Meson mass spectrum using the Cayley-Dickson algebra

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Abstract
From an injective map between the mass of the meson 16-plet and the eigenvalue of the right multiplication in the Cayley-Dickson algebra, we obtain the mass formula as $2m_\mathcal{D} = m_\eta + m_\eta'$, which is in excellent agreement with experiment.

1 Introduction
In the standard quark model, mesons are bound states of a quark and anti-quark. Due to the difference between quark masses, the SU($N$) flavor symmetry is broken. Concerning the meson mass formula, the Gell-Mann–Okubo relation \cite{1,2} is well known. For $N = 3$, the Gell-Mann–Okubo formula for the pseudoscalar mesons is given by $3m_\eta \simeq 4m_K - m_\pi$, assuming no singlet-octet mixing, which, however, cannot be neglected due to the SU(3) symmetry breaking. It is difficult to estimate the singlet-octet mixing from the improvement of this formula (or other meson mass formula based on the dual resonance model \cite{3} in connection with string theory) by taking account of the quark-quark interaction mediated by the gluon exchange \cite{4}. It may remain difficult to obtain the singlet-octet mixing, even if the SU(4) meson 16-plet and SU(5) 25-plet are taken into account \cite{5,6}. For the meson 16-plet, for example, the model such that the SU(4) symmetry is broken but the SU(3) symmetry is exact, leads to a simple mass formula of the form $12m_\mathcal{D}^2 = 5m_{\bar{c}c}^2 + 7m_{\bar{s}s}^2$ \cite{5}, where $m_\mathcal{D}$ represents the average mass of $c\bar{u}, c\bar{d},$ and $c\bar{s}$; and $m_0$ stands for the average of the meson octet masses; due to the assumption of the exact SU(3) symmetry, the mass relation between the physical states $\eta$ and $\eta'$ cannot be obtained.

The aim of this paper is obtain a simple but exact mass formula by relating the meson $n$-plet (denoted by $n$ for brevity) to a vector space where some appropriate algebra is given. The basic technique was developed in Ref. \cite{7}, where the meson octet can be identified with a certain algebra. Here by algebra $\mathcal{A}$, we mean a hypercomplex system, that is, a vector space $V$ with a given multiplication $L_x : V \to V$ (with $x \in V$). Denote by $m_i$ (for $i = 1, \ldots, n$) the mass of
the meson $n$-plet, and $\lambda_i$ (for $i = 1, \ldots, n$) the eigenvalue of $L_x$ with $x \in V/V_0$, where $V_0 \subset V$ represents the subspace of $V$ such that $\dim(V/V_0) = n$ (so that $\dim V$ should be a multiple of $n$). The reason of dividing $V$ by $V_0$ is that we want to obtain a bijective map between the meson mass and the eigenvalues of $L_x$ (in an actual case, $L_x$ is replaced by its relative). Suppose that there is a bijective map $\phi : M \rightarrow \Lambda$, where $M = \bigcup_i \{m_i\}$ and $\Lambda = \bigcup_i \{\lambda_i\}$. Then it is found that there is a bijective map $\tilde{\phi} : /D2 \rightarrow V/V_0$, due to the existence of a bijective maps $f : /D2 \rightarrow M$ and $g : \Lambda \rightarrow V/V_0$, that is, $\tilde{\phi} = g \circ \phi \circ f$ [see Eq. (1)]. Conversely, if the algebra $\mathcal{A}$ is given (this means that the maps $\tilde{\phi}$ and $g$ are given), then the eigenvalue of (the relative of) $L_x$ is related to the meson mass through the relation of $\phi \circ f = g^{-1} \circ \tilde{\phi}$ (= given).

As an example, consider the meson octet $\mathfrak{B}_K$, which may be composed as

$$\mathfrak{B}_K = \eta \oplus \pi^0 \oplus (\pi^+, \pi^-) \oplus (K^+, K^-; K^0, \bar{K}^0).$$

Although there may be many algebras such that the map $\tilde{\phi} : /D2 \rightarrow V/V_0$ is injective, we have already chosen in Ref. [7] the Cayley-Dickson algebra with $V = \mathbb{R}^{64}$ and $V_0 = \mathbb{R}^8$.

The reason of adopting the Cayley-Dickson algebra is as follows. Originally, the flavor $\text{SU}(N)$ symmetry breaking is responsible for the meson mass difference. Recall that $\text{SU}(N)$ is one of the compact simple Lie groups, which are categorized into two types: classical and exceptional ones. The classical type represents the isometry transformation in the vector space over the real number field $\mathbb{R}$ ($= \mathcal{A}_0$), the complex number field $\mathbb{C}$ ($= \mathcal{A}_1$), and the Hamilton number (quaternion) field $\mathbb{H}$ ($= \mathcal{A}_2$). The isometry over the Cayley number (octonion) field $\mathbb{O}$ ($= \mathcal{A}_3$) is not a simple Lie group of a classical type, due to the lack of the associativity. However, due to the alternativity of $\mathbb{O}$, $\mathbb{O}$ is still related to the simple Lie group of an exceptional type as $G_2 \cong \text{Aut}(\mathbb{O})$, $F_4 \cong \text{Isom}(\mathbb{OP}^2)$, $E_6 \cong \text{Isom}(\mathbb{C} \otimes \mathbb{O})\mathbb{P}^2$, $E_7 \cong \text{Isom}(\mathbb{H} \otimes \mathbb{O})\mathbb{P}^2$, $E_8 \cong \text{Isom}(\mathbb{O} \otimes \mathbb{O})\mathbb{P}^2$, where $\text{Aut}$, $\text{Isom}$, and $\mathbb{K}\mathbb{P}^2$ represent the automorphism, isometry, and the projective plane over a (skew) field $\mathbb{K}$, respectively. Recall also that $\mathcal{A}_n$ is the $2^n$-dimensional Cayley-Dickson algebra over $\mathbb{R}$. The above mathematical facts imply that if part of the symmetry under simple Lie group is not broken, a subgroup of the isometry group over $\mathcal{A}_n$ (with $n \leq 3$) may still survive. Noticing that $\mathcal{A}_n$ is a subalgebra of $\mathcal{A}_{n'}$ (for $n < n'$), we may naturally choose $\mathcal{A}_n$ (with $n \geq 4$) as an algebra by which the Lie group symmetry breaking in the sense of Eq. (1) (actually, $\mathcal{A}_6 = \mathbb{R}^{64}$ is chosen for the meson octet $\mathfrak{B}$).

Furthermore, it should be mentioned why we deal with the meson $n$-plet, rather than the baryon $n$-plet. The reason is simple. Consider, for example, the
baryon octet, which is composed of $\Lambda, \Sigma^0, \Sigma^+, \Sigma^-, p, n, \Xi^0, \Xi^-$.

Difference from the meson octet, the baryon octet cannot be identified with $A_0/R^8$ because the baryon octet cannot be decomposed into $1 \oplus 1 \oplus 2 \oplus 4$, where the masses in 4 are doubly degenerate.

The outline of this paper is as follows. In Sec. 2, we briefly review the basic property of the Cayley-Dickson algebra, where the eigenvalues of (the relative of) $L_x$ are given. In Sec. 3, we apply to the pseudoscalar meson 16-plet the analogous map $\tilde{\phi}$ that can be applied to the meson octet, to finally obtain the mass formula $2m_D = m_{\eta c} + m_{\eta'}$, which is well verified by experiment. In Sec. 4, we make further application to the vector meson 16-plet and to the meson 25-plet. In Sec. 5, we give summary.

2 Cayley-Dickson algebra

In this section, we briefly review the basic property of the Cayley-Dickson algebra. The Cayley-Dickson algebra $A_n$ over the real number $R$ represents the algebra structure on $R^{2^n}$, which is given inductively. Let $x = (x_1, x_2), y = (y_1, y_2)$ be in $R^{2^n} = R^{2^{n-1}} \times R^{2^{n-1}}$. Then the multiplication $xy$ is given by

$$xy = (x_1y_1 - \bar{y}_2x_2, y_2x_1 + x_2\bar{y}_1), \quad \text{with } \bar{x} = (\bar{x}_1, -x_2).$$

For $n \leq 3$, $A_n$ corresponds to

$$A_0 = R, \quad A_1 = C, \quad A_2 = H, \quad A_3 = O,$$

which and only which are normed division algebras. The basic property of $A_n$ is summarized in Table 1. The Euclidean norm and inner product are given by $\|x\|^2 = \bar{x}x = \bar{x}x$ and $\langle x, y \rangle = \frac{1}{2}(\bar{y}x + \bar{x}y)$, respectively. Due to the flexibility of $A_n$, one obtains for all $x, y, z \in A_n$ the identities

$$\langle x, yz \rangle = \langle x, \bar{z}y \rangle = \langle \bar{y}x, z \rangle. \quad (3)$$

For further analysis of the algebra structure of $A_n$, it is convenient to define the left and right multiplications $L_x, R_x : A_n \rightarrow A_n$ by

$$L_x(y) = xy, \quad R_x(y) = yx$$

Table 1: Basic property of $A_n$ [9], where the commutator and associator are given by $[x, y] = xy - yx, [x, y, z] = (xy)z - x(yz)$, respectively.

| $n$ | Property  | Identity |
|-----|-----------|----------|
| 0   | Self conjugate | $x = \bar{x}$ |
| 0, 1| Commutative  | $[x, y] = 0$ |
| 0, 1, 2| Associative | $[x, y, z] = 0$ |
| 0, 1, 2, 3| Alternative | $[x, x, y] = 0$ |
| All | Flexible    | $[x, y, x] = 0$ |
for \( x \in \mathbb{A}_n \), fixed. If one tries to decompose the vector space \( \mathbb{R}^{2^n} \) into the eigenspaces of \( L_x \), one encounters an obstacle; the eigenvalue of \( L_x \) is not necessarily given by a real number so that the eigenspace cannot be given by a real number. To remove the obstacle, we deal with the eigenvalue of \( N_x := L_x \) instead of the eigenvalue of \( L_x \) itself. Since \( N_x \) is (real) symmetric, that is, \( \langle y, N_x(z) \rangle = \langle y, \bar{x}(xz) \rangle = \langle x, y \rangle = \langle xz, xy \rangle = \langle z, N_x(y) \rangle \) by Eq. (3), all the eigenvalues of \( N_x \) turn out to be real numbers, so that the vector space \( \mathbb{R}^{2^n} \) can be decomposed into the eigenspaces of \( N_x \).

Denote by \( S_n \) the set of the eigenvalues of \( N_x - \|x\|^2 \) for \( x = (x_1, x_2) \in \mathbb{A}_n = \mathbb{A}_{n-1} \times \mathbb{A}_{n-1} \). For \( n \leq 4 \), it is relatively easy to calculate \( S_n \) as

\[
S_1 = \{0, \ldots, 0\} \quad \text{(for } i = 0, 1, 2, 3\text{)},
\]

\[
S_4 = \{0, \ldots, 0; \pm \Delta^2, \ldots, \pm \Delta^2\},
\]

where \( \Delta = 2|x_1 \times x_2| := 2 \sqrt{\|x_1\|^2 \|x_2\|^2 - (x_1 \cdot x_2)^2} \), with the bold face letter \( x \) representing the imaginary part of \( x \), that is, \( x = x - \text{Re}(x) = \frac{1}{2}(x - \bar{x}) \). For \( n \geq 5 \), the calculation of \( S_n \) turns out to be so difficult that it may not be useful for a physical application. Before proceeding further, it should be noticed that \( S_n \) satisfies the following inclusion relation:

\[
\begin{aligned}
S_{n-1} &\subseteq S_n \quad \text{(for } n \leq 4\text{)}, \\
S_{n-1} &\not\subseteq S_n \quad \text{(for } n \geq 5\text{)}.
\end{aligned}
\]

The inclusion relation of \( S_{n-1} \subseteq S_n \) implies that half of the eigenspaces of \( N_x \) for \( x \in \mathbb{A}_n \) are given by the eigenspaces of \( N_x \) with \( x \) restricted to \( \mathbb{A}_{n-1} \), which is the subspace of \( \mathbb{A}_n \). In the present study, we extend to the map applied to the meson octet to the meson 16-plet, so that the inclusion relation of \( S_{n-1} \subseteq S_n \) should be satisfied. Otherwise, the original map \( \phi : \mathcal{B} \to \mathbb{A}_6/R^8 \), in itself, would be violated when applied to the meson 16-plet.

Now we obtain the necessary and sufficient condition for \( S_{n-1} \subseteq S_n \). Recall that for \( n \geq 5 \), \( \mathbb{A}_n \) is not given by a pair of alternative elements in \( \mathbb{A}_{n-1} \) (which is referred to as alternative entries, for short), where \( S_{n-1} \subseteq S_n \) does not hold in general. Thus it is necessary for \( S_{n-1} \subseteq S_n \) that an element in \( \mathbb{A}_n \) is given by alternative entries. Here, the alternative element is given by the following definition \([9]\):

**Definition 1** \( a \) in \( \mathbb{A}_n \) is an alternative element, if \( [a, a, x] = 0 \) holds for all \( x \) in \( \mathbb{A}_n \).

Recall that if all the elements in \( \mathbb{A}_n \) are alternative, then we simply call \( \mathbb{A}_n \) alternative. Fortunately, it is sufficient for \( S_{n-1} \subseteq S_n \) that an element in \( \mathbb{A}_n \) is given by alternative entries:

**Proposition 2** If an element in \( \mathbb{A}_n \) is given by a pair of alternative elements in \( \mathbb{A}_{n-1} \), then it follows that \( S_{n-1} \subseteq S_n \).
The proof, however, is somewhat complicated, and is referred to our previous work [7].

Once an element $x \in \mathcal{A}_n$ (for $n \geq 3$) is given by alternative entries, the eigen-polynomial for $N_x$ turns out to be an even function with quadruple degeneracy. In this case, $S_n$ can be written as

$$S_n = \bigcup_{i=1}^{4} \left( \tilde{S}_n \cup (-\tilde{S}_n) \right) \quad \text{(for } n \geq 3 \text{)}$$

where the element in the set $\tilde{S}_n$ represents the quadruply degenerated non-negative eigenvalues of $S_n$. Under an appropriate parameterization, $\tilde{S}_n$ is given by [7]

$$\tilde{S}_3 = \{0\},$$

$$\tilde{S}_4 \setminus \tilde{S}_3 = \{\Delta^2\},$$

$$\tilde{S}_5 \setminus \tilde{S}_4 = \{\Delta^2_+, \Delta^2_-\},$$

$$\tilde{S}_6 \setminus \tilde{S}_5 = \{\Delta^2_+, \Delta^2_+, \Delta^2_-, \Delta^2_-, \Delta^2_+, \Delta^2_-\},$$

$$\tilde{S}_7 \setminus \tilde{S}_6 = \{\Delta^2_+, \Delta^2_+, \Delta^2_+, \Delta^2_+, \Delta^2_-, \Delta^2_-\},$$

(4)

and so on, where $\Delta_{\pm \cdots \pm}^{k \text{ times}} = \Delta \cdot \cos(\pm \theta_k \pm \theta_{k-1} \cdots \pm \theta_1)$. The parameters $\theta_i$ $(i = 1, 2, \ldots, k)$ are introduced by the requirement from the alternative entries. Thus for $n \leq 4$, there is no such parameter in $S_n$, because all the elements in $\mathcal{A}_n$ are given by alternative entries.

3 Mass formula

In this section, we obtain a simple mass formula as $2m_{D_s} = m_{\eta_c} + m_{\eta'}$ by extending the injective map $\tilde{\phi}$ to the meson 16-plet. Comparing Eq. (2) and $\tilde{S}_6$ in Eq. (4), we readily find that there is an injective map $\phi \circ f : \mathcal{B} \to \tilde{S}_6$, so that there is a one-to-one correspondence between the meson octet $\mathcal{B}$ and $\mathcal{A}_6$ as

$$\tilde{\phi} : \mathcal{B} \to \mathcal{A}_6 / \mathbb{R}^8. \quad (5)$$

The reason of dividing $\mathcal{A}_6$ by $\mathbb{R}^8$ is due to the inclusion map $i$

$$i : \tilde{S}_n \hookrightarrow S_n.$$

To make the correspondence concrete, let $\theta : \mathbb{R} \to \mathbb{R}$ be defined as $\theta = j \circ \phi \circ f$ with $j : \mathbb{R} \to \mathbb{R}$ by $j(x) = \arccos \sqrt{x/\Delta^2}$. Then the correspondence between the meson octet $\mathcal{B}$ and the element in $\tilde{S}_6$ is summarized as in the left-hand column of Table[2] from which it is found that the parameter $\theta_1$ represents the difference between $u$ and $d$ quarks, and the parameter $\theta_2$ the difference between $s$ and $u$ (or $d$) quarks. At the present stage, we have no mass relation unless the map
\( \phi : M \to \tilde{S}_6 \) is specified.

If we take account of the meson 16-plet, which is constructed from the SU(4) flavor system, it is expected that an analogous map to Eq. (5) should hold: 
\[ \phi : 16 \to A_7 / R^8. \]
If so, we have an injective map \( \phi \circ f : 16 \setminus \emptyset \to \tilde{S}_7 \setminus \tilde{S}_6 \), where \( 16 \setminus \emptyset \) is given by (see Fig. 1)

\[ 16 \setminus \emptyset = (D^0, \bar{D}^0) \oplus (D^+, D^-) \oplus (D_s^+, D_s^-) \oplus \eta_c \oplus \eta'. \quad (6) \]

At first glance, there seems to be no such injective map \( \phi \circ f \). This is because while the elements in \( \tilde{S}_7 \setminus \tilde{S}_6 \) are doubly degenerate, the masses in \( 16 \setminus \emptyset \) are not necessarily so; the mass of \( \eta_c \) is not equal to that of \( \eta' \). However, a slight modification of the basis of the meson field of \( \eta_c \) and \( \eta' \) will lead to a desirable result. Notice that among the mesons in \( 16 \setminus \emptyset \), \( \eta_c \) and \( \eta' \) are the (only) mesons that have the same quantum numbers: zero electric charge, zero isospin, zero strangeness, and zero charm. Thus, it is physically possible to consider a superposed state of \( \eta_c \) and \( \eta' \). To make the two (orthogonal) superposed mesons have the same mass, \( \eta_c \oplus \eta' \) in Eq. (6) should be transformed to a “doublet” as

\[ \eta_c \oplus \eta' \longrightarrow (\eta_+, \eta_-), \]

**Table 2:** Correspondence between the meson 16-plet and the elements in \( \tilde{S}_7 \), where \( \theta \) is a parameter representing an element in \( \tilde{S}_n \) such that \( x = \Delta^2 \cos^2 \theta \) for \( x \in \tilde{S}_n \).

| \( \emptyset \setminus \emptyset \) | \( \emptyset \) | \( \emptyset \setminus \emptyset \) | \( \emptyset \) |
|-----------------|-------------|-----------------|-------------|
| \( \pi^0 \)     | 0           | \( D^0, \bar{D}^0 \) | \( \pm (\theta_3 - \theta_2 - \theta_1) \) |
| \( \pi^\pm \)   | \( \pm \theta_1 \) | \( D^\pm \)       | \( \pm (\theta_3 - \theta_2 + \theta_1) \) |
| \( K^\pm \)     | \( \pm (\theta_2 - \theta_1) \) | \( D_s^\pm \)      | \( \pm (\theta_3 + \theta_2 - \theta_1) \) |
| \( K^0, \bar{K}^0 \) | \( \pm (\theta_2 + \theta_1) \) | \( 1/\sqrt{2} (\eta_c \oplus \eta') \) | \( \pm (\theta_3 + \theta_2 + \theta_1) \) |
where \( \eta_\pm = \frac{1}{\sqrt{2}}(\eta_c \pm \eta') \). In this case, the \( \theta \)-assignment for \( \mathbb{1} \oplus \mathbb{8} \) is summarized in the right-hand column of Table 2.

To obtain a mass relation, it should be recalled that the parameter \( \theta_1 \) represents the mass difference between the \( u \) and \( d \) quarks. Considering the empirical relation of \( m_u \approx m_d \), we find that \( \theta_1 \approx 0 \) as long as the map \( \theta : \eta \to \mathbb{R} \) is continuous. In this case, we obtain \( m_{D_s} \approx m_{\eta_\pm} \) from \( \theta(D_s) \approx \theta(\eta_\pm) \), that is

\[
2m_{D_s} \approx m_{\eta_c} + m_{\eta'}.
\]

(7)

The recent experimental value of the meson mass (see Table 3) leads to

\[
\frac{m_{\eta_c} + m_{\eta'}}{2m_{D_s}} = 1.00033 \pm 0.00035,
\]

which indicates the validity of the relation Eq. (7). In evaluating the standard deviation, we have assumed that \( m_{\eta_c}, m_{\eta'}, \) and \( m_{D_s} \) are independent variables.

Table 3: Recent experimental value of the meson mass [10].

| Meson | Mass (MeV) |
|-------|------------|
| \( \eta_c \) | 2980.5 \pm 1.2 |
| \( \eta' \) | 957.78 \pm 0.06 |
| \( D_s \) | 1968.49 \pm 0.34 |

At the end of this section, we discuss the usefulness of Eq. (7) as an \( \eta' \)-including mass formula. In the standard quark model, however, it is not so easy a task to relate \( m_{\eta'} \) to other meson masses, due to several reasons. One is that \( \eta' \) is not a pure SU(3) singlet \( \eta_1 \), but a mixture with one of the SU(3) octet, \( \eta_8 \), through the relation

\[
\begin{pmatrix}
\eta' \\
\eta
\end{pmatrix} =
\begin{pmatrix}
\cos \vartheta & \sin \vartheta \\
-\sin \vartheta & \cos \vartheta
\end{pmatrix}
\begin{pmatrix}
\eta_1 \\
\eta_8
\end{pmatrix},
\]

(8)

where \( \eta_1 = \frac{1}{\sqrt{3}}(u\bar{u} + d\bar{d} + s\bar{s}) \) and \( \eta_8 = \frac{1}{\sqrt{6}}(u\bar{u} + d\bar{d} - 2s\bar{s}) \). To eliminate the mixing angle \( \vartheta \) from the theory, one of the orthodox methods is to introduce the pion and kaon masses, with the result known as the Schwinger relation [11]. However, the Schwinger relation, which holds under the assumption of the “ideal mixing” \( \eta' = s\bar{s} \) and \( \eta = \frac{1}{\sqrt{2}}(u\bar{u} + d\bar{d}) \), is not satisfactory for the pseudoscalar mesons, due to the large deviation from the ideal mixing. For the pseudoscalar mesons, the interaction between the different flavor (such as \( u\bar{u} \leftrightarrow s\bar{s} \)) cannot be neglected, so that the mass caused by this interaction is comparable to the constituent quark mass [12]. Taking these things into account, we find it somewhat marvelous that \( m_{\eta'} \) satisfies so simple a relation like Eq. (7).
4 Further application

In this section, we apply an analogous map $\tilde{\phi}$ to the vector meson 16-plet and to the meson 25-plet. However, it is found that further application to the vector 16-plet causes a delicate problem and that the application to the 25-plet is not viable as follows.

First, we deal with the case of vector meson 16-plet, where $\rho, K^*, \phi, \omega, J/\psi, D^*, D_s^*$ take the place of $\pi, K, \eta, \eta', \eta_c, D, D_s$, respectively. Suppose that the map $\phi$ can be applied to the vector meson 16-plet, the relation of Eq. (7) might be replaced by $2m_{D_s^*} \approx m_{J/\psi} + m_\omega$, which, however, is not satisfactory compared to Eq. (7). The failure of the relation of $2m_{D_s^*} \approx m_{J/\psi} + m_\omega$ can be interpreted as follows. In deriving Eq. (7), it should be recalled that we assume that the injective map $\theta : n \to \mathbb{R}$ is continuous. This implies that $\cos \theta$ should decrease monotonically with respect to the meson mass, that is, $\cos \theta(\varphi) \geq \cos \theta(\varphi') \iff m_\varphi \leq m_{\varphi'}$ for $\varphi, \varphi' \in n$ (see Table 2 where $m_{\omega^*}$ is the smallest). Thus the condition of $m_{\rho^0} < m_{\rho^\pm}$ is necessary for the relation of $2m_{D_s^*} \approx m_{J/\psi} + m_\omega$. Different from the pseudoscalar meson, it is quite a delicate problem to determine the sign of $\Delta m_{\rho} (\equiv m_{\rho^0} - m_{\rho^\pm})$. The Particle Data Group gives the values of $\Delta m_{\rho} = -0.7 \pm 0.8 \text{MeV}$ [10], while some theoretical considerations indicate that $-0.4 \text{MeV} < \Delta m_{\rho} < 0.7 \text{MeV}$ [13], $\Delta m_{\rho} = -0.02 \pm 0.02 \text{MeV}$ [14], $\Delta m_{\rho} = 0.62 \text{MeV}$ [15], and $\Delta m_{\rho} \approx 1 \text{MeV}$ [16]. As long as $\Delta m_{\rho}$ is positive, it is not necessary to hold a relation of $2m_{D_s^*} \approx m_{J/\psi} + m_\omega$ in the sense above. Conversely, the failure of the relation of $2m_{D_s^*} \approx m_{J/\psi} + m_\omega$ suggests the relation of $m_{\rho^0} > m_{\rho^\pm}$.

Finally, we deal with the meson 25-plet, where $b$ quark is taken into account. In this case, we cannot single out an enlarged algebra $\mathbb{A}_8/R^8$ due to $\dim(\mathbb{A}_8/R^8) = 32 (> 25)$. Thus, we can only make a decomposition as $25 = 16 \oplus \mathbb{E}_6 \oplus 1_b$, where $\mathbb{E}_6$ and $1_b (= b \bar{b})$ represent the $b$-quark related octet and singlet, respectively. Although the $\mathbb{E}_6$ can be identified with $(\mathbb{A}_7 \setminus \mathbb{A}_6)/R^8$ due to the injective map $\phi \circ f : \mathbb{E}_6 \to \mathbb{S}_7 \setminus \bar{\mathbb{S}}_6$ [this should be contrasted with the injective map $\phi \circ f : \mathbb{E} \to \bar{\mathbb{S}}_6$ for the original meson octet $\mathbb{E}$ in Eq. (2)], no new mass relation is obtainable as in the case of the original $\mathbb{E}$.

5 Summary

So far, we have obtained the mass formula Eq. (7) by identifying the pseudoscalar meson 16-plet with $\mathbb{A}_7/R^8$ through the injective map $\phi : 16 \to \mathbb{A}_7/R^8$. The point is that $\eta_c$ and $\eta'$ can be mixed to form a “doublet” $(\eta_c, \eta)$ so as to have the same mass. This mixture is possible because $\eta_c$ and $\eta'$ have the same quantum numbers as charge, isospin, strangeness, and charm. The resultant mass formula is well verified by experiment. The application of an analogous map $\tilde{\phi}$ to the vector meson 16-plet brings about quite a delicate problem in connection with the sign of $m_{\rho^0} - m_{\rho^\pm}$. Further application to the meson 25-plet is not viable due to the lack of an enlarged algebra.
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