Abstract
This paper is concerned with the problem of enhancing convection-cooling via active control of the incompressible velocity field, described by a stationary diffusion-convection model. This essentially leads to a bilinear optimal control problem. A rigorous proof of the existence of an optimal control is presented and the first order optimality conditions are derived for solving the control using a variational inequality. Moreover, the second order sufficient conditions are obtained to show the uniqueness of the optimal control when the control weight parameter is sufficiently large. Finally, numerical experiments are conducted utilizing finite elements methods together with nonlinear iterative schemes, to demonstrate and validate the effectiveness of our control design.

Keywords: convection-cooling, bilinear control, optimality conditions, variational inequality, numerical experiments

1. Introduction
Convection-cooling is the mechanism where heat is transferred from the hot object into the surrounding air or liquid. There are several factors determine the effectiveness of cooling, including temperature difference between the surrounding and the hot object, viscosity of the fluid (air or liquid), and ability of the fluid to move in response to the density difference, etc. There are two types of convectional cooling, namely the natural convection cooling and the forced air convection cooling (cf. [3, 2, 16]). In the natural cooling, the air surrounding the object transfers the heat away from the object and does not use any fans or blowers. In contrast, forced air convection cooling is used in designs where the enclosures or environment do not offer an effective natural cooling performance and areas where natural cooling is not effective. The forced air convection cooling is the most effective cooling method in many industrial applications. It can be designed to provide the required cooling performance while increasing the efficiency of the related components.

The current work utilizes an optimal control approach for the forced air convection-cooling. To be more precise, consider a stationary diffusion-convection model for a cooling application in an open bounded and connected domain \( \Omega \subset \mathbb{R}^d \), \( d = 2, 3 \), with a Lipschitz boundary \( \Gamma \). The velocity field is assumed to be divergence-free. The system of equations reads
\[
-\kappa \Delta T + \mathbf{v} \cdot \nabla T = f \quad \text{in} \quad \Omega \\
\nabla \cdot \mathbf{v} = 0,
\]
with Dirichlet boundary condition for temperature and no-slip boundary condition for velocity
\[
T|_\Gamma = 0, \quad \mathbf{v}|_\Gamma = 0,
\]
where \( T \) is the temperature, \( \kappa > 0 \) is the thermal diffusivity, \( \mathbf{v} \) is the velocity, and \( f \in L^\infty(\Omega) \) is the external body force. Linear controls, either internal (distributed) or boundary controls, of diffusion-convection equations and the corresponding numerical schemes have been well studied (cf. [4, 5, 6, 7, 8, 12, 15, 9, 21]). Our objective is aimed at investigating the optimal convection-cooling via active control of the flow velocity. In essence, this gives rise to a bilinear optimal control problem. For example, in high power applications, a cooling fan is used to blow and direct air towards the electronic components with or without heat sinks. Most power supply units have built-in...
fans that provide the required forced-air convectional cooling. Optimal control for enhancing heat transfer and fluid mixing or optic flow control via flow advection, governed by nonstationary diffusion-convection, has been discussed in (cf. [1, 13, 14, 18]). However, to solve the resulting nonlinear optimality system, one has to solve the governing system forward in time, coupled with the adjoint system backward in time together with a nonlinear optimality condition. This leads to extremely high computational costs and intractable problems. Some preliminary numerical results were obtained in [18] with simplified conditions. As a first step to tackle such a complex system, our current work will focus on the stationary case and present a rigorous theoretical and numerical study of the optimal control design.

Now denote the spatial average of temperature by

$$\langle T \rangle = \frac{1}{|\Omega|} \int_{\Omega} T \, dx.$$  

The objective is to minimize the variance of the temperature with optimal control cost, that is,

$$J(v) = \frac{1}{2} \| T - \langle T \rangle \|_{L^2}^2 + \frac{\gamma}{2} \| v \|_{U_{ad}}^2,$$

subject to (1.1)–(1.3), where $\gamma > 0$ is the control weight parameter and $U_{ad}$ stands for the set of admissible control. The choice of the set of admissible control is usually dependent on the physical properties and the need to establish the existence of an optimal control. Due to the advection term $v \cdot \nabla T$, the control map $v \mapsto T$ is bilinear and hence problem $(P)$ is non-convex. Establishing the existence of an optimal velocity field will involve a compactness argument associated with the control map. Moreover, in order to reduce the effects of rotation on the flow and the shear stress at the boundary in the cooling process, we consider to minimize the magnitude of the strain tensor (cf. [11]), which is equivalent to minimize $\| \nabla v \|_{L^2}$. To this end, we set

$$U_{ad} = \{ v \in H^1_0(\Omega): \nabla \cdot v = 0 \}$$

equipped with $H^1$-norm

$$\| v \|_{U_{ad}} = \| v \|_{H^1}.$$

The remainder of this paper is organized as follows. Section 2 focuses on the existence of an optimal solution to problem $(P)$. Sections 3 presents the first and second order optimality conditions for charactering and solving an optimal solution by using a variational inequality (cf. [17]). Moreover, it can be shown that there exists at most one optimal solutions if the control weight $\gamma$ is large enough. Section 4 discusses the numerical implementation of our control design, where the finite element formulation and nonlinear iterative solvers are used to construct our numerical schemes. In particular, the relation regarding the solutions of the optimality system associated with different values in $\kappa$ and $\gamma$ is established. This result provides a practical guidance for choosing these parameters in our numerical implementation. In Section 5 several numerical experiments are conducted to demonstrate the effectiveness of our control design for convection-cooling. Lastly, this paper concludes with potential problems for future work in Section 6.

In the sequel, the symbol $C$ denotes a generic positive constant, which is allowed to depend on the domain as well as on indicated parameters. Without ambiguous

2. Existence of an Optimal Solution

As a starting point to analyze problem $(P)$, we first recall some basic properties of the governing system (1.1)–(1.3). The following lemma will be often used in this paper.

**Lemma 2.1.** Let $w \in (H^1(\Omega))^d, d = 2, 3,$ and $\phi, \psi \in H^1(\Omega)$. Then we have

$$| \int_\Omega w \cdot \nabla \phi \psi \, dx | \leq \| w \|_{L^4} \| \nabla \phi \|_{L^2} \| \psi \|_{L^4} \leq C \| \nabla w \|_{L^2} \| \nabla \phi \|_{L^2} \| \nabla \psi \|_{L^2}. \quad (2.1)$$

Moreover, if $\nabla \cdot w = 0$ and $w|_{\Gamma} = 0$, then

$$\int_\Omega w \cdot \nabla \phi \psi \, dx = - \int_\Omega \phi w \cdot \nabla \psi \, dx. \quad (2.2)$$
Proof. Inequalities in (2.1) are direct results of H"older’s inequality and Sobolev embedding theorem (cf. [20]). To see (2.2), applying Stokes formula together with \( \nabla \cdot w = 0 \) and \( w|_{\Gamma} = 0 \) follows
\[
\int_{\Omega} w \cdot \nabla \phi \psi \, dx = \int_{\Omega} w \cdot \nabla (\phi \psi) \, dx - \int_{\Omega} \phi w \cdot \nabla \psi \, dx \\
= \int_{\Gamma} w \cdot n (\phi \psi) \, dx - \int_{\Omega} \nabla \cdot w \phi \psi \, dx - \int_{\Omega} \phi w \cdot \nabla \psi \, dx \\
= - \int_{\Omega} \phi w \cdot \nabla \psi \, dx.
\]

Lemma 2.2. Let \( f \in L^\infty(\Omega) \). For \( v \in L^2(\Omega) \) with \( \nabla \cdot v = 0 \) and \( v|_{\Gamma} = 0 \), there exists a unique weak solution to equation (1.1) with Dirichlet boundary condition \( T|_{\Gamma} = 0 \), which satisfies \( T \in H^1_0(\Omega) \cap L^\infty(\Omega) \). Moreover,
\[
\|T\|_{L^2} + \|\nabla T\|_{L^2} \leq \frac{C}{\kappa} \|f\|_{L^2} \tag{2.3}
\]
and
\[
\|T\|_{L^\infty} \leq C \|f\|_{L^\infty}, \tag{2.4}
\]
where \( C > 0 \) depends on \( \Omega \) but not on \( f \).

Proof. The existence of a unique solution follows the standard approaches for the elliptic equations (cf. [10]). To see (2.3), taking the inner product of (1.1) with \( T \) and integrating by parts using (1.3), we have
\[
\kappa \|\nabla T\|_{L^2}^2 = - \int_{\Omega} (v \cdot \nabla T) T \, dx + \int_{\Omega} f T \, dx \\
\leq - \frac{1}{2} \int_{\Omega} v \cdot \nabla (T^2) \, dx + \|f\|_{L^2} \|T\|_{L^2} \\
= - \frac{1}{2} \int_{\Gamma} v \cdot n T^2 \, dx + \frac{1}{2} \int_{\Omega} \nabla \cdot v T^2 \, dx + \|f\|_{L^2} \|T\|_{L^2} \\
\leq \|f\|_{L^2} \|T\|_{L^2} \leq c_0 \|f\|_{L^2} \|\nabla T\|_{L^2}, \tag{2.5}
\]
which follows
\[
\|\nabla T\|_{L^2} \leq \frac{c_0}{\kappa} \|f\|_{L^2}.
\]
Note that in (2.5) we have used Poncaré inequality \( \|T\|_{L^2} \leq c_0 \|\nabla T\|_{L^2} \), where \( c_0 > 0 \) is a constant dependent on domain \( \Omega \) but not \( f \).

Analogously, taking the inner product of (1.1) with \( T^{N-1} \) for a positive even integer \( N \) and then letting \( N \to \infty \) we get (2.4). In fact, a finer estimate of \( f \) in (2.4) can be achieved by using the Stampacchia theory. The reader is referred to [19] for details. This completes the proof.

To show the existence of an optimal control to problem (P), we first introduce the weak solution to (1.1)–(1.3).

Definition 2.3. Let \( f \in L^\infty(\Omega) \) and \( v \in U_{ad} \). \( T \in H^1_0(\Omega) \) is said to be a weak solution to system (1.1)–(1.3), if \( T \) satisfies
\[
\kappa (\nabla T, \nabla \psi) - (Tv, \nabla \psi) = (f, \psi), \quad \forall \psi \in H^1_0(\Omega). \tag{2.6}
\]

Theorem 2.4. For \( f \in L^\infty(\Omega) \), there exists an optimal velocity \( v \in U_{ad} \) to problem (P).

Proof. Since \( J \) is bounded from below, we may choose a minimizing sequence \( \{v_m\} \subset U_{ad} \) such that
\[
\lim_{m \to \infty} J(v_m) = \inf_{v \in U_{ad}} J(v). \tag{2.7}
\]
This also indicates that \( \{v_m\} \) is uniformly bounded in \( U_{ad} \), and hence there exists a weakly convergent subsequence, still denoted by \( \{v_m\} \), such that

\[
\begin{align*}
v_m & \to v^* \quad \text{weakly in } H^1(\Omega), \quad m \to \infty, \\
v_m & \to v^* \quad \text{strongly in } L^2(\Omega), \quad m \to \infty.
\end{align*}
\]  

Let \( \{T_m\} \) be the solutions corresponding to \( \{v_m\} \). Then \( \{T_m\} \) is uniformly bounded in \( H^1(\Omega) \cap L^\infty(\Omega) \) according to (2.3) and (2.4). Thus there exists a subsequence, still denoted by \( \{T_m\} \), satisfying

\[
\begin{align*}
T_m & \to T^* \quad \text{weakly in } H^1(\Omega), \quad m \to \infty, \\
T_m & \to T^* \quad \text{weakly* in } L^\infty(\Omega), \quad m \to \infty.
\end{align*}
\]  

Next we show that \( T^* \) is the solution corresponding to \( v^{opt} \) by Definition 2.3. Recall that \( v_m \) and \( T_m \) satisfy

\[
\kappa(\nabla T_m, \nabla \psi) - (T_m v_m, \nabla \psi) = (f, \psi), \quad \forall \psi \in H^1_0(\Omega),
\]  

with the help of (2.10), it is easy to pass to the limit in the first term on the left hand of (2.12). Next we show that applying (2.8), (2.9) and (2.11) makes passing to the limit in the nonlinear term \( v^* \) possible.

In fact, for the second term on the left hand of (2.12), we have for \( \psi \in H^1_0(\Omega) \),

\[
\begin{align*}
\left| \int_\Omega T_m v_m \cdot \nabla \psi \, dx - \int_\Omega T^* v^* \cdot \nabla \psi \, dx \right| & \leq \left| \int_\Omega T_m v_m \cdot \nabla \psi - T_m v^* \cdot \nabla \psi \, dx \right| \\
& \quad + \left| \int_\Omega T_m v^* \cdot \nabla \psi - T^* v^* \cdot \nabla \psi \, dx \right| \\
& = I_1 + I_2,
\end{align*}
\]  

where

\[
I_1 \leq \|T_m\|_{L^\infty} \|v_m - v^*\|_{L^2} \|\nabla \psi\|_{L^2} \to 0 \quad \text{as} \quad m \to \infty,
\]  

due to (2.9) and the uniform boundedness of \( \|T_m\|_{L^\infty} \). Moreover, \( I_2 \to 0 \) due to (2.11) and \( v^* \nabla \psi \in L^1(\Omega) \). Clearly, \( T^* \in H^1_0(\Omega) \) is the solution corresponding to \( v^* \) based on Definition 2.3.

Lastly, using the weakly lower semicontinuity property of norms yields

\[
\|v^*\|_{U_{ad}} \leq \liminf_{m \to \infty} \|v_m\|_{U_{ad}} \quad \text{and} \quad \|T^* - \langle T^* \rangle\|_{L^2} \leq \liminf_{m \to \infty} \|T_m - \langle T_m \rangle\|_{L^2}.
\]

In other words,

\[
J(v^*) \leq \liminf_{m \to \infty} J(v_m) = \inf_{v \in U_{ad}} J(v),
\]

which indicates that \( v^* \) is an optimal solution to problem (P). \( \square \)

3. Optimality Conditions

Now we derive the first-order necessary optimality conditions for problem (P) by using a variational inequality (cf. [17]), that is, if \( v \) is an optimal solution of problem (P), then

\[
J'(v) \cdot (\psi - v) \geq 0, \quad \psi \in U_{ad}.
\]  

To establish the Gâteaux differentiability of \( J(v) \), we first check the Gâteaux differentiability of \( T \) with respect to \( v \). Let \( z \) be the Gâteaux of \( T \) with respect to \( v \) in the direction of \( h \in U_{ad} \), i.e., \( z = T'(\psi v) \cdot h \). Then \( z \) satisfies

\[
-\kappa \Delta z + v \cdot \nabla z + h \cdot \nabla T = 0, \quad z|_\Gamma = 0.
\]  

\[
\begin{aligned}
-\kappa \Delta z + v \cdot \nabla z + h \cdot \nabla T = 0, \\
z|_\Gamma = 0.
\end{aligned}
\]
Using the $L^2$-estimate as in Lemma 2.2 with the help of Lemma 2.1 and (2.3), we get
\[ \kappa \| \nabla z \|_{L^2}^2 \leq | \int_{\Omega} (h \cdot \nabla T) z \, dx | \leq C \| \nabla h \|_{L^2} \| \nabla T \|_{L^2} \| \nabla z \|_{L^2}, \] (3.3)
which implies
\[ \| \nabla z \|_{L^2} \leq \frac{C}{\kappa} \| \nabla h \|_{L^2} \| \nabla T \|_{L^2} \leq \frac{C}{\kappa^2} \| f \|_{L^2} \| \nabla h \|_{L^2}. \] (3.4)
Therefore, $T(v)$ is Gâteaux differentiable for $v \in \mathcal{U}_{ad}$, so is $J(v)$.

Next, we use (3.1) to establish the first order optimality conditions for solving the optimal solution.

3.1. First Order Optimality Conditions

Let $A = -P \Delta$ be the Stokes operator with
\[ D(A) = \{ H^1_0(\Omega) \cap H^2(\Omega) : \nabla \cdot v = 0 \}, \]
where $P : L^2(\Omega) \to \{ v \in L^2(\Omega) : \nabla \cdot v = 0 \}$ is the Leray projector. Note that $A$ is a strictly positive and self-adjoint operator. Moreover, define $DT = T - \langle T \rangle$. Then the cost functional can be rewritten as
\[ J(v) = \frac{1}{2} (D^* DT, T) + \frac{\gamma}{2} (Av, v), \] (3.5)
where $D^*$ is the adjoint operator of $D$.

Remark 3.1. Here we present some basic properties of operator $D$. For any $T, \psi \in L^2(\Omega)$, since $\langle T \rangle$ and $\langle \psi \rangle$ are constants, we have
\[ \frac{1}{|\Omega|} \int_{\Omega} T(\psi) \, dx = \langle T \rangle \langle \psi \rangle = \frac{1}{|\Omega|} \int_{\Omega} \langle T \rangle \psi \, dx. \]
Therefore,
\[ (DT, \psi) = \int_{\Omega} (T - \langle T \rangle) \psi \, dx = \int_{\Omega} T \psi \, dx - \int_{\Omega} \langle T \rangle \psi \, dx \]
\[ = \int_{\Omega} T \psi \, dx - \int_{\Omega} T(\psi) \, dx = (T, \psi - \langle \psi \rangle) = (T, D\psi), \]
which says that $D$ is a self-adjoint operator, i.e., $D = D^*$. Moreover, since $\langle T - \langle T \rangle \rangle = 0$, it is straightforward to verify that $D^2 = D$. This implies that the operator norm $\| D \| \leq 1$.

Next, let $q$ be the adjoint state associated with $T$. Then it is easy to verify that $q$ satisfies
\[ -\kappa \Delta q - v \cdot \nabla q = D^* DT \quad \text{in} \; \Omega, \]
\[ q|_{\Gamma} = 0. \] (3.6)
Moreover, thanks to (2.3) and $\| D \| \leq 1$, we have
\[ \| \nabla q \|_{L^2} \leq \frac{C_0}{\kappa} \| T \|_{L^2} \leq \frac{C}{\kappa^2} \| f \|_{L^2}. \] (3.7)

The following theorem establishes the first order necessary optimality conditions for characterizing and solving the optimal solution.

Theorem 3.2. If $v^{opt}$ is an optimal solution to problem $(P)$ and $(T, q)$ is the corresponding solution to the governing system (1.1)–(1.3) and the adjoint system (3.6), then $v^{opt}$ satisfies
\[ v^{opt} = \frac{1}{\gamma} A^{-1}(q \nabla T). \] (3.8)

In other words, there exists $p \in L^2(\Omega)$ with $\int_{\Omega} p \, dx = 0$ such that
\[ -\gamma \Delta v^{opt} + \nabla p = q \nabla T. \] (3.9)
Proof. In light of (3.5), (3.6), and (2.2), the Gâteaux derivative of $J$ becomes
\[
J'(v) \cdot h = (D^* DT, z) + \gamma(Av, h)
\]
\[
= (-\kappa \Delta q - v \cdot \nabla q, z) + \gamma(Av, h)
\]
\[
= (q, -\kappa \Delta z + v \cdot \nabla z) + \gamma(Av, h).
\]
Using (3.2) we get
\[
J'(v) \cdot h = -(q, h \cdot \nabla T) + \gamma(Av, h).
\]
If $v^{opt}$ is the optimal solution, then $J'(v^{opt}) \cdot h \geq 0$ for any $h \in U_{ad}$. This implies
\[
\gamma Av^{opt} - q \nabla T = 0,
\]
which yields the desired result (3.8). \hfill \Box

3.2. Second Order Optimality Conditions

In this section, we shall show that the second Gâteaux derivative of $J(v)$ is coercive if the control weight $\gamma$ is sufficiently enough. As a result, the second order sufficient condition holds, and hence the optimal solution is unique.

Theorem 3.3. Let $v \in U_{ad}$. If $\gamma > 0$ is sufficiently large, then there exists some constant $\delta > 0$ such that
\[
J''(v) \cdot (h, h) \geq \delta \|h\|_{U_{ad}}, \quad h \in U_{ad}.
\]
Proof. Let $h_1, h_2 \in U_{ad}$ and $z_i = T'(v) \cdot h_i, i = 1, 2$. Then $z_i$ satisfies
\[
-\kappa \Delta z_i + v \cdot \nabla z_i + h_i \cdot \nabla T = 0 \quad \text{in } \Omega,
\]
\[
z_i|_{\Gamma} = 0.
\]
Moreover, let $Z = z_i'(v) \cdot h_2$. Then $Z$ satisfies
\[
-\kappa \Delta Z + h_2 \cdot \nabla z_1 + v \cdot \nabla Z + h_1 \cdot \nabla z_2 = 0 \quad \text{in } \Omega,
\]
\[
Z|_{\Gamma} = 0.
\]
Again applying the $L^2$-estimate for $Z$ and using (3.4), we can easily verify that
\[
\|\nabla Z\|_{L^2} \leq C \kappa \left(\|\nabla h_2\|_{L^2} \|\nabla z_1\|_{L^2} + \|\nabla h_1\|_{L^2} \|\nabla z_2\|_{L^2}\right)
\]
\[
\leq C \kappa^3 \|f\|_{L^2} \|\nabla h_1\|_{L^2} \|\nabla h_2\|_{L^2}.
\]
This implies that $T(v)$ is twice Gâteaux differentiable for $v \in U_{ad}$, so is $J(v)$.

Now differentiating $J'(v) \cdot h_1$ once again in the direction $h_2 \in U_{ad}$ gives
\[
J''(v) \cdot (h_1, h_2) = (D^* Dz_1, z_2) + (D^* Dz, T) + \gamma(Ah_2, h_1).
\]
To further address the second term involving $Z$, we take the inner product of (3.11) with $q$ and apply (2.2). We get
\[
-\kappa(Z, \Delta q) - (z_1, h_2 \cdot \nabla q) - (Z, v \cdot \nabla q) - (z_2, h_1 \cdot \nabla q) = 0.
\]
With the help of the adjoint equation (3.6), we have
\[
(z_1, h_2 \cdot \nabla q) + (z_2, h_1 \cdot \nabla q) = (Z, D^* DT),
\]
and thus
\[
J''(v) \cdot (h_1, h_2) = (D^* Dz_1, z_2) + (z_1, h_2 \cdot \nabla q) + (z_2, h_1 \cdot \nabla q) + \gamma(Ah_2, h_1).
\]
Consequently, setting $h_1 = h_2$ yields $z_1 = z_2$ and

$$J''(v) \cdot (h, h) = \|Dz\|_{L^2}^2 + 2(z, h \cdot \nabla q) + \gamma A^{1/2}h\|f\|_{L^2}^2.$$  (3.13)

Moreover, by (2.1), (3.4) and (3.7), we get

$$|\int_{\Omega} z h \cdot \nabla q \, dx| \leq C\|\nabla z\|_{L^2}\|\nabla h\|_{L^2}\|\nabla q\|_{L^2} \leq \frac{C}{k^2}\|f\|_{L^2}^2 A^{1/2}h\|f\|_{L^2}^2$$

and

$$\|Dz\|_{L^2} \leq C\|\nabla z\|_{L^2} \leq \frac{C}{k^2}\|f\|_{L^2}^2 A^{1/2}h\|f\|_{L^2}^2.$$  

As a result,

$$|J''(v) \cdot (h, h)| \leq \frac{C}{k^2}\|f\|_{L^2}^2 A^{1/2}h\|f\|_{L^2}^2 + \gamma A^{1/2}h\|f\|_{L^2}^2 = \left( \frac{C}{k^2}\|f\|_{L^2}^2 + \gamma \right)A^{1/2}h\|f\|_{L^2}^2$$  (3.14)

and

$$J''(v) \cdot (h, h) \geq -2|z, h \cdot \nabla q| + \gamma A^{1/2}h\|z\|_{L^2}^2 = \left( \gamma - \frac{C}{k^2}\|f\|_{L^2}^2 \right)A^{1/2}h\|f\|_{L^2}^2.$$  (3.15)

Therefore, if let $\gamma$ large enough such that

$$\gamma - \frac{C}{k^2}\|f\|_{L^2}^2 \geq \delta$$  (3.16)

for some $\delta > 0$, then (3.10) holds. 

**Lemma 3.4.** There exists a constant $C > 0$ such that

$$|(J''(v_1) - J''(v_2)) \cdot (h, h)| \leq \frac{C}{k^2}\|f\|_{L^2}^2 \|h\|_{H^1}^2,$$  (3.17)

for any $h, v_i \in U_{ad}, i = 1, 2.$

**Proof.** Let $h, v_i \in U_{ad}$ and $z_i = T_i(v_i) \cdot h, i = 1, 2.$ Here $T_i$ is the temperature corresponding to $v_i.$ Then $z_i$ satisfies

$$-\kappa \Delta z_i + v_i \cdot \nabla z_i + h \cdot \nabla T_i = 0 \quad \text{in} \quad \Omega,$$

$$z_i|_{\Gamma} = 0.$$  

According to (3.13), (3.4) and (3.7) we have

$$|(J''(v_1) - J''(v_2)) \cdot (h, h)| = \|Dz_1\|_{L^2}^2 + 2(z_1, h \cdot \nabla q_1) + \gamma A^{1/2}h\|z_1\|_{L^2}^2$$

$$- \left( \|Dz_2\|_{L^2}^2 + 2(z_2, h \cdot \nabla q_2) + \gamma A^{1/2}h\|z_2\|_{L^2}^2 \right)$$

$$= \|Dz_1\|_{L^2}^2 - \|Dz_2\|_{L^2}^2 + 2(z_1, h \cdot \nabla q_1) - 2(z_2, h \cdot \nabla q_2)$$

$$\leq C(\|\nabla z_1\|_{L^2} + \|\nabla z_2\|_{L^2}) + C\|\nabla z_1\|_{L^2}\|\nabla h\|_{L^2}\|\nabla q_1\|_{L^2}$$

$$+ C\|\nabla z_2\|_{L^2}\|\nabla h\|_{L^2}\|\nabla q_2\|_{L^2}$$

$$\leq \frac{C}{k^2}\|f\|_{L^2}^2 \|h\|_{H^1}^2,$$

which establishes the desired result. 

**Corollary 3.5 (SOSC).** Let $v_{opt}$ be an optimal solution to problem (P) and $\gamma$ satisfy (3.16). Then

$$J''(v_{opt}) \cdot (h, h) \geq \delta \|h\|_{L^2}, \quad h \in U_{ad}.$$  (3.18)

Moreover, there exists $\delta_0 > 0$ such that the quadratic growth condition holds

$$J(v_{opt}) + \delta_0\|v - v_{opt}\|_{U_{ad}}^2 \leq J(v), \quad v \in U_{ad}.$$  (3.19)
Proof. The second order sufficient condition follows immediately from coercivity (3.10) if \( \gamma \) satisfies (3.16). To see the gap between \( J(\mathbf{v}^{\text{opt}}) \) and \( J(\mathbf{v}) \) for any \( \mathbf{v} \in \mathcal{U}_{\text{ad}} \), we set \( h = \mathbf{v} - \mathbf{v}^{\text{opt}} \) and apply a Taylor expansion of \( J(\mathbf{v}) \) around \( \mathbf{v}^{\text{opt}} \) together with (3.17) and (3.18). We have

\[
J(\mathbf{v}) - J(\mathbf{v}^{\text{opt}}) = J'(\mathbf{v}^{\text{opt}}) \cdot (\mathbf{v} - \mathbf{v}^{\text{opt}}) + J''(\mathbf{v}^{\text{opt}}) : \left( (\mathbf{v} - \mathbf{v}^{\text{opt}})(\mathbf{v} - \mathbf{v}^{\text{opt}})^\top \right) \\
n\geq \delta \|\mathbf{v} - \mathbf{v}^{\text{opt}}\|_{H^1}^2 - \frac{C}{\kappa} \|f\|_{L^2}^2 \|\mathbf{v} - \mathbf{v}^{\text{opt}}\|_{H^1}^2 \\
\geq (\delta - \frac{C}{\kappa} \|f\|_{L^2}^2) \|\mathbf{v} - \mathbf{v}^{\text{opt}}\|_{H^1}^2,
\]

where \( \xi \in (0, 1) \). Therefore, if \( 0 < \delta_0 \leq \delta - \frac{C}{\kappa} \|f\|_{L^2}^2 \) and \( \delta > \frac{C}{\kappa} \|f\|_{L^2}^2 \), then (3.19) holds. As a result, the uniqueness of the minimizer is obtained.

**Corollary 3.6.** If \( \gamma \) satisfies (3.16), then there exists a unique minimizer to problem (P), which can be solved from the optimality system described by (3.17), (3.18), (3.19), and (3.18).

4. Numerical Implementation

In this section, we shall present a detailed numerical implementation of our control design for 2D convection-cooling system via divergence-free velocity field. Firstly, let us recall the nonlinear optimality systems consisting of the governing system (1.1), (1.3), the adjoint system (4.6), and the optimality condition (3.9), i.e.,

\[
\begin{cases}
-\kappa \Delta T + \mathbf{v} \cdot \nabla T = f & \text{in } \Omega, \quad \text{and} \quad T|_{\partial \Omega} = 0, \\
-\kappa \Delta q - \mathbf{v} \cdot \nabla q = D^* DT & \text{in } \Omega, \quad \text{and} \quad q|_{\partial \Omega} = 0, \\
-\gamma \Delta \mathbf{v} + \nabla p = q \nabla T - \mathbf{v} \cdot \mathbf{v} & \text{in } \Omega, \quad \text{and} \quad \mathbf{v}|_{\partial \Omega} = 0.
\end{cases}
\]

(4.1)

The following lemma establishes the relation between the diffusivity coefficient \( \kappa \) and the control weight parameter \( \gamma \), which indicates that it is sufficient to test the results when \( \kappa = 1 \). The results for other \( \kappa \) values can then be obtained by this relation.

**Lemma 4.1.** Let \( [T_{\gamma}, q_{\gamma}, \mathbf{v}_{\gamma}, p_{\gamma}] \) be the solution to (4.1) corresponding \( \kappa = 1 \) and \( \gamma \). Let \( [T_{\kappa, \tilde{\gamma}}, q_{\kappa, \tilde{\gamma}}, \mathbf{v}_{\kappa, \tilde{\gamma}}, p_{\kappa, \tilde{\gamma}}] \) be the solution to (4.1) corresponding \( \kappa \) and \( \tilde{\gamma} \) where \( \tilde{\gamma} = \frac{1}{\kappa^2} \gamma \). Then the following relation holds:

\[
T_{\kappa, \tilde{\gamma}} = \frac{1}{\kappa} T_{\gamma}, \quad q_{\kappa, \tilde{\gamma}} = \frac{1}{\kappa^2} q_{\gamma}, \quad \mathbf{v}_{\kappa, \tilde{\gamma}} = \kappa \mathbf{v}_{\gamma}, \quad \text{and} \quad p_{\kappa, \tilde{\gamma}} = \frac{1}{\kappa^3} p_{\gamma}.
\]

**Proof.** Based on (4.1), it is straightforward to verify that

\[
-\kappa \Delta T_{\kappa, \tilde{\gamma}} + \mathbf{v}_{\kappa, \tilde{\gamma}} \cdot \nabla T_{\kappa, \tilde{\gamma}} = -\Delta T_{\gamma} + \mathbf{v}_{\gamma} \cdot \nabla T_{\gamma} = f,
\]

\[
-\kappa \Delta q_{\kappa, \tilde{\gamma}} - \mathbf{v}_{\kappa, \tilde{\gamma}} \cdot \nabla q_{\kappa, \tilde{\gamma}} = \frac{1}{\kappa} (-\Delta q_{\gamma} - \mathbf{v}_{\gamma} \cdot \nabla q_{\gamma}) = \frac{1}{\kappa} D^* DT_{\gamma} = D^* DT_{\kappa, \tilde{\gamma}},
\]

and

\[
-\tilde{\gamma} \Delta \mathbf{v}_{\kappa, \tilde{\gamma}} + \nabla p_{\kappa, \tilde{\gamma}} = \frac{1}{\kappa^3} (-\gamma \Delta \mathbf{v}_{\gamma} + \nabla p_{\gamma}) = \frac{1}{\kappa^3} (q_{\gamma} \nabla T_{\gamma}) = q_{\kappa, \tilde{\gamma}} \nabla T_{\kappa, \tilde{\gamma}}.
\]

This completes the proof.

As a by product of the above lemma, we also have the following result

\[
J(\kappa, \tilde{\gamma}) = \frac{1}{\kappa^2} J(\gamma),
\]

and therefore,

\[
\frac{\log(J(\kappa, \tilde{\gamma}_1)/J(\kappa, \tilde{\gamma}_2))}{\log(\tilde{\gamma}_1/\tilde{\gamma}_2)} = \frac{\log(J(\gamma_1)/J(\gamma_2))}{\log(\gamma_1/\gamma_2)}.
\]
4.1. Finite Element Formulation

The weak formulation for the nonlinear system \((4.1)\) is to find \(T \in H^1_0(\Omega), q \in H^1_0(\Omega), \mathbf{v} \in [H^1_0(\Omega)]^2\) and \(p \in L^2(\Omega)\) such that:

\[
\begin{align*}
(k \nabla T, \nabla \phi) + (\mathbf{v} \cdot \nabla T, \phi) &= (f, \phi), \quad \forall \phi \in H^1_0, \\
(k \nabla q, \nabla \psi) - (\mathbf{v} \cdot \nabla q, \psi) - (DT, \phi) &= 0, \quad \forall \psi \in H^1_0, \\
(\gamma \nabla v_h, w) - (p, \nabla \cdot w) - (q \nabla T, w) &= 0, \quad \forall w \in [H^1_0(\Omega)]^2, \\
(\nabla \cdot \mathbf{v}, \theta) &= 0, \quad \forall \theta \in L^2(\Omega).
\end{align*}
\]

We aim to use finite element method to approximate the system. Let \(T_h\) be a partition of the domain \(\Omega\) consisting of triangles in two dimensions. For every element \(\tau \in T_h\), we denote by \(h_{\tau}\) its diameter and define the mesh size \(h = \max_{\tau \in T_h} h_{\tau}\) for \(T_h\). On the mesh \(T_h\), we define the continuous finite element spaces as follows,

\[
\begin{align*}
V_h &= \{ v \in H^1(\Omega) : v|_{\tau} \in P_2(\tau), \forall \tau \in T_h \}, \\
V^0_h &= \{ v \in [H^1(\Omega)]^2 : v|_{\tau} \in [P_2(\tau)]^2, \forall \tau \in T_h \}, \\
Q_h &= \{ q \in H^1(\Omega) \cap L^2_0(\Omega) : q|_{\tau} \in P_1(\tau), \forall \tau \in T_h \}.
\end{align*}
\]

Here \(P_\ell\) denotes the space of polynomials with degree less than or equal to \(\ell\) and \(L^2_0(\Omega) := \{ \theta \in L^2(\Omega) : \int_\Omega \theta \, dx = 0 \}\). The corresponding finite element spaces with homogeneous Dirichlet boundary condition are denoted by \(V^0_h\) and \(V^0_h\). For the Stokes solver, we apply the inf-sup stable Taylor-Hood element \([23, 22]\).

Below we introduce the bilinear and trilinear forms. For \(\phi, \psi \in V_h\), \(\mathbf{v}, \mathbf{w} \in V^0_h\), \(\theta \in Q_h\), let

\[
\begin{align*}
A(\phi, \psi) &= \sum_{\tau \in T_h} \int_\tau \kappa \nabla \phi \cdot \nabla \psi \, dx, \\
C(\mathbf{w}; \phi, \psi) &= \sum_{\tau \in T_h} \int_\tau (\mathbf{w} \cdot \nabla \phi) \psi \, dx, \\
D(\mathbf{v}, \mathbf{w}) &= \sum_{\tau \in T_h} \int_\tau \gamma \nabla \mathbf{v} : \nabla \mathbf{w} \, dx, \\
B(\mathbf{w}, \theta) &= \sum_{\tau \in T_h} \int_\tau \nabla \cdot \mathbf{w} \theta \, dx.
\end{align*}
\]

Now, we are ready to propose the finite element schemes for system \((4.1)\) with \(D^* DT = T - (T)\). The finite element scheme for the system \((4.1)\) is to solve: \(T_h \in V^0_h\), \(q_h \in V^0_h\), \(v_h \in V^0_h\), and \(p_h \in Q_h\), such that:

\[
\begin{align*}
A(T_h, \phi) - C(v_h; T_h, \phi) &= (f, \phi), \quad \forall \phi \in V^0_h, \\
A(q_h; \psi) + C(v_h; q_h, \psi) - (T_h - (T_h), \psi) &= 0, \quad \forall \psi \in V^0_h, \\
D(v_h, \mathbf{w}) - B(\mathbf{w}, p_h) - (q_h \nabla T_h, \mathbf{w}) &= 0, \quad \forall \mathbf{w} \in V^0_h, \\
B(\mathbf{w}, \theta) &= 0, \quad \forall \theta \in Q_h.
\end{align*}
\]

4.2. Picard and Newton iterative Solvers

Note that \((4.7)\) is a nonlinear system involving a nonlinear Stokes problem. To tackle the nonlinearity, we combine both the Picard and Newton iterative solvers to achieve the required computational efficiency.

For the Picard iterative method, we seek to find \((T^{k+1}, q^{k+1}, \mathbf{v}^{k+1}, p^{k+1})\) based on the previously given approximation \((T^k, q^k, \mathbf{v}^k, p^k)\). The idea simply replaces the unknown nonlinear terms by the known solutions in the previous step. The nonlinear system can be linearized as follows:

\[
\begin{align*}
-\kappa \Delta T^{k+1} + \mathbf{v}^k \cdot \nabla T^{k+1} &= f, \quad \text{and } T^{k+1}|_{\partial \Omega} = 0, \\
-\kappa \Delta q^{k+1} + \mathbf{v}^k \cdot \nabla q^{k+1} &= T^{k+1} - \frac{1}{|\Omega|} \int_\Omega T^{k+1}, \quad \text{and } q^{k+1}|_{\partial \Omega} = 0, \\
-\gamma \Delta v^{k+1} + \nabla p^{k+1} &= q^{k+1} \nabla T^{k+1}, \quad \nabla \cdot v^{k+1} = 0, \quad \text{and } v^{k+1}|_{\partial \Omega} = 0.
\end{align*}
\]
The finite element solution to (4.8) is then to find \((T_h^{k+1}, q_h^{k+1}, v_h^{k+1}, p_h^{k+1}) \in V_h^0 \times V_h^0 \times V_h^0 \times Q_h\) such that

\[
\begin{align*}
&A(T_h^{k+1}, \phi) - C(v_h^{k+1}; T_h^{k+1}, \phi) = (f, \phi), \quad \forall \phi \in V_h^0, \\
&A(q_h^{k+1}, \psi) + C(v_h^{k+1}; q_h^{k+1}, \psi) - (T_h^{k+1} - (T_h^{k+1}), \psi) = 0, \quad \forall \psi \in V_h^0, \\
&D(v_h^{k+1}, w) - B(w, p_h^{k+1}) - (q_h^{k+1} \nabla T_h^{k+1}, w) = 0, \quad \forall w \in V_h^0, \\
&B(v_h^{k+1}, \theta) = 0, \quad \forall \theta \in Q_h.
\end{align*}
\] (4.9)

Note that the system (4.9) can be solved sequentially. For the Picard method in the finite element scheme, we set the following initial guess: \((T_h^0, q_h^0, v_h^0, p_h^0)\) such that

\[
\begin{align*}
&v_h^0 = 0, \quad p_h^0 = 0, \\
&A(T_h^0, \phi) = (f, \phi), \quad \forall \phi \in V_h^0, \\
&A(q_h^0, \psi) = (T_h^0 - (T_h^0), \psi) = 0, \quad \forall \psi \in V_h^0.
\end{align*}
\] (4.10)

We now derive the formulation for the Newton method in the PDE level. Given an approximation to the solution field, \(\{T^k, q^k, v^k, p^k\}\), we aim to find a perturbation \(\{\delta T, \delta q, \delta v, \delta p\}\) so that

\[
\{T^{k+1}, q^{k+1}, v^{k+1}, p^{k+1}\} = \{T^k, q^k, v^k, p^k\} + \{\delta T, \delta q, \delta v, \delta p\}.
\]

and that

\[
\begin{align*}
-\kappa \Delta T^{k+1} + v^{k+1} \cdot \nabla T^{k+1} &= f, \forall x \in \Omega, \text{ and } T^{k+1}|_{\partial \Omega} = 0, \\
-\kappa \Delta q^{k+1} - v^{k+1} \cdot \nabla q^{k+1} - T^{k+1} + (T^{k+1}) &= 0, \forall x \in \Omega \text{ and } q^{k+1}|_{\partial \Omega} = 0, \\
-\gamma \Delta v^{k+1} + \nabla p^{k+1} - q^{k+1} \nabla T^{k+1} = 0, \quad \nabla \cdot v^{k+1} &= 0, \forall x \in \Omega \text{ and } v^{k+1}|_{\partial \Omega} = 0.
\end{align*}
\] (4.11)

This above PDE system is still a nonlinear system. The idea to obtain a linear system is to assume that \(\delta\)-quantities are sufficiently small so that we can linearize the problem with respect to those \(\delta\)-quantities using Taylor expansion. Eventually we obtain the following linear system by dropping the higher order nonlinear terms in terms of \(\delta\)-quantities.

\[
\begin{align*}
-\kappa \Delta T^{k+1} + v^{k+1} \cdot \nabla T^{k+1} + v^k \cdot \nabla T^k &= f + v^k \cdot \nabla T^k, \quad T^{k+1}|_{\partial \Omega} = 0, \\
-\kappa \Delta q^{k+1} - v^{k+1} \cdot \nabla q^{k+1} - v^k \cdot \nabla q^k - T^{k+1} + (T^{k+1}) &= -v^k \cdot \nabla q^k, \quad q^{k+1}|_{\partial \Omega} = 0, \\
-\gamma \Delta v^{k+1} + \nabla p^{k+1} - q^{k+1} \nabla T^k &= -q^k \nabla T^k, \quad v^{k+1}|_{\partial \Omega} = 0.
\end{align*}
\]

The finite element solution to (4.11) is then to find \((T_h^{k+1}, q_h^{k+1}, v_h^{k+1}, p_h^{k+1}) \in V_h^0 \times V_h^0 \times V_h^0 \times Q_h\) such that

\[
\begin{align*}
&A(T_h^{k+1}, \phi) + C(v_h^{k+1}; T_h^{k+1}, \phi) + C(v_h^k; T_h^k, \phi) = (f, \phi) + C(v_h^k; T_h^k, \phi), \quad \forall \phi \in V_h^0, \\
&A(q_h^{k+1}, \psi) - C(v_h^k; q_h^k, \psi) - C(v_h^{k+1}; q_h^{k+1}, \psi) - (T_h^{k+1} - (T_h^{k+1}), \psi) = -C(v_h^k; q_h^k, \psi), \quad \forall \psi \in V_h^0, \\
&D(v_h^{k+1}, w) - B(w, p_h^{k+1}) - (q_h^k \nabla T_h^k, w) - (q_h^{k+1} \nabla T_h^{k+1}, w) = - (q_h^k \nabla T_h^k, w), \quad \forall w \in V_h^0, \\
&B(v_h^{k+1}, \theta) = 0, \quad \forall \theta \in Q_h.
\end{align*}
\] (4.12)

**Remark 4.2.** Comparing to the Picard method, the Newton’s method has faster convergence rate. However, its initial condition should be chosen wisely. For Picard method, our numerical experiments show that it can yield a satisfactory initial solution for the Newton’s method very quickly. This suggests that we can use Picard method at the first stage to obtain a good initial guess and then apply Newton’s method to obtain the converged numerical solutions within several iterations.

### 4.3. Numerical Algorithm

In this subsection, we summarize our numerical method in the following algorithm.
Algorithm 4.1 Finite Element Scheme for system (4.1)

- Set $\epsilon_1 = 1E-3$, $\epsilon_2 = 1E-6$, $n_1 = 20$, and $n_2 = 50$.
- Set the initial guess $(T^0_h, q^0_h, v^0_h, p^0_h)$ as in (4.10).
- Compute the cost functional:
  \[ J_0 = \frac{\gamma \| \nabla v_h^0 \|_2^2}{2} + \frac{\| T_h^0 - \langle T_h^0 \rangle \|_2^2}{2}. \] (4.13)
  
- For $k = 0, \ldots, n_1$, perform the Picard iteration as below:
  - Solve $(T_h^{k+1}, q_h^{k+1}, v_h^{k+1}, p_h^{k+1}) \in V_h^0 \times V_h^0 \times V_h^0 \times Q_h$ for (4.9).
  - Compute the cost functional:
    \[ J_k = \frac{\gamma \| \nabla v_h^k \|_2^2}{2} + \frac{\| T_h^k - \langle T_h^k \rangle \|_2^2}{2}. \] (4.14)
  - If $\left| \frac{J_k - J_{k-1}}{J_{k-1}} \right| < \epsilon_1$, STOP and OUTPUT $T_h^k, q_h^k, v_h^k, p_h^k$.
- Set $(T_h^0, q_h^0, v_h^0, p_h^0) = (T_h^k, q_h^k, v_h^k, p_h^k)$.
- For $k = 0, \ldots, n_2$, perform the Newton iteration as below:
  - Solve $(T_h^{k+1}, q_h^{k+1}, v_h^{k+1}, p_h^{k+1}) \in V_h^0 \times V_h^0 \times V_h^0 \times Q_h$ for (4.12).
  - Compute the cost functional:
    \[ J_k = \frac{\gamma \| \nabla v_h^k \|_2^2}{2} + \frac{\| T_h^k - \langle T_h^k \rangle \|_2^2}{2}. \] (4.15)
  - If $\left| \frac{J_k - J_{k-1}}{J_{k-1}} \right| < \epsilon_2$, STOP and OUTPUT $T_h^k, q_h^k, v_h^k, p_h^k$.

5. Numerical Experiments

In this section, we shall present several numerical experiments by employing different heat source profiles to validate the proposed numerical schemes in Algorithm 4.1. The domain for all test problems are set to be the unit square, i.e., $\Omega = (0, 1) \times (0, 1)$. Thanks to Lemma 4.1, it is sufficient to test for one $\kappa$ value. Without loss of generality, we perform all our numerical tests only for $\kappa = 1$. The numerical experiments are performed using the FENICS package [24] on the uniform triangular mesh with $h = 1/100$.

Example 5.1. We first test a symmetric heat distribution. Let
\[ f(x, y) = 2\pi^2 \sin(\pi x) \sin(\pi y). \]
\textbf{Figure 1:} Example 5.1 Plot of temperature $T_h$ of $\kappa = 1.0$ for (a). Initial heat distribution $T^0_h$, and with (b). $\gamma = 3.6E-6$; (c). $\gamma = 8.5E-7$; (d). $\gamma = 3.9E-7$.

The initial heat distribution $T^0_h$ corresponding to $\mathbf{v} = 0$ is shown in Fig. 1a. The optimal heat distribution $T_h$ corresponding for $\gamma = 3.6E-6$, $8.5E-7$, and $3.9E-7$ are plotted in Fig. 1b-d. For the initial heat distribution, one can observe that the maximum of $T^0_h$ is 1.0. Thanks to advection effect, the “hot” region, which is at the center of the domain initially, is now spread out, but still inherits certain symmetric pattern. As a result, the heat distribution over the entire domain is evened out. Note that the maximum of $T_h$ is reduced to 9.8E-1, 7.8E-1, and 6.8E-1 corresponding to $\gamma = 3.6E-6$, $8.5E-7$, and $3.9E-7$, respectively. Also, it is shown from these plots that the smaller value in $\gamma$ (which indicates less penalty on the control), the more effective is the convection-cooling.
Figure 2: Example 5.1: Plot of velocity field $v_h$ for $\kappa = 1.0$ and (a). $\gamma = 3.6E-6$; (b). $\gamma = 8.5E-7$; (c). $\gamma = 3.9E-7$. Here, the color illustrates the magnitude of velocity $v_h$ and the vector plots the field of $v_h$.

Figure 3: Example 5.1: Plot of streamlines of $v_h$ for $\kappa = 1.0$ and (a). $\gamma = 3.6E-6$; (b). $\gamma = 8.5E-7$; (c). $\gamma = 3.9E-7$. Here, the color illustrates the magnitude of velocity $v_h$ and the curve plots the streamline of $v_h$.

On the other hand, as shown in Figs. 2-3, the optimal velocity fields $v_h$ and their streamlines computed by our algorithm for different $\gamma$ well preserve the divergence-free condition and also present symmetric patterns. This also explains the symmetric pattern of the temperature distribution shown in Fig. 1. Moreover, the patterns for $v_h$ are very similar for different $\gamma$ values. However, the magnitude of $v_h$ increases as the $\gamma$ value decreases.
Figure 4: Example 5.1. Illustration of results for $\kappa = 1.0$ (a). Plot of profiles in the cost functional with respect to $\gamma$ (here $\|T_h - \langle T_h \rangle\|^2/2 = 4.287E-2$); (b). Convergence results of the cost functional with respect to $\gamma$.

Next, we explore the behavior of the cost functional with respect to $\gamma \in [3.9E-7, 4.1E-6]$. In Fig. 4a, we plot the cost values versus various $\gamma$ values. It shows that smaller values in $\gamma$ lead to smaller cost functional values. When $\gamma = 4E-7$, we obtain $J_{\text{min}} = 2.60E-2$, which is 39% smaller than the initial value (which is 4.287E-2). In Fig. 4b, we plot the convergence rate of $J$ with respect to $\gamma$. It can be seen that the convergence rate gradually decreases from 0.35 to almost 0 as increasing the values in $\gamma$.

Example 5.2. In this example, we consider an asymmetric distribution of the hear source. Let

$$f(x, y) = 1000((x - 0.5)^2 + (y - 0.75)^2)x(1-x)y(1-y).$$
Figure 5: Example 5.2. Plot of optimal $T_h$ for $\kappa = 1.0$ and (a). Initial heat distribution $T_0^0$; (b). $\gamma = 1.8E-6$; (c). $\gamma = 8E-7$; (d). $\gamma = 4E-7$.

The initial heat distribution corresponding to $\gamma = 1.0$ and $v = 0$ is plotted in Fig. 5a. As shown in this figure, the maximum of $T_0^0$ is 4.6E-1. The optimal heat distributions corresponding various values in $\gamma$ are plotted in Fig. 5b-c. We observe similar results as in Example 5.1 i.e., the smaller value in $\gamma$ will yield the lower maximal of the optimal temperature.

The optimal vector fields and their streamlines are demonstrated in Fig. 6-7. The profiles of the cost functional are plotted in Fig. 8. For $\gamma = 4E-7$, we obtain the cost functional value $J_{\text{min}} = 6.76E-3$, which is 25% smaller than the initial value (which is 8.97E-3). In this case, we observe that the convergence rate gradually decreases from 0.22 to almost 0.
Figure 6: Example 5.2. Plot of temperature $T_h$ and vector field $v$ for $\kappa = 1.0$ and (a) $\gamma = 1.8E-6$; (b) $\gamma = 8E-7$; (c) $\gamma = 4E-7$. Here, the color illustrates the magnitude of velocity $v_h$ and the vector plots the field of $v_h$.

Figure 7: Example 5.2. Plot of temperature $T_h$ and vector field $v$ for $\kappa = 1.0$ and (a) $\gamma = 1.8E-6$; (b) $\gamma = 8E-7$; (c) $\gamma = 4E-7$. Here, the color illustrates the magnitude of velocity $v_h$ and the curve plots the streamline of $v_h$.

Figure 8: Example 5.2. Illustration of results for $\kappa = 1.0$: (a). Plot of profiles in the cost functional with respect to $\gamma$ (here $\|T_h^n - \langle T_h^n \rangle\|^2/2 = 8.97E-3$); (b). Convergence results of the cost functional with respect to $\gamma$. 

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Example 5.3. In this example, we continue to examine an asymmetric distribution of the heat source, where the heat source is centered at the upper right corner. We especially examine the behavior of the velocity field for such a heat distribution. Let 

\[ f(x, y) = 100 \exp(-100(x - 0.75)^2 - 100(y - 0.75)^2). \]

The initial heat distribution corresponding to \( \gamma = 1.0 \) and \( v = 0 \) is plotted in Fig. 9a. As shown in this figure, the maximum of \( T_h^0 \) is 7.7E-1. The numerical optimal solutions for heat distribution \( T_h \) are plotted in Fig. 9 for \( \gamma = 6\text{E-7}, 3.7\text{E-7}, \) and \( 3.3\text{E-7}. \) As we can observe in Fig. 12a, the maximum value of the heat distribution is reduced from \( \max T_h^0 = 0.77 \) to \( \max T_h = 0.6, \) \( \max T_h = 0.55, \) and \( \max T_h = 0.54 \) corresponding to \( \gamma = 6\text{E-7}, 3.7\text{E-7}, \) and \( 3.3\text{E-7}, \) respectively. Similar to former examples, smaller value in \( \gamma \) indicates a more effective cooling process.

Fig. 10-11 illustrate the velocity fields and the corresponding streamlines. Based on the direction fields we observe that for each case the velocity tends to “blow” the heat source further to the upper right corner, however due to divergence-free, the heat distribution is stretched toward to the cooler region. For this example, the velocity fields associated with different values of \( \gamma \) also share a similar pattern. The profiles of the cost functional are plotted in Fig. 12. For \( \gamma = 3.3\text{E-7}, \) we obtain the cost function value \( J_{\min} = 7.74\text{E-3}, \) which is 38\% smaller than the initial value (1.24E-2). In this case, we find that the convergence rate gradually decreases from 0.29 to almost 0.
Figure 10: Example 5.3 Plot of optimal $v_h$ for $\kappa = 1.0$ and (a). $\gamma = 6E-7$; (b). $\gamma = 3.7E-7$; (c). $\gamma = 3.3E-7$. Here, the color illustrates the magnitude of velocity $v_h$ and the vector plots the field of $v_h$.

Figure 11: Example 5.3 Plot of optimal $T_h$ for $\kappa = 1.0$ and (a). $\gamma = 6E-7$; (b). $\gamma = 3.7E-7$; (c). $\gamma = 3.3E-7$. Here, the color illustrates the magnitude of velocity $v_h$ and the curve plots the streamline of $v_h$.

Figure 12: Example 5.3 Illustration of results for $\kappa = 1.0$ (a). Plot of initial temperature $T^0_h$ (here $\|T^0_h - \langle T^0_h \rangle\|^2/2 = 1.24E-2$); (b) Plot of profiles in the cost functional with respect to $\gamma$; (c) Convergence results of the cost functional with respect to $\gamma$. 

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Example 5.4. In the last example, we consider that there is a heat source as well as a heat sink and examine how the velocity behaves in an environment with such heat distributions. Let

\[ f(x, y) = 75 \exp\left(-\frac{(9x - 2)^2}{4} - \frac{(9y - 2)^2}{4}\right) - 75 \exp\left(-\frac{(9x - 4)^2}{4} - \frac{(9y - 7)^2}{4}\right). \]

Figure 13: Example 5.4. Plot of optimal \( T_h \) for \( \kappa = 1.0 \) of (a). Initial heat distribution \( T_0^h \); and (b). \( \gamma = 5\times10^{-5} \); (c). \( \gamma = 1\times10^{-5} \); (d). \( \gamma = 6.9\times10^{-6} \).

The initial heat distribution corresponding to \( \gamma = 1.0 \) and \( v = 0 \) is plotted in Fig. 13a. As shown in this figure, the maximum and minimum values of \( T_0^h \) are 1.0 and -1.4, respectively. The numerical optimal solutions for heat distribution \( T_h \) are plotted in Fig. 13b-d for \( \gamma = 5\times10^{-5}, 1\times10^{-5}, \) and \( 6.9\times10^{-6} \). We observe that the upper and lower bounds of the initial temperature are reduced from \( T_{\text{min}} = -1.4 \) and \( T_{\text{max}} = 1 \) (shown in Fig. 16a) to \( (\min T_h = -1.3, \max T_h = 1.0), (\min T_h = -0.82, \max T_h = 0.76), \) and \( (\min T_h = -0.69, \max T_h = 0.95) \) with respective to \( \gamma = 5\times10^{-5}, 1\times10^{-5}, \) and \( 6.9\times10^{-6} \). Different to former examples, it is shown in Figs. 14-15 that the velocity profiles differ significantly for these three values of \( \gamma \). When \( \gamma = 5\times10^{-5}, \) as we can see in Figs. 15-14a, the velocity field seems to steer the cold region toward the hot region and thus the minimum value is increased from -1.4 to -1.3, however the maximum value remains at 1. When \( \gamma = 1\times10^{-5}, \) as shown in Figs. 15-14b, it seems that the cold and the hot regions are advected simultaneously, and hence both the maximum and minimum values are tuned. However, as one further reduces the value in \( \gamma \) from \( 5\times10^{-5} \) to \( 6.9\times10^{-6} \), the circulation between the cold and hot regions becomes
disproportional, which results in a smaller minimum value of the temperature but a higher maximum compared to the case with $\gamma=5\times10^{-5}$. This may be due to the disproportional steering effect of the velocity field shown in Figs. [15][14].

**Figure 14:** Example 5.4 Plot of optimal $v_h$ for $\kappa = 1.0$ and (a). $\gamma = 5\times10^{-5}$; (b). $\gamma = 1\times10^{-5}$; (c). $\gamma = 6.9\times10^{-6}$. Here, the color illustrates the magnitude of velocity $v_h$ and the vector plots the field of $v_h$.

Lastly, the convergence results are plotted in Fig. [16] Similar results as in the previous tests can be observed from these two figures. For $\gamma = 6.9\times10^{-6}$, the cost function $J_{\text{min}} = 9.17\times10^{-2}$, which is 29% smaller than the initial value ($1.29\times10^{-1}$). In this case, we observe that the convergence rate gradually decreases from 0.31 to almost 0.
6. Conclusion

In this paper, we address the optimal control design for convection-cooling via incompressible velocity field. We present rigorous theoretical analysis and conditions to establish the existence and uniqueness of the optimal control. Our numerical experiments demonstrate the effectiveness of the cooling process through flow advection. Moreover, we observe that in order to enhance the heat transfer, small values in $\gamma$ should be employed in the control design. Our next step is to extend the current results to the study of the non-stationary convection-cooling problem. Specially, we shall consider a more realistic setup by incorporating the flow dynamics into the velocity field, which will be controlled in real-time. How to construct effective numerical schemes to address such problems will be further investigated in our future work.

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