Function algebras on a 2-dimensional quantum complex plane

I Cohen and E Wagner

1 Instituto de Física y Matemáticas, Universidad Michoacana, Morelia, Mexico, and Centro de Ciencias Matemáticas, UNAM, Morelia, Mexico
2 Instituto de Física y Matemáticas, Universidad Michoacana, Morelia, Mexico

E-mail: ismaelcohen10@gmail.com elmar@ifm.umich.mx

Abstract. The well-behaved representations of the coordinate algebra of a 2-dimensional quantum complex plane are classified and a C*-algebra is defined which can be viewed as the algebra of continuous functions on the 2-dimensional quantum complex plane vanishing at infinity.

1. Introduction
The general purpose of this paper is to study non-compact quantum spaces in the C*-algebraic framework. Usually quantum spaces arising in Quantum Group Theory are given by generators and relations. The (*-)-algebra obtained in this way can then be viewed as the coordinate ring of polynomial functions on the quantum space. For compact quantum spaces, there is a general procedure to assign a unital C*-algebra to the quantum space: one considers the universal C*-norm defined as the supremum of the operator norms of all bounded *-representation of the coordinate ring and takes the closure with respect to this norm. Here, having a compact quantum space is essentially synonymous to the existence of the universal C*-norm.

The non-compact situation is characterized by the fact that the *-algebra admits unbounded *-representations and that the universal C*-norm might not exist. For this setting, S. L. Woronowicz developed a theory of C*-algebras generated by unbounded elements [10, 13]. However, this method is not constructive, the unbounded operators and the C*-algebra have to be given at the beginning, one only proves that the unbounded operators actually generate the C*-algebra. Since we are more interested in having an explicit C*-algebra at hand than proving technical details, we prefer to construct a non-commutative C*-algebra by analogy to the classical C*-algebra of continuous functions vanishing at infinity on the corresponding locally compact space. The analogy to the classical case involves concrete Hilbert space representations of the coordinate ring and can therefore be done only by a case to case study. In the present paper, we will do it for a 2-dimensional quantum complex plane, the 1-dimensional version has already been treated in [3, 9]. Similar construction of function algebras on non-compact quantum spaces can be found, for instance, in [1, 5, 6, 11, 12] but none of these papers touches the C*-algebra framework.

Let us briefly outline our construction. First we classify all well-behaved Hilbert space representations of the coordinate ring \( \mathcal{O}(\mathbb{C}^2_q) \). It is important to have knowledge about all possible representations because it turns out that different representations correspond to different domains of the quantum complex plane. The next step is to realize these representations on...
a function space ($L_2$-space) such that modulus of each generator (the non-negative self adjoint part in its polar decomposition) acts as a multiplication operator. Furthermore, the measures are chosen in such a way that the partial isometries from the polar decompositions are given on the same footing: they act as multiplicative $q$-shifts on functions. In this manner we obtain very simple commutation relations between the multiplication operators and the partial isometries. Then we consider an auxiliary *-algebra of bounded operators generated by continuous functions of the moduli of the generators (represented by multiplication operators) and powers of the partial isometries and their adjoints. For the interpretation as continuous functions on the 2-dimensional quantum complex plane vanishing at infinity, we require that the continuous functions belong to $C_0([0, \infty) \times [0, \infty))$ and that these functions, when evaluated at 0, do not depend on the phases (the partial isometries from the polar decompositions). Moreover, in order not to “miss any points”, we consider some sort of universal representation, where the involved measures have the largest possible support. Finally, the C*-algebra of continuous functions vanishing at infinity is defined by taking the C*-closure of the auxiliary algebra in the operator norm.

An advantage of our approach is that it allows a geometric interpretation of the different representations. As usual, a nontrivial 1-dimensional representation corresponds to a classical point, in our case to the origin of $\mathbb{C}_q^2$. Setting one generator to zero, we get a copy of $\mathbb{C}_q$ inserted into the quantum space $\mathbb{C}_q^2$. Last but not least, there is a family of faithful representations that describe a 2-dimensional quantum complex plane, where the copy $\mathbb{C}_q$ from the previous representation is shrunk to a point. Therefore, restricting oneself (as quite customary) to the family of faithful representations will not yield the whole 2-dimensional quantum complex plane.

2. Preliminaries

Throughout this paper, $q$ stands for a real number in the interval $(0, 1)$. The coordinate ring $\mathcal{O}(\mathbb{C}_q^2)$ of polynomial functions on 2-dimensional quantum complex plane is the *-algebra over $\mathbb{C}$ generated by $z_1$ and $z_2$ satisfying the (overcomplete) relations [4]

\begin{align*}
q z_1 z_2 &= q z_1 z_2 , \\
q z_1 z_2^* &= q z_2^* z_1 , \\
q z_2 z_2^* &= q z_2 z_2^* + (1-q^2) z_1 z_2^* .
\end{align*}

(1) (2) (3)

By a slight abuse of notation, we will use in the following sections the same letter to denote a generator of the coordinate ring $\mathcal{O}(\mathbb{C}_q^2)$ and its representation as a Hilbert space operator.

We adopt the convention that $\mathbb{N} = \{1, 2, \ldots \}$ and $\mathbb{N}_0 = \{0, 1, 2, \ldots \}$. Given an at most countable index set $I$ and a Hilbert space $\mathcal{H}_0$, consider the orthogonal sum $\mathcal{H} = \bigoplus_{i \in I} \mathcal{H}_0$. We write $\eta_i$ for the vector in $\mathcal{H}$ which has the element $\eta \in \mathcal{H}_0$ as its $i$-th component and zero otherwise. It is understood that $\eta_i = 0$ whenever $i \notin I$.

For a subset $A \subset [0, \infty)$, the indicator function $\chi_A : [0, \infty) \to \mathbb{C}$ is defined by

\[ \chi_A(t) := \begin{cases} 
1, & t \in A, \\
0, & t \notin A.
\end{cases} \]

(4)

3. Hilbert space representations of the 2-dimensional quantum complex plane

In this section, we give a complete description of “good” *-representations of $\mathcal{O}(\mathbb{C}_q^2)$. Here “good” means that, in order to avoid pathological cases, we impose in Definition 1 some natural regularity conditions on the unbounded operators. These representations will be called well-behaved, see [8]. To motivate the regularity conditions, we start with formal algebraic manipulations. These algebraic relations, together with the regularity conditions of Definition 1, will allow us to classify in Theorem 1 all well-behaved representations of $\mathcal{O}(\mathbb{C}_q^2)$.
Let $z_1$ and $z_2$ be densely defined closed operators on a Hilbert space $H$ satisfying the relations (1)–(3) on a common dense domain. Set $Q := z_2^* z_2$. From the relations (1)–(3) within the algebra $O(C_q^2)$, we get

$$z_1 Q = Q z_1, \quad z_1^* Q = Q z_1^*, \quad z_2 Q = q^2 Q z_2, \quad z_2^* Q = q^{-2} Q z_2^*, \quad (5)$$

and for any polynomial $p$ in one variable, (5) yields

$$z_1 p(Q) = p(Q) z_1, \quad z_1^* p(Q) = p(Q) z_1^*, \quad z_2 p(Q) = p(q^2 Q) z_2, \quad z_2^* p(Q) = p(q^{-2} Q) z_2^*. \quad (6)$$

Let us assume that (6) holds for all bounded Borel measurable functions on $\text{spec}(Q)$, where $p(Q) = \int p(\lambda) \, dE(\lambda)$ is defined by the spectral theorem with the unique projection-valued measure $E$ of $Q$. Then $\ker(Q) = E(\{0\}) H$ and $\ker(Q)^\perp = E((0, \infty)) H$ are invariant under the actions of the generators of $O(C_q^2)$. On $\ker(Q)$, we have $Q = z_2^* z_2 = 0$, thus $z_2 = z_2^* = 0$, and (3) becomes

$$z_1 z_1^* = q^2 z_1^* z_1. \quad (3')$$

On $\ker(Q)^\perp$, the operator $\sqrt{Q}^{-1} = \int \frac{1}{\sqrt{\lambda}} \, dE(\lambda)$ is well-defined. Consider at the moment the abstract element

$$w := \sqrt{Q}^{-1} z_1 = z_1 \sqrt{Q}^{-1}. \quad (7)$$

Inserting (7) into the second relation of (3) yields formally

$$ww^* - q^2 w^* w = -(1 - q^2) z_2^* z_2 Q^{-1} = -(1 - q^2). \quad (8)$$

Note that, by (5), we have $z_1^* z_1 z_2^* z_2 = z_2^* z_2 z_1^* z_1$. This relation together with (8), (6), (3'), and the first equation in (3) motivate the following definition of well-behaved $\ast$-representations of $O(C_q^2)$.

**Definition 1.** A well-behaved $\ast$-representations of $O(C_q^2)$ is given by densely defined closed operators $z_1$ and $z_2$ satisfying (1)–(3) on a common dense domain such that

(i) The self-adjoint operators $z_1^* z_1$ and $z_2^* z_2$ strongly commute.

(ii) $z_2$ is a $q$-normal operator, i.e., it satisfies the operator equation

$$z_2 z_2^* = q^2 z_2^* z_2. \quad (9)$$

(iii) For all bounded Borel measurable functions $f$ on $\text{spec}(Q)$, the operator relations

$$f(Q) z_1 \subset z_1 f(Q), \quad f(Q) z_1^* \subset z_1^* f(Q), \quad f(Q) z_2 \subset z_2 f(q^2 Q), \quad f(Q) z_2^* \subset z_2^* f(q^2 Q)$$

hold.

(iv) On $\ker(Q)$, $z_1$ is a $q$-normal operator, i.e.,

$$z_1 z_1^* = q^2 z_1^* z_1. \quad (10)$$

(v) On $\ker(Q)^\perp$, $z_1$ commutes with $\sqrt{Q}^{-1}$ and setting $w := \sqrt{Q}^{-1} z_1 = z_1 \sqrt{Q}^{-1}$ defines a densely defined closed operator fulfilling the operator equation

$$ww^* = q^2 w^* w - (1 - q^2). \quad (11)$$

Here, the equality of operators on both sides of the equations includes the equality of their domains.
The well-behaved representations of $q$-normal operators have been studied in [2] and [3]. By [3, Corollary 2.2], any $q$-normal operator $\zeta$ on a Hilbert space $\mathcal{G}$ admits the following representation:

$$\mathcal{G} = \ker(\zeta) \oplus (\oplus_{n \in \mathbb{Z}} \mathcal{G}_0), \quad \zeta = 0 \text{ on } \ker(\zeta), \quad \zeta g_n = q^n Z g_{n-1} \text{ on } \oplus_{n \in \mathbb{Z}} \mathcal{G}_0,$$

(10)

where $Z$ denotes a self-adjoint operator on $\mathcal{G}_0$ such that $\text{spec}(Z) \subset [q, 1]$ and $q$ is not an eigenvalue of $Z$.

Furthermore, the representations of operators $w$ satisfying (9) on a Hilbert space $\mathcal{G}$ have been classified in [5, Lemma 2.3]. It follows from this lemma that $\mathcal{G}$ can be written as a direct sum $\mathcal{G} = \oplus_{m \in \mathbb{N}} \mathcal{G}_0$, and the actions of $w$ and $w^*$ are determined by

$$w g_m = \sqrt{q^{-2m} - 1} g_{m+1}, \quad w^* g_m = \sqrt{q^{-2(m-1)} - 1} g_{m-1}, \quad g \in \mathcal{G}_0, \quad m \in \mathbb{N}.$$

(11)

Equations (10) and (11) are all we need for the classification of the well-behaved representations of $\mathcal{O}(\mathbb{C}_q^2)$.

**Theorem 1.** Any well-behaved Hilbert space representation of $\mathcal{O}(\mathbb{C}_q^2)$ is unitarily equivalent to a representation given by the following formulas: Let $\mathcal{H}_0$, $\mathcal{H}_{00}$ and $\mathcal{N}$ be Hilbert spaces, and let $A$ and $B$ be self-adjoint operators on $\mathcal{H}_0$ and $\mathcal{H}_{00}$, respectively, such that their spectrum belongs to $[q, 1]$ and $q$ is not an eigenvalue. Then the Hilbert space $\mathcal{H}$ of the representation decomposes into the direct sum

$$\mathcal{H} = \mathcal{N} \oplus (\oplus_{k \in \mathbb{Z}} \mathcal{H}_0) \oplus (\oplus_{n \in \mathbb{Z}} \oplus_{m \in \mathbb{N}} \mathcal{H}_{00}),$$

and the actions of $z_1$ and $z_2$ are determined by

$$z_1 = z_2 = 0 \text{ on } \mathcal{N}, \quad z_1 h_k = q^k A h_{k-1}, \quad z_2 = 0 \text{ on } \oplus_{k \in \mathbb{Z}} \mathcal{H}_0, \quad z_1 h_{n,m} = \sqrt{q^{-2m} - 1} q^m B h_{n,m+1}, \quad z_2 h_{n,m} = q^m B h_{n-1,m} \text{ on } \oplus_{n \in \mathbb{Z}} \oplus_{m \in \mathbb{N}} \mathcal{H}_{00}.$$

(12)

(13)

(14)

A common dense domain is obtained by considering the subspace of those elements of $\mathcal{H}$ which have at most a finite number of non-zero components in the direct sum. Only the representation (14) is faithful. A representation is irreducible if and only if one of the Hilbert spaces $\mathcal{H}_0$, $\mathcal{H}_{00}$ and $\mathcal{N}$ is isomorphic to $\mathbb{C}$ and the others are zero.

**Proof.** As the sets $\{0\}$ and $(0, \infty)$ are invariant under multiplication with powers of $q$, it follows from Definition 1(iii) that $\mathcal{K} := E((0)) \mathcal{H}$ and $\mathcal{G} := E((0, \infty)) \mathcal{H}$ are invariant under the actions of $z_1$ and $z_2$. Clearly, $\mathcal{H} = \mathcal{K} \oplus \mathcal{G}$. Since $\mathcal{K} = \ker(Q) = \ker(z_2^2 z_2)$, we have $z_2 = 0$ on $\mathcal{K}$. By Definition 1(iv), the restriction of $z_1$ to $\mathcal{K}$ is a $q$-normal operator, therefore its representation is given by (10). Setting $\mathcal{N} := \ker(z_1)$, $\mathcal{H}_0 := \mathcal{G}_0$ and $A := Z$, we obtain (12) y (13) from (10).

By Definition 1(ii) and the definition of $\mathcal{G}$, $z_2$ is a $q$-normal operator on $\mathcal{G}$ with $\ker(z_2) = \{0\}$. Therefore $z_2$ acts on $\mathcal{G} = \oplus_{n \in \mathbb{Z}} \mathcal{G}_0$ by the formulas on right hand side of (10). Note that

$$Q g_n = q^{2n} Z^2 g_n \text{ on } \mathcal{G}_n := \{g_n : g \in \mathcal{G}_0\},$$

(15)

with $\text{spec}(q^{2n} Z^2) \subset [q^{2n+2}, q^{2n}]$ and $q^{2n+2}$ is not an eigenvalue. Considering the disjoint union $(0, \infty) = \cup_{n \in \mathbb{Z}} (q^{2n+2}, q^{2n}]$, one readily sees that $\mathcal{G}_n = E((q^{2n+2}, q^{2n}]) \mathcal{G}$. From Definition 1(iii), it follows that

$$E((q^{2n+2}, q^{2n}]) z_1 \subset z_1 E((q^{2n+2}, q^{2n}]), \quad E((0, \infty) \setminus (q^{2n+2}, q^{2n}]) z_1 \subset z_1 E((0, \infty) \setminus (q^{2n+2}, q^{2n}]),$$

$$E((q^{2n+2}, q^{2n}]) z_2 \subset z_2 E((q^{2n+2}, q^{2n}]), \quad E((0, \infty) \setminus (q^{2n+2}, q^{2n}]) z_2 \subset z_2 E((0, \infty) \setminus (q^{2n+2}, q^{2n}]),$$

$$E((q^{2n+2}, q^{2n}]) z_1 z_2 \subset z_1 z_2 E((q^{2n+2}, q^{2n}]), \quad E((0, \infty) \setminus (q^{2n+2}, q^{2n}]) z_1 z_2 \subset z_1 z_2 E((0, \infty) \setminus (q^{2n+2}, q^{2n})).$$
and the same holds for \(z_1\) replaced by \(z_1^*\). Since \(Q^{-1}\) trivially commutes with \(E((q^{2(n+1)}, q^{2n}))\), we conclude that \(w := \sqrt{Q^{-1}}z_1\) and \(w^*\) leave \(\mathcal{G}_n\) invariant. On \(\mathcal{G}_n\), \(w\) still satisfies (9), thus its representation is given by (11). Therefore we can write \(\mathcal{G}_n = \oplus_{m \in \mathbb{N}} \mathcal{H}_{nm}\) and

\[
wh_{n,m} = \sqrt{q^{2m} - 1}h_{n,m+1},
\]

where \(h_{n,m}\) belongs to the \(m\)-th position in the direct sum \(\oplus_{m \in \mathbb{N}} \mathcal{H}_{nm}\). But \(\mathcal{G}_n\) is just a copy of \(\mathcal{G}_0\), so \(\mathcal{H}_{n0} = \mathcal{H}_{00}\) for all \(n \in \mathbb{Z}\). Equation (16) yields

\[
w^*wh_{n,m} = (q^{2m} - 1)h_{n,m},
\]

hence \(\mathcal{H}_{nm} := \{h_{n,m} : h \in \mathcal{H}_{00}\}\) is the eigenspace for the eigenvalue \(q^{2m} - 1\) of the restriction of \(w^*w\) to \(\mathcal{G}_n\). Definition 1(i) implies that \(w^*w\) and \(Q\) strongly commute. Therefore the restrictions of \(w^*w\) and \(Q\) to \(\mathcal{G}_n = E((q^{2n+2}, q^{2n}))\mathcal{G}_n\) also strongly commute. As a consequence, the self-adjoint operator \(Z\) from (15) leaves the eigenspaces \(\mathcal{H}_{nm}\) invariant. Denote the restriction of \(Z\) to \(\mathcal{H}_{00}\) by \(B\). Since \(\mathcal{H}_{nm}\) is an identical copy of \(\mathcal{H}_{00}\) in the \(m\)-th position of the direct sum \(\oplus_{m \in \mathbb{N}} \mathcal{H}_{nm}\), we get

\[
Zh_{n,m} = Bh_{n,m} \quad \text{for all} \quad h_{n,m} \in \mathcal{H}_{nm}.
\]

Moreover, \(B\) inherits the spectral properties from \(Z\) as required in Theorem 1. Finally, (10) and (18) give

\[
z_2h_{n,m} = q^nZh_{n-1,m} = q^nBh_{n-1,m}, \quad h_{n,m} \in \mathcal{H}_{nm},
\]

and from (15) and (16), we get

\[
z_1h_{n,m} = \sqrt{Q}wh_{n,m} = \sqrt{q^{2m} - 1}h_{n,m+1} = \sqrt{q^{2m} - 1}q^nBh_{n,m+1}
\]

for all \(h_{n,m} \in \mathcal{H}_{nm}\). This proves (14).

That the representation (14) is faithful follows from the fact that the representations of \(z_2\) and \(\omega\) are faithful, see [2] and [5], respectively. The statement about irreducible representations is obvious since writing any of the Hilbert spaces \(\mathcal{H}_0\), \(\mathcal{H}_{00}\) and \(N\) as an orthogonal sum of two non-zero subspaces will result in an orthogonal sum of non-trivial representations.

**4. Hilbert space representations on function spaces**

Note that the decomposition of \(\mathcal{H}\) in Theorem 1 is determined by the spectral properties of the self-adjoint operators \(Q\) and \(w^*w\). As well-known [7, Theorem VII.3], each self-adjoint operator \(T\) on a separable Hilbert space is unitarily equivalent to a direct sum of multiplication operators on \(L_2(\text{spec}(T), \mu)\). We will use this fact to realize the representations of Theorem 1 on \(L_2\)-spaces which will be the basis for studying function algebras on the 2-dimensional quantum complex plane in the next section.

The direct sum of Hilbert spaces \(\oplus_{n \in \mathbb{Z}} \oplus_{m \in \mathbb{N}} \mathcal{H}_{00}\) in Theorem 1 is isomorphic to the tensor product \(\ell_2(\mathbb{N}) \otimes (\oplus_{n \in \mathbb{Z}} \mathcal{H}_{00})\). Let \(\zeta\) be a \(q\)-normal operator acting on \(\oplus_{n \in \mathbb{N}} \mathcal{H}_{00}\) by the formulas on the right hand side of (10), and let \(\omega\) act on \(\ell_2(\mathbb{N})\) by the formulas in (11) with \(\mathcal{G}_0 = \mathbb{C}\). Then \(z_2\) from (19) and \(w\) from (16) can be written \(z_2 = id \otimes \zeta\) and \(w = \omega \otimes id\), respectively. It has been shown in [2, Theorem 1] that any \(q\)-normal operator \(\zeta\) is unitarily equivalent to a direct sum of operators of the following form: There exists a \(q\)-invariant Borel measure \(\mu\) on \([0, \infty)\) such that \(\zeta\) and \(\zeta^*\) act on \(\mathcal{H} = L_2([0, \infty), \mu)\) by

\[
\zeta f(t) = qt f(qt), \quad \zeta^* f(t) = tf(q^{-1}t), \quad f \in \text{dom}(\zeta) := \{h \in L_2([0, \infty), \mu) : \int t^2 |h(t)|^2 d\mu(t) < \infty\}.
\]

Here, the \(q\)-invariance of the measure means that \(\mu(qS) = \mu(S)\) for all Borel subsets \(S\) of \([0, \infty)\). Note that \(\ker(\zeta) = \{0\}\) if and only if \(\mu(\{0\}) = 0\). Therefore, in order to obtain a representation of the form (14), we have to assume that \(\mu(\{0\}) = 0\).
To turn \( \ell_2(\mathbb{N}) \) into an \( L_2 \)-space, we consider the operator \( ye := \sqrt{\omega^* \omega + 1} \) on \( \ell_2(\mathbb{N}) \). Denoting by \( \{ e_n : n \in \mathbb{N} \} \) the standard basis of \( \ell_2(\mathbb{N}) \), we have

\[
y e_n = q^{-n} e_n, \quad n \in \mathbb{N}.
\] (21)

Since the set of eigenvalues of \( y \) is discrete, \( y \) can be realized as a multiplication operator on \( L_2(\text{spec}(y), \sigma) \cong \ell_2(\mathbb{N}) \) by choosing the counting measure \( \sigma(\{ q^{-n} \}) = 1 \) on \( \text{spec}(y) \). Extending \( \sigma \) to a Borel measure on \([0, \infty)\) by setting \( \sigma([0, \infty) \setminus \text{spec}(y)) := 0 \), we get \( yg(s) = sg(s) \). The set

\[
\{ e_n := \chi_{\{ q^{-n} \}}(s) : n \in \mathbb{N} \}
\]
is an orthonormal basis of \( L_2(\text{spec}(y), \sigma) \), where \( \chi_{\{ q^{-n} \}} \) denotes the indicator function (4). Note that

\[
\chi_{\{ q^{-n} \}}(qs) = \chi_{\{ q^{-(n+1)} \}}(s) = e_{n+1} \quad \text{and} \quad \sqrt{(qs)^2 - 1} \chi_{\{ q^{-(n+1)} \}}(s) = \sqrt{q^{-2n} - 1} \chi_{\{ q^{-(n+1)} \}}(s),
\]
where we used \( f(t) \chi_{\{ t \}}(t) = f(t) \chi_{\{ t \}}(t) \) in the second equation. Hence

\[
\omega g(s) = \sqrt{(qs)^2 - 1} g(qs) \quad \text{and} \quad \omega^* g(s) = \sqrt{s^2 - 1} g(s^{-1})
\] (22)
for \( g \in \text{dom}(\omega) = \text{dom}(\omega^*) := \{ h \in L_2([0, \infty), \sigma) : \int s^2 |h(s)|^2 \, \text{d} \sigma(s) < \infty \} \). En particular,

\[
\omega^* e_1 = \sqrt{s^2 - 1} \chi_{\{ q^{-1} \}}(q^{-1}s) = \sqrt{1^2 - 1} \chi_{\{ q^{-1} \}}(q^{-1}s) = 0,
\] (23)
as required. Also, although \( ||\chi_{\{ 1 \}}|| = 0 \) and \( \chi_{\{ 1 \}}(qs) = \chi_{\{ q^{-1} \}}(s) = e_1 \), we have

\[
\sqrt{(qs)^2 - 1} \chi_{\{ 1 \}}(qs) = \sqrt{(qq^{-1})^2 - 1} \chi_{\{ 1 \}}(qs) = 0,
\]
so that (22) remains consistent.

Now, under the isomorphism \( L_2([0, \infty), \sigma) \otimes L_2([0, \infty), \mu) \cong L_2([0, \infty) \times [0, \infty), \sigma \otimes \mu) \), we obtain from (20) and (22) the following representation of \( z_1 = \sqrt{Q} w = \omega \otimes \sqrt{\zeta} \) and \( z_2 = \text{id} \otimes \zeta \),

\[
z_1 h(s, t) = \sqrt{(qs)^2 - 1} h(qs, t), \quad z_2 g(s, t) = qt g(s, qt),
\] (24)
where \( h \in \text{dom}(\omega) \otimes_{\text{alg}} \text{dom}(\zeta) \) and \( g \in L_2([0, \infty), \sigma) \otimes_{\text{alg}} \text{dom}(\zeta) \). To sum up, we have shown that the representations from (14) are unitarily equivalent to a direct sum of representations of the type described in (24).

Recall that \( N \oplus (\oplus_{k \in \mathbb{Z}} K_0) \) in Theorem 1 corresponds to the kernel of the \( q \)-normal operator \( z_2 \), and that \( q \)-normal operator in the representation (20) has a trivial kernel if and only if \( \mu(\{ 0 \}) = 0 \). Since \( \mu \) from the last paragraph was assumed to satisfy \( \mu(\{ 0 \}) = 0 \), we will now add a point measure \( \delta_0 \) centred at 0 to it. By unitary equivalence, we may assume that \( \delta_0(\{ 0 \}) = 1 \). Then the representation of \( z_1 \) on \( \oplus_{k \in \mathbb{Z}} K_0 \) is again unitarily equivalent to a direct sum of representations of the type (20). To realize these representations on our \( L_2 \)-space, we choose a \( q \)-invariant measure on \([0, \infty)\), say \( \nu \), assume again \( \nu(\{ 0 \}) = 0 \), take the product measure \( \nu \otimes \delta_0 \), and add it to \( \sigma \otimes \mu \). Then

\[
L_2([0, \infty) \times [0, \infty), \sigma \otimes \mu + \nu \otimes \delta_0) \cong L_2([0, \infty) \times [0, \infty), \sigma \otimes \mu) \oplus L_2([0, \infty) \times [0, \infty), \nu \otimes \delta_0),
\]
and on \( L_2([0, \infty) \times [0, \infty), \nu \otimes \delta_0) \), we have the representation

\[
z_1 h(s, t) = q s h(qs, t) = q \chi_{\{ 0 \}}(t) s h(qs, t), \quad z_2 g(s, t) = qt g(s, qt) = 0
\] (25)
for all $h$ such that $\int s^2 |h(0,s)|^2 \, dv(s) < \infty$ and for all $g$. Here, for functions depending on the second variable $t$, we used the fact that $\text{supp}(\delta_0) = \{0\}$. Again, by [2, Theorem 1] and the same argumentation as above, the representations from (13) are unitarily equivalent to a direct sum of representations of the type described in (25).

Finally, to obtain a non-trivial component $\mathcal{N} = \ker z_1 \cap \ker z_2$, we add to $\sigma \otimes \mu + \nu \otimes \delta_0$ the point measure $\epsilon \delta_0 \otimes \delta_0$, where $\epsilon = 0$ or $\epsilon = 1$ depending on whether $\mathcal{N} = \{0\}$ or $\mathcal{N} \neq \{0\}$. Summarizing, we have proven the following theorem:

**Theorem 2.** The Hilbert space representations of $\mathcal{O}(\mathbb{C}_q^2)$ from Theorem 1 are unitarily equivalent to a direct sum of representations of the following type: Let $\mu$ and $\nu$ be $q$-invariant Borel measures on $[0, \infty)$ such that $\mu(\{0\}) = \nu(\{0\}) = 0$. Denote by $\delta_0$ the Dirac measure centred at 0, and define a Borel measure $\sigma$ on $[0, \infty)$ by setting $\sigma(\{q^{-n}\}) := 1$ for all $n \in \mathbb{N}$ and $\sigma([0, \infty) \setminus \{q^{-n} : n \in \mathbb{N}\}) := 0$. For $\epsilon \in \{0, 1\}$, consider the Hilbert space

$$\mathcal{H} := \mathcal{L}_2([0, \infty) \times [0, \infty), \sigma \otimes \mu + \nu \otimes \delta_0 + \epsilon \delta_0 \otimes \delta_0),$$

and set

$$\text{dom}(z_1) := \{h \in \mathcal{H} : s \, h \in \mathcal{H} \text{ and } s \, \chi_{\{0\}}(t) \, h \in \mathcal{H}\}, \quad \text{dom}(z_2) := \{g \in \mathcal{H} : t \, g \in \mathcal{H}\},$$

where $\chi_{\{0\}}$ denotes the indicator function from (4). For $h \in \text{dom}(z_1)$ and $g \in \text{dom}(z_2)$, the actions of the generators of $\mathcal{O}(\mathbb{C}_q^2)$ are given by

$$z_1 h(s, t) = \sqrt{(qs)^2 - 1} h(qs, t) + q \chi_{\{0\}}(t) s h(qs, t), \quad z_2 g(s, t) = q t g(s, qt),$$

$$z_1^* h(s, t) = \sqrt{s^2 - 1} h(q^{-1} s, t) + \chi_{\{0\}}(t) s h(q^{-1} s, t), \quad z_2^* g(s, t) = t g(s, q^{-1} t).$$

Note that

$$\mathcal{H} = \mathcal{L}_2([0, \infty) \times [0, \infty), \epsilon \delta_0 \otimes \delta_0) \oplus \mathcal{L}_2([0, \infty) \times [0, \infty), \nu \otimes \delta_0) \oplus \mathcal{L}_2([0, \infty) \times [0, \infty), \sigma \otimes \mu)$$

$$= \mathcal{L}_2(\{0\} \times \{0\}, \epsilon \delta_0 \otimes \delta_0) \oplus \mathcal{L}_2([0, \infty) \times [0, \infty), \nu \otimes \delta_0) \oplus \mathcal{L}_2(\{q^{-n} : n \in \mathbb{N}\} \times [0, \infty), \sigma \otimes \mu)$$

and that the restriction of the representation (27) to one of the orthogonal components corresponds to one of the representations from (12)–(14). Of course, we could have formulated Theorem 2 for each of the orthogonal subspaces separately. The reason why we prefer to work with a single Hilbert space on the domain $[0, \infty) \times [0, \infty)$ will become clear in the next section.

5. **C*-algebra of continuous functions vanishing at infinity**

The aim of this section is to define a C*-algebra which can be viewed as the algebra of continuous functions on the 2-dimensional quantum complex plane vanishing at infinity. The definition will be motivated by a similar construction for the 1-dimensional quantum complex plane [3]. As a point of departure, we first look for an auxiliary *-algebra, where the commutation relations are considerable simple.

For the convenience of the reader, we recall the construction of the C*-algebra $C_0(\mathbb{C}_q)$ of continuous functions vanishing at infinity on the 1-dimensional quantum complex plane [3]. Given a representation of the type (20), consider the following *-subalgebra of $\mathcal{A}(\mathbb{C}_q^2([0, \infty), \mu))$:

$$\text{*-alg}\{C_0(\text{spec}(\{\zeta\}), U) := \{ \sum_{\text{finite}} f_k(|\zeta|) U^k : k \in \mathbb{Z}, f_k \in C_0(\text{spec}(\{\zeta\}), f_k(0) = 0 \text{ if } k \neq 0 \},$$
where \( \mu(\{0\}) = 0 \) and \( U \) denotes the unitary operator from the polar decomposition \( \zeta = U|\zeta| \). For all bounded continuous functions \( f \) on \( \text{spec}(|\zeta|) \), the operators \( f(|\zeta|) \) and \( U \) satisfy the commutation relation

\[
U f(|\zeta|) = f(q|\zeta|)U.
\]

In [3], a representation of the type (20) of a \( q \)-normal operator \( Z = U|Z| \) was said to be universal if \( \text{spec}(|Z|) = [0, \infty) \), or equivalently if \( \text{supp}(\mu) = [0, \infty) \). It has the universal property that

\[
*\text{-alg}\{C_0(\text{spec}(|Z|), U) \ni \sum_{f \in \text{finite}} f_k(|Z|)U^k \rightarrow \sum_{f \in \text{finite}} f_k(|\zeta|)U^k \in *\text{-alg}\{C_0(\text{spec}(|\zeta|), U) \}
\]

yields always a well-defined surjective *-homomorphism. Although the exact definition in [3] is slightly abstract, [3, Theorem 3.3] states that \( C_0(C_q) \) is isomorphic to the norm closure of \(*\text{-alg}\{C_0(\text{spec}(|Z|), U) \in B(L_2([0, \infty), \mu)) \).

Motivated by the previous description, we call a representation from Theorem 2 universal if \( \epsilon = 1 \) and \( \text{supp}(\mu) = \text{supp}(\nu) = [0, \infty) \). Such \( q \)-invariant measures can be obtained, for instance, by taking the Lebesgue measure \( \lambda \) on \([q, 1]\) and setting

\[
\mu(M) = \sum_{k \in \mathbb{Z}} \lambda(q^{-k}(M \cap (q^{k+1}, q^k])).
\]

Given a universal representation, consider the polar decompositions \( z_1 = U|z_1| \) and \( z_2 = V|z_2| \). For all \( h \in \text{dom}(|z_1|) = \text{dom}(z_1) \) and \( g \in \text{dom}(|z_2|) = \text{dom}(z_2) \), (27) and (28) imply

\[
|z_1|h(t, s) = (\chi_{[-1, \infty)}(s)\sqrt{s^2 - 1}t + s\chi_{[0]}(t))h(t, s), \quad |z_2|g(t, s) = tg(t, s).
\]

Since

\[
\text{ran}(|z_1|) = \ker(|z_1|) = \text{ran}(\chi_{(0, \infty)}(t)\chi_{[q^{-1}, \infty)}(s) + \chi_{[0]}(t)\chi_{(0, \infty)}(s)),
\]

\[
\text{ran}(|z_2|) = \ker(|z_2|) = \text{ran}(\chi_{(0, \infty)}(t)),
\]

it follows from (27) that

\[
Uh(s, t) = \left(\chi_{(0, \infty)}(t)\chi_{[q^{-1}, \infty)}(qs) + \chi_{[0]}(t)\chi_{(0, \infty)}(s)\right)h(qs, t),
\]

\[
Vh(s, t) = \chi_{(0, \infty)}(t)h(s, qt),
\]

for all \( h \in \mathcal{H} \), where we used \( \chi_{(0, \infty)}(qr) = \chi_{(0, \infty)}(r) \). Their adjoints act on \( \mathcal{H} \) by

\[
U^*h(s, t) = \left(\chi_{(0, \infty)}(t)\chi_{[q^{-1}, \infty)}(s) + \chi_{[0]}(t)\chi_{(0, \infty)}(s)\right)h(q^{-1}s, t),
\]

\[
V^*h(s, t) = \chi_{(0, \infty)}(t)h(s, q^{-1}t).
\]

From (33)–(36), we get

\[
U^*U = \chi_{(0, \infty)}(t)\chi_{[q^{-1}, \infty)}(s) + \chi_{[0]}(t)\chi_{(0, \infty)}(s),
\]

\[
UU^* = \chi_{(0, \infty)}(t)\chi_{[q^{-1}, \infty)}(qs) + \chi_{[0]}(t)\chi_{(0, \infty)}(s),
\]

\[
V^*V = \chi_{(0, \infty)}(t) = VV^*.
\]

Using again \( \chi_{(0, \infty)}(q^{\pm 1}t) = \chi_{(0, \infty)}(t) \) and \( \chi_{[0]}(q^{\pm 1}t) = \chi_{[0]}(t) \), one easily sees that

\[
UV = VU, \quad UV^* = V^*U, \quad U^*V = VU^*, \quad U^*V^* = V^*U^*.
\]
Considering Borel measurable functions \( f \) on \([0, \infty) \times [0, \infty)\) as multiplication operators by
\[
f h(s, t) := f(s, t) h(s, t),
\]
we obtain from (33)–(36) the following simple commutation relations:
\[
U f(s, t) = f(qs, t) U,
\]
\[
U^* f(s, t) = f(q^{-1}s, t) U^*,
\]
\[
V f(s, t) = f(s, qt) V,
\]
\[
V^* f(s, t) = f(s, q^{-1}t) V.
\]
(41)

(42)

In fact, the reason for choosing \( \zeta \) from (10) and \( y \) from (21) as multiplication operators was to obtain such simple commutation relations between functions and the phases from the polar decompositions of the generators of \( O(C^2_q) \). As a consequence,
\[
\text{Fun}(C^2_q) := \left\{ \sum_{\text{finite}} f_{nm}(s, t) U^{*n} V^{m} : f \in \mathcal{L}_\infty([0, \infty) \times [0, \infty)) \right\}
\]
(43)
is a \(*\)-subalgebra of \( B(\mathcal{H}) \), where
\[
U^{*n} := \left\{ \begin{array}{ll} U^n, & n \geq 0, \\ U^{-n}, & n < 0, \end{array} \right. \quad V^{m} := \left\{ \begin{array}{ll} V^n, & n \geq 0, \\ V^{-n}, & n < 0, \end{array} \right., \quad n \in \mathbb{Z}.
\]
(44)

Moreover, by (32) and the previous commutation relations, there exists for all \( k, l, m, n \in \mathbb{N}_0 \) a Borel measurable function \( p_{klmn} \) on \([0, \infty) \times [0, \infty)\) such that
\[
z_1^k z_2^l z_1^m z_2^n = p_{klmn}(s, t) U^{*k-1} V^{m-n}.
\]
(45)

Equations (43) and (45) are the motivation for the construction of the \( C^*\)-algebra of continuous functions on \( C^2_q \) vanishing at infinity. Before treating the quantum case, let us briefly review the classical \( C^*\)-algebra \( C_0(C^2) \). In analogy to the polar decomposition of the generators, write \( z_1 = e^{i\varphi} |z_1| \) and \( z_2 = e^{i\theta} |z_2| \). Let \( n, m \in \mathbb{Z} \). Given a function \( f_{nm} \in C_0([0, \infty) \times [0, \infty)) \), the assignment
\[
C^2 \ni (e^{i\varphi} |z_1|, e^{i\theta} |z_2|) \mapsto f_{nm}(|z_1|, |z_2|) e^{i\varphi n} e^{i\theta m} \in \mathbb{C}
\]
defines a function in \( C_0(C^2) \) if and only if
(a) \( f_{nm}(0, |z_2|) e^{i\varphi n} e^{i\theta m} \) does not depend on \( \varphi \) \( \implies f_{nm}(0, |z_2|) = 0 \) for \( n \neq 0 \),
(b) \( f_{nm}(|z_1|, 0) e^{i\varphi n} e^{i\theta m} \) does not depend on \( \theta \) \( \implies f_{nm}(|z_1|, 0) = 0 \) for \( m \neq 0 \).

Moreover, the following \(*\)-subalgebra of \( C_0(C^2) \),
\[
C_0(C^2) := \left\{ \sum_{\text{finite}} f_{nm}(|z_1|, |z_2|) e^{i\varphi n} e^{i\theta m} : f_{nm} \in C_0([0, \infty) \times [0, \infty)), \ n, m \in \mathbb{Z}, \ f_{nm}(0, |z_2|) = 0 \text{ if } n \neq 0, \ f_{nm}(|z_1|, 0) = 0 \text{ if } m \neq 0 \right\},
\]
(46)

separates the points of \( C^2 \). By the Stone–Weierstraß theorem, its norm closure yields \( C_0(C^2) \).

To pass from the classical to the quantum case, we start with a universal representation from Theorem 2. For all bounded continuous functions \( g \) on \([0, \infty)\), the operators \( g(|z_1|), g(|z_2|) \in B(\mathcal{H}) \) are well-defined by the spectral theorem, and (32) shows that
\[
g(|z_1|) h(s, t) = g(\chi_{[0,\infty)}(s) \sqrt{s^2 - 1} t + s \chi_{[0]}(t)) h(s, t), \quad g(|z_2|) h(s, t) = g(t) h(s, t).
\]
By the universality of the representation, we have \( \|g(|z_i|)\| = \|g\|_\infty, \ i = 1, 2 \). In particular, the norm does not depend on the chosen measures of a universal representation. Similarly, for \( f \in C_0([0, \infty) \times [0, \infty)) \), the formula
\[
f(|z_1|, |z_2|) h(s, t) := f(\chi_{[0,\infty)}(s) \sqrt{s^2 - 1} t + s \chi_{[0]}(t), t) h(s, t), \quad h \in \mathcal{H},
\]
(47)
yields a well-defined operator in \( B(\mathcal{H}) \) with \( \|f(|z_1|, |z_2|)\| = \|f\|_\infty \). These observations lead to the following definition of \( C_0(C^2_q) \).
Definition 2. Given a universal representation of $\mathcal{O}(\mathbb{C}_q^2)$ from Theorem 2, let $z_1 = U|z_1|$ and $z_2 = V|z_2|$ be the polar decompositions of $z_1$ and $z_2$, respectively. The C*-algebra $C_0(\mathbb{C}_q^2)$ of continuous functions on the two-dimensional quantum complex plane vanishing at infinity is defined as the norm closure of

$$C_0(\mathbb{C}_q^2) := \left\{ \sum_{nm} f_{nm}(x)|q^{-1},\infty\rangle(s)\sqrt{s^2-1}t + s\chi(x_0)(t) , t) U^# nV^# m : n, m \in \mathbb{Z}, \right\}$$

(48)

$$f_{nm} \in C_0([0, \infty) \times [0, \infty)), \quad f_{nm}(0, t) = 0 \text{ if } n \neq 0, \quad f_{nm}(s, 0) = 0 \text{ if } m \neq 0$$

in $\mathcal{B}(\mathcal{H})$.

Apart from the non-commutativity in (41) and (42), the main difference to the classical case is the unusual expression in the first argument of the function $f_{nm}$. However, if we look at the representation on the orthogonal components of (29) separately, our formulas have a natural geometric interpretation. First note that the function $h_{00}(s,t) := \chi(x_0)(s)\chi(x_0)(t) \in \mathcal{H}$ generates the 1-dimensional invariant subspace $\mathcal{L}_2([0, \infty) \times [0, \infty), \epsilon \delta_0 \otimes \delta_0)$, and the representation of $C_0(\mathbb{C}_q^2)$ on it reads

$$\sum_{nm} f_{nm}(x)|q^{-1},\infty\rangle(s)\sqrt{s^2-1}t + s\chi(x_0)(t) , t) U^# nV^# m h_{00}(s,t) = f_{00}(0,0) h_{00}(s,t)$$

(49)

since $f_{nm}(0,0) = 0$ if $n \neq 0$ or $m \neq 0$. Obviously, (49) corresponds to evaluating functions on 2-dimensional complex plane at $(0,0)$. This 1-dimensional representation describes the only classical point $(0,0)$ of $\mathbb{C}_q^2$.

Next, on $\mathcal{L}_2([0, \infty) \times [0, \infty), \nu \otimes \delta_0)$, we have $z_2 = 0$ and can thus write

$$\sum_{n,m} f_{nm}(x)|q^{-1},\infty\rangle(s)\sqrt{s^2-1}t + s\chi(x_0)(t) , t) U^# nV^# m = \sum_n f_{n0}(s,0) U^# n.$$ 

(50)

Recalling that $z_1$ acts on $\mathcal{L}_2([0, \infty) \times [0, \infty), \nu \otimes \delta_0)$ as a $q$-normal operator and comparing (50) with (30) shows that the restriction of $C_0(\mathbb{C}_q^2)$ to $\mathcal{L}_2([0, \infty) \times [0, \infty), \nu \otimes \delta_0)$ generates $C_0(\mathbb{C}_q^2)$. This representation corresponds to an inclusion $\mathbb{C}_q \times \{0\} \subset \mathbb{C}_q^2$.

Finally, on $\mathcal{L}_2([0, \infty) \times [0, \infty), \sigma \otimes \mu)$, we have $t = |z_2| > 0$ and $\chi(x_0)(s)\sqrt{s^2-1} = |\omega|$, see (22). Thus the representation of the functions from (48) can be written

$$\sum_{nm} f_{nm}(|\omega| t, t) U^# nV^# m.$$ 

Classically we get, for all $|\omega| \geq 0,$

$$\sum_{nm} f_{nm}(|\omega| t, t) e^{i\omega m} e^{i\theta m} = \sum_{nm} f_{nm}(0,0) e^{i\omega n} e^{i\theta m} = f_{00}(0,0).$$

Therefore these functions separate only the points of $\mathbb{C}_2 \setminus \mathbb{C} \times \{0\}$ and the whole subspace $\mathbb{C} \times \{0\}$ gets identified with the single point $(0,0)$. Geometrically, this corresponds to a 2-dimensional complex plane, where $\mathbb{C} \times \{0\}$ is shrunk to one point.

Arguing backwards, we can say that the representation from (24) corresponds to a 2-dimensional quantum complex plane, where $\mathbb{C}_q \times \{0\}$ is shrunk to a point, and that $\mathbb{C}_q \times \{0\}$ gets glued into this space by the representation (25). Moreover, the origin of the 2-dimensional quantum complex plane is the only classical point described by the 1-dimensional representation (49).
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