THE WAGNER CURVATURE TENSOR IN NONHOLONOMIC MECHANICS

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Abstract

We present the classical Wagner construction from 1935 of the curvature tensor for completely nonholonomic manifolds in both invariant and coordinate way. The starting point is the Shouten curvature tensor for nonholonomic connection introduced by Vranceanu and Shouten. We illustrate the construction on two mechanical examples: the case of a homogeneous disc rolling without sliding on a horizontal plane and the case of a homogeneous ball rolling without sliding on a fixed sphere. In the second case we study the conditions on the ratio of diameters of the ball and the sphere to obtain a flat space - with the Wagner curvature tensor equal zero.

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§1. Introduction

1.1. Historical overview

It is well known that the full difference between nonholonomic variational problems and nonholonomic mechanics was understood after Hertz [Hr]. The geometrization of nonholonomic mechanics started in late 20’ of the XX century, with works of Vranceanu, Synge and Shouten. Vranceanu defined the notion of nonholonomic structure on a manifold (see [Vr]). Synge and Shouten made the first steps toward the definition of the curvature in nonholonomic case (see [Sy, Sh]). It was Shouten who introduced the notion of partial, or nonholonomic connection. However, the highlights of that pioneers period of development of mechanically motivated nonholonomic geometry was the work of V. V. Wagner, published in several papers from 1935 till 1941 (see [Wa1, Wa2, Wa3]). Wagner constructed the curvature tensor as an extension of the Shouten tensor. This construction is performed in several steps, following the flag of the distribution. In that sense, the structure of nonholonomicity of given distribution is reflected in the Wagner construction. For those achievements, Wagner was awarded by Kazan University in 1937 (see [VG]).

The main aim of this paper is to present Wagner’s construction, both in invariant and coordinate way. The existence of Gorbatenko’s recent, modern review [Go] is very helpful in understanding original Wagner’s works. Since we want to follow the original Wagner ideas, there are some differences from Gorbatenko’s presentation.

We also give two mechanical examples. The first one is the problem of a homogeneous disc rolling without sliding on a horizontal plane and the second is the problem of a homogeneous ball rolling without sliding on a fixed sphere. In both cases we produced complete computations of the construction of the Wagner curvature tensor. Although the first problem is of degree 2 of nonholonomicity, and the second one is of degree 1, the computations in the second case are much more complicated.

The problem of homogeneous ball rolling without sliding on a fixed sphere is interesting because it gives a family of (3, 5)- problems depending on a parameter $k$, which is the ratio between the diameters of the ball and the sphere. We investigate the Wagner flatness in these cases, in terms of this parameter $k$.

Geometry of nonholonomic variational problems is intensively developing nowadays, (see [Ju, Mn, AS]) motivated by the Control Theory. As an important example, we mention the Agrachev curvature tensor and related invariants of Sub-Riemannian Geometry (see [AS]). These natural geometric constructions were developed further in [AZ1, AZ2], and Agrachev and Zelenko implied their theory to the situation of a homogeneous ball rolling without sliding on a fixed sphere. It appears that there exist $k$ for which their invariants are zero, exactly in the same cases where the Cartan tensor is zero (see [Ca, Mn]).

So, putting altogether, we can summarize the conclusion of this paper by saying that the Wagner construction of curvature tensor is natural, and essentially different from other natural constructions, such as the Cartan and the Agrachev curvatures.

1.2. Basic notions from nonholonomic geometry
Let us fix some basic notions from the theory of distributions [VG].

**Definition 1.** Let $TM = \bigcup_{x \in M} T_x M$, be the tangent bundle of a smooth $n$-dimensional manifold $M$. A sub-bundle $V = \bigcup_{x \in M} V_x$, where $V_x$ is a vector subspace of $T_x M$, smoothly dependent on points $x \in M$, is a distribution. If the manifold $M$ is connected $\dim V_x$ is called the dimension of the distribution.

A vector field $X$ on $M$ belongs to the distribution $V$ if $X(x) \subset V_x$. A curve $\gamma$ is admissible relatively to $V$, if the vector field $\dot{\gamma}$ belongs to $V$.

A differential system is a linear space of vector fields having a structure of $C^\infty(M)$ - module. Vector fields which belong to the distribution $V$ form a differential system $N(V)$.

A $k$-dimensional distribution $V$ is integrable if the manifold $M$ is foliated to $k$-dimensional sub-manifolds, having $V_x$ as the tangent space at the point $x$. According to the Frobenius theorem, $V$ is integrable if and only if the corresponding differential system $N(V)$ is involutive, i.e. if it is a Lie sub-algebra of Lie algebra of vector fields on $M$.

**Definition 2.** The flag of a differential system $N$ is a sequence of differential systems: $N_0 = N$, $N_1 = [N,N]$, ..., $N_l = [N_{l-1}, N]$, ....

The differential systems $N_i$ are not always differential systems of some distributions $V_i$, but if for every $i$, there exists $V_i$, such that $N_i = N(V_i)$, then there exists a flag of the distribution $V$: $V = V_0 \subset V_1$ .... Such distributions, which have flags, will be called regular. It is clear that the sequence $N(V_i)$ is going to stabilize, and there exists a number $r$ such that $N(V_{r-1}) \subset N(V_r) = N(V_{r+1})$.

**Definition 3.** If there exists a number $r$ such that $V_r = TM$, the distribution $V$ is called completely nonholonomic, and minimal such $r$ is the degree of nonholonomity of the distribution $V$.

We are going to consider only regular and completely nonholonomic distributions.

1.3. The equations of motion of mechanical nonholonomic systems

One of the basic references on nonholonomic mechanics is [NF], see also [AKN]. Let us consider nonholonomic mechanical system corresponding to a Riemannian manifold $(M,g)$, where $g$ is a metric defined by the kinetic energy. It is well-known that to every Riemannian metric $g$ on $M$ corresponds a connection $\nabla$ with the properties:

\begin{align*}
  &i) \quad \nabla_X g(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0, \\
  &ii) \quad T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y] = 0,
\end{align*}

where $X, Y, Z$ are smooth vector fields on $M$. This symmetric, metric connection is usually called the Levi-Chivita connection.
We assume that the distribution $V$ is defined by $(n - m)$ 1-forms $\omega_\alpha$; in local coordinates $q = (q^1, ..., q^n)$ on $M$

\begin{equation}
\omega_\rho(q)\dot{q}^i = a_\rho^i(q) \dot{q}^i = 0 \quad \rho = m + 1, ..., n \quad ; \quad i = 1, ..., n.
\end{equation}

**Definition 4.** A virtual displacement is a vector field $X$ on $M$, such that $\omega_\rho(X) = 0$, i.e. $X$ belongs to the differential system $N(V)$.

Differential equations of motion of a given mechanical system follow from the D’Alambert-Lagrange principle: trajectory $\gamma$ of given system is a solution of the equation

\begin{equation}
\langle \nabla_\dot{\gamma} \dot{\gamma} - Q, X \rangle = 0,
\end{equation}

where $X$ is an arbitrary virtual displacement, $Q$ a vector field of internal forces, and $\nabla$ is the metric connection for the metric $g$.

The vector field $R(x)$ on $M$, such that $R(x) \in V^\perp$, $V^\perp \oplus V = T_x M$, is called reaction of ideal nonholonomic connections. Equation (2) can be written in the form:

\begin{equation}
\nabla_\dot{\gamma} \dot{\gamma} - Q = R,
\omega_\alpha(\dot{\gamma}) = 0.
\end{equation}

If the system is potential, by introducing $L = T - U$, where $U$ is the potential energy of the system ($Q = -\text{grad} U$), then in local coordinates $q$ on $M$, equations (3) become:

\begin{equation}
\frac{d}{dt} \frac{\partial L}{\partial \dot{q}^i} - \frac{\partial L}{\partial q^i} = \tilde{R},
\omega_\alpha(\dot{q}) = 0.
\end{equation}

Now $\tilde{R}$ is a 1-form in $(V^\perp)$, and it can be represented as a linear combination of 1-forms $\omega^{m+1}, ..., \omega^n$ which define the distribution: $\tilde{R} = \sum_{\alpha=m+1}^{n} \lambda_\alpha \omega_\alpha$.

Suppose $e_1, ..., e_n$ are the vector fields on $M$, such that $e_1(x), ..., e_n(x)$ form a base of the vector space $T_x M$ at every point $x \in M$, and $e_1, ..., e_m$ generate the differential system $N(V)$. Express them through the coordinate vector fields:

\begin{equation}
e_i = A^j_i(q) \frac{\partial}{\partial q^j}, \quad i, j = 1, ..., n.
\end{equation}

Denote by $p$ a projection $p : TM \rightarrow V$ orthogonal according to the metric $g$. Corresponding homomorphism of $C^\infty$-modules of sections of $TM$ and $V$ will be also denoted by $p$:

\begin{equation}
p \left( \frac{\partial}{\partial q^i} \right) = p^a_i e_a, \quad a = 1, ..., m, \quad i = 1, ..., n.
\end{equation}

Projecting by $p$ the equations (3), from $R(x) \in V^\perp(x)$, we get $p(R) = 0$, and denote $p(Q) = \tilde{Q}$ we get

\begin{equation}
\nabla_\dot{\gamma} \dot{\gamma} = \tilde{Q}.
\end{equation}
where $\mathring{\nabla}$ is the projected connection. A relationship between coefficients $\tilde{\Gamma}^c_{ab}$ of the connection $\mathring{\nabla}$, defined by the formula

$$\nabla_{e_a} e_b = \tilde{\Gamma}^c_{ab} e_c$$

and the Christoffel symbols $\Gamma^k_{ij}$ of the connection $\nabla$ follows from

$$\nabla_{e_a} e_b = \tilde{\Gamma}^c_{ab} e_c = p (\nabla_{e_a} e_b) = p \left( \nabla_{A'_a} A'_b \frac{\partial}{\partial q^j} \right) = p \left( A'_a \frac{\partial A'_b}{\partial q^j} \frac{\partial}{\partial q^i} + A'_a \nabla_{e_a} \frac{\partial}{\partial q^i} \right) = A'_a \frac{\partial A'_b}{\partial q^j} \rho^j_{ij} e_c + A'_a A'_b \Gamma^k_{ij} \rho^j_{ik} e_c.$$

Thus we get

$$\tilde{\Gamma}^c_{ab} = \Gamma^k_{ij} A'_a A'_b \rho^c_{ij} + A'_a \frac{\partial A'_b}{\partial q^j} \rho^j_{ij}. \quad (6)$$

If the motion is taking place under the inertia ($Q = \mathring{Q} = 0$), the trajectories of nonholonomic mechanical problem are going to be geodesics for the projected connection $\mathring{\nabla}$. Equations (5) were derived by Vrancheanu and Shouten.

Note. The projected connection $\mathring{\nabla}$ is not a connection on the vector bundle $V$ over $M$, because the parallel transport is defined only along admissible curves. So, it is called partial or nonholonomic connection. (Exact definition follows in Section 2.2).

§2. THE SHOUTEN TENSOR

Let $V$ be a distribution on $M$. Denote $C^\infty(M)$- module of sections on $V$ by $\Gamma(V)$.

Definition 1. A nonholonomic connection on the sub-bundle $V$ of $TM$ is a map $\nabla : \Gamma(V) \times \Gamma(V) \to \Gamma(V)$ with the properties:

1) $\nabla_X (Y + Z) = \nabla_X Y + \nabla_X Z$

2) $\nabla_X (f \cdot Y) = X(f)Y + f \nabla_X Y$

3) $\nabla f X + g Y Z = f \nabla X Z + g \nabla Y Z$ for $X, Y, Z \in \Gamma(V); \ f, g \in C^\infty(M)$.

Having a morphism of vector bundles $p_0 : TM \to V$, formed by the projection on $V$, denote by $q_0 = 1 - p_0$ the projection on $W, V \oplus W = TM$.

Definition 2. The tensor field $T_V : \Gamma(V) \times \Gamma(V) \to \Gamma(V)$ defined in the following way:

$$T_V(X, Y) = \nabla_X Y - \nabla_Y X - p_0 [X, Y] \quad X, Y \in \Gamma(V)$$
is called the tensor of torsion for the connection $\nabla$.

Suppose there is a positively defined metric tensor $g$ on $V$:

$$g : \Gamma(V) \times \Gamma(V) \to C^\infty(M), \quad g(X,Y) = g(Y,X).$$

**Theorem 1.** Given a distribution $V$, with $p_0$ and $g$, there exists a unique nonholonomic connection $\nabla$ with the properties:

$$\begin{align*}
\text{i) } & \nabla_X g(Y,Z) = X(g(Y,Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_X Z) = 0 \\
\text{ii) } & T_V = 0.
\end{align*}$$

Theorem 1 is a generalization of a well-known theorem from differential geometry. A proof can be found in [Go].

The conditions (1) can be rewritten in the form:

$$\begin{align*}
\text{i) } & \nabla_X Y = \nabla_Y X + p_0[X,Y] \\
\text{ii) } & Z(g(X,Y)) = g(\nabla_Z X, Y) + g(X, \nabla_Z Y).
\end{align*}$$

By cyclic permutation of $X, Y, Z$ in (2 ii)) and by summation we get:

$$g(\nabla_X Y, Z) = \frac{1}{2} \{X(g(Y,Z)) + Y(g(Z,X)) - Z(g(X,Y)) + \\
+ g(Z,p_0[X,Y]) + g(Y,p_0[Z,X]) - g(X,p_0[Y,Z])\}.$$  \hspace{1cm}(3)

Let $q^i$, $(i = 1, \ldots, n)$ be local coordinates on $M$, such that the first $m$ coordinate vector fields $\frac{\partial}{\partial q^i}$ are projected by projection $p_0$ into vector fields $e_a$, $(a = 1, \ldots, m)$, generating the distribution $V$: $p_0(\frac{\partial}{\partial q^i}) = p_i^a(q)e_a$. Vector fields $e_a$ can be expressed in the basis $\frac{\partial}{\partial q^i}$ as $e_a = B_a^i \frac{\partial}{\partial q^i}$, with $B_a^ib^b_i = \delta_a^b$. Now we give coordinate expressions for the coefficients of the connection $\Gamma^c_{ab}$, defined as $\nabla_{e_a} e_b = \Gamma^c_{ab} e_c$. From (3) we get:

$$\Gamma^c_{ab} = \{e_a\} + g_{ac}g^{cd}\Omega^d_{bd} + g_{bc}g^{cd}\Omega^d_{ad} - \Omega^c_{ab},$$  \hspace{1cm}(4)

where $\Omega$ is obtained from $p_0[e_a, e_b] = -2\Omega^c_{ab}e_c$ as:

$$2\Omega^c_{ab} = p^c_\alpha e_a(B^\alpha_b) - p^c_\beta e_b(B^\beta_a),$$

and $\{e_a\} = \frac{1}{2}g^{ce}(e_a(g_{be}) + e_b(g_{ae}) - e_c(g_{ab}))$.

It was shown in Section 1.3 that the equations of a nonholonomic mechanical problem, without external forces, are geodesic equations for the connection $\nabla$. The connection $\nabla$ is obtained by projection on the sub-bundle $V$ of the Levi-Civita connection $\nabla$ for the metric $g$. The question is: what is a relationship between the connection $\nabla$ and the metric $\tilde{g}$, induced from $g$ on $V$.

**Proposition 1.** The connection $\nabla$, obtained by projecting metric torsion-less connection $\nabla$ for the metric $g$, is the metric torsion-less connection for the induced metric $\tilde{g}$ if the projector $p_0$ is orthogonal.
Proof. Let \( p_0 : TM \to V \) be the orthogonal projector.
a) We need to prove \( \tilde{\nabla} \tilde{g} = 0 \). For arbitrary \( X,Y,Z \in \Gamma(V) \) we have:

\[
(5) \quad \tilde{\nabla}X \tilde{g}(Y,Z) = X(\tilde{g}(Y,Z)) - \tilde{g}(\tilde{\nabla}_X Y, Z) - \tilde{g}(Y, \tilde{\nabla}_X Z).
\]

Since \( \tilde{g} \) is induced by \( g \), it follows that \( \tilde{g}(Y,Z) = g(Y,Z) \). In the same way, \( \tilde{\nabla}X Y = p_0 \nabla X Y = \nabla X Y - U \), where \( U \in \Gamma(V^\perp) \) is a vector field projected with \( p_0 \) into 0. From the orthogonality condition, \( U \) is orthogonal on \( X,Y \) and \( Z \) relatively to the metric \( g \), so we get: \( \tilde{g}(\tilde{\nabla}_X Y, Z) = g(\nabla X Y - U, Z) = g(\nabla X Y, Z) \). Similarly, \( \tilde{g}(Y, \tilde{\nabla}_X Z) = g(Y, \nabla X Z) \). Plugging into (5), we get:

\[
\tilde{\nabla}X \tilde{g}(Y,Z) = \nabla X g(Y,Z), \quad X,Y,Z \in \Gamma(V),
\]

and from the assumption \( \nabla g = 0 \) we get \( \tilde{\nabla} \tilde{g} = 0 \).

b) We need to show that the connection \( \tilde{\nabla} \) is torsion-less.

\[
T_{\tilde{\nabla}}(X,Y) = \tilde{\nabla}_X Y - \tilde{\nabla}_Y X - p_0[X,Y]
= p_0 \nabla X Y - p_0 \nabla Y X - p_0[X,Y] = p_0(\nabla X Y - \nabla Y X - [X,Y]),
\]

and since \( \nabla \) is free of torsion, the same is valid for \( \tilde{\nabla} \). \( \square \)

Note. Both the Wagner and the Shouten tensor, as we will see later, depend on the choice of the projector. Wagner defined curvature tensor for a metric which is defined on the distribution \( V \). If we start from some mechanical problem, then there is a metric on the whole \( TM \), which is afterwards induced on \( V \). According to the last Proposition, in order to get projected connection which is metric for the induced metric, one is obliged to choose the orthogonal projector. That means, that for mechanical systems there is a unique choice of a projector.

The problem of definition of the curvature tensor for nonholonomic connections was considered for the first time by Shouten. He defined the curvature tensor in the following way:

Definition 3. The Shouten tensor is a mapping \( K : \Gamma(V) \times \Gamma(V) \times \Gamma(V) \to \Gamma(V) \) defined by:

\[
(6) \quad K(X,Y)Z = \nabla X (\nabla Y Z) - \nabla Y (\nabla X Z) - \nabla_{p_0[X,Y]} Z - p_0[g_0[X,Y],Z],
\]

where \( X,Y,Z \in \Gamma(V) \).

To check that the Definition 3 is correct, one has to verify that \( K \) is of tensor nature, i.e. that it is linear on \( X,Y,Z \) relatively to the multiplication by smooth functions on \( M \). Really, by direct check [Go] we get:

\[
K(fX,Y)Z = f K(X,Y)Z,
K(X,Y)(fZ) = f K(X,Y)Z,
K(X,Y)Z = - K(Y,X)Z.
\]

In comparison to the curvature tensor for connections on \( M \), we see that Shouten tensor (6) has one term more, the last one in (6), and that in the third term \( p_0 \) appears. The last term gives a correction in order that \( K \) be a tensor. Note that
without that last term linearity for $Z$ relatively to the multiplication by smooth functions would not be satisfied.

A mapping $K(X,Y) : Z \to K(X,Y)Z$ is a morphism of $C^\infty(M)$-module $\Gamma(V)$. Since $K$ is anti-symmetric relatively to $X,Y$, a $C^\infty(M)$-linear mapping $\Gamma(K) : \Gamma(\wedge^2 V) \to \Gamma(\text{End}(V,V))$ can be corresponded to the Shouten tensor by the condition:

$$\Gamma(K)(X \wedge Y)Z = K(X,Y)Z, \quad X,Y,Z \in \Gamma(V),$$

where $\wedge^2 V$ is the space of bivectors.

§3. The Wagner tensor

3.1. The Wagner construction

Wagner constructed a curvature tensor starting from the integrability condition for the tensor equation $\nabla X = U$ where $U \in \text{End}(V,V), X \in \Gamma(V)$. If the curvature tensor is zero, then absolute parallelism should take place, i.e. a covariantly constant vector field in any direction should exist, which is equivalent to the integrability of the equations $\nabla X = 0$. Wagner noticed that if the degree of nonholonomicity is greater than 1, then the Shouten tensor does not satisfy the condition of absolute parallelism, and he suggested a correction. The idea is the following. One starts with some metric $g$ on $V$. The metric $g$ is going to be extended to each sub-bundle $V_i$ of the flag $V = V_0 \subset V_1 \subset \cdots \subset V_N = TM$. The next step, the connection on $V_i$ and the curvature tensor analogous to the Shouten tensor are going to be defined. In this way, in the $N$-th step, the curvature tensor which satisfies the absolute parallelism condition is constructed. The basic Wagner’s paper where this was performed is [Wa1].

Let a metric $g$ be defined on $k$-dimensional vector space $W$. Then a metric $g^\wedge$ on $\wedge^2 W$ is defined by the expression:

$$(1) \quad g^\wedge(x_1 \wedge y_1, x_2 \wedge y_2) = \begin{vmatrix} g(x_1, x_2) & g(x_1, y_2) \\ g(y_1, x_2) & g(y_1, y_2) \end{vmatrix}.$$

(The isomorphism $\varphi : \wedge^2 W^* \to (\wedge^2 W)^*$

$$\varphi(f \wedge g)(x \wedge y) = \omega(x,y) = f(x)g(y) - f(y)g(x).$$

is used here.)

**Lemma 1.** If $g$ is positively definite form on $W$, then $g^\wedge$ is also positively defined form on $\wedge^2 W$.

Consider a mapping

$$\Delta : \wedge^2 \Gamma(V) \to \Gamma(TM)/\Gamma(V),$$

defined by

$$\Delta(X \wedge Y) = [X,Y] \mod \Gamma(V), \quad X,Y \in \Gamma(V).$$
The mapping $\Delta$ is $C^\infty(M)$-linear:

$$\Delta(fX \wedge Y) = [fX, Y] \mod \Gamma(V) = \{-Y(f)X + f[X, Y]\} \mod \Gamma(V) = f[X, Y] \mod \Gamma(V) = f\Delta(X \wedge Y).$$

Observe that $\text{Im}(\Delta)$ is not always equal to $\Gamma(TM)/\Gamma(V)$, but it is its $C^\infty(M)$-submodule, and denote

$$\Gamma(V_i) = \{X \in \Gamma(TM)|X \mod \Gamma(V) \in \text{Im}(\Delta)\}.$$

So, we get a sequence of $C^\infty$ submodules $\Gamma(V_0) \subset \cdots \subset \Gamma(V_N) = \Gamma(TM)$, defined by:

$$\Gamma(V_i) = \{X \in \Gamma(TM)|X \mod \Gamma(V_{i-1}) \in \text{Im}(\Delta_{i-1})\},$$

where $V = V_0$, $\Delta = \Delta_0$. Note that the sequence of sub-bundles $V_0 \subset V_1 \subset \cdots \subset V_N = TM$ is a flag of the distribution $V$, and $N$ is the degree of nonholonomicity, since we reduced our attention to the case of regular distributions. The mapping $\Delta_i : \wedge^2 V_i \to TM/V_i$ is called the $i$-th tensor of nonholonomicity of the distribution $V$.

For every point $x \in M$, there is a factor space $V_{i+1,x}/V_{i,x}$ with the projection $\pi_i : V_{i+1,x} \to V_{i+1,x}/V_{i,x}$. Suppose the mappings $\theta_{i,x} : V_{i+1,x}/V_{i,x} \to R_{i,x}$ are defined, where $R_{i,x}$ are some sub-spaces, chosen transversely to $V_{i,x}$, so that $V_{i,x} \oplus R_{i,x} = V_{i+1,x}$. Mappings $q_i = \theta_i \cdot \pi_i$ and $p_i = 1_{V_{i+1}} - q_i$ are the projectors onto $R_i$ and $V_i$ respectively. Now we are going to extend the metric from $V$ to the whole $TM$.

**Theorem 1.** Let the distribution $V$ with metric $g$ and mappings $\theta_0, \ldots, \theta_{N-1}$ are given. Then there exists a unique metric tensor $G$ on $TM$, which satisfies the conditions:

1. $G|_V = g$.
2. In the direct sum $TM = V_0 \oplus R_0 \oplus \cdots \oplus R_{N-1}$ the components are mutually orthogonal.
3. $(G|R_i)^{-1} = \theta_i \cdot \Delta_i \cdot ((G|V_i)^\wedge)^{-1} \cdot (\theta_i \cdot \Delta_i)^*$.

**Proof.** For an arbitrary point $x$ on $M$ we have $T_x M = V_{0,x} \oplus R_{0,x} \oplus \cdots \oplus R_{N-1,x}$. Define $G|_{R_i,x} = g_{i+1,x}$ by the condition 3 of this Theorem. By the previous Lemma, $g_{0,x}^\wedge$ is a positively defined form on $\wedge^2 V_0$, so it is $(g_{0,x}^\wedge)^{-1}$ on $(\wedge^2 V_0)^\wedge$. The operation of conjugation preserves positive definitness, so we get that $g_{i+1,x}$ is also a positively definite form. By iterations we get that $g_{i+1,x}$ are positively definite. □

Coordinate expressions for the metric enlarged from $V_{i-1}$ to $V_i = V_{i-1} \oplus R_{i-1}$ are obtained in the following way. Let the vectors $e_{a-1}$ span $V_{i-1}$. Corresponding dual base denote by $e^{a-1}$. If $X_{a_i} e^{a_i}$ is a given 1-form on $R_{i-1}$, then:

$$\tilde{g}(X_{a_i} e^{a_i}) = \tilde{g}^{a_i b_i} X_{a_i} e_{b_i} = (\theta_{i-1} \cdot \Delta_{i-1}) (G_{V_{i-1}}^\wedge)^{-1} (\theta_{i-1} \cdot \Delta_{i-1})^* (X_{a_i} e^{a_i})$$

$$= (\theta_{i-1} \cdot \Delta_{i-1}) (G_{V_{i-1}}^\wedge)^{-1} (X_{a_i} M_{a_{i-1} b_{i-1}}^{i-1} e^{a_{i-1}} \wedge e^{b_{i-1}})$$

$$= (\theta_{i-1} \cdot \Delta_{i-1}) (g^\wedge)^{a_{i-1} b_{i-1} c_{i-1} d_{i-1}} (M_{a_{i-1} b_{i-1}}^{i-1} X_{a_i} e_{c_{i-1}} \wedge e_{d_{i-1}})$$

$$= (g^\wedge)^{a_{i-1} b_{i-1} c_{i-1} d_{i-1}} (X_{a_i} M_{a_{i-1} b_{i-1}}^{i-1} M_{c_{i-1} d_{i-1}}^{i-1} e_{b_i}).$$
where \( g^{\alpha_1 \ldots \alpha_{i-1} b_{i-1} c_{i-1} d_{i-1}} \) is the inverse metric tensor for \( g^{\Lambda} \) defined by (1), and 
\( M_{c_{i-1} d_{i-1}}^{b_{i-1}} \) are coordinate expressions for the \((i - 1)\)-th tensor of nonholonomicity \( \Delta_{i-1} \). It is obvious that
\[
g^{\alpha_{i-1} b_{i-1} c_{i-1} d_{i-1}} = \frac{1}{2} \left( g^{\alpha_{i-1} b_{i-1} c_{i-1}} g^{i_{b_{i-1}} d_{i-1}} - g^{i_{b_{i-1}} c_{i-1}} g^{i_{d_{i-1}} c_{i-1}} \right),
\]
so, finally we get
\[
\eta^{\alpha_{i-1} b_{i-1} c_{i-1} d_{i-1}} = M_{a_{i-1} b_{i-1}}^{a_{i-1}} M_{c_{i-1} d_{i-1}}^{c_{i-1}} \eta^{i_{a_{i-1}} c_{i-1} d_{i-1}}.
\]

Let us define morphism of vector bundles \( \mu_i : V_{i+1} \to \wedge^2 V_i \), by:
\[
(3) \quad \mu_i = (B^\Lambda_i)^{-1} \cdot (\theta_i \cdot \Delta_i)^* \cdot (G_{i+1}|_{R_i} \cdot \theta_i \cdot \pi_i).
\]
So, if \( X \in \Gamma(V_i) \), then \( \mu_i(X) = 0 \).

Now we get coordinate expressions for \( \mu_i \):
\[
\mu_i - 1 (e_a) = M_{a_{i-1} b_{i-1}}^{a_{i-1}} e_{a_{i-1}} \wedge e_{b_{i-1}}
\]
\[
= M_{c_{i-1} d_{i-1}}^{c_{i-1}} g^{i_{c_{i-1}} d_{i-1}} e_{a_{i-1}} \wedge e_{b_{i-1}}.
\]

Coordinate expressions for \( \mu_i \) and those for metrics are in the agreement with the original Wagner’s paper [Wa1].

We are ready to expose Wagner’s construction for the curvature tensor for nonholonomic systems.

Denote by \( \nabla_0 \) the connection for the metric \( g_0 \) on \( V_0 \), and by \( \nabla_1 \) the Shouten tensor. Define \( \square_1 : \Gamma(V_1) \times \Gamma(V_0) \to \Gamma(V_0) \) by:
\[
\square_1 X U = \nabla_0 p_1 X U + \nabla_0 (\mu_0(X))(U) + p_0 [q_0 X, U],
\]
and \( \square_1 : \wedge^2 V_1 \to \text{End}(V_0) \) by the condition:
\[
\Gamma(\square_1)(X \wedge Y)(U) = \frac{1}{2} X \square_1 Y U - \frac{1}{2} Y \square_1 X U - \frac{1}{2} p_{[X,Y]} U - p_0 [q_1 [X,Y] U],
\]
where \( X, Y \in \Gamma(V_1), U \in \Gamma(V_0) \).

Similarly, by induction: \( \square_1 : \Gamma(V_i) \times \Gamma(V_0) \to \Gamma(V_0) \)
\[
\square_1 X U = \frac{1}{2} p_{[X,Y]} U + \frac{1}{2} p_{[X,Y]} U - \frac{1}{2} p_{[X,Y]} U - \frac{1}{2} p_{[X,Y]} U - p_0 [q_1 [X,Y] U],
\]
\[
\square_1 : \wedge^2 V_i \to \text{End}(V_0) \quad X, Y \in \Gamma(V_i), U \in \Gamma(V_0),
\]
\[
\Gamma(\square_1)(X \wedge Y) U = \frac{1}{2} X \square_1 Y U - \frac{1}{2} Y \square_1 X U - \frac{1}{2} p_{[X,Y]} U - p_0 [q_i [X,Y] U],
\]
Finally for \( i = N \) we get:
\[
(4) \quad \nabla_0 X U = \nabla_0 p_{N-1} X U + \nabla_0 (\mu_{N-1}(X))(U) + p_0 [q_{N-1} X, U],
\]
\[
\begin{align*}
K^N : \wedge^2 V_N &\rightarrow \text{End}(V_0), \quad X,Y \in \Gamma(V_N), \quad U \in \Gamma(V_0), \\
\Gamma(N)(X \wedge Y)U &\equiv N X \square Y U - N Y \square X U - N (X,Y)U,
\end{align*}
\]

because \( p_N = \text{id} \), and \( q_N = 0 \).

**Theorem 2.** Mappings \( \square \), satisfy the following conditions:

1. \( \square f_X + g_Y U = f \square X U + g \square Y U

2. \( \square f U = X(f)U + f \square X U, \quad X,Y \in \Gamma(V_i) \)

3. \( \square \) is a linear connection on the vector bundle \( V \).

The proof follows by direct calculations.

Since \( \square \) is a connection on the vector bundle, according to the Theorem 2, we get that \( K^N \) is the curvature tensor of the vector bundle \( V \) over \( M \), relative to the connection \( \square \), and it is called the Wagner tensor of nonholonomic manifold.

**Note.** In [Go], the Wagner tensor is defined in a slightly different manner, as a mapping \( K^N : \wedge^2 \Gamma(V_N) \rightarrow \Gamma(\text{End}(V_{N-1})) \). The way presented here is in agreement with the original Wagner paper [Wa1], as it is going to be clear from the coordinate expressions given below.

### 3.2. Coordinate expressions for the Wagner tensor

Now we are going to derive the coordinate expressions for the Shouten tensor and the Wagner tensor. The Latin indices \( a_i \) run in the intervals \( 1, \ldots, n_i \), where \( n_i = \text{dim} V_i \), and Greek indices \( \alpha \) in the interval \( 1, \ldots, n \). Let \( e_a \) be vector fields spanning the distribution \( V \), and \( p_0 \) and \( q_0 \) the projectors to \( V \) and \( V^\perp \) respectively. The components of the Shouten tensor \( K^{d}_{abc} \) are derived from:

\[
K(e_a,e_b)(e_c) = K^{d}_{abc}e_d.
\]

Plugging into (2.6) and using the properties of the connection \( \nabla \) we get:

\[
(6) \quad \Lambda^{d}_{abc} = e_a(\Gamma^{d}_{bc}) - e_b(\Gamma^{d}_{ac}) + \Gamma^{d}_{ac} \Gamma^{e}_{bc} - \Gamma^{d}_{bc} \Gamma^{e}_{ac} + 2 \Omega^{e}_{abc} \Gamma^{d}_{bc} - M^{d}_{abc} \Lambda^{p}_{pe}.
\]

Coefficients \( \Lambda^{d}_{pe} \) are defined by \( p_0[e_p,e_c] = \Lambda^{d}_{pe}e_d \), \( p = m + 1, \ldots, n \) and \( M^{d}_{abc} \) are the components of the tensor of nonholonomicity \( \Delta \) defined by \( M^{d}_{abc}e_p = q_0[e_a,e_b] \).

Expressing \( e_a \) in the basis of coordinate vector fields \( \partial/\partial q_r \) as \( e_a = B^{i}_{a} \partial/\partial q_r \) and plugging into (6), we get coordinate expressions for the Shouten tensor, which coincide with those obtained in [Wa1].

**Denote by** \( \Pi^{i}_{a,b} \) **the components of the connection for** \( \square \) **defined by** \( \Pi^{i}_{a,b}e_c = \Pi^{i}_{a,b}e_c \), where the vector fields \( e_a \) span the distribution \( V_i \). So, we get:

\[
(7) \quad \Pi^{i}_{a,b} = p^{i}_{a_i} \Pi^{i-1}_{a_i-1,b_i-1} + M^{i-1}_{a_i} K^{i-1}_{a_i,b_i} + q^{i-1}_{a_i} \Lambda^{p}_{pb}
\]
In the same way we get coordinate expressions for $\mathbb{K}$:

\[
\mathbb{K}_{a,b,c} = e_{a_i} (\Pi_{b,c}^{d}) - e_{b_i} (\Pi_{a,c}^{d}) + \Pi_{a,c}^{d} \Pi_{b,c}^{e} - \Pi_{b,c}^{d} \Pi_{a,c}^{e} + 2 \Omega_{a_i}^{e} \Pi_{c}^{d} - M_{a_i,b_i}^{d} \Lambda_{pc}^{d}.
\]

Let $\mathcal{p}$ and $\mathcal{q}$ be the corresponding projectors to $V_i$ and $V_i^\perp$ and $\Omega_{a_i,b_i}^{c_i}$ is defined by

\[
2 \Omega_{a_i,b_i}^{c_i} e_{c_i} = -\mathcal{p}[e_{a_i}, e_{b_i}],
\]

while $M_{a_i,b_i}^{p_i}$ are the components of the $i$-th tensor of nonholonomicity, defined by (2).

Finally, for $i = N$, we get coordinate expressions for the Wagner tensor

\[
\mathbb{K}_{a_N,b_N,c} = e_{a_N} (\Pi_{b_N,c}^{d}) - e_{b_N} (\Pi_{a_N,c}^{d}) + \Pi_{a_N,c}^{d} \Pi_{b_N,c}^{e} - \Pi_{b_N,c}^{d} \Pi_{a_N,c}^{e} + 2 \Omega_{a_N,b_N}^{c_N} \Pi_{c_N}^{d}.
\]

The vector fields $e_{a_N}$ are now spanning the whole $TM$.

### 3.3. Absolute parallelism and the Wagner tensor

We start from the equation

\[
\nabla W = U, \quad U \in \Gamma(\text{End}(V)), \quad W \in \Gamma(V).
\]

The question is if for a given endomorphism $U$ and for every $X \in \Gamma(V)$, the equation:

\[
\nabla_X W = U_X
\]

has a solution. From (10) we get:

\[
\nabla_X \nabla_Y W - \nabla_Y \nabla_X W = \nabla_{p_0[X,Y]} W - p_0[g_0[X,Y], W] = \\
= \nabla_X U_Y - \nabla_Y U_X - U_{p_0[X,Y]} - p_0[g_0[X,Y], W].
\]

So, there exists $X \in \Gamma(V_1)$ such that:

\[
0 \mathbb{K}(\mu_0(X))(W) + p_0[g_0X,W] = U^\nabla(\mu_0(X))W
\]

where $U^\nabla(\mu_0(X)) = \nabla_X U_Y - \nabla_Y U_X - U_{p_0[X,Y]}$. Then:

\[
\nabla_{p_0[X]} W + 0 \mathbb{K}(\mu_0(X))(W) + p_0[g_0X,W] = \frac{1}{U_X} = U^\nabla(\mu_0(X)) + U_{p_0X}.
\]

The integrability conditions for the equation (10) are reduced to:

\[
\mathbb{K} W = U.
\]

In the same way, iteratively, we reduce the integrability condition for the equation (10) to the condition:

\[
\mathbb{K} W = U.
\]
Finally, for \( i = N \) we get:
\[
\square W = U.
\]
So:
\[
\begin{align*}
N \circ (X \wedge Y)(W) &= \square X \square Y W - \square Y \square X W - \square [X,Y] W \\
&= \square X U_Y - \square Y U_X - U_{[X,Y]}.
\end{align*}
\]
This equation is the integrability condition for the equation (10). Therefore, in the case \( U = 0 \), the necessary and sufficient condition for the existence of the vector fields parallel along any direction is that the Wagner tensor is equal to zero.

**§4. The rolling disc**

Now, we are going to illustrate the theory exposed before by calculating the Wagner tensors in two mechanical problems. In this section, we deal with a homogeneous disc of the unit mass and radius \( R \) rolling without sliding on a horizontal plane.

Note that we are going to present only basic steps of the calculations. As it is well known, the configuration space is \( M = \mathbb{R}^2 \times SO(3) \). For local coordinates we chose \( x \) and \( y \) as coordinates of the mass center of the disc, and the Euler angles \( \varphi, \psi, \theta \). Nonholonomic constraints follow from the condition that the velocity of the point of contact of the disc and the plane should be equal to zero. The two nonholonomic constraints are:
\[
\begin{align*}
\dot{x} + R \cos \varphi \dot{\psi} + R \cos \theta \cos \dot{\varphi} - R \sin \theta \sin \varphi \dot{\theta} &= 0, \\
\dot{y} + R \sin \varphi \dot{\psi} + R \cos \theta \sin \dot{\varphi} + R \sin \theta \cos \varphi \dot{\theta} &= 0.
\end{align*}
\]

Corresponding 1-forms which define the three-dimensional distribution \( V \) are:
\[
\begin{align*}
\omega_1 &= dx + R \cos \varphi d\psi + R \cos \theta \cos \varphi d\varphi - R \sin \theta \sin \varphi d\theta, \\
\omega_2 &= dy + R \sin \varphi d\psi + R \cos \theta \sin \varphi d\varphi + R \sin \theta \cos \varphi d\theta.
\end{align*}
\]

The vector fields which span the differential system \( N(V) \) are:
\[
\begin{align*}
e_1 &= R \cos \varphi \frac{\partial}{\partial x} + R \sin \varphi \frac{\partial}{\partial y} - \frac{\partial}{\partial \psi}, \\
e_2 &= \cos \theta \frac{\partial}{\partial \psi} - \frac{\partial}{\partial \varphi}, \\
e_3 &= R \sin \theta \sin \varphi \frac{\partial}{\partial x} - R \sin \theta \cos \varphi \frac{\partial}{\partial y} + \frac{\partial}{\partial \theta}.
\end{align*}
\]

First, let us calculate the degree of nonholonomicity of this mechanical system:
\[
\begin{align*}
[e_1, e_2] &= -R \sin \varphi \frac{\partial}{\partial x} + R \cos \varphi \frac{\partial}{\partial y} = T, \\
[e_1, e_3] &= 0, \\
[e_2, e_3] &= -\sin \theta e_1.
\end{align*}
\]
So, the distribution $V$ is nonintegrable, and the whole $TM$ is not generated in the first step. From:

\[
\begin{align*}
[e_1, e_2] = T, & \quad [e_1, e_3] = 0, \quad [e_2, e_3] = -\sin \theta e_1, \\
[e_1, T] = 0, & \quad [e_2, T] = R \cos \varphi \frac{\partial}{\partial x} + R \sin \varphi \frac{\partial}{\partial y} = U,
\end{align*}
\]

since $e_1, e_2, e_3, T, U$ span the tangent space in every point of $M$, the degree of nonholonomicity is 2.

It is well known that the kinetic energy of the system is:

\[
2T = \dot{x}^2 + \dot{y}^2 + (A \sin^2 \theta + C \cos^2 \theta) \dot{\varphi}^2 + 2C \cos \theta \dot{\psi} + C \dot{\psi}^2 + (A + R^2 \cos^2 \theta) \dot{\theta}^2
\]

where $A$ and $C$ are the principle central moments of inertia of the disc in the moving frame. This gives a metric on $M$:

\[
(g_{ij}) = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & A \sin^2 \theta + C \cos^2 \theta & C \cos \theta & 0 \\
0 & 0 & C \cos \theta & C & 0 \\
0 & 0 & 0 & 0 & A + R^2 \cos^2 \theta
\end{pmatrix}.
\]

As it was pointed out after the Proposition 2.1, in mechanical problems we chose the orthogonal projector $p_0$ from $TM$ onto $V$. The vector fields annulled by $p_0$ are:

\[
e_4 = -\sin \varphi (A + R^2 \cos^2 \theta) \frac{\partial}{\partial x} + \cos \varphi (A + R^2 \cos^2 \theta) \frac{\partial}{\partial y} + R \sin \theta \frac{\partial}{\partial \theta},
\]

\[
e_5 = C \cos \varphi \frac{\partial}{\partial x} + C \sin \varphi \frac{\partial}{\partial y} + R \frac{\partial}{\partial \psi}.
\]

The vector fields $e_a$ are expressed in the basis $\frac{\partial}{\partial x^i}$ by $e_a = B^i_a \frac{\partial}{\partial x^i}$. So we get:

\[
(B^i_a) = \begin{pmatrix}
R \cos \varphi & R \sin \varphi & 0 & -1 & 0 \\
0 & 0 & -1 & \cos \theta & 0 \\
R \sin \theta \sin \varphi & -R \sin \theta \cos \varphi & 0 & 0 & 1
\end{pmatrix}.
\]

From $p_0(\frac{\partial}{\partial x^i}) = p^a_i e_a$, we get the coordinates of the projector:

\[
(p^a_i) = \begin{pmatrix}
\frac{R \cos \varphi}{C + R^2} & 0 & \frac{R \sin \theta \sin \varphi}{A + R^2} \\
\frac{R \sin \varphi}{C + R^2} & 0 & -\frac{R \sin \theta \cos \varphi}{A + R^2} \\
\frac{-C \cos \theta}{C + R^2} & -1 & 0 \\
\frac{C \cos \theta}{C + R^2} & 0 & 0 \\
0 & 0 & \frac{A + R^2 \cos^2 \theta}{A + R^2}
\end{pmatrix}.
\]
Similarly, for \( q_0 \) we get:

\[
(q_1^p) = \begin{pmatrix}
- \sin \varphi \frac{A}{A+R^2} & \cos \varphi \frac{A}{A+R^2} \\
\cos \varphi \frac{A}{C+R^2} & - \sin \varphi \frac{A}{C+R^2} \\
0 & \frac{R \cos \theta}{C+R^2} \\
0 & \frac{R}{C+R^2} \\
\frac{R \sin \theta}{A+R^2} & 0
\end{pmatrix}.
\]

The induced metric \( g_{ab} \) on \( V \), is derived from \( g_{ij} \):

\[
(g_{ab}) = \begin{pmatrix}
R^2 + C & 0 & 0 \\
0 & A \sin^2 \theta & 0 \\
0 & 0 & A + R^2
\end{pmatrix}.
\]

Now we calculate the components of the connection \( \Gamma^c_{ab} \) for metric connection using coordinate expressions (2.4). We start with determining \( \{ \epsilon_{ab} \} \). The only nonzero coefficients are:

\[
\{2_{23}\} = \{3_{23}\} = \frac{\cos \theta}{\sin \theta}, \quad \{3_{22}\} = - \frac{A \sin \theta \cos \theta}{A + R^2}.
\]

The coefficients \( \Omega \) we derive from \( -2 \Omega^c_{ab} = p_0 [e_a, e_b] \). Having the expressions for the commutators of \( e_a \), it can easily be seen that nonzero elements are:

\[
\Omega^3_{12} = - \Omega^3_{21} = \frac{R^2 \sin \theta}{2(A + R^2)}, \quad \Omega^1_{23} = - \Omega^1_{12} = \frac{\sin \theta}{2}.
\]

From (2.4) we get the following nonzero components of the connection:

\[
\Gamma^1_{23} = - \frac{(2R^2 + C) \sin \theta}{2(C + R^2)}, \quad \Gamma^1_{32} = \frac{C \sin \theta}{2(C + R^2)}, \quad \Gamma^2_{23} = \Gamma^2_{32} = \frac{\cos \theta}{\sin \theta},
\]

\[
\Gamma^2_{13} = \frac{C}{2A \sin \theta}, \quad \Gamma^3_{12} = \frac{C \sin \theta}{2(A + R^2)},
\]

\[
\Gamma^3_{21} = \frac{(2R^2 + C) \sin \theta}{2(A + R^2)}, \quad \Gamma^3_{22} = - \frac{A \sin \theta \cos \theta}{A + R^2}.
\]

In order to get the components of the Shouten tensor (see (3.6)), we are calculating the coefficients \( \Lambda \). From:

\[
[e_4, e_1] = 0, \quad [e_4, e_2] = - \cos \varphi (A + R^2 \cos^2 \theta) \frac{\partial}{\partial x} - \sin \varphi (A + R^2 \cos^2 \theta) \frac{\partial}{\partial y} - R \sin^2 \theta \frac{\partial}{\partial \psi},
\]

\[
[e_4, e_3] = - R^2 \sin \varphi \cos \theta \sin \theta \frac{\partial}{\partial x} + R^2 \cos \varphi \cos \theta \sin \theta \frac{\partial}{\partial y} - R \cos \theta \frac{\partial}{\partial \theta}, \quad [e_5, e_1] = 0,
\]

\[
[e_5, e_2] = - C \sin \varphi \frac{\partial}{\partial x} + C \cos \varphi \frac{\partial}{\partial y}, \quad [e_5, e_3] = 0,
\]
we get:

\[ \Lambda_{42}^1 = \frac{-R(A + R^2 \cos^2 \theta - C \sin^2 \theta)}{C + R^2}, \quad \Lambda_{43}^3 = -R \cos \theta, \quad \Lambda_{52}^3 = \frac{-RC \sin \theta}{A + R^2}. \]

Similarly, for the components of the tensor of nonholonomicity we get:

\[ \begin{array}{l}
0 M_{42}^1 = \frac{R}{A + R^2}, \\
1 M_{24}^1 = \frac{A + R^2}{C + R^2},
\end{array} \]

where the projectors \( p_1 \) and \( q_1 \) to \( V_1 \) and \( V_1^\perp \) are used. Here \( V_1 \) is generated by the vector fields \( e_1, e_2, e_3, e_4 \):

\[
\begin{pmatrix}
\frac{R \cos \varphi}{C + R^2} & 0 & -\frac{R \sin \varphi \sin \varphi}{A + R^2} & \frac{\sin \varphi}{A + R^2} \\
\frac{R \sin \varphi}{C + R^2} & 0 & -\frac{R \sin \varphi \cos \varphi}{A + R^2} & \frac{\cos \varphi}{A + R^2} \\
-\frac{C \cos \theta}{C + R^2} & 0 & 0 & 0 \\
0 & 0 & \frac{A + R^2 \cos^2 \theta}{A + R^2} & \frac{R \sin \theta}{A + R^2}
\end{pmatrix}, \quad \begin{pmatrix}
\frac{C \sin \varphi}{C + R^2} \\
\frac{R \cos \theta}{C + R^2} \\
0 \\
0
\end{pmatrix}.
\]

Expansion of the metric from \( V_0 \) to \( V_1 \) is obtained from the coordinate expression:

\[ g_{ab}^1 = M_{ab}^1 M_{cd}^1 g^{cd} \]  \( g^{bd} \) as:

\[ g^{44} = \frac{2R^2}{(A + R^2)^2(C + R^2)A \sin^2 \theta}, \quad g_{44} = \frac{1}{g^{44}}. \]

Similarly, we get the coordinate expressions for the metric expanded on \( V_2 = TM \) by:

\[ g^{55} = \frac{4R^2}{A^2(C + R^2)^3 \sin^4 \theta}, \quad g_{55} = \frac{1}{g^{55}}. \]

From the expanded metric, as it was mentioned before, we get the components for the morphisms \( \mu_0 \) and \( \mu_1 \):

\[ \begin{array}{l}
0 M_{42}^1 = (M_{12}^4)^2 g^{11}, \\
22 \quad \frac{A + R^2}{2R}, \\
24 \quad \frac{C + R^2}{2(A + R^2)}.
\end{array} \]

Everything is prepared for calculation of the Wagner tensor. In the coordinate expressions for the Wagner tensor, the first two indices take values from 1 to 5, and the second two from 1 to 3. From the antisymmetry for the first two indexes, there are 90 independent components of the Wagner tensor. We are going to calculate three components. All calculations are performed in three steps: the first step is
the Shouten tensor, then the tensor $K^1_{121}$ on $V_1$, and finally the Wagner tensor. We are calculating only the necessary components.

We calculate the component $K^{2}_{451}$ of the Wagner tensor.

$$K^{2}_{451} = c_4(\Pi^2_{451}) - c_5(\Pi^2_{314}) + 2^{2}_{c_4}(\Pi^2_{514} - \Pi^2_{5c4})\Pi^2_{411},$$

$$\Pi^2_{514} = M^2_{5} K^{2}_{124}, \quad \Pi^2_{5c4} = M^2_{5} K^{2}_{124c},$$

$$\Pi^2_{411} = \Pi^2_{c_41} = M^2_{4} K^{2}_{121},$$

$$\Pi^2_{4c4} = \Pi^2_{4c4} = M^2_{4} K^{2}_{12c},$$

$$K^{2}_{241} = e_2(\Pi^2_{c_41}) - e_4(\Gamma^2_{c_41}) + \Gamma^2_{2d4} C_{21}^2 - \Pi^2_{23} \Gamma^2_{21},$$

$$K^{2}_{24c} = e_2(\Pi^2_{c_4c}) - e_4(\Gamma^2_{c_4c}) + \Gamma^2_{2d4} C_{12c}^2 - \Pi^2_{23} \Gamma^2_{21c},$$

$$\Pi^2_{c_4c} = M^2_{4} K^{2}_{12c},$$

So, for the component $K^{2}_{451}$, we need first the coordinate expressions for the components $K^{2}_{12c}$ of the Shouten tensor. From (3.6) we get:

$$K^{1}_{121} = 0, \quad K^{2}_{121} = -C(4R^2 + C)\frac{A}{4(A + R^2)},$$

$$K^{0}_{121} = 0, \quad K^{0}_{122} = \frac{4R^2 A + 4R^4 \cos^2 \theta + C^2 \sin^2 \theta}{4(A + R^2)(C + R^2)},$$

$$K^{2}_{122} = \frac{R^2 \cos \theta}{A + R^2}, \quad K^{0}_{122} = 0,$$

$$K^{0}_{123} = 0, \quad K^{0}_{123} = 0, \quad K^{0}_{123} = \frac{R^2 \cos \theta}{A + R^2}.$$

Similarly, we get:

$$\Pi^0_{c_41} = 0, \quad \Pi^0_{c_41} = 0, \quad \Pi^0_{c_41} = \frac{-C(4R^2 + C)}{4AR},$$

$$\Pi^1_{c_42} = R \cos \theta, \quad \Pi^1_{c_42} = 0, \quad \Pi^1_{c_42} = 0,$$

$$\Pi^1_{c_43} = 0, \quad \Pi^1_{c_43} = 0, \quad \Pi^1_{c_43} = 0.$$

Therefore:

$$K^{2}_{241} = 0, \quad K^{2}_{242} = 0, \quad K^{2}_{241} = 0,$$

$$K^{1}_{243} = \frac{8R^4 A \sin^2 \theta - 10R^2 C^2 \sin^2 \theta - C^3 \sin^2 \theta + 8R^2 AC \sin^2 \theta + 4R^2 AC - 8R^4 C \sin^2 \theta + 4R^4 C \cos^2 \theta}{8AR \sin \theta(C + R^2)},$$

So

$$\Pi^1_{51} = M^2_{5} K^{1}_{241} = 0, \quad \Pi^2_{51} = M^2_{5} K^{2}_{241} = 0, \quad \Pi^2_{52} = M^2_{5} K^{2}_{242} = 0.$$
Finally, we get
\[ K_{251}^2 = 0. \]

In the same way, we can calculate the other components of the Wagner tensor. For example, we are calculating also \( K_{121}^2 \) and \( K_{312}^3 \).

From
\[ K_{121}^2 = e_1(\Pi_{21}^2) - e_2(\Pi_{11}^2) + \Pi_{1c}^2 \Pi_{21}^c - \Pi_{2c}^2 \Pi_{11}^c + 2\Omega_{12}^a \Pi_{a21}^2, \]
we get:
\[ K_{121}^2 = \Gamma_{1c}^2 \Gamma_{21}^c + 2\Omega_{12}^a \Pi_{a21}^2, \]
and finally:
\[ K_{121}^2 = 0. \]

Similarly \( K_{133}^1 = \frac{c^2}{4A(R^2+C)} \).

§5. Ball rolling on the fixed sphere

Now we will give a construction of Wagner tensor for the system of a homogeneous ball of unit mass \( o \) rolling on the fixed sphere \( S^2 \). Denote the diameters of the ball and the sphere by \( r_2, r_1 \) respectively. This system has five degrees of freedom. Let us introduce the following coordinates: the spherical coordinates \( \alpha, \beta \) on \( S^2 \) and the Euler angles \( \psi, \varphi, \theta \) which determine position of the ball. Nonholonomic constraints are derived from the condition that velocity of the contact point is equal to zero. There are two independent nonholonomic constraints:

\[
(1 + k)\dot{\beta} + \sin(\psi - \alpha)\dot{\theta} - \sin \theta \cos(\psi - \alpha)\dot{\phi} = 0
\]
\[
(1 + k)\dot{\alpha} + \tan \beta \cos(\psi - \alpha)\dot{\theta} + [\tan \beta \sin \theta \sin(\psi - \alpha) - \cos \theta]\dot{\phi} - \dot{\psi} = 0,
\]
where \( k = r_1/r_2 \). So, we assume \( r_2 = 1 \). Corresponding 1-forms that define the three-dimensional distribution \( V \) are:
\[
\omega_1 = (1 + k)d\beta + \sin(\psi - \alpha)d\theta - \sin \theta \cos(\psi - \alpha)d\phi
\]
\[
\omega_2 = (1 + k)d\alpha + \tan \beta \cos(\psi - \alpha)d\theta + [\tan \beta \sin \theta \sin(\psi - \alpha) - \cos \theta]d\phi - d\psi = 0.
\]

Vector fields:
\[
X_1 = \frac{\partial}{\partial \alpha} + (1 + k)\frac{\partial}{\partial \psi}
\]
\[
X_2 = \tan \beta \sin \theta \frac{\partial}{\partial \alpha} - (1 + k)\sin \theta \cos(\psi - \alpha)\frac{\partial}{\partial \theta} - (1 + k)\sin(\psi - \alpha)\frac{\partial}{\partial \varphi}
\]
\[
+ (1 + k)\cos \theta \sin(\psi - \alpha)\frac{\partial}{\partial \psi}
\]
\[
X_3 = \sin \theta \frac{\partial}{\partial \beta} - (1 + k)\sin \theta \sin(\psi - \alpha)\frac{\partial}{\partial \theta} - (1 + k)\cos(\psi - \alpha)\frac{\partial}{\partial \varphi}
\]
\[
- (1 + k)\cos \theta \cos(\psi - \alpha)\frac{\partial}{\partial \psi}
\]
span the differential system $N(V)$. Since

$$[X_1, X_2] = (0, 0, k \cos \theta \cos (\psi - \alpha) (1 + k), -k \cos (\psi - \alpha) (1 + k),$$

$$k \sin \theta \sin (\psi - \alpha) (1 + k))$$

$$[X_1, X_3] = (0, 0, k \cos \theta \sin (\psi - \alpha) (1 + k), -k \sin (\psi - \alpha) (1 + k),$$

$$-k \sin \theta \cos (\psi - \alpha) (1 + k))$$

$$[X_2, X_3] = \left( \frac{-\sin^2 \theta + (1 + k) \sin \theta \sin (\psi - \alpha) \cos \theta \sin \beta \cos \beta}{\cos^2 \beta},ight.$$

$$\left. -\sin \theta \cos (\psi - \alpha) \cos \theta (1 + k),ight.$$\n
$$\left. \frac{(1 + k)^2 (2 \cos^2 \theta \cos \beta - \cos \beta) - (1 + k) \sin \theta \sin (\psi - \alpha) \sin \beta \cos \theta}{\cos \beta},ight.$$\n
$$\left. -\frac{(1 + k)^2 \cos \theta \cos \beta - (1 + k) \sin \beta \sin (\psi - \alpha)}{\cos \beta}. \right)$$

the degree of nonholonomicity is equal to one.

From the kinetic energy of the system:

$$2T = (1 + k)^2 (\dot{\beta}^2 + \cos^2 \beta \dot{\alpha}^2) + A \dot{\psi}^2 + \dot{\varphi}^2 + 2 \cos \theta \dot{\varphi} \dot{\psi},$$

where $A$ is the inertia momentum of the ball, the formula for the metric is derived

$$(g_{ij}) = \begin{pmatrix}
(1 + k)^2 \cos^2 \beta & 0 & 0 & 0 & 0 \\
0 & (1 + k)^2 & 0 & 0 & 0 \\
0 & 0 & A & A \cos \theta & 0 \\
0 & 0 & A \cos \theta & A & 0 \\
0 & 0 & 0 & 0 & A
\end{pmatrix}.$$ 

We choose the orthogonal projector $p_0$. The vector fields orthogonal to the distribution $V$ are:

$$X_4 = A \cos (\psi - \alpha) \frac{\partial}{\partial \alpha} + A \tan \beta \cos^2 \beta \sin (\psi - \alpha) \frac{\partial}{\partial \beta}$$

$$- (1 + k) \cos \beta \cos (\psi - \alpha) \frac{\partial}{\partial \psi} + (1 + k) \tan \beta \cos \beta \frac{\partial}{\partial \theta}$$

$$X_5 = A \sin \theta \frac{\partial}{\partial \beta} + (1 + k) \cos \theta \cos (\psi - \alpha) \frac{\partial}{\partial \psi}$$

$$- (1 + k) \cos (\psi - \alpha) \frac{\partial}{\partial \varphi} + (1 + k) \sin \theta \sin (\psi - \alpha) \frac{\partial}{\partial \theta}$$

So the induced metric on the distribution $V$ is

$$(g_{ab}) = \begin{pmatrix}
A + \cos^2 \beta & \sin \beta \cos \beta \sin \theta & 0 \\
\sin \beta \cos \beta \sin \theta & \sin^2 \theta (A + \sin^2 \beta) & 0 \\
0 & 0 & \sin^2 \theta (1 + A)
\end{pmatrix}.$$
Using formula (2.4) we get:

\[
\begin{align*}
\Gamma_{11}^3 &= \frac{\sin \beta \cos \beta}{\sin \theta (1 + A)}, \\
\Gamma_{12}^3 &= -\frac{1}{2} \frac{A k - A - 2 + 2 \cos^2 \beta}{1 + A}, \\
\Gamma_{13}^1 &= -\frac{1 + k \sin \theta \sin \beta \cos \beta}{(1 + A)}, \\
\Gamma_{13}^3 &= \frac{1}{2} \frac{A k - A + \cos^2 \beta k - 2 + \cos^2 \beta}{1 + A}, \\
\Gamma_{21}^3 &= \frac{1}{2} \frac{A + A k + 2 - 2 \cos^2 \beta}{1 + A}, \\
\Gamma_{22}^3 &= -(1 + k) \cos \theta \cos(\psi - \alpha), \\
\Gamma_{22}^2 &= \frac{(A + \sin^2 \beta) \cos \theta \sin \beta}{\cos \beta (1 + A)}, \\
\Gamma_{23}^1 &= \frac{k + 1 - A \sin^2 \theta + \cos^2 \beta - 1 + \cos^2 \beta \cos^2 \theta + \cos^2 \theta}{1 + A}, \\
\Gamma_{23}^2 &= -\frac{(2 A - (1 + k) \cos^2 \beta + 2) \sin \theta \sin \beta}{2 \cos \beta (1 + A)}, \\
\Gamma_{23}^3 &= -(1 + k) \cos \theta \cos(\psi - \alpha), \\
\Gamma_{31}^1 &= \frac{1}{2} \frac{(-1 + k) \cos \beta \sin \sin \theta}{1 + A}, \\
\Gamma_{31}^3 &= -(1 + k) \sin \theta \cos(\psi - \alpha), \\
\Gamma_{32}^1 &= \frac{1}{2} \frac{A + A k + \cos^2 \beta k - \cos^2 \beta + 2}{1 + A}, \\
\Gamma_{32}^2 &= \frac{1}{2} \frac{-2(1 + k)(1 + A) \sin(\psi) \cos \theta + (1 - k) \sin \theta \sin \cos \theta}{1 + A}, \\
\Gamma_{33}^3 &= -(1 + k) \sin(\psi) \cos \theta.
\end{align*}
\]

Other \( \Gamma \) are equal to zero. Some components of the Shouten tensor different from zero are:

\[
\begin{align*}
0^1_{121} &= 0^1_{122} = \frac{(k - 1)^2 A + 4 k^2 \sin \beta \cos \beta \sin \theta}{4(1 + A)^2}, \\
0^1_{121} &= -\frac{(1 + k^2)(2 A + A \cos^2 \beta) + 4 A k (1 + k) + 2 k (A^2 - A \cos^2 \beta + 2 k \cos^2 \beta)}{(1 + A)^2}, \\
0^3_{231} &= \frac{(-5 A + 2 A k + 3 A k^2 - 4) \cos \beta \sin \beta \sin \theta}{4(1 + A)^2}, \\
0^2_{133} &= -\frac{(1 + k^2) \sin \theta \sin \beta \cos \beta}{1 + A}.
\end{align*}
\]

The following components of the Shouten tensor are zero:

\[
\begin{align*}
0^3_{121} &= 0^3_{122} = 0^3_{123} = 0^2_{123} = 0^3_{131} = 0^2_{131} = 0^1_{132} = \\
0^3_{132} &= 0^3_{133} = 0^1_{231} = 0^3_{231} = 0^1_{232} = 0^3_{232} = 0^2_{232} = 0^3_{233} = 0.
\end{align*}
\]
Expansion of the metric is given by the following formulae:
\[
\begin{align*}
g^{44} &= \frac{2k^2}{A(A+1)^3 \cos^2 \beta \cos^2(\psi - \alpha)}, \\
g^{45} &= \frac{-2k^2 \sin \beta \sin(\psi - \alpha)}{A(A+1)^3 \sin \theta \cos \beta \cos(\psi - \alpha)}, \\
g^{55} &= \frac{k^2(1 - \cos^2 \beta \sin^2(\psi - \alpha))}{A(1+A)^3 \sin^2 \theta \cos^2(\psi - \alpha)}.
\end{align*}
\]

One of the components of the Wagner tensor is:
\[
K_{133}^1 = \frac{\sin^2 \theta \cos^2 \beta (k^2(A + 4 \sin^2 \beta) + 2Ak + A + 4 \cos^2 \beta)}{4(1+A)}.
\]

From the last formula we get

**Theorem 1.** For any \(k\) the Wagner curvature tensor is different from zero.

**Conclusion**

From the Theorem 5.1, it follows that the Wagner tensor is essentially different from the tensors constructed by Cartan [Ca] and Agrachev’s school [AS, AZ1, AZ2], since it doesn’t recognize the nilpotent case. A natural question is to find the theory of Jacobi fields which corresponds to the Wagner curvature.

At the end let us note that the paper [Ta] appeared very recently, dealing with geometrization of nonholonomic mechanics, based on some later Cartan’s work. The connections studied in [Ta] are generally not torsion-less.

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