NEVANLINNA THEORY ON COMPLETE KÄHLER MANIFOLDS

XIANJING DONG

Abstract. We study Nevanlinna theory on complete Kähler manifolds. As a consequence of the main result, we prove a defect relation of holomorphic mappings from complete Kähler manifolds of non-positive sectional curvature into complex projective manifolds under certain growth condition.

1. Introduction

Early in 1972, Carlson and Griffiths [6, 11] established the equi-distribution theory of holomorphic mappings from $\mathbb{C}^m$ into complex projective manifolds intersecting divisors. Later, Griffiths and King [10, 11] further generalized the theory from $\mathbb{C}^m$ to affine manifolds. Let us review this theory briefly.

Let $V$ be a complex projective manifold of complex dimension $m$, and let $L \to V$ be a positive line bundle over $V$. For a reduced divisor $D$ on $V$ and a holomorphic mapping $f : \mathbb{C}^m \to V$, we have the standard notations $T_f(r, L)$, $m_f(r, D)$ and $N_f(r, D)$ in Nevanlinna theory (see [16, 18] or Remark 2.1). Carlson and Griffiths proved the following Second Main Theorem

Theorem A. Let $L \to V$ be a positive line bundle and let a reduced divisor $D \in |L|$ be of simple normal crossing type. Let $f : \mathbb{C}^m \to V$ be a differentiably non-degenerate equi-dimensional holomorphic mapping. Then for any $\delta > 0$

$$T_f(r, L) + T_f(r, K_V) \leq N_f(r, D) + O(\log T_f(r, L) + \delta \log r)$$

holds for all $r > 1$ outside a set $E_\delta \subset (1, \infty)$ of finite Lebesgue measure.

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Theorem A was extended by Sakai [19] in terms of Kodaira dimension, and
generalized by Shiffman [20] in the singular divisor case. More investigations
were done by Wong, Lang, Cherry and Noguchi (see [10, 15, 16, 17, 18, 22]).

The purpose of this paper is to generalize Theorem A to complete Kähler
manifolds. Our approach is to combine stochastic method with Wong-Lang’s
technique. Recall that the Brownian motion method was first used by Carne
[7] in proving Nevanlinna’s Second Main Theorem of meromorphic functions
on \( \mathbb{C} \). Later, Atsuji [1, 2, 3, 4] developed this technique to study the Second
Main Theorem of meromorphic functions on complete Kähler manifolds. Re-
cently, Dong-He-Ru [9] re-visited this technique and provided a probabilistic
proof of Cartan’s Second Main Theorem of holomorphic curves.

We state the main result. For technical reasons, all manifolds (as domains)
are assumed to be open in this paper. Let \( M \) be a complete Kähler manifold
of non-positive sectional curvature with complex dimension \( m = \dim_{\mathbb{C}} V \). For
a holomorphic mapping \( f : M \to V \), one can extend the definition of classical
Nevanlinna’s functions (see Section 2) to Kähler manifold \( M \) naturally. Let
\( \text{Ric}_M \) be the Ricci curvature tensor of \( M \), set
\[
(1) \quad \kappa(t) = \frac{1}{2m-1} \min_{x \in B_0(t)} R_M(x),
\]
where \( R_M(x) \) is the pointwise lower bound of Ricci curvature defined by
\[
R_M(x) = \inf_{\xi \in T_x M, \|\xi\|=1} \text{Ric}_M(\xi, \xi).
\]

**Theorem 1.1.** Let \( L \to V \) be a positive line bundle and let a reduced divisor
\( D \in |L| \) be of simple normal crossing type. Let \( f : M \to V \) be a differentiably
non-degenerate equi-dimensional holomorphic mapping. Then
\[
T_f(r, L) + T_f(r, K_V) - N_f^{[1]}(r, D) \leq \frac{m+k}{2} \log T_f(r, L) + O \left( \log^+ \log T_f(r, L) - \kappa(r)r^2 + \log^+ \log r \right)
\]
holds for all \( r > 1 \) outside a set of finite Lebesgue measure, where \( k \) is the complexity of \( D \) defined by (5).

The term \( \kappa(r) \) appeared in Theorem 1.1 depends on the curvature of \( M \). Consider the simple case where \( M = \mathbb{C}^m \), we have \( \kappa(r) \equiv 0 \) and \( T_f(r, L) \geq O(\log r) \) as \( r \to \infty \). It yields from Theorem 1.1 that
\[
T_f(r, L) + T_f(r, K_V) - N_f^{[1]}(r, D) \leq \frac{m+k}{2} \log T_f(r, L) + \text{Lower order terms}
\]
So, Theorem 1.1 implies Theorem A. Coefficient \( (m+k)/2 \) before \( \log T_f(r, L) \)
is optimal. When \( m = 1 \), we have \( k = 1 \) and \( (m+k)/2 = 1 \). It is mentioned
that Ye [23] showed the estimate “1” is best.
As a consequence of Theorem 1.1, we derive a defect relation. Recall that the defect (without counting multiplicities) of $f$ with respect to $D$ is defined by

$$\Theta_f(D) := 1 - \limsup_{r \to \infty} \frac{N_f^{[1]}(r, D)}{T_f(r, L)}.$$ 

In general, we set for two holomorphic line bundles $L_1, L_2$ over $V$ that

$$\left[\frac{c_1(L_2)}{c_1(L_1)}\right] := \inf \left\{ s \in \mathbb{R} : \omega_2 < s\omega_1, \exists \omega_1 \in c_1(L_1), \exists \omega_2 \in c_1(L_2) \right\}.$$

Corollary 1.2. Assume the same conditions as in Theorem 1.1. If $f$ satisfies the growth condition

$$\liminf_{r \to \infty} \frac{\kappa(r)^2}{T_f(r, L)} = 0,$$

then

$$\Theta_f(D) \leq \left[\frac{c_1(K^*_V)}{c_1(L)}\right].$$

In particular, when $M = \mathbb{C}^m$, we derive Carlson-Griffiths’ defect relation.

2. Basic notations

2.1. Brownian motions.

We first introduce Brownian motions in Riemannian manifolds and notions of Nevanlinna’s functions, then we give the First Main Theorem of Nevanlinna theory.

Let $(M, g)$ be a Riemannian manifold with Laplace-Beltrami operator $\Delta_M$ associated to $g$. For $x \in M$, we denote by $B_x(r)$ the geodesic ball centered at $x$ with radius $r$, and denote by $S_x(r)$ the geodesic sphere centered at $x$ with radius $r$. By Sard’s theorem, $S_x(r)$ is a submanifold of $M$ for almost every $r > 0$. A Brownian motion $X_t$ in $M$ is a heat diffusion process generated by $\frac{1}{2}\Delta_M$ with transition density function $p(t, x, y)$ which is the minimal positive fundamental solution of the heat equation

$$\frac{\partial}{\partial t} u(t, x) - \frac{1}{2}\Delta_M u(t, x) = 0.$$ 

We denote by $\mathbb{P}_x$ the law of $X_t$ started at $x \in M$ and by $\mathbb{E}_x$ the corresponding expectation with respect to $\mathbb{P}_x$.

Co-area formula and Dynkin formula

Let $D$ be a bounded domain with smooth boundary $\partial D$ in $M$. Fix $x \in D$, we use $d\pi^D_x$ to denote the harmonic measure on $\partial D$ with respect to $x$. This measure is a probability measure. Set

$$\tau_D := \inf \{ t > 0 : X_t \notin D \}$$
which is a stopping time. Let $g_D(x, y)$ denote the Green function of $\Delta_M/2$ for $D$ with a pole at $x$ and Dirichlet boundary condition, namely

$$-\frac{1}{2}\Delta_M g_D(x, y) = \delta_x(y), \quad y \in D; \quad g_D(x, y) = 0, \quad y \in \partial D,$$

where $\delta_x$ is the Dirac function. For $\phi \in C_\delta(D)$ (space of bounded continuous functions on $D$), co-area formula [5] asserts that

$$E_x \left[ \int_0^{\tau_D} \phi(X_t) dt \right] = \int_D g_D(x, y) \phi(y) dV(y).$$

From Proposition 2.8 in [5], we also have the relation of harmonic measures and hitting times that

$$(2) \quad E_x [\psi(X_{\tau_D})] = \int_{\partial D} \psi(y) d\pi^D_x(y)$$

for any $\psi \in C(D)$. Since the expectation “$E_x$”, co-area formula and (2) still work in the case when $\phi$ or $\psi$ has a pluripolar set of singularities.

Let $u \in C^2(M)$ (space of bounded $C^2$-class functions on $M$), we have the famous Itô formula (see [1, 12, 13, 14])

$$u(X_t) - u(x) = B \left( \int_0^t \| \nabla_M u \|^2(X_s) ds \right) + \frac{1}{2} \int_0^t \Delta_M u(X_s) dt, \quad P_x \text{ - a.s.}$$

where $B_t$ is the standard Brownian motion in $\mathbb{R}$ and $\nabla_M$ is gradient operator on $M$. Take expectation of both sides of the above formula, it follows Dynkin formula (see [1, 14])

$$E_x [u(X_T)] - u(x) = \frac{1}{2} E_x \left[ \int_0^T \Delta_M u(X_t) dt \right]$$

for a stopping time $T$ such that each term makes sense. Noting that Dynkin formula still holds for $u \in C^2(M)$ if $T = \tau_D$. In further, it also works when $u$ is of a pluripolar set of singularities, particularly, for a plurisubharmonic function $u$.

### 2.2. Nevanlinna’s functions.

Let

$$f : M \rightarrow V$$

be a holomorphic mapping into a compact complex manifold $V$. Fix $o \in M$ as a reference point and denote by $g_r(o, x)$ the Green function of $\Delta_M/2$ for geodesic ball $B_r(o)$ with a pole at $o$ and Dirichlet boundary condition. For a $(1,1)$-form $\varphi$ on $M$, we use the following convenient notations

$$e_{\varphi}(x) := 2m \frac{\varphi \wedge \alpha^{m-1}}{\alpha^m}, \quad T(r, \varphi) := \frac{1}{2} \int_{B_r(o)} g_r(o, x) e_{\varphi}(x) dV(x),$$
where $dV$ is the Riemannian volume measure of $M$. For a $(1,1)$-form $\omega$ on $N$, the characteristic function of $f$ with respect to $\omega$ is defined by

$$T_f(r, \omega) := T(r, f^*\omega).$$

Let $L \to V$ be a holomorphic line bundle equipped with Hermitian metric $h$, the associated Chern form of $L$ is $c_1(L, h) := -dd^c \log h$. We define

$$T_f(r, L) := T_f(r, c_1(L, h))$$

up to a bounded term. A simple computation shows that

$$e_{f^*c_1(L, h)} = 2m \frac{f^*c_1(L, h) \wedge \alpha^{m-1}}{\alpha^m} = -\frac{1}{2} \Delta_M \log (h \circ f).$$

Set

$$\tau_r := \inf \{ t > 0 : X_t \notin B_o(r) \},$$

where $X_t$ is the Brownian motion in $M$ generated by $\Delta_M/2$ started at $o$. By co-area formula, we have

$$T_f(r, L) = \frac{1}{2} \mathbb{E}_o \left[ \int_0^{\tau_r} e_{f^*c_1(L, h)}(X_t) dt \right].$$

Let $\mathcal{R}_M := -dd^c \log \det (g_{ij})$ be the Ricci curvature form of $(M, g)$. We define the Ricci curvature term by

$$T(r, \mathcal{R}_M) := \frac{1}{2} \int_{B_o(r)} g_r(o, x) e_{\mathcal{R}_M}(x) dV(x)$$

$$= m \mathbb{E}_o \left[ \int_0^{\tau_r} \frac{\mathcal{R}_M \wedge \alpha^{m-1}}{\alpha^m} (X_t) dt \right]$$

$$= -\frac{1}{4} \mathbb{E}_o \left[ \int_0^{\tau_r} \Delta_M \log \det (g_{ij}(X_t)) dt \right].$$

Given $D \in |L|$, an effective divisor such that $s_D \in H^0(N, L)$, where $s_D$ is the canonical section defined by $D$. Since $V$ is compact, assume that $\|s_D\| < 1$.

The proximity function of $f$ with respect to $D$ is defined by

$$m_f(r, D) := \int_{S_o(r)} \log \frac{1}{\|s_D \circ f(x)\|} d\pi_o^r(x),$$

where $d\pi_o^r$ is the harmonic measure on $S_o(r)$ with respect to $o$. The relation between harmonic measure and hitting time implies that

$$m_f(r, D) = \mathbb{E}_o \left[ \log \frac{1}{\|s_D \circ f(X_{\tau_r})\|} \right].$$

The counting function of $f$ with respect to $D$ is defined by

$$N_f(r, D) := \frac{\pi^m}{(m-1)!} \int_{f^*D \cap B_o(r)} g_r(o, x) \alpha^{m-1},$$
where
\[ \alpha := \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^{m} g_{ij} dz_i \wedge d\bar{z}_j \]
is the Kähler metric form of \( M \) associated to \( g \). Writing \( s_D = \tilde{s}_D e^\alpha \) locally, where \( \{e_\alpha, U_\alpha\} \) is a local holomorphic frame of \((L,h)\) restricted to \( U_\alpha \). Then we have
\[ N_f(r, D) = \int_{B_o(r)} g_r(o,x) \frac{\pi^m}{(m-1)!} \int_{B_o(r)} g_r(o,x) \Delta_M \log |\tilde{s}_D \circ f|^2 dV(x). \]

**Remark 2.1.** The definitions of Nevanlinna’s functions in above are natural extensions of the classical ones. To see that, we recall the \( \mathbb{C}^m \) case:
\[ T_f(r, L) = m f(r, D) + N_f(r, D) + O(1). \]

Let \( M \) be a simply-connected complete Kähler manifold with non-positive sectional curvature. Let \( \kappa \) be defined by (1), then \( \kappa \) is a non-positive, non-increasing and continuous function on \([0, \infty)\). Consider the ODE
\[ G''(t) + \kappa(t)G(t) = 0; \quad G(0) = 0, \quad G'(0) = 1 \]
on \([0, \infty)\). Comparing (4) with \(y''(t) + \kappa(0)y(t) = 0\) provided with the same initial conditions, we see that \(G\) can be estimated simply as
\[
G(t) = t \quad \text{for } \kappa \equiv 0; \quad G(t) \geq t \quad \text{for } \kappa \not\equiv 0.
\]
This follows that (4)
\[
G(r) \geq r \quad \text{for } r \geq 0; \quad \int_1^r \frac{dt}{G^{2m-1}(t)} \leq \log r \quad \text{for } r \geq 1.
\]
On the other hand, we rewrite (3) in the form
\[
\log' G(t) \cdot \log' G'(t) = -\kappa(t).
\]
Since \(G(t) \geq t\) is increasing, then the decrease and non-positivity of \(\kappa\) imply that for each fixed \(t\), \(G\) must be satisfied one of the following two inequalities
\[
\log' G(t) \leq \sqrt{-\kappa(t)} \quad \text{for } t > 0; \quad \log' G'(t) \leq \sqrt{-\kappa(t)} \quad \text{for } t \geq 0.
\]
By virtue of \(G(t) \to 0\) as \(t \to 0\), by integration, \(G\) is bounded from above by (5)
\[
G(r) \leq r \exp(r\sqrt{-\kappa(r)}) \quad \text{for } r \geq 0.
\]
Before giving the Calculus Lemma, we introduce some lemmas.

**Lemma 3.1** ([4]). Let \(G(t)\) be defined in (3), and let \(\eta > 0\) be a constant. Then there exists a constant \(C > 0\) such that for \(r > \eta\) and \(x \in B_\eta(r) \setminus B_\eta(\eta)\), we have
\[
g_r(o, x) \int_\eta^r G^{1-2m}(t)dt \geq C \int_{r(x)}^r G^{1-2m}(t)dt.
\]

**Lemma 3.2** ([8]). We have
\[
d\pi_o^r(x) \leq \frac{1}{\omega_{2m-1}r^{2m-1}}d\sigma_r(x),
\]
where \(d\pi_o^r(x)\) is the harmonic measure on geodesic sphere \(S_\eta(r)\) with respect to \(o \in M\), \(d\sigma_r(x)\) is the induced volume measure on \(S_\eta(r)\) and \(\omega_{2m-1}\) is the Euclidean volume of unit sphere in \(\mathbb{R}^{2m}\).

**Lemma 3.3** (Borel Lemma, [18]). Let \(T\) be a strictly positive nondecreasing function of \(C^1\)-class on \((0, \infty)\). Let \(\gamma > 0\) be a number such that \(T(\gamma) \geq e\), and \(\phi\) be a strictly positive nondecreasing function such that
\[
c_\phi = \int_\epsilon^\infty \frac{1}{t\phi(t)}dt < \infty.
\]
Then, the inequality \(T'(r) \leq T(r)\phi(T(r))\) holds for all \(r \geq \gamma\) outside a set of Lebesgue measure not exceeding \(c_\phi\). In particular, if take \(\phi(t) = \log^{1+\delta} t\) for \(\delta > 0\), then
\[
T'(r) \leq T(r)\log^{1+\delta} T(r)
\]
holds for all \(r > 0\) outside a set \(E_\delta \subset (0, \infty)\) of finite Lebesgue measure.
We are ready to prove the following so-called Calculus Lemma

**Theorem 3.4** (Calculus Lemma). Let $\Gamma \geq 0$ be a locally integrable function on $M$ such that it is locally bounded at $o \in M$. Then for any $\delta > 0$, there exists a constant $C > 0$ independent of $\Gamma, \delta$, and a set $E_\delta \subset (1, \infty)$ of finite Lebesgue measure such that

$$\mathbb{E}_o[\Gamma(X_{\tau_r})] \leq \frac{F(\hat{\Gamma}, \kappa, \delta)e^{(2m-1)\tau\sqrt{-\kappa(r)}}}{C\omega_{2m-1}} \log r \mathbb{E}_o \left[ \int_0^{\tau_r} \Gamma(X_t)dt \right]$$

holds for $r > 1$ outside $E_\delta$, where $\kappa$ is defined by (1), $\omega_{2m-1}$ is the Euclidean volume of unit sphere in $\mathbb{R}^{2m}$ and $F$ is defined by

$$F(\hat{\Gamma}, \kappa, \delta) = \left\{ \log^+ \hat{\Gamma}(r) \log^+ \left( (r^{2m-1}e^{(2m-1)\tau\sqrt{-\kappa(r)}}\hat{\Gamma}(r) \left( \log^+ \hat{\Gamma}(r) \right)^{1+\delta} \right) \right\}^{1+\delta}$$

with

$$\hat{\Gamma}(r) = \frac{\log r}{C} \mathbb{E}_o \left[ \int_0^{\tau_r} \Gamma(X_t)dt \right].$$

Moreover, we have the estimate

$$\log F(\hat{\Gamma}, \kappa, \delta) \leq O \left( \log^+ \log \mathbb{E}_o \left[ \int_0^{\tau_r} \Gamma(X_t)dt \right] + \log^+ (r \sqrt{-\kappa(r)}) + \log^+ \log r \right).$$

**Proof.** Combining Lemma 3.1 with Lemma 5.2 and (1), we obtain

$$\mathbb{E}_o \left[ \int_0^{\tau_r} \Gamma(X_t)dt \right] = \int_{B_o(r)} g_r(o, x) \Gamma(x) dV(x)$$

$$= \int_0^r dt \int_{S_o(t)} g_r(o, x) \Gamma(x) d\sigma_t(x)$$

$$\geq C \int_0^r dt \int_{S_o(t)} G^{1-2m}(s) ds \int_{S_o(t)} \Gamma(x) d\sigma_t(x)$$

$$= \frac{C}{\log r} \int_0^r dt \int_t^r G^{1-2m}(s) ds \int_{S_o(t)} \Gamma(x) d\sigma_t(x)$$

and

$$\mathbb{E}_o \left[ \Gamma(X_{\tau_r}) \right] = \int_{S_o(r)} \Gamma(x) d\sigma_r^o(x) \leq \frac{1}{\omega_{2m-1}r^{2m-1}} \int_{S_o(r)} \Gamma(x) d\sigma_r(x),$$

where $d\sigma_r$ is the induced volume measure on $S_o(r)$. Thus, we have

$$\mathbb{E}_o \left[ \int_0^{\tau_r} \Gamma(X_t)dt \right] \geq \frac{C}{\log r} \int_0^r dt \int_t^r G^{1-2m}(s) ds \int_{S_o(t)} \Gamma(x) d\sigma_t(x)$$

and

$$\mathbb{E}_o \left[ \Gamma(X_{\tau_r}) \right] \leq \frac{1}{\omega_{2m-1}r^{2m-1}} \int_{S_o(r)} \Gamma(x) d\sigma_r(x).$$

(6)
Put
\[ \Gamma(r) = \int_0^r dt \int_t^r G^{1-2m}(s)ds \int_{S_o(t)} \Gamma(x) d\sigma_t(x). \]
Then
\[ \Gamma(r) \leq \frac{\log r}{C} \mathbb{E}_o \left[ \int_0^r \Gamma(X_t) dt \right] = \hat{\Gamma}(r). \]
Since
\[ \Gamma'(r) = G^{1-2m}(r) \int_0^r dt \int_{S_o(t)} \Gamma(x) d\sigma_t(x), \]
it yields from (6) that
\[ (7) \quad \mathbb{E}_o \left[ \Gamma(X_r) \right] \leq \frac{1}{\omega_{2m-1} r^{2m-1}} \frac{d}{dr} \left( \frac{\Gamma'(r)}{G^{1-2m}(r)} \right). \]
Using Borel Lemma (Lemma 3.3) twice, then for any \( \delta > 0 \)
\[ \frac{d}{dr} \left( \frac{\Gamma'(r)}{G^{1-2m}(r)} \right) \]
\[ \leq G^{2m-1}(r) \left\{ \log^+ \Gamma(r) \cdot \log^+ \left( G^{2m-1}(r) \Gamma(r) \left( \log^+ \Gamma(r) \right)^{1+\delta} \right) \right\}^{1+\delta} \hat{\Gamma}(r) \]
\[ \leq G^{2m-1}(r) \left\{ \log^+ \hat{\Gamma}(r) \cdot \log^+ \left( G^{2m-1}(r) \hat{\Gamma}(r) \left( \log^+ \hat{\Gamma}(r) \right)^{1+\delta} \right) \right\}^{1+\delta} \hat{\Gamma}(r) \]
\[ = \frac{F(\hat{\Gamma}, \kappa, \delta) G^{2m-1}(r) \log r}{C} \mathbb{E}_o \left[ \int_0^r \Gamma(X_t) dt \right] \]
holds outside a set \( E_\delta \subset (1, \infty) \) of finite Lebesgue measure. By this with (7) and (5), we have the desired inequality. Taking “log” before \( F \), the estimate is obtained. \( \Box \)

4. A PROOF OF THEOREM 1.1

4.1. Preparations.

Let \((M, g)\) be a \(m\)-dimensional simply-connected Kähler manifold of non-positive sectional curvature. The associated Kähler form is defined by
\[ \alpha = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^m g_{ij} dz_i \wedge d\bar{z}_j. \]
Let \( \varphi = \frac{\sqrt{-1}}{2\pi} \sum_{i,j=1}^m \varphi_{ij} dz_i \wedge d\bar{z}_j \) be a (1,1)-form on \( M \). We use the following convenient symbols
\[ \det(\varphi) := \det(\varphi_{ij}), \quad T_g(\varphi) := \sum_{i,j=1}^m g^{i\bar{j}} \varphi_{ij}, \]
where \((g^{i\bar{j}})\) is the inverse of \((g_{ij})\). It is trivial to show that \( T_g(\varphi) \) is globally defined on \( M \).
Lemma 4.1. We have
\[ \varphi \wedge \alpha^{m-1} = \frac{1}{m} T_g(\varphi) \alpha^m. \]

Proof. By a direct computation, it follows that
\[ \frac{\varphi \wedge \alpha^{m-1}}{\alpha^m} = \frac{1}{m} \sum_{i,j=1}^m \varphi_{ij} G_{ji} \frac{\bar{G}_{ij}}{\det(g_{st})}, \]
where
\[ G^* = \begin{pmatrix} G_{11} & \cdots & G_{m1} \\ \vdots & \ddots & \vdots \\ G_{1m} & \cdots & G_{mm} \end{pmatrix} \]
is the adjoint matrix of \( G = (g_{st}) \). Note \( g_{ij} = \frac{G_{ij}}{\det(g_{st})} \), hence we have
\[ \sum_{i,j=1}^m \frac{\varphi_{ij} G_{ji}}{\det(g_{st})} = \sum_{i,j=1}^m g_{ij} \varphi_{ij} = T_g(\varphi). \]
The proof is completed.

Lemma 4.2. If \( \varphi \) is Hermitian semi-positive, then
\[ \left( \frac{\det(\varphi)}{\det(g_{ij})} \right)^{\frac{1}{m}} \leq \frac{1}{m} T_g(\varphi). \]

Proof. Fix \( x \in M \), take local holomorphic coordinates \( z_1 \cdots, z_m \) around \( x \) such that \( g_{ij}(x) = \delta_{ij} \). At \( x \), the inequality is equivalent to
\[ (\det(\varphi))^{\frac{1}{m}} \leq \frac{1}{m} \text{tr}(\varphi) \]
which holds clearly. In fact, the linear algebra theory asserts that
\[ \det(\varphi) = \lambda_1 \cdots \lambda_m, \quad \text{tr}(\varphi) = \lambda_1 + \cdots + \lambda_m, \]
where \( \lambda_1, \cdots, \lambda_m \) are eigenvalues of \( (\varphi_{ij}) \). Since \( \varphi \) is Hermitian semi-positive, then \( \lambda_1, \cdots, \lambda_m \geq 0 \). The mean-value inequality implies the lemma.

Wong-Lang’s approach

Let \( V \) be a complex projective manifold and let \( L \to V \) be a positive line bundle. Let a reduced divisor \( D \in |L| \) be of simple normal crossing type, we write \( D = \sum_{j=1}^q D_j \) as the union of irreducible components, i.e., \( D_1, \cdots, D_q \) are irreducible and non-singular, moreover, at each point \( x \) of \( V \) there exists a local holomorphic coordinate neighborhood \( U(z_1, \cdots, z_m) \) of \( x \) such that
\[ D \cap U = \{ z_1 \cdots z_{k_x} = 0 \}, \quad 0 \leq k_x \leq m. \]
If \( k_x = 0 \), then it means that \( D \cap U = \emptyset \). Set
\[
(8) \quad k := \max_{x \in V} k_x,
\]
which is called the complexity of \( D \). Denote by \( s_j \) \((1 \leq j \leq q)\) the canonical section defined by \( D_j \). Clearly, \( s_D = s_1 \otimes \cdots \otimes s_q \) gives the canonical section defined by \( D \). Endowing \( L_{D_j} \) \((1 \leq j \leq q)\) with a Hermitian metric \( h_j \), which induces a natural Hermitian metric \( h \) on \( L \). Since \( L > 0 \), one may assume that \( c_1(L, h) = -dd^c \log h > 0 \). Define the singular volume form
\[
(9) \quad \Phi_{D, \lambda} := \frac{\Omega}{\prod_{j=1}^q \|s_j\|^{2(1-\lambda)}}, \quad \Omega = (-dd^c \log h)^m
\]
on \( V \), where \( 0 < \lambda < 1 \) is a constant. Set
\[
\eta_{D, \lambda} := (1 + q)\lambda c_1(L, h) + \sum_{j=1}^q dd^c \log(1 + \|s_j\|^{2\lambda}).
\]
Lang proved that

**Lemma 4.3 (Lemma 7.4, [15])**. There exists a number \( b > 0 \) depending only on \( D \), \( \Omega \) and \( c_1(L, h) \) such that
\[
\lambda^{m+k} \Phi_{D, \lambda} \leq b \eta_{D, \lambda}^m.
\]

**Estimate of \( \log F(\hat{\Gamma}, \kappa, \delta) \) with \( \Gamma = \xi^{1/m} \)**

Let \( f : M \to V \) be a differentiably non-degenerate equi-dimensional holomorphic mapping, i.e., the differential \( df \) has rank \( m \) at a point of \( M \). Write
\[
(10) \quad \Omega = a(w) \bigwedge_{j=1}^m \frac{\sqrt{-1}}{2\pi} dw_j \wedge d\bar{w}_j
\]
in a local holomorphic coordinate system \( w \). It follows that
\[
f^* \Omega = a(f)|J(f)|^2 \bigwedge_{j=1}^m \frac{\sqrt{-1}}{2\pi} dz_j \wedge d\bar{z}_j
\]
in a local holomorphic coordinate system \( z \) of \( M \), where \( J(f) \) is the Jacobian determinant of \( f \). Clearly, the zero divisor \( (J(f)) \) is globally defined. In fact, for \( x \in M \), given two local holomorphic coordinate systems \( z, \bar{z} \) near \( x \) and two local holomorphic coordinate systems \( w, \bar{w} \) near \( f(x) \), then
\[
J(f(z)) = \begin{vmatrix}
\frac{\partial(w_1, \ldots, w_m)}{\partial(z_1, \ldots, \bar{z}_m)} \\
\frac{\partial(w_1, \ldots, w_m)}{\partial(\bar{w}_1, \ldots, \bar{w}_m)} & \frac{\partial(\bar{w}_1, \ldots, \bar{w}_m)}{\partial(z_1, \ldots, \bar{z}_m)} & \frac{\partial(\bar{z}_1, \ldots, \bar{z}_m)}{\partial(z_1, \ldots, \bar{z}_m)} \\
\frac{\partial(w_1, \ldots, w_m)}{\partial(\bar{w}_1, \ldots, \bar{w}_m)} & \frac{\partial(\bar{w}_1, \ldots, \bar{w}_m)}{\partial(\bar{z}_1, \ldots, \bar{z}_m)} & \frac{\partial(\bar{z}_1, \ldots, \bar{z}_m)}{\partial(\bar{z}_1, \ldots, \bar{z}_m)}
\end{vmatrix}.
\]
We use Ram$_f$ to denote $(J(f))$, called the ramification divisor of $f$.

**Lemma 4.4.** Set $f^* \Phi_{D,\lambda} = \xi \alpha^m$, where $\Phi_{D,\lambda}$ is defined by (9). Then

$$\xi \frac{1}{m} \leq \frac{(q + 1)b^\frac{1}{m}}{2m\lambda^\frac{k}{m}} e_f^* c_1(L,h) + \frac{b^\frac{1}{m}}{4m\lambda^1 + \frac{k}{m}} \sum_{j=1}^{q} \Delta_M \log (1 + \|s_j \circ f\|^{2\lambda}).$$

**Proof.** By Lemma 4.1-Lemma 4.3, we directly compute that

$$\xi \frac{1}{m} \leq \frac{(q + 1)b^\frac{1}{m}}{2m\lambda^\frac{k}{m}} e_f^* c_1(L,h) + \frac{b^\frac{1}{m}}{4m\lambda^1 + \frac{k}{m}} \sum_{j=1}^{q} \Delta_M \log (1 + \|s_j \circ f\|^{2\lambda}).$$

where $b > 0$ is a suitable number independent of $\lambda$.

**Lemma 4.5.** There exists a number $b > 0$ independent of $\lambda$ such that

$$\mathbb{E}_o \left[ \int_0^r \xi \frac{1}{m}(X_t)dt \right] \leq \frac{(q + 1)b^\frac{1}{m}}{m\lambda^\frac{k}{m}} \left( (q + 1)T_f(r,L) + \frac{q \log 2}{2\lambda} \right)$$

holds for any constant $0 < \lambda < 1$. Moreover, $\lambda$ can be replaced by a function $\kappa$ satisfying $0 < \kappa(r) \leq c_0 < 1$. If take $\kappa(r) = 1/T_f(r,L)$, then there exists a number $B > 0$ such that

$$\mathbb{E}_o \left[ \int_0^r \xi \frac{1}{m}(X_t)dt \right] \leq BT^\frac{1}{m}(r,L)$$

for $r > 0$ large enough, where

$$B \geq (1 + q + \frac{q \log 2}{2}) \frac{b^\frac{1}{m}}{m}.$$

**Proof.** Using Lemma 4.5, we obtain

$$\mathbb{E}_o \left[ \int_0^r \xi \frac{1}{m}(X_t)dt \right] \leq \frac{(q + 1)b^\frac{1}{m}}{2m\lambda^\frac{k}{m}} \mathbb{E}_o \left[ \int_0^r e_f^* c_1(L,h)(X_t)dt \right]$$

$$+ \frac{b^\frac{1}{m}}{4m\lambda^1 + \frac{k}{m}} \sum_{j=1}^{q} \mathbb{E}_o \left[ \int_0^r \Delta_M \log (1 + \|s_j \circ f\|^{2\lambda})(X_t)dt \right].$$
where \( b \) is independent of \( \lambda \) with \( 0 < \lambda < 1 \). Observing that

\[
E_0 \left[ \int_0^r e_{f \cdot c_1(L,h)}(X_t) dt \right] = 2T_f(r, L)
\]

and Dynkin formula implies

\[
E_0 \left[ \int_0^r \Delta_M \log (1 + \|s_j \circ f(X_t)\|^{2\lambda}) dt \right] = 2E_0 \left[ \log (1 + \|s_j \circ f(X_t)\|^{2\lambda}) \right] - 2 \log (1 + \|s_j \circ f(o)\|^{2\lambda}) < 2 \log 2
\]

since the assumption that \( \|s_j\| < 1 \) for \( 1 \leq j \leq q \). Thus, we conclude that

\[
E_0 \left[ \int_0^r \xi^\frac{1}{m}(X_t) dt \right] \leq \frac{b^\frac{1}{m}}{m^\frac{1}{m}} \left( (q + 1)T_f(r, L) + \frac{q \log 2}{2\lambda} \right).
\]

The independence of \( b \) from \( \lambda \) implies that \( \lambda \) could be replaced by a function \( \kappa \) satisfying \( 0 < \kappa(r) \leq c_0 < 1 \). Since \( f \) is non-degenerate, then we have that \( T_f(r, L) > 1 \) when \( r \) is large enough. Replacing \( \lambda \) by \( 1/T_f(r, L) \), we conclude that

\[
E_0 \left[ \int_0^r \xi^\frac{1}{m}(X_t) dt \right] \leq \left( 1 + q + \frac{q}{2} \log 2 \right) \frac{b^\frac{1}{m}}{m} T_f^{1+\frac{1}{m}}(r, L)
\]

for \( r > 1 \) large enough. The proof is completed. \( \square \)

**Lemma 4.6.** Set \( \Gamma = \xi^\frac{1}{m} \), we have

\[
\log F(\hat{\Gamma}, \kappa, \delta) \leq O \left( \log^+ \log T_f(r, L) + \log^+ \left( r \sqrt{-\kappa(r)} \right) + \log^+ \log r \right).
\]

holds for \( r > 1 \) large enough, where \( F \) is defined in Lemma 3.4

**Proof.** Lemma 3.4 implies that

\[
(11) \quad \log F(\hat{\Gamma}, \kappa, \delta)
\]

\[
\leq O \left( \log^+ \log E_0 \left[ \int_0^r \xi^\frac{1}{m}(X_t) dt \right] + \log^+ \left( r \sqrt{-\kappa(r)} \right) + \log^+ \log r \right).
\]

Note by Lemma 4.5 that there exists a number \( B > 0 \) such that

\[
(12) \quad E_0 \left[ \int_0^r \xi^\frac{1}{m}(X_t) dt \right] \leq BT_f^{1+\frac{1}{m}}(r, L)
\]

for \( r > 1 \) large enough. Combining (11) with (12), we prove the lemma. \( \square \)

**Estimate of** \( T(r, \mathcal{R}_M) \)

Write \( \text{Ric}_M = \sum_{i,j} R_{ij} dz_i \otimes d\bar{z}_j \), where

\[
R_{ij} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log \det(g_{st}).
\]
Let \( s_M \) be the scalar curvature of \( M \) defined by

\[
s_M = \sum_{i,j=1}^{m} g_{i\bar{j}} R_{i\bar{j}},
\]

where \((g_{i\bar{j}})\) is the inverse of \((g_{i\bar{j}})\). By virtue of (13), we obtain

\[
s_M = -\frac{1}{4} \Delta_M \log \det(g_{s\bar{t}}).
\]

**Lemma 4.7.** We have

\[
s_M \geq m R_M.
\]

**Proof.** Fix any point \( x \in M \), we take a local holomorphic coordinate system \( z \) near \( x \) such that \( g_{i\bar{j}}(x) = \delta_{i\bar{j}} \). Then

\[
s_M(x) = \sum_{j=1}^{m} R_{j\bar{j}}(x) = \sum_{j=1}^{m} \text{Ric}_M(\frac{\partial}{\partial z_j}, \frac{\partial}{\partial \bar{z}_j})x \geq m R_M(x)
\]

which proves the lemma. \( \square \)

**Lemma 4.8.** We have

\[
\mathbb{E}_o[\tau_r] \leq \frac{2r^2}{2m - 1}.
\]

**Proof.** Let \( X_t \) be the Brownian motion in \( M \) started at \( o \neq o_1 \), where \( o_1 \in B_o(r) \). Let \( r_1(x) \) be the distance function of \( x \) from \( o_1 \). Apply Itô formula to \( r_1(x) \)

\[
(14) \quad r_1(X_t) - r_1(X_0) = B_t - L_t + \frac{1}{2} \int_0^t \Delta_M r_1(X_s) ds,
\]

here \( B_t \) is the standard Brownian motion in \( \mathbb{R} \), and \( L_t \) is a local time on cut locus of \( o \), an increasing process which increases only at cut loci of \( o \). Since \( M \) is simply connected and non-positively curved, then

\[
\Delta_M r_1(x) \geq \frac{2m - 1}{r_1(x)}, \quad L_t \equiv 0.
\]

By (14), we arrive at

\[
r_1(X_t) \geq B_t + \frac{2m - 1}{2} \int_0^t \frac{ds}{r_1(X_s)}.
\]

Let \( t = \tau_r \) and take expectation on both sides of the above inequality, then it yields that

\[
\max_{x \in S_o(r)} r_1(x) \geq \frac{(2m - 1) \mathbb{E}_o[\tau_r]}{2 \max_{x \in S_o(r)} r_1(x)}.
\]

Let \( o' \to o \), we are led to the conclusion. \( \square \)
Lemma 4.9. Let $\kappa$ be defined by (1). We have

$$T(r, \mathcal{R}_M) \geq 2m\kappa(r)r^2.$$ 

Proof. Lemma 4.7 implies that $0 \geq s_M \geq mR_M$. By co-area formula

$$T(r, \mathcal{R}_M) = -\frac{1}{4}E_o \left[ \int_0^{\tau_r} \Delta_M \log \det(g_{ij}(X_t)) dt \right]$$

$$= E_o \left[ \int_0^{\tau_r} s_M(X_t) dt \right] \geq mE_o \left[ \int_0^{\tau_r} R_M(X_t) dt \right]$$

$$\geq m(2m-1)\kappa(r)E_o[\tau_r].$$

Using Lemma 4.8 we show the lemma. \qed

4.2. Proof of Theorem 1.1

Consider the (analytic) universal covering

$$\pi: \tilde{M} \to M.$$ 

Via the pull-back of $\pi$, $\tilde{M}$ can be equipped with the induced metric from the metric of $M$. So, under this metric, $\tilde{M}$ becomes a simply-connected complete Kähler manifold of non-positive sectional curvature. Take a diffusion process $\tilde{X}_t$ in $\tilde{M}$ such that $X_t = \pi(\tilde{X}_t)$, where $X_t$ is the Brownian motion started at $o \in M$. Then $\tilde{X}_t$ is the Brownian motion generated by $\Delta_{\tilde{M}}/2$ induced from the pull-back metric. Let $\tilde{X}_t$ start at $\tilde{o} \in \tilde{M}$ with $o = \pi(\tilde{o})$, we have

$$E_o[\phi(X_t)] = E_{\tilde{o}}[\phi \circ \pi(\tilde{X}_t)]$$

for $\phi \in \mathcal{C}_b(M)$. Set

$$\tilde{\tau}_r = \inf \{ t > 0 : \tilde{X}_t \notin B_{\tilde{o}}(r) \},$$

where $B_{\tilde{o}}(r)$ is a geodesic ball centered at $\tilde{o}$ with radius $r$ in $\tilde{M}$. If necessary, one can extend the filtration in probability space where $(X_t, \mathbb{P}_o)$ are defined so that $\tilde{\tau}_r$ is a stopping time with respect to a filtration where the stochastic calculus of $X_t$ works. By the above arguments, we may assume $M$ is simply connected without loss of generality by lifting $f$ to the covering, see [4].

Proof. The equality

$$f^*\Phi_{D,\lambda} = \xi \alpha^m,$$

where $\Phi_{D,\lambda}$ is defined by (9), implies that

$$dd^c \log \xi = (1 - \lambda)f^*c_1(L, h) - f^*\text{Ric}\Omega + \mathcal{R}_M - (1 - \lambda)f^*D + \text{Ram}_f$$

$$\geq (1 - \lambda)f^*c_1(L, h) - f^*\text{Ric}\Omega + \mathcal{R}_M - \text{Supp} f^*D.$$
in the sense of currents. Thus,

\[ \frac{1}{4} \int_{B_o(r)} g_r(o, x) \Delta_M \log \xi(x) dV(x) \]

\[ \geq (1 - \lambda) T_f(r, L) + T_f(r, K_V) + T(r, \mathcal{R}_M) - N_f^{[1]}(r, D) + O(1), \]

where \( K_V \) is the canonical line bundle over \( V \). By Dynkin formula

\[
\frac{1}{2} \mathbb{E}_o \left[ \int_0^{r_o} \Delta_M \log \xi(X_t) dt \right] = \mathbb{E}_o \left[ \log \xi(X_{r_o}) \right] - \log \xi(o).
\]

By this with (15) to get

\[
\frac{1}{2} \mathbb{E}_o \left[ \log \xi(X_{r_o}) \right] \geq (1 - \lambda) T_f(r, L) + T_f(r, K_V) + T(r, \mathcal{R}_M) - N_f^{[1]}(r, D) + O(1).
\]

Take \( r_0 > 0 \) such that \( T_f(r, L) > 1 \) as \( r \geq r_0 \). Replacing \( \lambda \) by \( 1/T_f(r, L) \), we obtain

\[ \frac{1}{2} \mathbb{E}_o \left[ \log \xi(X_{r_o}) \right] \geq T_f(r, L) + T_f(r, K_V) + T(r, \mathcal{R}_M) - N_f^{[1]}(r, D) + O(1) \]

for \( r > 1 \) large enough. On the other hand, using Lemma 3.4, for any \( \delta > 0 \), there exists a set \( E'_\delta \subset (1, \infty) \) such that

\[
\frac{1}{2} \mathbb{E}_o \left[ \log \xi(X_{r_o}) \right] \leq m \log \mathbb{E}_o \left[ \xi^{1/m}(X_{r_o}) \right] 
\]

\[
\leq m \log \mathbb{E}_o \left[ \int_0^{r_o} \xi^{1/m}(X_t) dt \right] + m \log \frac{F(\hat{\Gamma}, \kappa, \delta)e^{(2m-1)r\sqrt{-\kappa(r)} - \log r}}{C\omega_{2m-1}} 
\]

\[
:= \frac{m}{2} (A_1 + A_2)
\]

holds for \( r > 1 \) outside \( E'_\delta \) with \( \Gamma = \xi^{1/m} \). For \( A_1 \), apply Lemma 4.3 to get

\[ A_1 \leq \frac{m + k}{m} \log T_f(r, L) + O(1) \]

as \( r > r_0 \). For \( A_2 \), by Lemma 4.6 we have

\[ A_2 \leq \log F(\hat{\Gamma}, \kappa, \delta) + (2m - 1)r\sqrt{-\kappa(r)} + \log^+ \log r + O(1) \]

\[ \leq O \left( \log^+ \log T_f(r, L) + \log^+ (r\sqrt{-\kappa(r)}) + \log^+ \log r \right) 
\]

\[ + (2m - 1)r\sqrt{-\kappa(r)} + \log^+ \log r + O(1) \]

\[ \leq O \left( \log^+ \log T_f(r, L) + r\sqrt{-\kappa(r)} + \log^+ \log r \right) \]
as $r > r_0$. Hence, it finally follows that
\[
\frac{1}{2} \mathbb{E}_\delta \left[ \log \xi(X_{\tau_r}) \right] 
\leq \frac{m + k}{2} \log T_f(r, L) + O \left( \log^+ \log T_f(r, L) + r \sqrt{-\kappa(r)} + \log^+ \log r \right)
\]
for $r > 1$ outside $E_\delta = E_\delta' \cup (0, r_0]$. By the above with (16) and Lemma 4.9, the theorem is proved. □

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Academy of Mathematics and Systems Science, Chinese Academy of Sciences, Beijing, 100190, P.R. China

Email address: xjdong@amss.ac.cn