A class function on the mapping class group of an orientable surface
and the Meyer cocycle

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Abstract

In this paper we define a $\mathbb{QP}^1$-valued class function on the mapping class group $\mathcal{M}_{g,2}$ of a surface $\Sigma_{g,2}$ of genus $g$ with two boundary components. Let $E$ be a $\Sigma_{g,2}$ bundle over a pair of pants $P$. Gluing to $E$ the product of an annulus and $P$ along the boundaries of each fiber, we obtain a closed surface bundle over $P$. We have another closed surface bundle by gluing to $E$ the product of $P$ and two disks.

The sign of our class function cobounds the 2-cocycle on $\mathcal{M}_{g,2}$ defined by the difference of the signature of these two surface bundles over $P$.

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0 Introduction

Let $\Sigma_{g,r}$ be a compact oriented surface of genus $g$ with $r$ boundary components. The mapping class group $\mathcal{M}_{g,r}$ is $\pi_0 \text{Diff}_+ (\Sigma_{g,r}, \partial \Sigma_{g,r})$ where $\text{Diff}_+ (\Sigma_{g,r}, \partial \Sigma_{g,r})$ is the group of orientation preserving diffeomorphisms of $\Sigma_{g,r}$ which restrict to the identity on the boundary $\partial \Sigma_{g,r}$. We simply denote $\Sigma_g := \Sigma_{g,0}$ and $\mathcal{M}_g := \mathcal{M}_{g,0}$. Harer[4] proved that $H^2 (\mathcal{M}_{g,r}; \mathbb{Z}) \cong \mathbb{Z} \quad g \geq 3, \ r \geq 0$, see also Korkmaz, Stipsicz[8]. Meyer[9] defined a cocycle $\tau_g \in Z^2 (\mathcal{M}_g; \mathbb{Z}) \ (g \geq 0)$ called the Meyer cocycle which represents four times generator of the second cohomology class when $g \geq 3$. Let $P := S^2 - \bigcup_{i=1}^3 \text{Int} D_i$ where $D_i \subset S^2$ is a disk, $\text{Int} D_i$ its interior in $S^2$, and $\alpha, \beta, \gamma \in \pi_1 (P)$ be the homotopy classes as shown in Figure 1. We consider a $\Sigma_{g,r}$ bundle $E_{g,r}^\varphi,\psi$ on the pair of pants $P$ which has monodromies $\varphi, \psi, (\psi \varphi)^{-1} \in \mathcal{M}_{g,r}$ along $\alpha, \beta, \gamma \in \pi_1 (P)$. The diffeomorphism type of $E_{g,r}^\varphi,\psi$ does not depend on the choice of representatives in the mapping classes $\varphi$ and $\psi$. The Meyer cocycle is defined by

$$\tau_g : \mathcal{M}_g \times \mathcal{M}_g \to \mathbb{Z}, \quad (\varphi, \psi) \mapsto \text{Sign} E_{g,r}^\varphi,\psi$$

where $\text{Sign} E_{g,r}^\varphi,\psi$ is the signature of the compact 4-manifold $E_{g,r}^\varphi,\psi$. For $k > 0$, it is known as Novikov additivity that when two compact oriented $4k$-manifolds are glued by an orientation reversing diffeomorphism of their boundaries, the signature of their union is the sum of their signature. When a pants decomposition of a closed 2-manifold is given, the signature of a $\Sigma_g$ bundle on the 2-manifold is the sum of the signature of the $\sigma_g$ bundles restricted to each pair of pants. Therefore, it is important to study the Meyer cocycle to calculate the signature of compact 4-manifolds. For $g = 1, 2$ the Meyer cocycle $\tau_g$ is a coboundary, and the cobounding function of this cocycle is calculated by several authors, for instance, Meyer[9], Atiyah[1], Kasagawa[6], Iida[5]. The Meyer cocycle is not a coboundary if genus $g \geq 3$, but the cocycle can be a coboundary when it is restricted to a certain subgroup, and calculated by Endo[2], Morifuji[10].

Let $I$ be the unit interval $[0, 1] \subset \mathbb{R}$. By sewing a pair of disks onto the surface $\Sigma_{g,2}$ along the boundary, we have $\Sigma_g$. For $h \in \text{Diff}_+ (\Sigma_{g,2}, \partial \Sigma_{g,2})$, if we extend $h$ by the identity on the pair of disks, we have a self-
diffeomorphism of $\Sigma_g$, we denote it $h \cup id_{\Sigma_{g+1}^2}$. By sewing an annulus $S^1 \times I$ onto the surface $\Sigma_{g+1}$ along the boundary, we have $\Sigma_{g+1}$. In the same way, if we extend $h \in \text{Diff}^+ (\Sigma_{g+1}, \partial \Sigma_{g+1})$ by the identity on the annulus, we have a self-diffeomorphism $h \cup id_{S^1 \times I}$.

Define the induced homomorphism on the mapping class group by

$$\theta : \mathcal{M}_{g,2} \to \mathcal{M}_g$$
$$[h] \mapsto [h \cup id_{\Sigma_{g+1}^2}]$$

and

$$\eta : \mathcal{M}_{g,2} \to \mathcal{M}_{g+1,0}$$
$$[h] \mapsto [h \cup id_{S^1 \times I}]$$

Harer[3][4] shows that $\theta$ and $\eta$ induce an isomorphism on the second homology classes when genus $g \geq 5$, so that $\tilde{\tau}_g = \eta^* \tau_{g+1} - \theta^* \tau_g$ is a coboundary. Powell[11] proved that the first cohomology group $H_1(\mathcal{M}_{g,r} ; \mathbb{Z})$ is trivial for $g \geq 3$ and $r \geq 0$, so by the universal coefficient theorem, it follows that the cobounding function of $\tilde{\tau}_g$ is unique.

In this paper we define a $\mathbb{QP}^1$-valued class function $m$ on the mapping class group $\mathcal{M}_{g,2}$ in an explicit way by using information of the first homology group of a mapping torus of $[h] \in \mathcal{M}_{g,2}$, and prove that the sign of the function $m$ cobounds the cocycle $\tilde{\tau}_g = \eta^* \tau_{g+1} - \theta^* \tau_g$. Especially it turns out that the cocycle $\tilde{\tau}_g$ is coboundary for any $g \geq 0$.

In section 1, we construct a class function $m$, prove some properties of this function, and calculate the image of the function. In section 2, we prove that the sign of this function cobounds the difference $\tilde{\tau}_g = \eta^* \tau_{g+1} - \theta^* \tau_g$. By the definition of the Meyer cocycle $\tau_g$, $\tilde{\tau}_g(\phi, \psi)$ is just the difference $\text{Sign} E_g^{\phi, \eta(\psi)} - \text{Sign} E_g^{\phi, \theta(\psi)}$, so that we calculate the difference by using the sign of the function $m$. Moreover we compute the other differences of signature $\text{Sign}(E_g^{\phi, \psi}) - \text{Sign}(E_g^{\phi, \theta(\psi)})$ and $\text{Sign}(E_g^{\phi, \eta(\psi)}) - \text{Sign}(E_g^{\phi, \psi})$ by the function $m$.

## 1 Class function $m : \mathcal{M}_{g,2} \to \mathbb{QP}^1$

In this section we define the class function on the mapping class group $\mathcal{M}_{g,2}$ stated in Introduction and describe some properties of the function including the nontriviality.

For $[p : q], [r : s] \in \mathbb{QP}^1$, we define an addition in $\mathbb{QP}^1$ by

$$[p : q] + [r : s] = \begin{cases} [pr : ps + qr], & \text{if } [p : q] \neq [0 : 1] \text{ or } [r : s] \neq [0 : 1] \\ [0 : 1], & \text{if } [p : q] = [r : s] = [0 : 1]. \end{cases}$$

The projective line $\mathbb{QP}^1$ forms an additive monoid under this operation with $[1 : 0]$ the zero element.

In this section, all (co)homology groups is with $\mathbb{Q}$ coefficients.

### 1.1 Construction of the class function

Before constructing the function, we prepare a fact about homology groups of compact 3-manifolds. Let $Y$ be a compact oriented 3-manifold with boundary $\partial Y$ and $i : \partial Y \hookrightarrow Y$ the inclusion map. Consider the commutative
Let $e \in \Sigma := \Sigma g, r$. We define the mapping torus of $\Sigma$ as a subspace of $X = \Sigma g, r$ by $X^\varphi := \Sigma g, r \times I / \sim$, $(x, 1) \sim (h(x), 0)$, and the equivalent class of $\pi([x, t]) = [t]$, where $[x, t] \in X^\varphi$ is the image under the inclusion homomorphism $H_1(S_k) \to H_1(\Sigma)$ of the fundamental homology class.

We consider $\Sigma$ as a subspace of $X$ by the embedding $\iota : \Sigma \hookrightarrow X : x \mapsto [x, 0]$. We choose points $p_1 \in S_1$, $p_2 \in S_2$, and $p \in S^1$, and orientation-preserving homeomorphisms $\iota_1 : S^1 \to S_1$ and $\iota_2 : S^1 \to S_2$. We define singular cochains $f_k : I \to (S_1 \amalg S_2) \times S^1 = \partial X$ for $k = 1, 2, 3, 4$ by

$$f_1(t) = (\iota_1(t), p), \quad f_2(t) = (\iota_2(t), p), \quad f_3(t) = (p_1, t), \quad \text{and} \quad f_4(t) = (p_2, t),$$

Let $e_k \in H_1(\partial X)$ be the homology class of $f_k$ for $k = 1, 2, 3, 4$. Then the set $\{e_1, e_2, e_3, e_4\}$ forms a basis for $H_1(\partial X)$.

Now we describe the kernel of the homomorphism $i_* : H_1(\partial X) \to H_1(X)$. Since $e_1$ and $e_2$ lie in the kernel of $(\pi|_{\partial X})_*$ and $\pi_*(e_3) = \pi_*(e_4) = [S^1] \in H_1(S^1)$, we have

$$\text{Ker } i_* \subset \text{Ker } (\pi_* i_*) = Q e_1 \oplus Q e_2 \oplus Q(e_3 - e_4).$$

By the definition of the map $f_k$, $(i \circ f_k)_*[S^1] = \iota_* [S_k]$, and so $i_*(e_1 + e_2) = \iota_* ([S_1] + [S_2]) \in H_1(X)$. Since $S_1 \cup S_2$ is the boundary of $\Sigma$, we have $[S_1] + [S_2] = 0 \in H_1(\Sigma)$. Hence

$$Q(e_1 + e_2) \subset \text{Ker } i_*.$$

As we saw at the beginning of this subsection, $\dim \text{Ker } i_* = \frac{1}{2} \dim H_1(\partial X) = 2$. It follows that $\text{Ker } i_* = Q(e_1 + e_2) \oplus Q(p(e_3 - e_4) + qe_1)$ for some $p, q \in Q$. Now we can define a class function.
Definition 1.1. For \( \phi \in M_{g, 2} \), we take \( p, q \in \mathbb{Q} \) such that \( \text{Ker} \ i_{\phi} = \mathbb{Q}(e_1 + e_2) \oplus \mathbb{Q}(p(e_3 - e_4) + qe_1) \). We define \( m : M_{g, 2} \to \mathbb{Q}P^1 \) by \( m(\phi) = [p : q] \).

Lemma 1.2. For \( \phi, \psi \in M_{g, 2} \),

\[
m(\psi \phi \psi^{-1}) = m(\phi).
\]

Proof. Define \( \Psi : X^\phi \to X^{\psi \phi \psi^{-1}} \) by \( \Psi(x, t) = (\psi(x), t) \). Then the following diagram commutes

\[
\begin{array}{ccc}
H_1(\partial X^\phi) & \xrightarrow{i_{\phi*}} & H_1(X^\phi) \\
\downarrow{\psi_*} & & \downarrow{\psi_*} \\
H_1(\partial X^{\psi \phi \psi^{-1}}) & \xrightarrow{i_{\psi \phi \psi^{-1}*}} & H_1(X^{\psi \phi \psi^{-1}}).
\end{array}
\]

We can see from the diagram, \( \Psi_* \) gives the natural isomorphism between \( \text{Ker}(H_1(\partial X^\phi) \to H_1(X^\phi)) \) and \( \text{Ker}(H_1(\partial X^{\psi \phi \psi^{-1}}) \to H_1(X^{\psi \phi \psi^{-1}})) \). Hence we have \( m(\psi \phi \psi^{-1}) = m(\phi) \).

1.2 Some properties and the nontriviality of the class function

By the Serre spectral sequence, we have the exact sequence

\[
0 \longrightarrow \text{Coker}(\phi_* - 1) \xrightarrow{i_*} H_1(X) \xrightarrow{\pi_*} H_1(S^1) \longrightarrow 0,
\]

where \( \text{Coker}(\phi_* - 1) \) is the cokernel of the homomorphism \( \phi_* : H_1(\Sigma) \to H_1(\Sigma) \).

Then we have a unique homomorphism \( j_\phi : \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}(e_3 - e_4) \to \text{Coker}(\phi_* - 1) \) such that the diagram with exact rows

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{Q}e_1 \oplus \mathbb{Q}e_2 \oplus \mathbb{Q}(e_3 - e_4) \\
\downarrow{j_\phi} & & \downarrow{i_*} \\
0 & \longrightarrow & \text{Coker}(\phi_* - 1)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & H_1(\partial X) \\
\downarrow{i_*} & & \downarrow{i_*} \\
0 & \longrightarrow & H_1(S^1)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & H_1(X) \\
\downarrow{j_*} & & \downarrow{j_*} \\
0 & \longrightarrow & H_1(S^1)
\end{array}
\]

\[
\begin{array}{ccc}
0 & \longrightarrow & 0
\end{array}
\]

commutes. By the diagram, we have

\[
\text{Ker} \ i_* = \text{Ker} \ j_\phi,
\]

and

\[
j_\phi(e_1) = -j_\phi(e_2) = [S_1] \in \text{Coker}(\phi_* - 1).
\]

Now we introduce a cochain \( \omega_l \in C^1(M_{g, 2}; H_1(\Sigma)) \) defined in [7]. On the fiber \( \Sigma = \pi^{-1}(0) \subset X \), pick a path \( l \) such that \( l(0) \in S_2 \) and \( l(1) \in S_1 \). Define \( \omega_l \) by

\[
\omega_l(\phi) := \phi(l) - l \in H_1(\Sigma).
\]

Then we have

Lemma 1.3.

\[
j_\phi(e_3 - e_4) = [\omega_l(\phi)] \in \text{Coker}(\phi_* - 1).
\]
Proof. Define a 2-chain $L : I \times I \to X$ by $L(s, t) = [l(s), t]$. Its boundary is given by $-i_*(e_3) + \varphi(l) + i_*(e_4) - l \in B_1(X)$. Hence,

$$i_*(e_3 - e_4) = i_*(\varphi(l) - l) \in H_1(X)$$

Since $i_*$ is injective, the lemma follows.

From the lemma, we see the homolopy class $[\omega_1(\varphi)] \in \text{Coker}(\varphi_* - 1)$ is independent of the choice of the path $l$. If $\omega_1(\varphi) = 0$, then $j_*(e_3 - e_4) = 0$.

Remark 1.4. If there exists a path $l$ from a point in $S_2$ to a point in $S_1$ which has no common point with the support of a representative of $\varphi \in \mathcal{M}_{g, 2}$, then $m(\varphi) = [1 : 0]$. In particular, $m(id) = [1 : 0]$, the zero element of the monoid $\mathbb{QP}^1$.

At the beginning of this section, we defined the commutative monoid structure on $\mathbb{QP}^1$. So integral multiples of $m(\varphi)$ are well-defined.

Proposition 1.5. If $\varphi \in \mathcal{M}_{g, 2}$ and $k \in \mathbb{Z}$, then

$$m(\varphi^k) = km(\varphi).$$

Proof. The proposition is trivial for $k = 0$ and $k = 1$. Assume $k \geq 2$.

Let $m(\varphi) = [p : q]$. By the definition of $j_*, p_\varphi(e_3 - e_4) = -q[S_1] \in \text{Coker}(\varphi_* - 1)$. Hence, there exists $v \in H_1(\Sigma)$ such that

$$p(\varphi(l) - l) = -q[S_1] + (\varphi_* - 1)v \in H_1(\Sigma)$$

Apply $\varphi^i$ ($i = 1, 2, \cdots, k - 1$) to the both sides of the equation and sum over $i$. Then

$$\sum_{i=1}^{k-1} p(\varphi^{i+1}(l) - \varphi^i(l)) = -\sum_{i=1}^{k-1} \{[S_1] + (\varphi_*^{i+1}(v) - \varphi_*^i(v))\},$$

that is

$$p(\varphi^k(l) - l) = -kq[S_1] + (\varphi_*^k - 1)v.$$

Hence, $m(\varphi^k) = [p : kq] = km(\varphi)$ for $k > 0$.

By applying $\varphi^{-1}$ to the equation $p(\varphi(l) - l) = -q[S_1] + (\varphi_* - 1)v$, we have

$$p(\varphi^{-1}(l) - l) = q[S_1] + (\varphi_*^{-1} - 1)v \in H_1(\Sigma).$$

Hence, $m(\varphi^{-1}) = [p : -q] = -m(\varphi)$. Since $m(\varphi^{-k}) = -m(\varphi^k) = -km(\varphi)$ for $k > 0$, the proposition follows for the case $k < 0$.

Now we compute the image of the function $m$. Especially we prove that $m$ is nontrivial.

Proposition 1.6. For $g \geq 1$, $m$ is surjective. For $g = 0$, $\text{Im}(m) = [1 : \mathbb{Z}]$. 
Proof. Suppose $g \geq 1$. We choose oriented simple closed curves $\alpha$, $\alpha'$, and $\beta$ and paths $l$ and $l'$ as shown in Figure 2. We denote the Dehn twists along a simple closed curve $C \subset \Sigma$ by $t_C$, and the homology class of $C$ by $[C]$. Then $[\alpha] + [\alpha'] + [\beta] = 0 \in H_1(\Sigma)$ since they bound a 2-chain. For $p \in \mathbb{Z}$, if we denote $\varphi := t_p \alpha t_{\alpha'} t_{\beta}^{-1}$, then

$$j_\varphi((p+1)(e_3 - e_4)) = \omega_1(\varphi) + p\omega_l(\varphi) = (t_p \alpha t_{\alpha'} t_{\beta}^{-1})(l) - l + p(t_p \alpha t_{\alpha'} t_{\beta}^{-1})(l') - l' = p([\alpha] + [\alpha'] + [\beta]) + [\beta] = [\beta] = [S_1].$$

Hence, $j_\varphi((p+1)(e_3 - e_4) - e_1) = 0$, so that

$$m(\varphi) = [p + 1 : -1].$$

By Proposition 2.5, we have

$$m(\varphi^{-q}) = -q[p + 1 : -1] = \begin{cases} [p + 1 : q], & \text{if } p \neq -1 \\ [0 : 1], & \text{if } p = -1. \end{cases} \quad (q \in \mathbb{Z})$$

Since $p$ and $q$ can run over all integers, we see $m$ is surjective for $g \geq 1$.

For $g = 0$, $\mathcal{M}_{0,2}$ is the infinite cyclic group generated by $t_\beta$. Since $m(t_\beta^{-q}) = [1 : q]$, we have $\text{Im}(m) = [1 : \mathbb{Z}]$. \qed
2 The difference of two Meyer cocycles $\eta^* \tau_{g+1}$ and $\theta^* \tau_g$

In this section (co)homology groups are with $\mathbb{Z}$ coefficient unless specified.

Let $g \geq 0$ be a positive integer. In Introduction, we defined the homomorphisms $\eta : M_{g,2} \to M_{g+1,0}$ and $\theta : M_{g,2} \to M_g$ to be the induced maps by sewing a pair of disks and by sewing an annulus onto the surface $\Sigma_{g,2}$ along their boundaries respectively. We denote the Meyer cocycle on the mapping class group of genus $g$ closed orientable surface $M_g$ by $\tau_g \in Z^2(M_g)$ and define $\tilde{\tau}_g \in Z^2(M_{g,2})$ to be the difference between the Meyer cocycles

$$\tilde{\tau}_g := \eta^* \tau_{g+1} - \theta^* \tau_g.$$ 

Let $P := S^2 - \coprod_{i=1}^3 D^2$. In this section, we prove the main theorem and calculate the changes of signature associated with sewing a pair of trivial disk bundles $P \times \coprod_{i=1}^2 D^2$ and sewing an trivial annulus bundles $P \times (S^1 \times I)$ onto $\Sigma_{g,2}$ bundle on the pair of pants $P$ along their boundaries. To state the main theorem, we define the sign of $[p : q] \in \mathbb{Q}P^1$ by

$$\text{sign}(p : q) := \begin{cases} 
1 & \text{if } pq > 0, \\
0 & \text{if } pq = 0, \\
-1 & \text{if } pq < 0.
\end{cases}$$

**Theorem 2.1.** For $\varphi, \psi \in M_{g,2}$, we define

$$\tilde{\phi}_g(\varphi) := \text{sign}(m(\varphi)).$$

Then $\tilde{\phi}_g$ cobounds the difference $\tilde{\tau}_g$ between the Meyer cocycles $\eta^* \tau_{g+1}$ and $\theta^* \tau_g$

$$\tilde{\tau}_g(\varphi, \psi) = \delta \tilde{\phi}_g(\varphi, \psi)$$

$$= \text{sign}(m(\varphi)) + \text{sign}(m(\psi)) + \text{sign}(m((\varphi \psi)^{-1})).$$

**Remark 2.2.** Let $k$ be an integer. By Lemma 2.2 and Proposition 2.5, $\tilde{\phi}_g$ has the properties

$$\tilde{\phi}_g(\psi \varphi \psi^{-1}) = \tilde{\phi}_g(\varphi),$$

$$\tilde{\phi}_g(\varphi^k) = \text{sign}(k) \tilde{\phi}_g(\varphi)$$

for any $g \geq 0$.

### 2.1 Proof of Main Theorem

In this subsection we prove Theorem 2.1.

In Introduction, we defined $E^\varphi_{g,r}$ as a $\Sigma_{g,r}$ bundle on the pair of pants $P$ which has monodromies $\varphi$, $\psi$, and $(\psi \varphi)^{-1} \in M_{g,r}$ along $\alpha$, $\beta$, and $\gamma \in \pi_1(P)$ respectively, and in Subsection 2.1, we defined $X_{g,r}^\varphi$ by the mapping torus of $\Sigma_{g,r} \times I / \sim$ where $(x,1) \sim (h(x),0)$ for $\varphi = [h] \in M_{g,r}$.

We consider

$$E^\eta(\varphi)_{g+1} = E^\varphi_{g,2} \cup (-S^1 \times I \times P),$$

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and
\[ X_{g+1}^\eta(\varphi) = X_{g,2}^\varphi \cup (-S^1 \times I \times S^1). \]

Define
\[
G : \partial D^2 \times I \to \{1\} \times S^1 \times I.
\]
\[
(x,t) \mapsto (1,x,\frac{1}{3}t)
\]

By the map \( G \), we can glue \( D^2 \times I \) to \( I \times S^1 \times I \) as shown in figure 3. Glue \( D^2 \times I \times \Sigma \) to \( I \times X_{g+1}^\eta(\varphi) = (I \times X_{g,2}^\varphi) \cup (-I \times S^1 \times I \times S^1) \) with the gluing map \( G \times \text{id} \). Denote
\[
\tilde{E}^\varphi,\psi := (I \times E_{g+1}^\eta(\varphi,\psi)) \cup (D^2 \times I \times \Sigma), \quad \text{and} \quad \tilde{X}^\varphi := (I \times X_{g+1}^\eta(\varphi)) \cup (D^2 \times I \times S^1).
\]

To prove main theorem, it suffices to prove Lemma 2.3 and Lemma 2.4 below.

**Lemma 2.3.**
\[
(\eta^* \tau_{g+1} - \theta^* \tau_g)(\varphi,\psi) = \text{Sign} \tilde{X}^\varphi + \text{Sign} \tilde{X}^\psi + \text{Sign} \tilde{X}^{(\varphi \psi)^{-1}} \quad \text{for} \ \varphi,\psi \in M_{g,2}, \ g \geq 0.
\]

**Lemma 2.4.**
\[
\text{Sign} \tilde{X}^\varphi = \text{sign}(m(\varphi)) \quad \text{for} \ \varphi \in M_{g,2}, \ g \geq 0.
\]

**proof of Lemma 3.3.** Note that
\[
X^\varphi = \tilde{E}^\varphi,\psi|_{\partial D^1}.
\]

Then we can see
\[
\partial \tilde{E}^\varphi,\psi = (\tilde{E}^\varphi,\psi|_{\partial D^1}) \cup (\tilde{E}^\varphi,\psi|_{\partial D^2}) \cup (\tilde{E}^\varphi,\psi|_{\partial D^3}) \cup E^\theta(\varphi,\theta(\psi)) \cup -E^\eta(\varphi,\eta(\psi))
\]
\[
= (\tilde{X}^\varphi \cup \tilde{X}^\psi \cup \tilde{X}^{(\varphi \psi)^{-1}}) \cup E^\theta(\varphi,\theta(\psi)) \cup -E^\eta(\varphi,\eta(\psi)).
\]
By Novikov Additivity, the fact \( \text{Sign} \, \partial \tilde{E}^{\varphi, \psi} = 0 \) implies
\[
\text{Sign}(E_{g+1}^{\eta(\varphi), \eta(\psi)}) - \text{Sign}(E_g^{\theta(\varphi), \theta(\psi)}) = \text{Sign} \, \tilde{X}^{\varphi} + \text{Sign} \, \tilde{X}^{\psi} + \text{Sign} \, \tilde{X}^{(\psi \varphi)^{-1}}.
\]
Notice that \( \tilde{X}^{(\psi \varphi)^{-1}} \) is diffeomorphic to \( \tilde{X}^{(\psi \varphi)^{-1}} \), so that \( \text{Sign} \, \tilde{X}^{(\psi \varphi)^{-1}} = \text{Sign} \, \tilde{X}^{(\psi \varphi)^{-1}} \). By the definition of the Meyer cocycle, we have
\[
\text{Sign}(E_{g+1}^{\eta(\varphi), \eta(\psi)}) = \eta^* \tau_{g+1}(\varphi, \psi), \quad \text{and} \quad \text{Sign}(E_g^{\theta(\varphi), \theta(\psi)}) = \theta^* \tau_g(\varphi, \psi).
\]
Define \( \tilde{\phi}(\varphi) = \text{Sign}(\tilde{X}^{\psi}) \), then we have \( \delta \tilde{\phi} = \eta^* \tau_{g+1} - \theta^* \tau_g \). We get the cobounding function \( \tilde{\phi} \).

**proof of Lemma 3.4.** Write simply \( X := X_{g+1}^{\eta(\varphi)}, \ X' := X_{g,2}^e, \) and \( Y := \tilde{X}^{\psi} = (I \times X) \cup (D^2 \times I \times S^1) \).

For \( i = 0, 1 \), define
\[
\begin{align*}
 j_i : \ X & \to I \times X \hookrightarrow Y, \\
 x & \mapsto (i, x)
\end{align*}
\]
where \( I \times X \hookrightarrow Y \) is a natural embedding. We will prove there is a exact sequence
\[
H_2(X') \xrightarrow{j_0 = j_1} H_2(Y) \xrightarrow{\partial} \ker(H_1(\partial X') \to H_1(X')) \xrightarrow{\partial} 0.
\]
Define \( Y_1 := I \times X' \) and \( Y_2 := (I \times S^1 \times I \times S^1) \cup (D^2 \times I \times S^1) \subset Y \), then
\[
Y_1 \simeq X', Y_2 \simeq S^1, Y_1 \cap Y_2 \simeq \partial X' = (S_1 \cup S_2) \times S^1.
\]

By the Mayer-Vietoris exact sequence, we have
\[
\begin{array}{cccc}
H_2(Y_1) \oplus H_2(Y_2) & \to & H_2(Y) & \to & H_1(Y_1 \cap Y_2) & \to & H_1(Y_1) \oplus H_1(Y_2) \quad \text{(exact)}.
\end{array}
\]

Denote the map \( H_1(\partial X') \to H_1(X') \oplus H_1(S^1) \) in the above diagram by \( h. \) the projection \( H_1(\partial X') \to H_1(S^1) \) to the second entry of \( h \) is the composite of inclusion homomorphism \( H_1(\partial X') \to H_1(X') \) and \( \pi_* : H_1(X') \to H_1(S^1) \). Therefore,
\[
\ker(H_1(\partial X') \to H_1(X') \oplus H_1(S^1)) = \ker(H_1(\partial X') \to H_1(X')).
\]
So the sequence is exact.

Next we construct the splitting \( H_2(Y; Q) = j_{*}, H_2(X'; Q) \oplus \ker(H_1(\partial X'; Q) \to H_1(X'; Q)) \). Note that there exist \( p, q \in Q \) such that
\[
\ker(H_1(\partial X'; Q) \to H_1(X'; Q)) = Q(e_1 + e_2) \oplus Q(p(e_3 - e_4) + qe_1)
\]
as in section 1. To construct the splitting, we choose elements of inverse images of \( e_1 + e_2, \ p(e_3 - e_4) + qe_1 \) under \( H_2(Y) \to H_1(\partial X') \). Define \( \iota_Y : \Sigma_{g+1} \to Y \) by
\[
\begin{array}{cccc}
\Sigma_{g+1} & \to & X & \to & I \times X & \hookrightarrow & Y, \\
x & \mapsto & (x, 0) & \mapsto & (0, x, 0)
\end{array}
\]

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then we have

\[ H_2(\hat{X}) \rightarrow H_1(Y_1 \cap Y_2) \rightarrow H_1(\partial X'), \]

\[ \iota_{Y^*}[\Sigma_g] \mapsto \partial_s \iota_{Y^*}[\Sigma_g] \mapsto e_1 + e_2 \]

so we choose \( \iota_{Y^*}[\Sigma_g] \) as an element of the inverse image of \( e_1 + e_2 \).

Next, we choose an element of the inverse image of \( p(e_3 - e_4) + qe_1 \). Since \( p(e_3 - e_4) + qe_1 \in \text{Ker}(H_1(\partial X'; \mathbb{Q}) \rightarrow H_1(X'; \mathbb{Q})) \), there exists a singular 2-cochain \( s \in C_2(X'; \mathbb{Q}) \) such that

\[ \partial s = p(f_3 - f_4) + qf_1 \in B_1(X'; \mathbb{Q}). \]

For \( i = 0, 1 \), define \( s'_{0i} : I \times S^1 \rightarrow I \times S^1 \times I \times S^1 \hookrightarrow Y_2 \) by \( s'_{0i}(s, t) = (i, 0, s, t) \). then

\[ [\partial s'_{0i}] = [j_if_3 - j_if_4] \in H_1(Y_1 \cap Y_2; \mathbb{Q}). \]

Define \( s'_{1i} : D^2 \rightarrow (-I \times S^1 \times I \times S^1) \cup (D^2 \times I \times S^1) \subset Y \) as shown in Figure 4 by

\[
\begin{align*}
\text{Image of } s'_{10} & \quad (I \times S^1 \times I \times 0) \cup (D^2 \times I \times 0) \\
\text{Image of } s'_{11} & \quad (I \times S^1 \times I \times 0) \cup (D^2 \times I \times 0)
\end{align*}
\]

Figure 4: Images of \( s'_{10} \) and \( s'_{11} \subset (I \times S^1 \times I \times 0) \cup (D^2 \times I \times 0) \subset Y \)

\[
\begin{align*}
& s'_{10}(x) = \begin{cases} 
(6x, 1, 0) & \in D^2 \times I \times S^1 \\
(2 - 6||x||, \frac{|x|}{||x||}, 0, 0) & \in I \times S^1 \times I \times S^1 \\
(0, 1 - ||x||, \frac{|x|}{||x||}, 0) & \in I \times I \times S^1
\end{cases} \quad (||x|| \leq \frac{4}{3}), \\
& s'_{11}(x, t) = \begin{cases} 
(\frac{2}{3}x, 0, 0) & \in D^2 \times I \times S^1 \\
(1, \frac{|x|}{||x||}, 1 - ||x||, 0) & \in I \times S^1 \times I \times S^1 \\
(\frac{4}{3} \leq ||x|| \leq 1)
\end{cases}
\end{align*}
\]

Then, we have \( [\partial s'_{1i}] = [j_if_3] \in H_1(Y_1 \cap Y_2; \mathbb{Q}). \)

Define \( s'_i = ps'_{0i} + qs'_{1i} \), then it follows that

\[ [\partial s'_i] = [j_i(p(f_3 - f_4) + qf_1)] \in H_1(Y_1 \cap Y_2; \mathbb{Q}), \]

so that we have \( [\partial(j_is - s'_i)] = 0 \in H_1(Y_1 \cap Y_2; \mathbb{Q}). \)

We see

\[
\begin{align*}
H_2(Y; \mathbb{Q}) & \rightarrow H_1(Y_1 \cap Y_2; \mathbb{Q}) \rightarrow H_3(\partial X'; \mathbb{Q}), \\
[j_is - s'_i] & \mapsto \partial_s [j_is - s'_i] \mapsto p(e_3 - e_4) + qe_1
\end{align*}
\]
so that we can choose \([j_is - s_i']\) as an element of the inverse image of \(p(e_3 - e_4) + qe_1\).

Now we calculate the intersection form of \(H_2(Y; \mathbb{Q})\). Define \(X''_1 = j_1(X) \cup (D^2 \times 0 \times S^3) \subset (I \times X) \cup (D^2 \times I \times S^3) \subset Y\), then \(X''_1\) is deformation retract of \(Y\). Hence, every element of \(H_2(Y; \mathbb{Q})\) is represented by a cocycle in \(X''_1\). Therefore, a cohomology class is included in the annihilator of intersection form in \(H_2(Y; \mathbb{Q})\) if it is represented by a cocycle which have no common point with \(X''_1\). We see

\[
j_0(X') \cap X''_1 = \emptyset, \quad \text{and} \quad \iota_Y(\Sigma_{g+1}) \cap X''_1 = \emptyset,
\]

so that \(\mathbb{Q}(e_1 + e_2)\) and \(j_0, H_2(X'; \mathbb{Q})\) are included in the annihilator of intersection form in \(H_2(Y; \mathbb{Q})\).

To describe the signature of \(Y\), it suffices to calculate the self-intersection number of \([j_is - s_i'] = p(e_3 - e_4) + qe_1\). The cocycle \(j_is - s_i'\) satisfies

\[
\begin{align*}
\text{Im}(j_0s) \cap (\text{Im}(j_1s) \cup \text{Im}(s_0') \cup \text{Im}(s_1')) & = \emptyset \\
\text{Im}(s_0') \cap (\text{Im}(j_1s) \cup \text{Im}(s_0')) & = \emptyset \\
\text{Im}(s_1') \cap (\text{Im}(j_1s) \cup \text{Im}(s_0') \cup \text{Im}(s_1')) & = \emptyset,
\end{align*}
\]

so that

\[
(j_0s - s_0) \cdot (j_1s - s_1') = (j_0s - (ps_0' + qs_1')) \cdot (j_1s - (ps_0' + qs_1'))
\]

\[
= ps_0' \cdot q's_1'.
\]

We can see \(s_0'\) and \(s_1'\) intersect only once positively. Hence, \(\text{Sign}(Y) = \text{Sign}(pq) = \text{Sign}(m(\varphi))\). \qed

\section{Wall’s Non-additivity Formula}

Wall derives the Novikov additivity for a more general case: two compact oriented smooth 4k-manifolds are glued along a common submanifolds, which itself have boundary, of the boundaries of the original manifolds.

We will give the specific case of his formula for \(k = 1\):

Let \(Z\) be a closed oriented smooth 2-manifold, \(X_-, X_0, X_+\) compact oriented smooth 3-manifolds with the boundaries \(\partial X_- = \partial X_0 = \partial X_+ = Z\), and \(Y_-, Y_+\) compact oriented smooth 4-manifolds with the boundaries \(\partial Y_- = X_- \cup_Z (\neg X_0), \partial Y_+ = X_0 \cup_Z (\neg X_+).\) Here we denote by \(M \cup_B (\neg N)\) the union of two manifolds \(M\) and \(N\) glued by orientation reversing diffeomorphism of their common boundaries \(\partial M = \partial N = B\). Let \(Y = Y_- \cup_{X_0} Y_+\) be the union of \(Y_-\) and \(Y_+\) glued along submanifolds \(X_0\) of their boundaries. Suppose \(Y\) is oriented by the induced orientation of \(Y_-\) and \(Y_+\).

Write \(V = H_1(Z; \mathbb{R})\); let \(A, B,\) and \(C\) be the kernels of the maps on first homology induce by the inclusions of \(Z\) in \(X_-\), \(X_0\) and \(X_+\) respectively.

We define

\[
W := \frac{B \cap (C + A)}{(B \cap C) + (B \cap A)},
\]

and a bilinear form \(\Psi\) by

\[
\Psi : W \times W \to \mathbb{R}, \quad (b, b') \mapsto b \cdot c'
\]
Here $c'$ is a element which satisfies $a' + b' + c' = 0$, and $b \cdot c'$ denote the intersection product of $b$ and $c'$.

Then $\Psi$ is independent of $c'$ and well-defined on $W$. Denote the signature of the bilinear form $\Psi$ by $\text{Sign}(V; BCA)$ and the signature of the compact oriented 4-manifold $M$ by $\text{Sign} M$. We are now ready to state the formula.

**Theorem 2.5** (Wall[12]). $\text{Sign} Y = \text{Sign} Y_- + \text{Sign} Y_+ - \text{Sign}(V; BCA)$.

### 2.3 The differences of signature $\text{Sign} E_g - \text{Sign} E_{g,2}$ and $\text{Sign} E_{g+1} - \text{Sign} E_{g,2}$

In this subsection, we calculate the difference of signature associated with sewing the trivial Disk bundles onto the $\Sigma_{g,2}$ bundle.

In Introduction, we defined $E_{g,r}^\varphi$ as a oriented $\Sigma_{g,r}$ bundle on $P$ which has monodromies $\varphi, \psi, (\psi \varphi)^{-1} \in \mathcal{M}_{g,r}$ along $\alpha, \beta, \gamma \in \pi_1(P)$. If we fix $\varphi, \psi \in \mathcal{M}_{g,2}$, we denote simply

$$E_{g,2} := E_{g,2}^\varphi, \quad E_g := E_{g,2}^\psi \varphi \theta(\psi), \quad E_{g+1} := E_{g+1}^\eta(\varphi) \theta(\psi) \quad (g \geq 0).$$

**Proposition 2.6.** $\text{Sign}(E_g) - \text{Sign}(E_{g,2}) = - \text{Sign}(m(\varphi) + m(\psi) + m((\varphi \psi)^{-1})) \quad (g \geq 0)$

**Proof.** $E_g$ is the union of $E_{g,2}$ and $E_D := (D^2 \amalg D^2) \times P$ glued along their boundaries. Using Non-additivity formula Theorem 2.5, we calculate $\text{Sign}(E_g) - \text{Sign}(E_{g,2})$.

Define $Y_-, Y_+, X_-, X_0, X_+$, and $Z$ by

$$Y_- := (\Pi_{j=1}^2 D^2) \times P, \quad Y_+ := E_{g,2},$$

$$X_- := (\Pi_{j=1}^2 D^2) \times \partial P, \quad X_+ := E_{g,2}|_{\partial P}, \quad X_0 := (\Pi_{j=1}^2 \partial D^2) \times P,$$

and $Z := (\Pi_{j=1}^2 \partial D^2) \times \partial P$, respectively.

Here, by the notation stated in subsection 1.1,

$$X_+ = E_{g,2}|_{\partial P} \cong X^\varphi \amalg X^\psi \amalg X^{(\psi \varphi)^{-1}}, \quad Z \cong \partial X^\varphi \amalg \partial X^\psi \amalg \partial X^{(\psi \varphi)^{-1}}.$$

Define $V, A, B, \text{and } C$ as stated in subsection 3.1.

Since $X^\varphi = X^\psi = X^{(\psi \varphi)^{-1}} = S^1 \times S^1$, we can choose the base of $H_1(\partial X^\varphi; \mathbb{R}), \ H_1(\partial X^\psi; \mathbb{R})$, and $H_1(\partial X^{(\psi \varphi)^{-1}}; \mathbb{R})$ as in section 1.1. Denote their base by $\{e_{11}, e_{12}, e_{13}, e_{14}\}, \{e_{21}, e_{22}, e_{23}, e_{24}\}, \{e_{31}, e_{32}, e_{33}, e_{34}\}$ respectively.

Since $Z = \partial X^\varphi \amalg \partial X^\psi \amalg \partial X^{(\psi \varphi)^{-1}}$, we think of $e_{ij}$ as an element of $H_1(Z; \mathbb{R})$. 


Denote $m(\varphi) = [a_1 : b_1]$, $m(\psi) = [a_2 : b_2]$, and $m((\psi\varphi)^{-1}) = [a_3 : b_3]$ respectively, then

\[ V = H_1(Z, R) = \bigoplus_{i=1}^{3} \bigoplus_{j=1}^{4} Re_{ij}, \]

\[ A = Re_{11} \oplus Re_{21} \oplus Re_{31} \oplus Re_{12} \oplus Re_{22} \oplus Re_{32}, \]

\[ B = R(e_{11} - e_{21}) \oplus R(e_{11} - e_{31}) \oplus R(e_{12} - e_{22}) \oplus R(e_{12} - e_{32}) \]

\[ \oplus R(e_{13} + e_{23} + e_{33} - e_{24} - e_{34} - m_1e_{11} + m_2e_{21} + m_3e_{31}), \]

\[ C = \bigoplus_{i=1}^{3} \begin{cases} 
  R(e_{i1} + e_{i2}) \oplus R(e_{i3} - e_{i4} + m_i e_{i1}) & \text{if } a_i \neq 0 \\
  Re_{i1} \oplus Re_{i2} & \text{if } a_i = 0.
\end{cases} \]

Here we denote $m_i := \frac{b_i}{a_i}$.

Hence,

\[ B \cap A = R(e_{11} - e_{21}) \oplus R(e_{12} - e_{22}) \oplus R(e_{11} - e_{31}) \oplus R(e_{12} - e_{32}), \]

\[ B \cap C = \begin{cases} 
  R(e_{11} - e_{21} + e_{12} - e_{22}) & \text{if } a_i \neq 0 \text{ for } i = 1, 2, 3 \\
  R(e_{11} - e_{31} + e_{12} - e_{32}) & \text{and } m_1 + m_2 + m_3 = 0, \\
  R(e_{13} + e_{23} + e_{33} - e_{24} - e_{34} - m_1e_{11} + m_2e_{21} + m_3e_{31}) & \text{and } m_1 + m_2 + m_3 \neq 0,
\end{cases} \]

\[ B \cap (C + A) = \begin{cases} 
  R(e_{11} - e_{21} + e_{12} - e_{22}) & \text{if } a_i = 0 \text{ for } i = 1, 2, 3 \\
  R(e_{11} - e_{12} + e_{21} - e_{22}) & \text{if } a_i = 0 \text{ otherwise}, \\
  R(e_{11} - e_{12} + e_{21} - e_{22}) \oplus R(e_{12} - e_{22}) & \text{if } a_i \neq 0 \text{ for } i = 1, 2, 3,
\end{cases} \]

By computing the signature of $\Psi$, we have

\[ \text{Sign}(V; BCA) = \begin{cases} 
  \text{Sign}(m_1 + m_2 + m_3) & \text{if } a_i \neq 0 \text{ for } i = 1, 2, 3, \\
  0 & \text{otherwise}.
\end{cases} \]

Hence,

\begin{align*}
\text{Sign}(V; BCA) &= \text{Sign}(m(\varphi) + m(\psi) + m((\psi\varphi)^{-1})) \\
&= \text{Sign}(m(\varphi) + m(\psi) + m((\varphi\psi)^{-1})).
\end{align*}
By Non-additivity formula, we have
$$\operatorname{Sign}(E_g) = \operatorname{Sign}(E_D) + \operatorname{Sign}(E_{g,2}) - \operatorname{Sign}(V;BCA).$$

Since $E_D$ is a trivial bundle $(D^2 \amalg D^2) \times P$, we have $\operatorname{Sign}(E_D) = 0$.
This completes the proof of the proposition.

By the theorem and Proposition 2.6, we can calculate the difference of signature $\operatorname{Sign}(E_g) - \operatorname{Sign}(E_{g,2})$.

**Corollary 2.7.** For $g \geq 0$,
$$\operatorname{Sign}(E_{g+1}) - \operatorname{Sign}(E_{g,2}) = \operatorname{Sign}(m(a)) + \operatorname{Sign}(m(b)) + \operatorname{Sign}(m((ab)^{-1}))$$
$$- \operatorname{Sign}(m(a) + m(b) + m((ab)^{-1})).$$

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