$L^p$-Green-tight measures of $L^p$-Kato class for symmetric Markov processes

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Abstract

In this paper, we introduce the notion of $L^p$-Green-tight measures of $L^p$-Kato class in the framework of symmetric Markov processes. The class of $L^p$-Green-tight measures of $L^p$-Kato class is defined by the $p$-th power of resolvent kernels. We first prove that under the $L^p$-Green tightness of the measure $\mu$, the embedding of extended Dirichlet space into $L^{2p}(E;\mu)$ is compact under the absolute continuity condition for transient Markov processes, which is an extension of recent seminal work by Takeda. Secondly, we prove the coincidence between two classes of $L^p$-Green-tightness, one is originally introduced by Zhao, and another one is invented by Chen. Finally, we prove that our class of $L^p$-Green-tight measures of $L^p$-Kato class coincides with the class of $L^p$-Green tight measures of Kato class in terms of Green kernel under the global heat kernel estimates. We apply our results to $d$-dimensional Brownian motion and rotationally symmetric relativistic $\alpha$-stable processes on $\mathbb{R}^d$.

Keywords: Dirichlet form, Markov process, $L^p$-Kato class measure, $L^p$-Dynkin class measure, $L^p$-Green-tight measures of $L^p$-Kato class, heat kernel, semigroup kernel, resolvent kernel, Green kernel, Stollman-Voigt inequality, Brownian motion, symmetric $\alpha$-stable process, relativistic $\alpha$-stable process.

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1 Introduction

The notion of Green-tightness for Kato class potential was introduced by Zhao [3] to consider the gaugeability for Feynman-Kac functionals and the subcriticality of Schrödinger operator $-\frac{1}{2}\Delta + V$ in the framework of $d$-dimensional Brownian motion with $d \geq 3$. Before Zhao [3], the gaugeability of Feynman-Kac functionals with Kato class potential has been considered for absorbing Brownian motions on bounded open domains (see Zhao [22]). Zhao [3] also clarified that Kato class potential for absorbing Brownian motion on a bounded open domain satisfies the Green-tightness condition in terms of the Green function of Dirichlet Laplacian on the domain. This was a motivation to formulate the notion of Green-tight measures of Kato class for transient symmetric Markov processes. However, the Green-tightness as introduced by Zhao [3] was not enough to develop the theory of gaugeability of Feynman-Kac functionals and subcriticality of Schrödinger operators for symmetric Markov processes. To overcome this difficulty, Chen-Song [11,12] gave a new notion of Green-tight smooth measures of Kato class in the strict sense in the framework of general $m$-symmetric transient Markov processes $X = (\Omega, X_t, \mathbb{P}_x)$ on a locally compact separable metric space $E$ having a positive Radon measure $m$ with full support satisfying the absolute continuity condition with respect to $m$. Moreover, in Chen [2], this new notion was refined with remaining value for the gaugeability of Feynman-Kac functionals and the subcriticality of Schrödinger operators. Here $X$ is said to possess the absolute continuity condition with respect to $m$ ((AC) in short) if for any Borel set $B$, $m(B) = 0$ implies $P_t(x, B) = \mathbb{P}_x(X_t \in B) = 0$ for all $t > 0$ and $x \in E$.

The refined new class introduced in [2] coincides with the class similarly as defined in [3] not only for $d$-dimensional Brownian motions with $d \geq 3$ but also rotationally symmetric $\alpha$-stable processes with $d > \alpha$. Chen [2] remarked that if the underlying measure $m$ of symmetric Markov process belongs to the Green-tight measures of Kato class in the original sense of Zhao, then it belongs to the new class provided the given process possesses the strong Feller property (see [2] Theorem 4.2). On the other

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hand, Kim-Kuwae [18, Lemma 4.1] proved that the both classes coincide provided the given symmetric Markov process \( X = (\Omega, X_t, \mathbb{P}_x) \) possesses the resolvent strong Feller property. Here \( X \) is said to possess the resolvent strong Feller property ((RSF) in short) (resp. strong Feller property ((SF) in short)) if \( R_x(\mathscr{B}(E)) \subset C_b(E) \) for some/any \( \alpha > 0 \) (resp. \( P_t(\mathscr{B}(E)) \subset C_b(E) \) for any \( t > 0 \)), where \( R_x f(x) = \mathbb{E}_x \left[ \int_0^\infty e^{-\alpha t} f(X_t) \, dt \right] = \int_0^\infty e^{-\alpha t} P_t f(x) \, dt \) and \( P_t f(x) = \mathbb{E}_x [f(X_t)] \) for \( f \in \mathscr{B}(E) \). Here \( \mathscr{B}(E) \) (resp. \( C_b(E) \)) denotes the family of all bounded Borel measurable (resp. continuous) functions on \( E \). It is known that the implication (SF) \( \implies \) (RSF) \( \implies \) (AC) holds.

Based on the coincidence of two classes of Green-tight measures of Kato class under (RSF), Takeda [23] proved that the semi-group \((P_t)_{t \geq 0}\) of \( X \) is a compact operator on \( L^2(\mathbb{E}; m) \) provided \( X \) belongs to the Class (T). Using the compactness of \( P_t \), Takeda proved that the embedding \( \mathscr{F} \hookrightarrow L^2(\mathbb{E}; m) \) is compact. Here \( X \) is said to belong to Class (T) if it satisfies that the underlying measure \( m \) of \( X \) belongs to the 1-order Green-tight measures of Kato class in the sense of [2] (denoted by \( m \in S^1_{CK,\infty}(X^{(1)}) \)). \( X \) is irreducible and it possesses (RSF). Here \( X^{(1)} \) denotes the 1-subprocess of \( X \) defined by \( X^{(1)} = (\Omega, X_t, \mathbb{P}_x^{(1)}) \) with \( \mathbb{P}_x^{(1)}(X_t) = e^{-I} \mathbb{P}_x(X_t) \) for all \( t > 0 \) and \( A \in \mathscr{B}(E) \). \( X \) is said to be irreducible ((I) in short) if \( B \in \mathscr{B}(E) \) satisfies \( P_t 1_B u = 1_B P_t u \) for all \( u \in L^2(\mathbb{E}; m) \cap \mathscr{B}(E) \) and \( t > 0 \), then \( m(B) = 0 \) or \( m(B') = 0 \) holds. Let \((\mathscr{E}, \mathscr{F})\) be the extended Dirichlet space of \( X \). If \( X \) is transient and \( \mu \) is a Dynkin class measure, Stollmann-Voigt’s inequality [27] implies the continuity of embedding \( \mathscr{F} \hookrightarrow L^2(\mathbb{E}; \mu) \). If \( X \) is transient and \( \mu \) is a 0-order Green-tight measure of Kato class in the sense of [2], this embedding is compact.

On the other hand, the notation of \( L^p \)-Kato class was proposed in [24] by the second named author to obtain the several probabilistic properties on the intersection measures. In [24], Stollmann-Voigt’s inequality [27] is extended to the measure \( \mu \) of \( L^p \)-Dynkin class, and it implies the continuity of the embedding \( \mathscr{F} \hookrightarrow L^{2p}(\mathbb{E}; \mu) \). The notion of \( L^p \)-Green-tight measures of \( L^p \)-Kato class in the sense of Zhao or Chen is a natural extension of usual Green-tight measures of Kato class in these senses, and inequality [27] is extended to the measures based on the coincidence of two classes of Green-tight measures of Kato class under (RSF) [21, Corollary 4.3]. The first part of Theorem 1.1(2) is an extension of recent seminal work [23] by Takeda on the case \( p = 1 \) for symmetric Markov processes in Class (T), and the second part of Theorem 1.1(2) is a slight extension of [23, Corollary 4.3].

Let \( S^p_{CK,\infty}(X) \) (resp. \( S^p_{CK,\infty}(X^{(1)}) \)) denote the class of 0-order (resp. 1-order) \( L^p \)-Green-tight measures of \( L^p \)-Kato class in the sense of Chen (see Definition 2.3 below).

Our first main theorem in this paper is the following, which is a natural extension of [23, Theorem 4.8] and [24, Corollary 4.3].

**Theorem 1.1** We have the following:

1. Suppose that \( X \) is transient and it satisfies (AC). Assume \( \mu \in S^p_{CK,\infty}(X) \), or \( \mu \in S^p_{K}(X) \) and \( m \in S^1_{CK,\infty}(X) \). Then \((\mathscr{E}, \mathscr{F})\) is compactly embedded into \( L^{2p}(\mathbb{E}; \mu) \).

2. Suppose that \( X \) satisfies (AC). Assume \( \mu \in S^p_{CK,\infty}(X^{(1)}) \), or \( \mu \in S^p_{K}(X) \) and \( m \in S^1_{CK,\infty}(X^{(1)}) \). Then \((\mathscr{E}, \mathscr{F})\) is compactly embedded into \( L^{2p}(\mathbb{E}; \mu) \).

The first part of Theorem 1.1(2) is an extension of recent seminal work [23] by Takeda on the case \( p = 1 \) for symmetric Markov processes in Class (T), and the second part of Theorem 1.1(2) is a slight extension of [23, Corollary 4.3].

Let \( S^p_{CK,\infty}(X) \) denote the class of 0-order \( L^p \)-semi-Green-tight measures of extended \( L^p \)-Kato class in the sense of Chen and \( S^p_{K}(X) \) the class of 0-order \( L^p \)-semi-Green-tight measures of extended \( L^p \)-Kato class in the sense of Zhao (see Definition 2.3 below).

Our second theorem is the following:
Theorem 1.2 Suppose that $X$ is transient and it possesses (RSF). Then we have

1. $S^p_{\infty}(X) = S^p_{CK_{\infty}}(X)$.  
2. $S^p_{K_{1}}(X) \cap S^p_{\infty}(X) = S^p_{CK_{1}}(X) \cap S^p_{\infty}(X)$.

This is an extension of [19, Lemma 4.1] which treats the case $p = 1$.

To state the third theorem, we need the following conditions on heat kernel global estimates: We consider $\nu, \beta \in [0, +\infty]$ and $t_0 \in [0, +\infty]$.

(A) 1 (Life time condition) $X$ has the following property that
$$ \lim_{t \to 0} \sup_{x \in E} P_t(\zeta \leq t) =: \gamma \in [0, 1]. $$
In particular, if $X$ is stochastically complete, that is, $X$ is conservative, then this condition is satisfied with $\gamma = 0$.

We fix an increasing positive function $V$ on $[0, +\infty]$.

(A) 2 (Bishop type inequality) Suppose that $r \mapsto V(r)/r^\nu$ is increasing or bounded, and $\sup_{x \in E} m(B_r(x)) \leq V(r)$ for all $r > 0$.

(A) 3 (Upper and lower estimates of heat kernel) Let $\Phi_i$ ($i = 1, 2$) be positive decreasing functions defined on $[0, +\infty]$ which may depend on $t_0$ if $t_0 < +\infty$, let $\nu, \beta > 0$ and assume that $\Phi_2$ satisfies the following condition $H(\Phi_2)$:
$$ \int_1^\infty \frac{(V(t) \vee t)^{\nu} \Phi_2(t)}{t} \, dt < +\infty $$
and $(\Phi E_{\nu, \beta})$: for any $x, y \in E$, $t \in [0, t_0]$
$$ \frac{1}{\nu/\beta} \Phi_1 \left( \frac{d(x, y)}{t^{1/\beta}} \right) \leq p_t(x, y) \leq \frac{1}{\nu/\beta} \Phi_2 \left( \frac{d(x, y)}{t^{1/\beta}} \right). $$

Definition 1.3 (Kato class $K^{p, \nu}_{\nu, \beta}$) Fix $\nu > 0$ and $\beta > 0$. For a positive Borel measure $\mu$ on $E$, $\mu$ is said to be of $L^p$-Kato (p-Kato in short) class relative to Green kernel (write $\mu \in K^{p, \nu}_{\nu, \beta}$) if
$$ \lim_{r \to 0} \sup_{x \in E} \int_{G(x, y)^{\nu}} G(x, y)^{p} \mu(dy) = 0 \text{ for } \nu \geq \beta, $$
and
$$ \sup_{x \in E} \int_{G(x, y)^{\nu} \leq 1} \mu(dy) < +\infty \text{ for } \nu < \beta, $$
where $G(x, y) := G(d(x, y))$ with
$$ G(r) := \begin{cases} \frac{\nu^{\beta}}{\log(r)} & \nu > \beta, \\ \nu^{-\beta} & \nu = \beta, \end{cases} \quad r \in [0, +\infty], $$
and $r \in [0, 1]$.

For a positive Borel measure $\mu$ on $E$, $\mu$ is said to be of local $L^p$-Kato (p-Kato in short) class relative to Green kernel (write $\mu \in K^{p, \nu}_{\nu, \beta}$) if $1_G \mu \in K^{p, \nu}_{\nu, \beta}$ for any relatively compact open set $G$. Clearly $K^{p, \nu}_{\nu, \beta} \subset K^{p, \nu}_{\nu, \beta}$. When $p = 1$, we write $K_{\nu, \beta}$ (resp. $K^{1, \nu}_{\nu, \beta}$) instead of $K^{1, \nu}_{\nu, \beta}$ (resp. $K^{1, \nu}_{\nu, \beta}$). It is shown in [21] Lemma 3.6] that any $\mu \in K^{1, \nu}_{\nu, \beta}$, hence any $\mu \in K_{\nu, \beta}$, is a Radon measure.

Definition 1.4 ($L^p$-Green-tight measures of $L^p$-Kato class, $K^{p, \nu, \beta}_{\nu, \beta}$) Assume $\nu > \beta$ and the upper estimate in (A) 3 holds with $t_0 = +\infty$. In this case $X$ is transient (see Section 5 below). A positive Borel measure $\mu$ is said to be a $L^p$-Green-tight measure of $L^p$-Kato class (in terms of Green kernel) if $\mu \in K^{p, \nu}_{\nu, \beta}$ and for some $o \in E$ it holds that
$$ \lim_{R \to \infty} \sup_{x \in E} \int_{G(x, y)^{\nu} \geq R} G(x, y)^{p} \mu(dy) = 0, $$
where $G(x, y) = 1/d(x, y)^{\nu-\beta}$. Denote by $K^{p, \nu, \beta}_{\nu, \beta}$ the class of all $L^p$-Green-tight measures of $L^p$-Kato class in terms of Green kernel.
Now we can state the fourth and fifth theorems.

**Theorem 1.5** Suppose that (A 1) (A 2) and (A 3) hold. We also assume $\nu > \beta$ and the upper estimate in (A 3) holds with $t_0 = +\infty$. Then we have the following:

1. Assume $\mu(E) < +\infty$, or $\mu \in S_{K,\infty}^{1}(X)$ with $p > 1$. Then $\mu \in S_{K}^{p}(X)$ is equivalent to $\mu \in S_{CK,\infty}^{p}(X)$.

2. It holds that $S_{CK,\infty}^{p}(X) = S_{K,\infty}^{p}(X)$. If further the lower estimate in (A 3) also holds with $t_0 = +\infty$, then $S_{CK,\infty}^{p}(X) = S_{K,\infty}^{p}(X) = K_{\nu,\beta}$.

Since the 1-subprocess $X^{(1)}$ is always transient, a similar result also holds.

**Theorem 1.6** Suppose that (A 1) (A 2) and (A 3) hold. Then we have the following:

1. Assume $\mu(E) < +\infty$, or $\mu \in S_{K,\infty}^{1}(X^{(1)})$ with $p > 1$. Then $\mu \in S_{K}^{p}(X)$ is equivalent to $\mu \in S_{CK,\infty}^{p}(X^{(1)})$.

2. It holds that $S_{CK,\infty}^{p}(X^{(1)}) = S_{K,\infty}^{p}(X^{(1)})$.

The constitution of this paper is as follows. In Section 2, we prepare our framework and explain several definitions. In Section 3, we give the proof of Theorem 1.1. In Section 4, we give the proof of Theorem 1.2. In Section 5, we give the proofs of Theorems 1.3 and 1.5. In Section 6, we investigate the various type of compact embeddings of Sobolev spaces in the framework of $d$-dimensional Brownian motion and rotationally symmetric relativistic $\alpha$-stable processes on $\mathbb{R}^d$.

## 2 Preliminary

For real numbers $a, b \in \mathbb{R}$, we set $a \vee b := \max\{a, b\}$ and $a \wedge b := \min\{a, b\}$. Let $(E, d)$ be a locally compact separable metric space and $m$ a positive Radon measure with full support. Let $E_B := E \cup \{\partial\}$ be the one-point compactification of $E$. For each $x \in E$ and $r > 0$, denote by $B_r(x) := \{y \in E : d(x, y) < r\}$ the open ball with center $x$ and radius $r$. We consider and fix a symmetric regular Dirichlet form $(\mathcal{E}, \mathcal{F})$ on $L^2(E; m)$. Then there exists a Hunt process $X = (\Omega, \mathcal{F}, \xi, \mathbb{P}, \mathbb{P})$ such that for each Borel $u \in L^2(E; m)$, $T_t u(x) = \mathbb{E}_x[u(X_t)]$ $\mathbb{P}$-a.e. $x \in E$ for all $t > 0$, where $(T_t)_{t > 0}$ is the semigroup associated with $(\mathcal{E}, \mathcal{F})$. Here $\xi := \inf\{t \geq 0 \mid X_t = \partial\}$ denotes the life time of $X$. For a Borel set $B$, we denote $\sigma_B := \inf\{t > 0 \mid X_t \in B\}$ (resp. $\tau_E := \inf\{t > 0 \mid X_t \notin E\}$) the first hitting time to $B$ (resp. first exit time from $B$). Throughout this paper, we assume that $X$ satisfies (AC). Under (AC), there exists a jointly measurable function $p_t(x, y)$ defined for all $(t, x, y) \in [0, +\infty) \times E \times E$ such that $\mathbb{E}_x[u(X_t)] = \int_E p_t(x, y)u(y)m(dy)$ for any $x \in E$, bounded Borel function $u$ and $t > 0$ (see [35], Theorem 2). $p_t(x, y)$ is said to be a semigroup kernel, or sometimes called a heat kernel of $X$ on the analogy of heat kernels of diffusions. Then $P_t$ can be extended to contractive semigroups on $L^p(E; m)$ for $p \geq 1$. The following are well-known:

1. $p_{t+s}(x, y) = \int_E p_t(x, z)p_s(z, y)m(dz)$ for all $x, y \in E$ and $t, s > 0$.

2. $P_t(x, dy) = p_t(x, y)m(dy)$ for all $x \in E$ and $t > 0$.

3. $\int_E p_t(x, y)m(dy) \leq 1$ for all $x \in E$ and $t > 0$.

We define $R_\alpha(x, y) := \int_0^{\infty} e^{-\alpha t}p_t(x, y)dt$, and $R(x, y) := R_0(x, y) = \int_0^{\infty} p_t(x, y)\nu(dy)$ for $\alpha > 0$, $x, y \in E$. $R_\alpha(x, y)$ (resp. $R(x, y)$) is called the $\alpha$-order resolvent kernel (resp. 0-order resolvent kernel).

Throughout this paper, we consider a constant $p \in [1, +\infty]$. 

**Definition 2.1** ($L^p$-Kato class $S_{K,\infty}^{p}(X)$, extended $L^p$-Kato class $S_{E_K,\infty}^{p}(X)$, $L^p$-Dynkin class $S_{D}^{p}(X)$) A positive Radon measure $\nu$ on $E$ is said to be of $L^p$-Kato class (write $\nu \in S_{K,\infty}^{p}(X)$) if

$$\lim_{\alpha \to \infty} \sup_{x \in E} \int_E R_\alpha(x, y)^p\nu(dy) = 0.$$ (2.1)
A positive Radon measure \( \nu \) on \( E \) is said to be of extended \( L^p \)-Kato class (write \( \nu \in S_{EK}^p(X) \)) if

\[
\lim_{\alpha \to \infty} \sup_{x \in E} \int_E R_\alpha(x, y)^p \nu(dy) < 1.
\]

A positive Radon measure \( \nu \) on \( E \) is said to be of \( L^p \)-Dynkin class (write \( \nu \in S_D^p \)) if

\[
\sup_{x \in E} \int_E R_\alpha(x, y)^p \nu(dy) < +\infty \quad \text{for some } \alpha > 0.
\]

We denote \( R_\alpha^p(x) := \int_E R_\alpha(x, y)^p \nu(dy) \). Then we see \( \nu \in S_K^p(X) \) (resp. \( \nu \in S_{EK}^p(X) \)) if and only if \( \lim_{\alpha \to \infty} \|R_\alpha^p\|_\infty = 0 \) (resp. \( \lim_{\alpha \to \infty} \|R_\alpha^p\|_\infty < 1 \)), and \( \nu \in S_D^p(X) \) if and only if \( \|R_\alpha^p \|_\infty < +\infty \) for some/all \( \alpha > 0 \) (see [25 Proposition 2.6]). Clearly, \( S_K^p(X) \subset S_{EK}^p(X) \subset S_D^p(X) \). A positive Radon measure \( \nu \) on \( E \) is said to be of local \( L^p \)-Kato class (write \( \nu \in S_{LK}^p(X) \)) if \( 1_{G} \nu \in S_K^p(X) \) for any relatively compact open set \( G \). When \( p = 1 \), we may write \( S_D(X) \) (resp. \( S_{K}(X), S_{EK}(X), S_{LK}(X) \)) instead of \( S_{D}^1(X) \) (resp. \( S_{K}^1(X), S_{EK}^1(X), S_{LK}^1(X) \)) for simplicity.

To the end of Section 4 we basically assume that \( X \) is transient.

**Definition 2.2 (\( L^p \)-Green-bounded measures)** A (positive) Radon measure \( \nu \) on \( E \) is said to be of \( L^p \)-Green-bounded if

\[
\|R^p \nu\|_\infty := \sup_{x \in E} R^p \nu(x) < +\infty,
\]

where

\[
R^p \nu(x) := \int_E R(x, y)^p \nu(dy).
\]

We denote by \( S_{D_0}^p(X) \) the class of \( L^p \)-Green-bounded measures.

We now introduce the notion of \( L^p \)-Green-tight measures of \( L^p \)-Kato class.

**Definition 2.3 (\( L^p \)-Green-tight measures of \( L^p \)-Kato class in the sense of Zhao)** A (positive) Radon measure \( \nu \) on \( E \) is said to be an \( L^p \)-Green-tight measure of \( L^p \)-Kato class in the sense of Zhao if \( \nu \in S_{D_0}^p(X) \cap S_{K}^p(X) \) and for any \( \varepsilon > 0 \) there exists a compact set \( K \) such that

\[
\|R^p 1_{K^C} \nu\|_\infty := \sup_{x \in E} R^p 1_{K^C} \nu(x) < \varepsilon.
\]

A (positive) Radon measure \( \nu \) on \( E \) is said to be an \( L^p \)-semi-Green-tight measure of extended \( L^p \)-Kato in the sense of Zhao if \( \nu \in S_{D_0}^p(X) \cap S_{EK}^p(X) \) and there exists a compact set \( K \) such that

\[
\|R^p 1_{K^C} \nu\|_\infty := \sup_{x \in E} R^p 1_{K^C} \nu(x) < 1.
\]

We denote by \( S_{K}^p(X) \) the class of \( L^p \)-Green-tight measures of \( L^p \)-Kato class in the sense of Zhao, and by \( S_{K_1}^p(X) \) the class of \( L^p \)-semi-Green-tight measures of extended \( L^p \)-Kato class in the sense of Zhao. Clearly, we see \( S_{K}^p(X) \subset S_{K_1}^p(X) \subset S_{D_0}^p(X) \cap S_{EK}^p(X) \).

**Definition 2.4 (\( L^p \)-Green-tight measures of \( L^p \)-Kato class in the sense of Chen)** A (positive) Radon measure \( \nu \) on \( E \) is said to be an \( L^p \)-Green-tight measure of \( L^p \)-Kato class in the sense of Chen if for any \( \varepsilon > 0 \) there exist a Borel set \( K = K(\varepsilon) \) with \( \nu(K) < +\infty \) and \( \delta = \delta(\varepsilon) > 0 \) such that

\[
\|R^p 1_{K \cup \cup B} \nu\|_\infty := \sup_{x \in E} R^p 1_{K \cup \cup B} \nu(x) < \varepsilon
\]

holds for any Borel subset \( B \) of \( K \) with \( \nu(B) < \delta \).

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A (positive) Radon measure $\nu$ on $E$ is said to be an $L^p$-semi-Green-tight measure of extended $L^p$-Kato in the sense of Chen if there exist a Borel set $K$ with $\nu(K) < +\infty$ and $\delta > 0$ such that

$$\|R^p 1_{K \cup_B} \nu\|_\infty := \sup_{x \in E} R^p 1_{K \cup_B} \nu(x) < 1$$

holds for any Borel subset $B$ of $K$ with $\nu(B) < \delta$. We denote by $S^p_{\text{CK}_{\infty}}(X)$ the class of $L^p$-Green-tight measures of $L^p$-Kato class in the sense of Chen, and by $S^p_{\text{CK}_1}(X)$ the class of $L^p$-semi-Green-tight measures of extended $L^p$-Kato class in the sense of Chen. Clearly, we see $S^p_{\text{CK}_{\infty}}(X) \subset S^p_{\text{CK}_1}(X)$.

**Remark 2.5** In the definitions for $S^p_{\text{CK}_{\infty}}(X)$ and $S^p_{\text{CK}_1}(X)$, the Borel set $K$ can be taken to be a closed (or open), and a compact set. This is remarked in [2, remark after Definition 2.2] in the case of $p = 1$. The same thing also holds for general $p$.

Now we state several propositions on $S^p_{\text{CK}_{\infty}}(X)$ and $S^p_{\text{CK}_1}(X)$.

**Proposition 2.6** Suppose that $X$ is transient. Then it holds that $S^p_{\text{CK}_1}(X) \subset S^p_{D_0}(X)$.

**Proof.** Suppose $\nu \in S^p_{\text{CK}_1}(X)$. Then there exist a Borel set $K$ with $\nu(K) < +\infty$ and $\delta > 0$ such that (2.9) holds for any Borel subset $B$ of $K$ with $\nu(B) < \delta$. The Borel set $K$ can be taken to be compact so that $K$ can be covered by finitely many Borel (relatively open in $K$) subsets $\{B_i\}_{i=1}^n$ of $K$ with $\nu(B_i) < \delta$. Therefore

$$\sup_{x \in E} \int_{E} R(x,y)^p \nu(dy) \leq \sum_{i=1}^{n} \sup_{x \in E} \int_{K \cup_i B} R(x,y)^p \nu(dy) < \ell < +\infty.$$

To prove the inclusions $S^p_{\text{CK}_{\infty}}(X) \subset S^p_K(X)$ and $S^p_{\text{CK}_1}(X) \subset S^p_{\text{CK}}(X)$, we need the following lemmas.

**Lemma 2.7** Let $f$ be a (nearly) Borel function on $E$. Suppose that for any $x \in E$ it holds that

$$P_x \left( f(X_0) \leq \lim_{t \to 0} f(X_t) \right) = 1.$$

Then $f$ is finely lower semi-continuous on $E$.

**Proof.** Set $B := \{ x \in E \mid f(x) > \beta \}$ for $\beta \in \mathbb{R}$. Then $B$ is a nearly Borel set. On the event $\{ \sigma_{E \setminus B} = 0 \}$, there exists a decreasing sequence $\{ t_n \}$ converging to 0 such that $X_{t_n} \in E \setminus B$ for all $n \in \mathbb{N}$, and then $\lim_{n \to 0} f(X_{t_n}) \leq \lim_{t \to 0} f(X_{t}) \leq \beta$. By combining this with the assumption, we have for any $x \in B$

$$P_x (\sigma_{E \setminus B} = 0) \leq P_x (f(X_0) \leq \beta) = P_x (x \in E \setminus B) = 0.$$

Thus we have $P_x (\sigma_{E \setminus B} > 0) = 1$ for all $x \in B$, i.e., $B$ is a finely open set. Therefore $f$ is finely lower semi-continuous.

**Lemma 2.8** For any $\nu \in S^p_{D_0}(X)$ (resp. $\nu \in S^p_{D_0}(X)$ in transient case), the $p$-potential function $R^p \nu$ (resp. $R^p \nu$) is Borel and finely lower semi-continuous on $E$.

**Proof.** First note that $(x,y) \mapsto R_{\alpha}(x,y)$ is $\mathcal{B}(E) \times \mathcal{B}(E)$-measurable in view of the joint measurability of $(t,x,y) \mapsto p_t(x,y)$. From this, we can deduce the Borel measurability of $x \mapsto \int_E R_{\alpha}(x,y)^p \nu(dy)$. Since $x \mapsto R_{\alpha}(x,y)$ is a Borel finely continuous function for a fixed $y \in E$, we have

$$P_x \left( \lim_{t \to 0} R_{\alpha}(X_t,y) \neq R_{\alpha}(X_0,y) \right) = 0.$$

Applying Fubini’s theorem to the jointly measurable function $1_{\{ \lim_{t \to 0} R_{\alpha}(X_t,y) \neq R_{\alpha}(X_0,y) \}}(\omega, y)$, we have

$$P_x \left( \lim_{t \to 0} R_{\alpha}(X_t,y) = R_{\alpha}(X_0,y) \quad \nu\text{-a.e. } y \in E \right) = 1.$$
This implies
\[
\int_E R_\alpha(x,y)^p \nu(dy) = \int_{t \to 0} \lim_{t \to 0} R_\alpha(x_t, y)^p \nu(dy) \\
\leq \lim_{t \to 0} \int_E R_\alpha(x_t, y)^p \nu(dy) \quad \text{P-a.s.}
\]
Therefore \(x \mapsto \int_E R_\alpha(x,y)^p \nu(dy)\) is finely lower semi-continuous by Lemma 2.7. The proof for the case \(\alpha = 0\) under the transience of \(X\) is similar. \(\square\)

**Proposition 2.9** Suppose that \(X\) is transient. Then we have the following inclusions.

1. \(S_{CK,\infty}^p(X) \subset S_{K}^p(X)\).
2. \(S_{CK_1}^p(X) \subset S_{EK}^p(X)\).

**Proof.**

1. Suppose \(\nu \in S_{CK,\infty}^p(X)\). Then for any \(\varepsilon > 0\), there exist a Borel set \(K = K(\varepsilon)\) with \(\nu(K) < +\infty\) and \(\delta = \delta(\varepsilon) > 0\) such that (2.8) holds for any subset \(B\) of \(K\) with \(\nu(B) < \delta\). We may assume that \(K\) is a compact set. Set \(B := \{x \in K \mid \int_B R_\alpha(x,y)^p \nu(dy) > \varepsilon\}\). Since \(\lim_{t \to \infty} \int_B R_\alpha(x,y)^p \nu(dy) = 0\) for each fixed \(x \in E\) and \(\nu(K) < +\infty\), we have \(\nu(B) < \delta\) for sufficiently large \(\alpha > 0\). Moreover, the set \(K \setminus B = \{x \in K \mid \int_E R_\alpha(x,y)^p \nu(dy) \leq \varepsilon\}\) is a finely closed Borel set by virtue of Lemma 2.8. Applying Frostman’s maximum principle

\[
\sup_{x \in E} R_\alpha^p 1_{K \setminus B} \nu(x) = \sup_{x \in K \setminus B} R_\alpha^p 1_{K \setminus B} \nu(x), \quad (2.10)
\]

which was proved by [23] (3.5) for \(\alpha = 0\), and its proof remains valid for general \(\alpha > 0\), we obtain

\[
\sup_{x \in E} R_\alpha^p \nu(x) \leq \sup_{x \in E} R_\alpha^p (1_{K \cup B} \nu)(x) + \sup_{x \in E} R_\alpha^p (1_{K \setminus B} \nu)(x) \\
\leq \varepsilon + \sup_{x \in K \setminus B} R_\alpha^p (1_{K \setminus B} \nu)(x) \\
\leq \varepsilon + \sup_{x \in K \setminus B} R_\alpha^p \nu(x) \leq 2\varepsilon.
\]

Hence \(\lim_{t \to \infty} \sup_{x \in E} R_\alpha^p \nu(x) = 0\).

2. Suppose \(\nu \in S_{CK_1}^p(X)\). Then there exist a Borel set \(K\) with \(\nu(K) < +\infty\) and \(\delta > 0\) such that (2.9) holds for any Borel subset \(B\) of \(K\) with \(\nu(B) < \delta\). We may assume that \(K\) is a compact set. For \(0 < \varepsilon < 1 - \sup_{B \subset K, \nu(B) < \delta} \|R_\alpha^p (1_{K \cup B} \nu)\|_\infty\), we set

\[
B := \left\{x \in K \mid \int_E R_\alpha(x,y)^p \nu(dy) > 1 - \varepsilon - \sup_{B \subset K, \nu(B) < \delta} \|R_\alpha^p (1_{K \cup B} \nu)\|_\infty\right\}.
\]

Then \(\nu(B) < \delta\) for sufficiently large \(\alpha > 0\) and \(K \setminus B\) is finely closed as proved in (1). Applying (2.10), we obtain

\[
\sup_{x \in E} R_\alpha^p \nu(x) \leq \sup_{x \in E} R_\alpha^p (1_{K \cup B} \nu)(x) + \sup_{x \in E} R_\alpha^p (1_{K \setminus B} \nu)(x) \\
\leq \sup_{B \subset K, \nu(B) < \delta} \|R_\alpha^p (1_{K \cup B} \nu)\|_\infty + \sup_{x \in K \setminus B} R_\alpha^p (1_{K \setminus B} \nu)(x) \\
\leq 1 - \varepsilon < 1.
\]

Hence \(\lim_{t \to \infty} \sup_{x \in E} R_\alpha^p \nu(x) < 1\). \(\square\)

**Remark 2.10** From Propositions 2.6 and 2.9 we have the following inclusions:
Proposition 3.1 (following is essentially proved in [25, Theorem 4.1]):

1. $S^p_{CK\infty}(X) \subset S^p_{K\infty}(X)$.
2. $S^p_{CK1}(X) \subset S^p_{K1}(X)$.

The following proposition is an extension of [13 Proposition 4.1].

Proposition 2.11 Suppose that $X$ is transient and assume $m \in S^p_{D0}(X)$. Then we have the following:

1. $S^p_{D0}(X) = S^p_{D1}(X)$.
2. $S^p_{K\infty}(X) = S^p_{K\infty}(X^{(1)})$.
3. $S^p_{CK\infty}(X) = S^p_{CK\infty}(X^{(1)})$.

Proof. Take a positive Radon measure $\nu$ on $E$ and fix $x \in X$. By Hölder’s inequality, we have

$$
\int_E \left( \int_E R(x, z) R_\alpha(z, y) m(dz) \right)^p \nu(dy)
= \int_E \left\{ \int_E \left( \int_E R(x, z) R_\alpha(z, y) m(dz) \right)^{p-1} R_\alpha(z, y) \nu(dy) \right\} R(x, z) m(dz)
\leq \left\{ \int_E \left( \int_E R(x, z) R_\alpha(z, y) m(dz) \right)^p \nu(dy) \right\}^{\frac{p}{p-1}} \| R^p_\alpha \nu \|_\infty \| Rm \|_\infty,
$$

that is, it holds that

$$
\left\{ \int_E \left( \int_E R(x, z) R_\alpha(z, y) m(dz) \right)^p \nu(dy) \right\}^{\frac{1}{p}} \leq \| R^p_\alpha \nu \|_\infty \| Rm \|_\infty.
$$

Thanks to the resolvent equation

$$
R(x, y) = R_\alpha(x, y) + \alpha \int_E R(x, z) R_\alpha(z, y) m(dz),
$$

we have

$$
R^p_\alpha(x, y) \frac{1}{p} = \left( \int_E R(x, y)^p \nu(dy) \right)^{\frac{1}{p}}
= \left( \int_E \left( R_\alpha(x, y) + \alpha \int_E R(x, z) R_\alpha(z, y) m(dz) \right)^p \nu(dy) \right)^{\frac{1}{p}}
\leq \left( \int_E R_\alpha(x, y) \nu(dy) \right)^{\frac{1}{p}} + \alpha \left( \int_E \left( \int_E R(x, z) R_\alpha(z, y) m(dz) \right)^p \nu(dy) \right)^{\frac{1}{p}}
\leq \| R^p_\alpha \nu \|_\infty \frac{1}{p} + \alpha \| R^p_\alpha \nu \|_\infty \| Rm \|_\infty,
$$

that is, $\| R^p \nu \|_\infty \leq (1 + \alpha \| Rm \|_\infty)^p \| R^p \nu \|_\infty$ holds. This implies the each assertion.

\[\square\]

3 Proof of Theorem [17]

At the beginning of the section, we state a useful inequality to estimate the $L^{2p}$-norm of functions. The following is essentially proved in [25 Theorem 4.1]:

Proposition 3.1 (p-version of Stollmann-Voigt’s inequality) Suppose that $X$ is transient and let $\nu \in S^p_{D0}(X)$. Then, it holds that

$$
\| u \|_{L^{2p}(E, m)}^2 \leq \| R^p \nu \|_\infty \mathcal{E}(u, u), \quad \text{for all} \quad u \in \mathcal{F}_e.
$$

(3.1)

In the following, we omit “p-version” and simply call this Stollmann-Voigt’s inequality.

Lemma 3.2 Set $\mathcal{A}_M := \{ u \in \mathcal{F}_e \mid \mathcal{E}(u, u) \leq M \}$. Suppose $m \in S^p_{CK\infty}(X)$. Then it holds that

$$
\lim_{L \to \infty} \sup_{u \in \mathcal{A}_M} \int_{\{ u \geq L \}} u^{2p} dm = 0.
$$

(3.2)
Proof. Fix $\varepsilon > 0$. Since $m \in S^p_{\mathcal{C}\mathcal{K}_\infty}(X) \subset S^p_{K_\infty}(X)$, there exist a compact set $K$ and $\delta > 0$ such that

$$\sup_{x \in E} \int_{K \cup B} R(x, y)^p m(dy) < \varepsilon$$

(3.3)

holds for any subset $B$ of $K$ with $m(B) < \delta$. By applying Stollmann-Voigt’s inequality \[[3,1]\], we see that for sufficiently large $L > 0$

$$\sup_{u \in \mathcal{A}_M} m(\{u^{2p} \geq L\}) \leq \frac{1}{L} \sup_{u \in \mathcal{A}_M} \int_E u^{2p} dm$$

$$\leq \frac{1}{L} \|R^p m\|_\infty \sup_{u \in \mathcal{A}_M} \mathcal{E}(u, u)^p \leq \frac{\|R^p m\|_\infty M_p}{L} < \delta.$$

We regard $B := \{u^{2p} \geq L\} \cap K$ and apply Stollmann-Voigt’s inequality \[(3.1)\] to $1_{K \cup B} m$. Then we have

$$\int_{\{u^{2p} > L\}} u^{2p} dm \leq \int_{K \cup B} u^{2p} dm \leq \|R^p 1_{K \cup B}\|_\infty \mathcal{E}(u, u)^p \leq M_p \varepsilon.$$

Hence the conclusion follows. \hfill $\square$

**Lemma 3.3** Suppose $m \in S^p_{\mathcal{C}\mathcal{K}_\infty}(X)$. If $\{g_n\}_{n=1}^\infty \subset \mathcal{F}$ is a sequence with $\sup_{n \in \mathbb{N}} \mathcal{E}(g_n, g_n) < +\infty$, and satisfies $g_n \to g$ m-a.e. (or in m-measure), then $g_n$ converges to $g$ in $L^{2p}(E;m)$. \hfill $\square$

**Proof.** Since $\{g_n\}$ is $\mathcal{E}$-bounded, it is $L^{2p}$-bounded by Stollmann-Voigt’s inequality \[(3.1)\]. The uniform integrability of $\{g_n^{2p}\}$ is obtained in Lemma \[3.2\] and hence Vitali’s Theorem (see \[26, Theorem 16.6\]) for example) gives the conclusion. \hfill $\square$

**Proposition 3.4** Let $p_t(x, \cdot)$ be the heat kernel of $X$. Then for each $t > 0$ and $x \in E$, $p_t(x, \cdot) \in \mathcal{F}$ and $\mathcal{E}(p_t(x, \cdot), p_t(x, \cdot)) \leq \frac{1}{t^2} p_t(x, x)$. Moreover, if $X$ is transient and $m \in S^p_{D_0}(X)$ for $p > 1$, then $R(p_t(x, \cdot)) \in \mathcal{F}$ for all $x \in E$.

**Proof.** It is easy to see that $p_t(x, \cdot) \in L^2(E;m)$ for $t > 0$ and $x \in E$, from

$$\int_E p_t(x, y)^2 m(dy) = p_{2t}(x, x) < +\infty.$$

Hence $p_t(x, \cdot) = P_{t/2}(P_{t/2}(x, \cdot)) \in \mathcal{F}$ and by \[23, Lemma 4.1\] we see

$$\mathcal{E}(p_t(x, \cdot), p_t(x, \cdot)) = \mathcal{E}(P_{t/2}(P_{t/2}(x, \cdot)), P_{t/2}(P_{t/2}(x, \cdot)))$$

$$\leq \frac{1}{t^2} (P_{t/2}(x, \cdot), P_{t/2}(x, \cdot))_m = \frac{1}{t^2} p_t(x, x) < +\infty.$$

Suppose further that $X$ is transient and $m \in S^p_{D_0}(X)$ for $p > 1$. Since $m \in S^p_{D_0}(X)$ for $p > 1$, by \[31, Theorem 1\], \[(3.1)\] implies the ultra-contractivity, i.e., there exists $C > 0$ such that

$$\|P_t\|_{L^1(E;m) \to L^\infty(E;m)} \leq C t^{-\frac{1}{p-1}}$$

for all $t > 0$, 

(3.4)

equivalently

$$p_t(x, y) \leq C t^{-\frac{1}{p-1}}$$

for all $x, y \in E$ and $t > 0$. 

(3.5)

From this, we have

$$\int_E R(p_t(x, \cdot))(y)p_t(x, y)m(dy) = \int_{2t}^{\infty} p_s(x, x)ds \leq C(p-1)(2t)^{-\frac{1}{p-1}} < +\infty$$

(3.6)

for all $x \in E$. Then

$$\sup_{\alpha > 0} \mathcal{E}(R_\alpha(p_t(\cdot, \cdot)), R_\alpha(p_t(\cdot, \cdot))) \leq \sup_{\alpha > 0} \mathcal{E}_\alpha(R_\alpha(p_t(\cdot, \cdot)), R_\alpha(p_t(\cdot, \cdot))) = \sup_{\alpha > 0} (p_t(\cdot, \cdot), R_\alpha(p_t(\cdot, \cdot)))_m$$

$$\leq (p_t(\cdot, \cdot), R(p_t(\cdot, \cdot)))_m < +\infty$$

so that $R(p_t(\cdot, \cdot)) \in \mathcal{F}$. \hfill $\square$
Lemma 3.5 Suppose m ∈ \(S^p_{\text{CK}_\infty}(X)\). If \(\{g_n\}_{n=1}^\infty \subset \mathcal{F}_c(\subset L^p(E;m))\) is an \(\mathcal{E}\)-bounded sequence, then there exists a subsequence \(\{g_{n_k}\}_{k=1}^\infty\) such that \(\{P_k g_{n_k}\}_{k=1}^\infty\) \(L^p(E;m)\)-converges.

Proof. First note that \(P_k(\mathcal{F}_c) \subset \mathcal{F}_c\) and \(\mathcal{E}(P_k g, P_l g) \leq \mathcal{E}(g, g)\) for \(g \in \mathcal{F}_c\) holds by \cite{16} Lemma 1.5.4. Suppose \(m \in S^p_{\text{CK}_\infty}(X)\) with \(p > 1\). In this case \(R(p_k(\cdot), \cdot) \in \mathcal{F}_c\) for all \(x \in E\) and \(t > 0\) by Lemma 5.3. Since \(\{g_n\} \subset \mathcal{F}_c\) is \(\mathcal{E}\)-bounded, there exist a subsequence \(\{g_{n_k}\}\) and \(g \in \mathcal{F}_c\) such that \(\{g_{n_k}\}\) \(\mathcal{E}\)-weakly converges to \(g\). Then

\[
P_k g_{n_k}(x) = \int_E p_k(x, y) g_{n_k}(y) m(\mathrm{d}y) = \mathcal{E}(\mathcal{E}(p_k(\cdot, \cdot)), g_{n_k})
\]

\[
\rightarrow \mathcal{E}((\mathcal{E}(p_k(\cdot, \cdot)), g) = \int_E p_k(x, y) g(y) m(\mathrm{d}y) = P_k g(x)
\]

as \(k \to \infty\) for all \(x \in E\). Next we suppose \(p = 1\) with \(m \in S^p_{\text{CK}_\infty}(X) \subset S^p_{\text{DK}_0}(X) = S^p_{\text{DK}^1}(X)\). In this case,

\[
\sup_{n \in \mathbb{N}} \|g_n\|^2_{L^p(E;m)} \leq \|Rm\|_{\infty} \sup_{n \in \mathbb{N}} \mathcal{E}(g_n, g_n) \leq +\infty
\]

implies that there exist a subsequence \(\{g_{n_k}\}\) and \(g \in L^p(E;m)\) such that \(\{g_{n_k}\}\) converges to \(g\) \(L^p(E;m)\)-weakly. Then

\[
P_k g_{n_k}(x) = \int_E p_k(x, y) g_{n_k}(y) m(\mathrm{d}y) \rightarrow \int_E p_k(x, y) g(y) m(\mathrm{d}y) = P_k g(x)
\]

as \(k \to \infty\) for all \(x \in E\). Therefore \(P_k g_{n_k} \rightarrow P_k g\) in \(L^p(E;m)\) in view of Lemma 3.3. □

Theorem 3.6 Suppose that \(X\) is transient and \(m \in S^p_{\text{CK}_\infty}(X)\). Then the semigroup \(P_t : \mathcal{F}_c \rightarrow L^p(E;m)\) is a compact operator.

Proof. Let \(\{g_n\} \subset \mathcal{F}_c\) be a sequence \(\mathcal{E}\)-weakly converges to \(g \in \mathcal{F}_c\). Then by Lemma 3.5, there exists a subsequence \(\{g_{n_k}\}_{k=1}^\infty\) such that \(\{P_k g_{n_k}\}_{k=1}^\infty\) is \(L^p(E;m)\)-convergent to some function \(h\). Since \(\{g_{n_k}\}\) \(\mathcal{E}\)-weakly converges to \(g\), there exists a subsequence \(\{g_{n_k}\}\) such that the Cesáro mean \(\frac{1}{N} \sum_{k=1}^N P_k g_{n_k}\) converges to \(g\) in \((\mathcal{F}_c, \mathcal{E})\) by Banach-Saks Theorem, hence \(\frac{1}{N} \sum_{k=1}^N P_k g_{n_k}\) converges to \(P_k g\) in \((\mathcal{F}_c, \mathcal{E})\), in particular, in \(L^p(E;m)\). Therefore \(h = P_k g\) in \(L^p(E;m)\). □

Theorem 3.7 We have the following:

1. Suppose that \(X\) is transient and let \(m \in S^p_{\text{CK}_\infty}(X)\). Then the embedding \(\mathcal{F}_c \hookrightarrow L^p(E;m)\) is compact.
2. Let \(m \in S^p_{\text{CK}_\infty}(X^{(1)})\). Then the embedding \(\mathcal{F} \hookrightarrow L^p(E;m)\) is compact.

Proof. (2) follows from (1). We only prove (1). Suppose that \(\{u_n\}_{n=1}^\infty \subset \mathcal{F}_c\) converges to \(u \in \mathcal{F}_c\) \(\mathcal{E}\)-weakly. We write \(u_n^{(k)} := (-k) \lor u_n \land k\) and \(u^{(k)} := (-k) \lor u \land k\) for \(k \in \mathbb{N}\). Then

\[
\lim_{n \to \infty} \|u - u_n\|_{L^p(E;m)} \leq \|u - u^{(k)}\|_{L^p(E;m)} + \lim_{n \to \infty} \|u_n - u^{(k)}\|_{L^p(E;m)} + \lim_{n \to \infty} \|u^{(k)} - u_n\|_{L^p(E;m)}. \tag{3.7}
\]

Regarding the first term of (3.7), we have

\[
\|u - u^{(k)}\|_{L^p(E;m)} \leq \int_{\{|u| \geq k\}} u^{2p} m \to 0 \quad \text{as} \quad k \to \infty. \tag{3.8}
\]

Regarding the second term of (3.7), we have

\[
\lim_{n \to \infty} \|u_n - u^{(k)}\|_{L^p(E;m)} \leq \sup_{n \in \mathbb{N}} \int_{\{|u_n| \geq k\}} u_n^{2p} m \to 0 \quad \text{as} \quad k \to \infty, \tag{3.9}
\]
where the convergence follows from the $\mathcal{B}$-boundedness of $\{u_n\}$ and Lemma 3.2. It remains to prove the convergence of the third term, that is, $\lim_{n \to \infty} \|u^{(k)} - u_n^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})} \to 0$ as $k \to \infty$.

Now, we have
\[
\|u^{(k)} - P_t u^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})} \leq \|u^{(k)} - P_t u^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})} \|u^{(k)} - P_t u^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})}^{p-1} \\
\leq \left( \sqrt{\mathcal{E}(u, u^{(k)})} \right)^{\frac{1}{2}} (2k)^{\frac{p-1}{p}} \\
\leq \left( \sqrt{\mathcal{E}(u, u)} \right)^{\frac{1}{2}} (2k)^{\frac{p-1}{p}} \to 0 \quad \text{as} \quad t \to 0. \tag{3.10}
\]

Similarly, we also have
\[
\lim_{n \to \infty} \|u_n^{(k)} - P_t u_n^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})} \leq \left( \sqrt{\sup_{n \in \mathbb{N}} \mathcal{E}(u, u^{(k)})} \right)^{\frac{1}{2}} (2k)^{\frac{p-1}{p}} \to 0 \quad \text{as} \quad t \to 0. \tag{3.11}
\]

By Theorem 3.6, we have $\lim_{n \to \infty} \|P_t u - P_t u_n\|_{L^{2p}(\mathcal{E}, \mathcal{F})} = 0$ for each $t > 0$, and combining this with (3.11), we have
\[
\lim_{n \to \infty} \|u_n^{(k)} - u^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})} \leq \lim_{n \to \infty} \|P_t u_n^{(k)} - P_t u^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})} + \lim_{n \to \infty} \|u_n^{(k)} - u^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})} \to 0 \quad \text{as} \quad k \to \infty. \tag{3.12}
\]

By combining (3.10), (3.11) and (3.12), we have
\[
\lim_{n \to \infty} \|u_n^{(k)} - u^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})} \leq \lim_{n \to \infty} \|u_n^{(k)} - u^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})} + \lim_{n \to \infty} \|P_t u_n^{(k)} - P_t u^{(k)}\|_{L^{2p}(\mathcal{E}, \mathcal{F})} \to 0 \quad \text{as} \quad k \to \infty.
\]

which converges to 0 by letting $t \to 0$ and then $k \to \infty$. Therefore we complete the proof.

\begin{proof}[Proof of Theorem 1.1] (2) is an easy consequence of (1). We only prove (1). Suppose first $\mu \in S^p_{\mathcal{C}, \infty}(\mathbf{X})$. Then $\mu \in S_1(\mathbf{X})$, that is, $\mu$ is a smooth measure in the strict sense with respect to $\mathbf{X}$. Let $A^\mu$ be a positive continuous additive functional in the strict sense associated with $\mu$ and $\mathcal{F}$ its fine support. Let $(\mathbf{X}, \mu)$ be the time changed process of $\mathbf{X}$ with respect to $A^\mu$ and $\hat{R}(x, y)$ be its 0-order resolvent kernel. It is proved in [25] proof of Lemma 3.4 that for all $x \in F$
\[
\hat{R}(x, y) = R(x, y) \quad \mu\text{-a.e. } y \in F. \tag{3.13}
\]

Then we see $\mu \in S^p_{\mathcal{C}, \infty}(\mathbf{X})$. Let $(\mathcal{E}, \mathcal{F}_e)$ be the extended Dirichlet space associated to the time changed process $(\mathbf{X}, \mu)$ (see [19] (6.2.7)]). $(\mathcal{E}, \mathcal{F}_e)$ is given by
\[
\left\{ \begin{array}{l}
\mathcal{F}_e = \{ \varphi = u|_Y \mu\text{-a.e. } | u \in \mathcal{F} \} \\
\mathcal{E}(\varphi, \psi) = \mathcal{E}(\mathcal{P} u, \mathcal{P} u) \quad \varphi \in \mathcal{F}_e, \psi = u|_Y \mu\text{-a.e.}, u \in \mathcal{F}_e.
\end{array} \right.
\]

Here $\mathcal{P}$ denotes the orthogonal projection on $\mathcal{H}_F := (\mathcal{F}_e \cap F)^\perp$ in the Hilbert space $(\mathcal{E}, \mathcal{F}_e)$, $\mathcal{P} u(x) := \mathcal{H} \mathbf{u}(x) = \mathbf{E}[\mathbf{u}(X_T)]$ and $Y = \mathbf{supp}[\mu]$ is the topological support of $\mu$. Here $\mathcal{F}_e \cap F := \{ u \in \mathcal{F} | \tilde{u} = 0 \text{ q.e. on } F \}$. Now suppose that $\{u_n\} \subset \mathcal{F}_e$ $\mathcal{B}$-weakly converges to $u \in \mathcal{F}_e$ in $(\mathcal{E}, \mathcal{F}_e)$. Let $\varphi_n, \varphi \in \mathcal{F}_e$ with $\varphi_n := u_n|_Y$ and $\varphi = u|_Y \mu$-a.e., and take any $\psi \in \mathcal{F}_e$ with $\psi = v|_Y \mu$-a.e. Thus we have
\[
\mathcal{E}(\varphi_n - \varphi, \psi) = \mathcal{E}(\mathcal{P} u_n - \mathcal{P} u, \mathcal{P} v) = \mathcal{E}(u_n - u, \mathcal{P} v) \to 0 \quad \text{as} \quad n \to \infty,
\]

that is, $\{\varphi_n\}$ $\mathcal{B}$-weakly converges to $\varphi$. Since $\mathcal{F}_e$ is compactly embedded into $L^{2p}(Y; \mu)$ by Theorem 3.7, we conclude
\[
\int_Y |u_n - u|^2 p d\mu = \int_Y |\varphi_n - \varphi|^2 p d\mu \to 0 \quad \text{as} \quad n \to \infty.
\]
Next we suppose \( \mu \in S^p_K(X) \) and \( m \in S^p_{DK}(X) \). Then \( m \in S^p_{DK}(X) \), hence \( \mathcal{F} = \mathcal{F} \) by \( \|u\|^2_{L^2(E,m)} \leq \|R_1\|_\infty \delta(u,u) \) for \( u \in \mathcal{F} \).

Applying the former result to this case, we can see the compact embedding \( \mathcal{F} \hookrightarrow L^2(E;m) \). Moreover, let \( \{u_n\} \subset \mathcal{F} = \mathcal{F} \) be an \( \delta \)-bounded sequence. Then there exist a subsequence \( \{u_{n_k}\} \) and \( u \in \mathcal{F} = \mathcal{F} \) such that \( \{u_{n_k}\} \) \( \delta \)-weakly and \( L^2(E;m) \)-strongly converges to \( u \). Since \( \mu \in S^p_K(X) \), we have

\[
\lim_{k,l \to \infty} \|u_{n_k} - u_{n_l}\|^2_{L^2(E,\mu)} \leq \lim_{k,l \to \infty} \|R_{\alpha} \mu\|_\infty \delta_{\alpha}(u_{n_k} - u_{n_l}, u_{n_k} - u_{n_l}) \\
\leq \|R_{\alpha} \mu\|_\infty \sup_{n \in \mathbb{N}} \delta(u_n,u_n) \to 0 \quad \text{as} \quad \alpha \to \infty.
\]

This implies the \( L^{2p}(E;\mu) \)-strong convergence of \( \{u_{n_k}\} \).

The next proposition is an addendum.

**Proposition 3.8** Suppose that the embedding \( \mathcal{F} \hookrightarrow L^{2p}(E;m) \) is continuous. Then the following statements are equivalent.

1. The embedding \( \mathcal{F} \hookrightarrow L^{2p}(E;m) \) is compact.
2. \( P_t : \mathcal{F} \to L^{2p}(E;m) \) is a compact operator for \( t > 0 \).
3. \( P_t : L^2(E;m) \to L^{2p}(E;m) \) is a compact operator for \( t > 0 \).

**Proof.** (3) \( \Rightarrow \) (2) is trivial. The proof of (2) \( \Rightarrow \) (1) is already done in the proof of Theorem 3.7(1) based on Theorem 3.6. We prove (1) \( \Rightarrow \) (3) only. Since \( \delta(P_t f, P_t f) \leq \frac{1}{t^2} \|f\|^2_{L^2(E;m)}, t > 0, f \in L^2(E;m) \) by [23, Lemma 4.1], \( P_t : L^2(E;m) \to \mathcal{F} \) is a bounded operator. Hence \( P_t : L^2(E;m) \to L^{2p}(E;m) \) is the composition of these operators so that it is compact. \( \square \)

### 4 Proof of Theorem 1.2

**Proof of Theorem 1.2.** By Remark 2.11, we already know \( S^p_{DK}(X) \subset S^p_{K infinite}(X) \) and \( S^p_{DK}(X) \subset S^p_{K infinite}(X) \) by Proposition 2.9. Suppose that \( X \) possesses (RSF).

We first prove (1). Take \( \nu \in S^p_{DK}(X) \). Then \( \nu \in S^p_{K}(X) \cap S^p_{DK}(X) \) and assume \( \nu \notin S^p_{DK}(X) \). Then there is an \( \varepsilon > 0 \) such that for any \( \delta > 0 \) and any compact set \( K \) with \( \sup_{x \in E} R^p(1_K, \nu)(x) < \varepsilon/2 \), there exists a Borel subset \( B \) of \( K \) with \( \nu(B) < \delta \) satisfying \( \sup_{x \in E} R^p(1_B, \nu)(x) \geq \frac{\varepsilon}{2} \). Let \( \{B_n\}_{n=1}^\infty \) be a sequence of such Borel subsets of \( K \) with \( \nu(B_n) < 1/2^n \). Define \( A_n := \bigcup_{k=n}^{\infty} B_k \). Then \( \nu(A_n) < 1/2^n - 1 \).

We have for any \( n \in \mathbb{N} \),

\[
\frac{\varepsilon}{2} \leq \sup_{x \in E} \int_{B_n} R(x,y)^p \nu(dy) \\
\leq \sup_{x \in E} \int_{A_n} R(x,y)^p \nu(dy) \\
= \sup_{M > 0} \sup_{x \in E} \int_{A_n} R(x,y)(R(x,y) \wedge M)^{p-1} \nu(dy).
\]

Now set

\[
a_n(M) := \sup_{x \in E} \int_{A_n} R(x,y)(R(x,y) \wedge M)^{p-1} \nu(dy) \leq M^{p-1}\|R1_K \nu\|_\infty,
\]

\[
a_\infty(M) := \lim_{n \to \infty} a_n(M),
\]

and apply Terkelsen’s minimax principle (see [38, Corollary 1]) for the continuous function \( u \to a_n(M) \) on the compact set \( \mathbb{N} \cup \{\infty\} \). Then we have

\[
\frac{\varepsilon}{2} \leq \lim_{n \to \infty} \sup_{M > 0} a_n(M) = \min_{n \in \mathbb{N} \cup \{\infty\}} \sup_{M > 0} a_n(M) = \sup_{M > 0} \min_{n \in \mathbb{N} \cup \{\infty\}} a_n(M) = \sup_{M > 0} \lim_{n \to \infty} a_n(M).
\]
We will see that \(\lim_{n \to \infty} a_n(M) = 0\) for any \(M > 0\), which gives a contradiction. Recall \(\nu \in S^p_K(X)\). Since \(\nu(K) < +\infty\), we have \(1_{K\nu} \in S^p_K(X)\). The function \(f_n(x) := M^{p-1} \int_{A_n} R(x, y) \nu(dy)\) is bounded and continuous in view of (RSF) for the time changed process \((X, 1_{K\nu})\) (see [23 Lemma 4.1]). Since \(\nu(\bigcap_{n=1}^{\infty} A_n) = 0\), we have \(f_n(x) \to 0\) as \(n \to \infty\) for each \(x \in E\). By use of Dini’s theorem, we have \(\lim_{n \to \infty} a_n(M) \leq \lim_{n \to \infty} \| f_n \| = 0\), because \(\| f_n \| = \sup_{x \in E} f_n(x) = \sup_{x \in K} f_n(x)\) in view of Frostman’s maximum principle. Hence we complete the proof of (1).

We next prove (2). Take \(\nu \in S^p_{K_1}(X) \cap S^p_{L_K}(X)\), then \(\nu \in S^p_{B_n}(X) \cap S^p_{E_K}(X)\) by the definition of \(S^p_{K_1}(X)\). Assume \(\nu \notin S^p_{C_1}(X)\). Then for any \(\delta > 0\) and any compact set \(K\) with \(\sup_{x \in E} R^p(1_{K\nu})(x) < 1\), there exists a Borel subset \(B\) of \(K\) with \(\nu(B) < \delta\) satisfying \(\sup_{x \in E} R^p(1_{B\nu})(x) \geq 1 - \sup_{x \in E} R^p(1_{K\nu})(x)\). Let \(\{B_n\}_{n=1}^{\infty}\) be a sequence of such Borel subsets of \(K\) with \(\nu(B_n) < 1/2^n\). Define \(A_n := \bigcup_{k=n}^{\infty} B_k\). Then \(\nu(A_n) < 1/2^{n-1}\). In the same way as in the proof of (1), we have for each \(n \in \mathbb{N}\),
\[
1 - \sup_{x \in E} R^p(1_{K\nu})(x) \leq \sup_{x \in E} \int_{B_n} R(x, y)^p \nu(dy) = \sup_{M > 0} \sup_{x \in E} \int_{A_n} R(x, y) (R(x, y) \wedge M)^{p-1} \nu(dy).
\]
Then one can obtain a similar contradiction as in the proof of (1) by replacing \(\varepsilon/2\) with \(1 - \sup_{x \in E} R^p(1_{K\nu})(x)\). Note that the time changed process \((X, 1_{K\nu})\) possesses (RSF), because \(1_{K\nu} \in S^p_{K_1}(X)\) implies \(1_{K\nu} \in S^p_K(X)\) by \(\nu(K) < +\infty\).

5 Proofs of Theorems 1.5 and 1.6

In this section we prove Theorems 1.5 and 1.6. First note that under the conditions in Theorem 1.5 we have the coincidence \(S^p_K(X) = K_{p, \beta}\) proved in [21]. Moreover, by use of [23 Lemma 4.3], \(X\) is transient provided \(\nu > \beta\) and the upper estimate in (A) holds with \(t_0 = +\infty\). Indeed, from [23 Lemma 4.3], we easily see that there exists \(C > 0\) such that \(R(x, y) \leq CG(x, y) \leq +\infty\) for \(x, y \in E\). Since \(m \in S^p_K(X) = K_{p, \beta}\), we have
\[
Rf(x) \leq C \int_{E} G(x, y)f(y)m(dy) = C \int_{B_{r}(x)} f(y) d(x, y)^{-p} \nu(dy) + C \int_{B_{r}(x)^c} f(y) d(x, y)^{-p} \nu(dy) \leq C\|f\|L^\infty \sup_{x \in E} \int_{B_{r}(x)} m(dy) d(x, y)^{-p} \nu(dy) + C\|f\|L^\infty \frac{1}{d(x, y)^{p-1}} < +\infty
\]
for any \(f \in L^1(E; m) \cap L^\infty(E; m)\) which are positive m-a.e. on \(E\). This implies the transience of \(X\) in the sense of [10] Lemma 1.5.1.

Proof of Theorem 1.5 (1). Note that [22] Lemma 4.3(3)] under these conditions gives that, there exists \(C > 0\) such that \(R(x, y) \leq CG(x, y)\) for all \(x, y \in E\). We already know \(S^p_{\infty K_\nu}(X) \subset S^p_K(X)\) by Proposition 2.9. Assume that \(\mu \in S^p_K(X)\) is a finite measure. Then for any \(A \subset \mathcal{B}(E)\),
\[
\int_{A} G(x, y)^p \mu(dy) \leq \int_{B_{r}(x)} G(x, y)^p \mu(dy) + \int_{A \setminus B_{r}(x)} G(x, y)^p \mu(dy) \leq \sup_{x \in E} \int_{B_{r}(x)} G(x, y)^p \mu(dy) + r^{p(\beta - \nu)} \mu(A).
\]
(5.1)
Fix \(\varepsilon > 0\). We choose small \(r > 0\) so that \(\sup_{x \in E} \int_{B_{r}(x)} G(x, y)^p \mu(dy) < \varepsilon/3\), a compact set \(K\) satisfying \(r^{p(\beta - \nu)} \mu(K^c) < \varepsilon/3\) and \(\delta > 0\) so that \(r^{p(\beta - \nu)} \delta < \varepsilon/3\). Applying (5.1) to \(A = K^c \cup B\), we have for \(B \subset K\) with \(\mu(B) < \delta\)
\[
\sup_{x \in E} \int_{K^c \cup B} G(x, y)^p \mu(dy) \leq \sup_{x \in E} \int_{B_{r}(x)} G(x, y)^p \mu(dy) + r^{p(\beta - \nu)} \mu(K^c \cup B) \leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
\]
Thus we have $\mu \in S^p_{CK,\infty}(X)$.

Next suppose $p > 1$ and $\mu \in S^p_K(X) \cap S^1_{K,\infty}(X)$. Then for any $A \in \mathcal{B}(E)$,

$$
\int_A G(x, y)^p \mu(dy) \leq \int_{B_r(z)} G(x, y)^p \mu(dy) + \int_{A \setminus B_r(z)} G(x, y)^p \mu(dy)
$$

$$
\leq \sup_{x \in E} \int_{B_r(z)} G(x, y)^p \mu(dy) + r^{p(\beta - \nu)} \sup_{x \in E} \int_{A \setminus B_r(z)} G(x, y) \mu(dy). \tag{5.2}
$$

Fix $\varepsilon > 0$. We choose small $r > 0$ so that $\sup_{x \in E} \int_{B_r(z)} G(x, y)^p \mu(dy) < \varepsilon/3$, a compact set $K$ satisfying $r^{p(\beta - \nu)} \sup_{x \in E} \int_{K^c} G(x, y) \mu(dy) < \varepsilon/3$, and $\delta > 0$ so that $r^{p(\beta - \nu)} \delta < \varepsilon/3$. Applying (5.2) to $A = K^c \cup B$, we have for $B \subset K$ with $\mu(B) < \delta$

$$
\int_{K^c \cup B} G(x, y)^p \mu(dy) \leq \sup_{x \in E} \int_{B_r(z)} G(x, y)^p \mu(dy) + r^{p(\beta - \nu)} \sup_{x \in E} \int_{(K^c \cup B) \setminus B_r(z)} G(x, y) \mu(dy)
$$

$$
\leq \sup_{x \in E} \int_{B_r(z)} G(x, y)^p \mu(dy) + r^{p(\beta - \nu)} \sup_{x \in E} \int_{K^c} G(x, y) \mu(dy)
$$

$$
+ r^{p(\beta - \nu)} \mu(B)
$$

$$
\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.
$$

Thus we have $\mu \in S^p_{CK,\infty}(X)$.

Proof of Theorem 1.5 (2). We first show the first half of the claim. By applying [23, Lemma 4.3(3)] with $t_0 = +\infty$, there exists $C > 0$ such that $R(x, y) \leq CG(x, y)$ for all $x, y \in E$. The inclusion $S^p_{CK,\infty}(X) \subset S^p_{K,\infty}(X)$ is given in Remark 2.10. It suffices to show the converse inclusion. Take $\mu \in S^p_{K,\infty}(X) \subset S^p_K(X)$. Then, for any $\varepsilon > 0$ there exists a compact set $K$ such that $\sup_{x \in E} \int_{K^c} R(x, y)^p \mu(dy) < \varepsilon/2$. Since $\mu \in S^p_K(X) = K^{p,\beta}$, for any $\varepsilon > 0$, there exists $r > 0$ such that

$$
\sup_{x \in E} \int_{B_r(z)} R(x, y)^p \mu(dy) \leq C \sup_{x \in E} \int_{B_r(z)} G(x, y)^p \mu(dy) < \frac{\varepsilon}{2}.
$$

Take a small $\delta > 0$ so that $\delta/r^{p(\nu - \beta)} < \varepsilon/2$. Then, for any Borel subset $B$ of $K$ with $\mu(B) < \delta$ we have

$$
\sup_{x \in E} \int_B R(x, y)^p \mu(dy) \leq C \sup_{x \in E} \int_B G(x, y)^p \mu(dy)
$$

$$
\leq C \left( \sup_{x \in E} \int_{B \cap B_r(z)} G(x, y)^p \mu(dy) + \sup_{x \in E} \int_{B \cap B_r(z)^c} G(x, y)^p \mu(dy) \right)
$$

$$
\leq C \left( \frac{\varepsilon}{2} + \frac{\mu(B)}{r^{p(\nu - \beta)}} \right) < C \varepsilon.
$$

So we obtain $\mu \in S^p_{CK,\infty}(X)$ for this case.

Next we suppose the lower estimate in (A\(\mu\)) also holds with $t_0 = +\infty$. Then there exist $C_1, C_2 > 0$ such that $C_1 G(x, y) \leq R(x, y) \leq C_2 G(x, y)$ for all $x, y \in E$ by applying [23, Lemmas 4.1(3) and 4.3(3)] with $t_0 = +\infty$. The inclusion $S^p_{CK,\infty}(X) \subset S^p_{K,\infty}(X)$ is given in Remark 2.10. To prove $S^p_{K,\infty}(X) \subset K^{p,\infty}$ let $o \in E$ and a compact set $K$. For sufficiently large $R > 0$ with $K \subset B_R(o)$ we have

$$
\sup_{x \in E} \int_{B_R(o)^c} G(x, y)^p \mu(dy) \leq C^{-1} \sup_{x \in E} \int_{K^c} R(x, y)^p \mu(dy),
$$

hence the inclusion $S^p_K(X) \subset K^{p,\infty}$ holds. It remains to prove the inclusion $K^{p,\infty} \subset S^p_{CK,\infty}(X)$. Take $\mu \in K^{p,\infty}_{\nu,\beta}$. Then for any $\varepsilon > 0$, there is $R > 0$ such that

$$
\sup_{x \in E} \int_{B_R(o)^c} G(x, y)^p \mu(dy) < \frac{\varepsilon}{2}.
$$
We choose small $r > 0$ so that $\sup_{x \in E} \int_{B_r(x)} G(x, y)^p \mu(dy) < \varepsilon/4$ and $\delta > 0$ so that $r^{p(\beta-\nu)} \delta < \varepsilon/4$. Then applying (5.11) to $A = B$, we have

$$\sup_{B \subset B_{R(0), \mu(B)} < \delta} \int_B G(x, y)^p \mu(dy) \leq \sup_{x \in E} \int_{B_{r(x)}} G(x, y)^p \mu(dy) + \sup_{B \subset B_{R(0), \mu(B)} < \delta} r^{p(\beta-\nu)} \mu(B) \leq \frac{\varepsilon}{4} + \frac{\varepsilon}{4} = \frac{\varepsilon}{2}.$$ 

Thus we have $\mu \in S^p_{\mathcal{C}_K}(X)$.

**Proof of Theorem 1.6 (1).** We already know $S^p_{\mathcal{C}_K}(X^{(1)}) \subset S^p_{\mathcal{C}_K}(X)$. Suppose that $\mu \in S^p_{\mathcal{C}_K}(X)$. Throughout the proof, we fix $\varepsilon > 0$, $\alpha > 0$ and $t \in [0, t_0]$.

(Case 1) $\mu(E) < +\infty$ and $\nu \geq \beta$: In this case, by the upper bound of (A3) we can see that for $d(z, y) \geq r$,

$$\int_0^t p_s(z, y)ds \leq \int_0^t \frac{1}{s^{1/\beta}} \Phi_2 \left( \frac{r}{s^{1/\beta}} \right) ds \leq \beta r^{\beta-\nu} \int_0^\infty u^{\nu-\beta-1} \Phi_2(u) du =: M(r) < +\infty$$

and then, we have for any $a > 0$,

$$\int_{d(z, y) > r} \int_{d(z, y) > r} \left( \int_0^t p_s(z, y)ds \right)^p p_a(x, z) m(dz) \mu(dy) \leq M(r)^p \mu(E).$$

We also have

$$\int_{d(z, y) > r} \int_{d(z, y) < r} \left( \int_0^t p_s(z, y)ds \right)^p p_a(x, z) m(dz) \mu(dy) \leq \sup_{z \in E} \int_{B_{r(z)}} \left( \int_0^t p_s(z, y)ds \right)^p \mu(dy)$$

and hence

$$\int_{d(z, y) > r} \left( \int_0^{a+t} p_s(z, y)ds \right)^p \mu(dy) \leq M(r)^p \mu(E) + \sup_{z \in E} \int_{B_{r(z)}} \left( \int_0^t p_s(z, y)ds \right)^p \mu(dy),$$

which concludes that

$$\left\{ \int_{d(z, y) > r} R_a(x, y)^p \mu(dy) \right\} \leq \sum_{n=0}^\infty e^{-\alpha nt} \left\{ \int_{d(z, y) > r} \left( \int_0^{(n+1)t} p_s(z, y)ds \right)^p \mu(dy) \right\} \leq \frac{1}{1 - e^{-\alpha t}} \left( M(r)^p \mu(E) + \sup_{z \in E} \int_{B_{r(z)}} \left( \int_0^t p_s(z, y)ds \right)^p \mu(dy) \right) \leq \left( 1 + 3 \left( \frac{1}{1 - e^{-\alpha t}} \right)^p \right) \varepsilon. \quad (5.3)$$

By [21] Theorem 4.1, $\mu \in S^p_{\mathcal{C}_K}(X)$ implies

$$\lim \sup_{r \to 0} \int_{x \in E \setminus B_{r(x)}} R_a(x, y)^p \mu(dy) = 0 \quad \text{and} \quad \lim \sup_{r \to 0} \int_{x \in E \setminus B_{r(x)}} \left( \int_0^t p_s(x, y)ds \right)^p \mu(dy) = 0.$$
So we can obtain $\mu \in S_{CR, \infty}^p(\mathbf{X}^{(1)})$ for this case.

(Case II) $\mu(E) < +\infty$ and $\nu < \beta$: In this case, we see that

$$\int_0^t p_s(z, y) ds \leq \Phi_2(0) \int_0^t \frac{1}{s^{\nu/\beta}} ds < +\infty.$$ 

By a similar calculation as to obtain (5.3), we have

$$\sup_{x \in E} \int_{K \cup B} R_\alpha(x, y)^p \mu(dy) \leq \left( \frac{1}{1 - e^{-at}} \right)^p \left( \Phi_2(0) \int_0^t \frac{1}{s^{\nu/\beta}} ds \right)^p \mu(K \cup B).$$

So we can obtain $\mu \in S_{CR, \infty}^p(\mathbf{X}^{(1)})$ for this case.

(Case III) $\mu \in S_{K, \infty}^p(\mathbf{X}^{(1)})$ with $p > 1$ and $\nu \geq \beta$: In this case, we have for any $a > 0$,

$$\int_{d(x, y) > a} \int_{d(z, y) \geq a} \left( \int_0^t p_s(z, y) ds \right)^p p_a(x, z) \mu(dx) \mu(dy) \leq M(r)^{p-1} \sup_{x \in E} \int_E \left( \int_0^t p_s(z, y) ds \right)^p \mu(dy)$$

$$= M(r)^{p-1} e^{at} \sup_{x \in E} \int_E R_\alpha(z, y) \mu(dy).$$

By a similar calculation to obtain (5.3), we have

$$\left\{ \sup_{x \in E} \int_{K \cup B} R_\alpha(x, y)^p \mu(dy) \right\}^{\frac{1}{p}} \leq \frac{1}{1 - e^{-at}} \left( M(r)^{p-1} e^{at} \sup_{x \in E} \int_E R_\alpha(z, y) \mu(dy) \right) \leq \left( 1 + 4 \left( \frac{1}{1 - e^{-at}} \right)^p \right) \varepsilon.$$ 

Now, choose small $r > 0$ so that $\sup_{x \in E} \int_{E(x)} R_\alpha(x, y)^p \mu(dy) < \varepsilon$ and $\sup_{x \in E} \int_{E(x)} \left( \int_0^t p_s(z, y) ds \right)^p \mu(dy) < \varepsilon$, a compact set $K$ satisfying $M(r)^{p-1} e^{at} \sup_{x \in E} \int_E R_\alpha(x, y) \mu(dy) < \varepsilon$ and $\delta > 0$ so that $M(r)^p \delta < \varepsilon$. Apply (5.3) by replacing $\mu$ with $1_{B_\mu}$, and apply (5.4) by replacing $\mu$ with $1_{K \cdot \mu}$. Then we have for any Borel set $B \subset K$ with $\mu(B) < \delta$

$$\sup_{x \in E} \int_{K \cup B} R_\alpha(x, y)^p \mu(dy) \leq \sup_{x \in E} \int_{E(x)} R_\alpha(x, y)^p \mu(dy) + \sup_{x \in E} \int_{E(x)} R_\alpha(x, y)^p 1_{B_\mu}(y) \mu(dy) + \sup_{x \in E} \int_{E(x)} R_\alpha(x, y)^p 1_{K \cdot \mu}(y) \mu(dy) \leq \left( 1 + 4 \left( \frac{1}{1 - e^{-at}} \right)^p \right) \varepsilon.$$ 

So we can obtain $\mu \in S_{CR, \infty}^p(\mathbf{X}^{(1)})$ for this case.

(Case IV) $\mu \in S_{K, \infty}^p(\mathbf{X}^{(1)})$ with $p > 1$ and $\nu < \beta$: In this case, we see

$$\int_0^t p_s(z, y) ds \leq \Phi_2(0) \int_0^t \frac{1}{s^{\nu/\beta}} ds < +\infty.$$ 

By a similar calculation as to obtain (5.3), we have

$$\sup_{x \in E} \int_{K \cdot \mu} R_\alpha(x, y)^p \mu(dy) \leq \left( \frac{1}{1 - e^{-at}} \right)^p \left( \Phi_2(0) \int_0^t \frac{1}{s^{\nu/\beta}} ds \right)^{p-1} e^{at} \sup_{x \in E} \int_{K \cdot \mu} R_\alpha(x, y) \mu(dy).$$

So we can obtain $\mu \in S_{CR, \infty}^p(\mathbf{X}^{(1)})$ for this case by the same manner as (Case III). 

□
Proof of Theorem 1.6 (2). Since $S_{CK}^p(K) \subset S_{K}^p(X)$ by Remark 2.10 it suffices to show $S_{CK}^p(X) \subset S_{K}^p(K)$. Take $\mu \in S_{K}^p(K)$ and $\mu \in S_{K}^p(X)$. Then, for any $\varepsilon > 0$ there exists a compact set $K$ such that

$$\sup_{z \in E} \int_{B_r(z)} R_1(z, y)^p \mu(dy) < \varepsilon$$

and

$$\sup_{z \in E} \int_{B_r(z)} \left( \int_0^t p_s(z, y) ds \right)^p \mu(dy) < \frac{\varepsilon}{2^+},$$

where $t \in [0, t_0]$ is a fixed small time. Take a small $r > 0$ so that $M(r)^p < 2$. Applying [21, Theorem 4.1] with $\alpha = 1$ by replacing $\mu$ with $1_B \mu$

$$\left( \int_{B \cap B_r(z)} R_1(x, y)^p \mu(dy) \right)^{\frac{1}{p}} \leq \frac{1}{1 - e^{-t}} \left\{ M(r)^p \mu(B) + \sup_{z \in E} \int_{B_r(z)} \left( \int_0^t p_s(z, y) ds \right)^p \mu(dy) \right\}^{\frac{1}{p}}.$$

Then we have

$$\int_B R_1(x, y)^p \mu(dy) \leq \int_{B \cap B_r(z)} R_1(x, y)^p \mu(dy) + \int_{B \cap B_r(z)} R_1(x, y)^p \mu(dy) \leq \varepsilon + \left( \frac{1}{1 - e^{-t}} \right)^p \left( \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \right) = \left( 1 + \left( \frac{1}{1 - e^{-t}} \right)^p \right) \varepsilon.$$

This concludes $\mu \in S_{CK}^p(X)$.

Next we suppose $\nu < \beta$. As noted in the proof of Theorem 1.6 (1) (Case II),

$$\int_0^t p_s(z, y) \mu(dy) \leq \Phi_2(0) \int_0^t \frac{1}{s^{\nu/\beta}} ds < +\infty.$$

By a similar calculation as to obtain [23] with $\alpha = 1$, we have

$$\sup_{x \in E} \int_B R_1(x, y)^p \mu(dy) \leq \left( \frac{1}{1 - e^{-t}} \right)^p \left( \Phi_2(0) \int_0^t \frac{1}{s^{\nu/\beta}} ds \right)^p \mu(B).$$

So we obtain $\mu \in S_{CK}^p(X)$ for this case. \[ \Box \]

6 Examples

Example 6.1 (Brownian motions on $\mathbb{R}^d$) Let $X = (\Omega, B \times \mathbb{R}_+)_{x \in \mathbb{R}^d}$ be a $d$-dimensional Brownian motion on $\mathbb{R}^d$. Consider $p \in [1, +\infty[$. We say that $\mu \in K_d^p$ (or $\mu \in K^p_{d+2}$) if and only if

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| < r} \frac{\mu(dy)}{|x-y|^{(d-2)p}} = 0 \quad \text{for} \quad d \geq 3,$n

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y| < r} (\log |x-y|^{-1})^p \mu(dy) = 0 \quad \text{for} \quad d = 2,$n

$$\sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq 1} \mu(dy) < +\infty \quad \text{for} \quad d = 1.$$n

We write $K_d$ instead of $K_d^p$ for $p = 1$. Then we have $K_d^p = S_{K}^p(X)$ by [25, Example 2.4]. The $d$-dimensional Lebesgue measure $\mu$ belongs to $K_d^p = S_{K}^p(X)$ if and only if $p \in [1, d/(d - 2) + 1]$ by [21, Theorem 3.2 or Corollary 4.4], where $d/(d - 2)_+ := d/(d - 2)$ if $d \geq 3$, $d/(d - 2)_+ := + \infty$ if $d = 1, 2$. For any non-negative bounded $g \in L^1(\mathbb{R}^d)$ the finite measure $gm$ also belongs to $S_{CK}^p(X)$ (to $S_{CK}^p(X)$ if $X$ is transient) by Theorem 1.5 under $p \in [1, d/(d - 2) + 1]$. The surface measure $\sigma_\partial$ on the $R$-sphere $\partial B_R(0)$ satisfies that $\sigma_\partial(B_r(x)) \leq C_d r^{d-1}$ for any $x \in \mathbb{R}^d$ and $r > 0$ with some $C_2 > 0$, and $\sigma_\partial(B_r(x)) \geq C_1 r^{d-1}$ for any $x \in \partial B_R(0)$ and $r \in (0, r_0]$ with some $C_1, r > 0$. Then we can conclude that by Theorem 1.5 and [21]
Theorem 3.2], \(\sigma_R \in S_{\mathcal{S}^{p}_{\infty}}(X^{(1)})\) holds if and only if \(p \in [1, (d-1)/(d-2)]\), where \((d-1)/(d-2) := (d-1)/2\) if \(d \geq 3\), \((d-1)/(d-2) := +\infty\) if \(d = 1, 2\). Moreover, \(\sigma_R \in S_{\mathcal{S}^{p}_{\infty}}(X)\) holds if and only if \(p \in [1, (d-1)/(d-2)]\) provided \(d \geq 3\).

We consider a non-empty connected open set \(D \subset \mathbb{R}^d\). The boundary point \(z \in \partial D\) is said to be regular if \(P_z(\tau_D = 0) = 1\). Denote by \((\partial D)\), the set of regular points in boundary. \(D\) is said to be regular if \((\partial D) = \partial D\). Let \(D\) be a connected open regular set of \(\mathbb{R}^d\). The absorbing Brownian motion \(X_D = (\Omega, F^D, \mathbb{P}_x)\) (or part process of \(X\) on \(D\)) is defined as the process killed upon leaving \(D\). Then \(X_D\) is an irreducible doubly Feller diffusion process on \(D\) (see [13]). If further \(D^p\) is non-polar, (in particular \(m(D^p) > 0\)), then \(X_D\) is transient in view of [13] Theorem 4.7.1 and Exercise 4.7.1. Let \(R^D(x, y)\) be the Green function with respect to \(X_D\). \(D\) is said to be Green-bounded if 
\[
\sup_{x \in D} \int_D R^D(x, y) \nu(dy) = \sup_{x \in D} E_x[\tau_D] < +\infty, \text{ equivalently } m \in S^{\mathbb{R}}(X_D)\),
\]
where \(m\) is the \(d\)-dimensional Lebesque measure on \(D\). If \(d = 1\) and \(D\) is not bounded, \(R^D \nu \in C_\infty(D)\) fails even for \(\nu(D) < +\infty\) (see [13] Example 1]). Since \(\sup_{x \in D} E_x[\tau_D] \leq \frac{d+2}{2d} \left(\frac{d+2}{2}\right)^{2/d} m(D)^{2/d}\) (see [13] Theorem 1.17]), \(m(D) < +\infty\), in particular the boundedness of \(D\), implies the Green-boundedness of \(D\). For a (positive) Radon measure \(\nu\) on \(\mathbb{R}^d\), if
\[
\begin{cases}
  d = 1, & D \text{ is bounded and } \nu(D) < +\infty \text{ or,} \\
  d = 2, & D \text{ is Green-bounded and } \nu \in S^{\mathbb{R}}_K(X) \text{ with } \nu(D) < +\infty \text{ or,} \\
  d \geq 3, & \nu \in S^{\mathbb{R}}_K(X) \text{ with } \nu(D) < +\infty,
\end{cases}
\]
then we can prove \((R^D)^p \nu := \int_D R^D(x, y)^p \nu(dy) \in C_b(D)\), and it belongs to \(C_\infty(D)\) provided \(D\) is a regular domain. Indeed, by [14] Theorem 2.6(ii)],
\[
R^D(x, y) \leq \left\{ \begin{array}{ll}
  \frac{1}{\pi} \log^+ |x - y|^{-1} + C & d = 2, \\
  C|x - y|^{-(d-2)} & d \geq 3,
\end{array} \right.
\]
(6.2)
where \(C\) is a positive constant depending on \(\|R^D m\|_\infty\) for \(d = 2\) (here we use the Green-boundedness of \(D\)) and on \(d\) for \(d \geq 3\). Moreover, there exists a positive sequence \(\alpha_n \to 0\) such that \(R^D(x, y) = R^D(x, y) \wedge n\) if \(|x - y| \geq \alpha_n\). Then we can calculate
\[
\sup_{x \in D} \int_D R^D(x, y)^p \nu(dy) - \int_D (R^D(x, y) \wedge n)^p \nu(dy) \leq \sup_{x \in D} \int_{\{|x - y| < \alpha_n\}} R^D(x, y)^p \nu(dy).
\]
(6.3)
The right-hand side of (6.3) uniformly converges to 0 as \(n \to \infty\), because of the estimate \(5.2\), \(\nu \in S^{\mathbb{R}}_K(X) = K^d\) and \(\nu(D) < +\infty\) for the case \(d = 2\). Under the conditions in (6.1), the function \(x \mapsto \int_D (R^D(x, y) \wedge n)^p \nu(dy)\) belongs to \(C_b(D)\) and in \(C_\infty(D)\) provided \(D\) is a regular domain, because of the extended continuity of \((x, y) \mapsto R^D(x, y)\) (see [14] Theorem 2.6(iii)]). The uniform convergence
\[
\lim_{n \to \infty} \lim_{n \to 0} \int_D R^D(x, y)^p \nu(dy) - \int_D (R^D(x, y) \wedge n)^p \nu(dy) = 0
\]
noted above implies the assertion for the case \(d \geq 2\). The proof of \((R^D)^p \nu \in C_\infty(D)\) for \(d = 1\) is clear from the expression
\[
\int_a^b R^D(x, y)^p \nu(dy) = \left(\frac{2(x - a)}{b - a}\right)^p \int_x^b (b - y)^p \nu(dy) + \left(\frac{2(b - x)}{b - a}\right)^p \int_a^x (y - a)^p \nu(dy)
\]
(6.4)
for \(D = [a, b]\). Hence, if \(\nu\) satisfies (6.1) and \(D\) is a regular domain, then one can obtain \(1_{D \nu} \in C_\infty(D)\).

For any compact set \(K\) of \(D\), the 0-order version of Frostman’s maximum principle (2.10) gives that \(\sup_{x \in D} (R^D)^p 1_{K \nu}(x) = \sup_{x \in D \setminus K} (R^D)^p 1_{K \nu}(x) \leq \sup_{x \in D \setminus K} (R^D)^p 1_{\nu}(x)\). Therefore
\[
1_{D \nu} \in S^{\mathbb{R}}_{\mathcal{S}^{p}_{\infty}}(X_D) = S^{\mathbb{R}}_{\mathcal{S}^{p}_{\infty}}(X_D),
\]
where the equality follows from Theorem 1.2 because $X_D$ possesses (RSF). It is proved in [21] Corollary 3.3 that $p(d - 2) < d$ is equivalent to $m \in S_{C}(X)$. From this, if
\[
\begin{cases}
  d = 1, & D \text{ is bounded or,} \\
  d \geq 2, & p(d - 2) < d \text{ with } m(D) < +\infty,
\end{cases}
\]
then $1_D m \in S_{K}^{p}(X_D) = S_{C,K}^{p}(X_D)$ provided $D$ is a regular domain. We relax (6.5) in the following way:
\[
\begin{cases}
  d = 1, & D \text{ is bounded or,} \\
  d \geq 2, & p(d - 2) < d \text{ with } \lim_{x \in D, |x| \to \infty} m(D \cap B_1(x)) = 0.
\end{cases}
\]
Therefore, by Theorem 1.1 we have the following:

**Theorem 6.2** Suppose that (6.6) is satisfied. Then $1_D m \in S_{K}^{p}(X_D) = S_{C,K}^{p}(X_D)$, hence the embedding
\[H_{1}^{p}(D) \hookrightarrow L^{2p}(D; m)\]
is compact. Moreover, if $D$ is Green-bounded, then $1_D m \in S_{K}^{p}(X_D) = S_{C,K}^{p}(X_D)$, hence the embedding
\[H_{1}^{p}(D)_{\text{e}} \hookrightarrow L^{2p}(D; m)\]
is compact.

**Proof.** The latter assertion follows from Proposition 2.1.1. So it suffices to prove the former assertion. Since $X_D$ possesses (RSF), we know $1_D m \in S_{K}^{p}(X_D) = S_{C,K}^{p}(X_D)$ under (6.6) by [29] Lemma 3.3. So there exists an increasing sequence $\{K_{l}\}$ of compact subsets of $D$ such that
\[
\lim_{l \to \infty} \sup_{x \in D \setminus K_{l}} R_{1}^{D}(x, y)m(dy) = 0.
\]
(6.7)

It is easy to see that there exist $C = C(d) > 0$ such that
\[
R_{1}^{D}(x, y) \leq R_{1}(x, y) \leq \begin{cases}
  e^{-\sqrt{2}(x-y)/\sqrt{2}} & d = 1, \\
  \frac{1}{\pi} \frac{1}{|x-y|^d} + \frac{1}{2\pi} & d = 2, \\
  \frac{1}{|x-y|^{d-2}} & d \geq 3,
\end{cases}
\]
for all $x, y \in D$. (6.8)

Indeed, for $d = 2$,
\[
R_{1}(x, y) \leq \int_{0}^{1-x-y^2} e^{-t} \frac{1}{2\pi t} e^{-|x-y|^2/2\pi t} dt + \int_{1-x-y^2}^{\infty} e^{-t} e^{-|x-y|^2/2\pi t} dt \\
\leq \int_{1-x-y^2}^{0} e^{-s} \frac{1}{2\pi s} e^{-|x-y|^2/2\pi s} ds + \frac{1}{2\pi} \frac{2}{|x-y|^{2}} \int_{1-x-y^2}^{\infty} e^{-t} dt \\
\leq \frac{1}{2\pi} + \frac{1}{\pi} \frac{1}{|x-y|^{2}}.
\]

Then there exists a decreasing sequence $\{\alpha_{n}\}$ converging to 0 such that $R_{1}^{D}(x, y) = R_{1}^{D}(x, y) \wedge n$ for $|x-y| \geq \alpha_{n}$. Indeed, we can choose $\alpha_{n} = (C_{3}/n)^{1/2}$ for $d \geq 3$, $\alpha_{n} = \sqrt{\frac{2}{2\pi n - 1}}$ for $d = 2$ with $n > 1/(2\pi)$, and $\alpha_{n}$ is arbitrary for $d = 1$. On the other hand,
\[
\sup_{x \in D \setminus K_{l}} R_{1}^{D}(x, y)^{p}(dy) - \sup_{x \in D \setminus K_{l}} R_{1}^{D}(x, y) \left( R_{1}^{D}(x, y) \wedge n \right)^{p-1} m(dy) \leq \sup_{x \in \mathbb{R}^{d}} \int_{|x-y| < \alpha_{n}} R_{1}(x, y)^{p}(dy).
\]
Since $m \in S^p_{\infty}(X) = K^p_{d,2}$, the right-hand side of (6.9) converges to 0 by [21] Theorem 4.1. Combining this with (6.7), we see

$$\limsup_{t \to \infty} \sup_{x \in D} \int_{D \setminus K_t} R^p_t(x,y)^p m(dy) = 0,$$

that is, $1_D m \in S^p_{\infty}((X^{(1)}_D))$. For $d = 1$ with bounded $D$, it is easy to see the same assertion by $1_D m \in S^p_{\infty}(X_D) = S^p_{C_{\infty}}(X_D)$ derived from the expression (6.4) by replacing $\nu$ with $m$. \hfill \square

Next we set

$$\mathcal{B}_0 := \left\{ B \in \mathcal{B}(\mathbb{R}^d) \mid \lim_{|x| \to \infty} m(B \cap B_1(x)) = 0 \right\}. \quad (6.10)$$

As noted in [29] (4.1), it holds that

$$\lim_{|x| \to \infty} m(B \cap B_R(x)) = 0 \quad \text{for any} \quad R > 0. \quad (6.11)$$

As proved in [29] Theorem 4.1], a domain $B$ belongs to $\mathcal{B}_0$ if and only if $m^B := 1_B m \in S^1_{\infty}(X^{(1)}) = S^1_{C_{\infty}}(X^{(1)})$ and if and only if $H^1(\mathbb{R}^d)$ is compactly embedded into $L^2(D)$ provided $d \geq 3$. In the following, we will give a $p$-extension of the fact.

**Lemma 6.3** For general $d \geq 1$, $B \in \mathcal{B}_0$ implies $m^B \in S^1_{\infty}(X^{(1)}) = S^1_{C_{\infty}}(X^{(1)})$.

**Proof.** By the ultra-contractivity $\|P_t\|_{L^1 \to L^\infty} \leq Ct^{-\frac{d}{q} - \delta}$ of $X$ and the estimate

$$\int_0^t P_s 1_{B \cap B_R(x)}(x)ds \leq E_x \left[ \int_0^t 1_{B_R(x)}(X_s)ds \right]$$

$$\leq E_x[(t - \tau_{B_R(x)})^+] = E_0[(t - \tau_{B_R(0)})^+]$$

$$\leq tP_0(\tau_{B_R(0)} \leq t),$$

we have

$$\int_0^t P_s 1_B(x)ds \leq \int_0^t P_s 1_{B \cap B_R(x)}(x)ds + \int_0^t P_s 1_{B \setminus B_R(x)}(x)ds$$

$$\leq C \int_0^t s^{-\frac{d}{q}}ds \cdot m(B \cap B_R(x))^\frac{1}{q} + tP_0(\tau_{B_R(0)} \leq t),$$

where $q > d/2$. So

$$\lim_{|x| \to \infty} \int_0^t P_{s + nt} 1_B(x)ds \leq (t + nt)P_0(\tau_{B_R(0)} \leq t + nt) \to 0 \quad \text{as} \quad R \to \infty.$$ 

Here we use the quasi-left continuity of $X$. From this,

$$R_1 1_B(x) = \sum_{n=0}^{\infty} e^{-nt} \int_0^t e^{-s} P_{s + nt} 1_B(x)ds$$

$$\leq \sum_{n=0}^{\infty} e^{-nt} \int_0^t e^{-s} P_{s + nt} 1_B(x)ds \to 0 \quad \text{as} \quad |x| \to \infty,$$

which implies $R_1 1_B \in C_{\infty}(\mathbb{R})$, hence $m^B \in S^1_{\infty}(X) = S^1_{C_{\infty}}(X)$. \hfill \square

Now we claim the following:

**Proposition 6.4** Suppose $p(d - 2) < d$ and $B \in \mathcal{B}_0$. Then $m^B \in S^p_{\infty}(X^{(1)}) = S^p_{C_{\infty}}(X^{(1)})$. In particular, $H^1(\mathbb{R}^d)$ is compactly embedded into $L^p(B)$. 

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Proof. It suffices to prove \(m^B \in S^p_{K_\infty}(X^{(1)})\) by Theorems 1.1 and 2.2. By (6.3), we have \(R_1(x,y) = R_1(x,y) \wedge n\) for \(|x-y| \geq \alpha_n\), where \(\alpha_n\) is the constant appeared as above. Then one can deduce the following estimate:

\[
\sup_{x \in \mathbb{R}^d} \int_{K_1^c} R_1(x,y)p_{m}(dy) - \sup_{x \in \mathbb{R}^d} \int_{K_1^c} R_1(x,y)R_1(x,y) \wedge n)^{p-1}m^{B}(dy) \leq \sup_{x \in \mathbb{R}^d} \int_{|x-y| < \alpha_n} R_1(x,y)^p m(dy).
\]

(6.12)

The right-hand side of (6.12) converges to 0 as \(n \to \infty\) from \(m \in S^p_K(X)\) under \(p(d - 2) < d\) by applying [21] Theorem 3.1] provided \(d \geq 2\). When \(d = 1\), the right-hand side of (6.12) is estimated above by \((1/\sqrt{2})p m(B_{\alpha_n}(0))\), which goes to 0 as \(n \to \infty\). Thus we can obtain the assertion.

**Theorem 6.5** Let \(D\) be a domain of \(\mathbb{R}^d\). Suppose \(p(d - 2) < d\). Then the following statements are equivalent:

1. \(D \in \mathcal{B}_0\),
2. \(1_D m \in S^p_{K_\infty}(X^{(1)}) = S^p_{CR_\infty}(X^{(1)})\),
3. \(H^1(\mathbb{R}^d)\) is compactly embedded into \(L^{2p}(D)\).

**Proof.** The condition \(p(d - 2) < d\) is only used to establish the continuity of the embedding \(H^1(\mathbb{R}^d) \hookrightarrow L^{2p}(D)\). We have already proved (1) \(\Rightarrow\) (2) \(\Rightarrow\) (3) in Proposition 6.4. The proof of the implication (3) \(\Rightarrow\) (1) is similar to [15] Chapter X, Lemma 6.11.

In the end of this example, we give the compactness of the Schrödinger semigroups, which is a \(p\)-version of [29] Theorem 5.2] under \(\alpha = 2\).

**Theorem 6.6** Let \(V\) be a positive Borel function on \(\mathbb{R}^d\) satisfying \(V m \in S^1_{LH}(X)\). Suppose that \(\{V \leq M\} \in \mathcal{B}_0\) for any \(M > 0\) and \(p(d - 2) < d\). Then the Schrödinger semigroup \(P^{-V}_t\) defined by

\[
P^{-V}_t f(x) = E_x \left[ e^{-\int_0^t V(X_s)ds} f(X_t) \right], \quad f \in L^2(\mathbb{R}^d) \cap \mathcal{B}(\mathbb{R}^d),
\]

(6.13)

forms a compact operator from \(L^2(\mathbb{R}^d)\) to \(L^{2p}(\mathbb{R}^d)\).

**Corollary 6.7** Let \(V\) be a positive continuous function on \(\mathbb{R}^d\) satisfying \(V m \in S^1_{LH}(X)\). Suppose that \(p(d - 2) < d\) and

\[
\lim_{|x| \to \infty} V(x) = +\infty.
\]

(6.14)

Then the Schrödinger semigroup \(P^{-V}_t\) defined in (6.13) forms a compact operator from \(L^2(\mathbb{R}^d)\) to \(L^{2p}(\mathbb{R}^d)\).

**Proof.** For any \(M > 0\), the sublevel set \(\{V \leq M\}\) is a compact set from (6.14), hence it belongs to \(\mathcal{B}_0\) automatically.

**Remark 6.8** Theorem 6.6 and Corollary 6.7 are not included in [24] Example 4.4]. Our conclusion of the compactness of \(P^{-V}_t\) is different from that in [24] Theorems 2.2 and 2.4.

**Proof of Theorem 6.6** When \(p = 1\), the assertion is nothing but [29] Theorem 5.2], which was done by showing \(m \in S^p_{K_\infty}(X^{-V} - 1) = S^1_{CR_\infty}(X^{-V} - 1)\) (note that its proof is valid for general \(d \geq 1\)). Here \(X^{-V}\) is the subprocess of \(X\) by \(\exp \left( -\int_0^t V(X_s)ds - t \right)\), which possesses (RSF) because of \(V m \in S^1_{LH}(X)\) (see [20] Corollary 6.1]). Let \(R^{-V}_t(x,y)\) be the Green kernel of \(X^{-V} - 1\). Then

\[
\sup_{x \in \mathbb{R}^d} \int_{K_1^c} R^{-V}_t(x,y)p_{m}(dy) - \sup_{x \in \mathbb{R}^d} \int_{K_1^c} R^{-V}_t(x,y)R^{-V}_t(x,y) \wedge n)^{p-1}m(dy) \leq \sup_{x \in \mathbb{R}^d} \int_{|x-y| < \alpha_n} R_1(x,y)^p m(dy).
\]

(6.15)

The right-hand side of (6.15) converges to 0 as \(n \to \infty\) as shown above. So we can obtain \(m \in S^p_{K_\infty}(X^{-V} - 1) = S^p_{CR_\infty}(X^{-V} - 1)\) from \(m \in S^1_{K_\infty}(X^{-V} - 1) = S^1_{CR_\infty}(X^{-V} - 1)\). Therefore, the assertion holds from Theorem 1.11. □
Example 6.9 (Symmetric Relativistic $\alpha$-stable Process) Take $0 < \alpha < 2$ and $m \geq 0$. Let $X = (\Omega, \mathcal{F}_t, \mathbb{P}_x)$ be a Lévy process on $\mathbb{R}^d$ with

$$E_0 \left[ e^{-\frac{1}{\alpha} \langle \xi, X_t \rangle} \right] = \exp \left( -t \left( \frac{|\xi|^2}{2} + \frac{m^{2/\alpha}}{\alpha/2} - m \right) \right).$$

If $m > 0$, it is called the relativistic $\alpha$-stable process with mass $m$ (see [11]). In particular, if $\alpha = 1$ and $m > 0$, it is called the free relativistic Hamiltonian process (see [3,6,17]). When $m = 0$, $X$ is nothing but the usual (rotationally) symmetric $\alpha$-stable process. It is known that $X$ is transient if and only if $d > 2$ under $m > 0$ or $d > \alpha$ under $m = 0$, and $X$ is a doubly Feller conservative process.

Let $(\mathcal{E}', \mathcal{F})$ be the Dirichlet form on $L^2(\mathbb{R}^d)$ associated with $X$. Using Fourier transform $\hat{f}(x) := \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{ix \cdot y} f(y) dy$, it follows from [16, Example 1.4.1] that

$$\mathcal{F} = \left\{ f \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d} |\hat{f}(\xi)|^2 \left( \frac{|\xi|^2}{2} + \frac{m^{2/\alpha}}{\alpha/2} - m \right) d\xi < +\infty \right\},$$

$$\mathcal{E}'(f, g) = \int_{\mathbb{R}^d} \hat{f}(\xi) \hat{g}(\xi) \left( \frac{|\xi|^2}{2} + \frac{m^{2/\alpha}}{\alpha/2} - m \right) d\xi$$

for $f, g \in \mathcal{F}$.

Since $X$ is a Lévy process, in view of [11] Corollary 7.16, there exists $C = C(m, \alpha) > 0$ such that

$$\mathcal{E}_t(f, f) \leq C(\|\nabla f\|_2^2 + \|f\|_2^2) \quad \text{for} \quad f \in H^1(\mathbb{R}^d).$$

It is shown in [10] that the corresponding jumping measure $J$ of $(\mathcal{E}', \mathcal{F})$ satisfies

$$J(dx) = J_m(x) dx \quad \text{with} \quad J_m(x, y) = A(d, -\alpha) \frac{\Psi(m^{1/\alpha}|x-y|)}{|x-y|^{d+\alpha}},$$

where $A(d, -\alpha) = \frac{\alpha^{d+\alpha} \Gamma(d+\alpha)}{2^{d+\alpha} \pi^{d/2} \Gamma(1-\frac{\alpha}{2})}$, and $\Psi(r) := I(r)/I(0)$ with

$$I(r) := \int_0^\infty 8 \frac{dr}{\hat{s}^2} e^{-\hat{s}^2} d\hat{s}$$

is a decreasing function satisfying $\Psi(r) \asymp e^{-r}(1 + r^{(d+\alpha-1)/2})$ near $r = +\infty$, and $\Psi(r) = 1 + \Psi'(0)r^2/2 + o(r^2)$ near $r = 0$. In particular,

$$\mathcal{F} = \left\{ f \in L^2(\mathbb{R}^d) \mid \int_{\mathbb{R}^d \times \mathbb{R}^d} |f(x) - f(y)|^2 J_m(x, y) dx dy < +\infty \right\},$$

$$\mathcal{E}'(f, g) = \frac{1}{2} \int_{\mathbb{R}^d \times \mathbb{R}^d} (f(x) - f(y))(g(x) - g(y)) J_m(x, y) dx dy \quad \text{for} \quad f, g \in \mathcal{F}.$$

Let $p_t(x,y)$ be the heat kernel of $X$. The following global heat kernel estimates are proved in [8, Theorem 2.1]: There exist $C_1, C_2 > 0$ such that

$$C_2^{-1} \Phi^m_{1/C_1}(t, x, y) \leq p_t(x,y) \leq C_2 \Phi^m_{C_1}(t, x, y) \quad \text{for all} \quad (t, x, y) \in ]0, +\infty[ \times \mathbb{R}^d \times \mathbb{R}^d,$$

where

$$\Phi^m_{C_1}(t, x, y) := \left\{ \begin{array}{ll}
  t^{-d/\alpha} \wedge t J_m(x, y), & t \in ]0, 1/m[, \\
  m^{d/\alpha - d/2} t^{-d/2} \exp \left( -C^{-1} \left( m^{1/\alpha}|x-y| \wedge m^{2/\alpha - 1} \frac{|x-y|^2}{4} \right) \right), & t \in ]1/m, +\infty[.
  \end{array} \right.$$
It is shown in [9, Theorem 1.2 and Example 2.4] or [4,5] Theorem 1.2 that $p_t(x,y)$ is jointly continuous in $(t,x,y) \in [0, +\infty] \times \mathbb{R}^d \times \mathbb{R}^d$. The $\beta$-order resolvent kernel $R_\beta(x,y) := \int_0^\infty e^{-\beta t} p_t(x,y) dt \in [0, +\infty]$ is also continuous in $(x,y) \in \mathbb{R}^d \times \mathbb{R}^d$.

We say that $\mu \in K_{d,\alpha}^p$ if and only if

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} \frac{\mu(dy)}{|x-y|^{(d-\alpha)p}} = 0 \quad \text{for} \quad d > \alpha,$$

$$\lim_{r \to 0} \sup_{x \in \mathbb{R}^d} \int_{|x-y|<r} (\log |x-y|)^{-1} \mu(dy) = 0 \quad \text{for} \quad d = \alpha = 1,$$

$$\sup_{x \in \mathbb{R}^d} \int_{|x-y| \leq 1} \mu(dy) < +\infty \quad \text{for} \quad d = 1 < \alpha.$$ We write $K_{d,\alpha}$ instead of $K_{d,\alpha}^1$ for $p = 1$. Then we have $K_{d,\alpha}^p = S^p_{K}(X)$ by [21, Theorem 3.1]. The $d$-dimensional Lebesgue measure $\nu$ belongs to $K_{d,\alpha}^p = S^p_{K}(X)$ if and only if $\nu \in \mathbb{R}^d$ [21, Theorem 3.2 or Corollary 4.4], and for any non-negative bounded $\nu \in L^1(\mathbb{R}^d)$ the finite measure $\nu$ also belongs to $S^p_{K_{\infty}}(X)$ (to $S^p_{C_{\infty}}(X)$ if $X$ is transient) by Theorem 4.3. Here $d/(d-\alpha)_+ := d/(d-\alpha)$ if $d > \alpha$ and $d/(d-\alpha)_+ := +\infty$ if $d \leq \alpha$. The surface measure $\sigma_{R}$ on the $R$-sphere $\partial B_R(0)$ satisfies that $\sigma_{R}(B_r(x)) \leq C_2 r^{d-1}$ for any $x \in \mathbb{R}^d$ and $r > 0$ with some $C_2 > 0$, and $\sigma_{R}(B_r(x)) \geq C_1 r^{d-1}$ for any $x \in \partial B_R(0)$ and $r \in [0,1]$ with some $C_1, C_2 > 0$. By Theorem 4.5 and [21, Theorem 3.1 and 3.2], we can conclude that $\nu \in S^p_{K_{\infty}}(X)$ holds if and only if $\nu \in \mathbb{R}^d \cap S^p_{K_{\infty}}(X)$ holds if and only if $\nu \in \mathbb{R}^d \cap S^p_{C_{\infty}}(X)$ holds if and only if $\nu \in \mathbb{R}^d \cap S^p_{C_{\infty}}(X)$ holds if and only if $\nu \in \mathbb{R}^d \cap S^p_{C_{\infty}}(X)$ holds if and only if $\nu \in \mathbb{R}^d \cap S^p_{C_{\infty}}(X)$ holds.

We consider a connected non-empty open set $D$ of $\mathbb{R}^d$. The notion of regular point in $\partial D$ is similarly defined as in Example 6.1. Denote by $\partial D$, the set of regular points in boundary. $D$ is said to be regular if $\partial D = \partial D$. The part process $X_D = (\Omega, \mathbb{F}_D, \mathbb{P}_D)$ of $X$ on $D$ is defined as the process killed upon leaving $D$. Let $R_D(x,y)$ be the Green function with respect to $X_D$. $D$ is said to be Green-bounded if $\sup_{x \in D} R_D(x,y)dy = \sup_{x \in D} \mathbb{E}_x[\tau_D] < +\infty$, equivalently $\nu \in S^p_{D}(X_D)$, where $\nu$ is the $d$-dimensional Lebesgue measure on $D$. By [19, Lemma 4.1], $\nu(D) < +\infty$ implies the Green-boundedness of $D$. For a (positive) Radon measure $\nu$ on $\mathbb{R}^d$, we consider the following conditions:

\begin{align*}
\text{for } m = 0, & \begin{cases} 
\quad d = 1 < \alpha, & D \text{ is bounded and } \nu(D) < +\infty, \\
\quad d = \alpha = 1, & D \text{ is Green-bounded and } \nu \in S^p_{K}(X) \text{ with } \nu(D) < +\infty, \\
\quad d > \alpha, & \nu \in S^p_{K}(X) \text{ with } \nu(D) < +\infty,
\end{cases} \quad (6.20) \\
\text{and} & \\
\text{for } m > 0, & \begin{cases} 
\quad d = 1 < \alpha, & D \text{ is bounded and } \nu(D) < +\infty, \\
\quad d = 1 \geq \alpha, \text{ or } d = 2, & D \text{ is Green-bounded and } \nu \in S^p_{K}(X) \text{ with } \nu(D) < +\infty, \\
\quad d \geq 3, & \nu \in S^p_{K}(X) \text{ with } \nu(D) < +\infty.
\end{cases} \quad (6.21)
\end{align*}

As we see the following, the difference of (6.20) and (6.21) comes from the order of $t$ in the upper heat kernels (6.15) and (6.19). It is unclear if $(x,y) \mapsto R^D(x,y)$ is extended continuous as in [14, Theorem 2.10(iii)]. So we do not know if $(R^D)^\nu \in C_0(D) \subset C_0(D)$ under the regularity of $D$ under (6.20) or (6.21). However, we can deduce the following:

**Proposition 6.10** Suppose that $\nu$ satisfies (6.20) and (6.21). Then $1_{D^c} \nu \in S^p_{K_{\infty}}(X_D) = S^p_{C_{\infty}}(X_D)$. 

**Proof.** Since $X_D$ possesses (RSF), it suffices to show $1_{D^c} \nu \in S^p_{K_{\infty}}(X_D)$ by Theorem 1.2. First we prove $1_{D^c} \nu \in S^p_{K}(X_D)$ under (6.20) and (6.21). Consider the case $d = 1 < \alpha$ in both cases. Since $\nu(D) < +\infty$, we see $1_{D^c} \nu \in K_{1,\alpha}^p = S^p_{K}(X)$ so that $1_{D^c} \nu \in S^p_{K}(X_D)$. In other cases, $1_{D^c} \nu \in S^p_{K}(X_D)$ follows from $\nu \in S^p_{K}(X)$. So it suffices to show the $L^p$-Green-tightness of $1_{D^c} \nu$ under $X_D$ in the sense of Zhao. To do this, we prove the $L^p$-Green-tightness of $1_{D^c} \nu$ under $X_D$ in the sense of Zhao. By use of the claims (C1)-(C4) in [19, proof of Theorem 4.1], we have that

$$R^D_{\beta}(x,y) = R^D_{\beta}(x,y) \wedge n \quad \text{for} \quad |x-y| \geq \alpha_n,$$

$$R^D_{\beta}(x,y) = R^D_{\beta}(x,y) \wedge n \quad \text{for} \quad |x-y| \geq \beta_n.$$
where $\alpha_n := \left( \frac{C_d}{n-c_2} \right)^{\frac{1}{n_a}}$ for $d > \alpha$, $n > C_2$, $\alpha_n := \exp \left( -\frac{n-C_2}{C_1} \right)$ for $d = \alpha = 1$, $n > C_2$, and $\alpha_n > 0$ is arbitrary for $d = 1 < \alpha$, and $\beta_n := \left( \frac{C_d}{n} \right)^{\frac{1}{n_a}}$ for $d > \alpha$ with $m = 0$ and $\beta_n := \left( \frac{C_d}{n} \right)^{\frac{1}{n_a}} \lor \left( \frac{C_d}{m^{\frac{1}{m}}} \right)^{\frac{1}{n_a}}$ for $d \geq 3$ with $m > 0$. Here $C_1, C_2$ and $C_3$ are the positive constants appeared in the claims (C1)-(C4) in [13] the proof of Theorem 4.1. $C_2$ depends on $\beta$ and $C_3$ depends on $m$. (Note that there is a typo in (C4); the upper bound of $R(x,y)$ for $m > 0$ and $d \geq 3$ should be $C_3(1 + m^{\frac{2m}{m^2}}|x-y|^{2-\alpha})/|x-y|^{d-\alpha}$.) From (6.22), for any $A \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\left| \sup_{x \in D} \int_A R^D_\beta (x,y)^p \nu(dy) - \sup_{x \in D} \int_A (R^D_\beta (x,y) \land n)^p \nu(dy) \right| \leq \sup_{x \in \mathbb{R}^d} \int_{\{|x-y| < \alpha_n\}} R_\beta (x,y)^p \nu(dy). \quad (6.24)$$

Since $\nu \in S^p_{K_\alpha}(X) = K_\alpha^p_\nu$, the right-hand side of (6.22) converges to 0 by [21] Theorem 3.1. Taking an increasing sequence $\{K_\ell\}$ of compact subsets of $D$, we have

$$\lim_{\ell \to \infty} \sup_{x \in D} \int_{D \setminus K_\ell} R^D_\beta (x,y)^p \nu(dy) \leq \lim_{\ell \to \infty} \sup_{x \in D} \int_{D \setminus K_\ell} (R^D_\beta (x,y) \land n)^p \nu(dy) + \sup_{x \in D} \int_{\{|x-y| < \alpha_n\}} R_\beta (x,y)^p \nu(dy) \leq \sup_{x \in \mathbb{R}^d} \int_{\{|x-y| < \alpha_n\}} R_\beta (x,y)^p \nu(dy) \to 0 \quad \text{as} \quad n \to \infty.$$

Thus $1_{D^\nu} \in S^p_{K_\alpha}(X)$. If $D$ is Green-bounded, i.e., $1_{D^\mu} \in S_{D_\alpha}(X_D)$, then we obtain $1_{D^\nu} \in S^p_{K_\alpha}(X_D)$ from Proposition 4.1.1. From now on, we consider the transient case, i.e., $d > \alpha$ with $m = 0$ or $d \geq 3$ with $m > 0$. In this case, by replacing (6.22) with (6.23), a similar estimate with (6.24) holds in the following manner: for any $A \in \mathcal{B}(\mathbb{R}^d)$, we have

$$\left| \sup_{x \in D} \int_A R^D_\beta (x,y)^p \nu(dy) - \sup_{x \in D} \int_A (R^D_\beta (x,y) \land n)^p \nu(dy) \right| \leq \sup_{x \in D} \int_{\{|x-y| < \alpha_n\}} R_\beta (x,y)^p \nu(dy). \quad (6.25)$$

The right-hand side of (6.25) converges to 0, because $\nu \in S^p_{K_\alpha}(X) = K_\alpha^p_\nu$ and the fourth claim (C4) yields $R(x,y) \leq C/|x-y|^{d-\alpha} = CG(x,y)$ for $|x-y| < \beta_n$ with small $\beta_n$. Taking an increasing sequence of compact sets as above

$$\lim_{\ell \to \infty} \sup_{x \in D} \int_{D \setminus K_\ell} R^D_\beta (x,y)^p \nu(dy) \leq \lim_{\ell \to \infty} \sup_{x \in D} \int_{D \setminus K_\ell} (R^D_\beta (x,y) \land n)^p \nu(dy) + \sup_{x \in D} \int_{\{|x-y| < \alpha_n\}} R_\beta (x,y)^p \nu(dy) \leq \sup_{x \in \mathbb{R}^d} \int_{\{|x-y| < \alpha_n\}} R_\beta (x,y)^p \nu(dy) \to 0 \quad \text{as} \quad n \to \infty.$$

Therefore $1_{D^\nu} \in S^p_{K_\alpha}(X_D) = S^p_{C_{D^\alpha}}(X_D)$. \hfill \qed

It is proved in [21] Corollary 3.3 that $p(d-\alpha) < d$ is equivalent to $\mu \in S^p_{K_\alpha}(X)$. From this, if

for $m = 0$, \begin{align*}
\begin{cases}
 d = 1 \leq \alpha, & D \text{ is bounded,} \\
 d > \alpha, & p(d-\alpha) < d \text{ with } m(D) < +\infty,
\end{cases}
\end{align*}

and

for $m > 0$, \begin{align*}
\begin{cases}
 d = 1, & p(d-\alpha) < d \text{ and } D \text{ is bounded,} \\
 d \geq 2, & p(d-\alpha) < d \text{ with } m(D) < +\infty,
\end{cases}
\end{align*}

then $\mu \in S^p_{K_\alpha}(X_D) = S^p_{C_{D^\alpha}}(X_D)$ provided $D$ is a regular domain. We relax (6.26) and (6.27) in the following:

for $m = 0$, \begin{align*}
\begin{cases}
 d = 1 \leq \alpha, & D \text{ is bounded,} \\
 d > \alpha, & p(d-\alpha) < d \text{ with } \lim_{x \to \infty} m(D \cap B_1(x)) = 0,
\end{cases}
\end{align*}

and

for $m > 0$, \begin{align*}
\begin{cases}
 d = 1, & p(d-\alpha) < d \text{ and } D \text{ is bounded,} \\
 d \geq 2, & p(d-\alpha) < d \text{ with } m(D) < +\infty,
\end{cases}
\end{align*}

then $\mu \in S^p_{K_\alpha}(X_D) = S^p_{C_{D^\alpha}}(X_D)$ provided $D$ is a regular domain. We relax (6.26) and (6.27) in the following:
and
\[
\text{for } m > 0, \quad \left\{ \begin{array}{ll} d = 1, & p(d - \alpha) < d \text{ and } D \text{ is bounded}, \\ d \geq 2, & p(d - \alpha) < d \text{ with } \lim_{x \in D, |x| \to \infty} m(D \cap B_t(x)) = 0. \end{array} \right. \tag{6.29}
\]

**Lemma 6.11** Suppose that \(\lim_{x \in D, |x| \to \infty} m(D \cap B_t(x)) = 0\). Then for the symmetric relativistic \(\alpha\)-stable process \(X\) the absorbing process \(X_D\) of \(X\) killed upon leaving \(D\) is in Class (T) defined in Section 7. In particular, \(1_D m \in S_{K,\infty}(X_D^{(1)}) = S_{CK,\infty}(X_D^{(1)})\).

**Proof.** The latter assertion follows from Proposition 2.11. So it suffices to prove the former assertion. The condition \(\text{is compact. Here (C1)-(C3) in } \left[19, \text{proof of Theorem 4.1}\right]\), we only note the following estimate; there exists \(S\) such that \(d \geq 2\), \(p(d - \alpha) < d\) with \(\lim_{x \in D, |x| \to \infty} m(D \cap B_t(x)) = 0\).

\[
\text{Proof.}\quad \text{The latter assertion follows from Proposition 2.11. So it suffices to prove the former assertion. The condition (C1)-(C3) in } \left[19, \text{proof of Theorem 4.1}\right]\text{, we only note the following estimate; there exists } S \text{ such that } d \geq 2, p(d - \alpha) < d \text{ with } \lim_{x \in D, |x| \to \infty} m(D \cap B_t(x)) = 0. \tag{6.29}
\]

Therefore, by Theorem 3.1 we have the following:

**Theorem 6.12** Suppose that \(\text{(6.28) and (6.29) are satisfied. Then } 1_D m \in S_{K,\infty}(X_D^{(1)}) = S_{CK,\infty}(X_D^{(1)}), \text{ in particular, the embedding }\)

\[
\mathcal{F}_D \hookrightarrow L^{2p}(D; m)
\]

is compact. Moreover, if \(D\) is Green-bounded, then \(1_D m \in S_{K,\infty}(X_D) = S_{CK,\infty}(X_D)\), in particular, the embedding

\[
(\mathcal{F}_D)_c \hookrightarrow L^{2p}(D; m)
\]

is compact. Here \((\mathcal{F}_D)_c\) is the extended Dirichlet space of \((\mathcal{E}, \mathcal{F}_D)\) on \(L^2(D)\).

**Proof.** The latter assertion follows from Proposition 2.11. So it suffices to prove the former assertion. The condition \(p(d - \alpha) < d\) in \(\text{(6.28) and (6.29) is equivalent to } m \in S_{K}(X)\), hence it implies \(1_D m \in S_{K}(X_D)\). By using Lemma 3.11 the proof is similar to the proof of Theorem 6.2. To do this, from claims (C1)-(C3) in \(\left[19, \text{proof of Theorem 4.1}\right]\) and \(\left[10, \text{proof of Lemma 6.2}\right]\), we only note the following estimate; there exists \(C > 0\) which depends on \(d, \alpha\) and \(m\) such that

\[
R_D^1(x, y) \leq R_1(x, y) \leq \begin{cases} C, & d = 1 < \alpha, \\ C \left(\frac{1}{|x - y|^2} + 1\right), & d = \alpha = 1, \\ C \left(\frac{1}{|x - y|^{d-\alpha}} + 1\right), & d > \alpha \end{cases}
\]

for all \(x, y \in D\). \(\square\)

Recall \(\mathcal{B}_0\) defined in \(\left[6.10\right]\). As proved in Lemma 3.3 we can prove that \(B \in \mathcal{B}_0\) implies \(m^B := 1_B m \in S_{K}(X^{(1)}) = S_{CK}(X^{(1)})\) by using \(\left[6.12\right]\) and \(\left[6.19\right]\). We now claim the following:

**Proposition 6.13** Suppose \(p(d - \alpha) < d\) and \(B \in \mathcal{B}_0\). Then \(m^B := 1_B m \in S_{K}(X^{(1)}) = S_{CK}(X^{(1)})\). In particular, \(\mathcal{F}\) is compactly embedded into \(L^{2p}(B)\).

**Proof.** It suffices to prove \(m^B := 1_B m \in S_{K}(X^{(1)}) = S_{CK}(X^{(1)})\) by Theorems 6.11 and 6.12. By \(\left[6.3\right]\), we have \(R_1(x, y) = R_1(x, y) \wedge n\) for \(|x - y| \geq \alpha_n\), where \(\alpha_n\) is a sequence converging to 0, which can be constructed based on the estimates in the proof of Theorem 6.12. Then one can deduce the following estimate:

\[
\left| \sup_{x \in \mathbb{R}^d} \int_{K_t^1} R_1(x, y)^p m^B(dy) - \sup_{x \in \mathbb{R}^d} \int_{K_t^1} R_1(x, y)(R_1(x, y) \wedge n)^{p-1} m^B(dy) \right| \leq \sup_{x \in \mathbb{R}^d} \int_{|x - y| < \alpha_n} R_1(x, y)^p m(dy). \tag{6.30}
\]
The right-hand side of (6.12) converges to 0 as \( n \to \infty \) from \( m \in S^p_K(X) \) under \( p(d-\alpha) < d \) by applying \[21\] Theorem 3.1 except \( d = 1 < \alpha \). When \( d = 1 < \alpha \), the right-hand side of (6.30) is estimated above by \( C(m)^p m(B_{\alpha_n}(0)) \), which goes to 0 as \( n \to \infty \). Here \( C(m) := \sup_{x,y \in \mathbb{R}^d} R_1(x,y) > 0 \) is the constant for the case \( d = 1 < \alpha \). Thus we can obtain the assertion.

**Corollary 6.14** Let \( D \) be a domain of \( \mathbb{R}^d \). Suppose \( p(d-\alpha) < d \). Then the following statements are equivalent:

1. \( D \in \mathcal{B}_0 \),
2. \( 1_D \, m \in S^{P}_{K_\infty}(X^{(1)}) = S^{P}_{CK_\infty}(X^{(1)}) \),
3. \( \mathcal{F} \) is compactly embedded into \( L^{2p}(D) \).

**Proof.** The condition \( p(d-\alpha) < d \) is only used to establish the continuity of the embedding \( \mathcal{F} \hookrightarrow L^{2p}(D) \). We have already proved (1) \( \implies \) (2) \( \implies \) (3) in Proposition 6.13. The proof of the implication (3) \( \implies \) (1) is similar to \[13\] Chapter X, Lemma 6.11 by using \[10\]. \( \square \).

In the end of this example, we give the compactness of the Schrödinger semigroups, which is a \( p \)-version of \[29\] Theorems 5.2] to symmetric relativistic \( \alpha \)-stable processes.

**Theorem 6.15** Let \( V \) be a positive Borel function on \( \mathbb{R}^d \) satisfying \( V \, m \in S^1_{LR}(X) \). Suppose that \( \{V \leq M\} \in \mathcal{B}_0 \) for any \( M > 0 \) and \( p(d-\alpha) < d \). Then the Schrödinger semigroup \( P_t^{-V} \) defined in (6.13) forms a compact operator from \( L^2(\mathbb{R}^d) \) to \( L^{2p}(\mathbb{R}^d) \).

**Corollary 6.16** Let \( V \) be a positive continuous function on \( \mathbb{R}^d \) satisfying \( V \, m \in S^1_{LR}(X) \). Suppose \( p(d-\alpha) < d \) and

\[
\lim_{|x| \to \infty} V(x) = +\infty.
\]

Then the Schrödinger semigroup \( P_t^{-V} \) defined in (6.13) forms a compact operator from \( L^2(\mathbb{R}^d) \) to \( L^{2p}(\mathbb{R}^d) \).

**Remark 6.17** Theorem 6.16 and Corollary 6.16 are not included in \[24\] Example 4.4, because we consider relativistic \( \alpha \)-stable process \( X \), and the compactness of \( P_t^{-V} \) is a different type from in \[24\] Theorems 2.2 and 2.4.

**Proof of Theorem 6.15** Let \( X^{-V-1} \) be the subprocess of \( \mathbf{X} \) by \( \exp \left( \int_0^t V(X_s) ds - t \right) \), which possesses (RSF) because of \( V \in S^1_{LR}(X) \). Though the framework of \[29\] Theorems 5.1 and 5.2] treats only the case \( m = 0 \), the proof remains valid for \( m \neq 0 \) in view of (6.18) and (6.19). Then the assertion for \( p = 1 \) is nothing but \[29\] Theorem 5.2, which was done by showing \( m \in S^1_{K_\infty}(X^{-V-1}) = S^1_{CK_\infty}(X^{-V-1}) \). Suppose \( p > 1 \) and let \( R_1^{-V}(x,y) \) be the Green kernel of \( X^{-V-1} \). Then

\[
\left| \sup_{x \in \mathbb{R}^d} \int_{K^p_1} R_1^{-V}(x,y)^p m(dy) - \sup_{x \in \mathbb{R}^d} \int_{K^p_1} R_1^{-V}(x,y)(R_1^{-V}(x,y) \wedge n)^{p-1} m(dy) \right| \leq \sup_{x \in \mathbb{R}^d} \int_{|x-y| < \alpha_n} R_1(x,y)^p m(dy).
\]

(6.31)

The right-hand side of (6.31) converges to 0 as \( n \to \infty \) as shown above. So we can obtain \( m \in S^p_{K_\infty}(X^{-V-1}) = S^p_{CK_\infty}(X^{-V-1}) \) from \( m \in S^1_{K_\infty}(X^{-V-1}) = S^1_{CK_\infty}(X^{-V-1}) \). Therefore, the assertion holds from Theorem 1.1. \( \square \)

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