Inhomogeneous quantum antiferromagnetism on periodic lattices

A. Jagannathan
Laboratoire de Physique des Solides, CNRS-UMR 8502, Université Paris-Sud, 91405 Orsay, France

R. Moessner
Laboratoire de Physique Théorique de l’Ecole Normale Supérieure, CNRS-UMR 8549, Paris, France

S. Wessel
Institut für Theoretische Physik III, Universität Stuttgart, 70550 Stuttgart, Germany

(Dated: March 23, 2022)

We study quantum antiferromagnets on two-dimensional bipartite lattices. We focus on local variations in the properties of the ordered phase which arise due to the presence of inequivalent sites or bonds in the lattice structure, using linear spin wave theory and quantum Monte Carlo methods. Our primary finding is that sites with a high coordination tend to have a low ordered moment, at odds with the simple intuition of high coordination favoring more robust Néel ordering. The lattices considered are the dice lattice, which is dual to the kagome, the CaVO lattice, an Archimedean lattice with two inequivalent bonds, and the crown lattice, a tiling of squares and rhombi with a greater variety of local environments. We present results for the onsite magnetizations and local bond expectation values for the spin-1/2 Heisenberg model on these lattices, and discuss the exactly soluble model of a Heisenberg star, which provides a simple analytical framework for understanding our lattice studies.

PACS numbers: PACS numbers: 75.10.Jm, 71.23.Ft, 71.27.+a

I. INTRODUCTION

The long-wavelength universal properties of antiferromagnets on bipartite lattices are well established [1]. In this paper, we ask to what extent local properties – the dependence of the ordered moment or bond strength on local coordination – display systematic behavior. To do this, we consider quantum Heisenberg antiferromagnets on various two-dimensional lattices. These lattices are bipartite, so that classical antiferromagnets form collinear Néel states at zero temperature; for quantum spins, this ordering persists, albeit with reduction of the order parameter due to quantum fluctuations. A review of results obtained for uniform Archimedean lattices (in which all sites – but not necessarily all bonds – are equivalent) is given in Ref. [2]. Generally, the size of the order parameter depends on dimensionality – Heisenberg Néel order is absent in $d = 1$ – and coordination, $z$: it is bigger for the square ($z = 4$) than for the honeycomb ($z = 3$) lattice.

Upon increasing the coordination $z$ the exchange ‘mean-field’ acting on a spin at a given lattice site grows. The larger this mean-field, the less effective quantum fluctuations are in reducing the staggered alignment of the spin at that site. For inhomogeneous lattices with sites of different coordination, this suggests that the local staggered magnetization should increase with sites coordination $z$. Analogously, one would expect average bond energies to decrease with $z$, as fluctuations transverse to the ordered moment become increasingly difficult to coordinate between a growing number of neighbors.

However, recent investigations on a two-dimensional, quasiperiodic system, the octagonal tiling, with six different local coordinations, $z = 3, 4, 5, 6, 7, 8$ lead to the result that the local staggered magnetization does not follow this expectation; instead, both the local staggered magnetization $m_i$ and the averaged local bond strength of all bonds connected to a given site, $\overline{\varepsilon}_i$, tend to decrease with increasing $z$ (cf. Sec. II for a formal definition of both $m_i$ and $\overline{\varepsilon}_i$). This property has been observed in quantum Monte Carlo simulations [3], in linear spin wave theory [4] and in a real space renormalization group approach based on the self-similar structure of the quasiperiod lattice [5]. The absence of translation invariance, however, has important consequences for the spectrum and eigenmodes of the spin Hamiltonian [6], and it is the invariance under scale transformation that determines the real space properties of the antiferromagnet [7, 8]. In order to assess the relevance of the absence of translation symmetry on the behavior of the local order parameter distribution, here we consider various inhomogeneous but periodic lattices.

In particular, we calculate the variations in $m_i$ and $\overline{\varepsilon}_i$ for lattices with small unit cells that contain inequivalent bonds and/or sites. We consider the dice lattice, the CaVO lattice and a lattice that to our knowledge has not been studied previously, the crown lattice, which is built from squares and rhombi. This structure shows similar local environments to those found in the octagonal tiling, but unlike the latter possesses translational invariance.

Our results on regular lattices are in qualitative agreement with those of the octagonal tiling. To provide a simple analytical explanation of the $z$-dependence of the local quantities, we appeal to the exactly soluble case of the Heisenberg star [6]. In particular, this resolves
the conundrum of a decreasing $m_i(z)$ by showing that, while the average moment $m_i^{av}$ increases with the average coordination $z^{av}$ of the lattice, an order-by-disorder type mechanism concentrates the reduction of the ordered moment to highly coordinated sites.

II. MODEL AND NUMERICAL METHODS

In the following we consider the nearest neighbor Heisenberg antiferromagnet, described by the Hamiltonian

$$H = J \sum_{\langle i,j \rangle} \vec{S}_i \cdot \vec{S}_j,$$

(1)

where the sum is taken over all bonds in the lattice. $J > 0$ is assumed to be independent of the local environments, and set equal to 1 in the rest of the paper. We employ a combination of linear spin wave theory \[3\] and quantum Monte Carlo simulations in our analysis. Details of the linear spin wave approach will be presented in the following section for each specific lattice.

The quantum Monte Carlo simulations were performed using the stochastic series expansion method \[8\] for finite lattices. In particular, for the dice lattice we considered systems with $N_x = 192, 432$ and 768 sites, for the CaVO lattice those with 64, 144, 265 and 748 sites, and for the crown lattice, systems with 112, 252, 448 and 700 sites. In each case the temperature was taken sufficiently low to obtain ground state expectation values. For a given site $i$ of the finite size lattice, the local value of the staggered magnetization is given by

$$m_i = \frac{1}{N_i} \sum_{j=1}^{N_i} \frac{3}{2} (-1)^{i+j} \langle \vec{S}_i \cdot \vec{S}_j \rangle,$$

(2)

where the sum extends over all the $N_i$ lattice sites $j$ which are equivalent to site $i$ with respect to the antiferromagnetic unit-cell. Finally, standard finite size scaling analysis was performed to obtain the local staggered magnetization in the thermodynamic limit \[3\]. For a given bond $\langle i,j \rangle$ on the lattice, we define the corresponding bond strength as $b_{i,j} = \langle \vec{S}_i \cdot \vec{S}_j \rangle$. Similarly, the averaged local bond strength at a site $i$ is

$$b_i = \frac{1}{z} \sum_{j=1}^{z} \langle |\vec{S}_i \cdot \vec{S}_j| \rangle,$$

(3)

where $z$ extends over all the $z$ sizes that are connected to site $i$. In the following we present the results of the quantum Monte Carlo simulations as well as those from linear spin wave theory for the different lattices.

III. LOCAL PROPERTIES OF TWO-DIMENSIONAL BIPARTITE LATTICES

III.A The Dice lattice

Our simplest example is the dice (or T3) lattice, Fig. 1, which has a three-site unit cell for which Eq. (1) is easily diagonalized in linear spin wave theory. This is a system with a net magnetic moment per unit cell and therefore it has a Goldstone mode with a quadratic dispersion as in a ferromagnet. For completeness, we present an outline of the spin wave calculation for this simple case. The magnetic and the structural cell unit cells are the same and have three sites, one on sublattice A with $z = 6$, and two on sublattice B, both with $z = 3$. One introduces Holstein-Primakoff operators of three types, corresponding to each of these sites: $a_i, b_i, c_i, (i = 1, ..., N$ where $N$ is the number of unit cells) along with their adjoints, obeying the appropriate bosonic commutation relations. Introducing Fourier-transformed operators $a_{\vec{k}} = \sqrt{N} \sum e^{i\vec{k} \cdot \vec{r}} a_i, b_{\vec{k}}(c_{\vec{k}}) = \frac{1}{\sqrt{N}} \sum e^{-i\vec{k} \cdot \vec{r}} b(c)i$, the linearized Hamiltonian is $H_{lin} = -JS(S+1) N_b + JS H^{(2)}$, with $N_b = 2N$, where

$$H^{(2)} = \sum (a_{\vec{k}}^\dagger b_{\vec{k}}, c_{\vec{k}}^\dagger)
\begin{pmatrix}
6 & A & A^*
\end{pmatrix}
\begin{pmatrix}
a_{\vec{k}}
b_{\vec{k}}^\dagger
c_{\vec{k}}^\dagger
\end{pmatrix}. $$

(4)

Here, $A(k_x, k_y) = e^{-i3(k_x/2)}(e^{i3k_y/2} + 2 \cos(\sqrt{3}k_z/2))$ ($k_x, k_y$ are expressed in dimensionless units). This Hamiltonian is diagonalized by a generalization of the standard case as described in \[7\] in order to find a set of bosonic operators $\alpha_{\vec{k}}, \beta_{\vec{k}}, \gamma_{\vec{k}}$ such that

$$H^{(2)} = \sum (\alpha_{\vec{k}}^\dagger \beta_{\vec{k}}, \gamma_{\vec{k}})
\begin{pmatrix}
\omega_1(\vec{k}) & 0 & 0 \\
0 & \omega_2(\vec{k}) & 0 \\
0 & 0 & \omega_3(\vec{k})
\end{pmatrix}
\begin{pmatrix}
\alpha_{\vec{k}} \\
\beta_{\vec{k}}^\dagger \\
\gamma_{\vec{k}}^\dagger
\end{pmatrix}. $$

(5)

An analytic solution is easily found and the results for the variations of $\omega_i$ along different symmetry directions...
are shown in Fig. 2. The flat branch $\omega_3 = 3$ corresponds to an excitation that lives on a subset of sites, situated on hexagons of B-sites (as illustrated in Fig. 1). In this linearized model, each of the six spins around the central site precesses in anti-phase with its neighbors. These states arise from the local topology, which gives rise, similarly, to the “Aharonov-Bohm cages” discussed in [3] for electrons subjected to a magnetic flux on the dice lattice. The ground state energy per site is $E_0/N_s = 0.363$ within linear spin wave theory, compared to the quantum Monte Carlo result of $-0.633$.

![FIG. 2: The three dice lattice energy bands along the main directions in k-space. Special points in the hexagonal Brillouin zone are O: origin, A: vertex, B: midpoint of an edge.](image)

The local staggered magnetization of A-sites, $m_A = S - \langle a_i^\dagger a_i \rangle$, and that of the B-sites, $m_B$, can be calculated from the transformation matrices $O_k$ that take the set $a_k^\dagger, b_k^\dagger, c_k^\dagger$ into the set $\hat{a}_k^\dagger, \hat{b}_k^\dagger, \hat{c}_k^\dagger$. One finds

$$m_A = \frac{1}{2} - \frac{1}{N} \sum_k \frac{3 - \omega_2(k)}{3 + 2\omega_2(k)} \approx 0.356, \quad (6)$$

$$m_B = \frac{1}{2} - \frac{1}{N} \sum_k \frac{3 - 2\omega_2(k)}{6 + 4\omega_2(k)} \approx 0.428. \quad (7)$$

The corresponding quantum Monte Carlo results are $0.3754(3)$, and $0.4381(2)$, respectively. Note that the quantum correction to the classical B-sublattice magnetization, $\delta m_B = S - m_B$ is half that of the A-sublattice correction $\delta m_A = S - m_A$. The local magnetization on the dice lattice thus follows the trend seen on the octagonal tiling, being smaller on sites with higher coordination number. As for the bond strengths, there is only one type of nearest neighbor bond in this system, $b = |(S_A S_B)| = 0.316$ in linear spin wave theory (quantum Monte Carlo result: 0.3189(1)).

### III.B The CaVO lattice

The second lattice considered here has four sites per unit cell, and twice that in the magnetic unit cell. This is the T11 or CaVO lattice (after the calcium vanadium oxide compound whose spins lie on the topological equivalent of this structure) discussed previously (c.f. discussion in Ref. [2]). We introduce a number from 1 through 4 for the four sites of a structural unit cell. The magnetic unit cell is doubled, with $x$ and $y$ axes rotated by 45° with respect to the original axes as shown in Fig. 3. We thus introduce the boson destruction operators $a_{i-4}$ ($b_{i-4}$) corresponding to sub-lattice A (B). The magnetic moment per magnetic unit cell is zero. The transformation using bosonic operators, linearization, and Fourier transformation for the sets of $a_i$ and $b_i$ are carried out similarly as for the dice lattice. For a system of $N$ magnetic unit cells, one finds

$$H_{lin} = -J S(S+1)N_b + JS H^{(2)},$$

where $H_1 = z_i \delta_{ij}$, $H_2$ is the adjoint of $H_2$ with

$$H_2 = \begin{pmatrix} 1 & z_y^* & z_x^* & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & z_x & z_y & 1 \end{pmatrix},$$

$$H_1 = z_i \delta_{ij}.$$ 

$z_i = \exp^{ik}$. The basis vectors are $c^T = (a_1, a_2, b_2, b_4, b_4, b_4)$. Numerical diagonalization of this Hamiltonian leads to $\alpha_k = O_k \alpha_k$, after discretizing the Brillouin zone. We obtain four branches $\omega_i$ ($i = 1, \ldots, 4$) shown in Fig. 4. There are dispersionless directions, $k_x = \pm k_y$ for excitations living on strips, such as the one shown in Fig. 4 of energy $E = 2\sqrt{2}$. The participating sites occur with larger (smaller) amplitudes as indicated by filled (open) circles.

The ground state energy per site for this system within linear spin wave theory, obtained by a discrete sum over the Brillouin zone, is $E_0/N_s = -0.5376$, in agreement with the value of Ueda et al. [10]. Ueda et al. also show that in this system, the Néel state corrected by spin waves has an energy very close to the energy of a plaquette resonating valence bond (PRVB) state. The ground state energy obtained in linear spin wave theory is seen to compare less well with the quantum Monte Carlo results, $-0.55367(2)$ than in the case of the dice lattice. Similar deviations between linear spin wave theory and quantum Monte Carlo results are also obtained for the local quantities: All sites in the CaVO lattice have the same value of the staggered magnetization, which is obtained within linear spin wave theory as $m_i = S - \langle a_i^\dagger a_i \rangle = S - \langle b_i^\dagger b_i \rangle \equiv m$. Our k-space sum extrapolates to $m \approx 0.213$, close to the value, 0.212, published in [10], but significantly larger than the quantum Monte Carlo result, 0.1805(2).

The increased relevance of quantum fluctuations, which are not captured appropriately within linear spin wave theory stems from the closeness of the system to a quantum critical point [14], beyond which the system enters a
plaquette phase with dominant singlet formations on the squares in Fig. 3. This is reflected in the enhanced bond strength on the plaquettes, \( b_p = 0.40731(6) \) compared to the remaining (dimer) bonds \( b_d = 0.2926(1) \). Linear spin wave theory does not fully capture this effect, yielding \( b_p = |-S^2 + S(1 - 2m + 2Re\langle a_3^b b_1^b \rangle)| = 0.365 \), and \( b_d = |-S^2 + S(1 - 2m + 2Re\langle a_1^b b_1^b \rangle)| = 0.344 \), respectively.

FIG. 3: The CaVO lattice, showing a state localized on a strip (open circles correspond to bigger amplitudes than the filled circles).

FIG. 4: The CaVO energy bands along the main symmetry directions. Special points in the square Brillouin zone are O: origin, X: midpoint of edge, Γ: vertex.

III.C The crown lattice

This is a square lattice system with seven sites per unit cell, labeled as shown in Fig. 5 and twice this number in the antiferromagnetic unit cell. Each unit has a mirror symmetry that reduces the number of distinct environments to four. This periodic structure has most, though not all, of the local environments present in the quasiperiodic octagonal tiling, and can in fact be obtained via projection from a four dimensional cubic lattice using the method for obtaining approximants of the tiling outlined in 112. Each of the sites occurs once on each sublattice, as for the CaVO lattice, and we introduce accordingly the Bogoliubov operators \( a_i, a_i^\dagger \) for sites of sublattice A and \( b_i, b_i^\dagger \) for sites of sublattice B \((i = 1, ..., 7)\). The linearized Hamiltonian is \( H_{lin} = -JS(S + 1)N_b + JS\langle H^2 \rangle \), where \( N_b = 14N \). \( H^2 \) takes the form given in Eq. (4) in a basis \( a_i = (a_1, a_2, ..., a_7, b_1^\dagger, b_2^\dagger, ..., b_7^\dagger) \) where

\[
(H_1)_{ij} = z_i\delta_{ij}; \begin{pmatrix} z \end{pmatrix} = (6, 3, 3, 4, 4, 5),
\]

and

\[
H_2 = \begin{pmatrix}
0 & z_y z_x & z_y z_x & z_x & z_y z_x & 1 & z_y \\
1 & 0 & 0 & 0 & 0 & 1 & z_y \\
1 & 0 & 0 & z_x & 1 & 0 \\
1 & 0 & 0 & z_x & 0 & 1 \\
1 & 0 & z_y & 1 & 0 & 0 & z_y \\
1 & 1 & 1 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 1 & z_x & 0
\end{pmatrix}.
\]

The 14 × 14 system is diagonalized to find the new basis set in terms of operators \( \alpha_i, \beta_i \) as in the preceding cases. The solution was obtained by discretizing \( \vec{k} \) in the Brillouin zone of the crystal, whose unit cell is defined in a coordinate system oriented at 45° with respect to the original axes as shown in Fig. 6. The seven distinct energy levels thus obtained are plotted in Fig. 6 for some of the main directions in \( \vec{k} \)-space.

There are no localized states, however there is a dispersionless direction, \( \vec{k}_y = 0 \), for excitations located on ribbons (or strips) which are localized in the \( x \) but extended in the \( y \) direction. As shown in Fig. 6, the sites with non-zero amplitudes lie on the corners of rows of rectangles (formed by the double squares). Each rectangle has a pair of sites of \( z = 3 \), and a pair of sites of \( z = 4 \), whose amplitudes are different (corresponding to open (closed) circles). This state corresponds to the energy band \( E \approx 3.85 \) which is, as can be seen in Fig. 6 flat in the \( OX \) direction.

The ground state energy per site for this system is found within spin wave theory to be \( E_0/N_s = -0.6475 \), close to the value obtained in spin wave theory for the quasiperiodic octagonal tiling, of \( E_0/N_s = -0.646 \). The quantum Monte Carlo results are \(-0.6602(1) \) for

FIG. 5: A unit cell of the crown lattice, with sites numbered from 1 to 7.
FIG. 6: The seven crown lattice energy bands along some of the main symmetry directions. Special points in the square Brillouin zone are O: origin, X: zone boundary along $k_x$ axis, $\Gamma$: the $\pi, \pi$ vertex.

FIG. 7: States on strips (see text). Amplitudes are different on the sites with open (filled) circles.

FIG. 8: Dependence of the local staggered magnetization $m_i$ on the coordination number $z$ for the inequivalent sites of the crown lattice. Quantum Monte Carlo data (open) are compared to linear spin wave theory results (full).

FIG. 9: Dependence of the local averaged bond strength $b_i$ on the coordination number $z$ for the inequivalent sites of the crown lattice. Quantum Monte Carlo data (open circles) are compared to linear spin wave theory results (full circles).

the crown lattice and $= -0.6581(1)$ for the octagonal tiling $[13]$, respectively.

The linear spin wave theory results for the local staggered magnetization are shown in Fig. 8 for different values of the coordination number $z$ of the sites, and compared to the quantum Monte Carlo results. We find that linear spin wave theory does rather well in describing this inhomogeneous system. Apart from an overall overestimation of the antiferromagnetic order, both the spin wave theory and quantum Monte Carlo results exhibit similar qualitative features: for $z = 3$ there are two different values of the staggered magnetization—the sites labeled 2 and 4 have a larger staggered magnetization than the site labeled 3 (more is said about this below). The case of $z = 4$ is particularly interesting, since these sites are found by quantum Monte Carlo to have the smallest value of $m_i$. As a result one finds a non-monotonic dependence of the local staggered magnetizations on $z$ (comparing $z = 4$ to $z = 5$). Such a non-monotonic dependence of the local staggered magnetization on $z$ was noticed in the quantum Monte Carlo [3] and numerical spin wave calculations for the octagonal tiling $[4]$. We will return to this phenomenon in the following section.

For the crown lattice, the number of bonds per structural unit cell is 14, but only 8 are inequivalent due to the mirror symmetry. From their respective strengths, we obtain the values of the averaged local bond strength shown as a function of $z$ in Fig. 9 where the results from linear spin wave theory are compared to the quantum Monte Carlo data. We again find a difference between the two types of $z = 3$ sites. Furthermore, we observe a monotonic decrease of $b_i$ with the coordination number $z$. This indicates that correlations beyond the nearest-neighbor distance are important for the non-monotonic behavior of the local staggered magnetization $m_i$ seen in Fig. 8.

IV. HEISENBERG STARS

In this section, we present a semiclassical theory for Heisenberg stars, which reconciles our finding of a mono-
tonically decreasing \( m_i(z) \) with the intuition that increasingly highly coordinated lattices (in \( d \geq 2 \)) should approach a saturated order parameter.

A Heisenberg star, studied in detail from a different perspective in Ref. [4] consists of a central site coupled to \( \zeta \) neighbors, which are mutually disconnected (Fig. 10). We denote the destruction operators of the Holstein-Primakoff bosons for the central spin, \( \vec{s} \), with \( a \), and those for the outer spins, \( \vec{S}_i \), with \( B_i, i = 1 \ldots \zeta \). We then have

\[
H = -\zeta S^2 + S \sum_{i=1}^{\zeta} a_i^* a_i + B_i^* B_i + a B_i + a_i^* B_i. \tag{11}
\]

Next, we carry out a unitary transformation, \( U \), between the \( b_i: b_i = U_{ij} B_j \), with \( U_{1i} = 1/\sqrt{\zeta} \), so that

\[
H = -\zeta S^2 + S \left[ \zeta a_i^* a_i + b_i^* b_i + \sqrt{\zeta}(ab_1 + a_i^* b_1) + \sum_{i=2}^{\zeta} b_i^* b_i \right]. \tag{12}
\]

All modes \( b_i \) with \( i \geq 2 \) correspond to precessions of outer spins around the central spin so that their net transverse component vanishes – the analogue of the local modes for the dice lattice mentioned above.

The origin of this phenomenon can be traced to the structure of the Hamiltonian (Eq. 11). Holstein-Primakoff bosons in a bipartite antiferromagnet are created only in pairs, one each on opposite ends of each bond. The higher a site’s coordination, the more anomalous terms in the Hamiltonian act to create such boson pairs. As the number of bosons quantifies the reduction in \( m_i \), high coordination thus implies low magnetization. This is reminiscent of the order-by-disorder effect encountered in a simple hopping problem, where one finds that in the ground state, particles are more likely to be found near highly coordinated sites. At the same time, a high exchange field (corresponding to large diagonal terms in the Hamiltonian) makes such bosons costly and suppresses their overall number – which is why high-

The above order-by-disorder mechanism is operative only when differently coordinated sites are located at opposite ends of a bond. If, for the sake of argument, one were to consider an inhomogeneous system formed by patching together large pieces of uniform lattices (say, square and honeycomb), the local \( m \) in the interior of each piece would of course reflect that of the relevant parent lattice – and in this case, \( m(z) \) would increase with \( z \). In other words, as placing a boson on a site involves placing a boson on one of a neighboring sites as well, \( m(z) \) is of course not a purely local quantity. The next-best guess would be that, for sites with the same \( z \), the magnetization will be lowest for those sites whose neighbors have the lowest average coordination. This is in keeping with the result that 0.3366(5) = \( m_2 > m_3 = 0.3214(5) \) for the crown lattice and also with the bimodal splitting of the local staggered magnetization observed for the five-fold coordinated sites on to quasiperiodic octagonal tilings [2, 4]. In other words, this analysis shows that the monotonicity in \( m_i(z) \) is a trend, not a law, in that details of the mode spectrum can in fact violate it.

We close this section with two side remarks. Firstly, note that the Heisenberg star for \( \zeta > 1 \) has a nonvanishing total magnetization, \( M \), which has enabled us to obtain a finite reduction of \( m \) due to quantum fluctuations, even though it is a 0-dimensional system. For \( \zeta = 1 \), when the the total magnetization vanishes, the corrections (Eq. 13) themselves diverge. Secondly, since \( M \) is invariant under the action of the Hamiltonian, it follows that \( S - m = \zeta(S - M) \). By the same token, simply dressing the Néel state on the dice lattice leads to \( S - m_A = 2(S - m_B) \), a result which holds not only for linear spin waves but also in our quantum Monte Carlo simulations.

FIG. 10: A Heisenberg star: a central spin is coupled to \( \zeta \) mutually un-coupled neighbors.

The lowering of the magnetization is due to the anomalous terms linking \( b_1 \) with \( a \). This part of the problem can be diagonalised with a standard Bogoliubov transformation matrix

\[
O = \begin{pmatrix} \cosh\theta & \sinh\theta \\ \sinh\theta & \cosh\theta \end{pmatrix},
\]

with

\[
\tanh(2\theta) = 2\sqrt{\zeta}/(\zeta + 1). \tag{14}
\]

This yields for the magnetization of the central and outer spins, respectively,

\[
m = S - 1/(\zeta - 1)
\]

\[
M = S - 1/|\zeta(\zeta - 1)|. \tag{15}
\]

Note the following two features. Firstly, as \( \zeta \) grows, the quantum reduction to the sublattice magnetization decreases, keeping with the idea that a high exchange field implies a reduction in the importance of quantum fluctuations. Secondly, however, it is systematically the highly coordinated central site which has the lower magnetization: at large \( \zeta \), the central \( m \) is reduced by \( 1/\zeta \), whereas the outer \( M \) are each only reduced by \( 1/\zeta^2 \).
V. DISCUSSION AND CONCLUSIONS

We have presented results for local properties of bipartite inhomogeneous quantum magnets. They lead us to conclude that in periodic systems containing inequivalent sites, there is in general a trend towards decreased values of $m_i$ and $b_i$ with increasing local coordination number. This trend was also previously observed in a quasiperiodic system. The results for the crown lattice, in particular, show a qualitative similarity with the results that were earlier obtained for the self-similar quasiperiodic octagonal tiling. We have provided a simple analytical model system, the Heisenberg star, the semiclassical solution of which explains this qualitative behavior. Whereas it is well known that quantum fluctuations can lead to a range of ordering phenomena in otherwise disordered magnets [14], it is interesting to ask whether a novel disordering effect can be obtained by increasing $z$ for a sublattice, as suggested by a decreasing $m_i(z)$. However, as we have shown above, an increase of $z$ on one sublattice will generically make the ordering more robust everywhere, although not uniformly so for all sublattices. This reconciles in a simple way the observed decrease of $m_i(z)$ with increasing $z$ with the naive expectation that the ordered moment $m_i$ should grow with the coordination of the lattice.

SW acknowledges LPS, Orsay and CEA Saclay for hospitality during this collaboration. The numerical calculations were performed at HRLS Stuttgart and NIC Jülich.

[1] P. M. Chaikin and T. C. Lubensky, Principles of Condensed Matter Physics, Cambridge University Press, 1995.
[2] J. Richter, J. Schulenberg and A. Honecker in Quantum magnetism in two dimensions: from semi-classical Neel order to magnetic disorder, Lect. Notes in Physics 645 Springer-Verlag Berlin Heidelberg 2004.
[3] S. Wessel, A. Jagannathan and S. Haas, Phys. Rev. Lett. 90, 177205 (2003).
[4] S. Wessel and I. Milat, Phys. Rev. B 71, 104427 (2005).
[5] A. Jagannathan, Phys. Rev. Lett. 92, 047202 (2004).
[6] J. Richter and A. Voigt, J. Phys. A: Math. Gen. 27, 1139 (1994).
[7] R. M. White, M. Sparks and I. Ortenburger, Phys. Rev. 139, A450 (1965).
[8] A. W. Sandvik, Phys. Rev. B 59, R14157 (1999).
[9] J. Vidal, R. Mosseri and B. Douçot, Phys. Rev. Lett. 81, 5888 (1998).
[10] K. Ueda, H. Kontani, M. Sigrist and P. A. Lee, Phys. Rev. Lett. 76, 1932 (1996).
[11] M. Troyer, H. Kontani and K. Ueda, Phys. Rev. Lett. 76, 3822 (1996).
[12] M. Duneau, R. Mosseri and Ch. Oguey, J. Phys. A 22, 4549 (1989).
[13] S. Wessel, Phys. Rev. Lett. 94, 029701 (2005).
[14] For a recent discussion, see S. R. Hassan and R. Moessner, Phys. Rev. B 73, 094443 (2006) and references therein.