Velocity memory effect without soft particles

B. Cvetković and D. Simić∗
Institute of Physics, University of Belgrade
Pregrevica 118, 11080 Belgrade, Serbia
February 1, 2022

Abstract

We study the behavior of geodesics in the plane-fronted wave background of the three-dimensional gravity with propagating torsion, which possesses only massive degrees of freedom. We discover the velocity memory effect, in contrast to the current belief that its existence is due to the presence of soft particles.

1 Introduction

Memory effect for gravitational waves was first discovered by Zeldovich and Polnarev [1] and got its name from Braginsky and Grishchuk [2]. The conclusion of [1, 2] is that massive test particles, initially at rest, will suffer permanent displacement after the passage of gravitational wave. For this reason this displacement is called memory effect.

The memory effect [1, 2] is described in linear approximation. A nonlinear contribution to the memory effect is discovered in reference [3], for less technical derivation see [4].

In the recent years, we have witnessed a great new discoveries connecting asymptotic symmetries, soft theorems, and displacement memory effect [5]. This line of reasoning applied to the black holes [6] offers new insights into black hole physics.

Permanent displacement implies that relative velocity of massive test particles is zero. This conclusion is questioned in references [7, 8], where velocity memory effect is derived on the contrary to displacement memory effect. The main result of [7, 8] is that passage of gravitational wave will be encoded not in the permanent displacement but in the nonzero relative velocity of test masses. For recent development on velocity memory effect see, [9], where, among other things, authors concluded that velocity memory effect is connected with soft gravitons.

The goal of this paper is to investigate if there is a memory effect for gravitational waves within the framework of Poincaré gauge theory [10, 11, 12]. We shall consider the solutions of three-dimensional (3D) gravity with propagating torsion [13, 14], a theory in which all modes are massive. If the soft modes are the ones that cause memory effect (according to the already mentioned conclusions in [9]) there should be no memory effect in this theory.

∗Email addresses: cbranislav@ipb.ac.rs, dsimic@ipb.ac.rs
Let us note that 3D general relativity (GR) is a topological theory and there are consequently no gravitational wave solutions in vacuum. The gravitational waves with torsion in 3D are solutions in which the metric function crucially depends on torsion [14], in the sense that in the absence of torsion, the metric function becomes trivial and the wave solution "disappears". This offers us an interesting opportunity to study the effects of torsion already at the level of geodesic motion of spinless particles.

The paper is organized as follows. First, we review the theory of gravity under consideration and its gravitational pp wave solutions. Next, we derive the geodesic equations in this pp wave space-time. Thereafter, we investigate the solutions of this equations. Unfortunately, the geodesic equations are not analytically solvable except in a very special case, which we have also analyzed; thus, we solved the geodesic equations numerically for some characteristic choices of the coefficients which appear in the gravitational wave solutions.

Our conventions are as follows. The Latin indices \((i, j, \ldots)\) refer to the local Lorentz (co)frame and run over \((0, 1, 2)\), \(b^i\) is the tetrad (one form), \(h_i\) is the dual basis (frame), such that \(h_i b^k = \delta^i_k\); the volume three form is \(\hat{\epsilon} = b^0 \wedge b^1 \wedge b^2\), the Hodge dual of a form \(\alpha\) is \(\star \alpha\), with \(\star 1 = \hat{\epsilon}\), totally antisymmetric tensor is defined by \(\star (b_i \wedge b_j \wedge b_k) = \varepsilon_{ijk}\) and normalized to \(\varepsilon_{012} = +1\); the exterior product of forms is implicit.

## 2 Riemannian pp waves

In this section, we give an overview of Riemannian 3D pp waves. For details see [13].

### 2.1 Geometry

The metric of pp waves can be written as

\[
\begin{align*}
    ds^2 &= du(Sdu + dv) - dy^2, \\
    S &= \frac{1}{2}H(u, y).
\end{align*}
\]

Next, we choose the tetrad field (coframe) in the form

\[
\begin{align*}
    b^0 := du, \quad b^1 := Sdu + dv, \quad b^2 := dy,
\end{align*}
\]

so that \(ds^2 = \eta_{ij} b^i \otimes b^j\), where \(\eta_{ij}\) is the half-null Minkowski metric

\[
\eta_{ij} = \begin{pmatrix}
    0 & 1 & 0 \\
    1 & 0 & 0 \\
    0 & 0 & -1
\end{pmatrix}.
\]

The corresponding dual frame \(h_i\) is given by

\[
\begin{align*}
    h_0 &= \partial_u - S\partial_v, \quad h_1 = \partial_v, \quad h_2 = \partial_y.
\end{align*}
\]

For the coordinate \(x^\alpha = y\) on the wave surface, we have

\[
\begin{align*}
    x^c &= b^c_\alpha x^\alpha = y, \quad \partial_c = h^\alpha_\alpha \partial_\alpha = \partial_y,
\end{align*}
\]
where \( c = 2 \).

Starting from the general formula for the Riemannian connection one form,

\[
\omega^{ij} := -\frac{1}{2} \left[ h^i \mathcal{J} db^j - h^j \mathcal{J} db^i - (h^i \mathcal{J} h^j \mathcal{J} db^k) b_k \right],
\]

one can find its explicit form; for \( i < j \), its nonvanishing component reads as

\[
\omega^{12} = -\partial_y \mathcal{S} b^0.
\] (2.3a)

Introducing the notation \( i = (A,a) \), where \( A = 0,1 \) and \( a = 2 \), one can rewrite \( \omega^{ij} \) in a more compact form as follows:

\[
\omega^{Ac} = k^A b^0 \partial_y S,
\] (2.3b)

where \( k^i = (0,1,0) \) is a null propagation vector, \( k^2 = 0 \).

The above connection defines the Riemannian curvature

\[
R^{ij} = 2 b^0 k^i Q^j,
\] (2.4a)

where

\[
Q^2 = \partial_y \mathcal{S} b^2
\] (2.4b)

The Ricci one form \( Ric^i := h_m \mathcal{J} Ric^m_i \) is given by

\[
Ric^i = b^0 k^i Q, \quad Q = h_c \mathcal{J} Q^c = \frac{1}{2} \partial_y H,
\] (2.5a)

and the scalar curvature vanishes

\[
R = 0.
\] (2.5b)

### 2.2 Dynamics

#### 2.2.1 pp waves in GR

Starting with the action \( I_0 = -\int d^4x (a_0 R + 2 \Lambda_0) \), one can derive the GR field equations in vacuum,

\[
2a_0 G^n_{\ i} = 0,
\] (2.6)

where \( G^n_{\ i} \) is the Einstein tensor. As a consequence, the metric function \( H \) must obey

\[
\partial_y H = 0.
\] (2.7)

However, the solution of this equation is trivial,

\[
H = C(u) + y D(u),
\]

since the corresponding radiation piece of curvature vanishes.
3 pp waves with torsion

3.1 Geometry of the ansatz

We assume that the form of the triad field (2.2) remains unchanged, whereas the connection is

\[ \omega^{ij} = \tilde{\omega}^{ij} + \frac{1}{2} \epsilon^{ij}_m k^m k^n b^n G, \]  
\[ G := S' + K. \]  

Here, the new term \( K = K(u, y) \) describes the effect of torsion, as follows:

\[ T^i := \nabla b^i = \frac{1}{2} K k^i k^m b^m. \]  

The only nonvanishing irreducible piece of \( T^i \) is its tensorial piece

\[ (1) T^i = T^i, \]

while the curvature is

\[ R^{ij} = \epsilon^{ijm} k^m k^{*n} b_n G', \]
\[ Ric^i = \frac{1}{2} k^i k^m b^m G', \]
\[ R = 0. \]  

The nonvanishing irreducible components of the curvature \( R^{ij} \) are

\[ (4) R^{ij} = \frac{1}{2} \epsilon^{ijm} k^m k^{*n} b_n pG'. \]

and the quadratic curvature invariant vanishes \( R^{ij} R_{ij} = 0 \). For details on irreducible decomposition of torsion and curvature see [15].

The geometric configuration defined by the triad field (2.2) and the connection (3.1) represents a generalized gravitational plane-fronted wave of GR\(_\Lambda\), or the torsion wave for short.

3.2 Massive torsion waves

The field equations take the following form [14]:

\[ a_0 G' - a_1 K' = 0, \quad \Lambda = 0, \]
\[ K'' + m^2 K = 0, \quad m^2 = \frac{a_0(a_1 - a_0)}{b_1 a_1}, \]  

with \( G = S' + K \) and \( S = H/2 \). The solution has a simple form,

\[ K = A(u) \cos my + B(u) \sin my, \]
\[ \frac{1}{2} H = \frac{a_1 - a_0}{a_0 m}(A(u) \sin my - B(u) \cos my) + h_1(u) + h_2(u) y. \]  

Disregarding the integration “constants” \( h_1 \) and \( h_2 \), the metric and the torsion functions, \( H \) and \( K \), become both periodic in \( y \). In the absence of torsion the metric function becomes trivial. This is an expected result since 3D general relativity is a theory which possesses no propagating degrees of freedom.

The vector field \( k = \partial_v \) is the Killing vector for both the metric and the torsion; moreover, it is a null and covariantly constant vector field. This allows us to consider the solution (3.2) as a generalized pp wave.

## 4 Geodesic motion

In this section, we shall examine the geodesic motion of particles in the field of the massive wave with torsion.

At first sight that might be puzzling, since the motion of spinless particles is not affected by torsion \([12, 16]\), and in the gravitational field, they follow geodesic lines, which are influenced by the Riemannian connection depending on the metric. However, gravitational waves with torsion in 3D are interesting solutions, which are intrinsically different from the well-known spherically symmetric (static or stationary) solutions of Poincaré gauge theory \([17]\) (for review, see \([18]\)). The metric of these solutions is “independent” of torsion in the sense that it represents Schwarzschild (or Schwarzschild anti de Sitter, Kerr etc.) metric and the motion of spinless particles is not affected by the presence of torsion. However, for the gravitational wave solution (3.2), metric crucially depends on torsion as we noted in the previous section. This offers us an interesting opportunity to study the effects of torsion already at the level of geodesic motion.

**Christoffel connection.** The nonvanishing components of Christoffel (torsion free) connection are given by

\[
\tilde{\Gamma}^v_{uu} = \frac{1}{2} \partial_u H, \\
\tilde{\Gamma}^v_{uy} = \frac{1}{2} H', \\
\tilde{\Gamma}^y_{uu} = \frac{1}{2} H'.
\]  

Let us mention that nontrivial contribution to metric function and consequently Christoffel connection stems from the presence of torsion.

**Geodesic equations.** The geodesic equation for \( u \) takes the expected form

\[
\frac{d^2 u}{d\lambda^2} = 0.
\]  

Therefore, without the loss of generality, we can assume \( u \equiv \lambda \).

The equation for \( y \) is given by

\[
\ddot{y} + \frac{1}{2} H' = 0,
\]  

or more explicitly

\[
\ddot{y} + \frac{a_1 - a_0}{a_0} (A \cos m y + B \sin m y) = 0.
\]
Finally, the equation for \( v \) reads as
\[
\ddot{v} + \frac{1}{2} \partial_u H + H' \dot{y} = 0 ,
\]
(4.4)

In the special case, when \( H \) does not explicitly depend on \( u \), the equation (4.4) can be integrated as
\[
\dot{v} + H y = C ,
\]
where \( C \) is a integration constant. Hence, consequently, we get:
\[
v = \int (C - H y) du .
\]
(4.5)

4.1 Exact solutions

Interestingly, the geodesic equations admit the existence of exact solutions in particular cases. The simplest case is when \( A(u) \) and \( B(u) \) are constants. In that case (4.3b) can be rewritten in the form
\[
\frac{1}{2} \frac{d\dot{y}^2}{dy} + \frac{a_1 - a_0}{a_0} (A \cos my + B \sin my) = 0 .
\]
If we impose initial conditions
\[
y(0) = 0 , \quad \dot{y}(0) = 0 ,
\]
(4.6)
by integrating the previous equation, we obtain
\[
\frac{1}{2} \dot{y}^2 + \frac{a_1 - a_0}{a_0 m} (A \sin my - B (\cos my - 1)) = 0 ,
\]
or equivalently:
\[
\frac{dy}{\sqrt{\tilde{A} \sin my - B (\cos my - 1)}} = du ,
\]
where
\[
\tilde{A} := \frac{2(a_0 - a_1)}{a_0 m} A \quad \text{and} \quad \tilde{B} := \frac{2(a_0 - a_1)}{a_0 m} B ,
\]
which after integration yields the following equation for \( y \):
\[
4i \sqrt{2\tilde{A}\tilde{B} - (\tilde{A}^2 + \tilde{B}^2) \frac{\cos \frac{my}{2}}{\sin^2 \frac{my}{4}}} \sin \frac{my}{2} \sqrt{\tan \frac{my}{4}} \times
\]
\[
\times F \left( i \text{Arcsinh} \left( \sqrt{\frac{\tilde{B} + \sqrt{\tilde{A}^2 + \tilde{B}^2}}{\sqrt{\tan \frac{my}{4}}}} \right) \left| \frac{\tilde{A}^2 + 2\tilde{B}(\tilde{B} - \sqrt{\tilde{A}^2 + \tilde{B}^2})}{\tilde{A}^2} \right) = u ,
\]
(4.7)
where \( F(\phi|k) \) represents the elliptic integral of the first kind \[19\].
The choice $\bar{A} \neq 0$, $B(u) = 0$ yields the following exact solution for $y(u)$:

$$y(u) = -\frac{2am\left(\frac{1}{2}\sqrt{Amu}\right) |2|}{m},$$

where $am(z|m)$ is a Jacobi amplitude function. $H$ does not explicitly depend on $u$ we get that $v$ is given by the expression (4.5).

The characteristic plots for particle position $y$ and velocity $\dot{y}$ (for $m = 2, \bar{A} = 1$) are shown in Fig. 1 and Fig 2, respectively.
4.2 Velocity memory effect

The velocity memory effect is present in the case when functions $A(u)$ and $B(u)$ vanish for large $u$.

**Shockwave case.** In the shock wave case, when functions $A(u) = 0$ and $B(u)$ vanish exponentially, for example $B(u) \sim e^{-(u-10)^2}$ numerical solutions of the geodesic equations lead to the plots for the particle position $y$ and $v$ shown in the Figure 3 and velocity $\dot{y}$ and $\dot{v}$ shown in the Figure 4.

**Slow fall off.** In the case when $A(u) = 0$ and $B(u) \sim 1/u$ numerical solutions lead to the following plots for the particle position $y$ and $v$ shown in the Figure 5 and velocity $\dot{y}$ and $\dot{v}$ shown in the Figure 6:
5 Discussion

We studied the geodesic motion in asymptotically flat pp wave space-time and we discovered the presence of velocity memory effect. The effect is present for the very fast falloff of the gravitational wave, as well as for the slow one. Analysis of this paper provides the first example of memory effect for gravitational waves with torsion. We demonstrated that torsion waves lead to the memory effect same as the torsion-less waves do. This is, also the first account of the memory effect in three dimensions to authors knowledge. It would be interesting to see is there a connection to BMS$_3$ symmetry. Because theory has no massless modes, without a doubt, we can conclude that there can be no soft particles responsible for the memory effect. Consequently, the belief that soft particles are responsible for the velocity memory effect is demonstrated to be incorrect. For the related work on massive gravity, see [20].

Intuitively, we can say that memory effect is due to energy transfer. Passing gravitational wave transfers energy to the test particle which after the passage of the gravitational wave continues to move with constant velocity, in which intensity is dictated by the amount of energy transferred. Looking at the memory effect in this way we conclude that displacement memory effect is not possible, except, maybe, in some special cases where the total amount of transferred energy would be zero. To make this intuitive discussion precise it is required to define energy in asymptotically flat space-times in a satisfying manner; this is left for further investigation.

The theory we considered is three-dimensional, while the four-dimensional case is realistic and relevant for applications. The next step in investigation is to study geodesic motion for massive gravitational waves with torsion in four dimensions. The metric of the gravitational waves with torsion in 4D has a nontrivial contribution stemming from the tensorial component of torsion [21] as in 3D, which affects geodesic motion. Consequently, it is expected that in 4D we shall obtain the velocity memory effect similar to the one noticed in 3D case. Also, there is a possible difference compared to the memory effect in general relativity, which, in principle, may be observable. This will be the possible experimental setup for detection of torsion.
Acknowledgments

This work was partially supported by the Serbian Science Foundation under Grant No. 171031.

References

[1] Ya. B. Zeldovich and A. G. Polnarev, Radiation of gravitational waves by a cluster of superdense stars, Astron. Zh. 51, 30 (1974) [Sov. Astron. 18 17 (1974)].

[2] V. B. Braginsky and L. P. Grishchuk, Kinematic resonance and the memory effect in free mass gravitational antennas, Zh. Eksp. Teor. Fiz. 89 744 (1985) [Sov. Phys. JETP 62, 427 (1985)].

[3] D. Christodoulou, Nonlinear Nature of Gravitation and Gravitational Wave Experiments, Phys. Rev. Lett. 67 1486 (1991).

[4] K. S. Thorne, Gravitational-wave bursts with memory: The Christodoulou effect, Phys. Rev. D 45 520, (1992).

[5] Temple He, V. Lysov, P. Mitra and A. Strominger, BMS supertranslations and Weinberg’s soft graviton theorem, JHEP 05 (2015) 151; A. Strominger and A. Zhiboedov, Gravitational memory, BMS supertranslations and soft theorems, JHEP 01 (2016) 086.

[6] S. W. Hawking, M. J. Perry and A. Strominger, Soft Hair on Black Holes, Phys. Rev. Lett 116, 231301 (2016).

[7] H. Bondi and F. A. E. Pirani, Gravitational waves in general relativity III. Exact plane waves, Proc. R. Soc. A 251, 519 (1959).

[8] L. P. Grishchuk and A. G. Polnarev, Gravitational wave pulses with velocity coded memory, Zh. Eksp. Teor. Fiz. 96 (1989) 1153 [Sov. Phys. JETP 69 (1989) 653].

[9] P.-M Zhang, C. Duval, G.W. Gibbons and P.A. Horvathy, The memory effect for plane gravitational waves, Phys. Lett. B 743 772 (2017); P. M Zhang, C. Duval, G.W. Gibbons and P.A. Horvathy, Soft gravitons and the memory effect for plane gravitational waves, Phys. Rev. D 96, 064013 (2017) no.6; P. M. Zhang, C. Duval, G.W. Gibbons and P. A. Horvathy, Velocity memory effect for polarized gravitational waves, JCAP 05 (2018), 030.

[10] M. Blagojević, Gravitation and Gauge Symmetries (IOP Publishing, Bristol, 2002); T. Ortín, Gravity and Strings (Cambridge University Press, Cambridge, United Kingdom, 2004).

[11] Yu. N. Obukhov, Poincaré gauge gravity: Selected topics, Int. J. Geom. Meth. Mod. Phys. 03, 95 (2006).
[12] *Gauge Theories of Gravitation, A Reader with Commentaries*, edited by M. Blagojević and F. W. Hehl, (Imperial College Press, London, 2013).

[13] M. Blagojević and B. Cvetković, 3D gravity with propagating torsion: The AdS sector, Phys. Rev. D 85, 104003 (2012).

[14] M. Blagojević and B. Cvetković, Gravitational waves with torsion in 3D, Phys. Rev. D 90, 044006 (2014).

[15] F. W. Hehl, J. D. McCrea, E. W. Mielke, and Y. Neeman, Metric-affine gauge theory of gravity: Field equations, Noether identities, world spinors, and breaking of dilation invariance, Phys. Rep. 258, 1 (1995).

[16] D. Puetzfeld and Yuri N. Obukhov, Probing non-Riemannian spacetime geometry, Phys. Lett A 372 6711 (2008).

[17] P. Baekler, A spherically symmetric vacuum solution of the quadratic Poincaré gauge field theory of gravitation with Newtonian and confinement potentials, Phys. Lett. 99 B (1981) 329-332;

J. D. McCrea, P. Baekler and M. Gürses, A Kerr-like solution of the Poincaré gauge field equations, Nuovo Cimento B 99 171 (1987);

[18] Yu. N. Obukhov, Exact solutions in Poincaré gauge gravity theory, Universe 5, 127 (2019).

[19] G. E. Andrews, R. Askey, and R. Roy, *Special Functions* (Cambridge University Press, United Kingdom, Cambridge, 1999);

Z. X. Wang and D. R. Guo, *Special Functions* (World Scientific, Singapore, 1989).

[20] E. Kilicarslan and B. Tekin, Graviton mass and memory, Eur. Phys. J. C 79 114 (2019).

[21] M. Blagojević and B. Cvetković, Generalized pp waves in Poincaré gauge theory, Phys. Rev. D 95, 104018 (2017);

M. Blagojević, B. Cvetković and Y. N. Obukhov, Generalized plane waves in Poincaré gauge theory of gravity, Phys. Rev. D 96, 064031 (2017).