Some Comments on $\mathcal{N} = 1$ Gauge Theories from Wrapped Branes

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Abstract

We discuss various aspects of gauge theories realized on the world-volume of wrapped branes. In particular we analyze the coupling of SYM operators to space-time fields both in $\mathcal{N} = 1$ and $\mathcal{N} = 2$ models and give a description of the gluino condensate in the Maldacena-Nuñez $\mathcal{N} = 1$ solution. We also explore the seven-dimensional BPS equations relevant for these solutions and their generalizations.

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1 Introduction

$\mathcal{N} = 1$ and $\mathcal{N} = 2$ SYM theories can be obtained as the low energy limit of wrapped five-branes in Type IIB superstring theory. In both cases, the five-branes are wrapped on a supersymmetric cycle of the ambient geometry and a partial twist \cite{1, 2} of the world-volume theory is necessary to preserve some supersymmetry. The amount of surviving supersymmetry depends on how the cycle is embedded in the ambient geometry. Information about the quantum field theory can be obtained with a geometric engineering approach \cite{3}. In AdS/CFT context, solutions for both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ pure SYM in four dimensions with gauge group $SU(N)$ have been found by exploiting seven dimensional gauged supergravity \cite{4, 5, 6}. The $\mathcal{N} = 2$ solution correctly reproduces the one-loop $\mathcal{N} = 2$ effective action but it is singular at the scale where, in QFT, instantonic corrections take over. This IR singularity could be possibly solved by some modifications of the enhançon mechanism \cite{7}. On the other hand, the $\mathcal{N} = 1$ Maldacena-Nuñez (MN) solution, upon inclusion of non-abelian background fields, is completely regular.

In this note, we address some aspects of these constructions. All solutions are asymptotic to the linear dilaton background, which is holographically dual to the little string theory associated with the five-branes. It makes sense then to ask whether and to which extent the AdS/CFT rules apply to such solutions. We will show that one can make a reasonable identification of dual operator-fields and asymptotic behaviors. In particular, in the $\mathcal{N} = 1$ case, we find that the space-time deformation which de-singularizes the MN solution is dual to the gaugino condensate. The asymptotic behavior of the latter can be explicitly computed and it exhibits the expected dependence on the QFT parameters. Our results closely parallels those found for the Klebanov-Strassler (KS) solution \cite{8, 9}. We also classify all operators that are involved in generalized solutions such as, for example, the $\mathcal{N} = 2 \to \mathcal{N} = 1$ breaking. Some relevant BPS equations for this case are collected in the Appendices, leaving the explicit solution for future work. Finally, we also explicitly write and solve the BPS equations for the MN solution in seven dimensions. They are written and solved in many papers in literature, but, curiously, never in the natural setting, which is seven dimensional gauged supergravity.

2 The relevant operators

We consider the general setting of wrapped five-branes with world-volume $R^4 \times S^2$, which can be easily adapted to describe both $\mathcal{N} = 1$ and $\mathcal{N} = 2$ theories. In the case of N flat NS5 branes the world-volume theory is a (1,1) six-dimensional SYM. The $SO(4)_R$ R-symmetry acts on the transverse space $R^4$, whose directions we label with an index $i = 1, ..., 4$. As well known, there are no covariantly constant spinors on $S^2$. Some
supersymmetry is preserved only if an abelian background field in $SO(4)$ is turned on in order to cancel the spin connection on $S^2$. From the schematic formula for the variation of a fermion:

$$\delta \Psi \sim D_\mu \epsilon = (\partial_\mu + \omega_\mu^\nu \gamma^\nu - A_\mu^{ij} \Gamma^{ij}) \epsilon,$$

we see that the surviving spinors are those satisfying the twist condition:

$$(\omega_\mu^\nu \gamma^\nu - A_\mu^{ij} \Gamma^{ij}) \epsilon = 0.$$

The relevant $U(1)$ fields are those of the decomposition $SO(4) \rightarrow U(1) \times U(1)$, $A^{(1)}$ and $A^{(2)}$. In the $\mathcal{N} = 2$ solution [4, 5] only $A^{(1)}$ is turned on, while in the $\mathcal{N} = 1$ MN solution [4], both $A^{(1)}$ and $A^{(2)}$ are turned on.

Let us discuss first the $\mathcal{N} = 2$ case. The relevant type IIB space-time fields are described by the $SO(4)$ gauged supergravity in seven dimensions [10]. The bosonic Lagrangian and the fermionic shifts are written in Appendix A. The gauged supergravity has a vacuum corresponding to a set of coincident five-branes. The theory contains ten scalar fields parameterizing a symmetric matrix $T_{ij}$. Before twisting, they are dual to the bilinear operator $T_{rX_iX_j}$, where $X_i$ are the six dimensional scalar fields. Upon compactification on $S^2$ and related twist, $X_1$ and $X_2$ get masses. The massless complex scalar $\phi = X_3 + i X_4$ parameterizes the $\mathcal{N} = 2$ moduli space. The $\mathcal{N} = 2$ solutions that have been found in [5, 6] always involve a diagonal $T_{ij}$ and only three scalars. In seven dimensions they have the following form [5, 6]:

$$\begin{align*}
    ds_7^2 &= e^{2f} (dx_4^2 + d\rho^2) + e^{2g} (d\theta^2 + \sin^2 \theta d\phi^2), \\
    A^{(1)} &= \frac{1}{2} \cos \theta d\phi, \\
    T_{ij} &= \text{diag}(e^{2\lambda_1}, e^{2\lambda_2}, e^{2\lambda_3}),
\end{align*}$$

where $g, f$ and $\lambda_i$ are functions of the radial coordinate $\rho$ [5, 6]. Moreover these 7d solutions can be explicitly lifted to ten dimensions [5, 6]. The effective action for a probe in these backgrounds can be written in manifest $\mathcal{N} = 2$ language with coupling constant:

$$\begin{align*}
    \text{Solution I: } \tau(z) &= \frac{Ni}{\pi} \log \frac{z}{\Lambda}, \\
    \text{Solution II: } \tau(z) &= \frac{Ni}{\pi} \left( \text{Arcosh} \frac{z}{2b} + \text{const} \right).\n\end{align*}$$

Solution I corresponds to $\lambda_2 = \lambda_3$, while in Solution II all the three scalars are different. As discussed in [5, 11], the presence of $\lambda_2, \lambda_3$ indicates that the operators $\text{Tr}\phi\bar{\phi}$ and $\text{Tr}\phi^2$ are turned on. These solutions indeed describe points in the Coulomb branch of the $\mathcal{N} = 2$ theory. $\text{Tr}\phi\bar{\phi}$ and $\text{Tr}\phi^2$ are dual to $\lambda_2 + \lambda_3$ and $\lambda_2 - \lambda_3$ respectively. This
is consistent with the fact that the rotationally invariant Solution I has $\lambda_2 = \lambda_3$ [1]. A detailed discussion of the asymptotic behavior of the fields can be found in [1].

Let us consider the fermionic fields. The six dimensional theory has fermions $\Psi = \Psi^+ + \Psi^-$ transforming in the representation $(4, 2) + (4', 2')$ of $SO(5, 1) \times SO(4)_R$. We have $\Psi^\pm = \pm \gamma_7 \Psi^\pm$. We choose the following basis for the six dimensional gamma matrices

$$\gamma^\mu = \gamma_{(4)}^\mu \otimes 1, \quad \gamma^{\theta, \phi, 7} = \gamma_{(4)}^5 \otimes \sigma_{123},$$

with conjugation matrix $C^{(6)} = C^{(4)} \otimes \sigma_2$. We write the $SO(4) = SU(2)^+ \otimes SU(2)^-$ action on spinors in a basis of sigma matrices for $\Psi^\pm$,

$$\Gamma^{12} \pm \Gamma^{34} = 2i\sigma_3^\pm, \quad \Gamma^{24} \pm \Gamma^{31} = 2i\sigma_1^\pm, \quad \Gamma^{14} \pm \Gamma^{23} = -2i\sigma_2^\pm. \quad (6)$$

Notice that the previously defined $A_{(1)}$ and $A_{(2)}$ are related to the $U(1)$ subgroups of $SU(2)^\pm$ by a change of basis. Since both $SO(5, 1)$ and $SO(4)_R$ act on spinors, a symplectic-Majorana condition can be imposed on the fermions, reducing the supersymmetry to the expected 16 real supercharges. The symplectic-Majorana condition reads $\Psi^\alpha = C_{(6)} \gamma_0^+ \Omega^\alpha \beta \Psi^\beta$, where the symplectic matrix is $\Omega^\pm = i\sigma_2^\pm$ for each $SU(2)^\pm$ group.

It is convenient to write the six dimensional spinors as two-by-two matrices on which the Lorentz $\sigma$’s (associated with the spin connection on $S^2$) act on the left and the gauge $\sigma^\pm$ on the right. With these conventions, the six dimensional fermions are given by:

$$\Psi = \Psi^+ \oplus \Psi^-, \quad \Psi^+ = \begin{pmatrix} p & q \\ iq^c & -ip^c \end{pmatrix}, \quad \Psi^- = \begin{pmatrix} \tilde{p} & \tilde{q} \\ iq^c & -ip^c \end{pmatrix}. \quad (7)$$

From $\gamma^7 = \gamma_{(4)}^5 \otimes \sigma_3$ we see that $p$ and $q$ are four dimensional Weyl spinors of positive chirality while $\tilde{p}$ and $\tilde{q}$ are four dimensional Weyl spinors of negative chirality.

| $\sigma_1$ | $\sigma_2$ | $\sigma_3$ | $\sigma_4$ |
|------------|------------|------------|------------|
| $U(1)_R = U(1)_{(2)}$ | 1 | -1 | -1 | 1 |
| $U(1)_J = U(1)_{S^2}$ | 1 | 1 | 1 | 1 |
| $U(1)_{(1)}$ | 1 | 1 | -1 | -1 |

Table 1: Charge assignment of the spinors.

The twisted $U(1)$ in the $N = 2$ solution is $\Gamma^{12} = i(\sigma_3^+ + \sigma_3^-)$ and the twist condition reads:

$$i\sigma_3^+ \Psi^+ = \gamma^{\theta, \phi} \Psi^+, \quad i\sigma_3^- \Psi^- = \gamma^{\theta, \phi} \Psi^-.$$

(8)
The spinors \( p \) and \( \tilde{p} \), which have the same charges under the twisted \( U(1) \) and the \( S^2 \) generator \( \gamma^{\theta\phi} \), remain massless in four dimensions. Analogously \( q \) and \( \tilde{q} \) are massive four dimensional fields. The four dimensional theory has an \( SU(2)_R \times U(1)_R \) R-symmetry. The two chiral spinors in the \( \mathcal{N} = 2 \) vector multiplet, \( \psi \) and \( \lambda \) (this one being the gaugino), have different charges under \( U(1)_R \) and \( U(1)_J \) (the abelian generator of \( SU(2)_R \)) \[12\]. While \( \psi \) has charges +1 and −1 respectively, \( \lambda \) has charge +1 under both of them. In the supergravity language \( U(1)_R \) is the untwisted \( U(1)_{(2)} \) \[5\]\[6\], so in our notation it is generated by \( \Gamma^{34} = i(\sigma_3^+ - \sigma_3^-) \). On the other hand \( U(1)_J \) is the isometry of the sphere on which we are wrapping the branes, so it is generated by \( \gamma^{\theta\phi} \). Now, it is easy to verify that \( p \) and \( \tilde{p} \) have both charge +1 with respect to \( \gamma^{\theta\phi} \), and charges +1 and −1 respectively under \( \Gamma^{34} \), so that we can safely identify them with \( \lambda \) and \( \bar{\psi} \).

We are interested in two special operators, the bilinears \( \lambda \lambda \) and \( \bar{\psi} \psi \). They can be easily obtained from the coupling:

\[
A^{\mu}_{ij} \bar{\Psi} \gamma^\mu \Gamma^{ij} \Psi. \tag{9}
\]

We choose the combination of gauge fields:

\[
A_\theta = \frac{1}{2} a \eta^+_2 + \frac{1}{2} \tilde{a} \eta^-_2,
\]

\[
A_\phi = \frac{1}{2} a \sin \theta \eta^+_3 + \frac{1}{2} \tilde{a} \sin \theta \eta^-_3. \tag{10}
\]

The \( \eta \) matrices are the generators of the \( SU(2)^\pm \) in the \( SO(4) \) notation and take the form:

\[
\eta^+_1 = \frac{1}{2} \begin{pmatrix}
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0 \\
0 & 0 & 0 & \pm 1 \\
0 & 0 & \mp 1 & 0
\end{pmatrix},
\]

\[
\eta^-_2 = \frac{1}{2} \begin{pmatrix}
0 & 0 & \mp 1 & 0 \\
0 & 0 & 0 & 1 \\
\pm 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 0
\end{pmatrix},
\]

\[
\eta^+_3 = \frac{1}{2} \begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 0 & \pm 1 & 0 \\
0 & \mp 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}. \tag{11}
\]

The calculation is straightforward. The \( q \) and \( \tilde{q} \) spinors get opposite contributions from the \( \theta \) and \( \phi \) components of \( (9) \) and they do not survive in the reduction. We then find that \( a \) and \( \tilde{a} \) couple to fermionic bilinears:

\[
a \bar{\lambda} \lambda, \quad \tilde{a} \bar{\psi} \psi. \tag{12}
\]

These two fermionic bilinears are necessary ingredients for the soft breaking \( \mathcal{N} = 2 \to \mathcal{N} = 1 \) and for the study of the gaugino condensate in \( \mathcal{N} = 1 \) theories.

### 3 The gaugino condensate

The previous formalism is easily adapted to \( \mathcal{N} = 1 \) theories and the MN solution. The relevant twisted \( U(1) \) field is now \( \sigma^+_3 = -i(\Gamma^{12} + \Gamma^{34})/2 \), so that both \( A^{(1)} \) and \( A^{(2)} \) are
turned on. The twist condition is now:

\[ i\sigma_3^+\Psi^+ = \gamma^{\theta\phi}\Psi^+. \]  

(13)

From this equation (see also Table 1), we easily see that the only massless fermionic field is \( p \sim \lambda \). The MN solution is reviewed in Appendix B. The fields that are turned on are the metric (then the fields \( f, g \) of equation (3)), the non-abelian gauge fields:

\[ A = \frac{1}{2}[\cos \theta d\phi \eta_1^+ + a(\rho) d\theta \eta_2^+ + a(\rho) \sin \theta d\phi \eta_3^+], \]

(14)

and the singlet scalar \( \lambda = \lambda_1 = \lambda_2 = \lambda_3 \). The solution reads:

\[ e^{2h} = \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}; \quad a = \frac{2\rho}{\sinh 2\rho}; \quad e^{10\lambda} = \frac{2e^h}{\sinh 2\rho}, \]

(15)

where \( h = g - f \). The ten dimensional solution describing N NS five-branes in the string frame is:

\[ ds_{st}^2 = dx_{(4)}^2 + \alpha' N [d\rho^2 + e^{2h(\rho)} (d\theta^2 + \sin^2 \theta d\phi^2)] + ds_{S^5}^2. \]

(16)

This solution is an appropriate description in the UV (\( \rho \gg 1 \)). In the IR the appropriate description is the one with D5-branes. In the UV the non-abelian gauge fields are asymptotically zero, and the behavior of the MN solution can be described by the following simpler solution of the BPS equations with only the abelian component of the gauge fields (the one defining the twist):

\[ e^{2h} = \rho, \quad 5\lambda = \frac{1}{4} \log \rho - \rho. \]

(17)

We are now ready to discuss the gaugino condensate in the MN solution. We see from formula (12) that the particular combination of non-abelian gauge fields \( a \) in the MN solution is dual to the gaugino condensate. It is remarkable that the space-time field that de-singularizes the solution is associated with the condensate, i.e. with the non-trivial IR dynamics of the \( \mathcal{N} = 1 \) SYM theory. A closely related situation has been noticed for the KS solution.

By adapting AdS/CFT rules, we can determine the value of the gaugino condensate from the asymptotic behavior of the dual supergravity field. We face various conceptual problems. The decoupling of SYM scale from the six-dimensional little string scale requires to go beyond the supergravity approximation. Moreover, for related reasons, the identification of the radial parameter with the energy scale in the MN solution is difficult and ambiguous. We nevertheless obtain a reasonable result by using a Born-Infeld analysis. From the BI action for a stack of D5-branes wrapped over a two-sphere we determine the four dimensional gauge coupling:

\[ \frac{1}{g_{YM}^2} = \frac{\tau_5}{2}(2\pi\alpha')^2 \int_{S^2} e^{-\Phi} \sqrt{g_{\mu\nu}g_{\phi\phi}}, \]

(18)
where \( \tau_5 = (2\pi)^{-5}(\alpha')^{-3} \) and where we used the conventions (see for example [15]):

\[
L = -\frac{1}{4g_Y^2} (F_{\mu
u}^a)^2 = -\frac{1}{2g_Y^2} Tr(F_{\mu\nu}F^{\mu\nu}).
\] (19)

From the UV behavior of the metric and the dilaton for the D5 solution (perform an S-duality in (16)) we can thus read the UV behavior of the gauge coupling:

\[
\frac{1}{g_{YM}^2} = \frac{N}{4\pi^2} \rho,
\]

which, compared with the gauge theory result

\[
\frac{1}{g_{YM}^2} = \frac{3N}{8\pi^2} \log(\mu/\Lambda),
\]

gives the radius/energy relation:

\[
\rho = \frac{3}{2} \log(\mu/\Lambda). \tag{20}
\]

The field \( a \) vanishes in the UV as \( a \sim \rho e^{-2\rho} \), that is, from (20), as:

\[
a \sim \left( \frac{\Lambda^3}{\mu^3} \right) \log \frac{\mu}{\Lambda}. \tag{21}
\]

This result has the expected dependence on the scale \( \Lambda \), since the gaugino condensate is an operator with (protected) dimension three. Moreover, under the chiral symmetry \( Z_{2N} \), which is a subgroup of \( U(1)_R \) [4], the condensate transforms with a phase, as expected [4]. A similar result was found for the KS solution [4]. In the AdS/CFT philosophy [16], the two independent solutions of the asymptotic second order equations of motion for a field are associated in QFT with a deformation by the dual operator (the dominant solution) and with a different vacuum of the theory where the dual operator has a VEV (the subdominant solution). Using the explicit second order equations for \( a \) in the asymptotic background, written for example in [17], one can explicitly check that the BPS equations single out the behavior of \( a \) appropriate for a condensate.

## 4 Conclusions

The analysis in Section 2 is useful for identifying the fields that are needed for more general solutions, such as the breaking \( \mathcal{N} = 2 \rightarrow \mathcal{N} = 1 \). We expect that both \( SU(2)^\pm \) non-abelian gauge fields are turned on, as in formula (10), \( a \) representing the gaugino condensate and \( \tilde{a} \) the soft breaking mass term for fermions. The corresponding mass term for the scalars should be \( (\lambda_2 + \lambda_3)/2 \). We can expect both operators \( Tr\phi^2 \) and \( Tr\phi\bar{\phi} \) to have a VEV [12, 18] in QFT. On the supergravity side this should correspond

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1. The gaugino condensate is BPS related to the tension of a domain wall, which can be easily computed in this model [3] using the IR form of the metric. It is however difficult to compare the two results.

2. In that case, the logarithmic correction was associated with the logarithmically varying AdS radius in the UV. In our case, the asymptotic solution should decompactify to six-dimensions. The fact that we nevertheless get a sensible four dimensional result could be related to a topological nature of the twisted theory on the two-sphere [3].
to a solution where all the fields ($\lambda_1, \lambda_2, \lambda_3, a, \tilde{a}$) are turned on. The BPS equations for this case are written in Appendix A. We leave their resolution for future work. It is not clear that a solution exists, because there are more equations than independent variables. Since we cannot rely on any consistent truncation argument, a more general ansatz for the scalar and gauge fields could be required. We notice, however, that a simplified model with $\lambda_2 = \lambda_3$ and $\beta = 0$ admits a solution, despite the fact that the number of equations is redundant. Even if this solution is IR singular, we could interpret its existence as a signal for the existence of the (expected) more general one. Details are presented in Appendix C.

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Appendix A: General equations

The relevant supergravity is the seven dimensional $SO(4)$ gauged supergravity [10]. It can be obtained by the maximally supersymmetric $SO(5)$ gauged supergravity performing a suitable singular limit on the scalars [19, 6]. We use the notation of [20]. The relevant terms of the bosonic lagrangian are:

$$2\kappa^2 e^{-1}\mathcal{L} = R + \frac{1}{2} m^2 (T^2 - 2 T_{ij} T^{ij}) - \text{Tr}(P_{\mu} P^\mu) - \frac{1}{2} (V_i V_j F_{ij}^\mu)^2.$$  \hspace{1cm} (22)

In the expression above, the signature of space-time is mostly plus, and $I$ and $i$ are respectively the gauge and composite $SO(4)$ indices. $V_i^I$ is the symmetric matrix for the ten scalar degrees of freedom parameterizing the $SL(4, R)/SO(4)$ coset space, and the $T$ matrix is defined as $T_{ij} = V_i^{-1} V_j^{-1} \delta_{IJ}$, $T = T_{ij} \delta_{ij}$. The kinetic term for the scalars $P_{\mu}$ is the symmetric part of $V_i^{-1} \mathcal{D}_{\mu} V_i^J = (Q_{\mu})_{[ij]} + (P_{\mu})_{(ij)}$, where the covariant derivatives are defined as $\mathcal{D}_{\mu} V^j_i = \partial_{\mu} V^j_i + 2 m A^j_i V^j_i$ on the scalars, $\mathcal{D}_{\mu} \psi = (\partial_{\mu} + \frac{1}{4} Q_{\mu ij} \Gamma^{ij} + \frac{1}{4} \omega^{\nu\mu\lambda} \gamma^\nu \gamma^\lambda) \psi$ on the spinors.

With an $SO(4)_R$ gauge rotation, $T_{ij}$ can be always diagonalized. In this appendix we write down the general formulae for the supersymmetry variations of fermions with only three diagonal scalars:

$$V_i^I = \text{diag}(e^{-\lambda_1}, e^{-\lambda_1}, e^{-\lambda_2}, e^{-\lambda_3}).$$  \hspace{1cm} (23)
We take the \( SO(4) = SU(2)^+ \times SU(2)^- \) gauge fields of the form:

\[
A = \alpha [\cos \theta \, d\phi \, \eta_1^+ + a(\rho) \, d\theta \, \eta_2^+ + b(\rho) \, \sin \theta \, d\phi \, \eta_3^+] + \\
\beta [\cos \theta \, d\phi \, \eta_1^- + \tilde{a}(\rho) \, d\theta \, \eta_2^- + \tilde{b}(\rho) \, \sin \theta \, d\phi \, \eta_3^-].
\]

(24)

The field strength is normalized as \( F = dA + 2m[A,A] \) and the \( \eta \) matrices are given in (11). The ansatz for the metric (in the Einstein frame) is:

\[
ds_7^2 = e^{2f}(dx_4^2 + d\rho^2) + e^{2g}(d\theta^2 + \sin^2 \theta d\phi^2).
\]

(25)

In this paper we do not use different notations for curved and flat indices. To pass from the former to the latters we must multiply \( \gamma_\phi, \gamma_\theta, \gamma_\rho, \chi \) by the inverse vielbein (\( \chi = 0, 1, 2, 3 \) labels the four dimensional coordinates). From (25) it follows that the non-trivial components of the spin connection are:

\[
\omega^{\lambda \rho} = f', \quad \omega^{\theta \rho} = g' e^{g-f}, \quad \omega^{\phi \rho} = g' e^{g-f} \sin \theta, \quad \omega^{\phi \theta} = \cos \theta,
\]

(26)

where the prime denotes a derivation with respect to \( \rho \).

The general form of the supersymmetry variations can be found in [20]:

\[
\delta \psi_\mu = \left[ D_\mu + \frac{1}{4} \gamma_\mu \gamma^\nu V_{i}^{-1} \partial_\nu V_{i}^{+} + \frac{1}{4} \Gamma_{i}^{ij} F_{\mu \lambda} \gamma^\lambda \right] \epsilon,
\]

\[
\delta (\Gamma_i \lambda_i) = \left[ \frac{m}{2} (T_{ij} - \frac{1}{5} T \delta_{ij}) \Gamma_i^{ij} + \frac{1}{2} \gamma_\mu P_{\mu ij} \Gamma_i^{ij} + \frac{1}{16} \gamma^{\mu \nu} (\Gamma_i^{kl} \Gamma_i^{kl} - \frac{1}{5} \Gamma^{kl} F_{\mu \nu}^{kl}) \right] \epsilon.
\]

(27)

with \( F_{\mu \lambda}^{ij} = V_{i}^{+} V_{j} F_{\mu \lambda}^{kl} \). Notice that the index \( \hat{i} \) is not summed over. Being the spinor \( \epsilon \) charged under \( SU(2)^+ \times SU(2)^- \), we separate the two components in \( \epsilon = \epsilon^+ \oplus \epsilon^- \). If we want to preserve \( \mathcal{N} = 1 \) supersymmetry, we can concentrate only on \( \epsilon^+ \). There are some constraints which come from the dependence on \( \theta \) in the fermionic variations. From the \( \frac{\cos \theta}{\sin \theta} \) terms in the gravitino shift, one gets \( b = 2 \, m \, \alpha \, a \) and \( \tilde{b} = 2 \, m \, \beta \, \tilde{a} \). The contribution in \( \cos \theta \) in the gravitino shift gives the twist condition:

\[
\left[ \gamma^{\phi \theta} + m[(\alpha + \beta) + \frac{1}{2} (\alpha - \beta) (e^{\lambda_2 - \lambda_3} + e^{\lambda_3 - \lambda_2})] i \sigma_3^+ \right] \epsilon^+ = 0.
\]

(28)

Eq. (28) takes a more illuminating form in the two interesting cases, a) pure \( \mathcal{N} = 1 \) with \( \beta = 0 \) and \( \lambda_2 = \lambda_3 = \lambda_1 \) (MN) and b) \( \mathcal{N} = 2 \rightarrow \mathcal{N} = 1 \) breaking with \( \alpha = \beta \):

\[
\left[ \gamma^{\phi \theta} + 2 m a i \sigma_3^+ \right] \epsilon^+ = 0,
\]

(29)

which implies \( 2 m \alpha = \pm 1 \), for consistency.
Finally, the remaining parts of the fermionic variations only depend on functions of \( \rho \) and we obtain the following BPS equations:

\[
\begin{align*}
\delta \psi_\chi & \rightarrow f' + x' = 0, \\
\delta \psi_\rho & \rightarrow \left[ \partial_\rho + \frac{1}{2} x' + \frac{1}{2} e^{-h} \gamma^\theta i \sigma^+_1 (a' \cosh z + \tilde{a}' \sinh z) \right] \epsilon^+ = 0, \\
\delta \psi_\phi & \rightarrow \left[ h' e^h + \gamma^\theta i \sigma^+_1 (a \cosh z \cosh y + \tilde{a} \sinh z \sinh y) + \frac{1}{2} \gamma^\theta i \sigma^+_1 (a' \cosh z + \tilde{a}' \sinh z) + \\
& \quad - \frac{1}{2} e^{-h} \gamma^\rho \left[ (a^2 - 1) \cosh y - (\tilde{a}^2 - 1) \sinh y \right] \right] \epsilon^+ = 0, \\
\delta \lambda_i & \rightarrow \left[ -e^{-y} \sinh 2z - z' \gamma^\rho + \gamma^\theta i \sigma^+_1 e^{-h} \left( a \sinh z \cosh y + \tilde{a} \cosh z \sinh y \right) + \\
& \quad + \frac{1}{2} \gamma^\theta i \sigma^+_1 e^{-h} \left( a' \sinh z + \tilde{a}' \cosh z \right) \right] \epsilon^+ = 0, \\
& \quad \left[ \frac{1}{5} (e^y + e^{-y} \cosh 2z) + \frac{1}{10} e^{-2h} \left( a^2 - 1 \right) \cosh y - (\tilde{a}^2 - 1) \sinh y \right] + \\
& \quad - y' \gamma^\rho - \frac{1}{5} \gamma^\theta i \sigma^+_1 e^{-h} \left( a' \cosh z + \tilde{a}' \sinh z \right) \right] \epsilon^+ = 0, \\
& \quad \left[ (e^y - e^{-y} \cosh 2z) - \frac{1}{2} e^{-2h} \left( a^2 - 1 \right) \sinh y - (\tilde{a}^2 - 1) \cosh y \right] + \\
& \quad - y' \gamma^\rho + 2 \gamma^\theta i \sigma^+_1 e^{-h} \left( a \cosh z \sinh y + \tilde{a} \sinh z \cosh y \right) \right] \epsilon^+ = 0,
\end{align*}
\]

where \( x = \lambda_1 + \frac{\lambda_2 + \lambda_3}{2}, y = \lambda_1 - \frac{\lambda_2 + \lambda_3}{2}, z = \frac{\lambda_2 - \lambda_3}{2} \) and \( h = g - f \) (the equation for the \( \theta \) component of the gravitino shift is equivalent to that for \( \phi \)). Notice that there are more equations than independent fields. Since we cannot rely on any consistent truncation argument, a more general ansatz for the scalar and gauge fields could be required.

**Appendix B: the Maldacena-Nuñez solution**

Let us review first the singular solution of [4], that has only one scalar and gauge group \( U(1)^{\dagger} \). We have to put \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda, a = b = \beta = 0 \) in the equations. We can take the explicit representation for the \( \gamma \) matrices: \( \gamma^\theta, \phi, \rho = \gamma_5 \otimes \sigma^{1,2,3} \); the symplectic-Majorana spinor \( \epsilon^+ \) will be as in (3), but now \( p \) and \( q \) are Dirac spinors and functions of \( \rho \). The twist (29) in this representation gives \( \alpha = 1/2, q = 0 \) (\( m = 1 \) in our units). The only non-vanishing equations are:

\[
\begin{align*}
h' &= \frac{1}{2} e^{-2h}, \\
\chi' &= -\frac{1}{5} + \frac{1}{20} e^{-2h}.
\end{align*}
\]

The solution is given in [4] (\( \lambda \) is related to the dilaton of reference [2] by \( \phi = 5\lambda \)):

\[
e^{2h} = \rho, \quad 5\lambda = \frac{1}{4} \log \rho - \rho.
\]

10
The non-singular solution, corresponding to one scalar and group $SU(2)^+$, can be obtained from the general equations letting $\lambda_1 = \lambda_2 = \lambda_3 = \lambda$ and $\beta = 0$. In this case the structure of the equations is no more linear. In fact, with the definitions:

$$
A = \frac{1}{2} h_e^h, \quad B = \frac{1}{2} a, \quad C = \frac{1}{4} e^{-h} (a^2 - 1), \quad D = -\frac{1}{4} a',
$$

(33)

the $\psi_\theta$ equation becomes:

$$
[\gamma^{\theta \mu} A + i B \sigma_1^+ + i \gamma^\phi C \sigma_3^+ + i \gamma^\rho D \sigma_1^+] \epsilon^+ = 0,
$$

(34)

that can be rewritten as:

$$
i \sigma_1^+ \gamma^{\rho \theta} \epsilon^+ = (\Delta + \Pi \gamma^\rho) \epsilon^+,
$$

(35)

with:

$$
\Delta = -\frac{AB - CD}{A^2 - C^2}, \quad \Pi = -\frac{AD - BC}{A^2 - C^2}.
$$

(36)

Multiplying (33) by $i \sigma_1^+ \gamma^{\rho \theta}$ we obtain the consistency relation:

$$
\Delta^2 - \Pi^2 = 1.
$$

(37)

The gaugino variation can also be cast in the form (35) with:

$$
\Delta = \frac{4 e^h + e^{-h} (a^2 - 1)}{2a'}, \quad \Pi = -\frac{10 e^h \lambda'}{a'},
$$

(38)

so that from consistency of (36) with (38) we obtain other two equations. Finally we have to take into account the $\psi_\rho$ variation, giving the $\rho$ dependence of the spinor and another equation:

$$
[2 + \frac{1}{2} e^{-2h} (a^2 - 1)] \Delta - \partial_\rho \Pi = 0.
$$

(39)

One can verify that all these equations are solved by the functions in [4]:

$$
e^{2h} = \rho \coth 2\rho - \frac{\rho^2}{\sinh^2 2\rho} - \frac{1}{4}, \quad a = \frac{2\rho}{\sinh 2\rho}, \quad e^{10\lambda} = \frac{2 e^h}{\sinh 2\rho}.
$$

(40)

Note that this system consists of four equations for three functions and, although there is no evident relation between them, nevertheless a solution exists.

**Appendix C: other solutions**

The previous equations are quite hard to solve due to their quadratic nature induced by the non abelian gauge field. In the abelian case $U(1)^+ \times U(1)^-$ with two or three scalars the equations reduce to the linear ones in [5, 6]. The next-to-simple example with
quadratic equations other than the one by Chamseddine-Volkov [21] is with $SU(2)^+$, $eta = 0$, and two scalars ($\lambda_1$ and $\lambda_2 = \lambda_3$). The four singlets of the inert $U(1)^-$ define a consistent truncation. They transform in the $1 + 3$ representation of $SU(2)^+$. With an $SU(2)^+$ gauge transformation, we can rotate the triplet to its third component $\lambda_2$. We can try to find a solution with only the fields in Appendix A. Compared with the MN solution we have an extra equation from the gaugino variation in the form (35) with:

$$\Delta = -\frac{4e^h - e^{-h}(a^2 - 1)}{4a}, \quad \Pi = \frac{e^h d' \coth d}{2a},$$

for a total of six equations for four variables. After the change of variables:

$$\frac{du}{d\rho} = \cosh d, \quad d \equiv \lambda_1 - \lambda_2,$$

four equations reduce to the MN ones. The two new gaugino equations give an equation for $d$ and an equation which only depends on $a$ and $h$ ($d$ cancels out) and that, quite remarkably, is automatically satisfied by the MN solution. Solving for $d$ we find:

$$d(u) = -\arcsinh \left( \frac{c_1}{2\sqrt{1 - 8u^2 - \cosh 4u + 4u \sinh 4u}} \right),$$

$c_1$ being an integration constant. This one doesn’t flow to the MN solution, because in the IR $u \sim \rho^{1/3}$. We explicitly checked that the second order equations of motion are satisfied. Using formulae in [19], we obtain the ten dimensional solution which presents a good type singularity (according to the criteria of [2]). The U.V. normalizable behavior of the solution indicates that it corresponds to the attempt of giving a VEV to scalar fields. Since these scalars are massive to begin with, we expect an instability in QFT that may explain the singular behavior of the supergravity solution.

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