ABSTRACT. We obtain sufficient conditions for a densely-defined operator on the Fock space to be bounded or compact. Under the boundedness condition we then characterize the compactness of the operator in terms of its Berezin transform.

1. INTRODUCTION

Let \( \mathbb{C} \) be the complex plane and \( \alpha \) be a positive parameter that is fixed throughout the paper. Let

\[
d\lambda_{\alpha}(z) = \frac{\alpha}{\pi} e^{-\alpha|z|^2} \, dA(z)
\]

be the Gaussian measure, where \( dA \) is the Euclidean area measure. A calculation with polar coordinates shows that \( d\lambda_{\alpha} \) is a probability measure.

The Fock space \( F^2_{\alpha} \) consists of all entire functions \( f \) in \( L^2(\mathbb{C}, d\lambda_{\alpha}) \). It is easy to show that \( F^2_{\alpha} \) is a closed subspace of \( L^2(\mathbb{C}, d\lambda_{\alpha}) \) and so is a Hilbert space with the inherited inner product

\[
\langle f, g \rangle = \int_{\mathbb{C}} f(z)\overline{g(z)} \, d\lambda_{\alpha}(z).
\]

In fact, \( F^2_{\alpha} \) is a reproducing kernel Hilbert space whose kernel function is given by

\[
K_w(z) = K(z, w) = e^{\alpha z\overline{w}}.
\]

The norm of functions in \( L^2(\mathbb{C}, d\lambda_{\alpha}) \) will simply be denoted by \( \|f\| \). The norm of functions in \( f \in L^p(\mathbb{C}, d\lambda_{\alpha}) \) will be denoted by \( \|f\|_p \).

We study linear operators (not necessarily bounded) on the Fock space. Throughout the paper we let \( \mathcal{D} \) denote the set of all finite...
linear combinations $f$ of kernel functions in $F^2_\alpha$: 

$$f(z) = \sum_{k=1}^{N} c_k e^{\alpha z w_k}.$$ 

It is well known that $\mathcal{D}$ is a dense linear subspace of $F^2_\alpha$. See [7] for example. We also assume that the domain of every linear operator that appears in the paper contains $\mathcal{D}$. Using the relation $\langle SK_z, K_w \rangle = \langle K_z, S^* K_w \rangle$ we see that we can also assume that the domain of $S^*$ contains $\mathcal{D}$ as well. One additional standing assumption we make is that the function $z \mapsto SK_z$ is conjugate analytic.

Our main focus here is the boundedness and compactness of operators on $F^2_\alpha$. To state our main results, we need to introduce a class of unitary operators on $F^2_\alpha$. More specifically, for any $z \in \mathbb{C}$, let $\varphi_z$ denote the analytic self-map of $\mathbb{C}$ defined by $\varphi_z(w) = z - w$, let $k_z$ denote the normalized reproducing kernel defined by 

$$k_z(w) = K(w, z)/\sqrt{K(z, z)} = e^{-\alpha |w|^2 + \alpha z \overline{w}},$$

and let $U_z$ denote the operator on $F^2_\alpha$ defined by $U_z f = f \circ \varphi_z k_z$. Each $k_z$ is a unit vector in $F^2_\alpha$. It follows easily from a change of variables that each $U_z$ is a self-adjoint unitary operator on $F^2_\alpha$. See [7].

For any $z \in \mathbb{C}$ and any linear operator $S$ on $F^2_\alpha$ let $S_z = U_z SU_z$. It is easy to check that each $U_z$ maps $\mathcal{D}$ onto $\mathcal{D}$ (see Lemma 7), so the domain of each $S_z$ contains $\mathcal{D}$ whenever the domain of $S$ contains $\mathcal{D}$.

Each operator $S$ on $F^2_\alpha$ also induces a function $\tilde{S}$ on $\mathbb{C}$, namely,

$$\tilde{S}(z) = \langle Sk_z, k_z \rangle, \quad z \in \mathbb{C}.$$ 

We call $\tilde{S}$ the Berezin transform of $S$. Since each $k_z$ is a unit vector, $\tilde{S}$ is bounded whenever $S$ is bounded, and $\|\tilde{S}\|_{\infty} \leq \|S\|$. Also, $k_z \to 0$ weakly in $F^2_\alpha$ as $z \to \infty$, so $\tilde{S}(z) \to 0$ as $z \to \infty$ whenever $S$ is compact on $F^2_\alpha$.

We can now state the main results of the paper.

**Theorem A.** If there exist some $p > 2$ and $C > 0$ such that $\|S_z 1\|_p \leq C$ for all $z \in \mathbb{C}$, then the operator $S$ is bounded on $F^2_\alpha$.

**Theorem B.** If there exists some $p > 2$ such that $\|S_z 1\|_p \to 0$ as $z \to \infty$, then $S$ is compact on $F^2_\alpha$.

**Theorem C.** Suppose that there exist some $p > 2$ and $C > 0$ such that $\|S_z 1\|_p \leq C$ for all $z \in \mathbb{C}$. Then $S$ is compact if and only if $\tilde{S}(z) \to 0$ as $z \to \infty$. 
As an example, we will apply these results to the study of Toeplitz operators on $F^2_\alpha$.

The condition $\|S_z 1\|_p \leq C$ was first introduced in [1] and further studied in [4]. An analogue of Theorem C was proved in [4] in the context of Bergman spaces on the unit disk. The papers [2, 3, 8] also explore the condition $\|S_z 1\|_p \leq C$.

Our approach here is different from those in the papers mentioned above, although a key idea from [1, 4] will be used. One of the novelties here is that there is no need for us to use Schur’s test.

A major difference exists between the Bergman space setting and the current one. More specifically, in the Bergman space setting, there is a certain cut-off requirement, namely, $p$ cannot be too close to 2. In fact, it was shown in [4] that $p$ must be greater than 3 in the case of operators on the Bergman space of the unit disk. However, the cut-off requirement disappears in the Fock space setting; any $p > 2$ will work. This is not entirely surprising; some similar situations were pointed out and explained in the book [7].

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2. A SUFFICIENT CONDITION FOR BOUNDEDNESS

We prove Theorem A in this section. The following lemma will be used several times in the paper.

**Lemma 1.** For any $p > 0$ we have

$$|f(z)| \leq \left( \frac{\beta}{\alpha} \right)^{\frac{1}{p}} \|f\|_p e^{\frac{\alpha}{2}|z|^2}$$

for all entire functions $f \in L^p(\mathbb{C}, d\lambda_\alpha)$ and $z \in \mathbb{C}$, where $\beta = 2\alpha/p$.

**Proof.** It is clear that

$$\|f\|_p^p = \frac{\alpha}{\pi} \int_{\mathbb{C}} |f(z)|^p e^{-\alpha|z|^2} dA(z) = \frac{\alpha}{\beta} \cdot \frac{\beta}{\pi} \int_{\mathbb{C}} \left|f(z)e^{-\frac{\alpha}{2}|z|^2}\right|^p dA(z).$$

The desired estimate then follows from Corollary 2.8 in [7].

We will also need the following estimate several times later on.

**Lemma 2.** Suppose $p > 2$ and $S$ is a linear operator on $F^2_\alpha$. Then

$$|\langle SK_w, K_z \rangle| \leq \|S_1\|_p e^{\frac{\alpha}{2}(|z|^2+|w|^2)-\sigma|z-w|^2}$$
for all \( w \) and \( z \) in the complex plane, where \( \beta = 2\alpha/p \) and \( \sigma = (\alpha - \beta)/2 \). Consequently, if \( \|S_w1\|_p \leq C \) for some constant \( C > 0 \) and all \( w \in \mathbb{C} \), then for the same constant \( C \) we have
\[
|\langle SK_w, K_z \rangle| \leq Ce^{\frac{\alpha}{2}(|z|^2 + |w|^2) - \sigma |z-w|^2}
\]
for all \( z \) and \( w \) in \( \mathbb{C} \).

\textbf{Proof.} Recall that
\[
S_w1(z) = (U_wSU_w1)(z) = (U_wSk_w)(z) = k_w(z)(Sk_w)(w - z).
\]
By Lemma 1, we have
\[
|k_w(z)(Sk_w)(w - z)| \leq \left( \frac{\beta}{\alpha} \right)^\frac{1}{p} \|S_w1\|_p e^{\frac{\beta}{2}|z|^2} \leq \|S_w1\|_p e^{\frac{\beta}{2}|z|^2}
\]
for all \( z \) and \( w \), where \( \beta = 2\alpha/p < \alpha \). Replacing \( z \) by \( w - z \), using
\[
Sk_w(z) = e^{-\frac{\alpha}{2}|w|^2}SK_w(z) = e^{-\frac{\alpha}{2}|w|^2}\langle SK_w, K_z \rangle,
\]
and simplifying the result, we obtain
\[
|\langle SK_w, K_z \rangle| \leq \|S_w1\|_p e^{\frac{\alpha}{2}(|z|^2 + |w|^2) - \sigma |z-w|^2}
\]
for all \( z \) and \( w \). \( \square \)

The following lemma shows that every linear operator on \( F^2_\alpha \) can be represented as an integral operator in a canonical way.

\textbf{Lemma 3.} Let \( S \) be a linear operator on \( F^2_\alpha \) and let \( T \) be the integral operator defined on \( L^2(\mathbb{C}, d\lambda_\alpha) \) by
\[
Tf(z) = \int_{\mathbb{C}} f(w)\langle SK_w, K_z \rangle \, d\lambda_\alpha(w).
\]
Then \( S \) is bounded on \( F^2_\alpha \) if and only if \( T \) is bounded on \( L^2(\mathbb{C}, d\lambda_\alpha) \). Furthermore, when either of them is bounded, \( S \) is equal to the restriction of \( T \) to \( F^2_\alpha \).

\textbf{Proof.} For any fixed \( z \in \mathbb{C} \), the function
\[
w \mapsto \langle K_z, SK_w \rangle = \langle S^* K_z, K_w \rangle = (S^* K_z)(w)
\]
is entire and belongs to \( F^2_\alpha \) for any fixed \( z \in \mathbb{C} \). Therefore, \( Tf = 0 \) for every \( f \in L^2(\mathbb{C}, d\lambda_\alpha) \oplus F^2_\alpha \).

If \( S \) is bounded on \( F^2_\alpha \) and \( f = K_a \) is the reproducing kernel at some point \( a \in \mathbb{C} \), then by the reproducing property of \( K_a \),
\[
Tf(z) = \int_{\mathbb{C}} K(w, a)\langle SK_w, K_z \rangle \, d\lambda_\alpha(w)
\]
\[
\int_{C} \langle S^*K_z, K_w \rangle K(a, w) \, d\lambda_{\alpha}(w)
\]

\[
= \langle S^*K_z, K_a \rangle = \langle SK_a, K_z \rangle
\]

\[
SK_a(z) = Sf(z).
\]

It follows that \( Tf = Sf \) on \( D \) and \( \| Tf \| \leq \| S \| \| f \| \) for all \( f \in D \). Combining this with the conclusion of the previous paragraph, we conclude that \( T \) is bounded on \( L^2(\mathbb{C}, d\lambda_{\alpha}) \) and \( S \) is equal to the restriction of \( T \) to \( F^2_\alpha \).

Conversely, if \( T \) is bounded on \( L^2(\mathbb{C}, d\lambda_{\alpha}) \) and \( f \in D \), then

\[
Tf(z) = \int_{C} f(w)S^*K_z(w) \, d\lambda_{\alpha}(w) = \langle f, S^*K_z \rangle = \langle Sf, K_z \rangle = Sf(z)
\]

for all \( z \in \mathbb{C} \). This shows that the restriction of \( T \) on \( D \) coincides with action of \( S \) there. Since \( D \) is dense in \( F^2_\alpha \) and \( T \) is bounded, we conclude that \( S \) extends to a bounded linear operator on \( F^2_\alpha \).

We can now prove Theorem A which is the main result of this section.

**Theorem 4.** Let \( S \) be a linear operator on \( F^2_\alpha \). If there are constants \( p > 2 \) and \( C > 0 \) such that \( \| S_z1 \| \leq C \) for all \( z \in \mathbb{C} \), then \( S \) is bounded on \( F^2_\alpha \) with \( \| S \| \leq (2pC)/(p - 2) \).

**Proof.** By Lemma 3, it suffices for us to show that the integral operator \( T \) defined by (1) is bounded on \( L^2(\mathbb{C}, d\lambda_{\alpha}) \).

By Lemma 2, for the same constant \( C \) and

\[
\sigma = \frac{\alpha - \beta}{2} = \frac{\alpha(p - 2)}{2p},
\]

we have

\[
|Tf(z)| \leq C \int_{C} |f(w)| e^{\frac{\alpha}{2}|z|^2 + |w|^2} - \sigma |z-w|^2 d\lambda_{\alpha}(w)
\]

for all \( z \in \mathbb{C} \). Rewrite this as

\[
F(z) \leq C_1 \int_{C} |f(w)| e^{-\frac{\sigma}{2}|w|^2} e^{-\sigma |z-w|^2} dA(w),
\]

where \( C_1 = C\alpha/\pi \) and

\[
F(z) = |Tf(z)| e^{-\frac{\sigma}{2}|z|^2}.
\]

By Hölder’s inequality

\[
F(z)^2 \leq C_1^2 \int_{C} |f(w)| e^{-\frac{\sigma}{2}|w|^2} \left( e^{-\sigma |z-w|^2} dA(w) \right) \int_{C} e^{-\sigma |z-w|^2} dA(w)
\]
\[ = C_2 \int_C \left| f(w)e^{-\frac{\sigma}{2}|w|^2} \right|^2 e^{-\sigma|z-w|^2} dA(w), \]

where

\[ C_2 = C_1^2 \int_C e^{-\sigma|u|^2} dA(u) = \frac{C_1^2 \pi}{\sigma}. \]

It follows from Fubini’s theorem and a change of variables that

\[ \int_C \left| T f(z)e^{-\frac{\sigma}{2}|z|^2} \right|^2 dA(z) \leq C_3 \int_C \left| f(w)e^{-\frac{\sigma}{2}|w|^2} \right|^2 dA(w), \]

where

\[ C_3 = C_2 \int_C e^{-\sigma|u|^2} dA(u) = \frac{C_2 \pi}{\sigma} = \left( \frac{2pC}{p-2} \right)^2. \]

This shows that the operator \( T \) is bounded on \( L^2(C, d\lambda_\alpha) \) and

\[ \|T\| \leq \frac{2p}{p-2} C. \]

Restricting \( T \) to the space \( F^2_\alpha \) then yields the desired result for \( S \). \( \square \)

Note that the proof above only depends on the pointwise estimate derived in Lemma 2, not the full assumption about the norms \( \|S_z1\|_p \).

3. SUFFICIENT CONDITIONS FOR COMPACTNESS

In this section we present two sufficient conditions for an operator on \( F^2_\alpha \) to be compact. The first condition is the little oh version of the condition in Theorem A, while the second condition is a natural deviation of the first one.

We begin with Theorem B, the companion result of Theorem A, which we restate as follows.

**Theorem 5.** Let \( S \) be a linear operator on \( F^2_\alpha \) and \( p > 2 \). If \( \|S_z1\|_p \to 0 \) as \( z \to \infty \), then \( S \) is compact on \( F^2_\alpha \).

**Proof.** It follows from our standing assumptions on \( S \) that the condition \( \|S_z1\|_p \to 0 \) as \( z \to \infty \) implies that \( \|S_z1\|_p \) is bounded in \( z \). Therefore, by Theorem 4, \( S \) is already bounded on \( F^2_\alpha \). By Lemma 3, it suffices for us to show that the integral operator \( T \) defined by (1) is compact on \( L^2(C, d\lambda_\alpha) \). We do this using an approximation argument.

For any \( r > 0 \) let us consider the integral operator \( T_r \) defined on \( L^2(C, d\lambda_\alpha) \) by

\[ T_r f(z) = \int_{|w| < r} f(w) \langle SK_w, K_z \rangle d\lambda_\alpha(w) \]
where $\chi_r$ is the characteristic function of the disk $\{z \in \mathbb{C} : |z| < r\}$. It follows easily from Lemma 2 that

$$\int_C \int_C |\chi_r(w)\langle SK_w, K_z \rangle|^2 d\lambda_\alpha(z) d\lambda_\alpha(w) < \infty.$$  

Thus each $T_r$ is Hilbert-Schmidt. In particular, each $T_r$ is compact on $L^2(\mathbb{C}, d\lambda_\alpha)$.

Let $D_r = T - T_r$. Then

$$D_rf(z) = \int_C f(w)(1 - \chi_r(w))\langle SK_w, K_z \rangle d\lambda_\alpha(w)$$

$$= \int_{|w| > r} f(w)\langle SK_w, K_z \rangle d\lambda_\alpha(w).$$

We are going to show that $\|D_r\| \to 0$ as $r \to \infty$, which would imply that $T$ is compact.

Given any $\varepsilon > 0$, choose a positive number $R$ such that $\|S_w^1\|_p < \varepsilon$ for all $|w| > R$. By Lemma 2, for any $r > R$ we have

$$|1 - \chi_r(w)||\langle SK_w, K_z \rangle| \leq \varepsilon e^{\frac{\beta(|z|^2 + |w|^2) - \sigma|z-w|^2}{2}}$$

for all $z$ and $w$ in $\mathbb{C}$ (just consider the cases $|w| \leq r$ and $|w| > r$ separately). It follows from the proof of Theorem 4 that there is a positive constant $C$, independent of $\varepsilon$ and $r$, such that $\|D_r\| \leq C\varepsilon$ for all $r > R$. This shows that $\|D_r\| \to 0$ as $r \to \infty$ and completes the proof of the theorem.

Recall from the definition of $S_z$ and $U_z$ that

$$S_z 1 = U_z SU_z 1 = U_z Sk_z, \quad z \in \mathbb{C}.$$  

Since each $U_z$ is a unitary operator on $F_\alpha^2$, the condition $\|S_z 1\| \leq C$ is the same as $\|Sk_z\| \leq C$. However, $U_z$ is not isometric on $L^p(\mathbb{C}, d\lambda_\alpha)$ when $p \neq 2$, so it is natural for us to consider the condition $\|Sk_z\|_p \leq C$.

**Proposition 6.** Let $S$ be a linear operator on $F_\alpha^2$ and $p > 2$. If there is a constant $C > 0$ such that $\|Sk_z\|_p \leq C$ and $\|S^*k_z\|_p \leq C$ for all $z \in \mathbb{C}$, then $S$ is Hilbert-Schmidt on $F_\alpha^2$. In particular, $S$ is compact.

**Proof.** By Lemma 1, the assumption on $\|Sk_w\|_p$ implies that there exists another positive constant $C$ such that

$$|(Sk_w)(z)| \leq Ce^{\frac{\beta}{2}|z|^2}, \quad z, w \in \mathbb{C},$$
where $\beta = 2\alpha/p < \alpha$. This can be rewritten as
\[ |\langle SK_w, K_z \rangle| \leq Ce^{\frac{\alpha}{2}|z|^2 + \frac{\beta}{2}|w|^2} \] (2)
for all $z$ and $w$. Similarly, the assumption on $\|S^*k_w\|_p$ implies that
\[ |\langle SK_w, K_z \rangle| \leq Ce^{\frac{\alpha}{2}|z|^2 + \frac{\beta}{2}|w|^2} \] (3)
for all $z$ and $w$.

Multiply the inequalities in (2) and (3) and then take the square root on both sides. The result is
\[ |\langle SK_w, K_z \rangle| \leq Ce^{\frac{\delta}{2}(|z|^2 + |w|^2)} \]
for all $z$ and $w$, where $\delta = (\alpha + \beta)/2 < \alpha/2$. It follows from this that
\[ \int_{\mathbb{C}} \int_{\mathbb{C}} |\langle SK_w, K_z \rangle|^2 \, d\lambda_{\alpha}(w) \, d\lambda_{\alpha}(z) < \infty, \]
so that the integral operator $T$ defined by
\[ Tf(z) = \int_{\mathbb{C}} f(w) \langle SK_w, K_z \rangle \, d\lambda_{\alpha}(w) \]
is Hilbert-Schmidt on $L^2(\mathbb{C}, d\lambda_{\alpha})$. Since $S$ is the restriction of $T$ on $F^2_{\alpha}$, we conclude that $S$ is Hilbert-Schmidt on $F^2_{\alpha}$. \hfill \Box

Once again, we only used the pointwise estimates deduced from the assumptions on $\|S_kz\|_p$ and $\|S^*k_z\|_p$.

4. COMPACTNESS VIA THE BEREZIN TRANSFORM

In this section we show that, under the assumption of Theorem A, the compactness of a linear operator on $F^2_{\alpha}$ can be characterized in terms of its Berezin transform.

**Lemma 7.** For any $a$ and $w$ in the complex plane we have
\[ U_aK_w = \overline{k_a(w)}K_{\varphi_a(w)}, \quad U_awk_w = \beta k_{\varphi_a(w)}, \quad \widetilde{S} \circ \varphi_a = \widetilde{S}_a, \]
where $\beta$ is a unimodular constant depending on $a$ and $w$.

**Proof.** The first identity follows from the definition of $U_a$ and the explicit form of the kernel function. The second identity follows from the first one with
\[ \beta = e^{\frac{\alpha}{2}(\varphi - \varphi_w)}. \]

By the definition of the Berezin transform, the definition of $S_a$, and the second identity that we have already proved, we have
\[ \widetilde{S}_a(w) = \langle S_awk_w, k_w \rangle = \langle U_awk_w, k_w \rangle = \langle SU_awk_w, k_w \rangle = |\beta|^2 \langle Sk_{\varphi_a(w)}, k_{\varphi_a(w)} \rangle \]
This proves the third identity. \(\square\)

**Lemma 8.** Let \(S\) be a linear operator on \(F_\alpha^2\). Suppose that there are constants \(p > 2\) and \(C > 0\) such that \(\|S_z 1\|_p \leq C\) for all \(z \in \mathbb{C}\). Then \(\tilde{S}(w) \to 0\) as \(w \to \infty\) if and only if for every (or some) \(2 < p' < p\) we have \(\|S_w 1\|_{p'} \to 0\) as \(w \to \infty\).

**Proof.** If for some \(p' \in (2, p)\) we have \(\|S_w 1\|_{p'} \to 0\) as \(w \to \infty\), then by Hölder’s inequality,

\[
\tilde{S}(w) = |\langle S_w 1, 1 \rangle| \leq \|S_w 1\|_{p'} \to 0
\]

as \(w \to \infty\).

Next, suppose \(\tilde{S}(w) \to 0\) as \(w \to \infty\) and fix any \(p' \in (2, p)\). We proceed to show that \(\|S_w 1\|_{p'} \to 0\) as \(w \to \infty\).

For any \(a\) and \(z\) we have

\[
\tilde{S}(\varphi_z(a)) = e^{-|a|^2} \langle S_z K_a, K_a \rangle,
\]

where

\[
K_a(u) = e^{2\alpha u} = \sum_{k=0}^{\infty} \frac{\alpha^k}{k!} u^k a^k.
\]

By the proof of Lemma 6.26 in [7], starting at line 4 from the bottom of page 240 and finishing at line 3 from the top of page 242, with \(\tilde{f}\) replaced by \(\tilde{S}\) and \(T_{f \circ \varphi_z}\) replaced by \(S_z\), we will have

\[
\lim_{z \to \infty} \langle S_z 1, z^n \rangle = 0
\]

for every \(n \geq 0\). Since the polynomials are dense in \(F_\alpha^2\), we conclude that \(S_z 1 \to 0\) weakly in \(F_\alpha^2\) as \(z \to \infty\). In particular, for every \(w \in \mathbb{C}\), \(S_z 1(w) \to 0\) as \(z \to \infty\).

Let \(s = p/p' > 1\) and choose \(t > 1\) such that \(1/s + 1/t = 1\). For any measurable set \(E\) we have

\[
\int_E |S_z 1(w)|^{p'} d\lambda_\alpha(w) \leq \left[ \int_E |S_z 1(w)|^p d\lambda_\alpha(w) \right]^{\frac{1}{s}} \left[ \int_E d\lambda_\alpha(w) \right]^{\frac{1}{t}} \leq \|S_z 1\|_p^{p'} [\lambda_\alpha(E)]^{\frac{1}{t}}.
\]

Since \(\|S_z 1\|_p \leq C\) for all \(z \in \mathbb{C}\), this shows that the family \(\{|S_z 1|^{p'} : z \in \mathbb{C}\}\) is uniformly integrable. By Vitali’s Theorem,

\[
\lim_{z \to \infty} \int_{\mathbb{C}} |S_z 1(w)|^{p'} d\lambda_\alpha(w) = 0.
\]

This completes the proof of the lemma. \(\square\)
We can now prove Theorem C, the main result of this section, which we restate as follows.

**Theorem 9.** Suppose $S$ is a linear operator on $\mathcal{F}_2^2$, $p > 2$, $C > 0$, and $\|S_z1\|_p \leq C$ for all $z \in \mathbb{C}$. Then $S$ is compact on $\mathcal{F}_2^2$ if and only if $\tilde{S}(z) \to 0$ as $z \to \infty$.

*Proof.* By Theorem 4, $S$ is bounded on $\mathcal{F}_2^2$. If $S$ is further compact, then $\tilde{S}(z) = \langle Sk_z, k_z \rangle \to 0$ as $z \to \infty$, because $k_z \to 0$ weakly in $\mathcal{F}_2^2$ as $z \to \infty$.

Conversely, if $\tilde{S}(z) \to 0$ as $z \to \infty$, it follows from Lemma 8 that $\|S_z1\|_{p'} \to 0$ as $z \to \infty$, where $p'$ is any fixed number strictly between 2 and $p$. This together with Theorem 5 then implies that $S$ is compact. \[\square\]

5. **AN APPLICATION TO TOEPLITZ OPERATORS**

Let $P : L^2(\mathbb{C}, d\lambda_\alpha) \to \mathcal{F}_2^2$ denote the orthogonal projection. If $\psi \in L^\infty(\mathbb{C})$, we can define a linear operator $T_\psi$ on $\mathcal{F}_2^2$ by $T_\psi f = P(\psi f)$. It is clear that $T_\psi$ is bounded and $\|T_\psi\| \leq \|\psi\|_\infty$. It is also easy to verify that

$$(T_\psi)_z = U_zT_\psi U_z = T_{\psi \circ \varphi_z}$$

for all $z \in \mathbb{C}$. In particular, $(T_\psi)_z1 = P(\psi \circ \varphi_z)$, or

$$(T_\psi)_z1(w) = \int_\mathbb{C} K(w, u)\psi(z - u) \, d\lambda_\alpha(u), \quad w \in \mathbb{C}.$$ 

It follows that

$$(T_\psi)_z1(w) \leq \|\psi\|_\infty \int_\mathbb{C} |e^{\alpha(w/2)}| \, d\lambda_\alpha(u)$$

$$= \|\psi\|_\infty \int_\mathbb{C} |e^{\alpha(w/2)}|^2 \, d\lambda_\alpha(u)$$

$$= \|\psi\|_\infty e^{\alpha(w/2)^2} = \|\psi\|_\infty e^{\frac{\alpha}{4}|w|^2}$$

for all $w \in \mathbb{C}$. This shows that

$$\sup_{z \in \mathbb{C}} \int_\mathbb{C} |(T_\psi)_z1|^p \, d\lambda_\alpha < \infty$$

whenever $0 < p < 4$. Therefore, the assumption in Theorem 9 is satisfied for each $p \in (2, 4)$. Consequently, we arrive at the well-known result that such a Toeplitz operator is compact if and only if its Berezin transform vanishes at $\infty$. See [2, 7].

Using the integral representation for the orthogonal projection, it is possible to define Toeplitz operators $T_\psi$ for functions $\psi$ that are not necessarily bounded. In particular, $T_\psi$ is well defined on $D$ whenever
ψ belongs to the space BMO used in Section 6.4 of [7]. For such a symbol function ψ, Lemma 6.25 of [7] states that
\[ |(T_\psi)_z1(w)| \leq Ce^{\frac{\alpha}{2}|w|^2}, \quad w \in \mathbb{C}, \]
whenever \( \tilde{\psi} \) is bounded, which implies that \( \| (T_\psi)_z1 \|_{p} \leq C \) for \( 2 < p < 4 \). This together with the arguments in the previous paragraph shows that, for such ψ, the operator \( T_\psi \) is bounded if and only if its Berezin transform is bounded; and \( T_\psi \) is compact if and only if its Berezin transform vanishes at \( \infty \). See [2, 7] again.

The Berezin transform of \( T_\psi \) is usually written as \( \tilde{\psi} \) or \( B_\alpha \psi \). It is easy to see that
\[ B_\alpha \psi(z) = \int_{\mathbb{C}} \psi(z - w) \, d\lambda_\alpha(w) = \frac{\alpha}{\pi} \int_{\mathbb{C}} \psi(w)e^{-\alpha|z-w|^2} \, d\lambda_\alpha(w) \]
for \( z \in \mathbb{C} \). See [7] for more information about the Berezin transform which is also called the heat transform in many articles.

The arguments above can also be extended to operators of the form \( T = T_{\psi_1} \cdots T_{\psi_n} \), where each \( \psi_k \) belongs to \( L^\infty(\mathbb{C}) \). In fact, in the case \( S = T_{\psi_1}T_{\psi_2} \), we have
\[ S_z1 = P[\psi_1 \circ \varphi_z(T_{\psi_2})_z1]. \]
Using the integral representation for the outside \( P \) and the pointwise estimate we already obtained for \( (T_{\psi_2})_z1 \), we arrive at
\[ |S_z1(w)| \leq C \int_{\mathbb{C}} e^{\frac{\alpha}{2}|w|^2} |K(w, u)| \, d\lambda_\alpha(u) \leq C_1 e^{\frac{\alpha}{2}|w|^2}. \]
This implies that
\[ \sup_z \| S_z1 \|_{p} < \infty, \quad 2 < p < 3. \]
More generally, if \( \sigma > 2 \), then
\[ \int_{\mathbb{C}} e^{\frac{\alpha}{\sigma}|u|^2} |K(w, u)| \, d\lambda_\alpha(u) \leq C e^{\frac{\alpha}{\sigma'}|w|^2}, \]
with
\[ \sigma' = 4 \left(1 - \frac{1}{\sigma}\right) > 2. \]
So by mathematical induction, each operator \( S = T_{\psi_1} \cdots T_{\psi_n} \) satisfies the pointwise estimate
\[ |S_z1(w)| \leq Ce^{\frac{\alpha}{\sigma'}|w|^2}, \quad w \in \mathbb{C}, \]
for some \( \sigma > 2 \). It follows that
\[ \sup_z \| S_z1 \|_{p} < \infty, \quad p \in (2, \sigma). \]
Going one step further, we can also extend the arguments above to operators on $F^2_\alpha$ that are finite sums of finite products of Toeplitz operators.

6. Further results and remarks

For any $p > 0$ the Fock space $F^p_\alpha$ is defined to be the set of all entire functions $f$ such that $f(z)e^{-\frac{\alpha}{2}|z|^2}$ belongs to $L^p(\mathbb{C},dA)$. The norm in $F^p_\alpha$ is defined by

$$\|f\|_{p,\alpha}^p = \frac{p\alpha}{2\pi} \int_{\mathbb{C}} \left| f(z)e^{-\frac{\alpha}{2}|z|^2}\right|^p dA(z).$$

It is clear that when $p = 2$, the definition here is consistent with the definition of $F^2_\alpha$ in the Introduction. More generally, we have

$$F^p_\alpha = H(\mathbb{C}) \cap L^p(\mathbb{C},d\lambda_\beta), \quad \beta = \frac{p\alpha}{2},$$

where $H(\mathbb{C})$ is the space of all entire functions. Equivalently,

$$H(\mathbb{C}) \cap L^p(\mathbb{C},d\lambda_\alpha) = F^p_\beta, \quad \beta = \frac{2\alpha}{p}.$$  

Although both $F^p_\alpha$ and $H(\mathbb{C}) \cap L^p(\mathbb{C},d\lambda_\alpha)$ are natural extensions of the Fock space $F^2_\alpha$, in most cases it is much more beneficial, more convenient, and more natural to use $F^p_\alpha$ instead of the other one. Of course there are exceptions, the results of this paper being one of them. Nevertheless, the following question still seems natural: What happens if we replaced the condition $\|S_1\|_p \leq C$ by the condition $\|S_1\|_{p,\alpha} \leq C$? We do not know the answer. But the techniques used in the paper would certainly not work, because the optimal pointwise estimate for functions in $F^p_\alpha$ is given by

$$|f(z)| \leq \|f\|_{p,\alpha} e^{\frac{\alpha}{2}|z|^2}, \quad z \in \mathbb{C}.$$  

See Corollary 2.8 in [7]. We needed a certain decrease in the exponent in order to perform the analysis in Sections 2–4.

In the case of $S = T_\psi$, where $\psi \in L^\infty(\mathbb{C})$, we already showed that

$$\sup_{z \in \mathbb{C}} \|(T_\psi)_{z,1}\|_p < \infty, \quad \sup_{z \in \mathbb{C}} \|(T_\psi^*)_{z,1}\|_p < \infty,$$

for $0 < p < 4$. On the other hand, for every $p \in [1,\infty)$, the projection $P$ is bounded from the space

$$L^p_\alpha(\mathbb{C}) = \left\{ f : f(z)e^{-\frac{\alpha}{2}|z|^2} \in L^p(\mathbb{C},dA) \right\}$$
onto the space $F^p_\alpha$; see [7] for example. It follows from this and the identity $(T_\psi)z_1 = P(\psi \circ \varphi_z)$ that
\[
\sup_{z \in \mathbb{C}} \|(T_\psi)z_1\|_{p,\alpha} < \infty, \quad \sup_{z \in \mathbb{C}} \|(T_\psi)z_1\|_{p,\alpha} < \infty,
\]
for all $1 \leq p < \infty$. Thus the condition $\|Sz_1\|_p \leq C$ appears stronger (or more difficult to satisfy) than the condition $\|Sz_1\|_{p,\alpha} \leq C$. This is easily confirmed by the elementary continuous embedding
\[
H(\mathbb{C}) \cap L^p(\mathbb{C}, d\lambda_\alpha) = F^p_\beta \subset F^p_\alpha,
\]
where $\beta = (2\alpha)/p < \alpha$ for $p > 2$.

The example in the previous section of Toeplitz operators on $F^2_\alpha$ induced by bounded symbols shows that the condition $\|Sz_1\|_p \leq C$ is a meaningful one. We just do not know what the weaker condition $\|Sz_1\|_{p,\alpha} \leq C$ would imply. But there is more we can say.

For each $z \in \mathbb{C}$ the operator $U_z$ is actually a surjective isometry on each $F^p_\alpha$, and $k_z$ is actually a unit vector in $F^p_\alpha$. Therefore, the condition $\|Sz_1\|_{p,\alpha} \leq C$ is the same as $\|Sk_z\|_{p,\alpha} \leq C$. If there exists a bounded linear operator $S$ on $F^p_\alpha$, $2 < p < \infty$, such that $S$ is not bounded on $F^2_\alpha$, then the condition $\|Sz_1\|_{p,\alpha} \leq C$ would not imply the boundedness of $S$ on $F^2_\alpha$. Although we do not have an example at hand, this seems very plausible to us.

Note that the proof of Theorem 4 amounts to showing that the integral operator $T$ defined by
\[
Tf(z) = \int_{\mathbb{C}} f(w)H(z, w) d\lambda_\alpha(w)
\]
is bounded on $L^2(\mathbb{C}, d\lambda_\alpha)$, where
\[
H(z, w) = e^{\frac{\alpha}{2}|z|^2 + |w|^2 - \sigma|z-w|^2}.
\]
Since $f \in L^2(\mathbb{C}, d\lambda_\alpha)$ if and only if the function $f(w)e^{-\frac{\alpha}{2}|w|^2}$ is in $L^2(\mathbb{C}, dA)$, and since
\[
e^{-\frac{\alpha}{2}|z|^2} Tf(z) = \frac{\alpha}{\pi} \int_{\mathbb{C}} \left[f(w)e^{-\frac{\alpha}{2}|w|^2}\right] e^{-\sigma|z-w|^2} dA(w),
\]
we see that the operator $T$ on $L^2(\mathbb{C}, d\lambda_\alpha)$ is unitarily equivalent to the Berezin transform $B_\sigma$ as an operator on $L^2(\mathbb{C}, dA)$. Recall that
\[
B_\sigma f(z) = \frac{\sigma}{\pi} \int_{\mathbb{C}} f(w)e^{-\sigma|z-w|^2} dA(w).
\]
The boundedness of $B_\sigma$ on $L^2(\mathbb{C}, dA)$ is actually a known result. See [7] for example.

A natural question here is the following: is the Berezin transform $B_\sigma$ compact on $L^2(\mathbb{C}, dA)$? Since the proof of Theorem 4 along with
the fact that \(|\langle T_\psi z, 1 \rangle_p \rangle_p \leq C\) for \(2 < p < 4\) shows that every Toeplitz operator \(T_\psi\) on \(F^2_{\alpha}, \psi \in L^\infty(\mathbb{C})\), is dominated by \(B_\sigma\) as an operator on \(L^2(\mathbb{C}, dA)\), and it is very easy to see that there are such Toeplitz operators that are not compact, we see that \(B_\sigma\) cannot possibly be compact on \(L^2(\mathbb{C}, dA)\). To see this more directly, we consider the sequence \(\{\chi_n\}\) of characteristic functions of the disks \(B(n, 1)\). It is easy to see that \(\{\chi_n\}\) converges to 0 weakly in \(L^2(\mathbb{C}, dA)\). But

\[
B_\sigma \chi_n(z) = \frac{\sigma}{\pi} \int_{B(0,1)} e^{-\sigma|z-n-w|^2} dA(w) = g(z-n),
\]

where

\[
g(z) = \frac{\sigma}{\pi} \int_{B(0,1)} e^{-\sigma|z-w|^2} dA(w).
\]

By translation invariance, the norm of each \(B_\sigma \chi_n\) in \(L^2(\mathbb{C}, dA)\) is equal to that of \(g\). Thus \(\|B_\sigma \chi_n\|_{L^2(\mathbb{C}, dA)} \not\to 0\) as \(n \to \infty\), so \(B_\sigma\) is not compact on \(L^2(\mathbb{C}, dA)\).

Our arguments can also be adapted to work for Bergman spaces on the unit ball \(B_n\) in \(\mathbb{C}^n\). More specifically, for any \(\alpha > -1\) we consider the weighted volume measure

\[
dv_\alpha(z) = c_\alpha (1 - |z|^2)^\alpha dv(z),
\]

where \(dv\) is ordinary volume measure on \(B_n\) and \(c_\alpha\) is a normalizing constant chosen so that \(v_\alpha(B_n) = 1\). For any \(p > 0\) the spaces

\[
A^p_\alpha = H(B_n) \cap L^p(B_n, dv_\alpha)
\]

are called (weighted) Bergman spaces, where \(H(B_n)\) is the space of all holomorphic functions on \(B_n\).

The space \(A^2_\alpha\) is a reproducing kernel Hilbert space whose reproducing kernel is given by

\[
K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.
\]

The normalized reproducing kernels are still defined by

\[
k_z(w) = \frac{K(w, z)}{\sqrt{K(z, z)}} = \frac{(1 - |z|^2)^{\frac{n+1+\alpha}{2}}}{(1 - \langle z, w \rangle)^{n+1+\alpha}}.
\]

For every \(z \in B_n\) there is also a canonical involutive automorphism \(\varphi_z\) of the unit ball \(B_n\), and an associated self-adjoint unitary operator \(U_z\) can be defined on \(A^2_\alpha\) by \(U_z f = f \circ \varphi_z k_z\). If \(S\) is a linear operator on \(A^2_\alpha\), not necessarily bounded, whose domain contains all finite linear combinations of kernel functions, then we can still consider \(S_z = U_z S U_z\).
The optimal pointwise estimate for functions in Bergman spaces is given by
\[ |f(z)| \leq \frac{\|f\|_{A_p^\alpha}}{(1 - |z|^2)^{\frac{n+1}{p}}}. \]
See [6] for this and the results quoted in the previous two paragraphs. It follows from the proof of Lemma 2 that the condition\[ \sup_{z \in \mathbb{B}_n} \|S_z 1\|_{A_p^\alpha} < \infty, \]
where \( p > 2 \), implies the inequality\[ |\langle SK_w, K_z \rangle| \leq \frac{C(1 - \langle z, w \rangle)^{\frac{2}{p-1}(n+1+\alpha)}}{(1 - |z|^2)^{\frac{n+1}{p}}(1 - |w|^2)^{\frac{n+1}{p}}}. \]
Our techniques here can be adapted to show that for\[ p > 2 + \frac{2n}{\alpha + 1}, \tag{4} \]
the condition \( \|S_z 1\|_{A_p^\alpha} \leq C \) implies that the operator \( S \) is bounded on \( A^2_\alpha \). Similarly, the condition\[ \lim_{|z| \to 1^-} \|S_z 1\|_{A_p^\alpha} = 0 \]
implies that the operator \( S \) is not only bounded but also compact on \( A^2_\alpha \). Furthermore, under the assumption \( \|S_z 1\|_{A_p^\alpha} \leq C \), the compactness of \( S \) on \( A^2_\alpha \) is equivalent to the vanishing of the Berezin transform of \( S \) on the unit sphere \( |z| = 1 \). We leave the details to the interested reader.

We point out that in the case when \( n = 1 \) and \( \alpha = 0 \), the restriction \( p > 4 \) in (4) is not as good as the optimal restriction \( p > 3 \) obtained in [4]. The discrepancy stems from the fact that our approach here only uses pointwise estimates derived from the assumption about norms, while the approach in [4] made full use of the assumption about norms.

We also mention that the conditions\[ \sup_{z \in \mathbb{B}_n} \|Sk_z\|_{A_p^\alpha} < \infty, \quad \sup_{z \in \mathbb{B}_n} \|S^* k_z\|_{A_p^\alpha} < \infty, \tag{5} \]
where \( p > 2 \), imply the inequality\[ |\langle SK_w, K_z \rangle| \leq \frac{C}{(1 - |z|^2)^{\frac{n+1+\alpha}{q}}(1 - |w|^2)^{\frac{n+1+\alpha}{q}}}, \]
where $q \in (2, p)$ is the exponent given by

$$\frac{1}{q} = \frac{1}{2} \left( \frac{1}{2} + \frac{1}{p} \right).$$

If $p$ and $\alpha$ satisfy

$$\alpha + 1 > \left( \frac{1}{2} + \frac{1}{p} \right) (n + 1 + \alpha),$$

then the conditions in (5) imply that $S$ is Hilbert-Schmidt on $A^2_\alpha$. Obviously, the dependence on $p$ and $\alpha$ in the Bergman space theory is much more delicate. Again, the interested reader can easily work out the details by following arguments in previous sections of this paper.

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