Nonlinear polarisation and dissipative correspondence between
low frequency fluid and gyrofluid equations

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Abstract

The correspondence between gyrofluid and low frequency fluid equations is examined. The lowest
order conservative effects in ExB advection, parallel dynamics, and curvature match trivially. The
principal concerns are polarisation fluxes, and dissipative parallel viscosity and parallel heat fluxes.
The emergence of the polarisation heat flux in the fluid model and its contribution to the energy
theorem is reviewed. It is shown that gyroviscosity and the polarisation fluxes are matched by the
finite gyroradius corrections to advection in the long wavelength limit, provided that the differences
between gyrocenter and particle representations is taken into account. The dissipative parallel
viscosity is matched by the residual thermal anisotropy in the gyrofluid model in the collision
dominated limit. The dissipative parallel heat flux is matched by the gyrofluid parallel heat flux
variables in the collision dominated limit. Hence, the gyrofluid equations are a complete superset
of the low frequency fluid equations.

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I. INTRODUCTION

Low frequency reduced fluid equations are used to treat a variety of phenomena in magnetised plasma dynamics, including turbulence \cite{1, 2} and tearing modes \cite{3}. The usual derivation path is to start with the Braginskii collisional fluid equations \cite{4}, and then to solve for the velocity in the Lorentz force rather than the inertia. To lowest order in the inertia/gyrofrequency ratio, balance among the principal forces is assumed, with pressure and the electric forces balancing the magnetic force. The result is a combination of E-cross-B and diamagnetic flow terms, arising from the electric field and pressure gradient, respectively. In the conventional magnetohydrodynamic limit (MHD: considering a single velocity for all species and a single, total pressure) the diamagnetic velocity is necessarily ordered small, but in general the electron parallel dynamics holds the electric and pressure forces to similar level. The latter is the “adiabatic response” which couples the electron pressure to the electric parallel current through pressure forces and compressional motion parallel to the background magnetic field. Hence the MHD ordering cannot be taken, and E-cross-B and diamagnetic flows are at similar level.

The correction due to the inertia becomes the polarisation drift, which is so called because it is opposite for electrons and ions. The main contribution in MHD ordering is due to the time dependence of the electric field. Nonlinear advection of the velocity field represents the polarisation nonlinearity. It is responsible for maintaining drift wave self sustained turbulence \cite{5, 6, 7} and also any Reynolds stress flow phenomena \cite{8, 9}. In general the diamagnetic contributions enter at the same order and give rise to what is generally called “gyroviscosity” — the cancellation of advection by the diamagnetic velocity in the equation of motion \cite{10}. Once it is established that the polarisation drift enters at all, it is necessary to keep it also in the ion temperature dynamics, since the dynamics of the temperature and density are at similar order. The logical chain to this starts with the adiabatic response, including the compression in the electron density equation, then noting the equality of the electron and ion densities and the similar order in compression of the polarisation and parallel currents, and finally the polarisation drift entering the ion density and temperature at similar order \cite{11}.

Less familiar is the same phenomenon concerning the heat flux. The Braginskii model starts with a drifting Maxwellian distribution, with not only arbitrary velocity but also
velocity gradient, to lowest order in the inertia/collision frequency ratio. At next order, velocity gradients appear but heat flux gradients do not. This effectively and implicitly assumes that heat fluxes are subthermal: the heat flux is assumed to be smaller than the pressure times the velocity (see also Ref. [12] for similar considerations regarding implicit assumptions on the electron inertia and the magnetic current). However, this is not true even in the diamagnetic flows and heat fluxes, which in the presence of temperature gradients are of similar strength. The implicit assumption of small heat fluxes breaks down completely. When the MHD velocity ordering also breaks down, it follows that the diamagnetic heat flux is of similar magnitude as the pressure times the E-cross-B velocity. This has been noted before, by a treatment showing that the heat fluxes must be kept in the gyroviscosity even to obtain the standard form of the polarisation current \[13\]. However, one has to go further and consider inertia in the formulation of the perpendicular heat flux itself \[14\]. This is one order higher in the moment hierarchy considered by Braginskii, which is why it is rarely considered. Nevertheless, polarisation enters the heat flux equation as the correction due to finite inertia upon the diamagnetic heat flux balance. Then, since the polarisation enters the density and temperature equations at the same order, and the polarisation heat flux and velocities are also of the same order, the polarisation heat flux should be considered in the temperature equation. We will review this herein as a preparation for establishing the correspondence between the gyrofluid and low frequency Braginskii equations. Ultimately, correspondence is found in the nonlinear advection effects only if the polarisation heat flux is kept in the fluid model.

These polarisation phenomena enter the gyrofluid equations differently. The gyrofluid equations have an entirely different derivation path \[15, 16\], starting with the gyrokinetic equation with the low frequency and small amplitude orderings already taken \[17\]. Polarisation enters the charge balance equation rather than the density and temperature equations, since the latter are for the gyrocenters and not the particles themselves. The polarisation density balances differences in the gyrocenter densities, maintaining quasineutrality \[18\]. The time derivative of this gyrokinetic polarisation equation gives a relation analogous to the current balance (equivalently, vorticity) equation in the fluid models, with the time derivative of the polarisation density being the same as the divergence of the polarisation current. Underlying this is the Lie transformation between particle and gyrocenter coordinates at the gyrokinetic level \[19\]. Moments over this transform give the equations describing
the particle and gyrocenter representations of the moment variables, corresponding to the fluid and gyrofluid models, respectively.

The gyrofluid equations are of significance because they allow treatment of this drift dynamics at arbitrary order in the finite gyroradius parameter (generally, the square of the perpendicular wavenumber normalised to the gyroradius). Tearing modes and reconnection involve inertial layers which are thinner than the ion gyroradius \[20\]. Tokamak edge turbulence has a vorticity spectrum which always reaches down below the ion gyroradius \[21\]. Treatment of these is generally beyond the limits of equations whose derivation assumes the gyroradii are all small. Nevertheless, the low frequency Braginskii equations have a systematic derivation, and it is desirable to know whether the gyrofluid equations correspond properly to these under the limits within which the Braginskii equations are perfectly valid. That task is the purpose of this work. In the linear MHD limit the correspondence between gyroviscosity and the finite gyroradius corrections in the ion density equation were already shown \[15\]. Examination of the nonlinear gyroviscous “force” in the MHD limit found certain correspondences \[22\]. Herein, we complete the correspondence in the fully two-fluid limit. It is recovered only if the polarisation heat flux is kept in the fluid model. Viewed another way, this effect has always been present in the version of the gyrofluid model which keeps perpendicular and parallel temperature moments \[15, 16\]. The correspondence question is completed by examining the dissipation model in the gyrofluid equations concerning viscosity and parallel heat fluxes. Ultimately, the gyrofluid equations are found to recover the low frequency Braginskii equations, in the Braginskii limits of long wavelengths, small heat fluxes, and complete collisional dominance.

The following sections respectively concern (II) the polarisation heat flux and its effect on the free energy theorem within the low frequency fluid equations, then (III) the correspondences concerning polarisation in the density and temperature equations including all the finite gyroradius nonlinearities, then (IV) the collisional viscosity effects including correspondence to the anisotropic corrections sometimes included in turbulence equations, and also the contribution of heat fluxes to the viscosity, and then (V) the parallel heat flux effects, whose correspondence is the easiest to show. The gyrofluid equations in question are from the most general GEM (Gyrofluid ElectroMagnetic) model \[23\]. They will be introduced piece by piece as needed. The collisional fluid equations are much better known — see, e.g, the recent model including the anisotropy effect in viscosity and the full polarisation...
velocity treatment in Ref. [24]. A concluding commentary section (VI) is given at the end.

II. POLARISATION INCLUDING THE HEAT FLUX

Low frequency fluid equations can be derived directly using the equation of motion for each species [4],

\[ nM \left( \frac{\partial u}{\partial t} + u \cdot \nabla u \right) + \nabla \cdot \Pi + \nabla p = nZe \left( E + \frac{u}{c} \times B \right) \]  \( (1) \)

solving for \( u \) in the Lorentz force term to lowest order in \( \omega/\Omega_c \ll 1 \) and assuming an electrostatic perpendicular electric field with potential \( \phi \) (justified by \( \omega \ll k_B v_A \); i.e., the dynamics is too slow for dynamical Alfvénic compression),

\[ u_{\perp}^{(0)} = \frac{c}{B^2} B \times \nabla \phi + \frac{1}{nZe} \frac{c}{B^2} B \times \nabla p \]  \( (2) \)

noting this gives solely the perpendicular component. The parallel component has its own equation, derived separately. The polarisation corrections are found by inserting this \( u_{\perp}^{(0)} \) form into the inertia terms,

\[ u_{\perp} = u_{\perp}^{(0)} + \frac{M}{Ze} \frac{c}{B^2} B \times \left( \frac{\partial u_{\perp}^{(0)}}{\partial t} + u_{\perp}^{(0)} \cdot \nabla u_{\perp}^{(0)} \right) + \frac{1}{nZe} \frac{c}{B^2} B \times \nabla \cdot \Pi(u_{\perp}^{(0)}) \]  \( (3) \)

assuming flute mode ordering wherein \( u_{\parallel} \nabla \ll u_{\perp} \cdot \nabla \). This is the standard version [10], usually behind the derivation of the equations in turbulence models. Alternatively, a systematic procedure splitting the velocity into solenoidal and parallel pieces [25], which can also include a potential-flow compressional piece [26], may be used. The solenoidal flow potential becomes \( \phi \) under MHD ordering or generally a combination of \( \phi \) and \( p \). If drift ordering [27, 28] is then taken, the equations become identical to the reduced forms. Drift ordering refers to the small amplitude but unity-order nonlinearity limit used in the turbulence models.

Application of drift ordering to the velocity and including the diamagnetic pieces in \( \Pi \) results in the following form [10, 13, 14],

\[ u_{\perp} = \frac{c}{B^2} B \times \nabla \phi + \frac{1}{nZe} \frac{c}{B^2} B \times \nabla p - \frac{Mc}{Ze B^2} \frac{d}{dt} \left( \nabla \phi + \frac{1}{nZe} \nabla p \right) \]  \( (4) \)

where the \( d/dt \) operator includes the nonlinear E-cross-B advection

\[ \frac{d}{dt} = \frac{\partial}{\partial t} + v_E \cdot \nabla \quad \quad v_E = \frac{c}{B^2} B \times \nabla \phi \]  \( (5) \)
The last term in Eq. (4) is the polarisation velocity, whose charge-flux divergence is given by

\[ \nabla \cdot nZe \mathbf{u}_p = -\nabla \cdot \frac{nMc^2}{B^2} \frac{d}{dt} \left( \nabla_\perp \phi + \frac{1}{nZe} \nabla_\perp p \right) \] (6)

In the conventional gyro-Bohm normalisation for a single component plasma with singly charged ions this becomes

\[ \nabla \cdot \mathbf{u}_p = -\nabla \cdot \frac{d}{dt} \nabla_\perp (\phi + p_i) \equiv -\nabla \cdot \frac{d}{dt} \nabla_\perp W \] (7)

where \( p_i = \tau_i (n_i + T_i) \) and \( \tau_i \) is the background ion/electron temperature ratio. The pressure gradient is linearised, and each species has its density and temperature normalised to its own background. The flux and velocity divergence enter the same way because of the normalisation and ordering. The total ion force potential is denoted as \( W \). The time scale inferred by the divergence of the velocity is normalised to the profile scale and the sound speed, \( L_\perp/c_s \), where \( c_s^2 = T_e/M_i \) and \( L_{Te} \) is usually used for \( L_\perp \). The double perpendicular derivative is normalised to the square of the drift scale, \( \rho_s^2 = c^2 T_e M_i/e^2 B^2 \), and it is useful to note that \( \tau_i \rho_s^2 = \rho_i^2 \), which makes the role of the finite gyroradius explicit. The above considerations constitute what is also called local ordering in the context of turbulence computation.

A similar treatment for the heat flux starts with the \( (Mv^2/2)v \) moment of the kinetic equation, analogous to the \( Mv \) moment and the equation of motion. The heat flux equation is given by

\[ \frac{d\mathbf{q}}{dt} + \frac{5}{2} \frac{p}{M} \nabla T = \frac{e}{Mc} \mathbf{q} \times \mathbf{B} \] (8)

under drift ordering (Eq. 11 of Ref. [14], after the diamagnetic cancellation is taken). Solving this to lowest order neglecting the inertial effects yields the diamagnetic heat flux,

\[ \mathbf{q}_\perp^{(0)} = \frac{5}{2} \frac{p}{Ze B^2} \mathbf{B} \times \nabla T \] (9)

Using the same ordering and normalisation conventions as for the velocity, we find the divergence of the polarisation correction for singly charged ions,

\[ \nabla \cdot \mathbf{q}_p = -\frac{5}{2} \tau_i \nabla \cdot \frac{d}{dt} \nabla_\perp T_i \] (10)

under gyro-Bohm normalisation (\( \mathbf{q} \) is normalised the same way as \( p\mathbf{u} \)). Clearly, this is the same order as the velocity polarisation divergence if the gradients of the state variables (potential, densities and temperatures) are all comparable. The polarisation velocity divergence
enters both the density and temperature equations, and the polarisation heat flux divergence enters the temperature equation. This sort of consistency is well known for the diamagnetic fluxes themselves, since with similar \((e/T)\nabla \phi\) and \(\nabla \log T\) and \(\nabla \log n\) they all enter at the same order with each other and with toroidal compression of the E-cross-B velocity \(\mathcal{K} \left[W + G\right]\) (cf. discussion and manipulations in Ref. \[31\]), but the same results concerning the polarisation fluxes was not widely known before Refs. \[13, 14\], and is still routinely missed by low frequency fluid models.

Under the above considerations we have the normalised ion density and temperature equations with polarisation divergences,

\[
\begin{align*}
\frac{dn_i}{dt} - \nabla \cdot d\frac{dt}{\nabla W + \nabla \|u\|} &= \mathcal{K}(W + G) \\
\frac{3}{2} \frac{d}{dt} \frac{dT_i}{dT} - \nabla \cdot d\frac{dt}{\nabla W - \frac{5}{2} \tau_i \nabla \cdot \frac{d}{dt} \nabla T_i + \nabla \| (u\| + q_i\|) &= \mathcal{K}(W + G) + \frac{5}{2} \tau_i \mathcal{K}(T_i)
\end{align*}
\]

The parallel velocity divergence is included in both equations and the heat flux divergence in the temperature equation. The terms denoted by \(\mathcal{K}\) are the remnant divergences of the E-cross-B and diamagnetic velocities (represented in total by \(W\) and diamagnetic specific heat flux (represented by \(5\tau_i T_i/2\), due to the inhomogeneous magnetic field, after the diamagnetic cancellation is taken in the temperature equation \[32\]. The curvature operator is then defined, e.g.,

\[
\mathcal{K}(\phi) = -\nabla \cdot \mathbf{v}_E = -\nabla \cdot \frac{c}{B^2} \mathbf{B} \times \nabla \phi
\]

in terms of the E-cross-B divergence. The quantity \(G\) in the ion density and temperature equations arises from thermal anisotropy. It is given by \[24\],

\[
G = \frac{0.96}{12\nu_i} \left[ \mathcal{K}(W) - 4\nabla \| u\| \right]
\]

in the collisional limit and represents viscous dissipation, with \(\nu_i\) the ion collision frequency normalised to \(c_s/L_{\perp}\).

For the electrons the convention is to neglect the mass everywhere except in parallel inertia (entering the parallel velocity and heat flux equations). The electron density equation is given by

\[
\frac{dn_e}{dt} + \nabla \| v_e \| = \mathcal{K}(\phi - p_e)
\]

in which anisotropy and polarisation (electron viscosity and inertia) are neglected. The electron pressure gradient is linearised in the same was as for the ions, with the minus sign
reflecting the normalised temperature/charge ratio. The quasineutrality condition is given by the subtraction of the two density equations and neglecting the space charge density, so that
\[
\nabla \cdot \frac{d}{dt} \nabla W = \nabla \| (u\| - v\|) - \mathcal{K}(p_e + p_i + G)
\] (16)
equivalently, \( \nabla \cdot \mathbf{J} = 0 \), whose three pieces are the polarisation, parallel, and diamagnetic divergences, respectively. We note that
\[
J\| = u\| - v\|
\] (17)
defines the parallel current (under the normalisation); this is usually used to eliminate \( v\| \) in favour of \( J\| \). We may further subtract this from the ion temperature equation to obtain
\[
\frac{3}{2} \frac{dT_i}{dt} - \frac{5}{2} \tau_i \frac{d}{dt} \nabla T_i + \nabla \| (v\| + q_i\|) = \mathcal{K}(\phi - p_e) + \frac{5}{2} \tau_i \mathcal{K}(T_i)
\] (18)
eliminating the polarisation divergence in the velocity but not the heat flux. In this equation, the explicit ion velocity divergences are replaced by the electron ones, but the ion heat flux divergences remain. These are the polarisation and diamagnetic heat flux terms, respectively the second and last terms in the line above.

A. Free energy in the fluid model

The complete set of equations in the fluid model is given by
\[
\nabla \cdot \frac{d}{dt} \nabla W = \nabla \| J\| - \mathcal{K}(p_e + p_i + G)
\] (19)
\[
\frac{dn_e}{dt} + \nabla \| v\| = \mathcal{K}(\phi - p_e)
\] (20)
\[
\frac{3}{2} \frac{dT_e}{dt} + \nabla \| (v\| + q_e\|) = \mathcal{K}(\phi - p_e) - \frac{5}{2} \mathcal{K}(T_e)
\] (21)
\[
\frac{3}{2} \frac{dT_i}{dt} - \frac{5}{2} \tau_i \frac{d}{dt} \nabla T_i + \nabla \| (v\| + q_i\|) = \mathcal{K}(\phi - p_e) + \frac{5}{2} \tau_i \mathcal{K}(T_i)
\] (22)
\[
\frac{du\|}{dt} + \nabla \| (p_e + p_i + 4G) = 0
\] (23)
\[
\beta_e \frac{\partial A\|}{\partial t} + \mu_e \frac{dJ\|}{dt} + \nabla \| (\phi - p_e) = -R_{ei}
\] (24)
where \( n_i \) and \( n_e \) are equivalent, \( v\| \) is given by \( u\| - J\| \), and the parallel heat fluxes \( q_e\| \) and \( q_i\| \) and the resistive dissipation \( R_{ei} \) are left undetermined (at this level they may be given their
Braginskii dissipative formulae \footnote{4}). The factor of $4G$ in the parallel momentum equation is also the result of anisotropy. Except for the retentions of the polarisation heat flux in Eq. (22) and the electron inertia in Eq. (24), these equations are the same as those given in Ref. \footnote{24}. The normalisation convention is the standard gyro-Bohm one, with $\nabla_\parallel$ normalised against $L_\perp$, not $qR$, which is why the un-scaled forms for

$$\beta_e = \frac{4\pi p_e}{B^2} \quad \mu_e = \frac{m_e}{M_D}$$

are used. The factor $\tau_i$ gives the background $T_i/T_e$ ratio. The pressures are linearised as above.

The free energy of the system is given by

$$E = \int dV \frac{1}{2} \left[ |\nabla_\perp W|^2 + (1 + \tau_i) n_e^2 + \frac{3}{2} T_e^2 + \frac{3}{2} \tau_i T_i^2 \right. \right.$$  

$$+ u_\parallel^2 + \beta_e \left| \nabla_\perp A_\parallel \right|^2 + \mu_e J_\parallel^2 + \frac{5}{2} |\tau_i \nabla_\perp T_i|^2 \right]$$

(26)

where $\int dV$ denotes complete spatial integration and now the $\tau_i$ factors are put in explicitly. Except for the last term, due to the polarisation heat flux, this has been analysed before \footnote{33, 34}. Insertion of Eqs. (19-24) into $\partial E/\partial t$ finds this time derivative to vanish except for the dissipative terms (here, there are no gradient source terms since the profile gradients are kept within the dependent variables of the model; without explicit sources this corresponds to decaying cases initialised with a finite profile and a random bath of fluctuations, as in Ref. \footnote{35}).

The last term in Eq. (26) represents the polarisation heat flux. If Eq. (22) is multiplied by $\tau_i T_i$ and integrated, the $\partial/\partial t$ terms yield the two terms in Eq. (26) explicitly dependent upon $T_i$. If Eq. (19) is multiplied by $W$ and integrated, the $\partial/\partial t$ term yields the term in Eq. (26) explicitly dependent upon $W$. The resulting term $W \nabla_\parallel J_\parallel$ is balanced by the contributions $\tau_i n_e$ and $\tau_i T_i$ times $\nabla_\parallel J_\parallel$ in Eqs. (20,22) and the contribution $J_\parallel \nabla_\parallel \phi$ coming from Eq. (24). And henceforth. The appearance of a heat flux term in the energy may be unfamiliar in a fluid model, but it is known from the gyrofluid model (cf. Ref. \footnote{36} and below) and also from treatments of extended fluid dynamics \footnote{37}. Due to the relation between the polarisation heat flux (as, up to coefficients, the time derivative of the curl of) the diamagnetic heat flux, the energy contribution is equivalent to $2/5$ times the square of the diamagnetic heat flux (up to normalisation). This is equivalent to the relation between the polarisation velocity and the E-cross-B and diamagnetic velocities, and the appearance of the square of the latter
in the form of the perpendicular kinetic energy. For general gradient driven turbulence, the heat flux and velocity pieces are of comparable magnitude.

The rest of the fluid model is given by the dissipation due to parallel heat fluxes and viscosity. The latter comes from temperature anisotropy, in this case $G$. Starting with a diagonal pressure tensor

$$\mathbf{P} = \text{diag}\{p_{\perp}, p_{\perp}, p_{\parallel}\} \tag{27}$$

we split it into isotropic and traceless parts,

$$\mathbf{P} = p\mathbf{g} + \Pi \quad \Pi = 2G \text{diag}\{-1, -1, 2\} \tag{28}$$

where $\mathbf{g}$ is the metric tensor, identifying $6G$ with the anisotropy $\Delta p = p_{\parallel} - p_{\perp}$ (see Section IV, below). With the factor of density common, $\Delta p$ is equivalent to $\Delta T$. The viscosity model sets collisional dissipation of $G$ against perpendicular and parallel velocity divergences as above. Similarly, the parallel heat flux formulae are given by setting their collisional dissipation against the corresponding parallel temperature gradients, e.g.,

$$\frac{5}{2}\kappa_i \nu_i q_i = -\frac{5}{2} T_i \nabla T_i \tag{29}$$

for ions, with the coefficient set such that the familiar formula \cite{4} with $\kappa_i = 3.9$ results.

The remainder of this paper is concerned with recovery of these formulae (polarisation effects in the vorticity and ion temperature equations, viscosity through anisotropy, and the parallel heat fluxes), from the gyrofluid model under the same ordering conventions as for this one.

### B. Interlude — bracket notation

In several treatments of the equations of turbulence in confined plasmas the nonlinearities are explicitly written in a form which makes their conservation properties obvious. Basically, e.g., $v_E \cdot \nabla n_e$ is written as $[\phi, n_e]$, where the bracket involves the perpendicular derivatives involved in the drift motion. It is variously written as

$$[\phi, n_e] = \frac{\partial \phi}{\partial x} \frac{\partial n_e}{\partial y} - \frac{\partial n_e}{\partial x} \frac{\partial \phi}{\partial y} \tag{30}$$

in slab or local fluxtube treatments, or as

$$[\phi, n_e] = \frac{1}{r} \left( \frac{\partial \phi}{\partial r} \frac{\partial n_e}{\partial \theta} - \frac{\partial n_e}{\partial r} \frac{\partial \phi}{\partial \theta} \right) \tag{31}$$
in “cylinder” treatments, which with the field aligning coordinate transformations 
\( x = r^2/a^2 \) and 
\( y_k = q(\theta - \theta_k) - \zeta \) and 
\( s = \theta \) with 
\( q = q(r) \) becomes

\[
[\phi, n_e] = \frac{2}{a^2} \left[ \frac{\partial \phi}{\partial x} \left( q \frac{\partial n_e}{\partial y} + \frac{\partial n_e}{\partial s} \right) - \frac{\partial n_e}{\partial x} \left( q \frac{\partial \phi}{\partial y} + \frac{\partial \phi}{\partial s} \right) \right]
\] (32)

at \( s = \theta_k \) since \( \partial y_k/\partial r \) vanishes there. Under fluxtube ordering \( \partial/\partial s \) is small compared to 
either \( \partial/\partial x \) or \( \partial/\partial y \) and the factor of \( q \) is replaced by a constant, and Eq. (32) reverts to
the form in Eq. (30) with a multiplier of \( 2q/a^2 \) which can be normalised away. More detail
on this is given in Ref. [38].

With the bracket notation we make use of the following properties

\[
\int dV \ [f, g] = 0 \quad f[f, g] = \frac{1}{2} [f^2, g] \quad g[f, g] = \frac{1}{2} [f, g^2]
\] (33)
in the manipulations below. That is, the bracket is a perfect divergence, and both energy
and entropy are conserved. Useful manipulations include

\[
\nabla^2_{\perp} [f, g] = \nabla \cdot [f, \nabla_{\perp} g] + \nabla \cdot [\nabla_{\perp} f, g]
\] (34)

\[
\nabla \cdot [f, \nabla_{\perp} g] = [\nabla_{\perp} f, \nabla_{\perp} g] + [f, \nabla^2_{\perp} g]
\] (35)

In any of these the perpendicular subscript may be regarded as understood, as is the con-
traction implied by \( [\nabla f, \nabla g] \).

C. Free energy, adiabatic response, and MHD ordering

The fluid model’s vorticity equation in this notation is

\[
\frac{d}{dt} \nabla^2_{\perp} W + [\nabla_{\perp} \phi, \nabla_{\perp} W] = \nabla_{\parallel} \cdot J_{\parallel} - \mathcal{K}(p_e + p_i + G)
\] (36)

where use is made of

\[
\frac{d}{dt} = \frac{\partial}{\partial t} + [\phi, ]
\] (37)

Since the bracket is antisymmetric, the second term in Eq. (36) is equivalent to \([\nabla_{\perp} \phi, \nabla_{\perp} p_i]\),
and hence this “gyroviscous correction” is a proper warm-ion effect. The polarisation terms
may be manipulated to show

\[
\frac{\partial}{\partial t} \nabla_{\perp}^2 W + \nabla \cdot [\phi, \nabla_{\perp} W] = \frac{\partial}{\partial t} \nabla_{\perp}^2 \phi + \nabla \cdot [W, \nabla_{\perp} \phi] + \nabla_{\perp}^2 \frac{dp_i}{dt}
\] (38)
Under MHD ordering the last term on the right hand side is dropped, as was done in Ref. [10], unfortunately without the explicit statement that this depends strictly on $\nabla_\perp p_i \ll n_e e \nabla_\perp \phi$ remaining valid (un-normalised units).

MHD ordering implies the lack of a $\tau_i \nabla || J ||$ term in the energetics. This can only be reconciled if $\nabla || J ||$ is neglected in all of the continuity equations, not just the one for $n_e$, since quasineutrality ties the ion and electron dynamics together. If $\nabla || J ||$ is neglected then $\nabla || p_e$ itself must be neglected in Eq. (24). Hence, we would be back not only to MHD ordering but to reduced MHD itself. This may be demonstrated by alternatively multiplying Eq. (19) by $\phi$ or $W$ and integrating, and observing the logical consequences. Either reduced MHD is taken in its entirety, or the consequences of the adiabatic response are taken to their conclusion. No intermediate version is energetically closed [11].

It is the adiabatic response in the electron dynamics which disallows the use of MHD ordering in this kind of turbulence. In Eq. (24) the two static force terms (those not dependent on $J ||$ or $A ||$) are often the largest, and even for edge turbulence there is a partial cancellation between them. Their difference determines $J ||$, mediated by induction, inertia, or resistivity according to whether finite $\beta_e$, finite $\mu_e$, or finite $\nu_e$ is the strongest. Once $\nabla || p_e$ is kept in Eq. (21) then $\nabla || J ||$ must be kept in Eq. (20), since in the energetics $n_e \nabla || J || + J || \nabla || n_e$ must become a total divergence. Since it is $p_e$ which appears in Eq. (24), then $\nabla || J ||$ must also be kept in Eq. (21). Now, the densities $n_e$ and $n_i$ are equivalent, so the appearance of $\nabla || J ||$ in Eq. (20) implies the appearance of $\nabla \cdot u_p$ in the equation for $n_i$, which is made explicit if Eqs. (20,19) are subtracted (eliminating the $\nabla || J ||$ term). To conserve against the implied $\tau_i \nabla || J ||$ term, the pressure effects in polarisation must be kept since Eq. (19) must be multiplied by $W$ rather than $\phi$, since $W$ contains $\tau_i n_e$. However, $W$ also contains $\tau_i T_i$. Hence, $\nabla \cdot u_p$ must also be kept in Eq. (22), wherein it has been replaced by $\nabla || J ||$. Only now is the energetic loop started by $\nabla || p_e$ in Eq. (24) closed, since now $-W \nabla || J ||$ closes against $\tau_i (n_i + T_i) \nabla || J ||$ from Eqs. (20,21,22) as well as against $-J || \nabla || \phi$ from Eq. (24). Hence the logical chain: adiabatic response in the equation for $J ||$, parallel compression in the equations for $n_e$ and $T_e$, quasineutrality, by which $\nabla || J ||$ implies $\nabla \cdot u_p$, and finally the qualitative similarity among densities and temperatures. All this is forced by the similarity in magnitude among $\nabla_\perp \{ \phi, n_e, T_e, T_i \}$, caused by the adiabatic response. Similar consequences may be found among the curvature terms — essentially, the retention of diamagnetic compression in the continuity equations also forces the complete two fluid version of ion polarisation. The
above analysis has been given for the local form of the equations; the corresponding one for the global form was given in Ref. [11].

D. Interlude — relation to extended fluid and extended MHD models

The low frequency fluid models are themselves distinct from what is called extended fluid dynamics or extended MHD [39, 40, 41, 42, 43, 44, 45, 46]. In the latter the low frequency is invoked to obtain expressions for heat fluxes and viscosities. These are not, however, expressed in terms of drifts and polarisation, but left in the native form with velocity and heat flux vectors. Specifically, the steps in Eqs. (1–4) are not taken. Moreover, explicit time dependence of the heat flux and dissipative viscosities are neglected (thermal anisotropy is assumed to be small – see below). Fluid drift theory replaces the vector forms with scalar quantities in the list of dependent variables; for example, $\phi$ and $p$ and $u_\parallel$ are the variables with which $u$ is described. Further to that are the gyrofluid models [15, 16, 23, 47], to be discussed below. No matter the complexity, extended fluid models and fluid drift models break down when $k_\perp \rho_i$ becomes unity or larger, as it always does in tokamak edge turbulence [21, 47, 48, 49]. Gyrofluid models are required to overcome this. Of course, since they have a different formulation, it is desired to know how well they recover the fluid forms when the latter are valid. That is the point of this work.

III. GYROFLUID FLR NONLINEARITIES AND FLUID GYROVISCOSITY

The gyrofluid model has a different structure from the fluid one — like the underlying gyrokinetic model, the moment variables (the model for the kinetic distribution function) are advanced independently for each species, and then the field equations (polarisation and induction) are solved for the electrostatic and parallel magnetic potentials. Analysis of the gyrofluid moment equations in the various limits proceeds the same way for each species. Due to the correspondences involved we concentrate mainly on the ions. Consideration of thermal forces at the end will then involve the electrons. This section is concerned with the gyroviscosity effects in the fluid model, meaning essentially all the differences in the polarisation between the general one and the MHD one (the latter involving $\phi$ only). We will show how these emerge naturally from the finite gyroradius (FLR) nonlinearities in the
ion gyrocenter density and temperature equations in the limit of small $k_\perp \rho_i \ll 1$.

The equations under consideration are for the density and the parallel and perpendicular temperatures (Eqs. 99,101,102 of Ref. [23]),

$$\frac{\partial n_i}{\partial t} + \left[ \phi_G, n_i \right] + \left[ \Omega_G, T_{i\perp} \right] + \nabla_\parallel u_\parallel = K \left( \phi_G + \frac{p_i|| + p_{i\perp} + \Omega_G}{2} \right)$$

(39)

$$\frac{1}{2} \frac{\partial T_{i\parallel}}{\partial t} + \frac{1}{2} \left[ \phi_G, T_{i\parallel} \right] + \nabla_\parallel (u_\parallel + q_{i\parallel}) = K \left( \frac{\phi_G + p_i|| + 2\tau_i T_{i\parallel}}{2} \right) - 2\nu_i G$$

(40)

$$\frac{\partial T_{i\perp}}{\partial t} + \left[ \phi_G, T_{i\perp} \right] + \left[ \Omega_G, n_i + 2T_{i\perp} \right] + \nabla_\parallel q_{i\parallel\perp} = \frac{1}{2} K \left( \frac{\phi_G + 4\Omega_G + p_{i\perp} + 3\tau_i T_{i\perp}}{2} \right) + 2\nu_i G$$

(41)

where in terms of the perpendicular and parallel pressures the isotropic one is $p_i = (p_{i\parallel} + 2p_{i\perp})/3$ and the difference is $G = (p_{i\parallel} - p_{i\perp})/6$, dissipated in the term proportional to $\nu_i G$. Normalisation is to a common background temperature $T_0$, with $\tau_i = T_i/ZT_0$ giving the temperature/charge ratio. The factor of $\tau_i$ is folded into the pressures. The heat fluxes are also broken up into parallel transport of perpendicular and parallel energy, the $(Mw_\perp^2/2)w_\parallel$ and $(Mw_\perp^2/2)w_\parallel$ moments, respectively, where $w$ is the kinetic velocity in the co-moving reference frame. In the gyrokinetic and gyrofluid models $w_\perp$ is not used directly, but as $w_\perp^2$, specifically, the magnetic moment $\mu = Mw_\perp^2/2B$, due to the low frequency ordering. These heat flux pieces are $q_{i\parallel\parallel}$ and $q_{i\perp\parallel}$, respectively. Finally, the potentials $\phi_G$ and $\Omega_G$ represent the FLR treatment. The Padé approximants are

$$\phi_G = \frac{\phi}{1 + b/2} \quad \Omega_G = \frac{-b^2 \phi/2}{(1 + b/2)^2}$$

(42)

in wavenumber space, with argument $b = k_\perp^2 \rho_i^2$. In the limit of $k_\perp^2 \to 0$ we have

$$- b \to \tau_i \nabla_\perp^2 \quad \phi_G \to (1 - b/2)\phi \quad \Omega_G \to (-b/2)\phi$$

(43)

which we will use to show correspondence. The notation of Ref. [23] is used, and the FLR treatment follows Refs. [15, 16] with the necessary modifications to restore free energy conservation as discussed in Ref. [23].

A. The isothermal version

To make the analysis easier to follow, we start with the isothermal model which neglects all considerations of temperature dynamics, including the anisotropy and heat fluxes. In
In this case the ion and electron density equations are given by

\[
\frac{\partial n_i}{\partial t} + [\phi_G, n_i] + \nabla \parallel u = K(\phi_G + \tau_in_i) \tag{44}
\]

\[
\frac{\partial n_e}{\partial t} + [\phi, n_e] + \nabla \parallel v = K(\phi - n_e) \tag{45}
\]

and they are related through the polarisation equation,

\[
\frac{n_e}{1 + b/2} + \tau_i^{-1} \frac{b}{1 + b} \phi = n_e \tag{46}
\]

using the Padé approximants (cf. Refs. \[16, 23\]). This determines the gyrocenter density \( n_i \) in terms of the particle density (equal to \( n_e \)) and the polarisation contribution (due to \( \phi \)), as

\[
n_i = (1 + b/2)n_e + \tau_i^{-1}b\phi \tag{47}
\]

expanding in powers of \( b \) and keeping the \( O(1) \) and \( O(b) \) terms. This is now converted into configuration space identifying \( b \) with \(-\tau_i \nabla^2_\perp \), leaving

\[
n_i = n_e - \nabla^2_\perp \phi - \frac{1}{2} \nabla^2_\perp p_i \tag{48}
\]

where \( p_i = \tau_in_e \). The FLR-corrected potential is given as

\[
\phi_G = (1 - b/2)\phi \rightarrow \phi + \frac{\tau_i}{2} \nabla^2_\perp \phi \tag{49}
\]

up to \( O(b) \). Here and below, \( \nabla^2_\perp \) is normalised against \( \rho_s^{-2} \), so that \( \rho_s^2 \nabla^2_\perp \) becomes \( \tau_i \nabla^2_\perp \).

Now we use the equations for \( n_e \) and \( n_i \) to find the vorticity equation, using these forms to eliminate \( n_i \) and \( \phi_G \) in terms of \( n_e \) and \( \phi \). First, the ion density equation becomes

\[
\frac{\partial}{\partial t} \left( n_e - \nabla^2_\perp \phi - \frac{1}{2} \nabla^2_\perp p_i \right) + [\phi, n_e] - [\phi, \nabla^2_\perp \phi] - \frac{1}{2} [\phi, \nabla^2_\perp p_i] + \frac{1}{2} [\nabla^2_\perp \phi, p_i] + \nabla \parallel u = K(\phi + p_i) \tag{50}
\]

where under \( K \) only the \( O(1) \) terms are kept. The terms under \( \partial/\partial t \) are from \( n_i \). The first three bracket terms are from \([\phi, n_i] \), and the last bracket term is from the difference \( \phi_G - \phi \). We manipulate the bracket terms involving \( \nabla^2_\perp \) as follows

\[
[\phi, \nabla^2_\perp \phi] = \nabla \cdot [\phi, \nabla \perp \phi] - [\nabla \perp \phi, \nabla \perp \phi] \tag{51}
\]

\[
[\phi, \nabla^2_\perp p_i] = \nabla \cdot [\phi, \nabla_\perp p_i] - [\nabla_\perp \phi, \nabla_\perp p_i] \tag{52}
\]
\[ \nabla^2 \phi, p_i = \nabla \cdot [\nabla \phi, \nabla p_i] - [\nabla \phi, \nabla \perp p_i] = \nabla^2 \phi_i - \nabla \cdot [\phi, \nabla \perp p_i] - [\nabla \phi, \nabla \perp p_i] \] (53)

so that with cancellations (noting also \([\nabla \phi, \nabla \perp \phi]\) vanishes) we obtain

\[
\frac{\partial}{\partial t} \left( n_e - \nabla^2 \phi - \frac{1}{2} \nabla^2 p_i \right) + [\phi, n_e] - \nabla \cdot [\phi, \nabla \phi] - \nabla \cdot [\phi, \nabla \perp p_i] + \frac{1}{2} \nabla^2 \phi_i \\
+ \nabla \parallel u\parallel = \mathcal{K}(\phi + p_i) 
\] (54)

Then, under \(\nabla^2\) we replace the bracket \([\phi, p_i]\) with \(-\partial p_i/\partial t\) noting all the other terms are \(O(b)\) corrections to the \(\nabla\parallel\) and \(\mathcal{K}\) terms, so that

\[
\frac{\partial}{\partial t} \left( n_e - \nabla^2 \phi - \frac{1}{2} \nabla^2 p_i \right) + [\phi, n_e] - \nabla \cdot [\phi, \nabla \phi] - \nabla \cdot [\phi, \nabla \perp p_i] \\
+ \nabla \parallel u\parallel = \mathcal{K}(\phi + p_i) 
\] (55)

Finally, we combine \(\partial/\partial t\) and \([\phi, .\) into \(d/\partial t\), and \(\phi\) and \(p_i\) into \(W\), obtaining

\[
\frac{dn_e}{dt} - \nabla \cdot \frac{d}{dt} \nabla \parallel W + \nabla \parallel u\parallel = \mathcal{K}(\phi + p_i) 
\] (56)

This is the ion density equation in the isothermal fluid model, wherein \(p_i = \tau_i n_e\) and by quasineutrality the particle (not gyrocenter) densities are equal. Subtraction of Eq. (56) from Eq. (20) in Section \(\text{II A}\) we find

\[
\nabla \cdot \frac{d}{dt} \nabla W = \nabla \parallel J\parallel - (1 + \tau_i) \mathcal{K}(n_e) 
\] (57)

where we have inserted \(u\parallel - v\parallel = J\parallel\) and \(p_e + p_i = (1 + \tau_i)n_e\). This is the same as Eq. (19) in Section \(\text{II A}\) under the isothermal gyro-Bohm normalised forms \(p_e = n_e\) and \(p_i = \tau_i n_e\) and \(G = 0\).

The rest of the isothermal fluid equations are

\[
\frac{dn_e}{dt} + \nabla \parallel (u\parallel - J\parallel) = \mathcal{K}(\phi - n_e) 
\] (58)

\[
\frac{du\parallel}{dt} + (1 + \tau_i) \nabla \parallel n_e = 0 
\] (59)

\[
\beta_e \frac{\partial A\parallel}{\partial t} + \mu_e \frac{dJ\parallel}{dt} + \nabla \parallel (\phi - n_e) = -0.51 \mu_e \nu_e J\parallel 
\] (60)

and these satisfy the energetics as given in Section \(\text{II A}\) without the temperature dynamics (the latter inclusive of \(G\) and the heat fluxes). The resistive dissipation includes the 0.51 coefficient from Ref. [4] and the collision frequency \(\nu_e\) is normalised against \(c_s/L\parallel\). What can be termed “gyroviscous correspondence” to the fluid model is thereby proved for the isothermal case, for any occurrence of nonlinearity within the local ordering.
B. With temperature dynamics

Now we return to the version of the model with all the temperature dynamics, involving $T_{i\parallel}$ and $T_{i\perp}$. We will make use of the particle representations of these variables. For the density the relation between the particle and gyrocenter representations is given by the polarisation equation, Eq. (46), whose thermal version is (Eq. 92 of Ref. [23]),

$$
\Gamma_1 n_i + \Gamma_2 T_{i\perp} + \frac{\Gamma_0 - 1}{\tau_i} \phi = n_e
$$

(61)

The $\Gamma_1$ and $\Gamma_2$ are the gyroaveraging operators. Eq. (61) is the closure approximation to the gyrokinetic polarisation equation,

$$
\sum_z \int dW \left[ eJ_0(\delta f) + e^2 J_0 F^M J_0 - \frac{F^M}{T} \phi \right] = 0
$$

(62)

where $\delta f$ denotes the distribution function, $\int dW$ represents integration over velocity space, the sum is over species, $J_0$ with argument $k_{\perp} v_{\perp} / \Omega$ acting upon $\delta f$ or $\phi$ is the orbit averaging operator, $F^M$ is the background Maxwellian, and $e$ and $T$ with $n$ and $M$ are the species constants giving the charge, background density and temperature, and mass. The polarisation equation comes from setting the particle charge density to zero [18], and the polarisation term itself ultimately arises from the transformation from gyrocenter to particle phase space [19]. The closure approximation to $\int dW F^M J_0$ is $\Gamma_1$, with argument $b$. The closure form is given by $\Gamma_1^{1/2}$ by correspondence to linear kinetic theory [15]. The second operator $\Gamma_2$ is given by the logarithmic derivative of $\Gamma_1$ with respect to $b$ no matter the form chosen for $\Gamma_1$, since

$$
T \frac{\partial \Gamma_1}{\partial T} = \int dW \left( \frac{\mu B}{T} - 1 \right) F^M J_0
$$

(63)

The Padé approximant forms for $\Gamma_1$ and $\Gamma_2$ are

$$
\Gamma_1 = \frac{1}{1 + b/2} \quad \Gamma_2 = \frac{-b^2/2}{(1 + b/2)^2}
$$

(64)

as given in Ref. [16]. Hence, the gyroaveraged potential and its FLR correction as given in Eq. (42) result from

$$
\phi_G = \Gamma_1 \phi \quad \Omega_G = \Gamma_2 \phi
$$

(65)

That these are the same operators as the ones in Eq. (61), and that $\phi_G$ and $\Omega_G$ are associated with $n_i$ and $T_{i\perp}$, are fundamentals underlying the free energy conservation of the model [23]. The form of the nonlinear terms in the $T_{i\perp}$-equation results from the next higher moment
with $\mu B$ and applying free energy conservation as a constraint. Further detail on this and FLR closure in general is given in Refs. [23, 50], which update Refs. [15, 16].

In these terms the particle (space) representations for the three state variables for the ions are

$$n_{sp} = \Gamma_1 n_i + \Gamma_2 T_{i\perp} + \frac{\Gamma_0 - 1}{\tau_i} \phi$$  \hspace{1cm} (66)

$$T_{i\parallel sp} = \Gamma_1 T_{i\parallel}$$  \hspace{1cm} (67)

$$T_{i\perp sp} = \Gamma_1 T_{i\perp} + \Gamma_2 (n_i + 2T_{i\perp}) + 2\frac{\Gamma_1 \Gamma_2}{\tau_i} \phi$$  \hspace{1cm} (68)

all arising from corresponding moments of Eq. (62). The first is the same as the polarisation equation. In the second, resulting from the $Mu_\parallel - T$ moment, the parallel and perpendicular velocity space integrals separate, and the $\phi$ piece vanishes. In the third, the moment of $\mu B - T$ over the $J_0$ operator gives rise to the same factor of $(\Gamma_1 + 2\Gamma_2)$ as in the nonlinearities in the $T_{i\parallel}$ equation (Eq. 41) itself, in addition to the term $\Gamma_1 \Gamma_2$ coming from the moment of $\mu B - T$ over the $J_0^2$ operator.

As in the isothermal case, the fluid equations are found by constructing the time derivatives of these variables in the particle (not gyrocenter) representation. The low-$k_\perp$ limit is taken, with the $O(b)$ corrections kept only in the nonlinear advection terms. The equations for $n_i$ and $T_{i\perp}$ occur together, ultimately due to the way $J_0$ through its $b$-dependence mixes the perpendicular moments. Up to $O(b)$ the particle representations are given by

$$n_{isp} = n_i + \nabla^2_{\perp} \phi + \frac{\tau_i}{2} \nabla^2_{\perp} (n_i + T_{i\perp})$$  \hspace{1cm} (69)

$$T_{i\perp sp} = T_{i\perp} + \nabla^2_{\perp} \phi + \frac{\tau_i}{2} \nabla^2_{\perp} (n_i + 3T_{i\perp})$$  \hspace{1cm} (70)

$$T_{i\parallel sp} = T_{i\parallel} + \frac{\tau_i}{2} \nabla^2_{\perp} T_{i\parallel}$$  \hspace{1cm} (71)

with the terms on the right sides understood to be in the gyrocenter representation. In the $O(b)$ terms the representations are equivalently the particle or gyrocenter ones, so that the inverses of Eqs. (69) (71) are given by

$$n_{igy} = n_i - \nabla^2_{\perp} \phi - \frac{\tau_i}{2} \nabla^2_{\perp} (n_i + T_{i\perp})$$  \hspace{1cm} (72)

$$T_{i\perp gy} = T_{i\perp} - \nabla^2_{\perp} \phi - \frac{\tau_i}{2} \nabla^2_{\perp} (n_i + 3T_{i\perp})$$  \hspace{1cm} (73)

$$T_{i\parallel gy} = T_{i\parallel} - \frac{\tau_i}{2} \nabla^2_{\perp} T_{i\parallel}$$  \hspace{1cm} (74)
with the terms on the right sides understood to be in the particle representation. The gyroreduced potentials in Eq. (65) by
\[ \phi_G = \phi + \frac{\tau_i}{2} \nabla_\perp^2 \phi \]
\[ \Omega_G = \frac{\tau_i}{2} \nabla_\perp^2 \phi \]
(75)
The partial time derivatives of Eqs. (72–74) are taken, and then Eqs. (39–40) are used to evaluate the right hand sides, as was done in Eq. (50) above.

For the density the result of the substitution is
\[ \frac{\partial}{\partial t} \left( n_i - \nabla_\perp^2 \phi - \frac{1}{2} \nabla_\perp^2 p_i \right) + [\phi, n_i] - [\phi, \nabla_\perp^2 \phi] - \frac{1}{2} [\phi, \nabla_\perp^2 p_i] + \frac{1}{2} [\nabla_\perp^2 \phi, p_i]
+ \nabla \cdot u = K \left( \phi + \frac{p_i \parallel + p_i \perp}{2} \right) \]
(76)
where \( \nabla_\perp^2 p_i = \tau_i \nabla_\perp^2 (n_i + T_i \parallel) \), linearised as before, represents the combining of the \( \nabla_\perp^2 n_i \) and \( \nabla_\perp^2 T_i \parallel \) terms. The manipulations of the \( \nabla_\perp^2 \) operators are done exactly as before, and the result is
\[ \frac{dn_i}{dt} - \nabla \cdot \frac{d}{dt} \nabla W + \nabla \cdot u = K \left( \phi + \frac{p_i \parallel + p_i \perp}{2} \right) \]
(77)
For the perpendicular temperature the result of the substitution is
\[ \frac{\partial}{\partial t} \left( T_i \parallel - \nabla_\perp^2 \phi - \frac{1}{2} \nabla_\perp^2 p_i - \tau_i \nabla_\perp^2 T_i \parallel \right)
+ [\phi, T_i \parallel] - [\phi, \nabla_\perp^2 \phi] - \frac{1}{2} [\phi, \nabla_\perp^2 p_i] - [\phi, \nabla_\perp^2 \tau_i T_i \parallel] + \frac{1}{2} [\nabla_\perp^2 \phi, p_i] + [\nabla_\perp^2 \phi, \tau_i T_i \parallel]
+ \nabla \cdot q_i \parallel = K \left( \phi + p_i \parallel + 3 \tau_i T_i \parallel \right) + 2 \nu_i G \]
(78)
where FLR corrections to the \( K \) terms are dropped as before. The nonlinear terms proportional to \( T_i \parallel \) arise from the extra factors of \( 2T_i \parallel \) in Eqs. (41,73). The manipulations of the \( \nabla_\perp^2 \) operators are done exactly as before, and the result is
\[ \frac{dT_i \parallel}{dt} - \nabla \cdot \frac{d}{dt} \nabla_W - 2 \tau_i \nabla \cdot \frac{d}{dt} \nabla_T + \nabla \cdot q_i \parallel = K \left( \phi + \frac{p_i \parallel + 3 \tau_i T_i \parallel}{2} \right) + 2 \nu_i G \]
(79)
with the last of the nonlinear time derivative terms representing the perpendicular part of the polarisation heat flux. For the parallel temperature the result of the substitution is
\[ \frac{\partial}{\partial t} \left( T_i \parallel - \frac{\tau_i}{2} \nabla_\perp^2 T_i \parallel \right)
+ [\phi, T_i \parallel] - \frac{\tau_i}{2} [\phi, \nabla_\perp^2 T_i \parallel] + \frac{\tau_i}{2} [\nabla_\perp^2 \phi, T_i \parallel]
+ 2 \nabla \parallel (u \parallel + q_i \parallel) = K \left( \phi + p_i \parallel + 2 \tau_i T_i \parallel \right) - 4 \nu_i G \]
(80)
where FLR corrections to the $K$ terms are dropped as before. The nonlinear terms proportional to $T_{i\parallel}$ arise from the extra factors of $T_{i\parallel}$ in Eqs. (40,74). The manipulations of the $\nabla^2$ operators are done exactly as before, and the result is

$$\frac{1}{2} \frac{dT_{i\parallel}}{dt} - \frac{\tau_i}{2} \nabla \cdot \frac{d}{dt} \nabla_{\perp} T_{i\parallel} + \nabla_{\parallel}(u_{\parallel} + q_{i\parallel}) = K \left( \frac{\phi + p_{i\parallel} + 2\tau_i T_{i\parallel}}{2} \right) - 2\nu_i G$$  (81)

with the last of the nonlinear time derivative terms representing the parallel part of the polarisation heat flux. The two temperature equations (Eqs. 79,81) are added to provide the final temperature equation

$$\frac{3}{2} \frac{dT_i}{dt} - \nabla \cdot \frac{d}{dt} \nabla_{\perp} W - \frac{5}{2} \tau_i \nabla \cdot \frac{d}{dt} \nabla_{\perp} T_i + \nabla_{\parallel}(u_{\parallel} + q_{i\parallel}) = K \left( \frac{\phi + p_i + \frac{5}{2} \tau_i T_i + 2G}{2} \right)$$  (82)

where we use $T_i = (2T_{i\perp} + T_{i\parallel})/3$ and $G = \tau_i(T_{i\parallel} - T_{i\perp})/6$ and $q_i = q_{i\perp} + q_{i\parallel}$. The anisotropy dissipation term cancels. The first factor of $G$ is from the pressures (diamagnetic flow in the fluid model), and the second is from the temperatures (diamagnetic heat fluxes) and is neglected in the fluid model. In terms of $p_i$ and $G$ the density equation (Eq. 77) becomes

$$\frac{dn_i}{dt} - \nabla \cdot \frac{d}{dt} \nabla_{\perp} W + \nabla_{\parallel} u_{\parallel} = K(\phi + p_i + G)$$  (83)

Subtraction of Eq. (83) from Eq. (20) recovers Eq. (19) above, and then addition of Eq. (19) to Eq. (82) recovers Eq. (22) above (except for the second factor of $G$ which the fluid model doesn’t keep), and the correspondence in the nonlinear polarisation terms is thereby proved.

IV. GYROFLUID TEMPERATURE ANISOTROPY AND FLUID PARALLEL VISCOSITY

We now turn to the less obscure parts of the correspondence between gyrofluid and fluid equations. The general pressure tensor arises from

$$P = \int dW mwwf(w)$$  (84)

where the integration is over velocity space and $w$ is the random kinetic velocity in the co-moving frame with fluid velocity $u$. In the Braginskii fluid equations $f$ is assumed to be a Maxwellian $f_0$ with variable density $n$ and temperature $T$, and also flow $u$, where all of $n$, $T$, $u$ are arbitrarily variable, i.e., including all dynamics as well as the background
(reduction to low frequency equations under drift ordering comes later). Then, the corrections to \( f_0 \) are considered to be of the form \( f_1 = \Phi(w)f_0 \), and \( \Phi \) solved for in terms of Sonine polynomials, with arbitrarily large collision frequency and small gyroradius, as well as small mean free path (i.e., all gradients are assumed to represent small corrections to local thermodynamic equilibrium, LTE). Besides the specific conductive heat flux \( (q/nT) \), the temperature anisotropy is assumed to be small. Hence \( P \) is split in terms of an isotropic part and a trace-free correction,

\[
P = pg + \Pi
\]

(85)

where \( g \) is the metric tensor. The diagonal elements of \( \Pi \) represent the parallel viscosity. These pressure contributions may be written, separately from any non-diagonal \( \Pi \) contributions, as

\[
P = p_\perp g + (p_\parallel - p_\perp)bb
\]

(86)

where \( b = B/B \) is the magnetic unit vector, hence

\[
\Pi = \Delta pbb - \frac{1}{3} \Delta pg
\]

(87)

with isotropic pressure and deviation given by

\[
p = \frac{2p_\perp + p_\parallel}{3} \quad \Delta p = p_\parallel - p_\perp
\]

(88)

We also have

\[
p_\parallel = p + \frac{2}{3} \Delta p \quad p_\perp = p - \frac{1}{3} \Delta p
\]

(89)

to assist the evaluation of gyrofluid combinations.

It is important to note that with the inhomogeneous magnetic field \( \nabla \cdot (bb) \) contributes to the general divergence \( \nabla \cdot P \). Then, in the reduction to low frequency we obtain

\[
b \cdot (\nabla \cdot P) = \nabla_\parallel p_\parallel - \Delta p \nabla_\parallel \log B
\]

(90)

for the parallel pressure force, which is covered by the parallel gradient and magnetic pumping terms in the gyrofluid moment equation for the parallel velocity. In the perpendicular drifts we make the same approximations as in the gyrofluid model itself:

\[
\nabla \cdot \frac{c}{B^2} B \times B \nabla p \to -2 \nabla \log B \cdot \frac{c}{B^2} B \times B \nabla p \quad b \cdot \nabla b \to \nabla \log B
\]

(91)
and hence also
\[ \nabla \cdot \frac{c}{B^2} \mathbf{B} \times (\mathbf{b} \cdot \nabla) \rightarrow -2 \nabla \log B \cdot \frac{c}{B^2} \mathbf{B} \times \nabla \log B = 0 \quad (92) \]

We can then find
\[ \nabla \cdot \frac{c}{B^2} \mathbf{B} \times (\nabla \cdot \mathbf{P}) \rightarrow -\nabla \log B \cdot \frac{c}{B^2} \mathbf{B} \times \nabla (p_\parallel + p_\perp) \quad (93) \]

which is covered by the same curvature terms in the gyrofluid moment equation for the density. The gyrofluid model then also includes FLR effects, wherein \( \nabla \cdot \phi \) becomes \( \nabla \cdot \phi_G \) and under derivatives and in the magnetic pumping terms \( \tau_i T_{i\perp} \) becomes \( \tau_i T_{i\perp} + \Omega_G \).

Hence, the pressure/temperature anisotropy enters the continuity equations for ions the same way as in the fluid model, although via a different route: grad-B and curvature drifts for gyrocenters rather than the pressure tensor for particles. It remains to obtain the anisotropy itself. The gyrofluid equations for \( T_{i\perp} \) and \( T_{i\parallel} \) for the ion species are given by
\[
\frac{1}{2} \frac{d T_{i\perp}}{dt} + [\phi_G, T_{i\perp}] + B \nabla \cdot \left( \frac{u_{\parallel} + q_{i\parallel}}{B} \right) = K \left( \frac{\phi_G + \tau_i n_i + 3 \tau_i T_{i\parallel}}{2} \right) - \frac{\nu_i}{3 \pi_i} [\tau_i (T_{i\parallel} - T_{i\perp}) - \Omega_G] \quad (94)
\]
\[
\frac{\partial T_{i\parallel}}{\partial t} + [\phi_G, T_{i\parallel}] + [\Omega_G, (n_i + 2 T_{i\perp})] + B \nabla \cdot \left( \frac{q_{i\perp}}{B} \right) = K \left( \frac{\phi_G + \tau_i n_i + 4 \tau_i T_{i\parallel} + 4 \Omega_G}{2} \right) + \frac{\nu_i}{3 \pi_i} [\tau_i (T_{i\parallel} - T_{i\perp}) - \Omega_G] \quad (95)
\]

where the \( \nabla \cdot \log B \) terms give the magnetic pumping of anisotropy and the \( \nu_i \) terms its collisional dissipation, and \( \pi_i \) is a numerical constant which we eventually adjust to obtain correspondence. If we add these equations the total temperature equation results, and the magnetic pumping and dissipation cancel, leaving
\[
\frac{3}{2} \frac{dT_i}{dt} + (\text{FLR}) + B \nabla \cdot \left( \frac{u_{\parallel} + q_{i}}{B} \right) = K \left( \phi + \tau_i n_i + \frac{7}{2} \tau_i T_i + \frac{1}{3} \frac{\pi_i}{\Omega_G} \right) \quad (96)
\]

where “FLR” denotes the FLR corrections which eventually become the polarisation terms as established above and \( T_i = (2 T_{i\perp} + T_{i\parallel})/3 \) and \( q_i = q_{i\parallel} + q_{i\perp} \) as before. The anisotropy is \( \Delta T = T_{i\parallel} - T_{i\perp} \). If we instead subtract Eq. (95) from twice Eq. (94) forming an equation for \( \Delta T \), we find
\[
\frac{\partial \Delta T}{\partial t} + [\phi_G, \Delta T] - [\Omega_G, (n_i + 2 T_{i\perp})] + B \nabla \cdot \left( \frac{2 q_{i\parallel} - q_{i\perp}}{B} \right) - 3 (u_{\parallel} + q_{i\perp}) \nabla \cdot \log B = -K \left( \frac{\phi_G - \phi + \Omega_G}{2} - \frac{8}{3} \tau_i \Delta T \right) - \frac{\nu_i}{\pi_i} \Omega_G
\]
\[
= \frac{1}{2} K \left( \phi + \tau_i n_i + \tau_i T_i \right) - 2 B \nabla \cdot \left( \frac{u_{\parallel}}{B} \right) - \frac{\nu_i}{\pi_i} \tau_i \Delta T \quad (97)
\]

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having arranged terms such that the collisional and velocity divergence terms are on the right side and the nonlinearities, magnetic pumping, FLR, anisotropy in curvature terms, and heat flux effects are on the left side. The Braginskii assumptions are essentially that the left side terms are small, even though that is obviously not the case in the curvature terms, as pointed out before [13].

If the Braginskii assumptions are taken, then we find

\[
\tau_i \Delta T = \frac{\pi_i}{2\nu_i} \left[ K (\phi + \tau_i n_i + \tau_i T_i) - 4B \nabla \parallel u \right] \tag{98}
\]

which is equivalent to

\[
G = \frac{\pi_i}{12\nu_i} \left[ (\nabla \log B^2 \cdot u_\perp) - 4(\nabla \cdot u_\parallel b) \right] \tag{99}
\]

This is the form given by Ref. [24], and correspondence is thereby proved. It is important to note, however, that this regime is never reached even in deep edge turbulence. Tokamak edge turbulence typically has \(\nu_i\) about two orders of magnitude slower than nonlinear advection. Worse than this, the nonlinear advection is the largest effect in Eq. (97), larger than any of \(c_s/R\) or \(c_s/qR\) or \(\nu_i\), even for zonal flows. Hence the dissipation of ion flows cannot be properly modelled by \(K(G)\) in the vorticity equation or \(\nabla_\parallel G\) in the parallel velocity equation, with either \(\nu_i\) or \(c_s/qR\) or \(c_s/R\) as the controlling frequency. These forms will overestimate the dissipation effects and will enforce a particular phase shift between viscosity and the variables determining the ion flow (\(\phi, n_i, T_i, \) and \(u_\parallel\)). Use of the collisional form with \(\nu_i\) will very strongly overestimate ion flow damping in tokamak edge regimes, ultimately corrupting any investigation of bifurcation dynamics. Nevertheless, we set \(\pi_i = 0.96\) for ions and \(\pi_e = 0.73\) for electrons to obtain correspondence to the Braginskii regime in the gyrofluid equations.

V. GYROFLUID HEAT FLUXES AND THE COLLISIONAL FLUID LIMIT

The simplest correspondence is in the heat flux equations. In the gyrofluid model the parallel heat fluxes (parallel and perpendicular energy components) are dynamical variables with their own equations. However, the Braginskii limit assumes that all of the time scales involved in advection, divergences, dissipation, etc., of the heat fluxes are slow, with the exception of collisional dissipation. The dissipation balances the forcing represented by the
temperature gradient. When the parallel heat flux components (Eqs. 103, 104 of Ref. 23) are added to form the equation for \( q_i \parallel = q_i \parallel \parallel + q_i \perp \parallel \) the anisotropy dissipation effects cancel, leaving

\[
\mu_i \frac{dq_i \parallel}{dt} + (\text{FLR}) + (\text{LD}) - \mu_i \tau_i \kappa (2u \parallel + 4q_i \parallel \parallel + 3q_i \perp \parallel)
\]

\[
= -\tau_i \nabla \parallel \left( \frac{3}{2} T_i \parallel + T_i \perp \right) - \frac{5/2}{\kappa_i \mu_i \tau_i \nu_i q_i \parallel}
\]

(100)

where \( \kappa_i \) is the thermal conduction coefficient, and “FLR” denotes the FLR corrections (including the appearance of \( \Omega G \) under \( \nabla \parallel \)) and “LD” the Landau damping dissipation. All of the terms on the left side scale with advection or are slower. The terms on the right side are the ones left after the Braginskii ordering is taken — assuming that \( \nu_i \) overpowers advection. Neglecting the temperature anisotropy the factors of 5/2 cancel and we have

\[
q_i \parallel = \frac{-\kappa_i \nu_i}{\mu_i \nu_i} \nabla \parallel T_i
\]

(101)

which is the Braginskii formula. We set \( \kappa_i = 3.9 \) to set the quantitative correspondence.

For electrons we additionally have the mixing of the moments under the collisional dissipation. The re-expression of the thermal force as such was given in Ref. [51], whose two salient equations in the fluid model are

\[
\beta_e \frac{\partial A_i \parallel}{\partial t} + \mu_e \frac{dJ_i \parallel}{dt} = \nabla \parallel (n_e + T_e - \phi) - \mu_e \nu_e \left[ \eta J_i \parallel + \frac{\alpha_e}{\kappa_e} (q_e \parallel + \alpha_e J_i \parallel) \right]
\]

(102)

\[
\mu_e \frac{dq_e \parallel}{dt} + (\text{LD}) = -\frac{5}{2} \nabla \parallel T_e - \frac{5/2}{\kappa_e \mu_e \nu_e} (q_e \parallel + \alpha_e J_i \parallel)
\]

(103)

where the coefficients \( \eta, \kappa_e, \alpha_e \) are for resistivity, thermal conduction, and the thermal force, respectively. If the nonlinear advection and Landau damping are assumed small in Eq. (103) then the Braginskii formula

\[
q_e \parallel + \alpha_e J_i \parallel = -\frac{\kappa_e}{\mu_e \nu_e} \nabla \parallel T_e
\]

(104)

with \( \kappa_e = 3.2 \) and \( \alpha_e = 0.71 \) for pure hydrogen, is recovered. Then, insertion of this into Eq. (102) gives

\[
\beta_e \frac{\partial A_i \parallel}{\partial t} + \mu_e \frac{dJ_i \parallel}{dt} = \nabla \parallel (n_e + T_e - \phi) + \alpha_e \nabla \parallel T_e - \eta \mu_e \nu_e J_i \parallel
\]

(105)

which recovers the Braginskii Ohm’s law if \( \eta = 0.51 \) is chosen.
This dissipation model was built into the electron gyrofluid moment equations in order to obtain this correspondence, in both Refs. 23, 47. The electron heat flux equations are

\[
\mu_e \frac{\partial q_e^\parallel}{\partial t} + \mu_e a_{le} q_e^\parallel + \mu_e \left[ \partial \phi_e, q_e^\parallel \right] = -\frac{3}{2} \nabla^\parallel T_e^\parallel - \mu_e \mathcal{K} \left( \frac{3v_e^\parallel + 8q_e^\parallel}{2} \right)
- \frac{(5/2)}{\kappa_e} \mu_e \nu_e \left( q_e^\parallel + 0.6 \alpha_e J_e^\parallel \right) + 1.28 \nu_e \left( q_e^\parallel - 1.5 q_e^\perp \right)
\]

\[
(106)
\]

\[
\mu_e \frac{\partial q_e^\perp}{\partial t} + \mu_e a_{le} q_e^\perp + \mu_e \left[ \partial \phi_e, q_e^\perp \right] + \mu_e \left[ \Omega_e, (v_e^\parallel + 2q_e^\perp) \right]
= -\nabla^\parallel (T_e^\perp - \Omega_e) - \mu_e \mathcal{K} \left( \frac{v_e^\parallel + 6q_e^\perp}{2} \right) - \left( T_e^\perp - T_e^\parallel - \Omega_e \right) \nabla^\parallel \log B
- \frac{(5/2)}{\kappa_e} \mu_e \nu_e \left( q_e^\perp + 0.4 \alpha_e J_e^\perp \right) - 1.28 \nu_e \left( q_e^\parallel - 1.5 q_e^\perp \right)
\]

\[
(107)
\]

from Eqs. (103,104) with additions in Eqs. (114,118,119) of Ref. 23. Electron FLR corrections are kept, with \( \phi_e \) and \( \Omega_e \) the corresponding potentials. Adding these to form the total, neglecting FLR effects, magnetic pumping, and curvature terms, we find

\[
\mu_e \frac{dq_e^\parallel}{dt} + (LD) = -\nabla^\parallel \left( \frac{3}{2} T_e^\parallel + T_e^\perp \right) - \frac{(5/2)}{\kappa_e} \mu_e \nu_e \left( q_e^\parallel + \alpha_e J_e^\parallel \right)
\]

\[
(108)
\]

noting the anisotropy dissipation terms cancel. Now assuming \( \nu_e \) overcomes nonlinear advection or Landau damping, and neglecting \( \Delta T \), we find

\[
q_e^\parallel + \alpha_e J_e^\parallel = -\frac{\kappa_e}{\mu_e \nu_e} \nabla^\parallel T_e
\]

\[
(109)
\]

which is the same as Eq. (104) above, i.e., the Braginskii formula.

The electron gyrofluid parallel velocity equation is

\[
\beta_e \frac{\partial A^\parallel}{\partial t} - \mu_e \frac{\partial v_e^\parallel}{\partial t} - \mu_e \left[ \partial \phi_e, v_e^\parallel \right] - \mu_e \left[ \Omega_e, q_e^\perp \right]
= -\nabla^\parallel \left( \phi_e - n_e - T_e^\parallel \right) - \mu_e \mathcal{K} \left( \frac{4v_e^\parallel + 2q_e^\parallel + q_e^\perp}{2} \right)
- \left( \Omega_e - T_e^\perp + T_e^\parallel \right) \nabla^\parallel \log B - \mu_e \nu_e \left[ \eta J_e^\parallel + \frac{\alpha_e}{\kappa_e} \left( q_e^\parallel + q_e^\perp + \alpha_e J_e^\parallel \right) \right]
\]

\[
(110)
\]

from Eq. (100) with additions in Eq. (115) of Ref. 23. Neglecting FLR effects, magnetic pumping, curvature terms, adding \( q_e^\parallel + q_e^\perp = q_e \), replacing \( T_e^\parallel \) by \( T_e \) (neglecting \( \Delta T \)) and setting \( v^\parallel = -J^\parallel \) (effectively neglecting finite \( \mu_e \) corrections), we find

\[
\beta_e \frac{\partial A^\parallel}{\partial t} + \mu_e \frac{d J_e^\parallel}{dt} = \nabla^\parallel \left( n_e + T_e - \phi \right) - \mu_e \nu_e \left[ \eta J_e^\parallel + \frac{\alpha_e}{\kappa_e} \left( q_e + \alpha_e J_e^\parallel \right) \right]
\]

\[
(111)
\]
which is the same as Eq. (102) above. Then going to the Braginskii limit by inserting $q_{e\parallel}$ from Eq. (104), we find Eq. (105), which is the Braginskii version.

It has been pointed out that curvature terms should appear in the fluid model’s equations for $J_{\parallel}$ and $u_{\parallel}$ [13]. This is indeed the case, but in that event the involved terms are the same as the ones in the gyrofluid model. Correspondence in the heat fluxes and the Ohm’s law is thereby proved.

VI. CONCLUSIONS

This work has considered the low frequency Braginskii fluid drift equations on the one hand and the electromagnetic, transcollisional gyrofluid equations on the other. In both cases the model is comprehensive enough to treat temperature dynamics of both species and has a free energy functional which is conserved in the absence of dissipation and external drive. By considering the entirety of the Braginskii limit — collision frequency larger than advection or transit frequency, specific heat flux smaller than fluid velocity, and small gyroradius — the sets of equations and each one’s conserved free energy have been shown to be one and the same. Whether or not the Braginskii limit is ever reached is a separate question. But the low frequency Braginskii fluid drift equations have been shown by correspondence to be a fully contained subset of the electromagnetic, transcollisional gyrofluid equations.

One substantial advantage of the gyrofluid model is the fact that in all the terms involving derivatives, only scalar quantities are involved. Instead of vector or tensor components, it has gyroreduced potentials or charge densities which involve Hermitian operators (enabling the free energy conservation). This leads to numerical schemes which are easier to formulate. All the nonlinear terms have the Poisson bracket structure, for which the Arakawa spatial discretisation scheme is uniquely suited [52]. Coupled with a timestep that is highly accurate, requires only one evaluation of the terms per step, and is stable for waves [53], we have the best scheme found so far for this type of microturbulence in magnetised plasmas [54]. Its use for the gyrofluid equations is detailed in Ref. [23]. With these advantages together with the correspondence to the Braginskii fluid drift equations, some of the mystery surrounding the efficacy or validity of the gyrofluid model for tokamak edge turbulence should be alleviated.
[1] M. Wakatani and A. Hasegawa, Phys. Fluids 27, 611 (1984).
[2] R. E. Waltz, Phys. Fluids 28, 577 (1985).
[3] D. A. Monticello and R. B. White, Phys. Fluids 23, 366 (1980).
[4] S. I. Braginskii, Rev. Plasma Phys. 1, 205 (1965).
[5] B. Scott, Phys. Fluids B 4, 2468 (1992).
[6] S. Camargo, D. Biskamp, and B. Scott, 2, 48 (1995).
[7] B. Scott, New J. Phys. 4, 52 (2002).
[8] P. Diamond and Y. Kim, Phys. Fluids B 3, 1626 (1991).
[9] B. Scott, Plasma Phys. Contr. Fusion 34, 1977 (1992).
[10] F. L. Hinton and C. W. Horton, Phys. Fluids 14, 116 (1971).
[11] B. Scott, Phys. Plasmas 10, 963 (2003).
[12] A. B. Hassam, Phys. Fluids 23, 38 (1980).
[13] A. Smolyakov, Can. J. Phys. 76, 321 (1997).
[14] I. O. Pogutse, A. I. Smolyakov, and A. Hirose, J. Plasma Phys. 60, 133 (1998).
[15] W. Dorland and G. Hammett, Phys. Fluids B 5, 812 (1993).
[16] M. A. Beer and G. Hammett, Phys. Plasmas 3, 4046 (1996).
[17] E. A. Frieman and L. Chen, Phys. Fluids 25, 502 (1982).
[18] W. W. Lee, Phys. Fluids 26, 556 (1983).
[19] T. S. Hahm, Phys. Fluids 31, 2670 (1988).
[20] B. Coppi, Phys. Rev. Lett. 12, 417 (1964).
[21] B. Scott, Plasma Phys. Contr. Fusion 45, A385 (2003).
[22] E. V. Belova, Phys. Plasmas 8, 3936 (2001).
[23] B. Scott, Phys. Plasmas 12, 102307 (2005), arXiv:physics/0501124.
[24] B. N. Rogers, J. F. Drake, and A. Zeiler, Phys. Rev. Lett. 81, 4396 (1998).
[25] W. Park, D. A. Monticello, and R. B. White, Phys. Fluids 27, 137 (1984).
[26] W. Park, D. A. Monticello, and T. K. Chu, Phys. Fluids 30, 285 (1987).
[27] P. Rutherford and E. A. Frieman, Phys. Fluids 11, 569 (1968).
[28] J. B. Taylor and R. J. Hastie, Plasma Phys. 10, 479 (1968).
[29] H. Nordman and J. Weiland, Nucl. Fusion 29, 251 (1989).
[30] H. Nordman, J. Weiland, and A. Jarmén, Nucl. Fusion 30, 983 (1990).
[31] J. Weiland, Collective Modes in Inhomogeneous Plasmas (Institute of Physics Publishing, 1999).
[32] S.-T. Tsai, F. W. Perkins, and T. H. Stix, Phys. Fluids 13, 2108 (1970).
[33] B. Scott, Contrib. Plasma Phys. 38, 171 (1998).
[34] B. Scott, Plasma Phys. Contr. Fusion 40, 823 (1998).
[35] B. Scott, Phys. Plasmas 12, 082305 (2005).
[36] H. Sugama, T. Watanabe, and W. Horton, Phys. Plasmas 8, 2617 (2001).
[37] S. Sieniutycz and R. S. Berry, Phys. Rev. E 65, 046132 (2002).
[38] B. Scott, Phys. Plasmas 8, 447 (2001).
[39] J. J. Ramos, Phys. Plasmas 12, 052102 (2005).
[40] S. C. Jardin and J. A. Breslau, Phys. Plasmas 12, 056101 (2005).
[41] P. J. Catto and A. N. Simakov, Phys. Plasmas 11, 90 (2004).
[42] A. N. Simakov and P. J. Catto, Phys. Plasmas 10, 4744 (2003).
[43] C. R. Sovinec, T. A. Gianakon, E. D. Held, S. E. Kruger, and D. D. Schnack, Phys. Plasmas 10, 1727 (2003).
[44] L. E. Sugiyama and W. Park, Phys. Plasmas 7, 4644 (2000).
[45] W. Park, E. V. Belova, G. Y. Fu, X. Z. Tang, H. R. Strauss, and L. E. Sugiyama, Phys. Plasmas 6, 1796 (1999).
[46] Z. Chang and J. D. Callen, Phys. Fluids B 4, 1167 (1992).
[47] B. Scott, Phys. Plasmas 7, 1845 (2000).
[48] B. Scott, Plasma Phys. Contr. Fusion 48, B277 (2006).
[49] B. Scott, Plasma Phys. Contr. Fusion 49, S25 (2007).
[50] B. Scott, Derivation via free energy conservation constraints of gyrofluid equations with finite-gyroradius electromagnetic nonlinearities, submitted to Phys Plasmas, arXiv:0710.4899 (2007).
[51] B. Scott, Plasma Phys. Contr. Fusion 39, 1635 (1997).
[52] A. Arakawa, J. Comput. Phys. 1, 119 (1966), repr. vol 135 (1997) 103.
[53] G. E. Karniadakis, M. Israeli, and S. A. Orszag, J. Comput. Phys. 97, 414 (1991).
[54] V. Naulin, Phys. Plasmas 10, 4016 (2003).