Uniqueness of the hyperspaces \( C(p, X) \) in the class of trees

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Abstract

Given a continuum \( X \) and \( p \in X \), we will consider the hyperspace \( C(p, X) \) of all subcontinua of \( X \) containing \( p \). Given a family of continua \( C \), a continuum \( X \in C \) and \( p \in X \), we say that \((X, p)\) has unique hyperspace \( C(p, X) \) relative to \( C \) if for each \( Y \in C \) and \( q \in Y \) such that \( C(p, X) \) and \( C(q, Y) \) are homeomorphic, then there is an homeomorphism between \( X \) and \( Y \) sending \( p \) to \( q \). In this paper we study some topological and geometric properties about the structure of \( C(p, X) \) when \( X \) is a tree, being the main result that \((X, p)\) has unique hyperspace \( C(p, X) \) relative to the class of trees.

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1. Introduction

A continuum is a nonempty compact connected metric space. Given a continuum \( X \), by a hyperspace of \( X \) we mean a specified collection of subsets of \( X \). In the literature, some of the most studied hyperspaces are the

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following:

\[ 2^X = \{ A \subset X : A \text{ is nonempty and closed} \}, \]
\[ C(X) = \{ A \subset X : A \text{ is nonempty, connected and closed} \}, \]
\[ F_n(X) = \{ A \subset X : A \text{ has at most } n \text{ points} \}, \text{ where } n \in \mathbb{N}, \]
\[ C(p, X) = \{ A \in C(X) : p \in A \}, \text{ where } p \in X. \]

The collection \( 2^X \) is called the \textit{hyperspace of closed subsets} of \( X \) whereas that \( C(X) \) is called the \textit{hyperspace of subcontinua} of \( X \). These hyperspaces are considered with the Hausdorff metric, see [11, p. 1].

Given a finite collection \( K_1, \ldots, K_r \) of subsets of \( X \), \( \langle K_1, \ldots, K_r \rangle \), denotes the following subset of \( 2^X \):

\[ \left\{ A \in 2^X : A \subset \bigcup_{i=1}^{r} K_i, A \cap K_i \neq \emptyset \text{ for each } i \in \{1, \ldots, r\} \right\}. \]

It is known that the family of all subsets of \( 2^X \) of the form \( \langle K_1, \ldots, K_r \rangle \), where each \( K_i \) is an open subset of \( X \), forms a basis for a topology for \( 2^X \) (see [11, Theorem 0.11, p. 9]) called the \textit{Vietoris Topology}. The Vietoris topology and the topology induced by the Hausdorff metric coincide (see [11, Theorem 0.13, p. 10]). The hyperspaces \( C(X) \), \( F_n(X) \) and \( C(p, X) \) are considered as subspaces of \( 2^X \).

The topological structure of the hyperspaces \( C(p, X) \) has been recently studied, for example, in [1], [2], [10], [13], [14], and [15], being useful to characterize classes of continua.

In general, given a hyperspace \( \mathcal{H}(X) \in \{ 2^X, C(X), F_n(X) \} \), we say that \( X \) has unique hyperspace \( \mathcal{H}(X) \) if for each continuum \( Y \) such that \( \mathcal{H}(X) \) is homeomorphic to \( \mathcal{H}(Y) \), it holds that \( X \) is homeomorphic to \( Y \). This concept has served to characterize classes of continua through the structure of their hyperspaces, see for example [3], [4], [6] and [7].

In a similar setting, given a family of continua \( C \), a continuum \( X \in C \) and \( p \in X \), we say that \( (X, p) \) has unique hyperspace \( C(p, X) \) in (or relative to) \( C \) if for each \( Y \in C \) and \( q \in Y \) such that \( C(p, X) \) is homeomorphic to \( C(q, Y) \), there is an homeomorphism \( h : X \to Y \) such that \( h(p) = q \) (compare with the continua \( C \)-determined given in [3, p. 12] or [9]). The main goal in this paper is to show that for each tree \( X \) and \( p \in X \), \( (X, p) \) has unique hyperspace \( C(p, X) \) in the class of trees (see Theorem 4.14).
The following examples show that the uniqueness of hyperspace $C(p, X)$ in the general setting is false, being our results the best that can be obtained in the class of trees.

**Example 1.1.** In [13, Lemma 3.19] it was proved that a continuum $Y$ is hereditarily indecomposable if and only if for each $q \in Y$, $C(q, Y)$ is an arc. Also in [13, Theorem 3.17] it was proved that if $X$ is an arc and $p \in E(X)$, $C(p, X)$ is an arc. It follows that $(X, p)$ has no unique hyperspace in the class of continua.

**Example 1.2.** By using [13, Observation 5.18 and Theorem 5.21], is easy to see that the Knaster continuum $Z$ with two end points $a$ and $b$ (see [8, p. 205]) satisfy that $C(a, Z)$ and $C(b, Z)$ are arcs and $C(q, Z)$ is a 2-cell for each $q \in Z \setminus \{a, b\}$. This continuum shows that if $X$ is an arc and $p \in X$, then $(X, p)$ has no unique hyperspace in the class of continua.

**Example 1.3.** If $X$ is a tree and $p \in X$, then there exist a continuum $Y$ and $q \in Y$, such that $Y$ is not a tree and $C(p, X)$ is homeomorphic to $C(q, Y)$. To get such a continuum, let $Z$ be a Knaster continuum with two end points $a$ and $b$ and let $T$ be a simple triod with vertex $p \in T$. Choose an end point $x \in T$ and attach $Z$ to $T$ by identifying $x$ with $a$. Let $Y$ be the space we get, $q := [p] \in Y$. It is not difficult to see that $C(p, T)$ is homeomorphic to $C(q, Y)$. This shows that trees have no unique hyperspace $C(p, X)$ in the class of continua.

**Example 1.4.** If $X$ is an arc with end points $a$ and $b$, and $Y$ is a simple closed curve, then $C(p, X)$ is homeomorphic to $C(q, Y)$ for each $p \in X \setminus \{a, b\}$ and $q \in Y$. This shows that if $p \in X \setminus \{a, b\}$ then $(X, p)$ has no unique hyperspace in the class of finite graphs.

2. Definitions and preliminaries

By a **finite graph** we mean a continuum $X$ which can be written as the union of finitely many arcs, any two of which are either disjoint or intersect only in one or both of their end points. A **tree** is a finite graph without simple closed curves. Given a positive integer $n$, a **simple $n$-od** is a finite graph, denoted by $T_n$, which is the union of $n$ arcs emanating from a single point, $v$, and otherwise disjoint from each another. The point $v$ is called the **vertex** of the simple $n$-od. A simple 3-od, $T_3$, will be called a **simple triod**.
A \textit{n-cell}, $I^n$, is any space homeomorphic to $[0, 1]^n$. For a subset $A$ of $X$ we denote by $|A|$ the cardinality of $A$ and we use the symbols $\text{int}(A)$ and $\overline{A}$ to denotes the interior and the closure of $A$ in $X$, respectively.

Given a finite graph $X$, $p \in X$ and a positive integer $n$, we say that $p$ is of \textit{order} $n$ in $X$, denoted by $\text{ord}(p, X) = n$, if $p$ has a closed neighborhood which is homeomorphic to a simple $n$-od having $p$ as the vertex. If $\text{ord}(p, X) = 1$ the point $p$ is called an \textit{end point} of $X$. The set of all end points of $X$ will be denoted by $E(X)$. If $\text{ord}(p, X) = 2$ the point $p$ is called an \textit{ordinary point} of $X$. The set of all ordinary points of $X$ will be denoted by $O(X)$.

A point $p \in X$ is a \textit{ramification point} of $X$ if $\text{ord}(p, X) \geq 3$. The set of all ramification points of $X$ will be denoted by $R(X)$. The \textit{vertices} of a finite graph $X$ will be the end points and the ramification points of $X$, we denote by $V(X)$ the set of all vertices of $X$. An \textit{edge} will be an arc joining two elements of $V(X)$ and containing no more than two points of that set; the set of edges of $X$ will be denoted by $\text{edge}(X)$.

A \textit{subtree} $Y$ of the tree $X$ is a subcontinuum of $X$ such that if $e \in \text{edge}(Y)$ then $e \in \text{edge}(X)$, i.e., every edge in $Y$ should be an edge in $X$, it is avoid to use only a part of an edge of $X$ when constructing a subtree $Y$ of $X$.

In this paper, \textit{dimension} means inductive dimension as defined in \cite{11, (0.44), p. 21}. The symbol $\text{dim}(X)$ will be used to denote the dimension of the space $X$.

3. About the cells in $C(p, X)$

From now on $X$ will denote a tree and $p \in X$.

Given $n \in \mathbb{N}$, we denote by $U_n(X)$ the following set

$$\{ A \in C(p, X) : A \text{ has a neighborhood in } C(p, X) \text{ homeomorphic to } I^n \},$$

we set $U(X) = \bigcup_{n \in \mathbb{N}} U_n(X)$, and as customary, $\pi_0(U(X))$ denote the set of connected components of $U(X)$.

Given a tree $X$, which is not an arc nor a simple $n$-od, we consider the subcontinuum

$$T(X) = \bigcup \{ e \in \text{edge}(X) : e \cap E(X) = \emptyset \}.$$

For each $x \in E(X)$ there is exactly one point $r_x \in R(X)$ such that the arc joining $x$ with $r_x$, $\overline{xr_x}$, is an edge of $X$. With this notation the continuum
$T(X)$ can be described as the closure of

$$X \setminus \bigcup_{x \in E(X)} X^x.$$ 

The key idea is to remove all arcs incident in any end point of $X$ (see the picture below).

It is clear that the following statements hold.

- $T(X)$ is a subtree of $X$ and $R(T(X)) \subset R(X) \subset T(X)$.
- $x \in R(X)$ is stills a ramification point of $T(X)$ if and only if there is at least three different elements $y, z, w \in R(X)$ such that the arcs $yx, zx$ and $wx$ are edges of $X$.
- For each $z \in E(T(X))$ holds

$$|\{e \in \text{edge}(X) \setminus \text{edge}(T(X)) : z \text{ is an end point of } e\}| \geq 2.$$ 

Let $\text{Sub}_p(T(X))$ the collection of all subtrees of $T(X)$ containing $p$. One of the main goals of this section is to establish a one to one correspondence between $\text{Sub}_p(T(X))$ and the components of $U(X)$. In order to achieve this, let $Y \in \text{Sub}_p(T(X))$ and for each $e \in \text{edge}(X) \setminus \text{edge}(Y)$ such that $e \cap Y \neq \emptyset$, the intersection $e \cap Y$ consist of exactly one point, so we can pick a homeomorphism $h_e : [0, 1] \rightarrow e$ in such a way that $h_e(0) \in Y$.

We define the following set of $C(p, X)$

$$U_Y := \left\{ Y \cup \bigcup h_e(t_e) : e \in \text{edge}(X) \setminus \text{edge}(Y), e \cap Y \neq \emptyset, t_e \in (0, 1) \right\}.$$
Observe that \( U_Y \subset C(Y, X) := \{ A \in C(X) : Y \subset A \} \). On the other hand, notice that if the set \( \{ e \in \text{edge}(X) \setminus \text{edge}(Y), e \cap Y \neq \emptyset \} \) is numbered as \( \{ e_1, \ldots, e_k \} \) then \( U_Y = \{ Y \cup A \in C(X) : A \in (\text{int}(e_1), \ldots, \text{int}(e_k)) \} \), thus \( U_Y \) is homeomorphic to \( \prod_{i=1}^k \text{int}(e_i) \); these two facts shows the following result.

**Proposition 3.1.** Let \( Y \in \text{Sub}_p(T(X)), A \in U_Y \) and \( n = |E(A)| \). Then

\( a) \quad n = |\{ e \in \text{edge}(X) \setminus \text{edge}(Y) : e \cap Y \neq \emptyset \}| \) and \( U_Y \subset U_n(X) \).
\( b) \quad \) For each \( B \in U_Y, Y = \bigcup \{ A \in \text{Sub}_p(T(X)) : A \subset B \} \), consequently, \( Y \) is the largest element in \( \text{Sub}_p(T(X)) \) contained in \( B \).
\( c) \quad \) If \( Y' \in \text{Sub}_p(T(X)) \) is such that \( U_Y \cap U_{Y'} \neq \emptyset \), then \( Y = Y' \).
\( d) \quad U_Y \) is open and connected in \( U(X) \).

Next we establish some consequences of the previous proposition.

According to (c) and (d), given \( Y \in \text{Sub}_p(T(X)) \), \( U_Y \) is a component of \( U(X) \), therefore we get a function \( \phi_X : \text{Sub}_p(T(X)) \to \pi_0(U(X)) \), given by \( \phi_X(Y) = U_Y \).

**Corollary 3.2.** The function \( \phi_X \) is bijective.

**Proof.** By (c) from Proposition 3.1, we have that \( \phi_X \) is one to one. In order to prove that \( \phi_X \) is surjective, it is enough to see that for each \( B \in U(X) \) there exists \( Y \in \text{Sub}_p(T(X)) \) such that \( B \in U_Y \). Let \( Y = \bigcup \{ A \in \text{Sub}_p(T(X)) : A \subset B \} \). It is clear that \( Y \in C(p, X) \cap \text{Sub}_p(T(X)) \) and \( B \in U_Y \).

Since \( U_Y \) is a component of \( U(X) \) there exists \( n \in \mathbb{N} \) such that \( U_Y \subset U_n(X) \), thus \( \dim(U_Y) = n \). We have the following monotony property of \( \phi_X \).

**Corollary 3.3.** Let \( Y, Y' \in \text{Sub}_p(T(X)) \). If \( Y \subset Y' \) then \( \dim(\phi_X(Y)) \leq \dim(\phi_X(Y')) \).

**Proof.** Using (a) from Proposition 3.1, we get

\[
\dim U_Y = |\{ e \in \text{edge}(X) \setminus \text{edge}(Y) : e \cap Y \neq \emptyset \}|
\leq |\{ e \in \text{edge}(X) \setminus \text{edge}(Y') : e \cap Y' \neq \emptyset \}| = \dim U_{Y'}.
\]

\( \square \)
Corollary 3.4. The following holds

\[ \text{ord}(p, X) = \min \{ \dim(Y) : Y \in \text{Sub}_p(T(X)) \} = \dim U_{\{p\}}, \]
\[ |E(X)| = \max \{ \dim(Y) : Y \in \text{Sub}_p(T(X)) \} = \dim U_{T(X)}. \]

Moreover \( U_{\text{ord}(p, X)}(X) = U_{\{p\}} \) and \( U_{|E(X)|}(X) = U_{T(X)}. \)

Proof. The right-hand side equalities are trivial from previous corollary. The left-hand side equalities are also an easy consequence of (a) from Proposition 3.1. \[ \square \]

4. Unique hyperspace \( C(p, X) \) of trees

We will denote by \( \mathcal{T} \) the class of trees, by \( \mathcal{I} \) the class of arcs, by \( \mathcal{N} \) the class of simple \( n \)-ods and by \( \hat{\mathcal{T}} = \mathcal{T} \setminus (\mathcal{I} \cup \mathcal{N}) \).

The following result is a slight modification of \cite[Corollary 3.4, p. 45]{1}, and its proof is essentially the same.

Proposition 4.1. Let \( X, Y \in \mathcal{T} \) and \( p \in X, q \in Y \). If \( C(p, X) \) is homeomorphic to \( C(q, Y) \), then \( \text{ord}(p, X) = \text{ord}(q, Y) \).

Definition 4.2. Let \( n \in \mathbb{N} \). A continuum \( Y \) is an \( n \)-od if there exists \( K \in C(Y) \) such that \( Y \setminus K \) has at least \( n \) components. Further, we will say that \( K \) is a core of the \( n \)-od. If \( n = 3 \), \( Y \) is called a triod.

Theorem 4.3. If \( X \in \mathcal{I} \) and \( p \in X \), then \((X, p)\) has unique hyperspace in \( \mathcal{T} \).

Proof. If \( X \) is an arc with end points \( a \) and \( b \) and \( p \in X \), then \( C(p, X) \) is an arc if \( p \in \{a, b\} \) and \( C(p, X) \) is a two cell if \( p \in X \setminus \{a, b\} \) (see \cite[Theorem 3.17, p. 265]{13}). By \cite[Lemma 3.15, p. 264]{13}, if \( Y \) is a continuum and \( q \in Y \) is such that \( C(q, Y) \) is an arc or a two cell, then \( q \) is not in a core of a triod in \( Y \), thus in the case that \( Y \in \mathcal{T}, Y \) must be an arc. By Proposition 4.1, it is easy to see that \( C(p, X) \) homeomorphic to \( C(q, Y) \) implies that there exists an homeomorphism \( h : X \to Y \) sending \( p \) to \( q \). \[ \square \]

Theorem 4.4. If \( X \in \mathcal{N} \) and \( p \in X \), then \((X, p)\) has unique hyperspace in \( \mathcal{T} \).
Proof. Let \( p \in X \) and suppose that \( Y \in \mathcal{T} \) and \( q \in Y \) is such that \( C(p, X) \) is homeomorphic to \( C(q, Y) \). We consider the following two cases.

Case 1. If \( p \in R(X) \). By Proposition 4.4, \( q \in R(Y) \) and \( \text{ord}(p, X) = \text{ord}(q, Y) \). If \( n = \text{ord}(p, X) \), then \( C(p, X) \) and \( C(q, Y) \) are \( n \)-cells, thus \( Y \) is a simple \( n \)-od with vertex \( q \).

Case 2. If \( q \in E(X) \cup O(X) \). By Proposition 4.4, \( q \in E(Y) \cup O(X) \). Consider \( l_q \) and \( l_q \) the edges in \( X \) and \( Y \), containing \( p \) and \( q \), respectively. Let \( x \in (V(X) \setminus \{p\}) \cap l_q \) and \( y \in (V(Y) \setminus \{q\}) \cap l_q \). By [1, Proposition 3.7, p. 45], we have that \( \text{ord}(x, X) = \text{ord}(y, Y) \). If \( Y \) contains a point \( r \in R(Y) \setminus \{y\} \), then \( C(q, Y) \) contains an \( (\text{ord}(q, Y) + \text{ord}(r, Y)) \)-cell, but this is impossible because \( \text{dim}(C(q, Y)) = \text{dim}(C(p, X)) = \text{ord}(x, X) = \text{ord}(y, Y) \). Therefore, \( Y \) is an simple \( \text{ord}(x, X) \)-od.

In both cases the existence of an homeomorphism \( h : X \to Y \) sending \( p \) to \( q \) is trivial. \( \square \)

Notation. Let \( X \in \hat{T} \). Suppose that \( G \in \text{Sub}_p(T(X)) \) consists of the edges \( e_1, \ldots, e_k \) and let \( f_1, \ldots, f_m \) be the edges of \( X \setminus G \) incident on \( G \). We will use the notation

\[
U_G = [G; f_1, \ldots, f_m] = [e_1, \ldots, e_k; f_1, \ldots, f_m],
\]

to describe the component \( \phi(G) \) of \( \mathcal{U}(X) \).

We will need the following lemma:

Lemma 4.5. Let \( X \in \hat{T} \). Let \( G = e_1 \cup \ldots \cup e_k \subset T(X) \) be a collection of edges and let \( f_1, \ldots, f_m \) be the edges of \( X \setminus G \) incident on \( G \). Then \( m \geq 2 \).

Proof. By induction on \( k \). For \( k = 1 \), let \( e = \overline{p_1p_2} \), we can assume \( p_2 \neq p \). Then \( \text{ord}(p_2, X) \geq 3 \), since all the vertices \( T(X) \) are ramification points; in other words, there are at least two incident edges on \( e = \overline{p_1p_2} \).

Next, assume the lemma holds for \( k \) and consider \( G = e_1 \cup \ldots \cup e_k \cup e_{k+1} \), since \( T(X) \) is a tree we can choose, without loss of generality, \( e_{k+1} \) such that it intersects at most one of \( e_1, \ldots, e_k \). Let \( e_{k+1} = f_1, \ldots, f_m \) be the edges incident on \( G \), by hypothesis \( m \geq 2 \). Furthermore let the point \( p_2 \) be the extreme of \( e_{k+1} \) which is not in \( G \), once again we have \( \text{ord}(p_2, X) \geq 3 \), that is, the edges incident on \( G' = e_1 \cup \ldots \cup e_k \cup e_{k+1} \) are \( m - 1 + \text{ord}(p_2, X) - 1 \geq m + 1 \geq 2 \). This finishes the induction and the proof of the lemma. \( \square \)
Let $X \in \hat{T}$. Let $G, G' \in \text{Sub}_p(T(X))$, suppose there is an $e \in \text{edge}(G)$ such that $e \cap G' = \emptyset$ (i.e. $e$ is not an edge of $G'$ nor it is incident on $G'$). Then $\overline{U}_G \cap \overline{U}_{G'} = \emptyset$.

Proof. Let $f_1, \ldots, f_m$ be the edges of $X \setminus G$ incident on $G$, similarly, let $f'_1, \ldots, f'_m$ be the edges of $X \setminus G'$ incident on $G'$. Note that an element $K' \in \overline{U}_{G'}$ is of the form $K' = G' \cup \left( \bigcup_{i=1}^{m'} \{ A_i : A_i \in C(f'_i) \} \right)$, similarly an element $K \in \overline{U}_G$ is of the form $K = G \cup \left( \bigcup_{j=1}^{m} \{ A_j : A_j \in C(f_j) \} \right)$, therefore $e \subset K$, however there is no way $e \subset K'$, since $e$ is not an edge of $G'$ nor it is incident on $G'$. Hence $\overline{U}_G$ and $\overline{U}_{G'}$ cannot have elements in common. \hfill \Box

The case when all the edges of $G$ are edges of $G'$ or are incident on $G'$ and vice versa is covered in the following:

Proposition 4.7. Let $X \in \hat{T}$. Consider $G, G' \in \text{Sub}_p(T(X))$ and let

- $e_1, \ldots, e_k$ be the edges of $G \cap G'$.
- $e_{k+1}, \ldots, e_{k+l}$ be the edges of $G$ which are not in $G'$.
- $e'_{k+1}, \ldots, e'_{k+l'}$ be the edges of $G'$ which are not in $G$.
- $d_1, \ldots, d_n$ be the edges of $X \setminus (G \cup G')$ incident on $G \cap G'$.
- $f_1, \ldots, f_m$ be the edges of $X \setminus (G \cup G')$ incident on $G$ but not on $G'$.
- $f'_1, \ldots, f'_m$ be the edges of $X \setminus (G \cup G')$ incident on $G'$ but not on $G$.

Suppose that $e'_{k+1}, \ldots, e'_{k+l'}$ are incident on $G$ and $e_{k+1}, \ldots, e_{k+l}$ are incident on $G'$. Then

1. $\overline{U}_G \cap \overline{U}_{G'} \neq \emptyset$.
2. $\dim(\overline{U}_G \cap \overline{U}_{G'}) = n$.

Proof. Note that, $G$ consists of the edges $e_1, \ldots, e_k, \ldots e_{k+l}$ and the edges incident on $G$ are $d_1, \ldots, d_n, e'_{k+1} \ldots e'_{k+l'}, f'_1, \ldots, f'_m$. Similarly for $G'$, thus,

$$U_G = [e_1, \ldots, e_k, e_{k+1}, \ldots, e_{k+l}; d_1, \ldots, d_n, e'_{k+1}, \ldots, e'_{k+l'}, f'_1, \ldots, f'_m],$$

$$U_{G'} = [e'_1, \ldots, e'_k, e'_{k+1}, \ldots, e'_{k+l'}; d_1, \ldots, d_n, e_{k+1}, \ldots, e_{k+l}, f'_1, \ldots, f'_m].$$

The following pictures should make clear the above notation.
In the case of the above subtrees, the edges are as follows:

Therefore, \( \dim(U_G) = n + l' + m \) and \( \dim(U_{G'}) = n + l + m' \). Moreover, it can be seen from this description that \( \overline{U}_G \cap \overline{U}_{G'} \) consists of elements of the form \( K = \overline{G} \cup \overline{G'} \cup \bigcup_{k=1}^{n} \{ A_k : A_k \in C(d_k) \} \). Hence \( \dim(\overline{U}_G \cap \overline{U}_{G'}) = n \). □

As an important consequence of the description given above, we have:
Proposition 4.8. Let $X \in \hat{T}$. Consider $G, G' \in \text{Sub}_p(T(X))$ such that $G' = G \cup e$, for some $e \in \text{edge}(T(X))$. Then

1. $\overline{U}_G \cap \overline{U}_{G'} \neq \emptyset$.

2. $\dim(\overline{U}_G \cap \overline{U}_{G'}) = \dim(U_G) - 1$.

3. $\dim(U_{G'}) = \dim(U_G) + \text{ord}(p', X) - 2$, where $p'$ is the vertex of $e$ which is not in $G$.

And vice versa, if $\overline{U}_G \cap \overline{U}_{G'} \neq \emptyset$ and $\dim(\overline{U}_G \cap \overline{U}_{G'}) = \dim(U_G) - 1$, then there exists an unique $e \in \text{edge}(T(X))$ such that $G' = G \cup e$.

Proof. Using the notation of the previous proposition we have in this case that $G = G \cap G' = \{e_1, \ldots, e_k\}$, $e = e_{k+1}'$ is the only edge of $G'$ incident on $G$, $d_1, \ldots, d_n$ are the edges incident on $G$, there are no $f$’s and $e, f_1', \ldots, f_{m'}'$ are the edges incident on $p'$. Hence,

$$U_G = [e_1, \ldots, e_k; d_1, \ldots, d_n, e],$$

$$U_{G'} = [e_1, \ldots, e_k, e; d_1, \ldots, d_n, f_1', \ldots, f_{m'}'] .$$

From the previous it can be seen that $\overline{U}_G \cap \overline{U}_{G'}$ consists of elements of the form $K = G' \cup \bigcup_{i=1}^{m'} \{A_i : A_i \in C(d_i)\}$. Thus, $\overline{U}_G \cap \overline{U}_{G'} \neq \emptyset$ and $\dim(\overline{U}_G \cap \overline{U}_{G'}) = n$.

Moreover,

$$\dim(U_G) = n + 1,$$

$$\dim(U_{G'}) = n + m' = \dim(U_G) - 1 + \text{ord}(p', X) - 1 .$$

This concludes the first part.

For the converse, first observe that by Lemma 4.6 and the condition $\overline{U}_G \cap \overline{U}_{G'} \neq \emptyset$, all the edges of $G \setminus G'$ are incident on $G'$ and all the edges of $G' \setminus G$ are incident on $G$; thus satisfying the condition of the previous proposition. Then the condition $\dim(\overline{U}_G \cap \overline{U}_{G'}) = \dim(U_G) - 1$ translates to $n = n + l' + m - 1$, that is, $l' + m = 1$. We only have two possibilities $(l', m) = (1, 0)$ or $(l', m) = (0, 1)$. The first case is precisely when $e = e_{k+1}'$ is the only edge of $G'$ which is not in $G$, and $m = 0$ implies that $G \subset G'$. The second case, when $m = 1$, is not possible due to Lemma 4.5. Therefore the only possibility is $(l', m) = (1, 0)$, so $G' = G \cup e$. Since $G'$ is given, then the edge $e$ is unique. \qed
Definition 4.9. Let \( p, p' \in X \) be vertices in \( X \in \mathcal{T} \), we will denote by \( \overrightarrow{pp'} \) the smallest connected subtree containing \( p \) and \( p' \).

Proposition 4.10. Let \( X, Y \in \mathcal{T}, p \in X \) and \( q \in Y \). If \( h : C(p, X) \to C(q, Y) \) is an homeomorphism then for each vertex \( p' \in T(X) \) there exists a unique \( q' \in T(Y) \) such that \( h(U_{pp'}) = U_{qq'} \). Moreover, if \( pp' \) consists of the edges \( e_i = \overrightarrow{p_{i-1}p_i}, i = 1, \ldots, n \), with \( p = p_0 \) and \( qq' \) consists of the edges \( f_i = \overrightarrow{q_{i-1}q_i}, i = 1, \ldots, m \), with \( q = q_0 \), then \( n = m \) and \( h(U_{pp'}) = U_{qq'} \) for \( i = 0, 1, \ldots, n \).

Proof. By induction on \( n \). The case \( n = 0 \) follows straightforward from Corollary 3.4. The case \( n = 1 \) is covered by Proposition 4.8 more precisely, consider an edge \( e = \overrightarrow{pp'} \subset T(X) \) and let \( G = \{p\} \) and \( G' = G \cup e \), then \( U_G \cap U_G' \neq \emptyset \) and \( \dim(U_G \cap U_G') = \dim(U_G) - 1 \). These conditions are preserved under the homeomorphism \( h \), that is, \( h(U_G) \cap h(U_G') \neq \emptyset \) and \( \dim(h(U_G) \cap h(U_G')) = \dim(h(U_G)) - 1 \) and we know \( h(U_G) = U_{\{q\}} \). Since \( \phi_Y \) is bijective there exists \( F' \in \text{Sub}_p(T(Y)) \) such that \( \phi_Y(F') = U_{F'} = h(U_{G'}) \).

By the second part of Proposition 4.8 applied to \( U_{\{q\}} \) and \( U_{F'} \), there exists an unique edge \( f' \in \text{edge}(T(Y)) \) such that \( F' = \{q\} \cup f' \), in other words, there is a unique vertex \( q' \in T(X) \) and an edge \( f' = \overrightarrow{qq'} \subset T(Y) \) such that \( h(U_{pp'}) = U_{qq'} \).

Now suppose the proposition holds for \( n \) and consider the subtree \( \overrightarrow{pp'} \) having edges \( e_i = \overrightarrow{p_{i-1}p_i}, i = 1, 2, \ldots, n + 1 \). Once again we rely completely on Proposition 4.8 let \( G = e_1 \cup \ldots \cup e_n, G' = G \cup e_{n+1} \), then \( U_G \cap U_G' \neq \emptyset \) and \( \dim(U_G \cap U_G') = \dim(U_G) - 1 \). These conditions are preserved under the homeomorphism \( h \), that is, \( h(U_G) \cap h(U_G') \neq \emptyset \) and \( \dim(h(U_G) \cap h(U_G')) = \dim(h(U_G)) - 1 \) and we know by hypothesis of induction that \( h(U_{pp'}) = U_{qq'} \) for \( i = 0, 1, \ldots, n \); in particular, \( h(U_G) = U_{qq'} \).

As above, let \( F' \in \text{Sub}_p(T(Y)) \) such that \( U_{F'} = h(U_{G'}) \), by Proposition 4.8 there exists an edge \( f' \in \text{edge}(T(Y)) \) such that \( F' = \overrightarrow{qq'} \cup f' \), in other words, there is a unique vertex \( q' \in T(X) \) and an edge \( f' = \overrightarrow{qq'} \subset T(Y) \) (for some \( j = 0, 1, \ldots, n \)) such that \( h(U_{pp'}) = U_{qq'} \). Now, it is left to prove that \( j = n \).

This is a consequence of Lemma 4.6 indeed, for \( i = 0, 1, \ldots, n-1 \), let \( G_i = \{p_0\} \cup e_1 \cup \ldots \cup e_i \) and \( G' = \{p_0\} \cup e_1 \cup \ldots \cup e_n \cup e_{n+1} \). Note that \( e_{n+1} \) is not incident on \( G_i \), then \( U_{G_i} \cap U_{G'} = \emptyset \), this condition is preserved under \( h \), that is, \( U_{G_i} \cap h(U_{G'}) = h(U_{G_i}) \cap h(U_{G'}) = \emptyset \) (for \( i = 0, 1, \ldots, n-1 \)) this means that the edge \( f' = \overrightarrow{qq_{n+1}} \) is not incident on any of \( q_0, \ldots, q_{n-1} \) and therefore it must be incident on \( q_n \) and then \( h(U_{pp'}) = U_{qq'} \) for each \( i = 0, 1, \ldots, n + 1 \). \( \square \)
Finally we put all the pieces together to prove one of the main theorems of this paper.

**Theorem 4.11.** Let $X, Y \in \mathcal{T}$, $p \in R(X)$ and $q \in Y$. If $h : C(p, X) \to C(q, Y)$ is an homeomorphism then there exists an homeomorphism $\hat{h} : X \to Y$ with $\hat{h}(p) = q$.

**Proof.** We will prove the following stronger statement: there is an isomorphism $\hat{h} : X \to Y$ such that $\hat{h}(p) = q$ (this implies the existence of the desired homeomorphism). The Proposition 4.10 allows us to define a function $h : V(T(X)) \to V(T(Y))$ given by $h(p') = q'$ if $h(U_{pp'}) = U_{qq'}$. The same construction applied to $h^{-1}$ provides us with its inverse, therefore $h : V(T(X)) \to V(T(Y))$ is a bijection. It is left to prove that it preserves edges. Formally, consider an edge $e = \overline{pp'} \subset T(X)$ and without lost of generality assume that the path $\overline{pp'}$ consists of edges $e_i = \overline{p_{i-1}p_i}$, $i = 1, \ldots, n$, with $p = p_0$, $p' = p_{n-1}$ and $p'' = p_n$, then by Proposition 4.10 we know that the path $\overline{qq''}$ consists of the edges $f_i = \overline{q_{i-1}q_i}$, $i = 1, \ldots, n$, with $q_i = h(p_i)$ and $q = q_0$, $q' = q_{n-1}$ and $q'' = q_n$. Hence, there is an edge between $q' = h(p')$ and $q'' = h(p'')$.

Finally we need to prove that $h : V(T(X)) \to V(T(Y))$ can be extended to all of $V(X)$. Indeed, consider a point $p' \in E(X)$ then there is an edge $e = \overline{pn}p'$ with $p_n \in T(X)$. Let $l(p_n) = |\{p' \in E(X) : \text{there is an edge } e = \overline{pn}p' \subset X\}|$, we will prove $l(p_n) = l(h(p_n))$. Consider the path $\overline{pp_n}$ consisting of edges $e_i = \overline{p_{i-1}p_i}$, $i = 1, \ldots, n$, with $p = p_0$. We can count $l(p_n)$ as follows:

$$l(p_n) = \text{ord}(p_n, X) - \text{ord}(p_n, T(X))$$
$$= \text{dim}(U_{G'}) - \text{dim}(U_{G}) + 2 - \text{ord}(p_n, T(X)),$$

where in the second line we have used part 3 of Proposition 4.8 for $G = \overline{pp_n}$ and $G' = \overline{pp'}$. Next, it is already known that $h : V(T(X)) \to V(T(Y))$ is a bijection preserving edges, which, together with $h$ being a homeomorphism and preserving dimensions implies that

$$l(p_n) = \text{dim}(U_{G'}) - \text{dim}(U_{G}) + 2 - \text{ord}(p_n, T(X))$$
$$= \text{dim}(h(U_{G'})) - \text{dim}(h(U_{G})) + 2 - \text{ord}(q_n, T(Y))$$
$$= \text{dim}(U_{G'}) - \text{dim}(U_{q_{n-1}}) + 2 - \text{ord}(q_n, T(Y))$$
$$= \text{ord}(q_n, Y) - \text{ord}(q_n, T(Y))$$
$$= l(q_n)$$
$$= |\{q' \in E(Y) : \text{there is an edge } f = \overline{qnq'} \subset Y\}|,$$
where \( q_{n-1} = h(p_{n-1}) \) and \( q_n = h(p_n) \). Thus we have established the existence of a bijection between the sets \( \{ p' \in E(X) : \text{there is an edge } e = \overline{p_np'} \} \) and \( \{ q' \in E(Y) : \text{there is an edge } f = \overline{q_nq'} \} \), by the construction this bijection preserves incidence relations. Therefore we get a full bijection \( h : V(X) \to V(Y) \) preserving the edge relations of the graphs and with \( h(p) = q \).  

To complete our discussion now we will extend the previous result to the case when \( p \in O(X) \cup E(X) \). In order to do so we need the following lemma.

**Lemma 4.12.** Let \( X, Y \in \mathcal{T} \), \( p \in R(X) \) and \( q \in R(Y) \). If there exists edges \( l_p \) in \( X \) and \( l_q \) in \( Y \), containing \( p \) and \( q \), respectively, such that \( l_p \cap E(X) \neq \emptyset \), \( l_q \cap E(Y) \neq \emptyset \) and \( C(p, (X \setminus l_p) \cup \{p\}) \) is homeomorphic to \( C(q, (Y \setminus l_q) \cup \{q\}) \), then \( C(p, X) \) is homeomorphic to \( C(q, Y) \).

**Proof.** Let \( h : C(p, (X \setminus l_p) \cup \{p\}) \to C(q, (Y \setminus l_q) \cup \{q\}) \) be a homeomorphism. Take \( f : [0, 1] \to l_p \) and \( g : [0, 1] \to l_q \) two homeomorphisms such that \( f(0) = p \) and \( g(0) = q \). It is easy to see that the map \( H : C(p, X) \to C(q, Y) \) given by

\[
H(A) = h(A \cap ((X \setminus l_p) \cup \{p\})) \cup g(f^{-1}(A \cap l_p)), \quad \text{for each } A \in C(p, X)
\]

is a homeomorphism.

**Theorem 4.13.** Let \( X \in \mathcal{T} \) and \( p \in O(X) \cup E(X) \). Then \( (X, p) \) has unique hyperspace \( C(p, X) \) in \( \mathcal{T} \).

**Proof.** By Theorems 4.3 and 4.4 we can suppose that \( X \in \hat{\mathcal{T}} \). Let \( p \in O(X) \) and suppose that \( Y \in \mathcal{T} \) and \( q \in Y \) is such that \( C(p, X) \) is homeomorphic to \( C(q, Y) \). By Proposition 4.1 \( \text{ord}(X, p) = \text{ord}(Y, q) \). Since \( X \) can be embedded into the euclidean space \( \mathbb{R}^3 \), suppose that \( X \) is a subspace of \( \mathbb{R}^4 \) such that \( X \subset \mathbb{R}^3 \times \{0\} \) and \( p = (0, 0, 0, 0) \). Similarly for \( Y \). Consider \( X' = X \cup ((0, 0, 0) \times [0, 1]) \) and \( Y' = Y \cup ((0, 0, 0) \times [0, 1]) \). By Lemma 4.12, \( C(p, X') \) is homeomorphic to \( C(q, Y') \) and using Theorem 4.11 we obtain that there exists an homeomorphism between \( X' \) and \( Y' \) sending \( p \) to \( q \), thus \( X \) and \( Y \) are homeomorphic.

The case when \( p \in E(X) \) reduces to the case when \( p \in O(X) \) by attaching one arc at \( p \) and one arc at \( q \) becoming these ordinary points.

Finally we state the main result of this paper which summarizes Theorems 4.11 and 4.13.
Theorem 4.14. Let $X \in \mathcal{T}$ and $p \in X$. Then $(X, p)$ has unique hyperspace $C(p, X)$ in the class $\mathcal{T}$.

The next is an application of the previous theorem.

Given a continuum $X$, we consider the hyperspace of $C(C(X))$,

$$K(X) = \{C(x, X) : x \in X\}.$$

In [1], it defines in $K(X)$ the following equivalence relation, $C(p, X) \sim C(q, X)$ if and only if $C(p, X)$ is homeomorphic to $C(q, X)$. Given a positive integer $n$ and a continuum $X$, it is say that $K(X)$ has size $n$ if the quotient $K(X)/\sim$ has cardinality $n$. In [1], the authors shown that in the class of graphs $X$, the size of $K(X)$ can be different of the homogeneity degree of $X$. As a consequence of Theorem 4.14, in the following result, we obtain a partial solution of [1, Problem 5.4, p. 49].

Corollary 4.15. Let $X$ be a tree. Then the size of $K(X)$ is equal to the homogeneity degree of $X$.

To finish this paper we present some open problems.

Question 4.16. If $X$ is a dendrite and $p \in X$, has $(X, p)$ unique hyperspace $C(p, X)$ in the class of dendrites?

Question 4.17. If $X$ is a dendroid and $p \in X$, has $(X, p)$ unique hyperspace $C(p, X)$ in the class of dendroids?

Question 4.18. Regarding Example 1.3, are there a continuum $X$ and $p \in X$ such that $(X, p)$ has unique hyperspace in the class of continua?

Question 4.19. Are there a continuum $X$ and two points $p, q \in X$ such that $(X, p)$ has unique hyperspace $C(p, X)$, whereas $(X, q)$ does not has it?

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