Restrictions on torsion-spinor field theory

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Torsion propagation and torsion-spin coupling are studied in the perspective of the Velo-Zwanziger method of analysis; specifically, we write the most extensive dynamics of the torsion tensor and the most exhaustive coupling that is permitted between torsion and spinors, and check the compatibility with constraints and hyperbolicity and causality of field equations: we find that some components of torsion and many terms of the torsion-spin interaction will be restricted away and as a consequence we will present the most general theory that is compatible with all restrictions.

I. INTRODUCTION

It is almost a century that Einsteinian gravity has been complemented with torsion [1–3], and many decades that its importance for the coupling to the spin of spinors was recognized [4, 5] (see also [6] for recent review/overview).

But while the Cartan extension of Riemannian geometry is now well known, the Sciama-Kibble completion of Einsteinian gravitation is still mainly focused on its most straightforward generalization of Einstein gravity where torsion is not allowed to propagate: as torsion is meant to be a physical field then it must have propagation, and as a consequence second-order derivative torsional terms have to be included into the action. In parallel, also the torsional coupling to the spin of spinors is still essentially focused on its simplest form: in the perspective of studying the torsion-spin interaction most in general, then all torsion-spin terms are to be included in the action too.

The problem we now face is that, in principle, we shall find an infinity of such terms, unless some concepts would intervene to restrain this profusion: a first-level solution is to notice that because the spinor field equations have first-order derivatives in the spinor fields, then the inclusion of higher-than-first-order derivatives of the spinorial field would make no sense, and hence we should restrict ourselves to products of first-order derivative spinor fields and torsion (terms of mass-dimension 5 are not renormalizable, but despite this, they have been allowed in papers such as the famous [7, 8], and so we will retain the right to study them in full); with mass-dimension 5 allowed, it becomes possible to still include several terms, therefore a second-level solution would be to assess which of these terms are acceptable within the Velo-Zwanziger method of analysis [9, 10]. We shall see that at this point, a few terms alone will still be compatible with all restrictions.

So in this paper, we will first of all make the inventory of terms that can be included in the action: this means all possible terms that are quadratic in the derivatives of torsion, beside the usual Dirac term containing first-order derivatives of spinor field, as well as all the interaction terms given by the products of derivatives of spinor and torsion, but also products of spinors and squared torsion down to the usual products of spinors and torsion; once this is done, we will proceed in removing all those terms that do not cope with the restrictions that are imposed by the implementation of the Velo-Zwanziger analysis.

The Velo-Zwanziger analysis ([9, 10]) in its full extent goes as follows: given a system of field equations, check whether they may develop supplementary conditions that would produce a mismatch between degrees of freedom and independent field equations; for all fields for which there is no such a mismatch, so that their field equations are well-defined, consider such field equations with only the highest-order derivatives, and then make the replacement \( \nabla_{\alpha} \rightarrow n_{\alpha} \), getting an algebraic equation in the form \( AF = 0 \) where \( F \) is the field in exam: det\( A = 0 \) gives the so-called characteristic equation, and whose solutions are such that \( n_{\alpha} \) must be real to ensure that the original field equations be hyperbolic and then \( n_{\alpha} \) must be space-like to ensure that the original field equations be causal.

As we are going to see, there will only be a few terms remaining after that this analysis is implemented.

II. TORSION AND ITS PROPAGATION

To begin we introduce the torsion and its propagation.

The torsion tensor \( Q_{\rho\mu\nu} \) is a tensor of order three antisymmetric in two of its indices: as a consequence, it is always possible to decompose it according to

\[
Q_{\rho\mu\nu} = \frac{1}{3}(g_{\rho\nu}Q_{\alpha} - g_{\rho\mu}Q_{\nu}) + \frac{1}{6}W^\alpha\varepsilon_{\alpha\rho\mu\nu} + T_{\rho\mu\nu}
\]

where \( Q_{\alpha} = Q^\alpha_{\rho\nu} \) is the trace and \( W^\alpha = Q_{\rho\mu\nu}\varepsilon_{\rho\mu\nu\alpha} \) is the completely antisymmetric part and where \( T_{\rho\mu\nu} \) such that it is \( T_{\rho\nu} = 0 \) and \( T_{\rho\mu\nu}\varepsilon_{\rho\mu\nu\alpha} = 0 \) is called non-completely antisymmetric irreducible part of torsion; the three parts are mutually independent and as such it becomes possible to study their propagations independently, and so simply.

The propagation of the completely antisymmetric part or its dual axial-vector \( W^\alpha \) and its curl \( (\partial W)_{\alpha\rho} \) has been studied in [11]: there it was found that \( W^\alpha \) has the same dynamical properties of an axial-vector massive field, and that is an axial-vector Proca field, quite generally indeed.

For the trace vector \( Q^\alpha \) and its curl \( (\partial Q)_{\alpha\rho} \) we should study the propagation now: because the dynamics cannot be distinguished by the Velo-Zwanziger analysis only on the bases of parity-evenness or parity-oddness, then it is not surprising that performing on \( Q^\alpha \) the Velo-Zwanziger
analysis we find that $Q^\alpha$ has the same dynamical properties of a vector massive field, and that is a vector Proca field, also quite generally. One may think at $Q^\alpha$ as some sort of massive electrodynamic field in a good analogy.

The propagation of the non-completely antisymmetric irreducible part of torsion is trickier: the Lagrangian has to be formed with squares of derivatives of $T_{\mu\nu\rho}$ and as a quick inventory of all indices combination reveals, there are only three possible scalar terms given by

\[
\mathcal{L} = \frac{1}{2} \nabla_\alpha T_{\mu\nu\sigma} \nabla^\alpha T^{\mu\nu\sigma} + \frac{3}{2} A \nabla_\alpha T^{\alpha\mu\nu} \nabla^\beta T_{\beta\mu\nu} + 3 B \nabla_\alpha T^{\alpha\mu\nu} \nabla^\beta T_{\beta\mu\nu} - \frac{1}{2} M^2 T_{\mu\nu\sigma} T^{\mu\nu\sigma} + \mathcal{L}_{\text{matter}}
\]

with a mass term and a source. Its variation gives

\[
\nabla^2 T_{\mu\nu\rho} + A(2 \nabla_\mu \nabla^\beta T_{\beta\nu\rho} - \nabla_\nu \nabla^\rho T_{\rho\mu\nu} + \nabla_\rho \nabla^\mu T_{\mu\nu\rho} - \nabla^\mu \nabla_\nu \nabla^\rho T_{\rho\mu\nu} + \nabla^\nu \nabla_\rho \nabla^\mu T_{\mu\nu\rho} - \nabla^\rho \nabla_\mu \nabla^\nu T_{\nu\rho\mu}) + B(3 \nabla_\nu \nabla^\beta T_{\beta\mu\rho} - 3 \nabla_\rho \nabla^\beta T_{\beta\mu\nu} - \nabla^\beta T_{\alpha\beta\nu} g_{\mu\rho} + \nabla^\beta T_{\alpha\beta\rho} g_{\mu\nu} + \nabla_\mu \nabla^\rho T_{\rho\nu\sigma} + \nabla_\nu \nabla^\rho T_{\rho\mu\sigma} - \nabla_\rho \nabla^\mu T_{\mu\nu\sigma}) + M^2 T_{\mu\nu\rho} = S_{\mu\nu\rho}
\]

where $S_{\mu\nu\rho}$ is the source tensor obtained from the matter Lagrangian: notice that like $T_{\mu\nu\rho}$ its field equations also have a vanishing trace and a vanishing dual.

These field equations must now be restricted in terms of the Velo-Zwanziger analysis. In terms of this analysis, the first thing to do is to study the free case, which means taking also flat space-time: in such a circumstance all of covariant derivatives commute and hence the divergence with respect to the first index gives rise to

\[
\nabla^2 \nabla_\mu T_{\mu\nu\rho} (1 + 2A + B) + B(\nabla_\mu \nabla_\nu \nabla^{\rho} T_{\rho\beta\sigma} - \nabla_\beta \nabla_\rho T_{\sigma\nu\rho} + \nabla_\nu \nabla^\rho T_{\rho\mu\nu} - \nabla^\rho \nabla_\nu \nabla^\sigma T_{\rho\mu\nu}) + M^2 \nabla_\mu T_{\mu\nu\rho} = \nabla_\mu S_{\mu\nu\rho}
\]

in which third-order derivatives have arisen; on the other hand however, third-order derivatives are not acceptable, especially in a constraint: their removal requires a restriction of the type $B = 0$ and $1 + 2A = 0$ to hold identically.

With these restrictions, the field equations (3) become

\[
\nabla^2 T_{\mu\nu\rho} - \frac{1}{2} (2 \nabla_\mu \nabla^\beta T_{\beta\nu\rho} - \nabla_\nu \nabla^\rho T_{\rho\mu\nu} + \nabla_\rho \nabla^\mu T_{\mu\nu\rho} - \nabla^\mu \nabla_\nu \nabla^\rho T_{\rho\mu\nu}) + \nabla^\rho T_{\beta\mu\nu} g_{\beta\sigma} + \nabla^\rho T_{\beta\nu\mu} g_{\beta\sigma} + M^2 T_{\mu\nu\rho} = S_{\mu\nu\rho}
\]

with no free parameter apart the mass of the field and a coupling constant contained in the source.

They develop the divergence

\[
M^2 \nabla_\mu T_{\mu\nu\rho} = \nabla_\mu S_{\mu\nu\rho}
\]

with no higher-order derivative and so a true constraint.

But the field equations develop also the divergence

\[
\nabla^2 \nabla_\mu T_{\mu\nu\rho} - \frac{1}{2} (2 \nabla_\mu \nabla^\beta T_{\beta\nu\rho} + \nabla_\nu \nabla^\rho T_{\rho\mu\nu} + \nabla_\rho \nabla^\mu T_{\mu\nu\rho} + \nabla^\beta T_{\alpha\beta\nu} g_{\mu\rho} + \nabla^\beta T_{\alpha\beta\rho} g_{\mu\nu} + \nabla_\mu \nabla^\rho T_{\rho\nu\sigma} + \nabla_\nu \nabla^\rho T_{\rho\mu\sigma} + \nabla_\mu T_{\mu\nu\rho} + M^2 \nabla_\mu T_{\mu\nu\rho} = \nabla_\mu S_{\mu\nu\rho}
\]

where again third-order derivatives have arisen although now the absence of free parameters leaves us without any freedom to adjust the coefficients so to remove them.

As these third-order derivatives will remain, the constraint will be preserved as a field equation, and the field will be over-determined and therefore unacceptable.

As a consequence, we are forced to the conclusion that the non-completely antisymmetric irreducible part of the torsion tensor is not well defined as a physical field.

This conclusion stands in line with the trend emerging from the Velo-Zwanziger analysis in [10]: while the scalar field is always well defined, spin-1 fields start to display consistency issues and as the spin goes higher the consistency problems tend to increase. In this paper, Velo and Zwanziger find that whereas spin-1 fields are still rather manageable, spin-2 fields require a number of constraints to be arbitrarily implemented for good position.

Here we are treating the non-completely antisymmetric irreducible part of torsion, a spin-3 field, and coming from the underlying background, and thus with no freedom for adjustment, and we regard in this rigidity the reason why such a part of torsion cannot have propagation.

### III. TORSION WITH SPIN AND THEIR INTERACTIONS

Having dismissed such a non-completely antisymmetric part of torsion as not well defined in its propagation and thus as not physical, the trace part $Q^\alpha$ and the dual of the completely antisymmetric part $W^\alpha$ will be the only fields we shall consider: their dynamics are given by the vector and axial-vector massive Proca field equations.

As for matter fields we will only consider the spinorial field $\psi$ (with $\overline{\psi}$ as conjugate) defined upon introduction of the Clifford algebra $\gamma^a$ from which $[\gamma^a, \gamma^b] = 4 \epsilon^{abc}\gamma^c$ and the implicit $2\sigma_{ab} = \epsilon_{abcd}\pi^{(a}c^{b)}$ are the relations defining the generators $\sigma^{ab}$ of the spinor group and the parity-odd matrix $\pi$ (which is merely the matrix usually indicated as gamma with an index five but in a notation in which the useless index is not in display): writing $\nabla_\mu \psi$ as the covariant derivative of spinor, we have that its dynamics is given by the usual Dirac Lagrangian, which is known to be well defined for the propagation of the spinors, and for spinors having no restrictions whatsoever.

With $Q^\alpha$ and $W^\alpha$ as well as $\overline{\psi}$ and $\psi$ and all combinations of the $\gamma^a$ matrices, we can now come up with all the possible torsion-spinor interaction terms: those involving the coupling of spinors to the axial-vector torsion $W^\alpha$ are found in [12]; again, because such a list of terms is quite independent on the field being an axial-vector or a vector, one may expect that very similar terms would appear for the vector torsion $Q^\alpha$ as well. This is indeed what shall happen; however, there are also properties that depend on the parity of the fields, and so some additional terms with products of vector $Q^\alpha$ and axial-vector $W^\alpha$ must be expected too. As an additional remark, we specify that all throughout this work we are going to consider only a
Lagrangian that display an explicit parity invariance.

The Lagrangian in this case is therefore given by

$$\mathcal{L} = -\frac{1}{4}(\partial W)^2 + \frac{1}{2}M_\mu^2 W^2 - \frac{1}{4}(\partial Q)^2 + \frac{1}{2}M_\mu^2 Q^2 +$$

$$+ i\bar{\psi} \gamma^\mu \nabla_\mu \psi - m \bar{\psi} \psi -$$

$$- X_W \bar{\psi} \gamma^\mu \pi W_{\mu} - X_Q \bar{\psi} \gamma^\mu \pi Q_{\mu} -$$

$$- \bar{\psi} \psi (B_W W^2 + B_Q Q^2) +$$

$$+ S_W \bar{\psi} \gamma^\mu \pi W Q + A_W \bar{\psi} \gamma^\mu \pi W Q_{\mu} +$$

$$+ R_W \bar{\psi} \gamma^\mu \pi W + D_W \bar{\psi} \gamma^\mu \pi W_{\mu} +$$

$$+ R_Q \bar{\psi} \gamma^\mu \pi Q + D_Q \bar{\psi} \gamma^\mu \pi Q_{\mu} +$$

$$+ Y_W \bar{\psi} \gamma^\mu \pi \nabla_\mu \psi + D_W \nabla_\mu \psi Q_{\mu} +$$

$$+ Y_Q \bar{\psi} \gamma^\mu \pi \nabla_\mu \psi Q_{\mu} +$$

$$+ Y_Q \bar{\psi} \gamma^\mu \pi \nabla_\mu \psi - \nabla_\mu \gamma^\mu \pi \psi Q_{\mu}$$

(8)

with 14 coupling constants. This will be our Lagrangian. Its variation would yield the field equations given by

$$\nabla_{\mu} (\partial W)^{\mu\nu} + (M_W^2 - 2B_W \bar{\psi} \psi) W^{\nu} = \nabla_{\mu} \bar{\psi} \gamma^\mu \pi \psi -$$

$$- S_W \bar{\psi} \gamma^\mu \pi W^{\mu} + 2A_W \bar{\psi} \gamma^\mu \pi W^{\mu} +$$

$$+ R_W \nabla_{\mu} \bar{\psi} \gamma^\mu \pi W_{\mu} + D_W \nabla_\mu \bar{\psi} \gamma^\mu \pi W_{\mu} -$$

$$- Y_W \nabla_{\mu} \bar{\psi} \gamma^\mu \pi \nabla_\mu \psi -$$

$$- Y_Q \nabla_{\mu} \bar{\psi} \gamma^\mu \pi \nabla_\mu \psi -$$

$$- Y_Q \nabla_{\mu} \bar{\psi} \gamma^\mu \pi \nabla_\mu \psi -$$

(9)

and

$$\nabla_{\mu} (\partial Q)^{\mu\nu} + (M_Q^2 - 2B_Q \bar{\psi} \psi) Q^{\nu} = Q \bar{\psi} \gamma^\mu \pi \psi -$$

$$- S_W \bar{\psi} \gamma^\mu \pi W^{\mu} - 2A_W \bar{\psi} \gamma^\mu \pi W^{\mu} +$$

$$+ R_Q \nabla_{\mu} \bar{\psi} \gamma^\mu \pi W_{\mu} + D_Q \nabla_\mu \bar{\psi} \gamma^\mu \pi W_{\mu} -$$

$$- Y_Q \nabla_{\mu} \bar{\psi} \gamma^\mu \pi \nabla_\mu \psi -$$

$$- Y_Q \nabla_{\mu} \bar{\psi} \gamma^\mu \pi \nabla_\mu \psi -$$

(10)

for the axial-vector and vector torsion with

$$i \gamma^\mu \nabla_\mu \psi - (m + B_W W^2 + R_Q Q^2) \psi -$$

$$- 2Y_W i \pi W_\mu \nabla_\mu \psi + Y_W' W^2 \nabla_\mu \psi -$$

$$- 2Y_Q \epsilon_\mu^\nu \nabla_\mu \psi + Y_Q' Q^2 \nabla_\mu \psi -$$

$$- X_W W_\mu \gamma^\mu \pi W_{\mu} - X_Q Q_\mu \gamma^\mu \pi Q_{\mu} +$$

$$+ S_W i \pi W_\mu \psi + A_W i \pi W_\mu Q_{\mu} +$$

$$+ (R_W + \frac{1}{2} Y_W) (\partial W)_{\mu\nu} \pi W^{\mu\nu} +$$

$$+ (D_W - \frac{1}{2} Y_W') \nabla_\mu W^\mu \pi W_{\mu} +$$

$$+ (R_Q - \frac{1}{2} Y_Q) (\partial Q)_{\mu\nu} \pi Q^{\mu\nu} +$$

$$+ (D_Q + \frac{1}{2} Y_Q') \nabla_\mu Q^\mu \psi = 0$$

(11)

for the spinor field. These are the field equations that we are going to employ in the rest of the present paper.

To perform the Velo-Zwanziger analysis we have to get the field equations with only the highest-order derivatives of the spinor and replace $\nabla_\mu \rightarrow n_\mu$, then obtaining

$$\det |\gamma^\mu n_\mu + 2Y_W \pi \sigma^{\mu\nu} n_\mu W_{\nu} - iY_W' \pi n_\mu W_{\mu} -$$

$$- 2iY_Q \sigma^{\mu\nu} n_\mu Q_{\nu} + Y_Q' n_\mu Q_{\mu}| = 0$$

(12)

as characteristic equation: its explicit form can be calculated very straightforwardly and it results as

$$|n^2|^2 [(1 + W^2)Y_W + Y_Q^2 Q^2]^2 +$$

$$+ 4|Y_Q|^2 |Y_W|^2 [(Q^2 W^2 - Q^2 W^2)]^2 +$$

$$+ |n^2|^{-2} [Y_Q^2 + Y_W^2]^2 \cdot$$

$$\cdot (1 - W^2) |Y_W|^2 + Y_Q^2 Q^2) |Q - n|^2 -$$

$$- 2|Y_Q^2 + Y_W^2|^2 \cdot$$

$$\cdot (1 + W^2) |Y_W|^2 + Y_Q^2 Q^2) W^n +$$

$$+ 8Y_Q Y_W (Y_W - Y_Q Y_W) \cdot$$

$$\cdot (Q - n) (W - n) - Q (W) +$$

$$+ 2[|Y_Q|^2 |Y_W|^2 + |Y_Q|^2 Y_W^2 -$$

$$- |Y_Q^2 |Y_W|^2 - |Y_Q|^2 Y_W^2 -$$

$$- 4Y_Q Y_W Y_Q (Q)^2 W - Q n^2 +$$

$$+ (|Y_Q|^2 + |Y_W|^2) Q^n|^2 +$$

$$+ (|Y_W|^2 + |Y_Q|^2) |W - n|^4 = 0$$

(13)

which now has to be discussed in specific circumstances corresponding to special cases that can always be taken.

A first thing to notice is that in circumstances in which torsion were to be weak, then (13) could be approximated down to the simpler characteristic equation given by

$$|n^2|^2 - 2n^2 |(Y_Q^2 + |Y_W|^2)| Q - n^2 +$$

$$+ |(Y_Q^2 + |Y_W|^2)| W - n^2 +$$

$$+ 2[|Y_Q|^2 |Y_W|^2 + |Y_Q|^2 Y_W^2 -$$

$$- |Y_Q^2 |Y_W|^2 - |Y_Q|^2 Y_W^2 -$$

$$- 4Y_Q Y_W Y_Q (Q)^2 W - Q n^2 +$$

$$+ (|Y_Q|^2 + |Y_W|^2) Q^n|^4 +$$

$$+ (|Y_W|^2 + |Y_Q|^2) |W - n|^4 \approx 0$$

(14)

which is easier to manipulate: if $Q$ were much smaller or at least smaller than $W$ then we would have

$$|n^2|^2 - 2n^2 |(Y_Q^2 + |Y_W|^2)| W - n^2 +$$

$$+ |(Y_Q^2 + |Y_W|^2)| W - n^2 +$$

admitting the only solution

$$n^2 \approx |(Y_Q^2 + |Y_W|^2)| W - n^2$$

(16)

for which the wave fronts are out of the light-cone unless we get $Y_W = Y_Q = 0$ identically; then (14) becomes

$$|n^2|^2 - 2n^2 |(Y_Q^2 + |Y_W|^2)| Q - n^2 +$$

$$+ |(Y_Q^2 + |Y_W|^2)| Q - n^2 +$$

admitting the only solution

$$n^2 \approx |(Y_Q^2 + |Y_W|^2)| Q - n^2$$

(18)

for which the wave fronts are out of the light-cone unless we get $Q = Y_Q = 0$ identically too. Consequently, we can
see that (14) reduces to the form $n^2 \approx 0$ and for which all wave fronts are within the interior of the light-cone.

As a consequence of this, the causal propagation does impose a restriction on the structure of the higher-order interactions, with field equations then being reduced to

$$
\nabla_\mu (\partial W)^{\mu\nu} + (M_W^2 - 2B_W \psi W) W^{\nu} - X_W \psi \gamma^\nu \psi - 
- S_W Q \psi \psi Q^{\nu} + 2A_W Q \psi + 
+ 2R_W \nabla_\mu (\psi \pi^{\mu\nu} \psi) + D_W \nabla^{\nu} (\psi \psi) 
$$

(19)

and

$$
\nabla_\mu (\partial Q)^{\mu\nu} + (M_Q^2 - 2B_Q \psi W Q) Q^{\nu} - X_Q \psi \gamma^\nu \psi - 
- S_W Q \psi W \psi Q^{\nu} + 2A_W W \psi \pi^{\mu\nu} \psi + 
+ 2R_Q \nabla_\mu (\psi \pi^{\mu\nu} \psi) + D_Q \nabla^{\nu} (\psi \psi) 
$$

(20)

for the axial-vector and vector torsion with

$$
i \gamma^a \nabla_\psi \psi - (m + B_W W^2 + R_Q Q^2) \psi - 
- X_W \nabla_\mu \pi^{\mu\nu} \psi - X_Q \gamma^\nu \psi + 
+ S_W Q \psi W \psi Q^{\nu} + 2A_W W \psi \pi^{\mu\nu} \psi + 
+ R_W (\partial W)^{\mu\nu} \pi^{\mu\nu} \psi + D_W \nabla_\mu \pi^{\mu\nu} \psi + 
+ R_Q (\partial Q)^{\mu\nu} \pi^{\mu\nu} \psi + D_Q \nabla^{\nu} Q^{\mu\nu} \psi = 0
$$

(21)

for the spinor field. So far as the Velo-Zwanziger analysis is concerned, field equations cannot be any more general. Nonetheless, special cases are indeed possible.

### IV. CONSTANT TORSION

Up to now we have seen how the Velo-Zwanziger analysis restricts the structure of the field equations. However, there are specific situations in which further reductions are implementable: in fact, one of the physical situations in which these field equations can be used is in the study of possible Lorentz symmetry violations [7, 8].

In these papers, the authors consider Lagrangians such as the one we have examined here, to assess whether, for some instance, Lorentz symmetry may be violated: they show that for constant torsion there may be the breaking of some Lorentz transformation. Nevertheless, a constant torsion cannot be assumed and instead it would have to be obtained as solution of the field equations.

In [7, 8] the torsional field equations are never studied, nor even presented, but here we have, and therefore such a study can be done: constant torsion would be compatible with the torsion field equations whenever

$$(M_W^2 - 2B_W \psi W) W^{\nu} = X_W \psi \gamma^\nu \psi - 
- S_W Q \psi \psi Q^{\nu} + 2A_W Q \psi + 
+ 2R_W \nabla_\mu (\psi \pi^{\mu\nu} \psi) + D_W \nabla^{\nu} (\psi \psi) 
$$

(22)

and

$$(M_Q^2 - 2B_Q \psi W Q) Q^{\nu} = X_Q \psi \gamma^\nu \psi - 
- S_W Q \psi W \psi Q^{\nu} + 2A_W W \psi \pi^{\mu\nu} \psi + 
+ 2R_Q \nabla_\mu (\psi \pi^{\mu\nu} \psi) + D_Q \nabla^{\nu} (\psi \psi) 
$$

(23)

which would convert the Lagrangian (8) into

$$
\mathcal{L} = i \gamma^\mu \nabla_\mu \psi - m \psi - 
- \frac{1}{2} X_W \psi \gamma^\mu \psi W^{\mu} - \frac{1}{2} X_Q \psi \gamma^\mu \psi Q^{\mu}
$$

(24)

as it can be checked with a straightforward substitution.

Thus, constant torsion would reduce the Lagrangian to the one we have without higher-order mass-dimensional terms at all. So if the Lagrangian considered in [7] was to be taken in deep examination, it would become evident that there is no Lorentz symmetry violation, or at least that there would be no Lorentz symmetry violation apart from one that would also be present with the standard renormalizable Lagrangian in the most general case.

We conclude therefore that the assumption of constant torsion exceeds the boundary of its applicability.

### V. CONCLUSION

In this paper, we have considered the Lagrangian that would arise from allowing all propagating torsional terms as well as all consistent interactions between torsion and spinors fields; after writing the Lagrangian, we proceeded in finding the field equations, studying them in terms of the Velo-Zwanziger method, removing all the terms that are found to be inconsistent with such an analysis.

As a first result, we found that, of the three irreducible parts of torsion, which we indicated with $T$, $Q$, and $W$, a number of restrictions took place for the non-completely antisymmetric irreducible component: we could not write field equations compatibly with the requirement that the number of degrees of freedom must match the number of independent field equations. This circumstance is consistent with the consideration that the higher the spin of a field the more difficult is to define its field equations.

Having ruled out that the non-completely antisymmetric irreducible component $T$ we have been left with the two vector components $Q$ and $W$ and as a consequence we built the Lagrangian with these two alone coupling to the spinor field: after obtaining the field equations, we computed the characteristic equations witnessing that it was always possible to find situations where acausal propagation would arise unless four coefficients were zero identically, and in doing so we eventually established the most general set of field equations compatible with all restrictions imposed by the Velo-Zwanziger analysis.

We finally considered works such as those of Kostelecky and co-workers about Lorentz symmetry violations based on the assumption of constant torsion, proving that when such models are studied in detail it is clear that the assumption of constant torsion is non-sensical.

At the end of this analysis, we can thus summarize the novelties of the extended Lagrangian (8) in these: terms proportional to the constants $B_W$ and $B_Q$ can be seen as corrections to the masses of torsion and as the mass of the spinor field; terms proportional to $R_W$ and $R_Q$ are those related to the coupling of torsion to the electric-like and
magnetic-like di-pole spinor quantities; $D_W$ and $D_Q$ are the scalar analogous of the two we just mentioned, with coupling to the scalar and pseudo-scalar spinor quantities that would make them a mass and a pseudo-mass for the spinor field; terms in $S_{WQ}$ and $A_{WQ}$ instead do not have a clear interpretation because they describe a new type of interaction involving both torsion and the spinor field, although it is clear that the former is a pseudo-mass term and the latter is an electric-like di-pole interaction.

We have not studied what this new type of interaction could give as effect, leaving it to following works.

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