Quantization of Edge Currents Along Magnetic Interfaces: A $K$-Theory Approach

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Abstract
The purpose of this paper is to investigate the propagation of topological currents along magnetic interfaces (also known as magnetic walls) of a two-dimensional material. We consider tight-binding magnetic models associated to generic magnetic multi-interfaces and describe the $K$-theoretical setting in which a bulk-interface duality can be derived. Then, the (trivial) case of a localized magnetic field and the (non trivial) case of the Iwatsuka magnetic field are considered in full detail. This is a pedagogical preparatory work that aims to anticipate the study of more complicated multi-interface magnetic systems.

Keywords Magnetic interfaces · Iwatsuka Hamiltonian · Edge currents · K-theory

Mathematics Subject Classification (2010) Primary 81R60 · Secondary 58B34 · 81R15 · 46L80

1 Introduction

The purpose of this paper is to investigate the propagation of topological currents along magnetic interfaces (also known as magnetic walls) of a two-dimensional material. A magnetic interface is a thin region of space (ideally a one-dimensional curve) which separates the material in two parts subjected to perpendicular uniform magnetic fields of distinct intensity. The typical example of magnetic interface is provided by the Iwatsuka magnetic field [30], which is a magnetic field oriented orthogonally to the plane with a (monotone) strength function $B_1 : \mathbb{R}^2 \to \mathbb{R}$ that does not depend on the $y$-direction and such that $\lim_{x \to \pm \infty} B_1(x) = b_\pm$ where $b_-$ and $b_+$ are distinct values. The response of the system to an Iwatsuka magnetic field is the production of a current flowing in the $y$ direction along the interface (fixed around $x = 0$). Such a current is carried by extended states localized near the interface which are
the quantum analog of the classical “snake orbits” [13, Fig. 6.1]. Such “snake orbit states” have received considerable attention in physics literature [23, 42, 45, 46, 54, 55].

A very interesting and important property of the current carried by the “snake orbit states” along the interface is its topological quantization. For the Iwatsuka magnetic field, this fact has been rigorously proved in [16] for continuous models and in [36] for discrete (or tight-binding) models. In both cases the crucial result can be stated by considering the “full” Hamiltonian \( \hat{h} \), which contains the full information about the Iwatsuka magnetic field, and two “asymptotic” Hamiltonians \( \{ \hat{h}_-, \hat{h}_+ \} \) depending only on the constant magnetic fields \( b_- \) and \( b_+ \), respectively. Accordingly, the pair \( \{ \hat{h}_-, \hat{h}_+ \} \) carries information only about the asymptotic behavior of the Iwatsuka magnetic field. Let \( \Delta \) be an energy domain that sits inside a non-trivial gap of \( \text{Spec}(\hat{h}_-) \cup \text{Spec}(\hat{h}_+) \) (asymptotic open gap hypothesis). Let \( J_\Delta(\Delta) \) be the interface current carried by the extended “snake orbit states” of \( \hat{h} \) of energy \( \Delta \) and \( \sigma_\Delta(\Delta) := eJ_\Delta(\Delta) \) the associated interface conductance (\( e > 0 \) is the magnitude of the electron charge). Then, it holds true that

\[
\sigma_\Delta(\Delta) = \frac{e^2}{\hbar} (N_+ - N_-), \quad N_\pm \in \mathbb{Z},
\]

where \( \hbar \) is the Planck’s constant. The integers \( N_\pm \) are related to the topology of the asymptotic system. Let \( \mu \in \Delta \) be a given Fermi energy and \( p_{\mu, \pm} := \chi(-\infty, \mu](\hat{h}_{\pm}) \) the Fermi (spectral) projections in the gap of the asymptotic Hamiltonians. Then, \( N_\pm := \text{Ch}(p_{\mu, \pm}) \) are the Chern numbers of such projections. Since it is well known that the Chern numbers of the Fermi projections provide the values of the Hall conductance (in unit of \( e^2/h \)) in the bulk of a quantum Hall system [4, 7, 38, 60], one can rewrite (1.1) in the form

\[
\sigma_\Delta(\Delta) = \sigma_{b_+}(\mu) - \sigma_{b_-}(\mu),
\]

where \( \sigma_{b_\pm}(\mu) \) are the asymptotic Hall conductances at energy \( \mu \) (cf. equation (3.13)). If one interprets the collection \( \sigma_{\text{bulk}}(\mu) := \{ \sigma_{b_\pm}(\mu) \} \) as a description of the conductance of the “bulk system” described by the Hamiltonians \( \{ \hat{h}_-, \hat{h}_+ \} \) (cf. Definition 3.17), then equation (1.2) expresses the bulk-interface duality which is a generalization, or rather a manifestation, of the celebrated bulk-boundary correspondence [20, 27, 33]. The latter is a phenomenon ubiquitous in the physics of topological insulators [28, 32, 43].

The derivation of (1.1) provided in [16] only considered Landau operators and a restricted class of perturbations, and relies on techniques from functional analysis and spectral theory. The proof of (1.1) contained in [36] works for a large family of discrete models with random perturbations and is based on an index theorem of Noether-Gohberg-Krein type. In this work we will present an alternative proof of (1.1) which makes use of \( K \)-theory for \( C^* \)-algebras. Indeed, it is well established in the literature that the bulk-boundary correspondence in condensed matter can be successfully explained and studied inside a \( K \)-theoretical framework [8, 11, 34, 49, 58]. For that reason it seems surprising that problems concerning magnetic interfaces have never been studied before with \( K \)-theoretical technique. One of the primary aims of this work is to try to fill this gap.

We focus our study on magnetic tight-binding models without considering random perturbations. Accordingly, the setting in which we derive our results is quite close to that of [36]. The fact that we are not considering random potentials does not constitute any loss of information as long as \( K \)-theory is used. In fact, random potentials are usually introduced by families of covariant operators parametrized by a configuration space \( \Omega_{\text{random}} \) which is generally assumed to be contractible (see e.g. [49, Definition 2.4.1]). Therefore, in view of its contractibility, \( \Omega_{\text{random}} \) does not add any information to the underlying \( K \)-theory and
can be replaced with a singleton. The latter fact is equivalent to consider only non-random systems. The decision to focus on tight-binding models is motivated by the desire to avoid the technicalities necessary to deal with continuous models in favor of bigger clarity in the description of the algebraic and topological aspects. However, it is worth mentioning that the bulk-boundary correspondence can be treated with $K$-theoretical methods even for continuous models [11, 34, 40].

It is important to point out that the aim of this works goes beyond the goal of proving the equation (1.1) by using a different technique. Indeed, the main purpose is to set a precise mathematical framework for the analysis of general multi-interface magnetic systems. In a nutshell a magnetic system with a multi-interface of order $N$ is a two-dimensional material divided in $N + 1$ (asymptotic) regions where the magnetic field has constant strength. The Iwatsuka magnetic field provides an example of magnetic interface of order $N = 1$, but more general examples can easily be imagined. One of the major contribution of this work is to show that systems with magnetic multi-interfaces can be naturally studied by using $K$-theory. A primary role in our analysis is played by the magnetic flux $f_B := e^{iB}$ associated with the magnetic field $B$. The magnetic flux, together with all its $\mathbb{Z}^2$-translations, defines a commutative $C^*$-algebra which, in view of the Gelfand isomorphism, can be represented as the set of continuous functions $\mathcal{C}(\Omega_B)$ over the compact Hausdorff space $\Omega_B$. The latter is known as the magnetic hull. The magnetic hull, endowed with the action of $\mathbb{Z}^2$, provides a topological dynamical system which encodes the structure of the magnetic multi-interface through its stationary points at infinity. It is worth pointing out that the topology of $\Omega_B$ is usually not trivial and influences substantially the $K$-theory of the magnetic systems under consideration (in contrast to the role of random perturbations). The idea of the use of the magnetic hull for the study of the topological phases of magnetic multi-interface systems seems to be (to the best of our knowledge) a new contribution in the literature concerning topological insulators. On the other hand, similar ideas have already been used successfully in a totally different context [24].

This work is written with a pedagogical spirit. Along with original results and ideas, it contains an important amount of review material (e.g. the rich appendix section). The purpose of this “stylistic” choice relies in the hope that this work can be preparatory for further investigations. In fact, if on the one hand the description of the $K$-theoretical picture for general multi-interface magnetic systems has been developed in a comprehensive form in this work, on the other hand the proof of the bulk-interface duality has been obtained only for the special case of the Iwatsuka magnetic field (other than that the trivial case of a localized magnetic field). The study of more general interfaces will require a deeper and more specific analysis and probably will benefit from the “pioneering” results described in [59], as well as some ideas from the papers [21, 41, 48] where boundary currents along non-straight boundaries are studied.

Structure of the paper and overview of the results. Sect. 2 is devoted to the construction of the magnetic $C^*$-algebra associated to a generic magnetic field $B : \mathbb{Z}^2 \to \mathbb{R}$ and to the study of the related integro-differential structure. In more details, in Sect. 2.1 we introduce the notion of vector potential $A_B$ associated to $B$, in Sect. 2.2 we describe the magnetic translations and the flux operator associated to $A_B$ and in Sect. 2.3 we define the magnetic $C^*$-algebra $\mathcal{A}_{A_B}$ generated by the magnetic translations. Sect. 2.4 describes the Fourier theory for the algebra $\mathcal{A}_{A_B}$ which is the natural adaptation of the classical Fourier theory for tori. Sect. 2.5 deals with the differential structure of $\mathcal{A}_{A_B}$ and the description of its smooth pre-$C^*$-algebra $\mathcal{A}_{A_B}^\infty$. The new concept of the magnetic hull $\Omega_B$ and the associated dynamical system are investigated in Sect. 2.6 along with the construction of possible integrals (or
traces) for \( \mathcal{A}_{AB} \). Sect. 3 contains the basic definitions and the main results for the framework in which the \( K \)-theory of the magnetic \( C^* \)-algebra \( \mathcal{A}_{AB} \) can be studied. The notions of the evaluation homomorphism and the interface algebra \( \mathcal{I} \subset \mathcal{A}_{AB} \) are introduced in Sect. 3.1. In Sect. 3.2 it is shown that the evaluation homomorphism and the interface algebra fit into a short exact sequence (cf. Theorem 3.6) which is reminiscent of the classical Toeplitz extension for the torus algebra. In Sect. 3.3 the interconnection between the structure at infinity of the dynamical system generated by the magnetic hull \( \Omega_B \) and the existence of suitable evaluation homomorphisms is investigated (cf. Proposition 3.11). This analysis paves the way for the rigorous definition of magnetic multi-interface and the associated bulk algebra \( \mathcal{A}_{\text{bulk}} \) (cf. Definition 3.13). Sect. 3.4 contains the description of the six-term sequence in \( K \)-theory for a magnetic multi-interface system described by the magnetic \( C^* \)-algebra \( \mathcal{A}_{AB} \), the interface algebra \( \mathcal{I} \) and the bulk algebra \( \mathcal{A}_{\text{bulk}} \). Finally, Sect. 3.5 provides the notion of bulk and interface currents and the crucial formula for the bulk-interface correspondence (equation (3.23)). Sect. 4 contains the detailed study of the magnetic interface generated by the Iwatsuka magnetic field and the proof of formula (1.1). In more detail, Sect. 4.1 contains the presentation of the Toeplitz extension associated to the Iwatsuka magnetic field along with a useful characterization of the evaluation homomorphism (cf. Remark 4.1). Sect. 4.2 contains a precise description of the interface algebra in terms of a relevant projection (cf. Proposition 4.6) and Sect. 4.3 provides a short discussion about the (non) splitting property of the Toeplitz extension for the Iwatsuka magnetic field. The \( K \)-theory of the Iwatsuka \( C^* \)-algebra \( \mathcal{A}_I \) is described in full detail in Sect. 4.4 (cf. Theorem 4.10) and the six-term exact sequence associated to the Toeplitz extension of \( \mathcal{A}_I \) is completely solved in Sect. 4.5. This fact provides a finer understanding of the \( K \)-theory of \( \mathcal{A}_I \) (cf. Theorem 4.15). Finally, the bulk-interface correspondence for the Iwatsuka \( C^* \)-algebra and the proof of the formula (1.1) are contained in Sect. 4.6 (cf. Theorem 4.19). Let us emphasize once again that this work has been written with the aim of being pedagogical and self-consistent (as far as possible). For this reason a significant amount of supporting material has been included in the Appendices A, B, C, D. The experienced reader can certainly skip this part. Also, the “study case” of a localized magnetic field \( B_{\Lambda} \) has been discussed throughout the work in various examples as an aid to better fix the proposed concepts. As a pay-off we got the generalization of certain results described in [19].

2 The (Tight-Binding) Magnetic \( C^* \)-Algebra

In this section we will describe the algebra of operators on \( \ell^2(\mathbb{Z}^2) \) that describe the kinematics of charged particles subjected to a generic orthogonal magnetic field in the tight-binding approximation. This algebra, called magnetic \( C^* \)-algebra, is a suitable generalization of the noncommutative torus [14, 15, 22], and has the structure of a crossed product \( C^* \)-algebra [14, 47] (cf. Appendix A).

2.1 Magnetic Fields and Vector Potentials

In the tight-binding approximation the (two-dimensional) position space is \( \mathbb{Z}^2 \). An orthogonal magnetic field is described by a function \( B : \mathbb{Z}^2 \rightarrow \mathbb{R} \). However, to describe the quantum dynamics in presence of a magnetic field we first need to introduce the notion of vector potential in the discrete setting. For that, let us fix first some notation. We will denote by \( n := (n_1, n_2) \) the generic point of \( \mathbb{Z}^2 \) and with \( e_1 := (1, 0) \) and \( e_2 := (0, 1) \) the canonical generators of \( \mathbb{Z}^2 \). Moreover, we will view \( \mathbb{Z}^2 \) as the vertices of an oriented graph, with oriented edges given by the oriented line segments \( (n - e_j, n) \) between nearest vertices.
Definition 2.1 (Tight-binding vector potential) Let $B : \mathbb{Z}^2 \to \mathbb{R}$ be a magnetic field. A magnetic potential for $B$ is a real-valued function

$$A_B : \mathbb{Z}^2 \times \mathbb{Z}^2 \to \mathbb{R}$$

satisfying:

(i) $A_B(n, m) = 0$ for $n, m \in \mathbb{Z}^2$ such that $|n - m| \neq 1$;
(ii) $A_B(m, n) = -A_B(n, m)$ for all $n, m \in \mathbb{Z}^2$;
(iii) $B(n) = C[A_B](n)$ for all $n \in \mathbb{Z}^2$ where

$$C[A_B](n) := A_B(n, n - e_1) + A_B(n - e_1, n - e_1 - e_2) + A_B(n - e_1 - e_2, n - e_2) + A_B(n - e_2, n)$$

is the (counterclockwise) circulation of $A_B$ along the boundary of the unit cell of $\mathbb{Z}^2$ with right upper corner centered in $n$.

From [19, Proposition 1] we know that every magnetic field $B$ admits infinite (gauge) equivalent vector potentials and every two potentials $A_B$ and $A'_B$ associated with the same magnetic field $B$ are related by a gauge function $G : \mathbb{Z}^2 \to \mathbb{R}$ according to

$$A'_B(n, m) = A_B(n, m) + G(n) - G(m), \quad |n - m| = 1.$$  (2.1)

In this case a simple computation shows that $C[A_B] = C[A'_B]$.

The following examples will be will be considered throughout the work.

Example 2.2 (Constant magnetic field) A constant magnetic field of strength $b$ is described by the function

$$B_b(n) := b, \quad \forall n \in \mathbb{Z}^2.$$  (2.2)

Among the infinite vector potentials associated to the constant magnetic field $B_b$, there is just one of utility in this work, which is the so-called Landau potential defined by

$$A^L_B(n, n - e_j) := \delta_{j, 1} n_2 b, \quad \forall n \in \mathbb{Z}^2.$$  (2.3)

A straightforward computation shows that $C[A^L_B] = B_b$.

Example 2.3 (Iwatsuka magnetic field) The Iwatsuka magnetic field [30] models systems with a one-dimensional interface between two constant magnetic fields. In the tight-binding approximation it is defined by

$$B_I(n) := b_- \delta_-(n) + b_0 \delta_0(n) + b_+ \delta_+(n), \quad \forall n \in \mathbb{Z}^2$$  (2.4)

where $b_-, b_0, b_+ \in \mathbb{R}$, and the three functions $\delta_\pm, \delta_0$ are defined by

$$\delta_\pm(n_1, n_2) := \begin{cases} 1 & \text{if } \pm n_1 > 0 \\ 0 & \text{otherwise} \end{cases}, \quad \delta_0(n_1, n_2) := \begin{cases} 1 & \text{if } n_1 = 0 \\ 0 & \text{otherwise} \end{cases}.$$
The magnetic field \( B_I \) describes a one-dimensional interface along the vertical line defined by \( n_1 = 0 \). It’s worth mentioning that to obtain non-trivial topological effect it will be necessary to impose the constraint
\[
b_+ - b_- \neq 2\pi \mathbb{Z}.
\]

The simplest choice for a vector potential, and the only one of importance for this work, producing the magnetic field \( B_I \) is the so-called Landau-Iwatsuka potential defined by
\[
A_I(n, n - e_j) := \delta_{j,1} n_2 B_I(n), \quad \forall n \in \mathbb{Z}^2.
\]

A direct computation shows that
\[
\mathcal{C}[A_I](n) = n_2 B_I(n) - (n_2 - 1) B_I(n - e_2) = B_I(n)
\]
in view of the fact that \( B_I(n - e_2) = B_I(n) \) for all \( n \in \mathbb{Z}^2 \).

**Example 2.4** (Localized magnetic field) Let \( \mathcal{P}_0(\mathbb{Z}^2) \) be the collection of bounded subsets of \( \mathbb{Z}^2 \). For every \( \Lambda \in \mathcal{P}_0(\mathbb{Z}^2) \) let \( \delta_\Lambda \) be the characteristic function defined by
\[
\delta_\Lambda(n) := \begin{cases} 
1 & \text{if } n \in \Lambda \\
0 & \text{otherwise} 
\end{cases}.
\]

A localized magnetic field of strength \( b \in \mathbb{R} \) is described by the function
\[
B_\Lambda(n) := b \delta_\Lambda(n), \quad \forall n \in \mathbb{Z}^2.
\]

Observe that \( B_\Lambda = \sum_{\lambda \in \Lambda} B_{\{\lambda\}} \) and that a simple vector potential for \( B_{\{\lambda\}} \) is provided by the so-called half-line potential
\[
A_{\{\lambda\}}(n, n - e_j) := b \delta_{j,1} \sum_{t=0}^{+\infty} \delta_{\{\lambda+te_2\}}(n).
\]

By linearity of the circulation one gets that \( A_\Lambda := \sum_{\lambda \in \Lambda} A_{\{\lambda\}} \) provides a vector potential for the localized magnetic field \( B_\Lambda \). Observe that \( A_\Lambda \) is well defined in view of the finiteness of the sum in \( \lambda \).

### 2.2 The Magnetic Translations

Let \( s_1 \) and \( s_2 \) be the canonical shift operators defined on the Hilbert space \( \ell^2(\mathbb{Z}^2) \) by
\[
(s_j \psi)(n) := \psi(n - e_j), \quad j = 1, 2, \quad \psi \in \ell^2(\mathbb{Z}^2).
\]

Let \( B \) a magnetic field with associated vector potential \( A_B \). The magnetic phases in the gauge \( A_B \) are the unitary operators \( \eta_{A_B,1} \) and \( \eta_{A_B,2} \) defined by
\[
(\eta_{A_B,j} \psi)(n) := e^{i A_B(n,n-e_j)} \psi(n), \quad \psi \in \ell^2(\mathbb{Z}^2).
\]

The magnetic translations (in the gauge \( A_B \)) are the “twisted” shift operators defined by \( s_{A_B,j} := \eta_{A_B,j} s_j \), or more explicitly as
\[
(s_{A_B,j} \psi)(n) := e^{i A_B(n,n-e_j)} \psi(n - e_j), \quad \psi \in \ell^2(\mathbb{Z}^2).
\]
The magnetic translations are composition of unitary operators, hence one has that $s_{AB,j}^* = s_{AB,j}^{-1}$ with adjoint given by

$$(s_{AB,j}^*\psi)(n) := e^{iA_B(n,n+e_j)}\psi(n+e_j), \quad \psi \in \ell^2(\mathbb{Z}^2).$$

A direct computation shows that

$$s_{AB,1}s_{AB,2}s_{AB,1}^*s_{AB,2}^* = \hat{f}_B$$

(2.7)

where the flux operator $\hat{f}_B$ acts as

$$(\hat{f}_B\psi)(n) := e^{iC[A_B](n)}\psi(n) = e^{iB(n)}\psi(n), \quad \psi \in \ell^2(\mathbb{Z}^2).$$

In the case $A_B = 0$ the magnetic translations reduce to the shift operators and the flux operator becomes the identity $1$.

It is worth noting that the flux operator $\hat{f}_B$ only depends on the magnetic field $B = C[A_B]$ and not on the specific choice of the potential $A_B$. Let $A'_{B}$ be a second vector potential for $B$ related to $A_B$ by the gauge transform $G$ according to (2.1). Then, it is straightforward to check that the magnetic translations associated to $A_B$ and $A'_{B}$ are related by the unitary equivalence

$$s_{A'_{B},j} = e^{-iG}s_{AB,j}e^{iG}, \quad j = 1, 2.$$}

Therefore, the change of the gauge translates to a unitary equivalence at levels of the magnetic translations.

**Example 2.5** (Magnetic translations for a constant magnetic field) In the case of a constant magnetic field of strength $b$ the associated magnetic translations (in the Landau gauge) are given by

$$\begin{cases}
(s_{b,1}\psi)(n) := e^{inzb}\psi(n-e_1), & \psi \in \ell^2(\mathbb{Z}^2) \\
(s_{b,2}\psi)(n) := \psi(n-e_2)
\end{cases}$$

(2.8)

Evidently, $s_{b,2} = s_2$ coincides with the standard shift operator in the direction $e_2$. In this specific case equation (2.7) can be rewritten as

$$s_{b,1}s_{b,2} = e^{ib}s_{b,2}s_{b,1}$$

(2.9)

and the flux operator coincides with the multiplication by the constant phase $e^{ib}$. Equation (2.9) provides the canonical commutation rule for the noncommutative torus (cf. [22, Chap. 12]).

**Example 2.6** (Iwatsuka magnetic translations) The magnetic translations associated to a Iwatsuka magnetic field (2.4) can be easily described in the Landau-Iwatsuka potential (2.5). With this choice one obtains the two twisted translations

$$\begin{cases}
(s_{I_{1}}\psi)(n) := e^{inzB_I(n)}\psi(n-e_1), & \psi \in \ell^2(\mathbb{Z}^2) \\
(s_{I_{2}}\psi)(n) := \psi(n-e_2)
\end{cases}$$

(2.10)
Exactly as in the case of the constant magnetic field, one has that \( s_{1,2} = s_2 \) coincides with the standard shift operator in the direction \( e_2 \). Therefore, the difference between (2.8) and (2.10) lies totally in the translation along \( e_1 \). The flux operator

\[
(\mathcal{F}_1 \psi)(n_1, n_2) := \begin{cases} 
\exp \left( i b - \psi(n_1, n_2) \right) & \text{if } n_1 < 0 \\
\exp \left( i b_0 \right) \psi(n_1, n_2) & \text{if } n_1 = 0 \\
\exp \left( i b + \psi(n_1, n_2) \right) & \text{if } n_1 > 0 
\end{cases}
\]

implies the commutation relation (2.7) for the Iwatsuka magnetic translations.

**Example 2.7** (Magnetic translations for a localized field) The magnetic translations associated to a localized magnetic field \( B/\Lambda_1 \) of the type (2.6) can be defined exactly as in the previous examples by using the vector potential \( A/\Lambda_1 \). Also in this case one obtains that the magnetic translation in the direction \( e_2 \) coincides with the standard shift operator. An explicit computation shows that in this case the flux operator is given by

\[
\mathcal{F}_\Lambda := \left( \exp \left( i b - 1 \right) \right) p_\Lambda + 1,
\]

where the projection \( p_\Lambda \) is the multiplication operator by the function \( \delta/\Lambda_1 \) defined in Example 2.4.

Let \( \ell^\infty(\mathbb{Z}^2) \) be the von Neumann algebra of bounded sequences\(^1\) on \( \mathbb{Z}^2 \). The group \( \mathbb{Z}^2 \) acts on \( \ell^\infty(\mathbb{Z}^2) \) by translations as follows: Let \( \gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^2 \) and \( g \in \ell^\infty(\mathbb{Z}^2) \). Then \( \tau_\gamma(g) \in \ell^\infty(\mathbb{Z}^2) \) is defined by

\[
\tau_\gamma(g)(n) := g(n - \gamma_1 e_1 - \gamma_2 e_2), \quad \forall \ n \in \mathbb{Z}^2.
\]

The map \( \gamma \mapsto \tau_\gamma \) defines a representation of \( \mathbb{Z}^2 \) by automorphisms of \( \ell^\infty(\mathbb{Z}^2) \).

Let \( B(\ell^2(\mathbb{Z}^2)) \) be the algebra of bounded operators on \( \ell^2(\mathbb{Z}^2) \). The algebra \( \ell^\infty(\mathbb{Z}^2) \) is canonically identified with the von Neumann sub-algebra \( \mathcal{M} \) of bounded multiplication operators on \( \ell^2(\mathbb{Z}^2) \). More precisely one has that

\[
\mathcal{M} := \left\{ m_g \mid g \in \ell^\infty(\mathbb{Z}^2) \right\}
\]

where the multiplication operator \( m_g \in B(\ell^2(\mathbb{Z}^2)) \) is defined by

\[
(m_g \psi)(n) := g(n) \psi(n), \quad \psi \in \ell^2(\mathbb{Z}^2).
\]

Observe that the magnetic phases \( \eta_{A_B, j} \) and the flux operator \( \mathcal{F}_B \) are elements of \( \mathcal{M} \). In particular one has that \( \mathcal{F}_B = m_{e_i B} \). The following result follows from a direct computation:

**Lemma 2.8** For all \( \gamma \in \mathbb{Z}^2 \) and \( g \in \ell^\infty(\mathbb{Z}^2) \) it holds true that

\[
(m_{\tau_\gamma(g)}(s_{A_B, 1}))^{\gamma_1} (s_{A_B, 2})^{\gamma_2} m_g (s_{A_B, 2})^{-\gamma_2} (s_{A_B, 1})^{-\gamma_1} = (s_{A_B, 2})^{\gamma_2} (s_{A_B, 1})^{\gamma_1} m_g (s_{A_B, 1})^{-\gamma_1} (s_{A_B, 2})^{-\gamma_2}
\]

independently of the vector potential \( A_B \).

\(^1\)Due to the discreteness of \( \mathbb{Z}^2 \) every function on \( \mathbb{Z}^2 \) is automatically continuous. This provides the identification \( C_b(\mathbb{Z}^2) \equiv \ell^\infty(\mathbb{Z}^2) \) where the symbol \( C_b(X) \) denotes the algebra of continuous bounded functions on \( X \).
Lemma 2.8 states that, independently of the magnetic field, the magnetic translations can be used to implement the induced action (still denoted with the same symbol)

\[ \mathbb{Z}^2 \ni \gamma \mapsto \tau_\gamma \in \text{Aut}(\mathcal{M}) \]
defined by

\[ \tau_\gamma(m_g) := m_{\tau_\gamma(g)} \]
for every \( g \in \ell^\infty(\mathbb{Z}^2) \).

### 2.3 Construction of the Magnetic C*-Algebra

Let \( s_{AB,1} \) and \( s_{AB,2} \) be the magnetic translations associated to the magnetic field \( B \) through the vector potential \( A_B \).

**Definition 2.9** (The tight-binding magnetic C*-algebra) The magnetic C*-algebra \( A_{AB} \) of the magnetic field \( B \) (in the gauge \( A_B \)) is the unital C*-subalgebra of \( B(\ell^2(\mathbb{Z}^2)) \) generated by \( s_{AB,1} \) and \( s_{AB,2} \), i.e.

\[ A_{AB} := C^* \left( s_{AB,1}, s_{AB,2} \right). \]

In more detail \( A_{AB} \) is constructed by closing the collection of the Laurent polynomials in \( s_{AB,1} \) and \( s_{AB,2} \) with respect to the operator norm. Since \( A_{AB} \) is unital we will use the convention \((s_{AB,j})^0 = 1, j = 1, 2\).

**Example 2.10** (Noncommutative torus, Iwatsuka algebra and flux tubes) In the case of a constant magnetic field of strength \( b \) the associated magnetic C*-algebra will be denoted simply with

\[ A_b := C^* \left( s_{b,1}, s_{b,2} \right) \]
where the magnetic translations \( s_{b,1} \) and \( s_{b,2} \) are described in Example 2.5. Indeed \( A_b \) turns out to be a faithful representation of the noncommutative torus \( \mathbb{A}_{\theta b} \) with deformation parameter \( \theta_b := b(2\pi)^{-1} \) on the Hilbert space \( \ell^2(\mathbb{Z}^2) \) (cf. [22, Chap. 12]). The magnetic C*-algebra associated to the Iwatsuka magnetic field \( B_I \) (2.4) is defined by

\[ A_I := C^* \left( s_{I,1}, s_{I,2} \right) \]
where the magnetic translations \( s_{I,1} \) and \( s_{I,2} \) are defined in Example 2.6. We will refer to \( A_I \) as the Iwatsuka C*-algebra. The magnetic C*-algebra associated to a localized magnetic field is

\[ A_\Lambda := C^* \left( s_{\Lambda,1}, s_{\Lambda,2} \right) \]
where the magnetic translations \( s_{\Lambda,1} \) and \( s_{\Lambda,2} \) are defined through the vector potential \( A_\Lambda \) described in Example 2.4. The special case \( \Lambda = \{ \lambda \} \) has been considered in [19] and, by analogy, one could refer to \( A_\Lambda \) as the (extended) flux-tube C*-algebra.
The C*-algebra $A_{AB}$ contains a relevant commutative C*-subalgebra $F_B$. From (2.7) we know that the flux operator $f_B = m_{eB}$ is an element of $A_{AB}$. In view of Lemma 2.8, also every translated element $\tau_\gamma(f_B) := m_{e\gamma(e_B)}$, with $\gamma \in \mathbb{Z}^2$, belongs to $A_{AB}$. As a result one can define the C*-algebra generated by these elements, i.e.

$$F_B := C^*(\tau_\gamma(f_B), \gamma \in \mathbb{Z}^2) \subset A_{AB}. \quad (2.13)$$

It turns out that $F_B$ is a commutative unital C*-algebra. Therefore, in view of the Gelfand isomorphism, one has $F_B \simeq C(\Omega_B)$, where $\Omega_B$ is a compact Hausdorff space. We will refer to $\Omega_B$ as the hull of the magnetic field $B$ (cf. Section 2.6 for a more accurate description of this space). It is worth pointing out that the commutative C*-algebra $F_B$ only depends on the magnetic field $B$ and not on the particular vector potential $A_B$. More precisely, if $A_B$ and $A'_B$ are two distinct vector potentials for the same magnetic field $B$, then the C*-algebras $A_{AB}$ and $A'_{AB}$ are in general different, even though they are gauge equivalent $A_{A_B} = e^{-iG}(A_{AB})e^{iG}$. Nevertheless $F_B \subset A_{AB} \cap A'_{AB}$, since the elements in $F_B$ are gauge invariant. In this sense the commutative C*-subalgebra $F_B$ encodes all the information about the magnetic field $B$.

A finite monomial in $A_{AB}$ is an element of the type

$$u := (s_{AB,1})^{r_1} (s_{AB,2})^{r_2} \cdots (s_{AB,1})^{r_d} (s_{AB,2})^{s_d} \quad (2.14)$$

with $d \in \mathbb{N}$ and $r = (r_1, \ldots, r_d), s = (s_1, \ldots, s_d) \in \mathbb{Z}^d$. In view of (2.7) and Lemma 2.8 the monomial $u$ can always be rearranged in the form

$$u = f_{r,s} (s_{AB,1})^{\mid r \mid} (s_{AB,2})^{\mid s \mid} \quad (2.15)$$

where $\mid r \mid := r_1 + \cdots + r_d, \mid s \mid := s_1 + \cdots + s_d$ and $f_{r,s}$ is a the (unitary) element of $F_B$ uniquely defined by (2.14) and (2.15). From its very definition it follows that $A_{AB}$ is linearly generated by the family of monomials (2.14), or equivalently (2.15). This observation allows to define several relevant dense *-subalgebras of $A_{AB}$.

The first dense *-subalgebra is denoted with $A^0_{AB}$ and is defined as the collection of the finite linear combination of monomials of the type (2.14), or equivalently of type (2.15). We will refer to $A^0_{AB}$ as the subalgebra of the noncommutative polynomials.

For the second dense *-subalgebra we need to introduce the operator-valued Schwartz space $S(\mathbb{Z}^2, F_B)$ made by the rapidly descending sequences

$$\{g_{r,s}\} := \{g_{r,s} \mid (r, s) \in \mathbb{Z}^2\} \subset F_B$$

such that

$$r_k(\{g_{r,s}\})^2 := \sup_{(r,s) \in \mathbb{Z}^2} (1 + r^2 + s^2)^k \|g_{r,s}\|^2 < \infty, \quad (2.16)$$

for all $k \in \mathbb{N}_0$. It turns out that the system of seminorms (2.16) endows $S(\mathbb{Z}^2, F_B)$ with the structure of a Fréchet space. In the (standard) case $F_B = C$ we will use the short notation $S(\mathbb{Z}^2)$ instead of $S(\mathbb{Z}^2, C)$. The smooth *-subalgebra $A^\infty_{AB}$ is then defined as

$$A^\infty_{AB} := \left\{ a_g := \sum_{(r,s) \in \mathbb{Z}^2} g_{r,s} (s_{AB,1})^r (s_{AB,2})^s \right\} \quad (2.17)$$

One can easily check the chain of inclusions $A^0_{AB} \subset A^\infty_{AB} \subset A_{AB}$.
Proposition 2.11 \( \mathcal{A}^\infty_{AB} \) is a Fréchet space with respect to the topology induced by the system of norms

\[
|||a|||_k := r_k(\{g_{r,s}\}), \quad k \in \mathbb{N}_0.
\]

**Proof** The map \( g_{r,s} \mapsto a^r \) which associates to a rapidly descending sequence in \( S(\mathbb{Z}^2, \mathcal{F}_B) \) an element of \( \mathcal{A}^\infty_{AB} \) is a bijection. Surjectivity follows from the definition while injectivity is a consequence of Theorem 2.14. Therefore, the map \( g_{r,s} \mapsto a^r \) defines an isomorphism of Fréchet spaces once \( \mathcal{A}^\infty_{AB} \) is endowed with the topology induced by the norms \( ||| \cdot |||_k \). \( \square \)

### 2.4 Fourier Theory

In the case \( B = 0 \) the magnetic algebra is isomorphic to \( C(\mathbb{T}^2) \) (cf. Example 2.10) and for this algebra a very rich Fourier theory is available [25, 31]. In this section we will show that some of the results of the classical Fourier theory extend to the algebra \( \mathcal{A}_{AB} \). Similar results are also described in [15, Section VIII.2].

Let \( n_1 \) and \( n_2 \) be the position operators acting on \( \ell^2(\mathbb{Z}^2) \) by

\[
(n_j \psi)(n_1, n_2) := n_j \psi(n_1, n_2), \quad j = 1, 2, \quad \psi \in \ell^2(\mathbb{Z}^2).
\]

Let \( \theta := (\theta_1, \theta_2) \) be the coordinates of \( \mathbb{T}^2 \simeq [0, 2\pi)^2 \) and consider the unitary operator \( \mathbf{w}_\theta := e^{-i\theta n} = e^{-i(\theta_1 n_1 + \theta_2 n_2)} \) defined by

\[
(\mathbf{w}_\theta \psi)(n_1, n_2) := e^{-i(\theta_1 n_1 + \theta_2 n_2)} \psi(n_1, n_2), \quad \psi \in \ell^2(\mathbb{Z}^2).
\]

The unitary operators \( \mathbf{w}_\theta \) allow to define a family of inner automorphisms \( \alpha_\theta : B(\ell^2(\mathbb{Z}^2)) \rightarrow B(\ell^2(\mathbb{Z}^2)) \) given by

\[
\alpha_\theta(a) := \mathbf{w}_\theta a \mathbf{w}_\theta^*, \quad a \in B(\ell^2(\mathbb{Z}^2)). \tag{2.18}
\]

A straightforward check shows that the map \( \mathbb{T}^2 \ni \theta \mapsto \alpha_\theta \in \text{Aut}(B(\ell^2(\mathbb{Z}^2))) \) defines an action of the group \( \mathbb{T}^2 \) on \( B(\ell^2(\mathbb{Z}^2)) \). However, this action is not continuous on the whole algebra \( B(\ell^2(\mathbb{Z}^2)) \). For that consider the parity operator \( \mathbf{\varphi} \) defined by \( (\varphi \psi)(n) = \psi(-n) \). Since \( \alpha_\theta(\varphi) = \mathbf{w}_{2\varphi} \varphi \), it follows that \( |||\alpha_\theta(\varphi) - \varphi||| = |||\mathbf{w}_{2\varphi} - 1||| = 2 \) whenever one of \( \theta_1 \) and \( \theta_2 \) are irrational. Things go differently if the action of \( \mathbb{T}^2 \) is restricted to \( \mathcal{A}_{AB} \).

**Proposition 2.12** The prescription \( (2.18) \) defines a continuous group action \( \mathbb{T}^2 \ni \theta \mapsto \alpha_\theta \in \text{Aut}(\mathcal{A}_{AB}) \).

**Proof** A direct computation shows that

\[
\alpha_\theta(g) = g \quad \forall \ g \in \mathcal{F}_B, \tag{2.19}
\]

independently of \( \theta = (\theta_1, \theta_2) \in \mathbb{T}^2 \), and

\[
\alpha_\theta((s_{AB,1})^r (s_{AB,2})^s) = e^{-i(r\theta_1 + s\theta_2)} (s_{AB,1})^r (s_{AB,2})^s \tag{2.20}
\]

for all \( (r, s) \in \mathbb{Z}^2 \). The relations \( (2.19) \) and \( (2.20) \) along with the definition \( (2.17) \) of \( \mathcal{A}^\infty_{AB} \) imply that \( \alpha_\theta(A^\infty_{AB}) = A^\infty_{AB} \) for all \( \theta \in \mathbb{T}^2 \). Finally, the density of \( \mathcal{A}^\infty_{AB} \) and the fact that \( \alpha_\theta \) is norm-preserving imply that \( \alpha_\theta(A_{AB}) = A_{AB} \), namely \( \alpha_\theta \in \text{Aut}(A_{AB}) \) for all \( \theta \in \mathbb{T}^2 \).
Let us prove now the continuity of the group action. Let \( a = \sum_{(r,s) \in \mathbb{Z}^2} g_{r,s} (s_{A'B},1)^r (s_{A'B},2)^s \) according to (2.17). Then
\[
\| \alpha_\theta (a) - a \| \leq \sum_{(r,s) \in \mathbb{Z}^2} \left| e^{-i (r \theta_1 + s \theta_2)} - 1 \right| \| g_{r,s} \| \leq 2 \sum_{(r,s) \in \mathbb{Z}^2} \| g_{r,s} \|
\]
and from the dominated convergence theorem (for series) it follows that
\[
\lim_{\theta \to 0} \| \alpha_\theta (a) - a \| \leq \sum_{(r,s) \in \mathbb{Z}^2} \lim_{\theta \to 0} \left| e^{-i (r \theta_1 + s \theta_2)} - 1 \right| \| g_{r,s} \| = 0 \tag{2.21}
\]
for all \( a \in A_{AB}^\infty \).

Now, let \( b \in A_{AB} \) be a generic element and \( \varepsilon > 0 \). By density it exists an \( a \in A_{AB}^\infty \) such that \( \| b - a \| < \varepsilon / 2 \). Moreover,
\[
\| \alpha_\theta (b) - b \| \leq \| \alpha_\theta (a) - a \| + \| \alpha_\theta (b - a) - (b - a) \|
\]
< \( \| \alpha_\theta (a) - a \| + \varepsilon \).

Therefore, from (2.21) it follows that \( \lim_{\theta \to 0} \| \alpha_\theta (b) - b \| < \varepsilon \), independently of \( \varepsilon > 0 \) and for all \( b \in A_{AB} \). This concludes the proof. \( \Box \)

Let
\[
\text{Inv}_{T^2}(A_{AB}) := \{ a \in A_{AB} \mid \alpha_\theta (a) = a , \ \forall \ \theta \in T^2 \}
\]
be the set of invariant elements of \( A_{AB} \). From (2.19) one gets that \( F_B \subseteq \text{Inv}_{T^2}(A_{AB}) \). The next goal is to characterize \( \text{Inv}_{T^2}(A_{AB}) \). For that let us denote with \( d \mu (\theta) := (2\pi)^{-2} d\theta \) the normalized Haar measure on \( T^2 \) and consider the averaging
\[
\langle a \rangle := \int_{T^2} d \mu (\theta) \alpha_\theta (a) , \quad a \in A_{AB}
\]
where the integral is meant in the Bochner sense. From the invariance of the Haar measure it follows that \( \langle a \rangle \in \text{Inv}_{T^2}(A_{AB}) \) by construction. Moreover, \( \langle a \rangle = a \) if and only if \( a \in \text{Inv}_{T^2}(A_{AB}) \). This means that every element of \( \text{Inv}_{T^2}(A_{AB}) \) can be always represented as the averaging of some element in \( A_{AB} \). The next result characterizes the set of invariant elements.

**Lemma 2.13** It holds true that
\[
\text{Inv}_{T^2}(A_{AB}) = F_B .
\]

**Proof** Since we already know that \( F_B \subseteq \text{Inv}_{T^2}(A_{AB}) \) we only need to prove the opposite inclusion. Since every element in \( \text{Inv}_{T^2}(A_{AB}) \) can be always represented as the averaging of some element in \( A_{AB} \) it is enough to prove that \( \langle a \rangle \in F_B \) for all \( a \in A_{AB} \). Since the map \( a \mapsto \langle a \rangle \) is continuous, i.e. \( \| \langle a \rangle \| \leq \| a \| \), and \( F_B \) is closed, it is sufficient to prove that the averaging of the monomials (2.15) takes value in \( F_B \). Based on (2.19) and (2.20), a direct computation shows that
\[
\langle g (s_{A'B},1)^r (s_{A'B},2)^s \rangle = g (s_{A'B},1)^r (s_{A'B},1)^s \int_{T^2} d \mu (\theta) \ e^{-i (r \theta_1 + s \theta_2)}
\]
\[
= g \delta_{r,0} \delta_{s,0} \tag{2.22}
\]
for all \( g \in \mathcal{F}_B \) and for all \((r,s) \in \mathbb{Z}^2 \). This completes the proof. \( \square \)

We are now in position to prove that every element of \( \mathcal{A}_{AB} \) can be represented as a Fourier-type series in the generating monomials (2.15). To make precise this statement, we need to introduce some notation. Given a \( a \in \mathcal{A}_{AB} \) let us define the \( \mathcal{F}_B \)-valued coefficients

\[
\hat{a}_{r,s} := \langle a(s_{AB},2)^{-s}(s_{AB},1)^{-r} \rangle
\]

\[
= \left( \int_{\mathbb{T}^2} d\mu(\theta) \ e^{i(r\theta_1+s\theta_2)} \ a_\theta(a) \right) (s_{AB},2)^{-s}(s_{AB},1)^{-r}.
\] (2.23)

To every box \( \Lambda_N := [-N, N]^2 \cap \mathbb{Z}^2 \) with \( N \in \mathbb{N} \), we associate the Cesàro mean

\[
\sigma_N(a) := \sum_{(r,s) \in \Lambda_N} \left( 1 - \frac{|r|}{N+1} \right) \left( 1 - \frac{|s|}{N+1} \right) \hat{a}_{r,s} (s_{AB},1)^r (s_{AB},2)^s.
\] (2.24)

**Theorem 2.14 (Fourier expansion - Cesàro mean)** 2 For every element \( a \in \mathcal{A}_{AB} \) it holds true that

\[
\lim_{N \to \infty} \| \sigma_N(a) - a \| = 0.
\]

**Proof** By combining (2.23) and (2.24) one gets

\[
\sigma_N(a) = \int_{\mathbb{T}^2} d\mu(\theta) \ K_N(\theta) \ a_\theta(a)
\]

where

\[
K_N(\theta) := \sum_{(r,s) \in \Lambda_N} \left( 1 - \frac{|r|}{N+1} \right) \left( 1 - \frac{|s|}{N+1} \right) e^{i(r\theta_1+s\theta_2)}
\]

\[
= F_N(\theta_1) F_N(\theta_2)
\]

and

\[
F_N(\theta_j) := \sum_{k=-N}^{N} \left( 1 - \frac{|k|}{N+1} \right) e^{i k \theta_j} = \frac{1}{N+1} \left( \frac{\sin(N\theta_j + \frac{\theta_j}{2})}{\sin(\frac{\theta_j}{2})} \right)^2
\]

is the Fejér kernel, with \( j = 1, 2 \) [31, Chapter I, Sect. 2.5] or [25, Chapter I, Sect. 3.1.3]. Since \( (2\pi)^{-1} \int_0^{2\pi} d\theta_j F_N(\theta_j) = 1 \), and consequently \( \int_{\mathbb{T}^2} d\mu(\theta) K_N(\theta) = 1 \), one gets that

\[
\sigma_N(a) - a = \int_{\mathbb{T}^2} d\mu(\theta) \ K_N(\theta) [a_\theta(a) - a].
\]

\(^2\)The proof of this theorem is adapted from [62, Theorem 5.5.7].
Using the identity
\[ \alpha(\theta_1,0) - \alpha = \alpha(\theta_1,0) - \alpha + \alpha(0,\theta_2) - \alpha(0,\theta_2) = \alpha(\theta_1,0) - \alpha + \alpha(0,\theta_2) - \alpha(0,\theta_2) \]
and the fact that the \( T^2 \)-action is isometric one gets
\[ \| \sigma_N(a) - a \| \leq \int_0^{2\pi} \frac{d\theta_1}{2\pi} F_N(\theta_1) \| \alpha(\theta_1,0)(a) - a \| + \int_0^{2\pi} \frac{d\theta_2}{2\pi} F_N(\theta_2) \| \alpha(0,\theta_2)(a) - a \| . \]

Since the functions \( f_1(\theta_1) := \| \alpha(\theta_1,0)(a) - a \| \) and \( f_2(\theta_2) := \| \alpha(0,\theta_2)(a) - a \| \) are continuous with \( f_1(0) = f_2(0) \) and the Fejér kernel is a summability kernel \([31, \text{Chapter I, Sect. 2.2}]\) one obtains that the two integrals on the right go to zero when \( N \to \infty \). This concludes the proof. □

Theorem 2.14 states that every element of \( a \in A_{A_B} \) can be approximated by the sequence \( \sigma_N(a) \in A_{A_B} \) obtained from the “Fourier coefficients” \( \hat{a}_{r,s} \). It follows that two elements with the same \( F_{A_B} \)-valued coefficients are identical.

**Corollary 2.15** Let \( a \in A_{A_B} \). Then \( a = 0 \) if and only if \( \hat{a}_{r,s} = 0 \) for all \( (r, s) \in \mathbb{Z}^2 \).

**Remark 2.16** (Cesàro vs. uniform convergence) By observing that
\[ K_N(\theta) = \frac{1}{(N+1)^2} \sum_{n_1=0}^{N} \sum_{n_2=0}^{N} D_{(n_1,n_2)}(\theta) , \]
where
\[ D_{(n_1,n_2)}(\theta) := \sum_{(r,s) \in \Lambda_{(n_1,n_2)}} e^{i(r\theta_1 + s\theta_2)} = \frac{\sin (n_1\theta_1 + \frac{\theta_1}{2}) \sin (n_1\theta_2 + \frac{\theta_2}{2})}{\sin (\frac{\theta_1}{2}) \sin (\frac{\theta_2}{2})} \]
is the Dirichlet kernel of the rectangular domain \( \Lambda_{(n_1,n_2)} := ([-n_1,n_1] \times [-n_2,n_2]) \cap \mathbb{Z}^2 \), one can rewrite (2.24) in the form
\[ \sigma_N(a) = \frac{1}{(N+1)^2} \sum_{(n_1,n_2) \in \Lambda_N} S_{(n_1,n_2)}(a) \]
(2.25)
where
\[ S_{(n_1,n_2)}(a) := \sum_{(r,s) \in \Lambda_{(n_1,n_2)}} \hat{a}_{r,s} (s_{A_B,1})^r (s_{A_B,2})^s \]
is the partial Fourier-type expansion of \( a \). Therefore, Theorem 2.14 provides a justification of the series representation
\[ a = \lim_{(n_1,n_2) \to \infty} S_{(n_1,n_2)}(a) := \sum_{(r,s) \in \mathbb{Z}^2} \hat{a}_{r,s} (s_{A_B,1})^r (s_{A_B,2})^s \]
where the symbol \( \lim \) means that the limit must be understood in the sense of Cesàro, as given by equation (2.25). This is the best that one can generally hope for a generic element

\( \square \) Springer
\( a \in \mathcal{A}_{AB} \). Indeed, let \( f \in C(\mathbb{T}) \) be the Fejér-type function constructed as in [31, Chapter II, Sect. 2.1]. Then, the sequence of the partial Fourier-type expansions of the element \( f(s^{AB}_1) \in \mathcal{A}_{AB} \) cannot be convergent in norm.

It is useful to characterize the collection of elements of \( \mathcal{A}_{AB} \) having an absolutely convergent Fourier series of \( \mathcal{F}_B \)-valued coefficients. More precisely, let us introduce the space

\[
\mathcal{A}^{ac}_{AB} := \left\{ a \in \mathcal{A}_{AB} \left| \sum_{(r,s) \in \mathbb{Z}^2} \| \hat{a}_{r,s} \| < \infty \right. \right\}
\]

where the coefficients \( \hat{a}_{r,s} \) are defined by (2.23). Since \( \mathcal{A}^\infty_{AB} \subset \mathcal{A}^{ac}_{AB} \subset \mathcal{A}_{AB} \) it follows that \( \mathcal{A}^{ac}_{AB} \) is dense in \( \mathcal{A}_{AB} \). The main properties of \( \mathcal{A}^{ac}_{AB} \) are described in the next result.

**Proposition 2.17** The space \( \mathcal{A}^{ac}_{AB} \), endowed with the norm \( \| \cdot \|_1 \), is a Banach \( \ast \)-algebra isomorphic to \( \ell^1(\mathbb{Z}^2, \mathcal{F}_B) \). In particular every \( a \in \mathcal{A}^{ac}_{AB} \) agrees with its Fourier-type expansion, i.e.

\[
a = \sum_{(r,s) \in \mathbb{Z}^2} \hat{a}_{r,s} (s_{AB,1})^r (s_{AB,2})^s
\]

**Proof** Every \( a \in \mathcal{A}^{ac}_{AB} \) defines an element \( \{ \hat{a}_{r,s} \} \in \ell^1(\mathbb{Z}^2, \mathcal{F}_B) \) by definition. Moreover, the map \( a \mapsto \{ \hat{a}_{r,s} \} \) is injective in view of Corollary 2.15. The surjectivity follows by observing that every \( \{ \hat{a}_{r,s} \} \in \ell^1(\mathbb{Z}^2, \mathcal{F}_B) \) defines an element

\[
a := \lim_{N \to \infty} \sum_{(r,s) \in \Lambda_N} \hat{a}_{r,s} (s_{AB,1})^r (s_{AB,2})^s \in \mathcal{A}^{ac}_{AB},
\]

with \( \mathcal{F}_B \)-valued coefficients \( \{ \hat{a}_{r,s} \} \). Finally, a straightforward computation as in [31, Chapter I, Sect. 6.1] shows that \( \mathcal{A}^{ac}_{AB} \) is closed under the operations inherited by the \( \ast \)-algebraic structure of \( \mathcal{A}_{AB} \). \( \square \)

### 2.5 Spatial Derivations and Differential Structure

Proposition 2.12 can be reinterpreted in the jargon of the theory of \( C_0 \)-(semi)groups [10, Chap. 3] by saying that the map \( \theta \mapsto \alpha(\theta) \) defines a strongly continuous \( \mathbb{T}^2 \)-action on the \( C^\ast \)-algebra \( \mathcal{A}_{AB} \) by automorphisms [10, Corollary 3.1.8]. This allows us to introduce the infinitesimal generators \( \nabla_1 \) and \( \nabla_2 \) defined by

\[
\nabla_1(a) := \lim_{\theta_1 \to 0} \frac{\alpha(\theta_1,0)(a) - a}{\theta_1}, \\
\nabla_2(a) := \lim_{\theta_2 \to 0} \frac{\alpha(0,\theta_2)(a) - a}{\theta_2}
\]

for suitable elements \( a \in \mathcal{A}_{AB} \) [10, Definition 3.1.5]. Indeed, \( \nabla_1 \) and \( \nabla_2 \) are unbounded linear maps on \( \mathcal{A}_{AB} \), defined on dense domains \( D(\nabla_1) \) and \( D(\nabla_2) \), respectively [10, Proposition 3.1.6]. Moreover, they are (symmetric) derivations [10, Definition 3.2.21], in the sense that

\[
\nabla_j(a^*) = \nabla_j(a)^*; \\
\nabla_j(ab) = a \nabla_j(b) + \nabla_j(a) b,
\]

(2.26)
Since the subalgebra $\mathcal{F}_B$ is invariant under the action $\alpha_{\theta}$ it follows that

$$\nabla_1(g) = \nabla_2(g) = 0, \quad \forall g \in \mathcal{F}_B.$$  \hfill (2.27)

Moreover, a direct computation shows

$$\nabla_1((s_{AB,1})^r (s_{AB,2})^s) = -ir (s_{AB,1})^r (s_{AB,2})^s, \quad \forall (r, s) \in \mathbb{Z}^2.$$  \hfill (2.28)

In particular, one can check that

$$\nabla_j \left( g (s_{AB,1})^r (s_{AB,2})^s \right) = i \left[ g (s_{AB,1})^r (s_{AB,2})^s, n_j \right],$$  \hfill (2.29)

where $[ , ]$ denotes the commutator. Indeed equation (2.29) is a special case of a more general result [10, Definition 3.2.55], which justifies the name of spatial derivation for $\nabla_1$ and $\nabla_2$.

From (2.27) and (2.28) it follows that

$$\mathcal{A}^0_{AB} \subset \mathcal{A}_B^\infty \subset \mathcal{D}(\nabla_1) \cap \mathcal{D}(\nabla_2).$$

Moreover, the elements of $\mathcal{A}_B^\infty$ support several iterated derivations. Let $\nabla^a_j := \nabla_j \circ \cdots \circ \nabla_j$ be the $a$-times iteration of the derivation $\nabla_j$. Since the group $\mathbb{T}^2$ is abelian, it follows that $\nabla_1 \circ \nabla_2 = \nabla_2 \circ \nabla_1$ whenever the product of the derivatives is well defined. It follows that the expression $\nabla^a_1 \nabla^b_2$, for $a, b \in \mathbb{N}_0$, is not ambiguous in suitable domains like $\mathcal{A}^0_{AB}$ or $\mathcal{A}_B^\infty$. Let us introduce the spaces

$$\mathcal{C}^k(A_{AB}) := \overline{\mathcal{A}^0_{AB} \mathcal{I}_k},$$

obtained by closing the noncommutative polynomials with respect to the norm

$$\|a\|_k := \sum_{i=0}^k \sum_{a+b=i} \|\nabla^a_1 \nabla^b_2(a)\|.$$

A standard argument shows that $a \in \mathcal{C}^k(A_{AB})$ if and only if $\nabla^a_1 \nabla^b_2(a) \in \mathcal{A}_{AB}$ is well defined for all $a, b \in \mathbb{N}_0$ such that $a + b \leq k$, namely

$$\mathcal{C}^k(A_{AB}) = \left\{ a \in \mathcal{A}_{AB} \mid \theta \mapsto \alpha_{\theta}(a) \text{ is } k\text{-differentiable} \right\}.$$

The regularity of an element is reflected on the decay property of its $\mathcal{F}_B$-valued coefficients. This is the content of the next result.

**Lemma 2.18** 3 Let $a \in \mathcal{C}^k(A_{AB})$, then

$$\sup_{(r, s) \in \mathbb{Z}^2} (1 + r^2 + s^2)^k \|\hat{a}_{r,s}\|^2 < \infty$$  \hfill (2.30)

3The results provided in Lemma 2.18 are not optimal, in general. For instance, in the case of a zero magnetic field described in Example 2.10 one can replace (2.30) with $(1 + r^2 + s^2)^k \|\hat{a}_{r,s}\|^2 \to 0$ when $(r, s) \to \infty$ [25, Theorem 3.3.9]. Moreover, the absolute convergence of the series of coefficients is generally guaranteed by a degree of regularity weaker than $k > 2$ [25, Theorem 3.3.16]. However, for the purposes of this work we will not need such a kind of generalization.
where the $\hat{\alpha}_{r,s}$ are defined by (2.23). In particular
\[ C^k(A_{AB}) \subset \mathcal{A}^{a,c} \]
for all $k > 2$.

**Proof** Let $a, b \in \mathbb{N}_0$ such that $a + b \leq k$. Then $\nabla_i^a \nabla_j^b(a) \in A_{AB}$ and we can calculate the $F_B$-valued coefficients according to (2.23). An iterated integration by parts provides
\[ \nabla_i^a \nabla_j^b(a)_{r,s} = \left( \int_{\mathbb{T}^2} d\mu(\theta) \ e^{i(\theta_1 + s\theta_2)} \ \alpha_\theta(\nabla_i^a \nabla_j^b(a)) \right) (s_{AB,2})^{-r} (s_{AB,1})^{-s} \]
\[ = (-i)^{a+b} r^{a} s^{b} \left( \int_{\mathbb{T}^2} d\mu(\theta) \ e^{i(\theta_1 + s\theta_2)} \ \alpha_\theta(a) \right) (s_{AB,2})^{-r} (s_{AB,1})^{-s} \]
\[ = (-i)^{a+b} r^{a} s^{b} \hat{\alpha}_{r,s} \ . \]

Since $\|\nabla_i^a \nabla_j^b(a)_{r,s}\| \leq \|\nabla_i^a \nabla_j^b(a)\| =: C_{a,b}$ for all $(r, s) \in \mathbb{Z}^2$, we can define $C := \max_{a+b=k} \{C_{a,b}\}$. It follows that $r^{2a} s^{2b} \|\hat{\alpha}_{r,s}\|^2 \leq C^2$ for all $a, b$ such that $a + b = k$. Then, by using the formula for the binomial expansion one gets
\[ (r^2 + s^2)^k \|\hat{\alpha}_{r,s}\|^2 \leq 2^k C^2 \ . \quad (2.31) \]

From (2.31), a second application of the formula for the binomial expansion provides (2.30) with bound given by $4^k C^2$. From (2.31) one gets
\[ \sum_{(r,s)\in\mathbb{Z}^2} \|\hat{\alpha}_{r,s}\| \leq \|\hat{\alpha}_{0,0}\| + 2^k C^2 \sum_{(r,s)\in\mathbb{Z}^2 \setminus \{0,0\}} \frac{1}{(r^2 + s^2)\frac{1}{2}} \]
\[ = \|\hat{\alpha}_{0,0}\| + 2^{k+1} C^2 \left( 2 \sum_{r=1}^{+\infty} \sum_{s=1}^{+\infty} \frac{1}{(r^2 + s^2)^\frac{1}{2}} + \sum_{r=1}^{+\infty} \frac{1}{r^k} + \sum_{s=1}^{+\infty} \frac{1}{s^k} \right) \]
\[ = \|\hat{\alpha}_{0,0}\| + 2^{k+2} C^2 \left( \sum_{r=1}^{+\infty} \sum_{s=1}^{+\infty} \frac{1}{(r^2 + s^2)^\frac{1}{2}} + \sum_{r=1}^{+\infty} \frac{1}{r^k} \right) \]
\[ \leq \|\hat{\alpha}_{0,0}\| + 2^{\frac{k}{2} + 2} C^2 \left( \sum_{r=1}^{+\infty} \frac{1}{r^2} \right) \left( \sum_{s=1}^{+\infty} \frac{1}{s^2} \right) + \sum_{r=1}^{+\infty} \frac{1}{r^k} \]
where in the last inequality we used $2rs \leq r^2 + s^2$. This concludes the proof. \hfill \Box

The space of the smooth elements is defined by
\[ C^\infty(A_{AB}) := \bigcap_{k\in\mathbb{N}_0} C^k(A_{AB}) \ . \]
For $a \in C^\infty(A_{AB})$ the map $\theta \mapsto \alpha_\theta(a)$ turns out to be smooth.

**Proposition 2.19** The dense subalgebra $\mathcal{A}^\infty_{AB}$ defined by (2.17) coincides with the algebra of the smooth elements with respect to the $\mathbb{T}^2$-action, i.e.
\[ \mathcal{A}^\infty_{AB} = C^\infty(A_{AB}) \ . \]
Proof Let \( a \in A_{A_B}^\infty \). Then the computation of the \( F_B \)-valued coefficients of \( \nabla_1^a \nabla_2^b (a) \) provided in the proof of Lemma 2.18 shows that \( \nabla_1^a \nabla_2^b (a) \in A_{A_B}^{a+2} \) for all \( a, b \in \mathbb{N}_0 \). This implies that \( A_{A_B}^\infty \subseteq C^k (A_{A_B}) \) for all \( k \in \mathbb{N}_0 \), and so \( A_{A_B}^\infty \subseteq C^\infty (A_{A_B}) \). On the other hand it is also true that \( C^\infty (A_{A_B}) \subseteq A_{A_B}^\infty \). In fact, if \( a \in C^\infty (A_{A_B}) \) then (2.30) applies for all \( k \in \mathbb{N}_0 \), showing that \( a \in A_{A_B}^\infty \). This concludes the proof. \( \square \)

The last result justifies the name of smooth algebra for \( A_{A_B}^\infty \). Let us recall that a pre-\( C^* \)-algebra is a dense subalgebra of a \( C^* \)-algebra which is stable under holomorphic functional calculus (see [22, Definition 3.26]).

**Proposition 2.20** The smooth algebra \( A_{A_B}^\infty \) defined by (2.17) is a unital Fréchet pre-\( C^* \)-algebra of \( A_{A_B}^\infty \).

**Proof** Since \( \mathbb{T}^2 \) is a Lie group, the criterion established in [22, Proposition 3.45] applies proving the claim. \( \square \)

The Fréchet topology of the pre-\( C^* \)-algebra \( A_{A_B}^\infty \) is provided by the system of norms described in Proposition 2.11.

### 2.6 Magnetic Hull and Integration Theory

The first task of this section is to define the magnetic hull by following the construction sketched in [5, Sect. 2.4]. Then, we will construct traces on the magnetic algebra by following the ideas of [15, VIII.3].

Let \( B : \mathbb{Z}^2 \to \mathbb{R} \) be a magnetic field and \( f_B \in \ell^\infty (\mathbb{Z}^2) \) the function defined by \( f_B (n) := e^{i B (n)} \) for all \( n \in \mathbb{Z}^2 \). The natural discrete topology of \( \mathbb{Z}^2 \) implies that \( \ell^\infty (\mathbb{Z}^2) = \mathcal{C}_0 (\mathbb{Z}^2) \) (cf. Note 1) and equation (2.12) provides the \( C^* \)-algebra \( \mathcal{C}_0 (\mathbb{Z}^2) \) with the continuous \( \mathbb{Z}^2 \)-action \( \gamma \mapsto \tau_\gamma \). It is worth recalling that the Gelfand-Naimark theorem [22, Theorem 1.4] provides the isomorphism \( \mathcal{C}_0 (\mathbb{Z}^2) \simeq \mathcal{C} (\beta \mathbb{Z}^2) \) where \( \beta \mathbb{Z}^2 \) is the Stone-Čech compactification of \( \mathbb{Z}^2 \) [22, Sect. 1.3]. In particular, one has a canonical inclusion \( \mathbb{Z}^2 \hookrightarrow \beta \mathbb{Z}^2 \), which identifies the lattice \( \mathbb{Z}^2 \) with an open and dense subset of \( \beta \mathbb{Z}^2 \).

Let

\[
\mathcal{C} (f_B, \mathbb{Z}^2) := C^* \left( \tau_\gamma (f_B), \gamma \in \mathbb{Z}^2 \right)
\]

be the \( C^* \)-subalgebra of \( \mathcal{C}_0 (\mathbb{Z}^2) \) generated by the \( \mathbb{Z}^2 \)-translated of \( f_B \) and its complex conjugated. It turns out that there is an isomorphism \( \mathcal{C} (f_B, \mathbb{Z}^2) \simeq F_B \) under the identification of \( \mathcal{C}_0 (\mathbb{Z}^2) \) with the von Neumann algebra \( \mathcal{M} \) of bounded multiplication operators described in Sect. 2.2. As anticipated in Sect. 2.3, the Gelfand-Naimark theorem provides the isomorphism \( \mathcal{C} (f_B, \mathbb{Z}^2) \simeq \mathcal{C} (\Omega_B) \) where \( \Omega_B \) is a compact Hausdorff space. Since \( \mathcal{C} (f_B, \mathbb{Z}^2) \) is generated by a countable family, it follows that it is separable (i.e. it has a countable and dense subset) and in turn \( \Omega_B \) is second countable and metrizable as a separable complete metric space\(^4\) [22, Proposition 1.11] (see also [3, Sect. 2.2]). We will refer to the topological space \( \Omega_B \) as the hull of the magnetic field \( B \), or the magnetic hull for short.

Actually, \( \Omega_B \) is built as the Gelfand spectrum of \( \mathcal{C} (f_B, \mathbb{Z}^2) \), namely the set of characters defined as the \( * \)-homomorphisms \( \omega : \mathcal{C} (f_B, \mathbb{Z}^2) \to \mathbb{C} \). As a consequence, \( \mathbb{Z}^2 \) acts by duality on \( \Omega_B \). For every \( \gamma \in \mathbb{Z}^2 \) let \( \tau_\gamma^*: \Omega_B \to \Omega_B \) be the map defined on \( \omega \in \Omega_B \) as \( \tau_\gamma^*(\omega)(g) := \omega(g) \).

\(^4\) Indeed, \( \Omega_B \) is a compact Polish space.
\( \omega(\tau_\gamma(g)) \) for all \( g \in \mathcal{C}(f_B, \mathbb{Z}^2) \). It is straightforward to show that \( \tau_\gamma^+ \in \text{Homeo}(\Omega_B) \) are homeomorphisms of \( \Omega_B \) and that the mapping \( \gamma \mapsto \tau_\gamma^+ \) provides a continuous \( \mathbb{Z}^2 \)-action by homeomorphisms. As a result \( (\Omega_B, \tau^+, \mathbb{Z}^2) \) is a genuine topological dynamical system (see e.g. [61, Chap. 5]). In \( \Omega_B \) there is a remarkable point \( \omega_0 \), called the evaluation at 0, defined by \( \omega_0(g) := g(0) \) for all \( g \in \mathcal{C}(f_B, \mathbb{Z}^2) \). Let \( \omega_\gamma := \tau_\gamma^+(\omega_0) = \omega_0 \circ \tau_\gamma^- \) be the \( \gamma \)-translated of \( \omega_0 \) and \( \text{Orb}(\omega_0) := \{ \omega_\gamma \in \Omega_B \mid \gamma \in \mathbb{Z}^2 \} \) the \( \mathbb{Z}^2 \)-orbit of \( \omega_0 \). The next result provides a relevant property of the dynamical system \( (\Omega_B, \tau^+, \mathbb{Z}^2) \).

**Proposition 2.21** The \( \mathbb{Z}^2 \)-orbit of \( \omega_0 \) is dense, i.e.

\[
\text{Orb}(\omega_0) = \Omega_B.
\]

**Proof** In view of the Gelfand-Na\v{r}ímark isomorphism \( \mathcal{C}_b(\mathbb{Z}^2) \simeq \mathcal{C}(\beta \mathbb{Z}^2) \), the Gelfand spectrum of \( \mathcal{C}_b(\mathbb{Z}^2) \) can be identified with the Stone-\v{C}ech compactification \( \beta \mathbb{Z}^2 \). The inclusion \( j : \mathcal{C}(f_B, \mathbb{Z}^2) \hookrightarrow \mathcal{C}_b(\mathbb{Z}^2) \) provides, by duality, a continuous map \( j' : \beta \mathbb{Z}^2 \to \Omega_B \) defined by \( j'(\tilde{\omega}) := \tilde{\omega} \circ j \) where \( \tilde{\omega} \in \beta \mathbb{Z}^2 \) is a character of \( \mathcal{C}_b(\mathbb{Z}^2) \). More precisely, \( j'(\tilde{\omega}) \) is by definition the restriction of the character \( \tilde{\omega} \) to the subalgebra \( \mathcal{C}(f_B, \mathbb{Z}^2) \). On the other hand, every character \( \omega \in \mathcal{C}(f_B, \mathbb{Z}^2) \) admits an (not necessarily unique) extension \( \tilde{\omega} \) to a character of \( \mathcal{C}_b(\mathbb{Z}^2) \) [10, Proposition 2.3.24]. As a result, it turns out that \( j' \) is a continuous surjection. Therefore, if \( X \subset \beta \mathbb{Z}^2 \) is dense in \( \beta \mathbb{Z}^2 \) then \( j'(X) \subset \Omega_B \) is dense in \( \Omega_B \). In view of the Riesz-Markov-Kakutani representation theorem [56, Theorem IV.14], the Gelfand spectrum of \( \mathcal{C}_b(\mathbb{Z}^2) \) consists of the evaluations (Dirac measures) at the points of \( \beta \mathbb{Z}^2 \). Since \( \mathbb{Z}^2 \) is identified with a dense open subset of \( \beta \mathbb{Z}^2 \), it follows that the set of characters \( \{ \tilde{\omega}_\gamma \mid \gamma \in \mathbb{Z}^2 \}, \) defined by \( \tilde{\omega}_\gamma(f) := f(\gamma) \) for \( f \in \mathcal{C}_b(\mathbb{Z}^2) \), is dense in the Gelfand spectrum of \( \mathcal{C}_b(\mathbb{Z}^2) \). On the other hand, it holds true that \( \omega_\gamma = j'(\tilde{\omega}_\gamma) \), and consequently

\[
j'(\{ \tilde{\omega}_\gamma \mid \gamma \in \mathbb{Z}^2 \}) = \text{Orb}(\omega_0) .
\]

The last equality proves the density of \( \text{Orb}(\omega_0) \).

For a given \( g \in \mathcal{C}(f_B, \mathbb{Z}^2) \) the Gelfand transform \( \hat{g} \in \mathcal{C}(\Omega_B) \) is defined by the equation \( \hat{g}(\omega) := \omega(g) \) for all \( \omega \in \Omega_B \). The density of the orbit of \( \omega_0 \) implies that the Gelfand transform is entirely defined by the equation \( \hat{g}(\gamma) = \omega_\gamma(g) = \hat{g}(\tau_\gamma^+(\omega_0)) \) for all \( \gamma \in \mathbb{Z}^2 \).

**Remark 2.22** (Topological transitivity) Proposition 2.21 can be rephrased by saying that the dynamical system \( (\Omega_B, \tau^+, \mathbb{Z}^2) \) is topologically transitive [61, Definition 5.6]. As a consequence, every invariant element of \( \mathcal{C}(\Omega_B) \) is automatically constant [61, Theorem 5.14]. In our specific setting (\( \Omega_B \) compact and second countable) the notion of topological transitivity for \( (\Omega_B, \tau^+, \mathbb{Z}^2) \) is equivalent to the following property: Whenever \( U \) and \( V \) are nonempty open subsets of \( \Omega_B \), then there exists a \( \gamma \in \mathbb{Z}^2 \) such that \( \tau_\gamma^+(U) \cap V \neq \emptyset \) [61, Theorem 5.8]. The latter, is the usual definition of topological transitivity in the context of the general theory of topological dynamical systems (see e.g. [1, 35] and references therein).

The subsets \( \text{Orb}(\omega_0) \) and \( \partial \Omega_B := \Omega_B \setminus \text{Orb}(\omega_0) \) are disjoint and \( \tau^+ \)-invariant by construction. Moreover, \( \partial \Omega_B \) is nowhere dense [61, Theorem 5.8] and is contained in the subset of non-wandering points of the dynamical system [61, Theorem 5.6]. Let \( \text{Mes}_{1, \tau^+}(\Omega_B) \) be the set of the normalized and \( \tau^+ \)-invariant regular Borel measures\(^5\) of the dynamical system

---

\(^5\)By the Riesz-Markov-Kakutani representation theorem [56, Theorem IV.14], \( \text{Mes}_{1, \tau^+}(\Omega_B) \) provides the space of \( \tau^+ \)-invariant states of the Abelian \( C^* \)-algebra \( \mathcal{C}(\Omega_B) \).

---
$(\Omega_B, \tau^*, \mathbb{Z}^2)$. It is well known that $\text{Mes}_{1, \tau^*}(\Omega_B)$ is a non-empty, convex and compact set (i.e. a Choquet simplex) whose extreme points are exactly the ergodic measures \cite{Corollary 6.9.1 & Theorem 6.10}. Let $\text{Erg}(\Omega_B)$ be the subset of the ergodic probability measures of $(\Omega_B, \tau^*, \mathbb{Z}^2)$. It is worth recalling that an ergodic measure $\mathbb{P} \in \text{Erg}(\Omega_B)$ is characterized by the dichotomy $\mathbb{P}(X) = 1$ or $\mathbb{P}(X) = 0$ for every $\tau^*$-invariant subset $X \subseteq \Omega_B$. A $\mathbb{P} \in \text{Erg}(\Omega_B)$ such that $\mathbb{P}(\partial \Omega_B) = 1$ will be called a measure at infinity.

**Example 2.23** (Magnetic hull for a constant magnetic field) In the case of a constant magnetic field of strength $b$ one has that $f_b(n) := e^{i b} \delta_n$ for all $n \in \mathbb{Z}^2$ (see Example 2.2) and accordingly $\mathcal{C}(f_b, \mathbb{Z}^2) = \mathbb{C}$. Therefore the associated magnetic hull $\Omega_b \simeq \{ \omega_0 \}$ is a singleton (or one point set) on which the $\tau^*$-action is trivial. The unique normalized ergodic measure on $\Omega_b$ is entirely specified by $\mathbb{P}(\{ \omega_0 \}) = 1$.

**Example 2.24** (Iwatsuka magnetic hull) In the case of the Iwatsuka magnetic field (2.4) one has

$$ f_1 := e^{i B_1} = e^{i b - \delta_-} + e^{i b_0} \delta_0 + e^{i b_+ \delta_+}. $$

Let us assume $b_- - b_+ \notin 2 \pi \mathbb{Z}$. From the definition one has that $\tau_{\omega_0}(f_1) = f_1$ and $\tau_{(q, r_2)}(f_1) \neq f_1$ for all $q \in \mathbb{Z} \setminus \{0\}$. This means that the $\mathbb{Z}^2$-action on $f_1$ indeed reduces to a $\mathbb{Z}$-action. It follows that the Gelfand isomorphism $\mathcal{C}(f_1, \mathbb{Z}^2) \simeq \mathcal{C}(\Omega_1)$ is provided by the Iwatsuka magnetic hull

$$ \Omega_1 \simeq \mathbb{Z} \cup \{-\infty\} \cup \{+\infty\}, \quad (2.33) $$

given by the two-point compactification of $\mathbb{Z}$. The inclusion $\mathbb{Z} \ni q \mapsto \omega_q \in \Omega_1$ is given by the evaluation at finite distance defined by

$$ \omega_q(\tau_\gamma(f_1)) := f_1((q - \gamma_1)e_1 - \gamma_2 e_2) $$

for every $\gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^2$. The two limit points $\{ \pm \infty \}$ are identified with the evaluations at infinity $\omega_{\pm \infty} \in \Omega_1$ defined by

$$ \omega_{\pm \infty}(\tau_\gamma(f_1)) := e^{i b_{\pm}} $$

for every $\gamma \in \mathbb{Z}^2$. From the construction it follows that $\mathbb{Z} \simeq \text{Orb}(\omega_0)$ and in turn $\{ \pm \infty \} \simeq \partial \Omega_1$. Therefore, equation (2.33) provides a decomposition of $\Omega_1$ in three invariant subsets. Since $\mathbb{Z}^2$ acts on $\text{Orb}(\omega_0)$ as a one dimensional shift it follows that $\text{Orb}(\omega_0)$ is made by wandering points \cite[Definition 5.5]{61}. As a consequence every ergodic measure $\mathbb{P} \in \text{Erg}(\Omega_1)$ necessarily must satisfy $\mathbb{P}(\text{Orb}(\omega_0)) = 0$ \cite[Theorem 6.15]{61}. This implies that the set $\text{Erg}(\Omega_1) = \{ \mathbb{P}_{\pm \infty} \}$ is made by two ergodic measures at infinity specified by the condition $\mathbb{P}_{\pm \infty}(\{ \pm \infty \}) = 1$.

**Example 2.25** (Magnetic hull for a localized magnetic field) In the case of a localized magnetic field (2.6) one has

$$ f_\Lambda := e^{i B_\Lambda} = (e^{i b} - 1) \delta_\Lambda + 1. $$

Observe that $\tau_\gamma(f_\Lambda) = f_{\gamma + \Lambda} \neq f_\Lambda$ for every $\gamma \in \mathbb{Z}^2 \setminus \{0\}$. In this case the Gelfand isomorphism $\mathcal{C}(f_\Lambda, \mathbb{Z}^2) \simeq \mathcal{C}(\Omega_\Lambda)$ is given by the localized magnetic hull

$$ \Omega_\Lambda \simeq \mathbb{Z}^2 \cup \{\infty\}, \quad (2.34) $$

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given by the one-point compactification of \( \mathbb{Z}^2 \). The inclusion \( \mathbb{Z}^2 \ni \xi \mapsto \omega_\xi \in \Omega_\Lambda \) is given by the evaluation at finite distance defined by

\[
\omega_\xi \left( \tau_\gamma(f) \right) := f(\xi - \gamma)
\]

for every \( \gamma \in \mathbb{Z}^2 \). The limit point \( \{ \infty \} \) is identified with the evaluation at infinity \( \omega_\infty \in \Omega_\Lambda \) given by

\[
\omega_\infty \left( \tau_\gamma(f) \right) := 1
\]

for every \( \gamma \in \mathbb{Z}^2 \). From the construction it follows that \( \mathbb{Z}^2 \cong \text{Orb}(\omega_0) \) and \( \{ \infty \} \cong \partial \Omega_\Lambda \) are the two invariant subsets of \( \Omega_\Lambda \). Since \( \text{Orb}(\omega_0) \) is made of wandering points under the action of \( \mathbb{Z}^2 \) it follows that \( \text{Erg}(\Omega_\Lambda) = \{ \omega_\infty \} \) where the measure at infinity \( \mathbb{P}_\infty \) is specified by \( \mathbb{P}_\infty(\{ \infty \}) = 1 \).

The ergodic measures of \( (\Omega_B, \tau^*, \mathbb{Z}^2) \) play a crucial role for the construction of the integration theory of the magnetic algebra \( A_{AB} \). Let us start by the following fact.

**Lemma 2.26** Under the isomorphism \( \iota : F_B \rightarrow C(\Omega_B) \) every invariant measure \( \mathbb{P} \in \text{Mes}_{1,\tau^*}(\Omega_B) \) defines a trace \( t_\mathbb{P} \) on \( F_B \) through the formula

\[
t_\mathbb{P}(g) := \int_{\Omega_B} d\mathbb{P}(\omega) \iota(g)(\omega), \quad g \in F_B.
\]

The trace \( t_\mathbb{P} \) is \( \mathbb{Z}^2 \)-invariant in the sense that

\[
t_\mathbb{P} \left( \tau_\gamma(g) \right) = t_\mathbb{P}(g), \quad \forall \gamma = (\gamma_1, \gamma_2) \in \mathbb{Z}^2
\]

where \( \tau_\gamma(g) := (s_{AB,1})^{\gamma_1}(s_{AB,2})^{\gamma_2} g(s_{AB,2})^{-\gamma_2}(s_{AB,1})^{-\gamma_1} \)

**Proof** The claim follows immediately by noting that the integration with respect to \( \mathbb{P} \) defines an invariant trace on \( C(\Omega_B) \). \( \square \)

We are now in position to construct the integration theory of the magnetic algebra \( A_{AB} \). Let \( \mathbb{P} \in \text{Mes}_{1,\tau^*}(\Omega_B) \) be an invariant measure and define the map \( \mathcal{T}_\mathbb{P} : A_{AB} \rightarrow \mathbb{C} \) by

\[
\mathcal{T}_\mathbb{P}(a) := t_\mathbb{P}(\hat{a}_{0,0}), \quad a \in A_{AB}
\]

where the \( F_B \)-valued coefficient \( \hat{a}_{0,0} \) is defined by (2.23).

**Proposition 2.27** The map \( \mathcal{T}_\mathbb{P} : A_{AB} \rightarrow \mathbb{C} \) defined by (2.35) is a tracial state of the \( C^* \)-algebra \( A_{AB} \). Moreover, it holds true that:

(i) \( \mathcal{T}_\mathbb{P}(\nabla_j(a)) = 0 \) for all \( a \in C^1(A_{AB}) \) and \( j = 1, 2 \);

(ii) \( \mathcal{T}_\mathbb{P}(b \nabla_j(a)) = -\mathcal{T}_\mathbb{P}(a \nabla_j(b)) \) for all \( a, b \in C^1(A_{AB}) \) and \( j = 1, 2 \).

**Proof** The map \( \mathcal{T}_\mathbb{P} \) is evidently linear (composition of linear maps) and normalized, i.e. \( \mathcal{T}_\mathbb{P}(1) = 1 \). The positivity follows by observing that

\[
(\widehat{a^*a})_{0,0} = \int_{\mathbb{T}^2} d\mu(\theta) \alpha_\theta(a^*) \alpha_\theta(a) \geq 0
\]
and consequently \( \mathcal{T}_\varphi(\alpha^*) = t_\varphi((\alpha^*)_{0,0}) \geq 0 \) since \( t_\varphi \) is a state trace (hence positive) on \( \mathcal{F}_B \). Since \( \mathcal{T}_\varphi \) is linear and positive then it is automatically continuous [10, Proposition 2.3.11]. To prove the cyclic property of the trace let us consider two monomials \( u_j := g_j (s_{AB,1})^{r_j} (s_{AB,2})^{s_j} \) with \( g_j \in \mathcal{F}_B \) and \( j = 1, 2 \). Observe that

\[
\begin{align*}
  u_1 u_2 =\ g_1 \tau_{(r_1,s_1)}(g_2) (s_{AB,1})^{r_1} (s_{AB,2})^{s_1} (s_{AB,1})^{r_2} (s_{AB,2})^{s_2}
\end{align*}
\]

where \( \tau_{(r_1,s_1)}(g_2) := (s_{AB,1})^{r_1} (s_{AB,2})^{s_1} g_2 (s_{AB,2})^{-s_1} (s_{AB,1})^{-r_1} \) and by mimicking the computation of (2.22) one gets

\[
\begin{align*}
  (u_1 u_2)_{0,0} = g_1 \tau_{(r_1,s_1)}(g_2) (s_{AB,1})^{r_1} (s_{AB,2})^{s_1} (s_{AB,1})^{-r_1} (s_{AB,2})^{-s_1} \delta_{r_1,-r_2} \delta_{s_1,-s_2}.
\end{align*}
\]

A similar argument provides

\[
\begin{align*}
  (u_2 u_1)_{0,0} = \tau_{(-r_1,-s_1)}(g_1) g_2 (s_{AB,1})^{-r_1} (s_{AB,2})^{-s_1} (s_{AB,1})^{r_1} (s_{AB,2})^{s_1} \delta_{r_1,-r_2} \delta_{s_1,-s_2}.
\end{align*}
\]

An iterated application of the commutation relation (2.7) provides

\[
\begin{align*}
  (s_{AB,1})^{r_1} (s_{AB,2})^{s_1} (s_{AB,1})^{-r_1} (s_{AB,2})^{-s_1} =: \xi_{(r_1,s_1)} \in \mathcal{F}_B
\end{align*}
\]

where \( \xi_{(r_1,s_1)} \) is given by a product of suitable translations of \( f_B \). This implies

\[
\begin{align*}
  (u_1 u_2)_{0,0} = \xi_{(r_1,s_1)} \ g_1 \tau_{(r_1,s_1)}(g_2) \delta_{r_1,-r_2} \delta_{s_1,-s_2}
\end{align*}
\]

and

\[
\begin{align*}
  (u_2 u_1)_{0,0} = \tau_{(-r_1,-s_1)}(\xi_{(r_1,s_1)}) \tau_{(-r_1,-s_1)}(g_1) g_2 \delta_{r_1,-r_2} \delta_{s_1,-s_2} = \tau_{(-r_1,-s_1)}((u_1 u_2)_{0,0}).
\end{align*}
\]

From the invariance property of Lemma 2.26 it follows that

\[
\begin{align*}
  t_\varphi((u_1 u_2)_{0,0}) = t_\varphi((u_2 u_1)_{0,0})
\end{align*}
\]

and in turn \( \mathcal{T}_\varphi(u_1 u_2) = \mathcal{T}_\varphi(u_2 u_1) \) for all pair of monomials \( u_1, u_2 \). It turns out that \( \mathcal{T}_\varphi \) satisfies the cyclic property of the trace on the dense subalgebra \( A^0_{AB} \) of the noncommutative polynomials, and by continuity on the whole algebra \( A_{AB} \). Property (i) follows from the computation at the beginning of the proof of Lemma 2.18 which provides \( \overline{\nabla}(\alpha)_{0,0} = 0 \) for \( j = 1, 2 \). Property (ii) follows by the application of property (i) along with the Leibniz’s rule (2.26).

The trace property of the map \( \mathcal{T}_\varphi \) is guaranteed by the invariance property of the measure \( \varphi \). The ergodicity of \( \varphi \) plays a role for the physical interpretation of \( \mathcal{T}_\varphi \). For the next result we need to introduce some notation. Let \( \{ \Lambda_i \}_{i \in \mathbb{N}} \subset \mathcal{P}_0(\mathbb{Z}^2) \) a sequence of bounded subsets of cardinality \( |\Lambda_i| \). The family \( \{ \Lambda_i \}_{i \in \mathbb{N}} \) is a Følner sequence [26] if: (i) it is increasing, i.e. \( \Lambda_i \subseteq \Lambda_{i+1} \) for all \( i \in \mathbb{N} \); (ii) it is exhaustive, i.e. \( \Lambda_i \not\supset \mathbb{Z}^2 \); (iii) it meets the Følner condition. i.e.

\[
\lim_{i \to \infty} \frac{|(\gamma + \Lambda_i) \Delta \Lambda_i|}{|\Lambda_i|} = 0, \quad \forall \gamma \in \mathbb{Z}^2,
\]

where \( \gamma + \Lambda_i \) is the \( \gamma \)-translated of \( \Lambda_i \) and \( \Delta \) is the symmetric difference.
Let $\mathbb{P} \in \text{Erg}(\Omega_B)$ be an ergodic measure and $\{\Lambda_i\}_{i \in \mathbb{N}}$ a Følner sequence. The Birkhoff’s Ergodic Theorem [61, Lemma 6.13] assures that there exists a Borelian subset $Y \subseteq \Omega_B$ such that $\mathbb{P}(Y) = 1$ and

$$t_{\mathbb{P}}(g) = \lim_{i \to \infty} \frac{1}{|\Lambda_i|} \sum_{\gamma \in \Lambda_i} \iota(g)(\tau_{\gamma}(\omega)) , \quad \forall \omega \in Y , \quad \forall g \in \mathcal{F}_B .$$

By observing that $\iota(g) \circ \tau_{\gamma}^* = \iota(\tau_{-\gamma}(g))$ and recalling the definition of the trace $\mathcal{T}_\mathbb{P}$ given by (2.35) one gets

$$\mathcal{T}_\mathbb{P}(a) = \lim_{i \to \infty} \frac{1}{|\Lambda_i|} \sum_{\gamma \in \Lambda_i} \iota(\tau_{-\gamma}(\hat{a}_{0,0}))(\omega) , \quad \forall \omega \in Y , \quad \forall a \in \mathcal{A}_{AB} .$$

Finally, by observing that the extraction of the $\mathcal{F}_B$-valued coefficient commutes with the translations one gets

$$\mathcal{T}_\mathbb{P}(a) = \lim_{i \to \infty} \frac{1}{|\Lambda_i|} \sum_{\gamma \in \Lambda_i} \iota(\tau_{-\gamma}(a))(\omega) , \quad \forall \omega \in Y , \quad \forall a \in \mathcal{A}_{AB} .$$

The latter formula becomes physically relevant when $Y$ is an invariant singleton. In view of Proposition 2.21 there are two possibilities in this regard: either $Y = \{\omega_0\} = \Omega_B$ coincides with the whole magnetic hull as in the case of the constant magnetic field (see Example 2.23) or $Y = \{\omega_\ast\} \subseteq \partial \Omega_B$ is an invariant limit point as in the case of the Iwatsuka magnetic field (see Example 2.24). In the first case one gets a well-known result, sometimes known as Shubin formula (see e.g. [6, Sect. 4] and references therein).

**Proposition 2.28 (Trace per unit volume)** Assume that $\Omega_B = \{\omega_0\}$ and let $\mathbb{P}$ be the (ergodic) measure supported on $\{\omega_0\}$. Let $\{\Lambda_i\}_{i \in \mathbb{N}}$ be a Følner sequence and for every $\Lambda_i$ let $p_{\Lambda_i}$ be the associated projection defined by $(p_{\Lambda_i} \psi)(n) = \delta_{\Lambda_i}(n) \psi(n)$ for all $\psi \in l^2(\mathbb{Z}^2)$. Then, it holds true that

$$\mathcal{T}_\mathbb{P}(a) := \lim_{i \to \infty} \frac{1}{|\Lambda_i|} \text{Tr}_{l^2(\mathbb{Z}^2)}(p_{\Lambda_i} a p_{\Lambda_i}) , \quad \forall a \in \mathcal{A}_{AB} .$$

**Proof** Let $\psi_\gamma \in l^2(\mathbb{Z}^2)$ be the normalized vector defined by $\psi_\gamma(n) := \delta_{n,\gamma}$. The key of the proof is the sequence of equalities

$$\iota(\tau_{-\gamma}(a))(\omega_0) = \langle \psi_0, \tau_{-\gamma}(a) \psi_0 \rangle = \langle \psi_\gamma, a \psi_\gamma \rangle , \quad \forall \gamma \in \mathbb{Z}^2 . \quad (2.36)$$

The second equality is straightforward while the first equality follows from $\iota(\hat{b}_{0,0})(\omega_0) = \langle \psi_0, b \psi_0 \rangle$ for all $b \in \mathcal{A}_{AB}$. Indeed, since $\omega_0$ identifies the evaluation at $0 \in \mathbb{Z}^2$, one infers from (2.24) that $\iota(\hat{b}_{0,0})(\omega_0) = \langle \psi_0, \sigma_N(b) \psi_0 \rangle$ for all $N \in \mathbb{N}$. The continuity of the scalar product when $N \to +\infty$ concludes the argument. \hfill $\square$

The case $Y = \{\omega_\ast\} \subseteq \partial \Omega_B$ will be discussed in Remark 3.14 with a special emphasis on the Iwatsuka magnetic field.

### 3 Magnetic Interfaces, Toeplitz Extensions and $K$-Theory

In this section we will show that magnetic $C^*$-algebras are Toeplitz-type extensions of its interface subalgebras. This observation will be used to study the related $K$-theory.
3.1 Evaluation Homomorphisms and Interface Algebra

Let $B_1$ and $B_2$ be two magnetic fields with associated vector potentials $A_{B_1}$ and $A_{B_2}$, respectively. In this section we will study a family of $C^*$-homomorphisms between the magnetic algebras $A_{B_1}$ and $A_{B_2}$ which will be of central importance in the rest of the work.

**Definition 3.1 (Evaluation homomorphisms)** A $C^*$-homomorphism $ev : A_{B_1} \to A_{B_2}$ such that

$$
\begin{align*}
\text{ev}(g_{A_{B_1},1}) &:= g_{A_{B_2},1} \\
\text{ev}(g_{A_{B_1},2}) &:= g_{A_{B_2},2},
\end{align*}
$$

will be called an *evaluation homomorphism* from $A_{B_1}$ to $A_{B_2}$.

Let $f_{B_1}$ and $f_{B_2}$ be the flux operators of the magnetic algebras $A_{B_1}$ and $A_{B_2}$, respectively. If $ev : A_{B_1} \to A_{B_2}$ is an evaluation homomorphism then from (2.7) it follows that $ev(f_{B_1}) = f_{B_2}$. More in general, one can check that

$$
ev(\tau_\gamma(f_{B_1})) = \tau_\gamma(f_{B_2}), \quad \forall \gamma \in \mathbb{Z}^2$$

(3.1)

where, with a little abuse of notation, $\tau_\gamma$ denotes the $\mathbb{Z}^2$-action described in Sect. 2.3, for both algebras.

**Lemma 3.2** Let $ev : A_{B_1} \to A_{B_2}$ be an evaluation homomorphism. Then

$$
ev|_{F_{B_1}} : F_{B_1} \longrightarrow F_{B_2}
$$

restricts to a surjective $C^*$-homomorphism.

**Proof** Let $F_{B_j}^0 \subset F_{B_j}$, $j = 1, 2$, be the dense subalgebra generated by the finite polynomials in the generators $\tau_\gamma(f_{B_j})$. From (3.1) it follows that $F_{B_2}^0 \subseteq \text{ev}(F_{B_1}^0) \subset F_{B_2}$. Since by assumption $ev$ is a $C^*$-homomorphism and $F_{B_1}^0$ is dense, one gets that

$$
F_{B_2}^0 \subseteq \text{ev}(F_{B_1}^0) \subset \text{ev}(F_{B_1}) \subseteq \overline{\text{ev}(F_{B_1})} \subseteq F_{B_2}.
$$

From the chain of inclusions above it follows that $ev(F_{B_1})$ is a $C^*$-subalgebra of $F_{B_2}$ [10, Proposition 2.3.1] which contains the dense set $F_{B_2}^0$. This implies that: (i) the restriction $ev|_{F_{B_1}}$ is well defined, and (ii) $ev(F_{B_1}) = F_{B_2}$, i.e. the surjectivity of the map. \(\square\)

Since $ev|_{F_{B_1}}$ is a well defined $C^*$-homomorphism between $F_{B_1}$ and $F_{B_2}$ it follows that $\text{Ker}(ev|_{F_{B_1}}) \subset F_{B_1}$ is a closed (two-sided) ideal.

**Definition 3.3 (Interface algebra)** Let $ev : A_{B_1} \to A_{B_2}$ be an evaluation homomorphism. The *interface algebra* $I \subset A_{B_1}$ is the closed two-sided ideal generated in $A_{B_1}$ by $\text{Ker}(ev|_{F_{B_1}})$.

In other words the $I$ coincides with the $C^*$-subalgebra of $A_{B_1}$ generated by elements of the type $aqb$ with $q \in \text{Ker}(ev|_{F_{B_1}})$ and $a, b \in A_{B_1}$. This justifies the following notation

$$
I := A_{B_1} \text{Ker}(ev|_{F_{B_1}}) A_{B_1}.
$$

Let $C(S^1)$ be the $C^*$-algebra of the continuous functions on the unit circle $S^1 \simeq \mathbb{R}/2\pi \mathbb{Z}$. Springer
Definition 3.4 (Localized and straight-line interfaces) Let $\mathcal{I}$ be the interface algebra associated to a given evaluation homomorphism. We will say that the interface is localized if $\mathcal{I} = \mathcal{K}(l^2(\mathbb{Z}^2))$. The interface will be called straight-line if $\mathcal{I} \simeq C(S^1) \otimes \mathcal{K}(l^2(\mathbb{Z}^2))$ up to a unitary transformation.

The motivation for the terminology introduced in Definition 3.4 will be clarified in part in the next example and in part in Sect. 4.1.

Example 3.5 (Interface algebra for a localized magnetic field) According to the notations introduced in Example 2.4, Example 2.7 and Example 2.10, let $\mathcal{A}_\Lambda$ be the magnetic algebra associated to a localized magnetic field $B_\Lambda$ and $\mathcal{A}_0$ be the algebra associated to a (constant) zero magnetic field. The map defined by $\text{ev}(\mathfrak{s}_{\Lambda, j}) = \mathfrak{s}_j$, where $\mathfrak{s}_j$ are the canonical shift operators, extends to a $C^*$-homomorphism $\text{ev} : \mathcal{A}_\Lambda \rightarrow \mathcal{A}_0$ (see the proof of Proposition 3.11). Therefore, from (3.1) one has that $\text{ev}(f_\Lambda) = 1$, and in turn

$$\text{ev}(p_\Lambda) = \text{ev} \left( (e^{ib} - 1)^{-1}(f_\Lambda - 1) \right) = 0, \quad b \neq 2\pi \mathbb{Z}.$$  

This shows that $p_\Lambda \in \mathcal{I}$ is an element of the interface algebra. Moreover, by acting with the magnetic translations $\mathfrak{s}_{\Lambda, j}$ one obtains that also $p_{\gamma + j} \in \mathcal{I}$ for every $\gamma \in \mathbb{Z}^2$. This fact is the key observation to conclude that $\mathcal{I} = \mathcal{K}(l^2(\mathbb{Z}^2))$. Let us start by the simplest case of a flux tube supported on a single site $\Lambda := \{\lambda_0\}$. In this case the latter observation implies that every rank-one projection $p_{\gamma + 1}$ is in $\mathcal{I}$ and this is enough to conclude that $\mathcal{I}$ is the $C^*$-algebra of compact operators (see [19, Appendix B] for the details). In the more general case $|\Lambda| > 1$ one can show that it is always possible to build a projector in $\mathcal{I}$ supported in a single point. Let $\lambda_0, \lambda \in \Lambda$ be two distinct points and $\gamma_0 := \lambda_0 - \lambda$. Then $\lambda_0 \in \gamma_0 + \Lambda$ and $p_\Lambda (1 - p_{\gamma_0 + \Lambda})$ is a projection in $\mathcal{I}$ which projects on a subset $\Lambda' \subset \Lambda$ where the strict inclusion is justified by the fact that $\lambda_0 \notin \Lambda'$. By iterating the procedure a sufficient number of times one ends with a projection on a singleton. In summary, a localized magnetic field always provides a localized interface in the sense of Definition 3.4.

3.2 Toeplitz Extensions by an Interface

Let $\mathcal{A}, \mathcal{B}$ and $\mathcal{C}$ three $C^*$-algebras fitting into the short exact sequence

$$0 \rightarrow \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \rightarrow 0. \quad (3.2)$$

In such a case we will say that $\mathcal{B}$ is the Toeplitz extension of $\mathcal{A}$ by $\mathcal{C}$. For a simple and complete review of the theory of extension of $C^*$-algebras we refer to [63, Chap. 3]. It is worth pointing out that we are proposing the use of the expression Toeplitz extension in a somehow (ultra) generalized sense. Indeed the original notion of Toeplitz extension refers to a very specific example of extension of $C^*$-algebras, see e.g. [44, Sect. 3.5] or [63, Exercise 3.6]. However, such a generalized use of the name Toeplitz extension is becoming common in condensed matter community (see the recent review [2] and references therein) and we decided to adopt it.

The main aim of this section is to show that an evaluation homomorphism automatically provides a Toeplitz extension.

Theorem 3.6 Every evaluation homomorphism $\text{ev} : \mathcal{A}_{\mathcal{A}_1} \rightarrow \mathcal{A}_{\mathcal{A}_2}$ fits into the short exact sequence

$$0 \rightarrow \mathcal{I} \xrightarrow{i} \mathcal{A}_{\mathcal{A}_1} \xrightarrow{\text{ev}} \mathcal{A}_{\mathcal{A}_2} \rightarrow 0. \quad (3.3)$$


where $\mathcal{I}$ is the related interface algebra and $\iota$ is the (natural) inclusion map.

**Proof** The map $\iota$ is injective by definition. Therefore, to complete the proof we need to prove that the evaluation homomorphism is surjective and that $\ker(\ev) = \mathcal{I}$. The surjectivity is a consequence of Lemma 3.2 which ensures $\ev(A_{AB_1}^0) = A_{AB_2}^0$ (or equivalently $\ev(A_{AB_1}^\infty) = A_{AB_2}^\infty$). Then, as in the proof of Lemma 3.2, the chain of inclusions

$$A_{AB_2}^0 = \ev(A_{AB_1}^0) \subseteq \ev(A_{AB_1}) \subseteq \ev(A_{AB_1}^\infty) \subseteq A_{AB_2}$$

implies $\ev(A_{AB_1}) = A_{AB_2}$. The description of the kernel of $\ev$ is a consequence of Corollary 2.15 which guarantees that $a \in \ker(\ev)$ if and only if all the $F_{B_2}$-coefficients of $\ev(a)$ are zero. From the definition (2.23), the linearity of the integral and the fact that the evaluation homomorphism $\ev$ commutes (by construction) with the family of automorphisms $\alpha_\theta$, it follows that $\ev(\hat{\alpha})_{r,s} = \ev(\tilde{\alpha}_{r,s})$. Then $a \in \ker(\ev)$ if and only if $\ev(\hat{\alpha}_{r,s}) = 0$ for all $(r, s) \in \mathbb{Z}^2$. This implies that $a \in \ker(\ev)$ if and only if $\sigma_N(a) \in \mathcal{I}$ for every $N \in \mathbb{N}$, where $\sigma_N(a)$ is the Cesàro mean (2.24) which converges to $a$. Since $\mathcal{I}$ is a closed ideal it follows that $a \in \ker(\ev)$ if and only if $a \in \mathcal{I}$.

By using the terminology introduced at the beginning of this section we will say that $A_{AB_1}$ is the Toeplitz extension of the interface $\mathcal{I}$ by the final (or bulk\(^6\)) algebra $A_{AB_2}$.

**Corollary 3.7** Let $1$ be the unit of $A_{AB_1}$. Then $1 \in \mathcal{I}$ if and only if $\ev = 0$.

**Proof** Since $\mathcal{I}$ is an ideal one has that $1 \in \mathcal{I}$ if and only if $\mathcal{I} = A_{AB_1}$.

**Example 3.8** (Localized interface and discrete spectrum) In the case of a localized interface $\mathcal{I} = \mathcal{K}(\ell^2(\mathbb{Z}^2))$ (like in Example 3.5) the short exact sequence (3.3) provides the isomorphism

$$A_{AB_2} \simeq A_{AB_1} / \mathcal{K}(\ell^2(\mathbb{Z}^2)).$$

This means that the elements of $A_{AB_1}$ are compact perturbations of elements of the (bulk) algebra $A_{AB_2}$. Since $\mathcal{K}(\mathcal{H})$ is an essential ideal in $\mathcal{B}(\mathcal{H})$ for any separable Hilbert space $\mathcal{H}$ (see e.g. [44, Example 3.1.2]), $\mathcal{K}(\ell^2(\mathbb{Z}^2))$ is an essential ideal of $A_{AB_1}$ and it follows that the short exact sequence (3.3) is essential [63, Definition 3.2.1]. The isomorphism above is useful to analyze the spectrum of elements $a \in A_{AB_1}$. In fact it holds true that the evaluation $\ev(a) \in A_{AB_2}$ contains the information about the essential spectrum $\sigma_{\text{ess}}(a)$ while the discrete spectrum $\sigma_d(a)$ is generated by the part of $a$ which belongs to the interface. Usually, the discrete spectrum of $a$ is located in the gaps of the spectrum of $\ev(a)$.

The short exact sequence (3.2) is called split exact if there exists a $C^*$-homomorphism (the lifting map) $\beta' : \mathcal{C} \to \mathcal{B}$ such that $\beta \circ \beta' = \text{Id}_\mathcal{C}$. In such a case both $\alpha(A)$ and $\beta'(C)$ are $C^*$-subalgebras of $\mathcal{B}$ and $\mathcal{B} = \alpha(A) \oplus \beta'(C)$ is the Banach space direct sum of these two $C^*$-subalgebras. It is worth pointing out that this is not the same of the direct sum of $C^*$-algebras. The latter condition is stronger and requires that also $\beta'(C)$ is an ideal in $\mathcal{B}$. When this extra condition holds true one has that $\mathcal{B} = \alpha(A) \oplus \beta'(C)$ is an orthogonal direct sum of $C^*$-algebras.

---

\(^6\)The use of this terminology will be clarified in Definition 3.13.
Example 3.9 (Toeplitz extension for a localized magnetic field) From Example 3.5 one infers that a localized magnetic field provides the Toeplitz extension
\[ 0 \rightarrow K(\ell^2(\mathbb{Z}^2)) \overset{j}{\rightarrow} A_{\Lambda} \overset{ev}{\rightarrow} A_0 \rightarrow 0. \] (3.4)

By adapting the argument in [19, Proposition 2], and up to gauge transformation, one can show that the differences \( s_{A,j} - s_j = (\eta_{A,j} - 1)s_j \) are compact operators, proving that \( K(\ell^2(\mathbb{Z}^2)) \subset A_{\Lambda} \). It follows that \( A_{\Lambda} \) coincides with the \( C^* \)-algebra generated by \( K(\ell^2(\mathbb{Z}^2)) \) and \( A_0 [19, \text{Theorem 11}] \) and, as a consequence, the Toeplitz extension (3.4) is split exact in view of the (identity) homomorphism \( j : A_0 \hookrightarrow A_{\Lambda} \) such that \( ev \circ j \) is the identity on \( A_{AB_2} \) and which provides a splitting of the linear space structure of \( A_{AB_2} \). This fact implies that \( A_{\Lambda} = K(\ell^2(\mathbb{Z}^2)) + A_0 \) as direct sum of Banach spaces.

Example 3.9 is somehow special since the Toeplitz extensions (3.3) considered in this work will be not split exact in general. Nevertheless, in many cases it is possible to show that there exists a linear map (not a \( C^* \)-homomorphism) \( j : A_{AB_2} \rightarrow A_{AB_1} \) such that \( ev \circ j \) is the identity on \( A_{AB_2} \) and which provides a splitting of the linear space structure of \( A_{AB_1} \). This is made possible by the fact that the magnetic algebras are generated by monomials in the magnetic translations. An extended discussion of this aspect can be found in [49, Sect. 3.2.2].

### 3.3 Evaluation Homomorphisms and Dynamics

In the previous section we described the consequences of having an evaluation homomorphism between two magnetic algebras. In this section we will analyze the relation between the existence of evaluation homomorphisms and the dynamical properties of the dynamical systems generated by the magnetic hulls. As a result we will provide a generalized definition of magnetic multi-interface based on purely dynamical properties of the magnetic hulls.

Let \((\Omega_{B_1}, \tau^*, \mathbb{Z}^2)\) and \((\Omega_{B_2}, \tau^*, \mathbb{Z}^2)\) be the two topological dynamical systems associated to the magnetic fields \( B_1 \) and \( B_2 \), respectively. An equivariant map from \( \Omega_{B_2} \) to \( \Omega_{B_1} \) is a continuous function \( \phi^* : \Omega_{B_2} \rightarrow \Omega_{B_1} \) such that
\[ \phi^* \circ \tau_{\gamma}^* = \tau_{\phi^*(\gamma)}^* \circ \phi^*, \quad \forall \gamma \in \mathbb{Z}^2. \]

**Proposition 3.10** Every evaluation homomorphism \( ev : A_{AB_1} \rightarrow A_{AB_2} \) defines an injective closed equivariant map \( \phi^* : \Omega_{B_2} \rightarrow \Omega_{B_1} \).

**Proof** Let us consider the Gelfand transforms \( \mathcal{G}_j : \mathcal{F}_{B_j} \rightarrow \mathcal{C}(\Omega_{B_j}) \), with \( j = 1, 2 \). The map \( \phi : \mathcal{C}(\Omega_{B_1}) \rightarrow \mathcal{C}(\Omega_{B_2}) \) defined by
\[ \phi := \mathcal{G}_2 \circ ev|_{\mathcal{F}_{B_1}} \circ \mathcal{G}_1^{-1} \]
is the composition of surjective \( C^* \)-homomorphisms, hence it is a surjective \( C^* \)-homomorphism. By duality, \( \phi \) induces a continuous map \( \phi^* : \Omega_{B_2} \rightarrow \Omega_{B_1} \) defined by
\[ \phi^*(\omega) := \omega \circ \phi. \]

Indeed, if \( \omega \in \Omega_2 \) is meant as a character of \( \mathcal{C}(\Omega_{B_2}) \), then \( \phi^*(\omega) \) is a character of \( \mathcal{C}(\Omega_{B_1}) \), hence a point of \( \Omega_1 \). The surjectivity of \( \phi \) implies the injectivity of \( \phi^* \). Indeed, \( \phi^*(\omega_1) = \phi^*(\omega_2) \) implies that \( \omega_1(\hat{g}) = \omega_2(\hat{g}) \) for all \( \hat{g} \in \mathcal{C}(\Omega_{B_2}) \) which is exactly \( \omega_1 = \omega_2 \). Finally \( \phi^* \) is closed in view of the Closed Map Lemma [39, Lemma 4.25] since \( \Omega_{B_1} \) and \( \Omega_{B_2} \) are both compact Hausdorff spaces. \( \square \)
Let us recall that a continuous closed injection between topological spaces is usually called a (topological) embedding. Let \( \phi^* : \Omega_{B_2} \hookrightarrow \Omega_{B_1} \) be the equivariant embedding of Proposition (3.10). The subset \( \Omega_s := \phi^*(\Omega_{B_2}) \) is evidently a closed invariant subset of \( \Omega_{B_1} \) and \( (\Omega_s, \tau^*, \mathbb{Z}^2) \) becomes a dynamical subsystem of \( (\Omega_{B_1}, \tau^*, \mathbb{Z}^2) \). Moreover

\[
\Omega_s = \phi^* \left( \text{Orb}(\omega_0) \right) = \text{Orb}(\omega_s)
\]

where \( \omega_s := \phi^*(\omega_0) \) and \( \omega_0 \in \Omega_{B_2} \) is the evaluation at 0. In conclusion, Proposition (3.10) states that every evaluation homomorphism identifies (up to isomorphisms) a dynamical subsystem of the initial magnetic hull. However, in view of Proposition 2.21, the only possibilities for a closed and invariant subset \( \Omega_s \) are \( \Omega_s = \Omega_{B_1} \) or \( \Omega_s \subseteq \partial \Omega_{B_1} = \Omega_{B_1} \setminus \text{Orb}(\omega_0') \) where we are denoting with \( \omega_0' \) the evaluation at 0 for \( \Omega_{B_1} \). The first case corresponds to \( \omega_0' \in \Omega_s \) while the second refers to \( \omega_0' \notin \Omega_s \). If \( \omega_0' \in \Omega_s \), then \( \phi^* \) is an isomorphism and, as a consequence of Proposition 3.10 and the short exact sequence of Theorem 3.6, this is equivalent to the isomorphism \( A_{A_{B_1}} \cong A_{A_{B_2}} \). This case will be called trivial as opposite to the non trivial case in which \( \phi^* \) defines a proper dynamical subsystem of the initial dynamical system. The next result provides a sort of inverse implication of Proposition (3.10).

**Proposition 3.11** Let \( A_{A_B} \) be a magnetic algebra and \( (\Omega_B, \tau^*, \mathbb{Z}^2) \) the topological dynamical system associated to its magnetic hull. Let \( \Omega_s \subseteq \partial \Omega_B \) be a proper invariant closed subset. Assume that \( \Omega_s = \text{Orb}(\omega_s) \) for some \( \omega_s \in \partial \Omega_B \). Then, there is a magnetic algebra \( A_{A_{B_2}} \) with magnetic hull \( \Omega_s \) and an evaluation homomorphism \( \text{ev} : A_{A_B} \to A_{A_{B_2}} \).

**Proof** Let \( \phi : \mathcal{C}(\Omega_B) \to \mathcal{C}(\Omega_s) \) be the surjective restriction \( C^* \)-homomorphism defined by \( \phi(g) := \hat{g}|_{\Omega_s} \) for all \( g \in \mathcal{C}(\Omega_B) \). Let \( f_B \) be the Gelfand transform of the generator \( f_B \) of \( \mathcal{C}(f_B, \mathbb{Z}^2) \) and define the function \( f_{B_2} : \mathbb{Z}^2 \to \mathbb{C} \) by

\[
f_{B_2}(\gamma) := \hat{f}_B(\tau_\gamma^*(\omega_s)) , \quad \gamma \in \mathbb{Z}^2.
\]

By the Gelfand isomorphism one obtains that \( \mathcal{C}(f_{B_2}, \mathbb{Z}^2) \cong \mathcal{C}(\Omega_s) \). The function \( f_{B_2} \) provides a magnetic flux with an associated (non unique) magnetic field \( B_2 : \mathbb{Z}^2 \to \mathbb{R} \). Let \( A_{B_2} \) be a suitable vector potential for \( B_2 \) and \( A_{A_{B_2}} \) the associated magnetic algebra. Let \( F_{B_2} \subset A_{A_{B_2}} \) be the abelian subalgebra generated by the magnetic flux \( \hat{f}_{B_2} := m_{f_{B_2}} \). The surjective \( C^* \)-homomorphism \( \phi \) and the Gelfand isomorphism provide a surjective \( C^* \)-homomorphism \( \hat{\text{ev}} : F_B \to F_{B_2} \) characterized by \( \hat{\text{ev}}(f_B) = \hat{f}_{B_2} \). It turns out that the map \( \text{ev} : A_{A_B} \to A_{A_{B_2}} \) defined by

\[
\text{ev} \left( \sum_{(r,s) \in \mathbb{Z}^2} g_{r,s} (s_{A_{B_1}})^r (s_{A_{B_2}})^s \right) = \sum_{(r,s) \in \mathbb{Z}^2} \hat{\text{ev}}(g_{r,s}) (s_{A_{B_2}})^r (s_{A_{B_2}})^s
\]

is a \(*\)-homomorphism of pre-\( C^* \)-algebras (Proposition 2.20). Therefore, the claim follows from [22, Lemma 3.41]. \( \square \)

**Remark 3.12** (Non-uniqueness of the magnetic field) The magnetic algebra \( A_{A_{B_2}} \) which enters in Proposition 3.11 is not unique for two reasons. First of all \( A_{A_{B_2}} \) depends on the election of a vector potential \( A_{B_2} \) for the magnetic field \( B_2 \) and this involves the election of gauge. However, magnetic algebras related to different gauges are unitarily equivalent as discussed in Sect. 2.3. The second source of ambiguity is more subtle and is related with
the determination of the magnetic field $B_s$ from the magnetic flux $f_{B_s}$. Indeed, the natural candidate would be $B_s = -i \log(f_{B_s})$ but the logarithm is not univocally defined in the complex plane. In particular, given a magnetic field $B_s$ compatible with the magnetic flux $f_{B_s}$ and a (not necessarily bounded) function $\zeta: \mathbb{Z}^2 \to \mathbb{Z}$ one gets that $B'_s := B_s + 2\pi \zeta$ provides the same magnetic flux. A way to solve this ambiguity is to fix the convention that $B_s := \text{Arg}(f_{B_s}) \in [0, 2\pi)$ give the principal argument of the flux $f_{B_s}$. This correspond to a sort of minimal growth assumption for the magnetic field at infinity and we will use this convention in the rest of this work.

We are now in position to introduce a key definition for this work.

**Definition 3.13 (Magnetic multi-interface)** A system subjected to a magnetic field $B: \mathbb{Z}^2 \to [0, 2\pi)$ and with the boundary of the magnetic hull given by a finite collection of invariant points

$$\partial \Omega_B = \{\omega_{s,1}, \ldots, \omega_{s,N+1}\}$$

will be called a $N$-interface magnetic system. In this one can introduce the Toeplitz extension given by

$$0 \longrightarrow \mathcal{I} \xrightarrow{\iota} \mathcal{A}_{A_B} \xrightarrow{\text{ev}} \mathcal{A}_{\text{bulk}} \longrightarrow 0 \quad (3.5)$$

where $\mathcal{A}_{A_B}$ is any magnetic algebra associated to the magnetic field $B$, the map $\text{ev}$ is the direct sum of $N+1$ evaluation maps associated to each $\omega_{s,j}$ for $j = 1, \ldots, N+1$ as in 3.11, and the bulk algebra is then defined by

$$\mathcal{A}_{\text{bulk}} := \mathcal{A}_{b_1} \oplus \cdots \oplus \mathcal{A}_{b_{N+1}}, \quad (3.6)$$

where $\mathcal{A}_{b_j}$ is constructed as in Example 2.10 with respect to the constant magnetic field of strength $b_j := \text{Arg}(\hat{f}_B(\omega_{s,j}))$ for every $j = 1, \ldots, N+1$ and $\hat{f}_B$ is the Gelfand transform of the flux function $f_B$ as described in the proof of Proposition 3.11. Finally the evaluation map and the interface algebra $\mathcal{I}$ are completely specified by

$$\text{ev}(\hat{f}_B) := (e^{ib_1} \mathbf{1}, \ldots, e^{ib_{N+1}} \mathbf{1})$$

as discussed in Sect. 3.1.

As showed in Example 2.25 and Example 3.5, a localized magnetic field provides an example of a magnetic interface with order $N = 0$. On the other hand Example 2.24 shows that the Iwatsuka magnetic field provides an example of magnetic interface of order $N = 1$. The case of the Iwatsuka magnetic field will be discussed extensively in Sect. 4.

**Remark 3.14 (Ergodic traces for the Iwatsuka algebra)** Let $\{\omega_s\} \subset \partial \Omega_B$ an invariant singleton of the magnetic hull $\Omega_B$, $\mathcal{P}_s$ the ergodic measure supported on it and $\mathcal{F}_{\mathcal{P}_s}$ the associated ergodic trace as defined by (2.35). Let $b_s := \text{Arg}(\hat{f}_B(\omega_s))$ be the associated constant magnetic field and $\text{ev}_s : \mathcal{A}_{A_B} \to \mathcal{A}_{b_s}$ the evaluation map constructed in Proposition 3.11. A straightforward check shows that

$$\iota \left( \tau_{-\gamma}(\mathbf{a})_{0,0} \right)(\omega_s) = \iota \left( \tau_{-\gamma}(\text{ev}_s(\mathbf{a}))_{0,0} \right)(\omega'_0), \quad \mathbf{a} \in \mathcal{A}_{A_B}$$
where \( \{ \omega'_0 \} = \Omega_{b_1} \) is the singleton which defines the magnetic hull of the constant magnetic field algebra (see Example 2.23). As a consequence, one can use the argument in Proposition 2.28 to represent \( \mathcal{T}_{P_*} \) as a trace per unit volume, i.e.

\[
\mathcal{T}_{P_*}(a) := \lim_{i \to \infty} \frac{1}{|\Lambda_i|} \text{Tr}_{\mathbb{C} \times \mathbb{Z}^2}(p_{\Lambda_i} \sigma_{\mathbb{C} \times \mathbb{Z}^2}(a) p_{\Lambda_i}), \quad \forall \ a \in A_{AB}.
\]

In the special case of the Iwatsuka algebra equation (3.7) provides an explicit prescription for the computation of the traces \( \mathcal{T}_{P_{\pm \infty}} \) associated to the two ergodic measures \( P_{\pm \infty} \) described in Example 2.24.

### 3.4 The K-Theory of Magnetic Interfaces

In this section we will discuss some aspects of the K-theory of magnetic interfaces. There is a large literature concerning the K-theory for \( C^* \)-algebras. We will refer to the classic monographs [9, 22, 44, 63] as well as [49] for a stronger connection with condensed matter problems.

Let us recall that for each Toeplitz extension of type (3.2) there is an associated six-term sequence in K-theory [63, Theorem 9.3.2]. Therefore, there is a six-term sequence for every magnetic Toeplitz extension of type (3.3) or (3.5). We will focus here on the latter case concerning a magnetic multi-interface.

From the exact sequence (3.5) one obtains the six-term sequence

\[
\begin{array}{cccccc}
K_0(\mathcal{I}) & \xrightarrow{i_*} & K_0(A_{AB}) & \xrightarrow{\text{ev}_*} & K_0(A_{\text{bulk}}) \\
\uparrow\text{ind} & & \downarrow\text{exp} & & \\
K_1(A_{\text{bulk}}) & \xleftarrow{\text{ev}_*} & K_1(A_{AB}) & \xleftarrow{i_*} & K_1(\mathcal{I})
\end{array}
\]

where the canonical maps ind and exp are called index map and exponential map respectively. The role of the six-term sequence (3.8) is twofold: first of all it allows to reconstruct the K-theory of \( A_{AB} \) from the knowledge of the K-theory of \( \mathcal{I} \) and \( A_{\text{bulk}} \); secondly it defines how the K-theory of \( A_{AB} \) intertwines the K-theories of \( \mathcal{I} \) and \( A_{\text{bulk}} \) through the maps ind and exp. The latter aspect is the core of the topological interpretation of the bulk-boundary correspondence in condensed matter [49].

The K-theory of the bulk algebra \( A_{\text{bulk}} \) can be easily computed since \( A_{\text{bulk}} \) is an orthogonal direct sum of noncommutative tori (cf. Example 2.10) and the K-theory of the noncommutative torus is well-known (cf. Appendix D). In particular one gets

\[
K_0(A_{\text{bulk}}) = \bigoplus_{j=1}^{N+1} K_0(A_{b_j}) \cong \bigoplus_{j=1}^{N+1} \mathbb{Z}^2,
\]

\[
K_1(A_{\text{bulk}}) = \bigoplus_{j=1}^{N+1} K_1(A_{b_j}) \cong \bigoplus_{j=1}^{N+1} \mathbb{Z}^2.
\]

The K-theory of the interface algebra requires a preliminary observation. In fact, by Corollary 3.7, when ev is not trivial then \( \mathcal{I} \) does not contain the unit 1 of \( A_{AB} \) (although it can be unital with a different unit). In such a case, \( K_0(\mathcal{I}) \) is meant as the kernel of the
induced map $K_0(I^+) \to K_0(\mathbb{C})$, while $K_1(I) = K_1(I^+)$ [63, Sect. 6.2 & Example 7.1.11 (5)]. The main case of interest for this work is when there exists a unitary equivalence

$$I \cong I_0 \otimes \mathcal{K}(\mathcal{H}_{\text{red}})$$

(3.9)

where $I_0$ is a unital and abelian $C^*$-algebra and $\mathcal{K}(\mathcal{H}_{\text{red}})$ is the $C^*$-algebra of compact operators on the (reduced) separable Hilbert space $\mathcal{H}_{\text{red}}$. In such case one has

$$K_j(I) \cong K_j(I_0), \quad j = 0, 1,$$

because of the stability property of $K$-theory [63, Corollary 6.2.11 & Corollary 7.1.9].

The ansatz (3.9) imposes a quite strong condition on the geometry of the interface. To handle more general geometries like corners, the ansatz (3.9) must be modified in the form discussed in [59].

**Example 3.15 (six-term sequence for a localized magnetic field)** The six-term sequence associated to the Toeplitz extension (3.4) for a localized magnetic field can be easily computed by observing that in this case the interface algebra has the form $I \cong \mathbb{C} \otimes \mathcal{K}$ (cf. Example 3.5) and in turn its $K$-theory is given by

$$K_0(I) \cong \mathbb{Z}, \quad K_1(I) \cong \mathbb{Z},$$

Moreover, with the same argument used in the proof of [19, Theorem 12] one gets $K_0(A_\lambda) = \mathbb{Z}^3$ and $K_1(A_\lambda) = \mathbb{Z}^2$.

The case of straight-line interface (Definition 3.4) will be relevant in Sect. 4. Its $K$-theory is described below.

**Proposition 3.16 ($K$-theory for the straight-line interface)** In the case of a straight-line interface $I \cong C(S^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$ the $K$-theory is given by

$$K_0(I) \cong \mathbb{Z}, \quad K_1(I) \cong \mathbb{Z}.$$

**Proof** The result follows from the stability property of $K$-theory along with $K_0(C(S^1)) \cong \mathbb{Z}[1]$ and $K_1(C(S^1)) \cong \mathbb{Z}[u]$ where $u(k) = e^{ik}$ [63, Sect. 6.5].

### 3.5 Bulk and Interface Currents

Let $A_{AB}$ be a magnetic algebra endowed with the trace $\mathcal{T}_P$ associated to an ergodic measure $P \in \text{Erg}(\Omega_{AB})$ as discussed in Sect. 2.6. Given a differentiable projection $p \in C^1(A_{AB})$, the (generalized) transverse Hall conductance associated to $p$ is defined by

$$\sigma_{B,P}(p) := \frac{e^2}{\hbar} \text{Ch}_{B,P}(p)$$

(3.10)

where $e$ is the electron charge, $\hbar = 2\pi\hbar$ is the Planck’s constant and the dimensionless part, known as *Chern number*, is given by

$$\text{Ch}_{B,P}(p) := i2\pi \mathcal{T}_P(p[\nabla_1(p), \nabla_2(p)])$$

(3.11)

The projection $p$ is usually obtained as the spectral projection into a gap of a self-adjoint element (Hamiltonian) of $A_{AB}$ and represents the ground state of the system as described...
by the Fermi-Dirac distribution in the limit of the temperature \( T = 0 \) and chemical potential

(Fermi energy) sited into the gap. The quantity (3.10) enters in the (microscopic) Ohm’s law

\[
J_\perp = \sigma_{B, \mathcal{F}}(p) \ E
\]  

which describes the transverse current density \( J_\perp \) generated in the material as a response to the external electric perturbation \( E \). The expression (3.12) is usually known as \textit{Kubo’s formula} and is obtained in the linear response approximation. There are countless derivations of the Kubo’s formula (3.12) in the literature. For our aims we will refer to [7, 57] for the case of a constant magnetic field and to [18] for more general cases.

In the case of a constant magnetic field \( B \) of strength \( b \) there is a unique ergodic measure (\textit{cf. Example 2.23}) and the associated trace, simply denoted with \( \mathcal{F} \), is given by the trace per unit volume as proved in Proposition 2.28. Therefore, it is appropriate to rewrite equations (3.10) and (3.11) with the lighter notation

\[
\sigma_b(p) = \frac{e^2}{h} \text{Ch}_b(p) \tag{3.13}
\]

In particular, the map \( \text{Ch}_b \) can be obtained from the trilinear map \( \xi_b : C^1(A_b)^{\times 3} \rightarrow \mathbb{C} \), defined by

\[
\xi_b(a_0, a_1, a_2) := i2\pi \mathcal{F}(a_0(\nabla_1(a_1)\nabla_2(a_2) - \nabla_2(a_1)\nabla_1(a_2))), \tag{3.14}
\]

according to \( \text{Ch}_b(p) = \xi_b(p, p, p) \). Formula (3.14) is crucial in the study of the topology of the algebra \( A_b \) (which coincides with the noncommutative torus). In fact, as discussed in [14, Chap. 3], [22, Chap. 12] or [49, Chap. 5] among others, it turns out that the map \( \xi_b \) is a cyclic 2-cocycle of the \( C^* \)-algebra \( A_b \) and therefore defines a class \( [\xi_b] \) in the odd cyclic cohomology of (a dense subalgebra of) \( A_b \). Moreover, \( \text{Ch}_b(p) \) provides the values of the canonical bilinear pairing between the class \([p] \in K_0(A_b)\) of \( p \) and the class \([\xi_b] \). This fact is represented in the formula

\[
\text{Ch}_b(p) = \langle [p], [\xi_b] \rangle \in \mathbb{Z} \tag{3.15}
\]

which conveys the information that the value of \( \text{Ch}_b(p) \) only depends on the class \([p]\). The integrality of \( \text{Ch}_b(p) \) is the celebrated \textit{index theorem} for the even \( K \)-theory [14, Sect. 3.3, Corollary 16]. Equation (3.15) along with (3.10) provides the quantization (in units of \( e^2h^{-1} \)) of the transverse Hall conductance for a constant magnetic field [7, 60].

The conductance for the bulk algebra 3.6 can be defined (by linearity) from the case of a constant magnetic field.

\textbf{Definition 3.17 (Bulk transverse conductance)} Let \( A_{\text{bulk}} \) be the bulk algebra defined in equation (3.6) and \( p := (p_1, \ldots, p_{N+1}) \) a projection in \( C^1(A_{\text{bulk}}) \). The \textit{bulk transverse conductance} for the projection \( p \) is given by the collection

\[
\sigma_{\text{bulk}}(p) := \{\sigma_{b_1}(p_1), \ldots, \sigma_{b_{N+1}}(p_{N+1})\}
\]

where every \( \sigma_{b_j}(p_j) \) is defined by (3.13).

Let us now consider the current associated with the interface algebra \( \mathcal{I} \). We will focus on the case described by the ansatz (3.9) which postulates the existence of the unitary equivalence \( \varpi \) between \( \mathcal{I} \) and \( \mathcal{I}_0 \otimes K(H_{\text{red}}) \), and we will assume that the unital and abelian
$C^*$-algebra $I_0$ is endowed with a faithful (normalized) trace $\tau_0$ and a suitable (unbounded) derivation $\delta_0$ which meet the compatibility condition $\tau_0 \circ \delta_0 = 0$. In this circumstance one can define a faithful lower-semicontinuous trace $\mathcal{T}_I$ on $I$ through the prescription

$$\mathcal{T}_I(a) := \tau_0 \otimes \text{Tr}_{\mathcal{H}_{\text{red}}} (\sigma(a)) , \quad a \in \mathcal{D}_I$$

where the ideal $\mathcal{D}_I \subset I$ is defined by $\mathcal{D}_I := \sigma^{-1} (\mathcal{I}_0 \otimes \mathcal{L}^1 (\mathcal{H}_{\text{red}}))$ and $\mathcal{L}^1 (\mathcal{H}_{\text{red}})$ is the ideal of trace class operators on $\mathcal{H}_{\text{red}}$. Similarly, one can endow $I$ with the derivation $\nabla_I$ given by

$$\nabla_I(a) := \delta_0 \otimes \text{Id}_\mathcal{K} (\sigma(a)) , \quad a \in \mathcal{C}_I^1$$

where $\mathcal{C}_I^k := \sigma^{-1} (\mathcal{C}_k (\mathcal{I}_0) \otimes \mathcal{K}(\mathcal{H}_{\text{red}}))$ for every $k \in \mathbb{N}$. Therefore, such a derivation can be extended to the unitalization $I^+$ by the prescription $\nabla_I(1) = 0$. With these structures one can define the map

$$W_I(u) := i \mathcal{T}_I ((u^* - 1) \nabla_I (u - 1)) = i \mathcal{T}_I (u^* \nabla_I (u)) \quad (3.16)$$

for every unitary operator $u \in I^+$ such that $u - 1 \in \mathcal{C}_I^1 \cap \mathcal{D}_I$. The map $W_I$ is known as the (non-commutative) winding number of $u$.

**Example 3.18 (Triviality of the winding number in the localized case)** According to Example (3.5) the structure of the interface algebra in the case of a localized magnetic field is given by $I \simeq \mathbb{C} \otimes \mathcal{K} (\ell^2 (\mathbb{Z}^2))$ and therefore it satisfies the ansatz (3.9). However, in view of the simple structure of $I_0 = \mathbb{C}$ one has that the only faithful (normalized) trace $\tau_0$ is the identity $\tau_0(a) = a$ and the only derivation $\delta_0$ is the null-map $\delta_0(a) = 0$ for all $a \in \mathbb{C}$. As a consequence the associated trace on $I$ coincides with the canonical trace of the Hilbert space $\ell^2 (\mathbb{Z}^2)$, while there is no non-trivial derivation compatible with the ansatz (3.9). In view of that one has that $W_I = 0$ identically in the case of a localized magnetic field.

**Definition 3.19 (Interface conductance)** Let $I$ be an interface algebra of type 3.9 endowed with the derivation $\nabla_I$ and the trace $\mathcal{T}_I$. Let $u \in I^+$ be a unitary operator such that $u - 1 \in \mathcal{C}_I^1 \cap \mathcal{D}_I$. The interface conductance associated to $u$ is defined by

$$\sigma_I(u) := \frac{e^2}{\hbar} W_I(u) . \quad (3.17)$$

The terminology used in Definition 3.19 is justified by the fact that from a physical point of view $\sigma_I(u)$ provides the proportionality coefficient for the current that flows along the interface when the system is prepared in a specific configuration described by $u$. (cf. [58] or [49, Sect. 7.1]). To clarify this important point we need some intermediate concepts.

Let us call magnetic Hamiltonians the self-adjoint elements of $\mathcal{A}_{AB}$. Let $\hat{h} \in \mathcal{A}_{AB}$ be a magnetic Hamiltonian and $\hbar := ev(\hat{h}) \in \mathcal{A}_{\text{bulk}}$ its image in the bulk algebra. By construction the bulk Hamiltonian $h = (h_1, \ldots, h_{N+1})$ is made by a $N + 1$-upla of suitable self-adjoint elements of the constant magnetic field algebras $\mathcal{A}_{B_j}$ and its spectrum is given by $\text{Spec}(h) = \bigcup_{j=1}^{N+1} \text{Spec}(h_j)$.

**Definition 3.20 (Non-trivial bulk gap)** The magnetic Hamiltonian $\hat{h} \in \mathcal{A}_{AB}$ has a non-trivial bulk gap if there is a compact set $\Delta \in \mathbb{R}$ such that

$$\min \text{Spec}(h) < \min \Delta < \max \Delta < \max \text{Spec}(h)$$
and $\Delta \cap \text{Spec}(\mathfrak{h}) = \emptyset$.

According to the above definition $\Delta$ lies inside a non-trivial spectral gap of the bulk Hamiltonian $\mathfrak{h}$ and for every chemical potential $\mu \in \Delta$ the Fermi projection

$$p_\mu = (p_{\mu,1}, \ldots, p_{\mu,N+1}) \in \mathcal{A}_{\text{bulk}} , \quad p_{\mu,j} := \chi_{(-\infty,\mu)}(\mathfrak{h}_j) \in \mathcal{A}_{\mathfrak{h}_j}$$

is an element of the bulk algebra. If the magnetic Hamiltonian is smooth, i.e. $\hat{\mathfrak{h}} \in \mathcal{A}_{\text{bulk}}^{\infty}$, then also $\mathfrak{h} \in \mathcal{A}_{\mathfrak{h}}^{\infty}$ (the evaluation map preserves the regularity), and in turn $p_\mu \in \mathcal{A}_{\text{bulk}}^{\infty}$ since $\mathcal{A}_{\text{bulk}}^{\infty}$ is closed under holomorphic functional calculus. Let $[p_\mu] = [(p_{\mu,1}, \ldots, p_{\mu,N+1})] \in K_0(\mathcal{A}_{\text{bulk}})$ be the class of the Fermi projection in the $K_0$-group of $\mathcal{A}_{\text{bulk}}$. As a first step let us point out that one can compute the image of $[p_\mu]$ inside $K_1(\mathcal{I})$ under the exponential map. Indeed, using the exact same proof as in [49, Proposition 4.3.1], and the additive notation for the $K_1$-group, one gets the next result.

**Proposition 3.21** Assume that the magnetic Hamiltonian $\hat{\mathfrak{h}} \in \mathcal{A}_{\mathfrak{h}}$ has a non-trivial bulk gap detected by $\Delta$. Let $g : \mathbb{R} \to [0,1]$ be a non-decreasing (smooth) function such that $g = 0$ below $\Delta$ and $g = 1$ above $\Delta$ and consider the unitary operator

$$u_\Delta := e^{i2\pi g(\hat{\mathfrak{h}})} . \quad (3.18)$$

Then $u_\Delta \in \mathcal{I}^+$ and

$$\exp([p_\mu]) = -[u_\Delta] \in K_1(\mathcal{I}) .$$

In the case $\hat{\mathfrak{h}} \in \mathcal{A}_{\mathfrak{h}}^{\infty}$ it follows from the construction that $u_\Delta \in \mathcal{I}^+ \cap \mathcal{A}_{\mathfrak{h}}^{\infty}$ acquires the same regularity. It is worth noting that the element $1 - u_\Delta$ can be constructed entirely from the spectral subspace of $\hat{\mathfrak{h}}$ corresponding to the bulk insulating gap $\Delta$. Indeed, the support of the function $e^{i2\pi g} - 1$ is contained inside the region $\Delta$ which lies in the insulating gap.

**Remark 3.22** (Gap closing as a topological obstruction) The condition $[u_\Delta] \neq 0$ (cf. Note 7) measures the obstruction to lift the Fermi projection $p_\mu \in \mathcal{A}_{\text{bulk}}$ to a true projection in $\mathcal{A}_{\mathfrak{h}} \otimes \text{Mat}_N(\mathbb{C})$ (for some $N$ large enough). From the construction emerges that this obstruction detects the presence of spectrum of $\hat{\mathfrak{h}}$ inside $\Delta$ which is generated by the existence of the magnetic interface. Since the election of $\Delta$ inside the bulk gap is totally arbitrary, and the Fermi projection does not depend on the specific $\mu$ inside the bulk gap, one gets that for any given $\Delta$ the related element $1 - g(\hat{\mathfrak{h}})$ is a self-adjoint lift of the Fermi projection. This implies immediately that the condition $[u_\Delta] \neq 0$ guarantees the complete closure of the bulk gap due to the presence of the magnetic interface.

Let $g$ as in the claim of Proposition 3.21. The derivative $g'$ is non-negative, supported in $\Delta$ and normalized in the sense that $\|g'\|_{L^1} = 1$. By construction the element $g'(\hat{\mathfrak{h}})$ satisfies the condition $\text{ev}(g'(\hat{\mathfrak{h}})) = g'(\text{ev}(\hat{\mathfrak{h}})) = 0$ and so $g'(\hat{\mathfrak{h}}) \in \mathcal{I}$ is an element of the interface algebra. Moreover $g'(\hat{\mathfrak{h}})$ can be regarded as a density matrix which describes a state of the system with energy distributed in the region $\Delta$. Now, let us assume that the derivation $\nabla_Z$ can be extended to class of sufficiently regular elements of $\mathcal{A}_{\mathfrak{h}}$. For instance, in many physical

\(^7\)In terms of the additive notation of $K_1(\mathcal{I})$, the trivial element is $[1] = 0$ and $-[u] = [u^*]$ denotes the inverse of $[u]$.
situations the derivation \( \nabla_\mathcal{I} \) can be identified with a suitable function of the derivations \( \nabla_1, \nabla_2 \) of \( \mathcal{A}_{A_B} \) and in such a case \( \nabla_\mathcal{I}(\hat{h}) \) makes sense whenever \( \hat{h} \in C^1(\mathcal{A}_{A_B}) \). The Iwatsuka interface provides a very simple example of this situation where \( \nabla_\mathcal{I} \) can be identified with \( \nabla_2 \). The same happens in the standard scenario of the bulk-edge correspondence [49]. A more complicated situation where a suitable identification of \( \nabla_\mathcal{I} \) requires both \( \nabla_1, \nabla_2 \) is described in [59]. Assuming that \( \nabla_\mathcal{I}(\hat{h}) \) makes sense for a suitable \( \hat{h} \in \mathcal{A}_{A_B} \), then \( \hat{h}^{-1}\nabla_\mathcal{I}(\hat{h}) \) can be physically interpreted as the velocity operator along the interface, one in turn

\[
J_\mathcal{I}(\Delta) := -\frac{e}{\hbar} \beta_\mathcal{I} \left( g'(\hat{h}) \nabla_\mathcal{I}(\hat{h}) \right) \tag{3.19}
\]

represents the current density along the interface carried by the “extended states” in \( \Delta \) and, as a consequence, \( \sigma_\mathcal{I} = eJ_\mathcal{I} \) provides the associated conductance (we are assuming that \( e > 0 \) is the magnitude of the electron charge). The connection between the latter formula and Definition 3.19 is provided by the following result originally proved in [58].

**Proposition 3.23**  Let \( \hat{h} \in \mathcal{A}_{A_B} \) such that \( \nabla_\mathcal{I}(\hat{h}) \) is well defined (in the sense discussed above) and \( g'(\hat{h})\nabla_\mathcal{I}(\hat{h}) \in \mathcal{D}_\mathcal{I} \). Then, it holds true that

\[
\beta_\mathcal{I} \left( g'(\hat{h}) \nabla_\mathcal{I}(\hat{h}) \right) = -\frac{1}{2\pi} W_\mathcal{I}(u_\Delta) .
\]

**Proof** The result can be obtained by adapting step by step the proof of [49, Proposition 7.1.2]. Indeed the proof is purely algebraic and only uses the properties of the trace \( \beta_\mathcal{I} \) and the derivation \( \nabla_\mathcal{I} \) assumed by hypothesis at the beginning of this section. \( \square \)

By combining definition 3.19 (which is motivated by physics) with Proposition 3.23 one gets that the interface conductance generated by the “extended states” in \( \Delta \) is given by

\[
\sigma_\mathcal{I}(\Delta) := \frac{e^2}{\hbar} W_\mathcal{I}(u_\Delta) . \tag{3.20}
\]

This equation justifies the “abstract” Definition 3.19.

The relevance of Definition 3.19 lies in its topological interpretation. Consider the map \( \eta_\mathcal{I} : (C^1(\mathcal{I}) \cap \mathcal{D}_\mathcal{I}) \times ^2 \rightarrow \mathbb{C} \), defined by

\[
\eta_\mathcal{I}(b_0, b_1) := i \beta_\mathcal{I} \left( b_0 \nabla_\mathcal{I}(b_1) \right) . \tag{3.21}
\]

In view of the properties of \( \beta_\mathcal{I} \) and \( \nabla_\mathcal{I} \) assumed by hypothesis, \( \eta_\mathcal{I} \) turns out to be a cyclic 1-cocycle and therefore defines a class \([\eta_\mathcal{I}]\) in the odd cyclic cohomology of (a dense subalgebra of) \( \mathcal{I} \) [14, Chap. 3]. Moreover, \( W_\mathcal{I}(u) \) provides the values of the canonical bilinear pairing between the class \([u] \in K_1(\mathcal{I})\) of the unitary \( u \in \mathcal{I}^+ \) and the class \([\eta_\mathcal{I}]\). This fact is represented in the formula

\[
W_\mathcal{I}(u) = \langle [u], [\eta_\mathcal{I}] \rangle \tag{3.22}
\]

which conveys the information that the value of \( W_\mathcal{I}(u) \) only depends on the class \([u]\). In particular, by combining together Proposition 3.21 and equation (3.22) one gets

\[
\sigma_\mathcal{I}(\Delta) := \frac{e^2}{h} \langle [u_\Delta], [\eta_\mathcal{I}] \rangle = -\frac{e^2}{h} \left( \exp((p_\mu)), [\eta_\mathcal{I}] \right) . \tag{3.23}
\]

Equation (3.23) is the topological essence of the bulk-interface duality and will be used in Sects. 4.6 to prove equation (1.1) in the case of the Iwatsuka magnetic field (cf. Theorem 4.19).
4 The Iwatsuka C*-Algebra

This section is devoted to the detailed study of the Toeplitz extension and the $K$-theory of the Iwatsuka C*-algebra. The magnetic translations associated to the Iwatsuka magnetic field has been described in Example 2.6 and the Iwatsuka C*-algebra has been defined in Example 2.10.

4.1 Toeplitz Extension for the Iwatsuka Magnetic Field

The simplest examples of a magnetic multi-interface system as described in Definition 3.13 is provided by the Iwatsuka magnetic $B_I$ defined by (2.4) with the conditions

\[ b_- - b_+ \not\in 2\pi \mathbb{Z}. \]  (4.1)

In fact, according to the content of Example 2.24 one has that the boundary of the Iwatsuka magnetic hull $\Omega_I$ can be represented as $\partial \Omega_I = \{\omega_{-\infty}, \omega_{+\infty}\}$ with $\omega_{\pm\infty}$ two distinct invariant points. As a consequence the associated Toeplitz extension is given by

\[ 0 \rightarrow \mathcal{I} \xrightarrow{i} \mathcal{A}_I \xrightarrow{ev} \mathcal{A}_{\text{bulk}} \rightarrow 0 \]  (4.2)

with bulk algebra given by

\[ \mathcal{A}_{\text{bulk}} := \mathcal{A}_{b_-} \oplus \mathcal{A}_{b_+} \]  (4.3)

and evaluation map defined by

\[ ev(s_{I,1}) := (s_{b_-}, s_{b_+}) \]
\[ ev(s_{I,2}) := (s_{b_-}, s_{b_+}) \]  (4.4)
\[ ev(f_I) := (e^{ib_-} - 1, e^{ib_+} - 1) \]

where $s_{I,1}$ and $s_{I,2}$ are the Iwatsuka magnetic translations and $f_I$ is the associated flux operator as defined in Example 2.6.

Remark 4.1 (Interpretation of the evaluation map) It is worth interpreting the action of the evaluation map on the commutative subalgebra $F_I$ generated by $f_I$ (cf. Equation (2.13)) as a generalized limit. Let $g \in F_I$ and $g \in C(\Omega_I)$ its Gelfand transform as a continuous function on the hull $\Omega_I$. According to the discussion in Example 2.24, $C(\Omega_I)$ coincides with the $C^*$-subalgebra of $C_b(\mathbb{Z}^2)$ of sequences that admit left and right limits. Then, it follows that

\[ ev(g) = \left( \lim_{s \to -\infty} g(a, s), \lim_{s \to +\infty} g(b, s) \right) \]  (4.5)

for every $a, b \in \mathbb{Z}$ and $g(n) := g(\omega_n)$ for every $n \in \mathbb{Z}^2$. It is immediate to check that equation (4.5) holds true on the dense subalgebra of $F_I$ generated by finite linear combinations of elements of the type

\[ (s_{I,1})^r (s_{I,2})^s f_I (s_{I,1})^{-r} (s_{I,1})^{-s} = (s_{I,1})^r f_I (s_{I,1})^{-r}, \]

and the result follows from a standard density argument.
4.2 Interface Algebra for the Iwatsuka Magnetic Field

The Iwatsuka $C^*$-algebra $\mathcal{A}_I$ contains several interesting projections. Let us introduce the projections

\[
(p_{\pm} \psi)(n) := \delta_{\pm}(n) \psi(n), \quad \psi \in \ell^2(\mathbb{Z}^2)
\]

where the functions $\delta_{\pm}$ and $\delta_0$ are defined in Example 2.3.

**Lemma 4.2** Under the assumption (4.1) the projections $p_{\pm}$ and $p_0$ are elements of $\mathcal{A}_I$.

**Proof** The identity $1$ and the flux operator $f_I$ defined by (2.11) are elements of $\mathcal{A}_I$. Let us start with the case $b_0 \neq b_+$ and $b_0 \neq b_-$. A straightforward computation shows that

\[
p_0 = \left( e^{i b_-} - e^{i b_0} \right)^{-1} \left( e^{i b_+} - e^{i b_0} \right)^{-1} \left( e^{i b_-} 1 - f_I \right) \left( e^{i b_+} 1 - f_I \right),
\]

hence $p_0 \in \mathcal{A}_I$. Similarly, one can check that

\[
p_{\pm} = \left( e^{i b_+} - e^{i b_-} \right)^{-1} \left( e^{i b_+} 1 - f_I \right) (1 - p_0).
\]

Let us assume now $b_0 = b_+$ and consider the projection $p_{\geq} := p_0 + p_+$. One can check as above that

\[
p_{\geq} = \left( e^{i b_-} - e^{i b_+} \right)^{-1} \left( e^{i b_-} 1 - f_I \right)
\]

\[
p_{\leq} = \left( e^{i b_+} - e^{i b_-} \right)^{-1} \left( e^{i b_+} 1 - f_I \right)
\]

are both elements of $\mathcal{A}_I$. Moreover, the equality

\[
p_0 = p_{\geq} - s_{1,1} p_{\geq} s_{1,1}^* (4.7)
\]

shows that also $p_0 \in \mathcal{A}_I$. Finally $p_+ = p_{\geq} - p_0$. The case $b_0 = b_-$ is similar.

For every $j \in \mathbb{Z}$ let us introduce the projection

\[
(p_j \psi)(n) := \delta_0(n - je_1) \psi(n), \quad \psi \in \ell^2(\mathbb{Z}^2).
\]

From the definition it follows that $p_j$ is the translation of $p_0$ along the vertical line located at $n_1 = j$. The projections $p_j$ are mutually orthogonal.

**Corollary 4.3** Under the assumption (4.1) it holds true that $p_j \in \mathcal{A}_I$ for all $j \in \mathbb{Z}$.

**Proof** From Lemma 4.2 we know that $p_0 \in \mathcal{A}_I$. Moreover, a direct computation shows that

\[
p_j = \begin{cases} (s_{1,1})^j p_0 (s_{1,1}^*)^j & \text{if } j > 0 \\ (s_{1,1}^*)^j p_0 (s_{1,1})^j & \text{if } j < 0. \end{cases} \quad (4.8)
\]

This completes the proof.
From (4.8) one gets the useful formula

\[ p_j s_{l,1} = s_{l,1} p_{j-1}, \quad j \in \mathbb{Z}. \]  

(4.9)

The next result provides a first step for the description of the evaluation map.

**Lemma 4.4** Under the assumption (4.1) it holds true that

\[ \text{ev}(p_+) = (0, 1) \]

(4.10)

\[ \text{ev}(p_-) = (1, 0) \]

and

\[ \text{ev}(p_j) = (0, 0), \quad \forall \ j \in \mathbb{Z}. \]  

(4.11)

**Proof** Let us start with the case \( b_0 \neq b_+ \) and \( b_0 \neq b_- \). Then the result follows from the last equation in (4.4), the formulas for \( p_\pm \) and \( p_j \) in Lemma 4.2 and Corollary 4.3 along with the fact that \( \text{ev} \) is a \( C^* \)-homomorphism. In the case \( b_0 = b_+ \) one obtains from (4.6) that \( \text{ev}(p_+) = (0, 1) \) and \( \text{ev}(p_-) = (1, 0) \). Moreover, from (4.7) one gets that

\[ \text{ev}(p_j) = (0, 0) - (0, s_{b_+,1}1 s_{b_+,1}^*) = 0. \]

The case \( b_0 = b_- \) is similar. \( \square \)

Let \( \Sigma \subset \mathbb{Z} \) be a finite subset and

\[ p_{\Sigma} := \bigoplus_{j \in \Sigma} p_j. \]  

(4.12)

The next result is a direct consequence of Lemma 4.4.

**Corollary 4.5** Under the assumption (4.1) it holds true that

\[ \text{ev}(ap_{\Sigma}b) = 0 \]

for all \( a, b \in A_I \) and for all finite subset \( \Sigma \subset \mathbb{Z} \).

Elements of the type \( ap_{\Lambda}b \) can be considered as “localized” operators (in the direction \( e_1 \)) and Corollary 4.5 establishes that localized elements are in the kernel of the evaluation map, namely they are elements of the interface algebra \( I \) in view of Theorem 3.6. We are now in position to provide a useful characterization of the interface algebra.

**Proposition 4.6** The interface algebra \( I \) is the closed two-sided ideal of \( A_I \) generated by \( p_0 \), i.e.

\[ I = A_I p_0 A_I := \text{span} \{ ap_0 b \mid a, b \in A_I \}. \]

**Proof** A comparison with Definition 3.3 shows the claim is equivalent to state that \( p_0 \) generates \( \text{Ker}(\text{ev}|_{F_I}) \). From Corollary 4.5 one gets that \( p_j \in \text{Ker}(\text{ev}|_{F_I}) \subset F_I \) for every \( j \in \mathbb{Z} \). Springer
A close look at the construction of $\mathcal{F}_I$ shows that every $g \in \mathcal{F}_I$ admits the (unique) representation
\[
g = \sum_{j \in \mathbb{Z}} g_j p_j ,
\]
where the sequence $\{g_j\} \in C_b(\mathbb{Z})$ admits left and right limits (cf. Remark 4.1). Therefore, one gets that $g \in \text{Ker}(\text{ev}|_{\mathcal{F}_I})$ if and only if the associated sequence $\{g_j\}$ vanishes at infinity, i.e. if and only if $\{g_j\} \in C_0(\mathbb{Z})$. The proof is completed by observing that $C_0(\mathbb{Z})$ is the uniform closure of the sequences with compact support on $\mathbb{Z}$. □

The Iwatsuka magnetic field is constant in one direction and therefore one can use the (magnetic) Bloch-Floquet transform \([37]\) to study the interface algebra. Consider
\[
V_f := e^{i f(n_1) s_2}
\]
defined through the function
\[
f(m) := \begin{cases} mb_+ & \text{if } m \geq 0 \\ (m + 1)b_+ - b_0 & \text{if } m < 0 , \end{cases}
\]
and let $\mathcal{U}_B$ be the associated (magnetic) Bloch-Floquet transform first defined on the compactly supported sequences $\psi \in \mathcal{C}_c(\mathbb{Z}^2)$ by
\[
(\mathcal{U}_B \psi)_k(m) := \sum_{\gamma \in \mathbb{Z}} e^{-i \gamma k} V_f^\gamma \psi(m, 0) \quad m \in \mathbb{Z}
\]
and then extended by continuity to $\ell^2(\mathbb{Z}^2)$. One can show that $\mathcal{U}_B$ provides a unitary equivalence
\[
\mathcal{U}_B : \ell^2(\mathbb{Z}^2) \longrightarrow \int_{\mathbb{S}^1} dk \, \ell^2(\mathbb{Z}) \simeq L^2(\mathbb{S}^1) \otimes \ell^2(\mathbb{Z}) .
\] (4.13)
Moreover, one can check that
\[
\begin{align*}
\mathcal{S}_{A_B, 1} & \mapsto s := \mathcal{U}_B \mathcal{S}_{A_B, 1} \mathcal{U}_B^{-1} = \int_{\mathbb{S}^1} dk \, s \simeq 1 \otimes s \\
\mathcal{S}_{A_B, 2} & \mapsto t := \mathcal{U}_B \mathcal{S}_{A_B, 2} \mathcal{U}_B^{-1} = \int_{\mathbb{S}^1} dk \, e^{ik} e^{-i f(n)} \simeq e^{ik} \otimes e^{-i f(n)}
\end{align*}
\]
where $s$ and $n$ are the usual shift and position operator on $\ell^2(\mathbb{Z})$. Also, it holds true that
\[
\mathcal{U}_B \mathcal{P}_B \mathcal{U}_B^{-1} = \mathcal{S} \otimes \mathcal{S} \simeq 1 \otimes i \mathcal{B}(n)
\]
where $\mathcal{B}(n)$ denotes the multiplication operator on $\ell^2(\mathbb{Z})$ given by the restriction of the magnetic field $B(m) := B(m, 0)$. As a consequence of the formulas above one gets that the Bloch-Floquet transform maps $\mathcal{I}$ as a subalgebra of $\mathcal{C}(\mathbb{S}^1) \otimes \mathcal{B}(\ell^2(\mathbb{Z}))$.

**Proposition 4.7** It holds true that $\mathcal{U}_B \mathcal{U}_B^{-1} = \mathcal{C}(\mathbb{S}^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$. In particular the Iwatsuka interface algebra is a straight-line according to Definition 3.4.

**Proof** A direct computation shows that $\mathcal{U}_B \mathcal{P}_B \mathcal{U}_B^{-1} = 1 \otimes \pi_j$ where $\pi_j$ is the rank-one projection on $\ell^2(\mathbb{Z})$ defined by $(\pi_j \phi)(m) := \delta_{m, j} \phi(m)$. Since $\mathcal{U}_B(\mathcal{S}_{1, 2} \mathcal{P}_B) \mathcal{U}_B^{-1} = \mathcal{U}_B(\mathcal{S}_{1, 2}) \mathcal{P}_B \mathcal{U}_B^{-1}$
is proportional to $e^{ink} \otimes \pi_j$ up to a phase factor one gets that $g \otimes \pi_j \in \mathcal{U}_B \mathcal{U}_B^{-1}$ for every $g \in \mathcal{C}(S^1)$ and $j \in \mathbb{Z}$. Acting with powers of $\mathcal{U}_B \mathcal{U}_B^{-1}$ on the latter elements one gets that also $g \otimes \pi_{i,j} \in \mathcal{U}_B \mathcal{U}_B^{-1}$ where $\pi_{i,j}$ is the rank-one operator defined by $(\pi_{i,j} \phi)(m) := \delta_{m,j} \phi(i)$. The result follows by observing that the rank-one operators generate the compact operators.

Following the procedure described in Sect. 3.5 we can use Proposition 4.7 to equip $\mathcal{I}$ with a derivation and a trace. The natural (unbounded) derivation on $\mathcal{C}(S^1)$ is $\delta_0 := -\frac{d}{dk}$ (with its natural domain $\mathcal{C}^1(S^1)$). With this sign convention a comparison with (2.28) provides

$$
\delta_0 \otimes \text{Id}_K \left( \mathcal{U}_B a \mathcal{U}_B^{-1} \right) = \nabla_2(a) = i [a, n_2]
$$

for differentiable elements $a \in \mathcal{I}$. Therefore we obtain that the interface derivation is given by $\nabla_{\mathcal{I}} := i [\cdot, n_2]$. Similarly the natural trace on $\mathcal{C}(S^1)$ is given by $\tau_0 := \int_{S^1} dk$ where $dk$ is the normalized Haar measure. Since $\tau_0 (e^{i n k}) = \delta_{n,0}$ one gets that

$$
\tau_0 \otimes \text{Tr}_{\ell^2(\mathbb{Z})} \left( \mathcal{U}_B a \mathcal{U}_B^{-1} \right) = \text{Tr}_{\ell^2(\mathbb{Z})}(q_0 a q_0)
$$

where the projection $q_0$ is given by $(q_0 \psi)(n, m) = \delta_{m,0} \psi(n, m)$ and $a \in \mathcal{I}$ is any suitable integrable elements. In this way one can define the interface trace as

$$
\mathcal{I}_\mathcal{I}(a) := \text{Tr}_{\ell^2(\mathbb{Z})}(q_0 a q_0) = \sum_{n \in \mathbb{Z}} (n, 0) a |n, 0\rangle \langle n, 0|
$$

(4.14)

where the Dirac notation in the right-hand side turns out to be particularly useful.

### 4.3 Linear Space Splitting of the Toeplitz Extension

The Toeplitz extension for the Iwatsuka magnetic field admits a natural splitting of the linear space structure which turns out to be useful in applications. Such a fact has already been anticipated at the end of Sect. 3.2.

Let us start by recalling that $\mathcal{A}_{\text{bulk}}$ is generated, as $*$-linear space, by the linear combinations of monomials of the type $(s^r_{b_{s-1}, s^q_{b_{s-2}}, s^p_{b_{s+1}}, s^q_{b_{s+2}}})$ with $r, s, p, q \in \mathbb{Z}$. Consider the linear map $j : \mathcal{A}_{\text{bulk}} \to \mathcal{A}_I$ initially defined on the monomials by

$$
j(s^r_{b_{s-1}, s^q_{b_{s-2}}, s^p_{b_{s+1}}, s^q_{b_{s+2}}}) := p_- s^r_{l_{1,1}} s^q_{l_{1,2}} p_- + p_+ s^p_{l_{1,1}} s^q_{l_{1,2}} p_+
$$

(4.15)

and then extended linearly to $\mathcal{A}_{\text{bulk}}$. Such a map is well defined because both $\mathcal{A}_{\text{bulk}}$ and $\mathcal{A}_I$ are spanned as Banach spaces by the families of respective monomials. From its very definition it follows that $\text{ev} \circ j = \text{Id}_{\mathcal{A}_{\text{bulk}}}$, namely $j$ provides a splitting of the linear structures. It follows that

$$
\mathcal{A}_I = \mathcal{I} + j(\mathcal{A}_{\text{bulk}})
$$

as direct sum of linear spaces [63, Proposition 3.1.3].

It is worth noting that the linear map $j$ defined by (4.15) cannot be extended to a $C^*$-homomorphism. For instance, a direct computation shows that

$$
j(1, s_{b_{s+1}}) j(1, s^*_{b_{s+1}}) - j(1, 1) = p_+ (s_{b_{s+1}} p_+ s^*_{b_{s+1}} - 1) p_- = -p_1
$$
since \( s_{b_+,1} p_+ s_{b_+,1}^* = p_+ - p_1 \). On the other hand,

\[
J(1, s_{b_+,1}^*) J(1, s_{b_+,1}) - J(1, 1) = p_+ (s_{b_+,1}^* p_+ s_{b_+,1} - 1) p_+ = 0
\]
due to \( s_{b_+,1} p_+ s_{b_+,1} = p_+ + p_0 \).

**Remark 4.8** (Failure of the \( C^* \)-splitting) A linear splitting is the best that we can do since the existence of a \( C^* \)-lifting would imply the short exact sequence (see [63, Proposition 8.2.2] or [22, Proposition 3.29])

\[
0 \longrightarrow K_0(\mathcal{I}) \overset{\iota_\ast}{\longrightarrow} K_0(\mathcal{A}_I) \overset{\text{ev}_\ast}{\longrightarrow} K_0(\mathcal{A}_\text{bulk}) \longrightarrow 0,
\]

(4.16)
at the level of the \( K_0 \)-groups. However, it will be proved that \( K_0(\mathcal{I}) \cong \mathbb{Z} \) (Proposition 4.11) and \( \iota_\ast = 0 \) (Remark 4.13) making the short exact sequence (4.16) just impossible.

### 4.4 \( K \)-Theory for the Iwatsuka \( C^* \)-Algebra

In this section we will provide a preliminary study of the \( K \)-theory of the Iwatsuka \( C^* \)-algebra which will be complemented in the next Sect. 4.5. We will make use of the fact that the \( C^* \)-algebra \( \mathcal{A}_I \) can be represented as an iterated crossed product with \( \mathbb{Z} \) (see Appendix A), and in turn we will exploit the Pimsner-Voiculescu exact sequence (Appendix C) described in [51] or in [9, Chapter V].

By adapting the notation of Appendix A we have the isomorphisms

\[
\mathcal{A}_I \cong \mathcal{Y}_{l,1} \rtimes_{\alpha_2} \mathbb{Z}, \quad \mathcal{Y}_{l,1} := \mathcal{F}_l \rtimes_{\alpha_1} \mathbb{Z}
\]

where \( \mathcal{F}_l \) is the \( C^* \)-algebra generated by the flux operator \( f_l \) according to (2.13), the automorphism \( \alpha_1 \) is defined by \( \alpha_1(g) := s_{l,1} g s_{l,1}^* \) for every \( g \in \mathcal{F}_l \) and the automorphism \( \alpha_2 \) is defined by \( \alpha_2(g s_{l,1}^r) := s_{l,2} g s_{l,1}^r s_{l,2}^* \) for every \( g \in \mathcal{F}_l \) and \( r \in \mathbb{N}_0 \).

The \( K \)-theory of the \( C^* \)-algebra \( \mathcal{F}_l \) is calculated in Appendix B and is given by

\[
K_0(\mathcal{F}_l) = \left( \bigoplus_{j \in \mathbb{Z}} \mathbb{Z}[p_j] \right) \oplus \mathbb{Z}[p_-] \oplus \mathbb{Z}[p_+], \quad K_1(\mathcal{F}_l) = 0.
\]

(4.17)

The \( K \)-theory of the first crossed product \( \mathcal{Y}_{l,1} \) can be computed from the Pimsner-Voiculescu exact sequence

\[
\begin{array}{cccc}
K_0(\mathcal{F}_l) & \overset{\beta_{l,\ast}}{\longrightarrow} & K_0(\mathcal{Y}_{l,1}) & \overset{\text{ev}_\ast}{\longrightarrow} & K_0(\mathcal{A}_\text{bulk}) & \longrightarrow 0, \\
\partial_1 & | & \downarrow \partial_0 \\
K_1(\mathcal{Y}_{l,1}) & \overset{\iota_\ast}{\longleftarrow} & K_1(\mathcal{F}_l) & \overset{\beta_{l,\ast}}{\longleftarrow} & K_1(\mathcal{F}_l) & \\
\end{array}
\]

(4.18)

where the connecting maps \( \beta_{l,\ast} \) and \( \iota_\ast \) and the boundary maps \( \partial_0 \) and \( \partial_1 \) are described in Appendix C.
Proposition 4.9 Consider the six-term exact sequence (4.18). Then, it holds true that:

(i) The image and kernel of the map $\beta_{1,*}: K_0(\mathcal{F}_1) \to K_0(\mathcal{F}_1)$ are given by

$$\text{Im}(\beta_{1,*}) = \bigoplus_{j \in \mathbb{Z}} \mathbb{Z}[p_j], \quad \text{Ker}(\beta_{1,*}) = \mathbb{Z}[1].$$

(ii) $\partial_1[\mathfrak{s}_{1,1}] = -[1]$. Consequently,

$$K_0(\mathcal{Y}_1) = \mathbb{Z}[p_-] \oplus \mathbb{Z}[p_+], \quad K_1(\mathcal{Y}_1) = \mathbb{Z}[\mathfrak{s}_{1,1}].$$

Proof For (i) let us recall that $\beta_{1,*} = \text{Id}_{*-} - \alpha_{1,*}^{-1}$, as described in Appendix C. Therefore, one gets

$$\beta_{1,*}([p_j]) = [p_j - \mathfrak{s}_{1,1}^* p_j \mathfrak{s}_{1,1}] = [p_j] - [p_{j-1}],$$

$$\beta_{1,*}([p_-]) = [p_- - \mathfrak{s}_{1,1}^* p_- \mathfrak{s}_{1,1}] = [p_-],$$

$$\beta_{1,*}([p_+]) = [p_+ - \mathfrak{s}_{1,1}^* p_+ \mathfrak{s}_{1,1}] = -[p_0].$$

It follows that the image of $\beta_{1,*}$ is $\bigoplus_{j \in \mathbb{Z}} \mathbb{Z}[p_j]$ and

$$\beta_{1,*} \left( n_- [p_-] + n_+ [p_+] + \sum_{j=-M}^{M} n_j [p_j] \right) = 0$$

has a non-trivial solution if and only if $n_- = n_0 = n_+$, and $n_j = 0$ in all other cases. As a consequence one has that the kernel of $\beta_{1,*}$ is generated by $[p_-] + [p_0] + [p_+] = [1]$. For (ii) let us recall that the boundary map $\partial_1 := \kappa_{1,1}^{-1} \circ \text{ind}$ is the composition of the index map $\text{ind}: K_1(\mathcal{Y}_1) \to K_0(\mathcal{F}_1 \otimes \mathcal{K})$ associated to the Toeplitz extension (C.1) and the inverse of the stabilization isomorphism $\kappa_0: K_0(\mathcal{F}_1) \to K_0(\mathcal{F}_1 \otimes \mathcal{K})$ induced by the identification $g \mapsto g \otimes \pi_0$ (here $\pi_0 \in \mathcal{K}$ is any fixed rank-one projection). The isometry $V := \mathfrak{s}_{1,1} \otimes v$ which generates the Toeplitz algebra $\mathcal{T}_{\mathfrak{s}_{1,1}}$ together with $\mathcal{F}_1 \otimes 1$ verifies the condition $\psi(V) = \mathfrak{s}_{1,1}$. Therefore, $V$ provides a lift of $\mathfrak{s}_{1,1}$ by an isometry. Consider the unitary matrix

$$w(\mathfrak{s}_{1,1}) := \begin{pmatrix} V & P \\ 0 & V^* \end{pmatrix} \in \text{Mat}_2(\mathcal{T}_{\mathfrak{s}_{1,1}})$$

where $P := 1 - VV^*$. By construction $w(\mathfrak{s}_{1,1})$ is a lift of $\text{diag}(\mathfrak{s}_{1,1}, \mathfrak{s}_{1,1}^*)$ and $[\text{diag}(\mathfrak{s}_{1,1}, \mathfrak{s}_{1,1}^*)] = [1]$ as a class in $K_1(\mathcal{Y}_{1,1})$. As a consequence we can construct the index map according to [63, Definition 8.1.1] and after an explicit computation one gets

$$\text{ind}([\mathfrak{s}_{1,1}]) = \varphi_*^{-1}([1 - V^*V] - [1 - VV^*])$$

$$= \varphi_*^{-1}([0] - [P]) = -[1 \otimes \pi_0]$$

where in the last equality we used the property $\varphi(1 \otimes \pi_0) = P$. By using the isomorphism $\kappa_0$, one finally gets $\partial_1[\mathfrak{s}_{1,1}] = -[1]$. Since $K_1(\mathcal{F}_1) = 0$, it follows that $\partial_1$ provides an isomorphism between $K_1(\mathcal{Y}_{1,1})$ and $\text{Ker}(\beta_{1,*})$. In view of (i) and (ii) one infers that $K_1(\mathcal{Y}_1) = \mathbb{Z}[\mathfrak{s}_{1,1}]$. Again, $K_1(\mathcal{F}_1) = 0$ implies the surjectivity of $\iota_*: K_0(\mathcal{F}_1) \to K_0(\mathcal{Y}_{1,1})$ and so $K_0(\mathcal{Y}_{1,1}) \simeq K_0(\mathcal{F}_1)/\text{Im}(\beta_{1,*}) = \mathbb{Z}[p_-] \oplus \mathbb{Z}[p_+].$
For the $K$-theory of the second crossed product $A_l \simeq Y_{l,1} \rtimes \omega_2 \mathbb{Z}$ we need the Pimsner-Voiculescu exact sequence

$$K_0(Y_{l,1}) \xrightarrow{\beta_{2,+}} K_0(Y_{l,1}) \xrightarrow{\iota_*} K_0(A_l) \xrightarrow{\partial_1} K_1(A_l) \xrightarrow{\iota_*} K_0(Y_{l,1}) \xrightarrow{\partial_0} K_1(Y_{l,1}),$$

(4.19)

**Theorem 4.10 (K-theory of the lwutsuka $C^*$-algebra I)** Consider the six-term exact sequence (4.19). Then, it holds true that

(i) Both maps $\beta_{2,+} : K_j(Y_{l,1}) \to K_j(Y_{l,1})$, with $j = 1, 2$, vanish;

(ii) The map $\partial_1$ verifies

$$\partial_1([p_- s_{l,2} + p_0 + p_+]) = -[p_-],$$

$$\partial_1([p_- + p_0 + p_+ s_{l,2}]) = -[p_+];$$

(iii) There exists $N \in \mathbb{N}$ and a projection $p_1 \in A_l \otimes \text{Mat}_N(\mathbb{C})$ such that

$$\partial_0[p_1] = [s_{l,1}].$$

Consequently,

$$K_0(A_l) = \mathbb{Z}[p_-] \oplus \mathbb{Z}[p_-] \oplus \mathbb{Z}[p_1]$$

$$K_1(A_l) = \mathbb{Z}[w_{l,-}] \oplus \mathbb{Z}[w_{l,+}] \oplus \mathbb{Z}[s_{l,1}]$$

where $w_{l,\pm} := 1 + p_\pm (s_{l,2} - 1)$.

**Proof** For (i) it is enough to note that $\alpha_2(p_l) = p_l$ for $l \in \mathbb{Z} \cup \{\pm\}$ and

$$\alpha_{2,+}^{-1}[s_{l,1}] = [s_{l,2}^* s_{l,1} s_{l,2}] = [(s_{l,2}^* [s_{l,2}]) s_{l,1}] = [f_1] s_{l,1} = [s_{l,1}],$$

since $[f_1] = [1]$ in $K_1(Y_{l,1})$. As a consequence $\beta_{2,+} = \text{Id}_+ - \alpha_{2,+}^{-1} = 0$. For (ii) let us observe that the isometry $V = s_{l,2} \otimes v \in T_{a_2}$ satisfies $\psi((p_+ \otimes 1)V) = p_+ s_{l,2}$. It follows that $W := (p_- \otimes 1)V + (p_0 + p_+) \otimes 1$ is an isometry which provides a lift of $p_- s_{l,2} + p_0 + p_+$ in $T_{a_2}$. The index map of the latter element can be computed as in the proof of Proposition 4.9 and after some computation one gets

$$\text{ind}([p_- s_{l,2} + p_0 + p_+]) = \varphi_*^{-1}([1 - W^* W] - [1 - W W^*]) = \varphi_*^{-1}([0] - [p_- \otimes P]) = -[p_- \otimes \pi_0],$$

where we used $P := 1 - V V^*$ and $\varphi(p_- \otimes \pi_0) = p_- \otimes P$. After recalling that $\partial_1 := k^{-1} \circ \text{ind}$, with $\kappa_0$ stabilization isomorphism, one gets the first equation in (ii). The derivation of the second equation is identical. Item (iii) follows from the fact that $\beta_{2,+} = 0$ implies the surjectivity of the map $\partial_0$ and so there must be a projection $p_1 \in A_l \otimes \text{Mat}_N(\mathbb{C})$ and $M \leq N$ such that

$$\partial_0([p_1] - M[1]) = [s_{l,1}]$$
(see [63, Proposition 6.2.7]). Now, since $[1] = [p_-] + [p_+] \in \text{Im}(\iota_s)$ it follows that $
abla_0[p_1] = [s_{1,1}]$. The exactness of the sequence (4.19) along with $\beta_{2,s} = 0$ implies $K_j(A_i) = \iota_s(K_j(Y_{1,1})) \oplus \partial_j^{-1}(K_{j+1}(Y_{1,1}))$ with $j = 0, 1$ (mod. 2). This concludes the proof.

Observe that the proof for the existence of the element $p_1$ works as well for the case $\nabla_0(p_1^*) = [s_{1,1}^*] = -[s_{1,1}]$. Any projection $p_1$ with the property

$$\nabla_0(p_1) \in \{[s_{1,1}], [s_{1,1}^*]\}$$

will be called a Powers-Rieffel-Iwatsuka projection or simply a PRI-projection. We can say a little more about $p_1$. From its very definition one has that $\exp[p_1] = [(s_{1,1} - 1) \otimes \pi_0 + 1]$ where $\exp$ is the actual exponential map associated with the Toeplitz exact sequence (C.1).

### 4.5 Six-Term Exact Sequence for the Iwatsuka $C^*$-Algebra

We are now in position to study the six-term exact sequence associated with the Toeplitz extension for the Iwatsuka magnetic field (4.2). This is given by

$$
\begin{align*}
K_0(\mathcal{I}) & \xrightarrow{\iota_s} K_0(A_i) \xrightarrow{\exp} K_0(A_{\text{bulk}}) \\
& \quad \xrightarrow{\text{ind}} K_1(A_{\text{bulk}}) \xleftarrow{\iota_s} K_1(A_i) \xleftarrow{\text{ev}_s} K_1(\mathcal{I})
\end{align*}
$$

The $K$-theory of the bulk algebra is explicitly given by

$$
\begin{align*}
K_0(A_{\text{bulk}}) & = \mathbb{Z}[(1,0)] \oplus \mathbb{Z}[(p_{\theta_-},0)] \oplus \mathbb{Z}[(0,1)] \oplus \mathbb{Z}[(0,p_{\theta_+})], \\
K_1(A_{\text{bulk}}) & = \mathbb{Z}[(s_{\theta_-,-1},1)] \oplus \mathbb{Z}[(s_{\theta_-,-2},1)] \oplus \mathbb{Z}[(1,s_{\theta_+,-2})] \oplus \mathbb{Z}[(1,s_{\theta_+,-2})],
\end{align*}
$$

where $p_{\theta_{\pm}}$ are the Powers-Rieffel projections of the $C^*$-algebras $A_{\theta_{\pm}}$, respectively (cf. Appendix D).

The description of the $K$-theory of the interface algebra follows from Proposition 3.16 and Proposition 4.7.

**Proposition 4.11** It holds true that

$$
K_0(\mathcal{I}) = \mathbb{Z}[p_0], \quad K_1(\mathcal{I}) = \mathbb{Z}[w_\mathcal{I}],
$$

where $w_\mathcal{I} := p_- + p_0s_{1,2} + p_+ = 1 + p_0(s_{1,2} - 1) \in \mathcal{I}^+$.

**Proof** Let us start with the $K_0$-group. As showed in the proof of Proposition 3.16 the generator of $K_0(\mathcal{C}(S^1))$ is the constant function $1$. The group isomorphism $K_0(\mathcal{C}(S^1)) \simeq K_0(\mathcal{C}(S^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z})))$ is induced by the $C^*$-homomorphism $\mu : \mathcal{C}(S^1) \to \mathcal{C}(S^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z}))$ defined by $\mu : g \mapsto g \otimes \pi_0$ where $\pi_0$ is the projection on $\ell^2(\mathbb{Z})$ defined by $(\pi_0\phi)(m) := \delta_{m,0}\phi(m)$ [63, Corollary 6.2.11]. The result follows by observing that $\mathcal{U}_B^{-1}(1 \otimes \pi_0)\mathcal{U}_B = p_0$ where $\mathcal{U}_B$ is the magnetic Bloch-Floquet transform used in Proposition 4.7. The argument for the $K_1$-group follows a similar structure. We already know that the generator of $K_1(\mathcal{C}(S^1))$ is the exponential function $e^{ik}$ and the isomorphism $K_1(\mathcal{C}(S^1)) \simeq K_1(\mathcal{C}(S^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z})))$ is induced by the same isomorphism $\mu$ defined above [53, Proposition 8.2.8].

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However, since the $K_1$ is computed from the unitalization of the related $C^*$-algebra one needs to promote $e^{ik} \otimes \pi_0$ to a unitary in $(\mathcal{C}(\mathbb{S}^1) \otimes \mathcal{K}(\ell^2(\mathbb{Z})))^\perp$. This can be done through the map

$$e^{ik} \otimes \pi_0 \mapsto e^{ik} \otimes \pi_0 - 1 \otimes \pi_0 + 1 \otimes 1$$

as described in [53, Proposition 8.1.6]. As a result one has the generator of the $K_1$-group can be identified with the class of $(e^{ik} - 1) \otimes \pi_0 + 1 \otimes 1$. Finally, the magnetic Bloch-Floquet transform

$$\mathcal{U}_B^{-1}((e^{ik} - 1) \otimes \pi_0 + 1 \otimes 1) \mathcal{U}_B = V_f p_0 - p_0 + 1$$

along with the identities $V_f p_0 = s_2 p_0 = s_{1,2} p_0$ and $1 - p_0 = p_- + p_+$ provides the desired result.

We now have all the ingredients to study the vertical homomorphisms of the diagram (4.20). Let us start with the index map.

**Proposition 4.12** The image of the generators of $K_1(A_{\text{bulk}})$ under the map $\text{ind}$ in diagram (4.20) are given by

$$\text{ind}([(s_{b-1,2}, 1)]) = \text{ind}([(1, s_{b+,2})]) = 0,$$

$$\text{ind}([(s_{b-1,1}, 1)]) = -\text{ind}([(1, s_{b+,1})]) = [p_0].$$  \hspace{1cm} (4.21)

Consequently the index map is surjective.

**Proof** Let us construct the index map according to [63, Definition 8.1.1] for the set of generators $A \in \{(s_{b-1,1}), (s_{b-2,1}), (1, s_{b+,1}), (1, s_{b+,2})\} \subset A_{\text{bulk}}$ of the $K_1$-group of $A_{\text{bulk}}$. Let $j$ as in (4.15) and define the map

$$w(A) := \begin{pmatrix} f(A) & 1 - f(A) f(A)^* \\ 1 - f(A)^* f(A) & f(A)^* \end{pmatrix} \in \text{Mat}_2(A_{\text{f}}).$$

A direct check shows that $j(A) \in A_{\text{f}}$ is a partial isometry for every $A$ in the generator set, indeed

$$j(s_{b-1,1}, 1) j(s_{b-1,1})^* = 1 - p_0,$$

$$j(s_{b-2,1}, 1) j(s_{b-2,1})^* = 1 - p_0,$$

$$j(1, s_{b+,1}) j(1, s_{b+,1})^* = 1 - (p_0 + p_1),$$

$$j(1, s_{b+,2}) j(1, s_{b+,2})^* = 1 - p_0,$$

and, on the other hand,

$$j(s_{b-1,1}, 1)^* j(s_{b-1,1}) = 1 - (p_0 + p_-),$$

$$j(s_{b-2,1}, 1)^* j(s_{b-2,1}) = 1 - p_0,$$

$$j(1, s_{b+,1})^* j(1, s_{b+,1}) = 1 - p_0,$$

$$j(1, s_{b+,2})^* j(1, s_{b+,2}) = 1 - p_0.$$

(4.23)
As a consequence one can check that \( w(A) \) is a unitary operator for every generator \( A \). Moreover \( \text{ev}(w(A)) = \text{diag}(A, A^*) \) showing that \( w(A) \) is a \textit{unitary lift} of \( \text{diag}(A, A^*) \). Finally \( [\text{diag}(A, A^*)] \simeq [1] \) as a class in the \( K_1 \)-group. With all these data we can compute the index map of each generators according to \( \text{ind}(\{A\}) := [w(A)P_1 w(A)^*] − [P_1] \) where \( P_1 := \text{diag}(1, 0) \). An explicit computation provides

\[
\text{ind}(\{A\}) = \begin{bmatrix} j(A)j(A)^* & 0 \\ 0 & 1 − j(A)^*j(A) \end{bmatrix} − \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 − j(A)^*j(A) \end{bmatrix} − \begin{bmatrix} 1 − j(A)^*j(A) & 0 \\ 0 & 0 \end{bmatrix} = [1 − j(A)^*j(A)] − [1 − j(A)j(A)^*]
\]

where the second and third equality are understood in the sense of the \( K_0 \)-group. The equations (4.21) follow from the latter formula along with the computations (4.22) and (4.23) and the observation that, in view of (4.8), \( p_j \) is unitarily equivalent to \( p_0 \) for every \( j ∈ \mathbb{Z} \). The latter fact implies \( [p_j] = [p_k] \) for every pair \( j, k ∈ \mathbb{Z} \) as elements of \( K_0(\mathcal{A}_I) \). Consequently, \( \text{ind}(\{s_{0, −1}, I\}) = [p_0] \), and \([p_0]\) is the generators of \( K_0(\mathcal{I}) \). This shows that the map \( \text{ind} \) is surjective.

\[\square\]

\textbf{Remark 4.13} As a consequence of Proposition 4.12 and the exactness of diagram 4.20 one infers that the map \( \iota_\ast : K_0(\mathcal{I}) → K_0(\mathcal{A}_I) \) is just the zero map. This implies that \( (\iota_\ast([p_j])) = [p_j] = 0 \) as element of \( K_0(\mathcal{A}_I) \). This fact is in agreement with the description of \( K_0(\mathcal{A}_I) \) in Theorem 4.10 and can be justified by the following direct argument: From \( p_0 = s_{1,1}^* p_+ s_{1,1} − p_+ \) one gets \( [p_0] = [s_{1,1}^* p_+ s_{1,1}] − [p_+] = [p_+] − [p_+] = 0 \) and \([p_0] = [p_j]\) for every \( j ∈ \mathbb{Z} \) as justified at the end of the proof of Proposition 4.12.

Now we are in position to study the exponential map of diagram (4.20).

\textbf{Proposition 4.14} The map \( \exp \) in diagram (4.20) is surjective. Moreover, it holds true that

\[
\begin{align*}
\exp(\{(1, 0)\}) &= \exp(\{(0, 1)\}) = [1] = 0, \\
\exp(\{(0, p_{θ_+})\}) &= −\exp(\{(p_{θ_−}, 0)\}) = −[w_\mu].
\end{align*}
\]

where the additive notation for the group \( K_1(\mathcal{I}) \) is used (cf. Note 7) and \([w_\mu]\) denotes the generator of \( K_1(\mathcal{I}) \) defined in Proposition 4.11.

\textbf{Proof} The surjectivity of the exponential map can be deduced directly by the exactness of the diagram (4.20). Since \( \text{Ker}(\exp) \simeq K_0(\mathcal{A}_I) \simeq \mathbb{Z}^3 \) and \( K_0(\mathcal{A}_{\text{bulk}}) \simeq \mathbb{Z}^4 \) it follows that there is an element \([q] ∈ K_0(\mathcal{A}_{\text{bulk}}) \) such that \( \exp([q]) = m[p_− + s_{1,2}p_0 + p_+] \in K_1(\mathcal{I}) \) for some \( m ∈ \mathbb{Z} \setminus \{0\} \). However, by observing that \( K_1(\mathcal{A}_I) \simeq \mathbb{Z}^3 \) is torsion-free one infers that \( m = ±1 \) are the only admissible values. In both cases the exponential map turns out to be surjective. Now we can prove formulas (4.24). The construction of the exponential map is described in [63, Definition 9.3.1 & Exercise 9.E]. The first step is to construct appropriate lifts of the representatives of the elements of the group \( K_0(\mathcal{A}_{\text{bulk}}) \). Let us start with the two generators \((1, 0)\) and \((0, 1)\). From Lemma 4.4 we get that suitable self-adjoint lifts are given by \( \text{lift}(1, 0) := p_- \) and \( \text{lift}(0, 1) := p_+ \). Moreover, since \( p_± \) are genuine projections one gets \( e^{−12\pi p_±} = 1 \in \mathcal{I}^+ \). As a consequence, one gets the first equation in (4.24). For the second set of equations we need to construct explicitly the element \([q]\) introduced abstractly above.
We will follow quite closely the strategy in [51, pp. 114-116]. Let us start with the Powers-Rieffel projection (cf. Appendix D)

\[ p_{\theta+} = s_{b+1}^* \partial_1 + \partial_0 + \partial_1 s_{b+1} \in \mathcal{A}_{b+} \]

where \( \partial_1 := g(s_2) \) and \( \partial_0 := f(s_2) \) are self-adjoint elements of \( \mathcal{A}_{b+} \cap \mathcal{A}_1 \) in view of \( s_2 = s_{b+2} = s_{1,2} \). Consider the self-adjoint lift of \((0, p_{\theta+})\) given by

\[ q_+ = v_+^* \partial_1 + \partial_0 p_\geq + \partial_1 v_+ , \]

where \( v_+ := s_{1,1} p_\geq = s_{b+1} p_\geq \) and \( p_\geq := p_0 + p_+ \). It is worth remembering that \([\partial_i, p_0] = [\partial_i, p_+] = 0 \) for \( i = 0, 1 \). A direct computation shows that

\[ q_+^2 = q_+ - \partial_1^2 p_0 = q_+ + (\partial_0^2 - \partial_0) \mathcal{L} p_0 . \]  

(4.25)

The first equality in (4.25) is justified by the relations

\[ \partial_1 v_+ \partial_1 v_+ = v_+ (s_{b+1}^* \partial_1 s_{b+1} \partial_1) v_+ = 0 , \]

\[ \partial_0 p_\geq \partial_1 v_+ + \partial_1 v_+ \partial_0 p_\geq = (\partial_0 \partial_1 + \partial_1 s_{b+1} \partial_0 s_{b+1}^*) v_+ = \partial_1 v_+ , \]

\[ \partial_0^2 p_\geq + v_+^* \partial_1 \partial_1 v_+ + \partial_1 v_+ v_+^* \partial_1 = p_\geq (\partial_0^2 + s_{b+1}^* \partial_1 s_{b+1} + \partial_1^2) p_\geq - \partial_1^2 p_0 \]

\[ = \partial_0 p_\geq - \partial_1^2 p_0 \]

deduced from (D.2). The second equality in (4.25) follows from (D.3) where \( \mathcal{L} := \mathcal{L}(\partial_1) \) is the support projection of \( \partial_1 \) (in the von Neumann algebra generated by \( \mathcal{A}_{b+} \)). An inductive argument, based on the identities

\[ q_+ p_0 = \partial_0 p_1 + \partial_1 s_{b+1} p_0 , \quad \partial_1 s_{b+1} \mathcal{L} = 0 \]

and the commutation relations \([\mathcal{L}, \partial_1] = 0 = [\mathcal{L}, p_0] \), provides

\[ q_+^N = q_+ + (\partial_0^N - \partial_0) \mathcal{L} p_0 = (q_+ - \partial_0 \mathcal{L} p_0) + (\partial_0 \mathcal{L})^N p_0 . \]  

(4.26)

Equation (4.26) facilitates the computation of the exponential of \( q_+ \). Indeed, one immediately gets

\[ e^{-i2\pi q_+} = \sum_{N=0}^{+\infty} (-i2\pi)^N q_+^N N! = (e^{-i2\pi} - 1)(q_+ - \partial_0 \mathcal{L} p_0) + e^{-i2\pi} \partial_0 \mathcal{L} p_0 + (1 - p_0) \]

\[ = p_- + e^{-i2\pi} \partial_0 \mathcal{L} p_0 + p_+ . \]

Finally, by using the homotopy \( e^{-i2\pi \partial_0 \mathcal{L}} \sim s_{b+2}^* = s_{1,2}^* \) described in Lemma D.1 one obtains \( \exp\{((0, p_{\theta+}))\} = -[p_- + s_{1,2} p_0 + p_+] \). The proof for \((p_{\theta-}, 0)\) proceeds in a similar way by considering the Powers-Rieffel projection\(^8\)

\[ p_{\theta-} = s_{b-1}^* \partial_1' + \partial_0' + \partial_1' s_{b-1} \in \mathcal{A}_{b-} \]

\(^8\)Observe that the set of self-adjoint operators \([\partial_0, \partial_1]\) which defines \( p_{\theta+} \) is in principle different from the set of self-adjoint operators \([\partial_0', \partial_1']\) which defines \( p_{\theta-} \).
and the lift
\[ q_- = v_- s_1 I + d_0 p_- + d_1 v_- , \]
where \( v_- := p_-s_{1,1} = p_-s_{b_{-1}}. \) This time, a direct computation provides
\[ q_-^2 = q_- - (s_{b_{-1}}d(s_{b_{-1}}))^2 p_- = q_- + (d_0^2 - d_0) \mathcal{L}' p_- , \]
where now \( \mathcal{L}' := \mathcal{L}'(s_{b_{-1}}d(s_{b_{-1}})) \) is the support projection of \( s_{b_{-1}}d(s_{b_{-1}}). \) After an induction one gets
\[ q_-^N = q_- + (d_0^N - d_0) \mathcal{L}' p_- = (q_- - d_0 \mathcal{L}' p_-) + (d_0 \mathcal{L}') N p_- , \]
and the exponential of \( q_- \) is given by
\[ e^{-i2\pi q_-} = e^{-i2\pi d_0 \mathcal{L}' p_-} + (1 - p_-) . \]
The homotopy argument \( e^{-i2\pi d_0 \mathcal{L}' p_-} \simeq s_{b_{-2}} = s_{1,2} \) provided in Lemma D.2 provides \( \exp((p_{b_{-2}})) = [s_{1,2}p_{-1} + (1 - p_{-1})]. \) To finish the proof, let us consider the operator \( \tau := s_{1,1}p_{-1} + s_{1,2}^2 p_0 + (1 - p_{-1} - p_0). \) This is an unitary involution in \( \mathcal{I}' \), i.e. \( \tau = \tau^{-1} = \tau^* . \) This implies that \( [\tau] = [1] \) is the trivial element of \( K_1(\mathcal{I}) \simeq \mathbb{Z} \) which is torsion-free. As a consequence
\[
[s_{1,2}p_{-1} + (1 - p_{-1})] = [\tau] + [s_{1,2}p_{-1} + (1 - p_{-1})] + [\tau] \\
= [\tau(s_{1,2}p_{-1} + (1 - p_{-1})\tau] \\
= [e^{i\theta_0} s_{1,2}p_0 + (1 - p_0)] \\
= [s_{1,2}p_0 + (1 - p_0)]
\]
where we used \( \tau p_{-1} = p_0, \tau s_{1,2}p_{-1} = \int_B s_{1,2}p_0 = e^{i\theta_0} s_{1,2}p_0 \) and the fact that \( e^{i\theta_0} s_{1,2} \) is connected to \( s_{1,2} \) by the homotopy \( [0, 1] \ni \theta \mapsto e^{i(1-\theta)h_0} s_{1,2}. \)

The surjectivity of the index map (Proposition 4.12) and of the exponential map (Proposition 4.14) implies that the two maps \( \iota_\star \) in the diagram (3.8) are just the zero maps. After replacing \( \iota_\star = 0 \) in (3.8) one obtains the short exact sequences
\[
0 \longrightarrow K_0(\mathcal{A}_1) \xrightarrow{\ev} K_0(\mathcal{A}_{\text{bulk}}) \xrightarrow{\exp} K_1(\mathcal{I}) \longrightarrow 0 , \\
0 \longrightarrow K_1(\mathcal{A}_1) \xrightarrow{\ev} K_1(\mathcal{A}_{\text{bulk}}) \xrightarrow{\ind} K_0(\mathcal{I}) \longrightarrow 0 .
\]
As a result, one gets further information about the structure of the \( K \)-theory of the Iwatsuka \( C^* \)-algebra.

**Theorem 4.15 (K-theory of the Iwatsuka C*-algebra II)** It holds true that
\[
K_0(\mathcal{A}_{\text{bulk}}) = \ev_*(K_0(\mathcal{A}_1)) \oplus \psi_{\exp}(K_1(\mathcal{I})) , \\
K_1(\mathcal{A}_{\text{bulk}}) = \ev_*(K_1(\mathcal{A}_1)) \oplus \psi_{\ind}(K_0(\mathcal{I}))
\]
where \( \psi_{\exp} \) and \( \psi_{\ind} \) are suitable lifts of the exponential map and of the index map, respectively.
**Proof** The two short exact sequences are of the form

\[ 0 \rightarrow \mathbb{Z}^3 \xrightarrow{\alpha} \mathbb{Z}^4 \xrightarrow{\beta} \mathbb{Z} \rightarrow 0 \]

meaning that \( \mathbb{Z}^4 \) is an abelian extension of \( \mathbb{Z} \) by \( \mathbb{Z}^3 \). The possible extensions are classified by \( \text{Ext}_\mathbb{Z}(\mathbb{Z}, \mathbb{Z}^3) = 0 \) [29, Chapter III], meaning that only the trivial extension is possible. This in particular ensures the existence of the lifts \( \psi \) and \( \psi_{\text{ind}} \).

**Remark 4.16** We can provide a more precise presentation of \( K_1(A_{\text{bulk}}) \) by combining Theorem 4.15 with the computation of the map \( \text{ev}_s \) and Proposition 4.12. One gets that

\[
\text{ev}_s(K_1(A_I)) = \mathbb{Z}[(s_{b_-}, 1)] + \mathbb{Z}[(1, s_{b_+})] + \mathbb{Z}[(s_{b_-}, s_{b_+})],
\]

\[
\psi_{\text{ind}}(K_0(I)) = \mathbb{Z}[(s_{b_-}, 1)],
\]

where \([(s_{b_-}, s_{b_+})] = [(s_{b_-}, 1)] + [(1, s_{b_+})] \) in the sense of the \( K_1 \)-group and the (non-unique) lift \( \psi_{\text{ind}} \) has been chosen as \( \psi_{\text{ind}}([p_0]) := [(s_{b_-}, 1)] \). A similar analysis for \( K_0(A_{\text{bulk}}) \) provides

\[
\text{ev}_s(K_0(A_I)) = \mathbb{Z}[(1, 0)] + \mathbb{Z}[(0, 1)] + \mathbb{Z}[(p_{\theta_-}, p_{\theta_+})],
\]

\[
\psi_{\text{exp}}(K_1(I)) = \mathbb{Z}[(0, p_{\theta_+})],
\]

where the (non-unique) lift \( \psi_{\text{exp}} \) is defined by \( \psi_{\text{exp}}([p_- + s_{l_2}p_0 + p_+]) := (0, p_{\theta_+}) \). Finally, we are in position to say something more about the Powers-Rieffel-Iwatsuka projection \( p_I \in A_I \otimes \text{Mat}_N(\mathbb{C}) \) introduced short after Theorem 4.10. First consider \( p_I \in A_I \otimes \text{Mat}_N(\mathbb{C}) \) and \( I_M \) the identity matrix in \( \text{Mat}_M(A_{\text{bulk}}) \) with \( M \leq N \), such that

\[
\text{ev}_s([p_I] - [I_M]) = [(p_{\theta_-}, p_{\theta_+})].
\]

This relation is satisfied in view of the surjectivity of \( \text{ev}_s \) and the standard picture of \( K_0 \)-group [63, Proposition 6.2.7]. It follows that

\[
\text{ev}_s([p_I]) = [(p_{\theta_-}, p_{\theta_+})] + M[1],
\]

and consequently \([(p_-), [p_+], [p_I]] \) is a set of generators for \( K_0(A_I) \). The fact that \( [p_I] \) is the third generator of \( K_0(A_I) \) can be used in the six-term exact sequence 4.19 which provides \( \partial_0[p_I] \in [\pm [s_{l_1}]] \), showing that \( p_I \) is actually a PRI-projection. It is interesting to note that even though neither \([(p_{\theta_-}, 0)] \) nor \([(0, p_{\theta_+})] \) can be lifted into a projection, the existence of the PRI-projection implies that the matrix \( \text{diag}((p_{\theta_-}, p_{\theta_+}), I_M) \) \( \in \text{Mat}_{M+1}(A_{\text{bulk}}) \), can actually be lifted into a PRI-projection.

### 4.6 Bulk-Interface Correspondence for the Iwatsuka C*-Algebra

Let us start with a preliminary result which is a direct consequence of Proposition 4.14.

**Lemma 4.17** Let \( \theta = (p_-, p_+) \in A_{\text{bulk}} \) be a projection and \( [p] \in K_0(A_{\text{bulk}}) \) the related class in the \( K_0 \)-group. Let \( N_{\pm} := \text{Ch}_{b_{\pm}}(p_{\pm}) \in \mathbb{Z} \) be the Chern numbers of \( p_{\pm} \) defined by (3.15). Then,

\[
\exp([p]) = (N_- - N_+)[w_Z]
\]

where \([w_Z]\) is the generator of \( K_1(I) \) defined in Proposition 4.11.
Proof In terms of the generators of $K_0(A_{\text{bulk}})$ one has that
\[
[p] = M_- [(1, 0)] + M_+ [(0, 1)] + N_- [(p_{\theta_-}, 0)] + N_+ [(0, p_{\theta_+})]
\]
with $M_\pm, N_\pm \in \mathbb{Z}$ suitable integers. The discussion at the end of Appendix D justifies $N_\pm := \text{Ch}_{p_\pm}(p_\pm)$. Finally, by using that the map exp is a group homomorphism along with formulas (4.24), one gets the result.

For the next result we need the winding number $W_I$ defined by (3.16). The derivation and the trace on $I$, needed to build $W_I$, are described in Sect. 4.2

Lemma 4.18 Let $w_I \in I^+$ be the unitary operator defined in Proposition 4.11. Then, it holds true that
\[
W_I(w_I) = 1
\]

Proof An explicit computation provides
\[
(w_I^*-1) \nabla_I(w_I-1) = \ i [p_0 (s_{1,2} - 1)] \left[ p_0 (s_{1,2} - 1), n_2 \right] = i p_0 (s_{1,2} - 1) [s_{1,2}, n_2]
\]
\[
= -i p_0 (s_{1,2} - 1) s_{1,2}
\]
\[
= i p_0 (s_{1,2} - 1).
\]
By applying formula (4.14) one gets
\[
\mathcal{T}_I((w_I^*-1) \nabla_I(w_I-1)) = -i \mathcal{T}_I(p_0 q_0) = -i
\]
since $q_0 s_{1,2} q_0 = 0$. This completes the proof.

We are now in position to provide our main result, namely the proof of equation 1.1.

Theorem 4.19 (Bulk-interface duality for the Iwatsuka magnetic field) Let $\hat{h} \in C^k(A_I)$ (for some $k \geq 1$) be a magnetic Hamiltonian with non-trivial bulk gap detected by $\Delta$ (cf. Definition 3.20). Let $g : \mathbb{R} \to [0, 1]$ be a non-decreasing (smooth) function such that $g = 0$ below $\Delta$ and $g = 1$ above $\Delta$ and consider the unitary operator $u_\Delta := e^{i 2 \pi g(\hat{h})}$ and the associated interface conductance (cf. Definition 3.19)
\[
\sigma_I(\Delta) = \frac{e^2}{\hbar} W_I(u_\Delta).
\]
Let $h := \text{ev}(\hat{h}) = (h_-, h_+ \in C^k(A_{\text{bulk}})$ be the bulk Hamiltonian and for a given Fermi energy inside the bulk gap $\mu \in \Delta$ let $p_{\mu} \equiv (p_{\mu}, -, p_{\mu}, +)$ with $p_{\mu, \pm} := \chi(-\infty, \mu)(h_\pm)$ be the associated Fermi projections. Denote with $N_\pm := \text{Ch}(p_{\mu, \pm}) \in \mathbb{Z}$ the Chern numbers of such projectors. Then it holds true that
\[
\sigma_I(\Delta) = \frac{e^2}{\hbar} (N_+ - N_-).
\]
Proof We can compute \( \sigma_\Delta (\Delta) \) with the topological formula (3.23). From Lemma 4.17 and the bilinearity of the canonical pairing between \( K_1(\mathcal{I}) \) and \( HC^1(\mathcal{I}) \) one obtains

\[
< \exp(\{ p_\mu \}), [\eta_I] > = (N_- - N_+) < [w_I], [\eta_I] > .
\]

Then, equation (3.22) and Lemma 4.18 provide

\[
< \exp(\{ p_\mu \}), [\eta_I] > = (N_- - N_+) W_I(w_I) = N_- - N_+ .
\]

This concludes the proof. \( \square \)

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Appendix A: Crossed Product Structure

In Sect. 2.3 we provided an explicit construction of the magnetic \( C^* \)-algebra \( A_{A_B} \) associated to a vector potential \( A_B \) for the magnetic field \( B : \mathbb{Z}^2 \rightarrow \mathbb{R} \). In this section we will show that the magnetic \( C^* \)-algebra \( A_{A_B} \) is the “concrete” realization of an abstract twisted crossed product \( C^* \)-algebra over \( \mathbb{Z}^2 \). For the general theory of the crossed product \( C^* \)-algebras we will refer to classic monographs [47, 64]. The special case of discrete crossed product \( C^* \)-algebras, which is the most related to our construction, is discussed in detail in [15, Chapter VIII].

Let \( (\mathcal{C}(\Omega_B), \tau, \mathbb{Z}^2) \) be the \( C^* \)-dynamical system associated with the dynamical system \( (\Omega_B, \tau^*, \mathbb{Z}^2) \) described in Sect. 2.6. Consider a pair of (abstract) unitary elements \( u_1, u_2 \) and for every \( \gamma := (\gamma_1, \gamma_2) \in \mathbb{Z}^2 \) set \( u_\gamma := u_{\gamma_1} u_{\gamma_2} \). Let \( \mathcal{C}(\Omega_B)[\mathbb{Z}^2] \) be the set of finite sums

\[
G := \sum_{\gamma \in \Lambda} g_\gamma u_\gamma
\]

with \( g_\gamma \in \mathcal{C}(\Omega_B) \) for all \( \gamma \in \Lambda \) and \( \Lambda \in \mathcal{P}_0(\mathbb{Z}^2) \) a finite subset of \( \mathbb{Z}^2 \). The product in \( \mathcal{C}(\Omega_B)[\mathbb{Z}^2] \) is defined by the rules

\[
u_2 u_1 = f_B u_1 u_2 , \quad u_\gamma g u_{-\gamma} = \tau_\gamma(g)
\]

for every \( \gamma \in \mathbb{Z}^2 \) and \( g \in \mathcal{C}(\Omega_B) \), where \( f_B := e^{iB} \in \mathcal{C}(\Omega_B) \) is the magnetic phase associated to \( B \). The involution is provided by

\[
g u_\gamma = u_{-\gamma} g = \tau_{-\gamma}(g) u_{-\gamma} .
\]

Endowed with these operations \( \mathcal{C}(\Omega_B)[\mathbb{Z}^2] \) acquires the structure of a unital \( \ast \)-algebra. Moreover it can be completed to a Banach \( \ast \)-algebra with respect to the norm

\[
\| G \|_1 := \sum_{\gamma \in \Lambda} \| g_\gamma \|_\infty .
\]

The (universal) enveloping \( C^* \)-algebra [17, Sect. 2.7] obtained from this Banach \( \ast \)-algebra is called the \( B \)-twisted crossed product of \( \mathcal{C}(\Omega_B) \) and is denoted with \( \mathcal{C}(\Omega_B) \rtimes_{\tau, B} \mathbb{Z}^2 \).
In order to better understand the twisted structure of the crossed product $C(\Omega_B) \rtimes_{\tau,B} \mathbb{Z}^2$

one can observe that the mapping $\gamma \mapsto u_{\gamma}$ provides a projective (abstract) unitary representation of $\mathbb{Z}^2$ defined by

$$u_{\gamma} u_{\xi} = \Theta_B(\gamma, \xi) u_{\gamma + \xi}, \quad \forall \, \gamma, \xi \in \mathbb{Z}^2,$$

with phase given by

$$\Theta_B(\gamma, \xi) := \prod_{\nu \in \Lambda(\gamma, \xi)} \tau_{\nu}(f_B)$$

and the product is extended on the finite cell

$$\Lambda(\gamma, \xi) := \left( [\gamma_1, \gamma_1 + \xi_1 - 1] \times [0, \gamma_2 - 1] \right) \cap \mathbb{Z}^2.$$ 

The map $\Theta_B : \mathbb{Z}^2 \times \mathbb{Z}^2 \to U(C(\Omega_B))$ takes value on the unitary elements of $C(\Omega_B)$ and a direct check shows that it satisfies the cocycle condition

$$\Theta_B(\gamma + \xi, \zeta) \Theta_B(\gamma, \xi) = \tau_{\gamma}(\Theta_B(\xi, \zeta)) \Theta_B(\gamma, \xi + \zeta)$$

for all $\gamma, \xi, \zeta \in \mathbb{Z}^2$.

Given a vector potential $A_B$ for the magnetic field $B$ one can consider the representation $\pi_{A_B} : C(\Omega_B) \rtimes_{\tau,B} \mathbb{Z}^2 \to B(\ell^2(\mathbb{Z}^2))$ defined by

$$\pi_{A_B}(u_{\gamma}) := (s_{A_B,1})^{\gamma_1} (s_{A_B,2})^{\gamma_2}, \quad \forall \gamma \in \mathbb{Z}^2, \quad \forall g \in C(\Omega_B)$$

where $s_{A_B,1}, s_{A_B,2}$ are the magnetic translations defined in Sect. 2.2 and $\iota^{-1}(g) \in F_B$ is given by the isomorphism defined in Lemma 2.26. The map $\pi_{A_B}$ coincides with the tensor product of the isomorphism $\iota^{-1}$ with the $B$-twisted (left) regular representation of $\mathbb{Z}^2$. This representation turns out to be faithful (cf. [15, p. 218]) and as a consequence one gets

$$A_{A_B} = \pi_{A_B} \left( C(\Omega_B) \rtimes_{\tau,B} \mathbb{Z}^2 \right).$$

It is also interesting to observe that $A_{A_B}$ can be represented as an iterated crossed product algebra as discussed in [49, Sect. 3.1.1]. Indeed one can check that

$$C(\Omega_B) \rtimes_{\tau,B} \mathbb{Z}^2 \cong \mathcal{Y}_{B,j} \rtimes_{\alpha_k} \mathbb{Z}, \quad \mathcal{Y}_{B,j} := F_B \rtimes_{\alpha_j} \mathbb{Z},$$

where $\{j, k\} = \{1, 2\}$, the crossed product algebra $\mathcal{Y}_{B,j}$ is generated by $F_B$ and $u_j$ along with the relation

$$\alpha_j(g) := u_j g u_j^* := \begin{cases} \tau_{(1,0)}(g) & \text{if } j = 1 \\ \tau_{(0,1)}(g) & \text{if } j = 2 \end{cases} =: \tau_j(g)$$

for all $g \in C(\Omega_B)$ and the crossed product algebra $\mathcal{Y}_{B,j} \rtimes_{\alpha_k} \mathbb{Z}$ is generated by $\mathcal{Y}_{B,j}$ and $u_k$ along with the relation $\alpha_k(g u_j) := u_k(g u_j) u_k^* = \tau_k(g) f_B u_j$ for all $g \in F_B$.

In the special case the magnetic field $B$ is constant in the $n_2$-direction (as in the case of the Iwatsuka magnetic field) it follows that the $\alpha_2$-action is trivial, meaning that it reduces
to the identity $\alpha_2(g) = g$ for all $g \in C^*(\Omega_R)$. In this situation the crossed product algebra $Y_{b,2}$ acquires the following very simple structure

$$Y_{b,2} \simeq C^*(\Omega_R) \otimes C^*_\ell(\mathbb{Z}) \simeq C^*(\Omega_R) \otimes \mathcal{C}(S^1) \simeq (\mathcal{C}(\Omega_R) \times S^1)$$

The first isomorphism involves the (reduced) group $C^*$-algebra $C^*_\ell(\mathbb{Z})$ and is proved in [64, Lemma 2.73] (along with the nuclearity of the various $C^*$-algebras). The isomorphism $C^*_\ell(\mathbb{Z}) \simeq C(S^1)$ is a consequence of the Pontryagin duality [15, Proposition VII.1.1].

### Appendix B: The $K$-Theory of the Iwatsuka Magnetic Hull

Let $\Omega_I = \mathbb{Z} \cup (-\infty) \cup (+\infty)$ be the Iwatsuka magnetic hull described in Example 2.24 and consider the short exact sequence

$$0 \longrightarrow C_0(\mathbb{Z}) \xrightarrow{i} \mathcal{C}(\Omega_I) \xrightarrow{\text{ev}} \mathbb{C} \oplus \mathbb{C} \longrightarrow 0 \quad (B.1)$$

where $C_0(\mathbb{Z})$ is the $C^*$-algebra of sequences vanishing at infinity, $i$ is the inclusion homomorphism and the evaluation homomorphism $\text{ev}$ compute the left and right limits of elements in $\mathcal{C}(\Omega_I)$. The sequence (B.1) is split exact in view of the homomorphism

$$\mathbb{C} \oplus \mathbb{C} \ni (\ell_-, \ell_+) \xrightarrow{j} c(\ell_-, \ell_+) \in \mathcal{C}(\Omega_I)$$

where the element $c(\ell_-, \ell_+)$ is specified by

$$c(\ell_-, \ell_+)(n) := \begin{cases} \ell_- & \text{if } n < 0 \\ \ell_+ & \text{if } n \geq 0. \end{cases}$$

Then, it follows that [63, Corollary 8.2.2]

$$K_j(\mathcal{C}(\Omega_I)) \simeq K_j(\mathcal{C}_0(\mathbb{Z})) \oplus K_j(\mathbb{C} \oplus \mathbb{C}), \quad j = 1, 2.$$

The $K$-theory of $\mathbb{C} \oplus \mathbb{C}$ is easily calculated as $K_0(\mathbb{C} \oplus \mathbb{C}) = \mathbb{Z} \oplus \mathbb{Z}$ and $K_1(\mathbb{C} \oplus \mathbb{C}) = 0$. The $K$-theory of $\mathcal{C}_0(\mathbb{Z})$ is given by $K_0(\mathcal{C}_0(\mathbb{Z})) = \mathbb{Z}^\oplus \mathbb{Z}$ and $K_1(\mathcal{C}_0(\mathbb{Z})) = 0$. The latter fact follows from the isomorphism $K_j(\mathcal{C}_0(\mathbb{Z})) \simeq K^j_{\text{top}}(\mathbb{Z}) \simeq K^j_{\text{top}}(\ast)^{\oplus \mathbb{Z}}$ between the algebraic and the topological $K$-theory [12, Theorem 5]. Another way of achieving the same result is to consider the Pontryagin duality $S^1 = \hat{\mathbb{Z}}$ and the isomorphism $\mathcal{C}_0(\mathbb{Z}) \simeq C^*(S^1)$ where $C^*_\ell(S^1)$ is the (reduced) group algebra of the circle [15, Proposition VII.1.1]. Therefore, one has that $K_0(C^*_\ell(S^1)) \simeq \text{Rep}(S^1) \simeq \mathbb{Z}^{\oplus \mathbb{Z}}$ and $K_0(C^*_\ell(S^1)) \simeq 0$ where $\text{Rep}(S^1)$ denotes the complex representation ring of $S^1$ [12, Sect. 7]. The generators of $K_j(\mathcal{C}_0(\mathbb{Z}))$ are the classes $[\pi_i], \ i \in \mathbb{Z}$, of the projections $\pi_i(n) := \delta_{i,n}$. After putting all the information together, we can describe the $K$-theory of the Iwatsuka magnetic hull as

$$K_0(\mathcal{C}(\Omega_I)) \simeq \bigoplus_{i \in \mathbb{Z}} \mathbb{Z}[\pi_i] \oplus \mathbb{Z}[\pi_-] \oplus \mathbb{Z}[\pi_+], \quad K_1(\mathcal{C}(\Omega_I)) = 0,$$

where $\pi_- := j((1, 0))$ and $\pi_+ := j((0, 1)) - \pi_0$ are the projections at infinity.
Appendix C: The Pimsner-Voiculescu Exact Sequence

In this section we will provide a brief overview on the Pimsner-Voiculescu six-term exact sequence which is the main tool to compute the $K$-theory for crossed product $C^*$-algebras by $\mathbb{Z}$. For the interested reader we refer to the original work [51] and the monograph [9, Chapter V].

Let $\mathcal{Y}$ be a $C^*$-algebra, $\alpha \in \text{Aut}(\mathcal{Y})$ and automorphism and $\mathcal{Y} \rtimes_\alpha \mathbb{Z}$ the crossed product generated by $\mathcal{Y}$ and the unitary $u$ with the relation

$$\alpha(a) = uau^*, \quad \forall a \in \mathcal{Y}.$$ 

The first step of the construction is to define an appropriate short exact sequence of $C^*$-algebras. This is done by considering the tensor product $\mathcal{Y} \otimes \mathcal{K}$, where $\mathcal{K}$ denotes the $C^*$-algebra of compact operators, and the $C^*$-algebras. This is done by considering the tensor product $\mathcal{Y} \otimes \mathcal{C}$ where $\mathcal{C}$ is the (non-trivial) self-adjoint projection given by $P := 1 - VV^* = 1 \otimes (1 - vv^*)$ and $e_{j,k}$ are the rank one operators which generates $\mathcal{K}$. Then, there exists a short exact sequence of $C^*$-algebras

$$0 \longrightarrow \mathcal{Y} \otimes \mathcal{K} \xrightarrow{\varphi} \mathcal{T}_\alpha \xrightarrow{\psi} \mathcal{Y} \rtimes_\alpha \mathbb{Z} \longrightarrow 0,$$

where the map $\psi : \mathcal{T}_\alpha \rightarrow \mathcal{Y} \rtimes_\alpha \mathbb{Z}$ defined by

$$\psi(a \otimes 1) := a, \quad \psi(V) := u.$$

For this reason $\mathcal{T}_\alpha$ is called the Toeplitz extension of the stabilized algebra $\mathcal{Y} \otimes \mathcal{K}$ by the crossed product $\mathcal{Y} \rtimes_\alpha \mathbb{Z}$.

The Pimsner-Voiculescu (six-term) exact sequence is a cyclic sequence which connects the $K$-theory of $\mathcal{Y}$ and $\mathcal{Y} \rtimes_\alpha \mathbb{Z}$, and is given by

$$
\begin{array}{ccc}
K_0(\mathcal{Y}) & \xrightarrow{\beta_*} & K_0(\mathcal{Y}) \\
\downarrow \partial_1 & \xrightarrow{\iota_*} & K_0(\mathcal{Y} \rtimes_\alpha \mathbb{Z}) \\
K_1(\mathcal{Y} \rtimes_\alpha \mathbb{Z}) & \xleftarrow{\iota_*} & K_1(\mathcal{Y})
\end{array}
$$

and it is worth pointing out that this is not exactly the standard six-term exact sequence associated with the short exact sequence (C.1), although it is closely related. The maps $\iota_*$ are induced by the canonical inclusion $\iota : \mathcal{Y} \hookrightarrow \mathcal{Y} \rtimes_\alpha \mathbb{Z}$ and the maps $\beta_*$ are induced by the map $\beta : \mathcal{Y} \rightarrow \mathcal{Y}$ defined as $\beta := 1d - \alpha^{-1}$. The vertical maps are related with the index and the exponential maps for the standard six-term exact sequence in $K$-theory emerging from the short exact sequence (C.1) (cf. [63, Theorem 9.3.2]). More precisely one has that $\partial_0 := k_0^{-1} \circ \text{ind}$ and $\partial_1 := k_0^{-1} \circ \exp$ where $\text{ind} : K_1(\mathcal{Y} \rtimes_\alpha \mathbb{Z}) \rightarrow K_0(\mathcal{Y} \otimes \mathcal{K})$ and $\exp : K_0(\mathcal{Y} \rtimes_\alpha \mathbb{Z}) \rightarrow K_1(\mathcal{Y} \otimes \mathcal{K})$ are the usual index and the exponential maps related to the short exact sequence (C.1) and $k_0 : K_j(\mathcal{Y}) \rightarrow K_j(\mathcal{Y} \otimes \mathcal{K})$, with $j = 0, 1$, is the stabilization isomorphism induced by $a \mapsto a \otimes e_{0,0}$ for every $a \in \mathcal{Y}$. 

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Appendix D: $K$-Theory for a Constant Magnetic Field

In this section the $K$-theory of the magnetic $C^*$-algebra $\mathcal{A}_b$ associated with a constant field of strength $b$ will be described. The key observation is that $\mathcal{A}_b$ is a faithful representation of the noncommutative torus $\mathcal{A}_{\theta_b}$ provided that $\theta_b := b(2\pi)^{-1}$. Let us observe that $b$ enters in the definition of $\mathcal{A}_{\theta_b}$ only modulo $2\pi$. For this reason, without loss of generality, we can assume $0 < \theta_b < 1$ as the condition for a non trivial magnetic field. The $K$-theory of the noncommutative torus $\mathcal{A}_{\theta_b}$ has been investigated in [50–52] and is described in several textbooks like [63, Sect. 12.3] or [22, Chap. 12]. As a consequence of the isomorphism $\mathcal{A}_b \simeq \mathcal{A}_{\theta_b}$ we get

$$K_0(\mathcal{A}_b) = \mathbb{Z}[1] \oplus \mathbb{Z}[p_{\theta_b}] \simeq \mathbb{Z}^2,$$

$$K_1(\mathcal{A}_b) = \mathbb{Z}[s_{b,1}] \oplus \mathbb{Z}[s_{b,2}] \simeq \mathbb{Z}^2.$$  \hspace{1cm} (D.1)

The generators of the $K$-theory of $\mathcal{A}_b$ are quite explicit except for the projection $p_{\theta_b} \in \mathcal{A}_b$ which is known as Powers-Rieffel projection. Our next task is to provide a presentation of $p_{\theta_b}$ optimized for the aims of this work. We will set

$$p_{\theta_b} := s_{b,1}^\ast \partial_1 + \partial_0 + \partial_1 s_{b,1},$$

where $\partial_1 := g(s_2)$ and $\partial_0 := f(s_2)$ are suitable self-adjoint elements of $C^*(s_2) \subset \mathcal{A}_b$. Here we are using the coincidence $s_2 = s_{b,2}$ between the ordinary shift and magnetic translation in view of the election of the Landau gauge for the constant magnetic field. The requirement for $p_{\theta_b}$ of being a projection is automatically satisfied if the following relations hold true:

$$(s_{b,1}^\ast \partial_1) \partial_1 = 0,$$

$$\partial_1 (\partial_0 + s_{b,1} \partial_0 s_{b,1}^\ast) = \partial_1,$$

$$\partial_0^2 + \partial_1^2 + (s_{b,1}^\ast \partial_1 s_{b,1})^2 = \partial_0.$$  \hspace{1cm} (D.2)

The way of implementing these relation is by the isomorphism (induced by the Fourier transform) $C^*(s_2) \simeq C_{per}([0,1])$ where on the right-hand side one has the $C^*$-algebra of continuous function on $[0,1]$ with periodic boundary conditions, i.e. $f(0) = f(1)$. Under this isomorphism $s_2 \mapsto e$ where $e(k) := e^{ik}$ and $s_{b,1} s_{b,1}^\ast = e^{ib}$ $s_2 \mapsto e(\cdot + \theta_b)$ Consider a $0 < \delta < \theta_b$ such that $\theta_b + \delta < 1$ and the function $f$ such that

$$f(e(k)) := \frac{k}{\delta} \chi_{[0,\delta]}(k) + \chi_{(\delta, \theta_b]}(k) \left( 1 + \frac{\theta_b - k}{\delta} \right) \chi_{[\theta_b, \theta_b + \delta]}(k).$$

Define

$$g(e(k)) := \sqrt{f(e(k))(1 - f(e(k)))} \chi_{[0,\delta]}(k) = \sqrt{\frac{k}{\delta} \left( 1 - \frac{k}{\delta} \right)} \chi_{[0,\delta]}(k).$$

One can check that by using $f$ and $g$ above to define $\partial_0$ and $\partial_1$ respectively, then the conditions (D.2) are automatically verified. The crucial identities are $s_{b,1}^\ast \partial_1 s_{b,1} \mapsto g(e(\cdot - \theta_b))$ (which is supported in $[\theta_b, \theta_b + \delta]$) and $s_{b,1} \partial_0 s_{b,1}^\ast \mapsto f(e(\cdot + \theta_b))$ (which must be defined periodically on $[0,1]$). Let us point out that with a standard “smoothing argument” it is possible to replace the continuous function $f$ and $g$ with smooth functions. This implies that it is possible to define the Powers-Rieffel projection inside the smooth algebra $\mathcal{A}_b^\infty$. 

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The relations \((D.2)\) provide other useful identities. Let \(\mathcal{L} := \mathcal{L}(\partial_1)\) be the support projection of \(\partial_1\) (in the von Neumann algebra generated by \(A_b\)). This is by definition the smallest projection such that \(\mathcal{L} \partial_1 = \partial_1 \mathcal{L}\). It is immediate to conclude that \(\mathcal{L}\) is mapped into the characteristic function on the support of \(g \circ e\) under the isomorphism used above, i.e. \(\mathcal{L} \mapsto \chi_{[0,1]}\). Combining \(\mathcal{L}\) with the first relation in \((D.2)\) one gets \((s^*_b,1, \partial_1 s_b,1) \mathcal{L} = 0\). This relation combined with the third equation in \((D.2)\) provides
\[
\partial^2_1 = \mathcal{L}(\partial_0 - \partial^2_0) = (\partial_0 - \partial^2_0) \mathcal{L}.
\]
(D.3)

For the next result we need to recall that two unitary operators \(u_0, u_1 \in A_b\) are said to be homotopic equivalent, denoted \(u_0 \sim u_1\), if there is a continuous map \([0,1] \ni t \mapsto u(t) \in A_b\) such that \(u(0) = u_0, u(1) = u_1\) and \(u(t)\) is unitary for every \(t \in [0,1]\).

**Lemma D.1** The unitary operators \(e^{-i2\pi \partial_0} \mathcal{L}\) and \(s^*_b,2\) are homotopic equivalent in \(A_b\), i.e. \(e^{-i2\pi \partial_0} \mathcal{L} \sim s^*_b,2\).

**Proof** In view of the isomorphism \(C^*(s_2) \cong C_{\text{per}}([0,1])\) it is enough to find an homotopy between the functions \(t(k) := e^{-i2\pi \frac{k}{2} \chi_{[0,1]}(k)}\) and \(e(k) := e^{-i2\pi k}\). Such an homotopy is explicitly given by
\[
[0,1] \ni t \mapsto u_t(k) := e^{-i2\pi \frac{k}{2\pi} (1 - t) \chi_{[0,1]}(k)}
\]
and this completes the proof. \(\Box\)

Let \(\mathcal{L}'\) be the support projection of the shifted operator \(s^*_b,1, \partial_1 s_b,1\). It turns out that \(\mathcal{L}'\) is isomorphically mapped into the characteristic function \(\chi_{\mathcal{L}[\theta_b,\theta_b + \delta]}\).

**Lemma D.2** The unitary operators \(e^{-i2\pi \partial_0} \mathcal{L}'\) and \(s^*_b,2\) are homotopic equivalent in \(A_b\), i.e. \(e^{-i2\pi \partial_0} \mathcal{L}' \sim s^*_b,2\).

**Proof** As above it is enough to find an homotopy between the functions \(t'(k) := e^{-i2\pi \left(1 + \frac{\theta_b}{2}\right) \chi_{\mathcal{L}[\theta_b,\theta_b + \delta]}(k)}\) and \(e(k) := e^{i2\pi k}\). Such an homotopy is explicitly given by
\[
[0,1] \ni t \mapsto u'_t(k) := e^{-i2\pi \left(\frac{1-t}{2\pi} \left(1 - \theta_b - \delta - \frac{k}{2}\right)\right) \chi_{\mathcal{L}[1-t\theta_b,1-t\theta_b + \delta - \frac{k}{2}]}(k)}
\]
and this completes the proof. \(\Box\)

Let us end this appendix with a more precise description of the \(K_0\)-group of \(A_b\). Let \([p] \in K_0(A_b)\). Then, from the first equation of \((D.1)\) one infers the existence of \(M, N \in \mathbb{Z}\) such that
\[
[p] \simeq M [1] + N [p_{\theta_b}]\text{.}
\]
The number \(N\) can be deduced by using the pairing \((3.15)\) along with \(\langle [1], [\xi_b] \rangle = 0\) and \(\langle [p_{\theta_b}], [\xi_b] \rangle = 1\). This implies that \(N = \text{Ch}_{\theta_b}(p)\). The number \(M\) can be deduced from the pairing \(\tau : K_0(A_b) \to \mathbb{Z} + \theta_b \mathbb{Z}\) induced by the trace, i.e. \(\tau([p]) := \mathcal{F}(p)\). Since \(\tau([1]) = 1\) and \(\tau([p_{\theta_b}]) = \theta_b\) one gets that \(M = \mathcal{F}(p) - N \theta_b\).
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