Weak-Field Limit of a Kaluza-Klein Model with a Nonlinear Perfect Fluid#

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Abstract—The main purpose of our paper is to construct a viable Kaluza–Klein model satisfying the observational constraints. To this end, we investigate a six-dimensional model with spherical compactification of the internal space. Background matter is considered in the form of a perfect fluid with nonlinear equations of state both in the external/our and internal spaces, and the model is set to include an additional bare cosmological constant $\Lambda_6$. In the weak-field approximation, the background is perturbed by a pressureless gravitating mass that is a static pointlike particle. The nonlinearity of the equations of state of the perfect fluid makes it possible to solve simultaneously a number of problems. The requirement that the post-Newtonian parameter $\gamma$ be equal to 1 in this configuration, first, ensures compatibility with the gravitational tests in the Solar system (deflection of light and time delay of radar echoes) at the same level of accuracy as General Relativity. Second, it translates into the absence of internal space variations, so that the gravitational potential exactly coincides with the Newtonian one, securing the absence of a fifth force. Third, the gravitating mass remains pressureless in the external space, as in the standard approach to nonrelativistic astrophysical objects and, meanwhile, acquires an effective tension in the internal space.

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1. INTRODUCTION

Despite the fact that the idea of multidimensionality of space-time is about a hundred years old (starting with the pioneering works by Th. Kaluza and O. Klein [1]), it yet remains popular in modern physical science. On the one hand, this is related to the ongoing attempts to construct theories of unification of physical interactions such as superstrings, supergravity and M-theory which have the most self-consistent formulation in space-time with extra dimensions (see, e.g., [2]). On the other hand, it is encouraged by the efforts to solve the problems of dark matter and dark energy as well as the hierarchy problem by introducing additional dimensions [3, 4]. Obviously, such multidimensional theories should not contradict the experimental data, in particular, the well-known gravitational tests in the Solar system: deflection of light, time delay of radar echoes, and perihelion precession of Mercury.

We investigated this problem in a number of our previous papers [5–12], and one of the main conclusions of these papers is that, in the case of Ricci-flat internal spaces, the model with a pressureless gravitating mass contradicts the gravitational tests [5]. The negative result follows as a consequence of massless fluctuations of the internal space that generate the fifth force [13]. In order to avoid these fluctuations, the gravitating mass should acquire a negative pressure (i.e., tension) a la black strings/black branes [9]. Once the tension is introduced, the model achieves agreement with the gravitational tests for the parameterized post-Newtonian (PPN) parameter $\gamma$ at the same level of accuracy as General Relativity. Nonetheless, the physical reason for a negative relativistic pressure remains unclear, and such a requirement appears to be the main problem of this model.

In the case of spherical compactification of the internal space, the situation is handled differently; i.e., to provoke curved background, one first needs to introduce background matter [8]. For this type of models, it is possible to achieve agreement with gravitational tests in two cases. First, similarly to models with toroidal compactification, we can exclude fluctuations of the internal space with the help of tension (a la black strings/black branes) [9]. Second, in the absence of tension, we may attain suffi-
ciently short ranges of interaction for the fifth force for a large enough mass of radions [8]. At first glance, it seems that the latter option well meets the necessary requirements; it is in agreement with the gravitational tests and, additionally, the gravitating mass lacks tension in the internal space. However, fluctuations of the background matter concentrate around the gravitating mass, and the bare gravitating gravitational tests and, additionally, the gravitating sufficiently short ranges of interaction for the fifth force achieve agreement with observations while effective relativistic pressure in the external space. Certainly, this contradicts the observations (e.g., the Sun does not have such a relativistic pressure), and the only way to avoid such a problem is to introduce, again, tension in the internal space [10, 14].

In [10], an interesting example of a model with spherical compactification was demonstrated; a bare gravitating mass without tension was shown to achieve agreement with observations while effective relativistic pressure in the external/our space was successfully avoided. The result was obtained through fine tuning between the components of a multicomponent background perfect fluid with linear equations of state. An interesting additional aspect of the model was the emergence of an effective tension for the initially pressureless bare gravitating mass. This example led to the next natural question: Is it possible to build a model that has all the properties listed above, but contains a single-component background perfect fluid instead? The present paper gives an affirmative answer. Here, rather than some multicomponent background perfect fluid, we consider just one, called a non-linear perfect fluid, which has nonlinear equations of state in both the external and internal spaces. We demonstrate that this model has all positive properties inherent to the multicomponent perfect fluid case and similar to the model in [10], and fine tuning of the constituent parameters appears to be the price for this result.

The paper is structured as follows: In Section 2, we describe the background model and present metric perturbations together with pressure and energy density fluctuations of the background matter upon introduction of the gravitating source. In Section 3, first-order metric corrections are determined, and fine tuning conditions are imposed on these corrections for the purposes of compatibility with the gravitational tests. The main results and consequences of the fine tuning of parameters are briefly summarized in the concluding Section 4.

2. THE BACKGROUND AND PERTURBED MODELS

We consider the Kaluza-Klein model with spherical compactification of the internal space. The background static metric

\[ ds^2 = g^{(0)}_{ik} dx^i dx^k \]

is defined on a product manifold \( M = M_4 \times S^2 \), where \( M_4 \) is four-dimensional Minkowski space-time, and \( S^2 \) is a two-dimensional sphere of radius \( a \). We also include in the model a bare multidimensional cosmological constant \( \Lambda_6 \). The metrics (1) should satisfy the Einstein equation

\[ R_{ik} - \frac{1}{2} R g_{ik} - \kappa \Lambda_6 g_{ik} = \kappa T_{ik}, \]

where \( \kappa = 2S_6 G_6 / c^4 \) with the total solid angle \( S_6 = 2\pi^5 / \Gamma(5/2) = 8\pi^2 / 3 \), and \( G_6 \) is the gravitational constant in the six-dimensional space-time.

The form of background matter can be obtained from Eq. (2) after substitution of the metric (1):

\[ T_{ik}^{(0)} = \begin{cases} 1/(\kappa a^2) - \Lambda_6 g_{ik}^{(0)} & \text{for } i, k = 0, \ldots, 3; \\ -\Lambda_6 g_{ik}^{(0)} & \text{for } i, k = 4, 5. \end{cases} \]

Such an energy-momentum tensor can be written in the form of a perfect fluid:

\[ (T_{ik}^{(0)}) = \text{diag}(\bar{\epsilon}, -\bar{p}_0, -\bar{p}_0, -\bar{p}_0, -\bar{p}_1, -\bar{p}_1), \]

where we introduce the background parameters

\[ \bar{\epsilon} \equiv \frac{1}{\kappa a^2} - \Lambda_6, \quad \bar{p}_0 \equiv -\left( \frac{1}{\kappa a^2} - \Lambda_6 \right), \quad \bar{p}_1 \equiv \Lambda_6. \]

In what follows, the upper bar always denotes background values.

The energy-momentum tensor in the form (3) or (4) corresponds to the background value of a perfect fluid with nonlinear equations of state in the external and internal spaces: \( p(\bar{\epsilon}) = (p_0(\bar{\epsilon}), p_1(\bar{\epsilon})) \). Up to linear perturbations, this equation of state reads

\[ p(\bar{\epsilon}) = (p_0(\bar{\epsilon}), p_1(\bar{\epsilon})) + \left( \frac{\partial p_0}{\partial \bar{\epsilon}} \right) \delta \bar{\epsilon} + O(\delta \bar{\epsilon}^2) \]

\[ \equiv (\bar{\omega}_0, \bar{\omega}_1) \bar{\epsilon} + (\omega_0, \omega_1) \delta \bar{\epsilon} + O(\delta \bar{\epsilon}^2). \]

Taking into account that \( \bar{p}_0 \equiv p_0(\bar{\epsilon}) \) and \( \bar{p}_1 \equiv p_1(\bar{\epsilon}) \), Eqs. (3) and (4) for the background parameters \( \bar{\omega}_0 \) and \( \bar{\omega}_1 \) yield

\[ \bar{\omega}_0 = -1, \quad \bar{\omega}_1 = \frac{\Lambda_6}{1/(\kappa a^2) - \Lambda_6}. \]

The new parameters \( \omega_0 \) and \( \omega_1 \) denote the first derivatives of pressures in the external and internal spaces that are calculated for the background value of the perfect fluid energy density \( \bar{\epsilon} \). Therefore, pressure
fluctuations in the external and internal spaces can be expressed via fluctuations of the energy density $\delta\varepsilon$ of the perfect fluids as follows:

$$\delta p_0 = \omega_0 \delta\varepsilon, \quad \delta p_1 = \omega_1 \delta\varepsilon.$$  

(8)

It is worth noting that in the case of nonlinear perfect fluids in general, $\omega_0 \neq \omega_0$ and $\omega_1 \neq \omega_1$.

In our model, we assume that the fluctuations (8) are caused by a gravitating mass in the form of a static pointlike particle. This mass is pressureless in both the external and internal spaces. Accordingly, the only nonzero component of its energy density is $\hat{T}_0^0 = \hat{\rho} c^2$, where $\hat{\rho}$ is the mass density of the gravitating object. In the case of uniform smearing of the gravitating mass over the internal space, the multidimensional $\hat{\rho}$ and the three-dimensional $\rho_3$ mass densities are related as $\hat{\rho} = \rho_3/V_2$, where $V_2 = 4\pi a^2$ is the volume of the internal space; $\hat{\rho}_3(r_3) = m\delta(r_3)$ and $r_3 = |r_3| = \sqrt{x^2 + y^2 + z^2}$.

The total perturbed energy-momentum tensor is

$$T^i_k = \hat{T}^i_k + \hat{T}^i_k,$$  

(9)

where

$$\hat{T}^i_k = \begin{cases} 
(\bar{\varepsilon} + \delta\varepsilon)\delta^i_k & \text{for } i, k = 0, \\
-(\bar{p}_0 + \delta p_0)\delta^i_k & \text{for } i, k = 1, 2, 3, \\
-(\bar{\rho}_1 + \delta p_1)\delta^i_k & \text{for } i, k = 4, 5.
\end{cases}$$

(10)

The gravitating mass also perturbs the background metric (1). Since the gravitating source is spherically symmetric and pressure in each factor space is isotropic, the perturbed metric preserves its block-diagonal form (for mathematical justification of this claim, see [12]):

$$ds^2 = [1 + A^1(x, y, z)] c^2 dt^2$$

$$- [1 - B^1(x, y, z)] (dx^2 + dy^2 + dz^2)$$

$$- [a^2 - E^1(x, y, z)] (d\xi^2 + \sin^2 \xi d\eta^2).$$

(11)

We suppose that $\hat{\rho}, \bar{\varepsilon}, \delta p_0, \delta p_1, A^1, B^1$ and $E^1$ are of the same order of smallness.

Our task now is to determine the first-order corrections to the metric coefficients $A^1, B^1$ and $E^1$. To perform it, we should substitute the perturbed metric (11) and the perturbed energy-momentum tensor (9) in the Einstein equation

$$R_{ik} = \kappa \left( T_{ik} - \frac{1}{4} T g_{ik} - \frac{1}{2} \Delta g_{ik} \right).$$

(12)

3. SOLUTION OF THE PERTURBED EINSTEIN EQUATION

First, we consider the diagonal components of the Einstein equation (12). To get the expressions for the components of the Ricci tensor, we can use the results of Appendix A in [8]. Then, the 00-component of (12) is

$$\frac{1}{2} \Delta_3 A^1 = \frac{3\kappa}{4} \left( \delta\varepsilon + \delta p_0 + \hat{\rho} c^2 \right) + \frac{\kappa}{2} \delta p_1. \quad (13)$$

Similarly, for the 11-, 22- and 33- components we have

$$\frac{1}{2} \Delta_3 B^1 = \frac{\kappa}{4} \left( \delta\varepsilon + \delta p_0 + \hat{\rho} c^2 \right) - \frac{\kappa}{2} \delta p_1, \quad (14)$$

and the 44- and 55- components read

$$\frac{1}{2} \Delta_3 E^1 = -\frac{1}{a^2} E^1 + \frac{\kappa}{4} \left[ a^2 \left( \delta\varepsilon - 3\delta p_0 + \hat{\rho} c^2 \right) \right]$$

$$+ \frac{\kappa}{2} a^2 \delta p_1. \quad (15)$$

According to Appendices A and B in [8], the off-diagonal components are either identically zero or finally result in the relation

$$-A^1 + B^1 + \frac{2}{a^2} E^1 = 0. \quad (16)$$

Taking this relation into account, we get

$$\Delta_3 E^1 = \frac{a^2}{2} \left( \Delta_3 A^1 - \Delta_3 B^1 \right)$$

$$= \frac{a^2}{2} \kappa \left[ \delta\varepsilon + \delta p_0 + \hat{\rho} c^2 + 2\delta p_1 \right], \quad (17)$$

and comparing the results with the expression in (15), we obtain a relationship between fluctuations $E^1$ and $\delta p_0$ as

$$\frac{E^1}{a^2} = -\kappa \delta p_0. \quad (18)$$

It is important to stress that the function $E^1$ describes variations of the internal space, i.e., the variations responsible for the appearance of the fifth force that may lead to a contradiction with the gravitational tests. Now, we will demonstrate how we can avoid this problem in the present model.

Let us assume that, similarly to General Relativity, the PPN parameter $\gamma = 1$. To get it, we should put $B^1/A^1 = 1 [15]$; this is our fine tuning condition. As follows from (13) and (14), this takes place if

$$\delta\varepsilon + \delta p_0 + 2\delta p_1 + \hat{\rho} c^2 = 0, \quad (19)$$

which indicates that

$$\delta\varepsilon = \frac{-1}{1 + \omega_0 + 2\omega_1} \hat{\rho} c^2, \quad (20)$$

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taking into account also the expressions in (8). For the condition (19), Eq. (17) takes the form
\[ \Delta_3 E^1 = 0, \]  
with the trivial solution
\[ E^1 = 0. \]  
Hence, the internal space excitation does not take place (a la black string), and there is no such problem as the fifth force. As follows from (18), this is possible for
\[ \delta p_0 = 0 \Rightarrow \omega_0 = 0, \]  
which means that the background perfect fluid is in the extremum position of the equation of state in the external space: \( (\partial p_0/\partial \varepsilon)_\varepsilon = \omega_0 = 0 \). Then, Eq. (20) reads
\[ \delta \varepsilon = \frac{-1}{1 + 2\omega_1} \tilde{\rho} c^2, \quad \omega_1 \neq -1/2. \]  
It can be easily seen that the particular case \( \omega_1 = 0 \Rightarrow \delta \varepsilon = -\tilde{\rho} c^2 \) results in the trivial model with \( A^1 = B^1 = 0 \) and hence is excluded. If \( \omega_1 \neq 0 \), then in accordance with the condition (19), the gravitational potential coincides with the Newtonian one:
\[ A^1 = B^1 = \frac{2 \varphi N}{c^2}, \quad \varphi N = -\frac{G N m_{\text{grav}}}{r_3}. \]  
The gravitating mass \( m_{\text{grav}} \) in the above expression relates to the bare mass \( m \) through
\[ m_{\text{grav}} = m \frac{2 \omega_1}{1 + 2 \omega_1}, \]  
and the Newtonian gravitational constant relates to its multidimensional counterpart via
\[ 4 \pi G_N = \frac{S_6}{V_2} \tilde{G}_6. \]  
A similar relation between the Newtonian and multidimensional gravitational constants was obtained, e.g., in [8]. Equation (25) demonstrates that the condition for positiveness of the gravitating mass also requires positiveness of the parameter \( \omega_1 \).

Eventually, we get a model which, first, does not contradict the observations with respect to the PPN parameter \( \gamma \), and, second, since \( \delta p_0 = 0 \), prevents the gravitating mass from picking up an effective relativistic pressure in the external space. Namely, the gravitating object remains pressureless in our space, as it should. For any \( \omega_1 \neq -1/2, 0 \); however, the gravitating mass acquires an effective tension in the internal space. To show it, we should note that since \( \delta \varepsilon \propto -\tilde{\rho} c^2 \), fluctuations of the background matter concentrate around the gravitating mass. Then, its effective energy density reads
\[ \delta \varepsilon_{\text{eff}} = \delta \varepsilon + \tilde{\rho} c^2 = -2 \omega_1 \delta \varepsilon, \]  
\( \forall \omega_1 \neq -1/2, 0, \)
and for the effective equation of state parameter we get
\[ \omega_{1\text{eff}} = \frac{\delta p_1}{\delta \varepsilon_{\text{eff}}} = \frac{-1}{2} < 0, \]  
which corresponds to the black strings/branes’ equation of state. As we already mentioned in the Introduction, the physical origin of such a type of tensions for the black strings/branes is unclear. Our model proposes a possible mechanism for the occurrence of tension in the internal space for astrophysical objects which initially had negligibly small pressure (as compared to the energy density) in all dimensions.

4. CONCLUSION

In the present paper, we have considered the Kaluza-Klein model where the internal space is subject to spherical compactification. As the background matter providing such compactification, we have introduced the six-dimensional bare cosmological constant together with a perfect fluid that has nonlinear equations of state both in the external and internal spaces. We have fine-tuned the parameters of the model with the requirement that the PPN parameter \( \gamma \) be equal to 1, similarly to General Relativity, which is in very good agreement with observations. In this case, the fluctuations of the internal space proved to be absent, and a fifth force did not arise. Additionally, the gravitating mass remained pressureless in the external/our space. This is an important point since inside astrophysical objects similar to our Sun, the pressure is much smaller than the energy density, and a negligible pressure is a good approximation for the calculation of PPN parameters [5, 15, 16]. The appearance of an effective tension for the gravitating mass in the internal space stands out as a very interesting feature of this model since it presents a possible mechanism for the emergence of tension for initially tensionless gravitating objects.

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