Rigidity of Gradient Shrinking Ricci Solitons

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Abstract

We prove that a gradient shrinking Ricci soliton with fourth order divergence-free Riemannian tensor is rigid. For the 4-dimensional case, we show that any gradient shrinking Ricci soliton with fourth order divergence-free Riemannian tensor is either Einstein, or a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^4$, $\mathbb{R}^2 \times S^2$ or the round cylinder $\mathbb{R} \times S^3$. Under the condition of fourth order divergence-free Weyl tensor, we have the same results.

Keywords: Rigidity; Gradient shrinking Ricci soliton; Riemannian curvature; Weyl tensor.

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1 Introduction

A complete Riemannian manifold $(M^n, g, f)$ is called a gradient Ricci soliton if there exists a smooth function $f$ on $M^n$ such that the Ricci tensor $Ric$ of the metric $g$ satisfies the equation

$$Ric + \nabla^2 f = \lambda g$$ (1.1)
for some constant $\lambda$. For $\lambda > 0$ the Ricci soliton is shrinking, for $\lambda = 0$ it is steady and for $\lambda < 0$ expanding.

The classification of gradient Ricci solitons has been a subject of interest for many people in recent years. For four-dimensional gradient Ricci solitons, A. Naber [12] showed that a four-dimensional non-compact shrinking Ricci soliton with bounded nonnegative Riemannian curvature is a finite quotient of $\mathbb{R}^4$, $\mathbb{R}^2 \times S^2$ or $\mathbb{R} \times S^3$. X. Chen and Y. Wang [6] classified four-dimensional anti-self dual gradient steady and shrinking Ricci solitons. More generally, J.Y. Wu, P. Wu and W. Wylie [17] proved that a four-dimensional gradient shrinking Ricci soliton with half harmonic Weyl tensor (i.e. $\text{div} W^\pm = 0$) is either Einstein or a finite quotient of $\mathbb{R}^4$, $\mathbb{R}^2 \times S^2$ or $\mathbb{R} \times S^3$.

For $n$-dimensional gradient Ricci solitons, M. Eminenti, G. La Nave and C. Mantegazza [7] proved that an $n$-dimensional compact shrinking Ricci soliton with vanishing Weyl tensor is a finite quotient of $S^n$. More generally, P. Peterson and W. Wylie [14] showed that a gradient shrinking Ricci soliton with vanishing Weyl tensor is a finite quotient of $\mathbb{R}^n$, $S^{n-1} \times \mathbb{R}$, or $S^n$ by assuming $\int_M |\text{Ric}|^2 e^{-f} < \infty$. The integral assumption was proven to be true for gradient shrinking Ricci solitons (see Theorem 1.1 of [11]). Without additional assumptions, Z. H. Zhang [18] obtained the same classification of gradient shrinking Ricci solitons with vanishing Weyl tensor.

H. D. Cao and Q. Chen [1] introduced the covariant 3-tensor $D$, i.e.

$$D_{ijk} = \frac{1}{n-2}(R_{jk} \nabla_i f - R_{ik} \nabla_j f) + \frac{1}{2(n-1)(n-2)}(g_{jk} \nabla_i R - g_{ik} \nabla_j R)$$

$$- \frac{R}{(n-1)(n-2)}(g_{jk} \nabla_i f - g_{ik} \nabla_j f).$$

to study the classification of locally conformally flat gradient steady solitons. The vanishing of $D$ is a crucial ingredient in their classification results. They [2] proved that a compact gradient shrinking Ricci solitons with $D = 0$ is Einstein. Moreover, they showed that a four-dimensional complete non-compact Bach-flat gradient shrinking Ricci soliton is a finite quotient of $\mathbb{R}^4$ or $\mathbb{R} \times S^3$. More generally, they proved that a $n$-dimensional ($n \geq 5$) complete non-compact Bach-flat gradient shrinking Ricci soliton with is a finite quotient of $\mathbb{R}^n$ or $\mathbb{R} \times N^{n-1}$, where $N$ is an $(n-1)$-dimensional Einstein manifold.

H. D. Cao and Q. Chen [2] proved that a Bach-flat gradient shrinking Ricci soliton has vanishing $D$, where the Bach tensor is given by

$$B_{ij} = \frac{1}{n-3} \nabla_k \nabla_l W_{ikjl} + \frac{1}{n-2} R_{kl} W_{ikjl}.$$
Moreover, they showed that the 3-tensor $D$ is closely related to Cotton tensor, i.e.

$$C_{ijk} = \nabla_i R_{jk} - \nabla_j R_{ik} - \frac{1}{2(n-1)} (g_{jk} \nabla_i R - g_{ik} \nabla_j R),$$

and Weyl tensor, i.e.

$$W_{ijkl} = R_{ijkl} - \frac{1}{n-2} (g_{ik} R_{jl} - g_{il} R_{jk} - g_{jk} R_{il} + g_{jl} R_{ik}) + \frac{R}{(n-1)(n-2)} (g_{ik} g_{jl} - g_{il} g_{jk})$$

by

$$D_{ijk} = C_{ijk} + W_{ijkl} \nabla_l f.$$

M. Fernández-López and E. García-Río [8] proved that the compact Ricci soliton is rigid if and only if it has harmonic Weyl tensor. For the complete non-compact case, O. Munteanu and N. Sesum [11] showed that a gradient shrinking Ricci soliton with harmonic Weyl tensor is rigid.

In 2016, G. Catino, P. Mastrolia and D. D. Monticelli [4] proved that the gradient shrinking Ricci soliton is rigid if $\text{div}^4 W = 0$. In their paper, $\text{div}^4$ is defined by $\text{div}^4 W = \nabla_k \nabla_j \nabla_i \nabla_l W_{ijkl}$. They showed that $\text{div}^4 W = 0$ if and only if $\text{div}^3 C = 0$, where $\text{div}^3 C = \nabla_i \nabla_j \nabla_k C_{ijk}$. Then, they proved that $\text{div}^3 C = 0$ implies $C = 0$. The rigidity result follows.

S. Tachibana [15] proved that a compact orientable Riemannian manifold with $Rm > 0$ and $\text{div} Rm = 0$ is a space of constant curvature. P. Peterson and W. Wylie [13] proved that a compact shrinking gradient Ricci soliton is Einstein if $\int_M \text{Ric}(\nabla f, \nabla f) \leq 0$. They also showed that a gradient Ricci soliton is rigid if and only if it has constant scalar curvature and is radially flat.

In order to state our results precisely, we introduce the following definitions for the Riemannian curvature:

$$(\text{div} Rm)_{ijkl} := \nabla_i R_{ijkl},$$

$$(\text{div}^2 Rm)_{ik} := \nabla_j \nabla_i R_{ijkl},$$

$$(\text{div}^3 Rm)_i := \nabla_k \nabla_j \nabla_l R_{ijkl},$$

$$(\text{div}^4 Rm) := \nabla_i \nabla_k \nabla_j \nabla_l R_{ijkl}.$$
For the Weyl curvature tensor, we define:

\[
(div W)_{ijk} := \nabla_l W_{ijkl},
\]
\[
(div^2 W)_{ik} := \nabla_j \nabla_l W_{ijkl},
\]
\[
(div^3 W)_i := \nabla_k \nabla_j \nabla_l W_{ijkl},
\]
\[
div^4 W := \nabla_i \nabla_k \nabla_j \nabla_l W_{ijkl}.
\]

Our main results are the following theorems for gradient shrinking Ricci solitons:

**Theorem 1.1** Let \((M^n, g)\) be a gradient shrinking Ricci soliton with (1.1). If \(div^4 Rm = 0\), then \((M^n, g)\) is rigid.

**Theorem 1.2** Let \((M^n, g)\) be a gradient shrinking Ricci soliton with (1.1). If \(div^3 Rm(\nabla f) = 0\), then \((M^n, g)\) is rigid.

G. Catino, P. Mastrolia and D. D. Monticelli [4] proved that a gradient shrinking Ricci soliton with \(div^4 W = 0\) is rigid. We will give a different proof in Section 8 Appendix. Moreover, we have the following result:

**Theorem 1.3** Let \((M^n, g)\) be a gradient shrinking Ricci soliton with (1.1). If \(div^3 W(\nabla f) = 0\), then \((M^n, g)\) is rigid.

For the 4-dimensional case, we have the following classification theorems:

**Theorem 1.4** Let \((M^4, g)\) be a 4-dimensional gradient shrinking Ricci soliton with (1.1). If \(div^4 Rm = 0\), then \((M^4, g)\) is either

(i) Einstein, or

(ii) a finite quotient of the Gaussian shrinking soliton \(\mathbb{R}^4, \mathbb{R}^2 \times \mathbb{S}^2\) or the round cylinder \(\mathbb{R} \times \mathbb{S}^3\).

**Theorem 1.5** Let \((M^4, g)\) be a 4-dimensional gradient shrinking Ricci soliton with (1.1). If \(div^3 Rm(\nabla f) = 0\), then \((M^4, g)\) is either

(i) Einstein, or

(ii) a finite quotient of the Gaussian shrinking soliton \(\mathbb{R}^4, \mathbb{R}^2 \times \mathbb{S}^2\) or the round cylinder \(\mathbb{R} \times \mathbb{S}^3\).

**Theorem 1.6** Let \((M^4, g)\) be a 4-dimensional gradient shrinking Ricci soliton with (1.1). If \(div^3 W(\nabla f) = 0\), then \((M^4, g)\) is either
(i) Einstein, or
(ii) a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^4$, $\mathbb{R}^2 \times S^2$ or the round cylinder $\mathbb{R} \times S^3$.

**Remark 1.1** As it will be clear from the proof, the scalar assumptions on the vanishing of $\text{div}^4 Rm$, $\text{div}^3 Rm(\nabla f)$, and $\text{div}^3 W(\nabla f)$ in all the above theorems can be trivially relaxed to a (suitable) inequality. To be precise, Theorem 1.1 and Theorem 1.4 hold just assuming $\text{div}^4 Rm \geq 0$. Under the condition of $\text{div}^3 Rm(\nabla f) \geq 0$, Theorem 1.2 and Theorem 1.5 still hold. Moreover, Theorem 1.3 and Theorem 1.6 hold for $\text{div}^3 W(\nabla f) \geq 0$.

The rest of this paper is organized as follows. In Section 2, we recall some background material and prove some formulas which will be needed in the proof of the main theorems. In Section 3, we will prove that the compact gradient Ricci soliton with fourth divergence-free Riemannian tensor is Einstein. The proof makes use of a rigid theorem obtained by P. Peterson and W. Wylie [13]. In Section 4, we will deal with the complete noncompact case of Theorem 1.1. In Section 5, we give a direct proof of Theorem 1.2. We first prove divergence formulas of the Weyl tensor in Section 6, then we will prove Theorem 1.3. Finally, in Section 7 we will finish the proof of Theorems 1.4 to 1.6.

## 2 Preliminaries

First of all, we present some basic facts of gradient shrinking Ricci solitons.

**Proposition 2.1** ([7,10,11,14]) Let $(M^n, g)$ be a gradient shrinking Ricci soliton with (1.1), we have the following identities.

\[
\begin{align*}
\nabla_i R_{ijkl} &= \nabla_j R_{ik} - \nabla_i R_{jk}, \quad (2.2) \\
\nabla R &= 2\text{div}Ric, \quad (2.3) \\
R_{ijkl} \nabla_l f &= \nabla_i R_{ijkl}, \quad (2.4) \\
\nabla_i (R_{ijkl} e^{-f}) &= 0, \quad (2.5) \\
R_{jl} \nabla_l f &= \nabla_i R_{jl}, \quad (2.6) \\
\nabla_i (R_{jl} e^{-f}) &= 0, \quad (2.7)
\end{align*}
\]
\[ \nabla R = 2Ric(\nabla f, \cdot), \quad (2.8) \]
\[ \Delta f R_{ik} = 2\lambda R_{ik} - 2R_{ijkl}R_{jl}, \quad (2.9) \]
\[ \Delta f R = 2\lambda R - 2|Ric|^2, \quad (2.10) \]

where \( \Delta f := \Delta - \nabla \nabla f, \)
\[ \Delta f |Ric|^2 = 4\lambda |Ric|^2 - 4Rm(Ric, Ric) + 2|\nabla Ric|^2, \quad (2.11) \]

where \( Rm(Ric, Ric) = R_{ijkl}R_{ik}R_{jl}, \) and
\[ R + |\nabla f|^2 - 2\lambda f = Const. \quad (2.12) \]

Next we prove the following formulas for gradient shrinking Ricci soliton with (1.1).

**Proposition 2.2** Let \((M^n, g)\) be a gradient shrinking Ricci soliton with (1.1), we have the following identities.

\[ (\text{div}^2 Rm)_{ik} = 2\lambda R_{ik} + \nabla_i R_{ik} \nabla_j f - \frac{1}{2} \nabla_i \nabla_k R - R_{ik}^2 - R_{ijkl}R_{jl}, \quad (2.13) \]
\[ (\text{div}^3 Rm)_i = -R_{ijkl} \nabla_k R_{jl}, \quad (2.14) \]
and
\[ \text{div}^4 Rm = \nabla_i R_{jk} \nabla_k R_{jl} - |\nabla Ric|^2 - R_{ijkl} \nabla_k \nabla_l R_{jl}. \quad (2.15) \]

**Proof.** By direct computation,

\[ (\text{div}^2 Rm)_{ik} = \nabla_j \nabla_i R_{ijkl} \]
\[ = \Delta R_{ik} - \nabla_j \nabla_i R_{jk} \]
\[ = \Delta f R_{ik} + \nabla_i R_{ik} \nabla_j f - \nabla_i \nabla_j R_{jk} + R_{ijkl} R_{jl} - R_{ik}^2 \]
\[ = 2\lambda R_{ik} - 2R_{ijkl} R_{jl} + \nabla_i R_{ik} \nabla_j f - \frac{1}{2} \nabla_i \nabla_k R + R_{ijkl} R_{jl} - R_{ik}^2 \]
\[ = 2\lambda R_{ik} - R_{ijkl} R_{jl} + \nabla_i R_{ik} \nabla_j f - \frac{1}{2} \nabla_i \nabla_k R - R_{ik}^2, \]

where we used (2.2) in the second equality. Moreover, we used (2.3) and (2.9) in the fourth equality.
Using (2.13), we have

\[
(d\text{iv}^3 Rm)_i = \nabla_k \nabla_j \nabla_l R_{ijkl}
\]

\[
= \nabla_k (2\lambda R_{ik} - R_{ijkl} R_{jl} + \nabla_l R_{ik} \nabla_l f - \frac{1}{2} \nabla_i \nabla_k R - R^{jk}_{ik})
\]

\[
= \lambda \nabla_i R - \nabla_k R_{ijkl} R_{jl} - R_{ijkl} \nabla_k R_{jl} + \nabla_l R_{ik} \nabla_l f + \nabla_k \nabla_l R_{ik} \nabla_l f
\]

\[-\frac{1}{2} \nabla_k \nabla_i \nabla_k R - R_{ij} \nabla_k R_{kj} - R_{kj} \nabla_k R_{ij}
\]

\[
= \lambda \nabla_i R + (\nabla_j R_{il} - \nabla_i R_{jl}) R_{jl} - R_{ijkl} \nabla_k R_{jl} + \nabla_l R_{ik} (\lambda g_{kl} - R_{kl})
\]

\[+(\nabla_l \nabla_k R_{ik} + R_{ij} R_{ij} + R_{klij} R_{jk}) \nabla_l f - \frac{1}{2} \nabla_i \Delta f - R_{ij} \nabla_l R_{ij} - \frac{1}{2} \nabla_i (\nabla_k R \nabla_k f)
\]

\[-\frac{1}{2} R_{ij} \nabla_j R - \frac{1}{2} R_{ij} \nabla_j R - R_{klij} \nabla_k R_{ij}
\]

\[
= \lambda \nabla_i R + R_{jl} \nabla_j R_{il} - \frac{1}{2} \nabla_i |\text{Ric}|^2 - R_{ijkl} \nabla_k R_{jl} + \frac{\lambda}{2} \nabla_i R
\]

\[-R_{kl} \nabla_l R_{ik} + \frac{1}{2} \nabla_i \nabla_l R \nabla_l f + \frac{1}{2} R_{ij} \nabla_j R + R_{jk} \nabla_l R_{ijkl}
\]

\[-\frac{\lambda}{2} \nabla_i R + \nabla_l |\text{Ric}|^2 - \frac{1}{2} \nabla_i \nabla_l R \nabla_l f - \frac{1}{2} \nabla_l R \nabla_i \nabla_l f
\]

\[-R_{ij} \nabla_j R - R_{klij} \nabla_k R_{ij}
\]

\[
= \frac{1}{2} \nabla_i |\text{Ric}|^2 - R_{ijkl} \nabla_k R_{jl} + \frac{\lambda}{2} \nabla_i R - \frac{1}{2} R_{ik} \nabla_k R + R_{jk} \nabla_j R_{ik}
\]

\[-\frac{1}{2} \nabla_i |\text{Ric}|^2 - \frac{\lambda}{2} \nabla_i R + \frac{1}{2} R_{il} \nabla_l R - R_{klij} \nabla_k R_{ij}
\]

\[
= -R_{ijkl} \nabla_k R_{jl},
\]

where we used (2.3) in the third equality, used (2.2) and (1.1) in the fourth equality. Moreover, we used (2.4), (2.8) and (2.10) in the fifth equality. In the sixth equality, we used (1.1) and (2.2).

It follows from (2.14) that

\[d\text{iv}^4 Rm = \nabla_i \nabla_k \nabla_j \nabla_l R_{ijkl}
\]

\[= -\nabla_i R_{ijkl} \nabla_k R_{jl} - R_{ijkl} \nabla_i \nabla_k R_{jl}
\]

\[= (\nabla_i R_{jk} - \nabla_k R_{jl}) \nabla_i \nabla_k R_{jl} - R_{ijkl} \nabla_i \nabla_k R_{jl}
\]

\[= \nabla_i R_{jk} \nabla_i \nabla_k R_{jl} - |\nabla\text{Ric}|^2 - R_{ijkl} \nabla_i \nabla_k R_{jl},
\]

where we used (2.2) in the third equality. □
Remark 2.1 It follows from (2.13) that \( \text{div}^2 Rm \) is a symmetric 2-tensor. Therefore, we have the following identities.

\[
\begin{align*}
\text{(div}^2 Rm)_{ik} &= \nabla_j \nabla_l R_{ijkl} = \nabla_l \nabla_j R_{ijkl}, \\
\text{(div}^3 Rm)_i &= \nabla_k \nabla_j \nabla_l R_{ijkl} = \nabla_k \nabla_i \nabla_j R_{ijkl} = \nabla_k \nabla_l \nabla_j R_{ijkl}, \\
\text{and} \\
div^4 Rm &= \nabla_i \nabla_k \nabla_j \nabla_l R_{ijkl} = \nabla_i \nabla_k \nabla_l \nabla_j R_{ijkl} = \nabla_i \nabla_l \nabla_j R_{ijkl}.
\end{align*}
\]

Finally, we list following results that will be needed in the proof of the main theorems.

**Lemma 2.1** Let \((M^n, g)\) be a complete gradient shrinking soliton with (1.1). Then it has nonnegative scalar curvature \( R \geq 0 \).

**Remark 2.2** Lemma 2.1 is a special case of a more general result of B. L. Chen [5] which states that \( R \geq 0 \) for any ancient solution to the Ricci flow.

**Lemma 2.2** (H. D. Cao and D. Zhou [3]) Let \((M^n, g)\) be a complete gradient shrinking soliton with (1.1). Then,

(i) the potential function \( f \) satisfies the estimates

\[
\frac{1}{4}(r(x) - c_1)^2 \leq f(x) \leq \frac{1}{4}(r(x) + c_2)^2, \tag{2.16}
\]

where \( r(x) = d(x_0, x) \) is the distance function from some fixed point \( x_0 \in M \), \( c_1 \) and \( c_2 \) are positive constants depending only on \( n \) and the geometry of \( g \) on the unit ball \( B(x_0, 1) \);

(ii) there exists some constant \( C > 0 \) such that

\[
\text{Vol}(B(x_0, s)) \leq Cs^n \tag{2.17}
\]

for \( s > 0 \) sufficiently large.

**Lemma 2.3** (P. Petersen and W. Wylie [13]) A shrinking compact gradient soliton is rigid with trivial \( f \) if

\[
\int_M \text{Ric}(\nabla f, \nabla f) \leq 0. \tag{2.18}
\]
Lemma 2.4 (P. Petersen and W. Wylie [13]) A gradient soliton is rigid if and only if it has constant scalar curvature and is radially flat, that is, $\sec(E, \nabla f) = 0$.

Remark 2.3 The condition of $\text{div} Rm = 0$ is stronger than $\sec(E, \nabla f) = 0$.

Lemma 2.5 (O. Munteanu and N. Sesum [11]) For any complete gradient shrinking Ricci soliton with (1), we have

$$\int_M |Ric|^2 e^{-\alpha f} < +\infty$$

for any $\alpha > 0$.

Lemma 2.6 (O. Munteanu and N. Sesum [11]) Let $(M, g)$ be a gradient shrinking Ricci soliton. If for some $\beta < 1$ we have $\int_M |Rm|^2 e^{-\beta f} < +\infty$, then the following identity holds.

$$\int_M |\text{div} Rm|^2 e^{-f} = \int_M |\nabla Ric|^2 e^{-f} < +\infty.$$

(2.20)

3 The Compact Case of Theorem 1.1

In this section, we prove the compact case of Theorem 1.1:

Theorem 3.1 Let $(M^n, g)$ be a compact gradient shrinking Ricci soliton with (1.1). If $\text{div}^4 Rm = 0$, then $(M^n, g)$ is Einstein.

The first step in proving Theorem 3.1 is to obtain the following integral equation.

Lemma 3.1 Let $(M^n, g)$ be a compact gradient shrinking Ricci soliton with (1.1), then

$$\int_M \nabla_l R_{jk} \nabla_k R_{lj} e^{-f} = \frac{1}{2} \int_M |\nabla Ric|^2 e^{-f}.$$

(3.21)
Proof. Calculating directly, we have
\[
\begin{align*}
\int_M \nabla_l R_{jk} \nabla_k R_{jl} e^{-f} &= -\int_M R_{jk} \nabla_l \nabla_k R_{jl} e^{-f} + \int_M R_{jk} \nabla_k R_{jl} \nabla_l f e^{-f} \\
&= -\int_M R_{jk} (\nabla_k \nabla_l R_{jl} + R_{jl} R_{pk} + R_{lkji} R_{il}) e^{-f} \\
&\quad + \int_M R_{jk} \nabla_k (R_{jl} \nabla_l f) e^{-f} - \int_M R_{jk} R_{jl} \nabla_k \nabla_l f e^{-f} \\
&= -\int_M R_{jk} (R_{jl} R_{pk} + R_{lkji} R_{il}) e^{-f} - \int_M R_{jk} R_{jl} (\lambda g_{kl} - R_{kl}) e^{-f} \\
&= -\int_M \text{tr Ric}^3 e^{-f} + \int_M Rm(\text{Ric}, \text{Ric}) e^{-f} - \lambda \int_M |\text{Ric}|^2 e^{-f} + \int_M \text{tr Ric}^3 e^{-f} \\
&= \int_M Rm(\text{Ric}, \text{Ric}) e^{-f} - \lambda \int_M |\text{Ric}|^2 e^{-f}, \tag{3.22}
\end{align*}
\]
where we used (2.6) and (1.1) in the third equality.

Applying (2.11) to (3.22), we obtain
\[
\begin{align*}
\int_M \nabla_l R_{jk} \nabla_k R_{jl} e^{-f} &= -\frac{1}{4} \int_M \Delta_f |\text{Ric}|^2 e^{-f} + \frac{1}{2} \int_M |\nabla \text{Ric}|^2 e^{-f} \\
&= -\frac{1}{4} \int_M (\Delta |\text{Ric}|^2 - \nabla_f |\text{Ric}|^2) e^{-f} + \frac{1}{2} \int_M |\nabla \text{Ric}|^2 e^{-f} \\
&= -\frac{1}{4} \int_M \nabla_f |\text{Ric}|^2 e^{-f} + \frac{1}{4} \int_M \nabla_f |\text{Ric}|^2 e^{-f} + \frac{1}{2} \int_M |\nabla \text{Ric}|^2 e^{-f} \\
&= \frac{1}{2} \int_M |\nabla \text{Ric}|^2 e^{-f}.
\end{align*}
\]
Now we are ready to prove Theorem 3.1.

Proof of Theorem 3.1:
Integrating (2.15), we obtain

\[ \int_M \text{div}^4 Rm e^{-f} = \int_M \nabla_i R_{jk} \nabla_k R_{jl} e^{-f} - \int_M |\nabla Ric|^2 e^{-f} - \int_M R_{ijkl} \nabla^i \nabla^k R_{jl} e^{-f} = \frac{1}{2} \int_M |\nabla Ric|^2 e^{-f} - \int_M |\nabla Ric|^2 e^{-f} = \frac{1}{2} \int_M |\nabla Ric|^2 e^{-f}, \]  

(3.23)

where we used Lemma 3.1 and (2.5) in the second equality.

Since \( \text{div}^4 Rm = 0 \), it follows from (3.23) that \( \int_M |\nabla Ric|^2 e^{-f} = 0 \), i.e. \( |\nabla Ric| = 0 \text{ a.e.} \). Note that any gradient shrinking Ricci soliton is analytic in harmonic coordinates, we have \( |\nabla Ric| = 0 \) on \( M \).

By direct computation, we have

\[ 0 \leq |\nabla Ric - \frac{\nabla R}{n} g|^2 = |\nabla Ric|^2 - \frac{|\nabla R|^2}{n} = -\frac{|\nabla R|^2}{n} \leq 0. \]

Therefore, \( R \) is a constant on \( M \). It follows from (2.8) that \( \text{Ric}(\nabla f, \nabla f) = \frac{1}{2} \langle \nabla R, \nabla f \rangle = 0 \). By Lemma 2.3, \( (M^n, g) \) is rigid. The compactness of \( (M^n, g) \) implies that \( (M^n, g) \) is Einstein. \( \square \)

4 The Complete Non-compact Case of Theorem 1.1

In this section, we prove the complete non-compact case of Theorem 1.1:

**Theorem 4.1** Let \( (M^n, g) \) be a complete non-compact gradient shrinking Ricci soliton with (1.1). If \( \text{div}^4 Rm = 0 \), then \( (M^n, g) \) is rigid.

The first step in proving Theorem 4.1 is to obtain the following integral inequality.

**Lemma 4.1** Let \( (M^n, g) \) be a complete non-compact gradient shrinking Ricci soliton with (1.1). For every \( C^2 \) function \( \phi : \mathbb{R}_+ \to \mathbb{R} \) with \( \phi(f) \) having com-
pact support in $M$ and some constant $c > 0$, we have
\[
\int_M \nabla_i R_{jk} \nabla_k R_{jl} \phi^2(f) e^{-f} \leq c \int_M |\text{Ric}|^2 |\nabla f|^2 (\phi')^2 e^{-f} + \frac{3}{4} \int_M |\nabla \text{Ric}|^2 \phi^2(f) e^{-f}.
\]
(4.24)

**Proof.** By direct computation, we have
\[
\int_M \nabla_i R_{jk} \nabla_k R_{jl} \phi^2(f) e^{-f}
= -\int_M R_{jk} \nabla_i (\nabla R_{jl}) \phi^2(f) e^{-f} - \int_M R_{jk} \nabla k R_{jl} \nabla_t \phi^2(f) e^{-f} + \int_M R_{jk} \nabla_k R_{jl} \nabla_t \phi^2(f) e^{-f}
= -\int_M R_{jk} \nabla_i (\nabla R_{jl}) \phi^2(f) e^{-f} - \int_M R_{jk} \nabla k R_{jl} \nabla_t \phi^2(f) e^{-f} + \int_M R_{jk} \nabla_k R_{jl} \nabla_t \phi \phi' e^{-f}
+ \int_M R_{jk} \nabla_k (R_{jl} \nabla_t \phi) \phi^2(f) e^{-f} - \int_M R_{jk} \nabla k R_{jl} \nabla_t \phi \phi' e^{-f}
= -\int_M R_{jk} (R_{jl} R_{pk} + R_{lkji} R_{il}) \phi^2(f) e^{-f} - 2 \int_M R_{jk} \nabla k R_{jl} \nabla_t \phi \phi' e^{-f}
- \lambda \int_M \nabla_k |\text{Ric}|^2 \phi^2(f) e^{-f} + \int_M \nabla_k |\text{Ric}|^2 (\phi')^2 e^{-f}
\]
(4.25)

where we used (2.7) and (1.1) in the third equality.
Applying (2.11) to (4.25), we obtain

\[
\int_M \nabla_i R_{jk} \nabla_k R_{jl} \phi^2(f) e^{-f} = -2 \int_M R_{jk} \nabla_k R_{jl} \phi \phi' e^{-f} - \frac{1}{4} \int_M \Delta_f |\text{Ric}|^2 \phi^2(f) e^{-f} + \frac{1}{2} \int_M |\nabla \text{Ric}|^2 \phi^2(f) e^{-f}
\]

\[
= -2 \int_M R_{jk} \nabla_k R_{jl} \phi \phi' e^{-f} - \frac{1}{4} \int_M \Delta |\text{Ric}|^2 \phi^2(f) e^{-f}
\]

\[
+ \frac{1}{4} \int_M \nabla \phi \phi' e^{-f} - \frac{1}{4} \int_M \nabla |\text{Ric}|^2 \phi^2(f) e^{-f}
\]

\[
= -2 \int_M R_{jk} \nabla_k R_{jl} \phi \phi' e^{-f} + \frac{1}{2} \int_M \nabla \phi \phi' e^{-f} + \frac{1}{2} \int_M |\text{Ric}|^2 \phi^2(f) e^{-f}
\]

\[
= -2 \int_M R_{jk} \nabla_k R_{jl} \phi \phi' e^{-f} + \int_M R_{ik} \nabla_k R_{jk} \phi \phi' e^{-f}
\]

\[
+ \frac{1}{2} \int_M |\nabla \text{Ric}|^2 \phi^2(f) e^{-f}
\]

\[
\leq c \int_M |\text{Ric}| |\nabla f| |\nabla \text{Ric}| |\phi||\phi'| e^{-f} + \frac{1}{2} \int_M |\nabla |\text{Ric}|^2 \phi^2(f) e^{-f}
\]

\[
\leq c \int_M |\text{Ric}|^2 |\nabla f|^2 (\phi')^2 e^{-f} + \frac{3}{4} \int_M |\nabla \text{Ric}|^2 \phi^2(f) e^{-f}
\]

for some constant \( c > 0 \).

□

Lemma 4.2 Let \((M^n, g)\) be a complete non-compact gradient shrinking Ricci soliton with (1.1). For every \( C^2 \) function \( \varphi : \mathbb{R}_+ \to \mathbb{R} \) with \( \varphi(f) \) having compact support in \( M \), we have

\[
- \int_M R_{ijkl} \nabla_i \nabla_k R_{jl} \varphi(f) e^{-f} = \int_M (|\nabla \text{Ric}|^2 - \nabla_i R_{kj} \nabla_k R_{jl}) \varphi' e^{-f}. \tag{4.26}
\]
Proof. By direct computation, we have

\[ -\int_M R_{ijkl} \nabla_i \nabla_k R_{jl} \phi(f)e^{-f} = \int_M R_{ijkl} \nabla_k R_{jl} \phi(e^{-f}) \]

\[ = \int_M \nabla_i R_{ijkl} \nabla_k R_{jl} \phi' e^{-f} \]

\[ = \int_M (\nabla_k R_{jl} - \nabla_l R_{kj}) \nabla_k R_{jl} \phi' e^{-f} \]

\[ = \int_M (|\nabla Riem|^2 - \nabla_l R_{kj} \nabla_k R_{jl}) \phi' e^{-f}, \]

where we used (2.5), (2.4) and (2.2) in the first, second and third equality, respectively.

Now we are ready to prove Theorem 4.1.

Proof of Theorem 4.1:
Let \( \phi : \mathbb{R}_+ \to \mathbb{R} \) be a \( C^2 \) function with \( \phi = 1 \) on \((0, s]\), \( \phi = 0 \) on \([2s, \infty) \) and \( -\frac{c}{2} \leq \phi'(t) \leq 0 \) on \((s, 2s) \) for some constant \( c > 0 \). Define

\[ D(r) := \{ x \in M | f(x) \leq r \}. \]

By Lemma 4.2, we have

\[ -\int_M R_{ijkl} \nabla_i \nabla_k R_{jl} \phi^2(f)e^{-f} = \int_M (|\nabla Riem|^2 - \nabla_l R_{kj} \nabla_k R_{jl}) \phi^2 e^{-f} \]

\[ = 2 \int_M (|\nabla Riem|^2 - \nabla_l R_{kj} \nabla_k R_{jl}) \phi \phi' e^{-f} \]

\[ \leq 0. \] (4.27)

Integrating (2.15) and using Lemma 4.1 and (4.27), we have

\[ \int_M \text{div}^4 Rm \phi^2(f)e^{-f} \]

\[ = \int_M \nabla_i R_{ijkl} \nabla_k R_{jl} \phi^2(f)e^{-f} - \int_M |\nabla Riem|^2 \phi^2(f)e^{-f} - \int_M R_{ijkl} \nabla_i \nabla_k R_{jl} \phi^2(f)e^{-f} \]

\[ \leq c \int_M |Riem|^2 |\nabla f|^2 (\phi')^2 e^{-f} + \frac{3}{4} \int_M |\nabla Riem|^2 \phi^2(f)e^{-f} - \int_M |\nabla Riem|^2 \phi^2(f)e^{-f} \]

\[ \leq \frac{c}{s^2} \int_{D(2s) \setminus D(s)} |Riem|^2 |\nabla f|^2 e^{-f} - \frac{1}{4} \int_M |\nabla Riem|^2 \phi^2(f)e^{-f}. \] (4.28)
It follows from Lemma 2.1, (2.12), (2.16) and Lemma 2.5 that
\[
\int_M |\text{Ric}|^2 |\nabla f|^2 e^{-f} \leq \int_M |\text{Ric}|^2 e^{-\alpha f} < +\infty
\]
for some \( \alpha \in (0, 1] \). Therefore,
\[
\frac{c}{s^2} \int_{D(2s) \setminus D(s)} |\text{Ric}|^2 |\nabla f|^2 e^{-f} \to 0
\]
as \( s \to +\infty \).

By taking \( r \to +\infty \) in (4.28), we obtain \( \int_M |\nabla \text{Ric}|^2 e^{-f} = 0 \). Since \( \int_M |\nabla \text{Ric}|^2 e^{-f} < +\infty \), it follows from (2.20) that
\[
\int_M |\text{div} Rm|^2 e^{-f} = \int_M |\nabla \text{Ric}|^2 e^{-f} = 0.
\]

Hence, \( |\text{div} Rm| = |\nabla \text{Ric}| = 0 \text{ a.e.} \). Note that any gradient shrinking Ricci soliton is analytic in harmonic coordinates, we have \( |\text{div} Rm| = |\nabla \text{Ric}| = 0 \) on \( M \).

It is clear \( \text{div} Rm = 0 \) implies that \( M^n \) is radially flat.

By direct computation, we have
\[
0 \leq |\nabla \text{Ric} - \frac{\nabla R}{n} g|^2 = |\nabla \text{Ric}|^2 - \frac{|\nabla R|^2}{n} = -\frac{|\nabla R|^2}{n} \leq 0.
\]

Therefore, \( R \) is a constant on \( M \).

Since \( M^n \) is radially flat and has constant scalar curvature, it follows from Lemma 2.4 that \( (M^n, g) \) is rigid.

Theorem 1.1 follows by combining Theorem 3.1 and Theorem 4.1. \( \Box \)

5 The proof of Theorem 1.2

In this section, we give a direct proof of Theorem 1.2.

**Theorem 5.1** Let \( (M^n, g) \) be a gradient shrinking Ricci soliton with (1.1). If \( \text{div}^3 Rm(\nabla f) = 0 \), then \( (M^n, g) \) is rigid.
Proof. From (2.14), we have

\[
div^3 Rm(\nabla f) = \nabla_k \nabla_j \nabla_i R_{ijkl} \nabla_i f
\]
\[
= -R_{ijkl} \nabla_i R_{jkl}
\]
\[
= \frac{1}{2} (\nabla_i R_{ijkl})(\nabla_i R_{jkl} - \nabla_j R_{ikl})
\]
\[
= -\frac{1}{2} |div Rm|^2,
\]
where we used (2.4) in the third equality and (2.2) in the last.

Since \(div^3 Rm(\nabla f) = 0\), \(div Rm = 0\). It follows that \(M\) is radially flat. Moreover, we have

\[
\nabla_i R = 2\nabla_i R_{ik} = -2g^{jk} \nabla_i R_{ijkl} = 0,
\]
i.e. \(R\) is a constant on \(M\).

Since \(M^n\) is radially flat and has constant scalar curvature, it follows from Lemma 2.4 that \((M^n, g)\) is rigid. \(\square\)

6 Under the condition of Weyl tensor

In this section, we prove Theorems 1.3. The first step is to obtain the following formulas.

Proposition 6.1 Let \((M^n, g)\) \((n \geq 3)\) be a gradient shrinking Ricci soliton with (1.1), we have the following identities.

\[
(divW)_{ijk} = \frac{n-3}{n-2} (divRm)_{ijk} - \frac{n-3}{2(n-1)(n-2)} (g_{ik} \nabla_j R - g_{jk} \nabla_i R), \quad (6.29)
\]
\[
(div^2 Rm)_k = \frac{n-3}{n-2} (div^2 Rm)_k - \frac{n-3}{2(n-1)(n-2)} (g_{ik} \Delta R - \nabla_k \nabla_i R), \quad (6.30)
\]
\[
(div^3 W)_i = \frac{n-3}{n-2} (div^3 Rm)_i + \frac{n-3}{2(n-1)(n-2)} R_{ik} \nabla_k R, \quad (6.31)
\]
and

\[
div^4 W = \frac{n-3}{n-2} div^4 Rm + \frac{n-3}{2(n-1)(n-2)} \left( \frac{1}{2} |\nabla R|^2 + R_{ik} \nabla_i \nabla_k R \right). \quad (6.32)
\]
Proof. By direct computation,

\[(div W)_{ijk} = \nabla_l W_{ijkl}\]

\[= \nabla_l R_{ijkl} - \frac{1}{n-2} \left( g_{ik} \nabla_l R_{jk} - \nabla_i R_{jk} - g_{jk} \nabla_l R_{il} + \nabla_j R_{ik} \right) \]

\[+ \frac{1}{(n-1)(n-2)} \left( g_{ik} \nabla_j R - g_{jk} \nabla_i R \right) \]

\[= \nabla_l R_{ijkl} - \frac{1}{n-2} \nabla_l R_{ijkl} \]

\[- \frac{1}{2(n-2)} \left( g_{ik} \nabla_j R - g_{jk} \nabla_i R \right) + \frac{1}{(n-1)(n-2)} \left( g_{ik} \nabla_j R - g_{jk} \nabla_i R \right) \]

\[= \frac{n-3}{n-2} \nabla_l R_{ijkl} - \frac{n-3}{2(n-1)(n-2)} \left( g_{ik} \nabla_j R - g_{jk} \nabla_i R \right), \]

where we used (2.8) in the second equality.

It follows from (6.29) that

\[(div^2 W)_{ik} = \nabla_j \nabla_l W_{ijkl}\]

\[= \frac{n-3}{n-2} \nabla_j \nabla_l R_{ijkl} - \frac{n-3}{2(n-1)(n-2)} \left( g_{ik} \nabla_j R - \nabla_k \nabla_i R \right), \]

By (6.30), we have

\[(div^3 W)_i = \nabla_k \nabla_j \nabla_l W_{ijkl}\]

\[= \frac{n-3}{n-2} \nabla_k \nabla_j \nabla_l R_{ijkl} - \frac{n-3}{2(n-1)(n-2)} \left( \nabla_i \Delta R - \nabla_k \nabla_i R \right) \]

\[= \frac{n-3}{n-2} \nabla_k \nabla_j \nabla_l R_{ijkl} + \frac{n-3}{2(n-1)(n-2)} R_{ik} \nabla_k R, \]

From (6.31), we have

\[div^4 W = \nabla_i \nabla_k \nabla_j \nabla_l W_{ijkl}\]

\[= \frac{n-3}{n-2} \nabla_i \nabla_k \nabla_j \nabla_l R_{ijkl} + \frac{n-3}{2(n-1)(n-2)} \nabla_i \left( R_{ik} \nabla_k R \right) \]

\[= \frac{n-3}{n-2} \nabla_i \nabla_k \nabla_j \nabla_l R_{ijkl} \]

\[+ \frac{n-3}{2(n-1)(n-2)} \left( |\nabla R|^2 + R_{ik} \nabla_i \nabla_k R \right), \]
As a corollary of Proposition 6.1, we have

**Corollary 6.1** Let \((M^n, g)\) \((n \geq 3)\) be a gradient shrinking Ricci soliton with (1.1), we have the following identities.

\[
(div W)_{ijk} = \frac{n-3}{n-2}(\nabla_j R_{ik} - \nabla_i R_{jk}) - \frac{n-3}{2(n-1)(n-2)}(g_{ik} \nabla_j R - g_{jk} \nabla_i R),
\]

(6.33)

\[
(div^2 W)_{ik} = \frac{n-3}{n-2}(2\lambda R_{ik} + \nabla \nabla_f R_{ik} - R_{ik}^2 - R_{ijkl} R_{jl}) - \frac{n-3}{2(n-1)} \nabla_i \nabla_k R
\]

\[\quad \quad \quad - \frac{n-3}{2(n-1)(n-2)}(\nabla \nabla_f R + 2\lambda R - 2|\text{Ric}|^2)g_{ik},\]

(6.34)

\[
(div^3 W)_i = \frac{n-3}{n-2} R_{ijkl} \nabla_k R_{jl} + \frac{n-3}{2(n-1)(n-2)} R_{ik} \nabla_k R,
\]

(6.35)

and

\[
div^4 W = \frac{n-3}{n-2}(\nabla_i R_{jk} \nabla_k R_{jl} - |\nabla \text{Ric}|^2 - R_{ijkl} \nabla_i \nabla_k R_{jl})
\]

\[\quad \quad \quad + \frac{n-3}{2(n-1)(n-2)} \left(\frac{1}{2} |\nabla R|^2 + R_{ik} \nabla_i \nabla_k R\right).
\]

(6.36)

**Proof.** Applying (2.2) to (6.29), we obtain (6.33).

Applying (2.10) and (2.13) to (6.30), we can get (6.34).

Applying (2.14) to (6.31), we have (6.35).

Applying (2.15) to (6.32), we have (6.36). □

Next, we prove that a gradient shrinking Ricci soliton with \(div^3 W(\nabla f) = 0\) is rigid.

**Theorem 6.1** Let \((M^n, g)\) be a gradient shrinking Ricci soliton with (1.1). If \(div^3 W(\nabla f) = 0\), then \((M^n, g)\) is rigid.
Proof. By (6.35), we have
\[
\begin{align*}
div^3 W(\nabla f) & = \frac{n-3}{n-2} R_{ijkl} \nabla_k R_{jl} \nabla_i f + \frac{n-3}{2(n-1)(n-2)} R_{ik} \nabla_k R \nabla_i f \\
 & = \frac{n-3}{2(n-2)} (\nabla_i R_{ijkl})(\nabla_i R_{jk} - \nabla_k R_{ji}) + \frac{n-3}{4(n-1)(n-2)} |\nabla R|^2 \\
 & = -\frac{n-3}{2(n-2)} |\text{div} Rm|^2 + \frac{n-3}{4(n-1)(n-2)} |\nabla R|^2,
\end{align*}
\]  
(6.37)
where we used (2.4) and (2.8) in the second equality and (2.2) in the last.

It follows from (2.8) that $|\nabla R|^2 \leq 4|\text{Ric}|^2 |\nabla f|^2$. By Lemma 2.1, (2.12), (2.16) and Lemma 2.5, we have
\[
\int_M |\nabla R|^2 e^{-f} \leq 4 \int_M |\text{Ric}|^2 |\nabla f|^2 e^{-f} \leq \int_M |\text{Ric}|^2 e^{-\alpha f} < +\infty,
\]
where for some constant $\alpha \in (0, 1]$.

Integrating (6.37) and using the condition of $\text{div}^3 W(\nabla f) = 0$, we obtain
\[
\int_M |\text{div} Rm|^2 e^{-f} = \frac{1}{2(n-1)} \int_M |\nabla R|^2 e^{-f} < +\infty.
\]

It follows from (2.20) that
\[
\begin{align*}
\int_M |\nabla \text{Ric}|^2 e^{-f} & = \int_M |\text{div} Rm|^2 e^{-f} \\
 & = \frac{1}{2(n-1)} \int_M |\nabla R|^2 e^{-f} \\
 & \leq \frac{n}{2(n-1)} \int_M |\nabla \text{Ric}|^2 e^{-f},
\end{align*}
\]  
(6.38)
where we used $|\nabla R|^2 \leq n|\nabla \text{Ric}|^2$.

Note that $\frac{n}{2(n-1)} < 1$, we conclude from (6.38) that
\[
\int_M |\text{div} Rm|^2 e^{-f} = \int_M |\nabla R|^2 e^{-f} = 0,
\]
i.e., $|\text{div} Rm| = |\nabla R| = 0$ a.e. .

Since any gradient shrinking Ricci soliton is analytic in harmonic coordinates, $|\text{div} Rm| = 0$ on $M$. It follows that $M$ is radially flat. Moreover, $|\nabla R| = 0$ on $M$, i.e., $R$ is a constant on $M$. By Lemma 2.4, $(M^n, g)$ is rigid.$\square
7 Four-dimensional Case

We prove Theorems 1.4 to 1.6 in this section. From Theorems 1.1 to 1.3, we only need to show the following classification theorem.

**Theorem 7.1** Let \((M^4, g)\) be a 4-dimensional rigid gradient shrinking Ricci soliton with (1.1), then \((M^4, g)\) is either

(i) Einstein, or

(ii) a finite quotient of the Gaussian shrinking soliton \(\mathbb{R}^4\), \(\mathbb{R}^2 \times S^2\) or the round cylinder \(\mathbb{R} \times S^3\).

Before we prove Theorem 7.1, we present some results that are needed in the proof of Theorem 7.1.

**Lemma 7.1** (M. Fernández-López and E. García-Río [9]) Let \((M^n, g, f)\) be an \(n\)-dimensional gradient shrinking Ricci soliton with constant scalar curvature, then \(R \in \{0, \lambda, \cdot, \cdot, \cdot, \cdot, (n-1)\lambda, n\lambda\}\).

**Lemma 7.2** (M. Fernández-López and E. García-Río [9]) No complete gradient shrinking Ricci soliton may exist with \(R = \lambda\).

Now we are ready to prove Theorem 7.1.

**Proof of Theorem 7.1:**

Note that \((M^4, g)\) is rigid, i.e., it is a finite quotient of \(\mathbb{R}^k \times N^{4-k}\), where \(N\) is an Einstein manifold and \(k \in \{0, 1, 2, 3, 4\}\). It follows that \(M^4\) has constant scalar curvature. Moreover, Lemma 7.1 and Lemma 7.2 imply that \(R \in \{0, 2\lambda, 3\lambda, 4\lambda\}\).

We denote by \(\{e_i\}_{i=1}^4\) a local orthonormal frame of \(M^4\) with \(e_1 = \frac{\nabla f}{|\nabla f|}\). Moreover, We use \(\{\alpha_i\}_{i=1}^4\) to represent eigenvalues of the Ricci tensor with corresponding orthonormal eigenvectors \(\{e_i\}_{i=1}^4\).

In the following, we divide the arguments into four cases:

- **Case 1:** \(R \equiv 0\). In this case, \((M^4, g, f)\) is a finite quotient of the Gaussian soliton \(\mathbb{R}^4\).
- **Case 2:** \(R \equiv 2\lambda\). In this case, we have

\[
(\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \{\left(\frac{\lambda}{2}, \frac{\lambda}{2}, \frac{\lambda}{2}, \frac{\lambda}{2}\right), (0, \frac{2\lambda}{3}, \frac{2\lambda}{3}, \frac{2\lambda}{3}), (0, 0, \lambda, \lambda), (0, 0, 0, 2\lambda)\}.
\]

It follows from (2.10) that \(|Ric|^2 = \lambda R = 2\lambda^2\). Therefore, \((\alpha_1, \alpha_2, \alpha_3, \alpha_4) = (0, 0, \lambda, \lambda)\). The rigidity of \((M^4, g)\) implies that it is a finite quotient of \(\mathbb{R}^2 \times N^2\).
with positive scalar curvature. It is clear that \( N^2 \) has to be \( \mathbb{S}^2 \). Therefore, \((M^4, g)\) is a finite quotient of \( \mathbb{R}^2 \times \mathbb{S}^2 \).

- Case 3: \( R \equiv 3 \lambda \). In this case, we have
  \[
  (\alpha_1, \alpha_2, \alpha_3, \alpha_4) \in \left\{ \left( \frac{3\lambda}{4}, \frac{3\lambda}{4}, \frac{3\lambda}{2}, \frac{3\lambda}{2} \right), (0, \lambda, \lambda, \lambda), (0, 0, \frac{3\lambda}{2}, \frac{3\lambda}{2}), (0, 0, 0, 3\lambda) \right\}.
  \]
  It follows from (2.10) that \( |Ric|^2 = \lambda R = 3\lambda^2 \). Therefore, \((M^4, g)\) is a finite quotient of \( \mathbb{R} \times N^3 \), where \( N^3 \) is Einstein with positive scalar curvature. It is clear that \( N^3 \) has to be \( \mathbb{S}^3 \). Therefore, \((M^4, g)\) is a finite quotient of \( \mathbb{R} \times \mathbb{S}^3 \).

- Case 4: \( R \equiv 4 \lambda \). In this case, \((M^4, g)\) is Einstein with \( Ric = \lambda g \).

We conclude that \((M^4, g)\) is either Einstein or a finite quotient of \( \mathbb{R}^4 \), \( \mathbb{R}^2 \times \mathbb{S}^2 \) or \( \mathbb{R} \times \mathbb{S}^3 \). \( \square \)

8 Appendix

G. Catino, P. Mastrolia and D. D. Monticelli [4] defined the fourth order divergence of Weyl tensor \( div^4 W \) to be \( \nabla_k \nabla_j \nabla_l \nabla_i W_{ikjl} \). Moreover, they proved that a gradient shrinking Ricci soliton with \( div^4 W = 0 \) is rigid. It is clear from their proof that this result holds for \( \nabla_j \nabla_l W_{ijkl} \leq 0 \).

Remark 8.1 The definition of \( div^4 W \) in G. Catino, P. Mastrolia and D. D. Monticelli [4] differs from ours by a minus sign. To be more precise, we have

\[
\nabla_k \nabla_j \nabla_l \nabla_i W_{ikjl} = \nabla_j \nabla_k \nabla_l \nabla_i W_{ijkl} = -\nabla_j \nabla_k \nabla_l \nabla_i W_{ijkl} = -\nabla_i \nabla_k \nabla_l \nabla_j W_{ijkl}. \tag{8.39}
\]

It follows from (6.30) that \( \nabla_j \nabla_l W_{ijkl} \) is symmetric on \( i \) and \( k \), then it is also symmetric on \( j \) and \( l \), i.e.,

\[
\nabla_j \nabla_l W_{ijkl} = \nabla_l \nabla_j W_{ijkl}. \tag{8.40}
\]

Combining (8.39) and (8.40), we have

\[
\nabla_k \nabla_j \nabla_l \nabla_i W_{ikjl} = -\nabla_i \nabla_k \nabla_l \nabla_j W_{ijkl}.
\]

It is clear from (6.32) that

\[
div^4 W = \frac{n-3}{n-2} div^4 Rm + \frac{n-3}{2(n-1)(n-2)} \left( \frac{1}{2} |\nabla R|^2 + R_{ik} \nabla_l \nabla_k R \right).
\]
The following theorems were proved by G. Catino, P. Mastrolia and D. D. Monticelli \[4\], we give a different proof here.

**Theorem 8.1** Let \((M^n, g)\) be a compact gradient shrinking Ricci soliton with \((1.1)\). If \(\text{div}^4 W = 0\), then \((M^n, g)\) is Einstein.

**Proof.** Integrating (6.36), we have

\[
\int_M \text{div}^4 W e^{-f} = \frac{n-3}{n-2} \int_M (\nabla_i R_{jk} \nabla_k R_{jl} - |\nabla \text{Ric}|^2 - R_{ijkl} \nabla_i \nabla_k R_{jl}) e^{-f} \\
+ \frac{n-3}{2(n-1)(n-2)} \int_M (\frac{1}{2} |\nabla R|^2 + R_{ik} \nabla_i \nabla_k R) e^{-f} \\
= - \frac{n-3}{2(n-2)} \int_M |\nabla \text{Ric}|^2 e^{-f} + \frac{n-3}{4(n-1)(n-2)} \int_M |\nabla R|^2 e^{-f} \\
\leq - \frac{n-3}{4n(n-1)} \int_M |\nabla R|^2 e^{-f},
\]

where we used Lemma 3.1, (2.5) and (2.7) in the second equality. Moreover, we used \(|\nabla R|^2 \leq n |\nabla \text{Ric}|^2\) in the inequality.

Since \(\text{div}^4 W = 0\), it follows from (8.41) that \(\nabla R = 0\) a.e. . Note that any gradient shrinking Ricci soliton is analytic in harmonic coordinates, we have \(\nabla R = 0\) on \(M\), i.e., \(R\) is a constant on \(M\). Therefore, \(\text{Ric}(\nabla f, \nabla f) = \frac{1}{2} (\nabla R, \nabla f) = 0\). By Lemma 2.3, \((M^n, g)\) is Einstein. \(\square\)

**Theorem 8.2** Let \((M^n, g)\) be a complete non-compact gradient shrinking Ricci soliton with \((1.1)\). If \(\text{div}^4 W = 0\), then \((M^n, g)\) is rigid.

**Proof.** Let \(\phi : \mathbb{R}_+ \rightarrow \mathbb{R}\) be a \(C^2\) function with \(\phi = 1\) on \((0, s]\), \(\phi = 0\) on \([2s, \infty)\) and \(-\frac{c}{2} \leq \phi'(t) \leq 0\) on \((s, 2s)\) for some constant \(c > 0\). Define \(D(r) := \{x \in M | f(x) \leq r\}\).
Integrating (6.37) we have

\[
\int_M \text{div}^4 W \phi^2(f) e^{-f} = \frac{n-3}{n-2} \int_M (\nabla_i R_{jk} \nabla_k R_{jl} - |\nabla \text{Ric}|^2 - R_{ijkl} \nabla_i \nabla_k R_{jl}) \phi^2(f) e^{-f} \\
+ \frac{n-3}{2(n-1)(n-2)} \int_M \left( \frac{1}{2} |\nabla R|^2 + R_{ij} \nabla_i \nabla_j R \right) \phi^2(f) e^{-f} \leq c \int_{D(2s) \setminus D(s)} |\nabla R|^2 |\nabla f|^2 (\phi')^2 e^{-f} - \frac{n-3}{4(n-2)} \int_M |\nabla \text{Ric}|^2 \phi^2(f) e^{-f} \\
+ \frac{n-3}{4(n-1)(n-2)} \int_M |\nabla^2 \phi| e^{-f} - \frac{n-3}{(n-1)(n-2)} \int_M R_{ij} \nabla_i \nabla_j (\phi R) e^{-f} \leq c \int_{D(2s) \setminus D(s)} |\nabla R|^2 |\nabla f|^2 e^{-f} + \frac{n-3}{3(n-1)(n-2)} \int_M |\nabla R|^2 \phi^2(f) e^{-f} \\
- \frac{n-3}{2(n-2)} \int_M |\nabla \text{Ric}|^2 \phi^2(f) e^{-f},
\]
(8.42)

where we used Lemma 4.1 and Lemma 4.2 in the first inequality.

Applying \( \text{div}^4 W = 0 \) and \( |\nabla \text{Ric}| \geq \frac{|\nabla R|^2}{n} \) to (8.42), we obtain

\[
0 \leq c \int_{D(2s) \setminus D(s)} |\nabla R|^2 |\nabla f|^2 e^{-f} - \frac{(n-3)^2}{6n(n-1)(n-2)} \int_M |\nabla R|^2 \phi^2(f) e^{-f}
\]
(8.43)

It follows from Lemma 2.1, (2.12), (2.16) and Lemma 2.5 that

\[
\int_M |\nabla R|^2 |\nabla f|^2 e^{-f} \leq \int_M |\nabla R|^2 e^{-\alpha f} < +\infty
\]

for some \( \alpha \in (0, 1] \). Therefore,

\[
\frac{c}{s^2} \int_{D(2s) \setminus D(s)} |\nabla R|^2 |\nabla f|^2 e^{-f} \rightarrow 0
\]
as \( s \rightarrow +\infty \).

By taking \( r \rightarrow +\infty \) in (8.43), we obtain \( \int_M |\nabla R|^2 e^{-f} = 0 \). It follows that \( \nabla R = 0 \) a.e. Note that any gradient shrinking Ricci soliton is analytic in harmonic coordinates, we have \( \nabla R = 0 \) on \( M \), i.e., \( R \) is a constant on \( M \).
By taking \( r \to +\infty \) in (8.43) and using \( \text{div}^4 W = 0 \) and \( |\nabla R| = 0 \), we obtain
\[
\int_M |\nabla \text{Ric}|^2 e^{-f} = 0.
\]
Since \( \int_M |\nabla \text{Ric}|^2 e^{-f} < +\infty \), it follows (2.20) that
\[
\int_M |\text{div}\text{Rm}|^2 e^{-f} = \int_M |\nabla \text{Ric}|^2 e^{-f} = 0.
\]
Hence, \( |\text{div}\text{Rm}| = 0 \) a.e. Note that any gradient shrinking Ricci soliton is analytic in harmonic coordinates, we have \( |\text{div}\text{Rm}| = 0 \) on \( M \).

It is clear \( \text{div}\text{Rm} = 0 \) implies that \( M^n \) is radially flat.

Since \( M^n \) is radially flat and has constant scalar curvature, it follows from Lemma 2.4 that \( (M^n, g) \) is rigid.

From Theorem 7.1, Theorem 8.1 and Theorem 8.2, we have a classification theorem of 4-dimensional gradient shrinking Ricci solitons with \( \text{div}^4 W = 0 \):

**Theorem 8.3** Let \( (M^4, g) \) be a 4-dimensional gradient shrinking Ricci soliton with (1.1). If \( \text{div}^4 W = 0 \), then \( (M^4, g) \) is either
(i) Einstein, or
(ii) a finite quotient of the Gaussian shrinking soliton \( \mathbb{R}^4 \), \( \mathbb{R}^2 \times \mathbb{S}^2 \) or the round cylinder \( \mathbb{R} \times \mathbb{S}^3 \).

**Remark 8.2** It is clear from the proof that Theorems 8.1 to 8.3 hold for \( \text{div}^4 W \geq 0 \). Moreover, it follows from (8.39) that Theorem 8.1 to 8.3 still hold if indices of \( \text{div}^4 W \) permutate.

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