Some Results of The Class of Functions with Bounded Radius Rotation

Yaşar Polatoğlu, Yasemin Kahramaner and Arzu Yemişçi Şen

Abstract

Let $A$ be the family of functions $f(z) = z + a_2z^2 + ...$ which are analytic in the open unit disc $D = \{ z : |z| < 1 \}$, and denote by $P$ of functions $p(z) = z + p_1z + p_2z^2 + ...$ analytic in $D$ such that $p(z)$ is in $P$ if and only if

$$p(z) \prec \frac{1 + z}{1 - z} \Leftrightarrow p(z) = \frac{1 + \phi(z)}{1 - \phi(z)},$$

for some Schwarz function $\phi(z)$ and every $z \in D$.

Let $f(z)$ be an element of $A$, and satisfies the condition

$$zf'(z) = \left( \frac{k}{4} + \frac{1}{2} \right)p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right)p_2(z)$$

where $p_1(z), p_2(z) \in P$ and $k \geq 2$, then $f(z)$ is called function with bounded radius rotation. The class of such functions is denoted by $R_k$. This class is generalization of starlike functions.

The main purpose is to give some properties of the class $R_k$.

1 Introduction

Let $\Omega$ be the family of functions $\phi(z)$ which are analytic in $\mathbb{D}$ and satisfy the conditions $\phi(0) = 0$, $|\phi(z)| < 1$ for all $z \in \mathbb{D}$. If $f_1(z)$ and $f_2(z)$ are analytic functions in $\mathbb{D}$, then we say that $f_1(z)$ is subordinate to $f_2(z)$, written as $f_1(z) \prec f_2(z)$ if there exists a Schwarz function $\phi \in \Omega$ such that $f_1(z) = f_2(\phi(z))$, $z \in \mathbb{D}$. We also note that if $f_2$ univalent in $\mathbb{D}$, then $f_1(z) \prec f_2(z)$ if and only if $f_1(0) = f_2(0)$, $f_1(\mathbb{D}) \subset f_2(\mathbb{D})$ implies $f_1(\mathbb{D}_r) \subset f_2(\mathbb{D}_r)$, where

2010 Mathematics Subject Classification: 30C45

Key words and phrases: Bounded radius rotation, bounded boundary rotation, distortion theorem, growth theorem and coefficient inequality.
\[ D_r = \{ z : |z| < r, 0 < r < 1 \} \] (see [2]). Denote by \( \mathcal{P} \) the family of functions \( p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \) analytic in \( \mathbb{D} \) such that \( p \) is in \( \mathcal{P} \) if and only if
\[
p(z) < \frac{1 + z}{1 - z} \iff p(z) = \frac{1 + w(z)}{1 - w(z)}, z \in \mathbb{D}
\]

(1.1)

Let \( f(z) \) be an element of \( \mathcal{A} \). Then \( f(z) \) is called convex or starlike if it maps \( \mathbb{D} \) onto a convex or starlike region, respectively. Corresponding classes are denoted by \( \mathcal{C} \) and \( S^* \). It is well known that \( \mathcal{C} \subset S^* \), that both are subclasses of the univalent functions and have the following analytical representations.
\[
f(z) \in \mathcal{C} \iff \Re \left( 1 + z \frac{f''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D}
\]
and
\[
f(z) \in S^* \iff \Re \left( z \frac{f''(z)}{f'(z)} \right) > 0, \quad z \in \mathbb{D}
\]

(1.2) (1.3)

More on these class can be found in [2]. Let \( f(z) \) be an element of \( \mathcal{A} \). If there is a function \( g(z) \) in \( \mathcal{C} \) such that
\[
\Re \left( \frac{f'(z)}{g'(z)} \right) > 0, \quad z \in \mathbb{D}
\]
then \( f(z) \) is called close-to-convex function in \( \mathbb{D} \) and the class of such functions is denoted by \( \mathcal{CC} \).

A function analytic and locally univalent in a given simply connected domain is said to be of bounded boundary rotation if its range has bounded boundary rotation which is defined as the total variation of the direction angle of the tangent to the boundary curve under a complete circuit. Let \( V_k \) denote the class of functions \( f(z) \in \mathcal{A} \) which maps \( \mathbb{D} \) conformally onto an image domain of boundary rotation at most \( k\pi \). The class of functions of bounded boundary rotation was introduced by Loewner [3] in 1917 and was developed by Paatero [5, 6] who systematically developed their properties and made an exhaustive study of the class \( V_k \). Paatero has shown that \( f(z) \in V_k \) if and only if
\[
f'(z) = \exp \left[ - \int_0^{2\pi} \log (1 - ze^{-it}) \, d\mu(t) \right],
\]

(1.5)

where \( \mu(t) \) is real-valued function of bounded variation for which
\[
\int_0^{2\pi} d\mu(t) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k
\]

(1.6)
for fixed $k \geq 2$ it can also be expressed as

$$
\int_0^{2\pi} \left| \frac{\text{Re} (zf'(z))'}{f'(z)} \right| d\theta \leq 2k\pi, \quad z = re^{i\theta}. \tag{1.7}
$$

Clearly, if $k_1 < k_2$ then $V_{k_1} \subset V_{k_2}$ that is the class $V_k$ obviously expands on $k$ increases. $V_2$ is the class of $C$ of convex univalent functions. Paatero showed that $V_4 \subset \mathcal{S}$, where $\mathcal{S}$ is the class of normalized univalent functions. Later Pinchuk proved that $V_k$ are close-to convex functions in $D$ if $2 \leq k \leq 4 \tag{1.11}$.

Let $R_k$ denote the class of analytic functions $f$ of the form $f(z) = z + a_2z^2 + a_3z^3 + \ldots$ having the representation

$$
f(z) = z\text{Exp} \left[ -\int_0^{2\pi} \log \left( 1 - ze^{-it} \right) d\mu(t) \right], \tag{1.8}
$$

where $\mu(t)$ is given in (1.6). We note that the class $R_k$ was introduced by Pinchuk and Pinchuk showed that Alexander type relation between the classes $V_k$ and $R_k$ exists,

$$
f \in V_k \Leftrightarrow zf'(z) \in R_k \tag{1.9}
$$

$R_k$ consists of those function $f(z)$ which satisfy

$$
\int_0^{2\pi} \left| \text{Re}(re^{i\theta} f'(re^{i\theta})) \right| d\theta \leq k\pi, \quad z = re^{i\theta}. \tag{1.10}
$$

Geometrically, the condition is that the total variation of angle between radius vector $f(re^{i\theta})$ makes with positive real axis is bounded $k\pi$. Thus, $R_k$ is the class of functions of bounded radius rotation bounded by $k\pi$, therefore $R_k$ generalizes the starlike functions.

$P_k$ denote the class of functions $p(0) = 1$ analytic in $D$ and having representation

$$
p(z) = \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t) \tag{1.11}
$$

where $\mu(t)$ is given in (1.6). Clearly, $P_2 = P$ where $P$ is the class of analytic functions with positive real part. For more details see \cite{7}. From (1.11), one can easily find that $p(z) \in P_k$ can also written by

$$
p(z) = \left( \frac{k}{4} + \frac{1}{2} \right) p_1(z) - \left( \frac{k}{4} - \frac{1}{2} \right) p_2(z), \quad z \in D \tag{1.12}
$$

where $p_1(z), p_2(z) \in P$. Pinchuk \cite{7} has shown that the classes $V_k$ and $R_k$ can be defined by using the class $P_k$ as gives below

$$
f \in V_k \Leftrightarrow \frac{(zf'(z))'}{f'(z)} \in P_k \tag{1.13}
$$
and
\[ f \in R_k \iff \frac{zf'(z)}{f(z)} \in P_k \] (1.14)

At the same time, we note that \( V_k \) generalizes of convex functions.

## 2 Main Results

**Lemma 2.1.** Let \( p(z) \) be an element of \( P_k \), then
\[ \left| p(z) - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2} \] (2.1)

**Proof.** Let \( f(z) \) be an element of \( h(z) \in V_k \). Using (1.13), we can write
\[ p(z) = 1 + \frac{f''(z)}{f'(z)}, p(z) \in P_k \] (2.2)

On the other hand M.S. Robertson \[ 8 \] proved that if \( f(z) \in V_k \), then
\[ \left| z f''(z) f'(z) - \frac{2r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2} \] (2.3)

Therefore the relation can be written in the following form,
\[ \left| (1 + z f''(z)) - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2} \] (2.4)

Using the definition of the class \( V_k \), we obtain (2.1). \( \square \)

**Theorem 2.2.** Let \( f(z) \) be an element of \( R_k \), then
\[ \frac{1}{(1-r)^{2+\frac{1}{2}}(1+r)^{-\frac{1}{2}}(2h)} \leq |f(z)| \leq \frac{r}{(1-r)^{2+\frac{1}{2}}(1+r)^{\frac{1}{2}}} \] (2.5)

\[ \frac{1 - kr + r^2}{(1-r)^{2+\frac{1}{2}}(1+r)^{2+\frac{1}{2}}} \leq |f'(z)| \leq \frac{1 + kr + r^2}{(1-r)^{2+\frac{1}{2}}(1+r)^{2+\frac{1}{2}}} \] (2.6)

**Proof.** Using the definition of \( R_k \), then we can write
\[ \left| z f'(z) f(z) - \frac{1 + r^2}{1 - r^2} \right| \leq \frac{kr}{1 - r^2} \] (2.7)

This inequality can be written in the following form,
\[ \frac{1 - kr + r^2}{1 - r^2} \leq \text{Re} \frac{f'(z)}{f(z)} \leq \frac{1 + kr + r^2}{1 - r^2} \] (2.8)
On the other hand, we have
\[ \text{Re} \left( \frac{f'(z)}{f(z)} \right) = r \frac{\partial}{\partial r} \log |f(z)| \] (2.9)

Thus we have
\[ \frac{1 - kr + r^2}{1 - r^2} \leq \frac{\partial}{\partial r} \log |f(z)| \leq \frac{1 + kr + r^2}{1 - r^2} \] (2.10)

Integrating both sides (2.10), we get (2.5). The inequality (2.7) can be written in the form
\[ \frac{1 - kr + r^2}{1 - r^2} \leq \left| \frac{f'(z)}{f(z)} \right| \leq \frac{1 + kr + r^2}{1 - r^2} \] (2.11)

In this step, if we use (2.5), we obtain (2.6).

**Corollary 2.3.** For \( k = 2 \) in (2.5), we obtain
\[ \frac{r}{(1 + r)^2} \leq |f(z)| \leq \frac{r}{(1 - r)^2} \]

*This is well known growth theorem for starlike functions [2].*

**Corollary 2.4.** For \( k = 2 \) in (2.6), we obtain
\[ \frac{1 - r}{(1 + r)^3} \leq |f'(z)| \leq \frac{1 + r}{(1 - r)^3} \]

*This is well known distortion theorem for starlike functions [2].*

**Corollary 2.5.** The radius of starlikeness of \( R_k \) is
\[ R_{S^*} = \frac{k - \sqrt{k^2 - 4}}{2}, k \geq 2 \] (2.12)

**Proof.** Since
\[ \text{Re} \left( z \frac{f'(z)}{f(z)} \right) > \frac{1 - kr + r^2}{1 - r^2} \]

Hence for \( R < R_{S^*} \) the left hand side of the preceding inequality is positive which implies (2.12). We note that all results are sharp because of extremal function is
\[ f_*(z) = \frac{z(1-z)^{k-1}}{(1+z)^{k+1}} \]
Indeed,
\[
\frac{z f'(z)}{f(z)} = \frac{1 - kz + z^2}{1 - z^2} = \left(\frac{k}{4} + \frac{1}{2}\right) \frac{1 + z}{1 - z} - \left(\frac{k}{4} - \frac{1}{2}\right) \frac{1 - z}{1 + z}
\]

Thus, \( f(z) \in R_k \) and \( f(z) \) is extremal function.

**Lemma 2.6.** Let \( p(z) = 1 + p_1 z + p_2 z^2 + \ldots \) be an element of \( \mathcal{P}_k \), then
\[
|p_n| \leq k
\]

**Proof.** Method I. Since \( p(z) \in \mathcal{P}_k \), then we have
\[
p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z)
= \left(\frac{k}{4} + \frac{1}{2}\right) (1 + a_1 z + a_2 z^2 + \ldots) - \left(\frac{k}{4} - \frac{1}{2}\right) (1 + b_1 z + b_2 z^2 + \ldots)
\]
Then we have
\[
p_n = \left(\frac{k}{4} + \frac{1}{2}\right) a_n - \left(\frac{k}{4} - \frac{1}{2}\right) b_n
\]
Thus
\[
|p_n| = \left| \left(\frac{k}{4} + \frac{1}{2}\right) a_n - \left(\frac{k}{4} - \frac{1}{2}\right) b_n \right|
\leq \left(\frac{k}{4} + \frac{1}{2}\right) |a_n| + \left(\frac{k}{4} - \frac{1}{2}\right) |b_n|
\leq \left(\frac{k}{4} + \frac{1}{2}\right) 2 + \left(\frac{k}{4} - \frac{1}{2}\right) 2
\]

This shows that,
\[
|p_n| \leq k
\]

Method II. Since \( p(z) \in \mathcal{P}_k \), then \( p(z) \) can be written in the form
\[
p(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t)
\]
and
\[
\int_0^{2\pi} d\mu(t) = 2\pi \quad \text{and} \quad \int_0^{2\pi} |d\mu(t)| \leq k\pi.
\]
Then
\[
p(z) = 1 + p_1 z + p_2 z^2 + \ldots = \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-it}}{1 - ze^{-it}} d\mu(t)
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + ze^{-it} - ze^{-it} + ze^{-it}}{1 - ze^{-it}} d\mu(t)
\]
\[
= \frac{1}{2\pi} \int_0^{2\pi} \left(1 - \frac{2ze^{-it}}{1 - ze^{-it}}\right) d\mu(t)
\]

is obtained.

We note that this lemma was proved first by K.I. Noor [4] (Method II).

\[\square\]

**Theorem 2.7.** Let \( f(z) \) be an element of \( R_k \), then

\[
|a_n| \leq \frac{1}{(n-1)!} \prod_{\nu=0}^{n-2} (k + \nu)
\]  \hspace{1cm} (2.13)

**Proof.** Since \( f(z) \in R_k \), then we have

\[
z \frac{f'(z)}{f(z)} = p(z)
\]

where \( p(z) \in \mathcal{P}_k \). Thus

\[
z f'(z) = f(z)p(z)
\]

Comparing the coefficients in both sides of \( zf'(z) = f(z)p(z) \), we obtain the recursion formula

\[
a_n = \frac{1}{n-1} \sum_{\nu=1}^{n-1} p_{n-\nu} a_\nu, \quad n \geq 2
\]

and therefore by Lemma 2.6,

\[
|a_n| = \frac{k}{n-1} \sum_{\nu=1}^{n-1} |a_\nu|
\]

Induction shows that

\[
|a_n| \leq \frac{1}{(n-1)!} \prod_{\nu=0}^{n-2} (k + \nu).
\]

\[\square\]
Corollary 2.8. For \( k = 2 \), we obtain \(|a_n| \leq n\). This inequality is well known coefficient inequality for starlike functions.

Indeed,

\[
|a_n| \leq \frac{1}{(n-1)!} \prod_{\nu=0}^{n-2} (k + \nu) = \frac{k(k+1)(k+2)\ldots(k+(n-2))}{(n-1)!}.
\]

If we take \( k = 2 \),

\[
|a_n| \leq \frac{2.3.4\ldots(n-2).(n-1)n}{(n-1)!} = n.
\]

Corollary 2.9. Let \( f(z) \) be an element of \( V_k \), then

\[
|a_n| \leq \frac{1}{n!} \prod_{\nu=0}^{n-2} (k + \nu) \tag{2.14}
\]

Proof. Using the theorem of Pinchuk

\[
f(z) \in V_k \iff zf'(z) \in R_k
\]

we get (2.14).

Corollary 2.10. For \( k = 2 \), we obtain \(|a_n| \leq 1\). This inequality is well known coefficient inequality for convex functions.

We note that all these inequalities are sharp because extremal function is,

\[
f_*(z) = \frac{z(1-z)^{k-1}}{(1+z)^{\frac{k}{2}+1}}.
\]

References

[1] D.A. Brannan, On functions bounded boundary rotation I, Proc. Edinburgh Math. Soc. 16 (1969), 339-347.

[2] A.W. Goodman, Univalent functions Volume I and Volume II, Mariner Pub. Co. Inc. Tampa Florida, 1984.

[3] C.Loewner, Untersuchungen über die Verzerrung bei konformen Abbildungen des Einheitskreises \(|z| < 1\), die durch Funktionen mit nicht verschwindender Ableitung geliefert werden, Ber. Verh. Sächs. Gess. Wiss. Leipzig, 69 (1917), 89-106.
[4] K.I. Noor, *On generalization of close-to-convexity*, International Journal of Mathematics and Mathematical Sciences Volume 6 (1983), Issue 2, 327-333.

[5] V. Paatero, *Über die konforme Abbildung von Gebieten deren Ränder von beschränkter Drehung sind*, Ann. Acad. Sci. Fenn. Ser., A 33, (1931), 1-77.

[6] V. Paatero, *Über Gebiete von beschränkter Randdrehung*, Ann. Acad. Sci. Fenn. Ser., A 37, (1933), 1-20.

[7] B. Pinchuk, *Functions with bounded boundary rotation*, Isr. J. Math., 10 (1971), 7-16.

[8] M.S. Robertson, *Coefficients of functions with bounded boundary rotation*, Canad. J. Math., 21 (1969), 1477-1482

Yaşar Polatoglu
Department of Mathematics and Computer Sciences
İstanbul Kültür University, İstanbul, Turkey
e-mail: y.polatoglu@iku.edu.tr

Yasemin Kahramaner
Department of Mathematics,
İstanbul Ticaret University, İstanbul, Turkey
e-mail: ykahramaner@iticu.edu.tr

Arzu Yemisci Şen
Department of Mathematics and Computer Sciences
İstanbul Kültür University, İstanbul, Turkey
e-mail: