A SURFACE WITH DISCONTINUOUS ISOPERIMETRIC PROFILE

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Abstract. We show that there is a complete connected 2-dimensional Riemannian manifold with discontinuous isoperimetric profile, answering a question of Nardulli and Pansu.

1. Introduction

If $(M^n, g)$ is a Riemannian manifold the isoperimetric profile function of $M^n$ is a function $I_M : (0, vol(M)) \rightarrow \mathbb{R}^+$ defined by:

$$I_M(t) = \inf_{\Omega} \{vol_{n-1}(\partial \Omega) : \Omega \subset M^n, \ vol_{n}(\Omega) = t\}$$

where $\Omega$ ranges over all regions of $M^n$ with smooth boundary.

It is easy to see that $I_M(t)$ is upper semi-continuous (just add or take away a small ball with smooth boundary from a given region).

The continuity of the isoperimetric profile function is established in a number of cases: Gallot [3] showed that it is continuous for compact manifolds and Nardulli-Russo [5] showed that this holds also for manifolds of finite volume. Rittoré [7] showed that $I_M$ is continuous for Hadamard manifolds and complete non-compact manifolds of strictly positive sectional curvature. We refer to [7] and [4] for a more complete exposition of the cases where the continuity of $I_M$ is established and related questions.

Adams, Morgan and Nardulli observed that $I_M(t)$ is not necessarily continuous for manifolds with density (see Frank Morgan’s blog, [1]).

Nardulli and Pansu [4] constructed an example of a Riemannian 3-manifold with discontinuous $I_M(t)$ and asked whether there is such an example in dimension 2.

We answer this question below. Our construction is based on expander graphs. We note that P. Buser [2] has used in the past expanders as ‘blueprints’ for constructing surfaces.

2. The example

We follow the same method as [1, 4]. Given some constants $a, b > 0$ it is enough to find a sequence of closed surfaces $S_n$ such that area$(S_n) = \ldots$
\[ a + \tau_n \rightarrow 0 \text{ such that } I_{S_n}(a) = I_{S_n}(\tau_n) > b. \] Indeed we may join the surfaces \( S_n \) by tubes of negligible area to obtain a surface \( S \) such that \( I_S(a + \tau_n) \rightarrow 0 \) but \( I_S(a) > b \).

We explain now how to construct \( S_n \).

Our construction relies on the existence of expander graphs. We recall the definition of expanders. Let \( \Gamma = (V,E) \) be a graph. For \( S, T \subseteq V \) denote the set of all edges between \( S \) and \( T \) by \( E(S, T) = \{(u, v) : u \in S, v \in T, (u, v) \in E\} \).

**Definition.** The edge boundary of a set \( S \subseteq V \), denoted \( \partial S \) is defined as \( \partial S = E(S, S^c) \).

A \( k \)-regular graph \( \Gamma = (V,E) \) is called a \( c \)-expander graph if for all \( S \subseteq V \) with \( |S| \leq |V|/2 \), \( |\partial S| \geq c|S| \).

Pinsker [6] has shown that there is a \( c > 0 \) such that for any \( n \) large enough there is a 3-regular \( c \)-expander graph with \( n \)-vertices.

In the course of the proof that follows we will need to show several inequalities; as obtaining best constants is irrelevant for our construction we will always be using inequalities that are easy to state and verify rather than optimal ones.

Consider a 3-regular \( c \)-expander graph \( \Gamma_n \) with \( n^2 + n \) vertices. We give a way to replace this graph by a Riemannian surface. For each vertex \( v \) we pick a Euclidean 3-sphere \( S_v \) of radius \( 1/n \). Recall that the area of this sphere is \( 4\pi(1/n)^2 \). If \( l \) is a great circle of \( S_v \) we pick 3 equidistant points \( e_1, e_2, e_3 \) on \( l \) and we consider 3 spherical caps \( D_{1v}, D_{2v}, D_{3v} \) on \( S_v \) with centers \( e_1, e_2, e_3 \) and heights equal to \( \frac{1}{10n} \).

Clearly these spherical caps are disjoint and it is easy to see that the distance between any two of them on the sphere is greater than \( 1/n \).

Indeed if \( D_i^v \) intersects \( l \) at \( x_i, x_i' \) and \( \theta = \angle x_i O e_i \) (where \( O \) is the center of \( S_v \)) then \( \cos \theta = 9/10 > \cos(\pi/6) \) so the angle \( \angle x_i O x_j \) is greater than \( \pi/3 \) hence \( d(x_i, x_j) > 1/n \) on the sphere (for \( i \neq j \)).

Let \( C_{1v}, C_{2v}, C_{3v} \) be the boundary curves of \( D_{1v}, D_{2v}, D_{3v} \) respectively. We note that
\[
\text{length}(C_{iv}) \geq 1/n.
\]

We remove the open spherical caps \( D_{1v}, D_{2v}, D_{3v} \) from \( S_v \) and we still denote this sphere with holes by \( S_v \) to keep notation simple. We note that
\[
\text{area}(S_v) = \frac{4\pi}{n^2} - 3 \cdot \frac{4\pi}{10n^2} = \frac{18\pi}{5n^2}.
\]
We set \( a = \frac{36\pi}{10} \), so \( \text{area}(S_v) = a/n^2 \).
Now to each edge $E_i$ ($i = 1, 2, 3$) in $\Gamma$ leaving $v$ we associate the boundary curve $C^i_v$. If an edge $e$ joins the vertices $v, w$ of $\Gamma$ we identify the corresponding boundary curves of the spheres with holes $S_v, S_w$ by an isometry. This gluing produces a closed surface $S_n$ with
\[
\text{area}(S_n) = \frac{a(n^2 + n)}{n^2} = a + \frac{a}{n}.
\]
We set $\tau_n = \frac{a}{n}$. We note that $S_n$ is not a smooth manifold but it is easy to slightly modify the metric to make it smooth and this modification with have a negligible effect on the following calculations, so we will pretend for the moment that $S_n$ is a smooth manifold. We set $b = \min(\frac{1}{10}, \frac{c}{4})$.

**Proposition 2.1.** Let $\Omega$ be a smooth region in $S_n$ with $\text{area}(\Omega) = \tau_n$. Then
\[
\text{length}(\partial \Omega) \geq b.
\]

**Proof.** We distinguish two types of spheres with holes $S_v$.

*Type 1:* $\text{length}(\Omega \cap (C^1_v \cup C^2_v \cup C^3_v)) \geq \frac{5}{2} \text{length}(C^1_v)$ and

*Type 2:* where the opposite inequality holds.

Let’s denote by $\Omega_1$ the intersection of $\Omega$ with all type 1 spheres and $\Omega_2$ its intersection with type 2 spheres.

**Lemma 2.2.** Let $S_v$ be a type 2 sphere and let $\Omega_v = \Omega \cap S_v$ and $\partial \Omega_v = \partial \Omega \cap S_v$. We have then
\[
\text{area}(\Omega_v) \leq \frac{8\pi}{n} \text{length}(\partial \Omega_v).
\]

**Proof.** Note that $\text{area} \Omega_v \leq \text{area} S_v = a/n^2$.

Clearly it suffices to prove the inequality for all connected components $U$ of $\Omega_v$, namely it suffices to show:
\[
\text{area}(U) \leq \frac{8\pi}{n} \text{length}(\partial \Omega_v \cap \partial U).
\]

We consider the connected components of $\partial \Omega_v$. If there is a connected component $\alpha$ that is an arc with end points in two distinct $C^i_v$, $C^j_v$ then $\text{length}(\alpha) \geq 1/n$ so the lemma holds as
\[
\frac{a}{n^2} \leq \frac{8\pi}{n} \frac{1}{n}.
\]
Similarly if there is a connected component $\alpha$ that is a simple closed curve which bounds a region of $\Omega_v$ that contains some $C^i_v$ then $\text{length}(\alpha) \geq 1/n$ and the lemma holds.
It follows that there are 3 possibilities for the connected components of $\Omega_v$:

1. Regions $U$ homeomorphic to a disc with $\partial U \subseteq \partial \Omega_v$. By the isoperimetric inequality of the sphere the minimal boundary length for such regions is for spherical caps. However the corresponding spherical cap has area less than the sphere with holes $S_v$ so it has height $h$ less than $\frac{19}{10n}$. Recall that the area of a spherical cap is $2\pi rh$ where $r$ is the radius of the sphere and $h$ the height. It follows that the length of the boundary of the spherical cap is greater or equal to $h/2$ so

$$\text{area}(U) \leq \frac{2\pi h}{n} \leq \frac{4\pi \text{length}(\partial U)}{n}$$

and the inequality holds.

2. Regions $U$ homeomorphic to a disc with $\partial U = \alpha \cup \beta$ where $\alpha, \beta$ are arcs with the same endpoints and $\alpha \subseteq \partial \Omega_v, \beta \subseteq C_i^v$ for some $i$. Then clearly $\text{length}(\alpha) > \text{length}(\beta)$ and applying the isoperimetric inequality of the sphere again we have

$$\text{area}(U) \leq \frac{8\pi \text{length}(\partial U)}{n}$$

and the inequality holds.

3. Regions $U$ homeomorphic to a sphere with 3 holes and $\partial U = \alpha \cup \beta$ where $\alpha$ is a union of arcs in $\partial \Omega_v$ with endpoints on the $C_i^v$’s and $\beta$ is a union of arcs contained in the $C_i^v$’s. Since $S_v$ is of type 2

$$\text{length}(\alpha) \geq \frac{1}{2n}$$

while

$$\text{area}(U) \leq \text{area}(S_v) = \frac{a}{n^2}$$

and the inequality holds in this case too.

We note that at least one of $\text{area}(\Omega_1), \text{area}(\Omega_2)$ is greater than or equal to $\tau_n/2$. We will show that in both cases the proposition holds.

Let’s say that $\text{area}(\Omega_2) \geq \tau_n/2$. If $a_v$ is the area of $S_v \cap \Omega_1$ and $\ell_v$ is the length of $\partial \Omega_1 \cap S_v$ then

$$\tau_n/2 \leq \sum a_v \leq \frac{8\pi}{n} \sum \ell_v$$

so

$$\text{length}(\partial \Omega) \geq \sum \ell_v > \frac{1}{10} \geq b.$$

Let’s say now that $\text{area}(\Omega_1) \geq \tau_n/2$. Let $T$ be the set of vertices of $\Gamma_n$ that correspond to spheres with holes of type 1. Clearly $T$ contains
at least $n/2$ vertices. We may assume also that $T$ contains less than $rac{n^2+n}{2}$ vertices. Indeed if $T$ contains more than $\frac{n^2+n}{2}$ vertices then there are at least $n^2/10$ vertices $v$ of type 1 such that $\text{area}(S_v \cap \Omega_1) < \frac{\text{area}(S_v)}{2}$. By the isoperimetric inequality of the sphere we have that $\text{length}(\partial \Omega_1) \cap S_v \geq 1/n$ so

$$\text{length}(\partial (\Omega_1)) \geq n/10$$

from which the proposition clearly follows.

By our hypothesis that $\Gamma_n$ is a sequence of $c$-expander graphs there are at least $\frac{cn}{2}$ edges in the boundary of $T$. If $(v, w)$ is such a vertex then

$$\text{length}(\Omega \cap C^i_v) \geq \frac{1}{2} \text{length}(C^i_v)$$

$C^i_v$ is identified to $C^i_w$ for some $i, j \in \{1, 2, 3\}$ and $S_v$ is of type 1 while $S_w$ is of type 2. It follows that $\partial \Omega \cap S_w$ has length at least $\text{length}(C^i_w)/2$ so more than $1/2n$. Hence

$$\text{length}(\partial \Omega) \geq \frac{c}{4} \geq b$$

from which the proposition follows.

□

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