The good, the bad and the ugly coherent states through polynomial Heisenberg algebras

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Abstract
Second degree polynomial Heisenberg algebras are realized through the harmonic oscillator Hamiltonian, together with two deformed ladder operators chosen as the third powers of the standard annihilation and creation operators. The corresponding solutions to the Painlevé IV equation are easily found. Moreover, three different sets of eigenstates of the deformed annihilation operator are constructed, called the good, the bad and the ugly coherent states. Some physical properties of such states will be as well studied.

1 Introduction
Polynomial Heisenberg algebras (PHA) of second degree are interesting deformations of the Heisenberg-Weyl algebra. In a differential representation they can be realized by one-dimensional Schrödinger Hamiltonians, together with a pair of third order ladder operators. In fact, when looking for the most general Hamiltonian ruled by such algebraic structure, it turns out that the potential depends on solutions to a non-linear second-order ordinary differential equation called Painlevé IV (PIV) equation. Reciprocally, if one has Hamiltonians with third-order differential ladder operators, then it is possible to design a simple algorithm for generating solutions to such an equation, by identifying just the associated extremal states [1,2].

On the other hand, it is important to look for the simplest systems ruled by second degree PHA, such that the corresponding extremal states satisfy the boundary conditions for being eigenfunctions of the Hamiltonian [3,4]. This is the main subject to be addressed in this work. Indeed, it will be shown that the harmonic oscillator Hamiltonian, together with deformed ladder operators which are the third powers of the standard annihilation and creation operators, will define a second degree PHA with such properties (Section 2). The three solutions of the PIV equation associated to this deformed algebra will be derived at the same Section. The corresponding coherent states (CS) as well as their properties, will be studied in Section 3 while Section 4 will contain our conclusions.
2 Second degree PHA for the harmonic oscillator

There are several ways to realize the second degree PHA through the harmonic oscillator. Here, we look for realizations such that the three extremal states are eigenfunctions of $H$ and, thus, we can generate from them three infinite ladders of eigenfunctions and eigenvalues \[3\]. Let us consider then the deformed ladder operators,

$$a_g = a^3, \quad a_g^+ = (a^+)^3. \tag{1}$$

The operator set \{ $H, a_g, a_g^+$ \} gives place to a second degree PHA, since

$$[H, a_g] = -3a_g, \quad [H, a_g^+] = 3a_g^+, \quad [a_g, a_g^+] = N(H + 3) - N(H), \tag{2}$$

where the analogue of the number operator reads:

$$N(H) = a_g^+ a_g = (H - \frac{1}{2}) (H - \frac{3}{2}) (H - \frac{5}{2}). \tag{3}$$

Three extremal state energies are identified, $E_j = E_{j-1} = j - \frac{1}{2}$, $j = 1, 2, 3$, with eigenvectors given by:

$$|\psi_{E_j}\rangle \equiv |\psi_0^j\rangle = |j - 1\rangle, \quad j = 1, 2, 3, \tag{4}$$

where $|j - 1\rangle$, $j = 1, 2, 3$ are the first three energy eigenstates of the harmonic oscillator in Fock notation. Departing from them, by acting $a_g^+$ iteratively, we can construct three independent ladders of energy eigenstates. The eigenvalues associated to the $j$-th ladder are $E_{jn} = E_j + 3n$, $n = 0, 1, \ldots$, $j = 1, 2, 3$, and the corresponding eigenstates become

$$|\psi_{E_{jn}}\rangle = |3n + j - 1\rangle = \sqrt{\frac{(j-1)!}{(3n+j-1)!}} (a_g^+)^n |j - 1\rangle, \quad j = 1, 2, 3. \tag{5}$$

The spectrum of $H$ thus takes the form:

$$\text{Sp}(H) = \{E_{j0}^1, E_{j1}^1, \ldots\} \cup \{E_{j0}^2, E_{j1}^2, \ldots\} \cup \{E_{j0}^3, E_{j1}^3, \ldots\}, \tag{6}$$

which is the harmonic oscillator spectrum seen from a new viewpoint: the Hilbert space is the direct sum of three orthogonal supplementary subspaces, $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$, each one of them containing one ladder, which is represented in Figure 1.

Since \{ $H, a_g, a_g^+$ \} generate a second degree PHA, there is a link with the PIV equation \[1\,2\]:

$$\frac{d^2g}{dy^2} = \frac{1}{2g} \left( \frac{dg}{dy} \right)^2 + \frac{3}{2} g^3 + 4yg^2 + 2(y^2 - a)g + \frac{b}{g}, \tag{7}$$

which allows us to find some of its solutions. We just need to supply the three extremal states and their associated energies, in our case $\psi_{E_j}(x) = \langle x | j - 1 \rangle$, $E_j = j - 1/2$, $j = 1, 2, 3$. The PIV solution and its parameters turn out to be given by:

$$g(y) = -y - \frac{d}{dy} [\ln \phi_1(y)], \quad a = \tilde{E}_2 + \tilde{E}_3 + 2\tilde{E}_1 - 1, \quad b = -2 \left( \tilde{E}_2 - \tilde{E}_3 \right)^2, \tag{8}$$
where \( \phi_1(y) \) is the first extremal state for the previous ordering (the ground state), and \( y = \sqrt{3}x \), \( \tilde{\mathcal{E}}_j = \mathcal{E}_j/3 \), \( j = 1, 2, 3 \) are the changes required to fit the spacing of levels of our system (\( \Delta E = 3 \)) with the standard spacing (\( \Delta E = 1 \)) used in [1][2]. Since the first label can be assigned to any extremal state, we can find indeed three PIV solutions, whose explicit expressions and corresponding parameters become:

\[
\begin{align*}
g(y) & = -2y/3, \quad a = 0, \quad b = -2/9, \\
g(y) & = -2y/3 - 1/y, \quad a = -1, \quad b = -8/9, \\
g(y) & = -2y/3 - 4y/(2y^2 - 3), \quad a = -2, \quad b = -2/9.
\end{align*}
\]

### 3 Coherent states

Let us consider now the CS as eigenstates of the deformed annihilation operator:

\[
a_g |\alpha\rangle_j = \alpha |\alpha\rangle_j, \quad j = 0, 1, 2,
\]

with \( |\alpha\rangle = \sum_{n=0}^{\infty} C_n |3n + j\rangle \). Following a standard procedure, we arrive at:

\[
|\alpha\rangle_j = \frac{1}{\sqrt{\sum_{n=0}^{\infty} \frac{\alpha^n}{(3n+j)!} (3n+j)!}} \sum_{n=0}^{\infty} \frac{\alpha^n}{(3n+j)!} |3n + j\rangle.
\]
Several important quantities for the CS $|\alpha\rangle_j$ can be obtained straightforwardly:

$$
\langle x \rangle_j = \langle p \rangle_j = 0,
\langle x^2 \rangle_j = \langle p^2 \rangle_j = (\Delta x)_j (\Delta p)_j = \langle H \rangle_j = |a|\alpha\rangle_j |^2 + \frac{1}{2},
$$

where

$$
|a|\alpha\rangle_j |^2 = \begin{cases}
\sum_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(3r)!} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n+2}}{(3n+2)!} & \text{for } j=0, \\
\sum_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(3r+1)!} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(3n)!} & \text{for } j=1, \\
\sum_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(3r+2)!} \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{(3n+1)!} & \text{for } j=2.
\end{cases}
$$

Plots of the uncertainty products $(\Delta x)_j (\Delta p)_j$ for $j = 0, 1, 2$ are shown in Figure 2.

It is important to explore the completeness relation in each subspace $\mathcal{H}_j$:

$$
\int |\alpha\rangle_j \langle \alpha| d\mu_j(\alpha) = I_j, \quad j = 0, 1, 2,
$$

where $I_j$ is the identity operator on $\mathcal{H}_j$ and

$$
d\mu_j(\alpha) = \frac{1}{\pi|\alpha|} \left( \sum_{r=0}^{\infty} \frac{|\alpha|^{2r}}{(3r+j)!} \right) f_j(|\alpha|^2) d|\alpha| d\varphi.
$$

If $f_j(x)$ satisfies $\int_0^\infty x^{n-1} f_j(x) dx = \Gamma(3n + j + 1)$, thus any state vector can be decomposed in terms of our CS.
Figure 3: Probability densities for the good, the bad and the ugly CS (left, right, down respectively).
Finally, the time evolution of a coherent state is quite simple, \( U(t)\alpha \rangle_j = e^{-i(j+\frac{1}{2})t}\alpha(t) \rangle_j, \alpha(t) = \alpha e^{-3it} \).

Let us consider next the non-normalized coherent states:
\[
|z\rangle = \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle, \quad |z\rangle_j = \sum_{n=0}^{\infty} \frac{z^{3n+j}}{\sqrt{(3n+j)!}} |3n+j\rangle, \quad \alpha = z^3.
\] (18)

The first state in Eq. (18) is a standard CS while the second ones are the good, the bad and the ugly CS, also named three-photon CS [5]. Equation (16) ensures that \( |ze^{i2\pi j/3}\rangle \) can be written in terms of \( |z\rangle_j, j = 0, 1, 2 \) [3]. Reciprocally, we can express \( |z\rangle_j \) in terms of \( |ze^{i2\pi j/3}\rangle, j = 0, 1, 2 \):
\[
|z\rangle_0 = N_0 (|z\rangle + |e^{i2\pi/3}z\rangle + |e^{i4\pi/3}z\rangle),
\]
\[
|z\rangle_1 = N_1 (|z\rangle - e^{i\pi/3}|e^{i2\pi/3}z\rangle + e^{i2\pi/3}|e^{i4\pi/3}z\rangle),
\]
\[
|z\rangle_2 = N_2 (|z\rangle + e^{i2\pi/3}|e^{i2\pi/3}z\rangle + e^{i4\pi/3}|e^{i4\pi/3}z\rangle),
\] (19, 20, 21)
i.e., the good, the bad and the ugly CS are superpositions of standard CS with complex labels \( ze^{i2\pi j/3} \) defining an equilateral triangle on the complex plane. Expressions (19–21) are used to build the wave packets associated to \( |z(t)\rangle_j, j = 0, 1, 2 \), whose probability densities as functions of \( x \) and \( t \) are shown in Figure 3 [3].

As we can see, the probability densities are periodic in time, with a period \((2\pi/3)\) equal to one third of the period for a classical motion for the oscillator. This implies that the good, the bad and the ugly CS cannot describe semi-classical situations, i.e., they are intrinsically quantum states. It is worth to notice the existence of some other states which are strongly quantum, e.g., the even and odd CS [4–7].

4 Conclusions

We have explored a realization of the second degree PHA in which the generators are the harmonic oscillator Hamiltonian and the ladder operators \( a_g = a^3, a_g^+ = (a^+)^3 \). The three associated extremal states become physical eigenstates of \( H \), and the ladders generated from them are of infinite length. In addition, these extremal states supply some solutions to the PIV equation. The search of the eigenstates of \( a_g \) leads to three different sets, which here have been called the good, the bad and the ugly CS. Their period turns out to be a fraction \((1/3)\) of the original period \((2\pi)\) for the oscillator, indicating the strong quantum nature of such states. They could be important to describe the kind of interaction matter-radiation appearing in the so-called multiphoton quantum optics [8].
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