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The hyperbolic polygons of type \((\epsilon, n)\) and Möbius transformations

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Abstract: An \(n\)-sided hyperbolic polygon of type \((\epsilon, n)\) is a hyperbolic polygon with ordered interior angles \(\frac{\pi}{2} + \epsilon, \theta_1, \theta_2, \ldots, \theta_{n-2}, \frac{\pi}{2} - \epsilon\), where \(0 < \epsilon < \frac{\pi}{2}\) and \(0 < \theta_i < \pi\) satisfying

\[
\sum_{i=1}^{n-2} \theta_i + \left(\frac{\pi}{2} + \epsilon\right) + \left(\frac{\pi}{2} - \epsilon\right) < (n-2)\pi
\]

and \(\theta_i + \theta_{i+1} \neq \pi\) \((1 \leq i \leq n-3)\), \(\theta_1 + (\frac{\pi}{2} + \epsilon) \neq \pi\), \(\theta_{n-2} + (\frac{\pi}{2} - \epsilon) \neq \pi\). In this paper, we present a new characterization of Möbius transformations by using \(n\)-sided hyperbolic polygons of type \((\epsilon, n)\). Our proofs are based on a geometric approach.

Keywords: conformal mapping, Möbius transformations, hyperbolic polygons

MSC 2010: 30C20, 30C35, 30F45, 51M10, 51M15

1 Introduction

A Möbius transformation \(f : \mathbb{C} \rightarrow \mathbb{C}\) is a map defined by \(f(z) = \frac{az + b}{cz + d}\), where \(a, b, c, d \in \mathbb{C}\) with \(ad - bc \neq 0\). They are the automorphisms of extended complex plane \(\overline{\mathbb{C}}\) and define the Möbius transformation group \(M(\mathbb{C})\) with respect to composition. Möbius transformations are also directly conformal homeomorphisms of \(\overline{\mathbb{C}}\) onto itself and they have beautiful properties. For example, a map is Möbius if and only if it preserves cross ratios. As for geometric aspect, circle-preserving is another important characterization of Möbius transformations. The following result is one of the most famous theorems for Möbius transformations:

Theorem 1. [1] If \(f : \mathbb{C} \rightarrow \mathbb{C}\) is a circle preserving map, then \(f\) is a Möbius transformation if and only if \(f\) is a bijection.

The transformations \(f(z) = \frac{az + b}{cz + d}\), where \(a, b, c, d \in \mathbb{C}\) with \(ad - bc \neq 0\) are known as conjugate Möbius transformations of \(\mathbb{C}\). It is easy to see that each conjugate Möbius transformation \(f\) is the composition of complex conjugation with a Möbius transformation. Since the complex conjugate transformation and Möbius transformations are homeomorphisms of \(\overline{\mathbb{C}}\) onto itself (complex conjugation is given by reflection in the plane through \(\mathbb{R} \cup \{\infty\}\)), conjugate Möbius transformations are homeomorphisms of \(\overline{\mathbb{C}}\) onto itself. Notice that the composition of a conjugate Möbius transformation with a Möbius transformation is a conjugate Möbius transformation and composition of two conjugate Möbius transformations is a Möbius transformation. There is topological distinction between Möbius transformations and conjugate Möbius transformations: Möbius transformations preserve the orientation of \(\overline{\mathbb{C}}\), whereas conjugate Möbius transformations reverse it. To see

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more details about conjugate Möbius transformations, we refer the reader to [2]. The following definitions are well known and fundamental in hyperbolic geometry.

**Definition 2.** [3] A Lambert quadrilateral is a hyperbolic quadrilateral with ordered interior angles $\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{2}$ and $\theta$, where $0 < \theta < \frac{\pi}{2}$.

**Definition 3.** [3] A Saccheri quadrilateral is a hyperbolic quadrilateral with ordered interior angles $\frac{\pi}{2}, \frac{\pi}{2}, \theta, \theta$, where $0 < \theta < \frac{\pi}{2}$.

A Möbius invariant property is naturally related to hyperbolic geometry. To see the characteristics of Möbius transformations involving Lambert quadrilaterals and Saccheri quadrilaterals, we refer the reader to [4]. Moreover, there are many characterizations of Möbius transformations by using various hyperbolic polygons; see, for instance, [5–7].

In [8, 9], O. Demirel presented some characterizations of Möbius transformations by using new classes of geometric hyperbolic objects called “degenerate Lambert quadrilaterals” and “degenerate Saccheri quadrilaterals”, respectively.

**Definition 4.** [9] A degenerate Lambert quadrilateral is a hyperbolic convex quadrilateral with ordered interior angles $\frac{\pi}{2} + e, \frac{\pi}{2}, \frac{\pi}{2} - e, \theta$, where $0 < \theta < \frac{\pi}{2}$ and $0 < e < \frac{\pi}{2} - \theta$.

**Theorem 5.** [9] Let $f : B^2 \rightarrow B^2$ be a surjective transformation. Then $f$ is a Möbius transformation or a conjugate Möbius transformation if and only if $f$ preserves all $e$-Lambert quadrilaterals.

**Definition 6.** [8] A degenerate Saccheri quadrilateral is a hyperbolic convex quadrilateral with ordered angles $\frac{\pi}{2} - e, \frac{\pi}{2} + e, \theta, \theta$, where $0 < \theta < \frac{\pi}{2}$ and $0 < e < \frac{\pi}{2} - \theta$.

**Theorem 7.** [8] Let $f : B^2 \rightarrow B^2$ be a surjective transformation. Then $f$ is a Möbius transformation or a conjugate Möbius transformation if and only if $f$ preserves all $e$-Saccheri quadrilaterals.

In the theorems above, $B^2$ is the open unit disc in the complex plane. Naturally, one may wonder whether the counterpart of Theorem 7 exists for hyperbolic polygons instead of using degenerate Saccheri quadrilaterals. Before giving the affirmative answer of this question let us state the following definition:

**Definition 8.** Let $n$ be a positive integer satisfying $n \geq 5$. An $n$-sided hyperbolic polygon of type $(e, n)$ is a convex hyperbolic polygon with ordered interior angles $\frac{\pi}{2} + e, \theta_1, \theta_2, \ldots, \theta_{n-2}, \frac{\pi}{2} - e$ or $\frac{\pi}{2} - e, \theta_1, \theta_2, \ldots, \theta_{n-2}, \frac{\pi}{2} + e$, where $0 < e < \frac{\pi}{2}$ and $0 < \theta_1 < \pi$ satisfying

$$\sum_{i=1}^{n-2} \theta_i + \left( \frac{\pi}{2} + e \right) + \left( \frac{\pi}{2} - e \right) < (n-2)\pi,$$

and $\theta_i + \theta_{i+1} \neq \pi (1 \leq i \leq n-3), \theta_1 + (\frac{\pi}{2} + e) \neq \pi, \theta_{n-2} + (\frac{\pi}{2} - e) \neq \pi$. Notice that the sides of the $n$-sided hyperbolic polygon of type $(e, n)$ we mentioned here are hyperbolic line segments.

The existence of $n$-sided hyperbolic polygons of type $(e, n)$ is clear by the following result:

**Lemma 9.** [3] Let $(\theta_1, \theta_2, \cdots, \theta_n)$ be any ordered $n$-tuple with $0 \leq \theta_j < \pi, j = 1, 2, \ldots, n$. Then there exists a polygon $P$ with interior angles $\theta_1, \theta_2, \ldots, \theta_n$, occurring in this order around $\partial P$, if and only if $\sum_{j=1}^{n} \theta_j < (n-2)\pi$.

This paper presents a new characterization of Möbius transformations by use of mappings which preserve $n$-sided hyperbolic polygons of type $(e, n)$. To do so, we need Carathéodory’s theorem which plays a major role in our results. C. Carathéodory [10] proved that every arbitrary one-to-one correspondence between the points of a circular disc $C$ and a bounded point set $C’$ such that which circles lying completely in $C$ are transformed into circles lying in $C’$ must always be either a Möbius transformation or a conjugate Möbius transformation.
Throughout the paper we denote by $X'$ the image of $X$ under $f$, by $[P, Q]$ the geodesic segment between points $P$ and $Q$, by $PQ$ the geodesic through points $P$ and $Q$, by $PQR$ the hyperbolic triangle with vertices $P$, $Q$ and $R$, by $\angle PQR$ the angle between $[P, Q]$ and $[P, R]$ and by $d_H(P, Q)$ the hyperbolic distance between points $P$ and $Q$. We consider the hyperbolic plane $B^2 = \{ z \in \mathbb{C} : |z| < 1 \}$ with length differential $ds^2 = \frac{1}{(1-|z|^2)^2}$. The Poincaré disc model of hyperbolic geometry is built on $B^2$; more precisely the points of this model are points of $B^2$ and the hyperbolic lines of this model are Euclidean semicircular arcs that intersect the boundary of $B^2$ orthogonally including diameters of $B^2$. Given two distant hyperbolic lines that intersect at a point, the measure of the angle between these hyperbolic lines is defined by the Euclidean tangents at the common point.

2 A characterization of Möbius transformations by use of hyperbolic polygons of type $(\epsilon, n)$

The assertion $f$ preserves $n$-sided hyperbolic polygons $A_1A_2 \cdots A_n$ of type $(\epsilon, n)$, $n \geq 5$, with ordered interior angles $\frac{\pi}{2} - \epsilon, \theta_1, \theta_2, \ldots, \theta_{n-2}, \frac{\pi}{2} + \epsilon$ means that the image of $A_1A_2 \cdots A_n$ under $f$ is again an $n$-sided hyperbolic polygon $A'_1A'_2 \cdots A'_n$, with ordered interior angles $\frac{\pi}{2} - \epsilon, \theta'_1, \theta'_2, \ldots, \theta'_{n-2}, \frac{\pi}{2} + \epsilon$ and if $P$ is a point on any side of $A_1A_2 \cdots A_n$, then $P'$ is a point on any side of $A'_1A'_2 \cdots A'_n$.

**Lemma 10.** Let $f : B^2 \rightarrow B^2$ be a mapping which preserves $n$-sided hyperbolic polygons of type $(\epsilon, n)$ for all $0 < \epsilon < \frac{\pi}{2}$. Then $f$ is injective.

**Proof.** Let $A_1$ and $A_2$ be two distinct points in $B^2$. It is clear that there exists an $(2n-4)$-sided hyperbolic regular polygon $(n > 4, n \in \mathbb{N})$, say $A_1A_2 \cdots A_{2n-4}$. By $\beta$ denote the interior angles of $A_1A_2 \cdots A_{2n-4}$. Let $M$ and $N$ be the midpoints of $[A_{2n-4}, A_1]$ and $[A_{n-2}, A_{n-1}]$, respectively. Then the hyperbolic polygons $MA_1A_2 \cdots A_{n-2}N$ and $NA_{n-1}A_n \cdots A_{2n-4}M$ are $n$-sided hyperbolic polygons satisfying $MN \perp A_1A_{2n-4}$ and $MN \perp A_{n-2}A_{n-1}$. Let $P$ be a point on $[M, A_1]$ and let $Q$ be a point on $[N, A_{n-1}]$ satisfying $d_H(P, A_1) = d_H(Q, A_{n-1})$. By $\psi$ denote the angle $\angle QPA_1$. Since $A_1A_2A_3 \cdots A_{2n-4}$ is an $(2n-4)$-sided hyperbolic regular polygon, we immediately get $\angle QPA_{n-1} = \psi$. If $\psi > \frac{\pi}{2}$ let’s denote $\psi = \frac{\pi}{2} + a$ and if $\psi < \frac{\pi}{2}$ let’s denote $\psi = \frac{\pi}{2} - a$. Hence we see that $PA_1A_2 \cdots A_{n-2}Q$ is an $n$-sided hyperbolic polygon of type $(a, n)$. By assumption, we obtain that $P'A'_1A'_2 \cdots A'_{n-2}Q'$ is also an $n$-sided hyperbolic polygon of type $(a, n)$, which implies $A'_1 \neq A'_2$. Thus $f$ is injective. □

**Lemma 11.** Let $f : B^2 \rightarrow B^2$ be a mapping which preserves $n$-sided hyperbolic polygons of type $(\epsilon, n)$ for all $0 < \epsilon < \frac{\pi}{2}$. Then $f$ preserves the collinearity and betweenness property of the points.

**Proof.** Let $P$ and $Q$ be two distinct points in $B^2$ and assume that $S$ is an interior point of $[P, Q]$. Let $\Delta$ be the set of all $n$-sided hyperbolic polygons of type $(\epsilon, n)$ such that the points $P$ and $Q$ are two adjacent vertices of these hyperbolic polygons. Then $S$ belongs to all elements of $\Delta$. By the property of $f$, the images of the elements of $\Delta$ are $n$-sided hyperbolic polygons of type $(\epsilon, n)$ whose vertices contain $P'$ and $Q'$. Moreover, the images of the elements of $\Delta$ must contain $S'$. Since $f$ is injective by **Lemma 10**, we get $P' \neq S' \neq Q'$. Therefore, $S'$ must be an interior point of $[P', Q']$, which implies that $f$ preserves the collinearity and betweenness of the points. □

**Lemma 12.** Let $f : B^2 \rightarrow B^2$ be a mapping which preserves $n$-sided hyperbolic polygons of type $(\epsilon, n)$ for all $0 < \epsilon < \frac{\pi}{2}$. Then $f$ preserves the angles $\frac{\pi}{2} + \epsilon$ and $\frac{\pi}{2} - \epsilon$.

**Proof.** Let $A_1A_2 \cdots A_n$ be an $n$-sided hyperbolic polygon of type $(\epsilon, n)$ such that $\angle A_{i-1}A_iA_{i+1} = \frac{\pi}{2} - \epsilon, \angle A_{i-1}A_iA_{i+1} = \frac{\pi}{2} - \epsilon$. Let us denote the midpoint of $A_1$ and $A_n$ by $M$. Clearly, the hyperbolic line $A_1A_2$ and the complex unit disc $B^2$ meet at two points, say $P$ and $Q$. Assume $A_1 \in [P, A_2]$ and $\angle MA_2Q = \rho$. Let $X$ be a point on $[P, A_2]$ moving from $P$ to $A_2$. It is easy to
see that if $X$ moves from $P$ to $A_2$, the angle $\angle MXQ$ increases continuously from 0 to $\pi$. Let $H$ be a point on $[A_1, A_2]$ such that $\angle MHA_2 < \frac{\pi}{2}$. Now take a point on the hyperbolic line $AnAn_1$, say $S$, satisfying $A_n \in [S, An_1]$ and $d_H(S, An) = d_H(H, A_1)$. It is easy to see that $d_H(M, A_1) = d_H(M, An)$, $\angle SA_nM = \angle HA_1M$, and $d_H(An, H) = d_H(An, S)$ hold, which indicates that the hyperbolic triangles $HA_1M$ and $SA_nM$ are congruent by the hyperbolic side-angle-side theorem. Hence we get $\angle A_1MH = \angle A_nMS$ and this yields that the points $H, M$ and $S$ must be collinear and $\angle MHA_1 = \angle MSAn$. Since $\angle MHA_2 < \frac{\pi}{2}$, we may assume the representation $\angle MHA_2 = \frac{\pi}{2} - \alpha$, where $0 < \alpha < \frac{\pi}{2}$, which implies $\angle MSAn = \frac{\pi}{2} + \alpha$. Notice that $\alpha$ must be less than $\pi$ since $\frac{\pi}{2} - \epsilon < \frac{\pi}{2} - \alpha$. Therefore, one can easily see that $HA_2 \cdots An_1S$ is an $n$-sided hyperbolic polygon of type $(a, n)$. By the property of $f$, the images of the hyperbolic polygons $A_1A_2 \cdots An$ and $HA_2 \cdots An_1S$ are $n$-sided hyperbolic polygons of type $(c, n)$ and type $(a, n)$, respectively. The hyperbolic polygons $A_1'A_2' \cdots An_1'S'$ have $n - 2$ common angles, which implies $\angle A_n'A_1'A_2' = \frac{\pi}{2} - \alpha$ and $\angle S'H'A_2' = \frac{\pi}{2} + \alpha$. Now assume $\angle A_n'A_1'A_2' = \frac{\pi}{2} + \alpha$. By Lemma 11, we get $H' \in [A_1', A_2']$, $M' \in [A_1', A_n']$ and $M' \in [H', S']$. Since $\angle S'H'A_2' = \frac{\pi}{2} - \alpha$ and $\alpha < \epsilon$ hold, we get that the sum of the measures of interior angles of the hyperbolic triangle $M'A_1'H'$ is $\angle A_1'M'H' + \left(\frac{\pi}{2} + \alpha\right) + \left(\frac{\pi}{2} + \alpha\right)$, which is greater than $\pi$. This is not possible in hyperbolic geometry and so we get $\angle A_n'A_1'A_2' = \frac{\pi}{2} - \epsilon$, which yields $\angle A_n'A_1'A_2' = \frac{\pi}{2} - \epsilon$.}

\textbf{Lemma 13.} Let $f : B^2 \to B^2$ be a mapping which preserves $n$-sided hyperbolic polygons of type $(c, n)$ for all $0 < \epsilon < \frac{\pi}{2}$. Then $f$ preserves the hyperbolic distance.

\textbf{Proof.} Let $P$ and $Q$ be two distinct points in $B^2$. Take a point $S$ such that $PQS$ forms a hyperbolic equilateral triangle. By $\beta$ denote its angles $\angle PQS = \angle QSP = \angle SPQ = \beta$. Since $\beta < \frac{\pi}{2}$ let’s use the representation $\beta = \frac{\pi}{2} - \alpha$ with $0 < \alpha < \frac{\pi}{2}$. By Lemma 9, there exists an $n$-sided hyperbolic polygon of type $(a, n)$, say $A_1A_2 \cdots An$, such that $\angle A_1A_2A_3 = \gamma_2$, $\angle A_2A_3A_4 = \gamma_3$, \ldots, $\angle An-2An-1An = \gamma_{n-1}$, $\angle An-1AnA_1 = \frac{\pi}{2} + \alpha$ and $\angle AnA_1A_2 = \frac{\pi}{2} - \alpha$. Then the angle $\angle A_nA_1A_2$ of the hyperbolic polygon $A_1A_2 \cdots An$ can be moved to the point $P$ by using an appropriate hyperbolic isometry $g$ such that $g(A_2) \in [P, S]$ (or $S \in [P, g(A_2)]$) and $g(An) \in [P, Q]$ (or $Q \in [P, g(An)]$). Since $f$ preserves the angles $\frac{\pi}{2} + \alpha$ and $\frac{\pi}{2} - \alpha$ of $n$-sided hyperbolic polygons of type $(c, n)$ for all $0 < \epsilon < \frac{\pi}{2}$ by Lemma 12, we get

$$\pi - \alpha = \angle SPQ = \angle g(An)g(A_1)g(A_2) = \angle g(A_n)'g(A_1)'g(A_2)' = \angle S'P'Q',$$

which implies $\angle PQS = \angle P'Q'S'$ and $\angle QSP = \angle Q'S'P'$. Because of the fact that the angles at the vertices of a hyperbolic triangle determine its lengths, we get $d_H(P, Q) = d_H(P', Q')$; see [11, 12].

\textbf{Corollary 14.} Let $f : B^2 \to B^2$ be a mapping which preserves $n$-sided hyperbolic polygons of type $(c, n)$ for all $0 < \epsilon < \frac{\pi}{2}$. Then $f$ preserves the measures of all angles.

\textbf{Theorem 15.} Let $f : B^2 \to B^2$ be a surjective transformation. Then $f$ is a Möbius transformation or a conjugate Möbius transformation if and only if $f$ preserves $n$-sided hyperbolic polygons of type $(c, n)$ for all $0 < \epsilon < \frac{\pi}{2}$.

\textbf{Proof.} The “only if” part is clear since $f$ is an isometry. Conversely, we may assume that $f$ preserves all $n$-sided hyperbolic polygons of type $(c, n)$ in $B^2$ and $f(O) = O$ by composing an hyperbolic isometry if necessary. Let us take two distinct points in $B^2$ and denote them by $x, y$. By Lemma 13, immediately we get $d_H(O, x) = d_H(O, x')$ and $d_H(O, y) = d_H(O, y')$, namely $|x| = |x'|$ and $|y| = |y'|$, where $|\cdot|$ denotes the Euclidean norm. Therefore, we get $|x - y| = |x' - y'|$ since $f$ preserves angular sizes by Corollary 14. As

$$2\langle x, y \rangle = |x|^2 + |y|^2 - |x - y|^2 = |x'|^2 + |y'|^2 - |x' - y'|^2 = 2\langle x', y' \rangle,$$

$f$ preserves the Euclidean inner-product this implies that $f$ is a restriction of an orthogonal transformation on $B^2$, that is, $f$ is a Möbius transformation or a conjugate Möbius transformation by Carathéodory’s theorem. \hfill $\Box$

\textbf{Corollary 16.} Let $f : B^2 \to B^2$ be a conformal (angle preserving with sign) surjective transformation. Then $f$ is a Möbius transformation if and only if $f$ preserves $n$-sided hyperbolic polygons of type $(c, n)$ for all $0 < \epsilon < \frac{\pi}{2}$. \hfill $\Box$
Corollary 17. Let \( f : B^2 \rightarrow B^2 \) be an angle reversing surjective transformation. Then \( f \) is a conjugate Möbius transformation if and only if \( f \) preserves \( n \)-sided hyperbolic polygons of type \((e, n)\) for all \( 0 < e < \frac{\pi}{2} \).

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