Spin angular momentum of gravitational wave interference

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Abstract
Spin angular momentum (SAM) is an important feature for wave systems, prominent in various properties like spin-momentum locking of wave propagations. Here, we study the SAM of gravitational waves in general relativity based on the Noether theorem in classical field theory. We demonstrate gravitational wave spin in various interference cases and evanescent waves, which is manifested as locally rotating metric perturbations, driving particles on geodesic spheroid locally deformed in elliptical trajectories. For non-polarized gravitational wave modes of zero SAM, their superpositions can induce nonzero density of SAM with interfered patterns. The evanescent gravitational wave shows clear SAM, which is also a consequence of wave interference between propagating and transverse evanescent components. The spin-momentum locking relations are clearly present for all different interference cases of wave modes and components based on general relativity.

1. Introduction
Spin angular momentum (SAM) related concepts and topological mechanisms are the cornerstone of modern physics [1–4], which are responsible for tremendous intriguing phenomena. Recently, the interest about the SAM-related topological properties is being reinvigorated from quantum matters to classical wave systems [5–8], be it light [9], elastic wave [10], acoustic wave [11–13], as well as topological equatorial wave in geophysical systems [14, 15]. SAM across different wave systems share the similar mathematical expressions and endow similar physical properties.

These progress indicates that the SAM could be an intrinsic property for general wave systems, so that we expect similar ideas also apply to gravitational waves (GWs). Some early discussions for GW systems include localizing angular momentum [16], spin Hall effect [17], orbital angular momentum (OAM) [18], and evanescent waves with transverse spin [19]. In this work we will focus on the classical SAM and associated interference properties of GWs.

Following the successful detection of GW signal [20] and multi-messenger [21, 22], GW has now been an experimentally testable subject. The experimental study for properties of GW as a wave system, e.g. SAM and energy, will soon be in order. In addition to ground based detectors [23, 24], the space interferometers [25] and pulsar-timing arrays [26] will bring tremendous information on a large range of GW spectrum. Analogous to other wave systems, in addition to the simplest plane waves, studying scenarios with richer components, such as wave interference and evanescent waves, will become more and more important.

The stellar mass binary coalescence signals stay in the sensitive band of current detectors only for a few seconds and thus they usually do not interfere steadily. Nevertheless, future space-based detectors will detect sources with prolonged inspiral signals in-band, like intermediate or extreme mass ratio inspirals [27], existing for several months or years. Interference would be common for this kind of systems, which has effects with SAM. On the other hand, evanescent wave is also a generic system where nontrivial effects of SAM emerge. A recent work [19] discusses it in the framework of ‘Maxwellian form’ [28]. We here discuss...
the GW interference, evanescent GW as well as their associated SAMs based on the well tested theory of general relativity.

Since the study of wave SAM is bringing together multiple fields of research, to avoid confusion in terminology, we clarify here the usage of the words ‘SAM’, ‘polarization’ and ‘spin’ in study of GW, light wave and others. Traditionally, ‘polarization’ is used for classical waves whereas ‘spin’ is for quantum particles. The polarizations of GW are well used as textbook concepts [29, 30] and in GW detection and theoretical studies [20, 21], which are mathematical descriptions of different modes. ‘SAM’, along with OAM, is well defined as part of Noether current of rotation operation. They are related to the real physical quantity: angular momentum as well as torque, which makes them more physical and more useful than the mathematical definition of polarization. SAM and OAM have been widely used in study of light and other waves [31], showing that SAM can be non-zero for cases of unpolarized waves. Recently people also start to use SAM and OAM in the study of GW [16, 18]. Meanwhile, the word ‘spin’ of classical waves is endowed with different meanings in different contexts. In optics this usually means the intrinsic angular momentum of electromagnetic wave [9], sometimes short for ‘SAM’ of optic waves. While, for GW some people regard spin as polarization since quantized graviton picture is adopted [17] while some use spin as short for SAM [19]. Here, we choose SAM as the terminology throughout the work.

This article is organized as follows. We first describe the SAM for GWs in general relativity in section 2. Then we describe the interference between GWs and demonstrate the associated SAM in section 3. We study the SAM of evanescent GW based on general relativity in section 4. Finally, we conclude with discussions in section 5.

2. Spin angular momentum of gravitational waves in general relativity

Calculating the SAM of GW as the Noether current is discussed in textbooks [29, 30] within the standard framework in classical field theory. We briefly review the key elements in this section and give a representation based on spin-2 matrix.

The Einstein–Hilbert action for general relativity reads:

$$I = \frac{-c^2}{16\pi G} \int d^4x \sqrt{-g} R,$$  \hspace{1cm} (1)

where $g$ is the determinant of the metric $g_{\mu\nu}$, $R$ is the Ricci scalar, $c$ and $G$ are the light speed and gravitational constant. To facilitate derivation, we do an integration by part [32], removing all second-order derivatives:

$$I = \frac{-c^2}{16\pi G} \int d^4x \sqrt{-g} G + \text{boundary terms},$$  \hspace{1cm} (2)

where $\tilde{G} = g^{\mu\nu}(\Gamma^\beta_{\mu\alpha} \Gamma^\alpha_{\nu\beta} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha})$ with $\Gamma^\alpha_{\mu\beta}$ the Christoffel connection. Therefore, we can identify the Lagrangian density for the metric field in general relativity, as

$$\mathcal{L} = \frac{c^2}{16\pi G} \sqrt{-g} g^{\mu\nu}(\Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha} - \Gamma^\alpha_{\mu\beta} \Gamma^\beta_{\nu\alpha}),$$  \hspace{1cm} (3)

which will readily lead to the Einstein–Hilbert field equations when inserted into the Euler–Lagrange equation $\partial_\mu (\frac{\partial \mathcal{L}}{\partial (\partial_\mu R)}) + \partial_\mu (\frac{\partial \mathcal{L}}{\partial \partial_\mu R}) = 0$. Note that the index denoted by Greek letters ($\alpha$, $\beta$, $\mu$, $\nu$, ...) runs on four spacetime coordinates and Latin letters ($i$, $j$, $k$, $l$, ...) appearing latter will run on three spatial ones.

To describe the GW in the linearized gravity, we can perturb the metric field as $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $|h_{\mu\nu}| \ll 1$ and $\eta_{\mu\nu}$ is the background flat metric. Following the Noether’s theorem, one can obtain the SAM density corresponding to the above Lagrangian density of GWs (see appendix A for details), as

$$s_i = \frac{c^2}{16\pi G} \epsilon^i_{\beta\gamma} h_{\beta\gamma} \partial_\mu h_{\mu\nu}.$$  \hspace{1cm} (4)

For harmonic waves, we consider the complex version of GW tensor $h_{ij} \propto e^{-i\omega t}$ by using an over tilde to denote the complex tensor, which relates to the physical GW perturbation tensor by $h_{ij} = \text{Re}(\tilde{h}_{ij}) = \frac{1}{2}(\tilde{h}_{ij} + \tilde{h}_{ji})$ with $\dagger$ the Hermitian conjugate. Then the SAM density in the $i$th direction can be written as:

$$s_i = -i \frac{\omega c^2}{32\pi G} \epsilon_{ijk} \tilde{h}_{jk}.$$  \hspace{1cm} (5)

By denoting $[\tilde{h}^\dagger \times \tilde{h}]_i := \epsilon_{ijk} \tilde{h}^\dagger_{jk} \tilde{h}_{kl}$, we have the expression $s_i = \frac{\omega c^2}{32\pi G} \text{Im}[\tilde{h}^\dagger \times \tilde{h}]_i$. It is worthy noting that this expression shares the similar form as SAMs in other distinct wave systems, such as the electromagnetic wave spin [9, 31], the elastic wave spin [10], and the acoustic spin [11–13].
One subtlety in Noether approach is that the above expression is gauge-dependent, and a gauge independent Noether current does not exist for general relativity. Only the time-average is meaningful. As mentioned in reference [30], we can say that the SAM (and also the energy) can only be localized to the accuracy of a few periods. So when computing the SAM, we should always do a time average. As we will find in the following sections, specialized to the cases we consider, equation (4) is time-independent, i.e. itself the time-averaged result.

We could also express this SAM of GWs in terms of spin matrices to imply the spin-2 nature of GWs. To do this we choose the spinorial notation [33], so that \( \hbar_{ij} = \frac{\sqrt{\sqrt{G}}} {\sqrt{G^{(0)}}} \phi_{ij} \), where the GW is represented by a rank-4 spinor \( \phi_{ABCD} \), where capital Latin letters (A, B, C, D) run on (0, 1) to represent the spinor indexes, and \( \{S_{00}, S_{01}, S_{11}\} = -i\sigma_y \{\sigma_x, \sigma_y, \sigma_z\} \) with \( \sigma \); the Pauli matrix. \( \phi_{ABCD} \) is symmetric in exchange of any indexes, i.e. the permutation invariance, and therefore is fully determined by its five components \( \phi_{0000}, \phi_{0001}, \phi_{0011}, \phi_{1111} \). We can rearrange it into a five-dimensional vector:

\[
\tilde{\mathbf{h}} = \left[ \sqrt{\Phi_{0000}}, \sqrt{\Phi_{0001}}, \sqrt{\Phi_{0011}}, \sqrt{\Phi_{0111}} \right]^T,
\]

where the numbers under the square root are combination numbers \((\frac{1}{2})_i, (\frac{1}{2})_i, (\frac{1}{2})_i, (\frac{1}{2})_i\), counting the recurring times of each components. By this definition the norm is conserved:

\[
\mathbf{1} = \sqrt{\phi} \cdot \sqrt{\phi} = \mathbf{1},
\]

satisfying the commutation relation \([\mathbf{J}_i, \mathbf{J}_j] = i\hbar \epsilon_{ijk} \mathbf{J}_k\).

Note that the Planck constant \(\hbar\) merely appears as a convention in the definition of spin matrices \(\mathbf{J}_i\), and throughout the paper we are only discussing the SAM of GWs in classical general relativity. The more compact equation (7) is equivalent to equation (5). This bilinear GW SAM also shares the similar form as in other distinct wave systems, such as the electromagnetic wave spin [9, 31], the elastic wave spin [10], and the acoustic spin [11–13], but here with different spin-2 matrices \(\mathbf{J}_i\) instead of spin-1 matrices therein. As in a quantized gauge theory, the electromagnetic wave can be quantized into photon leading to the photonic quantum spin operators [34], while this quantization may not be directly applicable to the gravity in general relativity.

3. Wave interference induced spin

Using the notation above, we would like to discuss the physical picture of SAM endowed by GW interferences composed of multiple plane waves. Rigorously speaking, we are discussing such kind of solution to Einstein’s equation: \( \tilde{h}_{\mu \nu}^{(n)} = h_{\mu \nu}^{(1)} + h_{\mu \nu}^{(2)} + h_{\mu \nu}^{(3)} + \cdots \), where each component \( h_{\mu \nu}^{(n)} \propto e^{i\kappa_{\mu \nu}} \) is a
Figure 1. Motion patterns excited by GW spin induced by interference. (a) A cross mode GW along x axis combines another cross mode GW along y axis with phase difference $\pi/2$ into a total GW with local rotation field, which forms SAM along z axis. The red dots and black curves show local motion and trajectory of point particles following the GW-induced metric evolution. The orange spheroid shows the instantaneous shape formed by the motion of multiple points originally on a sphere. The red two-way arrow indicates the principle axis of this spheroid. (b) A plus mode GW along x axis interferes with another plus mode GW along $\hat{x} + \hat{y}$ with phase difference $\pi/2$ to form a locally rotated GW field, with a SAM along z axis. (c) A plus mode GW along x axis and a cross mode GW along y axis with phase difference $\pi/2$ are interfered into a rotated field with SAM along y axis. (d) Circular polarized GW of nonzero SAMs can also be synthesized as the interference between plus and cross modes traveling along the same (or opposite) direction with phase difference $\pm \pi/2$. (The spheroids indicate spatial components of the metric perturbation. See appendix C for details).

Figure 2. Distribution of GW SAM induced by interference. (a) When a cross mode plane GW traveling along x axis interfere with another cross mode plane GW along y axis. The interference pattern will be formed on the x–y plane, forming a non-zero SAM component along z axis. (b) When a plus mode plane GW traveling along x axis interfere with another plus mode plane GW with an angle of $\pi/4$. The interference pattern will be formed on the x–y plane, forming a non-zero SAM component along z axis. (c) When a plus mode plane GW traveling along x axis interfere with a cross mode plane GW along y axis. The interference pattern will be formed on the x–y plane, forming a non-zero SAM component along y axis.

plane wave solution with wave vector $k_\mu^{(n)}$ in transverse traceless (TT) gauge (thus $\tilde{h}^\mu_{total \nu}$ also satisfy TT gauge). Such interference wave solution is different from that of single plane wave in many perspectives, since they do not have a single well-defined wave vector.

Some nontrivial effects will emerge when considering interference solutions. For instance, the SAM will have hybrid components, as in the case of elastic wave [10]. Take double wave interference as an example, the total SAM is proportional to $\tilde{s}_{int} \propto (\tilde{h}^{(1)} + \tilde{h}^{(2)})^\dagger \tilde{J} (\tilde{h}^{(1)} + \tilde{h}^{(2)})$, where $\tilde{h}^{(n)}$ is related to $\tilde{h}_{total}^{\mu\nu}$ through equation (6) and appendix B. Due to the bi-linearity of the spin-2 operator $\tilde{J}$, besides the SAM of each component $\tilde{h}^{(1)} \tilde{J} \tilde{h}^{(2)}$, we see an additional hybrid SAM part:

$$\tilde{s}_{h} = \frac{c^2}{2\pi G a^2 h} \text{Re}[\tilde{h}^{(1)} \tilde{J} \tilde{h}^{(2)}].$$

(11)

Therefore, even when each component of GWs does not process any SAM, the interferences among them are able to produce nonzero hybrid SAM.
In general relativity, each GW component can hold two different independent modes: the cross mode (×) and plus mode (+). Considering the wave interference and geometric properties of GWs, the hybrid spin $\tilde{\mathbf{s}}_h$ can be decomposed further into (a) cross-mode wave spin; (b) plus-mode wave spin; (c) mixed-mode wave spin, expressed as

$$\tilde{\mathbf{s}}_h = \tilde{\mathbf{s}}_{\times \times} + \tilde{\mathbf{s}}_{+ +} + \tilde{\mathbf{s}}_{+ \times}.$$  \hfill (12)

The physical mechanism behind these interfered GW SAMs can be understood as the non-trivial rotation projection from one mode to another mode due to the geometrical differences \[10\]. As such, one can exploit the GW interference to synthesize the chirally (in general, elliptically) polarized local field to obtain non-zero SAMs, even when each wave component is linear polarized.

To exemplify the interference-induced SAM, let us consider a plane GW $h_{ij}^{(1)}$ propagating along $x$ axis:

$$\tilde{h}_{ij}^{(1)} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \tilde{h}_{11}^{(1)} & \tilde{h}_{12}^{(1)} \\ 0 & \tilde{h}_{12}^{(1)} & -\tilde{h}_{11}^{(1)} \end{bmatrix},$$  \hfill (13)

and another plane GW $h_{ij}^{(2)}$ propagating in another direction in $x$–$y$ plane, forming an angle of $\theta$ with $x$ axis:

$$\tilde{h}_{ij}^{(2)} = \begin{bmatrix} \tilde{h}_{11}^{(2)} \sin^2 \theta & -\frac{3}{4} \tilde{h}_{12}^{(2)} \sin(2\theta) & -\tilde{h}_{11}^{(2)} \sin \theta \\ -\frac{3}{4} \tilde{h}_{12}^{(2)} \sin(2\theta) & \tilde{h}_{12}^{(2)} \cos^2 \theta & \tilde{h}_{11}^{(2)} \cos^2 \theta \\ -\tilde{h}_{11}^{(2)} \sin \theta & \tilde{h}_{12}^{(2)} \cos^2 \theta & -\tilde{h}_{11}^{(2)} \sin \theta \end{bmatrix},$$  \hfill (14)

where $\tilde{h}_{ij}^{(n)}$ are complex functions of plane waves: $\tilde{h}_{ij}^{(n)} \propto \exp[i(\tilde{\omega}^{(n)} \vec{x} - \omega t)]$ with $\vec{x} = (x^1, x^2, x^3)$, i.e. $(x, y, z)$.

Substituting $\tilde{h}_{ij} = \tilde{h}_{ij}^{(1)} + \tilde{h}_{ij}^{(2)}$ into equation \((5)\) leads to the hybrid SAM of interfered GW (following the equation \((7)\) will give us the same results in what follows). The existence of SAM can be manifested by the geodesic motion of a series of particles. These particles are at rest on a sphere when GW is absent. When GW comes, the sphere is deformed into a spheroid, and the major axis is ‘spinning’ if the GW have nonzero SAM. We here consider three special cases, as illustrated in figure \(1\).

- **Cross-mode GW spin by cross–cross interference**: given $\tilde{h}_{1+}^{(1)} = \tilde{h}_{1+}^{(2)} = 0$, $\tilde{h}_{1-}^{(1)} = A \exp[i(\phi_1 - \omega t)]$ with $\phi_1 = \tilde{x}^1 + \phi_0^1$, $\tilde{h}_{1-}^{(2)} = A \exp[i(\phi_2 - \omega t)]$ with $\phi_2 = \tilde{x}^2 + \phi_0^2$, $h_1^{(1,2)}$ do not carry SAM individually. The hybrid SAM is contributed by interference between cross modes of different directions. We get the non-vanishing component of SAM along $z$ axis, as:

$$s_z = \frac{A^2 \omega^2}{16\pi G} \sin \theta \sin(\phi_2 - \phi_1).$$  \hfill (15)

$\theta$ is the angle between two waves and the terms in the bracket is actually the phase difference. More compactly, the hybrid GW SAM reads

$$\tilde{z}_{\times \times} = \frac{A^2 \omega^2}{16\pi G} \sin(\phi_2 - \phi_1) \frac{\tilde{k}^{(1)} \times \tilde{k}^{(2)}}{|\tilde{k}^{(1)}| |\tilde{k}^{(2)}|},$$  \hfill (16)

in this case. The geodesic motion caused by GW SAM and the corresponding spatial distribution of $s_z$ for $\theta = \pi/2$, $\phi_0^1 - \phi_0^2 = 0$ case are illustrated in the figures \(1(a)\) and \(2(a)\), respectively.

- **Plus-mode GW spin by plus–plus interference**: given $\tilde{h}_{1+}^{(1)} = \tilde{h}_{1+}^{(2)} = 0$, $\tilde{h}_{1-}^{(1)} = A \exp[i(\phi_1 - \omega t)]$ with $\phi_1 = \tilde{x}^1 + \phi_0^1$, $\tilde{h}_{1-}^{(2)} = A \exp[i(\phi_2 - \omega t)]$ with $\phi_2 = \tilde{x}^2 + \phi_0^2$. $h_1^{(1,2)}$ do not carry SAM individually. We then obtain the only non-vanishing component of interference SAM, also along $z$ axis, as:

$$s_z = \frac{3A^2 \omega^2}{64\pi G} \sin 2\theta \sin(\phi_2 - \phi_1).$$  \hfill (17)

In a more compact form, it reads:

$$\tilde{z}_{++} = \frac{3A^2 \omega^2}{32\pi G} \cos \theta \sin(\phi_2 - \phi_1) \frac{\tilde{k}^{(1)} \times \tilde{k}^{(2)}}{|\tilde{k}^{(1)}| |\tilde{k}^{(2)}|}.$$  \hfill (18)

The geodesic motion excited by this GW SAM and the spatial distribution profile of $s_z$ for $\theta = \pi/4$, $\phi_0^1 - \phi_0^2 = 0$ case are illustrated in the figures \(1(b)\) and \(2(b)\), respectively.

- **Mixed-mode GW spin by plus–cross interference**: setting $\tilde{h}_{1+}^{(1)} = \tilde{h}_{1+}^{(2)} = 0$, $\tilde{h}_{1-}^{(1)} = A \exp[i(\phi_1 - \omega t)]$ with $\phi_1 = \tilde{x}^1 + \phi_0^1$, $\tilde{h}_{1-}^{(2)} = A \exp[i(\phi_2 - \omega t)]$ with $\phi_2 = \tilde{x}^2 + \phi_0^2$, then only the interference
between plus mode and cross mode contribute to the hybrid SAM. Thus, we get the only non-vanishing component of SAM is in the same direction as $\hat{k}^{(2)}$ of the cross mode:

$$\tilde{z}_{+x} = \frac{A^2\omega c^2}{16\pi G} \sin \theta \sin(\phi_2 - \phi_1) \hat{k}^{(2)}. \quad (19)$$

For the case of orthogonal interference, $\theta = \pi/2$, $\phi_2 - \phi_1 = 0$, the wave vector $\hat{k}^{(2)}$ of $\tilde{z}_{+x}$ is along $y$ direction. The geodesic motion of this GW SAM and the associate distribution profile of $s_x$ are illustrated in the figures 1(c) and 2(c), respectively. We note that different from the first two interference cases that cross- or plus-mode GW spin are transverse SAM, perpendicular to the GW direction, this mixed-mode SAM has the same direction along with the cross-mode GW component.

It is worth noting that when `+' mode and `×' mode GWs are propagating in the same direction, i.e. along z axis as depicted in figure 1(d), we obtain the chirally polarized field with mixed-mode interfered spin: $s_x = \frac{A^2\omega c^2}{8\pi G} \sin(\phi_0^0 - \phi_0^x)$. This mixed-mode GW induced by interference becomes a left(right)-circular polarized wave when $\phi_0^x - \phi_0^x = \pm \pi/2$. As such, the ratio between SAM density and GW energy density $\tau_{00} = \frac{A^2\omega c^2}{16\pi G}$ is a constant $s_x/\tau_{00} = \pm 2/\omega$, indicating again the spin-2 nature of GWs. The SAM direction of this case will be locked longitudinally with the propagation direction, i.e. the tight spin-momentum locking of certain helicity.

4. SAM in evanescent gravitational waves

Besides the propagating wave cases, spatially confined evanescent waves will also possess similar rich properties about SAM. In general, evanescent waves have been playing important roles in many prominent phenomena for diverse wave systems, like the spin-momentum locking in electromagnetic waves [9, 35], elastic waves [10], and acoustic waves [11–13]. The evanescent GW has also been discussed recently [19] and was discussed to exist near binary sources. They also notice that for the coherent superposition of two propagating GW plane waves arriving simultaneously at one detector, the combined polarization may be locally (not globally) identical to that of an evanescent GW, in analogy to the case of electromagnetic evanescent wave vs two-wave interference [31]. Here we apply the general formula of GW SAM to the evanescent case, taking the evanescent GW SAM as the consequence of interference between the propagating component and the orthogonal-confined evanescent component. Note that our results is based on the general relativity and Noether’s theorem in classical field theory.

For evanescent GWs, the metric perturbation $h_{ij}$ is given by:

$$h_{ij} = \text{Re}[H_{ij} e^{ik \cdot x - i\omega t}], \quad (20)$$

where $H_{ij}$ is the complex-valued polarization matrix. As shown by [19], requiring $h_{ij}$ to satisfy TT gauge means that the polarization tensor satisfies:

$$H_{ij} = h_{++} \begin{pmatrix} \kappa^2 & 0 & -i\alpha\kappa \\ 0 & -1 & 0 \\ -i\alpha\kappa & 0 & -\alpha^2 \end{pmatrix} + h_{××} \begin{pmatrix} \kappa & 0 & 0 \\ 0 & 0 & -i\alpha \\ 0 & -i\alpha & 0 \end{pmatrix}, \quad (21)$$

with wave-vector $k = \omega/c(\alpha, 0, \kappa)$ where $\alpha, \kappa$ are real and satisfying $\kappa^2 - \alpha^2 = 1$. Note that coordinates are aligned for convenience so that the wave vector is propagating in z-direction and decaying in x-direction. $\kappa$ denotes the propagating component while $\alpha$ denotes the evanescent component. Inserting the perturbation metric equation (20) into the SAM expression equation (4), we obtain:

$$s_x = \frac{c^2}{8\pi G} \omega c^3 e^{-2n\omega c/\omega} \text{Re}[h_{+} h_{+}^*],$$

$$s_y = \frac{c^2}{16\pi G} \omega c^3 e^{-2n\omega c/\omega} ((\kappa^2 + \alpha^2)|h_{+}|^2 + |h_{×}|^2), \quad (22)$$

$$s_z = \frac{c^2}{8\pi G} \omega c^3 e^{-2n\omega c/\omega} \text{Im}[h_{+} h_{+}^*],$$

where $h_{+}^*$ is the complex conjugate of $h_{+}$. The SAM exhibits one longitudinal spin component $s_z$ along with the propagation direction and two transverse spin components $s_x$ and $s_y$ perpendicular to the propagating direction. From the expressions, one see clearly that the longitudinal spin $s_z$ relates to the helicity, resulting from the interference between the propagating components of $h_{+}$ and $h_{×}$ modes. The transverse spin $s_x$ results from the interference between the propagating wave component and the orthogonal-confined
evanescent wave component, so that $s_j$ is proportional to the product $\alpha \kappa$. The other transverse spin $s_x$, along the wave vector decaying direction, has additional contributions from the interference between two orthogonal transverse evanescent wave components. When we reverse the propagating wave component $\kappa$, both transverse spin $s_j$ and longitudinal spin $s_y$ reverse the sign, showing the tight spin-momentum locking, but with $s_x$ keeping intact.

To further simplify the math and magnify the physics, we can further express the SAM density in terms of the energy density. Let us calculate $\tau_{00}$, the energy density endowed by the evanescent wave, i.e. the 00 component of energy–momentum tensor $\tau_{\mu\nu} = \delta_{\mu0}L - \frac{\partial L}{\partial (\partial_0 h_{\alpha0})}\partial_0 h_{\alpha0}$. For GWs in general relativity, the Lagrangian takes the form equation (3) and the energy density reads

$$\tau_{00} = \frac{c^2}{64\pi G} \left( (\partial_0 h_{ij})^2 + (\partial_i h_{0j})^2 - \partial_j h_{0i}\partial_0 h_{ij} \right).$$

(23)

Note that the index denoted by small Latin letters $(i,j,k,l,\ldots)$ runs on $(1,2,3)$ or say $(x,y,z)$. Inserting the evanescent wave equation (20) and integrating the oscillation part, we have the time-averaged energy density:

$$\langle \tau_{00} \rangle = \frac{c^2}{32\pi G} \omega^2 \kappa^2 e^{-2i\omega/c} (|h_+|^2 + |h_x|^2).$$

(24)

As analogous to the electromagnetic wave [31], we define three Stokes parameters

$$\rho = \frac{|h_+|^2 - |h_x|^2}{|h_+|^2 + |h_x|^2}, \quad \sigma = \frac{2 \Im[h_+ h_x^*]}{|h_+|^2 + |h_x|^2}, \quad \chi = \frac{2 \Re[h_+ h_x^*]}{|h_+|^2 + |h_x|^2}.$$ 

(25)

Then the SAM density for the evanescent GW in terms of the energy density, is re-expressed as:

$$s_x = \frac{2\chi\alpha^2 \langle \tau_{00} \rangle}{\omega \kappa^2}, \quad s_y = \frac{2\alpha \langle \tau_{00} \rangle}{\kappa \omega} (\kappa^2 + \alpha^2 \rho), \quad s_z = \frac{2\sigma \kappa \langle \tau_{00} \rangle}{\omega},$$

(26)

where the factor 2 implies the spin-2 nature of GWs, and $\sigma$ is reminiscent of the helicity coefficient. Semi-classically, if the GW energy $\langle \tau_{00} \rangle$ is quantized as $\sim h\omega$, then each quantum of GW will possess $2\sigma h\kappa$ as the longitudinal spin. When the decaying evanescent component vanishes $\alpha \to 0$, the real wave vector will recover as $\kappa \to 0$ so that the SAM will reduce to the helicity $|s^z|/\langle \tau_{00} \rangle = 2\sigma/\omega$, the same as the circularly polarized case discussed in figure 1(d).

Analogous to reference [31] for evanescent electromagnetic wave, the evanescent GW SAM equation (26) can take a more compact form, as:

$$\vec{s} = \frac{2\langle \tau_{00} \rangle}{\omega} \left( \sigma \kappa^2 \frac{\omega}{\epsilon} \left| \frac{\Re[\hat{k}]}{\Re[\hat{k}]} \right|^2 + (\kappa^2 + \alpha^2 \rho) \frac{\Re[\hat{k}] \times \Im[\hat{k}]}{\left| \Re[\hat{k}] \right|^2} + \chi \alpha^2 \omega \frac{\Im[\hat{k}]}{\left| \Re[\hat{k}] \right|^2} \right).$$

(27)

The first term is the longitudinal spin $s_x$ along the propagation direction related to the helicity, which can be regarded as the result from the interference between the plus- and cross-modes. The second term is the transverse spin $s_y$ perpendicular to the propagating direction $\Re[\hat{k}]$ and decaying direction $\Im[\hat{k}]$, which can be regarded as the consequence of interference between the propagating wave component and the orthogonal-confined evanescent wave component. The last term is $s_z$, parallel to the decaying direction, which as we will see in the following is also a consequence of interference.

Let us decompose the evanescent GW $H_{ij}$ into five parts of propagating plane wave components:

$$H_{ij} = h_+ e^+_i(\tilde{z}) + \alpha^2 h_+ e^+_y(\tilde{y}) - i\alpha h_+ e^+_y(\tilde{y}) + \kappa h_x e^+_y(\tilde{z})$$

$$- i\omega h_x e^+_y(\tilde{x}),$$

(28)

where $e^+_\tilde{z}(\tilde{z}), e^+\tilde{y}(\tilde{y}), e^\tilde{x}(\tilde{x}), e^-\tilde{z}(\tilde{z}), e^-\tilde{x}(\tilde{x})$ are the five polarizations.
Based on what we have discussed in section 3, three pairs of coherent superposition can produce the interfered transverse SAM in \( \hat{x} \) direction: \( e^+(\hat{\tilde{z}}) \) with \( e^- (\hat{x}) \), \( e^{-} (\hat{\tilde{y}}) \) with \( e^{+} (\hat{x}) \), and \( e^{+} (\hat{\tilde{z}}) \) with \( e^{-} (\hat{\tilde{y}}) \). Specifically, using equation (19), interference between plus and cross polarizations will give rise to SAM along the wave vector of cross polarization. As such, the interferences of \( e^{+} (\hat{\tilde{z}}) \) with \( e^{-} (\hat{x}) \), and \( e^{+} (\hat{\tilde{y}}) \) with \( e^{-} (\hat{x}) \) will produce SAM in \( \hat{x} \) direction. Also, following equation (16), interference between cross polarizations, i.e. \( e^{+} (\hat{\tilde{z}}) \) with \( e^{-} (\hat{\tilde{y}}) \), leads to SAM in the direction of \( \hat{\tilde{y}} \times \hat{\tilde{z}} = \hat{x} \). Similar arguments will apply to other interference-induced GW SAM.

Furthermore, we would like to point out that our equation (27) (equivalently, equations (22) and (26)) is distinct from the equation (6) in reference [19], since the presence of additional SAM along the wave vector decaying direction \( \text{Im}[\hat{k}] \). This is attributed to the two special polarizations of GW, plus and cross modes with \( \pi/4 \) azimuth angle difference, which will produce complex effective components in the evanescent case, see equation (28). Also our result is not symmetric between \( h_+ \) and \( h_\times \), as controlled by the first Stokes parameter \( \rho \), which is expected because the decaying wave vector breaks the symmetry between plus and cross modes in the evanescent case. The reason of those difference relies on the fact that the present results about the SAM of GW are based on the general relativity in its classical form, i.e. the conventional Lagrangian density from Einstein–Hilbert action. Reference [19] used a modified version of 'Maxwellian form' [28], whose Lagrangian is established using the argument of duplex symmetry. Though both Lagrangians lead to the same equations of motion, their Noether currents are different, which is expected by the classical field theory, in analog to the case of transverse spin in evanescent electromagnetic waves [31].

5. Conclusion and discussion

The SAM of GWs is an important part of gravitational wave physics. In this work, we have discussed the classical SAM of GWs in general relativity. We have shown that the non-trivial gravitational wave spin is essentially a local chiral polarized field and can be constructed by wave interferences. For interference of multiple plane waves, additional SAM has been induced and coupled to the wave vector. For interference of two plane waves with induced SAM, local particles follow a peculiar three dimensional pattern of motion as shown in figure 1. This could be important in both detecting GWs and understanding the associated physical effect in stellar medium. The gravitational SAM and associated torque properties could be a powerful tool for studying unknown cosmological physics, or even trapping and guiding bodies, as in the study of GW OAM [18]. The study of SAM is also of theoretical importance as a bridge between classical and quantum theories. Further explorations on the quantized theory of global spin operator [34] of GW and possible skyrmionic behavior [36] will be interesting in the future.

We here discuss the chance for observing the interfered GW that will be the testbed for the interfered GW SAM. Binary coalescence signals are short and thus unlikely to be coincident. Though interference may happen for lensed sources [37, 38], the wave vectors for the two interfering waves in gravitational lens system are almost identical and thus challenging to detect. But with future space-based detectors like LISA, extreme-mass ratio inspirals (EMRIs) are promising sources based on event rate estimation [39–41]. If two EMRIs of comparable amplitude from different sky locations arrive at the same period of time, interference will be significant.

We give a rough estimation on the likelihood of interference: based on the model M1 of [39], the number of EMRI expected to be seen by LISA in its 2 years lifetime with MBH mass \( 10^5 M_\odot < M < 10^{5.5} M_\odot \), is \( \sim 260 \). In the final stage of EMRI, the frequency is fixed by the innermost stable circular orbit and thus similar for different signals. The time period for the final stage is \( \sim 0.02 \) years. Thus the chance for two events being coincident is \( p_1 \sim 0.02 \) assuming the events are uniformly distributed in time (suppose that two independent events, each last for 0.02 years, may start at any time in the 2 years, so the probability for them to co-exist is \( p_1 = 1 - (2 \cdot 0.02)^2 \sim 0.02 \)). For the amplitude to be comparable, let the difference in the red shift distance be \( \delta z < 0.1 \). Suppose LISA can detect events up to red shift distance \( z < 5 \). The chance is \( p_2 = 1 - (\frac{2.5 - \delta z}{5})^2 \sim 0.04 \) assuming uniform distribution in distance (this makes sense because although phase space is larger for further distances, signal is weaker). Therefore, for the \( (260) \) \( \cdot 33670 \) pairs in the 260 events, we have at least \( 33670 \cdot p_1 \cdot p_2 \sim 27 \) pairs expected to interfere. Then for the 230 events with MBH in the range \( 10^5 M_\odot < M < 10^6 M_\odot \), we have \( (230) \cdot p_1 \cdot p_2 \sim 21 \) pairs expected to interfere.

Our results also have been applied to the evanescent case of complex wave vector. When gravitational wave is damping along the direction of the imaginary wave vector part, the transverse spin can be originated by the interference between the propagation and evanescent components, forming a tight spin-momentum locking, analogous to the electromagnetic waves [35]. The evanescent wave will be common for gravitational wave sourced by binary systems. Further analysis on the merger process and the quasi-normal ringdowns may supply more information for evanescent GWs.
The discrepancy between SAM given by the presently used Einstein–Hilbert action and the dual symmetric Lagrangian used in reference [19] is very similar to what was previously studied in electromagnetism. In electromagnetism, if we use the classical Lagrangian \( \mathcal{L}_{EM} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \), we would get an SAM expression \( s_i = \epsilon_{\alpha\beta\gamma} \partial_\alpha A_\beta A_\gamma \) which is twice the electric part of ‘true’ SAM as given by the dual symmetric Lagrangian [42]. Similarly, if we follow duplex symmetric Lagrangian for GWs [28], we would get additional terms due to the new ‘dual field’ canceling terms proportional to \( \rho \) and \( \chi \). The present equation (27) using Einstein–Hilbert action will be twice the electric part of duplex SAM. However, whether the dual symmetry between gravitational analogue of ‘electric’ field and ‘magnetic’ field should be promoted as a fundamental principle of gravity is still open to be tested. Without further physical input, like experiment measuring the SAM of GWs, we cannot argue which one represents the physically relevant quantity. Therefore, it seems detecting the GW SAM will be promising to test the different theories of GWs.

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Appendix A. Derivation of spin angular momentum density

In general, for a field with any indexes \( \psi_{\alpha_1\alpha_2...} \) under a transformation \( U_{\alpha_1\alpha_2...} \) (acting on the field as \( [U_{\alpha_1\alpha_2...}]_{\rho_1\rho_2...} \)), the conserved current is:

\[
\rho^\mu_{\alpha_1\alpha_2...} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi_{\rho_1\rho_2...})} [U_{\alpha_1\alpha_2...}]_{\rho_1\rho_2...} \psi_{\alpha_1\alpha_2...}, \tag{A1}
\]

according to Noether’s theorem. The SAM is the conserved current in rotational transformation. Consider the gravitational perturbation field \( h_{\mu\nu} \) with Lagrangian given by equation (3) in Lorentz transformation \( \tilde{M}_{\alpha\beta} \) (including rotation). The conserved current \( S^\mu_{\alpha\beta} \) is given by:

\[
S^\mu_{\alpha\beta} = \frac{\partial \mathcal{L}}{\partial (\partial_\mu h_{\rho\sigma})} [\tilde{M}_{\alpha\beta}]^\lambda_\rho h_{\lambda\sigma}, \tag{A2}
\]

where \( [\tilde{M}_{\alpha\beta}]^\lambda_\rho \) is the generator of Lorentz group in tensorial representation. The relation between \( S^\mu_{\alpha\beta} \) and the usual three-dimensional notion of spin \( s_x, s_y, s_z \) is \( s_x = S^0_{12}, s_y = S^0_{13}, s_z = S^0_{23} \). So we only consider \( S^0_{\alpha\beta} \) components of the SAM tensor.

We first rewrite \( \tilde{G} \) so that the dependency on \( \partial_\mu h_{\rho\sigma} \) appears explicitly: \( \tilde{G} = g^{\mu\nu} g^{\sigma\rho} g^{\beta\chi} (\Gamma_{\gamma\rho\lambda} \Gamma_{\delta\sigma\nu} - \Gamma_{\gamma\rho\sigma} \Gamma_{\delta\mu\lambda}) \), where the Christoffel symbol here is given by:

\[
\Gamma_{\rho\sigma\lambda} = \frac{1}{2} (\partial_\rho h_{\sigma\lambda} + \partial_\sigma h_{\rho\lambda} - \partial_\lambda h_{\rho\sigma}). \tag{A3}
\]

Then by direct calculation we get:

\[
\frac{\partial \mathcal{L}}{\partial (\partial_\mu h_{\rho\sigma})} = -\frac{\sqrt{-g}}{32\pi} \left( 2\Gamma^{\rho\sigma\tau} + 2\Gamma^{\rho\tau\sigma} - 2\Gamma^{\rho\sigma\tau} - 2g^{\rho\sigma} \Gamma_{\tau\mu} + g^{\rho\sigma} \Gamma_{\beta}^{\mu\tau} - g^{\rho\sigma} \Gamma_{\tau}^{\mu\beta} \right). \tag{A4}
\]

Substituting the definition of Christoffel symbol \( \Gamma \) into equation above, we shall get a rather lengthy expression, but in the TT gauge most terms vanish. Imposing the TT gauge condition \( (h_{\mu\tau} = 0, \partial_\mu h_{\rho\sigma} = 0, h_{kk} = 0) \), we find that the result simplifies to:

\[
\frac{\partial \mathcal{L}}{\partial (\partial_\mu h_{\rho\sigma})} = \sqrt{\frac{\sqrt{-g}}{32\pi} \partial_\nu h^{\mu\sigma}}. \tag{A5}
\]

The generator of Lorentz group \( \tilde{M} \) is given by

\[
[\tilde{M}_{\alpha\beta}]^\mu_\rho = [M_{\alpha\beta}]^\mu_\rho g_{\sigma\nu} + g^{\mu\rho} \left[ M_{\alpha\beta} \right]_\nu^\sigma, \tag{A6}
\]

where the generator for vector representation is \( [\tilde{M}_{\alpha\beta}]^\mu_\rho = -g^{\mu\rho} g_{\beta\nu} + g^{\mu\rho} g_{\alpha\nu} \).

Substituting equations (A5) and (A6) into (A2), we finally obtain:

\[
S^0_{\alpha\beta} = \frac{\sqrt{\frac{\sqrt{-g}}{32\pi}}}{16\pi G} (\partial_\nu h_{\alpha\beta} h^{\lambda\nu} - \partial_\nu h_{\lambda\nu} h_{\alpha\beta}). \tag{A7}
\]
The higher order terms in the determinant $g = -1 + O(h^2)$ can be ignored. Our result is consistent with reference [16] for flat background. The usual three-dimensional notion of SAM density $\vec{s} = (s_x, s_y, s_z)$ is related to the tensor $S_{\mu \nu}$ by $s_i = \frac{1}{2} \epsilon_{ijk} S_{\mu \nu}$. Therefore we have the SAM density endowed by GWs, as:

$$s_i = \frac{c^2}{16\pi G} \epsilon_{ijk} h_{ij} \partial_k h_{kl}. \quad (A8)$$

**Appendix B. Spinorial formalism**

Here, we briefly go over the spinorial formalism and provide the detailed relation between $\phi_{ABCD}$ and $\tilde{h}_{ij}$.

**B.1. Definition of gravitational spinor**

Conventionally, one defines the self-dual Riemann tensor as [43]:

$$G_{\mu \nu \alpha \beta} = R_{\mu \nu \alpha \beta} + \frac{i}{2} \epsilon_{\mu \nu \alpha \beta} R. \quad (B1)$$

A natural representation of this tensor through the rank-4 spinor $\phi_{ABCD}$ is:

$$G_{\mu \nu \alpha \beta} = S_{\mu \nu}^{AB} C_{\alpha \beta}^{CD} \phi_{ABCD}, \quad (B2)$$

where $S_{\mu \nu}^{AB}$ is symmetric in the two lower indexes and

$$\{S_{01, 02, 03}\} = -i \sigma_y \{\sigma_x, \sigma_y, \sigma_z\}, \quad (B3)$$
$$\{S_{23, 31, 12}\} = i \{S_{01, 02, 03}\}. \quad (B4)$$

More explicitly:

$$S_{01} = -i \sigma_y \sigma_x = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (B5)$$
$$S_{02} = -i \sigma_y \sigma_y = \begin{bmatrix} -i & 0 \\ 0 & -i \end{bmatrix}, \quad (B6)$$
$$S_{03} = -i \sigma_y \sigma_z = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad (B7)$$
$$S_{23} = i S_{01} = \begin{bmatrix} -i & 0 \\ 0 & i \end{bmatrix}, \quad (B8)$$
$$S_{31} = i S_{02} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad (B9)$$
$$S_{12} = i S_{03} = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}. \quad (B10)$$

The $S_{\mu \nu}^{AB}$ means the $(A, B)$ element of the matrix. For instance $S_{03} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, we have $S_{03}^{00} = 0, S_{03}^{01} = 1, S_{03}^{10} = 1, S_{03}^{11} = 0$.

For components of the self-dual Riemann tensor, we have:

$$G_{0101} = \phi_{0000} - 2 \phi_{0011} + \phi_{1111},$$
$$G_{0102} = i \phi_{0000} - i \phi_{1111},$$
$$G_{0103} = 2 \phi_{0011} - 2 \phi_{0000},$$
$$G_{0202} = - \phi_{0000} - 2 \phi_{0011} - \phi_{1111},$$
$$G_{0203} = -2i \phi_{0000} - 2i \phi_{0011},$$
$$G_{0303} = 4 \phi_{0011}. \quad (B11)$$

Using standard spinorial formalism, the relation between Riemann tensor $R_{\mu \nu \alpha \beta}$ and a completely symmetric spinor $\phi_{ABCD}$ is [18, 33, 43]:

$$R_{\mu \nu \alpha \beta} = \text{Re}(S_{\mu \nu}^{AB} C_{\alpha \beta}^{CD} \phi_{ABCD}) = \text{Re}(G_{\mu \nu \alpha \beta}). \quad (B12)$$
The relations among components are:

\[ R_{0001} = \Re(\phi_{0000} - 2\phi_{0011} + \phi_{1111}), \]
\[ R_{0102} = \Re(\imath\phi_{0000} - \imath\phi_{1111}), \]
\[ R_{003} = \Re(2\phi_{0011} - 2\phi_{0000}), \]
\[ R_{0202} = \Re(-\phi_{0000} - 2\phi_{0011} - \phi_{1111}), \]
\[ R_{0203} = \Re(-2\imath\phi_{0001} - 2\phi_{0011}), \]
\[ R_{0003} = \Re(4\phi_{0001}). \]  

(B13)

In TT gauge, given the harmonic gravitational wave, then:

\[ R_{00ij} = \frac{\omega^2}{2} h_{ij}. \]  

(B14)

The general solution to gravitational wave equation, represented by spinorial formalism, is:

\[ \phi_{ABCD} = D_A D_B D_C D_D \chi(r, t) \]  

(B15)

where \( D_a = \frac{1}{2}(\partial_t - \partial_x, D_1 = -\partial_x - \imath\partial_y) \) and \( \chi(r, t) \) is an arbitrary generating function. For a circular polarized harmonic plane wave traveling along \( z \) direction \( \chi(r, t) = \frac{\omega}{c} e^{-\imath\omega t - \imath k z} \), the only non-vanishing part of \( \phi_{ABCD} \) is \( \phi_{0000} = \frac{\omega^2}{2} e^{-\imath\omega t - \imath k z} \). Then the non-vanishing parts of Riemann tensors are \( R_{0010} = \Re(\phi_{0000}) = \frac{\omega^2}{2} \cos(\omega t - k z), R_{0002} = \Re(\imath\phi_{0000}) = \frac{\omega^2}{2} \sin(\omega t - k z) \). Using the relation \( R_{00ij} = \frac{\omega^2}{2} h_{ij} \), we recover the familiar ‘plus’ and ‘cross’ modes \( h_+ = h_{11} = -h_{22} = \cos(\omega t - k z), h_\times = h_{12} = h_{21} = \sin(\omega t - k z) \).

B.2. Relation between \( \phi_{ABCD} \) and \( \tilde{h}_{ij} \)

Using the relation between Riemann tensor \( R_{\alpha\mu,\beta} \) and the metric perturbation \( h_{\mu\nu} \) (here we temporarily set \( c = 1 \) for convenience):

\[ R_{\alpha\mu,\beta} = \frac{1}{2} \left( h_{\alpha\nu,\beta} + h_{\mu\beta,\alpha} - h_{\mu\alpha,\beta} - h_{\alpha\beta,\mu} \right). \]  

(B16)

Since \( R_{00ij} = \frac{\omega^2}{2} h_{ij} \), we obtain the relation between \( G_{00i} \) and \( h_{ij} \):

\[ G_{00i} = R_{00i} + \frac{i}{2} h_{0\alpha}\delta_{\alpha i} R_{00\beta} \]
\[ = R_{00i} + \frac{i}{2} h_{0\alpha}(h_{\alpha\beta,0} - h_{\beta\alpha,0}) \]
\[ = \frac{\omega^2}{2} h_{ij} + i\epsilon_{\alpha i} h_{\beta,\alpha} + \frac{c}{2} \epsilon_{\alpha i} h_{\beta,0}. \]  

(B17)

(B18)

Then, we would like to alternatively relate the complex GW tensor to the conventional real one (and recover the speed of light \( c \)): \( \tilde{h}_{ij} = h_{ij} + \frac{c}{2} \epsilon_{\alpha i} h_{\beta,\alpha} \). Similar to \( R_{00ij} = \frac{\omega^2}{2} h_{ij} \), we will have

\[ G_{00i} = \frac{\omega^2}{2} \tilde{h}_{ij}. \]  

(B20)

As such, the relation between \( \tilde{h}_{ij} \) and \( \phi_{ABCD} \) is directly given by \( \tilde{h}_{ij} = \frac{1}{c^2} S_{AB} S_{CD} \phi_{ABCD}, \) with elements:

\[ \tilde{h}_{11} = \frac{2}{\omega^2}(\phi_{0000} - 2\phi_{0011} + \phi_{1111}), \]  

(B21)

\[ \tilde{h}_{12} = \frac{2}{\omega^2}(i\phi_{0000} - i\phi_{1111}), \]  

(B22)

\[ \tilde{h}_{13} = \frac{2}{\omega^2}(2\phi_{0011} - 2\phi_{0000}), \]  

(B23)

\[ \tilde{h}_{22} = \frac{2}{\omega^2}(-\phi_{0000} - 2\phi_{0011} - \phi_{1111}), \]  

(B24)

\[ \tilde{h}_{23} = \frac{2}{\omega^2}(-2i\phi_{0001} - 2i\phi_{0011}), \]  

(B25)

\[ \tilde{h}_{33} = \frac{2}{\omega^2}(4\phi_{0001}). \]  

(B26)
and conversely:

\[ \phi_{0000} = \omega^2 \left( \frac{\tilde{h}_{11}}{2} + \frac{\tilde{h}_{33}}{4} - \frac{i \tilde{h}_{12}}{2} \right), \]  

(B27)

\[ \phi_{0001} = \omega^2 \left( -\frac{\tilde{h}_{13}}{4} + \frac{i \tilde{h}_{23}}{4} \right), \]  

(B28)

\[ \phi_{0011} = \omega^2 \left( \frac{1}{4} \tilde{h}_{33} \right), \]  

(B29)

\[ \phi_{0111} = \omega^2 \left( \frac{\tilde{h}_{13}}{4} + \frac{\tilde{h}_{23}}{4} \right), \]  

(B30)

\[ \phi_{1111} = \omega^2 \left( \frac{1}{2} \tilde{h}_{12} + \frac{\tilde{h}_{33}}{4} + \frac{i \tilde{h}_{12}}{2} \right). \]  

(B31)

### Appendix C. Illustration of gravitational wave by geodesic spheroid

Unless otherwise specified, we use the geometrized unit \((G = c = 1)\) in this appendix part. Consider the motion of particles near a spatially fixed point, the displacement \(\xi^\mu\) away from that fixed point (reference geodesics) is governed by geodesic deviation equation:

\[ \frac{d^2 \xi^\mu}{d\tau^2} = -R^\mu_{\nu\rho\sigma} u^\nu u^\rho \xi^\sigma = -\frac{\omega^2}{2} h^\mu_{\nu} \xi^\nu, \]  

(C1)

where we used the property that \(R^\mu_{\nu\rho\sigma} = -\frac{\omega^2}{2} h^\mu_{\nu}\) for harmonic GW in TT gauge. \(u^\mu = (1, 0, 0, 0)\) is the four-velocity of a spatially fixed point. Since \(h^\mu_{\nu} \ll 1\), \(\xi^\mu - \xi^\mu(\tau = 0)\) is therefore at the first order \(\propto O(h)\).

Ignoring \(O(h^2)\) term and reducing the equation to three-dimension, we have the particle motion induced by GW, as:

\[ \xi^i(t) - \xi^i(0) = \frac{1}{2} h^i_{(t)}(t) \xi^i(0). \]  

(C2)

Then we examine the shape formed by motion of a group of particles. Consider particles that are originally located on a sphere: \(\xi^i(0) = [x_0, y_0, z_0]^T\) with \(x_0^2 + y_0^2 + z_0^2 = r_0^2\), the motion of these particles is given by:

\[ \xi^i(t) = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \delta_y + \frac{1}{2} h_{y} \end{bmatrix} \begin{bmatrix} x_0 \\ y_0 \\ z_0 \end{bmatrix}. \]  

(C3)

Considering \(x_0^2 + y_0^2 + z_0^2 = r_0^2\) we get:

\[ r_0^2 = [x, y, z] \begin{pmatrix} \delta_y + \frac{1}{2} h_{y} \end{pmatrix}^{-1, T} \begin{pmatrix} \delta_y + \frac{1}{2} h_{y} \end{pmatrix}^{-1} \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \]  

(C4)

which describes the surface formed by these particles. Although this is not a surface of spheroid, after ignoring \(O(h^3)\) and higher order terms, we can identify this shape as a spheroid corresponding to the quadratic from \((\delta_y + \frac{1}{2} h_{y})^{-1, T}(\delta_y + \frac{1}{2} h_{y})^{-1} \approx (\delta_y - \frac{1}{2} h_{y})^2(\delta_y - \frac{1}{2} h_{y}) \approx \delta_y - h_{y}(t)\). The orange 'spheroids' in figure 1 are related to the corresponding \(h_{y}\) by the above relation. This is what we call 'the geodesic spheroid'.

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