GAUGED HETEROTIC SIGMA-MODELS

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ABSTRACT

The gauging of isometries in general sigma-models which include fermionic terms which represent the interaction of strings with background Yang-Mills fields is considered. Gauging is possible only if certain obstructions are absent. The quantum gauge anomaly is discussed, and the (1,0) supersymmetric generalisation of the gauged action given.
Non-linear sigma-models are important two-dimensional field theories and those that are conformally invariant describe the propagation of a string in a curved space-time [1]. Gauging such sigma-models can give a construction of new conformal field theories, and gauged Wess-Zumino-Witten models provide a lagrangian formulation of the coset construction [2]. More recently, duality symmetry in string theory has been formulated in terms of gauged sigma-models [3], and this has led to the proposal of non-abelian generalisations of duality symmetry [4]. Supersymmetric gauged sigma-models have also been used to construct (p,q) supersymmetric integrable models [5]. In [6,7,8], the gauging of general non-linear sigma-models with Wess-Zumino terms was shown to be possible only if certain obstructions were absent, and the gauged action was given. The (p,q) supersymmetric generalisations of these sigma-models were constructed in [9]. The purpose of this paper is to extend these results to the case of sigma-models with fermionic terms, representing the interaction of strings with background Yang-Mills fields $C_i^{AB}$, in addition to the metric $g_{ij}$, anti-symmetric tensor gauge field $b_{ij}$ and dilaton $\Phi$. In particular, the (1,0) supersymmetric version of such terms describes the propagation of heterotic strings in backgrounds with non-trivial gauge fields [10] and the gauged version can be used to formulate the effect of duality transformations on Yang-Mills fields [12].

The action for a bosonic two dimensional sigma-model with Wess-Zumino term and Fradkin-Tseytlin term is

$$S_0 = \frac{1}{2} \int d^2x \sqrt{h} \left( g_{ij} \partial^\mu \phi^i \partial^\nu \phi^j + \epsilon^{\mu\nu\sigma} b_{ij} \partial^\mu \phi^i \partial^\nu \phi^j + \Phi R \right)$$

(1)

where the $D$-dimensional target space $M$ has metric $g_{ij}(\phi)$, coordinates $\phi^i$ ($i = 1, ... D$) and torsion three-form $H$ given in terms of a potential $b_{ij}$ by $H_{ijk} = \frac{3}{2} \partial^i [b_{jk}]$. It is invariant under the transformation

$$\delta \phi^i = e \lambda^a \xi^i_a$$

(2)

where $\lambda^a$ ($a = 1, ... n$) are infinitesimal constant parameters, $e$ is a constant and $\xi^i_a$
are a set of vector fields on $M$ provided the Lie derivative with respect to $\xi_a$ of $g_{ij}$, $\Phi$ and $H_{ijk}$ vanish, which will be the case if

$$\nabla_{(i\xi_j)a} = 0$$

(3)

(where $\xi_{ia} = g_{ij} \xi^j_a$)

$$\xi^i_a \partial_i \Phi = 0$$

(4)

and $\xi^i_a H_{ijk}$ is an exact two-form, i.e. there is some (globally defined) set of Lie-algebra-valued one-forms $v_a$, defined up to the addition of a closed form, with components $v_{ia}$ such that

$$\xi^i_a H_{ijk} = \partial_j v_{k|a}$$

(5)

Then $\xi_a^i$ are Killing vectors which can be taken to generate some $n$-dimensional isometry group $G$ satisfying

$$[\xi_a, \xi_b] \equiv L_a \xi_b = f^{abc} \xi_c$$

(6)

where $f^{abc}$ are the structure constants of $G$ and $L_a$ denotes the Lie derivative with respect to $\xi_a$.

The vanishing of the Lie derivative of $H_{ijk}$ implies that

$$L_a b_{ij} = \partial_{[i} \Lambda_{j]a}$$

(7)

for some $\Lambda_{ia}$ so that the variation of $b_{ij}$ can be cancelled by an anti-symmetric tensor gauge transformation, or equivalently, the variation of the $b$-term in (1) is a total derivative. If $\Lambda_{ia} = 0$, then the symmetry can be gauged by minimal coupling, i.e. by replacing the derivatives $\partial_{\mu}$ in (1) by gauge-covariant derivatives
\[ D_\mu, \text{ where} \]
\[ D_\mu \phi^i = \partial_\mu \phi^i - e A^a_\mu \xi^i_a \] (8)

and the gauge field \( A^a_\mu \) transforms as
\[ \delta A^a_\mu = \partial_\mu \lambda^a + e f^a_{bc} A^b_\mu \lambda^c \] (9)

In the case in which \( \Lambda_{ia} \neq 0 \), minimal coupling is not sufficient and the gauging is as given in [6,7,8]. The gauging can in principle be given in terms of the \( \Lambda_{ia} \) given in (7), but these are not vector fields in general since the \( b_{ij} \) are not tensors but are connections, and it is more convenient to work in terms of the covariant \( v_{ia} \).

Gauging of the isometry symmetry (2) is possible only if [6,7,8] (i) the \( v_{ia} \) can be chosen to be equivariant, \( i.e. \) chosen so that
\[ \mathcal{L}_a v_{ib} = f_{ab}^c v_{ic} \] (10)

and (ii) if
\[ c_{(ab)} = 0 \] (11)

where
\[ c_{ab} = v_{ia} \xi^i_b, \] (12)

If these two conditions are satisfied, then the gauged action is
\[ S_G = \frac{1}{2} \int d^2 x \sqrt{h} \left\{ g_{ij} D_\mu \phi^i D^\mu \phi^j + \Phi R \right\} \]
\[ + \frac{1}{2} \int d^2 x \sqrt{h} \epsilon^{\mu \nu} \left( b_{ij} \partial_\mu \phi^i \partial_\nu \phi^j + 2e A^a_\mu v_{ia} \partial_\nu \phi^i - e^2 c_{[ab]}(\phi) A^a_\mu A^b_\nu \right) \] (13)

This can be rewritten as [6,8]
\[ S_G = \frac{1}{2} \int d^2 x \sqrt{h} \left\{ g_{ij} D_\mu \phi^i D^\mu \phi^j + \Phi R \right\} \]
\[ + \int \left[ \frac{1}{3} H_{ijk} D_\phi^i D_\phi^j D_\phi^k + \frac{e}{2} v_{ia} D_\phi^i F^a \right] \] (14)

where \( Y \) is a three-manifold whose boundary is the world-sheet \( X \) and the field
strength two-form is $F^a = dA^a - \frac{1}{2} ef_{bc}^a A^b A^c$. As usual, the fields $\phi^i, A^a_{\mu}$ on $X$ are extended to fields on $Y$ in the second term in (14). If (10) is satisfied but (11) is not, then the action (13) is not gauge-invariant, but satisfies

$$\delta S = e^2 \int d^2 x \sqrt{h} \epsilon^{\mu \nu} c_{(ab)} A^a_{\mu} (\partial_{\nu} \lambda^b)$$

which is proportional to the consistent chiral anomaly in two dimensions.

Suppose now that one adds a fermionic term of the form [10]

$$S_f = \frac{i}{2} \int d^2 x \sqrt{h} \psi_A (\nabla_+ \psi)_A$$

where $\psi_A$ are chiral Majorana world-sheet spinor fields that are also sections of an $O(N)$ vector bundle over $M$ with connection $C_i^{AB}(\phi), C_i^{AB} = -C_i^{BA}$, and fibre metric $\delta_{AB}$, which is used to raise and lower the $O(N)$ vector indices $A, B, \ldots = 1, \ldots, N$. The covariant derivative is

$$(\nabla_\mu \psi)_A = \partial_\mu \psi_A - \frac{1}{2} \omega_\mu \psi_A - \partial_\mu \phi^j C_i^{AB} \psi_B$$

where $\omega_{\mu} = \frac{1}{2} \epsilon_{ab} \omega_{\mu}^{ab}, \omega_{ab}^{ab}$ is the world-sheet spin-connection and $\nabla_\pm = e^\mu_{\pm} \nabla_\mu$ where $e_a^\mu (a = \pm)$ are zweibeins, with $e_{\pm}^\mu = \pm \epsilon_{\mu \nu} e^\nu_{\pm}$. This is formally invariant under the $O(N)$ gauge transformations

$$\delta \psi_A = M_A^B \psi_B, \quad \delta C_i = \partial_i M - [C_i, M]$$

with parameter $M_A^B(\phi)$. Under an arbitrary variation of the fields, the action (16) changes by

$$\delta S_f = i \int d^2 x \sqrt{h} \left\{ (\Delta \psi_A)(\nabla_+ \psi)_A - \frac{1}{2} \delta \phi^i \partial_+ \phi^j G_{ij}^{AB} \psi_A \psi_B \right\}$$

where the field strength is

$$G_{ij} = \partial_i C_j - \partial_j C_i - [C_i, C_j]$$
and the covariant variation is defined by

$$\Delta \psi_A = \delta \psi_A - \delta \phi^i C_i^{AB} \psi_B$$  \hfill (21)

The transformation (2) for constant \( \lambda \) will lead to a symmetry of the action \( S_f \) if the connection \( C_i \) is invariant up to a gauge transformation:

$$\mathcal{L}_\alpha C_i = \nabla_i \kappa_a$$  \hfill (22)

for some \( \kappa_a^{AB}(\phi) \), as the variation of the action can then be cancelled by an \( O(N) \) transformation of \( \psi_A \), \( \delta \psi_A = \lambda^a \kappa_a^{AB} \psi_B \). This condition has been discussed in [11], where particular attention is paid to global aspects. It can be reformulated covariantly as follows. The condition for there to be an isometry symmetry is that the field strength satisfy

$$\xi^i_a G^{AB}_{ij} = \nabla_i \mu_a^{AB}$$  \hfill (23)

for some \( \mu_a^{AB}(\phi) \). This is equivalent to (22) with

$$\mu_a = \kappa_a - \xi^i_a C_i$$  \hfill (24)

The \( \kappa_a \) are not \( O(N) \)-covariant (they transform as a connection), and it is more convenient to work with the \( \mu_a^{AB} \) defined by (23), which transform covariantly under \( O(N) \) transformations

$$\delta \mu_a = [M, \mu_a]$$  \hfill (25)

just as for the Wess-Zumino term it was better to work with the \( v_{ia} \) rather than the \( \Lambda_{ia} \). If (23) is satisfied, then the action (16) is invariant under the rigid transformations given by (2) and

$$\Delta \psi_A = e \lambda^a \mu_a^{AB} \psi_B$$  \hfill (26)

Under the transformations given by (2),(26), with local \( \lambda(x) \), the action \( S_f \)
varies by

\[ \delta S_f = e \int d^2 x \sqrt{h} \partial_+ \lambda^a J_{a_-} \]  

(27)

where \( J_{a_-} \) is the Noether current

\[ J_{a_-} = \frac{i}{2} (\psi_A \mu^A_B \psi_B) \]  

(28)

This variation can be cancelled by adding the Noether coupling \(-eA_+J_-\) to obtain

\[ S_g = \frac{i}{2} \int d^2 x \sqrt{h} \left( \psi_A (\nabla_+ \psi)_A - eA^a_+ (\psi \mu_a \psi) \right) \]  

(29)

where

\[ (\nabla_+ \psi)_A = (\nabla_+ \psi)_A - eA^a_+ \mu^A_B \psi_B = \partial_+ \psi_A - A^{AB}_+ \psi_B, \]  

\[ A^{AB}_\mu = eA^a_\mu \mu^A_B + C_i^{AB} \partial_\mu \phi^i \]  

(30)

This action is then fully gauge-invariant provided the \( \mu_a \) are equivariant, \( i.e. \) they satisfy

\[ \hat{\mathcal{L}}_{a} \mu_b - [\mu_a, \mu_b] = f_{ab}^c \mu_c \]  

(31)

where \( \hat{\mathcal{L}}_a \) is the gauge-covariant Lie derivative, which for tensors \( T_{ij...}^{AB} \) transforming according to the adjoint of \( O(N) \) is given by

\[ \hat{\mathcal{L}}_a T_{ij...}^{AB} = \mathcal{L}_a T_{ij...}^{AB} - \hat{\xi}_a[T_{ij...}]^{AB} \]  

(32)

This is the analogue of the equivariance condition (10).

To summarise, the action (16) is invariant under rigid isometries provided that the field-strength satisfies (23) for some \( \mu_a \), and this can be promoted to a local
symmetry provided the $\mu_a$ satisfy the equivariance condition (31), in which case the gauged action is (29). Note that (23) is equivalent to the condition that

$$\mathcal{L}_a G_{ij} = [\mu_a, G_{ij}]$$

for some $\mu_a$, so that the gauge-covariant Lie derivative of the field-strength vanishes up to a gauge transformation. The action (29) is also invariant under the $O(N)$ transformations (18),(25).

As an example, consider the case in which the vector bundle is the tangent bundle and $C^{AB}_i$ is the torsion-free spin-connection, where $A, B$ are now tangent space indices with respect to a target space vielbein $E^A_A$. Then $G^{AB}_{ij} = R_{ijAB}$ where $R_{ijAB}$ is the curvature tensor and (23) is automatically satisfied with $\mu^{AB}_a$ proportional to $E^A_A E^B_B \nabla_i \xi^j$.

The gauge variation of the connection $\mathcal{A}_+$ defined by (30) is

$$\delta \mathcal{A}^{AB}_+ = \partial_+ \lambda^{AB} - [\mathcal{A}, \lambda]^{AB}, \quad \lambda^{AB} \equiv \lambda^a \kappa^{AB}_a$$

so that the action (29) is manifestly invariant under the transformations (34) and $\delta \psi_A = -e \lambda^{AB} \psi_B$, which is equivalent to (26). Quantum mechanically, the gauge symmetry of the chiral fermion action (29) is anomalous, with the variation of the effective action proportional to

$$\Delta = \int d^2 x \sqrt{h} \mathcal{A}^{AB}_+ \partial_\lambda^{AB}$$

Adding a counterterm proportional to $\text{Tr} (\mathcal{A}_+ \mathcal{A}_-)$, the anomaly becomes proportional to

$$\Delta = \int \mathcal{A}^{AB} d\lambda^{AB}$$

where $\mathcal{A}^{AB} = \mathcal{A}^{AB}_\mu dx^\mu$. When rewritten in terms of $A^a$ and $\lambda^a$, this contains $d\phi$ and $d\phi d\phi$ terms.
It is straightforward to extend these results to the (1,0) supersymmetric sigma-model. For a flat world-sheet with $h_{\mu\nu} = \eta_{\mu\nu}$, flat (1,0) superspace has coordinates $x^+, x^-$ and $\theta$ and flat superspace derivative $D$ with $D^2 = i\partial_+$. The (1,0) supersymmetric generalisation of (1) is given by [10]

$$S = -i \left[ \int d^2 x \, d\theta \, g_{ij} D\phi^i \partial_- \phi^j + \int d^2 x \, dt \, d\theta \, H_{ijk} \partial_t \phi^i D\phi^j \partial_- \phi^k \right],$$

(37)

where $\phi$ is now a superfield, i.e. a map from the (1,0) superspace into $M$. The transformations (2), now involving superfields, are rigid symmetries of the action (37) provided that the vector fields $\xi^i_\alpha$ satisfy the same conditions as in the bosonic case. To promote these rigid symmetries to local ones, it is necessary to introduce a (1,0) super Yang-Mills multiplet. This is described by gauge superfields $A(x, \theta), A_+(x, \theta), A_-(x, \theta)$ which can be used to define gauge-covariant derivatives $\nabla, \nabla_-, \nabla_+$. These are taken to satisfy the constraints [13]

$$[\nabla, \nabla] = 2i\nabla_+, \quad [\nabla, \nabla_-] = W, \quad [\nabla_+, \nabla_-] = F, \quad (38)$$

with all other super-commutators equal to zero. In equation (38), the field strength $W^a$ is an unconstrained superfield while the Bianchi identities imply that $F^a$ is proportional to $DW$. Acting on sigma-model fields, these become (setting $e = -1$)

$$\nabla \phi^i = D\phi^i + A^a \xi^i_\alpha(\phi), \quad [\nabla, \nabla_-] \phi^i = W^a \xi^i_\alpha(\phi)$$

(39)

etc. Gauging is possible if and only if it is possible for the corresponding bosonic model, in which case the action for the gauged (1,0) sigma model is

$$S = -i \int d^2 x \, d\theta \left[ g_{ij} \nabla \phi^i \nabla_- \phi^j + b_{ij} D\phi^i \partial_- \phi^j - A^a u_{ia} \partial_- \phi^i + A_+^a u_{ia} D\phi^i + A_+^a A_-^b c_{[ab]} \right],$$

(40)

The (1,0) supersymmetric generalisation of (16) is [10]

$$S = \frac{1}{2} \int d^2 x \, d\theta \psi_A \tilde{D}\psi_A$$

(41)
where

\[ \tilde{D}\psi_A = D\psi_A - D\phi^i C_i^{AB} \psi_B \]  

(42)

and \( \psi_A \) are now fermionic superfields. Again, this is invariant under rigid isometry symmetries if and only if the connection \( C_i(\phi) \) satisfies (23) for some \( \mu_a \), and the rigid symmetry can be gauged if \( \mu_a \) is equivariant (31), in which case the gauged action is the \( (1,0) \) supersymmetrisation of (29), given by

\[ S = \frac{1}{2} \int d^2x d\theta \psi_A \tilde{D}\psi_A \]  

(43)

where

\[ \tilde{D}\psi_A = \tilde{D}\psi_A + A^a \mu_a^{AB} \psi_B \]  

(44)

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