Large polar caps for twisted magnetosphere of rotating magnetars

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ABSTRACT

The magnetic field of magnetars may be twisted compared with that of normal pulsars. Previous works mainly discussed magnetic energy release in the closed field line regions of magnetars. For a twisted magnetic field, the field lines will inflate in the radial direction. Similar to normal pulsars, the idea of light cylinder radius is introduced. More field lines will cross the light cylinder and become open for a twisted magnetic field. Therefore, magnetars may have a large polar cap, which may correspond to the hot spot during outburst. Particle flow in the open field line regions will result in the untwisting of the magnetic field. Magnetic energy release in the open field line regions can be calculated. The model calculations can catch the general trend of magnetar outburst decay: decreasing X-ray luminosity, shrinking hot spot etc. For magnetic energy release in the open field line regions, the geometry will the same for different outburst in one magnetar.

Key words: stars: magnetar – stars: neutron – pulsars: individual (XTE J1810–197; Swift J1822.3–1606)

1 INTRODUCTION

Magnetars may be young and high magnetic field neutron stars (Duncan & Thompson 1992; Kaspi & Beloborodov 2017). Observationally, they show various kinds of activities: bursts (including giant flares), outbursts, and variation of timing properties (Esposito et al. 2018). During the outburst, the magnetar persistent X-ray luminosity may be enhanced and decays subsequently. During the outburst decay phase, the magnetar may show a shrinking hot spot, appearance of radio emission and subsequent cessation, decreasing torque, softening X-ray spectra, and simple pulse profile etc (Coti Zelati et al. 2018; Esposito et al. 2018). The physical reason accounting for the magnetar activities may be that magnetars have twisted magnetic fields (Thompson et al. 2002). The twist may be bumped from interior in the neutron star. This internal strong magnetic field can also contribute to the magnetic energy release (Vigano et al. 2013).

Magnetars typically have a rotational period about $P \sim 10$ s. For such a long rotational period, the expected neutron star polar cap radius is very small, only about 50 m. Therefore, the closed field line regions of magnetars are mainly discussed previously (Thompson et al. 2002; Beloborodov 2009; Pavan et al. 2009). However, this argument (magnetars have a small polar cap) is based on a pure dipole magnetic field. The magnetar’s magnetic field may be highly twisted. For a twisted dipole magnetic field, it will inflate radially (Wolfson 1995; Thompson et al. 2002). Similar to the case of normal pulsar, the light cylinder radius may be introduced to define the boundary between closed field lines and open field lines. Then, for a twisted dipole field, as the field lines inflate in the radial direction, more field lines will cross the light cylinder and become open. Therefore, magnetars may have a large polar cap despite their long rotational period.

The particle flow in the open field line regions will be enhanced due to a large polar cap. This may also result in subsequent untwisting of the twisted magnetic field. Therefore, the open field line regions may provide an independent channel for the magnetic energy release in magnetars.

In Section 2, the numerical and analytical solutions of a twisted dipole field is presented. For a twisted magnetosphere of a rotating magnetar, the magnetar may have a large polar cap. This is demonstrated in Section 3. The consequences of a large polar of magnetar is presented in Section 4. Comparisons with observations are discussed in Section 5. Section 6 is the conclusion.

2 AXISYMMETRIC FORCE-FREE EQUILIBRIUM

2.1 Basic equations

The axisymmetric force-free equilibrium has been studied by many authors (Wolfson 1995; Thompson et al. 2002; Pa-
van et al. 2009). The symbolic system of Wolfson (1995) is adopted in the following. The magnetosphere of pulsars and magnetars may be in the force-free equilibrium state. Assuming axisymmetric condition, the magnetosphere is described by the Grad-Shafranov equation. Following the prescription of Wolfson (1995) and references therein, in spherical coordinate, the Grad-Shafranov equation is
\[
\frac{\partial^2 A}{\partial r^2} + \frac{1 - x^2}{r^2} \frac{\partial^2 A}{\partial x^2} + F(A) \frac{dF}{dA} = 0, \tag{1}
\]
where \(r\) is the dimensionless radial coordinate (in units of the neutron star radius), \(x = \cos \theta\) (\(\theta\) is the polar angle), \(A = A(r, \theta)\) is the flux function, and \(F(A)\) is a yet undetermined function. The magnetic field is related to the flux function as (Wolfson 1995; Pavan et al. 2009)
\[
B = \frac{1}{r \sin \theta} \left[ \frac{1}{r} \frac{\partial A}{\partial \theta} - \frac{\partial A}{\partial r} \right]. \tag{2}
\]
If the function \(F(A)\) is in the form \(F(A) = \lambda A^{1+1/n}\) (\(\lambda\) and \(n\) are numerical parameters), then equation (1) can be solved by separation of variables. The flux function will have the following form
\[
A = r^{-n} f(x). \tag{3}
\]
From equation (3), the parameter \(n\) reflects the radial dependence of the flux function. From the relationship of magnetic field and flux function (equation (2)), the radial dependence of the magnetic field will be \(B(r) \propto r^{-(2+\nu)}\). In general, the radial dependence of the magnetic field is different from the magnetic dipole field. The undetermined function \(f(x)\) can be viewed as the dimensionless flux function. By the separation of variables, equation (1) is reduced to an ordinary differential equation
\[
(1 - x^2) f''(x) + n(n + 1) f(x) + \lambda^2 \left( 1 + \frac{1}{n} \right) f^{1+2/n}(x) = 0. \tag{4}
\]
When \(\lambda = 0\) and \(n = 1\), this corresponds to the magnetic dipole field. When \(\lambda\) is different from zero, this means the presence of toroidal field, and the magnetic field is twisted compared with the dipole case. The polar axis should be a field line, this gives the boundary condition of equation (4): \(f(\pm 1) = 0\). The boundary value requires that there is an eigenvalue of \(n\) for a given \(\lambda\) in equation (4). For a twisted dipole field, the flux conservation gives the initial condition: \(f(0) = 1\). Symmetry about the equatorial plane gives the second initial condition: \(f'(0) = 0\). Starting from the two initial conditions, and considering the boundary condition requirement, equation (4) can be solved numerically. For consideration near the polar cap regions, it is found that there are analytical solutions to equation (4).

2.2 Analytical solutions

For \(\lambda = 0\) and \(n = 1\), the solution of equation (4) is the magnetic dipole field: \(f_0(x) = 1 - x^2\). For an decreasing \(n\) from \(n = 1\) to \(n = 0\), the magnetic field evolves from the dipole configuration to the split monopole. During this process, \(\lambda\) will also be nonzero. For values of \(1 - n \ll 1\), an expansion around \(\lambda = 0\) and \(n = 1\) may be made. Following the treatment of Pavan et al. (2009), denote \(\Delta n = n - 1\),

\[
f(x) = f_0(x) + f_1(x) \Delta n, \quad \lambda^2 = \kappa \Delta n.
\]
The expansion is made for \(\lambda^2\) because it is \(\lambda^2\) which appeared in equation (4). The calculation is straightforward and similar to that of Pavan et al. (2009). It is found that \(\lambda^2 = (35/16)(1 - n)\), or
\[
\lambda = \sqrt{\frac{35}{16}(1 - n)}, \tag{5}
\]
where the positive root of \(\lambda^2\) is chosen, which means the field line in the southern hemisphere of the neutron star will twist eastward (compare with field lines in the northern hemisphere). If during the definition of angular shear, a minus sign is absorbed, then the field line in the southern hemisphere will twist westward. This is the picture shown in figure 2 in Wolfson (1995) and figure 3 in Thompson et al. (2002). The dimensionless flux function now is
\[
f(x) = f_0(x) \left[ 1 - \frac{22 - 5x^2}{32} x^2(1 - n) \right]. \tag{6}
\]
Figure 1 shows the comparison of analytical approximation with the numerical solution of equation (4). For \(n\) not so small, the analytical solution is roughly consistent with the numerical calculations.

For the polar cap region, the polar angle is relatively small, e.g., \(\theta \leq 0.3\), and \(x = \cos \theta > 0.9\). This will correspond to a hot spot of 3 km if observed, which is the typical observed value (Enoto et al. 2017). In this case, the analytical solution can match the numerical solutions quite well, even for \(n\) as small as \(n = 0.1\), see figure 2. This feature of equation (6) ensures that we can use the analytical solutions when calculating the geometry of the polar cap regions. Once the magnetic flux is obtained, the calculation of magnetic field, shear, current density, and magnetic energy free energy can be calculated straightforward. In the following, we will try to provide analytical approximations when possible. The corresponding numerical results will also be shown for comparison in several cases.

2.3 Maximum twist of magnetic field lines

In the above self-similar solutions, the footpoints of the magnetic field lines will move not only in the latitudinal direction but also in the longitudinal direction. The longitudinal
movement will result in the twist of the magnetic field lines. For a specific field line starts from \(x_1 = \cos \theta_1\) in the northern hemisphere and ends at \(x_2 = \cos \theta_2\) in the southern hemisphere, the angular shear is (Wolfson 1995)

\[
\Delta \phi = \frac{\lambda}{n} \int_{x_2}^{x_1} \frac{f^{1/n}(x)}{1 - x^2} \, dx,
\]

where during the definition of the twist a minus sign is absorbed. According to the definition of the twist, the field line starts at the north pole and ends at the south pole will have the maximum twist. Therefore, the maximum twist for a given equilibrium configuration is (Thompson et al. 2002)

\[
\Delta \phi_{\text{max}} = \frac{2\lambda}{n} \int_0^1 \frac{f^{1/n}(x)}{1 - x^2} \, dx.
\]

For \(n \approx 1\) and \(f(x) \approx 1 - x^2\), the maximum twist is

\[
\Delta \phi_{\text{max}} = 2\lambda.
\]

Figure 3 shows that the analytical approximation is quite accurate for a large range of parameter space. In this figure, the analytical approximation is from equation (9), where \(\lambda\) is adopted from that of equation (5). The numerical calculation results from the definition of maximum twist (equation (8)), where \(f(x)\) is solved numerical from equation (4).

For a specific equilibrium configuration, there is a one to one correspondence for the three parameters: \(n, \lambda,\) and \(\Delta \phi_{\text{max}}\). In previous works (e.g., Thompson et al. 2002), the maximum twist is also used to characterize the state of the magnetic field lines. However, during the definition of the maximum twist, it is assumed that the central magnetar does not rotate. Therefore, all the field lines are closed field lines and there is no open field lines. For real magnetars, they are rotating neutron stars. The field lines near the polar cap regions will become open field lines. Therefore, the definition of the maximum twist will be no longer valid mathematically. In reality, as shown below, magnetar can have large polar caps. This will further make the definition of the maximum twist inappropriate. Therefore, the parameter \(n\) may be a better parameter to describe the state of a twisted magnetic dipole field. Physically, the parameter \(n\) reflects the radial dependences of the magnetic field.

### 3 LARGE POLAR CAPS

In modeling the twisted magnetosphere of magnetars, the effect of rotation is neglected in several previous works (Thompson et al. 2002; Beloborodov 2009). Therefore, in these works, the central magnetar has no open field lines and no polar caps. This kind of models may be dubbed as “twisted magnetosphere of non-rotating magnetars”. For a typical rotational period of 10 seconds, the polar cap region is very small for a dipolar magnetic field. This is the taken to be the reason why rotation is neglected in previous works. However, For a twisted dipole field, the field line geometry is no longer dipolar. They tend to inflate in the radial direction. If in analogy with normal pulsars, the open field line is defined as those that pass through the light cylinder radius, then more field lines will become open due to the inflation of field lines in the radial direction. Therefore, for “twisted magnetosphere of rotating magnetars”, the central magnetar can have more open field lines and large polar caps.

For a constant flux \(A = r^{-n} f(x) = \text{constant}\) (equation (3)), it corresponds to the projection of magnetic field lines in the \(r - \theta\) plane. The dimension flux function \(f(x)\) has the largest value at the equator (figure 1), therefore, the maximum radial extension of the magnetic field line is also reached in the equatorial plane. The last closed field line can be defined as those with maximum radial extension equal to the light cylinder radius: \(r_{\text{max}} = R_{\text{lc}} = P c/(2\pi)\), where \(P\) the magnetar rotational period, and \(c\) is the speed of light. The intersection of the last closed field line with the neutron star surface defines the boundary of the polar cap region. Since the flux function is a constant, then

\[
\frac{f(0)}{R_{\text{lc}}^n} = \frac{f(x_{\text{pc}})}{R_{\text{lc}}^n},
\]

where \(x_{\text{pc}}\) is the angular radius of the polar cap, and \(R\) is the neutron star radius. According to the conservation of magnetic flux \(f(0) = 1\), therefore the angular radius of the polar cap is determined by

\[
f(x_{\text{pc}}) = \left(\frac{R}{R_{\text{lc}}}\right)^n.
\]

Note that \(f(x_{\text{pc}})\) is also the fraction of magnetic flux of the polar cap region. A typical magnetar is assumed to have rotational period of 10 s, and a neutron star radius of 10 km.
For a magnetic dipole field, the parameter $n$ is $n = 1$. Then the polar cap will have a fractional magnetic flux of $2 \times 10^{-5}$. For a magnetic dipole field, the dimensionless flux function is: $f_0(x) = 1 - x^2 = 1 - \cos^2 \theta = \sin^2 \theta$. Then the angular radius of the polar cap is: $\theta_{pc} = \sqrt{R/R_c}$. This is the result in normal pulsars (Goldreich–Julian 1969; Ruderman & Sutherland 1975). Typically, the polar cap radius is only about 50 meters.

From equation (11), for a twisted dipole field with $n < 1$, the polar cap will have larger fractional magnetic flux. Furthermore, in equation (11), only the value of $f(x)$ at the polar cap region is required. From figure 2, near the polar cap region, the analytical solution (equation (6)) is quite accurate for a large range of $n$. In the polar cap region with $x \approx 1$, the flux function can be further simplified: $f(x) = f_0(x)[1 - ((22 - 5x^2)/32)x^2(1 - n)] = f_0(x)[(15 + 17n)/32]$. The angular radius of the polar cap for a twisted dipole field is

$$\sin \theta_{pc} = \frac{(R/R_c)^n}{\sqrt{(15 + 17n)/32}}.$$  

(12)

Figure 4 shows the polar cap angular radius of a magnetar with twisted dipole field. For a large radius of $\theta_{pc} = 0.1$ or $\theta_{pc} = 0.2$, the corresponding polar cap radius is about 1 km or 2 km. If the polar cap is heated by accelerated particles, a hot spot of 1 – 2 km may be seen observationally.

Equation (12) reduces to the magnetic dipole case for $n = 1$. If the magnetar’s magnetic field is twisted during a star quake, during the untwisting process, the parameter $n$ evolves from $n < 1$ to $n = 1$. From figure 4, during the untwisting process, the polar cap will decrease with time. Observationally, this may correspond to the shrinking hot spot of magnetars during outbursts.

4 PARTICLE FLOW IN THE OPEN FIELD LINE REGIONS AND UNTWISTING

4.1 Magnetic free energy

For a twisted dipole field, the appearance of the toroidal component of the magnetic field means there are additional magnetic free energy compared with pure dipole case (Beloborodov 2009). For a twisted dipole field with radial dependence $B(r) \propto r^{-(2+n)}$, most of the magnetic energy are stored in the vicinity of the neutron star. Using the tensor virial theorem, the total magnetic energy is determined by the distribution of magnetic field at the surface of the neutron star$^1$ (Wolfson 1995)

$$E_{\text{mf}} = \frac{3}{2} \int_0^1 (B_c^2 - B_0^2 - B_{\phi}^2) dx.$$  

(13)

The total magnetic energy is expressed in units of the total magnetic energy of the dipole magnetic field: $(1/12)B_c^2R^4$ (Thompson et al. 2002). Then the magnetic free energy is

$$E_{\text{mf}} = E_B - 1 = \frac{3}{2} \int_0^1 (B_c^2 - B_0^2 - B_{\phi}^2) dx - 1.$$  

(14)

When there are open field line regions, the total magnetic free energy may be modified. The above treatment will be employed as the approximation to the real case. By solving equation (4) numerically, and from the expression of magnetic field (equation (2)), the corresponding magnetic free energy as a function of parameter $n$ is shown in figure 5. For a pure dipole field, $n = 1$, the magnetic free energy is zero. For a split monopole, $n = 0$, the magnetic free energy is 0.5. Therefore, a not too bad analytical guess for the magnetic free energy is: $E_{\text{mf}} = 0.5(1 - n)$. It is found that the following analytical expression fits the numerical results better

$$E_{\text{mf}} = 0.5(1 - n)^{1.5}.$$  

(15)

This analytical fitting will make relevant discussions and calculations much simpler.

4.2 Particle flow in the open field line regions

For normal pulsars, the particle flow, acceleration and radiation process mainly takes in the open field line regions (Goldreich & Julian 1969; Ruderman & Sutherland 1975; Cheng et al. 1986; Du et al. 2010). For “twisted magneto-sphere of non-rotating magnetars”, the rotation of central magnetar is neglected from the starting point. All the field lines are closed field lines. There are no open field lines. Only

$^1$ The numerical coefficient in equation (11) of Wolfson (1995) should be 3/2 instead of 2/3.
where the magnetic field lines. Similar to equation (19), the magnetic free energy and results in the untwisting of the open field line regions will also consume energy due to the twist of magnetic field lines. The particle acceleration potential is the potential drop between the edge of the polar cap and the magnetic pole (Ruderman & Sutherland 1975)

\[ \Delta V_{\text{max}} = \frac{\Omega R^2 B_p}{2c} \sin^2 \theta_{pc}, \]  

where \( \Omega \) is the angular velocity of the central neutron star. For normal pulsars, with \( \sin^2 \theta_{pc} = R/\rho_{GJ} \), the maximum acceleration potential is \( \Delta V_{\text{max}} = \frac{\Omega^2 R^2 B_p}{(2c^2)} \). The current through one polar cap is (Harding et al. 1999; Tong et al. 2013).

\[ I_{pc} = \pi R_{pc}^2 \rho_{GJ} c, \]  

where \( R_{pc} \) is the polar cap radius, \( \rho_{GJ} = \Omega B_p/(2\pi c) \) is the so-called Goldreich-Julian density (Goldreich-Julian 1969). Then the energy loss rate due to the particle flow is (i.e. particle luminosity)

\[ \dot{E}_{\text{p,dipole}} = 2I_{pc} \Delta V_{\text{max}} = \frac{\Omega^4 R^4 B_p^2}{2c^3}. \]

This result is the similar to that of magnetic dipole radiation, except for a different numerical factor.

For a twisted dipole field, the particle flow in the open field line regions will be modified mainly due a larger polar cap (equation (12)). For a larger polar cap compared with that of the pure dipole case, the corresponding maximum acceleration potential (equation (16)) and current (equation (17)) will be larger. This will result in a higher energy loss rate due to the particle flow. This enhanced particle flow is due to the twist of magnetic field lines. The particle acceleration and flow in the open field line regions will also consume the magnetic free energy and results in the untwisting of the magnetic field lines. Similar to equation (18), the corresponding particle luminosity for a twisted dipole field is

\[ \dot{E}_{\text{p, twist}} = \frac{\Omega^2 R^4 B_p^2}{2c^2} \sin^4 \theta_{pc}, \]

where \( \theta_{pc} \approx \sin \theta_{pc} \) is used for small values of \( \theta_{pc} \) (i.e. polar cap regions). By colliding with the neutron star surface, this particle luminosity may result in a hot spot on the neutron star and be converted into X-ray luminosity of the magnetar. As there may be multipole fields near the magnetar surface (Tiengo et al. 2013), a significant amount of the particle flow may be trapped by the multipole field. The physics may be similar to that in the closed field line regions (Beloborodov 2009). By colliding with the neutron star surface, a high conversion efficiency from the particle luminosity to the X-ray luminosity may be resulted. For a smaller conversion efficiency, it will only affect the normalization of the X-ray flux. The decaying pattern is the same. By choosing a higher magnetic field, a higher X-ray luminosity can be obtained again. For normal pulsars, the particle acceleration and flow is still an unsolved question (Zhang et al. 2000; Kou & Tong 2015). The corresponding acceleration potential in various gap models is different from the maximum acceleration potential. The flowing particle density can also deviates from the Goldreich-Julian density. The above treatment can be viewed as the strong particle flow case. When the twisted dipole field is relaxed back to the pure dipole case, the polar cap (equation (12)) will return back to the dipole case. The corresponding particle luminosity (equation (19)) will also return back the dipole case. In this way, the magnetosphere of magnetars and normal pulsars may be unified. This is merit of the above treatment.

There are other possibilities for both the current and the acceleration potential. The acceleration potential is assumed to equal to the maximum acceleration potential. Physically, the acceleration potential may be nearly a constant around \( 10^{12} - 10^{13} \text{ V} \) (Ruderman & Sutherland 1975; Contopoulos & Spitkovsky 2006; Medin & Lai 2010). For a Goldreich-Julian current and a constant acceleration potential, the corresponding particle luminosity is proportional to \( \Omega^2 \). This will result in braking index of one in the case of rotation-powered pulsars (Xu & Qiao 2001; Kou & Tong 2015). The current is assumed equal to the Goldreich-Julian current in the above treatment. Another way for obtaining the polar cap current is through its relation with the toroidal magnetic field \( B_\phi \). The quasi-steady polar cap current is \( I_{pc} = (c/2)B_\phi |\gamma = \theta_{pc} R_{pc} \) (Harding et al. 1999; Beloborodov 2009). The poloidal magnetic field may be caused by the twist of the magnetic field or the effect of rotation. For a twisted magnetosphere of rotating magnetars, the poloidal magnetic field will be stronger, and the polar cap radius will be larger. This will result in an enhanced polar cap current compared with the normal pulsar case. For a twisted magnetic field, \( B_\phi \propto F(A) \propto \lambda \). When the twist vanishes, the poloidal magnetic field due to twist vanishes. The polar cap current will return back to the normal pulsar case. For this choice of polar cap current, there are also two possibilities of acceleration potential. Therefore, there are at least four kinds of combinations for the two choices of polar cap current and two choices of acceleration potential. These are the parameter spaces for the physics in the open field line regions of and magnetars. When employing the above particle luminosity formula (equation (19)), we should bear in mind these freedoms of the model.

4.3 Untwisting

According to energy conservation, the untwisting of the twisted dipole field is governed by the following equation

\[ \frac{dE_{\text{tot}}}{dt} = -\dot{E}_{\text{p, twist}}. \]

It should be noted that the magnetic free energy is expressed in units of the magnetic energy of a pure dipole field.

For typical parameter space, the particle luminosity is much higher than the magnetar’s rotational energy loss rate (see below). Furthermore, this enhanced particle luminosity is due to a large polar cap. A large polar cap will result
in a higher particle flux and higher maximum acceleration potential, and finally a higher particle luminosity. Equation (19) is obtained similar to that of normal pulsars. In the absence of twist, it will return back to the normal pulsar case (equation (18)). The particle luminosity can also be obtained similar to that of magnetar closed field line regions (Beloborodov 2009). The polar cap current is determined by the toroidal magnetic field $B_\lambda \propto \lambda$, which is determined by the twist of the magnetic field. Similar particle acceleration process may occur for this polar cap current, either similar to magnetar closed field line regions (Beloborodov 2009) or similar to normal pulsar polar cap regions (Medin & Lai 2010). Therefore, the particle luminosity should be of magnetic origin (instead of rotational origin, Lyutikov 2013). This is one basic assumption of our model. Different modeling of the particle luminosity will only result in quantitative differences.

5 DISCUSSION

Observationally, the magnetar outbursts show a variety of changes: flux decay, shrinking hot spot, decreasing temperature, softening spectra, simpler pulse profile etc. The outburst may contain magnetic energy release from both the neutron star crust (Vigano et al. 2013) or the magnetosphere (Thompson et al. 2002). Previous works considered the energy release in the closed field line regions (Beloborodov 2009). As shown above, the open field line regions may also contribute to the magnetic energy release. In below, the outburst of the first transient magnetar XTE J1810−197 is taken as an example. We will show to what degree the particle luminosity will decrease and the untwisting timescale will increase with time during the untwisting process. Years after the outburst, it will keep a magnetospheric configuration about $n \approx 0.93$, until the next outburst. The magnetic energy release rate is about $10^{33} B_{14}^2$ erg s$^{-1}$. It may contribute to the quiescent luminosity of magnetars. Observationally, e.g. for XTE J1810−197, the quiescent luminosity contains a warm component plus a cold component from the whole neutron star surface (Alford & Halpern 2016). The warm component may due to continued magnetic energy release during the quiescent state.

- If a magnetar is induced to show outburst, e.g. by starquake, with initial parameter $n < 0.93$ (or $\Delta \phi_{\text{max}} > 0.8$), then its untwisting timescale will increase with time during the untwisting process. After sometime, it will return to a magnetosphere without twist. In this case, the maximum energy release rate due to the untwisting magnetosphere is $10^{33} B_{14}^2$ erg s$^{-1}$. For young high magnetic field pulsars, their rotational energy loss rate will be significantly larger than this. Therefore, the magnetic field powered activities in this case will be insignificant. Only when the initial $n$ parameter is smaller than 0.93 (maximum twist larger than 0.8), the magnetic activities may be significant. In this case, the untwisting behavior will be similar to that of magnetars. Up to now, two high magnetic field pulsar showed magnetar-like activities (Gavriil et al. 2008; Archibald et al. 2016; Gogus et al. 2016).

By solving Equation (20), the theoretical flux decay can be compared with the hot spot of XTE J1810−197. From the total released energy and peak flux during the outburst, a surface magnetic field at the magnetic pole is $B_{14} = 0.17$. This will be the polar dipole magnetic field strength when there is no twist. This value of magnetic field is for an energy conversion efficiency of unity. For a significantly smaller conversion efficiency (e.g., similar to that of normal pulsar $\sim 10^{-5}$), by choosing a higher magnetic field (e.g., 30 times higher), a high X-ray luminosity can be obtained again. The flux evolution time scale and evolution pattern is not affected by the energy conversion efficiency. For a parameter $n = 0.58$ at the time of the first XMM-Newton observation of XTE J1810−197 (MJD 52890.6), Figure 7 shows the hot spot luminosity of XTE J1810−197. Similar to Figure 4 in Alford & Halpern (2016), the luminosities here are bolometric luminosities. The model calculations can catch the general trend of hot spot luminosity decay. At later time, the model calculation seems to over estimate the hot spot luminosity. However, at later time, the hot spot transforms to a warm-hot component (Alford & Halpern 2016). Figure 8 shows the long term flux decay of XTE J1810−197. The first seven observational data in Figure 8 are the same as that in Figure 7. The later observational data points in Figure 8 are the warm-hot component in Alford & Halpern (2016). The theoretical long term luminosity is about $10^{33}$ erg s$^{-1}$. It can explain the warm-hot component of XTE J1810-197 during the quiescent state.

\[ E_{\text{mf}} \approx 10^{34} B_{14}^2 \text{ erg}, \]
\[ \text{particle luminosity is about } E_{\text{p, twist}} \approx 10^{37} B_{14}^2 \text{ erg s}^{-1}, \]
\[ \text{and the magnetic energy decay timescale (i.e. untwisting timescale) is about } \tau \sim E_{\text{mf}} / E_{\text{p, twist}} \sim 0.3 \text{ yr}. \]
Figure 6. Magnetic field untwisting timescale as a function of $n$. The untwisting timescale peaks at about $n = 0.93$.

Figure 7. Luminosity of the hot spot component of XTE J1810$-$197. The blue squares are observations (Alford & Halpern 2016), the solid line is the model calculations.

Figure 8. Long term luminosity decay of XTE J1810$-$197. Similar to Figure 7, the blue squares are observations (Alford & Halpern 2016), the solid line is the model calculations.

5.2 Shrinking hot spot and temperature

Observationally, XTE J1810$-$197 shows a transition from three blackbody spectra to two blackbody spectra (Alford & Halpern 2016). It is possible that the central neutron star has a hot spot. The hot spot has a temperature gradient with the rest of the neutron star surface. The magnetic energy release deposited onto the polar cap region may be diffused to the rest part of the neutron star surface. Figure 9 shows the hot spot area as a function of time. Observationally, in order to explain the X-ray spectra of XTE J1810$-$197, a large neutron star radius is required $R \approx 30$ km (Alford & Halpern 2016). In obtaining Figure 9, a neutron star radius of $R = 30$ km is chosen. Again, the theoretical curve can not match exactly the observations. But it can catch the general trend of the observations. Later, when the hot spot transits to the warm-hot component, the warm-hot area is too large to be the emitting area from the polar cap regions. The temperature of the hot spot can also be calculated theoretically. During the untwisting process, the acceleration potential for each particles decreases. Then the hot spot spot temperature is also expected to decrease with time. The theoretical hot spot temperature is about 1 keV. While the observed temperature is about 0.6 keV (Alford & Halpern 2016). Considering that the heat in the hot spot will diffuse to the rest part of the neutron star, it is natural that the theoretical hot spot temperature (not considering the diffusion process) is higher than the observations.

5.3 Geometry and spin-down torque

Here we considered the untwisting due to particle flow in the open field line regions of a globally twisted dipole field. In reality, the magnetar may contain higher order multipole fields. The magnetic energy release may also occur in the closed field line regions (Beloborodov 2009) or in the neutron star crust (Vigano et al. 2013). One merit of considering particle flow in the open field line regions is that the geometry of hot spot will always be the same. If the radio emission also originates from the large polar caps of the twisted dipole field, then its magnetic field geometry is also not expected to change significant, except for a different twist at the onset of different outbursts. The revival of the magnetar PSR J1622$-$4950 (Camilo et al. 2018) and XTE J1810$-$197 (Gotthelf et al. 2019) both may require the same magnetic field geometry with the previous outburst.

For a dipole magnetic field, the twist will result in a large polar cap, enhanced particle flow, stronger magnetic field at the light cylinder radius, and the presence of a strong toroidal field. All these aspects will contribute to a larger

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3 In the previous calculations, when the neutron star radius is relevant, it is taken as the typical value of $R = 10$ km.
torque than the pure dipole case. This may explain why the required magnetic field of XTE J1810$-$197 is smaller than its characteristic magnetic field. During the untwisting process, a decreasing torque is expected. This is in general consistent with the timing observations of XTE J1810$-$197 (Camilo et al. 2016; Levin et al. 2019).

For a twisted dipole field, the light cylinder radius determines the boundary of the polar cap region. A large polar cap will result in a stronger particle flow. This strong particle flow may also result in the opening of the magnetic field lines at a smaller radius than the light cylinder radius ( Harding et al. 1999; Tong et al. 2013). A smaller opening radius will again result in a larger polar cap. Therefore, the torque due to the particle flow in the case of twisted magnetic field should be treated in a self-consistent way. This will be the future works.

5.4 Statistical properties of magnetar outbursts

The outburst observations of magnetars are very diverse and rich (Esposito et al. 2018; Coti Zelati et al. 2018). The above calculations for XTE J1810$-$197 is only one example. From Equation (19), the particle luminosity is proportional to the square of the polar cap area. This may result in a X-ray luminosity proportional to the square of the hot spot area during the outburst decay of the magnetar: $L_x \propto A^2$, where $A$ is the hot spot area. This relation is for the case of maximum acceleration potential. For a constant acceleration potential, the particle luminosity will be proportional to the polar cap area. This may result in a X-ray luminosity proportional to the hot spot area: $L_x \propto A$. The real case may lie between these two cases. Therefore, a correlation between the X-ray luminosity and hot spot area will be: $L_x \propto A^\alpha$, where the power law coefficient $1 < \alpha < 2$ is expected. Long term flux decay of the magnetar Swift J1822.3$-$1606 found a power coefficient $\alpha < 2$ (Scholz et al. 2014).

As been discussed above, long after the outburst, the magnetar may enter into a quiescent state with $n = 0.93$ and typical magnetic energy release rate $\sim 10^{32} B_7^2 \text{erg s}^{-1}$. However, years after the outburst, the decay time scale is long enough that the magnetar may already be considered as in the quiescent state. Therefore, the corresponding theoretical quiescent luminosity will be $> 10^{33} B_7^2 \text{erg s}^{-1}$. A general correlation between magnetar quiescent luminosity and dipole magnetic field is: $L_{x,q} \propto B_7^p$. For the case of a constant acceleration potential, the correlation will be: $L_{x,q} \propto B_7$. Therefore, the power law index between one and two is expected for $L_{x,q} \propto B_7^\beta$, where $1 < \beta < 2$. Observationally, such a correlation is indeed found by Coti Zelati et al. (2018). However, when interpreting the observations, two cautions should be made: (i) In observations, the quiescent luminosity may include both the magnetic energy release and the a cold component from the whole neutron star surface (e.g. XTE J1810$-$197, Alford & Halpern 2016). (ii) The neutron star spindown torque may be significantly enhanced compared with the dipole case. The characteristic dipole magnetic field may just be a measure of the total torque. It can be significantly larger than the true dipole magnetic field (Tong et al. 2013). Albeit with these two uncertainties, we think that the correlation between magnetar quiescent luminosity and dipole magnetic field is another evidence that are consistent with our model.

5.5 Comparison with Beloborodov (2009)

In Thompson et al. (2002), a globally twisted magnetosphere is considered. While in Beloborodov (2009), a locally twisted magnetic field is explored. There is no open field lines. In Beloborodov (2009), magnetic energy release in the closed field lines regions is considered. In our model, the magnetar may have a large polar cap due the twist of the dipole magnetic field. Possible magnetic energy release in the open field line regions are considered. This will also result in untwisting of the magnetic field. Therefore, our model provides an independent channel for magnetic energy release in the case of magnetars. Magnetic energy release may be at work simultaneously in both the closed field line regions and open field line regions.

For magnetic energy release in the closed field line regions and the same magnetar, different outbursts may happen at different locations. While for magnetic energy release in the open field line regions, different outburst will always happen at the polar cap regions. Therefore if magnetic energy release occurs in the open field line regions, the magnetar will exhibit the same geometry during different outbursts. The recurrent outburst of XTE J1810$-$197 may help us to solve this problem observationally, at least for this source.

6 CONCLUSION

Previously, the polar cap of magnetars is thought to be very small due to their long rotational period ($P \sim 10\text{s}$). Magnetic energy release in closed field line region are mostly discussed. Considering that the magnetar’s magnetic field may be twisted and a twisted magnetic field tend to inflate in the radial direction. This will result in a large polar cap for magnetars. This is our major finding when considering the twisted magnetosphere of rotating magnetars. The consequences of a large polar cap are particle flow in the open field line regions and untwisting of the magnetic field etc. Our model calculations can catch the general trend of magnetar outburst decay.

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REFERENCES

Alford J. A. J., Halpern J. P., 2016, ApJ, 818, 122
Archibald R. F., Kaspi V. M., Tendulkar S. P., et al., 2016, ApJ, 829, L21
Beloborodov A. M., 2009, ApJ, 703, 1044
Cheng K. S., Ho C., Ruderman M. A., 1986, ApJ, 300, 500
Camilo F., Ransom S. M., Halpern J. P., et al., 2016, ApJ, 820, 110
Camilo F., Scholz P., Scrylak M., et al., 2018, ApJ, 856, 180
Contopoulos I., Spitkovsky A., 2006, ApJ, 643, 1139
Coti Zelati F., Rea N., Pons J. A., et al., 2018, MNRAS, 474, 961
Du Y. J., Qiao G. J., Han J. L., et al., 2010, MNRAS, 406, 2671
Duncan R. C., Thompson C., 1992, ApJL, 392, L9

MNRAS 000, 1–9 (2015)
