Nonlinear transforms of momenta and Planck scale limit

A. Chakrabarti

Centre de Physique Théorique*, Ecole Polytechnique, 91128 Palaiseau Cedex, France.
e-mail chakra@cpht.polytechnique.fr

Abstract

Starting with the generators of the Poincaré group for arbitrary mass \( (m) \) and spin \( (s) \) a nonunitary transformation is implemented to obtain momenta with an absolute Planck scale limit. In the rest frame (for \( m > 0 \)) the transformed energy coincides with the standard one, both being \( m \). As the latter tends to infinity under Lorentz transformations the former tends to a finite upper limit \( m \coth(lm) = l^{-1} + O(l) \) where \( l \) is the Planck length and the mass-dependent nonleading terms vanish exactly for zero rest mass. The invariant \( m^2 \) is conserved for the transformed momenta. The speed of light continues to be the absolute scale for velocities. We study various aspects of the kinematics in which two absolute scales have been introduced in this specific fashion. Precession of polarization and transformed position operators are among them. A deformation of the Poincaré algebra to the \( SO(4,1) \) deSitter one permits the implementation of our transformation in the latter case. A supersymmetric extension of the Poincaré algebra is also studied in this context.

*Laboratoire Propre du CNRS UPR A.0014
1 Introduction

Possible modifications of special relativity introducing, in addition to the velocity of light, a second invariant scale corresponding to the Planck energy (the inverse of the Planck length) have been studied in numerous recent papers exploring various aspects [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]. The titles of these papers (citing other relevant sources) convey some idea of the topics addressed. Among the papers cited above our work can be compared most directly, concerning both analogies and crucial differences, with the work of Magueijo and Smolin [4, 5]. Like them we introduce the nonlinear constructions via a nonunitary transformation. But (unlike all the foregoing studies) we introduce spin at the outset. To be able to do so adequately we start with an irreducible representation $[m, s]$ of the Poincaré group of positive rest mass ($m > 0$) and an arbitrary integer or half-integer spin $s$. The momentum generators are thus constrained to satisfy

$$P^\mu P_\mu = P^2_0 - \vec{P}^2 = m^2 \tag{1}$$

It will be implicit henceforth that (with positive squareroot and $\nabla_i$ denoting derivative with respect to $P_i$)

$$P_0 = \sqrt{\vec{P}^2 + m^2}, \quad \nabla P_0 = \frac{\vec{P}}{P_0} \tag{2}$$

Introducing $(2s + 1) \times (2s + 1)$ spin matrices $\vec{S}$ satisfying

$$[S_i, S_j] = i\epsilon_{ijk}S_k \tag{3}$$

the generators of pure rotations ($\vec{J}$) and those of pure Lorentz transformations ($\vec{K}$) can be represented as

$$\vec{J} = -i\vec{P} \times \nabla + \vec{S} \tag{4}$$

$$\vec{K} = -iP_0\nabla - \frac{\vec{P} \times \vec{S}}{P_0 + m} \tag{5}$$

Let us briefly note the following points [12] for later use.

(a): The first term of $\vec{K}$ should not be symmetrized. The hermiticity of $\vec{K}$ and the relation to Newton-Wigner position operators are discussed in Ref.12 (from eqn. (2.18) onwards).

(b): For $m = 0$, the last term of $\vec{K}$ is not well-defined only for energy-momenta $(0, 0, 0, 0)$, namely at the tip of the light cone which is never in the same orbit with

$$p^2 = 0, \quad p_0 \neq 0$$

Hence excluding massless particles with strictly zero energy one can consistently use (5) with $m = 0$. We will present below results for $m = 0$ obtained systematically in this fashion.
An explicit unitary transformation to Wigner’s construction in terms of the little group $E_2$ has been presented elsewhere (see the discussion and the references in Ref. 12 from eqn. (2.21) onwards). There it is explained how, despite the presence of three spin components in (4) and (5) one finally deals with only one conserved helicity component for $m = 0$.

(c): The canonical form given by (4) and (5) is valid for any spin. The Pauli-Lubansky 4-vector is obtained by contracting the dual of the tensor $(\mathbf{J}, \mathbf{K})$ by $P_\mu$ or equivalently as

$$ W_\mu = -i[P_\mu, \mathbf{K} \cdot \mathbf{J}] $$(6)

$$ = (\mathbf{P} \cdot \mathbf{J}, P_0 \mathbf{J} - \mathbf{P} \times \mathbf{K}) $$(7)

$$ = (\mathbf{P} \cdot \mathbf{S}, m \mathbf{S} + (P_0 + m)^{-1}(\mathbf{P} \cdot \mathbf{S}) \mathbf{P}) $$(8)

This satisfies

$$ W_\mu W^\mu = -m^2 \mathbf{S}^2 $$(9)

The relation, for $s = 1/2$, with the Dirac representation and the Dirac equation are indicated in Ref. 12 (from eqn. (2.49) onwards). The relevant transformation relating the two representations diagonalizes the Dirac mass operator $(\gamma \cdot p)$.

Before introducing the explicit form of the transformation $V$ (to be presented below) let us note the following aspect.

Having explicitly constructed all the transforms

$$ V(P_0, \mathbf{P}, \mathbf{J}, \mathbf{K})V^{-1} $$

(10)

one can study the set

$$ V(\mathbf{J}, \mathbf{K})V^{-1}, (P_0, \mathbf{P}) $$

or, in a complementary fashion, the set

$$ (\mathbf{J}, \mathbf{K}), V(P_0, \mathbf{P})V^{-1} $$

In the latter case one conserves the explicit representation of the Lorentz algebra and associates physical significance with the transformed momenta. We will adopt the latter approach below (providing however complete results for (10)). This will furnish, in the terminology of Ref. 7, an example of DSR2 theories with bounded energy and momenta. The inverse formulae, giving the standard momenta in terms of the transformed ones are obtained very simply.
2 The Transformation:

Let

$$V = \exp(-lP_0 \hat{P} \cdot \hat{\nabla})$$  \hspace{1cm} (11)

where $0 < l \ll 1$, and $P_0 = \sqrt{\vec{P}^2 + m^2}$.

In fact, one may assume that in our chosen units ($c = 1$) $l$ is the Planck length which is more generally

$$l_P = \sqrt{\frac{\hbar G}{c^3}}$$  \hspace{1cm} (12)

As compared to the corresponding operator

$$U^{-1} = \exp(-l_P P_0^\mu \partial_\mu)$$  \hspace{1cm} (13)

of Ref.4, we have kept only the three space components in the scalar product but still with the factor $P_0$ rather than $P = |\vec{P}|$. This is crucial for the remarkable properties obtained below for arbitrary spin.

Implementing, consistently with (2), for any $f$,

$$[\hat{\nabla}, P_0]f = \frac{\vec{P}}{P_0}f$$

one obtains

$$\vec{P} \equiv V \vec{P} V^{-1} = \frac{m\vec{P}}{sh(lm)P_0 + ch(lm)m}$$  \hspace{1cm} (14)

$$P_0 \equiv VP_0 V^{-1} = m \frac{ch(lm)P_0 + sh(lm)m}{sh(lm)P_0 + ch(lm)m}$$  \hspace{1cm} (15)

satisfying

$$P_0^2 - \vec{P}^2 = P_0^2 - \vec{P}^2 = m^2$$  \hspace{1cm} (16)

$$V \vec{J} V^{-1} = \vec{J} = -i\vec{P} \times \nabla + S$$  \hspace{1cm} (17)

$$V \vec{K} V^{-1} = -ich(lm)P_0 \vec{\nabla} - ish(lm)(m\vec{\nabla} + m^{-1}\vec{P}(\vec{P} \cdot \vec{\nabla})) - e^{-lm} \frac{\vec{P} \times \vec{S}}{P_0 + m}$$  \hspace{1cm} (18)

Note the simplicity of the spin dependent part on the right of (18). This corresponds to

$$V \frac{\vec{P}}{P_0 + m} V^{-1} = e^{-lm} \frac{\vec{P}}{P_0 + m}$$  \hspace{1cm} (19)
Hence, squaring each side and using (1) one obtains

\[ \frac{V P_0 - m}{P_0 + m} V^{-1} = e^{-2lm} \frac{P_0 - m}{P_0 + m} \]  

(20)

Thus one readily obtains (15) and hence also (14). If, without knowing (15) beforehand, one proceeds directly to compute the power series

\[ P_0 = V P_0 V^{-1} = P_0 - l[P_0(\vec{P} \cdot \vec{\nabla}), P_0] + l^2[P_0(\vec{P} \cdot \vec{\nabla})[P_0(\vec{P} \cdot \vec{\nabla}), P_0]] - \ldots \]  

(21)

one obtains

\[ P_0 = P_0 - l(P_0^2 - m^2) + l^2 P_0(P_0^2 - m^2) - \ldots \]  

(22)

The series is difficult to sum up. On the other hand, developing (15) in powers of \( l \) one easily reproduces (22). One similarly obtains for the modulus of the momentum

\[ P = P - lPP_0 + \frac{1}{2} l^2 P(P_0^2 + P^2) - \ldots \]  

(23)

It is easy to verify that (14) and (15) can be inverted by simply changing the sign of \( l \). Thus

\[ P_0 = m \frac{ch(lm)P_0 - sh(lm)m}{-sh(lm)P_0 + ch(lm)m} \]  

(24)

and so on. This is consistent with the invariance of \( (P_0(\vec{P} \cdot \vec{\nabla})) \) under the transformation.

Let us now consider momentum eigenstates and denote the eigenvalues of \( P_\mu \) and \( P_\mu' \) by \( p_\mu \) and \( p'_\mu \) respectively. Then

\[ p'_0 = m \frac{coth(lm)p_0 + m}{p_0 + coth(lm)m} \]  

(25)

Hence, since we are considering positive \( p_0, m \) and \( l \),

\[ p'_0 < coth(lm)m \]

Similarly for the modulus of the momentum one obtains

\[ p'_0 < \frac{m}{sh(lm)} \]

Thus our transformation, valid for arbitrary spin, indeed leads to an invariant energy scale. This is the crucial property. For

\[ p_0 = m, \quad p'_0 = m \]

and as

\[ p_0 \to \infty, \quad p'_0 \to coth(lm)m \]
from below. *For all observers* \( p'_0 \) *remains bounded*. Starting together with \( p_0 \) in the rest frame \( p'_0 \) lags progressively behind as the former increases to finally encounter the barrier \( m \coth(lm) \). In powers of \( l \) one obtains

\[
p'_0 < l^{-1} + \frac{1}{3} m^2 l + O(l^2) \tag{26}
\]

and

\[
p' < l^{-1} - \frac{1}{6} m^2 l + O(l^2) \tag{27}
\]

The modulus of the transformed velocity, quite consistently with our chosen units (\( c = 1 \)), has the high energy limit, for \( p_0 \to \infty \), given by

\[
\frac{p'}{p_0} \to \frac{1}{ch(lm)} = 1 - \frac{1}{2} l^2 m^2 < 1 \tag{28}
\]

The limit 1 is attained exactly for \( m = 0 \). This will be seen more precisely immediately below.

### 3 Zero rest mass:

As explained in note (b) following eqn.(5), the essential results for \( m = 0 \) can be obtained (rather than starting again with \( m = 0 \) in \( V \)) easily and directly from our previous ones. Thus for \( m \to 0 \) one obtains from (14) and (15)

\[
\vec{P}' = \frac{\vec{P}}{lP_0 + 1} \tag{29}
\]

and

\[
P_0 = \frac{P_0}{lP_0 + 1} \tag{30}
\]

satisfying evidently (like \( P_0 \) and \( \vec{P} \))

\[
P_0^2 - \vec{P}'^2 = 0
\]

Now as compared to the inequality following (25), again for all parameters positive,

\[
p'_0 = \frac{p_0}{lP_0 + 1} < l^{-1} \tag{31}
\]

As

\[
p_0 \to \infty, \quad p'_0 \to l^{-1}
\]

from below. And as compared to (28),

\[
\frac{p'}{p'_0} = \frac{p}{p_0} = 1 \tag{32}
\]
Thus, considering all masses and the system

\[ (\vec{J}, \vec{K}, P_0, \vec{P}) \]

one indeed implements two absolute scales, one for velocity \( c = 1 \) and one for energy. The leading term for the limiting energy is always \( l^{-1} \). This becomes exact for zero rest mass. For positive mass the exact result is provided by (25).

\section{4 Precession of polarization:}

Since \( V \) commutes with \( \vec{S} \) the standard results for precession of polarization are conserved. (See the complete discussion in Ref.12.) They can however be reexpressed in terms of \((P_0, \vec{P})\) if so desired. Thus under an infinitesimal Lorentz transformation of velocity \( \tanh \chi \) (\( \to \chi \)) parallel to the unit vector \( \hat{n} \), the change

\[
\delta \vec{S} = i[\chi \hat{n}, \vec{K}, \vec{S}] = -\chi (\hat{n} \times \vec{P}) \times \vec{S} / P_0 + m = -\chi e^{lm}(\hat{n} \times \vec{P}) \times \vec{S} / P_0 + m
\]

Thus the formal expression is altered by a simple overall factor \( e^{lm} = 1 + O(l) \). In Ref.12 it is explained how (34) leads to the famous Thomas factor \( \frac{1}{2} \). We will not go further into such topics in the present study.

We indicate below very briefly possible generalizations of our study in two different directions.

\section{5 Deformation of Poincaré to \( SO(4, 1) \) deSitter algebra:}

The Lorentz algebra has two invariants,

\[
(\vec{K} \cdot \vec{J}), \quad (\vec{K}^2 - \vec{J}^2)
\]

As pointed out before (see eqns.(6) to (9)), commuting \( P_\mu \) with the first one leads to \( W_\mu \) giving the spin. Commutation of \( P_\mu \) with the second leads to the homogenous \( SO(4, 1) \) algebra where (along with the Lorentz \( SO(3, 1) \) generators and \( \mu = (0, 1, 2, 3) \))

\[
L_{\mu\delta} = \frac{i}{M} ([\vec{K}^2 - \vec{J}^2], P_\mu] + \lambda P_\mu \tag{34}
\]
Here \( M = (P_\mu P_\mu)^{\frac{1}{2}} \) is the mass operator and \( \lambda \) is an arbitrary parameter. Starting with an irreducible space \([m, s]\) (with \( m > 0 \), say) one can compute explicitly the actions of \( L_{\mu 5} \) on the states using (36). Elsewhere [13] we have studied (36) in a more general context using however the Lorentz basis. Here we only point out that \((VL_{\mu 5}V^{-1})\) is obtained directly from our foregoing results. A detailed study is beyond the scope of this paper.

### 6 A supersymmetric extension:

A simple supersymmetric extension [14] permitting a ready implementation of our transformation can be obtained as follows. (Previous sources are cited in [14].) One starts with two fermionic operators satisfying (for \( i = 1, 2 \))

\[
[a_i, a_j]_+ = 0 = [a_i^\dagger, a_j^\dagger]_+, \quad [a_i, a_j^\dagger]_+ = \delta_{ij}
\]

One defines \( Q = (Q_1, Q_2) \) and the adjoint \( Q^\dagger \) (a column with two rows) as

\[
Q^\dagger = \frac{1}{\sqrt{2(P_0 + M)}}[P_0 + M] + \tau \cdot \vec{P}]a^\dagger
\]  

(35)

Then in terms of Pauli matrices

\[
[Q^\dagger, Q]_+ = \tau_0 P_0 + \tau \cdot \vec{P}
\]  

(36)

This compact notation implies symmetrization of each term of the \( 2 \times 2 \) matrix \( Q^\dagger Q \). Thus, for example, at the top right one obtains \( Q_1^\dagger Q_2 + Q_2 Q_1^\dagger = P_1 - iP_2 \)

A Majorana spinor is provided by \((Q_1, Q_2, Q_2^\dagger, -Q_1^\dagger)\).

Next one defines

\[
\overrightarrow{\Sigma} = \frac{1}{2}(a \vec{\tau} a^\dagger)
\]  

(37)

Adding the spin operator \( \overrightarrow{\Sigma} \) to \( \overrightarrow{S} \) define

\[
\overrightarrow{J} = -i \vec{P} \times \overrightarrow{\Sigma} + (\overrightarrow{S} + \overrightarrow{\Sigma})
\]  

(38)

\[
\overrightarrow{K} = -iP_0 \overrightarrow{\Sigma} - \vec{P} \times (\overrightarrow{S} + \overrightarrow{\Sigma}) \overrightarrow{P_0 + m}
\]  

(39)

Now \((P_\mu, \overrightarrow{K}, \overrightarrow{J})\) continue to satisfy the Poincaré algebra along with

\[
[\overrightarrow{J}, Q^\dagger] = -\frac{1}{2} \vec{\tau} Q^\dagger
\]  

(40)

\[
[\overrightarrow{K}, Q^\dagger] = -\frac{i}{2} \vec{\tau} Q^\dagger
\]  

(41)
Thus (38), (42), (43) together complete the supersymmetric extension. Various aspects are studied in Ref. 14 citing other sources. Here we only note that \( \Sigma \) commutes with \( V \) and denoting

\[
\tilde{Q} = VQV^{-1}
\]

\[
\tilde{Q}^\dagger = \frac{1}{\sqrt{2(P^0 + M)}}[(P_0 + M) + \vec{\tau} \cdot \vec{P}]a^\dagger
\] (42)

and

\[
[\tilde{Q}^\dagger, \tilde{Q}]_+ = \tau_0 P_0 + \vec{\tau} \cdot \vec{P}
\] (43)

Thus our transformation can be readily implemented for such an extension. A more detailed study is beyond the scope of this paper.

7 Gradient operators for \( \vec{P} \):

One obtains for transforms of \( \vec{\nabla} \), consistently with (15) and (18) and with \( P_0 \) given by (15),

\[
\vec{\xi} = \frac{1}{P_0} \left( (ch(lm)P_0 + sh(lm)m)\vec{\nabla} + m^{-1}sh(lm)\vec{P} (\vec{P} \cdot \vec{\nabla}) \right)
\] (44)

where

\[
\vec{\xi} \equiv V\vec{\nabla}V^{-1}
\]

Hence, for such \( \xi_i \),

\[
[\xi_i, P_j] = \delta_{ij}
\] (45)

and

\[
[\xi_i, \xi_j] = 0
\] (46)

Substituting for \( \xi_i \)

\[
\xi_i' = \xi_i + V f_i V^{-1}
\]

where \( f_i \) depends only on the momenta conserves (47) but not necessarily (48) unless

\[
(\partial_i f_j - \partial_j f_i) = 0
\]

In particular, starting with the localizing and hermitian Newton-Wigner position operators (Ref. 12 from eqn. (2.18) onwards), namely,

\[
\vec{X} = i\vec{\nabla} - \frac{\vec{P}}{2P_0^2}
\] (47)
one obtains

\[ V^X V^{-1} = i \xi - \frac{\vec{P}}{2P_0^2} \]

(48)

The components continue to satisfy (47) and (48) (with a factor \(i\)).

We will not attempt to explore here whether other choices can lead to interesting noncommutative Hopf algebras for the coordinates. Both noncommutative (Ref.9 and sources cited) and commutative [5] space-times have been proposed in the context of Planck scale limits of momenta. In our formalism, apart from the commutativity of (48), the time \(t\) remains a parameter (\(P_0\) being given by (2)).

8 Conclusion:

For all mass and spin we have obtained nonlinear functions of the standard momenta possessing a Planck scale limit. Our construction exhibits that such a property is quite consistent with a fixed velocity of light, time remaining a parameter and commuting position operators corresponding to those for the nonlinear momenta. Even if one deliberately seeks a different formalism violating such properties, comparison with our formalism will provide a deeper understanding.

Due to the fact that the new momenta are introduced via a relatively simple conjugation, by our \(V\), all relevant properties are obtained fairly easily and systematically. This has permitted a ready passage to deSitter \(SO(4,1)\) and to a supersymmetric extension as well.

Elsewhere [15] we have presented explicit constructions for the generators of the Poincaré group for spacelike momenta and for lightlike momenta with continuous spin. We just mention that they have strong analogies with those introduced here for the timelike case and thus may suggest how our transformation can be adapted to those cases.

References

[1] G. Amelino-Camelia, Relativity in space-times with short-distance structure governed by an observer-independent (Planckian) length scale, Int.Jour.Mod.Phys. D11, 35, 2002 [gr-qc 0012051]; Testable scenarios for relativity with minimum length, Phys.Lett. \textbf{B510}, 255 (2001)

[2] G.Amelino-Camelia,J.Ellis,N.E.Mavromatos,D.V.Nanopoulos and S.Sarkar, Tests of quantum gravity from observations of \(\gamma\)-ray bursts, Nature, \textbf{393}, 763 (1998)

[3] G.Amelino-Camelia, Special treatment, Nature, \textbf{418}, 34 (2002)

[4] J.Magueijo and L.Smolin, Lorentz invariance with an invariant energy scale, Phys.Rev.Lett. \textbf{88}, 190403 (2002)
[5] J.Magueijo and L.Smolin, Generalized Lorentz invariance with an invariant energy scale, gr-qc/0207085

[6] J.Lukierski and A.Nowicki, Doubly special relativity versus $\kappa$-deformation of relativistic kinematics, hep-th/0203065

[7] J.Lukierski and A.Nowicki, Four classes of modified relativistic symmetry transformations, hep-th/0210111

[8] J.Kowalski-Glikman and S.Nowak, Doubly Special Relativity theories as different bases for $\kappa$-Poincaré algebra, hep-th/0203040

[9] J.Kowalski-Glikman and S.Nowak, Non-commutative space-time of Doubly Special Relativity theories, hep-th/0204245

[10] S.Judes and M.Visser, Conservation laws in ”Doubly Special Relativity”, gr-qc/0205067

[11] J.Kowalski-Glikman, DeSitter space as an arena for Doubly Special Relativity, hep-th/0207279

[12] A.Chakrabarti, Wigner rotations and precession of polarization, Fortschr.Phys. 36, 863, (1988)

(The review paper[12] provides a concise and systematic presentation of results derived in a series of previous papers by the present author, citing original sources. Relevant papers not included here are cited separately.)

[13] A.Chakrabarti, A class of representations of $IU(n)$ and $IO(n)$ algebras and respective deformations to $U(n,1)$ and $O(n,1)$, Jour.Math.Phys. 9, 2087 (1968)

[14] A.Chakrabarti, Remarks on the supersymmetry algebra, Phys.Rev. D11, 3054 (1975)

[15] A.Chakrabarti, Remarks on lightlike continuous spin and spacelike representations of the Poincaré group, Jour.Math.Phys. 12, 1813 (1971)