On the Global Structure of Normal Forms for Slow-Fast Hamiltonian Systems

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Abstract

In the framework of Lie transform and the global method of averaging, the normal forms of a multidimensional slow-fast Hamiltonian system are studied in the case when the flow of the unperturbed (fast) system is periodic and the induced $S^1$-action is not necessarily free and trivial. An intrinsic splitting of the second term in a $S^1$-invariant normal form of first order is derived in terms of the Hannay-Berry connection associated with the periodic flow.

1 Introduction

In this paper, in the context of normal form, we deal with a class of so-called slow-fast Hamiltonian systems \[1, 16\] of the form

\[
\begin{align*}
\dot{y} &= -\frac{\partial H}{\partial x}, \quad \dot{x} = \frac{\partial H}{\partial y}, \\
\dot{p} &= -\varepsilon \frac{\partial H}{\partial q}, \quad \dot{q} = \varepsilon \frac{\partial H}{\partial p},
\end{align*}
\]

(1.1)

(1.2)

where \((y, x) \in \mathbb{R}^{2r}, (p, q) \in \mathbb{R}^{2k}\) and \(\varepsilon\) is a small perturbation parameter. System \((1.1), (1.2)\) is Hamiltonian relative to a function \(H = H(p, q, y, x)\) and the \(\varepsilon\)-dependent Poisson bracket \(\{,\} = \{,\}_0 + \varepsilon \{,\}_1\) on \(\mathbb{R}^{2r} \times \mathbb{R}^{2k}\). From the viewpoint of Hamiltonian perturbation theory, this splitting of the Poisson bracket into a slow and a fast part leads to the following unusual feature of the perturbed model: the unperturbed (fast) system and the perturbation are Hamiltonian relative to the different (nonisomorphic) Poisson structures. In this situation, one can expect that the perturbation effects are not only represented by correction terms in the Hamiltonian but also related to the rescaling of Poisson brackets.
We assume that the flow of the unperturbed system is periodic and hence induces an $S^1$-action which is canonical relative to the fast Poisson bracket $\{,\}_0$. For $r > 1$, such a situation occurs in the case when the unperturbed motion is described by a family of systems which are superintegrable in the noncommutative sense [17]. The normalization question comes from the fact that the $S^1$-action does not respect the perturbation vector field of system (1.1), (1.2) since the slow Poisson bracket $\{,\}_1$ is not $S^1$-invariant in general. The traditional averaging procedure [4, 16] works within domains of (generalized) action-angles variables where the original $S^1$-action is trivial. We are interested in the global structure of normal forms for system (1.1), (1.2) in the general case when the $S^1$-action is not necessarily free and trivial. Our approach is based on the global averaging technique on $S^1$-manifolds [2, 6, 15] which refers to the flow on a phase space rather than to a local coordinate description. One of the important tools here is the Hannay-Berry connection [13, 14] associated to the $S^1$-action which naturally arises in the averaging procedure for symplectic and Poisson structures [19, 20, 21]. We show that the Hamiltonian vector field of system (1.1), (1.2) can be transformed by a near-identity mapping to an $S^1$-invariant normal form of first order whose second term splits with respect the Hannay-Berry connection into two parts $P_{\text{hor}}$ and $P_{\text{ver}}$ with the following properties. The vertical component $P_{\text{ver}}$ is a Hamiltonian vector field relative to the fast Poisson bracket $\{,\}_0$ and an $S^1$-invariant function which interpreted as a first correction to the Hamiltonian $H$. This interpretation is motivated by the fact that system (1.1), (1.2) can be approximated by a Hamiltonian system with $S^1$-symmetry on a phase space equipped with a corrected Poisson bracket. The horizontal component $P_{\text{hor}}$ involves the horizontal lift of the Poisson tensor on the slow $(p, q)$-space which satisfies the Jacobi identity only in the case when the curvature of the Hannay-Berry connection is zero. Therefore, in general, $P_{\text{hor}}$ does not inherit any natural Hamiltonian structure. These results are applied to the construction of approximate first integrals of system (1.1), (1.2) and illustrated by some examples.

2 Averaging and Integrating Operators

In this section, we collect some facts concerning algebraic properties of the averaging procedure on general $S^1$-manifolds. For more details, see, for example, [2, 6, 15].

Suppose that on a manifold $M$ we are given a complete vector field $\mathcal{T}$ with $2\pi$-periodic flow, $\text{Fl}_t \mathcal{T}^{2\pi} = \text{Fl}_t \mathcal{T}$. Then, we have an action on $M$ of the circle $S^1 = \mathbb{R} / 2\pi \mathbb{Z}$ with infinitesimal generator $\mathcal{T}$. Let us associated to this $S^1$-action the following operations. Denote by $\mathcal{T}^k_s(M)$ be the space of all tensor fields on $M$ of type $(k, m)$ and by $\mathcal{L}_\mathcal{T} : \mathcal{T}^k_s(M) \to \mathcal{T}^k_s(M)$ the Lie derivative along $\mathcal{T}$. For every tensor field $A \in \mathcal{T}^k_s(M)$, its average with respect to the $S^1$-action is a tensor field $\langle A \rangle \in \mathcal{T}^k_s(M)$ of the same type which is defined by

$$\langle A \rangle := \frac{1}{2\pi} \int_0^{2\pi} (\text{Fl}_t \mathcal{T})^* A \text{d}t.$$  \hfill (2.1)
This formula gives the global averaging operator associated to the $S^1$-action on $M$. A tensor field $A \in T^k_s(M)$ is said to be invariant with respect to the $S^1$-action if $(Fl^t_Υ)^* A = A$ ($\forall t \in \mathbb{R}$) or, equivalently, $\mathcal{L}_Υ A = 0$. In terms of the $S^1$-average of $A$ the $S^1$-invariance condition reads $A = \langle A \rangle$.

Introduce also the $\mathbb{R}$-linear operator $\mathcal{S} : T^k_s(M) \to T^k_s(M)$ given by

$$\mathcal{S}(A) := \frac{1}{2\pi} \int_0^{2\pi} (t - \pi)(Fl^t_Υ)^* A dt.$$  \hspace{1cm} (2.2)

Then, we have the following important algebraic identities involving the operators $\mathcal{L}_Υ$, $\langle \rangle$ and $\mathcal{S}$.

**Lemma 2.1** For every $A \in T^k_s(M)$, the following identities hold

$$\mathcal{L}_Υ \circ \mathcal{S}(A) = A - \langle A \rangle,$$  \hspace{1cm} (2.3)

$$\langle \mathcal{L}_Υ(A) \rangle = \mathcal{L}_Υ \langle A \rangle = 0,$$  \hspace{1cm} (2.4)

$$\langle \mathcal{S}(A) \rangle = \mathcal{S}(\langle A \rangle) = 0.$$  \hspace{1cm} (2.5)

For a given $A$, one can think of (2.3) as a homological equation involving the Lie derivative along $Υ$. Then, this equation admits a solution with zero average of the form $\mathcal{S}(A)$. In this context, it is natural to call $\mathcal{S}$ an integrating operator.

Remark also that operators (2.1) and (2.2) are well-defined on the exterior algebras of multivector fields and differential forms on $M$. Together with the Lie derivative, these operators are natural with respect to the exterior derivative $d$ on $M$, that is,

$$\mathcal{S}(dω) = d(\mathcal{S}(ω)) \text{ and } d(\langle ω \rangle) = \langle dω \rangle$$

for any $k$-form $ω$ on $M$. Moreover, we have the similar properties with respect to the interior product. Recall that the interior product of a 1-form $α$ and $k$-vector field $A$ on $M$ is a $(k-1)$-vector field $i_α A$ defined by $(i_α A)(α_1, ..., α_{k-1}) = A(α, α_1, ..., α_{k-1})$. If $α$ is an $S^1$-invariant 1-form, then

$$\langle i_α A \rangle = i_α \langle A \rangle \text{ and } \mathcal{S}(i_α A) = i_α \mathcal{S}(A)$$

for an arbitrary $k$-vector field $A$.

\section{Setting of the Problem}

Consider the phase space $\mathbb{R}^{2r}_{y,x} \times \mathbb{R}^{2k}_{p,q}$ endowed with the $\varepsilon$-dependent Poisson bracket

$$\{, \} = \{, \}_0 + \varepsilon \{, \}_1,$$  \hspace{1cm} (3.1)

where $\{, \}_0$ and $\{, \}_1$ denote the natural lifts of the canonical Poisson brackets on the factors $\mathbb{R}^{2r}_{y,x}$ and $\mathbb{R}^{2k}_{p,q}$, respectively. Suppose we start with slow-fast Hamiltonian system \[1\] \[2\] associated to a smooth function $H = H(p, q, y, x)$. As was mentioned, this system is Hamiltonian relative to Poisson bracket (3.1) and
the function $H$. The corresponding Hamiltonian vector field $X_H$ is represented as follows

$$X_H = X_H^{(0)} + \varepsilon X_H^{(1)},$$

where the unperturbed vector field $X_H^{(0)}$ and the perturbation vector field $X_H^{(1)}$ are Hamiltonian relative to $H$ and the Poisson brackets $\{ , \}_0$ and $\{ , \}_1$, respectively,

$$X_H^{(0)} = -\frac{\partial H}{\partial x} \cdot \frac{\partial}{\partial y} + \frac{\partial H}{\partial y} \cdot \frac{\partial}{\partial x}, \quad (3.2)$$

$$X_H^{(1)} = -\frac{\partial H}{\partial q} \cdot \frac{\partial}{\partial p} + \frac{\partial H}{\partial p} \cdot \frac{\partial}{\partial q}. \quad (3.3)$$

We assume that the unperturbed system admits an invariant open domain $M \subseteq \mathbb{R}^2_{y,x} \times \mathbb{R}^2_{p,q}$ such that the flow $\text{Fl}^{t}_{X_H^{(0)}}$ of $X_H^{(0)}$ is periodic on $M$ with frequency function $\omega \in C^\infty(M), \omega \rangle 0$. This means that $\text{Fl}^{t+T(m)}_{X_H^{(0)}}(m) = \text{Fl}^{t}_{X_H^{(0)}}(m)$ for all $t \in \mathbb{R}$ and $m \in M$. Here $T = \frac{2\pi}{\omega}$ is the period function. Then, the flow of the vector field

$$\Upsilon := \frac{1}{\omega} X_H^{(0)} \quad (3.4)$$

is $2\pi$-periodic and hence $\Upsilon$ is an infinitesimal generator of the $S^1$-action on $M$.

It is clear that the frequency function $\omega$ and the Hamiltonian $H$ are $S^1$-invariant. Moreover, by the period-energy relation [5, 10] for periodic Hamiltonian flows, we have the equality

$$(d_y H + d_x H) \wedge (d_y \omega + d_x \omega) = 0, \quad (3.5)$$

where $d_y$ and $d_x$ denote the partial exterior derivatives on $M$ with respect to the fast variables $y$ and $x$, respectively. Relation (3.5) means that, for a fixed $(p, q)$, the frequency function $\omega$ is constant along the intersection of a level set of $H$ and the slice $\mathbb{R}^2_{y,x} \times \{(p, q)\}$. It is also easy to see from relation (3.5) that the $S^1$-action is canonical with respect to the bracket $\{ , \}_0$. On the other hand, as we will show below (see Lemma 4.7), the $S^1$-action does not preserve the slow Poisson bracket $\{ , \}_1$, in general. Therefore, the perturbation vector field $X_H^{(1)}$ is not necessarily $S^1$-invariant. This fact rises the normalization question: in the class of near-identity mappings on $M$, bring the Hamiltonian vector field $X_H$ to an $S^1$-invariant normal form of desired order in $\varepsilon$.

### 4 The Hannay-Berry Connection Associated to the $S^1$-Action

To formulate our main results, we need some preliminary facts related to the averaging procedure on phase spaces with $S^1$-symmetry, [2, 9, 13, 14, 20, 21].

Throughout this section, we will use operators the $\langle \rangle$ and $S$ in [21] and [22] which are associated to the $S^1$-action with infinitesimal generator (3.4).
Lemma 4.1 The $\mathbb{S}^1$-action associated to the periodic flow of $X_H^{(0)}$ is Hamiltonian relative to the fast Poisson bracket $\{,\}_0$,

$$Y = X_H^{(0)},$$

where the momentum map $J \in C^\infty(M)$ is given by

$$J = \frac{1}{\omega} i_{X_H^{(0)}}(ydx).$$

Moreover,

$$\langle \frac{\partial J}{\partial p^i} \rangle = \langle \frac{\partial J}{\partial q^i} \rangle = 0$$

for $i = 1, \ldots, k$.

Proof. Let $\eta = ydx$. Then, $J = i_Y(\eta)$ and

$$i_{X_H^{(0)}}d\eta = -dH + d_pH + d_qH.$$ 

Using property (2.4), the $\mathbb{S}^1$-invariance of $dH$ and Cartan’s formula, we get

$$dJ = d(i_Y(\eta)) = L_Y(\eta) - i_Y(d\eta)
= -\frac{1}{\omega}(i_{X_H^{(0)}}d\eta) = \frac{1}{\omega}(dH - \langle d_pH \rangle - \langle d_qH \rangle).$$

From here, taking into account that the 1-forms $dp^i, dq^i$ are $\mathbb{S}^1$-invariant, we deduce the relations

$$dyJ = \frac{1}{\omega}dyH, \quad dxJ = \frac{1}{\omega}dxH,$$

$$d_pJ = \frac{1}{\omega}(d_pH - \langle d_pH \rangle), \quad d_qJ = \frac{1}{\omega}(d_qH - \langle d_qH \rangle)$$

which imply (4.1) and (4.3). ■

As a consequence of (3.5) (4.4) and (4.5), we get the following fact.

Corollary 4.2 The differentials $dH$ and $dJ$ are linear independent on $M$ if and only if

$$\langle d_pH \rangle + \langle d_qH \rangle \neq 0.$$ 

Remark 4.3 If the $\mathbb{S}^1$-action is free on $M$, then formula (4.2) gives the standard action along the periodic orbits of $X_H^{(0)}$.

Now, using the momentum map $J$ and operator (2.2), we define the 1-form $\Theta = \Theta^p_i dp^i + \Theta^q_i dq^i$ on $M$ with coefficients

$$\Theta^p_i := S(\frac{\partial J}{\partial p^i}), \quad \Theta^q_i := S(\frac{\partial J}{\partial q^i}).$$

It follows from property (2.5) that

$$\langle \Theta^p_i \rangle = \langle \Theta^q_i \rangle = 0$$

(4.8)
for \( i = 1, \ldots, k \). Here and throughout the remainder of the text, the summation on repeated indices will be understood.

Consider the pre-symplectic 2-form \( dy \wedge dx \) on \( \mathbb{R}^{2r} \times \mathbb{R}^{2k} \) associated to the fast Poisson bracket \( \{ , \} \). The following lemma shows that the differential of the 1-form \( \Theta \) measures the deviation of \( dy \wedge dx \) from the property of being invariant with respect to the \( S^1 \)-action.

**Lemma 4.4** The average \( S^1 \)-average of the 2-form \( dy \wedge dx \) has the following representation on \( M \):

\[
\langle dy \wedge dx \rangle = dy \wedge dx - d\Theta. \tag{4.9}
\]

**Proof.** First, we observe that the closed 2-form \( \sigma = dy \wedge dx \) satisfies the relation

\[
\sigma = \langle \sigma \rangle + d \circ i_{\Upsilon} S(\sigma) \tag{4.10}
\]

Indeed, property (2.3) together with Cartan's formula yields

\[
d \circ i_{\Upsilon} S(\sigma) = L_{\Upsilon}(S(\sigma)) = \sigma - \langle \sigma \rangle,
\]

By (4.1) we have

\[
i_{\Upsilon} \sigma = -(dyJ + dxJ) = -dJ + (dpJ + dqJ)
\]

and consequently,

\[
i_{\Upsilon} S(\sigma) = S(i_{\Upsilon} \sigma) = -S(dJ) + S(dpJ + dqJ) = -dJ + \Theta.
\]

Putting this equality into (4.10), we get (4.9). \( \blacksquare \)

Introduce now the following vector fields on \( M \):

\[
hor^p_i := \frac{\partial}{\partial p^i} + X^{(0)}_{\Theta^p_i}, \quad hor^q_i := \frac{\partial}{\partial q^i} + X^{(0)}_{\Theta^q_i}. \tag{4.11}
\]

**Lemma 4.5** The following identities hold

\[
\mathcal{L}_{\text{hor}^p_i} J = \mathcal{L}_{\text{hor}^q_i} J = 0, \tag{4.12}
\]

\[
[\text{hor}^p_i, Y] = [\text{hor}^q_i, Y] = 0, \tag{4.13}
\]

for all \( i = 1, \ldots, k \).

**Proof.** Definition (4.7) and property (2.3) imply that

\[
\mathcal{L}_Y \Theta^p_i = \frac{\partial J}{\partial p^i}, \quad \mathcal{L}_Y \Theta^q_i = \frac{\partial J}{\partial q^i}.
\]

Moreover, property (4.1) shows that \( \mathcal{L}_{X^{(0)}_J} J = \{ F, J \}_0 = -\mathcal{L}_Y F \) for any \( F \in C^\infty(M) \). Using above relations and (4.11), we obtain (4.12). Next, if \( Y \) is an Poisson vector field \( Y \) of the bracket \( \{ , \}_0 \), then

\[
[Y, X^{(0)}_J] = X^{(0)}_{\mathcal{L}_Y J}.
\]
Combining this identity for Poisson vector fields hor\textsubscript{p}\textsuperscript{i} and hor\textsubscript{q}\textsuperscript{i} with equalities (4.12), we justify (4.13).

Therefore, it follows from (4.12) and (4.13) that vector fields in (4.11) are S\textsuperscript{1}-invariant and have the momentum map \(J\) as a common first integral. The following consequence of Lemma 4.5 gives us an alternative definition of hor\textsubscript{p}\textsuperscript{i} and hor\textsubscript{q}\textsuperscript{i}.

**Corollary 4.6** The vector fields in (4.11) coincide with the S\textsuperscript{1}-averages of the coordinate vector fields associated to the slow variables,

\[
\text{hor}\textsubscript{p}\textsuperscript{i} = \langle \frac{\partial}{\partial p^i} \rangle, \quad \text{hor}\textsubscript{q}\textsuperscript{i} = \langle \frac{\partial}{\partial q^i} \rangle
\]

for all \(i = 1, ..., k\).

**Proof.** By (4.1), the S\textsuperscript{1}-action is Hamiltonian relative to the Poisson bracket \{,\}\textsubscript{0} and hence for any \(F \in C^\infty(M)\), the S\textsuperscript{1}-average of the Hamiltonian vector \(X\textsubscript{F}^{(0)}\) is given by

\[
\langle X\textsubscript{F}^{(0)} \rangle = X\textsubscript{F}^{(0)}
\]

In particular, the condition \(\langle F \rangle = 0\) implies that \(\langle X\textsubscript{F}^{(0)} \rangle = 0\). Then, it follows from (4.8) that

\[
\langle X\textsubscript{Θ}\textsubscript{p}\textsuperscript{i} \rangle = \langle X\textsubscript{Θ}\textsubscript{q}\textsuperscript{i} \rangle = 0
\]

These relations and the S\textsuperscript{1}-invariance of vector fields (4.11) imply (4.14). ■

Let us think of the domain \(M \subset \mathbb{R}^{2r}_y \times \mathbb{R}^{2k}_{p,q}\) as the total space of a trivial symplectic bundle whose base is the projection of \(M\) to the “slow” \((p,q)\)-space and the fibers are given by the intersections of \(M\) with slices \(\mathbb{R}^{2r}_y \times \{(p,q)\}\). The S\textsuperscript{1}-action leave invariant the fibers whose tangent spaces form the vertical distribution \(V = \text{Span}\{\frac{\partial}{\partial y^i}, \frac{\partial}{\partial x^i}\}\). Denote by \(H\) the distribution on \(M\) spanned by vector fields hor\textsubscript{p}\textsuperscript{i} and hor\textsubscript{q}\textsuperscript{i} in (4.11) for \(i = 1, ..., k\). Then, we have the S\textsuperscript{1}-invariant splitting

\[
TM = V \oplus H.
\]

Relations (4.14) show that the horizontal distribution \(H\) gives the Hannay-Berry connection in the sense of [13, 14]. This connection is obtained by the averaging of the trivial connection on \(M\) with respect to the S\textsuperscript{1}-action associated to the periodic flow of \(X\textsubscript{H}^{(0)}\). The horizontal lifts of vector fields on the base with respect to the Hannay-Berry are just given by (4.11). The vector fields tangent to the distributions \(V\) and \(H\) are said to be vertical and horizontal, respectively. The curvature of the Hannay-Berry connection is zero if and only if the horizontal distribution \(H\) is integrable. This happens in the case when the vector fields in (4.11) pairwise commute.

Now, let us consider the following S\textsuperscript{1}-invariant bivector field on \(M\):

\[
\Pi_\Theta := \text{hor}\textsubscript{p}\textsuperscript{i} \wedge \text{hor}\textsubscript{q}\textsuperscript{i}
\]

which is just the horizontal lift of the Poisson tensor on \(\mathbb{R}^{2k}_{p,q}\) with respect to splitting (4.17).
Lemma 4.7. The $S^1$-average of the Poisson tensor of the slow Poisson bracket $\{,\}_1$ has the representation
\[(\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}) = \Pi_\Theta - \mathcal{L}_V(\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}), \tag{4.19}\]
where $V$ is a vector field on $M$ given by
\[V := \frac{1}{2} i_\Theta(\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q}) \equiv \frac{1}{2} \left( \Theta^p \frac{\partial}{\partial q} - \Theta^q \frac{\partial}{\partial p} \right). \tag{4.20}\]

Proof. By straightforward but lengthy calculations, we verify the following identity
\[\frac{\partial}{\partial p} \wedge \frac{\partial}{\partial q} = \Pi_\Theta - \frac{1}{2} \text{hor}^p \wedge X^{(0)}_\Theta + \frac{1}{2} \text{hor}^q \wedge X^{(0)}_\Theta \tag{4.21}\]
\[- \mathcal{L}_V(\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}).
\]
The first term on the right hand side of this equality is $S^1$-invariant. The $S^1$-average of the corresponding second and third terms is zero because of properties (4.14) and (4.16). Finally, taking into account that the Poisson tensor $\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}$ is $S^1$-invariant and averaging the both sides of (4.21), we get decomposition (4.19). \hfill \blacksquare

5 An Intrinsic Splitting of Normal Forms

Here, we apply the (non-canonical) Lie transform method to the perturbed Hamiltonian vector field of system (1.1), (1.2). Taking into account that the normalization procedure contains a certain freedom of formulation, we show how to fix this freedom to get an intrinsic splitting of a first order normal form.

We say that an open domain $N$ in $M$ is admissible if its closure $\overline{N}$ is compact and invariant with respect to the $S^1$-action. By a near-identity transformation we mean a smooth family of mappings $\mathcal{T}_\varepsilon : N \to M$, $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$ such that $\mathcal{T}_0 = \text{id}$ and $\mathcal{T}_\varepsilon$ is a diffeomorphism onto its image.

Theorem 5.1. Assume that the flow of the unperturbed Hamiltonian vector field $X^{(0)}_H$ is periodic with frequency function $\omega$. Then, for every admissible domain $N \subset M$ and small enough $\varepsilon$, there exists a near identity transformation $\mathcal{T}_\varepsilon : N \to M$ which brings the Hamiltonian vector field $X_H = X^{(0)}_H + \varepsilon X^{(1)}_H$ of slow-fast system (1.1), (1.2) to the following $S^1$-invariant normal form of first order:
\[\mathcal{T}_\varepsilon^* X_H = X^{(0)}_H + \varepsilon (P_{\text{hor}} + P_{\text{ver}}) + O(\varepsilon^2), \tag{5.1}\]
where the horizontal $P_{\text{hor}}$ and vertical $P_{\text{ver}}$ vector fields on $M$ are given by
\[P_{\text{hor}} := i_{dH} \Pi_\Theta, \quad P_{\text{ver}} := X^{(0)}_{(K)}\]
and

\[ K := \frac{1}{2} \left( S \left( \frac{\partial J}{\partial p^i} \right) \frac{\partial H}{\partial q^i} - S \left( \frac{\partial J}{\partial q^i} \right) \frac{\partial H}{\partial p^i} \right). \tag{5.2} \]

**Proof.** First, let us apply a general normalization result [2] to the perturbation vector field \( X_H \). Let

\[ Z = \frac{1}{\omega} S(X_H^{(1)}) + \frac{1}{\omega^3} S^2(\mathcal{L}_{X_H^{(1)}}\omega)X_H^{(0)} + Y, \tag{5.3} \]

where \( Y \) is an arbitrary \( S^1 \)-invariant vector field on \( M \). Denote by

\[ T_\varepsilon = \text{Fl}_{t=\varepsilon}^Z \tag{5.4} \]

the time-\( \varepsilon \) flow of the vector field \( Z \). Then, for small enough \( \varepsilon \), the near-identity transformation \( T_\varepsilon \) sends \( X_H \) to the following first order normal form [2]:

\[ T_\varepsilon X_H = X_H^{(0)} + \varepsilon \left( X_H^{(1)} \right) + \frac{1}{\omega} \mathcal{L}_Y(\omega)X_H^{(0)} + O(\varepsilon^2). \tag{5.5} \]

Next, let us choose an \( S^1 \)-invariant vector field \( Y \) in a such a way that the second terms in normal forms (5.5) and (5.1) coincide. It follows from representation \( X_H^{(1)} = i_{dH}(\frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}) \) that

\[ \langle X_H^{(1)} \rangle = i_{dH}(\frac{\partial}{\partial q} \wedge \frac{\partial}{\partial p}). \tag{5.6} \]

On the other hand, taking into account that

\[ [(V), X_H^{(0)}] = [(V), \omega Y] = (\mathcal{L}_{(V)}\omega)Y = \frac{1}{\omega}(\mathcal{L}_{(V)}\omega)X_H^{(0)}, \]

by the standard properties of the Lie derivative [1], we obtain

\[ i_{dH} \circ \mathcal{L}_{(V)}(\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial x}) = -X_H^{(0)} + [(V), X_H^{(0)}] \]

\[ = -X_H^{(0)} + \frac{1}{\omega}(\mathcal{L}_{(V)}\omega)X_H^{(0)}. \]

Combining this relation with (4.19) and (5.6), we get the following representation

\[ \langle X_H^{(1)} \rangle = P_{\text{hor}} + P_{\text{ver}} - \frac{1}{\omega}(\mathcal{L}_{(V)}\omega)X_H^{(0)}. \tag{5.7} \]

Using (4.20), we verify that

\[ \mathcal{L}_V H = K \tag{5.8} \]

and hence \( \mathcal{L}_{(V)} H = \langle \mathcal{L}_V H \rangle = \langle K \rangle \). Finally, the desired choice of \( Y \) in (5.3) is that \( Y = (V) \). \( \blacksquare \)

The horizontal and vertical components of the second term in normal form (5.1) possess the following properties. The vertical component \( P_{\text{ver}} \) is a Hamiltonian vector field relative to the slow Poisson bracket \( \{,\}_0 \) and the function
which can be interpreted as an $S^1$-invariant correction of first order to the Hamiltonian $H$ (see, Theorem 6.2). The horizontal component $P_{\text{hor}}$ involves the $H$-horizontal lift $\Pi_0$ of the Poisson tensor on the $(p, q)$-space which satisfies the Jacobi identity only in the case when the horizontal distribution $H$ is integrable \cite{7, 20}. Therefore, in general, $P_{\text{hor}}$ does not inherit any natural Hamiltonian structure from $X_H$.

Moreover, from (5.7) and (5.8), we derive the following relationship between the $S^1$-average $\langle X_H^{(1)} \rangle$ of the perturbation vector field and the second term in normal form (5.1).

**Corollary 5.2** The averaged perturbation vector field has the representation

$$\langle X_H^{(1)} \rangle = P_{\text{hor}} + P_{\text{ver}} + gX_H^{(0)},$$

where

$$g = -\frac{1}{2\omega} \left( S(\frac{\partial J}{\partial p}) \frac{\partial \omega}{\partial q} - S(\frac{\partial J}{\partial q}) \frac{\partial \omega}{\partial p} \right).$$

Remark, that last term on the right hand side of (5.9) is not Hamiltonian relative to the bracket $\{\cdot, \cdot\}$, in general.

Property (4.12) yields $\mathcal{L}_{P_{\text{hor}}} J = 0$. Moreover, $\mathcal{L}_{P_{\text{ver}}} J = \mathcal{L}_{\langle K \rangle} J = -\mathcal{L}_{\mathcal{T}} \langle K \rangle = 0$ and $\mathcal{L}_{gX_H^{(0)}} J = g\omega \mathcal{L}_{\mathcal{T}} J = 0$. Therefore, we arrive at the following fact.

**Corollary 5.3** The momentum map $J \ (4.2)$ is a first integral of the averaged perturbation vector field,

$$\mathcal{L}_{\langle X_H^{(1)} \rangle} J = 0.$$  

(5.10)

**Remark 5.4** The momentum map $J$ is uniquely determined by condition (2.1) up to adding a smooth function $f = f(p, q)$. But, such a renormalization of $J$ does not preserve property (5.10).

**Remark 5.5** Suppose that the $S^1$-action associated to the periodic flow of $X_H^{(0)}$ is free on $M$ and not necessarily trivial. Then, it follows from (5.10) and the periodic averaging theorem \cite{3, 18} that action $J \ (4.2)$ is an adiabatic invariant of slow-fast Hamiltonian system \cite{1, 2}, that is, $\left| J(\mathcal{F}^t_{X_H^{(0)}} (m)) - J(m) \right| = O(\varepsilon)$, for $m \in \mathbb{N}$ and $t \sim \frac{1}{\varepsilon}$. This is just the contents of the classical adiabatic theorem \cite{4, 16} which is usually formulated in the case $r = 1$ and for domains of action-angle variables.

The following fact can be useful in theory of semiclassical quantization of slow-fast Hamiltonian systems \cite{12}.

**Proposition 5.6** Under hypothesis of Theorem 5.1, the function

$$F = J - \frac{\varepsilon}{\omega} S(\{H, J\})$$

is an approximate first integral on $M$ of slow-fast Hamiltonian system \cite{4, 11}, \cite{1, 2} in the sense that

$$\mathcal{L}_{X_H} F = O(\varepsilon^2).$$

(5.12)
**Proof.** For a function $F = F_0 + \varepsilon F_1$ condition (5.12) holds if and only if the functions $F_0$ and $F_1$ are solutions to the following equations

\[ \mathcal{L}_{X_H^{(0)}} F_0 = 0, \quad (5.13) \]

\[ \mathcal{L}_{X_H^{(0)}} F_1 = -\mathcal{L}_{X_H^{(1)}} F_0. \quad (5.14) \]

If we put $F_0 = J$, then (5.13) is satisfied because of the $S^1$-invariance of $H$. In terms of the infinitesimal generator $\Upsilon$, equation (5.14) for $F_1$ is written as

\[ \mathcal{L}_\Upsilon F_1 = -\frac{1}{\omega} \mathcal{L}_{X_H^{(1)}} J. \quad (5.15) \]

By the identity $\langle \mathcal{L}_{X_H^{(1)}} J \rangle = \mathcal{L}_{(X_H^{(1)}) J}$, the solvability condition of equation (5.15) just coincides with (5.10). Finally, equality (2.3) shows that a particular solution to (5.15) is given by the formula $F_1 = -\frac{1}{\omega} S(\mathcal{L}_{X_H^{(0)}} J_1) = -\frac{1}{\omega} S(\{H, J\}_1)$. □

6. **An Approximate Hamiltonian Model with $S^1$-Symmetry**

Here, under hypothesis of Theorem 5.1, we give an alternative derivation of normal form splitting in (5.1) by applying a normalization procedure to the Poisson bracket (3.1) and the Hamiltonian. In the first step, by means of a near-identity transformation $\Phi_\varepsilon$, we correct original Poisson bracket (3.1) to get an $S^1$-invariant one. In the second step, a canonical averaging transformation is applied to the transformed Hamiltonian $H \circ \Phi_\varepsilon$.

First, we recall some facts concerning to the averaging procedure for symplectic and Poisson structures. Consider the symplectic form associated to Poisson bracket (3.1):

\[ \Omega = \frac{1}{\varepsilon} dp \wedge dq + dy \wedge dx. \]

Then, by (5.1) the $S^1$-average of $\Omega$ is given by the formula

\[ \langle \Omega \rangle = \Omega - d\Theta. \]

**Lemma 6.1** Let $N \subset M$ be an admissible domain. Then, for sufficiently small $\varepsilon \neq 0$, the $S^1$-average $\langle \Omega \rangle$ is a symplectic form on $N$. Moreover, there exists a near-identity transformation $\Phi_\varepsilon : N \to M$ which is a symplectomorphism between $\Omega$ and $\langle \Omega \rangle$,

\[ \Phi_\varepsilon^* \Omega = \langle \Omega \rangle \quad (6.1) \]

The proof of this lemma is based on the minimal coupling procedure and the Moser homotopy method, see, for example, [11, 20, 21]. Here, we recall an algorithm of the construction of $\Phi_\varepsilon$. Let us associate to the 1-form $\Theta$ the following $\lambda$-parameter family of 2-forms on $M$:

\[ \delta_\Theta^\lambda := dp \Theta + dq \Theta + \frac{(1 - \lambda)}{2} \{\Theta \wedge \Theta\}_0, \quad (6.2) \]
where
\[
\{\Theta \wedge \Theta\}_0 := \{\Theta^p \wedge \Theta^p\}_0 dp^i \wedge dp^j + 2\{\Theta^p \wedge \Theta^q\}_0 dq^i \wedge dq^j
+ \{\Theta^q \wedge \Theta^q\}_0 dq^i \wedge dq^j.
\]

Notice that the vanishing of the form $\delta_0^H$ (called the Hamiltonian 2-form of the Hannay-Berry connection) provides the integrability of the horizontal distribution $\mathbb{H}$. Let $W_\lambda$ be a time-dependent horizontal vector field on $N$ which is uniquely determined by the equation
\[
i_{W_\lambda} (dp \wedge dq - \varepsilon(1 - \lambda)\delta_\lambda^H) = -\varepsilon \Theta.
\]

Here, we use the fact: for small enough $\varepsilon$ and $\lambda \in [0, 1]$, the 2-form on the left hand side of (6.3) is nondegenerate on $\bar{N}$ along the horizontal distribution $\mathbb{H}$. Then, the symplectomorphism $\Phi_\varepsilon$ in (6.1) is defined as the time-1 flow of $W_\lambda$,
\[
\Phi_\varepsilon = Fl_{W_\lambda} |_{\lambda=1}.
\]

Denote by $\{,\}^{\text{inv}}$ the nondegenerate Poisson bracket associated to the symplectic form $(\Omega)$ on $N$. Then, the bracket $\{,\}^{\text{inv}}$ is $S^1$-invariant and has the decomposition
\[
\{F, G\}^{\text{inv}} = \{F, G\}_0 + \varepsilon \Pi_\Theta (dF, dG) + O(\varepsilon^2),
\]
where the bivector field $\Pi_\Theta$ is given by (4.18).

**Theorem 6.2** Under the hypothesis of Theorem 5.1, for small enough $\varepsilon$, there exists a near-identity transformation $\tilde{T}_\varepsilon : N \to M$ with the following properties:

(a) $\tilde{T}_\varepsilon$ is a Poisson isomorphism between Poisson brackets $\{,\}^{\text{inv}}$ and $\{,\}$;

(b) the transformed Hamiltonian is of the form
\[
H \circ \tilde{T}_\varepsilon = H + \varepsilon (K) + O(\varepsilon^2),
\]
where $K$ is just given by (5.2).

**Proof.** Applying transformation (6.4) to the original Hamiltonian system $(M, \{,\}, H)$, we get the following one
\[
(N, \{,\}^{\text{inv}}, H \circ \Phi_\varepsilon = H + \varepsilon H_1 + O(\varepsilon^2)),
\]
where the correction term $H_1$ is not necessarily $S^1$-invariant. To put the Hamiltonian in (6.6) to an $S^1$-invariant normal form of first order, we apply a canonical transformation defined as the time-$\varepsilon$ flow of the Hamiltonian vector field $\tilde{X}_G$ relative to the Poisson bracket $\{,\}^{\text{inv}}$ and a function $G$ which satisfies the homological equation
\[
\{H, G\}^{\text{inv}} = H_1 - \langle H_1 \rangle.
\]

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In terms of the integrating operator, a particular solution to this equation is represented as $G = S(H_1)$. Expanding equation (6.3) and the transformation $\Phi_\varepsilon$ at $\varepsilon = 0$, we show that the $S^1$-averages of $H_1$ and $K$ coincide, $\langle H_1 \rangle = \langle K \rangle$. Finally, we conclude that the desired normalization transformation is defined as the composition

$$\tilde{T}_\varepsilon = \Phi_\varepsilon \circ \text{Fl}^\varepsilon_{\tilde X_{S(H_1)}}.$$  \hspace{1cm} (6.8)

Consider the model $S^1$-invariant Hamiltonian system

$$(N, \{, \}^{\text{inv}}, H + \varepsilon \langle K \rangle) \hspace{1cm} (6.9)$$

Property (4.13) imply that the infinitesimal generator $\Upsilon$ is Hamiltonian relative to the Poisson bracket $\{, \}^{\text{inv}}$ and $J, \Upsilon = \tilde X J$. Therefore, the $S^1$-action is Hamiltonian on $(N, \{, \}^{\text{inv}})$ with momentum map $J$. It follows that the truncated Hamiltonian $H + \varepsilon \langle K \rangle$ and $J$ Poisson commute. For small $\varepsilon$, these functions are independent if $H$ satisfies condition (4.6).

**Corollary 6.3** Normalization transformation (6.8) carries the original slow-fast Hamiltonian system $(M, \{, \}, H)$ into a system which is $\varepsilon^2$-close to the Hamiltonian model with $S^1$-symmetry (6.9).

Moreover, an easy verification, by using (6.5), shows that the first order term in the Taylor expansion at $\varepsilon = 0$ of the Hamiltonian vector field of system (6.9) coincides with the normal form of first order in (5.1).

**Remark 6.4** In fact, by the standard Deprit normal form argument [6, 8] and by the fact that homological equation of the type (6.7) is solvable, one can extend $\tilde{T}_\varepsilon$ to a normalization transformation of arbitrary order $n \geq 2$ in $\varepsilon$. This means that one can correct formula (5.11) to get an approximate first integral of system (1.1), (1.2) which satisfies condition (5.12) mod $O(\varepsilon^n)$.

**Remark 6.5** Theorem 5.1 and Theorem 6.2 carry a general character and can be directly generalized to a class of slow-fast Hamiltonian systems on a phase space $M = M_0 \times M_1$ which is the product of an exact (fast) symplectic manifold $M_0$ and an arbitrary (slow) symplectic manifold $M_1$ (see, also [24, 27]).

7 The Quadratic Case

To illustrate our general results, we consider the particular case when $r = 1$ and the Hamiltonian $H$ is a quadratic function in the fast variables $z = (y, x) \in \mathbb{R}^2$. Let us associated to every matrix-valued function $A \in \text{sl}(2, \mathbb{R}) \otimes C^\infty(\mathbb{R}^{2k})$ the following function

$$Q_A = -\frac{1}{2} J A z \cdot z,$$
where \( \mathbf{J} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \). Then, the Hamiltonian vector field relative to the bracket \( \{ \cdot, \cdot \} \) and \( Q_\mathbf{A} \) is given by \( X_{Q_\mathbf{A}}^{(0)} = \mathbf{A} z \cdot \frac{\partial}{\partial z} \). Consider slow-fast Hamiltonian system (1.1), (1.2) with Hamiltonian of the form \( H = h + \omega Q_\mathbf{A} \)

for some smooth functions \( h = h(p, q) \), \( \omega = \omega(p, q) > 0 \) and \( \mathbf{A} = \mathbf{A}(p, q) \in \mathfrak{sl}(2; \mathbb{R}) \). We assume that \( \det \mathbf{A} = 1 \) on a certain open domain in \( \mathbb{R}^2 \). Then, the flow of \( X_{Q_\mathbf{A}}^{(0)} \) is periodic with frequency function \( \omega \) and the associated \( S^1 \)-action is given by the linear \( 2\pi \)-periodic flow \( F^t = \cos t \mathbf{I} + \sin t \mathbf{A} \). The corresponding momentum map (4.2) is \( J = Q_\mathbf{A} \). In this case, operators in (2.1) and (2.2) possess the following properties.

Lemma 7.1 For arbitrary \( \mathbf{B}, \mathbf{C} \in \mathfrak{sl}(2, \mathbb{R}) \otimes C^\infty(\mathbb{R}^{2k}_{p, q}) \), the following identities hold

\[
\langle Q_\mathbf{B} \rangle = \frac{1}{2} Q_\mathbf{B} - \mathbf{A} \mathbf{B} \mathbf{A}, \tag{7.1}
\]

\[
S(Q_\mathbf{B}) = \frac{1}{4} Q_{[\mathbf{A}, \mathbf{B}]}, \tag{7.2}
\]

\[
\langle Q_\mathbf{B} Q_\mathbf{C} \rangle = \frac{1}{4} Q_\mathbf{B} - \mathbf{A} \mathbf{B} Q_\mathbf{C} - \mathbf{A} \mathbf{C} \mathbf{A} + \frac{1}{8} Q_{[\mathbf{B}, \mathbf{A}]} Q_{[\mathbf{C}, \mathbf{A}]} \tag{7.3}
\]

Using the identities \( \mathbf{A}^{-1} = -\mathbf{A} \) and \( J \mathbf{A} = -\mathbf{A}^T \mathbf{J} \), one can verify identities (7.1)-(7.3) by a direct computation. As a consequence of (7.2), we get that the components of 1-form \( \Theta \) in (4.7) are given by the formulas

\[
\Theta^p_i = \frac{1}{2} Q_{\mathbf{A} \frac{\partial}{\partial p_i}} \quad \Theta^q_i = \frac{1}{2} Q_{\mathbf{A} \frac{\partial}{\partial q_i}}.
\]

In this case, it is easy to see that \( \delta_0^0 = 0 \) and hence the curvature of the Hannay-Berry connection is zero. Combining above relations with (7.1), we show that the \( S^1 \)-invariant function \( \langle K \rangle \) in (5.2) is represented as follows

\[
\langle K \rangle = \frac{\omega}{4} \left( Q_{\mathbf{A} \frac{\partial}{\partial p}} Q_{\mathbf{A} \frac{\partial}{\partial p}} - Q_{\mathbf{A} \frac{\partial}{\partial q}} Q_{\mathbf{A} \frac{\partial}{\partial q}} \right).
\]

Finally, an easy computation by using (7.2) and (7.3) shows that the approximate first integral \( F \) (5.11) is written in the form

\[
F = Q_\mathbf{A} - \frac{\epsilon}{4\omega} \left( Q_{[\mathbf{A}, \mathbf{B}]} + Q_\mathbf{A} Q_{[\mathbf{A}, \mathbf{C}]} \right),
\]

where

\[
\mathbf{B} := \{ h, \mathbf{A} \}_1, \quad \mathbf{C} := \{ \omega, \mathbf{A} \}_1
\]

and the Poisson bracket between \( h \) and \( \mathbf{A} \) is defined entry by entry.

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