QUASI-CHAPLYGIN SYSTEMS AND NONHOLONOMIC RIGID BODY DYNAMICS

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Abstract. We show that the Suslov nonholonomic rigid body problem studied in [10, 13, 26] can be regarded almost everywhere as a generalized Chaplygin system. Furthermore, this provides a new example of a multidimensional nonholonomic system which can be reduced to a Hamiltonian form by means of Chaplygin reducing multiplier. Since we deal with Chaplygin systems in the local sense, the invariant manifolds of the integrable examples are not necessary tori.

1. Introduction

We start the paper with the definition of nonholonomic Chaplygin systems, their reductions and Hamiltonization.

Chaplygin Systems. Suppose we are given a natural nonholonomic system on the n-dimensional Riemannian manifold \((N, \kappa)\) with local coordinates \(x_i\), Lagrangian \(l(x, \dot{x}) = \frac{1}{2} \sum \kappa_{ij} \dot{x}_i \dot{x}_j - V(x)\) and \(k\)-dimensional distribution \(D \subset TN\) describing kinematic constraints: a curve \(x(t)\) is said to satisfy the constraints if \(\dot{x}(t) \in D_{x(t)}\) for all \(t\). The trajectory of the system \(x(t)\) that satisfies the constraints is a solution to the Lagrange–d’Alembert equation

\[
\sum_{i=1}^{n} \left( \frac{\partial l}{\partial \dot{x}_i} - \frac{d}{dt} \frac{\partial l}{\partial x_i} \right) \eta_i = 0, \quad \text{for all } \eta \in D_x.
\]

Assume that \(N\) has a principal bundle structure \(\pi: N \rightarrow Q = N/\mathfrak{g}\) with respect to the left action of a \((n-k)\)-dimensional Lie group \(\mathfrak{g}\) and \(D\) is the collection of horizontal spaces of a principal connection. Given a vector \(X_x \in T_x N\), we have the decomposition \(X_x = X^h_x + X^v_x\), where \(X^h_x \in D_x\), \(X^v_x \in V_x\). Here \(V_x\) is tangent space to the fiber \(\mathfrak{g} \cdot x\) (the vertical space at \(x\)).

Further, suppose that the Lagrangian \(l\) is also \(\mathfrak{g}\)-invariant, i.e., \(v\) is a \(\mathfrak{g}\)-invariant function and \(\mathfrak{g}\) acts by isometries on Riemannian manifold \((N, \kappa)\). Then the constrained Lagrangian \(l_c(x, \dot{x}) = l(x, \dot{x}^h)\) induces a well defined reduced Lagrangian \(L: TQ \rightarrow \mathbb{R}\) via identification \(TQ \approx D/\mathfrak{g}\). The reduced Lagrangian \(L\) is of the natural mechanical type as well, the corresponding kinetic energy (metric) and the potential energy will be denoted by \(\kappa_D\) and \(V\) respectively.

As a result, equation (1) is \(\mathfrak{g}\)-invariant and defines a reduced Lagrange–d’Alembert system on the tangent bundle \(TQ\) (for the details see [15, 4, 6]). After the Legendre transformation it can be rewritten as the following first-order dynamical system on
\[ T^*Q \]
\[ \dot{q}_i = \frac{\partial H}{\partial \dot{q}_i}, \quad \dot{P}_i = -\frac{\partial H}{\partial q_i} + \Pi_i(q, P) \quad i = 1, \ldots, k, \]
where \( q = (q_1, \ldots, q_k) \) are local coordinates on the base space \( Q \) and \( P_i = \partial L/\partial \dot{q}_i \), \( i = 1, \ldots, k \) are conjugate momenta. The Hamiltonian \( H = \frac{1}{2} \sum x_{ij}^D P_i P_j + V(q) \) is the Legendre transformation of the reduced Lagrangian \( L = \frac{1}{2} \sum x_{ij} \dot{q}_i \dot{q}_j - V(q) \). The functions \( \Pi_i \) are quadratic in momenta and depend on the curvature of the principal connection and the metric \( \kappa \).

The system \( (N, l, D, \Theta) \) is called a (generalized) Chaplygin system (see Koiller [15]), as a generalization of classical Chaplygin systems with Abelian symmetries [8].

**Chaplygin Reducing Multiplier.** Let \( \mathcal{N}(q) \) be a differentiable nonvanishing function on \( Q \). Then, under the time substitution \( d\tau = \mathcal{N}(q) dt \) the following commutative diagram holds

\[
\begin{array}{ccc}
TQ \{q, \dot{q}\} & \xrightarrow{q'=\dot{q}/\mathcal{N}(q)} & TQ \{q, q'\} \\
\downarrow^p=\kappa_{Dq} & & \downarrow^p=\kappa_{Dq'} \\
T^*Q \{q, P\} & \xrightarrow{p=\mathcal{N}P} & T^*Q \{q, p\}.
\end{array}
\]

Here \( q' = dq/d\tau \). In the coordinates \( \{q, q'\} \) and \( \{q, p\} \), \( L \) and \( H \) take the forms

\[ L^* = \frac{1}{2} \sum N^2 \kappa_{Dij} q'_i q'_j - V(q) \quad \text{and} \quad H^* = \frac{1}{2} \sum \frac{1}{N^2} \kappa_{ij}^2 P_i P_j + V(q), \]
respectively. We look for a factor \( \mathcal{N}(q) \) such that after the above time substitution the equations (2) take the form

\[ q'_i = \frac{\partial H^*}{\partial P_i}, \quad \dot{P}_i = -\frac{\partial H^*}{\partial q_i}, \quad i = 1, \ldots, k. \]

That is, they become Hamiltonian with respect to the symplectic form \( \Omega = \sum dp_i \wedge dq_i = \mathcal{N}(q) \Omega + \sum P_i d\mathcal{N} \wedge dq_i \), where \( \Omega = \sum dP_i \wedge dq_i \) is the canonical symplectic form on \( T^*Q \). In nonholonomic mechanics the factor \( \mathcal{N} \) is known as the *reducing multiplier*.

It appears that non-existence of an invariant measure of the reduced system (2) is an obstruction to its reducibility to a Hamiltonian form. Namely, suppose that the original system (2) is transformed to the Hamiltonian form with a reducing multiplier \( \mathcal{N} \). Then the system has the invariant measure \( \mathcal{N}(q)^k \Omega^k \) (see [22] [11] [6]). According to the celebrated *Chaplygin’s reducibility theorem* (see [8] [7] [11] or section III.12 in [18]), for \( k = 2 \), the above statement can also be inverted: the existence of the invariant measure with the density \( \mathcal{N}(q) \) implies that in the new time \( d\tau = \mathcal{N}(q) dt \), the system (2) gets the Hamiltonian form (4).

Necessary and sufficient conditions for the existence of an invariant measure of the reduced system are given by Castrigion, Cortes, de Leon and Martin de Diego [6]. Recently, a nontrivial example of a nonholonomic system (multidimensional generalization of the Veselova problem on nonholonomic rigid body motion [25] [10]) for which the Chaplygin reducibility theorem is applicable for any dimension is given by Fedorov and Jovanović [11].

Note that there is an alternative (but equivalent) description of the method of reducing multiplier. The system (2) can be written as \( \Omega_{nb}(X_H, \cdot) = dH(\cdot) \), where
\( \Omega_{nh} = \Omega + \Xi \) is a nondegenerate 2-form. Then the Chaplygin multiplier is a function \( N \) such that the form \( N \Omega_{nh} \) is closed (see [22, 6, 9]).

Contents of the Paper. In Section 2 we introduce the notion of quasi-Chaplygin systems. In Section 3 we give a brief description of the Suslov problem and show that it can be considered as a quasi-Chaplygin system. Furthermore, this provides an example which can be reduced to a Hamiltonian form via Chaplygin reducing multiplier. In this sense, the complete integrability of the reduced Suslov problem can be defined in the natural way (Section 4). The Hamiltonian description explains the solvability of multidimensional Kharlamova, Klebsh–Tisserand and Lagrange cases obtained in [13]. The topology of invariant manifolds of the Kharlamova and Klebsh–Tisserand cases is studied in Section 5. Finally, we note that the Lagrange case can be treated as a generalized Chaplygin system in two different ways. In the first approach the reduced system becomes Hamiltonian after the time rescaling, while in the second approach the reduced system is already an integrable Hamiltonian system, namely the multidimensional spherical pendulum.

2. Quasi-Chaplygin Systems

Several nonholonomic mechanical systems have classically been regarded as Chaplygin systems in certain properly chosen local coordinates (an example is the well known Chaplygin skate), although, globally, they are not Chaplygin systems in the sense of the above definition (e.g., see [18]).

Definition 1. With the above notation, we say that \((N, l, D, \mathfrak{G})\) is a quasi-Chaplygin system, if we allow that the sum \(D_x + V_x\) does not span the tangent space \(T_x N\) on some \(\mathfrak{G}\)-invariant subvariety \(S \subset N\).

As for the Chaplygin systems, the Lagrange–d’Alembert equation (1) is \(\mathfrak{G}\)-invariant and reduces to the quotient space \(D/\mathfrak{G}\) (e.g., see [17]). The later has a structure of the \(\mathbb{R}^k\)-vector bundle \(\mathbb{R}^k \to D/\mathfrak{G} \to Q = N/\mathfrak{G}\), which is not, in general, diffeomorphic to \(TQ\). Outside \(N \setminus S\), we can treat the system as a usual Chaplygin system that reduces to \((D|_{N \setminus S})/\mathfrak{G} \approx T(Q \setminus (S/\mathfrak{G}))\). The jumping in the rank of the distribution \(D_x + V_x\) leads to several interesting properties of the system (see examples given below).

Locally, the reduced system can be derived by the use of Poincaré–Chetayev (or Bolzano–Hamel) equations. Consider an open \(\mathfrak{G}\)-invariant set \(U \subset N\) with local coordinates \(x = (q, g)\), in which the \(\mathfrak{G}\)-action is simply \(a \cdot (q, g) = (aq, ag)\), \(a \in \mathfrak{G}\). The Lagrangian \(l\) is \(\mathfrak{G}\)-invariant. Whence \(l(x, \dot{x}) = l(q, \dot{q}, g^{-1}\dot{g})\).

Let \(X_1, \ldots, X_n\) be linearly independent \(\mathfrak{G}\)-invariant vector fields on \(U\). Then the commutators \([X_i, X_j]\) can be written in the basis \(X_1, \ldots, X_n\): \([X_i, X_j] = \sum_k c_{ij}^k X_k\), where structural coefficients \(c_{ij}^k = c_{ij}^k(q)\) are \(\mathfrak{G}\)-invariant functions. Let \(\omega_1, \ldots, \omega_n\) be the quasi-velocities defined by \(\dot{x} = (\dot{q}, \dot{g}) = \sum \omega_i X_i\). From the definition of \(\omega_i\), one get the relations \(\dot{q}_i = \sum_j A_{ij} \omega_j\), where coefficients \(A_{ij} = A_{ij}(q)\) are also \(\mathfrak{G}\)-invariant functions.

Now, write the Lagrangian as a function of \(q\) and \(\omega\): \(\tilde{l}(q, \omega) = l(q, \dot{q}, g^{-1}\dot{g})\). Further, suppose that the distribution \(D\) is spanned by \(X_1, \ldots, X_k\). Then, the
constraints are
\[ \omega_{k+1} = 0, \ldots, \omega_n = 0 \]
and \( \{q_1, \ldots, q_k, \omega_1, \ldots, \omega_k\} \) can be regarded as local coordinates on \( D/\mathfrak{g} \). With the above notation, the Poincaré–Chetayev equations of the system (e.g., see [18, 9]):
\[
\frac{d}{dt} \left( \frac{\partial \hat{h}}{\partial \omega_i} \right) = \sum_{l=1}^{n} \sum_{j=1}^{k} c_{lj}(q) \frac{\partial \hat{h}}{\partial \omega_j} \omega_j + X_i(\hat{l}), \quad i = 1, \ldots, k,
\]
together with the kinematic equations \( \dot{q}_i = \sum_{j=1}^{k} A_{ij}(q) \omega_j \), form a closed system in variables \( \{q_1, \ldots, q_k, \omega_1, \ldots, \omega_k\} \). Note that, contrary to Chaplygin systems, the matrix \( (A_{ij}(q))_{1 \leq i,j \leq k} \) does not need to be invertible for all \( q \).

3. SUSLOV PROBLEM AS A QUASI-CHAPLYGIN SYSTEM

**Suslov Problem.** Consider the motion of an \( n \)-dimensional rigid body around a fixed point \( O \) in the \( n \)-dimensional Euclidean vector space \( \mathcal{V}, \langle \cdot, \cdot \rangle \). Let \( \mathcal{E}_1, \ldots, \mathcal{E}_n \) be the orthonormal frame fixed in the body and \( e_1, \ldots, e_n \) be the orthonormal frame fixed in the space. The configuration space of the system is the Lie group \( SO(n) \); the element \( g \in SO(n) \) maps the moving coordinate system to the fixed one. We use the following usual matrix notation. Let
\[
E_1 = (1, 0, \ldots, 0)^t, \ldots, E_n = (0, \ldots, 0, 1)^t.
\]
We take \( \{\mathcal{E}_1, \ldots, \mathcal{E}_n\} \) for the base of \( \mathcal{V} \). Then \( E_1, \ldots, E_n \) and
\[
e_1 = (e_{11}, \ldots, e_{1n})^t, \ldots, e_n = (e_{n1}, \ldots, e_{nn})^t, \quad e_{ij} = (e_i, \mathcal{E}_j)
\]
will be the coordinate expressions of \( \mathcal{E}_1, \ldots, \mathcal{E}_n \) and of \( e_1, \ldots, e_n \), respectively. The matrix \( g \in SO(n) \) maps the vectors in the moving frame to the same vectors regarded in the fixed frame. Therefore \( E_i = g \cdot e_i \) and \( g = (e_1, \ldots, e_n)^t \), i.e., \( g_{ij} = e_{ij} \). Note that we can consider the components of the vectors \( e_1, \ldots, e_n \) as redundant coordinates on \( SO(n) \).

For a path \( g(t) \in SO(n) \), the angular velocity of the body is defined by \( \omega(t) = g^{-1} \dot{g}(t) \in so(n) \). From the conditions \( 0 = \dot{E}_i = \dot{g} \cdot e_i + g \cdot \dot{e}_i \), we find that \( e_1, \ldots, e_n \) satisfy the Poisson equations
\[
\dot{e}_i = -\omega \cdot e_i, \quad i = 1, \ldots, n.
\]

The kinetic energy of the rigid body is a left-invariant function on \( TSO(n) \) of the form \( \frac{1}{2} \langle \mathcal{I} \omega, \omega \rangle \), where \( \mathcal{I} : so(n) \rightarrow so(n) \) is a non-degenerate inertia operator and \( \langle \cdot, \cdot \rangle \) denotes the Killing metric on \( so(n) \). For a “physical” rigid body, \( \mathcal{I} \omega \) has the form \( I \omega + \omega I \), where \( I \) is a symmetric \( n \times n \) matrix called mass tensor (see [10]). Further, suppose that the body is placed in a potential field that is invariant with respect to the orthogonal transformations which fix \( e_n \). In our notation this means that the Lagrangian has the form
\[
l(g, \dot{g}) = \frac{1}{2} \langle \mathcal{I} \omega, \omega \rangle - v(e_n).
\]

The Suslov problem describes the motion of a rigid body with the left-invariant constraints (see Fedorov and Kozlov [10])
\[
\langle \omega, E_i \wedge E_j \rangle = 0, \quad 1 \leq i < j \leq n - 1.
\]
Equivalently, we can say that the velocity $\dot{g}$ belong to the left-invariant distribution $D_g = g \cdot \mathfrak{d} \subset T_g SO(n)$, where

$$\mathfrak{d} = \text{span}\{E_1 \wedge E_n, \ldots, E_{n-1} \wedge E_n\}.$$ 

The motion of the system is described by the Lagrange–d’Alembert equation (1), which in the left trivialization takes the form of the Euler–Poincaré–Suslov (EPS) equation (see [10])

$$\left\langle \frac{d}{dt} (I \omega) - [I \omega, \omega] - \frac{\partial \nu}{\partial c_n} \wedge e_n, \eta \right\rangle = 0, \quad \text{for all} \quad \eta \in \mathfrak{d}. \quad (7)$$

The EPS equation, together with the Poisson equations (4) and the constraints (5) completely describe the motion of the Suslov problem in the variables $\{e_1, \ldots, e_n, \omega\}$. For $n = 3$ these are the equations of the classical Suslov problem with the following nonholonomic constrain: the projection of the angular velocity to the vector $e_3$ equals zero (see [23, 16, 1]).

**Geometry of the Constraints.** Let $H \cong SO(n-1)$ be the subgroup of $SO(n)$ with the Lie algebra $\mathfrak{h} = \mathfrak{d}^\perp \cong \text{so}(n-1)$:

$$\begin{pmatrix} \text{SO}(n-1) & 0^t \\ 0 & 1 \end{pmatrix}, \quad 0 = (0, \ldots, 0).$$

Through the paper we shall simply write $SO(n-1)$ instead of $H$. We have the following simple geometrical lemma.

**Lemma 1.** The left action of $SO(n-1)$ represents the rotations of a body around the vector $e_n$ fixed in the space, while the right action represents the rotations around the vector $e_n$ fixed in the body.

Consider the left action of $SO(n-1)$ on $SO(n)$ and the principal bundle

$$\begin{align*}
\text{SO}(n-1) & \rightarrow \text{SO}(n) \\
S^{n-1} = \text{SO}(n)/\text{SO}(n-1) & \rightarrow \pi.
\end{align*} \quad (8)$$

According to Lemma 1, the sphere $S^{n-1}$ (the space of cosets $\text{SO}(n-1) \cdot g$) can be identified with the vector $e_{n}$ via bijection $e_{n} \leftrightarrow \text{SO}(n-1) \cdot (e_1, \ldots, e_n)^t$.

The distribution $D \subset TSO(n)$ is invariant with respect to the left $\text{SO}(n-1)$-action on TSO(n), but it cannot be regarded as a connection of the principal bundle $S^{n-1}$. However $D$ can be seen as a connection almost everywhere on $SO(n)$.

**Theorem 2.** The distribution $D$ (6) can be regarded as a principal connection of the bundle $S^{n-1}$ outside the submanifold $\{e_{nn} = 0\}$.

**Proof.** Since we deal with the left $\text{SO}(n-1)$-action, the vertical distribution $V$ is right invariant; $V_g = \text{so}(n-1) \cdot g$. We shall prove that $D_g$ and $V_g$ span the tangent space at $g$ outside the submanifold $\{e_{nn} = 0\}$.

We have:

$$D_g + V_g \neq T_g SO(n) \iff \mathfrak{d} + g^{-1} \cdot \text{so}(n-1) \cdot g \neq \text{so}(n)$$

$$\iff g \cdot \mathfrak{d} \cdot g^{-1} + \text{so}(n-1) \neq \text{so}(n)$$

$$\iff g \cdot X \cdot g^{-1} \in \text{so}(n-1), \quad \text{for some} \quad X \in \mathfrak{d}. \quad (9)$$
Suppose that there is a vector $X \in \mathfrak{d}$ such that $g \cdot X \cdot g^{-1} \in \text{so}(n-1)$. Let $\bar{g} = g^{-1}$, $E_i = \bar{g} \cdot \bar{e}_i$. If $X = x_1 E_1 \wedge E_n + \cdots + x_{n-1} E_{n-1} \wedge E_n$, then, by using $\bar{g}^{-1} \cdot E_i \wedge E_j \cdot \bar{g} = \bar{e}_i \wedge \bar{e}_j$, we get

$$
\bar{g}^{-1} \cdot X \cdot \bar{g} = x_1 \bar{e}_1 \wedge \bar{e}_n + \cdots + x_{n-1} \bar{e}_{n-1} \wedge \bar{e}_n = Y \wedge \bar{e}_n \in \text{so}(n-1),
$$

where $Y = x_1 \bar{e}_1 + \cdots + x_{n-1} \bar{e}_{n-1}$. On the other hand, $Y \wedge \bar{e}_n$ belongs to $\text{so}(n-1)$ if and only if

$$
Y_n = x_1 \bar{e}_{1n} + \cdots + x_{n-1} \bar{e}_{n-1,n} = 0 \quad \text{and} \quad \bar{e}_{nn} = 0.
$$

But $\bar{e}_{nn} = \varepsilon_{nn}$, hence we proved that (9) implies $\varepsilon_{nn} = 0$.

To prove the opposite statement, we just note that the relation (10) considered as an equation in the variables $x_i$ always has a solution. Let us choose $x_1$ such that (10) holds. Then, if $\varepsilon_{nn} = \varepsilon_{nn} = 0$, we conclude that the element $X = x_1 E_1 \wedge E_n + \cdots + x_{n-1} E_{n-1} \wedge E_n \in \mathfrak{d}$ satisfies $\bar{g}^{-1} \cdot X \cdot \bar{g} = g \cdot X \cdot g^{-1} \in \text{so}(n-1)$. The theorem is proved. □

**Reduced Suslov Equation.** The Suslov equations (7), (11), (12) are $SO(n-1)$-invariant and equations (7), (11) together with $\varepsilon_n = -\omega \cdot \varepsilon_n$ can be viewed as their $SO(n-1)$-reduction to

$$
S^{n-1} \times \mathbb{R}^{n-1} \{ \varepsilon_n, \omega \} \approx D/\text{SO}(n-1).
$$

For the simplicity, denote the vector $\varepsilon_n$ by $q = (q_1, \ldots, q_n)$. Further, suppose that $\mathcal{I}$ preserves the decomposition $\text{so}(n) = \text{so}(n-1)+\mathfrak{d}$. Note that $(\text{so}(n), \text{so}(n-1))$ is a symmetric pair, i.e., $[\mathfrak{d}, \mathfrak{d}] \subset \text{so}(n-1)$. Therefore, if $\mathfrak{d}$ is an eigenspace of $\mathcal{I}$, then, in view of the condition $\text{pr}_\mathfrak{d}[\mathfrak{d}, \mathfrak{d}] = 0$, the EPS equation (7) and the Poisson equation for $\varepsilon_n$ become

$$
(\mathcal{I} \omega)_{in} = \frac{\partial v}{\partial q_i} q_n - \frac{\partial v}{\partial q_n} q_i,
$$

$$
\dot{q}_i = -\omega_{in} q_n, \quad i = 1, \ldots, n-1, \quad \dot{q}_n = \sum_{i=1}^{n-1} \omega_{in} q_i.
$$

Contrary to the general case, the reduced Suslov problem (11), (12) preserves the standard measure in the variables $\{q_1, \ldots, q_n, \omega_{1n}, \omega_{2n}, \ldots, \omega_{n-1,n}\}$. By the Euler–Jacobi theorem (e.g., see [1]), for the integrability (more precisely, solvability by quadratures) of the reduced problem (11), (12), we need $2n-5$ additional integrals which are independent of the energy integral $\frac{1}{2}(\mathcal{I} \omega, \omega) + v(q)$. In particular, for $n = 3$ we need only one additional integral (see [14, 16, 19]).

However, as it is shown in [18], we do not need $2n-5$ integrals to solve the multidimensional variants of Kharlamova, Klebs–Tisserand and Lagrange cases of the Suslov problem. Below we will give the natural definition of a complete integrability of (11), (12) such that the examples studied in [13] provide the simplest completely integrable cases. Note that, in general, it is still not clear how to define the notion of a complete integrability (in the sense of the Liouville theorem) of nonholonomic systems (see [16, 20]).

**Chaplygin Reduction.** The Lagrangian function $l$ and the distribution $D$ are both invariant with respect to the left $SO(n-1)$-action on $TSO(n)$. As follows from Theorem 2, the multidimensional Suslov problem is a quasi-Chaplygin system. Further, after the appropriate time rescaling, the system becomes Hamiltonian.
The reduced space $(SO(n) \setminus \{e_{nn} = 0\})/SO(n-1)$ is the union of two half-spheres with $q_n > 0$ and $q_n < 0$. On the half-spheres we can use coordinates $q = (q_1, \ldots, q_{n-1})$ within the ball

$$B = \{q \in \mathbb{R}^{n-1} \mid (q, q) = q_1^2 + \cdots + q_{n-1}^2 < 1\}.$$  

Using (13) we can write down the reduced Lagrangian and Hamiltonian

$$L_\pm = \frac{1}{2} \frac{1}{1 - (q, q)} (J\dot{q}, \dot{q}) - V_\pm(q), \quad H_\pm = \frac{1}{2} (1 - (q, q))(AP, P) + V_\pm(q),$$

where $V_\pm(q) = v(q, q_n)$, $q_n = \pm \sqrt{1 - (q, q)}$, the metric $J$ is given by $J_{ij} = I_{in,jn}$, $P = \frac{1}{q^n} J\dot{q}$ and $A = J^{-1}$.

We do not need to find the curvature of the connection to find the reduced system. In view of (14), after straightforward computations, we can get:

**Theorem 3.** Suppose that the inertia operator $I$ preserves the decomposition so(n) = so(n-1) + $\mathfrak{d}$. After the Chaplygin reduction outside submanifold $\{e_{nn} = 0\}$, the multidimensional Suslov problem (7), (4), (6) takes the following form on $T^*B$

$$\dot{q} = \frac{\partial H_\pm}{\partial P}, \quad \dot{P} = -\frac{\partial H_\pm}{\partial q} + (AP, P)P - (P, AP)q.$$  

Under the time substitution $\tau = q_n dt$ and an appropriate change of momenta $p = q_n P$, the reduced system becomes a Hamiltonian system describing a motion of a particle within the ball $B$,

$$\dot{q}' = \frac{\partial H^*_\pm}{\partial P} = Ap, \quad \dot{p}' = -\frac{\partial H^*_\pm}{\partial q}' = -\frac{\partial V}{\partial q} \iff Jq'' = -\frac{\partial V}{\partial q},$$

where $H^*_\pm(q, p) = \frac{1}{2}(Ap, p) + V_\pm(q)$.

**LL Systems as a Quasi-Chaplygin Systems.** We can consider Suslov-type problems on other Lie groups as nonholonomic systems with left-invariant Lagrangians and left-invariant nonintegrable distributions (see [15] and references therein). Suppose we are given a natural nonholonomic system with a left-invariant Lagrangian $l$ and a left-invariant distribution $D$ on a Lie group $G$ (so called LL system). Let $\mathfrak{d}$ be the restriction of $D$ to $\mathfrak{g} = T_1G$. Futher, suppose that there is a subgroup $H$ of $G$ such that for its Lie algebra $\mathfrak{h}$ and $\mathfrak{d}$ hold $\mathfrak{g} = \mathfrak{h} + \mathfrak{d}$, $\mathfrak{h} \cap \mathfrak{d} = 0$. Then LL system is a quasi-Chaplygin system $(G, l, D, H)$ with respect to the left action of $H$. The system reduces to the dim $\mathfrak{d}$-dimensional vector bundle over the homogeneous space $G/H$. As for the Suslov problem, the subvariety $S \subset G$ where the vertical spaces $V_g = \mathfrak{h} \cdot g$ and the distribution $D_g = g \cdot \mathfrak{d}$ do not span the tangent spaces $T_gG$ is given by the equation $g \cdot \mathfrak{d} \cdot g^{-1} + \mathfrak{h} \neq \mathfrak{g}$.

4. INTEGRABILITY

In view of (14), (15), we can identify

$$-\omega_{in} \longleftrightarrow (Ap)_i, \quad i = 1, \ldots, n - 1.$$  

and write (11), (12) in the form

$$\dot{p} = -q_n \frac{\partial v}{\partial q} + \frac{\partial v}{\partial q_n} p, \quad \dot{q} = q_n Ap, \quad \dot{q}_n = -(Ap, q).$$
The phase space $M = S^{n-1}\{q\} \times \mathbb{R}^{n-1}\{p\}$ of the reduced Suslov problem (17) has the following natural decomposition

$$M = S^{n-1}\{q\} \times \mathbb{R}^{n-1}\{p\} \approx T^*\bar{B} \cup_{\Lambda} T^*\bar{B},$$

where $\Lambda$ be the boundary of $T^*\bar{B}$.

According to Theorem 3 in the new time $\tau$ and outside the domain $\Lambda$ the dynamics is Hamiltonian. Note that for $q_n > 0$, time $\tau$ has the same direction as $t$, while for $q_n < 0$ the direction is opposite. Therefore, the closure of flows of Hamiltonian vector fields $X_{H^+}$ and $X_{H^-}$ to $T^*\bar{B}$ recovers the dynamic of (17) in the following manner. Let

$$\Sigma = \left\{ (q, p) \in \Lambda \mid (Ap, q) = 0, \frac{\partial v}{\partial q_n}\bigg|_{q=(q,0)} = 0 \right\}$$

be the set of equilibria points of the system (17) with $q_n = 0$.

Let $(q_0, p_0) \in \Lambda \setminus \Sigma$ and let $(q_+(\tau), p_+(\tau))$ and $(q_-(\tau), p_-(\tau))$ be the trajectories of Hamiltonian vector fields $X_{H^+}$ and $X_{H^-}$ such that

$$\lim_{\tau \to +\tau_0} (q_+(\tau), p_+(\tau)) = \lim_{\tau \to -\tau_0} (q_-(\tau), p_-(\tau)) = (q_0, p_0).$$

Then, inverting the quadrature

$$\int_{\tau_0}^{\tau} \frac{\pm d\tau}{\sqrt{1 - (q_{\pm}(\tau), q_{\pm}(\tau))}} = t - t_0,$$

we find $t = t(\tau)$ and the dynamics of $q = (q, q_n)$ in the original time $t$.

The points $(q, p) \in \Sigma$ that correspond to equilibrium points of (17) are reached in an infinite time $t$. They not need to be the equilibrium points of vector fields $X_{H^+}$ and $X_{H^-}$.

**Definition 2.** We shall say that the reduced Suslov problem (17), is **completely integrable** if one can find $n - 1$ independent smooth integrals $f_i : M \to \mathbb{R}$, which after substitutions $q_n = \pm \sqrt{1 - (q, q)}$ Poisson commute between themselves.

![Figure 1](https://via.placeholder.com/150)

**Figure 1.** (a) Complete integrability (b) Kharlamova case

In particular, we can take $f_1 = \frac{1}{2}(Ap, p) + v(q)$. In the case of integrability, due to the Liouville theorem, one can (locally) find trajectories $(q_{\pm}(\tau), p_{\pm}(\tau))$ by quadratures (e.g., see [1]). On the other hand, the topological part of the theorem
can not be directly applied. Namely, if a compact connected component \( \mathcal{L}_c \) of the regular invariant set

\[
\mathcal{M}_c = \{(q, p) \in \mathcal{M} \mid f_i(q, p) = c_i, \ i = 1, \ldots, n - 1\}
\]

does not intersect \( \Lambda \), then \( \mathcal{L}_c \) is an \((n - 1)\)-dimensional torus with a uniform quasi-periodic dynamics in \( \tau \) and therefore with a non-uniform quasi-periodic motion in the original time \( t \). However, if \( \mathcal{L}_c \) intersect \( \Lambda \), it may have a quite complicate topology (as an illustration, see Figure 1a). In the three-dimensional case, the topological structure of invariant manifolds of several integrable variants of the Suslov problem was studied by Tatarinov [23] and Okuneva [19, 20].

**Reconstruction.** To reconstruct the motion \((q(t), \dot{q}(t))\) on the whole phase space \( D \), we have to solve the Poisson equations (4) for \( e_1, \ldots, e_{n-1} \), i.e., to find all trajectories in \( D \) that projects to the given trajectory \((e_n(t), \omega(t)) = (q(t), Ap(t))\). If \((e_n(t_0), \omega(t_0))\) is an equilibrium point or if \((e_n(t), \omega(t))\) is a periodic orbit, then the invariant set \( \pi^{-1}\{(e_n(t), \omega(t))\} \subset D \) is foliated with invariant tori of maximal dimension rank \( SO(n - 1) \) or rank \( SO(n - 1) + 1 \), respectively (e.g., see Zenkov and Bloch [20] and references therein).

5. **Examples of Topology of Invariant Manifolds**

**Systems with the Time Symmetry.** Suppose that the potential satisfy the condition \( V_+(q) = V_-(q) = V(q) \). Then the reduced Suslov equations (17), (18), i.e., are invariant with respect to the transformation \( t \mapsto -t \), \( q_n \mapsto -q_n \), and \( \Sigma \) is simply given by

\[
\Sigma = \{(p, q) \in \Lambda \mid (Ap, q) = 0\}.
\]

Thus, in the time \( t \), we are going along the trajectory \((q_+(\tau), p_+(\tau))\) of \( X_{H_+} \) until we reach \( \Lambda \setminus \Sigma \). Then we continue to go along the same trajectory, but in the opposite direction. If we reach again the boundary \( \Lambda \setminus \Sigma \), we get a closed trajectory.

As an example, consider the Suslov problem with a rigid body inertial operator \( I \omega = I_\omega + \omega I \), \( I = \text{diag}(I_1, \ldots, I_n) \) and a quadratic potential \( v(q) = C_1q_1 + \cdots + C_{n-1}q_{n-1} + \frac{1}{2}(B_1q_1^2 + \cdots + B_nq_n^2) \). Then

\[
H^* = H^*_\pm = \frac{1}{2} \left( \frac{1}{I_1 + I_n} p_1^2 + \cdots + \frac{1}{I_{n-1} + I_n} p_{n-1}^2 \right) + C_1q_1 + \cdots + C_{n-1}q_{n-1} + \frac{1}{2}((B_1 - B_n)q_1^2 + \cdots + (B_{n-1} - B_n)q_{n-1}^2).
\]

The functions

\[
f_i(q, p) = p_i^2 + 2C_i(I_i + I_n)q_i + (I_i + I_n)(B_i - B_n)q_i^2, \quad i = 1, \ldots, n - 1
\]

Poisson commute and they also commute with the Hamiltonian (17). Moreover, they are smooth functions on \( \mathcal{M} \). Thus, the system (17) is completely integrable.

Furthermore, we can consider the flow of \( X_{H^*_0} \) and integrals \( f_i \) on the whole symplectic linear space \( \mathbb{R}^{n-1}\{q\} \times \mathbb{R}^{n-1}\{p\} \). Let \( \mathcal{T}_c = \{f_i = c_i\} \subset \mathbb{R}^{2n-2} \) be an invariant submanifold of \( X_{H^*_0} \). Then the invariant set \( \mathcal{M}_c = \{f_i = c_i\} \subset \mathcal{M} \) is a two-fold covering of

\[
\mathcal{T}_c^* = \mathcal{M}_c \cap \{q_1^2 + \cdots + q_{n-1}^2 \leq 1\}
\]

determined by the multivalued function \( q_n = \pm \sqrt{1 - \langle q, q \rangle} \). The branching points of the covering correspond to zeros of \( q_n \), i.e., to \( \partial \mathcal{T}_c^* \approx \Lambda \cap \mathcal{M}_c \).
The Kharlamova Case. Let \( v(q) = C_1 q_1 + \cdots + C_{n-1} q_{n-1} \) (multidimensional Kharlamova case). This is a potential for a rigid body placed in a homogeneous gravitation field with the mass center orthogonal to \( \xi_n \). The trajectories of the system can be found by quadratures and can be expressed in terms of elliptic functions of time \( t \) (see Jovanović [13]).

A generic invariant set \( T_c \) is diffeomorphic to \( \mathbb{R}^{n-1} \) and \( T^*_c \) is a disjoint union of \( 2^l \) copies of \( (n-1) \)-dimensional closed balls (the number \( l \), \( 0 \leq l \leq n-1 \) depends on the choice of constants \( c_i \)). Therefore, connected components of \( T_c \) are spheres. Trajectories of the vector field \( X_{H*} \) pass through \( \Lambda \). Therefore, apart from a finite number of equilibrium points and their asymptotic trajectories, all the trajectories of the Suslov problem in the original time \( t \) will be closed (Figure 1b). Thus we get

**Theorem 4.** In the Kharlamova case, the phase space \( M \) of the reduced problem is almost everywhere foliated with \( (n-1) \)-dimensional spheres. The distribution \( D \) is filled up with conditionally-periodic trajectories of maximal dimension \([n/(n-1)] + 1\).

Klebsh–Tisserand Case. For generic values of the constants \( C_i \) and \( B_i \) the topology of invariant manifolds is much more complicated. Consider the quadratic potential \( v(q) = \frac{1}{2}(B_1 q_1^2 + \cdots + B_n q_n^2) \) (multidimensional Klebsh–Tisserand case [13]) with \( B_i > B_n \), \( i = 1, \ldots, n-1 \). Then \( T_c \) is the \( (n-1) \)-dimensional torus \( S_{c_1} \times \cdots \times S_{c_{n-1}} \), where \( S_{c_i} = \{f_i = c_i\} \) are circles in the planes \( \mathbb{R}^2 \{p_i, q_i\} \). (If some of \( c_i \) vanish, then the dimension of the tori decreases.) Let \( \varphi_i \) be angular variables on the circles \( S_{c_i} \):

\[
\begin{align*}
q_i &= \sqrt{c_i} \cos \varphi_i, & p_i &= \sqrt{c_i} \sin \varphi_i, & \kappa_i &= (I_i + I_n)(B_i - B_n), & i &= 1, \ldots, n-1,
\end{align*}
\]

Then the Hamilton equations on \( T_c \) take the form

\[
\frac{d\varphi_i}{dt} = \Omega_i = \sqrt{\frac{B_i - B_n}{I_i + I_n}}, \quad i = 1, \ldots, n-1
\]

and the subset \( T^*_c \subset T_c \) is given by equation

\[
\frac{c_1}{\kappa_1} \sin^2 \varphi_1 + \cdots + \frac{c_{n-1}}{\kappa_{n-1}} \sin^2 \varphi_{n-1} \leq 1.
\]

Using the relation \( 21 \), we can describe the topological structure of the invariant manifolds \( M_c \) of the Klebsh–Tisserand case. Namely, the condition

\[
\frac{c_1}{\kappa_1} + \cdots + \frac{c_{n-1}}{\kappa_{n-1}} < 1
\]

implies that \( q_n \neq 0 \) on \( T_c \). As a result, the following theorem holds (see [13]).

**Theorem 5.** In the domain defined by the condition \( 22 \), invariant manifolds \( T_c \) are disjoint unions of two \( l \)-dimensional tori \( (l \leq n-1) \) with non-uniform conditionally-periodic motions.

For \( \frac{c_1}{\kappa_1} + \cdots + \frac{c_{n-1}}{\kappa_{n-1}} \geq 1 \), the boundary of \( T^*_c \) is not empty and there are several cases depending of the values of \( c_i \). We quote some of them.

(i) \( 0 < c_i < \kappa_1, \ldots, 0 < c_{n-1} < \kappa_{n-1} \).

In this case \( T^*_c \) is obtained by removing a disjoint union of \( 2^{n-1} \) open balls \( B_i \) from the torus \( T_c \). The invariant set \( M_c \) is diffeomorphic to

\[
(T_c^{n-1} \setminus \cup_i B_i) \cup \cup_i \partial B_i \cup (T_c^{n-1} \setminus \cup_i B_i).
\]
For $n = 3$ this manifold is a sphere with five handles (see Figure 2).

(ii) $c_1 > \kappa_1, \ldots, c_{n-1} > \kappa_{n-1}$.

$\mathcal{T}_c$ is a disjoint union of $2^{n-1}$ closed balls. Similarly to the Kharlamova case, $\mathcal{M}_c$ is diffeomorphic to disjoint union of $2^{n-1}$ spheres $S^{n-1}$.

(iii) $c_1 > \kappa_1, 0 < c_2 < \kappa_2, \ldots, 0 < c_{n-1} < \kappa_{n-1}$.

The subset $\mathcal{T}_c$ is diffeomorphic to two copies of $T^{n-2} \times [0,1]$. As a result, $\mathcal{M}_c$ is a two-fold covering of $2T^{n-2} \times [0,1]$ with branching point on the boundaries $T^{n-2} \times \{0,1\}$, i.e., $\mathcal{M}_c$ is a disjoint union of two $(n-1)$-dimensional tori $T^{n-1}$ (Figure 2).

Figure 2. Klebsh–Tisserand Case

Analysis of other cases goes along similar lines. It is desirable to have a complete description of the bifurcation diagram of the mapping $F = (f_1, \ldots, f_{n-1}) : \mathcal{M} \to \mathbb{R}^{n-1}$, as given for $n = 3$ in [19].

For generic values of the frequencies $\Omega_i$, the trajectories of (20) are dense in $\mathcal{T}_c$ and, in the domain $\frac{c_1}{\kappa_1} + \cdots + \frac{c_{n-1}}{\kappa_{n-1}} \geq 1$, a generic trajectory intersects the boundary $\partial\mathcal{T}_c \approx \Lambda \cap \mathcal{M}_c$. Therefore, in the original time $t$ almost all the trajectories are closed. Thus, concerning the reconstruction problem, it would be interesting to describe qualitative behavior of the motions over invariant sets $\mathcal{M}_c$, where $\frac{c_1}{\kappa_1} + \cdots + \frac{c_{n-1}}{\kappa_{n-1}} < 1$. A similar problem for the Fedorov–Kozlov integrable case of the Suslov problem (the original constraints [0] are relaxed and $v(e_n) \equiv 0$) is studied in [20].

**Lagrange Case.** Consider the EPS equations [7], [11], [3] describing the motion of a dynamically symmetric heavy rigid body. Namely, assume that $I = \text{diag}(I_1, \ldots, I_1, I_n)$ and that the mass center lies on the symmetry axis $E_n$. This gives us a nonholonomic version of the multidimensional *Lagrange top* considered by Beljaev [3]. The Lagrangian of the system has the form

$$
(23) \quad l(g, \dot{g}) = I_1 \langle \text{pr}_{so(n-1)} \omega, \text{pr}_{so(n-1)} \omega \rangle + \frac{1}{2} (J_1 + I_n) \langle \text{pr}_{\phi} \omega, \text{pr}_{\phi} \omega \rangle - \epsilon e_{nn}.
$$

The reduced system (17) is integrable in view of the Euler–Jacobi theorem and in the sense of Definition 2 as well. Namely, the functions $f_{ij} = q_i p_j - g_j p_i : \mathcal{M} \to \mathbb{R}$ are integrals of (17). There are $2n - 5$ independent integrals of this form, which, together with the energy $\frac{1}{2}(J_1 + I_n)(p, p) + \epsilon q_n$, implies that the phase
space $\mathcal{M}$ is foliated with two-dimensional invariant manifolds. Since the system has an invariant measure, it is integrable by the Euler–Jacobi theorem [1]. On the other hand, the functions $f_{ij}, H^*_{\pm}$ ensure the non-commutative integrability (and, therefore, the usual complete integrability [5]) of the Hamiltonian flows $X_{H^*_{\pm}}$ within $T^*B$.

The geometrical meaning of the integrals $f_{ij}$ is that the trajectories $q(\tau)$ take place over invariant two-dimensional planes, depending on the initial conditions. After fixing the plane, the problem becomes the usual two-dimensional problem of the motion of a particle in a central potential force field.

**Reduction to the Spherical Pendulum.** The nonconstrained Lagrange top system is completely integrable (see [3], the Lax pair for the system is given in [21]). It appears that the Suslov problem (7), (4), (6) can be regarded as a subsystem of the Lagrange top. Namely, the Lagrangian [23] is invariant with respect to both left and right actions of the Lie subgroup $SO(n - 1)$. Let us consider the right action. (Note that $D$ is a principal connection of the bundle [3] given by the right action of $SO(n - 1)$.) The momentum map of the system is $\phi(g, \dot{g}) = 2I_1 \text{pr}_{so(n-1)} \omega$. This gives us the conservation law $\frac{d}{dt} \text{pr}_{so(n-1)} \omega = 0$. In particular, the distribution $D = \phi^{-1}(0)$ is an invariant submanifold of the Lagrange top system.

Therefore, the right $SO(n - 1)$-Chaplygin reduction of the Suslov problem to $T^*S^{n-1}$ coincides with the Lagrange–Routh reduction of the Lagrange top for zero value of the momentum mapping (see [1], page 87, Theorem 13). According to Lemma 11 the sphere $S^{n-1} = SO(n)/SO(n-1)$ can be identified with the positions of the vector $\mathbf{e}_n$ considered in the fixed frame. The reduced Lagrangian has the form

$$L = \frac{1}{2} (I_1 + I_n) (\dot{e}_1^2 + \cdots + \dot{e}_n^2) - \epsilon e_{nn}.$$ 

Hence, the reduced system is integrable: it is the multidimensional spherical pendulum.

**Acknowledgments.** The first author (Yu.F.) acknowledges the support of grant BFM 2003-09504-C02-02 of Spanish Ministry of Science and Technology. The second author (B.J.) was supported by the Serbian Ministry of Science, Project ”Geometry and Topology of Manifolds and Integrable Dynamical Systems”.

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