On the global behavior of solutions of the Beltrami equations

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Abstract. In this paper, the estimate for growth of homeomorphic solutions of the Beltrami equation at infinity is obtained, provided that the dilatation quotient has a global finite mean oscillation.

MSC 2020. 30C62+31A15

Keywords. Beltrami equations, ring $Q$–homeomorphisms, modulus, capacity.

1 Introduction

Let $D$ be a domain in the complex plane $\mathbb{C}$, i.e., a connected and open subset of $\mathbb{C}$, and let $\mu : D \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. (almost everywhere) in $D$. The Beltrami equation is the equation of the form

$$f_\tau = \mu(z)f_z$$

where $f_\tau = \overline{\partial}f = (f_x + if_y)/2$, $f_z = \partial f = (f_x - if_y)/2$, $z = x + iy$, and $f_x$ and $f_y$ are partial derivatives of $f$ in $x$ and $y$, correspondingly. The function $\mu$ is called the complex coefficient and

$$K_\mu(z) = \frac{1 + |\mu(z)|}{1 - |\mu(z)|}$$

the dilatation quotient for the equation (1.1). The Beltrami equation (1.1) is said to be degenerate if $\text{ess sup} K_\mu(z) = \infty$. The existence theorem for homeomorphic $W^{1,1}_{\text{loc}}$ solutions was established to many degenerate Beltrami equations, see, e.g., related references in the recent monographs [3], [10], [7]; cf. also [6], [14] – [18].

Recall that the (conformal) modulus of a family $\Gamma$ of curves $\gamma$ in $\mathbb{C}$ is the quantity

$$M(\Gamma) = \inf_{\rho \in \text{adm} \Gamma} \int_{\mathbb{C}} \rho^2(z) \, dx \, dy$$

(1.3)
where a Borel function $\rho : \mathbb{C} \to [0, \infty]$ is admissible for $\Gamma$, write $\rho \in \text{adm } \Gamma$, if
\[
\int_{\gamma} \rho \, ds \geq 1 \quad \forall \gamma \in \Gamma \tag{1.4}
\]
where $s$ is a natural parameter of the length on $\gamma$.

Throughout this paper,
\[
B(z_0, r) = \{z \in \mathbb{C} : |z - z_0| < r\},
\]
\[
S(z_0, r) = \{z \in \mathbb{C} : |z - z_0| = r\},
\]
and
\[
A(z_0, r_1, r_2) = \{z \in \mathbb{C} : r_1 < |z - z_0| < r_2\}.
\]
Let $E, F \subset \overline{\mathbb{C}}$ be arbitrary sets. Denote by $\Delta(E, F, D)$ a family of all curves $\gamma : [a, b] \to \overline{\mathbb{C}}$ joining $E$ and $F$ in $D$, i.e., $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ as $t \in (a, b)$.

Here a condenser is a pair $E = (A, C)$ where $A \subset \mathbb{C}$ is open and $C$ is a non-empty compact set contained in $A$. $E$ is a ringlike condenser if $B = A \setminus C$ is a ring, i.e., if $B$ is a domain whose complement $\mathbb{C} \setminus B$ has exactly two components where $\mathbb{C} = \mathbb{C} \cup \{\infty\}$ is the one-point compactification of $\mathbb{C}$. $E$ is a bounded condenser if $A$ is bounded. A condenser $E = (A, C)$ is said to be in a domain $G$ if $A \subset G$.

The following lemma is immediate.

**Lemma 1.** If $f : G \to \mathbb{C}$ is open and $E = (A, C)$ is a condenser in $G$, then $(fA, fC)$ is a condenser in $fG$.

In the above situation we denote $fE = (fA, fC)$.

Let $E = (A, C)$ be a condenser. We set
\[
\text{cap } E = \text{cap } (A, C) = \inf_{u \in W_0(E)} \int_A |\nabla u|^2 \, dxdy
\]
and call it the capacity of the condenser $E$. The set $W_0(E) = W_0(A, C)$ is the family of nonnegative functions $u : A \to \mathbb{R}$ such that $u \in C_0(A)$, $u(z) \geq 1$ for $z \in C$, and $u$ is absolutely continuous on lines (ACL). In the above formula
\[
|\nabla u| = \sqrt{\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2}.
\]
We mention some properties of the capacity of a condenser. It was proven in ([20], Theorem 1) that
\[ \text{cap} \mathcal{E} = M(\Delta(\partial A, \partial C; A \setminus C)) , \tag{1.5} \]
where \( \Delta(\partial A, \partial C; A \setminus C) \) denotes the set of all continuous curves joining the boundaries \( \partial A \) and \( \partial C \) in \( A \setminus C \).

Moreover, the following estimate is known:
\[ \text{cap} \mathcal{E} \geq \frac{4\pi}{\log \frac{m(A)}{m(C)}} \tag{1.6} \]
(see, e.g., (8.8) in [11]).

The following notion is motivated by the ring definition of Gehring for quasiconformal mappings, see, e.g., [5], introduced first in the plane, see [17], and extended later on to the space case in [13], see also Chapters 7 and 11 in [10], cf. [1], [2], [4], [12].

Given a domain \( D \) in \( \mathbb{C} \), a (Lebesgue) measurable function \( Q : D \to [0, \infty] \), \( z_0 \in D \), a homeomorphism \( f : D \to \mathbb{C} \) is said to be a \( Q \)-homeomorphism at the point \( z_0 \) if
\[ M(f(\Delta(S_1, S_2, \mathcal{A}(z_0, r_1, r_2)))) \leq \int_{\mathcal{A}(z_0, r_1, r_2)} Q(z) \cdot \eta^2(|z - z_0|) \, dx \, dy \tag{1.7} \]
for every ring \( \mathcal{A}(z_0, r_1, r_2) \) and the circles \( S_i = S(z_0, r_i) \), \( i = 1, 2 \), where \( 0 < r_1 < r_2 < r_0 := \text{dist} (z_0, \partial D) \), and every measurable function \( \eta : (r_1, r_2) \to [0, \infty] \) such that
\[ \int_{r_1}^{r_2} \eta(r) \, dr = 1 . \]
\( f \) is called a \( Q \)-homeomorphism in the domain \( D \) if \( f \) is a \( Q \)-homeomorphism at every point \( z_0 \in D \).

The following statement was first proved in [9], Theorem 3.1, cf. also Corollary 3.1 in [19].

**Proposition 1.** Let \( f \) be a homeomorphic \( W^{1,1}_{\text{loc}} \) solution of the Beltrami equation (1.1). Then \( f \) is a ring \( Q \)-homeomorphism at each point \( z_0 \in D \) with \( Q(z) = K_{\mu}(z) \).
2 GFMO functions

Similarly to [8] (cf. also [14], [16]), we say that a function \( \varphi: \mathbb{C} \to \mathbb{R} \) has \textit{global finite mean oscillation at a point} \( z_0 \in \mathbb{C} \), abbr. \( \varphi \in GFMO(z_0) \), if

\[
\limsup_{R \to \infty} \frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} |\varphi(z) - \overline{\varphi}_R| \, dx \, dy < \infty, \quad (2.1)
\]

where

\[
\overline{\varphi}_R = \frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} \varphi(z) \, dx \, dy
\]

is the mean value of the function \( \varphi(z) \) over \( B(z_0, R), R > 0 \). Here \( B(z_0, R) = \{ z \in \mathbb{C} : |z - z_0| < R \} \), and condition (2.1) includes the assumption that \( \varphi \) is integrable in \( B(z_0, R) \) for \( R > 0 \).

**Proposition 2.** If, for some collection of numbers \( \varphi_R \in \mathbb{R}, R \in [r_0, +\infty), r_0 > 0 \),

\[
\limsup_{R \to \infty} \frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} |\varphi(z) - \varphi_R| \, dx \, dy < \infty,
\]

then \( \varphi \) has \textit{global finite mean oscillation at} \( z_0 \).

**Proof.** Indeed, by the triangle inequality,

\[
\frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} |\varphi(z) - \overline{\varphi}_R| \, dx \, dy \leq
\]

\[
\leq \frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} |\varphi(z) - \varphi_R| \, dx \, dy + |\varphi_R - \overline{\varphi}_R| \leq
\]

\[
\leq \frac{2}{m(B(z_0, R))} \int_{B(z_0, R)} |\varphi(z) - \varphi_R| \, dx \, dy.
\]

**Corollary 1.** If, for a point \( z_0 \in \mathbb{C} \),

\[
\limsup_{R \to \infty} \frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} |\varphi(z) - \varphi(z_0)| \, dx \, dy < \infty,
\]

then \( \varphi \) has \textit{global finite mean oscillation at} \( z_0 \).
Corollary 2. If, for a point \( z_0 \in \mathbb{C} \),
\[
\limsup_{R \to \infty} \frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} |\varphi(z)| \, dx \, dy < \infty,
\]
then \( \varphi \) has global finite mean oscillation at \( z_0 \).

Lemma 2. Let \( z_0 \in \mathbb{C} \). If a nonnegative function \( \varphi: \mathbb{C} \to \mathbb{R} \) has global finite mean oscillation at \( z_0 \) and \( \varphi \) is integrable in \( B(z_0, e) \), then, for \( R > e^e \),
\[
\int_{A(z_0, e, R)} \frac{\varphi(z) \, dx \, dy}{(|z - z_0| \log |z - z_0|)^2} \leq C \cdot \log \log R,
\]
where
\[
C = \frac{\pi}{6} ((24 + \pi^2)e^2 \delta_\infty + 2\pi^2 \varphi_0),
\]
\( \varphi_0 \) is the mean value of \( \varphi \) over the disk \( B(z_0, e) \) and
\[
\delta_\infty = \delta_\infty(\varphi) = \sup_{R \in (e^e, +\infty)} \frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} |\varphi(z) - \varphi_R| \, dx \, dy
\]
is the maximal dispersion of \( \varphi \).

Proof. Let \( R > e^e, r_k = e^k, A_k = \{ z \in \mathbb{C} : r_k \leq |z - z_0| < r_{k+1} \} \).
Clearly,
\[
\delta_\infty = \sup_{R \in (e^e, +\infty)} \frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} |\varphi(z) - \varphi_R| \, dx \, dy < \infty,
\]
\( B_k = B(z_0, r_k) \) and let \( \varphi_k \) be the mean value of \( \varphi(z) \) over \( B_k \), \( k = 1, 2, \ldots \).
Take a natural number \( N \) such that \( R \in [r_N, r_{N+1}) \).
Then \( A(z_0, e, R) \subset \Delta(R) = \bigcup_{k=1}^{N} A_k \) and
\[
I(R) = \int_{\Delta(R)} \varphi(z) \alpha(|z - z_0|) \, dx \, dy \leq |S_1(R)| + S_2(R),
\]
\[
\alpha(t) = \frac{1}{(t \log t)^2},
\]
\[
S_1(R) = \sum_{k=1}^{N} \int_{A_k} (\varphi(z) - \varphi_{k+1}) \alpha(|z - z_0|) \, dx \, dy,
\]
and
\[ S_2(R) = \sum_{k=1}^{N} \varphi_{k+1} \int_{A_k} \alpha(|z - z_0|) \, dx \, dy. \]

Since \( \mathcal{A}_k \subset B_{k+1} \), \( \frac{1}{|z - z_0|} \leq \frac{\pi e^2}{m(B_{k+1})} \) for \( z \in \mathcal{A}_k \) and \( \log |z - z_0| > k \) in \( \mathcal{A}_k \), then
\[ |S_1(R)| \leq \pi e^2 \sum_{k=1}^{N} \frac{1}{k^2} \cdot \frac{1}{m(B_{k+1})} \int_{B_{k+1}} |\varphi(z) - \varphi_{k+1}| \, dx \, dy \leq \pi e^2 \delta_\infty \sum_{k=1}^{N} \frac{1}{k^2} \leq \frac{\pi^3 e^2 \delta_\infty}{6}. \]

Now,
\[ \int_{A_k} \alpha(|z - z_0|) \, dx \, dy \leq \frac{1}{k^2} \int_{A_k} \frac{dx \, dy}{|z - z_0|^2} = \frac{2\pi}{k^2}. \]

Moreover,
\[ |\varphi_{k-1} - \varphi_k| = \left| \frac{1}{m(B_{k-1})} \int_{B_{k-1}} \varphi(z) \, dx \, dy - \frac{1}{m(B_{k-1})} \int_{B_{k-1}} \varphi_k \, dx \, dy \right| \leq \frac{1}{m(B_{k-1})} \int_{B_{k-1}} |\varphi(z) - \varphi_k| \, dx \, dy \leq \frac{e^2}{m(B_k)} \int_{B_k} |\varphi(z) - \varphi_k| \, dx \, dy \leq e^2 \delta_\infty, \]
and by the triangle inequality, for \( k \geq 1 \)
\[ \varphi_{k+1} = |\varphi_{k+1}| = \left| \varphi_1 + \sum_{l=2}^{k+1} (\varphi_l - \varphi_{l-1}) \right| \leq |\varphi_1| + \sum_{l=2}^{k+1} |\varphi_l - \varphi_{l-1}| \leq |\varphi_1| + e^2 \delta_\infty k. \]

Hence,
\[ S_2(R) = |S_2(R)| \leq 2\pi \sum_{k=1}^{N} \frac{\varphi_{k+1}}{k^2} \leq 2\pi \sum_{k=1}^{N} \frac{\varphi_1 + e^2 \delta_\infty k}{k^2} \leq 2\pi \varphi_1 \sum_{k=1}^{\infty} \frac{1}{k^2} + 2\pi e^2 \delta_\infty \sum_{k=1}^{N} \frac{1}{k} = \]
6
\[ \frac{\pi^3 \varphi_1}{3} + 2\pi e^2 \delta_{\infty} \sum_{k=1}^{N} \frac{1}{k}. \]

But

\[ \sum_{k=2}^{N} \frac{1}{k} < \int_{1}^{N} \frac{dt}{t} = \log N \]

and, for \( R > r_N \),

\[ N = \log r_N < \log R. \]

Consequently,

\[ \sum_{k=1}^{N} \frac{1}{k} < 1 + \log \log R \]

and thus, for \( R \in (e^e, +\infty) \)

\[ I(R) \leq \frac{\pi^3 e^2 \delta_{\infty}}{6} + \frac{\pi^3 \varphi_1}{3} + 2\pi e^2 \delta_0 (1 + \log \log R) = \]

\[ = \left( \frac{\pi^3 e^2 \delta_{\infty} + 12\pi e^2 \delta_{\infty} + 2\pi^3 \varphi_1}{6 \log \log R} + 2\pi e^2 \delta_{\infty} \right) \log \log R \leq \]

\[ \leq \frac{\pi}{6} ((24 + \pi^2) e^2 \delta_{\infty} + 2\pi^2 \varphi_1) \log \log R. \]

Finally,

\[ \int_{A(z_0, e, R)} |\varphi(z)| \frac{dxdy}{(|z - z_0| \log |z - z_0|)^2} \leq I(R) \leq \]

\[ \leq \frac{\pi}{6} ((24 + \pi^2) e^2 \delta_{\infty} + 2\pi^2 \varphi_1) \log \log R. \]

3 The behavior of homeomorphic solutions of the Beltrami equations at infinity

Set

\[ l_f(z_0, e) = \min_{|z - z_0| = e} |f(z) - f(z_0)|, \]

\[ \delta_{\infty} = \delta_{\infty}(K_\mu, z_0) = \]

\[ = \sup_{R \in (e^e, +\infty)} \frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} |K_\mu(z) - K_{\mu, z_0}(R)| dxdy, \]

\[ K_{\mu, z_0}(R) = \frac{1}{m(B(z_0, R))} \int_{B(z_0, R)} K_\mu(z) dxdy, \quad k_0 = K_{\mu, z_0}(e). \]
Theorem 1. Let $\mu : \mathbb{C} \to \mathbb{C}$ be a measurable function with $|\mu(z)| < 1$ a.e. and $f : \mathbb{C} \to \mathbb{C}$ be a homeomorphic $W^{1,1}_{\text{loc}}$ solution of the Beltrami equation (1.1). If $K_\mu \in GFMO(z_0)$, $z_0 \in \mathbb{C}$, then

$$\liminf_{R \to \infty} \max_{|z-z_0| = R} \frac{|f(z) - f(z_0)|}{(\log R)^2} \geq l_f(z_0, e),$$

(3.1)

where $C = \frac{\pi}{6}((24 + \pi^2)e^2 \delta_\infty + 2\pi^2 k_0)$.

Proof. Consider the ring $A(R) = A(z_0, e, R)$, with $R > e^e$. Set $E = (B(z_0, R), B(z_0, e))$. Then, by lemma 1 $fE = (fB(z_0, R), fB(z_0, e))$ is a condenser in $\mathbb{C}$, according (1.5),

$$\text{cap} (fB(z_0, R), fB(z_0, e)) = M(\Delta(\partial fB(z_0, e), \partial fB(z_0, R); fA(R)))$$

and, in view of the homeomorphism of $f$,

$$\Delta(\partial fB(z_0, e), \partial fB(z_0, R); fA(R)) = f\Delta(\partial B(z_0, e), \partial B(z_0, R); A(R)).$$

By Proposition 1 $f$ is a ring $Q$-homeomorphism with $Q = K_\mu(z)$

$$\text{cap} (fB(z_0, R), fB(z_0, e)) \leq \int_{A(R)} K_\mu(z) \eta^2(|z-z_0|) \, dx \, dy$$

(3.2)

for every measurable function $\eta : (e, R) \to [0, +\infty]$ such that

$$\int_{e}^{R} \eta(t) \, dt = 1.$$

Choosing in (3.2) $\eta(t) = \frac{1}{\log \log \log R}$, we obtain

$$\text{cap} (fB(z_0, R), fB(z_0, e)) \leq \frac{1}{(\log \log R)^2} \cdot \int_{A(R)} \frac{K_\mu(z) \, dx \, dy}{(|z-z_0| \log |z-z_0|)^2}.$$

Since $K_\mu \in GFMO(z_0)$, then by lemma 2

$$\text{cap} (fB(z_0, R), fB(z_0, e)) \leq \frac{C}{\log \log R},$$

(3.3)

where $C = \frac{\pi}{6}((24 + \pi^2)e^2 \delta_\infty + 2\pi^2 k_0)$. On the other hand, by (1.6), we have

$$\text{cap} (fB(z_0, R), fB(z_0, e)) \geq \frac{4\pi}{\log \frac{m(fB(z_0, R))}{m(fB(z_0, e))}}.$$

(3.4)
Combining (3.3) and (3.4), we obtain
\[\frac{4\pi}{\log \frac{m(fB(z_0, R))}{m(fB(z_0, e))}} \leq \frac{C}{\log \log R}.\]

This gives
\[m(fB(z_0, e)) \leq \frac{m(fB(z_0, R))}{(\log R)^{\frac{2\pi}{C}}}.\]

Using the inequalities
\[\pi \left( \min_{|z-z_0|=\epsilon} |f(z) - f(z_0)| \right)^2 \leq m(fB(z_0, e)) \leq m(fB(z_0, R)) \leq \pi \left( \max_{|z-z_0|=R} |f(z) - f(z_0)| \right)^2,
\]
we obtain
\[\min_{|z-z_0|=\epsilon} |f(z) - f(z_0)| \leq \frac{\max_{|z-z_0|=R} |f(z) - f(z_0)|}{(\log R)^{\frac{2\pi}{C}}}. \tag{3.5}\]

Set
\[l_f(z_0, e) = \min_{|z-z_0|=\epsilon} |f(z) - f(z_0)|.\]

Passing to the lower limit as \(R \to \infty\) in (3.5), we obtain relation (3.1).

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