ADIABATIC LIMITS ON RIEmannian SOl-MANIFOLDS

ANDREY A. YAKOVLEV

Abstract. We obtain an asymptotic formula for the spectrum distribution function of the Laplace operator on a compact Riemannian Sol-manifold in the adiabatic limit determined by a one-dimensional foliation defined by the orbits of a left-invariant flow.

The paper is devoted to investigation of adiabatic limits on Riemannian Sol-manifolds. We understand adiabatic limits in the sense, which was introduced by Witten in [1]. More precisely, let \((M, \mathcal{F})\) be a closed foliated manifold equipped with a Riemannian metric \(g\). Thus, the tangent bundle \(TM\) of \(M\) is represented as a direct sum

\[ TM = F \oplus H, \]

where \(F = T\mathcal{F}\) is the tangent bundle of \(\mathcal{F}\) and \(H = F^\perp\) the orthogonal complement of \(F\). Let \(g_F\) and \(g_H\) denote the restriction of the metric \(g\) to \(F\) and \(H\), respectively. Therefore, \(g = g_F + g_H\). Define a one-parameter family of Riemannian metrics on \(M\) by the formula

\[ g_\varepsilon = g_F + \varepsilon^{-2}g_H, \quad \varepsilon > 0. \]

Investigation of various properties of the family of Riemannian manifolds \((M, g_\varepsilon)\) as \(\varepsilon \to 0\) will be called by passage to adiabatic limit.

Recall [2] that the group \(\text{Sol}\) is the solvable Lie subgroup of the Lie group \(\text{GL}(3, \mathbb{R})\), which consists of all matrices of the form:

\[
\gamma(u, v, w) = \begin{pmatrix}
e^w & 0 & u \\
0 & e^{-w} & v \\
0 & 0 & 1
\end{pmatrix}, \quad (u, v, w) \in \mathbb{R}^3.
\]

The Lie algebra \(\mathfrak{sol}\) of \(\text{Sol}\) is the Lie subalgebra of the Lie algebra \(\text{gl}(3, \mathbb{R})\), which consists of all matrices of the form

\[
X(u, v, w) = \begin{pmatrix}
w & 0 & u \\
0 & -w & v \\
0 & 0 & 0
\end{pmatrix}, \quad (u, v, w) \in \mathbb{R}^3.
\]

Let \(A \in \text{SL}(2, \mathbb{Z})\) and \(|\text{tr} A| > 2\). Denote by \(\lambda\) and \(\lambda^{-1}\) the eigenvalues of \(A\) and assume that \(\lambda > 1\). Define a vectors \((c_1^1, c_1^2), (c_2^1, c_2^2)\) by the equation

\[
A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix}^{-1} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} \begin{pmatrix} c_1^1 & c_2^1 \\ c_1^2 & c_2^2 \end{pmatrix}.
\]

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Definition 1. A Riemannian Sol-manifold is a compact manifold $M^3_A = G_A \backslash \text{Sol}$ equipped with a Riemannian metric $g$, where:

- $G_A$ is the uniform discrete subgroup of the Lie group $\text{Sol}$, which consists of all $\gamma(u, v, w) \in \text{Sol}$ such that
  $$(u, v) \in \Gamma := \{k(c_1^1, c_1^2) + l(c_2^1, c_2^2), \quad k, l \in \mathbb{Z}\},$$
  $w = m \ln \lambda, \quad m \in \mathbb{Z},$

- $g$ is a Riemannian metric on $M^3_A$ whose lift on $\text{Sol}$ is invariant under left translations by elements of $\text{Sol}$ (such metrics will be called locally left-invariant).

A locally left-invariant metric $g$ is uniquely determined by its value at the identity $\gamma(0, 0, 0)$ of $\text{Sol}$, and, therefore, is given by a symmetric positive definite $3 \times 3$-matrix.

Let $\alpha \in \mathbb{R}$. Consider the left-invariant vector field on $\text{Sol}$ associated with $X(1, \alpha, 0) \in \mathfrak{sol}$. The orbits of the corresponding vector field on $M^3_A$ define a one-dimensional foliation $\mathcal{F}$. The leaf of $\mathcal{F}$ through $G_A \gamma(u, v, w) \in M^3_A$ is given by

$$L_{GA\gamma(u,v,w)} = \{G_A \gamma(u + e^w t, v + \alpha e^{-w} t, w) \in M^3_A : t \in \mathbb{R}\}.$$

Suppose that a locally left-invariant metric $g$ correspond to the identity matrix. Consider the adiabatic limit associated with the Riemannian Sol-manifold $(M^3_A, g)$ and the foliation $\mathcal{F}$. Denote by $\Delta_\varepsilon$ the Laplace-Beltrami operator on $M^3_A$ associated with the metric $g_\varepsilon$ given by (1). For any $\varepsilon > 0$ the spectrum of $\Delta_\varepsilon$ consists of eigenvalues of finite multiplicity:

$$0 = \lambda_0(\varepsilon) < \lambda_1(\varepsilon) \leq \ldots, \lambda_j(\varepsilon) \to +\infty \quad j \to \infty.$$

The main result of the paper is a computation of the asymptotics of the spectrum distribution function

$$N_\varepsilon(t) = \#\{i : \lambda_i(\varepsilon) \leq t\}$$

of the operator $\Delta_\varepsilon$ in the adiabatic limit, that is, when $t \in \mathbb{R}$ is fixed and $\varepsilon \to 0$.

**Theorem 2.** For any $t > 0$, the following asymptotic formulae hold:

1. For $\alpha \neq 0$

   $$N_\varepsilon(t) = \frac{1}{4\pi^2} t^2 \varepsilon^{-2} + o(\varepsilon^{-2}), \quad \varepsilon \to 0.$$

2. For $\alpha = 0$

   $$N_\varepsilon(t) = \frac{1}{6\pi^2} t^2 \varepsilon^{-2} + o(\varepsilon^{-2}), \quad \varepsilon \to 0.$$
The asymptotic behavior of the spectrum distribution function for the Laplace operator in the adiabatic limit was studied earlier in [3] for Riemannian foliations and in [4] for one-dimensional foliations on Riemannian Heisenberg manifolds (see also [5]). In all cases, the function $N_\varepsilon(t)$ has order $\varepsilon^{-q}$, where $q$ is the codimension of the foliation (in our case $q = 2$), but the coefficients of $\varepsilon^{-q}$ are different in each case. Observe also that, in each case, the asymptotic formula for $N_\varepsilon(t)$ in the adiabatic limit is different from the classical Weyl formula, which describes asymptotic behavior of $N_\varepsilon(t)$ as $t \to \infty$ (cf. [6]).

For $\alpha \neq 0$, the proof of the theorem uses the calculation of the spectrum of the Laplace operator on a Riemannian Sol-manifold given in [6], which continues the investigation of the geodesic flow on a Riemannian Sol-manifold started in [7] and [8], and semiclassical spectral asymptotics [9] for the modified Mathieu operator

$$H_\varepsilon = -\varepsilon^2 \frac{d^2}{dx^2} + a \cosh(2\mu x), \quad x \in \mathbb{R}.$$

In the case $\alpha = 0$, the foliation is Riemannian, and the metric is bundle-like, and, therefore, we can use the asymptotic formula obtained in [3].

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Department of Mathematics, Ufa State Aviation Technical University, 12 K. Marx str., 450000 Ufa, Russia
E-mail address: yakovlevandrey@yandex.ru