A method of self-consistent three-dimensional solution of the magnetohydrostatic equations using magnetogram data

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Abstract. A technique is proposed for constructing three-dimensional solutions complying with the self-consistent magnetohydrostatic (MHS) equations and with observations along the line of sight of the magnetic field at the photosphere. The technique is a generalization of a paper (Rudenko, 1999) to a potential-field approximation. The solution of the problem under consideration lends a representation of the magnetic field in terms of a scalar function, with its subsequent harmonic expansion in terms of a spherical functional basis that satisfies specified boundary conditions. A numerical realization of the proposed method is expected to permit a real-time modelling of three-dimensional magnetic field, temperature, pressure and density distributions.

1. Introduction

In a paper (Rudenko, 1999) it was suggested a method to model a three-dimensional distribution of the above-photospheric magnetic field, base on solving the Laplace boundary-value problem that corresponds to actual one-component magnetic measurements at the photospheric level. Boundary conditions on the lower sphere were specified in (Rudenko, 1999) by a distribution of the projection of the magnetic field onto a certain fixed direction corresponding to the line of sight at the time of measurement. In this paper, the same boundary conditions are used in solving a boundary-value problem for a certain class of self-consistent magnetohydrostatic (MHS) equations admitting of analytic solutions. Developing self-consistent physical models of chromospheric and coronal plasmas is one of the major tasks of theoretical astrophysics. Using the MHS approximation is justified when it is intended to study the current state of the medium or relatively slowly occurring macroscopic plasma processes. Also, because of the non-linearity of the MHS equations, the problem of seeking all possible solutions is extremely complicated. Several approaches to constructing self-consistent MHS models are known to date, which permit analytic solutions to be determined. These approaches are based on some not fundamentally differing preliminary assumptions about the way in which the solutions behave. An outline of the approaches and research results on this subject may
be found in ([Low, 1985], [Low, 1991], [Low, 1992], [Low, 1993a], [Low, 1993b], [Bogdan and Low, 1986], and [Neukirch, 1995]). The class of the models studied thus far is relatively narrow to pretend to a generality of the description of the real states of magnetic plasma. Nevertheless, these models, when used, would be expected to give at least qualitative insight into the possible correlations between the fundamental physical plasma characteristics. There is no question that the MHS modelling is much higher in information content than the simplest modelling in the potential-field approximation of the magnetic field in extensive use to date.

An important factor as to the extrapolation of the magnetic field and other physical characteristics in terms of a particular physical model is the ability to derive physical solutions that satisfy boundary conditions corresponding to actual measurements. Only then can we expect that the result of an extrapolation will represent the facts, and only then can we select – on the basis of many comparisons – the most adequate physical model for use in the study. Three methods of specifying boundary conditions are customarily used to extrapolate magnetic fields in the potential-field approximation: from the radial component $B_r$ (Neumann’s classical boundary-value problem); from the component $B_l$ (projection along the direction corresponding to the orientation of the line of sight at the time when each point crosses the surface of the central meridian); and from the component $B_d$ (projection along the line of sight corresponding to the time of measurement ([Rudenko, 1999])). Only the second and third methods of specifying boundary conditions correspond to actual magnetic field observations. More specifically, the second method gives only an averaged picture of the magnetic configuration (to accumulate boundary conditions requires a full series of daily observations over the course of a complete rotation). The third method appears to be the most correct as only its application can provide the best fit to a current state of the magnetic field. As in the case of the potential-field approximation, of important significance is the statement and solution of a boundary-value problem in the MHS approximation with boundary conditions of the third type (from the component $B_d$). In this paper such a setting of the problem is realized for MHS models as suggested in ([Neukirch, 1995]) where analytic solutions were obtained for MHS models satisfying boundary conditions of the 1st type (from the component $B_r$). In this paper we consider the same MHS approximations, yet for different boundary conditions. We restrict the discussion only to the analytic solution of the problem formulated. It can be used in constructing a numerical algorithm for calculating a current state of solar plasma, and this issue will be the subject of further investigation. The problems associated
with scarcity of data from the averted side of the Sun will not be discussed here as they were covered in considerable detail in a previous paper (Rudenko, 1999) for the potential-field case.

2. Setting up the problem, and the method of solution.

Here we take advantage of the physical models whose substantiation and physical aspects are reported in (Neukirch, 1995). We restrict ourselves only to a brief formulation of the problem in the form of equations. From the logical standpoint, the problem is conveniently broken up into three parts: one is the system of basic equations; the other involves imposing additional requirements on the function of current density (strictly speaking, a concrete definition of the physical model); and the third includes specifying the boundary conditions. The system of MHS equations is

\[ j \times B - \nabla p - \rho \nabla \psi = 0, \]  
\[ \nabla \times B = \mu_0 j, \]  
\[ \nabla \cdot B = 0. \]

Here \( B, j, p, \rho \) and \( \psi \) are magnetic induction, current density, pressure, density, and gravitational potential, respectively.

As shown in (Low, 1991) and (Neukirch, 1995), the current density that allows pressure and density from (1) to be expressed explicitly in terms of the magnetic field has the form:

\[ j = \alpha B + \nabla F(\nabla \cdot B, \psi) \times \nabla \psi, \]

where \( F \) is an arbitrary function of its arguments. As is (Neukirch, 1995), we consider a simplified model: \( \alpha = \text{const} \), and

\[ F = k(\psi) \frac{GM}{r^3} \cdot B = \xi(r) \frac{r^3}{GM} \cdot B, \quad \xi(r) = k(\psi) \left( \frac{GM}{r^3} \right)^2. \]

Here the gravitational potential is specified as

\[ \psi = -\frac{GM}{r}, \quad \nabla \psi = -\frac{GM}{r^3} r, \]

with \( M \) being the mass of the central body. For \( \xi(r) \), the same variants as in ((Neukirch, 1995), (40)) can be considered.
\begin{align*}
\mu_0 \xi(r) = \begin{cases} 
k/r^2, & \text{case I}, \\
1/r^2 - 1/(r + a)^2, & \text{case II}, \\
1/r^2 + k \exp(-2r/L) - q, & \text{case III},
\end{cases}
\end{align*}

where \( k, q, \) and \( L \) are constants.

To formulate the boundary conditions we introduced in the direction selected a fixed unit vector \( \mathbf{d} \), corresponding to the direction of the line of sight, and require the fulfillment of the conditions

\begin{align*}
\mathbf{d} \cdot \mathbf{B}_{|r=R} &= B_d(\theta, \phi), \\
\mathbf{B}_{|r=\infty} &\to 0.
\end{align*}

Here \( B_d(\theta, \phi) \) is some specified distribution on a spherical surface of radius \( R \).

Formulas (2-9) are sufficient for a full determination of the magnetic field in the region of space \( r > R \). The hydrostatic balance equation (1) describes pressure and density variations associated with the magnetic field. At this point we shall not give the formulas describing the specific relation of these quantities to the magnetic field; their derivation and particular representation may be found in ((Neukirch, 1995), (25)-(31)). In ((Neukirch, 1995), see (24)), an analytic expression was obtained for the magnetic field in terms of the coefficient of spherical harmonic expansion of the scalar product \( (\mathbf{r} \cdot \mathbf{B}) \). It can be demonstrated that the magnetic field satisfying (2-5) may be represented in a more general (than in (Neukirch, 1995)) form in terms of a single scalar function. It is easy to verify by mere substitution that the magnetic field that is expressed as

\begin{equation}
\mathbf{B} = \left\{ -\mathbf{r} \triangle + \frac{\mu_0}{L} \mathbf{L} + \nabla \left( \frac{\mathbf{r}}{r} \cdot \nabla \right) \mathbf{r} \right\} \chi
\end{equation}

in terms of a certain scalar function \( \chi \) satisfying the equation

\begin{equation}
(\triangle + \frac{\xi}{L^2} + \frac{\alpha^2}{L^2}) \chi = 0
\end{equation}

satisfies equations (2-5). In (10, 11), \( \overline{\alpha} = \mu_0 \alpha, \overline{\xi} = \mu_0 \xi, \) and \( \mathbf{L} \) is the angular moment operator (see (Neukirch, 1995))

\begin{equation}
\mathbf{L} = \frac{1}{i} \mathbf{r} \times \nabla.
\end{equation}

Note that equation (11) for \( \chi \) is identical to the equation for \( (\mathbf{r} \cdot \mathbf{B}) \) in (Neukirch, 1995). Hence the initial problem can be reformulated as a boundary-value problem for \( \chi \), satisfying (11) and the boundary conditions (8), (9).
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To solve the boundary-value problem formulated we introduce a spherical system of coordinates \((r, \theta, \phi)\) connected with the unit vector \(\mathbf{d}\) (as done in (Rudenko, 1999)). Let the axis \(z\) that correspond to the selected spherical system of coordinate, be directed along \(\mathbf{d}\), and let the axis \(x\), for definiteness, be taken to lie in the plane produced by the axis \(z\) and the heliographic axis \(z'\). Finally, let the function \(\chi\) be represented as an expansion into a series in terms of spherical harmonics of the function

\[
\Psi (r, \theta, \phi) = R \sum_{l=1}^{\infty} \sum_{m=-l}^{l} c_l^m f_l(r) \tilde{P}_l^{|m|} (\cos \theta) e^{im\phi},
\]

where \(c_l^m\) are the desired complex expansion coefficients

\[
c_l^{-m} = \overline{c_l^m},
\]

\[
\tilde{P}_l^m = P_l^m / \sqrt{2\pi w_l^m},
\]

\(P_l^m\) is the Legendre function (see (Abramowitz and Stegun, 1964)), and

\[
w_l^m = \int_{-1}^{1} [P_l^m (u)]^2 du = (l + 1/2)^{-1}(l + m)!/(l - m)!.\]

The expression (13) should be regarded as an exact solution if the summation with respect to the index \(l\) is made ad infinitum, or as a finite approximation if the summation is limited to the value of \(L\) (the main index of expansion). Substitution of (13) into equation (11) gives an equation which must be satisfied by the function \(f_l(r)\):

\[
\left( \frac{d^2}{dr^2} - l(l + 1) \left( \frac{1}{r^2} - \overline{\xi}(r) \right) + \overline{\alpha} \right) r f_l(r) = 0.
\]

For the sake of simplicity, we did not introduced one further summation into (13) - with respect to the indices of two independent solutions of equation (16). For the three cases of the dependences \(\overline{\xi}(r)\) in (7), the solitons of equation (16) are expressed in terms of different variants of Bessel functions (for the explicit expression see in ((Neukirch, 1995), (40)-(44)). Out of them we may always choose linear combinations satisfying the upper boundary conditions (9). Moreover, for the case of \(\alpha = 0\) we can impose on some surface \(|r| = R_w > R\) the boundary conditions of a radial magnetic field. Such conditions are used very widely in the technique for extrapolating a potential magnetic field in modelling the conditions on the source surface. As will be apparent from the ultimate expansion for a full magnetic field vector, such conditions can be obtained by satisfying the equality

\[
\Psi (r, \theta, \phi) = R \sum_{l=1}^{\infty} \sum_{m=-l}^{l} c_l^m f_l(r) \tilde{P}_l^{|m|} (\cos \theta) e^{im\phi},
\]
\[
\frac{d}{dr} \left( a_1 f_1^{(1)}(r) + a_2 f_1^{(2)}(r) \right) = 0, \\
f_l(r) = a_1 f_l^{(1)}(r) + a_2 f_l^{(2)}(r).
\]

When \( \alpha \neq 0 \) such conditions cannot be satisfied in principle; however, if \( \alpha \) is small, the condition (17) might still be used. In this case the field will be approximately a radial one. In any case, here \( f_l(r) \) is taken to mean the solutions of equation (16), satisfying some of the upper boundary conditions. Explicit expressions for them can always be obtained. Upon substituting the expansion (13) into the expression for the magnetic field (10), we arrive at the expansion of the full magnetic field vector

\[
\mathbf{B}(r) = R \sum_{l=1}^{\infty} \sum_{m=-l}^{l} c_l^m \left\{ \frac{r}{r^2} l(l+1) + \frac{\bar{r}}{r} L + \nabla_\perp \frac{d}{dr} r \right\} f_l \bar{P}_l^{|m|} (\cos \theta) e^{im\phi}.
\]

This expansion coincides exactly in its form with the expansion in ((Neukirch, 1995), (24)). It can be used to calculate the magnetic field once its coefficients are determined. Note that by virtue of the identity \( (\mathbf{r} \cdot \mathbf{B}) = L^2 \chi \), the coefficients \( c_l^m \) are those of the initial expansion \( (\mathbf{r} \cdot \mathbf{B}) \) that is represented by the expression ((Neukirch, 1995), (17)).

To solve the boundary-value problem formulated, an explicit form of the expansion of the component \( (\mathbf{d} \cdot \mathbf{B}) \) is required. Using the representation of the operators \( L \) and \( \nabla_\perp \) in a spherical system of coordinates

\[
L = \frac{1}{i} \left( \mathbf{e}_\phi \frac{\partial}{\partial \theta} - \mathbf{e}_\theta \frac{\partial}{\sin \theta \partial \phi} \right); \\
\nabla_\perp = \frac{\mathbf{e}_\theta}{r} \frac{\partial}{\partial \theta} + \frac{\mathbf{e}_\phi}{r \sin \theta} \frac{\partial}{\partial \phi},
\]

we obtain

\[
\mathbf{d} \cdot \mathbf{B} = R \sum_{l=1}^{\infty} \sum_{m=-l}^{l} c_l^m \left\{ \frac{u}{r} l(l+1) - \bar{\alpha} i m + (1 - u^2) \frac{d}{du} \frac{1}{r} \frac{d}{dr} \right\} f_l \bar{P}_l^{|m|} (u) e^{im\phi}.
\]

Here \( \mathbf{e}_\phi \) and \( \mathbf{e}_\theta \) are unit vectors of the spherical system of coordinates, and \( u = \cos \theta \). To determine the coefficients of the initial expansion we take advantage of the projection scheme from ((Rudenko, 1999)). We multiply the equality (20) by the factor \( \bar{P}_l^{|m|} (\cos \theta) e^{-im\phi} \cos \theta \sin \theta d\phi d\theta \), and integrate it on a sphere \( |\mathbf{r}| = R \) over angular variables:
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\[ \pi \int_0^{2\pi} B_d(\theta, \varphi) \bar{P}_l^{|m|}(\cos \theta) e^{-im\phi} \cos \theta \sin \theta d\phi d\theta = \]

\[ 2\pi \sum_{l=1}^{\infty} c_l^m \int_{-1}^{1} \left\{ u^2 k(k + 1) - i\pi R m u \right\} \bar{P}_k^{|m|}(u) \bar{P}_l^{|m|}(u) \]

\[ -u(u^2 - 1) g_k \frac{d\bar{P}_k^{|m|}}{du} \bar{P}_l^{|m|}(u) \right\} du \]

Here \( g_k = \left[ \frac{d}{dr} r f_k(r) \right] |_{r=R} \) It is also assumed that \( f_k(R) = 1 \), which can always be done. Using the orthogonality property of Legendre polynomials and the known recurrence relations for expressions of the form \( uP_n^m \) and \((u^2 - 1) dP_n^m / du \) ((Abramowitz and Stegun, 1964)), and introducing the designation for the left-hand side of the equality (21) \( b_l^m \) (these quantities will be referred to as the weight coefficients) we can bring the expression (21) into the form:

\[ b_l^m = a_{-2}^m c_{l-2}^m + a_{-1}^m c_{l-1}^m + a_0^m c_l^m + a_1^m c_{l+1}^m + a_2^m c_{l+2}^m, \]

where

\[ a_{-2}^m = \frac{w_{l+2}^m}{w_{l-2}^m} \frac{(l-m-1)(l+m)(l-2)}{(2l-3)(2l+1)} (l - 1 - g_{l-2}), \]

\[ a_{-1}^m = -i\alpha R m \frac{w_{l+1}^m}{w_{l-1}^m} \frac{(l-m)}{(2l-1)}, \]

\[ a_0^m = \frac{w_{l+1}^m}{w_{l-1}^m} \frac{(l-m+1)^2}{2l+1} (l + 1 - g_l) + \frac{w_{l+1}^m}{w_{l+1}^m} \frac{(l+m)^2}{(2l+1)} (l + 1)(l + g_l), \]

\[ a_1^m = -i\alpha R m \frac{w_{l+2}^m}{w_{l+1}^m} \frac{(l+m+1)}{(2l+3)}, \]

\[ a_2^m = \frac{w_{l+3}^m}{w_{l+1}^m} \frac{(l-m+1)(l+m+2)(l+3)}{(2l+5)(2l+1)} (l + 2 + g_{l+2}). \]

In view of the property of (14), when determining the coefficients \( c_l^m \), it is sufficient to use equations (22) for positive \( m \) only. The equalities of (22) can be broken up into \( m \) independent systems of equations, each of which can be represented in the matrix form (for a finite expansion to \( L \))

\[ A_{ij}^m C_j^m = B_i^m, \]

where
\begin{align*}
A_{ij}^m &= \delta_{j+2}^i a_{m,2j+m-2} + \delta_{j+1}^i a_{m,2j+m-2} + \delta_{j}^i a_{0,2j+m-2} \\
&+ \delta_{j-1}^i a_{m,2j+m-2} + \delta_{j-2}^i a_{m,2j+m-2}, \\
C_j^m &= c_{2j+m-2}^m, B_i^m = b_{2i+m-2}^m, 1 \leq i, j \leq (L - m)/2 + 1,
\delta_j^i &= \begin{cases} 
1, & i = j \\
0, & i \neq j
\end{cases}
\end{align*}

Hence, by determining the coefficients \(c\) from equation (24), we solve the problem formulated. For each value of \(m\), the matrices \(\hat{A}_m\) in (24) have a pentadiagonal form. Since the selected values of \(\alpha\) are always small and the behavior of the elements on even diagonals is similar to the behavior of the respective elements of the matrix derived in \(((\text{Rudenko, 1999}))\), one might expect that calculations of this vector equation will also present no problems.

3. Conclusions

The formulas obtained in this study for determining the expansion coefficients of the magnetostatic solution can be used in further investigation in calculations using real daily magnetograms. As shown in \(((\text{Rudenko, 1999}))\), this technique can only work well with sufficiently high resolution data, such as magnetograms from Kitt Peak Observatory. Technically, the realization of the numerical code of the solution outlined above is not too different in its complexity from the code of calculating an analogous problem for the potential-field model. One might expect good results from extrapolating solar plasma parameters throughout the visible disk at the time of observation, to the heights of about the solar radius \(((\text{Rudenko, 1999}))\).

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