beta(q_\alpha)) \rangle}{\det \omega_I \omega_J(p_a)} \quad Z_M[\delta](\Omega) \equiv \frac{\vartheta[\delta]^5}{Z^{15}} \quad (13.3)$$

Here, $S_C$ denotes the supercurrent of the compactified theory, and $Z$ is the chiral boson partition function. The actual cosmological constant is obtained by assembling contributions from left and right movers. For Type II theories, we have

$$\Lambda_{\text{II}} = \int_{M_2} \sum_{\delta, \bar{\delta}} \sum_M M(\delta, f; \bar{\delta}, \bar{f}) \times d\mu_C[\delta](\Omega, f) \times d\mu_C[\bar{\delta}](\bar{\Omega}, \bar{f}) \quad (13.4)$$

and an analogous formula may be derived for the Heterotic strings. The matrix $M(\delta, f; \bar{\delta}, \bar{f})$ represents a Hermitian metric on the space of superconformal blocks and must be chosen consistently with modular invariance. Implementation of these results on specific orbifold compactification models will be deferred to \[37\].
14 Comments on Higher Loops

The case of genus 2 is of particular importance because it is the lowest order where odd supermoduli play a non-trivial role. The case of genus higher than 2 is expected to be significantly more difficult, since a number of simplifying features, special to genus 2, will be then absent. Reliable explicit calculations are not available at the present time, and we shall limit our discussion to a few speculative remarks.

The most encouraging fact for string perturbation theory in higher genus is that the chiral splitting procedure of [9, 26] holds for any genus. In particular, the super period matrix exists for any genus, and is given by

$$\hat{\Omega}_{IJ} = \Omega_{IJ} - \frac{i}{8\pi} \int_{\Sigma} d^2 z \int_{\Sigma} d^2 w \, \omega_I(z) \chi_z^+ \hat{S}_\delta(z, w) \chi_w^+ \omega_J(w)$$

(14.1)

where the modified Szego kernel $\hat{S}_\delta(z, w)$ is defined recursively by the relation

$$\partial_z \hat{S}_\delta(z, w) + \frac{1}{8\pi} \chi_z^+ \int_{\sigma} d^2 x \partial_z \partial_x \ln E(z, x) \chi_x^+ \hat{S}_\delta(x, w) = 2\pi \delta(z, w)$$

(14.2)

The key projection onto the super period matrix is thus well-defined for any genus.

In genus 3 and higher, new technical and conceptual difficulties arise. First, the parametrization of the fiber of the projection onto the super period matrix will be more complicated. Second, the Dirac operator may develop zero modes even for even spin structures. For example, in genus 3, the number of zero modes jumps from 0 to 2 on the complex codimension 1 subvariety of genus 3 hyperelliptic surfaces. Therefore, the relation between the period matrix $\Omega_{IJ}$ and the super period matrix $\hat{\Omega}_{IJ}$ may become singular at hyperelliptic Riemann surfaces.† We expect however that the full chiral string measure will remain smooth at this locus, because the fermion determinant factor produces multiple zeros precisely at the same locus. Actually, a related mechanism is known to occur for the bosonic string measure in higher genus. While the bosonic string measure is known to be singularity-free inside moduli space [56], (see also [57]) it has apparent singularities when expressed in terms of $\vartheta$-functions and modular forms. In fact, its denominator in such an expression is the product $\Psi_{18}(\Omega)$ of all 36 even $\vartheta$-characteristics, which vanishes along the locus of hyperelliptic curves [58]. But, as can be anticipated on general ground, the corresponding poles are cancelled by the remaining factors in the measure.

Another difficulty is the issue of Schottky relations. Naively, it appears that the gauge-fixing procedure requires both $\Omega_{IJ}$ and $\hat{\Omega}_{IJ}$ to be the period matrices of two-dimensional bosonic geometries. While such a requirement may seem difficult to satisfy in general,

†The importance of this fact was stressed to us by Edward Witten.
we note that here, the deformation from $\hat{\Omega}_{IJ}$ to $\Omega_{IJ}$ consists purely of soul elements. At least for genus 3, outside of a lower dimensional subvariety, any positive definite symmetric matrix $\hat{\Omega}_{IJ}$ is the period matrix of a Riemann surface. One may therefore expect that the gauge fixing procedure presented here will extend at least to the case of genus 3.

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References

[1] E. D’Hoker and D.H. Phong, “Two-Loop Superstrings I, Main Formulas”, Phys. Lett. B529 (2002) 241-255.

[2] E. D’Hoker and D.H. Phong, “Two-Loop Superstrings II, The chiral Measure on Moduli Space”, Nucl. Phys. B636 (2002) 3-60.

[3] E. D’Hoker and D.H. Phong, “Two-Loop Superstrings III, Slice Independence and Absence of Ambiguities”, Nucl. Phys. B636 (2002) 61-79.

[4] E. D’Hoker and D.H. Phong, “Two-Loop Superstrings IV, The Cosmological Constant and Modular Forms”, Nucl. Phys. B639 (2002) 129-181.

[5] M.B. Green, J.H. Schwarz, “Supersymmetrical string theories” Phys Lett B109 (1982) 444.

[6] D.J. Gross, J.A. Harvey, E.J. Martinec and R. Rohm, “Heterotic String Theory (II). The interacting heterotic string”, Nucl. Phys. B267 (1986) 75.

[7] D. Friedan, E. Martinec, and S. Shenker, “Conformal invariance, supersymmetry, and string theory”, Nucl. Phys. B271 (1986) 93.

[8] E. D’Hoker and D.H. Phong, “Loop amplitudes for the fermionic string”, Nucl. Phys. B278 (1986) 225;
G. Moore, P. Nelson, J. Polchinski, “Strings and supermoduli”, Phys Lett B169 (1986) 47.

[9] E. D’Hoker and D.H. Phong, “The geometry of string perturbation theory”, Rev. Modern Physics 60 (1988) 917-1065.

[10] E. Verlinde and H. Verlinde, “Multiloop calculations in covariant superstring theory”, Phys. Lett. B192 (1987) 95-102.

[11] H. Verlinde, “A note on the integral over fermionic supermoduli”, Utrecht Preprint No. THU-87/26 (1987) unpublished.

[12] G. Moore, J. Harris, P. Nelson, and I.M. Singer, “Modular forms and the cosmological constant”, Phys. Lett. B178 (1986) 167-173;
E. Gava and R. Iengo, “Modular Invariance and the Two Loop Vanishing of the Cosmological Constant”, Phys. Lett. B207 (1988) 283;
O. Lechtenfeld, “Factorization and modular invariance of multiloop superstring amplitudes in the unitary gauge”, Nucl. Phys. B338 (1990) 403-414.

[13] S. Mandelstam, “Interacting string picture of the fermionic string”, in Workshop on Unified String Theories, eds. M. Green and D. Gross (1986) World Scientific, 577;
S. Mandelstam, “The n loop string amplitude: Explicit formulas, finiteness and absence of ambiguities,” Phys. Lett. B277, 82 (1992).

[14] J. Atick, G. Moore, and A. Sen, “Some global issues in string perturbation theory”, Nucl. Phys. B308 (1988) 1; “Catoptric tadpoles”, Nucl. Phys. B307 (1988) 221-273.
[15] O. Lechtenfeld and A. Parkes, “On the Vanishing of the genus 2 Superstring Vacuum Amplitude”, Phys. Lett. B202 (1988) 75;
A. Parkes, “The Two-Loop Superstring Vacuum Amplitude and Canonical Divisors”, Phys. Lett. B217 (1989) 458;
O. Lechtenfeld and W. Lerche, “On Non-renormalization Theorems for Four-Dimensional Superstrings”, Phys. Lett. B227 (1989) 373;
T. Ortin, “Genus 2 Heterotic String Cosmological Constant”, Nucl. Phys. B387 (1992) 280;
O. Lechtenfeld and A. Parkes, “On covariant multiloop superstring amplitudes”, Nucl. Phys. B332 (1990) 39-82.

[16] A. Morozov, “Pointwise Vanishing of Two-Loop Contributions to 1, 2, 3 Point Functions in Superstring Theories”, Nucl. Phys. B318 (1989) 137;
A. Morozov, “On the two-loop contribution to the superstring four-point function”, Phys. Lett. B209 (1988) 473-476.

[17] R. Iengo and C.J. Zhu, “Notes on Non-renormalization Theorem in Superstring Theories”, Phys. Lett. B212 (1988) 309;
R. Iengo and C.-J. Zhu, “Two loop computation of the four-particle amplitude in the heterotic string”, Phys. Lett. B212 (1988) 313;
C. J. Zhu, “Two Loop Computations In Superstring Theories,” Int. J. Mod. Phys. A4 (1989) 3877; “Factorization Of A Two Loop Four Particle Amplitude In Superstring Theories,” Phys. Lett. B221 (1989) 27.

[18] L. Alvarez-Gaumé, C. Gomez, G. Moore, P. Nelson, and C. Vafa, “Fermionic strings in the operator formalism”, Nucl. Phys. B311 (1988) 333;
R. Pettorino and F. Pezzella, “On the (B,C) System Contribution to Superstring Amplitudes”, Phys. Lett. B255 (1991) 223.

[19] P. Di Vecchia, K. Hornfeck, M. Frau, A. Lerda, S. Sciuto, “BRST Invariant Operator Formalism for the Superstring”, Phys. Lett. B205 (1988) 250;
P. Di Vecchia, K. Hornfeck, M. Frau, A. Lerda, S. Sciuto, “N String, G Loop Vertex for the Fermionic String”, Phys. Lett. B211 (1988) 301.

[20] A. Neveu and P. West, “Group theoretic approach to the superstring and its supermoduli” Nucl. Phys. B311 (1988) 79.

[21] O. Yasuda, “Factorization of a two loop Four Point Superstring Amplitude”, Phys. Rev. Lett. 60 (1988) 1688; erratum-ibid 61 (1988) 1678;
A. Morozov and A. Perelomov, “A Note on Multiloop Calculations for Superstrings in the NSR Formulation”, Int. J. Mod. Phys. A4 (1989) 1773;
I.G. Koh, D. Lüst, S. Theisen, “Factorization Properties of Genus Two Bosonic and Fermionic String Partition Functions”, Phys. Lett. B208 (1988) 433;
O. Lechtenfeld, “On Finiteness of the Superstring”, Nucl. Phys. B322 (1989) 82;
O. Yasuda, “Multiloop Modular Invariance of D=10 Type II Superstring Theory”, Nucl. Phys. B318 (1989) 397.
[22] J. Rabin, “Teichmuller Deformations Of Superriemann Surfaces” Phys. Lett. B190 (1987) 40.
L. Crane and J. M. Rabin, “Superriemann Surfaces: Uniformization And Teichmuller Theory,” Commun. Math. Phys. 113, 601 (1988).
K. Aoki, “Heat Kernels And Superdeterminants Of Laplace Operators On Superriemann Surfaces,” Commun. Math. Phys. 117, 405 (1988).
A. Baranov, A.S. Schwarz, Yu.I. Manin, and I.V. Frolov, “A superanalog of the Selberg trace formula and multiloop contributions for fermionic strings”, Commun. Math. Phys. 111 (1987) 373-392;
M. Martellini, P. Teofilato, “Global structure of the superstring partition function and resolution of the supermoduli measure ambiguity”, Phys. Lett. B211 (1988) 293;
S. Giddings and P. Nelson, “The geometry of super Riemann surfaces”, Commun. Math. Phys. 116 (1988) 607.

[23] J. Atick, J. Rabin, and A. Sen, “An ambiguity in fermionic string perturbation theory”, Nucl. Phys. B299 (1988) 279-294.

[24] G. Moore and A. Morozov, “Some remarks on two-loop string calculations”, Nucl. Phys. B306 (1988) 387-404.

[25] H. La and P. Nelson, “Unambiguous fermionic string amplitudes”, Phys. Rev. Lett. 63 (1989) 24-27.

[26] E. D’Hoker and D.H. Phong, “Conformal scalar fields and chiral splitting on super Riemann surfaces”, Commun. Math. Physics 125 (1989) 469-513.

[27] P.S. Howe, “Super Weyl transformations in two dimensions”, J. Phys. A12 (1979) 393.

[28] E. Martinec, “Superspace geometry of superstrings”, Phys. Rev. D28 (1983) 2604;
D. Friedan, “Notes on string theory and two-dimensional conformal field theory”, in Unified String Theories: Proceedings of the 1985 Santa Barbara Workshop M.B. Green and D.J. Gross eds, World Scientific 1986;
S. Chaudhuri, H. Kawai and H. Tye, Phys. Rev. D36 (1987) 1148.

[29] A. Belavin and V. Knizhnik, “Algebraic geometry and the theory of quantized strings”, Phys. Lett. B168 (1986) 201;

[30] E. D’Hoker and D.H. Phong, “Superholomorphic anomalies and supermoduli space”, Nucl. Phys. B292 (1987) 317.

[31] K. Aoki, E. D’Hoker and D.H. Phong, “Unitarity of Closed Superstring Perturbation Theory”, Nucl. Phys. B342 (1990) 149.

[32] E. D’Hoker and D.H. Phong, “Superstrings, Super Riemann Surfaces, And Supermoduli Space,” UCLA-89-TEP-32 preprint and in Symposia Mathematica, String Theory, Vol XXXIII, Academic Press London and New York, 1990
[33] F. Gliozzi, J. Scherk and D. Olive, “Supersymmetry, supergravity theories and the dual spinor model”, Nucl. Phys. B122 (1977) 253;
E. Witten, in Symposium on Anomalies, Geometry, Topology, W.A. Bardeen and A.R. White eds, World Scientific (1985) p 61;
N. Seiberg and E. Witten, Nucl. Phys. B276 (1986) 272.

[34] E. D’Hoker and D.H. Phong, “Two-Loop Superstrings V, Scattering Amplitudes, the 1-, 2- and 3-point functions”, in preparation.

[35] E. D’Hoker and D.H. Phong, “Momentum analyticity and finiteness of the one-loop superstring amplitude”, Phys. Rev. Lett. 70 (1993) 3692-3695;
E. D’Hoker and D.H. Phong, “The box graph in superstring theory”, Nucl. Phys. B440 (1995) 24-94;
E. D’Hoker and D.H. Phong, “Dispersion relations in string theory,” Theor. Math. Phys. 98, 306 (1994) [arXiv:hep-th/9404128].

[36] E. D’Hoker and D.H. Phong, “Two-Loop Superstrings VI, Scattering Amplitudes: the 4-point function”, in preparation.

[37] K. Aoki, E. D’Hoker and D.H. Phong, (2002) in preparation.

[38] P. Ramond, “Dual Theory for Free Fermions”, Phys. Rev D3 (1971) 2415;
A. Neveu, J.H. Schwarz, “Factorizable Dual Model of Pions”, Nucl. Phys. B31 (1971) 86.

[39] S. Deser and B. Zumino, “A complete action for the spinning string”, Phys. Lett. B65 (1976) 369;
L. Brink, P. Di Vecchia and P. Howe, “A Locally supersymmetric and Reparametrization invariant action for the spinning string”, Phys. Lett. B65 (1976) 471.

[40] A.M. Polyakov, “Quantum Geometry of Fermionic strings”, Phys. Lett. B103 (1981) 211.

[41] E. D’Hoker, D.H. Phong, “Vertex Operators for Closed Strings”, Phys Rev D35 (1987) 3890.

[42] J.I. Igusa, Theta Functions, Springer Verlag, 1972;
J.I. Igusa, “On Siegel modular forms of genus two”, Amer. J. Math. 84 (1962) 175; “On the graded ring of theta-constants”, Amer. J. Math. 86 (1964) 219.

[43] E. Martinec, “Non-renormalization Theorems and Fermionic String Finiteness”, Phys. Lett. B171 (1986) 189.

[44] M. Grisaru, W. Siegel and M. Rocek, “Improved Methods for Supergraphs”, Nucl. Phys. B159 (1979) 429;
M.A. Shifman, A.I. Vainshtein, Nucl. Phys. B277 (1986) 456; ibid B359 (1991) 571;
N. Seiberg, “Naturalness versus supersymmetric non-renormalization theorems”, Phys. Lett. B318 (1993) 469; Phys. Rev. D49 (1994) 6857.
D. Amati, K. Konishi, Y. Meurice, G.C. Rossi and G. Veneziano, Phys. Rep. 162 (1988) 169;
K. Intriligator and N. Seiberg, “Lectures on supersymmetric gauge theories and electric-
magnetic duality”, Nucl. Phys. Proc. Suppl. 45BC (1996) 1, hep-th/9509066;
E. D’Hoker and D. H. Phong, “Lectures on supersymmetric Yang-Mills theory and integrable
systems,” in Theoretical Physics at the End of the Twentieth Century, Lecture Notes of the
CRM Summer School, Banff, Alberta, Y. Saint-Aubin and L. Vinet eds., Springer (2002);
[arXiv:hep-th/9912271].

O. Aharony, S.S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, “Large N field theories, string
theory and gravity.” Phys. Rept. 323, 183 (2000); [arXiv:hep-th/9905111];
E. D’Hoker and D. Z. Freedman, “Supersymmetric gauge theories and the AdS/CFT cor-
respondence,” TASI-2001 Lecture Notes; [arXiv:hep-th/0201253].

J. Fay Theta Functions on Riemann surfaces, Springer Lecture Notes in Mathematics, No
352, Springer Berlin (1973).

E. Verlinde and H. Verlinde, “Chiral Bosonization, determinants and the string partition
function”, Nucl. Phys. B288 (1987) 357;
E. Verlinde and H. Verlinde, “Superstring Perturbation Theory”, In Superstrings 88, M.
Green et al. Eds., World Scientific Publ. Co. Singapore (1989) 222-240.

D.J. Gross and E. Witten, “Superstring Modifications of Einstein’s Equations”, Nucl. Phys.
B277 (1986) 1;
M. T. Grisaru, A. E. van de Ven and D. Zanon, “Four Loop Divergences For The N=1 Su-
persymmetric Nonlinear Sigma Model In Two-Dimensions,” Nucl. Phys. B277, 409 (1986).

M.B. Green, M. Gutperle and P. Vanhove, “One loop in Eleven Dimensions”, Phys. Lett.
B409 (1997) 177, hep-th/9706175;
M.B. Green, H. Kwon and P. Vanhove, “Two loops in Eleven Dimensions”, Phys. Rev. D61
(2000) 104010, hep-th/9910055;
M.B. Green and P. Vanhove, “The Low Energy Expansion of the one loop Type II super-
string”, Phys. Rev. D61 (2000) 104010, hep-th/9910056.

N.A. Ober and B. Pioline, “U-duality and M-theory”, Phys. Rept. 318 (1999) 113; hep-
th/9809039;
R. Iengo and C.-J. Zhu, “Explicit modular invariant two-loop superstring amplitude relevant
for $R^4$”, J. High E. Phys. 06 (1999) 011.

K. Peeters, P. Vanhove and A. Westerberg, “Chiral splitting and world-sheet gravitinos in higher-derivative string amplitudes,” Class. Quant. Grav. 19, 2699 (2002); [arXiv:hep-
th/0112157];
K. Peeters, P. Vanhove and A. Westerberg, “Supersymmetric higher-derivative actions in ten and eleven dimensions, the associated superalgebras and their formulation in superspace,”
Class. Quant. Grav. 18, 843 (2001); [arXiv:hep-th/0010167].

S. Deser and B. Zumino, “Broken Supersymmetry and Supergravity”, Phys. Rev. Lett. 38
(1977) 1433;
E. Cremmer, B. Julia, J. Scherk, S. Ferrara, L. Girardello and P. van Nieuwenhuizen, “Spontaneous Symmetry Breaking and Higgs Effect in Supergravity without cosmological constant”, Nucl. Phys. B147 (1979) 105;
E. Cremmer, S. Ferrara, C. Kounnas and D. V. Nanopoulos, “Naturally Vanishing Cosmological Constant In N=1 Supergravity,” Phys. Lett. B133, 61 (1983);
A.B. Lahanas and D.V. Nanopoulos, “The Road to no-scale supergravity”, Phys. Rept. 145 (1987) 1;
L. Andrianopoli, R. D'Auria, S. Ferrara and M. A. Lledo, “Gauging of flat groups in four dimensional supergravity,” JHEP 0207, 010 (2002) [arXiv:hep-th/0203206].

[54] S. Kachru, J. Kumar, and E. Silverstein, “Vacuum energy cancellation in a nonsupersymmetric string”, Phys. Rev D59 (1999) 106004, hep-th/9807076;
S. Kachru and E. Silverstein, “Self-dual nonsupersymmetric Type II string compactifications”, JHEP 9811:001 (1998) hep-th/9808056;
S. Kachru and E. Silverstein, “On vanishing two loop cosmological constants in nonsupersymmetric strings”, JHEP 9901:004 (1999) hep-th/9810129;

[55] R. Iengo and C.J. Zhu, “Evidence for Non-Vanishing Cosmological Constant in Non-SUSY Superstring Models”, JHEP 0004:028 (2000), hep-th/9912074.

[56] E. D’Hoker and D.H. Phong, “Multiloop amplitudes for the bosonic Polyakov string”, Nucl. Phys. B 269 (1986) 205
G. Moore and P. Nelson, “Absence of global anomalies, Measure for Moduli,” Nucl. Phys. B 266 58.

[57] G. Moore, “Modular forms and twoloop string physics”, Phys. Lett. B176 (1986) 369;
E. D’Hoker and D.H. Phong, “On determinants of Laplacians on Riemann surfaces” Comm. Math. Phys. 104 (1986) 537;
A. Morozov, “Explicit formulae for one-, two-, three- and four-point loop amplitudes”, Phys. Lett. B184 (1987) 171-176.
S. Wolpert, “Asymptotics of the spectrum and the Selberg zeta function on the space of Riemann surfaces”, Commun. Math. Phys. 112 (1987) 283-315.

[58] A. Belavin, V. Knizhnik, A. Morozov, and A. Perelomov, “Two and three loop amplitudes in the bosonic string theory”, Phys. Lett. 177 (1986) 324-328