TOPICAL REVIEW

Some properties of sandpile models as prototype of self-organized critical systems

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Abstract

This paper is devoted to the recent advances in self-organized criticality (SOC), and the concepts. The paper contains three parts; in the first part we present some examples of SOC systems, in the second part we add some comments concerning its relation to logarithmic conformal field theory, and in the third part we report on the application of SOC concepts to various systems ranging from cumulus clouds to 2D electron gases.

1. Introduction

Back, Tang and Wiesenfeld (BTW) created the notion of self-organized criticality (SOC) in 1987 [1], which stimulated a huge number of works to explore its various aspects. It forms now a large class of critical phenomena. SOC systems show critical properties without tuning of any external parameter, for which the BTW sandpile model was the first prototype. Since the advent of the theory of SOC, many authors have concentrated on it with analytical and numerical tools to explain its various features theoretically. Due to the huge literature during 30 past years, the newcomers in the field may be lost in the specialized papers that have been written in a very technical level, which avoid them to follow the main ideas in the field. This paper has been devoted to help a better understanding of the main ideas and the new progresses in SOC systems. It also contains some new contributions of the authors, especially extending the SOC ideas to cumulus clouds, two-dimensional electron gas (2DEG) and porous media. Based on this, the paper has two objectives: (I) reviewing the analytical and numerical progresses of the SOC systems which is tutorial, and (II) presenting some recent advances in the field.

Dhar was the first to discover the Abelian structure of sandpiles models [2]. This model has a simple dynamic, but because of its attractiveness, a lot of work has been done in this model. [3–11]. Thanks to conformal field theory (CFT), it is known that the BTW model is described by the \( c = -2 \) CFT class, \( c \) being the central charge [4]. The fact that the external frontiers of avalanches in the BTW model are the loop-erased random walks (LERW) [12] can be realized considering the long-standing relationship between CFT and Schramm-Loewner evolution (SLE). More precisely both of them are described by the diffusivity parameter \( \kappa = 2 \) [13, 14].

In this paper, after introducing the original work of BTW, we explore some properties of the model, with an emphasis on the dynamics of the avalanches in sandpiles. As a matter of the first objective (I), first introduce some natural systems that show SOC, and then we discuss about its relation to the logarithmic conformal field theory (LCFT), making it possible to obtain some essential properties of the model. In the last part of the paper we concentrate on the second objective, which is reviewing some new aspects of SOC system mostly observed by the authors. This part has two sub-parts: 1- natural systems, like cumulus clouds and fluid propagation in the porous media, as new systems that show SOC, and 2- the models that examine the SOC dynamics in various systems, like SOC in small world networks, vibrating piles, invasion BTW model, SOC in imperfect supports, and 2DEG. In the latter case the SOC techniques are used to explain the 1/f noise and metal-insulator transition in 2DEG.
This paper is arranged as follows: in the next section, we introduce shortly the SOC in Nature and explain briefly the examples and avalanche dynamics and the basic ingredients. Section 3 is devoted to the definition of the BTW model and other variants. We will present relation to the logarithmic conformal field theory, including the ghost-free fields and W-Algebra in the section 4. Various applications of SOC concepts to natural processes is presented in section 5.

2. SOC in Nature

In nature, there are circumstances that unlike the thermal critical systems, no external parameter needs to be adjusted in order to reach and maintain in criticality, which is called self-organized critical (SOC) systems. Such a system automatically reaches and organizes itself in a critical state [15]. These systems, which are usually open and absorb and dissipate energy, change with time but their general properties are almost unchanged on observed time scales. They need external energy to compensate for the dissipation.

Explaining the '1/\nnoise' phenomena seen often in the nature was the main goal of the BTW model, like rain fall [16], sun flares [17, 18], real piles of rice and other objects [19, 20], earthquake [21–28], forest fire [29], and clouds [30–34]. The aim of this section is to introduce these natural phenomena as a motivation for analyzing the models of SOC.

2.1. Examples

2.1.1. SOC in Earthquake

One of the most popular examples of the SOC systems is an earthquake for which the frequency of earthquakes (N) with energy (E) follows Gutenberg-Richter’s power-law relation as follows [35, 36]

\[ N = aE^{-b} \] (1)

Here, N is the number of earthquakes with the energy of E, the constant number a is the measure of size and the amount of vibrational activity, and b is a critical exponent being often between 0.8 and 1.2 [37]. Also, the number of earthquakes is related to area A as follows

\[ N = cA^{-d} \] (2)

where \( d \approx 2.40 \) [29] is another exponent. SOC in the earthquake is reported in many cases [21]. Many SOC models have been developed to capture the physics of earthquakes, like block-spring models [37, 38], sandpile based models [21, 26, 27], and sandpile on earthquake network [39]. In these models, the dynamics are predicted to be avalanche-like, based on a local stimulation (by increasing the local stress and tension), and the spread of stress throughout the system.

Quenched-disorder based models and dynamical-instability models are two strategies used to describe seismic observations. The former one attributes the seismic activity, and also the power-law behaviors to the disorder of the material and the geometric structure of the Earth [40], whereas the second takes into account the dynamic non-uniform random forces arising from the fault dynamics [25]. Despite of the success of these strategies, still questions raises concerning the type of the seismic activity whether it is based on a single or a combined scheme.

The application of SOC ideas for earthquakes is very efficient and provides realistic results [21, 26, 27]. The Olami-Feder–Christensen earthquake model model [26] is a two-dimensional coupled map lattice model which is known as a simplified version of the Burridge-Knopoff spring-block model [41] for earthquakes. This model is famous and attracted much attention for it serves as a paradigm for nonconservative SOC systems, and also reproducing the most important statistical property of real earthquakes [26, 42], and also Omori’s law [43, 44], and the statistics of foreshocks and aftershocks [45].

One of the most important issues about critical self-organizing phenomena such as earthquakes, etc is the predictability which helps to prevent the possible damages. In a 1994, Pepke and Carlson [46] used the m8-algorithm introduced by Keilis-Borok et al.,[47] to predict large and destructive events by examining previous events based on spatial and temporal distributions. Algorithm m8 is based on the hypothesis that small-scale regional seismicity can be used to predict the large events. There, the main focus was on a set of models that, at that time were famous to have similarities to seismic activity. These (four) models are: Bak, Tang and Wiesenfeld (BTW) sandpile model [48], Olami, Feder and Christensen (OFC) [26], Chen, Bak and obukov (CBO) [49] and Uniform Burridge and Knopoff (UBK) [50].

The UBK equation is a nonlinear wave equation in the form of

\[ \frac{\partial^2 U}{\partial t^2} = \frac{\partial^4 U}{\partial x^4} - U - \phi(U) + \nu t \] (3)
where $U(x, t)$ represents the relative displacement of the opposite sides of a homogeneous fault as a function of position $x$ and time $t$. Here $\nu$ is the slow shear variable that stimulates the relative motion of the plates and instability. The key instability leading to chaotic behavior is a velocity-weakening, stick-slip friction law $\phi(U)$. Unlike this model, all other models ignore the details of the dynamics of inertia and the laws of friction. The BTW model will be described in full below, and the OFC model has already been introduced. The CBO model, like the BTW model, keeps the stress away from the boundary. In addition, although it is not randomly routed, it contains a random element. After the toppling of each site, its threshold stress is reset to a random value chosen uniformly from $[0, 1]$. The correlations observed in the results of the studies show (do not prove) that it is possible to predict real systems. Among the mentioned models, the weakest predictive performance is related to BTW and CBO and the highest level of predictability is related to UBK and OFC. Bak and Paczuski further reported the unpredictability of the BTW model [51], Geller et al. [52] also reached to a same result for the earthquakes. However, efforts to make predictions continued and various algorithms have been introduced and developed. In [53] the authors claimed to predict large avalanches through analyzing the statistics the minimally stable sites (MSS), defined as the sites that become unstable by a single stimulus. Based on their proposed algorithm working with the heights and avalanche masses they showed that large events are predictable before which the cluster of MSS becomes dense. Another try for making prediction is [54] which introduces a prediction algorithm based on maximum likelihood principle prefabricated structures. When the probability measure of the avalanche sizes is Gaussian, the large leading events are shown to be more predictable than the cases where the measure is the underlying probability distribution function is power-law. In [55] the Abelian sandpile model (ASM, for which the distribution of the size of avalanches behaves like $P(s) \propto s^{-\tau}$ in the limit $L \to \infty$, where $-1.293 < \tau < -1.05$ [56–59]) was studied and showed interestingly that the large avalanches repel each other, which helps the prediction of them. More precisely by considering atime series of avalanche size and using a decision variable, the occurrence of particularly large avalanche is predicted. Using the Boolean series ($X_t$) that considers all avalanches as events which exceed a threshold $\eta$ defined as

$$
\begin{cases}
X_t = 1 & s_t > \eta \\
X_t = 0 & \text{else}
\end{cases}
$$

the authors search for the predictibility of the large events, i.e. the events with $X = 1$. To do so, a decision variable $y_i = \sum_{k=1}^n a_k b_{i-k}$ is introduced for each time step (note that $y_{i+1} = a^2 y_i + a s_i$) and $0 < a < 1$, based on which one can show that $y_i$ is independent of $s_i$ for $k \leq i$. The optimum values for $a$ were proposed in [55]. It was shown that the conditional probability $f(y) \equiv P(X = 1|y) \propto \exp(-b(\eta) y)$ where $b(\eta) \approx \frac{1}{\eta_{max}}$ is $\eta$-dependent decay rate where the numerical constant $c$ depends on $L$.

In a recent study, the ideas of SOC models were applied to the Rigan earthquake [39], in which a close relationship was observed between the dynamics of the SOC model and the real data of the earthquake. To understand this relationship, the stress propagation was investigated. The cause of this stress is the tectonic motion of the continental plates. The tectonic motion of the continental plates occurs with a slow process, but the corresponding stress propagation occurs in different rates and with different sizes.

### 2.1.2. SOC in Forest Fire

Forest fire is another natural phenomenon that shows SOC behaviors, like power-law and scaling behaviors [60, 61]. Various surveys of firefighting data in different parts of the United States and Australia have shown a range of the size critical exponents between 1.3 and 1.5, depending on the area [29, 61]. The SOC structure of forest fire was discovered by Malamud et al. [61]. Many models have emerged in order to capture the physics of this phenomenon. Among them, an important one is the Drossel-Schwabl model [62, 63] which in some limit gives acceptable exponents.

One of the models for the forest is Drossel-schwabl model [62–65]. This model is a $d$-dimensional lattice model with a lattice length $L$. The sites of this system can have three states at any time, empty, green, or burnt, at the beginning of which only two states are assumed to be empty or green. The lattice state is updated at any time by the following rules:

1. The burning site will be vacated in the next time,
2. The site where the green tree is located will catch fire at the next time if at least one of its nearest neighbor is burning, otherwise, it fires spontaneously with lightning probability $f$,
3. A tree with probability $p$ grows in an empty cite.

Starting from an initial tree configuration, one The model can become critical only in the limit $p \to 0$ and $f/p \to 0$. In this limit, the length of the correlation is divergent [66]. The latter provides the conditions under which the time scale of tree growth and burning out the forest clusters are well-separated. The scale of the average size of clusters is controlled by the growth rate of $\theta \equiv \frac{p}{f}$. In the simulations, at each time step $\theta$ sites are
randomly chosen for being occupied (if it is already occupied nothing happens, otherwise it turns to occupied). Then a randomly chosen site is ignited so that all sites in the connected cluster to which the start site belongs burn. This model becomes critical in the limit $\theta \to \infty$, although there has been much discussion whether this model is SOC or not [64].

2.1.3. SOC in Sun Flares
After the observation of R. C. Carrington and R. Hodgson in 1859 on the solar flares in white light, much attention has been paid to this problem. The cause of solar activities is the presence of a solar magnetic field in the hot plasma around the outer layer of the Sun or the convective zone created by the collision of particles. The convective zone is the highest inner layer of the Sun that extends from the radiative zone to the surface of the Sun. This area is made up of convective effervescent cells. The magnetic flux forms active regions which include sunspot [18, 67].

Price et al [68] criticized the hypothesis of chaos in solar activity and found no evidence for a low-dimensional deterministic nonlinear process using analysis of the sunspot number time series. Continuing this critical trend, the theory of Self-Organized Criticality (SOC) was proposed as the basic mechanism for explaining solar activity [69, 70]. According to the concept of SOC theory, the solar corona operates in a self-organized critical state, while the solar flares constitute random avalanche events with a power-law profile like the earthquake process. L.P. Karakatsanis and G.P. Pavlos [68] presented new results that strongly support the concept of low dimensional chaos and SOC theory. In this study, they found a co-existence between the self-organized critical state according to the SOC theory and low dimensional chaotic dynamics underlying to the solar activity. These results are obtained by nonlinear analysis of the sunspot index. For the original signal, the largest Lyapunov exponents were found to be zero, while the correlation integral profile was similar to the alternative data slope profile, showing a high-dimensional stochastic process and a critical state based on SOC theory [48].

2.1.4. SOC in Rain-Falls
Rainfall is another natural example for SOC that shows the power-law behavior in the distribution size and durations [16, 71]. In this case, stable stimulation is provided by the heat of the Sun, which causes the oceans to evaporate, for which rain relaxation with the continuous and uninterrupted event, rain. Peters et al reported a scale-free fluctuations of the accumulated water column and also a power-law behavior for the distribution of the size $M$. Two new exponents were estimated to be 1.36 for the distribution size, and 1.42 for the distribution of drought. To understand the other quantity that was shown to be in power-law, let us define the rain-fall rate $q(u) \equiv \sum_{i} n_{i} V_{i} u_{i}$, where $n_{i}$ is the the density number of droplets with a volume of $V_{i}$ that reaches the Earth at the speed of $u_{i}$. Then it was shown that

$$R(\tau) / S(\tau) \sim \tau^H,$$

where

$$R(\tau) \equiv \max_{t \leq \tau} X(t, \tau) - \min_{t \leq \tau} X(t, \tau)$$

and $X(t, \tau) \equiv \sum_{u_{i}} q(u) - (\langle q \rangle)$, and $(\langle q \rangle) \equiv \frac{1}{\tau} \sum_{t=1}^{\tau} q(t) \Delta t$ [16]. Self-organized criticality in rainfalls was observed in many other studies [72–76], which stimulated many theoretical studies on the subject [77–81].

2.1.5. SOC in Clouds
Self-affinity and scaling properties in clouds have been found from satellite images [82], and in particular, in cumulus clouds [83] on several scales. Various observables were shown to exhibit scaling behavior, like the area-perimeter relation [82–88], the nearest neighbor spacing [89], the rainfall time series [90], cloud droplets [91], and the distribution function of geometrical quantities [92–95]. After these observations, and considering the multi-fractality of clouds [82, 83, 86, 96–99], attempts for classifying clouds into universality classes were carried out based on cloud field statistics [100–102] and cloud morphology [103]. The self-organized criticality in the atmosphere was first detected by Peters by analyzing the precipitation [104] and developed further for atmospheric convective organization [105]. The fractal dimension of the perimeter of self-organized vortices shown to be near $\frac{1}{3}$ using the quasi-geostrophic vorticity equation. The area perimeter relation with the $D = 1.37 \pm 0.02$ of cirrus, and $D = 1.18 \pm 0.05$ for cumulonimbus tropical clouds. These fractal dimensions are in agreement with the relative turbulent diffusion model, predicting 1.35, which is also confirmed by means of some other observations.

As a main building block of the atmosphere dynamics, turbulence seems to be essential in the dynamics and formation of clouds. Analysis of the images from the fair weather cumulus clouds reveals that they additionally exhibit self-organized criticality degrees of freedom, leading us to use the term SOC turbulent state. Observations (in our submitted paper) support the fact that this system, when projected to 2D, demonstrates
conformal symmetry compatible with $c = -2$ conformal field theory, in contrast to 2D turbulence which is $c = 0$ conformal field theory. Using a mix of turbulence and cellular automata, namely, the coupled map lattice model [106, 107], one obtains the same exponents as the observations. Also, in a separate (unpublished yet) work we developed a 2D monte Carlo based stochastic model including the competition between avalanche dynamics and cohesive energy between water droplets that generates the same properties. The fractal geometry of clouds was seen in many real observations, like the multi-fractality structure of clouds, universality classes of cloud fields, analysis rainfall time series, nearest-neighbor spacing statistics, cumulus cloud morphology, "variable" and "steady" cloudy regions for warm continental cumulus cloud, fractal analysis of high-resolution cloud droplet measurements, the fractal dimension of noctilucent clouds, the fractal dimension of convective clouds around Delhi, scaling properties of clouds, self-similarity of clouds in the intertropical convergence zone, depending on the equivalent black body temperature.

2.1.6. SOC in real piles

SOC has been observed in labs for real piles, like pile of beads [108], rice pile [109–113], and other granular piles with various aspect ratio [114]. It has been observed that for all of the cases, a critical angle $\theta_c$ (called the threshold angle) plays a dominants role [4], which cause a separation of time scales. This angel depends on the structural details of the constituent grains. A sandpile with local slopes less than $\theta_c$ anywhere is stable, and adding a small amount of sand causes a small reaction, but if this operation results in a slope larger than $\theta_c$, then an avalanche is formed, which sometimes is as large as the system in size. For a pile with an average slope a less than $\theta_c$, the avalanches sizes are medium, and rarely large, whereas when the it approaches $\theta_c$ from bellow, then the avalanches become large in size and duration and sometimes a catastrophic avalanche takes place that affects the entire system. Bak, Tang and Wiesenfeld (BTW) [48] were the first to provide an example of these systems, by introducing an automatic cellular model which shows critical properties without tuning of any parameter. The argument presented by BTW, if applied to a growing conical pile (adding sand grains is interpreted as local stimulations) is interpreted in spirit as the fact that when the average slope of the pile reaches to a critical slope ($\theta_c$) then the network of minimally stable sites network percolates causing the system to show critical properties, called the critical state which is also characterized by a steady-state, where the average amount of input and output energy are equal. Sands are added to the system at a slow rate but leaves the system in a completely irregular manner over long periods at unpredictable intervals, which is a charactercistic of the SOC steady state. For example, some power-law behaviors are found for various observables in the system, and also the correlation length (which is estimated as the point where the green function falls off rapidly) becomes of the order of system size (infinite in the thermodynamic limit). The dynamic of the BTW model which is defined on the finite $d$-dimensional lattice is based on avalanches that arise as a result of local topplings of sand columns (see following sections). The separation of time scales is achieved by considering a threshold above which the column becomes unstable and topples. This system needs some time to pass from transient regime to the recurrent regime which is characterized by steady state properties.

The rice pile is another example of natural system showing self-organized criticality. Two pieces of glass with a specific diameters are put..., and the rice grains are poured on top of them in an spatial interval $L$, unit the system becomes stable. Then one pours colored grains in a similar way and follow their movement so that their traces and also their exit time $T$ can be recorded. The following distribution function was then proposed

$$ P(T, L) = L^{-\beta} F\left(\frac{T}{L} \right) $$

where $\nu$ and $\beta$ are the critical exponents.

There are other examples of SOC systems, like river basins [115, 116], in air pollution [117], in climate change [118], in brain plastisity [119], in stock markets [120, 121], in magnetosphere [122], in midlatitude geomagnetic activity [123], in magnetohydrodynamics [124], in Kardar-Parizi-Zhang growth model [125] in Bean state in YBCO thin films [126], in granular systems [127], and in much more systems [128] which are out of scope of this paper.

2.2. Avalanche Dynamics and the Basic Ingredients

It is a common belief that avalanche dynamics are the underlying mechanism that is responsible for SOC behaviors. Although normal diffusive transport and branching are believed to be very basic ingredients of SOC, it was shown in [129] some new evidences suggesting that the avalanches in the SOC can be referred to as linear random branchless walks. The scaling behaviors in this case is different from the branch avalanche. The sandpiles model introduced by BTW [1] is an example of this system, including avalanche-based dynamics in which the system is slowly stimulated, i.e. subjected to small external perturbations. Large events in these systems, which are the result of these small stimuli, occur less frequently and on a larger scale, i.e. the energy is gradually absorbed and is excreted out on a larger scale. An interesting feature in these systems is that in the
steady-state, where the average amount of energy input and output is equal, the system exhibits critical properties, which is a SOC state. Dhar discovered the Abelian structure of the sandpile model, which was named the Abelian sandpile model (ASM) [25], which has been the subject of much work due to its attractiveness. One can mention the following for instance: the different height and cluster probabilities [40], the and avalanche distribution [130], and also its the connection to the other models like the spanning trees [131], the ghost model [132, 133], and the q-state Potts model [134]. For more information, one can see [4, 135]. A logarithmic conformal field theory with central charge \( c = -2 \) [8, 132, 133] case describe this model, as well as the Schramm–Loewner evolution (SLE) with the diffusivity parameter \( \kappa = 2 \) [133].

3. The BTW model

We consider the BTW model on a d-dimensional square lattice with length \( L \) and neighborhood number \( z = 2d \), which for each site, we assign a height variable \( z_i \) that can have values in the range \([1, z]\). This model is defined by local topplings as a consequence of adding sand grains in a randomly selected site \( i \), i.e. \( z_i \rightarrow z_i + 1 \). More precisely, if this operation causes instability of the site \( z_j > z_{th} \equiv z \), the site loses 2d grains of sand and transfers one sand to each of its neighbors, and as a result of this transfer, neighboring sites may become unstable, so that they will continue to topple until no unstable site remains in the system. The chain of topplings that occurs is in each relaxation is called an avalanche. During this process, in boundary sites one or two sand grains (depending on the position of the boundary site) are removed from the system. After reaching a stable configuration (the avalanche is finished), the process is repeated starting from another random site for grain injection. The toppling rule for the site \( i \) can also be written in the form \( z_j \rightarrow z_j + \Delta_{ij} \), where

\[
\Delta_{ij} = \begin{cases} 
-z & \text{if } j = i \\
+1 & \text{if } j \text{ and } i \text{ are neighbors} \\
0 & \text{otherwise}
\end{cases}
\] (8)

which is discrete Laplacian operator.

3.1. Transient v.s. Recurrent Configurations

Let us suppose that we start from a random height configuration. Then during the system evolution, many configurations come about, some of which are transient, meaning that they do not occur again, and some of which are recurrent. In fact, the primitive configurations are transient, during which the average height grows with time (let us define the time as the number of injections). This linear growth cannot definitely last infinitely, and the system saturates at some stage (becomes stationary), after which the average height becomes nearly constant, meaning that on average energy input and output are the same. Recurrent states live in this regime. The total number of recurrent configurations is \( \det \Delta \), see SEC. 3.2.4. One important question is how we can identify a configuration to be recurrent or transient. Fortunately, there are tests that do this for us. One of our tests is the lack of forbidden subconfigurations (FSC) which does not exist in a recurrent configuration. FSCs can be identified simply by the requirement that they can never be created by the addition of sand and relaxation, if not already present in the initial state [4]. We introduce two following tests for checking transient or recurrent configurations:

A: We use the instability test of boundary site for the given configuration as follows: we add sand to any boundary site of configuration and we allow the system to evolve until achieving stable configuration. If the new configuration is the same as the first configuration, then the original configuration has been recurrent.

B: The second test is called the burning algorithm. This test is a dynamic one in which we burns sites one by one. Let us characterize this dynamical algorithm with a ‘time’ \( t \), although the order of burnings doesn’t matter. Firstly at \( t = 0 \), we assume that all the sites are un-burned. In the next step \( (t = 1) \), we burn all the sites whose heights are higher than the number of their un-burnt neighbors. We go on to burn the sites until the time in which no candidate for burning remains. If all the sites get burnt at the end of the burning process, the desired configuration is recurrent [2, 4].

3.2. Other Sandpiles

3.2.1. Manna Model

The BTW model is a representative member of the BTW universality class. A relevant question is how one can change the details of this model to change its universality class. Stochasticity is one candidate to do this, which was tested for the first time by Manna by introducing a two-state SOC system, known also as the Manna model [136]. It includes randomness in the toppling rule, i.e. in a two-dimensional system if a site has more than one sand, it is unstable and topples. During a toppling, a direction is chosen randomly (each direction is chosen with the probability of 0.5) and all of the grains are distributed in this direction (no sand grain is transferred to the
other direction). This model is interpreted as a realization of a system with particles experiencing a local infinite repulsive force between each other.

Much attention has been paid to identify whether the Manna model belongs to the BTW universality class or not [57, 58, 137–143], which is still open. The most challenging problem to this end is the accurate determination of exponents (controlling the finite-size effects, controlling the noise, etc.).

3.2.2. Zhang Model
The Zhang model [144, 145] indicates the continuous state of the BTW model, and the height of each site is considered to be a real number. The dynamics governing this model is such that at any moment, a continuous and random value between zero and one is added to the site. If its height exceeds a critical value, then that site is unstable, and its value is distributed to nearby sites, and its grain content becomes zero. This continues until all sites become stable. This model does not have an Abelian property, because the amount transferred in each toppling to neighboring sites depends on the initial value [4].

3.2.3. Oriented/Directional Abelian Sandpile Model
If we define an ASM on a directional lattice (e.g. the tilted square lattice), then we have an oriented/directional ASM. In this model, the movement of the sand is in a certain direction and the threshold height is commonly considered to be unity. Consider as an example a tilted site, where any site has two equivalent neighbors in its bottom. Let us call the time direction as the top to bottom direction, and the space direction as the left to right direction, and suppose that the sand grains move only towards the bottom, i.e. the time direction. By adding one sand grain in a random site, after which the number of sand grains reaches to two (becomes unstable), then a sand grain moves randomly to one of its bottom sites [4]. The exponents here depend on the direction of the propagation, i.e. the time or the space direction. There are two general categories, i.e. stochastic and Abelian versions in this class with different exponents [146]. The fact that the exponents are different, shows that the stochasticity plays a dominant role in this case. It is different from the case for the BTW and the Manna models, where the distinction of their universality classes is under debate.

3.2.4. The General Undirected Abelian Sandpile Model: Relation to Spanning Trees
In sandpiles, the Abelian property is that the order of topplings in an avalanche does not matter. This means that if one topples first the ith site and then jth site, or interchanges their order nothing changes in the final configuration. Here we describe a general set up for ASM. In this general case, a graph with \(N\) sites is considered each of which has its own height \(z_i\) which is a positive integer lower than \(z^*\) as the threshold height. At each time step, we select a random site and add a grain to it. Then the \(N \times N\) matrix \(\Delta\) gives us the toppling rule, for which the thresholds of toppling are different for each site, i.e. \(\{z_i,\Delta_{ij}\}_{i,j=1}^{N}\). A site \(i\) whose height is greater than the threshold topples, causing the heights are updated according to \(z_i \rightarrow z_i - \Delta_{ij}\) for any \(j\). We often consider \(z_i^* = \Delta_{ii}\), although it is not necessary. The matrix \(\Delta\) must meet the following conditions [4]:

1. for every \(i\), \(\Delta_{ii} > 0\).
2. For every pair \(i = j\), \(\Delta_{ij} \leq 0\).
3. for every \(j\), \(\sum_i \Delta_{ij} \geq 0\).
4. There is at least one site \(i\), which is \(\sum_j \Delta_{ij} > 0\) called dissipative sites.

Consider a general ASM on a graph with \(N\) sites with thresholds \(\{z_i^*\}_{i=1}^{N}\), the dynamics of which is defined in SEC. 3.2.4 with the toppling matrix \(\Delta\). To investigate its correspondence to spanning trees, we should represent the toppling rules by a graph, say \(G\) with \(N + 1\) nodes, the extra node is labeled as 0. We let the number of links between \(i\) and \(j\) (both being non-zero) equals \(|\Delta_{ij}|\), and the number of links between any \(i = 0\) site to the node 0 is \(w_i \equiv \sum_j \Delta_{ij}\). Note that for bulk sites \(w_i = 0\), and for the boundary sites \(w_i\) is the number of wasted sand grains. In fact the 0 node represents the ‘waste’ or sink site into which the sand grains waste. The burning algorithm is just the same as the method that explained above, this time on the graph \(G\). In the dynamical process of burning, let label the time at which ith site is burnable by \(\tau_i\). Then for any burnable site \(i\), at time \(\tau_i\) we have \(z_i > \zeta_i\), where \(\zeta_i = \sum_j (\tau_j - \Delta_{ij})\) and primed summation is over all unburnt neighbors \(j\) of \(i\) at time \(\tau_j\), and also \(z_i < \zeta_i\) for \(t = \tau_j - 1\). Let us show the number of burnt neighbors of \(i\) at \(t = \tau_i\) by \(r_i\). Obviously at time \(t = \tau_j - 1\), \(z_i \leq \zeta_i + K\), where \(K\) is the number of distinct links connecting i to its r_i unburnt sites at time \(\tau_j\). One says that the fire reaches the site \(i\) by one of these \(K\) bonds, for which there is one possibility for \(K = 1\). For \(K > 1\) one selects the link by which the fire reaches \(i\) by ordering them in some preference which is not important in the final result. It is then shown that all sites in the lattice are connected to the sink site 0, and there is no loop in the resulting graph. The set of these edges (links) forms a spanning tree on the graph \(G\).
Having found the corresponding tree on the graph, one can determine the number of the recurrent configurations by counting the number of spanning trees in a general graph which is given in terms of the minors of the incident matrix of the graph. This matrix $\Delta_{\text{INC}}$ is obtained by adding one row and column to $\Delta$ so that $\sum_i \Delta_i \theta = 0$ and $\Delta_{\text{INC}} \equiv \Delta^\mu \Delta_i$. The resulting number of recurrent configurations is then shown to be $\det \Delta$.

### 4. Relation to the Logarithmic conformal field theory

In a famous and important study, Majumdar and Dhar showed that undirected ASM for any arbitrary finite graph is equivalent to $c = -2$ conformal field theory (CFT) in the scaling limit [147]. For this, they used the $q$-state Potts model in the limit $q \to 0$, which is equivalent spanning trees, and $c = -2$ CFT [148]. Their starting point is to note that the spanning trees is equivalent to $q \to 0$ Potts model, which is constructed using the graph representation of it. Let us consider the following Hamiltonian which contains Potts spins $\sigma_i = 1, 2, \ldots, q$:

$$H = q^2 K \sum_j \Delta_i \delta(\sigma_i, \sigma_j)$$

(9)

for fixed $K$. It results to the following partition function in $q \to 0$ limit

$$Z = q^{2 + M} \det \Delta + \text{higher orders in } q$$

(10)

where the higher orders get away taking the limit $q \to 0$. For the BTW model in the SOC state in an $L \times M$ square lattice, $\Delta_{\text{INC}}$ is 4 for $r = r', -1$ when $r$ and $r'$ are nearest neighbors, and zero otherwise. $\Delta$ is in fact the toppling matrix, see Sec. 3.2.4. For periodic boundary conditions in say $y$ direction (with length $M$) they found that

$$\det \Delta = \prod_{l=0}^{M-1} \left( \lambda_{2}^{l+1} (2\pi l / M) - \lambda_{-3}^{l+1} (2\pi l / M) \right)$$

(11)

where $\lambda_{2}(x) = 2 - \cos x \pm \sqrt{(1 - \cos x)(3 - \cos x)}$. One can compare this result with the results of the transfer matrix technique, for which a system of length $L$ is transferred along the $y$ direction. Showing the logarithm of the corresponding eigenvalues with $\mu_n$, $n = 1, 2, \ldots, 2^M$, the resulting partition function is of the form

$$Z = \sum_n c_n \exp(\mu_n L),$$

(12)

where $c_n$ are constants that have to do with the boundary conditions. To make connection to CFTs, we use a simple trick: we calculate the largest eigenvalue of the transfer matrix whose logarithm is related to the leading term of the free energy of the system. For this note that the free energy is proportional to the logarithm of the partition function. To calculate the spectrum of the transfer matrix, we compare equation (11) with equation (12), and write $2^M$ eigenvalues of the transfer matrix for the ASM problem,

$$\mu_n = \sum_{l=0}^{M-1} \epsilon_l \ln \left( \lambda_{2}^{l+1} (2\pi l / M) \right), \quad \epsilon_l = \pm 1$$

(13)

where $n = (\epsilon_1, \epsilon_2, \ldots, \epsilon_M)$ shows a configuration of $\epsilon_i$’s, from which it is obvious that we have $2^M$ eigenvalues corresponding to various choices of $\epsilon_i$’s. Let us define $\mu_{\text{max}}$ as the state corresponding to all $\epsilon_i$’s equal to $+1$, which is the largest eigenvalue. $\mu_{\text{max}}$ is estimated to be the leading free energy in the system. For large $M$ values (thermodynamic limit) we find that

$$\mu_{\text{max}} = \alpha M - \frac{2\pi}{6M} + O(1 / M^2)$$

(14)

where $\alpha = \int_{0}^{2\pi} \frac{dk}{2\pi} \ln|\lambda_{+}(k)|$. Using the fact that in a CFT (finite size scaling theory in critical systems) [149], the corrections to the bulk free energy should vary as $\sigma c / 6M$, $c$ being the central charge of CFT. Majumdar and Dhar found that ASM corresponds to a CFT with $c = -2$, which is shown to be a logarithmic CFT. Especially the energy-energy correlation in $q$-state Potts model decays with distance $r$ like $r^{-2+2\epsilon}$, where

$$x_{T} = \frac{1 + y}{2 - y}$$

(15)

where $y = \frac{1}{2} \cos^{-1}\left(\frac{1}{\sqrt{q}}\right)$. For $q \to 0$ we have $x_T = 2$, which is a known result for height-height correlation is sandpiles [147].

There is another more straightforward way to show that this result is consistent. Using the number of recurrent states in sandpiles (which is $\det \Delta$ as stated above), a connection is established with the free ghost field. From the properties of the Grassmann algebra, one can easily shown that $\det \Delta = \int_{0}^{1} \frac{d\theta}{2\pi} \exp \left[ \int d^2 z \partial \bar{\theta}(z) \bar{\theta}(z) \right]$, where $\theta$ and $\bar{\theta}$ are independent Grassmann variables, and $\partial \equiv \frac{\partial}{\partial \theta}$ and $\bar{\partial} \equiv \frac{\partial}{\partial \bar{\theta}}$. For the convenience of readers, a brief explanation of Grassmann algebra is provided in the appendix. Therefore the connection to the free ghost field is established defined by the following action
\[
S = \int d^2 z \partial \theta (z) \bar{\partial} \bar{\theta} (\bar{z}) = \frac{1}{2\pi} \int \varepsilon_{\alpha\beta} \partial \theta^\alpha \bar{\partial} \bar{\theta}^\beta, \tag{16}
\]
where the pair of free grassmanian scalar fields are defined as \( \theta^\alpha = (\theta, \bar{\theta}) \), and \( \varepsilon_{\alpha\beta} \) is the canonical symplectic form, \( \varepsilon_{12} = +1 \), and \( \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \). Grassman fields and their derivatives can be used to express other correlations and probabilities of sandpiles [3].

4.1. Ghost Free Fields and W-Algebra
In this section, we turn to the continuum limit of the sandpile model, which is a nonunitary field theory. The existence of ghost fields in a field theory means the existence of a state with a negative amplitudes, or in the other words, nonunitarity of the model. It is shown in [147] that it is a \( c = -2 \) conformal field theory (CFT) which is logarithmic. The term logarithmic is used here due to appearing logarithmic correlations in CFT, i.e. LCFT. In these theories, each primary filed has a logarithmic partner, which are identified using the operator product expansions (OPE) [151]. The OPEs of logarithmic CFTs are anomalous in the sense that a non-diagonalizable Jordan form is seen, i.e. in the OPEs with the energy momentum the primary field is present along with its logarithmic partner, see below. It can be shown that this is equivalent to have conformal dimension with nilpotent numbers [152, 153]. In this way, the techniques in ordinary CFTs can be easily used for LCFTs, but this time with nilpotent exponents. In ordinary CFT, the OPE of the energy-momentum tensor and a primary field is

\[
\text{the following OPE with the energy-momentum tensor}
\]

where the pair of free grassmanian scalar fields are defined as \( \theta^\alpha = (\theta, \bar{\theta}) \), and \( \varepsilon_{\alpha\beta} \) is the canonical symplectic form, \( \varepsilon_{12} = +1 \), and \( \varepsilon_{\alpha\beta} = -\varepsilon_{\beta\alpha} \). Grassman fields and their derivatives can be used to express other correlations and probabilities of sandpiles [3].
coordinate \( z \)
\[
\theta^a(z) = \sum_{n=0} \theta^a_n z^{-n} + \theta^a_0 \log(z) + \xi^a,
\]
(24)

where \( \theta_n \) and \( \bar{\theta}_n \) are modes. The absence of zero-mode \( \xi \) and \( \bar{\xi} \) in the action (equation (16)) leads to vanishing of the expectation value of unity, i.e. \( \int_{(0, \theta)} e^z = 0 \) for both twisted and untwisted sectors. In the above equation, \( n \) is an integer number for the untwisted sector and it is a half-integer number for the twisted sector. It is appropriate to insert the zero modes \( \bar{\xi} \) and \( \bar{x} \) in the expectations in order to avoid vanishing the correlation functions involving \( \bar{\theta} \) fields. For example, the two-point correlation function is
\[
\langle \theta^a(z) \bar{\theta}^\beta(\omega) \bar{\xi}^\alpha \rangle = e^{\alpha \beta} \log |z - \omega|
\]
(25)
The \( \theta \) fields are not primary, but their derivative \( \partial \theta \) and \( \bar{\partial} \bar{\theta} \) are:
\[
\langle \partial \theta^a(z) \bar{\partial} \bar{\theta}^\beta(\omega) \rangle = e^{a \beta} \frac{1}{2(z - \omega)^2}.
\]
(26)
The energy-momentum tensor is \( T(z) = 2: \partial \theta \bar{\partial} \bar{\theta}: \) resulting to the following OPE with the central charge \( c = -2 \):
\[
T(z)T(\omega) = \frac{-1}{(z - \omega)^4} + \frac{2 T(\omega)}{(z - \omega)^2} + \frac{\partial T(\omega)}{z - \omega} + \ldots
\]
(27)

One can also find the same central charge using the explicit form of Virasoro operators \( L_n \) which is an expansion in terms of the modes
\[
L_n = 2 \sum_m a_m a_{n-m};
\]
(28)

where
\[
a_n = \begin{cases} n \theta_n & n \neq 0 \\ -\bar{\theta}_0 & n = 0 \end{cases}
\]
(29)

and \( : \) means the normal ordering. One can easily check that \( T(z) = \sum_n z^{-n-2} L_n \). One may try to build the Fock space using the modes and a vacuum state. One should however take into account that in this case, the Verma module is staggered, meaning that a descendant of the primary field may be itself a primary field having its own Verma module [155]. There are three important representations \( (\bar{R}_0, R_0, R) \) in \( c = -2 \) models, the two representations \( R_0, R_1 \) are the highest weight representations (explored in the following) but \( R \) is local representation whose amplitudes are local. The \( R_0 \) representation contains the identity operator \( I \) and the field \( \bar{I} \equiv -2: \theta \bar{\theta}: \) (with zero conformal dimension) whose OPE with \( T \) is shown to be
\[
T(z)I(\omega) = \frac{I}{(z - \omega)^2} + \frac{\bar{I}}{z - \omega} + \ldots
\]
(30)

showing that \( \bar{I} \) is the logarithmic partner of \( I \). The \( R_1 \) representation includes \( \phi^\alpha = \partial \theta^\alpha \) and \( \psi^\alpha = :\partial \theta^\alpha \bar{I}: \) carrying the conformal weight \( (1, 0) \) and the OPE
\[
T(z)\phi^\alpha(\omega) = \frac{\theta^\alpha(\omega)}{2(z - \omega)^3} + \frac{\psi^\alpha(\omega) + \psi^\alpha(\omega)}{(z - \omega)^2}
\]
(31)

which is anomalous OPE, having an extra singular term (the first term in the right-hand side). This makes \( R_1 \) representation more complex than \( R_0 \), and its operator content is larger. In fact \( W \)-algebras come about in this representation which works with operators at level three [156]
\[
W^+ = \partial^2 \theta \partial \bar{\theta}
\]
\[
W^0 = \frac{1}{2} (\partial \theta \bar{\partial} \bar{\theta} + \partial \bar{\theta} \partial \theta)
\]
\[
W^- = \partial \bar{\theta} \partial \theta.
\]
(32)

Note that this set is isospin one with spins \(+1, 0\) and \(-1\) for first, second and third fields respectively (note that conformal weights are \((1, 0), (1, 1)\) and \(0, 1\) respectively) [157]. Their OPEs are
Where \( g^{ij} \) is the metric on the isospin one representation, \( g^+ = g^- = 2 \) and \( g^{00} = -1 \), and \( f_k^{ij} \) are the structure constants of \( SL(2) \). One can write the \( W \) algebra, by using the above OPEs. Following Gaberdiel and Kausch [156], we have:

\[
\begin{align*}
[L_m, W_n^i] &= (2m - n) W_{m+n}^i \\
[W_m^i, W_n^j] &= g^{ij} (2m - n) \Lambda_{m+n} \\
&+ \frac{1}{20} (m - n)(2m^2 + 2n^2 - mn - 8) L_{m+n} \\
&- \frac{1}{120} m(m^2 - 1)(m^2 - 4) \delta_{m+n} \\
&+ f_k^{ij} \left( \frac{5}{14} (2m^2 + 2n^2 - 3mn - 4) W_{m+n}^k + \frac{12}{5} V_{m+n}^k \right),
\end{align*}
\]

(34)

Where \( \Lambda =: T^a = -\frac{1}{10} \partial^2 T \) and \( V^n =: TW^n = -\frac{1}{14} \partial^2 W^n \) are quasiprimary normal ordered fields. This \( W \) algebra is different from Zamolodchikov’s \( W \) algebra [158], because \( f_k^{ij} \) is different. By using the above \( W \) algebra and Gaberdiel and Kausch [156] found the null vectors, from which, using the zero mode the following equation for highest weight field \( \phi \) were found [156, 159]:

\[
L_0^2 (8L_0 + 1)(8L_0 - 3)(L_0 - 1) \phi = 0.
\]

(35)

implying that \( h \) must be from the set \( \{0, -\frac{1}{8}, \frac{1}{8}, 1\} \) which are represented by \( V_h \). For \( V_0 \) we have \( L_0^2 \phi = 0 \) which is satisfied for \( I \) and \( \bar{I} \) that is the only logarithmic highest weight representation of \( c = -2 \), and the other logarithmic representations are not highest weight (see \( R_1 \) representation for example). Two of the three remaining highest weight representations are related to the twisted sector and the other is nontwisted. For a more complete reference see [160].

### 5. Application of Sandpiles to Natural Processes

In this section, we turn to the application of SOC concepts to real systems. All subjects considered here are simulations (except a part for SOC in clouds). Most parts of this section are carried out by the authors of this paper.
5.1. SOC in Fluid Propagation in Porous Media

The fluid propagation in porous media (involving avalanche-type dynamics) is a complex procedure, resulting in various interesting patterns. The avalanches arise from a non-linearity in the laws governing the dynamics of fluid in this system, namely the critical fluid saturation (CFS), shown by $S_c$. Above $S_c$, the fluid freely overflows into neighboring areas under a pressure gradient. More precisely, the fluid does not have a macroscopic transfer to neighboring areas, and the fluid is static in a specific area (consisting of many pores) of the reservoir until the liquid saturation reaches $S_{CFS}$, after which the small droplets of fluid collect in the pores and the liquid is able to move to neighboring areas. In terms of the porous media parameters, one common choice for the relation between the relative permeability of phase $\alpha$ (shown by $k_{\alpha\text{res}}$) and the saturation of that phase ($S_\alpha$) is

$$ k_{\alpha\text{res}} = \begin{cases} S_\alpha - S_C & S_\alpha \geq S_C \\ 0 & S_\alpha < S_C \end{cases} $$

which shows the above-mentioned dynamics. In [134], it was shown that the set of Darcy equations (known as the reservoir flow or RF model) for two-phase propagation in porous media is very similar to the BTW dynamics, except that the former is directional (the fluid moves in the direction of the pressure gradient), whereas the latter is not. In this work, the ordinary BTW model has been used defined on a percolation lattice which realizes a porous media (uncorrelated) in which some points are occupied with a probability of $p$ and the others are unoccupied (with the probability $1-p$). Fig.1 shows a lattice site with three occupied neighbors and one unoccupied neighbor and shows the toppling rule. To find the relationship between this model and the Darcy model, the Schramm-Loewner evolution theory (SLE) was used. To investigate the behavior of this model with respect to $p$, the fractal dimension was also used as a standard statistical analysis in their domain walls. The results of the SLE theory (figure 2, in which where $\xi$ is a driving function with a continuous real value) shows at $p = p_c$, that the two critical models have the same diffusivity parameter and are believed to be in the same universality class. Note that $\xi$ is proportional to a one-dimensional Brownian motion for the conformal invariant curves ($\xi = \sqrt{\kappa} B_\kappa$, $\kappa$ being the diffusivity parameter) [161]. The two models are compatible with the Ising universality class at $p = p_c$.

In the natural systems, the permeable pores of the porous medium are not completely independent, and hence the correlation of vacant spaces depends on the dynamics of the assumed process. In [162], the zero Ising magnetic field is considered as the source for the correlations of the occupied regions. In this study the artificial temperature $T$ controls the strength of the correlations. In the critical temperature $T_c$, the defined model is numerically shown to be in the 2D Self-avoiding walk (ASW) universal class. From a theoretical point of view, the mixing of two conformal symmetric models, which can be examined within the Zamolodchikov $\gamma$-theorem, is very attractive.

In another work [163] Najafi et al. concentrated on the SOC dynamics on 3D correlated porous media. It was found evidence for a new nonequilibrium universality class that is reached by changing the geometry of the underlying graph upon which the model is defined. This might be applicable to experiments with spatial flow patterns of transport in heterogeneous porous media [164].

5.2. SOC in Cumulus Clouds

As stated in the previous sections, there are many papers reporting on the fractal structure of the clouds, some of which are based on SOC. Convective clouds (such as cumulus clouds as a member of the cumuliform clouds)
When the relative humidity reaches 100%, then condensation to the wet phase decreases and humidity rises. At a threshold, named as lowest condensation level, the atmosphere cannot precipitate. The height of the cloud and temperature gradient. When they grow into the congestus or cumulonimbus clouds, they are more probable of other types of clouds, such as cumulonimbus when in fair air conditions. These clouds which may appear in lines or in clusters, in low-level clouds, less than 2 km in altitude unless they are more vertical (cumulus congestus form). Cumulonimbus clouds form via atmospheric convection as air warmed by the surface begins to rise, resulting in the temperature decrease and humidity rise. At a threshold, named as lowest condensation level (LCL), in which the relative humidity reaches 100%, then condensation to the wet phase (known as wet-adiabatic phase) starts. The released latent heat (due to condensation) warms up the air parcel, resulting to further convection. At LCL, the nucleation process starts on various nuclei present in the air. The process of formation of raindrops and rainfall has been explained successfully by Langmuir [130].

Although the liquid water density within a cumulus cloud changes with height above the cloud base [135] (for the non-precipitating clouds the concentration of droplets ranges from 23 to 1300 droplets per cubic centimeter [165]), the density can be thought of as being approximately constant throughout the cloud. The height of the cumulus clouds depends on the amount of moisture in the thermal that forms the cloud, and humid air will generally result in a lower cloud base. In stable air conditions in which their vertical growth is not high, they are considered to be effectively two-dimensional.

In places, cumulus clouds can have holes where there are no water droplets [165]. This fact causes to create the fractal structures that it provides powerful tools to classify them in terms of the circumstances in which they form. The fractal structure of clouds has been reported in some previous works [166–168].

The self-organized criticality in the atmosphere and clouds was first detected by Peters et al by analyzing the precipitation [104]. They used satellite data and define a critical value of water vapor as a tuning parameter, and precipitation as the order parameter shows a non-equilibrium continuous phase transition to a regime of strong atmospheric convection and precipitation.

In an unpolished work, we uncovered the SOC state of clouds directly by analyzing some earth to sky images of cumulus clouds under fair air conditions. The analysis of the level lines of two-dimensional cloud images shows that the clouds are very close to the loop-erased random walkers (LERW) traces with $D_l^e = 5/4$.

**Figure 3.** SEC. 5.2: (a) The log–log plot of trace lengths $l$ in terms of $L$ (the box linear size). The dashed line is a linear fit with slope $D_l^b = 1.248 \pm 0.006$. Upper inset is the end–to–end distance $R$ in terms of $N$, and the lower inset is a semi-log plot of the loop green function in terms of $r$, with the exponent $\nu = 0.81 \pm 0.01$. (b) Ensemble average of $\log l$ in terms of $\log r$ ($\langle \rangle$ means the ensemble average) with slope $D_l^g = 1.22 \pm 0.02$. The log–log plot of the distribution function of $r$ and $l$ are shown in the upper and lower insets, with exponents $r_1 = 2.12 \pm 0.03$ and $r_2 = 2.38 \pm 0.02$ respectively.
5.3. Vibrating Piles

A very important question concerning the SOC model is its stability against external manipulations, like vibrations. As stated above, some experiments have also been done to test the critical properties of real piles in the presence of external vibrations [169] which were modeled and simulated [170]. Recently we have simulated the BTW model under vibration conditions, affecting the toppling rules in one direction, namely the x-direction. To this end, we manipulated the toppling rules, which depend on time. The toppling matrix was considered to be

$$\Delta_{i,j,i',j'}(\epsilon_0 \sin \omega t) = \begin{cases} 
4n & i = i', j = j' \\
-n & i = i', j = j' \pm 1 \\
-n & i = i' + 1, j = j' \\
-n & i = i' - 1, j = j' \\
0 & \text{other}
\end{cases}$$  \hspace{1cm} (37)

where $\omega = 2\pi / T$ is the angular frequency, $T$ is the time period, and $\epsilon_0$ is the vibration strength parameter. This toppling rule states that the system is vibrating in the x-direction making the model anisotropic. The properties of the model were investigated in terms of $T$ and $\epsilon_0$. We uncovered that the exponents run with $\omega$ and $\epsilon_0$. Importantly increasing the strength of vibrations makes the avalanches more smooth with exponents that depend on the (time and space) directions.

5.4. Invasion Sandpile Model

In the previous section, we provided evidence that the Darcy model has similarities with the BTW model at the critical occupation. Invasion percolation (IP) is another natural choice, in which one phase invades the other phase towards the production well. In fact, IP [171] is a standard two-phase model including two immiscible wet and non-wet phase in porous medium [172, 173] where the wet phase invades the non-wet phase and liquids advance through the cavity with the lowest threshold. Despite of its huge applications and studies, this model is lacking the concept of critical saturation defined in the previous sections. To do this, one considers a two-dimensional $L \times L$ square lattice and assign two random integer numbers $h_r$ and $h_b$ to each lattice site, which represents the number of red and blue sands in the site and represents the two wet and non-wet phases respectively. The random numbers are uniformly selected from the set $\{1, 2, 3, \ldots, h_{th}\}$. The height threshold $h_{th}$ is considered to be 20 (note that its exact value does not change the results).

In this model, three conditions are considered for the stability of a column of sand grains. A site $i$ is unstable if at least one of the following conditions is violated, and the site is considered to be stable if all conditions are met:

A: $h_r(i) \leq h_{th}$,
B: $h_b(i) \leq h_{th}$,
C: $h_r(i) + h_b(i) \leq H_0$

In the above terms, $h_{th}$ represents CFS. The third condition expresses the fact that in the set of Darcy equations, an auxiliary equation is considered that the sum of two-phase saturations $S_w + S_o$ is a constant related to capillary forces.
is selected and red or blue sand is randomly added to it. Let us suppose that it is red. Then if \( h_r \) becomes unstable it topples. The dynamic is as follows: all \( h_j \)'s and \( h_b \) are first randomly selected from a uniform distribution, so that no unstable site exist. Then a random site is selected and red or blue sand is randomly added to it. Let us suppose that it is red. Then if \( h_r(i) > h_{th} \), then \( h_r(i) \rightarrow h_r(i-1) \) and \( h_b(j) \rightarrow h_b(j) + 1 \) where \( j \) is the neighbor of \( i \) with the lowest red-grain content. The same is true for blue grains. If condition \( h_r(i) + h_b(i) > H_0 \) is met, then \( h_r(i) \rightarrow h_r(i-1) \) and \( h_b(j) \rightarrow h_b(j) + 1 \) where \( j \) is the neighbor of \( i \) with the lowest x-grain content, and \( x \) is randomly chosen to be \( r \) (red) or \( b \) (blue). Figure 5 is to clarify how the species invade each other, realizing the condition C. This rule realizes the limited pore capacity due to which one species pushes another species, making the system invasive [171]. In this model, the total particle volume does not exceed a specific threshold, the third condition C expressing the stability of the sites indicates this fact.

After the main site become stable, the neighboring sites are checked for stability and instability, and the toppling process is repeated until all sites are stable again. The overall process is called an avalanche. In this model, like the regular BTW model, the grains leave the system via the boundaries [48].

This model has two types of avalanches; the first type includes only one species (red or blue), but in the second one is the total avalanche including the two species. In this model, two different regimes are also observed, which are in accordance with the fractal dimension 5/4 observed for the external perimeter of 2D BTW model [175]. For the small avalanches however, \( D_f^{(1)} = 1.47 \pm 0.02 \) that it is shown in Fig. (5). A crossover point \( x^* \), which was observed for all geometrical quantities like the gyration radius \( r \), avalanche size \( s \), and avalanche mass \( m \) was shown to exist between these two regimes which was obtained using \( R^2 \) test [176] that is

\[
\tau_f(L \to \infty) = 0.95 \pm 0.05
\]

\[
\beta = 1.32 \pm 0.02
\]

\[
\nu = 1.25 \pm 0.01
\]

pressure. In the dynamic of this model, once a site \( i \) becomes unstable it topples. The dynamic is as follows: all \( h_r \) and \( h_b \) are first randomly selected from a uniform distribution, so that no unstable site exist. Then a random site is selected and red or blue sand is randomly added to it. Let us suppose that it is red. Then if \( h_r(i) > h_{th} \), then \( h_r(i) \rightarrow h_r(i-1) \) and \( h_b(j) \rightarrow h_b(j) + 1 \) where \( j \) is the neighbor of \( i \) with the lowest red-grain content. The same is true for blue grains. If condition \( h_r(i) + h_b(i) > H_0 \) is met, then \( h_r(i) \rightarrow h_r(i-1) \) and \( h_b(j) \rightarrow h_b(j) + 1 \) where \( j \) is the neighbor of \( i \) with the lowest x-grain content, and \( x \) is randomly chosen to be \( r \) (red) or \( b \) (blue). Figure 5 is to clarify how the species invade each other, realizing the condition C. This rule realizes the limited pore capacity due to which one species pushes another species, making the system invasive [171]. In this model, the total particle volume does not exceed a specific threshold, the third condition C expressing the stability of the sites indicates this fact.

After the main site become stable, the neighboring sites are checked for stability and instability, and the toppling process is repeated until all sites are stable again. The overall process is called an avalanche. In this model, like the regular BTW model, the grains leave the system via the boundaries [48].

This model has two types of avalanches; the first type includes only one species (red or blue), but in the second one is the total avalanche including the two species. In this model, two different regimes are also observed, which are in accordance with the fractal dimension 5/4 observed for the external perimeter of 2D BTW model [175]. For the small avalanches however, \( D_f^{(1)} = 1.47 \pm 0.02 \) that it is shown in Fig. (5). A crossover point \( x^* \), which was observed for all geometrical quantities like the gyration radius \( r \), avalanche size \( s \), and avalanche mass \( m \) was shown to exist between these two regimes which was obtained using \( R^2 \) test [176] that is

\[
\tau_f(L \to \infty) = 0.95 \pm 0.05
\]

\[
\beta = 1.32 \pm 0.02
\]

\[
\nu = 1.25 \pm 0.01
\]

---

**Table 1.** SEC: 5.4: The \( \tau \) and \( \nu \) exponents from the finite-size analysis, \( \tau_f \) and \( \tau_{fi} \) for \( m, s, l, r \) corresponding to the red avalanches. Reproduced from [174]. © IOP Publishing Ltd. All rights reserved. The last row contains the exponents for the 2D BTW model for the sake of comparison with \( \tau_f(L \to \infty) \) [133, 177].

| quantity | \( m \) | \( s \) | \( l \) | \( r \) |
|----------|--------|--------|--------|--------|
| \( \tau_f(L \to \infty) \) | 1.04 ± 0.04 | 0.95 ± 0.03 | 2.5 ± 0.03 | 3.1 ± 0.1 |
| \( \tau_{fi}(L \to \infty) \) | 1.32 ± 0.02 | 1.26 ± 0.04 | 1.63 ± 0.03 | 1.8 ± 0.1 |
| \( \beta \) | -- | -- | 1.87 ± 0.05 | 1.58 ± 0.03 |
| \( \nu \) | -- | -- | 1.21 ± 0.03 | 0.95 ± 0.03 |
| \( \tau_{2D BTW} \) | 1.33 ± 0.01 | 1.29 ± 0.01 | 1.25 ± 0.03 | 1.66 ± 0.01 |

---

**Table 2.** SEC: 5.4: The exponents \( \beta, \nu, \tau_f, \) and \( \tau_{2D} \) for \( m, s, l, r \) corresponding to the two-species avalanches. Reproduced from [174]. © IOP Publishing Ltd. All rights reserved.

| quantity | \( m \) | \( s \) | \( l \) | \( r \) |
|----------|--------|--------|--------|--------|
| \( \tau_f(L \to \infty) \) | 0.95 ± 0.05 | 0.90 ± 0.05 | -- | 1.61 ± 0.05 |
| \( \tau_{2D} \) | 1.32 ± 0.02 | 1.25 ± 0.03 | 1.50 ± 0.03 | 2.0 ± 0.1 |
| \( \beta \) | -- | -- | 1.90 ± 0.05 | 1.78 ± 0.03 |
| \( \nu \) | -- | -- | 1.18 ± 0.03 | 0.95 ± 0.03 |
shown in Fig. (5), table (1) shows all exponent for this measures in one-species avalanches regime. For instance Fig. (5)) shows the distribution function of the loop length for one species, for which the fractal dimension of large avalanches corresponds to the two-dimensional BTW model, but for small avalanches \( D_f = 1.31 \pm 0.01 \). table (2) shows all exponent for this measures in two-species avalanches regime. For further information see [174].

5.5. SOC in Exitable Complex Networks

Probably the most important reason for considering of the BTW dynamics on top of the complex networks has been the important observation of Beggs et. al. [180] in which it was shown that the propagation of spontaneous activity in cortical networks is self-organized critical phenomena governed by avalanches much similar to the BTW model. The structural and functional characteristics of these models are very important and finding the conditions in which the system exhibits critical behavior [181] is the most important achievement of examining these systems. Different time series are considered to investigate critical behaviors and power-law behavior [119]. The time series based on the series of topplings in the SOC model are often considered in different random link network, such a scale-free networks [182, 183], multiplex networks [182], optimized scale-free network on Euclidean space [184], Watts-Strogatz small-worlds [185], directed small-world networks [186], and on Scale-free Networks with preferential sand distribution [187].

An example of the network that is embedded in the Euclidean space is a graph with random links with finite range interaction (RLFRI), i.e. two nodes are connected if their distance is smaller than a control parameter \( R \). Therefore the topology of this graph is tuned by \( n \) and \( R \), where \( n \) is the number of links per site. Then dynamic is defined via the following toppling matrix

\[
\Delta_{ij} = \begin{cases} 
    z_i & \text{if } i = j \\
    -1 & \text{if } i \text{ and } j \text{ are connected} \\
    0 & \text{other} 
\end{cases}
\]

where \( z_i \) is the degree of node \( i \).

The other system of interest is the small world networks that are interpolation between regular and random networks. In this system, there are regular links between neighbors, as well as long links between randomly selected sites. Let us define the connection matrix \((L(i,j))\) which is unity (zer) if two sites \( i \) and \( j \) are connected (disconnected), the distribution of which is considered to be uniform for long connections. In the absence of the link removal (which is another recepie of small world networks) the total number of links in node \( i \) in three dimensions is given by \( z_i(t) = 6 + \sum L(i,j) \). \( \alpha \) (the percent of long range links) is defined in this model as follows:

\[
\alpha \equiv 100 \times \frac{1/2 \sum_{i,j} L(i,j)}{\text{# total regular links}}.
\]

In the sandpile dynamics in this network, if the height of a node exceeds \( z_i \) it topples just like the instructions given above according to the rule \( h(i) \rightarrow h(i) - \Delta_{ij} \) in which:

![Figure 6. SEC: 5.5: (a) The plot of \( M_1 \) in terms of \( R_1 \) and the corresponding exponents \( \gamma_{(6,0)} \). Upper inset: \( \gamma_{(6,0)}^{UV} \) and \( \gamma_{(6,0)}^{IR} \). Lower inset: the cross-over radius \( R^* \) in terms of \( \alpha \) with the exponent 0.5 \( \pm \) 0.03. (b) The same as (a) for various lattice sizes \( L \). The finite size dependent (UV and IR) slopes \( \gamma_{(6,0)} \) have been shown in the inset. Reprinted figure with permission from [178], Copyright (2018) by the American Physical Society.](image)
some steps sandpile on a lattice constructed by a host media is the percolation model, which, in a most popular scheme realizes the permeability of each part of the host of the system by the following toppling matrix

\[
\Delta_{ij} = \begin{cases} 
-1 & \text{if and j are neighbors or } L(i, j) > 0 \\
    1 & i = j \\
    0 & \text{other.}
\end{cases}
\] (40)

Figure 6(a) shows the (\(\alpha\) dependent) fractal dimensional of 3D mass for different \(\alpha\) values, which is the scaling exponent between \(M_\alpha\) and \(R_\alpha\) [188]. The graphs show a smooth crossover point which separates the small (UV) and large (IR) scales behaviors. The upper inset shows that the exponent is \(\alpha\) dependent, whereas the lower one shows the run of the crossover point by \(\alpha\). For \(r < R_\alpha^*\), we have the normal BTW behavior, but for \(r > R_\alpha^*\) it is different and non-universal since it depends on the finite size. It was argued that the behaviors of this model can be understood in terms of the dissipative BTW model which is equivalent to the massive ghost action with the mass \(m^3\) defined as the number of sand grains dissipated in each toppling, where the correlation is shown to decal like \(R_\alpha^* \sim m^{-1}\) [189]. In the inset of 6(b), constant trend of \(m_{UV}\) is depicted for \(\alpha = 1\) in terms of the lattice size (L). \(m_{UV}\) have a universal behavior but \(m_{IR}\) varies considerably with lattice size.

5.6. SOC in Imperfect Supports

Definitely all critical phenomena do not take place in perfect media, and always some irregularities and imperfections are present in the system. Whether such disorders are relevant, i.e. bring the original regular model to a new critical universality class is an unanswered question yet, which should be addressed by simulations or by analytical approaches. A very popular model for realizing the imperfections and generally disorders in the host media is the percolation model, which, in a most popular scheme realizes the permeability of each part of the system by occupied and unoccupied sites. The examples are filling the vacuum of clusters with nanoparticles of ferromagnetic fluids [190–193], the loop-erased random walk on the percolation lattices (which is related to watersheds [194]) and fluid propagation in porous media [171, 195–197]. In the mathematical point of view this set up can be considered also as mixing two statistical models, one model as a dynamic model and the other as the host of the first model. The interaction between these two statistical models leads to critical behaviors.

Consider a lattice whose sites are impenetrable with probability \(p\) and impermeable with \(1-p\), and define sandpile on top of the lattice. If the permeability pattern is uncorrelated, then one can use the percolation theory for describing the lattice. Let us define the number of active neighbors of site \(i\) as \(z_i\) and the grains are governed by the following toppling matrix (see figure 1)

\[
\Delta_{ij} = \begin{cases} 
    z_i & i = j \\
    -1 & i, j \text{ are neighbors, and } j \text{ is permeable} \\
    0 & \text{other}
\end{cases}
\] (41)

The movement of the sand grains only on the spanning permeable cluster, and investigate the properties of sandpile on a lattice constructed by a fixed \(p\). Sand grains are added randomly throughout the sample. After a some steps (depending on \(p\)), the system saturates in a state where the mean height is nearly constant. The analysis of the gyration radius and the loop length of the external frontier of avalanches, the cluster mass, and the size of avalanches in this model were analyzed in [134, 198], the scaling exponents of which are shown in figure 7.
Note that at $p = 1$ one recovers the BTW universality class whereas for $p_c < p < 1$ the behavior is different. Especially at $p = p_c$, the fractal dimension of 2D Ising model is found i.e. $D^{perc} = \frac{11}{4}$. The question of how the information in $d + 1$ is projected to $d$ subsystem is an essential question. The 2D images of clouds, and the 2D images of CMB map are important questions in this respect. The projection is the simplest way to reduce the dimensionality, and track how statistical properties change. In [198] the sandpile model on the three dimensional lattice with percolation type disorder was analyzed, whose 2D projection was shown to exhibit non-trivial critical properties, especially in the vicinity of $p_c$.

The above analysis was for the case where the imperfections are uncorrelated. Turning on the correlations for the spatial configuration of the imperfections over the lattice makes the results different. The diluteness pattern in the host in the latter case is tuned by the Ising model on a cubic lattice where the spin configuration $s$ gives us the diluteness pattern, i.e. $s = \pm 1$ is for active site and $s = -1$ is for inactive site and the temperature controls the correlations. The analysis is restricted to the ferromagnetic phase of the Ising model to have spanning active sites clusters.

Based on the determination of the fractal dimension of the external perimeter of the avalanches, and the Schramm–Loewner evolution, it was suggested in [199] that the BTW on the critical percolation, results to critical Ising universality class, and also the BTW model on the Ising-correlated percolation lattice results to self-avoiding walk universality class.

More interesting is the BTW model on the three-dimensional Ising correlated lattice, for which the magnetic phase transition is not accompanied by a percolation transition, allowing us to measure the properties of the model across the transition point, which is $T_c \approx 4.51$, i.e. the spin clusters of the 3D Ising model on the cubic lattice percolate at any temperature. The implementation of the sandpile model is just like above. This line of thinking has already been done on the other systems [179, 188, 200, 201]. It was shown that the average avalanche size changes abruptly in the transition point, see figure 8(a) in which $\zeta(T) \equiv f_{perc} - f(T)$, where $f(T) = \frac{m(T)}{N(T)}$, $m(T)$ being the number of topplings, $N(T)$ the number of sites in the spanning (majority) spin cluster, and $f_{perc} \equiv f(T = \infty)$. Many interesting properties of $\zeta(T)$ were analyzed, like the finite size scaling hypothesis, and various scaling behaviors.

5.7. Diffusive Sandpiles

One of the important ingredients of sandpile models is the threshold which causes the separation of time scales. What happen when the sandpiles are able to diffuse, meaning that the sand grains are able to move from a higher site to a neighboring site with lowest number of grains, which is expected in natural systems. When a site randomly selected and checked for more stable configuration, a local smoothing (or equivalently diffusion to neighboring site) takes place [203]. Suppose the selected i site has neighbors $i_1, i_2, i_3$ and $i_4$ and the $i_{max}$ is the site with maximal height $h(i_{\text{max}}) = \text{Max} \{ h_i \}$, $i_{min}$ is the site with minimum height $h(i_{\text{min}}) = \text{Min} \{ h_i \}$, then during a local smoothing operation in which $\delta h_i \equiv \text{int}[h(i_{\text{max}}) - h(i)]/2 > 0$ grains are transferred from site $i_{\text{max}}$ to i, and $\delta h_2 \equiv \text{int}[(h(i) - h(i_{\text{min}})]/2 > 0$ grains are transferred from site $i$ to $i_{\text{min}}$. If more than one site is of the same height, the site into which the grains enter is randomly selected. The sites which are unstable topple and the sites which are chosen for local smoothings moderate their local height gradient. When the grains are
lubricated in a sandpile such that they have the chance to slip to the neighboring sites, the above mentioned dynamics is expecting.

In the mean-field (MF) level one may consider a square lattice with \( N = L^2 \) sites and 4L boundary sites, and follow the dynamics at \( T \)th injection, where the number of sites that were toppled in the avalanche is \( A(T) \) and the average height is \( \bar{h}_t \). When one unit of energy is added to the system, the average energy increases by \( 1/N \). Some energy is also dissipated from the boundaries. It is argued in [203] that at the mean field level

\[
\bar{h}(T + 1) = \bar{h}(T) + \frac{1}{N}(1 - f(\zeta)),
\]

where \( f(\zeta) \) is a quantity which only depends on \( \zeta \) (the average number of local smoothing per avalanche). Therefore \( \bar{h} \) has grown linearly with time reaching to some threshold (\( \bar{h} \rightarrow \bar{h}_t \)), at which a spanning avalanche takes place, and the system overall relaxes.

The figure 9(a) demonstrates that there are two different regimes in early times and large times. In accordance with the mean field arguments some spanning avalanches are observed. A bifurcation point is observed in this graph, that can be viewed more directly in figure 9(b).

5.8. propagation of electrons in 2D electron gas (2DEG)
The resistors through which a current passes show voltage fluctuations with a power spectrum density proportional to a power of the inverse frequency for both ac and dc current. The presence of \( 1/f \) noise (power-law behavior of the power spectrum) in condensed matter systems reflects the internal structure (scale invariance) of the systems under study. The fact that the shape of the noise spectrum is almost identical in metals, semiconductors, semi-metals, three-dimensional materials, quasi-one-dimensional materials, and even in superconducting quantum interference devices shows that the noise should not depend on the precise details of the models that describe materials. Experimental measurements show that the power spectrum is \( s(f) \propto f^{-\alpha} \), where \( 0.8 < \alpha < 1.4 \) in a large frequency range [204]. Webb and Gershenson showed that the deterministic particle dynamics do not play role in \( 1/f \) noise [205]. The following questions are essential:

Is there a general formula for describing at least part of the \( 1/f \) noise?

Is \( 1/f \) noise a surface effect or a volumetric effect?

Is the fluctuations in the number of carriers responsible for it or it is due to the mobility of electrons?

The theoretical and empirical answer to the first question is no. In fact, none of these questions have a common practical answer. Much attention has been paid to these issues [206–208]. In the references [207, 208] some evidences has been provided in favor of the following scaling relation

\[
S_R(f)/R^2 = \alpha_H/N_c f,
\]

which is known as the Hooge formula. In this relation \( \alpha_H \approx 2 \times 10^{-3} \), \( N_c \) is the number of load carriers in a homogeneous sample and \( R \) is the resistance and \( S_R(f) \) is the density of the power spectrum of the fluctuations of the resistance at frequency. In fact this relation serves as a test for any microscopic model which is designed to
capture the physics of $1/f$ noise in condensed matter systems. The studies on Au-films doped with a sufficient amount of impurities [209] and semiconductors [210], show that $\alpha_H$ is an adjustable parameter [211]. The main strategies people use are Temperature fluctuations [212, 213], Mcwhorter model in which one overlaps a large number of Lorentz spectra to produce the power-law dependence, the Dutta-Horn approach [214] where a two-state model is used to describe the phenomena [212], and the symmetry measurement technique, for a good review see [212].

An interesting approach in this area is the classical percolation of electrons. Percolation picture has been proved to be useful for the interpretation of low-dimensional condensed matter systems, like the finite-size power-law conductivity of 2DEG, the self-averaging and percolation prescription of 2DEG. In [215] the author has proposed a model according to which the percolation of electrons with avalanche dynamics can be a source for MIT of a two-dimensional electron gas at zero magnetic fields, called the semi-localization of electrons, which is related to the percolation-non-percolation phase transition, although it differs from conventional diffusion theory. The percolative phase has the property $\frac{d}{dT}\sigma < 0$ ($\sigma \equiv$ the conductivity) which is the

---

**Figure 10.** (Color Online) (a) Schematic diagram of the division of a 2D lattice into multiple hexagons. Inside the hexagons, we have a pure quantum electron gas. Transition between cells occurs in a semi-classical manner. The green points of the electron and the red points show the impurities. (b) Schematic adjustment of 2D electron gas surrounded by charge tanks. In this case, the same division has been done. Electrons can enter and exit the 2D system at any random point (with some energy considerations) [215].

**Figure 11.** (a) Transfer of electrons to neighboring cells. (b) The schematic pattern of motion of an electron in a virtual lattice. Gray cells are border sites (external sites with $\mu = \mu_0$) from which electrons can leave the system. The black circle is where the electron is injected, and the gray circles represent places that have been unstable and relaxed through the charge transfer chain [215].
characteristics of the metallic phase. Interestingly this MIT occurs in the diffusion regime of 2DEG and therefore has nothing to do with the Anderson localization. In this model, a two-dimensional electron gas in contact with some electronic reservoirs are considered. In random systems, the most important scales are the phase relaxation length \( l_0 = \sqrt{D\tau_0} \), \( D \) being diffusion coefficient and \( \tau_0 \) being the phase relaxation time, and mean free path due to the electron-electron or the electron-phonon interactions \( l \). According to the phase relaxation time, one considers two scales for the dynamics for electrons in the system: \( l \ll r \ll l_0 \) and \( r \gg l_0 \). In the diffusive phase the position of the electron \( r \) changes with time \( t \) effectively by \( r \sim \sqrt{Dt} \) and \( D \) is the diffusion coefficient. A full treatment concerning the quantum phase of electrons was presented in [215], where it was shown that in some limits the picture can be semi-classical, since quantum fluctuations in this scale do not play a vital role, i.e. one can use the classical Boltzmann transport equation. Then 2DEG can be treated as a semi-classical system with some cells inside which pure quantum mechanics is governing, but for the transport of the particles to neighboring regions some semi-classical rules can be developed as was done in [215]. In this model, electrons are dispersed through the system according to the energy content (temperature-dependent) as well as the chemical potentials of the cells. In some cases, some electrons can reach from one side to the opposite boundary, which is called percolated.

The schematic set up can be found in Fig 10 and Fig 11. The energy functional of each quantum cell is not hard to calculate, using of which the chemical potential is obtained to be

\[
\mu_i = k_B T \ln(e^{\mu_i} - 1) + UT^i(h_i - IZ, T^i)
\]

in which \( U = \frac{2m_e \omega^2}{8\hbar^2} r^2 \), \( I = \sinh^{-1}(1) \), \( r^2 = \frac{E_{\text{kin}}}{\hbar^2} \), \( h_i = \frac{N_i}{T} \) and \( i \) stands for the \( i \)th cell. The effect of randomness of \( Z_i \)'s (that are supposed to be random noise with an uniform probability measure), which captures the on-site (diagonal) disorder is investigated. It is also proved that the transition probability between two cells is given by

\[
\frac{\mu_{N+1}(V, T)}{\rho_N(V, T)} = \exp[\beta(\mu_0 - (A_{N+1}(V, T) - A_N(V, T))]
\]

\[
\approx \exp[\beta(\mu_0 - \mu_N(V, T))].
\]

(45)

in which \( A(V, T) \) is the Helmholtz free energy and \( \mu_N(V, T) = \frac{\partial A}{\partial S} \). The probability of adding a particle to the \( i \)th cell of the system is shown to be proportional to \( \exp[-\beta \mu_i] \), in which \( \mu_i = \mu_i - \mu_0 \) and \( \mu_0 \) is the average chemical potential of the system, whereas the probability of the transition between two sites (say cell 1 \( \rightarrow \) cell 2) is obtained by

\[
\text{relative probability} = e^{-\beta(\mu_2 - \mu_1)}
\]

(46)

for which the following relation is used:

\[
\mu_2 - \mu_1 = k_B T \ln\left(\frac{e^{\mu_2} - 1}{e^{\mu_1} - 1}\right) + UT^1(h_2 - h_1) - IT^1(Z_2 - Z_1).
\]

(47)

In the stationary state \( \mu_0 = \mu_N \) is the equation of the equilibrium state, where \( \mu_0 \) is the chemical potential of the total system. Fig 11(a) shows the transition rule for a local cell, for which the Metropolis Monte Carlo was used. If the mentioned site is unstable, it has the potential to release electrons to the neighbors, and the first candidate for this charge transfer is the neighbor with the smallest chemical potential \( \mu \). The same process continues until the system reaches equilibrium.

Using this model, many aspects of the observations on 2DEG were found, like the power-law behavior at the critical temperature, and also absence of universality in high disordered systems [216]. Especially a 1/f noise has been observed in this system with an exponent in the range [0.4 – 0.6]. The power-law behavior for the power spectrum is not restricted to the transition point. This observation for 2DEG based on a dynamical model is a new one which shows that 1/f noise in these systems can be attributed to a particular form of a self-organized critical dynamics for the electrons.

5.9. SOC in sliding charge-density waves

The local phases of charge-density waves (CDW) is described by (the generalizations of) the nonlinear Fukuyama-Lee-Rice model [217–219] with random potentials, which shows depinning transition as a function of the driving force [220]. In this model the elastic medium is pulled through a random potential by an elastic driving force, and a pinned-moving phase transition is found at \( F_T \), where some power-law behaviors are observed [220]. The relationship between the BTW model and the sliding pinned density waves has been investigated by Narayan and Middleton in 4 – \( \epsilon \) dimensions. The automaton models studied are a charge-density wave model and a sandpile model with periodic boundary conditions; these models are found to have the same critical behavior, associated with diverging avalanche sizes. The authors found a numerical agreement for the polarization and the diverging length and time scale with the analytical treatment. The resulting exponents are found to be different above and below threshold [220]. The [221] gives evidences that the functional
which have been shown separately in the table. After [179]

| τ(α = 0)  | 1.34(4) | ... | 2.53(5) | ... | ... |
| τ₁       | 1.37(4) | 1.6(2) | 2.07(5) | 2.8(5) | 1.33(1) | 1.66(2) |
| τ₂(α = 1) | 3.46(5) | 3.62(9) | 5.98(7) | 6.1(7) | ... | ... |
| τ₃        | 0.17(3) | 0.19(9) | 0.18(1) | 0.18(3) | ... | ... |
| cut(α = 1) | 2843  | 4601  | 11.2(5)  | 13.9(4) | 3073  | 6629  |
| γcut     | 1.28(9) | 1.23(5) | 0.42(8)  | 0.49(7)  | 1.57(3)  | 1.06(3)  |

6. Conclusion

In this paper, we reviewed the SOC concepts in various systems. First, we presented some examples, including the systems that show SOC, like earthquake, rain falling, etc. In the second part, we presented the evidence showing that the BTW sandpile model is \( \alpha = -2 \) LCFT, and is tied to W-algebras. The simulation results for SOC in various systems were presented in the last part. There we considered the SOC in fluid propagation in porous media, in cumulus clouds, in an excitable random system, in imperfect supports, and in 2DEG. We also considered vibrating ASM, invasion sandpile model, and diffusive sandpiles.

Data availability statement

All data that support the findings of this study are included within the article (and any supplementary files).

Appendix A. A brief explanation about Grassmann algebra

In this appendix, we give a brief explanation of Grassmann algebra and Gaussian integration (both bosonic and fermionic) and mention some of the main features. Gaussian merger The bosonic is actually the Gaussian merger on \( R^n \) and \( C^n \). Also in this category, we can express the integration of Grassmann and Grassman-Berezin algebra (formionics) and explain the Gaussian formulation integration formulas that the reader can refer to ([225, 226]) for a comprehensive study. A Grassmann algebra is defined by a set of generators, which we denote by \( \{ \xi_\alpha \}, \alpha = 1, \ldots, n \). These generators anticommute: \( \xi_\alpha \xi_\beta + \xi_\beta \xi_\alpha = 0 \), so that, in particular: \( \xi_\alpha^* = 0 \). The distinctive results of the generators give rise to suppose we have the hypothetical set: \( \{ 1, \xi_\alpha, \xi_\alpha^*, \xi_\alpha \xi_\beta, \xi_\alpha^* \xi_\beta, \ldots, \xi_\alpha^n \} \) that \( \alpha_1 < \alpha_2 < \ldots < \alpha_n \) each generator can produce two possibilities of 0 or 1, so that grassmann algebra with \( n \) generator is \( 2^n \). Defines the following properties of the compositions in grassmann algebra:

- \( (\xi_\alpha)^* = \xi_\alpha^* \)
- \( (\xi_\alpha^*)^* = \xi_\alpha \)
- \( (\lambda \xi_\alpha)^* = \lambda^* \xi_\alpha^* \)
- \( (\xi_\alpha, \xi_\beta, \ldots, \xi_\alpha^*)^* = \xi_\alpha^* \ldots \xi_\alpha \)

that \( \lambda \) is complex number. For convenience, we will define two generators and discuss the \( \{ 1, \xi, \xi^*, \xi \xi^* \} \). Any analytic function defined on this algebra is linear function: \( f(\xi) = f_0 + f_1 \xi \). The coherent state representation of an operator in the grassmann algebra will be a function of \( \xi^* \) and \( \xi \) and must have the form \( A(\xi^*, \xi) = a_0 + a_1 \xi + a_2 \xi^* + a_{12} \xi \xi^* \). About derivation:
\[
\frac{\partial}{\partial \xi}(\xi^* \xi) = \frac{\partial}{\partial \xi}(-\xi^* \xi) = -\xi^* \\
\frac{\partial}{\partial \xi} A(\xi^*, \xi) = a_i - a_{22} \xi^* \\
\frac{\partial}{\partial \xi^*} A(\xi^*, \xi) = \bar{a}_i + a_{22} \xi \\
\frac{\partial}{\partial \xi} \frac{\partial}{\partial \xi^*} A(\xi^*, \xi) = -a_{22} = -\frac{\partial}{\partial \xi^*} \frac{\partial}{\partial \xi} A(\xi^*, \xi)
\]

The operators \( \frac{\partial}{\partial \xi} \) and \( \frac{\partial}{\partial \xi^*} \) are anticommuting. About the integral:

\[
\int d\xi_1 = 0 \\
\int d\xi^*_1 = 0 \\
\int d\xi = 1 \\
\int d\xi^* \xi^* = 1 \\
\int d\xi^* f(\xi) = f_0 \\
\int d\xi A(\xi^*, \xi) = \int d\xi (a_0 + a_1 \xi + \bar{a}_1 \xi^* + a_{22} \xi^* \xi) = a_0 - a_{22} \xi^* \\
\int d\xi^* A(\xi^*, \xi) = \bar{a}_0 + a_{22} \xi \\
\int d\xi^* d\xi A(\xi^*, \xi) = -a_{22} = -\int d\xi d\xi^* A(\xi^*, \xi)
\]

Consider the definition of a Grassmann \( \delta \)-function by:

\[
\delta(\xi, \xi') = \int d\eta \epsilon(\eta, \xi, \xi') = \int d\eta (1 - \eta(\xi - \xi')) = -(\xi - \xi') \\
\int d\xi^* \delta(\xi, \xi') f(\xi') = -\int d\xi^*(\xi - \xi')(f_0 + f_0\xi) = f_0 + f_0\xi = f(\xi)
\]

A \( m \times n \) matrix with \( m \) rows and \( n \) columns. When we keep all the columns from state \( A_{[m]} = A_{[m]}(j) \) to \( J \subseteq [n] \), when we keep all the rows from state \( A_{[n]} = A_{[n]}(m) \) which is \( I \subseteq [m] \). If \( A \) is invertible, we denote by \( A_{-T} \) the matrix \( (A^{-1})^T = (A^T)^{-1} \). If \( A = (a_{ij})_{i,j=1}^n \) is an \( n \times n \) matrix, we define its permanent

\[
\text{per}A = \sum a_{i_1 \sigma_1} \ldots a_{i_n \sigma_n}, \quad (A1)
\]

and its determinant

\[
\det A = \sum_{\sigma \in S_n} \text{sgn}(\sigma) a_{i_1 \sigma_1} \ldots a_{i_n \sigma_n}, \quad (A2)
\]

Some of the properties of determinant are \( \det I = 1 \), \( \det(AB) = \det(A) \det(B) \) and Cauchy-Binet formula that show \( \det(AB) = \sum_{\sigma \subseteq [m]} \det(A_{[\sigma]}) \det(B_{[\sigma]}) \) that \( A \) is \( m \times n \) matrix and \( B \) is \( n \times m \) matrix. One of the other important properties is adjugate matrix, if \( A \) be \( n \times n \) matrix, we have \( \text{adj}A_{ij} = (\epsilon^{-1})^T + \det A_{[i]}(\epsilon) \) then \( \text{adj}A = A \text{adj}A = (\det A)I \), \( A^{-1} = (\det A)^{-1} \text{adj}A \). We know that \( A \) is invertible in the hypothetical circle \( R^{n \times n} \), if and only if \( \det A \) be invertible in \( R \). About the matrix properties, there are many cases, some of which are listed below.

Jacobi’s identity for \( I, J \subseteq [n], |I| = |J| = k \): \( \det((A^T)^{-1})_{I,J} = (\det A)^{-1} \epsilon(I, J)(\det A_{I,F}) \) where \( \epsilon(I, J) = (-1)^{\sum_{a \subseteq I \cup J} |a|} \). Multi-row laplace expansion for each fixed set of rows \( I \subseteq [n] \) with \( |I| = k \):

\[
\det A = \sum_{I \subseteq [n], |I| = k} \epsilon(I, J)(\det A_{I,F}) (A3)
\]

For a very brief introduction of Hafnian, if \( A = (a_{ij}^m)_{i,j=1}^m \) and be \( A \) symmetric matrix \( 2m \times 2m \) with entries in \( R \):

\[
hf A = \sum_{\sigma \in S_{2m}} \prod_{i \neq j} a_{ij}^\sigma. \quad \text{About pfaffian, if } A = (a_{ij}^m)_{i,j=1}^m \text{ and be } A \text{ antisymmetric matrix } 2m \times 2m:
\]

\[
\text{pf} A = \sum_{\sigma \in S_{2m}} \epsilon(\tilde{M}, \tilde{M}_0) \prod_{i \neq j} c_{i,j}^\sigma a_{ij}^\sigma. \quad \text{For } I, J \subseteq [n] \text{ with } |I| = |J| = k, A \text{ and } B \text{ are } n \times n \text{ matrix and } M \text{ is invertible } n \times n \text{ matrix, } l \text{ is nonnegative integer: }
\]

\[
\int \tau_{ij}^k (\int (\tilde{M} \cdot (\tilde{B}_n))_{ij,F} (\tilde{M}^T \tilde{M}_n)^{^{F}} d\xi)^k = k! b_{ijk}^k \det(M) \det((AM^{-T}B)^{^{T}})_{ij,F}.
\]
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