Breaking the hegemony of the triangle method in clique detection

Mirosław Kowaluk\textsuperscript{1} and Andrzej Lingas\textsuperscript{2}

\textsuperscript{1} Institute of Informatics, University of Warsaw, Warsaw, Poland. kowaluk@mimuw.edu.pl
\textsuperscript{2} Department of Computer Science, Lund University, 22100 Lund, Sweden.
Andrzej.Lingas@cs.lth.se

Abstract. We consider the fundamental problem of detecting/counting copies of a fixed pattern graph in a host graph. The recent progress on this problem has not included complete pattern graphs, i.e., cliques (and their complements, i.e., edge-free pattern graphs, in the induced setting). The fastest algorithms for the aforementioned patterns are based on a straightforward reduction to triangle detection/counting. We provide an alternative method of detection/counting copies of fixed size cliques based on a multi-dimensional matrix product. It is at least as time efficient as the triangle method in cases of $K_4$ and $K_5$. The complexity of the multi-dimensional matrix product is of interest in its own rights. We provide also another alternative method for detection/counting $K_r$ copies, again time efficient for $r \in \{4, 5\}$.

1 Introduction

The problems of detecting, finding, counting or listing subgraphs or induced subgraphs of a host graph that are isomorphic to a pattern graph are basic in graph algorithms. They are generally termed as subgraph isomorphism and induced subgraph isomorphism problems, respectively. Such well-known NP-hard problems as the independent set, clique, Hamiltonian cycle or Hamiltonian path can be regarded as their special cases.

Recent examples of applications of different variants of subgraph isomorphism include among other things [7,19]: a comparison of biomolecular networks by their so-called motifs [2], an analysis of social networks by counting the number of copies of a small pattern graph [16], graph matching constraints in automatic design of processor systems [18], and the detection of communication patterns between intruders in network security [17]. In the aforementioned applications, the pattern graphs are typically of fixed size which allows for polynomial-time solutions.
At the beginning of 80s, Itai and Rodeh [11] presented the following straightforward reduction of not only triangle detection but also triangle counting to fast matrix multiplication. Let $A$ be the $0-1$ adjacency matrix of the host graph $G$ on $n$ vertices (see Preliminaries). Consider the matrix product $C = A \times A$. Note that $C[i, j] = \sum_{l=1}^{n} A[i, l]A[l, j]$ is the number of two-edge paths connecting the vertices $i$ and $j$. Hence, if $\{i, j\}$ is an edge of $G$ then $C[i, j]$ is the number of triangles in $G$ including the edge $\{i, j\}$. Consequently, the number of triangles in an $n$-vertex graph can be reported in $O(n^\omega)$ time, where $\omega$ is the exponent of fast matrix multiplication for matrices of size $n \times n$. (Recently, Alman and Vassilevska Williams have shown that $\omega \leq 2.3729$ [1].)

A few years later, Necetril and Poljak [15] showed an efficient reduction of detection and counting copies of any pattern graph both in the standard and induced case to the aforementioned method for triangle detection and counting. The idea is to divide the pattern graph into three almost equal parts and to build an auxiliary graph on copies of subgraphs isomorphic to one of three parts. Then, the triangle detection/counting method is run on the auxiliary graph. Two decades later, Eisenbrand and Grandoni [6] (cf. [12]) refined this general triangle method by using fast algorithms for rectangular matrix multiplication instead of those for square matrix multiplication. For a pattern graph on $r \geq 3$ vertices and a host graph on $n$ vertices, the (refined) general triangle method runs in time $O(n^{\omega(\lfloor r/3 \rfloor, \lceil (r-1)/3 \rceil, \lceil r/3 \rceil)})$ [6,12,15], where $\omega(p, q, s)$ denotes the exponent of fast matrix multiplication for rectangular matrices of size $n^p \times n^q$ and $n^q \times n^s$, respectively [14]. For example, it is known that $\omega(1, 2, 1) \leq 3.257$ [14].

Up to now, the general triangle method remains the fastest known universal method for the detection and counting standard and induced copies of fixed pattern graphs. In the recent two decades, there has been a real progress in the design of efficient algorithms for detection and even counting of fixed pattern graphs both in the standard [7,13] and induced case [4,5,7,15]. Among other things, the progress has been based on the use of equations between the numbers of copies of different fixed patterns in the host graph [5,12,13,19] and randomization [5,17,19]. Unfortunately, this progress has not included complete pattern graphs, i.e., $K_r$ graphs (and their complements, i.e., edge-free pattern graphs in the induced setting). For the aforementioned pattern graphs, the generalized triangle method remains the fastest known one.
In this paper, we consider another universal method that in fact can be viewed as another type of generalization of the classic algorithm for triangle detection and counting due to Itai and Rodeh. We can rephrase the description of their algorithm as follows. At the beginning, we form a list of subgraphs isomorphic to $K_2$ (i.e., edges) and then for each subgraph on the list we count the number of vertices outside it that are adjacent to both vertices of the subgraph, in other words, we count the number of extensions of the subgraph to a clique on three vertices. The latter task can be done efficiently by squaring the adjacency matrix of the host graph. We can generalize the algorithm to include detection/counting $K_r$ copies, $r \geq 3$, by replacing $K_2$ with $K_{r-1}$ and using a $(r-1)$-dimensional product of $r-1$ copies of the adjacency matrix (see Section 3 for the definition) instead of squaring the matrix. Listing the subgraphs of the host graph takes $O(n^{r-1})$ time so the overall time required by the alternative method is $O(n^{r-1} + n^{\omega(r-1)})$, where $\omega_k$ is the exponent of fast $k$-dimensional product of $k n \times n$ matrices. On the other hand, we show in particular that $\omega_k \leq \omega([k/2], 1, [k/2])$. Hence, our alternative method in particular computes the number of $K_4$ copies in an $n$-vertex graph in $O(n^{\omega_2})$ time and the number of $K_5$ copies in $O(n^{\omega_2})$ time. Also, if the input graph contains a copy of $K_4$ or $K_5$ respectively then a copy of $K_4$ can be found in the graph in $\tilde{O}(n^{\omega_2})$ time while that of $K_5$ in $\tilde{O}(n^{\omega_2})$ time by a slightly modified alternative method. Thus, our upper time bounds for $K_4$ and $K_5$ at least match those for $K_4$ and $K_5$ yielded by the generalized triangle method [6]. If $\omega_k < \omega([k/2], 1, [k/2])$ for $k$ equal to 3 or 4 then we would get a breakthrough in detection/counting of $K_4$ or $K_5$, respectively. For $K_r$, where $r \geq 6$, the generalized triangle method asymptotically subsumes our alternative method and for $K_3$ the methods coincide.

We provide also another alternative method for detection/counting $K_r$ copies, where $r \geq 3$. It starts from listing all $K_{r-2}$ copies and then it tries to extend them by two vertices to $K_r$ copies. Again, the method is time efficient for $r \leq 5$. Finally, in order to obtain a method for detection/counting $K_r$ copies that could compete with the generalized triangle method for $r \geq 6$, we consider a generalization of our alternative methods. Similarly, it starts from listing all $K_q$ copies, where $q < r - 1$, and then it tries to extend them by $r - q$ vertices to form $K_r$ copies. However, to perform the extension step efficiently, we need to split the extending
vertex sets in two almost equal parts, so the generalized method can be also regarded as a variant of the generalized triangle one.

1.1 Paper organization

In the next section, the basic matrix and graph notation used in the paper is presented. Section 3 is devoted to the $k$-dimensional matrix product of $k$ matrices, in particular its definition and upper time bounds on the product in terms of those for fast rectangular matrix multiplication. In Section 4, the alternative method for detection/counting copies of fixed cliques in a host graph relying on the multi-dimensional matrix product is presented and analyzed. Section 5 presents shortly another alternative method for detection/counting $K_r$ copies while Section 6 is devoted to a generalization of the alternative methods. We conclude with open problems.

2 Preliminaries

For a positive integer $r$, we shall denote the set of positive integers not greater than $r$ by $[r]$.

For a matrix $D$, $D^T$ denotes its transpose. For positive real numbers $p$, $q$, $s$, $\omega(p, q, s)$ denotes the exponent of fast matrix multiplication for rectangular matrices of size $n^p \times n^q$ and $n^q \times n^s$, respectively. For convenience, $\omega = \omega(1, 1, 1)$.

Let $\alpha$ stand for $\sup \{0 \leq q \leq 1 : \omega(1, q, 1) = 2 + o(1)\}$. The following recent lower bound on $\alpha$ is due to Le Gall and Urrutia [9].

**Fact 1** The inequality $\alpha > 0.31389$ holds [9].

A *witness* for a non-zero entry $C[i, j]$ of the Boolean matrix product $C$ of a Boolean $p \times q$ matrix $A$ and a Boolean $q \times s$ matrix $B$ is any index $\ell \in [q]$ such that $A[i, \ell]$ and $B[\ell, j]$ are equal to 1.

The *witness problem* is to report a witness for each non-zero entry of the Boolean matrix product of the two input matrices.

Alon and Naor provided a solution to the witness problem for square Boolean matrices [3] which is almost equally fast as that for square matrix multiplication [1]. It can be easily generalized to include the Boolean product of two rectangular Boolean matrices of sizes $n^p \times n^q$ and $n^q \times n^s$,
respectively. The asymptotic matrix multiplication time $n^\omega$ is replaced by $n^{\omega(p,q,s)}$ in the generalization.

**Fact 2** For positive $p, q, s$, the witness problem for the Boolean matrix product of an $n^p \times n^q$ Boolean matrix with an $n^q \times n^s$ Boolean matrix can be solved (deterministically) in $\tilde{O}(n^{\omega(p,q,s)})$ time.

We shall consider only simple undirected graphs.

A subgraph of the graph $G = (V, E)$ is a graph $H = (V_H, E_H)$ such that $V_H \subseteq V$ and $E_H \subseteq E$.

An induced subgraph of the graph $G = (V, E)$ is a graph $H = (V_H, E_H)$ such that $V_H \subseteq V$ and $E_H = E \cap (V_H \times V_H)$. A subgraph of $G$ induced by $S \subseteq V$ is a graph $F = (V_F, E_F)$ such that $V_F = S$ and $E_F = E \cap (S \times S)$. It is denoted by $G[S]$.

For simplicity, we shall refer to a subgraph of a graph $G$ that is isomorphic to $K_r$ as a copy of $K_r$ in $G$ or just $K_r$ copy in $G$.

The adjacency matrix $A$ of a graph $G = (V, E)$ is the $0-1$ matrix such that $n = |V|$ and for $1 \leq i, j \leq n, A[i, j] = 1$ if and only if $\{i, j\} \in E$.

### 3 Multi-dimensional matrix product

**Definition 1.** For $k$ $n \times n$ matrices $A_q$, $q = 1, ..., k$, (arithmetic or Boolean, respectively) their $k$-dimensional (arithmetic or Boolean, respectively) matrix product $D$ is defined by

$$D[i_1, i_2, ..., i_k] = \sum_{\ell=1}^{n} A_1[i_1, \ell]A_2[i_2, \ell]...A_k[i_k, \ell],$$

where $i_j \in [n]$ for $j = 1, ..., k$. The exponent of fast $k$-dimensional (arithmetic) matrix product of $k$ $n \times n$ matrices is denoted by $\omega_k$.

In the Boolean case, a witness for a non-zero entry $D[i_1, i_2, ..., i_k]$ of the $k$-dimensional Boolean matrix product is any index $\ell \in [n]$ such that $A_1[i_1, \ell]A_2[i_2, \ell]...A_k[i_k, \ell]$ is equal to (Boolean) $1$. The witness problem for the $k$-dimensional Boolean matrix product is to report a witness for each non-zero entry of the product.

Note that in particular the $2$-dimensional matrix product of the matrices $A_1$ and $A_2$ coincides with the standard matrix product of $A_1$ and $(A_2)^T$ which yields $\omega_2 = \omega$.
Lemma 1. Let $k, k_1, k_2$ be three positive integers such that $k = k_1 + k_2$. Both in the arithmetic and Boolean case, the $k$-dimensional matrix product of $n \times n$ matrices can be computed in $O(n^{\omega(k_1,1,k_2)})$ time, consequently $\omega_k \leq \omega(k_1,1,k_2)$. Also, in the Boolean case, the witness problem for the $k$-dimensional matrix product can be solved in $\tilde{O}(n^{\omega(k_1,1,k_2)})$ time.

Proof. To prove the first part, it is sufficient to consider the arithmetic case as the Boolean one trivially reduces to it.

Let $A_1, \ldots, A_k$ be the input matrices. Form an $n^{k_1} \times n$ matrix $A$ whose rows are indexed by $k_1$-tuples of indices in $[n]$ and whose columns are indexed by indices in $[n]$ such that $A[i_1 \ldots i_{k_1}, \ell] = A_1[i_1, \ell] \ldots A_{k_1}[i_{k_1}, \ell]$. Similarly, form an $n^{k_2} \times n$ matrix $B$ whose rows are indexed by $k_2$-tuples of indices in $[n]$ and whose columns are indexed by indices in $[n]$ such that $B[j_1 \ldots j_{k_2}, \ell] = A_{k_1+1}[j_1, \ell] \ldots A_k[j_{k_2}, \ell]$. Compute the rectangular matrix product $C$ of the matrix $A$ with the matrix $B^T$. By the definitions, the $D[i_1, \ldots, i_{k_1}, i_{k_1+1}, \ldots, i_k]$ entry of the product of the input matrices $A_1, \ldots, A_k$ is equal to the entry $C[i_1 \ldots i_{k_1}, i_{k_1+1} \ldots i_k]$. The matrices $A$, $B$ can be formed in $O(n^{k_1+1} + n^{k_2+1})$ time, i.e., $O(n^k)$ time, while the product $C$ can be computed in $O(n^{\omega(k_1,1,k_2)})$ time.

To prove the second part of the lemma it is sufficient to consider Boolean versions of the matrices $A$, $B$, $C$ and use Fact 2.

By combining Lemma 1 with Fact 1, we obtain the following corollary.

Corollary 1. For even $k \geq 8$, $\omega_k = k + o(1)$.

Proof. We obtain the following chain of qualities on the asymptotic time required by the $k$-dimensional matrix product using Lemma 1 and Fact 1:

$$n^{\omega(k/2,1,k/2)} = \left(n^{k/2}\right)^{\omega(1,2/k,1)} = \left(n^{k/2}\right)^{2+o(1)} = n^{k+o(1)}.$$ 

□

4 Clique detection

The following algorithm is a straightforward generalization of that due to Itai and Rodeh for triangle counting [14] to include $K_r$ counting, for $r \geq 3$. 

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Algorithm 1

1. form a list \( L \) of all \( K_{r-1} \) copies in \( G \)
2. \( t \leftarrow 0 \)
3. for each \( C \in L \) do
   increase \( t \) by the number of vertices in \( G \) that are adjacent to all vertices of \( C \)
4. return \( t/r \)

The correctness of Algorithm 1 follows from the fact that the number of \( K_r \) copies including a given copy \( C \) of \( K_{r-1} \) in the host graph is equal to the number of vertices outside \( C \) in the graph that are adjacent to all vertices in \( C \) and that a copy of \( K_r \) includes exactly \( r \) distinct copies of \( K_{r-1} \) in the graph.

The first step of Algorithm 1 can be implemented in \( O(n^{r-1}) \) time. We can use the \((r-1)\)-dimensional matrix product to implement the third step by using the next lemma immediately following from the definition of the product.

Lemma 2. Let \( D \) be the \( k \)-dimensional matrix product of \( k \) copies of the adjacency matrix of the input graph \( G \) on \( n \) vertices. Then, for any \( k \) tuple \( i_1, i_2, \ldots, i_k \) of vertices of \( G \), the number of vertices in \( G \) adjacent to each vertex in the \( k \) tuple is equal to \( D[i_1, i_2, \ldots, i_k] \).

By the discussion and Lemma 2 we obtain the following theorem.

Theorem 1. The number of \( K_r \) copies in the input graph on \( n \) vertices can be computed (by Algorithm 1) in \( O(n^{r-1} + n^{\omega r - 1}) \) time.

By Lemma 1 we obtain the following corollary from Theorem 1, matching the upper time bounds on the detection/counting copies of \( K_4 \) and \( K_5 \) established in [6].

Corollary 2. The number of \( K_4 \) copies in an \( n \)-vertex graph can be computed (by Algorithm 1) in \( O(n^{\omega(2,1,1)}) \) time while the number of \( K_5 \) copies in \( O(n^{\omega(2,1,2)}) \) time. Also, if the input graph contains a copy of \( K_4 \) or \( K_5 \) respectively then a copy of \( K_4 \) can be found in the graph in \( \tilde{O}(n^{\omega(2,1,1)}) \) time while that of \( K_5 \) in \( \tilde{O}(n^{\omega(2,1,2)}) \) time (by a modification of Algorithm 1).
5 Another alternative method for $K_r$ detection/counting

The basic idea of our alternative method for detection/counting copies of $K_r$ presented in the previous section is to list copies of $K_{r-1}$ and then extend them by single vertices to form copies of $K_r$ if possible. In this section, we present a similar method based on the idea of extending $K_{r-2}$ copies by pairs of vertices if possible.

This simple method for detection/counting copies of $K_r$, where $r \geq 3$, in a host graph $G = (V, E)$ on $n$ vertices is as follows. First, we form a list $L$ of all $K_{r-2}$ copies in $O(n^{r-2})$ time. Then, for each $H$ in $L$, we compute the set $S(H)$ of vertices which extend $H$ to a copy of $K_{r-1}$ in $G$. It takes totally $O(n \times n^{r-2})$ time. Next, we form a $0 \times |V|$ matrix $B$ whose rows correspond to $v \in V$ and whose columns correspond to $H \in L$ such that $B[v, H] = 1$ if and only if $v \in S(H)$. Then, we compute the matrix product $C$ of $B$ with its transpose $B^T$ in $O(n^{\omega(1, r-2, 1)})$ time. Note that $C[v, u]$ is equal to the number of copies of $K_{r-2}$ in $G$ that can be extended to a copy of $K_{r-1}$ in $G$ both by $v$ and $u$. Now, it is sufficient to check for each non-zero entry $C[v, u]$ if in the adjacency matrix $A$ of $G$, for the corresponding entry $A[v, u] = 1$ holds. Simply, then the pair of vertices $v, u$ extending the same $C[v, u]$ copies of $K_{r-2}$ in $G$ to pairs of $K_{r-1}$ copies in $G$ is adjacent so $C[v, u]$ copies of $K_r$ occur in $G$. More concisely, we can describe this method as follows under the assumptions that $r \geq 3$, $G = (V, E)$ is the input graph and $A$ is its adjacency matrix.

Algorithm 2

1. $L \leftarrow$ a list of all $K_{r-2}$ copies in $G$
2. for $H \in L$ do
   S($H$) $\leftarrow$ the set of vertices extending $H$ to a copy of $K_{r-1}$ in $G$
3. initialize a $0 \times |V| \times |L|$ matrix $B$
4. for $v \in V \land H \in L$ do
   if $v \in S(H)$ then $B[v, H] \leftarrow 1$ else $B[v, H] \leftarrow 0$
5. $C \leftarrow B \times B^T$
6. $t \leftarrow 0$
7. for $\{v, u\} \subset V$ do
   if $A[v, u] = 1$ then $t \leftarrow t + C[v, u]$
8. return $t / \binom{r}{2}$
As each edge \( \{v, u\} \in E \) occurs in \( C[v, u] \) copies of \( K_r \) in \( G \) it contributes \( C[v, u] \) to \( t \). On the other hand, \( K_r \) has \( \binom{r}{2} \) edges. Hence, the final value of \( t \) divided by \( \binom{r}{2} \) yields the number of \( K_r \) copies in \( G \). By the discussion and \( \omega(1, r - 2, 1) \geq r - 1 \), we obtain the following theorem.

**Theorem 2.** Algorithm 2 computes the number of \( K_r \) copies in an \( n \)-vertex graph in \( O(n^{\omega(1, r - 2, 1)}) \) time.

**Corollary 3.** Algorithm 2 computes the number of \( K_4 \) copies in \( O(n^{\omega(1, 2, 1)}) \) time while the number of \( K_5 \) copies in \( O(n^{\omega(1, 3, 1)}) \) time.

Again, we can use Fact 2 to modify Algorithm 2 to find a copy of \( K_r \) in \( \tilde{O}(n^{\omega(1, r - 2, 1)}) \) time in the graph in case it contains copies of \( K_r \).

### 6 A generalization of the alternative methods

Our two alternative methods for detection/counting \( K_r \) copies at least match the generalized triangle method for \( r \leq 5 \) but they are asymptotically subsumed by the latter method for larger \( r \). In this section, we present a generalization of our two alternative methods that for appropriate parameters is competitive even for \( r \) larger than 5. The basic idea of the generalization is to start from listing copies of \( K_q \) in the host graph, where \( r - q \geq 2 \), and then to detect extensions of the \( K_q \) copies by \( r - q \geq 2 \) vertices to \( K_r \) copies in the graph. To perform the latter task efficiently, we split such an extension into two almost equal parts, so this generalized method can be also regarded as a variant of the triangle one.

The generalized method for detecting copies of \( K_r \) in a host graph \( G = (V, E) \) on \( n \) vertices presented in this section is as follows. First, we form a list \( L \) of all \( K_q \) copies in \( O(n^q) \) time. Then, for each \( H \) in \( L \), we compute the set \( S(H) \) of vertices which extend \( H \) to a copy of \( K_{q+1} \) in \( G \). It takes totally \( O(n \times n^q) \) time. Now, to find extensions of the \( K_q \) copies by \( r - q \) vertices to form \( K_r \) copies, we set \( r_1 \) to \( \lceil \frac{r - q}{2} \rceil \) and \( r_2 \) to \( \lfloor \frac{r - q}{2} \rfloor \). Next, for each \( H \in L \) and \( i \in [2] \), we form a list \( L_i(H) \) of all \( K_{r_i} \) copies in \( G[S(H)] \). It takes totally \( O(n^q + r_1) \) time. Then, for \( i \in [2] \), we create a \( 0 - 1 \) matrix \( B_i \) whose rows correspond to sets \( s_i \) of \( r_i \) vertices and whose columns correspond to \( H \in L \) such that \( B_i[s_i, H] = 1 \) if and only if there is a copy of \( K_{r_i} \), whose vertex set is \( s_i \), in \( L_i(H) \). Again, this takes totally \( O(n^q + r_1) \) time. Now, it is sufficient to compute the matrix
product $C$ of $B_1$ with $B_2^T$ and check if there is a non-zero entry $C[s_1, s_2]$, where $s_1 \cup s_2$ induces a copy of $K_{r-q}$ in the graph. Simply, then all vertices in the induced $(r - q)$-clique have to be adjacent to the same $H \in L$, so they jointly with the vertices of $H$ induce a copy of $K_r$ in $G$.

The computation of the matrix product $C$ takes $O(n^{\omega(\lceil r-q/2 \rceil, q, \lfloor r-q/2 \rfloor)})$ time, and the checking of the matrix product $O(n^{r+q})$ time, i.e., $O(n^{r-q})$ time.

More concisely, we can describe this method as follows under the assumptions that $r \geq 3$, $q \in [r-2]$, and $G = (V, E)$ is the input graph.

**Algorithm 3**

1. $L \leftarrow$ a list of all $K_q$ copies in $G$
2. **for** $H \in L$ do
   - $S(H) \leftarrow$ the set of vertices extending $H$ to a copy of $K_{q+1}$ in $G$
3. $r_1 \leftarrow \lceil \frac{r-q}{2} \rceil$
4. $r_2 \leftarrow \lfloor \frac{r-q}{2} \rfloor$
5. **for** $H \in L \land i \in [2]$ do
   - $L_i(H) \leftarrow$ a list of all $K_{r_i}$ copies in $G[S(H)]$
6. **for** $i \in [2]$ do
   - form a 0–1 matrix $B_i$ whose rows correspond to sets $s_i$ of $r_i$ vertices and whose columns correspond to $H \in L$ such that $B_i[s_i, H] = 1$ iff $G[s_i]$ is a copy of $K_{r_i}$ in $L_i(H)$
7. $C \leftarrow B_1 \times B_2^T$
8. **for** each $r_1$-vertex subset $s_1$ and each $r_2$-vertex subset $s_2$ do
   - if $C[s_1, s_2] = 1$ and $G[s_1 \cup s_2]$ is a $(q-r)$-clique then return YES and stop
9. return NO

By the discussion and $\omega(\lceil \frac{r-q}{2} \rceil, q, \lfloor \frac{r-q}{2} \rfloor) \geq \lceil \frac{r-q}{2} \rceil + q$, we obtain the following theorem.

**Theorem 3.** Let $q \in [r-2]$. Algorithm 3 detects a copy of $K_r$ in an $n$-vertex graph in $O(n^{\omega(\lceil r-q/2 \rceil, q, \lfloor r-q/2 \rfloor)})$ time.

Algorithm 3 can be refined to return the number of $K_r$ copies in the input graph. We can also use Fact 2 to modify Algorithm 3 to find a copy of $K_r$ in $\tilde{O}(n^{\omega(\lceil r-q/2 \rceil, q, \lfloor r-q/2 \rfloor)})$ in the graph in case it contains copies of $K_r$.

### 7 Open problems

It is an intriguing open problem if the upper bounds in terms of rectangular matrix multiplication on the $k$-dimensional matrix product of $k$
square matrices yielded by Lemma 1 are asymptotically tight. In other words, the question is if \( \omega_k = \min_{k' = 1}^{k - 1} \omega(k', 1, k - k') \) holds or more specifically if \( \omega_k = \omega([k/2], 1, |k/2|) \)? If this was not the case for \( k \) equal to 3 or 4 then we would get a breakthrough in detection/counting of \( K_4 \) or \( K_5 \), respectively.

An argument for the inequality \( \omega_k < \omega(k_1, 1, k_2) \), for positive integers \( k_1, k_2 \) satisfying \( k = k_1 + k_2 \), is that in the context of the efficient reduction in the proof of Lemma 1 the rectangular matrix product seems more general than the \( k \)-dimensional one. A reverse efficient reduction seems to be possible only under very special assumptions. However, proving such an inequality would be extremely hard as it would imply \( \omega(k_1, 1, k_2) > k \) and in consequence \( \omega > 2 \) by the straightforward reduction of the rectangular matrix product to the square one. On the other hand, this does not exclude the possibility of establishing better upper bounds on \( \omega_k \) than those known on \( \omega(k_1, 1, k_2) \).

Our alternative methods for detection/counting \( K_r \) copies are competitive and promising for \( r \leq 5 \). It is also an interesting question if there is a truly alternative method for detection/counting \( K_r \) copies that could at least match the generalized triangle method for \( r > 5 \)?

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