THE NEVO-ZIMMER INTERMEDIATE FACTOR THEOREM
OVER LOCAL FIELDS

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Abstract. The Nevo-Zimmer intermediate factor theorem classifies the possible $G$-spaces $Y$ with $X \times G/P \to Y \to X$, where $G$ is a higher rank semisimple Lie group, $P$ is a minimal parabolic and $X$ is a $G$-space with invariant probability measure.

An important corollary is the Stuck-Zimmer theorem, which states that an ergodic action of a higher rank Kazhdan semisimple Lie group with an invariant probability measure is either transitive or free, up to a null set.

We present a different proof of the first theorem, that allows us to extend these two well-known theorems to linear groups over an arbitrary local field.

1. Introduction

Let $k$ be a local field and let $G$ be a connected semisimple algebraic $k$-group without $k$-anisotropic almost $k$-simple subgroups. Assume for the moment that $G$ is algebraically simply-connected.

1.1. The Nevo-Zimmer intermediate factor theorem. Our central task is to prove the following generalized version of the Nevo-Zimmer intermediate factor theorem over local fields:

Theorem 1.1. With $k$ and $G$ as above, assume that $\operatorname{rank}_k(G) \geq 2$. Let $X$ be an irreducible ergodic $G_k$-space with finite invariant measure, and let $P$ be a minimal parabolic $k$-subgroup. If $Y$ is an ergodic $G_k$-space for which there exist measure preserving $G_k$-equivariant maps $X \times G_k/P_k \to Y \to X$ whose composition is the projection, then there is a parabolic $k$-subgroup $Q$ containing $P$ such that

$$Y \cong X \times G_k/Q_k$$

as a $G_k$-space in such a way that the maps $X \times G_k/P_k \to Y$ and $Y \to X$ are identified with the natural one.

A slightly more general formulation of the above theorem, without assuming that $G$ is simply connected, is given below as Theorem 4.1.

The real case $k = \mathbb{R}$ of the above is essentially the intermediate factor theorem for real Lie groups, which first appeared in Zimmer’s work [18]. The proof given in [18] is a natural generalization of Margulis’ proof of the factor theorem (see [10]).

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1The reader is referred to subsection 2.2 for several definitions regarding algebraic $k$-groups.

2The reader is referred to subsection 2.1 for the needed definitions regarding measurable group actions.
However, as was pointed out in section 4 of [13], that proof contains an imprecise argument.

An additional treatment of the real case is given in a series of papers by Nevo and Zimmer (see [12] and [13]). In fact, a much stronger version - namely the generalized intermediate factor theorem - is proved in [13]. This approach uses ideas of Furstenberg such as the interplay between stationary and $P$-invariant measures, and diverges from the original approach of Margulis in [10].

We mention that a corrected proof along the lines of [18] was given by Nevo and Zimmer in [11], but this work remained unpublished.

Our chief motivation was to prove Theorem 1.1 as closely as possible to the lines of [18] and [10], while extending the result to local fields. Indeed, the factor theorem [10] is already given in this (indeed, even greater) generality. We remark that our approach differs from that of [11].

In fact, the real case of Theorem 1.1 implies the real Lie group version, and the same remark applies to Theorem 1.2 as well - this follows from the fact that every connected semisimple real Lie group with trivial center is isomorphic to $G_k^0$ for some connected algebraic $\mathbb{R}$-group (see 3.1.6 in [19]). Moreover for every almost $k$-simple subgroup $H$, its group of $k$-points $G_k$ is compact if and only if $G$ is $k$-anisotropic. This reduction is discussed in more detail in section 5.

An important corollary of the intermediate factor theorem is the theorem of Stuck and Zimmer (see section 1.2 below). Moreover observe that the factor theorem is a consequence of the intermediate factor theorem, and that the normal subgroup theorem for groups having property $(T)$ is a consequence of the Stuck-Zimmer theorem. So these four results are related, implying and generalizing each other.

We mention that the Nevo-Zimmer intermediate factor theorem has been generalized in several directions since it first appeared - see for example the works of Bader-Shalom [3], Creutz-Peterson [6] and the above mentioned [13].

To overcome the problem that was encountered in [18] we introduce a certain structure, namely the Effros Borel $\sigma$-algebra, on the space of subalgebras of a given measure algebra that appears in a product of measure spaces. Working over a general local field introduces some technical difficulties which are related to the possible discrepancy between $G_k$ and $G_k^+$.  

1.2. The Stuck-Zimmer theorem. The following is a formulation of the theorem in our generality:

**Theorem 1.2.** With $k$ and $G$ as above, assume that $\text{rank}_k(G) \geq 2$ and $G_k$ has property $(T)$. Then every faithful, properly ergodic, irreducible and finite measure preserving action of $G_k$ is essentially free.

Again, a slightly more general formulation, without assuming simple connectivity of $G$, is given below in Theorem 5.1.

For example, as $G = SL_n$ is simply-connected, the above theorem holds for the groups $SL_n(\mathbb{R})$, $SL_n(\mathbb{Q}_p)$ and $SL_n(\mathbb{F}_p((t)))$ for every prime $p$ whenever $n \geq 3$.

The classical theorem of Stuck and Zimmer (see [16]) is the analogous statement for actions of semisimple real Lie groups of real rank at least 2 with property $(T)$, having finite center and no compact factors.

The proof given in [16] is deduced in quite general fashion from the Nevo-Zimmer intermediate factor theorem. In fact, the possibility of the current generalization is already suggested there (see section 5.3 of [16]). Therefore our contribution
to Theorem 1.2 is little more that its restatement in the above form, and the application of Theorem 1.1.

We remark that $G_k$ has property $(T)$ whenever $\text{rank}_k(H) \geq 2$ for every almost $k$-simple subgroup $H$ of $G$, and this is in fact a necessary condition unless $k$ is $\mathbb{R}$ or $\mathbb{C}$.

1.3. Applications of the Stuck-Zimmer theorem. A possible viewpoint on Theorem 1.2 is that it provides an alternative between two different types of actions, in the following sense:

Let $G$ be a group satisfying the hypothesis of Theorem 1.2, and let $X$ be a faithful ergodic and irreducible (standard) $G$-space. There are two possibilities, depending on whether the action is essentially transitive.

If $G$ acts essentially transitively, pick some $x_0 \in X$ in the conull orbit. It is well known that $G_0 = \text{Stab}_G(x_0)$ is a closed subgroup of $G$ (see e.g. 2.1.20 of Zimmer [19]). In particular, $X \cong G/G_0$ as $G$-spaces. It now follows from the Borel Density Theorem (see chapter II of [9]) and the faithfulness of the action that $G_0$ is discrete and in particular a lattice in $G$ (see Lemma 3.5 of [16]).

Alternatively, the $G$-action is properly ergodic. The Stuck-Zimmer theorem implies that the action is essentially free. So essentially free ergodic actions are in a sense complementary to actions on $G/\Gamma$ where $\Gamma$ is a lattice in $G$. In particular, there are generalizations of the classical rigidity theorems to this situation - see chapter 5.2 of [19].

A context in which theorem 1.2 has an interesting application is that of invariant random subgroups (IRS’s) - see chapter 2 of [1], [2] or [5] for additional information.

Given a second-countable locally-compact group $G$, an invariant random subgroup is a probability measure on the space $\text{Sub}(G)$ of closed subgroups of $G$ with the Chaubuty topology (for a discussion of this topology see the remark in subsection 3.3 and the remark there) that is invariant under the $G$-action via conjugation. Obvious examples of IRS’s include $\delta_N$ - a Dirac measure supported on a normal subgroup $N \vartriangleleft G$, and $\mu_\Gamma$ - where $\Gamma$ is a lattice in $G$ and $\mu_\Gamma$ is the push-forward of the measure on $G/\Gamma$.

In [1] the Stuck-Zimmer theorem is used to prove that these are the only possible invariant random subgroups for a higher rank Lie group.

**Corollary 1.3.** Let $G$ be an algebraic $k$-group as above, where $k$ is either $\mathbb{R}$ or $\mathbb{Q}_p$. In particular, assume that $k$-$\text{rank}(G) \geq 2$ and that $G_k$ has property $(T)$. Then every non-atomic irreducible IRS of $G_k$ must be of the form $\mu_\Gamma$ for some irreducible lattice $\Gamma \leq G$.

The real case is Theorem 4.1 of [1], and the $p$-adic case is deduced from Theorem 1.2 in exactly the same way, given a corresponding smoothness result (see section 6 below).

Indeed, a stronger classification result is proved in [1] for reducible IRS’s, relying on the reducible case of the Stuck-Zimmer theorem, and is applied towards a study of asymptotic geometry of locally symmetric spaces.

We remark that contrary to the above, rank one groups have a great variety of wild invariant random subgroups, see e.g. [5].

**Contents.** These notes are organized as follows.
In section 2 we recall terminology regarding \( k \)-algebraic groups and their subgroup structure, as well as generalities regarding measurable group actions. Moreover we mention a relevant result of Margulis on contracting automorphisms and discuss the subgroup \( G_+^k \) of \( G_k \). In section 3 we discuss measure algebras on product spaces, and in Theorem 3.8 construct a measurable mapping into the space of all subalgebras of a given measure algebra with the Effros Borel structure. In the last two sections we prove the two main theorems over local fields - the Nevo-Zimmer intermediate factor theorem in section 4 and the Stuck-Zimmer theorem in section 5. As mentioned above, in section 5 we observe that the proof in \cite{16} extends to our situation as well. Finally in section 6 we give the smoothness result required to extend Corollary 1.3 to the \( p \)-adic case.

2. \( k \)-ALGEBRAIC GROUPS AND THEIR ACTIONS

In this section we recall for the reader’s convenience some facts and notations regarding probability measure preserving group actions and the structure theory of \( k \)-algebraic groups.

In what follows \( k \) denotes an arbitrary local field.

2.1. Measurable group actions. Let \( G \) be second countable locally compact group. A \( G \)-space is a standard \( \sigma \)-finite Borel measure space \((X, \mu)\) with a Borel measurable quasi-invariant action \( G \times X \rightarrow X \).

Consider a given \( G \)-space \( X \) and the associated \( G \)-action.

The action is ergodic if every measurable \( G \)-invariant subset of \( X \) is either null or conull. It is essentially transitive if there exists a conull orbit. Clearly an essentially transitive action is ergodic. An ergodic action is properly ergodic if it is not essentially transitive, or equivalently every \( G \)-orbit has measure 0. An ergodic action is irreducible if every non-central normal subgroup acts ergodically. The action is faithful if for every \( g \in G \), \( gx \neq x \) holds for \( x \in X \) of positive measure. Finally, the action is essentially free if \( \mu \)-a.e. \( x \in X \) has trivial stabilizer in \( G \).

We will also require the following somewhat less standard definitions:

Assume that \( H, K \) are two subgroups of \( G \). The \( K \)-action on \( X \) is \( H \)-ergodic, if every \( K \)-invariant subset \( E \subset X \) satisfies \( HE = E \). Similarly, \( X \) is \( H \)-irreducible if every non-central normal subgroup acts \( H \)-ergodically.

Let \( H \) be a subgroup of \( G \), and assume that \( X \) is an \( H \)-space. Consider the \( G \)-space \( G \times X/H \), where \( H \) acts on \( G \) from the right and on \( X \) from the left, and with the push-forward of the product measure. This is called the induced action from \( H \) to \( G \), and it is ergodic if and only if the original \( H \)-action is ergodic.

2.2. Subgroup structure of semisimple \( k \)-groups. An algebraic group \( G \) is said to be defined over \( k \), or simply a \( k \)-group, if the underlying variety as well as the multiplication and inverse maps are defined over \( k \) (see 0.10 of \cite{9} or \cite{4} for details).

Consider a connected semisimple \( k \)-group \( G \) and let \( G = G_k \) denote its group of \( k \)-points. Note that \( G \) is naturally a locally compact topological group (see 3.1 of \cite{14}) and so the definitions of \cite{24} apply. \( G \) is said to be \( k \)-anisotropic if \( \text{rank}_k(G) = 0 \), and this is the case if and only if \( G \) is compact (see 3.1 of \cite{14}). Moreover \( G \) is said to be simply-connected if every central isogeny from a connected algebraic group into \( G \) is an algebraic group isomorphism. For example, \( \text{SL}_n \) is simply-connected.
We recall the terminology from [9, 10] regarding several subgroups of particular interest. Let $S$ be a maximal $k$-split torus in $G$, and let $P$ be a minimal parabolic $k$-subgroup containing $S$. We choose an ordering on the set of roots $\Phi = \Phi (S, G)$ which corresponds to $P$ and denote by $\Delta$ the set of simple roots with respect to this ordering. For every subset $\theta \subset \Delta$ we consider (see I.1.2 of [9]):

- The torus $S_\theta$ which is the connected component of the intersection of the kernels of all $\alpha \in \theta$.
- The standard parabolic $k$-subgroup $P_\theta$ corresponding to $\theta$ and defined by the condition that it contains $P$ and that $Z_G (S_\theta)$ is the reductive Levi component of $P_\theta$.
- The opposite parabolic $\overline{P}_\theta$ such that $P_\theta \cap \overline{P}_\theta$ is the Levi $Z_G (S_\theta)$.
- The unipotent radicals $U_\theta = R_u (P_\theta)$ and $\overline{U}_\theta = R_u (\overline{P}_\theta)$.

Note that $P = P_\emptyset$ and $G = P_\Delta$. For each $\theta \subset \Delta$ we denote the corresponding groups of $k$-points by

$$P_\theta = (P_\theta)_k, \overline{P}_\theta = (\overline{P}_\theta)_k, S_\theta = (S_\theta)_k, V_\theta = (V_\theta)_k, \overline{V}_\theta = (\overline{V}_\theta)_k$$

groups without subscripts ($P, V$ etc.) will refer to $\theta = \emptyset$. For each $\theta$ write $L_\theta = P_\theta \cap \overline{V}$. Then $\overline{V}$ decomposes as $\overline{V} = L_\theta \times \overline{V}_\theta$. We quote the following lemma (IV.2.2 of [10]):

**Lemma 2.1.** Let $\theta \subset \Delta$. Then the map $\varepsilon : \overline{V}_\theta \times P_\theta \to G$ sending $(v, p) \in \overline{V}_\theta \times P_\theta$ to $v p^{-1} \in G$ is a homeomorphism onto its image $\overline{V}_\theta \cdot P_\theta$, which is open in $G$. Moreover $\overline{V}_\theta \cdot P_\theta$ is conull in $G$ and the Haar measure of $G$ is in the same measure class as the image of the product measure on $\overline{V}_\theta \times P_\theta$ under $\varepsilon$.

Lemma 2.1 implies that as $G$-spaces we have $G/P_\theta \cong \overline{V}_\theta$, and that the natural map $G/P \to G/P_\theta \cong \overline{V}_\theta$ corresponds under this isomorphism to the projection $\overline{V} \to \overline{V}_\theta$ with respect to the decomposition $\overline{V} = \overline{V}_\theta \times L_\theta$.

In particular $\overline{V} \cong G/P$ as Borel $G$-spaces. For certain elements of $G$ it is easy to describe the action explicitly. Let $u \in \overline{V} \cong G/P$. Observe that if $v \in \overline{V} \subset G$ then $v u = v u$, and similarly if $s \in S \subset G$ then $s u = s u$. $s u = s u$.

2.3. **Contracting automorphisms.** Recall that a contraction of a topological group is an automorphism $\varphi$ such that for every compact set $K \subset G$ and every neighborhood of the identity $U$ there exists $n \in \mathbb{N}$ such that $\varphi^n (K) \subset U$.

In the situation of 2.2 given a subset $\theta \subset \Delta$ we can find an element $s \in S_\theta$ such that $\text{Im} \ (s)^{-1}$ is contracting on $\overline{V}_\theta$ and is the identity of $L_\theta$ (see II.3.1 in [9]). Note that $\text{Im} \ (s)^{-1}$ is a non-trivial automorphism of $\overline{V}$ provided that $\theta \neq \emptyset$.

Given measurable subsets $E_n$ for $n \in \mathbb{N}$ and $E$ of the $\sigma$-finite measure space $(X, \mu)$, say that $E_n$ converge to $E$ in measure if for every measurable $K \subset X$ with $\mu (K) < \infty$ we have

$$\lim_n \mu ((E_n \cap K) \Delta (E \cap K)) = 0$$

For a measurable subset $E \subset \overline{V}$ and given $\theta \subset \Delta$ define

$$\psi_\theta (E) = \overline{V}_\theta (E \cap L_\theta)$$

For example, $\psi_\emptyset (E)$ is either $\overline{V}$ or $\emptyset$ depending on whether $e \in E$ while $\psi_\Delta (E) = E$. The following lemma (see IV.2.5 in [10] or 8.2.8 in [19]) controls the image of measurable subsets of $\overline{V}$ under iterations of $\text{Im} \ (s)$:
Lemma 2.2. Given \( \theta \) and an element \( s \in S_{\theta} \) as above, if \( C \subset V \) is measurable then for almost all \( u \in V \) we have
\[
\text{Inn}(s^n)(uC) = s^n \cdot uC \cdot s^{-n} \longrightarrow \psi_{\theta}(uC)
\]
where the convergence is understood in measure.

2.4. The group \( G^+_k \) and Mautner’s lemma. Let \( G \) be a connected semisimple algebraic \( k \)-group without \( k \)-anisotropic almost \( k \)-simple subgroups. Consider the subgroup \( G^+_k \) of \( G_k \) generated by the unipotent \( k \)-split subgroups of \( G_k \). The following properties of \( G^+_k \) are discussed in sections I.1.5 and I.2.3 of [9]:

Proposition 2.3. Assume that \( k \) is a local field and let \( G \) be as above. Then:

1. If \( G \) is simply-connected then \( G^+_k = G_k \).
2. Every subgroup of finite index in \( G_k \) contains \( G^+_k \).
3. For \( \text{char}(k) = 0 \) we have that \( G^+_k \) is a normal subgroup of finite index in \( G_k \) and \( G_k/G^+_k \) is abelian.
4. For \( k = \mathbb{R} \) we have \( G^+_k = G^0_k \), where \( G^0_k \) is the identity component of \( G_k \) considered as a real Lie group.

The discrepancy between \( G_k \) and \( G^+_k \) is a source of some technical difficulties for us, which disappear for simply-connected groups. For this reason Theorems 1.2 and 1.3 were stated with this additional assumption.

The group \( G^+ \) enters the picture due to the following, which is an immediate corollary of Mautner’s lemma (see II.3.3 of [9]):

Lemma 2.4. Let \( X \) be a \( G^+_k \)-irreducible \( G \)-space with finite \( G \)-invariant measure, and let \( s \in S \) be such that \( \text{Inn}(s)^{-1} \) is contracting on some \( V_{\theta} \) with \( \theta \neq \Delta \). Then the \( s \)-action is \( G^+_k \)-ergodic.

We will also need the following lemma, which follows easily from the definitions of an induced action and \( G^+_k \)-irreducibility.

Lemma 2.5. Consider a group \( H \) such that \( G^+_k \leq H \leq G_k \) and let \( X \) be a \( H \)-space. If \( X \) is \( G^+_k \)-irreducible, then the induced action of \( G_k \) is \( G^+_k \)-irreducible.

3. Measure algebras and the Effros topology

A central tool in the proofs of the factor and the intermediate factor theorems is that of measure algebras. Indeed, both theorems admit an equivalent formulation in that language. The restatement relies on Mackey’s point realization and its extensions to \( G \)-spaces.

Therefore we therefore study measure algebras in the current section, in particular those of product spaces. Indeed, we construct a topology on a certain collection of measure sub-algebras of a given algebra. The relevant Borel structure is the Effros Borel space, which is defined on the collection of closed subsets of an arbitrary topological space.

To be precise, we will be interested in understanding the measure algebras associated to the following situation - given three \( G \)-spaces \( X, Y \) and \( Z \) we have the measure-preserving \( G \)-maps \( f : X \times Z \to Y \) and \( g : Y \to X \) whose composition is the projection. For the purposes of Theorem 4.1 we will only require the case \( Z = G/p \), but currently we discuss the general case. Our goal is to prove Theorem 3.8 below.
3.1. Measure algebras. Given a measure space \((X, \mu)\) we let \(\Sigma(X)\) denote the \(\sigma\)-algebra of \(X\). The corresponding measure algebra \(B(X)\) is the Boolean algebra of all measurable subsets upto a.e. equivalence. Namely, \(B(X)\) is \(\Sigma(X)\) modulo the ideal of null sets. \(B(X)\) clearly depends on the class of \(\mu\).

Consider the \(G\)-map \(f : X \times Z \to Y\). Passing to measure algebras we have the inclusion \(B(Y) \subset B(X \times Z)\).

**Definition 3.1.** For a measurable set \(C \in \Sigma(Y)\) and for each \(x \in X\), define \(C_x \in \Sigma(Z)\) by \(C_x = \{z \in Z : (x, z) \in f^{-1}(C)\}\). Similarly, define the sub-\(\sigma\)-algebra \(B_x\) of \(\Sigma(Z)\) by \(B_x = \{C_x : C \in B(Y)\}\). Abusing notations, we denote by \(B_x\) also the corresponding sub-measure-algebra of \(B(Z)\) for any \(x \in X\), and consider \(C_x\) as an element of \(B_x\).

Note that we insist on making the above definition at the \(\sigma\)-algebra and not measure algebra level, so that every \(B_x\) is meaningfully defined. In other words, we assume that all involved \(\sigma\)-algebras are the standard Borel ones (and not the completions thereof).

Given an element \(C\) of \(B(Y)\) let \(\tilde{C} \in \Sigma(Y)\) be any representative of its equivalence class. The choice of \(\tilde{C}\) determines in turn a member \(C_x\) of the measure algebra \(B_x\) for every \(x \in X\). Note that a different \(\tilde{C}\) still gives the same \(C_x\) for \(\mu\)-almost every \(x\). We will denote this situation \(C = f^0 C_x\) while keeping in mind the ambiguity involved.

Every element \(g\) of \(G\) defines an isomorphism of \(Z\) as a measurable space, and so acts on \(\Sigma(Z)\) via Boolean automorphisms. Since \(\Sigma(Y) \subset \Sigma(X \times Z)\) we have \(gC_x = C_{gx}\) for every \(C \in \Sigma(Y)\), \(g \in G\) and \(x \in X\). Similarly, since the \(G\)-action is measure class preserving it descends to \(B(Z)\) and \(gB_x = B_{gx}\) for every \(g \in G\), \(x \in X\) by definition.

3.2. The topology of convergence in measure on \(B(X)\). There is a well-known metric on the measure algebra \(B(X)\), where \(X\) is an \(\sigma\)-finite \(G\)-space. For every pair \(E, F \in B(X)\) set \(d(E, F) = \mu(E \Delta F)\), where \(\mu\) is some fixed finite measure \(\mu\) in the quasi-invariant measure class on \(X\). \(d\) is clearly a metric, and it defines the topology of convergence in measure on \(B(X)\) described above.

**Remark.** The use of this topology was suggested by Nevo and Zimmer in section 4 of [13].

**Proposition 3.2.** Every sub-Boolean-algebra \(B \subset B(X)\) is complete with respect to \(d\). Therefore every \(B\) is closed in the topology of convergence in measure. Moreover \(B(X)\) with the convergence in measure topology is second countable and Polish.

**Proof.** For the completeness of every sub-algebra \(B\), see Proposition 2.30 of [13]. Therefore \(B(X)\) is metric and complete with respect to \(d\).

Since \((X, \mu)\) is a standard Borel probability space, \(\mu\) is regular. We see that \(B(X)\) is separable by considering the countable collection of all finite unions of basic open sets in \(X\). Being separable and metric, \(B(X)\) is second countable. We conclude that \(X\) is Polish.

**Remark 3.3.** \(B(X)\) with the above topology is not compact. For example, assume that \(X \cong [0, 1]\) and consider the sequence \(f_n = \chi_{E_n}\) for \(n \geq 1\) where \(E_n \subset [0, 1]\) is the subset of all real numbers with 1 at the \(n\)-th position of their terminating
binary expansion. Then \( f_n \) has no converging subsequence. By a similar argument, \( B(X) \) is not locally compact.

3.3. The Effros Borel space and Effros topology. Let \( \mathcal{X} \) be an arbitrary topological space, and let \( \mathcal{C} = \mathcal{C}(\mathcal{X}) \) denote the collection of all closed subsets of \( \mathcal{X} \). Endow \( \mathcal{C} \) with the \( \sigma \)-algebra generated by the sets
\[
M_U = \{ F \in \mathcal{C} : F \cap U = \emptyset \}
\]
for every open \( U \) in \( \mathcal{X} \). The space \( \mathcal{C} \) with the resulting \( \sigma \)-algebra is the Effros Borel space of \( \mathcal{X} \).

**Proposition 3.4.** If \( \mathcal{X} \) is Polish then the Effros Borel space \( \mathcal{C}(\mathcal{X}) \) is standard. Moreover if \( \{ U_n \} \) is a countable basis for \( \mathcal{X} \) then \( M_{U_n} \) generates the Effros \( \sigma \)-algebra.

**Proof.** See Theorem 12.6 on page 75 of [8]. \( \square \)

It will be convenient for us to have in mind a topology on \( \mathcal{C}(\mathcal{X}) \) whose Borel structure is Effros:

**Definition 3.5.** The Effros topology on \( \mathcal{C} = \mathcal{C}(\mathcal{X}) \) is the topology generated by the basis consisting of the sets defining the Effros Borel structure, namely \( M_U \) for every open \( U \) in \( \mathcal{X} \).

It is easy to verify that this is indeed a basis: \( \mathcal{X} = M_\emptyset \) is in itself a basis member, and \( M_U \cap M_V = M_{U \cup V} \) for \( U \) and \( V \) open in \( \mathcal{X} \). The motivation for introducing the Effros topology comes from the following useful fact:

**Lemma 3.6.** Let \( F_n \in \mathcal{C} \) and \( x_n \in F_n \) for every \( n \in \mathbb{N} \) such that \( x_n \to x \in \mathcal{X} \) and \( F_n \to F \in \mathcal{C} \) where the second convergence is Effros. Then \( x \in F \).

**Proof.** \( x \notin F \) is impossible, since it would imply \( x_n \in F^c \) and hence \( F_n \notin M_{F^c} \) for all large \( n \) while \( F \in M_{F^c} \). \( \square \)

**Remark 3.7.** Recall that the Chaubuty topology on \( \mathcal{C}(\mathcal{X}) \) where \( \mathcal{X} \) is an arbitrary topological space is generated by the sub-basis given by
\[
\{ F \in \mathcal{C} : F \cap U \neq \emptyset \} \quad \text{and} \quad \{ F \in \mathcal{C} : F \cap K = \emptyset \}
\]
where \( U \) is any open and \( K \) is any compact in \( \mathcal{X} \). This topology has the advantage of being compact. However, the analog of Lemma 3.6 only holds for the Chaubuty topology whenever \( \mathcal{X} \) is locally compact, which is not true in our case of interest (see Remark 3.3).

Note that the basic Effros-open sets are Chaubuty-closed. Moreover if \( \mathcal{X} \) is metric, it is easy to see that the Borel structure for the Chaubuty topology is Effros. To summarize, the Effros topology mainly serves here a notational purpose, namely to be able to state Lemma 3.6 in the natural language of convergence.

Our goal is to topologize the family of subalgebras \( \{ B_x \}_{x \in \mathcal{X}} \) of \( B(Z) \) where \( Z \) is some \( G \)-space. Specialize the above to \( \mathcal{X} = B(Z) \) with the topology of convergence in measure. Now, by Proposition 3.2 we have that \( B_x \in \mathcal{C}(\mathcal{X}) \) for every \( x \in \mathcal{X} \). The desired topology is obtained by regarding \( \mathcal{C}(\mathcal{X}) \) with the Effros topology and \( \{ B_x \} \) with the induced subspace topology. Note that by Propositions 3.2 and 3.4 the Borel structure on \( \mathcal{C}(\mathcal{X}) \) is standard.

The central result of this section is the following:
Theorem 3.8. Let \( f : X \times Z \to Y \) and \( g : Y \to X \) be a pair of measurable \( G \)-maps whose composition is the projection on \( X \). Define a map \( \iota : X \to C = C(B(Z)) \) such that \( \iota(x) = B_x \). Then \( \iota \) is Borel measurable.

The proof of Theorem 3.8 will depend on disintegrating the involved measures. Namely, consider

\[
(X \times Z, \Sigma(X) \times \Sigma(Z), \eta \times \mu) \overset{f}{\to} (Y, \Sigma(Y), \lambda)
\]

where we have indicated the respective \( \sigma \)-algebras and measures. To simplify set \( m = \eta \times \mu \).

For every \( x \in X \) denote \( \mu_x = \delta_x \times \mu \) and \( \lambda_x = f_* \mu_x \) where \( \delta_x \) is the Dirac measure on \( x \). From Fubini’s theorem on \( X \times Z \) and as \( f_* m = \lambda \) it follows that \( \lambda = \int_X \lambda_x d\eta(x) \) is a disintegration of \( \lambda \) over \( \eta \). Since all the involved measure spaces are standard, we may form the disintegration \( m = \int_Y m_y d\eta(y) \) where \( \lambda \)-a.e. \( m_y \) is a probability measure on \( X \times Z \). By definition for every measurable subset \( E \subset X \times Z \) the map \( y \mapsto m_y(E) \) is \( \Sigma(Y) \)-measurable and the following equality holds

\[
m(E) = \int_Y m_y(E) d\lambda(y) = \int_X \left( \int_Y m_y(E) d\lambda_x(y) \right) d\eta(x)
\]

On the other hand, by Fubini’s theorem we have \( m(E) = \int_X \mu_x(E) d\eta(x) \).

These two functions under the integral over \( X \) are \( \Sigma(X) \)-measurable, and we claim that they are \( \eta \)-a.e. equal for any fixed \( E \). Otherwise, wlog the \( \Sigma(X) \)-measurable set

\[
X_0 = \left\{ x : \int_Y m_y(E) \lambda_x(y) > \mu_x(E) \right\}
\]

has \( \eta(X_0) > 0 \), but then we arrive at a contradiction by considering the \( \Sigma(X) \times \Sigma(Z) \)-measurable set \( E_0 = E \cap (X_0 \times Z) \). It follows that for any fixed \( E \) we have \( \int_Y m_y(E) \lambda_x(y) = \mu_x(E) \) for \( \eta \)-a.e. \( x \in X \). Let \( \mathcal{E} \) be a countable generating family for \( \Sigma(X) \times \Sigma(Z) \), such that \( \mathcal{E} \) is also closed under (finite) intersections. Then we can find a conull set in \( X \) such that the above equality holds for every \( E \in \mathcal{E} \) - from now on implicitly restrict attention to this conull set. Hence by the monotone class theorem we get the formula

\[
\mu_x = \int_Y m_y d\lambda_x
\]

Having established the required machinery, we turn to the proof of Theorem 3.8

Proof of 3.8. By Propositions 3.2 and 3.4 it is enough to show that \( \iota^{-1} M_U \) is measurable for every \( U \in \mathcal{U} \) where \( \mathcal{U} \) is a countable basis for \( B(Z) \). In fact, we will show that each complement \( \iota^{-1} (M_U)^c \) is measurable.

We may assume that \( U = B_x(A) \) for some \( A \in B(Z) \) and \( \varepsilon > 0 \). Note that

\[
x \notin \iota^{-1} M_U \iff B_x \notin M_U \iff B_x \cap U \neq \emptyset \iff d(B_x, A) < \varepsilon
\]

where \( d \) is a metric generating the convergence in measure topology on \( Z \). So we need to find a condition for \( d(B_x, A) < \varepsilon \) that depends measurably on \( x \). Consider

\[
\Phi_A = \left\{ y \in Y : m_y(A) \geq \frac{1}{2} \right\}
\]

so that \( \Phi_A \) consists of all the \( y \in Y \) such that \( A \) meets the fiber \( f^{-1}(y) \) at more than half of its \( m_y \)-measure. Then \( \Phi_A \) is \( \Sigma(Y) \)-measurable since \( m_y \) is a disintegration of \( m \).
For every \( x \in X \) define
\[
C(x) = \{ z : f(x, z) \in \Phi_A \}
\]
and note that \( C(x) \) is a measurable subset of \( Z \). Now, let \( X_1 \subset X \) be the following set, which is \( \Sigma(X) \)-measurable due to Fubini’s Theorem
\[
X_1 = \{ x : d(C(x), A) < \varepsilon \}
\]
We claim that \( X_1 = (\iota^{-1}M_U)^{\varepsilon} \).

If \( x \in X_1 \) we have by definition that \( d(C(x), A) < \varepsilon \). Since \( \Phi_A \) is \( \Sigma(Y) \)-measurable, we have \( C(x) \in B_x \) and so \( d(B_x, A) < \varepsilon \) as well.

Conversely, suppose that \( d(B_x, A) < \varepsilon \) holds for a given \( x \in X \) so that we can find \( D(x) \in B_x \) that satisfies \( d(D(x), A) < \varepsilon \). On the other hand, let \( C(x) \in B_x \) be as above. Recall that \( m_y \) is a probability measure supported on the fiber \( f^{-1}(y) \) for \( \lambda \)-a.e. \( y \). Clearly, this implies that each of \( C(x) \) and \( D(x) \) is either \( m_y \)-null or \( m_y \)-conull. In particular, \( C(x) \) is \( m_y \)-null if and only if \( m_y(A) < \frac{1}{\varepsilon} \). This gives the estimate
\[
m_y(C(x) \Delta A) \leq m_y(D(x) \Delta A)
\]
for a.e. \( y \in Y \). Integrating over \( \lambda_x \) we obtain that
\[
d(C(x), A) = \mu(C(x) \Delta A) \leq \mu(D(x) \Delta A) = d(D(x), A) < \varepsilon
\]
and so \( x \in X_1 \) by definition, and the proof is complete. \( \square \)

We have constructed a Borel map \( \iota : X \to C \), where \( C \) is given the Effros topology.

From the above remarks this map is \( G \)-equivariant.

Remark. In fact, we only found a conull measurable subset \( X' \subset X \) such that the restriction of \( \iota \) to \( X' \) is measurable. But by a known result (see 2.2.16 of \([19]\)) we may assume that \( X' \) is \( G \)-invariant, and so we implicitly assume that \( X' \) is all of \( X \).

4. The Nevo-Zimmer Intermediate Factor Theorem

The goal of this section is to prove the Nevo-Zimmer intermediate factor theorem over local fields:

**Theorem 4.1.** Let \( k \) be a local field with \( \text{char}(k) = 0 \), and let \( G \) be a connected semisimple algebraic \( k \)-group without \( k \)-anisotropic almost \( k \)-simple subgroups. Assume that \( \text{rank}_k(G) \geq 2 \) and denote \( G = G_k, G^+ = G^+_k \). Let \( X \) be an ergodic and \( G^+ \)-irreducible \( G \)-space with finite invariant measure, and let \( P \) be a minimal parabolic \( k \)-subgroup. If \( Y \) is an ergodic \( G \)-space for which there exist measure preserving \( G \)-maps \( X \times G/P_k \to Y \to X \) whose composition is the projection, then there exists a parabolic \( k \)-subgroup \( Q \) containing \( P \) such that \( Y \cong X \times G/Q_k \) as a \( G \)-space in such a way that the maps \( X \times G/P_k \to Y \) and \( Y \to X \) are identified with the natural ones.

The above formulation is slightly more general than that given in the introduction (see Theorem \([12]\)), since we no longer assume that \( G \) is simply-connected. Note that \( G^+ \)-irreducibility is a weaker condition than irreducibility. We remark that if \( \text{char}(k) \neq 0 \) but \( G \) is indeed simply-connected, our proof still applies (see \([2, 3]\)). Moreover, if \( G^+ \) is assumed to equal \( G \) much of the following becomes technically simpler to a large extent.

The motivation for what follows could be perhaps better appreciated in light of the proof of the factor theorem (see \([10] \) or chapter 8 of \([19] \)). We retain the notations and assumptions of Theorem \([4, 1]\) throughout the current section.
4.1. The main lemma. The main lemma (see below) is just the natural generalization of Margulis’ Lemma 1.14.1 of [10] to the context of the intermediate factor theorem (see also Lemma 4.3 of [18]). We retain the notations and assumptions of Theorem 4.1.

Fix some $\theta \subseteq \Delta$ and let $s \in S_{\theta}$ be an element such that $\text{Inn}(s)^{-1}$ is contracting on $V_{\theta}$. We study the space $X \times G$ with the $G \times \mathbb{Z}$-action given by $(h, n) \cdot (x, g) = (hx, hgs^{-n})$.

**Proposition 4.2.** Every $G \times \mathbb{Z}$ invariant subset $E \subset X \times G$ under the above action is of the form $\cup_{a \in G/G^+} (aE_0 \times aG^+)$ for some $G^+$-invariant subset $E_0 \subset X$.

**Proof.** Since the $G$-action on $X$ is $G^+$-irreducible and finite measure preserving, it follows from Mautner’s lemma (reproduced here as Lemma 2.4) that the $s$-action on $X$ is $G^+$-ergodic as well. Let $E \subset X \times G$ be a measurable subset invariant under the given action of $G \times \mathbb{Z}$. In particular $E = \pi^{-1}(\overline{E})$ where $\pi$ is the natural projection of $X \times G$ on $X \times G/\mathbb{Z}$ and $\overline{E}$ is $G$-invariant. For $g \mathbb{Z} \in G/\mathbb{Z}$ denote

$$\overline{E}_g = \{ x \in X : (x, g\mathbb{Z}) \in E \}$$

$G$-invariance of $\overline{E}$ implies that $h\overline{E}_g = \overline{E}_{hg}$ for every $h \in G$ which in turn gives $s\overline{E}_e = \overline{E}_e$ and $G^+ \overline{E}_e = \overline{E}_e$. Set $E_0 = \overline{E}_e$. The result follows from the fact that $G^+ \triangleleft G$ (see Proposition 2.3). \qed

Consider $B(G/r)$ with the convergence in measure topology. Let $\mathcal{C}$ be the collection of all closed subsets of $B(G/r)$ with the Effros topology. In Theorem 4.8 we have constructed a $G$-equivariant Borel map $\iota : X \to \mathcal{C}$ where $\iota x = B_x$. From now on we regard $X$ with the topology induced from $\iota$.

Given an open subset $U \subset X$ define the subset $Z_U \subset X \times G$ by

$$Z_U = \left\{ (x, g) \in U \times G : gG^+ \supset \{ hgs^{-n} \} \quad \text{where } n \geq 0 \text{ and } hx \in U \right\} \quad \text{II} \quad (U^c \times G)$$

**Proposition 4.3.** For every open subset $U \subset X$, $Z_U$ is conull in $X \times G$.

**Proof.** We may assume that $\eta(U) > 0$, where is $\eta$ is the probability measure on $X$. Let $\mu$ be the right Haar measure on $G$ and $\mathcal{V} = \{ V_i \}$ be a countable basis for $G^+$. Fix some coset representative $a \in G/G^+$. Every $V_i$ is open and so $(\eta \times \mu)(U \times aV_i) > 0$. Consider the $G \times \mathbb{Z}$-action on $X \times G$ via

$$(h, n) \cdot (x, g) = (hx, hgs^{-n})$$

Note that $W_i = (G \times \mathbb{Z}^+) \cdot (U \times aV_i)$ satisfies $(\text{id}_G \times s)W_i \subset W_i$, and since the action is measure class preserving this implies that $(\text{id}_G \times s)W_i = W_i$ (up to a null set) and that $W_i$ is $G \times \mathbb{Z}$-invariant. It follows from Proposition 4.2 that $W_i$ contains $U \times aG^+$ (up to a null set). So the intersection $W = \cap_i W_i$ contains $U \times aG^+$ as well. This means that a.e. $(x, g) \in X \times aG^+$ meets the $G \times \mathbb{Z}^+$-orbit of $U \times aV_i$ for every $i$, or in other words that for a.e. $(x, g) \in X \times aG^+$ there are $h \in G, n \in \mathbb{Z}$ such that $hx \in U$ and $hgs^{-n} \in aV_i$ for every $V_i \in \mathcal{V}$. Since $a$ was arbitrary this proves that $Z_U$ is conull. \qed

The following main lemma is used below to complete the proof of Theorem 4.1 along the lines of Zimmer in [18].
Lemma 4.4. Given \( \theta \subseteq \Delta \) let \( P \subset P_0 \subset G \) be a parabolic k-subgroup, let \( C \in B(Y) \) and write \( C = \bigoplus C_x \). Then for almost all \((x, u) \in X \times V\) we have \( g \cdot \psi_\theta(uC_x) \in B_x \) for all \( g \in G^+ \).

Proof. We let \( U = \{W_i\} \) denote a countable basis for \( B(G/P) \) (see Proposition 3.2). Consider the following countable collection of open subsets of \( X \) — for every \( l \geq 1 \) and \( n_1, \ldots, n_l \) define \( U_{n_1, \ldots, n_l} \subset X \) by

\[
U_{n_1, \ldots, n_l} = \{ x : B_x \cap W_{n_i} = \emptyset \ \forall i = 1, \ldots, l \}
\]

where each \( M_{W_{n_i}} \subset C \) is the corresponding basic Effros open set. From Proposition 4.3 the subset \( Z_U \) of \( X \times G \) is conull for every such \( U = U_{n_1, \ldots, n_l} \), and let \( Z \subset X \times G \) be the intersection over all \( Z_U \) of this form. Then \( Z \) is conull in \( X \times G \) as well.

Now let \( Z_1 = Z \cap X \times V \). In fact, \( Z_1 \) is conull in \( X \times V \) with the measure \( \eta \times \mu_V \) where \( \mu_V \) is the Haar measure on \( V \). This is proved using the argument of lemma 1.9 in [10].

Recall that we identify \( G/P \) with \( V \), and so we have a corresponding action of \( G \) on \( V \). Similarly, we identify \( B(G/P) \) with \( B(V) \). Let

\[
Z_2 = \{ (x, u) \in X \times V : s^n uC_x s^{-n} \text{ converges in measure } \eta_V \text{ to } \psi_\theta(uC_x) \}
\]

which is conull by Lemma 2.2 and Fubini’s theorem.

We claim that the assertion of the lemma holds for the conull set \( Z_1 \cap Z_2 \). Fix \( g \in G^+ \) and consider some \((x, u) \in Z_1 \cap Z_2 \). Let

\[
\alpha(x) = \{ n \in \mathbb{N} : lx \in M_{U_n} \} = \{ \alpha_1(x), \alpha_2(x), \ldots \}
\]

and define \( U_n(x) = U_{\alpha_1(x), \ldots, \alpha_n(x)} \). Clearly \( U_n(x) \) is a decreasing sequence of neighborhoods of \( x \).

Since \((x, u) \in Z_1 \) and \( V \subset G^+ \) we can choose \( h_n \in G \) such that \( h_nus^{-m_n} \rightarrow g \) with \( m_n \rightarrow \infty \) and such that \( h_nx \in U_n(x) \) for every \( n \). Denote \( g_n = h_nus^{-m_n} \) so that \( h_n = g_ns^{m_n}u \) and \( g_n \rightarrow g \). Since \((x, u) \in Z_2 \) we have

\[
h_nC_x = g_ns^{m_n}uC_x = g_n\left(s^{m_n}uC_x s^{-m_n}\right) \rightarrow g\psi_\theta(uC_x)
\]

so \( h_nC_x \) has a limit in measure. Denote the limit by \( C \). On the other hand clearly \( h_nx \rightarrow x \) and so by Lemma 4.6 we have \( C \in b_{x} \), as required.

\[\square\]

4.2. Proof of the Nevo-Zimmer intermediate factor theorem. We are ready to apply lemma 4.4 and complete the proof of theorem 4.1. This is essentially the proof given in [18] and we reproduce it here for the sake of completeness.

We make use of the fact that a \( G \)-equivariant Boolean isomorphism of measure algebras lifts to a \( G \)-equivariant measure space isomorphism between conull subsets of the original spaces (see B.7 in [19]). So it is enough to prove the following assertion: every \( G \)-invariant sub-Boolean-\( \sigma \)-algebra \( B \) with \( B(X) \subset B \subset B(X \times G/P) \) equals \( B(X \times G/P_0) \) for some \( \theta \subset \Delta \).

Proof of Theorem 4.1. Consider a \( G \)-invariant sub-Boolean-\( \sigma \)-algebra \( B \) lying in between \( B(X) \) and \( B(X \times G/P) \) as above. Let \( B_0 \) be a maximal Boolean \( \sigma \)-algebra such that \( B(X) \subset B_0 \subset B \) and \( B_0 = B(X \times G/P_0) \) for some \( \theta \subset \Delta \). We want to show that \( B_0 = B \), so assume to the contrary that there exists some \( C \in B \setminus B_0 \). It
is certainly possible that $P_0 = G$ but as $B_0$ is a proper subalgebra we must have $P \subseteq P_0$.

Write $C = \bigcup C_x$ where $C_x$ satisfies $C_x \in B_x$ (see Section 3 and Definition 5). Since $C \notin B_0$ but $C \in B (X \times G/p)$ we may assume that $C_x \notin B (G/p_k)$ for $x$ in a set of positive measure (see e.g. lemma 5.5 of [17]). Recall that $P_0$ is generated by the parabolic $k$-subgroups $P(b)$ for $b \in \Theta$, i.e. the minimal parabolic $k$-subgroups $P'$ contained in $P_0$ such that $P \leq P'$ (see 1.1.2.4 of [9]). Observe that for a measurable subset $E \in B (G/p)$ we have $E \in B (G/p_k)$ if and only if the preimage $\overline{E}$ of $E$ in $B (G)$ satisfies $\overline{E}P_0 = \overline{E}$. The two previous facts imply that there is a fixed $b \in \Delta$ such that $C_x \notin B (G/p_k)$ for $x$ in a set of positive measure.

We emphasize that the rank $k (G) \geq 2$ assumption is used at this point to imply that $P_0$ is a proper subgroup of $G$. This is needed to apply Lemma 4.4 or more concretely to establish the existence of some $s \in S_\Theta$ such that $\text{Inn} (s)^{-1}$ is contracting on $\overline{v}_\Theta$.

Recall the identifications of $B (G/p_k) \subset B (G/p)$ with $B (\overline{V}_b) \subset B (\overline{V})$. Moreover $\overline{V} = L_b \times \overline{V}_b$ and the second inclusion corresponds to the pullback of the projection $\overline{V} \to \overline{V}_b$. So we have for $x$ in a set of positive measure that $uC_x \cap L_b$ is neither null or conull for $u \in \overline{V}$ in a positive measure subset of $\overline{V}$. We may now apply Lemma 4.4 and Fubini’s theorem to obtain some particular $u \in \overline{V}$ such that

1. $uC_x \cap L_b$ is neither null or conull in $\overline{V}_b$ for $x$ in a set of positive measure.
2. for almost all $x$, $g\psi_b (uC_x) \notin B (G/p_k)$ and in particular $\psi_b (uC_x) \notin B (G/p_k)$.

Note that condition (1) implies that for $x$ in a set of positive measure, $\psi_b (uC_x) \notin B (G/p_k)$ and in particular $\psi_b (uC_x) \notin B (G/p_k)$.

Let $B_1 (x)$ be the algebra generated by $B (G/p_k)$ and $g\psi_b (uC_x)$ for every $g \in G^+$. Note that on the one hand from (2) above $B (G/p_k) \subset B_1 (x) \subset B_x$ for a.e. $x$, while on the other hand from (1) above $B_1 (x)$ strictly contains $B (G/p_k)$ for $x$ in a subset of positive measure. Moreover $B_1 (x)$ is clearly a $G^+$-invariant subalgebra of $B (G/p_k)$ whenever it is defined. But every such $G^+$-invariant subalgebra must be of the form $B (G/p_\theta')$ for some $\theta' \in \Delta$ (see lemma 1.13 of [10]). Now $\Delta$ is finite, and so there exists a subset $X_1 \subset X$ of positive measure such that $B_1 (x) = B (G/p_\theta')$ for some fixed $\theta' \subset \Delta$ with $\theta' \subset \Theta$.

We have established that $B (G/p_\theta') \subset B_x$ for every $x \in X_1$. Recall that $gB_x = B_{gx}$ and so $X_1$ is in fact $G$-invariant. From ergodicity, it must be conull and so $B (G/p_\theta') \subset B_x$ holds for a.e. $x$. But this contradicts the maximality of $B_0$, and we’re done.

\[ \square \]

5. The Stuck-Zimmer theorem

We formulate and discuss the Stuck-Zimmer theorem over local fields. Note that we allow actions of a subgroup of $G$, as long as it contains $G^+$.

**Theorem 5.1.** Let $k$ be a local field with $\text{char} (k) = 0$, and let $G$ be a connected semisimple algebraic $k$-group without $k$-anisotropic almost $k$-simple subgroups. Assume that $\text{rank}_k (G) \geq 2$ and $G_k$ has property $(T)$. Denote $G = G_k$, $G^+ = G_k^+$ and let $X$ be a $H$-space where $H$ is a group with $G^+ \leq H \leq G$. If the $H$-action is faithful, properly ergodic, $G^+$-irreducible and finite measure preserving then it is essentially free.

We remark that $G^+$-irreducibility is weaker than irreducibility, and that the proof still applies if $\text{char} (k) \neq 0$ but $G$ is simply-connected. Note that Lemma 5.3
allows to slightly weaken the requirement that \( H \) is acting \( G^+ \)-irreducibly in the situation where the \( H \)-action in question is the restriction of an \( G \)-action.

The Stuck-Zimmer theorem for real semisimple Lie groups without compact factors and with finite center follows from the case \( k = \mathbb{R} \). To see this, first note that we may assume that the center is trivial (see [16]). Next, recall that every connected semisimple real Lie group with trivial center is isomorphic to \( G_0^0 \) for some connected algebraic \( \mathbb{R} \)-group \( G \) (see 3.1.6 in [19]). Moreover \( G_k \) is compact if and only if \( G \) is \( k \)-anisotropic (see e.g. I.2.3.6 of [9]). Finally for \( k = \mathbb{R} \) we have \( G_0^0 = G_0^0 \) (see proposition 2.3) and so Theorem 5.1 applies.

5.1. Proof of the Stuck-Zimmer theorem. Let \( G, G, G^+ \) and \( H \) be as in the statement of Theorem 5.1. We need to show that every faithful properly ergodic and \( G^+ \)-irreducible \( H \)-space with finite invariant measure is essentially free.

The first step is to show that we may assume \( H = G \). Given a \( H \)-space \( X \) as above, consider the induced action of \( G \) on \( X' \). It is easy to see that the induced action is also faithful, properly ergodic and finite measure preserving. Moreover, from Lemma 2.5 the \( G \) action on \( X' \) is \( G^+ \)-irreducible. If the theorem is known to hold for the full \( G \)-action, then a.e. point of \( X' \) has trivial stabilizer in \( G \), and in particular a.e. point of \( X \) has trivial stabilizer in \( H \).

From now on we follow [16]. We sketch the coarse outline of the proof and observe that it carries over to the local field case. We work over \( k \), while keeping in mind that the original theorem is the case \( k = \mathbb{R} \). For the complete details the reader is referred to [16].

Consider a \( G \)-space \((X, \mu)\) as above. The starting point is the following (lemma 1.5 of [16]):

**Lemma 5.2.** Let \( G \) be a second-countable group with property \((T)\), and let \( X \) be an ergodic \( G \)-space with finite invariant measure. Then the action of \( G \) on \( X \) is weakly amenable if and only if the action is essentially transitive.

For discussion of amenable and weakly amenable actions the reader is referred to [16, 19]. Since \( G \) has property \((T)\), we may assume that the action is not weakly amenable.

Next, we let \( P = P_k \) where \( P \) is a minimal parabolic \( k \)-subgroup of \( G \). We equip \( G/P \) with a quasi-invariant measure \( \eta \). \( P \) is solvable and hence amenable, and so the \( G \) action on \( G/P \) is amenable (see section 4.3 in [19]). From this fact combined with the contra-positive of the weak amenability of the action one obtains a measure space \((Y, \nu)\) and \( G \)-equivariant maps

\[
G/P \times X \xrightarrow{f} Y \xrightarrow{p} X
\]
such that \( p \circ f \) is the projection on \( X \) (for a general formulation of this argument, see Proposition 2.4.5 of [6]). Furthermore \( f \) and \( p \) are measure preserving with respect to the measures \( \eta \times \mu, \nu \) and \( \mu \) on \( G/P \times X, Y \) and \( X \) respectively.

We are now in a situation to apply the Nevo-Zimmer intermediate factor theorem - theorem 1.1 of [13] for \( k = \mathbb{R} \), or Theorem 4.1 for arbitrary \( k \). As a result we may assume that \( Y \cong G/Q \times X \) where \( Q = Q_k \) for a parabolic \( k \)-subgroup with \( P \subset Q \), that \( f \) and \( p \) are the natural maps and that the \( G \)-action on \( Y \) is the product action. Furthermore \( Q \) is a proper subgroup of \( G \) - this follows from the lack of weak amenability.
To complete the proof we consider the stabilizers $G_x$ for $x \in X$. Again, from the definition of a weak amenable action we obtain that for a.e. $x \in X$ the group $G_x$ acts trivially on $p^{-1}(x)$. Hence

$$G_x \subset H_x = \bigcap_{gQ} gQg^{-1}$$

where the intersection is over a.e. $gQ \in G/Q$. By standard arguments $H_x$ is a normal subgroup of $G$, and is in fact proper since $Q \subset G$. So a.e. stabilizer $G_x$ is contained in a proper normal subgroup, and we conclude using the faithfulness and ergodicity of the action (see [10] for details).

We observe that up to Theorem 5.1 over arbitrary $k$, which was proved in [10] indeed generalizes immediately to $k$.

5.2. Restricted actions. The following straightforward observation allows to slightly weaken the assumptions of Theorem 5.1 in particular cases.

Lemma 5.3. Let $X$ be an ergodic $G_k$-space with invariant probability measure and let $H \triangleleft G_k$ be a subgroup such that $H$ acts $G_k^+$-ergodically. If $K \leq H$ is a subgroup of finite index such that $K \triangleleft G_k$, then $K$ acts $G_k^+$-ergodically as well.

Proof. Choose finite sets of coset representatives $h_1, \ldots, h_m$ and $g_1, \ldots, g_n$ for $H/K$ and $G_k^+ / G_k^+$, respectively.

Consider a $K$-invariant measurable subset $E \subset X$ of positive measure, namely $KE = E$. Since $K \triangleleft H$ it follows that $\cup_i h_i E$ is $H$-invariant. By assumption $\cup_i h_i E$ is $G_k^+$-invariant, and by a similar argument we have that $\cup_{i,j} g_i h_j E = X$. In particular every $K$-invariant subset $E$ must have $\mu(E) \geq 1/mn$.

Since $K \triangleleft G_k$, if $E$ is $K$-invariant then so is every translate $gE$ for $g \in G_k$. From the above, there is some $E_0 \subset E$ of measure $\mu(E_0) = \frac{1}{m} \geq \frac{1}{nm}$ such that the partition generated by all the $gE$ for $g \in G_k$ consists of finitely many disjoint copies $\gamma_1 E_0, \ldots, \gamma_l E_0$ for $\gamma_1, \ldots, \gamma_l \in G_k$. We may assume that $E = \cup_{i=1}^l \gamma_i E_0$ for some $1 \leq k \leq l$. Note that we obtain a map $G_k \to \text{Sym}_l$, and in particular the set-wise stabilizer of $E$ is a finite index subgroup of $G_k$. But from 2. of Proposition 2.3 we have that $G_k^+ E = E$ as required. \hfill \Box

6. A smoothness result in the $p$-adic case

In section 1.3 of the introduction we recalled that, as corollary of the Shalom-Zimmer theorem over the reals, a classification of invariant random subgroups in higher rank Lie groups was given in [1]. In order to extend this to the $p$-adic case, we need the following technical result. The corresponding fact is known to hold over $\mathbb{R}$ (see 3.2 of [10]).

Proposition 6.1. Let $G$ be a $Q_p$-group and denote $G = \mathbb{G}_{Q_p}$. Let $\Gamma \subset \text{Sub}(G)$ be a lattice subgroup. Then the $G$-orbit of $\Gamma$ is closed in $\text{Sub}(G)$.

Proof. It is well known that being a lattice in a $p$-adic Lie group, $\Gamma$ is cocompact (This follows from the fact that $G$ admits a compact open subgroup $U$ without any non-trivial torsion elements (see II.4.9 of [15]). Therefore $U$ acts freely on $G \setminus \Gamma$, and indeed the latter is a disjoint union of $U$-orbits each having the same volume).

Therefore the $G$-orbit of $\Gamma$ is the image of the compact space $G \setminus \Gamma$ under the orbit map $g \Gamma \mapsto g \Gamma g^{-1}$, and since $\text{Sub}(G)$ is Hausdorff it suffices to show that this map is continuous. In other words, given a sequence $g_n \Gamma \in G \setminus \Gamma$ that converges
to \( g_0 \Gamma \in G \setminus \Gamma \) we need to show that the subgroups \( g_n \Gamma g_n^{-1} \) Chaubuty converge to \( g_0 \Gamma g_0^{-1} \). There are two types of Chaubuty basis sets to check:

First, if \( g_0 \Gamma g_0^{-1} \cap U = \emptyset \) for some open \( U \subset G \) then we clearly have that \( g_n \Gamma g_n^{-1} \) intersects \( U \) as well for all large \( n \) by continuity of conjugation.

Next consider the case where \( g_0 \Gamma g_0^{-1} \cap K = \emptyset \) for some compact \( K \subset G \), and assume towards contradiction that \( g_{n_k} \Gamma g_{n_k}^{-1} \cap K \neq \emptyset \) for some subsequence \( n_k \) tending to infinity. To be precise let \( \gamma_k \in \Gamma \) be such that \( g_{n_k} \gamma_k g_{n_k}^{-1} \in K \). We may choose some symmetric compact open \( V \subset G \) with \( e \in V \), such that \( g_n \in V \) \( g_0 \Gamma \) holds for all large \( n \). Therefore every \( \gamma_k \) belongs to the compact set \( g_0^{-1} V K V g_0 \) and since \( \Gamma \) is discrete there are only finitely many different such \( \gamma_k \)'s. By passing to a further subsequence we may assume that \( \gamma_k = \gamma \in \Gamma \) is fixed. This implies that \( g_{n_k} \gamma g_{n_k}^{-1} \in K \) for all \( k \) while \( g_0 \gamma g_0^{-1} \notin K \), which is a contradiction.

The above shows that the \( G_{Q_p} \)-action on the space of lattices in \( \text{Sub}(G_{Q_p}) \) is smooth, is the sense that the quotient has a countably separated Borel structure (see section 2.1 of \cite{19}). This is the precisely the required ingredient to complete Corollary 1.3 exactly as in Theorem 4.1 of \cite{1}.

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