Moderate deviation principle for stochastic reaction-diffusion systems with multiplicative noise and non-Lipschitz reaction

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1. Introduction

Consider the following stochastic reaction-diffusion system:

\[
\frac{\partial u_i(t, \xi)}{\partial t} = A_i u_i(t, \xi) + f_i(\xi, u_1(t, \xi), \ldots, u_r(t, \xi)) + \sum_{j=1}^r g_{ij}(\xi, u_1(t, \xi), \ldots, u_r(t, \xi)) Q_j \frac{\partial w_j}{\partial t}(t, \xi),
\]

\[t \geq 0, \ \xi \in \tilde{\Omega},\]

\[u_i(0, \xi) = x_i(\xi), \ \xi \in \tilde{\Omega},\]

\[B^i u_i(t, \xi) = 0, \ t \geq 0, \ \xi \in \partial \Omega, \ 1 \leq i \leq r,\]

with \(\varepsilon \geq 0\), where \(\tilde{\Omega}\) is a bounded open set of \(\mathbb{R}^d\), with \(d \geq 1\), having a \(C^\infty\) boundary. For each \(i = 1, \ldots, r, A_i(\xi, D) = \sum_{k=1}^d \frac{\partial}{\partial \xi_k} (a_{ik}(\xi)) \frac{\partial}{\partial \xi_k}, \ \xi \in \tilde{\Omega}\), where the coefficients \(a_{ik}(\xi)\) are taken in \(C^\infty(\tilde{\Omega})\), the matrices \(a^i(\xi) := [a_{ik}(\xi)]_{k=1}^d\) are non-negative and symmetric for each \(\xi \in \tilde{\Omega}\) and fulfill a uniform ellipticity condition, i.e.,

\[\inf_{\xi \in \tilde{\Omega}} \sum_{k=1}^d a^i(\xi) v_k v_k \geq \lambda_i |v|^2, \ \forall \in \mathbb{R}^d, \ \text{for some positive constants} \ \lambda_i, \ \text{and coefficients} \ b^i_k \ \text{are continuous. The operators} \ B^i \ \text{act on the boundary of} \ \Omega \ \text{and are assumed to be either of Dirichlet or of co-normal type. The linear operators} \ Q_j \ \text{are bounded on} \ L^2(\Omega) \ \text{and may be equal to the identity operator if} \ d = 1. \]

Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space with an increasing family \(\{\mathcal{F}_t\}_{0 \leq t \leq T}\) of the sub-\(\sigma\)-fields of \(\mathcal{F}\) satisfying the usual conditions. The noises \(\frac{\partial w_j}{\partial t}\) are independent cylindrical Wiener processes on \((\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})\).

Moderate deviations trace back to Cramér (1938) established expansions of tail probabilities about sums of independent random variables defying the normal distribution. Comparing to large deviation, which offers a accurate estimate associated with the law of large number, moderate deviation is commonly applied to provide further estimates concerning the central limit theorem and the law of iterated logarithm.

Let \(u^\varepsilon\) be the solution of a small perturbation of the equation of (1), that is

\[
\frac{\partial u^\varepsilon(t, \xi)}{\partial t} = A_i u^\varepsilon(t, \xi) + f_i(\xi, u^\varepsilon_1(t, \xi), \ldots, u^\varepsilon_r(t, \xi)) + \sum_{j=1}^r g_{ij}(\xi, u^\varepsilon_1(t, \xi), \ldots, u^\varepsilon_r(t, \xi)) Q_j \frac{\partial w_j}{\partial t}(t, \xi),
\]

\[t \geq 0, \ \xi \in \tilde{\Omega},\]

\[u_i^\varepsilon(0, \xi) = x_i(\xi), \ \xi \in \tilde{\Omega},\]

\[B^i u^\varepsilon_i(t, \xi) = 0, \ t \geq 0, \ \xi \in \partial \Omega, \ 1 \leq i \leq r.\]

The solution of Equation (2) will tend to the solution, denoted by \(u^0\), of the following deterministic equation, as \(\varepsilon \rightarrow 0\),

\[
\frac{\partial u^0_i(t, \xi)}{\partial t} = A_i u^0_i(t, \xi) + f_i(\xi, u^0_1(t, \xi), \ldots, u^0_r(t, \xi)) + \sum_{j=1}^r g_{ij}(\xi, u^0_1(t, \xi), \ldots, u^0_r(t, \xi)) Q_j \frac{\partial w_j}{\partial t}(t, \xi),
\]

\[t \geq 0, \ \xi \in \tilde{\Omega},\]

\[u^0_i(0, \xi) = x_i(\xi), \ \xi \in \tilde{\Omega},\]

\[B^i u^0_i(t, \xi) = 0, \ t \geq 0, \ \xi \in \partial \Omega, \ 1 \leq i \leq r.\]

Assume that the mapping \(f := (f_1, \ldots, f_r) : \tilde{\Omega} \times \mathbb{R}^r \rightarrow \mathbb{R}^r\) is locally Lipschitz-continuous and has polynomial growth. The mapping \(g := [g_{ij}] : \tilde{\Omega} \times \mathbb{R}^r \rightarrow L(\mathbb{R}^r)\) is...
globally Lipschitz, without any assumption of boundedness and non-degeneracy. The existence and uniqueness of the solution for Equation (1) is proved in Cerrai (2003). The existence and uniqueness of the solutions for Equation (2) and Equation (3) can be established as that of Equation (1). In Cerrai and Röckner (2004), Cerrai and Röckner had shown that the process \( \{u^\varepsilon\}_{\varepsilon>0} \) obeys a large deviation principle (abbreviated LDP) in \( C([0,T];E) \), for any \( T > 0 \). Also, they established the large deviations for invariant measures for \( \{u^\varepsilon\}_{\varepsilon>0} \) in Cerrai and Röckner (2005). Based on these works, in this paper, we then consider the precise deviations of \( u^\varepsilon \) from \( u^0 \) as \( \varepsilon \to 0 \) in the following way: the asymptotic behaviour of the trajectory

\[
(u^\varepsilon - u^0)(t)/(\sqrt{\varepsilon} \lambda(\varepsilon)), \quad t \in [0,T].
\]

We emphasize that

1. it is related to central limit theorem (abbreviated CLT) when \( \lambda(\varepsilon) = 1 \), see Theorem 3.4 below;
2. it provides moderate deviation principle (MDP for short, cf. Dembo & Zeitouni, 2000) when \( \lambda(\varepsilon) \) is the deviation scale verifying

\[
\lambda(\varepsilon) \to +\infty, \quad \sqrt{\varepsilon} \lambda(\varepsilon) \to 0, \quad \varepsilon \to 0, \quad (4)
\]

see Theorem 4.3 below. In the sequel, we assume that (4) is in place.

Moderate deviation is an intermediate estimation between the large deviation with deviation scale \( \lambda(\varepsilon) = 1/\sqrt{\varepsilon} \) and the CLT with \( \lambda(\varepsilon) = 1 \). The MDP provides rates of convergence and a way to construct asymptotic confidence interval, see Gao and Zhao (2011) and the references therein. In recent years, there is an increasing interest on the study of MDP. For independent and identically distributed random sequences, Chen (1990, 1991, 1993) and Ledoux (1992) found the necessary and sufficient conditions for MDP. Moderate deviation was discussed by Djellout and Guillin (2001), Gao (1996, 2003), and Wu (1995, 1999) for Markov processes as well as Guillin and Liptser (2006) for diffusion processes. For mean field interacting particle models, MDP was worked by Douc et al. (2001). Wang and Zhang (2015) and Wang et al. (2015) discussed moderate deviations for stochastic reaction-diffusion equation and 2D stochastic Navier-Stokes equations, respectively. Recently, there are several works on moderate deviations for SPDEs with jump, see Budhiraja et al. (2016) and Dong et al. (2017).

In this paper, we prove a CLT and establish a MDP for the class of reaction–diffusion systems of the form (1). Since we consider the solutions in spaces of continuous functions and there is no Itô formula, we will work directly on heat kernels. Also, because of the weak assumptions on drift and diffusion coefficients, we have to deal with the estimates on the solutions delicately.

The organization of this paper is as follows. The assumptions will be given in Section 2. Section 3 is devoted to prove the CLT. In Section 4, we establish the MDP. Throughout this paper, \( c \) and \( c_0 \) are positive constants independent of \( \varepsilon \) and those values may be different from line to line.

2. Preliminaries

In this section, we formulate the equation and state the precise conditions on the coefficients.

Let \( H \) be the separable Hilbert space \( L^2(\Omega;\mathbb{R}^r) \) endowed with the scalar product \( |x|_H = \sum_1^T |x|_{L^2(\Omega)} \). We shall denote by \( A \) the realization in \( H \) of the differential operator \( A := (A_1, \ldots, A_r) \), endowed with the boundary conditions \( B := (B_1, \ldots, B_r) \), i.e. \( D(A) = \{ x \in W^{2,2}(\Omega;\mathbb{R}^r) : B_i(D) x = 0 \} \). Let \( E := D(A)^{C(\Omega;\mathbb{R}^r)} \times \cdots \times D(A_r)^{C(\Omega;\mathbb{R}^r)} \) and \( D(A)^{C(\Omega;\mathbb{R}^r)} \) coincides with \( C(\Omega;\mathbb{R}) \) or \( C_0(\Omega;\mathbb{R}) \).

We set \( L_1(\xi, D) := \sum_{i=1}^d (b_i^j(\xi) - \sum_{k=1}^d \frac{\partial}{\partial x_k} \cdot a_{i k}^j(\xi)) \frac{\partial}{\partial x_k}, \quad \xi \in \hat{\Omega} \), and define \( C_i := A_i - L_i \). Denoted by \( C \) the realization in \( H \) of the second-order elliptic operator \( C := (C_1, \ldots, C_r) \), endowed with the boundary conditions \( B \). In Davies (1989), it is proved that \( C \) is a non-positive and self-adjoint operator which generates an analytic semigroup \( e^{tC} \) with dense domain given by \( e^{tC} = (e^{tC_1}, \ldots, e^{tC_r}) \), where \( C_i \) is the realization in \( L^2(\Omega) \) of \( C_i \) with the boundary condition \( B_i \). Moreover, denoted by \( L_p \) the realization of the operator \( \mathcal{L} \) in \( L^p(\Omega;\mathbb{R}^r) \) for any \( p \geq 1 \). We see that \( L_p x = L_q x, \quad x \in D(L_p) \cap D(L_q) \) and denote all of them by \( L \) if there is no confusion.

The covariance operator \( Q \) of the Wiener process \( W(\cdot) \) is a positive symmetric, trace class operator on \( H \). Let \( H_0 := Q^{1/2} H \). Then \( H_0 \) is a Hilbert space with the inner product

\[
\langle u, v \rangle_0 = \langle Q^{-1/2} u, Q^{-1/2} v \rangle_H, \quad \forall u, v \in H_0.
\]

Let \( | \cdot |_0 \) denote the norm on \( H_0 \). Clearly, the embedding of \( H_0 \) in \( H \) is Hilbert-Schmidt, since \( Q \) is a trace class operator. Let \( L_Q(H_0; H) \) be the space of linear operators \( S \) such that \( SQ^{1/2} \) is a Hilbert-Schmidt operator from \( H \) to \( H \). Define the norm on the space \( L_Q(H_0; H) \) by \( |S|_{L_Q} = \sqrt{\text{Tr}(S Q^2)} \). Set \( L_{10}(H_0; H) \) the space of all bounded linear operators from \( H_0 \) into \( H \) and denote by \( | \cdot |_{L_{10}(H_0; H)} \) its norm.

The hypothesis concerns the eigenvalues of \( A \):

**Hypothesis 2.1:** The complete orthonormal system of \( H \) which diagonalizes \( A \) is equi-bounded in the sup-norm.

Assume that \( Q := (Q_1, \ldots, Q_r) : H \to H \) is a bounded linear operator which satisfies:
Hypothesis 2.2: \( Q \) is non-negative and diagonal with respect to the complete orthonormal basis which diagonalizes \( A \), with eigenvalues \( \{\lambda_n\} \). Moreover, if \( d \geq 2 \), there exists \( q (q < \infty \) if \( d = 2 \) and \( q < \frac{4d}{d-2} \) if \( d > 2 \)) such that \( \|Q\|_q := (\sum_{k=1}^{\infty} \lambda_k^q) \frac{1}{q} < \infty \).

The assumptions on the coefficients \( f \) and \( g \) are as follows.

Hypothesis 2.3: The mapping \( g : [0, \infty) \times \hat{O} \times \mathbb{R}^r \rightarrow \mathcal{L}(\mathbb{R}^r) \) is continuous and \( g(t, \xi, \cdot) : \mathbb{R}^r \rightarrow \mathcal{L}(\mathbb{R}^r) \) is Lipschitz-continuous, uniformly with respect to \( t, \xi \in \hat{O} \) and \( t \) in bounded sets of \( [0, \infty) \), i.e., for some \( \Psi \in L^\infty([0, \infty), \sup_{\xi \in \hat{O}} \sup_{\sigma, \rho \in \mathbb{R}^r, \sigma \neq \rho} \|g(t, \xi, \sigma) - g(t, \xi, \rho)\|_{C(\mathbb{R}^r)} \leq \Psi(t), t \geq 0 \).

We set for every \( x, y \in E \) and \( t \geq 0 \), \( G(t, x, y)(\xi) := g(t, x, y)(\xi) \), and for \( f = (f_1, \ldots, f_r) \), define for every \( t \geq 0 \) the composition operator \( F(t, \cdot) \) by setting, for any \( x : \hat{O} \rightarrow \mathbb{R}^r, F(t, x)(\xi) := f(t, x(\xi)), \xi \in \hat{O} \).

Hypothesis 2.4: (1) The mapping \( F(t) : E \rightarrow E \) is locally Lipschitz-continuous, uniformly for \( t \geq 0 \), and there exists \( m \geq 1 \) and \( \Phi \in L^\infty([0, \infty) \) such that \( |F(t, x)|_E \leq \Phi(t)(1 + |x|^{\frac{p}{2}}), x \in E, t \geq 0 \).

(2) There exists \( \Lambda \in L^\infty([0, \infty) \) such that, for every \( x, h \in E \), and for some \( \delta_h \in \partial|h|_E = \{h^* \in E^*; |h^*|_E = 1, (h^*, h)_E = |h|_E, t \geq 0, (F(t, x) + h) - F(t, x), \delta_h)_E \leq \Lambda(t)(1 + |h|_E + |x|_E) \).

One of the following two conditions holds:

(a) \( \sup_{\xi \in \hat{O}} |g(t, \xi, \sigma)|_{\mathcal{L}(\mathbb{R}^r)} \leq \beta(t)(1 + |\sigma|^{\frac{m}{2}}), \sigma \in \mathbb{R}^r, t \geq 0, \) where \( m \geq 1 \) is as in Hypothesis 2.4 (1) and \( \beta \in L^\infty([0, \infty) \).

(b) There exists some \( a > 0 \), \( m \geq 1 \) and \( \beta_2 \in L^\infty([0, \infty) \) such that, for every \( x, h \in E \) and for some \( \delta_h \in \partial|h|_E, (F(t, x) + h) - F(t, x), \delta_h)_E \leq -a|h|_E^{\frac{m}{2}} + \beta_2(t)(1 + |x|^{\frac{m}{2}}), t \geq 0 \).

Then the system (1) can be rewritten in the following type:

\[
\begin{align*}
\begin{cases}
\mathrm{d}u(t) = Au(t) \mathrm{d}t + F(t, u(t)) \mathrm{d}t + G(t, u(t)) \mathrm{d}W(t), \\
u(s) = x,
\end{cases}
\end{align*}
\]

where the operator \( A \) can be written as \( A = C + L \).

3. Central limit theorem

In this section, our task is to establish the CLT. We will need some results in Cerrai (2003), and we list them here for the convenience of readers.

Proposition 3.1: Under Hypotheses 2.1–2.4 on the coefficients, define the following mappings

\[
\psi(f)(t) := \int_s^t e^{(t-r)C}L(f)(r) \, \mathrm{d}r, \quad t \in [s, T].
\]

Then, (1) (the inequality (3.7) in Cerrai, 2003) for any \( p \geq 1 \) and \( f \in C([s, t]; D(L)) \)

\[
|\psi(f)|_E \leq C \int_s^t (t - r) \cdot 1^{\frac{1}{1+p}} |f(r)|_E \, \mathrm{d}r
\]

where \( c (c > 0) \) is any constant;

(2) (the inequality (4.6) in Cerrai, 2003) there exists \( p_\ast \geq 1 \) such that \( \gamma \) maps \( L^p(\Omega; C([0, T]; E)) \) into itself for any \( p \geq p_\ast \) and for any \( u, v \in L^p(\Omega; C([0, T]; E)) \)

\[
|\gamma(u) - \gamma(v)|_{L^p(T)} \leq c_\gamma L(T)|u - v|_{L^p(T)},
\]

for some continuous increasing function \( c_\gamma \) such that \( c_\gamma(s) = 0 \).

Proposition 3.2 (Theorem 5.3 in Cerrai, 2003):

Under Hypotheses 2.1–2.4 on the coefficients, for any \( x \in E \) and for any \( p \geq 1 \) and \( T > 0 \) such a problem admits a unique mild solution in \( L^p(\Omega; C([0, T]; E)) \cap L^\infty(s, T; E) \) which is the Banach space of all adapted processes \( u \in C([0, T]; E) \cap L^\infty(s, T; E) \), such that \( |u|_{L^p(T)} < \infty \), where

\[
|u|_{L^p(T)} := \left[ \mathbb{E} \sup_{t \in [s, T]} |u(t)|_E^p \right]^{\frac{1}{p}}.
\]

Moreover, \( |u|_{L^p(T)} \leq c_\gamma p(T)(1 + |x|_E) \).

Similarly as the proof of (11), under Hypotheses 2.1–2.4, we have \( |u' - u'_0||_{L^p(T)} \leq c_\gamma p(T)(1 + |x|_E) \).

The following result is concerned with the convergence of \( u' \) as \( \varepsilon \to 0 \).

Proposition 3.3: Under Hypotheses 2.1–2.4, there exists a constant \( \varepsilon_0 > 0 \) such that, for all \( 0 < \varepsilon \leq \varepsilon_0 \)

\[
|u' - u'_0||_{L^p(T)} \leq c^2 \varepsilon_0.
\]

Proof: Define

\[
F_n(t, x)(\xi) = \begin{cases} \frac{F(t, x)(\xi), \quad \text{if } |x| \leq n,} \end{cases}
\]

\[
\frac{F(t, nx/|x|)(\xi), \quad \text{if } |x| > n.}
\]

It is immediate to check that \( F_n \) is Lipschitz-continuous in \( E \). Then if we consider

\[
\begin{align*}
\begin{cases}
\mathrm{d}u_n(t) = Au_n(t) \mathrm{d}t + F_n(t, u_n(t)) \mathrm{d}t \\
u_n(s) = x
\end{cases}
\end{align*}
\]

and

\[
\begin{align*}
\begin{cases}
\mathrm{d}u_n^0(t) = Au_n^0(t) \mathrm{d}t + F_n(t, u_n^0(t)) \mathrm{d}t \\
u_n^0(s) = x
\end{cases}
\end{align*}
\]

there exist unique mild solutions \( u_n \) and \( u_n^0 \) in \( L^p(\Omega; C([s, T]; E)) \cap L^\infty(s, T; E) \) for Equations (14) and (15), respectively. Define

\[
\Lambda_n(u' - u'_0)(t) := \int_s^t e^{(t-r)C}F_n(r, u'_0(r)) \, \mathrm{d}W(r).
\]
So,
\[
(u_n^0 - u_n^0)(t) = \psi(u_n^0 - u_n^0)(t) + \Lambda_n(u_n^0 - u_n^0)(t) + \gamma(u_n^0)(t).
\]
From (9),
\[
|\psi(u^e - u^0)|_E \leq C \int_s^t \left[ |(t - r) \wedge 1| \frac{1}{1 + s} + |(u^e - u^0)(r)|_E \right] dr \\
\leq C \int_s^t \left[ |(t - r) \wedge 1| \frac{1}{1 + s} \right] dr \sup_{t \in [s, r]} |(u^e - u^0)(r)|_E \\
\leq c^0 \psi(t) \sup_{t \in [s, r]} |(u^e - u^0)(r)|_E, \tag{16}
\]
and one obtains
\[
\sup_{t \in [s, r]} |(u^e_n - u^0_n)(r)|_E \\
\leq c^0 \psi(t) \sup_{t \in [s, r]} |(u^e_n - u^0_n)(r)|_E + \sup_{t \in [s, r]} |\gamma(u^0_n)(r)|_E \\
+ c_n(s + t_0) \int_s^t \sup_{r \in [s, r]} |(u^e_n - u^0_n)(r')|_E dr.
\]
Gronwall's inequality implies
\[
\sup_{t \in [s, r + t_0]} |(u^e_n - u^0_n)(r)|_E \\
\leq 2 \sup_{t \in [s, s + t_0]} |\gamma(u^0_n)(r)|_E \\
+ 2c_n(s + t_0) \int_s^{r + t_0} \sup_{r' \in [s, r]} |(u^e_n - u^0_n)(r')|_E dr.
\]
Therefore, due to (10),
\[
|u^e_n - u^0_n|_{L_{s, s + t_0, \phi}(E)} = \left[ \mathbb{E} \sup_{t \in [s, s + t_0]} |(u^e_n - u^0_n)(r)|_E \right]^\frac{1}{2} \\
\leq c(s + t_0, n) \sup_{t \in [s, s + t_0]} |\gamma(u^0_n)(r)|_{L_{s, s + t_0, \phi}(E)} \\
\leq c(s + t_0, n) \sqrt{c_{\epsilon, p} |u^0_n|_{L_{s, s + t_0, \phi}(E)} \cdot \sqrt{\epsilon}}.
\]
Next we can repeat the same arguments in the intervals [s + t_0, s + 2t_0], [s + 2t_0, s + 3t_0] and so on and for any T > s,
\[
|u^e_n - u^0_n|_{L_{s, T, \phi}(E)} \leq c_{\epsilon, p}(T) \sqrt{\epsilon}, \tag{17}
\]
with the help of (11) for u^e_n and u^0_n.

Define \( r_n = \inf\{ t \geq s : |(u^e_n - u^0_n)(t)|_E = n \} \) and \( r = \inf = +\infty \). We can see that \( r_n \) is non-decreasing and denote \( \tau := \sup_{n \in \mathbb{N}} r_n \). So (17) with \( p = 1 \) yields
\[
P(\tau < \infty) = \lim_{T \to \infty} \lim_{n \to \infty} P(\tau \leq T) = \lim_{T \to \infty} \lim_{n \to \infty} P(\sup_{t \in [s, T]} |(u^e_n - u^0_n)(t)|_E \geq n) \\
\leq \lim_{T \to \infty} \lim_{n \to \infty} \frac{1}{n} \mathbb{E}[ \sup_{t \in [s, T]} |(u^e_n - u^0_n)(t)|_E \geq n] = 0,
\]
i.e. \( P(\tau = \infty) = 1 \). Thus, Fatou’s lemma implies
\[
\mathbb{E}[ \sup_{t \in [s, T]} |(u^e_n - u^0_n)(r)|_E ] \\
= \mathbb{E}[ \lim_{n \to \infty} \sup_{t \in [s, T]} |(u^e_n - u^0_n)(r)|_E ]_{T \leq r_n} \\
= \mathbb{E}[ \lim_{n \to \infty} \sup_{t \in [s, T]} |(u^e_n - u^0_n)(r)|_E ]_{T \leq r_n} \\
\leq \lim_{n \to \infty} \mathbb{E}[ \sup_{t \in [s, T]} |(u^e_n - u^0_n)(r)|_E ]_{T \leq r_n} \\
\leq (c_{\epsilon, p} \sqrt{\epsilon})^p, \tag{18}
\]
which completes the proof. ■

Let \( V^e(t) := (u^e - u^0)/\sqrt{\epsilon} \) be the solution of the following SPDE:
\[
dV^e(t) = AV^e(t) \, dt \\
+ [F(t, u^e(t)) - F(t, u^0(t))]/\sqrt{\epsilon} \, dt \\
+ G(t, u^0(t)) Q dW(t),
\]
with initial value \( V^e(0) = 0 \). Let \( V^0 \) be the solution of the following SPDE:
\[
dV^0(t) = AV^0(t) \, dt + F'(u^0(t)) V^0(t) \, dt \\
+ G(t, u^0(t)) Q dW(t),
\]
\( V^0(0) = 0 \),
where we need an additional assumption about \( F' \).

**Hypothesis 3.1:** \( F' : [0, T] \times E \to L(E) \) is Fréchet derivative of \( F \) w.r.t. the second variable satisfying
\[
|F'(t, u) - F'(t, h)|_E \leq F'_0 |u - h| + F'_1 |t| \text{ for some positive } F'_0, F'_1 \text{ and there exists } \Lambda' \in L^\infty_0 (0, \infty) \text{ such that, for each } x, h \in E \text{ and } t \geq 0, (F'(t, x + h) - F'(t, x), \delta_h)_E \leq \Lambda'(t)(1 + |h|_E + |x|_E). \]

We now come to the main result of this section.

**Theorem 3.4 (Central Limit Theorem):** Under Hypotheses 2.1–3.1, \( V^e(t) \) converges to \( V^0 \) in the space \( L^p(\Omega; C([0, T]; E)) \) in probability, that is, there exists a constant \( \epsilon_0 > 0 \) such that, for all \( 0 < \epsilon \leq \epsilon_0 \),
\[
|V^e - V^0|_{L_{s, t, \phi}(E)} \leq c\sqrt{\epsilon}. \tag{19}
\]
\textbf{Proof:} Define
\[
F'_n(t, x)(\xi) = \begin{cases} 
F(t, x)(\xi), & \text{if } |x| \leq n, \\
F(t, nx/|x|)(\xi), & \text{if } |x| > n,
\end{cases}
\]
and we see that $F'_n$ is Lipschitz-continuous in $E$. Then if we consider
\[
dV_n^e(t) = AV_n^e(t) \, dt \\
+ \left[ F_n(t, u^e(t)) - F_n(t, u^0(t)) \right]/\sqrt{\varepsilon} \, dr \\
+ G(t, u^e(t))Q \, dW(t), \quad V_n^e(0) = 0
\]
and
\[
dV_n^0(t) = AV_n^0(t) \, dt + F_n(t, u^0(t))V_n^0(t) \, dt \\
+ G(t, u^0(t))Q \, dW(t), \quad V_n^0(0) = 0.
\]
There exists unique mild solutions $V_n^e$ and $V_n^0$ in $L^p(\Omega; C([0, T]; E))$, respectively. Define the following mappings
\[
\psi(V^e - V^0)(t) := \int_s^t e^{(t-r)C}L(V^e - V^0)(r) \, dr;
\]
\[
\Lambda_n(V^e - V^0)(t) := \int_s^t e^{(t-r)C} \left[ \left[ F_n(r, u^e(r)) - F_n(r, u^0(r)) \right]/\sqrt{\varepsilon} - F'_n(r, u^0(r)) \right] \, dr;
\]
\[
\gamma(V^e - V^0)(t) := \int_s^t e^{(t-r)C} \left[ G(r, u^e(r)) - G(r, u^0(r)) \right] Q \, dW_r.
\]
Then
\[
(V_n^e - V_n^0)(t) = \psi(V_n^e - V_n^0)(t) + \Lambda_n(V_n^e - V_n^0)(t) \\
+ \gamma(V_n^e - V_n^0)(t).
\]
Due to inequality (16), one obtains
\[
\sup_{r \in [s,t]} |(V_n^e - V_n^0)(r)|_E \\
\leq c^\psi(t) \sup_{r \in [s,t]} |(V^e - V^0)(r)|_E \\
+ \sup_{r \in [s,t]} |\Lambda_n(V^e - V^0)(r)|_E \\
+ \sup_{r \in [s,t]} |\gamma(V^e - V^0)(r)|_E.
\]
If we take $t_1 > 0$ such that $c^\psi(s + t_1) < \frac{1}{2}$, this yields
\[
\sup_{r \in [s,t_1]} |(V_n^e - V_n^0)(r)|_E \\
\leq 2 \sup_{r \in [s,t_1]} |\Lambda_n(V_n^e - V_n^0)(r)|_E \\
+ 2 \sup_{r \in [s,t_1]} |\gamma(V_n^e - V_n^0)(r)|_E. \tag{20}
\]
Consider
\[
[F_n(r, u^e(r)) - F_n(r, u^0(r))] / \sqrt{\varepsilon} - F'_n(r, u^0(r))V_n^0 \\
= [F_n(r, u^e(r)) - F_n(r, u^0(r))] / \sqrt{\varepsilon} + F'_n(r, u^0(r)) \\
\times [V_n^e - V_n^0 - V_n^0] \\
= F'_n(r, u^0(r))[V_n^e - V_n^0] \\
+ [F_n(r, u^e(r)) - F_n(r, u^0(r))] (u_n^e - u_n^0) / \sqrt{\varepsilon}.
\]
There exists a random field $\eta^e(\epsilon, r)$ taking values in $(0, 1)$ such that
\[
F_n(r, u^e(r)) - F_n(r, u^0(r)) \\
= F'_n(r, u^0(r)) (u_n^e - u_n^0) - \eta^e(u_n^e - u_n^0).
\]
Then,
\[
[F_n(r, u^e(r)) - F_n(r, u^0(r))] / \sqrt{\varepsilon} - F'_n(r, u^0(r))V_n^0 \\
= F'_n(r, u^0(r))[V_n^e - V_n^0] \\
+ [F_n(r, u^e(r)) - F_n(r, u^0(r))] - F'_n(r, u^0(r)) \\
\times (u_n^e - u_n^0) / \sqrt{\varepsilon}.
\]
In view of last inequality, the Lipschitz continuity of $F'_n$ and Hypothesis 3.1, it yields
\[
\sup_{r \in [s,t]} |\Lambda_n(V_n^e - V_n^0)(r)|_E \\
\leq c_n(t) \int_s^t \sup_{r \in [s,t]} |F_n(u^e(r)) - F_n(u^0(r))|_E \, dr \\
+ c_n(t) \int_s^t \sup_{r \in [s,t]} |(u_n^e - u_n^0)(r)|^2 / \sqrt{\varepsilon} \, dr \\
\leq c_n(t) \int_s^t \sup_{r \in [s,t]} (F'_n(u^0(r))[V_n^e - V_n^0] \\
\times |(V_n^e - V_n^0)(r)|_E \, dr \\
+ c_n(t) \int_s^t \sup_{r \in [s,t]} |(u_n^e - u_n^0)(r)|^2 / \sqrt{\varepsilon} \, dr. \tag{21}
\]
Inequalities (20), (21) and Gronwall’s inequality implies
\[
\sup_{r \in [s,s+t_1]} |(V_n^e - V_n^0)(r)|_E \\
\leq 2c_n(s + t_1) \int_s^{s+t_1} \sup_{r \in [s,t]} |(u_n^e - u_n^0)(r)|^2 / \sqrt{\varepsilon} \, dr \\
x \exp \left\{ \int_s^{s+t_1} 2c_n(s+t_1) \right\}. 
\]
Furthermore,
\[
\mathbb{E}\left[2c_n(s + t_1)\int_{s}^{s + t_1} \sup_{r \in [s, r]} |(u_n^r - u_0^n)(r^\varepsilon)|_E^2 \frac{dr}{\sqrt{\varepsilon}}\right] + 2\sup_{r \in [s, s + t_1]} |\gamma(V_n^r - V_0^n)(r)|_E^2 \right]\}
\]
\[
\leq 2c_n(s + t_1) \left\{ \mathbb{E}\left[\int_{s}^{s + t_1} \sup_{r \in [s, r]} |(u_n^r - u_0^n)(r^\varepsilon)|_E^2 \frac{dr}{\sqrt{\varepsilon}}\right] \right\}^\frac{1}{p} + 2\sqrt{\varepsilon} dr \left\{ \gamma(V_n^r - V_0^n)(r)|_E \right\}^\frac{1}{p}
\]
\[
\leq 2c_n(s + t_1) \int_{s}^{s + t_1} |u_n^r - u_0^n|^2_{L_{s,t_1}^2(E)} \frac{dr}{\sqrt{\varepsilon}}
\]
\[
+ 2\gamma(V_n^r - V_0^n)_{L_{s,t_1}^2(E)}
\]
\[
\leq 2c_n(s + t_1) \int_{s}^{s + t_1} |\sqrt{\varepsilon} dr|_E^2 \frac{dr}{\sqrt{\varepsilon}}
\]
\[
+ 2\gamma(V_n^r - V_0^n)_{L_{s,t_1}^2(E)}
\]
\[
= 2c_n(s + t_1)\sqrt{t_1} \varepsilon + 2c_n(s + t_1)
\]
\[
\times |V_n^r - V_0^n|_{L_{s,t_1}^2(E)},
\]

According to inequality (17). Also, \(\exp\left[\int_{s}^{s + t_1} 2c_n(s + t_1) \sup_{r \in [s, r]} (F_0^r |u_0^n|_E + F_1^r) dr\right]\) is finitely controlled by some positive constant \(M\) from \(|u_0^n|_E \leq c_2\varepsilon(s + t_1)(1 + |x|_E)\).

Then, we take \(t_2 \in (0, t_1)\) such that \(2c_n(s + t_1) < \frac{1}{2}\),
\[
|(V_n^r - V_0^n)|_{L_{s,t_2}^2(E)} \leq 2M \cdot 2c_n(s + t_1)\sqrt{t_1} \varepsilon + 2c_n(s + t_1)
\]
\[
\times |V_n^r - V_0^n|_{L_{s,t_1}^2(E)}.
\]

Then we can repeat the same arguments in the intervals \([s + t_1, s + 2t_1], [s + 2t_1, s + 3t_1]\) and so on and for any \(T > s,\)
\[
|V_n^r - V_0^n|_{L_{s,T}^2(E)} \leq c\sqrt{\varepsilon},
\]
with the help of (11).

Similarly as inequality (18) replacing \(u^\varepsilon - u^0\) with \(V^\varepsilon - V^0\), the proof is completed. □

4. Moderate deviations

Let \(Z^\varepsilon = (u^\varepsilon - u^0)/\sqrt{\varepsilon}\lambda(\varepsilon)).\) Then \(Z^\varepsilon\) satisfies the following SPDE:
\[
dZ^\varepsilon(t) = AZ^\varepsilon(t) dt + [F(t, u^0(t) + \sqrt{\varepsilon}\lambda(\varepsilon)) dt + (\varepsilon\lambda(\varepsilon) Z^\varepsilon(t) - F(t, u^0(t)))]/\sqrt{\varepsilon}\lambda(\varepsilon) dt + G(t, u^0(t)) Q dW(t),
\]
with initial value \(Z^\varepsilon(0) = 0\). This equation admits a unique solution \(Z^\varepsilon = \Gamma^\varepsilon(W(\cdot)),\) where \(\Gamma^\varepsilon\) stands for the solution functional from \(C([0, T]; H)\) into \(E\).

In this part, we will prove that \(Z^\varepsilon\) satisfies an LDP on \(L^2(\Omega; C([0, T]; E))\) with \(\lambda(\varepsilon)\) satisfying (4). This special type of LDP is usually called the MDP of \(u^\varepsilon\) (cf. Dembo & Zeitouni, 2000).

Firstly we recall the general criteria for an LDP given in Budhiraja and Dupuis (2000). Let \(E\) be a Polish space with the Borel \(\sigma\)-field \(B(E)\).

**Definition 4.1 (Rate function):** A function \(I : E \to [0, \infty]\) is called a rate function on \(E\), if for each \(M < \infty\), the level set \(\{x \in E : I(x) \leq M\}\) is a compact subset of \(E\).

**Definition 4.2 (LDP):** Let \(I\) be a rate function on \(E\). A family \(\{Z^\varepsilon\}\) of \(E\)-valued random elements is said to satisfy the LDP on \(E\) with rate function \(I\), if the following two conditions hold.

(a) (Large deviation upper bound) For each closed subset \(F\) of \(E\),
\[
\lim_{\varepsilon \to 0} \sup x \in F \log \mathbb{P}(Z^\varepsilon \in F) \leq - \inf x \in F I(x).
\]
(b) (Large deviation lower bound) For each open subset \(G\) of \(E\),
\[
\lim_{\varepsilon \to 0} \inf x \in G \log \mathbb{P}(Z^\varepsilon \in G) \geq - \inf x \in G I(x).
\]

Next, we introduce the Skeleton Equations. The Cameron-Martin space associated with the Wiener process \(\{W(t), t \in [0, T]\}\) is given by
\[
\mathcal{H}_0 := \left\{ h : [0, T] \to H : h\text{ is absolutely continuous and } \int_0^T |\dot{h}(s)|_0^2 ds < +\infty \right\}.
\]

The space \(\mathcal{H}_0\) is a Hilbert space with inner product \((h_1, h_2)_{\mathcal{H}_0} := \int_0^T \langle \dot{h}_1(s), \dot{h}_2(s) \rangle_0 ds).\) Let \(\mathcal{A}\) denote the class of \(\mathcal{H}_0\)-valued \(\{\mathcal{F}_t\}_t\)-predictable processes \(\phi\) belonging to \(\mathcal{H}_0\) a.s.. Let \(S_N = \{h \in \mathcal{H}_0 : \int_0^T |\dot{h}(s)|_0^2 ds \leq N\}.\) The set \(S_N\) endowed with the weak topology is a Polish space. Define \(\mathcal{A}_N = \{\phi \in \mathcal{A} : \phi(\omega) \in S_N, \mathbb{P}-\text{a.s.}\}\).

For any \(h \in \mathcal{H}_0\), consider the deterministic equation
\[
dX^h(t) = AX^h(t) dt + F(t, u^0(t))X^h(t) dt + G(t, u^0(t)) Q h(t) dt,
\]
with initial value \(X^h(0) = 0\) and for any \(\phi^\varepsilon \in \mathcal{A}\), consider
\[
dX^\varepsilon(t) = AX^\varepsilon(t) dt + [F(t, u^0(t) + \sqrt{\varepsilon}\lambda(\varepsilon) X^\varepsilon(t))] dt
\]
\[
+ G(t, u^0(t)) / (\sqrt{\varepsilon}\lambda(\varepsilon)) Q dW(t) + G(t, u^0(t)) \sqrt{\varepsilon}\lambda(\varepsilon) X^\varepsilon(t)) Q dW(t)
\]
\[
+ G(t, u^0(t)) \sqrt{\varepsilon}\lambda(\varepsilon) X^\varepsilon(t)) Q dW(t),
\]

with initial value \(X^\varepsilon(0) = 0\). This equation admits a unique solution \(X^\varepsilon = \Gamma^\varepsilon(W(\cdot)),\) where \(\Gamma^\varepsilon\) stands for the solution functional from \(C([0, T]; H)\) into \(E\).
\( X^\varepsilon(0) = 0. \) \hspace{1cm} (26)

Now we are ready to state the second main result.

**Theorem 4.3 (Moderate Deviation Principle):** Under Hypotheses 2.1–3.1, \( Z^\varepsilon \) obeys an LDP on \( L^p(\Omega; C([0, T]; E)) \) with speed \( \lambda^2(\varepsilon) \) and with rate function \( I \) given by

\[
I(g) := \inf_{\{h \in \mathcal{H}_0; g = \Gamma^\varepsilon(h) \}} \left\{ \frac{1}{2} \int_0^T |h(s)|^2 \, ds \right\}, \quad \forall g \in L^2(\Omega; C([0, T]; E)),
\]

with the convention \( \inf \{0\} = \infty \).

We will adopt the following weak convergence method to prove the MDP.

**Theorem 4.4 (Budhiraja & Dupuis, 2000):** For \( \varepsilon > 0 \), let \( \Gamma^\varepsilon \) be a measurable mapping from \( C([0, T]; H) \) into \( \mathcal{E} \). Let \( Y^\varepsilon := \Gamma^\varepsilon(W(\cdot)) \). Suppose that \( \{\Gamma^\varepsilon\}_{\varepsilon > 0} \) satisfies the following assumptions: there exists a measurable map \( \Gamma^0 : C([0, T]; H) \rightarrow \mathcal{E} \) such that

(a) for every \( N < +\infty \) and any family \( \{h^\varepsilon; \varepsilon > 0\} \subset \mathcal{A}_N \) satisfying that \( h^\varepsilon \) converge in distribution as \( \varepsilon \rightarrow 0 \), \( \Gamma^\varepsilon(W(\cdot) + \int_0^T h^\varepsilon(s) \, ds / \sqrt{\varepsilon}) \) converges in distribution to \( \Gamma^0(\int_0^T h(s) \, ds) \) as \( \varepsilon \rightarrow 0 \);

(b) for every \( N < +\infty \), the set \( \{\Gamma^0(\int_0^T h(s) \, ds); h \in \mathcal{S}_N\} \) is a compact subset of \( \mathcal{E} \).

Then the family \( \{Y^\varepsilon\}_{\varepsilon > 0} \) satisfies an LDP in \( \mathcal{E} \) with the rate function \( I \) given by

\[
I(g) := \inf_{\{h \in \mathcal{H}_0; g = \Gamma^\varepsilon(h) \}} \left\{ \frac{1}{2} \int_0^T |h(s)|^2 \, ds \right\}, \quad g \in \mathcal{E},
\]

with the convention \( \inf \{0\} = \infty \).

For \( h \in \mathcal{H}_0 \), set \( \Gamma^0(\int_0^T h(s) \, ds) := X^h \) and define \( \Gamma^\varepsilon \) satisfying

\[
\Gamma^\varepsilon \left( W(\cdot) + \lambda(\varepsilon) \int_0^T \phi^\varepsilon(s) \, ds \right) = X^\varepsilon.
\]

To prove Theorem 4.3, we only need to verify the following two propositions according to Theorem 4.4.

**Proposition 4.5:** Under the same conditions as Theorem 4.3, for every fixed \( N \in \mathbb{N} \), let \( \phi^\varepsilon, \phi \in \mathcal{A}_N \) be such that \( \phi^\varepsilon \) converges in distribution to \( \phi \) as \( \varepsilon \rightarrow 0 \). Then \( \Gamma^\varepsilon(W(\cdot) + \lambda(\varepsilon) \int_0^T \phi^\varepsilon(s) \, ds) \) converges in distribution to \( \Gamma^0(\int_0^T \phi(s) \, ds) \) in \( C([0, T]; H) \cap L^2([0, T]; V) \) as \( \varepsilon \rightarrow 0 \).

**Proposition 4.6:** Under the same conditions as Theorem 4.3, for every positive number \( N < \infty \), the family \( K_N := \{\Gamma^0(\int_0^T h(s) \, ds); h \in \mathcal{S}_N\} \) is compact in \( L^p(\Omega; C([0, T]; E)) \).

We start to prove Proposition 4.5 and we need the following lemma and inequalities.

**Lemma 4.1:** Under the same conditions as Theorem 4.3, for any \( \phi \in \mathcal{H}_0 \) and \( \phi^\varepsilon \in \mathcal{A}_N \), (25) and (26) replacing \( F \) and \( F^\varepsilon \) with \( F_n \) and \( F_n^\varepsilon \), respectively, admit the unique solutions, respectively, \( X_n^\phi, X_n^\varepsilon \) in \( L^p(\Omega; C([0, T]; E)) \). Moreover, for any \( N > 0 \) and \( p \geq 2 \), there exist constants \( c_n,N,T \) and \( \varepsilon_0 > 0 \) such that for any \( \phi \in \mathcal{S}_N \), \( \phi^\varepsilon \in \mathcal{A}_N \),

\[
\sup_{\varepsilon \in (0, \varepsilon_0]} |X_n^\phi|_{L^p(\Omega, T)} \leq c_{n,N,T} \quad (27)
\]

and

\[
\sup_{\varepsilon \in (0, \varepsilon_0]} |X_n^\varepsilon|_{L^p(\Omega, T)} \leq c_{n,N,T}. \quad (28)
\]

**Proof:** Similarly as Theorem 2.4 in Chueshov and Millet (2010), the existence and uniqueness of the solution can be proved. Here, we will prove inequalities (27) and (28).

Define the following mappings

\[
\psi(X^\varepsilon)(t) := \int_s^t e^{(r-s)C} X^\varepsilon(r) \, dr;
\]

\[
\Lambda_n(X^\varepsilon)(t) := \int_s^t e^{(r-s)C} \left( X^\varepsilon(r) \phi^\varepsilon(r) \right) \, dr;
\]

\[
\gamma(X^\varepsilon)(t) := \int_s^t e^{(r-s)C} G(r, u^\varepsilon(r)) \left( X^\varepsilon(r) \phi^\varepsilon(r) \right) \, dr;
\]

\[
\gamma^\varepsilon(X^\varepsilon)(t) := \int_s^t e^{(r-s)C} G(r, u^0(r)) \left( X^\varepsilon(r) \phi^\varepsilon(r) \right) \, dr.
\]

Then

\[
X_n^\varepsilon(t) = \psi(X_n^\phi(t)) + \Lambda_n(X_n^\varepsilon(t)) \gamma(X_n^\varepsilon(t)) + \gamma^\varepsilon(X_n^\varepsilon(t)).
\]

In view of the Lipschitz continuity of \( F_n \) and inequality (16), one obtains

\[
\sup_{r \in [s, t]} |X_n^\phi(r)|_E \leq c^\phi(t) \sup_{r \in [s, t]} |X_n^\varepsilon(r)|_E \int_s^t dr
\]

\[
+ c_n(T) \int_s^t \sup_{r \in [s, r]} |X_n^\varepsilon(r')|_E \, dr
\]
If we take \( t_3 > 0 \) such that \( c_1^\phi (s + t_3) < \frac{1}{2} \), this yields
\[
\sup_{r \in [s, t_3]} |X_n^\phi(r)| \leq 2c_2(T) \int_s^{s + t_3} \sup_{r \in [s, r]} |X_n^\phi(r)| dr \leq 2c_2(T) t_3 + 2 \sup_{r \in [s, t_3]} |\gamma(X_n^\phi(r))| \leq 2c_2(T) t_3 + 2 \sup_{r \in [s, t_3]} |\gamma^\phi(X_n^\phi(r))|.
\]

Then,
\[
|X_n^\phi|_{L^{s + t_3}([0, T])} \leq 2c_2(T) (|X_n^\phi|_{L^2([0, T])} + |\gamma^\phi(X_n^\phi)|_{L^2([0, T])}).
\]

Furthermore,
\[
|\gamma^\phi(X_n^\phi)|_{L^2([0, T])} \leq c_3^\phi (s + t_3) |u_0^\phi| + \sqrt{\varepsilon\lambda(\varepsilon)} |X_n^\phi|_{L^{s + t_3}([0, T])} \cdot N
\]

as similar proof as Theorem 4.2 in Cerrai (2003); due to (4.6) and the proof in Theorem 5.3 in Cerrai (2003), it yields
\[
|\gamma(X_n^\phi)|_{L^{s + t_3}([0, T])} \leq c_4^\phi (s + t_3) |u_0^\phi| + \sqrt{\varepsilon\lambda(\varepsilon)} |X_n^\phi|_{L^{s + t_3}([0, T])} \cdot N.
\]

Therefore, for some constant \( c_{n,N,s,t_3} \),
\[
|X_n^\phi|_{L^{s + t_3}([0, T])} \leq c_{n,N,s + t_3}.
\]

Next we can repeat the same arguments in the intervals \( [s + t_3, s + 2t_3] \), \( [s + 2t_3, s + 3t_3] \) and so on and for any \( T > s \), there exists some constant \( c_{n,N,T} \),
\[
|X_n^\phi|_{L^{s + T}([0, T])} \leq c_{n,N,T}.
\]

Here choosing \( \varepsilon \) small enough, we get (27).

The proof of (28) is very similar to that of (27), we omit it here. The proof of this lemma is complete. \( \blacksquare \)

Under the conditions of Lemma 4.1, we have the following three inequalities. Let \( G^n_M(t) := \{ \omega : (\sup_{0 \leq s \leq t} |X_n^\phi(s)|^2) \leq M \} \). Then, from inequality (29),
\[
[\mathbb{E}1_{G^n_M(t)} |\gamma(X_n^\phi(r))|_{L^2([s, t])}^\frac{1}{2}]
\]

\[
\leq 1_{G^n_M(t)} G^n_M(t)|u_0^\phi| + \sqrt{\varepsilon\lambda(\varepsilon)} |X_n^\phi|_{L^{s + t_3}([0, T])}/\lambda(\varepsilon);\]

(31)

and
\[
\left[ \mathbb{E}1_{G^n_M(t)} \sup_{r \in [s, T]} \left( \int_s^t e^{-(r-t)C} G(u_0^\phi(t)) Q[\phi^\varepsilon(r)] dr \right)^{\frac{1}{2}} \right] \to 0, \quad \text{as} \quad \varepsilon \to 0.
\]

(33)

**Proof of Proposition 4.5.** Let \( Z_\varepsilon := X^\phi - X^\phi \), where \( X^\phi \) is the solution of (25) replaced \( h \) by \( \phi \). Then \( Z_\varepsilon(0) = 0 \) and \( Z_\varepsilon = X^\phi_n - X^\phi_n \), where \( X^\phi_n, X^\phi_n \) satisfy, respectively,
\[
dX_n^\phi(t) = AX_n^\phi(t) dt + F_n(\omega^\phi(t)) X_n^\phi(t) dt + \sigma(t, u^\phi(t)) Q\phi^\varepsilon(t) dt,
\]

with initial value \( X_n^\phi(0) = 0 \) and
\[
dX_n^\phi(t) = AX_n^\phi(t) dt + [F_n(t, u^0(t)) + \sqrt{\varepsilon\lambda(\varepsilon)} X_n^\phi(t)] dt,
\]

\[
\psi(Z_\varepsilon(t)) := \int_s^t e^{(t-r)C} L(Z_\varepsilon(r)) dr;
\]

\[
\Lambda_n(Z_\varepsilon(t)) := \int_s^t e^{(t-r)C} [\Lambda_n^\phi(t, u^0(r) + \sqrt{\varepsilon\lambda(\varepsilon)} X^\phi(r))]
\]

(32)
\[ \gamma(X_n)(t) := \int_s^t e^{(t-r)-C}G(u^n_0(r) + \sqrt{\varepsilon \lambda(e)}X^n(r) + G(u^n_0(r)Q\phi\lambda) dr. \]

Then

\[ Z^n_n(t) = \psi(Z^n_n(t)) + \Lambda_n(Z^n_n(t)) \]

Thus,

\[ \sup_{r \in [s,t]} |Z^n_n(r)|_E \]

\[ \leq c^n \sup_{r \in [s,t]} |Z^n_n(r)|_E + \sup_{r \in [s,t]} |\Lambda_n(Z^n_n(r))|_E \]

There exists a random field \( \eta^e(\rho, x) \) taking values in (0, 1) such that

\[ [F_n(u^n_0 + \sqrt{\varepsilon \lambda(e)}X^n_0) - F_n(u^n_0)]/(\sqrt{\varepsilon \lambda(e)}) = F_n(u^n_0 + \eta^e \sqrt{\varepsilon \lambda(e)}X^n_0) \cdot X^n_0 - F_n(u^n_0)X^n_0 \]

\[ = [F_n(u^n_0 + \eta^e \sqrt{\varepsilon \lambda(e)}X^n_0) - F_n(u^n_0)] \cdot X^n_0 + F_n(u^n_0)Z^n_n, \]

which yields

\[ \sup_{r \in [s,t]} |\Lambda_n(Z^n_n(r))|_E \]

\[ \leq c_n(t) \sup_{r \in [s,t]} |X^n_n(r')|_E \cdot \sqrt{\varepsilon \lambda(e)} dr \]

\[ + c_n(t) \sup_{r \in [s,t]} |F_n(u^n_0(r'))| \cdot |Z^n_n(r')|_E dr \]

\[ \leq c_n(t) \sup_{r \in [s,t]} |X^n_n(r')|_E \cdot \sqrt{\varepsilon \lambda(e)} dr \]

\[ + c_n(t) \sup_{r \in [s,t]} (F_0[u^n_0(r')] + F_1) |Z^n_n(r')|_E dr. \]

We take \( t_4 > 0 \) such that \( c^n \sup_{s + t_4} < 1/2 \). One obtains

\[ \sup_{r \in [s,t+t_4]} |Z^n_n(r)|_E \]

\[ \leq 2 \sup_{r \in [s,t+t_4]} |\Lambda_n(Z^n_n(r))|_E + 2 \sup_{r \in [s,t+t_4]} |\gamma(X^n_n(r))|_E \]

Gronwall’s inequality implies

\[ \sup_{r \in [s,t+t_4]} |Z^n_n(r)|_E \leq 2c_n(s + t_4) \int_s^{s+t_4} |Z^n_n(r')|_E \cdot \sqrt{\varepsilon \lambda(e)} dr \]

\[ + \sup_{r \in [s,t+t_4]} |\gamma(X^n_n(r))|_E \]

\[ + \sup_{r \in [s,t+t_4]} |\phi(X^n_n(r))| \cdot \exp \left\{ \int_s^{s+t_4} 2c_n(s + t_4) \right\}. \]

Thus,

\[ |Z^n_n|_{L_{s+t+t_4}^p(E)} = \left\{ \left[ E \sup_{r \in [s,t+t_4]} |Z^n_n(r)|_E \right]^\frac{1}{p} \right\} \]

\[ \leq 2c_n(s + t_4) \int_s^{s+t_4} |Z^n_n(r')|_E \cdot \sqrt{\varepsilon \lambda(e)} dr \]

\[ + \sup_{r \in [s,t+t_4]} |\gamma(X^n_n)|_{L_{s+t+t_4}^p(E)} \]

\[ + \sup_{r \in [s,t+t_4]} |\phi(X^n_n)|_{L_{s+t+t_4}^p(E)} \]

\[ \times \sup_{r \in [s,t+t_4]} (F_0[u^n_0(r')] + F_1) dr \}

With the help of inequalities (30)–(33), we get

\[ \left[ E \mathbf{1}_{\mathfrak{M}_T(s+t_4)} \sup_{r \in [s,t+t_4]} |Z^n_n(r)|_E \right]^\frac{1}{p} \rightarrow 0, \quad \text{as} \quad \varepsilon \rightarrow 0. \]

(34)

Then we can repeat the same arguments in the intervals \([s + t_4, s + 2t_4], [s + 2t_4, s + 3t_4] \) and so on and for any \( T > s \),

\[ \left[ E \mathbf{1}_{\mathfrak{M}_T(T)} \sup_{r \in [s,T]} |Z^n_n(r)|_E \right]^\frac{1}{p} \rightarrow 0, \quad \text{as} \quad \varepsilon \rightarrow 0. \]

(35)

Furthermore, by Chebysev inequality and (30),

\[ \mathbb{P}\left((\mathfrak{M}_T(T))^c\right) \leq c_{n,N,T}/M. \]

(36)

Finally,

\[ |Z^n_n|_{L_{s+t_4}^p(E)} \leq \left[ E \mathbf{1}_{\mathfrak{M}_T(T)} \sup_{r \in [s,T]} |Z^n_n(r)|_E \right]^\frac{1}{p} \]

\[ + \left[ E \mathbf{1}_{\mathfrak{M}_T(T)^c} \sup_{r \in [s,T]} |Z^n_n(r)|_E \right]^\frac{1}{p} \]

\[ \leq \left[ E \mathbf{1}_{\mathfrak{M}_T(T)} \sup_{r \in [s,T]} |Z^n_n(r)|_E \right]^\frac{1}{p} \]

\[ + \left[ E \mathbf{1}_{\mathfrak{M}_T(T)^c} \right]^\frac{1}{p} \cdot |Z^n_n|_{L_{s+t_4}^p(E)} \]
\[
\begin{align*}
&= [\mathbb{E} \mathbf{1}_{G_M(T)} \sup_{r \in [s,T]} |Z^n_\varepsilon(r)\|_E^p]^{\frac{1}{p}} \\
&\quad + \mathbb{P}(\{(G^c_M(T))^\varepsilon\} \cdot |Z^n_\varepsilon|_{L_{s,T}p}(E),
\end{align*}
\]

Taking \(M_0 > 0\) and for \(M > M_0\), \(\mathbb{P}(\{(G^c_M(T))^\varepsilon\}) \leq \frac{1}{2}\). Then,

\[
|Z^n_\varepsilon|_{L_{s,T}p}(E) \leq 2[E \mathbf{1}_{G_M(T)} \sup_{r \in [s,T]} |Z^n_\varepsilon(r)\|_E^p]^{\frac{1}{p}} \to 0,
\]
as \(\varepsilon \to 0\).

Similarly as inequality (18), replacing \(u^\varepsilon - u^0\) with \(Z^n_\varepsilon\),
the proof of the proposition is completed.

**Proof of Proposition 4.6:** The proof is similar as that of Proposition 4.5 and easier. The proof will be omitted.

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