The structure of the spectrum of anomalous dimensions in the $N$–vector model in $4 - \epsilon$ dimensions

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Abstract

In a recent publication we have investigated the spectrum of anomalous dimensions for arbitrary composite operators in the critical $N$–vector model in $4 - \epsilon$ dimensions. We could establish properties like upper and lower bounds for the anomalous dimensions in one–loop order. In this paper we extend these results and explicitly derive parts of the one–loop spectrum of anomalous dimensions. This analysis becomes possible by an explicit representation of the conformal symmetry group on the operator algebra. Still the structure of the spectrum of anomalous dimensions is quite complicated and does generally not resemble the algebraic structures familiar from two dimensional conformal field theories.

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1 Introduction

The renormalization of composite operators is a classical problem in quantum field theory \[17\]. In a recent publication \[5\] we have investigated some aspects of this problem in the $N$–vector model in $4-\epsilon$ dimensions. We found a one–loop order solution for the spectrum of anomalous dimensions in terms of a “two–particle” interaction operator $V$ acting on a Hilbert space $C$ isomorphic to the space of composite operators. This hermitean operator $V$ encodes all the information about the one–loop spectrum of anomalous dimensions and the corresponding scaling eigenoperators. Therefore one is interested to know as much as possible about the explicit structure of the spectrum of $V$. However in ref. \[5\] we have only been able to diagonalize $V$ explicitly for composite operators consisting of two or three elementary fields. In this paper we extend these results and derive anomalous dimensions for larger classes of composite operators.

A complete classification of the spectrum of anomalous dimensions is possible in two dimensional conformal field theories. It is an interesting question to find out how much of the algebraic structures there can also be found in $d$ dimensional conformal field theories for $d > 2$. But it is well known that beyond two dimensions conformal symmetry yields less stringent conditions. Therefore many questions in $d$ dimensional conformal field theory still remain open. Non–perturbative treatments generally generate complicated series of equations that cannot be easily solved.

In contrast we are working within the framework of one–loop order perturbation theory. It is in principle a straightforward problem to diagonalize our operator $V$ and to calculate anomalous dimensions for whatever composite operators one is interested in. Conformal symmetry enters here only as the symmetry group of $V$: One can restrict attention to conformal invariant operators (CIOs) as is well known from conformal field theory. However the dimensions of the vector spaces of mixing CIOs can become arbitrarily large for a large number of gradients in the composite operators. Thus the well–known problem of operator mixing occurs. One has to resort to numerical diagonalizations of $V$ and finds a rather complicated spectrum of non–rational anomalous dimensions. This spectrum seems to show little resemblance to the familiar algebraic structures from two dimensional conformal field theories.

Nevertheless some properties of the spectrum can be understood analytically. It emerges a picture that in many spaces of mixing CIOs there are operators with smallest and rational anomalous dimensions (“ground states”), and “excitations” with typically non–rational eigenvalues above. The anomalous dimensions of the “ground states” can
be derived in closed form. In particular for a sufficiently large number of gradients there is a highly degenerate subspace of CIOs with vanishing anomalous dimensions in one–loop order. This agrees with Parisi’s intuitive conjecture that a high spin “effectively” separates space–time points \[ \text{(1)}. \]

The structure of this paper is as follows. Sects. 2 and 3 give a short summary of the main results of our previous publication \[ \text{(2)} \] and define our notation. In sect. 2 the operator \( V \) is introduced that generates the spectrum of anomalous dimensions. We also present the explicit representation of the conformal symmetry group on the operator algebra. Sect. 3 is concerned with the explicit diagonalization of \( V \) for composite operators consisting of \( n = 2 \) and \( n = 3 \) fields. Some remarks on the \( N \to \infty \) limit of the spectrum follow in sect. 4. Explicit values for the anomalous dimensions can be given if the composite operators contain less than six gradients. This is shown in sect. 5. In sect. 6 some improved lower limits for the general spectrum problem are established.

Sects. 7 and 8 are concerned with the more complicated parts of the spectrum of anomalous dimensions for a large number of gradients. In sect. 7 we allow for arbitrary SO(4) spin structure, whereas the important class of composite operators with \( l \) gradients and SO(4) spin structure \((l/2, l/2)\) is treated in more detail in sect. 8. In work on conformal field theory one frequently considers this spin structure only since only this type of CIOs appears in the operator product expansion of two scalar operators (see ref. \[ \text{(12)} \]). One can view the spectrum problem for such CIOs as an energy spectrum for a certain quantum mechanical system of bosons interacting with \( \delta \)–potentials. Finally sect. 9 deals with the particularly intriguing highly degenerate class of CIOs with vanishing anomalous dimensions in one–loop order. A short summary of the results is presented in sect. 10.

2 The spectrum–generating operator \( V \)

We briefly sum up the main results of our earlier paper \[ \text{(3)} \]. An attempt is made to make the presentation here self–consistent, for more details the reader is referred to the earlier work.

The \( N \)–vector model of the field \( \vec{\phi} = (\phi_1, \ldots, \phi_n) \) in \( 4 - \epsilon \) dimensions is governed by the Lagrangian

\[
\mathcal{L} = \frac{1}{2} (\partial \vec{\phi})^2 + \frac{1}{2} m_0^2 \vec{\phi}^2 + \frac{g_0}{4!} (\vec{\phi}^2)^2. \tag{1}
\]
Composite local operators in this model consisting of \( n \) elementary fields and gradients acting on these fields are represented as polynomials of fields \( \Phi_i^{(j;m_1,m_2)} \), \( j = 0, 1, \ldots \) and \( m_1, m_2 = -j/2, j/2 + 1, \ldots, j/2 \). Here \( \Phi_i^{(j;m_1,m_2)} \) is an elementary field \( \phi_i \) with \( j \) gradients

\[
\Phi_i^{(j;m_1,m_2)} \overset{\text{def}}{=} h^{(m_1,m_2)}_{\alpha_1 \ldots \alpha_j} \partial_{\alpha_1} \ldots \partial_{\alpha_j} \phi_i(0),
\]

where \( h^{(m_1,m_2)}_{\alpha_1 \ldots \alpha_j} \) is a symmetric and traceless SO(4) tensor belonging to the irreducible representation \((j/2, j/2)\) of SO(4). \( h^{(m_1,m_2)}_{\alpha_1 \ldots \alpha_j} \) can be constructed via the one to one correspondence with homogeneous harmonic polynomials of degree \( j \)

\[
H_j^{(m_1,m_2)}(x) = h^{(m_1,m_2)}_{\alpha_1 \ldots \alpha_j} x_{\alpha_1} \cdot \ldots \cdot x_{\alpha_j}
\]

with the generating functional

\[
(u \cdot x)^j = \sum_{m_1,m_2=-j/2}^{j/2} 2^j j! \sqrt{\binom{j}{m_1+j/2} \binom{j}{m_2+j/2}} H_j^{(m_1,m_2)}(x) \cdot t^{m_1+j/2} s^{m_2+j/2}
\]

and

\[
u = (i - i t s, - i t - i s, - t + s, 1 + t s).
\]

We define a creation operator \( a_i^{(j;m_1,m_2)} \) corresponding to multiplication with the field \( \Phi_i^{(j;m_1,m_2)} \) and an annihilation operator \( a_i^{(j;m_1,m_2)} \) corresponding to the derivative \( \partial/\partial \Phi_i^{(j;m_1,m_2)} \) with Bose commutation relations

\[
[a_i^{(j;m_1,m_2)}, a_j^{(j';m_1',m_2')}^\dagger] = \delta_{ii'} \delta_{jj'} \delta_{m_1 m_1'} \delta_{m_2 m_2'}.
\]

\( a_i^{(j;m_1,m_2)} \) and \( a_i^{(j;m_1,m_2)} \) operate on a Hilbert space \( \mathcal{C} \) with vacuum \( |\Omega> \) that can therefore be thought of as the space of composite operators. Notice that by omitting terms like \( \Delta \phi_i \) in our construction of \( \mathcal{C} \) we have removed a class of redundant operators that is of no physical interest (compare ref. [5]).

As we have shown in our first paper, scaling eigenoperators and anomalous dimensions of the critical model are in one–loop order eigenvectors and eigenvalues of a two–particle interaction operator

\[
V_N = \frac{1}{2} \sum_{d,d',e,e'=1}^N \sum_Q v_Q \cdot (\delta_{ee'} \delta_{dd'} + \delta_{ed} \delta_{e'd} + \delta_{ed'} \delta_{e'd'}) \times a_e^{(j;m_1,m_2)} \delta_{j'}^{(j;m_1',m_2')} a_d^{(j;m_1,n_2)} a_d^{(j';m_1',n_2')}
\]

The sum \( \sum_Q \) runs over all SO(4) quantum numbers of the creation and annihilation operators. The interaction kernel consists of a product of four SO(3) Clebsch–Gordan
An eigenvector $|\psi> \rangle$ of $V_N$ with $n$ fields and $l$ gradients
\[ V_N |\psi> \rangle = \alpha |\psi> \rangle \] (9)
corresponds to an eigenoperator in one–loop order with anomalous dimension
\[ \lambda = \epsilon \cdot \frac{\alpha}{N+8} + O(\epsilon^2). \] (10)
The critical exponent of this eigenoperator is
\[ x = l + n \left(1 - \frac{\epsilon}{2}\right) + \lambda \] (11)
and the full scaling dimension
\[ y = d - x = 4 - \epsilon - l - n \left(1 - \frac{\epsilon}{2}\right) - \lambda. \] (12)
Operators are relevant, marginal or irrelevant for $y > 0$, $y = 0$, $y < 0$ respectively.
In the sequel we will frequently restrict ourselves to the scalar $\phi^4$–theory ($N = 1$) or to operators that are completely symmetric and traceless with respect to the $O(N)$ indices. In both cases one can omit the $N$–vector indices of the creation and annihilation operators and simply write instead of (7)
\[ V = \frac{1}{2} \sum_Q v_Q a^{(j;m_1,m_2)} a^{(j';m'_1,m'_2)} a^{(l;n_1,n_2)} a^{(l';n'_1,n'_2)}. \] (13)
Anomalous dimensions for eigenoperators with eigenvalue $\alpha$ are then
\[ \lambda = \epsilon \cdot \mu \cdot \alpha + O(\epsilon^2). \] (14)
with
\[ \mu = \begin{cases} 
1/3 & \text{for } N = 1 \\
2/(N+8) & \text{for } O(N) \text{ symmetric and traceless tensors.}
\end{cases} \] (15)

Eigenoperators in the scalar theory therefore generate \( O(N) \) symmetric and traceless eigenoperators
\[ \Phi(j^{(1)},m_{1}^{(1)},m_{2}^{(1)}) \ldots \Phi(j^{(n)},m_{1}^{(n)},m_{2}^{(n)}) + \ldots \]
\[ \rightarrow t_{i_{1} \ldots i_{n}} \left( \Phi(j^{(1)},m_{1}^{(1)},m_{2}^{(1)}) \ldots \Phi(j^{(n)},m_{1}^{(n)},m_{2}^{(n)}) + \ldots \right), \]
where \( t_{i_{1} \ldots i_{n}} \) is a symmetric and traceless tensor. Notice that all \( O(N) \) totally antisymmetric composite operators are annihilated by \( V_{N} \) and have vanishing anomalous dimensions in one-loop order. We will ignore this uninteresting class of operators in the sequel.

It has already been mentioned in our previous paper that \( V_{N} \) is a hermitean positive semi-definite operator. Hence its eigenvalues are real positive numbers that can only make canonically irrelevant operators even more irrelevant according to (12). In contrast to some \( 2 + \epsilon \) expansions [6, 7, 11, 13, 11, 13, 2] the stability of the nontrivial fixed point in \( 4 - \epsilon \) dimensions is in one-loop order not endangered by high-gradient operators.

The spatial symmetry group of \( V_{N} \) is a representation of the conformal group in four dimensions \( O(5,1) \) on the operator algebra \( \mathcal{C} \). Conformal symmetry should of course be expected at the critical point of a second order phase transition. The 15 generators of \( SO(5,1) \) can conveniently be expressed as
\[ \partial_{st} = \sum_{i=1}^{N} \sum_{(j,m_{1},m_{2})} \sqrt{(j/2 + 1 + s m_{1})(j/2 + 1 + t m_{2})} a_{i}^{(j+1;m_{1}+s/2,m_{2}+t/2)} a_{i}^{(j;m_{1},m_{2})} \]
\[ \partial_{st}^{\dagger} = \sum_{i=1}^{N} \sum_{(j,m_{1},m_{2})} \sqrt{(j/2 + 1 + s m_{1})(j/2 + 1 + t m_{2})} a_{i}^{(j;m_{1},m_{2})} a_{i}^{(j+1;m_{1}+s/2,m_{2}+t/2)} \]
\[ J_{s}^{1} = \sum_{i=1}^{N} \sum_{(j,m_{1},m_{2})} \sqrt{(j/2 - s m_{1})(j/2 + 1 + s m_{1})} a_{i}^{(j+1;m_{1}+s,m_{2})} a_{i}^{(j;m_{1},m_{2})} \]
\[ J_{s}^{1} = \sum_{i=1}^{N} \sum_{(j,m_{1},m_{2})} m_{1} a_{i}^{(j;m_{1},m_{2})} a_{i}^{(j;m_{1},m_{2})} \] (17)
\[ J_{s}^{2} = \sum_{i=1}^{N} \sum_{(j,m_{1},m_{2})} \sqrt{(j/2 - t m_{2})(j/2 + 1 + t m_{2})} a_{i}^{(j;m_{1},m_{2}+t)} a_{i}^{(j;m_{1},m_{2})} \]
\[ J_{s}^{2} = \sum_{i=1}^{N} \sum_{(j,m_{1},m_{2})} m_{2} a_{i}^{(j;m_{1},m_{2})} a_{i}^{(j;m_{1},m_{2})} \]
$$J_0 = \sum_{i=1}^{N} \sum_{(m_1,m_2)} (j/2 + 1/2) \ a_i^{(j;m_1,m_2)} \ d_i^{(j;m_1,m_2)}$$

with $s, t = -1, +1$. For later purposes we also introduce the linear combinations

$$\begin{align*}
J_{++}^1 &= J_0 + J_z^1, & J_{--}^1 &= J_0 - J_z^1, & J_{+-}^1 &= J_+^1, & J_{-+}^1 &= J_-^1, \\
J_{++}^2 &= J_0 + J_z^2, & J_{--}^2 &= J_0 - J_z^2, & J_{+-}^2 &= J_+^2, & J_{-+}^2 &= J_-^2.
\end{align*}$$

(18)

Now it is clearly sufficient to restrict our attention to conformal invariant operators (CIOs) annihilated by the generators of special conformal transformations

$$\partial_{st}^\dagger |\psi> = 0 \quad \forall s, t = -1, +1$$

(19)

since all derivative operators $\partial_{st_1} \ldots \partial_{st_p} |\psi>$ reproduce the anomalous dimension of $|\psi>$. Considering only CIOs reduces the dimensions of the spaces of mixing operators considerably and renders the diagonalization problem of $V_N$ more tractable.

The vector space of CIOs with $n$ fields, $l$ gradients and spin structure $(j_1, j_2)$ in the two SO(3) sectors of SO(4) will be denoted by $C[n, (l; j_1, j_2)]$, where $j_1, j_2 = l/2, l/2 - 1, \ldots, 1/2$ or 0. Since eq. (17) gives an explicit representation of the generators of special conformal transformations $\partial_{st}^\dagger$, it is in principle straightforward to construct CIOs in a given $C[n, (l; j_1, j_2)]$. If one applies a generator $\partial_{st}^\dagger$ on a not conformally invariant operator, the result is a composite operator with one gradient less than the original operator. Thus repeated application of the generators of special conformal transformations finally yields a conformal invariant operator that is mapped to zero by all $\partial_{st}^\dagger$. This immediately provides a method for constructing CIOs that is simple to implement if the dimension of a space of CIOs is small.

**Example:**

For $n = 2$ fields one finds e.g. exactly one $O(N)$ scalar CIO $T^{(l)}$ with an even number of fields $l$. $T^{(l)}$ transforms as a tensor with spin structure $(l/2, l/2)$ under SO(4)–rotations. Its components $T^{(l)(m_1,m_2)}$

$$J_{z}^{1,2} T^{(l)(m_1,m_2)} = m_{1,2} T^{(l)(m_1,m_2)}$$

(20)

can be generated from $T^{(l)(l/2,l/2)}$ by application of $J_{-}^{1,2}$

$$T^{(l)(m_1,m_2)} = \frac{(l/2 + m_1)! (l/2 + m_2)!}{l!^2 (l - 2m_1 - 1)!! (l - 2m_2 - 1)!!} \ (J_{-}^{1})^{l/2-m_1} \ (J_{-}^{2})^{l/2-m_2} T^{(l)(l/2,l/2)}$$

(21)
where
\[ T^{(l/l_2/l_2)} = \frac{1}{2} \sum_{k=0}^{l} (-1)^k \binom{l}{k} \phi^{(k; k/2, k/2)} \cdot \phi^{(l-k; l/2 - k/2, l/2 - k/2)}. \] (22)

Notice that \( T^{(2)} \) is proportional to the stress tensor.

Usually we do not explicitly write down the form of the scaling eigenoperators in the sequel.

If \( \text{dim} \mathcal{C}[n, (l; j_1, j_2)] \) is large the problem of finding all CIOs is harder. We come back to this problem in sect. 7. For the important case \( \mathcal{C}[n, (l; l/2, l/2)] \) we give a complete and constructive solution in sect. 8.

3 Conformal invariant operators with \( n \leq 3 \) fields

For two and three fields the spectrum of CIOs has already been worked out in our previous paper [5]. The results are summed up in tables 1 to 3. All the CIOs in these tables have SO(4) spin structure \((l/2, l/2)\).

However for \( n = 3 \) fields one can in general also construct CIOs with a different SO(4) spin structure, or more than one CIO with spin structure \((l/2, l/2)\). Since the spectrum–generating operator \( V \) acts like a projection operator (except for a slight complication for the last case in table 3) on the three particle space, these additional CIOs always have vanishing anomalous dimensions in order \( \epsilon \). In general it is quite difficult to give an explicit formula for the number of these additional CIOs. At least for \( N = 1 \) (or \( O(N) \) completely symmetric and traceless tensors) with SO(4) spin structure \((l/2, l/2)\), we will establish the following generating function in sect. 7
\[ \sum_{l=0}^{\infty} (\text{dim} \mathcal{C}[n = 3, (l; l/2, l/2)]) \cdot x^l = \frac{1}{(1 - x^2)(1 - x^3)}. \] (23)

Thus there are \((\text{dim} \mathcal{C}[n = 3, (l; l/2, l/2)] - 1)\) CIOs with zero eigenvalues besides the CIOs with non–zero eigenvalues from tables 2 and 3. For large \( l \) this number increases approximately linearly with \( l \), therefore vanishing anomalous dimensions dominate the spectrum for \( l \to \infty \).

Let us mention that we have compared our anomalous dimensions for CIOs with Young frame \[
\begin{array}{c}
|\rule{0.1cm}{0.1cm}|
|\rule{0.1cm}{0.1cm}|
|\rule{0.1cm}{0.1cm}|
\end{array}
\] for \( l = 0, 2, 3, 4, 5, 6 \) against results obtained by Lang and Rühl in a \( 1/N \)–expansion [10]. Their results give the exact dependence on the dimension \( 2 < d < 4 \) in order \( 1/N \) and can therefore be expanded in \( \epsilon \) for \( d = 4 - \epsilon \). Whereas our results are obtained in first order in \( \epsilon \) but give the full \( N \)–dependence. We have found complete agreement in order \( \epsilon/N \).
4 The large-N limit

One can easily solve the large $N$ limit of the spectrum problem for $V_N$. This serves only as another consistency check of our approach since the order $O(N^0)$ contributions of the anomalous dimensions are trivial in a $1/N$–expansion.

First one notices that the action of $V_N$ on a composite operator $|\psi_i>$ can be split up in a term linear in $N$ and a constant term

$$V_N |\psi_i> = N \cdot \sum_j \gamma_{ij} |\psi_j> + \sum_j \gamma'_{ij} |\psi_j>,$$

where the mixing matrices $\gamma_{ij}, \gamma'_{ij}$ are independent of $N$ and all composite operators normalized. The $O(N^0)$ term $\epsilon \cdot \alpha$ in the anomalous dimension of an eigenoperator in this double limit ($d \to 4, N \to \infty$)

$$\lambda = \epsilon \alpha + O(1/N) + O(\epsilon^2).$$

is simply the respective eigenvalue of $\gamma_{ij}$. But the terms linear in $N$ in eq. (24) come solely from $V_N$ acting on composite operators like

$$S = \prod_{i=1}^k \left( \sum_Q c_Q \tilde{\Phi}^{(j(i);m_1^{(i)};m_2^{(i)})} \cdot \tilde{\Phi}^{(j(i)^r;m_1^{(i)^r};m_2^{(i)^r})} \right)$$

with $Q = (j^{(i)};m_1^{(i)},m_2^{(i)}), (j^{(i)^r};m_1^{(i)^r},m_2^{(i)^r})$ and

$$V_N (S) = \prod_{i=1}^k V_N \left( \sum_Q c_Q \tilde{\Phi}^{(j(i);m_1^{(i)};m_2^{(i)})} \cdot \tilde{\Phi}^{(j(i)^r;m_1^{(i)^r};m_2^{(i)^r})} \right) + O(N^0).$$

Now $V_N$ acts as a projection operator on each product in (27)

$$V_N \left( \tilde{\Phi}^{(j^{(i)};m_1^{(i)};m_2^{(i)})} \cdot \tilde{\Phi}^{(j^{(i)^r};m_1^{(i)^r};m_2^{(i)^r})} \right) \propto p(\vec{\partial}) (\vec{\phi}^2)$$

where $p(\vec{\partial})$ is a certain homogeneous polynomial of degree $j^{(i)} + j^{(i)^r}$ in the derivatives. Trivially we have for all such homogeneous polynomials

$$V_N \left( p(\vec{\partial}) \vec{\phi}^2 \right) = (N + 2) p(\vec{\partial}) \vec{\phi}^2.$$

Therefore eigenoperators in the double limit $d \to 4, N \to \infty$ are of type

$$O = \left[ p_1(\vec{\partial}) \vec{\phi}^2 \right] \left[ p_2(\vec{\partial}) \vec{\phi}^2 \right] \cdot \ldots \cdot \left[ p_{n_0}(\vec{\partial}) \vec{\phi}^2 \right] \cdot S,$$

with
where each term of $S$ from eq. (26)

$$
\left( \sum_Q c_Q \Phi^{(j(i):m_1^{(i)},m_2^{(i)})} \cdot \Phi^{(j(i'):m_1^{(i')},m_2^{(i')})} \right)
$$

(31)

is as a vector in $C$ orthogonal to $\Phi^2$ and all total derivatives thereof. Also each product of $O$ with a traceless composite operator $C_{i_1...i_p}$ consisting of $p$ elementary fields $\Phi^{(j(m_1,m_2)}}$ is an eigenoperator in this double limit. The anomalous dimension of $O$ depends only on the number $n_0$ of factors $\Phi^2$ or total derivatives of $\Phi^2$

$$
\lambda = \epsilon \cdot n_0 + O(1/N) + O(\epsilon^2)
$$

(32)

$$
\Rightarrow \text{Critical exponent } x = (n - 2n_0) \cdot \left( \frac{d}{2} - 1 \right) + l + 2n_0 + O(1/N) + O(\epsilon^2).
$$

As should be expected this is consistent with the spherical model limit.

It is, however, not too surprising that the $1/N$–corrections in (25) are very involved. In fact a $1/N$–expansion beyond the zeroth order term does here not provide any additional simplifications and is already equivalent to the full problem of diagonalizing $V_N$.

In the remainder of this paper we therefore keep the exact $N$–dependence. But since we are mainly interested in the spatial symmetry, we restrict the discussion to $O(N)$ tensors consisting of $n$ elementary fields with the Young frame

$$
\begin{array}{ccc}
1 & \ldots & n
\end{array}
$$

(33)

that is to $O(N)$ completely symmetric and traceless tensors. By virtue of eq. (15) we need thus in fact only discuss the spectrum of anomalous dimensions in the scalar $\phi^4$–theory: Eq. (16) yields the generalization from $N = 1$ to $O(N)$ tensors of type (33) and the anomalous dimensions are related according to eq. (14).

5 Conformal invariant operators with $l \leq 5$ gradients

As already mentioned, operator mixing due to the increasing number of conformal invariant operators makes the diagonalization problem in general intractable for a large number of gradients $l$. There are typically no “simple” rational anomalous dimensions besides the ones belonging to the lower bound as established in the next section.
For the scalar $\phi^4$–theory or $O(N)$ tensor with the transformation properties (33), however, the spaces of mixing CIOs $C[n, (l; j_1, j_2)]$ have maximum dimension two for $l \leq 5$ gradients. All eigenvalues can easily be derived explicitly in this case. For the sake of completeness these eigenvalues $\alpha$ are summed up in table 4. The corresponding anomalous dimensions follow from eq. (15). Again we have checked a few randomly chosen anomalous dimensions against results obtained in the $1/N$–expansion by Lang and Rühl [9, 10] and always found consistency.

6 Lower limits for the spectrum

Since the spectrum–generating operator $V$ is positive semi–definite as shown in our previous paper [3], zero is an obvious lower limit for the eigenvalues $\alpha$ of $V$ and hence for the anomalous dimensions $\lambda$ in (12). However zero is not a strict lower limit for the spectrum and can be improved for $n$ larger than approximately $l/2$.

In order to establish a more stringent lower bound one first notices that

$$C = \sum_{s,t} \partial_{st} \partial_{st}^\dagger - \frac{1}{2} \sum_{s,t} J_{st}^1 J_{ts}^1 - \frac{1}{2} \sum_{s,t} J_{st}^2 J_{ts}^2 + 4J_0$$

is a Casimir operator of the group $SO(5,1)$ from (17). Then

$$L = \frac{1}{6} \left( C + \frac{7}{2} \hat{\nu}(\hat{\nu} - 1) - \frac{3}{2} \hat{\nu} \right)$$

also commutes with all generators of $SO(5,1)$ and with the two–particle interaction operator $V$ from (13). Here $\hat{\nu}$ is an operator that counts the number of fields

$$\hat{\nu} = \sum_{(j;m_1,m_2)} a^{(j;m_1,m_2)} \dagger a^{(j;m_1,m_2)}.$$  

(36)

$V$ and $L$ can be simultaneously diagonalized and since $[L, \partial_{st}] = 0$ we need only consider CIOs as usual. Acting on $C[n, (l; j_1, j_2)]$ the Casimir operator $C$ is equal to

$$-j_1 (j_1 + 1) - j_2 (j_2 + 1) - \frac{(n + l)^2}{2} + 2(n + l),$$

therefore

$$L\bigg|_{C[n,(l;j_1,j_2)]} = \frac{1}{2} n (n - 1) - \frac{1}{6} n l - \frac{1}{12} l (l - 4) - \frac{1}{6} j_1 (j_1 + 1) - \frac{1}{6} j_2 (j_2 + 1).$$

(38)

Eq. (38) represents the lower limit that we want to establish: Because of

$$L \Phi^{(0,0,0)} = 0$$

(39)
Let us briefly sketch this proof of \( \mathcal{L} \) being a strict lower bound. Essentially one has to show that for \( n \geq l \geq 4 \) and spin structure \((j, j)\), that is equal spin in both SO(3) sectors, there always exists an eigenoperator with an eigenvalue given by eq. (38). \( \mathcal{L} \) is thus a strict lower limit for this class of composite operators.

We can go a step further and show that for \( n \geq l \geq 4 \) and spin structure \((j, j)\), that is equal spin in both SO(3) sectors, there always exists an eigenoperator with an eigenvalue given by eq. (38). Now in fact this eigenoperator turns out to have the following general structure

\[
O = (\Phi(0;0,0))^{n-l} \sum_{\{\alpha_i\}} O_{\alpha_1...\alpha_4}^{(0,0,0)} [\alpha_1, \alpha_2, \alpha_3, \alpha_4]
\]

\[
+ (\Phi(0;0,0))^{n+1-l} \sum_{m_1, m_2 = -1} O_{\beta_1...\beta_4}^{(2,m_1,m_2)} [\beta_1, \beta_2, \beta_3, \beta_4]
\]

\[
+ \text{other terms with higher powers of } \Phi(0,0,0),
\]

where we have used the compact notation

\[
[\alpha_1, \alpha_2, \alpha_3, \alpha_4] = (\Phi(1;-1/2,-1/2))^{\alpha_1} (\Phi(1;1/2,1/2))^{\alpha_2} (\Phi(1;1/2,-1/2))^{\alpha_3} (\Phi(1;1/2,1/2))^{\alpha_4}.
\]

The important thing to notice is that all the terms not written out in eq. (42) cannot be mapped on \( (\Phi(0;0,0))^{n-l} \sum_{\{\alpha_i\}} \ldots \) by \( V \). Hence we only need to know something about the coefficients \( O_{\alpha_1...\alpha_4}^{(0,0,0)} \) and \( O_{\beta_1...\beta_4}^{(2,m_1,m_2)} \). This information can be obtained by using \( O \in \mathcal{C}[n, (l; j, j)] \), for example by requiring

\[
J_2^+ \vert O \rangle = J_+^2 \vert O \rangle = j \vert O \rangle
\]

\[
\partial_{st} \vert O \rangle = J_+^1 \vert O \rangle = J_+^2 \vert O \rangle = 0.
\]
From eqs. (44) and (45) one concludes after some calculation

\[
O_{\alpha_1...\alpha_4}^{(0,0,0)} = \begin{cases} 
(-1)^{\alpha} \binom{l/2-j}{\alpha} & \text{if } \alpha = 0, \ldots, l/2-j \text{ exists with} \alpha_1 = l/2-j-\alpha, \alpha_2 = \alpha_3 = \alpha, \alpha_4 = l/2+j-\alpha \\
0 & \text{otherwise}
\end{cases}
\]

where we have normalized \(O_{l/2-j,0,0,l/2+j}^{(0,0,0)} = 1\) and

\[
\begin{align*}
2O_{l/2-j,0,0,l/2+j-2}^{(2;1,1)} + O_{l/2-j-1,0,0,l/2+j-2}^{(2;0,0)} &= -\frac{l}{2} - j \\
2O_{l/2-j-1,0,0,l/2+j}^{(2;-1,1)} + O_{l/2-j-1,0,0,l/2+j-1}^{(2;0,0)} &= -\frac{l}{2} + j.
\end{align*}
\]

This information is sufficient to derive

\[
V|O> = \left( \frac{1}{2} n (n-1) - \frac{1}{6} n l - \frac{1}{12} l (l-4) - \frac{1}{3} j (j+1) \right) \\
\times \left( \phi^{(0,0,0)} \right)^{n-l} \left[ l/2-j,0,0,l/2+j \right] + \text{other terms}
\]

So if a CIO with structure (42) exists, its eigenvalue must be

\[
\alpha = \frac{1}{2} n (n-1) - \frac{1}{6} n l - \frac{1}{12} l (l-4) - \frac{1}{3} j (j+1)
\]

equal to the lower limit from eq. (38).

Finally one can show that CIOs of type (42) with spin structure \((l/2,l/2)\) exist for \(n \geq l, l \neq 1\). In the case of non–maximum spin in the SO(3) sectors \((j,j), j \neq l/2\) they exist for \(n \geq l \geq 4\).

Thus the lower bound (38) is strict in the classes:

\begin{itemize}
  \item \(C[n,(l;l/2,l/2)]\) for \(n \geq l, l \neq 1\)
  \item \(C[n,(l;j,j)]\) for \(n \geq l \geq 4, j \neq l/2\)
\end{itemize}

Notice that the above proof cannot be extended to CIOs with unequal spin \(j_1 \neq j_2\) in the SO(3) sectors. This agrees with the fact that in general (38) is not a strict limit in this case as one can see in the next section from explicit diagonalizations of \(V\).\footnote{Essentially the proof fails because one can show \(O_{n_1,\ldots,n_4}^{(0,0,0)} \equiv 0\) for \(j_1 \neq j_2\).}
7 More complex CIOs

Besides the lowest eigenvalues established in the previous section, the spectrum of anomalous dimensions consists mainly of irrational numbers for \( n \geq 4, l \geq 6 \). In order to investigate this spectrum, one first has to construct the various CIOs in a given space \( C[n, (l; j_1, j_2)] \). In the language of conformal field theory this amounts to determining the field content of the theory.

Now the problem of constructing all CIOs is nontrivial for a large number of gradients. We have applied the following technique. Operators \( P_{st} \), \( s, t = \pm 1 \) are introduced

\[
P_{st} = \sum_{k=0}^{\infty} \frac{1}{k!} \left( -\frac{a^{(1; s/2, t/2)\dagger}}{a^{(0; 0)\dagger}} \right)^k (\partial_{st}^\dagger)^k
\]

with the following properties

\[
[P_{st}, P_{s't'}] = 0, \quad P_{st}^2 = P_{st} \quad \text{for } s \neq s' \text{ or } t \neq t'
\]

\[
\partial_{st} P_{s't'} = \begin{cases} P_{s't'} \partial_{st} \partial_{st}^\dagger & \text{for } s \neq s' \text{ or } t \neq t' \\ 0 & \text{for } s = s' \text{ and } t = t' \end{cases}
\]

Let us comment about the creation operators \( a^{(0; 0)\dagger} \) in the denominator of eq. (50) later. The construction of CIOs then runs as follows. First a set of states of \( |\psi'\rangle \) is constructed by application of polynomials of creation operators onto the vacuum without use of any operators \( a^{(1; s/2, t/2)\dagger} \). This is done is such a way that states with the desired quantum numbers \( [n, (l; j_1, j_2)] \) are obtained. Then the states

\[
|\psi\rangle = \left( \prod_{s, t = \pm 1} P_{st} \right) |\psi'\rangle
\]

are conformal invariant due to the property (51). Note that for a given \( l \) it is sufficient to sum up to \( k = l \) in eq. (50). If \( n \geq l \) then obviously no negative powers of \( a^{(0; 0)\dagger} \) appear in \( |\psi\rangle \). However since polynomials of \( a^{(1; s/2, t/2)\dagger} \) generate only states with \( j_1 = j_2 \), it is in fact sufficient that \( n \geq [l - 1/2 |j_1 - j_2|] \). Here \( [l - 1/2 |j_1 - j_2|] \) is the integer part of \( l - 1/2 |j_1 - j_2| \). Since \( a^{(0; 0)\dagger} \) commutes with \( \partial_{st}^\dagger \) one may multiply the conformal invariant states with arbitrary powers of \( a^{(0; 0)\dagger} \).

If the original states \( |\psi'\rangle \) are linearly independent then also the states \( |\psi\rangle \) have this property. This is due to the fact that \( |\psi\rangle \) and \( |\psi'\rangle \) differ only by states which contain at least one factor \( a^{(1; s/2, t/2)\dagger} \). On the other hand a sum of products each of
which contains at least one factor $a^{(1,s/2,t/2)\dagger}$ acting on the vacuum is not conformally invariant. Suppose the term with the smallest number of factors $a^{(1,s/2,t/2)\dagger}$ has $k$ such factors. Then at least one of the operators obtained by application of $\partial_{s't'}$ contains a term with $k - 1$ such factors. Thus the space $\mathcal{C}[n,(l;j_1,j_2)]$ for $n \geq \lceil l - 1/2 |j_1 - j_2| \rceil$ is spanned by the operators $|\psi\rangle$ from eq. (52).

We now come to the numerical solution of the eigenvalue problem of $V$. One finds that generally the only “simple” eigenvalues can be written like $\alpha = 1/2 n(n-1) + n \ a + b$ with rational numbers $a$ and $b$. Besides the trivial quadratic dependence on the number of fields, these eigenvalues are linear functions of $n$. Notice that the ground state eigenvalues (38) are of just this linear type. We have therefore solved the eigenvalue problem for a general number of fields $n \geq \lceil l - 1/2 |j_1 - j_2| \rceil$ for $l \leq 8$, but only present the eigenvalues of this simple type in tables 5 and 6. That is we list the degeneracy of the ground state eigenvalues and additional linear eigenvalues.

The actual calculation was done as follows. It is not necessary to construct the conformal invariant operators $|\psi\rangle$ from eq. (52). Instead one leaves out all contributions which contain $a^{(1,s/2,t/2)\dagger}$ in $V$ and applies this to $|\psi'\rangle$. Similarly the part of $V$ which destroys $\Phi^{(1,s/2,t/2)}$ is applied to

$$\frac{1}{2} a^{(1,s/2,t/2)\dagger} a^{(1,s'/2,t'/2)\dagger} a^{(0,0,0)\dagger} a^{(0,0,0)\dagger} \partial_{s't'} \partial_{s't'} |\psi'\rangle.$$  (54)

This saves considerable memory and computing time and still allows us to determine the eigenvalues. In this way we loose explicit hermitecity, but since the calculation was done for general $n$, it would be difficult to make use of this property anyway. Finally the eigenvalues of $1/2 n(n-1) + n \ A + B$ with matrices $A$ and $B$ have been determined.

Table 5 lists the field content of our model for $l \leq 8$ gradients and for $n \geq \lceil l - 1/2 |j_1 - j_2| \rceil$ fields constructed via eq. (52). The additional quantum number $\pi$ for $j_1 = j_2$ corresponds to the discrete symmetry that interchanges the two SO(3) representations

$$a^{(j;m_1,m_2)} \leftrightarrow a^{(j;m_2,m_1)}.$$  (55)

CIOs can be symmetric ($\pi = +$) or antisymmetric ($\pi = -$) with respect to this symmetry. Obviously $\pi$ makes only sense for operators with the same spin in both SO(3) sectors $j_1 = j_2$. From table 5 one sees that some ground state eigenvalues
occur with remarkably high degeneracies. For example in $C[n, (8; 1, 1, +)]$ one finds a fourfold degeneracy (in $C[n, (10; 1, 1, +)]$ the degeneracy is even fivefold). We know no explanation for this feature. In table 6 the additional eigenvalues of linear type are listed for $l \leq 8$. If one goes to an even larger number of gradients, the number of these additional linear eigenvalues decreases again. There might only be a finite set of them though we cannot prove that.

8 Spatially symmetric and traceless tensors

The simplest type of conformal invariant operators that are spatially symmetric and traceless tensors is studied in more detail in this section. This class of CIOs $C[n, (l; l/2, l/2)]$ is especially important since it is the only type that appears in an operator product expansion of two scalar operators [12]. In particular we can go beyond the results of the last section and derive all CIOs in closed form.

First one notices that a composite operator in $C[n, (l; l/2, l/2)]$ can be expressed as

$$h^{(m_1, m_2)}_{\alpha_1 \ldots \alpha_l} \left( \partial_{\alpha_1} \cdots \partial_{\alpha_{j_1}} \phi \cdots \partial_{\alpha_{l-j_1+1}} \cdots \partial_{\alpha_{l}} \phi + \ldots \right)$$

(56)

with a symmetric and traceless tensor $h^{(m_1, m_2)}_{\alpha_1 \ldots \alpha_l}$. Creation and annihilation operators need only have one index corresponding to the number of derivatives acting on a field $\phi$

$$[a_j, a_j^\dagger] = \delta_{jj'}.$$  

(57)

It is straightforward to show that $V$ from eq. (13) takes the following simple structure on this space

$$V = \frac{1}{2} \sum_{J=0}^{\infty} \frac{1}{J+1} \sum_{j,k=0}^{J} a_j^\dagger a_{j-j} a_k a_{J-k}. $$

(58)

Eigenvalues of $V$ and anomalous dimensions are again connected by eq. (14) and (15). It will be quite remarkable to see what a complicated spectrum is generated by such a seemingly innocent operator.

The symmetry group of $V$ left over from the full SO(5,1) is SO(2,1) generated by

$$D = \sum_{j=0}^{\infty} (j + 1) a_{j+1}^\dagger a_j$$

$$D^\dagger = \sum_{j=0}^{\infty} (j + 1) a_j^\dagger a_{j+1}$$

$$S = \sum_{j=0}^{\infty} (j + 1/2) a_j^\dagger a_j$$

(59)
with commutators
\[
[S, D] = D, \quad [S, D^\dagger] = -D^\dagger, \quad [D^\dagger, D] = 2S. \tag{60}
\]

In order to construct the CIOs, one now introduces the one to one mapping between a composite operator
\[
|\psi\rangle = \sum_{\{j_i\}} c_{j_1...j_n} a_{j_1}^\dagger \cdots a_{j_n}^\dagger |\Omega\rangle \tag{61}
\]
and a symmetric homogeneous polynomial of degree \(l\) in \(n\) variables
\[
p_\psi(x_1, \ldots, x_n) = \sum_{\{j_i\}} c_{j_1...j_n} x_1^{j_1} \cdots x_n^{j_n}. \tag{62}
\]
The coefficients \(c_{j_1...j_n}\) can be assumed totally symmetric. It is easy to see that conformal invariance of \(|\psi\rangle\) translates into translation invariance of the polynomial \(p_\psi(x_1, \ldots, x_n)\)
\[
D^\dagger |\psi\rangle = 0 \iff \left( \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_n} \right) p_\psi(x_1, \ldots, x_n) = 0. \tag{63}
\]
A basis \(b^{(k)}(x_1, \ldots, x_n)\) for the vector space of symmetric translation invariant polynomials of degree \(l\) is generated by products
\[
b^{(k)}(x_1, \ldots, x_n) = [q_2(x_1, \ldots, x_n)]^{(k)}_2 \cdot [q_3(x_1, \ldots, x_n)]^{(k)}_3 \cdot \cdots \cdot [q_{n-1}(x_1, \ldots, x_n)]^{(k)}_{n-1}, \tag{64}
\]
where
\[
2i_2^{(k)} + 3i_3^{(k)} + \cdots + (n - 1)i_{n-1}^{(k)} = l, \quad i_j^{(k)} \geq 0 \tag{65}
\]
and
\[
q_m(x_1, \ldots, x_n) = \sum_{\pi \in S_n} s_m(x_{\pi(1)}, \ldots, x_{\pi(m)}),
\]
\[
s_m(y_1, \ldots, y_m) = \prod_{j=1}^{m} \left( \sum_{h=1}^{m} y_h - (m - 1) y_j \right). \tag{66}
\]
The different partitions of \(l\) in (65) yield all linear independent basis vectors \(b^{(k)}(x_1, \ldots, x_n)\), i.e. generate \(C[n, (l; l/2, l/2)]\). The combinatorial problem of finding all partitions (65) is well-known [3] and the answer given in terms of a generating function
\[
\sum_{l=0}^{\infty} \left( \dim C[n, (l; l/2, l/2)] \right) \cdot x^l = \frac{1}{\prod_{i=2}^{n} (1 - x^i)}. \tag{67}
\]
For large $l$ one can prove the following asymptotic behaviour by expanding the polynomial on the right hand side of eq. (67) and approximating its coefficients

$$\dim C[n, (l; l/2, l/2)] \simeq \frac{1}{n! (n-2)!} \left( l + \frac{(n-1)(n+2)}{4} \right)^{n-2}.$$  \hspace{1cm} (68)

Thus for $n \geq 4$ the dimensions $\dim C[n, (l; l/2, l/2)]$ increase quickly with $l$, making the operator mixing problem very hard. The field content of the $N$–vector model in $4 - \epsilon$ dimensions therefore becomes extremely large for many gradients.

We have used a computer program to generate the polynomials $b^{(k)}(x_1, \ldots, x_n)$ and thus the corresponding CIOs. Then a diagonalization of $V$ from eq. (58) was done in these spaces. The results for $n \leq 12$, $l \leq 12$ are summed up in table 7. This table is not meant to scare the reader, but shall mainly give some idea of the complexity of the spectrum of anomalous dimensions encoded by $V$. This is of course even more true if one allows for arbitrary SO(4) spin structure.

Looking at table 7 more closely one finds in particular:

• The lower bounds established in sect. 5, eq. (41)

$$V \Big|_{C[n, (l; l/2, l/2)]} \geq \frac{1}{2} n (n-1) - \frac{1}{6} l (l + n - 1)$$  \hspace{1cm} (69)

and this bound is strict for $n \geq l, l \neq 1$.

• For $n$ fixed, $l \rightarrow \infty$ the smallest eigenvalues seem to converge to 0. This will be explained in the next section and is connected with the intuitive idea from Parisi that a high spin “effectively” separates space–time points [1].

• The eigenvalues for $n = 4$ and an odd number of gradients $l$ are just the rational numbers appearing in the spectrum for $n = 3$ fields, plus the additional eigenvalue 1. This eigenvalue 1 also appears degenerate for even larger values of $l$ (e.g. twofold degenerate for $l = 13$). We know no explanation for this feature.

9 Eigenvalues 0 in the spectrum

The case of vanishing anomalous dimensions in one–loop order is especially interesting since besides their canonical dimension, the corresponding eigenoperators are the “least” irrelevant. This might be of importance for operator product expansions on the light cone, see below. Besides it is also possible to give a concise classification of such eigenoperators with eigenvalue 0.
In our previous work \cite{5} we already established the following expression for \( V \)
\[
V = \frac{1}{2} \sum_{(L,M_1,M_2)} \frac{(-1)^{L+M_1+M_2}}{(L+1-k)k!^2} \sum_{k=0}^{\left\lfloor \frac{L}{2} \right\rfloor} (L - 2k)! (L - 2k + 1)!
\]
\[
\cdot H_{L-2k}^{(M_1,M_2)}(\ad \vec{P}) \left( \frac{\ad P}{2} \right)^k [a^{(0,0,0)} \dagger a^{(0,0,0)} \dagger].
\]
\[
\cdot H_{L-2k}^{(-M_1,-M_2)}(\ad \vec{K}) \left( \frac{\ad K}{2} \right)^k [a^{(0,0,0)} a^{(0,0,0)}].
\]  

Here \( \ad P[X] \) denotes the commutator \([P, X]\) and
\[
\vec{P} = (\partial_{++} - \partial_{--}, \partial_{+-} + \partial_{-+}, i(\partial_{+-} - \partial_{-+}), -i(\partial_{++} + \partial_{--}))
\]
\[
K_{\mu} = -P^\dagger_{\mu}.
\]  

In order to be an eigenoperator with eigenvalue 0, a conformal invariant operator \( |\psi> \) must obviously possess a vanishing matrix element \(<\psi| V |\psi> = 0\). According to \[70\] this is equivalent to \( a^{(0,0,0)} a^{(0,0,0)} |\psi> = 0 \), that is the composite operator must not contain any factors \( \phi^2 \) without derivatives acting on it
\[
V |\psi> = 0 \iff a^{(0,0,0)} a^{(0,0,0)} |\psi> = 0
\]  

for CIOs (\( \partial^\dagger \psi = 0 \)).

In the case of spatially symmetric and traceless tensors discussed in the previous section, the above statement can be made more precise. One easily shows that the action of \( V \) on \( |\psi> \) can be represented on \( p_{\psi}(x_1, \ldots, x_n) \) from eq. \[62\] by
\[
V p_{\psi}(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} V^{ij} p_{\psi}(x_1, \ldots, x_n),
\]  

where
\[
V^{ij} p_{\psi}(x_1, \ldots, x_n)
\]
\[
= \int_0^1 du \ p_{\psi}(x_1, \ldots, x_{i-1}, u x_i + (1-u) x_j, x_{i+1}, \ldots, x_{j-1}, u x_i + (1-u) x_j, x_{j+1}, \ldots, x_n).
\]  

A sufficient condition for \( V |\psi> = 0 \) is then
\[
\forall i, j \quad x_i = x_j \implies p_{\psi}(x_1, \ldots, x_n) = 0.
\]  

In the appendix we show that this is in fact also a necessary condition: The non–local interaction \[74\] is equivalent to a \( \delta \)–interaction potential for a quantum mechanical system of \( n \) bosons living on the twofold covering of the homogeneous space.
SO(2,1)/SO(2). A necessary and sufficient condition for eigenvalues 0 is that the wave function vanishes if any two coordinates coincide, which just turns out to be equivalent to eq. (75).

Eq. (75) has to be solved for translation invariant polynomials. But then the solutions are simply Laughlin’s polynomials \[11\] for bosons
\[
p(x_1, \ldots, x_n) = \left( \prod_{i<j}^{n} (x_i - x_j)^2 \right) \cdot q(x_1, \ldots, x_n),
\]
where \(q(x_1, \ldots, x_n)\) can be an arbitrary homogeneous symmetric and translation invariant polynomial. Thus one finds CIOs with vanishing anomalous dimensions for
\[
l = n(n-1) \quad \left[ q(x_1, \ldots, x_n) = 1 \right]
\]
\[
l = n(n-1) + 2 \quad \left[ q(x_1, \ldots, x_n) = \sum_{i,j=1}^{n} (x_i - x_j)^2 \right]
\]
\[
l = n(n-1) + 3 \quad \left[ q(x_1, \ldots, x_n) = \sum_{i,j,k=1}^{n} (x_i + x_j - 2x_k)(x_j + x_k - 2x_i)(x_k + x_i - 2x_j) \right]
\]
\[
\vdots
\]
The number of eigenvalues zero in \(C[n, (l; l/2, l/2)]\) is just the number of linear independent polynomials \(q(x_1, \ldots, x_n)\) in (76), and this number is simply \(\dim C[n, (l_0; l_0/2, l_0/2)]\) given by eq. (77) with \(l_0 = l - n(n-1)\). The degeneracy in the subspace of CIOs with vanishing anomalous dimension in one–loop order therefore becomes arbitrarily large for \(l \to \infty\). In fact one has according to eq. (38)
\[
\frac{\# \text{ Eigenvalues } 0 \text{ in } C[n, (l; l/2, l/2)]}{\dim C[n, (l; l/2, l/2)]} \xrightarrow{l \to \infty} 1.
\]
For large enough \(l\) “almost all” eigenvalues are zero for any fixed number of elementary fields \(n\).

Now already Parisi has conjectured that for composite operators like \(\phi \partial_{\mu_1} \ldots \partial_{\mu_l} \phi\) in the limit of large spin \(l \to \infty\), no further subtractions besides the renormalization of the field \(\phi\) are necessary (compare ref. \[1\]). An intuitive argument would be to say that a large angular momentum “effectively” separates the two space–time points. From our results in this section we can extend this statement to an arbitrary number of space–time points, at least in one–loop order. Thereby even higher twist contributions in an
operator product expansion on the light cone are (in one–loop order) “dominated” by their canonical scaling behaviour.

10 Conclusions

In this paper we have investigated the structure of the one–loop spectrum of anomalous dimensions in the $N$–vector model in $4 – \epsilon$ dimensions. For a small number of gradients ($l \leq 5$) or elementary fields ($n \leq 3$) in the composite operators one can derive explicitly the anomalous dimensions of the scaling eigenoperators. Neglecting some obvious complications due to the $O(N)$ degrees of freedom of the model by either setting $N = 1$ or by considering completely symmetric traceless $O(N)$ tensors only, these anomalous dimensions are rational numbers. Let us mention that wherever we have compared these results with results obtained by Lang and Rühl in a $1/N$–expansion using operator product expansion techniques [8, 9, 10], we have found agreement.

However the simple structure of the spectrum of anomalous dimensions does not extend to more complex composite operators ($l > 5$ or $n > 3$). The essential reason for this turns out to be the increasing dimensionality of the spaces of mixing conformal invariant operators (CIOs), compare e.g. Table 5 or eq. (67). One does not find a closed expression for the eigenvalues which are in general non–rational numbers. This complicated structure even in one–loop order is quite different from the familiar algebraic structures in two dimensional conformal field theories. Clearly life in $d = 4 – \epsilon$ dimensions is considerably more complicated than in $d = 2$ dimensions since conformal symmetry yields less stringent conditions. In the language of two dimensional conformal field theory one would say that the field content here in $d = 4 – \epsilon$ dimensions is infinite with seemingly no underlying algebraic structure.

In our opinion this is not remedied by the fact that some features in this complicated spectrum can be understood analytically. For quite general classes of CIOs we have been able to work out the smallest anomalous dimensions in a given space of mixing CIOs explicitly and to characterize the corresponding scaling eigenoperator (“ground states”). In particular for a large enough number of gradients for a fixed number of fields, one finds a highly degenerate subspace of CIOs with vanishing anomalous dimensions in one–loop order. This agrees with Parisi’s conjecture that a high spin “effectively” separates space–time points in the sense that no additional subtractions are necessary to renormalize the composite operator.

Finally let us mention some of the remaining problems. First of all we would not
expect that much more can be learned explicitly about the eigenvalues in the spectrum of anomalous dimensions beyond the “ground state” properties that we have discussed in this paper. In principle one can of course argue that all the relevant information is encoded in the relatively simple spectrum–generating operator $V_N$ anyway. A remarkable observation in the spectrum is certainly the infinite number of degenerate eigenvalues, especially with vanishing anomalous dimension in order $\epsilon$. One can always wonder whether there are deeper physical reasons (symmetries) responsible for such degeneracies. We do not consider this very likely and intend to investigate whether these degeneracies are lifted in two–loop order in a subsequent publication.

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A Representation as a local interaction problem

A more intuitive understanding of the spectrum of anomalous dimensions for spatially symmetric and traceless tensors can be gained by looking at \( V \) in eq. (58) from a different viewpoint: \( V \) can be regarded as a local interaction potential for a quantum mechanical system of bosons. This interpretation is particularly useful for saying something about the \( l \to \infty \) limit of the spectrum.

As already mentioned in sect. 7, eq. (74) unfortunately corresponds to a non–local interaction. In order to have a “physical” local interaction potential we have to work with wave functions in a suitable two dimensional configuration space \( S \). Considering the \( \text{SO}(2,1) \) symmetry of \( V \) it is not surprising that this space \( S \) is provided by the homogeneous space \( \text{SO}(2,1)/\text{SO}(2) \) — or to be precise its twofold covering. \( S \) can be parametrized as the twofold covering of the upper nappe of a hyperboloid embedded in \( \mathbb{R}^3 \) with metric

\[
\begin{pmatrix}
-1 & 0 & 0 \\
0 & -1 & 0 \\
0 & 0 & 1
\end{pmatrix},
\]

(78)
and parametrized by

\[
\begin{align*}
y_1 &= \sinh \tau \cos \alpha \\
y_2 &= \sinh \tau \sin \alpha \\
y_3 &= \cosh \tau.
\end{align*}
\]

(79)

The \( \text{SO}(2,1) \) symmetry group is realized on \( S \) by generators of rotations (compare ref. [14])

\[
H_+ = -A_1 - i A_2 = e^{-i \alpha} \left( -\frac{1}{2 \sinh \tau} - \frac{\partial}{\partial \tau} + i \coth \tau \frac{\partial}{\partial \alpha} \right)
\]

\[
H_- = -A_1 + i A_2 = e^{i \alpha} \left( \frac{1}{2 \sinh \tau} - \frac{\partial}{\partial \tau} - i \coth \tau \frac{\partial}{\partial \alpha} \right)
\]

(80)

\[
H_3 = i A_3 = i \frac{\partial}{\partial \alpha}.
\]
corresponding to \( D, -D^\dagger, S \) from eq. (60).

In order to represent the spectrum problem of \( V \) as a quantum mechanical problem, one maps a composite operator \(|\psi\rangle \in C[n, (l;1/2,1/2)]\) on an \( n \)–particle wave function
for bosons on \( S \) defined via
\[
|\psi\rangle = \sum_{\{j_i\}} c_{j_1...j_n} a^\dagger_{j_1} \ldots a^\dagger_{j_n} |\Omega\rangle \quad \rightarrow \quad \Psi(y_1, \ldots, y_n) = \sum_{\{j_i\}} c_{j_1...j_n} \sigma_{j_1}(y_1) \cdot \ldots \cdot \sigma_{j_n}(y_n). \tag{81}
\]
\( \sigma_j(y) \) are the one–particle wave functions
\[
\sigma_j(y(\tau, \alpha)) = e^{-i(j+1/2)\alpha} \frac{1}{\cosh \frac{\tau}{2}} \left( \tanh \frac{\tau}{2} \right)^j \tag{82}
\]
and \( 0 \leq \tau < \infty, 0 \leq \alpha < 4\pi \) in the notation of eq. (79). \( \Psi(y_1, \ldots, y_n) \) has the same transformation properties under \( H_+, H_-, H_3 \) as \( |\psi\rangle \) under \( D, -D^\dagger, S \).

Notice that the wave functions constructed in this manner do not span a complete basis on \( S \). In fact they span a basis in the subspace with eigenvalue \( 1/4 \) of the Laplace operator
\[
\Delta_S = -A_1^2 - A_2^2 + A_3^2 \tag{83}
\]
with \( A_1, A_2, A_3 \) from (80). In general the eigenvalues of \( \Delta_S \) are \( (1/4 + \rho^2) \) for the odd representations that we are using (\( \epsilon = 1/2 \) in the notation of ref. [14]). Therefore the wave functions \( \Psi(y_1, \ldots, y_n) \) here have lowest “kinetic energy” on \( S \).

It turns out that it is just a \( \delta \)–potential interaction on \( S \) that mimicks the respective action of \( V \) on \( |\psi\rangle \):
\[
|\psi\rangle \xrightarrow{\text{eq. (81)}} \Psi(y_1, \ldots, y_n) \quad \rightarrow \quad V |\psi\rangle \xrightarrow{\text{eq. (83)}} H \Psi(y_1, \ldots, y_n) \tag{84}
\]
The quantum mechanical Hamiltonian \( H \) is a sum of two–particle \( \delta \)–interaction potentials
\[
h(y_i, y_j) = \frac{1}{8\pi} \delta(||y_i(\tau_i, \psi_i) - y_j(\tau_j, \psi_j)||) \\
= \frac{1}{8\pi} \frac{1}{\sinh \tau_i} \delta(\tau_i - \tau_j) \delta(\alpha_i - \alpha_j) \tag{85}
\]
in the sense that
\[
H \Psi(y_1, \ldots, y_n) \overset{\text{def}}{=} \sum_{i,j=1}^n \sum_{l_1,l_2=0}^\infty \sigma_{l_1}(y_i) \sigma_{l_2}(y_j) \int_S dS(y_i') dS(y_j') \\
\times \sigma_{l_1}(y_i') \sigma_{l_2}(y_j') h(y_i', y_j') \Psi(y_1, \ldots, y_i', \ldots, y_j', \ldots, y_n) \tag{86}
\]
with the induced surface element
\[
\int_S dS(y(\tau, \alpha)) = \int_0^\infty d\tau \ \sinh \tau \int_0^{4\pi} d\alpha. \tag{87}
\]
From this viewpoint it is obvious that a necessary and sufficient condition for a vanishing eigenenergy of $H$ is that the wave function $\Psi(y_1, \ldots, y_n)$ vanishes if any two coordinates coincide. By virtue of

$$\Psi(\tau_1, \alpha_1; \ldots; \tau_n, \alpha_n) = \frac{e^{-\frac{i}{2}(\alpha_1 + \cdots + \alpha_n)}}{\cosh \frac{\tau_1}{2} \cdot \cdots \cdot \cosh \frac{\tau_n}{2}} \cdot p_\psi \left( e^{-i \alpha_1 \tanh \frac{\tau_1}{2}} \cdot \cdots \cdot e^{-i \alpha_n \tanh \frac{\tau_n}{2}} \right)$$

equivalent to condition (75) in sect. 8.

Finally one can wonder whether there are more common features between our problem here and the fractional quantum Hall effect (FQHE) besides the use of Laughlin wave functions. Loosely speaking this is true to some extent: In a certain sense the restriction of wave functions on $S$ to those with eigenvalue $1/4$ of the Laplace operator $\Delta_S$ resembles the restriction to the lowest Landau level in the FQHE. And instead of fermions in the plane or on a two dimensional sphere interacting with Coulomb potentials we are interested in bosons living on the twofold covering of a hyperboloid with a $\delta$–interaction. As a consequence of this $\delta$–interaction the ground state energy vanishes exactly below the threshold $n(n-1) \leq \ell$ (with the exception of $n(n-1) = l+1$). Similarly the ground state energy per particle vanishes in the FQHE below a certain filling factor $\frac{l}{2}$. 

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Tables:

| # Gradients | O(N) transformation properties | Anomalous dimension λ |
|-------------|---------------------------------|-----------------------|
| $l = 0$     | Scalar                          | $\epsilon \cdot \frac{N+2}{N+8} + O(\epsilon^2)$ |
| $l \geq 1$, odd | for $N \geq 2$               | $\epsilon \cdot \frac{2}{N+8} + O(\epsilon^2)$ |
| $l \geq 2$, even | Scalar (Notation: $T^{(l)}$) | $O(\epsilon^2)$ |
|             | for $N \geq 2$               | $O(\epsilon^2)$ |

Table 1: The (trivial) spectrum of anomalous dimensions for conformal invariant operators with $n = 2$ fields. Notice that $T^{(2)}$ is proportional to the stress tensor.

| # Gradients | Anomalous dimension λ |
|-------------|-----------------------|
| $l = 0$     | $\epsilon + O(\epsilon^2)$ |
| $l \geq 2$  | $\frac{\epsilon}{3} \cdot \left(1 + (-1)^l \frac{2}{l+1}\right) + O(\epsilon^2)$ |

Table 2: The non–zero anomalous dimensions in the spectrum for $n = 3$ fields and $N = 1$ component.
## CIOs with \( n = 3 \) fields for \( N \geq 2 \) components

| # Gradients | O(N) transformation properties | Anomalous dimension \( \lambda \) |
|-------------|--------------------------------|----------------------------------|
| \( l = 0 \) | \( \epsilon \cdot \frac{6}{N+8} + O(\varepsilon^2) \) | \( \epsilon + O(\varepsilon^2) \) |
|             | (two indices are contracted)   |                                  |
| \( l = 1 \) | \( \epsilon \cdot \frac{3}{N+8} + O(\varepsilon^2) \) |                                  |
| \( l \geq 2 \) | \( \epsilon \cdot \frac{N+2}{N+8} + O(\varepsilon^2) \) | \( \epsilon \cdot \frac{2}{N+8} \left(1 + (-1)^l \frac{2}{l+1}\right) + O(\varepsilon^2) \) |
|             | \( \epsilon \cdot \frac{1}{N+8} \left(2 - (-1)^l \frac{2}{l+1}\right) + O(\varepsilon^2) \) |                                  |

Eigenvalues of

\[
\frac{\varepsilon}{N+8} \begin{pmatrix}
N+2 & 1 + (-1)^l \frac{2}{l+1} \\
(N+2) \cdot (-1)^l \frac{2}{l+1} & 2 + (-1)^l \frac{4}{l+1}
\end{pmatrix} + O(\varepsilon^2)
\]

Table 3: The non–zero anomalous dimensions in the spectrum for \( n = 3 \) fields and \( N \geq 2 \) components.
### CIOs for $N = 1$ or with $O(N)$ structure \(^{(33)}\)

| # Gradients | # Fields | SO(4) spin structure | Eigenvalues $\alpha$ of $V$ |
|-------------|----------|----------------------|----------------------------|
| $l = 0$     | $n \geq 1$ | $(0,0)$              | $1/2 n(n - 1)$             |
| $l = 2$     | $n \geq 2$ | $(1,1)$              | $1/2 n(n - 1) - 1/3 n - 1/3$ |
| $l = 3$     | $n \geq 3$ | $(3/2,3/2)$          | $1/2 n(n - 1) - 1/2 n - 1$ |
| $l = 4$     | $n \geq 2$ | $(2,2)$              | $1/2 n(n - 1) - 3/5 n + 1/5$ |
|             | $n \geq 4$ | $(2,2)$              | $1/2 n(n - 1) - 2/3 n - 2$ |
|             | $n \geq 3$ | $(2,0),(0,2)$        | $1/2 n(n - 1) - 2/3 n - 1$ |
|             | $n \geq 4$ | $(1,1)$              | $1/2 n(n - 1) - 2/3 n - 2/3$ |
|             | $n \geq 4$ | $(0,0)$              | $1/2 n(n - 1) - 2/3 n$     |
| $l = 5$     | $n \geq 3$ | $(5/2,5/2)$          | $1/2 n(n - 1) - 2/3 n - 1/3$ |
|             | $n \geq 5$ | $(5/2,5/2)$          | $1/2 n(n - 1) - 5/6 n - 10/3$ |
|             | $n \geq 3$ | $(5/2,3/2),(3/2,5/2)$ | $1/2 n(n - 1) - 5/6 n - 1/2$ |
|             | $n \geq 4$ | $(5/2,1/2),(1/2,5/2)$ | $1/2 n(n - 1) - 5/6 n - 2$ |
|             | $n \geq 5$ | $(3/2,3/2)$          | $1/2 n(n - 1) - 5/6 n - 5/3$ |
|             | $n \geq 4$ | $(3/2,1/2),(1/2,3/2)$ | $1/2 n(n - 1) - 5/6 n - 7/6$ |
|             | $n \geq 5$ | $(1/2,1/2)$          | $1/2 n(n - 1) - 5/6 n - 2/3$ |

Table 4: Eigenvalues $\alpha$ of $V$ from eq. \(^{(13)}\). The above list of conformal invariant operators is complete for $l \leq 5$ gradients.
| $l$ | $(j_1, j_2, \pi)$ | # CIOs | # Ground states |
|-----|------------------|--------|-----------------|
| 0   | (0,0,+)         | 1      | 1               |
|     | (1,1,+)         | 1      | 1               |
|     | (3/2,3/2,+)    | 1      | 1               |
| 4   | (2,2,+)         | 2      | 1               |
|     | (2,0)           | 1      | 1               |
|     | (1,1,+)         | 1      | 1               |
|     | (0,0,+)         | 1      | 1               |
| 5   | (5/2,5/2,+)    | 2      | 1               |
|     | (5/2,3/2)      | 1      | 0               |
|     | (5/2,1/2)      | 1      | 1               |
|     | (3/2,3/2,+)    | 1      | 1               |
|     | (3/2,1/2)      | 1      | 1               |
|     | (1,2,1/2,+)    | 1      | 1               |
| 6   | (3,3,+)         | 4      | 1               |
|     | (3,2)           | 1      | 0               |
|     | (3,1)           | 3      | 1               |
|     | (2,2,+)         | 3      | 1               |
|     | (2,1)           | 2      | 1               |
|     | (1,1,+)         | 4      | 3               |
|     | (0,0,+)         | 2      | 2               |

| $l$ | $(j_1, j_2, \pi)$ | # CIOs | # Ground states |
|-----|------------------|--------|-----------------|
| 7   | (7/2,7/2,+)     | 4      | 1               |
|     | (7/2,5/2)       | 3      | 0               |
|     | (7/2,3/2)       | 4      | 1               |
|     | (7/2,1/2)       | 2      | 0               |
|     | (5/2,5/2,+)     | 4      | 1               |
|     | (5/2,3/2)       | 5      | 1               |
|     | (5/2,1/2)       | 3      | 1               |
| 8   | (4,+,)          | 7      | 1               |
|     | (4,3)           | 4      | 0               |
|     | (4,2)           | 8      | 1               |
|     | (4,1)           | 3      | 0               |
|     | (4,0)           | 4      | 0               |
|     | (3,3,+)         | 8      | 1               |
|     | (3,2)           | 9      | 1               |
|     | (3,1)           | 8      | 1               |
|     | (3,0)           | 2      | 0               |
|     | (2,2,+)         | 14     | 3               |
|     | (2,2,−)         | 2      | 0               |
|     | (2,1)           | 9      | 2               |
|     | (2,0)           | 7      | 2               |
|     | (1,1,+)         | 10     | 4               |
|     | (1,1,−)         | 1      | 0               |
|     | (1,0)           | 2      | 0               |
|     | (0,0,+)         | 5      | 3               |

Table 5: A complete list of CIOs for $l \leq 8$, $n \geq |l - 1/2| |j_1 - j_2|$. CIOs with $j_1 \neq j_2$ are only listed once for $j_1 > j_2$. # Ground states is the number of ground state eigenvalues eq. (38) in the spectrum for the respective quantum numbers.
Table 6: Linear eigenvalues $\alpha$ different from the ground state eigenvalues for $l \leq 8$, $n \geq \lfloor l - 1/2 \rfloor |j_1 - j_2|$. All these eigenvalues are non-degenerate.
CIOs in $\mathcal{C}[n, (l; l/2, l/2)]$

| $n$ | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 9   | 10  | 11  | 12  |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| 0   | 1   | 3   | 6   | 10  | 15  | 21  | 28  | 36  | 45  | 55  | 66  |
| 1   | 0   | 5/3 | 13/3| 8   | 38/3| 55/3| 25  | 98/3| 124/3| 51 | 185/3|
| 2   | 0   | 7/5 | 14/3| 9   | 43/3| 62/3| 28  | 109/3| 137/3| 56 | 158/3|
| 3   | 0   | 1/2 | 13/2| 11  | 33/2| 23  | 61/2| 39  | 97/2| 59  |
| 4   | 0   | 2/3 | 5/2 | 16  | 67/3| 89/3| 38  | 142/3| 173/3| 59 | 243/5|
| 5   | 0   | 1.49| 4.51| 4   | 15  | 22  | 30  | 39  | 49  |     |
| 6   | 0   | 2.01| 5.11| 8.49| 13.45| 19.42| 26.38| 34.34| 43.31| 53.28|
| 7   | 0   | 3.54| 6.77| 9.25| 14.40| 20.56| 27.71| 35.86| 45  | 55.14|
| 8   | 0   | 11.01| 16.24| 22.47| 29.71| 37.95| 47.20| 57.45 |
| 9   | 0   | 21/2 | 35/3| 37/2 | 37/2 | 79/3 | 211/6| 45  | 56.85|
| 10  | 0   | 0.34 | 2.11| 4.33 | 9.10 | 8   | 44/3| 67/3 | 31 | 122/3|
| 11  | 0   | 1.57 | 5.77| 10.42| 14.85| 21.57| 29.27| 37.95| 47.62|
| 12  | 0   | 3.85 | 6.24| 10.47| 12.94| 18.95| 25.98| 34  | 43.03| 53.06|
| 13  | 2/3 | 5/2 | 16 | 67/3 | 89/3 | 38 | 142/3| 173/3 |
| 14  | 0   | 4.91 | 8.92| 13.93| 19.94| 26.95| 34  | 43.03| 53.06|
| 15  | 0   | 3.84 | 6.24| 10.47| 12.94| 18.95| 25.98| 34  | 43.03| 53.06|
| 16  | 0   | 1.57 | 5.77| 10.42| 14.85| 21.57| 29.27| 37.95| 47.62|

Table 7: Spectrum of $V$ from eq. (58) for $n, l \leq 12$. Numbers with decimal points are approximate numerical results and no rational numbers, otherwise the given eigenvalues are exact. \#... denotes the dimension of the respective space $\mathcal{C}[n, (l; l/2, l/2)]$ if not all eigenvalues are written explicitly.
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