Magnus Expansion and Three-Neutrino Oscillations in Matter

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We present a semi-analytical derivation of the survival probability of solar neutrinos in the three generation scheme, based on the Magnus approximation of the evolution operator of a three level system, and assuming a mass hierarchy among neutrino mass eigenstates. We have used an exponential profile for the solar electron density in our approximation. The different interesting density regions that appear throughout the propagation are analyzed. Finally, some comments on the allowed regions in the solar neutrino parameter space are addressed.

PACS numbers: 14.60.Pq, 12.15.Ff, 26.65.+t

I. INTRODUCTION

The need to introduce the three generations of neutrinos into a computation of the transition probabilities of these particles while traversing a medium has been recognized long ago to try to accomodate the observations of different experiments studying neutrino oscillations. Analytical treatments of three neutrino oscillations in matter with varying density in the three generation scheme have been studied in the past aiming to deduce expressions for the oscillation probabilities of one type of neutrino into another. It has been shown, first in the case of two generations, and later in the case of three generations, that the differential equation describing the evolution of the neutrino state in an exponentially varying density profile could be solved analytically in terms of confluent hypergeometric functions. Corrections to the mixing parameters in matter calculated as series expansions have been performed by Freund, and a different perturbative analysis has been done by Narayan in . Global analyses of the recent experimental data have been extensively studied both in the two and three generations cases. In a previous paper by D’Olivo and Oteo, an approximate expression to the evolution operator using the Magnus expansion was found, only for the non-adiabatic regime in the exponentially varying density profile. In this paper we present a complete semi-analytical computation of the evolution operator for neutrinos in the same density profile, using the Magnus expansion approximation which will work properly in the case of adiabatic and non-adiabatic evolution of the neutrino state. This paper describes some important results of the M. C. Thesis work presented by Luis G. Cabral-Rosetti in . Let denote the Hamiltonian of a quantum system and be the time evolution operator satisfying the Schrödinger equation

\[
\frac{ih}{\partial t} U = HU , \quad U(t_0, t_0) = I . \quad (1)
\]

When is independent of time, or more generally when \( [\int_{t_0}^t dt' H(t'), H(t)] = 0 \), the solution of Eq. (1) is formally

\[
U = \text{Exp}[ -i/\hbar \int_{t_0}^t dt' H(t') ] .
\]

Then it is natural to ask whether a solution of the form \( U = \text{Exp}\Omega \) would always be possible. A method for finding such a true exponential solution (without time ordering) is supplied by the Magnus Expansion (ME) . The Magnus operator \( \Omega = iU \) satisfies a differential equation which in turn is solved through a series expansion: \( \Omega = \sum_{n=1}^{\infty} \Omega_n \), where each term \( \Omega_n \) is of order \( h^{-n} \). The first two contributions are explicitly given by

\[
\Omega_1 = -\frac{i}{\hbar} \int_{t_0}^t dt_1 H(t_1) ,
\]

\[
\Omega_2 = -\frac{i}{2\hbar^2} \int_{t_0}^t dt_1 \int_{t_0}^{t_1} dt_2 [H(t_1), H(t_2)] . \quad (2)
\]

Recursive methods to obtain the successive terms have been extensively worked out in the literature . Because of the anti-Hermitian character of every \( \Omega_n \), each
approximate time-evolution operator obtained as \( U \approx U_k = \exp(\sum_{n=1}^{k} \Omega_n) \) will be unitary. Here we use the first Magnus approximant to obtain (approximate) analytical solutions to the problem of 3 neutrinos oscillations in a medium with varying density as the Sun.

![Figure 1: Behaviour of the matter mixing angles as functions of the effective potential \( V(t) \).](image)

**II. SURVIVAL PROBABILITY FOR ELECTRON NEUTRINOS**

From the dependency of \( \sin 2\theta_{12}(t) \), and \( \sin 2\theta_{13}(t) \) on \( V(t) \) (see Ref. [10] and [11] for all details), we can distinguish five interesting regions where the oscillations behave differently.

1. **Region of Extremely Low Density**

   If \( 0 \leq V(t) \ll \Delta_2 \), both mixing angles, \( \theta_{12}^\odot(t) \), and \( \theta_{13}^\odot(t) \), are close to their values in vacuum (see Fig.1), this is \( \theta_{12}^\odot(t) \to \theta_{12}^\odot \); \( \theta_{13}^\odot(t) \to \theta_{13}^\odot \), giving

   \[
   P_\alpha(t) \to c_{12}^2 c_{13}^2, \quad \text{and} \quad P_\gamma(t) \to s_{13}^2, \tag{3}
   \]

   leading us to the well known result of vacuum oscillations of three neutrinos

   \[
   \langle P(\nu_e \to \nu_e) \rangle = c_{12}^4 c_{13}^4 + s_{12}^4 c_{13}^4 + s_{13}^4, \tag{4}
   \]

2. **Low Density Resonance Region**

   For \( V(t) \approx \Delta_2 \), \( \theta_{12}^\odot(t) \) is still close to its vacuum value (\( \theta_{12}^\odot(t) \approx \theta_{12}^\odot \)), while \( \theta_{13}^\odot(t) \) is at the low density resonance, i.e. \( \theta_{13}^\odot(t) \approx \frac{\pi}{2} \), making

   \[
   P_\alpha(t) \to c^2 \Pi(t) \cos^2 \theta_{12}^\odot(t) \cos^2 \theta_{13}^\odot(t) \sin^2 \theta_{12}^\odot(t) \cos^2 \theta_{13}^\odot(t), \tag{5}
   \]

   and

   \[
   P_\gamma(t) \to \sin^2 \theta_{13}^\odot(t), \tag{6}
   \]

   which leads to the expression

   \[
   \langle P(\nu_e \to \nu_e) \rangle = \frac{1}{2} + \frac{1}{2} \left( 1 - 2P_c^I \right) \cos 2\theta_{12}(t_0) \cos 2\theta_{12}^\odot, \tag{7}
   \]

   where

   \[
   \cos 2\theta_{12}(t_0) = \frac{V_i - V(t_0)}{\sqrt{(V_i - V(t_0))^2 + 2m_i^2}}, \tag{8}
   \]

   and

   \[
   P_c^I = \sin^2 \left( \left( \theta_{12}(t_0) - \theta_{12}^\odot(T) \right) \exp (-\kappa_i) \right). \tag{9}
   \]

   ![Figure 2: The survival probability for an electron neutrino coming from the Sun vs. \( \Delta m^2_{21}/2E \), with the parameters (a) \( \sin^2 2\theta_{12} = 0.1, \sin^2 2\theta_{13} = 0.01 \ y R = 10^9 \); (b) \( \sin^2 2\theta_{12} = 0.3, \sin^2 2\theta_{13} = 0.1 \ y R = 10^9 \).](image)

   The quantity \( P_c^I \) represents the transition probability between the states \( \nu_{2m} \ y \nu_{1m} \), and has been derived by D’Olivo [11] in the case of oscillations between two neutrino species. The adiabatic result can be recovered by setting \( P_c^I = 0 \). For \( \kappa_i \gg 1 \), \( P_c^I \) is exponentially suppressed as expected in the asymptotic regime. On the other hand, for \( \kappa_i < 1 \) there are significant corrections to the adiabatic approximation which reduce the effect of the resonant transition. Eq. (7) was first obtained by Parke [12] for two species using the Landau-Zener approximation for the crossing probability: \( P_c^I = \exp \left( -\frac{\pi}{2} \kappa_i \right) \). In the extreme non adiabatic case, \( \Delta_2 \to 0 \), making \( \kappa_i \to 0 \), and from Eq. (3) it follows that \( \theta_{13}^\odot(t_0) \to \frac{\pi}{2} \), making \( P_c^I = \cos^2 \theta_{12}^\odot \). Introducing this value of \( P_c^I \) in Eq. (7) we recover the vacuum result for two neutrinos:

   \[
   \langle P(\nu_e \to \nu_e) \rangle = 1 - \frac{1}{2} \sin^2 2\theta_{12}, \tag{10}
   \]

   This should be contrasted with the result

   \[
   \langle P(\nu_e \to \nu_e) \rangle = \cos^2 \theta_{12}^\odot, \tag{11}
   \]

   predicted by the Landau-Zener formula (and, in general, by any result derived from the Dykhne’s
formula), which deviates from the correct limit given by Eq. (11), when $\theta_{12}$ is large. The correct value for the extreme nonadiabatic case has been derived before, under the assumption that the transition between the adiabatic eigenstates occurs instantaneously at the time $t = t_1$. 

$$P_{\nu_e}(T) \approx \cos^2 \theta_{13}(t_0) \cos^2 \theta_{12}(t_0),$$

(14)

and

$$P_{\nu_{\mu}}(t) \approx \cos^2 \Pi(t) \sin^2 \theta_{13}^m(t_0),$$

(15)

leading to

$$\langle P_{\nu_e \to \nu_{\mu}} \rangle = \frac{1}{2} + \frac{1}{2} \left( 1 - 2P_h \right) \cos 2\theta_{13}^m(t_0) \cos 2\theta_{13},$$

(16)

where

$$\cos 2\theta_{13}^m(t_0) = \frac{V_h - V(t_0)}{\sqrt{(V(t_0) - V_h)^2 + B_h^2}},$$

(17)

and

$$P_h = \sin^2 \left( \theta_{13}^m(t_0) - \theta_{13}^m(T) \right) \exp \left( -\kappa_h \right),$$

(18)

which again implies the two neutrino oscillations.

Figure 3: The survival probability for an electron neutrino coming from the Sun vs. $\delta m^2_{21}/2E$, with the parameters (a) $\sin^2 2\theta_{12} = 0.1$, $\sin^2 2\theta_{13} = 0.001$ and $R = 100$. (b) $\sin^2 2\theta_{12} = 0.1$, $\sin^2 2\theta_{13} = 0.3$ and $R = 10^3$.

However, in this case the corresponding result for $P^l_c$ approaches $\cos^2 \theta_{12}$ as $\langle \delta m^2 \rangle^2$ instead of linearly, as in Eq. (11). Non adiabatic effects start to become important in this region, when $\kappa_i$ is comparable to 1, whenever the neutrinos cross the low density resonance. If $E < \delta m^2_{21}/2E \cos 2\theta_{12}$, $P^l_c = 0$, and the propagation of neutrinos will be adiabatic for $\kappa_i < 1$. For this reason, the asymptotic exponential expression ($\hbar \to 0$) for $P^l_c$, must be modified by hand to consider this situation. An effective way of implementing such modification is to multiply $P^l_c$ by a step function $\Theta(V(t_0) - \Delta m^2 \cos 2\theta_{12})$, in such a way that the transition probability vanishes if a neutrino is produced after the resonance. It is worth noting that such modification is not necessary with the Magnus result Eq. (9) given that, as a function of $\delta m^2_{21}/2E$, the difference $\theta_{12}^m(t_0) - \theta_{12}^m(T)$ behaves as a continuous step.

(3) Intermediate Density Region

For $\Delta_{21} \ll V(t) \ll \Delta_{31}$, the mixing angles in matter are $\theta_{12}^m(t) \approx \frac{\pi}{2}$, and $\theta_{13}^m(t) \approx \theta_{13}$, making

$$P_{\alpha}(t) \approx c_{12}^2 c_{13}^2,$$

(12)

and

$$P_{\gamma}(t) \approx s_{13}^2,$$

(13)

giving the result shown in Eq. (15), provided there exists a clear separation between the resonance regions: $\Delta_{31} \gg \Delta_{21}$.

(4) High Density Resonance Region

When $V(t) \approx \Delta_{31}$, the mixing angle $\theta_{13}^m(t)$ is approximately equal to $\frac{\pi}{2}$, while $\theta_{12}^m(t)$ is at its high density resonance value of $\theta_{13}^m(t) \approx \frac{\pi}{4}$. In this case

$$P_{\alpha}(t) \to \cos^2 \theta_{13}^m(t_0) \cos^2 \theta_{12}^m(t_0),$$

(14)

and

$$P_{\gamma}(t) \to \cos^2 \Pi(t) \sin^2 \theta_{13}^m(t_0),$$

(15)

leading to

$$\langle P_{\nu_e \to \nu_{\mu}} \rangle = \frac{1}{2} + \frac{1}{2} \left( 1 - 2P_h \right) \cos 2\theta_{13}^m(t_0) \cos 2\theta_{13},$$

(16)

where

$$\cos 2\theta_{13}^m(t_0) = \frac{V_h - V(t_0)}{\sqrt{(V(t_0) - V_h)^2 + B_h^2}},$$

(17)

and

$$P_h = \sin^2 \left( \theta_{13}^m(t_0) - \theta_{13}^m(T) \right) \exp \left( -\kappa_h \right),$$

(18)

which again implies the two neutrino oscillations. The crossing probability between the instantaneous eigenstates $\nu_{2\mu}$ and $\nu_{2\nu}$ given by $P_h$ is perfectly analogue to that studied in the low density resonance region. We can recover the adiabatic case if we set $P_h = 0$, and for $\kappa_i \gg 1$, $P_h$ is exponentially suppressed. The extreme nonadiabatic case requires $\Delta_{31} \to 0$, and from Eq. (9) we have $\theta_{13}^m(t_0) \to \frac{\pi}{4}$ leading to $P_h = \cos^2 \theta_{13}$. All these observations will lead us to an expression of the form
of Eq. (10) with \( \theta_{12} \) replaced by \( \theta_{13} \). Non adiabatic effects become important when \( \kappa_{\alpha} \approx 1 \) provided the neutrino crosses the \textit{High density resonance}. If \( E < \delta \frac{m_{\nu}^{2}}{2 V(t)} (\cos 2 \theta_{13} - R^{-1} \sin^{2} \theta_{12} \cos 2 \theta_{13}) \), with \( R = \Delta_{31}/\Delta_{21} \), no transitions between the instantaneous eigenstates can occur \((P_{\nu}^{h} = 0)\), and the propagation is adiabatic for \( \kappa_{\alpha} < 1 \). Again, the difference \( \theta_{13}^{m}(t_{0}) - \theta_{13}^{e}(T) \) behaves as a continuous step giving \( P_{\nu}^{e} \) the appropriate behaviour.

![Figure 5: Overlap of the allowed regions for the experiments SAGE+GALLEX (0.46 < \( P < 0.66 \)) and SNO+Super-Kamiokande (0.25 < \( P < 0.43 \)). \( R = S = 250 \).](image)

\( \text{(5) Extremely High Density Region} \)

For \( V(t) \gg \Delta_{31} \) the \((\nu_{e})\) oscillations are strongly suppressed due to the fact that \( \theta_{13}^{m}(t) \approx \frac{\pi}{2} \), and \( \theta_{13}^{e}(t) \approx \frac{\pi}{2} \), making

\[
P_{\nu}(t) \rightarrow c_{12}^{2} c_{13}^{2}
\]

and

\[
P_{\gamma}(t) \rightarrow s_{12}^{2},
\]

giving the result of Eq. (11).

\[
\langle P(\nu_{e} \rightarrow \nu_{e}) \rangle = c_{12}^{2} c_{13}^{4} + s_{12}^{4} c_{13}^{4} + s_{13}^{4},
\]

Plots of the \( \nu_{e} \) survival probability \( \langle P(\nu_{e} \rightarrow \nu_{e}) \rangle \) as a function of \( \frac{4 m_{\nu}^{2}}{2 E} \) are shown in Figs.2–3, for different values of \( \sin^{2}2\theta_{12}, \sin^{2}2\theta_{13}, \) and \( R \). We use the exponential profile \( N_{\nu}(r) = 245 \exp(\frac{-10.54}{R_{\odot}}) N_{A} cm^{-3} \), where \( N_{A} \) is the Avogadro’s number, \( r \) is the radial distance measured from the center of the Sun, and \( R_{\odot} \) is the solar radius \( (R_{\odot} = 6.96 \times 10^{8} Km) \). Except for those regions close to the center or the surface, this is a good approximation of the electron density in the Sun. We further assume that the \( \nu_{e} \) are produced at \( r_{0} = 0.08635 R_{\odot} \), where \( N_{e} \) is the central electron density predicted by the Standard Solar Model (SSM). In Fig. 4 we show the allowed region in the \( \Delta_{31} vs \sin^{2}2\theta_{13} \) plane obtained by simply plotting the points in the plane with survival probabilities lying within the \( 1\sigma \) range of values extracted from recent joint analyses of the results of SNO and Super-Kamiokande \( (18) \). To produce this region we used the estimated value \( P_{\nu} = (0.34 \pm 0.05) \) for the survival probability, and an average neutrino energy of 5 MeV. We computed similar regions using the survival probabilities estimated by the experiments SAGE, GALLEX and HOMESTAKE, taking the average energies for this experiments as \( (11) \), and looked for an overlap of these regions in the parameter plane (see Fig. 5). No intention to give statistical significance to this region exists, but only it is shown that our result it is consistent with those achieved by rigorous analyses.

Acknowledgments

This work has been supported in part by DGAPA-UNAM under Grant PAPIIT project No. IN109001 and in part by CoNaCyT (México) under Grant No. 130307-E.

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