Two-wavelet theory in Weinstein setting

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Abstract

In this paper we introduce the notion of a Weinstein two-wavelet. Then we establish and prove the resolution of the identity formula for the Weinstein continuous wavelet transform. Next, we give results on Calderón's type reproducing formula in the context of the Weinstein two-wavelet.

Keywords. Weinstein operator; Weinstein wavelet transform; Weinstein two-wavelet transform; Calderón’s type reproducing formula.

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1 Introduction

The Weinstein operator $\Delta_{W,\alpha}^d$ defined on $\mathbb{R}_{+}^{d+1} = \mathbb{R}^d \times (0, \infty)$, by

$$\Delta_{W,\alpha}^d = \sum_{j=1}^{d+1} \frac{\partial^2}{\partial x_j^2} + \frac{2\alpha + 1}{x_{d+1}} \frac{\partial}{\partial x_{d+1}} = \Delta_d + L_{\alpha}, \quad \alpha > -1/2,$$

where $\Delta_d$ is the Laplacian operator for the $d$ first variables and $L_{\alpha}$ is the Bessel operator for the last variable defined on $(0, \infty)$ by

$$L_{\alpha} u = \frac{\partial^2 u}{\partial x_{d+1}^2} + \frac{2\alpha + 1}{x_{d+1}} \frac{\partial u}{\partial x_{d+1}}.$$

The Weinstein operator $\Delta_{W,\alpha}^d$ has several applications in pure and applied mathematics, especially in fluid mechanics [2] [27].

Very recently, many authors have been investigating the behaviour of the Weinstein transform [25] with respect to several problems already studied for the classical Fourier transform. For instance, Heisenberg-type inequalities [17], Littlewood-Paley g-function [19], Shapiro and Hardyâ€”Sobolev type inequalities [16, 18], Paley-Wiener theorem [9], Uncertainty principles [12, 21, 24], multiplier Weinstein operator [20], wavelet and continuous wavelet transform [4, 11], Wigner transform and localization operators [22, 23], and so forth...

In the classical setting, the notion of wavelets was first introduced by Morlet in connection with his study of seismic traces and the mathematical foundations were
given by Grossmann and Morlet [6]. Later, Meyer and many other mathematicians recognized many classical results of this theory [8, 14]. Classical wavelets have wide applications, ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics [3, 5, 7].

Recently, the theory of wavelets and continuous wavelet transform has been extended and generalized in the context of differential-differences operators [4, 10, 11, 13]. Wavelet analysis has attracted attention for its ability to analyze rapidly changing transient signals. Any application using the Fourier like transform can be formulated using wavelets to provide more accurately localized temporal and frequency information. The reason for the extension from one wavelet to two wavelets comes from using wavelets to provide more accurately localized temporal and frequency information. The extra degree of flexibility in signal analysis and imaging when the localization is recognized many classical results of this theory [8, 14]. Classical wavelets have wide applications, ranging from signal analysis in geophysics and acoustics to quantum theory and pure mathematics [3, 5, 7].

Using the harmonic analysis associated with the Weinstein operator (generalized translation operators, generalized convolution, Weinstein transform, ...) and the same idea as for the classical case, we define and study in this paper the notion of a translation operators, generalized convolution, Weinstein transform, ...) and the same by extending one wavelet to two wavelets in Weinstein setting.

Next, we prove for the pair \((\varphi, \psi)\) an inversion formula for all \(f \in L^1_\alpha(\mathbb{R}^{d+1}_+)\) (resp. \(L^1_\alpha(\mathbb{R}^{d+1}+\cap L^\infty(\mathbb{R}^{d+1}))\), of the form

\[
f(y) = \frac{1}{C_{\varphi, \psi}} \int_0^\infty \int_{\mathbb{R}^{d+1}_+} \Phi^W_\varphi(f)(a, x) \psi_{a, x}(y) d\mu_\alpha(a, x) d\mu_\alpha(a, x),
\]

where for each \(y \in \mathbb{R}^{d+1}_+\) both the inner integral and the outer integral are absolutely convergent, but eventually not the double integral. In the end, we prove for the pair \((\varphi, \psi)\) a Calderón’s type formulas. For \(\varphi\) and \(\psi\) be two Weinstein wavelets in \(L^2_\alpha(\mathbb{R}^{d+1}_+)^2\) such that \((\varphi, \psi)\) be a Weinstein two-wavelet, \(C_{\varphi, \psi} \neq 0\) and \(F_W(\varphi)\), and \(F_W(\psi)\) are in \(L^\infty(\mathbb{R}^{d+1}_+)^2\). Then for all \(f \in L^2_\alpha(\mathbb{R}^{d+1}_+)^2\), the function

\[
f_{\gamma, \delta}(y) = \frac{1}{C_{\varphi, \psi}} \int_{\gamma}^{\delta} \int_{\mathbb{R}^{d+1}_+} \Phi^W_\varphi(f)(a, x) \psi_{a, x}(y) d\mu_\alpha(a, x) d\mu_\alpha(a, x), \quad y \in \mathbb{R}^{d+1}_+
\]

is constant for almost all \(\xi \in \mathbb{R}^{d+1}_+\). We prove for this Weinstein two-wavelet a Parseval type formula

\[
\int_{\mathcal{X}} \Phi^W_\varphi(f)(a, x) \overline{\Phi^W_\psi(g)(a, x)} d\mu_\alpha(a, x) = C_{\varphi, \psi} \int_{\mathbb{R}^{d+1}_+} f(x) g(x) d\mu_\alpha(x),
\]

for all \(\varphi\) and \(\psi\) in \(L^2_\alpha(\mathbb{R}^{d+1}_+)^2\) where \(\Phi^W_\varphi\) is a Weinstein continuous wavelet transform on \(\mathbb{R}^{d+1}_+\) defined for regular functions \(f\) on \(\mathbb{R}^{d+1}_+\) in [11] by

\[
\forall(a, x) \in \mathcal{X}, \quad \Phi^W_\varphi(f)(a, x) = \int_{\mathbb{R}^{d+1}_+} f(y) \overline{\varphi_{a, x}(y)} d\mu_\alpha(y) = \langle f, \varphi_{a, x}\rangle_{\alpha, 2}.
\]
belongs to $L_2^2(\mathbb{R}_+^{d+1})$ and satisfies
\[
\lim_{(\gamma, \delta) \to (0, \infty)} \|f_{\gamma, \delta} - f\|_{\alpha, 2} = 0.
\]

This paper is organized as follows. In Section 2, we recall some properties of harmonic analysis for the Weinstein operators. In Section 3, we define the two-wavelet in Weinstein in setting and prove for it a Parseval type formula and we establish for it an inversion formula. In last Section, we introduce a Caldéron’s type reproducing formula.

2 Preliminaires

For all $\lambda = (\lambda_1, \ldots, \lambda_{d+1}) \in \mathbb{C}^{d+1}$, the system
\[
\begin{align*}
\frac{\partial^2 u}{\partial x_j^2}(x) &= -\lambda_j^2 u(x), \quad \text{if } 1 \leq j \leq d \\
L_\alpha u(x) &= -\lambda_{d+1}^2 u(x), \\
u(0) &= 1, \quad \frac{\partial u}{\partial x_{d+1}}(0) = 0, \quad \frac{\partial u}{\partial x_j}(0) = -i\lambda_j, \quad \text{if } 1 \leq j \leq d
\end{align*}
\]
has a unique solution denoted by $\Lambda_\alpha^d(\lambda, \cdot)$, and given by
\[
\Lambda_\alpha^d(\lambda, x) = e^{-i<x', \lambda>} j_\alpha(x_{d+1}\lambda_{d+1})
\]
where $x = (x', x_{d+1})$, $x'_d = (x_1, x_2, \ldots, x_d)$, $\lambda = (\lambda', \lambda_{d+1})$, $\lambda'_d = (\lambda_1, \lambda_2, \ldots, \lambda_d)$ and $j_\alpha$ is the normalized Bessel function of index $\alpha$ defined by
\[
j_\alpha(x) = \Gamma(\alpha + 1) \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^k k! \Gamma(\alpha + k + 1)}.
\]
The function $(\lambda, x) \mapsto \Lambda_\alpha^d(\lambda, x)$ is called the Weinstein kernel and has a unique extension to $\mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$, and satisfied the following properties.

(i) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we have
\[
\Lambda_\alpha^d(\lambda, x) = \Lambda_\alpha^d(x, \lambda).
\]

(ii) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we have
\[
\Lambda_\alpha^d(\lambda, -x) = \Lambda_\alpha^d(-\lambda, x).
\]

(iii) For all $(\lambda, x) \in \mathbb{C}^{d+1} \times \mathbb{C}^{d+1}$ we get
\[
\Lambda_\alpha^d(\lambda, 0) = 1.
\]

(iv) For all $\nu \in \mathbb{N}^{d+1}$, $x \in \mathbb{R}^{d+1}$ and $\lambda \in \mathbb{C}^{d+1}$ we have
\[
|D_\lambda^\nu \Lambda_\alpha^d(\lambda, x)| \leq \|x\|^{||\nu||} e^{\|x\|\|\text{Im}\lambda\|}.
\]
where \( D_\nu = \partial_\nu / (\partial_1^{\nu_1} \ldots \partial_{d+1}^{\nu_{d+1}}) \) and \(|\nu| = \nu_1 + \ldots + \nu_{d+1}\). In particular, for all \((\lambda, x) \in \mathbb{R}^{d+1} \times \mathbb{R}^{d+1}\), we have
\[
|\Lambda_\alpha^d(\lambda, x)| \leq 1. \tag{2.3}
\]

In the following we denote by

(i) \(-\lambda = (-\lambda', \lambda_{d+1})\)

(ii) \(C_*(\mathbb{R}^{d+1})\), the space of continuous functions on \(\mathbb{R}^{d+1}\), even with respect to the last variable.

(iii) \(S_*(\mathbb{R}^{d+1})\), the space of the \(C^\infty\) functions, even with respect to the last variable, and rapidly decreasing together with their derivatives.

(iv) \(S_*(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1})\), the Schwartz space of rapidly decreasing functions on \(\mathbb{R}^{d+1} \times \mathbb{R}^{d+1}\) even with respect to the last two variables.

(v) \(D_*(\mathbb{R}^{d+1})\), the space of \(C^\infty\)-functions on \(\mathbb{R}^{d+1}\) which are of compact support, even with respect to the last variable.

(vi) \(L^p_\alpha(\mathbb{R}^{d+1}_+), 1 \leq p \leq \infty\), the space of measurable functions \(f\) on \(\mathbb{R}^{d+1}_+\) such that
\[
\|f\|_{\alpha,p} = \left( \int_{\mathbb{R}^{d+1}_+} |f(x)|^p \, d\mu_\alpha(x) \right)^{1/p} < \infty, \quad p \in [1, \infty),
\]
\[
\|f\|_{\alpha,\infty} = \operatorname{ess sup}_{x \in \mathbb{R}^{d+1}_+} |f(x)| < \infty,
\]
where \(d\mu_\alpha(x)\) is the measure on \(\mathbb{R}^{d+1}_+ = \mathbb{R}^d \times (0, \infty)\) given by
\[
d\mu_\alpha(x) = \frac{x_2^{2\alpha+1}}{(2\pi)^{d/2} \Gamma^2(\alpha + 1)} \, dx.
\]

For a radial function \(\varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+)\) the function \(\tilde{\varphi}\) defined on \(\mathbb{R}_+\) such that \(\varphi(x) = \tilde{\varphi}(|x|)\), for all \(x \in \mathbb{R}^{d+1}_+\), is integrable with respect to the measure \(r^{2\alpha+d+1}dr\), and we have
\[
\int_{\mathbb{R}^{d+1}_+} \varphi(x) \, d\mu_\alpha(x) = a_\alpha \int_0^\infty \tilde{\varphi}(r) r^{2\alpha+d+1}dr, \tag{2.4}
\]
where
\[
a_\alpha = \frac{1}{2^{\alpha+\frac{d}{2}} \Gamma(\alpha + \frac{d}{2} + 1)}.
\]

The Weinstein transform generalizing the usual Fourier transform, is given for \(\varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+)\) and \(\lambda \in \mathbb{R}^{d+1}_+\), by
\[
\mathcal{F}_W(\varphi)(\lambda) = \int_{\mathbb{R}^{d+1}_+} \varphi(x) \Lambda_\alpha^d(x, \lambda) \, d\mu_\alpha(x), \tag{2.5}
\]

We list some known basic properties of the Weinstein transform as follows. For the proofs, we refer \[11\] [13].
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(i) For all \( \varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+) \), the function \( \mathcal{F}_W(\varphi) \) is continuous on \( \mathbb{R}^{d+1}_+ \) and we have
\[
\|\mathcal{F}_W\varphi\|_{\alpha,\infty} \leq \|\varphi\|_{\alpha,1}.
\] (2.6)

(ii) The Weinstein transform is a topological isomorphism from \( \mathcal{S}_*(\mathbb{R}^{d+1}) \) onto itself. The inverse transform is given by
\[
\mathcal{F}_W^{-1}\varphi(\lambda) = \mathcal{F}_W\varphi(-\lambda), \quad \text{for all } \lambda \in \mathbb{R}^{d+1}_+.
\] (2.7)

(iii) For all \( f \in \mathcal{D}_*(\mathbb{R}^{d+1}) \) (resp. \( \mathcal{S}_*(\mathbb{R}^{d+1}) \)), we have the following relations
\[
\forall \lambda \in \mathbb{R}^{d+1}_+, \quad \mathcal{F}_W(\overline{f})(\lambda) = \overline{\mathcal{F}_W(f)(\lambda)},
\] (2.8)
\[
\forall \lambda \in \mathbb{R}^{d+1}_+, \quad \mathcal{F}_W(f)(\lambda) = \mathcal{F}_W(\overline{f})(-\lambda),
\] (2.9)
where \( \overline{f} \) is the function defined by
\[
\forall \lambda \in \mathbb{R}^{d+1}_+, \quad \overline{f}(\lambda) = f(-\lambda).
\]

(iv) Parseval’s formula: For all \( \varphi, \phi \in \mathcal{S}_*(\mathbb{R}^{d+1}) \), we have
\[
\int_{\mathbb{R}^{d+1}_+} \varphi(x)\overline{\phi(x)}d\mu_\alpha(x) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}_W(\varphi)(x)\overline{\mathcal{F}_W(\phi)(x)}d\mu_\alpha(x).
\] (2.10)

(v) Plancherel’s formula: For all \( \varphi \in L^2_\alpha(\mathbb{R}^{d+1}_+) \), we have
\[
\|\mathcal{F}_W\varphi\|_{\alpha,2} = \|\varphi\|_{\alpha,2}.
\] (2.11)

(vi) Plancherel Theorem: The Weinstein transform \( \mathcal{F}_W \) extends uniquely to an isometric isomorphism on \( L^2_\alpha(\mathbb{R}^{d+1}_+) \).

(vii) Inversion formula: Let \( \varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+) \) such that \( \mathcal{F}_W\varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+) \), then we have
\[
\varphi(\lambda) = \int_{\mathbb{R}^{d+1}_+} \mathcal{F}_W\varphi(x)\Lambda_\alpha^d(-\lambda, x)d\mu_\alpha(x), \quad \text{a.e. } \lambda \in \mathbb{R}^{d+1}_+.
\] (2.12)

Using relations (2.6) and (2.11) with Marcinkiewicz’s interpolation theorem [25], we deduce that for every \( \varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+) \) for all \( 1 \leq p \leq 2 \), the function \( \mathcal{F}_W(\varphi) \in L^q_\alpha(\mathbb{R}^{d+1}_+) \), \( q = p/(p-1) \), and
\[
\|\mathcal{F}_W\varphi\|_{\alpha,q} \leq \|\varphi\|_{\alpha,p}.
\] (2.13)

**Definition 2.1.** The translation operator \( \tau^\alpha_x \), \( x \in \mathbb{R}^{d+1}_+ \) associated with the Weinstein operator \( \Delta^d_{W,\alpha} \), is defined for a continuous function \( \varphi \) on \( \mathbb{R}^{d+1}_+ \), which is even with respect to the last variable and for all \( y \in \mathbb{R}^{d+1}_+ \) by
\[
\tau^\alpha_x \varphi(y) = C_\alpha \int_0^\pi \varphi(x + y\sqrt{x^2_{d+1}} + y^2_{d+1} + 2x_{d+1}y_{d+1}\cos \theta)(\sin \theta)^{2\alpha}d\theta,
\]
with
\[
C_\alpha = \frac{\Gamma(\alpha + 1)}{\sqrt{\pi}\Gamma(\alpha + 1/2)}.
\]
By using the Weinstein kernel, we can also define a generalized translation, for a
function \( \varphi \in S_\alpha(\mathbb{R}^{d+1}) \) and \( y \in \mathbb{R}^{d+1}_+ \) the generalized translation \( \tau_x^\alpha \varphi \) is defined by the
following relation

\[
F_W(\tau_x^\alpha \varphi)(y) = \Lambda^d(x, y) F_W(\varphi)(y).
\]  

(2.14)

In the following proposition, we give some properties of the Weinstein translation
operator:

**Proposition 2.2.** The translation operator \( \tau_x^\alpha \), \( x \in \mathbb{R}^{d+1}_+ \) satisfies the following properties.

i). For \( \varphi \in C_\alpha(\mathbb{R}^{d+1}) \), we have for all \( x, y \in \mathbb{R}^{d+1}_+ \)

\[
\tau_x^\alpha \varphi(y) = \tau_y^\alpha \varphi(x) \text{ and } \tau_0^\alpha \varphi = \varphi.
\]  

(2.15)

ii). Let \( \varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+) \), \( 1 \leq p \leq \infty \) and \( x \in \mathbb{R}^{d+1}_+ \). Then \( \tau_x^\alpha \varphi \) belongs to \( L^p_\alpha(\mathbb{R}^{d+1}_+) \) and we have

\[
\|\tau_x^\alpha \varphi\|_{a,p} \leq \|\varphi\|_{a,p}.
\]  

(2.16)

**Proposition 2.3.** Let \( \varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+) \). Then for all \( x \in \mathbb{R}^{d+1}_+ \),

\[
\int_{\mathbb{R}^{d+1}_+} \tau_x^\alpha \varphi(y) d\mu_\alpha(y) = \int_{\mathbb{R}^{d+1}_+} \varphi(y) d\mu_\alpha(y).
\]  

(2.17)

**Proof.** The result comes from combination identities (2.12) and (2.14). \( \square \)

By using the generalized translation, we define the generalized convolution product
\( \varphi \ast \psi \) of the functions \( \varphi, \psi \in L^1_\alpha(\mathbb{R}^{d+1}_+) \) as follows

\[
\varphi \ast \psi(x) = \int_{\mathbb{R}^{d+1}_+} \tau_x^\alpha \varphi(-y) \psi(y) d\mu_\alpha(y).
\]  

(2.18)

This convolution is commutative and associative, and it satisfies the following properties.

**Proposition 2.4.** i) For all \( \varphi, \psi \in L^1_\alpha(\mathbb{R}^{d+1}_+) \), \( \text{resp. } \varphi, \psi \in S_\alpha(\mathbb{R}^{d+1}_+) \), then \( \varphi \ast \psi \in L^1_\alpha(\mathbb{R}^{d+1}_+) \), \( \text{resp. } \varphi \ast \psi \in S_\alpha(\mathbb{R}^{d+1}_+) \) and we have

\[
F_W(\varphi \ast \psi) = F_W(\varphi) F_W(\psi).
\]  

(2.19)

ii) Let \( p, q, r \in [1, \infty] \), such that \( \frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1 \). Then for all \( \varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+) \) and
\( \psi \in L^q_\alpha(\mathbb{R}^{d+1}_+) \) the function \( \varphi \ast \psi \) belongs to \( L^r_\alpha(\mathbb{R}^{d+1}_+) \) and we have

\[
\|\varphi \ast \psi\|_{a,r} \leq \|\varphi\|_{a,p} \|\psi\|_{a,q}.
\]  

(2.20)

iii) Let \( \varphi, \psi \in L^2_\alpha(\mathbb{R}^{d+1}_+) \). Then

\[
\varphi \ast \psi = F_W^{-1} \left( F_W(\varphi) F_W(\psi) \right).
\]  

(2.21)

iv) Let \( \varphi, \psi \in L^2_\alpha(\mathbb{R}^{d+1}_+) \). Then \( \varphi \ast \psi \) belongs to \( L^2_\alpha(\mathbb{R}^{d+1}_+) \) if and only if \( F_W(\varphi) F_W(\psi) \) belongs to \( L^2_\alpha(\mathbb{R}^{d+1}_+) \) and we have

\[
F_W(\varphi \ast \psi) = F_W(\varphi) F_W(\psi).
\]  

(2.22)

v) Let \( \varphi, \psi \in L^2_\alpha(\mathbb{R}^{d+1}_+) \). Then

\[
\|\varphi \ast \psi\|_{a,2} = \| F_W(\varphi) F_W(\psi) \|_{a,2};
\]  

(2.23)

where both sides are finite or infinite.
3 Weinstein two-wavelet theory

In the following, we denote by 
\[ X = \{(a, x) : x \in \mathbb{R}^{d+1} \text{ and } a > 0\}. \]

\[ L^p_\alpha(X), \ p \in [1, \infty] \] the space of measurable functions \( \varphi \) on \( X \) such that 
\[ \|\varphi\|_{L^p_\alpha(X)} := \left( \int_X |\varphi(a, x)|^p d\mu_\alpha(a, x) \right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \]
\[ \|\varphi\|_{L^\infty_\alpha(X)} = \text{ess sup}_{(a, x) \in X} |\varphi(a, x)| < \infty, \]

where the measure \( \mu_\alpha(a, x) \) is defined on \( X \) by 
\[ d\mu_\alpha(a, x) = \frac{d\mu_\alpha(x)da}{a^{2\alpha+d+3}}. \]

**Definition 3.1.** [4] A classical wavelet on \( \mathbb{R}^{d+1}_+ \) is a measurable function \( \varphi \) on \( \mathbb{R}^{d+1}_+ \) satisfying for almost all \( \xi \in \mathbb{R}^{d+1}_+ \), the condition
\[ 0 < C_\varphi = \int_0^\infty |F_W(\varphi)(a\xi)|^2 da < \infty. \]  
(3.1)

We extend the notion of the wavelet to the two-wavelet in Weinstein setting as follows

**Definition 3.2.** Let \( \varphi \) and \( \psi \) be in \( L^2_\alpha(\mathbb{R}^{d+1}_+) \). We say that the pair \( (\varphi, \psi) \) is a Weinstein two-wavelet on \( \mathbb{R}^{d+1}_+ \) if the following integral
\[ C_{\varphi, \psi} = \int_0^\infty F_W(\psi)(a\xi)\overline{F_W(\varphi)(a\xi)} \frac{da}{a} \]
(3.2)
is constant for almost all \( \xi \in \mathbb{R}^{d+1}_+ \) and we call the number \( C_{\varphi, \psi} \) the Weinstein two-wavelet constant associated to the functions \( \varphi \) and \( \psi \).

It is to highlight that if \( u \) is a Weinstein wavelet then the pair \( (\varphi, \psi) \) is a Weinstein two-wavelet, and \( C_{\varphi, \psi} \) coincides with \( C_\varphi \).

Let \( a > 0 \) and \( \varphi \) be a measurable function. We consider the function \( \varphi_a \) defined by
\[ \forall x \in \mathbb{R}^{d+1}_+, \quad \varphi_a(x) = \frac{1}{a^{2\alpha+d+2}} \varphi \left( \frac{x}{a} \right). \]  
(3.3)

**Proposition 3.3.** 1. Let \( a > 0 \) and \( \varphi \in L^p_\alpha(\mathbb{R}^{d+1}_+) \), \( p \in [1, \infty] \). The function \( \varphi_a \) belongs to \( L^p_\alpha(\mathbb{R}^{d+1}_+) \) and we have
\[ \|\varphi_a\|_{L^p_\alpha} = a^{(2\alpha+d+2)(\frac{1}{p}-1)} \|\varphi\|_{L^p_\alpha}. \]  
(3.4)

2. Let \( a > 0 \) and \( \varphi \in L^1_\alpha(\mathbb{R}^{d+1}_+) \cup L^2_\alpha(\mathbb{R}^{d+1}_+) \). Then, we have
\[ F_W(\varphi_a)(\xi) = F_W(\varphi)(a\xi), \quad \xi \in \mathbb{R}^{d+1}_+. \]  
(3.5)
For $a > 0$ and $\varphi \in L^2_0(\mathbb{R}^{d+1}_+)$, we consider the family $\varphi_{a,x}$, $x \in \mathbb{R}^{d+1}_+$ of Weinstein wavelets on $\mathbb{R}^{d+1}_+$ in $L^2_0(\mathbb{R}^{d+1}_+)$ defined by

$$\forall y \in \mathbb{R}^{d+1}_+, \quad \varphi_{a,x} = a^{\alpha+\frac{d}{2}} \varphi(y).$$

**Remark 3.4.**

1. Let $\varphi$ be a function in $L^2_0(\mathbb{R}^{d+1}_+)$, then we have

$$\forall (a,x) \in \mathcal{X}, \quad \|\varphi_{a,x}\|_{\alpha,2} \leq \|\varphi\|_{\alpha,2}.$$  

2. Let $p \in [1, \infty]$ and $\varphi$ be a function in $L^p_0(\mathbb{R}^{d+1}_+)$, then we have

$$\forall (a,x) \in \mathcal{X}, \quad \|\varphi_{a,x}\|_{\alpha,p} \leq a^{(2\alpha+d+2)(\frac{1}{p} - \frac{1}{2})}\|\varphi\|_{\alpha,p}.$$  

**Definition 3.5.** [11] Let $\varphi$ be a Weinstein wavelet on $\mathbb{R}^{d+1}_+$ in $L^2_0(\mathbb{R}^{d+1}_+)$. The Weinstein continuous wavelet transform $\Phi^W_\varphi$ on $\mathbb{R}^{d+1}_+$ is defined for regular functions $f$ on $\mathbb{R}^{d+1}_+$ by

$$\forall (a,x) \in \mathcal{X}, \quad \Phi^W_\varphi(f)(a,x) = \int_{\mathbb{R}^{d+1}_+} f(y)\overline{\varphi_{a,x}(y)}d\mu_\alpha(y) = \langle f, \varphi_{a,x} \rangle_{\alpha,2}. $$

This transform can also be written in the form

$$\Phi^W_\varphi(f)(a,x) = a^{\alpha+\frac{d}{2}} \overline{\hat{f}} \ast \overline{\varphi}(x).$$  

**Remark 3.6.**

1. Let $\varphi$ be a function in $L^p_0(\mathbb{R}^{d+1}_+)$, and let $f$ be a function in $L^p_0(\mathbb{R}^{d+1}_+)$, with $p \in [1, \infty]$, we define the Weinstein continuous wavelet transform $\Phi^W_\varphi(f)$ by the relation (3.14).

2. Let $\varphi$ be a Weinstein wavelet on $\mathbb{R}^{d+1}_+$ in $L^2_0(\mathbb{R}^{d+1}_+)$. Then from the relations (3.7) and (3.9), we have for all $f \in L^2_0(\mathbb{R}^{d+1}_+)$

$$\|\Phi^W_\varphi(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,2}\|\varphi\|_{\alpha,2}. $$

3. Let $\varphi$ be a function in $L^p_0(\mathbb{R}^{d+1}_+)$, with $p \in [1, \infty]$, then from the inequality (2.20) and the identity (3.16), we have for all $f \in L^p_0(\mathbb{R}^{d+1}_+)$

$$\|\Phi^W_\varphi(f)\|_{\alpha,\infty} \leq \|f\|_{\alpha,\infty}\|\varphi\|_{\alpha,p}. $$

The following Theorem generalizes the Parseval’s formula for the continuous Weinstein wavelet transform proved by Mejjaoli [11].

**Theorem 3.7.** Let $(\varphi, \psi)$ be a Weinstein two-wavelet. Then for all $f$ and $g$ in $L^2_0(\mathbb{R}^{d+1}_+)$, we have the following Parseval type formula

$$\int_{\mathcal{X}} \Phi^W_\varphi(f)(a,x)\overline{\Phi^W_\psi(g)(a,x)}d\mu_\alpha(a,x) = C_{\varphi,\psi} \int_{\mathbb{R}^{d+1}_+} f(x)\overline{g(x)}d\mu_\alpha(x),$$

where $C_{\varphi,\psi}$ is the Weinstein two-wavelet constant associated to the functions $\varphi$ and $\psi$ given by the identity (3.2).
Proof. According to Fubini’s Theorem, the relation (3.10) and Parseval’s formula for the Weinstein transform (2.10), we obtain
\[
\int_X \Phi^W_\varphi (f)(a, x) \Phi^W_\psi (g)(a, x) d\mu_\alpha(a, x)
= \int_0^\infty \int_{\mathbb{R}^{d+1}} \tilde{f} \ast \overline{\varphi}_a(x) \tilde{g} \ast \overline{\psi}_a(x) da^2 d\mu_\alpha(a, x)
= \int_0^\infty \int_{\mathbb{R}^{d+1}} \tilde{f} \ast \overline{\varphi}_a(x) \tilde{g} \ast \overline{\psi}_a(x) d\mu_\alpha(a) da
= \int_0^\infty \int_{\mathbb{R}^{d+1}} F_W(\tilde{f}) (\xi) \overline{F_W(\tilde{g}) (\xi)} F_W(\overline{\varphi}_a)(\xi) \overline{F_W(\overline{\psi}_a)(\xi)} d\mu_\alpha(\xi) da
= \int_{\mathbb{R}^{d+1}} F_W(\tilde{f}) (\xi) \overline{F_W(\tilde{g}) (\xi)} \left( \int_0^\infty F_W(\overline{\varphi})(a\xi) F_W(\psi)(-a\xi) \frac{da}{a} \right) d\mu_\alpha(\xi).
\]
Moreover, using the relations (2.8) and (2.9), we conclude that
\[
\int_X \Phi^W_\varphi (f)(a, x) \Phi^W_\psi (g)(a, x) d\mu_\alpha(a, x)
= \int_{\mathbb{R}^{d+1}} F_W(f)(\xi) \overline{F_W(g)(\xi)} \left( \int_0^\infty F_W(\varphi)(a\xi) F_W(\psi)(a\xi) \frac{da}{a} \right) d\mu_\alpha(\xi)
= C_{\varphi, \psi} \int_{\mathbb{R}^{d+1}} F_W(f)(\xi) \overline{F_W(g)(\xi)} d\mu_\alpha(\xi).
\]
Finally, we get the desired result using the Parseval’s formula for the Weinstein transform (2.10).

In the particular case of the previous theorem when \( \varphi = \psi \) and \( f = g \), we obtain the following PlancherelâĂŹs formula for the Weinstein continuous wavelet transform provided in [11]
\[
\int_X |\Phi^W_\varphi (f)(a, x)|^2 d\mu_\alpha(a, x) = C_{\varphi} \int_{\mathbb{R}^{d+1}} |f(x)|^2 d\mu_\alpha(x), \quad (3.14)
\]
where
\[
C_{\varphi} = C_{\varphi, \varphi} = \int_0^\infty |F_W(\varphi)(a\xi)|^2 \frac{da}{a}. \quad (3.15)
\]
From the Parseval type formula in Theorem 3.7 we deduce the following orthogonality result.

**Corollary 3.8.** Let \( (\varphi, \psi) \) be a Weinstein two-wavelet. Then we have the following assertion: If the Weinstein two-wavelet constant \( C_{\varphi, \psi} = 0 \), then \( \Phi^W_\varphi (L^2_\alpha(\mathbb{R}^{d+1})) \) and \( \Phi^W_\psi (L^2_\alpha(\mathbb{R}^{d+1})) \) are orthogonal.

**Theorem 3.9.** (Inversion formula) Let \( (\varphi, \psi) \) be a Weinstein two-wavelet. For all \( f \in L^1_\alpha(\mathbb{R}^{d+1}) \) (resp. \( L^2_\alpha(\mathbb{R}^{d+1}) \)) such that \( F_W(f) \) belongs to \( f \in L^1_\alpha(\mathbb{R}^{d+1}) \) (resp. \( L^2_\alpha(\mathbb{R}^{d+1}) \cap L^\infty(\mathbb{R}^{d+1}) \)), we have
\[
f(y) = \frac{1}{C_{\varphi, \psi}} \int_0^\infty \int_{\mathbb{R}^{d+1}} \Phi^W_\varphi (f)(a, x) \psi_{a,x}(y) d\mu_\alpha(a, x), \quad a.e. \quad (3.16)
\]
where for each \( y \in \mathbb{R}^{d+1} \) both the inner integral and the outer integral are absolutely convergent, but eventually not the double integral.

**Proof.** Using similar ideas as in the proof in [26, Theorem 6.III.3], we obtain the relation (3.10). \( \square \)

### 4 Calderón’s reproducing formulas

The main result of this section is to establish a Calderón’s type formulas for the Weinstein two-wavelet transform under the following assumptions:

- **(A1)** Let \( \varphi \) and \( \psi \) be two Weinstein wavelets in \( L^2_\alpha(\mathbb{R}^{d+1}_+) \) such that \((\varphi, \psi)\) be a Weinstein two-wavelet, and \( \mathcal{F}_W(\varphi) \) and \( \mathcal{F}_W(\psi) \) are in \( L^\infty_\alpha(\mathbb{R}_+^{d+1}) \).

- **(A2)** The Weinstein two-wavelet constant \( C_{\varphi,\psi} \neq 0 \).

**Theorem 4.1.** (Calderón’s type formulas) Let \( \varphi \) and \( \psi \) be two-Weinstein wavelets satisfying the assumptions \((A_1)\) and \((A_2)\) and \( 0 < \gamma < \delta < \infty \). Then for all \( f \in L^2_\alpha(\mathbb{R}^{d+1}_+) \), the function

\[
\lf_{\gamma, \delta}(y) = \frac{1}{C_{\varphi, \psi}} \int_\gamma^\delta \int_{\mathbb{R}^{d+1}_+} \phi(a,x)\psi_a(y) d\mu(a)(x) \frac{da}{a^{2\alpha+d+3}}, \quad y \in \mathbb{R}^{d+1}_+ \tag{4.1}
\]

belongs to \( L^2_\alpha(\mathbb{R}^{d+1}_+) \) and satisfies

\[
\lim_{(\gamma, \delta) \to (0^+, \infty)} \|f_{\gamma, \delta} - f\|_{\alpha,2} = 0. \tag{4.2}
\]

In the order to prove this theorem we need the following Lemmas.

**Lemma 4.2.** Let \( \varphi \) and \( \psi \) be two-Weinstein wavelets satisfying the assumptions \((A_1)\) and \((A_2)\) and \( f \in L^2_\alpha(\mathbb{R}^{d+1}_+) \). Then we have the following assertions:

(i) The functions \( \lf (f \ast \varphi_a) \ast \psi_a(\xi) = \mathcal{F}_W(f)(\xi)\mathcal{F}_W(\varphi_a)(\xi)\mathcal{F}_W(\psi_a)(\xi), \quad \xi \in \mathbb{R}^{d+1}_+ \tag{4.3} \)

(ii) We have the following inequality

\[
\|\mathcal{F}_W(\lf \ast \overline{\varphi_a}) \ast \psi_a\|_{\alpha,2} \leq \|f\|_{\alpha,2}\|\mathcal{F}_W(\varphi)\|_{\alpha,\infty}\|\mathcal{F}_W(\psi)\|_{\alpha,\infty}. \tag{4.4}
\]

**Proof.** (i) According to the relations (2.8), (2.9) and Proposition 2.4 (v) we obtain

\[
\mathcal{F}_W((\lf \ast \overline{\varphi_a})(\xi)) = \mathcal{F}_W(\lf)(\xi)\mathcal{F}_W(\varphi_a)(\xi) = \lf(\varphi_a)(\xi),
\]

Hence, we have

\[
\mathcal{F}_W((\lf \ast \overline{\varphi_a})(\xi)) = \mathcal{F}_W(\lf)(\xi)\mathcal{F}_W(\varphi_a)(\xi). \tag{4.5}
\]
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After that, we consider the function $G$ defined on $\mathbb{R}^{d+1}$ by

$$G(\xi) = \mathcal{F}_W((\hat{f} * \varphi_a)\cdot(\hat{\psi}_a))(\xi).$$

Therefore

$$\mathcal{F}_W((\hat{f} * \varphi_a)\cdot(\hat{\psi}_a))(\xi) = \mathcal{F}_W(G \cdot(\hat{\psi}_a))(\xi), \quad \xi \in \mathbb{R}^{d+1}.$$

According to Proposition 2.4 (v), we deduce that the function $G$ belongs to $L^2_\alpha(\mathbb{R}^{d+1})$ and we have

$$\mathcal{F}_W(G \cdot(\hat{\psi}_a))(\xi) = \mathcal{F}_W(G \cdot(\hat{\psi}_a))(\xi), \quad \xi \in \mathbb{R}^{d+1}.$$  \hspace{0.5cm} (4.6)

Finally, we obtain the result by combining the relations (4.5) and (4.6).

(ii) From the assertion (i), we have

$$\int_{\mathbb{R}^{d+1}} |\mathcal{F}_W((\hat{f} * \varphi_a)\cdot(\hat{\psi}_a))(\xi)|^2 d\mu_\alpha(x) = \int_{\mathbb{R}^{d+1}} |\mathcal{F}_W(f)(\xi)|^2 |\mathcal{F}_W(\varphi_a)(\xi)|^2 |\mathcal{F}_W(\psi_a)(\xi)|^2 d\mu_\alpha(x).$$

Therefore, according to Plancherel formula for Weinstein transform (2.11) and the fact that $\mathcal{F}_W(\varphi_a)$ and $\mathcal{F}_W(\psi_a)$ belongs to $L^\infty_\alpha(\mathbb{R}^{d+1})$, we deduce that

$$\|\mathcal{F}_W((\hat{f} * \varphi_a)\cdot(\hat{\psi}_a)\cdot(\hat{\psi}_a))\|_{a,2} \leq \|f\|_{a,2} \|\mathcal{F}_W(\varphi_a)\|_{a,\infty} \|\mathcal{F}_W(\psi_a)\|_{a,\infty}. $$

In the end, we conclude the result from the relation (3.5).

Lemma 4.3. Let $\varphi$ and $\psi$ be two-Weinstein wavelets satisfying the assumptions $(A_1)$ and $(A_2)$. Then the function $K_{\gamma,\delta}$ defined as follows

$$K_{\gamma,\delta}(\xi) = \frac{1}{C_{\varphi,\psi}} \int_{\gamma}^{\delta} \frac{\mathcal{F}_W(\varphi_a)(\xi) \mathcal{F}_W(\psi_a)(\xi)}{a} da, \quad \xi \in \mathbb{R}^{d+1} \hspace{0.5cm} (4.7)$$

satisfies, for almost all $\xi \in \mathbb{R}^{d+1}$:

$$0 < K_{\gamma,\delta}(\xi) \leq \frac{\sqrt{C_\varphi C_\psi}}{C_{\varphi,\psi}}, $$ \hspace{0.5cm} (4.8)

and

$$\lim_{(\gamma,\delta) \to (0,\infty)} K_{\gamma,\delta}(\xi) = 1. \hspace{0.5cm} (4.9)$$

Proof. According to the Cauchy-Schwarz inequality and the identity (3.15), we have for almost all $\xi \in \mathbb{R}^{d+1}$

$$|K_{\gamma,\delta}(\xi)| \leq \frac{1}{C_{\varphi,\psi}} \left( \int_{\gamma}^{\delta} |\mathcal{F}_W(\varphi_a)(\xi)|^2 da \right)^{\frac{1}{2}} \left( \int_{\gamma}^{\delta} |\mathcal{F}_W(\psi_a)(\xi)|^2 da \right)^{\frac{1}{2}} \leq \frac{1}{C_{\varphi,\psi}} \left( \int_{0}^{\infty} |\mathcal{F}_W(\varphi_a)(\xi)|^2 da \right)^{\frac{1}{2}} \left( \int_{0}^{\infty} |\mathcal{F}_W(\psi_a)(\xi)|^2 da \right)^{\frac{1}{2}} = \frac{\sqrt{C_\varphi C_\psi}}{C_{\varphi,\psi}}.$$  \hspace{0.5cm} \hspace{0.5cm}

On the other hand, it’s clear that for almost all $\xi \in \mathbb{R}^{d+1}$

$$\lim_{(\gamma,\delta) \to (0,\infty)} K_{\gamma,\delta}(\xi) = 1.$$  \hspace{0.5cm} \hspace{0.5cm}

This completes the proof. \qed
Theorem 4.4. Let \( \varphi \) and \( \psi \) be two-Weinstein wavelets satisfying the assumptions \((A_1)\) and \((A_2)\). Then the function \( f_{\gamma, \delta} \) defined by the relation (4.1) belongs to \( L^2_\alpha(\mathbb{R}^{d+1}) \) and satisfies

\[
\mathcal{F}_W(f_{\gamma, \delta})(\xi) = \mathcal{F}_W(f)(\xi)K_{\gamma, \delta}(\xi), \quad \xi \in \mathbb{R}^{d+1},
\]

where \( K_{\gamma, \delta} \) is the function given by the relation (4.7).

Proof. In a first step, we try to show that \( f_{\gamma, \delta} \) belongs to \( L^2_\alpha(\mathbb{R}^{d+1}) \). From the definition of the Weinstein continuous wavelet transform (3.9) and the relations (2.15) and (3.6) we get

\[
f_{\gamma, \delta}(y) = \frac{1}{C_{\varphi, \psi}} \int_\gamma \int_{\mathbb{R}^{d+1}} (\hat{f} \ast \overline{\varphi}_a)(x) \tau^\alpha_y \psi_a(x) d\mu_\alpha(x) \, da_a.
\]

On the other hand, from the definition of the convolution product associated to the Weinstein transform given by the relation (2.18), we have

\[
\int_{\mathbb{R}^{d+1}} (\hat{f} \ast \overline{\varphi}_a)(x) \tau^\alpha_y \psi_a(x) d\mu_\alpha(x) = \int_{\mathbb{R}^{d+1}} (\hat{f} \ast \overline{\psi}_a)(x) \tau^\alpha_y \psi_a(-x) d\mu_\alpha(x)
\]

\[
= (\hat{f} \ast \overline{\varphi}_a) \ast \psi_a(y).
\]

So, we have

\[
f_{\gamma, \delta}(y) = \frac{1}{C_{\varphi, \psi}} \int_\gamma (\hat{f} \ast \overline{\varphi}_a) \ast \psi_a(y) \, da_a.
\]

By using Hölder’s inequality for the measure \( da_a \), we obtain

\[
|f_{\gamma, \delta}(y)|^2 \leq \frac{1}{|\varphi, \psi|^2} \left( \int_\gamma \left( \int_{\mathbb{R}^{d+1}} |(\hat{f} \ast \overline{\varphi}_a) \ast \psi_a(y)|^2 \, da_a \right) \, da_a \right) \cdot
\]

Applying Fubuni-Tonelli’s theorem, after integrating the previous inequality, we get

\[
\int_{\mathbb{R}^{d+1}} |f_{\gamma, \delta}(y)|^2 \, d\mu_\alpha(y) \leq \frac{1}{|\varphi, \psi|^2} \left( \int_\gamma \left( \int_{\mathbb{R}^{d+1}} \, d\mu_\alpha(y) \right) \, da_a \right) \cdot
\]

According to the Parseval’s formula for the Weinstein transform (2.10) and the assertion \((i)\) of the Lemma 4.2, we deduce that

\[
\int_{\mathbb{R}^{d+1}} |f_{\gamma, \delta}(y)|^2 \, d\mu_\alpha(y) \leq \frac{1}{|\varphi, \psi|^2} \left( \int_\gamma \left( \int_{\mathbb{R}^{d+1}} \right) \, d\mu_\alpha(y) \right) \, da_a \cdot
\]

On the other hand, from the identity (3.15) and the relation (3.5), we have

\[
\int_\gamma |\mathcal{F}_W(\varphi_a)(\xi)|^2 |\mathcal{F}_W(\psi_a)(\xi)|^2 \, da_a \leq C_{\varphi} \|\mathcal{F}_W(\varphi)\|_{\alpha, \infty}.
\]
By applying Hölder’s inequality to the right hand side of the previous equality, we obtain
\[
\int_{\mathbb{R}^{d+1}} |f_{\gamma,\delta}(y)|^2 d\mu_\alpha(y) \leq \frac{C_\psi}{|C_{\varphi,\psi}|^2} \left( \int_{\gamma}^\delta \frac{d\alpha}{\alpha} \right) \|F_W(\varphi)\|_{\alpha,\infty} \|F_W(f)\|_{\alpha,2}.
\]

Finally, from the Plancherel’s formula for the Weinstein transform \([2.11]\), we get
\[
\int_{\mathbb{R}^{d+1}} |f_{\gamma,\delta}(y)|^2 d\mu_\alpha(y) \leq \frac{C_\psi}{|C_{\varphi,\psi}|^2} \left( \int_{\gamma}^\delta \frac{d\alpha}{\alpha} \right) \|F_W(\varphi)\|_{\alpha,\infty} \|f\|_{\alpha,2} \leq \infty,
\]
which implies that \(f_{\gamma,\delta}\) belongs to \(L^2_\alpha(\mathbb{R}^{d+1})\).

In the second step, we prove the relation \((4.10)\). Let \(h \in \mathcal{S}_*(\mathbb{R}^{d+1})\), then \(F_{W}^{-1}(h)\) belongs to \(\mathcal{S}_*(\mathbb{R}^{d+1})\) and from the relation \((4.11)\), we have
\[
\int_{\mathbb{R}^{d+1}} f_{\gamma,\delta}(y)\overline{F_{W}^{-1}(h)}(y) d\mu_\alpha(y)
= \int_{\mathbb{R}^{d+1}} \left( \frac{1}{|C_{\varphi,\psi}|} \int_{\gamma}^\delta (\hat{f} \ast \overline{\varphi_a} \ast \psi_a)(y) \frac{d\alpha}{\alpha} \right) \overline{F_{W}^{-1}(h)}(y) d\mu_\alpha(y).
\]

Consider that
\[
\frac{1}{|C_{\varphi,\psi}|} \int_{\mathbb{R}^{d+1}} \int_{\gamma}^\delta \left| (\hat{f} \ast \overline{\varphi_a} \ast \psi_a)(y) \right| \frac{d\alpha}{\alpha} d\mu_\alpha(y) = \frac{1}{|C_{\varphi,\psi}|} \int_{\gamma}^\delta \int_{\mathbb{R}^{d+1}} \left| (\hat{f} \ast \overline{\varphi_a} \ast \psi_a)(y) \right| \frac{d\alpha}{\alpha} d\mu_\alpha(y).
\]

By applying Hölder’s inequality to the right hand side of the previous equality, we obtain
\[
\frac{1}{|C_{\varphi,\psi}|} \int_{\gamma}^\delta \left( \int_{\mathbb{R}^{d+1}} \left| (\hat{f} \ast \overline{\varphi_a} \ast \psi_a)(y) \right| d\mu_\alpha(y) \right) \frac{d\alpha}{\alpha} \leq \frac{1}{|C_{\varphi,\psi}|} \int_{\gamma}^\delta \left( \int_{\mathbb{R}^{d+1}} \left| (\hat{f} \ast \overline{\varphi_a} \ast \psi_a)(y) \right| d\mu_\alpha(y) \right) \frac{d\alpha}{\alpha}.
\]

From the assertion \((ii)\) of Lemma\([4.2]\) and Plancherel’s formula for Weinstein transform \([2.11]\), we get
\[
\frac{1}{|C_{\varphi,\psi}|} \int_{\gamma}^\delta \left( \int_{\mathbb{R}^{d+1}} \left| (\hat{f} \ast \overline{\varphi_a} \ast \psi_a)(y) \right| d\mu_\alpha(y) \right) \frac{d\alpha}{\alpha} \leq \frac{1}{|C_{\varphi,\psi}|} \left( \int_{\gamma}^\delta \frac{d\alpha}{\alpha} \right) \|F_W(\varphi)\|_{\alpha,\infty} \|F_W(f)\|_{\alpha,\infty} \|h\|_{\alpha,2} \|f\|_{\alpha,2}.
\]

Then, from Fubini’s theorem, the right hand side of the relation \((4.12)\) can also be written in the form
\[
\frac{1}{|C_{\varphi,\psi}|} \int_{\gamma}^\delta \left( \int_{\mathbb{R}^{d+1}} (\hat{f} \ast \overline{\varphi_a} \ast \psi_a)(y) \overline{F_{W}^{-1}(h)}(y) d\mu_\alpha(y) \right) \frac{d\alpha}{\alpha}.
\]
Moreover, according to the Parseval’s formula for the Weinstein transform (2.10) and the assertion (i) of the Lemma 4.2, the previous integral becomes

\[
\frac{1}{C_{\varphi, \psi}} \int_{\gamma} \left( \int_{\mathbb{R}^{d+1}} F_W(f)(\xi) \overline{F_W(\varphi_a)(\xi)} F_W(\psi_a)(\xi) h(\xi) d\mu_\alpha(\xi) \right) \frac{da}{a}.
\]

By applying Fubini’s theorem to this last integral, we have

\[
\int_{\mathbb{R}^{d+1}} F_W(f)(\xi) \left( \frac{1}{C_{\varphi, \psi}} \int_{\gamma} \overline{F_W(\varphi_a)(\xi)} F_W(\psi_a)(\xi) \frac{da}{a} \right) h(\xi) d\mu_\alpha(\xi)
= \int_{\mathbb{R}^{d+1}} F_W(f)(\xi) K_{\gamma,\delta}(\xi) h(\xi) d\mu_\alpha(\xi).
\]

(4.13)

On the other hand, by applying the Parseval’s formula for the Weinstein transform (2.10) to the left hand side of the relation (4.12), it takes the integral form

\[
\int_{\mathbb{R}^{d+1}} F_W(f_{\gamma,\delta})(\xi) h(\xi) d\mu_\alpha(\xi).
\]

(4.14)

Finally, from the relations (4.13) and (4.14), we obtain for all \( h \) in \( \mathcal{S}(\mathbb{R}^{d+1}) \)

\[
\int_{\mathbb{R}^{d+1}} (F_W(f_{\gamma,\delta})(\xi) - F_W(f)(\xi) K_{\gamma,\delta}(\xi)) h(\xi) d\mu_\alpha(\xi) = 0,
\]

and we deduce that

\[
F_W(f_{\gamma,\delta})(\xi) = F_W(f)(\xi) K_{\gamma,\delta}(\xi), \quad \xi \in \mathbb{R}^{d+1}.
\]

We now return to the proof of the Theorem 4.1.

**Proof of Theorem 4.1.** At first, from Theorem 4.4, the function \( f_{\gamma,\delta} \) belongs to \( L^2_{\alpha}(\mathbb{R}^{d+1}) \). Next, according to Plancherel’s formula for Weinstein transform (2.11) and Theorem 4.4, we get

\[
||f_{\gamma,\delta} - f||_{\alpha,2} = \int_{\mathbb{R}^{d+1}} |F_W(f_{\gamma,\delta} - f)(\xi)|^2 \alpha(\xi)
= \int_{\mathbb{R}^{d+1}} |F_W(f)(K_{\gamma,\delta}(\xi) - 1)|^2 \alpha(\xi)
= \int_{\mathbb{R}^{d+1}} |F_W(f)|^2 |(1 - K_{\gamma,\delta}(\xi))|^2 \alpha(\xi).
\]

Another time, according to Theorem 4.4 we have for almost all \( \xi \in \mathbb{R}^{d+1} \)

\[
\lim_{(\gamma,\delta) \to (0,\infty)} |F_W(f)|^2 |(1 - K_{\gamma,\delta}(\xi))|^2 = 0,
\]

and there exists a positive constant \( C \) such that

\[
\lim_{(\gamma,\delta) \to (0,\infty)} |F_W(f)|^2 |(1 - K_{\gamma,\delta}(\xi))|^2 \leq C |F_W(f)|^2,
\]

with \( |F_W(f)|^2 \) is in \( L^1_{\alpha}(\mathbb{R}^{d+1}) \). Thus, we conclude the relation (4.2) from the dominated convergence theorem.  \( \square \)
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