NOTES ON MIXED TEICHMÜLLER MOTIVES

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Abstract
As a higher genus version of universal mixed elliptic motives by Hain and Matsumoto, we consider mixed Teichmüller motives as certain motivic local systems on the moduli space of pointed curves. We show that the category of mixed Teichmüller motives is equivalent to a full subcategory of a certain product category of mixed Tate motives over \(\mathbb{Z}\) and universal mixed elliptic motives. Furthermore, we show that unipotent fundamental torsors for the universal open curve give rise to a pro-object in the category of mixed Teichmüller motives. Our results can give a realization of motivic correlators proposed by Goncharov.

1. Introduction
The aim of this paper is to study mixed Teichmüller motives defined as a higher genus version of mixed Tate motives (unramified) over \(\mathbb{Z}\) (see Deligne-Goncharov [8] for the definitive reference) and universal mixed elliptic motives by Hain and Matsumoto [16]. For nonnegative integers \(g, n\) such that \(n, 2g - 2 + n > 0\), let \(M_{g,n}\) denote the moduli stack over \(\mathbb{Z}\) of proper smooth curves of genus \(g\) with \(n\) tangent vectors, and \(M_{g,n}^{an}\) denote the associated complex orbifold. Then we consider mixed Teichmüller motives for \(M_{g,n}\) as motivic unipotent local systems on \(M_{g,n}^{an}\) such that their pure parts are generated by the cohomology groups (with Tate twists) of corresponding curves, and that they give mixed Tate motives over \(\mathbb{Z}\) on the points at infinity. From the viewpoint of (arithmetic) algebraic geometry, it seems interesting to study the relationship with sheaves of mixed motives over \(M_{g,n}\) given by Voevodsky, Ayoub and others [33, 1].

One of main results of this paper describes the structure of the category of mixed Teichmüller motives using the theory of Teichmüller groupoid for \(M_{g,n}\) defined as its fundamental groupoid with tangential base points at infinity (cf. [2, 3, 10, 13, 17, 21, 27, 30]). We describe the category of mixed Teichmüller motives in terms of Teichmüller’s Lego game proposed by Grothendieck [13]. Namely, we show that the category of mixed Teichmüller motives is equivalent to a full subcategory of a certain product category of mixed Tate motives over \(\mathbb{Z}\) and universal mixed elliptic motives. As its application, by works of Brown [5], Terasoma [32], Deligne-Goncharov [8] and Hain-Matsumoto [16] on mixed Tate motives over \(\mathbb{Z}\) and universal mixed elliptic motives, one can describe the motivic Galois group of the category of mixed Teichmüller motives.

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Another main result gives important examples of mixed Teichmüller motives. Denote by $\mathcal{C}$ the universal curve over $\mathcal{M}_{g,\vec{n}}$ with sections (and tangential vectors) $\sigma_1, \ldots, \sigma_n$, and put

$$\mathcal{C}^o = \mathcal{C} - \bigcup_{k=1}^n \text{Im}(\sigma_k).$$

Then for $i, j \in \{1, \ldots, n\}$, the sets $\pi_1(C^o_s; (\sigma_i)_s, (\sigma_j)_s) \ (s \in \mathcal{M}_{g,\vec{n}})$ of homotopy classes of paths from $(\sigma_i)_s$ to $(\sigma_j)_s$ give a motivic local system on $\mathcal{M}_{g,\vec{n}}$ as a torsor over $\varprojlim \mathbb{Q}[\pi_1(C^o_s)]/I^m$, where $I$ denotes the augmentation ideal of $\mathbb{Q}[\pi_1(C^o_s)]$. We show that this local system gives rise to a pro-object in the category of mixed Teichmüller motives for $\mathcal{M}_{g,\vec{n}}$. Therefore, one has the associated motivic monodromy representation which substantially gives the motivic correlators proposed by Goncharov [11] under the existence of the abelian category of mixed motives over fields.

2. Generalized Tate curve

2.1. Degenerate curve

We review the well known correspondence between certain graphs and degenerate pointed curves, where a (pointed) curve is called degenerate if it is a stable (pointed) curve and the normalization of its irreducible components are all projective (pointed) lines. A graph $\Delta = (V, E, T)$ means a collection of 3 finite sets $V$ of vertices, $E$ of edges, $T$ of tails and 2 boundary maps

$$b : T \to V, \quad b : E \longrightarrow (V \cup \{\text{unordered pairs of elements of } V\})$$

such that the geometric realization of $\Delta$ is connected. A graph $\Delta$ is called stable if its each vertex has degree $\geq 3$, i.e. has at least 3 branches. Then for a degenerate pointed curve, its dual graph $\Delta = (V, E, T)$ is given by the correspondence:

$$V \longleftrightarrow \{\text{irreducible components of the curve}\},$$

$$E \longleftrightarrow \{\text{singular points on the curve}\},$$

$$T \longleftrightarrow \{\text{marked points on the curve}\}$$

such that an edge (resp. a tail) of $\Delta$ has a vertex as its boundary if the corresponding singular (resp. marked) point belongs to the corresponding component. Denote by $\sharp X$ the number of elements of a finite set $X$. Under fixing a bijection $\nu : T \sim \{1, \ldots, \sharp T\}$, which we call a numbering of $T$, a stable graph $\Delta = (V, E, T)$ becomes the dual graph of a degenerate $\sharp T$-pointed curve of genus rank$_\mathbb{Z}H_1(\Delta, \mathbb{Z})$ and that each tail $h \in T$ corresponds to the $\nu(h)$th marked point. In particular, a stable graph without tail is the dual graph of a degenerate (unpointed) curve by this correspondence. If $\Delta$ is trivalent, i.e. any vertex of $\Delta$ has just 3 branches, then a degenerate $\sharp T$-pointed curve with dual graph $\Delta$ is maximally degenerate.

An orientation of $\Delta = (V, E, T)$ means giving an orientation of each $e \in E$. Under an orientation of $\Delta$, denote by

$$\pm E = \{e, -e \mid e \in E\}$$
the set of oriented edges, and by $v_h$ the terminal vertex of $h \in \pm E$ (resp. the boundary vertex $b(h)$ of $h \in T$). If $e$ is a loop, then in fact $v_e = v_{-e}$. For each $h \in \pm E$, let $|h| \in E$ be the edge $h$ without orientation.

2.2. Universal Schottky group

Let $\Delta = (V, E)$ be a stable graph without tail. Fix an orientation of $\Delta$, and take a subset $E$ of $\pm E$ whose complement $E_\infty$ satisfies the condition that

$$E_\infty \cap \{-h \mid h \in E_\infty\} = \emptyset,$$

and that $v_h \neq v_{h'}$ for any distinct $h, h' \in E_\infty$. We attach variables $\alpha_h$ for $h \in E$ and $q_e = q_{-e}$ for $e \in E$. Let $A_0$ be the $\mathbb{Z}$-algebra generated by $\alpha_h$ ($h \in E$), $1/(\alpha_e - \alpha_{-e})$ ($e, -e \in E$) and $1/(\alpha_h - \alpha_{h'})$ ($h, h' \in E$ with $h \neq h'$ and $v_h = v_{h'}$), and let

$$A = A_0[[q_e \mid e \in E]], \quad B = A \left[ \prod_{e \in E} q_e^{-1} \right].$$

According to [19, Section 2], we construct the universal Schottky group $\Gamma$ associated with oriented $\Delta$ and $E$ as follows. For $h \in \pm E$, let $\phi_h$ be the element of $PGL_2(B) = GL_2(B)/B^\infty$ given by

$$\phi_h = \frac{1}{\alpha_h - \alpha_{-h}} \begin{pmatrix} \alpha_h - \alpha_{-h} q_h & -\alpha_h \alpha_{-h} (1 - q_h) \\ 1 - q_h & -\alpha_{-h} + \alpha_h q_h \end{pmatrix} \mod(B^\infty),$$

where $\alpha_h$ (resp. $\alpha_{-h}$) means $\infty$ if $h$ (resp. $-h$) belongs to $E_\infty$. Then

$$\frac{\phi_h(z) - \alpha_h}{z - \alpha_h} = \frac{\phi_h(z) - \alpha_{-h}}{z - \alpha_{-h}} \quad (z \in \mathbb{P}^1),$$

where $PGL_2$ acts on $\mathbb{P}^1$ by linear fractional transformation.

2.3. Generalized Tate curve

For any reduced path $\rho = h(1) \cdot h(2) \cdots h(l)$ which is the product of oriented edges $h(1), \ldots, h(l)$ such that $h(i) \neq -h(i + 1)$ and $v_{h(i)} = v_{-h(i+1)}$, one can associate an element $\rho^*$ of $PGL_2(B)$ having reduced expression $\phi_{h(l)} \phi_{h(l-1)} \cdots \phi_{h(1)}$. Fix a base point $v_b$ of $V$, and consider the fundamental group $\pi_1(\Delta, v_b)$ which is a free group of rank $g = \text{rank}_\mathbb{Z} H_1(\Delta, \mathbb{Z})$. Then the correspondence $\rho \mapsto \rho^*$ gives an injective anti-homomorphism $\pi_1(\Delta, v_b) \to PGL_2(B)$ whose image is denoted by $\Gamma$. As is shown in [19, Proposition 1.3], each $\gamma \in \Gamma$ has its attractive (resp. repulsive) fixed point $a$ (resp. $a'$) in $\mathbb{P}^1(\Omega)$ and its multiplier $b \in \sum_{e \in E} A q_e$ which satisfy that

$$\frac{\gamma(z) - a}{z - a} = b \frac{\gamma(z) - a'}{z - a'} \quad (z \in \mathbb{P}^1(\Omega)).$$

It is shown in [19, Section 3] (and had been shown in [22, Section 2] when $\Delta$ is trivalent and has no loop) that for any stable graph $\Delta = (V, E)$ without tail, there exists a stable curve $C_\Delta$ of genus $g$ over $A$ which satisfies the following:
• The closed fiber $C_\Delta \otimes_A A_0$ of $C_\Delta$ obtained by substituting $q_e = 0$ ($e \in E$) becomes the degenerate curve over $A_0$ with dual graph $\Delta$ which is obtained from the collection of $P_v := \mathbb{P}^1_{A_0}$ ($v \in V$) by identifying the points $\alpha_e \in P_{v_e}$ and $\alpha_{-e} \in P_{v_{-e}}$ ($e \in E$), where $\alpha_h$ denotes $\infty$ if $h \in E_\infty$.

• $C_\Delta$ gives a universal deformation of degenerate curves with dual graph $\Delta$, i.e. if $R$ is a noetherian and normal complete local ring with residue field $k$, and $C$ is a stable curve over $R$ with nonsingular generic fiber such that the closed fiber $C \otimes_R k$ is a degenerate curve with dual graph $\Delta$, in which all double points are $k$-rational, then there exists a ring homomorphism $A \to R$ giving $C_\Delta \otimes_A R \cong C$.

• $C_\Delta \otimes_A B$ is smooth over $B$ and is Mumford uniformized (cf. [28]) by $\Gamma$.

• Let $\alpha_h$ ($h \in \mathcal{E}$) be complex numbers such that $\alpha_e \neq \alpha_{-e}$ and that $\alpha_h \neq \alpha_{h'}$ if $h \neq h'$ and $v_h = v_{h'}$. Then for nonzero complex numbers $q_e$ ($e \in E$) with sufficiently small absolute value, $C_\Delta$ becomes a Riemann surface which is Schottky uniformized (cf. [31]) by the Schottky group $\Gamma$ over $C$.

We apply the above result to the construction of a uniformized deformation of a degenerate pointed curve (see [22, Section 2, Theorems 1 and 10] when the degenerate pointed curve consists of smooth pointed projective lines). Let $\Delta = (V,E,T)$ be a stable graph with numbering $\nu$ of $T$. We define its extension $\tilde{\Delta} = (\tilde{V},\tilde{E})$ as a stable graph without tail by adding a vertex with a loop to the end distinct from $v_h$ for each tail $h \in T$. Then from the uniformized curve associated with $\tilde{\Delta}$, by substituting 0 for the deformation parameters which correspond to $e \in \tilde{E} - E$ and by replacing the singular projective lines which correspond to $v \in \tilde{V} - V$ with marked points, one has the required universal deformation.

3. Teichmüller groupoid

3.1. Moduli space of pointed curves

We review fundamental facts on the moduli space of pointed curves and its compactification [9, 24, 23]. Let $g$ and $n$ be nonnegative integers such that $n$ and $2g - 2 + n$ are positive. As in the introduction, $M_{g,n}$ (resp. $M_{g,\vec{n}}$) denote the moduli stacks over $\mathbb{Z}$ classifying proper smooth curves of genus $g$ with $n$ marked points (resp. $n$ marked points with nonzero tangent vectors). Then $M_{g,\vec{n}}$ becomes naturally a principal $(\mathbb{G}_m)^n$-bundle on $M_{g,n}$. Furthermore, let $\overline{M}_{g,n}$ denote the Deligne-Mumford-Knudsen compactification of $M_{g,n}$ which is defined as the moduli stack over $\mathbb{Z}$ classifying stable curves of genus $g$ with $n$ marked points, and $\overline{M}_{g,\vec{n}}$ denote the $(\mathbb{A}^1)^n$-bundle on $\overline{M}_{g,n}$ obtained by considering (possibly zero) tangent vectors on marked points. For these moduli stacks $M_{g,n}$ and $\overline{M}_{g,n}$, $M_{g,n}^{an}$ and $\overline{M}_{g,n}^{an}$ denote the associated complex orbifolds. A point at infinity on $M_{g,n}$ (resp. $M_{g,\vec{n}}$) is a point on $\overline{M}_{g,n}$ (resp. $\overline{M}_{g,\vec{n}}$) which corresponds to a maximally degenerate $n$-pointed curve, and a tangential point at infinity is a point at infinity with tangential structure over $\mathbb{Z}$.
3.2. Coordinates on moduli space

To obtain explicit local coordinates on the above moduli stacks, we will rigidify a coordinate on each projective line appearing as an irreducible component of the base degenerate curve. In the maximally degenerate case, this process is considered in [22] using the notion of “tangential structure”. A rigidification of an oriented stable graph $\Delta = (V, E, T)$ with numbering $\nu$ of $T$ means a collection $\tau = (\tau_v)_{v \in V}$ of injective maps

$$\tau_v : \{0, 1, \infty\} \to \{h \in \pm E \cup T \mid v_h = v\}$$

such that $\tau_v(a) \neq -\tau_v'(a)$ for any $a \in \{0, 1, \infty\}$ and distinct elements $v, v' \in V$ with $\tau_v(a), \tau_v'(a) \in \pm E$. One can see that any stable graph has a rigidification by the induction on the number of edges and tails. Denote by $A(\Delta, \tau)$ the ring of formal power series ring of $q_e (e \in E)$ over the $\mathbb{Z}$-algebra which is generated by $\alpha_h (h \in E)$, $1/(\alpha_e - \alpha_{-e}) (e, -e \in E - T)$ and $1/(\alpha_h - \alpha_{h'}) (h, h' \in E$ with $h \neq h'$ and $v_h = v_{h'})$, where $E = \pm E \cup T - \bigcup_{v \in V} \text{Im}(\tau_v)$, $\alpha_h = a$ for $h = \tau_v(a)$ ($a \in \{0, 1\}$) and $\alpha_h$ are variables. Then as is stated above (see also [22, 2.3.9]), there exists a stable $\sharp T$-pointed curve $C(\Delta, \tau)$ over $A(\Delta, \tau)$ which is obtained as the quotient by $\pi_1(\Delta)$ of the glued scheme of pointed projective lines associated with the universal cover of $\Delta$. Therefore, $C(\Delta, \tau)$ gives a universal deformation by $q_e (e \in E)$ of the degenerate $\sharp T$-pointed curve with dual graph $\Delta$.

Note that if one takes another rigidification, then $q_e (e \in E)$ may give different deformation parameters of the degenerate pointed curve, and these parameters associated with distinct rigidifications can be compared (cf. [20, Section 2]).

Let $\tau$ be a rigidification of an oriented stable graph $\Delta = (V, E, T)$ with numbering of $T$, and put

$$\mathcal{E}_\tau = \pm E \cup T - \bigcup_{v \in V} \text{Im}(\tau_v).$$

Then $\alpha_h (h \in \mathcal{E}_\tau)$ and $q_e (e \in E)$ give effective parameters of the moduli and the deformation of degenerate $\sharp T$-pointed curves with dual graph $\Delta$ respectively. Therefore, if we put $g = \text{rank}_\mathbb{Z} H_1(\Delta, \mathbb{Z})$ and $n = \sharp T$, then

$$(\alpha_h (h \in \mathcal{E}_\tau), q_e (e \in E))$$

gives a system of formal coordinates on an étale neighborhood of $Z_\Delta$, where $Z_\Delta$ denotes the substack of $\overline{\mathcal{M}}_{g,n}$ classifying degenerate $n$-pointed curves with dual graph $\Delta$. Furthermore, by the above result, this system gives local coordinates on an étale neighborhood of the complex orbifold $Z^\text{an}_\Delta$ associated with $Z_\Delta$.

3.3. Teichmüller groupoid

Let $\mathcal{M}^\text{an}_{g,n}$ denote the complex orbifold associated with $\mathcal{M}_{g,n}$ as above. Then the Teichmüller groupoid for $\mathcal{M}_{g,n}$ is defined as the fundamental groupoid for $\mathcal{M}^\text{an}_{g,n}$ with tangential base points which correspond to maximally degenerate pointed curves. Its
fundamental paths called basic moves are half-Dehn twists, fusing moves and simple moves defined as follows.

Let $\Delta = (V, E, T)$ be a trivalent graph as above, and assume that $\Delta$ is trivalent. Then for any rigidification $\tau$ of $\Delta$, $\pm E \cup T = \bigcup_{v \in V} \text{Im}(\tau_v)$, and hence $A_{(\Delta, \tau)}$ is the formal power series ring over $\mathbb{Z}$ of $3g - 3 + n$ variables $q_e$ ($e \in E$). First, the half-Dehn twist $\delta_{e/2}$ associated with $e$ is defined as the deformation of the pointed Riemann surface corresponding to $C_\Delta$ by $q_e \mapsto -q_e$. Second, a fusing move (or associative move, A-move) is defined to be different degeneration processes of a 4-fold Riemann sphere. A fusing move changes $(\Delta, e)$ to another trivalent graph $(\Delta', e')$ such that $\Delta, \Delta'$ become the same graph, which we denote by $\Delta''$. We denote this move by $\varphi(e, e')$. In [20, Section 2], this move is constructed using $C_{\Delta''}$. Finally, simple move (or S-move) is defined to be different degeneration processes of a 1-fold complex torus.

Then as the completeness theorem called in [27], the following assertion is conjectured in [13] and shown in [2, 3, 10, 17, 27, 30] (especially in [30, Sections 7 and 8] using the notion of quilt-decompositions of Riemann surfaces).

**Completeness Theorem**
The Teichmüller groupoid is generated by half-Dehn twists, fusing moves and simple moves with relations induced from $\mathcal{M}_{0,\vec{3}}, \mathcal{M}_{0,\vec{5}}, \mathcal{M}_{1,\vec{1}}$ and $\mathcal{M}_{1,\vec{2}}$.

4. Mixed Teichmüller motives

4.1. Definition of mixed Teichmüller motives
Let $\mathbb{H} = (\mathbb{H}^{\text{Be}}, \mathbb{H}^{\text{dR}})$ denote the (pure) motivic local system on $\mathcal{M}_{g,\vec{n}}$ associated with $R^1\pi_*\mathbb{Q}$ for the universal proper smooth curve $\pi : C \to \mathcal{M}_{g,\vec{n}}$. Namely, $\mathbb{H}^{\text{Be}}$ is the $\mathbb{Q}$-local system on $\mathcal{M}_{g,\vec{n}}$ given by $\mathbb{H}^{\text{Be}} = H^1_{\text{Be}}(C_s, \mathbb{Q})$ ($s \in \mathcal{M}_{g,\vec{n}}$), and $\mathbb{H}^{\text{dR}}$ is the vector bundle on $\mathcal{M}_{g,\vec{n}/\mathbb{Q}} = \mathcal{M}_{g,\vec{n}} \otimes \mathbb{Q}$ with Gauss-Manin connection given by $\mathbb{H}^{\text{dR}} = H^1_{\text{dR}}(C_s)$ ($s \in \mathcal{M}_{g,\vec{n}/\mathbb{Q}}$) with canonical isomorphism $\mathbb{H}^{\text{dR}} \otimes \mathbb{C} \cong \mathbb{H}^{\text{dR}} \otimes \mathbb{C}$.

Definition. A mixed Teichmüller motive $\mathbb{V}$ (over $\mathbb{Z}$) for $\mathcal{M}_{g,\vec{n}}$ is defined as a unipotent motivic local system

$$\mathbb{V} = \left( \mathbb{V}^{\text{Be}}, \mathbb{V}^{\text{dR}} \right),$$

namely

(i) $\mathbb{V}^{\text{Be}}$ is a $\mathbb{Q}$-local system on $\mathcal{M}_{g,\vec{n}}$,

(ii) $\mathbb{V}^{\text{dR}}$ is a filtered vector bundle $\mathbb{V}$ on $\overline{\mathcal{M}}_{g,\vec{n}/\mathbb{Q}}$ with Hodge filtration $F^*$ and flat connection

$$\nabla : \mathbb{V} \to \mathbb{V} \otimes \Omega^1_{\overline{\mathcal{M}}_{g,\vec{n}/\mathbb{Q}}} \left( \log \left( \mathcal{D}_{g,\vec{n}/\mathbb{Q}} \right) \right); \mathcal{D}_{g,\vec{n}} = \overline{\mathcal{M}}_{g,\vec{n}} - \mathcal{M}_{g,\vec{n}}$$
which has nilpotent residue along each component of $\mathcal{D}_{g,\vec{n}/\mathbb{Q}}$ and satisfies the Griffith transversality:

$$\nabla : F^p \mathcal{V} \to F^{p-1} \mathcal{V} \otimes \Omega^1_{\mathcal{M}_{g,\vec{n}/\mathbb{Q}}} \left( \log \left( \mathcal{D}_{g,\vec{n}/\mathbb{Q}} \right) \right)$$

such that

(iii) there exists an increasing weight filtration $W_\bullet$ of $\mathcal{V}$ such that

$$\bigcup_m W_m \mathcal{V} = \mathcal{V}, \quad \bigcap_m W_m \mathcal{V} = \{0\}$$

and that each weight graded quotient $\text{Gr}_W^m \mathcal{V}$ is isomorphic to a direct sum of subquotients of copies (with multiplicities) of $H \otimes (m+2r)$.

(iv) there exists a comparison isomorphism between $(\mathcal{V}^{\text{dR}}, W_\bullet) \otimes \mathcal{O}_{\mathcal{M}_{g,\vec{n}}}$ with the canonical extension of the filtered flat bundle $(\mathcal{V}^{\text{Be}}, W_\bullet) \otimes \mathcal{O}_{\mathcal{M}_{g,\vec{n}}}$ to $\mathcal{M}_{g,\vec{n}}$.

Furthermore, $\mathcal{V}$ is required to have the structure of mixed Tate motives over $\mathbb{Z}$ on points at infinity as follows:

(v) for each tangential point $t$ at infinity on $\mathcal{M}_{g,\vec{n}}$, there exists a mixed Tate motive $V(t)$ over $\mathbb{Z}$ satisfying

- the fiber $\mathcal{V}^{\text{dR}}_t$ over $t$ gives the de Rham realization $V(t)^{\text{dR}}$ of $V(t)$,
- the monodromy representation associated with $\mathcal{V}^{\text{Be}}$ gives a homomorphism $\pi_1(\mathcal{M}_{g,\vec{n}}^\text{an}; t) \to \text{Aut}(V(t)^{\text{Be}})$, where $V(t)^{\text{Be}}$ denotes the Betti realization of $V(t)$,
- the fiber of the isomorphism in (iv) over $t$ gives the comparison isomorphism $(V(t)^{\text{dR}}, W_\bullet) \otimes \mathbb{C} \cong (V(t)^{\text{Be}}, W_\bullet) \otimes \mathbb{C}$ for the motive $V(t)$,

(vi) for each prime number $l$ and tangential points $t_1, t_2$ at infinity on $\mathcal{M}_{g,\vec{n}}$, the monodromy isomorphism $V(t_1)^{\text{Be}} \sim V(t_2)^{\text{Be}}$ along a path in $\mathcal{M}_{g,\vec{n}}^\text{an}$ from $t_1$ to $t_2$ gives rise to a $\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-equivariant isomorphism $V(t_1)^l \sim V(t_2)^l$, where $V(t)^l$ denotes the $l$-adic realization of $V(t)$.

The category of mixed Teichmüller motives for $\mathcal{M}_{g,\vec{n}}$ is a $\mathbb{Q}$-linear neutral tannakian category which we denote by $\text{MTeM}_{g,\vec{n}}$.

A mixed Tate local system on $(\mathbb{G}_m)^d$ is defined as a unipotent motivic local system on $(\mathbb{G}_m)^d$ which satisfy the above conditions (i)–(v) replacing $\mathcal{M}_{g,\vec{n}}, \mathcal{M}_{g,\vec{n}}$ by $(\mathbb{G}_m)^d$, $(\mathbb{A}^1)^d$ respectively. A mixed Teichmüller motive for $\mathcal{M}_{g,\vec{n}}$ gives a mixed Tate local system on $(\mathbb{G}_m)^{3g-3+2n}$ around each point at infinity.

PROPOSITION 4.1
The category of mixed Tate local systems on $(\mathbb{G}_m)^d$ is equivalent to the category of mixed Tate motives over $\mathbb{Z}$.

Proof
First, assume that $d = 1$, and let $V$ be a mixed Tate local system on $\mathbb{G}_m = \mathbb{P}^1 - \{0, \infty\}$. Extend $V$ to a unipotent $\mathbb{Q}$-local system on $\mathbb{P}^1 - \{0, 1, \infty\}$ with trivial monodromy around 1 which we denote by the same symbol. Then by a result of Brown [5], one may assume that the fiber $V$ of $V$ over $t = 0\bar{1}$ is derived from the motivic fundamental group of $\mathbb{P}^1 - \{0, 1, \infty\}$. Since $V$ gives a torsor of $V_t$ by a closed path around 0, it is an extension of $\mathbb{Q}(0)$ by $V_t$ as a unipotent $\mathbb{Q}$-local system on $\mathbb{P}^1 - \{0, 1, \infty\}$. Therefore, $V$ becomes the Betti trivialization of a mixed Tate motive over $\mathbb{Z}$, and hence the assertion for any $d$ follows from induction. □

4.2. Monodromy for mixed Teichmüller motives
Let $\Delta$ be a trivalent graph of type $(g, n)$, namely satisfying $\text{rank}_{\mathbb{Z}} H_1(\Delta, \mathbb{Z}) = g$ and having a bijection $\{\text{tails of } \Delta\} \sim \{1, \ldots, n\}$. Let $\Delta'$ be a trivalent subtree of $\Delta$ with $m$ tails with numbering from 1 to $m$, and denote by $C'$ the (possibly nonconnected) maximally degenerate pointed curve with dual graph $\Delta - (\Delta'$ without tails). Then by attaching $C'$ to stable curves over $\overline{\mathcal{M}}_{0, m}$ under this numbering, one has a morphism

$$\iota_{\Delta, \Delta'} : \overline{\mathcal{M}}_{0, m} \to \overline{\mathcal{M}}_{g, \vec{n}}.$$ 

PROPOSITION 4.2
Let $V$ be a mixed Teichmüller motive for $\mathcal{M}_{g, \vec{n}}$, and denote by $V$ the associated vector bundle on $\overline{\mathcal{M}}_{g, \vec{n}}/\mathbb{Q}$. Then the pullback $(\iota_{\Delta, \Delta'})^*(V)$ of $V$ by $\iota_{\Delta, \Delta'}$ becomes a trivial bundle on $\overline{\mathcal{M}}_{0, m}$, and this trivialization is unique up to a change of basis of a fiber of $V$. Furthermore, if $W$ is a mixed Teichmüller submotive of $V$, then for the vector bundle $W$ associated with $W$, the trivialization of $(\iota_{\Delta, \Delta'})^*(W)$ is compatible with that of $(\iota_{\Delta, \Delta'})^*(V)$.

Proof
Denote the canonical extension of $\mathbb{H}$ to $\overline{\mathcal{M}}_{g, \vec{n}}$ by the same symbol. Then $(\iota_{\Delta, \Delta'})^*(\mathbb{H})$ has the weight filtration:

$$0 \to \mathbb{Q}^{\oplus(2g+n-m)} \to (\iota_{\Delta, \Delta'})^*(\mathbb{H}) \to \mathbb{Q}(-1)^{\oplus(m-1)} \to 0.$$ 

Since $\overline{\mathcal{M}}_{0, m}$ is rational, $H^1(\overline{\mathcal{M}}_{0, m}, O_{\overline{\mathcal{M}}_{0, m}}) = \{0\}$, and hence $(\iota_{\Delta, \Delta'})^*(\mathbb{H})$ is trivialized. Therefore, by the above condition (iii), $(\iota_{\Delta, \Delta'})^*(V)$ is also trivialized. The uniqueness follows from that any morphism from the complete variety $\overline{\mathcal{M}}_{0, m}$ to the affine variety $GL_r$ ($r = \text{rank}(V)$) becomes a constant map.

Let $W$ be as above. Then $(\iota_{\Delta, \Delta'})^*(W)$ and $(\iota_{\Delta, \Delta'})^*(V/W)$ are trivialized, and hence by the vanishing of $H^1(\overline{\mathcal{M}}_{0, m}, O_{\overline{\mathcal{M}}_{0, m}})$, $(\iota_{\Delta, \Delta'})^*(V)$ is uniquely trivialized in a compatible way with the above trivializations. □

COROLLARY 4.3
Let $V$ and $V'$ be as above. Then $V$ can be trivialized along all fusing moves in a unique way up to a change of basis of a fiber of $V$. Furthermore, if $W$ is a mixed Teichmüller submotive of $V$, then for the vector bundle $W$ associated with $W$, the trivialization of $W$ is compatible with that of $V$.

**Proof**

By the completeness theorem, the points at infinity on $\mathcal{M}_{g,n}^{an}$, are connected by compositions of fusing moves and half-Dehn twists with relations induced from $\mathcal{M}_{0,4}^{an}$ and $\mathcal{M}_{0,5}^{an}$. Then the assertion follows from Proposition 4.2.

For elements $A, B$ of $\text{End}_C(C^r)$, we consider the differential equation

$$G'(t) = \left(\frac{A}{t} + \frac{B}{t-1}\right) G(t) \quad (0 < t < 1),$$

and take its $\text{End}_C(C^r)$-valued solutions $G_i(t)$ $(i = 0, 1)$ which are normalized in the sense that

$$\lim_{t \to 0} \frac{G_0(t)}{t^A} = \lim_{t \to 1} \frac{G_1(t)}{(1-t)^B} = 1.$$  

Then the connection matrix $\Phi(A, B)$ is defined as $G_1(t) - G_0(t) \cdot t$.

**THEOREM 4.4**

Denote by $\delta^1_\ell/2$ and $\varphi(e, e')$ the half Dehn twist and the fusing move respectively given in 3.3. For a mixed Teichmüller motive $V$, let $V$ be the associated vector bundle with connection $\nabla$ on $\mathcal{M}_{g,n}^{an}/\mathbb{Q}$. Then under a trivialization of $V$ by $C^r$ along $\varphi(e, e')$, the monodromy of $\nabla$ along $\delta^1_\ell/2$ and $\varphi(e, e')$ are expressed as $\exp \left(\pi \sqrt{-1} \cdot \text{Res}_e(\nabla)\right)$ and $\Phi(\text{Res}_e(\nabla), \text{Res}_{e'}(\nabla))$ respectively, where $\text{Res}_e(\nabla)$ denotes the residue of $\nabla$ around $g_e = 0$.

**Proof**

This assertion follows from [21, Proposition 1 and Theorem 2].

**4.3. Category of mixed Teichmüller motives**

Let $\mathcal{T}_{g,n}$ be the set of trivalent graphs of type $(g, n)$ which is identified with the set of points at infinity on $\mathcal{M}_{g,n}$, and $\mathcal{L}_{g,n}$ be the set of loops in all trivalent graphs of type $(g, n)$. Then there is a natural map $\mathcal{L}_{g,n} \to \mathcal{T}_{g,n}$ which sends each $\ell \in \mathcal{L}_{g,n}$ to the element $t_\ell$ of $\mathcal{T}_{g,n}$ containing $\ell$. Let $V$ be a mixed Teichmüller motive for $\mathcal{M}_{g,n}$. Then for each $t \in \mathcal{T}_{g,n}$, $V_t$ denotes the mixed Tate motive over $\mathbb{Z}$ given by the fiber of $V$ at the point at infinity corresponding to $t$. By the definition of universal mixed elliptic motives [16], for each $\ell \in \mathcal{L}_{g,n}$, the restriction of $V$ to the space $\mathcal{M}_{1,1}$ corresponding to $\ell$ gives rise to a universal mixed elliptic motive which we denote by $V_{t_\ell}$.

We define the $(g,n)$-type product category $\text{PTE}_{g,n}$ of mixed Tate motives over $\mathbb{Z}$ and universal mixed elliptic motives as follows:

- Each object consists of mixed Tate local systems $\mathbb{V}(t)$ on $(\mathbb{G}_m)^{3g-3+2n}$ indexed by $t \in \mathcal{T}_{g,n}$ and universal mixed elliptic motives $\mathbb{V}(\ell)$ indexed by $\ell \in \mathcal{L}_{g,n}$ with...
identifications $V(t)^{dR} \cong V$ as $\mathbb{Q}$-vector spaces for a fixed $\mathbb{Q}$-vector space $V$. Furthermore, for each $\ell \in \mathcal{L}_g,n$, the fiber of $V(\ell)$ over the point at infinity on $\mathcal{M}_{1,\overline{1}}$ is identified with the restriction of $V(t)$.  

- Each morphism $(V(t), V(l)) \to (V'(t), V'(l))$ consists of morphisms $V(t) \to V'(t)$ ($t \in T_{g,n}$) as mixed Tate motives over $\mathbb{Z}$ and those $V(l) \to V'(l)$ ($\ell \in \mathcal{L}_{g,n}$) as universal mixed elliptic motives which are compatible with the above structure of these objects.

**Theorem 4.5**

By the functor
\[
V \mapsto (V_t (t \in T_{g,n}), V_{\ell} (\ell \in \mathcal{L}_{g,n})),
\]
$\text{MTeM}_{g,\overline{n}}$ is equivalent to a full subcategory of $\text{PTE}_{g,\overline{n}}$.

**Proof**

By Corollary 4.3, $(V_t, V_{\ell})$ becomes an object of $\text{PTE}_{g,\overline{n}}$. Therefore, to prove the assertion, it is enough to show that for any object $V = (V^\text{Be}, V^\text{dR} = (V, V))$ of $\text{MTeM}_{g,\overline{n}}$,
\[
\text{Hom}_{\text{MTeM}_{g,\overline{n}}} (\mathbb{Q}(0), V) \to \text{Hom}_{\text{PTE}_{g,\overline{n}}} (\mathbb{Q}(0), V)
\]
is an isomorphism. Let $\psi : \mathbb{Q}(0) \to V$ be a nonzero homomorphism in $\text{PTE}_{g,\overline{n}}$. Then for the trivialization $V$ of $\mathcal{V}$, the subspace of $V$ corresponding to $\psi$ is stable under the action of $\text{Res}_e(\mathcal{V})$, and hence by Theorem 4.4, $\psi$ gives a 1-dimensional trivial subbundle of $V^\text{dR}$ along all fusing moves. By definition, $\psi$ also corresponds to a 1-dimensional trivial subbundle of $V^\text{dR}$ along each simple move. Therefore, by the completeness theorem, there exists uniquely a 1-dimensional trivial subbundle of $V^\text{dR}$ corresponding to $\psi$ on $\overline{\mathcal{V}}_{g,\overline{n}}$ which gives nonzero homomorphism $\mathbb{Q}(0) \to V$ in $\text{MTeM}_{g,\overline{n}}$. □

The $\mathbb{Q}$-linear neutral tannakian categories $\text{MTeM}_{g,\overline{n}}$ and $\text{PTE}_{g,\overline{n}}$ are identified with the categories of representations of affine group schemes which we will denote by $\pi_1 (\text{MTeM}_{g,\overline{n}})$ and $\pi_1 (\text{PTE}_{g,\overline{n}})$ respectively, and call the motivic Galois groups.

**Corollary 4.6**

There exists a natural surjective homomorphism
\[
\pi_1 (\text{PTE}_{g,\overline{n}}) \to \pi_1 (\text{MTeM}_{g,\overline{n}}).
\]

**Proof**

The assertion follows from Theorem 4.5 which states that $\text{MTeM}_{g,\overline{n}}$ is a full subcategory of $\text{PTE}_{g,\overline{n}}$. □

**5. Motivic fundamental torsors**

**5.1. Fundamental torsors of curves**

Assume that $n$ is a positive integer, and let $C^o$ be an algebraic curve over a subfield $K$ of $\mathbb{C}$ which is obtained from a proper smooth curve $C$ of genus $g$ by removing $n$
points. Note that $C^\circ$ is not complete, and hence its first de Rham cohomology group $H^1_{\text{dR}}(C^\circ)$ has a basis $B_{C^\circ}$ consisting of $2g + n - 1$ meromorphic 1-forms on $C$ of the first or second kind which may have poles outside $C^\circ$. For $w_1, ..., w_m \in B_{C^\circ}$, we define a $\mathcal{D}$-module $D(w_1, ..., w_m)$ on $C^\circ$ whose underlying bundle is given by the trivial bundle $K^{m+1} \times C^\circ$ with connection $d - \sum_{i=1}^m e_{i,i+1} w_i$, where $e_{i,j}$ denotes the square matrix of degree $m+1$ whose $(k, l)$-entry is $\delta_{kl} \cdot \delta_{ij}$ (where $\delta_{ij}$ denotes Kronecker’s delta). We consider the tannakian subcategory of $\mathcal{D}$-modules on $C^\circ$ generated by $D(w_1, ..., w_m)$ ($w_1, ..., w_m \in B_{C^\circ}$). Since the underlying bundles of objects in this category are trivial, for each $K$-rational (tangential) point $x$ on $C^\circ$, one can define the fiber functor on this category by taking the (trivial) fibers at $x$. Denote by $\pi_1^{\text{dR}}(C^\circ; x)$ the tannakian fundamental group of this category which is a pro-finite algebraic group over $K$, and by $A^{\text{dR}}(C^\circ; x)$ the enveloping algebra of the Lie algebra $\text{Lie}(\pi_1^{\text{dR}}(C^\circ; x))$.

Let $(C^\circ)^{\text{an}}$ be a Riemann surface associated with $C^\circ \otimes_K \mathbb{C}$, and for each (tangential) points $x, y$ on $C^\circ$, denote by $\pi_1((C^\circ)^{\text{an}}; x, y)$ the set of homotopy classes of paths from $x$ to $y$ in $(C^\circ)^{\text{an}}$. When $x = y$, $\pi_1((C^\circ)^{\text{an}}; x, y)$ becomes the fundamental group $\pi_1((C^\circ)^{\text{an}}; x)$ of $(C^\circ)^{\text{an}}$ with base point $x$. We consider the tannakian category of unipotent local systems on $(C^\circ)^{\text{an}}$ with fiber functor obtained from taking the fiber at $x$. Then it is shown in [7] that its tannakian fundamental group $\pi_1^{\text{Be}}((C^\circ)^{\text{an}}; x)$ is a pro-algebraic group over $\mathbb{Q}$, and the associated enveloping algebra $A^{\text{Be}}((C^\circ)^{\text{an}}; x)$ of $\text{Lie}(\pi_1^{\text{Be}}((C^\circ)^{\text{an}}; x))$ is isomorphic to

$$\lim_{m \to \infty} \mathbb{Q}[\pi_1((C^\circ)^{\text{an}}; x)] / \Gamma^m,$$

where $\Gamma$ denotes the augmentation ideal of the group ring $\mathbb{Q}[\pi_1((C^\circ)^{\text{an}}; x)]$. Since $\pi_1((C^\circ)^{\text{an}}; x)$ is a free group of rank $2g + n - 1$, $A^{\text{Be}}((C^\circ)^{\text{an}}; x)$ becomes the ring of noncommutative formal power series over $\mathbb{Q}$ in $2g + n - 1$ variables.

**PROPOSITION 5.1**

For a $K$-rational (tangential) point $x$ on $C^\circ$, there exists a canonical isomorphism

$$A^{\text{dR}}(C^\circ; x) \otimes_K \mathbb{C} \cong A^{\text{Be}}((C^\circ)^{\text{an}}; x) \otimes_{\mathbb{Q}} \mathbb{C}.$$

Consequently, $\text{Lie}(\pi_1^{\text{dR}}(C^\circ; x)) \otimes_K \mathbb{C}$ is isomorphic to $\text{Lie}(\pi_1^{\text{Be}}((C^\circ)^{\text{an}}; x)) \otimes_{\mathbb{Q}} \mathbb{C}$, and $\pi_1^{\text{dR}}(C^\circ; x) \otimes_K \mathbb{C}$ is isomorphic to $\pi_1^{\text{Be}}((C^\circ)^{\text{an}}; x) \otimes_{\mathbb{Q}} \mathbb{C}$.

**Proof**

By associating local systems on $(C^\circ)^{\text{an}}$ with $D(w_1, ..., w_m)$ ($w_i \in B_{C^\circ}$), one has a group homomorphism

$$\pi_1^{\text{Be}}((C^\circ)^{\text{an}}; x) \otimes_{\mathbb{Q}} \mathbb{C} \to \pi_1^{\text{dR}}(C^\circ; x) \otimes_K \mathbb{C},$$

which gives a ring homomorphism

$$A^{\text{Be}}((C^\circ)^{\text{an}}; x) \otimes_{\mathbb{Q}} \mathbb{C} \to A^{\text{dR}}(C^\circ; x) \otimes_K \mathbb{C}.$$

Let $B_m((C^\circ)^{\text{an}})$ be the $\mathbb{C}$-vector space of iterated integrals spanned by

$$\int w_1 \cdots w_r \ (w_i \in B_{C^\circ}, \ r \leq m),$$

11
and $H^0(B_m((C^o)_{an}); x)$ be the space consisting of elements of $B_m((C^o)_{an})$ whose restriction to $\{ \text{loops in } (C^o)_{an} \text{ based at } x \}$ is homotopy functional. Then it is shown in [15, (5.3)] that $H^0(B_m((C^o)_{an}); x)$ is the dual space of $\mathbb{C}[\pi_1((C^o)_{an}; x)]/I^{m+1}$, and that by taking leading terms of iterated integrals, one has an exact sequence

$$0 \to H^0(B_{m-1}((C^o)_{an}); x) \to H^0(B_m((C^o)_{an}); x) \to H^1_{\text{dR}}((C^o)_{an})^{\otimes m}.$$  

Since $H^1_{\text{dR}}(C^o) \otimes_{\mathbb{K}} \mathbb{C} \cong H^1_{\text{dR}}((C^o)_{an}) \cong H^1_{\text{Be}}((C^o)_{an}) \otimes_{\mathbb{Q}} \mathbb{C}$, one has a ring homomorphism

$$A^\text{dR}(C^o; x) \otimes_{\mathbb{K}} \mathbb{C} \to A^\text{Be}((C^o)_{an}; x) \otimes_{\mathbb{Q}} \mathbb{C}$$

which is an inverse map to the above map, and hence this ring homomorphism is an isomorphism. The remaining assertions follow from that the tannakian fundamental group $\pi_1^\text{Be}(\mathbb{Z} = \text{dR, Be})$ and its Lie algebra are given as the subsets of $A^\sharp$ which consist of grouplike elements and primitive elements in $A^\sharp$ respectively. $\square$

For $K$-rational (tangential) points $x, y$ on $C^o$, denote by $\pi_1^\text{dR}(C^o; x, y)$ the tannakian fundamental left (resp. right) torsor over $\pi_1^\text{dR}(C^o; x)$ (resp. $\pi_1^\text{dR}(C^o; y)$). Then one has the associated torsors $\text{Lie}(\pi_1^\text{dR}(C^o; x, y))$ and $A^\text{dR}(C^o; x, y)$ over $\text{Lie}(\pi_1^\text{dR}(C^o; *))$ and $A^\text{dR}(C^o; *)$ ($* = x, y$) respectively. Similarly one can define

$$\pi_1^\text{Be}((C^o)_{an}; x, y), ~ \text{Lie}(\pi_1^\text{Be}((C^o)_{an}; x, y)) , ~ A^\text{Be}((C^o)_{an}; x, y)$$

as the torsors over $\pi_1^\text{Be}((C^o)_{an}; *), ~ \text{Lie}(\pi_1^\text{Be}((C^o)_{an}; *)) , ~ A^\text{Be}((C^o)_{an}; *)$ respectively. Especially, $\pi_1^\text{Be}((C^o)_{an}; x, y)$ is the completion of the $\mathbb{Q}$-vector space $\mathbb{Q}[\pi_1((C^o)_{an}; x, y)]$ generated by $\pi_1((C^o)_{an}; x, y)$ with respect to $I^m(\mathbb{Q}[\pi_1((C^o)_{an}; x, y)])$ ($m \geq 1$). Then Proposition 5.1 implies:

PROPOSITION 5.2

For a $K$-rational (tangential) point $x, y$ on $C^o$, there exists a canonical isomorphism $\pi_1^\text{dR}(C^o; x, y) \otimes_K \mathbb{C} \cong \pi_1^\text{Be}((C^o)_{an}; x, y) \otimes_{\mathbb{Q}} \mathbb{C}$ which gives canonical isomorphisms:

$$\text{Lie}(\pi_1^\text{dR}(C^o; x, y)) \otimes_K \mathbb{C} \cong \text{Lie}(\pi_1^\text{Be}((C^o)_{an}; x, y)) \otimes_{\mathbb{Q}} \mathbb{C},$$

$$A^\text{dR}(C^o; x, y) \otimes_K \mathbb{C} \cong A^\text{Be}((C^o)_{an}; x, y) \otimes_{\mathbb{Q}} \mathbb{C}.$$  

Remark

For a generalized Tate curve, the isomorphism in Proposition 5.2 is a higher genus version of elliptic polylogarithms [4, 6, 25, 26], and is described by the iterated integrals of canonical meromorphic 1-forms given in [18].

5.2. Motivic sheaves of fundamental torsors

Let $\pi : \mathcal{C} \to \mathcal{M}_{g, \vec{n}}$ denote the universal curve with sections $\sigma_1, \ldots, \sigma_n$,

$$\mathcal{C}^o = \mathcal{C} - \bigcup_{k=1}^n \text{Im}(\sigma_k)$$

12
denote the associated open curve and \((C_s = \pi^{-1}(s); (\sigma_1)_s, \ldots, (\sigma_n)_s)\) denote these fibers at \(s \in M_{g, \bar{n}}\).

**THEOREM 5.3**

For \(i, j \in \{1, \ldots, n\}\), \((\sigma_i)_s\) and \((\sigma_j)_s\) denote tangential points on \(C_s^o\). Then

\[
A(C^o; \sigma_i, \sigma_j) = \left( A^{Be}(C_s^o; (\sigma_i)_s, (\sigma_j)_s), A^{dr}(C_s^o; (\sigma_i)_s, (\sigma_j)_s) \right)
\]

gives rise to a pro-object in the category of mixed Teichmüller motives for \(M_{g, \bar{n}}\).

**Proof**

Let \(A^{Be}\) denote the local system on \(M_{g, \bar{n}}^{an}\) given by \(A^{Be}(C_s^o; (\sigma_i)_s, (\sigma_j)_s)\) whose monodromy corresponds to the natural action of the mapping class group \(\pi_1\left(M_{g, \bar{n}}^{an}\right)\) on \(\pi_1((C_s^o)^{an})\). Hence the action of Dehn twists on \(A^{Be}\) is unipotent. Let \(A^{dr}\) denote the vector bundle on \(M_{g, \bar{n}}/\mathbb{Q}\) given by \(A^{dr}(C_s^o; (\sigma_i)_s, (\sigma_j)_s)\). Then by the comparison isomorphism given in Proposition 5.2, there exists the canonical extension of \(A^{dr}\) to \(\overline{M}_{g, \bar{n}}/\mathbb{Q}\) for which the associated flat connection on \(A^{dr}\) has nilpotent residue along each component of \(\mathcal{D}_{g, \bar{n}}/\mathbb{Q}\). Following Hain [15], we recall the construction of the Hodge filtration \(F^p\) and the weight filtration \(W_l\) on \(A = (A^{Be}, A^{dr})\). Put \(D_s = C_s - C_s^o\), and denote by \(E_s^\bullet = E^\bullet(C_s(\log(D_s)))\) the complex of \(C^\infty\) forms on \((C_s)^{an}\) with logarithmic singularities along \(D_s\). Then \(H^0(B_m(E_s^\bullet))\) is canonically isomorphic to the dual space of \(\mathbb{C}[\pi_1((C_s^o)^{an})]/I^m\), and the Hodge, weight filtrations on \(A^{Be} \otimes \mathbb{C}\) (and hence on \(A^{dr} \otimes \mathbb{C}\)) are given by the filtrations

\[
\begin{align*}
F^p E_s^\bullet &= \{\text{forms with } \geq p \text{ holomorphic 1-forms on } C(\log(D_s))\}, \\
W_l E_s^\bullet &= \{\text{forms with } \leq l \text{ 1-forms having logarithmic pole at } D_s\}
\end{align*}
\]

of \(E_s^\bullet\) respectively. Especially, \(F^p\) is spanned by iterated integrals \(\int w_1 \cdots w_r\) for at least \(p\) holomorphic 1-forms on \(C(\log(D_s))\). If \(v\) is a holomorphic tangent vector at \(s \in M_{g, \bar{n}}^{an}\), then the associated derivative \(\nabla_v\) for the above connection on \(A^{dr}\) satisfies

\[
\nabla_v \left( \int w_1 \cdots w_r \right) = \int \nabla_v \left( \int w_1 \cdots w_{r-1} \right) w_r + \int \left( \int w_1 \cdots w_{r-1} \right) \nabla_v(w_r),
\]

and hence \(\nabla_v\) satisfies the Griffith transversality. Since there are a canonical exact sequence

\[
0 \to \mathbb{Q}(1)^{\oplus(n-1)} \to H^1_{Be}((C_s^o)^{an}, \mathbb{Q}) \cong I/I^2 \to H^1_{Be}(C_s^o, \mathbb{Q}) \to 0
\]

and a surjection \((I/I^2)^{\otimes m} \to I^m/I^{m+1},\) each weight graded quotient of \(A\) is isomorphic to a direct sum of subquotients of copies of \(\mathbb{H}^{\otimes(m+2r)}(r)\). Therefore, \(A\) gives a unipotent motivic local system on \(M_{g, \bar{n}}\).

For a tangential point \(t\) at infinity on \(M_{g, \bar{n}}\), \(\pi_1((C_s^o)^{an})\) is the amalgamated product of copies of \(\Pi = \pi_1(P^1_\mathbb{C} - \{0, 1, \infty\})\) for the trivalent graph associated with \(t\). Then
the limit group structure of $\pi_1((C^\circ_s)_{\text{an}})$ is described by the vertex groups $\Pi$ and the edge groups $G_m$, and hence the limit motivic structure of $\mathcal{A} = (\mathcal{A}^{\text{Be}}, \mathcal{A}^{\text{dR}})$ becomes a mixed Tate motive over $\mathbb{Z}$. Let $l$ be a prime number. Then by the theory of algebraic fundamental groups [12], $\mathcal{A}^l = \mathcal{A}^{\text{Be}} \otimes \mathbb{Q}_l$ gives a $l$-adic sheaf with Galois action. Furthermore, by the theorem of van Kampen on fundamental groups of Riemann surfaces and its arithmetic version given by Grothendieck-Murre [14] and Nakamura [29], the $l$-adic geometric fundamental group $\pi^l_1(C \circ t)$ is the amalgamated product of copies of $\pi^l_1(\mathbb{P}^1 - \{0,1,\infty\})$ with Galois action. Therefore, $\mathcal{A}$ gives mixed Tate motives over $\mathbb{Z}$ on points at infinity, and hence is a pro-object in the category of mixed Teichmüller motives. □

5.3. Polylogarithmic motive

We construct the polylogarithmic motive which is a pro-object in the category of universal mixed Teichmüller motives. Let the notation be as in 5.2. Then for each $i \in \{1, \ldots, n\}$,

$$L_s = \text{Lie} \left( \pi^\text{Be}_1(C^\circ_s; (\sigma_i)_s) \right) \quad (s \in \mathcal{M}^\text{an}_{g,\vec{n}})$$

gives a $\mathbb{Q}$-local system $L$ on $\mathcal{M}^\text{an}_{g,\vec{n}}$ whose central series filtration is

$$L^1_s = L_s, \quad L^{k+1}_s = \left[ L^k_s, L_s \right].$$

We define the logarithmic local system $\text{Log}$ as $L^2 / \left[ L^2, L^2 \right]$, and the polylogarithmic local system $\text{Pol}$ as $L / \left[ L^2, L^2 \right]$. Then one has a natural exact sequence

$$0 \to \text{Log} \to \text{Pol} \to \pi^*(\mathcal{H}) \to 0,$$

where $\mathcal{H}_s = H^\text{Be}_1(C^\circ_s, \mathbb{Q})$. By Theorem 5.3, one can see the following:

THEOREM 5.4

For each $i \in \{1, \ldots, n\}$, the $\text{Pol}$ gives rise to a pro-object in the category of mixed Teichmüller motives for $\mathcal{M}_{g,\vec{n}}$.

5.4. Motivic monodromy and correlator

Let $s$ be a point or tangential point at infinity on $\mathcal{M}_{g,\vec{n}/\mathbb{Q}}$, and take $i,j \in \{1, \ldots, n\}$. Since the Betti realization of the fiber over $s$ gives a fiber functor of the category of mixed Teichmüller motives for $\mathcal{M}_{g,\vec{n}}$, by Theorem 5.3, one has the associated group homomorphism

$$\pi_1(\text{MTeM}_{g,\vec{n}}) \to \text{Aut} \left( \mathcal{A}^\text{Be}_{s;i,j} \right); \quad \mathcal{A}^\text{Be}_{s;i,j} = \mathcal{A}^\text{Be}_s(C^\circ_s; (\sigma_i)_s, (\sigma_j)_s).$$

We call this homomorphism a motivic monodromy representation of $\pi_1(\text{MTeM}_{g,\vec{n}})$.

Assume that $s$ is a point at infinity. Then combining the homomorphism in Corollary 4.6, one has the representation

$$\pi_1(\text{PTE}_{g,\vec{n}}) \to \text{Aut} \left( \mathcal{A}^\text{Be}_{s;i,j} \right)$$
whose description can be reduced to the case when \((g, n) = (0, 3), (1, 1)\) by the theorem of van Kampen on fundamental groups of Riemann surfaces.

In [11, 1.11 and Section 10], Goncharov construct the motivic correlator map under the existence of the abelian category of mixed motives over a field. Using our results, one has the motivic correlator map as follows. Assume that \(s \in \mathcal{M}_{g, \vec{n}}\) and \(i = j\), and put \(C = \mathcal{C}_s\) and \(S^* = \{(\sigma_k)_s | k \neq i\}\). Denote by \(\mathcal{CL}_{\text{ie}_{C,s}^S}\), the cyclic envelope of the tensor algebra of \(H^1_{\text{Be}}(C^{an}, \mathbb{Q}) \oplus \mathbb{Q}[S^*]\) modulo shuffle relations. Then by [11, (1.28) and Proposition 8.5], the motivic monodromy representation \(\pi_1(\text{MTeM}_{g, \vec{n}}) \to \text{Aut}(A_{s,i,i}^{\text{Be}})\) gives rise to the motivic correlator map

\[
\text{Cor}_{\text{MTeM}} : \mathcal{CL}_{\text{ie}_{C,s}^S}(1) \to \mathcal{L}_{\text{ie}_{g,\vec{n}}},
\]

where \(\mathcal{L}_{\text{ie}_{g,\vec{n}}}\) is a Lie coalgebra defined as the dual to the Lie algebra of \(\pi_1(\text{MTeM}_{g, \vec{n}})\). Goncharov’s motivic correlator map substantially becomes \(\text{Cor}_{\text{MTeM}}\) since its Hodge realization with period map give the Hodge correlator map

\[
\text{Cor}_{\text{Hod}} : \mathcal{CL}_{\text{ie}_{C,s}^S}(1) \to \mathbb{C}
\]

constructed in [11, 3.2].

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