Sum of Divisors Function And The Largest Integer Function Over The Shifted Primes

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Abstract: Let $x \geq 1$ be a large number, let $[x] = x - \{x\}$ be the largest integer function, and let $\sigma(n)$ be the sum of divisors function. This note presents the first proof of the asymptotic formula for the average order $\sum_{p \leq x} \sigma([x/p]) = c_0 x \log \log x + O(x)$ over the primes, where $c_0 > 0$ is a constant. More generally, $\sum_{p \leq x} \sigma([x/(p + a)]) = c_0 x \log \log x + O(x)$ for any fixed integer $a$.

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1 Introduction

A series of results have been proved for the finite sums $\sum_{n \leq x} f([x/n])$ over the integers, see [2], [3], [8], et alii. This note introduces the analytic techniques for evaluating the finite sums $\sum_{p \leq x} f([x/p])$ over the primes. This elementary methods ably handle these finite sums for multiplicative functions $f$ defined by Dirichlet convolutions $f(n) = \sum_{d|n} g(n/d)$, where $f, g : \mathbb{N} \rightarrow \mathbb{C}$, and the rates of growth, approximately $f(n) \gg n(\log n)^b$, for some $b \in \mathbb{Z}$. These elementary methods are efficient, produce very short proofs, and sharp error terms. The asymptotic formula for the fractional sum $\sum_{p \leq x} \varphi([x/p])$ is assembled in [4], and the asymptotic formula for the fractional sum $\sum_{p \leq x} \sigma([x/p])$ of the sum of divisors function $\sigma$ in Theorem 2.1, is evaluated here. This is a new result in the literature.
2 Sum of Divisors Function Over The Primes

The result in this section deals with the sum of divisors function \( \sigma(n) = \sum_{d|n} 1/d \) composed with the largest integer function \( \lfloor z \rfloor = z - \{z\} \). The function \( \sigma : \mathbb{N} \to \mathbb{N} \) is multiplicative and satisfies the growth condition \( \sigma(n) \gg n \). The first asymptotic formula for the fractional sum of divisor function over the primes is given below.

**Theorem 2.1.** If \( x \geq 1 \) is a large number, then,

\[
\sum_{p \leq x} \sigma \left( \left\lfloor \frac{x}{p} \right\rfloor \right) = c_0 x \log \log x + c_1 x + O \left( \text{li}(x) \log \log x \right),
\]

where \( c_0 = \zeta(2) \), and \( c_1 = B_1 \zeta(2) \) are constants.

**Proof.** Use the identity \( \sigma(n) = \sum_{d|n} 1/d \) to rewrite the finite sum, and switch the order of summation:

\[
\sum_{p \leq x} \sigma \left( \left\lfloor \frac{x}{p} \right\rfloor \right) = \sum_{p \leq x} \left\lfloor \frac{x}{p} \right\rfloor \sum_{d|\lfloor x/p \rfloor} \frac{1}{d}
\]

Apply Lemma 4.6 to remove the congruence on the inner sum index, and break it up into two subsums. Specifically,

\[
\sum_{d \leq x} \frac{1}{d} \sum_{p \leq x} \left\lfloor \frac{x}{p} \right\rfloor = \sum_{d \leq x} \frac{1}{d} \sum_{p \leq x} \left\lfloor \frac{x}{p} \right\rfloor \sum_{0 \leq a \leq d-1} e^{2\pi i a x/d}
\]

The first sum \( S(x) \) is computed in Lemma 4.1 and the sum \( T(x) \) is computed in Lemma 4.7. Summing these expressions

\[
\sum_{p \leq x} \sigma \left( \left\lfloor \frac{x}{p} \right\rfloor \right) = S(x) + T(x)
\]

where \( c_0 = \zeta(2) \), and \( c_1 = B_1 \zeta(2) \), \( c_2 = (1 - \gamma) \zeta(2) \) are constants, and \( c > 0 \) is an absolute constant. ■

The constants occurring in the above expression are the followings.

1. \( \zeta(2) = \frac{\pi^2}{6} = 1.644934066848226436472415 \ldots \), the zeta constant,

2. \( B_1 = \lim_{x \to \infty} \frac{1}{x} \sum_{p \leq x} \log \log x = 0.261497212847642783755426 \ldots \), Mertens constant,
3. \( \gamma = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} \frac{1}{n} - \log x = 0.577215664901532860606512 \ldots \), Euler constant,

4. \( c_1 = B_1 \zeta(2) = 0.430145673798949331799597 \ldots \),

5. \( c_2 = (1 - \gamma) \zeta(2) = 0.695452355333244911926851 \ldots \).

The standard proof for the average order over the primes

\[
\sum_{p \leq x} \sigma(p) = c_3 \text{li}(x^2) + O \left( x^2 e^{-c \sqrt{\log x}} \right),
\]

where \( c > 0 \) is an absolute constant, this follows from the prime number theorem, [6, Theorem 6.9], and partial summation. And the standard proof for the average order over the shifted primes

\[
\sum_{p \leq x} \sigma(p - 1) = \frac{315 \zeta(3)}{2 \pi^4} x + O \left( \frac{x}{(\log x)^{0.999}} \right)
\]

was proved in [3].

3 The Sum \( S(x) \)

The detailed and elementary evaluation of the asymptotic formula for the finite sum \( S(x) \) occurring in [2] are recorded in this section. Both conditional and unconditional results are provided. For a real number \( z \in \mathbb{R} \), the largest integer function is defined by \([z] = z - \{z\}\)

**Lemma 3.1.** Let \( x \geq 1 \) be a large number. Then,

\[
\sum_{d \leq x} \frac{1}{d^2} \sum_{p \leq x} \left\lfloor \frac{x}{p} \right\rfloor = c_0 x \log \log x + c_1 x + c_2 \text{li}(x) + O \left( x e^{-c \sqrt{\log x}} \right),
\]

where \( c_0 = \zeta(2) \), and \( c_1 = B_1 \zeta(2) \), \( c_2 = (1 - \gamma) \zeta(2) \) are constants, and \( c > 0 \) is an absolute constant.

**Proof.** Expand the bracket and evaluate the two subsums. Specifically,

\[
S(x) = \sum_{d \leq x} \frac{1}{d^2} \sum_{p \leq x} \left\lfloor \frac{x}{p} \right\rfloor = x \sum_{d \leq x} \frac{1}{d^2} \sum_{p \leq x} 1 - x \sum_{d \leq x} \frac{1}{d^2} \sum_{p \leq x} \left\lfloor \frac{x}{p} \right\rfloor = S_0(x) - S_1(x).
\]

Substituting the standard asymptotic for the zeta constant \( \sum_{n \leq x} 1/n^2 \), and the prime harmonic sum \( \sum_{p \leq x} 1/p \), return the followings.

\[
S_0(x) = x \sum_{d \leq x} \frac{1}{d^2} \sum_{p \leq x} \frac{1}{p} = x \left( \frac{1}{\zeta(2)} + O \left( \frac{1}{x} \right) \right) \left( \log \log x + B_1 + O \left( e^{-c \sqrt{\log x}} \right) \right)
\]

\[
= c_0 x \log \log x + c_1 x + O \left( x e^{-c \sqrt{\log x}} \right),
\]
where \( B_1 > 0 \) is Mertens constant, \( c > 0 \) is an absolute constant, \( c_0 = \zeta(2) \), and \( c_1 = B_1 \zeta(2) \).

Likewise, substituting the standard asymptotics for the zeta constant \( \sum_{n \leq x} 1/n^2 \), and the prime number theorem for fractional parts sum \( \sum_{p \leq x} [x/p] \), see [7], [6] Exercise 1g, p. 248], return the followings.

\[
S_1(x) = \sum_{d \leq x} \frac{1}{d^2} \sum_{p \leq x} \left\{ \frac{x}{p} \right\} = \left( \frac{1}{\zeta(2)} + O \left( \frac{1}{x} \right) \right) \left( (1 - \gamma) \text{li}(x) + O \left( x e^{-c \sqrt{\log x}} \right) \right) = c_2 \text{li}(x) + O \left( x e^{-c \sqrt{\log x}} \right),
\]

where \( \text{li}(x) \) is the logarithm integral, \( \gamma \) is the Euler constant, \( c_2 = (1 - \gamma) \zeta(2) \), and \( c > 0 \) is an absolute constant. Subtracting the expressions (8) and (9) yields

\[
S(x) = S_0(x) - S_1(x) = c_0 x \log \log x + c_1 x - c_0 \text{li}(x) + O \left( x e^{-c \sqrt{\log x}} \right),
\]

where \( c_0 = \zeta(2) \), and \( c_1 = B_1 \zeta(2) \), \( c_2 = (1 - \gamma) \zeta(2) \), and \( c > 0 \) is an absolute constant. ■

Lemma 3.2. Assume the RH. Let \( x \geq 1 \) be a large number. Then,

\[
\sum_{d \leq x} \frac{1}{d^2} \sum_{p \leq x} \left\{ \frac{x}{p} \right\} \sum_{0 < a \leq d - 1} e^{i 2\pi a [x/p] / d} = O \left( \text{li}(x) \log \log x \right).
\]

where \( c_0 = \zeta(2) \), and \( c_1 = B_1 \zeta(2) \), \( c_2 = (1 - \gamma) \zeta(2) \) are constants.

Proof. Everything remain the same as the previous proof. However, the unconditional error term is replaced with the conditional error term. ■

4 The Sum \( T(x) \)

The detailed evaluation of the asymptotic formula for the finite sum \( T(x) \) occurring in (2) are recorded in this section. For a real number \( z \in \mathbb{R} \), the fractional function is defined by \( \{ z \} = z - [z] \).

Lemma 4.1. Let \( x \geq 1 \) be a large number. Then,

\[
\sum_{d \leq x} \frac{1}{d^2} \sum_{p \leq x} \left\{ \frac{x}{p} \right\} \sum_{0 < a \leq d - 1} e^{i 2\pi a [x/p] / d} = O \left( \text{li}(x) \log \log x \right).
\]

Proof. Let \( \pi(x) = \#\{ \text{prime } p \leq x \} \) be the primes counting function, let \( \text{li}(x) \) be the logarithm integral, and let \( p_k \) be the \( k \)th prime in increasing order. The sequence of values

\[
\left\{ \frac{x}{p_k} \right\} = \left\{ \frac{x}{p_{k+1}} \right\} = \cdots = \left\{ \frac{x}{p_{k+r}} \right\}
\]

arises from the sequence of primes \( x/(n+1) \leq p_k, p_{k+1}, \ldots, p_{k+r} \leq x/n \). Therefore, the value \( m = [x/p] \geq 1 \) is repeated

\[
\pi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) - \pi \left( \left\lfloor \frac{x}{n+1} \right\rfloor \right) = \frac{\text{li}(x)}{n(n+1)} + O \left( \frac{x}{n} e^{-c \sqrt{\log x}} \right)
\]
times as \( p \) ranges over the prime values in the interval \([x/(n+1), x/n]\), see Exercise 6.1 in Section 6. Hence, substituting (13) into the triple sum \( T(x) \), and reordering it yield

\[
T(x) = \sum_{p \leq x} \frac{x}{p} \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi m/d}
\]

(14)

\[
= \sum_{p \leq x} \left( \frac{\text{li}(x)}{n(n+1)} + O\left( \frac{x}{n}e^{-c\sqrt{\log x}} \right) \right) \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi m/d}
\]

\[
= \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi m/d}
\]

\[
+ O\left( xe^{-c\sqrt{\log x}} \sum_{n \leq x} \frac{1}{n} \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi m/d} \right)
\]

\[
= T_0(x) + T_1(x).
\]

The finite subsums \( T_0(x) \) computed in Lemma 4.2, and \( T_1(x) \), computed in Lemma 4.5. Summing yields

\[
T(x) = T_0(x) + T_1(x) = O(\text{li}(x) \log \log x) + O\left( xe^{-c\sqrt{\log x}} \right)
\]

(15)

\[
= O(\text{li}(x) \log \log x),
\]

where \( c > 0 \) is an absolute constant.

\[\square\]

### 4.1 The Sum \( T_0(x) \)

**Lemma 4.2.** Let \( x \geq 1 \) be a large number. Then,

\[
\text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi m/d} = O(\text{li}(x) \log \log x).
\]

**Proof.** Partition the finite sum into two finite subsums

\[
T_0(x) = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi m/d}
\]

(16)

\[
+ \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi m/d}
\]

\[
= T_{00}(x) + T_{01}(x).
\]

The finite subsums \( T_{00}(x) \) is estimated in Lemma 4.3, and \( T_{01}(x) \) is estimated in Lemma 4.4. These finite sums correspond to the subsets of integers \( p \leq x \) such that \( d \mid m \), and \( d \nmid m \), respectively. Summing yields

\[
T_0(x) = T_{00}(x) + T_{01}(x) = O(\text{li}(x) \log \log x).
\]

(17)

\[\square\]
4.2 The Sum $T_{00}(x)$

**Lemma 4.3.** Let $x \geq 1$ be a large number, let $[x] = x - \{x\}$ be the largest integer function, and $m = [x/p] \leq [x/n] \leq x$. Then,

$$\text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{2\pi i am/d} = O(\text{li}(x) \log \log x).$$

(18)

**Proof.** The set of values $m = [x/p] \leq [x/n] \leq x$ such that $d \mid m$. Evaluating the incomplete function returns

$$T_{00}(x) = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x, d \mid m} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{2\pi i am/d}$$

(19)

$$= \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x, d \mid m} \frac{1}{d^2} \cdot (d - 1)$$

$$= \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x, d \mid m} \frac{1}{d} - \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x, d \mid [x/n]} \frac{1}{d^2}$$

$$= T_{20}(x) + T_{21}(x).$$

The first term has the upper bound

$$T_{20}(x) = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x, d \mid m} \frac{1}{d} = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \mid [x/n]} \frac{1}{d} \ll x \log \log x.$$  

(20)

This follows from upper bound of the sum of divisors function

$$\sum_{d \leq x, d \mid m} \frac{1}{d} = \sum_{d \mid m} \frac{\sigma(m)}{m} \leq 2 \log \log x,$$

(21)

where $m = [x/p] \leq [x/n] \leq x$. The second term has the upper bound

$$T_{21}(x) = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x, d \mid [x/n]} \frac{1}{d^2} \ll \text{li}(x).$$

(22)

Summing yields $T_{00}(x) = T_{20}(x) + T_{21}(x) = O(\text{li}(x) \log \log x)$.

4.3 The Sum $T_{01}(x)$

**Lemma 4.4.** Let $x \geq 1$ be a large number. Then,

$$\text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x, d \mid m} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{2\pi i am/d} = O(\text{li}(x)).$$

Proof. The set of values $m = [x/p] \leq [x/n] \leq x$ such that $d \nmid m$. Evaluating the incomplete
indicator function returns

\[ T_1(x) = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x \atop d|m} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi am/d} \]

\[ = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x \atop d|m} \frac{1}{d^2} \cdot (-1) \]

\[ = \text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} (c_1(n) + O(1)) \]

\[ = O(\text{li}(x)), \]

where \(|c_1(n)| < 2\) depends on \(n\).

### 4.4 The Sum \(T_1(x)\)

**Lemma 4.5.** If \(x \geq 1\) is a large number, then,

\[ xe^{-c_0 \sqrt{\log x}} \sum_{n \leq x} \frac{1}{n} \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi am/d} = O\left(xe^{-c \sqrt{\log x}}\right), \]

where \(c_0 > 0\) and \(c > 0\) are absolute constants.

**Proof.** The absolute value provides an upper bound:

\[ |T_1(x)| = \left| xe^{-c_0 \sqrt{\log x}} \sum_{n \leq x} \frac{1}{n} \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi am/d} \right| \]

\[ \leq xe^{-c_0 \sqrt{\log x}} \sum_{n \leq x} \frac{1}{n} \sum_{d \leq x} \frac{1}{d} \]

\[ = O\left((x \log^2 x)e^{-c_0 \sqrt{\log x}}\right) \]

\[ = O\left(xe^{-c \sqrt{\log x}}\right), \]

where \(c_0 > 0\) and \(c > 0\) are absolute constants.

**Lemma 4.6.** Let \(x \geq 1\) be a large number, and let \(1 \leq d, m, n \leq x\) be integers. Then,

\[ \frac{1}{d} \sum_{0 \leq a \leq d-1} e^{i2\pi am/d} = \begin{cases} 1 & \text{if } d \mid m, \\ 0 & \text{if } d \nmid m, \end{cases} \]

### 5 Numerical Data

Small numerical tables were generated by an online computer algebra system to estimate the constant, the range of numbers \(x \leq 10^6\) is limited by the wi-fi bandwidth. The error term is defined by

\[ E(x) = \sum_{p \leq x} \sigma([x/p]) - c_0 x \log \log x - c_1 x, \]

where \(c_0 = \zeta(2)\), and \(c_1 = B_1\zeta(2)\) are constants.
Table 1: Numerical Data For $\sum_{p \leq x} \sigma([x/p])$.

| $x$  | $\pi(x)$ | $\sum_{p \leq x} \sigma([x/p])$ | $c_0 x \log \log x + c_1 x$ | Error $E(x)$ |
|------|---------|-------------------------------|-----------------------------|-------------|
| 10   | 4       | 14                            | 18.02                       | 4.02        |
| 100  | 25      | 277                           | 294.22                      | 17.22       |
| 1000 | 168     | 3852                          | 3609.21                     | -242.79     |
| 10000| 1229    | 45843                         | 40824.25                    | -5018.74    |
| 100000|9592   | 481903                        | 444948.14                   | -36954.86   |
| 1000000|78498| 5412077                       | 4749388.38                  | -662688.62  |

6 Problems

**Exercise 6.1.** Let $\pi(x) = \#\{\text{prime } p \leq x\}$, and let $\text{li}(x)$ be the logarithm integral. Show that

$$\pi \left( \left\lfloor \frac{x}{n} \right\rfloor \right) - \pi \left( \left\lfloor \frac{x}{n+1} \right\rfloor \right) = \frac{\text{li}(x)}{n(n+1)} + O \left( \frac{x}{n} e^{-c \sqrt{\log x}} \right),$$

where $c > 0$ is an absolute constant.

**Exercise 6.2.** Sharpen the proof of Lemma ???. Specifically, show that

$$\text{li}(x) \sum_{n \leq x} \frac{1}{n(n+1)} \sum_{d \leq x} \frac{1}{d^2} \sum_{0 < a \leq d-1} e^{i2\pi am/d} = a_0 \text{li}(x) \log \log x + a_1 \text{li}(x) + O \left( xe^{-c \sqrt{\log x}} \right),$$

where $\text{li}(x)$ be the logarithm integral, $a_0 \neq 0$ and $a_1 \neq 0$ are constants, and $c > 0$ is an absolute constant.

**Exercise 6.3.** Let $n \geq 1$ be an integer. Determine the closed form evaluations of the finite sums

$$\sum_{d|n} \frac{1}{d^2} =?, \quad \sum_{d|n} \frac{1}{d^3} =?, \quad \sum_{d|n} \frac{1}{d^4} =?, \quad \sum_{d|n} \frac{1}{d^5} =?, \ldots,$$

Hint: Consider the generalized sum of divisors function $\sigma_s(n) = \sum_{d|n} d^s$.

**Exercise 6.4.** Let $n \geq 1$ be an integer, and let $\mu$ be the Mobius function. Determine the closed form evaluations of the finite sums

$$\sum_{d|n} \frac{\mu(d)}{d^2} =?, \quad \sum_{d|n} \frac{\mu(d)}{d^3} =?, \quad \sum_{d|n} \frac{\mu(d)}{d^4} =?, \quad \sum_{d|n} \frac{\mu(d)}{d^5} =?, \ldots,$$

Hint: Consider the (Jordan function) generalized Euler function $\varphi_s(n) = \prod_{p|n} (1 - p^{-s})$.

References

[1] Apostol, Tom M. *Introduction to analytic number theory*. Undergraduate Texts in Mathematics. Springer-Verlag, New York-Heidelberg, 1976.

[2] Olivier Bordelles, Randell Heyman, Igor E. Shparlinski. *On a sum involving the Euler function*, http://arxiv.org/abs/1808.00188.

[3] Carella, N. A. *Average Orders of the Euler Phi Function, The Dedekind Psi Function, The Sum of Divisors Function, And The Largest Integer Function*. http://arxiv.org/abs/2101.02248.
[4] Carella, N. A. *Euler Totient Function Over The Shifted Primes.* http://arxiv.org/abs/2105.00790.

[5] Linnik, Y. V. *The dispersion method in binary additive problems.* Izdat. Leningrad Univ., Leningrad, 1961.

[6] Montgomery, Hugh L.; Vaughan, Robert C. *Multiplicative number theory. I. Classical theory.* Cambridge University Press, Cambridge, 2007.

[7] Pillichshammer, Friedrich. *Euler’s constant and averages of fractional parts.* Amer. Math. Monthly 117 (2010), no. 1, 78-83.

[8] Zhao, F.; Wu, J. *On a sum involving the sum of divisors function.* J. Math. Art. ID 5574465, 7 pp.