On Hollow acts

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Abstract

In this paper, we complete an early result by R. Khosravi and M. Roueentan on the hollow act and obtained new properties and characterizations for this notion continues the account of hollow act for commutative monoids begun on 2019. This notion (hollow acts) represents a dual notion to the uniform acts which were submit by M. Roueentan and M. Sedaghatjoo on 2017. An S-act $M_S$ is referred to as hollow act if every subact $N_S$ of $M_S$ is small where a subact $N_S$ is called small (or superfluous) in $M_S$ if for every subact $H_S$ of $M_S$, $N_S \cup H_S = M_S$ implies $H_S = M_S$. Equally, we reformulate this definition in another words as follows: an S-act $M_S$ is referred to as hollow if whenever $N_1, N_2$ are subacts of $M_S$ and $N_1 \cup N_2 = M_S$ implies either $N_1 = M_S$ or $N_2 = M_S$. Conditions under which subacts are inheriting the property of the Hollow act have been examined. As well as the condition on the quotient subact to be Hollow act was studied. Other properties of the small subact differ from those properties which were early studied by the author are investigated. In the interests of simplicity, the relationship between hollow acts and cyclic act is considered. As a consequence, conditions to coincide those classes are shown. Ultimately, the notion of the Hollow act was used to study when the endomorphism monoid will be local.

Keywords: Hollow acts, small subacts, local acts, lifting acts, S-acts, Monoid, cyclic acts.

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1. INTRODUCTION

The notion of hollow modules has been studied extensively in many papers (see for example [1,2,3]) because of its important subject. As for as, hollow acts, it was first studied by R. Khosravi and M. Roueentan on August 2019[4]. In this paper, we complete early results on hollow acts by R. Khosravi and
M. Roueentan and obtained new properties and characterizations for this notion. Besides, a reformulation for the definition of hollow acts was clarified. In everywhere of this paper, every S-acts is unitary right S-acts with zero element \( \Theta \) represented by \( M_S \) where \( S \) is commutative monoid. We refer the reader to the references for more details pertaining to S act which are used here (for basic definitions and terminology) ([5,6,7,8,9,10]).

Let \( M_S \) be an S-act and \( N_S \) be any subact of \( M_S \), then \( N_S \) defines Rees congruence \( \rho_N \) on \( M \), by setting \( a \rho_N a' \) if \( a, a' \in N_S \) or \( a = a' \). The resulting factor act is referred to as Rees factor of \( M_S \) by subact \( N_S \) and it represented by \( M_S/N_S \) [5,p.52]. A subact \( B_S \) of an S-act \( M_S \) is referred to as coessential subact of \( A_S \) in \( M_S \) if \( A_S/B_S \) is small in \( M_S/B_S \) [11].

An S-act \( M_S \) is referred to as \( \Theta \)-simple act if it contains no subacts other than \( M_S \) and one element subact. Besides, \( M_S \) is referred to as simple if it contains no subacts other than \( M_S \) itself ([5],p50). Let \( S \) be a semigroup. A nonempty subset \( K \) of \( S \) is called left ideal of \( S \) if \( SK \subseteq K \); a right ideal of \( S \) if \( KS \subseteq K \); an ideal of \( S \) if \( SK \subseteq K \) and \( KS \subseteq K \) [5,p.20]. An element \( s \) of a semigroup \( S \) is referred to as nilpotent if there exists \( n \in \mathbb{N} \) such that \( S^n = z \in S \) where \( z \) is a (right) zero of \( S \). A semigroup \( S \) is referred to as nilpotent if all elements of \( S \) are (right) nilpotent [5, p.29]. A proper subact \( N_S \) of an S-act \( M_S \) is referred to as maximal if for each subact \( K_S \) of \( M_S \) with \( N_S \subseteq K_S \subseteq M_S \) implies either \( K_S = N_S \) or \( K_S = M_S \) [12]. A right S-act is referred to as local if it contains exactly one maximal subact, also, a monoid \( S \) is also called right (left) ideal [4]. An S-act \( A_S \) is referred to as cyclic (or principal) act if it is generated by one element and it denotes by \( A_S = \langle u \rangle \) where \( u \in A_S \), then \( A_S = uS \) ([5],P.63). An S-act \( M_S \) is called decomposable if there exist two subacts \( A_S, B_S \) of \( M_S \) such that \( M_S = A_S \cup B_S \) and \( A_S \cap B_S = \Theta \). In this case, \( A \cup B \) is referred to as decomposition of \( M_S \). Otherwise \( M_S \) is referred to as indecomposable ([5],p.65). Every cyclic act is indecomposable. An S-act \( M_S \) is called semisimple if and only if every subact of \( M_S \) is a retract or it is union of simple subacts [13].

A subact \( N \) of a right S-act \( M_S \) is referred to as small (or superfluous) in \( M_S \) in case of for every subact \( H \) of \( M_S \), \( N \cup H = M_S \) implies \( H = M_S \) [14]. A subact \( B_S \) of an S-act \( M_S \) is called coessential subact of \( A_S \) in \( M_S \) if \( A_S/B_S \) is small in \( M_S/B_S \) [11]. An S-act \( M_S \) is referred to as lifting, if for every subact \( N_S \) of \( M_S \) contains a retract \( H_S \) of \( M_S \) such that
In other words, an S-act $M_S$ is referred to as **lifting** or satisfies (D$_1$), if for every subact $N_S$ of $M_S$ there exists a retract $H_S$ of $M_S$ such that $H_S$ is a coessential subact of $N_S$ in $M_S$.

In [15], M. Roueentan and M. Sedaghatjoo introduced the notion of uniform acts which were submitted on 2017. In this paper, the author adopt a dually notion for its which is referred to as hollow acts. An S-act $M_S$ is referred to as **hollow** act if whenever $N_1, N_2$ are subacts of $M_S$ and $N_1 \cup N_2 = M_S$ implies that either $N_1 = M_S$ or $N_2 = M_S$. Against this backdrop, this paper aims to give a comprehensive study of the class of hollow acts. As well as, the act theory is signifies a generalization of module theory, thereby it is of interest to know how far the old theories of hollow module extend to this new situation.

This paper is consists of four sections. In section 2, we begin by showing some general properties of hollow acts. We prove that if an S-act $M_S$ is D$_1$-act (lifting act) and $M_1, M_2$ are hollow acts, then $M_S$ is disjoint union of hollow acts (theorem(2.11)). Also, basic facts about hollow were established, where if an S-act is hollow and $\text{Rad}(M) \neq M_S$, then $M_S$ is local (corollary(2.12). In Section 3, we will be concerned with the relationship of hollow acts and some concepts such as, local of the endomorphism monoid, nilpotent ideal, cyclic acts, and maximal ideals. In this way, it is given some conditions under which subacts of hollow acts will be hollow. Section 4 is devoted to discussion of this work. Finally, section 5 illustrates the conclusions of this paper.

### 2. Hollow acts

Motivated by [16], P.J.Fleury who defined the hollow module, we want to extend this notion to S-act as shown below, but before we refer that in [4], R. Khosravi and M. Roueentan defined hollow act on 2019, where an S-act $A_S$ is said to be hollow if every its proper subact is superfluous. For this reason, we reformulate this definition in another form as shown in the following:

**Definition (2.1):** An S-act $M_S$ is referred to as **hollow** act if whenever $N_1, N_2$ are subacts of $M_S$ and $N_1 \cup N_2 = M_S$ implies that either $N_1 = M_S$ or $N_2 = M_S$.

**Lemma (2.2):** Let $M_S$ and $N_S$ be two S-acts and $f$ be S-epimorphism from $M_S$ into $N_S$. If $A_S$ be small subact of $M_S$, then $f(A_S)$ is small subact of $N_S$.

**Proof:** Let $f(A_S) \cup B_S = N_S$, to prove $B_S = N_S$. Let $m \in M_S$, so $f(m) \in f(A_S) \cup B_S$ which implies that either $f(m) \in f(A_S)$ or $f(m) \in B_S$. If $f(m) \in f(A_S)$, then, we have $m \in A_S$. If $f(m) \in B_S$, then $m \in f^{-1}(B_S)$. For this reason, we obtain that $M_S = A_S \cup f^{-1}(B_S)$, but $A_S$ is small subact of $N_S$. 

$N_S/H_S$ is small in $M_S/H_S$ [11].
MS, so $M_S = f^{-1}(B_S)$, and then $f(M) = f(f^{-1}(B_S)) = N_S$. Hence, $f(A_S)$ is subact of $f(M_S)$ and $f(M_S)$ is subact of $B_S$ (that is $f(A_S) \subseteq f(M_S) \subseteq B_S$). Therefore, we get $B_S = f(A_S) \cup B_S = N_S$.

**Proposition (2.3):** Let $AS, BS, MS, NS$ be S-acts such that $A_S \subseteq B_S \subseteq M_S \subseteq N_S$ and $B_S$ is small in $M_S$. Then $A_S$ is small in $N_S$.

**Proof:** Let $A_S \cup H_S = N_S$. Then $B_S \cup H_S = N$ and so, we have $B_S \cup (H_S \cap M_S) = M_S$. Since $B_S$ is small in $M_S$, then we obtain $H_S \cap M_S = M_S$ and so $M_S \subseteq H_S$ and since by assumption $A_S \subseteq M_S$. Thus, we have $A_S \subseteq H_S$ and then $H_S = A_S \cup H_S = N_S$ which is required.

**Corollary (2.4):** Let $A_S, B_S, M_S$ be S-acts such that $A_S \subseteq B_S \subseteq M_S$ and $B_S$ is small in $M_S$. Then $A_S$ is small in $M_S$.

In [4], R. Khosravi and M. Roueentan defined radical of S-act by using maximal subacts (radical of act is the intersection of all maximal subacts), here we give equivalently definition for radical but by using small subact as follows:

**Definition (2.5):** Let $M_S$ be S-act. Then the radical of the act $M_S$ is the union of all small subacts of $M_S$. Accordingly, we mention that by $Rad(M)$. A monoid for which $Rad(M) = \Theta$ for every right S-act $M_S$ is called a right V-monoid.

**Remarks and Examples (2.6):**

1. If $M_S$ is semisimple, then $\Theta$ is the only small subact of $M_S$.

   **Proof:** Let $N_S$ be subact of $M_S$, so there exists $U$ subact of $M_S$ with $N_S \cup U_S = M_S$. If $N_S$ is small subact of $M_S$ then $U_S = M_S$ and so $N_S = \Theta$. If $M_S$ has no maximal subact ($Rad(M) = M_S$), then all finitely generated subacts of $M_S$ are small in $M_S$ and this means $M_S$ be finitely supplemented but not hollow.

2. $Z_6$ is not hollow act and $Z$ as a $Z$-act is hollow act.

3. In a Noetherian act $rad(M)$ is itself is small subacts.

4. A finite sum of small subacts of $M_S$ is always small in $M_S$.

5. If we take $C$ (complex numbers) and $Q$ (rational numbers) as acts over $Z$ (integer numbers), then first one is hollow while the second one is not.

6. Every hollow act is coextending act.

**Definition (2.7):** A subact $N_S$ of S-act $M_S$ is called lies above subact $L_S$ in $M_S$ if the Rees factor $N_S / L_S$ is small in $M_S / L_S$.

**Definition (2.8):** An S-act $M_S$ is called (D1)-act if for every subact $N_S$ of $M_S$, there
is a decomposition $M_S = M_1 \cup M_2$ such that $M_1$ is subact of $A_S$ and $(M_2 \cap A_S)$ is small subact of $M_S$.

**Definition (2.9):** An $S$-act $M_S$ is said to be have property $(P_2)$, if for every coessentialy finitely generated subact $N_S$ of $M_S$ such that $M_1$ is subact of $N_S$, $(N_S \cap M_2)$ is small subact of $M_S$ then we can get a decomposition $M_S = M_1 \cup M_2$.

**Definition (2.10):** An $S$-act $M_S$ is called c-f-lifting if every subact of $M_S$ which contained coessentialy in a finitely generated subact lies a bove retract. Therefore if $M_S$ have $D_1$ property ($M_S$ is lifting act) then it is c- f-lifting. Every c-f-lifting act is satisfy $(P_2)$ property.

**Theorem (2.11):** Let $M_S$ be $(D_1)$-act and let $M_1$ and $M_2$ are hollow acts, then $M_S$ is a disjoint union of hollow acts.

**Proof:** Since $M_S$ is $(D_1)$-act, then $M_S$ is lifting act and so $M_S$ is c-f-lifting. Now, by $(P_2)$ property of $M_S$, we have that $M_S$ is a disjoint union of hollow acts $M_S = M_1 \cup M_2$.

**Corollary (2.12):** Let $M_S$ is an $S$-act. If $M_S$ is hollow and radical of $M_S$ not equal to $M_S$ ($\text{Rad}(M) \neq M_S$), then $M_S$ is local.

**Proposition (2.13):** The epimorphic image of hollow $S$-act is hollow.

**Proof:** Let $M_S$, $M'$ be two $S$-acts, $M_S$ be hollow and $f: M_S \rightarrow M'$ be an $S$-epimorphism. To show that $M'$ is hollow. Let $A_S$ be a proper subact of $M'$. Then $f^{-1}(A)$ is a proper subact of $M_S$ and Since $M_S$ is hollow act, so $f^{-1}(A)$ is small in $M_S$. By lemma (2.2), $f(f^{-1}(A))$ is small in $M'$. Therefore $A$ is small in $M'$ and hence $M'$ is hollow act.

The following corollaries explain under which subacts will be hollow:

**Corollary (2.14):** A retract subact of a hollow act is also hollow.

**Proof:** Let $M_S$ be hollow act and $N_S$ be retract of $M_S$. Let $A_S$ be proper subact of $N_S$, then $A_S$ be proper subact of $N_S$. Since $M_S$ is hollow act, so $A_S$ is small in $M_S$. To prove $A_S$ is small in $N_S$, in similar way as in proof of lemma(2.2) and by taking epimorphism is the projection map($\pi: M_S \rightarrow N_S$).

**Corollary (2.15):** Let $M_S$ be an $S$-act. If $M_S$ is a hollow act, then Rees factor $M_S/N_S$ is a hollow act for every proper subact $N_S$ of $M_S$.

Before we give the next theorem which illustrates the characterization of hollow acts, we need the following concepts and lemma:
**Definition (2.16):** Let $M_S$ be $S$-act. Then $M_S$ is referred to as a **lifting** act (or satisfies (D1)) if for any subact $N_S$ of $M_S$, there exists a retract $A_S$ of $M_S$ such that $A_S$ is subact of $N_S$ and the Rees factor $N_S/A_S$ is small in $M_S/A_S$.

**Definition (2.17):** Let $M_S$ be $S$-act and $A_S$, $B_S$ be two subacts, where $B_S \subseteq A_S \subseteq M_S$. $B_S$ is referred to as a coessential subact of $A_S$ in $M_S$ if the Rees factor $A_S/B_S$ is small in $M_S/B_S$.

**Definition (2.18):** A subact $A_S$ of $M_S$ is called coclosed in $M_S$ if $A_S$ has no proper coessential subact. Equivalently, a subact $A_S$ of $M_S$ is referred to as coclosed, if whenever $A_S/B_S$ is small in $M_S/B_S$, it implies that $A_S = B_S$ for every subact $B_S$ of $M_S$.

**Lemma (2.19):** Let $S$ be $V$-monoid. An $S$-act $M_S$ is lifting if and only if it is semisimple.

**Proof:** Assume that $M_S$ is lifting act. Since $S$ is $V$-monoid, so $rad(M)=\Theta$. As radical of the act $M_S$ is the union of all small subacts of $M_S$. Thus, the only small subact of $M_S$ is zero subact ($\Theta$).

Let $N_S$ be any subact of $M_S$, by definition of lifting act (2.16), there exists a retract $A_S$ of $M_S$ such that $A_S$ is subact of $N_S$ and the Rees factor $N_S/A_S$ is small in $M_S/A_S$. Since the only small subact of $M_S$ is zero subact by the proof, so $N_S/A_S = \Theta$ which implies that $N_S = A_S$ and this means that $N_S$ is retract of $M_S$. Therefore, any subact of $M_S$ is a retract of $M_S$ and $M_S$ is semisimple act. The other direction is obvious.

**Theorem (2.20):** Let $S$ be a commutative Noetherian monoid and let $M_S$ be an $S$-act satisfying the following:

1. Every simple right $S$-act is injective.
2. $M_S$ is semisimple $S$-act.
3. $M_S$ is indecomposable $S$-act. Then $M$ is hollow act.

**Proof:** Since every simple act $M_S$ over any monoid $S$ is injective then $S$ is $V$-monoid, but $M_S$ is semisimple then by lemma(2.19) $M_S$ is lifting act therefore by definition(2.16) for all $N_S$ subact of $M_S$, there is a decomposition $M_S = H_S \oplus L_S$ such that $H_S$ is subact of $N_S$ and $(N_S \bigcap H_S)$ is small subact of $M_S$, but $S$ is commutative Noetherian monoid with indecomposable property implies $M_S$ is hollow act by property (P1) (definition(2.10)).

The next theorem gave the converse of corollary (2.12)

**Theorem (2.21):** Let $M_S$ be an $S$-act. If $M_S$ is local, then $M_S$ is hollow act.

**Proof:** Let $M_S$ be local act and $N_S$ is any subact of $M_S$. By definition of local act we can say that every proper subact $N_S$ of $M_S$ implies $N_S$ subset of radical of $M_S$ and radical of $M_S$ is small in $M_S$. 
Then $N_S$ is small in $M_S$ and by definition of hollow act we get the required.

In [4], R. Khosravi and M. Roueentan defined supplement subact, here we give another definition for supplement subact as follows:

**Definition (2.22):** A subact $N_S$ of $S$-act $M_S$ is referred to as supplement of a subact $H_S$ in $M_S$ if $M_S = N_S \cup H_S$ and $N_S \cap H_S$ is small in $N_S$. An $S$-act $M_S$ is called supplement if every subact has supplement in $M_S$.

**Theorem (2.23):** Let $M_S$ be a nonzero act such that it satisfying the following:

1. $\text{Rad}(M) \neq M_S$.
2. Every subact of $M_S$ lies over retract of $M_S$.
3. $M_S$ is indecomposable act. Then $M_S$ is hollow.

**Proof:** We must prove that $M_S$ is local act. Let $M_S$ be a nonzero act such that $M_S$ satisfying conditions 2 and 3. Let $N_S$ be a proper subact of $M_S$. Then $M_S = A_S \cup B_S$, $A_S$ subact of $N_S$ and $(B_S \cap N_S)$ is small in $M_S$. Since $A_S \neq M_S$, $B_S = M_S$ and $N_S = (B_S \cap N_S)$ is subact of $\text{rad}(M)$ then radical of $M_S$ is the unique maximal subact of $M_S$. Thus $M_S$ is local and by theorem (2.21) $M_S$ is hollow act.

**Theorem (2.24):** Let $M_S$ be an $S$-act. If $M_S$ is hollow act, then $M_S$ is supplemented act.

**Proof:** Let $M_S$ be an $S$-act and $N_S$ be a subact of $M_S$. Then $N_S \cup M_S = M_S$. By hypothesis, $(N_S \cap M_S) = N_S$ which is small in $M_S$. Hence $M_S$ is supplemented act.

**Proposition (2.25):** Let $M_S$ be an $S$-act and $N_S$ be a nonzero subact of $M_S$ which is hollow, then either $N_S$ is small subact of $M_S$ or coclosed subact of $M_S$, but not both.

**Proof:** Assume that $N_S$ is not coclosed subact of $M_S$, so there exist $B_S$ is subact of $N_S$. But $N_S$ is hollow and hence $B_S$ is small in $N_S$ and $N_S / B_S$ is small in $M_S / B_S$. Therefore, $N_S$ is small in $M_S$. If $N_S$ is coclosed subact of $M_S$, and it is small in $M_S$, then $N_S / \theta$ is small in $M_S / \theta$, and hence $N_S = \theta$, which is a contradiction.

The following propositions clarified under which condition inherits the property of hollow act:

**Proposition (2.26):** Let $M_S$ be a hollow act and $N_S$ be a subact of $M_S$ such that $N_S / A_S$ is retract of $M_S / A_S$ for each proper subact $A_S$ of $N_S$. Then $N_S$ is hollow.

**Proof:** Let $H_S$ be a proper subact of $N_S$. Then $H_S$ is subact of $M_S$ and hence $H_S$ is small subact of $M_S$ (since $M_S$ is hollow act). To prove $H_S$ is small subact of $N_S$. Let $A_S$ be a proper subact of $H_S$. Then $H_S / A_S$ is small subact of $M_S / A_S$ (since Rees factor $M_S / A_S$ is hollow act by
corollary (2.15)). By hypothesis \( \frac{N_S}{A_S} \) is a retract of \( \frac{M_S}{A_S} \). Therefore by corollary (2.14) \( \frac{N_S}{A_S} \) is hollow and so \( \frac{H_S}{A_S} \) is small in \( \frac{N_S}{A_S} \). For this reason \( HS \) is small subact of \( NS \).

**Proposition (2.27):** Every nonzero coclosed subact of a hollow act is hollow.

**Proof:** Let \( M_S \) be a hollow act and let \( N_S \) be a nonzero subact of \( M_S \) such that \( N_S \) is nonzero coclosed subact of \( M_S \). To explain that \( N_S \) is hollow. Let \( A_S \) be subact of \( N_S \) in \( M_S \), then \( A_S \) be subact of \( M_S \) and so \( A_S \) is small in \( M_S \). To prove \( A_S \) is small subact of \( N_S \). Since epimorphic image of \( M_S \) is hollow act by proposition (2.13), so for any subact \( B_S \) of \( A_S \), we obtain that \( \frac{A_S}{B_S} \) is small subact of \( \frac{M_S}{B_S} \) and also we have \( \frac{N_S}{B_S} \) is small subact of \( \frac{M_S}{B_S} \) but \( N_S \) is coclosed in \( M_S \), so we have \( N_S = B_S \) for every subact \( B_S \) of \( M_S \) with \( B_S \subseteq N_S \). But \( B_S \) is subact of \( A_S \), and hence \( A_S = N_S \). Then \( A_S \) is small in \( N_S \). Thus \( N_S \) is a hollow act.

### 3. Relationship of hollow acts with some concepts

In this section, we demonstrate the relationship of hollow acts with some concepts close to it such as cyclic act, local monoid, maximal ideal, finitely generated.

**Theorem (3.1):** If \( M_S \) is a finitely generated hollow \( S \)-act, then \( M_S \) is cyclic. In addition, if \( M_S \cong \frac{S}{I} \), where \( \frac{S}{I} \) is the Rees factor monoid, then \( I \) is contained in a unique maximal right ideal \( X \). Conversely, if \( I \) is contained in a unique maximal right ideal, then \( \frac{S}{I} \) is hollow.

**Proof:** Assume that \( m_1, m_2, \ldots, m_n \) form a set of generators for \( M_S \). Then \( M = m_1S \cup m_2S \cup \ldots \cup m_nS \). Since every proper subact of \( M_S \) is small, so we delete the retract sequentially, then we obtain \( M_S = m_iS \) for some \( i \). Thus \( M \) is cyclic. Now, let \( M_S \cong \frac{S}{I} \), where \( I \) is the annihilator of \( m_i \). To prove that \( I \) is contained in a unique maximal left ideal, assume that \( I \) is contained in two maximal ideals, say \( X_1, X_2 \). Then \( X_1 \cup X_2 = S \), so union of image of \( X_1 \) and \( X_2 \) is equal to \( S/I \). But it is obvious that \( \frac{X_1}{I} \) and \( \frac{X_2}{I} \) are proper subacts of \( \frac{S}{I} \) and hence, we have a contradiction. Thus \( I \) is contained in unique maximal left ideal. Lastly, assume that \( I \) contained uniquely in the maximal left ideal \( X \). Then any subact of \( \frac{S}{I} \) corresponds to left ideal of \( S \) containing \( I \). All proper ideals must contained in \( X \), so any finite union of proper subacts of \( \frac{S}{I} \) is contained \( \in \frac{S}{I} \).

**Corollary (3.2):** If \( M_S \) is hollow and finitely generated, then \( M_S \) contains a unique maximal subact.

**Proof:** By theorem (3.1), we have \( M_S \cong \frac{S}{I} \), where \( I \) is contained in unique maximal right ideal.
X of S. So any proper subact of \( S/I \) must be contained in \( X/I \) which is the unique maximal subact of \( X/I \). Let \( M_S \) be \( S- \) act and \( X_1, X_2 \) be subacts of \( M_S \). Then \( X_1 \) and \( X_2 \) are correlated by equivalent relation if \( X_1 \cup H = M_S \) if and only if \( X_2 \cup H = M_S \) for some subact \( H \) of \( M_S \). Now, if \( S/I \) is hollow, then \( I \) must be correlated to the maximal right ideal \( X \) which contains it in \( S \), for if \( X \) is maximal right ideal which contains \( S \), then \( I \cup H = S \) for sure implies \( X \cup H = S \). On the other hand, if \( X \cup H = S \), then, taking the canonical image of \( X \) and \( H \) in \( S/I \), we have \( X/I \cup (H/I) = S/I \). Since \( X/I \) is small in \( S/I \), we obtain \( (H/I)/I = S/I \) or \( H/I \) is \( S \). Conversely, if \( I \) is correlated to another maximal right ideal \( Y \), then we have \( I \subseteq Y \), from the relationship \( X \cup Y = S \). But since \( I \subseteq Y \), we have \( I \subseteq Y \).

**Theorem (3.3):** Let \( I \) be right ideal of \( S \). Then \( I \) is contained in a unique maximal right ideal \( X \) if and only if \( aS \cup X = S \) for all \( a \in S \) and \( a \notin X \).

**Proof:** Let \( I \) be right ideal and \( X \) be unique maximal ideal which contain \( I \). To prove \( aS \cup X = S \) for all \( a \in S \) and \( a \notin X \). Let \( a \notin X \), then \( X \cup aS \cup X = S \), but \( X \) is maximal, so \( aS \cup X = S \). Conversely, let \( aS \cup X = S \) for all \( a \in S \) and \( a \notin H \), where \( H \) be any ideal contain \( I \). Let \( H \subseteq L \subseteq S \) and let \( a \in L \), \( a \notin H \). Then, \( S = aS \cup X \cup L \cup H \subseteq L \subset S \). Thus \( L = S \). Hence \( H \) is maximal ideal. To prove it is unique, let \( A \) be another maximal ideal which contain \( I \) and subset of \( S \), (that is \( H \subseteq A \subseteq S \)) but \( H \) is maximal, so \( H = A \).

The following theorem illustrate that there are several of right ideals which are contained uniquely in maximal ideals.

**Theorem (3.4):** If \( I \) is right ideal of \( S \) and it is contained uniquely in the maximal right ideal \( X \), then \( I^n \) is contained uniquely in \( X \).

**Proof:** Assume that \( I^2 \) is contained in the maximal right ideal \( X' \neq X \). Then there is \( x_1 \in X', x_1 \notin X \), so \( x_1S \cup I = S \). Thus, there is \( s \in S \) and \( x_1s \in x_1S \cup I \). Let \( i' \in I \), then \( x_1s_i' \in x_1S \cup I \) and \( x_1s_i' \in X' \). Therefore, \( I \subseteq X' \) which is contradiction.

**Corollary (3.5):** If \( I \) is a right ideal of \( S \) which is contained uniquely in a maximal right ideal, \( X \), then \( I^n \) is contained uniquely in \( X \) for all \( n \geq 1 \).

**Corollary (3.6):** If \( I \) is maximal right ideal of \( S \), then \( I^n \) is contained uniquely in \( X \) for all \( n \geq 1 \).

**Corollary (3.7):** If \( I \) is a right ideal of \( S \) which contains some power of the maximal right ideal \( X \), then \( I \) is contained uniquely in \( X \).

Recall that nil semigroup \( S \) if all elements of \( S \) are nilpotent, where an element \( s \) of a
semigroup S is called nilpotent if there exists $n \in \mathbb{N}$ such that $s^n = z$, where z is zero of $S.([5], p.29)$.

In [17], I.N. Herstein define nilpotent ideal which motivates us to extend this notion to act theory, as shown below:

**Definition (3.8):** An ideal I, of a monoid is said to be a nilpotent ideal, if there exists a natural number k such that $I^k = \Theta$.

The following corollary explains an essential point which is whenever a monoid S will be local:

**Corollary (3.9):** If I is a nilpotent right ideal of S which is contained uniquely in a maximal right ideal X, then S is local.

**Proposition (3.10):** Let $S/I$ be hollow and $x \in E(I)$. Then, the endomorphism induced by x on $S/I$ is epic if and only if $x \notin X$ where X is the unique maximal right ideal containing I.

**Proof:** The endomorphism induced by x is epic if and only if $S/I = xS/I$. But this is true if and only if $xS = I = S$ which is true if and only if $x \notin X$.

**Proposition (3.11):** If $S/I$ is hollow, where I is an ideal of S, then $\text{End}(S/I)$ is local.

**Proof:** It is obvious that $\text{End}(S/I) \cong S/I$ is a monoid and $X/I$ is unique maximal ideal of $S/I$, where X is unique maximal ideal containing I.

4. Discussion

In this section, we try to give meaning of the results which were obtained in this paper by us. One of these results, it is in the theorem (2.11) such that it was clarified that $M_S$ is disjoint union of hollow act (this means that $M_S = M_1 \cup M_2$), but it was need some conditions for example; if an S-act $M_S$ is D1-act(lifting act) and $M_1, M_2$ are hollow acts. As for the corollary (2.12), it was said a major point which was $M_S$ is local, but also need conditions; firstly if $M_S$ is hollow act and secondly; if $\text{Rad}(M_S) \neq M_S$. Ultimately, proposition(2.13) was elucidated when the image of hollow act is hollow and then obtained it was satisfied if the homomorphism is epimorphism. In addition, corollary(2.14) and corollary(2.15) illustrated interesting results about the conditions for subacts to inheriting the property of hollow act, and obtained two major points; subact is hollow if; firstly, retract of hollow act, secondly, if $M_S$ is hollow, then the Rees factor subact $M_S/N_S$ is hollow. As for the proposition (2.27) explained another condition on subact to be hollow which was every nonzero coclosed of hollow act will be hollow. Besides, proposition(2.26) illustrated further conditions, one of them $N_S/A_S$ is retract of $M_S/A_S$ for each proper subact $A_S$ of $N_S$ and the other $M_S$ is hollow act. Furthermore, in the theorem(2.20) and
theorem(2.23), we proved that when an S-act will be hollow and reached it was satisfied if there are some conditions such that: a) Every simple right act is injective, b) M_S is semisimple, c) M_S is indecomposable and these conditions followed by a monoid S is commutative Noetherian, in addition the following conditions will be satisfied on any monoid a) \( \text{Rad}(M_S) \neq M_S \), b) every subact lies over retract of M_S, c) M_S is indecomposable and then obtained what we want( M_S is hollow act). As for as theorem (2.21) was explained the converse of corollary(2.12) is true such that it is said that M_S is hollow, if it is local. Also, the relationship of hollow acts and supplement act was demonstrated in theorem(2.24) where it is stated that M_S is supplement act, if it is hollow. Proposition(2.25) showed that a subact N of S-act M_S will be either small or coclosed subact but not both when some conditions occurred for example; M_S is hollow S-act and a subact N of M_S is nonzero(this means N \neq \emptyset). In the theorem(3.1), we found the relationship between finitely generated hollow act and cyclic act and obtained that M_S will be cyclic act, if it is finitely generated hollow act. Also, corollary (3.2) gave the condition on any act to have contained maximal subact, such that M_S contains unique maximal subact, if M_S is hollow act and finitely generated. Corollary (3.5) clarified there are several right ideals contained in maximal ideal and have the form \( P/I \), when I is right ideal and contained uniquely in maximal ideal. An essential result was illustrated in corollary (3.9) such that it was shown that a monoid S will be local if I is nilpotent of S and contained uniquely in maximal ideal. Another important result from our sight was demonstrated in proposition(3.10) such that it was said the endomorphism of I induced by X on \( S/I \) will be epimorphism if the following conditions satisfied; x \in X, where X is maximal ideal contain I, x \in E(I), and \( S/I \) is hollow. Finally, proposition(3.11) showed that End(\( S/I \)) will be local under certain condition which was \( S/I \) is hollow such that I is an ideal of S.

5. Conclusions
From previous theorem, examples, remarks, theorems and propositions, we can highlight some major points as follows: One of these points, it is in the theorem (2.11) such that it was clarified if an S-act M_S is D1- act(lifting act) and M_1,M_2 are hollow acts, then M_S is disjoint union of hollow act (this means that M_S=M_1 \cup M_2). As for the corollary(2.12), it was demonstrated major point which was M_S is local if it is hollow and Rad(M_S) \neq M_S. Also, proposition(2.13) was elucidated that epimorphic image of hollow act is hollow. In addition, corollary(2.14) and corollary(2.15) illustrates interesting results about the conditions for subacts to inheriting the
property of hollow act, and obtained two points which are: firstly, retract of hollow act is hollow, secondly, if $M_S$ is hollow, then the Rees factor $M_S/N_S$ is hollow. As well as, in the proposition (2.27) explains another condition on subacts to be hollow which was every nonzero coclosed of hollow act is hollow. In addition, proposition(2.26) illustrated further condition on subact where a subact $N$ of $M_S$ is hollow under a condition that the Rees factor $N_S/A_S$ is retract of $M_S/A_S$ for each proper subact $A_S$ of $N_S$ and S-act $M_S$ is hollow act. As for the theorem (2.20) and theorem(2.23), we proved that an S-act is hollow if it is satisfied some conditions such that: a) Every simple right act is injective, b) $M_S$ is semisimple, c) $M_S$ is indecomposable and these condition satisfied when $S$ is commutative Noetherian monoid, but the following conditions on any monoid, a) $\text{Rad}(M_S) \neq M_S$, b) every subact lies over retract of $M_S$, c) $M_S$ is indecomposable, then $M_S$ will be hollow act respectively. Furthermore, theorem (2.21) was explained that the converse of corollary (2.12) is true where it was clarified if $M_S$ is local, then $M_S$ is hollow. The relationship of hollow acts and supplement act was demonstrated in theorem (2.24) where it was said that if $M_S$ is hollow, then it is supplement act. Proposition(2.25) showed that when $M_S$ is hollow S-act and $N \neq \Theta$ subact of $M_S$, then $N$ is either small or coclosed subact but not both. In the theorem (3.1), we found the relationship between finitely generated hollow act and cyclic act, where it was proved if $M_S$ is finitely generated hollow act, then it is cyclic. Corollary (3.2) gave the condition on any act to have contained maximal subact which was if $M_S$ is hollow and finitely generated, then $M_S$ contains unique maximal subact. Corollary (3.5) clarified if $I$ is right ideal and contained uniquely in maximal ideal, then there are several right ideal contained in maximal ideal and have the form $I^n$. An essential result was illustrated in corollary (3.9) such that it was shown that a monoid $S$ is local if $I$ is nilpotent of $S$ and contained uniquely in maximal ideal. Another important result was demonstrated in proposition(3.10) where it was gave the conditions on $S/I$ such that the endomorphism of $I$ induced by $X$ on $S/I$ is epimorphism and it was if $x \notin X$, where $X$ is maximal ideal contain $I$ , $x \in E(I)$ and $S/I$ is hollow. Finally, proposition (3.11) showed that $\text{End}(S/I)$ is local monoid under certain condition which was $S/I$ is hollow such that $I$ is an ideal.

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