Abstract We prove some general density statements about the subgroup of invertible points on intermediate jacobians; namely those points in the Abel-Jacobi image of nullhomologous algebraic cycles on projective algebraic manifolds.

Key words: Abel-Jacobi map, intermediate jacobian, normal function, Chow group; 1991 Mathematics Subject Classification: Primary 14C25, Secondary 14C30, 14C35.
Dynamics of Special Points on Intermediate Jacobians

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1 Introduction

Let $X/\mathbb{C}$ be a projective algebraic manifold, $\text{CH}^r(X)$ the Chow group of codimension $r$ algebraic cycles on $X$ (with respect to the equivalence relation of rational equivalence), and $\text{CH}^r_{\text{hom}}(X)$ the subgroup of cycles that are nullhomologous under the cycle class map to singular cohomology with $\mathbb{Z}$-coefficients. Largely in relation to the celebrated Hodge conjecture, as well as with regard to equivalence relations on algebraic cycles, the Griffiths Abel-Jacobi map

$$\Phi_r : \text{CH}^r_{\text{hom}}(X) \to J^r(X)^{\text{Carlson}} \simeq \text{Ext}^1_{\text{MHS}}(\mathbb{Z}(0), H^{2r-1}(X, \mathbb{Z}(r))),$$

has been a focus of attention for the past 60 years. The role of the Abel-Jacobi map in connection to the celebrated Hodge conjecture began with the work of Lefschetz in his proof of his famous “Lefschetz (1, 1) theorem”; and which inspired Griffiths to develop his program of updating Lefschetz’s ideas as a general line of attack on the Hodge conjecture (see [Z1], as well as [L, Lec. 6, 12, 14]). To this day, a precise statement about what the image of $\Phi_r$ is in general seems rather elusive. What we do know is that there are examples where the image of $\Phi_r$ can be a countable set (even infinite dimensional over $\mathbb{Q}$) [Gr], [Cl], or completely torsion [G]. One can ask whether the image of $\Phi_r$
is always dense in \( J^r(X) \), but even that is unlikely to be true in light of some results in the literature inspired by some of the conjectures in [G-H]. In this paper, we seek to come up with a general statement about the image of \( \Phi \), which however is modest, is indeed better than no statement at all. There are two key ideas exploited in this paper, viz., the business of Lefschetz pencils and associated normal functions, and the classical Kronecker’s theorem (see [H-W] (Chapter XXIII)), which we state in the following form:

**Theorem 1.1.** Let \( A = \mathbb{R}^n/\mathbb{Z}^n \) be a compact real torus of dimension \( n \). For a point \( p = (x_1, x_2, ..., x_n) \in A \), \( \mathbb{Z}p = \{ kp : k \in \mathbb{Z} \} \) is dense in \( A \) if and only if \( 1, x_1, x_2, ..., x_n \) are linearly independent over \( \mathbb{Q} \). In particular, the set

\[
\{ p \in A : \mathbb{Z}p \text{ is not dense in } A \}
\]

is of the first Baire category.

The main results are stated in Theorem 3.1 and Corollaries 3.2 and 3.4 below.

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**2 Some preliminaries**

All integral cohomology is intended modulo torsion. Let \( X/\mathbb{C} \) be a projective algebraic manifold of dimension \( 2m \) and \( \{ X_t \}_{t \in \mathbb{P}^1} \) a Lefschetz pencil of hyperplane sections of \( X \) arising from a given polarization on \( X \). Let \( D := \bigcap_{t \in \mathbb{P}^1} X_t \) be the (smooth) base locus and \( \overline{X} = B_D(X) \), the blow-up. One has a diagram:

\[
\begin{array}{ccc}
\overline{X}_U & \hookrightarrow & \overline{X} \\
\rho_U & & \downarrow \rho \\
U & \xrightarrow{j} & \mathbb{P}^1,
\end{array}
\]

where \( \Sigma := \mathbb{P}^1 \setminus U = \{ t_1, ..., t_M \} \) is the singular set, viz., where the fibers are singular Lefschetz hyperplane sections. One has a short exact sequence of sheaves

\[
0 \rightarrow j_* H^{2m-1} \rho_{U,*} \mathbb{Z} \rightarrow \mathcal{F}^{m,*} \rightarrow \mathcal{J} \rightarrow 0,
\]

where

\[
\mathcal{F}^{m,*} = \mathcal{O}_{\mathbb{P}^1} \left( \prod_{t \in \mathbb{P}^1} \frac{H^{2m-1}(X_t, \mathbb{C})}{F_{m,H^{2m-1}}(X_t, \mathbb{C})} \right) \quad \text{(canonical extension)},
\]
and where the cokernel sheaf \( \mathcal{J} \) is the sheaf of germs of normal functions. The canonical (sometimes called the privileged) extension \( \mathcal{F}^{m,*} \) of the vector bundle

\[
\mathcal{F}^{m,*} := \mathcal{O}_U \left( \prod_{t \in U} \frac{H^{2m-1}(X_t, \mathbb{C})}{F^m H^{2m-1}(X_t, \mathbb{C})} \right),
\]

is introduced in [Z1] (as well as in the references cited there). It plays a role in the required limiting behaviour of the group \( H^0(\mathbb{P}^1, \mathcal{J}) \) of normal functions “at the boundary”, viz., at \( \Sigma \). Roughly speaking then, a normal function \( \nu \in H^0(\mathbb{P}^1, \mathcal{J}) \) is a holomorphic cross-section,

\[
\nu : \mathbb{P}^1 \to \prod_{t \in \mathbb{P}^1} J^m(X_t),
\]

where for \( t \in \Sigma \), \( J^m(X_t) \) are certain “generalized” intermediate jacobians, and where \( \nu \) is locally liftable to a section of \( \mathcal{F}^{m,*} \). The results in [Z1, Cor. 4.52] show that (2) induces a short exact sequence:

\[
0 \to J^m(X) \to H^0(\mathbb{P}^1, \mathcal{J}) \xrightarrow{\delta} H^1(\mathbb{P}^1, j_* R^{2m-1} \rho_{U,*} \mathbb{Z})^{(m,m)} \to 0,
\]

where it is also shown that with respect to the aforementioned polarization of \( X \) defining primitive cohomology,

\[
H^1(\mathbb{P}^1, j_* R^{2m-1} \rho_{U,*} \mathbb{Z}) \simeq \text{Prim}^{2m}(X, \mathbb{C}) \bigoplus H^2_{v}^{m-2}(D, \mathbb{C}),
\]

where \( H^2_{v}^{m-2}(D, \mathbb{C}) = \ker (H^2_{m-2}(D, \mathbb{C}) \to H^{2m+2}(X, \mathbb{C})) \), (induced by the inclusion \( D \hookrightarrow X \), and where \( H^1(\mathbb{P}^1, j_* R^{2m-1} \rho_{U,*} \mathbb{Z})^{(m,m)} \) are the integral classes of Hodge type \((m,m)\) in \( H^1(\mathbb{P}^1, j_* R^{2m-1} \rho_{U,*} \mathbb{Z}) \), and the fixed part \( J^m(X) \) is the Griffiths intermediate jacobian of \( X \). It should be pointed out that there is an intrinsically defined Hodge structure on the space \( H^1(\mathbb{P}^1, j_* R^{2m-1} \rho_{U,*} \mathbb{C}) \) and that (4) is an isomorphism of Hodge structures [Z2]. For \( t \in U \), the Lefschetz theory guarantees an orthogonal decomposition

\[
H^{2m-1}(X_t, \mathbb{C}) = H^{2m-1}(X, \mathbb{C}) \oplus H^2_{v}^{m-1}(X_t, \mathbb{C}),
\]

where by the weak Lefschetz theorem, \( H^{2m-1}(X, \mathbb{C}) \) is identified with its image \( H^{2m-1}(X, \mathbb{C}) \hookrightarrow H^{2m-1}(X_t, \mathbb{C}) \) and integrally speaking, \( H^2_{v}^{m-1}(X_t, \mathbb{Z}) \) is the space generated by the vanishing cocycles \( \{\delta_1, \ldots, \delta_M\} \) (cf. [L, Lec. 6, p. 71]). For fixed \( t \in U \), we put

\[
J^m_v(X_t) = \text{Ext}^1_{\text{MHS}} \left( \mathbb{Z}(0), H^2_{v}^{m-1}(X_t, \mathbb{Z}(m)) \right).
\]

\footnote{There is also a horizontality condition attached to the definition of normal functions of families of projective algebraic manifolds, which automatically holds in the Lefschetz pencil situation (see [Z1, Thm. 4.57]).}
For each $t_i \in \Sigma$, we recall the Picard-Lefschetz transformation $T_i$, and formula $T_i(\gamma) = \gamma + (-1)^m(\gamma, \delta_i)\delta_i$, where $(\delta_i, \delta_j) := (\delta_i, \delta_j)_X \in \mathbb{Z}$ is the cup product on $X_i$ (followed by the trace). Note that a lattice in $H^{2m-1}_v(X_t, \mathbb{Z})$ (i.e. defining a basis of $H^{2m-1}_v(X_t, \mathbb{Q})$), is given (up to relabelling) by a suitable subset $\{\delta_1, \ldots, \delta_{2g}\}$. However we are going to choose our lattice generators $\{\delta_1, \ldots, \delta_{2g}\}$ more carefully as follows:

- Given $\delta_1$, choose $\delta_2$ such that $(\delta_1, \delta_2) \neq 0$. Since $(\delta_j) = 0 \forall j = 1, \ldots, M$, it follows that $\{\delta_1, \delta_2\}$ are $\mathbb{Q}$-independent.

- Next, we argue inductively on $k$ with $1 \leq k \leq 2g-1$, that $\{\delta_1, \ldots, \delta_k, \delta_{k+1}\}$ are $\mathbb{Q}$-independent and $(\delta_\ell, \delta_{k+1}) \neq 0$ for some $\ell \in \{1, \ldots, k\}$.

Indeed if (5) failed to hold for any or all such $k$, then in light of the Picard-Lefschetz formula, $\{\delta_1, \ldots, \delta_M\}$ would not be conjugate under the monodromy group action [L, Lec. 6, p. 71].

### 3 Main results

The class $\delta(\nu) \in H^1(\mathbb{P}^1, j_*R^{2m-1}\rho_{U,*}\mathbb{Z})^{(m,m)}$ is called the topological invariant or the cohomology class of the normal function $\nu$.

**Theorem 3.1.** Let $\nu \in H^0(\mathbb{P}^1, \mathcal{J})$ be a normal function with nontrivial cohomology class, i.e., satisfying $\delta(\nu) \neq 0$. Then for very general $t \in U$, the subgroup $\langle \nu(t) \rangle \subset J^m_v(X_t)$ generated by $\nu(t)$, is dense in the strong topology. In particular, the family of rational curves in the manifold (see [Z1], Prop. 2.9):

$$J := \coprod_{t \in \mathbb{P}^1} J^m_v(X_t),$$

(viz., the images of non-constant holomorphic maps $\mathbb{P}^1 \to J$), is dense in the strong topology.

**Proof.** From the Picard-Lefschetz formula,

$$N_i = \log T_i = (T_i - I), \text{ using } (T_i - I)^2 = 0.$$

Now let $\nu \in H^0(\mathbb{P}^1, \mathcal{J})$ and $\omega \in H^0(\mathbb{P}^1, \mathcal{F}^m)$ be given. Note that

$$\nu : \mathbb{P}^1 \to J,$$

defines a rational curve on $J$. Next, the images

$$\{[\delta_1], \ldots, [\delta_{2g}]\} \text{ in } F_m^*H^{2m-1}_v(X_t, \mathbb{C}) := H^{2m-1}_v(X_t, \mathbb{C})/F^mH^{2m-1}_v(X_t, \mathbb{C}),$$
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define a lattice. In terms of this lattice and modulo the fixed part $J^m(X)$, a local lifting of $\nu$ is given by $\sum_{j=1}^{2g} x_j(t)[\delta_j]$, for suitable real-valued functions \{x_j(t)\}, multivalued on $U$. Let $T_i\nu(\omega(t))$ be the result of analytic continuation of $\nu(\omega(t))$ counterclockwise in $\mathbb{P}^1$ about $t_i$ and $N_i\nu(\omega(t)) = T_i\nu(\omega(t)) - \nu(\omega(t))$. About $t_i$, we pick up a period $$N_i\nu(\omega(t)) = c_i \int_{\delta_i} \omega(t),$$ for some $c_i \in \mathbb{Z}$, dependent only on $\nu$ (not on $\omega$), where we identify $\delta_i$ with its corresponding homology vanishing cycle via Poincaré duality. Likewise in terms of the lattice description, $$N_i\nu(\omega(t)) = \sum_{j=1}^{2g} N_i(x_j(t)) \int_{\delta_j} \omega(t) + (-1)^m \left( \sum_{j=1}^{2g} T_i(x_j(t)) (\delta_j, \delta_i) \right) \cdot \int_{\delta_i} \omega(t).$$ Thus $$c_i = N_i(x_i(t)) + (-1)^m \sum_{j=1}^{2g} T_i(x_j(t)) (\delta_j, \delta_i), \quad (1)$$ and $$N_i(x_j(t)) = 0 \text{ for all } i \neq j. \quad (2)$$ Hence $T_i(x_j(t)) = x_j(t)$ for all $i \neq j$ and further, using $(\delta_i, \delta_i) = 0$, we can rewrite equation (1) as: $$c_i = N_i(x_i(t)) + (-1)^m \sum_{j=1}^{2g} x_j(t) (\delta_j, \delta_i). \quad (3)$$ Note that if $N_i(x_i(t)) = 0$ for all $i$, then from the linear system in (3), $x_i(t) \in \mathbb{Q}$ for all $i$, and so $\delta(\nu) = 0 \in H^1(\mathbb{P}^1, j_* \mathbb{R}^{2m-1} \rho_{U, *} \mathbb{Q})$. Now suppose that we have a nontrivial relation: $$\sum_{j=1}^{2g} \lambda_j x_j(t) = \lambda_0, \text{ for some } \lambda_i \in \mathbb{Q}, \forall i, t \in U. \quad (4)$$ Then by (2) and (4) we have $$\lambda_i N_i(x_i(t)) = \sum_{j=1}^{2g} \lambda_j N_i(x_j(t)) = 0.$$
So \( \lambda_i \neq 0 \Rightarrow N_i(x_i(t)) = 0 \). Let us assume for the moment that \( \lambda_1 \neq 0 \). Then 
\( N_1(x_1(t)) = 0 \), hence from (3):

\[
(-1)^m c_1 = (\delta_2, \delta_1)x_2(t) + (\delta_3, \delta_1)x_3(t) + \cdots + (\delta_{2g}, \delta_1)x_{2g}(t),
\]

and applying \( N_2 \) and (5) we arrive at 
\[
0 = N_2(c_1) = (\delta_2, \delta_1)N_2(x_2(t)) \Rightarrow N_2(x_2(t)) = 0.
\]

Hence again from (3):

\[
(-1)^m c_2 = (\delta_1, \delta_2)x_1(t) + (\delta_3, \delta_2)x_3(t) + \cdots + (\delta_{2g}, \delta_2)x_{2g}(t).
\]

Applying \( N_3 \) to both equations (5) and (6), and (5) we arrive at

\[
(0, 0) = (N_3(c_1), N_3(c_2)) = ((\delta_3, \delta_1), (\delta_3, \delta_2)) \cdot N_3(x_3(t)) \Rightarrow N_3(x_3(t)) = 0,
\]

and so on. Now it may happen that \( \lambda_1 = 0 \). Since \((\lambda_1, ..., \lambda_{2g}) \neq (0, ..., 0)\) we can assume that \( \lambda_{\ell_1} \neq 0 \) for some \( 1 \leq \ell_1 \leq 2g \). Thus by (4), \( N_{\ell_1}(x_{\ell_1}(t)) = 0 \) and accordingly by (3):

\[
(-1)^m c_{\ell_1} = \sum_{j=1}^{2g} (\delta_j, \delta_{\ell_1})x_j(t).
\]

By (5), \( (\delta_{\ell_2}, \delta_{\ell_1}) \neq 0 \) for some \( 1 \leq \ell_2 < \ell_1 \) (assuming \( \ell_1 > 1 \)). Applying \( N_{\ell_2} \) to (7), we arrive at \( N_{\ell_2}(x_{\ell_2}(t)) = 0 \), and hence again by (3):

\[
(-1)^m c_{\ell_2} = \sum_{j=1}^{2g} (\delta_j, \delta_{\ell_2})x_j(t).
\]

Again by (5), \( (\delta_{\ell_3}, \delta_{\ell_2}) \neq 0 \) for some \( 1 \leq \ell_3 < \ell_2 \) (assuming \( \ell_2 > 1 \)), and thus we can repeat this process until we get \( N_i(x_i(t)) = 0 \). This puts in the situation of equation (5), where the same arguments imply that \( N_i(x_i(t)) = 0 \) for all \( i = 1, ..., 2g \).

**Corollary 3.2.** Let \( V \) be a general quintic threefold. Then the image of the Abel-Jacobi map \( AJ : \text{CH}^2_{\text{hom}}(V) \rightarrow J^2(V) \) is a countable dense subset of \( J^2(V) \).

**Proof.** Let \( X \subset \mathbb{P}^5 \) be the Fermat quintic fourfold, and \( \{ X_t \}_{t \in \mathbb{P}^1} \) a Lefschetz pencil of hyperplane sections of \( X \). We will assume the notation given in diagram (1). For the Fermat quintic, it is easy to check that

\[
H^1(\mathbb{P}^1, R^3\nu_*\mathbb{Q})^{(2,2)} \neq 0,
\]

so by the sequence in (3), there exists \( \nu \in H^0(\mathbb{P}^1, \mathcal{J}) \) such that \( \delta(\nu) \neq 0 \in H^1(\mathbb{P}^1, R^3\nu_*\mathbb{Q}) \) (this being related to Griffiths’ famous example [Gr]). Thus by Theorem 3.1 and for general \( t \in \mathbb{P}^1 \), the Abel-Jacobi image is dense in \( J^2(X_t) \). But it is well known that the lines in
for general $t \in \mathbb{P}^1$, deform in the universal family of quintic threefolds in $\mathbb{P}^4$. The corollary follows from this.

Remark 3.3. In light of the conjectures in [G-H], Corollary 3.2 most likely does not generalize to higher degree general hypersurface threefolds. However there is a different kind of generalization that probably holds. Namely, let $S$ be the universal family of smooth threefolds $\{V_t\}_{t \in S}$ of degree $d$ say in $\mathbb{P}^4$. Put

$$J_S := \coprod_{t \in S} J^2(V_t),$$

and

$$J^2_{S, \text{inv}} := \text{Image} \left( \coprod_{t \in S} \text{CH}^2_{\text{hom}}(V_t) \xrightarrow{\text{Abel-Jacobi}} J_S \right).$$

Then in the strong topology, we anticipate that $J^2_{S, \text{inv}} \subset J_S$ is a dense subset.

In this direction, we have the following general result.

Corollary 3.4. Let $\coprod_{\lambda \in S_0} W_\lambda \rightarrow S_0$ be a smooth proper family of $2m$-dimensional projective varieties in some $\mathbb{P}^N$ with the following property:

There exists a dense subset $\Sigma \subset S_0$ such that $\lambda \in \Sigma \Rightarrow \text{Prim}_{\text{alg}}^{m,m}(W_\lambda, \mathbb{Q}) \neq 0$, where Prim is primitive cohomology with respect to the embedding $W_\lambda \subset \mathbb{P}^N$. Further, let us assume that $H^{2m-1}(W_\lambda, \mathbb{Q})_{\pi_1(S_0)} = H^{2m-1}(W_\lambda, \mathbb{Q})$ and let

$$T := \{ t := (c, \lambda) \in \mathbb{P}^N \times S_0 \mid V_t := \mathbb{P}^{N-1}_c \cap W_\lambda \text{ smooth, } \& \dim V_t = 2m-1 \},$$

with corresponding $J^m_{T, \text{inv}} \subset J^m_T$ (where this Jacobian space only involves the orthogonal complement of the fixed part of a corresponding variation of Hodge structure). Then in the strong topology $J^m_{T, \text{inv}}$ is dense in $J^m_T$.

Proof. This easily follows from the techniques of this section and is left to the reader. □

Remark 3.5. (i) The following is obvious, but certainly merits mentioning: Let us assume given the setting and assumptions in Corollary 3.4, and further assume that for all $\lambda \in \Sigma$ and general $c$ with $t = (c, \lambda) \in T$, the $m$-th Q-Griffiths group $\{ \text{CH}^m_{\text{hom}}(V_t) / \text{CH}^m_{\text{alg}}(V_t) \} \otimes \mathbb{Q} = 0$. Then $J^m_T$ is a family of Abelian varieties. (The general Hodge conjecture would imply in this situation that $J^m_{T, \text{inv}} = J^m_T$, but we don’t yet know this.)

(ii) Our results say nothing about the arithmetic nature of the invertible points on the Jacobians. Matt Kerr pointed out to us Proposition 124 in [K-P, p. 92], which appears to be related to our results, and may have some potential in this direction; albeit it is unclear how to move forward with this.
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