Self-excited vibration of whole vehicle with multiple limit cycles induced by shimmy of front wheels

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Abstract
Front wheel shimmy may induce the self-excited vibration of whole vehicle with multiple limit cycles. This kind of shimmy is difficult to control and can do great harm to the vehicle. In order to obtain the mechanism of this phenomenon, a 7-DOF dynamic model of whole vehicle self-excited vibration induced by the shimmy of front wheels which take the nonlinear factor of tire lateral force and the dry friction force in suspension and steering system into consideration was established. By using Hopf bifurcation theorem and central manifold theorem, the existence and stability of limit cycles are qualitatively determined. By means of numerical analysis, the self-excited vibration with multiple limit cycles is found. The results show that in the speed range of 50–185 km/h, front wheels shimmy induced the self-excited vibration of whole vehicle and the amplitude of system’s vibration decreases with the increase of vehicle speed; in the speed range of 50–83 km/h and 149–185 km/h, the self-excited vibration with multiple limit cycles occurs; in the speed range of 83–149 km/h, the self-excited vibration with single limit cycle occurs.

Keywords
Shimmy of front wheels, multiple limit cycles, Hopf bifurcation, center manifold theorem, self-excited vibration

Introduction
Shimmy of front wheels is the phenomenon of front wheels which continue to vibrate around the kingpin when the vehicle is running on a flat road. It will cause the car body to sway, resulting in whole vehicle self-excited vibration, and the vehicle wound appear serpentine trajectory. This will seriously affect the vehicle’s handling stability, ride comfort and safety. The self-excited vibration with multiple limit cycles means that the induced vibration’s amplitude may be different under different initial excitations.

Over the years, scholars from all over the world have conducted extensive and in-depth research on the shimmy of front wheels from a linear perspective, and have achieved rich results.¹-³

However, the vehicle is a multiple degree-of-freedom (DOF) nonlinear system, and its nonlinear dynamic behavior becomes more complicated as the vehicle speed increases. Therefore, in recent years, some scholars have studied the mechanism of shimmy of front wheels and its effect on motion stability of the vehicle from a nonlinear perspective. A three-DOF shimmy model of steering system in dependent suspension vehicle was established in literature,⁴ which took clearance and dry friction into consideration. The numerical calculations showed that both factors would induce front wheels shimmy system to experience multiple limit cycles vibration. A front wheel shimmy model of independent suspension vehicle was established in literature,⁵ which took steering clearance and dry friction into consideration. The numerical calculations showed that the clearance and dry...
friction induced the system to experience supercritical Hopf bifurcation. A front wheels shimmy model of independent suspension vehicle was established in literature\textsuperscript{6}, which took dry friction and nonlinear characteristics of the tires into consideration. The numerical calculations showed that dry friction induced the front wheels to experience shimmy with multiple limit cycles. Motion differential equations of motorcycle were established in literature\textsuperscript{7}, in which shimmy of front wheels and its effect on the motion stability of motorcycle were studied. Based on a certain type of trailer, a 24-DOF dynamic model was established in literature\textsuperscript{8}, in which the dynamic behavior of vehicle instability in a high speed was studied. Stability equations of motion of trailers were established in literature\textsuperscript{9}, in which the effect of system parameters on high-speed shimmy and handling stability of vehicle were studied. Equations of motion of semi-trailers were established in literature\textsuperscript{10}, in which the transverse self-excited vibration of semi-trailers and its effect on handling stability of vehicles were studied. Considering the shimmy of steering system, a 3-DOF dynamic equations of vehicle motion stability were established in literature\textsuperscript{11}, in which the effect of steering system’s parameters on handling stability of vehicles were studied. The results show that the shimmy of front wheels induced vehicle system to experience a supercritical Hopf bifurcation behavior. But the author did not find the self-excited vibration with multiple limit cycles. We preliminarily studied the self-excited vibration and multiple limit cycles shimmy of front wheels in the early research\textsuperscript{12–14}, but we did not study the self-excited vibration of whole vehicle.

In summary, the above research on shimmy can be divided into two categories. One is that dynamic model was confined to the shimmy system of front wheels and the shimmy system of whole vehicle was not studied. The other is that the dynamic model of whole vehicle shimmy system was established but the self-excited vibration with multiple limit cycles is not found, which may be related to the matching of system parameters and the selection of dry friction model. Based on the above deficiencies, this paper establishes a 7-DOF dynamic model of whole vehicle self-excited vibration induced by the shimmy of front wheels. The qualitative analysis method is used to determine the Hopf bifurcation of the system, and the numerical analysis method is used to find the multiple limit cycles of the system, which enriches the knowledge of dynamics of the vehicle shimmy system.

Establish the dynamic model of vehicle shimmy

Dynamic model of vehicle

Choose a certain vehicle as the sample vehicle. The vehicle’s front suspension is McPherson suspension. The steering system is a disconnected steering mechanism with rockers and it has steering gears of recirculating ball type, as shown in Figure 1(a).

The whole vehicle is simplified as a 7-DOF mechanical model, which is shown in Figure 1 (a) to (c). Seven degrees of freedom are, respectively, the left and right wheels swing angle around its kingpin $\theta_1$ and $\theta_2$, the intermediate rocker arm angle around its axis of rotation $\theta_3$, the steering wheel angle $\theta_4$, sideslip angle $\beta$, yaw speed $\omega$, and the body roll angle $\phi$. In the model, $O_1$ is barycenter and $O_2$ is roll center. The following assumptions were made before differential equations of vehicle self-excited shimmy system were established:

1. The steering system of the vehicle uses an off-ackermann steering linkage with an intermediate steering arm. The left and right trapezoidal arms, suspension and tire structures are axisymmetrical with respect to X-axis. Therefore, it is assumed that the rotational inertia, mass, stiffness, damping and other structural parameters of

Figure 1. Mechanical model of vehicle steering. (a) Front suspension and steering mechanism; (b) planar motion and (c) roll motion.
the left and right wheels and the left and right trapezoidal arms are the same, and the lateral forces of the left and right rear wheels are approximately equal.

2. Only the dry friction in the motion pair of the suspension and the steering system are considered, and the collision force in the motion pair clearance is ignored. Only the dry friction at the kingpin is considered and the dry friction in the motion pair of the suspension and the steering system are equivalent to be at the kingpin.

3. The vehicle’s front suspension is McPherson suspension. When the wheel is bouncing up and down, the displacement of the wheel camber freedom is small, and the corresponding gyroscopic effect is weak. Therefore, only the freedom of swinging around the kingpin is considered for the two front wheels.

4. The vehicle runs on a straight road at a constant velocity.

**Differential equations of vehicle self-excited vibration system**

Based on mechanical model in Figure 1(a) to (c) and the above-mentioned assumptions, differential equations of vibration of steering system and vehicle motion were established as follows by using Lagrange’s equation:

Shimmy equation of left wheel around kingpin

\[
J_1 \ddot{\theta}_1 + k_1 \dot{\theta}_1 - k_1 k_3 \dot{\theta}_3 + (c_1 + c_1) \dot{\theta}_1 - c_1 k_3 \dot{\theta}_3 + (F_{z1} - F_1g) \sin(\cos \gamma - R \sin \gamma) \cos \theta_1 \\
+ \sin^2 \gamma (\sin \gamma + R \cos \gamma) \sin \dot{\theta}_1 + F_{z1} [D_1 \cos \alpha + \sin(\sin \gamma + R \cos \gamma)] \\
+ F_{z1} (\cos \gamma - R \cos \gamma) \cos \alpha + M_{f1} = 0
\]  

(1)

Shimmy equation of right wheel around kingpin

\[
J_2 \ddot{\theta}_2 + k_2 \dot{\theta}_2 - k_2 k_3 \dot{\theta}_3 + (c_2 + c_2) \dot{\theta}_2 - c_2 k_3 \dot{\theta}_3 + (F_{z2} - F_2g) \sin(\cos \gamma - R \sin \gamma) \cos \theta_2 \\
+ \sin^2 \gamma (\sin \gamma + R \cos \gamma) \sin \dot{\theta}_2 + F_{z2} [D_1 \cos \alpha + \sin(\sin \gamma + R \cos \gamma)] \\
+ F_{z2} (\cos \gamma - R \cos \gamma) \cos \alpha + M_{f2} = 0
\]  

(2)

Shimmy equation of rocker

\[
J_3 \ddot{\theta}_3 + (k_2 k_3^2 + k_1 k_3^2 + k_3) \dot{\theta}_3 + k_1 k_3 \dot{\theta}_1 - k_2 k_3 \dot{\theta}_2 + (c_1 k_3^2 + c_2 k_3^2 + c_3 + 3 c_3) \dot{\theta}_3 - c_1 k_3 \dot{\theta}_1 \\
- c_2 k_3 \dot{\theta}_2 - k_2 k_3 \dot{\theta}_4 - c_3 k_3 \dot{\theta}_4 = 0
\]  

(3)

Shimmy equation of steering assembly

\[
J_4 \ddot{\theta}_4 + (k_3 k_3^2 + k_4) \dot{\theta}_4 - k_3 k_3 \dot{\theta}_3 - c_3 k_3 \dot{\theta}_3 + (c_3 k_3^2 + c_4 + 4 c_4) \dot{\theta}_4 = 0
\]  

(4)

Vehicle sliding sideways equation

\[
u (\dot{\beta} + \omega) - m_s h_s \dot{\phi} = F_{y1} \cos \theta_1 + F_{y2} \cos \theta_2 + 2 F_{y3}
\]

(5)

Motion equation of vehicle yaw

\[
I_2 \ddot{\phi} = l_1 F_{y1} \cdot \cos \theta_1 + l_1 F_{y2} \cdot \cos \theta_2 - 2 l_2 F_{y3}
\]

(6)

Motion equation of vehicle roll

\[
I_4 \ddot{\phi} + c_\phi \ddot{\phi} + k_\phi \phi = m_s h_s (\dot{\beta} + \omega) + m_s g h_s \phi
\]

(7)

In these equations, \( J_1 \) and \( J_2 \) are moments of inertia of left and right front wheel assembly around the kingpin, respectively; \( J_3 \) is equivalent moment of inertia of intermediate rocker arm assembly about its axis of rotation; \( J_4 \) is equivalent moment of inertia of steering gear assembly converting to its input; \( F_{y1} \) and \( F_{y2} \) are perpendicular
forces from the ground on the left and right front wheel; \( F_{x1} \) and \( F_{x2} \) are longitudinal forces from the ground on the left and right front wheel; \( F_{y1} \) and \( F_{y2} \) are transverse forces from the ground on the left and right front wheel; \( M_{f1} \) and \( M_{f2} \) are equivalent dry friction torques of left and right front wheel around the kingpin; \( c_{1f} \) and \( c_{2f} \) are equivalent angles damping coefficient of left and right front wheel assemblies around the kingpin; \( c_{3} \) is the equivalent angle damping coefficient of intermediate rocker arm assembly about its axis of rotation; \( c_{4} \) is the equivalent angle damping coefficient of steering gear assembly converting to its input; \( k_{1} \) and \( k_{2} \) are equivalent connection angular stiffness between left and right front wheels and intermediate rocker arm; \( k_{5} \) is equivalent connection angular stiffness between intermediate rocker arm and steering gear; \( c_{1} \) and \( c_{2} \) are equivalent connection angular damping coefficients between left and right front wheels and intermediate rocker arm; \( c_{3s} \) is equivalent connection angular damping coefficient between intermediate rocker arm and steering gear; \( c_{4s} \) is equivalent connection angular damping coefficient between steering gears and steering wheel; \( m_{lg} \) and \( m_{rg} \) are loads of left and right wheels; \( f \) is rolling resistance coefficient of the tire; \( \gamma \) is wheel camber; \( R \) is rolling radius of the wheel; \( z \) is inclination angle of kingpin; \( k_{a} \) is amplification coefficient between left and right front wheels and intermediate rocker arm; \( k_{p} \) is amplification coefficient between intermediate rocker arm and steering gears; \( D_{l} \) is pneumatic trail; \( l \) is effective length of steering knuckle; \( l_{f} \) is the front wheelbase; \( l_{r} \) is the rear wheelbase; \( m \) is mass of the vehicle; \( u \) is vehicle speed; \( m_{s} \) is mass on sprung; \( \delta_{f} \) is steering angle of front wheel; \( \delta_{r} \) is steering angle of rear wheel; \( k_{p} \) is scaling coefficient of rotary angle of front and rear wheels; \( F_{yr} \) is lateral force of rear wheel; \( I_{x} \) is moment of inertia of the vehicle around the yaw axis(Z); \( I_{g} \) is moment of inertia of the vehicle around the roll axis(X); \( c_{p} \) is Roll angle damping; \( h_{s} \) is the distance of centroid of mass on sprung to the roll axis and it is the difference between the centroid height of mass on sprung \( h \) and the height of the roll axis \( h_{p} \); and \( k_{p} \) is angle stiffness of roll.

**Dynamics model of tire**

**Vertical load model of tire.** According to the elastic energy change when the tire has a radial deformation, perpendicular force from the ground on the left and right front wheel is deduced when steering system experiences shimmy.

\[
\begin{align*}
F_{z1} &= m_{lg} - k_{5}(l_{c}\cos\gamma - R_{s}\sin\gamma)\sin\gamma\sin\theta_{l} + k_{5}\sin^{2}\alpha(l_{c}\sin\gamma + R_{s}\cos\gamma)(1 - \cos\theta_{l}) \\
F_{z2} &= m_{rg} - k_{5}(l_{c}\cos\gamma - R_{s}\sin\gamma)\sin\gamma\sin\theta_{r} + k_{5}\sin^{2}\alpha(l_{c}\sin\gamma + R_{s}\cos\gamma)(1 - \cos\theta_{r})
\end{align*}
\]

(8)

In the equation, \( k_{5} \) is vertical stiffness of the tire; \( F_{z1} \) is vertical load of left front wheel; \( F_{z2} \) is vertical load of right front wheel. The rear vertical load is assumed approximately unchanged and it is defined as \( F_{zr} \).

**Selection of lateral force model of tire.** Lateral force of tire is the main aspect affecting the vehicle’s transverse stability. Pacejka’s magic formula tire model has some advantages. For example, its form is simple, it is easy to calculate, and it has commonality among different tire fitting parameters and so on. In this paper, the reduced magic formula is chosen for lateral force of tire\(^{15}\)

\[
F_{yi} = D_{i}\sin(C_{i} \cdot \arctan(B_{i}\alpha_{i} - E_{i}(B_{i}\alpha_{i} - \arctan(B_{i}\alpha_{i}))))
\]

(9)

Among it

\[
\begin{align*}
B_{i} &= a_{3} \cdot \sin(a_{4} \cdot (\arctan(a_{5} \cdot F_{zi}))) \\
C_{i} &= 1.3 \\
D_{i} &= a_{1} \cdot F_{zi}^{2} + a_{2} \cdot F_{zi} \\
E_{i} &= a_{6} \cdot F_{zi}^{2} + a_{7} \cdot F_{zi} + a_{8}
\end{align*}
\]

(10)

In the equation, \( \alpha_{i} \) (i = 1, 2, r) separately represent left front wheel, right front wheel, and rear wheel). \( B_{i}, C_{i}, D_{i} \) and \( E_{i} \) represent separately stiffness factor, shape factor, crest factor, and curvature factor. \( a_{1}, a_{2}, a_{3}, a_{4}, a_{5}, a_{6}, a_{7} \) and \( a_{8} \) are parameters obtained by experiment fitting. The value is shown as Table 1. Figure 2 shows the curve which is between cornering force and
sideslip angle of front and rear tire under an approximately static equilibrium state, and calculated by equation (9) and Table 1.

The tire’s sideslip angle, which is required when calculating how front wheel shimmy induces vehicle self-excited vibration, is given by equations (11) and (12).

The front wheel will have self-excited vibration when vehicle runs because of the presence of dry friction in joint clearance of suspension and steering system. The sideslip angle of tire cannot be expressed by the vehicle’s motion state directly. Sideslip characteristic of front wheel is described by choosing a first-order approximation tension wire theory. The rolling constraint equation of left and right tires of front axle was established.4

\[
\begin{align*}
\dot{x}_1 + \frac{u}{\sigma} * x_1 + \frac{u}{\sigma} * \theta_1 - \frac{a}{\sigma} * \dot{\theta}_1 &= 0 \quad (i = 1, \text{ 2 expresses left and right separately}) \\
\dot{x}_2 + \frac{u}{\sigma} * x_2 + \frac{u}{\sigma} * \theta_2 - \frac{a}{\sigma} * \dot{\theta}_2 &= 0
\end{align*}
\]

(11)

\( a \) is semi-length of tire blot; \( \sigma \) is relaxation length of tire.

Considering the vehicle’s motion state, the expression of sideslip angle of rear wheel is as follows

\[
z_r = \arctan\left(\frac{\beta - l_2 * \omega}{u}\right) \quad (l_2 \text{is the rear wheelbase})
\]

(12)

**Selection of dry friction model**

In this paper, we choose to use the Stefanski-Wojewoda static dry friction model.16 The dry friction models commonly used in vehicle shimmy system include Coulomb dry friction model17 and Strubeck dry friction model,18 because these two models are simple in mathematical form (the friction coefficient in Coulomb dry friction model is constant and the Strubeck dry friction model is in exponent form), which is convenient for computer numerical calculation. Stefanski-Wojewoda dry friction model has been neglected by scholars because of its complex mathematical form (piecewise function form), which results in large amount of numerical calculation. But in this paper, we mainly consider dry friction of ball joints between suspension and steering mechanism and dry friction of steering system, and the relative speed in the kinematic pair is close to zero. When the relative speed of two objects is close to zero, frictional lag and negative slope of friction occur, which are called frictional lag effect and Strubeck effect. These two effects are widely considered to be closely related to unstable phenomena such as shimmy in engineering. Compared with other static dry friction models, the Stefanski-Wojewoda dry friction model can well
describe both frictional lag effect and Stribeck effect when the relative speed is close to zero. Therefore, this paper uses Stefanski-Wojewoda dry friction model.

The mathematical expression of Stefanski-Wojewoda dry friction model is as follows

\[ M_f = M_c \cdot f \]

\[ f = \begin{cases} 
  f_s \text{sign}(v), & f_s < f_d+ \cup \text{sign}\left(\frac{dv}{dt} \cdot v\right) > 0 \\
  f_d+\text{sign}(v), & f_s > f_d- \cup \text{sign}\left(\frac{dv}{dt} \cdot v\right) > 0 \\
  f_d-\text{sign}(v), & \text{sign}\left(\frac{dv}{dt} \cdot v\right) < 0 
\end{cases} \]  

(14)

\[ f_s = f_s + \Delta f_s \frac{|v_A|}{|v| + |v|}; \quad g \left(\frac{v}{\frac{dv}{dt}}\right) = \frac{1}{1 + \left(\frac{v}{\frac{dv}{dt}}\right)^2}; \quad f_{d+} = f_c \cdot \left(1 + \frac{f_s(v) - f_c}{f_c} \cdot g\left(\frac{v}{\frac{dv}{dt}}\right)\right) \cdot v = \dot{\theta} \cdot r; \]

\[ f_d = \frac{k}{2N_c} v^2 - f_0; \quad f_s = 2f_c - f_s; \quad f_d+ = f_c \cdot \left(1 + \frac{f_s(v) - f_c}{f_c} \cdot g\left(\frac{v}{\frac{dv}{dt}}\right)\right) \cdot v = \dot{\theta} \cdot r; \]

In the equation, \( v \) is relative velocity, \( f_s \) is dynamic friction coefficient, \( f_s \) is static friction factor, \( \Delta f_s \) is adjustment parameter of maximum static friction force, \( v_A \) is adjustment parameter of average speed, \( \tau \) is hysteresis time, \( M_c \) is amplitude coefficient of equivalent dry friction torque. The curve between dry friction coefficient \( f \) and relative velocity \( v \) is shown as Figure 3 and it is calculated by Stefanski-Wojewoda dry friction model and parameter in Table 2.

The parameter values of the sample vehicle are shown in Table 3.

### Table 2. Parameters for calculating dry friction.

| Parameter | \( f_s \) | \( f_c \) | \( \tau \) | \( v_A \) | \( \Delta f_s \) |
|-----------|------------|------------|-----------|----------|-------------|
| Value     | 0.6        | 0.4        | 0.002     | 0.2      | 0.003       |

Figure 3. Graph of Stefanski-Wojewoda dry friction model.

**Hopf bifurcation qualitative analysis of the shimmy system**

The existence analysis and stability analysis of Hopf bifurcation in a nonlinear system are the most fundamental and important method in studying the dynamics of a nonlinear system. In this section, Hurwitz criterion\(^{19}\) is applied to investigate the existence of Hopf bifurcation in the shimmy system of the whole vehicle. The center manifold
Whereas the equilibrium point of the system is unstable when $u = 50.2 \text{ km/h}$ and contains quadratic and $G(X)$ contains quadratic and cubic nonlinear terms.

According to Hurwitz criterion, the equilibrium point of the system is stable when $u = u_1$ or when $u = u_2$. Whereas the equilibrium point of the system is unstable when $u_1 < u < u_2$. And when the critical speeds are $u_1 = 50.2 \text{ km/h}$ and $u_2 = 185.3 \text{ km/h}$, the eigenvalues of the Jacobian matrix are shown in Table 4.

According to Table 4, the Jacobian matrix of the system has a pair of purely imaginary eigenvalues when $u = u_1$ and $u = u_2$, and the other eigenvalues have negative real parts. According to the Hopf bifurcation theorem, we can get the critical speeds $u_1 = 50.2 \text{ km/h}$ and $u_2 = 185.3 \text{ km/h}$ as the bifurcation points of the systems, and a two-dimensional center manifold exists. When $u_1 < u < u_2$, positive real part eigenvalues exist, and the shimmy occurs in

### Table 3. The vehicle parameter values.

| parameter | Value       | parameter | value       | parameter | value       |
|-----------|-------------|-----------|-------------|-----------|-------------|
| $J_1, J_2$ | 15 (kg.m$^2$/rad) | $c_{3y}$  | 10 (N.m.s/rad) | $J_3$     | 6 (kg.m$^2$/rad) |
| $c_{d1}$  | 10 (N.m.s/rad)  | $l_4$     | 12 (kg.m$^2$/rad) | $c_1$     | 20 (N.m.s/rad)  |
| $c_{d2}$  | 50 (N.m.s/rad)  | $R$       | 0.35 (m)     | $c_2$     | 20 (N.m.s/rad)  |
| $f$       | 0.015        | $c_d$     | 60 (N.m.s/rad) | $k_r$     | 1.37         |
| $k_1, k_2$ | 300000 (N.m/rad) | $k_p$     | 0.049        | $k_3$     | 60000 (N.m/rad) |
| $D_1$     | 0.06 (m)    | $k_4$     | 70000 (N.m/rad) | $l$       | 0.15 (m)     |
| $k_5$     | 1200000 (N.m/rad) | $m_{1g}, m_{2g}$ | 3500 (N) | $c_{11, c_{21}}$ | 20 (N.m.s/rad) |
| $\sigma$  | 0.22 (m)   | $\gamma$ | 4(°)         | $h_{20}$  | 0.087 (m)    |
| $m$       | 1340 (kg)  | $l_4$     | 584 (kg.m$^2$/rad) | $m_2$     | 1200 (kg)    |
| $l_2$     | 2048.1 (kg.m$^2$/rad) | $k_0$    | 48900 (N.m/rad) | $F_{cr}$  | 3200 (N)    |
| $c_{60}$  | 4476 (N.m/s/rad) | $M_c$     | 20 (N.m)     | $h$       | 0.542 (m)   |
| $l_1$     | 1.21 (m)  | $l_2$     | 1.47 (m)     |           |             |

### Table 4. Eigenvalues of Jacobi matrix $A$ corresponding to the critical vehicle speeds.

| Eigenvalues corresponding to $u_1$ | Eigenvalues corresponding to $u_2$ |
|-----------------------------------|-----------------------------------|
| $\lambda_1 = 1.24i$              | $\lambda_8 = -2.92 + 76.4i$      |
| $\lambda_2 = -1.24i$              | $\lambda_9 = -2.92 - 76.4i$      |
| $\lambda_3 = -14.2 + 171i$        | $\lambda_{10} = -1.66 + 53.9i$   |
| $\lambda_4 = -14.2 - 171i$        | $\lambda_{11} = -1.66 - 53.9i$   |
| $\lambda_5 = -35.4 + 153i$        | $\lambda_{12} = -6.2 + 8.86i$    |
| $\lambda_6 = -35.4 - 153i$        | $\lambda_{13} = -6.2 - 8.86i$    |
| $\lambda_7 = -81.1$               | $\lambda_{14} = -4.39$           |

where $A$ is the Jacobian matrix in the equilibrium point vicinity of the system, and $G(X)$ contains quadratic and cubic nonlinear terms.

### Existence analysis of Hopf bifurcation

Make

$$X = (x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}).$$

According to the theorem of nonlinear dynamics, the equilibrium point of system $X_0$ is $(0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0)$. If the vehicle speed $u$ is taken as a bifurcation parameter, equations (1) to (14) can be expressed as the following static equations

$$\dot{X} = AX + G(X), X \in R^{14}$$

(15)
the original system and produces limit cycles. When \( u<u_I \) or \( u>u_2 \), all the eigenvalues of \( \Lambda \) have negative real parts, and the original shimmy system is stable asymptotically and eventually becomes balanced.

**Stability analysis of limit cycles**

The center manifold approach is applied to reduce the original system from a 14-dimensional system to a two-dimensional system and to determine the stability of the original system in two-dimensional systems. Assume \( \mu_k=u-u_k \) (\( k=I,2 \)), and \( \mu_k \) is an increment in the speed of the bifurcation parameter at the bifurcation point and has a minimum value. Using the non-singular transformation \( X=PU \), where \( U \) has the same dimensions as \( X \), \( P \) is composed of the real and imaginary parts of all the eigenvectors, and \( \Lambda=P^{-1}AP \) is a \( 14 \times 14 \) diagonal matrix. Equation (15) is converted into

\[
\dot{U} = \Lambda U + P^{-1}G(PU, \mu_k)
\]

(16)

Using the center manifold theorem to reduce dimension, the center manifold \( W^s \) of the expansion system represented by equation (16) is tangent to the plane of \((y_1, y_2, \mu_k)\) at the singular point \((X_0, u_0)\), assuming the center manifold

\[
y_i = h_i(y_1, y_2) = h_{i1}y_1^2 + h_{i2}y_2^2 + h_{i3}y_1^2y_2 + h_{i4}y_1^2y_2 + h_{i5}y_1^2y_2, \quad i = 3, 4, \ldots, 13, 14
\]

(17)

Substitute equation (17) into equation (16) and then combine with equation (18), after which the coefficients of the same item on both sides of the equation are compared using the software Maple. In solving the linear equations, the coefficients of \( h_i \) (\( y_1, y_2, \mu_k \) (\( i = 3, 4, \ldots, 13, 14 \)) can be obtained and brought into the previous two equations of equation (16). After simplifying the equation, reduction equations can be obtained when \( u_I = 50.2 \text{ km/h} \) or \( u_2 = 185.3 \text{ km/h} \) at the center manifold.

In the vicinity of \( u_I = 50.2 \text{ km/h} \), the reduced equation is

\[
\dot{y}_1 = -9.3767y_2 - 0.6205\mu_1y_1 + 0.4936\mu_1y_2 - 0.01265\mu_1y_1^3 - 0.008495\mu_1y_1^2y_2 - 0.001426\mu_1y_1^2y_2^2 + 1.9378e^{-8}\mu_1y_2^3 - 0.0233y_1^3 - 0.04624y_1^2y_2 - 0.02721y_1y_2^2 - 0.03849\mu_1^2y_1 - 0.001737y_1^3 + 0.0004941\mu_1^2y_2^2
\]

(19)

\[
\dot{y}_2 = 9.3767y_1 - 0.4825\mu_1y_1 + 0.6323\mu_2y_2 + 0.003823\mu_1y_1^3 + 0.02567\mu_1y_2^3 \ldots
\]

(20)

In the vicinity of \( u_2 = 185.3 \text{ km/h} \), the reduced equation is

\[
\dot{y}_1 = -11.3334y_2 - 0.5596\mu_2y_1 + 0.3117\mu_2y_2 - 0.005651\mu_2y_1^3 - 0.02198\mu_2y_1^2y_2 - 0.006328\mu_2y_1^2y_2^2 - 3.6681e^{-7}\mu_2y_2^3 - 0.02072y_1^3 - 0.2903y_1^2y_2 - 0.02579y_1y_2^2 - 0.00808\mu_2^2y_1^2 - 0.07096y_1^3 - 0.0003041\mu_2^2y_2^2
\]

(21)

According to the Hopf bifurcation theorem of planar system, equations (19) and (20) can be turned into the Hopf bifurcation paradigm under the polar coordinates

\[
\begin{aligned}
\dot{\rho} &= c\mu \rho + \alpha \rho^3 \\
\dot{\theta} &= \omega + \epsilon \mu + b\rho^2
\end{aligned}
\]
when \( c = 0 \), the Hopf bifurcation is the first type of degenerate Hopf bifurcation which mean that the existence of limit cycle may not be unique in the system.\(^{21}\)

According to equations (19) to (21), we can get the following:

In the vicinity of \( u = u_1 = 50.2 \text{ km/h} \)

\[
\dot{\rho} = (0.004601 \mu_1 - 0.0656) \rho^3 - 0.6205 \mu_1 \rho \tag{22}
\]

In the vicinity of \( u = u_2 = 185.3 \text{ km/h} \)

\[
\dot{\rho} = (0.002661 \mu_2 + 0.7457) \rho^3 - 0.5596 \mu_2 \rho \tag{23}
\]

The bifurcation diagram of the system equilibrium point \( X_0 \) at each critical velocity is obtained from equations (22) and (23), as shown in Figure 4.

According to Figure 4, Hopf bifurcations are supercritical at the critical speeds \( u_1 \) and \( u_2 \). When \( u < u_1 = 50.2 \text{ km/h} \) (that \( \mu_1 < 0 \)), shimmy does not occur in this system, which mean that the system is stable and equilibrium point \( X_0 \) is a stable focus. When \( u_1 < u < u_2 \) (that \( \mu_1 > 0 \) and \( \mu_2 < 0 \)), shimmy occurs in the system, the equilibrium point \( X_0 \) is has unstable focus, and a stable limit cycle appears. The phenomenon of the mutation of a stable focus into a stable limit cycle is called the limit cycle hard to produce or stable hard loss. When \( u > u_2 = 185.3 \text{ km/h} \) (that \( \mu_2 > 0 \)), the equilibrium point turns into a stable focus again, the limit cycle disappears, the shimmy phenomenon disappears, and the system tends to be stable. Therefore, the front wheels shimmy is a typical self-excited vibration.

According to equations (22) and (23), we can get \( u = u_1 = 50.2 \text{ km/h}, c = -0.6205 \neq 0; \) and when \( u = u_2 = 185.3 \text{ km/h}, c = -0.5596 \neq 0. \) This means that the Hopf bifurcation is not a degenerate Hopf bifurcation. Therefore, through the qualitative analysis above, we only found that the system has a single limit cycle self-excited vibration. However, in the third section of the paper, we found that the system has the self-excited vibration with multiple limit cycles. The possible reasons will be analyzed after the third section of numerical analysis.

**Calculation and analysis of multiple limit cycles characteristic of the system**

**Calculation and analysis of bifurcation characteristic of the system**

Based on the above-established dynamic differential equations (1) to (14), the paper used four-order Runge-Kutta to make numerical calculation aimed at searching multiple limit cycles bifurcation characteristic of self-excited vibration.

In order to study the characteristic of vehicle self-excited vibration with multiple limit cycles under different speeds, the paper used forward speed of the vehicle as bifurcation parameters. The paper made numerical calculation when the initial excitations of front wheel’s angle are \( 0.2^\circ \) and \( 10^\circ \), respectively. Through the numerical calculation, we can get that the shimmy of front wheel induced the self-excited vibration of the whole vehicle with multiple limit cycles, as shown in Figure 5(a1), (b1), (c1), and (d1).
Figure 5. Self-excited vibration speed bifurcation diagram of the whole vehicle and front wheels. (a1) Bifurcation diagram of $\theta_1$ about vehicle speed; (a2) three-dimensional phase diagram of $\theta_1 - \theta'_1 - u$; (b1) bifurcation diagram of $\omega$ about vehicle speed; (b2) three-dimensional phase diagram of $\omega - \omega' - u$; (c1) bifurcation diagram of $\beta$ about vehicle speed; (c2) three-dimensional phase diagram of $\beta - \beta' - u$; (d1) bifurcation diagram of $\phi$ about vehicle speed; (d2) three-dimensional phase diagram of $\phi - \phi' - u$.

Figure 5(a2), (b2), (c2), (d2) shows the lateral sectional view of Figure 5(a1), (b1), (c1), (d2) at speeds of 30 km/h, 60 km/h, 90 km/h, 120 km/h, 150 km/h and 180 km/h, respectively.

Table 5 shows the speed range of no limit cycle, single limit cycle and multiple limit cycles in Figure 5. It can be seen from Table 5 that the speed ranges of front wheel shimmy ($h_1$) are completely consistent with the speed range of whole vehicle self-excited vibration ($x, b_0/C_0 u$). Therefore, it can be considered that the self-excited vibration of whole vehicle ($x, b_0/C_0 u$) was induced by front wheels shimmy ($h_1$). And the self-excited vibration of system occurs only in a certain speed range: In the range of vehicle speed $u = 2–50$ km/h and $u > 185$ km/h, the front wheels shimmy and the induced vehicle self-excited vibration’s amplitude are very small, so it can be considered
that there is no self-excited vibration at this time; when \( u = 50–83 \) km/h and \( u = 149–185 \) km/h, the self-excited vibration with multiple limit cycles occurred; when \( u = 83–149 \) km/h, the self-excited vibration with single limit cycle occurred. In the speed ranges of self-excited vibration occurring (\( u = 50–185 \) km/h), the amplitude of system’s self-excited vibration decreases with the increase of vehicle speed.

The critical bifurcation speed is just the actual speed of the self-excited vibration of the vehicle with multiple degrees of freedom. When the vehicle is at the upper critical bifurcation speed, the Hopf bifurcation occurs in the whole vehicle system, which is represented by the vibration with limit cycles. As to \( \theta_1 \), front wheels have a constant amplitude swing around kingpin. As to \( \omega \), the vehicle yaw angle has a constant amplitude swing around yaw axis. As to \( \beta \), the vehicle has a constant amplitude swing around Y-axis. As to \( \phi \), the suspended portion has a constant amplitude swing around the rolling axis. When the speed of the vehicle increases to the lower critical bifurcation speed, the limit cycles disappears, the self-excited vibration of the vehicle with multiple degrees of freedom disappears, and the vehicle returns to stable driving conditions.

**Multiple limit cycles characteristic of the system**

To further analyze the characteristic of self-excited vibration with multiple limit cycles induced by front wheels shimmy, the paper made calculations and analysis for the vibration with multiple limit cycles’ characteristic of main variables which describe the characteristic of vehicle self-excited vibration when \( u = 60 \) km/h. As a result, multiple limit cycles’ phase diagrams of each variable are shown as Figure 6 and the comparison of its multiple limit cycles’ amplitude are shown in Table 6.

According to Figure 6(a), it can be got that the front wheel shimmy produces three limit cycles with different amplitudes when the front wheel is under different initial excitation: a large stable limit cycle and a small stable limit.
cycle, and there is an unstable limit cycle between the two stable limit cycles. And the unstable limit cycle was obtained by the numerical approximation method. According to Figure 6(b) to (d), the phase diagrams of $\omega$, $\beta$ and $\phi$ show two stable limit cycles with different amplitudes. There must be one unstable limit cycle between the two stable limit cycles. But the unstable limit cycle cannot be solved by knowing methods because in this model the initial excitation is from the front wheel. When the vehicle is subjected to an excitation’s amplitude smaller than the excitation that causes the unstable limit cycle self-excited vibration, the system do self-excited vibration with the smaller limit cycle; when the system is subjected to an excitation’s amplitude greater than the excitation, that causes an unstable limit cycle self-excited vibration, the system has self-excited vibration with the bigger limit cycle.

In this section, the self-excited vibration with multiple limit cycles of the system was found by numerical analysis. However, in the qualitative analysis of the Hopf bifurcation qualitative analysis of the shimmy system section, we did not find evidence of the degenerate Hopf bifurcation occurring which is inconsistent with the conclusions in this section. The possible reasons are as follows.

The existence, stability and numerical characteristics of limit cycles have been well studied, but it is still a difficult problem to determine the number of limit cycles. (The second part of Hilbert’s 16th Problem is concerned with the number and relative distributions of limit cycles of planar polynomial systems. Smale in his paper titled ‘Mathematical problems for the 21st century’ posed the problem again.) At present, the number of limit cycles can only be judged for some low-dimensional systems. There are two main strong non-linear factors in the vehicle shimmy system: (1) The strong non-linearity of tires; (2) the strong non-linearity of dry friction in suspension/steering system. In the process of qualitative analysis, it is necessary to reduce the dimension of the original system. It is necessary to carry out Taylor series transformation on these two nonlinear factors, and discard the high-order terms, thus losing some nonlinear factors. The occurrence of the limit cycle is very dependent on the nonlinear factors of the system itself. After losing this part of the nonlinear factors, the system is likely to be reduced from the original system with multiple limit cycles to the system with single limit cycle. However, in the numerical analysis, we did not lose the nonlinear components of tire lateral force and dry friction of suspension/steering system in the model, and found the phenomenon of multiple limit cycles. It can be seen that the reason why the degenerate Hopf bifurcation is not found in Section 2 may indeed be that the system reduces the dimension and loses the high-order component of the two nonlinear factors.

**Conclusion**

1. Considering the nonlinear factor of tire lateral force and the dry friction force of suspension/steering system, a mechanical model of vehicle self-excited vibration and 7-DOF system differential equations was established.
2. Reducing the 14-dimensional system into 2-dimensional system by means of center manifold theorem. By using Hopf bifurcation theorem, the existence and stability of limit cycles are qualitatively determined.
3. By means of numerical analysis, the self-excited vibration with multiple limit cycles is found. The results show that: in the speed range of 50–185 km/h, front wheels shimmy induce the self-excited vibration of the whole vehicle and the amplitude of system’s shimmy decreases with the increase of vehicle speed; in the speed range of 50–83 km/h and 149–185 km/h, the self-excited vibration with multiple limit cycles occurs; in the speed range of 83–149 km/h, the self-excited vibration with single limit cycle occurs.
4. The paper provides a theoretical reference for the vehicle development process to predict and avoid vehicle self-excited vibration induced by front wheels shimmy. The main content of next step is to find accurate numerical solution methods of unstable limit cycle when vehicle has self-excited vibrations with multiple limit cycles and to inhibit this self-excited vibration phenomenon by matching the vehicle system parameters.

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