ON BERNSTEIN- AND MARCINKIEWICZ-TYPE INEQUALITIES ON MULTIVARIATE $C^\alpha$-DOMAINS

FENG DAI, ANDRÁS KROÓ, AND ANDRIY PRYMAK

Abstract. We prove new Bernstein and Markov type inequalities in $L^p$ spaces associated with the normal and the tangential derivatives on the boundary of a general compact $C^\alpha$-domain with $1 \leq \alpha \leq 2$. These estimates are also applied to establish Marcinkiewicz type inequalities for discretization of $L^p$ norms of algebraic polynomials on $C^\alpha$-domains with asymptotically optimal number of function samples used.

1. Introduction and main results

The classical Bernstein and Markov inequalities for univariate algebraic polynomials $f$ of degree $n$ give the following sharp upper bounds for their derivatives:

$$\|\sqrt{1-x^2} f'(x)\|_{C[-1,1]} \leq n \|f\|_{C[-1,1]}, \quad \|f''\|_{C[-1,1]} \leq n^2 \|f\|_{C[-1,1]}.$$  

The above estimates were extended to the $d$ dimensional unit ball $B^d := \{x \in \mathbb{R}^d : |x| \leq 1\}$ by Sarantopoulos [12] (Bernstein type estimate) and Kellogg [7] (Markov type estimate). Namely, it was shown therein that

$$\|\sqrt{1-|x|^2} Df(x)\|_{C(B^d)} \leq n \|f\|_{C(B^d)}, \quad \|Df\|_{C(B^d)} \leq n^2 \|f\|_{C(B^d)},$$

where $|x| = \left(\sum_{1 \leq j \leq d} x_j^2\right)^{1/2}$ and $Df := |\nabla f| = \left(\sum_{j=1}^d \left(\frac{\partial f}{\partial x_j}\right)^2\right)^{1/2}$ is the $\ell_2$ norm of the gradient $\nabla f$ of $f \in \Pi_n^d$. Here $\Pi_n^d$ stands for the space of real algebraic polynomials of $d$ variables and total degree at most $n$. These inequalities have been instrumental in proving various results in approximation theory, and, as a result, they have been generalized and improved in many directions, in particular for various norms and multivariate domains. A crucial feature of the Bernstein-Markov estimates (1.1) lies in the fact that the magnitude of derivatives is of order $n^2$ uniformly on the domain, and of order $n$ pointwise if a proper distance from the boundary of the domain is enforced. In the multivariate setting this phenomenon becomes more intriguing in the sense that both the order $n^2$ and the “proper distance from the boundary” are affected by the smoothness of the domain. In addition, the study of tangential Bernstein-Markov inequalities is particularly important, since they play a significant role in the theory of the so called optimal meshes (see the corresponding definitions in Section 7 below), and the Marcinkiewicz-Zygmund type inequalities. In this respect let us mention that on $C^\alpha$ domains $D$, $1 \leq \alpha \leq 2$, the magnitude of tangential derivatives of $p \in \Pi_n^d$ was shown to be of order $n^{\frac{\alpha}{2}}$ uniformly on the domain, see [8]. In addition, the tangential derivatives at a given point $x \in D$ are of order $n$ if they are weighted by the quantity $d(x)^{\frac{\alpha}{2} - \frac{1}{2}}$ with $d(x)$ being the distance from $x$ to the boundary of the domain ([9]).

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In a recent paper [4] the authors proved a new tangential Bernstein type inequality in $L^p$ spaces for algebraic polynomials on a general compact $C^2$ domain. As an important application of this inequality, a Marcinkiewicz type inequality with asymptotically optimal number of sample points for discretization of $L^p$ norms of algebraic polynomials on $C^2$ domains was given. The main goal of the present paper is to extend the results of [4] to $C^\alpha$ domains with $1 \leq \alpha < 2$ (see Definition 5.2 for the precise definition of $C^\alpha$-domains). Similar to the case of uniform norm for $C^\alpha$ domains, $1 \leq \alpha < 2$, both the order of the tangential derivatives in $L^p$ Markov type estimates and the measure of distance from the boundary of the domain in tangential $L^p$ Bernstein type upper bounds are affected by the quantity $\alpha$ characterizing the smoothness of the domain.

In this introduction, we shall describe our main results and some basic notations. Necessary details and appropriate definitions will be given in later part of the paper. Given a nonempty set $E \subset \mathbb{R}^d$, we denote by $\text{dist}(\xi, E)$ the distance from a point $\xi \in \mathbb{R}^d$ to a set $E \subset \mathbb{R}^d$; that is, dist$(\xi, E) := \inf_{\eta \in E} \|\xi - \eta\|$, (we define dist$(\xi, E) = 1$ if $E = \emptyset$), where $\|\cdot\|$ denotes the Euclidean norm. Let $\Pi^d_n$ denote the space of all real algebraic polynomials in $d$ variables of total degree at most $n$. Given $\xi \in \mathbb{R}^d$, we denote by $\partial_\xi f = (\xi \cdot \nabla)f$ the directional derivative of $f$ along the direction of $\xi$. Throughout this paper, the letter $c$ denotes a generic constant whose value may change from line to line, and the notation $A \sim B$ means that there exists a constant $c > 0$, called the constant of equivalence, such that $c^{-1}B \leq A \leq cB$.

Let $\Omega \subset \mathbb{R}^d$ be a compact $C^\alpha$-domain with $1 \leq \alpha \leq 2$ and a nonempty boundary $\Gamma = \partial \Omega$. Denote by $n_\eta$ the outer unit normal vector to $\Gamma$ at $\eta \in \Gamma$. For $\xi \in \Omega$, $f \in C^\infty(\Omega)$, we define

$$|\nabla_{\text{tan}, \eta} f(\xi)| := \max\left\{|\partial_\tau f(\xi)| : \tau \in S^{d-1}, \tau \cdot n_\eta = 0\right\}, \eta \in \Gamma,$$

and

$$(1.2) \quad D_{n, \mu} f(\xi) := \max\left\{|\nabla_{\text{tan}, \eta} f(\xi)| : \eta \in \partial \Omega, \|\eta - \xi\| \leq \mu \varphi_{n, \Gamma}(\xi)^{2/\alpha}\right\}, \xi \in \Omega,$$

where $\mu \geq 1$ is a parameter, and

$$\varphi_{n, \Gamma}(\xi) := \sqrt{\text{dist}(\xi, \Gamma)} + n^{-1}, \quad n = 1, 2, \ldots, \xi \in \Omega.$$

In this paper, we will prove the following generalization of the $L^p$ tangential Bernstein-Markov type inequality from $C^2$ to $C^\alpha$ domains:

**Theorem 1.1.** For $0 < p < \infty$, any $f \in \Pi^d_n$ and $\mu > 1$, we have

$$\left\|\varphi_{n, \Gamma}\right\|^{2-1} D_{n, \mu} f \right\|_{L^p(\Omega)} \leq C(\mu, \Omega, p)n \|f\|_{L^p(\Omega)},$$

and

$$(1.3) \quad \|D_{n, \mu} f\|_{L^p(\Omega)} \leq C(\mu, \Omega, p)n^{2/\alpha} \|f\|_{L^p(\Omega)}.$$
we provide an example of a $C^\alpha$ domain $\Omega$ for which the asymptotic upper bound in \(1.3\) cannot be improved.

Finally, as an application, we show in Sections 7 ($d=2$) and 8 ($d>2$) that, similar to \cite{4}, our results yield a Marcinkiewicz type inequality with an asymptotically optimal number of sample points for discretizing $L^p$ norms of algebraic polynomials on $C^\alpha$ domains in $\mathbb{R}^d$ with $2-\frac{2}{d}<\alpha\leq 2$. More precisely, we will prove the following Marcinkiewicz type inequalities:

\textbf{Theorem 1.2.} If $2-\frac{2}{d}<\alpha\leq 2$, then for any positive integer $n$, there exists a partition $\Omega = \bigcup_{1\leq j\leq N} \Omega_j$ of $\Omega$ with $N \lesssim c_\alpha n^d$ such that for every $\xi_j \in \Omega_j$, each $f \in \Pi_n^d$ and $d-1<p<\infty$, we have

$$\frac{1}{2} \sum_{j=1}^{N} |\Omega_j||f(\xi_j)|^p \leq \int_{\Omega} |f(\xi)|^p d\xi \leq 2 \sum_{j=1}^{N} |\Omega_j||f(\xi_j)|^p.$$  

Of particular interest is the case of $d=2$, where Theorem \(1.3\) provides Marcinkiewicz type inequalities for $L^p$ norms on a compact $C^\alpha$-domain for all $1<\alpha\leq 2$ and $1<p<\infty$. Theorem \(1.2\) is proven in Section 7 for $d=2$ and in Section 8 for $d\geq 3$.

2. \textbf{Tangential Bernstein’s inequality on $C^2$ domains of special type}

In this section we will verify a tangential Bernstein type inequality on certain $C^2$ graph domains which will be the starting point of our further considerations. The proof follows closely \cite{4} Sect. 4.2. We repeat all the steps here for completeness because we need to track the dependence of the constants involved on the geometry of the domain. In addition, we need to introduce a certain weight into the $L^p$ norm which will be crucial below. This weighted approach will require applications of the so called doubling weights. Recall that a non-negative measurable function $w : I \rightarrow \mathbb{R}$ on an interval $I \subset \mathbb{R}$ is a doubling weight with constant $\beta$ if for any interval $J \subset I$

$$\int_{2J \cap I} w(x) dx \leq \beta \int_J w(x) dx,$$

where $2J$ denotes the interval with the length twice that of $J$ and the same midpoint as $J$, see, e.g. \cite{11} Sect. 2. We will need the following one-dimensional $L^p$ Bernstein inequality with doubling weights which can be found in \cite{11} Th. 7.3 or \cite{6} Th. 3.1. If $w$ is a doubling weight on $[-1,1]$ with the constant $\beta$, then for any $0<p<\infty$, and any algebraic polynomial $f \in \Pi_n^d$

$$\int_{-1}^{1} (1-x^2)^{p/2}|f'(x)|^pw(x) dx \leq Cn^p \int_{-1}^{1} |f(x)|^pw(x) dx,$$  

where $C$ depends only on $p$ and $\beta$.

Let $g \in C^2[-1,2]$, and $M := \|g''\|_{C[-1,2]} + 18$. Define

$$G := \{(x,y) : \ 0 \leq x \leq 1, \ 0 \leq g(x) - y \leq 1\},$$

and

$$G_* := \{(x,y) : \ -1 \leq x \leq 2, \ g(x) - 4 \leq y \leq g(x)\}.$$  

For $(x,y) \in G_*$, we set $\tau_x := (1, g'(x)) \in \mathbb{R}^2$, and $\delta(x,y) := g(x) - y$. Furthermore, let us denote by $\partial_u$ the operator $u \cdot \nabla$ of directional derivative for each $u \in \mathbb{R}^d$.

\textbf{Theorem 2.1.} For any $f \in \Pi_n^d$, $0<p<\infty$ and $0<\epsilon\leq 1$,

$$\left(\iiint_{G} |\partial_{\tau_x} f(x,y)|^p(\epsilon + \delta(x,y))^{-\frac{1}{2}} d\tau_x dx d\epsilon\right)^{1/p} \leq C_p \sqrt{M} n \left(\iiint_{G_*} |f(x,y)|^p(\epsilon + \delta(x,y))^{-\frac{1}{2}} d\tau_x dx d\epsilon\right)^{1/p},$$  

with $\tau_x := (1, g'(x)) \in \mathbb{R}^2$, and $\delta(x,y) := g(x) - y$. Furthermore, let us denote by $\partial_u$ the operator $u \cdot \nabla$ of directional derivative for each $u \in \mathbb{R}^d$.\]
where \(C_p > 0\) is a constant depending only on \(p\).

For the proof of Theorem 2.1 we need to introduce some necessary notations. Set \(r_0 := \sqrt{2/M}\), and define
\[
E := \{(z, t) \in \mathbb{R}^2 : z, z + t \in [-1, 2], \ |t| \leq r_0\}.
\]
We also write \(E = E^+ \cup E^\ast\), where
\[
E^+ = \{(z, t) \in E : t \geq 0\} \quad \text{and} \quad E^\ast = \{(z, t) \in E : t \leq 0\}.
\]
For each fixed \(z \in [-1, 2]\), define
\[
Q_z(t) := g(z) + g'(z)t - \frac{A}{2}t^2, \quad t \in \mathbb{R},
\]
where \(A\) is a fixed constant between \(5M/2\) and \(3M\). By Taylor’s theorem, we have that
\[
g(z + t) - Q_z(t) = \int_0^{\pm t} [A + g''(u)](z + t - u) \, du, \quad (z, t) \in E.
\]

Next, we define the differentiable mapping \(\Phi : E \rightarrow \mathbb{R}^2\) by
\[
\Phi(z, t) = (x, y) := (z + t, Q_z(t)), \quad (z, t) \in E.
\]
Denote by \(\Phi_+\) and \(\Phi_-\) the restrictions of \(\Phi\) on the sets \(E^+\) and \(E^-\) respectively.

The following lemma summarizes some useful properties of the mapping \(\Phi\), which will play a crucial role in the proof of Theorem 2.1.

**Lemma 2.2.** (i) For each \((z, t) \in E\), we have that \(\Phi(z, t) \in G_\ast\) and
\[
\left| \det(J_\Phi(z, t)) \right| = \frac{\partial(x, y)}{\partial(z, t)} = (A + g''(z))|t|,
\]
where \(J_\Phi\) denotes the Jacobian matrix of the mapping \(\Phi\).
(ii) Both the mappings \(\Phi_+ : E^+ \rightarrow \mathbb{R}^2\) and \(\Phi_- : E^- \rightarrow \mathbb{R}^2\) are injective.
(iii) For each \((x, y) \in G\), there exists a unique \((z, t) \in E^+\) such that \(0 \leq t \leq r_1 := 2/\sqrt{3M}\) and \(\Phi(z, t) = (x, y)\).

**Proof.** (i) Let \((x, y) = \Phi(z, t)\) with \((z, t) \in E\). Then \(x = z + t \in [-1, 2]\) and by \((2.2)\), we have
\[
g(x) - y = g(z + t) - Q_z(t) = \int_0^{\pm t} (g''(u) + A)(z + t - u) \, du,
\]
which implies that \(g(x) \geq y\). Since \(|t| \leq \sqrt{2/M}\), we deduce from \((2.4)\) that \(g(x) - y \leq 2Mt^2 \leq 4\), which implies that \((x, y) \in G_\ast\). Finally, we point out that \((2.4)\) follows directly from \((2.3)\) and straightforward calculations.

(ii) Assume that \(\Phi(z_1, t_1) = \Phi(z_2, t_2)\) for some \((z_1, t_1), (z_2, t_2) \in E\) with \(t_1t_2 \geq 0\) and \(t_2 \geq t_1\). Then \(z_1 + t_1 = z_2 + t_2\) and \(Q_{z_1}(t_1) = Q_{z_2}(t_2)\). We will show that \(t_1 = t_2\), which in turn imply \(z_1 = z_2\). Setting \(\bar{x} = z_1 + t_1\), we obtain from \((2.5)\) that for \(i = 1, 2,\)
\[
g(\bar{x}) - Q_{z_i}(t_i) = g(z_i + t_i) - Q_{z_i}(t_i) = \int_0^{t_i} [g''(\bar{x} - v) + A]v \, dv,
\]
which implies that
\[
\int_{t_1}^{t_2} (g''(\bar{x} - v) + A) \, dv = 0.
\]
Since \(g''(\bar{x} - v) + A \geq \frac{A}{2} > 0\) and \(v\) doesn’t change sign on the interval \([t_1, t_2]\), \((2.6)\) implies that \(t_1 = t_2\). This proves that both \(\Phi_+\) and \(\Phi_-\) are injective.
(iii) Given \((x, y) \in G\), we define the function \(h : [0, r_1] \to \mathbb{R}\) by \(h(t) := g(x) - Q_{x,t}(t)\). Note that the function \(h\) is well defined on \([0, r_1]\) since \(0 \leq x \leq 1\) and \(-1 < -r_1 \leq x - t \leq 1\) for any \(t \in [0, r_1]\). Moreover, using (2.2) with \(z = x - t\), we have
\[
h(t) = \int_{x-t}^{x} (g''(u) + A)(x - u) \, du.
\]
Thus, \(h\) is a continuous function on \([0, r_1]\) satisfying that \(h(0) = 0\) and
\[
h(r_1) \geq \frac{3}{2} M \int_0^{r_1} v \, dv = \frac{3}{4} M r_1^2 = 1.
\]
Since \(0 \leq g(x) - y \leq 1\), it follows by the intermediate value theorem that there exists \(t_0 \in [0, r_1]\) such that \(h(t_0) = g(x) - y\). This implies that \(y = Q_{x-t_0}(t_0)\) and \((x, y) = \Phi(x - t_0, t_0)\).

**Proof of Theorem 2.1** Let \(v_x(x, y) := (\varepsilon + \delta(x, y))^{-\frac{1}{2}}\) for \((x, y) \in G_x\). Performing the change of variables \(x = z + t\) and \(y = Q_z(t)\), and using Lemma 2.2, we obtain
\[
I := \left( \int_G \left| \partial_{z+I}f(z, t, Q_z(t)) \right|^p v_x(x, y) \, dx \, dy \right)^{1/p}
\]
\[
= \left( \int_{\Phi^{-1}(G)} \left| \partial_{z+I}f(z + t, Q_z(t)) \right|^p v_x(z + t, Q_z(t)) (A + g''(z)) t \, dz \, dt \right)^{1/p}.
\]
A straightforward calculation shows that
\[
\partial_{z+I}f(z + t, Q_z(t)) = \frac{d}{dt} \left[ f(z + t, Q_z(t)) \right] + \frac{1}{2} w(z, t) \partial_2 f(z + t, Q_z(t)),
\]
where \(w(z, t) = g'(z + t) - g'(z) + At\). It then follows that \(I \leq C_p(I_1 + I_2)\), where
\[
I_1 := \left( \int_{\Phi^{-1}(G)} \left| \frac{d}{dt} [f(z + t, Q_z(t))] \right|^p v_x(z + t, Q_z(t)) (A + g''(z)) t \, dz \, dt \right)^{1/p},
\]
\[
I_2 := \left( \int_{\Phi^{-1}(G)} \left| w(z, t) \right|^p |\partial_2 f(z + t, Q_z(t))|^p v_x(z + t, Q_z(t)) (A + g''(z)) t \, dz \, dt \right)^{1/p}.
\]
To estimate the double integral \(I_2\), we perform the change of variables \(x = z + t\) and \(y = Q_z(t)\) once again to obtain that
\[
I_2 = \left( \int_G |u(x, y)|^p |\partial_2 f(x, y)|^p v_x(x, y) \, dx \, dy \right)^{1/p},
\]
where \(u\) is a function on \(\Phi(E^+)\) given by
\[
u \circ \Phi(z, t) = w(z, t), \quad (z, t) \in E^+.
\]
Note that according to Lemma 2.2 (iii), the function \(u\) is well defined on \(G\). We further claim that
\[
|u(x, y)| \leq C \sqrt{M} \sqrt{g(x) - y}, \quad \forall (x, y) \in G.
\]
Indeed, using Lemma 2.2 (ii), we may write \(x = z + t\) and \(y = Q_z(t)\) with \((z, t) \in E^+\). We then have that
\[
|u(x, y)| = |w(z, t)| = |g'(z + t) - g'(z) + At| \leq CM|t|.
\]
On the other hand, using (2.2), we have that
\[
g(x) - y = g(z + t) - Q_z(t) = \int_z^{z+t} (g''(u) + A)(z + t - u) \, du \in \mathbb{R} CM|t|^2.
\]
Combining these last two estimates, we deduce the claim (2.9).
Now using (2.8) and (2.9), we obtain
\[ I_2 \leq C \sqrt{M} \left( \int_0^1 \int_{g(x)-1}^{g(x)} (\sqrt{g(x) - y})^p |\partial_x f(x, y)|^p v_\varepsilon(x, y) \, dy \right)^{1/p} \, dx. \]

Note that for each fixed \( x \in [0, 1] \), the function
\[ y \mapsto v_\varepsilon(x, y) = (g(x) - y)^{-\frac{1}{p}} \]
is a doubling weight on the interval \([g(x) - 4, g(x)]\) with a doubling constant independent of \( \varepsilon \in (0, 1) \). Thus, applying (2.11) w.r.t. \( y \) on the interval \([g(x) - 4, g(x)]\), we have that
\[ I_2 \leq C \sqrt{M} \left( \int_0^1 \int_{g(x)-4}^{g(x)} |f(x, y)|^p v_\varepsilon(x, y) \, dy \, dx \right)^{1/p}. \]

It remains to estimate the integral \( I_1 \). Recall that \( r_0 = \sqrt{2/M} > r_1 = \sqrt{4/(3M)} \). Since \( r_0 \in (0, \frac{1}{4}) \), it’s easily seen from Lemma 2.2 (iii) that
\[ \Phi_+^{-1}(G) \subset [-r_1, 1] \times [0, r_1] \subset [-r_0, 1] \times [-r_0, r_0] \subset E. \]

By the equality from (2.10) and the choice of \( A \), we also have
\[ (\varepsilon + 2Mt^2)^{-\frac{1}{2}} \leq v_\varepsilon(x, y) \leq \left( \varepsilon + \frac{3}{4}M t^2 \right)^{-\frac{1}{2}}, \]
and both sides are equivalent up to a constant factor of \( (\varepsilon + Mt^2)^{-\frac{1}{2}} \). Thus, using (2.7), we obtain
\[ I_1 \leq CM^{1/p} \left( \int_{-r_1}^{r_1} \int_0^{r_1} \left| f(z + t, Q_z(t)) \right|^p (\varepsilon + Mt^2)^{-\frac{1}{2}} |t| \, dt \right)^{1/p}. \]

A straightforward calculation shows that for any \( \varepsilon' > 0, |t|(\varepsilon' + t^2)^{-\frac{1}{2}} \) is a doubling weight on \([-1, 1]\) with a doubling constant independent of \( \varepsilon' \), which also implies that the doubling constant of \( |t|(\varepsilon + Mt^2)^{-\frac{1}{2}} \) is independent of both \( M \) and \( \varepsilon > 0 \). Now using (2.11) w.r.t. \( t \) on the interval \([-r_0, r_0]\), we obtain
\[ I_1 \leq CM^{1/r_0} \left( \int_{-r_0}^{r_0} \int_{-r_0}^{r_0} \left| f(z + t, Q_z(t)) \right|^p |t|(\varepsilon + Mt^2)^{-\frac{1}{2}} \, dt \right)^{1/p} \leq CM^{\frac{1}{r_0}} \left( \int_E \left| f(z + t, Q_z(t)) \right|^p |t|(\varepsilon + Mt^2)^{-\frac{1}{2}} \, dz \right)^{1/p}. \]

Splitting this last double integral into two parts \( \int_{E^+} + \int_{E^-} \), and applying the change of variables \( x = z + t \) and \( y = Q_z(t) \) to each of them separately, we obtain that
\[ I_1 \leq C_p M^{1/2} \int_{E^+} \int_{E^-} |f(z + t, Q_z(t))|^p (A + g''(z))|t| v_\varepsilon(z + t, Q_z(t)) \, dz \, dt \right)^{1/p} \]
\[ \leq C_p M^{1/2} \left[ \int_{\Phi(E^+)} |f(x, y)|^p v_\varepsilon(x, y) \, dx \, dy + \int_{\Phi(E^-)} |f(x, y)|^p v_\varepsilon(x, y) \, dx \, dy \right]^{1/p} \]
\[ \leq C_p M^{1/2} \left( \int_{-1}^{2} \int_{g(x)-4}^{g(x)} |f(x, y)|^p \frac{1}{\sqrt{\varepsilon + \delta(x, y)}} \, dy \, dx \right)^{1/p}, \]
where the last step uses Lemma 2.2 (i).
3. TANGENTIAL BERNSTEIN’S INEQUALITY ON $C^\alpha$ DOMAINS OF SPECIAL TYPE

The next step of our construction is to proceed from the $C^2$ to $C^\alpha$ graph domains in $\mathbb{R}^2$. This will be done using approximation of $C^\alpha$ functions by their Steklov transform. Let $g : [-4, 4] \to \mathbb{R}$ be a $C^\alpha$-function with constant $L > 1$ for some $1 \leq \alpha \leq 2$; that is, $g \in C^{1,2}[-4,4]$ and $|g'(x+t) - g'(x)| \leq L|t|^{\alpha-1}$ whenever $x, x+t \in [-4,4].$

Define

$G := \{(x,y) : \ 0 \leq x \leq 1, \ g(x) - 1 \leq y \leq g(x)\},$

$G_* := \{(x,y) : \ -1 \leq x \leq 2, \ g(x) - 8 \leq y \leq g(x)\},$

$\tau_2 := (1, g'(x)) \text{ and } \delta_n(x,y) = \delta(x,y) + \frac{1}{n^2} = g(x) - y + \frac{1}{n^2}, \quad (x,y) \in G.$

Here we point out that the set $G_*$ defined here is different from the set $G^*$ introduced in the last section. We will keep these assumptions and notations throughout this section. We will show that Theorem 2.1 implies the following:

**Theorem 3.1.** Let $\gamma = \frac{1}{2} - \frac{1}{2}$. Then for any $f \in \Pi_n^2$ and $0 < p < \infty$,

$$\left( \iint_{G} (\delta_n(x,y))^{\gamma p} |\partial_{\tau_2} f(x,y)|^p \ dy \ dx \right)^{1/p} \leq C_p L^\frac{p}{\alpha} \|f\|_{L^p(G_*).}$$

The proof of this theorem relies on the following lemma:

**Lemma 3.2.** For any $0 < b \leq 1$, $0 < p < \infty$, and $f \in \Pi_n^2$, we have

$$\left( \int_0^1 \left[ \int_{\frac{g(x)}{b-1}} |\partial_{\tau_2} f(x,y)|^p \frac{dy}{\sqrt{\delta(x,y) + b}} \right] dx \right)^{1/p} \leq C(p,\alpha)L^\frac{b}{2} b^{\frac{\alpha}{p}} n \left( \int_0^2 \left[ \int_{\frac{g(x)}{b-1}} (f(x,y))^p \frac{dy}{\sqrt{\delta(x,y) + b}} \right] dx \right)^{1/p}.$$

**Proof.** Let $0 < \delta < 1$ be a small parameter to be specified shortly. For the function $g$ its Steklov transform is given by

$$g_\delta(x) := \frac{1}{4\delta^2} \int_{[-\delta,\delta]^2} g(x + u + v)dudv, \quad -1 \leq x \leq 2.$$

Then $g_\delta \in C^2[-1,2]$, and moreover, for any $x \in [-1,2]$, we have

$$|g(x) - g_\delta(x)| \leq \frac{1}{8\delta^2} \int_{[-\delta,\delta]^2} \int_0^{u+v} \left( g'(x + s) - g'(x - s) \right) dsdudv \leq \bar{C}L\delta^\alpha,$$

$$|g'(x) - g'_\delta(x)| \leq \frac{1}{4\delta^2} \int_{[-\delta,\delta]^2} |g'(x + u + v) - g'(x)| dudv \leq \bar{C}L\delta^{\alpha-1},$$

$$|g''_\delta(x)| \leq \frac{1}{4\delta^2} \int_{-\delta}^{\delta} |g'(x + u + \delta) - g'(x + u - \delta)| du \leq \bar{C}L\delta^{\alpha-2}.$$
Thus,\[
\left(\int_G |\partial_x f(x, y)|^p \frac{dx dy}{\sqrt{\delta(x, y) + b}}\right)^{1/p} \leq C\left(J_1 + L^{1/\alpha} J_2\right),
\]
where\[
J_1 := \left(\int_0^1 \left[ \int_{g(x)-1}^{g(x)-1 + \frac{\delta}{b}} |\partial_x f(x, y)|^p \frac{dy}{\sqrt{\delta(x, y) + b}}\right] dx\right)^{1/p},
\]
and\[
J_2 := b^{1-\frac{1}{p}} \left(\int_0^1 \left[ \int_{g(x)-1}^{g(x)} |\partial_x f(x, y)|^p \frac{dy}{\sqrt{\delta(x, y) + b}}\right] dx\right)^{1/p}.
\]
On the one hand, for each fixed \(x \in [0, 1]\) and each \(0 < b \leq 1\), the function\[
y \mapsto (g(x) - y + b)^{-\frac{1}{p}} = (\delta(x, y) + b)^{-\frac{1}{p}}
\]
is a doubling weight on \([g(x) - 2, g(x)]\) with the doubling constant independent of \(b\). Thus, using (2.1), we obtain\[
J_2 \leq Cnb^{\frac{1}{p} - \frac{1}{2}} \left(\int_0^1 \left[ \int_{g(x)-1}^{g(x)+\frac{\delta}{b}} |f(x, y)|^p \frac{dy}{\sqrt{\delta(x, y) + b}}\right] dx\right)^{1/p}.
\]
On the other hand, since \(\|g - g_b\|_\infty \leq \sqrt{L}\delta^\alpha = \frac{\delta}{b}\), we have\[
J_1 \leq C\left(\int_0^1 \left[ \int_{g(x)-1}^{g(x)+\frac{\delta}{b}} |\partial_x f(x, y)|^p \frac{dy}{\sqrt{\delta(x, y) + b}}\right] dx\right)^{1/p}.
\]
Using Theorem 2.1 with \(M = CL\delta^{\alpha-2}\), we obtain\[
J_1 \leq C\sqrt{\ln n} b^{\frac{1}{p} - \frac{1}{2}} n \left(\int_0^1 \left[ \int_{g(x)-1}^{g(x)+\frac{\delta}{b}} |f(x, y)|^p \frac{dy}{\sqrt{\delta(x, y) + b}}\right] dx\right)^{1/p}.
\]
Putting the above estimates together, we get the desired inequality. \(\Box\)

\textbf{Proof of Theorem 3.1} Since\[
\partial_x f(x, y) = \partial_1 f(x, y) + g'(x) \partial_2 f(x, y)
\]
is a polynomial of degree at most \(n\) in the variable \(y\), we obtain by the Remez inequality (see [11, (7.17)] and [6, Sect. 7]) that\[
\int_G (\delta_n(x, y))^{\gamma p}\left|\partial_x f(x, y)\right|^p \ dx \ dy \leq C \int_0^1 \int_{g(x)-1}^{g(x)+\frac{\delta}{n^2}} \left(\delta(x, y)\right)^{\gamma p}\left|\partial_x f(x, y)\right|^p \ dy \ dx
\]
\[
\leq C \sum_{1 \leq j < 2 \log_2 n} \left(\frac{2^j}{n^2}\right) \gamma p \int_0^1 \int_{g(x)-1 - \frac{2^j}{n^2}}^{g(x)-1 + \frac{2^j}{n^2}} \left|\partial_x f(x, y)\right|^p \ dy \ dx
\]
\[
\leq C \sum_{1 \leq j < 2 \log_2 n} \left(\frac{2^j}{n^2}\right) \gamma p + 1/2 \int_0^1 \int_{g(x)-1 - \frac{2^j}{n^2}}^{g(x)-1 + \frac{2^j}{n^2}} \left|\partial_x f(x, y)\right|^p \left(\delta(x, y)\right)^{-1/2} \ dy \ dx.
\]
For each integer $1 \leq j < 2 \log_2 n$, we apply Lemma 3.2 to $b = \frac{2j+1}{n^2}$ and the function $g(x) - \frac{2j}{n^2}$ instead of $g$. We then obtain
\[
\int_G (\delta_n(x, y))^{\gamma p} |\partial f(x, y)|^p dxdy
\leq CL^\frac{\pi}{n^2} \sum_{1 \leq j < 2 \log_2 n} \left( \frac{n^2}{2^j} \right)^{\gamma p} \left( \frac{2j}{n^2} \right)^{\gamma p+1/2} \int_{g(x)-8}^{g(x)-1} \left[ \int_0^{\frac{s}{\delta(x,y)}} |f(x, y)|^p \left( \delta(x, y) \right)^{-1/2} dy \right] dx
\leq CL^\frac{\pi}{n^2} \int_{g(x)-8}^{g(x)+2} \left[ \int_0^{\frac{s}{\delta(x,y)}} |f(x, y)|^p \sum_{1 \leq j \leq \log_2(2n^2\delta(x,y))} \left( \frac{2j}{n^2\delta(x,y)} \right)^{1/2} dy \right] dx
\leq CL^\frac{\pi}{n^2} \int_{g(x)-8}^{g(x)+2} |f(x, y)|^p dydx.
\]

\[\square\]

4. TANGENTIAL BERNSTEIN’S INEQUALITY: HIGHER-DIMENSIONAL CASE

In order to establish the corresponding result on a general $C^\alpha$-domain, we need to formulate a slightly stronger version of the tangential Bernstein inequality on high dimensional domains of special type. Let us first introduce the general set up for the rest of this section.

We will write a point in $\mathbb{R}^d$ in the form $(x, y)$ with $x \in \mathbb{R}^{d-1}$ and $y = x_d \in \mathbb{R}$. Sometimes for the sake of simplicity, we also use a Greek letter to denote a point in $\mathbb{R}^d$ and write it in the form $\xi = (\xi_x, \xi_y)$ with $\xi_x \in \mathbb{R}^{d-1}$ and $\xi_y \in \mathbb{R}$.

Let $D_1 := [0, 1]^{d-1}$ and $D_2 = [-1, 2]^{d-1}$, where $d \geq 3$. Let $1 \leq \alpha \leq 2$, and let $g$ be a $C^\alpha$-function on $[-4, 4]^{d-1}$ with constant $L > 1$:
\[
\| \nabla g(x + t) - \nabla g(x) \| \leq L \| t \|^\alpha - 1 \quad \text{whenever } x, x + t \in [-4, 4]^{d-1},
\]
where $\| \cdot \|$ denotes the Euclidean norm. Define
\[
G := \{(x, y) : x \in D_1, g(x) - 1 \leq y \leq g(x)\},
\]
and
\[
G_* := \{(x, y) : x \in D_2, g(x) - 8 \leq y \leq g(x)\}.
\]
We call the set
\[
\partial' G := \{(x, g(x)) : x \in D_1\},
\]
the essential boundary of $G$. We also define $\partial' G_* := \{(x, g(x)) : x \in D_2\}$.

Given a point $x_0 \in D_1$, we denote by $S_{x_0}$ the space of all tangent vectors to the surface $y = g(x)$, $x \in D_2$ at the point $(x_0, g(x_0))$; namely,
\[
S_{x_0} := \{\xi \in \mathbb{R}^d : \nabla g(x_0), -1\} = 0\}.
\]

Let $e_1 = (1, 0, \cdots, 0), \cdots, e_d = (0, \cdots, 0, 1)$ denote the standard canonical basis of $\mathbb{R}^d$. For $j = 1, 2, \cdots, d - 1$, we define
\[
(4.1) \quad \xi_j(x_0) := e_j + \partial g(x_0)e_d.
\]
Geometrically, $\xi_j(x_0)$ is a tangent vector to the essential boundary $\partial' G$ at the point $(x_0, g(x_0))$ that is parallel to the $x_jx_d$-coordinate plane. It is easily seen that
\[
(4.2) \quad S_{x_0} = \{(\eta, \eta \cdot \nabla g(x_0)) : \eta \in \mathbb{R}^{d-1}\} = \text{span}\{\xi_j(x_0) : 1 \leq j \leq d - 1\}.
\]
Next, for \( \xi = (\xi_x, \xi_y) \in G \), we define
\[
\delta_n(\xi) := g(\xi_x) - \xi_y + \frac{1}{n^2}, \quad n = 1, 2, \ldots,
\]
and set, for \( \mu > 1 \),
\[
\Xi_{n, \mu, \alpha}(\xi) := \left\{ u \in D_1 : \|u - \xi_x\| \leq \mu(\delta_n(\xi))^{1/\alpha} \right\}.
\]
For \( f \in C^1(G_\ast) \) and \( u \in D_1 \), define
\[
|\nabla_{\tan, u} f(\xi)| := \max_{\eta \in S_n} |\partial_\eta f(\xi)|, \quad \xi \in G,
\]
where \( S^{d-1} \) denotes the unit sphere of \( \mathbb{R}^d \). Using (4.2), we have
\[
|\nabla_{\tan, u} f(\xi)| \sim \sum_{j=1}^{d-1} |\partial_{\xi_j(u)} f(\xi)|, \quad \xi \in G, \quad u \in D_1.
\]
Note that if \( \xi = (\xi_x, \xi_y) \in \partial^c G \) and \( u = \xi_x \), then \( |\nabla_{\tan, \xi_x} f(\xi)| \) is the length of the tangential gradient of \( f \) at \( \xi \). In the following result, we choose \( u \) to be an arbitrarily given point in a certain neighborhood of \( \xi_x \).

**Theorem 4.1.** Let \( \gamma = \frac{1}{\alpha} - \frac{1}{2} \), and \( 0 < p < \infty \). Then for any \( f \in \Pi_n^2 \) and parameter \( \mu > 1 \), we have
\[
\left\| (\delta_n(\xi))^{\gamma} \max_{u \in \Xi_{n, \mu, \alpha}(\xi)} |\nabla_{\tan, u} f(\xi)| \right\|_{L^p(G; \xi)} \leq C(\mu, L, p)n\|f\|_{L^p(G)}.
\]

Our goal in this section is to prove Theorem 4.1. We need the following lemma.

**Lemma 4.2.** If \( 1 \leq j \leq d - 1 \), \( 0 < p \leq \infty \) and \( \gamma = \frac{1}{\alpha} - \frac{1}{2} \), then for any \( f \in \Pi_n^d \), we have
\[
\left( \int_G (\delta_n(x, y))^{\gamma p} |\partial_{\xi_j(x, y)} f(x, y)|^p dxdy \right)^{1/p} \leq C_L^{p/\alpha} n\|f\|_{L^p(G)}.
\]

**Proof.** For simplicity, we assume that \( d = 3 \) and \( p < \infty \). (The proof below works equally well for \( d > 3 \) or \( p = \infty \).) Without loss of generality, we may also assume that \( j = 1 \). First, by Fubini’s theorem, we have
\[
\iint_G (\delta_n(x, y))^{\gamma p} \left|\partial_{\xi_1(x_1, x_2)} f(x_1, x_2, y)\right|^p dxdy = \int_0^1 I(x_2) dx_2,
\]
where
\[
I(x_2) := \int_0^1 \int_{g(x_1, x_2)} (\partial_1 + \partial_1 g(x_1, x_2) \partial_3) f(x_1, x_2, y)^p (g(x_1, x_2) - y + n^{-2})^{\gamma p} dy dx_1.
\]
For each fixed \( x_2 \in [0, 1] \), applying Theorem 3.1 to the function \( g(\cdot, x_2) \) and the polynomial \( f(\cdot, x_2, \cdot) \), we obtain
\[
I(x_2) \leq C L^{p/\alpha} n^{p/2} \int_{-1}^2 \int_{g(x_1, x_2)} f(x_1, x_2, y)^p dy dx_1.
\]
Integrating this last inequality over \( x_2 \in [-1, 2] \), we deduce the stated estimate in Lemma 4.2. \( \square \)

**Proof of Theorem 4.1.** For \( (x, y) \in G \) and \( u \in D_1 \), we have
\[
\partial_{\xi_j(u)} = \partial_{\xi_j(x)} + \left( \partial_j g(u) - \partial_j g(x) \right) \partial_u.
\]
It follows that for each \((x, y) \in G\),
\[
\max_{u \in \Xi_{n, \mu, \alpha}(x, y)} \left| \partial_{\xi_{i}(u)} f(x, y) \right| 
\leq \left| \partial_{\xi_{i}(x)} f(x, y) \right| + C \| u - x \|^{\alpha - 1} |\partial_d f(x, y)|
\leq \left| \partial_{\xi_{i}(x)} f(x, y) \right| + C_{\mu} L \delta_n(x, y)^{\alpha - 1/\alpha} |\partial_d f(x, y)|.
\]

Using Lemma 4.2 we have
\[
\left( \int_G (\delta_n(x, y))^{\gamma p} |\partial_{\xi_{i}(x)} f(x, y)|^p dxdy \right)^{1/p} 
\leq C_p L^2 n \| f \|_{L^p(G)}.
\]

On the other hand, we have
\[
\left( \int_G |\delta_n(x, y)|^{1 + \frac{1}{p} + \gamma p} |\partial_d f(x, y)|^p dxdy \right)^{1/p}
= \left( \int_{D_1} \int_{g(x) - 1}^{g(x)} (\sqrt{\delta_n(x, y)})^p |\partial_d f(x, y)|^p dy dx \right)^{1/p} 
\leq C(p)n \| f \|_{L^p(G)},
\]
where we used the Bernstein-Markov inequality for univariate algebraic polynomials in the last step. Putting the above together, and noticing that
\[
|\nabla_{tan,u} f| \sim \max_{1 \leq i \leq d - 1} |\partial_{\xi_{i}(u)} f|,
\]
we obtain the desired estimate. \( \square \)

5. TANGENTIAL \( L^p \) BERNSTEIN-MARKOV INEQUALITIES ON GENERAL \( C^{\alpha} \) DOMAINS

In this section, we will extend the tangential Bernstein type inequality established in the last section to a more general \( C^{\alpha} \)-domain with \( 1 \leq \alpha \leq 2 \). Our goal is to prove Theorem 1.1 (see Theorem 5.3 and Corollary 5.4 below).

Let us first write the tangential Bernstein inequality in Theorem 4.1 in an equivalent form, which can be easily extended to a general \( C^{\alpha} \)-domain. With the notations and assumptions of the last section, we have the following lemma, which will play a crucial role in our discussion.

**Lemma 5.1.** [4, Lemma 3.2] Let \( \Gamma' = \{(x, g(x)) : x \in D_2\} \). If \( \xi = (\xi_x, \xi_y) \in G \), then
\[
c_*(g(\xi_x) - \xi_y) \leq \text{dist}(\xi, \Gamma') \leq g(\xi_x) - \xi_y,
\]
where \( \text{dist}(\xi, \Gamma') = \inf_{\eta \in \Gamma'} \| \xi - \eta \| \), and \( c_* = \frac{1}{3\sqrt{1+\|\nabla g\|_2}} \) and \( \|\nabla g\|_2 = \max_{x \in D_2} \|\nabla g(x)\| \).

**Lemma 5.1** was proved in [4, Lemma 3.2] for \( C^2 \)-domains, but the proof there uses only the \( C^1 \) condition. For the sake of completeness, we copy the proof here.

**Proof.** Let \( \xi = (\xi_x, \xi_y) \in G \). Since \( (\xi_x, g(\xi_x)) \in \partial G \subset \Gamma' \), we have
\[
\text{dist}(\xi, \Gamma') \leq \| (\xi_x, \xi_y) - (\xi_x, g(\xi_x)) \| = g(\xi_x) - \xi_y.
\]

It remains to prove the inverse inequality,
\[
\text{dist}(\xi, \Gamma') \geq c_*(g(\xi_x) - \xi_y).
\]

Let \( (x, g(x)) \in \Gamma' \) be such that
\[
\text{dist}(\xi, \Gamma') = \| \xi - (x, g(x)) \|
\]
Since
\[
\text{dist}(\xi, \Gamma') \geq \| x - \xi_x \|
\]
Recall also that for $\|x - \xi_x\| \geq c_*(g(\xi_x) - \xi_y)$. Thus, without loss of generality, we may assume that $\|x - \xi_x\| < c_*(g(\xi_x) - \xi_y)$. We then write
\[
\|\xi - (x, g(x))\|^2 = \|\xi_x - x\|^2 + |\xi_y - g(\xi_x)|^2 + |g(\xi) - g(x)|^2 + 2(\xi_y - g(\xi_x)) \cdot (g(\xi_x) - g(x))
\]
(5.2)

Thus, using (5.2), we obtain
\[
\|\xi_x - x\|^2 + |g(\xi_x) - g(x)|^2 + 2|\xi_y - g(\xi_x)| \cdot (g(\xi_x) - g(x)) \\
\leq (1 + \|\nabla g\|^2_\infty)\|\xi_x - x\|^2 + 2\|\nabla g\|_\infty\|g(\xi_x) - \xi_y\|\|\xi_x - x\| \\
\leq \left|c_2^2 (1 + \|\nabla g\|^2_\infty) + 2\|\nabla g\|_\infty\right| c_1 (g(\xi_x) - \xi_y)^2 \leq \frac{7}{9} (g(\xi_x) - \xi_y)^2.
\]

Thus, using (5.2), we obtain
\[
\text{dist}(\xi, \Gamma')^2 = \|\xi - (x, g(x))\|^2 \geq \frac{2}{9} |\xi_y - g(\xi_x)|^2,
\]
which implies the desired lower estimate (5.1). This completes the proof of Lemma 5.1.

Next, we recall that for $u \in D_1$,
\[
|\nabla_{\tan, u} f(\xi)| = \max_{\eta \in \Sigma, \xi \in \Sigma_p - 1} |\partial_{\eta} f(\xi)|, \quad \xi \in G.
\]

For convenience, we also define
\[
|\nabla_{\tan, (u, g(u))} f(\xi)| := |\nabla_{\tan, u} f(\xi)|, \quad u \in D_1, \quad \xi \in G.
\]

Let $\Gamma' = \partial' G_s$. By Lemma 5.1, we have that for each $\xi = (\xi_x, \xi_y) \in G$,
\[
\varphi_n(\xi) := \sqrt{g(\xi_x) - \xi_y} + \frac{1}{n} \sim \sqrt{\text{dist}(\xi, \Gamma')} + \frac{1}{n} := \varphi_n,\Gamma'(\xi).
\]

Recall also that for $\mu > 1$, $\Xi_{n, \mu, \alpha} := \left\{ u \in D_1 : \|u - \xi_x\| \leq \mu |\varphi_n(\xi)|^{2/\alpha} \right\}$.

If $u \in \Xi_{n, \mu, \alpha}(\xi)$ and $\xi = (\xi_x, \xi_y)$, then $\eta = (u, g(u)) \in \partial' G$, and
\[
|\eta - \xi| \leq \|u - \xi_x\| + |g(u) - g(\xi_x)| + |g(\xi_x) - \xi_y| \\
\leq C\left(\|u - \xi_x\| + \text{dist}(\xi, \Gamma')\right) \leq C\mu\left(\varphi_n,\Gamma'(\xi)^{2/\alpha} + \text{dist}(\xi, \Gamma')\right).
\]

Since $\Gamma'$ is a bounded set and $\alpha \geq 1$, we have
\[
\text{dist}(\xi, \Gamma') \leq C\text{dist}(\xi, \Gamma')^{1/\alpha} \leq C\varphi_n,\Gamma'(\xi)^{2/\alpha}.
\]

Thus,
\[
\left\{ (u, g(u)) : u \in \Xi_{n, \mu, \alpha}(\xi) \right\} \subset \left\{ \eta \in \Gamma' : \|\eta - \xi\| \leq C\mu \varphi_n,\Gamma'(\xi)^{2/\alpha} \right\}.
\]

The corresponding inverse relation with a possibly different value of $\mu$ holds as well. Thus, we can reformulate the tangential Bernstein inequality (4.4) equivalently as follows:
\[
\left\| (\varphi_n,\Gamma')^{2/\alpha} D_{n, \mu, \xi} f \right\|_{L^p(G')} \leq C_{p, n} \|f\|_{L^p(G')},
\]

where
\[
D_{n, \mu, f}(\xi) := \max \left\{ |\nabla_{\tan, \eta, f}(\xi)| : \eta \in \partial' G', \|\eta - \xi\| \leq \mu \varphi_n,\Gamma'(\xi)^{2/\alpha} \right\}.
\]

This last version of tangential Bernstein inequality can be easily extended to a more general $C^\alpha$ domain.
In the sequel, we assume \( \Omega \subset \mathbb{R}^d \) is a compact \( C^\alpha \)-domain with \( 1 \leq \alpha \leq 2 \), whose precise definition is given as follows. For \( r > 0 \) and \( \xi \in \mathbb{R}^d \), we define \( \ell_\alpha \)-balls by

\[
B^\alpha(\xi, r) := \{ \eta \in \mathbb{R}^d : \| \eta - \xi \|_\alpha < r \},
\]

where \( \| \eta \|_\alpha := (\sum_{j=1}^d |\eta_j|^\alpha)^{1/\alpha} \) for \( \eta = (\eta_1, \cdots, \eta_d) \in \mathbb{R}^d \). Hence \( B^2(\xi, r) \) stands for the usual Euclidian balls.

**Definition 5.2.** Let \( 1 \leq \alpha \leq 2 \). A bounded set \( \Omega \) in \( \mathbb{R}^d \) is called a \( C^\alpha \)-domain if there exist a positive constant \( \kappa_0 \), and a finite cover of the boundary \( \partial \Omega \) by open sets \( \{ U_j \}_{j=1}^J \) in \( \mathbb{R}^d \) such that

(i) for each \( 1 \leq j \leq J \), there exists a function \( \Phi_j \in C^\alpha(\mathbb{R}^d) \) such that

\[
U_j \cap \partial \Omega = \{ \xi \in U_j : \Phi_j(\xi) = 0 \} \quad \text{and} \quad \nabla \Phi_j(\xi) \neq 0, \quad \forall \xi \in U_j \cap \partial \Omega;
\]

(ii) for each \( \xi \in \partial \Omega \) there exist affine transforms \( A_1, A_2 \) of \( \mathbb{R}^d \) with \( \det A_i \geq \kappa_0 \) and \( A_i(e_1) = \xi \), \( i = 1, 2 \), such that

\[
A_1(B^\alpha(0, 1)) \subset \Omega \quad \text{and} \quad A_2(B^\alpha(0, 1)) \subset \mathbb{R}^d \setminus \Omega,
\]

where \( e_1 = (1, 0, \ldots, 0) \) is the first standard basis vector.

Condition (ii) of this definition is needed to ensure that for any point of the boundary both the domain and its complement contain an \( \ell_\alpha \)-ball with “vertex” at this point. This is a generalization of the “rolling ball” property \([1] \) Def. 1.1(ii)] for \( \alpha = 2 \), see also \([1] \) Rem. 1.2.

Let \( \Gamma = \partial \Omega \). Denote by \( n_\eta \) the outer unit normal vector to \( \Gamma \) at \( \eta \in \Gamma \). For \( \xi \in \Omega, f \in C^\infty(\Omega) \), we define

\[
|\nabla_{\tan, \eta} f(\xi)| := \max \left\{ |\partial_\tau f(\xi)| : \tau \in \mathbb{S}^{d-1}, \tau \cdot n_\eta = 0 \right\}, \quad \eta \in \Gamma.
\]

Furthermore, given a parameter \( \mu \geq \left( \text{diam}(\Omega) + 1 \right)^2 \), we define

\[
D_{n, \mu} f(\xi) := \max \left\{ |\nabla_{\tan, \eta} f(\xi)| : \eta \in \partial \Omega, \| \eta - \xi \| \leq \mu \varphi_{n, \Gamma}(\xi)^{2/\alpha} \right\}, \quad \xi \in \Omega,
\]

where \( \varphi_{n, \Gamma}(\xi) := \sqrt{\text{dist}(\xi, \Gamma)} + n^{-1}, \quad n = 1, 2, \ldots, \xi \in \Omega \).

With the above notation, we can then state our main tangential \( L^p \) Bernstein type inequality on general \( C^\alpha \)-domains as follows:

**Theorem 5.3.** Let \( \gamma = \frac{1}{\alpha} - \frac{1}{2}, \) and \( 0 < p < \infty \). Then for any \( f \in \Pi^\alpha_n \) and parameter \( \mu > 1 \), we have

\[
\left\| (\varphi_{n, \Gamma})^{2\gamma} D_{n, \mu} f \right\|_{L^p(\Omega)} \leq C(\mu, \Omega, p)n\| f \|_{L^p(\Omega)}.
\]

Since general \( C^\alpha \) domains can be covered by \( C^\alpha \) domains of special type, Theorem 5.3 can be deduced directly from Theorem 4.1 and Lemma 5.1. Since the proof is very close to the proof in Section 6 of [2], we skip the details here.

Clearly, \( \varphi_{n, \Gamma}(\xi) \geq n^{-1} \) that is using the lower bound \( (\varphi_{n, \Gamma})^{2\gamma} \geq n^{-\frac{\alpha}{2}+1} \) in the above theorem yields the next tangential \( L^p \) Markov type inequality on general \( C^\alpha \)-domains:

**Corollary 5.4.** For any \( 0 < p < \infty, f \in \Pi^\alpha_n \) and parameter \( \mu > 1 \), we have

\[
\left\| D_{n, \mu} f \right\|_{L^p(\Omega)} \leq C(\mu, \Omega, p)n^{\frac{2}{\alpha}}\| f \|_{L^p(\Omega)}.
\]
6. Sharpness of the Tangential $L^p$ Markov Inequality on General $C^\alpha$ Domains

Corollary 5.3 asserts that $L^p$ norms of tangential derivatives of polynomials of degree $n$ on general $C^\alpha, 1 \leq \alpha \leq 2$ domains are of order $n^{\frac{\alpha}{2.}}$. For $p = \infty$ this upper bound is known to be sharp, see [5]. It is considerably harder to verify $L^p$ lower bounds in case when $0 < p < \infty$. This is the question we will settle in this section.

Let us consider $C^\alpha, 1 < \alpha \leq 2$ domain $D \subset \mathbb{R}^d$. In what follows affine images of the $\ell_\alpha$ unit ball $B^\alpha(0, 1)$ will be called $\ell_\alpha$ ellipsoids, with images of the standard basis vectors $e_j := (\delta_{i,j})_{1 \leq i \leq d}, 1 \leq j \leq d$ being the vertices of these ellipsoids. Recall that when $D \subset \mathbb{R}^d$ is a $C^\alpha, 1 < \alpha \leq 2$ domain then for any $x \in \partial D$ on its boundary there exists an inscribed $\ell_\alpha$ ellipsoid $E \subset D$ for which $x \in \partial E$. Clearly in order to prove lower bounds for tangential Markov type inequality for $C^\alpha, 1 < \alpha \leq 2$ domains we need to ensure that this domain is not in a higher Lip class. We will accomplish this by assuming that the boundary of the domain contains an exactly $C^\alpha$ point $y \in \partial D$ such that for some $\delta > 0$ there exist a superscribed $\ell_\alpha$ ellipsoid $E_1$ with vertex at $y$ so that $D \cap B^2(y, \delta) \subset E_1$. Given this definition we have the next general converse to the tangential Markov type inequality given by Corollary 5.3.

**Theorem 6.1.** Let $1 \leq p < \infty, n \in \mathbb{N}$. Assume that $C^\alpha, 1 < \alpha \leq 2$ domain $D \subset \mathbb{R}^d$ contains an exactly $C^\alpha$ point. Then there exists $f \in \Pi^d_n$ such that

$$\|D_n f\|_{L^p(D)} \geq c(D, p)n^{\frac{\alpha}{2}}\|f\|_{L^p(D)}.$$

**Proof.** Performing a proper affine map we can assume without the loss of generality that $e_d = (0, ..., 0, 1) \in \partial D$ is an exactly $C^\alpha$ point on the boundary and $D \cap B^2(e_d, \delta) \subset B^\alpha(0, 1)$ is the corresponding superscribed $\ell_\alpha$ ellipsoid. In addition, there exists an inscribed $\ell_\alpha$ ellipsoid $E \subset D$ for which $e_d \in \partial E$. The existence of such inscribed ellipsoid $E$ is ensured by condition (ii) of Definition 5.2. Then evidently $e_d$ must be the outer normal to $E$ at $e_d$ and it easily follows that for any $(x, y) \in D \cap B(e_d, \delta), x = (x_1, ..., x_{d-1}) \in \mathbb{R}^{d-1}, y \in \mathbb{R}$ we have with proper $c_1, c_2 > 0$

$$c_1(|x_1|^\alpha + ... + |x_{d-1}|^\alpha) \leq 1 - y \leq c_2(|x_1|^\alpha + ... + |x_{d-1}|^\alpha).$$

Consider the polynomial

$$Q(x, y) = x_1J_n(y)g_n(x, y) \in \Pi^d_{(2b+1)n+1}, x = (x_1, ..., x_{d-1}) \in \mathbb{R}^{d-1}, y \in \mathbb{R}$$

where $J_n(y) = J_n^{(\beta)}(y), y \in \mathbb{R}$ is the $n$-th Jacobi polynomial with parameter $\beta$ to be specified below and we set $g_n(x, y) := \left(1 - |x|^2 - (1 - y)^2\right)^b$ where $T$ stands for the diameter of the domain $D$ and the integer $b \in \mathbb{N}$ will be specified below. Note that since $e_d = (0, ..., 0, 1) \in \partial D$ it follows that $|g_n| \leq 1$ on $D$.

Then by [13, 4.21.7, p.63],

$$J_n'(y) = (n/2 + \beta + 1/2)f_{n-1}^{(\beta+1, \beta+1)}(y).$$

Let us give a lower bound for $\int_{D} |D_n Q|^p$. Clearly it follows from (6.1) the boundary $\partial D$ in $B(e_d, \delta)$ is given by the surface by $y = f(x), \nabla f \in \text{Lip}(\alpha - 1)$ with

$$1 - c_2(|x_1|^\alpha + ... + |x_{d-1}|^\alpha) \leq f(x) \leq 1 - c_1(|x_1|^\alpha + ... + |x_{d-1}|^\alpha), \nabla f(0) = 0.$$

Moreover, since $(\nabla f, -1)$ is a normal to $\partial D$ it follows that the tangent plane is spanned by $e_j + \frac{\partial f}{\partial x_j}e_d, 1 \leq j \leq d - 1$. Therefore setting $\partial^1 g := \frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_1} \frac{\partial g}{\partial x_d}$ we have that $|D_n g| \geq |\partial^1 g|, \forall y$. Furthermore

$$\partial^1 Q(x, y) = \partial^1 (x_1 J_n(y))g_n(x, y) + x_1 J_n(y) \partial^1 g_n(x, y) =$$

$$= \left(J_n(y) + \frac{\partial f}{\partial x_1} J_n'(y)x_1\right) g_n(x, y) + x_1 J_n(y) \partial^1 g_n(x, y).$$
Set \( D_a := \{ (x, y) \in D : 1 - \frac{a}{n} \leq y \leq 1 \} \) with \( 0 < a < 1 \) to be properly chosen below. First let us note that in \( D_a \) we have that \( 1 - y \sim \frac{a}{n}, |x| \sim \frac{a}{n^{1/\alpha}} \) yielding \( g_n \sim c, |\partial^1 g_n| \leq cn. \) Moreover by (6.3) we have for any \((x, y) \in D_a, |x| \leq c_3(1 - y)^{1/\alpha} \leq c_3 \left( \frac{a}{n} \right)^{1/\alpha} \) and \( \left| \frac{\partial f}{\partial x_1} x_1 \right| \leq c|x|^\alpha \). Hence by the previous relation

\[
|D_n| \geq |\partial^1 Q| \geq c |J_n(y)| - c \left| \frac{\partial f}{\partial x_1} J_n(y)x_1 \right| - cn|x_1 J_n(y)| \geq c |J_n(y)| - \frac{c_0}{n^2} |J_n(y)| - c \alpha |J_n(y)|.
\]

Now we need to recall that \( \|J_n^{(\beta, \gamma)}\|_{[c^{-1}, 1]} \sim n^\beta \sim J_n^{(\beta, \gamma)}(1) \). Therefore by the Markov inequality and relation (6.2) it follows that uniformly with respect to \( a \in [0, 1] \) and any \( y \in [1 - \frac{a}{n}, 1] \) we have

\[
|J_n(y)| \sim n^\beta, \quad |J_n(y)| \sim n^{\beta + 2}.
\]

Hence we can properly choose \( a > 0 \) so that the above estimate yields

\[
|D_n Q| \geq |\partial^1 Q| \geq cn^\beta, \quad (x, y) \in D_a.
\]

Thus we obtain the next lower bound for the integral of tangential derivative

\[
(6.4) \quad \int_D |D_n Q|^p \geq \int_{D_a} |D_n Q|^p \geq cn^{\beta p - 2 + \frac{2d + 2}{n}}.
\]

Next we need an upper bound for \( \int_D |Q|^p \). Since \( 0 \leq g_n \leq 1, (x, y) \in B(e_d, \delta) \) it follows by (6.1) that

\[
\int_{D \cap B(e_d, \delta)} |Q|^p \leq \int_{D \cap B(e_d, \delta)} |x|^p |J_n(y)|^p \leq \int_{1 - \delta}^{1} \int_{|x| \leq c_3(1 - y)^{1/\alpha}} |x|^p |J_n(y)|^p dx dy
\]

\[
\leq c \int_{1 - \delta}^{1} (1 - y)^{2\mu + \beta} |J_n(y)|^p dy.
\]

Now we will apply asymptotic properties of the Jacobi polynomial \( J_n = J_n^{(\beta, \gamma)}(x), \beta > -1 \) verified in [13], (7.34.1), (7.34.4), p. 173. It is essentially shown there that

\[
(6.5) \quad \int_{\delta}^{1} (1 - x)^\mu |J_n(x)|^p dx \sim n^{-2\mu - 2 + \beta p}, \quad 2\mu < \beta p - 2 + \frac{p}{2}.
\]

(In fact, in [13] these asymptotic relations are verified for \( p = 1 \) but they follow analogously for any \( p \geq 1 \).) Using this result with \( \mu := \frac{\beta + d - 1}{\alpha} \) and any \( \beta > 2d + 2 \) we obtain from the previous estimate

\[
(6.6) \quad \int_{D \cap B^2(e_d, \delta)} |Q|^p \leq cn^{-2\mu - 2d + 2 + \beta p}.
\]

On the other hand using that \( 0 \leq g_n \leq (1 - \delta^2)^{bn}, (x, y) \in D \setminus B^2(e_d, \delta) \) and \( |x_1 J_n| \leq M^n, (x, y) \in D \) with some \( M > 0 \) depending on the domain \( D \) it follows that \( |Q(x, y)| = |x_1 J_n(y) g_n(x, y)| \leq (1 - \delta^2)^{bn} M^n, (x, y) \in D \setminus B^2(e_d, \delta) \). Hence we can choose a proper \( b > 0 \) so that \( |Q(x, y)| \leq \gamma^n, (x, y) \in D \setminus B^2(e_d, \delta) \) with some \( 0 < \gamma < 1 \). Combining this observation with the upper bound (6.6) obviously implies that

\[
(6.7) \quad \int_D |Q|^p \leq cn^{-2\mu - 2d + 2 + \beta p}.
\]

Finally, lower bound (6.4) together with the upper bound (6.7) yield that

\[
\frac{\int_D |D_n Q|^p}{\int_D |Q|^p} \geq cn^\frac{2d}{\alpha}.
\]

Now taking the \( p \)-th root of the last estimate completes the proof of the theorem.
7. Marcinkiewicz type inequalities on \(C^\alpha\) domains: the case of \(d = 2\)

Given a subspace \(U \subset L^p(K)\) the Marcinkiewicz-Zygmund type problem for \(1 \leq p < \infty\) consists in finding discrete point sets \(Y_N = \{x_1, \ldots, x_N\} \subset K\) and corresponding positive weights \(w_j > 0, 1 \leq j \leq N\) such that for any \(g \in U\) we have

\[
(7.1) \quad c_1 \sum_{1 \leq j \leq N} w_j |g(x_j)|^p \leq \|g\|_{L^p(K)}^p \leq c_2 \sum_{1 \leq j \leq N} w_j |g(x_j)|^p
\]

with some constants \(c_1, c_2 > 0\) depending only on \(p, d\) and \(K\). In case when \(p = \infty\) the above relation is replaced by

\[
\|g\|_{C(K)} \leq c \max_{1 \leq j \leq N} |g(x_j)|, \quad g \in U.
\]

Evidently we must have \(N \geq \dim U\) in order for these estimates to be possible for every \(g \in U\). These equivalence relations turned out to be an effective tool used for the discretization of the \(L^p\) norms of trigonometric polynomials which is widely applied in the study of the convergence of Fourier series, Lagrange and Hermite interpolation, positive quadrature formulas, scattered data interpolation, see for instance \([10]\) for a survey on the univariate Marcinkiewicz-Zygmund type inequalities. Naturally it is crucial to find discrete point sets \(Y_N = \{x_1, \ldots, x_N\} \subset K\) of possibly minimal cardinality \(N\). When \(U = \Pi_n^d\), and the domain \(K \subset \mathbb{R}^d\) has nonempty interior, we have \(\dim \Pi_n^d \sim n^d\) and therefore asymptotically optimal discrete points sets for \(\Pi_n^d\) must be of order \(n^d\). For \(p = \infty\) such discrete points sets are called optimal meshes. Some general assertions concerning the existence of proper discrete point sets of cardinality \(\sim n^d \log^m n\) can be found in \([3]\) for \(1 \leq p \leq 2\) (\(m = 3\)) and \([2]\) for \(p = \infty\) (\(m = d\)). However, besides involving extra log factors the above mentioned results do not provide explicit construction for discrete point sets. On the other hand, an application of tangential Bernstein type inequalities makes it possible to give explicit construction of discrete point sets of asymptotically optimal cardinality. This approach was used in \([9]\) in case when \(p = \infty\) where existence of optimal meshes was verified for any \(C^\alpha, 2 - \frac{2}{d} < \alpha \leq 2\) domain. In addition, tangential Bernstein type inequalities were applied in \([4]\) in order to verify \(L^p, 1 \leq p < \infty\) Marcinkiewicz type inequalities for \(\Pi_n^d\) with \(\sim n^d\) points in \(C^2\) domains. The aim of the last two sections of this paper is to apply the tangential \(L^p\) Bernstein-Markov inequalities verified on general \(C^\alpha\) domains in the previous sections in order to provide asymptotically optimal Marcinkiewicz type inequalities for \(C^\alpha, 2 - \frac{2}{d} < \alpha < 2\) domains.

In this section we will prove Theorem \([12]\) for the case \(d = 2\), which we reformulate as follows.

**Theorem 7.1.** If \(\Omega \subset \mathbb{R}^2\) is a compact \(C^\alpha\)-domain with \(1 < \alpha < 2\), then for any positive integer \(n\), there exists a partition \(\Omega = \bigcup_{1 \leq j \leq N} \Omega_j\) of \(\Omega\) with \(N \leq cn^2\) such that for every \((x_j, y_j) \in \Omega_j\), each \(f \in \Pi_n^d\) and \(1 \leq p < \infty\), we have

\[
\frac{1}{2} \sum_{j=1}^{N} |\Omega_j||f(x_j, y_j)|^p \leq \int\int_{\Omega} |f(x, y)|^p \, dx \, dy \leq 2 \sum_{j=1}^{N} |\Omega_j||f(x_j, y_j)|^p.
\]

**Remark 7.2.** For \(\alpha = 1\), the stated result with \(N \leq cn^2 \log n\) remains true, as can be seen from the proof below.

Since general \(C^\alpha\) domains can be covered by \(C^\alpha\) domains of special type, it suffices to consider domains of special type.

\(^1\)Note that if \(g\) is an algebraic polynomial of degree \(\leq n/2\), then \(g^2\) is an algebraic polynomial of degree \(\leq n\). Hence, if \((\ref{1.4})\) is valid for algebraic polynomials of degree \(\leq n\) and some \(p\), then \((\ref{1.4})\) is valid for algebraic polynomials of degree \(\leq n/2\) and \(2p\). Therefore, the Marcinkiewicz-Zygmund type inequalities for algebraic polynomials for \(1 \leq p \leq 2\) automatically extend to the full range \(1 \leq p < \infty\) with the same order of dependence of \(N\) on \(n\).
Let \( g : [-2, 2] \to \mathbb{R} \) be a continuously differentiable function on \([-2, 2]\) satisfying
\[
|g'(x + t) - g'(x)| \leq L|t|^\alpha - 1 \quad \text{whenever } x, x + t \in [-2, 2],
\]
where \( L > 1 \) is a constant. Define
\[
G := \left\{ (x, y) : \quad 0 \leq x \leq 1, \quad g(x) - \frac{1}{4} \leq y \leq g(x) \right\},
\]
and
\[
G_* := \left\{ (x, y) : \quad -2 \leq x \leq 2, \quad g(x) - 2 \leq y \leq g(x) \right\}.
\]
For \((x, y) \in G_*\), we set \( \delta(x, y) := g(x) - y \) and
\[
\delta_n(x, y) = \delta(x, y) + \frac{1}{n^2} = g(x) - y + \frac{1}{n^2}.
\]
As verified in Theorem 3.1 for any \( f \in \Pi_2^N \) and \( 0 < p < \infty \),
\[
\left( \int_0^1 \int_{g(x)-1}^{g(x)} (\delta_n(x, y))^\gamma p \partial_{t_n} f(x, y)^p \, dy \, dx \right)^{1/p} \leq C_p L^2 n \| f \|_{L^p(G_*)},
\]
where \( \gamma := \frac{1}{p} - \frac{1}{2} \), and \( t_n := (1, g'(x)) \).

To formulate our main proposition, we define, for a bounded function \( f \) on \( G_* \) and a set \( A \subset G_* \),
\[
\text{osc}(f; A) := \sup_{\xi, \xi' \in A} |f(\xi) - f(\xi')|.
\]
Using the Bernstein inequality (7.2), we may obtain

**Proposition 7.3.** For any \( 0 < \epsilon < 1 \), \( 1 < \alpha \leq 2 \) and \( n \in \mathbb{N} \), there exists a partition \( G = \bigcup_{j=1}^N G_j \) with \( N \leq c_\alpha n^2 \) so that for every \( f \in \Pi_2^N \) and \( 1 \leq p < \infty \),
\[
\sum_{j=1}^N \|G_j\| \text{osc}(f; G_j)^p \leq e^p \int_G \|f(x, y)p\, dx \, dy.
\]

Using the argument of [4], Theorem 7.4 follows directly from Proposition 7.3. Indeed, following the proof of Lemma 7.5 of [4], and using (7.2) and the fact that \( \Omega \) can be covered by finitely many domains of special type, we can find a partition \( \Omega = \bigcup_{j=1}^N \Omega_j \) with \( N \leq c_\epsilon n^2 \) so that for every \( f \in \Pi_2^N \) and \( 1 \leq p < \infty \),
\[
\sum_{j=1}^N \|\Omega_j\| \text{osc}(f; \Omega_j)^p \leq C(\omega) e^p \int_\Omega \|f(x, y)p\, dx \, dy.
\]

We then use the following elementary inequality,
\[
|a^p - b^p| \leq C_p|a - b|^p + \delta b^p, \quad \forall a, b > 0, \quad \delta \in (0, 1),
\]
and obtain that for any \((x_j, y_j) \in G_j\) and any \( f \in \Pi_2^N \),
\[
\left| \sum_{j=1}^N |f(x_j, y_j)|^p \Omega_j - \int_\Omega |f(x, y)|^p \, dx \, dy \right|
\leq C_p \epsilon^{1-p} \sum_{j=1}^N \int_{\Omega_j} |f(x_j, y_j) - f(x, y)|^p \, dx \, dy + \epsilon \int_\Omega |f(x, y)|^p \, dx \, dy
\leq C_p(\Omega) \epsilon \int_\Omega |f(x, y)|^p \, dx \, dy.
\]

Theorem 7.4 then follows by choosing \( \epsilon \in (0, 1) \) so that \( C_p(\Omega) \epsilon \leq \frac{1}{4} \).
The rest of this note is devoted to the proof of Proposition 7.3. We need the following lemma for algebraic polynomials of one variable.

**Lemma 7.4.** Let \( \beta \geq -\frac{1}{2} \) and let
\[
x_j := \frac{j^2}{4m^2}, \quad j = 0, 1, \ldots, m.
\]
Then for every \( f \in \Pi_n \) with \( n \leq m \), and each \( 1 \leq p < \infty \), we have
\[
\left( \frac{1}{m} \sum_{j=1}^{m} (x_j^{\beta + \frac{1}{2}} + m^{-1}) \max_{x \in [x_{j-1}, x_j]} |f(x)|^p \right)^{\frac{1}{p}} \leq C \left( \int_0^1 |f(x)|^p x^\beta dx \right)^{\frac{1}{p}},
\]
where \( C > 0 \) is a constant depending only on \( \beta \).

**Proof.** This lemma can be obtained by a slight modification of the proof of Theorem 3.1 of [11]. Write
\[
x_j = \sin^2 \frac{\theta_j}{2}, \quad \text{where} \quad \theta_j \in [0, \pi/3], \quad j = 0, 1, \ldots, m.
\]
For \( f \in \Pi_n \), let
\[
T_n(\theta) = f(\sin^2 \frac{\theta}{2}) = f \left( \frac{1 - \cos \theta}{2} \right), \quad \theta \in \mathbb{R}.
\]
Then \( T_n \) is an even trigonometric polynomial of degree at most \( n \), and
\[
\frac{1}{m} \leq \theta_j - \theta_{j-1} \leq \frac{2}{m}, \quad j = 1, 2, \ldots, m.
\]
Now setting \( I_j = [\theta_{j-1}, \theta_j] \), we have that for each \( \theta \in I_j \),
\[
|T_n(\theta)| \leq \int_{I_j} |T_n'(s)| ds + \frac{1}{|I_j|} \int_{I_j} |T_n(s)| ds,
\]
which implies
\[
\max_{\theta \in I_j} |T_n(\theta)|^p \leq C_p m^{-(p-1)} \int_{I_j} |T_n'(s)|^p ds + C_p m \int_{I_j} |T_n(s)|^p ds.
\]
Since
\[
\sin \frac{\theta_j}{2} \leq \min_{\theta \in I_j} \sin \frac{\theta}{2} + m^{-1}, \quad 1 \leq j \leq m,
\]
it follows that
\[
\left( \frac{1}{m} \sum_{j=1}^{m} (x_j^{\beta + \frac{1}{2}} + m^{-1}) \max_{x \in [x_{j-1}, x_j]} |f(x)|^p \right)^{\frac{1}{p}} = \left( \frac{1}{m} \sum_{j=1}^{m} \left[ \left( \sin \frac{\theta_j}{2} \right)^{2\beta+1} + \frac{1}{m} \right] \max_{\theta \in I_j} |T_n(\theta)|^p \right)^{1/p} \leq C_\beta (S_1 + S_2),
\]
where
\[
S_1 := m^{-1} \left( \int_0^{\pi/3} |T_n'(\theta)|^p \left[ \left( \sin \frac{\theta}{2} \right)^{2\beta+1} + \frac{1}{m} \right] d\theta \right)^{1/p},
\]
\[
S_2 := \left( \int_0^{\pi/3} |T_n(\theta)|^p \left[ \left( \sin \frac{\theta}{2} \right)^{2\beta+1} + \frac{1}{m} \right] d\theta \right)^{1/p}.
\]
For the integral $S_1$, we use the weighted Bernstein inequality for trigonometric polynomials (see [11, Theorem 4.1]) to obtain
\[
S_1 \leq m^{-1} \left( \frac{2}{\sqrt{3}} \int_0^{\pi/3} |T_n(\theta)|^p \left[ \left( \sin \frac{\theta}{2} \right)^{2\beta+1} \cos \theta + \frac{1}{m} \right] d\theta \right)^{1/p}
\]
\[
\leq C \left( \int_0^\pi |T_n(\theta)|^p \left[ \left( \sin \frac{\theta}{2} \right)^{2\beta+1} \cos \theta + \frac{1}{m} \right] d\theta \right)^{1/p},
\]
which, using the Schur-type inequality for trigonometric polynomials (see [11, (3.3)]), is estimated above by
\[
C \left( \int_0^\pi |T_n(\theta)|^p \left( \sin \frac{\theta}{2} \right)^{2\beta+1} \cos \theta \right)^{1/p} = C \left( \int_0^1 |f(x)|^p x^\beta dx \right)^{1/p}.
\]
The integral $S_2$ can be estimated similarly:
\[
S_2 \leq \left( \frac{2}{\sqrt{3}} \int_0^{\pi/3} |T_n(\theta)|^p \left( \sin \frac{\theta}{2} \right)^{2\beta+1} \cos \theta + \frac{1}{m} \right] d\theta \right)^{1/p} \leq C \left( \int_0^1 |f(x)|^p x^\beta dx \right)^{1/p}.
\]
Putting the above together, we deduce the inequality stated in the lemma. □

We are now in a position to prove Proposition 7.3.

Proof of Proposition 7.3

Let $m > 2n$ be an integer such that $\frac{1}{m} \sim \frac{2}{m^2}$. Let $z_j := \frac{2}{m^2}$ for $j = 0, 1, \ldots, m$, and let $x_{i,j} := \frac{i}{N_j}$ for $1 \leq j \leq m$ and $0 \leq i \leq N_j$, where $N_j$ is an integer $\geq n$ to be specified later. We then define a partition $G = \bigcup_{j=1}^{m} \bigcup_{i=1}^{N_j} I_{i,j}$ as follows:
\[
I_{i,j} := \{(x, y) \in G : x_{i-1,j} \leq x \leq x_{i,j}, \quad z_{j-1} \leq g(x) - y \leq z_j, \quad j = 1, 2, \ldots, m, \quad i = 1, 2, \ldots, N_j\}.
\]
Note that
\[
|I_{i,j}| = (x_{i,j} - x_{i-1,j})(z_j - z_{j-1}) = \frac{2j - 1}{4m^2 N_j}, \quad 1 \leq i \leq N_j, \quad 1 \leq j \leq m.
\]
Our aim is to show that
\[
\sum_{j=1}^{m} \sum_{i=1}^{N_j} |I_{i,j}| \max_{f \in \Pi_n^2} \left| \int_G |f(x, y)|^p dxdy \right| \leq \epsilon^p.
\]
To show (8.3), we define $F(x, z) := f(x, g(x) - z)$ for $-2 \leq x \leq 2$ and $z \in \mathbb{R}$. Then $f(x, y) = F(x, g(x) - y)$, and
\[
\text{osc}(f; I_{i,j}) := \max_{x, x' \in [x_{i-1,j}, x_{i,j}]} \max_{z, z' \in [z_{j-1}, z_j]} |F(x, z) - F(x', z')| \leq 2 \sup_{z \in [z_{j-1}, z_j]} \sup_{x \in [x_{i-1,j}, x_{i,j}]} \left| F(x, z) - N_j \int_{x_{i-1,j}}^{x_{i,j}} F(u, z_j) du \right| \leq 2 \left[ a_{i,j}(f) + b_{i,j}(f) \right].
\]
where
\[ a_{i,j}(f) := \sup_{x \in [x_{i-1,j}, x_{i,j}]} \left| F(x, z) - N_j \int_{x_{i-1,j}}^{x_{i,j}} F(u, z) \, du \right|, \]
\[ b_{i,j}(f) := N_j \sup_{z \in [z_{j-1}, z_j]} \left| \int_{x_{i-1,j}}^{x_{i,j}} [F(u, z) - F(u, z_j)] \, du \right|. \]

It follows that
\[ \sum_{j=1}^{m} \sum_{i=1}^{N_j} |I_{i,j}| \osc(f; I_{i,j})^p \leq 4^p \left( \Sigma_1 + \Sigma_2 \right), \]
where
\[ \Sigma_1 := \sum_{j=1}^{m} \sum_{i=1}^{N_j} |I_{i,j}| |a_{i,j}(f)|^p \text{ and } \Sigma_2 := \sum_{j=1}^{m} \sum_{i=1}^{N_j} |I_{i,j}| |b_{i,j}(f)|^p. \]

To estimate the sums \( \Sigma_1 \) and \( \Sigma_2 \), we first note that
\[ |a_{i,j}(f)|^p \leq \left( \int_{x_{i-1,j}}^{x_{i,j}} \sup_{z \in [z_{j-1}, z_j]} |\partial_1 F(v, z)| \, dv \right)^p \leq N_j^{1-p} \int_{x_{i-1,j}}^{x_{i,j}} \sup_{z \in [z_{j-1}, z_j]} |\partial_1 F(v, z)|^p \, dv, \]
\[ |b_{i,j}(f)|^p \leq \left( N_j \int_{z_{j-1}}^{z_j} \int_{x_{i-1,j}}^{x_{i,j}} |\partial_2 F(u, z)| \, du \, dz \right)^p \]
\[ \leq N_j \left( \frac{j}{4m^2} \right)^{p-1} \int_{z_{j-1}}^{z_j} \int_{x_{i-1,j}}^{x_{i,j}} |\partial_2 F(u, z)|^p \, du \, dz. \]

For the sum \( \Sigma_1 \), using (8.3) and (7.7), we have
\[ \Sigma_1 \leq \frac{C}{m^2} \sum_{j=1}^{m} \sum_{i=1}^{N_j} j N_j^{1-p} \sup_{z \in [z_{j-1}, z_j]} |\partial_1 F(v, z)|^p \, dv \]
\[ = \frac{C}{m^2} \int_0^1 \left[ \sum_{j=1}^{m} j N_j^{1-p} \sup_{z \in [z_{j-1}, z_j]} |\partial_1 F(v, z)|^p \right] \, dv. \]

Now we choose \( N_j \in \mathbb{N} \) so that
\[ m \leq N_j \sim m \left( \frac{m}{j} \right)^{2\gamma} \sim \frac{m}{j^2}, \]
where \( \gamma = \frac{1}{\alpha} - \frac{1}{2} \). On one hand, since \( 1 < \alpha \leq 2 \) and \( m \sim n \), we have \( 2\gamma = \frac{2}{\alpha} - 1 < 1 \), and the number of sets in the partition \( G = \bigcup_{I_{i,j}} I_{i,j} \) equals to
\[ \sum_{j=1}^{m} N_j \leq C m^{2\gamma+1} \sum_{j=1}^{m} j^{-2\gamma} \sim m^2 \sim n^2. \]

On the other hand, since the function
\[ \partial_1 F(v, z) = (\partial_1 + g'(v)\partial_2) f(v, g(v) - z) = (\partial_1 f)(v, g(v) - z) \]
\[ This is the only place where the condition \( \alpha > 1 \) is required. In the case of \( \alpha = 1 \), we have \( 2\gamma = 1 \) and \( \sum_{j=1}^{m} N_j \sim n^2 \log n \).
is an algebraic polynomial of degree at most \( n \) in the variable \( z \) for each fixed \( v \in [-2, 2] \), we obtain from Lemma 7.4 with \( \beta = \gamma p \) that for each \( v \in [0, 1] \),

\[
\sum_{j=1}^{m} \frac{1}{m^2} jN_j^{-p} \sup_{z \in \{z \leq \tau \}} |\partial_1 F(v, z)|^p \leq C_p \frac{1}{m^p} \sum_{j=1}^{m} \frac{z^{j/2}}{m} \sup_{z \in \{z \leq \tau \}} |\partial_1 F(v, z)|^p dz \leq C_p \frac{1}{m^p} \int_0^1 z^{j/2} |\partial_1 F(v, z)|^p dz.
\]

This together with (7.3) and the Bernstein inequality (7.2) implies

\[
\Sigma_1 \leq C_p m^{-p} \int_0^1 \int_{g(v)}^{g(z)} |\partial_1 f(x, y)|^p \delta(x, y)^\gamma p dy dx \leq C_p \left( \frac{m}{n} \right)^p \|f\|_{L^p(G_\star)}^p.
\]

Finally, for the sum \( \Sigma_2 \), using (7.3) and (8.3), we obtain

\[
\Sigma_2 \leq C_p \sum_{j=1}^{m} \int_{x_{j-1}}^{x_j} \int_{z_{j-1}}^{z_j} |\partial_2 F(u, z)|^p dz du = C_p \frac{1}{m^p} \sum_{j=1}^{m} \left( \frac{j}{m} \right)^p \int_{x_{j-1}}^{x_j} \int_{z_{j-1}}^{z_j} |\partial_2 F(u, z)|^p dz du \leq C_p m^{-p} \int_0^1 \int_{x_{j-1}}^{x_j} |\partial_2 F(u, z)|^p \delta(x, y)^\gamma p dy dx \leq C_p m^{-p} \int_0^1 |\partial_2 F(u, z)|^p dz du.
\]

Since \( F(u, z) = f(u, g(u) - z) \) is an algebraic polynomial of the variable \( z \) of degree at most \( n \), using the univariate Markov-Bernstein-type inequality (111 Theorem 7.3), we obtain

\[
\Sigma_2 \leq C_p \left( \frac{m}{n} \right)^p \int_0^1 \int_0^2 |F(u, z)|^p dz du \leq C_p \left( \frac{m}{n} \right)^p \|f\|_{L^p(G_\star)}^p.
\]

Putting the above together, and taking into account the fact that \( m \sim \frac{2}{p} \), we obtain

\[
\sum_{j=1}^{m} \sum_{i=1}^{N_j} |I_{i,j}| \sup_{z \in \{z \leq \tau \}} |\partial_1 F(v, z)|^p \leq 4^p \left( \Sigma_1 + \Sigma_2 \right) \leq C_p \left( \frac{m}{n} \right)^p \|f\|_{L^p(G_\star)}^p \leq \varepsilon^p \|f\|_{L^p(G_\star)}^p.
\]

\[\square\]

8. Marcinkiewicz Type Inequalities on \( C^\alpha \) Domains: Higher-Dimensional Case

In this section, we assume \( d \geq 3 \). The main goal of this last part of the paper is to prove Theorem 1.2 for \( d \geq 3 \), namely, the following Marcinkiewicz type inequalities.

**Theorem 8.1.** If \( \Omega \subset \mathbb{R}^d \) is a compact \( C^\alpha \)-domain with \( 2 - \frac{2}{d} < \alpha \leq 2 \), then for any positive integer \( n \), there exists a partition \( \Omega = \bigcup_{1 \leq j \leq N} \Omega_j \) of \( \Omega \) with \( N \leq n d^d \) such that for every \( \xi_j \in \Omega_j \), each \( f \in \Pi_n^d \) and \( d - 1 < p < \infty \), we have

\[
\frac{1}{2} \sum_{j=1}^{N} |\Omega_j| |f(\xi)|^p \leq \int_{\Omega} |f(\xi)|^p d\xi \leq 2 \sum_{j=1}^{N} |\Omega_j| |f(\xi)|^p.
\]
Let \( g: [-2d, 2d]^{d-1} \to \mathbb{R} \) be a continuously differentiable function on \([-2d, 2d]^{d-1}\) satisfying
\[
|\nabla g(x + t) - \nabla g(x)| \leq L|t|^\alpha - 1
\]
whenever \( x, x + t \in [-2d, 2d]^{d-1} \), where \( L > 1 \) is a constant. Define
\[
G := \left\{ (x, y): \ x \in [0, 1]^{d-1}, \ g(x) - \frac{1}{4} \leq y \leq g(x) \right\},
\]
and
\[
G_* := \left\{ (x, y): \ x \in [-2d, 2d]^{d-1}, \ g(x) - 2 \leq y \leq g(x) \right\}.
\]
For \((x, y) \in G_*\), we set \( \delta(x, y) := g(x) - y \) and
\[
\delta_n(x, y) = \delta(x, y) + \frac{1}{n^2} = g(x) - y + \frac{1}{n^2}.
\]
Let \( f \in \Pi_n^d \). Define \( F(x, z) := f(x, g(x) - z) \) for \( x \in [-2d, 2d]^{d-1} \) and \( z \in \mathbb{R} \). Then
\[
\partial_j F(x, z) = \partial_{\xi_j} f(x, g(x) - z), \ j = 1, 2, \ldots, d - 1,
\]
where \( \xi_j(x) = \epsilon_j + \partial_j g(x) \epsilon_d \). Let \( \gamma = \frac{1}{4} - \frac{1}{p} \) and \( 1 \leq p < \infty \). Recall that by Theorem 4.1
\[
\int_{[0, 1]^{d-1}} \int_0^1 z^{\gamma p} |\nabla_x F(x, z)|^p \ d z dx \leq C(\mu, L)^p n^p \| f \|^p_{L^p(G_*)}.
\]
For the proof of Theorem 8.1, it is enough to show

**Proposition 8.2.** For any \( 0 < \epsilon < 1, \ \frac{2(d-1)}{d} < \alpha \leq 2 \) and \( n \in \mathbb{N} \), there exists a partition \( G = \bigcup_{j=1}^N G_j \) with \( N \leq c_{\alpha, \epsilon} n^d \) so that for every \( f \in \Pi_n^d \) and \( d - 1 < p < \infty \),
\[
\sum_{j=1}^N |G_j| \ |\text{osc}(f; G_j)|^p \leq \epsilon^p \int_{G_*} |f(\xi)|^p d\xi
\]

**Proof.** Let \( m > 2n \) be an integer such that \( \frac{1}{m} \sim \frac{\epsilon}{n} \). Let \( z_j := \frac{j^2}{4m^2} \) for \( j = 0, 1, \ldots, m \). Let \( N_j \geq m \) be an integer such that
\[
N_j \sim m \left( \frac{m}{j} \right)^{2\gamma}, \ \ j = 1, 2, \ldots, m.
\]
We partition the cube \([0, 1]^{d-1}\) into pairwise disjoint sub-cubes \( Q_{i,j}, \ i \in \Lambda_j \) of equal side length \( 1/N_j \), where \#\( \Lambda_j = N_j^{-d} \). We then define a partition \( G = \bigcup_{j=1}^m \bigcup_{i \in \Lambda_j} I_{i,j} \) as follows:
\[
I_{i,j} := \left\{ (x, y) \in G: \ x \in Q_{i,j}, \ z_{j-1} \leq g(x) - y \leq z_j \right\},
\]
\( j = 1, 2, \ldots, m, \ i \in \Lambda_j \).
Clearly, the number of sets in the partition \( G = \bigcup_{i,j} I_{i,j} \) equals to
\[
\sum_{j=1}^m N_j^{-d} \leq C m^{(2\gamma+1)(d-1)} \sum_{j=1}^m j^{-(2\gamma)(d-1)} \sim m^d,
\]
where the last step uses the assumption \( \frac{2(d-1)}{d} < \alpha \) so that
\[
2\gamma(d-1) = \left( \frac{2}{\alpha} - 1 \right)(d-1) < 1.
\]
Our aim is to show that
\[
\sum_{j=1}^m \sum_{i \in \Lambda_j} |I_{i,j}| \ |\text{osc}(f; I_{i,j})|^p \leq \epsilon^p \int_{G_*} |f(x, y)|^p dx dy, \ \ f \in \Pi_n^d,
\]
where

\[(8.4) \quad |I_{i,j}| = |Q_{i,j}||z_j - z_{j-1}| = \frac{2j - 1}{4m^2N_j^{d-1}}, \quad i \in \Lambda_j, \quad 1 \leq j \leq m.\]

Note that

\[
\text{osc}(f; I_{i,j}) := \max_{x, x' \in Q_{i,j}} |F(x, z) - F(x', z')|
\]

\[
\leq 2 \sup_{z \in [z_{j-1}, z_j]} \sup_{x \in Q_{i,j}} |F(x, z) - \int_{Q_{i,j}} F(u, z) \, du|
\]

\[
\leq 2 \left[ a_{i,j}(f) + b_{i,j}(f) \right],
\]

where

\[
a_{i,j}(f) := \sup_{x \in Q_{i,j}} \left| F(x, z) - \int_{Q_{i,j}} F(u, z) \, du \right|
\]

\[
b_{i,j}(f) := \sup_{z \in [z_{j-1}, z_j]} \left| \int_{Q_{i,j}} [F(u, z) - F(u, z_j)] \, du \right|
\]

Given a cube \(Q\) in \(\mathbb{R}^{d-1}\), we denote by \(Q^*\) the cube with the same center as \(Q\) but \(d\) times the length of \(Q\). For the term \(a_{i,j}(f)\), using the pointwise Poincaré inequality (see, for instance, [13] p.11) and Hölder’s inequality, we obtain that for \(p > d - 1\),

\[
|a_{i,j}(f)|^p \leq \sup_{x \in Q_{i,j}} \left( \int_{Q^*_{i,j}} \frac{\|\nabla u F(u, z)\|}{|u - x|^{d-2}} \, du \right)^p
\]

\[
\leq CN_j^{-(p-d+1)} \int_{Q^*_{i,j}} \sup_{z \in [z_{j-1}, z_j]} |\nabla u F(u, z)|^p \, du.
\]

For the term \(b_{i,j}(f)\), we have

\[
|b_{i,j}(f)|^p \leq \left( \int_{z_{j-1}}^{z_j} \int_{Q_{i,j}} |\partial_d F(u, z)| \, du dz \right)^p
\]

\[
\leq N_j^{(d-1)} \left( \frac{j}{4m^2} \right)^p \int_{z_{j-1}}^{z_j} \int_{Q_{i,j}} |\partial_d F(u, z)|^p \, du dz.
\]

Thus,

\[
\sum_{j=1}^{m} \sum_{i \in \Lambda_j} |I_{i,j}| \text{osc}(f; I_{i,j})|^p \leq C \left( \Sigma_1 + \Sigma_2 \right),
\]

where

\[
\Sigma_1 := \frac{1}{m^p} \sum_{j=1}^{m} \sum_{i \in \Lambda_j} \frac{z_j^{p+\frac{1}{2}}}{m} \int_{Q^*_{i,j}} \sup_{z \in [z_{j-1}, z_j]} |\nabla u F(u, z)|^p \, du
\]

\[
\leq C \int_{[-d,d]^{d-1}} \frac{1}{m^p} \sum_{j=1}^{m} \frac{z_j^{p+\frac{1}{2}}}{m} \sup_{z \in [z_{j-1}, z_j]} |\nabla u F(u, z)|^p \, du,
\]

\[
\Sigma_2 := \sum_{j=1}^{m} \sum_{i \in \Lambda_j} \left( \frac{j}{4m^2} \right)^p \int_{z_{j-1}}^{z_j} \int_{Q_{i,j}} |\partial_d F(u, z)|^p \, du dz = \sum_{j=1}^{m} \left( \frac{j}{4m^2} \right)^p \int_{z_{j-1}}^{z_j} \int_{[0,1]^{d-1}} |\partial_d F(u, z)|^p \, du dz
\]
Note that for each $1 \leq j \leq d - 1$, the function
\[
\partial_j F(u, z) = (\partial_j + \partial_j g(u) \partial_d) f(v, g(v) - z)
\]
is an algebraic polynomial of degree at most $n$ in the variable $z$ for each fixed $v \in [-d, d]^{d-1}$, we obtain from Lemma 7.4 with $\beta = \gamma p$ that for each $u \in [-d, d]^{d-1}$,
\[
\frac{1}{m^p} \sum_{j=1}^{m} \frac{\frac{1}{2} + \gamma p}{m} \sup_{z \in [z_{j-1}, z_j]} |\nabla_u F(u, z)|^p \leq C p \frac{1}{m^p} \int_0^1 z^\gamma |\nabla_u F(u, z)|^p dz,
\]
which, using the tangential Bernstein inequality, implies
\[
\Sigma_1 \leq C p m^{-p} \int_{[-d, d]^{d-1}} \int_{g(x)-1}^{g(x)} |\partial_d F(u, z)|^p dz \, du \leq C p \left( \frac{n}{m} \right)^p \|f\|_{L^p(G_\gamma)}^p.
\]
Finally, for the sum $\Sigma_2$, we have
\[
\Sigma_2 \leq C p \sum_{j=1}^{m} \left( \frac{j}{m} \right)^p \int_{[0,1]^{d-1}} \int_{z_{j-1}}^{z_j} |\partial_d F(u, z)|^p dz \, du
\]
\[
\leq C p \int_{[0,1]^{d-1}} \left[ \sum_{j=1}^{m} \int_{z_{j-1}}^{z_j} |\partial_d F(u, z)|^p \left( \sqrt{z} + \frac{1}{m} \right)^p dz \right] du
\]
\[
= C p \int_{[0,1]^{d-1}} \left[ \int_0^{1/4} |\partial_d F(u, z)|^p \left( \sqrt{z} + \frac{1}{m} \right)^p dz \right] du
\]
\[
\leq C p m^{-p} \int_{[0,1]^{d-1}} \left[ \int_0^{1/4} |\partial_d F(u, z)|^p (\sqrt{z})^p dz \, du + C p m^{-2p} \int_{[0,1]^{d-1}} \int_0^{1/4} |\partial_d F(u, z)|^p dz \, du \right].
\]
Since $F(u, z) = f(u, g(u) - z)$ is an algebraic polynomial of the variable $z$ of degree at most $n$, using the univariate Markov-Bernstein-type inequality (Theorem 7.3), we obtain
\[
\Sigma_2 \leq C p \left( \frac{n}{m} \right)^p \int_{[0,1]^{d-1}} \int_0^2 |F(u, z)|^p dz \, du \leq C p \left( \frac{n}{m} \right)^p \|f\|_{L^p(G_\gamma)}^p.
\]
Putting the above tougher, and taking into account the fact that $m \sim \frac{n}{\varepsilon}$, we obtain
\[
\sum_{j=1}^{m} \sum_{i \in \Lambda_j} |I_{i,j}| \text{osc}(f; I_{i,j})^p \leq C p \left( \frac{n}{m} \right)^p \|f\|_{L^p(G_\gamma)}^p \leq \varepsilon p \|f\|_{L^p(G_\gamma)}^p.
\]

\[\square\]

**Remark 8.3.** For arbitrary $\varepsilon > 0$, the Marcinkiewicz-Zygmund inequalities established in Theorem 7.1 and Theorem 8.1 are valid with $\frac{1}{2}$ and $2$ replaced with $(1 - \varepsilon)$ and $(1 + \varepsilon)$, respectively, and $c_\alpha$ in the bound on $N$ allowed to depend on $\varepsilon$ as well. The technique of the proof is exactly the same.

We would like to conclude this paper with the following open question: is it possible to extend Theorem 7.1 to $\alpha = 1$ and Theorem 8.1 to $1 \leq \alpha \leq 2 - \frac{2}{d}$ without allowing any additional logarithmic factors in the bound on $N$?
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Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta T6G 2G1, Canada

Email address: fda@ualberta.ca

Alfréd Rényi Institute of Mathematics, and Department of Analysis, Budapest University of Technology and Economics, Budapest, Hungary

Email address: kroo@renyi.hu

Department of Mathematics, University of Manitoba, Winnipeg, MB, R3T2N2, Canada

Email address: prymak@gmail.com