Connectivity of a Family of Bilateral Agreement Random Graphs

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Abstract

Bilateral agreement based random undirected graphs were introduced and analyzed by La and Kabkab [9]. The construction of the graph with n vertices in this model uses a (random) preference order on other n − 1 vertices and each vertex only prefers the top k other vertices using its own preference order; in general, k can be a function of n. An edge is constructed in the ensuing graph if and only if both vertices of a potential edge prefer each other. This random graph is a generalization of the random kth-nearest neighbor graphs of Cooper and Frieze [3] that only consider unilateral preferences of the vertices. The authors in [10] studied the emergence of a giant component and its size in this new random graph family in the limit of n going to infinity when k is finite. Connectivity properties of this random graph family have not yet been completely analyzed: for example, in their original paper [9], La and Kabkab conjectured that for k(t) = t log n, with high probability connectivity happens when t > 1 and the graph is disconnected for t < 1. We prove this conjecture. Further, we also prove an asymptote for the average degree of this graph for large n and k.

1 Introduction

Graphs are powerful mathematical tools used for modeling objects in a wide variety of topics [3], [8], [11]: from the Internet and chemistry to social science and economics. These mathematical objects have been studied using algebraic, combinatorial, and stochastic methods. Studying graphs using a stochastic framework [2], [4], [12] has a long history starting from when Gilbert introduced his random graph model in [7] in 1959, and Erdős and Rényi introduced the well-known Erdős-Rényi model in their celebrated paper [6] in 1960. These fundamental random graph models have been extensively analyzed over the last 50 years.

An Erdős-Rényi random graph with n vertices is characterized by a parameter 0 ≤ p(n) ≤ 1 indicating the probability of existence of each potential undirected edge between two vertices. Edges then appear independently based on Bernoulli coin tosses with no intrinsic preference between the edges on the part of the vertices. One question that immediately arises by specifying the parameter p(n) is whether the

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Erdős-Rényi graphs are connected when $n \to \infty$. In particular, Erdős and Rényi \cite{2}, \cite{4}, \cite{6}, \cite{12} prove that if $p(n) = t \log(n)/n$, with high probability, the graph is connected for $t > 1$ and disconnected for $t < 1$; note that the mean degree in this setting is $t \log(n)$. In fact, the refined result is that if (for large enough $n$) $p(n) = (\log(n) + \epsilon)/n$, then the graph is connected with probability $e^{-e^{-\epsilon}}$.

Cooper and Frieze in \cite{3} described a new family of random graphs based on preferences constructed using a distance measure. They considered the complete graph $K_n$ and assigned independent uniform $[0,1]$ random variables as distances to all edges. They then keep the $k(n)$ shortest edges incident to each vertex. In this model, the concept of distance induces a preference order on all edges. Indeed, the shorter an edge is the higher its preference is. Since the distances are $i.i.d.$, the preference order on all the edges will be chosen uniformly over the set of all permutations. We say a vertex proposes an edge if the edge belongs to the set of the $k(n)$ most-preferred edges incident to it. Note that in this model, an edge is kept if at least one of its endpoints proposes it, which is a unilateral perspective.

Preference relations of the sort used by Cooper and Frieze create a complicated structure on any resulting graph. For example, keeping edges based on their ranking even in the unilateral mode as done by Cooper and Frieze, induces an edge dependency which is not the case for the Erdős-Rényi model. Cooper and Frieze \cite{3} studied the connectivity of their model and proved that the graph is connected with high probability for $k \geq 3$. They also provided upper and lower bounds for the probability of connectivity as $n \to \infty$ when $k = 2$.

Just like the Cooper-Frieze unilaterally proposed random graphs, bilateral agreement random graphs introduced by La and Kabkab \cite{9}, are constructed based on the same parameter $k(n)$, which is the maximum number of other vertices each vertex wants as its neighbors. Considering vertices as agents, once again, all agents have their own preferences on the potential edges with others via a priority or preference order over other agents. Then, in contrast to both the Erdős-Rényi and the Cooper-Frieze models, an edge is drawn if and only if each end vertex has the other vertex in its $k(n)$ preferred vertices, i.e., if and only if both end vertices propose an edge, which then leads to bilateral agreement.

One can interpret the bilateral agreement model as a network formation process conducted via a game (selfish objective maximization) among agents. These kinds of network formation processes \cite{2}, \cite{3}, \cite{11} have been studied in social science and economics. Assume there are $n$ agents that are aware of their own benefits from the potential pairwise contracts with others where the benefit is assessed via an appropriate reward, cost or even distance measure. Then each agent agrees on its own $k(n)$ most profitable contracts. Finally, each contract will be concluded if and only if both parties mutually agree to it, and the graph of pairwise connections will be generated. The interpretation above highlights the importance of bilateral agreement random graphs from a game-theoretic (i.e., individual agent oriented) point of view.

A core question studied in the La and Kabkab paper \cite{9} is the determination of the choice of $k(n)$ that results in the graphs produced being connected. The authors showed the following results hold with high probability: 1) if $k(n) > C \log(n)$ for $C = 2.4625$, then the graph is connected; and 2) $k(n) < c \log(n)$ for $c = 0.5$, then the graph has isolated vertices and so is not connected. Furthermore, using extensive simulation-based experimentation, they also conjectured that the connectivity threshold was exactly $k(n) = \log(n)$, surprisingly, which is the same threshold for connectivity of the Erdős-Rényi random graphs family in terms of the mean degree; note, however, that $k(n)$ is the maximum degree in the La-Kabkab random graph family. In this
paper, we show the following:

1. In Lemma 4.6 we show that the expected number of isolated vertices increases without bound if $k \leq \log(n) - 3\sqrt{\log(n) \log \log(n)}$; see Section 4.

2. Using Lemma 4.6 and additional results, in Theorem 4.1 we prove that graphs from the La-Kabkab random graph family are not connected with high probability if $k \leq \log(n) + t'(\log \log(n))\sqrt{\log(n)}$ for $t' < -1/2$; see Section 4.

3. Theorem 5.3 then establishes that graphs from the La-Kabkab random graph family are connected with high probability if $k \geq \log(n) + t'(\log \log(n))\sqrt{\log(n)}$ for $t' > 1/2$; see Section 5.

4. Finally, in Theorem 6.2 assuming that $k$ grows to infinity with $k = o(n^{1/3})$, we show that the mean degree for graphs from the La-Kabkab random graph family asymptotically behaves like $k - \sqrt{\frac{k}{\pi} + \frac{1}{8\sqrt{k}}} + o\left(\frac{1}{\sqrt{k}}\right)$, i.e., the mean degree is asymptotically close to the maximum possible degree; see Section 6.

Whereas Theorem 4.1 and Theorem 5.3 establish the conjecture by La and Kabkab, they still do not precisely identify the connectivity threshold in comparison to existing results on the Erdős-Rényi random graph family [4], [12]. We believe this arises due to our use of analytically simpler sufficient conditions to prove both results, and hence, a different analysis could overcome this lacuna.

A different but related line of work from the connectivity question is that of Moharrami et al. [10], where they introduced a new branching process called the Erlang Weighted Tree (EWT) as the local weak limit of the bilateral agreement random graphs of La and Kabkab in the sparse regime. In [10] the parameter $k(n)$ is a (finite) random parameter $k$ for each vertex that is independently chosen and identically distributed with a distribution on $\mathbb{N}$ with finite mean. The authors then studied the degree distribution of the root vertex and its average. We will return to this specific result of [10] in Section 6 when we discuss the asymptotics of the mean degree. The authors of [10] also discuss the probability of extinction of an EWT, and conjecture its relevance to the size of the giant component [4], [6], [12] of the bilateral agreement random graphs of La and Kabkab, when there is one.

## 2 Mathematical Model

Consider the complete undirected graph $K_n$ with vertices $[n] = \{1, 2, \cdots, n\}$ and edges $\{(i, j) : 1 \leq i, j \leq n, i \neq j\}$ where we assume $(i, j) = (j, i)$. Let $k(n)$ be an integer such that $1 \leq k(n) \leq n$; henceforth, to avoid cumbersome notation, we will use $k$ instead of $k(n)$. We assign independently and identically distributed random variables called priority scores to all edges of $K_n$. For any explicit calculations, one can assume they are uniformly distributed in $[0, 1]$ or exponentially distributed with parameter 1; this holds because we will only be interested in the order statistics. We denote the score of edge $(i, j)$ by $V(i, j) = V(j, i)$. The set of all scores of the edges of vertex $i \in [n]$ is denoted by $\mathcal{V}_i = \{V(i, 1), \cdots, V(i, i-1), V(i, i+1), \cdots, V(i, n)\}$, and all the associated edges by $\mathcal{E}_i = \{(i, 1), \cdots, (i, i-1), (i, i+1), \cdots, (i, n)\}$. Without loss of generality, we can also assume that the scores are distinct, then $(R_i^j)_{1 \leq j \leq n-1}$ represents an order on $[n] \setminus \{i\} = \{1, 2, \cdots, i-1, i+1, \cdots, n\}$ based on $V(i, j)$ values.
In other words, for each \(1 \leq i \leq n\), the random vector \(R_i = (R_{i1}^1, R_{i2}^2, \ldots, R_{in}^n)\) is a permutation of \([n] \setminus \{i\}\) in which
\[
V(i, R_{i1}^1) > V(i, R_{i2}^2) > \cdots > V(i, R_{in}^n).
\]

As the scores are chosen \(i.i.d\), the distribution of the random vector \(R_i\) is uniform among all permutations of \([n] \setminus \{i\}\) as it only depends on the order-statistics. In general, the scores also impose a permutation over all the edges. This plays an important role in defining the bilateral agreement random graphs.

Let \(V(i, j)\) be realized for all edges \((i, j)\) and parameter \(k\) be fixed. Then we can determine two different classes of random graphs on vertices \([n]\). We first define the notion of agreement. If \(V(i, j)\) is among the \(k\) largest scores in \(\mathcal{V}_i\), i.e., \(j \in R_i^\leq k := \{R_{i1}^1, R_{i2}^2, \ldots, R_{ik}^k\}\) or \((i, j) \in E_i^\leq k := \{(i, R_{i1}^1), \ldots, (i, R_{ik}^k)\}\), we say that vertex \(i\) proposes edge \((i, j)\). The first model introduced by Cooper and Frieze [3] let the edge \((i, j)\) be present if at least one of \(i\) or \(j\) proposes \((i, j)\). We denote this class of random graphs by \(F(n, k)\), which we call the class of unilaterally proposed graphs. The second model described by La and Kabkab [4] requires the proposal by both vertices for an edge to appear, which we deem as a bilateral agreement. We denote this class by \(G(n, k)\) and we call it the class of bilateral agreement random graphs.

Both \(F(n, k), G(n, k)\) only depend on the order of \(V(i, j)\) not the values. This is the consequence of the scores being \(i.i.d\), which results in a uniformly drawn permutation among all permutations of edges of \(K_n\). Assigning a random variable to each edge only helps us to understand the extent of independence in this problem. However, in the analysis of any underlying probability, we will always refer to the permutation viewpoint of the construction of these random graphs.

In the following sections, we will analyze the connectivity of \(G(n, k)\) when \(k\) is around \(\log(n)\) as \(n \to \infty\). More precisely, we will prove that when \(k = t \log(n)\) with high probability this graph will be connected for \(t > 1\) and disconnected for \(t < 1\), and also attempt to determine the connectivity threshold with greater precision.

### 3 Asymptotically Equivalent Distribution

In our analysis we will use the abstraction of an infinite urn model. The next two lemmas allow us to link the probability of an event in finite permutations space to an event in the infinite urn model. In essence, we will show that the appropriate probabilities converge to a negative multinomial distribution. Before delving into details, we will consider the set of all permutations with order restrictions over some elements. The notation \(x \succ y\) states that \(x\) appears earlier than \(y\) in the permutation. We need to work on the set of all permutations with some order restrictions. For instance, whenever we say the set of permutations of \(\{1, 2, 3\}\) with order restrictions \(\{1 \succ 2, 1 \succ 3\}\), this narrows down the the set of all \(3! = 6\) permutations to \(\{(1, 2, 3), (1, 3, 2)\}\).

**Lemma 3.1.** Let \(n \in \mathbb{Z}\) grow to infinity. Assume \(s = o(n^{1/4})\) is a positive integer depending on \(n\). Moreover, \(m, m_0, m_1, \cdots, m_s\) are positive integer variables depending on \(n\) and \(M = m_0 + m_1 + \cdots + m_s\). Suppose there are \(m_i\) distinct objects of type \(i\) for \(0 \leq i \leq s\). Furthermore, \(m = o(n^{1/4})\) and for each \(i\), \(m_i = h_i n + o(n^{1/4})\) where \(h_i > 0\).

We consider a uniformly random permutation of all \(M\) objects with some order restrictions which are only within type 0. Let \(X\) denote the number of objects of type
that lie at the first \( m \) places of this permutation. The law of \( X \) for \( j < m \) is given by the following formula:

\[
p_j := \mathbb{P}\{X = j\} = \binom{m}{j} \left( \prod_{i=0}^{j-1} (m_0 - l_i) \right) \left( \prod_{i=0}^{m-j-1} (M - m_0 - l_2) \right) \frac{(M - m)!}{M!}
\]

which is asymptotically given by

\[
\binom{m}{j} \frac{h_0^j(h_1 + \cdots + h_s)^{m-j}}{(h_0 + h_1 + \cdots + h_s)^m} (1 + o(n^{-1/2}))
\]

as \( n \to \infty \).

**Proof.** To compute \( p_j \) we will count the number of permutations with \( j \) elements of type 0 at the first \( m \) place then divide it by the number all permutations. Any given order restrictions on objects of type 0 make both numerator and denominator of this fraction divided by the number of symmetries. As a result, we can assume no restriction on objects of type 0.

First part is straightforward by choosing those \( j \) places at the first \( m \) observations for objects of type 0 and \( m_0 - j \) places in the remaining part. Next, counting the number of desired arrangements for objects of type 0 and other types we arrive at the following formula for the probability

\[
p_j = \frac{\binom{m}{j} (M-m)_{m_0-j} M_0! (M - m_0)!}{M!}.
\]

This yields the first part by a few lines of algebra.

For the second part, we can write

\[
p_j = \frac{\prod_{i=0}^{j-1} (m_0 - l_i) \prod_{i=0}^{m-j-1} (M - m_0 - l_2) \times (M - m)!}{M!}
\]

\[
= \frac{\prod_{i=1}^{j} m_0 - j + i}{M - m + i} \left( \prod_{i=1}^{m-j} M - m_0 - m + j + i \right)
\]

\[
= \frac{(h_0 n + o(n^{1/4}))^j ((h_1 + \cdots + h_s) n + o(n^{1/4}))^{m-j}}{((h_0 + h_1 + \cdots + h_s) n + o(n^{1/4}))^m}
\]

\[
= \frac{h_0^j(h_1 + \cdots + h_s)^{m-j}}{(h_0 + h_1 + \cdots + h_s)^m}
\]

\[
\times \left( 1 + (1/h_0) o(n^{-3/4}) \right)^j \left( 1 + (1/(h_1 + \cdots + h_s)) o(n^{-3/4}) \right)^{m-j}
\]

\[
= \frac{h_0^j(h_1 + \cdots + h_s)^{m-j}}{(h_0 + h_1 + \cdots + h_s)^m} (1 + o(n^{-3/4}))^m
\]

\[
= \frac{h_0^j(h_1 + \cdots + h_s)^{m-j}}{(h_0 + h_1 + \cdots + h_s)^m} (1 + o(n^{-3/4})) o(n^{1/4})
\]

\[
= \frac{h_0^j(h_1 + \cdots + h_s)^{m-j}}{(h_0 + h_1 + \cdots + h_s)^m} (1 + o(n^{-1/2})).
\]

This completes the proof. \qed
Lemma 3.2. Under the same assumption of Lemma 3.1 let $i_j = o(n^{1/4}) < m_j$ be positive integers for each $j$. Suppose vector $X$ represents the number of appearance of each type before the first observation of type 0 in a uniformly chosen permutation. Then

$$
\mathbb{P}\{X = (i_1, \ldots, i_s)\} = \frac{m_0 \binom{m_1}{i_1} \cdots \binom{m_s}{i_s} (i_1 + \cdots + i_s)! (M_i - i_1 - \cdots - i_s - 1)!}{M!}
$$

which asymptotically is

$$
\frac{h_0 h_1 s \cdots h_s s (i_1 + i_2 + \cdots + i_s)! (i_1 + i_2 + \cdots + i_s)!}{(i_1 + i_2 + \cdots + i_s)!} (1 + o(n^{-1/4})).
$$

Proof. It is easy to check that the number of permutations of $M$ objects where at the first $i_1 + i_2 + \cdots + i_s$ places there are exactly $i_j$ elements of type $j$ for each $1 \leq j \leq k$, and the first observation of type 0 happens at the $(i_1 + \cdots + i_s + 1)$-th position is as follows

$$
m_0 \binom{m_1}{i_1} \cdots \binom{m_s}{i_s} (i_1 + \cdots + i_s)! (M_i - i_1 - \cdots - i_s - 1)! (M - m_0 - i_1 - \cdots - i_k)!
$$

Therefore, using $h_i \geq 1, h_0 + h_1 + \cdots + h_s \geq s$ the probability of this event is as follows:

$$
p_{i_1, \ldots, i_s} = \frac{m_0 \binom{m_1}{i_1} \cdots \binom{m_s}{i_s} (i_1 + \cdots + i_s)! (M_i - i_1 - \cdots - i_s - 1)!}{M!}
$$

$$
= \frac{m_0 \binom{m_1}{i_1} \cdots \binom{m_s}{i_s} (i_1 + \cdots + i_s)! (M - m_0 - i_1 - \cdots - i_s - 1)!}{(h_0 n + o(n^{1/4}))(h_1 n + o(n^{1/4})) \cdots (h_s n + o(n^{1/4})) (M - m_0 - i_1 - \cdots - i_s - 1)!}
$$

$$
= \left(\frac{h_0 h_1 s \cdots h_s s (i_1 + i_2 + \cdots + i_s)! (i_1 + i_2 + \cdots + i_s)!}{(h_0 + h_1 + \cdots + h_s) (i_1 + i_2 + \cdots + i_s)! (i_1 + i_2 + \cdots + i_s)!}\right) \left(1 + o(n^{-3/4})\right)^{i_1 + \cdots + i_s + 1}
$$

This finishes the proof.

Lemmas 3.1 and 3.2 demonstrate that a negative multinomial distribution is the limit for the distribution of number of other types before the first appearance of a given type. As mentioned earlier, this distributional convergence will help us greatly in our analysis. The special case required in this paper fixes all $h_i = 1$. Therefore, we state the following lemma.

Lemma 3.3. Same as Lemma 3.1, there are $m_i$ objects of type $i$ where $0 \leq i \leq s$ and $s = o(n^{1/4})$. Let $m_i = n + o(n^{1/4})$ and $m = o(n^{1/4})$ grow to infinity with an extra condition that $s = o(m)$. Consider a uniformly random permutation of all objects
with some order restrictions only among objects of type 0. We denote the number of objects of type 0 that within the first \(m\) places of this permutation by \(X\). We have the following bound for the lower tail of \(X\)

\[
\mathbb{P}\{X \leq t\} \leq \exp\left(\frac{-mt}{s+1} + t \log \left(\frac{m}{ls}\right) + t + \log(t+1)\right) (1 + o(n^{-1/2}))
\]

where \(t \leq m/s\). In particular, if \(s, t = \sqrt{k} + O(1)\) and \(m = (s - b)(k - ls - 1)\) where \(k = o(n^{1/6})\) goes to infinity and \(b, l = O(1)\), for sufficiently large \(n\) we have

\[
\mathbb{P}\{X \leq t\} \leq \exp\left(-k + \frac{1}{2}\sqrt{k}\log(k) + O(\sqrt{k})\right)
\]

**Proof.** Let \(k > s\) (\(s\) constant or non-constant). It follows from \(t \leq m/s\) that \(\frac{\binom{m}{j}}{(s+1)^m}\) is increasing for \(0 \leq j \leq t\). Using this fact and Lemma 3.1 with \(h \equiv 1\) we have

\[
P\{X \leq t\} = \sum_{j=0}^{t} \binom{m}{j} \frac{s^{m-j}}{(s+1)^m} (1 + o(n^{-1/2}))
\]

\[
\leq (t+1) \frac{m}{s} \left(1 - \frac{1}{s+1}\right)^m (1 + o(n^{-1/2}))
\]

\[
= \frac{t+1}{s} \left(1 - \frac{1}{s+1}\right)^{m/(s+1)} (1 + o(n^{-1/2}))
\]

\[
\leq \frac{t+1}{s} \left(\frac{s}{s+1}\right) \exp\left(-\frac{m}{s+1}\right) (1 + o(n^{-1/2}))
\]

\[
\leq \frac{t+1}{s} \frac{m^t}{t!} \exp\left(-\frac{m}{s+1}\right) (1 + o(n^{-1/2})).
\]

One can use Striling’s approximation \(t! > (t/e)^t\) to get

\[
P\{X \leq t\} \leq \exp\left(\frac{-mt}{s+1} + t \log \left(\frac{m}{ls}\right) + t + \log(t+1)\right) (1 + o(n^{-1/2}))
\]

(a)

Setting \(s, t = \sqrt{k} + O(1)\) and \(m = (s - b)(k - ls - 1)\), it follows that

\[
\frac{-m}{s+1} + t + \log(t+1) = -k + \frac{(bl + 1)k}{s+1} + s + t + \log(t+1) + O(1)
\]

\[
= -k + O(\sqrt{k})
\]

Additionally,

\[
t \log \left(\frac{m}{ls}\right) = \left(\sqrt{k} + O(1)\right) \log \left(\frac{k^{3/2} - (b + l)k + O(\sqrt{k})}{k + O(\sqrt{k})}\right)
\]

\[
= \left(\sqrt{k} + O(1)\right) \log \left(\sqrt{k} + O(1)\right)
\]

\[
= \frac{1}{2} \sqrt{k} \log(k) + O(\sqrt{k}.
\]

This completes the proof.
4 Disconnectivity of graph for $t < 1$

This section begins by recalling the negative multinomial distribution. In the special case that we need here, we present an urn model to interpret the distribution. Suppose we have a sequence of i.i.d. random variables with $k + 1$ possible outcomes chosen uniformly at random. We describe these outcomes as the type of objects. Therefore, there are $k + 1$ types. This sequence stops as soon as the first object of type $k + 1$ occurs in the sequence. The probability that there are $i_1$ objects of type 1, $i_2$ objects of type 2, ..., $i_k$ objects of type $k$, and finally the $n-$th observation which is of type $k + 1$, follows the negative multinomial distribution below

$$f(i_1, i_2, \cdots, i_k) = \frac{1}{(k+1)^{i_1+i_2+\cdots+i_k+1}} \frac{(i_1 + i_2 + \cdots + i_k)!}{i_1!i_2!\cdots i_k!}, \quad (2)$$

in which $i_t \geq 0$ for $1 \leq t \leq k$ and $i_1 + i_2 + \cdots + i_k = n - 1$.

We now introduce some lemmas that help with any required computation.

Lemma 4.1. With the definition (2)

$$\sum_{i_1+i_2+\cdots+i_k=n-1} f(i_1, i_2, \cdots, i_k) = \frac{1}{k+1} \left(\frac{k}{k+1}\right)^{n-1}$$

Proof. There is an algebraic proof to this lemma using the multinomial expansion formula. However, we give a simpler probabilistic way. This sum is over the probability of all possible arrangements of $n$ items in which all the first $n-1$ items are of type $1, 2, \cdots, k$ and the last one is of type $k + 1$. Hence, the probability of this event is $\left(\frac{k}{k+1}\right)^{n-1}(\frac{1}{k+1})$ due to the independent choice made in each position.

Lemma 4.2. Let $0 < \delta < 1$. Then:

$$\sum_{k^2(1+\delta)<i_1+i_2+\cdots+i_k<4(k+1)^2} f(i_1, i_2, \cdots, i_k) \geq \sum_{k^2(1+\delta)<i_1+i_2+\cdots+i_k<4(k+1)^2} f(i_1, i_2, \cdots, i_k) - e^{-4(k+1)}$$

Proof.

$$r = \sum_{k^2(1+\delta)<i_1+i_2+\cdots+i_k} f(i_1, i_2, \cdots, i_k) - \sum_{k^2(1+\delta)<i_1+i_2+\cdots+i_k<4(k+1)^2} f(i_1, i_2, \cdots, i_k)$$

$$= \sum_{4(k+1)^2 \leq i_1+i_2+\cdots+i_k} f(i_1, i_2, \cdots, i_k)$$

$$= \sum_{n=4(k+1)^2}^{\infty} \frac{1}{k+1} \left(\frac{k}{k+1}\right)^n$$

$$= \left(\frac{k}{k+1}\right)^{4(k+1)^2}$$

$$= \left(1 - \frac{1}{k+1}\right)^{(k+1)^2} 4(k+1)$$

$$\leq e^{-4(k+1)},$$

and the result follows.
Lemma 4.3. Let $0 < \delta < 1$. Then:

$$\sum_{k^2(1+\delta) < i_1 + i_2 + \cdots + i_k < 4(k+1)^2, \forall j, i_j \geq k} f(i_1, i_2, \cdots, i_k) \geq (1 - ke^{-\frac{k\delta^2}{2}}) \sum_{k^2(1+\delta) < i_1 + i_2 + \cdots + i_k < 4(k+1)^2} f(i_1, i_2, \cdots, i_k)$$

Proof. Assume $n = i_1 + \cdots + i_k + 1 > k^2(1 + \delta)$ is a fixed number. Then, at each step prior to the $n$-th observation, an outcome of type $m$ has a Bernoulli distribution with parameter $\frac{1}{k}$. Denote this random variable by $X_m^{(j)}$ where $j$ refers to the step number. For an arbitrary $1 \leq m \leq k$, we have $\mathbb{E}(X_m^{(j)}) = \frac{1}{k}$ and $\text{Var}(X_m^{(j)}) = \frac{k-1}{k^2}$. If $i_m < k$, then

$$\frac{X_m^{(1)} + \cdots + X_m^{(n-1)}}{n-1} < \frac{k}{k^2(1+\delta)} = \frac{1}{k(1+\delta)}.$$  

However, the expected value of the time average above is $\frac{1}{k}$, and $\frac{1}{k(1+\delta)} - \frac{1}{k} = -\frac{\delta}{k(1+\delta)}$, so this is a (lower) tail event. Now, we apply the Bernstein inequality:

$$\mathbb{P}\{i_m < k | i_1 + \cdots + i_k = n-1\} \leq \mathbb{P}\left\{\frac{X_m^{(1)} + \cdots + X_m^{(n-1)}}{n-1} < \frac{1}{k(1+\delta)}\right\}$$

$$\leq \exp\left(\frac{-(n-1) \times \frac{\delta^2}{k^2(1+\delta)^2}}{\frac{2(k-1)}{k} + \frac{2\delta}{3k(\delta+1)}}\right)$$

$$\leq \exp\left(\frac{-k^2(1+\delta) \times \frac{\delta^2}{k^2(1+\delta)^2}}{\frac{2(k-1)}{k} + \frac{2\delta}{3k(\delta+1)}}\right)$$

$$= \exp\left(\frac{-3k^2\delta^2}{6(k-1)(\delta+1) + 2\delta k}\right)$$

$$\leq \exp\left(\frac{-k\delta^2}{5}\right).$$

Using the union bound yields

$$\mathbb{P}\left\{\forall j : i_j \geq k \left| \sum_{m=1}^{k} i_m = n-1 \right\} \geq 1 - k \exp\left(\frac{-k\delta^2}{5}\right).$$

Therefore,

$$\sum_{k^2(1+\delta) < \sum_{i_m < 4(k+1)^2, \forall j, i_j \geq k}} f(i_1, i_2, \cdots, i_k)$$

$$= \sum_{k^2(1+\delta) < n < 4(k+1)^2} \sum_{i_m = n-1} f(i_1, i_2, \cdots, i_k)$$

$$= \sum_{k^2(1+\delta) < n < 4(k+1)^2} \left(\frac{k+1}{k}\right)^n \mathbb{P}\{\forall j : i_j \geq k, \sum_{m=1}^{k} i_m = n-1\}$$

$$= \sum_{k^2(1+\delta) < n < 4(k+1)^2} \left(\frac{k+1}{k}\right)^n \mathbb{P}\{\sum_{m=1}^{k} i_m = n-1\} \mathbb{P}\{\forall j : i_j \geq k \left| \sum_{m=1}^{k} i_m = n-1 \right\}$$
We know that

\[ \sum_{k^2(1+\delta) < n < 4(k+1)^2} \left( \frac{k+1}{k} \right)^n \mathbb{P}\{ \sum i_m = n-1 \} \left( 1 - k \exp \left( \frac{-k\delta^2}{5} \right) \right) \]

\[ = \left( 1 - k \exp \left( \frac{-k\delta^2}{5} \right) \right) \sum_{k^2(1+\delta) < \sum i_m < 4(k+1)^2} f(i_1, i_2, \ldots, i_k). \]

This completes the proof.

**Lemma 4.4.** Let \( k \) be a positive integer and \( 0 \leq \delta \leq 1 \) depends on \( k \) such that for sufficiently large \( k \) we have \( \delta \geq \sqrt{\frac{6 \log(k)}{k}} \). The following lower bound holds

\[ \sum_{k^2(1+\delta) < i_1 + i_2 + \ldots + i_k < 4(k+1)^2, \forall j, i_j \geq k} f(i_1, i_2, \ldots, i_k) \geq O(e^{-k(\delta+1)}) \]

as \( k \to \infty \).

**Proof.** Using Lemmas 4.3, 4.2, 4.1

\[ \sum_{k^2(1+\delta) < i_1 + i_2 + \ldots + i_k < 4(k+1)^2, \forall j, i_j \geq k} f(i_1, i_2, \ldots, i_k) \]

\[ \geq \left( 1 - ke^{-\frac{k\delta^2}{5}} \right) \sum_{k^2(1+\delta) < i_1 + i_2 + \ldots + i_k} f(i_1, i_2, \ldots, i_k) \]

\[ \geq \left( 1 - ke^{-\frac{k\delta^2}{5}} \right) \left( \sum_{k^2(1+\delta) < i_1 + i_2 + \ldots + i_k} f(i_1, i_2, \ldots, i_k) - e^{-4(k+1)} \right) \]

\[ = \left( 1 - ke^{-\frac{k\delta^2}{5}} \right) \left( \sum_{N=k^2(1+\delta)+1}^{\infty} \frac{1}{k+1} \frac{k}{k+1}^{N-1} - e^{-4(k+1)} \right) \]

\[ = \left( 1 - ke^{-\frac{k\delta^2}{5}} \right) \left( 1 - \frac{1}{k+1} \right)^{k^2(\delta+1)} - e^{-4(k+1)} \]

We know that \( (1 - \frac{1}{k+1})^k \approx e^{-1} \) (more precisely \( (1 - \frac{1}{k+1})^k > e^{-1} \)) and \( 1 - ke^{-\frac{k\delta^2}{5}} \geq 1 - \frac{1}{\sqrt{k}} \to 1 \) as \( k \to \infty \). Hence, the mentioned sum is asymptotically greater than:

\[ e^{-k(\delta+1)} - e^{-4(k+1)} \]

Furthermore, \( 0 < \delta < 1 \) causes \( e^{-4(k+1)} \) to vanish faster than \( e^{-k(\delta+1)} \).

We are now in a position to relate the lemmas above to the dis connectivity result. Assume \( I_j \) represents the event of vertex \( j \) being isolated. We want to show that with high probability there exists a \( j \) such that \( I_j = 1 \).

**Lemma 4.5.** For the random graph model \( \mathbb{G}(n, k) \) when \( k = o(\sqrt{n}) \) and \( k \to \infty \) we have

\[ \mathbb{P}\{ I_1 = 1 \} \geq O(e^{-k(1+\delta)}) \]

for any \( \delta = \delta(k) \) with \( \sqrt{\frac{6 \log(k)}{k}} \leq \delta(k) < 1 \).
In particular, if we have \( k \in \{ j \} \), we have \( 1 \leq j \leq n - 1 \). Then, for vertex 1 to be isolated, it is sufficient that for each \( j \in \{ 2, 3, \ldots, k + 1 \} \) there exist at least \( k \) elements of \( E_j \) (except \((1, j))\) appearing before all elements of \( E_1 \) in the permutation. Strictly speaking, if \( k \) elements of \( E_j \) appear before \((1, j + 1)\), then that would be necessary and sufficient, but we consider a stricter condition which insists that \( k \) elements of \( E_j \) appear before all elements of \( E_1 \). This way, vertex \( j \) does not agree on edges \( \{1, j\} \) for \( 2 \leq j \leq k + 1 \) and vertex 1 also does not agree upon edges \((1, j)\) for \( k + 2 \leq j \leq n \).

To make this condition even stronger, we insist on no intersections. Define for \( i \in \{1, 2, \ldots, k + 1\} \) the order over \( E \) as mentioned earlier, we can look at each configuration of independent random variables \( \{V(i, j)\}_{i,j} \) as a permutation over all edges \( E \). Without loss of generality, one can assume vertex labels are sorted according to the scores of edges connecting to vertex 1. In other words, \( V(1, 2) > V(1, 3) > \cdots > V(1, n) \), or equivalently \( R_{i}^{j} = j + 1 \) for \( 1 \leq j \leq n - 1 \). Then, in order to determine the order over \( E \), from what we explained earlier, the extra condition \( i \) guarantees that vertex 1 is isolated. Applying Lemma \( \ref{lemma:isolated} \) then allows us to conclude that

\[
P\{I_1 = 1\} \geq O(e^{-k(1+\delta)}),
\]

which completes the proof.

**Lemma 4.6.** For the random graph model \( G(n, k) \) where \( k \leq \log(n) - 3\sqrt{\log(n)\log\log(n)} \), we have

\[
\mathbb{E}\{\text{number of isolated vertices}\} \rightarrow \infty
\]

In particular, if \( k = \lceil t\log(n) \rceil \) for \( t < 1 \), the statement is true.

**Proof.** Let \( \delta = \sqrt{\frac{6\log(k)}{k}} \), according to Lemma \( \ref{lemma:isolated} \) for sufficiently large \( n \) and \( k \), we have

\[
\mathbb{E}\{\text{number of isolated vertices}\} = \mathbb{E}\left\{ \sum_{i=1}^{n} I_i \right\} = n\mathbb{E}\{I_1\} = n\mathbb{P}\{I_1 = 1\} \geq n\exp(-k(1+\delta)) = \exp(\log(n) - k - \sqrt{6k\log(k)}) \geq \exp(3\sqrt{\log(n)\log\log(n)} - \sqrt{6k\log(k)}) \geq \exp((3 - \sqrt{6})(\sqrt{\log(n)\log\log(n)})),
\]

wherein the right hand side goes to infinity as \( n \to \infty \).
Furthermore, we assume that the event where there is a vertex 1 such that 2 or one of its -most-preferred vertices of it belongs to the -most-preferred vertices of 1. We also assume that the -most-preferred of vertices 1 and 2 are disjoint and none of them is among the other’s -most-preferred vertices. Note that this event implies that the hop distance between nodes 1 and 2 is in \{2, 3\}.

- \(B_{1\rightarrow 2} := \{\exists i \in R_{1}^{\leq k} \text{ such that } (R_{2}^{\leq k} \cup \{2\}) \cap R_{1}^{\leq k} \neq \phi \} \setminus (B_{1;2} \cup B_{2;1} \cup B_{1\cup 2})\).
- \(B_{2\rightarrow 1} := \{\exists i \in R_{2}^{\leq k} \text{ such that } (R_{1}^{\leq k} \cup \{1\}) \cap R_{2}^{\leq k} \neq \phi \} \setminus (B_{1;2} \cup B_{2;1} \cup B_{1\cup 2})\).

This represents the event where there is a vertex 1 within the -most-preferred vertices of 2 such that 2 or one of its -most-preferred vertices of it belongs to the -most-preferred vertices of 2. Again, this event implies that the hop distance between nodes 1 and 2 is in \{2, 3\}. Obviously \(\tilde{B}_{1\rightarrow 2} \subset B_{1\rightarrow 2}\).

- \(B_{2\rightarrow 1} := \{\exists i \in R_{2}^{\leq k} \text{ such that } (R_{1}^{\leq k} \cup \{1\}) \cap R_{2}^{\leq k} \neq \phi \} \setminus (B_{1;2} \cup B_{2;1} \cup B_{1\cup 2})\).

This represents the event where there is a vertex 2 within the -most-preferred vertices of 1 such that 2 or one of its -most-preferred vertices of it belongs to the -most-preferred vertices of 1. We also assume that the -most-preferred of vertices 1 and 2 are disjoint and none of them is among the other’s -most-preferred vertices. Again, this event implies that the hop distance between nodes 1 and 2 is in \{2, 3\}. Obviously \(\tilde{B}_{2\rightarrow 1} \subset B_{2\rightarrow 1}\).

- \(B := B_{1;2} \cup B_{2;1} \cup B_{1\cup 2} \cup B_{1\rightarrow 2} \cup B_{2\rightarrow 1}\).

For ease of presentation, we omit the value 1 from the events \(\{I_1 = 1\}, \{I_2 = 1\}\) or \(\{I_1 I_2 = 1\}\): for example, we will use the shorthand \(\mathbb{P}\{I_1 I_2\} := \mathbb{P}\{I_1 I_2 = 1\}\). Furthermore, we assume \(k < \log(n)\) for the remainder of this section. We start with bounding the event \(B\).
Lemma 4.7. We have the following bound for $B$

\[ \mathbb{P}\{B\} \leq O(k^3/n). \]

Proof. First, note that $\mathbb{P}\{B_{1,2}\} = \mathbb{P}\{B_{2,1}\} = \frac{1}{n-1}$. Now, the event $B_{1,2} \setminus (B_{1,2} \cup B_{2,1})$ can be related to the problem of finding the probability of having an intersection for two completely random subsets of $\{3, \ldots, n\}$ of size $k$, which is a much simpler problem. We will now lower bound the probability of $(B_{1,2} \setminus (B_{1,2} \cup B_{2,1}))^c$ using the fact that for each $i$, $\mathcal{R}^{\leq k}_i$ is uniform among all subsets of size $k$.

Counting the number of ways to choose a pair of two disjoint subsets of size $k$ yields:

\[ \frac{1}{2} \binom{n-2}{k} \binom{n-k-2}{k}, \]

and for choosing two arbitrary subsets the number is:

\[ \frac{1}{2} \binom{n-1}{k}^2. \]

Therefore, the probability of not having intersection is

\[ \mathbb{P}\{(B_{1,2} \setminus (B_{1,2} \cup B_{2,1}))^c\} = \left(\frac{n-2}{n} \right) \left(\frac{n-k-2}{k} \right) \left(\frac{n-1}{k} \right)^2 = \frac{(n-k-2)^2(n-k-3)(n-k-4) \cdots (n-2k-2)}{(n-1)^2(n-2)(n-3) \cdots (n-k-1)} \geq \left(1 - \frac{2k+1}{n-1}\right)^k \]

\[ = 1 - O(k^2/n). \]

This implies $\mathbb{P}\{(B_{1,2} \setminus (B_{1,2} \cup B_{2,1}))\} \leq O(k^2/n)$; therefore,

\[ \mathbb{P}\{B_{1,2} \cup B_{1,2} \cup B_{2,1}\} = \mathbb{P}\{(B_{1,2} \setminus (B_{1,2} \cup B_{2,1}))\} + \mathbb{P}\{B_{1,2} \cup B_{2,1}\} \leq O(k^2/n). \]

To estimate the probability $B_{1 \rightarrow 2}$ we look at $\{\exists i \in \mathcal{R}^{\leq k}_i$ such that $(\mathcal{R}^{\leq k}_i \cup \{2\}) \cap \mathcal{R}^{\leq k}_i \neq \emptyset\}$ given $(B_{1,2} \cup B_{1,2} \cup B_{2,1})^c$ instead; note that the probability estimate increases with this conditioning. Let $i \in \mathcal{R}^{\leq k}_i$ be a vertex. A similar argument to (3) follows that the probability where for all $z \in \mathcal{R}^{\leq k}_i$: $z \notin \{2\} \cup \mathcal{R}^{\leq k}_2$ is bounded below by $(1 - \frac{k+1}{n-1})^k$. As a result, using union bound, with probability at most $k(1 - (1 - \frac{k+1}{n-1})^k)$ there exists $i \in \mathcal{R}^{\leq k}_i$ without this property. Hence

\[ \mathbb{P}\{B_{1 \rightarrow 2}\} \leq k \left(1 - \left(1 - \frac{k+1}{n-1}\right)^k\right) = O(k^3/n), \]

(5)

The same is true for $B_{2 \rightarrow 1}$. From (4), (5) we establish the desired bound. \qed

The following Lemma shows that $B_{1 \rightarrow 2} \cup B_{2 \rightarrow 1}$ is very small; consequently, one can suppose $(B_{1 \rightarrow 2} \cup B_{2 \rightarrow 1})^c$ holds. This way we can control the size of both intersections $(\mathcal{R}^{\leq k}_2 \cup \{2\}) \cap \mathcal{R}^{\leq k}_i$, and $(\mathcal{R}^{\leq k}_i \cup \{1\}) \cap \mathcal{R}^{\leq k}_i$. 


Lemma 4.8. The following bound for the probability of $B_{1\rightarrow 2} \cup B_{2\rightarrow 1}$ holds.

$$\mathbb{P}\{ B_{1\rightarrow 2} \cup B_{2\rightarrow 1} \} \leq O(k^5/n^2).$$

Proof. We again look at $\{ \exists i \in \mathcal{R}_i^{\leq k} \text{ such that } |(\mathcal{R}_i^{\leq k} \cup \{2\}) \cap \mathcal{R}_i^{\leq k}| \geq 2 \}$ given $(B_{1U2} \cup B_{1;2} \cup B_{2;1})^c$ instead. For an arbitrary $i \in \mathcal{R}_i^{\leq k}$, we already saw in the proof of Lemma 4.7 that the probability of $| (\mathcal{R}_i^{\leq k} \cup \{2\}) \cap \mathcal{R}_i^{\leq k} | = 0$ is bounded below by $(1 - \frac{k+1}{n-1})^k$. We now lower bound the probability of $| (\mathcal{R}_i^{\leq k} \cup \{2\}) \cap \mathcal{R}_i^{\leq k} | = 1$. The number of ways to determine $\mathcal{R}_i^{\leq k}$ in order to having only one vertex in common with $(\mathcal{R}_2^{\leq k} \cup \{2\})$ is $(k+1)^{n-k-2}$; therefore, the probability becomes

$$\frac{(k+1)(n-k-2)}{(n-1)} \geq \frac{k(k+1)}{n} \left(1 - \frac{2k-1}{n-1}\right)^k.$$ 

We now apply the union bound to obtain

$$\mathbb{P}\{ B_{1\rightarrow 2} \} \leq k \left(1 - \left(1 - \frac{k+1}{n-1}\right)^k - \frac{k(k+1)}{n} \left(1 - \frac{2k-1}{n-1}\right)^k\right) \leq O(k^5/n^2).$$

A similar argument for $B_{2\rightarrow 1}$ finishes the proof. \square

Lemma 4.9. The following two probability bounds hold:

$$\mathbb{P}\{ I_1 \cap (B_{1;2} \cup B_{1U2} \cup B_{1\rightarrow 2})\cap B_{i\rightarrow 2}^c \} \leq \exp \left( - \log(n) - k + \frac{1}{2} \sqrt{k\log(k)} + O(\sqrt{k}) \right),$$

$$\mathbb{P}\{ I_2 \cap (B_{2;1} \cup B_{1U2} \cup B_{2\rightarrow 2})\cap B_{i\rightarrow 2}^c \} \leq \exp \left( - \log(n) - k + \frac{1}{2} \sqrt{k\log(k)} + O(\sqrt{k}) \right).$$

Proof. We prove the first one, and then the second one will follow by symmetry. Let $s = \lfloor \sqrt{n} \rfloor$ and $B_{i\rightarrow 2}^c$ holds; for any $i \in \mathcal{R}_i^{\leq s}$ there is at most one element $t_i \in \mathcal{R}_2^{\leq k}$ such that $t_i \in \mathcal{R}_i^{\leq k}$ too. Define $\mathcal{E'} = \mathcal{E} \setminus \{(i, j) : j \in \mathcal{R}_i^{\leq s} \cup \mathcal{R}_2^{\leq k} \cup \{1, 2\} \}$. Therefore, if we consider the following union

$$\mathcal{E}' = \mathcal{E}_1 \bigcup_{i \in \mathcal{R}_i^{\leq s}} \mathcal{E}'_i,$$

then it will be a partition in which $|\mathcal{E}_1| = n - 1$, and $|\mathcal{E}'_i| = n - k - s - 3$ for all $i \in \mathcal{R}_i^{\leq s}$. Define $\mathcal{E}_1$ objects to be of type 0 and $\mathcal{E}'_i$ objects to be of type $i$ for $i \in \mathcal{R}_i^{\leq s}$. One can easily see that each realization of $G(n, k)$ induces a uniformly random permutation over $\mathcal{E}'$ which has only a fixed order on the first $s$ or $s + 1$ elements of $\mathcal{E}_1$, i.e., type 0.

A necessary condition for vertex 1 to be isolated given that $B_{1;2} \cup B_{1U2} \cup B_{1\rightarrow 2}$ and $B_{i\rightarrow 2}^c$ hold, is that for all $i(\neq 2) \in \mathcal{R}_i^{\leq s}$, there should be at least $k$ edges of $\mathcal{E}_i$ appearing before the $s^{th}$ element of $\mathcal{E}_1$. Excluding possible edges to $\mathcal{R}_i^{\leq s}$, $t_i \in \mathcal{R}_2^{\leq k}$, and $\{1, 2, i\}$ implies that there must be at least $k - s - 4$ edges of $\mathcal{E}'_i$ before the $s^{th}$ element of $\mathcal{E}_1$. As a result, in the first $(s - 1)(k - s - 4)$ elements of this permutation there are at most $s - 1$ elements of type 0. Lemma 3.3 bounds the probability of the event above by

$$\mathbb{P}\{ I_1 | (B_{1;2} \cup B_{1U2} \cup B_{1\rightarrow 2}) \cap B_{i\rightarrow 2}^c \} \leq \exp \left( - k + \frac{1}{2} \sqrt{k\log(k)} + O(\sqrt{k}) \right).$$
We also have
\[ \mathbb{P}\{ (B_{1:2} \cup B_{1:2} \cup B_{1:2}) \mid B_{1:2}^c \} \leq \frac{\mathbb{P}\{ B_{1:2} \cup B_{1:2} \cup B_{1:2} \}}{\mathbb{P}\{ B_{1:2}^c \}} \leq O(k^3/n). \]

And finally,
\[ \mathbb{P}\{ I_1 \cap (B_{1:2} \cup B_{1:2} \cup B_{1:2}) \mid B_{1:2}^c \} = \mathbb{P}\{ I_1 \cap (B_{1:2} \cup B_{1:2} \cup B_{1:2}) \cap B_{1:2}^c \} \]
\[ \leq \exp \left( -\log(n) - k + \frac{1}{2} \sqrt{k \log(k)} + O(\sqrt{k}) \right), \]

which concludes the proof.

\[ \square \]

**Lemma 4.10.** The following holds
\[ \mathbb{P}\{ I_1 I_2 \cap B \} \leq \exp \left( -\log(n) - k + \frac{1}{2} \sqrt{k \log(k)} + O(\sqrt{k}) \right) \]

**Proof.**
\[ \mathbb{P}\{ I_1 I_2 \cap B \} \leq \mathbb{P}\{ I_1 I_2 \cap (B_{1:2} \cup B_{1:2} \cup B_{1:2}) \} + \mathbb{P}\{ I_1 I_2 \cap (B_{2:1} \cup B_{1:2} \cup B_{2:1}) \}\]
\[ \leq \mathbb{P}\{ I_1 \cap (B_{1:2} \cup B_{1:2} \cup B_{1:2}) \cap B_{1:2} \} \]
\[ + \mathbb{P}\{ I_1 \cap (B_{1:2} \cup B_{1:2} \cup B_{1:2}) \cap B_{1:2}^c \} \]
\[ + \mathbb{P}\{ I_2 \cap (B_{2:1} \cup B_{1:2} \cup B_{2:1}) \cap B_{2:1} \} \]
\[ + \mathbb{P}\{ I_2 \cap (B_{2:1} \cup B_{1:2} \cup B_{2:1}) \cap B_{2:1}^c \} \]
\[ \leq \mathbb{P}\{ B_{1:2} \cup B_{2:1} \} \]
\[ + \mathbb{P}\{ I_1 \cap (B_{1:2} \cup B_{1:2} \cup B_{1:2}) \mid B_{1:2} \cap B_{1:2}^c \} \]
\[ \leq O(k^3/n^2) + \mathbb{P}\{ I_1 \cap (B_{1:2} \cup B_{1:2} \cup B_{1:2}) \mid B_{1:2}^c \} \] (Lemma 4.8)
\[ + \mathbb{P}\{ I_2 \cap (B_{2:1} \cup B_{1:2} \cup B_{2:1}) \mid B_{2:1}^c \} \]
\[ \leq O(k^3/n^2) + 2 \exp \left( -\log(n) - k + \frac{1}{2} \sqrt{k \log(k)} + O(\sqrt{k}) \right) \] (Lemma 4.9)
\[ = \exp \left( -\log(n) - k + \frac{1}{2} \sqrt{k \log(k)} + O(\sqrt{k}) \right), \]

which establishes the bound.

\[ \square \]

**Lemma 4.11.** We have the following conditional independence
\[ \mathbb{P}\{ I_1 I_2 \mid B^c \} = \mathbb{P}\{ I_1 \mid B^c \} \mathbb{P}\{ I_2 \mid B^c \}. \]

**Proof.** Note that \( I_1 \) holds whenever for each \( 1 \leq i \leq k \) there are at least \( k \) elements \( j \in [n] \setminus \{1, R_1^*\} \) such that \( V(R_1^*, 1) > V(R_1^*, j) \). Assuming \( B^c \), one can see this is equivalent to existing at least \( k \) elements \( j_1 \in [n] \setminus \{1, 2\} \setminus \{R_1^*, \ldots, R_2^*\} \) with this property. Repeating this argument for \( I_2 \), there must be at least \( k \) elements \( j_2 \in [n] \setminus \{1, 2\} \setminus \{R_1^*, \ldots, R_2^*\} \) with \( V(R_2^*, 2) > V(R_2^*, j) \) for all \( 1 \leq j \leq k \). Ruling out the possibility of choosing \( j_2, j_1 \) from \( [1, 2], \{R_1^*, R_1^*, \ldots, R_k^*\}, \) and \( \{R_2^*, \ldots, R_k^*\} \) implies that there is no common edge among \((R_1^*, 1), (R_1^*, j_1), (R_2^*, 2), (R_2^*, j_2)\). Since the scores are chosen independently, and the inequality conditions for the isolation of vertex 1 and those of vertex 2 are disjoint, we obtain the conditional independence.

\[ \square \]
Lemma 4.12. Let \( t' < -1/2 \) be a real number. Assume \( k \) grows to infinity as \( n \to \infty \) with \( k \leq \log(n) + t' \log \log(n) \sqrt{\log(n)} \). Then we have

\[
\limsup_{n \to \infty} \frac{\mathbb{E}\{I_1 I_2\}}{\mathbb{E}\{I_1\}^2} \leq 1.
\]

**Proof.** We begin with this equality

\[
\frac{\mathbb{P}\{I_1\}}{\mathbb{P}\{B^c\}} = \frac{\mathbb{P}\{I_1 \cap B^c\}}{\mathbb{P}\{B^c\}} + \frac{\mathbb{P}\{I_1 \cap B\}}{\mathbb{P}\{B^c\}}. \tag{6}
\]

Hence,

\[
\frac{1}{\mathbb{P}\{B^c\}} = \frac{\mathbb{P}\{I_1 \cap B^c\}}{\mathbb{P}\{I_1\}} \frac{\mathbb{P}\{B\}}{\mathbb{P}\{B^c\}} + \frac{\mathbb{P}\{I_1 \cap B\}}{\mathbb{P}\{I_1\} \mathbb{P}\{B^c\}} = \frac{\mathbb{P}\{I_1\}}{\mathbb{P}\{I_1\}} \frac{\mathbb{P}\{B\}}{\mathbb{P}\{B^c\}} + \frac{\mathbb{P}\{I_1 \cap B\}}{\mathbb{P}\{I_1\} \mathbb{P}\{B^c\}}. \tag{7}
\]

Lemmas 4.5 and 4.7 imply

\[
\frac{\mathbb{P}\{I_1 \cap B\}}{\mathbb{P}\{I_1\}} \leq O\left(\frac{k^3}{n}\right)O\left(e^{k(1 + \sqrt{\log(k)})}\right)
= O(\exp(k + \sqrt{6k \log(k)} + 3 \log(k) - \log(n)))
\leq O(\exp(t' \log \log(n) \sqrt{\log(n)} + \sqrt{6k \log(k)} + 3 \log(k)))
\leq O(\exp(t' \log \log(k) \sqrt{k} + \sqrt{6k \log(k)} + 3 \log(k))) \tag{8}
\]

\[
\to k \to \infty, 0,
\]

where we use the fact that \( k \leq \log(n) \). Considering the fact \( \frac{1}{\mathbb{P}\{B^c\}} \to 1 \), equation (7) implies that

\[
\frac{\mathbb{P}\{I_1 \cap B^c\}}{\mathbb{P}\{I_1\}} \to 1. \tag{9}
\]

It follows from Lemmas 4.10, 4.11 that

\[
\mathbb{P}\{I_1 I_2\} = \mathbb{P}\{I_1 I_2 \cap B^c\} + \mathbb{P}\{I_1 I_2 \cap B\}
\leq \mathbb{P}\{I_1 I_2 \cap B^c\} + \exp\left(-\log(n) - k + \frac{1}{2} \sqrt{k} \log(k) + O(\sqrt{k})\right)
\leq \mathbb{P}\{I_1 I_2 \cap B^c\} + \exp\left(-\log(n) - k + \frac{1}{2} \sqrt{k} \log(k) + O(\sqrt{k})\right)
\leq \frac{\mathbb{P}\{I_1 \cap B^c\} \mathbb{P}\{I_2 \cap B^c\} + \exp\left(-\log(n) - k + \frac{1}{2} \sqrt{k} \log(k) + O(\sqrt{k})\right)}{\mathbb{P}\{I_1\} \mathbb{P}\{I_2\}}. \tag{10}
\]

Dividing both sides of (10) by \( \mathbb{P}\{I_1\} \mathbb{P}\{I_2\} = \mathbb{P}\{I_1\}^2 \) and using Lemma 4.5 we have

\[
\frac{\mathbb{P}\{I_1 I_2\}}{\mathbb{P}\{I_1\} \mathbb{P}\{I_2\}} \leq \frac{\mathbb{P}\{I_1 \cap B^c\} \mathbb{P}\{I_2 \cap B^c\}}{\mathbb{P}\{I_1\} \mathbb{P}\{I_2\}} + \frac{\exp\left(-\log(n) - k + \frac{1}{2} \sqrt{k} \log(k) + O(\sqrt{k})\right)}{\mathbb{P}\{I_1\} \mathbb{P}\{I_2\}}
\leq \frac{\mathbb{P}\{I_1 \cap B^c\} \mathbb{P}\{I_2 \cap B^c\}}{\mathbb{P}\{I_1\} \mathbb{P}\{I_2\}} + \frac{\exp\left(-\log(n) - k + \frac{1}{2} \sqrt{k} \log(k) + O(\sqrt{k})\right)}{\exp\left(-2k(1 + \frac{1}{\sqrt{k} \log(k)})\right)}.
\]
\[ \begin{align*}
\leq & \frac{\mathbb{P}\{I_1|B^c\}\mathbb{P}\{I_2|B^c\}}{\mathbb{P}\{I_1\}\mathbb{P}\{I_2\}} + \exp \left( -\log(n) + k + \frac{1}{2} \sqrt{k \log(k)} + \sqrt{24k \log(k)} + O(\sqrt{k}) \right) \\
\leq & \frac{\mathbb{P}\{I_1|B^c\}\mathbb{P}\{I_2|B^c\}}{\mathbb{P}\{I_1\}\mathbb{P}\{I_2\}} \\
& + \exp \left( t' \log(n) \sqrt{\log(n)} + \frac{1}{2} \sqrt{k \log(k)} + \sqrt{24k \log(k)} + O(\sqrt{k}) \right) \\
\leq & \frac{\mathbb{P}\{I_1|B^c\}\mathbb{P}\{I_2|B^c\}}{\mathbb{P}\{I_1\}\mathbb{P}\{I_2\}} + \exp \left( \frac{1}{2} + t' \right) \log(k) \sqrt{k} + \sqrt{24k \log(k)} + O(\sqrt{k}) \right). \tag{11}
\end{align*} \]

Equation (11) guarantees the first term in right hand side converges to 1. The second term vanishes as \( k \to \infty \) as a result of \( t' < -0.5 \). This completes the proof. \( \square \)

**Theorem 4.1.** The random graph model \( G(n, k) \) with \( k \leq \log(n) + t' \log(n) \sqrt{\log(n)} \) for \( t' < -\frac{1}{2} \) is not connected with high probability. In particular, if \( k = \lfloor t \log(n) \rfloor \) for \( 0 < t < 1 \), the graph is disconnected with high probability.

**Proof.** Having both Lemmas 4.6 and 4.12 one can apply the second moment method. We begin with

\[ \mathbb{E} \left\{ \left( \sum_{i=1}^{n} I_i \right)^2 \right\} - \mathbb{E} \left\{ \sum_{i=1}^{n} I_i \right\}^2 = \text{Var} \left\{ \sum_{i=1}^{n} I_i \right\} \]

\[ \geq \left( 0 - \mathbb{E} \left\{ \sum_{i=1}^{n} I_i \right\} \right)^2 \mathbb{P} \left\{ \sum_{i=1}^{n} I_i = 0 \right\}. \tag{12} \]

Therefore,

\[ \mathbb{P} \left\{ \sum_{i=1}^{n} I_i = 0 \right\} \leq \frac{\mathbb{E} \left\{ \left( \sum_{i=1}^{n} I_i \right)^2 \right\} - \mathbb{E} \left\{ \sum_{i=1}^{n} I_i \right\}^2}{\mathbb{E} \left\{ \sum_{i=1}^{n} I_i \right\}^2} \]

\[ \leq \frac{\mathbb{E} \left\{ \left( \sum_{i=1}^{n} I_i \right)^2 \right\}}{\mathbb{E} \left\{ \sum_{i=1}^{n} I_i \right\}^2} - 1 \]

\[ = \frac{\sum_{i=1}^{n} \mathbb{E} \{ I_i^2 \} + \sum_{i,j, i \neq j} \mathbb{E} \{ I_i I_j \}}{n^2 \mathbb{E} \{ I_i \}^2} - 1 \]

\[ = \frac{n \mathbb{E} \{ I_1 \} + (n - 1) \mathbb{E} \{ I_1 I_2 \}}{n^2 \mathbb{E} \{ I_1 \}^2} - 1 \]

\[ = \frac{1}{n \mathbb{E} \{ I_1 \}} + \frac{(n - 1) \mathbb{E} \{ I_1 I_2 \}}{n^2 \mathbb{E} \{ I_1 \}^2} - 1. \tag{13} \]

Now it follows from Lemma 4.6 that the first fraction vanishes and from Lemma 4.12 that the second fraction will be sufficiently close to 1 (or less than 1) for large \( n \). Hence, the probability of having no isolated vertex converges to 0. \( \square \)

## 5 Connectivity for \( t > 1 \)

A common method to prove connectivity of random graphs is by locating (with high-probability) a Erdős-Rényi sub-graph consisting of all the vertices. Then, the celebrated result of Erdős and Rényi [6] guarantees connectivity if the connectivity probability is \( p = \frac{t \log(n)}{n} \) where \( t > 1 \). This method used in [9] proves the connectivity
of $G(n,k)$ for $k = t\log(n)$ where $t > C = 2.4625$. The connection between Erdős-Rényi and $G(n,k)$ is made through a concentration property of order statistics. The authors of [9] assume that the edges are independently scored by $\exp(1)$ random variables which results in having a connection for all edges of scores greater than equal to zero with high probability. Therefore, if $L_{i,j}$ is a random variable of law $\exp(1)$, let $W$ defines a partition in which each realization of $G$ has only a fixed order on the first $s$ elements. We can assume vertex $1$ of $G$ as in Lemma 4.9. Let $E$ be such a partition, then

$$P\left\{ L_{i,j} > \log\left(\frac{n-1}{t\log(n)}\right) + \sqrt{\frac{2}{t}} \right\} = \frac{t\log(n)}{n-1} e^{-\sqrt{\frac{t}{n}}}.$$ 

It can be easily shown that $te^{-\sqrt{\frac{t}{n}}} > 1$ for $t > 2.4625$. This provides an independent possibility of connection for all edges with the probability of $t' \log(n)/(n-1)$ for a $t' = te^{-\sqrt{\frac{2}{t}}}$ greater then 1. Thereafter, the connectivity result of Erdős-Rényi random graphs [4], [6], [12] implies the connectivity of $G(n,k)$.

We prove the connectivity of $G(n,k)$ for all $t > 1$ in two steps. First, we rule out the possibility of having components of size $O(1)$. Second, we apply the idea from [9] described above to find an Erdős-Rényi graph with all vertices contained in $G(n,k)$. As opposed to [9], we do not restrict $t$ to find a $t' > 1$. Hence, the resulting Erdős-Rényi graph is not necessarily connected. However, by modifying the original proof of the connectivity of Erdős-Rényi graphs for $t' > 1$, we can prove a weaker result for an arbitrary $t' < 1$ which then helps us deduce the connectivity result.

**Theorem 5.1.** Let $\kappa \geq 0$ be a non-negative integer. Consider random graphs of class $G(n,k)$ where $k \geq \log(n) + t' \log(n)\sqrt{\log(n)}$ for a real number $t' > \frac{1}{2}$. Then

$$P\{\exists a vertex of degree less than or equal to \kappa\} \to 0$$

as $n \to \infty$.

**Proof.** We can assume vertex $1$ has degree less than $\kappa$. We will use the same technique as in Lemma 4.3. Let $s = \lfloor \sqrt{k}\rfloor \leq k$. For $i \in R_1^{\leq s}$ define $E'_1 = E_i \setminus \{i,j\} : j \in R_1^{\leq s}$. Thereafter, if we consider the following union

$$E' = E_1 \bigcup_{i \in R_1^{\leq s}} E'_i,$$

it will be a partition in which $|E_1| = n-1$, and $|E'_i| = n-s-1$ for all $i \in R_1^{\leq s}$. Define $E_1$ objects of type 0 and $E'_i$ objects of type $i$ for $i \in R_1^{\leq s}$. One can easily see that each realization of $G(n,k)$ induces a uniformly random permutation over $E'$ which has only a fixed order on the first $s$ or $s+1$ elements of $E_1$, i.e., type 0.

A necessary condition for vertex $1$ to be of degree at most $\kappa$ is that for at least $s - \kappa$ elements $i \in R_1^{\leq s}$, there should be at least $k$ edges of $E_i$ appearing before the $s$th element of $E_1$. This implies there must be at least $k - s - 1$ edges of $E'_i$ before the $s$th element of $E'_1$. As a result, in the first $(s-\kappa)(k-s-1)$ elements of this permutation there are at most $s-1$ elements of type 0. Lemma 3.3 bounds the probability of the event above by

$$\exp\left(-k + \frac{1}{2}\sqrt{k\log(k)} + O(\sqrt{k})\right)$$

Then, using the union bound for sufficiently large $n$ we have:

$$P\{\exists v : \deg(v) \leq \kappa\} \leq n \times P\{\text{vertex 1 is of degree at most } \kappa\}$$
\[ \leq n \exp \left( -k + \frac{1}{2} \sqrt{k \log(k) + O(\sqrt{k})} \right) \]
\[ = \exp \left( \log(n) - k + \frac{1}{2} \sqrt{k \log(k) + O(\sqrt{k})} \right) \]

If \( k > O(\log(n)) \), then the right hand side obviously vanishes. In case of \( k = O(\log(n)) \), we use the fact that \( k - \frac{1}{2} \sqrt{k \log(k)} \) is increasing for sufficiently large \( k \), and replace \( k \) by \( \log(n) + t' \log \log(n) \sqrt{\log(n)} \) to obtain
\[ \exp \left( \left( \frac{1}{2} - t' \right) \log \log(n) \sqrt{\log(n)} + O(\sqrt{\log(n)}) \right) \to 0. \]

Hence, the result holds. \( \square \)

Theorem 5.1 excludes the possibility of having components of size \( O(1) \). In other words, the following corollary holds.

**Corollary 5.1.** Consider the random graphs model of class \( G(n, k) \) with \( k \geq \log(n) + t' \log \log(n) \sqrt{\log(n)} \) for a real number \( t' > \frac{1}{2} \). Then
\[ \mathbb{P}\left\{ \exists \text{ a component of size at most } \kappa \right\} \to 0 \]
for any fixed \( \kappa \) as \( n \to \infty \). In particular when \( k = \lfloor t \log(n) \rfloor \) for \( t > 1 \) the limit above is still true.

**Proof.** In case of having a component of size \( \kappa \) there most be a vertex of degree at most \( \kappa \) which is impossible with high probability due to Theorem 5.1. \( \square \)

Next, we establish a proof based on finding an Erdős-Rényi sub-graph and super-graph of \( G(n, k) \) with both containing all the vertices. Recall that the order statistics of a set of \( i.i.d. \) random variables \( \{X_1, X_2, \ldots, X_n\} \) is defined as their non-increasing rearrangement \( X^{(1)} \geq X^{(2)} \geq \cdots \geq X^{(n)} \). Recall that the \( k^{th} \) order statistic of \( V_i \) is denoted by \( V(i, R^k_i) \).

**Theorem 5.2.** For the random graph model \( G(n, k) \) with \( k = \lfloor t \log(n) \rfloor \) there is no component of size \( \lfloor \frac{\kappa + 1}{2} \rfloor \leq r \leq \lfloor n/2 \rfloor \) with high probability.

**Proof.** First, we note Lemma A.1 in [9] which states that if we choose scores \( V(i, j) \) independently distributed as exponential random variables with parameter 1, and define:
\[ A_n = \left\{ \forall 1 \leq i \leq n : V(i, R^k_i) \in \left( \log \left( \frac{n-1}{t \log(n)} \right) - \sqrt{2}, \log \left( \frac{n-1}{t \log(n)} \right) + \sqrt{2} \right) \right\}, \]
then \( \mathbb{P}\{A_n\} \to 1 \) as \( n \to \infty \). We let \( \underline{t} = \log \left( \frac{n-1}{t \log(n)} \right) - \sqrt{2} \) and \( \bar{t} = \log \left( \frac{n-1}{t \log(n)} \right) + \sqrt{2} \).

Next, note that
\[ \bar{p} := \mathbb{P}\{V(i, j) > \bar{t}\} = \frac{t \log(n)}{n-1} e^{-\sqrt{2}} \approx \frac{0.2431 t \log(n)}{n-1}, \]
\[ \underline{p} := \mathbb{P}\{V(i, j) > \underline{t}\} = \frac{t \log(n)}{n-1} e^{\sqrt{2}} \approx \frac{4.1133 t \log(n)}{n-1}. \]

Moreover, if the graph satisfies \( A_n \) condition (which holds whp), then \( V(i, j) > \bar{t} \) implies that the edge \( \{i, j\} \) exists and \( V(i, j) < \underline{t} \) implies that the edge \( \{i, j\} \) does not exist.
Let the graph satisfy $A_n$. In order to have a component of size $r$, one must choose $r$ vertices. The connectivity within this component implies that it should contains at least a tree. Moreover, according to Cayley’s formula there are $r^{r-2}$ trees with $r$ vertices. Hence, the $r - 1$ edges of the tree must have scores not less than $\bar{r}$ which happens with probability $\bar{r}^{r-1}$ due to independence. On the other hand, we require those $r$ vertices not to be connected to the rest of graph. This implies the scores for the intermediate edges between the component and the rest of graph must be less than $\bar{l}$. Note that both of these are necessary conditions. Counting the number of possible components yields the following:

$$\Pi = \mathbb{P}\{\exists \text{ a component of size between } \left[\frac{8.24}{t}\right] \text{ and } [n/2]\}$$

$$\leq \sum_{r=\left[\frac{8.24}{t}\right]}^{[n/2]} \left(\begin{array}{c} n \\ r \end{array}\right) r^{r-2} \bar{r}^{r-1} (1 - \bar{p})^{r(n-r)}.$$  

Substituting $\left(\frac{n}{r}\right) < \frac{n}{r}$, $r! \approx \sqrt{2\pi r} \left(\frac{r}{e}\right)^r$ implies

$$\Pi < \frac{1}{\sqrt{2\pi}} \sum_{r=\left[\frac{8.24}{t}\right]}^{[n/2]} n^r r^{-r-1/2} e^r r^{-2} \bar{r}^{r-1} (1 - \bar{p})^{r(n-r)}$$

$$< \frac{1}{\sqrt{2\pi}} \sum_{r=\left[\frac{8.24}{t}\right]}^{[n/2]} e^{r} n^r \bar{r}^{-5/2} p^r e^{-r(n-r)} \bar{p} \quad \text{(using } 1 - x < e^{-x})$$

$$< \frac{n}{\sqrt{2\pi}} \sum_{r=\left[\frac{8.24}{t}\right]}^{[n/2]} e^r e^{\log(\log(n) - n\bar{p}/2)} \quad \text{(using } 1/p < n, n - r > n/2, r^{-5/2} < 1)$$

$$< \frac{n}{\sqrt{2\pi}} \sum_{r=\left[\frac{8.24}{t}\right]}^{[n/2]} e^{r(1+\log(\log(n) - n\bar{p}/2))}.$$  

As $t$ is a fixed number, the dominant term in $1 + \log(\log(n) - n\bar{p}/2)$ is $-n\bar{p}/2 \approx -0.1216\log(n)$. As a result, for sufficiently large $n$, $1+\log(\log(n) - n\bar{p}/2) < -0.1215t\log(n)$. Thus, we have

$$\Pi < \frac{n}{\sqrt{2\pi}} \sum_{r=\left[\frac{8.24}{t}\right]}^{\infty} e^{-0.1215tr \log(n)}$$

$$= \frac{n}{\sqrt{2\pi}} \sum_{r=\left[\frac{8.24}{t}\right]}^{\infty} n^{-0.1215r}$$

$$= \frac{n^{-0.1215\left[\frac{8.24}{t}\right]}}{\sqrt{2\pi}(1 - n^{-0.1215t})}$$

$$\leq \frac{n^{-0.00116}}{\sqrt{2\pi}(1 - n^{-0.1215t})} \xrightarrow{n \to \infty} 0.$$  

Therefore, with high probability there is no component of size between 10 and $n/2$.  

The interesting fact about Theorem 5.2 is that it holds for any positive $t$. This means that if $k \approx t\log(n) = \Theta(\log(n))$, one should focus attention on components of size less than $\left[\frac{8.24}{t}\right] = O(1)$ to study the disconnectivity. Since Corollary 5.1 addresses components of size $O(1)$ and we conclude the main theorem.
Theorem 5.3. The random graph model $\mathbb{G}(n, k)$ with $k \geq \log(n) + t' \log \log(n) \sqrt{\log(n)}$ for a real number $t' > \frac{1}{2}$ is connected with high probability. In particular, if $k = \left\lfloor t \log(n) \right\rfloor$ for $t > 1$, connectivity holds with high probability.

Proof. If a graph from class $\mathbb{G}(n, k)$ is disconnected, then it should have at least one component of size $s$ where $1 \leq s \leq \lfloor n/2 \rfloor$. Since $\left\lceil \frac{8.24}{1} \right\rceil = 9$, Theorem 5.2 rules out the possibility of having components of size between 9 and $\lfloor n/2 \rfloor$ with high probability. Theorem 5.1 guarantees that the probability of having components of size at most 8 vanishes, when $n$ grows to infinity. Therefore, the graph will be connected with high probability.

6 Average Degree

Here, we discuss another set of results for $\mathbb{G}(n, k)$ random graphs. The degree sequence is bounded above by $k$. Therefore, average degree is also bounded by $k$. However, we will show that this number is very close to $k$. We also specify the error term. Finally we compare our result to the sparse case in [10]. Before presenting the main theorem we recall the negative binomial distribution. Consider an urn having infinite number of red and blue balls and each time we draw red with probability $p$. The probability of having $j$ blue balls appeared before the $k^{th}$ red ball is a negative binomial distribution given by the following formula

$$P\{X_k = j\} = \binom{k + j - 1}{j} p^k (1 - p)^j.$$

The theorems below provide an approximation for average degree.

Lemma 6.1.

$$\sum_{j=0}^{k-1} \binom{k + j - 1}{j} \frac{1}{2^{k+j}} = \frac{1}{2}$$

Proof. Let $X_k$ be a negative binomial random variable with $p = \frac{1}{2}$. Note that $P\{X_k < k\} = \sum_{j=0}^{k-1} \binom{k + j - 1}{j} \frac{1}{2^{k+j}}$ is the probability of observing the $k^{th}$ red ball earlier than the $k^{th}$ blue ball in the infinite urn model. By symmetry this value must be $1/2$.

Lemma 6.2.

$$\sum_{j=0}^{k-1} \binom{k + j - 1}{j} \left( \frac{j}{2^{k+j}} \right) = \frac{k}{2} - \frac{(2k - 1)}{2^{2k-1}} \binom{2k - 2}{k - 1}$$

Proof. Using Lemma 6.1 one can write the sum above as follows

$$A = \sum_{j=0}^{k-1} \binom{k + j - 1}{j} \left( \frac{j}{2^{k+j}} \right) = \sum_{j=1}^{k-1} \binom{k + j - 2}{j - 1} \left( \frac{k + j - 1}{2^{k+j}} \right) = \left( \frac{k}{2} \right) \sum_{j=1}^{k-1} \binom{k + j - 2}{j - 1} \frac{1}{2^{j+k-1}} + \left( \frac{1}{2} \right) \sum_{j=1}^{k-1} (j - 1) \binom{k + j - 2}{j - 1} \frac{1}{2^{j+k-1}} = \left( \frac{k}{2} \right) \left[ 1 - \binom{2k - 2}{k - 1} \frac{1}{2^{2k-1}} \right] + \left( \frac{1}{2} \right) \left[ A - (k - 1) \binom{2k - 2}{k - 1} \frac{1}{2^{2k-1}} \right]$$
\[
= \frac{k}{4} + \frac{A}{2} - \frac{(2k - 1)}{2^{2k-2}} \left( \frac{2k - 2}{k - 1} \right).
\]

We now solve the resulting equation for \( A \) to get the desired expression.

**Theorem 6.1.** In the model \( G(n, k) \) suppose \( D \) represents the degree of a randomly chosen vertex. If \( k = o(\sqrt{n}) \) then we have the following asymptotic

\[
\mathbb{E}\{D\} = k - \frac{(2k - 1)}{2^{2k-1}} \left( \frac{2k - 2}{k - 1} \right) (O(k^2/n) + 1) \tag{14}
\]

**Proof.** Without loss of generality assume the randomly chosen vertex is vertex 1 and \( R_i = i + 1 \) for all \( 1 \leq i \leq n - 1 \). Denote the indicator function of vertex 1 being connected to vertex \( i + 1 \) by \( E_i \). In order for vertex 1 not to be connected to vertex \( i + 1 \) there must be at least \( k \) elements of \( E_{i+1} \) before the \( i \)th element of \( E_1 \) which is \( \{1, i + 1\} \). Hence, there could be \( 0, 1, \ldots, i - 1 \) elements of \( E_1 \) before the \( k \)th element of \( E_{i+1} \). With the same idea used in Lemma 5.1 one can prove that the probability of \( j \) elements of \( E_1 \) appear before the \( k \)th element of \( E_{i+1} \) for \( j \leq k \) is:

\[
\binom{k + j - 1}{j} \binom{n + O(k)}{2n + O(k)}^{k+j} = \binom{k + j - 1}{j} \left( \frac{1}{2} \right)^{k+j} (1 + O(k^2/n))
\]

Varying \( 0 \leq j \leq i - 1 \) we conclude that for each \( 1 \leq i \leq k \) one has

\[
\mathbb{P}\{E_i = 0\} = \left[ \sum_{j=0}^{i-1} \binom{k + j - 1}{j} \frac{1}{2^{k+j}} \right] (1 + O(k^2/n))
\]

We now compute \( \mathbb{E}\{D\} \) using Lemmas 6.1 and 6.2 as follows

\[
\mathbb{E}\{D\} = \sum_{i=1}^{n-1} \mathbb{E}\{E_i\}
\]

\[
= \sum_{i=1}^{n-1} \mathbb{P}\{E_i = 1\}
\]

\[
= \sum_{i=1}^{n-1} \mathbb{P}\{E_i = 1\}
\]

\[
= k - \sum_{i=1}^{k} \mathbb{P}\{E_i = 0\}
\]

\[
= k - \sum_{i=1}^{k} \left[ \sum_{j=0}^{i-1} \binom{k + j - 1}{j} \frac{1}{2^{k+j}} \right] (O(k^2/n) + 1)
\]

\[
= k - \left[ \sum_{j=0}^{k-1} \binom{k + j - 1}{j} \frac{k - j}{2^{k+j}} \right] (O(k^2/n) + 1)
\]

\[
= k - \left[ \frac{k}{2} - \sum_{j=0}^{k-1} \binom{k + j - 1}{j} \frac{j}{2^{k+j}} \right] (O(k^2/n) + 1)
\]

\[
= k - \left[ \frac{(2k - 1)}{2^{2k-1}} \left( \frac{2k - 2}{k - 1} \right) \right] (O(k^2/n) + 1).
\]

This completes the proof. \( \blacksquare \)
Remark: As mentioned earlier, the work in [10] considers the preference threshold to be a random variable independently chosen per vertex using a distribution $P$ over $\mathbb{N}^0$ with finite mean instead of just a fixed $k$ for the potential number of neighbors of a randomly chosen vertex. Further, [10, Theorem 5.1] specifies the following formula for the average degree in the limit of $n$ going to infinity

$$
\mathbb{E}\{D\} = \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} P(i)P(j) \int_{0}^{\infty} \tilde{F}_i(x)\tilde{F}_j(x)dx
$$

(15)

where $\tilde{F}_i$ denotes the complementary cumulative distribution function of Erlang$(\cdot; i, 1)$ (the Erlang distribution of shape $i$ and rate 1).

In the case of bilateral agreement with preference threshold parameter to be a fixed $k$ the probability distribution $P$ becomes a delta mass function on $k$, i.e., $P(i) = 1$ if $i = k$, and $P(i) = 0$ otherwise. In addition, the mean being finite as $n$ goes to infinity, implies that $k$ must be finite and cannot grow to infinity. The sum (15) now simplifies to

$$
\mathbb{E}\{D\} = \int_{0}^{\infty} \tilde{F}_k(x)^2dx,
$$

(16)

Then using $\tilde{F}_k(x) = \sum_{i=0}^{k-1} \frac{1}{i!}e^{-x}x^i$, distributing the square, and exchanging the finite sum with integral, we obtain

$$
\mathbb{E}\{D\} = \sum_{0 \leq i,j \leq k-1} \int_{0}^{\infty} \frac{x^{i+j}}{i!j!}e^{-2x}dx

= \sum_{0 \leq i,j \leq k-1} \int_{0}^{\infty} \frac{(2x)^{i+j}}{2^{i+j}i!j!}e^{-2x}dx

= \sum_{0 \leq i,j \leq k-1} \frac{\Gamma(i+j+1)}{2^{i+j+1}i!j!}

= \frac{1}{2} \sum_{0 \leq i,j \leq k-1} \frac{1}{2^{i+j}} \binom{i+j}{i}.
$$

(17)

We substitute $s = i + j$ and use the notation of $X_k$ for a negative binomial random variable with parameter $p = 1/2$ representing the probability of appearing $k^{th}$ red ball after observing $X_k$ blue balls. Then we arrive at

$$
\mathbb{E}\{D\} = \frac{1}{2} \left[ \sum_{s=0}^{k-1} \sum_{i=0}^{s} \frac{1}{2^s} \binom{s}{i} + \sum_{s=k}^{k-2} \sum_{i=s-k+1}^{2k-2} \frac{1}{2^s} \binom{s}{i} \right]

= \frac{1}{2} \left[ k + \sum_{s=0}^{k-2} \left( 1 - 2 \sum_{i=0}^{s} \frac{1}{2^{s+k}} \binom{s+k}{i} \right) \right]

= \frac{1}{2} \left[ k + \sum_{s=0}^{k-2} (1 - 2\mathbb{P}\{X_k \leq s\}) \right]

= \frac{1}{2} \left[ k + \sum_{s=0}^{k-2} (2\mathbb{P}\{X_k > s\} - 1) \right]

= \frac{1}{2} \left[ 1 + 2\mathbb{E}\{X_k 1_{X_k \leq k-1}\} + 2(k-1)\mathbb{P}\{X_k \geq k\} \right]

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\[
\frac{1}{2} \left[ 1 + \sum_{j=0}^{k-1} \binom{k+j-1}{j} \frac{2j}{2^{k+j}} + 2(k-1)(1 - P\{X_k < k\}) \right] \]
\[
= \frac{k}{2} + \sum_{j=0}^{k-1} \binom{k+j-1}{j} \frac{j}{2^{k+j}} \quad \text{(Using Lemma 6.1)}
\]
\[
= k - \left[ \frac{(2k-1)}{2^{2k-1}} \binom{2k-2}{k-1} \right]. \quad \text{(Using Lemma 6.2)}
\]

(18)

Therefore, we reproduce the same formula for the average degree as in Theorem 6.1 using the result in [10]. We discussed this remark with the assumption of sparseness. However, Theorem 6.1 only needs \( k = o(\sqrt{n}) \) which covers both the sparse and non-sparse regimes.

Next we determine the asymptotic behavior of the mean degree as both \( n \) and \( k \) go to infinity.

**Theorem 6.2.** In the model \( \mathbb{G}(n, k) \), assume that \( k = o(n^{1/3}) \) grows to infinity as \( n \) goes to infinity. Suppose \( D \) represents the degree of a randomly chosen vertex. We have the following asymptotic

\[
\mathbb{E}\{D\} = k - \sqrt{\frac{k}{\pi}} + \frac{1}{8\sqrt{\pi k}} + o\left(\frac{1}{\sqrt{k}}\right).
\]

**Proof.** We simply use Stirling’s approximation \( n! = \sqrt{2\pi n} (\frac{n}{e})^n \left(1 + \frac{1}{12n} + o\left(\frac{1}{n}\right)\right) \) to find an asymptotic expression for \( \binom{2k-2}{k-1} \)

\[
\binom{2k-2}{k-1} = \frac{(2k-2)!}{(k-1)!^2} = \sqrt{2\pi(2k-2)} \left(\frac{2k-2}{e}\right)^{2k-2} \left(1 + \frac{1}{12(2k-2)} + o\left(\frac{1}{k}\right)\right)
\]
\[
= 2\pi(k-1) \left(\frac{k-1}{e}\right)^{2k-2} \left(1 + \frac{1}{12(k-1)} + o\left(\frac{1}{k}\right)\right)
\]
\[
= \sqrt{2\pi(2k-2)} 2^{2k-2} \left(1 + \frac{1}{12(2k-2)} + o\left(\frac{1}{k}\right)\right)
\]
\[
= \frac{2\pi(k-1)}{\sqrt{\pi(k-1)}} \left(1 + \frac{1}{12(k-1)} + o\left(\frac{1}{k}\right)\right)
\]
\[
= \left(\frac{2^{2k-2}}{\sqrt{\pi(k-1)}}\right) \left(1 - \frac{1}{8(k-1)} + o\left(\frac{1}{k}\right)\right).
\]

Now Theorem 6.1 implies

\[
\mathbb{E}\{D\} = k - \left[ \frac{(2k-1)}{2^{2k-1}} \binom{2k-2}{k-1} \right] (O(k^2/n) + 1)
\]
\[
= k - \left[ \frac{(2k-1)}{2^{2k-1}} \left(\frac{2^{2k-2}}{\sqrt{\pi(k-1)}}\right) \left(1 - \frac{1}{8(k-1)} + o\left(\frac{1}{k}\right)\right) \right] (O(k^2/n) + 1)
\]
\[
= k - \left[ \left(\sqrt{\frac{k-1}{\pi}} + \frac{1}{2\sqrt{\pi(k-1)}}\right) \left(1 - \frac{1}{8(k-1)} + o\left(\frac{1}{k}\right)\right) \right] (O(k^2/n) + 1)
\]

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\[ k - \frac{1}{\pi} \sqrt{k - 1} - \frac{3}{8\sqrt{\pi(k-1)}} + o\left(\frac{1}{\sqrt{k}}\right) \]
\[ = k - \frac{1}{\pi} \sqrt{k} + \frac{1}{8\sqrt{\pi k}} + o\left(\frac{1}{\sqrt{k}}\right), \]
which proves the result. \qed

7 Conclusions

We studied bilateral agreement random graph family and presented a complete proof of the connectivity threshold conjecture for this family by La and Kabkab \cite{9}, namely, if \( k = t \log(n) \), then (with high probability) connectivity holds if \( t > 1 \), and the graph is disconnected if \( t < 1 \). The proof for disconnectivity result is established using the second moment method in which we proved that the expected number of isolated vertices grows to infinity and showed that asymptotically any pair of vertices is isolated in an independent fashion, i.e. \( \mathbb{E}\{I_1 I_2\} \approx \mathbb{E}\{I_1\} \mathbb{E}\{I_2\} \). Therefore, with high probability there exists an isolated vertex which yields the disconnectivity result.

For the connectivity part, we took the same approach as in La and Kabkab \cite{9} to find Erdős-Rényi sub and super random graphs with all vertices associated with the original graph. This analysis then excludes the possibility of having components with \( r \) vertices for \( 10 \leq r \leq n/2 \). Independently, we also ruled out the existence of components of size \( O(1) \). At the end, we discussed the average degree for this random graph family and presented an asymptotic for it.

As discussed earlier, our results are refinements of the connectivity conjecture of La and Kabkab \cite{9} by establishing higher orders for the connectivity threshold. In particular, when \( k = \log(n) + t' \log \log(n) \sqrt{\log(n)} \), we show that connectivity holds for \( t' > \frac{1}{2} \), and the graph is disconnected when \( t' < -\frac{1}{2} \) (both with high probability). However, these results still do not completely determine the connectivity threshold, and determining the precise connectivity threshold, even in terms of the precise value of \( t' \in (-1/2, 1/2) \), is an interesting avenue for future research.

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