A LOW-RANK APPROXIMATION OF TENSORS AND THE TOPOLOGICAL GROUP STRUCTURE OF INVERTIBLE MATRICES

R.N. GUMEROV AND A.S. SHARAFUTDINOV

Abstract. By a tensor we mean an element of a tensor product of vector spaces over a field. Up to a choice of bases in factors of tensor products, every tensor may be coordinatized, that is, represented as an array consisting of numbers. This note is concerned with properties of the tensor rank that is a natural generalization of the matrix rank. The topological group structure of invertible matrices is involved in this study. The multilinear matrix multiplication is discussed from a viewpoint of transformation groups. We treat a low-rank tensor approximation in finite-dimensional tensor products. It is shown that the problem on determining a best rank-

Introduction

Tensors are ubiquitous in sciences. The subject of tensors is an active research area in mathematics and its applications (see, for example, [1, 2, 3] and references therein).

This note is devoted to the tensor rank and a low-rank approximation of tensors. The tensor rank can be considered as a measure of complexity of tensors. Therefore, one is often required to find an approximation of a given tensor by tensors with lower tensor ranks. In particular, the best low-rank approximation problem for tensors is of great interest in the statistical analysis of multiway data (see, for example, references in [4, p. 1085]). As is known, in general, the best low-rank approximation problem for tensors is ill-posed [5, 4].

A part of motivation for this work comes from our study the complexity of tensors in homological complexes of Banach spaces [5, 7]. The main part of motivation comes from results in [4, 8] on tensors in finite-dimensional spaces. In this note we consider tensors in finite-dimensional spaces with the Euclidean topology. Properties of those tensors are closely related to the topological group structure of invertible matrices. Here, we deal with the natural topological group action on a space of tensors. We show the ill-posedness of the best rank-

The note contains Introduction and two sections. Section 1 contains preliminaries and properties of the tensor rank. We show that the triple consisting of the

2010 Mathematics Subject Classification. 15A03, 15A60, 15A69, 41A28, 41A63, 49M27, 62H25, 68P05.

Key words and phrases. approximation by matrices with simple spectra, group action, low-rank tensor approximation, norm on a tensor space, open mapping, simple spectrum of a matrix, tensor rank, topological group of invertible matrices, topological transformation group.
Cartesian product of general linear groups, the space of tensors and the multilinear matrix multiplication is a topological transformation group. In Section 2 we prove that every element in the tensor space $\mathbb{C}^{n \times n \times 2}$ may be approximated arbitrarily closely by elements whose tensor ranks are equal to $n$.

Introduction and Section 2 were written by R. N. Gumerov. Section 1 was written by A. S. Sharafutdinov. The results of Section 2 were proved by R. N. Gumerov.

1. Tensor rank and its properties

As usual, $\mathbb{N}$ stands for the set of all natural numbers. In the sequel, $l, m, n \in \mathbb{N}$ and $l, m, n \geq 2$.

Throughout the note, $\mathbb{F}$ will denote either the field of complex numbers $\mathbb{C}$ or the field of real numbers $\mathbb{R}$. For an element $x \in \mathbb{F}^l$ we use the notation $x = (x_1, \ldots, x_l)^T$, where $x_i \in \mathbb{F}, i = 1, \ldots, l$.

We denote by $\mathbb{F}^{l \times m}$, or by $M_{l,m}(\mathbb{F})$, the linear space of all matrices $A = (a_{ij})$ of size $l \times m$, where $a_{ij} \in \mathbb{F}, i = 1, \ldots, l, \ j = 1, \ldots, m$. The space of all square matrices of order $n$ over the field $\mathbb{F}$ is denoted by $M_n(\mathbb{F})$. The general linear group of degree $n$, that is, the group of invertible matrices in $M_n(\mathbb{F})$, is denoted by $GL_n(\mathbb{F})$. The symbol $E_n$ stands for the identity matrix in $M_n(\mathbb{F})$.

For $x = (x_1, \ldots, x_l)^T \in \mathbb{F}^l$ and $y = (y_1, \ldots, y_m)^T \in \mathbb{F}^m$, the matrix $x \otimes y \in \mathbb{F}^{l \times m}$ is given by

$$x \otimes y = (x_i y_j), \quad \text{where} \quad i = 1, \ldots, l, \ j = 1, \ldots, m.$$  

Let $\mathbb{F}^{l\times m\times n}$ be the linear space of all arrays $A = (a_{ijk})$ of size $l \times m \times n$, where $a_{ijk} \in \mathbb{F}, i = 1, \ldots, l, \ j = 1, \ldots, m, \ k = 1, \ldots, n$. For a tensor $A = (a_{ijk}) \in \mathbb{F}^{l\times m\times n}$ we also use the following notation:

$$A = [A_1| \ldots |A_n],$$

where, for every $r = 1, \ldots, n$, the slice $A_r$ is defined by

$$A_r = (a_{ijr}) \in \mathbb{F}^{l \times m}, \quad \text{where} \quad i = 1, \ldots, l, \ j = 1, \ldots, m.$$

For $x = (x_1, \ldots, x_l)^T \in \mathbb{F}^l$, $y = (y_1, \ldots, y_m)^T \in \mathbb{F}^m$, $z = (z_1, \ldots, z_n)^T \in \mathbb{F}^n$, we define the array $x \otimes y \otimes z \in \mathbb{F}^{l \times m \times n}$ by

$$x \otimes y \otimes z = (x_i y_j z_k), \quad \text{where} \quad i = 1, \ldots, l, \ j = 1, \ldots, m, \ k = 1, \ldots, n.$$  

Consider the bilinear mapping $\theta$ and the trilinear mapping $\tau$ defined as follows:

$$\theta: \mathbb{F}^l \times \mathbb{F}^m \rightarrow \mathbb{F}^{l \times m}: (x, y) \mapsto x \otimes y;$$

$$\tau: \mathbb{F}^l \times \mathbb{F}^m \times \mathbb{F}^n \rightarrow \mathbb{F}^{l \times m \times n}: (x, y, z) \mapsto x \otimes y \otimes z.$$

It is well known that the pairs $(\mathbb{F}^{l \times m}, \theta)$ and $(\mathbb{F}^{l \times m \times n}, \tau)$ are the tensor products for the corresponding linear spaces. In what follows, elements of the spaces $\mathbb{F}^{l \times m}$ and $\mathbb{F}^{l \times m \times n}$ are called tensors.

For the basics of algebraic tensor products we refer the reader, for instance, to [2] Part I, Ch. 3, [10] Ch. 1 and [11] Ch. 2, § 7.

Both of the $l_1$-norms on the linear spaces $\mathbb{F}^{m \times n}$ and $\mathbb{F}^{l \times m \times n}$ will be denoted by the same symbol $\| \cdot \|_1$. We recall that the value of the $l_1$-norm at a tensor $A$ is defined as the sum of absolute values of all entries in $A$.

All norms on a space of tensors are equivalent and generate the same topology that is called the Euclidean topology. The convergence of a tensor sequence $\{A_t\} = \{(a_{ijk}^t)\} \subset \mathbb{F}^{l \times m \times n}$, $t \in \mathbb{N}$,
to a tensor $A = (a_{ijk}) \in \mathbb{F}^{l \times m \times n}$ with respect to this topology is exactly the entrywise convergence, that is, for every fixed triple of indices $i = 1, \ldots, l, j = 1, \ldots, m, k = 1, \ldots, n$, one has the equality

$$\lim_{t \to +\infty} a_{ijk}^t = a_{ijk}. $$

In the sequel, we consider spaces of tensors endowed with the Euclidean topologies. The general linear group $GL_n(\mathbb{F})$ is a topological group with respect to that topology.

**Definition 1.** Tensors $A \in \mathbb{F}^{l \times m}$ and $B \in \mathbb{F}^{l \times m \times n}$ are said to be elementary (or decomposable) if $A = a \otimes b$ and $B = \mathbf{x} \otimes \mathbf{y} \otimes \mathbf{z}$ for some vectors $a, \mathbf{x} \in \mathbb{F}^l, b, \mathbf{y} \in \mathbb{F}^m$ and $\mathbf{z} \in \mathbb{F}^n$.

**Definition 2.** A tensor $A \in \mathbb{F}^{l \times m}$ or a tensor $B \in \mathbb{F}^{l \times m \times n}$ has the tensor rank $r$ if it can be written as a sum of $r$ elementary tensors, but no fewer. We will use the notation $\text{rank}(A)$ (or $\text{rank}_\mathbb{C}(A)$) for the tensor rank of $A$. Therefore, we may write

$$\text{rank}(B) = \min \left\{ r \mid B = \sum_{i=1}^r \mathbf{x}_i \otimes \mathbf{y}_i \otimes \mathbf{z}_i, \quad \text{where} \quad \mathbf{x}_i \in \mathbb{F}^l, \mathbf{y}_i \in \mathbb{F}^m, \mathbf{z}_i \in \mathbb{F}^n \right\}. $$

As is well known, for $A \in \mathbb{F}^{l \times m}$, the tensor rank $\text{rank}(A)$ is exactly the matrix rank and, for $A \in \mathbb{R}^{l \times m}$, the equality $\text{rank}_\mathbb{R}(A) = \text{rank}_\mathbb{C}(A)$ is valid. On the other hand, the tensor rank $\text{rank}(A)$, where $A \in \mathbb{F}^{l \times m \times n}$, depends on a field $\mathbb{F}$. It is clear, that the inequality $\text{rank}_\mathbb{C}(A) \leq \text{rank}_\mathbb{R}(A)$ holds.

**Example.** Let $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & -1 \\ -1 & 0 & 0 \end{bmatrix}$. It can be shown that $\text{rank}_\mathbb{R}(A) = 3$ and $\text{rank}_\mathbb{C}(A) = 2$ (see [2, Example 3.44]).

Further, we introduce the topological group that is the Cartesian product of general linear groups

$$GL_{l,m,n}(\mathbb{F}) := GL_l(\mathbb{F}) \times GL_m(\mathbb{F}) \times GL_n(\mathbb{F})$$

and consider a $GL_{l,m,n}(\mathbb{F})$-action on the space $\mathbb{F}^{l \times m \times n}$ (see also [3, Section 2.1]).

For the notions and facts in the theory of topological transformation groups we refer the reader, for example, to [12] and [13].

Let us take elements $A \in \mathbb{F}^{l \times m \times n}$ and $(L, M, N) \in GL_{l,m,n}(\mathbb{F})$ given as follows:

$$A = (a_{ijk}), \quad L = (\lambda_{pi}), \quad M = (\mu_{qj}), \quad N = (\nu_{rk}). $$

The tensor $A$ is transformed into the tensor $B = (L, M, N) \cdot A \in \mathbb{F}^{l \times m \times n}$ by the rule:

$$B = (b_{pqr}) \in \mathbb{F}^{l \times m \times n}, \quad \text{where} \quad b_{pqr} = \sum_{i,j,k=1}^{l,m,n} \lambda_{pi} \mu_{qj} \nu_{rk} a_{ijk}. $$

Thus, we have the mapping called the multilinear matrix multiplication

$$\Phi : GL_{l,m,n}(\mathbb{F}) \times \mathbb{F}^{l \times m \times n} \rightarrow \mathbb{F}^{l \times m \times n} : ((L, M, N), A) \mapsto (L, M, N) \cdot A,$$

which was studied in [3, Sections 2.1, 2.2 and 2.5].

Below the mapping $\Phi$ is considered from a viewpoint of transformation groups.

**Proposition 1.** The following properties are fulfilled:
1) the triple \((GL_{l,m,n}(F), \mathbb{F}^{l \times m \times n}, \Phi)\) is a topological transformation group;

2) every orbit for the \(GL_{l,m,n}(F)\)-action consists of elements of the same tensor rank;

3) the group action of \(GL_{l,m,n}(F)\) on \(\mathbb{F}^{l \times m \times n}\) is non-effective;

4) the space \(\mathbb{F}^{l \times m \times n}\) is non-homogeneous under the \(GL_{l,m,n}(F)\)-action.

**Proof.** 1) We show only the continuity of the multilinear matrix multiplication \(\Phi\).

To this end, we take sequences \(\{(L_t, M_t, N_t)\} \subset GL_{l,m,n}(F)\) and \(\{A_t\} \subset \mathbb{F}^{l \times m \times n}\), \(t \in \mathbb{N}\), that converge to \((L, M, N) \in GL_{l,m,n}(F)\) and \(A \in \mathbb{F}^{l \times m \times n}\), respectively. Hence, we have the coordinatewise convergence:

\[
\lim_{t \to +\infty} L_t = L, \quad \lim_{t \to +\infty} M_t = M, \quad \lim_{t \to +\infty} N_t = N.
\]

We introduce the following notations:

\[
L_t = (\lambda_{pi}^t), \quad M_t = (\mu_{qj}^t), \quad N_t = (\nu_{rk}^t), \quad A_t = (a_{ijk}^t);
\]

\[
L = (\lambda_{pi}), \quad M = (\mu_{qj}), \quad N = (\nu_{rk}), \quad A = (a_{ijk}).
\]

Then we have the equalities for entries of tensors:

\[
\lim_{t \to +\infty} \lambda_{pi}^t = \lambda_{pi}, \quad \lim_{t \to +\infty} \mu_{qj}^t = \mu_{qj}, \quad \lim_{t \to +\infty} \nu_{rk}^t = \nu_{rk}, \quad \lim_{t \to +\infty} a_{ijk}^t = a_{ijk}.
\]

Further, we define the constants

\[
M_1 = \sup_{p,i,k} |\lambda_{pi}^t|, \quad M_2 = \sup_{q,j,t} |\mu_{qj}^t|, \quad M_3 = \sup_{r,k,t} |\nu_{rk}^t|, \quad M_4 = \sup_{i,j,k,t} |a_{ijk}^t|.
\]

In addition, let us set \((L_t, M_t, N_t) \cdot A_t = (b_{ijk}^t)\) and \((L_t, M_t, N_t) \cdot A = (b_{ijk})\).

Then, for every \(\varepsilon > 0\), there exists \(T \in \mathbb{N}\) such that for all \(t > T\) we have the following inequalities:

\[
|b_{pqr}^t - b_{pqr}| \leq \sum_{i,j,k=1}^{l,m,n} \lambda_{pi}^t \mu_{qj}^t \nu_{rk}^t a_{ijk}^t - \sum_{i,j,k=1}^{l,m,n} \lambda_{pi}^t \mu_{qj}^t \nu_{rk}^t a_{ijk}^t \leq \sum_{i,j,k=1}^{l,m,n} \lambda_{pi}^t \mu_{qj}^t \nu_{rk}^t a_{ijk}^t - \lambda_{pi}^t \mu_{qj}^t \nu_{rk}^t a_{ijk}^t \leq \sum_{i,j,k=1}^{l,m,n} \left( |\lambda_{pi}^t - \lambda_{pi}^t| \mu_{qj}^t \nu_{rk}^t a_{ijk}^t + |\mu_{qj}^t - \mu_{qj}^t| \lambda_{pi}^t \nu_{rk}^t a_{ijk}^t \right) + \sum_{i,j,k=1}^{l,m,n} \left( |\nu_{rk}^t - \nu_{rk}^t| \lambda_{pi}^t \mu_{qj}^t a_{ijk}^t + |a_{ijk}^t - a_{ijk}^t| \lambda_{pi}^t \mu_{qj}^t \nu_{rk}^t \right) \leq \sum_{i,j,k=1}^{l,m,n} \left( \frac{\varepsilon M_2 M_3 M_4}{4lmn} + \frac{\varepsilon M_1 M_3 M_4}{4lmn} + \frac{\varepsilon M_1 M_2 M_4}{4lmn} + \frac{\varepsilon M_1 M_2 M_3}{4lmn} \right) \leq \sum_{i,j,k=1}^{l,m,n} \frac{\varepsilon}{lmn} \leq \varepsilon.
\]

Hence, the sequence \(\Phi((L_t, M_t, N_t), A_t)\) converges to the tensor \(\Phi((L, M, N), A)\), as required.

2) See the proof of Lemma 2.3(2) in [4].
Take \((L, M, N) = (\alpha E_l, \beta E_m, \gamma E_n)\) with arbitrary scalars \(\alpha, \beta, \gamma \in \mathbb{F}\) satisfying the condition \(\alpha \beta \gamma = 1\). Then, for every \(A = (a_{ijk}) \in \mathbb{F}^{l \times m \times n}\), we have

\[
\Phi((L, M, N), A) = (\alpha \beta \gamma a_{ijk}) = (a_{ijk}) = \Phi((E_l, E_m, E_n), A).
\]

This shows that the action is non-effective.

4) Consider tensors \(A = x_1 \otimes y_1 \otimes z_1\) and \(B = x_1 \otimes y_1 \otimes z_1 + x_2 \otimes y_2 \otimes z_2\), where \(\{x_1, x_2\} \subset \mathbb{F}^l, \{y_1, y_2\} \subset \mathbb{F}^m\) and \(\{z_1, z_2\} \subset \mathbb{F}^n\) are pairs of linear independent vectors. Then, one has \(\text{rank}(A) = 1\) and \(\text{rank}(B) = 2\). Using item 2), we obtain the desired conclusion. \(\Box\)

We have the following proposition on the semicontinuity of the tensor rank (see [4, Proposition 4.3, Theorem 4.10]).

**Proposition 2.** The following properties are fulfilled:

1) for every \(r \leq \min(l, m)\), the set \(S_r(l, m) := \{A \in \mathbb{F}^{l \times m} | \text{rank}(A) \leq r\}\) is closed.

2) there exists \(r\) such that the set \(S_r(l, m, n) := \{B \in \mathbb{F}^{l \times m \times n} | \text{rank}(B) \leq r\}\) is not closed.

2. The topological group of invertible matrices and approximations of matrices and tensors

In this section we consider a low-rank approximation of tensors in the space \(\mathbb{C}^{n \times n \times 2}\).

To this end, first, we formulate Bi’s criterion for square-type tensors in the space \(\mathbb{C}^{m \times m \times n}\) (see [14, Proposition 2.5]).

**Proposition 3.** Let \(A = [A_1| \ldots |A_n]\) be a tensor in \(\mathbb{C}^{m \times m \times n}\), where \(n \geq 2\), and let \(A_1 \in \mathbb{C}^{m \times m}\) be a nonsingular matrix. Then the tensor rank of \(A\) is equal to \(m\) if and only if the matrices \(A_2 A_1^{-1}, \ldots, A_n A_1^{-1}\) can be diagonalized simultaneously.

Second, we recall some algebraic and topological definitions and facts about square matrices.

An eigenvalue of a matrix is said to be simple if its algebraic multiplicity equals one. The spectrum of a matrix is said to be simple provided that all eigenvalues of a given matrix are simple. In other words, if all eigenvalues are pairwise distinct. Certainly, a matrix with a simple spectrum is diagonalizable.

We recall that a mapping \(f : X \to Y\) between two topological spaces is said to be open, if for any open set \(O\) in \(X\) the image \(f(O)\) is open in \(Y\). For example, if a mapping \(f : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})\) is a surjective continuous homomorphism then it is open [13, Theorem 5.29].

Making use of the topological group structure of the general linear group \(GL_n(\mathbb{C})\), one can prove the following statement [3, Proposition 4].

**Proposition 4.** Let \(f_1, f_2, \ldots, f_k : GL_n(\mathbb{C}) \to GL_n(\mathbb{C})\) be a finite family of self-mappings of the general linear group. Assume that at least one of these mappings is open with respect to the Euclidean topology. Let \(A_1, A_2, \ldots, A_k\) be arbitrary matrices in \(M_n(\mathbb{C})\) and let \(\| \cdot \|\) be a norm on \(M_n(\mathbb{C})\). Then for every \(\varepsilon > 0\) there exists a finite family \(A_1, A_2, \ldots, A_k\) consisting of invertible matrices with simple spectra such that the inequalities

\[
\|A_1 - A_1\| < \varepsilon, \|A_2 - A_2\| < \varepsilon, \ldots, \|A_k - A_k\| < \varepsilon
\]

hold.
hold and the product matrix
\[ f_1(A_{1\varepsilon})f_2(A_{2\varepsilon})\cdots f_k(A_{k\varepsilon}) \]
has a simple spectrum.

For the case of two mappings, we put \( f_1 \) and \( f_2 \) to be the identity mapping and the inverse mapping respectively:
\[
f_1 : GL_n(\mathbb{C}) \to GL_n(\mathbb{C}) : X \mapsto X; \quad f_2 : GL_n(\mathbb{C}) \to GL_n(\mathbb{C}) : X \mapsto X^{-1}.
\]
Obviously, both of these mappings are open with respect to the Euclidean topology in \( GL_n(\mathbb{C}) \). Therefore, as a consequence of the preceding proposition, we have

**Corollary 1.** Let \( A \) and \( B \) be matrices \( M_n(\mathbb{C}) \) and let \( \| \cdot \| \) be a norm on \( M_n(\mathbb{C}) \). Then for every \( \varepsilon > 0 \) there exists a pair of matrices \( A_{\varepsilon} \) and \( B_{\varepsilon} \) in \( GL_n(\mathbb{C}) \) with simple spectra such that the inequalities
\[
\| A - A_{\varepsilon} \| < \varepsilon \quad \text{and} \quad \| B - B_{\varepsilon} \| < \varepsilon
\]
hold and the product matrix \( A_{\varepsilon}B_{\varepsilon}^{-1} \) has a simple spectrum.

It is worth noting that one can use Corollary 1 for estimating the tensor rank of inverse matrices in the case when given matrices are the factors of the Kronecker products (see [8]).

We make use of the above-mentioned results to prove the following assertion.

**Proposition 5.** Let \( A \) be a tensor in \( \mathbb{C}^{n \times n \times 2} \) and let \( \| \cdot \| \) be a norm on \( \mathbb{C}^{n \times n \times 2} \). Then the equality
\[
\inf \{ \| A - B \| : B \in \mathbb{C}^{n \times n \times 2} \quad \text{and} \quad \text{rank}(B) = n \} = 0
\]
holds, that is, the tensor \( A \) may be approximated by tensors whose tensor ranks are equal to \( n \).

**Proof.** We set \( A = [A_1 | A_2] \), where \( A_1 \) and \( A_2 \) are square matrices of size \( n \times n \).

Let us fix \( \varepsilon > 0 \). Using Corollary 1, we take two invertible matrices \( B_1 \) and \( B_2 \) of size \( n \times n \) such that the inequalities
\[
\| A_1 - B_1 \|_1 < \frac{\varepsilon}{2} \quad \text{and} \quad \| A_2 - B_2 \|_1 < \frac{\varepsilon}{2}
\]
hold and the product matrix \( B_2B_1^{-1} \) has a simple spectrum.

Consider the tensor \( B = [B_1 | B_2] \). By Bi’s criterion, since the matrix \( B_2B_1^{-1} \) is diagonalizable, the tensor rank of \( B \) is equal to \( n \). Moreover, we have the following estimation:
\[
\| A - B \|_1 = \| A_1 - B_1 \|_1 + \| A_2 - B_2 \|_1 < \varepsilon.
\]

In view of the equivalence of the norms \( \| \cdot \| \) and \( \| \cdot \|_1 \), the rest is clear. □

It is known (see [10], [17] Theorem 4.3), that the maximal value of the tensor rank on the space \( \mathbb{C}^{n \times n \times 2} \) is given by
\[
mrank(n, n, 2) := \max \{ \text{rank}(A) : A \in \mathbb{C}^{n \times n \times 2} \} = n + \left\lfloor \frac{n}{2} \right\rfloor,
\]
where the symbol \( \lfloor \cdot \rfloor \) means the integer part of a real number. Therefore \( mrank(2n, 2n, 2) = 3n \) for every \( n \in \mathbb{N} \). This fact together with the Proposition 5 guarantees that, generally speaking, the tensor rank can leap an arbitrary large gap (see also [4] Section 4.5]). More precisely, we have
Corollary 2. Let \( n \in \mathbb{N} \). There exists a tensor \( A \in \mathbb{C}^{2n \times 2n \times 2} \) with \( \text{rank} (A) = 3n \) and a sequence of tensors \( \{ A_k \} \subset \mathbb{C}^{2n \times 2n \times 2} \), \( k \in \mathbb{N} \), such that \( \text{rank} (A_k) = 2n \) for every \( k \in \mathbb{N} \) and
\[
\lim_{k \to +\infty} A_k = A,
\]
where the limit is taken in the Euclidean topology.

Finally, we can conclude that the tensor rank is not semicontinuous on the tensor space \( \mathbb{C}^{n \times n \times 2} \) endowed with the Euclidean topology.

Corollary 3. Let \( n \geq 2 \). In the tensor space \( \mathbb{C}^{n \times n \times 2} \) endowed with the Euclidean topology the set of tensors
\[
\{ T \in \mathbb{C}^{n \times n \times 2} \mid \text{rank} (T) \leq n \}
\]
is not closed and its closure coincides with the whole space \( \mathbb{C}^{n \times n \times 2} \).

Acknowledgments

The authors are grateful to participants of the seminar on functional and numerical analysis "Tensor Analysis" at Kazan Federal University, the seminar "Quantum Functional Analysis and Its Applications" at Kazan State Power Engineering University and International Conference "Probability Theory and Mathematical Statistics" (November 7–10, 2017, Kazan) for helpful discussions of the content of this note.

References

1. Helemskii A.Ya. Quantum functional analysis: non-coordinate approach. University lecture series, V.56. – Providence, Rhode Island: Amer. Math. Soc., 2010. - 241 p.
2. Hackbusch W. Tensor spaces and numerical tensor calculus. – Berlin: Springer, 2012. –500 p.
3. Landsberg J.M. Tensors: geometry and applications. Graduate studies in mathematics, V.128. – Providence, Rhode Island: Amer. Math. Soc., 2012. - 439 p.
4. De Silva V., Lim L.-H. Tensor rank and the ill-posedness of the best low-rank approximation problem // SIAM Journal on Matrix Analysis and Applications – 2008. – V.30. – No.3 – P. 1084–1127.
5. Bini D., Capovani M., Lotti G. and Romani F. \( O(n^2 \cdot 7799) \) complexity for \( n \times n \) approximate matrix multiplication // Inform. Process. Lett.–1979.—V. 8.— P. 234–235.
6. Gumerov R. N. Homological dimension of radical algebras of Beurling type with a rapidly decreasing weight // Moscow Univ. Math. Bull. –1988.–V. 43. – 5.–P. 24–28.
7. Gumerov R. N. On a geometric test for nontriviality of the simplicial and cyclic cohomology of Banach algebras// Moscow Univ. Math. Bull.—1991.—V. 46.– 3.– P. 52–53.
8. Tyryshnikov E. E. Tensor ranks for the inversion of tensor-product binomials // J. Comput. Appl. Math.—2010.—V. 234.— 11.— P. 3170–3174.
9. Gumerov R.N, Vidanov S.I. Approximation by matrices with simple spectra // Lobachevskii J.Math.— 2016.—V. 37. – 3.–P. 240–243.
10. Greub W. H. Multilinear algebra. – Berlin, Heidelberg: Springer-Verlag, 1967. – 225 p.
11. Helemskii A.Ya. Lectures and exercises on functional analysis. Translations of Mathematical Monographs, V.233.– Providence: Amer. Math. Soc., 2006. –470 p.
12. Bredon G.E. Introduction to compact transformation groups. – New York: Academic Press, 1972. – 461 p.
13. Vries, J, de Elements of topological dynamics. – Dordrecht: Springer,1993.–762 p.
14. Sakata T., Sumi T., Miyazaki M. Algebraic and computational aspects of real tensor ranks. – Tokyo: Springer, 2016. – 108 p.
15. Hewitt E., Ross K. A. Abstract harmonic analysis Vol. 1. - Berlin: Springer,1963. – 519 p.
16. Ja’Ja’ J. Optimal evaluation of a pair of bilinear forms // SIAM J. on Computing.— 1979.—V. 8. – 3.– P. 281–293.
17. Sumi T., Miyazaki M., Sakata T. Rank of 3-tensors with 2 slices and Kronecker canonical forms // Linear Algebra Appl.— 2009.—V. 431. – 10.– P. 1858–1868.
