A detailed comparison study of first order and higher order shear deformation theories in the analysis of laminated composite plate

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Abstract. The primary aim of the present work is to calculate and compare the response of composite plate using first order and higher order shear deformation theories. The present study initially attempts to develop a finite element formulation for handling the analysis of laminated composite plates. The current study elaborately discusses the formulation that makes an easy programming even for a beginner in this field. Presently, mathematical formulation and Matlab coding using First Order Shear Deformation Theory (FSDT) and Higher Order Shear Deformation Theory (HSDT) had done. Results obtained were compared with the available literature. Parametric study also conducted to clearly understand the variation in results obtained from both FSDT and HSDT.

1. Introduction

Plates form an essential part of many aerospace, marine, and automobile structures. Aircraft and spacecraft structures consist of a large number of flat and curved panel type structural elements. Increased usage of composite laminated plates in crucial structures demands the development of precise theoretical models to predict their response. Plates may be classified into three groups according to the ratio of length/thickness as thick, moderately thick, and thin plates. The behaviour of thin plate structures has been the subject of a number of investigations. The above classification is, of course, conditional because the reference of the plate to one or another group depends on the accuracy of analysis, type of loading, boundary conditions, etc. Thus due to some reason the behaviour of plate response may vary as small deflection or large deflection. The large deflection theory assumes that the deflections are sufficiently large (they can be comparable with the plate thickness or larger), but they should remain small relative to the other dimensions of the plate (except for its thickness). It should be also noted that the deflections of the plate are not assumed to be small, compared with its thickness, but at the same time still sufficiently small to justify an application of the simplified formulas for the plate curvatures. Finally, the large deflection theory deals with finite deflections. However, the relative deformations (strains) are assumed to be small quantities.

Different materials can be combined on a microscopic scale, such as in alloying of metals to form plate like structures, but the resulting material is, for all practical purposes, macroscopically homogeneous, i.e. the components cannot be distinguished by the naked eye and essentially acts
together. The word composite in the term of composite material signifies two or more materials are combined on a macroscopic scale to form useful third material. The benefit of composite materials is that, if properly designed, they generally show the best qualities of their components or constituents and frequently certain qualities that neither constituent owns. Also, in many cases, use of composites is more efficient. For example, in aircraft industry, most of the research work is to look for the ways to lower the overall weight of the aircraft without reducing the stiffness and strength of its components. In the past few decades, astonishing advances in sciences and technology have motivated researches to work on new structural materials. The development of composite materials has improved the performance and reliability of structural system. Aerospace structure engineering application requires an accurate prediction of system behavior of structure made up of composites. In the context of optimum design of aircraft components, it is necessary to have a fundamental understanding of their deformation characteristics. In the present work, bending behavior of laminated composite plates will be studied using first and higher order shear deformation theories in detail. The primary aim of the present study is to make a suitable solution technique with finite element method for bending analysis of a laminated composite plate using FSDT and HSDT and find out the variation in results obtained from both FSDT and HSDT. The important goal of the current study is to demonstrate elaborately the formulation that makes an easy programming.

2. Literature review

A few significant works which used FEM are incorporated in this paragraph. These important works using FEM helps for the readers who are learning these FEM concepts and doing formulations using any shear deformation theory. Zienkiewicz [1] studied structural behaviour using FEM. He discussed in detail about von Karman nonlinearity and geometric stiffness matrix associated with the membrane forces. Reddy [2] has described in detail about the laminated composite plates. Analytical and finite element derivations are discussed by Reddy [2] in detail. Solutions for bending, buckling, and vibration are also presented. He presented a good description of the mechanics and associated finite element models of laminated composite structures. Agarwal et al. [3] and Jones [4] presented in detail the fundamental and advance topics related to composite structures. Bhavikatti [5] has discussed the finite element concept and applications to simple structures in detail. Also, application of isoparametric concept to complex problems is discussed. Finite element formulations are made clear by solving simple problems by hand calculation. Sreehari and Maiti [6] presented in detail the fundamental and advanced topics related to composite structures. Many works are available with descriptions on the computational aspects of FEM. Chandrupatla and Belegundu [7], Ferreria [8], Cook et al. [9], and Kwon and Beng [10] discussed in detail about the finite element coding with numerous examples. Literatures with FEM have employed various shear deformation theories for finding solutions, like classical, first-order, and third-order plate theories.

3. Mathematical formulation

Consider a laminated plate, as in figure 1 as in reference [2], comprising of \( N \) orthotropic layers with the principle material co-ordinates \((x_1^k, x_2^k, x_3^k)\) of the \( k^{th} \) lamina oriented at an angle \( \theta^k \) to the laminate co-ordinate, \( x \). The length, width, and thickness of plate are \( a, b, \) and \( h \) respectively. The co-ordinate system has its origin at the corner of the plate on the mid plane. The \( z \)-axis is taken positive downward from the mid plane.
Figure 1. Coordinate system and layer numbering used for a laminated plate.

3.1 Displacement field

A simple higher order shear deformation theory in which transverse shear strains are assumed to be parabolically distributed across the plate thickness is considered initially. The displacement components are assumed to be in the form:

\[
\begin{bmatrix}
    u(x, y, z, t) \\
    v(x, y, z, t) \\
    w(x, y, z, t)
\end{bmatrix} =
\begin{bmatrix}
    u_0(x, y, z, t) \\
    v_0(x, y, z, t) \\
    w_0(x, y, z, t)
\end{bmatrix} + z \begin{bmatrix}
    \phi_x(x, y, t) \\
    \phi_y(x, y, t)
\end{bmatrix} + z^2 \begin{bmatrix}
    \beta_x(x, y, t) \\
    \beta_y(x, y, t)
\end{bmatrix} + z^3 \begin{bmatrix}
    \psi_x(x, y, t) \\
    \psi_y(x, y, t)
\end{bmatrix}
\]

(1)

Where \(u, v, \) and \(w\) are the displacement components in \(x, y, \) and \(z\) directions respectively; \(u_0, v_0, w_0\) are the displacements of a point on the mid plane \((x, y, 0)\). \(\phi, \phi, \beta, \beta, \psi, \psi\) are the rotations of the cross-section perpendicular to \(x\) and \(y\) axes respectively. The parameters \(\beta, \beta, \psi, \psi\) are the higher order terms in Taylor’s series expansion and they represent higher order transverse cross-sectional modes. For the case of FSDT, displacement field will be as shown below,

\[
\begin{bmatrix}
    u(x, y, z, t) \\
    v(x, y, z, t) \\
    w(x, y, z, t)
\end{bmatrix} =
\begin{bmatrix}
    u_0(x, y, z, t) \\
    v_0(x, y, z, t) \\
    w_0(x, y, z, t)
\end{bmatrix} + z \begin{bmatrix}
    \phi_x(x, y, t) \\
    \phi_y(x, y, t)
\end{bmatrix}
\]

3.2 Strain-displacement relations

The linear strain-displacement relations are used in formulating the governing differential equations and are given as:

\[
\varepsilon_{xx} = \frac{\partial u}{\partial x}, \varepsilon_{yy} = \frac{\partial v}{\partial y}, \gamma_{xy} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}, \gamma_{yx} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \gamma_{yz} = \frac{\partial w}{\partial z} + \frac{\partial u}{\partial x}
\]

(2)
3.3 Constitutive relations

For laminate composed of orthotropic layers, with their $x_i x_j$-plane oriented arbitrarily with $xy$-plane ($x_3 = 0$), the transverse stresses ($\sigma_{2x}$, $\sigma_{3y}$) are also zero. An orthotropic material is characterized by nine elastic moduli and has three planes of elastic symmetry. Under the assumption that material behaves linearly elastic, the constitutive relation for each lamina can be written as:

$$\{\sigma\} = [Q] \{\varepsilon\}$$

Where the components of Compliance matrix, $Q$ are expressed in terms of material properties and are given by the equation

$$Q_{11} = \frac{E_{11}}{(1 - v_{12} v_{23})}, \quad Q_{22} = \frac{E_{22}}{(1 - v_{12} v_{23})}, \quad Q_{12} = \frac{v_{11} E_{11}}{(1 - v_{12} v_{23})},$$

$$Q_{44} = G_{23}, \quad Q_{55} = G_{13}, \quad Q_{66} = G_{12}$$

Stress-strain relations in the local co-ordinate system can be expressed as:

$$\begin{bmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\tau_{12} \\
\tau_{23} \\
\tau_{31}
\end{bmatrix} = \begin{bmatrix}
Q_{11} & Q_{12} & Q_{16} & 0 & 0 \\
Q_{12} & Q_{22} & Q_{26} & 0 & 0 \\
Q_{16} & Q_{26} & Q_{66} & 0 & 0 \\
0 & 0 & 0 & Q_{44} & Q_{45} \\
0 & 0 & 0 & Q_{45} & Q_{55}
\end{bmatrix} \begin{bmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\gamma_{12} \\
\gamma_{23} \\
\gamma_{31}
\end{bmatrix}$$

Where, $Q_{ij}$'s are transformed reduced stiffness coefficients and expressed as:

$$Q_{11} = \begin{bmatrix}
m^2 & 2 m n & n^2 & 4 m^2 n^2 & 0 & 0 \\
2 m n & m^2 + n^2 & m^2 n^2 & -4 m^2 n^2 & 0 & 0 \\
m^2 n & -m^2 n & -m^2 n & 4 m^2 n^2 & 0 & 0 \\
4 m^2 n^2 & 4 m^2 n^2 & 4 m^2 n^2 & 0 & 0 \\
0 & 0 & 0 & m^2 & n^2 \\
0 & 0 & 0 & n^2 & m^2
\end{bmatrix}$$

where $m = \cos \theta$ and $n = \sin \theta$

3.4 Formulation for finite element method for FSDT

The strain–displacement relations given above are written using FSDT as:

$$\varepsilon_{xx} = \varepsilon_x = \frac{\partial u}{\partial x} + \varepsilon_x^t + z k_x$$

$$\varepsilon_{yy} = \varepsilon_y = \frac{\partial v}{\partial y} + \varepsilon_y^t + z k_y$$

$$\gamma_{xy} = \varepsilon_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} + \varepsilon_{xy}^t + z k_{xy}$$

$$\gamma_{xz} = \varepsilon_{xz} = \frac{\partial u}{\partial z}$$

$$\gamma_{yz} = \varepsilon_{yz} = \frac{\partial v}{\partial z}$$

$$\gamma_{zx} = \varepsilon_{zx} = \frac{\partial u}{\partial x}$$

$$\gamma_{zy} = \varepsilon_{zy} = \frac{\partial v}{\partial x}$$
The linear strain vector given in above equation can also be expressed in terms of midplane strain vector, \( \{\varepsilon\} \)

\[
\{\varepsilon\}_{x=1} = [T]_{x=1} \{\varepsilon\}_{x=1}
\]

Where,

\[
[T] = \begin{bmatrix}
1 & 0 & 0 & z & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & z & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{bmatrix}
\]

Also,

\[
\{\varepsilon\}_{x=1} = [L]_{x=1} \{\Delta\}_{x=1}
\]

Where,

\[
\{\Delta\} = \begin{bmatrix}
u_0 & w_0 & \phi_x & \phi_y
\end{bmatrix}^T
\]

and

\[
[L] = \begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 & 0 \\
0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\
0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial y} \\
0 & 0 & 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\
0 & 0 & 0 & \frac{\partial}{\partial x} & 0 \\
0 & 0 & 0 & \frac{\partial}{\partial x} & 1
\end{bmatrix}
\]

\[
D = T^T \overline{Q} T
\]

\[
D = \begin{bmatrix}
\overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} & z\overline{Q}_{11} & z\overline{Q}_{12} & z\overline{Q}_{16} & 0 & 0 \\
\overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} & z\overline{Q}_{22} & z\overline{Q}_{26} & 0 & 0 & 0 \\
\overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} & z\overline{Q}_{16} & z\overline{Q}_{26} & z\overline{Q}_{66} & 0 & 0 \\
z\overline{Q}_{11} & z\overline{Q}_{12} & z\overline{Q}_{16} & z^2\overline{Q}_{11} & z^2\overline{Q}_{12} & z^2\overline{Q}_{16} & 0 & 0 \\
z\overline{Q}_{12} & z\overline{Q}_{22} & z\overline{Q}_{26} & z^2\overline{Q}_{22} & z^2\overline{Q}_{26} & 0 & 0 & 0 \\
z\overline{Q}_{16} & z\overline{Q}_{26} & z\overline{Q}_{66} & z^2\overline{Q}_{16} & z^2\overline{Q}_{26} & z^2\overline{Q}_{66} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]
3.5 Introducing the HSDT

There are nine dependent unknowns in the displacement field given by equation (1). The number of dependent unknowns can be reduced by imposing the traction-free boundary conditions given by equations below on the top and bottom faces of the laminate.

\[ \sigma_{ys}(x, y, \pm h / 2) = 0 \quad \text{and} \quad \sigma_{yz}(x, y, \pm h / 2) = 0 \quad \sigma_{ys} \]

If the transverse shear stresses are to vanish at the bounding planes of the plate \((z = \pm h / 2)\), the transverse shear strains, \(\gamma_{yz}\) and \(\gamma_{xy}\) must also vanish there, i.e.,

\[ \gamma_{yz}(x, y, \pm h / 2) = 0 \quad \text{and} \quad \gamma_{xy}(x, y, \pm h / 2) = 0 \]

Using strain-displacement relations given by equations (2) and displacement field given by equation (1) in above equation, parameters \(\beta_x, \beta_y, \psi_x, \psi_y\) can be determined in the form:

\[
\psi_x = \frac{-4}{3h^2} \left[ \phi_x + \frac{\partial w}{\partial x} \right], \quad \psi_y = \frac{-4}{3h^2} \left[ \phi_y + \frac{\partial w}{\partial y} \right], \quad \beta_x = \beta_y = 0
\]

Using this equation, the displacement field given in equation (1) can now be expressed in terms of five dependent unknowns \((u_0, v_0, w_0, \phi_x, \phi_y)\). The modified displacement field is now written in terms of \((u_0, v_0, w_0, \phi_x, \phi_y)\).

\[
\begin{pmatrix}
  u(x, y, z, t) \\
  v(x, y, z, t) \\
  w(x, y, z, t)
\end{pmatrix} =
\begin{pmatrix}
  u_0(x, y, z, t) \\
  v_0(x, y, z, t) \\
  w_0(x, y, z, t)
\end{pmatrix} + z \begin{pmatrix}
  \phi_x(x, y, t) \\
  \phi_y(x, y, t) \\
  0
\end{pmatrix} - c_1 z^3 \begin{pmatrix}
  \frac{\partial w_0}{\partial x} \\
  \frac{\partial w_0}{\partial y} \\
  0
\end{pmatrix}
\]

The significance of constant \(c_1\) is that it facilitates the representation of FSDT and HSDT through same equation. For \(c_1 = 4 / 3h^2\), equation is the case of HSDT which contains the same unknown parameters as in the case of FSDT. For \(c_1 = 0\), equation above is for the case of FSDT. Equation (1) can be now written as:

\[
\begin{align*}
\{ \varepsilon_{xx} \} &= \{ \varepsilon_{xx}^0 \} + z \{ \varepsilon_{xx}^1 \} + z^2 \{ \varepsilon_{xx}^3 \} \\
\{ \varepsilon_{yy} \} &= \{ \varepsilon_{yy}^0 \} + z \{ \varepsilon_{yy}^1 \} + z^2 \{ \varepsilon_{yy}^3 \} \\
\{ \gamma_{yy} \} &= \{ \gamma_{yy}^0 \} + z \{ \gamma_{yy}^1 \} + z^2 \{ \gamma_{yy}^2 \} \\
\{ \gamma_{xy} \} &= \{ \gamma_{xy}^0 \} + z \{ \gamma_{xy}^1 \} + z^2 \{ \gamma_{xy}^2 \}
\end{align*}
\]

Introducing \(c_1 = 4 / 3h^2\) and \(c_2 = 3c_1\).
Where,
\[
\begin{align*}
\begin{bmatrix}
\epsilon_{xx} & \epsilon_{yy} & \gamma_{xy} \\
\epsilon_{yy} & \epsilon_{zz} & \gamma_{yz} \\
\gamma_{xy} & \gamma_{yz} & \gamma_{zz}
\end{bmatrix} &=
\begin{bmatrix}
\frac{\partial u}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial w}{\partial z} \\
\frac{\partial v}{\partial x} & \frac{\partial \phi_x}{\partial y} & \frac{\partial \phi_y}{\partial z} \\
\frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} & \frac{\partial \phi_y}{\partial x} + \frac{\partial \phi_z}{\partial y} & \frac{\partial \phi_z}{\partial x} + \frac{\partial \phi_x}{\partial y}
\end{bmatrix}
\end{align*}
\]
and
\[
\begin{align*}
\begin{bmatrix}
\phi_x \\
\phi_y \\
\phi_z \\
\phi_x \\
\phi_y \\
\phi_z
\end{bmatrix} &=
\begin{bmatrix}
\phi_x \\
\phi_y \\
\phi_z \\
\phi_x \\
\phi_y \\
\phi_z
\end{bmatrix}
\end{align*}
\]

\[\begin{align*}
\phi_x &= \epsilon_x + \frac{\partial w}{\partial y} \\
\phi_y &= \epsilon_y + \frac{\partial w}{\partial x} \\
\phi_z &= \epsilon_z + \frac{\partial w}{\partial x}
\end{align*}\]

3.6 Formulation for finite element method for HSDT

The strain–displacement relations are written using HSDT as:

\[
\begin{align*}
\epsilon_{xx} &= \frac{\partial u}{\partial x} = \epsilon_x^0 + z k^0_x - c_1 k^1_x z^1 \\
\epsilon_{yy} &= \frac{\partial v}{\partial y} = \epsilon_y^0 + z k^0_y - c_1 k^1_y z^1 \\
\gamma_{xy} &= \frac{\partial w}{\partial y} + \frac{\partial v}{\partial x} = \epsilon_{xy}^0 + z k^0_y - c_1 k^1_y z^1 \\
\gamma_{yz} &= \frac{\partial w}{\partial x} + \frac{\partial u}{\partial y} = \epsilon_{yz}^0 + z k^0_y - c_1 k^1_y z^1 \\
\gamma_{zy} &= \frac{\partial u}{\partial x} = \epsilon_{zy}^0 - 3 c_1 k^1_y z^1 \\
\gamma_{zz} &= \frac{\partial w}{\partial x} = \epsilon_{zz}^0 - 3 c_1 k^1_y z^1
\end{align*}
\]

where,

\[
\begin{align*}
\epsilon_x^0 &= \frac{\partial u}{\partial x} \\
\epsilon_y^0 &= \frac{\partial v}{\partial y} \\
\epsilon_z^0 &= \frac{\partial w}{\partial z} \\
k_1^1 &= \frac{\partial \phi_x}{\partial x} + \frac{\partial \theta_y}{\partial x} \\
k_2^1 &= \frac{\partial \phi_y}{\partial y} + \frac{\partial \phi_z}{\partial y} \\
k_3^1 &= \phi_x + \theta_y
\end{align*}
\]

Similarly as above, the linear strain vector given in above equation can also be expressed in terms of midplane strain vector, \(\{\epsilon\}\)

\[
\{\epsilon\}_{E-1} = [T]_{E-1}\{\epsilon\}_{E-1}
\]

\[
\{\epsilon\} = \begin{bmatrix}
\epsilon_x^0 & \epsilon_y^0 & k_1^0 & k_2^0 & k_3^0 & \epsilon_x^1 & \epsilon_y^1 & k_1^1 & k_2^1 & k_3^1 & \epsilon_z^0 & k_1^2 & k_2^2
\end{bmatrix}^T
\]
\[
[r] = \begin{bmatrix}
1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & z & 0 & 0 & z^3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}
\]

\[
\{\bar{e}\} = [L]_{13x1}, \{\Lambda\}_{1x1}
\]

Where,
\[
\{\Lambda\} = \{u_0, v_0, w_0, \phi_x, \phi_y, \theta_x, \theta_y\}^T
\]

and,
\[
\begin{bmatrix}
\frac{\partial}{\partial x} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 \\
\frac{\partial}{\partial y} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} \\
0 & 0 & \frac{\partial}{\partial y} & 0 & \frac{\partial}{\partial x} & 0 & \frac{\partial}{\partial x} \\
0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 & 0 \\
0 & 0 & \frac{\partial}{\partial y} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{\partial}{\partial x} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\partial}{\partial y} & 0 & 0 \\
\end{bmatrix}
\]

\[
D = \bar{T}^T \bar{Q} \bar{T}
\]

### 3.7 Potential energy of the laminate

The present analysis involves structural displacement due to external mechanical loading. The total energy of the system can thus be considered as the strain energy due to mechanical loading. Thus, the potential energy of the laminated composite plate undergoing deformation is given as Potential Energy=Strain Energy

\[
U = \frac{1}{2} \int \{\bar{e}\}^T \{\bar{\sigma}\} dV
\]

The stress strain relation can be written as:

\[
\{\bar{\sigma}\} = [\bar{Q}] \{\bar{e}\}
\]

Using the equations the equation for potential energy can be written as

\[
U = \frac{1}{2} \int (\{\bar{e}\}^T \{\bar{Q}\} \{\bar{e}\}) dV = \frac{1}{2} \int \bar{e}^T \bar{T}^T \bar{Q} \bar{T} \bar{e} dV
\]
Or

\[ U = \frac{1}{2} \int \{ \bar{e} \}^T \{ D \} \{ \bar{e} \} dA \]

Where

\[ \{ D \} = \sum_{z=1}^{N} \int_{z-1}^{z} \{ T \}^T \{ \bar{e} \} \{ T \} dz \]

Thus the expression for potential energy becomes

\[ U = \frac{1}{2} \int \{ \Lambda \}^T \{ L \}^T \{ D \} \{ L \} \{ \Lambda \} dA \]

### 3.8 Solution method

Solution methodologies for present analysis are presented. Also the implementation of finite element method with 8- noded isoparametric elements is presented.

The domain is divided into number of sub-domains that are known as finite elements. These elements are connected at various nodes. For the finite element analysis,

\[ U = \sum_{e=1}^{N_E} U^{(e)} \]

\[ U = \sum_{e=1}^{N_E} \frac{1}{2} \int \{ \Lambda \}^{(e)T} \{ L \}^T \{ D \} \{ L \} \{ \Lambda \}^{(e)} dA \]

Where \( N_E \) is the number of elements used for meshing the plate

The displacement vector \( \Delta \) can be written in terms of shape functions \( N_i \) and displacement vector, \( q \) for an element as \( \{ \Delta \}^{(e)} = [N_i]^{(e)} \{ q \}^{(e)} \)

On substituting, element potential energy can be written as

\[ U = \sum_{e=1}^{N_E} \frac{1}{2} \int \{ q \}^{(e)T} \{ N \}^{(e)T} \{ L \}^T \{ D \} \{ L \} \{ N \}^{(e)} \{ q \}^{(e)} dA \]

Element potential energy can be written as

\[ U^{(e)} = \frac{1}{2} \int \{ q \}^{(e)T} \{ B \}^{(e)T} \{ D \} \{ B \}^{(e)} \{ q \}^{(e)} dA^{(e)} \]

\[ \{ B \}^{(e)} = [L][N]^{(e)} \]

Where \( \{ B \}^{(e)} = [B_1 \ B_2 \ B_3 \ \cdots \ \ B_{NN}] \)

Element bending stiffness matrix is defined as

\[ K^{(e)} = \int B^{(e)T} D B^{(e)} dA^{(e)} \]

Thus finally the elemental potential energy can be written as

\[ U^{(e)} = \frac{1}{2} \{ q \}^{(e)T} K^{(e)} \{ q \}^{(e)} \]
Now, $K^{(e)}$ is computed numerically by transforming the existing coordinate system to natural coordinate system $\xi$ and $\eta$, and then can be written as:

$$K^{(e)} = \int_{-1}^{1} \int_{-1}^{1} B^T D B \det J d\xi d\eta$$

Where, $J$ is the Jacobian Matrix and is given by

$$J = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial x}{\partial \eta} \\ \frac{\partial y}{\partial \xi} & \frac{\partial y}{\partial \eta} \end{bmatrix}$$

When numerical integration is adopted, the element matrix of equation becomes:

$$K^{(e)} = \frac{1}{2} \sum_{p=1}^{n} \sum_{q=1}^{n} W_p W_q B^T D B \det J$$

Where $W_p, W_q$ are the weights used in the Gaussian quadrature and work done:

$$W = \iint W_p W_q dxdy$$

4. Results and discussions

Finite element method’s results for laminated composite plates are obtained by analyzing the formulation explained in previous section and programming in MATLAB. An eight noded $C_0$, isoparametric element has been employed for discretization of the laminate. For the FSDT, a shear correction factor 5/6 has been used. Based on convergence study, a $(12 \times 12)$ mesh has been used in most cases of later study. In all problems considered, the individual layers are taken to be of equal thickness. A variety of problems is studied and is compared the result to the existing results. The finite element method provides a numerical solution to a complex problem, it may therefore be expected that the solution must converge to the exact solution under certain circumstances. It can be shown that the displacement formulation of the method leads to be upper bound to the actual stiffness of the structure. Hence as the mesh is made finer, the solution should converge to the correct result. Non dimensional results are presented. The non-dimensionality used for transverse deflection is

$$\bar{w} = w \left( \frac{100h^3 E_z}{q_0d^4} \right)$$

Convergence of the solution with refinement in mesh for four layered, symmetric and anti-symmetric cross-ply laminate with $a/h$ ratio 10 to 100 is shown in table 1. Similarly convergence of solution with refinement in mesh for four layered, symmetric and anti-symmetric angle-ply laminate with $a/h$ ratio 10 to 100 is shown in table 2. As the number of mesh increases the convergence of the results is found to be fairly accurate (also indicated as percentage variation in table 3). It is clear from the obtained results that thick plates have high deflection and deflection becomes almost constant after $a/h$ ratio of 40. As the number of layers increases, the deflection becomes almost constant. From the table 1 and 2, it can be concluded that deflections decreases as the $a/h$ ratio is increased or number of layers are increased.
Table 1. Non-dimensional central deflection for symmetric and antisymmetric cross-ply, simply supported-1, subjected to sinusoidal load.

| a/h | Mesh size | 0/90/90/0 | 0/90/0/90 |
|-----|-----------|----------|----------|
| 10  | 2x2       | 0.61078599 | 0.63179162 |
|     | 4x4       | 0.66083444 | 0.67825433 |
|     | 6x6       | 0.66220033 | 0.67967253 |
|     | 8x8       | 0.66240051 | 0.67989676 |
|     | 12x12     | 0.66246599 | 0.67996510 |
|     | 16x16     | 0.66247432 | 0.67973649 |
| 20  | 2x2       | 0.46912938 | 0.53907672 |
|     | 4x4       | 0.49103859 | 0.54983221 |
|     | 6x6       | 0.49103351 | 0.54978842 |
|     | 8x8       | 0.49101430 | 0.54976667 |
|     | 12x12     | 0.49100115 | 0.54975900 |
|     | 16x16     | 0.49099687 | 0.54975188 |
| 30  | 2x2       | 0.43560223 | 0.49658543 |
|     | 4x4       | 0.45819491 | 0.52598234 |
|     | 6x6       | 0.45800222 | 0.52573235 |
|     | 8x8       | 0.45794732 | 0.52566923 |
|     | 12x12     | 0.45792116 | 0.52561919 |
|     | 16x16     | 0.45791485 | 0.52563615 |
| 40  | 2x2       | 0.41897476 | 0.47759340 |
|     | 4x4       | 0.44652651 | 0.51756527 |
|     | 6x6       | 0.44632058 | 0.51739001 |
|     | 8x8       | 0.44625650 | 0.51723449 |
|     | 12x12     | 0.44622675 | 0.51720223 |
|     | 16x16     | 0.44621994 | 0.51719562 |
| 50  | 2x2       | 0.40730782 | 0.45964784 |
|     | 4x4       | 0.44105045 | 0.51359891 |
|     | 6x6       | 0.44088891 | 0.51340659 |
|     | 8x8       | 0.44082313 | 0.51329800 |
|     | 12x12     | 0.44079241 | 0.51329577 |
|     | 16x16     | 0.44078554 | 0.51328882 |
| 60  | 2x2       | 0.39765708 | 0.44250435 |
|     | 4x4       | 0.43801956 | 0.51137254 |
|     | 6x6       | 0.43792978 | 0.51128330 |
|     | 8x8       | 0.43786524 | 0.51120817 |
|     | 12x12     | 0.43783448 | 0.51117365 |
|     | 16x16     | 0.43782769 | 0.51116658 |
| 70  | 2x2       | 0.38909467 | 0.42602464 |
|     | 4x4       | 0.43613903 | 0.50995674 |
|     | 6x6       | 0.43614101 | 0.50999967 |
|     | 8x8       | 0.43607919 | 0.50992837 |
|     | 12x12     | 0.43604881 | 0.50989400 |
|     | 16x16     | 0.43604216 | 0.50988990 |
| 80  | 2x2       | 0.38130352 | 0.41013982 |
|     | 4x4       | 0.43486581 | 0.50896293 |
|     | 6x6       | 0.43497686 | 0.50916328 |
|     | 8x8       | 0.43491874 | 0.50909726 |
|     | 12x12     | 0.43488090 | 0.50906339 |
|     | 16x16     | 0.43488242 | 0.50905632 |
| 90  | 2x2       | 0.37417221 | 0.39489880 |
|     | 4x4       | 0.43393949 | 0.50820534 |
|     | 6x6       | 0.43417607 | 0.50836602 |
|     | 8x8       | 0.43412339 | 0.50832790 |
|     | 12x12     | 0.43409321 | 0.50849385 |
|     | 16x16     | 0.43408689 | 0.50848685 |
| 100 | 2x2       | 0.36765554 | 0.38038078 |
|     | 4x4       | 0.43322246 | 0.50758606 |
|     | 6x6       | 0.43360838 | 0.50817094 |
|     | 8x8       | 0.43355225 | 0.50811870 |
|     | 12x12     | 0.43352378 | 0.50808641 |
|     | 16x16     | 0.43351762 | 0.50807949 |
Table 2. Non-dimensional transverse deflection for angle-ply, simply supported-2, sinusoidal load.

| a/h | Mesh size | -45/45/45/-45 | -45/45/-45 |
|-----|-----------|----------------|------------|
| 10  | 4x4       | 0.4970194      | 0.4546066  |
|     | 6x6       | 0.4989648      | 0.4553509  |
|     | 8x8       | 0.4989408      | 0.4552053  |
|     | 10x10     | 0.4988180      | 0.4550849  |
|     | 12x12     | 0.4987020      | 0.4550050  |
| 20  | 4x4       | 0.3541954      | 0.3257407  |
|     | 6x6       | 0.3563154      | 0.3252877  |
|     | 8x8       | 0.3564358      | 0.3249972  |
|     | 10x10     | 0.3563869      | 0.3248496  |
|     | 12x12     | 0.3563067      | 0.3247644  |
| 30  | 4x4       | 0.3247229      | 0.3016496  |
|     | 6x6       | 0.3282545      | 0.3011180  |
|     | 8x8       | 0.3287686      | 0.3008477  |
|     | 10x10     | 0.3288858      | 0.3007129  |
|     | 12x12     | 0.3288884      | 0.3006348  |
| 40  | 4x4       | 0.3127310      | 0.2930945  |
|     | 6x6       | 0.3177112      | 0.2926053  |
|     | 8x8       | 0.3186273      | 0.2923696  |
|     | 10x10     | 0.3189232      | 0.2922512  |
|     | 12x12     | 0.3190184      | 0.2921813  |
| 50  | 4x4       | 0.3059923      | 0.2890616  |
|     | 6x6       | 0.3123527      | 0.2886311  |
|     | 8x8       | 0.3136399      | 0.2884272  |
|     | 10x10     | 0.3141066      | 0.2883240  |
|     | 12x12     | 0.3142943      | 0.2882620  |
| 60  | 4x4       | 0.3014056      | 0.2868218  |
|     | 6x6       | 0.3090950      | 0.2864505  |
|     | 8x8       | 0.3107194      | 0.2862729  |
|     | 10x10     | 0.3113414      | 0.2861828  |
|     | 12x12     | 0.3116166      | 0.2861278  |
| 70  | 4x4       | 0.2978740      | 0.2854335  |
|     | 6x6       | 0.3068662      | 0.2851214  |
|     | 8x8       | 0.3088003      | 0.2849651  |
|     | 10x10     | 0.3095609      | 0.2848858  |
|     | 12x12     | 0.3099158      | 0.2848370  |
| 80  | 4x4       | 0.2949249      | 0.2844999  |
|     | 6x6       | 0.3052113      | 0.2842489  |
|     | 8x8       | 0.3074330      | 0.2841101  |
|     | 10x10     | 0.3083169      | 0.2840397  |
|     | 12x12     | 0.3087436      | 0.2839961  |
| 90  | 4x4       | 0.2923261      | 0.2838297  |
|     | 6x6       | 0.3039046      | 0.2836439  |
|     | 8x8       | 0.3063993      | 0.2835195  |
|     | 10x10     | 0.3073939      | 0.2834564  |
|     | 12x12     | 0.3078847      | 0.2834172  |
| 100 | 4x4       | 0.2899556      | 0.2833214  |
|     | 6x6       | 0.3028248      | 0.2832059  |
|     | 8x8       | 0.3055820      | 0.2830939  |
|     | 10x10     | 0.3066771      | 0.2830367  |
|     | 12x12     | 0.3072254      | 0.2830013  |
Table 3. Convergence of FEM solution for different mesh size for symmetric cross-ply laminate denoting the percentage change in last two values.

| Lamination Scheme | a/h | Mesh size | Percentage change in last two values |
|-------------------|-----|-----------|-------------------------------------|
|                   |     | 2x2       | 4x4       | 6x6       | 8x8       |                                      |
| 0/90/90/0         | 10  | 0.6107    | 0.6608    | 0.6622    | 0.6624    | -0.0320%                             |
|                   | 30  | 0.4356    | 0.4581    | 0.4580    | 0.4579    | 0.0218%                              |
|                   | 100 | 0.3676    | 0.4332    | 0.4336    | 0.4335    | 0.0236%                              |
| 0/90/0            | 10  | 0.6781    | 0.7383    | 0.7395    | 0.7396    | -0.0134%                             |
|                   | 30  | 0.4398    | 0.4672    | 0.4670    | 0.4669    | 0.0214%                              |
|                   | 100 | 0.3969    | 0.4340    | 0.4344    | 0.4344    | 0.0%                                  |

The Matlab coding is now being extended to higher order theories (as presented in Table 4) and presently the values of non-dimensional central deflections under the action of transverse loads are got. From the results obtained now in HSDT, it is clear that the third-order theory (TSDT) gives more accurate results for deflections when compared to the first-order shear deformation plate theory with $K = 5/6$. It is known that the shear correction factor $K$ depends on the lamina properties and the stacking sequence. The fact that no shear correction coefficients are needed in the third-order theory makes it more convenient to use. In general, the equilibrium-derived transverse shear stresses compare more favourably with the elasticity solution than those obtained from the constitutive equations for equivalent single-layer theories. There is quite difference between FSDT and HSDT results for thick plates while for thin plates both the theories predict similar behaviour. Effect of transverse shear strain is thus noticed. Figure 2 contains plots of non-dimensionalized centre deflection to thickness ratio $a/h$ for a square, symmetric cross-ply laminate (0/90/90/0) under sinusoidal distributed load. Compared to the elasticity solution, the third-order theory underpredicts deflection by less while the first-order theory underpredicts by higher amounts. When the plate is thick, the difference between FSDT and HSDT values are large. This means that the plate behaviour prediction will be more unsafe by FSDT than by HSDT in such cases.

Table 4. Non-dimensionalised deflection for FSDT and HSDT.

| a/h   | Source | Non-dimensionalised deflection*100 | Reference [2] |
|-------|--------|-----------------------------------|---------------|
| 10    | FSDT   | 0.6624                            | 0.663         |
|       | TSDT   | 0.7141                            | 0.715         |
| 20    | FSDT   | 0.4909                            | 0.4912        |
|       | TSDT   | 0.5055                            | 0.506         |
| 100   | FSDT   | 0.4335                            | 0.4337        |
|       | TSDT   | 0.4343                            | 0.434         |
|       | CLPT   | 0.431                             |               |
5. Conclusion

Detailed bending analysis of a laminated composite plate has been studied using FSDT and HSDT. Formulations based on both FSDT and HSDT explained in a detailed manner. Matlab codes for complete FSDT and HSDT analysis has done. The codes are providing satisfactory results when compared with references. The convergence is obtaining in composite plate analysis. The results of non-dimensionalized central transverse deflection various conditions are calculated and compared with published results available in literature. A very good agreement of the results obtained by present method with reference solutions, shows that the formulation and programming is robust, effective and highly accurate. As the $a/h$ ratio is increased, the non-dimensional transverse deflection decreases. And it is observed that for $a/h$ ratio greater than 40, the deflection becomes almost constant. The difference between FSDT and HSDT values are decreases as the thickness of plate decreases. Thus the current study elaborately discussed the FEM formulation that makes an easy programming even for a beginner in this field and presented some significant results for the structural responses of composite plates.
6. References

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