Lense-Thirring Precession in Plebański-Demiański spacetimes

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An exact expression of Lense-Thirring precession rate is derived for non-extremal and extremal Plebański-Demiański spacetimes. This formula is used to find the exact Lense-Thirring precession rate in various axisymmetric spacetimes, like: Kerr, Kerr-Newman, Kerr-de Sitter etc. We also show, if the Kerr parameter vanishes in Plebański-Demiański(PD) spacetime, the Lense-Thirring precession does not vanish due to the existence of NUT charge. To derive the LT precession rate in extremal Plebański-Demiański we first derive the general extremal condition for PD spacetimes. This general result could be applied to get the extremal limit in any stationary and axisymmetric spacetimes.

I. INTRODUCTION

The axisymmetric vacuum solutions of the Einstein equations are used to describe the various characteristics of different spacetimes. The most important and physical spacetime is Kerr spacetime\[^1\], describes the rotating black hole which possesses a finite angular momentum \(J\). The Kerr spacetime with a finite charge \(Q\) is expressed as Kerr-Newman black hole. Actually, these all spacetimes may include the Cosmological constant \(\Lambda\) by which term the complexity arises in calculations. Without this particular constant the spacetimes possess the two horizons, we mean, event horizon and Cauchy horizon. But, the presence of the cosmological constant leads to possess an extra horizon - the cosmological horizon. All these spacetimes can be taken as the special cases of the most general axisymmetric spacetime of Petrov type D which is first given by Plebański and Demiański (PD)\[^2\]. This spacetime contains the seven parameters - acceleration, mass \((M)\), Kerr parameter \((a, \text{angular momentum per unit mass})\), electric charge \((Q_e)\), magnetic charge \((Q_m)\) NUT parameter \((n)\) and Cosmological constant. At present, this is the most general axially symmetric vacuum solution of Einstein field equation. It is well-known to us that any axisymmetric and stationary spacetime with angular momentum (rotation) are known to exhibit an effect called Lense-Thirring (LT) precession whereby locally inertial frames are dragged along the rotating spacetime, making any test gyroscope in such spacetimes precess with a certain frequency called the LT precession frequency \[^3\]. More generally, we can say that frame-dragging effect is the property of all stationary spacetimes which may or may not be axisymmetric\[^4\]. We have also discussed this special feature in detail in our previous paper\[^5\]. In that paper, we have showed that only Kerr parameter is not responsible for the LT precession, NUT parameter is equally important to continue the frame-dragging effect \[^5\]. It has been shown by Hackmann and Lämmerzahl that LT precession vanishes (eqn.45 of \[^6\]) in PD spacetimes (with vanishing acceleration of the gravitating source), if the Kerr parameter \(a = 0\). But, it is not the actual case. As we have shown our previous paper, the same thing is also happened in the PD spacetimes with zero angular momentum \((J = a = 0)\), the LT precession does not vanish due to the presence of NUT charge \(n\) (angular momentum monopole\[^7\]).

Our aim is to derive the exact LT precession rates in non-extremal and extremal Plebański-Demiański spacetimes without invoking the weak field approximation. So, we organize the paper as follows. In section II, we review the general Lense-Thirring precession formula in stationary and axisymmetric spacetimes and derive the exact Lense-Thirring precession rate in Plebański-Demiański(PD) spacetimes with vanishing acceleration of the gravitating source and discuss the exact LT precession rates in some other stationary and axisymmetric spacetimes as the special cases of PD spacetimes. If the Kerr parameter vanishes in PD spacetimes, the frame-dragging effect does not vanish due to the existence of NUT charge. It is shown in a subsections of sec II, as a special case of LT precession in PD spacetimes. In sec. III, we derive the more general extremal condition for PD spacetimes and discuss about the exact LT precession rates in PD spacetimes and also other various extremal axisymmetric spacetimes as the special cases of PD spacetimes. A short discussion closes the paper which is in sec.IV.

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II. NON-EXTREMAL CASE

A. Plebański-Demiański (PD) spacetimes

The PD spacetime is the most general axially symmetric vacuum solution of Einstein equation, at present. The line element of six parameters (as we take acceleration vanishes) PD spacetimes can be written as (taking, $G = c = 1$)[6],

\[
\frac{ds^2}{p^2} = \frac{-\Delta}{p^2}(dt - A d\phi)^2 + \frac{\mu^2}{\Delta}dr^2 + \frac{\Xi}{p^2}d\theta^2 + \Xi \sin^2 \theta (dt - B d\phi)^2
\]  

(1)

where,

\[
p^2 = r^2 + (n - a \cos \theta)^2,
A = a \sin^2 \theta + 2n \cos \theta, B = r^2 + a^2 + n^2
\]

\[
\Delta = (r^2 + a^2 - n^2)(1 - \frac{1}{\ell^2}(r^2 + 3n^2)) - 2Mr + Q^2 + Q_m^2 - \frac{4n^2r^2}{\ell^2}
\]

\[
\Xi = 1 + \frac{a^2 \cos^2 \theta}{\ell^2} - \frac{4an \cos \theta}{\ell^2}
\]

$\ell^2 = \Lambda$ denotes the Cosmological constant divided by three, represents the Plebański-Demiański-de-Sitter (PD-dS) spacetimes and if $\ell^2$ is replaced by $-\ell^2$, it represents the PD-AdS spacetimes. So, our metric $g_{\mu \nu}$ is thus following

\[
g_{\mu \nu} = \begin{pmatrix}
-\frac{1}{p^2}(\Delta - a^2 \Xi \sin^2 \theta) & 0 & 0 & \frac{1}{p^2}(A \Delta - aB \Xi \sin^2 \theta) \\
0 & \frac{\Delta}{p^2} & 0 & 0 \\
0 & 0 & \frac{\Xi}{p^2} & 0 \\
\frac{1}{p^2}(A \Delta - aB \Xi \sin^2 \theta) & 0 & 0 & \frac{1}{p^2}(-A^2 \Delta + B^2 \Xi \sin^2 \theta)
\end{pmatrix}
\]  

(3)

In our previous paper[5], we have already discussed about the exact expression of LT precession rate[4]. This is applicable for any non-accelerating stationary spacetimes. In this paper, we are going to derive the exact LT precession rate for non-accelerating PD spacetimes and also some others axially symmetric spacetimes like this. The expression for LT precession rate in non-accelerating, stationary and axisymmetric spacetime can be written as,

\[
\Omega_{LT} = \frac{1}{2}\sqrt{-g} \left[ \left( g_{\theta \phi, r} - \frac{g_{\theta \phi}}{g_{\theta \theta}} g_{\theta \theta, r} \right) \partial_r - \left( g_{\theta \phi, \theta} - \frac{g_{\theta \phi}}{g_{\theta \theta}} g_{\theta \theta, \theta} \right) \partial_{\theta} \right]
\]  

(4)

which is same as eqn.(12) of [5]. The various metric components can be read off from above metric (3). Likewise,

\[
\sqrt{-g} = p^2 \sin \theta
\]  

(5)

Substituting the metric components into eqn.(4) we can easily get the LT precession rate in PD spacetimes. But, there is a problem in that formulation as the precession formula is in co-ordinate basis. So, we should transform the precession frequency formula from the coordinate basis to the orthonormal ‘Copernican’ basis: first note that

\[
\Omega_{LT} = \Omega^r \partial_r + \Omega^\theta \partial_{\theta}
\]

\[
\Omega_{LT}^2 = g_{rr}(\Omega^r)^2 + g_{\theta \theta}(\Omega^\theta)^2
\]  

(6)

(7)

Next, in the orthonormal ‘Copernican’ basis at rest in the rotating spacetime, with our choice of polar coordinates, $\Omega_{LT}$ can be written as

\[
\tilde{\Omega}_{LT} = \sqrt{g_{rr}} \Omega^r \hat{r} + \sqrt{g_{\theta \theta}} \Omega^\theta \hat{\theta}
\]  

(8)

where, $\hat{\theta}$ is the unit vector along the direction $\theta$. Our final result of LT precession in non-accelerating PD spacetime is then,

\[
\tilde{\Omega}_{LT}^{PD} = \frac{\sqrt{\Delta}}{p} \left[ \frac{a(\Xi \cos \theta + (2n - a \cos \theta) \Delta \sin^2 \theta)}{\Delta - a^2 \Xi \sin^2 \theta} - \frac{a \cos \theta - n}{p^2} \right] \hat{r}
\]

\[
+ \frac{\sqrt{\Xi}}{p} \frac{a \sin \theta}{\Delta - a^2 \Xi \sin^2 \theta} \left[ \frac{r - M - \frac{\Xi}{\Delta}(a^2 + 2r^2 + 6n^2)}{\Delta - a^2 \Xi \sin^2 \theta} - \frac{r}{p^2} \right] \hat{\theta}
\]

(9)

Now, from the above expression we can easily derive the LT precession rates for various axisymmetric stationary spacetimes as the special cases of PD spacetime.
B. Special cases

(a) Schwarzschild and Schwarzschild-de-Sitter spacetimes: As the Schwarzschild and Schwarzschild-de-Sitter spacetimes both are static and $a = \Lambda = Q_c = Q_m = n = 0$, the inertial frames are not dragged along it. So, we can’t see any LT effect in these spacetimes. This is very well-known feature of static spacetime.

(b) Kerr spacetimes: LT precession rate for non-extremal Kerr spacetimes is already discussed in detail in our previous paper\cite{ref5}. Setting $\Lambda = Q_c = Q_m = n = 0$ in eqn.\((\text{(9)})\) we can recover our result (eqn.(19) of \cite{ref5}) which is applicable for Kerr spacetimes.

(c) Kerr-Newman spacetime: Rotating black hole spacetimes with electric charge $Q_c$ and magnetic charge $Q_m$ is described by Kerr-Newman metric, which is quite important in General Relativity. Setting $\Lambda = n = 0$ in eqn.\((\text{(9)})\), we can easily get the LT precession in Kerr-Newman spacetime. It is thus following (taking $Q_c^2 + Q_m^2 = Q^2$),

\[
\bar{\Omega}_{LT}^{KN} = \frac{a}{\rho^3(\rho^2 - 2Mr + Q^2)} \left[ \sqrt{\Delta}(2Mr - Q^2) \cos \theta \hat{r} + (M(2r^2 - \rho^2) + rQ^2) \sin \theta \hat{\theta} \right]
\]  

(10)

In the Kerr-Newman spacetime,

\[
\Delta = r^2 - 2Mr + a^2 + Q^2 \quad \text{and} \quad \rho^2 = r^2 + a^2 \cos^2 \theta
\]  

(11)

From the expression \((\text{(10)})\) of LT precession in Kerr-Newman spacetime we can see that at the polar plane, the LT precession vanishes for the orbit $r = \frac{Q^2}{2M}$, though the spacetime is rotating \((a \neq 0)\). So, if a gyroscope rotates in the polar orbit of radius $r = \frac{Q^2}{2M}$ in this spacetime, the gyroscope does not experience any frame-dragging effect. So, if any experiment is performed in future by which we can’t see any LT precession in that spacetime, it may be happened that the specified spacetime is Kerr-Newman black hole and the gyroscope is rotating at the polar orbit whose radius is $r = \frac{Q^2}{2M}$. This is a very interesting feature of the Kerr-Newman geometry. Though the spacetime is rotating with the angular momentum $J$, the nearby frames are not dragged along it. Without this particular orbit the LT precession is continued in everywhere in this spacetimes.

(d) Kerr-de-Sitter spacetimes: Kerr-de-Sitter spacetime is more realistic, when we do not neglect the Cosmological constant parameter (though its value is very small, it may be very useful in some cases, where we need to very precise calculation). Setting $n = Q_c = Q_m = 0$, we get the following expression for Kerr-de-Sitter spacetime.

\[
\bar{\Omega}_{LT}^{KdS} = \frac{a}{\rho^3(\rho^2 - 2Mr - \frac{1}{12}(a^4 + a^2r^2 - a^4 \sin^2 \theta \cos^2 \theta))} \left[ \sqrt{\Delta}(2Mr + \frac{1}{\ell^2} \rho^4) \cos \theta \hat{r} + \sqrt{\Xi}[M(2r^2 - \rho^2) + \frac{r}{\ell^2}(a^4 + a^2r^2 - a^4 \sin^2 \theta \cos^2 \theta - \rho^2(a^2 + 2r^2))] \sin \theta \hat{\theta} \right]
\]  

(12)

Where,

\[
\Delta = (r^2 + a^2) \left(1 - \frac{r^2}{\ell^2}\right) - 2Mr \quad \text{and} \quad \Xi = 1 + \frac{a^2}{\ell^2} \cos^2 \theta
\]  

(13)

C. Non-vanishing Lense-Thirring precession in ‘zero angular momentum’ Plebański-Demiański spacetimes

This subsection can be regarded as a special case of non-accelerating Plebański-Demiański spacetime in where we take that PD spacetime is not rotating, we mean the Kerr parameter $a = 0$. In a very recent paper, Hackmann and Lämmerzahl shows that Lense-Thirring effect vanishes (eqn. no (45) of \cite{ref6}) due to the vanishing Kerr parameter. But, we can see easily from the eqn.\((\text{(9)})\) that if $a$ vanishes in PD spacetime, the LT precession rate will be

\[
\bar{\Omega}_{LT}^{PD}|_{a=0} = \frac{n \sqrt{\Delta}|_{a=0}}{\rho^3} \hat{r}
\]  

(14)
where,
\[
\Delta|_{a=0} = (r^2 - n^2) \left(1 - \frac{1}{\ell^2} (r^2 + 3n^2)\right) - 2Mr + Q_c^2 + Q_m^2 - \frac{4n^2r^2}{\ell^2}
\]
and,
\[
p^2 = r^2 + n^2
\]
(15)

So, it is not necessary to vanish the LT precession with vanishing Kerr parameter. The above expression reveals that NUT charge \( n \) is responsible for the LT precession in ‘zero angular momentum’ PD spacetimes. Here, \( M \) represents the “gravitoelectric mass” or ‘mass’ and \( n \) represents the “gravitomagnetic mass” or ‘dual’ (or ‘magnetic’) mass[8] of this spacetime. It is obvious that the spacetime is not invariant under time reversal \( t \to -t \), signifying that it must have a sort of ‘rotational sense’ which is analogous to a magnetic monopole in electrodynamics. One is thus led to the conclusion that the source of the nonvanishing LT precession is this “rotational sense” arising from a nonvanishing NUT charge. Without the NUT charge, the spacetime is clearly hypersurface orthogonal and frame-dragging effects vanish, as already mentioned in detail in our previous paper (sec. II of[5]). This ‘dual’ mass has been investigated in detail in ref. [9, 10] and it is also referred as an ‘angular momentum monopole’ [7] in Taub-NUT spacetime. This implies that the inertial frame dragging seen here in such a spacetime can be identified as a gravitomagnetic effect.

In that particular paper[6], Hackmann and Lämmerzahl investigates the timelike geodesic equations in the PD spacetimes. The orbital plane precession frequency \( (\Omega_\phi - \Omega_\theta) \) is computed, following the earlier work of ref. [11–13], and a vanishing result ensues. This result is then interpreted as a signature for a null LT precession in the ‘zero angular momentum’ PD spacetime.

We would like to say that what we have focused on in this paper is quite different from the ‘orbital plane precession’ considered in [6]. Using a ‘Copernican’ frame, we calculate the precession of a gyroscope which is moving in an arbitrary integral curve (not necessarily geodesic). Within this frame, an untorqued gyro in a stationary but not static spacetime held fixed by a support force applied to its center of mass, undergoes LT precession. Since the Copernican frame does not rotate (by construction) relative to the inertial frames at asymptotic infinity (“fixed stars”), the observed precession rate in the Copernican frame also gives the precession rate of the gyro relative to the fixed stars. It is thus, more an intrinsic property of the classical spin of the spacetime (as an untorqued gyro must necessarily possess), in the sense of a dual mass, rather than an orbital plane precession effect for timelike geodesics in a Taub-NUT spacetime.

In our case, we consider the gyroscope equation [4] in an arbitrary integral curve
\[
\nabla_u S = < S, a > u
\]
where, \( a = \nabla_u u \) is the acceleration, \( u \) is the four velocity and \( S \) indicates the spacelike classical spin four vector \( S^a = (0, \vec{S}) \) of the gyroscope. For geodesics \( a = 0 \Rightarrow \nabla_u S = 0 \).

In contrast, Hackmann and Lämmerzahl [6] consider the behaviour of massive test particles with vanishing spin \( S = 0 \), and compute the orbital plane precession rate for such particles, obtaining a vanishing result. We are thus led to conclude that because two different situations are being considered, there is no inconsistency between our results and theirs.

We note that the detailed analyses on LT precession in Kerr-Taub-NUT, Taub-NUT[14–16] and massless Taub-NUT spacetimes has been done in[5].

III. EXTREMEAL CASE

A. Extremal Plebański-Demiański Spacetime

In this section, we would like to describe the LT precession in extremal Plebański-Demiański spacetime, whose non-extremal case is already described in the previous section. To get the extremal limit in PD spacetimes we should first determine the radius of the horizons \( r_h \) which can be determined by setting \( \Delta|_{r=r_h} = 0 \). We can make a comparison of coefficients in

\[
\Delta = -\frac{1}{\ell^2} r^4 - \left(1 - \frac{a^2}{\ell^2} - \frac{6n^2}{\ell^2}\right) r^2 - 2Mr + \left[(a^2 - n^2) \left(1 - \frac{3n^2}{\ell^2}\right) + Q_c^2 + Q_m^2\right]
\]

\[
= -\frac{1}{\ell^2} [r^4 + (a^2 + 6n^2 - \ell^2)r^2 + 2Mr^2r + b]
\]

\[
= -\frac{1}{\ell^2} \Pi_{r=1}^4 (r - r_{hi})
\]
(17)
where,

\[ b = (a^2 - n^2)(3n^2 - \ell^2) - \ell^2(Q_c^2 + Q_m^2) \]  

(18)

and \( r_{hi}(i = 1, 2, 3, 4) \) denotes the zeros of \( \Delta \). From this comparison we can conclude that for the PD-AdS (when \( \Lambda \) is negative) black hole, there are two separated positive horizons at most, and \( \Delta \) is positive outside the outer horizon of the PD black hole. In the same way, we can conclude for the PD-dS (when \( \Lambda \) is positive) black hole that there are three separated positive horizons at most, and \( \Delta \) is negative outside the outer horizon of the black hole. Both of the above cases, the when two horizons of the PD black hole coincide, the black hole is extremal [17].

If we consider the extremal PD black hole, we have to make a comparison of coefficients in

\[ \Delta = (r - x)^2(a_2r^2 + a_1r + a_0) \]

(19)

\[ = \frac{1}{r^2}[r^4 + (a^2 + 6n^2 - \ell^2)r^2 + 2Mr\ell^2 + b] \]

with \( a_0, a_1, a_2 \) being real[6]. From this comparison we can get the following for PD (“AdS”) spacetime,

\[ \frac{b_A}{x^2} - 3x^2 = a^2 + 6n^2 + \ell^2 \]

(20)

\[ x^3 - \frac{b_A}{x} = -M\ell^2 \]

(21)

where, \( b_A \) represents the value of \( b \) at PD (“AdS”) spacetimes.

\[ b_A = (a^2 - n^2)(3n^2 + \ell^2) + \ell^2(Q_c^2 + Q_m^2) \]

(22)

Solving equation (20) for \( x \), we get

\[ x = \sqrt[3]{\frac{1}{6} \left[ -(\ell^2 + a^2 + 6n^2) + \sqrt{(\ell^2 + a^2 + 6n^2)^2 + 12b_A} \right]} \]

(23)

Similarly, we can obtain for PD (“dS”) black hole ,

\[ \frac{b}{x^2} - 3x^2 = a^2 + 6n^2 - \ell^2 \]

(24)

\[ x^3 - \frac{b}{x} = M\ell^2 \]

(25)

In these equation \( x \) is positive and related to the coincided horizon of the extremal PD (for “dS”) black hole.

Solving equation (24) for \( x \), we get

\[ x_+ = \sqrt[3]{\frac{1}{6} \left[ \ell^2 - a^2 - 6n^2 + \sqrt{(\ell^2 - a^2 - 6n^2)^2 + 12b} \right]} \]

(26)

\[ x_- = \sqrt[3]{\frac{1}{6} \left[ \ell^2 - a^2 - 6n^2 - \sqrt{(\ell^2 - a^2 - 6n^2)^2 + 12b} \right]} \]

(27)

where, \( x_+ \) and \( x_- \) indicate the outer horizon and inner horizon, respectively. This can be seen by calculating

\[ \frac{d^2\Delta}{dr^2} \bigg|_{r=x_+} = -\frac{2}{\ell^2} \sqrt{(\ell^2 - a^2 - 6n^2)^2 + 12b} \]

(28)

and

\[ \frac{d^2\Delta}{dr^2} \bigg|_{r=x_-} = \frac{2}{\ell^2} \sqrt{(\ell^2 - a^2 - 6n^2)^2 + 12b} \]

(29)

For the PD (“dS”) black hole, on the outer extremal horizon, \( \frac{d\Delta}{dr} = 0 \) and \( \frac{d^2\Delta}{dr^2} < 0 \) and on the inner extremal horizon \( \frac{d^2\Delta}{dr^2} > 0 \). Now, we can solve \( M \) and \( a \) from the two equations (24, 25).

\[ M = \frac{x \left[ x^4 + 2x^2(3n^2 - \ell^2) + (3n^2 - \ell^2)(7n^2 - \ell^2) + \ell^2(Q_c^2 + Q_m^2) \right]}{\ell^2(\ell^2 + x^2 - 3n^2)} \]

(30)

\[ a_c^2 = \frac{3x^4 + (6n^2 - \ell^2)x^2 + n^2(3n^2 - \ell^2) + \ell^2(Q_c^2 + Q_m^2)}{3n^2 - \ell^2 - x^2} \]

(31)
From the above values of $a_e^2$ and $M$, we get

$$a_e^2 = -M\ell^2 \frac{[3x^4 + (6n^2 - \ell^2)x^2 + n^2(3n^2 - \ell^2) + \ell^2(Q_e^2 + Q_m^2)]}{x[x^4 + 2x^2(3n^2 - \ell^2) + (3n^2 - \ell^2)(7n^2 - \ell^2) + \ell^2(Q_e^2 + Q_m^2)]} \quad (32)$$

The ranges of $x$ and $a_e$ are determined from the following expressions

$$x^2 < \left(\frac{\ell^2 + \ell \sqrt{\ell^2 - 12Q_e^2}}{6} - n^2\right) \quad (33)$$

$$0 < a_e^2 < \left[7\ell^2 - 24n^2\right) - \sqrt{(7\ell^2 - 24n^2)^2 - (\ell^4 - 12\ell^2Q_e^2 + 24Q_m^2)}\right] \quad (34)$$

Due to the presence of Cosmological constant, there exist four roots of $x$ in eqn. (31). When Cosmological constant $\frac{1}{r} \to 0$, eqn.(30) and eqn.(31) reduces to

$$x = M \quad (35)$$

and,

$$a_e^2 = x^2 + n^2 - Q_e^2 - Q_m^2 \quad (36)$$

or,

$$a_e^2 = M^2 + n^2 - Q_e^2 - Q_m^2 \quad (37)$$

respectively.

Now, the line element of extremal PD spacetimes can be written as,

$$ds^2 = -\frac{\Delta_e}{p_e^2}(dt - A_e d\phi)^2 + \frac{p_e^2}{\Delta_e} dr^2 + \frac{p_e^2}{\Delta_e} d\theta^2 + \frac{\Xi_e}{p_e} \sin^2 \theta (a_e dt - B_e d\phi)^2 \quad (38)$$

and, the final LT precession rate in extremal PD spacetime is,

$$\mathcal{G}_{LT}^{ePD} = \sqrt{\Delta_e} \left[ \frac{a_e(\Xi_e \cos \theta + (2n - a_e \cos \theta) \frac{p_e}{\ell} \sin^2 \theta)}{\Delta_e - a_e^2 \Xi_e \sin^2 \theta} - \frac{a_e \cos \theta - n}{p_e^2} \right] \frac{\hat{r}}{r} + \sqrt{\Xi_e} \frac{a_e \sin \theta}{p_e} \left[ \frac{r - M - \frac{p_e^2}{\ell^2}(a_e^2 + 2r^2 + 6n^2)}{\Delta_e - a_e^2 \Xi_e \sin^2 \theta} - \frac{r}{p_e^2} \right] \frac{\hat{\theta}}{r} \quad (39)$$

where,

$$\Delta_e = -\frac{1}{\ell^2} (r - x)^2 \left(r^2 + 2rx + \frac{b}{x^2}\right), \quad \Xi_e = 1 + \frac{a_e^2}{\ell^2} \cos^2 \theta - \frac{4a_e n}{\ell^2} \cos \theta$$

$$p_e = r^2 + (n - a_e \cos \theta)^2, \quad A_e = a_e \sin^2 \theta + 2n \cos \theta, \quad B_e = r^2 + a_e^2 + n^2 \quad (40)$$

and, the value of $a_e$ is determined from the eqn.(31) and the range of $x$ and $a_e$ (‘$e’$ stands for the extremal case) are determined from the eqn. (33) and (34), respectively. It could be noted that, for the extremal PD (“dS”), spacetimes, there are upper limiting values for angular momentum and extremal horizon of the black hole. Substituting all the above mentioned values and ranges in eqn.(39), we get the exact LT precession rate in extremal PD spacetimes.

B. Extremal Kerr Spacetime

Substituting $a_e = M$ in equation (19) of [5]) or substituting $a = M$ and $\Lambda = Q_e = Q_m = n = 0$ in equation (39) we can easily get the LT precession rate in extremal Kerr spacetime.

$$\mathcal{G}_{LT}^{eK} = \frac{M^2[2r(r - M) \cos \theta \hat{r} + (r^2 - M^2 \cos^2 \theta) \sin \theta \hat{\theta}]}{(r^2 + M^2 \cos^2 \theta)^2 \hat{r}^2 (r^2 - 2Mr + M^2 \cos^2 \theta)} \quad (41)$$
The above result is also coming from the eqn. (39), the general LT precession rate for extremal PD spacetime.

Case I: On the polar region i.e. \( \theta = 0 \) the \( \Omega_{LT} \) becomes

\[
\Omega_{LT}^K = \frac{2M^2r}{(r^2 + M^2)^{3/2}(r - M)}
\]

(42)

Case II: On the equator i.e. \( \theta = \pi/2 \), the \( \Omega_{LT} \) becomes

\[
\Omega_{LT}^K = \frac{M^2}{r^2(r - 2M)}
\]

(43)

It could be easily seen that \( \Omega_{LT} \) is diverges at \( r = M \). Since \( r = M \) is the only direct ISCO in extremal Kerr geometry which coincides with the principal null geodesic generator of the horizon [18] and it is also the radius of the horizon which is a null surface. The general LT precession formula is derived only considering that the observer is rest in a timelike Killing vector field. We have not incorporated the LT effect for any null geodesics. So, our formula is valid for \( r > M \) (outside the horizon).

C. Extremal Kerr-Newman Spacetime

Substituting \( a^2 = M^2 - (Q^2 + Q^2_m) = M^2 - Q^2 \) and \( \Lambda = n = 0 \) in equation (10), we can obtain the following LT precession rate for extremal KN blackhole

\[
\tilde{\Omega}_{LT}^{KN} = \frac{\sqrt{M^2 - Q^2}}{\rho^3(\rho^2 - 2Mr + Q^2)} \left[ (r - M)(2Mr - Q^2)\cos \theta \hat{r} + (M(2r^2 - \rho^2) + rQ^2)\sin \theta \hat{\theta} \right]
\]

(44)

where

\[
\rho^2 = r^2 + (M^2 - Q^2)\cos^2 \theta
\]

(45)

From the above expression (eqn. 44), we can make a similar comment again, like the extremal Kerr-Newman black hole that the gyroscope which is rotating in a polar orbit of radius \( r = \frac{Q^2}{2M} \), cannot experience any frame-dragging effect. Apparently, it seems that this argument is also true for the gyroscope which is rotating at \( r = M \) orbit. But, this is not true. Because, this is the horizon of extremal Kerr-Newman spacetime. So, \( r = M \) is a null surface. The general formula which we have considered in our whole paper, is valid only in timelike spacetimes (outside the horizon), not any null or spacelike regions.

D. Extremal Kerr-de Sitter Spacetime

Extremal Kerr-de Sitter spacetime is interesting because it involves the Cosmological constant. Setting \( n = Q_e = Q_m = 0 \) in eqn. (9), we can find the following expression of LT precession at extremal Kerr-de Sitter spacetimes

\[
\tilde{\Omega}_{LT}^{KdS} = \frac{a_e}{\rho^3(\rho^2 - 2Mr - \frac{1}{\ell^2}(a_e^2 + a_e^2r^2 - a_e^2\sin^2 \theta \cos^2 \theta))} \left[ \sqrt{\Delta_e}(2Mr + \frac{\rho^4}{\ell^2}) \cos \theta \hat{r} + \sqrt{\Xi_e}[M(2r^2 - \rho^2) + \frac{r}{\ell^2}\{a_e^2 + a_e^2r^2 - a_e^2\sin^2 \theta \cos^2 \theta - \rho^2(a_e^2 + 2r^2)\}]\sin \theta \hat{\theta} \right]
\]

(46)

where,

\[
\rho^2 = r^2 + a_e^2 \cos^2 \theta, \quad \Xi_e = 1 + \frac{a_e^2}{\ell^2} \cos^2 \theta
\]

(47)

and

\[
\Delta_e = -\frac{1}{\ell^2}(r - x)^2 \left( r^2 + 2xr - \frac{a_e^2\ell^2}{x^2} \right)
\]

Using equations (31, 30), we obtain the value for \( a \) and \( M \) are

\[
a_e^2 = \frac{(\ell^2 - 3x^2)x^2}{(\ell^2 + x^2)}
\]

\[
M = \frac{x(\ell^2 - x^2)^2}{\ell^2(\ell^2 + x^2)}
\]

(48)
where the horizons are at

\[ x_+ = \sqrt{\frac{1}{6} \left[ \ell^2 - a^2 + \sqrt{(\ell^2 - a^2)^2 - 12a^2\ell^2} \right]} \]

\[ x_- = \sqrt{\frac{1}{6} \left[ \ell^2 - a^2 - \sqrt{(\ell^2 - a^2)^2 - 12a^2\ell^2} \right]} \]  

(49)

where, \( x_+ \) and \( x_- \) indicate the outer horizon and inner horizon. The range of \( a_+^2 \) and \( x \) are the following.

\[ 0 < a_+^2 < (7 - 4\sqrt{3})\ell^2 \quad \text{and} \quad x^2 < \ell^2/3 \]  

(50)

which is already discussed in [17]. Substituting all the above values in eqn.(46) and taking the ranges of \( a_+ \) and \( x \), we get the exact LT precession rate in extremal Kerr-dS spacetimes.

E. Extremal Kerr-Taub-NUT spacetime

To derive the extremal limit in Kerr-Taub-NUT spacetime we set,

\[ \Delta = r^2 - 2Mr + a^2 - n^2 = 0 \]  

(51)

Solving for \( r \), we get two horizons which are located at \( r_+ = M \pm \sqrt{M^2 + n^2 - a^2} \). So, we get the extremal condition \( (r_+ = r_-) \) for Kerr-Taub-NUT spacetimes is \( a_+^2 = M^2 + n^2 \). If we set \( Q_e = Q_m = \Lambda = 0 \) and \( M^2 + n^2 = a_+^2 \) in eqn. (9), we get the following exact LT precession rate at extremal Kerr-Taub-NUT spacetime.

\[
\Omega_{LT}^{eKTN} = \frac{(r - M)}{p} \left[ \frac{\sqrt{M^2 + n^2} \cos \theta}{p^2 - 2Mr - n^2} - \frac{\sqrt{M^2 + n^2} \cos \theta - n}{p^2} \right] \dot{\varphi} \\
+ \frac{\sqrt{M^2 + n^2} \sin \theta}{p} \left[ \frac{r - M}{p^2 - 2Mr - n^2} - \frac{r - n}{p^2} \right] \dot{\theta} 
\]  

(52)

Where \( p^2 = r^2 + (n \mp \sqrt{M^2 + n^2} \cos \theta)^2 \).

IV. DISCUSSIONS

In this work we have explicitly derived the Lense Thirring precession frequencies for extremal and non-extremal Plebański-Demiański(PD) spacetime. The PD family of solutions are the solutions of Einstein field equations which contain a number of well known black hole solutions. Among them including Schwarzschild, Schwarzschild-de-Sitter, Kerr, Kerr-ads, Kerr-Newman, Kerr-Taub-NUT etc. We observe that LT precession frequency strongly depends upon the different parameters like mass \( M \), spin \( a \), Cosmological constant \( \Lambda \), NUT charge \( n \), electric charge \( Q_e \) and magnetic charge \( Q_m \). An interesting point that LT precession occurs solely due to the “dual mass”. This “dual mass” is equivalent to angular momentum monopole \( (n) \) of NUT spacetime. For our completeness we have also deduced the LT precession for extremal PD spacetime and also for others axisymmetric PD-like spacetimes. To get the LT precession rate in extremal Kerr-Taub-NUT-de-Sitter spacetime, the basic procedure is same as the PD spacetimes with the additional requirement is \( Q_e^2 + Q_m^2 = 0 \) in equations (30) and (31) and the range of \( x^2 < (\frac{\ell^2}{3} - n^2) \) and \( a_+^2 \) has a range of \( 0 < a_+^2 < (7\ell^2 - 24n^2) - 4\sqrt{3(\ell^2 - 3n^2)(\ell^2 - 4n^2)} \). As there is no any valid extremal condition at NUT spacetime (with \( M \) or without \( M \)) we could not get any real LT precession rate due to frame-dragging effect. Since, the direct ISCO coincides with the principal null geodesic generator[18] in extremal Kerr spacetimes and Kerr-Newman spacetimes, we are unable to discuss the LT precession at that particular geodesic. So this formula is not valid for the domain of \( r \leq M \) for the extremal Kerr and Kerr-Newman spacetimes. Here, our formula is valid only for \( r > M \). The general formula for LT precession in stationary spacetime is valid only outside the horizon, as the observer is in timelike Killing vector field. The formula is not valid on the horizon and inside the horizon. We will discuss about this problem in near future.

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