Enhancement of the escape time in metastable states with colored noise

A.Fiasconaro*, D.Valenti, B.Spagnolo

INFM and Dipartimento di Fisica e Tecnologie Relative
Viale delle Scienze - 90128 Palermo, Italy

today

Abstract

We present a study of the escape time from a metastable state in the presence of colored noise, generated by Ornstein-Uhlenbeck process. We analyze the role of the correlated noise and of unstable initial conditions of an overdamped Brownian particle on the enhancement of the average escape time as a function of the noise intensity. We observe the noise enhanced stability (NES) effect for all the initial unstable states and for all values of the correlation time \( \tau_c \) investigated. We can distinguish two dynamical regimes characterized by: (a) a weak correlated noise and (b) a strong correlated noise, depending on the value of \( \tau_c \) with respect to the relaxation time. With increasing \( \tau_c \) we find: (i) a shift of the maximum of the average escape time towards higher values of noise intensity and an enhancement of the value of this maximum; (ii) a broadening of the NES region, which becomes very large in the strong colored noise regime; (iii) in this regime (b), the absence of the peculiar initial condition \( x_c \), which separates the set of the initial unstable states lying into the divergency region from those which give only a nonmonotonic behavior of the average escape time.

1 Introduction

Nonlinear relaxation decay of physical systems from an initial unstable or metastable state involves fundamental aspects of non-equilibrium statistical mechanics. The investigation of nonlinear dynamics and instabilities in systems away from equilibrium in fact has led to some counterintuitive and resonance-like phenomena. Among these we cite the stochastic resonance [1], the resonant activation [2], the noise induced phase transitions [3]. Recent theoretical investigations have shown that the average escape time from metastable states in fluctuating potentials has a nonmonotonic behavior as a function of the noise intensity [4]. This resonance-like behaviour, which contradicts the monotonic behaviour predicted by the Kramers theory [7,8,9], is the NES phenomenon: the stability of metastable or unstable states can be enhanced by the noise and the average life time of the metastable state is greater than the deterministic decay time.

The inclusion of realistic noise sources with a finite correlation time in modelling dynamical systems can impact both the stationary and the dynamic features of nonlinear systems. For metastable thermal equilibrium systems it has been demonstrated that colored thermal noise can substantially modify the diffusive barrier transmission [8]. For bistable systems the colored noise driven escape rate for small and large correlation time has been derived in ref. [10]. A rich and enormous literature on escape processes driven by colored noise was produced in the 80’s. As an example of this literature concerning colored noise we cite the escape from metastable states [11], the MFPT from a marginal state [12] and the decay of an unstable state [13]. In this work we present a study of the average decay time of an overdamped Brownian particle subject to a cubic potential with a metastable state. We focus on the role of different unstable initial conditions and of colored noise in the average escape time. The effect of color on the transient dynamics of the escape process is strictly related to the characteristic time scale of the system, i.e. the relaxation time inside the metastable state \( \tau_r \). For correlation time values less than the relaxation time \( (\tau_c < \tau_r) \), the system ”sees” the noise source as white noise and the dynamical regime of the Brownian particle is like the white noise dynamics with a shift of all curves calculated at different initial positions towards higher values of noise intensity. For strong color, i.e. for values of correlation time larger than the relaxation time \( (\tau_c > \tau_r) \) nontrivial results are obtained. Particularly we obtain: (i) a big shift of the behaviours of the average escape times towards higher noise intensities; (ii) an enhancement of the value of the average escape time maximum and a broadening of the NES region in the plane \((\tau, D)\), which becomes very large for high values of \(\tau_c\); (iii) the absence of the peculiar initial position \( x_c \), found in a previous study [14], which separates the set of the initial unstable states producing divergency from those which give only a nonmonotonic behavior of the average escape time. This is the relevant result of this paper. For all the initial unstable states we find an enhancement of the average escape time.

*E-mail adress: afiasconaro@gip.dft.unipa.it
with respect to the deterministic decay time and in the strong correlated noise regime \( \tau_c >> \tau_r \) a nonmonotonic behavior of the average escape time.

2 The model

The starting point of our study is the Langevin equation

\[
\dot{x} = - \frac{\partial U(x)}{\partial x} + \eta(t)
\]  

(1)

where \( \eta(t) \) is the Ornstein-Uhlenbeck process \[14, 15\]

\[
d\eta = -k \eta dt + k \sqrt{D} dW(t)
\]  

(2)

where \( W(t) \) is the Wigner process giving a Gaussian noise with the usual statistical properties: \( \langle \xi(t) \rangle = 0 \) and \( \langle \xi(t)\xi(t+\tau) \rangle = \delta(\tau) \). The integration of Eq. (2) yields

\[
\eta(t) = \eta(0) e^{-kt} + k \sqrt{D} \int_0^t e^{-k(t-t')} dW(t')
\]  

(3)

which for \( \eta(0) = 0 \) and for \( t \to \infty \) gives

\[
\langle \eta(t) \rangle = 0,
\]

(4)

\[
\langle \eta(t)\eta(t+\tau) \rangle = \frac{kD}{2} e^{-k\tau},
\]

(5)

where \( 1/k = \tau_c \) is the correlation time of the process. The system of stochastic differential equations (1) and (2), which represent a two-dimensional Markovian process, is equivalent to a non-Markovian Langevin equation driven with additive Gaussian correlated noise with \( \eta(t) \) obeying the properties (4) and (5).

It is possible to show that the Eq. (2) for the OU process gives the correct limit \( \lim_{\tau_c \to 0} \eta(t) = \sqrt{D}\xi(t) \), i.e. the white noise term. In fact

\[
\lim_{\tau_c \to 0} \eta(t) = 2 \sqrt{D} \int_0^t \lim_{\tau_c \to 0} \frac{e^{-(t-t')/\tau_c}}{2\tau_c} dW(t')
\]

(6)

where we use the property of \( \delta \) function

\[
\int_{-\infty}^{+\infty} dx \delta(x-x_o) f(x) = f(x_o),
\]

(7)

and the factor 2 disappears because of the integration range. The stationary correlation function of the Ornstein-Uhlenbeck process \[14\] gives in the limit \( \tau_c \to 0 \) the correlation function of the white noise

\[
\lim_{\tau_c \to 0} \langle \eta(t)\eta(t+\tau) \rangle = \lim_{\tau_c \to 0} D e^{-\tau/\tau_c} = D\delta(\tau).
\]

(8)

The potential \( U(x) \) used in Eq. (1) is a cubic one

\[
U(x) = ax^2 - bx^3,
\]

(9)

with \( a = 0.3, b = 0.2 \). The potential profile has a local stable state at \( x_0 = 0 \) and an unstable state at \( x_0 = 1 \) (see Fig [11].

The relaxation time for the metastable state at \( x_0 = 0 \) is

\[
\tau_r = \left[ \frac{d^2U(x)}{dx^2} \right]_{x=0} = 2a,
\]

(10)

which is the characteristic time scale of our system.
Figure 1: The cubic potential $U(x)$ the various initial positions investigated (dots); $x_c$ is the critical initial position found in the case of white noise, which remains only for weak colored noise ($\tau_c < \tau_r$). The absorbing boundary is $x_F = 20$.

3 Results and Comments

The calculations of the average escape time as a function of the colored noise intensity have been performed by averaging over 5000 realizations the stochastic differential equation \[ \text{(1)} \]. The absorbing boundary for the escape process is put on $x_F = 20$. For all the initial unstable states (see Fig.1) and all the correlation times evaluated we find an enhancement of the average escape time with respect to the deterministic time as a function of the noise intensity. We can observe the characteristic behavior of the NES effect for unstable states with white noise in Fig.2a \[ \text{[6]} \] where we also find the comparison with the calculation performed with colored noise with $\tau_c = 0.01$ (Fig.2b). The behaviours of the average escape time as a function of colored noise intensity with the others values of $\tau_c$ are shown in Fig.3.

Figure 2: a) Log-Log plot of the Mean First Passage Time as a function of noise intensity in the case of white noise for the five initial positions investigated in this paper (see Fig.1, namely: $x_o = 1.1 \div 1.9$, with steps of 0.2 \[ \text{[6]} \]); b) MFPT for $\tau_c = 0.01$ and the same initial positions.

We clearly observe two dynamical regimes: (a) weak colored noise ($0 < \tau_c < \tau_r$) and (b) strong colored noise ($\tau_c > \tau_r$). Starting our observation from Fig.3 we see that the qualitative behavior of MFPT with white noise is
recovered with \( \tau_c = 0.1 \), until \( \tau_c \simeq \tau_r = 0.6 \). That is an enhancement of the average escape time with respect to the deterministic decay time with a divergent behaviour for \( x_{max} < x_0 < x_c \) and a non monotonic behavior for \( x_0 \geq x_c \).

Increasing the value of the correlation time \( \tau_c \geq \tau_r \) we observe a shift of the maximum towards higher values of the noise intensity and an enhancement of the value of the average escape maximum. Moreover we observe a broadening of the NES region, which becomes very large (up to twelve order of magnitude) for high values of the correlation time \( \tau_c \) (see Fig.3 with \( \tau_c = 100 \)). The unexpected result is that increasing the correlation time the divergent behaviour tends to disappear, as shown in Fig.3 for \( \tau_c = 1.0 \) and initial position \( x_0 = 1.73 \), and disappear for all the initial positions investigated for \( \tau_c = 10 \) and \( \tau_c = 100 \). For high values of the noise intensity all the

![Figure 3: Log-Log plot of the Mean First Passage Time as a function of noise intensity for the five initial positions investigated (see Fig.1) and for four values of the correlation times \( \tau_c \). In the x-axes we report only the order of magnitude in powers of ten.](image)

graphs show a monotonic decrease behavior as a function of noise intensity collapsing in a unique curve, according to the behavior predicted by the Kramers theory \[7, 8\]. Moreover the slope of this limit curve becomes flatter increasing the correlation time. This means that the NES effect involves more and more orders of magnitude of the noise intensity. The effect of the colored noise is therefore to delay the escape process or in other words to enhance more and more the stability of the metastable state.

### 4 Conclusions

In this work we analyzed the effect of the colored noise, generated by an OU process, on the enhancement of the average escape time in a cubic potential with a metastable state. We analyze different initial unstable states with \( x_0 > x_c \), where \( x_c \) is the crossover between the divergent behaviour and the nonmonotonic behaviour of the average escape time \[4, 6\]. We obtain NES effect for all the initial positions investigated and an enhancement of the NES effect for increasing values of correlation times. The results obtained for a particle moving in a cubic potential are...
quite general, because we always obtain NES effect when a particle is initially located just to the right of a local potential maximum, with a local minimum or metastable state in its left side and the global escape region in its right side.

From a comparison with analogue calculation with white noise, we observe the following similarity and differences for increasing values of the correlation times: 1) the peculiar point $x_c$ found for white noise separating divergent behaviours from non-monotonic behaviour, is present only in the weak colored regime $\tau_c < \tau_r$; 2) all the divergencies present in the case of white noise for all the unstable initial position in the range $x_{max} < x_0 < x_c$ tend to transform in pure NES effect, i.e. without divergencies, for increasing values of $\tau_c$; 3) the region in the plane $(\tau, D)$ of the NES effect increases up to 12 order of magnitude for increasing values of $\tau_c$ (strong colored noise regime).

Nonmonotonic behavior of the mean escape time as a function of noise intensity therefore is a noise-induced effect for nonlinear nonequilibrium systems with metastable states which is enhanced when we consider realistic noise sources with finite correlation time.

In experiments real noise sources are correlated with a finite correlation time. As a consequence the NES effect can be observed at higher noise intensities with respect to the idealized white noise case. Therefore the theoretical results obtained using white noise source are only a good approximation for low values of the correlation time. The enhancement and the shift of the NES region towards higher values of the noise intensity allow to reveal experimentally the NES effect using a suitable correlation time $\tau_c$.

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