The problem of the structure of quasi-stationary electromagnetic field of an infinite solenoid induction loaded with a metal cylinder of constant conductance

A Gerasimov and A Kirpichnikov
Kazan National Research Technological University 420015, Karl Marx str., 68 Kazan, Russia
Kazan National Research Technological University 420015, Karl Marx str., 68 Kazan, Russia

gerasimov@kstu.ru

Abstract. An exact solution of the problem of distribution of the magnetic and electrical field intensities in space around the metal cylinder placed inside an infinite solenoid, with alternating current flowing through it, is obtained. The solution is derived for Maxwell equations, describing the cylindrically symmetric electromagnetic field outside the metal cylinder for complex values of the magnetic and electric components of the electromagnetic field. A comparison between the derived exact solution and an approximate one is drawn. Their asymptotic convergence is shown.

1. Introduction

The classical electrodynamics faces with the problem of determining the distribution of the magnetic and electric field intensities around a metal cylinder with conductance \( \sigma \), placed inside an infinite solenoid, through which an alternating current with frequency \( \omega \) is passed. The problems of such kind have to be solved in the design of devices for inductive heating of metals and high-frequency inductive plasmatrons.

An approximate solution of this problem in the complex formulation is of the form [1]:

when \( R_1 \leq r \leq R_2 \)

\[
H_z = H_z^a(0) \left[ -\text{bei}(R_1) + i\text{ber}(R_1) \right]
\]

\[
E_\phi = \sqrt{\frac{\alpha}{\beta}} H_z^a(0) \left[ i\text{ber}(\sqrt{\alpha\beta}R_1) - \text{bei}(\sqrt{\alpha\beta}R_1) \right] \frac{R_1}{r} - \frac{\omega}{2\epsilon c} H_z^a(0) \left[ (r^2 - R_1^2) \text{ber}(\sqrt{\alpha\beta}R_1) + i\text{bei}(\sqrt{\alpha\beta}R_1) \right];
\]

when \( r \geq R_2 \)

\[
E_\phi = \sqrt{\frac{\alpha}{\beta}} H_z^a(0) \left[ i\text{ber}(\sqrt{\alpha\beta}R_1) - \text{bei}(\sqrt{\alpha\beta}R_1) \right] \frac{R_1}{r} - \frac{\omega}{2\epsilon c} H_z^a(0) \left[ (R_2^2 - R_1^2) \text{ber}(\sqrt{\alpha\beta}R_2) + i\text{bei}(\sqrt{\alpha\beta}R_2) \right]
\]

Here \( H_z \) and \( E_\phi \) are complex amplitudes of the longitudinal magnetic and azimuthal electric fields, respectively; \( c \) is the velocity of light in vacuum; \( R_1 \) is the radius of a cylinder; \( R_2 \) is the radius of a
solenoid; $H_z^0(0)$ is the amplitude of the magnetic field in the center of the cylinder; ber and bei are zero-order Kelvin functions of the first kind; $\alpha = \frac{\omega}{c}$; $\beta = \frac{4\pi \sigma}{c}$.

No magnetic field is present beyond the solenoid.

An exact solution of this problem is of particular interest from both practical and pure scientific points of view.

It is appropriate to recall that the problem of distribution of the magnetic and electric field inside the metal cylinder was first solved by J.J. Thomson in ref. [2].

The derived by J.J. Thomson formula for complex intensity of the axial magnetic field is of the following form:

$$H_z(r) = H_z^0 \left[ -ber\left(\sqrt{\alpha \beta} r\right) + i bei\left(\sqrt{\alpha \beta} r\right) \right].$$

Within the framework of this model, the distribution of the azimuthal electric field inside the cylinder can be written as

$$E_\phi = \sqrt{\frac{\alpha}{\beta}} H_z^0 \left[ ber\left(\sqrt{\alpha \beta} r\right) - i bei\left(\sqrt{\alpha \beta} r\right) \right].$$

2. Theoretical part
Let us find not the approximate but the exact solution of the problem of distribution of the electric and magnetic fields beyond the metal cylinder.

We can write the set of Maxwell equations, describing the cylindrically symmetric electromagnetic field for the complex values of the magnetic $H_z(r,t) = H_z(r) \exp(i\omega t)$ and electric $E_\phi(r,t) = E_\phi(r) \exp(i\omega t)$ components of the electromagnetic field. Finally, we have the set of equations for complex amplitudes $H_z(r)$ and $E_\phi(r)$:

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r E_\phi \right) = -i \alpha H_z;$$

$$\frac{\partial H_z}{\partial r} = i \alpha E_\phi .$$

By raising the order of the set, we obtain

$$\frac{\partial^2 E_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial E_\phi}{\partial r} - \frac{E_\phi}{r} = i \alpha \beta E_\phi;$$

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r H_z \right) = i \beta \alpha H_z ;$$

or

$$\frac{\partial^2 H_z}{\partial r^2} + \frac{1}{r} \frac{\partial H_z}{\partial r} - i \alpha \beta H_z = 0;$$

$$\frac{\partial^2 E_\phi}{\partial r^2} + \frac{1}{r} \frac{\partial E_\phi}{\partial r} - \left( i \alpha \beta + \frac{1}{r^2} \right) E_\phi = 0.$$

For the area $R_2 \geq r \geq R_1$, the solution of the derived set of equations, apparently, is of the following form:

$$H_z(r) = AI_0(\alpha r) + BK_0(\alpha r); \quad E_\phi(r) = -i [AI_1(\alpha r) + BK_1(\alpha r)],$$

where $I_0, I_1(x)$ и $K_0, K_1(x)$ are the modified Bessel and MacDonald functions of the zeroth and first orders, respectively.
Constants $A$ and $B$ are found from the condition of joining the electric and magnetic fields on the boundary $R_1$:

$$H_z(-R_1) = H_z(+R_1); \quad E_\varphi(-R_1) = E_\varphi(+R_1).$$

As a result, the solution inside this region can be written as

$$H_z(r) = \frac{H_z^\infty(0)}{\Delta} \times \left\{ \Phi_1(r) e^{i(\alpha \beta R_1)} - \Psi_1(r) e^{-i(\alpha \beta R_1)} \right\} + i \left\{ \Phi_1(r) e^{i(\alpha \beta R_1)} - \Psi_1(r) e^{-i(\alpha \beta R_1)} \right\},$$

(6)

$$E_\varphi(r) = \frac{H_z^\infty(0)}{\Delta} \times \left\{ \Phi_2(r) e^{i(\alpha \beta R_1)} - \Psi_2(r) e^{-i(\alpha \beta R_1)} \right\} + i \left\{ \Phi_2(r) e^{i(\alpha \beta R_1)} - \Psi_2(r) e^{-i(\alpha \beta R_1)} \right\},$$

(7)

where

$$\Phi_1(r) = I_0(r) K_1(R_1) - I_1(r) K_0(r); \quad \Psi_1(r) = \frac{\alpha}{\beta} [I_0(r) K_0(R_1) - I_0(R_1) K_0(r)],$$

$$\Phi_2(r) = \sqrt{\frac{\alpha}{\beta}} [I_0(r) K_1(R_1) - I_1(r) K_0(r)]; \quad \Psi_2(r) = [I_1(R_1) K_0(r) - I_0(r) K_1(r)];$$

and

$$\Delta = \Phi_1(R_1) = \Phi_2(R_1) = I_0(R_1) K_1(R_1) - I_1(R_1) K_0(R_1).$$

The solution for the electric field outside the solenoid (when $r \geq R_2$) is of the form

$$E_\varphi(r) = C I_1(\alpha r) + DK_1(\alpha r).$$

Constant $C = 0$ follows from apparent physical grounds, while constant $D$ is found from the condition of joining the electric field on the boundary $R_2$:

$$E_\varphi(-R_2) = E_\varphi(+R_2).$$

Finally, the solution in this area takes on the form

$$E_\varphi(r) = E_\varphi(R_2) \times \frac{K_1(r)}{K_1(R_2)} \times \frac{H_z^\infty(0)}{\Delta} \times \left\{ \Phi_2(r) e^{i(\alpha \beta R_1)} - \Psi_2(r) e^{-i(\alpha \beta R_1)} \right\} + i \left\{ \Phi_2(r) e^{i(\alpha \beta R_1)} - \Psi_2(r) e^{-i(\alpha \beta R_1)} \right\} \times \frac{K_1(r)}{K_1(R_2)},$$

(8)

3. Results

Figures 1 and 2 show the distribution of amplitudes of the longitudinal magnetic and azimuthal electric fields of particular practical interest, that have been calculated from formulae (6)-(8). Here are also the results of an approximate solution [1] calculated from formulae (1)-(3). Figures 1 and 2 also demonstrate the results of calculations for the J.J. Thomson’s solution inside the cylinder made according to formulae (4) and (5).

The data have been calculated for the case of $H_z^\infty(0) = 24.8$ Oe, $f = 1.76$ MHz of the high-frequency induction device VChI-11-60 with power of 60 kW, $R_1 = 3.3$ cm, $R_2 = 4$ cm, as described in ref. [3-7].
As one may see from Figs. 1 and 2, in the area $R_1 \leq r \leq R_2$ the exact solution passes higher than the approximate one. Therefore, approximate calculations of the distribution of the electromagnetic field near the solenoid appear to be severalfold underestimated as compared with the exact solution. This fact should be taken into account in design of high-frequency induction devices for thermal heating of conducting media. With increasing radial coordinate, both solutions approach each other asymptotically, which is demonstrated in Fig. 3.
4. Conclusion

The results obtained in the present work can be used by specialists in the field of high-frequency induction electrothermics and for designing the energy facilities based on the principle of induction heating of conducting media.

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