Extensional Uniformity for Boolean Circuits*

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Abstract. Imposing an extensional uniformity condition on a non-uniform circuit complexity class \( \mathcal{C} \) means simply intersecting \( \mathcal{C} \) with a uniform class \( \mathcal{L} \). By contrast, the usual intensional uniformity conditions require that a resource-bounded machine be able to exhibit the circuits in the circuit family defining \( \mathcal{C} \). We say that \( (\mathcal{C}, \mathcal{L}) \) has the Uniformity Duality Property if the extensionally uniform class \( \mathcal{C} \cap \mathcal{L} \) can be captured intensionally by means of adding so-called \( \mathcal{L} \)-numerical predicates to the first-order descriptive complexity apparatus describing the connection language of the circuit family defining \( \mathcal{C} \).

This paper exhibits positive instances and negative instances of the Uniformity Duality Property.

Keywords: Boolean circuits, uniformity, descriptive complexity.

1 Introduction

A family \( \{C_n\}_{n \geq 1} \) of Boolean circuits is uniform if the way in which \( C_{n+1} \) can differ from \( C_n \) is restricted. Generally, uniformity is imposed by requiring that some form of a resource-bounded constructor on input \( n \) be able to fully or partially describe \( C_n \) (see [1, 5, 8, 14, 19] or refer to [22] for an overview). Circuit-based language classes can then be compared with classes that are based on a finite computing mechanism such as a Turing machine.

Recall the gist of descriptive complexity. Consider the set of words \( w \in \{a, b\}^* \) having no \( b \) at an even position. This language is described by the \( \text{FO}[<, \text{Even}] \) formula \( \neg \exists i (\text{Even}(i) \land P_b(i)) \). In such a first-order formula, the variables range over positions in \( w \), a predicate \( P_\sigma \) for \( \sigma \in \{a, b\} \) holds at \( i \) iff \( w_i = \sigma \), and a numerical predicate, such as the obvious 1-ary \( \text{Even} \) predicate here, holds at its arguments iff these arguments fulfill the specific relation.

The following viewpoint has emerged [3, 5, 6] over two decades: when a circuit-based language class is characterized using first-order descriptive complexity, the circuit uniformity conditions spring up in the logic in the form of restrictions on the set of numerical predicates allowed.

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As a well studied example [5, 12], $\text{FO}[\langle, +, \times\rangle] = \text{DLOGTIME-uniform AC}^0 \subsetneq \text{non-uniform AC}^0 = \text{FO[arb]}$, where the latter class is the class of languages definable by first-order formulae entitled to arbitrary numerical predicates (we use a logic and the set of languages it captures interchangeably when this brings no confusion).

In a related vein but with a different emphasis, Straubing [21] presents a beautiful account of the relationship between automata theory, formal logic and (non-uniform) circuit complexity. Straubing concludes by expressing the proven fact that $\text{AC}^0 \subsetneq \text{ACC}^0$ and the celebrated conjectures that $\text{AC}^0[q] \subsetneq \text{ACC}^0$ and that $\text{ACC}^0 \subsetneq \text{NC}^1$ as instances of the following conjecture concerning the class $\text{REG}$ of regular languages:

$$\text{Q[arb]} \cap \text{REG} = \text{Q[reg]}. \quad (1)$$

In Straubing’s instances, $\text{Q}$ is an appropriate set of quantifiers chosen from $\{\exists\} \cup \{\exists^{(q,r)} : 0 \leq r < q\}$ and $\text{reg}$ is the set of regular numerical predicates, that is, the set of those numerical predicates of arbitrary arity definable in a formal sense by finite automata. We stress the point of view that intersecting $\{\exists\}[\text{arb}] = \text{FO[arb]}$ with $\text{REG}$ to form $\text{FO[arb]} \cap \text{REG}$ in conjecture (1) amounts to imposing uniformity on the non-uniform class $\text{FO[arb]}$. And once again, imposing uniformity has the effect of restricting the numerical predicates: it is a proven fact that $\text{FO[arb]} \cap \text{REG} = \text{FO[reg]}$, and conjecture (1) expresses the hope that this phenomenon extends from $\{\exists\}$ to other $\text{Q}$, which would determine much of the internal structure of $\text{NC}^1$. We ask:

1. Does the duality between uniformity in a circuit-based class and numerical predicates in its logical characterization extend beyond $\text{NC}^1$?
2. What would play the role of the regular numerical predicates in such a duality?
3. Could such a duality help understanding classes such as the context-free languages in $\text{AC}^0$?

To tackle the first question, we note that intersecting with $\text{REG}$ is just one out of many possible ways in which one can “impose uniformity”. Indeed, if $\mathcal{L}$ is any uniform language class, one can replace $\text{Q[arb]} \cap \text{REG}$ by $\text{Q[arb]} \cap \mathcal{L}$ to get another uniform subclass of $\text{Q[arb]}$. For example, consider any “formal language class” (in the loose terminology used by Lange when discussing language theory versus complexity theory [14]), such as the class $\text{CFL}$ of context-free languages. Undoubtedly, $\text{CFL}$ is a uniform class of languages. Therefore, the class $\text{Q[arb]} \cap \text{CFL}$ is another uniform class well worth comparing with $\text{Q[<, +]}$ or $\text{Q[<, +, \times]}$. Of course, $\text{FO[arb]} \cap \text{CFL}$ is none other than the poorly understood class $\text{AC}^0 \cap \text{CFL}$, and when $\text{Q}$ is a quantifier given by some word problem of a nonsolvable group, $(\text{FO} + \{Q\})[\text{arb}] \cap \text{CFL}$ is the poorly understood class $\text{NC}^1 \cap \text{CFL}$ alluded to 20 years ago [11].

The present paper thus considers classes $\text{Q[arb]} \cap \mathcal{L}$ for various $\text{Q}$ and $\mathcal{L}$. To explain its title, we note that the constructor-based approach defines uniform classes by specifying their properties: such definitions are intensional definitions.