Abstract

The goal of this article is to extend a theorem of Lurie

\[ \text{Sh}_A(X) = \text{Fun}(\text{Exit}_A(X), S) \]

representing constructible sheaves with values in \( S \), the \( \infty \)-category of spaces, on a stratified space \( X \) with poset of strata \( A \), as functors from the exit paths \( \infty \)-category \( \text{Exit}_A(X) \) to \( S \). Lurie's representation theorem works provided \( A \) satisfy the ascending chain condition. This typically rules out infinite dimensional examples of stratified space.

Building on it and with the help of a stratified homotopy invariance theorem from Haine, we show that when \( X \) is a nice enough \( A \)-stratified space and when \( A \) is itself stratified \( A_{\leq 0} \subset A_{\leq 1} \subset \cdots \subset A \) by posets satisfying the ascending chain condition,

\[ \text{Hyp}_A(X) = \text{Fun}(\text{Exit}_A(X), S) \]

the \( \infty \)-category of \( A \)-constructible hypersheaves on \( X \) is represented by functors from the exit paths \( \infty \)-category of \( X \).

There are two types of nice stratified spaces on which this extended representation theorem applies: conically stratified spaces and spaces that are sequential colimits of conically stratified spaces. Examples of application include the metric and the topological exponentials of a Fréchet manifold, locally countable simplicial complexes and more generally, locally countable cylindrically normal CW-complexes.
When a topological space is nice enough, its category of locally constant sheaves of sets is equivalent to the category of representations of its fundamental groupoid. The fundamental groupoid $X$ has objects the points of $X$ and arrows, the homotopy classes of continuous paths between two points in $X$ up to homotopy. Going further, Lurie has shown that the $\infty$-category of locally constant sheaves of spaces is equivalent to the $\infty$-category of representations of $\text{Sing}(X)$, the simplicial set of maps $\Delta^n \to X$, which is a model for the fundamental $\infty$-groupoid of $X$ [1, A.2.15].

This representation theorem can be further extended to stratified spaces. When $X$ is a stratified space with poset of strata $A$, one can consider $A$-constructible sheaves on $X$: sheaves whose restriction to each stratum $X_a$ is locally constant. In order to represent those constructible sheaves, the simplicial set $\text{Sing}(X)$ needs to be adapted to take into account the stratification of $X$. Following an idea of Treumann [2], Lurie considered the simplicial subset $\text{Exit}_A(X) \subset \text{Sing}(X)$ where paths are only allowed to immediately escape a deeper stratum and never return. When the stratification is conical this simplicial subset is an $\infty$-category, the $\text{exit paths } \infty$-category of $X$. With such a setup, the representation theorem for $A$-constructible sheaves on $X$ holds provided $A$ satisfies the ascending chain condition.

**Definition** (Ascending chain condition). A poset $A$ is said to satisfy the ascending chain condition if $A$ does not admit a chain $a_0 < a_1 < \cdots$ of infinite length.

This condition typically excludes stratified spaces of infinite dimension. For example, one may think of infinite dimensional simplicial complexes which are not locally finite or of any space $X$ with a filtration by dimension

$$X_{\leq 0} \subset X_{\leq 1} \subset \cdots \subset X_{\leq n} \subset \cdots \subset X$$

which happens every time $X$ is for example the colimit $\text{lim}_{\mu \in \omega} X_{\leq \mu}$. Our goal is to extend the representation theorem to a large class of posets $A$ that do not satisfy the ascending chain condition and which includes in particular the poset $\omega = \{0 < 1 < 2 < \cdots\}$.

There are two obstacles to a generalisation of the representation theorem. The first one has to do with hypercompleteness of sheaves, a phenomenon that starts appearing only in the $\infty$-world. We shall dedicate a section to the differences between sheaves and hypersheaves [$\S$ 1.2]. Having built a continuous map

$$\text{Fun}(\text{Exit}_A(X), S) \to \text{Sh}_A(X)$$

because every functor in $\text{Fun}(\text{Exit}_A(X), S)$ is the limit of its truncation tower, it follows that its image must be a hypersheaf. There is thus no hope of
representing all $A$-constructible sheaves in general and we shall instead focus on the full subcategory of $A$-constructible hypersheaves. Notice that when $A$ satisfies the ascending chain condition, all $A$-constructible sheaves are already hypersheaves.

Constructible hypersheaves have already been used by Lurie to describe the equivalence between locally constant factorisation algebras on a finite dimensional manifold $M$ and $E_M$-algebras. A key tool in the proof of Lurie is the use of the metric exponential of $M$, the metric space of finite subsets of $M$, which is naturally stratified by the cardinality of the subsets. Lurie notes that he had to add an hypercompletion hypothesis on the constructible sheaves on the exponential because the stratifying poset did not satisfy the ascending chain condition [1, 3.3.12]. On the other side of the equation, Cepek has shown that the combinatoric of the exit paths $\infty$-category of the exponential of $\mathbb{R}^n$ is also related to the one of $E_n$-algebras [3]. The study of the exponential of a manifold is a major motivation for extending the representation theorem and we shall give more details about this particular example at the end of the article [§ 4.3].

The second obstacle has to do with the proof of the representation theorem itself. It uses an induction on the depth of the stratification and a poset $A$ admits a depth function if and only if it satisfies the ascending chain condition. To circumvent this issue, we shall make use of the functoriality of the equivalence of $\infty$-categories in the representation theorem. We shall then work with posets $A$ which are themselves stratified

$$A_{\leq 0} \subset A_{\leq 1} \subset \cdots \subset A_{\leq n} \subset \cdots \subset A$$

by posets satisfying the ascending chain condition. In such a case, the stratified space $X$ inherits a filtration

$$X_{\leq 0} \subset X_{\leq 1} \subset \cdots \subset X_{\leq n} \subset \cdots \subset X$$

by closed stratified subspaces. We are thus considering posets that are strict ind-objects in the category of posets with the ascending chain condition. In another perspective, Barwick, Glasman and Haine have considered spaces (and $\infty$-toposes) stratified over profinite posets [4].

The functoriality of the exit paths $\infty$-category is easily addressed as it commutes with filtered colimits, since simplicies in $\text{Exit}_A(X)$ are only allowed to visit a finite number of strata. Undertaking the functoriality of the $\infty$-category of constructible hypersheaves is the real task here. We shall show that the canonical dévissage map

$$\text{Hyp}_A(X) \to \lim_{n<\omega} \text{Sh}_{A_{\leq n}}(X_{\leq n})$$

is an equivalence in two special cases: when $X$ is conically $A$-stratified and when the topology on $X$ coincides with the colimit topology $\lim_{n<\omega} X_{\leq n}$. In particular, one can replace the topology of a conically stratified $X$ with the colimit topology and keep the same $\infty$-category of constructible hypersheaves. This is coherent with the fact that the $\infty$-category of exit paths does not see the global topology of $X$, as every map $\Delta^n \to X$ in $\text{Exit}_A(X)$ is required to visit only a finite number of strata. These two sets of conditions are usually incompatible as explained in an impossibility theorem [2.14]. We then obtain two versions of the representation theorem.
Theorem [3.12] (conical case). Let $X$ be a paracompact conically $A$-stratified space such that each stratum of $X$ be locally of singular shape and $A$ be $\omega$-stratified with $A_{\leq n}$ satisfying the ascending chain condition, for each $n < \omega$. Then the $\infty$-category of exit paths $\operatorname{Exit}_A(X)$ represents

$$\operatorname{Fun}(\operatorname{Exit}_A(X), S) = \operatorname{Hyp}_A(X)$$

the $\infty$-category of $A$-constructible hypersheaves on $X$.

Theorem [3.13] (colimit case). Let $X$ be an $A$-stratified space, colimit of a sequence of closed stratified embeddings of paracompact conically stratified spaces over posets satisfying the ascending chain condition and whose strata are locally of singular shape. The $\infty$-category of exit paths $\operatorname{Exit}_A(X)$ represents

$$\operatorname{Fun}(\operatorname{Exit}_A(X), S) = \operatorname{Hyp}_A(X) = \operatorname{Sh}_A(X)$$

the $\infty$-category of $A$-constructible sheaves and all $A$-constructible sheaves on $X$ are hypersheaves.

The dévissage theorem in the conical case relies heavily on a stratified homotopy invariance theorem from Haine [5, 2.3]. However, it is neither constructible sheaves nor constructible hypersheaves that are invariant but hyperconstructible hypersheaves. This other notion of constructibility stems from the difference in functoriality between sheaves and hypersheaves.

For this reason, we shall dedicate the first section to the definitions of all the types of sheaves and constructibilities that we have mentioned so far. In the second section, we shall see that for general types of spaces constructible sheaves, constructible hypersheaves and hyperconstructible hypersheaves do coincide. The third section is dedicated to the exit paths $\infty$-category and the proof of the extended representation theorem. The last section shall present some examples that this new representation theorem allows us to consider: the metric and the topological exponentials of a Fréchet manifold, locally countable simplicial complexes and more generally locally countable cylindrically normal CW-complexes.

1 DIFFERENT NOTIONS OF CONSTRUCTIBILITY

We start by presenting the different characters at play: stratified spaces, hyperconstructible hypersheaves, constructible hypersheaves and constructible sheaves.

1.1 Stratified spaces

There exists many different non-equivalent notions of stratified spaces in topology. Here we shall use a very general one, following Lurie [1, A.5.1].

Definition 1.1 (Stratified space). A stratified topological space is the data of continuous map $f : X \to A$ where $A$ is a poset viewed as a topological space by defining $U \subset A$ to be open if and only if it is closed upwards.

A morphism of stratified spaces is a commutative square

$$
\begin{array}{ccc}
X & \longrightarrow & Y \\
\downarrow & & \downarrow \\
A & \longrightarrow & B
\end{array}
$$
where the top map is continuous and the bottom map is a poset map.

For each \( a \in A \), we shall let \( X_a \) denote the \( a \)-stratum, preimage \( f^{-1}(a) \). We shall denote by \( s_a \), the embedding \( X_a \subset X \). We shall also let \( X_{\leq a} \) denote the fibre product \( X \times_{A_{\leq a}} A \).

**Remark 1.2.** If \( A \) is itself stratified by a poset \( \Lambda \), we thus get two maps \( X \to A \to \Lambda \) and for every \( \lambda \in \Lambda \), \( X_{\leq \lambda} \) is naturally stratified over \( A_{\leq \lambda} \).

We shall mainly be interested in stratified maps over an identity morphism \( A = B \). Indeed we shall essentially focus on stratified homotopy equivalences. Also, if \( A \) is a subposet of \( B \), then one can see \( X \) as being \( B \)-stratified without loss of generality.

**Definition 1.3 (Stratified homotopy equivalence).** Let \( X \to A \) and \( Y \to A \) be two \( A \)-stratified spaces. An \( A \)-stratified homotopy between two \( A \)-stratified map \( f, g : X \to Y \) is an \( A \)-stratified map \( h : [0,1] \times X \to Y \) such that \( h(0,-) = f \) and \( h(1,-) = g \).

We shall say that an \( A \)-stratified map \( f : X \to Y \) is an \( A \)-stratified homotopy equivalence if there exists an \( A \)-stratified map \( g : Y \to X \) and \( A \)-stratified homotopies between \( f \circ g \) and \( \text{id}_Y \) on one hand, and between \( g \circ f \) and \( \text{id}_X \) on the other hand.

### 1.2 Recollections on hypersheaves

In the classical theory of sheaves of sets, it is well known that isomorphisms can be detected on stalks. This is no longer true for sheaves in the \( \infty \)-categorical world and this leads to a separation into two equally interesting objects: sheaves and hypersheaves. The main difference here to which we shall pay special attention is the difference in functoriality: for sheaves one uses pullbacks but for hypersheaves one needs to use hyperpullbacks. There are several equivalent definitions of hypersheaves [1, A.1.9]. Since we are only interested in the case of hypersheaves on topological spaces, we shall choose the most convenient definition.

**Definition 1.4 (Hypersheaf).** Given a topological space \( X \), we shall denote by \( \text{Sh}(X) \) the \( \infty \)-category of sheaves (of spaces) on \( X \) and by \( \text{Hyp}(X) \) the full subcategory of hypersheaves: sheaves that are local with respect to maps \( \mathcal{F} \to \mathcal{G} \) inducing an equivalence \( \mathcal{F}_x \to \mathcal{G}_x \) on stalks, for every point \( x \in X \).

The inclusion \( \text{Hyp}(X) \subset \text{Sh}(X) \) admits a left exact reflector, which sends a sheaf \( \mathcal{F} \) to its hypercompletion \( \widehat{\mathcal{F}} \). It is such that the canonical map \( \mathcal{F}_x \to \widehat{\mathcal{F}}_x \) is an equivalence for every point \( x \in X \).

Categories of sheaves admit the following functoriality: if \( f : X \to Y \) is a continuous map, it induces an adjunction

\[
\begin{array}{ccc}
\text{Sh}(X) & \xrightarrow{f^*} & \text{Sh}(Y) \\
\xleftarrow{f_*} & & \\
\end{array}
\]

between the \( \infty \)-categories of sheaves on \( X \) and sheaves on \( Y \). The right adjoint \( f_* \) preserves hypersheaves but not the left adjoint \( f^* \).

**Definition 1.5 (Hyperpullback).** Let \( f : X \to Y \) be a continuous map and let

\[
\begin{array}{ccc}
\text{Sh}(Y) & \xrightarrow{\widehat{f}} & \text{Hyp}(X) \\
\end{array}
\]
denote the hyperpullback along $f$, obtained by first using $f^*$ and then hyper-completing.

By construction, we obtain an adjunction

$$\begin{array}{c}
\text{Hyp}(X) \\
\text{Hyp}(Y)
\end{array} \xrightarrow{\bar{f}} \xleftarrow{f} \xrightarrow{\bar{f}}$$

similar to the sheaf case.

**Remark 1.6.** Given two continuous maps $f : X \to Y$ and $g : Y \to Z$ and a hypersheaf $\mathcal{H}$ on $Z$, the canonical map

$$\overset{(gf)^*}{\longrightarrow} \overset{\bar{f}^* \bar{g}^*}{\longrightarrow} \mathcal{H}$$

is an equivalence. This stem from the fact that $(gf)_* = g_* f_*$.

In addition, there is a useful case where hyperpullbacks and pullbacks coincide for hypersheaves.

**Lemma 1.7** [1, A.3.6]. Let $f : X \to Y$ be a continuous map. Assume $f^*$ admits a left adjoint $f_!$, then for every hypersheaf $\mathcal{H}$ the canonical map

$$f^* \mathcal{H} \longrightarrow \bar{f}^* \mathcal{H}$$

is an equivalence.

This happens for example, when $f$ is an open embedding.

### 1.3 Locally (hyper)constant (hyper)sheaves

Because sheaves and hypersheaves do not share the same functoriality, there are two possible candidates to extend the traditional notion of locally constant sheaves of sets to the $\infty$-category world: locally constant sheaves and locally hyperconstant hypersheaves.

Let $X$ be a topological space and let $\pi : X \to \ast$ denote the projection to the point.

**Definition 1.8** (Constant sheaf). A constant sheaf on $X$ is a sheaf of the form $\pi^* K$ for some $K \in S$, where $S$ is the $\infty$-category of spaces.

**Definition 1.9** (locally constant sheaf). A sheaf $\mathcal{F}$ is said to be locally constant if there exists an open covering $\{j_i : U_i \subset X\}_{i \in I}$ of $X$ such that $j_i^* \mathcal{F}$ is constant for each $i \in I$.

We shall denote by $\text{Sh}_{\text{loc}}(X) \subset \text{Sh}(X)$ the full subcategory of locally constant sheaves on $X$.

**Definition 1.10** (Hyperconstant hypersheaf). A hyperconstant hypersheaf on $X$ is a hypersheaf of the form $\bar{\pi}^* K$ for some $K \in S$.

**Definition 1.11** (Locally hyperconstant hypersheaf). We shall say that an hypersheaf $\mathcal{H}$ is locally hyperconstant if there exists an open covering $\{j_i : U_i \subset X\}_{i \in I}$ of $X$ such that the restriction $j_i^* \mathcal{H}$ is a hyperconstant hypersheaf for each $i \in I$.

We shall denote by $\text{Hyp}_{\text{loc-hyp}}(X) \subset \text{Hyp}(X)$ the full subcategory of locally hyperconstant hypersheaves on $X$. 
**Warning 1.12.** Even though, one has an inclusion \( \text{Hyp}(X) \subset \text{Sh}(X) \), one does not have \( \text{Hyp}_{\text{loc-hyp}}(X) \subset \text{Sh}_{\text{loc}}(X) \).

The two main results that we shall use in this article emanate from Lurie and Haine. Both authors use a different (but equivalent) definition for locally constant sheaves [5, 1.4] and locally hyperconstant hypersheaves [1, A.2.12]: the one of locally constant sheaves on an \( \infty \)-topos. In what follows, we spend some time explaining why their definition agrees with the one we have just given.

**Definition 1.13** (Locally constant sheaves on an \( \infty \)-topos). For an \( \infty \)-topos \( \mathcal{X} \), let \( \pi : \mathcal{X} \to \ast \) be a final map. A constant sheaf on \( \mathcal{X} \), is a sheaf of the form \( \pi^*K \) for some \( K \in \mathbb{S} \).

A sheaf \( \mathcal{F} \) on \( \mathcal{X} \) is locally constant if there is a small family of étale maps \( \{ j_i : U_i \to \mathcal{X} \}_{i \in I} \) such that \( \coprod_{i \in I} U_i \to \mathcal{X} \) be an effective epimorphism and \( j_i^*\mathcal{F} \) be a constant sheaf on \( U_i \) for every \( i \in I \).

**Notation 1.14.** Let us denote by \( \mathcal{O}(X) \) the (nerve of the) frame of open subsets \( U \subset X \) of a topological space \( X \) and let us denote by \( \mathcal{O}(\mathcal{X}) \), the frame of open subtoposes \( U \subset \mathcal{X} \) of an \( \infty \)-topos \( \mathcal{X} \). We shall also denote by \( \mathcal{E}(\mathcal{X}) \) the \( \infty \)-category of étale maps over \( \mathcal{X} \).

**Lemma 1.15.** The frame of open subtoposes of an \( \infty \)-topos \( \mathcal{X} \)

\[
\mathcal{O}(\mathcal{X}) \xrightarrow{\text{eff}} \mathcal{E}(\mathcal{X})
\]

is a left exact reflexive localisation of \( \mathcal{E}(\mathcal{X}) \).

**Proof.** Open subtoposes correspond to the \((-1)\)-truncated objects of \( \mathcal{E}(\mathcal{X}) \), it then is a reflexive subcategory of \( \mathcal{E}(\mathcal{X}) \) [6, 5.5.6.18]. Moreover, a morphism \( f \) in \( \mathcal{E}(\mathcal{X}) \) becomes invertible in \( \mathcal{O}(\mathcal{X}) \) if and only if \( f \) is an effective epimorphism [6, 6.2.3.5(1)]. Effective epimorphisms are stable under pull-backs [6, 6.2.3.15] and thus, the localisation functor preserves finite limits [6, 6.2.1.1]. \( \Box \)

**Lemma 1.16.** Let \( X \) be a topological space, with associated \( \infty \)-topos \( \mathcal{X} \) and hypercomplete subtopos \( \widehat{\mathcal{X}} \subset \mathcal{X} \).

Then, the maps sending an open \( U \subset X \) to the open subtoposes \( U \subset \mathcal{X} \) and \( \widehat{U} \subset \widehat{\mathcal{X}} \) induce equivalences

\[
\mathcal{O}(X) = \mathcal{O}(\mathcal{X}) = \mathcal{O}(\widehat{\mathcal{X}})
\]

between the frames of open subsets of \( X \), open subtoposes of \( \mathcal{X} \) and open subtoposes of \( \widehat{\mathcal{X}} \).

**Proof.** For every \( \infty \)-topos \( \mathcal{X} \), open subtoposes \( U \subset \mathcal{X} \) correspond to subsheaves \( 1_U \) of the terminal sheaf \( 1_{\mathcal{X}} \).

To every open \( U \subset X \) corresponds the characteristic sheaf \( 1_U \subset 1_X \) on \( X \) where \( 1_U(V) \) is punctual whenever \( V \subset U \) and is empty otherwise. Conversely, let \( \mathcal{F} \subset 1_X \) be a subsheaf of the terminal sheaf. Because \( \mathcal{F} \) is a sheaf, if \( \mathcal{F}(U) \) and \( \mathcal{F}(V) \) are both non-empty, the value \( \mathcal{F}(U \cup V) \) must also be non-empty. Thus there is a biggest open subset \( U \subset X \) for which \( \mathcal{F}(U) \) is non-empty. By direct inspection \( \mathcal{F} = 1_U \). We thus have \( \mathcal{O}(X) = \mathcal{O}(\mathcal{X}) \).

Since hypercompletion \( \text{Sh}(X) \to \text{Hyp}(X) \) is a left exact reflexive localisation functor, it induces a left exact reflexive localisation functor \( \mathcal{O}(\mathcal{X}) \to \mathcal{O}(\widehat{\mathcal{X}}) \). Lastly, since the terminal sheaf \( 1_X \) is truncated, every subsheaf is also truncated and thus hypercomplete [6, 6.5.1.14]. So, \( \mathcal{O}(\mathcal{X}) = \mathcal{O}(\widehat{\mathcal{X}}) \). \( \Box \)
Lemma 1.17. Let $X$ be a topological space with associated $\infty$-topos $\mathcal{X}$. Then for every étale map $\mathcal{V} \to \mathcal{X}$, there exists an effective epimorphism

$$\coprod_{i \in I} \mathcal{U}_i \to \mathcal{V}$$

over $\mathcal{X}$ where $I$ is a small set and each $\mathcal{U}_i$ is an open subtopos of $\mathcal{X}$.

Proof. By construction of $\mathcal{X}$, the open subtoposes $\mathcal{U} \subset \mathcal{X}$ form a dense subcategory of the $\infty$-category of étale maps over $\mathcal{X}$. Thus given any étale map $\mathcal{V} \to \mathcal{X}$, the canonical map

$$\coprod_{U \subset X} \mathcal{U} \to \mathcal{V}$$

over $\mathcal{X}$ is an effective epimorphism. Indeed, this can be checked using injectivity of the pullback of subobjects [6, 6.2.3.10]. Since subobjects of coproducts can be identified with products of subobjects [6, 6.2.3.9], we may start by letting let $\mathcal{Y}, \mathcal{Z} \subset \mathcal{V}$ be two open subtoposes of $\mathcal{V}$ such that $\mathcal{Y} \times_{\mathcal{V}} \mathcal{U} = \mathcal{Z} \times_{\mathcal{V}} \mathcal{U}$ for every $\mathcal{U} \to \mathcal{V}$ over $\mathcal{X}$. Then

$$\mathcal{Y} = \mathcal{Y} \times_{\mathcal{V}} \mathcal{V} = \mathcal{Y} \times_{\mathcal{V}} \lim_{\mathcal{U} \to \mathcal{V}} \mathcal{U}$$

(by density)

$$= \lim_{\mathcal{U} \to \mathcal{V}} \mathcal{Y} \times_{\mathcal{V}} \mathcal{U}$$

(by universality of colimits)

$$= \lim_{\mathcal{U} \to \mathcal{V}} \mathcal{Z} \times_{\mathcal{V}} \mathcal{U}$$

(by assumption)

$$= \mathcal{Z}$$

(by symmetry)

one gets $\mathcal{Y} = \mathcal{Z}$. \qed

Proposition 1.18 (Translation). Let $X$ be a topological space, with associated $\infty$-topos $\mathcal{X}$ and hypercomplete subtopos $\mathcal{X} \subset X$.

Then,

$$\mathcal{S}\mathcal{h}_{\text{loc}}(X) = \mathcal{S}\mathcal{h}_{\text{loc}}(\mathcal{X})$$

locally constant sheaves on $X$ are the locally constant sheaves on $\mathcal{X}$.

Likewise,

$$\mathcal{H}\mathcal{ypr}_{\text{loc-hyp}}(X) = \mathcal{S}\mathcal{h}_{\text{loc}}(\mathcal{X})$$

locally hyperconstant hypersheaves on $X$ are the locally constant sheaves on $\mathcal{X}$.

Proof. Since open embeddings are étale maps, it is clear that locally constant sheaves on $X$ are locally constant on $\mathcal{X}$.

Let us look at the reverse direction. Let $\mathcal{F}$ be a locally constant sheaf on $\mathcal{X}$ and let $\coprod_{i \in I} \mathcal{V}_i \to \mathcal{X}$ be an étale effective epimorphism such that the pullback of $\mathcal{F}$ to each $\mathcal{V}_i$ is constant.

By the previous lemma, for every $i \in I$ there exists a covering $\coprod_{j \in J} \mathcal{U}_{ij} \to \mathcal{V}_i$ over $\mathcal{X}$, by open subtoposes of $\mathcal{X}$. As small coproducts of effective epimorphisms are again effective epimorphisms [6, 6.2.3.11] and the composition of two effective epimorphisms is again an epimorphism [6, 6.2.3.12], we get an effective epimorphism

$$\coprod_{i \in I} \coprod_{j \in J} \mathcal{U}_{ij} \to \coprod_{i \in I} \mathcal{V}_i$$

over $\mathcal{X}$. Then

$$\coprod_{i \in I} \mathcal{U}_i \to \mathcal{V}$$

is an effective epimorphism. Indeed, this can be checked using injectivity of the pullback of subobjects [6, 6.2.3.10]. Since subobjects of coproducts can be identified with products of subobjects [6, 6.2.3.9], we may start by letting let $\mathcal{Y}, \mathcal{Z} \subset \mathcal{V}$ be two open subtoposes of $\mathcal{V}$ such that $\mathcal{Y} \times_{\mathcal{V}} \mathcal{U} = \mathcal{Z} \times_{\mathcal{V}} \mathcal{U}$ for every $\mathcal{U} \to \mathcal{V}$ over $\mathcal{X}$. Then

$$\mathcal{Y} = \mathcal{Y} \times_{\mathcal{V}} \mathcal{V}$$

$$= \mathcal{Y} \times_{\mathcal{V}} \lim_{\mathcal{U} \to \mathcal{V}} \mathcal{U}$$

(by density)

$$= \lim_{\mathcal{U} \to \mathcal{V}} \mathcal{Y} \times_{\mathcal{V}} \mathcal{U}$$

(by universality of colimits)

$$= \lim_{\mathcal{U} \to \mathcal{V}} \mathcal{Z} \times_{\mathcal{V}} \mathcal{U}$$

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one gets $\mathcal{Y} = \mathcal{Z}$. \qed

Proposition 1.18 (Translation). Let $X$ be a topological space, with associated $\infty$-topos $\mathcal{X}$ and hypercomplete subtopos $\mathcal{X} \subset X$.

Then,

$$\mathcal{S}\mathcal{h}_{\text{loc}}(X) = \mathcal{S}\mathcal{h}_{\text{loc}}(\mathcal{X})$$

locally constant sheaves on $X$ are the locally constant sheaves on $\mathcal{X}$.

Likewise,

$$\mathcal{H}\mathcal{ypr}_{\text{loc-hyp}}(X) = \mathcal{S}\mathcal{h}_{\text{loc}}(\mathcal{X})$$

locally hyperconstant hypersheaves on $X$ are the locally constant sheaves on $\mathcal{X}$.
covering \(X\). As the pullback of a constant sheaf is again a constant sheaf, the restriction of \(F\) to each \(U_{ij}\) is constant. Finally, since effective epimorphisms are preserved by left exact functors, this one is sent to an effective epimorphism in \(\mathcal{O}(X)\) [1.15] which is just \(\mathcal{O}(X)\) [1.16], that is \(\bigcup_{i \in I} U_{ij}, U_{ij} = X\).

Now for \(\tilde{X}\). Since it is a subtopos of \(X\), every étale map with codomain \(\tilde{X}\) is of the form \(\tilde{V} \to \tilde{X}\) with \(V \to X\) an étale map. If \(\bigcup_{i \in I} \tilde{V}_i \to \tilde{X}\) is an effective epimorphism, then as hypercompletion is left exact, we get an effective epimorphism
\[
\bigcup_{i \in I} \bigcup_{j \in J} U_{ij} \to \bigcup_{i \in I} \tilde{V}_i \to \tilde{X}
\]
by the same argument as above. In addition to the previous arguments, we add that for any open \(U \subset X\), constant sheaves on \(\tilde{U}\) correspond to hyperconstant sheaves on \(U\).

**Remark 1.19** (Terminology). Since locally hyperconstant hypersheaves on \(X\) are the locally constant sheaves on \(\tilde{X}\), it has naturally led Haine to call them ‘locally constant hypersheaves’. We had to change this terminology in order to distinguish between hypersheaves that are locally hyperconstant and hypersheaves that are locally constant (as sheaves).

### 1.4 (Hyper)constructible (hyper)sheaves

Continuing with the two possible functorialities, we shall obtain constructible sheaves and hyperconstructible hypersheaves.

**Definition 1.20** (Constructible sheaves). A sheaf of spaces \(F\) on an \(A\)-stratified space \(X\) is said to be \(A\)-constructible if its restriction \(s^*_a F\) to each stratum \(X_a\) is locally constant for every \(a \in A\).

We shall denote by \(\text{Sh}_A(X) \subset \text{Sh}(X)\) the full subcategory of \(A\)-constructible sheaves on \(X\).

**Remark 1.21.** By construction, given a stratified map \(f: X \to Y\) between an \(A\)-stratified space and a \(B\)-stratified space, the pullback functor
\[
\begin{array}{ccc}
\text{Sh}(Y) & \xrightarrow{f^*} & \text{Sh}(X) \\
\downarrow & & \downarrow \\
\text{Sh}_B(Y) & \xrightarrow{f^*} & \text{Sh}_A(X)
\end{array}
\]
preserves constructible sheaves.

**Definition 1.22** (Hyperconstructible hypersheaves). A sheaf \(F\) on \(X\) shall be called \(A\)-hyperconstructible if the hyperrestriction \(s^*_a F\) is locally hyperconstant for every \(a \in A\).

We shall denote by \(\text{Hyp}_{A,\text{hyp}}(X) \subset \text{Hyp}(X)\) the full subcategory of \(A\)-hyperconstructible hypersheaves.

**Remark 1.23** (Terminology). Following a previous remark [1.19] for the terminology about locally constant hypersheaves, what we have chosen to call hyperconstructible hypersheaves are the constructible hypersheaves of Haine. We had to change the terminology in order to distinguish between hypersheaves that are hyperconstructible and those that are constructible (as sheaves).
Warning 1.24. Here again $\text{Hyp}_{A,\text{hyp}}(X) \not\subseteq \text{Sh}_A(X)$, a priori.

Remark 1.25. By construction, given a stratified map $f: X \to Y$ between an $A$-stratified space and a $B$-stratified space, the hyperpullback functor

$$\begin{array}{ccc}
\text{Hyp}(Y) & \xrightarrow{\sim} & \text{Hyp}(X) \\
\downarrow & & \downarrow \\
\text{Hyp}_{B,\text{hyp}}(Y) & \xrightarrow{\sim} & \text{Hyp}_{A,\text{hyp}}(X)
\end{array}$$

preserves hyperconstructible hypersheaves.

The main distinctive feature of hyperconstructible hypersheaves is their invariance under stratified homotopy equivalences. It makes hyperconstructible hypersheaves the natural $\infty$-analogue of the usual theory of locally constant sheaves and constructible sheaves with values in sets.

Theorem 1.26 [5, 2.3]. An $A$-stratified homotopy equivalence $f: X \to Y$ between two $A$-stratified spaces induces an equivalence

$$\text{Hyp}_{A,\text{hyp}}(Y) \xrightarrow{\sim} \text{Hyp}_{A,\text{hyp}}(X)$$

between their $\infty$-categories of $A$-hyperconstructible hypersheaves.

1.5 Constructible hypersheaves

We have warned the reader that one does not have an inclusion $\text{Hyp}_{A,\text{hyp}}(X) \subset \text{Sh}_A(X)$ due to the different functorialities between constructibility and hyper-constructibility.

They are in fact related by a correspondence

$$\begin{array}{ccc}
\text{Hyp}_A(X) & \xrightarrow{\sim} & \text{Hyp}_{A,\text{hyp}}(X) \\
\downarrow & & \downarrow \\
\text{Sh}_A(X) & \xrightarrow{\sim} & \text{Hyp}_{A,\text{hyp}}(X)
\end{array}$$

via the $\infty$-category of $A$-constructible hypersheaves. This third $\infty$-category shall become the may object of study in this article. The left inclusion is obvious, the right one requires a lemma.

Lemma 1.27. Let $X$ be a topological space. Then

$$\text{Hyp}_{\text{loc}}(X) \subset \text{Hyp}_{\text{loc-hyp}}(X)$$

locally constant hypersheaves are locally hyperconstant.

More generally, if $X$ is $A$-stratified, then

$$\text{Hyp}_A(X) \subset \text{Hyp}_{A,\text{hyp}}(X)$$

$A$-constructible hypersheaves on $X$ are $A$-hyperconstructible.
Proof. Assume $\mathcal{H}$ be a locally constant constant hypersheaf. Then there exists an open covering $j_i : U_i \subset X$ such that $j_i^* \mathcal{H}$ be a constant sheaf. But since $j_i$ is an open embedding, $j_i^* \mathcal{H}$ is a hypersheaf. A sheaf which is both constant and a hypersheaf is hyperconstant. constant if and only if By construction, the hypercompletion of a constant sheaf is a constant hypersheaf, so $j_i^* \mathcal{F}$ is a constant hypersheaf and $\mathcal{F}$ is hyperlocally constant.

The case of an $A$-constructible sheaf now follows from the functoriality of hypercompleted pullbacks [1.6]: for every $a \in A$, by assumption $s_a^* \mathcal{H}$ is locally constant, so let $j_{a,i} : U_i \subset X_a$ be an open covering so that $j_{a,i}^* s_a^* \mathcal{F}$ is constant. Then $(s_a j_{a,i})^* \mathcal{H} = j_{a,i}^* s_a^* \mathcal{H}$ is a constant hypersheaf. \[\square\]

Remark 1.28. There is another correspondence relating constructible sheaves and hyperconstructible hypersheaves,

\[
\begin{array}{ccc}
\text{Sh}_A(X) & \overset{\psi_X}{\longrightarrow} & \text{Hyp}_{A,\text{hyp}}(X) \\
\downarrow & & \downarrow \\
\text{Sh}_{A,\text{hyp}}(X)
\end{array}
\]

it is the $\infty$-category of hyperconstructible sheaves. But we shall not use it.

2 COINCIDENCES

In this section we shall show that in some general cases, the $\infty$-categories of hyperconstructible hypersheaves, constructible hypersheaves and constructible sheaves, actually coincide.

2.1 Stratum case

We start with the case of a single stratum. Lurie has introduced the notion of topological space ‘locally of singular shape’ [1, A.4.15]. These are spaces $X$ for which the counit map $|\text{Sing}(U)| \to U$ is a shape equivalence for every open $U \subset X$. Letting $\pi_X : X \to \ast$ denote the projection to the point, if $X$ is locally of singular shape then the singular sheaf $\pi_X^* \text{Sing}(X)$ admits a canonical global section $1_X \to \pi_X^* \text{Sing}(X)$. This canonical section allows the definition of a functor

\[
\begin{array}{ccc}
S/\text{Sing}(X) & \overset{\psi_X}{\longrightarrow} & \text{Sh}(X) \\
\downarrow & & \downarrow \\
K & \longrightarrow & \pi_X^* K \times_{\pi_X^* \text{Sing}(X)} 1_X
\end{array}
\]

which is fully faithful and whose image is equivalent to the subcategory of locally constant sheaves on $X$ [1, A.2.15].

When $X$ is locally of singular shape, the pullback functor $\pi_X^*$ admits a left adjoint $(\pi_X)_!$ [1, A.2.8] and so does $\psi_X$. In particular, $\psi_X$ preserves all small limits; it also preserves all small colimits.

Proposition 2.1 (Coincidence, stratum case). Let $X$ be a space which is locally of singular shape. Then

$$\text{Hyp}_{\text{loc-hyp}}(X) = \text{Hyp}_{\text{loc}}(X) = \text{Sh}_{\text{loc}}(X)$$

locally hyperconstant hypersheaves are locally constant and locally constant sheaves are hypersheaves.
Proof. Since truncation towers converge in $S/Sing(X)$ and $\psi_X$ commutes with limits, all locally constant sheaves on $X$ are hypersheaves [1, A.2.17]. Since every open $U \subset X$ is again locally of constant shape, the notions of constant sheaves and hyperconstant sheaves on $U$ coincide. By ripple effect, locally hyperconstant hypersheaves on $X$ are locally constant. 

Being locally of singular shape also gives locally constant sheaves limits and colimits, they are computed as in the $\infty$-category of sheaves.

**Lemma 2.2.** Let $X$ be locally of singular shape. The $\infty$-category $\text{Sh}_{\text{loc}}(X)$ admits all small limits and colimits. In addition, the inclusion $\text{Sh}_{\text{loc}}(X) \subset \text{Sh}(X)$ preserves small limits and small colimits.

Proof. The slice $\infty$-category $S/Sing(X)$ has all small limits and colimits and $\psi_X$ preserves small limits and small colimits.

2.2 Conical case

Let $X$ be an $A$-stratified space. The first thing one can ask of $X$ to have some coincidence theorem is that each of $X$ be locally of singular shape. This is enough, for example, to guarantee that one can compute finite limits and small colimits of $A$-constructible sheaves on $X$.

**Lemma 2.3.** Let $X$ be an $A$-stratified space. Assume that the stratum $X_a \subset X$ is locally of singular shape for each $a \in A$. The $\infty$-category of $A$-constructible sheaves on $X$ admits finite limits and small colimits. Moreover, the inclusion $\text{Sh}_A(X) \subset \text{Sh}(X)$ preserves finite limits and small colimits.

Proof. For each $a \in A$, the functor $s_a^*: \text{Sh}(X) \to \text{Sh}(X_a)$ preserves small colimits and finite limits. Since $\text{Sh}_{\text{loc}}(X_a) \subset \text{Sh}(X_a)$ preserves all small limits and colimits, it follows that a finite limit or a small colimit of $A$-constructible sheaves on $X$ is again $A$-constructible.

But it is not enough to guarantee any coincidence between the different notions of constructibility. Indeed, one needs to add a gluing assumption of the strata together.

The one we shall use here is the conicality introduced by Lurie [1, A.5.5]; it is a less demanding condition than most other stratification hypothesis used in topology. A space is conically stratified when each point admits a coordinate decomposition with on one side, a local coordinate dependant on the stratum and on the other side a radial coordinate describing a neighbourhood of the point around the stratum.

**Definition 2.4 (Open cone).** For a topological space $X$, the open cone of $X$ is the set

$$C(X) := \{0\} \sqcup (R^*_+ \times X)$$

with topology defined as follows: A subset $U \subset C(X)$ is open if and only if $U \cap (R^*_+ \times X)$ is open, and if $0 \in U$, then $(0, \varepsilon) \times X \subset U$ for some positive real number $\varepsilon$.

If $X$ is stratified over a poset $A$, then $C(X)$ is naturally stratified over the poset $A^*$ obtained from $A$ by adding a new element smaller than every other element of $A$.  

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**Warning 2.5.** One should not confuse the cone on $X$ with the collapsed rectangle defined as the quotient $\mathbb{R}_+ \times X / \{0\} \times X$. When $X$ is compact and separated, the cone on $X$ and the collapsed rectangle on $X$ are homeomorphic. This is no longer true in the general case: the cone on the open interval $(0, 1)$ can be embedded in $\mathbb{R}^2$, whereas the collapsed rectangle on $(0, 1)$ is not metrizable.

If $(X, d)$ is a metric space, the topology of the cone $C(X)$ is metrizable by letting $d((\lambda, x), (\mu, y)) = \max(|\lambda - \mu|, d(x, y))$ and by adding $d((0, \lambda, x)) = \lambda$.

**Definition 2.6 (Conically stratified space).** Let $f : X \to A$ be a stratified topological space. Let $a \in A$ and let $U \subset X_a$ be an open subset. We shall say that an open $V \subset X$ is a conical extension of $U$ if there exists a stratified space $L$ over $A_{a<}$ such that $V$ is homeomorphic to $U \times C(L)$ over the poset map $A_{a<} = A_{a\leq} \subset A$. We shall say that $X$ is conically $A$-stratified if for every $a \in A$ every point $x \in X_a$ admits an open neighbourhood in $X_a$ that can be conically extended to $X$.

**Argument 2.7 (Reduction).** Let us gather some properties of conically stratified spaces that shall be used as a core reduction argument in the proofs.

- If a conically $A$-stratified space $X$ is paracompact, then each stratum $X_a$ can be covered with opens $U$ admitting a paracompact conical extension [1, A.5.16]. As a consequence, for any local problem on paracompact $X$, one can assume that $X = X_a \times C(L)$;

- In a paracompact space $X_a \times C(L)$, the closed subspace $X_a \subset X_a \times C(L)$ is also paracompact, thus $F_{a}$ open subsets $W \subset X_a$ form a basis of paracompact open subsets, stable under intersection [6, 7.1.1.1]. Moreover, if $W \subset X_a$ is an open $F_{a}$, then $W \times C(L) \subset X_a \times C(L)$ is again an open $F_{a}$ of $X_a \times C(L)$ and is again paracompact;

- For a paracompact $W$, the closed subspace $W \subset W \times C(L)$ admits a basis of open neighbourhoods $W \subset V \subset W \times C(L)$ all of which are homeomorphic to $W \times C(L)$ as stratified spaces [1, A.5.12];

- Combining the above arguments, every point $x \in X$ admits a basis of conical open neighbourhoods $V_x \equiv U_x \times C(L)$.

**Proposition 2.8 (Coincidence, conical case).** Let $X$ be a paracompact $A$-stratified space, such that each stratum $X_a$ be locally of singular shape, for $a \in A$. Then, for
every $A$-hyperconstructible sheaf $\mathcal{H}$ on $X$, the canonical map $s_a^*\mathcal{H} \to \hat{s}_a^*\mathcal{H}$ is an equivalence on $X_a$, for each $a \in A$ and thus

$$\text{Hyp}_{A\text{-hyp}}(X) = \text{Hyp}_A(X)$$

A-hyperconstructible hypersheaves on $X$ are $A$-constructible.

**Proof.** Constructible hypersheaves are hyperconstructible [1.27]. Also, since each stratum $X_a$ is locally of singular shape, for $a \in A$, locally hyperconstant hypersheaves on $X_a$ coincide with locally constant sheaves on $X_a$ [2.1]. It shall then be enough to show that for every $A$-hyperconstructible hypersheaf $\mathcal{H}$ the canonical map $a : s_a^*\mathcal{H} \to \hat{s}_a^*\mathcal{H}$ is an equivalence on $X_a$.

This question is local on $X_a$, so by the reduction arguments for conically stratified space [2.7], we can reduce to the case were $X_a = X_a \times C(L)$. Continuing the reduction, it is enough to show that $\alpha(W)$ is an equivalence on each $F_\sigma$-open subset $W \subset X_a$ since this is a basis stable under intersection. For each such $W$, $W \times C(L)$ is again paracompact and we can thus reduce to show that $\alpha(W)$ is an equivalence in the case where $X = W \times C(L)$.

Now because $X$ is paracompact, $s_a^*\mathcal{H}(W) = \lim_{W \subset V} \mathcal{H}(V) [6, 7.1.5.6]$. Because $W$ is paracompact, the neighbourhoods $V$ can be taken homeomorphic to $W \times C(L)$ as stratified spaces [2.7]. In such a case, by homotopy invariance [1.26], the restriction map $\mathcal{H}(V) \to \hat{s}_a^*\mathcal{H}(W)$ is an equivalence, from which we finally get that $\alpha(W)$ is an equivalence. \qed

### 2.3 $\mathcal{D}$-spaces

We have seen that on conically $A$-stratified spaces, hyperconstructibility coincides with constructibility for hypersheaves. In order to add coincidence with constructible sheaves, as in the stratum case, one needs to add an assumption on the poset allowing induction on depth. Namely, one needs to assume that the poset satisfy the ascending chain condition.

Adding the ascending chain condition to the poset $A$, we get a type of spaces that shall become the building brick of the next construction; we thus give it a name.

**Definition 2.9** (D-space). A $\mathcal{D}$-space (for ‘good’ depth stratified space) is an $A$-stratified space $X$ such that

- $X$ is paracompact;
- the stratum $X_a \subset X$ is locally of singular shape, for each $a \in A$;
- $X$ is conically $A$-stratified;
- $A$ satisfies the ascending chain condition.

**Remark 2.10.** Any $C^0$-stratified space in the sense of Ayala-Francis-Tannaka is a $\mathcal{D}$-space [7, 2.1.15].

**Proposition 2.11** (Coincidence, $\mathcal{D}$-space case). Let $X \rightarrow A$ be a $\mathcal{D}$-space, then

$$\text{Hyp}_{A\text{-hyp}}(X) = \text{Hyp}_A(X) = \text{Sh}_A(X)$$

$A$-constructible sheaves on $X$ are hypersheaves and $A$-hyperconstructible hypersheaves are $A$-constructible.

**Proof.** The first equality follows from the conical case [2.8]. The second can be proven by induction on the depth of $A$ [1, A.5.9]. \qed
2.4 Colimit and conical $\mathcal{D}_\omega$-spaces

We now turn to the main object of study: a class of stratified spaces on which to extend the representation theorem. The idea is to consider to those posets $A$ which do not satisfy the ascending chain condition but which can be obtained as a countable union of closed subposets satisfying the ascending chain condition.

**Definition 2.12 ($\mathcal{D}_\omega$-space).** A $\mathcal{D}_\omega$-space is a stratified space $X \to A$ such that

- $X$ is paracompact;
- $A$ is $\omega$-stratified;
- $X_{\leq n} \to A_{\leq n}$ is a $\mathcal{D}$-space for every $n < \omega$.

We shall say that $X \to A \to \omega$ is a **conical $\mathcal{D}_\omega$-space** if $X$ is conically $A$-stratified; **colimit $\mathcal{D}_\omega$-space** if $X$ coincides with the colimit $X_{<\omega} := \lim_{n<\omega} X_{\leq n}$.

One may legitimately ask: why divide $\mathcal{D}_\omega$-spaces into two categories? This is because the very topology of the cone is often incompatible with a colimit topology. For example, if $L$ is an $\omega$-stratified, then there is a continuous bijection $\lim_{n<\omega} C(L_{\leq n}) \to C(L)$ which is not a homeomorphism in the general case. This leads to an impossibility theorem, where the two conditions become mutually exclusive.

**Remark 2.13.** Let $X_0 \hookrightarrow \cdots \hookrightarrow X_p \hookrightarrow \cdots$ be a sequence of closed embeddings between $T_1$ topological spaces and let $X$ denote its colimit. Then every morphism $K \to X$ with $K$ compact factors through one $X_p \subset X$ [8, 2.4.2].

As a consequence, a sequence $(x_n)_{n \in \mathbb{N}}$ in $X$, converges only if it is bounded.

**Theorem 2.14 (Impossibility).** Assume $X$ be an $A$-stratified space such that

- $X$ be $T_1$ and not empty;
- $X_{\leq a} \subset X$ have empty interior for each $a \in A$;
- $X = \lim_{a \in A} X_{\leq a}$;
- $A$ contain an ascending chain,

then $X$ is not conically stratified.

**Proof.** One can assume that $A = \omega$ and that $X_0$ is not empty without loss of generality. Let $x \in X_{\leq 0}$, if $X$ is conically stratified, then there exists $Z$ and $Y$ such that $Z \times C(Y)$ is stratifiedly homeomorphic to an open neighbourhood $U_x$ of $x$. Since each $X_{\leq n}$ has empty interior, $U_x$ is not contained in any of them. One can thus find a sequence in $U_x$ whose image in $\omega$ is strictly increasing. Let $(y_n)$ be the corresponding coordinate in $Y$ of this sequence. Then for any $z \in Z$, the sequence $(z, (\lambda_n, y_n))$ converges to $(z, 0)$ in $Z \times C(Y)$ for any sequence $\lambda_n \to 0$ but cannot converge in $X$ since it is not bounded in the stratification [2.13].

Nevertheless, conical and colimit $\mathcal{D}_\omega$-spaces are very close; one can always change the global topology of a conical $\mathcal{D}_\omega$-space to make it become a colimit $\mathcal{D}_\omega$-space.
Lemma 2.15. Let $X \rightarrow A \rightarrow \omega$ be a $\mathbb{D}_\omega$-space. Then $X_{<\omega} \rightarrow A \rightarrow \omega$ is a colimit $\mathbb{D}_\omega$-space. Moreover, if $Y \rightarrow X$ is a continuous map making $Y \rightarrow X \rightarrow A \rightarrow \omega$ into a $\mathbb{D}_\omega$-space, then one gets a continuous map $Y_{<\omega} \rightarrow X_{<\omega}$ over $A$.

Proof. The only non-trivial thing to check is the paracompactness. But since by hypothesis each $X_{<n}$ is paracompact, the space $X_{<\omega}$ is built as a sequential colimit of closed embeddings of paracompact spaces and is thus paracompact [9, 8.2].

Lemma 2.16. Let $X \rightarrow A \rightarrow \omega$ be a $\mathbb{D}_\omega$-space. For each $n<\omega$, let us denote by $j(n)$ the closed stratified embedding $X_{<n} \subset X$. Then for each $A_{<n}$-constructible sheaf $\mathcal{F}$ on $X_{<n}$,

$$\mathcal{F} \in \text{Sh}_{A_{<n}}(X_{<n}) \iff j(n).\mathcal{F} \in \text{Hyp}_A(X)$$

its pushforward $j(n)_*\mathcal{F}$ is an $A$-constructible hypersheaf on $X$.

Proof. Let $a \in A$. Either $X_a \cap X_{<n} = \emptyset$, in which case $s_a^j(n)_*\mathcal{F}$ is the initial sheaf by proper base change [6, 7.3.2.13], or $X_a \subset X_{<n}$, in which case one has $s_a^j(n)_*\mathcal{F} = s_a^*\mathcal{F}$ again by proper base change and is locally constant by hypothesis.

Since by assumption $X_{<n}$ is a $\mathbb{D}$-space, $\mathcal{F}$ is a hypersheaf [2.11] and as pushforwards preserve hypersheaves, $j(n)_*\mathcal{F}$ is also a hypersheaf. □

Theorem 2.17 (Dévissage, colimit case). Let $X \rightarrow A \rightarrow \omega$ be a colimit $\mathbb{D}_\omega$-space. The inclusion maps $j(n): X_{<n} \subset X$ induce an adjunction

$$
\begin{array}{ccc}
\text{Sh}(X) & \xleftarrow{j^*} & \text{lim}_{n<\omega} \text{Sh}(X_{<n}) \\
\downarrow{j_*} & & \downarrow{\longrightarrow}
\end{array}
$$

which is an equivalence of $\infty$-categories and which reduces to an equivalence

$$
\text{Sh}_A(X) = \text{lim}_{n<\omega} \text{Sh}_{A_{<n}}(X_{<n})
$$

between the $\infty$-category of $A$-constructible sheaves on $X$ and the inverse limit of the $\infty$-categories of constructible sheaves on each $X_{<n}$. Moreover,

$$
\text{Hyp}_A(X) = \text{Sh}_A(X)
$$

$A$-constructible sheaves on $X$ are hypersheaves.

Proof. The fact that $j^* \dashv j_*$ is an equivalence of $\infty$-categories follows from the fact that $X$ is the colimit of a sequence of closed embeddings of paracompact spaces [6, 7.1.5.8]. Then the restriction to the subcategories of constructible sheaves follow directly. Finally, since since $X_{<n}$ is a $\mathbb{D}$-space for every $n<\omega$, all $A_{<n}$-constructible sheaves on $X_{<n}$ are hypersheaves [2.11] and as limits of hypersheaves are again hypersheaves, we get that all $A$-constructible sheaves on $X$ are hypersheaves. □

This theorem for colimit $\mathbb{D}_\omega$-spaces, together with the coincidence proposition for conically stratified spaces shall let us see that constructible hypersheaves (which are a priori not functorial) inherit the functoriality of hyperconstructible hypersheaves. In particular it shall also inherit its homotopy invariance.
Corollary 2.18 (Functoriality of constructible hypersheaves). Let \( X \to A \to \omega \) be a colimit or a conical \( \mathcal{D}_\omega \)-space and let \( Y \to B \) be a stratified space. Let \( f : X \to Y \) be a stratified map. Then for any \( B \)-constructible hypersheaf \( \mathcal{H} \)

\[
\mathcal{H} \in \text{Hyp}_B(Y) \iff \hat{f}^*\mathcal{H} \in \text{Hyp}_A(X)
\]

its hyperpullback \( \hat{f}^*\mathcal{H} \) is \( A \)-constructible.

In particular, if \( f \) is a stratified homotopy equivalence between two conical or colimit \( \mathcal{D}_\omega \)-spaces, then \( \hat{f}^* \) induces an equivalence between the \( \infty \)-categories of constructible hypersheaves.

Proof. When \( X \) is a conical \( \mathcal{D}_\omega \)-space: then since \( \mathcal{H} \) is a \( B \)-constructible hypersheaf, it is \( B \)-hyperconstructible [1.27]. The hyperpullback \( \hat{f}^*\mathcal{H} \) is then \( A \)-hyperconstructible. But since \( X \) is conically stratified, it is also \( A \)-constructible by coincidence [2.8].

When \( X \) is a colimit \( \mathcal{D}_\omega \)-space, then \( f^*\mathcal{H} \) is \( A \)-constructible because \( \mathcal{H} \) is \( B \)-constructible. By the previous theorem \( f^*\mathcal{H} \) is then a hypersheaf and so, the canonical map \( f^*\mathcal{H} \to \hat{f}^*\mathcal{H} \) is an equivalence of sheaves on \( X \).

As a consequence, when restricted to colimit and conical \( \mathcal{D}_\omega \)-spaces, constructible hypersheaves form a subfunctor of the functor of hyperconstructible hypersheaves [2.8]. It is thus invariant under stratified homotopy equivalences [1.26]. \(\square\)

Theorem 2.19 (Dévissage, conical case). Let \( X \to A \to \omega \) be a \( \mathcal{D}_\omega \)-space and assume that \( X \) is either a conical or a colimit \( \mathcal{D}_\omega \)-space. The inclusion maps \( j(n) : X_{\leq n} \subset X \) induce an adjunction

\[
\begin{array}{ccc}
\text{Sh}(X) & \xleftarrow{j^*} & \lim_{n<\omega} \text{Sh}(X_{\leq n}) \\
\xrightarrow{j_*} & & \\
\end{array}
\]

which reduces to an equivalence

\[
\text{Hyp}_A(X) = \lim_{n<\omega} \text{Sh}_{A_{\leq n}}(X_{\leq n})
\]

between the \( \infty \)-category of \( A \)-constructible hypersheaves on \( X \) and the inverse limit of the \( \infty \)-categories of constructible sheaves on each \( X_{\leq n} \).

Proof. Using the dévissage theorem for the colimit case, we shall identify the right hand side of the equivalence with the \( \infty \)-category of \( A \)-constructible sheaves on \( X_{<\omega} \) and see the adjunction \( j^* \dashv j_* \) as stemming from the canonical map \( j : X_{<\omega} \to X \). For a sheaf \( \mathcal{F} \) on \( X_{<\omega} \), we shall denote by \( \mathcal{F}_{\leq n} \) its restriction to \( X_{\leq n} \) for each \( n < \omega \) and by \( \mathcal{F}_{\leq a} \) its restriction to \( X_{\leq a} \) for each \( a \in A \).

We shall start by showing that for every \( A \)-constructible sheaf \( \mathcal{F} \) on \( X_{<\omega} \) the map \( \psi_{\mathcal{F}} : s_a^*j_*\mathcal{F} \to \mathcal{F}_{\leq a} \) is an equivalence for each \( a \in A \). This is of local nature, thus by the reduction arguments [2.7], one can assume that \( X = X_a \times C(L) \). Continuing the reduction, it shall then suffice to show that \( \psi(W) \) is an equivalence for every \( F_a \) open subset \( W \subset X_a \), since this is a basis stable under intersection. For each such \( W \), \( W \times C(L) \) is again paracompact and we thus reduce to show that \( \psi(X_a) \) is an equivalence.

We shall prove by induction on \( k < \omega \) that for every \( \mathcal{F} \) the map \( \psi_{\mathcal{F}}(X_k) \) is \( k \)-connective. For the case \( k = 0 \), the canonical map

\[
\pi_0 j_*\mathcal{F}(X) = \pi_0 \left( \lim_{n<\omega} j_*(\mathcal{F}_{\leq n}(X)) \right) \longrightarrow \lim_{n<\omega} \pi_0 j_*(\mathcal{F}_{\leq n}(X))
\]
We may thus reduce to the case where the theorem in the conical case \[2.8\], we see that for a conically stratified deformation retract \(X \times C(L) \to X_a\) gives us \(j(n)_\ast \mathcal{F}_\omega(X) = \mathcal{F}_a(X_a)\). We deduce that \(\pi_0 j_\ast \mathcal{F}(X) \to \pi_0 \mathcal{F}_a(X_a)\) is surjective, and thus that \(\pi_0 \psi_\mathcal{F}(X_a)\), through which the surjection factors, is surjective.

Assume that we have shown that \(\psi_\mathcal{F}(X_a)\) is \(k\)-connective for every \(A\)-constructible \(\mathcal{F}\), for some \(k < \omega\). To show that \(\psi_\mathcal{F}(X_a)\) is \((k + 1)\)-connective, it is equivalent to show that for every section \(\eta \in \mathcal{F}_a(X_a)\), the induced map

\[
\ast \times j_\ast \mathcal{F}(X_a) \ast \xrightarrow{\psi'} \ast \times \mathcal{F}(X_a) \ast
\]

is \(k\)-connective. Because \(X\) is paracompact and \(X_a \subset X\) is a closed embedding, there exists an open neighbourhood \(V\) of \(X_a\) in \(X\) on which \(\eta\) can be extend to a section \(\overline{\eta}\) \([6, 7.1.5.5]\). Since \(X_a\) is paracompact, shrinking \(V\) if necessary, one can assume that \(V\) is homeomorphic to \(X_a \times C(L)\) as a stratified space \([2.7]\).

We may thus reduce to the case where \(\overline{\eta}\) is a global section of \(j_\ast \mathcal{F}\) on \(X\). The induced map \(1_{X_{\omega}} \to \mathcal{F}\) allows us to define \(\mathcal{G} := 1 \times \mathcal{F} 1\). The sheaf \(\mathcal{G}\) is again \(A\)-constructible since it is a finite limit of \(A\)-constructible sheaves \([2.3]\). By left exactness, one has \(\psi' = \psi_\mathcal{G}(X_a)\) which is then \(k\)-connective by the induction hypothesis.

We deduce that \(j_\ast \mathcal{F}\) is \(A\)-constructible. Since hypersheaves are stable under pushforwards and all \(A\)-constructible sheaves on \(X_{<\omega}\) are hypersheaves, \(j_\ast \mathcal{F}\) is an \(A\)-constructible hypersheaf.

So, \(j^\ast j_\ast \mathcal{F}\) is also \(A\)-constructible and thus, a hypersheaf. We also deduce that for every point \(x \in X_{<\omega}\), the counit map \((j^\ast j_\ast \mathcal{F})_x \to \mathcal{F}_x\) is an equivalence and thus that \(j^\ast j_\ast \mathcal{F} \to \mathcal{F}\) is an equivalence.

We now show that for every \(A\)-constructible hypersheaf \(\mathcal{H}\) on \(X\), the unit map \(v: \mathcal{H} \to j^\ast j_\ast \mathcal{H}\) is an equivalence. Since \(\mathcal{H}\) is a hypersheaf, one needs to show that \(v_x: \mathcal{H}_x \to (j^\ast j_\ast \mathcal{H})_x\) is an equivalence for every \(x \in X\). For this it is enough to show that \(v(V_x): \mathcal{H}(V_x) \to j^\ast j_\ast \mathcal{H}(V_x)\) is an equivalence for a basis of open subsets \(x \in V_x \subset X\). By the conical nature of \(X\), it is possible to select \(V_x \cong U_x \times C(L)\) \([2.7]\). We then reduce to showing that \(v(X)\) is an equivalence in the case where \(X = X_a \times C(L)\).

Let \(p\) be the unique natural number such that \(X_a \subset X_p\). Since

\[
j^\ast j_\ast \mathcal{H}(X_a \times C(L)) = \lim_{\longrightarrow \in \omega} \mathcal{H}_{\leq n}(X_a \times C(L_{\leq n}))
\]

it shall be enough to see that \(\mathcal{H}(X_a \times C(L)) \to \mathcal{H}_{\leq n}(X_a \times C(L_{\leq n}))\) is an equivalence for every \(n \geq p\). One has a commutative diagram of restriction maps

\[
\begin{array}{ccc}
\mathcal{H}(X_a \times C(L)) & \xrightarrow{\simeq} & \mathcal{H}_{\leq n}(X_a \times C(L_{\leq n})) \\
\downarrow \mathcal{s}_\ast \mathcal{H}(X_a) & & \downarrow \mathcal{s}_\ast \mathcal{H}(X_a) \quad [2.8]
\end{array}
\]

for which both of the vertical arrows are equivalences by homotopy invariance \([1.26]\). We conclude using the two-out-of-three property of equivalences.

**Remark 2.20.** Combining the two dévissage theorems with the coincidence theorem in the conical case \([2.8]\), we see that for a conically \(A\)-stratified \(D_\omega\)-space, one has the following coincidences

\[
\text{Hyp}_A^{\text{hyp}}(X) = \text{Hyp}_A(X) = \text{Hyp}_A(X_{<\omega}) = \text{Sh}_A(X_{<\omega})
\]
which can be extended with
\[ \text{Hyp}_{A\text{-hyp}}(X_{<\omega}) = \text{Hyp}_{A}(X_{<\omega}) \]
a coincidence between \( A \)-hyperconstructible hypersheaves on \( X_{<\omega} \) and \( A \)-constructible hypersheaves on \( X \). Contrarily to the conical case, this last equality might not happen systematically for colimit \( D_\omega \)-spaces. However, one can show that this is the case whenever the colimit \( D_\omega \)-space arises from a conical \( D_\omega \)-space.

3 REPRESENTATION VIA EXIT PATHS

In this section, we extend the representation theorem for constructible sheaves on \( D \)-spaces given by Lurie, to a representation theorem for constructible hypersheaves on colimit or conical \( D_\omega \)-spaces.

3.1 Exit paths \( \infty \)-category

**Definition 3.1.** Let \( p < \omega \), we shall view the topological simplex
\[ |\Delta^p| = \{(t_0,\ldots,t_p) \in [0,1]^{p+1} : t_0 + \cdots + t_p = 1\} \]
as a stratified space over the poset \( \{0 < \cdots < p\} \) with \( |\Delta^p|_{\leq i} \) being the set of tuples \( (t_0,\ldots,t_i,0,\ldots,0) \) for every \( i \leq p \).

Given a stratified space \( X \to A \), we let \( \text{Exit}_A(X) \) denote the simplicial set whose \( p \)-simplicies are the stratified maps \( |\Delta^p| \to X \), for every \( p < \omega \).

**Theorem 3.2 [1, A.6.4].** Let \( X \) be a conically \( A \)-stratified space. Then \( \text{Exit}_A(X) \) is an \( \infty \)-category.

**Proposition 3.3.** Let \( X \to A \) be a stratified space. Assume \( A \) be also stratified over a filtered poset \( \Lambda \) and assume that \( X_{\leq \lambda} \) be conically \( A_{\leq \lambda} \)-stratified space for each \( \lambda \in \Lambda \). Then, the simplicial set \( \text{Exit}_A(X) \) is an \( \infty \)-category and the canonical isomorphism of simplicial sets
\[ \lim_{\lambda \in \Lambda} \text{Exit}_{A_{\leq \lambda}}(X_{\leq \lambda}) = \text{Exit}_A(X) \]
exhibits \( \text{Exit}_A(X) \) as a colimit of the diagram \( \{\text{Exit}_{A_{\leq \lambda}}(X_{\leq \lambda})\}_{\lambda \in \Lambda} \) in the \( \infty \)-category of \( \infty \)-categories.

**Proof.** We notice that \( \text{Exit}_{A_{\leq \lambda}}(X_{\leq \lambda}) \to \text{Exit}_A(X) \) is a monomorphism for every \( \lambda \in \Lambda \), thus the canonical map \( \lim_{\lambda \in \Lambda} \text{Exit}_{A_{\leq \lambda}}(X_{\leq \lambda}) \to \text{Exit}_A(X) \) is also a monomorphism. It is an isomorphism because, as \( \Lambda \) is filtered, any map of posets \( \{0 < \cdots < n\} \to A \) must factor through \( A_{<\lambda} \) for some \( \lambda \in \Lambda \) and by definition, an \( n \)-simplex of \( \text{Exit}_A(X) \) is a stratified map \( |\Delta^n| \to X \), which must then factor through \( X_{\leq \lambda} \) for some \( \lambda \in \Lambda \).

To show that \( \text{Exit}_A(X) \) is an \( \infty \)-category, we need to check that it has the right lifting property

\[
\begin{array}{ccc}
\Lambda^n & \rightarrow & \text{Exit}_A(X) = \lim_{\lambda \in \Lambda} \text{Exit}_{A_{\leq \lambda}}(X_{\leq \lambda}) \\
\downarrow & & \downarrow \\
\Delta^n & \rightarrow & * \\
\end{array}
\]
with respect to inner horns inclusions. Because inner horns \( \Lambda^n_i \) are finite simplicial set for every \( n < \omega \) and every \( 0 < i < n \), and \( \Lambda \) is filtered, any map \( \Lambda^n_i \to \text{Exit}_A(X) \) must factor through \( \text{Exit}_{A \leq \lambda}(X_{\leq \lambda}) \) for some \( \lambda \in A \). By assumption \( X_{\leq \lambda} \) is conically \( A \leq \lambda \)-stratified and a lift to a map \( \Delta^n \to \text{Exit}_{A \leq \lambda}(X_{\leq \lambda}) \subset \text{Exit}_A(X) \) exists by the previous theorem.

Since marked simplicial sets form a simplicial model category \([6, 3.1.4.4]\), in order to show that \( \text{Exit}_A(X) \) is a colimit of the diagram \( \{ \text{Exit}_{A \leq \lambda}(X_{\leq \lambda}) \} \lambda \in \Lambda \) in the \( \infty \)-category of \( \infty \)-categories, it will be enough to show that \( \text{Exit}_A(X) \) is a homotopy colimit of the diagram \( \{ \text{Exit}_{A \leq \lambda}(X_{\leq \lambda})^{\circ} \} \lambda \in \Lambda \)\([6, 3.1.4.1 & 4.2.4.1]\). This is the case since the model category of marked simplicial sets admits cofibrant generators with \( \omega \)-small domains and codomains (the marked simplicial maps, with underlying simplicial map \( \Delta^n \subset \Delta^n \)), and filtered colimits in such cases coincide with homotopy colimits \([10, 7.3]\).

**Corollary 3.4.** For any \( D_\omega \)-space \( X \), one has an equality

\[ \text{Exit}_A(X) = \text{Exit}_A(X_{<\omega}) \]

between the exit paths \( \infty \)-categories of \( X \) and \( X_{<\omega} \).

In other words, the exit paths \( \infty \)-category does not depend on the global topology of the space.

### 3.2 Representation theorem

Let us recall how the \( \infty \)-category \( \mathbf{Fun}(\text{Exit}_A(X), S) \) represents \( A \)-constructible sheaves in the case where \( X \) is a \( D \)-space \([1, A.10]\). First the \( \infty \)-category of functors \( \mathbf{Fun}(\text{Exit}_A(X), S) \) can be replaced by \( \mathbf{N}(A_X^\circ) \), the \( \infty \)-category associated to the simplicially enriched category of fibrant-cofibrant objects of the category \( \mathbf{Set}/\text{Exit}_A(X) \) endowed with the covariant model structure, via

\[
\mathbf{Fun}(\text{Exit}_A(X), S) \xleftarrow{\mathbf{N}} \mathbf{N}\left( \mathbf{Set}_\Delta^{\mathbf{c}[\text{Exit}_A(X)]} \right) \xrightarrow{\mathbf{N}} \mathbf{N}(A_X^\circ)
\]

a chain of equivalences \([6, 2.2.1.2 & 4.2.4.4]\), where the left functor is a forgetful functor and the right functor is the unstraightening functor. One can then define a functor \( \mathcal{O}(X)^{op} \times A_X^\circ \to \mathbf{Set}_\Delta^\circ \)

\[
(U, Y) \mapsto \mathbf{Fun}_{\text{Exit}_A(X)}(\text{Exit}_A(U), Y)
\]

which will induce a functor

\[
\mathbf{N}(A_X^\circ) \xrightarrow{\mathbf{Ψ}_X} \mathbf{PSh}(X)
\]

with values in the \( \infty \)-category of presheaves on \( X \).

**Theorem 3.5** \([1, A.10.5, A.10.10 & A.10.3]\). Let \( X \to A \) be a \( D \)-space. The functor \( \mathbf{Ψ}_X : \mathbf{N}(A_X^\circ) \to \mathbf{PSh}(X) \) is fully faithful and its image is equivalent to the subcategory of \( A \)-constructible sheaves on \( X \).

We shall now extend this theorem to the case where \( X \) is a \( D_\omega \)-space.

**Lemma 3.6.** Let \( X \) be an \( A \)-stratified space, then \( \mathbf{N}(A_X^\circ) \) is a presentable \( \infty \)-category and the functor \( \mathbf{Ψ}_X \) is a right adjoint.
Proof. The simplicial model category $A_X$ is combinatorial [6, 2.1.4.6], so its associated ∞-category is presentable [6, A.3.7.6].

In view of the adjoint functor theorem [6, 5.5.2.9], it shall be enough to show that $\Psi_X$ is both continuous and accessible.

For every open subset $U \subset X$, the functor $Y \mapsto \text{Fun}_{\text{Exit}_A}(X)(\text{Exit}_A(U), Y)$ from $A_X^\circ$ to $\text{Set}^\circ$ preserves homotopy limits since $A_X$ is a simplicial model category and $\text{Exit}_A(U)$ is cofibrant (as all objects are). As a consequence the functor $Y \mapsto \Psi_X(Y)(U)$ preserves all small limits [6, 4.2.4.1 & 4.2.3.14]. It follows that $\Psi_X$ preserves all small limits because limits in presheaves ∞-categories are computed pointwise [6, 5.1.2.3].

Using the same arguments, to prove that $\Psi_X$ is accessible, it shall be enough to find a cardinal $\kappa$ such that the functors $Y \mapsto \Psi_X(Y)(U)$ commute with $\kappa$-filtered homotopy colimits for every $U \subset X$. Since $A_X$ is combinatorial, there exists a cardinal $\kappa$ such that all $\kappa$-filtered colimits be homotopy colimits [10, 7.3]. Enlarging $\kappa$ if necessary, we can also demand that $\text{Exit}_A(U)$ be $\kappa$-small for every open subset $U \subset X$. As a consequence, $Y \mapsto \Psi_X(Y)(U)$ commutes with all (homotopy) $\kappa$-filtered colimits for every open $U \subset X$. □

Lemma 3.7. Let $X$ be a $D_\omega$-space. For each $n < \omega$, denote by $j(n)$ the inclusion of $X_{<n}$ into $X$. One has a bireflexive simplicial model localisation

$$
A_X \xleftarrow{j(n)_!} A_{X_{<n}} \xrightarrow{j(n)_*} A_{X_{<n}}
$$

where $j(n)_!$ is the forgetful functor, where

$$j(n)^*(Y) := \text{Exit}_{A_{<n}}(X_{<n}) \times_{\text{Exit}_A(X)} Y$$

and where both categories are endowed with the covariant model structure.

Proof. The existence of the right adjoint to $j(n)_!$, can be obtained by the adjoint functor theorem, using the presentability of $A_X$ and the universality of colimits in $A_{X_{<n}}$. The obvious fully faithfulness of the left adjoint $j(n)_*$ implies the fully faithfulness of the right adjoint $j(n)_!$.

The forgetful functor $j(n)_!$ has an obvious simplicial enrichment. Moreover, one has evident isomorphisms $j(n)_!(\Delta^p \times Y) = \Delta^p \times j(n)_!(Y)$ natural in $Y \in A_{X_{<n}}$ for every $p < \omega$. This is enough to endow $j(n)^*$ and the adjunction $j(n)_! \dashv j(n)^*$ with a simplicial enrichment [11, 3.7.10]. Likewise one has isomorphisms $j(n)^*(\Delta^p \times Y) = \Delta^p \times j(n)^*(Y)$ natural in $Y \in A_X$ for every $p < \omega$, giving $j(n)^*$, and the adjunction $j(n)^* \dashv j(n)_!$, a simplicial enrichment.

In this kind of setup, the pair $j(n)_! \dashv j(n)^*$ is always a model adjunction [6, 2.1.4.10]. The embedding $\text{Exit}_{A_{<n}}(X_{<n}) \subset \text{Exit}_A(X)$ is a right fibration. This follows from the fact that if a stratified path ends in $X_{<n}$, then all the path must be in $X_{<n}$. As a consequence, the adjunction $j(n)^* \dashv j(n)_!$, is a model adjunction [12, 11.2]. □

As a consequence, both functors $j(n)^*$ and $j(n)_!$, preserve fibrant objects by the previous lemma and induce a reflexive localisation

$$
N(A_X^\circ) \xleftarrow{j(n)_*} N(A_{X_{<n}}^\circ) \xrightarrow{j(n)_!} N(A_{X_{<n}}^\circ)
$$
between the associated ∞-categories [6, 5.2.4.5]. These adjunctions induce an adjunction
\[
\begin{array}{ccc}
\mathbb{N}(A_X^g) & \xleftarrow{j^*} & \lim_{n<\omega} \mathbb{N}(A_{X_{\leq n}}^g) \\
\downarrow{j_*} & & \downarrow \\
\end{array}
\]
where \(j^*\) is given by the family of the functors \(j(n)^*\) and where
\[
j_*(\{Y_{\leq n}\}) := \lim_{n<\omega} j(n)_* Y_{\leq n}
\]
for any \(\{Y_{\leq n}\} \in \lim_{n<\omega} \mathbb{N}(A_{X_{\leq n}}^g).

**Lemma 3.8.** Let \(C\) be an ∞-category and consider two towers of monomorphisms
\[
X_0 \hookrightarrow X_1 \hookrightarrow \ldots \hookrightarrow X_n \hookrightarrow \ldots \hookrightarrow X \quad \text{and} \quad Y_0 \hookrightarrow Y_1 \hookrightarrow \ldots \hookrightarrow Y_n \hookrightarrow \ldots \hookrightarrow Y \quad \text{in} \quad C.
\]
Assume that we have also given a morphism \(X \to Y\) and equivalences \(X_n \simeq Y_n\) together with natural transformations making the square
\[
\begin{array}{ccc}
X_n & \xrightarrow{=} & Y_n \\
\downarrow & & \downarrow \\
X & \xrightarrow{} & Y
\end{array}
\]
commute, for every \(n<\omega\). This data is enough to induce a commutative square
\[
\begin{array}{ccc}
\lim_{n<\omega} X_n & \xrightarrow{=} & \lim_{n<\omega} Y_n \\
\downarrow & & \downarrow \\
X & \xrightarrow{} & Y
\end{array}
\]
whose top map is an equivalence and whose vertical maps are the canonical projections.

**Proof.** By composition with \(X \to Y\), we get a commutative triangle
\[
\begin{array}{ccc}
\lim_{n<\omega} X_n & \xrightarrow{=} & \lim_{n<\omega} Y_n \\
\downarrow & & \downarrow \\
X & \xrightarrow{} & Y
\end{array}
\]
letting us reduce to the case where \(X = Y\).

Up to an equivalence, we can assume that the ∞-category of subobjects of \(Y\) is a 0-category [6, 2.3.4.18]. We thus reduce to the case where the maps \(X_n \hookrightarrow Y\) and \(Y_n \hookrightarrow Y\) are equal for every \(n<\omega\) and the result is trivial. \(\square\)

**Remark 3.9.** We shall use this lemma in the special case where \(C\) is the ∞-category of presentable ∞-categories and right adjoints. In this case fully faithful right adjoint functors are monomorphisms [13, 5.7]. As a consequence, reflexive localisation functors are epimorphisms in the ∞-category of presentable ∞-categories and left adjoint functors [6, 5.5.3.4].

**Proposition 3.10** (Dévissage). Let \(X\) be a \(\mathcal{D}_{\omega}\)-space. The adjunction
\[
\begin{array}{ccc}
\mathbb{N}(A_X^g) & \xleftarrow{j^*} & \lim_{n<\omega} \mathbb{N}(A_{X_{\leq n}}^g) \\
\downarrow{j_*} & & \downarrow \\
\end{array}
\]
induced by the pairs \(j(n)^* \dashv j(n)_*\), is an equivalence of ∞-categories.
Proof. Let $\mathcal{C}^{\operatorname{Exit}_A(X)}$ denote the category of simplicially enriched functors from $\mathcal{C}^{\operatorname{Exit}_A(X)}$ to $\mathcal{A}$ endowed with the projective model structure. The diagram of right model adjoints

$$
\begin{array}{ccc}
\mathcal{C}^{\operatorname{Exit}_A(X)} & \xrightarrow{\text{unstraightening}} & \mathcal{A}_X \\
\downarrow j(n)^* & & \downarrow j(n)^*
\end{array}
$$

is commutative [6, 2.2.1.1].

Moreover, all functors are simplicially enriched [6, 2.2.2.12] and the commutation holds at the enriched level. Let us see why. The enrichment for the unstraightening functor stems from the following sequence: for any two simplicial functors $F$ and $G$ and any simplicial set $K$, any natural transformation $F \otimes K \to G$ induces a natural transformation

$$
\alpha(F,G,K) \colon \operatorname{Un}(F) \times K \to \operatorname{Un}(F \otimes K)
$$

while the enrichment for both pullback functors comes straight from commutation with tensoring with simplicial sets. Using this fact, one gets that $j(n)^* \alpha(F,G,K)$ is isomorphic to $\alpha(j(n)^* F, j(n)^* G, K)$.

Because $\operatorname{Exit}_{A_{\leq n}}(X_{\leq n}) \subset \operatorname{Exit}_A(X)$ is a monomorphism, the associated simplicial functor $\mathcal{C}^{\operatorname{Exit}_{A_{\leq n}}(X_{\leq n})} \to \mathcal{C}^{\operatorname{Exit}_A(X)}$ is a cofibration [6, 2.2.5.1], as a consequence the associated pullback functor $j(n)^*$ preserves cofibrant objects [6, A.3.3.9]. Then, all functors in this square preserve fibrant-cofibrant objects and one gets

$$
\begin{array}{ccc}
\mathcal{C}^{\operatorname{Exit}_A(X)} & \xrightarrow{\text{unstraightening}} & \mathcal{A}_X \\
\downarrow j(n)^* & & \downarrow j(n)^*
\end{array}
$$

an equivalence between the two induced pullback functors at the $\infty$-category level [6, 2.2.3.11]. One also has a commutative diagram

$$
\begin{array}{ccc}
\mathcal{C}^{\operatorname{Exit}_A(X)} & \xrightarrow{\text{Fun}} & \mathcal{A}_X \\
\downarrow j(n)^* & & \downarrow j(n)^*
\end{array}
$$

letting us identify $j(n)^*$ with the pullback at the level of $\infty$-categories of functors [6, 4.2.4.4].

We then reduce to show that the canonical map

$$
\begin{array}{ccc}
\mathcal{C}^{\operatorname{Exit}_A(X)} & \xrightarrow{\text{Fun}} & \mathcal{A}_X \\
\downarrow j(n)^* & & \downarrow j(n)^*
\end{array}
$$

is an equivalence [3.8, 3.9], which follows from the fact that $\operatorname{Exit}_A(X)$ is a colimit of the diagram $\{\operatorname{Exit}_{A_{\leq n}}(X_{\leq n})\}_{n<\omega}$ in the $\infty$-category of $\infty$-categories [3.3].
Theorem 3.11. Let \( X \to A \to \omega \) be a \( D_\omega \)-space and assume that \( X \) be either a conical or a colimit \( D_\omega \)-space.

The functor \( \Psi_X \) is fully faithful and induces an equivalence of \( \infty \)-categories

\[
\mathsf{N}(A_X^0) = \mathsf{Hyp}_A(X)
\]

between \( \mathsf{N}(A_X^0) \) and the \( \infty \)-category of \( A \)-constructible hypersheaves on \( X \).

Proof. For each \( n < \omega \) and each open subset \( U \subset X \) one has \( j(n)^* \mathsf{Exit}_A(U) = \mathsf{Exit}_{A_{X_{\leq n}}}(U \cap X_{\leq n}) \), implying that for each left fibration \( Y \) over \( \mathsf{Exit}_{A_{X_{\leq n}}}(X_{\leq n}) \), \( \theta(U, j(n)^* Y) = \theta(U \cap X_{\leq n}, Y) \) since \( j(n)^* \dashv j(n)_* \) is a simplicially enriched adjunction [3.7]. As a consequence, the square

\[
\begin{array}{ccc}
\mathsf{N}(A_X) & \xrightarrow{\Psi_X} & \mathsf{PSh}(X) \\
\downarrow{j(n)} & & \downarrow{j(n)_*} \\
\mathsf{N}(A_{X_{\leq n}}) & \xrightarrow{\Psi_{X_{\leq n}}} & \mathsf{PSh}(X_{\leq n})
\end{array}
\]

commutes up to a natural equivalence. In fact, because \( X_{\leq n} \) is a \( D \)-space and \( j(n)_* \) preserves sheaves among presheaves and constructible sheaves among sheaves [2.16], we have an induced

\[
\begin{array}{ccc}
\mathsf{N}(A_X) & \xrightarrow{\Psi_X} & \mathsf{PSh}(X) \\
\downarrow{j(n)} & & \downarrow{j(n)_*} \\
\mathsf{N}(A_{X_{\leq n}}) & \xrightarrow{\Psi_{X_{\leq n}}} & \mathsf{Sh}_{A_{X_{\leq n}}}(X_{\leq n})
\end{array}
\]

commutative square of right adjoints [3.6] by the representation theorem for \( D \)-spaces [3.5].

One thus gets a commutative diagram

\[
\begin{array}{ccc}
\lim_{n<\omega} \mathsf{N}(A_{X_{\leq n}}) & \xrightarrow{\Psi_{X_{\leq n}}} & \lim_{n<\omega} \mathsf{Sh}_{A_{X_{\leq n}}}(X_{\leq n}) \\
\downarrow{j_*} & & \downarrow{j_*} \\
\mathsf{PSh}(X) & \xrightarrow{\Psi_X} & \mathsf{PSh}(X)
\end{array}
\]

where the colimits are taken in the \( \infty \)-category of presentable \( \infty \)-categories and right adjoints [3.8, 3.9]. By the representation theorem for \( D \)-spaces [3.5], the bottom map is an equivalence of \( \infty \)-categories. By dévissage [3.10], the left arrow is an equivalence of \( \infty \)-categories. Indeed, colimits in the \( \infty \)-category of presentable \( \infty \)-categories and right adjoints are canonically equivalent to the associated limit in the \( \infty \)-category of presentable \( \infty \)-categories and left adjoints [6, 5.5.3.4]. Using the same argument, one deduces that the right arrow is fully faithful and its image is the subcategory of \( A \)-constructible hypersheaves again by dévissage [2.17, 2.19]. \( \square \)

Corollary 3.12 (Representation theorem, conical case). Let \( X \to A \to \omega \) be a conical \( D_\omega \)-space. Then the \( \infty \)-category of exit paths \( \mathsf{Exit}_A(X) \) represents

\[
\mathsf{Fun}(\mathsf{Exit}_A(X), S) = \mathsf{Hyp}_A(X)
\]

the \( \infty \)-category of \( A \)-constructible hypersheaves on \( X \).
Corollary 3.13 (Representation theorem, colimit case). Let \( X \to A \to \omega \) be a colimit \( \mathcal{D}_\omega \)-space. The \( \infty \)-category of exit paths \( \text{Exit}_A(X) \)

\[
\text{Fun}(\text{Exit}_A(X), S) = \text{Hyp}_A(X) = \text{Sh}_A(X)
\]

represents the \( \infty \)-category of \( A \)-contractible sheaves.

Remark 3.14. It is natural to conjecture that the equivalences \( \Psi_X \) are part of a natural equivalence of functors. In particular, we should have commutative squares

\[
\begin{array}{ccc}
\text{Fun}(\text{Exit}_A(Y), S) & \xrightarrow{\text{Exit}_A(f)^*} & \text{Fun}(\text{Exit}_A(X), S) \\
\downarrow^\mu & & \downarrow^\mu \\
\text{Hyp}_A(Y) & \xrightarrow{\text{Hyp}_A(f)^*} & \text{Hyp}_A(X)
\end{array}
\]

whenever given a conical or colimit \( \mathcal{D}_\omega \)-space \( Y \to A \to \omega \) and a continuous map \( f : X \to Y \) letting \( X \to Y \to A \to \omega \) be a conical or colimit \( \mathcal{D}_\omega \)-space.

The existence of these commutative squares has been shown by Lurie in the case where \( X \) and \( Y \) are \( \mathcal{D} \)-spaces [1, A.10.16]. Using a similar proof, one can show their existence in the case of conical or colimit \( \mathcal{D}_\omega \)-spaces.

3.3 Homotopy invariance of exit paths

Ayala, Francis and Rozenblyum have shown that the exit paths \( \infty \)-category functor is fully faithful between the \( \infty \)-category of smooth stratified spaces and the \( \infty \)-category of \( \infty \)-categories [14]. We prove here a modest extension to \( \mathcal{D}_\omega \)-spaces, showing that the exit paths functor is homotopy invariant. This homotopy invariance result invites us to think that the stratified homotopy hypothesis for smooth stratified spaces could be extended to stratified spaces whose poset of strata does not satisfy the ascending chain condition.

Lemma 3.15. Let \( f : \mathcal{C} \to \mathcal{D} \) be a functor between two small idempotent complete \( \infty \)-categories. If the pullback functor

\[
\text{Fun}(\mathcal{D}, S) \xrightarrow{f^*} \text{Fun}(\mathcal{C}, S)
\]

is an equivalence of \( \infty \)-categories, then so is \( f \).

Proof. Let \( f_! \) denote the left adjoint to \( f^* \). The following square

\[
\begin{array}{ccc}
\text{Fun}(\mathcal{C}, S) & \xrightarrow{f} & \text{Fun}(\mathcal{D}, S) \\
\text{C}^{\text{op}} & \xrightarrow{\delta_C} & \mathcal{D}^{\text{op}} \\
\end{array}
\]

is commutative [6, 5.2.6.3]. By assumption, \( f^* \) is an equivalence of \( \infty \)-categories and so is its left adjoint \( f_! \). Then, \( f_! \) induces an equivalence of \( \infty \)-categories between the respective full subcategories of completely compact objects. By idempotent completeness, these full subcategories are equivalent to the full subcategories of representable functors [6, 5.1.6.8]. As a consequence \( f^{\text{op}} \) and thus \( f \) are equivalences of \( \infty \)-categories.

Lemma 3.16. Let \( X \) be an \( A \)-stratified space, then \( \text{Exit}_A(X) \) is an idempotent complete \( \infty \)-category.
Proof. An idempotent stratified path in $X$ must stay in the same stratum, in which case it is invertible and thus trivial.

**Theorem 3.17.** Let $\Lambda$ be a filtered poset, $A$ be a $\Lambda$-stratified poset, $Y$ be an $A$-stratified space and let $f : X \to Y$ be a continuous map.

Assume that for every $\lambda \in \Lambda$, both $X_{\leq \lambda} \to A_{\leq \lambda}$ and $Y_{\leq \lambda} \to A_{\leq \lambda}$ be $\mathbb{D}$-spaces. If $f$ is an $A$-stratified homotopy equivalence, then the functor

$$\text{Exit}_A(X) \xrightarrow{\text{Exit}_A(f)} \text{Exit}_A(Y)$$

is an equivalence of $\infty$-categories.

**Proof.** Since $f$ is an $A$-stratified homotopy equivalence, it induces an $A_{\leq \lambda}$-stratified homotopy equivalence $f_{\leq \lambda} : X_{\leq \lambda} \simeq Y_{\leq \lambda}$ for each $\lambda \in \Lambda$. By homotopy invariance, the pullback functor $(f_{\leq \lambda})^* : \text{Sh}_{A_{\leq \lambda}}(Y_{\leq \lambda}) \to \text{Sh}_{A_{\leq \lambda}}(X_{\leq \lambda})$ is an equivalence of $\infty$-categories [2.8, 2.11, 1.26]. As a consequence, the pullback functor $\text{Exit}_{A_{\leq \lambda}}(f_{\leq \lambda})^*$ from $\text{Fun}(\text{Exit}_{A_{\leq \lambda}}(Y_{\leq \lambda}), S)$ to $\text{Fun}(\text{Exit}_{A_{\leq \lambda}}(X_{\leq \lambda}), S)$ is also an equivalence of $\infty$-categories [3.14]. Then by the previous lemmas, $\text{Exit}_{A_{\leq \lambda}}(f_{\leq \lambda})$ is an equivalence between the $\infty$-category $\text{Exit}_{A_{\leq \lambda}}(X_{\leq \lambda})$ and $\text{Exit}_{A_{\leq \lambda}}(Y_{\leq \lambda})$. Taking the colimit [3.3], we get that $\text{Exit}_A(f)$ is an equivalence of $\infty$-categories.

### 4 EXAMPLES OF APPLICATION

We shall give examples of applications of the representation theorem for constructible hypersheaves in the case where the stratifying poset does not satisfy the ascending chain condition. Simplicial complexes and CW-complexes are natural examples of colimit $\mathbb{D}_\omega$-spaces. We shall then describe a conical example of $\mathbb{D}_\omega$-space, the metric exponential.

One common point of these examples is that each stratum is locally homeomorphic to a locally convex topological vector space and we shall need to know that such spaces are locally of singular shape.

**Lemma 4.1.** Let $X$ be a topological space locally homeomorphic to a locally convex real topological vector space $V$. Then $X$ is locally of singular shape.

**Proof.** By assumption $X$ is covered by spaces homeomorphic to $V$, so it shall be enough to show that $V$ is locally of singular shape [1, A.4.16]. Let $U \subset V$ be an open subset of $V$ and let $A$ be the set of all convex open subsets of $U$. Since $V$ is locally convex $U = \bigcup_{C \in A} C$. Moreover for any finite $B \subset A$, the intersection $\bigcap_{C \in B} C$ is still convex. Lastly, any convex subset in a topological vector space is contractible and thus of singular shape [1, A.4.11]. This is enough to show that $U$ is also of singular shape [1, A.4.14].

#### 4.1 Simplicial complexes

Let $X$ be a simplicial complex. Its constitutive simplices naturally give $X$ a structure of stratified space $X \to S$.

Lurie has shown that when $X$ is locally finite, then it is conically $S$-stratified [1, A.7.3]. The poset of simplices $S$ also satisfies the ascending chain condition. Moreover, $\text{Exit}_S(X)$ is always an $\infty$-category [1, A.7.4] and the canonical map $\text{Exit}_S(X) \to \text{N}(S)$ is an equivalence of $\infty$-categories [1, A.7.5].
If $X$ is a locally finite simplicial complex, it is then a $D$-space and the representation theorem applies. We shall extend this to the class of all locally countable simplicial complexes.

**Definition 4.2.** We shall say that a simplicial complex $X$ is locally countable if each vertex $v \in X$ belongs to at most a countable number of edges of $X$.

**Theorem 4.3.** Let $X$ be a locally countable simplicial complex with poset of simplices $S$. One can represent $S$-constructible sheaves on $X$

$$\text{Fun}(\mathcal{N}(S), S) = \text{Sh}_S(X)$$

as functors from $\mathcal{N}(S)$ to $S$.

**Proof.** For each vertex $v$ let $E^v$ be the set of edges $e$ for which $v \in e$. By assumption $E^v$ is countable so one can find an exhaustion by finite subsets

$$E^v_{\xi_0} \subset \cdots \subset E^v_{\xi_n} \subset \cdots \subset E^v$$

for each vertex $v$. For each $n < \omega$, let $X_{\xi_n} \subset X$ denote the biggest simplicial subcomplex for which each edge $e \in X_{\xi_n}$ is such that $v \in e \implies e \in E^v_{\xi_n}$. By construction the poset $S_{\xi_n}$ of simplices of $X_{\xi_n}$ is a downward closed subposet of $S$ and $X \to S$ is exhausted by the subcomplexes $X_{\xi_n} \to S_{\xi_n}$ which are all $D$-spaces as discussed earlier.

Every simplicial complex is paracompact and each simplex is locally of singular shape [4.1]. Thus $X \to S$ is a colimit $D_\omega$-space and the representation theorem applies [3.13].

### 4.2 CW-complexes

The case of CW-complexes resembles the one of simplicial complexes: the set of cells $C$ of a CW-complex $X$ can be made into a poset by letting $d \leq c$ whenever $d \subset c$. However, this poset is generally odd with cells attached to other cells but isolated in the poset. It is then natural to restrict one’s attention to normal CW-complexes.

**Definition 4.4.** A CW-complex $X$ with set of cells $C$ is said to be normal if for every $c \in C$, its boundary is a again a union of cells of $X$.

We shall say that a normal CW-complex $X$ is locally finite whenever $C_{v<}$ is finite for every 0-cell $v$, and we shall say that $X$ is locally countable if instead $C_{v<}$ is countable.

The last remaining obstacle is the conicality of the stratification as locally finite normal CW-complexes might not be conically stratified. Tamaki has introduced the notion of cylindrically normal CW-complex [15, 4.1] and with Tanaka, they have shown that for finite cylindrically normal CW-complexes, the cell stratification is conical [16, 1.7]. Since conicality is a local condition, this immediately extends to locally finite normal CW-complexes. Regular CW-complexes and, real or complex projective spaces are examples of cylindrically normal CW-complexes [15, 4.3].

**Theorem 4.5.** Let $X$ be a locally countable cylindrically normal CW-complex with poset of cells $C$. Then $\text{Exit}_C(X)$ is an $\infty$-category

$$\text{Fun}(\text{Exit}_C(X), S) = \text{Sh}_C(X)$$

representing $C$-constructible sheaves on $X$. 27
Proof. Every CW-complex is paracompact and each cell being homeomorphic to a locally convex topological vector space, it is locally of singular shape \[4.1\]. By assumption if \( X \) is locally countable, for every 0-cell \( \nu \), one can find an exhaustion of \( C_{\nu<} \)

\[ C_{\nu<}^0 \subset \cdots C_{\nu<}^n \subset \cdots \]

by finite subsets. Let \( X_{\leq n} \subset X \) denote the biggest subcomplex such that for each cell \( e \in X_{\leq n} \), if \( \nu \leq e \), then \( e \in C_{\nu<}^n \). By construction \( X_{\leq n} \) is locally finite and cylindrically normal, so it is conically stratified by the discussion above. In addition, because the border of every cell \( e \in X \) is a finite union of cells of lower dimension, a direct induction shows that each cell \( e \) belongs to one of the \( X_{\leq n} \). It follows that \( X \) is a colimit \( D_\omega \)-space and the representation theorem applies \[3.13\]. \( \square \)

4.3 The exponentials

Given a set \( X \), the exponential on \( X \) is the set \( \text{Exp}(X) \) of all finite subset \( S \subset X \). If \( (X,d) \) is a metric space, the formula

\[
D(S,T) := \max \left\{ \max_{s \in S} \min_{t \in T} d(s,t), \min_{t \in T} \max_{s \in S} d(s,t) \right\}
\]

canonically endows the exponential with a generalised metric. That is, a metric space in which two points can be at infinite distance from one another. Here this is the case of the empty configuration: \( D(\emptyset, S) = +\infty \) if \( S \) is not empty.

The set \( \text{Exp}(X) \) when endowed with its metric \( D \) is the metric exponential \( \text{Exp}_M(X) \). The metric exponential is canonically stratified over the poset \( \omega^* = \{0\} \cup \{1 < 2 < \cdots\} \), with a finite subset \( S \subset X \) being sent to its cardinality. Replacing the metric topology of \( \text{Exp}_M(X) \) with the colimit one arising from this stratification, one obtains

\[
\text{Exp}_T(X) := \lim_{\longrightarrow \omega^*} \text{Exp}_M^{<\omega}(X)
\]

the topological exponential. When \( M \) is a Fréchet manifold, one can show that the metric exponential is conically stratified \[17\]. It follows that \( \text{Exp}_M(M) \) is a conical \( D_\omega \)-space and \( \text{Exp}_T(M) \) is its associated colimit \( D_\omega \)-space. In particular

\[
\text{Hyp}_{\omega^*} (\text{Exp}_M(M)) = \text{Sh}_{\omega^*} (\text{Exp}_T(M)) = \text{Fun}(\text{Exit}_{\omega^*}(\text{Exp}_M(X)), S)
\]

both have the same \( \infty \)-category of constructible hypersheaves and these are representable using exit paths.

Constructible cosheaves on the exponential form a key ingredient in the theory of locally constant factorisation algebras. Constructible hypersheaves on the metric exponential are for example used by Lurie, to bridge locally constant factorisation algebras on a finite dimensional manifold \( M \) with the theory of \( E_M \)-algebras \[1, 3.6.10\].

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