Kondo Destruction in RKKY-Coupled Kondo Lattice and Multi-Impurity Systems

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In a Kondo lattice, the spin exchange coupling between a local spin and the conduction electrons acquires nonlocal contributions due to conduction electron scattering from surrounding local spins and the subsequent RKKY interaction. It leads to a hitherto unrecognized interference of Kondo screening and the RKKY interaction beyond the Doniach scenario. We develop a renormalization group theory for the RKKY-modified Kondo vertex. The Kondo temperature, \(T_K(y)\), is suppressed in a universal way, controlled by the dimensionless RKKY coupling parameter \(y\). Complete spin screening ceases to exist beyond a critical RKKY strength \(y_c\), even in the absence of magnetic ordering. At this breakdown point, \(T_K(y)\) remains nonzero and is not defined for larger RKKY couplings, \(y > y_c\). The results are in quantitative agreement with STM spectroscopy experiments on tunable two-impurity Kondo systems. The possible implications for quantum critical scenarios in heavy-fermion systems are discussed.

The concept of fermionic quasiparticles existing even in strongly interacting many-body systems is fundamental for a wealth of phenomena summarized under the term Fermi liquid physics. In heavy-fermion systems \cite{1}, quasiparticles with a large effective mass are formed by the Kondo effect \cite{2}. The conditions under which these heavy quasiparticles disintegrate near a quantum phase transition (QPT) have been an important, intensively debated and still open issue for many years \cite{1}.

The heavy Fermi liquid, like any other Fermi liquid, may undergo a spin density-wave (SDW) instability, leading to critical fluctuations of the magnetic order parameter but leaving the heavy quasiparticles intact. This scenario is well described by the pioneering works of Hertz, Moriya and Millis \cite{3,4}. However, early on Doniach pointed out \cite{5} that the Kondo spin screening of the local moments should eventually cease and give way to magnetic order, when the RKKY coupling energy between the local moments \cite{1,2} becomes larger than the characteristic energy scale for Kondo singlet formation, the Kondo temperature \(T_K\). It is generally believed that the Kondo destruction is driven by the critical fluctuations near a QPT. Several mechanisms have been proposed, invoking different types of fluctuations, including critical fluctuations of the local magnetization coupling to the fermionic quasiparticles (local quantum criticality) \cite{6,7} and Fermi surface fluctuations self-consistently generated by the Kondo destruction \cite{8}. Most recently, a scenario of critical quasiparticles with diverging effective mass and a singular interaction, induced by critical antiferromagnetic fluctuations, has been put forward \cite{9,10}. Intriguing in its generality, it does, however, not invoke Kondo physics.

Here, we show that the heavy-electron quasiparticles can be destroyed by the RKKY interaction even without critical fluctuations. This occurs because of a hitherto unrecognized feedback effect: in a Kondo lattice or multi-impurity system, the RKKY interaction, parametrized by a dimensionless coupling \(y\), reduces the Kondo screening energy scale \(T_K(y)\). This reduction implies an increase of the local spin susceptibility at low temperatures \(T\), \(\chi(T = 0) \sim 1/T_K(y)\), which in turn increases the effective RKKY coupling. We derive this effect and analyze it by a renormalization group (RG) treatment. In particular, we calculate the temperature scale for Kondo singlet formation in a Kondo lattice, \(T_K(y)\). It is suppressed with increasing \(y\) in a universal way. Beyond a critical RKKY coupling \(y_c\), complete Kondo singlet formation ceases to exist. However, at this breakdown point \(T_K(y_c)\) remains finite, and the suppression with respect to the single-impurity Kondo scale takes a universal value, \(T_K(y_c)/T_K(0) = 1/e\), where \(e = 2.718...\) is Euler’s constant. These findings are consistent with conformal field theory results \cite{16,17} and in quantitative agreement with STM spectroscopy experiments on tunable RKKY-coupled two-impurity Kondo systems \cite{18,19}.

The present results directly apply to cases where long-range order does not play a role, that is, two-impurity Kondo systems \cite{18,20}, compounds where the magnetic ordering does not occur at the Kondo breakdown point \cite{21}, and temperatures sufficiently above the Néel temperature \cite{22}. They will set the stage for a complete theory of heavy-fermion quantum criticality by including critical order-parameter fluctuations either of the incompletely screened magnetic moments or of an impending SDW instability.

The model. — We consider the Kondo lattice model

\[
H = \sum_{k,\sigma} \epsilon_k c_k^{\dagger} c_{k\sigma} + J_0 \sum_{i} \mathbf{S}_i \cdot \mathbf{s}_i \tag{1}
\]

where \(c_{k\sigma}, c_k^{\dagger}\) denote the conduction (c) electron operators with dispersion \(\epsilon_k\). \(\mathbf{S}_i\) are the local spin operators at the lattice sites \(\mathbf{x}_i\), exchange coupled to the conduction electron spins \(\mathbf{s}_i = \sum_{\sigma,\sigma'} c_{i\sigma}^{\dagger} \sigma_{\sigma'} c_{i\sigma'}\) via an on-site,
antiferromagnetically coupling 

antiferromagnetic coupling

The formation of the strong-coupling Kondo singlet, which is the origin of the heavy-Fermion state, is signalled by a RG divergence of the spin-scattering vertex operator \( \Gamma_{\text{RKKY}} \) between \( c \) electrons and an \( f \) spin. In the case of multiple Kondo sites, this vertex acquires nonlocal contributions in addition to the local coupling \( J \) at a site \( i \), because a \( c \) electron can scatter from a distant Kondo site \( j \neq i \), and the spin flip at that site is transferred to the \( f \) spin at site \( i \) via the RKKY interaction. In this way, \( \Gamma_{ff} \) will influence the RG flow of \( \Gamma_{cf} \), even though it is not renormalized itself. The corresponding diagrams are shown in Fig. 1 (a). As seen from the figure, such a nonlocal scattering process necessarily involves the exact, local dynamical \( f \)-spin susceptibility \( \chi_f(i\Omega) \) on site \( j \). The resulting \( c-f \) vertex \( \Gamma_{cf} \) has the structure of a nonlocal Heisenberg coupling in spin space. The exchange diagram, \( \gamma_{\text{RKKY}}^{(d)} \) in Fig. 1 (a), contributes only a subleading logarithmic term as compared to \( \gamma_{\text{RKKY}}^{(c)} \).

In particular, it does not alter the universal \( T_K(y) \) suppression derived below and can, therefore, be neglected. To leading (linear) order in the RKKY coupling, \( \Gamma_{cf} \) thus reads (in Matsubara representation),

\[
\Gamma_{cf} = \left[ J\delta_{ij} + \gamma_{\text{RKKY}}^{(d)}(\mathbf{r}_{ij}, i\Omega) \right] \mathbf{S}_i \cdot \mathbf{s}_j
\]

where \( \mathbf{r}_{ij} = \mathbf{x}_i - \mathbf{x}_j \), is the distance vector between the sites \( i \) and \( j \), and \( \Omega \) is the energy transferred in the scattering process. \( \chi_{\text{RKKY}}(\mathbf{r}_{ij}, i\Omega) \) is the \( c \) electron density correlation function between sites \( i \) and \( j \) [bubble of solid lines in Fig. 1 (a)] and \( \tilde{\chi}_f(i\Omega) \) is the \( f \) spin susceptibility in the expression derived below and can, therefore, be neglected. To leading (linear) order in the RKKY coupling, \( \Gamma_{cf} \) thus reads (in Matsubara representation),

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\]
ansatz solution \[27\] in terms of the Kondo scale \(T_K\). It has a \(T = 0\) value \(\chi_f(0) \propto 1/T_K\) and crosses over to the \(1/T\) behavior of a free spin for \(T > T_K\). These features can be modeled in the retarded and advanced, local, dynamical f-spin susceptibility \(\chi_f(\Omega \pm i0)\) as

\[
\chi_f(\Omega \pm i0) = \frac{(g_f\mu_B)^2W}{\pi T_K \sqrt{1 + (\Omega/T_K)^2}} \left( 1 \pm \frac{2i}{\pi} \text{arsinh} \left( \frac{\Omega}{T_K} \right) \right)
\]

where \(W\) is the Wilson ratio, and the imaginary part is implied by the Kramers-Kronig relation.

We now derive the one-loop RG equation for the \(c - f\) vertex \(\Gamma_{cf}\), including RKKY-induced, nonlocal contributions. The one-loop spin vertex function is shown diagrammatically in Fig. 1 (b). Using Eq. (2), the sum of these two diagrams is up to linear order in the RKKY coupling

\[
Y(r_{ij}, i\omega) = -J_T T \sum_{\Omega} \left[ J \delta_{ij} + \gamma^{(d)}_{\text{RKKY}}(r_{ij}, i\Omega) + \gamma^{(d)}_{\text{RKKY}}(r_{ij}, -i\Omega) \right]
\times \left[ G_c(r_{ij}, i\omega - i\Omega) - G_c(r_{ij}, i\omega + i\Omega) \right] G_f(i\Omega).
\]

Here, \(\omega\) is the energy of the incoming conduction electrons, \(G_c(r_{ij}, i\omega + i\Omega)\) is the single-particle c-electron propagator from the incoming to the outgoing site. For example, for an isotropic system, \(G_c(r, \omega + i\Omega) = \pi N(\omega) e^{\pm ik_F(r + \omega)r}/(\epsilon_F + \omega)r\), with the bare density of states \(N(\omega)\), and \(k_F(\epsilon_F + \omega)\) the modulus of the momentum corresponding to the energy \(\omega\).

For the low-energy physics, the vertex renormalization for \(c\) electrons at the Fermi surface is required. This means setting the energy \(i\omega \to \omega = 0 + i0\) and Fourier transforming the total vertex \(Y(r_{ij}, i\omega)\) with respect to the incoming and outgoing \(c\) electron coordinates, \(x_j\), \(x_i\), and taking its Fourier component for momenta at the Fermi surface \(k_F\), see Ref. [24]. Note that at the Fermi energy \(Y(k_F, 0)\) is real, even though the RKKY-induced, dynamical vertex \(\gamma^{(d)}_{\text{RKKY}}(\pm i\Omega)\) appearing in Eq. (4) is complex valued \[24\]. This ensures the total vertex operator of the renormalized Hamiltonian is Hermitian. By analytic continuation, the Matsubara summation in Eq. (4) becomes an integration over the intermediate \(c\) electron energy from the lower and upper band cutoff \(D\) to the Fermi energy \((\Omega = 0)\). The coupling constant renormalization is then obtained in the standard way by requiring that \(Y(k_F, 0)\) be invariant under an infinitesimal reduction of the running band cutoff \(D\). Note that the band cutoff appears in both, the intermediate electron propagator \(G_c\) and in \(\chi_c\). However, differentiation of the latter does not contribute to the logarithmic RG flow. This leads to the one-loop RG equation \[24\]

\[
\frac{dg}{d\ln D} = -2g^2 \left( 1 - y g_0 \right) \frac{D_0}{T_K} \frac{1}{\sqrt{1 + (D/T_K)^2}} \right),
\]

where we have introduced the dimensionless couplings \(g = N(0)J\), \(g_0 = N(0)J_0\), and \(D_0\) is the bare band cutoff. The first term on the right-hand side of Eq. (5) is the on-site contribution to the differential coupling renormalization (the \(\beta\) function), while the second term represents the RKKY contribution. It is seen that \(\chi_f\), as in Eq. (3), induces a soft cutoff on the scale \(T_K\) and the characteristic \(1/T_K^2\) dependence to the RG flow of this contribution, where \(T_K\) is the Kondo scale on the surrounding Kondo sites. The dimensionless coefficient \(y = -\frac{8W}{\pi^2} \sum_{j \neq i} e^{-\frac{\text{i}(k_F r_{ij})}{N(0)^2}} G_c^R(r_{ij}, \Omega = 0) \chi_c(r_{ij}, \Omega = 0)\) arises from the Fourier transform \(Y(k_F, 0)\) and parametrizes the RKKY coupling strength. The summation in Eq. (6) runs over all positions \(j \neq i\) of Kondo sites in the system. It is important to note that \(y\) is generically positive \[24\], even though the RKKY correlations \(\chi_c(r_{ij}, 0)\) may be ferro- or antiferromagnetic. For instance, for an isotropic and dense system with lattice constant \(a (k_F a \ll 1)\), the summation in Eq. (6) can be approximated by an integral, and with the substitution \(x = 2k_F|r_{ij}|\), \(y\) can be expressed as

\[
y \approx \frac{8W}{\pi^2} \int_{k_F a}^{\infty} dx \left( 1 - \cos x \right) \frac{x \cos x - \sin x}{x^3} > 0.
\]

As a consequence, the RKKY correlations reduce the \(g\)-renormalization in Eq. (4), irrespective of the sign of the sign of \(\chi_c(r_{ij}, 0)\), as one would physically expect.

Universal suppression of the Kondo scale. – The RG [24] can be integrated analytically [24]. The Kondo scale for singlet formation on site \(i\) is defined as the running
cutoff value where the $c-f$ coupling $g$ diverges. By equivalence of all Kondo sites, this is equal to the Kondo scale $T_K$ on the surrounding sites $j \neq i$, which appears as a parameter in the $\beta$ function on the right-hand side of Eq. (9). This implies an implicit equation for the Kondo scale $T_K = T_K(y)$ in a Kondo lattice, and that it depends on the RKKY parameter $y$

$$\frac{T_K(y)}{T_K(0)} = \exp\left(-y \frac{g_0^2 D_0}{T_K(y)}\right).$$  
(8)

Here, $T_K(0) = D_0 \exp(-1/2g_0)$ is the single-ion Kondo scale without RKKY coupling, and $\alpha = \ln(\sqrt{2} + 1)$. By the rescaling, $u = T_K(y)/(y\alpha g_0^2 D_0)$, $y_c = T_K(0)/(\alpha \alpha g_0^2 D_0)$, Eq. (9) takes the universal form ($e$ is Euler’s constant),

$$\frac{y}{ey_c} u = e^{-1/u}. \quad \text{(9)}$$

Its solution can be expressed in terms of the Lambert $W$ function $\text{Lambert W}$ as $u(y) = -1/W(-y/ey_c)$. The inset of Fig. 2 visualizes solving Eq. (9) graphically. It shows that Eq. (9) has solutions only for $y \leq y_c$. This means that $y_c$ marks a Kondo breakdown point beyond which the RG does not scale to strong coupling; i.e., a Kondo singlet is not formed for $y > y_c$ even at the lowest energies. Using the above definitions, the RKKY-induced suppression of the Kondo lattice temperature reads $T_K(y)/T_K(0) = u(y)y/(ey_c) = -y/[ey_c W(-y/ey_c)]$. It is shown in Fig. 2.

In particular, at the breakdown point it vanishes discontinuously and takes the finite, universal value (see inset of Fig. 2),

$$\frac{T_K(y_c)}{T_K(0)} = \frac{1}{e} \approx 0.368.$$

We emphasize that the RKKY parameter $y$ depends on details of the conduction band structure, including band renormalizations caused by the Kondo effect (coupling to the heavy-fermion band). It also depends on the spatial arrangement of Kondo sites. Subleading contributions to $\Gamma_{cf}$ may modify the form of the cutoff function in the RG (11) and thus the universal parameter $\alpha$. However, all this does not affect the universal dependence of $T_K(y)$ on $y$ given by Eq. (9).

The critical RKKY parameter, as defined before Eq. (9), can be expressed solely in terms of the single-ion Kondo scale

$$y_c = \frac{4}{\alpha e} \tau_K (\ln\tau_K)^2$$

with $\tau_K = T_K(0)/D_0$. Note that $[\text{via } T_K(0) = D_0 \exp(-1/2g_0) \text{ and } N(0) = 1/(2D_0)]$ this is equivalent to Doniach’s breakdown criterion $\left(\text{Ref. } 6\right)$, $N(0)y_c g_0^2 = T_K(0)$ up to a factor of $O(1)$. However, the present theory goes beyond the Doniach scenario in that it predicts the behavior of $T_K(y)$.

**Comparison with experiments.** – The present theory applies directly to two-impurity Kondo systems and can be compared to corresponding STM experiments (13, 19). In Ref. 18, the Kondo scale has been extracted as the line width of the (hybridization-split) Kondo-Fano resonance. In this experimental setup, the RKKY parameter $y$ is proportional to the overlap of tip and surface $e$ electron wave functions and thus, depends exponentially on the tip-surface separation $z$. $y = y_c \exp[-(z - z_0)/\xi]$. Identifying the experimentally observed breakdown point $z = z_0$ with the Kondo breakdown point, the only adjustable parameters are a scale factor $\xi$ of the $z$ coordinate and $T_K(0)$, which is the resonance width at large separation, $z = 300 \text{ pm}$. The agreement between theory and experiment is striking, as shown in Fig. 3. In particular, at the breakdown point $T_K(y_c)/T_K(0)$ coincides accurately with the prediction without any adjustable parameter. In the STM experiment of Ref. 19, the strongest observed suppression ratio is $T_K(y)/T_K(0) = 46 K/110 K \approx 0.42$, again in excellent agreement with the strongest theoretical suppression of $1/e$, considering that in Ref. 19 the RKKY coupling $y$ cannot be varied continuously. The detailed analysis of that experiment will be published elsewhere.

**Discussion and conclusion.** – We have derived a perturbative renormalization group theory for the interference of Kondo singlet formation and RKKY interaction in Kondo lattice and multi-impurity systems, assuming that magnetic ordering is suppressed, e.g., by frustration. The equivalence of the $c-f$ vertices on all Kondo sites is reminiscent of a dynamical mean-field theory treatment; however, it goes beyond the latter in taking the nonlocal RKKY contributions into account. Equations (8) or (9)...
represent a mathematical definition of the energy scale for Kondo singlet formation in a Kondo lattice, i.e., of the Kondo lattice temperature \( T_K(y_c) \). The theory predicts a universal suppression of \( T_K(y_c) \) and a breakdown of complete Kondo screening at a critical RKKY parameter \( y = y_c \). At the breakdown point, the Kondo scale takes a finite, universal value, \( T_K(y_c)/T_K(0) = 1/e \approx 0.368 \), and vanishes discontinuously for \( y > y_c \). In the Anderson lattice, by contrast to the Kondo lattice, the locality of the \( f \) spin no longer strictly holds, but our approach should still be valid in this case. The parameter-free, quantitative agreement of this behavior with different spectroscopic experiments strongly supports that the present theory captures the essential physics of the Kondo-RKKY interplay.

The results may have profound relevance for heavy-fermion magnetic QPTs. In an unfrustrated lattice, the partially screened local moments existing for \( y > y_c \) undergo a second-order magnetic ordering transition at sufficiently low temperature. This will also imply a power law divergence of the \( c \) electron correlation \( \chi_c \) in Eq. \( 2 \). We have checked the effect of such a magnetic instability, induced either by the ordering of remnant local moments or by a \( c \)-electron SDW instability: the breakdown ratio \( T_K(y_c)/T_K(0) \) will be altered, but must remain nonzero. The reason is that the inflection point of the exponential function on the right-hand side of Eq. \( 9 \) (see Fig. \( 2 \)) is not changed by such a divergence and, therefore, the solution ceases to exist at a finite value of \( T_K(y_c) \). This points to an important conjecture about a possible, new quantum critical scenario with Kondo destruction: the Kondo spectral weight may vanish continuously at the QCP, while the Kondo scale \( T_K(y_c) \) (resonance width) remains finite, both as observed experimentally in Ref. \( 18 \). Such a scenario may reconcile apparently contradictory experimental results in that it may fulfill dynamical scaling, even though \( T_K(y_c) \) is finite at the QCP. The present theory sets the stage for constructing a complete theory of magnetic ordering and RKKY-induced Kondo destruction.

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In this supplement we provide details of (1) the calculation of the spin scattering vertex between local $4f$ spins and conduction electrons, including RKKY contributions and (2) the derivation of the one-loop RG equation and its solution.

I. F-SPIN – CONDUCTION ELECTRON VERTEX $\hat{\gamma}_{cf}$

The elementary $f$–spin – $c$–electron vertex with coupling constant $J_{0}$ is defined via the Kondo Hamiltonian,

$$H = \sum_{k,\sigma} \epsilon_{k} c_{k\sigma}^{\dagger} c_{k\sigma} + J_{0} \sum_{i} S_{i} \cdot s_{i},$$

with the notation used in the main text. The direct ($d$) and exchange ($x$) parts of the RKKY-induced vertex can be written as the product of a distance and energy dependent function $\Lambda_{RKKY}(r_{ij}, i\Omega)$ and an operator in spin space,$^{(d/x)}$}

$$\hat{\gamma}_{RKKY}^{(d/x)} = \Lambda_{RKKY}(r_{ij}, i\Omega) \hat{\gamma}^{(d/x)},$$

where $\hat{I}$ is the unit operator in spin space, $\epsilon_{abc}$ the totally antisymmetric tensor and $\delta_{ab}$ the Kronecker-$\delta$. This results in a nonlocal Heisenberg coupling between sites $i$ and $j$.

$$\gamma_{RKKY}^{(d)}_{\alpha\beta,\kappa\lambda} = 4 \sum_{a,b,c=x,y,z} \sum_{\gamma,\delta,\mu,\nu=1} (\sigma_{a\gamma}^{\alpha\beta} s_{\kappa\lambda}^{a}) (\sigma_{b\delta}^{\gamma\nu} s_{\mu}^{b}) (\sigma_{c\nu}^{\beta\mu} s_{\delta}^{c})$$

$$\gamma_{RKKY}^{(x)}_{\alpha\beta,\kappa\lambda} = -2 \sum_{a,b,c=x,y,z} \sum_{\gamma,\delta,\mu,\nu=1} (\sigma_{a\gamma}^{\alpha\beta} s_{\kappa\lambda}^{a}) (\sigma_{b\delta}^{\gamma\nu} s_{\mu}^{b}) (\sigma_{c\nu}^{\beta\mu} s_{\delta}^{c}).$$

with $c$–electron spin indices $\alpha$, $\beta$, $\gamma$, $\delta$, and $f$–spin indices $\kappa$, $\lambda$, $\mu$, $\nu$, as shown in Fig. S1. The spin summations can be performed using the spin algebra

$$\sum_{\gamma=1}^{2} \sigma_{\alpha\gamma}^{a} \sigma_{\gamma\beta}^{b} = \sum_{c=x,y,z} i \epsilon_{abc} \sigma_{\alpha\beta}^{c} + \delta_{ab} \hat{I}_{\alpha\beta},$$

B. Energy dependence

With the energy variables as defined in Fig. S2 the energy dependent functions in Eq. (S2) read in Matsubara representation (the Matsubara indices are suppressed for simplicity of notation, i.e. $i\Omega \equiv i\Omega_{n}$, etc.).
\[ \Lambda_{RKKY}^{(d)}(r_{ij}, i\Omega) = J J_0^2 \chi_c(r_{ij}, i\Omega) \tilde{\chi}_f(i\Omega) \]  
\[ \Lambda_{RKKY}^{(x)}(r_{ij}, i\omega, i\Omega) = -J J_0^2 T \sum_i G_c(r_{ij}, i\omega + i\varepsilon) G_c(r_{ij}, i\omega + i\varepsilon + i\Omega) \tilde{\chi}_f(i\varepsilon) \]  
(\text{S8})

(\text{S9})

where

\[ \chi_c(r_{ij}, i\Omega) = -T \sum_i G_c(r_{ij}, i\varepsilon) G_c(r_{ij}, i\varepsilon + i\Omega) \]  
(\text{S10})

and \( \tilde{\chi}_f(i\varepsilon) = \chi_f(i\varepsilon)/(g_{LHB})^2 \), with \( \chi_f(i\varepsilon) \) the full, single-impurity \( f \)-spin susceptibility whose temperature dependence is known from Bethe ansatz (see main text). In an isotropic system in \( d \) dimensions, the retarded conduction electron Green’s function \( G_c \) as well as the susceptibilities \( \chi_c \) and \( \tilde{\chi}_f \) at temperature \( T = 0 \) are calculated in position space as,

\[ G_c(r, \omega \pm i0) = -\frac{\pi N(\omega)}{\frac{e^{\pm ik_F(\omega + \varepsilon)}r}{k_F(\omega + \varepsilon)}} + O\left(\frac{\Omega}{\varepsilon_F}\right)^2 \]  
(\text{S11})

\[ \chi_c(r_{ij}, \Omega + i0) = \left[ N(0) \frac{\sin(x) - x \cos(x)}{4x^4} \right] + O\left(\frac{\Omega}{\varepsilon_F}\right)^2 \]  
(\text{S12})

\[ \tilde{\chi}_f(\Omega + i0) = \frac{W}{\pi D_0} \frac{D_0}{T_K} \frac{1}{\sqrt{1 + (\Omega/T_K)^2}} \left( 1 \pm \frac{2i}{\pi} \arcsinh \frac{\Omega}{T_K} \right) \]  
with \( A := \frac{W}{\pi D_0} \).  
(\text{S13})

Here, \( \varepsilon_F \) and \( k_F \) are the Fermi energy and Fermi wavenumber, respectively, \( x = 2k_F r, N(0) \) the conduction electron density of states at the Fermi energy, and \( D_0 \) the bare band cutoff.

For the renormalization of the total \( c-f \) vertex for \( c \)-electrons at the Fermi energy, the contributions \( \Lambda_{RKKY}^{(d)} \), \( \Lambda_{RKKY}^{(x)} \) must be calculated for real frequencies, \( i\Omega \rightarrow \Omega \pm i0 \), \( i\omega \rightarrow \omega + i0 \), and for electrons at the Fermi energy, i.e., \( \omega = 0 \). In this limit, only the real parts of \( \Lambda_{RKKY}^{(d)} \), \( \Lambda_{RKKY}^{(x)} \) contribute to the vertex renormalization, as shown below. In order to analyze their importance for the RG flow, we will expand them in terms of the small parameter \( T_K/D_0 \). In the following, the real part of a complex function will be denoted by a prime ′ and the imaginary part by a double-prime ″.

### Direct contribution

Since in \( \Lambda_{RKKY}^{(d)} \) [Eq. (\text{S8})], \( \chi_c(i\Omega) \) and \( \tilde{\chi}_f(i\Omega) \) appear as a product and \( \chi_f(\Omega) \) cuts off the energy transfer \( \Omega \) at the scale \( T_K \ll \varepsilon_F \approx D_0 \), the electron polarization \( \chi_c(\Omega) \) contributes only in the limit \( \Omega \ll \varepsilon_F \) where it is real-valued, as seen in Eq. (\text{S12}). Using Eq. (\text{S12}) and Eq. (\text{S13}), the real part of the direct RKKY-induced vertex contribution reads,

\[ \Lambda_{RKKY}^{(d)}(r_{ij}, \Omega + i0) = J J_0^2 \mathcal{R}(r_{ij}) AN(0) \frac{D_0}{T_K} \frac{1}{\sqrt{1 + (\Omega/T_K)^2}} + O\left(\frac{\Omega}{D_0}\right)^2, \]  
(\text{S14})

where

\[ R(r_{ij}) = \frac{\sin(x) - x \cos(x)}{4x^4}, \quad x = 2k_F r \]  

(S15)

is a spatially oscillating function.

**Exchange contribution**

In order to analyze the size of \( \Lambda_{RKKY}^{(x)} \) in terms of \( T_K/D_0 \), it is sufficient to evaluate it for a particle-hole symmetric conduction band and for \( r_{ij} = 0 \), since the \( T_K/D_0 \) dependence is induced by the on-site susceptibility \( \chi_f(i\Omega) \). The dependence on \( T_K/D_0 \) can be changed by the frequency convolution involved in \( \Lambda_{RKKY}^{(x)} \), but does not depend on details of the conduction band and distance dependent terms. (The general calculation is possible as well, but considerably more lengthy \([2]\).) We use the short-hand notation for the momentum-integrated e–electron Green’s function, \( G_e(r = 0, \omega \pm i0) = G(\omega) = G'(\omega) + iG''(\omega) \), and assume a flat density of states \( N(\omega) \), with the upper and lower band cutoff symmetric about \( \varepsilon_F \), i.e.,

\[ G^{R/A'}(\omega) = \pm \frac{\pi}{2D_0} \Theta(D_0 - |\omega|) \]  

(S16)

\[ G^{R/A'}(\omega) = \frac{1}{2D_0} \ln \left| \frac{D_0 + \omega}{D_0 - \omega} \right| = \frac{\omega}{D_0} + O \left( \left( \frac{\omega}{D_0} \right) \right) . \]  

(S17)

Furthermore, at \( T = 0 \) the Fermi and Bose distribution functions are, \( f(\varepsilon) = -b(\varepsilon) = \Theta(-\varepsilon) \).

\[ \Lambda_{RKKY}^{(x)}(0, 0, \Omega + i0) \]  

then reads,

\[ -J_0 \frac{\pi}{4} \left\{ \int \frac{d\varepsilon}{\pi} [f(\varepsilon)G^{A''}(\varepsilon)G^{R'}(\varepsilon + \Omega) + f(\varepsilon + \Omega)G^{A'}(\varepsilon)G^{A''}(\varepsilon + \Omega)] \chi_f^R(\varepsilon) \right\} (S18) \]

With the above definitions, the four terms in this expression are evaluated in an elementary way, using the substitution \( \varepsilon_F/T_K = x = \sinh u \),

\[ \int \frac{d\varepsilon}{\pi} f(\varepsilon)G^{A''}(\varepsilon)G^{R'}(\varepsilon + \Omega)\chi_f^R(\varepsilon) = AN(0) \frac{T_K}{D_0} \left[ 1 - \sqrt{1 + \left( \frac{D_0}{T_K} \right)^2 + \frac{\Omega}{T_K} \arcsinh \left( \frac{D_0}{T_K} \right)} \right] \]

(S19)

\[ \int \frac{d\varepsilon}{\pi} f(\varepsilon + \Omega)G^{A'}(\varepsilon)G^{A''}(\varepsilon + \Omega)\chi_f^R(\varepsilon) \leq AN(0) + O \left( \frac{T_K}{D_0} \right) \]  

(S20)

\[ \int \frac{d\varepsilon}{\pi} f(\varepsilon)G^{R'}(\varepsilon)G^{R'}(\varepsilon + \Omega)\chi_f^{R''}(\varepsilon) = -\frac{4}{\pi^2} AN(0) \left( \frac{1}{2} + \frac{\Omega}{D_0} \right) \arcsinh \left( \frac{D_0}{T_K} \right) + O \left( \frac{T_K}{D_0} \right) \]  

(S21)

\[ \int \frac{d\varepsilon}{\pi} f(\varepsilon + \Omega)G^{A''}(\varepsilon)G^{A''}(\varepsilon + \Omega)\chi_f^{R''}(\varepsilon) = \frac{\pi}{4} AN(0) \left[ -\arcsinh \left( \frac{\Omega}{T_K} \right) + \arcsinh \left( \min \left( \frac{\Omega}{T_K}, \frac{D_0 + \Omega}{T_K} \right) \right) \right] \]

\[ \leq \frac{\pi}{4} AN(0) + O \left( \frac{T_K}{D_0} \right) \]  

(S22)
Comparing Eqs. (S18)–(S22) with Eq. (S14) shows that all terms of $\Lambda^{(d)}_{RKKY}(\Omega)$ are subleading compared to $\Lambda^{(d)}_{RKKY}(\Omega)$ by at least a factor $(T_K/D_0)\ln(T_K/D_0)$ for all transferred energies $\Omega$. Hence, it can be neglected in the RG flow. Combining the results of spin and energy dependence, Eqs. (S2), (S6) and (S14), one obtains the total RKKY-induced $c-f$ vertex as,

$$
\hat{\gamma}^{(d)}_{RKKY}(r_{ij}, i\Omega) = 2(1 - \delta_{ij}) \chi_c(r_{ij}, i\Omega) \chi_f(i\Omega) s_i \cdot s_j
$$

(S23)

or

$$
\text{Re} \hat{\gamma}^{(d)}_{RKKY}(r_{ij}, \Omega + i0) = 2J J_0 AN(0) (1 - \delta_{ij}) R(r_{ij}) \frac{D_0}{T_K} \frac{1}{\sqrt{1 + (\Omega/T_K)^2}} s_i \cdot s_j
$$

(S24)

II. PERTURBATIVE RENORMALIZATION GROUP

A. One-loop RG equation

The derivation of the RG equation follows the well-known procedure of perturbative coupling constant renormalization [1], however performed for the nonlocal $c-f$ vertex including RKKY contributions. The amplitude of the sum of the diagrams contributing to the one-loop renormalization of the $c-f$ vertex reads in Matsubara representation (c.f. Fig. 1 (b) and Eq. (4) of the main paper),

$$
Y(r_{ij}, i\omega) = -T \sum_{i\Omega} \left[ J^2 \delta_{ij} + J \gamma^{(d)}_{RKKY}(r_{ij}, i\Omega) + J \gamma^{(d)}_{RKKY}(r_{ij}, -i\Omega) \right] G_c(r_{ij}, i\omega - i\Omega) G_f(i\Omega)
$$

(S25)

$$
+ T \sum_{i\Omega} \left[ J^2 \delta_{ij} + J \gamma^{(d)}_{RKKY}(r_{ij}, i\Omega) + J \gamma^{(d)}_{RKKY}(r_{ij}, -i\Omega) \right] G_c(r_{ij}, i\omega + i\Omega) G_f(i\Omega)
$$

This is a nonlocal function of the ingoing and outgoing coordinates of $c$-electrons, $x_j, x_i$. For the low-energy strong coupling fixed point the coupling constant for $c$-electrons at the Fermi energy must be renormalized, i.e., for excitation energy $\omega = 0$ and momentum on the Fermi surface, $k_F$. Hence, the coupling constant renormalization is given by the Fourier transform of $Y(r_{ij}, \omega = 0 + i0)$ with respect to the ingoing or outgoing coordinates, $x_j, x_i$, taken for momenta $k_{in}$, $k_{out}$ on the Fermi surface. In a lattice system, translation invariance implies the conservation of in- and outgoing momenta, $k_{in} = k_{out} = k_F$. This yields,

$$
Y(k_F, i\omega) = -J^2 T \sum_{i\Omega} \left[ G_c(r_{ij} \rightarrow 0, i\omega - i\Omega) G_f(i\Omega) - G_c(r_{ij} \rightarrow 0, i\omega + i\Omega) G_f(i\Omega) \right]
$$

(S26)

$$
- J T \sum_{i\Omega} \sum_{j} e^{i k_F r_{ij}} \left[ \gamma^{(d)}_{RKKY}(r_{ij}, i\Omega) + \gamma^{(d)}_{RKKY}(r_{ij}, -i\Omega) \right] G_c(r_{ij}, i\omega - i\Omega) G_f(i\Omega)
$$

$$
+ J T \sum_{i\Omega} \sum_{j} e^{-i k_F r_{ij}} \left[ \gamma^{(d)}_{RKKY}(r_{ij}, i\Omega) + \gamma^{(d)}_{RKKY}(r_{ij}, -i\Omega) \right] G_c(r_{ij}, i\omega + i\Omega) G_f(i\Omega)
$$

where the $j-$summation runs over all lattice sites $j$ which are occupied by a Kondo ion. Using the symmetry $\gamma^{(d)}_{RKKY}(r_{ij}, -\Omega) = \gamma^{(d)*}_{RKKY}(r_{ij}, \Omega)$, and the pseudofermion propagator $G_f(i\Omega) = 1/i\Omega$, projected onto the physical Hilbert space [3], the frequency summations can be performed to yield,

$$
Y(k_F, \omega = 0) = -2J^2 N(0) \left[ \int_{-T}^{T} d\Omega \frac{d\Omega}{\Omega} - \int_{-D}^{-T} d\Omega \frac{d\Omega}{\Omega} \right]
$$

(S27)

$$
- 2J \frac{N}{\pi} \int_{-T}^{T} d\Omega \text{Im} \left[ \sum_{j} e^{i k_F r_{ij}} \gamma^{(d)}_{RKKY}(r_{ij}, \Omega) G_c(r_{ij}, \Omega - i0) \right]
$$

Where $D$ is the running band cutoff. The change of $Y(k_F, \omega = 0)$ under an infinitesimal, logarithmic cutoff reduction $d\ln D$ represents the renormalization of the $c-f$ coupling constant. That is, the renormalization group equation is obtained as,

$$
\frac{dg}{d\ln D} = \frac{dY(k_F, \omega = 0) N(0)}{d\ln D} = -2g^2 - \frac{4}{\pi} g \text{Im} \left[ \sum_{j} e^{-i k_F r_{ij}} \gamma^{(d)}_{RKKY}(r_{ij}, D) G_c(r_{ij}, D + i0) \right]
$$

(S28)
where \( g = JN(0), g_0 = J_0 N(0) \) are the dimensionless couplings. Defining the RKKY coupling parameter as,
\[
y = -\frac{8W}{\pi^2} \text{Im} \sum_{j \neq i} \frac{e^{-ik_F r_{ij}}}{N(0)^2} G_c^R(r_{ij}, \Omega = 0) \chi_c(r_{ij}, \Omega = 0),
\]
the RG equation takes the form,
\[
\frac{dg}{d \ln D} = -2g^2 \left[1 - yg_0^2 \frac{D_0}{T_K} \right].
\] (S29)

It naturally reduces to the single-impurity Kondo RG equation, if the RKKY interaction is switched off \((y = 0)\). In a dense Kondo lattice with lattice constant \( a \) in \( d = 3 \) dimensions, \( k_F a \ll 1 \), the lattice summation in Eq. (S29) can be approximated by an integral, and \( y \) becomes,
\[
y \approx \left(\frac{2W}{k_F a}\right)^3 \int_{k_F a}^{\infty} dx \left(1 - \cos x\right) \frac{x \cos x - \sin x}{x^4} > 0.
\] (S31)

\( y \) depends sensitively on \( k_F a \). It represents the dependence on the lattice structure of the Kondo ions \((a)\) and on the band filling \((k_f)\). For illustration we show in Fig. S3 the integrand of the expression for \( y \), Eq. (S31). It is also seen that \( y > 0 \), i.e., the RKKY coupling always reduces the effective coupling strength between conduction electrons and local \( f \)-spins, irrespective of the oscillating sign of the RKKY correlations, Eq. (S12).

This is physically expected, since a ferro- as well as an antiferromagnetic coupling of an \( f \)-spin to the neighboring \( f \)-spins will always reduce the local spin fluctuations and, therefore, the Kondo singlet formation. For a non-translational invariant systems, like two- or multi-impurity Kondo systems, the expression for \( y \) will somewhat differ from Eq. (S29), since in- and outgoing momenta are not conserved. However, this does not change the form of the RG equation (S30).

### B. Integration of the RG equation

The RG equation Eq. (S30) is readily integrated by separation of variables,
\[
- \int_{g_0}^g \frac{dg}{g^2} = 2 \int_{\ln D_0}^{\ln D'} d\ln D' = 2yg_0^2 \frac{D_0}{T_K} \int_{D_0/T_K}^{D/T_K} \frac{dx}{x} \sqrt{1 + x^2},
\]
or
\[
\frac{1}{g} - \frac{1}{g_0} = 2 \ln \left(\frac{D}{D_0}\right) - yg_0^2 \frac{D_0}{T_K} \ln \left(\frac{\sqrt{1+(D/T_K)^2} - 1}{\sqrt{1+(D/T_K)^2} + 1}\right).
\] (S33)

where we have used \( D_0/T_K \gg 1 \) in the last expression.

The Kondo scale is defined as the value of the running cutoff \( D \) where \( g \) diverges, i.e., \( g \to \infty \) when \( D \to T_K \). This yields the defining equation for the Kondo scale which, hence, depends on the RKKY parameter, \( T_K \equiv T_K(y) \),
\[
\frac{T_K(y)}{T_K(0)} = \exp \left(-yg_0^2 \frac{D_0}{T_K(y)}\right),
\] (S35)

with \( \alpha = \ln(\sqrt{2} + 1) \).

[1] A. C. Hewson, *The Kondo Problem to Heavy Fermions*, Cambridge University Press, Cambridge, UK (1993).

[2] A. Nejati, *Quantum phase transitions in multi-impurity and lattice Kondo systems*, PhD thesis, University of Bonn, Germany (2016).

[3] J. Kroha and P. Wölfle, Fermi and non-Fermi liquid behavior in quantum impurity systems: conserving slave boson theory, Acta Phys. Polonica B 29, 3781 (1998); [arXiv:cond-mat/9811074](http://arxiv.org/abs/cond-mat/9811074).