CONNECTIVE ALGEBRAIC K-THEORY

SHOUXIN DAI AND MARC LEVINE

Abstract. We examine the theory of connective algebraic K-theory, \( \mathcal{C}K \), defined by taking the \(-1\) connective cover of algebraic K-theory with respect to Voevodsky’s slice tower in the motivic stable homotopy category. We extend \( \mathcal{C}K \) to a bi-graded oriented duality theory \((\mathcal{C}K^\prime_\ast, \mathcal{C}K^\ast_\ast)\) in case the base scheme is the spectrum of a field \( k \) of characteristic zero. The homology theory \( \mathcal{C}K^\prime_\ast \) may be viewed as connective algebraic \( G \)-theory. We identify \( \mathcal{C}K^\prime_{2n, n}(X) \) for \( X \) a finite type \( k \)-scheme with the image of \( K_0(\mathcal{M}(n))(X) \) in \( K_0(\mathcal{M}(n+1))(X) \), where \( \mathcal{M}(n) \) is the abelian category of coherent sheaves on \( X \) with support in dimension at most \( n \); this agrees with the \((2n, n)\) part of the theory of connective algebraic K-theory defined by Cai. We also show that the classifying map from algebraic cobordism identifies \( \mathcal{C}K^\prime_{2n, n} \) with the universal oriented Borel-Morel homology theory \( \Omega^\mathcal{C}K_\ast := \Omega_{\ast} \otimes L\mathbb{Z}[\beta] \) having formal group law \( u + v - \beta uv \) with coefficient ring \( \mathbb{Z}[\beta] \). As an application, we show that every pure dimension \( d \) finite type \( k \)-scheme has a well-defined fundamental class \([X]_{\mathcal{C}K}\) in \( \Omega^\mathcal{C}K_d(X) \), and this fundamental class is functorial with respect to pull-back for lci morphisms. Furthermore, the fundamental class \([X]_{\mathcal{C}K}\) maps to the usual fundamental classes \([X]_{\text{Chow}}\), resp. \([X]_K\) under the natural maps

\[\Omega^\mathcal{C}K_\ast \to K_0(\beta, \beta^{-1}); \quad \Omega^\mathcal{C}K_\ast \to \text{CH}_\ast\]

given by inverting \( \beta \), resp. moding out by \( \beta \).

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Introduction

In topology, the theory of connective $K$-theory is represented by the $-1$-connected cover $ku$ of the topological $K$-theory spectrum $KU$. In the setting of motivic stable homotopy theory over a base-scheme $S$, Voevodsky [20] has constructed the algebraic analog of $KU$, namely the algebraic $K$-theory spectrum $K_S$, which represents Quillen’s algebraic $K$-theory on the category of smooth $S$-schemes, assuming that $S$ itself is a regular scheme (see also [21]), in that there are natural isomorphisms $K^{a,b}_S(X) \cong K_{2b-a}(X)$ for $X$ a smooth finite type $S$-scheme.

There are a number of possible notions of connectivity in the motivic stable homotopy category over $S$, $\mathcal{SH}(S)$, but one that has proved quite useful is given by using $T$-connectivity ($T = \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}$), in the sense of the tower of localizing subcategories

$$\cdots \subset \Sigma_{a+1}^{n+1} \mathcal{SH}^{\text{eff}}(S) \subset \Sigma_n \mathcal{SH}^{\text{eff}}(S) \subset \cdots \subset \mathcal{SH}(S),$$

where $\mathcal{SH}^{\text{eff}}(S)$ is the localizing subcategory of $\mathcal{SH}(S)$ generated by the $T$-suspension spectra of smooth $S$-schemes. The associated truncation functors give rise to Voevodsky’s slice tower

$$\cdots \to f_{n+1}E \to f_nE \to \cdots \to E$$

for each $E \in \mathcal{SH}(S)$, with $f_nE \to E$ the universal morphism from an object in $\Sigma_{T}^{n+1} \mathcal{SH}^{\text{eff}}(S)$ to $E$; one could call this the “$T$-$(n-1)$-connected cover” of $E$. It is thus natural to define connective algebraic $K$-theory as the bi-graded cohomology theory represented by $f_0K_S$.

In this paper we study connective algebraic $K$-theory, $\mathcal{CK}^{*,*}$ and its associated oriented homology theory, $\mathcal{CK}'^{*,*}$, this latter for $S = \text{Spec } k$, $k$ a characteristic zero field (see however remark 5.7). The oriented homology theory $\mathcal{CK}'^{*,*}$ is the connective analog of $G$-theory, that is, the $K$-theory of coherent sheaves rather than vector bundles. The canonical map $\mathcal{CK}^{*,*} \to K^{*,*}$ induces the map $\mathcal{CK}'^{*,*} \to G^{*,*}$ (where $G_{a,b}(X) := G_{a-2b}(X)$) and we elucidate here how the connective versions refine the non-connective ones.

Cai [1] has defined a bi-graded oriented Borel-Moore homology theory, which he calls connective algebraic $K$-theory, by using the Quillen-Gersten spectral sequence. Concretely, he defines the group $CK^{a,b}(X)$ as the image of $K_{2b-a}(\mathcal{M}^{a}(X))$ in $K_{2b-a}(\mathcal{M}^{a-1}(X))$, where $\mathcal{M}^{a}(X)$ is the category of coherent sheaves on $X$ supported in codimension at least $a$ (this is for $X$ smooth or at least equi-dimensional over a field $k$; in the general case, one indexes using dimension giving the theory $\mathcal{CK}'_{a,b}$). Cai verifies the properties of an oriented Borel-Moore homology theory for $CK^{*,*}$. It turns out this Cai’s theory does not in general agree with the one given by motivic homotopy theory, but it does agree in the portion corresponding to $K_0$ or $G_0$ (see theorem 5.7 and remark 5.8).

Besides comparison results of this type and other structural properties of connective $K$-theory, our main result is a comparison with algebraic cobordism. There is a canonical natural transformation

$$\Omega_*(X) \otimes_\mathbb{L} \mathbb{Z}[\beta] \to \mathcal{CK}^{*,*}_2(X)$$

which, in the case of characteristic zero, is an isomorphism for all quasi-projective $k$-schemes $X$ (see theorem 6.3). This allows us to use the natural fundamental classes in $G$-theory, namely, the class of the structure sheaf, to define a fundamental class
\([X] \in \Omega_d(X) \otimes_{\mathbb{L}} \mathbb{Z}[\beta]\) for \(X\) of pure dimension \(d\) over \(k\), which is functorial with respect to pull-back by l.c.i. morphisms. Note that such fundamental classes in \(\Omega_d(X)\) do not exist in general \([12]\).

We let \(\mathbf{Spc}\) and \(\mathbf{Spc}_*\) denote the categories of simplicial sets and pointed simplicial sets, respectively, with homotopy categories \(\mathcal{H}\) and \(\mathcal{H}_*\). \(\mathbf{Spt}\) the category of spectra (for the usual suspension operator \(\Sigma := (-) \wedge S^1\)) and \(\mathcal{SH}\) the stable homotopy category.

For a scheme \(S\), \(\mathbf{Sch}/S\) will denote the category of quasi-projective schemes over \(S\), \(\mathbf{Sm}/S\) the full subcategory of smooth quasi-projective schemes over \(S\). \(\mathbf{Spc}(S)\), and \(\mathbf{Spc}_*(S)\) the categories of pre sheaves on \(\mathbf{Sm}/S\) with values in \(\mathbf{Spc}, \mathbf{Spc}_*, \mathbf{Spt}_{S^1}(S)\) the category of \(S^1\)-spectra over \(S\), this being the category of presheaves of spectra on \(\mathbf{Sm}/S\). We let \(\mathbf{Spt}_T(S)\) denote the category of \(T\)-spectra in \(\mathbf{Spc}_*(S)\), with \(T := \mathbb{A}^1/\mathbb{A}^1 \setminus \{0\}\), and \(\mathbf{Spt}_T^\Sigma(S)\) the category of symmetric \(T\)-spectra. We have as well the category of \(S^1 - \mathbb{G}_m\) bi-spectra objects in \(\mathbf{Spc}_*(S)\), denoted \(\mathbf{Spt}_T^\Sigma(S)\).

The categories \(\mathbf{Spc}(S), \mathbf{Spc}_*(S), \mathbf{Spt}_{S^1}(S), \mathbf{Spt}_T(S), \mathbf{Spt}_{s,g}(S)\) and \(\mathbf{Spt}_T^\Sigma(S)\) all have so-called \textit{motivic} model structures (the original source for the unstable definition the composition \(\Sigma\)).

We denote by \(\mathbb{G}_m\) the pointed \(S\)-scheme \((\mathbb{A}^1_k - 0, 1)\). \(\mathbb{P}^1_k\) will denote the pointed \(S\)-scheme \((\mathbb{A}^1_k - 0, 1)\), with base-point \(1\).

\(\text{Ord}\) is the category of finite ordered sets, we let \([n] \in \text{Ord}\) denote the set \([0, \ldots, n]\) with the standard ordering. We let \(\mathbb{L}\) denote the Lazard ring, that is, the coefficient ring of the universal rank one commutative formal group law \(F_n \in \mathbb{L}[u, v]\). We let \(\mathbb{L}^1\) denote \(\mathbb{L}\) with the grading determined by \(\deg a_{ij} = 1 - i - j\) if \(F_n(u, v) = u + v + \sum_{i,j \geq 1} a_{ij} u^i v^j\) and let \(\mathbb{L}_n\) denote \(\mathbb{L}\) with the opposite grading \(\mathbb{L}_n := \mathbb{L}^{-n}\).

1. \(K\)-theory and connective \(K\)-theory

We work in the motivic stable homotopy category \(\mathcal{SH}(S)\) over a regular base-scheme \(S\); we will rather quickly pass to the case \(S = \text{Spec} k\) for a field \(k\).

We have the Tate-Postnikov tower

\[
\ldots \to f_{n+1} \to f_n \to \ldots \to \text{id}
\]

of endofunctors of \(\mathcal{SH}(S)\) associated to the inclusions of full localizing subcategories

\[
\cdots \subset \Sigma^m_{\mathcal{SH}}(S) \subset \sum_{n} \mathcal{SH}(S) \subset \cdots \subset \mathcal{SH}(S).
\]

with \(\Sigma^m_{\mathcal{SH}}(S)\) the full localizing subcategory of \(\mathcal{SH}(S)\) generated by the objects \(\Sigma^m_{X} X\), for \(X \in \mathbf{Sm}/S\), \(m \geq n\). Letting \(i_n : \Sigma^m_{\mathcal{SH}}(S) \to \mathcal{SH}(S)\) be the inclusion, \(i_n\) admits the right adjoint \(r_n : \mathcal{SH}(S) \to \Sigma^m_{\mathcal{SH}}(S)\) and \(f_n\), by definition the composition \(i_n \circ r_n\); the natural transformation \(f_n \to \text{id}\) is the counit of the adjunction. There are functors \(s_n, n \in \mathbb{Z}\) which give a distinguished triangle \(f_{n+1} \to f_n \to s_n \to f_{n+1}[1]\); one can show that these distinguished triangles are unique up to unique natural isomorphism. In addition, we have the canonical
isomorphism
\[ f_{n+1} \circ \Sigma_T \cong \Sigma_T \circ f_n. \]
For details, we refer the reader to \cite{3, 26, 27}.

Remark 1.1. By \cite{5} proposition 5.3, if \( \mathcal{E} \) is given a lifting to an \( E_\infty \) ring object in \( \text{Spt}_{\Sigma}^\infty(\mathcal{S}) \), then \( f_0 \mathcal{E} \) has a canonical lifting to an \( E_\infty \) ring object in \( \text{Spt}_{\Sigma}^\infty(\mathcal{S}) \). In particular, \( f_0 \mathcal{E} \) is itself a commutative unital monoid in \( \mathcal{S} \mathcal{H}(\mathcal{S}) \) and thus defines a bi-graded cohomology theory on \( \text{Sm}/S \) in the sense of Panin \cite{13} definition 2.0.1).

Following \cite{25} lemma 2.5, theorem 4.1] we have a commutative unital monoid object \( \mathcal{K}_S \) of \( \text{Spt}_{\Sigma}^\infty(\mathcal{S}) \) representing algebraic \( K \)-theory in \( \mathcal{S} \mathcal{H}(\mathcal{S}) \), in the sense that, for each \( X \in \text{Sm}/S \) and open subscheme \( U \), there is an isomorphism \( \text{Hom}(\Sigma^\infty_\mathcal{E} X/U, \mathcal{K}_S) \cong K(X, X \setminus U) \) in \( \mathcal{S} \mathcal{H}(\mathcal{S}) \), natural in pairs \((X, U)\). Here \( \text{Hom}(\cdot, \cdot) \) is the \( \text{Spt} \)-valued enriched \( \text{Hom} \) and \( K(X, X \setminus U) \) is the algebraic \( K \)-theory spectrum of the category of perfect complexes on \( X \) with support on \( X \setminus U \). Here we use our standing assumption that \( S \) is regular; for general \( S \), \( \mathcal{K}_S \) represents homotopy invariant \( K \)-theory \( KH \).

To fix the conventions, the image of \( \mathcal{K}_S \) in \( \mathcal{S} \mathcal{H}(\mathcal{S}) \) under the forgetful functor is isomorphic in \( \mathcal{S} \mathcal{H}(\mathcal{S}) \) to a \( T \)-spectrum of the form \((\mathcal{K}_S, \mathcal{K}_S, \ldots)\), with \( \mathcal{K}_S \) a commutative unital monoid in \( \mathcal{H}_\bullet(\mathcal{S}) \) representing \( K \)-theory on \( \text{Sm}/S \). The structure map \( \mathcal{K}_S \wedge T \to \mathcal{K}_S \) is isomorphic in \( \mathcal{H}_\bullet(\mathcal{S}) \) to the composition
\[ \mathcal{K}_S \wedge T \xrightarrow{id \wedge \sigma} \mathcal{K}_S \wedge \mathbb{P}^1 \xrightarrow{id \wedge \gamma} \mathcal{K}_S \wedge \mathcal{K}_S \xrightarrow{\mu} \mathcal{K}_S \]
where \( \mu \) is the multiplication \( \rho : T \to \mathbb{P}^1 \) is the standard isomorphism and \( \gamma : \mathbb{P}^1 \to \mathcal{K}_S \) represents the class \([O] - [O(-1)]\) in \( K_0(\mathbb{P}^1) \). This agrees with the convention in \cite{25}, and is the negative of the convention used in \cite{21}. We will often write \( \mathcal{K}_S \) for the image of \( \mathcal{K}_S \) in \( \mathcal{S} \mathcal{H}(\mathcal{S}) \), with the context determining the meaning.

Remark 1.2. 1. The fact that the shift operator leaves the \( T \)-spectrum \((\mathcal{K}_S, \mathcal{K}_S, \ldots)\) unchanged gives us the Bott periodicity isomorphism \( \Sigma_T \mathcal{K}_S \cong \mathcal{K}_S \) in \( \mathcal{S} \mathcal{H}(\mathcal{S}) \). We note that the map \( \gamma : \mathbb{P}^1 \to \mathcal{K}_S \) thus gives rise to an element \([\gamma] \in K^{2,1}(\mathbb{P}^1)\) which corresponds to the unit \( 1 \in K^{0,0}(\mathcal{S}) \) under the suspension isomorphism \( K^{2,1}(\mathbb{P}^1) \cong K^{3,0}(\mathcal{S}) \).

2. Let \( t_K \in K^{1,1}_S(\mathbb{G}_m) \) be the element corresponding to \( 1 \in K^{3,0}(\mathcal{S}) \) under the suspension isomorphism \( K^{1,1}_S(\mathbb{G}_m) \cong K^{0,0}_S(\mathcal{S}) \). Thus \( \Sigma_ST_K \cong [\gamma] \). Under the isomorphisms \( K^1_1(\mathbb{G}_m) \cong K^{1,1}(\mathbb{G}_m) \), \( K_0(\mathbb{P}^1) \cong K^{2,1}(\mathbb{P}^1)(\mathbb{P}^1) \), the isomorphism \( K^{1,1}_S(\mathbb{G}_m) \cong K^{2,1}_S(\mathbb{G}_m) \) is identified with the boundary map in the Mayer-Vietoris sequence
\[ \cdots \to K_1(\mathbb{A}^1) \oplus K_1(\mathbb{A}^1) \to K_1(\mathbb{A}^1 \setminus \{0\}) \xrightarrow{\partial} K_0(\mathbb{P}^1) \to \cdots \]
Let \([t] \in K_1(\mathbb{A}^1 \setminus \{0\})\) be the image of the canonical coordinate \( t \) on \( \mathbb{A}^1 \setminus \{0\} = \text{Spec} \mathcal{O}_S[t, t^{-1}] \). Then \( \partial[t] = 1 - [O(-1)] \) (at least up to a universal sign, see \cite{24} §7, lemma 5.16), and hence \( t_K = [t^{\pm 1}] \).

Definition 1.3. Let \( CK_S \) be the object \( f_0 \mathcal{K}_S \) of \( \mathcal{S} \mathcal{H}(\mathcal{S}) \), with canonical map \( \rho : CK_S \to \mathcal{K}_S \). Connective algebraic \( K \)-theory over \( S \) is the bi-graded cohomology theory on \( \text{Sm}/S \) represented by \( CK_S \). We call \( CK_S \) the connective algebraic \( K \)-theory \( T \)-spectrum (over \( S \)).
Remark 1.4. The fact that $\mathcal{C} K_S$ lifts to an $E_\infty$-ring object in $\text{Spt}^S_{\text{eff}}(S)$ is noted in [3 §6.2].

Lemma 1.5. Suppose $s_0K_S$ satisfies $(s_0K_S)^{a,0}(X) = 0$ for $a \neq 0$ and for all $X \in \text{Sm}/S$. For $X \in \text{Sm}/S$, consider the natural map $\rho^{a,1}(X): CK_S^{a,1}(X) \to K_S^{a,1}(X)$. Then $\rho^{2,1}$ is injective, $\rho^{3,1}$ is surjective and $\rho^{a,1}$ is an isomorphism for $a \neq 2, 3$.

Proof. By Bott periodicity, we have $\Sigma T f_0K_S = f_1\Sigma T K_S = f_1K_S$. Thus $CK_S^{a,1}(X) = (f_1K)^{a-2,0}(X)$. Since $\Sigma^\infty X_+$ is in $SH^{e/1}(S)$, we have $K^{e,0}(X) = (f_0K_S)^{e,0}(X)$, and we thus have a long exact sequence

$$\ldots \to (s_0K_S)^{a-3,0}(X) \to CK_S^{a,1}(X) \xrightarrow{\Delta_+} K^{a,1}(X) \to (s_0K_S)^{a-2,0}(X) \to \ldots$$

Our assumption $(s_0K_S)^{a,0}(X) = 0$ for $a \neq 0$ yields the desired result. \hfill $\square$

Remark 1.6. For $S = \text{Spec } k$, $k$ a perfect field, we know by [23, theorem 6.6] that $s_0K_S$ is isomorphic in $SH(k)$ to the $T$-spectrum representing motivic cohomology, $HZ$. We also know that $HZ^{e,0}(X) = H^n(X_{\text{Zar}}, \mathbb{Z})$ for $X \in \text{Sm}/k$, hence the assumption in lemma 1.5 is satisfied. In addition, the map $K_S^{2,1}(X) = K_0(X) \to HZ^{0,0}(X)$ is just the rank homomorphism, hence surjective. As $K_S^{e,1}(X) = K_{e-a}(X) = 0$ for $a > 2$, we see that $CK_S^{a,1}(X) = 0$ for $a > 2$ as well.

The functors $f_n$ and $s_n$ are compatible with pull-back by smooth morphisms [23, theorem 2.12, remark 2.13], and for $f: T \to S$ smooth, we have $f^*K_S \cong K_T$. Thus $(s_0K_S)^{a,0}(X) = H^n(X_{\text{Zar}}, \mathbb{Z})$ for $X \in \text{Sm}/S$ if $S$ is smooth over a perfect field. Thus, for $S$ smooth over a perfect field, the map $CK_S^{a,1}(X) \to K_S^{a,1}(X) = K_{2-a}(X)$ is an isomorphism for $a < 2$, an injection for $a = 2$ and $CK_S^{a,1}(X) = 0$ for $a > 2$.

Remark 1.7. For $n \geq 1$, let $\mathbb{P}^n$ denote the $S$-scheme $\mathbb{P}^n_S$, pointed by $(1 : 1 : 0 \ldots : 0)$ and let $\mathbb{P}^{\infty}$ denote the colimit (in $\text{Spec}_S(S)$) of the $\mathbb{P}^n_S$ under the linear embeddings $(x_0 : \ldots : x_n) \mapsto (x_0 : \ldots : x_n : 0)$. Recall from [20, definition 1.2] that an orientation on a commutative ring spectrum $E$ in $SH(S)$ is given by an element $c \in E^{2,1}(\mathbb{P}^1)$ such that the restriction $c|_{\mathbb{P}^n} \in E^{2,1}(\mathbb{P}^n) = E^{2,1}(T)$ is $\Sigma_T(1)$ (we use the opposite sign convention from loc. cit.).

Consider the sequence of elements $1 - [O_{\mathbb{P}^n}(-1)] \in K^{2,1}_{\mathbb{P}^n}(\mathbb{P}^n) = K_0(\mathbb{P}^n_S)$, which are clearly compatible with respect to restriction via the hyperplane embeddings $\mathbb{P}^n \to \mathbb{P}^{n+1}$. As the sequence $n \to K^{2,1}_{\mathbb{P}^n}(\mathbb{P}^n_S) = K_{1}(\mathbb{P}^n_S)$ satisfies the Mittag-Leffler condition (all the restriction maps are surjective), this defines a unique element $c_E \in K^{2,1}_{\mathbb{P}^{\infty}}(\mathbb{P}^{\infty})$, giving the standard orientation for $K_S$; the fact that $c_E$ is an orientation follows from remark 1.2.

Remark 1.8. The algebraic cobordism spectrum $MGL$ has been studied in [20]. $MGL$ is the $T$-spectrum $(MGL_0, MGL_1, \ldots)$ with $MGL_n$ the Thom space $Th(E_n)$, with $E_n \to BGL_n$ the universal $n$-plane bundle. $MGL_S$ is a commutative ring spectrum object (i.e. a commutative unital monoid) in $SH(S)$ with an orientation $c_{MGL} \in MGL^{2,1}(\mathbb{P}^{\infty})$ given by the diagram

$$\begin{array}{ccc}
E_1 & \longrightarrow & \text{Th}(E_1) = MGL_1 \\
\downarrow & & \\
\mathbb{P}^{\infty}
\end{array}$$

noting that $E_1 \to BGL_1 = \mathbb{P}^{\infty}$ is an isomorphism in $SH(S)$. The main result, theorem 1.1, of [20] is the universality of $(MGL, c_{MGL})$ in case $S = \text{Spec } k$: For
Thus the map \( k_{S}^{0,0}(\mathbb{P}^n_S) \to (\mathbb{P}^n_S) \) is the rank homomorphism \( K_0(\mathbb{P}^n_S) \to H^0(S, \mathbb{Z}) \) (up to sign). Then there is a unique element \( c_{CK} \in \mathcal{CK}^{2,1}(\mathbb{P}^\infty) \) mapping to \( c_K \) under the canonical map \( \mathcal{CK}^{2,1}(\mathbb{P}^\infty) \to \mathcal{K}^{2,1}(\mathbb{P}^\infty) \). Furthermore, \( c_{CK} \in \mathcal{CK}^{2,1}(\mathbb{P}^\infty) \) defines an orientation for \( CK_1 \).

**Proof.** By our assumptions on \( S \), we have the exact sequence

\[
0 \to \mathcal{CK}^{2,1}(\mathbb{P}^n_S) \to K_0(\mathbb{P}^n_S) \xrightarrow{\text{rank}} H^0(S, \mathbb{Z}) \to 0
\]

Thus \( \mathcal{CK}^{2,1}(\mathbb{P}^n_S) \to K^{2,1}(\mathbb{P}^n_S) \) is an isomorphism. Furthermore, the canonical map \( \mathcal{CK}^{2,1}(\mathbb{P}^n_S) \to \mathcal{K}^{2,1}(\mathbb{P}^n_S) = K_1(\mathbb{P}^n_S) \) is an isomorphism. By the projective bundle formula, the projective system of groups \( n \to K_1(\mathbb{P}^n_S) \) satisfies the Mittag-Leffler condition. Thus, passing to the limit over \( n \) gives us the isomorphism

\[
\mathcal{CK}^{2,1}(\mathbb{P}^\infty) \cong \mathcal{K}^{2,1}(\mathbb{P}^\infty)
\]

and the orientation \( c_K = (1 - [\mathcal{O}_{\mathbb{P}^n_S}(1)])_n \in \mathcal{K}^{2,1}(\mathbb{P}^\infty) \) gives us the element \( c_{CK} \in \mathcal{CK}^{2,1}(\mathbb{P}^\infty) \). The fact that \( \sum_{j \in P_1} (1_{\mathcal{K}_S}) = c_{K|P^1} \) implies \( \sum_{j \in P_1} (1_{\mathcal{K}_S}) = c_{CK|P^1} \), hence \( c_{CK} \) is an orientation.

**Remark 1.10.** By remark 1.9, if \( S \) is smooth over a perfect field, then the hypotheses of lemma 1.9 are fulfilled and hence \( CK_S \) has a unique orientation mapping to the standard orientation of \( K_S \).

2. **AN EXPLICIT MODEL**

We take \( S = \text{Spec } k \), \( k \) a perfect field. Let \( \Delta^0 \) be the cosimplicial scheme \( n \mapsto \Delta^n \), with

\[
\Delta^n := \text{Spec } k[t_0, \ldots, t_n]/\sum_i t_i - 1.
\]

The morphism \( \Delta(g) : \Delta^n \to \Delta^m \) associated to \( g : [n] \to [m] \) is given by

\[
g^*(t_i) = \sum_{j \in g^{-1}(i)} t_j
\]

where as usual the sum over the empty index set is 0. A face of \( \Delta^m \) is a closed subset \( F \) defined by equations of the form \( t_i = \ldots = t_r = 0 \).

We briefly recall the construction of the homotopy coneive tower associated to a presheaf of \( S^1 \)-spectra on \( \text{Sm}/k \). For \( X \in \text{Sm}/k \) and \( n, m \geq 0 \) integers, let \( S_X^0(m) \) denote the set of closed subsets \( W \) of \( X \times \Delta^m \) such that

\[
\text{codim}_{X \times F} W \cap X \times F \geq n
\]

for all faces \( F \) of \( \Delta^m \). For \( E \in \text{Spt}_{S^1}(k) \), we let

\[
E^{(n)}(X, m) := \text{hocolim}_{W \in S_X^0(m)} E^W(X \times \Delta^m).
\]

This gives us the simplicial spectrum \( m \mapsto E^{(n)}(X, m) \), and the associated total spectrum \( E^{(n)}(X) \). This construction is contravariantly functorial in \( X \) for equi-dimensional morphisms. Letting \( \text{Sm}/k \subset \text{Sm}/k \) denote the subcategory of \( \text{Sm}/k \) with the same objects, and with morphisms the smooth morphisms, sending \( X \) to
$E^{(n)}(X)$ defines a presheaf of spectra on $\text{Sm} // k$. It was shown in [9, theorem 4.1.1] that there are models $\tilde{E}^{(n)}(X)$ for $E^{(n)}(X)$ so that $X \to \tilde{E}^{(n)}(X)$ extends to a presheaf of spectra $\tilde{E}^{(n)}$ on $\text{Sm} // k$, isomorphic to $X \mapsto E^{(n)}(X)$ on $\text{Sm} // k$. The main result of [9] is:

**Theorem 2.1** ([9, theorem 7.1.1]). Let $E \in \text{Spt}_{S^1}(k)$ be quasi-fibrant. There is a natural isomorphism in $\text{SH}_{S^1}(k)$

$$f_n E \cong \tilde{E}^{(n)}.$$

Here *quasi-fibrant* means that a fibrant replacement $E \to E^{fib}$ gives a weak equivalence $E(X) \to E^{fib}(X)$ for all $X \in \text{Sm} // k$.

For a category $C$, we let $C^*$ be $C$ with a final object $*$ adjoined. Let $I$ be a finite category and $\mathcal{X} : I \to \text{Sm} // k^*$ an $I$-diagram in $\text{Sm} // k$, that is, a functor. For a presheaf of spectra $E$ on $\text{Sm} // k$, define $E(\mathcal{X})$ as

$$E(\mathcal{X}) := \text{holim}_{I^{op}} E \circ \mathcal{X}^{op},$$

where $E(*)$ is defined to be the initial object in $\text{Spt}$. For an $I$-diagram $\mathcal{X} : I \to \text{Sm} // k^*$, we have the $I$-diagram $\Sigma^\infty_+ \mathcal{X}_+ : I \to \text{Spt}_T(k)$ defined by $\Sigma^\infty_+ \mathcal{X}_+(i) := \Sigma^\infty_+ \mathcal{X}(i)+$ if $\mathcal{X}(i)$ is in $\text{Sm} // k$, and setting $\Sigma^\infty_+ *+$ equal to the final $T$-spectrum $(pt, pt, \ldots)$. We similarly define $\Sigma^\infty_+ \mathcal{X}_+ : I \to \text{Spt}_{S^1}(k)$.

**Example 2.2.** Let $X$ be in $\text{Sm} // k$ and $j : U \to X$ an open immersion with closed complement $Z$. Let $I$ be the category

$$0 \xrightarrow{a} 1 \xrightarrow{\ast} 1$$

and let $X/U : I \to \text{Sm} // k$ be the diagram $0 \to U$, $1 \to X$, $\ast \to \ast$, $a \to j$. Then for $E$ a presheaf of spectra on $\text{Sm} // k$, $E(X/U)$ is just the homotopy fiber of $j_* : E(X) \to E(U)$. Similarly, if $I$ is the one-point category $0$ and $\mathcal{X}$ is the functor $\mathcal{X} : 0 \to \text{Sm} // k$ with $\mathcal{X}(0) = X$, we have a canonical isomorphism $E(X) \cong E(\mathcal{X})$ in $\text{SH}$.

**Lemma 2.3.** Let $\mathcal{X} : I \to \text{Sm} // k^*$ be a finite diagram of smooth $k$-schemes (possibly with $\mathcal{X}(i) = \ast$ for some values $i \in I$) and take $E \in \text{SH}(k)$. Let $E \in \text{Spt}_{S^1}(k)$ be a fibrant model for $\Omega^\infty_+ E \in \text{SH}_{S^1}(k)$. Then $\text{Hom}(\text{holim}_I \Sigma^\infty_+ \mathcal{X}_+, f_n E) \in \text{SH}$ is represented by the spectrum $E^{(n)}(\mathcal{X})$.

**Proof.** The adjunction

$$\Sigma^\infty_+ : \text{SH}_{S^1}(k) \leftrightarrow \text{SH}(k) : \Omega^\infty_+$$

gives the isomorphism in $\text{SH}$

$$\text{Hom}(\text{holim}_I \Sigma^\infty_+ \mathcal{X}_+, f_n E) \cong \text{Hom}(\text{holim}_I \Sigma^\infty_+ \mathcal{X}_+, \Omega^\infty_+ f_n E).$$

It follows from [9, theorem 7.1.1, theorem 9.0.3] that we have the isomorphism in $\text{SH}_{S^1}(k)$

$$\Omega^\infty_+ f_n E \cong f_n \Omega^\infty_+ E = f_n E.$$
We thus have the isomorphisms in $\mathcal{SH}$

\[
\text{Hom}(\operatorname{hocolim}_I \Sigma^\infty_+ X_+, \Omega_T^\infty f_n E) \cong \text{Hom}(\operatorname{hocolim}_I \Sigma^\infty_+ X_+, f_n E) \\
\cong \operatorname{holim}_I \text{Hom}(\Sigma^\infty_+ X_+, f_n E) \\
\cong \operatorname{holim}_I f_n E \circ \mathcal{X}^{\text{op}}.
\]

We give the category of $I$-diagrams in $\mathbf{Spt}$ the projective model structure, with weak equivalences the pointwise ones, and let $\text{Ho}(\mathbf{Spt}_I)$ denote the homotopy category. By [9, theorem 7.1.1], we have the isomorphism in $\text{Ho}(\mathbf{Spt}_I)$

\[
f_n E \circ \mathcal{X}^{\text{op}} \cong E^{(n)} \circ \mathcal{X}^{\text{op}},
\]

giving the isomorphism in $\mathcal{SH}$

\[
\text{Hom}(\operatorname{hocolim}_I \Sigma^\infty_+ X_+, \Omega_T^\infty f_n E) \cong \operatorname{holim}_I E^{(n)} \circ \mathcal{X}^{\text{op}} = E^{(n)}(\mathcal{X}).
\]

We let $K \in \mathbf{Spt}_{S^1}(k)$ be the presheaf of spectra given by sending $X$ to the Quillen-Waldhausen spectrum $K(X)$ representing the algebraic $K$-theory of $X$. We note that $K$ is a quasi-fibrant object of $\mathbf{Spt}_{S^1}(k)$.

**Proposition 2.4.** Let $\mathcal{X} : I \rightarrow \mathbf{Sm}//k^*$ be a finite diagram of smooth $k$-schemes as in lemma 2.3 Take $X$ in $\mathbf{Sm}//k$. There is a canonical isomorphism

\[
\mathcal{C}K^{p,q}(\mathcal{X}) \cong \pi_{2q-p} K^{(q)}(X), p, q \in \mathbb{Z},
\]

natural in $\mathcal{X}$.

**Proof.** We make $X \mapsto \pi_{2q-p} K^{(q)}(X)$ a functor in $X$ by using the functorial model $K^{(q)}$ for $K^{(q)}$ and the canonical isomorphism $\pi_* K^{(q)}(X) \cong \pi_* K^{(q)}(X)$.

Using lemma 2.3 and a variety of adjunctions and definitions, we have the sequence of isomorphisms

\[
\mathcal{C}K^{p,q}(\mathcal{X}) = \text{Hom}_{\mathcal{SH}(k)}(\operatorname{hocolim}_I \Sigma^\infty_+ \mathcal{X}_+, \Sigma^p q \mathcal{C}K) \\
= \text{Hom}_{\mathcal{SH}(k)}(\operatorname{hocolim}_I \Sigma^\infty_+ \mathcal{X}_+, \Sigma^p q f_0 \mathcal{K}) \\
= \text{Hom}_{\mathcal{SH}(k)}(\operatorname{hocolim}_I \Sigma^\infty_+ \mathcal{X}_+, \Sigma^p q \Sigma^2 q f_0 \mathcal{K}) \\
= \text{Hom}_{\mathcal{SH}(k)}(\operatorname{hocolim}_I \Sigma^\infty_+ \mathcal{X}_+, \Sigma^p q \Sigma^2 q f_0 \mathcal{K}) \\
= \text{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_+ \Sigma^2 q f_0 \mathcal{K}) \\
= \text{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_+ \Sigma^2 q f_0 \mathcal{K}) \\
= \text{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_+ \Sigma^2 q f_0 \mathcal{K}) \\
= \text{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_+ \Sigma^2 q f_0 \mathcal{K}) \\
= \text{Hom}_{\mathcal{SH}(k)}(\Sigma^\infty_+ \Sigma^2 q f_0 \mathcal{K}) \\
= \pi_{2q-p} K^{(q)}(X).
\]

**Corollary 2.5.** Let $X$ be smooth over $k$. Then $\mathcal{C}K^{2n,n}(X)$ is equal to $K_0(X)$ for $n \leq 0$. For $n > 0$, $\mathcal{C}K^{2n,n}(X)$ is determined by the exact sequence

\[
K_0^{(n)}(X, 1) \xrightarrow{\delta_1 - \delta_2} K_0(\mathcal{M}_X^{(n)}) \rightarrow \mathcal{C}K^{2n,n}(X) \rightarrow 0.
\]

□
In particular,
\[ \mathcal{K}^{2n,n}(k) \to \mathcal{K}^{2n,n}(k) = K_0(k) = \mathbb{Z} \]
is an isomorphism for \( n \leq 0 \); for \( n > 0, \mathcal{K}^{2n,n}(k) = 0 \).

**Proof.** Viewing \( X \) as the one-point diagram \( \ast \to X \), it follows from proposition\(^2\) that we have a canonical isomorphism \( \mathcal{K}^{2n,n}(X) \cong \pi_0(K^{(n)}(X)) \). For \( n \leq 0 \), \( K^{(n)}(X) \) is weakly equivalent to \( K(X) \). For \( n > 0 \), \( K^{(n)}(X, m) \) is -1 connected, since \( K^W(Y) \) is so for all closed \( W \subset Y \), \( Y \in \text{Sm}/k \). This gives the presentation for \( \mathcal{K}^{2n,n}(X) = \pi_0(K^{(n)}(X)) \). \( \square \)

For \( X \in \text{Sm}/k \) with generic point \( \eta \) we define \( \mathcal{K}^{a,b}(\eta) \) as the stalk at \( \eta \) of the spectrum of the simplicial spectrum corresponding to the unit \( 1 \)

\[ \eta \]

by proposition \ref{prop2.4}.

**Corollary 2.6.** Let \( \eta \) be a generic point of some \( X \in \text{Sch}/k \). Then \( \mathcal{K}^{p,q}(\eta) = 0 \) for \( p > q \).

**Proof.** By proposition \ref{prop2.4} \( \mathcal{K}^{p,q}(\eta) \cong \pi_{2q-p}K^{(q)}(\eta) \). But \( K^{(q)}(\eta) \) is the total spectrum of the simplicial spectrum \( m \mapsto K^{(q)}(\eta, m) \), with \( K^{(q)}(\eta, m) := \text{hocolim}_{W \in S^\Delta_k(m)} K^W(\eta \times \Delta^m) \).

As \( \eta \times \Delta^m \cong \Delta^m_{(\eta)} \), it follows that each closed subset \( W \) of \( \eta \times \Delta^m \) has codim \( W \leq m \), and hence \( K^{(q)}(\eta, m) \) is the 0-spectrum if \( m < q \). Furthermore, the \( K \)-theory presheaf is a presheaf of -1 connected spectra and for each open immersion \( j : U \to V \) in \( \text{Sm}/k \), the restriction map \( j^*K_0(V) \to K_0(U) \) is surjective. Thus \( K^W(\eta \times \Delta^m) \) is a -1 connected spectrum for each \( m \). Using the strongly convergent spectral sequence \( E^1_{a,b} = \pi_bK^{(q)}(\eta, a) \Rightarrow \pi_{a+b}K^{(q)}(\eta) \), we see that \( \pi_nK^{(q)}(\eta) = 0 \) for \( n < q \), hence \( \mathcal{K}^{p,q}(\eta) \cong \pi_{2q-p}K^{(q)}(\eta) = 0 \) for \( p > q \). \( \square \)

We conclude this section with a discussion of the functor \( \mathcal{K}^{2q-1,q} \). Let \( E \in \text{SH}(S) \) represent a bi-graded cohomology theory. Let \( t_E \in E^{1,1}(\mathbb{G}_m) \) be the element corresponding to the unit \( 1 \in E^{0,0}(S) \) under the suspension isomorphism. By functoriality, \( t_E \) gives a map of pointed sets

\[ t_E(X) : \mathcal{O}^*_X(X) \to E^{1,1}(X) ; \]

if \( E \) admits an orientation \( c_E \in E^{2,1}(\mathbb{P}^\infty) \) (which we will from now on assume), then \( t_E(X) \) is a group homomorphism\(^1\) Using the \( E^{*,*}(S) \)-module structure on \( E^{*,*}(X) \), \( t_E(X) \) extends to a map of \( E^{*,*}(S) \)-modules

\[ t_E(X) : E^{2*,*}(S) \otimes \mathcal{O}^*_X(X) \to E^{2*,1,*+1}(X) . \]

\(^1\)Letting \( S \) denote the sphere spectrum and writing \( [a] := t_0(a), \) this follows from the identity \( [ab] = [a] + [b] + H[a][b] \) \((H : S \otimes \mathbb{G}_m \to S \) the stable Hopf map\) and the fact that \( H \) goes to zero in any oriented theory \( E \). Both these facts are proven by Morel in [17, §0].
Lemma 2.7. Suppose $S = \text{Spec } k$. Let $\eta$ be a generic point of some $X \in \text{Sm}/k$.

1. Take $E = K$. Then $K^{2\ast,\ast}(k) \cong \mathbb{Z}[\beta, \beta^{-1}]$, $\deg \beta = -1$ and $t_K(\eta) : K^{2\ast,\ast}(k) \otimes \mathbb{Z} \rightarrow K^{2\ast+1,\ast+1}(\eta)$ is an isomorphism.

2. Take $E = C K$. Then $C K^{2\ast,\ast}(k) \cong \mathbb{Z}[\beta]$, $\deg \beta = -1$ and $t_{C K}(\eta) : C K^{2\ast,\ast}(k) \otimes \mathbb{Z} \rightarrow C K^{2\ast+1,\ast+1}(\eta)$ is an isomorphism.

3. Take $E = \text{MGL}$ and suppose $k$ has characteristic zero. Then $\text{MGL}^{2\ast,\ast}(k)$ is canonically isomorphic to the Lazard ring $\mathbb{L}^\ast$, $t_{\text{MGL}}(\eta) : k(\eta)^\times \rightarrow \text{MGL}^{1,1}(\eta)$ is an isomorphism and $t_{\text{MGL}}(\eta) : \text{MGL}^{2\ast,\ast}(k) \otimes \mathbb{Z} k(\eta)^\times \rightarrow \text{MGL}^{2\ast+1,\ast+1}(\eta)$ is surjective.

Proof. We have already seen in remark 1.2 that under the isomorphism $K^{1,1}(\mathbb{G}_m) \cong K_1(\mathbb{G}_m)$, $t_K$ goes to the class of the canonical unit $t$. By functoriality, the map $t_K : k(\eta)^\times \rightarrow K_1(\eta) = K^{1,1}(\eta)$ is the usual isomorphism $k(\eta)^\times \cong K_1(\eta)$ sending $x \in k(\eta)^\times$ to the class of the automorphism $x^\epsilon : k(\eta) \rightarrow k(\eta)$, where $\epsilon = \pm 1$ is universal choice of sign (independent of $k$ or $\eta$).

The isomorphism $K^{2\ast,\ast}(k) \cong \mathbb{Z}[\beta, \beta^{-1}]$ follows from the Bott periodicity isomorphism $K^{2n,n}(k) \cong K_0(k) \cong \mathbb{Z}$. Since $\beta$ is invertible, the fact that $t_K : k(\eta)^\times \rightarrow K_1(\eta)$ is an isomorphism implies that $t_K(\eta) : K^{2\ast,\ast}(k) \otimes \mathbb{Z} k(\eta)^\times \rightarrow K^{2\ast+1,\ast+1}(\eta)$ is an isomorphism.

For (2), it follows from the universal property of $f_0 \rightarrow \text{id}$ that $C K^{a,b}(X) \rightarrow C K^{a,b}(X)$ is an isomorphism for all $b \leq 0$, $a \in \mathbb{Z}$, $X \in \text{Sm}/k$. In particular, $C K^{2b-1,1,b}(X) = K_1(X)$ for $b \leq 0$. For $b > 1$, $C K^{2b-1,1,b}(\eta) = 0$ by corollary 2.6 and $C K^{1,1}(X) \rightarrow C K^{1,1}(X)$ is an isomorphism by lemma 1.5. Similarly, the map $C K^{2n,n}(X) \rightarrow C K^{2n,n}(X) = \mathbb{Z} \beta^{-n}$ is an isomorphism for $n \leq 0$ and by corollary 2.6 $C K^{2n,n}(\eta) = 0$ for $n > 0$. Thus the map $C K^{2\ast,\ast}(k) \rightarrow C K^{2\ast,\ast}(k)$ identifies $C K^{2\ast,\ast}(k)$ with the subring $\mathbb{Z}[\beta]$ of $K^{2\ast,\ast}(k) = \mathbb{Z}[\beta, \beta^{-1}]$. Putting this all together, (1) implies (2).

For (3), the orientation $\gamma_K$ gives the canonical morphism of oriented ring $T$-spectra $\gamma_K : \text{MGL} \rightarrow C K[20, \text{thm}1.1]$, inducing the commutative diagram

$$
\begin{array}{ccc}
\mathcal{O}_X(X)^\times & \xrightarrow{t_{\text{MGL}}} & \text{MGL}^{1,1}(X) \\
\downarrow_{t_K} & & \downarrow_{\gamma_K} \\
K^{1,1}(X) & & 
\end{array}
$$

As $t_K : k(\eta)^\times \rightarrow K_1(\eta)$ is an isomorphism, it follows that $t_{\text{MGL}} : k(\eta)^\times \rightarrow \text{MGL}^{1,1}(\eta)$ is injective. The isomorphism $\mathbb{L}^* \rightarrow \text{MGL}^{2\ast,\ast}(k)$ and the surjectivity of $t_{\text{MGL}} : \mathbb{L}^* \otimes k(\eta)^\times \rightarrow \text{MGL}^{2\ast+1,\ast+1}(k(\eta))$ follow from the Hopkins-Morel spectral sequence [10, 14].

3. Oriented duality theories

Recall from [10, §1] the category $\mathbf{SP}/k$ of smooth pairs over $k$, with objects $(M, X), M \in \text{Sm}/k$ and $X \subseteq M$ a closed subset; a morphism $f : (M, X) \rightarrow (N, Y)$ is a morphism $f : M \rightarrow N$ in $\text{Sm}/k$ such that $f^{-1}(Y) \subseteq X$. We let $\mathbf{Sch}/k$ denote the category of quasi-projective $k$-schemes; for a full subcategory $V$ of $\mathbf{Sch}/k$, let $\mathcal{V}'$ be the subcategory of $\mathcal{V}$ with the same objects and morphisms the projective morphisms.
Building on work of Mocanu [16] and Panin [19], we have defined in [10, definition 3.1] the notion of a bi-graded oriented duality theory \((H, A)\) on \(\text{Sch}/k\). Here \(A\) is a bi-graded oriented cohomology theory on \(\text{SP}/k\), \((M, X) \mapsto A_X^{\ast, \ast}(M)\), and \(H\) is a functor from \(\text{Sch}'/k\) to bi-graded abelian groups. The oriented cohomology theory \(A\) satisfies the axioms listed in [10, definitions 1.2, 1.5]. In particular, \((M, X) \mapsto A_X^{\ast, \ast}(M)\) admits a long exact sequence
\[
\ldots \to A_X^{\ast, \ast}(M) \to A_X^{\ast, \ast}(M) \to A_X^{\ast, \ast}(M) \to \ldots
\]
where for instance \(A_X^{\ast, \ast}(M) := A_X^{\ast, \ast}(M)\) and the boundary map \(\partial\) is part of the data. In addition, there is an excision property and a homotopy invariance property. The ring structure is given by external products and pull-back by the diagonal. The orientation is given by a collection of isomorphisms \(\text{Th}_X: A_X(M) \to A_X(E)\), for \((M, E) \in \text{SP}/k\) and \(E \to M\) a vector bundle, satisfying the axioms of [19, def. 3.1.1]. We extend some of the results of [19] in [10, theorem 1.12, corollary 1.13] to show that the data of an orientation is equivalent to giving well-behaved push-forward maps \(f_* : A_X(M) \to A_Y(N)\) for \((M, X), (N, Y) \in \text{SP}/k\), with the meaning of “well-behaved” detailed in [10, §1].

The homology theory \(H\) comes with restriction maps \(j^*: H_{\ast, \ast}(X) \to H_{\ast, \ast}(U)\) for each open immersion \(j: U \to X\) in \(\text{Sch}/k\), external products \(\times: H_{\ast, \ast}(X) \otimes H_{\ast, \ast}(Y) \to H_{\ast, \ast}(X \times Y)\), boundary maps \(\partial_{X, Y}: H_{\ast, \ast}(X \times Y) \to H_{\ast-1, \ast}(Y)\) for each closed subset \(Y \subset X\), isomorphisms \(\alpha_{M, X}: H_{\ast, \ast}(X) \to A_X^{2m-\ast, m-\ast}(M)\) for each \((M, X) \in \text{SP}/k\), \(m = \dim_k M\), and finally cap product maps
\[
f^*(\cdot) \cap: A_X^{\ast, \ast, \ast}(M) \otimes H_{\ast, \ast}(Y) \to H_{\ast-\ast, \ast-\ast, \ast}(Y \cap f^{-1}(X))
\]
for \((M, X) \in \text{SP}/k\), \(f: Y \to X\) a morphism in \(\text{Sch}/k\). These satisfy a number of axioms and compatibilities (see [10] §3 for details), which essentially say that a structure for \(A_X^{\ast, \ast}(M)\) is compatible with the corresponding structure for \(H_{\ast, \ast}(X)\) via the isomorphism \(\alpha_{M, X}\). Roughly speaking, this is saying that a particular structure for \(A_X^{\ast, \ast}(M)\) depends only on \(X\) and not the choice of embedding \(X \hookrightarrow M\).

**Remark 3.1.** Let \(L \to Y\) be a line bundle on some \(Y \in \text{Sm}/k\) with 0-section \(0: Y \to L\). For an oriented cohomology theory \(A\) one has the element
\[
c_1^A(L) := 0^*(0_{\ast}(1_1^A)),
\]
where \(1_1^A \in A^0(Y)\) is the unit element. As pointed out in [19, corollary 3.3.8], or as noted in [10, remark 1.17], for line bundles \(L, M\) on some \(Y \in \text{Sm}/k\), the elements \(c_1(L), c_1(M) \in A^1(Y)\) are nilpotent, and commute with one another, hence for each power series \(F(u, v) \in A^\ast(k)[[u, v]]\) the evaluation \(F(c_1(L), c_1(M))\) gives a well-defined element of \(A^\ast(Y)\). In addition, the cohomology theory \(A\) has a unique associated formal group law \(F_A(u, v) \in A^\ast(k)[[u, v]]\) with
\[
F_A(c_1(L), c_1(M)) = c_1(L \otimes M)
\]
for all line bundles \(L, M\) on \(Y \in \text{Sm}/k\).

For an \(X \in \text{Sch}/k\) with line bundle \(L \to X\), the quasi-projectivity of \(X\) implies that \(X\) admits a closed immersion \(i: X \to M\) for some \(M \in \text{Sm}/k\) such that \(L\) extends to a line bundle \(L\) over \(M\). One can then define
\[
\widehat{c}_1(L): H_0(X) \to H_{-1}(X)
\]
via the product
\[
(-) \cdot c_1(L): A_X^\ast(M) \to A_X^{\ast+1}(M)
\]
and the isomorphisms $H_*(X) \cong A_X^{d_0-s}(M)$. One shows that this is independent of the choice of $(M, L)$, giving the well-defined operator $c_1(L)$.

The main example of oriented duality theory $(H, A)$ is given by an oriented $T$-ring spectrum $E$ in $SH(k)$, assuming $k$ is a field admitting resolution of singularities (e.g., characteristic zero), defined by taking

$$E_X^{a,b}(M) := \text{Hom}_{SH(k)}(\Sigma_2^n(M/M \setminus X), \Sigma^{a,b}E),$$

i.e., the usual bi-graded cohomology with supports. For each $X \in \text{Sch}/k$, choose a closed immersion of $X$ into a smooth $M$ and set $E'_X^{a,b}(X) := E_X^{2m-a,m-b}(M)$, where $m = \dim_k M$. The fact that $(M, X) \mapsto E_X^{*,*}(M)$ defines an oriented bi-graded ring cohomology theory is proved just as in the case of $E = \text{MGL}$, which was discussed in [10] §4; the main point is Panin’s theorem [19, theorem 3.7.4], which says that an orientation for $E$ (in the sense of remark 1.7) defines an orientation in the sense of ring cohomology theories for the bi-graded $E$-cohomology with supports.

The fact that the formula given above for the homology theory $E_\ast \ast$ is well-defined and extends to make $(E'_X \ast(-), E_X^{*,*}(-))$ a bi-graded oriented duality theory is [10] theorem 3.4. The essential point is to show that the cohomology with support $E_X^{d_0-s,d_0-s}(M)$, for $X \mapsto M$ a closed immersion of some $X$ in a smooth $M$ of dimension $d$, depends (up to canonical isomorphism) only on $X$, and similarly, given a projective morphism $f : Y \to X$ in $\text{Sch}/k$, there are smooth pairs $(M, X), (N, Y)$, an extension of $F$ to a morphism $F : N \to M$ and the map $F_* : E_Y^{2dN-s,dN-s}(N) \to E_X^{2dM-s,dM-s}(M)$ is independent (via the canonical isomorphisms $E_\ast_\ast(Y) \cong E_Y^{2dN-s,dN-s}(N), E_\ast_\ast(X) \cong E_X^{2dM-s,dM-s}(M)$) of the choices. The other structures for $E_\ast \ast(-)$ are defined similarly via the $E$-cohomology with supports, and one has the corresponding independence of any choices.

It follows directly from the construction of $E'$ that the assignment $(E, c_E) \mapsto (E', E)$ is functorial in the oriented cohomology theory $(E, c_E)$. In particular, let $ch : \text{MGL} \to E$ be a morphism of oriented cohomology theories, that is, $ch$ is a morphism in $SH(k)$, compatible with the ring-object structures of $\text{MGL}$ and $E$, and compatible with $1$st Chern classes. Then we have an extension of $ch$ to a natural transformation of oriented duality theories

$$(ch', ch) : (\text{MGL}', \text{MGL}) \to (E', E)$$

Remark 3.2. As shown in lemma 2.7, the coefficient rings for $K$ and $LK$ are $K^{2*}(k) = \mathbb{Z}[\beta, \beta^{-1}]$ and $LK^{2*}(k) = \mathbb{Z}[\beta]$, respectively, with $\beta$ having degree $-1$. For $K$, the orientation $c_K$ restricted to $\mathbb{P}^n$ is given by the class of $1 - [O(-1)] \in K_0(\mathbb{P}^n) \cong K^{2,1}(\mathbb{P}^n)$. It follows (by functoriality and Jouanolou’s trick) that for a line bundle $L$ on some $X \in \text{Sm}/k$, the $1$st Chern class is given by $c_K^1(L) = \beta^{-1}(1 - [L^{-1}])$ (where we consider $1, [L^{-1}] \in K_0(\mathbb{P}^1) = K_0(X)$). A direct calculation gives the formal group law for $(K_2^{2*}, K^{2*})$ as $(F_K(u, v) = u + v - \beta \cdot uv, Z[\beta, \beta^{-1}])$. Since the orientation for $K$ lifts to that of $LK$, it follows that the formal group law for $(LK_2^{2*},LK^{2*})$ is $(u + v - \beta \cdot uv, \mathbb{Z}[\beta])$.

4. Algebraic cobordism and oriented duality theories

We recall the theory of algebraic cobordism $X \mapsto 
\Omega_*(X), X \in \text{Sch}/k$. For each $X \in \text{Sch}/k, \Omega_n(X)$ is an abelian group with generators $(f : Y \to X), Y \in \text{Sm}/k$ irreducible of dimension $n$ over $k$ and $f : Y \to X$ a projective morphism. $\Omega_*$ is the
universal oriented Borel-Moore homology theory on $\text{Sch}/k$; this consists of the data of a functor from $\text{Sch}/k'$ to graded abelian groups, external products, first Chern class operators $c_1(L) : \Omega_*(X) \to \Omega_{*-1}(X)$ for $L \to X$ a line bundle, and pull-back maps $g^* : \Omega_* (X) \to \Omega_{*-d}(Y)$ for each l.c.i. morphism $g : Y \to X$ of relative dimension $d$. These of course satisfy a number of compatibilities and additional axioms.

For an oriented duality theory $(H, A)$ on $\text{Sch}/k$ and $Y$ in $\text{Sm}/k$ of dimension $d$ over $k$, the fundamental class $[Y]_{H, A} \in H_d(Y)$ is the image of the unit $1_Y \in A^0(Y)$ under the inverse of the isomorphism $\alpha_Y : H_d(Y) \to A^0(Y)$. For an oriented Borel-Moore homology theory $B$ on $\text{Sch}/k$, we similarly have the fundamental class $[Y]_B \in B_d(Y)$ defined by $[Y]_B := p^*(1)$, where $1 \in B_0(\text{Spec } k)$ is the unit and $p : Y \to \text{Spec } k$ the structure morphism.

We recall the following result from [10]:

**Proposition 4.1** ([10] propositions 4.2, 4.4, 4.5). Let $k$ be a field admitting resolution of singularities and let $(H, A)$ be a $\mathbb{Z}$-graded oriented duality theory on $\text{Sch}/k$.

1. There is a unique natural transformation $\vartheta_H : \Omega_\cdot \to H_\cdot$ of functors $\text{Sch}/k' \to \text{GrAb}$, such that $\vartheta_H(Y)$ is compatible with fundamental classes for $Y$ in $\text{Sm}/k$. In addition, $\vartheta_H$ is compatible with pull-back maps for open immersions in $\text{Sch}/k$, with 1st Chern class operators, with external products and with cap products.

2. For $Y \in \text{Sm}/k$, the map $\vartheta^A(Y) : \Omega^*(Y) \to A^*(Y)$ induced by $\vartheta_H$, the identity $\Omega^*(Y) = \Omega_{\dim Y - *}(Y)$ and the isomorphism $\alpha_Y : H_{\dim Y - *}(Y) \to A^*(Y)$ is a ring homomorphism and is compatible with pull-back maps for arbitrary morphisms in $\text{Sm}/k$. Finally, one has

$$\vartheta^A(Y)(c^\Omega_1(L)) = c^A_1(L)$$

for each line bundle $L \to Y$.

**Remark 4.2.** We have already noted that one has a formal group law $F_A(u, v) \in A^*(k)[[u, v]]$ associated to the oriented cohomology theory $A$. Similarly, for each oriented Borel-Moore homology theory $B$ on $\text{Sch}/k$, there is an associated formal group law $F_B(u, v) \in B_*(k)[[u, v]]$, characterised by the identity $F_B(c_1(L), c_1(M)) = c_1(L \otimes M)$ for each pair of line bundles $L, M$ on some $Y \in \text{Sm}/k$ (this follows from [15] corollary 4.1.8, proposition 5.2.1, proposition 5.2.6]). Letting $\varphi_A : L^* \to A^*(k)$, $\varphi_B : L^* \to B^*(k)$ denote the classifying maps associated to $F_A$, $F_B$, respectively, suppose that $A$ extends to an oriented duality theory $(H, A)$. Then

$$\vartheta^A(F_\Omega) = F_A.$$  

Indeed, $F_A$ is characterised by identity $F_A(c^\Omega_1(L), c^\Omega_1(M)) = c^A_1(L \otimes M)$ for each pair of line bundles $L, M$ on some $Y \in \text{Sm}/k$, and since $\vartheta^A(c^\Omega_1(N)) = c^A_1(N)$ for each line bundle $N \to Z, Z \in \text{Sm}/k$, the fact that $F_\Omega(c^\Omega_1(L), c^\Omega_1(M)) = c^\Omega_1(L \otimes M)$ combined with proposition [4.42] yields the identity (4.1).

Via the universal property of the Lazard ring, the relation (4.1) is equivalent to the identity

$$\vartheta^A \circ \varphi_\Omega = \varphi_A.$$  

Finally, we recall that the classifying map $\varphi_\Omega : \mathbb{L}_\cdot \to \Omega_\cdot(k)$ is an isomorphism [15 theorem 1.2.7].
Corollary 4.3. Let \( (\mathcal{E}, c_{\mathcal{E}}) \) be a pair consisting of a commutative unital monoid object \( \mathcal{E} \in SH(k) \) with orientation \( c_{\mathcal{E}} \), and let \( (\mathcal{E}_{*,*}^*, \mathcal{E}^{*,*}) \) be the corresponding bi-graded oriented duality theory. There is a unique natural transformation
\[
\vartheta_{(\mathcal{E}, c_{\mathcal{E}})} : \Omega_* \to \mathcal{E}_{2*,*}^{*}
\]
of functors \( \text{Sch}/k' \to \text{GrAb} \), such that \( \vartheta_{(\mathcal{E}, c_{\mathcal{E}})}(Y) \) is compatible with fundamental classes for \( Y \in \text{Sm}/k \). In addition, \( \vartheta_{(\mathcal{E}, c_{\mathcal{E}})} \) is compatible with pull-back maps for open immersions in \( \text{Sch}/k \), 1st Chern class operators, external products and cap products. For \( Y \in \text{Sm}/k \), the map \( \vartheta^{2*}(Y) : \Omega^{2*}(Y) \to \mathcal{E}^{2*,*}(Y) \) induced by \( \vartheta_{(\mathcal{E}, c_{\mathcal{E}})} \) is a ring homomorphism and is compatible with pull-back maps for arbitrary morphisms in \( \text{Sm}/k \), and satisfies
\[
\vartheta_{(\mathcal{E}, c_{\mathcal{E}})}(Y)(c_1^0(L)) = c_1^0(L)
\]
for each line bundle \( L \to Y \).

Remark 4.4. By [15 lemma 2.5.11], \( \Omega_*(X) \) is generated as an abelian group by the cobordism cycles \( f : Y \to X \), \( Y \in \text{Sm}/k \) irreducible, \( f : Y \to X \) a projective morphism. Furthermore, the identity \( (f : Y \to X) = f_*(\langle Y \rangle_\Omega) \) holds in \( \Omega_{\dim Y}(X) \). Thus \( \vartheta_{(\mathcal{E}, c_{\mathcal{E}})} \) is characterized by the formula
\[
\vartheta_{(\mathcal{E}, c_{\mathcal{E}})}(f : Y \to X) := f^{2*}(\langle Y \rangle_{\mathcal{E}^*,*}).
\]

We may apply corollary 4.3 in the universal case: \( \mathcal{E} = \text{MGL} \) with its canonical orientation. This gives us the natural transformation
\[
\vartheta_{\text{MGL}} : \Omega_* \to \text{MGL}_{2*,*}^{*}
\]
(4.3)
Theorem 4.5 ([11 theorem 3.1]). Assume that \( k \) is a field of characteristic zero. Then the natural transformation (4.3) is an isomorphism.

Remark 4.6. This result relies on the Hopkins-Morel spectral sequence, see [6,7].

In the course of the proof, we proved another result which we will be using here. Let \( X \) be in \( \text{Sch}/k \) and let \( d = d_X := \max \dim X_\eta \), as \( X' \) runs over the irreducible components of \( X \). We define \( \text{MGL}_{2*,*}^{2*}(X) \) by
\[
\text{MGL}_{2*,*}^{2*}(X) := \lim_{W \to W'} \text{MGL}_{2*,*}^{2*}(W)
\]
as \( W \) runs over all (reduced) closed subschemes of \( X \) which contain no dimension \( d \) generic point of \( X \); \( \Omega_{2*}(X) \) is defined similarly. The natural transformation \( \vartheta_{\text{MGL}} \) gives rise to the commutative diagram
\[
\begin{array}{ccc}
\text{MGL}_{2*,*}^{2*}(X) & \overset{i_*}{\longrightarrow} & \text{MGL}_{2*,*}^{2*}(X) \\
\vartheta_{\text{MGL}} & \overset{j_*}{\longrightarrow} & \oplus_{\eta \in X(d)} \Omega_*(k(\eta)) \\
\end{array}
\]
with exact rows and with all vertical arrows isomorphisms. As \( (\text{MGL}', \text{MGL}) \) is an oriented duality theory, the bottom line extends to the long exact sequence
\[
\ldots \to \oplus_{\eta \in X(d)} \text{MGL}_{2*,*}^{2*+1}(k(\eta)) \overset{\partial}{\longrightarrow} \text{MGL}_{2*,*}^{2*}(X) \overset{i_*}{\longrightarrow} \text{MGL}_{2*,*}^{2*}(X) \overset{j_*}{\longrightarrow} \oplus_{\eta \in X(d)} \text{MGL}_{2*,*}^{2*}(k(\eta)) \to 0.
\]
Furthermore, the Hopkins-Morel spectral sequence \[ E_2^{p,q} := L^{-q} \otimes H^{p-q}(Y, \mathbb{Z}(n + q)) \implies MGL^{p+q,n}(Y) \]
gives a surjection for each \( \eta \in X(d) \)
\[ t_{MGL}(\eta) : L_{s-d+1} \otimes k(\eta)^\times \to MGL'_{2s+1,\ast}(k(\eta)) \]
(see lemma \[ \[ \text{(4.6)} \]\]). We have constructed in \[ \[ \text{[11, \S 6]} \]\] a group homomorphism
\[ \text{Div} : L_{s-d+1} \otimes \bigoplus_{\eta \in X(d)} \mathbb{Z}[k(\eta)^\times] \to \Omega^{(1)}_\ast(X) \]
with \( \vartheta^{(1)} \circ \text{Div} = \partial \circ \bigoplus_{\eta} t_{MGL}(\eta). \) Since the maps \( \vartheta^{(1)} \) and \( \vartheta(X) \) are isomorphisms, the map \( \text{Div} \) factors through the surjection
\[ L_{s-d+1} \otimes \bigoplus_{\eta \in X(d)} \mathbb{Z}[k(\eta)^\times] \to L_{s-d+1} \otimes \bigoplus_{\eta \in X(d)} k(\eta)^\times, \]
we have the exact sequence
\[ \bigoplus_{\eta \in X(d)} L_{s-d+1} \otimes k(\eta)^\times \xrightarrow{\text{Div}} \Omega^{(1)}_\ast(X) \xrightarrow{\vartheta^{(1)}} \Omega_\ast(X) \xrightarrow{j^\ast} \bigoplus_{\eta \in X(d)} \Omega_\ast(k(\eta)) \to 0 \]
and the extension of diagram \[ \[ \text{(4.4)} \]\] to the commutative diagram
\[ \bigoplus_{\eta \in X(d)} L_{s-d+1} \otimes k(\eta)^\times \xrightarrow{\text{Div}_{MGL}} MGL'^{(1)}_{2s,\ast}(X) \xrightarrow{j^\ast} MGL'_{2s+1,\ast}(X) \xrightarrow{j} \bigoplus_{\eta} MGL'_{2s,\ast}(k(\eta)) \to 0 \]
with exact rows and vertical arrows isomorphisms. Here \( \text{div}_{MGL} := \partial \circ \bigoplus_{\eta} t_{MGL}(\eta). \)

5. \( K \)-THEORY

We apply the machinery described in \[ \[ \text{[3]} \]\] to \( K \)-theory and connective \( K \)-theory, with the orientations given by remark \[ \[ \text{[12]} \]\] (for \( K \)-theory) and lemma \[ \[ \text{[19]} \]\] (for connective \( K \)-theory). This gives us the oriented duality theories \( (K_{s,\ast}', \mathcal{C}K^{\ast,\ast}) \) and \( (\mathcal{C}K_{s,\ast}', \mathcal{C}K^{\ast,\ast}) \); the canonical morphism \( \mathcal{C}K \to K \) is compatible with the orientations and thus gives the map of oriented duality theories \( (\mathcal{C}K_{s,\ast}', \mathcal{C}K^{\ast,\ast}) \to (K_{s,\ast}', \mathcal{C}K^{\ast,\ast}) \).

We can in fact describe \( \mathcal{C}K_{a,b}(X) \) more explicitly, with the help of the models \( K^{(n)} \) for \( f_n K \) and Quillen’s localization theorem.

Indeed, for a smooth pair \( (M, X) \), Quillen’s localization theorem gives us the natural homotopy fiber sequence
\[ G(X) \to K(M) \to K(M \setminus X), \]
where \( G(X) \) is the Quillen-Waldhausen \( K \)-theory spectrum of the abelian category of coherent sheaves on \( X \). This, together with “algebraic Bott-periodicity”
\[ K^{a,b}(X) \cong K_{2b-a}(X) \]
gives us the canonical identifications
\[ K'_{a,b}(X) \cong G_{b-2a}(X); \ K'^{a,b}(M) \cong K_X^{b-a}(M) \]
where \( K_X(M) \) is as usual the homotopy fiber of the restriction map \( K(M) \to K(M \setminus X) \). The construction of the simplicial spectrum \( m \mapsto K^{(n)}(X, m) \) can be carried through replacing \( K \)-theory with \( G \)-theory, that is, the \( K \)-theory of the category.
of coherent sheaves. For details, we refer the reader to [13, §8.4] and [14]; for the reader’s convenience, we give a brief sketch of the construction here.

We denote the category of coherent sheaves on a scheme $Y$ by $\mathcal{M}_Y$. Let $X$ be a finite type $k$-scheme. There is a technical problem in the construction of the simplicial spectrum $m \mapsto G^{(n)}(X, m)$, due to the fact that, for $g : \Delta^p \to \Delta^m$ an inclusion of a face of $\Delta^m$, the corresponding restriction map

$$(\text{id}_X \times g)^* : \mathcal{M}_{X \times \Delta^m} \to \mathcal{M}_{X \times \Delta^p}$$

is not exact, hence does not lead to a functor on the corresponding $K$-theory spectra. To get around this problem, one replaces $\mathcal{M}_{X \times \Delta^m}$ with the corresponding sub-category $\mathcal{M}_{X \times \Delta^m, \partial}$ of coherent sheaves which are Tor-independent with respect to the inclusions of faces; for $U \subset X \times \Delta^m$ an open subset, we let $\mathcal{M}_{U, \partial}$ be the similarly defined full subcategory of $\mathcal{M}_U$. Letting $G(X, m)$ be the corresponding $K$-theory spectrum of the exact category $\mathcal{M}_{X \times \Delta^m, \partial}$, we have the simplicial spectrum $m \mapsto G(X, m)$.

Since we are not assuming that $X$ is equi-dimensional over $k$, it is more convenient to index by dimension rather than codimension. Let $S^X_n(m)$ denote the set of closed subsets $W$ of $X \times \Delta^m$ such that

$$\dim W \cap X \times F \leq n + p$$

for all faces $F \cong \Delta^p$ of $\Delta^n$, and let

$$G_{(n)}(X, m) := \operatorname{hocolim}_{W \in S^X_n(m)} G_W(X, m)$$

where $G_W(X, m)$ is by definition the homotopy fiber of the restriction map

$$K(\mathcal{M}_{X \times \Delta^m, \partial}) \to K(\mathcal{M}_{X \times \Delta^m \setminus W, \partial}).$$

This gives us the simplicial spectrum $m \mapsto G_{(n)}(X, m)$ with total spectrum $G_{(n)}(X)$, and the corresponding tower of spectra

$$\ldots \to G_{(n)}(X) \to G_{(n+1)}(X) \to \ldots \to G_{(\dim X)}(X) = G_{(\dim X+1)}(X, m) = \ldots$$

The identification of $G(X) := K(\mathcal{M}_X)$ with the 0-simplices in $G_{(\dim X)}(X)$ gives us the map of spectra

$$G(X) \to G_{(\dim X)}(X),$$

which, using the homotopy invariance property of $G$-theory, is easily seen to be a weak equivalence.

It is easy to see that $X \mapsto G_{(n)}(X)$ is contravariantly functorial with respect to flat maps of finite type $k$-schemes. As for the usual $G$-theory construction, there is a canonical push-forward functor

$$i_* : G_{(n)}(W) \to G_{(n)}(X)$$

for $i : W \to X$ a closed immersion; this extends to a push forward map for arbitrary projective morphisms, but only in the homotopy category $\mathcal{SH}$ as one needs to replace the various categories of coherent sheaves with the corresponding subcategories which have no higher direct images with respect to the given projective morphism (see [14, §4] for details).

The localization theorem for the spectra $G_{(n)}$ from [13, corollary 8.12] is

**Theorem 5.1.** Let $j : U \to X$ be an open immersion of finite type $k$-schemes with closed complement $i : W \to X$. Then the sequence

$$G_{(n)}(W) \xrightarrow{i_*} G_{(n)}(X) \xrightarrow{j^*} G_{(n)}(U)$$
is a homotopy fiber sequence.

**Corollary 5.2.** Let $X$ be in $\text{Sch}/k$.

1. There is a canonical isomorphism
   
   $$a_{a,b}^X: CK_{a,b}^r(X) \rightarrow \pi_{a-2b}(G_{(b)}(X))$$

   compatible with push-forward for projective morphisms and pull-back for open immersions. In particular, we have isomorphisms
   
   $$a_{2b,b}^X: CK_{2b,b}^r(X) \cong \pi_0(G_{(b)}(X))$$

   for all $b \in \mathbb{Z}$, compatible with push-forward for projective morphisms and pull-back for open immersions.

2. For $X \in \text{Sm}/k$ of dimension $d$, the isomorphism $a_{a,b}^X$ is compatible with the isomorphism of proposition 2.4 via the weak equivalence $G_{(b)}(X) \rightarrow K^{(d-b)}(X)$ and the isomorphism $\alpha_X: CK^{2d-a,d-b}(X) \rightarrow CK_{a,b}^r(X)$.

**Proof.** For each $q$, let $K_{X}^{(q)}(M)$ denote the homotopy fiber of the restriction map $j^*: K^{(q)}(M) \rightarrow K^{(q)}(M \setminus X)$.

Using example 2.2 and proposition 2.4 for the diagram

$$
\begin{array}{ccc}
M \setminus X & \longrightarrow & M \\
\downarrow & & \\
\ast
\end{array}
$$

gives us the canonical isomorphism

$$CK_{a,b}^r(X) \xrightarrow{a_{M,X}} CK_X^{2m-a,m-b}(M) \cong \pi_{a-2b}(K_X^{(m-b)}(M)).$$

By the localization theorem, we have the canonical weak equivalence

$$\beta_{X,M}: G_{(b)}(X) \rightarrow \text{fib}(G_{(b)}(M) \rightarrow G_{(b)}(M \setminus X))$$

induced by the functors $(i_X \times \text{id})_*: M_X \times \Delta^n \rightarrow M_{M \times \Delta^n}$ and the canonical isomorphism of $(j_{M \setminus X} \times \text{id})^* \circ (i \times \text{id})_* \circ \Delta^n$ with the 0-functor. The resolution theorem gives a canonical weak equivalence

$$K_X^{(m-b)}(M) = \text{fib}(K^{(m-b)}(M) \rightarrow K^{(m-b)}(M \setminus X)) \xrightarrow{\gamma_{X,M}} \text{fib}(G_{(b)}(M) \rightarrow G_{(b)}(M \setminus X)).$$

This defines the isomorphism in $\mathcal{SH}$

$$\delta_{X,M} := \beta_{X,M}^{-1} \circ \gamma_{X,M}: K_X^{(m-b)}(M) \rightarrow G_{(b)}(X).$$

Both $\gamma_{X,M}$ and $\beta_{X,M}$ are compatible with pull-back by flat maps $f: M' \rightarrow M$ (with $X' := f^{-1}(X)$), hence $\delta$ is compatible with pull-back by open immersions.
Suppose we have a closed immersion \( i : X' \to X \). We use the same ambient smooth scheme \( M \), giving the map of pairs \( \text{id}_M : (X, M) \to (X', M) \). The push-forward by \( i \) on \( K^{(m-b)}_X(M) \) is by definition the map on the homotopy fibers induced by the commutative diagram

\[
\begin{array}{ccc}
K^{(m-b)}_X(M) & \xrightarrow{u^*} & K^{(m-b)}_X(M \setminus X') \\
\text{id} & & w^* \\
K^{(m-b)}_X(M) & \xrightarrow{v^*} & K^{(m-b)}_X(M \setminus X')
\end{array}
\]

where \( u, v, w \) are the evident open immersions. It is not hard to check that the diagram

\[
\begin{array}{ccc}
G_{(b)}(X') & \xrightarrow{i_{X',*}} & G_{(b)}(M) \\
\downarrow i_* & & \downarrow w^* \\
G_{(b)}(X) & \xrightarrow{i_{X,*}} & G_{(b)}(M) \xrightarrow{v_*} G_{(b)}(M \setminus X)
\end{array}
\]

commutes up to canonical homotopy. From this it follows that \( i_*^G \circ \delta_{X,M} = \delta_{X',M} \circ i_*^K \).

For both \( K \)-theory and \( G \)-theory, the push-forward for a projection \( p : Y \times \mathbb{P}^n \to Y \) is compatible with the projection onto the factor \([\mathcal{O}_{Y \times \mathbb{P}^n}]\) in the respective projective bundle formulas with bases \([\mathcal{O}(-i)]\), \( i = 0, \ldots, n \). From this it is not hard to show that \( \delta_{X,M} \) is compatible with \( p_* \), and hence compatible with push-forward by an arbitrary projective morphism, completing the proof of (1).

The assertion (2) follows directly from the construction of \( \delta_{X,M} \). \( \square \)

Remark 5.3. Of course, Quillen’s localization theorem tells us that the homology theory \( \mathcal{K}'_{s,*} \) associated to \( K \)-theory is just \( G \)-theory. It follows from corollary 5.2 that the canonical natural transformation \( \mathcal{K}'_{a,b} \to \mathcal{K}'_{a,b} \) is given by the map

\[
\pi_{a-2b}G_{(b)}(X) \to \pi_{a-2b}G(X) = G_{a-2b}(X)
\]

induced by the canonical map \( G_{(b)}X(\cdot) \to G_{(\dim X)}X(\sim) \sim G(X) \).

Our main interest is in the “geometric” portion \( \mathcal{K}'_{2s,*} \) of the theory \( \mathcal{K}'_{s,*} \). For a finite type \( k \)-scheme \( X \), and integer \( b \geq 0 \), we have the full subcategory \( \mathcal{M}_{(b)}(X) \) of \( \mathcal{M}_X \) with objects the coherent sheaves \( \mathcal{F} \) on \( X \) with \( \dim_k \text{supp} \mathcal{F} \geq b \). The following result improves upon corollary 5.2.

Theorem 5.4. Let \( X \) be a finite type \( k \)-scheme. There is a canonical isomorphism

\[
a_n^X : \mathcal{K}'_{2n,n}(X) \to \text{im}(K_0(\mathcal{M}(n))(X) \to \mathcal{M}(n+1)(X)),
\]

compatible with pull-back by open immersions and push-forward by projective morphisms. Furthermore \( \mathcal{K}'_{a,b}(X) = 0 \) for \( a < 2b \).

Proof. By corollary 5.2 we have

\[
\mathcal{K}'_{a,b}(X) \cong \pi_{a-2b}(G_{(b)}(X)).
\]

Moreover, \( G_{(b)}(X) \) is the simplicial spectrum \( m \mapsto G_{(b)}(X, m) \) and

\[
G_{(b)}(X, m) = \text{hocolim}_{W \in \mathcal{S}_b(m)} G_W(\mathcal{M}(X \times \Delta^m, \partial)).
\]
We showed in [13 lemma 8.7] that for all open \( U \subset X \times \Delta^m \), the inclusion
\[
\mathcal{M}(U, \partial) \rightarrow \mathcal{M}_U
\]
induces a weak equivalence on the \( K \)-theory spectra. Thus \( G_W(\mathcal{M}(X \times \Delta^m, \partial)) \) is weakly equivalent to the homotopy fiber of the restriction map
\[
j^* : G(X \times \Delta^m) \rightarrow G(X \times \Delta^m \setminus W)
\]
which by Quillen’s localization theorem is in turn weakly equivalent to \( G(W) \). In particular, this shows that \( G_{(0)}(X, m) \) is -1 connected and we have a strongly convergent spectral sequence
\[
E^1_{p,q} = \pi_{p+q} G_{(b)}(X, p) \Rightarrow \pi_{p+q} G_{(b)}(X).
\]
As \( E^1_{p,q} = 0 \) for \( p + q < 0 \), this shows that \( \mathcal{C}K_{a,b}^{'}(X) = 0 \) for \( a < 2b \), as claimed.

For the assertion on \( \mathcal{C}K_{2n,n}(X) \), the spectral sequence gives the right exact sequence
\[
\pi_0(\mathcal{C}K_{n}(X, 1)) \xrightarrow{\partial} \pi_0(\mathcal{C}K_{n}(X, 0)) \rightarrow \mathcal{C}K_{2n,n}(X) \rightarrow 0.
\]
By definition, \( \mathcal{C}K_{n}(X, 0) = K(\mathcal{M}(n)(X)) \), so we need to show that the image \( M \) of \( \pi_0(\mathcal{C}K_{n}(X, 1)) \) under \( \partial \) is the same as the kernel \( N \) of the map
\[
K_0(\mathcal{M}(n)(X)) \rightarrow K_0(\mathcal{M}(n+1)(X)).
\]
We note that \( \mathcal{M}(n)(X) \) is a Serre subcategory of the abelian category \( \mathcal{M}(n+1)(X) \); let \( \mathcal{M}_{(n+1/n)}(X) \) denote the quotient category. By Quillen’s localization theorem, we have the exact sequence
\[
\ldots \rightarrow K_1(\mathcal{M}_{(n+1/n)}(X)) \xrightarrow{\partial} K_0(\mathcal{M}_{(n)}(X)) \rightarrow K_0(\mathcal{M}_{(n+1)}(X)) \rightarrow \ldots.
\]
Furthermore, by devissage, we have
\[
K_1(\mathcal{M}_{(n+1/n)}(X)) \cong \bigoplus_{x \in X_{(n)}} K_1(k(x)) = \bigoplus_{x \in X_{(n)}} k(x)^{\times}
\]
where \( X_{(n)} \) is the set of dimension \( n \) points of \( X \). Finally, the boundary map \( K_1(\mathcal{M}_{(n+1/n)}(X)) \rightarrow K_0(\mathcal{M}_{(n)}(X)) \) can be described as follows: Take \( y \in X_{(n+1)} \) and let \( Y \subset X \) be the reduced closure of \( y \). Take \( f \in k(y)^{\times} \) and let \( Y' \subset Y \times \mathbb{P}^1 \) be the reduced closure of the graph of \( f \). For \( t \in \mathbb{P}^1 \), let \( \mathcal{O}_{Y'}(t) := \mathcal{O}_{Y} \otimes_{\mathcal{O}_{\mathbb{P}^1}} k(t) \). Then
\[
\partial(f \in k(y)^{\times}) = p_{1*}([\mathcal{O}_{Y'}(0)] - [\mathcal{O}_{Y'}(\infty)]).
\]
Noting that \( \mathcal{O}_{Y'}(t) \) is an \( \mathcal{O}_{X \times \mathbb{P}^1} \)-module, we can consider \( p_{1*}([\mathcal{O}_{Y'}(0)] \) as simply \( [\mathcal{O}_{Y'}(0)] \) via the isomorphism \( p_1 : X \times 0 \rightarrow X \), and similarly for \( [\mathcal{O}_{Y'}(\infty)] \). Finally, as \( f \) is in \( k(y)^{\times} \) and \( Y \) has dimension \( n + 1 \), it follows that \( \mathcal{O}_{Y'}(0) \) and \( \mathcal{O}_{Y'}(\infty) \) have support of dimension at most \( n \).

Given such a \( Y \) and \( f \in k(y)^{\times} \), we may identify \( X \times \Delta^1 \) with the open subscheme \( X \times (\mathbb{P}^1 \setminus \{1\}) \) of \( X \times \mathbb{P}^1 \) via the open immersion \( j : \Delta^1 \rightarrow \mathbb{P}^1 \) defined by \( j(t_0, t_1) := t_1/t_0 \). Then \( (id \times j)^{-1}(\mathcal{O}_{Y'}) \) is a coherent sheaf in \( \mathcal{M}_{(n)}(X \times \Delta^1, \partial) \) and
\[
\partial(([id \times j]^{-1}(\mathcal{O}_{Y'}))] = \pm \partial(f \in k(y)^{\times}).
\]
Thus, \( M \supset N \).

For the reverse inclusion, take \( \mathcal{F} \in \mathcal{M}_{(n)}(X \times \Delta^1, \partial) \), let \( Z \) be the support of \( \mathcal{F} \), let \( Y \subset X \) be the closure of \( p_1(Z) \) and let \( W \subset X \times \Delta^1 \) be \( p_1^{-1}(Y) \); let \( i : Z \rightarrow W \) be the inclusion. By devissage, we may assume that \( \mathcal{F} \) is an \( \mathcal{O}_Z \)-module. But
Corollary 5.5. Take \( X \in \text{Sch}/k \) and let \( d \) be an integer with \( \dim_k X \leq d \). Then the natural map

\[
\gamma_n^X : \text{CK}_{2n,n}(X) \to K_{2n,n}(X) = G_0(X)
\]

is an isomorphism for \( n \geq d \) and is injective for \( n = d - 1 \).

Proof. The identity \( K_{2n,n}(X) = G_0(X) \) follows from the fact that the \( T \)-spectrum \( K \) represents Quillen \( K \)-theory on \( \text{Sm}/k \) and Quillen’s localization theorem identifies the homotopy fiber of \( K(M) \to K(M \setminus X) \) with the \( G \)-theory spectrum \( G(X) \), for \( M \in \text{Sm}/k \) and \( i : X \to M \) a closed immersion.

For \( n \geq d - 1 \), theorem [5,4] gives the identification

\[
\pi_0(G_{(n)}(X)) \cong \text{CK}_{2n,n}(X) \cong \text{im}(K_0(\mathcal{M}_{(n)}(X)) \to G_0(X)),
\]

the first isomorphism being induced by the localization fiber sequence

\[
G_{(n)}(X) \to K_{(n)}(M) \xrightarrow{i^*} K_{(n)}(M \setminus X)
\]

for a given closed immersion \( i : X \to M, M \in \text{Sm}/k, \) and the second isomorphism the computation of \( \text{CK}_{2n,n}(X) \) given in theorem [5,4]. We need only check that via this second isomorphism and the identity \( K_{2n,n}(X) = G_0(X) \), the canonical map

\[
\rho : \text{CK}_{2n,n}(X) \to K_{2n,n}(X)
\]

transforms to the evident map \( \text{im}(K_0(\mathcal{M}_{(n)}(X)) \to G_0(X)) \to G_0(X) \), which is clearly injective for all \( n \), and an isomorphism for \( n \geq d \).

We have the commutative square

\[
\begin{array}{ccc}
K_{(n)}(M) & \xrightarrow{i^*} & K_{(n)}(M \setminus X) \\
\downarrow & & \downarrow \\
K_{(d')} (M) & \xrightarrow{j^*} & K_{(d')} (M \setminus X)
\end{array}
\]

where \( d' \) is any integer with \( d' \geq n, d' \geq \dim_k M \). We have as well canonical weak equivalences \( K(M) \to K_{(d')} (M), K(M \setminus X) \to K_{(d')} (M \setminus X) \), induced by the identification of \( K(M) \) with the 0-spectrum in the simplicial spectrum \( m \to K_{(d')} (M, m) \), and similarly for \( M \setminus X \). This shows that the natural map \( \text{CK}_{2n,n}(X) \to K_{2n,n}(X) \) is given by the canonical map

\[
\pi_0(G_{(n)}(X)) \to \pi_0(G_{(d')}(X)) = G_0(X).
\]

Via the computation of theorem [5,4] this is just the evident map

\[
\text{im}(K_0(\mathcal{M}_{(n)}(X)) \to G_0(X)) \to G_0(X),
\]

as desired. \( \Box \)
Remarks 5.6. 1. For $X$ smooth of dimension $d$, we may take the identity closed immersion to define $\mathcal{K}'_{a,n}(X)$, so that

$$\mathcal{K}'_{a,b}(X) = \mathcal{K}'_{2d-a,d-b}(X)$$

In particular, letting $\mathcal{M}^{(b)}(X) \subset \mathcal{M}_X$ be the full subcategory of coherent sheaves on $X$ with support in codimension at least $b$, theorem 5.4 shows that

$$\mathcal{K}'_{2n,n}(X) = \text{im}(K_0(\mathcal{M}^{(n)}(X)) \rightarrow K_0(\mathcal{M}^{(n-1)}(X))).$$

In fact, the argument for theorem 5.4 works to show this identity for $k$ an arbitrary perfect field, we do not need resolution of singularities here.

2. Cai \cite{Cai1} has defined a theory of “connective higher $K$-theory” as a bi-graded oriented cohomology theory on $\text{Sm}/k$. Denoting this theory as $\mathcal{K}'_{a,b}$, Cai defines $\mathcal{K}'_{a,b}$ as

$$\mathcal{K}'_{a,b}(X) := \text{im}(K_{2b-a}(\mathcal{M}^{(b)}(X)) \rightarrow \mathcal{M}^{(b-1)}(X))$$

Thus we have

$$\mathcal{K}'_{2n,n}(X) = \mathcal{K}'_{2n,n}(X)$$

for $X \in \text{Sm}/k$.

We conclude this section with a description of the 1st Chern class operators on $\mathcal{K}'_{2n,n}(X)$. Let $L$ be a line bundle on $X$, and let $i : X \rightarrow M$ be a closed immersion with $M \in \text{Sm}/k$ such that $L$ extends to a line bundle $L$ on $M$. We note that the 1st Chern class operator $\hat{c}_1(L)$ on $\mathcal{K}'_{2n,n}(X)$ is given by the product with $c_1(L) \in \mathcal{K}'_{-1,0}(M)$ on $\mathcal{K}'_{2n,n}(M)$. As $\mathcal{K}'_{2n,n}(M) \rightarrow \mathcal{K}'_{2n,n}(M) = K_0(M)$ is injective (corollary 5.5), it follows that the operators $\hat{c}_1(L)$ satisfy the multiplicative formal group law:

$$\hat{c}_1(L \otimes L') = \hat{c}_1(L) + \hat{c}_1(L') - \beta \hat{c}_1(L) \circ \hat{c}_1(L')$$

where $\beta \in \mathcal{K}'_{1,0}(k)$ is the “Bott element”, i.e., the element corresponding to 1 in $\mathbb{Z}$ under the sequence of isomorphisms

$$\mathcal{K}'_{-1,0}(k) = \mathcal{K}'_{-2,1}(k) \rightarrow \mathcal{K}'_{-2,1}(k) \cong K_0(k) \xrightarrow{\text{dim}_k} \mathbb{Z}.$$

Thus, we need only describe $\hat{c}_1(L)$ for $L$ which is very ample on $X$.

Let $\alpha$ be in $\mathcal{K}'_{2n,n}(X)$ for some $n$. If $n \geq \text{dim}_k X + 1$, then both $\mathcal{K}'_{2n,n}(X)$ and $\mathcal{K}'_{2n-2,n-1}(X)$ are equal to $G_0(X)$ via the canonical map $\mathcal{K}'_{2n-2,n-1}(X) \rightarrow G_0(X)$, so $\hat{c}_1(L)$ is given by the 1st Chern class operator on $\mathcal{K}'_{2n,n}(X) = G_0(X)$. This is turned in by multiplication by the $K$-theory 1st Chern class via the $K_0(X)$-module structure on $G_0(X)$, that is,

$$\hat{c}_1^{G_0}(L)(\alpha) = (1 - L^{-1}) \cdot \alpha.$$

In general, we have our description

$$\mathcal{K}'_{2n,n}(X) \cong \text{im}(K_0(\mathcal{M}(n)(X)) \rightarrow K_0(\mathcal{M}(n+1)(X))),$$

compatible with projective push-forward. Thus, for each $\alpha \in \mathcal{K}'_{2n,n}(X)$, there is a closed immersion $i : X' \rightarrow X$ in $\textbf{Sch}/k$, with $\text{dim}_k X' \leq n$, and an element $\alpha' \in \mathcal{K}'_{2n,n}(X')$ with $i_*(\alpha') = \alpha$. Since

$$\hat{c}_1(L)(i_* \alpha') = i_*(\hat{c}_1(i^* L)(\alpha')),$$

we reduce to the case $n \geq \text{dim}_k X$; similarly, we may assume $X$ is irreducible. By our above computation, the only remaining case is $n = \text{dim}_k X$. 


In this case, $\mathcal{C}K'_{2n,n}(X) = G_0(X)$ and for $\alpha \in \mathcal{C}K'_{2n,n}(X)$, $\tilde{c}_1(\alpha) \in \mathcal{C}K'_{2n-2,n-1}(X)$ is characterised by the identity

$$(1 - L^{-1}) \cdot \alpha = \text{im}(\tilde{c}_1(\alpha) \in G_0(X)),$$

via the inclusion (corollary 5.5) $\mathcal{C}K'_{2n-2,n-1}(X) \to G_0(X)$. We now use the assumption that $L$ is very ample. Write $\alpha = \sum n_i [F_i]$, where the $F_i$ are coherent sheaves on $X$, $n_i$ are integers and $[-]$ denotes class in $G_0(X)$. Choose a section $s$ of $L$ that is locally a non-zero divisor on each $F_i$ and on $\mathcal{O}_X$; this is possible by Kleiman’s transversality theorem and the fact that $L$ is very ample. Let $H \subset X$ be the closed subschema defined by $s$, giving us the short exact sequence of sheaves on $X$

$$0 \to L^{-1} \to \mathcal{O}_X \to \mathcal{O}_H \to 0$$

For each $i$, the tensor product $F_i \otimes \mathcal{O}_H$ is an element of $\mathcal{M}_{(n-1)}(X)$ (its support is contained in $H$), giving the well-defined class $[F_i \otimes \mathcal{O}_H]$ in $K_0(\mathcal{M}_{(n-1)}(X))$. In addition, the above exact sequence and the fact that $s$ is a non-zero divisor on $F_i$ gives the identity

$$[F_i \otimes \mathcal{O}_H] = (1 - L^{-1}) \cdot [F_i] \in G_0(X)$$

for each $i$ and hence $\tilde{c}_1(\alpha) \in \mathcal{C}K'_{2n-2,n-1}(X)$ is given by $\sum n_i [F_i \otimes \mathcal{O}_H]$.

**Remark 5.7.** Set $CG_{a,b}(X) := \pi_{a-2b}G_b(X)$, for $X \in \text{Sch}/k$. From the proof of corollary 5.2 we have isomorphisms $\delta_{X,M} : \pi_{a-2b}(K_X^{m-b}(M)) \to \pi_{a-2b}(G_b(X))$ for $X$ a closed subscheme of $M \in \text{Sm}/k$, $m = \dim M$. Combined with the natural isomorphisms $\mathcal{C}K_{X,n}^{2m-a,m-b}(M) \cong \pi_{a-2b}(K_X^{m-b}(M))$ given by theorem 2.3 and lemma 2.2, the properties of $G_b(\cdot)$ discussed in this section show that we have defined an oriented duality theory $(CG_{a,*}, \mathcal{C}K^{*,*})$ on $\text{Sch}/k$, for $k$ an arbitrary perfect field, except that we have not defined a cap product structure of $\mathcal{C}K^{*,*}$ on $CG_{a,*}$. Possibly this could be supplied by the method used by Cai in [11, §6.3] to define pull back maps in his theory $\mathcal{C}K^{*,*}$ for regular embeddings.

The results of this section can be interpreted as saying that, in case $k$ admits resolution of singularities, the oriented duality theory $(\mathcal{C}K'_{a,*}, \mathcal{C}K^{*,*})$ on $\text{Sch}/k$ defined using the results of [11] is isomorphic to $(CG_{a,*}, \mathcal{C}K^{*,*})$ (neglecting the cap product structure).

## 6. The Comparison Map

We consider the classifying map

$$\theta_{\mathcal{C}K} : \Omega_* \to \mathcal{C}K'_{2,*}$$

constructed in proposition 4.1. Let $\varphi_{\mathcal{C}K} : L_* \to \mathbb{Z}[\beta]$ be the ring homomorphism classifying the formal group law $(u + v - \beta uv, \mathbb{Z}[\beta])$ of the theory $\mathcal{C}K$ and $\varphi_{\Omega} : L_* \to \Omega_*(k)$ the classifying map for the formal group law for $\Omega_*$. For each $X \in \text{Sch}/k$, the external products make $\Omega_*(X)$ into an $\Omega_*(k)$-module and thus via $\varphi_{\Omega}$ an $L_*$-module. It is easy to check that the various structures for $\Omega_*$ make the assignment $X \mapsto \Omega_*(X) \otimes_{L_*} \mathbb{Z}[\beta]$ into an oriented Borel-Moore homology theory on $\text{Sch}/k$, which we denote by $\Omega_* \otimes_{L_*} \mathbb{Z}[\beta]$; the universality of $\Omega_*$ makes $\Omega_* \otimes_{L_*} \mathbb{Z}[\beta]$ the universal oriented Borel-Moore homology theory on $\text{Sch}/k$ having formal group law $(u + v - \beta uv, \mathbb{Z}[\beta])$.

**Lemma 6.1.** $\theta_{\mathcal{C}K}$ factors through the surjection

$$\Omega_* \to \Omega_* \otimes_{L_*} \mathbb{Z}[\beta].$$
Proof. As the natural transformation $\theta_{CK}$ is compatible with external products, $\theta_{CK}(k)$ is a ring homomorphism and $\theta_{CK}(X)$ is a map of $L_* - Z[\beta]$-modules. Using the universal property of $\Omega_* \otimes_{L_*} Z[\beta]$, we need only show that

$$\theta_{CK}(k) \circ \varphi_{\Omega} : L_* \to Z[\beta]$$

is equal to the classifying homomorphism $\varphi_{CK}$. But this is a general property of the classifying map $\theta_{H,A} : \Omega_* \to H_*$ for any oriented duality theory $(H, A)$, as we have already noted in remark 4.2. □

In [15, theorem 1.2.18] it is shown that the classifying map of oriented cohomology theories on $\text{Sm}/k$

$$\vartheta_{K_0} : \Omega_* \otimes_{L_*} Z[\beta, \beta^{-1}] \to K_0[\beta, \beta^{-1}]$$

is an isomorphism, where $L \to Z[\beta, \beta^{-1}]$ is the classifying map for the formal group law $(u + v - \beta uv, Z[\beta, \beta^{-1}])$. Dai extended this result, showing in [2] that the classifying map

$$\vartheta_G_0 : \Omega_* \otimes_{L_*} Z[\beta, \beta^{-1}] \to G_0[\beta, \beta^{-1}]$$

is an isomorphism of oriented Borel-Moore homology theories on $\text{Sch}/k$. We now prove the analogous result for connective $K$-theory:

**Definition 6.2.** For $X \in \text{Sch}/k$, write $\Omega_{CK}^*(X)$ for $\Omega_*^*(X) \otimes_{L_*} Z[\beta]$.

**Theorem 6.3.** The induced classifying map

$$\theta_{CK} : \Omega_{CK}^* \to CK_{2*}^*$$

is an isomorphism.

Proof. For $X \in \text{Sch}/k$, we consider $\theta_{CK}$ as a map of presheaves on $X_{\text{Zar}}$, so we may evaluate at the generic points of $X$. For $\eta$ a dimension $d$ generic point of $X$, it follows from the description of $CK_{2*}^*$ given by theorem 5.4 that

$$CK_{2n,n}^*(\eta) = \begin{cases} K_0(k(\eta)) & \text{for } n \geq d \\ 0 & \text{for } n < d. \end{cases}$$

By [15] corollary 4.4.3], the pull-back map $p_X^* : \Omega_*^*(k) \to \Omega_{*+d}(k(\eta))$ is an isomorphism, so by [15] theorem 1.2.7] we have the isomorphism of $L_*$-modules

$$\Omega_*^*(k(\eta)) \cong L_{*-d}.$$

From this it follows easily that

$$\theta_{CK}(k(\eta)) : \Omega_{CK}^*(k(\eta)) \to CK_{2*,*}^*(k(\eta))$$

is an isomorphism. In particular, $\theta_{CK}(X)$ is an isomorphism for all reduced $X$ of dimension $0$ over $k$. As the inclusion $X_{\text{red}} \to X$ induces an isomorphism on $\Omega_*$ and $CK_{2*,*}^*$, the theorem is proven for all $X$ of dimension $0$.

We proceed by induction on the maximum $d$ of the dimensions of the components of $X$; we may assume that $X$ is reduced. We use the constructions and notations from theorem 4.5 and the discussion following that theorem. We let $CK_{2*,*}^{(1)}(X)$ be the inductive limit

$$CK_{2*,*}^{(1)}(X) := \lim_{W} CK_{2*,*}^*(W)$$

as $W$ runs over all (reduced) closed subschemes of $X$ which contain no dimension $d$ generic point of $X$. This, together with the map $\text{Div}$ defined following theorem 4.5.
and the localization exact sequence for $\mathcal{CK}'_{s, s}$ gives us the commutative diagram with exact rows

\[(6.1)\]

$$
\oplus_{\eta \in X(d)} \mathbb{L}_{s-d+1} \otimes k(\eta)^{\times} \xrightarrow{\text{Div}} \Omega^{(1)}_s(\mathbb{X}) \xrightarrow{i_*} \Omega^s(\mathbb{X}) \xrightarrow{j^*} \oplus_{\eta \in X(d)} \Omega^s(k(\eta)) \quad 0
$$

$$
\oplus_{\eta \in X(d)} \mathcal{CK}'_{2s+1, s}(k(\eta)) \xrightarrow{\vartheta} \mathcal{CK}'_{2s+1, s}(X) \xrightarrow{\theta} \oplus_{\eta \in X(d)} \mathcal{CK}'_{2s+1, s}(k(\eta)) \quad 0
$$

We apply $\mathbb{Z}[\beta] \otimes_{\mathbb{L}} (-)$ to the top row in (6.1). By lemma (6.1) the vertical maps in (6.1) descend to give the commutative diagram

\[(6.2)\]

$$
\oplus_{\eta \in X(d)} \mathbb{Z}[\beta]_{s-d+1} \otimes k(\eta)^{\times} \xrightarrow{\text{Div}_{\mathbb{CK}}} \Omega^{CK(1)}_s(\mathbb{X}) \xrightarrow{i_*} \Omega^{CK}_s(\mathbb{X}) \xrightarrow{j^*} \oplus_{\eta \in X(d)} \Omega^{CK}_s(k(\eta)) \quad 0
$$

$$
\oplus_{\eta \in X(d)} \mathcal{CK}'_{2s+1, s}(k(\eta)) \xrightarrow{\vartheta} \mathcal{CK}'_{2s+1, s}(X) \xrightarrow{\theta} \oplus_{\eta \in X(d)} \mathcal{CK}'_{2s+1, s}(k(\eta)) \quad 0
$$

By induction on $d$, the map $\vartheta^{(1)}$ is an isomorphism; we have already seen that $\vartheta$ is an isomorphism. We note that the bottom row is a sequence of $\mathbb{Z}[\beta]$-modules via the isomorphism $\mathcal{CK}_{2s, s}(k) \cong \mathbb{Z}[\beta]$ and the $\mathcal{CK}_{2s, s}(k)$-module structure given by external products.

Take $\eta \in X(d)$. Then $\mathcal{CK}'_{2s+1, s}(k(\eta)) \cong \mathcal{CK}_{2d-2s-1, d-1}(k(\eta))$. Via lemma 2.7 we have the isomorphism of $\mathbb{Z}[\beta]$-modules

$$
t_{\mathbb{CK}} : \mathbb{Z}[\beta]_{s-d+1} \otimes k(\eta)^{\times} \to \mathcal{CK}'_{2s+1, s}(k(\eta)).
$$

Putting this into the diagram (6.2) gives us the commutative diagram

\[(6.3)\]

$$
\oplus_{\eta \in X(d)} \mathbb{Z}[\beta]_{s-d+1} \otimes k(\eta)^{\times} \xrightarrow{\text{Div}_{\mathbb{CK}}} \mathbb{Z}[\beta]^{CK(1)}_s(\mathbb{X}) \xrightarrow{i_*} \mathbb{Z}[\beta]^{CK}_s(\mathbb{X}) \xrightarrow{j^*} \oplus_{\eta \in X(d)} \mathbb{Z}[\beta]^{CK}_s(k(\eta)) \quad 0
$$

$$
\oplus_{\eta \in X(d)} \mathbb{Z}[\beta]_{s-d+1} \otimes k(\eta)^{\times} \xrightarrow{\text{Div}_{\mathbb{CK}}} \mathcal{CK}'_{2s+1, s}(X) \xrightarrow{\theta} \mathcal{CK}'_{2s+1, s}(X) \xrightarrow{\theta} \oplus_{\eta \in X(d)} \mathcal{CK}'_{2s+1, s}(k(\eta)) \quad 0
$$

with the bottom row exact and the top row a complex.

We claim the identity map $\mathbb{Z}[\beta] \otimes k(\eta)^{\times} \to \mathbb{Z}[\beta] \otimes k(\eta)^{\times}$ fills in the diagram (6.3) to a (up to sign) commutative diagram. Assuming this claim, it follows by a diagram chase that the top row is exact and the map $\vartheta(X)$ is an isomorphism.

To prove the claim, the orientation $c_{\mathbb{CK}}$ for $\mathbb{CK}$ and the universal property of MGL gives the canonical map of oriented cohomology theories

$$
\rho_{\mathbb{CK}} : (\text{MGL}, c_{\text{MGL}}) \to (\mathbb{CK}, c_{\mathbb{CK}})
$$

which in turn gives us the map of bi-graded oriented duality theories

$$
\rho_{\mathbb{CK}} : (\text{MGL}_{s, s}, \text{MGL}^s) \to (\mathcal{CK}'_{s, s}, \mathcal{CK}'^s).
$$

It follows from the characterization of $\theta_{\text{MGL}}$, $\theta_{\mathbb{CK}}$ given in remark [4.4] that

$$
\theta_{\mathbb{CK}} = \rho_{\mathbb{CK}} \circ \theta_{\text{MGL}}.
$$
As discussed at the end of [2] and in lemma [2.7] the orientations \( c_{\text{MGL}} \), \( c_{\text{CK}} \) give rise to canonical elements

\[
t_{\text{MGL}} \in \text{MGL}^{1,1}(\mathbb{G}_m), \quad t_{\text{CK}} \in \text{CK}^{1,1}(\mathbb{G}_m).
\]

These in turn give by functoriality canonical homomorphisms for each \( X \in \text{Sm}/k \)

\[
t_{\text{MGL}}(X) : \mathcal{O}_X^\otimes(X) \to \text{MGL}^{1,1}(X), \quad t_{\text{CK}}(X) : \mathcal{O}_X^\otimes(X) \to \text{CK}^{1,1}(X)
\]

with \( t_{\text{CK}}(X) = \rho_{\text{CK}}(X) \circ t_{\text{CK}} \). We extend \( t_{\text{MGL}} \) to

\[
t_{\text{MGL}} : \mathcal{O}_X^\otimes(X) \otimes \mathbb{L}^* \to \text{MGL}^{2*+1,1*+1}(X)
\]

using the \( \mathbb{L}^* \)-module structure, and similarly have the extension of \( t_{\text{CK}}(X) \) to

\[
t_{\text{CK}} : \mathcal{O}_X^\otimes(X) \otimes \mathbb{Z}[\beta]^* \to \text{CK}^{2*+1,1*+1}(X).
\]

The map \( k(\eta)^\times \to \text{MGL}^{1,1}(k(\eta)) \) arising in the Hopkins-Morel spectral sequence is the map \( t_{\text{MGL}}(k(\eta)) \). As \( \rho_{\text{CK}} \) is a map of \( \mathbb{L}^* \)-\( \mathbb{Z}[\beta] \) modules, we have the commutative diagram

\[
\begin{align*}
\oplus_{\eta \in X(d)} \mathbb{L}_{s-d+1} \otimes k(\eta)^\times & \xrightarrow{t_{\text{MGL}}} \oplus_{\eta \in X(d)} \text{MGL}^{2*+1,1*+1}(k(\eta)) \xrightarrow{\partial} \text{MGL}^{2*+1,1*+1}(X) \\
\oplus_{\eta \in X(d)} \mathbb{Z}[\beta]_{s-d+1} \otimes k(\eta)^\times & \xrightarrow{t_{\text{CK}}} \text{CK}^{2*+1,1*+1}(X) \xrightarrow{\partial} \text{CK}^{2*+1,1*+1}(X)
\end{align*}
\]

where \( \pi \) is induced by the classifying map \( \mathbb{L}^* \to \mathbb{Z}[\beta] \). The map \( \text{div}_{\text{MGL}} \) in diagram (4.10) is the composition \( \partial \circ t_{\text{MGL}} \) in the diagram above. Defining

\[
\text{div}_{\text{CK}} : \oplus_{\eta \in X(d)} \mathbb{Z}[\beta]_{s-d+1} \otimes k(\eta)^\times \to \text{CK}^{2*+1,1*+1}(X)
\]

as \( \text{div}_{\text{CK}} := \partial \circ t_{\text{CK}} \) gives us the commutative diagram

\[
\begin{align*}
\oplus_{\eta \in X(d)} \mathbb{L}_{s-d+1} \otimes k(\eta)^\times & \xrightarrow{\text{div}_{\text{MGL}}} \text{MGL}^{2*+1,1*+1}(X) \\
\oplus_{\eta \in X(d)} \mathbb{Z}[\beta]_{s-d+1} \otimes k(\eta)^\times & \xrightarrow{\text{div}_{\text{CK}}} \text{CK}^{2*+1,1*+1}(X)
\end{align*}
\]

patching in the left-hand square in the commutative diagram (4.6) yields the commutative diagram

\[
(6.4)
\begin{align*}
\oplus_{\eta} \mathbb{L}_{s-d+1} \otimes k(\eta)^\times & \xrightarrow{\text{Div}} \Omega^{(1)}_*(X) \\
\oplus_{\eta} \mathbb{Z}[\beta]_{s-d+1} \otimes k(\eta)^\times & \xrightarrow{\text{Div}_{\text{CK}}} \text{CK}^{2*+1,1*+1}(X)
\end{align*}
\]

As \( \text{Div}_{\text{CK}} : \mathbb{Z}[\beta] \otimes k(\eta)^\times \to \Omega^{(1)}_*(X)_{\text{CK}} \) is just the map formed by applying the functor \( (-) \otimes_\mathbb{L} \mathbb{Z}[\beta] \) to \( \text{Div} : \mathbb{L} \otimes k(\eta)^\times \to \Omega^{(1)}_*(X) \), the desired commutativity follows from the commutativity of (6.4). \( \square \)
Corollary 6.4. Take $X \in \text{Sch}/k$, and let $d$ be an integer with $d \geq \dim_k X$. Then

$$\gamma_n^X \circ \theta_{\text{CK}}(X) : \Omega_{n}^{\text{CK}}(X) \to G_0(X)$$

is an isomorphism for $n = d$ and an injection for $n = d - 1$.

Proof. This follows from theorem 6.3 and corollary 5.5 □

7. LCI PULL-BACKS AND FUNDAMENTAL CLASSES

Theorem 6.3 gives us the isomorphism

$$\theta_{\text{CK}} : \Omega_{*}^{\text{CK}}(X) \to \text{CK}_{2*}^{\prime}(X)$$

compactible with projective push forward, pull-back by open immersions, external products and 1st Chern class operators. However, $\Omega_{*}^{\text{CK}}$ is an oriented Borel-Moore homology theory, hence has in addition to these structures pull-back maps for arbitrary l. c. i. morphisms. The isomorphism $\theta_{\text{CK}}$ thus endows the homology theory $\text{CK}_{2*}^{\prime}$ with l. c. i. pull-backs. The theory $X \mapsto \text{CK}_{2n,n}^{\prime}(X) = G_0(X)$ has l. c. i. pull-backs as well; we proceed to compare the two. As the l. c. i. pull-backs in $\Omega_{*}$ are defined using the classes of simple normal crossing divisors, we first need a result about these classes.

Fix an oriented cohomology theory $A^*$ with formal group law $F = F_A$. As usual, we use the notation $u + Fv$ for $F(u, v)$ and more generally write $[n_1]_F u_1 + \ldots + F[n_r]_F u_r$ for the power series $F_{n_1, \ldots, n_r}(u_1, \ldots, u_r)$ that expresses the evident sum operation. We may write $F_{n_1, \ldots, n_r}(u_1, \ldots, u_r)$ in the following form

$$F_{n_1, \ldots, n_r}(u_1, \ldots, u_r) = \sum_{I \subset \{1, \ldots, r\}, I \neq \emptyset} G_{I}^{n_1, \ldots, n_r}(u) \cdot u_I$$

with the $G_{I}^{n_1, \ldots, n_r}(u) \in A_*(k)[[u_1, \ldots, u_r]]$ and $u_I = \prod_{i \in I} u_i$. If we assume (which we will) that $G_{I}^{n_1, \ldots, n_r}(u)$ is not divisible by any $u_j$ with $j \notin I$, then the expression is unique (for instance, $G_{\{\}}^{n_1, \ldots, n_r}(u) = n_j$).

Now let $D = \sum_{i=1}^{r} n_i D_i$ be an effective simple normal crossing divisor on some $Y \in \text{Sm}/k$, with each $D_i$ smooth and irreducible. As usual, for $I \subset \{1, \ldots, r\}$, let $D_I := \cap_{i \in I} D_i$, and let $i_I : D_I \to |D|$, $i : |D| \to Y$ be the inclusions. Let $L_j = i^* O_Y(D_I)$; for $I = \{i_1, \ldots, i_m\}$, let $c_1(L_I)$ stand for the $m$-tuple $(c_1(L_{i_1}), \ldots, c_1(L_{i_m}))$. Suppose $Y$ has dimension $n$. The divisor class of $D$, $[D \to |D|]_A^{\text{sm}} \in A_{n-1}(|D|)$ is defined as

$$[D \to |D|]_A^{\text{sm}} := \sum_{I \subset \{1, \ldots, r\}, I \neq \emptyset} i_{I}^* G_{I}^{n_1, \ldots, n_r}(c_1(L_I))(1_{D_I})$$

Lemma 7.1. For $A_* = G_0[\beta, \beta^{-1}]$, we have

$$[D \to |D|]_A^{G_0} = [O_D]$$

in $G_0(|D|)$.

Proof. We write $[D \to |D|]$ for $[D \to |D|]_A^{G_0}$. and proceed by induction on $\sum_i n_i$, where $D = \sum_{i=1}^{r} n_i D_i$. For $D$ a smooth irreducible divisor, $[D \to |D|] = 1_D = [O_D]$, which takes care of the case $\sum_i n_i = 1$.

For the general case, let $D' := D - D_r = \sum_{i=1}^{r-1} n_i D_i + (n_r - 1) D_r$, and let $j : |D'| \to |D|$, $i_r : D_r \to |D|$, $i : |D| \to Y$ be the inclusions.
The formal group law for $G_0[β, β^{-1}]$ is given by $F(u, v) = u + v - βuv \in \mathbb{Z}[β, β^{-1}][[u, v]]$. From this, an elementary computation yields

\[(D | D) = j_*(D' | D') + i_*(D | D) - β \cdot \tilde{c}_1(k^*O_Y(D)) (j_*(D' | D')).\]

Let $\mathcal{I}_D, \mathcal{I}_{D'}, \mathcal{I}_{D''}$ be the respective ideal sheaves in $O_Y$. We have the exact sequence of $O_D$-modules

$0 \to \mathcal{I}_{D''}/\mathcal{I}_D \to O_D \to O_{D'} \to 0$

Furthermore, the multiplication map defines an isomorphism of $O_{D'}$-modules

$O_{D'} \otimes_{O_Y} \mathcal{I}_{D''} \cong \mathcal{I}_D/\mathcal{I}_D$.

This gives us the identities in $G_0(D)[β, β^{-1}]$:

$[O_D] = O_Y(-D) \cdot j_*[O_{D'}] + i_*[O_{D''}]$

$\quad = j_*[O_{D'}] + i_*[O_{D''}] - (1 - O_Y(-D)) \cdot j_*[O_{D'}]$

$\quad = j_*[O_{D'}] + i_*[O_{D''}] - \beta \tilde{c}_1(k^*O_Y(D)) (j_*[O_{D'}])$, the last identity following from

$\tilde{c}_1(L)(α) = β^{-1}(1 - L^{-1}) \cdot α$

for $L$ a line bundle on $|D|$ and $α \in G_0(|D|)[β, β^{-1}]$. Combined with (7.1) and our induction hypothesis, this proves the lemma.

\[\square\]

**Proposition 7.2.** Let $f : Y \to X$ be an l. c. i. morphism of relative dimension $d$ in $\text{Sch}/k$. Then the diagram

\[
\begin{array}{ccc}
\Omega^{CK}_{n}(X) & \xrightarrow{γ_0θ} & G_0(X) \\
\downarrow f^* & & \downarrow f^* \\
\Omega^{CK}_{n+d}(Y) & \xrightarrow{γ_0θ} & G_0(Y)
\end{array}
\]

commutes.

\[\text{Proof.}\] It suffices to consider two cases: (a) $f$ a regular embedding, (b) $f$ a smooth morphism.

We first note that $\Omega^{CK}_{n}(X)$ is generated by the classes $[g : Z \to X]$ with $Z \in \text{Sm}/k$ irreducible of dimension $n$ over $k$, and $g$ a projective morphism; here $[g : Z \to X]$ stands for the element $g_*([1_Z])$, with $[1_Z]$ the unit in the graded ring $Ω^{CK}_{n}(Z)$.

The image of $[g : Z \to X]$ is given by the same formula, that is

$(γ_0θ)([g : Z \to X]) = Rg_*(O_Z)$

where $Rg_* : G_0(Z) \to G_0(X)$ is the usual projective push-forward map on $G_0$:

$Rg_*([F]) := \sum_i (-1)^i [R^i g_* F]$. We now compute in cases (a) and (b):

**Case (a):** $f : Y \to X$ smooth. Then $f^*[g : Z \to X] = [p_2 : Z \times_X Y \to Y]$ and

$R_{p_2*}[O_{Z \times_X Y}] = R_{p_2*}(p_2^*[O_Z])$

$= f^*(Rg_*[O_Z])$

the last identity being the base-change isomorphism for the flat morphism $f$. 
Case (a): \( f : Y \to X \) a regular embedding, say of codimension \( c \). Since \( X \) is quasi-projective, we can factor \( f \) through a sequence of regular codimension one embeddings
\[
Y = X_c \hookrightarrow X_{c-1} \hookrightarrow \ldots \hookrightarrow X_1 \hookrightarrow X;
\]
this reduces us to the case \( c = 1 \), that is, \( f : Y \to X \) is a Cartier divisor on \( X \). In this case, \( \Omega_{X/K}^c(X) \) is generated by three types of elements (with \( g, Z \) as before):
1. \( g : Z \to X \) with \( g(Z) \subset Y \).
2. \( g : Z \to X \) such that \( g^{-1}(Y) \) is a simple normal crossing divisor on \( Z \).
3. \( g^{-1}(Y) = \emptyset \).

In case (3), \( f^*([g : Z \to X]) = 0 \) by definition, and clearly \( f^*(Rg_*(\mathcal{O}_Z)) \) is zero as well. In case (1), let \( L = f^*(\mathcal{O}_X(Y)) \) and let \( \tilde{g} : Z \to Y \) be the unique map with \( f \circ \tilde{g} = g \). Then by definition
\[
f^*([g : Z \to X]) := \check{c}_1(L)([\tilde{g} : Z \to Y]).
\]

On the \( G \)-theory side, let \( \mathcal{F} \) be a coherent sheaf on \( Y \). Then \( Rf_*[\mathcal{F}] = f_*\mathcal{F} \), since \( f \) is finite, and \( f^*([f_*\mathcal{F}]) := [\mathcal{F}] - [\text{Tor}_1^{\mathcal{O}_X}(f_*\mathcal{F}, \mathcal{O}_Y)] \). Via the exact sequence
\[
0 \to \mathcal{O}_X(-Y) \to \mathcal{O}_X \to \mathcal{O}_Y \to 0
\]
we see that
\[
[\mathcal{F}] - [\text{Tor}_1^{\mathcal{O}_X}(f_*\mathcal{F}, \mathcal{O}_Y)] = (1 - L^{-1}) \cdot [\mathcal{F}]
\]
in \( G_0(\mathcal{Y}) \), so \( f^*([f_*\mathcal{F}]) = (1 - L^{-1}) \cdot [\mathcal{F}] \). Applying this to the coherent sheaves \( R^0\tilde{g}_*\mathcal{O}_Z \) and noting that \( \check{c}_1^{G_0}(L) \) is multiplication by \( (1 - L^{-1}) \) shows that
\[
(\gamma \circ \theta)(f^*[g : Z \to X]) = f^*(Rg_*(\mathcal{O}_Z))
\]
as desired.

Finally, in case (2), the diagram
\[
\begin{array}{ccc}
g^{-1}(Y) & \xrightarrow{\bar{f}} & Z \\
\downarrow{\tilde{g}} & & \downarrow{g} \\
Y & \xrightarrow{f} & X
\end{array}
\]
is cartesian and Tor-independent, hence we have the base-change identity
\[
f^*Rg_*(\mathcal{O}_Z) = R\tilde{g}_*(\bar{f}^*[\mathcal{O}_Z]) = R\tilde{g}_*([\mathcal{O}_{g^{-1}(Y)}]).
\]
Thus it suffices to show that
\[
[\mathcal{O}_{g^{-1}(Y)}] = (\gamma \circ \theta)(\bar{f}^*(1z)) \tag{7.2}
\]
in \( G_0(g^{-1}(Y)) \). But by definition, we have \( \bar{f}^*(1z) = [g^{-1}(Y) \to [g^{-1}(Y)]] \), so the identity (7.2) follows from lemma (7.1).

**Definition 7.3.** Let \( X \in \text{Sch}/k \) have pure dimension \( d \). The *fundamental class* \( [X]_{CK} \in \Omega_{X/K}^d(X) \) is the element corresponding to \( [\mathcal{O}_X] \in G_d(0) \) under the isomorphism \( \Omega_{X/K}^d(X) \to G_d(0) \).

**Theorem 7.4.** Let \( f : Y \to X \) be an l.c.i. morphism in \( \text{Sch}/k \). Assume that \( X \) has pure dimension \( d_X \) and \( Y \) has pure dimension \( d_Y \). then \( f^*([X]_{CK}) = [Y]_{CK} \) in \( \Omega_{d_Y/K}^d(Y) \).
Proof. If $f$ is smooth, then $f^*[\mathcal{O}_X] = [\mathcal{O}_Y]$ in $G_0(Y)$. If $f$ is a regular embedding, then similarly $f^*[\mathcal{O}_X] = [\mathcal{O}_Y]$ in $G_0(Y)$. Thus, for an arbitrary l.c.i.-morphism $f$, $f^*[\mathcal{O}_X] = [\mathcal{O}_Y]$ in $G_0(Y)$. The theorem thus follows from the definition of the fundamental class, corollary [6.4] and proposition [7.2].

Let $\Omega^*_X := \Omega_* \otimes \mathbb{L}[\beta, \beta^{-1}]$ and $\Omega^+_X := \Omega_* \otimes \mathbb{L}[\beta]$, the first tensor product defined via the classifying map for the formal group law $(u + v - \beta uv, Z[\beta, \beta^{-1}], \text{the additive formal group law})$, the second for the additive formal group law $(u + v, Z)$. Dai’s theorem [2] states that the classifying map $\Omega^*_X \to G_0[\beta, \beta^{-1}]$ is an isomorphism, while it is shown in [15, theorem 4.5.1] that the classifying map $\Omega^+_X \to \text{CH}_*$ is an isomorphism.

In particular, inverting $\beta$ induces the map

$$\vartheta_+ : \Omega^+_X \to \Omega^*_X,$$

which after the isomorphisms $\Omega^*_X \cong G_0[\beta, \beta^{-1}] \cong K'_{2n,*}$ and $\Omega^+_X \cong CK'_{2n,*}$, is just the canonical map $\rho : CK'_{2n,*} \to K'_{2n,*}$. Similarly, taking the quotient by $\beta$ defines the map

$$\vartheta_0 : \Omega^+_X \to \Omega^+_X,$$

which is the same as the classifying map for the additive theory $\text{CH}_*$.

Both $G_0[\beta, \beta^{-1}]$ and $\text{CH}_*$ admit fundamental classes for equi-dimensional finite type $k$-schemes, functorial with respect to pull-back by l.c.i. morphisms. For $X \in \text{Sch}/k$ of dimension $d$ over $k$, the fundamental class $[X]_G \in G_0(X)[\beta, \beta^{-1}]_d$ is $\beta^d[O_X]$, where $[O_X] \in G_0(X)$ is the class of the structure sheaf. For $\text{CH}_*$, $[X]_{\text{CH}}$ is the cycle class of the scheme $X$; concretely, this is $\sum_{i=1}^{r} n_i [X_i]$ where $X$ has irreducible components $X_1, \ldots, X_r$ and $n_i$ is the length of the local ring $O_{X_i, \eta_i}$ at the generic point $\eta_i \in X_i$.

**Proposition 7.5.** Let $X$ be an equi-dimensional finite type $k$-scheme. Then after canonical identifications $\Omega^*_X \cong G_0[\beta, \beta^{-1}], \Omega^+_X \cong \text{CH}_*$, we have

$$\vartheta_0 ([X]_{CK}) = [X]_G, \quad \vartheta_+ ([X]_{CK}) = [X]_{CH}.$$

**Proof.** Suppose $X$ has dimension $d$ over $k$. For $G$-theory, it follows by the definition of $[X]_{CK}$ that the corresponding element $[X]_{CK} \in CK'_{2d,d}(X)$ maps to $[O_X] \in K'_{2d,d}(X) = G_0(X)$ under the canonical map $\rho : CK' \to K'$. This proves the result for $G$-theory.

For $\text{CH}_*$, the fundamental class $[X]_{\text{CH}}$ is determined by its restriction to each generic point of $X$, so we may assume that $X$ is irreducible and $X_{\text{red}}$ is smooth over $k$. Write $[X]_{\text{CH}} = n [X_{\text{red}}]_{\text{CH}}$. Shrinking $X$ if necessary, we may assume that $[O_X] = n [O_{X_{\text{red}}}]$ in $G_0(X)$, and thus $[X]_{CK} = n [X_{\text{red}}]_{CK}$ in $\Omega^0_{CK}(X) = \Omega^0_{CK}(X_{\text{red}})$. This reduces us to the case of $X \in \text{Sm}/k$. But then we already have the fundamental class $X$ in any oriented cohomology theory $A$, given by

$$[X]_A := p^*(1) \in A^0(X),$$

where $p : X \to \text{Spec} k$ is the structure morphism and $1 \in A^0(k)$ is the unit. Since every natural transformation of oriented cohomology theories preserves the unit and is compatible with pull-back for arbitrary morphisms in $\text{Sm}/k$, we have $\vartheta_+ ([X]_{CK}) = [X]_{CH}$, as desired. □
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