Zero-error feedback capacity via dynamic programming

Lei Zhao and Haim Permuter

Abstract

In this paper, we study the zero-error capacity for finite state channels with feedback when channel state information is known to both the transmitter and the receiver. We prove that the zero-error capacity in this case can be obtained through the solution of a dynamic programming problem. Each iteration of the dynamic programming provides lower and upper bounds on the zero-error capacity, and in the limit, the lower bound coincides with the zero-error feedback capacity. Furthermore, a sufficient condition for solving the dynamic programming problem is provided through a fixed-point equation. Analytical solutions for several examples are provided.

Index Terms

Bellman equations, competitive Markov decision processes, dynamic programming, feedback capacity, fixed-point equation, infinite-horizon average reward, stochastic games, zero-error capacity.

I. INTRODUCTION

In 1956, Shannon [1] introduced the concept of zero-error communication, which requires that the probability of error in decoding any message transmitted through the channel to be zero. Although the zero-error capacity for general channels remains an unsolved problem (see [2] for a comprehensive survey of zero-error information theory), Shannon [1] showed that for discrete memoryless channels (DMC) with feedback the zero-error capacity is either zero (if any two inputs can generate a common output) or equal to:

$$C_{0}^{FB} = \max_{P_X} \log_2 \left[ \max_y \sum_{x \in G(y)} P_X(x) \right]^{-1},$$

where $P_X$ is the channel input distribution, $y$ is an output realization of the channel, and $G(y)$ is the set of inputs that have a positive probability of generating the output $y$, i.e., $G(y) \triangleq \{ x : P_{Y|X}(y|x) > 0 \}$. The achievability proof of (1) is based on a determinist scheme rather than on a random coding scheme, as used for showing the achievability of regular capacity.

In this paper, we study the zero-error feedback capacity for finite state channels (FSC), a family of channels with memory. We make the assumptions that channel state information (CSI) is available both to the transmitter and to the receiver. In this case, we solve the zero-error capacity that depends only on the topological properties of the channel. A similar setup has been used by Chen and Berger [3], who solved the regular channel capacity by finding
the optimal stationary and nonstationary input processes that maximize the long-term directed mutual information. In [4] and [5], the zero-error capacity of the chemical channel with feedback was derived. The chemical channel is a special case of FSCs. With feedback, the transmitter knows the state of the chemical channel while the receiver does not, which is different from our setup. Other related work can be found in [6], which addresses the zero-error capacity for compound channels.

The remaining of the paper is organized as follows. In Section II, we introduce the channel model and the dynamic programming problem formulation. In Section III we use a finite-horizon dynamic programming (DP) to provide a condition for the channel to have zero zero-error capacity. In Section IV we define an infinite-horizon average reward DP problem and link its solution the the zero-error capacity. In Sections V and VI we prove the converse and direct parts respectively. In Section VII we explain how to evaluate the infinite-horizon average reward DP; in particular, we provide a sequence of lower and upper bounds that are easy to compute and prove the Bellman equation theorem for the particular DP, namely, a fixed-point equation that is a sufficient condition for verifying the optimality of a solution. In Section VIII we evaluate and then find analytically the zero error feedback capacity of several examples.

II. CHANNEL MODEL AND PRELIMINARIES

We use calligraphic letter $\mathcal{X}$ to denote the alphabet and $|\mathcal{X}|$ to denote the cardinality of the alphabet. Subscripts and superscripts are used to denote vectors in the following way: $x^j = (x_1, ..., x_j)$ and $x_i^j = (x_i, ..., x_j)$ for $i \leq j$.

Next we introduce the channel model and the DP formulation.

A. Channel model and zero-error capacity definition

An FSC [7, ch. 4] is a channel that, at each time index, has a state which belongs to a finite set $\mathcal{S}$ and has the property that, given the current input and state, the output and the next state is independent of the past inputs, outputs and states, i.e.,

$$p(y_t, s_{t+1}|x_t^t, s_1^t) = p(y_t, s_{t+1}|x_t, s_t).$$

For simplicity, we assume that the channel has the same input alphabet $\mathcal{X}$ and the same output alphabet $\mathcal{Y}$ for all states. The alphabets $\mathcal{X}$ and $\mathcal{Y}$ are both finite. Without loss of generality, we can assume that $\mathcal{X} = \{1, 2, ..., |\mathcal{X}|\}$. We consider the communication setting shown in Fig. 1 where the state of the channel is known to the encoder and to the decoder.

An $(M, n)$ zero-error feedback code of length $n$ is defined as a sequence of encoding mappings $x_t(m, y^{t-1}, s^t)$ and a decoding function $\hat{m} = g(y^n, S^{n+1})$, where a message $m$ is selected from a set $\{1, ..., M\}$. The probability of error is required to be zero, i.e., $\Pr\{g(y^n, S^{n+1}) \neq m | \text{message } m \text{ is sent}\} = 0$ for all messages $m \in \{1, 2, ..., M\}$. We emphasize that the size of the message set $M$ does not depend on the initial state of the channel; hence, the probability of error decoding needs to be zero for any initial state.

**Definition 1** A rate $R$ is **achievable** if there exists an $(M, n)$ zero-error feedback code such that $R \leq \frac{\log M}{n}$. 
Definition 2 The operational zero-error capacity of an FSC is defined as the supreme of all achievable rates.

Throughout this paper we use the following alternative and equivalent definition of the operational zero-error capacity.

Definition 3 Let $M(n, s)$ be the maximum number of messages that can be transmitted with zero error in $n$ transmissions when the initial state of the channel is $s \in \mathcal{S}$. Define

$$a_n = \min_{s \in \mathcal{S}} \log_2 M(n, s).$$

The operational zero-error capacity is given by:

$$C_0 \triangleq \lim_{n \to \infty} \frac{a_n}{n} = \lim_{n \to \infty} \frac{\min_{s \in \mathcal{S}} \log_2 M(n, s)}{n},$$

where the limit is shown to exist.

Since the transmitter knows the state, the sequence $\{a_n\}$ is super additive, i.e., $a_{n+m} \geq a_n + a_m$ and $\frac{a_n}{n} \leq \|\mathcal{X}\|$. By Fekete’s lemma [8, Ch. 2.6], $\lim_{n \to \infty} \frac{a_n}{n}$ exists and is equal to $\sup \frac{a_n}{n}$. Note that $R \leq \lim_{n \to \infty} \frac{a_n}{n}$ holds for any achievable rate $R$, and any rate less than $\lim \frac{a_n}{n}$ is achievable, which are simple consequences of Definition 1. Thus, $\lim \frac{a_n}{n}$ defines the zero-error capacity.

B. Dynamic programming

For the standard Markov decision process (MDP), we have the dynamic programming equation [9, 10]:

$$U_n(s) = \max_{a \in \mathcal{A}(s)} \left\{ r(s, a) + \sum_{s' = 1}^{N} P(s'|s, a)U_{n-1}(s') \right\},$$

where $r(s, a)$ is the reward, given that we are at state $s \in \mathcal{S}$, and we perform action $a \in \mathcal{A}$. The term $U_n(s)$ is the total reward after $n$ steps (a.k.a. the ”reward-to-go” in $n$ steps) when we start at time $s$. The conditional distribution $P(s'|s, a)$ is the probability of the next state $s' \in \mathcal{S}$, given the current state $s \in \mathcal{S}$ and action $a \in \mathcal{A}$.
The dynamic programming equation that is associated in this paper with the zero-error capacity has the form

\[ U_n(s) = \max_{a \in A(s)} \min_{s' \in \mathcal{S}(s)} \{ r(s, a, s') + U_{n-1}(s') \}, \]  

(6)

where \( r(s, a, s') \) is the reward, given the current state \( s \), the action \( a \) and the next state \( s' \). The reward may be any real number, including \( \pm \infty \). The value \( U_n(s) \) is defined as before, i.e., the total reward in \( n \) steps when starting at state \( s \).

The DP equation in (6) may be viewed as a stochastic game [11], which is a.k.a competitive MDP [12], in which there are two asymmetric players. Player 1, the leader, takes an action \( a \in \mathcal{A}(s) \), which may depend on the current state and Player 2, the follower, determines the next state \( s' \in \mathcal{S} \). Player 2 sees the state of the game \( s \) and the action of player 1. In the zero-error capacity problem, Player 1 would be the user who designs the code to maximize the transmitted rate, and Player 2 would be Nature, which chooses the next state to minimize the transmitted rate.

### III. A SUFFICIENT AND NECESSARY CONDITION FOR \( C_0 = 0 \)

Shannon [1] showed that for a DMC, which is an FSC with only one state, if any two input letters have at least one common output, it is impossible to distinguish between two messages with zero-error. Using finite-horizon dynamic programming, we derive in this section a sufficient and necessary condition for an FSC to have \( C_0 = 0 \), i.e., the zero-error capacity is zero.

**Definition 4** Two input letters \( x_1 \) and \( x_2 \) are called **adjacent** at state \( s \) if there exists an output letter \( y \) and a state \( s' \) such that \( p(y, s'|x_1, s) > 0 \) and \( p(y, s'|x_2, s) > 0 \).

**Definition 5** A state \( s \) is **positive** if there exist two input letters that are not adjacent at state \( s \).

The intuition behind the result in this section is that if the channel undergoes only non-positive states during the transmission, we cannot distinguish between two messages based on the output sequence and the channel state sequence, since they could result from either message. To determine whether \( C_0 = 0 \), we form the following dynamic programming equation,

\[ V_n(s) = r(s) + \max_{x \in \mathcal{X}} \min_{s' \in \mathcal{S}(s, x)} V_{n-1}(s'), \]  

(7)

where \( V_0(s) = 0, \forall s \in \mathcal{S}, \mathcal{S}(s, x) = \{ s' : p(y, s'|x, s) > 0 \text{ for some } y \in \mathcal{Y} \} \), and reward \( r(s) = 1 \) if state \( s \) is positive, while \( r(s) = 0 \) if state \( s \) is not positive.

**Lemma 1** (*monotonicity of \( V_n(s) \)). The total reward \( V_n(s) \) is non-negative and non-decreasing in \( n \), i.e.,

\[ 0 \leq V_n(s) \leq V_{n+1}(s), \forall n = 1, 2, 3, ... \text{ and } s \in \mathcal{S}. \]  

(8)

**Proof**: Let \( \tilde{V}_n(s) = r(s) + \max_{x \in \mathcal{X}} \min_{s' \in \mathcal{S}(s, x)} \tilde{V}_{n-1}(s') \) and \( \tilde{V}_0(s) \geq V_0(s), \forall s \in \mathcal{S} \). Then, by induction, we have \( \tilde{V}_n(s) \geq V_n(s), \forall n = 1, 2, 3, ... \) and \( s \in \mathcal{S} \). Since \( r(s) \geq 0, \forall s \in \mathcal{S} \), then \( V_1(s) \geq 0 \). Let us define \( \tilde{V}_0(s) = V_1(s) \). Since \( \tilde{V}_0(s) \geq V_0(s) \), we obtain that \( \tilde{V}_n(s) \geq V_n(s) \), which means that \( V_{n+1}(s) \geq V_n(s) \geq 0 \). 

\[ \blacksquare \]
TABLE I

| Interpretation of the DP given in (7), which corresponds to determining whether $C_0 > 0$. |
|-----------------------------------|------------------------------------------------------------------------------------------------|
| The DP given in (7)               | Interpretation of the DP                                                                           |
| state $s$ of the DP               | state $s$ of the channel                                                                           |
| reward $r(s) = 1$                 | state $s$ is positive; at least one bit can be transmitted error-free                             |
| reward $r(s) = 0$                 | state $s$ is non positive; no bits can be transmitted error-free                                   |
| Player 1 takes action $x$ in order to maximize the reward of the DP | encoder chooses input $x$ in order to maximize the number of positive states visited             |
| Player 2 chooses next state in order to minimize the reward of the DP | Nature chooses next state and output to minimize the number of messages transmitted              |
| $V_n(s)$ - total reward in $n$ rounds, starting the game from state $s$ | number of positive states visited in $n$ usages of the channel starting at state $s$,            |

This DP can be viewed as a two-person game, where $V_n(s)$ is the game result after $n$ steps starting with initial state $s$. Player 1 chooses the input letter $x$, and Player 2 chooses the next state $s'$. Both players know the current state $s$, and the reward of the game is a function only of the current state only, i.e., $r(s)$. Player 1 makes the first play, and the two players make alternative plays thereafter. The goal of Player 1 is to maximize the number of times the channel visits a positive state, and Player 2 tries to minimize it. The interpretation of the DP as a stochastic game between the user and Nature is summarized in Table I.

The following lemma states that if the total reward of the stochastic game is zero after $n$ rounds with initial state $s$, i.e., $V_n(s) = 0$, then only one message can be sent error-free through $n$ uses of the channel with initial state $s$.

**Lemma 2** $V_n(s) = 0$ implies $M(n, s) = 1$ and $V_n(s) > 0$ implies $M(n, s) > 1$.

**Proof:** First, we observe that so as to send two or more messages in $n$ uses of the channel, a positive state should be visited with probability one. Once a positive state is visited, we can use two inputs that are not adjacent to transmit without error one bit (two messages). If a positive state is not visited, then there are no two inputs that can distinguish between two messages.

The stochastic game given in (7) verifies whether a positive state is visited with probability 1. In the stochastic game, the rewards $r(s) = 1$ and $r(s) = 0$ indicate that state $s$ is positive and non-positive, respectively, Player 1 is the encoder which wants to visit a positive state and Player 2 is Nature which chooses the output and the state such that a positive state will not be visited. A total reward $V_n(s) = 0$ implies that in $n$ transmissions with initial state $s$, with positive probability, the channel undergoes only non-positive states, regardless of the inputs. Thus $V_n(s) = 0$ implies $M(n, s) = 1$.

According to Lemma 1, $V_n(s)$ is non-negative and non-decreasing in $n$ for any $s \in S$. Thus, $\min_{s \in S} V_n(s)$ is also non-decreasing in $n$, and therefore $\lim_{n \to \infty} \min_{s \in S} V_n(s)$ is well defined (it may also be infinite). If $\lim_{n \to \infty} \min_{s \in S} V_n(s) = 0$, then $\min_{s \in S} V_n(s) = 0, \forall n$ and invoking Lemma 2 $\min_{s \in S} M(n, s) = 1$, which gives $C_0 = 0$ by definition. The next lemma states that to verify whether $\lim_{n \to \infty} \min_{s \in S} V_n(s) > 0$, it is enough to calculate a finite-horizon problem.
Lemma 3

\[ \lim_{n \to \infty} \min_{s \in S} V_n(s) = 0 \iff \min_{s \in S} V_{|S|}(s) = 0 \]

Proof: The \( \implies \) direction follows from Lemma 1, which states that for any \( s \in S \), \( V_n(s) \) is a non-negative and non-decreasing function in \( n \).

Now we prove the \( \iff \) direction. Define \( S_n \), the set of initial states for which the reward is zero after \( n \) rounds of the stochastic game, i.e., \( S_n = \{ s \in S : V_n(s) = 0 \} \). Note that \( S_{n+1} \subseteq S_n \), \( S_0 = S \) and \( S_1 = \{ s \in S : r(s) = 0 \} \).

First, we claim that there exists \( n^* \), \( 0 \leq n^* \leq |S| - 1 \), for which \( S_{n^*} = S_{n^*+1} \) must hold, where \( S_{n^*} \) is non-empty. Otherwise \( S_{n+1} \) has at least one less element than \( S_n \) for \( 0 \leq n \leq |S| - 1 \), and therefore \( S_{|S|} = \emptyset \). If \( S_{|S|} \) is empty, it means that \( \min_{s \in S} V_{|S|}(s) > 0 \), which contradicts our assumption.

The equality between \( S_{n^*} \) and \( S_{n^*+1} \) means that when the channel starts at some \( s \in S_{n^*+1} \), for any input letter \( x \), there exists an action of Player 2 such that the next state \( s' \) would satisfy \( s' \in S_{n^*} \). Define this strategy of Player 2 as a function \( A_2(\cdot, \cdot) : S_{n^*} \times X \to S_{n^*} \), namely, given \( s \in S_{n^*} \), and any input \( x \in X \), the next step \( s' \) depends on \( s \) and \( x \) by the function \( A_2(s, x) \) such that \( s' = A_2(s, x) \). We claim that \( S_{n^*+k} = S_{n^*}, \forall k \geq 0 \), i.e., once the set \( S_n \) stops shrinking, it will stay the same. To prove this, let us fix an arbitrary \( s \in S_{n^*+1} \). Since \( S_1 \subseteq S_n \), \( s \in S_1 \) and \( r(s) = 0 \). We have

\[ V_{n^*+2}(s) = r(s) + \max_{x \in X} \min_{s' \in S(s, x)} V_{n^*+1}(s') = 0 \]

(9)

Therefore \( S_{n^*+2} = S_{n^*+1} \). Repeating the same argument, we have \( S_{n^*+k} = S_{n^*}, \forall k \geq 0 \), which means that \( V_n(s^*) = 0 \), \( \forall n \). This completes the proof.

The following theorem state the necessary and sufficient condition for \( C_0 = 0 \) through the stochastic game.

Theorem 1 The zero-error capacity is positive if and only if the total reward \( \min_{s \in S} V_{|S|}(s) \) is positive, i.e.,

\[ \min_{s \in S} V_{|S|}(s) = 0 \iff C_0 = 0. \] (10)

Proof:

If \( \min_{s \in S} V_{|S|}(s) = 0 \), then according to Lemma 1 \( \lim_{n \to \infty} \min_{s \in S} V_n(s) = 0 \), and following Lemma 2 it follows that \( \min_n M(n, s) = 1 \) for any \( n \); hence \( C_0 = 0 \).

If \( \min_{s \in S} V_{|S|}(s) > 0 \), then in according to Lemma 2 \( \min_{s \in S} M(|S|, s) \geq 2 \), and following from the definition of zero-error capacity \( C_0 \geq \frac{1}{|S|} \).
IV. THE DYNAMIC PROGRAMMING PROBLEM ASSOCIATED WITH THE CHANNEL

In this section, we define a dynamic programming problem associated with the channel. The solution to the problem is later used to determine the feedback capacity of the channel.

Denote \( G(y, s'|s) = \{ x : x \in \mathcal{X}, p(y, s'|x, s) > 0 \} \), i.e., \( G(y, s'|s) \) is the set of input letters at state \( s \) that can drive the channel state to \( s' \) while yielding an output letter \( y \) with positive probability. Denote \( W(\cdot, \cdot) \) as a mapping \( \mathbb{Z}^+ \times \mathcal{S} \mapsto \mathbb{R}^+ \). Set \( W(0, s) = 1, \forall s \in \mathcal{S} \) as the initial value. Denote \( P_{X|S}(\cdot | \cdot) \) as a mapping \( \mathcal{X} \times \mathcal{S} \mapsto \mathbb{R}^+ \) such that for each \( s \in \mathcal{S} \), \( P_{X|S}(x|s) \) is a probability mass function (pmf) on \( \mathcal{X} \), i.e., \( \sum_{x \in \mathcal{X}} P_{X|S}(x|s) = 1 \), and \( P_{X|S}(x|s) \geq 0, \forall x \in \mathcal{X} \). The term \( W(\cdot, \cdot) \) is the solution to the problem defined iteratively by:

\[
W(n, s) = \max_{P_{X|S}(\cdot | s)} \min_{s' \in \mathcal{S}} \left\{ W(n-1, s') \left[ \max_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} P_{X|S}(x|s) \right]^{-1} \right\}
\]

\( \forall s \in \mathcal{S} \), and for \( n = 1, 2, 3, \ldots \)

We adopt the convention that \( \frac{1}{0} = \infty \), and, if \( G(y, s'|s) = \emptyset, \sum_{x \in \mathcal{X}} P_{X|S}(x|s) = 0 \). One property that can be verified from the definition and the initial value is that \( \forall n \geq 0, \forall s \in \mathcal{S}, W(n, s) \geq 1 \).

The main result of this paper is the following theorem:

**Theorem 2** If \( \min_{s \in \mathcal{S}} V_{|S|}(s) > 0 \),

\[
C_0 = \liminf_{n \to \infty} \frac{1}{n} \min_{s \in \mathcal{S}} \log_2 W(n, s);
\]

Otherwise \( C_0 = 0 \).

Before proving the theorem, let us verify that the zero-error capacity of a DMC [1, Theorem 7] is a special case of Theorem 2. Since a DMC is an FSC with only one state, \( V_{|S|}(s) = 0 \) means that the state is non-positive, i.e., “all pairs of input letters are adjacent”, as stated in [1, Theorem 7]. If \( V_{|S|}(s) > 0 \), for a DMC, define \( M(n) = M(n, s) \) and \( G(y) = G(y, s'|s) \).

\[
M(n, s) = \max_{P_{X|S}(\cdot | s)} \left\{ M(n-1) \left[ \max_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} P_{X|S}(x|s) \right]^{-1} \right\}
\]

\[
= M(n-1) \max_{P_X} \left[ \max_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} P_X(x) \right]^{-1}
\]

(13)

and

\[
C_0 = \liminf_{n \to \infty} \frac{1}{n} \log_2 M(n)
\]

\[
= \log_2 \left[ \max_{y \in \mathcal{Y}} \sum_{x \in \mathcal{X}} P_X(x) \right]^{-1}
\]

(14)

which is exactly the result for DMC in [1].
The converse and the direct parts of Theorem 2 are proved in Section V and Section VI, respectively.

V. CONVERSE

**Theorem 3 (Converse.)** \( M(n, s) \leq W(n, s), \forall n = 0, 1, 2, \ldots \) and \( \forall s \in S \).

**Proof:** We prove the theorem by induction. First, the inequality holds when \( n = 0 \).

Now, suppose \( M(k, s) \leq W(k, s) \) is true for \( k = 0, \ldots, n - 1 \) and \( \forall s \in S \). Fix an arbitrary initial state \( s_0 \). It is sufficient to show that \( M(n, s_0) \leq W(n, s_0) \) to prove the converse.

For a fixed zero-error code that has \( M(n, s_0) \) messages, we define

\[
M(n, s_0) = \text{number of messages with first transmitted letter } x \text{ when initial state is } s_0,
\]

\( f(x|s_0) = \frac{M(n, s_0)}{M(n, s_0)} \). \hspace{1cm} (15)

Note that \( f(\cdot|s_0) \) is a valid pmf.

After the first transmission, suppose the output is some \( y \in \mathcal{Y} \) and the channel goes to state \( s_1 \). We have \( \sum_{x \in G(y, s_1|s_0)} M(n, s_0) \) messages, each of which with positive probability gives output \( y \) and changes the state to \( s_1 \). To guarantee that the decoder can distinguish between these messages in the following \( n - 1 \) transmission, we must have \( \sum_{x \in G(y, s_1|s_0)} M(n, s_0) \) which yields

\[
M(n, s_0) \sum_{x \in G(y, s_1|s_0)} f(x|s_0) \leq M(n - 1, s_1).
\] \hspace{1cm} (16)

Since the above inequality must hold, \( \forall y \in \mathcal{Y} \), and \( \forall s_1 \in S \)

\[
M(n, s_0) \leq \min_{s_1 \in S} M(n - 1, s_1) \left[ \max_{y \in \mathcal{Y}} \sum_{x \in G(y, s_1|s_0)} f(x|s_0) \right]^{-1} \hspace{1cm} (17)
\]

Since we assumed \( M(n - 1, s) \leq W(n - 1, s) \) for all \( s \in S \),

\[
M(n, s_0) \leq \min_{s_1 \in S} W(n - 1, s_1) \left[ \max_{y \in \mathcal{Y}} \sum_{x \in G(y, s_1|s_0)} f(x|s_0) \right]^{-1} \hspace{1cm} (18)
\]

Using the iterative formula of \( W(n, s_0) \) given in (11) and the fact that \( f(\cdot|s_0) \) is a valid pmf, we obtain

\[
M(n, s_0) \leq W(n, s_0). \hspace{1cm} (19)
\]

Finally, since \( s_0 \) is arbitrarily fixed, we have \( M(n, s) \leq W(n, s), \forall s \in S \). By induction, the theorem is proved. \( \blacksquare \)

From the converse, Theorem 3 and the zero-error capacity definition 3, we have the following upper bound

\[
C_0 = \lim_{n \to \infty} \frac{\min_{s \in S} \log_2 M(n, s)}{n} \leq \lim_{n \to \infty} \frac{\min_{s \in S} \log_2 W(n, s)}{n}.
\] \hspace{1cm} (20)
VI. Direct Theorem

**Theorem 4** Assume \( \min_{s \in S} V_{|S|}(s) > 0 \), then for any initial state \( s \in S \) there exists an \( n_0 > 0 \) such that for \( n > n_0 \), \( |W(n, s)| \) messages can be transmitted with no more than \( n + |S| \log_2 L \), where \( L \) is a positive integer that does not depend on \( n \) and \( s \).

*Proof:* The direct part is proved using deterministic codes [1] rather than random codes. Let the solution and the maximizer in the \( k \)th iteration \( (k = 1, 2, \ldots, n) \) of (11) be \( W(k, \cdot) \) and \( P^{(k)}_{X|S} (\cdot | \cdot) \), respectively.

Suppose that at the first transmission the channel state is \( s_1 \) and the total number of messages transmitted through the channel is \( |W(n, s_1)| \). We divide the message set into \( |X| \) groups and transmit \( x = i \) for the messages in the \( i \)th group for the first transmission. Let \( m_i \) denote the number of messages in the \( i \)th group. By similar arguments to those in [1, p. 18], we can control the size of each group such that:

\[
\begin{align*}
\text{if } P^{(n)}_{X|S} (i | s_1) > 0, & \quad \frac{m_i}{|W(n, s_1)|} - P^{(n)}_{X|S} (i | s_1) \leq \frac{1}{|W(n, s_1)|}; \\
\text{if } P^{(n)}_{X|S} (i | s_1) = 0, & \quad m_i = 0.
\end{align*}
\]  

Both the transmitter and the receiver know how the messages are divided before the transmission. An arbitrary message \( m \in \{1, \ldots, |W(n, s_1)|\} \) is selected, and letter \( i \) is sent if \( m \) belongs to the \( i \)th group. The number of messages about which the receiver is uncertain before the first transmission is \( Z_1 = |M(n, s_1)| \).

After the first transmission, we obtain an output \( y_1 \), and the channel state changes to \( s_2 \). Denote \( Z_2 \) as the number of messages that are compatible with \( (y_1, s_2) \), i.e., when transmitting those messages, \( (y_1, s_2) \) is obtained with positive probability. \( Z_2 \) can be upper bounded in the following way:

\[
\begin{align*}
Z_2 &= \sum_{x \in G(y_1, s_2 | s_1)} m_x \\
&= \left| W(n, s_1) \right| \sum_{x \in G(y_1, s_2 | s_1)} \frac{m_x}{|W(n, s_1)|} \\
&\leq \left| W(n, s_1) \right| \sum_{x \in G(y_1, s_2 | s_1)} \left( P^{(n)}_{X|S} (x | s_1) + \frac{1}{|W(n, s_1)|} \right) \\
&\leq \left\{ \left| W(n, s_1) \right| \max_{y \in Y} \sum_{x \in G(y, s_2 | s_1)} \left( P^{(n)}_{X|S} (x | s_1) \right) \right\} + |X|.
\end{align*}
\]  

For convenience, let us define

\[
J^{(k)} (s, s') = \max_{y \in Y} \sum_{x \in G(y, s' | s)} P^{(k)}_{X|S} (x | s).
\]

Eq. (11) and (22) can be written, respectively, in terms of \( J^{(k)} (s, s') \) as:

\[
W(k, s) \leq W(k - 1, s') \left[ J^{(k)} (s, s') \right]^{-1}, \quad \forall k \in \mathbb{Z}^+, s \in S, s' \in S.
\]  

(24)
\[ Z_2 \leq [W(n, s_1)J^{(n)}(s_1, s_2) + |X|] \]
\[ \leq W(n - 1, s_2) + |X|, \tag{25} \]
where the last inequality is due to (24).

Since both transmitter and receiver know \( s_1 \) and \( s_2 \) and the transmitter knows the output \( y_1 \) through feedback, both of them know which messages are compatible with \((y_1, s_2)\). In the second transmission, the transmitter can further divide the remaining \( Z_2 \) messages into groups according to \( P^{(n-1)}_X(\cdot |s_2) \), similar to eq. (21). The way the messages are divided is known to the receiver. Suppose the output letter is \( y_2 \) and the state goes to \( s_3 \). Following the argument in the previous iteration, we have

\[ Z_3 \leq Z_2 J^{(n-1)}(s_2, s_3) + |X| \]
\[ \leq W(n - 1, s_2)J^{(n-1)}(s_2, s_3) + |X| \left( 1 + J^{(n-1)}(s_2, s_3) \right) \tag{26} \]
\[ \leq W(n - 2, s_3) + |X| \left( 1 + J^{(n-1)}(s_2, s_3) \right), \]
where steps (a) and (b) follow from (25) and (24), respectively.

As the transmission proceeds, the channel state evolves as \( s_1, \ldots, s_n, s_{n+1} \), and the output sequence is \( y_1, \ldots, y_n \). The transmitter divides the remaining uncertain messages according to \( P^{(k)}_X(\cdot |s_k) \) for each transmission. After the \( n \)th transmission, the number of messages remaining can be upper bounded as:

\[ Z_{n+1} \leq Z_n J^{(1)}(s_n, s_{n+1}) + |X| \]
\[ \leq 1 + |X| \left( 1 + J^{(1)}(s_n, s_{n+1}) + J^{(1)}(s_n, s_{n+1})J^{(2)}(s_{n-1}, s_n) + \cdots + \prod_{i=1}^{n-1} J^{(i)}(s_{n+1-i}, s_{n+2-i}) \right) \tag{27} \]

Using Ineq. (24) iteratively, we obtain

\[ W(k; s_{n+1-k}) \leq \left[ \prod_{i=1}^{k} J^{(i)}(s_{n+1-i}, s_{n+2-i}) \right]^{-1} \tag{28} \]

hence we can further upper bound \( Z_{n+1} \) as

\[ Z_{n+1} \leq 1 + |X| \left( 1 + \frac{1}{W(1, s_n)} + \frac{1}{W(2, s_{n-1})} + \cdots + \frac{1}{W(n - 1, s_2)} \right). \tag{29} \]

Recall the assumption of the theorem \( \min_{s \in S} V(|S|, s) > 0 \), which implies, via Theorem 1 that \( C_0 > 0 \), and follows from Theorem 3 we obtain that

\[ \lim \inf_{n \to \infty} \min_{s \in S} \frac{1}{n} \log M(n, s) > 0. \tag{30} \]

Hence, there exists \( \epsilon > 0 \) and an integer \( n_0 \) such that \( \forall s \in S, \forall n > n_0, W(n, s) \geq M(n, s) \geq 2^{en} \) (the first inequality is due to the converse proved in the previous section, and second inequality is due to (30)). Recall that \( M(n, s) \geq 1; \)
we can thus further upper bound $Z_{n+1}$ as

$$
Z_{n+1} \leq 1 + |\mathcal{X}| \left( 1 + \sum_{k=1}^{n_0} \frac{1}{W(k, s_{n+1-k})} + \sum_{k=n_0+1}^{\infty} 2^{-\epsilon n} \right)
$$

$$
\leq 1 + |\mathcal{X}| \left( n_0 + 1 + \sum_{k=n_0+1}^{\infty} 2^{-\epsilon n} \right)
$$

$$
= 1 + |\mathcal{X}| \left( n_0 + 1 + \frac{2^{-\epsilon(n_0+1)}}{1 - 2^{-\epsilon}} \right)
$$

$$
\triangleq L.
$$

Note that $L$ is finite and is independent of $n$ and $s_1$. This means that after $n$ transmissions, the number of messages about which the receiver is uncertain is not more than $L$.

The assumption that $\min_{s \in S} V_{|S|}(s) > 0$ implies that we can drive the channel to a positive state with probability 1 in less than $|S|$ transmissions. In a positive state, we can transmit 1 bit of information with zero-error; hence we can now conclude that there exists a zero-error code such that $[W(n, s)]$ messages can be transmitted with no more than $n + |S| \lceil \log_2 L \rceil$ transmissions.

Based on the direct theorem, it is straightforward to derive a lower bound on the zero-error capacity:

$$
C_0 \geq \lim_{n \to \infty} \inf_{s \in S} \min_{n + |S| \lceil \log_2 L \rceil} \frac{\log_2 W(n, s)}{n}
$$

$$
= \lim_{n \to \infty} \frac{1}{n} \min_{s \in S} \log_2 W(n, s),
$$

given the condition $\min_{s \in S} V_{|S|}(s) > 0$. Combining ineq. (20) and ineq. (32), we have proved eq. (12) thus Theorem 2.

VII. SOLVING THE DYNAMIC PROGRAMMING PROBLEM

Throughout this section, we assume that $\min_{s \in S} V_{|S|}(s) > 0$, i.e., we focus on channels with positive zero-error capacity. Let us first introduce a few definitions so that we can use the standard language of dynamic programming to rewrite Eq. (11) in the form of Eq. (6). Basically, we take $\log_2$ on both sides of Eq. (11). Define the value function as $J_n(s) = \log_2 W(n, s)$, the action as $a = P_{X|S}(\cdot|s)$, and the reward as

$$
r(s', a, s) = \log_2 \left( \max_{y \in Y} \sum_{x \in G(y, s')|s} P_{X|S}(x|s) \right)^{-1}.
$$

And the DP equation in (11) becomes simply

$$
J_n(s) = \max_{a \in A} \min_{s' \in S} \{r(s', a, s) + J_{n-1}(s')\}.
$$

where $A$ is the action space, $A = \{ f(x) : \sum_x f(x) = 1, f(x) \geq 0 \}$. 
Theorem 2 states that
\[
C_0 = \lim_{n \to \infty} \min_{s \in S} \frac{J_n(s)}{n}.
\] (35)

Define an operator \( T \) as follows,
\[
(T \circ J)(s) = \max_{a \in A(s)} \min_{s'} \{ r(s', a, s) + J(s') \}.
\] (36)

The DP equation can be rewritten in a compact form as follows,
\[
J_n(s) = (T \circ J_{n-1})(s),
\] (37)
with initial value \( J_0(s) = 0 \). We also denote \( T^n \) as applying operator \( T \) \( n \) times.

**Lemma 4** Let \( W \) and \( V \) denote two functions \( S \to \mathbb{R}^+ \). The following properties of \( T \) hold:

(a) If \( W(s) \geq V(s), \forall s \in S \), then \( T \circ W(s) \geq T \circ V(s) \forall s \in S \).
(b) If \( W(s) = V(s) + d \forall s \in S \), where \( d \) is a constant, then \( T \circ W(s) = T \circ V(s) + d, \forall s \in S \).

**Proof:** Both parts of the lemma follow directly from the definition of \( T \). \( \blacksquare \)

**Lemma 5** The following properties of \( J_n \) hold:

(a) The sequence \( \{ \min_s J_n(s) \} \) is sup-additive, i.e., \( \min_s J_{n+m}(s) \geq \min_s J_n(s) + \min_s J_m(s) \)
(b) The sequence \( \{ \max_s J_n(s) \} \) is sub-additive, i.e., \( \max_s J_{n+m}(s) \leq \max_s J_n(s) + \max_s J_m(s) \).

**Proof:** We prove the first property here. The proof of the second one is similar.
\[
\min_s J_{n+m}(s) = \min_s (T^n \circ J_m)(s)
\geq \min_s \left( T^n \circ \min_{s'} J_m(s') \right) (s)
= \min_s \left( T^n \circ [J_0 + \min_{s'} J_m(s')] \right) (s)
\leq \min_s (T^n \circ J_0)(s) + \min_{s'} J_m(s'),
\] (38)
where the steps (a) and (b) follow from parts (a) and (b) of Lemma 4 respectively. \( \blacksquare \)

**Theorem 5** The \( \lim \inf \) in Theorem 2 can be replaced by \( \lim \), i.e.,
\[
C_0 = \lim_{n \to \infty} \min_{s} \frac{J_n(s)}{n},
\] (39)
and for all \( n \in \mathbb{Z}^+ \) the following bounds hold
\[
\min_{s} \frac{J_n(s)}{n} \leq C_0 \leq \max_{s} \frac{J_n(s)}{n}.
\] (40)
Proof: Following Lemma 5 and Fekete’s lemma [8, Ch. 2.6], we obtain the following two limits:
\[
\lim_{n \to \infty} \min_s \frac{J_n(s)}{n} = \sup_n \min_s \frac{J_n(s)}{n},
\]
\[
\lim_{n \to \infty} \max_s \frac{J_n(s)}{n} = \inf_n \max_s \frac{J_n(s)}{n}.
\]
Finally, from Theorem 2 we obtain:
\[
\max_s \frac{J_k(s)}{n} \geq \lim_{n \to \infty} \max_s \frac{J_n(s)}{n} \geq C_0 = \lim_{n \to \infty} \min_s \frac{J_n(s)}{n} \geq \min_s \frac{J_k(s)}{k},
\]
for all \( k \in \mathbb{Z}^+ \).

Eq. (40) provides a numerical way to approximate \( C_0 \). We now alter to the case that an analytical solution in the limit can be obtained via Bellman equations.

**Theorem 6 (Bellman equation) If there exists a positive bounded function \( g : \mathcal{S} \to \mathbb{R}^+ \) and a constant \( \rho \) that satisfy
\[
g(s) + \rho = (T \circ g)(s)
\]
then \( \lim_{n \to \infty} \frac{1}{n} J_n(s) = \rho \).

**Proof:** Assume that there exists a positive bounded function \( g : \mathcal{S} \to \mathbb{R}^+ \) and a constant \( \rho \) that satisfy \( g(s) + \rho = (T \circ g)(s) \). Define \( g_0(s) = g(s) \), \( g_n(s) = T^n g_0(s) \). Since \( J_0(s) = 0 \leq g_0(s) \), then according to part (a) of Lemma 4, \( J_n(s) \leq g_n(s) \). Let \( d = \max_s g(s) \). Then \( J_0 + d \geq g_0 \). Hence, according to part (a) of Lemma 4, \( g_n(s) \leq J_n(s) + d \). Therefore we have,
\[
g_n(s) - d \leq J_n(s) \leq g_n(s).
\]
Finally, \( g(s) + \rho = (T \circ g)(s) \) implies that \( \lim_{n \to \infty} \frac{g_n(s)}{n} = \rho \); hence \( \lim_{n \to \infty} \frac{J_n(s)}{n} = \rho \).

**Remark:** \( \rho \) does not depend on the initial state, which hints that for some decomposable Markov chains, it is impossible to find a \( g : \mathcal{S} \to \mathbb{R}^+ \) and a constant \( \rho \) to satisfy the Bellman equation.

**VIII. Examples**

Here we provide three examples and solve them analytically. For the first two examples, we also find the regular feedback capacity using [3].

**Example 1** We consider the very simple example illustrated in Fig. [2]. The channel has two states. In state 0, the channel is a binary symmetric channel (BSC) with positive cross probability. In state 1, the channel is a BSC with 0 cross probability. Roughly speaking, in state 0, the channel is noisy, and, in state 1, the channel is noiseless. Suppose the channel state evolves as a Markov process and is independent of the input and output. If the current state is 0, the next channel state is 1 with certainty. If the state is 1, the channel goes to state 0 with probability \( p > 0 \) or stays at state 0 with probability \( 1 - p \). Thus, the channel stays in the noisy state a geometric length of time, and returns to the perfect state immediately.
Finding $C_0$ by calculating $W(n, s)$: for this channel $G(y, 0|0) = 0$, $G(y, 1|0) = \{0, 1\}$, $G(y, 0|1) = G(y, 1|1) = \{y\}$. Using eq. (11), we have the solution to the DP problem of the 1st iteration as

$$W(1, 0) = \max_{P_X|S(\cdot|0)} \min \{1, 1\} = 1$$

$$W(1, 1) = \max_{P_X|S(\cdot|1)} \left[ \max \{P_X|S(0|1), P_X|S(1|1)\} \right]^{-1} = 2.$$  \hfill (45)

For the 2nd iteration, we have

$$W(2, 0) = \max_{P_X|S(\cdot|0)} W(1, 1) \min \{1, 1\} = 2$$

$$W(2, 1) = \max_{P_X|S(\cdot|1)} W(1, 0) \left[ \max \{P_X|S(0|1), P_X|S(1|1)\} \right]^{-1} = 2.$$  \hfill (46)

By induction and some simple algebra, we obtain the solution to the DP problem at the $n$th iteration:

$$W(n, 0) = 2^{[n/2]}, \text{ and } W(n, 1) = 2^{[n/2]}.$$  \hfill (47)

Thus

$$C_0 = 1/2.$$  \hfill (48)

Alternatively, we can solve the example by finding a solution to Bellman equation (45).

Finding $C_0$ via Bellman equation: the Bellman equation for the channel is simply the following,

$$g(0) = g(1) - \rho,$$

$$g(1) = 1 + g(0) - \rho.$$  \hfill (49)

Using simple algebra we obtain $\rho = \frac{1}{2}$, $g(0) = v$, $g(1) = v + \frac{1}{2}$. We note that we can achieve the zero-error capacity with feedback and state information simply by transmitting 1 bit of information whenever the channel state is 1.

Finding the regular feedback capacity $C_f$: To calculate the regular capacity we use the result of Chen and Berger in [3, Theorem 6]. The theorem states that if the channel is strongly irreducible and strongly aperiodic, then the
capacity is
\[ C = \max_{P_{X|S}} \sum_{k=0}^{|S|-1} \pi_k I(X;Y|S = k), \]  
(50)

where \( \pi_k \) is the equilibrium distribution of state \( k \) induced by the input distribution \( P_{X|S} \).

The channel is strongly irreducible and strongly aperiodic if the matrix \( T \) that is defined as
\[ T(k, l) = \min_x \{ \Pr(S_i = l|X_k = x, S_{i-1} = k) \} \]
(51)
is irreducible and aperiodic for any \( x \in \mathcal{X} \). Since the transition probability of the state does not depend on the input, and since the state transition matrix is irreducible and aperiodic for any \( p < 1 \), the capacity is given by (50); hence
\[ C(p) = \max_{P_{X|S}} \pi_0 I(X;Y|S = 0) + \pi_1 I(X;Y|S = 1) \\
= \pi_1 \\
= \frac{1}{2 - p} \]  
(52)

![Fig. 3. Feedback capacity and zero-error feedback capacity of the channel in Example 1 for different values of \( p = \Pr(S = 1|S = 1) \).](image)

**Example 2** Let us consider another channel with two states as illustrated in Fig. 4. In state 0, the channel is a Z-channel. In state 1, the channel is a BSC with 0 cross probability. The next channel state is determined by the output. If the output is 0, the channel goes to state 0; if the output is 1, the channel goes to state 1; hence the regular feedback of the output includes the state information.

It is tempting to make full use of state 1, i.e., to transmit 1 bit of information, but as a consequence the channel goes to the undesirable state 0 half the time, and the rate would be only \( \frac{1}{2} \).

**Finding** \( C_0 \) **by calculating** \( W(n, s) \): For this channel, \( G(0, 0|0) = \{0\} \), \( G(1, 1|0) = \{0, 1\} \), \( G(0, 0|1) = \{0\} \),
The maximum is achieved by setting $P_{X|S}(0|0) = 0$. For initial state 1, we have

$$W(n, 1) = \max_{P_{X|S}(|1|)} \min \left\{ \frac{W(n - 1, 0)}{P_{X|S}(0|1)}, \frac{W(n - 1, 1)}{P_{X|S}(1|1)} \right\}$$

By setting $P(0|1) = \frac{W(n-2,1)}{W(n-2,1)+W(n-1,1)}$, the maximum is achieved. Recall $W(0, 1) = 1$. Notice that $W(1, 1) = 2$, which can be computed directly. Thus, both $W(n, 1)$ and $W(n, 0)$ are a Fibonacci sequences (with proper shifts).

Therefore, $\lim_{n \to \infty} \frac{\log W(n, 1)}{n} = \lim_{n \to \infty} \frac{\log W(n, 0)}{n} = \log_2 \frac{1 + \sqrt{5}}{2}$. From Theorem 2 we have

$$C_0 = \log_2 \frac{1 + \sqrt{5}}{2} \approx 0.6942,$$

which is the log of the golden ratio. Here, we list the first few values of $W(n, s)$ in Table II.

### Table II

| $s$ | $n$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|-----|-----|-----|-----|-----|-----|-----|
| 0   |     | 1   | 2   | 3   | 5   |     |
| 1   |     | 2   | 3   | 5   | 8   | 13  |

**Finding $C_0$ via a Bellman equation:** Since the channel input is binary, the actions are equivalent to two numbers:
\( p_0 = P_{X|S}(0|0), p_1 = P_{X|S}(0|1) \). Bellman’s equation become

\[
J(0) + \rho = \max_{0 \leq p_0 \leq 1} \min \left\{ \log \frac{2}{p_0} + J(0), J(1) \right\}
\]

\[
J(1) + \rho = \max_{0 \leq p_1 \leq 1} \min \left\{ \log \frac{1}{p_1} + J(0), \log \frac{1}{1 - p_1} + J(1) \right\}
\]

which implies that \( p_0 = 0 \) and

\[
J(0) = J(1) - \rho, \quad J(1) = J(0) + \log \frac{1}{p_1} - \rho,
\]

the solution of which is \( \rho = \log_2 \frac{\sqrt{5} + 1}{2}, p_1 = \frac{3 - \sqrt{5}}{2} \).

It is of interest to observe that starting at state 1, any binary sequence with length \( n \) and no consecutive 0’s can be transmitted with zero-error in \( n \) transmissions. The number of such sequences as a function of \( n \) is also a Fibonacci sequence. Since we can always transmit a 1 to drive the channel from state 0 to state 1, this is actually one way to achieve the zero-error capacity.

**Finding the regular feedback capacity \( C^f \):** This channel is not strongly irreducible, since the matrix transition \( P_{S_i|S_{i-1}, X=0} \) is not irreducible; hence, the stationarity of the optimal policy used by Chen and Berger [3] requires additional justification. By invoking theory on the infinite-horizon average-reward dynamic programming we show that a stationary policy achieves the optimum of the DP and hence Eq. (50) holds.

The feedback-capacity of the channel in Example 2 can be formulated according to [3] and [13] as:

\[
C = \lim_{N \to \infty} \frac{1}{N} \max_{\{X_n|S_n\}_{n=1}^N} \sum_{n=1}^N I(X_n; Y_n|S_n),
\]

and this is equivalent to an infinite-horizon average-reward DP with finite state space and compact actions where:

- the state of the DP is the state of the channels i.e., \( S_n \),
- the actions of the DP are the input distributions \( p_0 \in [0, 1] \) and \( p_1 \in [0, 1] \), where \( p_0 = P_{X|S}(0|0), p_1 = P_{X|S}(0|1) \).
- the reward at time \( n \) given that the state of the DP is 0 or 1 is \( I(X_n; Y_n|S_n = 0) = H_b(p_0 p) - p_0 H_b(p) \) or \( I(X_n; Y_n|S_n = 1) = H_b(p_1) \), respectively,
- the transition probability given the actions \( p_1 \) and \( p_2 \) is \( P_{S_n|S_{n-1}}(0|1) = p_1 \) and \( P_{S_n|S_{n-1}}(0|0) = p_0 p \).

Next, we claim that it is enough to consider the action \( p_1 \in [\epsilon, 1] \) for some \( \epsilon > 0 \). First we note that for \( \epsilon \leq \frac{1}{6} \)

\[
H(2\epsilon) > H(\epsilon) + \epsilon,
\]

since \( \frac{dH(x)}{dx} > 1 \) for \( x < \frac{1}{3} \).

Next we show that it is never optimal to have an action \( p_1 \leq \frac{1}{6} \). Let \( J_n(0) \) and \( J_n(1) \) be the maximum rewards to go in \( n \) steps starting at state 0 and 1, respectively, and let assume that the optimal action in state 1 is \( p_1^* < \frac{1}{6} \),
then
\[ J_n(0) \overset{(a)}{=} H(p_1^*) + (1 - p_1^*) J_{n-1}(0) + p_1^* J_{n-1}(1) \]
\[ \overset{(b)}{=} H(p_1) + (1 - 2p_1^*) J_{n-1}(0) + 2p_1^* J_{n-1}(1) + p_1^* (J_{n-1}(0) - J_{n-1}(1)) \]
\[ \overset{(c)}{\leq} H(p_1^*) + (1 - 2p_1^*) J_{n-1}(0) + 2p_1^* J_{n-1}(1) + p_1^* \]
\[ \overset{(d)}{<} H(2p_1) + (1 - 2p_1^*) J_{n-1}(0) + 2p_1^* J_{n-1}(1) + p_1^* (J_{n-1}(0) - J_{n-1}(1)) \]
\[ (60) \]

where step (a) follows from the dynamic programming formulation; step (b) follows from the fact that we added and subtracted \( p_1^* (J_{n-1}(0) - J_{n-1}(1)) \); and step (c) follows from the fact that \( J_{n-1}(0) - J_{n-1}(1) \leq 1 \); this is because we can choose \( p_0 = 0 \), which means that in one epoch time we can cause the state to change from 0 to 1 with probability 1, and the reward in one epoch time is always less than 1. Finally, step (d) follows from (59). Since step (d) corresponds to the action \( 2p_1^* \), it implies that an optimal policy would never include the action \( p_1^* < \frac{1}{3} \).

Now we invoke [9, Theorem 4.5] that states that if the reward is a continuous function of the actions, and for any action the corresponding state chain is irreducible (unchain), then the optimal policy is stationary. Since the reward function is continuous in \( p_0, p_1 \) and since for any \( p_0 \in [0, 1], p_1 \in [\frac{1}{3}, 1] \) the state process is a irreducible, we conclude that the optimal policy \( p_1^*, p_2^* \) is stationary (time-invariant), and therefore the capacity is given by (50).

![Capacity and zero-error capacity of the channel in Example 2 for different values of p = Pr{Y = 0|X = 0, S = 0}.

Fig. 5. Capacity and zero-error capacity of the channel in Example 2 for different values of \( p = \text{Pr}\{Y = 0|X = 0, S = 0\} \).

Now, using (50), we obtain that the regular feedback capacity as a function of \( p \) is
\[ C_f(p) = \max_{p_0, p_1} (\pi_0 (H_b(p_0 p) - p_0 H_b(p)) + \pi_1 H_b(p_1)), \]
\[ (61) \]
where \((\pi_0, \pi_1)\) are the equilibrium distributions given by \( \pi_0 = \frac{p_1}{1 + p_0^* - p_0 p} \) and \( \pi_1 = 1 - \pi_0 \). Fig. 5 shows a numerical evaluation (61) as a function of \( p \).

**Example 3** We consider here an example with three states with a trinary input and trinary output. The topology of the channel is depicted in Fig. 6. The channel conditional distribution \( P(s', y|x, s) \) has the form of \( P(s', y|x, s) = P(s'|x, s)P(y|x, s) \), where state \( s = 0 \) is a perfect state, \( s = 1 \) is a good state and \( s = 0 \) is a bad state; the states 1, 2, 3 can transmit \( \log 3, 1 \) and 0 bits with zero error probability.
We first evaluate the zero-error capacity numerically using the dynamic programming value iteration, i.e., Eq. (40), and then, using the numerical evaluation, we conjecture an analytical solution, which we verify via the Bellman equation.

Fig. 6. Channel topology of Example 3.

Evaluating $C_0$ using a value iteration algorithm: We calculated 50 iterations of the DP value iteration formula given in (34). The action space of player 1 is the stochastic matrix $P_{X|S}$, and we quantize each element of the stochastic matrix with a $10^{-4}$ resolution. Fig. 7 depicts the value of $\max_s J_n(s)$ and $\max_s J_n(s)$ which according to Theorem 5 are upper and lower bounds, respectively, on the zero-error capacity.

After 50 iterations, we obtain that the first player’s action $P_{X|S}$ is given by

$$P_{X|S} = \begin{bmatrix} 0.4656 & 0.3177 & 0.2167 \\ 0 & 0.3177 & 0.6823 \\ 0 & 0 & 1 \end{bmatrix},$$

and the the reward $J_{50}(s) - J_{49}(s)$, which is an estimate of the zero-error capacity, is $1.10283$ for all $s \in \{0, 1, 2\}$.

Analytical solution via Bellman equation: We conjecture that the optimal policy of Player 1 is a stochastic matrix of the form given in (62), i.e., $P_{X|S}(1|1) = P_{X|S}(1|0)$, and $P_{X|S}(0|0) = P_{X|S}(0|2) = P_{X|S}(1|2) = 0$. Based on
this assumptions and the notation \( a_0 \triangleq P_{X|S}(0|0) \) and \( a_1 \triangleq P_{X|S}(1|0) \), the Bellman equation becomes:

\[
\begin{align*}
\rho + J(0) &= \max_{a_0, a_1} \{ -\log a_0 + J(0), -\log a_1 + J(1), -\log(1 - a_1 - a_0) + J(2) \} \\
\rho + J(1) &= \max_{a_1} \{ -\log a_1 + J(2), -\log(1 - a_1) + J(0) \} \\
\rho + J(2) &= J(0).
\end{align*}
\] (63)

Using simple algebraic manipulation, we obtain that

\[
\begin{align*}
a_1 &= (1 - a_1)^3 \\
\rho &= \log \frac{1 - a_1}{a_1},
\end{align*}
\] (64)

which implies that \( a_1 = 1 + u - \frac{1}{3a_1} \), where \( u = \sqrt{-\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{1}{27}}} \), hence \( a_1 = 0.31767... \) and

\[
C_0 = -\log(1 - a_1) = 1.102926...
\] (65)

IX. Conclusions

We introduced a DP formulation for computing the zero-error feedback capacity for FSCs with state information at the decoder and encoder. The DP formulation, which can also be viewed as a stochastic game between two players, is a powerful tool that allows us to evaluate numerically the zero-error feedback capacity and in many cases as shown in the paper, to find an analytical solution via a fixed-point equation.

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