Supertransport by Superclimbing Dislocations in $^4$He: When All Dimensions Matter

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The unique superflow-through-solid effect observed in solid $^4$He and attributed to the quasi-one-dimensional superfluidity along the dislocation cores exhibits two extraordinary features: (i) an exponentially strong suppression of the flow by a moderate increase in pressure, and (ii) an unusual temperature dependence of the flow rate with no analogy to any known system and in contradiction with the standard Luttinger liquid paradigm. Based on $ab$ initio and model simulations, we argue that the two features are closely related: Thermal fluctuations of the shape of a superclimbing edge dislocation induce large, correlated, and asymmetric stress fields acting on the superfluid core. The critical flux is most sensitive to strong rare fluctuations and hereby acquires a sharp temperature dependence observed in experiments.

**Introduction.** A pure (free of $^3$He impurities) but structurally imperfect crystal of $^4$He is a highly non-trivial system demonstrating a variety of unique phenomena taking place at temperatures $T \lesssim 0.5K$ that likely persist down to absolute zero: (i) the superflow-through-solid (STS) 1,3,8. (ii) the anomalous isochoric compressibility (also known as the syringe effect), which is the thresholdless matter accumulation inside the solid in response to small chemical potential changes that always accompanies the STS 1, and (iii) the giant plasticity 4,9. All three features are attributed to highly unusual and essentially quantum properties of dislocations.

The STS effect is explained by superfluidity in the cores of certain dislocations, as established for both the screw and edge dislocations by $ab$ initio path integral simulations in Refs. 10,11: the original idea that dislocations in $^4$He might have superfluid cores goes back to the work by Shevchenko 12. The only existing scenario explaining the syringe effect is based on superclimb of edge dislocations 11,13. In contrast to the conventional climb assisted by pipe diffusion of thermally activated vacancies along the dislocation core 13,14 (viable only at high temperature), the superclimb is assisted by the superflow along the core. The syringe effect persisting down to low temperatures when thermal activation is no longer possible, as well as first-principle simulations of edge dislocations demonstrating superclimb, provide strong support to the minimalistic unified scenario behind all phenomena based on the (quantum-)rough edge dislocations with superfluid cores.

Since dislocation cores are quasi-one-dimensional objects, it is natural to expect that their superfluid properties fit the Luttinger liquid (LL) paradigm when at zero temperature $T$ the I-V curve is non-linear (sub Ohmic at small bias). In agreement with the LL theory, $ab$ initio simulations of dislocations with superfluid cores reveal that LL parameters remain temperature-independent at $T \lesssim 0.5K$ 15. These observations resulted in a widely shared point of view that supertransport of $^4$He atoms through the dislocation network is described by a bosonic LL 3,8. However, at low finite $T$, the initial part of the otherwise temperature-independent I-V curve in LL is supposed to acquire an Ohmic regime characterized by high conductivity diverging in the $T \rightarrow 0$ limit as a power law. In contrast, experiments consistently observe a different mysterious temperature dependence of the critical flux $F$ shown in Fig.1 and, apparently, force one to look beyond isolated dislocations and invoke properties of the

![FIG. 1: Critical flux $F(T)$ [normalized by $F_0 = F(T \approx 0)$] for differently grown $^4$He samples. All data points connected by thin lines are a guide to the eye and are taken from Refs. 3,8. The dotted line represents the master curve, Ref. 1, fitting the data collected from multiple samples. In contrast to Refs. 3,8, the data from Ref. 1 shows no spread within ~ 10%. All the data for $T < 0.5K$ are consistent with the stretched exponential law $\exp[-(T/T_0)^\alpha]$, $\alpha = 1–1.3$, predicted by our model: The thick solid lines are fits with $\alpha = 5/4$ and $T_0/4 \approx 0.20K$, $0.45K$ for the lowest and the highest data sets, respectively. 3,8.](image-url)
dislocation network (cf. Ref. [16]). In this Letter, we argue that a single-dislocation scenario, where $F$ is simply a product of the critical current $J$ along one dislocation and a number of dislocations, is nevertheless possible by paying attention to the higher-dimensional nature of the problem—dislocations are defects of a three-dimensional crystalline order. In this context, two key ingredients become crucial: (i) thermal fluctuations of the dislocation shape, and (ii) the exponential dependence of superfluid properties of the core on moderate changes in pressure (more generally, the stress field around the dislocation core)—as observed in the experiment [5] and in our simulations (see below).

Their combination results in rare intermittent fluctuations with dislocation segments having strongly suppressed superfluid stiffness in what otherwise is a robust superfluid along the core. Phase slips in such regions limit the flux $F$. No other one-dimensional system is known to exhibit a similar behavior.

Our scenario is supported by ab initio and model simulations and explains the leading behavior observed in experiments, see Fig. 1. In particular, we explain finite size limitations of ab initio simulations and reveal strong suppression of the superfluid stiffness with pressure and temperature. Of direct relevance to experiments is, however, the critical current, which is out of reach for ab initio methods. We corroborate the scenario by a simplified effective model, which captures, at least qualitatively, the STS experiments that puzzled the community for over a decade. Given the spread of data for samples with different geometry, size, and growth conditions, it is natural to expect that our modelling will need further refinements: a group of samples from Ref. [6] follow a stretched exponential law with parameters outside of the range supported by our current model. In particular, our model does not capture the low-T saturation behavior seen in these samples. It also does not account for the sub-Olmic dependence of the flux $F$ on the chemical potential bias $\Delta \mu$ [2, 0], which we leave for future work. The scope of the present Letter is hence limited to the unifying picture of the pressure and temperature dependence over a wide temperature range.

Scenario. In solid $^4$He the motion of atoms/vacancies along the dislocation core is best described by tunnelling in the periodic potential (see the sketch, Fig. 2) implying exponential sensitivity of the superfluid properties to the potential strength. Quantum roughness of the superfluid edge dislocation renders thermal fluctuations of their shape gapless and, thus, anomalously large compared to the situation when the Peierls potential localizes the core within a single potential minimum. Shape fluctuations of the edge produce inhomogeneous stress fields along the core, which, in turn, modify the local superfluid response. Exponential sensitivity of tunneling phenomena to external parameters, such as the local stress, amplifies the effect.

Imagine an instant shape of the dislocation line being quenched. Transport properties of the resulting system are best described by the strong-disorder scenario [17–19] when special attention is paid to the statistics of rare regions (outliers in the pressure/stress distribution) creating “bottlenecks” that might determine the current. The dynamical nature of thermal fluctuations does not allow us to take this analogy literally: e.g., phase transitions at finite $T$ are forbidden. Nevertheless, a sharp suppression of the superfluid density $n_s$ and flux with temperature, see Fig. 1, is possible.

The difference between the two properties is that $n_s(T)$ dependence is a purely thermodynamic effect based on a macroscopic number of connected regions with suppressed local superfluid response, or weak regions (WR), while the critical current $J_c$ (with $F \propto J_c$) is not. Because of the one-dimensional character of the system, $J_c$ is determined by a single WR along the line allowing phase slips. The microscopic description of phase slips at WR goes beyond the scope of this work. However, for qualitative comparison with the experiment all we need are the following two natural assumptions. (i) $J_c$ is a product of the local superfluid density $n_s(x)$ and the local critical velocity at WR. (ii) Phase slip is a ground-state, quantum-tunnelling phenomenon often called an instanton. Smaller values of $n_s$ lead to a smaller instanton action and, correspondingly, smaller critical velocities. Thus, we expect that $J_c$ scales as a certain power ($p > 1$) of the local superfluid density $n_s$ at the WR.

Here we do not consider thermal phase slips destroying the core superfluid in the thermally activated fashion [20] at higher temperatures. The corresponding activation energy $E_0$ is determined by $n_s(0)$ and the healing length $\xi$, and it can be estimated as $E_0 \approx \hbar^2 n_s(0)/(\xi m)$, where $m$ is the $^4$He atomic mass. Our ab initio simulations give $E_0 \sim 5 – 10 K$, which is much higher than the typical energy scale $\sim 0.5K$ observed experimentally in Refs. [1, 5, 6].

Ab initio simulations. The key assumption is the exponential sensitivity of $n_s$ to small changes in system parameters such as the crystal number density $n$. This was verified by ab initio worm algorithm simulations [21] similar to those reported in Refs. [11] but at higher densities (see the Supplemental Material [15]): Fig. 3 clearly shows that $n_s$ can be suppressed by nearly two orders of magnitude with only a 10% change in $n$. 

FIG. 2: Schematic visualization of shape fluctuations of the edge dislocation core in the climbing plane. The incomplete atomic plane is shown by the dark area with the core at its boundary. The superfluid density is confined to the core (between the dashed lines) and its local value strongly depends on the local pressure $P(r)$ (or stress).
The largest simulated system linear size $L$ cannot accommodate full-scale shape fluctuations of the dislocation because $L$ does not satisfy the requirement $L \gg D_0$, where $D_0 \sim 10\AA$ is the core diameter. However, rather than dealing with the shape fluctuations of a long dislocation line in an infinite ideal crystal, we can study the statistics of position fluctuations of a short dislocation—of length $L \sim 20\AA \gtrsim D_0$—within the box. Approximately, the dislocation can be viewed as composed of straight segments of length $L$, moving with respect to each other and interacting by elastic forces. The simulation box boundaries are formed by atoms with “frozen” spatial positions arranged to enforce the topology of the dislocation inside the box (see Ref. [15]). As a result, the interior is under stress gradients with the characteristic scale $L$. At the qualitative level, this arrangement mimics the effect of thermal shape fluctuations and allows us to study how the dislocation segment explores the non-uniform stress landscape and changes its superfluid properties depending on the position within the small box.

In the simulations, $n_s$ is calculated through the variance of the winding number $W$ [22]: $n_s = \hbar^2mLT\langle W^2 \rangle$, where $m$ is the particle mass, and $\langle \ldots \rangle$ stands for averaging over an ensemble of path-integral configurations. Large rare thermal fluctuations of the dislocation position are statistically insignificant in $\langle W^2 \rangle$ and this is why $n_s$ stays $T$-independent in short samples [15]. To reveal the effect of rare fluctuations one has to look at correlations between the $W^2$ and dislocation core position within the simulation cell, which can be readily done on the basis of the one-to-one correspondence between the edge dislocation position and the particle number $N$. This correspondence is the essence of the superclimb effect [11]: the position of the core with respect to the crystal boundary determines the number of extra atoms belonging to the incomplete atomic plane (the dark area in Fig. 2). Accordingly, fluctuations of the edge dislocation imply fluctuations of $N$. Note that there is no such a relation for the screw dislocation. Since $N$ is a constant of motion, its fluctuations are thermal, ensuring that we are studying finite-$T$ rather than zero-point effects.

In the numerical protocol, the parameters are chosen so that $N$ experiences substantial fluctuations. The statistics of $\langle W^2 \rangle$ is then collected separately for each particle number $N$. The corresponding quantity is denoted as $\langle W^2 \rangle_N$ and defines the $N$-dependent superfluid density $n_s(N) = \hbar^2mLT\langle W^2 \rangle_N$. If our scenario is correct, dramatic changes in $n_s(N)$ between typical and rare values of $N$ should occur. The results are presented in Fig. 4. For large deviations of $N$ from its expectation value $\langle N \rangle$ (that is, for large deviations of the core from its equilibrium position within the simulation cell), we observe significant (by a factor of $\sim 5$) changes in $n_s(N)$ (as compared with $n_s$ averaged over $N$) for a superclimbing edge dislocation. In contrast, the screw dislocation with the superfluid core experiencing similar fluctuations in $N$ (which, however, cause no core displacement) demonstrates no dependence of $n_s(N)$ on $N$. This dramatic difference nicely illustrates the key aspect of our scenario for edge dislocations—the dependence on $N$ is not due to particle density fluctuations within the superfluid core (as is the case for the screw dislocation), but due to the modification of the local crystalline environment around the climbing edge dislocation in the presence of the stress field gradients. The same conclusion follows from the direct comparison between the Figs. 4 and 5 where both screw and edge dislocations demonstrate the exponential.
suppression with the crystal density $n$. 

**Model simulations.** Qualitatively, the shape fluctuations of the superclimbing edge dislocation and their effect on the superflow along its core can be studied within the simplified effective model of an isotropic string in two dimensions. That is, we ignore the presence of the crystalline lattice, which induces anisotropy and the Peierls potential. The effective Hamiltonian reads (in the units when the interatomic distance $b$ and the ratio $\hbar/m$ equal to unity)

$$H[\varphi, y] = \frac{1}{2} \int_0^L dx \left\{ n_s(y'')|V_0 + \varphi'|^2 + G y'^2 \right\}, \quad (1)$$

where the superfluid phase field, $\varphi(x)$, and the dislocation displacement field, $y(x)$, are canonically conjugated variables. The term $G$ is the elastic deformation energy. We consider the limit of small deviations of the line from equilibrium $y(x) = 0$, that is, $|y'(x)| < 1$. Also, the line curvature $y''(x)$ must be much smaller than $1/D_0$. The kinetic energy part contains the average flux velocity $V_0$, and the local superfluid density

$$n_s(y'') = n_0 \exp(-gy''), \quad (2)$$

which depends exponentially on the shape fluctuations through the line curvature $y''$ (see Refs. [13, 15] for additional details). It is consistent with the dependence on pressure/density observed in the *ab initio* simulations presented in Figs. 3, [4].

Quantum mechanical simulations of Eq. (1) at finite $V_0$ suffer from the sign problem. However, for purposes of qualitative analysis it is sufficient to consider the classical version of (1) with explicit temperature-dependent ultraviolet cutoff $\Delta x$, on the wave lengths of excited modes for which quantization effects can be neglected. We implement the cutoff by working with the discretized version of (1) of linear size $L = \Delta x \gg 1$ and the non-compact field $\varphi$. Given that the spectrum of excitations is parabolic at small momenta [11], $\omega = \sqrt{n_0 G q^2}$, [11], we have $\Delta x \approx (n_0 G)^{1/3}/T^{1/2}$. This treatment is valid as long as $\Delta x \gg D_0$.

Brute-force classical simulations of (1) suffer from severe slowing down because optimal WR configurations introduce strong and highly non-local correlations between the fields $y$ and $\varphi$: A bump in the former is accompanied by a large gradient in the latter at the bump location with reduced $\varphi$-gradients everywhere else. Stochastic sampling of optimal WR configurations thus requires an enormous number of elementary local moves. However, for purposes of the semi-quantitative analysis the slowing down problem can be solved by implementing the least-energy approximation for the field $\varphi(x)$ when $\varphi(x)$ nothing but the optimal solution for a given configuration of $y(x)$. In practice, this approximation reduces to an effective energy functional that depends only on $y(x)$, which is then sampled stochastically. The optimal phase field solution corresponds to the constant current condition:

$$J = n_s(y''(x))[V_0 + \varphi'(x)], \quad dJ/dx = 0. \quad (3)$$

The superfluid density $n_s$ is computed from $n_s = \langle J \rangle/V_0$ in the $V_0 \to 0$ limit. The critical flux $F \propto J_c$ cannot be determined from equilibrium Monte Carlo simulations. Nevertheless, the origin of its $T$-dependence can be traced back to the statistics of the WR through the proposed dependence of $J_c$ on superfluid density in WR

$$J_c \propto \langle n_s^2 \rangle_{\text{WR}} \propto (\exp(-pgy''))_{\text{WR}}. \quad (4)$$

Here $\langle \rangle_{\text{WR}}$ denotes averaging over the WR. (Note that an alternative interpretation in terms of the Landau criterion for stability of the superflow is also possible [15]). The results of the simulations for $n_s$ and $J_c$ with $p = 2$ are presented in Fig. 5 (see also Fig. 1), with further technical details delegated to the Supplemental Material [15].

![Figure 5: Superfluid density $n_s$ (dashed lines are guides to an eye) for linear sizes $\tilde{L}$ specified next to each data set. The solid line is the fit by $\exp(-(T/T_o)^\alpha)$ with $\alpha = 5/2$ and $T_o = 0.737 \pm 0.005$. Inset: The critical current $J_c$ (normalized to unity at $T = 0$) for $\tilde{L} = 4, 10, 10^2, 10^3, 10^4, 10^7$ increasing from the highest to the lowest data set. The solid line is the fit of the $\tilde{L} = 10^4$ data by $J_c \propto \exp(-(T/T_o)^{5/4})$ with $T_o = 0.257 \pm 0.01$. All data for $J_c$ are consistent with $\alpha \approx 5/4 \pm 0.1$ for $0 < T < 0.6$ and the spread is due to $T_o$ varying by a factor $\sim 2$ (cf. Fig. 1).](image)

**Conclusions and outlook.** Superflow-through-solid experiments exhibit a highly unusual temperature dependence of the critical flux (see Fig. 1) that does not conform with the standard Luttinger liquid theory expected to work within the otherwise consistent and numerically corroborated picture of superfluidity confined to dislocation cores. Our scenario provides an explanation: the one-dimensional superfluid channel is an inseparable part of the crystalline host and its shape fluctuations induce pressure/stress and density fluctuations modifying properties of the edge dislocation core (see Fig. 4). Since superfluidity comes from tunneling motion of core atoms in the periodic crystal potential the net result is an exponential dependence of $n_s$ on small changes in $P$, and, consequently, in combination with the thermal fluctuations—on $T$.

The long-wave physics of rough superfluid edge dislocations is captured by the effective model [11]. It reveals the
effect of rare shape fluctuations on the superfluid density and emphasizes a nearly classical nature of these fluctuations up to two quantum effects: (i) the UV cutoff on the wavelength of shape fluctuations and (ii) the value of the largest ground-state flux limited by phase slips (this quantum-tunneling phenomenon still remains to be understood).

The sharp temperature dependence of the critical flux is attributed to a single weakest region along the dislocation. Phenomenological treatment based on the assumption that the critical flux scales as a certain power (larger than one) of the local superfluid density at the weakest element allows us to reproduce experimental results at $T \leq 0.5K$, see Fig. 1. At higher temperature the data in Fig. 1 decay faster; this can be accounted for by considering the contribution of bulk phonons to shape fluctuations, which leads to $\ln(F/F_0) \sim -T^2$ [15].

The quantum treatment of (1) is of fundamental interest on its own because it describes a new type of the quasi-1D superfluid—not accounted for by the Luttinger Liquid paradigm.

The inset in Fig. 5 demonstrates weak (logarithmic) suppression of the critical flux with increasing the dislocation length. A similar suppression of the flux with the sample size has been reported in Ref. [6]. This motivates further experimental studies of the STS effect in single crystals with variable distance between the Vycor "electrodes." If crystals can be grown with predominantly screw dislocations, the genuine Luttinger-liquid behavior can be revealed through the temperature independent supercritical flux (at low temperature).

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Supplemental Material

I. DISLOCATIONS WITH SUPERFLUID CORES IN SOLID $^4$HE

Superfluidity of the edge dislocation with its Burgers vector $b$ along the hcp axis has been found numerically in Ref. [11]. This dislocation splits into two partials separated by the basal fault of the type $E$ (see in Refs. [13]) and each partial is characterized by half of the Burgers vector $b$. Another partial, which we call here as $E_1$, is not related to any fault. Both partials, $E$ and $E_1$, were simulated by the worm algorithm [21]. There are also two more partial dislocations with the same (1/2 of) Burgers vector at the edges of two other hcp faults, $I_1$ and $I_2$ (see Refs. [13]), which will be analyzed elsewhere.

![FIG. 6: (Color online) Triangular lattice of atoms in the A layer (located at, say, $z = 0$). The adjacent B layer is shifted with respect to A by the vector $B = (1/2, 1/\sqrt{12}, \sqrt{2}/3)$ (solid arrow) in units of the closest distance between the atoms. The hcp structure is formed by the periodic sequence ABABAB... The the fcc structure is formed by the sequence ABCABC... with the C layer being shifted by the vector $C = (0, 1/\sqrt{3}, \sqrt{8}/3)$ (dashed arrow).](image)

![FIG. 7: (Color online) Full edge dislocations with the superfluid cores (thick solid lines) in the hcp solid $^4$He. There are three orientations of the cores, which were analyzed in this work: (a) Edge dislocation with the core along the [0,1,0] direction (Y-axis); (b) Edge dislocation with the core along the [1,0,0] direction (X-axis); (c) Screw dislocation along the [0,0,1] direction. The solid (red) arrow along the hcp symmetry ($\hat{z}$)-axis shows the Burgers vector $b = (0, 0, \sqrt{8}/3)$ for all three dislocations.](image)
At high pressure there is a transition to the fcc phase which corresponds to the ABCABCABC... stacking, see Fig. 6. The full edge and screw dislocations possessing superfluid cores as found in Refs. 10, 11, are shown schematically in Fig. 7. The structure of the edge partials is shown schematically in Fig. 8.

II. AB INITIO SIMULATIONS

First principles studies of small 4He samples we perform using the worm algorithm 21 by closely following the scheme described in previous works 10, 11. The initial configuration was prepared starting from atomic positions in the ideal hcp solid. In order to produce a partial edge dislocation the corresponding half of the atomic plane was removed and the configuration was annealed by purely classical simulations in the canonical ensemble. Given periodic boundary conditions (PBC), the sample contained two partials with opposite orientations of the Burgers vectors. In the grand canonical worm algorithm simulations these partials would be eliminated by either completely removing the incomplete atomic layer or by extending it over the whole sample. To prevent this from happening, all atoms outside the concentric cylinder of the radius $R_0 = 15\AA$ around the core located at the center of the sample have been frozen (after updating particle world lines in the canonical ensemble by treating them as distinguishable). Full quantum mechanical simulations with quantum exchanges and in the grand canonical ensemble were performed only for the particles inside the cylinder. A typical snapshot of the atomic configuration is shown in Fig. 9. The superfluid response is concentrated in the shaded areas surrounding the core. We did not detect any finite size effects in the density dependence of $n_s$ by simulating samples with the dislocation length $L = 19\AA$ and $L = 38\AA$. It should be mentioned that the frozen particles at the periphery of the sample slightly shift the melting line to a lower density.

Screw dislocation samples were prepared similarly—by adding the screw type displacement to the ideal hcp structure. In order to mitigate the effect of the non-PBC in the directions perpendicular to the dislocation, the wall of the frozen particles has been created as well, with its symmetry axis aligned with the Z-axis. Accordingly, the PBC were preserved along this axis.
Two key parameters characterizing the hydrodynamic response are the superfluid stiffness $\Lambda$ and the compressibility $\kappa$ defined as \[ \Lambda = \langle W^2 \rangle LT, \quad \kappa = \langle W_x^2 \rangle /(LT), \] (5)
where $W$ and $W_x$ are the space and time winding numbers of the particle worldlines, respectively. The superfluid density and stiffness are related by $n_s = m\Lambda/h^2$ with $m$ being the mass of the $^4\text{He}$ atom. In one dimensional systems, an important quantity is the Luttinger parameter
\[ K = \pi\sqrt{\Lambda K} = \pi\sqrt{\langle W^2 \rangle /\langle W_x^2 \rangle}. \] (6)

In the Luttinger liquid theory, it determines the stability of the superfluid phase against disorder and commensurate potentials with the typical critical value $K_c = 2$ for the latter below which the insulating behavior emerges.

At this juncture we note that, while the screw dislocation with the superfluid core \[10\] is well described within the Luttinger liquid paradigm with finite $K$ well above its critical value, the situation with the superclimbing edge dislocation \[11\] is radically different. Here $K \propto L$ diverges with dislocation length $L \to \infty$. This behavior is the indication that the Luttinger liquid description does not apply in general to the superclimbing dislocations, and observing finite $K$ in \textit{ab initio} simulations is a consequence of the dislocation confinement within the wall of frozen particles in a finite-size sample. Conversely, if the special condition for suppressing the quantum roughness is realized as described in Refs. \[23\], the superclimb can vanish at $T \to 0$ and, accordingly, the Luttinger liquid behavior will be restored in the thermodynamic limit.

Simulation results for $n_s$ of the edge and the screw dislocations are presented in Figs. 3 and 4 of the main text and in Fig. \[10\] here. It is worth noting that $n_s$ changes by more than one order of magnitude, while in the regime of strong superfluid (with $K > 2$) the sample density changes by only a few percent. At the same time, no significant temperature (up to $T = 1\text{K}$) or dislocation length dependencies were observed in our samples, see Fig. \[10\]. There is a difference between the protocols of how Fig. 3 in the main text and Fig. \[10\] shown here were obtained. In the former, the edge dislocation was kept at the center of the sample prepared at some initial density by fine-tuning the chemical potential. In the latter, Fig. \[10\] the edge dislocation in the same sample (for each $T$ and $L$) was pushed towards the perimeter formed by the frozen particles by increasing $\mu$. This resulted in varying the sample density without changing significantly the density inside the core.

As emphasized in the main text, $n_s$ of the screw dislocation is sensitive to the density of the crystalline environment rather than to the density inside its core. As can be seen from Fig. 4 of the main text, changing the particle number inside the core does not affect $n_s$ of the screw dislocation. At the same time, changing the overall sample density does suppress $n_s$ significantly (see Fig. 3 of the main text).

\section{Effective Classical Model for a Superclimbing Edge Dislocation}

Here we provide more details about the simplified model (1) introduced in the main text. Eq. (3) of the main text can be written as
\[ V_0 + \varphi' = \frac{J}{n_0} \exp(gy''). \]
(7)

Integrating both sides over $x$, given the condition $dJ/dx = 0$, and taking the PBC into account, we find
\[ J = \frac{n_0 V_0 L}{\int_0^L dx \exp(gy'')} \]
(8)

Accordingly, the shape dependent energy functional (1) in the main text is given by
\[ H = \frac{n_0 V_0^2 L^2}{2 \int_0^L dx \exp(gy'')} + \int_0^L dx G \frac{1}{2} (y'')^2. \]
(9)

It has been discretized by introducing the lattice with the period $\Delta x$ which depends on temperature as explained in the main text. Accordingly, the derivatives $y'$ and $y''$ were changed as $y' \to \hat{y}' = (y(i+1) - y(i))/\Delta x$ and $y'' \to \hat{y}'' = (y(i+2) + y(i) - 2y(i+1))/\Delta x^2$, where $i = 0, 1, ..., L-1$ are sites of the 1D lattice. The classical Monte Carlo simulations were conducted with the weight $\exp(-H/T)$ by proposing random changes of $y(i)$ within the constraint $|\hat{y}'(i)| < 1$. The quantities $n_s$ and $J_s$ were determined as described in the main text. The velocity $V_0$ has been chosen well below the critical speed $V_c$ above which the structural instability develops.

\section{Structural Instability and the Critical Current}

Straightforward analysis of Eq. (9) reveals its structural instability at finite $V_0$. Indeed, by expanding (9) in small $y' \to 0$ and $y'' \to 0$ one finds Eq. (1) as
\[ H \approx \int_0^L dx \left[ -\frac{g^2 n_0 V_0^2}{4} (y'')^2 + \frac{G}{2} (y')^2 \right], \]
(10)

where the linear term $\sim \int dx y'' = 0$ due to the periodic boundary condition. Since the first term here is negative, all Fourier harmonics of $y$ with momenta\[ q > q_c = \sqrt{\frac{2G}{n_0 V_0^2 g^2}} \]
(11)
become unstable at finite $V_0$. However, the effective longwave model is only valid at length scales much larger than the core diameter $D_0$, or, equivalently, at wave vectors $q \ll 1/D_0$. Thus, the instability emerges only at large enough $V_0$
\[ V_0 > V_c \gg \sqrt{\frac{2G}{n_0} D_0 g}. \]
(12)
This instability, which can be traced back to the Landau criterion, reduces the critical superflow velocity roughly by a factor \( \sim g^{-1} \ll 1 \) as compared to the speed of sound in solid \(^4\)He.

In the discrete classical version of the model with the lattice period \( \Delta x \gg D_0 \), the discrete momenta are given as \( q = 2\pi m / (\Delta x L) \), where \( L = N \Delta x \) and \( m = 0, 1, 2, \ldots, L - 1 \). Accordingly, the discrete Fourier images of \( \hat{y}' \) and \( y' \) become \( -4\sin^{2}(\pi m / L) y_m / (\Delta x)^{2} \) and \( 2i\sin(\pi m / L) y_m / \Delta x \), respectively, with \( y_m \) being the discrete Fourier image of \( y(i) \). The critical speed, then, becomes

\[
V_c = \sqrt{\frac{G \Delta x}{2n_0 g}}. \tag{13}
\]

It corresponds to the unstable harmonic with \( m = \bar{L}/2 \), that is, \( q_c = \pi/2 \).

In this work, we assume that the critical current is limited by quantum phase slips occurring at local velocities well below \( V_c \). The phenomenological condition is expressed by Eq. (4) in the main text where \( p > 1 \). This choice is motivated by the experimental data of Ref. \( [5] \) showing that the exponential dependence of \( F \) on pressure is much steeper than what we observe for \( n_s \) in \textit{ab initio} simulations presented in Fig. 3 (by a factor 5–10) and in Figs. 4 of the main text and Fig. 10 here (by a factor of 2–5).

B. The dependence of \( n_s \) on the dislocation line curvature

We start by relating the line curvature to the local change in the crystal density, \( \delta n(x) \), in the vicinity of the core \( [13] \). In the discretized version of the model, this relation is given by

\[
\delta n(x) = b y''(x), \quad b = \frac{(1 - 2\nu)b}{2\pi(1 - \nu)} \ln \left[ \frac{\Delta x}{D_0} \right]. \tag{14}
\]

where \( \nu \) stands for the Poisson coefficient, \( b \) is the Burgers vector, and \( D_0 \) is the short distance cutoff taken to be the dislocation core width \( D_0 \) (of about 10Å). The overall sign depends on the Burgers vector direction \( b \) and the direction of the displacement \( y \)—that is, whether atoms are added to or subtracted from the incomplete atomic plane. \textit{Ab initio} simulations established that \( n_s \) dependence on the density variation is exponential, see Eq. (2) in the main text. With the help of Eq. (14) we relate the exponential slope \( \hat{y} \) of the \( n_s \) vs \( \delta n \) dependence to the coefficient \( g \) (introduced in Eq.(2) of the main text) as

\[
g = \frac{b}{2\pi(1 - \nu)} \ln \left[ \frac{\Delta x}{D_0} \right] \hat{y}. \tag{15}
\]

In our system of units, \( \bar{h} = 1, m = 1, b = 1, k_B = 1 \), the values of \( \hat{y} \) in Figs. 3 and 4 of the main text are roughly \( \hat{y} \sim 40 - 50 \) and \( \hat{y} \sim 70 - 100 \), respectively (\( \hat{y} \sim 40 - 50 \) also in Fig. 10 above). Equation (15) then predicts that \( g \approx 6 - 10 \). In the simulations we have used a fixed value of \( g = 40/2\pi \approx 6.37 \) for the entire temperature range (that is, we dropped the factor \( 1/\nu \ln(\Delta x/D_0) \) in Eq. (14)).

As for other parameters, the line tension \( G \) is given by the \( C_{44} \) modulus of the solid \(^4\)He as \( G \approx 0.5 \) and in our units corresponds to temperature \( \approx 0.5 \). The value of \( n_0 \) was chosen from the middle of the data set shown in Fig. 10; \( n_0 \approx 1 - 2A^{-1} \), or 5 - 10K in the chosen units. The simulation data presented in the main text correspond to \( n_0 = 10K \).

C. Superfluid density and critical current.

The dependence \( \ln n_s \propto -T^\alpha \) with \( \alpha = 5/2 \) shown in Fig. 5 of the main text can be understood as follows. In the steady flow regime at small \( V_0 \), the kinetic energy is small compared to the elastic energy \( \sim G f dx (y')^2 \). Accordingly, the averaging of \( \int dx n_s(x)(y')^2 \) over fluctuations of \( y \) results in \( n_s \approx n_0 \exp[-g^2/(\Delta x)^2] \). The value of \( \alpha = 5/2 \) follows directly from \( \langle (y')^4 \rangle \propto T^2 \), and the typical temperature scale is given by \( T_s \approx T_0^{1/\alpha} \) where \( T_0 = G^{2/3}n_0^{3/4}g^{-2} \approx 0.47K \) for the chosen parameters.

The weakest region (WR) along the dislocation corresponds to a spike in \( y'' > 0 \) such that the local \( n_s \) (see Eq.(2) in the main text) is strongly suppressed. This can be realized despite the conditions \( \|y\| < 1, \|y''\| < 1/D \) as long as \( g \) is large enough. Within the model \( [13] \), a simplified picture of the WR is based on the shape fluctuation such that \( y(i) = 0 \) everywhere except for some site \( i = x_0 \), so that \( y(x_0) = -y' < 0 \). Accordingly, \( y'(x_0) = Y/\Delta x, \hat{y}'(x_0 - 1) = -Y/\Delta x, \hat{y}'(x_0 - 2) = -Y/(\Delta x)^2, \hat{y}'(x_0 - 1) = 2Y/(\Delta x)^2, \) and \( \hat{y}'(x_0) = -Y/(\Delta x)^2 \) (all other values are zero). Substituting these values into Eq. (9) we observe that the kinetic energy term is exponentially small, \( \propto \exp[-2gY/(\Delta x)^2] \ll 1 \) at \( i = x_0 - 1 \), while the elastic one, \( \approx G Y^2/\Delta x \), dominates in the configuration energy.

The probability for having large \( Y \) anywhere on the dislocation line can be estimated as \( \approx \exp[-GY^2/(\Delta x T)] \). This probability is of order one for

\[
Y \approx \Delta x^{1/2} \ln(\bar{L}) T/G \Delta x. \tag{16}
\]

The condition \( |y'| < 1 \) [used in Eqs. (1)–(3) of the main text] implies that \( Y \) cannot exceed \( \Delta x \).

The statistical average \( \langle \ldots \rangle_{WR} \) over WR in Eq. (4) of the main text has been conducted for \( p = 2 \) according to the following protocol: A configuration \( y(x) \) contributes to the average if the following condition

\[
\hat{y}''(i) > \gamma Y / (\Delta x)^2 \tag{17}
\]

is satisfied. The coefficient \( \gamma = 0.3 \) was chosen such that the inequality (17) was satisfied at about 10-20% of the
sites. We verified that increasing $\gamma$ by a factor of 3 in order to decrease the number of WR sites to just a few instances changes the characteristic temperature scale $T_\alpha$ by a factor of 2, but does not change the power $\alpha \approx 5/4$ in the temperature dependence $\ln J_c = -(T/T_\alpha)^{\alpha}$.

D. The bulk phonon mechanisms as a source of the fluctuations.

As mentioned in the main text, at $T > 0.5 - 0.6 K$, bulk phonons start making a significant contribution to fluctuations of $\delta n$ along the dislocation line, and, thus, suppress $n_s$. This contribution is estimated as $\langle (\delta n)^2 \rangle \approx (T/T_D)^4$ where $T_D$ is of the order of the Debye temperature $T_D \sim 30 K$. This effect alone would predict $\ln n_s \sim -\tilde{\gamma}^2 (T/T_D)^4$ for the equilibrium superfluid density and $\ln J_c \sim -\tilde{p}\tilde{\gamma}(T/T_D)^2\sqrt{\ln L}$ for the critical current. It appears that a crossover from $\alpha = 5/4$ to $\alpha = 2$ can explain the deviation of the experimental data points from the model fits at high temperature, see Fig. 1 in the main text. It is also worth mentioning that the bulk phonons mechanism will affect both the edge and screw dislocations.

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