Centered Hardy–Littlewood maximal operator on the real line: lower bounds

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March 16, 2022

Abstract

For $1 < p < \infty$ and $M$ the centered Hardy-Littlewood maximal operator on $\mathbb{R}$, we consider whether there is some $\varepsilon = \varepsilon(p) > 0$ such that $\|Mf\|_p \geq (1 + \varepsilon)\|f\|_p$. We prove this for $1 < p < 2$. For $2 \leq p < \infty$, we prove the inequality for indicator functions and for unimodal functions.

Résumé

Soient $1 < p < \infty$ et $M$ la fonction maximale de Hardy-Littlewood sur $\mathbb{R}$. Nous étudions l’existence d’un $\varepsilon = \varepsilon(p) > 0$ tel que $\|Mf\|_p \geq (1 + \varepsilon)\|f\|_p$. Nous l’établissons pour $1 < p < 2$. Pour $2 \leq p < \infty$, nous prouvons l’inégalité pour les fonctions indicatrices et les fonctions unimodales.

1 Introduction

Given a locally integrable real-valued function $f$ on $\mathbb{R}^n$ define its uncentered maximal function $M_u f(x)$ as follows

$$M_u f(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|dy,$$

where the supremum is taken over all balls $B$ in $\mathbb{R}^n$ containing the point $x$, and $|B|$ denotes the Lebesgue volume of $B$. In studying lower operator norms of the maximal function [4] A. Lerner raised the following question: given $1 < p < \infty$ can one find a constant $\varepsilon = \varepsilon(p) > 0$ such that

$$\|M_u f\|_{L^p(\mathbb{R}^n)} \geq (1 + \varepsilon)\|f\|_{L^p(\mathbb{R}^n)} \quad \text{for all} \quad f \in L^p(\mathbb{R}^n).$$

The affirmative answer was obtained in [2], i.e., the Lerner’s inequality (2) holds for all $1 < p < \infty$ and for any $n \geq 1$. The paper also studied the estimate (2) for other maximal
functions. For example, the lower bound (2) persists if one takes supremum in (1) over the shifts and dilates of a fixed centrally symmetric convex body $K$. Similar positive results have been obtained for dyadic maximal functions [5]; maximal functions defined over $\lambda$-dense family of sets, and almost centered maximal functions (see [2] for details).

The Lerner’s inequality for the centered maximal function

$$\|Mf\|_{L^p(\mathbb{R}^n)} \geq (1 + \varepsilon(p, n))\|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n), \quad Mf(x) = \sup_{r>0} \frac{1}{|B_r(x)|} \int_{B_r(x)} |f|, \quad (3)$$

where the supremum is taken over all balls centered at $x$, is an open question, and the full characterization of the pairs $(p, n)$, $n \geq 1$, and $1 < p < \infty$, for which (3) holds with some $\varepsilon(p, n) > 0$ and for all $f \in L^p(\mathbb{R}^n)$ seems to be unknown. If $n \geq 3$, and $p > \frac{n}{n-2}$ then one can show that $f(x) = \min\{|x|^{n-2}, 1\} \in L^p(\mathbb{R}^n)$, and $Mf(x) = f(x)$, as $f$ is the pointwise minimum of two superharmonic functions. This gives a counterexample to (3).

In fact, Korry [3] proved that the centered maximal operator does not have fixed points unless $n \geq 3$ and $p > \frac{n}{n-2}$, but a lack of fixed points does not imply that (3) holds. On the other hand for any $n \geq 1$, by comparing $Mf(x) \geq C(n)Mu f(x)$, and using the fact that $\|Mu f\|_{L^p(\mathbb{R}^n)} \geq (1 + \frac{B(n)}{p-1})^{1/p}\|f\|_{L^p(\mathbb{R}^n)}$ (see [2]), one can easily conclude that (3) holds true whenever $p$ is sufficiently close to 1. It is natural to ask what is the maximal $p_0(n)$ for which if $1 < p < p_0(n)$ then (3) holds.

1.1 New results

In this paper we study the case of dimension $n = 1$ and the centered Hardy–Littlewood maximal operator $M$. We obtain

Theorem 1. If $1 < p < 2$ and $n = 1$ then Lerner’s inequality (3) holds true, namely

$$\|Mf\|_p \geq \left( \frac{p}{2(p-1)} \right)^{1/p} \|f\|_p.$$

Theorem 2. For $n = 1$, and any $p, 1 < p < \infty$, inequality (3) holds true a) for the class of indicator functions with $\varepsilon(p, n) = 1/4p$, and b) for the class of unimodal functions, with $\varepsilon(p, n)$ not explicitly given.

2 Proof of the main results

2.1 Proof of Theorem 1

First we prove the following modification of the classical Riesz’s sunrise lemma (see Lemma 1 in [1]). Our proof is similar to the proof of the lemma.
Lemma 3. For a nonnegative continuous compactly supported $f$ and any $\lambda > 0$, we have

$$|\{Mf \geq \lambda\}| \geq \frac{1}{2\lambda} \int_{\{f \geq \lambda\}} f.$$ 

Proof. Define an auxiliary function $\varphi(x)$ via

$$\varphi(x) = \sup_{y < x} \int_y^x f(t) dt - 2\lambda(x - y).$$

Notice that if $f(x) > 2\lambda$ then $\varphi(x) > 0$. Indeed,

$$\varphi(x) = \sup_{y < x} (x - y) \left[ \frac{1}{x - y} \int_y^x f - 2\lambda \right] > 0, \quad (4)$$

because we can choose $y$ sufficiently close to $x$, and use the fact that $\lim_{y \to x} \frac{1}{x - y} \int_y^x f = f(x)$. On the other hand if $\varphi(x) > 0$, then $Mf(x) > \lambda$. Indeed, it follows from (4) that $\sup_{y < x} \frac{1}{x - y} \int_y^x f > 2\lambda$. Therefore

$$Mf(x) = \sup_{r > 0} \frac{1}{2r} \int_{x-r}^{x+r} f \geq \frac{1}{2(x - y)} \int_y^x f \geq \lambda.$$

Thus, we obtain

$$\{Mf \geq \lambda\} \supseteq \{f \geq \lambda\} \cup \{\varphi > 0\}; \quad \{f > 2\lambda\} \subseteq \{\varphi > 0\}. \quad (5)$$

Therefore, it follows that

$$|\{Mf \geq \lambda\}| \geq |\{\varphi > 0\}| + |\{\lambda \leq f \leq 2\lambda\} \setminus \{\varphi > 0\}| \geq \frac{1}{2\lambda} \int_{\{\varphi > 0\}} f + \frac{1}{2\lambda} \int_{\{\lambda \leq f \leq 2\lambda\} \setminus \{\varphi > 0\}} f \geq \frac{1}{2\lambda} \int_{\{f \geq \lambda\}} f.$$ 

\[
\quad \square
\]

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Take any continuous bounded compactly supported $f \geq 0$. By Lemma 3, for any $\lambda > 0$ we have

$$|\{Mf \geq \lambda\}| \geq \frac{1}{2\lambda} \int_{\mathbb{R}} f(x) \mathbb{1}_{[\lambda, \infty)}(f(x)) dx. \quad (7)$$
Finally we multiply both sides of (7) by $p\lambda^{p-1}$, and we integrate the obtained inequality in $\lambda$ on $(0, \infty)$, so we obtain

$$\int_{\mathbb{R}} (Mf)^p \geq \int_{0}^{\infty} \int_{\mathbb{R}} \frac{p\lambda^{p-2}}{2} f(x) \mathbb{1}_{[\lambda, \infty)}(f(x)) \, dx \, d\lambda = \frac{p}{2(p-1)} \int_{\mathbb{R}} f^p,$$

and $\frac{p}{2p-2} > 1$ precisely when $p < 2$. This finishes the proof of Theorem 1 for continuous compactly supported bounded nonnegative $f$. To obtain the inequality $\|Mf\|_p \geq \left(\frac{p}{2p-2}\right)^{1/p} \|f\|_p$ for an arbitrary nonnegative $f \in L^p(\mathbb{R})$ we can approximate $f$ in $L^p$ by a sequence of compactly supported smooth functions $f_n$, and use the fact that the operator $M$ is Lipschitz on $L^p$ (since it is bounded and subadditive).

Remark 4. The argument presented above is a certain modification of the classical Riesz’s sunrise lemma, and an adaptation of an argument of Lerner (see Section 4 in [4]). For $p$ less than about 1.53, it is possible to use Lerner’s result directly, together with the fact that $Mf \geq (Mu f)/2$. We need the modified sunrise lemma to get the result for all $p < 2$.

2.2 Proof of Theorem 2

2.2.1 Indicator functions

Proof of Theorem 2 for indicator functions $\mathbb{1}_E$. Let $\mathbb{1}_E \in L^p(\mathbb{R})$ and let $\hat{\delta} > 0$. We approximate $\mathbb{1}_E$ arbitrarily well in $L^p$ by a nonnegative continuous compactly supported function $f$. Then $f$ approximates $\mathbb{1}_E$ and $Mf$ also approximates $M\mathbb{1}_E$ to within some $\delta \ll \hat{\delta}$ in $L^p$.

For a.e. $x \in E$, we have $M\mathbb{1}_E(x) \geq 1$. Additionally, by Lemma 3, we have that

$$|\{Mf \geq 1/4\}| \geq 2 \int_{\{f \geq 1/4\}} f \geq 2 \int_{\{f \geq 1/4\} \cap E} \mathbb{1}_E - 2 \int_{E} |\mathbb{1}_E - f|.$$

By making $\delta$ is small, we can ensure that $\{|f - \mathbb{1}_E| \geq 3/4\}$ is small, so

$$2 \int_{\{f \geq 1/4\} \cap E} \mathbb{1}_E \geq 2|E| - \hat{\delta}/2.$$

Also, by Holder’s inequality, we can bound $\int_{E} |\mathbb{1}_E - f|$ in terms of $||\mathbb{1}_E - f||_p < \delta$. Thus, when $\delta$ is sufficiently small, we get

$$|\{Mf \geq 1/4\}| \geq 2|E| - \hat{\delta},$$

so there is a set of measure at least $|E| - \hat{\delta}$ on which $\mathbb{1}_E = 0$ and $Mf \geq 1/4$. If $\delta$ is sufficiently small, we have that $|\{Mf - M\mathbb{1}_E \geq \delta\}| < \delta$, so there is a set of measure $|E| - 2\delta$ on which $\mathbb{1}_E = 0$ and $M\mathbb{1}_E \geq 1/4 - \delta$. Taking $\delta \to 0$, we get

$$\|M\mathbb{1}_E\|_p^p \geq (1 + 1/4^p)\|\mathbb{1}_E\|_p^p.$$


2.2.2 Unimodal functions

Next we obtain lower bounds on $L^p$ norms of the maximal operator over the class of unimodal functions. By unimodal function $f \in L^p(\mathbb{R})$, $f \geq 0$, we mean any function which is increasing until some point $x_0$ and then decreasing. Without loss of generality we will assume that $x_0 = 0$.

Proof of Theorem 2 for unimodal functions. We can assume that $\|f \mathbb{1}_{\mathbb{R}^+}\|_p^p \geq \frac{1}{2} \|f\|_p^p$.

Let $\tilde{f} = f \mathbb{1}_{\mathbb{R}^+}$. We define $M^n = M \circ \cdots \circ M$ to be the $n$-th iterate of $M$. We will find an $n$, independent of $f$, such that $\|M^n \tilde{f}\|_p^p > 2^{p+1} \|\tilde{f}\|_p^p$, independent of the function $f$. First, for $x > 0$, let

$$a(x) = \min_{k \in \mathbb{Z}} 2^k, 2^k > x.$$ 

Then let

$$\psi(x) = \tilde{f}(a(x)),$$

that is $\psi \leq \tilde{f}$, and $\psi$ is a step function approximation from below. Then

$$2p^p = 2 \sum_{k \in \mathbb{Z}} 2^k \tilde{f}(2^{k+1})^p = \sum_{s \in \mathbb{Z}} 2^s \tilde{f}(2^s)^p \geq \|\tilde{f}\|_p^p$$

Now let

$$\bar{g}(x) = (1 - \sqrt{x}) \mathbb{1}_{(0,1]}(x).$$

Then for $0 < x \leq 9/8$, we have that

$$M \bar{g}(x) \geq \frac{1}{2x} \int_0^{2x} \bar{g}(y)dy \geq \frac{1}{2x} \int_0^{2x} 1 - \sqrt{y}dy = \frac{1}{2} \sqrt{2x} = \bar{g}(8x/9),$$

and for all $x \notin (0,9/8]$, we have $M \bar{g}(x) \geq 0 = \bar{g}(8x/9)$. Thus

$$M^n \bar{g}(x) \geq \bar{g} \left( \left( \frac{8}{9} \right)^n x \right),$$

so

$$\int_{\frac{1}{2}(9/8)^n}^{(9/8)^n} (M^n \mathbb{1}_{(0,1]})^p \geq \int_{\frac{1}{2}(9/8)^n}^{(9/8)^n} (M^n g)^p \geq (9/8)^n \int_{\frac{1}{2}}^1 \bar{g}^p = C_p(9/8)^n. \quad (8)$$

Note that for all $k \in \mathbb{Z}$, we have $\psi \geq \tilde{f}(2^{k+1}) \mathbb{1}_{(2^k,2^{k+1})}$. Thus

$$M^n \psi(x) \geq \tilde{f}(2^{k+1}) M^n \mathbb{1}_{(2^k,2^{k+1})}(x).$$

We will use this lower bound for varying values of $k$ for different $x$. We use (8) in the third inequality below, since $\mathbb{1}_{(2^k,2^{k+1})}$ is just a horizontal rescaling and translation of $\mathbb{1}_{(0,1]}$. We
have
\[
\|M^n \psi\|_p^p \geq \sum_{-\infty}^{\infty} \int_{2^k+(9/8)^n 2^{k-1}}^{2^{k+1}+(9/8)^n 2^k} (M^n \psi)^p \\
\geq \int_{-\infty}^{\infty} \tilde{f}(2^{k+1})^p \int_{2^k+(9/8)^n 2^{k-1}}^{2^{k+1}+(9/8)^n 2^k} (M^n \psi)^p \\
\geq \sum_{-\infty}^{\infty} \tilde{f}(2^{k+1})^p C_p(9/8)^n 2^k \\
= C_p(9/8)^n \|\psi\|_p^p \\
\geq \frac{1}{2} C_p(9/8)^n \|\tilde{f}\|_p^p,
\]
so by picking \( n = n(p) \) sufficiently large, we get
\[
\|M^n f\|_p^p \geq \|M^n \psi\|_p^p \geq 2^n \|\tilde{f}\|_p^p \geq 2^n \|f\|_p^p,
\]
so
\[
\|M^n f\|_p \geq 2 \|f\|_p. \tag{9}
\]
Now suppose that \( \|Mf - f\|_p < \tilde{\epsilon} \|f\|_p \) for some \( \tilde{\epsilon} \) to be chosen later. From the subadditivity of the maximal operator, it follows that \( \|M\phi_1 - M\phi_2\|_p \leq A_p \|\phi_1 - \phi_2\|_p \), so
\[
\|M^n f - f\|_p \leq \sum_{j=1}^{n} \|M^j f - M^{j-1} f\|_p \leq \sum_{j=1}^{n} A_p^{j-1} \|M f - f\|_p < \left( \tilde{\epsilon} \sum_{j=1}^{n} A_p^{j-1} \right) \|f\|_p
\]
which contradicts (9) for \( \tilde{\epsilon} = \tilde{\epsilon}(p) \) sufficiently small. Thus \( \|Mf - f\|_p \geq \tilde{\epsilon} \|f\|_p \), so
\[
\|Mf\|_p^p = \int (Mf)^p \geq \int f^p + (Mf - f)^p = \|f\|_p^p + \|Mf - f\|_p^p \geq (1 + \tilde{\epsilon}^p) \|f\|_p^p,
\]
which proves the theorem. \( \square \)

### 3 Concluding Remarks

Take any compactly supported bounded function \( f \geq 0 \) which is not identically zero. One can show that
\[
(9/8)^{1/p} \leq \lim \inf_{k \to \infty} \|M^k f\|_{L^p}^{1/k} \leq \lim \sup_{k \to \infty} \|M^k f\|_{L^p}^{1/k} \leq a_p, \tag{10}
\]
where the number \( a_p > 1 \) solves \( M(|x|^{-1/p}) = a_p |x|^{-1/p} \) (such an \( a_p \) can be seen to exist by a calculation, or by scaling considerations). In other words, the growth of \( \|M^k f\|_p \) is exponential which suggests that Theorem 1 is likely to be true for all \( 1 < p < \infty \). To show (10)
let us first illustrate the upper bound. Consider the function \( \tilde{f}(x) := f(Cx)/\|f\|_\infty \). For any fixed constant \( C \neq 0 \) one can easily see that \( \limsup_{k \to \infty} \|M_k f\|_p^{1/k} = \limsup_{k \to \infty} \|M_k \tilde{f}\|_p^{1/k} \). Therefore without loss of generality we can assume that \( f \leq 1 \) and the support of \( f \) is in \([-1, 1]\). Next, take any \( \delta \in (0, p - 1) \), and consider

\[
h(x) = \begin{cases} 
1 & |x| \leq 1, \\
|x|^{-1/(p-\delta)} & |x| > 1.
\end{cases}
\]

Clearly \( h \in L^p \), and \( f \leq h \). Since \( M(|x|^{-1/p}) = a_p|x|^{-1/p} \) it follows that \( Mh(x) \leq a_p - \delta h(x) \) for all \( x \in \mathbb{R} \). Thus

\[
\limsup_{k \to \infty} \|M_k f\|_p^{1/k} \leq \limsup_{k \to \infty} \|M_k h\|_p^{1/k} \leq a_p - \delta \limsup_{k \to \infty} \|h\|_p^{1/k} = a_p - \delta.
\]

Finally, taking \( \delta \to 0 \) gives the desired inequality.

To prove the lower bound, we have already seen that the function \( \bar{g}(x) = (1 - \sqrt{x})1_{(0,1]} \) satisfies

\[
M^n \bar{g}(x) \geq \bar{g} ((8/9)^n x),
\]

so we can obtain the growth \((9/8)^{n/p}\) for the function \( \bar{g}(x) \). Now it remains to notice that for any \( f \geq 0, f \in L^p \) not identically zero we can rescale and shift the function \( \bar{g} \) so that \( Mf(x) \geq A\bar{g}(Bx + C) \) for some constants \( A > 0, B, C \neq 0 \). This finishes the proof of the claim.

4 Acknowledgments

We are grateful to an anonymous referee for her/his helpful comments and suggestions.

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