Monads and Rank Three Vector Bundles on Quadrics

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Abstract
In this paper we give the classification of rank 3 vector bundles without "inner" cohomology on a quadric hypersurface $Q_n$ ($n > 3$) by studying the associated monads.

Introduction
A monad on $\mathbb{P}^n$ or, more generally, on a projective variety $X$, is a complex of three vector bundles

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0$$

such that $\alpha$ is injective as a map of vector bundles and $\beta$ is surjective. Monads have been studied by Horrocks, who proved (see [Ho] or [BH]) that every vector bundle on $\mathbb{P}^n$ is the homology of a suitable minimal monad. Throughout the paper we often use the Horrocks correspondence between a bundle $E$ on $\mathbb{P}^n$ ($n \geq 3$) and the corresponding minimal monad

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0,$$

where $A$ and $C$ are sums of line bundles and $B$ satisfies:

1. $H^1_*(B) = H^{n-1}_*(B) = 0$
2. $H^*_i(B) = H^*_i(E)$ \forall $i, 1 < i < n - 1$.

where $H^*_i(B)$ is defined as $\oplus_{k \in (\mathbb{Z})} H^i(\mathbb{P}^n, B(k))$.

This correspondence holds also on a projective variety $X$ (dim $X \geq 3$) if we fix a very ample line bundle $O_X(1)$. Indeed the proof of the result in ([BH] proposition 3) can be easily extended to $X$ (see [Ml]).

Rao, Mohan Kumar and Peterson have successfully used this tool to investigate the intermediate cohomology modules of a vector bundle on $\mathbb{P}^n$ and give cohomological splitting conditions (see [KPR1]).

This theorem makes a strong use of monads and of Horrocks’ splitting criterion which states the following:

Let $E$ be a vector bundle of rank $r$ on $\mathbb{P}^n$, $n \geq 2$ then $E$ splits if and only if it does not have intermediate cohomology (i.e. $H^1_*(E) = \ldots = H^{n-1}_*(E) = 0$).

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This criterion fails on more general varieties. In fact there exist non-split vector bundles on $X$ without intermediate cohomology. This bundles are called ACM bundles. Rao, Mohan Kumar and Peterson focused on bundles without inner cohomology (i.e. $H^2_*(E) = \ldots = H^{n-2}_*(E) = 0$) and showed that these bundles on $\mathbb{P}^n (n > 3)$ are split if the rank is small.

On a quadric hypersurface $Q_n$ the Horrocks criterion does not work, but there is a theorem that classifies all the ACM bundles (see [Kn]) as direct sums of line bundles and spinor bundles (up to a twist - for generalities about spinor bundles see [Ot2]).

In [Mi] we improve Ottaviani’s splitting criterion for vector bundles on a quadric hypersurface (see [Ot1] and [Ot3]) and obtain the equivalent of the result by Rao, Mohan Kumar and Peterson. Moreover we give the classification of rank 2 bundles without inner cohomology on $Q_n (n > 3)$. It surprisingly exactly agrees with the classification by Ancona, Peternell and Wisniewski of rank 2 Fano bundles (see [APW]).

We proved that for an indecomposable rank 2 bundle $E$ on $Q_4$ with $H^1_*(E) \neq 0$ and $H^2_*(E) = 0$, the only possible minimal monad with $A$ and $C$ different from zero is (up to a twist)

$$0 \rightarrow O \xrightarrow{\alpha'} S'(1) \oplus S''(1) \xrightarrow{\beta'} O(1) \rightarrow 0,$$

where $S'$ and $S''$ are the two spinor bundles on $Q_4$. The homology is the bundle $Z_4$ associated to the disjoint union of a plane in $\Lambda$ and a plane in $\Lambda'$, the two families of planes in $Q_4$ (see [AS]).

The kernel $G_4$ and the cokernel $P_4$ (the dual) of this monad are rank 3 bundles without inner cohomology and we have the two sequences

$$0 \rightarrow G_4 \rightarrow S'(1) \oplus S''(1) \rightarrow O(1) \rightarrow 0, \quad (1)$$

and

$$0 \rightarrow O \rightarrow S'(1) \oplus S''(1) \rightarrow P_4 \rightarrow 0. \quad (2)$$

On $Q_5$ there is only one spinor $S_5$ and the only possible minimal monad with $A$ and $C$ different from zero, for a rank 2 bundle without inner cohomology, is (up to a twist)

$$0 \rightarrow O \xrightarrow{\alpha''} S(1) \xrightarrow{\beta''} O(1) \rightarrow 0.$$

The kernel $G_5$ and the cokernel $P_5$ (the dual) of the monad are rank 3 bundles without inner cohomology and we have the two sequences

$$0 \rightarrow G_5 \rightarrow S(1) \rightarrow O(1) \rightarrow 0, \quad (3)$$

and

$$0 \rightarrow O \rightarrow S(1) \rightarrow P_5 \rightarrow 0. \quad (4)$$

The homology of the monad $Z_5$ is a Cayley bundle (see [Ot4] for generalities on Cayley bundles).

The bundle $Z_5$ appear also in [Ta] and [KPR2].

For $n > 5$, no non-split bundle of rank 2 on $Q_n$ exists with $H^2_*(E) = \ldots = H^{n-2}_*(E) = 0.$
The main aim of the present paper is the classification of rank three bundles without inner cohomology on \( \mathbb{P}^4 \) by studying the associated monads. We are able to prove that:

For a non-split rank 3 bundle \( E \) on \( \mathbb{P}^4 \) with \( H^2_2(E) = 0 \), the only possible minimal monads with \( A \) or \( C \) different from zero are (up to a twist) the sequences (1) and (2) and

\[
0 \rightarrow \mathcal{O} \xrightarrow{\alpha} S'(1) \oplus S''(1) \oplus \mathcal{O}(a) \xrightarrow{\beta} \mathcal{O}(1) \rightarrow 0,
\]

(5)

where \( a \) is an integer, \( \alpha = (\alpha', 0) \) and \( \beta = (\beta', 0) \).

This means that on \( \mathbb{P}^4 \) the only non-split rank 3 bundles without inner cohomology are the following:

the ACM bundles \( S' \oplus \mathcal{O}(a) \) and \( S'' \oplus \mathcal{O}(a) \), \( G_4 \), \( P_4 \) and \( Z_4 \oplus \mathcal{O}(a) \).

In particular \( G_4 \) and its dual are the only indecomposable rank 3 bundles without inner cohomology on \( \mathbb{P}^4 \).

By using monads again we can also understand the behavior of rank three bundles on \( \mathbb{P}^5 \) and also on \( \mathbb{P}^n \), \( (n > 5) \).

More precisely we can prove that:

For a non-split rank 3 bundle \( E \) on \( \mathbb{P}^5 \) without inner cohomology, the only possible minimal monad with \( A \) or \( C \) not zero are (up to a twist) the sequences (3) and (4) and

\[
0 \rightarrow \mathcal{O} \xrightarrow{\alpha} S_5(1) \oplus \mathcal{O}(a) \xrightarrow{\beta} \mathcal{O}(1) \rightarrow 0.
\]

(6)

where \( a \) is an integer, \( \alpha = (\alpha'', 0) \) and \( \beta = (\beta'', 0) \). This means that on \( \mathbb{P}^5 \) the only rank 3 bundles without inner cohomology are the following:

\( G_5 \), \( P_5 \) and \( Z_5 \oplus \mathcal{O}(a) \).

In particular \( G_5 \) and its dual are the only indecomposable rank 3 bundles without inner cohomology on \( \mathbb{P}^5 \).

For a non-split rank 3 bundle \( E \) on \( \mathbb{P}^6 \) without inner cohomology, we have four possible minimal monads:

\[
0 \rightarrow \mathcal{O} \xrightarrow{\alpha} S'_6(1) \rightarrow P'_6 \rightarrow 0,
\]

(6)

\[
0 \rightarrow \mathcal{O} \xrightarrow{\alpha} S''_6(1) \rightarrow P''_6 \rightarrow 0,
\]

(7)

\[
0 \rightarrow G'_6 \rightarrow S'_6(1) \rightarrow \mathcal{O}(1) \rightarrow 0,
\]

(8)

and

\[
0 \rightarrow G''_6 \rightarrow S''_6(1) \rightarrow \mathcal{O}(1) \rightarrow 0.
\]

(9)

These sequences appear for instance in \cite{Ot2} Theorem 3.5.

Therefore on \( \mathbb{P}^6 \) the only rank 3 bundles without inner cohomology are the following:

\( G'_6 \), \( G''_6 \), \( P'_6 \) and \( P''_6 \).

For \( n > 6 \), no non-split bundles of rank 3 in \( \mathbb{P}^n \) exist with \( H^2_2(E) = ... = H^{n-2}_n(E) = 0 \).

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1 Monads for Bundles on ACM Varieties

In this section $X$ denotes a nonsingular subcanonical, irreducible ACM projective variety. By this we mean that $X$ has a very ample line bundle $\mathcal{O}_X(1)$ such that $\omega_X \cong \mathcal{O}_X(a)$ for some $a \in \mathbb{Z}$ and the embedding of $X$ by $\mathcal{O}_X(1)$ is arithmetically Cohen-Macaulay. We will also assume that every line bundle on $X$ has the form $\mathcal{O}_X(k), k \in \mathbb{Z}$.

If $M$ is a finitely generated graded module over the homogeneous coordinate ring of $X$, we denote by $\beta_{ij}(M)$ and $\beta_i(M)$ the graded Betti numbers and total Betti numbers of $M$. We will mainly use $\beta_0^j(M)$ and $\beta_0(M)$ which give the number of minimal generators of $M$ in degree $j$ and the total number of minimal generators respectively.

We say that a bundle is non-split if it does not split as a direct sum of line bundles.

We say that a bundle is indecomposable if it does not split in two direct summands.

Definition 1.1. We will call bundle without inner cohomology a bundle $\mathcal{E}$ on $X$ with

$$H^2_*(\mathcal{E}) = \cdots = H^{n-2}_*(\mathcal{E}) = 0,$$

where $n = \dim X$.

We prove a theorem about monads for rank $r$ bundles:

Theorem 1.2. On $X$ of dimension $n$ with $n > 3$, any minimal monad

$$0 \rightarrow \mathcal{A} \overset{\alpha}{\rightarrow} \mathcal{B} \overset{\beta}{\rightarrow} \mathcal{C} \rightarrow 0,$$

such that $\mathcal{A}$ or $\mathcal{C}$ are not zero, for a rank $r$ ($r \geq 2$) bundle with $H^2_*(\mathcal{E}) = H^{n-2}_*(\mathcal{E}) = H^2_*(\wedge^2 \mathcal{E}) = H^2_*(\wedge^2 \mathcal{E}^\vee) = 0$, must satisfy the following conditions:

1. $H^1_*(\wedge^2 \mathcal{B}) \neq 0$, $\beta_0(H^1_*(\wedge^2 \mathcal{B})) \geq \beta_0(H^0_*(S_2 \mathcal{C}))$ and

$$\beta_{0j}(H^1_*(\wedge^2 \mathcal{B})) \geq \beta_{0j}(H^0_*(S_2 \mathcal{C})) \quad \forall j \in \mathbb{Z}, \text{ if } \mathcal{C} \text{ is not zero.}$$

2. $H^1_*(\wedge^2 \mathcal{B}^\vee) \neq 0$, $\beta_0(H^1_*(\wedge^2 \mathcal{B}^\vee)) \geq \beta_0(H^0_*(S_2 \mathcal{A}^\vee))$ and

$$\beta_{0j}(H^1_*(\wedge^2 \mathcal{B}^\vee)) \geq \beta_{0j}(H^0_*(S_2 \mathcal{A}^\vee)) \quad \forall j \in \mathbb{Z}, \text{ if } \mathcal{A} \text{ is not zero.}$$

3. $H^2_*(\wedge^2 \mathcal{B}) = H^2_*(\wedge^2 \mathcal{B}^\vee) = 0$.

Proof. First of all, since $X$ is ACM, the sheaf $\mathcal{O}_X$ does not have intermediate cohomology. Hence the same is true for $\mathcal{A}$ and $\mathcal{C}$ that are free $\mathcal{O}_X$ sheaves.

Let us now assume the existence of a minimal monad with $H^1_*(\wedge^2 \mathcal{B}) = 0$ and $\mathcal{C}$ not zero

$$0 \rightarrow \mathcal{A} \overset{\alpha}{\rightarrow} \mathcal{B} \overset{\beta}{\rightarrow} \mathcal{C} \rightarrow 0.$$

Then, if we call $\mathcal{G} = \ker \beta$, from the sequence

$$0 \rightarrow S_2 \mathcal{A} \rightarrow (\mathcal{A} \otimes \mathcal{G}) \rightarrow \wedge^2 \mathcal{G} \rightarrow \wedge^2 \mathcal{E} \rightarrow 0,$$
we have
\[ H^2_*(\wedge^2 \mathcal{G}) = H^2_*(A \otimes \mathcal{G}) = 0, \]
since \( H^2_*(\mathcal{B}) = H^2_*(\mathcal{G}) = H^2_*(\mathcal{E}) = 0 \) and \( H^2_*(\wedge^2 \mathcal{E}) = 0 \).
Moreover, from the sequence
\[ 0 \to \wedge^2 \mathcal{G} \to \wedge^2 \mathcal{B} \to \mathcal{B} \otimes \mathcal{C} \to \mathcal{S}_2^2 \mathcal{C} \to 0, \]
by passing to the exact sequence of maps on cohomology groups, since \( H^1_*(\wedge^2 \mathcal{B}) = H^2_*(\wedge^2 \mathcal{G}) = 0 \) we get
\[ H^0_*(\mathcal{B} \otimes \mathcal{C}) \to H^0_*(\mathcal{S}_2^2 \mathcal{C}) \to 0. \]
Now, if we call \( S_X \) the coordinate ring, we can say that \( H^0_*(\mathcal{S}_2^2 \mathcal{C}) \) is a free \( S_X \)-module hence projective, then there exists a map
\[ H^0_*(\mathcal{B} \otimes \mathcal{C}) \leftarrow H^0_*(\mathcal{S}_2^2 \mathcal{C}) \]
and this means that
\[ \mathcal{B} \otimes \mathcal{C} \to \mathcal{S}_2^2 \mathcal{C} \to 0 \]
splits.
But this map is obtained from \( \beta \) as \( b \otimes c \mapsto \beta(b)c \), so if it splits also \( \beta \) has to split and this violates the minimality of the monad. We can say something stronger.
Recall that if \( M \to N \to 0 \) is a surjection of finitely generated graded \( S_X \)-modules, then \( \beta_0(M) \geq \beta_0(N) \) and also \( \beta_{0j}(M) \geq \beta_{0j}(N) \) for any \( j \). Furthermore, if the inequality is strict, it means that a set of minimal generators of \( M \) (in degree \( j \) in the second case) can be chosen in such a way that one of generators in the set maps to zero.

From the sequence
\[ 0 \to \wedge^2 \mathcal{G} \to \wedge^2 \mathcal{B} \to \mathcal{B} \otimes \mathcal{C} \to \mathcal{S}_2^2 \mathcal{C} \to 0, \]
since \( H^2_*(\wedge^2 \mathcal{G}) = 0 \), we have a surjective map
\[ H^1_*(\wedge^2 \mathcal{B}) \to H^1_*(\Gamma) \to 0 \]
where \( \Gamma = \ker \gamma \), and then
\[ \beta_0(H^1_*(\wedge^2 \mathcal{B})) \geq \beta_0(H^1_*(\Gamma)). \]
On the other hand we have the sequence
\[ H^0_*(\mathcal{B} \otimes \mathcal{C}) \to H^0_*(\mathcal{S}_2^2 \mathcal{C}) \to H^1_*(\Gamma) \to 0; \]
so, if
\[ \beta_0(H^1_*(\wedge^2 \mathcal{B})) < \beta_0(H^0_*(\mathcal{S}_2^2 \mathcal{C})), \]
also
\[ \beta_0(H^1_*(\Gamma)) < \beta_0(H^0_*(\mathcal{S}_2^2 \mathcal{C})), \]
and some of the generators of \( H^0_*(\mathcal{S}_2^2 \mathcal{C}) \) must be in the image of \( \gamma \).
But \( \gamma \) is obtained from \( \beta \) as \( b \otimes c \mapsto \beta(b)c \), so also some generators of \( C \) must be in the image of \( \beta \) and this contradicts the minimality of the monad.
We conclude that not just \( H^1_*(\wedge^2 \mathcal{B}) \) has to be non zero but also
\[ \beta_0(H^1_*(\wedge^2 \mathcal{B})) \geq \beta_0(H^0_*(\mathcal{S}_2^2 \mathcal{C})). \]
If we fix the degree \( j \) we have that also the map \( H^0(\mathcal{B} \otimes \mathcal{C}(j)) \to H^0(S_2 \mathcal{C}(j)) \) and so we see that, \( \forall j \in \mathbb{Z} \)

\[
\beta_{0j}(H^1_*(\wedge^2 \mathcal{B})) \geq \beta_{0j}(H^0_*(S_2 \mathcal{C})).
\]

If \( \mathcal{A} = 0 \) the monad is simply

\[
0 \to \mathcal{E} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0,
\]

and, by using the sequence

\[
0 \to \wedge^2 \mathcal{E} \to \wedge^2 \mathcal{B} \to \mathcal{B} \otimes \mathcal{C} \to S_2 \mathcal{C} \to 0,
\]

since \( H^2_*(\wedge^2 \mathcal{E}) = 0 \), we can conclude as before.

Let us now assume the existence of a monad with \( \mathcal{A} \) not zero \( (H^1_*(\wedge^2 \mathcal{B}^\vee)) = 0 \), we use the dual sequences.

From

\[
0 \to S_2 \mathcal{C}^\vee \to (\mathcal{C}^\vee \otimes \mathcal{B}^\vee) \to \wedge^2 \mathcal{B}^\vee \to \wedge^2 \mathcal{G}^\vee \to 0,
\]

we have \( H^1_*(\wedge^2 \mathcal{G}^\vee) \cong H^1_*(\wedge^2 \mathcal{B}^\vee) \).

Moreover, from the sequence

\[
0 \to \wedge^2 \mathcal{E}^\vee \to \wedge^2 \mathcal{G}^\vee \to \mathcal{G}^\vee \otimes \mathcal{A}^\vee \to S_2 \mathcal{A}^\vee \to 0,
\]

by passing to the exact sequence of maps on cohomology groups, since \( H^2_*(\wedge^2 \mathcal{E}^\vee) = H^1_*(\wedge^2 \mathcal{G}^\vee) = 0 \) we get

\[
H^0_*(\mathcal{G}^\vee \otimes \mathcal{A}^\vee) \to H^0_*(S_2 \mathcal{A}^\vee) \to 0,
\]

and this violates the minimality of the monad as before.

We can also conclude that

\[
\beta_0(H^1_*(\wedge^2 \mathcal{B}^\vee)) = \beta_0(H^1_*(\wedge^2 \mathcal{G}^\vee)) \geq \beta_0(H^0_*(S_2 \mathcal{A}^\vee)).
\]

If we fix the degree \( j \) we have that also the map \( H^0(\mathcal{G}^\vee \otimes \mathcal{A}^\vee(j)) \to H^0(S_2 \mathcal{A}^\vee(j)) \) and so we see that, \( \forall j \in \mathbb{Z} \)

\[
\beta_{0j}(H^1_*(\wedge^2 \mathcal{B}^\vee)) \geq \beta_{0j}(H^0_*(S_2 \mathcal{A}^\vee)).
\]

If \( \mathcal{C} = 0 \) the monad is simply

\[
0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{E} \to 0,
\]

and, by using the sequence

\[
0 \to \wedge^2 \mathcal{E}^\vee \to \wedge^2 \mathcal{B}^\vee \to \mathcal{B}^\vee \otimes \mathcal{A}^\vee \to S_2 \mathcal{A}^\vee \to 0,
\]

since \( H^2_*(\wedge^2 \mathcal{E}^\vee) = 0 \), we can conclude as before.

The third condition comes from the sequences

\[
0 \to \wedge^2 \mathcal{G} \to \wedge^2 \mathcal{B} \to \mathcal{B} \otimes \mathcal{C} \to S_2 \mathcal{C} \to 0,
\]

and

\[
0 \to S_2 \mathcal{C}^\vee \to (\mathcal{C}^\vee \otimes \mathcal{B}^\vee) \to \wedge^2 \mathcal{B}^\vee \to \wedge^2 \mathcal{G}^\vee \to 0,
\]

since \( H^2_*(\wedge^2 \mathcal{G}) = H^2_*(\mathcal{B} \otimes \mathcal{C}) = H^2_*(\mathcal{C}^\vee \otimes \mathcal{B}^\vee) = H^2_*(\wedge^2 \mathcal{G}^\vee) = 0 \).
Remark 1.3. If \( r = 2 \) we have ([ML]1.6).

Remark 1.4. If \( r = 3 \) we don’t need the hypothesis \( H^2_*(\wedge^2 \mathcal{E}) = H^2_*(\wedge^2 \mathcal{E}^\vee) = 0 \) because \( H^2_*(\wedge^2 \mathcal{E}) = H^{n-2}_*(\mathcal{E}) = 0 \).

Remark 1.5. On \( \mathbb{P}^n \) we can say the following:
Let \( \mathcal{E} \) be a bundle without inner cohomology such that \( H^2_*(\wedge^2 \mathcal{E}) = H^2_*(\wedge^2 \mathcal{E}^\vee) = 0 \), then \( \mathcal{E} \) splits.
In fact if \( \mathcal{E} \) does not split the associated minimal monad has \( \mathcal{A} \) or \( \mathcal{C} \) different to zero. Since \( H^2_*(\mathcal{E}) = \ldots = H^{n-2}_*(\mathcal{E}) = 0 \), the bundles \( \mathcal{B} \) does not have intermediate cohomology and hence it splits. In particular \( H^1_*(\wedge^2 \mathcal{B}) = 0 \). By the above theorem this is a contradiction.
So the hypothesis \( H^2_*(\wedge^2 \mathcal{E}) = H^2_*(\wedge^2 \mathcal{E}^\vee) = 0 \) avoid the limitation of the rank in the Kumar, Peterson and Rao theorem (see [KPR1]).

We need also the following lemma:

Lemma 1.6. Let \( \mathcal{E} \) be a rank 2 on \( X \). If

\[
0 \to \mathcal{A} \xrightarrow{\alpha'} \mathcal{B} \xrightarrow{\beta'} \mathcal{C} \to 0,
\]

is a minimal monad for \( \mathcal{E} \), then

\[
0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \oplus \mathcal{O}(a) \xrightarrow{\beta} \mathcal{C} \to 0,
\]

where \( \alpha = (\alpha',0) \) and \( \beta = (\beta',0) \), is a minimal monad for \( \mathcal{E} \oplus \mathcal{O}(a) \).

2 Rank 3 Bundles without Inner Cohomology

We want now apply these results in order to classify the rank 3 bundles without inner cohomology on \( \mathcal{Q}_4 \):

Theorem 2.1. For a non-split rank 3 bundle \( \mathcal{E} \) on \( \mathcal{Q}_4 \) with

\( H^2_*(\mathcal{E}) = 0 \), the only possible minimal monads with \( \mathcal{A} \) or \( \mathcal{C} \) different from zero are (up to a twist) the sequences (1) and (2) and

\[
0 \to \mathcal{O} \to S'/(1) \oplus S''/(1) \oplus \mathcal{O}(a) \to \mathcal{O}(1) \to 0,
\]

(10)

where \( a \) is an integer, \( \alpha = (\alpha',0) \) and \( \beta = (\beta',0) \).

Proof. First of all consider a minimal monad for \( \mathcal{E} \),

\[
0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0,
\]

Since by construction, \( \mathcal{B} \) is an ACM bundle on \( \mathcal{Q}_4 \), then it has to be isomorphic to a direct sum of line bundles and spinor bundles twisted by some \( \mathcal{O}(t) \).
Since \( \mathcal{B} \) cannot be split without violating part 1 of (Theorem 1.2) which states that

\( H^1_*(\wedge^2 \mathcal{B}) \neq 0 \).
Hence at least a spinor bundle must appear in its decomposition. If just one copy of \( S' \) or one copy of \( S'' \) appears in \( B \), since
\[
\text{rank } S'' = \text{rank } S' = 2,
\]
and then \( \wedge^2 S' \) and \( \wedge^2 S'' \) are line bundles, also the bundle \( \wedge^2 B \) is ACM and again the condition
\[
H_1^2(\wedge^2 B) \neq 0,
\]
in (Theorem 1.2), is not satisfied. If it appears more than one copy of \( S' \) or more than one copy of \( S'' \) appears in \( B \), in the bundle \( \wedge^2 B \) it appears \((S' \otimes S')(t)\) or \((S'' \otimes S'')(t)\) appears and, since
\[
H^2_1(S' \otimes S') = H^2_1(S'' \otimes S'') = \mathbb{C},
\]
the condition
\[
H^2_*(\wedge^2 B) = 0
\]
in (Theorem 1.2), fails to be satisfied. So \( B \) must contain both \( S' \) and \( S'' \) with some twist and only one copy of each. We can conclude that \( B \) has to be of the form
\[
(\bigoplus_i \mathcal{O}(a_i) \oplus (S'(b)) \oplus (S''(c))).
\]
Let us notice furthermore that if \( H^1_*(\mathcal{E}) \) has more than 1 generator, \( \text{rank } C > 1 \) and \( H^0_*(S_2 \mathcal{C}) \) has at least 3 generators. But
\[
H^1_*(\wedge^2 B) \cong H^1_*(S' \otimes S'') = \mathbb{C}
\]
has just 1 generator and this is a contradiction because by (Theorem 1.2)
\[
\beta_0(H^1_*(\wedge^2 B)) \geq \beta_0(H^0_*(S_2 \mathcal{C})).
\]
This means that \( \text{rank } C = 1 \) or \( = 0 \).
Similarly, looking at dual sequence, we have that also \( \text{rank } A \) must be 1. If \( C = 0 \), we have the minimal monad
\[
0 \rightarrow \mathcal{O} \rightarrow S'(l) \oplus S''(m) \rightarrow \mathcal{E} \rightarrow 0.
\]
By computing \( c_4(S'(l) \oplus S''(m)) \) as in ([M1] Theorem 3.1) we see that \( l \) and \( m \) must be both equal to 1 and we have the monad (2). If \( A = 0 \) we see in the same way that we have the monad (1). At this point the only possible monads with \( A \) and \( C \) not zero, are like
\[
0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(a) \oplus S'(1+b) \oplus S''(1+c) \overset{\beta}{\rightarrow} \mathcal{O}(d) \rightarrow 0.
\]
where \( a, b \) and \( c \) are integer numbers.

Now, since
\[
\beta_{0j}(H^1_*(\wedge^2 B)) \geq \beta_{0j}(H^0_*(S_2 \mathcal{C}))
\]
\( \forall j \in \mathbb{Z} \) we see that \( 2 + b + c = 2d \).
Let assume that \( b \leq 0 \) then by the sequence
\[
0 \rightarrow S'' \rightarrow \mathcal{O}^4 \rightarrow S'(1) \rightarrow 0,
\]
we see that $S'(1+b)$ does not have global section. Moreover $O(a) \oplus S''(1+c)$ does not have nowhere vanishing section. In fact a section of $O(a)$ has zero locus of dimension 4 (if it is the zero map) or 3. If $a = 0$ it could be a scalar different to zero but this is against our assumption of minimality. Since the zero locus of a section of $S''(1+c)$ has dimension at least 2, we conclude that the zero locus of a section of $O(a) \oplus S''(1+c)$ must be not empty.

This means that the map $\alpha$ cannot be injective. If $c \leq 0$ we have the same contradiction.

We have, hence, that $b$ and $c$ must be positives. Let us consider now the dual monad twisted by $\alpha$

$$0 \to O \to O(d - a) \oplus S'(d - b) \oplus S''(d - c) \to \to O(d) \to 0.$$  

By the argument above we have that $d - b \geq 1$ and $d - c \geq 1$; but, since $2 = d - b + d - c$, it follows that $b = c$ and $d = b + 1$.

We have the map

$$\beta : O(a) \oplus S'(1 + b) \oplus S''(1 + b) \to O(1 + b).$$

Let us consider the restriction

$$\beta' : S'(1 + b) \oplus S''(1 + b) \to O(1 + b).$$

We know (by [MI]) that in general, we can find a surjective map

$$\gamma : S'(1 + b) \oplus S''(1 + b) \to O(1 + b)$$

since we have some standard rank two bundles obtained from a monad on $Q_4$. Hence the map $\gamma'$ gives a nowhere vanishing section of $S'' \oplus S''$, which thus has fourth Chern class 0. Hence in particular, some other map like $\beta'$ must give a section which is either nowhere vanishing, or which vanishes on a locus of dimension $\geq 1$. (It cannot vanish on a non empty zero dimensional set). However, if $\beta'$ vanishes on a locus of dimension $\geq 1$, then $\beta'$ itself cannot give a nowhere vanishing section since the map $O \to O(-a + 1 + b)$ is either zero or defines a hypersurface, by minimality.

Therefore $\beta'$ must be surjective (like a standard map $\gamma$).

By an easy computation we have the following claim:

If $\mathcal{E}$ is a rank two bundle on $Q_4$ with monad

$$0 \to O \to \mathcal{E}'(1) \oplus \mathcal{E}''(1) \to O(1) \to 0,$$

then $H^1(\mathcal{E}(-1)) = k$ and $H^1(\mathcal{E}(t)) = 0$ for every $t \neq -1$.

Hence on the level of global sections, $\beta'$ is surjective onto $O(1+b)$, except that the element 1 in degree $(-1 - b)$ is not in the image. By minimality, 1 cannot be in the image of $O(a)$. Hence $O(a)$ maps by $\beta$ to the image of $\beta'$ i.e. there exists $l \in S'(1 + b) \oplus S''(1 + b)$ such that $\beta(1,0) = \beta'(l)$. Therefore, after a change of basis, we may assume that $O(a)$ maps to zero. In fact, if we consider a map

$$\delta : O(a) \oplus S'(1 + b) \oplus S''(1 + b) \to O(a) \oplus S'(1 + b) \oplus S''(1 + b)$$
sending \((1,0)\) in \((1,-l)\), we have that
\[
\beta(\delta(1,0)) = \beta(1,-l) = \beta'(l) - \beta'(l) = 0.
\]

We have at this point the monad
\[
0 \to \mathcal{O} \xrightarrow{(h,\alpha')} \mathcal{O}(a) \oplus S'(1+b) \oplus S''(1+b) \xrightarrow{(0,\beta')} \mathcal{O}(1+b) \to 0.
\]

We want to prove that \(h\) must be the zero map and \(b\) must be zero.
If \(a \leq 0\), clearly \(h = 0\) and \(\alpha'\) is injective if and only if \(b = 0\).
If \(a > 0\) we consider the kernel of \(\beta\) \(\mathcal{O}(a) \oplus \mathcal{G}_4(b)\) and, from the exact sequence
\[
0 \to \mathcal{O} \to \mathcal{O}(a) \oplus \mathcal{G}_4(b) \to \mathcal{E} \to 0,
\]
we see that \(c_4(\mathcal{O}(a) \oplus \mathcal{G}_4(b))\) must be zero. But from
\[
0 \to \mathcal{G}_4(b) \to S'(1+b) \oplus S''(1+b) \to \mathcal{O}(1+b) \to 0,
\]
we see that \(c_3(\mathcal{G}_4(b)) = c_4(S'(1+b) \oplus S''(1+b)) \ast c_1(\mathcal{O}(1+b))^{-1} = 2(1+b+b^2)(b+b^2)(1+b)^{-1}\)
and so \(c_4(\mathcal{O}(a) \oplus \mathcal{G}_4(b)) = c_1(\mathcal{O}(a)) \ast c_3(\mathcal{G}_4(b)) = 0\) if and only if \(b = 0\).

Remark 2.2. On \(Q_4\) the only rank 3 bundles without inner cohomology are the ACM bundles, \(\mathcal{G}_4\), \(\mathcal{P}_4\) and \(\mathcal{Z}_4 \oplus \mathcal{O}(a)\).

Corollary 2.3. In higher dimension we have:

1. For a non-split rank 3 bundle \(\mathcal{E}\) on \(Q_5\) without inner cohomology, the only possible minimal monad with \(A\) or \(C\) not zero are (up to a twist) the sequences (3) and (4) and
\[
0 \to \mathcal{O} \to S_5(1) \oplus \mathcal{O}(a) \to \mathcal{O}(1) \to 0.
\]
where \(a\) is an integer, \(\alpha = (\alpha'',0)\) and \(\beta = (\beta'',0)\).

2. For a non-split rank 3 bundle \(\mathcal{E}\) on \(Q_6\) without inner cohomology, the only possible minimal monad with \(A\) or \(C\) not zero are (up to a twist) the sequences (6), (7), (8) and (9).

3. For \(n > 6\), no non-split bundle of rank 3 in \(Q_n\) exist with
\[
H^2_n(\mathcal{E}) = \ldots = H^{n-2}_n(\mathcal{E}) = 0.
\]

Proof. First of all let us notice that for \(n > 4\) there is not non-split ACM rank 3 bundles since the spinor bundles have rank greater than 3.
Let us assume then that \(H^1_n(\mathcal{E}) \neq 0\) or \(H^{n-1}_n(\mathcal{E}) \neq 0\) and let us see how many monads it is possible to find:

1. In a minimal monad for \(\mathcal{E}\) on \(Q_5\),
\[
0 \to \mathcal{A} \xrightarrow{\alpha} \mathcal{B} \xrightarrow{\beta} \mathcal{C} \to 0,
\]
\(\mathcal{B}\) is an ACM bundle on \(Q_5\); then it has to be isomorphic to a direct sum of line bundles and spinor bundles twisted by some \(\mathcal{O}(t)\),
Moreover, since $H^2_s(E) = 0$ and $H^3_s(E) = 0$, $E|_{Q_4} = F$ is a bundle with $H^2_s(F) = 0$ and by ([Ml] Lemma 1.2) his minimal monad is just the restriction of the minimal monad for $E$

$$0 \to A \xrightarrow{\alpha} B \xrightarrow{\beta} C \to 0.$$ 

For the theorem above, hence, this monad must be

$$0 \to \mathcal{O} \to S'(1) \oplus S''(1) \oplus \mathcal{O}(a) \to \mathcal{O}(1) \to 0.$$ 

Now, since

$$S'_5|_{Q_4} \cong S' \oplus S'',$$

the only bundle of the form

$$\bigoplus_i \mathcal{O}(a_i)) \oplus \bigoplus_j S_5(b_j))$$

having $S'(1) \oplus S''(1) \oplus \mathcal{O}(a)$ as restriction on $Q_4$ is $S_5(1) \oplus \mathcal{O}(a)$ and then if $A$ and $C$ are different to zero the claimed monad

$$0 \to \mathcal{O} \xrightarrow{\alpha} S_5(1) \oplus \mathcal{O}(a) \xrightarrow{\beta} \mathcal{O}(1) \to 0$$

where $\alpha = (a'', 0)$ and $\beta = (\beta'', 0)$, is the only possible.

If $A = 0$, we have the monad (3) and if $C = 0$, we have the monad (4).

2. In $Q_6$ we use the same argument. Let us consider a minimal monad for $E$.

If $A$ and $C$ are not zero the restriction of the monad on $Q_5$ must be

$$0 \to \mathcal{O} \xrightarrow{\alpha} S_5(1) \oplus \mathcal{O}(a) \xrightarrow{\beta} \mathcal{O}(1) \to 0.$$ 

Since $S'_6|_{Q_5} \cong S_5$ and also $S''_6|_{Q_5} \cong S_5$, we have two possible minimal monads:

$$0 \to \mathcal{O} \to S'_6(1) \oplus \mathcal{O}(a) \to \mathcal{O}(1) \to 0$$

and

$$0 \to \mathcal{O} \to S''_6(1) \oplus \mathcal{O}(a) \to \mathcal{O}(1) \to 0,$$

where the maps are of the form $(\gamma, 0)$.

In both the sequences the homology is a bundle $F \oplus \mathcal{O}(a)$ where $F$ is a rank 2 bundle without inner cohomology that by ([Ml] Cor. 3.4) cannot exist, so they cannot be the monads of a rank 3 bundles.

If $A$ or $C$ are zero the restriction of the minimal monad on $Q_5$ must be the minimal monad (3) or the minimal monad (4). We have four possible minimal monads:

$$0 \to \mathcal{O} \to S'_6(1) \to \mathcal{E} \to 0;$$

$$0 \to \mathcal{O} \to S''_6(1) \to \mathcal{E} \to 0;$$

$$0 \to \mathcal{E} \to S'_6(1) \to \mathcal{O}(1) \to 0;$$

and

$$0 \to \mathcal{E} \to S''_6(1) \to \mathcal{O}(1) \to 0.$$ 

These are precisely the sequences (6), (7), (8) and (9).
3. Let us consider a minimal monad for bundle without inner cohomology $\mathcal{E}$ on $\mathcal{Q}_7$:

$$0 \to A \to B \to C \to 0.$$ 

$B$ must be not split and ACM. Since $S_7^6 |_{\mathcal{Q}_5} \cong S'_6 \oplus S''_6$, the restriction of the monad on $\mathcal{Q}_6$ cannot be one of the sequence (6), (7), (8) and (9).

We can conclude that no non-split bundle of rank 3 in $\mathcal{Q}_7$ exists without inner cohomology.

Clearly also in higher dimension it is not possible to find any rank 3 bundle without inner cohomology.

\[\square\]

**Remark 2.4.** On $\mathcal{Q}_n$ ($n > 3$) the only rank 3 bundles without inner cohomology are the following:

1. for $n = 4$, the ACM bundles $S' \oplus \mathcal{O}(a)$ and $S'' \oplus \mathcal{O}(a)$, $G_4$, $P_4$ and $Z_4 \oplus \mathcal{O}(a)$.

2. For $n = 5$, $G_5$, $P_5$ and $Z_5 \oplus \mathcal{O}(a)$.

3. For $n = 6$, $G'_6$, $G''_6$, $P'_6$ and $P''_6$.

**Remark 2.5.** If we consider rank $r$ ($r \geq 4$) without inner cohomology we cannot have such a simple classification on $\mathcal{Q}_n$ ($n \geq 4$).

In fact let $\mathcal{H}$ be any ACM bundle of rank $r$ ($r > 4$) on $\mathcal{Q}_4$. The generic map

$$0 \to \mathcal{O}^{r-4} \xrightarrow{\alpha} \mathcal{H}$$

is injective, so the cokernel of $\alpha$ is a rank 4 bundle without inner cohomology.

This means there are many bundles without inner cohomology of rank $r$ ($r \geq 4$) on $\mathcal{Q}_n$ ($n \geq 4$).

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