Transitions Between Hall Plateaus and the Dimerization Transition of a Hubbard Chain

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Abstract

We show that the plateau transitions in the quantum Hall effect is the same as the dimerization transition of a half-filled, one dimensional, $U(2n)$ Hubbard model at $n = 0$. We address the properties of the latter by a combination of perturbative renormalization group and Monte Carlo simulations. Results on both critical and off-critical properties are presented.

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The integer plateau transitions in the quantum Hall effect have attracted considerable interests recently [1]. Aside from a recent speculation [2], the effect of electron-electron interaction on the transition remains largely unknown. Indeed, most of the recent theoretical works on this subject are based on numerical analyses of models of non-interacting electrons [1]. Nonetheless, they have generated a wealth of interesting results. Among them, a consensus on the value of the localization length exponent $\nu \approx 2.3$ has been reached [1]. Remarkably this value, obtained without considering electronic interactions, agrees excellently with the experimental findings [3]. This agreement is even surprising in view of the recently measured dynamical exponent $z = 1$ [4] instead of the non-interacting value 2.

Even within the non-interacting theory, our understanding of the plateau transition is still limited. In this paper, we show that the latter is equivalent to the zero temperature, dimerization transition of a half-filled, 2$n$-component Hubbard chain, with $n = 0$. Thus from now on we shall view physics from two different angles - one associated with the plateau transition, and the other with the dimerization transition. For examples, the two neighboring plateau phases correspond to the two dimer phases when the hopping integrals in the Hubbard model are allowed to alternate between neighboring bonds; Tuning the Fermi energy corresponds to adjusting the degree of staggering in the hopping integrals; The variance in the Aharonov-Bohm phases as electrons traverse the edges of the randomly distributed quantum Hall droplets is proportional to the Hubbard $U$.

Crudely speaking, the behaviors of the half-filled, $n = 0$ and $n = 1$ Hubbard chains are quite similar. For example, in both cases a nonzero single-particle gap exists for all value of $U/t$ (i.e. Hubbard interaction/ the averaged hopping matrix elements). For $n = 0$ this is the familiar statement that the disorder averaged single-particle Green’s function is always short-ranged. When the hopping strength alternates between neighboring bonds
(Fig.1b), both models exhibit nonzero spin gaps $\Delta_s$. The spin gap vanishes continuously as the translational symmetry in hopping is gradually restored. Specifically, if we define $R \equiv (t_1 - t_2)/(t_1 + t_2)$, where $t_1$ and $t_2$ are values of the alternating hopping matrix elements shown in Fig.1b, $\Delta_s \sim |R|^\nu$ as $R \to 0$. One of the differences between the $n = 0$ and $n = 1$ models shows up in the value of $\nu$. For $n = 1$, $\nu = 2/3$, while for $n = 0$, $\nu = 2.33 \pm 0.03$ (Fig.2, inset (a)). Furthermore, at $R = 0$ while the gapless spin excitations in $n = 1$ can be identified with those of the $n = 1$ free fermion chain, we found that it is not so for $n = 0$. There are also non-critical properties that differentiate between the two cases. For example, the spontaneous staggered magnetization (SSM) is zero for all $R$ for $n = 1$, whereas for $n = 0$ it is always nonzero (Fig. 3 inset(a))). The latter appears rather counter-intuitive, since it requires the existence of a SSM in the presence of a spin gap! However, when stated in terms of the plateau transition it becomes very natural — the density of states (DOS) stays nonzero across the transition.

We have computed the spin-spin correlation function (or the two-particle Green’s function in the original language) at $R = 0$ for the $n = 0$ Hubbard chain using Monte Carlo method. The results give for $x_1$, the scaling dimension of the spin, $x_1 \approx 0$ (Fig.3). This value is consistent with a non-critical DOS across the plateau transition. The calculation also determines the dimension, $x_2$, of the leading scaling operator formed by the operator product of two spins, being $x_2 = -0.60 \pm 0.02$ (Fig.3). From $x_2$ we can deduce the fractal dimension of the eigen-wavefunctions at $R = 0$ via $D(2) = 2 + x_2 = 1.40 \pm 0.02$. These values agree with those obtained in Refs.[10, 11] while differ somewhat from those of Refs.[12, 13]. In the literature there has been a misnomer, namely, taking $-x_2$ as the decay-exponent of the spin-spin correlation function (or the two-particle Green’s function). In Fig. 3, inset(b) we show the low temperature, equal-time spin-spin correlation for a Hubbard chain of a fixed
length $M$. One does see that it decays with an exponent $-x_2$. Only after studying the finite size scaling with respect to $M$, is one able to conclude that this decay is the property of the scaling function $\mathcal{F}$ in Eq.(7).

The Lyapunov exponents of the transfer matrix are often studied in *typical* long disordered cylindrical samples. In the Hubbard model language, it corresponds to Hubbard-Stratonovich (H-S) decouple the Hubbard interaction and study the single-particle excitation energies in a *specific* (but typical) H-S field at a very low temperature. The Lyapunov exponents $\mu_k$, $k = 1, 2, 3, \ldots$ are defined as $\mu_k \equiv M\Delta_k$, where $\Delta_k$ is the $k$th lowest single-particle excitation energy. For small $\mu_k$, we found $\mu_k = A + kB$ where $A$ and $B$ depend only on $U$ and $t$ (Fig. 2 inset (b)). This result is consistent with that the single-particle Greens function is conformal invariant under a typical H-S field. In addition, it suggests that the associated decay exponent, $\eta_{\text{typical}}$, is given by $\eta_{\text{typical}} = 2A/B = 0.78 \pm 0.02$ [14]. In the literature one often uses $\eta_{\text{typical}} = A/\pi$, and ignores the fact that the spin-wave velocity can deviate from unity [15]. We have explicitly checked that while $A/\pi$ changes with $U/t$, $2A/B$ does not. In a fixed H-S field, the decay exponent for the spin-spin correlation function is twice that of the single-particle one, thus $x_{1,\text{typical}} = \eta_{\text{typical}} = 0.78 \pm .02$. The departure of $x_{1,\text{typical}}$ from $x_1$ usually signifies the existence of an infinite hierarchy of primary operators $\{O_m\}$ formed by the product of $m$ $SU(2n)$ spin operators [16].

Now we step into the more technical part of the paper. We use the Chalker-Coddington network model to describe the plateau transition [2, 15]. Specifically, one considers a square lattice and draws arrows with a fixed chirality around half of the plaquettes (Fig. 1a) representing the semiclassical electron orbits at the edges of quantum Hall droplets. Quantum tunneling is only allowed at the vertices. Away from these vertices, the electrons propagate along the directed links while accumulating Aharonov-Bohm phases. In a previous paper
one of us recognized that if one replaces these link Aharonov-Bohm phases by random tunneling phases, the resulting network model is equivalent to an $SU(2n) \mid n \to 0$ quantum spin chain. We show below that the original network model of Ref. [15] is in fact a $U(2n) \mid n \to 0$ Hubbard chain.

A faithful representation of the network model is depicted in Fig. 1b, where electrons move along the zigzag paths and tunnel across the lines joining them. By adjusting the tunneling matrix elements, one can reproduce the original 2x2 scattering matrix at each tunneling vertex. From now on we take each zigzag path as a constant-$x$ curve and let $y$ measure the displacement along it. We choose the coordinates so that the tunneling vertices are at $(x, y) = \text{integers}$. The (non-interacting) Hamiltonian for the network model is then

$$H = \sum_{x} (-1)^x \int dy \psi^\dagger(x, y) \left[ \frac{\partial_y}{i} - w(x, y) \right] \psi(x, y) + \sum_{x,y} t'_{x,y} \left[ \psi^\dagger(x + 1, y) \psi(x, y) + \text{h.c} \right].$$  \hspace{1cm} (1)

Here $\psi$ is the electron annihilation operator, $t'_{x,y}$ are the tunneling matrix elements, and $w$ is a local random variable satisfying $\langle w(x, y) w(x', y') \rangle = U' \delta_{x,x'} \delta(y-y')$. Upon accumulation, the latter gives rise to the Aharonov-Bohm phases. As in transport theories, we write down the generating functional $Z[w] = \int D[\bar{\psi}, \psi] exp\{-S\}$ for the Green’s function at the Fermi energy ($E = 0$), where

$$S = \sum_{x,p} \int dy (ih) S_p \bar{\psi}_p(x, y) \psi_p(x, y) - H \mid_{\psi \to \psi, \psi^\dagger \to \bar{\psi}}. \hspace{1cm} (2)$$

In the above, the last term stands for replacing the field operators by Grassmann variables in Eq. (1), $p = +(-)$ generates the advanced (retarded) Green’s function, $S_p \equiv \text{sign}(p)$, and $h$ is a positive infinitesimal. Our strategy is to make transformations that turn Eq. (2) into a 1+1 dimensional Euclidean action of an interacting quantum theory in one space dimension [18]. First, we let $U' = U\epsilon$ and $t' = t\epsilon$ where $\epsilon \to 0$. (We have checked that doing so neither affects our ability to tune through the transition, nor does it change the universality class.)
By regarding $\epsilon$ as the lattice spacing in $y$-direction, we are able to write down an action that is continuous in $y$. Next, we let $\psi_p \rightarrow \psi_p(i\psi_p)$ and $\bar{\psi}_p \rightarrow -i\bar{\psi}_p(\bar{\psi}_p)$ for even (odd) $x$'s, and integrate out $w$ using the replica trick to obtain

$$
S_R = \int dy \sum_{x,a} \left[ \bar{\psi}_a(x,y) \partial_y \psi_a(x,y) + (-1)^x h S_p \bar{\psi}_a(x,y) \psi_a(x,y) + \frac{U}{2} \bar{\psi}_a(x,y) \psi_a(x,y) \right] + \int dy \sum_{x,a \neq b} \bar{\psi}_a(x,y) \psi_a(x,y) \bar{\psi}_b(x,y) \psi_b(x,y) + \sum_{x,y,a} t_{x,y} \left[ \bar{\psi}_a(x+1,y) \psi_a(x,y) + \text{h.c.} \right] .
$$

Here $a, b = (p\alpha), (p'\beta)$ with replica indices $\alpha, \beta = 1, ..., n$. In the following we shall take $t_{x,y}$ to be $y$-independent and satisfying $t_x = t_{x+2}$ \[19\]. To tune through the plateau transition we adjust $R \equiv (t_1 - t_2)/(t_1 + t_2)$. Now if we regard $y$ as the Euclidean time $\tau$, then $S_R$ can be viewed as the 1+1 dimensional action of a quantum theory with the following Hamiltonian

$$
H = \sum_{x,a} \left\{ -h(-1)^x S_p \psi_a^\dagger(x) \psi_a(x) - t_x \left[ \psi_a^\dagger(x+1) \psi_a(x) + \text{h.c.} \right] \right\} + \frac{U}{2} \sum_x \sum_a [\psi_a^\dagger(x) \psi_a(x)]^2 .
$$

Eq. (4) is the $U(2n)$ Hubbard Hamiltonian. Note that due to the transformations on $\psi$ and $\bar{\psi}$ the Green’s function of Eq.(2) is related to that of Eq.(4) via $G_p(x, x'; y, y') \rightarrow \theta(x, x') G_a(x, x'; \tau, \tau')$, where $\theta = -i(-1)^x$ if $x, x'$ are on the same sublattice and $\theta = 1$ otherwise. As a result, $G_p(x, x'; y, y') G_{p'}(x', x; y', y) \rightarrow -(1)^{x-x'} G_a(x, x'; \tau, \tau') G_b(x', x; \tau', \tau) .
$

The positive infinitesimal $h$ in Eq.(2) becomes a small staggered magnetic field in the Hubbard model. Hence, the DOS corresponds to the SSM.

For $n = 0$ Eq.(4) is particle-hole symmetric, thus the system is half-filled. At $R = 0$ there are gapless spin excitations. Since the latter in the half-filled $n = 1$ Hubbard model are known to be the same as those of the $U(2)$ free fermion model \[1\], we first check whether the same is true for $n = 0$. Thus we study the stability of the free fermion spin sector against a small Hubbard $U$. As usual we linearize the free-fermion dispersion within an energy window around the left(L) and right(R) Fermi points. Then we project the Hubbard interaction into this energy window. That amounts to keeping four scattering amplitudes:
$g_1$ (L-R backward), $g_2$ (L-R forward), $g_3$ (umklapp), and $g_4$ (L-L or R-R forward) [20]. We then perform a one-loop renormalization group calculation and obtain the following flow equations:

\[
\frac{dg_1}{dl} = -\frac{1}{2\pi} \left[ 2ng_1^2 + 2(n-1)g_3^2 \right], \quad \frac{dg_2}{dl} = \frac{1}{2\pi} \left[ g_3^2 - g_1^2 \right], \quad \frac{dg_3}{dl} = -\frac{1}{\pi} \left[ (2n-1)g_1 - 2g_2 \right] g_3,
\]

and as usual, $dg_4/dl = 0$. In the above we have set $t = 1$ and the bare $g$-values are $g_i = U, \ i = 1,\ldots, 4$. Eq. (5) shows that $n = 1$ is quite special. For $n \geq 1$ $g_1 \to 0$ upon renormalization while $g_2$ and $g_3$ flow to large values [21]. (At $n = 1, g_1 \to 0$ allows the solution of the spin sector of the Hubbard model in term of that of the free fermions.) However, for $n < 1$, all $g_{1,2,3}$ flow to large values upon renormalization. Therefore, the plateau transition is governed by a strong coupling fixed point of the Hubbard model at $n = 0$.

To pin down the strong-coupling fixed point Hamiltonian at $R = 0$, we have followed a recent method used by Nagaosa and Oshikawa [22] and showed that the half-filled $U(2n)$ Hubbard model is equivalent to a $U(2n)/U(n) \times U(n)$ non-linear $\sigma$-model [23], which in turn is equivalent to an $SU(2n)$ spin chain [17, 24]. To support that conclusion, we have compared the critical behaviors of the typical $\Delta_s$ for the $U(2n)$ Hubbard and $SU(2n)$ spin chains at $n = 0$. Specifically, we calculated the typical $\Delta_s$ for both models in long cylindrical systems with linear dimension $M$ and time dimension $L$ ($L >> M$) [25] using the transfer matrix method. The results are given in Fig. 2, where the thermodynamic $\Delta_s,\infty$ is determined such that the data of $(M\Delta_{s,M})$ for different $M$ fall onto a single scaling curve. In presenting the results we have rescaled the $\Delta_s$ values at $R = 0$, which amounts to adjusting a non-universal spin-wave velocity. The excellent agreement between the scaling curves lends support to the statement that these two models are indeed asymptotically equivalent.

In order to study the strong coupling fixed point at $R = 0$, we extend the quantum Monte Carlo algorithm [26] to general $n$. Specifically, we decouple the Hubbard interaction
in Eq. (4) via a spin-symmetric H-S transformation and trace out the Grassmann variables. The resulting partition function

\[ Z = \int \mathcal{D}[w] \text{det}^{2n} P[w] \]

is used to evaluate observables. The single-particle Green’s function

\[ G(x_t, x_{t’}) \equiv \langle \bar{\psi}(x, t) \psi(x’, t') \rangle \]

is given by,

\[ G(x_t, x_{t’}) = Z^{-1} \int \mathcal{D}[w] G_w(x_t, x_{t’}) \text{det}^{2n}[w] P[w], \tag{6} \]

where \( G_w \) denotes the single particle Green’s function computed in a H-S field \( \{w_i(\tau)\} \), \( \text{det}[w] \) is the fermion determinant and \( P[w] = \exp\{- \int d\tau \sum_i w_i(\tau)^2 / 2U\} / \mathcal{N} \) (\( \mathcal{N} \) is a normalization factor). Note that as \( n \to 0 \) the annealed average in Eq. (6) becomes a quenched one. Since in one dimension we expect the Hubbard model to have Lorentz invariance, we approach the zero temperature and infinite size limit by finite size scaling of a sequence of systems with equal dimensionless spatial and temporal size \( M \) \cite{25}, and \( h = \gamma / M^2 \), with \( \gamma \ll 1 \). This choice of \( h \) will be justified below.

An important and often raised issue is whether the network model is able to predict a nonzero DOS across the plateau transition. In terms of the Hubbard model, the issue concerns the existence of a SSM. The staggered magnetization is obtained from the Monte Carlo results of the single-particle Green’s function in Eq. (6). The results are shown in Fig. 3, inset(a) for \( R = 0 \) and 1 as a function of \( M \) at \( U/t = 4 \). They clearly suggest a nonzero DOS across the plateau transition.

To shed more light on the properties of the \( n = 0 \) Hubbard model at \( R = 0 \), we have computed the transverse spin-spin correlation function \( \Gamma(x) \equiv \langle S_{+\alpha}(x, t) S_{-\beta}(0, t) \rangle \) in finite systems. For \( 1 << |x| << M \), one can write down the scaling ansatz for \( \Gamma \) with the help of operator product expansion \cite{14},

\[ |\Gamma(x, h, M)| \sim \left( \frac{1}{|x|} \right)^{2x_1} \left( \frac{|x|}{M} \right)^{x_2} \mathcal{F}(hM^2), \tag{7} \]

where the exponents \( x_1 \) and \( x_2 \) were defined earlier. That \( h \) scales with \( M^{-2} \) in Eq. (7)
follows from the anticipation that the DOS is non-critical. The fact that data collapsing is achieved in Fig. 3 confirms the latter. The results are consistent Eq. (7) with $x_1 \approx 0$ and $x_2 = -0.60 \pm 0.02$. While the value of $x_1$ offers a consistency check on the notion of a non-critical DOS, that of $x_2$ determines the eigen-function fractal dimension at the transition through $D(2) = 2 + x_2 = 1.40 \pm 0.02$. Note that in the thermodynamic limit $h\Gamma(0)$ corresponds to the ensemble averaged inverse participation ratio $P^{(2)}(E = 0)$ introduced by Wegner. As a check we have also determined $D(2)$ from the scaling behavior $h\Gamma(0) \propto M^{-D(2)} \mathcal{G}(hM^2)$. The result is $D(2) = 1.43 \pm 0.04$.

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FIGURE CAPTIONS

**Fig. 1.** The network model (a) and its space(x)-time(τ) representation (b). In (a) plaquettes marked with “1” are occupied by Hall droplets. Also shown in (b) is the corresponding Hubbard chain.

**Fig. 2.** The scaling plots of the typical spin gap for the $n = 0$ Hubbard ($U/t = 4$) and spin chains ($M = 16 \rightarrow 256$). Inset: (a) The exponent $\nu$. (b) The spectrum of Lyapunov exponents.

**Fig. 3.** The scaling plot of $\Gamma$ in Eq.(7) achieved with $x_1 = 0$. $M = 192(\bigcirc)$, $160(\Box)$, $128(\bigtriangleup)$, $94(\triangle)$, $64(\triangledown)$, $48(+)$, $32(\times)$. The solid line gives a slope corresponding to $x_2 = -0.60 \pm 0.02$. Inset: (a) The size-dependence of the SSM in the Hubbard chain at critical point (circles) and for independent dimers (squares). (b) ln $|\Gamma(x)|$ for a $M = 128$ chain at a temperature $\beta t = 100$. 