Full Current Statistics of Incoherent "Cold Electrons"

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We evaluate the full current statistics (FCS) in the low dimensional (1D and 2D) diffusive conductors in the incoherent regime, $eV > E_{\text{Th}} = D/L^2$, $E_{\text{Th}}$ being the Thouless energy. It is shown that Coulomb interaction substantially enhances the probability of big current fluctuations for short conductors with $E_{\text{Th}} \gg 1/\tau_E$, $\tau_E$ being the energy relaxation time, leading to the exponential tails in the current distribution. The current fluctuations are most strong for low temperatures, provided $E_{\text{Th}} \sim \sqrt{eV/g}$ for 1D and $E_{\text{Th}} \sim (eV/g) \ln g$ for 2D, where $g$ is a dimensionless conductance and $\nu_1$ is a 1D density of states. The FCS in the "hot electron" regime is also discussed.

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The influence of the Coulomb interaction onto the transport properties of low-dimensional diffusive systems has been a subject of extensive research for more than twenty years [1,2]. Initially, the conductance only was a main object of study. The powerful alternative to that is to investigate a quantum noise [3,4], or, more generally, the full current statistics (FCS) [5].

In the short diffusive wires with $E_{\text{Th}} \gg eV$, where $E_{\text{Th}} = D/L^2$ is the Thouless energy, the shot noise equals to $S = 2|e|IF$, $F = 1/3$ being the Fano factor [6,7]. In this case the conductor is coherent and effectively zero-dimensional so that all effects of Coulomb interaction come from the external electromagnetic environment. It modifies the conductance, noise [8] and generally the FCS [8,10].

Much less is known about the role of Coulomb interaction onto the FCS in the quasi-1D and 2D diffusive systems, when $E_{\text{Th}} \ll eV$. Under this condition the inelastic electron-electron scattering inside the conductor is important. This subject has recently attracted the attention in Ref. [11,12,13], where the "hot electron" regime was discussed. In this regime $E_{\text{Th}} \ll 1/\tau_E$, $\tau_E$ being the energy relaxation time, and the electron distribution function relaxes to the local Fermi distribution. This changes the Fano factor $F$ from $1/3$ to $\sqrt{3}/4$ [4], that was confirmed experimentally [5].

The microscopic theory [5] of electron-electron interaction in low-dimensional disordered conductors predicts, however, two different time scales, $\tau_0$ and $\tau_E$, where $\tau_0 \ll \tau_E$ is a dephasing time (See Table I). It is usually believed [10] that classical phenomena described by the Boltzmann equation are governed by $\tau_E$. While the time $\tau_0$ manifests itself in essentially quantum-mechanical phenomena. Since the FCS is a classical quantity one might expect it to cross over between the coherent and the "hot electron" regime on the scale $E_{\text{Th}} \sim 1/\tau_E$.

In this paper we consider the FCS in the low dimensional ($d = 1, 2$) diffusive conductors, taking into account the Coulomb interaction. We show that the time $\tau_E$ is indeed responsible for the smooth crossover between the coherent and the "hot electron" limits if one considers the noise and the 3rd cumulant. However it is not the case for the higher order cumulants of charge transfer in the shot noise limit $eV \gg T$. Moreover, in this limit the smooth crossover in the FCS does not exist. The Coulomb interaction drastically enhances the probability of current fluctuations for short conductors $E_{\text{Th}} \gg 1/\tau_E$. In this regime the higher order cumulants are given by

$$\langle n^{2k,2k+1} \rangle \propto \frac{(eV)^{d/2}}{g} \left( \frac{eV}{\omega^*} \right)^{k-1-\frac{d}{2}}, \quad k > 2$$

where $(n^{i}) \gg 1$ is the average number of electrons transferred, $g \gg 1$ is a dimensionless conductance, $\omega^* = \max\{E_{\text{Th}}, T, e^*\}$ and the scale $e^*$ reads

$$e^*(V) \simeq \begin{cases} (eV)^2/g^2 E_{\text{Th}}, & 1D \\ eV \exp[-g E_{\text{Th}}/eV], & 2D. \end{cases}$$

Eq. (11) shows that each $(k+1)$-th cumulant of charge transfer is parametrically enhanced versus the $k$-th one by the large factor $eV/\omega^* \gg 1$. It also follows from Eq. (11) that the higher cumulants grow with increasing the voltage at $E_{\text{Th}} > 1/\tau^*$ and decay at $E_{\text{Th}} < 1/\tau^*$, where the new time scale $\tau^*(eV, T)$ is parametrically smaller than $\tau_E$, $\tau^* \ll \tau_E$. (See Table I). The current fluctuations are most strong, provided $T \lesssim E_{\text{Th}} \sim 1/\tau_0(V)$. Therefore at the strongly non-equilibrium situation the time $\tau_0$ rather than $\tau_E$ governs the crossover in the FCS between the coherent and the "hot electron" limits.

**Model and the effective action.** We consider a quasi-one-dimensional (1D) diffusive wire of a length $L$ and a quasi-two-dimensional (2D) film of a size $L \times L$, with dimensionless conductance $g \gg 1$ and diffusion coefficient

| $d$ | $1/\tau_E$ | $1/\tau_0$ | $1/\tau^*$, $T \gtrsim \tau_0^{-1}(V)$ |
|-----|------------|----------|-----------------|
| 1   | $(E/D)^{1/2} \nu_1^{-1}$ | $(E^2/Dv_1^2)^{1/3}$ | $(eV/T)^{1/2} \tau_0^{-1}(V)$ |
| 2   | $E/g$     | $(E/g) \ln g$ | $\ln(eV/T) \tau_0^{-1}(V)$ |

**TABLE I:** The electron scattering times for low-dimensional (1D and 2D) diffusive conductors, $E = \max\{T, eV\}$. At $T \lesssim \tau_0^{-1}(V)$, we get $\tau^* = \tau_0(V)$.
D. They are attached to two reservoirs with negligible external impedance which are kept at voltages \( \pm V/2 \). The current flows along \( z \) direction. We assume the incoherent regime, \( \max\{eV, T\} \gg E_{\text{Th}} \), and disregard the possible electron-phonon scattering, so that \( L \ll L_{e-\text{ph}} \).

Our goal is to evaluate the cumulant generating function (CGF) \( S(\chi) \). The Fourier transform of \( \exp(-S) \) with respect to the "counting field" \( \chi \) gives the current probability distribution \( P(I) \) (See \[3\]). The derivatives of \( S \) give the average value of current, shot-noise and higher order moments \( \langle n^k \rangle \) of charge transfer during the observation time \( t_0 \).

To evaluate the CGF, taking into account the Coulomb interaction, we have used the Keldysh technique and employed the low-energy field theory of the diffusive transport \[17\] with the action

\[
F[\chi, Q, A] = \frac{1}{8} g L^2 \int d^3r \text{Tr} \left( \nabla Q - i[A, Q] \right)^2 + (3)
\]

\[2\pi \nu_d \int d^3r \text{Tr} \left( i \partial_t Q \right) - \frac{i}{8\pi e^2} \int d t \int d^3r \left( A_1^2 - A_2^2 \right)
\]

Here \( A = \text{diag}(\mathbf{A}_1(t, r), \mathbf{A}_2(t, r)) \) is the \( 2 \times 2 \) matrix in Keldysh space, where \( \mathbf{A}_{1,2} \) stand for fluctuating vector potentials in the conductor. Hereafter we use a longitudinal photon field, \( \text{curl} \mathbf{A} = 0 \), thus neglecting the relativistic effects.

The matrix \( \hat{Q}(r, t_1, t_2) \) accounts for the electrode degree of freedom and obeys the semi-classical constrain \( \hat{Q}(r) \circ \hat{Q}(r) = \delta(t_1 - t_2) \). The action \( F \) depends on \( \chi \) via the boundary conditions imposed on the field \( Q \) at the boundaries with the left(L) and right(R) reservoirs \[18\]:

\[
Q|_{r=R} = \hat{G}_R \text{ and } Q|_{r=L} = \hat{G}_L(\chi) = \exp(i\chi \hat{\tau}_3/2)G_L \exp(-i\chi \hat{\tau}_3/2).
\]

Here \( G_{L,R} \) are the Keldysh Green functions in the leads.

With action \[4\] the CGF should be evaluated as a path integral over all possible realization \( \mathbf{A}_{1,2} \) and \( \hat{Q} \). We proceed along the lines of Ref. \[17\] and employ the parameterization \( Q = e^{iW} \hat{G} e^{-iW} \), \( WF + GW = 0 \). Here field \( W \) accounts for the rapid fluctuations of \( Q \) with typical frequencies \( \omega \gg E_{\text{Th}} \) and momenta \( q \gg 1/L \), while \( \hat{G}(\epsilon, r) \) is the stationary Green function varying in space on the scale \( \sim L \). First we integrate out the field \( W \) in the Gaussian approximation to obtain the nonlinear action \( \bar{F}(\chi, \hat{G}, \mathbf{A}) \) of the screened electromagnetic fluctuations. We keep only quadratic terms to \( \bar{F} \) that is equivalent to the random phase approximation (RPA). At the second step one can integrate the photon field \( \mathbf{A} \) and arrive to the effective action \( F_{\text{eff}}[\chi, \hat{G}] \). Then the saddle point approximation, \( \delta F_{\text{eff}}[\chi, \hat{G}] / \delta \hat{G} = 0 \) yields the kinetic equation for \( \hat{G}(\epsilon, r) \).

For the rest we restrict consideration to the universal limit of a short screening radius \( r^{-1} = (4\pi e^2 \nu_d)^{1/2} \gg \sqrt{eV/D} \). In this limit we get the answer

\[
F_{\text{eff}}[\chi, \hat{G}] = \frac{t_0}{8} g L^2 \int d^3r \int d \epsilon \frac{d}{2\pi} \text{Tr} \left( \nabla \hat{G}_\epsilon(r) \right)^2 + (4)
\]

\[
+ \frac{t_0}{2} \int d^3r \int d \omega \frac{d^2q}{(2\pi)^{d+1}} \ln \left| \text{Det} \left[ \Box^{-1}(r, q) \right] \right|
\]

where \( \Box_d \) is \( 2 \times 2 \) matrix operator in Keldysh space, corresponding to the non-equilibrium diffusion propagator:

\[
\mathcal{D}^{\alpha \beta}(r, q) = [Dq^2 \gamma^{\alpha \beta} + (5)
\]

\[
\frac{i}{4} \int dr \text{Tr} \left( \gamma^{\alpha \gamma \beta} - \gamma^{\alpha} \hat{G}_{\epsilon+\omega/2}(r) \gamma^{\beta} \hat{G}_{\epsilon-\omega/2}(r) \right)^{-1}
\]

with \( \gamma^\beta = \hat{1}, \gamma^1 = \hat{\tau}_3 \). To derive the action \[4\] we have used a local approximation, i.e. we neglected gradient corrections proportional to \( \langle \nabla \hat{G} \sim 1/L \rangle \ll \nabla \hat{W} \).

Minimizing the action \( F_{\text{eff}} \) under constrain \( \hat{G}(\epsilon, r)^2 = 1 \) one can obtain the non-linear kinetic equation for \( \hat{G}(\epsilon, r) \) in the form

\[
\left( \hat{G}_\epsilon(r) \right) \nabla^2 \hat{G}_\epsilon(r) = \left[ \mathcal{L}_\epsilon(r), \hat{G}_\epsilon(r) \right],
\]

\[
\mathcal{L}_\epsilon(r) = \frac{i}{8\nu_d} \sum_{\alpha, \beta} \int d \omega \frac{d^2q}{(2\pi)^{d+1}} D^{\alpha \beta}(r, q)
\]

\[
\times \left( \gamma^{\alpha} \hat{G}_{\epsilon-\omega}(r) \gamma^{\beta} + \gamma^{\beta} \hat{G}_{\epsilon+\omega}(r) \gamma^{\alpha} \right)
\]

is the matrix collision integral. This kinetic equation should be supplemented by the \( \chi \)-dependent boundary conditions at the interfaces with the leads. The action \[4\] is one of the central results of the paper. In the absence of "counting" field our matrix kinetic equation reduces to the standard kinetic equation for the distribution function with a singular kernel \( K(\omega) \sim \omega^{d/2-2} \) (See \[17\] \[16\] \[15\]). We also note that only real inelastic processes with energy transfer \( \omega \sim \max\{eV, T\} \) contribute to the action \[4\]. In the rest of the paper we consider the shot-noise limit \( eV \gg T \) only and proceed with

\textit{Incoherent "cold electron" regime, \( E_{\text{Th}} \gg 1/\tau_E \).} In this regime the collision term in kinetic equation is small.

FIG. 1: The sketch of voltage dependence of the \( 6^{\text{th}} \) cumulant of charge transfer. Left plot - 1D, conductance \( g = 100 \), right plot - 2D, \( g = 10^{8} \). The temperature changes from up to down, \( T/E_{\text{Th}} = 1, 2, 4, 8 \).
and one can try to find the Green function perturbatively around the coherent solution which obeys the Usadel equation $\nabla z \left( \hat{G}_e^0(z) \nabla z \hat{G}_o^0(z) \right) = 0$. In the first order in $1/(E_{Th} T_E)$ the CGF can be found by substituting $\hat{G}^0$ in the action $\hat{S}$. The main contribution comes from frequencies $T < \omega < eV$. After some algebra we obtain

$$ S(\chi) = -\frac{1}{8 \pi} e V g \theta_{\chi}^2 (\epsilon) + F_{\text{Coll}}(\chi). $$

(7)

Here $\theta_{\chi} = \ln(u + \sqrt{u^2 - 1})$, $u = 2 \epsilon_{\chi} - 1$ and

$$ F_{\text{Coll}} = \frac{t_0 L^d}{2} \int_0^\infty dz \int_{\omega^* < \omega < eV} d\omega d^d q \left\{ 1 - \frac{N_\omega}{\omega} \left( \frac{\Pi(\chi, z) \omega^2}{(D q^2)^2 + \omega^2} \right) \right\} $$

(8)

$$ \Pi(\chi, z) = -4 L(\chi) R(\chi) e^{i \chi} \left\{ 1 - L(\chi) R(\chi) + \frac{z L(\chi)}{(1 - z) R(\chi)} \right\} $$

(9)

where $N_\omega = (eV/|\omega| - 1)$, $L(\chi) = \sinh(1 - z) \theta_{\chi}$, $R(\chi) = \sinh z \theta_{\chi}$, and $\omega^* = \max\{E_{Th}, T\}$. The presence of Thouless energy in the low momentum cut-off $n^*$ is due to the fact that the lowest allowed momenta $q$ in the diffusion propagator equals to $1/L$, while $T$ takes into account the smearing of a step in the Fermi distribution.

The result with the above defined cut-off $\omega^*$ ceased to be valid at sufficiently high voltages. Indeed, substituting a zero order distribution function $f_0(\epsilon) = (1 - z) f_F(\epsilon - eV/2) + z f_F(\epsilon + eV/2)$ to the collision integral, one estimates the 1st order correction

$$ f_{(1)}(\epsilon_{\pm}) \sim \frac{L^2 D}{eV} \int_{\pi} eV d\omega \left( \frac{\epsilon_+ - \epsilon_0}{\omega^2} \right) $$

(10)

if $\epsilon_{\pm} = \epsilon \pm eV/2 \ll eV$ and $\epsilon_{\pm} > \max\{E_{Th}, T\}$. By virtue of Pauli principle this correction may not exceed unity, $\delta f_{(1)} \lesssim 1$, which is true for $\epsilon_{\pm} \gg \epsilon^*$ only, where the scale $\epsilon^*$ is given by Eq. $\mathbf{4}$. Therefore a simple perturbation theory is valid provided $\epsilon^* < \max\{E_{Th}, T\}$. Resolving this condition we obtain that it is the case of relatively short conductors, or small voltages, $E_{Th} > 1/\tau^*$. The time $\tau^*$ decreases with increasing the voltage and the 1st order perturbation theory finally breaks down at $E_{Th} < 1/\tau^*$. In this situation we can obtain the result up to the factor of order of unity using the cut-off $\omega^* \simeq \epsilon^*$ in Eq. $\mathbf{5}$. The point is that the role of higher orders terms in the perturbation series is to smear the step in the distribution function $f(\epsilon)$ around $\epsilon = \pm eV/2$ on the scale $\epsilon^*$, while at $\epsilon \gg \epsilon^*$ the 1st order perturbation theory is still applicable. Therefore the overall effect is similar to the increase of the temperature in the system.

The result with the cut-off $\omega^* = \max\{E_{Th}, T, \epsilon^*\}$ enables to evaluate all irreducible cumulants $\langle n^k \rangle = i^k \epsilon^k / \partial \chi^k S(\chi)$ of a number of electrons transferred. The expansion of $\Pi(\chi, z)$ in $\chi$ starts from $\chi^2$. Thus there is no correction to the current on the classical level. The interaction correction to the noise and to the 3rd cumulant is small by the parameter $1/(E_{Th} T_E)$ and it is dominated by inelastic collisions with the energy transfer $\omega \sim eV$. On the contrary, the leading contribution to the higher order cumulants is due to Coulomb interaction and it is dominated by quasi-elastic collisions with small energy transfers $\omega^* \lesssim \omega \ll eV$. Up to the numerical constant the result is given by Eq. $\mathbf{4}$. The sketch of the voltage dependence for the 6th cumulant at different temperatures is shown in Fig.1. The cumulants grow with voltage in the range $E_{Th} > 1/\tau^*$ and decay if $E_{Th} < 1/\tau^*$. The enhancement of higher order cumulants are most strong at small temperatures $T \lesssim E_{Th}$. In this case their maximum occurs at $eV/E_{Th} \approx g$ for 1D and at $eV/E_{Th} \approx g / \ln g$ for 2D.

With result $\mathbf{7}$ $\mathbf{8}$ $\mathbf{9}$ we can also explore the current probability distribution $P(I) = (2\pi)^{-1} \int_{-\infty}^{\infty} \exp\{-S(\chi) + i/(\Pi_0/e)\chi\}$ in the long time limit $\langle \Pi_0/e\rangle \gg 1$. The action $S(\chi)$ has two branch points $\chi = \pm i\gamma$, where $\gamma \sim (\omega^*/eV)^{1/2} \ll 1$. The points $\pm i\gamma$ give two threshold currents, $I^\pm = (e/\Pi_0)\delta S/\delta \chi|_{\chi = \pm i\gamma}$, which read

$$ (I^+ - \langle I \rangle)/\langle I \rangle = \pm i\gamma/3. $$

Provided $I^- < I < I^+$ the probability can be evaluated with a saddle point method. Due to the smallness of parameter $1/E_{Th} T_E$ we found that $P(I)$ only slightly deviates from the probability $P_0(I)$ of current fluctuations in the non-interacting limit. For larger current fluctuations the potential $\Omega(\chi) = -S(\chi) + i/(\Pi_0/e)\chi$ does not possess the saddle point any more and one should use the contour $C_0$ of a zero phase, $\text{Im} \Omega(\chi)|_{\chi \in C_0} = 0$ for the asymptotic analysis of the integral $P(I)$. This contour is panned by the branch point $\chi = \pm i\gamma$, that yields the exponential tails in the current probability distribution

$$ P(I) \approx \exp\{-S(\pm\gamma) - \gamma|\Pi_0/e\langle I \rangle\}, \ I < I^- \text{ or } I > I^+ $$

(11)

The results for the probability distribution are displayed in Fig.2. The Coulomb interaction does not affect the
Gaussian fluctuations. However the tails of $P(I)$ drastically differ from those in the absence of interaction.

The FCS of this type can be understood as the statistics of the random current of electron-hole pairs which are excited by the low frequency fluctuations of the electromagnetic field, produced by all other electrons in the system. To shed more light on this point we note that at small frequencies $\omega \ll eV$ the factor $N_\omega \equiv eV/|\omega|$ in Eq. (3) can be associated with non-equilibrium photon distribution. This makes the $F_{\text{col}}$ similar to the photocount statistics studied recently by Kindermann et al. in Ref. [19]. In the latter case each photon being transmitted through the waveguide and absorbed by the photo-detector produces a single count. In the given case a single absorption of a photon by the electron gas generates the random current pulse in the circuit with a zero mean value. The generating function of the current intensity is given by $\Pi(0)$. Due to photon bunching, $N_\omega \gg 1$, the pulse is strongly amplified thereby producing the long exponential tails in the probability distribution $P(I)$.

"Hot electron" regime, $E_{\text{Th}} \ll 1/\tau_E$. In this limit the collision term in the kinetic equations dominates. Therefore the limiting saddle point of the action $I$ should nullify the collision integral. To find such solution we note that the collision term in the action is invariant under the gauge transformation $G_\epsilon(r) = e^{-iK_\epsilon(r)}G_\epsilon(r)e^{iK_\epsilon(r)}$. Here $K_\epsilon(r) = \frac{1}{2}i\epsilon_0[\epsilon_0(r) + \beta(r)(\epsilon - \phi(r))]$ and $\gamma, \beta$, and $\phi$ are arbitrary functions in space. This leads to the conservation of a current density, and a density of the energy flow. As well known, the physical Green function $G(\epsilon, r)$ with a local Fermi distribution $f_\epsilon(r) = [e^{iK_\epsilon(r)}/T(\epsilon+1)]^{-1}$ nullifies the collision term in the conventional kinetic equation. Its gauge transform $\hat{G}_\epsilon(r)$ does the same for the generalized matrix kinetic equation.

The unknown functions $\phi, \gamma, T, \beta$ can be found from the extremum of the action $F_{\text{hot}}$ which is obtained by substitution of $\hat{G}_\epsilon(r)$ into the diffusive part of the action $F_{\text{eff}}$. The action $F_{\text{hot}}$ reads

$$F_{\text{hot}} = (2\pi)^{-1}g_0 \int_0^1 dz \{ -T(\nabla \gamma - \beta \nabla \phi)^2$$

$$+ (\nabla \gamma - \beta \nabla \phi) \nabla \phi - \frac{\gamma^2}{3} T^2 (\nabla \beta)^2 + \frac{\beta^2}{6} (\nabla T^2 \nabla \beta) \}$$

(12)

Here $T(z)$ and $\phi(z)$ are a local temperature and a chemical potential, while $\beta(z)$ and $\gamma(z)$ are their quantum counterparts. The action (12) implies boundary conditions: $\phi(z)|_{z=0,1} = \pm eV/2$, $T(z)|_{z=0,1} = T$, $\gamma(0) = \chi$ and $\gamma(1) = \beta(0) = \beta(1) = 0$. With the use of integrals of motion the Lagrange equations of this action can be reduced to two coupled second order differential equations for $T(z)$ and $\beta(z)$. We are not aware of their analytical solutions under non-zero $\chi$ and solved them numerically. The results for the probability distribution $P(I)$ are shown in Fig. 2. As in the previous section we have evaluated it with the use of the saddle point method. Fig. 2 shows that the probability of positive current fluctuations, $\Delta I > 0$, is enhanced in the "hot electron" limit as compared to the coherent regime, while the probability of negative fluctuations, $\Delta I < 0$, is affected in the lesser extent. We also note that the action (12) is equivalent to the actions of Ref. [12] [13], which were derived with the use of Boltzmann-Langevin approach. These actions transforms to $F_{\text{hot}}$ under appropriate change of variables.

To conclude we investigated the effect of Coulomb interaction onto the FCS in the one- and two-dimensional diffusive conductors. We have revealed the long exponential tails in the probability of non-equilibrium big current fluctuations in short conductors with $E_{\text{Th}} \gg 1/\tau_E$, $\tau_E$ being the energy relaxation time. These tails arise from the huge fluctuations of the current of electron-hole pairs which are excited by the low frequency fluctuations of the electromagnetic field, produced by all other electrons in the system.

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