Symmetries and conservation laws in the theory of plasticity

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Abstract. This article is a review of the main results that were obtained by the authors while studying equations of plasticity using symmetries and conservation laws from the year 1975 to the present day.

1. Introduction

L. V. Ovsyannikov, on the basis of S. Lie’s works, has developed the methodology of researching and solving of differential equations of mechanics using point symmetries [1]. For the first time these methods were effectively applied to differential equations of the theory of plasticity by B. D. Annin [2]. He found a group of point symmetries admitted by the equations of ideal plasticity in axisymmetric case, and provided a new exact solution of these equations. B. D. Annin gave a suggestion to investigate the group properties of spatial equations of the ideal plasticity theory to one of the authors of the present article, and to build new solutions for these equations. S. I. Senashov and his disciples keep working till now in this direction. As a result, groups of point symmetries of the main plasticity equations were found [2]. These groups were used to build the new classes of exact solutions which have mechanical applications. Also, conservation laws for the equations of plasticity in case of plane strain were found [3]. These laws were used to solve the main boundary value problems for the equations of plasticity in case of plane strain [4]. Nowadays, it is shown that the conservation laws can also be used to solve elastoplastic problems [5, 6]. All these results are described in the present article.

2. Main equations of the theory of plasticity

Assume \( x_1 = x, \ x_2 = y, \ x_3 = z \) is an orthogonal Cartesian coordinate system. The main equations of the theory of plasticity in stationary case under Mises yield criterion are written as

\[
\partial_j S_{ij} = \partial_i P, \quad S_{ij} = \lambda e_{ij} = \lambda \left( \partial_i u_j + \partial_j u_i \right), \quad S_{ij} S_{ij} = 2k^2, \quad i, j = 1, 2, 3, \tag{1}
\]
where \( e_{ij}, S_{ij} \) are components of velocity tensor and stress deviator, \( P \) is hydrostatic pressure, \( u_i, u_z, u_\lambda \) are components of velocity vector, \( k \) is a constant (pure shear yield point), \( \lambda \) is nonnegative function. Summation is intended over the repeated indices.

A group of point symmetries has been found for equations (1). It is generated by the following operators \([2]\):

\[
X_i = \partial_i, \quad Y_i = u_i, \quad T_i = x_\lambda \partial_{x_i} - x_z \partial_{u_i}, \quad N = x_i \partial_{x_i}, \quad M = u_i \partial_{u_i}, \quad Z_i = x_\lambda \partial_{x_i} - x_z \partial_{u_i} + u_\lambda \partial_{u_i} - u_i \partial_{u_\lambda}, \quad S = \partial_P.
\]

Four more operators \( T_z, T_\lambda, Z_z, Z_\lambda \) come out from \( T_i, Z_i \) by cyclic permutation of indices.

System of equations (1) in spatial case allows only classical conservation laws: mass conservation law, momentum conservation law and angular momentum conservation law.

### 3. Some classes of exact solutions

The presence of symmetries (2) permits to build new classes of exact solutions. In this way, the new classes of helical solutions were built \([2, 3]\):

\[
u_i = u_i (r, z + \alpha \theta), \quad P = P (r, z + \alpha \theta),
\]

where \( \alpha \) is a constant, \( r, \theta, z \) is cylindrical coordinate system; \( i = 1, 2, 3 \). These solutions can be used to analyse plastic flows in helical canals, and they also describe plastic flow of rods at complex loading. In particular, the following new class of solutions was found:

\[
u_i = \alpha x_i, \quad u_z = \alpha x_z, \quad u_\lambda = -2\alpha x_z + f(x_i, x_z),
\]

where \( \alpha \) is an arbitrary constant, \( f \) is an arbitrary solution of the minimal surfaces equation \([2]\). These solutions find applications in the analysis of plastic flows of material compressed by rigid plates.

### 4. Homotopy of the Prandtl solution

Consider equations of the theory of plasticity describing plane strain under Saint-Venant – Mises yield criterion. These equations are statically determinate, that is why it is sufficient to consider only equations relating components of stress deviator and hydrostatic pressure.

\[
\partial_1 S_{11} + \partial_2 S_{12} = \partial_1 P, \quad \partial_1 S_{12} + \partial_2 S_{22} = \partial_2 P, \quad (S_{11} - S_{22})^2 + 4S_{12}^2 = 4k^2.
\]

Equations (5), in contrast to the system (1), possess an infinite group of point symmetries, an infinite group of higher symmetries and an infinite series of conservation laws \([4]\). These symmetries were able to use for construction of new solutions of the equations (5), and the conservation laws were applied to solve the main boundary value problems for these equations \([4]\).

The existence of the infinite group of point symmetries permits to «deform» continuously one of the exact solutions of equations (5) into any other. During the process of deformation, an infinite series of exact solutions is obtained. This process is convenient to observe on the slip lines of the system (5). In this manner one can «deform» the Prandtl solution describing compression of a plastic layer with rigid plates

\[
\cos \left( \frac{S_{11} - S_{22}}{S_{12}} \right) = y, \quad P = -x + \sqrt{1 - y^2}
\]

into the Nadai solution that describes stress state around a circular hole (see figures 1–3).
Figure 1. Slip lines of the Prandtl solution.

Figure 2. Slip lines of the Nadai solution.

Figure 3. Slip lines of one of intermediate solutions. The hole has the form of Pascal’s limacon.

Picture 1 shows the slip lines of the Prandtl solution, Picture 2 displays the slip lines of the Nadai solution, and Picture 3 illustrates the slip lines of one of intermediate solutions [5–11]. Presence of the infinite series of conservation laws has allowed solving the main boundary value problems for the equations (5). Herewith, the solutions of these problems are given by explicit analytic formulas [5].

5. Application of the conservation laws to elastoplastic problems solving

Let describe, in concluding, some new results that are related to solving of elastoplastic problems and to building of an unknown elastoplastic boundary. Such boundaries were able to build for rods under torsion, and for deformed plates with holes.

5.1. Elastoplastic boundary in a rectilinear rod under torsion

Let consider the elastoplastic torsion of a straight rod with the cross-section limited by a convex contour $\Gamma$. At a rather large value of the torsion torque, an elastic zone $F$ and plastic zone $P$ arise in the rod (Figure 4).
Figure 4. Cross-section of a rod under torsion.

It is known that the plastic zone arises on the outer contour Γ; suppose that the plastic zone fully involves the outer contour. Assume L is the boundary line between the elastic and the plastic zones.

In the elastic zone, stress components satisfy the following equations:

\[ \partial_1 S_{13} + \partial_2 S_{23} = 0, \quad \partial_2 S_{13} - \partial_1 S_{23} = -2\theta, \]

in the plastic zone, these components satisfy the equations

\[ \partial_1 S_{13} + \partial_2 S_{23} = 0, \quad S_{13}^2 + S_{23}^2 = k^2, \]

and on the outer contour Γ the stress components satisfy the boundary condition

\[ S_{13}l + S_{23}m = 0 \]

and the yield criterion \( S_{13}^2 + S_{23}^2 = k^2 \). Here \( \theta \) is a torsion angle, \( l, m \) are components of the outer normal vector to the contour Γ.

From Relations (8) – (9) one can determine tensor components \( S_{13} \) and \( S_{23} \) on the contour Γ.

The following relations are derived from the conservation laws obtained in [12]:

\[ 2\pi S_{13}(x_0, y_0) = -\oint_{\Gamma} A_i dy - B_i dx, \]

and

\[ 2\pi S_{23}(x_0, y_0) = -\oint_{\Gamma} A_i dy - B_i dx, \]

where \( A_i, B_i (i = 1, 2) \) are the conserved current components.

Relations (10) – (11) permit to find the values \( S_{13} \) and \( S_{23} \) at all the inner points of the rod. Further, the yield criterion (8) is checked. The points, at which the expression in the right side part of the yield criterion (8) is less than \( k \), belong to the elastic zone, and the rest of the points belong to the plastic one. These calculations allow restoring the sought border \( L \) with any required accuracy. Examples of building of elastoplastic boundaries for various types of rolled sections are considered in [12]. The case of multiply connected cross-section for isotropic medium is given in [13].

5.2. Elastoplastic boundary in a cantilever of uniform cross-section

Consider a cantilever of a uniform cross-section limited by contour Γ. The cantilever is subjected to the concentrated force \( P \) on its free ending; the force is parallel to axis \( O\alpha \).

Under the action of a rather large value of the force \( P \), the elastic and plastic zones appear in the cantilever. Let \( l \) is the length of this cantilever, and let the plastic zone fully involves the contour Γ.

With these assumptions, the stress components satisfy the following system of equations in the elastic zone:

\[ \partial_1 S_{13} + \partial_2 S_{23} = -\frac{Px}{l}, \quad \partial_2 S_{13} - \partial_1 S_{23} = -\frac{Py}{l}, \]

(12)
and on the lateral surface, i.e. on the contour Γ, the conditions (8) and (9) are held. It means that the lateral surface is free from stresses and is in plastic state.

Formulas (10)–(11) that permit to find the values of stress components in arbitrary point \((x_0, y_0)\) of the cross-section were obtained in the previous paragraph. Inserting the values \(A_i, B_i\) into these formulas, one can find the values of the stress in the sought point \((x_0, y_0)\). The only remaining thing is to check whether this point is located in the plastic or elastic zone. If the value of the left side part of the yield criterion (8) is less than \(k\), then the point belongs to the elastic zone, otherwise the point is in the plastic zone.

5.3. Elastoplastic boundary in a rectangular plate weakened by circular holes

Consider now a rectangular plate of the length \(a = 10\) cm and the width \(b = 2\) cm weakened by two circular holes with radii \(r_1 = r_2 = 2.5\) mm (figure 5) [13].

![Figure 5. Rectangular plate weakened by two circular holes.](image)

One have to find an elastoplastic boundary for the domain closed by \(\Gamma: \Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3\), where \(\Gamma_1\) is a rectangular, \(\Gamma_2\) and \(\Gamma_3\) are circles. Contour \(\Gamma\) is in the plastic state.

Yield criterion for the plane stress state on the boundary of the plate is written in the following form:

\[
\sigma_x^2 + \sigma_y^2 - \sigma_x \sigma_y + 3\tau^2 = 3k^2,
\]

where \(\sigma_x, \sigma_y, \tau\) are components of the stress tensor for the plane stress state, \(k\) is a constant of plasticity.

Boundary conditions for the plate have the form

\[
\sigma_x n_1 + \tau n_2 = X, \quad \sigma_y n_1 + \tau n_2 = Y,
\]

where \(\{n_1, n_2\}\) is the unit normal vector to the contour \(\Gamma_i; X, Y\) are components of the vector of external forces.

The plate is subjected to a tensile load in the axis \(x\) direction and to a compressive one in direction of the axis \(y\).

Construction of the elastoplastic boundary for the considered plate contains three main stages.

- Laplace equation solving
  
  This stage lies in solving of Laplace equation \(\Delta F = 0\) with boundary conditions \(F|_\Gamma = \sigma_x + \sigma_y\) (here \(\sigma_x, \sigma_y\) are functions from (13)–(14), \(F = \sigma_x + \sigma_y\) is a harmonic function) and in finding the values of the function \(F\) in inner points of the domain bounded by \(\Gamma\).

- Application of conservation laws to find \(\sigma_x, \sigma_y, \tau\)
  
  The purpose of the stage is in finding of the functions \(\sigma_x, \sigma_y, \tau\) in each point \((x_0, y_0)\) of the considered domain by formulas derived using conservation laws method [13]:
\[
\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau}{\partial y} = 0, \quad \frac{\partial \tau}{\partial x} + \frac{\partial F}{\partial y} - \frac{\partial \sigma_y}{\partial y} = 0, \quad \sigma_y = F - \sigma_x. \tag{15}
\]

- Yield criterion verification

On this stage of the elastoplastic boundary construction, the yield criterion for each inner point of the domain bounded by the contour \( \Gamma \) is verified. If the value of the left side of the yield criterion (13) is less than \( 3k^2 \) for the stresses at the inner point, then this point is in the elastic zone; all the rest points, with stresses which do not satisfy this condition, are in the zone of plasticity of the examined domain.

The following figure shows the distributions of the points in elastic and plastic zones calculated for the plate:

![Figure 6. Distribution of elastic (crosses) and plastic (points) zones in the plate.](image)

Note that detailed information can be found in [13].

6. Conclusion

Investigation of the plasticity equations using the symmetries and the conservation laws reveals availability of application of these methods to studying and solving the other equations of solid mechanics.

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