Fragile minor-monotone parameters under a random edge perturbation

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Abstract

We conduct a quantitative analysis on the number of random edges required to be added to a base graph $H$ to significantly increase natural minor-monotone graph parameters in the resulting graph $R$. Specifically, we show that if $R$ is obtained from a connected graph $H$ by adding only a few random edges, the tree-width, genus, and Hadwiger number of $R$ become very large, irrespective of the structure of $H$.

Keywords: Random graphs, randomly perturbed graphs, tree-width, genus, Hadwiger number

1 Introduction

Since the seminal work of Erdős and Rényi [10], various random graph models have been introduced and investigated. The most well-known model is the binomial random graph $G(n, p)$, a graph on the vertex set $[n] := \{1, 2, \ldots, n\}$ where each unordered pair of distinct vertices is connected by an edge with probability $p \in (0, 1]$ independently. During more recent years, randomly perturbed graphs have received considerable attention (see [2, 6, 8, 16, 18, 20, 21, 27, 29] and references therein). In this paper, we study a randomly perturbed graph defined as follows: Given an $n$-vertex graph $H := H_n$, called the base graph, we form a new graph $R := H \cup G(n, p)$ by adding edges of $G(n, p)$ to $H$. We extend and strengthen a result of Dowden, Kang, and Krivelevich [9] about the ‘fragile genus’ property of $R$ to additional minor-monotone graph parameters.

Our first result says that the tree-width $tw(\cdot)$, the genus $g(\cdot)$, and the Hadwiger number $h(\cdot)$ are ‘fragile’ in the sense that adding only a few random edges to a base graph can cause a drastic increase in these parameters. To state this formally, we first introduce some necessary notations. Throughout the paper, we use the standard Landau notations for asymptotic orders, with all asymptotics considered as $n \to \infty$. For example, whenever we write $x = o(1)$ and $y = \omega(1)$, we mean that $x$ tends to 0 and $y$ tends to $\infty$ as $n \to \infty$. We say that an event $E$ holds with high probability (whp in short), if $E$ holds with probability tending to 0 as $n \to \infty$. We use log to denote the natural logarithm unless the base is explicitly stated.

\textbf{Theorem 1.1.} Assume that $p = p(n) \in (0, 1]$ and $\Delta := \Delta(n) \in [1, \infty)$ such that $n^2p = \omega(1)$, and $\Delta \leq n^2p/48000$. Let $H := H_n$ be an $n$-vertex connected graph with maximum degree at most $\Delta$, and let $R := H \cup G(n, p)$. Then whp the following hold:

\begin{itemize}
  \item $tw(R) = \omega(n)$.
  \item $g(R) = \omega(n)$.
  \item $h(R) = \omega(n)$.\end{itemize}
Lemma 1.4 (Key lemma). Let $C \geq 8$ be an absolute constant. Let $p = p(n) \in (0, 1]$ and $\Delta = \Delta(n) \in [1, \infty)$ be the parameters satisfying $p \leq 2/n$, $n^2p = \omega(1)$, and $\Delta \leq \frac{n^2p}{40000}$. Let $H := H_n$ be an $n$-vertex connected graph with maximum degree at most $\Delta$ and let $R := H \cup G(n, p)$. Then whp $R$ contains vertex-disjoint connected subgraphs $R_1, \ldots, R_m$ such that

1. $96\frac{C\Delta}{np} \leq |V(R_i)| \leq 192\frac{C\Delta}{np}$ for each $i \in [m]$;

2. $m \geq \frac{n^2p}{9600C\Delta}$.

Let us illustrate how Lemma 1.4 can be used, which is the main idea of Lemma 3.2 called the boosting lemma. Assume that the conditions and settings of Lemma 1.4 are given; in particular, let $H$ be a base graph with maximum degree $\Delta$ and let $R := H \cup G(n, p)$. Given an arbitrary minor-monotone
graph parameter \( f(\cdot) \), assume that we want to bound \( f(R) \) from below. We begin with a standard two-round exposure approach: Take two independent random graphs \( G(n, p_1) \) and \( G(n, p_2) \) with \( p_1 = p_2 = 1 - \sqrt{1 - \frac{p}{2}} \geq p/2 \) such that \( G(n, p) = G(n, p_1) \cup G(n, p_2) \). Applying Lemma 1.4 we show that whp \( H \cup G(n, p_1) \) contains ‘many’ vertex-disjoint connected subgraphs \( R_1, \ldots, R_{m'} \) of comparable sizes \( \Theta(\frac{np}{m}) \). We contract each of \( R_1, \ldots, R_{m'} \) into a single vertex and delete all the other vertices; thereby we obtain a large minor \( S \) of \( R \) and hence \( f(R) \geq f(S) \). Then for some \( m = \Theta(m') = \Theta(\frac{np}{m}) \) and \( q = 1 - \exp(-\Theta(\frac{np}{m})) \), we show that \( S \) contains a random graph \( G(m, q) \) as a subgraph, due to the edges of \( G(n, p_2) \) going between \( R_i \) and \( R_j \) for \( 1 \leq i \neq j \leq m' \). Thus, it follows

\[
f(R) \geq f(G(m, q)).
\]

What is important and interesting about this procedure is that the edge density can be ‘amplified’ so that \( G(m, q) \) is in a supercritical regime, e.g., with \( mq > 1 \), even if we would begin with a very sparse base graph \( H \) (such as a path) and a subcritical random graph \( G(n, p) \), e.g., with \( np < 1 \).

In the proof of Theorem 1.1, we apply Lemma 3.2 with certain choices of constants and parameters so that \( G(m, q) \) is in a supercritical regime. Then we use well-known facts that \( tw(\cdot) \), \( g(\cdot) \), and \( h(\cdot) \) are ‘large’ for \( G(m, q) \) in a supercritical regime.

The rest of the paper is organized as follows. In Section 2 we provide definitions and facts about minor-monotone graph parameters. In Section 3 we study the boosting lemma and prove Theorem 1.1 by using the boosting lemma while assuming Lemma 1.4. In Section 4 we prove Theorem 1.3. In Section 5 we prove several lemmas on finding many vertex-disjoint connected subgraphs. Using these lemmas, we prove Lemma 1.4 in Section 6. In Section 7 we discuss the sharpness of the results. Finally, in Section 8 we discuss Lemma 1.4 and Theorem 1.1b).

2 Preliminaries

Throughout this paper, every graph is simple and undirected, meaning that graphs do not have loops or parallel edges. Let \( \mathbb{N} \) be the set of positive integers. For \( N \in \mathbb{N} \), let \([N] := \{1, 2, \ldots, N\}\) be the set of positive integers less than or equal to \( N \).

2.1 Definitions of tree-width, genus, and Hadwiger number

A graph \( G' \) is a minor of a graph \( G \) if a graph isomorphic to \( G' \) can be obtained from \( G \) by deleting vertices or edges and contracting edges. A graph parameter \( f(\cdot) \) is minor-monotone if \( f(G') \leq f(G) \) whenever a graph \( G' \) is a minor of a graph \( G \). We will review four minor-monotone graph parameters.

The tree-width, introduced by Robertson and Seymour [31], is one of the most well-known and well-studied graph parameters in algorithmic and structural graph theory. A tree decomposition of a graph \( G \) is a pair \((T, (B_v)_{v \in V(T)})\) of a tree \( T \) and a collection \((B_v)_{v \in V(T)}\) of subsets \( B_v \subseteq V(G) \) such that the following conditions hold:

- \( \bigcup_{v \in V(T)} B_v = V(G) \);
- For every edge \( e = xy \in E(G) \), there exists a vertex \( v \) of \( T \) such that \( x, y \in B_v \);
- For every vertex \( x \in V(G) \), a subset \( \{v \in V(T) : x \in B_v\} \) induces a subtree of \( T \).

The width of a tree decomposition \((T, (B_v)_{v \in V(T)})\) is \( \max_{v \in V(T)} (|B_v| - 1) \). The tree-width \( tw(G) \) of a graph \( G \) is defined as the minimum width over all tree decompositions of \( G \).

The study of graph minors is motivated by the fact that for each surface \( \Sigma \), the set of graphs that can be embedded in \( \Sigma \) without crossings is minor-closed. The genus \( g(G) \) of a graph \( G \) is defined as the minimum number of handles to be added to the sphere such that \( G \) can be embedded without any crossings.
We are also interested in the size of the largest complete minor in a graph. The **Hadwiger number** \( h(G) \) of a graph \( G \) is defined as the maximum integer \( \ell \geq 0 \) such that the complete graph \( K_\ell \) is a minor of \( G \).

Note that adding a new vertex or a new edge to a graph increases its tree-width and Hadwiger number by at most one. Adding a new edge to a graph increases its genus by at most one. We have \( h(G) \leq \text{tw}(G) + 1 \), because \( \text{tw}(K_t) = t - 1 \). Ringel and Youngs \([30]\) showed that \( g(K_t) = \lceil (t - 3)(t - 4)/12 \rceil \) for an integer \( t \geq 3 \). Because the genus is minor-monotone, it follows that

\[
g(G) \geq \lceil (h(G) - 3)(h(G) - 4)/12 \rceil = \Omega(h(G)^2).
\]

In fact, various graph parameters are closely related; see an excellent survey by Harvey and Wood \([17]\).

### 2.2 Lower bounds for \( G(n, p) \)

We collect some known results about lower bounds of tree-width, genus, and Hadwiger number of \( G(n, p) \) from \([9, 12, 24, 28]\), which are restated so that we can use them directly in the proof of Theorem 1.1.

The following theorem shows that the tree-width of \( G(n, p) \) grows linearly when \( p = c/n \) for a fixed \( c > 1 \). This was initially a conjecture of Gao \([14]\) and was proved by Lee, Lee, and Oum \([24]\) in a stronger form in terms of rank-width. Later, Perarnau and Serra \([28]\) presented a direct proof. Since both tree-width and genus do not increase by taking subgraphs, we may replace the condition \( p = c/n \) with \( p \geq c/n \).

**Theorem 2.1** (Lee, Lee, and Oum \([24]\)). For any \( c > 1 \) and \( p \geq c/n \), there exists \( r = r(c) > 0 \) such that whp \( \text{tw}(G(n, p)) \geq r \cdot n \).

**Theorem 2.2** (Dowden, Kang, and Krivelevich \([9]\)). For any \( c > 1 \) and \( p \geq c/n \), there exists \( r = r(c) > 0 \) such that whp \( g(G(n, p)) \geq r \cdot n^2p \).

**Theorem 2.3** (Fountoulakis, Kühn, and Osthus \([12]\)).

(i) There exists \( r > 0 \) such that for any \( p \geq \frac{c}{4n} \), whp \( h(G(n, p)) \geq r \sqrt{n} \).

(ii) There exists \( C' > 0 \) such that for any \( C'/n \leq p \leq 1/2 \), whp \( h(G(n, p)) \geq \frac{n}{2\sqrt{\log_{1/(1-p)}(np)}} \).

(iii) For any \( \varepsilon > 0 \) and \( \frac{1+\varepsilon}{n} \leq p \leq 1 - \varepsilon \), whp \( h(G(n, p)) = \Theta \left( n/\sqrt{\log_{1/(1-p)}(np)} \right) \).

There are many other interesting results about these minor-monotone graph parameters of \( G(n, p) \); for the overview, see Table 1 in the appendix of this paper.

### 3 Perturbing a graph of small maximum degree

In this section, we aim to prove Theorem 1.1 assuming our key lemma, Lemma 1.4. Inspired by Theorems 2.1, 2.3, we focus on a lower bound of the form \( f(G(n, p)) \geq r \cdot \mathbb{E}(f(G(n, p))) \), leading to the following definition.

**Definition 3.1.** Let \( f \) be a graph parameter, and let \( \tilde{f} : \mathbb{N} \times [0, 1] \to \mathbb{R}_{\geq 0} \) be a function. Let \( c, r > 0 \). We say that \( f \) is \((c, r)\)-bounded from below by \( \tilde{f} \), if for any \( p : \mathbb{N} \to \mathbb{R} \) with \( p(n) \in [c/n, 1] \) for all large \( n \in \mathbb{N} \), whp

\[
f(G(n, p(n))) \geq r \cdot \tilde{f}(n, p(n)).
\]

The following **boosting lemma** tells us how much a minor-monotone graph parameter can increase when we add a few edges randomly to a connected base graph of small maximum degree. It will be applicable for various minor-monotone graph parameters. Its name, the boosting lemma, comes from typical situations in which the value of \( f(G(n, p)) \) is whp larger than the value of \( r \cdot \mathbb{E}(f(G(m, q))) \) with \( q \gg p \). The proof uses Lemma 1.4 and a standard two-round exposure argument.
**Lemma 3.2 (Boosting lemma).** Let $f$ be a minor-monotone graph parameter such that $f$ is $(c, r)$-bounded from below by $\tilde{f}$ for some $c > 1$ and $r \geq 0$. Let $C \geq 4c$ and assume that $p = p(n) \in (0, 1]$ and $\Delta = \Delta(n) \in [1, \infty)$ satisfy $p \leq 2/n$ and $n^2p = \omega(1)$, and $\Delta \leq \frac{n^2p}{19200C\Delta}$. Let

$$m = m(n) := \left\lceil \frac{n^2p}{19200C\Delta} \right\rceil, \quad M = M(n) := \frac{(96C\Delta)^2}{n^2p}, \quad q = q(n) := 1 - e^{-M}.$$  

Let $H$ be an $n$-vertex connected graph with maximum degree at most $\Delta$, and let $R := H \cup G(n, p)$. Then whp

$$f(R) \geq \begin{cases} r \cdot \tilde{f}(m, q) & \text{if } \Delta \leq \sqrt{n^2p\log(n^2p)}, \\ f(K_m) & \text{otherwise.} \end{cases}$$

**Proof of Lemma 3.2 assuming Lemma 3.1** First we take two independent random graphs $G(n, p_1)$ and $G(n, p_2)$ where $1 - p = (1 - p_1)(1 - p_2)$ with $p_1 = p_2 := 1 - \sqrt{1 - p} \geq p/2$. Then $G(n, p)$ has the same probability distribution as $G(n, p_1) \cup G(n, p_2)$. Let $R_0 := H \cup G(n, p_1)$. Then $R$ has the same probability distribution as $R_0 \cup G(n, p_2)$, so with slight abuse of notation we write $R = R_0 \cup G(n, p_2)$.

By Lemma 3.1 whp $R_0$ contains vertex-disjoint connected subgraphs $R_1, \ldots, R_{m'}$ such that

1. $96C\Delta(np_1)^{-1} \leq |V(R_i)| \leq 192C\Delta(np_1)^{-1}$ for each $i \in [m']$;
2. $m' \geq np_1/(9600C\Delta)$.

Next, we contract $R_i$ into a single vertex $s_i$ for each $i \in [m']$ and delete all other vertices in $R = R_0 \cup G(n, p_2)$. For each $1 \leq i < j \leq m'$, we connect $s_i$ and $s_j$ by an edge if and only if there exists an edge of $G(n, p_2)$ having one end in $R_i$ and another end in $R_j$; these edges form the edge set of a graph on $[m']$, which we call $S$. Observe that $S$ is a minor of $R$. Furthermore, for each $1 \leq i < j \leq m'$, the probability that an edge of $G(n, p_2)$ exists between $R_i$ and $R_j$ is

$$1 - (1 - p_2)^{|V(R_i)||V(R_j)|} \geq 1 - (1 - p_2)^{\frac{(96C\Delta)^2}{np_1}},$$

and by the inequalities $1 + x \leq e^x$ and $p_1 = 1 - \sqrt{1 - p} \leq p$, we have

$$\geq 1 - \exp\left(-p_2\frac{(96C\Delta)^2}{n^2p_1^2}\right) \geq 1 - \exp\left(-\frac{(96C\Delta)^2}{n^2p}\right) = 1 - e^{-M}.$$

Let $q := 1 - e^{-M}$. Since $p_1 \geq p/2$, we observe that

$$m' \geq \left\lceil \frac{n^2p}{19200C\Delta} \right\rceil = m.$$

Hence, $S$ contains a random graph $G(m, q)$ as a subgraph. Since $f$ is minor-monotone, whp

$$f(R) \geq f(S) \geq f(G(m, q)). \quad (3.1)$$

If $\Delta \leq \sqrt{n^2p\log(n^2p)}$, then

$$m \geq \frac{n^2p}{19200C\Delta} = \omega(1), \quad q = 1 - \exp\left(-\frac{(96C\Delta)^2}{n^2p}\right).$$

If $\frac{(96C\Delta)^2}{n^2p} \geq 1$, then $q \geq 1 - e^{-1}$ and therefore $mq = \omega(1) > c$ for all large $n$. If $\frac{(96C\Delta)^2}{n^2p} < 1$, then as $1 - e^{-x} \geq (1 - e^{-1})x$ for all $0 < x < 1$, we have

$$mq > m(1 - e^{-1})\frac{(96C\Delta)^2}{n^2p} \geq (1 - e^{-1})\frac{962C\Delta}{1920} \geq (1 - e^{-1})\frac{962C}{1920} > c.$$
So in both cases, \( q > c/m \) for all large \( n \). Since \( f \) is \((c, r)\)-bounded below by \( \tilde{f} \) and \( q > c/m \), by Definition 5.1 whp

\[
f(G(m, q)) \geq r \cdot \tilde{f}(m, q).
\]

On the other hand, if \( \Delta \geq \sqrt{n^2p \log(n^2p)} \), then \( 1 - q \leq (n^2p)^{-406C^2} \) for all large \( n \). Since \( m^2(1 - q) \leq (n^2p)^{2-406C^2} = o(1) \), whp the random graph \( G(m, q) \) is isomorphic to \( K_m \) and by (5.1), whp

\[
f(G(m, q)) \geq f(K_m).
\]

These two inequalities with (5.1) imply the desired conclusion. \[\square\]

Now it is straightforward to obtain Theorem 1.1 from the boosting lemma.

**Theorem 1.1.** Assume that \( p = p(n) \in (0, 1) \) and \( \Delta := \Delta(n) \in [1, \infty) \) such that \( n^2p = \omega(1) \), and \( \Delta \leq n^2p/48000 \). Let \( H := H_n \) be an \( n \)-vertex connected graph with maximum degree at most \( \Delta \), and let \( R := H \cup G(n, p) \). Then whp the following hold:

(a) \( \text{tw}(R) = \Omega \left( \text{tw}(H) + \min\left(\frac{n^2p}{\Delta}, n\right) \right) \);

(b) \( g(R) = \Omega \left( g(H) + \min\left(\left(\frac{n^2p}{\Delta}\right)^2, n^2p\right) \right) \);

(c) \( h(R) = \Omega \left( h(H) + \min\left(\sqrt{\frac{n^2p}{\log \Delta}}, \frac{n^2p}{\Delta \sqrt{\log \Delta}}\right) \right) \).

**Proof.** (a) Since \( \text{tw}(R) \geq \text{tw}(H) \), it is enough to prove that \( \text{tw}(R) \geq \Omega(n^2p/\Delta) \). Let \( \tilde{t}(x, y) := x \). By Theorem 2.1 there exists \( r > 0 \) such that \( \text{tw} \) is \((5/4, r)\)-bounded from below by \( \tilde{t} \).

If \( p > 2/n \), then whp \( \text{tw}(R) \geq \Omega(n^2p/\Delta) \). Thus, we may assume that \( p \leq 2/n \). By Lemma 3.2 with \( c = 5/4, C = 5, \) and \( m := \left\lceil \frac{n^2p}{19200\log \Delta} \right\rceil = \Omega(\frac{n^2p}{\Delta}) \), whp

\[
\text{tw}(R) \geq \begin{cases} 
\begin{array}{ll}
\min\left(\frac{n^2p}{\Delta}, n\right) & \text{if } \Delta \leq \sqrt{n^2p \log(n^2p)}, \\
\text{tw}(K_m) = m - 1 = \Omega\left(\frac{n^2p}{\Delta}\right) & \text{otherwise}.
\end{array}
\end{cases}
\]

(b) Since \( g(R) \geq g(H) \), it suffices to show that \( g(R) \geq \Omega(\min(n^2p, (n^2p/\Delta)^2)) \).

If \( p \geq 2/n \), then by Theorem 2.2 there exists \( r > 0 \) such that \( g(G(n, p)) \geq r \cdot n^2p \), and thus \( g(R) = \Omega(n^2p) \).

Now we assume that \( p \leq 2/n \). Let \( \tilde{g}(x, y) := x^2y \). By Theorem 2.2 there exists \( r > 0 \) such that the genus \( g \) is \((5/4, r)\)-bounded from below by \( \tilde{g} \). Applying Lemma 3.2 with \( c = 5/4, C = 5, \) and \( m := \left\lfloor \frac{n^2p}{19200\log \Delta} \right\rfloor = \Omega(\frac{n^2p}{\Delta}), M = (96C\Delta^2)/(n^2p) \), and \( q := 1 - e^{-M} \), we deduce that whp

\[
g(R) \geq \begin{cases} 
\begin{array}{ll}
r \cdot \tilde{g}(m, q) = r \cdot m^2q = \Omega\left(\left(\frac{n^2p}{\Delta}\right)^2 q\right) & \text{if } \Delta \leq \sqrt{n^2p \log(n^2p)}, \\
g(K_m) = \Omega(m^2) = \Omega\left(\frac{n^2p}{\Delta}\right)^2 & \text{otherwise}.
\end{array}
\end{cases}
\]

Furthermore, observing \( q \geq (1 - \frac{1}{e}) \min\left(1, \frac{(96C\Delta)^2}{n^2p}\right) \), we complete the proof.

(c) Since \( h(R) \geq h(H) \), it suffices to show that \( h(R) = \Omega\left(\min\left(\sqrt{\frac{n^2p}{\log \Delta}}, \frac{n^2p}{\Delta \sqrt{\log \Delta}}\right) \right) \). We choose \( r, C' \) so that Theorem 2.3(i)–(ii) hold and define

\[
\tilde{h}(n, p) := \begin{cases} 
\frac{r \sqrt{n}}{2^{\log_2(1/(1-p))(np)}} & \text{if } \frac{5}{10} \leq p < \frac{C'}{10}, \\
\frac{r}{2^{\log_2(1/(1-p))(np)}} & \text{if } \frac{C'}{10} \leq p \leq \frac{1}{2}, \\
\frac{r}{2^{\log_2(1/(1-p))(np)}} & \text{if } \frac{1}{2} < p \leq 1.
\end{cases}
\]

(3.2)
Then \( h(G(n, p)) \geq \tilde{h}(n, p) \) for \( \frac{5}{3n} \leq p \leq 1/2 \). For \( p \geq 1/2 \), by Theorem 2.3 (ii), we have that whp \( h(G(n, p)) \geq h(G(n, 1/2)) \geq 2\sqrt{\frac{n^2p}{\log 2}} = h(n, p) \). Thus, \( h \) is \((5/4, 1)\)-bounded from below by \( \tilde{h} \).

Let \( C = 5 \), \( M = \left(\frac{96C\Delta}{n^2p}\right)^2, m = \left\lceil \frac{n^2p}{19200C\Delta} \right\rceil \), and \( q = 1 - e^{-M} \). By Lemma 3.2 we conclude that

\[
h(R) \geq \begin{cases} 
\tilde{h}(m, q) & \text{if } \Delta \leq \sqrt{n^2p\log n^2p}, \\
h(K_m) = m & \text{otherwise.} 
\end{cases}
\tag{3.3}
\]

If \( \Delta > \sqrt{n^2p\log n^2p} \), then by (3.2) and (3.3), we have that whp \( h(R) \geq m = \Omega(n^2p/\Delta) \). Thus we may assume that \( \Delta \leq \sqrt{n^2p\log n^2p} \).

If \( q \leq 1/2 \), then \( M \leq \log 2, \Delta < \sqrt{n^2p}, \) and \( \frac{mM}{2\log 2} \geq 1.5\Delta \). If \( q < C'/m \), then \( C' \geq \frac{m}{\Delta} \). This implies that whp

\[
h(R) \geq \tilde{h}(m, q) = r\sqrt{m} \geq r\sqrt{\frac{n^2p}{19200C\Delta}} > r\sqrt{\frac{1.5n^2p}{19200C\Delta}} = \Omega\left(\sqrt{\frac{n^2p}{\log 2}}\right).
\]

If \( C'/m < q \leq 1/2 \), then again by (3.2) and (3.3), we have that whp

\[
h(R) \geq \tilde{h}(m, q) = \frac{m}{2\sqrt{\log 1/(1-q)(mq)}} = \Omega\left(\sqrt{\frac{n^2p}{\log 2}}\right).
\]

Now, assume that \( q > 1/2 \). Then we have \( \Delta \geq \frac{\log 2}{96C}\sqrt{n^2p} \) and \( \Delta \geq \frac{1}{\Delta} \left(\frac{\log 2}{96C}\sqrt{n^2p}\right)^2 = \Omega(m) \). In this case, by (3.2) and (3.3), we have that whp

\[
h(R) \geq h(G(m, q)) \geq \tilde{h}(m, q) = \frac{m}{2\sqrt{\log 2}} = \Omega\left(\frac{n^2p}{\Delta\sqrt{\log 2}}\right).
\]

This completes the proof.

Now we prove Corollary 1.2 using Theorem 1.1

**Corollary 1.2.** Let \( p, \Delta, H, \text{ and } R \) be as in Theorem 1.1. Then whp \( R \) contains

(i) a cycle of length \( \Omega(n^2p/\Delta) \);

(ii) all forests on \( O(n^2p/\Delta) \) vertices as minors.

**Proof.** The first statement follows from Theorem 1.1 (a) and a result of Birmele [5] saying that every graph with tree-width at least \( k \) contains a cycle of length more than \( k \).

The second statement is an immediate consequence of Theorem 1.1 (a) and a result of Bienstock, Robertson, and Seymour [4] stating that every graph with path-width at least \( k - 1 \) contains all \( k \)-vertex forests as minors. Note that the path-width of a graph is larger than or equal to its tree-width.

\[\square\]

### 4 Perturbing a graph of small path cover number

Recall that a path cover number of a graph \( G \) is the minimum number of vertex-disjoint paths covering all vertices of \( G \). In this section, we are going to prove Theorem 1.3. Let us first prove a useful lemma to find many disjoint paths.

**Lemma 4.1.** Let \( n, m, k \in \mathbb{N} \). If \( G \) is a graph on \( n \) vertices whose path cover number is at most \( m \), then it contains at least \( n/k - m \) vertex-disjoint paths each having exactly \( k \) vertices.
Proof. Let $G$ be an $n$-vertex graph and let $P_1, P_2, \ldots, P_m$ be vertex-disjoint paths covering all vertices of $G$. For each path $P_i$, we choose as many vertex-disjoint subpaths on exactly $k$ vertices as possible. Then all but $m(k-1)$ vertices of $V(G)$ can be covered by vertex-disjoint paths on $k$ vertices, hence there are at least
\[
\frac{n - m(k - 1)}{k} \geq \frac{n}{k} - m
\]
vertex-disjoint paths on exactly $k$ vertices.

Let us restate Theorem 1.3 and present its proof.

**Theorem 1.3.** Assume that $p = p(n) \in [0, 1]$ satisfies $n^2 p = \omega(1)$. Let $H := H_n$ be a (not necessarily connected) $n$-vertex graph whose path cover number is at most $n^2 p/20$. Let $G := G(n, p)$ and $R := H \cup G$. Then the following hold:

(a) $\text{tw}(R) = \Theta(\text{tw}(H) + \min(n^2 p, n))$;

(b) $g(R) = \Theta(g(H) + n^2 p)$;

(c) $h(R) = \begin{cases} 
\Omega(h(H) + \sqrt{n^2 p}) & \text{if } np < \frac{11}{20}, \\
\Omega(h(H) + h(G)) & \text{otherwise}.
\end{cases}$

**Proof.** Suppose that $p \geq \frac{11}{20}$, By Theorems 2.1 and 2.2, whp $\text{tw}(G) \geq \Omega(n)$ and $g(G) \geq \Omega(n^2 p)$. Then (a) holds because the tree-width of an $n$-vertex graph is at most $n - 1$ and $\text{tw}(R) \geq (\text{tw}(H) + \text{tw}(G))/2$. Since adding one edge to a graph can increase the genus by at most 1, we have $(g(H) + g(G))/2 \leq g(R) \leq g(H) + O(n^2 p)$ whp, proving (b). Observe that $h(R) \geq (h(H) + h(G))/2$ and therefore (c) holds.

Hence, we may assume that $p < \frac{11}{20}$. Let us first prove upper bounds for (a) and (b). Since adding one edge to a graph can increase the genus and the tree-width by at most 1, we have $\text{tw}(R) \leq \text{tw}(H) + O(n^2 p)$ and $g(R) \leq g(H) + O(n^2 p)$ whp and trivially $\text{tw}(R) \leq n - 1$, proving upper bounds for (a) and (b).

For lower bounds, as $\text{tw}(R) \geq \text{tw}(H)$, $g(R) \geq g(H)$, and $h(R) \geq h(H)$, it suffices to show that
\[
\text{tw}(R) \geq \Omega(\min(n^2 p, n)),
\]
\[
g(R) \geq \Omega(n^2 p),
\]
\[
h(R) \geq \Omega(\sqrt{n^2 p}).
\]

Let $k$ be an integer such that $\frac{29}{10np} \leq k \leq \frac{4}{np}$, and let $m := \lceil \frac{n^2 p}{20} \rceil$. Observe that such an integer $k$ exists because $p < \frac{11}{20}$. Furthermore, $m \geq \frac{n^2 p}{20}$. By Lemma 4.1 $H$ has at least $m$ vertex-disjoint paths $P_1, \ldots, P_m$, each having exactly $k$ vertices. For each $1 \leq i < j \leq m$, the probability that $R$ has an edge between $P_i$ and $P_j$ is
\[
q := 1 - (1 - p)^{k^2} \geq k^2 p - \frac{k^2(k^2 - 1)}{2} p^2 \geq pk^2 \left(1 - \frac{pk^2}{2}\right) \geq pk^2 \left(1 - \frac{16}{n^2 p}\right).
\]

Let $G' := G/E(P_1)/E(P_2)/ \cdots /E(P_m)$, which is a graph obtained from $G$ by contracting edges in $\bigcup_{i=1}^{m} E(P_i)$. Then $G'$ is a minor of $R$ and $G'$ contains $G(m, q)$ as a subgraph. Thus $f(R) \geq f(G(m, q))$ for any minor-monotone graph parameter $f(\cdot)$.

Since $n^2 p = \omega(1)$ and $m \geq \frac{n^2 p}{20}$, for all sufficiently large $n$, we have
\[
mq \geq \frac{n^2 p}{5} pk^2 \left(1 - \frac{16}{n^2 p}\right) \geq \frac{n^2 p}{5} \cdot \left(\frac{29}{10}\right)^2 \frac{1}{n^2 p} \left(1 - \frac{16}{n^2 p}\right) > \frac{5}{4}.
\]
Hence, by Theorems 2.2 and 2.3 whp
\[ \text{tw}(R) \geq \text{tw}(G(m, q)) \geq \Omega(m) \geq \Omega(n^2p), \]
\[ g(R) \geq g(G(m, q)) \geq \Omega(m) \geq \Omega(n^2p), \]
\[ h(R) \geq h(G(m, q)) \geq \Omega(\sqrt{m}) \geq \Omega(\sqrt{n^2p}), \]
as desired.

5 Finding many large vertex-disjoint connected subgraphs

We begin with a lemma based on the following classical result of Ajtai, Komlós, and Szemerédi [1] and Fernandez de la Vega [11].

**Lemma 5.1.** For every \( n \in \mathbb{N} \), the random graph \( G(n, 20/n) \) contains a path on at least \( n/5 \) vertices with probability at least \( 1 - e^{-n} \).

For completeness, we present a proof that is based on the idea of the proof in [23, Theorem 3.4 in Chapter 1]. We will use the following lemma.

**Lemma 5.2** (Ben-Eliezer, Krivelevich, and Sudakov [3, Lemma 4.4]). Let \( n, k \in \mathbb{N} \) and \( G \) be an \( n \)-vertex graph such that for every pair \( (X, Y) \) of disjoint sets of \( k \) vertices, there is at least one edge joining \( X \) and \( Y \). Then \( G \) has a path whose length is at least \( n - 2k + 1 \).

**Proof of Lemma 5.2** We may assume that \( n > 5 \). Let \( G := G(n, 20/n) \) and let \( k := \lceil 2n/5 \rceil \). Then \( k \leq 2n/5 + 1 \) and \( n - 2k + 2 \geq n/5 \). By the union bound, the probability that there exist two disjoint subsets \( X,Y \subseteq V(G) \) of size \( k \) with no edges between \( X \) and \( Y \) is at most
\[
\left( \frac{n}{k} \right)^k \left( 1 - \frac{20}{n} \right)^{k^2} \leq \left( \frac{en}{k} \right)^{2k} \cdot e^{-\frac{20k^2}{n}}
= \left( \frac{en}{k} \cdot e^{-\frac{10k^2}{n}} \right)^{2k} \leq \left( \frac{5e}{2} e^{-4} \right)^{2k} \leq \left( \frac{5}{2e^3} \right)^{4k/n} < e^{-n}.
\]
Thus, with probability at least \( 1 - e^{-n} \), there exists a path on at least \( n/5 \) vertices in \( G \) by Lemma 5.2.

Now we are going to prove lemmas, which allow us to find many large vertex-disjoint connected subgraphs of \( G(n, p) \).

**Lemma 5.3.** Let \( p = p(n) \in (0, 1] \), let \( u > 0 \) be a real number, and let \( G := G(n, p) \). If \( C = \{X_1, \ldots, X_m\} \) is a nonempty set of disjoint subsets of \( V(G) \) such that \( |X_i| \geq u \) for all \( 1 \leq i \leq m \), then with probability at least \( 1 - m \exp(-pu^2) \), \( G \) has an edge between \( X_i \) and \( X_{i+1} \) for all \( 1 \leq i < m \).

**Proof.** For each \( j \in [m-1] \), the probability that there are no edges of \( G \) between \( X_j \) and \( X_{j+1} \) is
\[
(1 - p)^{|X_j| |X_{j+1}|} \leq (1 - p)^{u^2} \leq \exp(-pu^2).
\]
Therefore the probability that there are no edges of \( G \) between \( X_j \) and \( X_{j+1} \) for some \( j \in \{1, 2, \ldots, m-1\} \) is at most \( m \exp(-pu^2) \) by the union bound.

**Lemma 5.4.** Let \( p = p(n) \in (0, 1] \), let \( u > 0 \) be a real number, and let \( G := G(n, p) \). If \( C \) is a nonempty set of disjoint subsets of \( V(G) \) such that
\begin{itemize}
  \item[(a)] \( |X| \geq u \) for all \( X \in C \), and
  \item[(b)] \( |C| \geq \max(40, 40/(pu^2)) \).
\end{itemize}
then with probability at least \(1 - e^{-|C|}\), the set \(C\) contains \(|C|/5\) distinct sets \(X_1, X_2, \ldots, X_m\) such that \(G\) has an edge between \(X_i\) and \(X_{i+1}\) for every \(1 \leq i < m\).

**Proof.** Let \(q := 1 - (1-p)u^2\), and let \(H\) be an auxiliary random graph such that \(V(H) = C\) and \(\{X, Y\} \in E(H)\) if and only if there exists an edge of \(G\) between \(X\) and \(Y\) for distinct \(X, Y \in C\). Then, the probability that there are no edges of \(G\) between \(X\) and \(Y\) is

\[
(1-p)^{|X|Y|} \leq (1-p)^{uw} = 1 - q \leq \exp(-pu^2),
\]

so we can regard \(G(|C|, q)\) as a subgraph of \(H\), coupling \(G(|C|, q)\) and \(H\). Now we claim that \(q \geq 20/|C|\). To see this, if \(pu^2 \geq \log 2\) then \(q \geq 1 - \exp(-pu^2) \geq 1/2 \geq 20/|C|\), as \(|C| \geq 40\). Otherwise if \(pu^2 < \log 2\), then \(q \geq 1 - \exp(-pu^2) \geq pu^2/2 \geq 20/|C|\), as \(|C| \geq 40/(pu^2)\).

Thus, by Lemma 5.1 with probability at least \(1 - e^{-|C|}\), \(H\) has a path on at least \(|C|/5\) vertices. Equivalently, with probability at least \(1 - e^{-|C|}\), there exist \(m \geq |C|/5\) distinct sets \(X_1, \ldots, X_m \in C\) such that \(G\) has an edge between \(X_i\) and \(X_{i+1}\) for every \(1 \leq i < m\), as desired. □

**Lemma 5.5.** Let \(k\) be a positive real. Let \(G\) be a graph and let \(T_1, T_2, \ldots, T_m\) be vertex-disjoint connected subgraphs of \(G\) such that \(V(G) = \bigcup_{i=1}^m V(T_i)\) and \(|V(T_i)| \leq k\) for every \(i \in [m]\). If \(G\) has an edge from \(T_i\) to \(T_{i+1}\) for every \(i \in [m-1]\), then \(G\) contains vertex-disjoint connected subgraphs \(R_1, R_2, \ldots, R_m\) such that \(k \leq |V(R_i)| \leq 2k\) for all \(i \in [m]\) and \(|V(G)| - \sum_{i=1}^m |V(R_i)| < k\).

**Proof.** We proceed by induction on \(|V(G)|\). The statement is trivial if \(|V(G)| < k\). Thus, we may assume that \(|V(G)| \geq k\) and therefore \(m \geq 1\). Let \(i\) be the minimum positive integer such that \(\sum_{j=1}^i |V(T_j)| \geq k\). By the assumption, \(\sum_{j=1}^i |V(T_j)| < 2k\). Let \(R_1 = T_1 \cup T_2 \cup \cdots \cup T_i\). We now apply the induction hypothesis to \(G - \bigcup_{j=1}^{i-1} V(T_j)\) to find the remaining \(R_2, R_3, \ldots, R_m\). □

6 Proof of the Key Lemma

In order to prove Lemma 1.4, we will first present a lemma which allows us to obtain many vertex-disjoint connected subgraphs of \(V(H \cup G(n, p))\) of size between \(\Omega(1/(np))\) and \(O(\Delta/(np))\), which cover almost all vertices in \(V(H \cup G(n, p))\). A similar lemma with essentially the same proof appears in several papers; for example, [22, Proposition 4.5] and [8, Lemma 3].

**Lemma 6.1** (Tree partitioning lemma). Let \(\ell \geq 2\), \(\Delta \geq 1\) be reals. Every connected graph \(G\) with the maximum degree at most \(\Delta\) admits a partition \(V_0, V_1, \ldots, V_s\) of its vertex set with \(s \geq 0\) such that

1. \(|V_0| < \ell\),
2. \(G[V_i]\) is connected for all \(0 \leq i \leq s\),
3. \(\ell \leq |V_i| < \ell \Delta\) for all \(1 \leq i \leq s\).

**Proof.** We proceed by induction on \(|V(G)|\). We may assume that \(G\) is a tree by taking a spanning tree. We may assume that \(|V(G)| \geq \ell\) because otherwise, we can take \(V_0 := V(G)\). If \(\Delta < 2\), then \(2 \leq \ell \leq |V(G)| \leq 2 < \ell \Delta\) and therefore we can take \(V_1 := V(G)\). Thus, we may assume that \(\Delta \geq 2\). Fix a vertex of degree 1 as a root. For each vertex \(v\) of \(T\), let \(T_v\) be the subtree of \(T\) rooted at \(v\) consisting of \(v\) and all of its descendants. Let \(w\) be a vertex such that \(|V(T_w)| \geq \ell\) and the distance from the root to \(w\) is maximum. Then \(w\) has at most \(\Delta - 1\) children. (Note that this is still true even if \(w\) is the root because \(1 \leq \Delta - 1\).) By the maximality of the distance from the root to \(w\), for every child \(u\) of \(w\), we have \(|V(T_u)| < \ell\) and therefore \(|V(T_w)| < (\Delta - 1)\ell + 1 \leq \ell \Delta\). Now \(G' := G - V(T_w)\) is connected and we apply the induction hypothesis to \(G'\) to find a partition \(V_0, V_1, \ldots, V_s\) of \(V(G')\). We take \(V_{s+1} := V(T_w)\). □

Now we restate our key lemma to be proved in this section.
Lemma 1.4 (Key lemma). Let \( C \geq 8 \) be an absolute constant. Let \( p = p(n) \in (0, 1) \) and \( \Delta = \Delta(n) \in [1, \infty) \) be the parameters satisfying \( p \leq 2/n, n^2p = \omega(1), \) and \( \Delta \leq \frac{n^2p}{\log^2 n} \). Let \( H := H_n \) be an \( n \)-vertex connected graph with maximum degree at most \( \Delta \) and let \( R := H \cup G(n, p) \). Then \( \text{whp} R \) contains vertex-disjoint connected subgraphs \( R_1, \ldots, R_m \) such that

1. \( 96\frac{C\Delta}{np} \leq |V(R_i)| \leq 192\frac{C\Delta}{np} \) for each \( i \in [m] \);

2. \( m \geq \frac{n^2p}{19200C} \).

We first sketch our proof strategy. First, by applying Lemma 6.1, we obtain many vertex-disjoint connected subgraphs of \( H \), called clusters, each having at least \( \ell \) vertices and at most \( \ell \Delta \) vertices. Our goal is to merge them using edges of \( G(n, p) \) to obtain many vertex-disjoint connected subgraphs, each having a similar number of vertices required by (1). To this end, we will conduct the following four steps.

S1 (Dyadic decomposition). We partition the set of clusters into levels such that the number of vertices in each cluster is within a factor 2 of each other in the same level.

S2 (Connecting clusters in each level). For each level in the dyadic decomposition, whp we can find a linear ordering of many clusters so that \( G(n, p) \) has an edge between consecutive clusters; see Claim 1.

S3 (Connecting cluster between consecutive levels). Discarding some clusters in each linear ordering, whp we can concatenate all linear orderings into a single linear ordering, where the union of clusters in the linear ordering contains \( \Omega(n) \) vertices and \( G(n, p) \) has an edge between consecutive clusters in the ordering; see Claim 2.

S4 (Merging consecutive clusters into connected subgraphs). We merge consecutive clusters into connected subgraphs on \( \Theta(\Delta/(np)) \) vertices to obtain \( \Theta(n^2p/\Delta) \) vertex-disjoint connected subgraphs.

Proof of Lemma 1.4 Let \( n \) be sufficiently large and \( \ell := 96\frac{C}{np} \). As \( \ell > 2 \), by Lemma 6.1, there is a collection \( \mathcal{F} \) of disjoint sets \( X_1, \ldots, X_s \subseteq V(H) \) satisfying the following.

(a) \( \sum_{i=1}^s |X_i| > |V(H)| - 96\frac{C}{np} \);

(b) \( H[X_i] \) is connected for each \( i \in [s] \);

(c) \( 96\frac{C}{np} \leq |X_i| < 96\frac{C}{np} \) for each \( i \in [s] \).

We call each \( X_i \) a cluster. For \( 1 \leq i \leq \lceil \log_2 \Delta \rceil \), let

\[
V_i := \{ S \in \mathcal{F} : 2^{i-1}\ell \leq |S| < 2^i\ell \}, \quad V_i := \bigcup_{S \in V_i} S, \quad u_i := 2^{i-1}\ell, \quad n_i := |V_i|,
\]

and

\[
c_i := \max \left( \frac{80}{nu_i}, \frac{n}{50 \log_2(n^2p)} \right) = \max \left( \frac{5n}{6 \cdot 2^{-i-1}C}, \frac{n}{50 \log_2(n^2p)} \right). \tag{6.1}
\]

Then, \( \bigcup_{i=1}^{\lceil \log_2 \Delta \rceil} V_i = \mathcal{F} \).

We define \( A := \{ i : 1 \leq i \leq \lceil \log_2 \Delta \rceil, |V_i| \geq c_i \} \) and call \( i \in A \) a level. Then there are at least \( n/2 \) vertices in \( \bigcup_{i \in A} V_i \), because

\[
\sum_{i \in A} |V_i| \leq \sum_{i=1}^{\lceil \log_2 \Delta \rceil} \frac{5n}{6 \cdot 2^{-i-1}C} + \frac{n}{50 \log_2(n^2p)} \cdot \lceil \log_2 \Delta \rceil \leq \frac{5n}{3C} + \frac{n}{50} \leq \frac{5n}{12}. \tag{6.2}
\]
where we use $C \geq 8$ for the last two inequalities. Since each element of $\mathcal{V}_i$ has less than $2u_i$ vertices, for each $i \in \mathcal{A}$,
\[
n_i \geq \frac{|V_i|}{2u_i} > \frac{c_i}{2u_i} > \frac{40}{p u_i^2} = \frac{40}{p n^2} \geq \frac{40}{p} \geq \frac{40}{p} \geq 40 n^2 p \\
100 u_i \log_2 (n^2 p) > (n^2 p)^{1/3} \log_2 (n^2 p) > 40
\]

Let $G := G(n, p)$.

**Claim 1.** Whp, for every $i \in \mathcal{A}$, there exist $m_i = [n_i/5]$ distinct sets $S_{i,1}, \ldots, S_{i,m_i} \in \mathcal{V}_i$ such that there is an edge of $G$ between $S_{i,j}$ and $S_{i,j+1}$ for every $j \in [m_i - 1]$.

**Proof of Claim 1.** If $u_i \geq (n/p)^{1/3}$, then $n_i = |\mathcal{V}_i| \leq n/4u_i \leq (n^2 p)^{1/3}$ and $p u_i^2 \geq (n^2 p)^{1/3}$. Then by Lemma 5.3, with probability at least $1 - (n^2 p)^{1/3} e^{-(n^2 p)^{1/3}}$, $G$ has an edge between $S_{i,j}$ and $S_{i,j+1}$ for every $1 \leq j < n_i$, where $S_{i,1}, \ldots, S_{i,n_i}$ is a fixed ordering of the subsets in $\mathcal{V}_i$.

If $u_i < (n/p)^{1/3}$, then $n_i = |\mathcal{V}_i| > |\mathcal{V}_i| / 2 > c_i / 2u_i$ because $|S| < 2u_i$ for every $S \in \mathcal{V}_i$. Since $c_i \geq 80 / (p u_i)$, we have $n_i \geq 40 / (p u_i^2)$. In addition, as $c_i \geq n / 50 \log_2 (n^2 p)$, we have
\[
n_i \geq \frac{n}{100 u_i \log_2 (n^2 p)} > \frac{(n^2 p)^{1/3} \log_2 (n^2 p)}{100 \log_2 (n^2 p)} > 40
\]
because $n^2 p = \omega(1)$. Thus, by Lemma 5.4, with probability at least $1 - e^{-n_i} \geq 1 - \exp(- (n^2 p)^{1/3} \log_2 (n^2 p))$, there exist $m_i = [n_i/5]$ distinct sets $S_{i,1}, \ldots, S_{i,m_i} \in \mathcal{V}_i$ such that there exists an edge of $G$ between $S_{i,j}$ and $S_{i,j+1}$ for every $j \in [m_i - 1]$.

Thus, since $|\mathcal{A}| \leq |\log_2 \Delta| \leq \log_2 (n^2 p)$, the claim follows with probability at least
\[
1 - |\mathcal{A}| \max \left( (n^2 p)^{1/3} e^{-(n^2 p)^{1/3}}, e^{- \frac{(n^2 p)^{1/3}}{\log_2 (n^2 p)}} \right) = 1 - o(1)
\]
by the union bound. $
\square$

We now fix the edges of $G[\mathcal{V}_i]$ for all $i \in \mathcal{A}$ satisfying Claim 1. Then for each level $i \in \mathcal{A}$, let $S_{i,1}, \ldots, S_{i,m_i} \in \mathcal{V}_i$ be $m_i$ distinct subsets such that there is an edge of $G$ going between $S_{i,j}$ and $S_{i,j+1}$. For each $i \in \mathcal{A}$, let $\mathcal{V}_i' := \{S_{i,1}, \ldots, S_{i,m_i}\}$ and $\mathcal{V}' := \bigcup_{j=1}^{m_i} S_{i,j}$. As $n^2 p = \omega(1)$, we may assume that $\ell = \frac{96^2 C}{np} \leq \frac{1}{36}$. Since each $S \in \mathcal{V}_i$ has size at most $2u_i$ and $m_i \geq n_i/5$, (6.1) and (6.2) imply
\[
|\mathcal{V}_i'| \geq \frac{1}{10} |\mathcal{V}_i| \geq \frac{n}{50 \log_2 (n^2 p)}, \quad (6.3)
\]
\[
\sum_{i \in \mathcal{A}} |\mathcal{V}_i'| \geq \frac{1}{10} \left( n - \ell - \sum_{i \in \mathcal{A}} |\mathcal{V}_i| \right) \geq \frac{n}{18}, \quad (6.4)
\]

**Claim 2.** Whp there exist distinct $T_{i_1}, \ldots, T_{i'} \in \mathcal{F}$ satisfying the following.

(1) $\sum_{i \in \mathcal{S}_i} |\mathcal{V}_i| \geq n/20$.

(2) For each $j \in [\mathcal{S}_i - 1]$, there exists an edge of $G$ between $T_j$ and $T_{j+1}$.

**Proof of Claim 2.** So far, we have fixed the edges of $G[\mathcal{V}_i]$ for all $i \in \mathcal{A}$. In this claim, we will only use the edges of $G[\mathcal{V}_i, \mathcal{V}_i']$ for distinct $i, i' \in \mathcal{A}$. This will ensure that each edge of $G$ is exposed at most once.

For each $i \in \mathcal{A}$, let $a_i$ be the minimum such that $L_{i}^1 := \bigcup_{j=1}^{a_i} S_{i,j}$ has at least $\frac{n}{20} |\mathcal{V}_i'|$ vertices. Similarly, let $b_i$ be the maximum such that $L_{i}^2 := \bigcup_{j=b_i}^{a_i} S_{i,j}$ has at least $\frac{n}{20} |\mathcal{V}_i'|$ vertices. Then $a_i \leq b_i$. To see this, if $a_i \geq b_i + 1$, then
\[
|\mathcal{V}_i'| = \sum_{k=1}^{m_i} |S_{i,k}| = \sum_{k=1}^{b_i-1} |S_{i,k}| + |S_{i,b_i}| + \sum_{k=b_i+1}^{m_i} |S_{i,k}| < |\mathcal{V}_i'|/20 + |S_{i,b_i}| + |\mathcal{V}_i'|/20,
\]

since $b_i - 1 \leq a_i - 2$. Thus, $|S_{i,b_i}| > 9|V'_i|/10$, which contradicts that $a_i$ is the minimum such that $\bigcup_{j=1}^{b_i} S_{i,j}$ has at least $|V'_i|/20$ vertices, as $\bigcup_{j=1}^{a_i} S_{i,j}$ already has at least $9|V'_i|/10$ vertices.

Clearly, $\sum_{j=1}^{b_i} |S_{i,j}| \geq |V'_i| - \sum_{j=1}^{a_i} |V'_i| - \sum_{j=1}^{a_i} |V'_i| \geq \frac{9}{10} |V'_i|$ for each $i \in A$.

Then by (6.3), we have $|L'_i| \geq \frac{n}{10000 \log_2(n^2p)}$. For $i, i' \in A$ with $i < i'$, exposing the edges of $G[V_i, V_{i'}]$, the probability that $G$ has no edge between $L'_i$ and $L'_i$ is at most

$$(1 - p)|L'_i||L'_i| \leq \exp \left(-\frac{n^2p}{10000 \cdot (\log_2(n^2p))^2}\right).$$

Let $A = \{i_1, \ldots, i_t\}$ with $i_1 < \cdots < i_t$. As $|A| \leq \log_2 \Delta \leq \log n^2p$ and $n^2p = o(1)$, a union bound implies that whp there is an edge of $G$ between $L'_i$ and $L'_j$ for all $j \in [t - 1]$. Hence whp, for each $j \in \{t\}$, there exist $b_i \leq \beta_j \leq m_i$ and $1 \leq \alpha_{j+1} \leq a_{j+1}$ such that $G$ has an edge between $S_{i_j, \beta_j}$ and $S_{i_{j+1}, \alpha_{j+1}}$.

(6.5)

Let $T_1, \ldots, T_{s'}$ be the following sequence of subsets in $F$

$$S_{i_1,1}, \ldots, S_{i_1, \beta_1}, S_{i_2, \alpha_2}, S_{i_2, \alpha_2+1}, \ldots, S_{i_2, \beta_2}, S_{i_3, \alpha_3}, \ldots, S_{i_{t-1}, \beta_{t-1}}, S_{i_t, \alpha_t}, S_{i_t, \alpha_t+1}, \ldots, S_{i_t, m_t}.$$ 

Then we have

$$\sum_{j=1}^{s'} |T_j| \geq \sum_{i \in A} \frac{9}{10} |V'_i| \geq \frac{n}{20},$$

and by (6.5), $G$ has an edge between $T_j$ and $T_{j+1}$ for all $j \in [s'-1]$, as desired.

Let $T_1, \ldots, T_{s'} \in F$ be subsets provided by Claim 2. For each $1 \leq i \leq s'$, the graph $H[T_i]$ is a connected subgraph of $H$ of at most $\ell \Delta$ vertices, and whp there exists an edge of $G$ between $T_j$ and $T_{j+1}$ for every $1 \leq j < s'$.

Recall that $R = H \cup G$. By Lemma 5.5, $R[T_1 \cup \cdots \cup T_{s'}]$ contains vertex-disjoint connected subgraphs $R_1, \ldots, R_m$ where $\ell \Delta \leq |V(R_i)| < 2\ell \Delta$ for all $i \in [m]$ and

$$m \cdot (2\ell \Delta) \geq \sum_{j=1}^{m} |V(R_i)| > \sum_{i=1}^{s'} |T_i| - \ell \Delta \quad \text{Claim 2} \quad \geq \frac{n}{20} - \ell \Delta.$$

Thus, since $\ell = \frac{96C}{np}$, we have

$$m \geq \frac{1}{2\ell \Delta} \left(\frac{n}{20} - \ell \Delta\right) = \frac{n^2p}{3840C \Delta} - \frac{1}{2} \geq \frac{n^2p}{9600C \Delta},$$

where the last inequality follows from $\Delta \leq n^2p(4800C)^{-1}$. This completes the proof of Lemma 1.4.

## 7 Sharpness of the results

In this section, we will show that the results are best possible. To this end, we need the following two lemmas, the first of which is straightforward.

**Lemma 7.1.** Let $G$ be a graph, and let $V_1, \ldots, V_t \subseteq V(G)$ be disjoint subsets such that

- $G[V_i]$ is a tree for each $i \in [t]$;
- there is at most one edge between $V_i$ and $V_j$ for every pair of distinct $i, j \in [t]$.

and let $G^*$ denote the graph obtained from $G$ by contracting each $V_i$ to a vertex. If $G^*$ is a forest, then so is $G$. 

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Lemma 7.2. Let $C > 0$ be an absolute constant, let $p = p(n) \in [0, 1]$, and let $1 \leq x = x(n) \leq n$ be an integer-valued function such that $x = o(n)$ and $npx = o(1)$. Let $t$ be an integer such that $t \leq Cn/x$, and let $B_1, B_2, \ldots, B_t$ be vertex-disjoint trees on at most $x$ vertices. Let $H_0$ be the disjoint union of $B_1, \ldots, B_t$, and let $R_0 := H_0 \cup G(n, p)$. Then whp $R_0$ is a forest.

Proof. We first claim that whp $R_0[B_i]$ is a tree for each $i \in [t]$. It suffices to prove that whp no edge of $G(n, p)$ lies inside $V(B_i)$ for any $i \in [t]$. To show this, for each $i \in [t]$ let $\mathcal{E}_i$ denote the event that there is an edge of $G(n, p)$ that lies inside $V(B_i)$; then $\mathbb{P}(\mathcal{E}_i) \leq px^2/2$, because the expected number of edges of $G(n, p)$ that lie inside $V(B_i)$ is at most $p|V(B_i)|^2/2 \leq px^2/2$. Thus, by a union bound we have

$$\mathbb{P}\left(\bigcup_{i=1}^t \mathcal{E}_i\right) \leq t \cdot \frac{px^2}{2} \leq \frac{Cnpx}{2} = o(1).$$

Next, we claim that whp $R_0$ has at most one edge between $B_i$ and $B_j$ for all $i \neq j \in [t]$. Equivalently, we show that whp there exists at most one edge of $G(n, p)$ between $B_i$ and $B_j$ for all $i \neq j \in [t]$. For any distinct pairs $(u_1, v_1), (u_2, v_2) \in V(B_i) \times V(B_j)$, let $\mathcal{E}_{u_1, v_1, u_2, v_2}$ be the event that both $u_1v_1$ and $u_2v_2$ appear in $G(n, p)$. Then, by a union bound we obtain

$$\mathbb{P}\left(\bigcup_{(u_1, v_1) \neq (u_2, v_2)} \mathcal{E}_{u_1, v_1, u_2, v_2}\right) \leq |V(B_i)|^2|V(B_j)|^2 p^2 \leq x^4 p^2.$$

Therefore the probability that $G(n, p)$ has at least two edges between $B_i$ and $B_j$ is at most $x^4 p^2$.

Since there are $\binom{t}{2}$ unordered pairs $i \neq j \in [t]$, the probability that $G(n, p)$ has two edges between $B_i$ and $B_j$ for some $i \neq j \in [t]$ is at most

$$\binom{t}{2} x^4 p^2 \leq \frac{1}{2} \left(\frac{Cn}{x}\right)^2 x^4 p^2 = \frac{1}{2} (Cnp)^2 = o(1).$$

Let $R^*_0$ be the graph obtained from $R_0 := H_0 \cup G(n, p)$ by contracting $B_i$ into a single vertex $v_i$ for each $i \in [t]$. Because the probability that $G(n, p)$ has an edge between $B_i$ and $B_j$ is at most

$$1 - (1 - p)^{|B_i||B_j|} \leq 1 - (1 - p)^{|B_i||B_j|} = p|B_i||B_j| \leq px^2,$$

we have $v_iv_j \in E(R^*_0)$ with probability at most $px^2 \leq npx = o(1)$. Since $R^*_0$ has at most $t$ vertices, the expected number of cycles of length $i$ in $R^*_0$ is at most

$$t^i \cdot (px^2)^i = (tpx^2)^i \leq (Cnp)^i.$$

Therefore, the expected number of cycles in $R^*_0$ is at most $\sum_{i \geq 3} (Cnp)^i = o(1)$. Thus, whp $R^*_0$ is a forest, and by Lemma [7.1] whp $R_0$ is also a forest. 

7.1 Examples

The following two examples show that it is necessary to assume $\Delta = O(n^2p)$ in Theorem 1.1 when $p \leq \varepsilon/n$ for some absolute constant $\varepsilon \in (0, 1)$.

Example 7.3. Assume $p = \Omega(1/n)$ with $p \leq \varepsilon/n$ for some $\varepsilon \in (0, 1)$. Let $H$ be an $n$-vertex star. Then we have $\Delta(H) = n - 1$ and $p = \varepsilon/n$ and whp $G(n, p)$ is outerplanar, and thus $H \cup G(n, p)$ is planar.

Example 7.4. Assume $\Delta = \omega(n^2p)$ and $p = o(1/n)$. Let $c(n)$ be a function with $c = c(n) = \omega(1)$, and assume that $\omega(n^2p) = \omega(1)$ and $p \leq 1/(cn)$. Let $cn^2p/4 \leq t = t(n) \leq cn^2p/2$ be an integer. Then $t \leq n/2$. Let $B_1, \ldots, B_t$ be vertex-disjoint stars on either $\left\lfloor \frac{n-1}{t} \right\rfloor$ or $\left\lceil \frac{n-1}{t} \right\rceil$ vertices such that each $B_i$ has a center vertex $v_i$, and

$$|V(B_1)| + \cdots + |V(B_t)| = n - 1.$$
Let \( x = x(n) = \lceil \frac{4}{cn} \rceil \). Then \( |V(B_i)| \leq \lfloor n/t \rfloor \leq x \) for each \( i \in [t] \). Since \( \frac{4}{cn} \geq 4 \), we have \( x \leq \frac{4}{cn} \) and so \( t \leq \frac{n^2p}{4} \leq \frac{n}{c} \). Let \( H \) be an \( n \)-vertex rooted tree obtained from the disjoint union of \( B_1, \ldots, B_t \) by adding a root vertex \( r \) that is adjacent to \( r_1, \ldots, r_t \). Let \( R := H \cup G(n, p) \). As \( p \leq 1/(cn) \), \( n^2p = \omega(1) \), and \( x \leq 5/(cnp) \), we have

\[
1 \leq x \leq \frac{5n}{c} \frac{1}{n^2p} = o(n) \quad \text{and} \quad npx \leq \frac{5}{c} = o(1).
\]

By Lemma\(^7\) we deduce that whp \( R - r \) is a forest. Thus, whp \( \text{tw}(R) \leq \text{tw}(R - r) + 1 \leq 2, g(R) = 0 \), and \( h(R) \leq h(R - r) + 1 \leq 3 \).

The following two examples show that Theorem\(^1\) is best possible for tree-width and the Hadwiger number up to \( \log(n^2p) \) factor when \( p \leq \varepsilon/n \) for some absolute constant \( \varepsilon \in (0, 1) \).

**Example 7.5.** Let \( \Delta \geq 3 \) be an integer. Assume \( p = \Omega(1/n) \) with \( p \leq \varepsilon/n \) for some \( \varepsilon \in (0, 1) \). Let \( H \) be an \( n \)-vertex tree obtained from a path on \( \lceil n/\Delta \rceil \) vertices by attaching either \( \Delta - 1 \) or \( \Delta - 2 \) leaves to each vertex of the path. Then the maximum degree of \( H \) is either \( \Delta \) or \( \Delta + 1 \). Let \( R := H \cup G(n, p) \).

We claim that whp

\[
\text{tw}(R) \leq 3 + n/\Delta \quad \text{and} \quad h(R) \leq 4 + n/\Delta.
\]

Let \( L \) be the set of leaves in \( H \). It is well known\(^{10} \) that whp every connected component of \( R[L] \) in \( G(n, p) \) has at most one cycle, hence \( \text{tw}(R[L]) \leq 2 \). Since deleting a vertex can decrease the tree-width by at most 1, whp \( \text{tw}(R) \leq \text{tw}(R[L]) + \lceil n/\Delta \rceil \leq 3 + n/\Delta \). The upper bound on \( h(R) \) follows from the inequality that \( h(R) \leq \text{tw}(R) + 1 \).

**Example 7.6.** Assume \( p = o(1/n) \). Let \( c = c(n) \) be a function with \( c(n) = \omega(1) \) and \( 3 \leq \Delta = O(n^2p) \) is an integer. We also assume that \( n^2p = \omega(1) \) and \( p \leq 1/(cn) \).

Let \( cn^2p/4 \leq t = t(n) \leq cn^2p/2 \) be an integer. Then \( t \leq n/2 \). Let \( P \) be a path on \( \lceil t/\Delta - 2 \rceil \) vertices and let \( v_1, \ldots, v_{\lfloor t/(\Delta - 2) \rfloor} \) be the vertices of \( P \). Let \( B_1, \ldots, B_t \) be trees on either \( \lceil n - |V(P)| \rceil \) or \( \lceil n - |V(P)| \rceil \) vertices, each having maximum degree at most 3 and satisfying

\[
|V(B_1)| + \cdots + |V(B_t)| = n - |V(P)|.
\]

Let \( x = x(n) = \lceil \frac{4}{cn} \rceil \). Then \( |V(B_i)| \leq \lfloor n/t \rfloor \leq x \) for each \( i \in [t] \). Since \( \frac{4}{cn} \geq 4 \), we have \( x \leq \frac{4}{cn} \) and so \( t \leq \frac{n^2p}{4} \leq \frac{n}{c} \). For each \( i \in [t] \), let \( u_i \in V(B_i) \) be a leaf of \( B_i \). Now we partition \( [t] \) into \( |V(P)| = \lfloor t/(\Delta - 2) \rfloor \) sets, \( I_1, \ldots, I_{\lfloor t/(\Delta - 2) \rfloor} \) such that \( |I_i| \leq \Delta - 2 \) for each \( 1 \leq i \leq \lfloor t/(\Delta - 2) \rfloor \).

Let \( H \) be an \( n \)-vertex tree obtained from the disjoint union of \( B_1, B_2, \ldots, B_t \) and \( P \) by adding edges \( v_iu_j \), for all \( 1 \leq i \leq \lfloor t/(\Delta - 2) \rfloor \) and \( j \in I_i \).

Then \( B_1, \ldots, B_t \) are the connected components of \( H - V(P) \) and the maximum degree of \( H \) is either \( \Delta - 1 \) or \( \Delta \), since \( |I_i| = \Delta - 2 \) for some \( i \in [t] \). Moreover, as \( p \leq 1/(cn) \), \( n^2p = \omega(1) \), and \( x \leq 5/(cnp) \), we have

\[
1 \leq x \leq \frac{5n}{c} \frac{1}{n^2p} = o(n) \quad \text{and} \quad npx \leq \frac{5}{c} = o(1).
\]

Let \( R := H \cup G(n, p) \). By Lemma\(^7\) we deduce that whp \( R - V(P) \) is a forest. We may assume that \( n \) is sufficiently large so that \( cn^2p/\Delta > 1 \), because \( c = \omega(1) \) and \( \Delta = O(n^2p) \). Hence whp \( R \) satisfies

\[
\text{tw}(R) \leq |V(P)| + \text{tw}(R - V(P)) \leq \left\lceil \frac{3t}{\Delta} \right\rceil + 1 \leq \left\lceil \frac{3cn^2p}{2\Delta} \right\rceil + 1 \leq \frac{3cn^2p}{\Delta} + 1,
\]

\[
h(R) \leq \text{tw}(R) + 1 \leq \frac{3cn^2p}{\Delta} + 2.
\]
7.2 Sharpness of Theorem 1.1

Examples 7.3–7.6 provide best possible bounds for genus, tree-width, and Hadwiger number for various ranges of \(p\) and \(\Delta\).

1. Examples 7.3 and 7.4 show that the maximum degree bound \(\Delta = O(n^2p)\) is necessary for Theorem 1.1.

2. The lower bound in Theorem 1.1(a) is best possible by Examples 7.5 and 7.6; one cannot improve Theorem 1.1(a) to obtain the tree-width \(\omega(n^2p/\Delta)\) when \(p \leq \varepsilon/n\) for some \(\varepsilon \in (0, 1)\).

3. The lower bound in Theorem 1.1(b) is best possible when \(\Delta = O(\sqrt{n^2p}).\) Indeed, since \(G(n, p)\) has \(O(n^2p)\) edges whp and the genus increases by at most one when adding an edge, it follows that \(g(R) \leq g(H) + O(n^2p) = \Theta(\max(g(H), n^2p))\) whp.

4. The bound in Theorem 1.1(c) is best possible up to logarithmic factor in \(n\). Consider a connected graph \(H\) of genus \(O(n^2p).\) Since \(g(R) \leq \Theta(\max(g(H), n^2p))\) whp and the genus of \(K_k\) is \(\Theta(k^2),\) whp we have

\[
h(R) \leq \Theta\left(\sqrt{\max(g(H), n^2p)}\right) = O(\sqrt{n^2p}).
\]

When \(\Delta = O(1),\) this shows that Theorem 1.1(c) is best possible. When \(\Delta = \Omega(\sqrt{n^2p\log n^2p})\), we can consider the graph \(H\) in Example 7.6 which ensures whp \(h(R) = \Theta(n^2p/\Delta)\) and shows that Theorem 1.1(c) is best possible.

7.3 Sharpness of Theorem 1.3

The following observation shows that Theorem 1.3 would fail if the path cover number of \(H\) is \(\omega(n^2p)\).

**Observation 7.7.** Let \(c = c(n)\) be a function with \(c(n) = \omega(1)\), and let \(p = p(n)\) be a function with \(p \leq 1/(cn)\) and \(n^2p = \omega(1).\) Then there exists an \(n\)-vertex graph \(H\) with the path cover number at most \(O(cn^2p)\) and maximum degree \(o(n^2p)\) such that whp both \(\text{tw}(H \cup G(n, p))\) and \(h(H \cup G(n, p))\) are at most \(o(n)\).

**Proof.** Let \(c^* = c^*(n)\) be a slowly increasing function such that \(c^* = \omega(1),\) \(c^* = o(n^2p),\) and \((c^*)^2 = o(c)\). Let \(\Delta^* = \lceil n^2p/c^* \rceil,\) and let \(H\) be an \(n\)-vertex tree in Example 7.6 such that \(R_1, \ldots, R_t\) are paths, and \(c^*\) and \(\Delta^*\) play the roles of \(c\) and \(\Delta\) respectively. Note that \(p \leq 1/(cn) \leq 1/(c^*n)\) for all sufficiently large \(n\).

Then \(H\) has maximum degree at most \(\Delta^* = o(n^2p)\) and has exactly \(t = \Theta(c^*n^2p)\) leaves. Thus, the path cover number of \(H\) is \(\Theta(c^*n^2p)\) (see [15 Claim 1.2]), which is at most \(O(cn^2p).\) However, whp both \(\text{tw}(H \cup G(n, p))\) and \(h(H \cup G(n, p))\) are at most \(\frac{3c^*n^2p}{\Delta} + 2 = o(c),\) as \((c^*)^2 = o(c).\)

\[\Box\]

7.4 Sharpness of Lemma 1.4

The key idea of Lemma 1.4 is to obtain \(m\) vertex-disjoint connected subgraphs \(R_1, \ldots, R_m\) so that even though we begin with a random graph \(G(n, p)\) in a subcritical regime \(np < \varepsilon\) for some absolute constant \(\varepsilon \in (0, 1)\), whp \(H \cup G(n, p)\) contains \(G(m, q)\) in a supercritical regime \(mq > 1 + \varepsilon'\) as a minor, for some absolute constant \(\varepsilon' > 0\) and some \(q = 1 - (1 - p)^{-\Theta(m^2)}\).

Since \(mq = \Omega(1)\) implies \(m = O(n^2p),\) one might think that the result \(m = \Theta(n^2p/\Delta)\) in Lemma 1.4 is weak if \(\Delta = \omega(1)\). However, we claim that the condition \(m = \Theta(n^2p/\Delta)\) is essentially best possible, as follows.

**Observation 7.8.** Let \(p = p(n) \leq \varepsilon/n\) for some absolute constant \(\varepsilon \in (0, 1)\) with \(n^2p = \omega(1)\). Let \(\zeta = \omega(1)\) be sufficiently smaller than \(n^2p\), and let \(\Delta = O(n^2p)\) which is sufficiently larger than \(\zeta\).
If \( p = \Omega(1/n) \), then let \( H \) be the \( n \)-vertex tree in Example 7.5 and let \( p' = \frac{1+\varepsilon}{2} - p \). Otherwise if \( p = o(1/n) \) then let \( H \) be the \( n \)-vertex tree in Example 7.6 where \( c = c(n) \) is chosen to satisfy \( c = o(\zeta) \), and let \( p' = p \).

For \( m = \Theta(\zeta n^2 p/\Delta) \), whp \( H' = H \cup G(n, p) \) does not have \( m \) vertex-disjoint connected subgraphs \( R_1, \ldots, R_m \) with \( |V(R_i)| = \Theta(n/m) \) for all \( i \in [m] \).

**Proof.** Observe that \( p' = \Theta(p) \) and \( p + p' \leq \frac{1+\varepsilon}{2} \) for \( \frac{1+\varepsilon}{2} \in (0, 1) \). For the contrary, suppose that whp \( H' = H \cup G(n, p) \) has \( m \) vertex-disjoint connected subgraphs \( R_1, \ldots, R_m \) with \( |V(R_i)| = \Theta(n/m) \) for all \( i \in [m] \). Let \( p^* = 1 - (1-p)(1-p') \leq p+p' \). We regard \( H \cup G(n, p^*) \) as a minor obtained by contracting these connected subgraphs. This strat-}

Thus, we may consider \( H' \cup G(n, p') \) containing \( G(m, q) \) as a minor. By Theorem 2.1 whp \( \text{tw}(G(m, q)) = \Omega(m) \), and therefore

\[
\text{tw}(H \cup G(n, p^*)) = \Omega(m) = \Omega \left( \frac{\zeta n^2 p}{\Delta} \right). \tag{7.2}
\]

Note that \( p^* \leq p + p' \leq \frac{1+\varepsilon}{2} \). If \( p^* = \Omega(1/n) \), then Example 7.5 shows that whp \( \text{tw}(H \cup G(n, p^*)) = O(n^2 p/\Delta) \), contradicting (7.2). On the other hand, if \( p^* = o(1/n) \) then Example 7.6 shows that whp \( \text{tw}(H \cup G(n, p^*)) = O(cn^2 p/\Delta) \), contradicting (7.2) as we chose \( c = o(\zeta) \).

8 Discussions

In Theorem 1.1b, if the maximum degree \( \Delta \) of the base graph \( H \) is \( O(\sqrt{n^2 p}) \), then the lower bound \( \Omega(n^2 p) \) for the genus of \( R \) is best possible. However, we could not prove whether our bound is best possible or can be improved when \( \Delta = \Omega(\sqrt{n^2 p}) \).

**Problem 1.** Determine the asymptotic behaviour of \( g(R) \) in Theorem 1.1c when \( \Delta = \omega \left( \sqrt{n^2 p} \right) \).

To obtain the lower bound in Theorem 1.1c, we found many connected subgraphs of comparable sizes and estimated the genus of a minor obtained by contracting these connected subgraphs. This strategy is effective for \( \Delta = O \left( \sqrt{n^2 p} \right) \), as each of these connected subgraphs contains only a few edges, which does not significantly affect our estimation. However, when \( \Delta = \Omega \left( \sqrt{n^2 p} \right) \), there may be many edges in each of these connected subgraphs, suggesting that a novel method is needed to account for these edges.

Finally, Lemma 3.2 deals with fragile minor-monotone graph parameters when a base graph is perturbed by a binomial random graph \( G(n, p) \). It can be generalized in two ways: When a base graph is perturbed by a random bipartite graph and by a signed graph; this will be carried out in [19].
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Appendix. Minor-monotone graph parameters of random graphs

The following table summarizes interesting results of minor-monotone graph parameters of random graphs.

| Parameters         | Values in $G(n, p)$ (whp) | Range of $p$                                      | Ref |
|--------------------|----------------------------|---------------------------------------------------|-----|
| Treewidth $tw$     | $tw \leq 2$                | $p \leq c/n \ (0 < c < 1)$                         |     |
|                    | $tw = \Theta(n)$           | $p \geq c/n \ (c > 1)$                            | [24]|
| Genus $g$          | $g = 0$                    | $p = n^{-1} - \omega(n^{-4/3})$                    | [25]|
|                    | $g = (1 + o(1))c^4n^4/3$   | $p = n^{-1} + c$ and $n^{-4/3} \ll s \ll n^{-1}$ | [9] |
|                    | $g = (1 + o(1))\mu(c)n^2p/2$| $np = c$ and $c > 1$, where $\lim_{c \to 1} \mu(c) = 0$ and $\lim_{c \to \infty} \mu(c) = 1/2$ | [9] |
|                    | $(1 - o(1))n^2p/4 \leq g \leq n^2p/4$| $1 \ll np = n^{o(1)}$                           |     |
|                    | $(1 + o(1)) \max \left(\frac{1}{12}, \frac{(1-1)}{4n^2+1}\right) n^2p$| $np = \Theta(n^{1/2})$                         | [32]|
|                    | $\leq g \leq (1 + o(1))\frac{(n^2)}{4n^2+2}$| $n^{\frac{1}{2}+\delta} \ll np \ll n^{\frac{1}{2}} (j \in \mathbb{N})$ | [32]|
|                    | $g = (1 + o(1))\frac{n^2}{n^2+p/12}$| $n^{\frac{1}{2}+\delta} \ll np \ll n^{\frac{1}{2}} (j \in \mathbb{N})$ | [32]|
| Hadwiger number $h$| $h = \Theta(c(n)^{3/2})$ | $p = n^{-1} + c(n)n^{-4/3}$, where $c(n) = \omega(1)$ but $c(n) = o(n^{1/3})$ | [13]|
|                    | $\delta(c)\sqrt{n} \leq h \leq 2\sqrt{cn}$ for some constant $\delta(c)$ | $p \geq c/n$ and $c > 1$                          | [12]|
|                    | $\frac{(1-\varepsilon)n}{\log_{1/(1-p)}(np)} \leq h \leq \frac{(1+\varepsilon)n}{\log_{1/(1-p)}(np)}$| $C(\varepsilon)/n \leq p \leq 1 - \varepsilon$ | [7,12]|

Table 1: Summary of minor-monotone parameters of random graphs.