Subalgebra generated by ad-locally nilpotent elements of Borcherds Generalized Kac-Moody Lie algebras

Shrawan Kumar

Department of Mathematics, University of North Carolina, Chapel Hill, NC 27599-3250, USA

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A B S T R A C T
We determine the Lie subalgebra $\mathfrak{g}_{\text{nil}}$ of a Borcherds symmetrizable generalized Kac-Moody Lie algebra $\mathfrak{g}$ generated by ad-locally nilpotent elements and show that it is ‘essentially’ the same as the Levi subalgebra of $\mathfrak{g}$ with its simple roots precisely the real simple roots of $\mathfrak{g}$.

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1. Introduction

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the symmetrizable Generalized Kac-Moody (GKM) algebra associated to a $\ell \times \ell$ matrix $A$ (cf. Section 2). Let

$$\mathfrak{g}_{\text{nil}}^\circ := \{ x \in \mathfrak{g} : \text{ad } x \text{ acts locally nilpotently on } \mathfrak{g} \},$$

and let $\mathfrak{g}_{\text{nil}} \subset \mathfrak{g}$ be the Lie subalgebra generated by $\mathfrak{g}_{\text{nil}}^\circ$. Then, we prove the following theorem (cf. Theorem 3.1):

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E-mail address: shrawan@email.unc.edu.

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Theorem. Let $g = g(A)$ be as above, where $\ell \geq 2$ and $A$ is indecomposable, i.e., the corresponding Dynkin diagram is connected. Then,

$$g'(B) \subset g_{\mathrm{null}} \subset g'(B) + h,$$

where $B \subset A$ is the submatrix parameterized by those $i$ such that $a_{i,i} = 2$, $h$ is the Cartan subalgebra and $g'(B)$ is the derived subalgebra of $g(B)$.

As shown in Remark 3.2, the assumption $\ell \geq 2$ in the above theorem is necessary in general.

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2. Basic definition

In this section, we recall the definition of Borcherds Generalized Kac-Moody Lie algebras $g$ (for short GKM algebras). For a more extensive treatment of $g$ and its properties, see Chapters 1, 11 of [3] and the papers [1] and [2].

Definition 2.1. Let $A = (a_{i,j})$ be a $\ell \times \ell$ matrix (for $\ell \geq 1$) with real entries, satisfying the following properties:

(P1) either $a_{i,i} = 2$ or $a_{i,i} \leq 0$,

(P2) $a_{i,j} \leq 0$ if $i \neq j$, and $a_{i,j} \in \mathbb{Z}$ if $a_{i,i} = 2$,

(P3) $a_{i,j} = 0$ if and only if $a_{j,i} = 0$.

Fix a realization of $A$, which is a triple $(h, \Pi, \Pi^\vee)$ consisting of a complex vector space $h$, $\Pi = \{\alpha_1, \ldots, \alpha_\ell\} \subset h^*$ and $\Pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_\ell^\vee\} \subset h$ are indexed subsets, satisfying the following three conditions:

(Q1) both sets $\Pi$ and $\Pi^\vee$ are linearly independent,

(Q2) $\alpha_j(\alpha_i^\vee) = a_{i,j}$, for all $i,j$,

(Q3) $\ell - \mathrm{rank} A = \dim h - \ell$.

By [3], Proposition 1.1, such a realization is unique up to an isomorphism of the triple.

Now, the Borcherds Generalized Kac-Moody Lie algebra (for short GKM algebra) $g(A)$ is defined as the Lie algebra generated by $\{e_i, f_i, h\}_{1 \leq i \leq \ell}$ subject to the following relations:

(R1) $[e_i, f_j] = \delta i j \alpha_i^\vee$, for all $i$,

(R2) $[h, h'] = 0$, for all $h, h' \in h$,

(R3) $[h, e_i] = \alpha_i(h)e_i$; $[h, f_i] = -\alpha_i(h)f_i$, for all $1 \leq i \leq \ell$ and $h \in h$,
The matrix $A$ (or the Lie algebra $\mathfrak{g}(A)$) is called symmetrizable if there exists an invertible diagonal matrix $D = \text{diag}(\varepsilon_1, \ldots, \varepsilon_\ell)$ such that the matrix $DA$ is symmetric.

3. Main theorem and its proof

**Theorem 3.1.** Let $\mathfrak{g} = \mathfrak{g}(A)$ be the symmetrizable GKM algebra associated to a $\ell \times \ell$ matrix $A$ as in the last section. Assume further that $\ell \geq 2$ and $A$ is indecomposable, i.e., the corresponding Dynkin diagram is connected. Let

$$\mathfrak{g}_{\text{nil}} := \{x \in \mathfrak{g} : \text{ad } x \text{ acts locally nilpotently on } \mathfrak{g}\},$$

and let $\mathfrak{g}_{\text{nil}} \subset \mathfrak{g}$ be the Lie subalgebra generated by $\mathfrak{g}_{\text{nil}}^0$. Then,

$$\mathfrak{g}'(B) \subset \mathfrak{g}_{\text{nil}} \subset \mathfrak{g}'(B) + \mathfrak{h},$$

where $B \subset A$ is the submatrix parameterized by those $i$ such that $a_{ii} = 2$, i.e., $\alpha_i$ is a real root and $\mathfrak{g}'(B)$ is the derived subalgebra of $\mathfrak{g}(B)$.

**Proof.** Consider the $\mathbb{Z}$-gradation of $\mathfrak{g}$ induced from a homomorphism $\theta : Q := \bigoplus_i \mathbb{Z} \alpha_i \to \mathbb{Z}$. Then, for any $x \in \mathfrak{g}_{\text{nil}}^0$, $x_+(\theta) \in \mathfrak{g}_{\text{nil}}^0$, where $x_+(\theta)$ is the top degree component of $x$ in the $\mathbb{Z}$-gradation of $\mathfrak{g}$ induced by $\theta$. To prove this, observe that for any $y \in \mathfrak{g}_\alpha$ (where $\mathfrak{g}_\alpha$ is the root space corresponding to the root $\alpha$ or 0),

$$(\text{ad } x)^n(y) = (\text{ad } x_+(\theta))^n(y) + \text{lower degree terms}.$$

Similarly, for $x \in \mathfrak{g}_{\text{nil}}^0$, $x_-(\theta) \in \mathfrak{g}_{\text{nil}}^0$, where $x_-(\theta)$ is the lowest degree component of $x$.

Further, given any nonzero $x \in \mathfrak{g}$, we can get a gradation $\theta_x : Q \to \mathbb{Z}$ as above (depending upon $x$) such that all the homogeneous degree components of $x$ (under $\theta_x$) belong to root spaces $\mathfrak{g}_\beta$. To prove this, write $x = \sum_j x_{\beta_j}$, where $\beta_j$ are distinct roots or zero, $x_{\beta_j} \in \mathfrak{g}_{\beta_j}$ and each $x_{\beta_j} \neq 0$. Consider the finite collection of weights: $\{\beta_j - \beta_k\}_{j \neq k} \subset \mathfrak{h}^*$. Now, we can find a vector $\gamma = \gamma_x \in Q^\ell = Q \otimes \mathbb{Z} Q$ such that for the standard dot product $(\cdot, \cdot)$ in $Q^\ell$,

$$\theta_x(\beta_j - \beta_k) := (\beta_j - \beta_k, \gamma) \neq 0, \text{ for any } j \neq k. \quad (1)$$

To prove the above equation, consider the $(\ell - 1)$-dimensional subspace $V_{j,k} \subset Q^\ell$ (for any $j \neq k$) perpendicular to $\beta_j - \beta_k$. Since the collection $\{\beta_j - \beta_k\}_{j \neq k}$ is finite, we can find a vector $\gamma$ such that the equation (1) is satisfied. We can further take $\gamma \in Q \simeq \mathbb{Z}^\ell$ by clearing the denominators.
So, if \( x \in \mathfrak{g}_{\text{nil}}^o \), then either \( x \) belongs to the center \( Z(\mathfrak{g}) \) of \( \mathfrak{g} \) or the root component \( x_\beta \in \mathfrak{g}_{\text{nil}}^o \) for some root \( \beta (\beta \neq 0) \). (To prove this: if \( x \) belongs to the Cartan subalgebra \( \mathfrak{h} \), then it will have to lie in \( Z(\mathfrak{g}) \) of \( \mathfrak{g} \) by [3], Proposition 1.6. But, if it does not lie in \( \mathfrak{h} \), then, as observed in the beginning of the proof by making a choice of \( \theta_x \) as above, \( x_+(\theta_x) \in \mathfrak{g}_{\text{nil}}^o \) for the top degree component \( x_+(\theta_x) \) of \( x \) in \( Z \)-gradation \( \theta_x \) of \( \mathfrak{g} \).) Moreover, if some nonzero root component of \( x \) belongs to the root space \( \mathfrak{g}_p \) such that \( \delta \) contains an imaginary simple root \( \delta_p \) (i.e., with \( a_{p,p} \leq 0 \)) with nonzero coefficient, we can assume that \( x_\delta \in \mathfrak{g}_{\text{nil}}^o \) (possibly with a different nonzero root component of \( x \) corresponding to a root containing an imaginary simple root with nonzero coefficient). This is achieved by taking \( \gamma \) as above but requiring \( \theta_x(\alpha_p) \) to be much larger for all the imaginary simple roots \( \alpha_p \) as compared to the values \( \theta_x(\alpha_q) \) for all the real simple roots \( \alpha_q \) (i.e., those with \( a_{q,q} = 2 \)).

By using the Cartan involution \( \omega \) of \( \mathfrak{g} \) (i.e., \( \omega(e_i) = -f_i, \omega(f_i) = -e_i, \omega(h) = -h \forall h \in \mathfrak{h} \), if needed, we can further assume that \( \delta \) is a positive root. Write

\[
\delta = \sum_p (m_p \alpha_p) + \sum_q (n_q \alpha_q), \text{ for } m_p, n_q \geq 0,
\]

where \( \alpha_p \) (resp. \( \alpha_q \)) run over all the imaginary (resp. real) simple roots. In particular, some \( m_p > 0 \). By [3], Exercise 11.21, the support \( \text{supp}(\delta) \) is connected. Assume first that \( \delta \) is not an imaginary simple root. Further, taking some \( W \)-translate (where \( W \) is the Weyl group of \( \mathfrak{g} \), cf. [3], §11.13), we can assume that \( \delta(\alpha_q^\vee) \leq 0 \) for all the real simple coroots \( \alpha_q^\vee \) (cf. [3], Identity 11.13.3). Now, with respect to the \( W \)-invariant symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathfrak{h}^* \) (cf. [3], §2.1),

\[
\langle \delta, \delta \rangle = \sum_p m_p \langle \delta, \alpha_p \rangle + \sum_q n_q \langle \delta, \alpha_q \rangle
\]

\[
= \sum_q n_q \langle \delta, \alpha_q \rangle + \sum_{p,q} m_p n_q \langle \alpha_q, \alpha_p \rangle + \sum_{p,p'} m_p m_{p'} \langle \alpha_{p'}, \alpha_p \rangle,
\]

where \( \alpha_{p'} \) also runs over imaginary simple roots. Now, by assumption,

\[
\langle \delta, \alpha_q \rangle \leq 0, \text{ for all the real simple roots.}
\]

For any imaginary simple root \( \alpha_p \) and any real simple root \( \alpha_q \), we have

\[
\langle \alpha_q, \alpha_p \rangle \leq 0, \text{ since } a_{p,q} \leq 0.
\]

Further, for imaginary simple roots \( \alpha_p, \alpha_{p'} \),

\[
\langle \alpha_{p'}, \alpha_p \rangle \leq 0, \text{ by [3], Identity 2.1.6.}
\]

Observe that we can take the normalizing factor \( \epsilon_i > 0 \) for each \( 1 \leq i \leq \ell \) as can be seen from the identity:
\[ \epsilon_i a_{i,j} = \epsilon_j a_{j,i}, \text{ for all } 1 \leq i, j \leq \ell, \]

where the diagonal matrix \( D = \text{diag}(\epsilon_1, \cdots, \epsilon_\ell) \) is such that \( DA \) is a symmetric matrix. Moreover, since there exists \( p \) with \( m_p \neq 0 \) and since \( \text{supp} \delta \) is connected and \( \delta \) is not a simple root, by [3], Identity 2.1.6,

\[
\langle \alpha_p, \alpha_p \rangle < 0, \text{ for some } p' \neq p \text{ with } m_{p'} \neq 0 \text{ and } \alpha_{p'} \text{ an imaginary simple root}
\]

or \( \langle \alpha_q, \alpha_p \rangle < 0 \) for some \( q \) with \( n_q \neq 0 \) and \( \alpha_q \) a real simple root. \( \cdots \) \( \cdots \) \( \cdots \) \( \cdots \)

Thus, combining the equations (2) - (6), we get:

\[ \langle \delta, \delta \rangle < 0. \]

By [3], Corollary 9.12, \( \oplus_{k>0} g_{k\delta} \) is a free Lie algebra on a basis of the form \( \oplus_{k>0} g^0_{k\delta} \), where

\[
g^0_{k\delta} := \{ x \in g_{k\delta} : \langle x, y \rangle = 0 \forall y \text{ in the Lie subalgebra generated by} \]

\[
g_{-\delta}, g_{-2\delta}, \ldots, g_{-(k-1)\delta}\}.
\]

Observe next that \( g_{k\delta} \neq 0 \) for any \( k > 0 \) by [3], Identity 11.13.3. If \( g_{\delta} \) is one dimensional, then so is \( g_{-\delta} \) and hence \( g^0_{2\delta} \neq 0 \). (To prove \( \dim g_{-\delta} = 1 \), observe that, due to the existence of the Cartan involution, \( \dim g_{\delta} = \dim g_{-\delta} \) for any root \( \delta \), cf. [3], Identity 1.3.5 and Theorem 11.13.1. Moreover, \( g_{-\delta} \) being one dimensional, the Lie subalgebra of \( g \) generated by \( g_{-\delta} \) is \( g_{-\delta} \) itself. Thus, \( g^0_{2\delta} \neq 0 \) by the definition.) Thus, \( \oplus_{k>0} g_{k\delta} \) is a free Lie algebra on at least 2 generators. If \( \dim g_{\delta} \geq 2 \), then \( \oplus_{k>0} g_{k\delta} \) is again a free Lie algebra on at least two generators (since \( g^0_{\delta} = g_{\delta} \)). Thus, \( \text{ad}(x_\delta) \) can not act locally nilpotently on \( \oplus_{k>0} g_{k\delta} \) and hence on \( g \) (since the enveloping algebra of a free Lie algebra is the tensor algebra on the same generators and now use [4], Identity (3) of Definition 1.3.2).

Now, let \( \delta = \alpha_p \) be an imaginary simple root. Then, again \( x_\delta = e_p \) can not act nilpotently on any \( e_i, i \neq p \) such that \( a_{i,p} \neq 0 \). (This is where we have used the assumption that \( A \) is indecomposable and \( \ell \geq 2 \).) To prove this, use [3], Identity 11.13.3 by observing that \( (n\alpha_p + \alpha_i) \in K \) for all \( n \geq 2 \) in the notation of [3].

Thus, we conclude that any \( x \in g^0_{\text{nil}} \) must be of the form \( x \in g(B) + h \). Hence,

\[ g_{\text{nil}} \subset g'(B) + h. \]

Further, by [4], Lemma 1.3.3(a) and the defining relations of \( g(A) \), \( e_i, f_i \in g^0_{\text{nil}} \) for any real simple root \( \alpha_i \). Thus,

\[ g'(B) \subset g_{\text{nil}}. \]

This proves the theorem. \( \square \)
Remark 3.2. (a) Define
\[
\mathfrak{g}'_{\text{nil}} := \{ x \in \mathfrak{g}' : \text{ad} x \text{ acts locally nilpotently on } \mathfrak{g}' \}
\]
and let \( \mathfrak{g}'_{\text{nil}} \subset \mathfrak{g}' \) be the Lie subalgebra generated by \( \mathfrak{g}'_{\text{nil}} \). Then, by the same proof as above,
\[
\mathfrak{g}'(B) \subset \mathfrak{g}'_{\text{nil}} \subset (\mathfrak{g}'(B) + \mathfrak{h}) \cap \mathfrak{g}'.
\]
(b) It is easy to see that the above theorem remains true in the case \( A \) is parameterized by \( \mathbb{N} \times \mathbb{N} \).
(c) For the \( 1 \times 1 \)-matrix \( A = (0) \), following [3], \S 2.9, \( \mathfrak{g}(A) = \mathfrak{h} \oplus \mathbb{C}e_1 \oplus \mathbb{C}f_1 \), where \( \mathfrak{h} = \mathbb{C}\alpha_1^{\vee} \oplus \mathbb{C}d \) and \( [e_1, f_1] = \alpha_1^{\vee}, [\alpha_1^{\vee}, \mathfrak{g}] = 0, [d, e_1] = e_1, [d, f_1] = -f_1 \). Thus, in this case, \( \mathfrak{g}_{\text{nil}} = \mathfrak{g}' \). Hence, the assumption \( \ell \geq 2 \) in the above theorem is necessary in general.
(d) One interesting consequence of the above theorem is that the only connected ‘reasonable’ group attached to a GKM algebra \( \mathfrak{g}(A) \) is the one coming from its subalgebra \( \mathfrak{g}(B) \) (up to an \( \mathfrak{h} \)-factor).

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