Abstract. Recently, Cappelletti-Montano, De Nicola, and Yudin proved a Hard Lefschetz theorem for the De Rham cohomology of compact Sasakian manifolds, and proposed an associated notion of Lefschetz contact manifolds. Inspired by their work, we prove in this paper a Hard Lefschetz theorem for compact $K$-contact manifolds using the formalism of odd dimensional symplectic geometry.

This result leads directly to a new, conceptually different proof of the above mentioned Hard Lefschetz theorem for Sasakian manifolds, and allows us to establish a sufficient and necessary condition for a finitely presentable group to be the fundamental group of a compact Lefschetz contact five manifold. We then use it to produce first examples of compact Lefschetz contact manifolds which do not support any Sasakian structures. To demonstrate the full strength of the Hard Lefschetz theorem for a Sasakian manifold, we also give a simple construction of simply-connected $K$-contact manifolds without any Sasakian structures in any dimension $\geq 9$. This in particular answers an open question asked by Boyer and late Galicki concerning the existence of such examples.

1. Introduction

Conceptually, Sasakian geometry can be thought as an odd-dimensional counterpart of Kähler geometry. It is naturally related to two Kähler geometries. On the one hand, the metric cones of Sasakian manifolds must be Kähler. On the other hand, for any Sasakian manifold, the one dimensional foliation defined by the Characteristic Reeb vector field is transversally Kähler.

The Hard Lefschetz theorem is a remarkable result in Kähler geometry which has many important applications. It was first discovered by Lefschetz [L24]. However, its first complete proof was given by Hodge [H52] using Hodge theory much later. In Sasakian geometry, a basic Hodge theory was developed in [Ka90] on basic forms. In particular, it implies that the basic cohomology of any compact Sasakian manifold satisfies the Hard Lefschetz property. The basic Hodge theory is a very useful tool in Sasakian
geometry. For instance, it plays an important role in the recent remarkable work by Futaki, Ono and Wang [FOW06] on the existence of Sasakian-Einstein structures on any toric Sasakian manifold with positive anticanonical Sasakian structure. However, the basic Hodge theory also has certain limitations, as it only works on basic forms. It is therefore very interesting to note that a Hard Lefschetz theorem for the De Rham cohomology group of a compact Sasakian manifold was established by Cappelletti-Montano, De Nicola, and Yudin in [CNY13] very recently.

In Kähler geometry, taking cup product with the cohomology class of the kähler 2-form naturally gives rise to Lefschetz maps between cohomology groups. In contrast, in Sasakian geometry, it is not obvious at all how to define Lefschetz maps appropriately. In [CNY13], the authors first prove the existence of Lefschetz maps by examining the spectral properties of the Laplacian associated to a compatible Sasakian metric very carefully, and then verify that these maps are independent of the choices of compatible Sasakian metrics.

More precisely, let $(M, \eta, g)$ be a $2n + 1$ dimensional compact Sasakian manifold with a contact one form $\eta$. It is shown in [CNY13] that for any $0 \leq k \leq n$, the map

\[ \text{Lef}_k : \Omega^k(M) \rightarrow \Omega^{2n+1-k}(M), \quad \alpha \mapsto \eta \wedge (d\eta)^{n-k} \wedge \alpha \]

sends harmonic forms to harmonic forms, and therefore induces a map

\[ \text{Lef}_k : H^k(M, \mathbb{R}) \rightarrow H^{2n+1-k}(M, \mathbb{R}) \]

at the cohomology level. However, this method does not extend to more general contact manifolds.

In a different direction, Zhenqi He studied the geometry of odd dimensional symplectic manifolds in his Ph.D thesis [He10]. In particular, odd dimensional symplectic manifolds contain contact manifolds as special examples. On any odd dimensional symplectic manifold, there is also a canonical Reeb vector field. Moreover, when the odd dimensional symplectic structure is given by a contact structure, the Reeb vector field of the odd symplectic manifold agrees with the Reeb vector field of the contact manifold. Among other things, He developed the symplectic Hodge theory of the basic cohomology with respect to the Reeb vector field.

Inspired by the main result established in [CNY13], we study in this paper the Hard Lefschetz property for $K$-contact manifolds. Throughout this paper, a $K$-contact manifold $(M, \eta)$ is said to satisfy the **transverse Hard Lefschetz property** if its basic cohomology (with respect to the canonical Reeb vector field induced by $\eta$) satisfies the Hard Lefschetz property. Among other things, we show that a compact $K$-contact manifold satisfies the Hard Lefschetz property in the sense of [CNY13] if and only if it satisfies the transverse Hard Lefschetz property. Our approach involves two important pieces of technology: a long exact sequence [BG08 Sec.7.2] which relates the basic cohomology to the De Rham cohomology of the $K$-contact
manifold itself, and the odd dimensional symplectic Hodge theory developed in [He10]. In this approach, the symplectic Hodge theory replaces the role of the Riemannian Hodge theory used in [CNY13], and allows us to show that if the transverse Hard Lefschetz property holds for a compact K-contact manifold, then for any $0 \leq k \leq n$, the Lefschetz map (1.1) is well defined.

On the one hand, due to El Kacimi-Alaoui’s Hard Lefschetz theorem [Ka90], a compact Sasakian manifold always satisfies the transverse Hard Lefschetz property. Therefore, the above mentioned result leads directly to a new proof of the main result established in [CNY13] from the viewpoint of symplectic Hodge theory. On the other hand, it also provides a large class of Lefschetz contact manifolds which are not necessarily Sasakian.

A’Campo and Kotschick [AK94] proved that any finitely presentable group is the fundamental group of a five dimensional compact contact manifold. Applying the main result in this paper to a Boothby-Wang fibration, we are able to show that a finitely presentable group is the fundamental group of a compact Lefschetz contact five manifold if and only if it admits a non-degenerate skew structure in the sense of [JR87]. We then use this observation to show the existence of Lefschetz contact manifolds with non-Lefschetz finite covering spaces. In particular, this provides first examples of compact Lefschetz contact manifolds with no Sasakian structures.

In symplectic geometry, many examples of symplectic manifolds without any Kähler structures have been constructed. It has been very well understood that the category of symplectic manifolds is much larger than the category of Kähler manifolds. In contrast, much less is known about the differences between K-contact manifolds and Sasakian manifolds. Indeed, in the Open Problem 7.4 of their monograph [BG08], Boyer and late Galicki asked the question of whether there exist examples of simply connected K-contact manifolds which do not support any Sasakian structures. Only very recently did Hajduk and Tralle [HT13] find out an answer to this question by constructing first such examples. Their methods depend on the theory of fat bundles developed by Sternberg, Weinstein, and Lerman, and yield examples in any dimension $\geq 11$, c.f. [HT13 Thm. 5.4].

Let $M$ be a $2n + 1$ dimensional compact Sasakian manifold. It has been long known that for any $0 \leq p \leq n$, the Betti number $b_p$ of $M$ must be even if p is odd, c.f. [BG08 Thm. 7.4.11]. Indeed, this fact was used in [HT13] to detect the non-existence of Sasakian structures. As observed in [CNY13 Thm. 5.2], today this important property on the Betti numbers of a Sasakian manifold can be easily derived as a consequence of the Hard Lefschetz property for a contact manifold. However, we note that the Hard Lefschetz property for a contact manifold can provide more information. To demonstrate the full strength of the Hard Lefschetz theorem for compact Sasakian manifolds, we give a simple construction of simply-connected K-contact manifolds without any Sasakian structures in any dimension $\geq 9$. In a forthcoming paper [L13], we will refine the symplectic Hodge theoretic
methods developed in the current paper, and construct more such examples.

This paper is organized as follows. Section 2 reviews the machinery developed in [He10] on odd dimensional symplectic Hodge theory. Section 3 collects some facts from contact and Sasakian geometry we need in this paper. Section 4 proves that a compact K-contact manifold satisfies the Hard Lefschetz property in the sense of [CNY13] if and only if it satisfies the transverse Hard Lefschetz property. Section 5 produces simply-connected examples of K-contact manifolds without any Sasakian structures. Section 6 establishes a necessary and sufficient condition for a finitely presentable group to be the fundamental group of a compact Lefschetz contact five manifold, and use it to show the existence of compact Lefschetz manifolds which do not support any Sasakian structures.

2. REVIEW OF ODD DIMENSIONAL SYMPLECTIC HODGE THEORY

In this section we present a brief review of background materials in odd dimensional symplectic Hodge theory. We refer to [He10] for more details on odd dimensional version of symplectic Hodge theory, and to [Bry88] and [Yan96] for general background on symplectic Hodge theory. We begin with the definition of odd-dimensional symplectic manifolds.

**Definition 2.1.** ([He10]) Suppose that $M$ is a manifold of dimension $2n + 1$ with a volume form $\Omega$ and a closed 2-form $\omega$ of maximum rank. Then the triple $(M, \omega, \Omega)$ is called an odd-dimensional symplectic manifold.

**Example 2.2.** A contact manifold $M$ with a contact form $\eta$ naturally gives rise to an odd-dimensional symplectic manifold $(M, \omega, \Omega)$ with $\omega = d\eta$ and $\Omega = \eta \wedge \frac{(d\eta)^n}{n!}$.

Throughout the rest of this section, we assume that $(M, \omega, \Omega)$ is a $2n + 1$ dimensional symplectic manifold as given in Definition 2.1. We observe that since $\omega$ is of maximum rank, $\ker \omega$ is a one dimensional foliation on $M$; moreover, there is a canonical vector field $\xi$, called the Reeb vector field, given by

$$\iota_\xi \omega = 0, \ \iota_\xi \Omega = \frac{\omega^n}{n!}.$$ 

We define the space of horizontal and basic forms, as well as basic De Rham cohomology group on $M$ as follows.

$$\Omega_{\text{hor}}(M) = \{\alpha \in \Omega(M) | \iota_\xi \alpha = 0\},$$

$$\Omega_{\text{bas}}(M) = \{\alpha \in \Omega(M) | \iota_\xi \alpha = 0, \mathcal{L}_\xi \alpha = 0\},$$

$$H^k_B(\mathbb{R}, M) = \frac{\ker d \cap \Omega^k_{\text{bas}}(M)}{\im d(\Omega^{k-1}_{\text{bas}}(M))}.$$ 

(2.1)
There are three important operators, the Lefschetz map $L$, the dual Lefschetz map $\Lambda$, and the degree counting map $H$ which are defined on basic forms as follows.

\begin{align}
L & : \Omega^*_{\text{bas}}(M) \to \Omega^{*+2}_{\text{bas}}(M), \quad \alpha \mapsto \alpha \wedge \omega, \\
\Lambda & : \Omega^*_{\text{bas}}(M) \to \Omega^{*-2}_{\text{bas}}(M), \quad \alpha \mapsto \ast L \ast \alpha, \\
H & : \Omega^k_{\text{bas}}(M) \to \Omega^k_{\text{bas}}(M), \quad H(\alpha) = (n-k)\alpha, \quad \alpha \in \Omega^k_{\text{bas}}(M).
\end{align}

The actions of $L$, $\Lambda$ and $H$ on $\Omega_{\text{bas}}(M)$ satisfy the following commutator relations.

\begin{align}
[\Lambda, L] = H, \quad [H, \Lambda] = 2\Lambda, \quad [H, L] = -2L.
\end{align}

Therefore, these three operators define a representation of the Lie algebra $\mathfrak{sl}(2)$ on $\Omega(M)$. Although the $\mathfrak{sl}_2$-module $\Omega(M)$ is infinite dimensional, there are only finitely many eigenvalues of the operator $H$. The $\mathfrak{sl}_2$-modules of this type are studied in great details in [Ma95] and [Yan96]. Among other things, we have the following results.

**Lemma 2.3.** Let $(M, \omega, \Omega)$ be a $2n + 1$ dimensional symplectic manifold. For any $0 \leq k \leq n$, $\alpha \in \Omega^k_{\text{bas}}(M)$ is said to be primitive if $L^{n-k+1}\alpha = 0$. Then we have that

a) a basic $k$-form $\alpha$ is primitive if and only if $\Lambda \alpha = 0$;

b) any differential form $\alpha_k \in \Omega^k_{\text{bas}}(M)$ admits a unique Lefschetz decomposition

\begin{align}
\alpha_k = \sum_{r \geq \max(\frac{k-n}{2}, 0)} \frac{L^r}{r!} \beta_{k-2r},
\end{align}

where $\beta_{k-2r}$ is a primitive basic form of degree $k - 2r$.

**Remark 2.4.** Throughout the rest of this paper, we will denote the space of primitive basic $k$-forms on $M$ by $P^k_{\text{bas}}(M)$.

The restriction of 2-form $\omega$ to the space of horizontal $k$-forms induces a non-degenerate pairing $G(\cdot, \cdot)$ on $\Omega^k_{\text{hor}}(M)$. On the space of horizontal forms, the symplectic Hodge star $\ast$ is defined as follows.

\[ \ast \alpha_k \wedge \beta_k = G(\alpha_k, \beta_k) \frac{\omega^n}{n!}, \]

where $\alpha_k, \beta_k \in \Omega^k_{\text{hor}}(M)$.

It is easy to check that the symplectic Hodge star operator maps basic forms to basic forms. So there is a symplectic Hodge star operator on the space of basic forms.

\[ \ast : \Omega^k_{\text{bas}}(M) \to \Omega^{2n-k}_{\text{bas}}(M). \]
The symplectic Hodge operator gives rise to the following symplectic Hodge adjoint operator of the exterior differential \( d \).

\[
\delta \alpha_k = (-1)^{k+1} \ast d \ast \alpha_k, \quad \alpha_k \in \Omega^k_{\text{bas}}(M).
\]

In this context, a basic form \( \alpha \) is said to be symplectic Harmonic if and only if \( d\alpha = \delta \alpha = 0 \).

**Definition 2.5.** ([He10]) The \( 2n + 1 \)-dimensional symplectic manifold \( M \) is said to satisfy the transverse Hard Lefschetz property if and only if for any \( 0 \leq k \leq n \), the Lefschetz map

\[
L^{n-k} : H^k_B(M) \to H^{2n-k}_B(M) \quad [\alpha]_B \mapsto [\omega^{n-k} \wedge \alpha]_B
\]

is an isomorphism.

**Remark 2.6.** We say that a one form \( \eta \in \Omega(M) \) is a connection 1-form if \( \iota_\xi \eta = 1 \) and if \( L_\xi \eta = 0 \). It is shown in [He10] that if \( M \) is compact, and if there is a connection 1-form on \( M \), then \( \omega^k \) always represents a non-trivial cohomology class in \( H^k_B(M, \mathbb{R}) \). Clearly, if \( M \) is a contact manifold with a contact one form \( \eta \), then \( \eta \) will be a connection 1-form on \( M \). If \( M \) is also compact, then \( \omega^k \) always represents a non-trivial cohomology class in \( H^k_B(M, \mathbb{R}) \).

Among other things, [He10] extended Mathieu’s theorem, as well as the symplectic \( d\delta \)-lemma, to the odd dimensional case,

**Theorem 2.7.** ([Ma95], [He10]) On a compact odd dimensional symplectic manifold \( M \), every basic De Rham cohomology class in \( H^*_B(M) \) admits a symplectic Harmonic representative if and only if the manifold satisfies the transverse Hard Lefschetz property.

**Theorem 2.8.** ([Mer98], [Gui01], [He10]) Assume that \( M \) is a compact odd dimensional symplectic manifold which satisfies the transverse Hard Lefschetz property, and which admits a connection one form. Then on the space of basic forms, we have the following result.

\[
\text{im} d \cap \ker \delta = \ker d \cap \text{im} \delta = \text{im} d \delta.
\]

Next, we present the primitive decomposition of the basic cohomology. We first define the basic version of the primitive cohomology as follows.

**Definition 2.9.** Let \((M, \omega, \Omega)\) be a \( 2n + 1 \)-dimensional symplectic manifold. For any \( 0 \leq r \leq n \), the \( r \)-th primitive basic cohomology group, \( PH^r_B(M, \mathbb{R}) \), is defined as follows.

\[
PH^r_B(M, \mathbb{R}) = \ker (L^{n-r+1} : H^r_B(M, \mathbb{R}) \to H^{2n-r+2}_B(M, \mathbb{R})).
\]

When the odd-dimensional symplectic manifold \( M \) satisfies the transverse Hard Lefschetz property, the following primitive decomposition holds for basic De Rham cohomology.
Theorem 2.10. (c.f. [Yan96]) Assume that $M$ has the transverse Hard Lefschetz property. Then

\begin{equation}
H^k_B(M, \mathbb{R}) = \bigoplus_r L^r H^{k-2r}_B(M, \mathbb{R}).
\end{equation}

The following result does not assume that $M$ has the transverse Hard Lefschetz property. Its proof is completely analogous to the case of even dimensional symplectic Hodge theory. We refer to [Yan96] for details.

Lemma 2.11. Any primitive cohomology class in $PH^r_B(M, \mathbb{R})$ is represented by a closed primitive basic form.

Finally, we collect here a few commutator relations which we will use later in this paper.

Lemma 2.12.

\[ [d, \Lambda] = \delta, \quad [\delta, L] = d, \quad [d\delta, L] = 0, \quad [d\delta, \Lambda] = 0. \]

3. REVIEW OF CONTACT AND SASAKIAN GEOMETRY

Let $(M, \eta)$ be a co-oriented contact manifold with a contact one form $\eta$. We say that $(M, \eta)$ is K-contact if there is an endomorphism $\Phi : TM \to TM$ such that the following conditions are satisfied.

1) $\Phi^2 = -\text{Id} + \xi \otimes \eta$, where $\xi$ is the Reeb vector field of $\eta$;

2) the contact one form $\eta$ is compatible with $\Phi$ in the sense that

\[ d\eta(\Phi(X), \Phi(Y)) = d\eta(X, Y) \]

for all $X$ and $Y$, moreover, $d\eta(\Phi(X), X) > 0$ for all non-zero $X \in \ker \eta$;

3) the Reeb field of $\eta$ is a Killing field with respect to the Riemannian metric defined by the formula

\[ g(X, Y) = d\eta(\Phi(X), Y) + \eta(X)\eta(Y). \]

Given a K-contact structure $(M, \eta, \Phi, g)$, one can define a metric cone

\[ (C(M), g_C) = (M \times \mathbb{R}_+, r^2 g + dr^2), \]

where $r$ is the radial coordinate. The K-contact structure $(M, \eta, \Phi)$ is called Sasakian if this metric cone is a Kähler manifold with Kähler form $\frac{1}{2} d(r^2 \eta)$.

Let $(M, \eta)$ be a contact manifold with contact one form $\eta$ and a characteristic Reeb vector $\xi$. We note that the basic cohomology on $M$ given in (2.1) in the context of odd dimensional symplectic geometry agrees with the usual basic cohomology with respect to the characteristic foliation on $M$. We need the following result from [BG08, Sec. 7.2], which plays an important role in our work.
Proposition 3.1. 1) On any K-contact manifold $M$, there is a long exact cohomology sequence

$$
\cdots \to H^0_b(M, \mathbb{R}) \xrightarrow{i_*} H^k(M, \mathbb{R}) \xrightarrow{j_*} H^{k-1}_b(M, \mathbb{R}) \xrightarrow{\wedge [d\eta]} H^{k+1}_b(M, \mathbb{R}) \xrightarrow{i_*} \cdots ,
$$

where $i_*$ is the map induced by the inclusion, and $j_*$ is the map induced by $\iota_\xi$.

2) If $M$ is a compact K-contact manifold of dimension $2n + 1$, then for any $r \geq 0$ the basic cohomology $H^r_b(M, \mathbb{R})$ is finite dimensional, and for $r > 2n$, the basic cohomology $H^r_b(M, \mathbb{R}) = 0$; moreover, for any $0 \leq r \leq 2n$, there is a non-degenerate pairing

$$
H^r_b(M, \mathbb{R}) \otimes H^{2n-r}_b(M, \mathbb{R}) \to \mathbb{R}.
$$

On a compact Sasakian manifold $M$, the following Hard Lefschetz theorem was proved by El Kacimi-Alaoui [Ka90].

Theorem 3.2. ([Ka90]) Let $(M, \eta, g)$ be a $2n + 1$ dimensional compact Sasakian manifold with a contact one form $\eta$ and a Sasakian metric $g$. Then $M$ satisfies the transverse Hard Lefschetz property.

Very recently, Cappelletti-Montano, De Nicola, and Yudin [CNY13] established a Hard Lefschetz theorem for the De Rham cohomology group of a compact Sasakian manifold.

Theorem 3.3. ([CNY13]) Let $(M, \eta, g)$ be a $2n + 1$ dimensional compact Sasakian manifold with a contact one form $\eta$ and a Sasakian metric $g$, and let $\Pi : \Omega^r(M) \to \Omega^r_{\text{har}}(M)$ be the projection onto the space of Harmonic forms. Then for any $0 \leq k \leq n$, the map

$$
\text{Lef}_k : H^k(M, \mathbb{R}) \to H^{2n+1-k}(M, \mathbb{R}), \ [\beta] \mapsto [\eta \wedge (d\eta)^{n-k} \wedge \Pi \beta]
$$

is an isomorphism. Moreover, for any $[\beta] \in H^k(M, \mathbb{R})$, and for any closed basic primitive $k$-form $\beta' \in [\beta]$, $[\eta \wedge (d\eta)^{n-k} \wedge \beta'] = \text{Lef}_k([\beta])$. In particular, the Lefschetz map $\text{Lef}_k$ does not depend on the choice of a compatible Sasakian metric.

This result motivates them to propose the following definition of the Hard Lefschetz property for a contact manifold.

Definition 3.4. Let $(M, \eta)$ be a $2n + 1$ dimensional compact contact manifold with a contact 1-form $\eta$. For any $0 \leq k \leq n$, define the Lefschetz relation between the cohomology group $H^k(M, \mathbb{R})$ and $H^{2n+1-k}(M, \mathbb{R})$ to be

$$
\mathcal{R}_{\text{Lef}_k} = \{([\beta], [\eta \wedge L^{n-k} \beta]) | \iota_\xi \beta = 0, d\beta = 0, L^{n-k} \beta = 0\}.
$$

If it is the graph of an isomorphism $\text{Lef}_k : H^k(M, \mathbb{R}) \to H^{2n+1-k}(M, \mathbb{R})$ for any $0 \leq k \leq n$, then the contact manifold $(M, \eta)$ is said to have the hard Lefschetz property.
4. Hard Lefschetz theorem for K-contact manifolds

Throughout this section, we assume \( (M, \eta) \) to be a \( 2n + 1 \) dimensional compact K-contact manifold with a contact 1-form \( \eta \) and a Reeb vector field \( \xi \). Set \( \omega = d\eta \), and \( \Omega = \eta \wedge \frac{\omega^n}{n!} \). Then \( (M, \omega, \Omega) \) is an odd dimensional symplectic manifold in the sense of Definition 2.1. We will use extensively the machinery from odd dimensional symplectic Hodge theory as we explained in Section 2.

**Lemma 4.1.** Let \( (M, \eta) \) be a \( 2n + 1 \) dimensional compact K-contact manifold with a contact 1-form \( \eta \). Assume that \( M \) satisfies the transverse Hard Lefschetz property. Then for any \( 0 \leq k \leq n \), the map

\[
i_\ast : H^k_B(M, \mathbb{R}) \to H^k(M, \mathbb{R})
\]

is surjective; moreover, its image equals

\[
\{ i_\ast [\alpha]_B \mid \alpha \in \Omega^k_{bas}(M), d\alpha = 0, \omega^{n-k+1} \wedge \alpha = 0 \}.
\]

As a result, the restriction map \( i_\ast : PH^k_B(M, \mathbb{R}) \to H^k(M, \mathbb{R}) \) is an isomorphism.

**Proof.** Consider the long exact sequence (3.1). By assumption, \( M \) satisfies the transverse Hard Lefschetz property. Thus the map

\[
H^i_B(M, \mathbb{R}) \overset{\wedge[\omega]}{\longrightarrow} H^{i+2}_B(M, \mathbb{R})
\]

is injective for any \( 0 \leq i \leq n - 1 \). It then follows from the exactness of the sequence (3.1) that the map

\[
i_\ast : H^k_B(M, \mathbb{R}) \to H^k(M, \mathbb{R})
\]

is surjective for any \( 0 \leq k \leq n \). This proves the first assertion in Lemma 4.1.

Since \( M \) satisfies the transverse Hard Lefschetz property, by Theorem 2.10,

\[
H^k_B(M, \mathbb{R}) = PH^k_B(M, \mathbb{R}) \oplus LH^{k-2}_B(M, \mathbb{R}).
\]

It is clear from the exactness of the sequence (3.1) that

\[
i_\ast \left( H^k_B(M, \mathbb{R}) \right) = i_\ast \left( PH^k_B(M, \mathbb{R}) \right), \quad \ker i_\ast \cap PH^k(M, \mathbb{R}) = 0.
\]

Therefore the restriction map \( i_\ast : PH^k_B(M, \mathbb{R}) \to H^k(M, \mathbb{R}) \) is an isomorphism. Now applying Lemma 2.11 it follows immediately that \( i_\ast \left( H^k_B(M, \mathbb{R}) \right) \) equals (4.1). This completes the proof of Lemma 4.1.

q.e.d.

**Remark 4.2.** The result proved in Lemma 4.1 is known to hold for compact Sasakian manifolds, c.f. [BG08, Prop. 7.4.13]. The traditional proof uses Riemannian Hodge theory associated to a compatible Sasakian metric.
We are ready to define the Lefschetz map on the cohomology groups. In [CNY13], such maps are introduced using Riemannian Hodge theory associated to a compatible Sasakian metric. In contrast, we define these maps here using the symplectic Hodge theory on the space of basic forms.

For any $0 \leq k \leq n$, define $\text{Lef}_k : H^k(M, \mathbb{R}) \to H^{2n+1-k}(M, \mathbb{R})$ as follows. For any cohomology class $[\gamma] \in H^k(M, \mathbb{R})$, by Lemma 4.1 there exists a closed primitive basic $k$-form $\alpha \in \mathcal{P}_{bas}^k(M)$ such that $i_\ast [\alpha]_B = [\gamma]$. Observe that $d (\eta \wedge L^{n-k} \wedge \alpha) = L^{n-k+1} \alpha = 0$. We define

$$
(4.2) \quad \text{Lef}_k[\gamma] = [\eta \wedge L^{n-k} \alpha].
$$

**Lemma 4.3.** Assume that $M$ satisfies the transverse Hard Lefschetz property. Then the map (4.2) does not depend on the choice of closed primitive basic forms.

**Proof.** Suppose that there are two closed primitive basic $k$-forms $\alpha_1$ and $\alpha_2$ such that $i_\ast [\alpha_1]_B = i_\ast [\alpha_2]_B \in H^k(M, \mathbb{R})$. It follows from the exactness of the sequence (3.1) that $[\alpha_1]_B = [\alpha_2]_B + [L[\beta]]_B$ for some closed basic $(k-2)$-form $\beta$. Since $M$ satisfies the transverse Hard Lefschetz property, by Theorem 2.7 one may well assume that $\beta$ is symplectic Harmonic.

Therefore, $\alpha_1 - \alpha_2 - L[\beta]$ is both $d$-exact and $\delta$-closed. By Theorem 2.8 the symplectic $d\delta$-lemma, there exists a basic $k$-form $\varphi$ such that

$$
(4.3) \quad \alpha_1 - \alpha_2 - L[\beta] = d\delta \varphi
$$

Lefschetz decompose $\beta$ and $\varphi$ as follows.

$$
\beta = \beta_{k-2} + L\beta_{k-4} + L^2\beta_{k-6} + \cdots
$$

$$
\varphi = \varphi_k + L\varphi_{k-2} + L^2\varphi_{k-4} + \cdots
$$

Here $\varphi_{k-i} \in \mathcal{P}_{bas}^{k-i}(M)$, $i = 0, 2, \cdots$, and $\beta_{k-i} \in \mathcal{P}_{bas}^{k-i}(M)$, $i = 2, 4, \cdots$. Since $d\delta$ commutes with $L$, it follows from (4.3) that

$$
\alpha_1 - \alpha_2 = d\delta \varphi_k + L(\beta_{k-2} + d\delta \varphi_{k-2}) + L^2(\beta_{k-4} + d\delta \varphi_{k-4}) \cdots.
$$

Since $d\delta$ commutes with $\wedge$, $d\delta$ maps primitive forms to primitive forms. It then follows from the uniqueness of the Lefschetz decomposition that

$$
\alpha_1 - \alpha_2 = d\delta \varphi_k.
$$

Observe that

$$
\eta \wedge (\omega^{n-k} \wedge (\alpha_1 - \alpha_2)) = \eta \wedge (\omega^{n-k} \wedge d\delta \varphi_k)
$$

$$
= -d (\eta \wedge \omega^{n-k} \wedge \delta \varphi_k) + (L^{n-k+1} \delta \varphi_k).
$$

Now using the commutator relation $[L, \delta] = -d$ repeatedly, it is clear that $L^{n-k+1} \delta \varphi_k$ must be $d$-exact, since $\varphi_k$ is a primitive $k$-form and so $L^{n-k+1} \varphi_k = 0$. It follows immediately that $\eta \wedge L^{n-k} (\alpha_1 - \alpha_2)$ must be $d$-exact. This completes the proof of Lemma 4.3.

.q.e.d.
**Theorem 4.4.** Let $M$ be a $2n + 1$ dimensional compact $K$-contact manifold with a contact one form $\eta$. Then it satisfies the Hard Lefschetz property if and only if it satisfies the transverse Hard Lefschetz property.

**Proof.** Step 1. Assume that $M$ satisfies the transverse Hard Lefschetz property. We show that $M$ satisfies the Hard Lefschetz property. Since $M$ is oriented and compact, in view of the Poincaré duality, it suffices to show that for any $0 \leq k \leq n$, the map given in (4.2) is injective.

Suppose that $Lef_k[\gamma] = [\eta \wedge L^{n-k}\alpha] = 0$, where $\alpha \in \mathcal{P}^k_{bas}(M)$ such that $d\alpha = 0, i_*[\alpha] = [\gamma]$. Since the group homomorphism $j_{2n+1-k}: H^{2n+1-k}(M, \mathbb{R}) \to H^k_{b}(M, \mathbb{R})$ is induced by $i_\ell$, it follows that

$$0 = j_{2n+1-k}(0) = j_{2n+1-k}([\eta \wedge (L^{n-k}\alpha)]) = [L^{n-k}\alpha]_B.$$  

Since $M$ has the transverse Hard Lefschetz property, $[\alpha]_B = 0$. Thus $[\gamma] = i_*([\alpha]_B) = 0$.

**Step 2.** Assume that $M$ satisfies the Hard Lefschetz property. We show that for any $0 \leq k \leq n$, the map

$$(4.4) \quad L^{n-k} : H^k_b(M) \to H^{2n-k}_b(M), \quad [\alpha]_B \mapsto [\omega^{n-k} \wedge \beta]$$

is an isomorphism by induction on $k$. By Part 2) in Proposition 3.1, it suffices to show that for any $0 \leq k \leq n$, the map (4.4) is injective.

By assumption, for any $0 \leq k \leq n$,

$$\mathcal{R}_{Lef_k} = \{(\beta, [\eta \wedge L^{n-k}\beta]) | i_*\beta = 0, d\beta = 0, L^{n-k+1}\beta = 0\}$$

is the graph of an isomorphism $Lef_k : H^k(M) \to H^{2n-k+1}(M)$.

For $0 \leq k \leq n$, consider the restriction map $i_* : PH^k_b(M, \mathbb{R}) \to H^k(M, \mathbb{R})$. Since $\mathcal{R}_{Lef_k}$ is the graph of a map, one sees that $i_*$ must be surjective. Furthermore, when $k = 0, 1$, for simple dimensional reasons, $PH^k_b(M, \mathbb{R}) = H^k_b(M, \mathbb{R})$ and that the map $i_* : PH^k_b(M, \mathbb{R}) \to H^k(M, \mathbb{R})$ is an isomorphism.

Now consider the long exact sequence (3.1) at stage $2n+1$. Since $H^k_b(M, \mathbb{R}) = 0$ when $k \geq 2n + 1$, we have that

$$(4.5) \quad \cdots \to 0 \overset{i_*}{\to} H^{2n+1}(M, \mathbb{R}) \overset{j_{2n+1}}{\to} H^{2n}_b(M, \mathbb{R}) \overset{\wedge [\omega]}{\to} 0 \to \cdots .$$

It follows that the map $j_{2n+1} : H^{2n+1}(M, \mathbb{R}) \to H^{2n}_b(M, \mathbb{R})$ is an isomorphism.

Suppose that there is $[\alpha]_B \in PH^k_b(M, \mathbb{R})$ such that $L^{n-k}[\alpha]_B = 0 \in H^{2n-k}_b(M, \mathbb{R})$. By Lemma 2.11, we may assume that $\alpha$ is a closed primitive basic $k$-form. Then for any closed primitive basic $k$-form $\beta$,

$$j_{2n+1}([\eta \wedge L^{n-k}\alpha \wedge \beta]) = L^{n-k}[\alpha]_B \wedge [\beta]_B = 0.$$  

Since $j_{2n+1}$ is an isomorphism, it follows that $Lef_k([i_*[\alpha]_B] \cup i_*[\beta]_B = [\eta \wedge L^{n-k}\alpha \wedge \beta] = 0$. Since $\beta$ is arbitrarily chosen, by the Poincaré duality, we must have $Lef_k[i_*[\alpha]] = 0$. Since $Lef_k$ is an isomorphism, $i_*[\alpha] = 0$. By the exactness of the sequence (3.1), $[\alpha]_B = L[\lambda]_B$ for some $[\lambda]_B \in H^{k-2}_b(M, \mathbb{R})$. 


For dimensional considerations, when $k = 0, 1$, we must have $[\alpha]_B = 0$. This proves that the map (4.4) is an isomorphism when $k = 0, 1$.

Assume that the map (4.4) is an isomorphism for any non-negative integer less than $k$. We first observe that the inductive hypothesis implies $H^k_B(M, \mathbb{R}) = PH^k_B(M, \mathbb{R}) + \text{im } L$. Indeed, by the inductive hypothesis, $L^{n-k+2}: H^k_B(M) \to H^{2n-k+2}_B(M)$ is an isomorphism. Therefore, for any $[\varphi]_B \in H^k_B(M)$,

$$L^{n-k+1}[\varphi]_B = L^{n-k+2}[\sigma]_B$$

for some $[\sigma]_B \in H^{k-2}_B(M)$. As a result, $L^{n-k+1}([\varphi]_B - L[\sigma]_B) = 0$. This implies that $[\varphi]_B - L[\sigma]_B \in PH^k_B(M)$ and so $[\varphi]_B \in PH^k_B(M, \mathbb{R}) + \text{im } L$.

Now suppose that $L^{n-k}([\alpha]_B + L[\sigma]_B) = 0$, where $[\alpha]_B \in PH^k_B(M, \mathbb{R})$ and $[\sigma] \in H^{k-2}_B(M, \mathbb{R})$. Then we must have $L^{n-k+1}([\alpha]_B + L[\sigma]_B) = L^{n-k+2}[\sigma]_B = 0$. It follows from our inductive hypothesis again that $[\sigma]_B = 0$. As a result, $L^{n-k}[\alpha]_B = 0$. By our previous work, we must have that $[\alpha]_B = L[\beta]_B$ for some $[\beta]_B \in H^{k-2}_B(M, \mathbb{R})$. Thus $L^{n-k}[\beta]_B = 0$. By our inductive hypothesis again, we have that $[\beta]_B = 0$ and so $[\alpha]_B = L[\beta]_B = 0$. This completes the proof that the map (4.4) is an isomorphism for any $0 \leq k \leq n$.

q.e.d.

**Corollary 4.5.** Assume that $M$ is a compact Sasakian manifold. Then $M$ must satisfy the Hard Lefschetz property (as given in Definition 3.4).

**Proof.** By Theorem 3.2, $M$ satisfies the transverse Hard Lefschetz property. As an immediate consequence of Theorem 4.4, we conclude that $M$ must satisfy the Hard Lefschetz property as given in Definition 3.4. q.e.d.

5. **Simply-connected K-contact manifolds without Sasakian structures**

It is well known that Boothby-Wang construction provides important examples of K-contact manifolds. In this section, we apply Theorem 4.4 to a Boothby-Wang fibration, and construct examples of simply-connected K-contact manifolds which do not support any Sasakian structures in any dimension $\geq 9$.

We first briefly review Boothby-Wang construction here, and refer to [BP76] for more details. A co-oriented contact structure on a $2n + 1$ dimensional compact manifold $P$ is said to be regular if it is given as the kernel of a contact one form $\eta$, whose Reeb field $\xi$ generates a free effective $S^1$ action on $P$. Under this assumption, $P$ is the total space of a principal circle bundle $\pi: P \to M := P/S^1$, and the base manifold $M$ is equipped with an integral symplectic form $\omega$ such that $\pi^* \omega = \text{d}\eta$. Conversely, let $(M, \omega)$ be a compact symplectic manifold with an integral symplectic form $\omega$, and let $\pi: P \to M$ be the principal circle bundle over $M$ with Euler class $[\omega]$ and a connection one form $\eta$. Then $\eta$ is a contact one form on $P$ whose characteristic Reeb vector field generates the right translations of the structure group
S\(^1\) of this bundle. It is easy to see that as a direct consequence of Theorem 4.4 we have the following result.

**Theorem 5.1.** Let \(\pi : P \to M\) be a Boothby-Wang fibration as we described above. Then \((P, \eta)\) satisfies the Lefschetz property if and only if the base symplectic manifold \((M, \omega)\) satisfies the Hard Lefschetz property.

We need the following result in [CNY13].

**Theorem 5.2.** (CNY13, Thm. 5.2) Let \(M\) be a compact Lefschetz contact manifold of dimension \(2n + 1\). Then for any \(0 \leq k \leq n\), there is a non-degenerate \(< \cdot, \cdot > : H^k(M, \mathbb{R}) \times H^k(M, \mathbb{R}) \to \mathbb{R}\) given by

\[
< x, y > = \int_M \text{Lef}_k(x) \cup y.
\]

Moreover, this pairing is symmetric when \(k\) is even, and skew-symmetric when \(k\) is odd.

Note that under the assumptions of Theorem 5.2, if \(0 \leq 2k \leq n\), we also have a Lefschetz map \(\text{Lef}_{2k} : H^{2k}(M, \mathbb{R}) \to H^{2n-2k+1}(M, \mathbb{R})\). This gives rise to a linear functional

\[
f : H^{2k}(M, \mathbb{R}) \to \mathbb{R}, \quad c \mapsto \int_M \text{Lef}_{2k}(c) \wedge \omega^k,
\]

where as before \(\omega = d\eta\). A textual reading of [CNY13, Thm. 5.2] gives the following refinement of Theorem 5.2

**Lemma 5.3.** Let \((M, \eta)\) be a compact \(2n + 1\) dimensional K-contact manifold with a contact one form \(\eta\). Assume that \((M, \eta)\) satisfies the transverse Hard Lefschetz property. Then for any \(k\) with \(0 \leq 2k \leq n\), the non-degenerate pairing given in (5.1) factors through the cup product, i.e., there exists a linear functional \(f : H^{2k}(M, \mathbb{R}) \to \mathbb{R}\) such that

\[
< x, y > = f(x \cup y), \quad \forall x, y \in H^k(M, \mathbb{R}).
\]

We need another result due to Gompf [Go95, Thm. 7.1].

**Theorem 5.4.** For any \(n \geq 3\), there exists a compact simply-connected symplectic manifold \(N\) of dimension \(2n\) which admits a cohomology class \([\alpha] \in H^2(N, \mathbb{R})\) such that \([\alpha] \cup [\beta] = 0\) for any \([\beta] \in H^2(N, \mathbb{R})\).

Next, we recall an useful result on the fundamental group of the total space of a Boothby-Wang fibration proved in [Ha13]. Let \(X\) be a compact and oriented manifold of dimension \(m\). We say that \(c \in H^2(X, \mathbb{Z})\) is indivisible if the map

\[
c \cup : H^{m-2}(X, \mathbb{Z}) \to H^m(X, \mathbb{Z})
\]

is surjective.

**Lemma 5.5.** (Ha13, Lemma 15) Let \(\pi : P \to M\) be a Boothby-Wang fibration, and let \(\omega\) be an integral symplectic form on \(M\) which represents the Euler class of the Boothby-Wang fibration. Then \(\pi_1(P) = \pi_1(M)\) if the Euler class \([\omega]\) is indivisible.
The following result is an immediate consequence of Lemma 5.5.

**Lemma 5.6.** (c.f. [Ha13 Sec. 5]) Suppose that \((M, \omega)\) is a compact symplectic manifold. Then there exists a Boothby-Wang fibration \(\pi: P \rightarrow M\) such that \(\pi_1(P) = \pi_1(M)\). Moreover, if \((M, \omega)\) satisfies the Hard Lefschetz property, then the total space \(P\) satisfies the Hard Lefschetz property.

**Proof.** In view of Theorem 5.1 and Lemma 5.5 it suffices to show that we can deform \(\omega\) to an integral symplectic form which represents an indivisible second cohomology class. For completeness, we review the argument here, which is basically what is given in [Go95 Observation 4.3]. Fix a Riemannian metric on \(M\). Then we can find a small \(\epsilon\)-ball \(B_\epsilon\) around the origin in the space of harmonic 2-forms on \(M\) such that every element in \(\omega + B_\epsilon\) is symplectic. Clearly, the set of classes in \(H^2(M, \mathbb{R})\) represented by these elements is open. Therefore there exists a symplectic form which represents a rational cohomology class. Multiplying it with an appropriate integer number \(N\) would provide us a symplectic form \(\omega_0\) which represents an integral class. Using the Poincaré duality over the integer coefficients, we can choose an appropriate integer number \(N\) such that the cohomology class \([\omega_0]\) is indivisible.

We are ready to prove the following theorem.

**Theorem 5.7.** For any \(n \geq 4\), there exists a compact \(K\)-contact manifold of dimension \(2n + 1\) which does not support any Sasakian structures.

**Proof.** Since \(n \geq 4\), by Theorem 5.4 there exists a compact \(2n\) dimensional simply-connected symplectic manifold \((N, \omega)\), which admits a cohomology class \([\alpha] \in H^2(N, \mathbb{R})\) such that \([\alpha] \cup [\beta] = 0\) for any \([\beta] \in H^2(N, \mathbb{R})\). By Lemma 5.6 we may assume that \([\omega]\) represents an indivisible class in \(H^2(N, \mathbb{Z})\). Therefore, there is a Boothby-Wang fibration \(\pi: M \rightarrow N\) such that the total space \(M\) is a compact simply-connected \(K\)-contact manifold of dimension \(2n + 1\).

Now consider the following portion of the Gysin sequence for the principal circle bundle \(\pi: M \rightarrow N\).

\[
\cdots \xrightarrow{\wedge [\omega]} H^2(N, \mathbb{R}) \xrightarrow{\pi^*} H^2(M, \mathbb{R}) \xrightarrow{\pi_*} H^1(N, \mathbb{R}) \xrightarrow{\wedge [\omega]} H^3(N, \mathbb{R}) \xrightarrow{\pi^*} \cdots,
\]

where \(\pi_*: H^*(M, \mathbb{R}) \rightarrow H^{*-1}(N, \mathbb{R})\) is the map induced by integration along the fibre.

Since \(N\) is simply-connected, it follows from the exactness of the sequence (5.3) that the map \(\pi^*: H^2(N, \mathbb{R}) \rightarrow H^2(M, \mathbb{R})\) must be surjective. Hence \(\pi^*[\alpha] \cup y = 0\) for any \(y \in H^2(M, \mathbb{R})\). Using the exactness of the sequence (5.3) again, we see that the kernel of \(\pi^*\) equals \(L(H^0(N, \mathbb{R}))\), where \(L: H^0(N, \mathbb{R}) \rightarrow H^2(N, \mathbb{R})\) is given by \(L(x) = [\omega] \cup x\). Therefore \([\alpha]\) does not lie in the kernel of the map \(\pi^*\). In particular, \(\pi^*[\alpha]\) represent a non-trivial cohomology class in \(H^2(M, \mathbb{R})\).
Assume that $M$ admits a Sasakian structure. Then by Theorem 5.2 and Lemma 5.3 there exits a non-degenerate pairing

$$<\cdot, \cdot>: H^2(M, \mathbb{R}) \times H^2(M, \mathbb{R}) \to \mathbb{R}$$

which factors through the cup product. However, it is clear that $\pi^*[\alpha] \neq 0$ must lie in the kernel of the pairing $<\cdot, \cdot>$. This contradiction shows that $M$ does not support any Sasakian structures.

q.e.d.

Remark 5.8. Let $M$ be a symplectic manifold with an integral symplectic form $\omega$, let $\pi: P \to M$ be the Boothby-Wang fibration with Euler class $[\omega]$, and let $\eta$ be the contact one form on $P$ such that $\pi^*\omega = d\eta$. By Theorem 5.1 if $M$ does not satisfy the Hard Lefschetz property, then there is no Sasakian metrics on $P$ which are compatible with $\eta$. In view of Theorem 5.4 this observation yields an example of a compact simply-connected seven-dimensional K-contact manifold $(P, \eta)$, which does not admit any Sasakian metric compatible with $\eta$. However, the author is not able to show that $P$ does not support any Sasakian metric at all.

6. FUNDAMENTAL GROUPS OF LEFSCHETZ CONTACT MANIFOLDS

In this section, we prove that a finitely presentable group is the fundamental group of a five dimensional compact Lefschetz contact manifold if and only if it admits a non-degenerate skew structure. We first review the notion of a non-degenerate skew structure on a discrete group due to Johnson and Rees.

Definition 6.1. ([JR87]) Let $G$ be a discrete group. A non-degenerate skew structure on $G$ is a non-degenerate skew bilinear form

$$<\cdot, \cdot>: H^1(G, \mathbb{R}) \times H^1(G, \mathbb{R}) \to \mathbb{R}$$

which factors through the cup product, that is, there exists a linear functional $\sigma: H^2(G, \mathbb{R}) \to \mathbb{R}$ so that $<a, b> = \sigma(a \cup b)$, for all $a, b \in H^1(G, \mathbb{R})$.

We are in a position to prove the following result on the fundamental group of a compact connected Lefschetz manifold.

Theorem 6.2. Suppose that $G$ is the fundamental group of a compact connected Lefschetz contact manifold of dimension $\geq 5$. Then $G$ must admit a non-degenerate skew structure.

Proof. The argument given by Johnson and Rees in [JR87, Thm. 1] can be easily adapted to our situation. For the convenience of the reader, we recall this argument here.
Let $P$ be a compact connected Lefschetz contact manifold with $\pi_1(P) = G$, and let $f : P \to K(G, 1)$ be a map inducing an isomorphism of fundamental groups. Identify $H^\ast(G, \mathbb{R})$ with $H^\ast(K(G, 1), \mathbb{R})$. Then by considerations in elementary homotopy theory, $f$ induces a map of cohomology rings $f^\ast : H^\ast(G, \mathbb{R}) \to H^\ast(P, \mathbb{R})$ which is an isomorphism in dimension 1, and injective in dimension 2. By Theorem 5.2 and Lemma 5.3, there is a non-degenerate skew symmetric pairing on $H^1(P, \mathbb{R})$ which factors through the cup product. Since $f^\ast : H^k(G, \mathbb{R}) \to H^k(P, \mathbb{R})$ is an isomorphism when $k = 1$, the non-degenerate skew symmetric pairing on $H^1(P, \mathbb{R})$ induces a non-degenerate skew symmetric pairing on $H^1(G, \mathbb{R})$. Since $f^\ast : H^\ast(G, \mathbb{R}) \to H^\ast(M, \mathbb{R})$ is a ring homomorphism, it is easy to see that the induced pairing also factors through the cup product. q.e.d.

**Remark 6.3.** Let $G$ be a finitely presentable group. If $b_2(G) = 0$ and $b_1(G) > 0$, then it is clear from Definition 6.1 that $G$ can not support any non-degenerate skew structure. It is observed in [Go95, Observation 7.4] that a group $G$ must satisfy the condition $b_2(G) = 0$ if it has a presentation of $k$ generators and $l$ relations for which $k - l = b_1(G)$. In particular, any non-trivial finitely generated free group can not be the fundamental group of a compact Lefschetz contact manifold.

**Corollary 6.4.** Let $G$ be the fundamental group of a compact connected Sasakian manifold of dimension $\geq 5$. Then every subgroup of finite index of $G$ must admit a non-degenerate skew structure.

**Proof.** This follows from Theorem 6.2 and the simple observation that the finite covering space of a compact connected Sasakian manifold is also a compact connected manifold. q.e.d.

Next we start to prove that the converse to Theorem 6.2 is also true.

We recall the following result on the fundamental group of a compact Lefschetz symplectic four manifold.

**Theorem 6.5.** ([L04]) Suppose that $G$ is a finitely presentable group. Then the following statements are equivalent.

1) $G$ admits a non-degenerate skew structure.

2) $G$ can be realized as the fundamental group of a compact symplectic four manifold with the Hard Lefschetz property.

We are ready to prove the following converse to Theorem 6.2.

**Theorem 6.6.** If $G$ is a finitely presented group that admits a non-degenerate skew structure. Then $G$ can be realized as the fundamental group of a five dimensional compact connected Lefschetz contact manifold.

**Proof.** By Theorem 6.5, there is a compact connected symplectic four manifold $N$ with the Hard Lefschetz property whose fundamental group is $G$. By Lemma 5.6, there is a Boothby-Wang fibration $\pi : P \to M$ such
that the total space $P$ satisfies the Hard Lefschetz property, and such that $\pi_1(P) = \pi_1(M) = G$.

$q.e.d.$

We need the following result of Johnson and Rees [JR87].

**Theorem 6.7.** Let $G_1, G_2$ be groups which both have at least one nontrivial finite quotient, and let $H$ be any group. Assume that $G = (G_1 \ast G_2) \times H$ admits a non-degenerate skew structure. Then $G$ has a subgroup of finite index which does not support any non-degenerate skew structure.

We are ready to produce examples of compact connected Lefschetz contact manifolds which has non-Lefschetz finite covering spaces. In view of Corollary 6.4, these manifolds do not support any Sasakian structures.

**Example 6.8.** For any given positive integer $k$, and any given positive composite numbers $m$ and $n$, set

$$G_{m,n,k} = (Z_m \ast Z_n) \times Z \times \ldots \times Z_{\underbrace{2k}_{2k}}.$$  

We first observe that $G_{m,n,k}$ admits a non-degenerate skew structure itself. This is because $H^i(G_{m,n,k}, \mathbb{R}) = H^i(Z \times \ldots \times Z_{\underbrace{2k}_{2k}}, \mathbb{R})$ for any $i \geq 1$, and because $Z \times \ldots \times Z_{\underbrace{2k}_{2k}}$ is the fundamental group of a Kähler manifold $T^{2k}$, and so must admit a non-degenerate skew structure (c.f. [JR87, Thm. 1]). Since both $Z_m$ and $Z_n$ have non-trivial finite quotients, by Theorem 6.7, $G_{m,n,k}$ has a subgroup $L$ of finite index which does not admit any non-degenerate skew structure.

Now by Theorem 6.2 there exists a compact connected Lefschetz contact manifold $P$ such that $\pi_1(P) = G_{m,n,k}$. Let $\tilde{P}$ be the finite covering space of $P$ with fundamental group $L$. Then by Theorem 6.2, $\tilde{P}$ does not support any contact structure which satisfies the Hard Lefschetz property.

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