Statistical properties of the critical current distribution in superconductor with fractal clusters of a normal phase are considered. It is found that there is the range of fractal dimensions in which the variance and expectation for this distribution increases infinitely. Simple technique of avoiding such a divergence by the use of truncated distributions is proposed. It is suggested that the most current-carrying capability of a superconductor can be achieved by modifying the cluster area distribution in such a way that the regime of giant variance of critical currents will be realized.

I. INTRODUCTION

Considerable recent attention is being drawn to the fractal behavior of magnetic flux in type-II superconductors [1]-[3]. High temperature superconductors (HTS) containing clusters of correlated defects [4,5] are of special interest in this field. The case of clusters with fractal boundaries provides new possibilities for increasing the critical current value [2,6]. By virtue of the capability to trap a magnetic flux, such clusters can appreciably modify magnetic and transport properties of superconductors [3,7,8]. The distribution of the critical currents in superconductor containing the normal phase clusters with fractal boundaries has unusual statistical properties, and it is these features that will interest us.

Let us consider a superconductor containing columnar inclusions of a normal phase, which are out of contact with one another. These inclusions may be formed by the fragments of different chemical compositions, as well as by the domains of the reduced superconducting order parameter. The similar columnar defects can readily be created during the film growth process [5,9,10]. In the course of the cooling below the critical temperature in the magnetic field (“field-cooling” regime) the magnetic flux will be trapped in the isolated clusters of a normal phase so the two-dimensional distribution of the flux will be created in such a superconducting structure. When the transport current is passed transversely to the magnetic field, this one is added to all the persistent currents, which circulate around the normal phase clusters and keep the trapped magnetic flux to be unchanged. By the cluster we mean a set of the columnar defects, which are united by the common trapped flux and are surrounded by the superconducting phase. Inasmuch as the distribution of the trapped magnetic flux is two-dimensional, instead of dealing with an extended object, which indeed the normal phase cluster is, we will consider its cross-section by the plane carrying a transport current. As was first found in Ref. [2], clusters of a normal phase can have fractal boundaries, and this feature has a significant effect on the dynamics of the trapped magnetic flux [3,8]. In the subsequent consideration we will suppose that the characteristic sizes of the normal phase clusters far exceed both the coherence length and the penetration depth. This assumption agrees well with the data on the cluster structure in YBCO films [2,3,10], as well as will allow us to highlight the role played by the cluster boundary in the magnetic flux trapping.

II. GIANT DISPERSION OF CRITICAL CURRENTS

Suppose that there is a superconducting percolation cluster in the plane of the film where a transport current flows. Such a structure provides for an effective pinning, because the magnetic flux is locked in finite clusters of a normal phase. When the transport current is increased, the trapped magnetic flux remains unchanged until the vortices start to break away from the clusters of pinning force weaker than the Lorentz force created by the current. As this takes place, the vortices must cross the surrounding superconducting space, and they will first do that through the weak links, which connect the normal phase clusters between themselves. Such weak links form easily in HTS characterized by an extremely short coherence length. Diverse structural defects, which would simply cause an additional scattering at long coherence length, give rise to the weak links in HTS. Weak links arise readily on twin boundaries, and magnetic
flux can easily move along them [11]. Whatever the microscopic nature of weak links could be, they form the channels for vortex transport. Accordingly to weak link configuration each normal phase cluster has its own value of the critical current, which contributes to the total distribution. By the critical current of the cluster we mean the current of depinning, that is to say, such a current at which the magnetic flux ceases to be held inside the cluster of a normal phase. The critical current distribution is related to the cluster area distribution, because the cluster of a larger size has more weak links over its boundary with the surrounding superconducting space, and thus the smaller current of depinning [3].

In the practically important case of YBCO films with columnar defects the exponential distribution of the cluster areas can be realized [2], which is the special case of gamma distribution. The exponential distribution has only one characteristic parameter (mean cluster size), so there are not many possibilities to modify the geometric morphological properties of the clusters in this simplest case. By contrast, in the case of gamma distribution there is an additional way for optimizing the cluster structure of the composite superconductor by the control of two independent parameters in the course of the film growth. One of the aims of the present work is to find how the cluster area distribution should be optimized in order to get the highest current-carrying capability of a superconductor. In Ref. [7] the critical current distribution was derived in the case of gamma distribution of fractal clusters areas, which has the following probability density

\[ w(A) = \frac{A^g \exp(-A/A_0)}{\Gamma(g+1) A_0^{g+1}} \]  

where \( \Gamma(\nu) \) is Euler gamma function, \( A \) is the cluster area, \( A_0 > 0 \) and \( g > -1 \) are the parameters of gamma distribution that control the mean area of the cluster \( A = (g+1)A_0 \) and its variance \( \sigma_A^2 = (g+1) A_0^2 \). For further consideration it is convenient to introduce the dimensionless area of the cluster \( a = A/\bar{A} \), for which the distribution function of Eq. (1) can be rewritten as:

\[ w(a) = \frac{(g+1)^{g+1}}{\Gamma(g+1)} a^g \exp(-(g+1)a) \]  

(2)

The mean dimensionless area of the cluster is equal to unity, whereas the variance is determined by one parameter only: \( \sigma_a^2 = 1/(g+1) \). The probability density of Eq. (2) is presented in Fig. 1 for the characteristic values of \( g \)-parameter. The critical current distribution has the following form [7]

\[ f(i) = \frac{2G^{g+1}}{D\Gamma(g+1)} i^{(2/D)g(g+1)} \exp\left(-Gi^{2/D}\right) \]  

(3)

where

\[ G \equiv \left( \frac{\theta^g}{\theta^{g+1} - (D/2) \exp(\theta) \Gamma(g+1,\theta)} \right)^\frac{1}{2} \]

\[ \theta \equiv g + 1 + \frac{D}{2} \]

\( \Gamma(\nu, z) \) is the complementary incomplete gamma function, \( i \equiv I/I_c \) is the dimensionless electric current, \( I_c = \alpha (A_0 G)^{-D/2} \) is this the critical current of the resistive transition, \( \alpha \) is the form factor, and \( D \) is the fractal dimension of the cluster perimeter. The value of \( D \) specifies the scaling relation \( P^{1/D} \propto A^{1/2} \) between perimeter \( P \) and area \( A \) of the cluster [12,13].

The probability density curve for critical current distribution of Eq. (3) has the skew bell-shaped form with inherent broad “tail” extended over the region of high currents (see Fig. 2). As may be seen from this graph, the critical current distribution spreads out with some shift to the right as \( g \)-parameter decreases. This broadening can be characterized by the standard deviation of critical currents

\[ \sigma_i = G^{\frac{g}{2}} \sqrt{\frac{\Gamma(g+1-D)}{\Gamma(g+1)} - \left( \frac{\Gamma(g+1-D/2)}{\Gamma(g+1)} \right)^2} \]  

(4)

The standard deviation grows nonlinearly with increase in the fractal dimension, as is illustrated in Fig. 3. The peculiarity of the distribution of Eq. (3) is that its variance becomes infinite in the range of fractal dimensions \( D \geq g + 1 \). The distributions with divergent variance are known in probability theory - the classic example of that
kind is Cauchy distribution [14]. However, such an anomalous feature of exponential-hyperbolic distribution of Eq. (3) is of special interest, inasmuch as the current-carrying capability of a superconductor would be expected to increase in the region of giant variance. Then the statistical distribution of critical currents has a very elongated “tail” containing the contributions from the clusters of the highest depinning currents.

The distribution of Eq. (3) has one even more striking feature: its mathematical expectation, which represents the mean critical current

$$\bar{I} = \frac{\Gamma(g + 1 - D/2)}{\Gamma(g + 1)} G^\frac{D}{2} \tag{5}$$

is also divergent in the range of fractal dimensions $D \geq 2(g + 1)$. At the same time, the mode of the distribution $mode f(i) = (G/\theta)^{D/2}$ remains finite for all possible values of the fractal dimension $1 \leq D \leq 2$.

The reason for divergence of the mean critical current consists in the behavior of the cluster area distribution of Eq. (2). As may be seen from Fig. 1, the graph of the distribution function of Eq. (2) takes essentially different shapes depending on the sign of $g$-parameter - from the skew unimodal curve (1) (at $g > 0$) to the monotonic curve (3) with hyperbolic singularity at zero point (at $g < 0$). In the borderline case of $g = 0$, which separates these different kinds of the functions, the distribution obeys an exponential law (curve (2)). It is just for negative values of $g$-parameter that the mean critical current diverges at $g < 0$. Nevertheless, the total area between the curve of the probability density (like curve (3) in Fig. 2) and the abscissa remains finite by virtue of the normalization requirement.

### III. TRUNCATED DISTRIBUTION OF CRITICAL CURRENTS

Obviously, the proper critical current cannot be infinitely high as well as the clusters of infinitesimal area do not really exist. There is the minimum value of the normal phase cluster area $A_m$, which is limited by the processes of the film growth. So in YBCO based composites, which were prepared by magnetron sputtering on sapphire substrates with a cerium oxide buffer sublayer [3], the sample value of minimum area of the normal phase cluster has been equal to $A_m = 2070 \text{ nm}^2$ at mean cluster area $\bar{A} = 76500 \text{ nm}^2$, that corresponds to the lower bound of the dimensionless area of the cluster $a_m \equiv A_m/\bar{A} = 0.027$. In view of this limitation, we will describe the distribution of the normal phase cluster areas by the truncated version of the probability density of Eq. (2):

$$w(a | a \geq a_m) = \frac{h(a - a_m)}{1 - W(a_m)} w(a) \tag{6}$$

where $\gamma(\nu, z)$ is the complementary gamma function, $h(x) = \begin{cases} 1 & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$ is the Heaviside step function,

$$W(a_m) = \int_{a_m}^{\infty} w(a) \, da = \frac{\gamma(g + 1, (g + 1) a_m)}{\Gamma(g + 1)} \tag{7}$$

is the truncation degree, which is equal to the probability $Pr \{ \forall a < a_m \}$ to find the cluster of area smaller than the least possible value $a_m$ in the initial population.

The expression of Eq. (6) gives the conditional distribution of probability, for which all the events of finding a cluster of area less than $a_m$ are excluded. Thus the truncation provides a natural way to fulfill our initial assumption that the cluster size has to be greater than the coherence and penetration lengths. New distribution of cluster areas gives rise to the truncated distribution of the critical currents:

$$f(i \mid i \leq i_m) = \frac{h(i_m - i)}{1 - W(a_m)} f(i) \tag{8}$$

where $i_m \equiv (G/ (g + 1) a_m)^{D/2}$ is the upper bound of the depinning current, which corresponds to the cluster of the least possible area $a_m$. Then, instead of Eqs. (4) and (5), the standard deviation and mean critical current are
\[ \sigma_i^* = G^2 \sqrt{\frac{\Gamma(g+1-D, (g+1) a_m)}{\Gamma(g+1, (g+1) a_m)}} - \left( \frac{\Gamma(g+1-D/2, (g+1) a_m)}{\Gamma(g+1, (g+1) a_m)} \right)^2 \]

\[ \bar{\sigma} = \frac{\Gamma(g+1-D/2, (g+1) a_m)}{\Gamma(g+1, (g+1) a_m)} G^2 \]

For the truncated distribution of Eq. (8) the possible values of depinning currents are bounded from above by the quantity \( \bar{\sigma} \), therefore the mean critical current as well as the variance do not diverge any more. Both of these characteristics are finite for any fractal dimensions, including the case of maximum fractality \( D = 2 \). The corresponding graphs for standard deviation are presented in Fig. 4. All the curves are calculated at \( g = -0.2 \) (as for the main curve (2) of the mean critical current in Fig. 5). In this case the variance for initial distribution of the critical currents is infinite, so no graph for \( W(a_m) = 0 \) is shown in Fig. 4 at all. The dependence of the mean critical current on the fractal dimension \( D \) in the case of truncated distribution is demonstrated in Fig. 6. The corresponding graphs in Figs. 4 and 6 are drawn for the same values of truncation degree. The truncation degree is related to the least possible area of the cluster by the equation (7). The values of \( W(a_m) \) and \( a_m \) involved in Figs. 4 and 6 are presented in the Table I.

The degree of truncation gives the probability measure of the number of normal phase clusters that have the area smaller than the lower bound \( a_m \) in the initial distribution. If the magnitude of \( W(a_m) \) is sufficiently small (no more than several percent), then the procedure of truncation scarcely affects the initial shape of the distribution and still enables the contribution from the clusters of zero area (therefore, of infinitely high depinning current) to be eliminated. It is interesting to note that the very similar situation occurs in analyzing the statistics of the areas of fractal islands in the ocean [12]. The island areas obey the Pareto distribution, which also has the hyperbolic singularity at zero point that causes certain of its moments to diverge. For exponential distribution of the cluster areas, which is valid in the above-mentioned case of YBCO films [3], the probability to find the cluster of area smaller than \( a_m \) in the sampling is equal to \( \Pr\{a < a_m\} = 2.7\% \) only. In principle, the truncation procedure could be made here, too, but there is no need for that, because the contribution of infinitesimally small clusters is finite at \( g = 0 \) (and equal to zero at \( g > 0 \)).

Referring to Figs. 4 and 6 it can be seen that the dependencies of the standard deviation and the mean critical current on the fractal dimension become smoother and smoother with increase in the degree of truncation. At the same time, the inherent tendency of initial distribution is still retained: as the fractal dimension increases, the critical current distribution broadens out (the variance grows, see Fig. 4), moving towards higher currents (the mean critical current grows, see Fig. 6). As may be seen from Fig. 2, this trend is further enhanced with decreasing \( g \)-parameter. The most current-carrying capability of a superconductor should be achieved when the clusters of small size, which have the highest currents of depinning, contribute maximally to the overall distribution of the critical currents. Such a situation takes place just in the region of giant variance of critical currents. So far, the least magnitude of \( g \)-parameter (equal to zero) has been realized in YBCO composites containing normal phase clusters of fractal dimension \( D = 1.44 \) [2]. The critical currents of superconducting films with such clusters are higher than usual [6,9,10]. It would thus be expected to further improve the current-carrying capability in superconductors containing normal phase clusters, which will be characterized by area distribution with negative magnitudes of \( g \)-parameter, especially at the most values of fractal dimensions.

Thus, it has been revealed that the distribution of the critical currents in superconductor with fractal clusters of a normal phase has the anomalous statistical properties, implying that its variance and expectation diverges in the certain range of fractal dimensions. It may be expected that the most current-carrying capability of a superconductor can be achieved by optimization of the cluster area distribution that involves reducing \( g \)-parameter with concurrent increasing the fractal dimension.

IV. ACKNOWLEDGEMENTS

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| Truncation degree $W(a_m)$ | Minimum area $a_m$ | Markers in Figs. 4, 6 |
|-----------------------------|-------------------|----------------------|
| 0.01%                       | $1.144 \times 10^{-5}$ | a                    |
| 0.1%                        | $2.034 \times 10^{-4}$ | b                    |
| 1%                          | $3.622 \times 10^{-3}$ | c                    |
FIG. 1. The distribution of the areas of normal phase clusters at different values of $g$-parameter. Curve (1) corresponds to the case of $g = 1$; curve (2) of $g = 0$; curve (3) of $g = -0.5$. The arrow indicates the mean cluster area.
FIG. 2. The critical current distribution for the fractal dimension of the cluster boundary $D = 1.5$. Curve (1) corresponds to the case of $g = 1$; curve (2) of $g = 0$; curve (3) of $g = -0.5$. 
FIG. 3. Influence of the fractal dimension of the cluster boundary on the standard deviation of critical currents. Curve (1) corresponds to the case of $g = 0.25$; curve (2) of $g = 0.5$; curve (3) of $g = 0.75$; curve (4) of $g = 1$. 
FIG. 4. Standard deviation graphs for truncated distribution of the critical currents at $g = -0.2$ with the different degree of truncation: curve (a) is drawn for $W(a_m) = 0.01\%$; curve (b) for $W(a_m) = 0.1\%$; curve (c) for $W(a_m) = 1\%$. 
FIG. 5. Influence of the fractal dimension of the cluster boundary on the mean critical current. Curve (1) corresponds to the case of $g = -0.4$; curve (2) of $g = -0.2$; curve (3) of $g = 0$; curve (4) of $g = 1$. 
FIG. 6. Mean critical current graphs for truncated distribution at $g = -0.2$ with the different degree of truncation: curve (2) is drawn for $W(a_m) = 0$; curve (a) for $W(a_m) = 0.01\%$; curve (b) for $W(a_m) = 0.1\%$; curve (c) for $W(a_m) = 1\%$. The dotted lines are given for truncated distribution, the solid line for initial distribution.