Spinning switches on a wreath product

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Abstract

In this paper, we attempt to classify an algebraic phenomenon on wreath products that can be seen as coming from a family of puzzles about switches on the corners of a spinning table. Such puzzles have been written about and generalized since they were first popularized by Martin Gardner in 1979. In this paper, we provide perhaps the fullest generalization yet, modeling both the switches and the spinning table as arbitrary finite groups combined via a wreath product. We classify large families of wreath products depending on whether or not they correspond to a solvable puzzle, completely classifying the puzzle in the case when the switches behave like abelian groups, constructing winning strategies for all wreath product that are $p$-groups, and providing novel examples for other puzzles where the switches behave like nonabelian groups, including the puzzle consisting of two interchangeable copies of the monster group $M$. Lastly, we provide a number of open questions and conjectures, and provide other suggestions of how to generalize some of these ideas further.

1 Overview and preliminaries

This paper is organized into six sections. This section, Section 1 provides a brief history of this genre of puzzles and introduces some of the first approaches to generalizing the puzzle further. Section 2 models these generalizations in the context of the wreath product, and formalizes the notation of puzzles being solvable. Section 3 explores situations where the puzzle does not have a winning strategy, and provides reductions that allow us to prove that entire families of puzzles are not solvable. Section 4 constructs a strategy for switches that behave like $p$-groups, and gives us ways of building strategies from smaller parts. Section 5 provides novel examples of puzzles that do not behave like $p$-groups, but still have winning strategies. Lastly, Section 6 provides further generalizations, and contains dozens of conjectures, open questions, and further directions.

1.1 History

Generalized spinning switches puzzles are a family of closely related puzzles that were first popularized by Martin Gardner in a puzzle called “The Rotating
Table” in the February 1979 edition of his column “Mathematical Games” \cite{2}. Gardner writes that he learned of the puzzle from Robert Tappay of Toronto who “believes it comes from the U.S.S.R.,” a history that is not especially forthcoming.

My preferred version of the puzzle appears in Peter Winkler’s 2004 book *Mathematical Puzzles A Connoisseur’s Collection* \cite{16}

Four identical, unlabeled switches are wired in series to a light bulb. The switches are simple buttons whose state cannot be directly observed, but can be changed by pushing; they are mounted on the corners of a rotatable square. At any point, you may push, simultaneously, any subset of the buttons, but then an adversary spins the square. Show that there is a deterministic algorithm that will enable you to turn on the bulb in at most some fixed number of steps.

(Winkler’s version will be a working example in many parts of this paper, so it is worth keeping in mind. An illustration can be found in Figure 1)

Over the last three decades, various authors have considered generalizations of this puzzle. Here, we build on those results and go further. The first place authors looked to generalize was suggested by Gardner himself. In his March 1979 column the following month, he provided the answer to the original puzzle and wrote

The problem can also be generalized by replacing glasses with objects that have more than two positions. Hence the rotating table leads into deep combinatorial questions that as far as I know have not yet been explored. \cite{3}

In 1993, Bar Yehuda, Etzion, and Moran \cite{17}. took on the challenge and developed a theory of the spinning switches puzzle where the switches behave like roulette wheels with a single “on” state. In this paper we take Gardner’s charge to its logical conclusion and consider switches that behave like arbitrary “objects that have more than two positions”.

Another generalization of this puzzle could look at other ways of “spinning” the switches. In 1995, Ehrenborg and Skinner \cite{1}. did this in a puzzle they call “Blind Bartender with Boxing Gloves”, that analyzed this puzzle while allowing the adversary to use an arbitrary, faithful group action to “scramble” the switches. We analyze our generalized switches within this same context.

This puzzle was re-popularized in 2019 when it appeared in “The Riddler” column from the publication FiveThirtyEight \cite{11}. Shortly after this, in 2022, Yuri Rabinovich synthesized Bar Yehuda and Ehrenborg’s results in a paper that modeled the collection of switches as a vector space over a finite field, and modeled the “spinning” or “scrambling” as a faithful, linear group action on this vector space.

For more background, see Sidana’s \cite{13} detailed overview of the history of this and related problems.


1.2 A solution to Winkler’s Spinning Switches puzzle

We will start by discussing the solution to Winkler’s version of the puzzle because the solution provides some insights and intuition for the techniques that we use later. Before solving the four-switch version of the puzzle, we will make Peter Winkler proud by beginning with a simpler, two-switch version.

Example 1. Suppose that we have two identical unlabeled switches on opposite corners of a square table, as in Figure 1.

Then we have a three-step solution for solving the problem. We start by toggling both switches simultaneously, and allow the adversary to spin the table. If this does not turn on the light, this means that the switches were (and still) are in different states.

Next, we toggle one of the two switches to ensure that the switches are both in the same state. If the light has not turned on, both must be in the off state.

The adversary spins the table once more, but to no avail. We know both switches are in the off state, so we toggle them both simultaneously, turning on the lightbulb.

In order to bootstrap the two-switch solution into a four-switch solution, we must notice two things:

1. First, if we can get two switches along each diagonal into the same state respectively, then we can solve the puzzle by toggling both diagonals (all

Figure 1: An illustration of Winkler’s Spinning Switches puzzle and a two-switch analog.
four switches), followed by both switches in a single diagonal, and lastly both diagonals again. In this (sub-)strategy, toggling both switches along a diagonal is equivalent to toggling a single switch in the two-switch analog.

2. Second, we can get both diagonals into the same state at some point by toggling a switch from each diagonal (two switches on any side of the square), followed by a single switch from one diagonal, followed by again toggling a switch from each diagonal.

We will interleave these strategies in a particular way, following the notation of Rabinovich [10].

Definition 2. Given two sequences $A = \{a_i\}_{i=1}^N$ and $B = \{b_i\}_{i=1}^M$, we can define the interleave operation as

$$A \odot B = (A, b_1, A, b_2, A, \ldots, b_M, A)$$

which has length $(M + 1)N + M = MN + M + N$.

Typically it is useful to interleave two strategies when $A$ solves the puzzle given that the switches are in a particular state, and $B$ gets the switches into that particular state. We also need $A$ not to “interrupt” what $B$ is doing. In the problem of four switches on a square table, $B$ will ensure that the switches are in the same state within each diagonal, and $A$ will turn on the light when that is the case. Moreover, $A$ does not change the state within either diagonal.

Proposition 3. There exists a fifteen-move strategy that guarantees that the light in Winkler’s puzzle turns on.

Proof. We begin by formalizing the two strategies. We will say that the first strategy $S_1$ where we toggle the two switches in a diagonal together will consist of the following three moves:

1. Switch all of the bulbs ($A$).
2. Switch the diagonal consisting of the upper-left and lower-right bulbs ($D$).
3. Switch all of the bulbs ($A$).

We will say that the second strategy $S_2$ where we get the two switches within each diagonal into the same state consists of the following three moves:

1. Switch both switches on the left side ($S$).
2. Switch one switch (1).
3. Switch both switches on the left side ($S$).
Then the 15 move strategy is

$$S_1 \ast S_2 = (A, D, A, S, A, D, A, 1, A, D, A, S, A, D, A)$$  \hspace{1cm} (3)

We will generalize this construction in Theorem 21 which offers a formal proof that this strategy works.

(It is worth briefly noting that $S_1 \ast S_2$ is the fourth Zimin word (also called a sequipower), an idea that comes up in the study of combinatorics on words.)

1.3 Generalizing switches

Two kinds of switches are considered by Bar Yehuda, Etzion, and Moran in 1993 [17]: switches with a single “on” position that behave like $n$-state roulettes ($Z_n$) and switches that behave like the finite field $\mathbb{F}_q$, both on a rotating $k$-gonal table. We generalize this notion further by considering switches that behave like arbitrary finite groups.

**Example 4.** In Figure 3, we provide a schematic for a switch that behaves like the symmetric group $S_3$. It consists of three identical-looking parts that need to be arranged in a particular order in order for the switch to be on.

We could also construct a switch that behaves like the dihedral group of the square, $D_8$. This switch might look like a flat, square prism that can slot into a square hole, such that only one of the $|D_8|=8$ rotations of the prism completes the circuit.

**Note 5.** One subtlety of using a group $G$ to model a switch is that both the “internal state” of a switch itself and the set of “moves” or changes are modeled by $G$. Therefore it may be useful to think of the state as the underlying set of $G$ where the moves act via a right group action of $G$ on itself.

The reason that it is appropriate to use a group to model a switch is because groups have many of the properties we would expect in a desirable switch.

**Note 6.** The axioms for a group $(G, \cdot)$ closely follow what we would expect from a switch.

1. (Closure) The group $(G, \cdot)$ is equipped with a binary operation, $\cdot : G \times G \to G$. That is, for all pairs of elements $g_1, g_2 \in G$ their product is in $G$

$$g_1 \cdot g_2 \in G.$$  \hspace{1cm} (4)

In the context of switches, this means that if the switch is in some state $g_1 \in G$ and the puzzle-solver applies the move $g_2 \in G$ to it, then the resulting state $g_1 \cdot g_2 \in G$ is in the set of possible states.
Figure 2: Part (a) shows a simple schematic for the components of a switch that behaves like $S_3$, the symmetric group on three letters. The three rectangles can be permuted arbitrarily, but only configuration (b) completes the circuit. All other configurations fail to complete the circuit (e.g., (c)).

2. (Identity) There exists an element $\text{id}_G \in G$ such that for all $g \in G$,

$$\text{id}_G \cdot g = g \cdot \text{id}_G = g.$$  \hspace{1cm} (5)

This axiom is useful because it means that the puzzle-solver can “do nothing” to a switch and leave it in whatever state it is in. Because the identity is a distinguished element in $G$, we will also use the convention that $\text{id}_G$ is the “on” or “winning” state for a given switch. (It is worth noting that all of the arguments work basically the same way regardless of which element is designated as the on state.)

3. (Inverses) For each element $g \in G$ there exists an inverse element $g^{-1} \in G$ such that

$$g \cdot g^{-1} = g^{-1} \cdot g = \text{id}_G.$$ \hspace{1cm} (6)

This axiom states that no matter what state a switch is in, there is a move that will transition it into the on state.

4. (Associativity) Given three elements $g_1, g_2, g_3 \in G$,

$$(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$$ \hspace{1cm} (7)

This axiom is not strictly necessary for modeling switches, but as we will see in a later definition, it gives us a convenient way to describe the conditions for a winning strategy. (In Subsection 6.3 we briefly discuss dropping the associativity axiom by considering switches that behave like quasigroups with identity.)
1.4 Generalizing spinning

We can also consider generalizations of “spinning” the switches. In particular, we will adopt the generalization from Ehrenborg and Skinner’s 1995 paper, which use arbitrary faithful group actions to permute the switches. In particular, they provide a criterion that determines which group actions yield a winning strategy in the case of a given number of “ordinary” switches (those that behave like $\mathbb{Z}_2$). Rabinovich 10 stretches these results a bit further and looks at faithful linear group actions on collections of switches that are modeled as a finite-dimensional vector space over a finite field. We build on this result in the context of more general switches.

2 A wreath product model

Recall that Peter Winkler’s Spinning Switches puzzle consists of four two-way switches on the corners of a rotating square table. The behavior of the switches are naturally modeled as $\mathbb{Z}_2$, and the rotating table is modeled as the cyclic group $C_4$. The abstraction that takes these two groups and creates a model for the underlying puzzle is the wreath product: that is, Winkler’s puzzle behaves like the wreath product of $\mathbb{Z}_2$ by $C_4$.

2.1 Modeling generalized spinning switches puzzles

We do not evoke wreath products arbitrarily: we use them because they are the right abstraction to model a generalized spinning switches puzzle where $G$ describes the behavior of the switches, $\Omega$ describes the positions of the switches, and the action of $H$ on $\Omega$ models the ways the adversary can permute the switches.

**Definition 7** (12). Let $G$ and $H$ be groups, let $\Omega$ be a finite $H$-set, and let $K = \prod_{\omega \in \Omega} G_\omega$, where $G_\omega \cong G$ for all $\omega \in \Omega$. Then the **wreath product** of $G$ by $H$ denoted by $G \wr H$, is the semidirect product of $K$ by $H$, where $H$ acts on $K$ by $h \cdot (g_\omega) = g_{h^{-1} \omega}$ for $h \in H$ and $g_\omega \in G_\omega$. The normal subgroup $K$ of $G \wr H$ is called the **base** of the wreath product.

The group operation is $(k,h) \cdot (k',h') = (k(h \cdot k'),hh')$

An element $(k,h) \in G \wr H$ represents a turn of the game: The puzzle-solver chooses an element of the base $k \in K$ to indicate how they want to modify each of their switches and then the adversary chooses $h \in H$ and acts with $h$ on $\Omega$ to permute the switches.

**Example 8.** Consider the setup in the Winkler’s Spinning Switches the puzzle, which consists of two-way switches ($\mathbb{Z}_2$) on the corners of a rotating square, $C_4 \cong \{0^\circ, 90^\circ, 180^\circ, 270^\circ\}$. The game itself corresponds to the wreath product $\mathbb{Z}_2 \wr C_4$. We will use the convention that the base of the wreath product, $K$, is ordered upper-left, upper-right, lower-right, lower-left, and that the group action is specified by degrees in the clockwise direction.
Consider the following two turns:

1. During the first turn, the puzzle-solver toggles the upper-left and lower-right switches, and the adversary rotates the table 90° clockwise. This is represented by the element

   \[ ((1, 0, 1, 0), 90°) \in \mathbb{Z}_2 \wr C_4. \]  

   (8)

2. During the second turn, the puzzle-solver toggles the upper-left switch, and the adversary rotates the table 90° clockwise. This is represented by the element

   \[ ((1, 0, 0, 0), 180°) \in \mathbb{Z}_2 \wr C_4. \]  

   (9)

As illustrated in Figure 3, the net result of these two turns is the same as a single turn where the puzzle-solver toggles the upper-left, upper-right, and lower-left switches and the adversary rotates the board 270° clockwise.

The multiplication under the wreath product agrees with this:

\[
((1, 0, 1, 0), 90°) \cdot ((1, 0, 0, 0), 180°) = ((1, 0, 1, 0) + 90° \cdot (1, 0, 0, 0), 90° + 180°) \\
= ((1, 0, 1, 1), 270°)
\]

As suggested earlier, it is occasionally useful to designate a particular state of the switches as the winning state. We will use the convention that the lightbulb turns on when all of the switches are equal to the identity, that is, \( \mathrm{id}_K \in K \).

It is worth noting, however, that the existence of a winning strategy does not depend on a particular choice of the winning state. Instead, we will see that a winning strategy is equivalent to a choice of moves that will walk over all of the possible configuration states, regardless of the choice of the adversary’s spin.

2.2 Surjective strategy

We will begin by formalizing the notation of a winning strategy in a generalized spinning switches puzzle. Informally, this is a sequence of moves that the puzzle-solver can make that will put the switches into every possible state, which
ensures that the winning state is reached regardless of the initial (hidden) state of the switches.

**Definition 9.** A **surjective strategy** for $G \wr H$ is a finite sequence of elements in the base $K$, \( \{k_i \in K\}_{i=1}^{N} \), such that for every sequence of elements in $H$, \( \{h_i \in H\}_{i=1}^{N} \),

\[
p\left(\bigwedge_{m_0}^{m_1} (k_1, h_1) \cdot (k_2, h_2) \cdots (k_i, h_i), \bigwedge_{m_2}^{m_3} (k_1, h_1) \cdot (k_2, h_2) \cdots (k_i, h_i) \right) = K.
\]

where $p : G \wr H \to K$ is the projection map from the wreath product onto its base.

This definition is useful because it puts the problem into purely algebraic terms. It is also useful because it abstracts away the initial state of the switches: regardless of the initial state $k \in K$, the existence of a surjective strategy means that its inverse $k^{-1} \in K$ appears in the sequence. (This follows the convention that $\text{id}_K \in K$ is designated as the winning state. If $k'$ is chosen to be the winning state, then the sequence must contain $k^{-1}k'$.)

**Proposition 10.** A finite sequence of moves is guaranteed to reach the winning state if and only if it is a surjective strategy.

**Proof.** Without loss of generality, we will say that the winning state for the switches is $\text{id}_K$.

We will begin by assuming that \( \{k_i \in K\}_{i=1}^{N} \) sequence of moves is guaranteed to reach the winning state, $\text{id}_K$. In the puzzle, we have an initial (hidden) state, $k \neq \text{id}_K$. Therefore, after the $i$-th move, the wreath product element that represents the state of the switches is

\[
p((k, \text{id}_H) \cdot (k_1, h_1) \cdot (k_2, h_2) \cdots (k_i, h_i)) = k \cdot p((k_1, h_1) \cdot (k_2, h_2) \cdots (k_i, h_i)),
\]

where the equality is due to associativity of the wreath product. We can factor out the first term because the “spin” is $\text{id}_H$, which acts trivially: $((k, \text{id}_H) \cdot (k', h') = (kk', h')$.

Say that $j$ is the index at which the puzzle-solver gets the switches into the winning state. Then

\[
k \cdot p((k_1, h_1) \cdot (k_2, h_2) \cdots (k_j, h_j)) = \text{id}_K
\]

\[
p((k_1, h_1) \cdot (k_2, h_2) \cdots (k_j, h_j)) = k^{-1}.
\]

Since $k \neq$ is arbitrary, there must exist such a $j$ for every $k \in K$ and adversarial sequence $\{h_i \in H\}_{i=1}^{N}$. This means that the projection of the sequence of partial products of $\{k_i \in K\}_{i=1}^{N}$ is a surjection onto $K$ and therefore is a surjective strategy.

Conversely, if $K$ is a surjective strategy, then for any initial state $k$ and sequence of adversarial moves $\{h_i \in H\}_{i=1}^{N}$, there exists some $j$ that satisfies Equation [13], and therefore reaches a winning state.

\[
\square
\]
It is also worth noting that this model can be thought of as a random model or an adversarial model: the sequence \( \{ h_i \in H \} \) can be chosen randomly at any point, or it can be chosen deterministically after the sequence \( \{ k_i \in K \} \) is specified.

### 2.3 Bounds on the length of surjective strategies

One useful consequence of this definition is that it is quite straightforward to prove certain propositions. For example, the minimum length for a surjective strategy has a simple lower bound.

**Proposition 11.** Every surjective strategy \( \{ k_i \in K \}_{i=1}^N \) is a sequence of length at least \( |K| - 1 \).

**Proof.** This follows from an application of the Pigeonhole Principle, because the set

\[
\{ \text{id}_{G \wr H}, (k_1, h_1), (k_1, h_1) \cdot (k_2, h_2), \ldots, (k_1, h_1) \cdot (k_2, h_2) \cdots (k_N, h_N) \}
\]

has at most \( N + 1 \) elements. In order for the projection to be equal to \( K \),

\[
p(\{ \text{id}_{G \wr H}, (k_1, h_1), (k_1, h_1) \cdot (k_2, h_2), \ldots, (k_1, h_1) \cdot (k_2, h_2) \cdots (k_N, h_N) \}) = K,
\]

it must be the case that \( N + 1 \geq |K| \). Therefore \( N \geq |K| - 1 \).

Minimal length surjective strategies are common, so we give them a name.

**Definition 12.** A **minimal surjective strategy** for \( G \wr H \) is a surjective strategy of length \( N = |K| - 1 \).

In practice, every wreath product known to the author to have a surjective strategy also has a known minimal surjective strategy. In Section 6, we ask whether this property always holds.

### 3 Reductions

In this section, we use three techniques to develop examples of generalized spinning switches puzzles that do not have surjective strategies: directly, by a reduction on switches, or by a reduction on spinning.

#### 3.1 Puzzles known to have no surjective strategies

Our richest collection of known puzzles without surjective strategies comes from a theorem of Rabinovich, which models switches as a vector space over a finite field.

**Theorem 13.** Assume that a finite “spinning” group \( H \) acts linearly and faithfully on a collection of switches that behave like a vector space \( V \) over a finite field \( \mathbb{F}_q \) of characteristic \( p \). Then the resulting puzzle has a surjective strategy if and only if \( H \) is a p-group.
It is worth noting that Rabinovich’s switches are less general than arbitrary finite groups, but the “spinning” is more general: in addition to permuting the switches, the group action might add linear combinations of them as well.

**Example 14.** By the theorem of Rabinovich [10], the game $\mathbb{Z}_2 \wr \mathbb{C}_3$ does not have a surjective strategy. In Rabinovich’s notation, the vector space of switches is $\mathbb{Z}_2^3$ over the field $\mathbb{F}_2 = \mathbb{Z}_2$, and $\mathbb{C}_3$ has 3 elements and therefore is not a 2-group.

The wreath product $\mathbb{Z}_2 \wr \mathbb{C}_3$ is perhaps the simplest example of a generalized spinning switches puzzle without a surjective strategy, so we will continue to use it as a basis of future examples.

### 3.2 Reductions on switches

With Theorem 13 providing a family of wreath products without surjective strategies, we now introduce a theorem that allows us to describe large families of wreath products that also do not have surjective strategies.

**Theorem 15.** If $G \wr H$ does not have a surjective strategy and there exists a group $G'$ and a surjective homomorphism $\varphi: G' \rightarrow G$, then $G' \wr H$ does not have a surjective strategy.

**Proof.** We will prove the contrapositive, and suppose that $G' \wr H$ has base $K'$ and a surjective strategy $\{k'_i \in K'\}_{i=1}^N$.

The homomorphism $\varphi: G' \rightarrow G$ extends coordinately to $\varphi: K' \rightarrow K$, which further extends in the first coordinate to $G \wr H$: $\varphi(k, h) := (\varphi(k), h)$.

It is necessary to verify that $\varphi: G' \wr H \rightarrow G \wr H$ is indeed a homomorphism.

\[
\varphi((k'_\alpha, h_\alpha)) \cdot \varphi((k'_\beta, h_\beta)) = (\varphi(k'_\alpha), h_\alpha) \cdot (\varphi(k'_\beta), h_\beta)
\]
\[
= (\varphi(k'_\alpha)(h_\alpha \cdot \varphi(k'_\beta)), h_\alpha h_\beta)
\]
\[
= (\varphi(k'_\alpha)\varphi(h_\alpha \cdot k'_\beta), h_\alpha h_\beta)
\]
\[
= (\varphi(k'_\alpha(h_\alpha \cdot k'_\beta)), h_\alpha h_\beta)
\]
\[
= \varphi((k'_\alpha(h_\alpha \cdot k'_\beta), h_\alpha h_\beta))
\]
\[
= \varphi((k'_\alpha, h_\alpha) \cdot (k'_\beta, h_\beta))
\]

Therefore the sequence $\{\varphi(k'_i) \in K\}_{i=1}^N$ is a surjective strategy on $G \wr H$, because the quotient map $\varphi: G' \rightarrow G$ (and thus $\varphi: K' \rightarrow K$) is injective. □

**Example 16.** We know that $\mathbb{Z}_2 \wr \mathbb{C}_3$ does not have a surjective strategy. This means that $\mathbb{Z}_6 \wr \mathbb{C}_3$ does not have a surjective strategy either, as illustrated in Figure 4.
3.3 Reductions on spinning

We can do two similar reductions on the “spinning” group of a wreath product. These theorems say that if a given wreath product $G \wr H$ does not have a surjective strategy, then a similar wreath product $G \wr H'$ with a “more complicated” spinning group $H'$ will not have a surjective strategy either.

**Theorem 17.** If $G \wr H$ does not have a surjective strategy, and $\phi: H \hookrightarrow H'$ is an embedding of $H$ into $H'$, then $G \wr H'$ does not have a surjective strategy.

**Proof.** Again we will prove the contrapositive. Assume that $G \wr H'$ does have a surjective strategy, $\{k_i\}_{i=1}^N$. Then by definition, for any sequence $\{h'_i\}_{i=1}^N$, the projection of the sequence

$$p((k_1, h'_1) \cdot (k_2, h'_2) \cdots (k_i, h'_i))_{i=1}^N = K,$$

and in particular this is true when $h'_i$ is restricted to be in the subgroup $\text{Im}(\phi) \leq H'$. Thus a surjective strategy for $G \wr H'$ is also a valid surjective strategy for $G \wr H$.

**Example 18.** Consider the wreath product $\mathbb{Z}_2 \wr_{\Omega_6} C_6$ where $\Omega_6$ consists of six switches on the corners of a hexagon as illustrated in Figure 5. We know that $\mathbb{Z}_2 \wr_{\Omega_6} C_3$ does not have a surjective strategy, by Theorem 13, therefore $\mathbb{Z}_2 \wr_{\Omega_6} C_6$ cannot have a surjective strategy either, by Theorem 17.

In the above example, we noted that $\mathbb{Z}_2 \wr_{\Omega_6} C_6$ does not have a surjective strategy because $\mathbb{Z}_2 \wr_{\Omega_6} C_3$ is known not to have one. However, there is another obstruction: the fact that the generalized spinning switches puzzle $\mathbb{Z}_2 \wr C_3$ on a triangle is known not to have a surjective strategy. That is, we cannot guarantee that the upright triangle (which is formed by taking “every other” switch) is ever in the all “on” state.
The following theorem abstracts this idea.

**Theorem 19.** Suppose that $H$ and $H'$ are groups, $\varphi : H \hookrightarrow H'$ is an embedding of $H$ into $H'$, and let

$$\text{Orb}(\omega) = \{\omega \cdot a : a \in \text{Im}(\varphi)\} \subseteq \Omega$$

be the (right) orbit of $\omega \in \Omega$ under $\text{Im}(\varphi)$. If $G \wr \text{Orb}(\omega) H$ does not have a surjective strategy, then $G \wr \Omega H'$ cannot have a surjective strategy either.

**Proof.** We start by making the contrapositive assumption that $G \wr \Omega H'$ has a surjective strategy $\{k_i \in K\}_{i=1}^N$, and we consider the projection $p_\omega : K \rightarrow K_\omega$ where

$$K = \prod_{\omega' \in \Omega} G_{\omega'} \quad \text{and} \quad K_\omega = \prod_{\omega' \in \text{Orb}(\omega)} G_{\omega'}.$$

Then $\{p_\omega(k_i) \in K_\omega\}_{i=1}^N$ is a surjective strategy for $G \wr \text{Orb}(\omega) H$, since the projection is a surjective map. \qed

**Example 20.** We know that $\mathbb{Z}_2 \wr C_3$ does not have a surjective strategy. This means that $\mathbb{Z}_2 \wr C_6$ does not have a surjective strategy either, as illustrated in Figure 6.

Now that we have proven that large families of wreath products do not have surjective strategies, it is worthwhile to construct families of wreath products that do have surjective strategies.

### 4 Surjective strategies on $p$-groups

In this section, we will develop a broad family of surjective strategies, namely those where $G$ and $H$ (and thus $G \wr H$) are $p$-groups.
Figure 6: We know that $\mathbb{Z}_2 \wr \Omega C_6$ cannot have a surjective strategy, because that would imply a surjective strategy for $\mathbb{Z}_2 \wr \Omega' C_3$, where $\Omega'$ is the orbit of the top switch rotations of multiples of $120^\circ$.

4.1 Surjective strategy decomposition

Our first constructive theorem provides a technique that can be used to construct surjective strategies for switches that behave like a group $G$ in terms of a normal group and its corresponding quotient group.

**Theorem 21.** The wreath product $G \wr H$ has a surjective strategy if there exists a normal subgroup $N \trianglelefteq G$ such that both $N \wr H$ and $G/N \wr H$ have surjective strategies.

**Proof.** Let $$S_{G/N} = \{k_i^{G/N} \in K_{G/N}\} \quad \text{and} \quad S_N = \{k_i^N \in K_N\}$$ denote the surjective strategies for $G/N \wr H$ and $N \wr H$ respectively.

We ultimately would like to interleave these two strategies, but $k_i^{G/N} \notin K_G$. To find the appropriate analog, we partition $G$ into $[G : N] = m$ right cosets of $N$, $$G = Ng_1 \sqcup Ng_2 \sqcup \cdots \sqcup Ng_m, \quad (19)$$ each with a chosen representative in $G$. Now we define a map $r : G/N \to G$ that chooses the chosen representative of the coset, and extends coordinatewise. We use this map to define a sequence $S = \{r(k_i^{G/N}) \in K_G\}$.

We claim that these two sequences interleaved, $S_N \circ S$, forms a surjective strategy for $G \wr H$. To prove this claim, we observe two facts:

1. Multiplying by elements of $S_N$ will not change cosets and will walk through every element of the current coset.

2. Multiplying by elements of $S$ will walk through all cosets.
Therefore the interleaved sequence will walk through all elements of each coset, and thus is surjective onto $K$. □

4.2 Construction of surjective strategies on $p$-groups

With the decomposition from Theorem 21 established, we can now construct a surjective strategy on all finite $p$-groups.

**Theorem 22.** If $H$ is a finite $p$-group that acts faithfully on $\Omega$, then the wreath product $G \wr H$ has a surjective strategy whenever $|G| = p^n$ for some $n$.

**Proof.** We will use the fact that if $|G| = p^n$, then either $G \cong \mathbb{Z}_p$ or $G$ is not simple.

If $n = 1$, then $G \cong \mathbb{Z}_p \cong \mathbb{F}_p$, so there exists a surjective strategy by Theorem 13. This is because $H$ permutes the coordinates of $V = \mathbb{F}_p^{[\Omega]} \cong K$, and so it is a linear action on the vector space, and so it has a surjective strategy by Theorem 13.

Otherwise, $G$ is not simple. This means that $G$ has a proper normal subgroup $N$ of order $|N| = p^t$ (with $0 < t < n$) and a quotient $G/N$ with order $|G/N| = p^{n-t}$. Therefore, by induction on the exponent, whenever $G$ and $H$ (and thus $G \wr H$) are $p$-groups, $G \wr H$ has a surjective strategy. □

This resolves the situation for switches that behave like $p$-groups on spinning groups that act faithfully and behave like $p$-groups. This allows us to fully classify the case of abelian switches.

4.3 A classification of puzzles with abelian switches

**Theorem 23.** If $G$ is an abelian group, and $H$ acts faithfully on $\Omega$, then $G \wr \Omega H$ has a surjective strategy if and only if $G$ and $H$ are both $p$-groups for the same prime $p$.

**Proof.** One direction is clear: if $G$ and $H$ are both $p$-groups for the same prime $p$, then $G \wr H$ has a surjective strategy by Theorem 22.

For the other direction, we will consider the contrapositive assumption, split into three cases, each time assuming that $G$ and $H$ are not both $p$-groups for the same prime $p$.

First, assume that $G$ is a $p$-group, but $H$ is not a $p$-group. Then by Sylow’s first theorem, there exists a subgroup $\hat{H} \leq H$ that is a $q$-group for some prime $q \neq p$. By Theorem 13, since $q \neq p$, $G \wr \hat{H}$ does not have a surjective strategy, thus it follows from the reduction on spinning given in Theorem 17 that $G \wr H$ does not have a surjective strategy either.

Second, assume that $H$ is a $p$-group, but $G$ is not a $p$-group. Similarly, by Sylow’s first theorem, there exists a subgroup $\hat{G} \leq G$ that is a $q$-group for some prime $q \neq p$. Since $G$ is abelian, $\hat{G}$ is a normal subgroup. By Theorem 13, since $q \neq p$, $G \wr H$ does not have a surjective strategy, thus it follows from the
reduction on switches given in Theorem \ref{thm:switch} that \( G \wr H \) does not have a surjective strategy either.

Lastly, assume that there does not exist any \( p \) such that \( G \) is a \( p \)-group or \( H \) is a \( p \)-group. Then \(|G|\) and \(|H|\) must have multiple prime divisors, so \( G \) has a normal subgroup, \( \overline{G} \), that is a \( q \)-group for some prime \( q \), and \( H \) has a normal subgroup \( \overline{H} \) that is an \( r \)-group for some prime \( r \neq q \). \( G \wr \overline{H} \) does not have a surjective strategy by Theorem \ref{thm:non-surjective} so \( G \wr H \) does not have a surjective strategy by Theorem \ref{thm:switch} so \( G \wr H \) does not have a surjective strategy by Theorem \ref{thm:non-surjective}.

4.4 A folklore conjecture

Here we note a conjecture from folklore, which—if true—implies that we have almost solved the problem in its full generality.

**Conjecture 24** (Folklore). *Almost all groups are 2-groups.*

There are both computational and theoretical bases for this conjecture. According to the On-Line Encyclopedia of Integer Sequences \cite{oeis}, there are \( A_{000001}(2^{10}) = 49487367289 \) groups of order \( 2^{10} \) and there are \( A_{063756}(2^{11} - 1) = 49910536613 \) groups of order less than \( 2^{11} \). This means that more than 99.15\% of the groups of order less than \( 2^{11} \) are of order \( 2^{10} \).

If this conjecture is true, then most types of switches have surjective strategies on most kinds of faithful finite group actions. Of course, while most finite groups may be 2-groups, most mathematicians are more interested in the other finite groups. This next section develops two families of examples of surjective strategies where the switches do not behave like \( p \)-groups.

5 Surjective strategies on other wreath products

Other authors have given surjective strategies for various configurations of generalized spinning switches strategies, but in all of these examples, the wreath products themselves are \( p \)-groups: that is, \(|G \wr \Omega H| = |G|^{|\Omega|} \cdot |H|\), where \( H \) acts faithfully. In this section we introduce two families of wreath products that have surjective strategies, but are not \( p \)-groups.

5.1 The trivial wreath product, \( G \wr 1 \)

The simplest—and least interesting—way to construct a wreath product with a surjective strategy is to remove the adversary (or randomness) altogether, by letting the spinning group be the trivial group \( H = 1 \). Additionally we will consider the case where \(|\Omega| = 1\), so there is only one switch. Because the adversary cannot “spin” the switches at all, the puzzle-solver has perfect information the entire time. We will see that whenever \( G \) is a finite group, \( G \wr 1 \cong G \) has not just one surjective strategy, but many.
Proposition 25. The wreath product $G \wr 1$ has $(|G| - 1)!$ minimal surjective strategies.

Proof. There are $(|G| - 1)!$ permutations of $G \setminus \{id_G\}$, and we claim each one corresponds to a minimal surjective strategy. Namely, if $(k_1, k_2, \ldots, k_{|G|-1})$ is such a permutation, then the sequence $\{k'_i\}_{i=1}^{[G]-1}$ where $k'_1 = k_1$ and $k'_i = k_{i-1}^{-1}k_i$ is a surjective strategy on $G \wr 1$.

Then we claim by induction that $(k'_1, id) \cdot (k'_2, id) \cdots (k'_j, id) = (k_j, id)$. By construction, the base case is true when $j = 1$. If the claim holds up to $j - 1$, then

$$
(k'_1, id) \cdot (k'_2, id) \cdots (k'_{j-1}, id)(k'_j, id) = (k_{j-1}, id)(k_{j-1}^{-1}k_j, id) = (k_j, id),
$$

as desired. Thus the projection of the partial products is

$$
p\left(\{id_{G\wr 1}, (k'_1, id), (k'_2, h_1), \cdots, (k'_j, h_1)\} \cdot (k'_2, h_2) \cdots (k'_j, id)\right)\equiv (k_{j-1}, id)
$$

$$
= p\left(\{id_{G\wr 1}, (k_1, id), (k_2, id), \cdots, (k'_{[G]-1}, id)\}\right)
$$

$$
= \{id_{G\wr 1}, k_1, k_2, \cdots, k_{[G]-1}\} = K,
$$

where $\{k_1, k_2, \ldots, k_{|G|-1}\}$ spans $G \setminus \{id_G\} \cong K \setminus \{id_K\}$ by assumption. 

5.2 Two interchangeable groups generated by involutions

In this section, we will construct a surjective strategy for the generalized spinning switches puzzle $G \wr H$ that consists of two switches each behaving like a group $G$, generated by involutions, together with a nontrivial spinning action $H$. (See, for example, Figure 2 which illustrates switches that behave like $S_3 = \langle (12), (13) \rangle$)

This strategy relies on the fact that because each generator is its own inverse, applying a generator to the first switch or the second switch has no effect on their difference.

This surjective strategy has two parts. The first part ensures that the two switches have every possible difference. The second part shows that we can get the first switch (with respect to the projection onto $K$) to take on every possible value without changing the difference between the switches.

Theorem 26. Suppose that $G$ is a finite group that can be generated by involutions. Then the generalized spinning switches puzzle $G \wr C_2$ consisting of two interchangeable copies of $G$ has a surjective strategy.
Proof. We start by writing $G$ in terms of its generators: $G = \langle t_1, t_2, \ldots, t_N \rangle$, where $t_i^{-1} = t_i$. Because this is the generating set, there exists a finite sequence of transpositions $(t_{i_1}, t_{i_2}, \ldots, t_{i_M})$ such that the partial products of the sequence generate $G$: 

$$G = \{ \text{id}_G, t_1, t_1t_2, \ldots, t_1t_2\ldots t_{i_M} \}.$$  

(24) 

We develop the strategy in two parts. First, we provide a strategy $A = \{ \alpha_i \in K \}$ such that for any adversarial sequence $\{ h_i \in H \}$ and element $g \in G$, there exists an $i \geq 0$ such that the $i$-th partial product, $(g_{i,1}, g_{i,2}) = p((\text{id}_K, \text{id}_H) \cdot (\alpha_1, h_1) \cdot (\alpha_2, h_2) \cdots (\alpha_i, h_i))$, has a difference of $g$. That is, $g_{i,1}g_{i,2}^{-1} = g$.

To do this, we define $\alpha_j = (t_{i_j}, \text{id}_G) \in K$, and notice that the difference of the coordinates is the same whether we add $\alpha_j$ or $(180^\circ) \cdot \alpha_j$ to an element $(g_1, g_2) \in K$:

$$g_1(g_2t_{i_j})^{-1} = g_1t_{i_j}^{-1}g_2^{-1} = (g_1t_{i_j})g_2^{-1}$$  

(25) 

Because the partial products of $t_{i_j}$ cover $G$, there exists some $t_{i_1}t_{i_2}\ldots t_{i_k} = g_1^{-1}gg_2$, so that

$$g_1(t_{i_1}t_{i_2}\ldots t_{i_k})g_2^{-1} = g_1(g_1^{-1}gg_2)g_2^{-1} = g,$$  

(26) 

as desired.

Next, we give a strategy $B = \{ \beta_i \}$ such that for any adversarial sequence $\{ h_i \in H \}$ and element $g \in G$, there exists an $i \geq 0$ such that the $i$-th partial product,

$$p((\text{id}_K, \text{id}_H) \cdot (\alpha_1, h_1) \cdot (\alpha_2, h_2) \cdots (\alpha_i, h_i))$$  

(27) 

has a first coordinate equal to $g$.

To do this, we define $\beta_j = (t_{i_j}, t_{i_j})$. This strategy is invariant up to actions of $H$, so we can see that regardless of the initial state $(g_1, g_2) \in K$, there exists some $k$ such that

$$\beta_1\beta_2\cdots \beta_k = (g_1^{-1}g, g_1^{-1}g)$$  

(28) 

and therefore $(g_1, g_2)\beta_1\beta_2\cdots \beta_k = (g, g_2g_1^{-1}g)$, as desired.

It is important to note that applying $\beta_j$ does not affect the difference:

$(g_1, g_2)\beta_j = (g_1t_{i_j}, g_2t_{i_j})$ has a difference of $(g_1t_{i_j})(g_2t_{i_j})^{-1} = g_1t_{i_j}t_{i_j}^{-1}g_2^{-1} = g_1g_2^{-1}$.

Now by interleaving these two strategies, we see that the partial products of $B \circ A$ hit every possible first letter and every possible difference for every first letter, therefore the projection of the set partial products of $B \circ A$ covers $K$ for all adversarial sequences, so $B \circ A$ is a surjective strategy.

We will illustrate this idea explicitly letting $G$ be the smallest nonabelian group of composite order, the symmetric group on three letters $S_3$, which is isomorphic to the dihedral group of the triangle, $D_6$.

**Example 27.** Note that $S_3 = \langle (12), (13) \rangle$ is generated by involutions, and that the partial products of the sequence $\langle (12), (13), (12), (13), (12) \rangle$ cover $S_3$. 

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As a bit of notation, for a permutation $\pi \in S_3$, let $\pi_1 = (\pi, \text{id}_{S_3}) \in K$ and $\pi_2 = (\pi, \pi) \in K$, corresponding to sequences B and A respectively in the above theorem. Then the following is a (minimal) surjective strategy on $S_3 \wr C_2$:

$$(12)_2(13)_2(12)_2(13)_2(12)_2$$

$$(13)_1$$

$$(12)_1$$

$$(12)_1$$

$$(12)_1$$

$$(13)_1$$

$$(12)_1$$

$$(12)_1$$

$$(12)_1$$

$$(12)_1$$

It is natural to ask which spinning groups $H$ correspond to to a generalized spinning switches puzzle $S_n \wr H$ with a surjective strategy. We can use a reduction on switches to rule out a large number of these.

**Proposition 28.** For $n > 1$, $S_n \wr H$ does not have a surjective strategy whenever $|H|$ is not a power of 2.

**Proof.** The alternating group $A_n$ is an index 2 subgroup of $S_n$, so $A_n$ is normal, and $S_n/A_n \cong \mathbb{Z}_2$. Since we know that $\mathbb{Z}_2 \wr H$ has no surjective strategy when $|H|$ is not a power of 2, by the reduction in Theorem 15, $S_n \wr H$ does not have a surjective strategy. \qed

Many groups are generated by transpositions, including 22 of the 26 sporadic simple groups and the alternating groups $A_5$ and $A_n$ for $n > 9$, as shown by Mazurov and Nuzhin respectively.

**Theorem 29.** [4] Let $G$ be one of the 26 sporadic simple groups. The group $G$ cannot be generated by three involutions two of which commute if and only if $G$ is isomorphic to $M_{11}$, $M_{22}$, $M_{23}$, or $M_{24}$.

**Theorem 30.** [5] The alternating group $A_n$ is generated by three involutions, two of which commute, if and only if $n \geq 9$ or $n = 5$.

Nuzhin provides other families of groups that are generated by three involutions, two of which commute, in subsequent papers [6, 7, 8].

Thus, we have characterized finite wreath products with surjective strategies in many cases: wreath products that are $p$-groups, trivial wreath products, and wreath products $G \wr C_2$ where $G$ is generated by involutions. In the next section, we provide even more general constructions and ask more specific questions.
6 Generalizations and open questions

In this section, we provide conjectures and suggest open questions about the structure of surjective strategies when they exist, further generalizations of spinning switches puzzles, and, lastly, introduce a notion of an infinite surjective strategy for infinite wreath products.

The ultimate open question is a full classification of finite wreath products with surjective strategies.

**Open Question 31.** What finite wreath products, $G \wr H$, have a surjective strategy?

6.1 Switches generated by elements of prime power order

In Theorem 26, we constructed a strategy for $G \wr C_2$, when $G$ can be generated by elements of order 2. We conjecture that there is a broader construction for the case where the adversary can act on the switches with a group of order $2^\ell$.

**Conjecture 32.** When $G$ is a finite group generated by involutions, and $H$ is a 2-group that acts faithfully on the set of switches, there exists a surjective strategy for $G \wr H$.

We can also ask about three switches that are generated by elements of order 3 on the corners of a triangular table. In particular, the alternating group is such a group, and we conjecture that it has a surjective strategy.

**Conjecture 33.** There exists a surjective strategy for $A_n \wr C_3$.

Putting these two conjectures together, we boldly predict a large family of wreath products with surjective strategies.

**Conjecture 34.** If $G$ can be generated by elements of order $p^n$, and $H$ is a $p$-group acting faithfully on the set of switches $\Omega$, then $G \wr \Omega H$ has a surjective strategy.

6.2 Palindromic surjective strategies

In all known examples, when there exists a surjective strategy $S$, we also know of a **palindromic** surjective strategy $S' = \{k'_i \in K\}_{i=1}^{N}$ such that $k'_i = k'_{N-i+1}$ for all $i$.

**Conjecture 35.** Whenever $G \wr H$ has a surjective strategy, it also has a palindromic surjective strategy.

If this conjecture is false, we suspect a counterexample can be found in the case of the trivial wreath product, $G \wr 1 \equiv G$. 
6.3 Quasigroup switches

In Section 1.3, we argued for modeling switches as finite groups because of some desirable properties:

1. Closure. Regardless of which state a switch is in, every move results in a valid state.

2. Identity. We do not have to move a switch on a given turn.

3. Inverses. If a switch is off, we can always turn it on.

In the list, we also included the axiom of associativity for three reasons: switches in practice typically have associativity, groups are easier to model than quasigroups with identity, and associativity makes defining a “surjective strategy” simpler.

However, one could design a non-associative switch and the puzzle would still be coherent. This is because the process of a generalized spinning switches puzzle is naturally “left associative” in the language of programming language theory: we are always “stacking” our next move onto the right. As such, it is worth noting the slightly more general way of modeling switches: as quasigroups with identity, called loops.

In particular, we are interested in the smallest loop that is not a group [14], which we denote as \( L = \{1, a, b, c, d\} \), and describe via its multiplication table, a Latin square of order 5. (Notice that \((ab)d = a \neq d = a(bd)\).)

\[
\begin{array}{c|cccc}
* & 1 & a & b & c & d \\
\hline
1 & 1 & a & b & c & d \\
a & a & 1 & c & d & b \\
b & b & d & 1 & a & c \\
c & c & b & d & 1 & a \\
d & d & c & a & b & 1 \\
\end{array}
\]

(29)

Conjecture 36. There exists a nontrivial adversarial group \( H \) such that the generalized spinning switches puzzle with switches that behave like \( L \) has a winning strategy for the puzzle-solver.

6.4 Expected number of turns

In practice, a puzzle-solver can get the switches into the winning state by playing randomly. Random play will eventually turn on the lightbulb with probability 1, due to the finite number of configurations and the law of large numbers.

Thus we drop the requirement of a finite strategy and ask about the expected value of the number of turns given various sequences of moves. Notice that this is an interesting question even (perhaps especially) in the context of generalized spinning switches puzzles that do not have a surjective strategy.

Winkler [15] notes in the solution “Spinning Switches”:
Although no fixed number of steps can guarantee turning the bulb on in the three-switch version [with two-way switches], a smart randomized algorithm can get the bulb on in at most \(5\frac{1}{2}\) steps on average, against any strategy by an adversary who sets the initial configuration and turns the platform \([15]\).

A basic model for computing the expected number of turns assumes that the initial hidden state \(k \in K\) is not the winning state \(\text{id}_K\), and that the adversary’s “spins” are independent and identically distributed uniformly random elements \(h_j \in H\).

**Proposition 37.** If the puzzle-solver chooses \(k_j \in K \setminus \{\text{id}_K\}\) uniformly at random (that is, never choosing the “do nothing” move) then the distribution of the resulting state after each turn will be uniformly distributed among the \(|K| - 1\) different states. The probability of the resulting state being the winning state is

\[
P(p((k_1, h_1) \ldots (k_j, h_j)) = k^{-1} \mid p((k_1, h_1) \ldots (k_{j-1}, h_{j-1}) \neq k^{-1})) = \frac{1}{|K| - 1},
\]

and the expected number of moves is \(|K| - 1\).

**Proof.** Because the new states are in 1-to-1 correspondence with the elements of \(K \setminus \{\text{id}_K\}\), and since \(k_j \in K \setminus \{\text{id}_K\}\) is chosen uniformly at random, the projection of the partial product \(p((k_1, h_1)(k_2, h_2) \ldots (k_j, h_j))\) is uniformly distributed among all elements of \(K \setminus \{p((k_1, h_1)(k_2, h_2) \ldots (k_{j-1}, h_{j-1}))\}\). The expected value is \(|K| - 1\) because the number of turns follows a geometric distribution with parameter \((|K| - 1)^{-1}\).

When a generalized spinning switches puzzle has a minimal surjective strategy, we can guarantee that we turn on the light within \(|K| - 1\) moves, so we can certainly solve in fewer than \(|K| - 1\) moves on average.

**Proposition 38.** If the generalized spinning switches puzzle, \(G \wr H\), has a minimal surjective strategy, then the expected number of moves is \(|K|/2\).

**Proof.** If there exists a surjective strategy of length \(|K| - 1\), then for each adversarial strategy \(\{h_i \in H\}_{i=1}^{[K]-1}\) the projection of the sequence of partial products of moves induces a permutation of \(K \setminus \{\text{id}_K\}\).

If the initial hidden state \(k\) is chosen uniformly at random, then \(k^{-1}\) is equally likely to occur at any position in this permutation, so the index of the winning state is uniform on \(\{1, 2, \ldots, [K] - 1\}\) and the expected number of moves is

\[
\frac{1 + 2 + \cdots + [K] - 1}{[K] - 1} = |K|/2.
\]

As this proposition suggests, we can always come up with a strategy that does better than the uniformly random strategy in Proposition \([37]\).
Proposition 39. For every generalized spinning switches puzzle, \( G \wr H \) such that \(|K| > 2\), there always exists a (perhaps infinite) sequence whose expected number of moves is strictly less than \(|K| - 1\).

Proof. When \(|K| > 2\), we can always improve on the random strategy in Proposition 37 by avoiding the move of \((g, \cdots, g) \in K\) followed by \((g^{-1}, \cdots, g^{-1}) \in K\), because the second move will put us into a previous state and so will turn on the lightbulb with probability 0.

We conjecture that this technique can be extended, and that the puzzle-solver can always do asymptotically better than randomly guessing.

Conjecture 40. There exists a constant \(\frac{1}{2} < c < 1\) such that for all (finite) wreath products \( G \wr H \) with sufficiently large \(|K|\), the expected number of moves is less than \(c|K|\).

6.5 Shortest surjective strategies

Based on all of the examples that we know of, we conjecture that there exist minimal surjective strategies whenever there exists a surjective strategy of any length.

Conjecture 41. Whenever \( G \wr H \) has a surjective strategy, it also has a minimal surjective strategy \( \{k_i \in K\}_{i=1}^{[K]-1} \).

On the other extreme, we have a weaker conjecture: whenever a wreath product has a surjective strategy, we can provide an upper bound for its shortest surjective strategy.

Conjecture 42. Let \( K/H \) be the set of equivalence classes of \( K \) up to the action of \( H \). Then if \( G \wr H \) has a surjective strategy, it always has a surjective strategy of length \( N < 2^{[K/H]-1} \).

6.6 Counting surjective strategies

The counting problem analog to the decision problem “does \( G \wr H \) have a surjective strategy” is of obvious interest to the author. It is interesting to count both the number of surjective strategies of length \( N \), and the number of such surjective strategies \emph{up to the action of} \( H \). We might also be interested in the number of palindromic surjective strategies, or the number of surjective strategies satisfying another desirable criterion.

In the case of the trivial wreath product, \( G \wr 1 \), we saw in Proposition 25 that there are \((|G| - 1)!\) surjective strategies. (And since the group action is trivial, there are also this many strategies up to group action.)

Open Question 43. Given a wreath product \( G \wr H \) how many surjective strategies of length \( N \) does it have? How many up to the action of \( H \)? How many are palindromic?
Proposition 44. The trivial wreath product $S_3 \wr 1$ has 12 palindromic surjective strategies of length 5:

$$(1\ 2),\ (1\ 3),\ (1\ 2),\ (1\ 3),\ (1\ 2).$$
$$(1\ 2),\ (2\ 3),\ (1\ 2),\ (2\ 3),\ (1\ 2).$$
$$(1\ 3),\ (1\ 2),\ (1\ 3),\ (1\ 2),\ (1\ 3).$$
$$(1\ 2\ 3),\ (1\ 2\ 3),\ (1\ 2),\ (1\ 2\ 3),\ (1\ 2).$$
$$(1\ 2\ 3),\ (1\ 2\ 3),\ (1\ 3),\ (1\ 2\ 3),\ (1\ 2\ 3).$$
$$(1\ 2\ 3),\ (1\ 2\ 3),\ (2\ 3),\ (1\ 2\ 3),\ (1\ 2\ 3).$$
$$(1\ 3\ 2),\ (1\ 3\ 2),\ (1\ 2),\ (1\ 3\ 2),\ (1\ 3\ 2).$$
$$(1\ 3\ 2),\ (1\ 3\ 2),\ (1\ 3),\ (1\ 3\ 2),\ (1\ 3\ 2).$$
$$(1\ 3\ 2),\ (1\ 3\ 2),\ (2\ 3),\ (1\ 3\ 2),\ (1\ 3\ 2).$$
$$(2\ 3),\ (1\ 2),\ (2\ 3),\ (1\ 2),\ (2\ 3).$$
$$(2\ 3),\ (1\ 3),\ (2\ 3),\ (1\ 3),\ (2\ 3).$$

Table 1: The 12 palindromic surjective strategies of length 5 on $S_3 \wr 1$.

Proof. The search space is small here, so this was computed naively by brute force. \qed

6.7 Restricted spinning

Another way that we could generalize a spinning switches puzzle is by restricting the adversary’s moves. For instance, we could modify the puzzle in such a way that the adversary can only spin the switches every $k$ turns. For every finite setup $G \wr H$, there exists $k \in \mathbb{N}$ such that the puzzle-solver can win. (For example, we can always take $k > |K|$ so that the puzzle-solver can do a walk of $K$.)

Open Question 45. How does one compute the minimum $k$ such that the puzzle solver has a surjective strategy of $G \wr H$ given that the adversary’s sequence $\{h_i \in H\}$ is constrained so that $h_i = \text{id}_H$ whenever $i \not\equiv 0 \mod k$?

6.8 Multiple winning states

One assumption that we made about our switches is that they have a single “on” state. Of course, we might conceive of a switch that has $m$ possible states and $k \leq m$ “on” states.

Open Question 46. Given a generalized spinning switches puzzle $G \wr H$ where each switch has a set of “on” states $O_G \subseteq G$, when is it possible for the puzzle-solver to have a finite strategy that guarantees the switches will get to a winning state?
6.9 Nonhomogeneous switches

Another way to generalize the spinning switches puzzle is by allowing different sorts of switches together in the same puzzle. For instance, we could imagine a square board containing four buttons: two 2-way switches, a 3-way switch, and a 5-way switch. Can a puzzle like this be solved? It is important that all of these buttons have the same “shape”, that is there is some group $G$ with a surjective homomorphism onto each of them.

One way to formalize this generalization is as follows:

**Definition 47.** A **nonhomogeneous generalized spinning switches puzzle** consists of a triple

- A wreath product $F \wr \Omega H$ of the free group on $k$ generators by a “rotation” group with a base denoted $K = \prod_{\omega \in \Omega} F_{k,\omega}$,
- a product of finite groups denoted $\hat{G} = \prod_{\omega \in \Omega} G_\omega$ where each group is specified by a presentation with $k$ generators and any number of relations: $G_\omega = \langle g_1^\omega, g_2^\omega, \ldots, g_k^\omega \mid R_\omega \rangle$, and
- a corresponding sequence of evaluation maps $e_\omega : F_{k,\omega} \to G_\omega$, that send generators in $F_{k,\omega}$ to the corresponding generators in $G_\omega$. This can be induced coordinatewise to a map $e_\Omega : K \to \hat{G}$.

When all of the copies of $G_\omega$ are isomorphic, this essentially simplifies to the original definition.

The analogous definition of a surjective strategy becomes more complicated.

**Definition 48.** Let $(F \wr \Omega H, \hat{G}, e_\Omega)$ be a nonhomogeneous generalized spinning switches puzzle.

Then a **nonhomogeneous surjective strategy** is a sequence $\{k_i \in K\}_{i=1}^N$ such that for each adversarial sequence $\{h_i \in H\}_{i=1}^N$ the induced projection/evaluation map $e_\Omega \circ p : F \wr \Omega H \to \hat{G}$ on the partial products of $\{(k_i, h_i) \in F_k\}$ covers $\hat{G}$.

**Proposition 49.** In the specific case that $\Omega = [n]$, $H \subseteq S_n$, and $\hat{G} = \prod_{\omega \in \Omega} G_\omega$ is a product of cyclic groups of pairwise coprime order, the nonhomogeneous generalized spinning switches puzzle has a surjective strategy, namely $\{(1, 1, \ldots, 1) \in K\}_{i=1}^{K'}$.

**Proof.** By the fundamental theorem of abelian groups, $\hat{G}$ is cyclic and is generated by $\hat{G} = \langle (1c_{k_1}, 1c_{k_2}, \ldots, 1c_{k_{\Omega}}) \rangle$. (32)

Thus, the sequence $\{(1, 1, \ldots, 1) \in K\}_{i=1}^{K'}$ is a surjective strategy because it is a fixed point under $H$, and its image is a generator of $\hat{G}$.

**Open Question 50.** Which nonhomogeneous generalized spinning switches puzzles have a nonhomogeneous surjective strategy?
6.10 Infinite surjective strategies

When we first introduced the notion of a surjective strategy in Definition 9, we defined it to be a finite sequence on finite wreath products. However, we can expand the definition to (countably) infinite wreath products by allowing for infinite sequences of moves in $K$. In particular, we can extend this definition to settings where switches have a countably infinite number of states, where there is a countably infinite number of switches, or both. To keep $K$ countable in the latter cases, we use the restricted wreath product, where $K \cong \bigoplus_{\omega \in \Omega} G_\omega$ is defined to be a direct sum instead of a direct product.

**Definition 51.** A **infinite surjective strategy** on an infinite wreath product $G \wr H$ is a sequence $\{k_i \in K\}_{i=1}^{\infty}$ such that for all $k \in K$ and all infinite sequences $\{h_i \in H\}_{i=1}^{\infty}$, there exists some $N \geq 0$ such that the projection

$$p((k_1,h_1) \cdot (k_2,h_2) \cdots (k_N,h_N)) = k^{-1}. \quad (33)$$

We claim, but do not prove that $G \wr \Omega C_2$ has an infinite surjective strategy in the following three settings:

1. $|\Omega| = 2$ and $G \cong (N_{\geq 0}, \wedge)$ where $\wedge$ is the bitwise XOR operator.
2. $\Omega \cong N_{\geq 0}$ as a set, and $K = \bigoplus_{i=0}^{\infty} Z_2^{(i)}$ where $C_2^{(i)} \cong Z_2$.
3. $\Omega \cong N_{\geq 0}$ as a set, and $K = \bigoplus_{i=0}^{\infty} (N_{\geq 0}, \wedge)$ for all $i$.

References

[1] Richard Ehrenborg and Chris M Skinner. “The Blind Bartender’s Problem”. In: Journal of Combinatorial Theory, Series A 70.2 (1995), pp. 249–266. issn: 0097-3165. doi: [https://doi.org/10.1016/0097-3165(95)90092-6](https://doi.org/10.1016/0097-3165(95)90092-6) url: [https://www.sciencedirect.com/science/article/pii/0097316595900926](https://www.sciencedirect.com/science/article/pii/0097316595900926).

[2] Martin Gardner. “Mathematical Games”. In: Scientific American 240.2 (1979), pp. 16–27. issn: 00368733, 19467087. url: [http://www.jstor.org/stable/24965105](http://www.jstor.org/stable/24965105) (visited on 04/20/2022).

[3] Martin Gardner. “Mathematical Games”. In: Scientific American 240.3 (1979), pp. 21–31. issn: 00368733, 19467087. url: [http://www.jstor.org/stable/24965144](http://www.jstor.org/stable/24965144) (visited on 04/20/2022).

[4] V. D. Mazurov. “On Generation of Sporadic Simple Groups by Three Involutions Two of Which Commute”. In: Siberian Mathematical Journal 44.1 (Jan. 2003), pp. 160–164. issn: 1573-9260. url: [https://doi.org/10.1023/A:1022028807652](https://doi.org/10.1023/A:1022028807652)
[5] Ya. N. Nuzhin. “Generating triples of involutions of alternating groups”. In: Mathematical Notes 51.4 (Apr. 1992), pp. 389–392. issn: 1573-8876. doi: 10.1007/BF01250552. url: https://doi.org/10.1007/BF01250552

[6] Ya. N. Nuzhin. “Generating triples of involutions of Chevalley groups over a finite field of characteristic 2”. In: Algebra i Logika 29.2 (1990), pp. 192–206, 261. issn: 0373-9252. doi: 10.1007/BF02001358. url: https://doi.org/10.1007/BF02001358

[7] Ya. N. Nuzhin. “Generating triples of involutions of Lie-type groups over a finite field of odd characteristic. I”. In: Algebra i Logika 36.1 (1997), pp. 77–96, 118. issn: 0373-9252. doi: 10.1007/BF02671953. url: https://doi.org/10.1007/BF02671953

[8] Ya. N. Nuzhin. “Generating triples of involutions of Lie-type groups over a finite field of odd characteristic. II”. In: Algebra i Logika 36.4 (1997), pp. 422–440, 479. issn: 0373-9252.

[9] OEIS Foundation Inc. The On-Line Encyclopedia of Integer Sequences. 2021. url: https://oeis.org/

[10] Yuri Rabinovich. “A generalization of the Blind Rotating Table game”. In: Information Processing Letters 176 (2022), p. 106233. issn: 0020-0190. doi: https://doi.org/10.1016/j.ipl.2021.106233. url: https://www.sciencedirect.com/science/article/pii/S0020019021001484

[11] Oliver Roeder. “The Riddler”. In: FiveThirtyEight (2019). url: https://fivethirtyeight.com/features/i-would-walk-500-miles-and-i-would-riddle-500-more/ (visited on 04/20/2022).

[12] Joseph J. Rotman. An Introduction to the Theory of Groups. New York: Springer, 1999.

[13] Tania Sidana. “Constacyclic codes over finite commutative chain rings”. PhD thesis. Indraprastha Institute of Information Technology, Delhi, 2020.

[14] Travis Willse. The unique loop (quasigroup with unit) L of order 5 satisfying $x^2 = 1$ for all $x \in L$. Mathematics Stack Exchange. url: https://math.stackexchange.com/q/1216907

[15] Peter Winkler. Mathematical Puzzles. CRC Press, 2021.

[16] Peter Winkler. Mathematical Puzzles: A Connoisseur’s Collection. Natick, Mass: AK Peters, 2004. isbn: 978-1568812014.

[17] Reuven Bar Yehuda, Tuvi Etzion, and Shlomo Moran. “Rotating-Table Games and Derivatives of Words”. In: Theor. Comput. Sci. 108 (1993), pp. 311–329.