THE LEAST GENERATING DEGREE OF SYMBOLIC POWERS OF FERMAT-LIKE IDEALS OF PLANES AND LINES ARRANGEMENTS

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Abstract. We compute explicitly the least degree of generators of almost all symbolic powers of the defining ideal of Fermat-like configuration of lines in $\mathbb{P}^3_\mathbb{C}$ and provide efficient bounds for remaining cases. We will also compute explicitly those numbers for ideal determining the singular locus of the arrangement of lines given by the pseudoreflection group $A_3$. As direct applications, we verify Chudnovsky’s(-like) Conjecture, Demailly’s(-like) Conjecture and Harbourne-Huneke Containment problem as well as calculate the Waldschmidt constant and (asymptotic) resurgence number.

1. Introduction

Let $n \geq 3$ and $I_n$ be the ideal in $\mathbb{C}[x, y, z, w]$ generated by:

\begin{align*}
    g_1 &= (x^n - y^n)(z^n - w^n)xy, \\
    g_2 &= (x^n - y^n)(z^n - w^n)zw \\
    g_3 &= (x^n - z^n)(y^n - w^n)xz, \\
    g_4 &= (x^n - z^n)(y^n - w^n)yw \\
    g_5 &= (x^n - w^n)(y^n - z^n)xw, \\
    g_6 &= (x^n - w^n)(y^n - z^n)yz
\end{align*}

which we will refer as Fermat-like ideal. This ideal corresponds to the restricted Fermat arrangement of planes in $\mathbb{P}^3$, where the correspondent variety is the union of all lines with multiplicity at least 3, i.e, there are at least 3 planes passing through each line. Fermat-like ideals were first introduced by Malara and Szpond in their work [MS17] in an effort to provide counterexamples in higher dimension to the famous containment $I^{(3)} \subseteq I^2$. It can be seen as an analog of Fermat ideals in higher dimension. For more information about Fermat ideals, we refer interested readers to [NS16], [DSTG13], [HS15], [Szp19], [Ngu21].

In this paper, we will discover many similarities of these ideals and Fermat ideals in terms of the least degree of generators of their symbolic powers, Waldschmidt constants, (asymptotic) resurgence numbers and especially, the fact that the verification to Harbourne-Huneke containment of them can also be decided purely by the knowledge of the aforementioned numerical invariants. This work is a continuation of our paper [Ngu21] where we answer those questions for Fermat ideals. Our results on Fermat-like ideals are the following:

**Theorem 1.1.** Let $n \geq 3$ be an integer and Fermat-like ideal $I_n$ in $\mathbb{C}[x, y, z, w]$ as described above. Then:

1. For $n \geq 5$, $\alpha(I_n^{(m)}) = 2nm$ for all $m \geq 3$.
2. $\alpha(I_4^{(m)}) = 8m$ for all $m \geq 3$ and $m \neq 5$.

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(3) \( \alpha(I_3^{(3k)}) = 18k \) for \( k \geq 1 \), \( \alpha(I_3^{(3k+1)}) = 18k + 8 \) and \( \alpha(I_3^{(3k+2)}) = 18k + 16 \) for \( k \geq 0 \).

(4) \( 4n + 2 \leq \alpha(I_n^{(2)}) \leq 4n + 4 = \alpha(I_n^2) \).

(5) \( \hat{\alpha}(I_n) = 2n \).

This gives the almost complete answer to the question of the least degree of generators of almost all symbolic powers of the defining ideal of Fermat-like configuration of lines in \( \mathbb{P}^3 \) and bounds only for the case when \( m = 2 \). Note that for each \( m, n \) the least generating degree of symbolic power of the Fermat-like ideal is twice as large as that of Fermat ideal in almost all cases (and should be in all cases as suggested by Macaulay2 computation \([GS]\)).

One important problem that has attracted a lot of attention recently is the containment problem, namely, to determine the set of pairs \((m, r)\) for which \( I_n^{(m)} \subseteq I_r \) for a given ideal \( I \). Following the celebrated results in \([ELS01]\) and \([HH02]\) that \( I_n^{(m)} \subseteq I_r \) whenever \( m \geq N r \), the resurgence \( \rho(I) \) is introduced in \([BH10]\), and the asymptotic resurgence \( \hat{\rho}(I) \) is introduced in \([GHVT13]\) in order to study the pairs \((r, m)\) numerically and has turned out to be very useful invariants. In term of these invariants, we prove the followings.

**Theorem 1.2.** For ideal \( I_n \) where \( n \geq 3 \):

1. \( \hat{\rho}(I_n) = \frac{n + 1}{n} \) for all \( n \geq 3 \).
2. \( \rho(I_3) = \frac{3}{2} \).
3. For \( n \geq 4 \), the resurgence number \( \rho(I_n) \) can only be one of the following numbers: \( \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{8}{5}, \) or \( \frac{9}{5} \).

Another motivation for computing the least degree of generators of their symbolic powers is to provide more evidence for ideals that satisfy Chudnovsky-like inequality and Demailly-like inequality as well as Harbourne-Huneke Containment and stable containment. This motivation stems from the previous work by Bisui, Grifo, Hà and the author, see \([BGHN20b, Section 3]\) and containment in \([BGHN20a]\). Similar to the Fermat ideals, the verification and failure of the containment for \( I_n \) can be checked purely by their numerical invariants including the least degree of generators of symbolic powers, the regularity, or the resurgence number and the maximal degree of generators. As direct applications, we will give an evidence to a similar question as question 3.1 in \([BGHN20b, Section 3]\), that all Fermat-like ideals satisfy the Harbourne-Huneke Containment, stable Harbourne Containment and some stronger containment.

**Corollary 1.3.** Fermat-like ideals verifies the following containment

1. Harbourne-Huneke containment (see \([HH13, Conjecture 2.1]\) for ideal of points)
\[
I_n^{(2r)} \subseteq m^r I_n, \quad \forall r
\]

2. Harbourne-Huneke stable containment (see \([HH13, Conjecture 4.1.5]\) for ideal of points)
\[
I_n^{(2r-1)} \subseteq m^{r-1} I_n, \quad \forall r \gg 0
\]

3. A stronger stable containment \( I_n^{(2r-2)} \subseteq m^r I_n^2, \) \( \forall r \gg 0 \)
As another direct application of these computations, we will give an evidence to question 3.2 in [BGHN20b, Section 3] that all Fermat-like ideals satisfy the Chudnovsky-like inequality and Demailly-like inequality.

**Corollary 1.4.** Fermat-like ideals satisfy Demailly-like inequality (and hence, Chudnovsky-like inequality)

\[ \hat{\alpha}(I_n) \geq \frac{\alpha(I_n^{(m)}) + h - 1}{m + h - 1} \]

for all \( m \), where \( h = 2 \).

Along the way, we give the free resolution, Castelnuovo-Mumford regularity formula for \( I_n \) and use them to improve the stable version of the above containment to a condition on \( r \) for \( I_3 \).

In previous work [Ngu21], we investigate the above questions for Fermat ideals. In particular, the Fermat ideal for 7 points can be seen as ideal of the singular locus of the arrangement of lines given by the pseudoreflection group \( G(2, 2, 3) = D_3 \). We continue to study the ideal corresponding to the group \( G(1, 1, 4) = A_3 \) in this paper and give some discussion for group \( G(2, 1, 3) = B_3 \). These are 3 groups that have correspondent ideals with small degree of generators, that had to be considered separately in the result involving Harbourne containment in [DS20, Proposition 6.3]. It turns out interestingly that the ideal corresponding to the group \( A_3 \) is the same as that of group \( D_3 \) in term of least degree of generators of symbolic powers, Waldschmidt constant, (asymptotic) resurgence numbers; and hence, satisfies Chudnovsky’s Conjecture and Demailly’s Conjecture as well as Harbourne-Huneke Containment, stable Harbourne Containment and some stronger containment.

**Theorem 1.5.** Let \( J \) be the ideal of the singular locus of the arrangement of lines given by the group \( G(1, 1, 4) = A_3 \). Then

1. \( \alpha(J^{(2k)}) = 5k \) for all \( k \geq 2 \) and \( \alpha(J^{(2)}) = 6 \).
2. \( \alpha(J^{(2k+1)}) = 5k + 3 \) for all \( k \geq 0 \).
3. \( \hat{\alpha}(J) = \frac{5}{2} \).
4. \( \rho(J) = \hat{\rho}(J) = \frac{6}{5} \).

Similar to the Fermat ideals and Fermat-like ideals, the verification and failure of the containment \( J \) can be checked purely by their numerical invariants including the least degree of generators of symbolic powers, the regularity, or the resurgence number and the maximal degree of generators.

**Corollary 1.6.** Ideal \( J \) verifies the following containment and stable containment

1. Harbourne-Huneke containment (see [HH13, Conjecture 2.1])
   \[ J^{(2r)} \subseteq m^r J, \quad \forall r \]

2. Harbourne-Huneke containment (see [HH13, Conjecture 4.1.5])
   \[ J^{(2r-1)} \subseteq m^{r-1} J, \quad \forall r \geq 1 \]

3. A stronger containment \( J^{(2r-2)} \subseteq m^r J^3, \quad \forall r \geq 5 \)
We work over the field of complex numbers but our results hold over any algebraically closed field of characteristic 0.

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2. Preliminary

Let \( R = \mathbb{C}[x_0, \ldots, x_N] \) be the homogeneous coordinate ring of \( \mathbb{P}^N \), and let \( m \) be its maximal homogeneous ideal. For a homogeneous ideal \( I \subseteq R \), let \( \alpha(I) \) denote the least degree of a homogeneous polynomial in \( I \), and let

\[
I^{(m)} := \bigcap_{p \in \text{Ass}(R/I)} I^m R_p \cap R
\]

denote its \( m \)-th symbolic power.

The Waldschmidt constant of \( I \) is defined to be the limit and turned out to be the infimum

\[
\alpha(I) := \lim_{m \to \infty} \frac{\alpha(I^{(m)})}{m} = \inf_{m \to \infty} \frac{\alpha(I^{(m)})}{m}
\]

In studying ideals defining sets of points in \( \mathbb{P}^N \), Chudnovsky’s Conjecture gives a lower bound for \( \alpha(I^{(m)}) \) in term of \( \alpha(I) \), \( N \) and \( m \) as follows.

**Conjecture 2.1** (Chudnovsky). Let \( I \) be the defining ideal of a set of points in \( \mathbb{P}^N \). Then, for all \( n \geq 1 \),

\[
\frac{\alpha(I^{(n)})}{n} \geq \frac{\alpha(I) + N - 1}{N}
\]

It is also natural to ask if the Chudnovsky-like inequality is still true for a homogeneous radical ideal \( I \), for example, if we replace \( N \) by bigheight \( h \) of \( I \). Many partial results are known for Chudnovsky’s Conjecture, for example, in [EV83, BH10, HH13, GHM13, Dum15, DTG17, FMX18, BGHN20a]. Recently, the conjecture was proved for a very general set of points in [DTG17, FMX18] and for a general set of sufficiently many points in [BGHN20a]. On the other hand, Chudnovsky-like inequality was verified for a set of very general lines in \( \mathbb{P}^3 \) in [DFSTG19]. The following generalization dues to Demailly [Dem82].

**Conjecture 2.2** (Demailly). Let \( I \) be the defining ideal of a set of points in \( \mathbb{P}^N \). Let \( m \in \mathbb{N} \) be any fixed integer. Then, for all \( n \geq 1 \),

\[
\frac{\alpha(I^{(n)})}{n} \geq \frac{\alpha(I^{(m)}) + N - 1}{m + N - 1}
\]

Demailly’s Conjecture for \( N = 2 \) was proved by Esnault and Viehweg [EV83]. In higher dimension, Demailly’s Conjecture holds for a very general set of sufficiently many points (the number of points depends on \( m \)) in [MSS18] and for a general set of \( k^N \) points in [CJ20]. In [BGHN20b], the conjecture was proved for a general set of sufficiently many points (also depends on \( m \)). We also raise a question [BGHN20b, Question 3.2] for ideal that verified Demailly-like inequality and show some examples of such ideals including: defining ideal of a codimension \( h \) star configuration in \( \mathbb{P}^N \), generic determinantal ideals, determinantal ideals...
of symmetric matrices and pfaffian ideals of skew symmetric matrices.

The resurgence number $\rho(I)$ is introduced in [BH10] as

$$\rho(I) := \sup \left\{ \frac{m}{r} : I^{(m)} \not\subseteq I^r \right\}$$

and the asymptotic resurgence $\hat{\rho}(I)$ is introduced in [GHVT13] as

$$\hat{\rho}(I) := \sup \left\{ \frac{m}{r} : I^{(mt)} \not\subseteq I^{rt} \text{ for } t > 0 \right\}$$

in effort to study the containment problem numerically, namely, the pairs $(m, r)$ such that $I^{(m)} \subseteq I^r$. It is a celebrated result in [ELS01] and [HH02] that $I^{(hr)} \subseteq I^r$ for codimension $h$ ideal $I$. In another effort to improve this containment for ideal of points and study Chudnovsky’s Conjecture and Demailly’s Conjecture, Harbourne and Huneke in [HH13] conjectured that the defining ideal $I$ for any set of points in $\mathbb{P}^N$ satisfies some stronger containment, namely, $I^{(Nm)} \subseteq m^{(N-1)}I^m$ and $I^{(Nm-N+1)} \subseteq m^{(m-1)(N-1)}I^m$, for all $m \geq 1$. We also raise a question [BGHN20b, Question 3.1] for other classes of ideals that satisfies some version of Harbourne-Huneke Containment (replacing $N$ by $h$) and showed some examples including the ideals mentioned in previous paragraph. Note that suitable version of stable Harbourne-Huneke Containment (or even infinitely many such containment) would imply Chudnovsky’s and Demailly’s Conjecture (or Demailly-like inequality), for example, see [BGHN20b]. We refer interested readers to [CHHVT20] for more information about the Waldschmidt constant, resurgence number, containment between symbolic and ordinary powers of ideals.

3. Ideal of restricted Fermat configuration of lines in $\mathbb{P}^3$

Recall that Fermat-like ideal $I_n$ is the ideal generated by

$g_1 = (x^n - y^n)(z^n - w^n)xy, \quad g_2 = (x^n - y^n)(z^n - w^n)zw$

$g_3 = (x^n - z^n)(y^n - w^n)xz, \quad g_4 = (x^n - z^n)(y^n - w^n)yw$

$g_5 = (x^n - w^n)(y^n - z^n)xw, \quad g_6 = (x^n - w^n)(y^n - z^n)yz$

From [MS17], geometrically, $I_n$ is the defining ideal of the union of lines with multiplicity at least 3 of the Fermat arrangement of flats (planes) in $\mathbb{P}^3$, that is defined by the vanishing of the polynomial

$$F_n = (x^n - y^n)(z^n - w^n)(x^n - z^n)(y^n - w^n)(x^n - w^n)(y^n - z^n)$$

There are $4n^2 + 6$ lines in the above restricted Fermat configuration of lines. Notice also that from this description, each of $6n$ planes $H_j$ passes through exactly $2n + 1$ lines, for example, the plane $x = \epsilon y$ passes through the line defined by $(x, y)$, $n$ lines defined by $(x - \epsilon y, x - \epsilon^k z)$ and $n$ lines defined by $(x - \epsilon y, y - \epsilon^k w)$ for $k = 0, \ldots, n - 1$ where $\epsilon$ in an $n$–root of 1.

Algebraically, let $f = (x^n - y^n)(z^n - w^n)$, $g = (x^n - z^n)(y^n - w^n)$, $h = (x^n - w^n)(y^n - z^n)$ and $K_n = (f, g)$, we can write

$$I_n = K_n \cap (x, y) \cap (x, z) \cap (x, w) \cap (y, z) \cap (y, w) \cap (z, w)$$

and since $f, g$ form a regular sequence, for any $m \geq 1$ we have

$$I_n^{(m)} = K_n^m \cap (x, y)^m \cap (x, z)^m \cap (x, w)^m \cap (y, z)^m \cap (y, w)^m \cap (z, w)^m$$
Recall our main strategy in previous work [Ngu21], we will study a subsequence of \( \alpha(I_n^{(m)}) \), which gives us information about \( \hat{\alpha}(I_n) \), then use this to calculate other \( \alpha(I_n^{(m)}) \).

**Proposition 3.1.** For \( n \geq 3 \), we have \( \hat{\alpha}(I_n) = 2n \).

**Proof.** For every \( m \geq 1 \), since \( I_n^{(m)} \subset K_n^m \), we have \( \alpha(I_n^{(m)}) \geq \alpha(K_n^m) = 2nm \). Thus, \( \hat{\alpha}(I_n) \geq 2n \). On the other hand, since \( h = g - f \in K_n \), we have \( F_n^m = f^m g^m h^m \in K_n^{3m} \). Furthermore, since \( nm \geq 3m \), \( F_n^m \in \cap(x, y)^{3m} \cap (x, z)^{3m} \cap (y, z)^{3m} \cap (y, w)^{3m} \cap (z, w)^{3m} \) so \( F_n^m \in I^{(3m)} \), hence \( \alpha(I_n^{(3m)}) \leq 6nm \) for every \( m \geq 1 \). Therefore \( \hat{\alpha}(I_n) \leq 2n \) and it follows that \( \hat{\alpha}(I_n) = 2n \). \( \square \)

Now following exactly the same argument as those of [Ngu21, Section 3] we have the following two results:

**Theorem 3.2.** For \( n \geq 5 \), we have

\[
\alpha(I_n^{(m)}) = 2nm
\]

for all \( m \geq 3 \).

**Proof.** The proof is identical to that of [Ngu21, Theorem 3.2]. \( \square \)

**Theorem 3.3.** For \( n = 4 \), for all \( m \geq 3 \) but \( m \neq 5 \),

\[
\alpha(I_4^{(m)}) = 8m
\]

**Proof.** The proof is identical to that of [Ngu21, Theorem 3.3]. For \( m = 5 \), we can check by the same argument that the element \( yzfg^2h^2 \in I_4^{(5)} \). Macaulay2 computation [GS] suggests that, in fact, \( \alpha(I_4^{(5)}) = 42 \). \( \square \)

Now we calculate the least degree of generators of symbolic powers for \( I_3 \).

**Theorem 3.4.** For \( n = 3 \) and for \( m \geq 1 \) we have the following

1. \( \alpha(I_3^{(3m)}) = 18m \) for \( m \geq 1 \),
2. \( \alpha(I_3^{(3m+1)}) = 18m + 8 \) for \( m \geq 0 \),
3. \( \alpha(I_3^{(3m+2)}) = 18m + 16 \) for \( m \geq 0 \).

**Proof.**

1. Since \( \hat{\alpha}(I_3) = 6 \), \( \alpha(I_3^{(3m)}) \geq 18m \) for \( m \geq 1 \). As we see earlier, \( F_3^m \in I_3^{(3m)} \) for \( m \geq 1 \) so \( \alpha(I_3^{(3m)}) \leq 18m \) and thus \( \alpha(I_3^{(3m)}) = 18m \) for \( m \geq 1 \).
2. Suppose that there is \( m \geq 1 \) such that \( \alpha(I_3^{(3m+1)}) \leq 18m + 7 \). Then there is a divisor \( D \) of degree \( 18m + 7 \) vanishing to order at least \( 3m + 1 \) along every line of \( 42 \) lines \( L_i \) in the restricted Fermat configuration. Intersecting \( D \) with any of the \( 18 \) planes \( H_j \), since each planes \( H_j \) contains exactly \( 7 \) lines, suppose that \( D \) doesn’t contain \( H_j \) then by the generalized Bezout Theorem, the intersection of \( D \) and \( H_j \) is of dimension \( 1 \) and degree \( 18m + 7 \), which is a contradiction since \( 18m + 7 < 7(3m + 1) \).

Thus we conclude that each \( H_j \) is a component of \( D \). Hence, there exists a divisor \( D' = D - \sum_{j=1}^{18} H_j \) of degree \( 18(m - 1) + 7 \) vanishing to order at least \( 3(m - 1) + 1 \) along every line \( L_i \). Repeating this argument \( m \) times we get a contradiction with \( \alpha(I_3) = 8 \). Thus \( \alpha(I_3^{(3m+1)}) \geq 18m + 8 \) for all \( m \geq 1 \). On the other hand, it easy to
see that, \( f^m g^m h^{m+1} yz \in I_3^{(3m+1)} \) so \( \alpha(I_3^{(3m+1)}) \leq 18m + 8 \) for all \( m \geq 1 \). Therefore, \( \alpha(I_3^{(3m+1)}) = 18m + 8 \) for all \( m \geq 1 \).

(3) By identical argument to previous case and the fact that \( \alpha(I_3^{(2)}) = 16 \), we have that \( \alpha(I_3^{(3m+2)}) \geq 18m + 16 \) for all \( m \geq 1 \). Since the element \( f^{m+1} g^{m+1} h^m xyzw \in I_3^{(3m+2)} \) we have that \( \alpha(I_3^{(3m+2)}) \leq 18m + 16 \) for all \( m \geq 1 \). \( \square \)

**Proposition 3.5.** For all \( n \geq 3 \), \( 4n + 2 \leq \alpha(I_n^{(2)}) \leq 4n + 4 \).

**Proof.** We know that \( \alpha(I_n^{(2)}) \leq \alpha(I_n^2) = 2(2n + 2) \). Now suppose that \( \alpha(I_n^{(2)}) \leq 4n + 1 \). Then there is a divisor \( D \) of degree \( 4n + 1 \) vanishing to order at least 2 along every line \( L_i \) in the restricted Fermat configuration. Since the intersection of \( D \) and any plane \( H_j \) consists of \( 2n + 1 \) lines to order at least 2, by generalized Bezout theorem, each \( H_j \) is a component of \( D \) because \( \deg(D) \cdot \deg(H_j) = 4n + 1 < 2(2n + 1) \). This is a contradiction since there are \( 6n \) planes and \( 6n > 4n + 1 = \deg(D) \) when \( n \geq 3 \). Therefore, \( 4n + 2 \leq \alpha(I_n^{(2)}) \leq 4n + 4 \) for all \( n \geq 3 \). \( \square \)

**Remark 3.6.** Macaulay2 computations [GS] for \( n \) small suggest that \( \alpha(I_n^{(2)}) = 2(2n + 2) \) for \( n \geq 3 \).

As mentioned earlier, in [BGHN20b, Question 3.2], we raise a question to study the Demailly-like bound for homogeneous ideals. We also give example of classes of ideals that satisfy Demailly-like inequality. From the above calculations, as a direct consequence, we see that the ideal \( I_n \) also gives an example for Demailly-like inequality.

**Corollary 3.7.** For \( n \geq 3 \), the ideal \( I_n \) satisfies Demailly-like inequality

\[
\hat{\alpha}(I_n) \geq \frac{\alpha(I_n^{(m)}) + h - 1}{m + h - 1}
\]

for all \( m \geq 1 \) where \( h \) is the big height of \( I_n \).

**Proof.** Direct from the above calculation with notice that \( h = 2 \). \( \square \)

We can easily compute the asymptotic resurgence number of \( I_n \) as follows.

**Corollary 3.8.** \( \hat{\rho}(I_n) = \frac{n + 1}{n} \).

**Proof.** It is clear from [GHVT13, Theorem 1.2] that

\[
\frac{2n + 2}{2n} = \frac{\alpha(I_n)}{\hat{\alpha}(I_n)} \leq \hat{\rho}(I_n) \leq \frac{\omega(I_n)}{\hat{\alpha}(I_n)} = \frac{2n + 2}{2n}.
\]

\( \square \)

The following containment are also direct consequences of the above calculations.

**Corollary 3.9.** For every \( n \geq 3 \), restricted Fermat configuration ideal verifies Harbourne-Huneke containment

\[
I_n^{(2r)} \subseteq \mathfrak{m}^r I_n, \quad \forall r
\]
Proof. Since $I_n$ has big height 2, $I_n^{(2r)} \subseteq I_n^r, \forall r$ so the above containment once again come from the fact that 
\[\alpha(I_n^{(2r)}) \geq r + \omega(I_n^r)\]
for all $n \geq 3$ and $r \geq 1$ (the case $r = 0$ is trivial). Indeed, $\alpha(I_n^{(2r)}) \geq 4rn > r + r(2n + 2) = r + \omega(I_n^r)$ for $n \geq 3$. \hfill \square

**Corollary 3.10.** For every $n \geq 3$, restricted Fermat configuration ideal verifies stable Harbourne-Huneke containment 
\[I_n^{(2r-1)} \subseteq m^{r-1}I_n^r, \quad \forall r \gg 0\]

**Proof.** Since $\rho(I_n) = \frac{n+1}{n} < 2$, by [HKZ20, Theorem 2.3] or [DD20, Corollary 4.1] we see that $\rho(I_n) < 2$ as well, i.e, $I_n$ has expected resurgence. Thus, by [Gri20, Remark 2.7] for all $r \gg 0$, $I_n^{(2r-1)} \subseteq I_n^r$, the above containment comes from the fact that 
\[\alpha(I_n^{(2r-1)}) \geq r - 1 + \omega(I_n^r)\]
for all $n \geq 3$ and $r \geq 3$. Indeed, we have $\alpha(I_n^{(2r-1)}) \geq (2r - 1)2n$ and $r - 1 + \omega(I_n^r) = r - 1 + r(2n + 2)$. Since $(2r - 1)2n \geq r - 1 + r(2n + 2) \iff 2n(r - 1) \geq 3r - 1$ and in the last inequality, $2n(r - 1) \geq 6(r - 1) \geq 3r - 1$ when $r \geq 3$, we get the desired inequality. \hfill \square

**Corollary 3.11.** For every $n \geq 3$, restricted Fermat configuration ideal verifies the stable containment 
\[I_n^{(2r-2)} \subseteq m^rI_n^r, \quad \forall r \gg 0\]

**Proof.** As before, since $I_n$ has expected resurgence, $I_n^{(2r-2)} \subseteq I_n^r$ for $r \gg 0$. On the other hand, the inequalities 
\[\alpha(I_n^{(2r-2)}) \geq r + \omega(I_n^r)\]
is true when $r \geq 4$. \hfill \square

We will compute here the resurgence number for $I_3$ by using an upper bound for regularity of its powers and use it to improve the above results of containment for $I_3$.

**Theorem 3.12.** $\rho(I_3) = \frac{3}{2}$.

**Proof.** By [MS17], $I_3^{(3)} \nsubseteq I_3^2$ hence, $\rho(I_3) \leq \frac{3}{2}$. We will show that for all $m, r$ such that 
\[\frac{m}{r} > \frac{3}{2}\]
we have $I^{(m)} \subseteq I^r$ and then it follows that $\rho(I_3) \leq \frac{3}{2}$. In fact, we will show that for all $m, r$ such that $\frac{m}{r} > \frac{3}{2}$ then $\alpha(I_3^{(m)}) \geq \text{reg}(I_3^r)$.

We will apply [Cha07, Theorem 0.6], which states the following:

Let $I$ be an homogeneous ideal of a polynomial ring $R$ over a field, generated in degrees at most $d$, such that $\dim(R/I) = 2$. Assume that $I_p \subseteq R_p$ is a complete intersection for every prime $I \subseteq p$ such that $\dim R/p = 2$. Then 
\[
\text{reg}(I^2) \leq \max\{2 \text{reg}(I), \text{reg}(I^{\text{sat}}) + 2d - 2\}
\]
and for $r \geq 3$
\[
\text{reg}(I^r) \leq \max\{3 \text{reg}(I) + (r - 3)d, \text{reg}(I^{sat}) + rd - 2\}
\]

Clearly, $\dim(R/I_3) = 2$. Since the set of associated primes of $I_3$ is also the set of its minimal primes which consists of 42 minimal primes $P_j$ where each minimal prime is the defining ideal of one of a line in the configuration. Since any prime ideal $P$ that contains $I_3$ must contain one $P_j$, such prime ideal $P$ with $\dim(R/P) = 2$ has to be $P_j$. Thus, for any prime $I \subset P$ such that $\dim(R/P) = 2$, $(I_3)_P$ is a complete intersection. Since $\text{reg}(I_3) = 11$, $\alpha(I_3) = 8$ and $I_3 = I_3^{sat}$, by [Cha07, Theorem 0.6] we have for $r \geq 3$:

\[
\text{reg}(I^r_3) \leq \max\{3 \text{reg}(I_3) + 8(r - 3), \text{reg}(I_3) + 8r - 2\} = \max\{33 + 8(r - 3), 11 + 8r - 2\} = 8r + 9
\]

and $\text{reg}(I^2_3) \leq \max\{22, 11 + 16 - 2\} = 25$. Thus, for every $r \geq 2$:

\[
\text{reg}(I^r_3) \leq 8r + 9
\]

Now for $\frac{m}{r} > \frac{3}{2}$, i.e, $2m \geq 3r + 1$, we have $\alpha(I^{(m)}_3) \geq 6m \geq 9r + 3$. Since $9r + 3 \geq 8r + 9 \geq \text{reg}(I^r_3)$ for all $r \geq 6$, it suffices now to check that for $r \leq 5$ and $2m \geq 3r + 1$ we still have $\alpha(I^{(m)}_3) \geq \text{reg}(I^r_3)$. Indeed,

- When $r = 1$, for $m \geq 2$, we have $\text{reg}(I_3) = 11 < 16 = \alpha(I^{(2)}_3) \leq \alpha(I^{(m)}_3)$.
- When $r = 2$, for $m \geq 4$, we have $\text{reg}(I^2_3) \leq 25 < 26 = \alpha(I^{(4)}_3) \leq \alpha(I^{(m)}_3)$.
- When $r = 3$, for $m \geq 5$, we have $\text{reg}(I^3_3) \leq 33 < 34 = \alpha(I^{(5)}_3) \leq \alpha(I^{(m)}_3)$.
- When $r = 4$, for $m \geq 7$, we have $\text{reg}(I^4_3) \leq 41 < 44 = \alpha(I^{(7)}_3) \leq \alpha(I^{(m)}_3)$.
- When $r = 5$, for $m \geq 8$, we have $\text{reg}(I^5_3) \leq 49 < 52 = \alpha(I^{(8)}_3) \leq \alpha(I^{(m)}_3)$.

\[\square\]

As a consequence, we can verify that $I_3$ satisfies Harbourne-Huneke Containment for $r \geq 3$

\textbf{Corollary 3.13.} Ideal $I_3$ verifies the following containment

1. $I^{(2r-1)}_3 \subseteq I^{r-1}_3$, $\forall r \geq 3$
2. $I^{(2r-2)}_3 \subseteq I^{r}_3$, $\forall r \geq 5$

\textbf{Proof.} Since $\rho(I_3) = \frac{3}{2}$, $I^{(2r-1)}_3 \subseteq I^r_3$ if and only if $r \geq 3$ and $I^{(2r-2)}_3 \subseteq I^r_3$ if and only if $r \geq 5$. The statements follows as we have the inequalities $\alpha(I^{(2r-1)}_3) \geq r - 1 + \omega(I^r_3)$ for all $r \geq 3$ and $\alpha(I^{(2r-2)}_3) \geq r + \omega(I^r_3)$ for all $r \geq 4$.$\square$

\textbf{Remark 3.14.} As shown in the proof, the first containment fails if $r \leq 2$ and the second fails if $r \leq 5$.

Now we calculate the regularity of ideal $I_n$ for all $n \geq 3$ and use this to investigate the resurgence number for $I_n$ where $n \geq 4$.

\textbf{Proposition 3.15.} For all $n \geq 3$, $R/I_n$ is Cohen-Macaulay and its minimal graded free resolution is

\[
0 \rightarrow R(-(2n + 3))^4 \oplus R(-4n) \rightarrow R(-(2n + 2))^6 \rightarrow R \rightarrow R/I_n \rightarrow 0
\]
where \( \varphi_1 = [g_1, g_2, g_3, g_4, g_5, g_6] \) and

\[
\varphi_2 = \begin{bmatrix}
-x & 0 & -w & 0 & x^{n-1}z^{n-1} \\
-x & 0 & 0 & 0 & -y^{n-1}z^{n-1} \\
-x & -w & 0 & 0 & -x^{n-1}y^{n-1} \\
0 & -y & z & 0 & -x^{n-1}w^{n-1} \\
0 & x & 0 & z & y^{n-1}w^{n-1} \\
0 & 0 & x & -y & -z^{n-1}w^{n-1}
\end{bmatrix}
\]

The maximal minors of \( \varphi_2 \) are multiple of \( g_1, \ldots, g_6 \) as in Hilbert-Burch theorem.

**Proof.** Direct calculation give the minors of \( \varphi_2 \) deleting row 1, 2, 3, 4, 5, 6 are \( 2g_3, -2g_6, 2g_1, -2g_5, 2g_4, -2g_2 \) respectively. Applying [Eis95, Theorem 18.18], we see that \( I_n \) is the ideal generated by \( 5 \times 5 \) minors of the \( 6 \times 5 \) matrix \( \varphi_2 \) and since \( I_n \) has codimension 2 which agrees with \( (6-5+1)(5-5+1) \), \( R/I_n \) is Cohen-Macaulay. In particular, \( \text{depth}(R/I_n) = 2 \) and by Auslander–Buchsbaum formula \( R/I_n \) has projective dimension 2. Thus by Hilbert-Burch theorem, \( R/I_n \) has such minimal graded free resolution.

The immediate consequence of the above free resolution is the regularity of ideal \( I_n \).

**Corollary 3.16.** For all \( n \geq 3 \), \( \text{reg}(I_n) = 4n - 1 \).

We now deduce the following containment as a step towards calculating the resurgence number of \( I_n \) for \( n \geq 4 \).

**Corollary 3.17.** If \( n \geq 4 \) then for all \( m, r \) such that \( \frac{m}{r} > \frac{3}{2} \) and \( r \geq 6 \), we have the inequality \( \alpha(I_n^{(m)}) \geq \text{reg}(I_n^r) \) which implies that \( I_n^{(m)} \subseteq I_n^r \).

**Proof.** By the same argument as that of in the proof of theorem 3.12, \( I_n \) satisfies all conditions of [Cha07, Theorem 0.6], so we have

\[
\text{reg}(I_n^r) \leq \max\{3(4n - 1) + (r - 3)(2n + 2), 4n - 1 + r(2n + 2) - 2\} = r(2n + 2) + 6n - 9
\]

for \( r \geq 3 \). Since \( \alpha(I_n^{(m)}) \geq 2nm \), for \( m, r \) such that \( \frac{m}{r} > \frac{3}{2} \), i.e. \( 2m \geq 3r + 1 \) we have \( \alpha(I_n^{(m)}) \geq n(3r + 1) \). Now the inequality

\[
n(3r + 1) \geq r(2n + 2) + 6n - 9
\]

is equivalent to

\[
r \geq \frac{5n - 9}{n - 2} = 5 + \frac{1}{n - 2}
\]

which is true for \( r \geq 6 \) and \( n \geq 4 \).

The resurgence number of \( I_n \) for \( n \geq 4 \) can be computed by checking 4 containment as the following corollary.

**Corollary 3.18.** For \( n \geq 4 \), the resurgence number \( \rho(I_n) \) can only be one of the following numbers: \( \frac{3}{2}, \frac{5}{3}, \frac{7}{4}, \frac{8}{5}, \) or \( \frac{9}{5} \). More precisely, among the four containment \( I_n^{(9)} \subseteq I_n^5, I_n^{(7)} \subseteq I_n^4, I_n^{(5)} \subseteq I_n^3 \) and \( I_n^{(8)} \subseteq I_n^5 \), the ratio \( \frac{m}{r} \) of the first containment (in that order) that fails is the exact value for \( \rho(I_n) \), otherwise, \( \rho(I_n) = \frac{3}{2} \).
Proof. We know that $I_n^{(3)} \not\subseteq I_n^2$ by [MS17], hence, $\rho(I_n) \leq \frac{3}{2}$. On the other hand, $I_n^{(m)} \subseteq I_n^r$ for all $m, r$ such that $\frac{m}{r} > \frac{3}{2}$ and $r \geq 6$. The containment are still true if $\frac{m}{r} > \frac{3}{2}$ for $r = 1, 2$ since $\frac{m}{r} > \frac{3}{2}$ implies that $m \geq 2r$ in these cases and $I_n$ has big height 2. On the other hand, if $r = 3$ then $\frac{m}{r} > \frac{3}{2}$ implies $m \geq 5$ and we know that $I_n^{(m)} \subseteq I_n^3$ for $m \geq 6$. Similarly, if $r = 4$ then $\frac{m}{r} > \frac{3}{2}$ implies $m \geq 7$ and $I_n^{(m)} \subseteq I_n^4$ for $m \geq 8$. Lastly, if $r = 5$ then $\frac{m}{r} > \frac{3}{2}$ implies $m \geq 8$ and $I_n^{(m)} \subseteq I_n^5$ for $m \geq 10$.

Thus, since $\frac{3}{2} < \frac{8}{5} < \frac{5}{3} < \frac{7}{4} < \frac{9}{5}$, to compute the exact value of $\rho(I_n)$, it suffices to examine whether the following four containment $I_n^{(9)} \subseteq I_n^5$, $I_n^{(7)} \subseteq I_n^3$, $I_n^{(5)} \subseteq I_n^3$, and $I_n^{(8)} \subseteq I_n^5$ fails or not in this order. In fact, if those containment all hold, then $I_n^{(m)} \subseteq I_n^r$ for all $m, r$ such that $\frac{m}{r} > \frac{3}{2}$ and it follows that $\rho(I_n) = \frac{3}{2}$. Otherwise, suppose $I_n^{(m)} \subseteq I_n^m$ is the first containment that fails among the above four containment, then $\rho(I_n) \geq \frac{m_0}{r_0}$. For all $m, r$ such that $\frac{m}{r} > \frac{m_0}{r_0} > \frac{3}{2}$, we know that either $I_n^{(m)} \subseteq I_n^r$ if $\frac{m}{r}$ is not among $\frac{5}{3}, \frac{7}{4}, \frac{8}{5}$ or $\frac{9}{5}$ by above result or $\frac{m}{r}$ is among $\frac{5}{3}, \frac{7}{4}, \frac{8}{5}$ or $\frac{9}{5}$ that is greater than $\frac{m_0}{r_0}$, in which $I_n^{(m)} \subseteq I_n^r$ hold as well. Either way we have $\rho(I_n) \leq \frac{m_0}{r_0}$, therefore $\rho(I_n) = \frac{m_0}{r_0}$. □

Remark 3.19. Macaulay2 calculations [GS] for $n, r$ small suggests that $\text{reg}(I_n^r) = (r + 1)\alpha(I_n) - 5 = (r + 1)(2n + 2) - 5$. If we can prove this then $\alpha(I_n^{(m)}) \geq \text{reg}(I_n^r)$ for all $\frac{m}{r} > \frac{3}{2}$. We also know that $I_n^{(3)} \not\subseteq I_n^2$ by [MS17]. Thus, in this case $\rho(I_n) = \frac{3}{2}$ for all $n \geq 3$.

| $n$ | $3$ | $4$ | $n \geq 5$ |
|-----|-----|-----|-----------|
| $m$ | 2k | 2k+1 | 2k+2 | 2 | 5 | $\geq 3$, $\neq \frac{5}{3}$ | 2 | $\geq 3$ |
| $\alpha(I_n^{(m)})$ | 16 | 18k | 18k+8 | 18k+16 | 20 | 42 | $4m$ | $\geq 4n + 2$, $\leq 4n + 4$ | $2nm$ |
| $\hat{\alpha}(I_n)$ | 3 | 4 | | | | | | |
| $\hat{\rho}(I_n)$ | $\frac{4}{3}$ | $\frac{5}{4}$ | | | | | | |
| $\rho(I_n)$ | $\frac{3}{2}$ | | | | | | | |

Table 1. Least degree of generators and other invariants related to symbolic powers of $I_n$.
In this section, we will deal with ideal $J = (yz(y-z), zx(z-x), xy(x-y))$, which is the ideal of the singular locus of the arrangement of lines given by the group $G(1, 1, 4) = A_3$. We will see that its least degree of generators of symbolic powers, Waldschmidt constant, (asymptotic) resurgence number are the same as those of group $D_3$, see [Ngu21, Section 4].

Geometrically, $J$ is the defining ideal of the singular locus of the line arrangement in $\mathbb{P}^2$ that consists of 6 lines $L_j$ whose equations are

$$x = 0, y = 0, z = 0, x = y, y = z, z = x$$

These 6 lines intersect at 7 points $P_i$ which are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1], [0 : 1 : 1], [1 : 0 : 1]$ and $[1 : 1 : 0]$ such that the first 4 points lie on 3 lines each and the rest lie on 2 lines each; and each line contains exactly 3 points.

Algebraically, we can write

$$J = ((y-z)(y+z-x), (x-y)(x+y-z)) \cap (x, y) \cap (y, z) \cap (z, x)$$

and

$$J^{(m)} = ((y-z)(y+z-x), (x-y)(x+y-z))^m \cap (x, y)^m \cap (y, z)^m \cap (z, x)^m \ \forall m$$

**Proposition 4.1.** For ideal $J$ we have the following

1. $\hat{\alpha}(J) = \frac{5}{2}$
2. $\alpha(J^{(2k)}) = 5k$, for all $k \geq 2$.
3. $\alpha(J^{(2k+1)}) = 5k + 3$, for all $k \geq 0$.
4. $\alpha(J^{(2)}) = 6$

**Proof.** By [FGH+17, Theorem 2.3], we can check that $\hat{\alpha}(J) \geq \frac{5}{2}$.

In particular, we have that $\alpha(J^{(2k)}) \geq 5k$ and $\alpha(J^{(2k+1)}) \geq 5k+3$, for all $k \geq 1$. We will show the reverse by showing there exists some element with the desired degree in the symbolic powers.

Denote $K = ((y-z)(y+z-x), (x-y)(x+y-z))$ and first notice that since

$$-(z-x)(z+x-y) = (y-z)(y+z-x) + (x-y)(x+y-z) \in K$$
we have that
\[(4.1)\quad 2x(x - y)(z - x) = (z - x)(x - y)(x + y - z) + (x - y)(z - x)(z + x - y) \in K\]

Similarly, \(y(y - z)(x - y), z(z - x)(y - z) \in K\). It follows that
\[(4.2)\quad (y - z)(z - x)xyz^2 = (y - z)(z - x)xy[z^2 - (x - y)^2] + (y - z)(z - x)xy(x - y)^2 \in K^2\]

since
\[(y - z)(z - x)xy[z^2 - (x - y)^2] = xy(y - z)(y + z - x)(z - x)(z + x - y) \in K^2\]

and by \(4.1\)
\[(y - z)(z - x)xy(x - y)^2 = x(x - y)(z - x)y(y - z)(x - y) \in K^2\]

By \(4.1\) we also have that
\[(4.3)\quad yz(y - z) = (y - z)z(y + z - x) - (y - z)z(z - x) \in K\]

We have the following cases

(1) Case 1: when \(m = 4k\). Consider the polynomial \(F = [(y - z)(z - x)xyz^2]^k(x - y)^2k(x + y - z)^2k\) that has degree 10k. By \(4.3\), \([(y - z)(z - x)xyz^2]^k \in K^{2k}\) so \(F \in K^{4k}\). On the other hand, \([(y - z)yz^2]^k \in (y, z)^{4k}, [(z - x)zx^2]^k \in (z, x)^{4k}\) and \((xy)^{2k}(x - y)^{2k} \in (x, y)^{4k}\). Thus
\[F \in K^{4k} \cap (x, y)^{4k} \cap (y, z)^{4k} \cap (x, z)^{4k} = J^{(4k)}, \forall k \geq 1\]

(2) Case 2: when \(m = 4k + 2\). We first show that
\[G = x^2y^2z^2(x - y)^2(y - z)^2(z - x)^2(x + y - z)(y + z - x)(z + x - y) \in J^{(6)}\]

In fact, \((x - y)(y - z)(z - x)(x + y - z)(y + z - x)(z + x - y) \in K^3\) and by \(4.3\) \(x^2y^2z^2(x - y)^2(y - z)^2(z - x)^2 \in K^3\). It is also clear that \(G \in (x, y)^6 \cap (y, z)^6 \cap (z, x)^6\).

The polynomial \(F = G[(y - z)(z - x)xyz^2]^k(x - y)^{2k}(x + y - z)^{2k}\) has degree \(10(k + 1) + 5\) and by case 1
\[F \in J^{(6)}J^{(4k)} \subseteq J^{(4(k+1)+2)}, \forall k \geq 0\]

(3) Case 3: when \(m = 4k + 1\). \(F = x^ky^{k+1}z^{2k+1}(y - z)^{k+1}(z - x)^k(x - y)^{2k}(x + y - z)^{2k}\) has degree \(10k + 3\). By \(4.2\) \(x^ky^kz^{2k}(y - z)^k(z - x)^k \in K^{2k}\), by \(4.3\) \(yz(y - z) \in K\) so \(F \in K^{4k+1}\). Similar to case 1, \(F \in \cap(x, y)^{4k+1} \cap (y, z)^{4k+1} \cap (z, x)^{4k+1}\) and hence,
\[F \in J^{(4k+1)}, \forall k \geq 0\]

(4) Case 4: when \(m = 4k + 3\). \(F = [(y - z)(z - x)xyz^2]^{k+1}(x - y)^{2k+1}(x + y - z)^{2k+1}\) has degree \(10k + 8\) and
\[F \in K^{4k+3} \cap (x, y)^{4k+3} \cap (y, z)^{4k+3} \cap (z, x)^{4k+3} = J^{(4k+3)}, \forall k \geq 0\]

Thus, \(\alpha(J^{(2k)}) \leq 5k\), for all \(k \geq 2\) and \(\alpha(J^{(2k+1)}) \leq 5k + 3\), for all \(k \geq 0\). It follows that statements (2) and (3) are true and by taking limit as \(k\) goes to \(\infty\), (1) follows as well. Part (4) can be checked directly by Macaulay2 or by Bezout theorem argument as follows: We know that \(\alpha(J^{(2)}) \leq \alpha(J^2) = 6\). Now suppose that \(\alpha(J^{(2)}) \leq 5\). Then there is a divisor \(D\) of degree 5 vanishing to order at least 2 at every point \(P_i\). Since the intersection of \(D\) and any line \(L_j\) consists of 3 points to order at least 2, we get a contradiction to Bezout theorem because \(\text{deg}(D), \text{deg}(L_j) = 5 < 2.3\).
Example 4.2. It is worth to point out that the first immediate application of the above calculations is the verification of Chudnovsky’s Conjecture and Demailly’s Conjecture, although the general case is already known from [EV83]. Ideal $J$ verifies:

1. Chudnovsky’s Conjecture $\hat{\alpha}(J) \geq \frac{\alpha(J) + 1}{2}$.
2. Demailly’s Conjecture $\hat{\alpha}(J) \geq \frac{\alpha(J^{(m)}) + 1}{m + 1}$ for all $m$.

Theorem 4.3. The resurgence number and asymptotic resurgence number are:

$$\rho(J) = \hat{\rho}(J) = \frac{6}{5}$$

Proof. Since $J$ has the same minimal degrees of generators of symbolic powers and same free resolutions of all its ordinary powers as those of the ideal of configuration of $D_3$, the proof is identical to that of ideal of configuration of $D_3$ [Ngu21, Proposition 4.2].

The following corollaries and remark also follow from the fact that ideal of configuration $A_3$ and $D_3$ share the same minimal degrees of generators of all symbolic powers, see [Ngu21, Section 4].

Corollary 4.4. Ideal $J$ verifies Harbourne-Huneke containment

1. $J^{(2r)} \subseteq \mathfrak{m}^r J^r$, $\forall r$.
2. $J^{(2r-1)} \subseteq \mathfrak{m}^{r-1} J^r$, $\forall r \geq 1$.

Remark 4.5. The above corollary gives a proof for the case $A_3$ in [DS20, Proposition 6.3].

In [BGHN20a, Example 3.7], we showed the stronger containment (which implies both Harbourne-Huneke) containment

$$J^{(2r-2)} \subseteq \mathfrak{m}^r J^r$$

for $r = 5$ (by Macaulay2) and thus for all $r \gg 0$ by our method. In particular, from the proof of [BGHN20a, Theorem 3.1], the containment hold for $r \geq 10$. Here we see that the containment hold for all $r \geq 5$.

Corollary 4.6. For every $n \geq 3$, Fermat configuration ideal verifies the following containment

$$J^{(2r-2)} \subseteq \mathfrak{m}^r J^r$$

Remark 4.7. For $r \leq 4$, the above containment fail with the same reason to that of [Ngu21, Remark 4.6].

| $m$ | $2k$ | $2k + 1$ |
|-----|------|----------|
| $\alpha(J^{(m)})$ | $6k$ | $5k + 3$ |
| $\hat{\alpha}(J)$ | $5/2$ |          |
| $\rho(J) = \hat{\rho}(J)$ | $6/5$ |          |

Table 2. Least degree of generators and other invariants related to symbolic powers of $J$
The table provide a complete answer to the question in the theory of Hermite interpolation, that is to determine the least degree of a homogeneous polynomial that vanishes to order $m$ at the 7 points of the given configuration in $\mathbb{P}^2$.

We end this section by calculating the invariant $\beta(J^{(m)})$. As introduced first in [HH13, Definition 2.2] for homogeneous ideals and later considered for ideal $I$ of a finite set of (fat) points in [HS15], $\beta(I)$ is set to be the smallest integer $t$ such that $I_t$ contains a regular sequence of length two, or equivalently, is the least degree $t$ such that the zero locus of $I_t$ is 0-dimensional. It is known that for ideals of fat points, $\beta(I) \leq \omega(I)$ [Ngu21, Proposition 3.9]. By the same argument to that of [Ngu21, Proposition 4.9] we have the following

**Proposition 4.8.** For all $m \geq 1$, $\beta(J^{(m)}) = 3m$ and $\omega(J^{(m)}) \geq 3m$.

**Proof.** The proof is identical to that of [Ngu21, Proposition 4.9] since $J^m$ is generated in degree $3m$ and each line in this configuration also passed through exactly 3 points in the configuration. \qed

**Remark 4.9.** Similar to Fermat ideals, it is suggested by Macaulay2 that $\omega(J^{(m)}) = 3m$.

5. An Additional Example

Consider ideal $J_2 = (yz(y^2 - z^2), zx(z^2 - x^2), xy(x^2 - y^2))$. Geometrically, $J$ is the defining ideal of the singular locus of the line arrangement in $\mathbb{P}^2$ corresponds to the group $B_3$, [DS20]. This arrangement consists of 9 lines $L_j$ whose equations are

$$x = 0, y = 0, z = 0, x = \pm y, y = \pm z, z = \pm x$$

These 9 lines intersect at 13 points $P_i$ which are $[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1], [-1 : 1 : 1], [1 : -1 : 1], [1 : 1 : -1], [0 : 1 : 1], [1 : 1 : 0], [0 : 1 : -1], [-1 : 0 : 1]$ and $[1 : -1 : 0]$ such that the first 3 points lie on 4 lines each, the next 4 points lie on 3 lines each and the rest lie on 2 lines each; and each line contains exactly 4 points.

Algebraically, we can write

$$J_2 = K_1 \cap K_2 \cap K_3 \cap K \cap (x, y) \cap (y, z) \cap (z, x)$$

where $K_1 = (x, y^2 - z^2), K_2 = (y, z^2 - x^2), K_3 = (z, x^2 - y^2)$ and $K = (x^2 - y^2, y^2 - z^2)$, or we can write

$$J_2 = K'_1 \cap K'_2 \cap K \cap (x, y) \cap (y, z) \cap (z, x)$$

where $K'_1 = ((y - z)(y + z - x), (x - y)(x + y - z)), K'_2 = (x + y + z, yz(y + z))$, then for all $m$

$$J^{(m)}_2 = K^{(m)}_1 \cap K^{(m)}_2 \cap K^{(m)}_3 \cap K^{(m)} \cap (x, y)^m \cap (y, z)^m \cap (z, x)^m$$

$$= K'^{(m)}_1 \cap K'^{(m)}_2 \cap K^{(m)} \cap (x, y)^m \cap (y, z)^m \cap (z, x)^m$$

Note that $K \cap (x, y) \cap (y, z) \cap (z, x)$ is the Fermat ideal with $n = 2$, so we immediately have that $\alpha(J_2) \geq \frac{5}{2}$. Moreover, computations with Macaulay2 show that $\alpha(J^{(6)}_2) = 21$, hence,
\( \alpha(J_2) \leq \frac{7}{2} \). It seems that \( \alpha(J_2) \) would be \( \frac{7}{2} \) but the \( \alpha(J_2^{(m)}) \) are tricky to deal with. On the other hand, \( \alpha(J_2^{(3)}) = 12 \), by [EV83], since \( J_2 \) satisfies Demailly’s Conjecture, we have:

\[
\hat{\alpha}(J_2) \geq \frac{\alpha(J_2^{(3)}) + 1}{3 + 1} = \frac{13}{4}
\]

Since \( \text{reg}(J_2) = 6 \), by [GHVT13, Theorem 1.2] we have:

\[
\frac{8}{7} \leq \frac{\alpha(J_2)}{\hat{\alpha}(J_2)} \leq \hat{\rho}(J_2) \leq \frac{\text{reg}(J_2)}{\hat{\alpha}(J_2)} \leq \frac{24}{13}
\]

In particular, \( J_2 \) has expected resurgence. Further computations by Macaulay2 show that, \( J_2^{(7)} \not\subset J_2^{(6)} \), thus, \( \frac{7}{6} \leq \rho(J_2) \). It is interesting to know if \( \rho(J_2) = \frac{7}{6} \).

More general, for \( n \geq 1 \), consider the ideal \( J_n = (yz(y^n - z^n), zx(z^n - x^n), xy(x^n - y^n)) \) which will capture \( J_2 \) and \( J = J_1 \). Geometrically, \( J_n \) corresponds to the configuration of \( 3n+3 \) lines that consists of all \( 3n \) lines in Fermat configuration and \( 3 \) lines \( x = 0, y = 0, z = 0 \); and \( n^2 + 3n + 3 \) intersection points. In particular, each line passes through \( n + 2 \) points. Similar to \( J_2 \), note that \( J_n \) is a subset of the Fermat ideals for \( n \geq 2 \), thus \( \hat{\alpha}(J_n) \geq n \) for \( n \geq 3 \).

**Remark 5.1.** Only by looking at the above lower bound, we easily see that \( J_n \) verifies Chudnovsky’s Conjecture \( \hat{\alpha}(J_n) \geq \frac{\alpha(J_n) + 1}{2} \) for all \( n \). In fact, the case \( n = 1 \) was verified in previous sections and \( n = 2 \) follows from \( \hat{\alpha}(J_2) \geq \frac{5}{2} \). When \( n \geq 3 \), \( \hat{\alpha}(J_n) \geq n \geq \frac{n + 2 + 1}{2} \).

On the other hand, by [NS16, Theorem 2.5], since \( J_n \) is a strict almost complete intersection ideal with minimal generators of degree \( n + 2 \) and its module syzygies is generated in degree \( 1 \) and \( n + 1 \), the minimal free resolution of \( J_n^{(r)} \) is:

\[
0 \rightarrow R(-(n + 2)(r + 1))^{(2)} \rightarrow R(-n + 2)r - (n + 1))^{(r + 1)} \rightarrow R(-n + 2)r ^{(r + 2)} \rightarrow J_n^{(r)} \rightarrow 0
\]

for any \( r \geq 2 \), in particular, \( \text{reg}(J_n^{(r)}) = (n + 2)r + n \) for all \( r \geq 2 \). It would be interesting to know if we can determine \( \alpha(J_n^{(m)}) \) and use them with the knowledge of \( \text{reg}(J_n^{(r)}) \) to verify Demailly’s Conjecture as well as Harbourne-Huneke Containment, stable Harbourne Containment as we did for Fermat ideals and Fermat-like ideals.

Back to ideal \( J_2 \), the following proposition (part (2)) gives a proof for the case \( B_3 \) in [DS20, Proposition 6.3].

**Proposition 5.2.** Ideal \( J_2 \) verifies Harbourne-Huneke containment

1. \( J_2^{(2r)} \subseteq m^rJ_2^r \), \( \forall r \).
2. \( J_2^{(2r+1)} \subseteq m^{r+1}J_2^r \), \( \forall r \geq 3 \).

**Proof.** (1) For all \( r \), \( J_2^{(2r)} \subseteq J_2^r \), hence, the containment follows since for all \( r \), we have

\[
\alpha(J_2^{(2r)}) \geq \frac{13r}{2} \geq r + 4r = r + \omega(J_2^r)
\]
(2) First, the containment $J_2^{(2r-1)} \subseteq J_2^r$ for $r \geq 3$ follows from the inequality

$$\alpha(J_2^{(2r-1)}) \geq \frac{13}{4}(2r-1) \geq 4r + 2 = \text{reg}(J_2^r)$$

for all $r \geq 3$. Thus, $J_2^{(2r-1)} \subseteq m^{r-1}J_2^r$ for $r \geq 3$ follows from:

$$\alpha(J_2^{(2r-1)}) \geq \frac{13}{4}(2r-1) \geq r - 1 + 4r = r + \omega(J_2^r)$$

for all $r \geq 2$.

\[\square\]

\textbf{Remark 5.3.} Macaulay2 shows that for $r = 2$, $\alpha(J_2^{(3)}) = 12$, hence, $\alpha(J_2^{(3)}) \geq 4.2 + 2 = \text{reg}(J_2^2)$. Therefore, $J_2^{(2r-1)} \subseteq J_2^r$ for $r \geq 2$ and since $\alpha(J_2^{(2r-1)}) \geq r + \omega(J_2^r)$ for all $r \geq 2$, $J_2^{(2r-1)} \subseteq m^{r-1}J_2^r$, for $r \geq 2$. The case $r = 1$ is obvious. Thus, $J_2^{(2r-1)} \subseteq m^{r-1}J_2^r$, for $r \geq 1$.

We end this section with the follow up to the discussion about the invariants $\beta$ and $\omega$ for Fermat ideals in [Ngu21] and $J = J_1$ in the previous section.

\textbf{Proposition 5.4.} For all $n \geq 1$ and $m \geq 1$, $\beta(J_n^{(m)}) = m(n + 2)$ and $\omega(J_n^{(m)}) \geq m(n + 2)$.

\textbf{Proof.} The proof is the same to that of [Ngu21, Proposition 3.10] with notice that $J_n^{m}$ is generated in degree $m(n + 2)$ and each line in this configuration passed through exactly $n + 2$ points in the configuration. \[\square\]

\textbf{Remark 5.5.} It is also suggested by Macaulay2 [GS] that $\omega(J_n^{(m)}) = m(n + 2)$ as in the case of Fermat ideals. It is interesting to know if $\omega(I^{(m)}) = \beta(I^{(m)})$ hold for what radical ideal of points $I$ in general.

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