ON THE HIGHER RANK NUMERICAL RANGE
OF THE SHIFT OPERATOR

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Abstract

For any $n$-by-$n$ complex matrix $T$ and any $1 \leq k \leq n$, let $\Lambda_k(T)$ be the set of all $\lambda \in \mathbb{C}$ such that $PTP = \lambda P$ for some rank-$k$ orthogonal projection $P$ be its higher rank-$k$ numerical range. It is shown that if $S_n$ is the $n$-dimensional shift on $\mathbb{C}^n$, then its rank-$k$ numerical range is the circular disc centered in zero and with radius $\cos\left(\frac{k\pi}{n+1}\right)$ if $1 < k \leq \left\lfloor \frac{n+1}{2} \right\rfloor$ and the empty set if $\left\lfloor \frac{n+1}{2} \right\rfloor < k \leq n$, where $[x]$ denote the integer part of $x$. This extends and refines previous results of Haagerup and de la Harpe [11] on the classical numerical range of the $n$-dimensional shift on $\mathbb{C}^n$. An interesting result for higher rank-$k$ numerical range of nilpotent operator is also established.

2010 Mathematics Subject Classification: 47A12, 47B35.

Keywords and phrases: operator theory, numerical radius, numerical range, higher rank numerical range, eigenvalues.

Submitted by Mubariz T. Karaev.

Received April 19, 2011

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1. Introduction

Let $\mathcal{H}$ be a complex separable Hilbert space and $\mathcal{B}(\mathcal{H})$ be the collection of all bounded linear operators on $\mathcal{H}$. The numerical range of an operator $T$ in $\mathcal{B}(\mathcal{H})$ is the subset

$$W(T) = \{ \langle Tx, x \rangle \in \mathbb{C}; x \in \mathcal{H}, \|x\| = 1 \},$$

of the plane, where $\langle ., . \rangle$ denotes the inner product in $\mathcal{H}$ and the numerical radius of $T$ is defined by

$$\omega_2(T) = \sup \{ |z|; z \in W(T) \}.$$  

We denote by $S$ the unilateral shift acting on the Hardy space $\mathbb{H}^2$ of the square summable analytic functions

$$S : \mathbb{H}^2 \to \mathbb{H}^2,$$

$$f \mapsto zf(z).$$

Beurling’s theorem implies that the nonzero invariant subspaces of $S$ are of the form $\phi \mathbb{H}^2$, where $\phi$ is some inner function. Let $S(\phi)$ denote the compressed of $S$ to the subspace $H(\phi) = \mathbb{H}^2 \Theta \phi \mathbb{H}^2$:

$$S(\phi)f(z) = P(zf(z)),$$

where $P$ denotes the orthogonal projection from $\mathbb{H}^2$ onto $H(\phi)$. We denote by $S^*(\phi)$ the adjoint of $S(\phi)$:

$$S^*(\phi) = S(\phi)^* = S^*_{|H(\phi)} = S^*_{|\ker(\phi(S)^*)}.$$  

The space $H(\phi)$ is a finite-dimensional exactly when $\phi$ is a finite Blaschke product. The numerical radius and numerical range of the model operator $S(\phi)$ seems to be important and have many applications. In [1], Badea and Cassier showed that, there is a relationship between numerical radius of $S(\phi)$ and Taylor coefficients of positive rational
functions on the torus and more recently in [8], the author gave an extension of this result. However, the evaluation of the numerical radius of $S(\phi)$ under an explicit form is always an open problem. The readers may consult [8] for an estimate of $S(\phi)$, where $\phi$ is a finite Blaschke product with unique zero. In the particular case, where $\phi(z) = z^n$, $S(\phi)$ is unitarily equivalent to $S_n$, where

$$S_n = \begin{pmatrix} 0 & \cdots & \cdots & 0 \\ 1 & \ddots & \cdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 1 & \cdots & \cdots & 0 \end{pmatrix}.$$ 

In [11], it is proved that $W(S_n)$ is the closed disc $D_n = \{z \in \mathbb{C}; |z| \leq \cos \left( \frac{\pi}{n+1} \right) \}$ and $\omega_2(S_n) = \cos \left( \frac{\pi}{n+1} \right)$ and more general

**Theorem 1.1 ([11]).** Let $T$ be an operator on $\mathcal{H}$ such that $T^n = 0$ for some $n \geq 2$. One has

$$\omega_2(T) \leq \|T\| \cos \left( \frac{\pi}{n+1} \right),$$

and $\omega_2(T) = \|T\| \cos \left( \frac{\pi}{n+1} \right)$ when $T$ is unitarily equivalent to $\|T\|S_n$.

In this mathematical note, we extend this result to the higher rank-$k$ numerical range of the shift. The notion of the higher rank-$k$ numerical range of an operator $T$ acting on a Hilbert space $\mathcal{H}$ of dimension at least $k$ is introduced in [5] and it is denoted by

$$\Lambda_k(T) = \{\lambda \in \mathbb{C}: PTP = \lambda P, \text{ for some rank-}k \text{ orthogonal projection } P\}.$$ 

The introduction of this notion was motivated by a problem in quantum error correction; see [4]. Note that if $P$ is a rank-1 orthogonal projection, then $P = x \otimes x$ for some unit $x \in \mathbb{C}^n$ and $PTP = \langle Tx, x \rangle P$. Thus when $k = 1$, this concept is reduces to the classical numerical range $W(T)$, which is well known to be convex by the Toeplitz-Hausdorff
theorem. We refer to [14] for a simple proof. In [7], it is conjectured that \( \Lambda_k(T) \) is convex, and reduced the convexity problem to the problem of showing that \( 0 \in \Lambda_k(T') \), where

\[
T' = \begin{pmatrix} I_k & X \\ Y & -I_k \end{pmatrix},
\]

for arbitrary \( X, Y \in \mathcal{M}_k \) (the algebra of \( k \times k \) complex matrix). They further reduced this problem to the existence of a Hermitian matrix \( H \) satisfying the matrix equation

\[
I_k + MH + HM^* - HRH = H,
\]

for arbitrary \( M \in \mathcal{M}_k \) and a positive definite \( R \in \mathcal{M}_k \). In [21], Woerdeman proved that Equation (1.1) is equivalent to Ricatti equation

\[
HRH - H(M^* - I_k / 2) - (M - I_k / 2)H - I_k = 0_k,
\]

and using the theory of Ricatti equations (see [13], Theorem 4), the Equation (1.2) is solvable, which prove the convexity of \( \Lambda_k(T) \). In [5], the authors showed that if \( \dim \mathcal{H} < \infty \) and \( T \in \mathcal{B}(\mathcal{H}) \) is a Hermitian matrix with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \), then the rank-\( k \) numerical range \( \Lambda_k(T) \) coincides with \( [\lambda_k, \lambda_{n+1-k}] \), which is a non-degenerate closed interval if \( \lambda_k < \lambda_{n+1-k} \), a singleton set if \( \lambda_k = \lambda_{n+1-k} \), and an empty set if \( \lambda_k > \lambda_{n+1-k} \). In [17], the authors proved that if \( \dim \mathcal{H} = n \), then

\[
\Lambda_k(T) = \bigcap_{\theta \in [0, 2\pi]} \left\{ \mu \in \mathbb{C} : e^{i\theta} \mu + e^{-i\theta} \bar{\mu} \leq \lambda_k(e^{i\theta} T + e^{-i\theta} T^*) \right\},
\]

for \( 1 \leq k \leq n \), where \( \lambda_k(H) \) denote the \( k \)-th largest eigenvalue of the Hermitian matrix \( H \in \mathcal{M}_n \). Thanks to this result, they establish that if \( \dim \mathcal{H} = n \) and \( T \in \mathcal{B}(\mathcal{H}) \) is a normal matrix with eigenvalues \( \lambda_1, \ldots, \lambda_n \), then

\[
\Lambda_k(T) = \bigcap_{1 \leq j_1 < \cdots < j_{n+1-k} \leq n} \text{conv}\{\lambda_{j_1}, \ldots, \lambda_{j_{n+1-k}}\}.
\]
We close this section by the following properties, which are easily checked. The readers may consult [4-7], [9], and [16].

(P1) For any \( a \) and \( b \in \mathbb{C} \), \( \Lambda_k(aT + bI) = a\Lambda_k(T) + b \).

(P2) \( \Lambda_k(T^*) = \overline{\Lambda_k(T)} \).

(P3) \( \Lambda_k(T \oplus S) \supseteq \Lambda_k(T) \cup \Lambda_k(S) \).

(P4) For any unitary \( U \in \mathcal{B}(\mathcal{H}) \), \( \Lambda_k(U^*TU) = \Lambda_k(T) \).

(P5) If \( T_0 \) is a compression of \( T \) on a subspace \( \mathcal{H}_0 \) of \( \mathcal{H} \) such that \( \dim \mathcal{H}_0 \geq k \), then \( \Lambda_k(T_0) \subseteq \Lambda_k(T) \).

(P6) \( W(T) \supseteq \Lambda_2(T) \supseteq \Lambda_3(T) \supseteq \ldots \).

Some results from [1] will be also developed in this context in a forthcoming paper.

2. Main Theorem

In the following theorem, we give the higher rank-\( k \) numerical range of the \( n \)-dimensional shift on \( \mathbb{C}^n \).

**Theorem 2.1.** For any \( n \geq 2 \) and \( 1 \leq k \leq n \), \( \Lambda_k(S_n) \) coincides with the circular disc \( \{ z \in \mathbb{C} : |z| \leq \cos \left( \frac{k\pi}{n+1} \right) \} \) if \( 1 \leq k \leq \left[ \frac{n+1}{2} \right] \) and the empty set if \( \left[ \frac{n+1}{2} \right] < k \leq n \).

**Proof.** First, observe that for \( 1 \leq k \leq n \),

\[
\Lambda_k(S_n) = \bigcap_{0 \in [0, 2\pi]} \left\{ \mu \in \mathbb{C} : e^{i\theta} \mu + e^{-i\theta} \bar{\mu} \leq \lambda_k(e^{i\theta} S_n + e^{-i\theta} S_n^*) \right\} 
\]

\[
= \bigcap_{0 \in [0, 2\pi]} \left\{ \mu \in \mathbb{C} : \mathcal{R}(e^{i\theta} \mu) \leq \frac{1}{2} \lambda_k(e^{i\theta} S_n + e^{-i\theta} S_n^*) \right\} 
\]

\[
= \bigcap_{0 \in [0, 2\pi]} \left\{ z \in \mathbb{C} : \mathcal{R}(z) \leq \frac{1}{2} \lambda_k(e^{i\theta} S_n + e^{-i\theta} S_n^*) \right\} 
\]  \hspace{1cm} (2.1)
Equations (2.1) and (2.2) are due to the fact that firstly \( S_n + S_n^* = D(\theta)^*(e^{i\theta} S_n + e^{-i\theta} S_n^*)D(\theta) \), where \( D(\theta) \) denotes the unitary diagonal matrix with entries \( e^{i\theta}, e^{2i\theta}, \ldots, e^{ni\theta} \), which implies that \( S_n + S_n^* \) and \( e^{i\theta} S_n + e^{-i\theta} S_n^* \) are unitarily equivalent for each \( \theta \) in \([0, 2\pi]\) (see [11], page 373) and secondly, the eigenvalues of \( S_n^* + S_n \) are \( \left( 2 \cos\left( \frac{m\pi}{n+1} \right) \right)_{1 \leq m \leq n} \). (We refer the readers to ([3], page 35) or ([10], page 67).) Now from (2.3), it follows that \( \Lambda_k(S_n) \) is the intersection of closed half planes. We note that \( \cos\left( \frac{k\pi}{n+1} \right) \) is positive, if and only if \( 1 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor \). We consider the following two cases:

**Case 1.** If \( 1 \leq k \leq \left\lfloor \frac{n+1}{2} \right\rfloor \), then \( \Lambda_k(S_n) \) is the circular disc \( \{ z \in \mathbb{C} : |z| \leq \cos\left( \frac{k\pi}{n+1} \right) \} \).

**Case 2.** If \( \left\lfloor \frac{n+1}{2} \right\rfloor < k \leq n \), using (2.3), it follows that

\[
\Lambda_k(S_n) \subseteq \left\{ z \in \mathbb{C} : \Re(z) \leq \cos\left( \frac{k\pi}{n+1} \right) \right\} \bigcap e^{i\pi} \left\{ z \in \mathbb{C} : \Re(z) \leq \cos\left( \frac{k\pi}{n+1} \right) \right\} = \emptyset.
\]

This completes the proof. \( \square \)

**Theorem 2.2.** For any integer \( k \geq 1 \),

\[ \Lambda_k(S) = D(0, 1). \]
Proof. Let a fixed $k \geq 1$, from (P5) and Theorem 2.1, we deduce that for all $n \geq k$, $D(0, \cos(\frac{k\pi}{n+1})) = \Lambda_k(S_n) \subseteq \Lambda_k(S)$. Let $n$ tends to $+\infty$, and we have $D(0, 1) \subseteq \Lambda_k(S)$. Now let $\lambda$ in $\Lambda_k(S)$, then there exists a rank-$k$ orthogonal projection $P$ such that $PSP = \lambda P$. Let denote by $U_0$, the unitary operator on $\mathbb{H}^2$ defined by $U_0(f)(z) = f(ze^{-i\theta})$, where $\theta \in [0, 2\pi[$. For every $f$ and $g$ in $\mathbb{H}^2$, we have

$$< e^{i\theta}Sf, g > = \int_0^{2\pi} e^{i(\theta+t)}f(e^{it})\overline{g(e^{it})} \frac{dt}{2\pi}$$

$$= \int_0^{2\pi} e^{is}f(e^{i(s-\theta)})\overline{g(e^{i(s-\theta)})} \frac{ds}{2\pi}$$

$$= < SU_0f, U_0g > .$$

This shows that $e^{i\theta}S = U_0^*SU_0$, for each $\theta \in [0, 2\pi[$.

Let denote by $Q$ the rank-$k$ orthogonal projection $U_0PU_0^*$, we can easily check that $QSQ = \lambda e^{i\theta}Q$, thus $\Lambda_k(S)$ is a circular disc centered in 0. On the other hand, if $1 \in \Lambda_k(S)$, then from (P6), $1 \in W(S)$. Thus, there exists a unitary $f \in \mathbb{H}^2$ such that $< Sf, f > = 1$. The known facts about when the Schwartz inequality becomes an equation implies that 1 is an eigenvalue for $S$ (cf. [12], Solution 212), which is absurd. The proof is now complete.

One easy consequence of Theorem 2.1 is the following:

Corollary 2.3. One has

$$W(S) = D(0, 1).$$

On the sequel of this paper, let denote by

$$\rho(k, d) = \begin{cases} 
  k/d, & \text{if } k/d \text{ is an integer}, \\
  \lfloor k/d \rfloor + 1, & \text{unless},
\end{cases}$$
where $k$ and $d$ are arbitrary integers. We convient that $\rho(k, d) = 1$ if $d = +\infty$, $\delta(k, d)$ denotes the remainder in the Euclidean division of $k$ by $d$. For $\zeta \geq 0$, $D(0, \zeta)$ denotes the open circular disc $\{z \in \mathbb{C} : |z| < \zeta\}$ and $\overline{D(0, \zeta)}$ its closure.

**Proposition 2.4.** Consider $I_\mathcal{H} \otimes S_n^*$ be the operator acting on the Hilbertian tensor product space $\mathcal{H} \otimes \mathbb{C}^n$ and $d = \dim \mathcal{H}$ (we convient that $d = +\infty$ if $\dim \mathcal{H} = +\infty$). Then for each $1 \leq k \leq nd$, we have

\[
\Lambda_k(I_\mathcal{H} \otimes S_n^*) = \begin{cases} 
D(0, \cos \frac{\rho(k, d)\pi}{n + 1}), & \text{if } 1 \leq \rho(k, d) \leq \left[\frac{n + 1}{2}\right], \\
0, & \text{if } \left[\frac{n + 1}{2}\right] < \rho(k, d) \leq n.
\end{cases}
\]

**Proof.** Firstly, assume that $1 \leq d < +\infty$, then

\[
\Lambda_k(I_\mathcal{H} \otimes S_n^*)
\]

\[
= \bigcap_{0 \in [0, 2\pi]} \left\{ \mu \in \mathbb{C} : e^{i\theta} \mu + e^{-i\theta} \overline{\mu} \leq \lambda_k (e^{i\theta} (I_d \otimes S_n^*) + e^{-i\theta} (I_d \otimes S_n^*)^*) \right\}
\]

\[
= \bigcap_{0 \in [0, 2\pi]} \left\{ \mu \in \mathbb{C} : e^{i\theta} \mu + e^{-i\theta} \overline{\mu} \leq \lambda_k (e^{i\theta} (I_d \otimes S_n^*) + e^{-i\theta} (I_d \otimes S_n^*)) \right\}
\]

\[
= \bigcap_{0 \in [0, 2\pi]} \left\{ \mu \in \mathbb{C} : e^{i\theta} \mu + e^{-i\theta} \overline{\mu} \leq \lambda_k (I_d \otimes (e^{i\theta} S_n + e^{-i\theta} S_n^*)) \right\}
\]

\[
= \bigcap_{0 \in [0, 2\pi]} \left\{ \mu \in \mathbb{C} : e^{i\theta} \mu + e^{-i\theta} \overline{\mu} \leq \lambda_k (\bigotimes_{l=1}^d (e^{i\theta} S_n + e^{-i\theta} S_n^*)) \right\}
\]

\[
= \bigcap_{0 \in [0, 2\pi]} e^{i\theta} \left\{ z \in \mathbb{C} : \text{Re}(z) \leq \lambda_k (\bigotimes_{l=1}^d (S_n + S_n^*)) \right\}
\]

\[
= \bigcap_{0 \in [0, 2\pi]} e^{i\theta} \left\{ z \in \mathbb{C} : \text{Re}(z) \leq \lambda_k (A_n) \right\}, \tag{2.4}
\]

\[

\]
where $A$ is the block matrix defined by

$$A_n = \begin{pmatrix} M_1 & M_2 & \cdots & M_{n-1} \\ M_2 & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ M_{n-1} & \vdots & \ddots & M_n \end{pmatrix} \in \mathcal{M}_{nd}(\mathbb{C}),$$

with

$$M_\mu = \begin{pmatrix} \cos\left(\frac{\mu \pi}{n+1}\right) \\ \vdots \\ \cos\left(\frac{\mu \pi}{n+1}\right) \end{pmatrix} \in \mathcal{M}_d(\mathbb{C}), \text{ for } 1 \leq \mu \leq n.$$

Thus, $\Lambda_k(I_{\mathcal{H}} \otimes S_n^*)$ is the intersection of closed half plane and it is easy to check from (2.4) that $\Lambda_k(I_{\mathcal{H}} \otimes S_n^*) = D(0, \lambda_k(A_n^*))$, if $\lambda_k(A_n) \geq 0$ and empty set unless. Note that the eigenvalues of $A_n$ are arranged in the decreasing order. To find the $k$-th largest eigenvalue of $A_n$, we will consider the Euclidean division of $k$ by $d$ as the following $k = \beta d + r$ with $0 \leq r < d - 1$ and $\beta \geq 0$. We consider the following two cases:

**Case 1.** If $r = 0$. We have $\lambda_k(A) = \cos\left(\frac{\beta \pi}{n+1}\right) = \cos\frac{\rho(k, d)\pi}{n+1}$.

**Case 2.** If $1 \leq r < d - 1$. Then $\lambda_k(A) = \cos\left(\frac{(\beta + 1)\pi}{n+1}\right) = \cos\frac{\rho(k, d)\pi}{n+1}$.

Thus,

$$\Lambda_k(I_{\mathcal{H}} \otimes S_n^*) = \begin{cases} D(0, \cos\frac{\rho(k, d)\pi}{n+1}), & \text{if } 1 \leq \rho(k, d) \leq \left\lfloor \frac{n+1}{2} \right\rfloor, \\ 0, & \text{if } \left\lfloor \frac{n+1}{2} \right\rfloor < \rho(k, d) \leq n. \end{cases}, \quad (2.5)$$

Now assume that $d = +\infty$. To complete the proof of this proposition, it is enough to prove that

$$\Lambda_k(I_{\mathcal{H}} \otimes S_n^*) = D(0, \cos\frac{\pi}{n+1}), \text{ for all } k \geq 1. \quad (2.6)$$
Firstly, note that (2.6) yields for $k = 1$, that is,

$$W(I_{\mathcal{H}} \otimes S_n^*) = D(0, \cos(\frac{\pi}{n+1})).$$

(2.7)

Indeed, it is clear from (P5) that $W(I_{\mathcal{H}} \otimes S_n^*) \supseteq W(S_n^*) = D(0, \cos(\frac{\pi}{n+1}))$.

For the proof of the second inclusion, let denote by $(\varepsilon_l)_{l \geq 0}$ an orthonormal basis of $\mathcal{H}$ and $(x_l)_{l \geq 0}$ a sequence in $\mathbb{C}$ such that $x = \sum_{l \geq 0} \varepsilon_l \otimes x_l$ is a unit vector in $\mathcal{H} \otimes \mathbb{C}^n$ (i.e., $\|x\|^2 = \sum_{l \geq 0} \|x_l\|^2 = 1$). Then

$$<(I_{\mathcal{H}} \otimes S_n^*)x, x> = \sum_{l \geq 0} \varepsilon_l \otimes x_l, \sum_{m \geq 0} \varepsilon_m \otimes x_m >$$

$$= \sum_{l, m \geq 0} <\varepsilon_l, \varepsilon_m> <S_n^*x_l, x_m>$$

$$= \sum_{l \geq 0} <S_n^*x_l, x_l> \cdot \sum_{m \geq 0} <\varepsilon_m, x_m> \cdot \sum_{l \geq 0} \|x_l\|^2 = \sum_{l \geq 0} \|S_n^*x_l\|^2. \tag{2.8}$$

Since $W(S_n^*)$ is a convex compact set, then (2.8) implies that $<I_{\mathcal{H}} \otimes S_n^*)x, x> \in W(S_n^*) = D(0, \cos(\frac{\pi}{n+1}))$. Thus $W(I_{\mathcal{H}} \otimes S_n^*) \subseteq D(0, \cos(\frac{\pi}{n+1}))$ and (2.7) holds. Now, from, respectively, (2.7), (P6), (P5), and (2.5), we infer that for each $p > k$, we have

$$D(0, \cos(\frac{\pi}{n+1})) = W(I_{\mathcal{H}} \otimes S_n^*) \supseteq \Lambda_k(I_{\mathcal{H}} \otimes S_n^*) \supseteq \Lambda_k(I_p \otimes S_n^*)$$

$$= D(0, \cos(\frac{\pi}{n+1})).$$

This completes the proof. \hfill \Box

Let $D_T = (I_N - T^*T)^{1/2}$ be the defect operator of $T$ and $D_T$ be the closed range of $D_T$. Let denote by $d = \dim D_T$. 

Theorem 2.5. Consider $T \in \mathcal{B}(\mathcal{H})$ such that $\|T\| \leq 1$ and $T^n = 0$. Then $\Lambda_k(T)$ is contained in the circular disc $\{z \in \mathbb{C} : |z| \leq \cos \frac{\rho(k, d)\pi}{n + 1}\}$ if $1 \leq \rho(k, d) \leq \left[\frac{n + 1}{2}\right]$ and an empty set if $\rho(k, d) > \left[\frac{n + 1}{2}\right]$.

Proof. If $T$ is a contraction with $T^n = 0$, then $T$ can be viewed as a compression of $I_{\mathcal{D}_T} \otimes S_n^*$ acting on the Hilbert space $\mathcal{D}_T \otimes \mathbb{C}^n$ (see [18], [2], [20], Theorem 1.2 in Chapter II). Consider the isometry $V : \mathcal{H} \to \mathcal{D}_T \otimes \mathbb{C}^n$, defined by

$$V(x) = \sum_{t=1}^{n} D_T T_t x \otimes e_t,$$

where $\{e_t\}_{t=1}^{n}$ is the canonical basis of $\mathbb{C}^n$. Note that

$$VTx = \sum_{t=1}^{n} D_T T_t x \otimes e_t = \sum_{t=1}^{n-1} D_T T_t x \otimes e_t = (I_{\mathcal{D}_T} \otimes S_n^*)Vx.$$ 

It follows that

$$T = V^* (I_{\mathcal{D}_T} \otimes S_n^*) V,$$

and from (P5),

$$\Lambda_k(T) = \Lambda_k(V^* (I_{\mathcal{D}_T} \otimes S_n^*) V) \subseteq \Lambda_k(I_{\mathcal{D}_T} \otimes S_n^*), \text{ for any } 1 \leq k \leq nd.$$ 

(2.9)

Therefore, if $1 \leq k \leq nd$, (2.9) and Proposition 2.4 implies that $\Lambda_k(T) \subseteq D(0, \cos \frac{\rho(k, d)\pi}{n + 1})$ if $1 \leq \rho(k, d) \leq \left[\frac{n + 1}{2}\right]$ and an empty set if $\left[\frac{n + 1}{2}\right] < \rho(k, d) \leq n$. Finally, if $k > nd$, $\Lambda_k(T) = \emptyset$ from (P6). \qed

The previous theorem establishes the connection between the higher rank-$k$ numerical range of nilpotent operators with nilpotency degree $n$ and the $n$-dimensional shift on $\mathbb{C}^n$. In the particular case, where $k = 1$ and using the fact that $\rho(1, d) = 1$ for all $1 \leq d \leq +\infty$, it becomes obvious to see
Corollary 2.6 (Haagerup and de la Harpe [11]). Consider \( T \in \mathcal{B}(\mathcal{H}) \) such that \( \|T\| \leq 1 \) and \( T^n = 0 \). Then we have \( \omega_2(T) \leq \cos\left( \frac{\pi}{n+1} \right) \).

In the next result, we will show that Theorem 2.5 generalizes to operator of class \( C_0 \). Recall that a completely non-unitary operator \( T \) is said to be of class \( C_0 \) [20] if \( u(T) = 0 \) for some nonzero function \( u \) in \( \mathbb{H}^\infty \). Among the functions \( u \) with this property, there is an inner one which divides all others. This function is denoted by \( m_T \), and is uniquely determined up to a nonstant factor of absolute value one. Note also that for the class \( C_0 \), the defect numbers of \( T \) and \( T^* \) are always the same (see [20], Theorem 5.2, page 267).

**Theorem 2.7.** If \( T \) is an operator of class \( C_0 \), then

\[
\Lambda_k(T) \subseteq \Lambda_k(I_{D_{T^*}} \otimes S(m_T)),
\]

for each \( k \geq 1 \).

**Proof.** Let us consider the isometry \( V : \mathcal{H} \to D_{T^*} \otimes \mathbb{H}^2 \), defined by

\[
V(x) = \sum_{j=0}^{\infty} D_{T^*} T^j x \otimes f_j,
\]

where \( (f_j)_{j \geq 0} \) is the orthonormal basis of \( \mathbb{H}^2 \) defined by \( f_j(z) = z^j \) for each \( j \geq 0 \). Note that (2.10) is a consequence of the fact that \( T^* \) tend strongly to zero. On the other hand, we have

\[
VT^*(x) = \sum_{j=0}^{\infty} D_{T^*} T^{*j}(x) \otimes S^*(f_j) = (I_{D_{T^*}} \otimes S^*)V(x).
\]

So \( VT^* = (I_{D_{T^*}} \otimes S^*)V \) and \( VT^* V^2 = (I_{D_{T^*}} \otimes S^*)^2 V \). By induction on \( n \), we obtain that \( VT^{*n} = (I_{D_{T^*}} \otimes S^*)^n V \) for each \( n \geq 0 \).
Consequently,
\[ Vh(T^*) = h(I_{D_{T^*}} \otimes S^*)V, \text{ for each } h \in \mathbb{H}^\infty. \] (2.11)

Now \( m_T(T) = 0 \) thus \( \overline{m_T}(T^*) = 0 \) and from (2.11), we have
\[ 0 = \overline{m_T}(I_{D_{T^*}} \otimes S^*)V = (I_{D_{T^*}} \otimes m_T(S^*))V. \]

So,
\[ \text{Im}(V) \subseteq \text{Ker}(I_{D_{T^*}} \otimes m_T(S^*)) = D_{T^*} \otimes H(m_T), \]

and the following diagram commutes:

It follows that
\[ VT^* = (I_{D_{T^*}} \otimes m_T(S^*))|_{D_{T^*} \otimes H(m_T)}V = (I_{D_{T^*}} \otimes S^*(m_T))V. \] (2.12)

In other words, \( T^* \) is the restriction of the sum of a number of copies of \( S^*(m_T) \), the number of copies being equal to the defect number of \( T \). An application of (P2) and (2.12) ends the proof. \( \square \)

**Corollary 2.8** ([2], Proposition 1). If \( T \) is an operator of class \( C_0 \), then \( W(T) \subseteq W(S(m_T)) \), and therefore, \( \omega_2(T) \leq \omega_2(S(m_T)) \).
Theorem 2.9. Let \( n, k, \) and \( q \) be arbitrary integers such that \( n \geq 2, 1 \leq k \leq n, \) and \( 2 \leq q \leq n - 1. \)

1. If \( \delta(n, q) = 0, \) \( S^n_q = \oplus_{i=1}^{q} S_{\rho(n,q)}. \)

2. If \( 1 \leq \delta(n, q) \leq q - 1, \) \( S^n_q = (\oplus_{i=1}^{\delta(n,q)} S_{\rho(n,q)}) \oplus (\oplus_{i=1}^{q-\delta(n,q)} S_{\rho(n,q)-1}). \)

3. If \( \delta(n, q) = 0, \)

\[
\Lambda_k(S^n_q) = \begin{cases} 
D(0, \cos \frac{\rho(k,q)\pi}{\rho(n,q) + 1}), & \text{if } 1 \leq \rho(k, q) \leq \left\lfloor \frac{\rho(n,q) + 1}{2} \right\rfloor, \\
0, & \text{unless.}
\end{cases}
\]

4. If \( 1 \leq \delta(n, q) \leq q - 1 \) and \( 1 \leq \delta(k, q) \leq \delta(n, q), \)

\[
\Lambda_k(S^n_q) = \begin{cases} 
D(0, \cos \frac{\rho(k,q)\pi}{\rho(n,q) + 1}), & \text{if } \rho(k, q) \leq \left\lfloor \frac{\rho(n,q) + 1}{2} \right\rfloor, \\
0, & \text{unless.}
\end{cases}
\]

5. If \( 1 \leq \delta(n, q) \leq q - 1 \) and \( \delta(k, q) = 0 \) or \( \delta(n, q) + 1 \leq \delta(k, q) \leq q - 1, \)

\[
\Lambda_k(S^n_q) = \begin{cases} 
D(0, \cos \frac{\rho(k,q)\pi}{\rho(n,q) + 1}), & \text{if } \rho(k, q) \leq \left\lfloor \frac{\rho(n,q) + 1}{2} \right\rfloor, \\
0, & \text{unless.}
\end{cases}
\]

Proof. Firstly, note that for \( 1 \leq s \leq n, \) we have

\[
S^n_q(e_s) = \begin{cases} 
e_{s+q}, & \text{if } 1 \leq s \leq n - q, \\
0, & \text{if } n - q < s \leq n.
\end{cases}
\]

Consider the Euclidean division of \( n \) by \( q. \) Then, there exists \( \alpha \geq 1 \) and \( 0 \leq r < q - 1 \) such that \( n = \alpha q + r. \)

Assume that \( r = 0, \) then \( n = \alpha q \) and necessarily \( \alpha \geq 2. \) For \( 1 \leq i \leq q, \) let denote by

\[
\mathcal{F}_i = \{e_{i+jq}; \ 0 \leq j \leq \alpha - 1\} \text{ and } \hat{\mathcal{F}}_i = \text{spam } \mathcal{F}_i.
\]
We claim that $\mathbb{C}^n = \bigotimes_{i=1}^q \hat{\mathcal{F}}_i$ and $S^n_q |_{\hat{\mathcal{F}}_i} = S_\alpha$. To prove the claim, note that $\sum_{i=1}^q \text{Card } \hat{\mathcal{F}}_i = n$. On the other hand, for $1 \leq i \neq i' \leq q$, $\mathcal{F}_i \cap \mathcal{F}_{i'} = 0$. Otherwise, there exist $0 \leq j, j' \leq \alpha - 1$ such that $i + jq = i' + j'q$, that is, $i' - i = q(j - j')$. We can assume that $i' > i$ and $j > j'$, thus $q = \frac{i' - i}{j - j'} \leq q - 1$, which is absurd. Thus $\mathbb{C}^n$ is the direct summand of $\hat{\mathcal{F}}_i$ for $1 \leq i \leq q$. Now from (2.13), for all $1 \leq i \leq q$ and $0 \leq j \leq \alpha - 1$, the compressed of $S^n_q$ to $\hat{\mathcal{F}}_i$ is given by

$$S^n_q(e_{i+jq}) = e_{i+(j+1)q}, \text{ if } 0 \leq j \leq \alpha - 2,$$

$$S^n_q(e_{i+\alpha q}) = 0,$$

which implies that $S^n_q |_{\hat{\mathcal{F}}_i}$ is unitarily equivalent to $S_\alpha$ and evidently $S^n_q$ is unitarily equivalent to $\bigotimes_{i=1}^q S_\alpha = I_q \otimes S_\alpha$. Then Proposition 2.4 implies that

$$\Lambda_k(S^n_q) = \begin{cases} D(0, \cos \frac{\rho(k,q)\pi}{\alpha + 1}), & \text{if } 1 \leq \rho(k,q) \leq \lfloor \frac{\alpha + 1}{2} \rfloor, \\ 0, & \text{unless.} \end{cases}$$

For the sequel of the proof, assume that $1 \leq r \leq q - 1$. For $1 \leq i \leq q$, let denote by

$$\mathcal{F}_i = \{ e_{i+jq}; 0 \leq j \leq \alpha \} \quad \text{and} \quad \hat{\mathcal{F}}_i = \text{span } \mathcal{F}_i, \quad \text{for } 1 \leq i \leq r,$$

$$\mathcal{G}_i = \{ e_{i+jq}; 0 \leq j \leq \alpha - 1 \} \quad \text{and} \quad \hat{\mathcal{G}}_i = \text{span } \mathcal{G}_i, \quad \text{for } r + 1 \leq i \leq q,$$

using the same scheme as before, we easily prove that

$$\mathbb{C}^n = \bigotimes_{i=1}^r \hat{\mathcal{F}}_i \bigoplus \bigotimes_{i=r+1}^q \hat{\mathcal{G}}_i,$$

(2.14)

$$S^n_q |_{\hat{\mathcal{F}}_i} = S_{\alpha+1},$$

(2.15)
\[ S_n^q | \tilde{\mathcal{G}}_i \rangle = S_\alpha, \quad (2.16) \]

which implies that
\[ S_n^q = \left( \bigoplus_{i=1}^r S_{\alpha+1} \right) \oplus \left( \bigoplus_{i=1}^q S_\alpha \right). \]

With respect to the basis \( \{ \mathcal{F}_i; 1 \leq i \leq r \} \cup \{ \mathcal{G}_i; r + 1 \leq i \leq q \} \) of \( \mathbb{C}^n \), \( S_n^q \) have this representation matrix
\[
\begin{pmatrix}
S_{\alpha+1} & & \\
& \ddots & \\
& & S_{\alpha+1}
\end{pmatrix}
\begin{pmatrix}
S_\alpha & & \\
& \ddots & \\
& & S_\alpha
\end{pmatrix},
\]

which implies that
\[ \Lambda_k(S_n^q) = \Lambda_k((I_r \otimes S_{\alpha+1}) \oplus (I_{q-r} \otimes S_\alpha)). \]

On the other hand,
\[ e^{i\theta}(I_r \otimes S_{\alpha+1}) \oplus (I_{q-r} \otimes S_\alpha) + e^{-i\theta}(I_r \otimes S_{\alpha+1}) \oplus (I_{q-r} \otimes S_\alpha)^* \]
\[ = e^{i\theta}(I_r \otimes S_{\alpha+1}) \oplus (I_{q-r} \otimes S_\alpha) + e^{-i\theta}(I_r \otimes S_{\alpha+1}^*) \oplus (I_{q-r} \otimes S_\alpha^*) \]
\[ = (I_r \otimes (e^{i\theta}S_{\alpha+1} + e^{-i\theta}S_{\alpha+1}^*)) \oplus (I_{q-r} \otimes (e^{i\theta}S_\alpha + e^{-i\theta}S_\alpha^*)) \]
\[ = \left( \bigoplus_{i=1}^r (e^{i\theta}S_{\alpha+1} + e^{-i\theta}S_{\alpha+1}^*) \right) \oplus \left( \bigoplus_{i=1}^q (e^{i\theta}S_\alpha + e^{-i\theta}S_\alpha^*) \right). \]

The eigenvalues of \( e^{i\theta}S_{\alpha+1} + e^{-i\theta}S_{\alpha+1}^* \) and \( e^{i\theta}S_\alpha + e^{-i\theta}S_\alpha^* \) are, respectively,
\[
\left\{ 2 \cos \left( \frac{\mu \pi}{\alpha + 2} \right) \right\}_{1 \leq \mu \leq \alpha+1} \quad \text{and} \quad \left\{ 2 \cos \left( \frac{\nu \pi}{\alpha + 1} \right) \right\}_{1 \leq \nu \leq \alpha}.
\]

Thus,
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\[ \Lambda_k(S^q_k) = \bigcap_{\theta \in [0, 2\pi]} e^{i\theta} \left\{ z \in \mathbb{C} : \Re(z) \leq \lambda_k(B) \right\}, \]

where \( B \) denotes this matrix representation

\[
\begin{pmatrix}
M_1 & & & \\
& N_1 & & \\
& & \ddots & \\
& & & M_\alpha \\
& N_\alpha & & \ddots \\
& & & & M_{\alpha+1}
\end{pmatrix}
\]

with

\[
M_\mu = \begin{pmatrix}
\cos\left(\frac{\mu\pi}{\alpha+2}\right) & & & \\
& \ddots & & \\
& & \cos\left(\frac{\mu\pi}{\alpha+2}\right)
\end{pmatrix} \in \mathcal{M}_\mu(\mathbb{C}), 1 \leq \mu \leq \alpha + 1,
\]

and

\[
N_\nu = \begin{pmatrix}
\cos\left(\frac{\nu\pi}{\alpha+1}\right) & & & \\
& \ddots & & \\
& & \cos\left(\frac{\nu\pi}{\alpha+1}\right)
\end{pmatrix} \in \mathcal{M}_{q-r}(\mathbb{C}), 1 \leq \nu \leq \alpha.
\]

Note that

\[
\cos\left(\frac{\pi}{\alpha+2}\right) > \cos\left(\frac{\pi}{\alpha+1}\right) > \cos\left(\frac{2\pi}{\alpha+2}\right) > \cos\left(\frac{2\pi}{\alpha+1}\right) > \cdots > \cos\left(\frac{\alpha\pi}{\alpha+1}\right) > \cos\left(\frac{(\alpha+1)\pi}{\alpha+2}\right),
\]

which implies that the eigenvalues in the last representation are arranged in the decreasing order. Now, consider the Euclidean division of \( k \) by \( q \) as the following \( k = \alpha'q + r' \) with \( 0 \leq r' < q - 1 \) and \( \alpha' > 0 \). We distinguish the following three cases:
Case 1. If $r' = 0$, $\lambda_k = \cos\left(\frac{\alpha' \pi}{\alpha + 1}\right)$, and

$$
\Lambda_k(S_n^q) = \begin{cases} 
D(0, \cos\left(\frac{\alpha' \pi}{\alpha + 1}\right)), & \text{if } \alpha' \leq \left[\frac{\alpha + 1}{2}\right], \\
0, & \text{unless.}
\end{cases}
$$

Case 2. If $1 \leq r' \leq r$, $\lambda_k = \cos\left(\frac{(\alpha' + 1)\pi}{\alpha + 2}\right)$, and

$$
\Lambda_k(S_n^q) = \begin{cases} 
D(0, \cos\left(\frac{(\alpha' + 1)\pi}{\alpha + 2}\right)), & \text{if } (\alpha' + 1) \leq \left[\frac{\alpha + 2}{2}\right], \\
0, & \text{unless.}
\end{cases}
$$

Case 3. If $r + 1 \leq r' \leq q - 1$, $\lambda_k = \cos\left(\frac{(\alpha' + 1)\pi}{\alpha + 1}\right)$, and

$$
\Lambda_k(S_n^q) = \begin{cases} 
D(0, \cos\left(\frac{(\alpha' + 1)\pi}{\alpha + 1}\right)), & \text{if } (\alpha' + 1) \leq \left[\frac{\alpha + 1}{2}\right], \\
0, & \text{unless.}
\end{cases}
$$

Acknowledgement

The author would like to express his gratitude to Gilles Cassier for his help and good advices.

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