ON LIE ALGEBRA ACTIONS

RICHARD H. CUSHMAN* AND JĘDRZEJ ŚNIATYCKI
Department of Mathematics and Statistics
University of Calgary
Calgary, AB, T2N 1N4, Canada

Abstract. In this paper we define an action of a Lie algebra on a smooth manifold. We get nearly the same results as those for group actions, when the flows of the symmetry vector fields are complete. We show that the orbit space of a Lie algebra action is a differential space. We discuss differential spaces occurring in the reduction of symmetries in integrable Hamiltonian systems.

1. Introduction. In this paper we define an action of a Lie algebra on a smooth manifold. We get nearly the same results when the flows of the symmetry vector fields are not complete as those for group actions, when the flows of the symmetry vector fields are complete. The most well known example of a Lie algebra action is the \( \mathfrak{so}(4) \) symmetry of the Kepler problem in classical mechanics, see [3, chpt 2 §3]. Our most important result is: the orbit space of a Lie algebra action is a differential space. Moreover, if the Lie algebra action is proper, then each point of the space of orbits has an open neighborhood, which is diffeomorphic to a subset of Euclidean space. We use these results to discuss differential spaces occurring in the reduction of symmetries in integrable Hamiltonian systems.

2. Basic definitions and properties. Let \( G \) be a connected Lie group with identity element \( e \) and Lie algebra \( \mathfrak{g} \). Let \( M \) be a smooth (Hausdorff) manifold with \( (\mathcal{X}(M), -\left[ \cdot, \cdot \right]) \) the Lie algebra of smooth vector fields on \( M \). The reason for the minus sign is to ensure that a Lie algebra action is a homomorphism of Lie algebras, see the proof of lemma 2.1.

A domain \( D \) is an open subset of \( G \times M \) such that for each \( p \in M \) the set \( D_p = \{ g \in G \mid (g, p) \in D \} \) is a connected open neighborhood of \( e \) in \( G \). A local action of \( G \) on \( M \) is a smooth mapping \( \Phi : D \subseteq G \times M \to M \), where \( D \) is a domain and

1) for each \( p \in M \), we have \( \Phi(e, p) = p \);
2) if \( (h, p) \in D \), \( (g, \Phi(h, p)) \in D \) and \( (gh, p) \in D \), then \( \Phi(gh, p) = \Phi(g, \Phi(h, p)) \).

For each \( (g, p) \in D \) let

\[ \Phi_p : D_p \subseteq G \to M : g \mapsto \Phi_p(g) = \Phi(g, p) \]

2010 Mathematics Subject Classification. Primary: 37J15, 37J45; Secondary: 22F05.

Key words and phrases. Lie algebra action, proper action, differential space, subcartesian, completely integrable.

* Corresponding author: R. H. Cushman.
and
\[ \Phi_g : D_g = \{ p \in M \mid (g, p) \in D \} \subseteq M \to M : p \mapsto \Phi_g(p) = \Phi(g, p). \]
From the definition of domain it follows that for each \((g, p) \in D\) the set \(D_g\) is an open subset of \(M\) containing \(p\). The definitions of domain and local \(G\)-action on \(M\) are taken from Palais [4].

**Example 1.** Here we give an example of another domain for a given local \(G\)-action \(\Phi : D \subseteq G \times M \to M\) on \(M\). Since \(G\) is a Lie group, the map \(\gamma^{-1} : G \to G : g \mapsto g^{-1}\) is continuous and open. Consequently, for each \(p \in M\) the set \(D_p^{-1} = \{ h \in G \mid h = g^{-1} \text{ for some } g \in D_p \}\) is a connected open subset of \(G\) containing \(e\), because \(D_p\) is. Thus \(D = \{(g, p) \in G \times M \mid g \in D_p \cap D_p^{-1} \text{ & } p \in M\}\) is a domain for the local \(G\)-action \(\Phi\) on \(M\).

Let \(\Phi\) be a local \(G\)-action on \(M\) with domain \(D\). The mapping
\[ \varphi : \mathfrak{g} \to \mathcal{X}(M) : \xi \mapsto \varphi(\xi) = X_\xi, \]
where \(X_\xi(p) = T_p \Phi_p \xi\) for every \(p \in M\) and every \(\xi \in \mathfrak{g}\), is the infinitesimal generator of \(\Phi\). A **Lie algebra action of \(\mathfrak{g}\) on \(M\)** is an homomorphism of the Lie algebra \((\mathfrak{g}, [\ , \ ])) into the Lie algebra \((\mathcal{X}(M), [\ , \ ]))

**Example 2.** Let \(X\) be a smooth vector field on \(M\). Then its flow \(\Phi : D \subseteq \mathbb{R} \times M \to M : (t, m) \mapsto (\exp t X)(m)\) is a local \(\mathbb{R}\)-action on \(M\) with domain \(D\). The mapping \(\varphi : \mathbb{R} \to \mathcal{X}(M) : \frac{d}{dt} \mapsto X\) is the infinitesimal generator of \(\Phi\).

**Lemma 2.1.** The infinitesimal generator \(\varphi\) of the local \(G\)-action \(\Phi\) on \(M\) with domain \(D\) is a \(\mathfrak{g}\)-action on \(M\).

**Proof.** This follows immediately because \([-[X_\xi, X_\eta](p) = X_{[\xi, \eta]}(p)\) for every \(\xi, \eta \in \mathfrak{g}\) and every \(p \in M\).

We have

**Theorem 2.2.** *(Lie’s second theorem)* Let \(\varphi\) be a \(\mathfrak{g}\)-action on \(M\). Then \(\varphi\) is the infinitesimal generator of a local \(G\)-action \(\Phi\) on \(M\) with domain \(D\).

**Proof.** See Palais [4, Thm XI, p.58].

**Example 3.** Let \(G\) be a Lie group with Lie algebra \((\mathfrak{g}, [\ , \ ]))\). Suppose that the mapping \(\psi : (\mathfrak{g}, [\ , \ ]) \to (\mathcal{X}(M), [\ , \ ])\) is an action of the Lie algebra \(\mathfrak{g}\) on the manifold \(M\), that is, \(\psi\) is a homomorphism of Lie algebras. By Lie’s second theorem, \(\psi\) is the infinitesimal generator of a local \(G\)-action \(\Psi : D \subseteq G \times M \to M\) with domain \(D\).

**Example 4.** Let \((\mathfrak{h}, [\ , \ ])\) be a Lie subalgebra of \((\mathfrak{g}, [\ , \ ]))\). Let \(H\) be the group generated by the neighborhood \(U = \exp \mathfrak{U} \subseteq G\) of \(e\), where \(\mathfrak{U}\) is an open neighborhood of \(0\) in \(\mathfrak{g}\), that is, every element of \(H\) is a finite product in \(G\) of elements of \(U\). Then \(H\) is a Lie subgroup of \(G\). Let \(\varphi : (\mathfrak{h}, [\ , \ ]) \to (\mathcal{X}(M), [\ , \ ])\) be an action of the Lie algebra \(\mathfrak{h}\) on \(M\) given by restricting the Lie algebra action \(\psi\) of \(\mathfrak{g}\) on \(M\) to its subalgebra \(\mathfrak{h}\). Then \(\varphi\) is a Lie algebra homomorphism. By Lie’s second theorem, \(\varphi\) is the infinitesimal generator of a local action of \(H\) on \(M\) given by \(\Phi : D \subseteq H \times M \to M\) with domain \(D\), which is the restriction of the local \(G\) action \(\Psi\) on \(M\) to \(D = D \cap (H \times M)\).
Let $\varphi$ be a $\mathfrak{g}$-action on $M$. For each $p \in M$ the orbit $O_p$ of $\varphi$ through $p$ is the orbit of the family $X_p = \{\varphi(\xi) \in \mathcal{X}(M) \mid \xi \in \mathfrak{g}\}$ of vector fields on $M$ through $p$. The orbit $O_p$ is formed by taking the union of the images of all piecewise smooth curves in $M$ passing through $p$, each of whose segments is an integral curve of a vector field in the family $X_p$. By Sussmann's first theorem, the orbit $O_p$ is a connected immersed submanifold of $M$, see [7, thm 4.1, p.179]. Note that $T_pO_p = \text{span}\{(\varphi(\xi))(p) \in T_pM \mid \xi \in \mathfrak{g}\}$.

3. Proper Lie algebra actions. Let $\varphi$ be a $\mathfrak{g}$-action on $M$. For each $p \in M$ let $\mathfrak{g}_p = \{\xi \in \mathfrak{g} \mid \varphi(\xi)(p) = 0\}$. Then $\mathfrak{g}_p$ is a Lie subalgebra of $\mathfrak{g}$ called the isotropy algebra at $p$ of the $\mathfrak{g}$-action $\varphi$. We say that the $\mathfrak{g}$-action $\varphi$ is proper if and only if the local $G$-action $\Phi$ on $M$ with domain $D$ generated by $\varphi$ is proper. In other words, the mapping

$$\text{gr } \Phi : D \subseteq G \times M \to M \times M : (g, p) \mapsto (p, \Phi_p(p))$$

is proper.

Example 5. Consider the vector field $X(x) = x^2 \frac{\partial}{\partial x}$ on $\mathbb{R}_{>0}$. As is easily checked, its flow is given by

$$\Phi : D = \{(t, x) \in \mathbb{R} \times \mathbb{R}_{>0} \mid x > 0 \& tx < 1\} \to \mathbb{R}_{>0} : (t, x) \mapsto \frac{1}{x - t}.$$ 

We now show that the local $\mathbb{R}$-action given by $\Phi$ is not proper. In other words, the map

$$\text{gr } \Phi : D \subseteq \mathbb{R} \times \mathbb{R}_{>0} \to \mathbb{R}_{>0} \times \mathbb{R}_{>0} : (t, x) \mapsto (x, \Phi(t, x))$$

is not proper. For $(t, x) \in D$ consider the compact box $K$ about $(x, \Phi(t, x))$ defined by $|x| \leq C_1$ and $|\Phi(t, x)| \leq C_2$ for some $C_1, C_2 \in \mathbb{R}_{>0}$. Then $(\text{gr } \Phi)^{-1}(K)$ is the subset of $\mathbb{R} \times K$ defined by $|x| \leq C_1$, $\left|\frac{1}{x - t}\right| \leq C_2$, $tx < 1$, and $x > 0$. For each $n \in \mathbb{Z}_{>0}$ consider $(t_n, x_n) = (C^{-1}(1 + \varepsilon_n), C) \in \mathbb{R} \times \mathbb{R}_{>0}$ such that $\varepsilon_n \leq -\frac{C}{C_2}$ and $0 < C \leq C_1$. The following argument shows that $(t_n, x_n) \in D$. By construction $0 < x_n = C \leq C_1$, $t_n x_n = C^{-1}(1 + \varepsilon_n) = 1 + \varepsilon_n < 1$, and $\left|\frac{1}{x_n - t_n}\right| > (1 + \varepsilon_n) = \frac{C}{C_2} \leq C_2$. Thus $(t_n, x_n) \in D$. Let $\varepsilon_n \searrow -\infty$. Then $(t_n, x_n) \searrow (-\infty, C)$. Hence $(\text{gr } \Phi)^{-1}(K)$ is unbounded and so is not compact. Thus the map $\text{gr } \Phi$ is not proper.

Example 6. Let $X$ be a complete vector field on $\mathbb{R}^2$ with the circle $C = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}$ an attracting limit cycle. Let $\varphi_t$ be the flow of $X$. If $(x_0, y_0) \in \mathbb{R}^2 \setminus C$, then the $\omega$-limit set of $(x_0, y_0)$ is $C$, that is, $\bigcap_{n=1}^{\infty} \varphi_n(x_0, y_0) = C$. Let $(x_c, y_c) \in C$ and let $K$ be a compact box in $\mathbb{R}^2$ about $(x_c, y_c)$. Then $\varphi_K(x_0, y_0) \cap K$ has a countable number of nonempty connected components $\varphi_{(t_m, t_m')}((x_0, y_0)) \cap K$, where $t_m < t_m' < t_{m+1} < \infty$ and $t_m \nearrow \infty$. Since the vector field $X$ is complete, its flow $\varphi_t$ generates an $\mathbb{R}$-action $\Phi$ on $\mathbb{R}^2$. Let

$$\text{gr } \Phi : \mathbb{R} \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2 : (t, x) \mapsto (x, \varphi_t(x)),$$

Then $(\text{gr } \Phi)^{-1}(K \times K)$ is not compact, since for all $m \in \mathbb{Z}_{>0}$ we have $\varphi_{(t_m, t_m')}((x_0, y_0))$ and $(x_0, y_0) \in K$ but $\bigcup_{m \in \mathbb{Z}_{>0}} (t_m, t_m') \subseteq \mathbb{R}$ is unbounded. Thus the $\mathbb{R}$-action $\Phi$ is not proper.

Lemma 3.1. Let $\varphi : \mathfrak{g} \to \mathcal{X}(M)$ be a proper $\mathfrak{g}$-action on $M$, which generates the local $G$-action $\Phi$ on $M$ with domain $D$. For each $p \in M$ the local isotropy group $G_p = \{g \in D_p \mid \Phi_g(p) = p\}$ with Lie algebra $\mathfrak{g}_p$ is a compact Lie subgroup of $G$. 
Proof. From the construction of the domain $D$ of the local $G$-action $\Phi$, it follows that $G_p$ is a subgroup of $G$. We only need show that $G_p$ is closed under multiplication. Suppose that $h, g \in G_p$ and $(gh, p) \in D$. Then $(h, p) \in D$ and $(g, \Phi_h(p)) \in D$, since $\Phi_h(p) = p$ and $(g, p) \in D$. From the definition of the domain $D$ it follows that $\Phi_{gh}(p) = \Phi_g(\Phi_h(p)) = \Phi_g(p) = p$. So $gh \in G_p$. Next we show that $G_p$ is a Lie group. Consider the set $(\text{gr } \Phi)^{-1}(p, p)$ for $p \in M$. Then $(\text{gr } \Phi)^{-1}(p, p) = \{(g, p) \in D \mid \Phi_g(p) = p\}$ is a compact subset of $D$, since $\{(p, p)\}$ is a compact subset of $M \times M$ and $\text{gr } \Phi$ is a proper mapping. Let $\pi_G : D \subseteq G \times M \rightarrow G : (g, p) \mapsto g$. Then $\pi_G((\text{gr } \Phi)^{-1}(p, p))$ is a compact subset $K$ of $G$, because the mapping $\pi_G$ is continuous. We now show that $K = G_p$. Suppose that $g \in K$. Then $(g, p) \in (\text{gr } \Phi)^{-1}(p, p) \subseteq D$, that is, $g \in D_p$. Moreover, $\Phi_g(p) = p$. So $g \in G_p$ by definition. Conversely, if $g \in G_p$, then $g \in D_p$ and $\Phi_g(p) = p$. So $(g, p) \in D$ and $\Phi_g(p) = p$, that is, $(g, p) \in (\text{gr } \Phi)^{-1}(p, p)$. Hence $g = \pi_G(g, p) \in \pi_G((\text{gr } \Phi)^{-1}(p, p)) = K$. Thus $G_p$ is a compact subset of $G$ and hence is a closed subset of $G$. So $G_p$ is a compact Lie subgroup of $G$ with Lie algebra $\mathfrak{g}_p$.

Lemma 3.2. If $\Phi : G \times M \rightarrow M$ is a proper action of $G$ on $M$, then its infinitesimal generator $\phi : \mathfrak{g} \rightarrow \mathcal{X}(M)$ is a proper $\mathfrak{g}$-action on $M$.

Proof. This follows immediately. \qed

Let $\Phi$ be the local $G$-action on $M$ generated by the $\mathfrak{g}$-action $\varphi$ with domain $D$. For each $p \in M$ let $O_p = \{\Phi_g(p) \in M \mid g \in D_p\}$ be the $G$-orbit of $\Phi$ on $M$.

In the next few results we compare the $G$-orbit $O_p$ through $p \in M$ of the local $G$-action on $M$ to the orbit $O_p$ through $p$ of the Lie algebra action of $\mathfrak{g}$ on $M$, which is its infinitesimal generator. If the local $G$-action is proper, then in theorem 3.6 we find that $O_p = O_p$. We know of no example of a local $G$-action where this equality does not hold.

Lemma 3.3. For each $p \in M$ the $G$-orbit $O_p$ is an open subset of the $X_\mathfrak{g}$-orbit $O_p$. Here the topology on $O_p$ is the topology induced from $M$.

Proof. Since $D$ is a domain for the local $G$-action $\Phi$ on $M$, for each $p \in M$ $D_p$ is a connected open subset of $G$ containing $e$. Then $O_p = \Phi_{D_p}(p)$. Let $q \in O_p$. Then there is a $q' \in D_p$ such that $\Phi_{q'}(p) = q$. Since $\exp : \mathfrak{g} \rightarrow G$ with $\exp 0 = e$ is a local diffeomorphism, there is an open neighborhood $\mathfrak{U}$ of 0 in $\mathfrak{g}$ (having previously chosen a norm on $\mathfrak{g}$ to define the topology of $\mathfrak{g}$) such that $U = \exp \mathfrak{U} \subseteq D_p$ is an open neighborhood of $q$ in $D_p$. Consequently, $V = \Phi_U(p) \subseteq O_p$ is an open subset of $M$ containing $q$, because $\Phi_p : G \rightarrow M : g \rightarrow \Phi_g(p)$ is an open mapping. Hence $V \cap O_p$ is an open subset of $O_p$. Let $q' \in V$. Since $V = \Phi_U(p)$ and $U = \exp \mathfrak{U}$, there is a $\xi \in \mathfrak{U} \subseteq \mathfrak{g}$ such that $q' = \Phi_{q'}(\exp \xi(p)) \in V$. Now $\gamma : [0, 1] \rightarrow M : t \mapsto \Phi_{q'}(\exp t\xi(p))$ is the integral curve of the vector field $\varphi(\text{Ad}_q \xi)$ on $M$ starting at $p$, because $[0, 1] \rightarrow M : t \mapsto \Phi_{q'}(\exp t\xi(p))$ is an integral curve of the vector field $\varphi(\xi)$ on $M$ starting at $p$ and $\varphi(\text{Ad}_q \xi)(p') = T_p\Phi_{q'}(\varphi(\xi))(\Phi_{q'}^{-1}(p'))$ for all $p' \in M$. Here $\varphi$ is the infinitesimal generator of the local $G$-action $\Phi$. Since $q' = \gamma(1)$, it follows that $q'$ lies on an integral curve of the vector field $\varphi(\text{Ad}_q \xi)$, where $\text{Ad}_q \xi \in \mathfrak{g}$, starting at $p$. So $q' \in O_p$. Hence $V \subseteq O_p$. Moreover, $V$ is an open subset of $O_p$, since $V = V \cap O_p$. Thus the orbit $O_p$ of the local $G$-action $\Phi$ is an open subset of the orbit $O_p$ of the family of vector fields $X_\mathfrak{g}$.

\qed
Lemma 3.5. Let \( \mathfrak{g} \) be a Lie algebra action on \( M \), which generates a local \( G \)-action \( \Phi \) with domain \( D \). For each \( p \in M \) the orbit \( O_p \) of the family \( X_\mathfrak{g} \) of vector fields on \( M \) is \( G \)-invariant.

Proof. Let \( q \in O_p \). Then there is a piecewise smooth curve \( \gamma \) in \( M \) joining \( p \) to \( q \) each of whose closed segments \( \gamma_i, 1 \leq i \leq n \), is an integral curve of some vector field \( X_{\xi_i}, \xi_i \in \mathfrak{g} \). Let \( g \in D_p \) and look at \( \Phi_g(\gamma) \). Then \( \Phi_g(\gamma) \) is a piecewise smooth curve whose segments are \( \Phi_g(\gamma_i), 1 \leq i \leq n \). But \( T_p\Phi_gX_{\xi}(\overline{p}) = X_{\text{Ad}_p\xi}(\Phi_g(\overline{p})) \) for every \( \overline{p} \in M \) and every \( \xi \in \mathfrak{g} \). So for every \( 1 \leq i \leq n \), the curves \( \Phi_g(\gamma_i) \) are closed segments of integral curves of \( X_{\text{Ad}_p\xi_i} \). Consequently, the image of \( \Phi_g(\gamma) \) lies in \( O_p \).

By definition \( \Phi_g(p) \in O_p \). By lemma 3.3 we have \( O_p \subseteq O_q \). Hence \( \Phi_g(p) \in O_p \). So there is a piecewise smooth curve \( \Gamma \) in \( M \) joining \( p \) to \( \Phi_g(p) \), whose segments \( \Gamma_j, 1 \leq j \leq m \) are integral curves of some \( X_{\eta_j} \) with \( \eta_j \in \mathfrak{g} \) for \( 1 \leq j \leq m \). Thus the image of the curve joining \( p \) to \( \Phi_g(q) \), formed by concatenating \( \Phi_g(\gamma) \) with \( \Gamma \) at \( \Phi_g(p) \), lies in \( O_p \).

Lemma 3.5. Suppose that the \( \mathfrak{g} \)-action \( \varphi \) on \( M \) is proper. Then for each \( p \in M \) the \( G \)-orbit \( O_p \) is a closed subset of the \( X_\mathfrak{g} \)-orbit \( O_p \).

Proof. Let \( p_n \in O_p \) and suppose that \( p_n \to g \) in \( O_p \subseteq M \). Then \( p_n = \Phi_{g_n}(p) = \Phi_{g_n}(p) = \Phi_{g_n}(p) \) for some \( g_n \in D_p \), because \( p_n \in O_p \). Since the local \( G \)-action \( \Phi \) is proper, the map \( g \mapsto \Phi_g \), see equation (1), is a proper map. Thus there is a subsequence \( \{ (g_{n_k}, p) \} \) in \( D \), which converges to \( (g, p) \in D \). So \( g \in D_p \). Moreover, \( \Phi_{g_n}(p) = \Phi_{g_n}(g) = q \), since \( g_{n_k} \to g \) and the mapping \( \Phi_p \) is continuous. Thus \( q \in O_p \). So \( O_p \) is a closed subset of \( O_p \).

From lemmas 3.3 and 3.5, it follows that \( O_p \) is a connected subset of \( O_p \). But \( O_p \) is connected. So \( O_p = O_q \). Thus we have proved

Theorem 3.6. For a proper \( \mathfrak{g} \)-action on \( M \), which generates a local \( G \)-action with domain \( D \), at each point \( p \) of \( M \) the \( G \)-orbit \( O_p \) is equal to the \( X_\mathfrak{g} \)-orbit \( O_p \).

4. Subcartesian orbit spaces. In this section our goal is to prove

Theorem 4.1. If \( \varphi \) is a proper \( \mathfrak{g} \)-action on \( M \), then the \( \mathfrak{g} \)-orbit space \( M/X_\mathfrak{g} \), with projection mapping \( \pi : M \to M/X_\mathfrak{g} : p \mapsto O_p \), is locally Euclidean differential space. In other words, every point of \( M/X_\mathfrak{g} \) has an open neighborhood, which is diffeomorphic to a subset\(^1\) of Euclidean space.

Our strategy for trying to prove the theorem 4.1 is to adapt the proof, given in [3, chpt VII, claim 3.19, p.327], that the differential space of orbits of a proper action of a Lie group on a smooth manifold is subcartesian,\(^2\) to the context of a local proper group action.

In example 7 below we give an example of an orbit space of a proper Lie algebra action, whose differential space topology is not Hausdorff.

Corollary 4.1.1. The differential space \( (M/X_\mathfrak{g}, C^\infty(M/X_\mathfrak{g})) \) is subcartesian if its differential space topology is Hausdorff.

\(^1\)When the subset in the definition of locally Euclidean is an open subset of Euclidean space, then a locally Euclidean differential space is a smooth manifold.

\(^2\)Aronszajn [1] defined subcartesian differential space to be a differential space such that every point has an open neighborhood which is diffeomorphic to a subset of Euclidean space and the differential space topology is Hausdorff. See also [6, def 2.1.12, p.12].
Proof. By theorem 4.1 the g-orbit space $M/X_g$ is locally diffeomorphic to a subset of some Euclidean space. By hypothesis the differential space topology of $M/X_g$ is Hausdorff. Hence the differential space $(M/X_p, C^\infty(M/X_g))$ is subcartesian. 

A slice at $p \in M$ for the g-action $\varphi$, which generates the local G-action $\Phi$ with domain $D$, is a local submanifold $S$ of $M$ through $p$ such that

1) $S$ is transverse and complementary to $O_p$ at $p$, that is, $T_pM = T_pO_p \oplus T_pS$;
2) for every $q \in S$ near $p$, the local submanifold $S$ is transverse to $O_q$, that is, $T_qM = T_qO_q + T_qS$;
3) $S$ is $G_p$-invariant;
4) for every $q \in S$ and every $g \in D_p \subseteq G$, if $\Phi_g(q) \in S$, then $g \in G_p$. Note that $g \in D_g$ if $q$ is close enough to $p$, because $D_g$ is an open neighborhood of $p$ in $M$.

**Theorem 4.2.** Let $\varphi : g \to \mathcal{X}(M)$ be a proper g-action. Then $\varphi$ has a slice.

**Proof.** Our proof follows that in [3, chpt VII, p.308]. Let $\Gamma$ be a Riemannian metric on $M$. For each $p \in M$ let $G_p$ be the local isotropy group of the proper local $G$-action $\Phi$ with domain $D$. Because $D$ is a domain, $D_p$ and thus $G_p$, is connected. Let $U_p^\circ = \text{int} \Phi_{G_p}(p)$. Since $G_p$ is connected and compact, $U_p^\circ$ is a $G_p$-invariant open neighborhood of $p$ in $M$. Averaging $\Gamma$ over $G_p$ gives a Riemannian metric on $M$, whose restriction $\gamma$ to $U_p^\circ$ is $G_p$-invariant.

For every $k \in G_p$ and every $g \in D_p$ we have

$$\Phi_k \circ \Phi_g(p) = \Phi_k \circ \Phi_g \circ \Phi_k^{-1}(p) = \Phi_{k^g k^{-1}}(p).$$

Thus for every $\xi \in g$ we get

$$T_p \Phi_k (T_p \Phi_k \xi) = \frac{d}{dt} \bigg|_{t=0} \Phi_k \left( \Phi_p(\exp t\xi(p)) \right) = \frac{d}{dt} \bigg|_{t=0} \Phi_k \exp t\xi_k^{-1}(p) = \frac{d}{dt} \bigg|_{t=0} \Phi_k \exp t\xi_k(p) = T_p \Phi_p(Ad_k \xi).$$

Consequently, for each $k \in G_p$, the map $T_p \Phi_k$ leaves $T_p O_p = \text{span}\{X_\xi(p) \in T_p M \mid \xi \in g\}$ invariant. With respect to the positive definite $G_p$-invariant inner product $\gamma_p$ on $T_p M$, we find that $V_p = (T_p O_p)^{\perp}$, the $\gamma_p$-orthogonal complement to $T_p O_p$ in $T_p M$, is $G_p$-invariant. For $\varepsilon > 0$ let $V^\varepsilon_p$ be an open $\gamma_p$-ball in $V_p \subseteq T_p M$ of radius $\varepsilon$ with center at $0 \in T_p M$. Let $S_p^\varepsilon = \exp_p V^\varepsilon_p \subseteq U_p^\circ$, where $\exp_p$ is the exponential map associated to the $G_p$-invariant Riemannian metric on $U_p$. Because $\gamma$ is $G_p$-invariant, the exponential map $\exp_p$ is $G_p$-equivariant, that is, $\exp_p(\Phi_g v_p) = \Phi_g(\exp_p v_p)$, for every $g \in D_p$ and every $v_p \in V^\varepsilon_p$. Thus $S_p^\varepsilon$ is $G_p$-invariant.

By construction $T_p M = T_p O_p \oplus T_p S_p^\varepsilon$. Since transversality is an open condition, we get $T_q M = T_q O_q + T_q S_q^\varepsilon$ for every $q \in U_p^\circ$, providing $\varepsilon$ is small enough.

To finish showing that $S_p^\varepsilon$ is a slice to the g-action at $p \in M$, we must show that the condition (4) holds. Suppose not. Then there is a sequence $\{p_j\}$ with $p_j \in S_p^{1/j}$ such that $p_j \to p$ and a sequence $\{g_j\}$ with $g_j \in D_p \setminus G_p$ such that $\Phi_{g_j}(p_j) \in S_p^{1/j}$. Because the local $G$-action is proper, there is a subsequence $\{g_{j_k}\}$, which converges to $g \in D_p$. Replacing $g_j$ by $g^{-1} g_j \in D_p$, we may assume that $g_{j_k} \to e$, but $g_{j_k} \notin G_p$. Let $H$ be a local Lie subgroup of $G$ whose Lie algebra is complementary to the Lie algebra $g_0$ of $G_p$ in $g$, the Lie algebra of $G$, see example 4 with $M = G$. Then $H \times G_p \to G : (h, k) \mapsto h \cdot k$ is a local diffeomorphism sending
an open neighborhood $V \times W$ in $H \times G_p$ of $(e, e)$ onto an open neighborhood of $e$ in $G$. Thus $g_j = h_j \cdot k_j$. As $g_j \in G \setminus G_p$ and $k_j \in G_p$, we deduce that $h_j \neq e$ for all $j$. We obtain $\Phi_{h_j}(p_j) = \Phi_{h_j}(\Phi_{k_j}(p_j)) \in S^{1/j}_p$. But $S^{1/j}_p$ is $G_p$-invariant. So $\Phi_{h_j}(p_j) \in S^{1/j}_p$. Thus $h_j = e$, since $\Phi_{g_j}(S^{1/j}_p) \cap S^{1/j}_p = \{p\}$ for all $g_j \in D_p \setminus G_p$. This is a contradiction. Hence the condition (4) holds for some $\varepsilon > 0$. Thus $S^\varepsilon_p$ is a slice at $p \in M$ to the proper $g$-action $\varphi$.

The map

$$\Theta : G_p \times T_pM \to T_pM : (g, v_p) \mapsto T_p\Phi_g v_p$$

is a linear local $G_p$-action on $T_pM$ with domain $D$. This action is generated by the Lie algebra action

$$\vartheta : \mathfrak{g}_p \to \mathfrak{gl}(T_pM, \mathbb{R}) : \xi \mapsto DX_\xi(p) = D\varphi(\xi)(p).$$

We check this. Since $\xi \in \mathfrak{g}_p$, we have $\varphi(\xi) \in \mathcal{X}(M)$. Let $v_p \in T_pM$. Then $v_p$ extends to a vector field $V$ on $M$ with $V(p) = v_p$. So $D\varphi(\xi)v_p = \text{adv}_{\varphi(\xi)}(v_p)$ maps $T_pM$ into itself because $\varphi(\xi)(p) = 0$. Clearly $\vartheta(\xi) = D\varphi(\xi)(p)$ is a linear mapping of $T_pM$ into itself. In fact

**Lemma 4.3.** The map $\vartheta$ in equation (2) is a Lie algebra homomorphism.

**Proof.** We use local coordinates for $TM$ near $(p, v_p)$. For each $\xi, \eta \in \mathfrak{g}_p$, we have

$$\vartheta([\xi, \eta])(p)v_p = D\varphi(\xi, \eta)(p)v_p = -D\varphi(\xi, \varphi(\eta))(p)v_p,$$

since $\varphi$ is a $\mathfrak{g}_p$-action

$$= -D\{(D\varphi(\xi)(p))\varphi(\eta)(p)v_p + D((D\varphi(\eta)(p))\varphi(\xi)(p)v_p \}
= -D\{(\vartheta(\xi)\varphi(\eta)(p)v_p + D(\varphi(\eta)\varphi(\xi)(p)v_p \}
= (\vartheta(\xi)\vartheta(\eta) - \vartheta(\eta)\vartheta(\xi))v_p = -[\vartheta(\xi), \vartheta(\eta)]v_p.$$
We show that $C^\infty(M)$ satisfies the conditions for a differential structure, see [6, chpt 2, p.15]. Before starting the proof, we recall the conditions defining the differential structure $C^\infty(M)$.

1. $M$ is a topological space and the family \{ $f^{-1}(I)$ \mid $f \in C^\infty(M)$ and $I$ is an open interval in $R$ \} is a subbasis for the topology of $M$.

2. If $f_1, \ldots, f_n \in C^{\infty}(M)$ and $F \in C^{\infty}(R^n)$, then $F(f_1, \ldots, f_n) \in C^{\infty}(M)$.

   Here we have $(F(f_1, \ldots, f_n))(x) = F(f_1(x), \ldots, f_n(x))$, for every $x \in R^n$.

3. If $f : M \to R$ is a function such that, for every $p \in M$ there exists an open neighborhood $U$ of $p$ in $M$ and a function $f_U \in C^{\infty}(M)$ satisfying $(f_U)|_U = f|_U$, then $f \in C^{\infty}(M)$.

Proof. (Theorem 4.5) Consider the topology on $M$ generated by the subbasis \{ $f^{-1}(I)$ \mid $I$ an open interval in $R$ and $f \in C^{\infty}(M)$ \}. This topology satisfies condition 1. Condition 2 is automatic. To verify condition 3 let $\bar{g} : M \to R$ be a function such that for every $p \in M$ there are functions $\bar{f}, \bar{f}_1, \ldots, \bar{f}_\ell \in C^{\infty}(M)$ and open intervals $I_1, \ldots, I_\ell \subseteq R$ such that

\[
\bar{p} \in U_{\bar{f}} = \bar{f}_1^{-1}(I_1) \cap \cdots \cap \bar{f}_\ell^{-1}(I_\ell)
\]

and

\[
\bar{g}|_{U_{\bar{f}}} = \bar{f}|_{U_{\bar{f}}}
\]  

(4)

Since $\bar{f}_1, \ldots, \bar{f}_\ell \in C^{\infty}(M)$, it follows that

\[
\pi^{-1}(U_{\bar{f}}) = \pi^{-1}(\bar{f}_1^{-1}(I_1) \cap \cdots \cap \bar{f}_\ell^{-1}(I_\ell))
\]

\[
= \pi^{-1}(\bar{f}_1^{-1}(I_1)) \cap \cdots \cap \pi^{-1}(\bar{f}_\ell^{-1}(I_\ell))
\]

\[
= (\bar{f}_1 \circ \pi)^{-1}(I_1) \cap \cdots \cap (\bar{f}_\ell \circ \pi)^{-1}(I_\ell)
\]

is open in $M$. Since \{ $U_{\bar{f}} \mid \bar{p} \in M$ \} is an open covering of $M$ and the projection mapping $\pi$ is surjective, it follows that \{ $\pi^{-1}(U_{\bar{f}}) \mid \bar{p} \in M$ \} is an open covering of $M$. From equation (4) it follows that $\bar{g}$ is a smooth function on $M$. Consequently, $\pi^*\bar{g}|_{\pi^{-1}(U_{\bar{f}})} = \pi^*\bar{f}|_{U_{\bar{f}}}$ is smooth. On $\pi^{-1}(U_{\bar{f}})$ we have $L_{\bar{f}(\xi)}(\pi^*\bar{g}|_{U_{\bar{f}}}) = 0$ for every $\xi \in g$. Hence $\pi^*\bar{g} \in C^{\infty}(M)^g$. So $C^{\infty}(M)$ is a differential structure.

Corollary 4.5.1. The projection mapping $\pi : (M, C^{\infty}(M)) \to (M, C^{\infty}(M))$ is a smooth mapping of differential spaces.

Proof. This follows immediately from the definitions.

In what follows we use the notation of the proof of the existence of a slice, see theorem 4.1.

Theorem 4.6. Let the $\mathfrak{g}$-action $\varphi$ on $M$ be proper and generate the local $G$-action $\Phi$ with domain $D$. Suppose that at each $p \in M$ the local isotropy group $G_p$ acts linearly on $V_p \subseteq T_pM$ via $G_p \times V_p \to V_p : (g, v_p) \mapsto T_{\varphi(p)}(g)v_p$. Then the differential spaces $(V_p/G_p, C^{\infty}(V_p/G_p))$ and $(M, C^{\infty}(M))$ are locally diffeomorphic.

Proof. Our proof follows [2, thm 5.1, p. 727–8]. Let $S_p = \exp_p(V_p^\varphi) \subseteq U_p^\varphi$ be a slice to the $\mathfrak{g}$-action at $p$ in $M$, where $U_p^\varphi$ is a $G_p$-invariant neighborhood of $p$ in $M$. Then $U_p = \pi(U_p^\varphi)$ is an open neighborhood of $p = \pi(p)$ in $M$ and $\pi : M \to M/X : p \mapsto O_p$.
is the projection map. The map $\lambda = \pi \circ \exp_p : V^\varepsilon_p \to \overline{U}_p^\varepsilon$ is continuous and $G_p$-invariant. Thus $\lambda = \rho^* \overline{\lambda}$, where $\overline{\lambda} : V^\varepsilon_p / G_p \subseteq V^\varepsilon_p / G_p \to \overline{U}_p^\varepsilon / \overline{M}$ is the map induced by $\lambda$ and $\rho : V^\varepsilon_p \to V^\varepsilon_p / G_p$ is the projection map. Then $\overline{\lambda}$ is a homeomorphism, since $\exp_p$ induces a homeomorphism between $V^\varepsilon_p / G_p$ and $S^\varepsilon_p / G_p$.

The map $\lambda$ is smooth. For suppose that $\overline{f} \in C^\infty(\overline{U}_p^\varepsilon)$. Then $f = \pi^* \overline{f} \in C^\infty(\pi^{-1}(\overline{U}_p^\varepsilon))$ is $g$ and hence $G_p$-invariant on $U_p^\varepsilon$. Hence on $S^\varepsilon_p \subseteq \pi^{-1}(\overline{p})$ the function $f|S^\varepsilon_p$ is smooth and $G_p$-invariant. Because $S^\varepsilon_p = \exp_p(V^\varepsilon_p)$ by construction and the mapping $\exp_p : V^\varepsilon_p \subseteq T_pM \to S^\varepsilon_p \subseteq U^\varepsilon_p \subseteq M$ is smooth, the function $(\exp_p)^* f$ on $V^\varepsilon_p$ is smooth and is $G_p$-invariant. Hence the mapping $\lambda : V^\varepsilon_p \to \overline{U}_p^\varepsilon$ is smooth. Since $\lambda$ is $G_p$-invariant, the induced mapping $\overline{\lambda} : V^\varepsilon_p / G_p \to \overline{U}_p^\varepsilon$ is smooth and invertible with a continuous inverse $\mu : \overline{U}_p^\varepsilon \to V^\varepsilon_p / G_p$.

We have to show that $\mu$ is a smooth mapping. Let $\overline{f} \in C^\infty(V^\varepsilon_p / G_p)$. Then $f = \rho^* \overline{f} \in C^\infty(V^\varepsilon_p)$, where $\rho : V^\varepsilon_p \subseteq T_pM \to V^\varepsilon_p / G_p$ is the projection mapping associated to the linear action of $G_p$ on $V_p$. This implies that $h = \theta^* f \in C^\infty(S^\varepsilon_p)$, where $\theta = \exp_p^{-1} : S^\varepsilon_p \subseteq U^\varepsilon_p \to V^\varepsilon_p \subseteq T_pM$. Since the function $h$ is $G_p$-invariant, it extends to a $G$-invariant function $h$ on $\Phi_G(S^\varepsilon_p)$. But $\Phi_G(S^\varepsilon_p) = \Phi_{\{g \in G \mid g \in D_p\}}(S^\varepsilon_p) = \cup_{q \in S^\varepsilon_p}O_q = X_g \cdot S^\varepsilon_p$. So $\overline{h}$ is $g$-invariant function on $X_g : S^\varepsilon_p$. Hence there is an induced smooth function $\overline{h}$ on $\pi(S^\varepsilon_p)$, that is, $\overline{h} \in C^\infty(\overline{U}_p^\varepsilon)$ such that $\pi^* \overline{h} = \overline{h}$.

Because $\overline{f}$ is a smooth function, $\mu^* \overline{f}$ is a continuous function on $\overline{U}_p^\varepsilon$. So $\pi^*(\mu^* \overline{f})$ is a continuous $g$-invariant function on $X_g : S^\varepsilon_p$. We need to show that $\pi^*(\mu^* \overline{f})$ is smooth. We argue as follows. Let $p' \in S^\varepsilon_p$ and $p'' = \Phi_g(p')$ with $g \in D_p$. Then

$$\overline{h}(p'') = \overline{h}(\Phi_g(p')) = h(p') = \theta^* f(p') = f(\theta(p')) = \rho^* \overline{f}(\theta(p')) = \overline{f}(\rho \theta(p')) = \overline{f}(\phi(p'')),$$

since $\phi = \rho \theta$. We now show that on $\Phi_{D_p}(S^\varepsilon_p)$ we have $\overline{\lambda} \phi = \pi$. For every $g \in D_p$ and $p'' = \Phi_g(p')$ with $p' \in S^\varepsilon_p$ we have

$$\overline{\lambda}(\phi(p'')) = \overline{\lambda}(\rho \theta(p'')) = \overline{\lambda}(\rho \theta \exp^{-1}_p(p'')) = \overline{\lambda}(\rho(T_{p'} \Phi_g \exp^{-1}_p(p'))),$$

since $p'' = \Phi_g(p')$ and $T_{p'} \Phi_g \exp^{-1}_p = \exp^{-1}_p \circ \Phi_g(p')$.

Consequently,

$$\overline{h}(p'') = \overline{h}(p') = \overline{f}(\rho \theta(p')) = \mu^* \overline{f}(\overline{\lambda} \rho \theta(p')) = \mu^* \overline{f}(\overline{\lambda} \phi(p')) = \mu^* \overline{f}(\pi(p')) = \mu^* \overline{f}(\pi(p')),$$

since $\overline{\lambda} \phi = \pi$ on $S^\varepsilon_p$.

Since $\overline{h} = \pi^* \overline{h}$, we get $\overline{h} = \mu^* \overline{f} \in C^\infty(\overline{U}_p^\varepsilon)$. This follows because the map $\pi^*$ is injective since $\pi$ is surjective. Hence the mapping $\mu = \overline{\lambda}^{-1} : \overline{U}_p^\varepsilon \to V^\varepsilon_p / G_p$ is smooth. Consequently, the differential spaces $(V^\varepsilon_p / G_p, C^\infty(V^\varepsilon_p / G_p))$ and $(\overline{U}_p^\varepsilon, C^\infty(\overline{U}_p^\varepsilon))$ are diffeomorphic. \(\square\)
Proof. (Theorem 4.1) The main theorem follows once we have shown that at each point \( p \in M \) the differential space \( (V^*_p/G_p, C^\infty(V^*_p/G_p)) \) is subcartesian. This is a consequence of the argument given in [3, chpt VII, p.324–327], since \( G_p \) is a compact Lie group, which acts via \( G_p \times V_p \to V_p : (g, v_p) \mapsto T_p \Phi_p g v_p \) in a linear fashion on the vector subspace \( V_p = T_p S^*_p \) of \( T_p M \). Using invariant theory, it follows that the image of the Hilbert map is orbit space \( V_p/G_p \), see [3, chpt VII, p.324]. Because the action is proper, this orbit space is a differential space with differential structure \( C^\infty(V_p/G_p) = C^\infty(V_p)^G_p \), which is diffeomorphic to a closed semialgebraic subset of some Euclidean space and hence is subcartesian, see [6, theorem 4.3.4].

Example 7. Let \( S^1 \) be the Lie group \( \mathbb{R}/2\pi\mathbb{Z} \) with Lie algebra \( \mathbb{R} \). Let \( M = TS^1 \setminus \{(0, 1), (\pi, 1)\} \subseteq S^1 \times \mathbb{R} = [0, 2\pi] \) mod \( 2\pi \times \mathbb{R} \) with coordinates \((\theta, p)\). Consider the Lie algebra action

\[
\varphi : \mathbb{R} \to \mathcal{X}(M) : \frac{\partial}{\partial t} \mapsto X(\theta, p) = \frac{\partial}{\partial \theta}.
\]

The flow of \( X \) on \( M \) is \( \varphi_t(\theta, p) = (\theta + t, p) \), when defined. An orbit \( O(\theta, p) \) of the \( \mathbb{R} \)-action \( \varphi \) through \((\theta, p) \in M \) is an integral of the vector field \( X \) through \((\theta, p) \). The orbits \( O(\theta, p) \) on \( M \) are exactly one item of the list: a circle \( S^1_{p_0} = \{(\theta, p_0) \in M \} \) when \( p_0 \neq 1 \); a circular arc \( A_1 = \{(\theta, 1) \in M \mid 0 < \theta < \pi \} \), when \( p = 1 \); and a circular arc \( A_2 = \{(\theta, 1) \in M \mid \pi < \theta < 2\pi \} \), when \( p = 1 \). Thus the space \( M/X \) of orbits of \( X \) is the disjoint union of \( \mathbb{R} \setminus \{1\} \) and two points \( q_0 \) and \( q_1 \). Let \( \pi : M \to M/X : (\theta, p) \mapsto O(\theta, p) \) be the projection map. A function \( f : M \to \mathbb{R} \) is smooth if and only if \( f = \pi^* T \) is smooth function on \( M \). Let \( C^\infty(M/X) \) be the set of smooth functions on \( M/X \). Since \( T \subseteq C^\infty(M/X) \), then \( C^\infty(M/X) \) is a differential structure on \( M/X \). The differential space topology on the differential space \((M/X, C^\infty(M/X))\) has a subbasis \( T^{-1}(I) \) where \( I \) is an open interval in \( \mathbb{R} \) and \( T \subseteq C^\infty(M/X) \). In the differential space topology on \( M/X \) there are no open neighborhoods \( U_0 \) and \( U_1 \) of \( q_0 \) and \( q_1 \), respectively, which have empty intersection. Thus the differential space topology is not Hausdorff. Consequently, the differential space \((M/X, C^\infty(M/X))\) is not subcartesian.

The Lie algebra action \( \varphi \) on \( M \) is the infinitesimal generator of the local \( S^1 \)-action

\[
\Phi : D \subseteq S^1 \times M \to M : (t, (\theta, p)) \mapsto (t + \theta, p)
\]
on \( M \) with domain \( D \). For each \((\theta, p) \in M \) the subset \( D_{(\theta, p)} \) of \( S^1 \) is connected open and contains the identity element.

Consider the \( S^1 \)-action

\[
\tilde{\Phi} : S^1 \times TS^1 \to TS^1 : (t, (\theta, p)) \mapsto (t + \theta, p).
\]

Then \( \tilde{\Phi}|_D = \Phi \). Because the Lie group \( S^1 \) is compact, the \( S^1 \)-action \( \tilde{\Phi} \) is proper. We now show that \( \varphi \) is a proper \( \mathbb{R} \)-action. Let \( \{(\theta_n, p_n)\} \) be a sequence in \( M \), which converges to \((\theta, p) \in M \) and for \( t_n \in D_{(\theta_n, p_n)} \subseteq S^1 \) let \( \{\Phi_{t_n}(\theta_n, p_n)\} \) be a sequence of points in \( M \), which converges to \((\theta, p) \in M \). Since \( S^1 \) is compact and the mapping

\[
gr \tilde{\Phi} : S^1 \times M \to M \times M : (t, (\theta, p)) \mapsto (t + \theta, p)
\]
is proper, there is a subsequence \( \{t_{n_k}\} \) in \( S^1 \) which converges to \( t \in S^1 \) and the sequence \( \{\Phi_{t_{n_k}}(\theta_n, p_n)\} \) converges to \( \tilde{\Phi}_t(\theta, p) = (\theta, p) \). So \( \Phi_t(\theta, p) = (\theta, p) \), since
((\theta, p) \in M). From \Phi_t(\theta, p) = (\tilde{\theta}, \tilde{p})$, it follows that $t \in D_{(\theta, p)}$. To see this we argue as follows. If $p \neq 1$, this is immediate since $t \in S^1$. If $p = 1$ and $t$ lies in the boundary $\partial D_{(\theta, p)}$ of the connected set $D_{(\theta, p)}$ containing the identity element of $S^1$, then either $\Phi_t(\theta, 1) = (0, 1)$ or $\Phi_t(\theta, 1) = (\pi, 1)$. But neither $(0, 1)$ or $(\pi, 1)$ lie in $M$. This contradicts the fact that $\Phi_t(\theta, 1) \in M$. Consequently, $t \in D_{(\theta, 1)}$.

Thus the subsequence $\{(t_n, (\theta_{n \pi} + p_{n \pi}))\}$ in $D$ converges to $(t, (\bar{\theta}, \bar{p})) \in D$. Hence the map $\text{gr} \, \Phi$ is proper, that is, the $\mathbb{R}$-action $\varphi$ is proper. Thus the $\mathbb{R}$-orbit space $(M/X, C^\infty(M/X))$ is properly

5. Hamiltonian Lie algebra actions. In this section we discuss the reduction of the Lie algebra symmetry of a proper Hamiltonian Lie algebra action.

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that $(M, \omega)$ is a smooth symplectic manifold with the usual Poisson bracket $\{\, , \}$ on its space $C^\infty(M)$ of smooth functions. Then $(C^\infty(M), \{\, , \})$ is a Lie algebra. A vector field $X_h$ on $(M, \omega)$ is Hamiltonian with Hamiltonian function $h \in C^\infty(M)$ if and only if $X_h = \omega(\cdot, \cdot) \varphi(\cdot) = \partial h$ on $M$. Let $\text{Ham}(M)$ be the set of Hamiltonian vector fields on $M$.

Thus for every $\xi \in \mathfrak{g}$ we have

$$- [X_f, X_h] = X_{\{f, h\}}, \quad \text{for every } f, h \in C^\infty(M),$$

(5)

if follows that $(\text{Ham}(M), [-, -])$ is a Lie algebra with Lie bracket $-[\, , \]$. A Hamiltonian Lie algebra action of $\mathfrak{g}$ on the symplectic manifold $(M, \omega)$ is a Lie algebra homomorphism

$$\varphi : (\mathfrak{g}, [-, -]) \to (C^\infty(M), \{\, , \}) : \xi \mapsto J^\xi.$$

(6)

**Theorem 5.1.** The mapping $J : M \to \mathfrak{g}^* : p \mapsto J(p)$, where $J(p) : \mathfrak{g} \to \mathbb{R} : \xi \mapsto J^\xi(p)$, is a momentum mapping for the $\mathfrak{g}$-action $\varphi$.

**Proof.** Since $\varphi$ is a linear mapping, for every $\xi, \eta \in \mathfrak{g}$ we have

$$J^{\xi + \eta} = \varphi(\xi + \eta) = \varphi(\xi) + \varphi(\eta) = J^\xi + J^\eta.$$

Thus for every $p \in M$, the mapping $J(p) : \mathfrak{g} \to \mathbb{R}$ is linear. Hence the mapping $J : M \to \mathfrak{g}^*$ is defined. $J$ is a momentum mapping, because $\varphi$ is a Lie algebra homomorphism.

**Lemma 5.2.** Associated to the Hamiltonian $\mathfrak{g}$-action $\varphi$ defined by (6) on $(M, \omega)$ is the Hamiltonian $\mathfrak{g}$-action.

$$\hat{\varphi} : (\mathfrak{g}, [-, -]) \to (\text{Ham}(M), [-, -]) : \xi \mapsto X_{\varphi(\xi)}.$$

**Proof.** $\hat{\varphi}$ is linear. It is a Lie algebra homomorphism because $\varphi$ is.

Suppose that $\varphi$ is a Hamiltonian $\mathfrak{g}$-action on $(M, \omega)$. Recall that a smooth function $f$ on $M$ is $\mathfrak{g}$-invariant if and only if for every $\xi \in \mathfrak{g}$ we have $\{f, \varphi(\xi)\} = 0$ on $M$. Let $C^\infty(M)^\mathfrak{g}$ be the space of smooth $\mathfrak{g}$-invariant functions on $M$.

**Lemma 5.3.** The space $(C^\infty(M)^\mathfrak{g}, \{\, , \})$ is a Poisson subalgebra of $(C^\infty(M), \{\, , \})$.

**Proof.** For every $\xi \in \mathfrak{g}$ and every $h_1, h_2 \in C^\infty(M)^\mathfrak{g}$ we have $\varphi(\xi) = J^\xi$ and

$$\{J^\xi, \{h_1, h_2\}\} = \{\{J^k, h_1\}, h_2\} + \{h_1, \{J^\xi, h_2\}\} = 0,$$

since $\{J^\xi, h_1\} = \{J^\xi, h_2\} = 0$ for $h_1, h_2 \in C^\infty(M)^\mathfrak{g}$.
So \( \{h_1, h_2\} \in C^\infty(M)^g \). Thus \( (C^\infty(M)^g, \{ , \}) \) is a Lie subalgebra of \( (C^\infty(M), \{ , \}) \). Now
\[
\text{ad}_{h_1 \cdot h_2} = (\text{ad}_{h_1} + \text{ad}_{h_2}) \cdot h_1 + h_1 \cdot \text{ad}_{h_2} \cdot h_2
\]
\[
= \{h_1, h_2\} \cdot h_1 + h_1 \cdot \{h_2, h_1\} = 0.
\]
So \( h_1 \cdot h_2 \in C^\infty(M)^g \). Thus \( (C^\infty(M)^g, \cdot) \) is a subalgebra of \( (C^\infty(M), \cdot) \). Let \( h \in C^\infty(M)^g \). Then
\[
\text{ad}_h(h_1 \cdot h_2) = \text{ad}_h h_1 \cdot h_2 + h_1 \cdot \text{ad}_h h_2 \in C^\infty(M)^g,
\]
since \( \text{ad}_h h_1 = \{h_1, h\} \) and \( \text{ad}_h h_2 = \{h_2, h\} \) lie in \( C^\infty(M)^g \) because \( h, h_1, \) and \( h_2 \) do and \( (C^\infty(M)^g, \cdot) \) is a subalgebra. Hence \( (C^\infty(M)^g, \{ , \}) \) is a Poisson algebra.

Let \( \varphi \) be a Hamiltonian \( g \)-action on \( (M, \omega) \). For each \( p \in M \) the orbit \( O_p \) of \( \varphi \) through \( p \) is the orbit of the family \( X_\# = \{\varphi(\xi) \in \text{Ham}(M) \mid \xi \in g\} \) of Hamiltonian vector fields on \( M \) through \( p \).

**Lemma 5.4.** \( f \in C^\infty(M)^g \) if and only if \( f \) is constant on the \( X_\# \)-orbit \( O_p \) in \( M \) for every \( p \in M \).

**Proof.** Suppose that \( f \in C^\infty(M)^g \). Then for every \( \xi \in g \) the function \( f \) is constant on the integral curve of the Hamiltonian vector field \( X_\#(\xi) \) through every \( p \in M \). Thus \( f \) is constant on the \( X_\# \)-orbit \( O_p \), because \( O_p \) is the image of piecewise smooth curves in \( M \) each of whose segments in an integral curve of \( X_\#(\eta) \) for some \( \eta \in g \). Conversely, suppose that the smooth function \( f \) on \( M \) is constant on the \( X_\# \)-orbit \( O_p \). Then it is constant on the integral curve \( X_\#(\xi) \) through \( p \) for every \( \xi \in g \). Hence \( \{f, \varphi(\xi)\}(p) = 0 \) for every \( p \in M \). In other words, \( f \in C^\infty(M)^g \).

Let \( \overline{M} = M/X_\# \) be the space of \( X_\# \)-orbits on \( M \) with projection mapping \( \pi : M \to \overline{M} : p \mapsto O_p \). If \( f \in C^\infty(M)^g \), then \( f \) is constant on every \( X_\# \)-orbit \( O_p \) for \( p \in M \). Hence there is an induced function \( \overline{f} \) on \( \overline{M} \) such that \( f = \pi^* \overline{f} \). Let \( C^\infty(\overline{M}) \) be the space of functions \( \overline{f} \) on \( \overline{M} \) such that \( \pi^* \overline{f} \in C^\infty(M)^g \). From theorem 4.5 it follows that \( C^\infty(\overline{M}) \) is a differential structure on \( \overline{M} \).

**Theorem 5.5.** The differential space \( (\overline{M}, C^\infty(\overline{M})) \) is Poisson.

**Proof.** We only need to define a Poisson bracket \( \{ , \}_{\overline{M}} \) on \( C^\infty(\overline{M}) \). We do this by requiring that
\[
\pi^* \{\overline{f}_1, \overline{f}_2\}_{\overline{M}} = \{\pi^* \overline{f}_1, \pi^* \overline{f}_2\},
\]
where \( \overline{f}_1, \overline{f}_2 \in C^\infty(\overline{M}) \) and \( \{ , \} \) is the Poisson bracket on \( C^\infty(M)^g \). To see that the bracket \( \{ , \}_{\overline{M}} \) is well defined, note that the right hand side of (7) is \( g \)-invariant. Hence there is a function \( h \in C^\infty(\overline{M}) \) such that \( \pi^* h = \{\pi^* \overline{f}_1, \pi^* \overline{f}_2\} \). Since \( \pi^* \) is injective, we obtain \( h = \{\overline{f}_1, \overline{f}_2\}_{\overline{M}} \). Thus the bracket \( \{ , \}_{\overline{M}} \) is well defined. It is straightforward to verify that \( (C^\infty(\overline{M}), \{ , \}_{\overline{M}}) \) is a Poisson algebra.

Using theorem 4.1 we obtain

**Corollary 5.5.1.** If the Hamiltonian \( g \)-action is proper, then the differential space \( (\overline{M}, C^\infty(\overline{M})) \) is locally Euclidean.
6. Integrable Hamiltonian systems. Suppose that \((M, \omega, \mathbf{F})\) is an integrable Hamiltonian system on the smooth 2n-dimensional symplectic manifold \((M, \omega)\), that is, the components of the integral mapping \(\mathbf{F} : M \to \mathbb{R}^n : p \mapsto (f_1(p), \ldots, f_n(p))\) Poisson commute, namely, \(\{f_i, f_j\} = 0\) and the rank of \(D\mathbf{F}(p)\) is \(n\) for all points \(p\) in an dense open subset \(U\) of \(M\). Let \((\mathfrak{g}, \{\cdot, \cdot\})\) be the Lie algebra with basis \(\{f_i\}_{i=1}^n\) and Lie bracket \(\{\cdot, \cdot\}\). Then \(\mathfrak{g}\) is abelian. Define a Hamiltonian Lie algebra action of \(\mathfrak{g}\) on \((M, \omega)\) by

\[
\varphi : \mathfrak{g} \to (C^\infty(M), \{\cdot, \cdot\}) : f_i \mapsto J^f_i = f_i.
\]

The \(\mathfrak{g}\)-action \(\varphi\) gives rise to the associated \(\mathfrak{g}\)-action

\[
\tilde{\varphi} : \mathfrak{g} \to (\text{Ham}(M), [\cdot, \cdot]) : f_i \mapsto X_{f_i}.
\]

Let \(X_\mathbf{F}\) be the family of Hamiltonian vector fields \(\{X_{f_i}\}, 1 \leq i \leq n\) on \((M, \omega)\). The collection \(C^\infty(M)^\mathfrak{g}\) of smooth \(\mathfrak{g}\)-invariant functions on \(M\) is the same as the collection \(C^\infty(M)^\mathbf{F}\) of smooth functions \(f\) which are invariant under the family \(X_\mathbf{F}\), that is, \(L_{X_{f_i}} f = 0\) for every \(1 \leq i \leq n\). From lemma 5.3 it follows that \((C^\infty(M)^\mathfrak{g}, \{\cdot, \cdot, \cdot\})\) is a Poisson subalgebra of \((C^\infty(M), \{\cdot, \cdot, \cdot\})\). As shown in theorem 4.5, \(C^\infty(M)^\mathbf{F}\) is a differential structure for the space \(M/X_\mathbf{F}\) of orbits of the \(\mathfrak{g}\)-action \(\varphi\). Equivalently, \(C^\infty(M)^\mathbf{F}\) is a differential structure for the space \(M/X_\mathbf{F}\) of orbits of the family \(X_\mathbf{F}\) of vector fields on \(M\).

We now look at the relation between the space of orbits \(M/X_\mathbf{F}\) of the family \(X_\mathbf{F}\) of Hamiltonian vector fields associated to the integrable Hamiltonian system \((M, \omega, \mathbf{F})\) and the image \(\mathbf{F}(M)\) of its integral map. Since \(\mathbf{F}(M) \subseteq \mathbb{R}^n\), we have a differential structure \(C^\infty_{\text{ind}}(\mathbf{F}(M))\) on \(\mathbf{F}(M)\) defined by \(g \in C^\infty_{\text{ind}}(\mathbf{F}(M))\) if and only if for every \(x \in \mathbf{F}(M)\) there is an open neighborhood \(V_x\) of \(x\) in \(\mathbb{R}^n\) and a function \(G_x \in C^\infty(\mathbb{R}^n)\) such that \(g|_{V_x \cap \mathbf{F}(M)} = G_x|_{V_x \cap \mathbf{F}(M)}\). The differential space \((\mathbf{F}(M), C^\infty_{\text{ind}}(\mathbf{F}(M)))\) is a differential subspace of \((\mathbb{R}^n, C^\infty(\mathbb{R}^n))\).

**Theorem 6.1.** The integral map

\[
\mathbf{F} : (M, C^\infty(M)) \to (\mathbf{F}(M), C^\infty_{\text{ind}}(\mathbf{F}(M)))
\]

is a smooth mapping of differential spaces.

**Proof.** Suppose that \(g \in C^\infty_{\text{ind}}(\mathbf{F}(M))\). We need to show that \(\mathbf{F}^*g \in C^\infty(M)\). By definition for every \(x \in \mathbf{F}(M)\) there is an open neighborhood \(V_x\) of \(x\) in \(\mathbb{R}^n\) and a function \(G_x \in C^\infty(\mathbb{R}^n)\) such that \(g|_{V_x \cap \mathbf{F}(M)} = G_x|_{V_x \cap \mathbf{F}(M)}\). Hence

\[
(\mathbf{F}^*g)|_{\mathbf{F}^{-1}(V_x \cap \mathbf{F}(M))} = (\mathbf{F}^*G_x)|_{\mathbf{F}^{-1}(V_x \cap \mathbf{F}(M))}.
\]

Since \(G_x \in C^\infty(\mathbb{R}^n)\) and \(\mathbf{F} : M \to \mathbf{F}(M) \subseteq \mathbb{R}^n\) is smooth, it follows that \(\mathbf{F}^*G_x = G_x \circ \mathbf{F}\) is smooth. Thus for every \(p \in \mathbf{F}(M)\) there is a \(x = \mathbf{F}(p) \in \mathbf{F}(M)\), an open neighborhood \(\mathbf{F}^{-1}(V_x)\) of \(p\) in \(M\), and a function \(\mathbf{F}^*G_x \in C^\infty(M)\) such that equation (8) holds. Hence \(\mathbf{F}^*g \in C^\infty(M)\).

For each \(p \in M\) let \(L_p\) be the connected component of the fiber \(\mathbf{F}^{-1}(\mathbf{F}(p))\) containing the point \(p\). Let \(N = \{L_p : p \in M\}\) and let \(\rho : M \to N : p \mapsto L_p\). The integral mapping \(\mathbf{F}\) induces the map

\[
\mu : N \to \mathbf{F}(M) \subseteq \mathbb{R}^n : L_p \mapsto \mathbf{F}(p).
\]

**Theorem 6.2.** For every \(p \in M\) the connected component \(L_p\) of the fiber \(\mathbf{F}^{-1}(\mathbf{F}(p))\) containing \(p\) is the orbit of the family \(X_\mathbf{F}\) through \(p\).
Proof. For each \(1 \leq i \leq n\) let \(\varphi_i^t\) be the local flow of the vector field \(X_i\) on \((M, \omega)\). Fix \(p_0 \in M\). Then \(f_j(\varphi_i^t(p_0)) = f_j(p_0)\) for every \(1 \leq j \leq n\). Hence the orbit of \(X_F\) through \(p_0\) is contained in \(F^{-1}(F(p_0))\). Since orbits of \(X_F\) are connected, they are the connected components of \(F^{-1}(F(p_0))\).

To finish the argument we must show that the orbits of \(X_F\) are open in the fibers of the integral mapping \(F\). Let \(O_p\) be the orbit of \(X_F\) through \(p\). Suppose that rank \(DF(p) = k\). By the implicit function theorem, there is a neighbourhood \(U\) of \(p\) in \(M\) such that \(U \cap F^{-1}(F(p))\) is a \(k\)-dimensional submanifold of \(M\). On the one hand, since \(O_p\) is a manifold contained in \(F^{-1}(F(p))\), it follows that its dimension is at most \(k\). On the other hand, rank \(DF(p) = k\) implies that there exist \(k\) vectors in \(\{X_{f_1}(p), \ldots, X_{f_n}(p)\}\) that are linearly independent at \(p\). Therefore, the dimension of the orbit \(O_p\) is at least \(k\). Hence, \(\dim O_p = k\), and there exists a neighbourhood \(U'\) of \(p\) in \(M\) such that \(U' \subseteq U\) and \(U' \cap O_p = U' \cap F^{-1}(F(p))\) is an open subset of \(F^{-1}(F(p_0))\). This holds for every \(p \in F^{-1}(F(p_0))\). So the orbits of \(X_F\) that are contained in \(F^{-1}(F(p_0))\) are open subsets of \(F^{-1}(F(p_0))\).

Theorem 6.2 enables us to identify the space \(M/X_F\) of orbits of the family \(X_F\) of vector fields on \(M\) with the space \(N\) of connected components of the fibers of the integral mapping \(F : M \to F(M) \subseteq \mathbb{R}^n\). The identification \((M/X_F) = N\) leads to the identification of the projection map \(\pi : M \to M/X_F\) with the map \(\rho : M \to N\). In papers on reduction of symmetries in Hamiltonian systems, the space \(M/X_F\) is called the orbit space and \(\pi : M \to M/X_F : p \mapsto O_p\) the orbit map, see [3]. In papers on completely integrable Hamiltonian systems, the space \(N\) of connected components of the fibers of the integral map \(F\) is called the base space, see Ratiu, et al. [5].

Since \(F\) is constant on the orbits of \(X_F\), which are connected components of the fibers of \(F\), it follows that the integral map \(F : M \to F(M) \subseteq \mathbb{R}^n\) factors into the composition of \(\pi : M \to M/X_F\) and the map

\[
\mu : M/X_F = N \to F(M) : L \mapsto F(p).
\]

In other words, \(F = \mu \circ \pi\).

**Corollary 6.2.1.** The mapping

\[
\mu : (M/X_F = N, C^\infty(M/X_F)) \to (F(M), C^\infty_{\text{ind}}(F(M)))
\]

is smooth.

**Proof.** Suppose that \(g \in C^\infty_{\text{ind}}(F(M))\). From theorem 6.1, we get \(F^*g \in C^\infty(M)\). Clearly \(F^*g = g_\circ F\) is constant on the fibers of \(F\). Since the orbits of \(X_F\) are connected subsets in the fibers of \(F\), it follows that \(F^*g \in C^\infty(M)\). Hence there is a function \(h \in C^\infty(M/X_F)\) such that \(F^*g = \pi^*h\). The equality \(F = \mu \circ \pi\) yields \(\pi^*h = (\mu \circ \pi)^*g = \pi^*(\mu^*g)\). Since \(\pi^* : C^\infty(M/X_F) \to C^\infty(M)\) is bijective, it follows that \(\mu^*g = h \in C^\infty(M/X_F)\). Hence the mapping \(\mu\) is smooth.

**Acknowledgment.** The authors wish to thank the referee for the useful comments and for improving the proof of theorem 5.5.

**References**

[1] N. Aronszajn, Subcartesian and subRiemannian spaces, Notices American Mathematical Society, 14 (1967), 111–111.

[2] R. Cushman and J. Śniatycki, Differential structure of orbit spaces, Canad. Math. J., 54 (2001), 715–755.
[3] R. H. Cushman and L. M. Bates, *Global Aspects of Classical Integrable Systems*, second edition, Birkhäuser, Basel, 2015.

[4] R. S. Palais, *A Global Formulation of the Lie Theory of Transformation Groups*, Memoir 22, American Mathematical Society, Providence, R.I. 1957.

[5] T. Ratiu, C. Wacheux and N. T. Zung, Convexity of singular affine structures and toric-focus integrable Hamiltonian systems, *arXiv:1706.01093v1*.

[6] J. Śniatycki, *Differential Geometry of Singular Spaces and Reduction of Symmetry*, Cambridge University Press, Cambridge, UK, 2013.

[7] H. Sussmann, Orbits of families of vector fields and foliations with singularities, *Trans. Amer. Math. Soc.*, 180 (1973), 171–188.

Received October 2017; revised July 2018.

*E-mail address: rcushman@ucalgary.ca*

*E-mail address: sniatycki@gmail.com*