Sigma Function as A Tau Function

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Abstract

The tau function corresponding to the affine ring of a certain plane algebraic curve, called \((n, s)\)-curve, embedded in the universal Grassmann manifold is studied. It is neatly expressed by the multivariate sigma function. This expression is in turn used to prove fundamental properties on the series expansion of the sigma function established in a previous paper in a different method.

1 Introduction

The purpose of this paper is to study the multivariate sigma function associated to certain plane algebraic curves, called \((n, s)\)\(-\)curves, by means of the tau function of the KP-hierarchy.

An \((n, s)\)\(-\)curve is a plane algebraic curve given by the equation,

\[
y^n = x^s + \sum_{si+jj < ns} \lambda_{ij} x^i y^j,
\]

where \(n, s\) are coprime and satisfy \(1 < n < s\). The sigma function associated to an \((n, s)\)\(-\)curve had been introduced by Buchstaber-Enolski-Leykin [3, 4] extending the Klein’s hyperelliptic [11, 12] and Weierstrass’ elliptic sigma functions. It is defined by modifying the Riemann’s the theta function in such a way that it becomes modular invariant [3].

If the genus of the curve is \(g\), it is a holomorphic function of \(g\) variables and has a remarkable algebraic properties. Namely its series expansion at the origin begins at the Schur function associated to the gap sequence at \(\infty\) and all the coefficients of the expansion are homogeneous polynomials of \(\{\lambda_{ij}\}\) with respect to certain degree. These properties have been proved in [19] by making an expression of the sigma function in terms of algebraic integrals generalizing the Klein’s formula [11, 12]. We remark that the general terms of the expansion are not known explicitly except for the elliptic case [25] and the case of genus two [1], where the recursion relations among expansion coefficients are explicitly given. Linear differential equations satisfied by sigma functions have been constructed in [1, 2]. It can be a base to study the series expansions in more general

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cases. In this paper we propose another observation on the series expansions on sigma functions.

It is well known that the Weierstrass’ elliptic functions \( \wp(u) \) and \( \wp'(u) \) uniformize the family of elliptic curves \( y^2 = 4x^3 - g_2x - g_3 \) even when \((g_2, g_3)\) is contained in the discriminant. Similarly the second and third logarithmic derivatives of the Klein’s hyperelliptic sigma function uniformize the family of affine Jacobians, that is, the Jacobian minus the theta divisor \( [3, 19, 18] \). Again they also give an uniformization of the singular fibers of the family. In particular the most degenerate fibers are uniformized by logarithmic derivatives of Schur polynomials. Those special properties of the abelian functions are the reflection of the algebraic properties mentioned above. In the case of \((n,s)\)-curves similar results are expected. Notice that this kind of degeneration structure from theta functions to trigonometric and rational functions are typical in the study of integrable systems.

Sato’s theory of KP-hierarchy associates a point \( U \) of the universal Grassmann manifold (UGM) a so called tau function \( \tau(t, \xi) \), where \( \xi \) is a frame of \( U \) \([23, 22]\). It is a solution of the KP-hierarchy in the bilinear form. Conversely any solution of the KP-hierarchy can be written as a tau function of some point of UGM. It is known that, for a point in a finite dimensional orbit of the KP-hierarchy, the tau function can be expressed by the Riemann’s theta function \([16, 14, 21, 13]\). Moreover, for a frame \( \xi \), the expansion of the tau function can be explicitly written as

\[
\tau(t, \xi) = \sum \xi_{\lambda}s_{\lambda}(t),
\]

where the sum runs over the set of partitions \( \lambda \), \( \xi_{\lambda} \) is the Plücker coordinate of \( \xi \) and \( s_{\lambda}(t) \) is the Schur function corresponding to \( \lambda \). Thus, if the frame \( \xi \) is known, the expansion of the corresponding tau function is given very explicitly.

Now we consider the affine ring \( A \) of an \((n,s)\)-curve. A basis of \( A \) as a vector space is given explicitly by a set of certain monomials of \( x \) and \( y \). The space \( A \) can be embedded in UGM using the local coordinate at \( \infty \). We show that, for the normalized frame \( \xi^A \) of the corresponding point of UGM, the tau function is given by

\[
\tau(t; \xi^A) = \exp \left( -\sum_{i=1}^{\infty} c_it_i + \frac{1}{2} \tilde{q}(t) \right) \sigma(Bt),
\]

(1)

where \( \tilde{q}(t) = \sum \tilde{q}_{ij}t_it_j \), \( B = (b_{ij})_{g\times \infty}, \ t = (t_1, t_2, ...) \) (see Theorems \([4, 8]\)). All constants \( c_i, \tilde{q}_{ij}, b_{ij} \) are homogeneous polynomials of \( \{\lambda_{kl}\} \) with rational coefficients. Using this formula we can deduce the above mentioned properties on the series expansion of a sigma function from those on the expansion of a tau function.

The crucial point to prove \([1]\) is the existence of a holomorphic one form which vanishes at \( \infty \) of order \( 2g-2 \). The square root of it plays the role of transforming half forms to functions \([19]\). The existence of such a form is specific to \((n,s)\)-curves.

Finally we remark that relations of tau and sigma functions are also discussed by C. Eilbeck, V. Enolski and J. Gibbons \([7, 8]\) extending the results of \([9]\). In \([8]\) a similar relation to \([1]\) is derived. The main difference between their formula and ours is that
the frame $\xi^A$ is described in terms of the derivatives of a tau function in the former while it is described by the expansion coefficients of monomials of $x$ and $y$ in the latter. They use their relation mainly to derive identities satisfied by Abelian functions. While we use (II) to give a general algebraic formula for the expansion coefficients of the tau function, the alternative to the sigma function. Therefore both results compensate to each other and combining them is effective for a further development. Some related subjects are also studied in [5].

The present paper is organized as follows. After the introduction fundamental properties of an $(n,s)$-curve is explained in section 2. In section 3 the KP-hierarchy and its reductions are reviewed. The Sato’s theory of KP hierarchy and UGM is reviewed in section 4. In section 5 the embedding of the affine ring of an $(n,s)$-curve to UGM is described. The construction of the sigma function is reviewed in section 7. The expression of the tau function corresponding to the affine ring embedded in UGM is described in section 8. In section 9 the properties of the series expansion of the sigma function are studied based on the formula in the previous section.

2 $(n, s)$-Curve

An $(n, s)$ curve is the plane algebraic curve defined by the equation $f(x, y) = 0$ with

$$f(x, y) = y^n - x^s - \sum_{ni + sj < ns} \lambda_{ij} x^i y^j,$$

where $n, s$ are nonnegative integers which are coprime and satisfy $1 < n < s \quad \mathbb{4, 1}$. We assume it non-singular and denote by $X$ the corresponding compact Riemann surface. Its genus is $g = 1/2(n - 1)(s - 1)$. Let $\pi: X \rightarrow \mathbb{P}^1$ be the projection to the $x$-coordinate, $(x, y) \rightarrow x$. Then $\pi^{-1}(\infty)$ consists of one point, which we denote by $\infty$, and is a branch point with the branching index $n$.

The affine ring $A$ of $X$ is by definition

$$A = \mathbb{C}[x, y]/\mathbb{C}[x, y]f.$$

Analytically it is isomorphic to the ring of meromorphic functions on $X$ which are holomorphic on $X - \{\infty\}$. For a meromorphic function $F$ on $X$ we denote $\text{ord } F$ the order of poles at $\infty$. Then

$$\text{ord } x = n, \quad \text{ord } y = s.$$

Let $\{f_i\}_{i=1}^{\infty}$ be the basis of $A$ as a vector space specified by the conditions:

(i) $f_i \in \{x^{m_1}y^{m_2} | m_1 \geq 0, n > m_2 \geq 0 \}$,

(ii) $\text{ord } f_i < \text{ord } f_{i+1}$ for $i \geq 1$.

Example $(n, s) = (2, 2g + 1)$: In this case $X$ is a hyperelliptic curve of genus $g$. We have
$$(f_1, f_2, \ldots) = (1, x, x^2, \ldots, x^g, y, x^{g+1}, xy, \ldots).$$

Let $w_1 < \cdots < w_g$ be the gap sequence at $\infty$. It is, by definition, given by
$$\mathbb{Z}_{\geq 0} \setminus \{\text{ord } f_i \mid i \geq 1\}.$$ In particular $w_1 = 1$ and $w_g = 2g - 1$.

### 3 KP-hierarchy

The following system of equations for a function $\tau(t)$, $t = (t_1, t_2, \ldots)$ is called the bilinear equations of the KP-hierarchy [6]:
$$\int_{k=\infty}^\infty \tau(t - [k^{-1}]) \tau(t' + [k^{-1}]) e^{\xi(t, k)} \, dk = 0,$$
$$\xi(t, k) = \sum_{i=1}^{\infty} t_i k^i, \quad [k] = (k, k^2/2, k^3/3, \ldots).$$ (2)

Notice that the equation (2) is invariant under the multiplication of the function of the form $c_0 \exp(\sum_{i=1}^{\infty} c_i t_i)$ to $\tau(t)$, where $c_i$ are constants.

The equations (2) can be rewritten using the Hirota derivative:
$$D^M \tau(t) \cdot \tau(t) = \partial^M_y \tau(t + y) \tau(t - y)|_{y=0},$$
where $D = (D_1, D_2, \ldots)$, $y = (y_1, y_2, \ldots)$, $M = (m_1, \ldots, m_l)$, $l \geq 0$, $D^M = D_1^{m_1} \cdots D_l^{m_l}$, $\partial^M_y = \partial^{m_1}_{y_1} \cdots \partial^{m_l}_{y_l}$. Let
$$e^{\xi(t, k)} = \sum_{j=0}^{\infty} p_j(t) k^j.$$ Then (2) is equivalent to
$$\sum_{j=0}^{\infty} p_j(-2y) p_{j+1}(\hat{D}) e^{\sum_{i=1}^{\infty} y_i D_i} \tau(t) \cdot \tau(t) = 0,$$ (3)
where $\hat{D} = (D_1, D_2/2, D_3/3, \ldots)$ [6]. The coefficient of $y_3$ gives the KP-equation in the bilinear form:
$$(D_1^4 + 3D_2^2 - 4D_1 D_3) \tau(t) \cdot \tau(t) = 0.$$ The system of equations obtained from (3) by setting $D_{jn} = 0$ for all $j \geq 1$ is called the n-reduced KP-hierarchy. For example the bilinear form of the KdV equation
$$(D_1^4 - 4D_1 D_3) \tau(t) \cdot \tau(t) = 0$$ is the first member of the 2-reduced KP-hierarchy and
$$(D_1^4 + 3D_2^2) \tau(t) \cdot \tau(t) = 0$$ is the first member of the 3-reduced KP-hierarchy (Boussinesq equation).

A solution $\tau(t)$ of the KP-hierarchy is a solution of the n-reduced KP-hierarchy if
$$\bar{\tau}(t) = c_0 e^{\sum_{i=1}^{\infty} c_i t_i} \tau(t)$$ does not depend on $\{t_{nj} \mid j \geq 1\}$ for some constants $c_i$. 4
4 Universal Grassmann Manifold

In this section we briefly review Sato’s theory of KP equation and the universal Grassmann manifold (UGM) following [23, 22] (see [20] for the English translation of [22]).

Let \( R = \mathbb{C}[[x]] \) be the ring of formal power series in \( x \) and \( \mathcal{E}_R = R((\partial^{-1})) \) the ring of microdifferential operators with the coefficients in \( R \):

\[
\mathcal{E}_R = \left\{ \sum_{-\infty < i < \infty} a_i(x) \partial^i \middle| a_i(x) \in R \right\}, \quad \partial = \frac{d}{dx}.
\]

Using the Leibnitz rule

\[
a(x) \partial^i = \sum_j (-1)^j \binom{i}{j} \partial^{i-j} a^{(j)}(x), \quad a^{(j)}(x) = \frac{d^j a(x)}{dx^j},
\]

\( \mathcal{E}_R \) can be described as the set of operators of the form \( \sum_{-\infty < i < \infty} \partial^i a_i(x) \) as well.

Let \( V = \mathcal{E}_R/\mathcal{E}_R x \simeq \mathbb{C}((\partial^{-1})) \) be the left \( \mathcal{E}_R \) module. We define the element \( e_i \) of \( V \) by

\[
e_i = \partial^{-i-1} \mod. \mathcal{E}_R x.
\]

The action of \( \mathcal{E}_R \) on \( e_i \) is given by

\[
\partial e_i = e_{i-1}, \quad xe_i = (i + 1)e_i.
\]

We define two subspaces of \( V \):

\[
V^\phi = \oplus_{i < 0} \mathbb{C}e_i, \quad V^{(0)} = \prod_{i > 0} \mathbb{C}e_i.
\]

Then we have the decomposition

\[
V = V^\phi \oplus V^{(0)}.
\]

For a subspace \( U \) of \( V \) let

\[
\pi_U : U \longrightarrow V/V^{(0)} \simeq V^\phi,
\]

be the composition of the inclusion \( U \hookrightarrow V \) and the natural projection \( V \twoheadrightarrow V/V^{(0)} \).

**Definition 1** The universal Grassmann manifold is the set of subspaces \( U \) of \( V \) such that \( \text{Ker} \pi_U, \text{Coker} \pi_U \) are finite dimensional and the index of \( \pi_U \) is zero:

\[
\text{index}(\pi_U) = \dim(\text{Ker} \pi_U) - \dim(\text{Coker} \pi_U) = 0.
\]
For a partition $\lambda = (\lambda_1, ..., \lambda_l)$ we define the Schur function $s_\lambda(t)$ by

$$s_\lambda(t) = \det(p_{\lambda_i - i + j}(t))_{1 \leq i, j \leq l}.$$

For a point $U$ of UGM, a frame $\xi$ of $U$ is a basis of $U$

$$\xi = (\xi_j)_{j<0} = (... , \xi_{-2} , \xi_{-1}) , \quad \xi_j = \sum_{i \in \mathbb{Z}} \xi_{ij} e_i,$$

such that, for $j << 0$,

$$\xi_{ij} = \begin{cases} 0 & i < j \\ 1 & i = j. \end{cases} \quad (4)$$

For a point $U$ of UGM there exists a unique sequence of integers $\rho_U = (\rho(i))_{i<0}$ and a unique frame $\xi$ of $U$ such that

$$\rho(-1) > \rho(-2) > \rho(-3) > \cdots , \quad \rho(i) = i \text{ for } i << 0, \quad (5)$$

$$\xi_{ij} = \begin{cases} 0 & i < \rho(j) \text{ or } i = \rho(j') \text{ for some } j' > j \\ 1 & i = \rho(j). \end{cases} \quad (6)$$

The frame satisfying the condition (6) is said to be normalized.

In terms of $\rho_U = (\rho(i))_{i<0}$ the index of $\pi_U$ is given by

$$\#\{i \mid \rho(i) \geq 0 \} - \#(\mathbb{Z}_{<0} \setminus \{i \mid \rho(i) < 0 \}),$$

where $\mathbb{Z}_{<0}$ is the set of negative integers.

In general a sequence of integers $\rho = (\rho(i))_{i<0}$ which satisfies (5) determines a partition $\lambda_\rho = (\lambda_1, \lambda_2, ...)$ by

$$\lambda_i = \rho(-i) + i.$$

**Example** If $\rho = (2, 0, -2, -4, -5, -6, ...)$ then $\lambda_\rho = \rho(-1, -2, -3, -4, ...) = (3, 2, 1, 0, ...) = (3, 2, 1)$.

**Definition 2** For a partition $\lambda$ the set of the points $U$ in UGM satisfying $\lambda_\rho = \lambda$ is denoted by $\text{UGM}^\lambda$.

The UGM is the disjoint union of $\text{UGM}^\lambda$'s:

$$\text{UGM} = \bigsqcup_\lambda \text{UGM}^\lambda.$$

Given a point $U$ of UGM and a sequence $\rho = (\rho(i))_{i<0}$ satisfying the condition (5), the Plücker coordinate $\xi_{\lambda_\rho}$ of the normalized frame $\xi$ is defined as the determinant of the $\mathbb{Z}_{<0} \times \mathbb{Z}_{<0}$ matrix:

$$\xi_{\lambda_\rho} = \det(\xi_{\rho(i)j})_{i,j<0}.$$
For two partitions \( \lambda = (\lambda_1, \lambda_2, \ldots) \) and \( \mu = (\mu_1, \mu_2, \ldots) \), we define \( \lambda \geq \mu \) if \( \lambda_i \geq \mu_i \) for any \( i \).

For \( U \in UGM^\lambda \) the Plücker coordinates satisfy

\[
\xi_\mu = \begin{cases} 
0 & \text{unless } \mu \geq \lambda \\
1 & \mu = \lambda.
\end{cases}
\]

**Definition 3** For \( U \in UGM \) let

\[
\tau(t; \xi) = \sum_{\mu \geq \lambda} \xi_\mu s_\mu(t),
\]

where \( \xi \) is the normalized frame of \( U \). The function \( \tau(t; \xi) \) and its constant multiple is called the tau function of \( U \).

**Theorem 1** [23, 22] For \( U \) in \( UGM^\lambda \) the tau function of \( U \) is a solution of the KP-hierarchy [2]. Conversely any solution \( \tau(t) \in \mathbb{C}[[t]] \) of (2) there exists a point of \( U \) of \( UGM \) such that \( \tau(t) \) is the tau function of \( U \).

The theorem follows from the fact [22, 23, 20] that, if we expand \( \tau(t) \) as

\[
\tau(t) = \sum_{\lambda} \xi_\lambda s_\lambda(t)
\]

with some set of constants \( \{\xi_\lambda\} \), the bilinear equation [2] is equivalent to the Plücker relations for \( \{\xi_\lambda\} \). Based on this theorem, the point of \( UGM \) corresponding to the solution \( \tau(t) \) of (2) is recovered through the wave function as follows.

Let \( K = \mathbb{C}((x)) \) be the field of formal Laurent series in \( x \) and \( \mathcal{E}_K = K((\partial^{-1})) \) the ring of microdifferential operators:

\[
\mathcal{E}_K = \left\{ \sum_{-\infty < i < \infty} a_i(x)\partial^i \mid a_i(x) \in K \right\}.
\]

**Definition 4** Let \( \mathcal{W} \) be the set of \( W \) in \( \mathcal{E}_K \) of the form

\[
W = \sum_{i \leq 0} w_i \partial^i, \quad w_0 = 1,
\]

satisfying the condition that there exist non-negative integers \( l, m \) such that

\[
x^lW, \quad W^{-1}x^m \in \mathcal{E}_R.
\]

Then

**Theorem 2** [23, 22] There is a bijective map \( \gamma : \mathcal{W} \longrightarrow UGM \) given by

\[
\gamma(W) = W^{-1}x^m V^\phi,
\]

where \( m \) is chosen as in Definition 4. The image \( \gamma(W) \) does not depend on the choice of \( m \).
Let \( \tau(t) \in \mathbb{C}[[t]] \) be a solution of the KP-hierarchy \(^{(2)}\). The wave function and the conjugate wave function are defined by

\[
\Psi(t; z) = \frac{\tau(t + [z])}{\tau(t)} \exp\left(-\sum_{i=1}^{\infty} t_i z^{-i}\right),
\]

\[
\bar{\Psi}(t; z) = \frac{\tau(t - [z])}{\tau(t)} \exp\left(\sum_{i=1}^{\infty} t_i z^{-i}\right).
\]

Due to the bilinear identity \(^{(2)}\) there exists \( W \in \mathcal{W} \) such that \( \Psi \) and \( \bar{\Psi} \) can be written as

\[
\Psi(t, z) = (W^*)^{-1} \exp\left(-\sum_{i=1}^{\infty} t_i z^{-i}\right),
\]

\[
\bar{\Psi}(t, z) = W \exp\left(\sum_{i=1}^{\infty} t_i z^{-i}\right),
\]

where \( x = t_1 \) and \( P^* = \sum (-\partial)^i a_i(x) \) is the formal adjoint of \( P = \sum a_i(x) \partial^i \). We have \( (P^*)^{-1} = (P^{-1})^* \) for an invertible \( P \in \mathcal{E}_K \).

The following lemma easily follows from the definition of \( s_\lambda(t) \).

**Lemma 1** \(^{(27)}\) For any partition \( \lambda = (\lambda_1, \lambda_2, ...) \)

\[
s_\lambda(t_1, 0, 0, ...) = d_\lambda t_1^{|\lambda|},
\]

\[
d_\lambda = \prod_{i<j} (\mu_i - \mu_j) / \prod_{i=1}^l \mu_i !,
\]

where \( l \) is taken large enough such that \( \lambda_i = 0 \), \( \mu_i = \lambda_i + l - i \) and \( |\lambda| = \sum_i \lambda_i \).

By the lemma and Theorem \(^{[1]}\) we see that \( \tau(x, 0, 0, ...) \) is not identically zero. Let \( m_0 \) be the order of zeros of \( \tau(x, 0, 0, ...) \) at \( x = 0 \) and \( m \geq m_0 \). Obviously we have \( x^m W(x, 0, ...) \in \mathcal{E}_R \) which implies \( W \in \mathcal{W} \). Then

**Theorem 3** \(^{(22, 23)}\)

\[
\tau(t) = \tau(t; \gamma(W(x, 0, ...))).
\]

Let us describe \( \gamma(W(x, 0, ...)) \) in terms of the wave function \( \Psi \).

**Proposition 1** Let

\[
x^m \Psi(x, 0, ...; z) = \sum_{i=0}^{\infty} \Psi_i(z) \frac{x^i}{i!}.
\]

Then we have

\[
(-1)^i W(x, 0, ...)^{-1} x^m e_{-1-i} = \Psi_i(\partial^{-1}) e_{-1},
\]

\[
\gamma(W(x, 0, ...)) = \text{Span}_\mathbb{C} \{\Psi_i(\partial^{-1}) e_{-1} \mid i \geq 0\},
\]

where \( \text{Span}_\mathbb{C} \{\cdots\} \) signifies the vector space generated by \( \{\cdots\} \).
Proof. Let
\[
x^m \Psi(x, 0, ..., z) = \sum_{i=0}^{\infty} \frac{x^i}{i!} \psi_i(z) e^{-x z} = \sum_{i=0}^{\infty} \frac{x^i}{i!} \psi_i(-\partial^{-1}) e^{-x z}.
\]

Then
\[
W(x, 0...)^{-1} x^m = \left( \sum_{i=0}^{\infty} \frac{x^i}{i!} \psi_i(-\partial^{-1}) \right)^* = \sum_{i=0}^{\infty} \psi_i(\partial^{-1}) \frac{x^i}{i!}.
\]
Thus
\[
W(x, 0...)^{-1} x^m e^{-1} = \sum_{j=0}^{\infty} \psi_j(\partial^{-1}) \frac{x^j}{j!} \partial^{i-1} e^{-1}
\]
\[
\quad = \sum_{j} (-1)^j \binom{i}{j} \psi_j(\partial^{-1}) \partial^{i-j} e^{-1}.
\] (8)

On the other hand
\[
\Psi_i(z) = \frac{\partial^i}{\partial x^i} (x^m \Psi(x, 0, ..., z))|_{x=0} = \sum_{j=0}^{i} \binom{i}{j} \psi_j(z) \left( -\frac{1}{z} \right)^{i-j}.
\] (9)

The assertion of the lemma follows from (8), (9) and Theorem 2.

5 Embedding of the Affine Ring to UGM

We can take a local coordinate \(z\) of \(X\) around \(\infty\) in such a way that
\[
x = \frac{1}{z^n}, \quad y = \frac{1}{z^g}(1 + O(z)).
\] (10)

Using the expansion in \(z\) define the embedding \(\iota : A \rightarrow V\) by
\[
\sum a_m z^m \mapsto \sum a_m e_{m+g-1}.
\]

Let \(U^A = \iota(A)\).

Lemma 2 The image \(U^A\) belongs to UGM.

Proof. Let \(0 = w_1^* < w_g^* < \cdots\) be non-gaps of \(X\) at \(\infty\), that is, \(\{w_i^*\} = \{\text{ord} f_i\}\). Then they satisfy
\[
\begin{align*}
  w_i^* &= g - 1 + i \quad \text{for } i \geq g + 1, \\
  \{w_1^*, ..., w_g^*\} \cup \{w_1, ..., w_g\} &= \{0, 1, ..., 2g - 1\}
\end{align*}
\] (11)
We have
\[ \dim(\ker \pi_U A) = \#\{ i \mid w_i^* \leq g - 1 \}, \quad \dim(\coker \pi_U A) = \#\{ i \mid w_i > g - 1 \}. \]

Then the equation \( (g - \dim(\ker \pi_U A)) + \dim(\coker \pi_U A) = g \).

Thus the index of \( \pi_U A \) is zero and \( U^A \) is in UGM.

Let
\[ \lambda(n, s) = (w_g, \ldots, w_1) - (g - 1, \ldots, 1, 0) \]
be the partition associated with the gap sequence. Then \( U^A \) belongs to \( UGM^{\lambda(n, s)} \) and the tau function \( \tau(t; \xi^A) \) has the expansion
\[ \tau(t; \xi^A) = s_{\lambda(n, s)}(t) + \sum_{\lambda > \lambda(n, s)} \xi_\lambda s_\lambda(t), \quad (12) \]
where \( \xi^A \) is the normalized frame of \( U^A \). The aim of the paper is to determine the analytic expression of \( \tau(t; \xi^A) \).

6 Sigma Functions

Let \( X \) be an \( (n, s) \) curve introduced in section 2 and \( \{du_{w_i}\} \) a basis of holomorphic one forms given by
\[ du_{w_i} = -\frac{f_{g+1-i}dx}{fy}, \]
where \( f_i \) is the monomial of \( x, y \) defined in section 2. We choose an algebraic fundamental form \( \tilde{\omega}(p_1, p_2) \) and decompose it as
\[ \tilde{\omega}(p_1, p_2) = d_{p_2} \Omega(p_1, p_2) + \sum_{i=1}^{g} du_{w_i}(p_1) dr_i(p_2), \]
where \( dr_i(p) \) is a second kind differential holomorphic outside \( \infty \). Here
\[ \Omega(p_1, p_2) = \sum_{i=0}^{n-1} y_i^{i+1} f_{x}(z,w) + f_{y}(z,w) dx_1, \]
\[ [\sum_{m \in \mathbb{Z}} a_m w^m]_+ = \sum_{m \geq 0} a_m w^m. \]

Then, with respect to the intersection form \( \circ \) defined by
\[ \omega \circ \eta = \text{Res}_{p=\infty}(\int^p \omega)\eta. \]
\{du_{w_i}, dr_j\} \text{ becomes symplectic:}

\begin{align*}
du_{w_i} \circ du_{w_j} &= dr_i \circ dr_j = 0, \\
\quad du_{w_i} \circ dr_j &= \delta_{ij}.
\end{align*}

(13)

Let us take a symplectic basis \(\{\alpha_i, \beta_j\}\) of the homology group of \(X\) and form the period matrices

\begin{align*}
2\omega_1 &= \left( \int_{\alpha_j} du_{w_i} \right), \\
2\omega_2 &= \left( \int_{\beta_j} du_{w_i} \right), \\
-2\eta_1 &= \left( \int_{\alpha_j} dr_i \right), \\
-2\eta_2 &= \left( \int_{\beta_j} dr_i \right), \\
\tau &= \omega_1^{-1}\omega_2.
\end{align*}

(14)

Let \(\delta = \delta' + \delta'', \delta', \delta'' \in \mathbb{R}^g\) be a representative of Riemann’s constant with respect to the choice \(\{\alpha_i, \beta_j\}, \infty\) and set \(\delta = t^{\prime}(\delta', \delta'') \in \mathbb{R}^{2g}\).

In general, for \(a, b \in \mathbb{R}^g\) and a point \(\tau\) of the Siegel upper half space of degree \(g\), the Riemann’s theta function is defined by

\[
\theta \left[ \begin{array}{c}
a \\
b
\end{array} \right] (z, \tau) = \sum_{m \in \mathbb{Z}^g} \exp(\pi i^t m \tau m + 2\pi i^t m z),
\]

where \(z = t(z_1, \ldots, z_g)\) [17].

**Definition 5** We define

\[
\hat{\sigma}(u) = \exp(\frac{1}{2}t^t u \eta_1 \omega_1^{-1} u) \theta[\delta](2\omega_1)^{-1} u, \tau),
\]

where \(u = t(u_{w_1}, \ldots, u_{w_g})\).

Notice that \(\hat{\sigma}(u)\) depends on the choice of \(\{du_{w_i}\}, \tilde{\omega}(p_1, p_2), \) and \(\{\alpha_i, \beta_j\}\). For the degrees of freedom on the choice \(\tilde{\omega}(p_1, p_2)\) see section 3.4 of [19].

Later we define the sigma function \(\sigma(u)\) by multiplying a certain constant to \(\hat{\sigma}(u)\) (Definition 6).

The function \(\hat{\sigma}(u)\) has the following transformation rule (see [19]).

**Proposition 2** [3] For \(m_1, m_2 \in \mathbb{Z}^g\)

\[
\hat{\sigma}(u + 2\omega_1 m_1 + 2\omega_2 m_2) = (-1)^{m_1 m_2 + 2t(\delta' m_1 - \delta'' m_2)} \exp \left( t^t(2\eta_1 m_1 + 2\eta_2 m_2)(u + \omega_1 m_1 + \omega_2 m_2) \right) \hat{\sigma}(u).
\]
7 Tau Function in Terms of Sigma Function

Let us take the local coordinate \( z \) of \( X \) around \( \infty \) as in (10) and consider the expansions in \( z \):

\[
du_{w_i} = \sum_{j=1}^{\infty} b_{ij} z^{j-1} dz, \]
\[
\hat{\omega}(p_1, p_2) = \left( \frac{1}{(z_1 - z_2)^2} + \sum_{i,j \geq 1} \hat{q}_{ij} z_1^{i-1} z_2^{j-1} \right) dz_1 dz_2, \]

By the definition of \( w_i, du_i \) and \( z \) we have \[19\]

\[
b_{ij} = \begin{cases} 0 & \text{if } j < w_i \\ 1 & \text{if } j = w_i. \end{cases} \quad (15)
\]

In particular we have

\[
du_w = z^{2g-2}(1 + \sum_{j>2g-2} b_{gj} z^{j-2g+1}) dz.
\]

Let

\[
\log z^{-(g-1)} \sqrt{\frac{du_w}{dz}} = \sum_{i=1}^{\infty} c_i z^i, \\
B = (b_{ij})_{g \times \infty}, \quad t = (t_1, t_2, ...), \\
\hat{q}(t) = \sum_{i,j=1}^{\infty} \hat{q}_{ij} t_i t_j.
\]

Then

**Theorem 4** (i) There exists a constant \( C \) such that

\[
\tau(t; \xi^A) = C \exp \left( -\sum_{i=1}^{\infty} c_i t_i + \frac{1}{2} \hat{q}(t) \right) \hat{\sigma}(Bt). \quad (16)
\]

(ii) The tau function \( \tau(t; \xi^A) \) satisfies the \( n \)-reduced KP-hierarchy.

\( \tilde{\text{Proof.} } \) We denote the right hand side of (21) by \( \hat{\tau}(t) \)

Let \( E(p_1, p_2) \) be the prime form \[10\]. Using the local coordinate \( z \) we define \( E(z_1, z_2), E(\infty, p) \) by

\[
E(p_1, p_2) = \frac{E(z_1, z_2)}{\sqrt{dz_1} \sqrt{dz_2}},
\]
\[
E(\infty, p_2) = \frac{E(0, z_2)}{\sqrt{dz_2}}.
\]
The normalized fundamental form \( \omega(p_1, p_2) \) is defined by
\[
\omega(p_1, p_2) = d_{p_1}d_{p_2} \log E(p_1, p_2).
\]
It has the expansion of the form
\[
\omega(p_1, p_2) = \left( \frac{1}{(z_1 - z_2)^2} + \sum_{i,j \geq 1} q_{ij} z_1^{i-1} z_2^{j-1} \right) dz_1 dz_2.
\]
Let \( \{dv_j\}_{j=1}^g \) be the normalized basis of holomorphic 1-forms and
\[
dv_i = \sum_{j=1}^\infty a_{ij} z_j^{j-1}
\]
its expansion near \( \infty \). We set
\[
\bar{A} = (a_{ij}), \quad q(t) = \sum_{i,j=1}^\infty q_{ij} t_i t_j.
\]
Then the following theorem is well known.

**Theorem 5** [14, 21] The function
\[
\tilde{\tau}(t) = \exp \left( \frac{1}{2} q(t) \right) \theta(\bar{A}t + \zeta)
\]
is a solution of the KP-hierarchy for any \( \zeta \in \mathbb{C}^g \).

Taking \( \zeta = \delta \), using the relations [19]
\[
2\omega_1 \bar{A} = B,
\]
\[
\tilde{\omega}(p_1, p_2) = \omega(p_1, p_2) - \sum_{i,j=1}^g (\eta_i \omega_1^{-1})_{ij} du_{w_i}(p_1) du_{w_j}(p_2), \quad (17)
\]
and the definition of \( \sigma \) in terms of the Riemann’s theta we easily see that \( \tilde{\tau}(t) \) is obtained from \( \tau(t) \) by multiplying a constant and the exponential of a linear function of \( t \). Thus \( \tilde{\tau}(t) \) is a solution of the KP-hierarchy.

In order to determine the point of UGM corresponding to \( \tilde{\tau}(t) \) we calculate the wave function.

Let \( d\tilde{r}_i \) be the normalized abelian differential of the second kind which means that it is holomorphic on \( X - \{ \infty \} \), has zero \( \alpha_j \) period for any \( j \) and it has the form
\[
d\tilde{r}_i = d \left( \frac{1}{z^i} + O(1) \right),
\]
near ∞. We set
\[ d\tilde{r}_i = d\tilde{r}_i + \sum_{k,l=1}^{g} b_{ki}(\eta_1\omega^{-1}_i)_{kl}du_{w_l}. \]

By calculation we have
\[
\Psi(t, z) = \hat{\tau}(t + [z]) \exp(-\sum_{i=1}^{\infty} t_i z^{-i})
\]
\[ = \sqrt{\frac{du_{w_g}}{dz}} \frac{z^g}{E(0, z)} \frac{\sigma(Bt + \int_{\infty}^{p} du)}{\sigma(Bt)} \exp \left( -\sum_{i=1}^{\infty} t_i \int_{\infty}^{p} d\tilde{r}_i - \frac{1}{2} \int_{\infty}^{p} t_i du \cdot \eta_1(\omega_1)^{-1} \cdot \int_{\infty}^{p} du \right). \]

This is simply a restatement of the known result ([14], [13]).

The following modification of the prime form had been introduced in [19]:
\[ \tilde{E}(\infty, p) = E(\infty, p) \sqrt{du_{w_g}} \exp \left( \frac{1}{2} \int_{\infty}^{p} t_i du \cdot \eta_1(\omega_1)^{-1} \cdot \int_{\infty}^{p} du \right). \]

Notice that this is not a half form but a multi-valued holomorphic function on X with zeros only at ∞. The expansion near ∞ is of the form
\[ \tilde{E}(\infty, p) = z^g(1 + O(z)). \]

Its transformation rule is determined in [19].

**Lemma 3** [19] Let \( \gamma \in \pi_1(X, \infty) \). Suppose that its image in \( H_1(X, \mathbb{Z}) \) is given by
\[ \sum_{i=1}^{g} m_{i, \alpha_1} + \sum_{i=1}^{g} m_{i, \beta_i}. \] Then
\[
\tilde{E}(\infty, \gamma(p))/\tilde{E}(\infty, p) = (-1)^{t_{m_1}m_2+2(t_{\delta' m_1}-t_{\delta m_2})} \exp \left( t_{2\eta_1 m_1 + 2\eta_2 m_2} \left( \int_{\infty}^{p} du + \omega_1 m_1 + \omega_2 m_2 \right) \right),
\]
where \( m_i = t(m_{i,1}, \ldots, m_{i,g}) \).

We rewrite \( \Psi \) using \( \tilde{E}(\infty, p) \) as
\[ \Psi(t, z) = \frac{z^g}{E(\infty, p)} \frac{\sigma(Bt + \int_{\infty}^{p} du)}{\sigma(Bt)} \exp \left( -\sum_{i=1}^{\infty} t_i \int_{\infty}^{p} d\tilde{r}_i \right). \]

**Lemma 4** The function \( z^{-g}\Psi(t, z) \) is \( \pi_1(X, \infty) \)-invariant and any coefficient of \( t_1^{m_1}t_2^{m_2} \ldots \) in the expansion of \( \sigma(Bt)z^{-g}\Psi(t, z) \) is in \( A \).

This lemma follows from
Lemma 5. We have
\[ \int_{\alpha_j} d\tilde{r}_i = (i'(2\eta_1)B)_{ij}, \quad \int_{\beta_j} d\tilde{r}_i = (i'(2\eta_2)B)_{ij}. \]

This lemma can be proved by a direct calculation using the definition of \( d\tilde{r}_i \).

Since \( \hat{\tau}(t) \) is a tau function of the KP-hierarchy, \( \hat{\tau}(x,0,...) \) is not identically zero. Let \( m \) be the order of zeros of \( \hat{\tau}(x,0,...) \) at \( x = 0 \) and
\[ x^m \Psi(x,0,...;z) = \sum_{i=0}^{\infty} \Psi_i(z)x^i. \]

Then
\[ z^{-g}\Psi_i(z) \in A. \]

Let
\[ z^{-g}\Psi_i(z) = \sum_{-\infty<<k<\infty} \psi_kz^k. \]

Then
\[ \Psi_i(\partial^{-1})e_{-1} = \sum_k \psi_k e_{-1+k} = t(z^{-g}\Psi_i(z)). \]

Thus the subspace \( U \) of \( V \) generated by \( \{ \Psi(\partial^{-1})e_{-1}\} \) is a subspace of \( U^A \). Since both \( U \) and \( U^A \) are in UGM, \( U = U^A \).

Next we prove that \( \tau(t;\xi^A) \) is a solution of the n-reduced KP-hierarchy.

Lemma 6
\[ \bar{q}_{ij} = 0 \quad \text{if } ij = 0 \mod n, \]
\[ b_{ij} = 0 \quad \text{if } j = 0 \mod n. \]

Proof. Firstly let us prove \( b_{ij} = 0 \) if \( j = 0 \mod n \). Notice that
\[ d\tilde{r}_{nk} = dx^k = d \left( \frac{1}{z^{nk}} \right). \]

Then
\[ d\tilde{r}_{nk} \circ du_{w_i} = \text{Res}_{p=\infty} \left( \int_{\alpha} d\tilde{r}_{nk}du_{w_i} \right) = b_{i,nk}. \]

On the other hand the left hand side is zero because \( d\tilde{r}_{nk} \) is an exact form.
Next we prove \( q_{ij} = 0 \) if \( i, j \) satisfy \( ij = 0 \mod n \). In fact \( q_{ij} \) can be obtained as the expansion of \( d\tilde{r}_i \) as (see the appendix of \[13\] for example)

\[
d\tilde{r}_i = d\left(\frac{1}{z^i} - \sum_{j=1}^{\infty} q_{ij} z^j\right).
\]

Then the assertion follows from \[18\] and the symmetry of \( q_{ij} \).

Finally \( \tilde{q}_{ij} = 0 \) for \( (i,j) \) satisfying \( ij = 0 \mod n \) follows from the relation \[17\] and the symmetry of \( \tilde{q}_{ij} \).

It follows from the lemma that \( \exp(\sum_{i \geq 1} c_i t_i) \tau(t; \xi^A) \) does not depend on \( t_{nk} \) for \( k \geq 1 \). Thus \( \tau(t; \xi^A) \) is a solution of the n-reduced KP-hierarchy.

**Corollary 1** The coefficients of \( x^l, l \geq 0 \), in the expansion of the function

\[
\frac{\sigma((x, 0, ...) + \int_\infty^p du)\exp \left(-x \int_0^p d\tilde{r}_1\right)}{E(\infty, p)}
\]

generate the affine ring \( A \) as a vector space.

**Remark.** In the case of \( g = 1 \) the above corollary tells that the coefficients of \( x^l, l \geq 0 \) of the Baker-Akhiezer function

\[
\frac{\sigma(u + x)}{\sigma(u)} \exp(-x \zeta(u))
\]

generate the space generated by \( \phi^{(i)}(u), i \geq 0 \) and \( 1 \). This fact can be easily checked and is well known.

## 8 Applications

In this section we study the series expansion of the sigma function as an application of Theorem 4.

Let \( u = (u_{w_1}, ..., u_{w_g}) \). We define the degrees of \( u_i \) and \( t_i \) to be \(-i:\)

\[
\deg u_i = \deg t_i = -i.
\]

**Theorem 6** For the constant \( C \) in Theorem 4 we have the following series expansion at \( u = 0 \):

\[
C\theta[\delta]\left((2\omega_1)^{-1} u, \tau\right) = s_{\lambda(n,s)}(u) + \cdots,
\]

where \( \cdots \) part contains lower degree terms than \( s_{\lambda(n,s)}(u) \).
Proof. Let $t^0 = (t_1, t_2, \ldots)$ in which $t_j = 0$ for $j \notin \{w_1, \ldots, w_g\}$ and $u = Bt^0$. Then, by (15), we have

$$u_{w_i} = t_{w_i} + \sum_{j=i+1}^{g} b_{iw_j} t_{w_j}.$$  

Inverting this we have

$$t_{w_i} = u_{w_i} + \sum_{j=i+1}^{g} b'_{iw_j} u_{w_j}. \quad (19)$$

Then the theorem follows from Theorem 4 and the expansion (12) of $\tau(t; \xi^4)$.

**Definition 6** The sigma function associated to the choice $(\{du_{w_i}\}, \tilde{\omega}, \{\alpha_i, \beta_j\})$ is defined by

$$\sigma(u) = \sigma(u|\{du_i\}, \tilde{\omega}(p_1, p_2), \{\alpha_i, \beta_j\}) = C\tilde{\sigma}(u),$$

where $C$ is the constant given in Theorem 6.

The sigma function inherits some remarkable properties from the tau function $\tau(t; \xi^4)$, the algebraic expansion and the modular invariance.

The symplectic group $Sp(2g, \mathbb{Z})$ acts on the set of canonical homology bases by

$$M \left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = \left( \begin{array}{cc} D & C \\ B & A \end{array} \right) \left( \begin{array}{c} \alpha \\ \beta \end{array} \right), \quad \text{for} \quad M = \left( \begin{array}{cc} A & B \\ C & D \end{array} \right) \in Sp(2g, \mathbb{Z}),$$

where

$$\left( \begin{array}{c} \alpha \\ \beta \end{array} \right) = t(\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g).$$

We assign degrees to the coefficients of $f(x, y)$ as

$$\deg \lambda_{ij} = ns - ni - sj.$$  

Then

**Theorem 7** (i) At $u = 0$ $\sigma(u)$ has the following expansion:

$$\sigma(u) = s_{\lambda(n, s)}(u) + \sum a_\gamma u^\gamma,$$

where $\gamma = (\gamma_1, \ldots, \gamma_g)$, $u^\gamma = u_{w_1}^{\gamma_1} \cdots u_{w_g}^{\gamma_g}$, $a_\gamma$ is a homogeneous polynomial of $\{\lambda_{ij}\}$ with the degree $-|\lambda(n, s)| + \sum_{i=1}^{g} \gamma_i w_i$ and the summation is taken for $\gamma$ with $\sum_{i=1}^{g} \gamma_i w_i > |\lambda(n, s)|$.

(ii) For $M \in Sp(2g, \mathbb{Z})$

$$\sigma(u|\{du_i\}, \tilde{\omega}, M^t(\alpha, \beta)) = \sigma(u|\{du_i\}, \tilde{\omega}, t(\alpha, \beta)).$$
Notice that the property (i) implies the property (ii). It is possible to study the modular transformation of the sigma function using that of Riemann’s theta function. However it is difficult to determine the 8-the root of unity part in that calculation.

We remark that Theorem 6 and 7 had been proved in [19] in a different way.

Proof of Theorem 7

By Lemma 15 of [19] \( b_{ij} \) is a homogeneous polynomial of \( \{ \lambda_{kl} \} \) of degree \(-w_i + j\) with the coefficient in \( \mathbb{Q} \). In particular \( \deg b_{ij} = -(2g - 1) + j \). It follows that \( c_i \) belongs to \( \mathbb{Q}[\{ \lambda_{kl} \}] \) and it is homogeneous of degree \(-i\). We assign degree \(-1\) to \( z \). Then \( \sum_{i \geq 1} c_i t_i \) is homogeneous of degree 0. Also \( \hat{q}_{ij} \) belongs to \( \mathbb{Q}[\{ \lambda_{kl} \}] \) and it is homogeneous of degree \( i + j \) by Lemma 15 of [19]. Thus \( \deg \hat{q}(t) = 0 \).

Let us calculate the degree of the Plücker coordinate \( \xi^A_{\lambda} \). Let \( \lambda \) and \( \lambda(n, s) \) correspond to \((\rho(i))_{i<0}\) and \((\rho_0(i))_{i<0}\) respectively and \(0 = w_1^* < w_2^* < \cdots\) the non-gaps of \( X \) at \( \infty \). Then

\[
\rho_0(-i) = -w_i^* + g - 1, \quad \lambda(n, s)_i = \rho_0(-i) + i. \tag{20}
\]

Let us expand \( f_j \) as

\[
f_j = \sum_{-\infty < i < \infty} a_{ij} z^i.
\]

Then

\[
\xi^A_j = \iota(f_j) = \sum a_{i-g+1,j} e_i.
\]

Thus

\[
\xi^A_{ij} = a_{i-g+1,j}.
\]

By Lemma 15 in [19] \( a_{ij} \in \mathbb{Q}[\{ \lambda_{kl} \}] \) and it is homogeneous of degree \( i + w^*_j \). Therefore

\[
\deg \xi^A_{ij} = i - g + 1 + w^*_j.
\]

Recall that

\[
\xi^A_{\lambda} = \det(\xi^A_{\rho(-i), -j})_{i, j \geq 1} = \sum_{\sigma \in S_m} \xi^A_{\rho(-\sigma_1), -1} \cdots \xi^A_{\rho(-\sigma_m), -m},
\]

where \( m \) is taken large enough so that \( \rho(i) = i \) for \( i < -m \). We have

\[
\deg \xi^A_{\rho(-\sigma_1), -1} \cdots \xi^A_{\rho(-\sigma_m), -m} = \sum_{i \geq 1} (\rho(-i) - g + 1 + w_i^*) = \sum_{i \geq 1} (\rho(-i) - \rho_0(-i)) = |\lambda| - |\lambda(n, s)|.
\]
Here we use (20) to eliminate \( w_i^* \). Consequently \( \xi_i^A \) is a homogeneous polynomial of \( \{\lambda_{ij}\} \) with the coefficients in \( \mathbb{Q} \) of degree \( |\lambda| - |\lambda(n,s)| \). In the expression (19) \( b'_{ij} \) has the same properties as \( b_{ij} \). Thus the theorem follows from Theorem 4 and (12).

We rephrase Theorem 4 (i) in terms of the sigma function:

**Theorem 8**

\[
\tau(t; \xi^A) = \exp \left( -\sum_{i=1}^{\infty} c_i t_i + \frac{1}{2} \hat{g}(t) \right) \sigma(Bt).
\] (21)

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**References**

[1] V. M. Buchstaber and D. V. Leykin, Addition laws on Jacobian varieties of plane algebraic curves, *Proc. Steklov Inst. of Math.* **251** (2005), 1-72.

[2] V. M. Buchstaber and D. V. Leykin, Solutions of the problem of differentiation of Abelian functions over parameters for families of \((n,s)\)-curves, *Funct. Anal. Appl.* **42** (2008), 268-278.

[3] V. M. Buchstaber, V. Z. Enolski and D. V. Leykin, Kleinian functions, hyperelliptic Jacobians and applications, in *Reviews in Math. and Math. Phys.* Vol.10, No.2, Gordon and Breach, London, 1997, 1-125.

[4] V. M. Buchstaber, V. Z. Enolski and D. V. Leykin, Rational analogue of Abelian functions, *Funct. Annal. Appl.* **33-2** (1999), 83-94.

[5] V.M. Buchstaber and S.Yu. Shorina, \( w \)-Function of the KdV hierarchy, in ”Geometry, topology and mathematical physics” *Amer. Math. Soc. Transl.* **212** (2004), 41-66.

[6] E. Date, M. Jimbo, M. Kashiwara and T. Miwa, Transformation groups for soliton equations, *Nonlinear Integrable Systems-Classical Theory and Quantum Theory*, M. Jimbo and T. Miwa (eds.), World Scientific, Singapore, 1983, 39-119.

[7] V.Z. Enolski, the talk at the conference ”Concrete theory of Abelian functions and its applications” held at Iwate University, Morioka, Japan, 2008.
[8] J.C. Eilbeck, V.Z. Enolski and J. Gibbons, Sigma, tau and abelian functions of algebraic curves, preprint.

[9] V.Z. Enolski and J. Harnad, Schur function expansions of KP tau functions associated to algebraic curves, preprint.

[10] J. Fay, *Theta functions on Riemann surfaces*, LNM 352, 1973, Springer.

[11] F. Klein, Ueber hyperelliptische Sigamafunctionen, *Math. Ann.** 27 (1886),341-464.

[12] F. Klein, Ueber hyperelliptische Sigamafunctionen (Zweiter Aufsatz), *Math. Ann.** 32 (1888), 351-380.

[13] N. Kawamoto, Y. Namikawa, A. Tsuchiya and Y. Yamada, Geometric realization of conformal field theory on Riemann surfaces, *Comm. Math. Phys.** 116 (1988) 247-308.

[14] I. Krichever, Methods of algebraic geometry in the theory of non-linear equations, *Russ. Math. Surv.** 32 (1977) 185-214.

[15] I.G. Macdonald, *Symmetric Functions and Hall Polynomials, second edition*, Oxford University Press, 1995.

[16] M. Mulase, Cohomological structure in soliton equations and jacobian varieties, *J. Diff. Geom.** 19 (1984) 403-430.

[17] D. Mumford, *Tata lectures on theta I*, Birkhauser, 1983.

[18] D. Mumford, *Tata lectures on theta II*, Birkhauser, 1983.

[19] A. Nakayashiki, Algebraic expressions of sigma functions for (n,s) curves, arXiv:0803.2083.

[20] M. Noumi and T. Takebe, Algebraic analysis of integrable hierarchies, in preparation.

[21] T. Shiota, Characterization of jacobian varieties in terms of soliton equations, *Inv. Math.** 83 (1986) 333-382.

[22] M. Sato, M. Noumi, Soliton equation and universal Grassmann manifold, Sophia University Kokyuroku in Math. 18 (1984) (in Japanese).

[23] M. Sato and Y. Sato, Soliton equations as dynamical systems on infinite dimensional Grassmann manifold, Nolinear Partial Differential Equations in Applied Sciences, P.D. Lax, H. Fujita and G. Strang (eds.), North-Holland, Amsterdam, and Kinokuniya, Tokyo, 1982, 259-271.
[24] G. Segal and G. Wilson, Loop groups and equations of KdV type, *Publ. Math. IHES* **61** (1985) 5-65.

[25] K. Weierstrass, Zur theorie der elliptischen funktionen, Mathematische Werke (Teubner, Berlin, 1894) Vol. 2, 245-255.