General relativistic plasma in higher dimensional spacetime

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\textbf{Abstract.} The well known \((3 + 1)\) decomposition of Thorne and Macdonald is invoked to write down the Einstein–Maxwell equations generalized to \((d + 1)\) dimensions and also to formulate the plasma equations in a flat Friedmann–Robertson–Walker-like spacetime in higher dimensions. Assuming an equation of state for the background metric we find solutions also as dispersion relations in different regimes of the universe in a unified manner, both for magnetized and for unmagnetized cold plasma. We find that for a free photon in an expanding background we get maximum redshift in 4D spacetime, while for a particular dimension this occurs in the pre-recombination era. Further wave propagation in magnetized plasma is possible for a restricted frequency range only, depending on the number of dimensions. It is worth pointing out that, unlike the case for the special relativistic result, this allowed range evolves with time. Interestingly the dielectric constant of the plasma media remains constant, not sharing the expansion of the background, which generalizes a similar 4D result of Holcomb and Tajima for a radiation background to the case of higher dimensions with cosmic matter obeying an equation of state. Further, like for the flat space static case, we observe the phenomenon of Faraday rotation in the higher dimensional case also.

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1. Introduction

While great strides have been made by general relativists in addressing the issues emerging from the recent observations in the field of astrophysics and cosmology, and despite the fact that more than 90% of the cosmic stuff in stellar interior and intergalactic spaces is made up of matter in the plasma state, the much sought after union between plasma dynamics and general relativity still remains elusive. Although we occasionally come across stray works like ‘Plasma suppression of large scale structure formation in the universe’ [1], and also even simulation techniques [2], where it is argued that the geometry of the spacetime can be purported to be structured by observing sound waves in primordial plasmas, the fact remains that as the field equations both in general relativity and in plasma dynamics are highly non-linear it is very difficult to obtain closed form solutions in physics. So either numerical relativity or a linearized approximation of the plasma equations is preferred, before going in for more complicated non-linear phenomena. If we briefly trace the thermal history of the universe, theorists believe that from approximately $t = 10^{-3}$ to 1 s at temperatures $T > 10^{10}$ K there was an electron–positron plasma at ultra-relativistic temperatures. With cooling, the plasma comprises mainly electron and hydrogen ions, perhaps with a small amount of helium and other light elements, in thermal equilibrium with photons. This period has come to be known as the radiation dominated era, because it is believed that the energy density of photons much exceeds that of matter [3]. As the temperature decreased further with expansion, at around $t \sim 10^{13}$ s, recombination of ionized hydrogen atoms took place with consequent decoupling of matter and radiation, and at a certain stage the universe became matter dominated. Consequently the temporal behaviour of the scale factor correspondingly changes with the evolution of matter at different timescales. However, the studies of the effects of this expansion on the electromagnetic interactions of the matter, including the longitudinal and transverse modes, have not so far received the attention they deserve.

Another area of current interest is the role of an external magnetic field in many astrophysical systems and cosmology, and possible sources for the field on different scales [4], although the origin of the field continues to evade plausible physical...
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explanation [5], so far. Granted that temperature (particularly for the early universe) is a known enemy of magnetism, there are no compelling reasons that magnetic fields should not have been present in the early universe either. Indeed, the presence of large scale magnetic fields in our observed universe is a well established experimental fact. Since the first evidence for them in diffuse astrophysical plasmas beyond the solar corona [4, 5], magnetic fields have been detected in our galaxy and in our local group through Zeeman splitting and through Faraday rotation measurements of linearly polarized radio waves. The Milky Way possesses a magnetic field whose strength is of the microgauss order, corresponding to an energy density roughly comparable with the energy density today stored in the cosmic microwave background radiation (CMBR) energy spectrum peaked around a frequency of 30 GHz. Faraday rotation measurements of radio waves from extra-galactic sources also suggest that various spiral galaxies are endowed with magnetic fields whose intensities are of the same order as that of the Milky Way. The existence of magnetic fields at even larger scales (intergalactic scale, present horizon scale, etc) cannot be excluded, but it is still quite debatable since, in principle, dispersion measurements (which estimate the electron density along the line of sight) cannot be applied in the intergalactic medium because of the absence of a pulsar signal. Given the fact that a seed field does exist, its amplification mechanism, through the so called dynamo effect, is relatively well understood [6] for an expanding cosmological model with the help of magnetohydrodynamical (MHD) equations [7]. On the other hand, when discussing a primordial magnetic field one should also bear in mind that a large value would create significant anisotropy of the background geometry [8] with the consequent impact on CMBR findings. As the large scale isotropy of the CMBR excludes that possibility fairly well there should be an efficient mechanism for rapidly damping that field.

On the other hand, there has been, of late, a resurgence of interest in physics in higher dimensional spacetime [9] in attempts to unify all the forces in Nature, to give a physical explanation of the current accelerating era of the universe without bringing in any hypothetical quintessential type of scalar field [10] by hand, in the newly fashionable area of brane cosmology [11] where the gravity is supposed to act in the bulk while other forces act in the physical 3D space. This has also received serious attention in the recently proposed induced matter theory pioneered by Wesson and others [12]. Most importantly, higher dimensional spacetime and cosmological plasma have one thing in common—both are very relevant in the context of the early universe. While early universe may be loosely viewed as the history of the evolution of matter in the plasma state, it can also be shown that starting from a higher dimensional phase, Einstein’s generalized field equations dictate results such that as time evolves the 3D space expands while the extra dimensions shrink until the Planckian length is reached, when some stabilizing mechanism (for example, quantum gravity, the Casimir effect, a repulsive potential) halts the shrinkage down to a very small length, say of Planckian size, so as to be invisible to the low energy physics at the moment. So the world around us appears manifestly three dimensional. But it should be emphasized that the timescales for the spontaneous self-compactification (SSC) and the onset of nucleosynthesis clearly differ, with SSC occurring much earlier. So many of our findings in MHD lose much of their relevance when working in higher dimensional spacetime. While the literature abounds with works on the effects of the expanding background on the matter distribution and vice versa, and also on a large
number of other physical processes, scant attention has been paid so far to addressing the issues resulting from the expanding universe on the propagation of, say, electromagnetic waves, and also the interactions in plasma media. To the authors’ knowledge, Holcomb and Tajima (HT) [13, 14], Banerjee et al [15] and later Dettmann et al [16] made important contributions in this regard. HT investigated the electromagnetic wave propagation in a radiation dominated and later in a matter dominated background, both with and without any plasma material, and also generalized it to the magnetized cases, both warm and cold. For the ultra-relativistic case it is observed that all the modes redshift at the same rate; i.e., photons are, so to say, self-similar. Although not explicitly pointed out in their works, this, to our mind, may be a direct consequence of the conformal flatness of the FRW metric chosen by them. Later Banerjee et al discussed this in a slightly more general way. Dettmann et al got similar results starting from a kinetic theory approach. HT’s findings for the matter dominated model, however, differ sharply from those of the first paper in the sense that while in the unmagnetized plasma case the photons redshift identically, here for the Alfvén waves the frequency redshifts in a bizarre fashion, unlike in the case of free photons, depending on the magnitudes of the plasma density and also the strength of the external magnetic field. On the other hand, Dettmann shows that even for the unmagnetized plasma the plasma oscillations and the photons do not share an identical temporal dependence if they are decoupled. They, however, treated the whole situation from kinetic energy considerations. In the present work we have investigated the plasma dynamics in an expanding higher dimensional background in a very general way. Taking a \((d + 1)\)-dimensional flat FRW type of metric as the background, which one of us [17] derived earlier in a different context, we first take the case of propagation of an electromagnetic wave in vacuum. To make things very general we assume an equation of state for the background of \(p = \gamma \rho\). Taking \(\gamma = 1/d\) (as we are dealing with a \((d + 1)\)-dimensional case) for the radiation and \(\gamma = 0\) for the dust case in the resulting solution, we observe that solutions are very similar to the special relativistic form except that the frequencies redshift depending on \(\gamma\) and the field amplitude is no longer a constant decaying with the expansion rate, reminiscent of the acoustic case where the damping occurs due to some form of dissipating mechanism. Here the expansion of the background, in a sense, takes the role of dissipation, causing this type of damping. It is observed that the redshift decreases with the number of dimensions, being maximum in the usual 4D case. Moreover, redshift is less in the post-recombination era compared to the early universe, as expected. After briefly carrying out the so called \((3 + 1)\) decomposition of the Maxwell’s equations generalized to \((d + 1)\)-dimensional spacetime in section 2, we investigate the propagation of an electromagnetic wave in vacuum in section 3. In section 4 the electrostatic oscillations are very briefly discussed for a cold plasma and the dispersion relations obtained both for transverse and for longitudinal modes. In section 5 we discuss in some detail the propagation of an electromagnetic wave in cold plasma in the presence of an ambient and homogeneous magnetic field, both parallel and perpendicular to the wave propagator \(k\). The presence of the magnetic field introduces newer and interesting modes of oscillation, creating left and right circularly polarized waves, resulting in the well known classical phenomenon of Faraday rotation. We also find that the wave propagation in the plasma is possible for a certain range of frequencies only and this range critically depends on the number of dimensions. The paper ends with a short discussion in section 6.
2. Field equations and the formalism

We extend here the $(3 + 1)$ decomposition of GR as formulated by Arnowitt, Deser and Misner (ADM) [18] to a higher dimensional spacetime of $(d + 1)$ dimensions. The ADM formalism was developed mainly to address the issue associated with numerical relativity and also quantization of gravity fields, especially when the spacetime has considerable symmetry. This work is connected with plasmaphysics in curved spacetime. To make use of the intuition from the known results of MHD in flat spacetime it is preferable to split the ordinary electromagnetic field tensor $F^{\mu\nu}$ into electric and magnetic fields $E$ and $B$ in terms of which the field equations are more familiar. As the split formalism has been extensively discussed and used in the literature [13, 14, 19] we shall only very briefly touch upon its salient features as extended to higher dimensions.

We define a set of observers (fiducial observers or FIDO) at rest in the space spanned by the hypersurfaces of constant universal time, having a $d$-velocity vector field, $n$ orthogonal to spatial slices. It is well known that for a rotation-free spacetime (as we are dealing with here)

$$n_{i;j} = \sigma_{ij} + \frac{1}{d}\theta\gamma_{ij} - a_{i}n_{j}$$

where $\sigma_{ij}$ is the shear of the Eulerian world lines given by

$$\sigma_{ij} \equiv \frac{1}{2} \left( n_{i;\mu}\gamma_{j}^{\mu} + n_{j;\mu}\gamma_{i}^{\mu} \right) - \frac{1}{d}\theta\gamma_{ij}.$$  

Here $\gamma_{ij}$ are $d$-metric spatial tensors,

$$a^{i} = n_{i}^{j}n^{j}$$

and

$$\theta = n_{i}^{i}(-K)$$

are the usual acceleration and expansion scalars, and $K$ is the trace of the extrinsic curvature, $K_{ii}^{\nu}$. For our metric we take the $(d + 1)$-dimensional generalized FRW spacetime as

$$ds^{2} = dt^{2} - A^{2}\left(dx^{2} + dy^{2} + dz^{2} + d\psi_{n}^{2}\right)$$

$(n = 5, 6, 7, \ldots, d)$ where $A \equiv A(t)$ is the scale function. For this particular metric two relevant quantities $\alpha = dr/dt$ (the lapse function—the rate of change of fiducial proper time with respect to that of universal time) and the shift vector $\beta$ (signifying how much spatial coordinates are shifted as one moves from one hypersurface to the other) naturally reduce to $\alpha = 1$ and $\beta = 0$.

In an earlier work [17] one of us extensively discussed the $(d + 1)$-dimensional isotropic and homogeneous spacetime and assuming an equation of state, $P = \gamma\rho$, found the scale factor as $(P = \text{pressure}, \rho = \text{energy density})$

$$A \sim t^{2/(d(1+\gamma))} = t^{n}, \quad n = \frac{2}{d(1+\gamma)}.$$


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With the extrinsic curvature scalar defined as

\[ K = -\theta = -d \frac{\dot{A}}{A} = -d \frac{n}{t} \]  

we finally write down the Maxwell equations [20]–[22] (see [20] for the (3 + 1) split for more details) generalized to \((d + 1)\) dimensions as

\[ \nabla \cdot E = 4\pi \rho_e \]  
\[ \nabla \cdot B = 0 \]  
\[ \frac{\partial E}{\partial t} = KE + cA^{-1} \nabla \times B - 4\pi J \]  
\[ \frac{\partial B}{\partial t} = KB - cA^{-1} \nabla \times E \]  
\[ \frac{\partial \rho_e}{\partial t} = K \rho_e - \nabla \cdot J \quad \text{(charge conservation)} \]  

and finally the particle equation of motion in \((d + 1)\) dimensions as

\[ \frac{DA^{d-1}p}{D\tau} = A^{d-1}q \left( E + A\frac{v}{c} \times B \right) \]  

or

\[ \frac{Dp}{D\tau} = \frac{d - 1}{d} Kp + q \left( E + A\frac{v}{c} \times B \right) \]  

where

\[ \frac{D}{D\tau} = \frac{1}{\alpha} (\partial_t + v \cdot \nabla) \]  

is the convective derivative and the \(d\)-momentum is

\[ p = m_e \Gamma v \]  

\((m_e\) is the rest mass, \(\Gamma\) is the boost factor, and \(v\) is the \(d\)-velocity\). We thus see that for the simple metric given by equation (5) the decomposed \((d + 1)\)-dimensional Maxwell equations closely mimic the flat space counterparts with some additional inputs from curved geometry (e.g., \(A\) and \(K\) terms).

Here \(\nabla \cdot\) and \(\nabla \times\) are the ordinary Minkowskian divergence and curl in Cartesian coordinates. In what follows we shall consider, for simplicity, the small amplitude linear theory such that the convective derivative simply reduces to ordinary derivative, \(d/dt\).
3. Electromagnetic waves in vacuum

With the set of equations split to the \((d + 1)\) formalism we are now in a position to attempt applications in varied plasma phenomena. To start with we discuss briefly the propagation of an electromagnetic wave in free space in the expanding background. Using equations (9)–(11) we get via \(\nabla \cdot \mathbf{E} = 0\) (for vacuum) the wave equation as

\[
A^2 \ddot{E} + (2d + 1) A \dot{A} \dot{E} + \left( d^2 \ddot{A}^2 + dA \dot{A} \right) E = c^2 \nabla^2 E. \tag{17}
\]

Assuming separation of variables in an electric field,

\[
E(x, t) = E_t E_r,
\]

we finally get

\[
\dot{E}_t + (2d + 1) \frac{\dot{A}}{A} \dot{E}_t + \left[ d \left( \frac{\ddot{A}}{A} + d \frac{\dot{A}^2}{A^2} \right) + \frac{k_i^2 c^2}{1 + n} \right] E_t = 0 \tag{19}
\]

which, through equation (6), finally reduces to

\[
t^2 \dot{E}_t + (2d + 1) n t \dot{E}_t + \left[ dn (dn + n - 1) + k_i^2 c^2 t^{2(1-n)} \right] E_t = 0 \tag{20}
\]

(here \(k_i\) is a separation constant) corresponding to some initial fiducial time \(t_i\). We shall subsequently see that \(k_i\) is also identified with the \(d\)-dimensional wavevector.

On the other hand, as we are dealing with a homogeneous world, the spatial equation remains unchanged, yielding a solution \(e^{i(k_i \cdot r)}\) as in the special theory of relativity. A little algebra shows that the time equation is reducible to a Bessel equation of order \(\frac{1}{2}\) as in the 4D case. Thus dimensionality or the equation of state has apparently no role in determining the order of the equation. Plugging everything together we get

\[
E = E_0 t^{(1-(2d+1)n)/2} H_{1/2}^{(2)} \left[ \frac{t^{1-n}}{1-n} k_i c \right] e^{i(k_i \cdot r)} \tag{21}
\]

where \(H_{1/2}^{(2)}\) is a Hankel function of order \(\frac{1}{2}\). Replacing the asymptotic form of \(H_{1/2}\) we get

\[
E = E_0 \sqrt{\frac{2k_i c d(1 + \gamma)}{\pi \{d(1 + \gamma) - 2\}}} t^{-2/(1+\gamma)} e^{-(ik_i c d(1+\gamma)/(d(1+\gamma)-2))} t^{d(d(1+\gamma)-2)/d(1+\gamma)} e^{i(k_i \cdot r)}. \tag{22}
\]

This represents a \(d\)-dimensional ‘damped’ harmonic wave such as one encounters in mechanical vibration. While in mechanical motion the damping occurs due to friction, here the expansion of the universe seemingly causes some sort of damping. For the pre-recombination era in \((d + 1)\)-dimensional spacetime, \(\gamma = 1/d\) and the damping factor is \(A^{-d}\). Hence the damping of the wave amplitude apparently decreases with the number of dimensions. This finding merits some explanation. We remarked earlier that the amplitude decay is somewhat geometrical in nature, because of the expansion and curvature. In that case, as the expansion rate decreases with the dimension, one expects the damping to be larger in 4D, but a brief inspection of the last relation shows that when we plug in the expression for \(A\) (equation (6)) the last relation further reduces to \(t^{-2d/(d+1)}\). So the damping actually increases in higher dimensional spacetime. On the other hand, for a fixed \(d\) the damping factor is \(t^{-2}\) for the post-recombination era (\(\gamma = 0\)
(see figure 1). Alternatively, for the case of a very large number of dimensions, the damping asymptotically reaches $t^{-2}$, a form set for the post-recombination era. It has not escaped our notice that the scaling of $E$ or $B$ for $\gamma = 0$ is independent of the number of dimensions, unlike the radiation case. So the amplitude factor gets increasingly damped as the universe ages. The fact that the damping of $E$ or $B$ increases with the number of dimensions in the early universe has a number of interesting theoretical implications. Firstly we mentioned in the introduction that in order for the universe to evolve isotropically according to the FRW model there should be efficient mechanisms for damping the primordial magnetic field as early as possible. In that respect the spacetime for higher dimensions (HD) has some inherent advantage over the standard 4D one in the sense that the damping is faster in HD. Secondly the magnetic field at small scales may influence the big bang nucleosynthesis and change the primordial abundances of light elements by significantly changing the expansion rate of the universe at the corresponding time. The success of the standard BBN scenario can provide an interesting set of bounds on the intensity of the magnetic field at that epoch [5], indirectly constraining the number of dimensions of the spacetime. At this stage it may not be out of place to call attention to the fact that most of the above findings are of theoretical nature only and it is not feasible to relate them to
current astrophysical data, because multidimensional cosmological models lose much of their relevance well before the onset of big bang nucleosynthesis and current observational findings can be explained for all practical purposes if the cosmological evolution is modelled through the standard four-dimensional spacetime.

If as usual we set $k_i c = \omega_i$ (the angular frequency of the wave at some initial time $t = t_i$), then the above equation may be rewritten as

$$E = E_0 \sqrt{\frac{2k_i cd(1+\gamma)}{\pi d(1+\gamma)-2}} t^{-2/(1+\gamma)} e^{-i\omega_d t d(1+\gamma)/(d(1+\gamma)-2)} e^{i k_i \cdot r}$$

where

$$\omega_d = \omega_i t^{-2/(d(1+\gamma))}$$  \hspace{1cm} (24)

gives a measure of the redshift of the photon due to background expansion. For the radiation dominated era, $\gamma = 1/d$, $\omega_d = \omega_i t^{-2/(d+1)}$, so the rate at which the frequency decreases is maximum in the 4D universe. Moreover damping is greater in the radiation era (see figure 2).
Returning again to equation (22) we see that the horizon for our metric is given by

\[
L_d = \int \frac{c \, dt}{t^{2/(d(1+\gamma))}} = \frac{d(1+\gamma)}{d(1+\gamma) - 2} c d^{(d(1+\gamma)-2)/d(1+\gamma)}
\]  

so the equation may be recast as

\[
E = E_0(x, t) e^{i(k \cdot r - k \cdot L_d)} = E_0(x, t) e^{i k \cdot (\hat{k} \cdot x - L)}
\]

very similar to the Newtonian result where the horizon is simply \( L = ct \).

4. Electrostatic oscillation

In this section we shall very briefly consider an electromagnetic wave in a two-component plasma. For simplicity we assume a small amplitude wave such that \( v \times B = 0 \), because it is of second order in perturbed quantities. This, in turn, allows us to neglect the motion of ions. Skipping intermediate mathematical steps for reasons of space, we get via equations (14) and (16) for our metric (5)

\[
v = \frac{iq t^{2/(d(1+\gamma))}}{m_e \Gamma \omega_i} E = \frac{ie}{m_e \Gamma \omega_d} E.
\]

The last equation is very similar to that for the flat space case except that here, \( \omega_d \) is not a constant, but shares the background expansion. When one uses the last equation in Maxwell equation (10), through \( J = n_0 q v \) we arrive at Coulomb’s law

\[
\nabla \times B = -i \omega_i c \epsilon E
\]

where

\[
\epsilon(\omega_d) = 1 - \frac{\omega_{pT}^2}{\omega_d^2}
\]

is the dielectric constant of the plasma medium with the suffix ‘T’ signifying transverse mode. \( \omega_{pT}^2 \) is related to the well known plasma frequency [23]:

\[
\omega_p^2 = \frac{b_d m_0 q^2}{m_e}, \quad \omega_{pT}^2 \sim \frac{\omega_p^2}{\Gamma}
\]

\[
b_d = \frac{2^{d/2} d^{d/2}}{(d - 1)!!} \quad (d \text{ even}),
\]

\[
= \frac{2^{(d+1)/2} \pi^{(d-1)/2}}{(d - 2)!!} \quad (d \text{ odd})
\]

Skipping mathematical details, it can also be shown that the transverse and the longitudinal modes give the dispersion relations as

\[
\omega_T^2 = \omega_p^2 + c^2 k^2
\]

\[
\omega_L^2 = \omega_p^2.
\]
These relations are fairly well known in the special relativistic case, excepting that here all the quantities depend on time as well as the total number of dimensions.

Apparently, equation (29) has the same Newtonian form, but both the frequencies depend on the scale factor, $A$, which, again, is a function of both the number of dimensions and the equation of state chosen. To end the section let us investigate the time dependence of the dielectric constant in equation (29). Now, $\omega_{\parallel}^2$ should share the time evolution of the background electron number density, $n_0$ (the inverse of volume of the universe), i.e., $n_0 \sim A^{-d} \sim t^{-2/(1+\gamma)}$. Again, from (27) we get $v \Gamma \sim A^{n(1-d)}$. Equation (10) further dictates that $v \sim A^{-1}$, which gives $\Gamma \sim A^{(2-d)}$. So, $\omega_{\parallel}^2 \sim n_0/\Gamma \sim A^{-2} \sim t^{-4/(d(1+\gamma))}$.

On the other hand, equation (24) implies that $\omega_{\perp}^2 \sim A^{-2}$; hence $\epsilon(\omega_{\perp})$ does not explicitly depend on time. This is a remarkable result in the sense that for a FRW type of metric the dielectric constant is a real constant irrespective of not only the total number of dimensions but also the equation of state, i.e., $\epsilon(\omega_{\perp})$ continues to remain constant throughout the evolution of the universe.

5. Electromagnetic oscillations in cold plasma

In this section we investigate the situation where a plasma in thermodynamic equilibrium is slightly disturbed through the passage of an electromagnetic wave. We assume that an external ambient magnetic field is also present. We do, however, assume the plasma medium to be cold so that the pressure can be neglected when considering the particle equation of motion. In stellar systems one often encounters situations where relaxation times are much larger than the age of the universe so that collisions (hence pressure) may be neglected. The effect of an electric field is not generally seriously considered because of the well known Debye shielding effect. The general problem of an electromagnetic wave propagating along an arbitrary direction with the external magnetic field is given by Appleton and Hartree in the Newtonian case when studying the propagation of radio waves in the ionosphere. Holcomb [14] studied for the FRW metric a specialized situation of the A–H equation in the dust case. Considering the fact that a general solution with arbitrary $\theta$ is very difficult to tackle in an expanding background with arbitrary number of dimensions we shall restrict ourselves to the cases where the electromagnetic wave propagates parallel and perpendicular to the magnetic field. However the topic is of great importance in astrophysics and space science, where electromagnetic wave propagation in magnetized plasma is very relevant.

Case I ($\vec{B} \parallel \vec{k}$): we assume that the external, uniform magnetic field and the wavevector $\vec{k}$ are both aligned along the $i$th direction (say the $z$ direction with $i = 3$) in the $d$-dimensional space, such that $\vec{k} = |k|\vec{e}_z$ and $\vec{B} = |B|\vec{e}_z$. As is customary for the analogous three-dimensional static space, we also assume that all the perturbed quantities have the same time dependence, given by equation (23), such that the linearized equation of motion (13) takes the form

$$i\omega_d m_e v = e\left(\frac{E}{c} + \frac{A^{\perp} B}{c} \times \vec{k}\right).$$

(Here $E$ is $\perp r$ to $k$ and considering that we are dealing with a $(d+1)$-dimensional spacetime it has components $E_1, E_2, E_3, E_4, \ldots, E_j$.)
Replacing $\partial E/\partial t$ via equation (23) by

$$\frac{\partial E}{\partial t} = -\left[ i\omega_d + \frac{2}{(1 + \gamma)J} \right] E$$

(33)

which, when plugged into equation (10), gives, after a long but fairly straightforward calculation, for $j = 1$,

$$\left( \nabla \times B \right)_1 = -\frac{i\omega_d}{c} A \left[ \left( 1 - \frac{\omega_p^2}{\omega_d^2} \right) E_1 - \frac{\omega_p^2}{\omega_d^2 - \omega_c^2} \omega_d E_1 + i \sum_{j=2}^{d} \frac{\omega_p^2}{\omega_d^2 - \omega_c^2} \omega_d E_j \right]$$

$$= -\frac{i\omega_d}{c} \left[ \left( 1 - \frac{\omega_p^2}{\omega_d^2 - \omega_c^2} \right) E_1 + i \sum_{j=2, j \neq 1}^{d} \frac{\omega_c}{\omega_d^2 - \omega_c^2} \omega_d E_j \right].$$

(34)

For the case $j = 3$ (i.e., along the direction of the magnetic field) it takes a simple form:

$$\left( \nabla \times B \right)_3 = -\frac{i\omega_d}{c} \left( 1 - \frac{\omega_p^2}{\omega_d^2} \right) E_3$$

(35)

and repeating the process for the remaining $(d - 2)$ components we can write for the $\mu$th component a tensorial relation:

$$\left( \nabla \times B \right)_\mu = -\frac{i\omega_d}{c} \epsilon_{\mu\nu} E_\nu$$

(36)

($\mu, \nu = 1, 2, 3, \ldots, d$), where $\epsilon_{\mu\nu}$ is a rank 2 skew symmetric tensor of order $d$. A brief inspection reveals that

$$\epsilon_{11} = \epsilon_{22} = \epsilon_{44} = \epsilon_{55} = \cdots = \epsilon_{dd} = 1 - \frac{\omega_p^2}{\omega_d^2 - \omega_c^2} = p_1 \text{(say)}$$

(37)

$$\epsilon_{12} = \epsilon_{14} = \epsilon_{15} = \cdots = \epsilon_{1d} = \frac{\omega_c}{\omega_d} \frac{\omega_p^2}{\omega_d^2 - \omega_c^2} = p_2$$

(38)

$$\epsilon_{31} = \epsilon_{32} = \epsilon_{34} = \cdots = \epsilon_{3d} = 0$$

(39)

$$\epsilon_{33} = 1 - \frac{\omega_p^2}{\omega_d^2} = p_3$$

(40)

so the $(d \times d)$ permittivity tensor comes out as

$$\epsilon_{\mu\nu} = \begin{pmatrix}
  p_1 & ip_2 & 0 & ip_2 & \cdots & ip_d \\
  -ip_2 & p_1 & 0 & ip_2 & \cdots & 0 \\
  0 & 0 & p_3 & 0 & \cdots & 0 \\
  -ip_2 & -ip_2 & 0 & p_1 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
  -ip_d & \cdots & \cdots & \cdots & \cdots & p_1 \\
\end{pmatrix}.$$ 

(41)

Here $\omega_p$ is the plasma frequency given by

$$\omega_p^2 = \frac{b_d n_0 e^2}{m_e}$$

(42)
and the electron cyclotron frequency is given by

$$\omega_c = \frac{eB}{m_ec^{2/(d(1+\gamma))}} \equiv \frac{\hat{B}}{m_ec}$$

(43)

where $\hat{B}$, the orthogonal magnitude of the ambient magnetic field, is given by $\hat{B} = |(B)_z(B)_z|^{1/2} = B_{t2/(d(1+\gamma))}$ for our system (see figure 3).

If the magnetic field is switched off ($\omega_c = 0$) equation (34) reduces to

$$(\nabla \times B)_j = -\frac{e\omega_i}{c}\epsilon E_j$$

(44)

very similar to the expression (28) of section 4. Thus the introduction of the magnetic field generates varied modes transforming the dielectric constant scalar $\epsilon$ in equation (35) to a second-rank tensor $\epsilon_{ij}$. Although equations (30)–(40) closely resemble the analogous expressions in Newtonian theory, the fact remains that all the frequencies now depend on time rather than being constant. Further the cyclotron frequency $\omega_c$ decays as $t^{-2(d-1)/(d(1+\gamma))}$, very similarly to the orthogonal component of the magnetic field.
If we take the curl of the equation (36) and replace \( \nabla \) by \( i k \hat{e}_z \), after a long but fairly straightforward calculation we are led to the matrix form

\[
\begin{pmatrix}
(1 - \frac{p_1}{n^2}) & -i\frac{p_2}{n^2} & 0 & 0 & 0 \\
-\frac{i p_2}{n^2} & (1 - \frac{p_1}{n^2}) & 0 & 0 & 0 \\
0 & 0 & (1 - \frac{p_1}{n^2}) & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & (1 - \frac{p_1}{n^2})
\end{pmatrix}
\begin{pmatrix}
E_1 \\
E_2 \\
E_3 \\
E_4 \\
E_d
\end{pmatrix} = \begin{pmatrix}
0 \\
0 \\
0 \\
0 \\
0
\end{pmatrix}
\]

(45)

where

\[
n^2 = \frac{c^2 k_L^2}{\omega_d^2}
\]

(46)

where \( n \) is the refractive index of the plasma medium.

Three modes are possible.

The first is the longitudinal mode characterized by \( E_1 = E_2 = E_4 = \cdots = E_d = 0 \) \( (E_3 \neq 0 \) and \( p_3 = 0 \). Since the displacement is along the \( z \) direction the magnetic field has no role to play and \( \omega_d = \omega_p \).

As we are more interested in the dynamics of the electromagnetic waves than the plasma oscillation as such, we take \( E_3 = 0 \) in equation (45). Setting the determinant of the resulting \((d - 1) \times (d - 1)\) matrix in equation (45) to zero we get

\[
n^2 = p_1 \pm p_2.
\]

(47)

The plus sign gives via equation (36)

\[
E_\mu - i E_\nu = 0 \quad (\mu \neq \nu \neq 3, i = \sqrt{-1})
\]

(48)

such that

\[
E_l = (\hat{e}_\mu - i \hat{e}_\nu) e^{i(k_L z - \omega_d t)}
\]

(49)

corresponding to a left circularly polarized wave.

The wavenumber \( k_L \) can be found from equations (37), (38), (46) and (47) as

\[
k_L = \frac{\omega_d}{c} \left[ 1 - \frac{\omega_p^2}{\omega_d(\omega_d + \omega_c)} \right]^{1/2}
\]

(50)

On the other hand for the minus sign in (47) we get

\[
E_R = (\hat{e}_\mu + i \hat{e}_\nu) e^{i(k_R z - \omega_d t)}
\]

(51)

representing a RCP wave with

\[
k_R = \frac{\omega_d}{c} \left[ 1 - \frac{\omega_p^2}{\omega_d(\omega_d - \omega_c)} \right]^{1/2}
\]

(52)

It is clear that the two eigenmodes have different phase and group velocities and, unlike for the former case, the right circularly polarized (RCP) wave has a resonance at \( \omega = \omega_f = \omega_c \) where the phase velocities vanish. The expressions so far closely resemble the ones found in the propagation of an electromagnetic wave with an ambient magnetic field in Newtonian mechanics. However, here all the quantities \( \omega_d, \omega_p \) etc depend on time,
a dependence modelled by the form of the line element, the number of dimensions and also the equation of state.

It should be noted that the time dependences of \( \omega_d \) and \( \omega_c \) are different, being \( \omega_d \sim t^{-2/(d(1+\gamma))} \) and \( \omega_c \sim t^{-2(d-1)/(d(1+\gamma))} \). So there is no fixed resonant frequency as in the Newtonian case; with time, it changes.

It also follows from equation (52) that for

\[
\omega_d = \omega_1 = \frac{1}{2} \left[ \omega_c + \sqrt{\omega_c^2 + 4 \omega_p^2} \right]
\]

the wavevector \( k \) vanishes and our analysis breaks down. So the wave propagates for \( \omega_d < \omega_c \) and \( \omega_1 < \omega_d \); otherwise it becomes evanescent. Moreover, as the temporal dependences of \( \omega_c \) and \( \omega_d \) are different, the magnitude of the allowed region changes. \( \omega_c \) decays more sharply with dimension than \( \omega_d \). Thus the propagation of the electromagnetic wave is more restricted in higher dimensions than the usual 4D.

Returning to the left circularly polarized (LCP) wave we see that the wavevector vanishes for

\[
\omega_d = \omega_2 = \frac{1}{2} \left[ -\omega_c + \sqrt{\omega_c^2 + 4 \omega_p^2} \right]
\]

and so the wave propagates for \( \omega_d > \omega_2 \).

From what has been discussed above it is tempting to look for Faraday rotation (see [4] for recent astrophysical data) like in the Newtonian case. Assuming that an electromagnetic wave traverses a distance \( z \) in a plasma medium with a magnetic field subject to the restriction on frequencies discussed above, the Faraday rotation is given by

\[
\theta = \frac{k_L - k_R}{2} z.
\]

It should be noted that one should revert to the physical coordinate rather than the co-moving one that we are considering here. Accordingly \( Z_{\text{ph}} = t^{-2/(d(1+\gamma))} \); \( Z_{\text{cm}} \) and \( \theta \) finally come out via equation (23):

\[
\theta = \frac{k_{L\text{i}} - k_{R\text{i}}}{2} Z_{\text{ph}}
\]

with no dependence on time. So apparently the number of dimensions and the equation of state have no impact on this classical result. It may be relevant that measurements of the radio waves from the extra-galactic sources suggest that various spiral galaxies are endowed with magnetic fields whose intensities are of the same order of magnitude as that of the Milky Way [4], i.e., of the microgauss order, corresponding to an energy density stored today in the CMBR energy spectrum peaked around a frequency of 30 GHz.

**Case II** \((\vec{B} \perp \vec{k})\): Let us very briefly consider the case of a plasma with a uniform magnetic field \( \vec{B} = B_0 \hat{e}_z \), through which an electromagnetic wave is propagating with propagation vector \( \vec{k} = k \hat{e}_x \), perpendicular to the magnetic field. Here two modes are possible. As the mathematical exercise closely resembles that for case I we shall totally skip intermediate steps and merely write the final form:
The first mode (called the ordinary wave), with displacements in the $z$ direction (i.e. $\parallel B$), has the dispersion relation
$$\omega_d^2 = \omega_p^2 + k^2 c^2,$$
(57)
as the magnetic field has no influence on motion parallel to itself. Equation (57) is exactly same as equation (30) for electrostatic oscillation.

The second mode (called the extraordinary wave), with displacements in the $(d-1)$-dimensional hypersurface $(\perp B)$, has the dispersion relation
$$\frac{c^2 k^2}{\omega_d^2} = 1 - \frac{\omega_p^2}{\omega_d^2} \frac{\omega_d^2 - \omega_p^2}{\omega_d^2 - \omega_p^2 - \omega_c^2}.$$ 
(58)

Before ending this section a final remark may be in order. We know that pulsars are rotating neutron stars giving out pulses of radio waves periodically, which are affected by the interstellar medium during their propagation in reaching us. If the interstellar medium has a component of magnetic field parallel to the propagation direction, then as shown earlier the plane of polarization will suffer Faraday rotation depending on the frequency, having a spread in the rotation angle. This spread may have, in principle at least, some imprint on the nature of the expanding universe.

6. Discussion

With the help of a $(3+1)$ formalism, the Einstein–Maxwell and the electrodynamical equations are written for a $(d+1)$-dimensional FRW-like spacetime in the presence of plasma, and linearized equations are solved for different phases of the universe. The analysis essentially generalizes to HD the well known results of Holcomb and Tajima. The salient features of our analysis may be summarized as follows:

(1) For a propagating wave in HD in vacuum the photons redshift most in 4D and for a fixed $d$ in the radiation dominated model.

(2) Although the plasma is sharing the expansion of the background the dielectric constant remains a true constant. So the photons are in a sense self-similar. This result was found earlier by Holcomb and Tajima. We here generalize this remarkable result to the case of extra-dimensional spacetime and also for a fluid obeying a general equation of state. It may be tempting to suggest that the fact that the classical flat space result of the constancy of the dielectric constant is carried over to a non-static curved background and that too in higher dimensions may be due to the conformal flatness of the particular metric analysed here. So one should proceed with caution as regards any far fetched generalization; and in other complicated spacetime this result may not be true.

(3) In the presence of an external magnetic field many interesting oscillation modes manifest themselves. A simplified Appleton–Hartree type of solution generalized to higher dimensions is obtained in curved spacetime. Only a selected range of frequencies are available for propagation here.

(4) The well known phenomenon of Faraday rotation is obtained.
To end, a final remark may be in order. The present work suffers from two serious shortcomings. For the sake of mathematical simplicity we work out everything assuming a linearized plasma theory. Conditions under which one may assume linearized plasma theory may well exist in Newtonian theory, but we are not able to clearly formulate this for the case of a non-flat spacetime, and that too when it is expanding. Secondly, most observational evidence suggests that even if one starts with a higher dimensional phase, the universe underwent the self-compactification transition much earlier than the epoch when the big bang nucleosynthesis set in. So although literature abounds with works (for example, higher dimensional black holes and their thermodynamics), studying the standard electromagnetic as well as MHD laws in the framework of multidimensions becomes somewhat suspect. In that sense our analysis is mainly of a purely theoretical nature without much direct physical application. In future work we should try to generalize these results to the realm of non-linear plasma and also attempt to relate some of our findings to known astrophysical data.

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References

[1] Chen P and Lai K-C, 2007 Phys. Rev. Lett. 99 231302 [SPIRES]
[2] Smoot G F, 2007 Rev. Mod. Phys. 79 1370 [SPIRES]
[3] Banerji S and Banerjee A, 2007 General Relativity and Cosmology (Amsterdam: Elsevier)
[4] Kronberg P P, 1994 Rep. Prog. Phys. 57 325
[5] Gasperini M, Giovannini M and Veneziano G, 1995 Phys. Rev. Lett. 75 3796 [SPIRES]
[6] Beck R, Brandenberg A, Moss D, Shukurov A A and Sokoloff D, 1996 Ann. Rev. Astron. Astrophys. 34 155 [SPIRES]
[7] Tsagas C G and Barrow J D, 1997 Class. Quantum Grav. 14 2539 [SPIRES]
[8] Zeldovich Y B, Ruzmaikin A A and Sokoloff D D, 1983 Magnetic Fields in Astrophysics (New York: McGraw-Hill)
[9] Hooper D and Profumo S, Dark matter and collider phenomenology of universal extra dimensions, 2007 Phys. Rep. 453 27 [SPIRES]
[10] Chatterjee S, Banerjee A and Zhang Z H, 2006 Int. J. Mod. Phys. A 21 4035 [SPIRES]
[11] Deb Nath U, Banerjee A and Chakraborty S, 2004 Class. Quantum Grav. 21 5609 [SPIRES]
[12] Wesson P S, 1999 Space Time Matter (Singapore: World Scientific)
[13] Holcomb K A and Tajima T, 1989 Phys. Rev. D 40 3809 [SPIRES]
[14] Holcomb K A, 1990 Astrophys. J. 362 381 [SPIRES]
[15] Banerjee A, Chatterjee S, Sil A and Banerjee N, 1994 Phys. Rev. D 50 1161 [SPIRES]
[16] Dettmann C P and Frankel N E, 1993 Phys. Rev. D 48 5655 [SPIRES]
[17] Chatterjee S and Bhui B, 1990 Mon. Not. R. Astron. Soc. 247 57
[18] Arnowitt R, Deser S and Misner C W, Gravitation, 1962 An Introduction to Current Research (New York: Wiley)
[19] Zhang X H, 1989 Phys. Rev. D 39 2933 [SPIRES]
[20] Macdonald D A and Thorne K, 1982 Mon. Not. R. Astron. Soc. 198 345
[21] Thorne K and Macdonald D A, 1982 Mon. Not. R. Astron. Soc. 198 339
[22] Evans C R and Hawley J F, 1988 Astrophys. J. 332 659 [SPIRES]
[23] Emelyanov V M, Nikitin Yu P, Rozenbal I L and Berkov A V, 1996 Phys. Rep. 143 1 [SPIRES]