Quantization of classical integrable systems
Part III:
systems in \( n \)-dimensional Euclidean space

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Abstract

In this paper we give examples of applications of general methods of quantization by symmetrization of classical integrable systems, which have been illustrated in two previous works by the same authors. We consider two classes of systems in \( n \) spatial dimensions, which respectively describe a point particle in a central force field and a freely rotating rigid body. In the former case, the application of the general methods to an integrable classical system leads in an almost straightforward way to the quasi-integrability of the corresponding quantum system. In the latter case instead, a modification of the symmetrization procedure is necessary in order to achieve quantum integrability for \( n = 6 \).

1 Introduction

In two previous papers of this series, we have introduced the concept of quasi-integrable quantum system [1], and we have established general methods to obtain examples of such systems starting from classical integrable systems [2]. Integrability of an operator \( \hat{H} \) is defined as the existence of a sufficiently large set \( \hat{F} \) of operators which commute with \( \hat{H} \), more exactly a quasi-integrable set \( \hat{F} \) of operators. The main source of integrable sets are Lie closed sets of operators commuting with \( \hat{H} \). Making the union of several such sets one can obtain an integrable set \( \hat{F} \). An important particular case of Lie closed set is a Lie algebra of operators. General methods for the constructions of integrable sets are based on the symmetrization of the products of operators, which correspond to the elements of an integrable set of functions for the classical system. In the present paper these methods will be applied to some important classes of integrable classical and quantum systems.

In section 2 we consider systems in a Euclidean space of arbitrary dimension \( n \), which describe a point particle in a central force field. The discussion is then extended to more general one-particle systems which are symmetric with respect to the group of rotations \( SO(n) \). For all these cases we construct various types of classical integrable sets of functions. These sets contain in general \( 2n - k \) elements, where \( k \) is equal to the number of elements in the central subset.
This number, for the various integral sets here considered, can take all possible values from 2 to $n$. Each integrable set can be applied to all systems whose Hamiltonian is an arbitrary function of the central elements. In all these cases we show that, by applying the general results of [2], one can obtain a corresponding integrable quantum systems with an equal number $k$ of central operators.

In section 3 we then consider a freely rotating rigid body in a Euclidean space of arbitrary dimension $n$. In the classical case, it is known that this system is completely integrable. By applying our scheme of noncommutative integrability [3, 4, 5], we find how the number $k$ of central integrals depends on the space dimension $n$ and on the properties of the set of generalized moments of inertia of the body. We then show that, for $n \leq 5$, the application of the general results of [2] leads in an almost straightforward way to the integrability of the corresponding quantum systems. However, for $n = 6$ we find that a modification of the symmetrization procedure is necessary in order to obtain a quasi-integrable set of operators. In fact, a Manakov polynomial of fourth order in the left-invariant momenta, which belongs to the central subset of the classical integrable set, does not commute with the Hamiltonian after symmetrization in the momenta. However, commutativity can be restored by adding to it a suitable second order polynomial. One can conjecture that analogous procedures can be applied also for $n > 6$.

2 One-particle systems in a $n$-dimensional central force field

It is well-known that several mechanical systems have been proven to be integrable both at the classical and at the quantum level. The integrability of various classes of systems is discussed for instance in [6, 7, 8, 9, 10, 11, 12]. These systems usually consist of point particles moving in a space of one or more dimensions, subjected to an external potential or mutually interacting via suitable two-particle potentials. In particular, in [13] a class of maximally superintegrable systems is studied, which includes as a particular case the hydrogen atom in $n$ dimensions. In this section we too shall consider systems in $n$ dimensions which generalize in some sense the hydrogen atom problem, although our aim will be partly different with respect to most of the cited investigations. We shall not in fact restrict ourselves to considering Hamiltonians which are the sum of the usual kinetic term and of a potential term dependent only on the position. We shall instead consider a generic invariant Hamiltonian with respect to the group of $n$-dimensional rotations, and we shall look for all the possible integrable sets of functions and operators which can be constructed for such an Hamiltonian. It is then obvious that each of these sets can also be associated with the entire class of integrable systems, whose Hamiltonian is expressible as a functions of the central elements of the set. In this way our approach leads to the systematic individuation of families of integrable systems in $n$ spatial dimensions. However, we shall not discuss the possible physical interpretation of the systems obtained with this method.
2.1 Classical particle in a central force field

The hamiltonian function of this system in an \( n \)-dimensional euclidean space has the following form:

\[
H = \frac{1}{2}p^2 + U(r)
\]  

(2.1)

where \( U \in C(0, +\infty) \). Here we use the notation

\[
r = \sqrt{x^2}, \quad x = (x_1, \ldots, x_n), \quad p = (p_1, \ldots, p_n).
\]

The first term on the right-hand side of (2.1) corresponds to the kinetic energy of the particle, and the second one corresponds to its potential energy. The configuration space \( K \) of this classical system is the \( n \)-dimensional euclidean linear space \( \mathbb{R}^n \), more exactly, \( K = \mathbb{R}^n \setminus \{0\} \). The group of orthogonal transformations \( G = SO(n) \) acts on this space. This action in an orthonormal basis in \( \mathbb{R}^n \) is defined by orthogonal matrices. The dimension of this Lie group is \( N = n(n-1)/2 \).

The action of this Lie group \( G \) transfers onto the cotangent bundle \( T^*\mathbb{R}^n = \mathbb{R}^{2n} \) to \( \mathbb{R}^n \). This bundle, without the cotangent space \( T^n\mathbb{R}^n \) to \( \mathbb{R}^n \) at the point 0, represents the phase space \( M = T^*K \) of the classical system. The action of this group \( G \) conserves the hamiltonian function \( H = H(x, p) = \frac{1}{2}p^2 + U(r) \) of the classical system.

Let us denote the Lie algebra of the group \( G \) as \( \mathfrak{g} = \mathfrak{so}(n) \). Each element \( a \) of \( \mathfrak{g} \) is associated with a vector field \( v_a \) on \( \mathbb{R}^n \). The corresponding vector field on the symplectic manifold \( M \subset T^*\mathbb{R}^n \) is hamiltonian with hamiltonian function \( P_a(m) = (p, v_a(x)), \) where \( p = m \in T_x^*\mathbb{R}^n \) is a linear form on \( T_x^*\mathbb{R}^n \). Let \( (v_{a1}(x), \ldots, v_{an}(x)) \) be the components of the vector field \( v_a \) in coordinates \( x = (x_1, \ldots, x_n) \); then \( P_a(p, x) = \sum p_i v_{ai}(x) \), where \( (p, x) \) are canonical coordinates on \( T^*\mathbb{R}^n \). In the considered case, in which the group is \( G = SO(n) \), an element \( a \) of the Lie algebra \( \mathfrak{g} \) is represented by a skew-symmetric matrix \( A = A_a \), and the vector \( v_a(x) \) at the point \( x = (x_1, \ldots, x_n) \) has the form \( v_a(x) = -A_a x \).

The action of the group \( G \) on \( T^*\mathbb{R}^n \) is a Poisson action, i.e., \( \{ P_a, P_b \} = P_{[a, b]} \), where \( a, b \) are any two elements of \( \mathfrak{g} \), and \([a, b] \) is the commutator defined in this algebra. Each system of cartesian coordinates in \( \mathbb{R}^n \) defines in the algebra \( \mathfrak{g} \) a basis whose elements are given in these coordinates by matrices \( D^{ij} \), \( 1 \leq i < j \leq n \), having a particularly simple form. These matrices have only two non-zero elements which are equal to \( \pm 1 \). Namely, \( D^{ij} = -D^{ji} = 1 \), where \( D^{ij} \) denotes the element of the matrix \( D^{ij} \) lying at the intersection of row \( k \) and column \( l \). Making use of Kronecker symbol \( \delta_{ij} \), we can write in general

\[
D^{ij}_{kl} = \delta_{ik}\delta_{lj} - \delta_{kj}\delta_{li}.
\]  

(2.2)

According to this formula, a matrix \( D^{ij} \) can naturally be defined also for \( i \geq j \), and we have \( D^{ij} = -D^{ji} \forall i, j = 1, \ldots, n \). The commutation relations between these matrices are

\[
[D^{ij}, D^{hk}] = -\delta_{ih}D^{jk} - \delta_{jh}D^{ik} + \delta_{ik}D^{jh} + \delta_{jk}D^{ih}.
\]  

(2.3)

Note that, whenever the commutator is nonzero, only one of the four terms on the right-hand side is different from zero.

Let \( P_{ij} \) denote the function \( P_a \), corresponding to the matrix \( A_a = D^{ij} \). Then

\[
P_{ij} = x_ip_j - x_jp_i.
\]  

(2.4)
Proof. Let us suppose that Lemma 2.2. We have is in involution with vector $G$ the group $P$ in involution with $H$ given in [1], with hamiltonian function $H$. Proposition 2.1. plane. $P$ that the set $F$ is thus an integrable set with two central integrals, and the system has 2 central functions, $\{ \} = 0$ 2. See above) it is easy to check that}

\[
\{P^2, P_{ij}\} = 0
\] 2.6 for any component $P_{ij}$ of vector $P$. It is clear that one can select $2n - 4$ components $L = (P_{i,j}, \ldots, P_{2n-i-1,2n-i-1})$ of this vector, such that the set $\Pi := (P^2, L)$ defines a regular map $\Pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-3}$ almost everywhere in $\mathbb{R}^{2n}_{x,P}$. This means that the rank of the map $\Pi$ is equal to $2n - 3$ almost everywhere. A possible choice is $L = (P_{13,14}, \ldots, P_{1n,1n}, P_{23,24}, \ldots, P_{2n,2n})$. Let us add to this set the hamiltonian function $H = H(x,p)$, and denote by $F = (H, P^2, L)$ the resulting set of $2n - 2$ functions. It is easy to see that this set $F$ is functionally independent almost everywhere, that is the set of critical points of the map defined by this set has zero measure, is a closed set and is nowhere dense. Using the relations

\[
\{x_i, P_{jk}\} = \delta_{ij}x_k - \delta_{ik}x_j, \\
\{p_i, P_{jk}\} = \delta_{ij}p_k - \delta_{ik}p_j
\] 2.7 2.8 for $i, j, k = 1, \ldots, n$, it is also easy to verify that

\[
\{r^2, P\} = 0, \quad \{p^2, P\} = 0,
\] 2.9 that is $\{r^2, P_{ik}\} = \{p^2, P_{jk}\} = 0$. Hence $\{H, P\} = 0$. Since $L \subset P$, this implies that the set $F$ has 2 central functions, $H$ and $P^2$. According to the definition given in [1], $F$ is thus an integrable set with two central integrals, and the system with hamiltonian function $H$ is globally integrable with set of invariants $F$. The conservation of $P$ implies that the orbit of the particle lies in a 2-dimensional plane.

**Proposition 2.1.** Let $(V_1, \ldots, V_l)$ and $(W_1, \ldots, W_s)$ be two sets of functionally independent functions on a 2n-dimensional symplectic manifold, such that $\{V_i, W_k\} = 0$ for $i = 1, \ldots, l$ and $k = 1, \ldots, s$. Then $l + s \leq 2n$.

Taking into account the above well-known result, we can describe the set of all integrable classical systems which are invariant with respect to the action of the group $G = SO(n)$ on $\mathbb{R}^{2n}_{x,p}$, i.e., the integrable systems whose hamiltonian $H$ is in involution with vector $P$.

**Lemma 2.2.** We have $\{H, P\} = 0$ if and only if locally $H = f(p^2, r, P^2)$.

**Proof.** Let us suppose that $H = f(p^2, r, P^2)$. Since the functions $r, p^2, P^2$ are in involution with $P$ (see above), we have $\{H, P\} = 0$.

Viceversa, let us suppose that there exists a function $H(x,p)$ such that $\{H, P\} = 0$ and that has not locally the form $H = f(p^2, r, P^2)$. In this case
we would have 4 functionally independent functions $p^2, r, P^2$ and $H$, which are in involution with the $2n - 3$ functionally independent functions of the set $\Pi$. Since the sum of the numbers of functions belonging to these two sets equals $2n + 1$, this would be in contradiction with proposition 2.1.

**Proposition 2.3.** In the real analytic case, a Hamiltonian function $H$ is in involution with $P$ and is integrable if and only if it has locally the following form: $H = f(p^2, r, P^2)$.

The situation considered in all the present article refers to the more general case of infinitely differentiable functions, that is of class $C^\infty$. In this case, if $\{H, P\} = 0$ and $H$ is integrable, then $H = f(p^2, r, P^2)$. Viceversa, if $H = f(p^2, r, P^2)$, where the 2-covector $\partial f/\partial (p_j^2, r) = (\partial f/\partial (p_j^2), \partial f/\partial r)$ is not zero, more exactly $\partial f/\partial (p_j^2, r) \neq 0$ almost everywhere, then $H$ is integrable and $\{H, P\} = 0$. Such systems are always integrable with $k = 2$, with central integrals $H$ and $P^2$. In certain cases it is possible to find an additional integral, and to have integrability with $k = 1$. For example in the case of the Newton potential, that is for $H = p^2/2 - \alpha/r$, and in the case of identical uncoupled oscillators, that is for $H = (p^2 + r^2)/2$. The system with Hamiltonian function $H$ which is a function only of the square angular momentum, more exactly $H = f(P^2)$, where $\partial f \neq 0$ almost everywhere, is integrable with $k = 1$.

Let us present the integrable sets $F = F(H)$ which correspond to these Hamiltonian functions. For $H = f(p^2, r, P^2)$, where the function $f$ is not locally functionally dependent only on $P^2$, one can take $F = (H; P^2, L)$, and therefore $k = 2$. For $H = p^2/2 - \alpha/r$ one can take $F = (H; P^2, L, A_1)$, where $A_1 = \sum_{j=1}^{n} P_j p_j - \alpha x_j/r$. For $H = (p^2 + r^2)/2$ one can take $F = (H; H_1, H_2, \ldots, H_{n-1}, P_2, P_3, \ldots, P_{n+1})$, where $H_i = (p_i^2 + x_i^2)/2$. For $H = f(P^2)$ one can take $F = (P^2; p^2, r, L)$. In all cases except the first one, we have $k = 1$.

**Proof.** According to the previous lemma, $\{H, P\} = 0$ if and only if $H = f(p^2, r, P^2)$.

If $\partial f/\partial (p_j^2, r) \neq 0$ almost everywhere, we can repeat the proof given at the beginning of section 2.1 for the case $H = p^2/2 - U(r)$. By adding the function $H$ to the set $\Pi = (P^2, L)$, we thus obtain a set of $2n - 2$ functions which defines almost everywhere a regular map and has two central integrals, $H$ and $P^2$. Therefore the system is integrable with $k = 2$. If instead $f = f(P^2)$, then the set $F = (P^2; p^2, r, L)$ is an integrable set with $k = 1$.

For the Kepler system, $H = p^2/2 - \alpha/r$, it is straightforward to verify that $\{H, A_1\} = 0$, where

$$A_i = \sum_{j=1}^{n} P_j p_j - \alpha x_i/r = \left( p^2 - \frac{\alpha}{r} \right) x_i - (x \cdot p)p_i$$

for $i = 1, \ldots, n$ [14]. Since vector $A$ lies in the plane of the orbit and satisfies $A^2 = 2P^2 H + \alpha^2$, obviously only one component of $A$ is functionally independent of the set $(H, P^2, L)$. Hence $F = (H; P^2, L, A_1)$ is an integrable set with $k = 1$. Of course, one has to keep in mind that the potential $U(x) = -\alpha/r$ is a solution of the Laplace equation $\sum_{i=1}^{n} \partial^2 U/\partial x_i^2 = 0$ only for $n = 3$.

The Hamiltonian $H = (p^2 + r^2)/2$ actually describes a set of resonators with equal frequencies, and will be considered again in a following paper.
In all the considered cases, the linear independence of the differentials of the functions \( F(H) \) can be checked directly by considering the corresponding jacobian matrices. In the analytic case, either \( H = f(P^2) \), or \( \partial f / \partial (p^2, r) \neq 0 \) almost everywhere: therefore integrability is always guaranteed.

**Remark 2.1.** In the previous proposition we have considered the potential \( \alpha/r \) for a particle in a space of arbitrary dimension \( n \). Of course one has to keep in mind that such a potential is a Green function for the \( n \)-dimensional Laplace operator only for \( n = 3 \).

Let us consider the case \( H = f(p^2, r, P^2) \), with \( \partial f / \partial (p^2, r) \neq 0 \) almost everywhere. We have seen that these systems are integrable with \( k = 2 \). This means that the typical invariant surface for the phase-flow of the system is a two-dimensional torus. It is however possible to find for the same systems also integrable sets with a larger number \( k \) of central integrals, up to the maximum possible number \( k = n \) which corresponds to standard Liouville integrability. In this way one can construct a larger class of integrable systems, which includes all systems whose hamiltonian is an arbitrary function of the central elements of the set. In general, such systems will no longer be invariant under the action of the whole group \( SO(n) \), but only of some subgroup of it.

For example, in the familiar case \( n = 3 \), one can take \( F = (H, P^2; P_{12}, P_{13}) \), with \( k = 2 \), but also \( F = (H, P^2, P_{12}) \), with \( k = 3 \), or in general \( F = (H, P^2, P_a) \), where \( P_a \) is the momentum associated with any arbitrary element \( a \in so(3) \). It follows that any system with hamiltonian \( K = g(p^2, r, P^2, P_a) \), where \( g \) is a function such that \( \partial g / \partial (p^2, r) \neq 0 \), is integrable with \( k = 3 \). Of course any such system is only invariant with respect to the one-parameter subgroup of the rotations around the axis associated with \( a \).

**Definition 2.1.** We say that two integrable sets are *functionally equivalent* if the elements of one set are locally functions of the elements of the other set.

Owing to the arbitrariness in the choice of the element \( a \in so(3) \), we see that, for the system with hamiltonian \( H = f(p^2, r, P^2) \), there exist infinitely many functionally inequivalent integrable sets with \( k = 3 \).

Let us now consider the case of arbitrary \( n \). Note first of all that in the integrable set described in proposition 2.3 all \( n \) coordinates of configuration space are treated on the same footing, since all possible choices of the \( 2n - 4 \) noncentral elements of the set actually lead to functionally equivalent integrable sets. More generally, under any transformation of the group \( SO(n) \), the set \( F \) with \( k = 2 \) is transformed into an equivalent set. One can however construct other integrable sets in the following way. One takes the function \( P^2 \) as central element, and then splits the set of \( n \) coordinates of configuration space into two arbitrary disjoint subsets. One takes as additional central elements the two functions, one for each of these two subsets, which are obtained by summing the squares of all the components of \( P \) acting on the coordinates of the subset. Then, for any of the two subsets of coordinates, one can proceed in two alternative ways. Either one takes as integral functions a suitable set \( L' \) of \( 2n' - 4 \) momenta acting on the coordinates of the subset, where \( n' \) is the number of such coordinates, or one splits again the subset into two arbitrary smaller disjoint subsets, and repeats the procedure. If one wishes, one can continue splitting the subsets into two parts, until one is left with only subsets consisting of either one or two space coordinates (the splitting of a set of two coordinates is ineffective...
with respect to the resulting integrable set). Coming back to the case \( n = 3 \), we see that the integrable system \( F = (H, P^2, P_{12}) \) considered above corresponds to the splitting of the set of coordinates \((x_1, x_2, x_3)\) into the two subsets \((x_1, x_2)\) and \((x_3)\).

The general procedure is described in a formal way by the following proposition.

**Proposition 2.4.** For any \( n \geq 2 \) and for any \( z = 1, \ldots, n - 1 \), it is possible to construct in a recursive manner sets \( Z_{n,z} \) and \( L_{n,z} \) of polynomial functions of degree \( \leq 2 \) in the variables \( P_n := (P_{ij}, 1 \leq i < j \leq n) \), with the following properties:

1. \( Z_{n,z} \) contains \( z \) elements,
2. \( L_{n,z} \) contains \( 2(n - z - 1) \) elements, all of degree 1 in \( P_n \),
3. the set \( \Pi_{n,z} := (Z_{n,z}, L_{n,z}) \) is functionally independent,
4. \( \{Z_{n,z}, \Pi_{n,z}\} = 0 \).

Given any set \( A \), it is useful to denote with \( \sharp A \) the number of its elements. Properties 1 and 2 can thus be written \( \sharp Z_{n,z} = z \) and \( \sharp L_{n,z} = 2(n - z - 1) \) respectively. Property 4 means that any element of \( Z_{n,z} \) is in involution with all elements of the set \( \Pi_{n,z} \).

**Proof.** According to proposition 2.3, for \( z = 1 \) one can take \( Z_{n,1} = P^2_n \) and form \( L_{n,1} \) by collecting \( 2n - 4 \) suitable elements of \( P_n \). For \( n = 2 \) we can only have \( z = 1 \), \( Z_{2,1} = (P_{12}) \) and \( L_{2,1} = \emptyset \). In order to construct sets \( Z_{n,z} \) and \( L_{n,z} \), with \( n > 2 \) and \( 2 \leq z \leq n - 1 \), we shall proceed by induction on \( n \).

Let us take \( m > 2 \) and suppose that, for all \( n = 2, \ldots, m - 1 \), we have constructed sets \( Z_{n,z} \) and \( L_{n,z} \) of polynomial functions of degree \( \leq 2 \) in \( P_n \), for all possible \( z = 1, \ldots, n - 1 \), satisfying properties 1–4 specified above. Let us split the set of indexes \( N_m := (1, \ldots, m) \) into two arbitrary nonempty disjoint subsets \( I_1 \) and \( I_2 \), such that \( \sharp I_1 = n_1 \), \( \sharp I_2 = n_2 \), \( n_1 + n_2 = m \) and \( N_m = I_1 \cup I_2 \). For \( k = 1, 2 \), consider the two sets of momenta \( P^{(k)} \subset P_m \), with \( P^{(k)} := (P_{ij}, i, j \in I_k, i < j) \) if \( 1 < n_k \leq m - 1 \), and \( P^{(k)} := \emptyset \) if \( n_k = 1 \). This means that the elements of \( P^{(k)} \) are the generators of the orthogonal transformations of the subspace \( \mathbb{R}^{n_k} \subset \mathbb{R}^m \) having set of coordinates \( (x_i, i \in I_k) \).

We have obviously \( \{P^{(1)}, P^{(2)}\} = 0 \). If \( 1 < n_k \leq m - 1 \), consider for any \( z_k \), with \( 1 \leq z_k \leq n_k - 1 \), the sets \( Z_{n_k,z_k}(P^{(k)}) \) and \( L_{n_k,z_k}(P^{(k)}) \), which are obtained from the set of polynomials \( Z_{n_k,z_k} \) and \( L_{n_k,z_k} \) by replacing the variables \( P_{n_k} := (P_{ij}, 1 \leq i < j \leq n_k) \) with \( P^{(k)} \). If instead \( n_k = 1 \), take \( z_k = 0 \), \( Z_{1,0}(P^{(k)}) := \emptyset \) and \( L_{1,0}(P^{(k)}) := \emptyset \). We thus have in all cases \( \sharp Z_{n_k,z_k}(P^{(k)}) = z_k \) and \( \sharp L_{n_k,z_k}(P^{(k)}) = 2(n_k - z_k - 1) \). Finally, take

\[
\begin{align*}
Z_{m,z} &= \left( P^2_m, Z_{n_1,z_1}(P^{(1)}), Z_{n_2,z_2}(P^{(2)}) \right), \\
L_{m,z} &= \left( L_{n_1,z_1}(P^{(1)}), L_{n_2,z_2}(P^{(2)}) \right),
\end{align*}
\]

where \( P^2_m := \sum_{1 \leq i < j \leq m} P^2_{ij} \). It is easy to see that the sets \( Z_{m,z} \) and \( L_{m,z} \) satisfy properties 1–4 above for \( n = m \). If \( n_1 = 1, z_1 = 0, n_2 = m - 1, z_2 = 1 \), we obtain \( z = 2 \). With any other choice of \( n_k \) and \( z_k \), \( k = 1, 2 \), \( z \) can assume any value from 3 to \( m - 1 \).
Let the hamiltonian of a system have the form \( H = f(p^2, r, P^2) \), with \( \partial f / \partial (p^2, r) \neq 0 \) almost everywhere. From proposition 2.4 it follows that the sets of functions \( F_{n,z} = (H, Z_{n,z}; L_{n,z}) \) are integrable sets, with subset of central elements \( (H, Z_{n,z}) \), for all \( z = 1, \ldots, n-1 \). We have \( \# F_{n,z} = 2n - k \) and \( k = z + 1 \). Hence the number \( k \) of central elements can take all values from \( k = 2 \) to \( k = n \).

It has to be noted that all integrable sets with \( k > 2 \), obtained by means of proposition 2.4, depend on the choice of a cartesian set of coordinates on \( \mathbb{R}^n \). This means that, to any such set of coordinates, it corresponds in general an inequivalent integrable set. If one performs a transformation of \( SO(n) \) on configuration space, then a given integrable set is transformed into an equivalent one only if the transformation leaves invariant all the central functions of the set.

As an example of application of proposition 2.4 in the two following tables we show explicitly some integrable sets which are obtained in the two cases \( n = 4 \) and \( n = 5 \) respectively. In these tables we use the notation \( P^2_{(123)} := P^2_{12} + P^2_{13} + P^2_{23} \) and \( P^2_{(1234)} := P^2_{12} + P^2_{13} + P^2_{14} + P^2_{23} + P^2_{24} + P^2_{34} \).

### Table 1: Integrable sets for \( H = f(p^2, r, P^2) \) and \( n = 4 \).

| \( F \)                                                                 | \( k \) |
|-------------------------------------------------------------------------|--------|
| \( (H, P^2; P_{13}, P_{14}, P_{23}, P_{24}) \)                         | 2      |
| \( (H, P^2, P^2_{(123)}, P_{12}, P_{13}) \)                            | 3      |
| \( (H, P^2, P^2_{(123)}, P_{12}) \)                                    | 4      |
| \( (H, P^2, P_{12}, P_{34}) \)                                         | 4      |

### Table 2: Integrable sets for \( H = f(p^2, r, P^2) \) and \( n = 5 \).

| \( F \)                                                                 | \( k \) |
|-------------------------------------------------------------------------|--------|
| \( (H, P^2; P_{13}, P_{14}, P_{15}, P_{23}, P_{24}, P_{25}) \)          | 2      |
| \( (H, P^2, P^2_{(1234)}; P_{13}, P_{14}, P_{23}, P_{24}) \)           | 3      |
| \( (H, P^2, P^2_{(1234)}, P^2_{(123)}; P_{13}, P_{23}) \)              | 4      |
| \( (H, P^2, P^2_{(1234)}, P^2_{(123)}; P_{12}) \)                      | 5      |
| \( (H, P^2, P^2_{(1234)}, P_{12}, P_{13}) \)                            | 5      |
| \( (H, P^2, P^2_{(123)}, P_{45}; P_{12}, P_{13}) \)                     | 4      |
| \( (H, P^2, P^2_{(123)}; P_{45}, P_{12}) \)                            | 5      |

#### 2.2 Quantum particle in a central force field

The hamiltonian operator \( \hat{H} \) of this system is obtained from the hamiltonian function (2.1) by standard quantization (see definition in [1]), which here simply consists in the substitution \( p \rightarrow \partial / \partial x \) and the the replacement of the multiplication of functions with the composition of corresponding operators, in symbols: \( \times \rightarrow \circ \). We thus obtain

\[
\hat{H} = \frac{1}{2} \hat{p}^2 + U(r),
\]

(2.10)
where
\[ r = \sqrt{x^2}, \quad x = (x_1, \ldots, x_n), \quad \hat{p} = (\hat{p}_1, \ldots, \hat{p}_n), \quad \hat{p}_i = \frac{\partial}{\partial x_i}. \]

The operator \( \hat{p}^2 \) is the Laplace operator in cartesian coordinates. We can proceed as for the classical system, and on the basis of the classical formulas we will obtain the corresponding formulas where functions are converted into operators and Poisson brackets into Lie brackets. In a similar way we will also verify the quasi-independence of operators. Let us fix cartesian coordinates in \( \mathbb{R}^n \) and consider the standard set of operators \( (\hat{x}, \hat{p}) \), where \( \hat{p} = \partial / \partial x \). This set is obtained by canonical quantization from the coordinates \( (x, p) \). Therefore, identifying \( (x, p) \) and \( (\hat{x}, \hat{p}) \) with the sets \( \mathcal{B} \) and \( \hat{\mathcal{B}} \) respectively, one can study the algebra of the polynomial functions of these operators by applying propositions 3.1, 4.1, and also remark 4.4 of [2]. This remark is useful in order to deal with arbitrary functions of \( r \).

Let us consider the operators \( \hat{P}_{ij} \), \( 1 \leq i < j \leq n \), obtained by symmetric quantization from the classical momenta \( \hat{P}_{ij} = x_i p_j - x_j p_i \). Taking into account the canonical commutation relations
\[
[x_i, x_j] = 0, \quad [\hat{p}_i, \hat{p}_j] = 0, \quad [\hat{p}_i, x_j] = \delta_{ij} \tag{2.11}
\]
for \( i, j = 1, \ldots, n \), where \( \delta_{ij} \) is the Kronecker symbol, we have that \( \hat{P}_{ij} = x_i \hat{p}_j - x_j \hat{p}_i \), i.e., these operators coincide with the standard quantization of momenta \( \hat{P}_{ij} \). From the quadratic dependence of \( \hat{P}_{ij} \) on \( (x, \hat{p}) \), and from proposition 4.1 (case b) of [2], it follows that the commutation relations among the operators \( \hat{P}_{ij} \) have the same form as the Poisson brackets (2.14) among the corresponding classical functions:
\[
[\hat{P}_{ij}, \hat{P}_{hk}] = -\delta_{ih} \hat{P}_{jk} - \delta_{jk} \hat{P}_{ih} + \delta_{ik} \hat{P}_{jh} + \delta_{jk} \hat{P}_{ih} \tag{2.12}
\]
Similarly, from (2.9) and proposition 4.1 of [2] it follows that
\[
[r^2, \hat{P}] = 0, \quad [\hat{p}^2, \hat{P}] = 0 \tag{2.13}
\]
It is also easy to verify that \([U(r), \hat{P}] = 0\) for any function \( U \), in accordance with the first of (2.13) and with remark 4.4 of [2]. We thus conclude that \([\hat{H}, \hat{P}] = 0\).

Using proposition 2.5 of [2], it is easy to check that the operator \( (P^2)^{\text{sym}} \), obtained by symmetrization with respect to \( (x, \hat{p}) \) of the square length \( P^2 \) of momentum \( P \), coincides with the operator \( \hat{P}^2 = \sum_{i < j} \hat{P}_{ij}^2 \) up to an additive constant. We have in fact \( \hat{P}^2 = (P^2)^{\text{sym}} + n(n - 1)/4 \). Since the additive constant \( n(n - 1)/4 \) is irrelevant for Lie brackets, from classical relations (2.9) and from proposition 4.1 (case b) of [2] we obtain
\[
[\hat{P}^2, \hat{P}_{ij}] = [((P^2)^{\text{sym}}, \hat{P}_{ij}] = 0 \tag{2.14}
\]

The quasi-independence of the set \( \hat{\Pi} = (\hat{P}^2, \hat{L}) \) of \( 2n - 3 \) operators follows from the functional independence of the corresponding set of symbols \( \Pi = (P^2, L) \), and from the homogeneity of these functions with respect to \( p \). In fact,
Lemma 3.24 of [1] implies that \( \{10\} \) are arbitrary functions of \( k \) \( \). Then the relation \( \hat{H} = (\hat{H}, P^2; L) \) is functionally independent. Since these functions are homogeneous with respect to \( p \), these functions are quasi-independent. Furthermore, since the main part of \( H \) does not depend on \( U \), the property of quasi-independence is true for arbitrary \( U \). Therefore, the quantum system with hamiltonian \( \hat{H} \) is quasi-integrable with integrable set \( F = (\hat{H}, P^2; L) \) and \( k = 2 \) central operators, \( \hat{H} \) and \( P^2 \).

It is possible to give a partial characterization of integrable quantum systems which are invariant with respect to the action of the group \( G = SO(n) \) on \( \mathbb{R}^{2n} \), i.e., the systems whose hamiltonian operator \( \hat{H} \) commutes with vector operator \( \hat{P} \).

**Proposition 2.5.** If the hamiltonian operator \( \hat{H} \) of a system has the form: \( \hat{H} = f(\hat{p}^2, P^2, g_1(r), \ldots, g_l(r)) \), where the function \( f \) is an arbitrary noncommutative polynomial in the \( l + 2 \) variables \( \{\hat{p}^2, P^2, g_1(r), \ldots, g_l(r)\} \), and \( g_1(r), \ldots, g_l(r) \) are arbitrary functions of \( r \), then \( [\hat{H}, \hat{P}] = 0 \).

Viceversa, let \( \hat{H} \) be an arbitrary operator of class \( O \) on \( K = \mathbb{R}^n \setminus \{0\} \), such that \([\hat{H}, \hat{P}] = 0 \). Then the symbol of its main part \( M \hat{H} \) with respect to linear momenta \( \hat{p} \) (see definitions in [7]) has the form \( MH = g(p^2, r, P^2) \), where \( g \) is a homogeneous polynomial in the two variables \( p^2, P^2 \), whose coefficients are arbitrary functions of \( r \) defined for all \( r > 0 \). If the polynomial \( g \) satisfies the condition \( \partial g/\partial p^2 \neq 0 \) almost everywhere, then the systems is quasi-integrable with \( k = 2 \), with central integrals \( \hat{H} \) and \( P^2 \). In certain cases it is possible to find an additional integral, and to have quasi-integrability with \( k = 1 \). This is the case for example for the Newton potential, that is for \( \hat{H} = \hat{p}^2/2 - \alpha/r \), and for identical uncoupled oscillators, that is for \( \hat{H} = (\hat{p}^2 + r^2)/2 \). The system with hamiltonian function \( \hat{H} \) which is a function only of the square angular momentum, more exactly \( \hat{H} = f(\hat{p}^2) \), where \( df \neq 0 \) almost everywhere, is quasi-integrable with \( k = 1 \).

Let us present the integrable sets of operators \( F = F(\hat{H}) \) which correspond to integrable quantum systems with these hamiltonian operators. For \( \hat{H} = f(\hat{p}^2, P^2, g_1(r), \ldots, g_l(r)) \), where the function \( f \) is not locally functionally dependent only on \( \hat{p}^2 \), one can take \( F = (\hat{H}, P^2; \hat{L}) \), and therefore \( k = 2 \). For \( \hat{H} = \hat{p}^2/2 - \alpha/r \) one can take \( F = (\hat{H}; P^2, \hat{L}, \hat{A}_1) \), where \( \hat{A}_1 = \sum_{j=2}^n (\hat{p}_j \hat{p}_j + \hat{p}_j \hat{L}_j)/2 - \alpha x_j/r \). For \( \hat{H} = (\hat{p}^2 + r^2)/2 \) one can take \( F = (\hat{H}; H_1, H_2, \ldots, H_{n-1}, \hat{P}_2, \hat{P}_3, \ldots, \hat{P}_n) \), where \( H_i = \frac{1}{2}(\hat{p}_i^2 + x_i^2) \). For \( \hat{H} = f(\hat{p}^2) \) one can take \( F = (\hat{p}^2; p^2, r, \hat{L}) \). In all cases except the first one, we have \( k = 1 \).

Since \( x \cdot \hat{p} = (\hat{p}^2r^2 - r^2\hat{p}^2)/4 - n/2 \), this proposition implies in particular that \( [x \cdot \hat{p}, \hat{P}] = 0 \). Note also the relation \( \hat{P}^2 = r^2\hat{p}^2 - (x \cdot \hat{p})^2 - (n - 2)x \cdot \hat{p} \), which can for instance be easily verified using proposition 3.2 of [1].

**Proof.** Let the operator \( \hat{H} \) of class \( O_K \) have the form \( \hat{H} = f(\hat{p}^2, P^2, g_1(r), \ldots, g_l(r)) \), where the functions \( f, g_1, \ldots, g_l \) have the properties specified in the proposition. Then the relation \([\hat{H}, \hat{P}] = 0 \) follows from \([13] \) and \([14] \).

Viceversa, let \( \hat{H} \) be an operator of class \( O \) such that \([\hat{H}, \hat{P}] = 0 \). Then lemma 3.24 of [1] implies that \( \{MH, P\} = 0 \), where \( H \) and \( P \) are the symbols of \( \hat{H} \) and \( \hat{P} \) respectively. Using lemma \([22] \) we thus obtain that \( MH = g(p^2, r, P^2) \),
where \( g \) is an arbitrary function of three variables. Furthermore, since \( p^2 \) and \( P^2 \) are both homogeneous polynomials of order 2 in \( p \), taking into account the definition of main part we obtain that \( g \) is a homogeneous polynomial in the two variables \((p^2, P^2)\), whose coefficients are arbitrary functions of \( r \) defined for all \( r > 0 \).

The proof of the remaining statements is similar to the proof of the corresponding statements of proposition 2.3. Let us consider, in particular, the hamiltonian \( \hat{H} = \frac{p^2}{2} - \alpha/r \) of the quantum Kepler system \([15]\). Using (2.13), together with proposition 2.1 and lemma 2.2 of \([2]\), it is easy to verify that 

\[
[\hat{H}, \hat{A}_i] = 0, \quad \hat{A}_i = \sum_{j=1}^{n} \hat{P}_{ij} \hat{p}_j - \frac{\alpha x_i}{r}, \quad i = 1, \ldots, n.
\]

We have used above the symbol \( \diamond \) to denote symmetrized products, as in \([2]\). Only one component of \( \hat{A} \) is quasi-independent of the set \((\hat{H}, \hat{P}^2, \hat{L})\). Note that we have in this case

\[
\hat{A}^2 = 2\hat{H} \left[ \hat{p}^2 - \left( \frac{n - 1}{2} \right)^2 \right] + \alpha^2.
\]

Finally, it is easy to check that the main parts of the operators of the sets considered in the last part of the proposition are functionally independent.

The commutation relations between operators, from which the integrability of the considered sets of operators has been established, have been derived exploiting the quadratic dependence of classical momenta \( P_{ij} \) on the canonical variables \((x, p)\). Let us now present an alternative proof of these relations, which is only based on the linear dependence of these momenta on impulses \( p \). We shall consider the quantization of an arbitrary vector field on configuration space \( K \), more exactly, the quantization of the hamiltonian function on \( T^*K \) which corresponds to the lifting of this field on \( T^*K \). These considerations are useful for the investigation of any linear operator which is invariant with respect to the phase flows of such vector fields on \( K \), independently of the assumption that these vector fields be linear.

Let \( P = (P_1, \ldots, P_l) \) be a set of functions on the symplectic manifold \( M = T^*K \), which are linear with respect to \( p \):

\[
P_i = v_i^0(x) + \langle p, v_i(x) \rangle, \quad (2.15)
\]

\( P_i : T^*K \to \mathbb{R} \). Such functions, in local coordinates \((x, p)\) induced by local coordinates \( x \) on \( K \), have the form \( P_i = P_i(x, p) = v_i^0(x) + \sum_{k=1}^{n} v_i^k(x)p_k \). Let us suppose that the linear combinations of these functions with constant coefficients form a Lie algebra \( g \) with Poisson brackets in the role of commutators, and that the functions of the set \( P \) form a basis of this algebra. Let us consider the set of operators \( \hat{P} = (\hat{P}_1, \ldots, \hat{P}_l) \) obtained by standard quantization from the set of functions \( P \), i.e.,

\[
\hat{P}_i := v_i^0(x) + \sum_{k=1}^{n} v_i^k(x) \frac{\partial}{\partial x_k}, \quad i = 1, \ldots, l. \quad (2.16)
\]
Then the linear combinations of these operators form also a Lie algebra with respect to the usual commutator of linear operators. Moreover, let us consider the linear map, from the original Lie algebra of functions on $M$ to the Lie algebra of operators, which is defined by the correspondence of sets $P \to \hat{P}$, obtained by standard quantization. This map is a isomorphism between these two Lie algebras, i.e., the linear map preserves commutators.

This fact is an obvious consequence of the following more general proposition. Let us consider the Lie algebra $\mathcal{V} = \text{Vect}_K(K \times \mathbb{R})$ of all vector fields defined on the direct product $K \times \mathbb{R} \ni (x, u)$, which do not depend on $u$. The commutator in this algebra is the usual Lie bracket of vector fields. In local coordinates $(x, u)$, $x = (x_1, \ldots, x_n)$, a vector field $V \in \mathcal{V}$ on the $(n + 1)$-dimensional manifold $K \times \mathbb{R}$ has the form $\dot{x} = v(x)$, $\dot{u} = v^0(x)$. Let us consider also the algebra $\mathcal{F}$ of all functions $P$ on the symplectic manifold $M = T^*K$ which are linear with respect to the impulse $p$, i.e., functions of the form (2.15). Let us also consider the Lie algebra $\mathcal{O}$ of all linear nonhomogeneous differential operators on $K$, i.e., operators of the form (2.16).

**Proposition 2.6.** There are canonical isomorphisms between these three Lie algebras $\mathcal{V}, \mathcal{F}, \mathcal{O}$. In local coordinates on $K$, the coefficients $v_0(x), v_1(x), \ldots, v_n(x)$ defining the elements of these algebras are conserved under these isomorphisms.

Let $V$ and $W$ be two vector fields of class $\mathcal{V} = \text{Vect}_K(K \times \mathbb{R})$. Let us indicate the functions and operators, associated with these fields, as $P_V, P_W$ and $\hat{P}_V, \hat{P}_W$, correspondingly. Then the conservation of commutators of these algebras under the considered isomorphisms can be written in the form

$$\{P_V, P_W\} = P_{[V,W]}, \quad [\hat{P}_V, \hat{P}_W] = \hat{P}_{[V,W]}.$$

The hamiltonian vector field $X_P$, defined by the hamiltonian function $P \in \mathcal{F}$, can be lowered by natural projection $\pi : T^*K \to K$. This means that $\pi_\ast(X_P(m)) \in T_xK$ does not depend on the choice of the point $m \in \pi^{-1}(x)$, where $X_P(m)$ is the vector field $X_P$ at point $m$, $\pi_\ast : TM \to TK$ is the derivative of the map $\pi$, and $T_xK$ is the tangent space to $K$ at point $x$. Suppose that $P = P_V$, where $V \in \mathcal{V}$. Since the elements of $\mathcal{V}$ do not depend on $u$, the vector field $V$ can be lowered onto configuration space $K$ via the natural projection $K \times \mathbb{R}_u \to K$. These two vector fields, obtained by projection on $K$ from $P_V$ and $V$ respectively, are coincident.

**Proof.** Both statements of this proposition, about the correspondence of commutators of the three Lie algebras and about the coincidence of the projections on $K$ of the two vector fields, can be easily checked by direct computation in local coordinates. In general, these statements are reformulations of simple well-known facts.

Since classical momenta $P_{ij}(p, x) = x_ip_j - x_jp_i$ are linearly dependent on classical impulses $p$, we can use the above proposition to deduce the commutation relations (2.12) for the operators $P_{ij}$ from the corresponding classical relations (2.5).

Relations (2.5) show that the set of momenta $P_{ij}$, $1 \leq i < j \leq n$, is a basis of the Lie algebra $\mathfrak{g} = \text{so}(n)$, which is isomorphic to the Lie algebra of all skew-symmetric matrices. A natural basis in this Lie algebra is formed by matrices $D^{ij}$, $1 \leq i < j \leq n$, defined by formula (2.2). The correspondence
$D^{ij} \to P_{ij}$ is extended to linear combinations of matrices $D^{ij}$ and functions $P_{ij}$ as an isomorphism of Lie algebras. The basis $P = (P_{ij}, 1 \leq i < j \leq n)$ of Lie algebra $g$ induces a dual set of coordinates on the co-algebra $g^*$. It is well-known that the function $P^2 : g^* \to \mathbb{R}$, $P^2 := \sum_{1 \leq i < j \leq n} P_{ij}^2$ is an invariant of the co-adjoint representation of $SO(n)$ on the co-algebra $g^*$, where $g = so(n)$. From corollary 4.4 of \cite{2}, it follows that $[(P^2)^{\text{sym}}_{P^2}, \hat{P}] = 0$, where $(P^2)^{\text{sym}}_{P^2}$ denotes the symmetrization with respect to $\hat{P}$ of polynomial $P^2$. But obviously $(P^2)^{\text{sym}} = \sum_{1 \leq i < j \leq n} P_{ij}^2 = \hat{P}^2$. We thus conclude that $[\hat{P}^2, \hat{P}] = 0$.

Let us prove now that $[\hat{r}^2, \hat{P}] = [j^2, \hat{P}] = 0$. Let us consider the set $B = (1, x, p, P)$ of $l := n(n-1)/2 + 2n + 1$ functions on $M^{2n} = T^*K$. From Poisson brackets relations (2.5), (2.7) and (2.8) it follows that this set of functions is a basis in a $l$-dimensional Lie algebra. The functions of set $B$ are linear non-homogeneous functions of $p$. Therefore proposition 2.6 implies that analogous commutation relations hold for the operators $B = (1, x, \hat{p}, \hat{P})$ obtained from the functions of set $B$ by standard quantization. We have in particular

$$[x_i, \hat{P}_{jk}] = \delta_{ij}x_k - \delta_{ik}x_j, \quad (2.17)$$
$$[\hat{p}_i, \hat{P}_{jk}] = \delta_{ij}\hat{p}_k - \delta_{ik}\hat{p}_j. \quad (2.18)$$

Then relations (2.13) can be derived from (2.9) using proposition 4.2, case b, of \cite{2}.

The following proposition is the quantum equivalent of proposition 2.4.

**Proposition 2.7.** For any $n \geq 2$ and for any $z = 1, \ldots, n-1$, it is possible to construct in a recursive manner sets $\hat{Z}_{n,z}$ and $\hat{L}_{n,z}$ of polynomial functions of degree $\leq 2$ in the variables $\hat{P}_n := (\hat{P}_{ij}, 1 \leq i < j \leq n)$, with the following properties:

1. $\hat{Z}_{n,z}$ contains $z$ elements,
2. $\hat{L}_{n,z}$ contains $2(n-z-1)$ elements, all of degree 1 in $\hat{P}_n$,
3. the set $\hat{\Pi}_{n,z} := (\hat{Z}_{n,z}, \hat{L}_{n,z})$ is quasi-independent,
4. $[\hat{Z}_{n,z}, \hat{\Pi}_{n,z}] = 0$.

**Proof.** Let $\hat{Z}_{n,z}$ and $\hat{L}_{n,z}$ be the polynomials obtained from the classical ones $Z_{n,z}$ and $L_{n,z}$ of proposition 2.4 by simply replacing their arguments $P_n$ with the quantized momenta $\hat{P}_n$. In this case symmetrization is unnecessary, since in these polynomials all monomials of degree 2 are squares of elements of $\hat{P}_n$. Since all elements of the set $\Pi_{n,z} = (Z_{n,z}, L_{n,z})$ are homogeneous polynomials in $p$, they coincide with the symbol of the main parts of the corresponding elements of $\hat{\Pi}_{n,z}$. Therefore the quasi-independence of $\hat{\Pi}_{n,z}$ follows from the functional independence of the classical set $\Pi_{n,z}$.

Point 4 can be proved just by repeating the proof of proposition 2.4 When the set of indexes $N_m := (1, \ldots, m)$ is split into two disjoint subsets $I_1$ and $I_2$, consider in fact the two sets of operators $\hat{P}^{(k)} \subset \hat{P}_m$, $k = 1, 2$, which are the standard quantization of the sets of momenta $P^{(k)}$. From the isomorphism between the two Lie algebras generated by the sets $P_m$ and $\hat{P}_m$ respectively, it follows that $[\hat{P}^{(1)}, \hat{P}^{(2)}] = 0$. Moreover, since $P_m$ is a Casimir function for the co-algebra $so(m)^*$, from corollary 4.4 of \cite{2} it follows that $[\hat{P}^{(2)}_m, Z_{n,z}^{(k)}(\hat{P}^{(k)})] = \mathbb{}$.
\[ [\hat{P}_m, \hat{L}_{n,z}(\hat{P}^k)] = 0 \text{ for } k = 1, 2. \] Hence point 4 is obtained by induction on \( n \), as in the classical case.

Let \( \hat{H} \) be an arbitrary operator of class \( \mathcal{O} \) on \( K = \mathbb{R}^n \setminus \{0\} \), such that \( [\hat{H}, \hat{P}] = 0 \). From proposition 2.4 it follows that \( MH = g(p^2, r, P^2) \), where the function \( g \) is a homogeneous polynomial in the two variables \( (p^2, P^2) \), with coefficients dependent on \( r \). If the polynomial \( g \) satisfies the condition \( \partial g / \partial (p^2, r) \neq 0 \) almost everywhere, from proposition 2.7 it follows that the sets of operators \( \hat{F}_{n,z} := (\hat{H}, \hat{Z}_{n,z}; \hat{L}_{n,z}) \) are quasi-integrable sets, with subset of central elements \( (\hat{H}, \hat{Z}_{n,z}) \), for all \( z = 1, \ldots, n-1 \). We have \( \sharp \hat{F}_{n,z} = 2n - k \) and \( k = z + 1 \). Hence the number \( k \) of central elements can take all values from \( k = 2 \) to \( k = n \).

3 Free rotation of an \( n \)-dimensional rigid body

The configuration space \( K \) of this system is the group \( SO(n) \) of orthogonal transformations of the \( n \)-dimensional euclidean space \( K = SO(n) \). Its phase space is \( M = T^*K = T^*SO(n) \). We shall present a detailed analysis of the integrability of the classical system, which distinguishes itself from other already existing investigations [16, 17, 18], in the fact that we apply here the concept of noncommutative integrability (see definition 3.16 of [1]), and analyze the dependence of the number \( k \) of central integrals on the properties of the set of generalized moments of inertia, more precisely on the presence in this set of subsets consisting of moments equal to each other. The results will then be applied to the investigation of the integrability of the corresponding quantum system. We begin however with a preliminary subsection on some properties of the group \( SO(n) \), most of which are more or less well-known, but which shall here be derived in a form and with a notation convenient for our present purposes.

3.1 Properties of left- and right-invariant vector fields on \( SO(n) \)

Any arbitrary Lie group \( G \) acts on itself by left and right shifts: for each element \( g \in G \) these are defined by the diffeomorphisms

\[
L_g : G \to G, \quad L_g h = gh,
\]

\[
R_g : G \to G, \quad R_g h = hg.
\]

Let us denote by \( T_g G \) the tangent space to \( G \) at point \( g \). Then the Lie algebra \( \mathfrak{g} \) associated with \( G \) can be identified with the tangent space \( \mathfrak{g} = T_e G \) at the neutral element \( e \) of the group \( G \). Let \( \{L_g\}_g \) and \( \{R_g\}_g \), respectively denote the derivatives of the maps \( L_g \) and \( R_g \) at \( e \). Each element \( a \in \mathfrak{g} \) defines two vector fields \( V^L_a \) and \( V^R_a \) on \( G \). At any point \( g \in G \) these vector fields are respectively defined as \( V^L_a(g) := (L_g)_*(a) \in T_g G \) and \( V^R_a(g) := (R_g)_*(a) \in T_g G \). We have

\[
(L_h)_* V^L_a(g) = (L_h)_*(L_g)_*(a) = (L_{hg})_*(a) = V^L_a(hg),
\]

\[
(R_h)_* V^R_a(g) = (R_h)_*(R_g)_*(a) = (R_{gh})_*(a) = V^R_a(gh).
\]

These relations mean that the vector field \( V^L_a \) (respectively \( V^R_a \)) is invariant with respect to the action of Lie group \( G \) on itself by left (respectively right) shifts.
Definition 3.1. For the above reason, the fields \( V^L_a \) and \( V^R_a \) are respectively called left-invariant and right-invariant vector field on \( G \) associated with the element \( a \in \mathfrak{g} \).

The Lie brackets of left-invariant vector fields respect the structure of the Lie algebra \( \mathfrak{g} \): \( [V^L_a, V^L_b] = V^L_{[a,b]} \), where \( a, b \in \mathfrak{g} \) and \([a,b]\) is their commutator in \( \mathfrak{g} \). An analogous result, although with a reversed sign, is true for right-invariant vector fields: \( [V^R_a, V^R_b] = -V^R_{[a,b]} \). We have also \( [V^L_a, V^R_b] = 0 \).

The Lie group \( G = SO(n) \) can be identified with the group of orthogonal matrices of size \( n \times n \), and its associated Lie algebra \( \mathfrak{g} = so(n) \) with the Lie algebra of skew-symmetric matrices of the same size. Then the vector field \( V^L_A \) (respectively \( V^R_A \)) corresponding to the skew-symmetric matrix \( A \) is defined by the system of differential equations \( \dot{X} = XA \) (respectively \( \dot{X} = AX \)), where \( X \) is an orthogonal matrix. The commutator of the Lie algebra \( \mathfrak{g} = so(n) \) is given by the usual commutator of matrices. We can therefore rewrite the above formulas for the Lie brackets between left- and right-invariant vector fields as

\[
[V^L_A, V^L_B] = V^L_{[A,B]}, \quad [V^R_A, V^R_B] = -V^R_{[A,B]}, \quad [V^L_A, V^R_B] = 0. \tag{3.1}
\]

We can also consider the classical impulses \( P^L_A, P^R_A : T^*M \to \mathbb{R} \), with \( M = SO(n) \), which are associated according to proposition 2.6 with the vector fields \( V^L_A \) and \( V^R_A \), for any \( A \in \mathfrak{g} \):

\[
P^L_A(m) = \langle p, V^L_A(X) \rangle, \quad P^R_A(m) = \langle p, V^R_A(X) \rangle, \tag{3.2}
\]

where \( p = m \in T^*_X G \). These impulses form a Lie algebra with respect to Poisson brackets, which is isomorphic to the Lie algebra of the corresponding vector fields. Therefore, from relations (3.1) we derive

\[
\{P^L_A, P^L_B\} = P^L_{[A,B]}, \quad \{P^R_A, P^R_B\} = -P^R_{[A,B]}, \quad \{P^L_A, P^R_B\} = 0. \tag{3.3}
\]

Let us consider the vector fields \( V^L_{ij}, V^R_{ij} \), which are associated with the matrices \( D^i_j \) defined by formula (2.2). We recall that the set \( (D^i_j, 1 \leq i < j \leq n) \) forms a basis of the Lie algebra \( \mathfrak{g} = so(n) = T_eG \). This implies that \( V^L_{ij}(g) \) and \( V^R_{ij}(g) \) are two bases of linear space \( T_gG \), for any \( g \in G \). From (2.2) and (3.1) we obtain that the Lie brackets between these vector fields have the form

\[
[V^L_{ij}, V^L_{hk}] = -\delta_{ik}V^L_{j} - \delta_{jk}V^L_{h} + \delta_{ih}V^L_{k} + \delta_{jh}V^L_{i}, \tag{3.4}
\]

\[
[V^R_{ij}, V^R_{hk}] = \delta_{ih}V^R_{j} + \delta_{jk}V^R_{h} - \delta_{ik}V^R_{j} - \delta_{jh}V^R_{i}, \tag{3.5}
\]

\[
[V^L_{ij}, V^R_{hk}] = 0. \tag{3.6}
\]

These vector fields are associated with the classical impulses

\[
P^L_{ij}(m) = \langle p, V^L_{ij}(X) \rangle, \quad P^R_{ij}(m) = \langle p, V^R_{ij}(X) \rangle. \tag{3.7}
\]

According to (3.3), the Poisson brackets between these functions have the same form as the Lie brackets (3.4)–(3.6) between the corresponding vector fields:

\[
\{P^L_{ij}, P^L_{hk}\} = -\delta_{ik}P^L_{j} - \delta_{jk}P^L_{h} + \delta_{ih}P^L_{k} + \delta_{jh}P^L_{i}, \tag{3.8}
\]

\[
\{P^R_{ij}, P^R_{hk}\} = \delta_{ih}P^R_{j} + \delta_{jk}P^R_{h} - \delta_{ik}P^R_{j} - \delta_{jh}P^R_{i}, \tag{3.9}
\]

\[
\{P^L_{ij}, P^R_{hk}\} = 0. \tag{3.10}
\]
In the following we shall often indicate with $P^L = P^L(m)$ and $P^R = P^R(m)$ the two skew-symmetric matrices having components $P^L_{ij}(m)$ and $P^R_{ij}(m)$ respectively, with $1 \leq i, j \leq n$. Since for any $A \in g$ we have $A = \sum_{i<j} A_{ij} D^{ij}$, from (3.3) we obtain $P^L_A = \sum_{i<j} A_{ij} P^L_{ij} = -\frac{1}{2} \sum_{i,j} P^L_{ij} A_{ji} = -\frac{1}{2} \text{Tr}(P^L A)$. Using the first of relations (3.3) we then obtain
\[ \{P^L_A, P^L_{hk}\} = -\frac{1}{2} \text{Tr}(P^L[A, D^{hk}]) = -\frac{1}{2} \text{Tr}(D^{hk}[P^L, A]) = [P^L, A]_{hk}, \] (3.11)
where the last member represents the element at row $h$ and column $k$ of the commutator of the two matrices $P^L$ and $A$. In a similar way we obtain
\[ \{P^R_A, P^R_{hk}\} = -[P^R, A]_{hk}. \] (3.12)

**Remark 3.1.** Since the $N$ vectors $V^R_{ij}(X)$, $1 \leq i < j \leq n$, form a linear basis of $T_X G$ for any $X \in G = SO(n)$, we see from formula (3.7) that the correspondence $p \mapsto P^L(m)$ is an isomorphism between the linear spaces $T_X G$ and $so(n)$. It follows that, for a generic $m \in M$, the matrix $P^L(m)$ is a “typical” $n \times n$ skew-symmetric matrix. The same fact is obviously true also for $P^R(m)$.

Let us represent the generic element $g$ of the group $G = SO(n)$ as an orthogonal $n \times n$ real matrix $X$. If we indicate as $\check{X}$ the transposed of the matrix $X$, so that $X_{\beta\gamma} := X_{\gamma\beta}$, the orthogonality of $X$ implies $X^{-1} = \check{X}$. For any skew-symmetric matrix $A = -\check{A}$, at any point $X \in SO(n)$ the tangent vector $\dot{X} = AX$ can also be written as $\dot{X} = XB$, where $B = X^{-1}AX$ is also a skew-symmetric matrix. Using this fact, it is easy to see that the basic left- and right-invariant vector fields introduced above are connected to each other by the relations
\[ V^R_{ij}(X) = \sum_{h,k=1}^n X_{ih} X_{jk} V^L_{hk}(X), \]
or in matrix notation
\[ V^R(X) = X V^L(X) \check{X}. \] (3.13)
Similarly, we have for the corresponding impulses
\[ P^R_{ij}(m) = \sum_{h,k=1}^n X_{ih} X_{jk} P^L_{hk}(m), \] (3.14)
or equivalently
\[ P^R(m) = X P^L(m) \check{X}, \] (3.15)
where $m \in T_X G$.

It is sometimes useful to indicate with $P^L$ and $P^R$ also the two sets of $N = n(n - 1)/2$ functions on $T^*G$ defined by formulas (3.7):
\[ P^L = (P^L_{ij}, 1 \leq i < j \leq n), \quad P^R = (P^R_{ij}, 1 \leq i < j \leq n). \] (3.16)
In this way the elements of the sets $P^L$ and $P^R$, evaluated at a point $m \in M$, just coincide with the independent elements of the two skew-symmetric matrices $P^L(m)$ and $P^R(m)$ respectively, which were introduced before formula (3.11). In the following, it should be clear from the context whether $P^L$ (or $P^R$) indicates a skew-symmetric matrix or the corresponding set of $N$ independent elements.
Let $F = (F_1, \ldots, F_r)$ be a set of $r$ real functions $F_i : M \to \mathbb{R}$, $i = 1, \ldots, r$, on a manifold $M$. As usual, we denote as rank $F$ at $m \in M$ the dimension of the image $F_*(T_m M)$ of tangent space $T_m M$ with respect to the derivative $F_*$ of the map $F : M \to \mathbb{R}^r$ defined by the set $F$. Let $dF = (dF_1, \ldots, dF_r)$ denote the set of the differentials of the elements of $F$ at the point $m$. It is well known that rank $F = \dim \text{Span} dF$, where $\text{Span} dF$ denotes the linear subspace of $T^*_m M$ spanned by the elements of $dF$.

Let $M = M^{2N}$ be a symplectic manifold. Let us denote with $\Pi$ the antisymmetric bilinear functional $\Pi : T^*_m M \times T^*_m M \to \mathbb{R}$ such that the Poisson bracket of any two functions $f, h : M \to \mathbb{R}$ is given at $m$ by the relation $\{f, h\} = \Pi(df, dh)$.

For any linear subspace $L \subseteq T^*_m M$, let us denote with $L^\perp$ the subspace skew-orthogonal to $L$ with respect to $\Pi$, i.e., $L^\perp = \{u \in T^*_m M : \Pi(u, w) = 0 \forall w \in L\}$. Since $\Pi$ is nondegenerate, we have $\dim L + \dim L^\perp = 2N$. From these facts we deduce the following

**Lemma 3.1.** Let $F$ be a set of functions on a symplectic manifold $M^{2N}$. Then
\[
\dim (\text{Span} dF)^\perp = 2N - \text{rank } F. \tag{3.17}
\]

**Proposition 3.2.** Let us consider the two set of functions $P^L$ and $P^R$ defined by formulas (3.10) and (3.7). On all $M = T^*G$, where $G = SO(n)$, we have
\[
\text{rank } P^L = \text{rank } P^R = N. \tag{3.18}
\]

In addition, the two subspaces $\text{Span } dP^L$ and $\text{Span } dP^R$ of $T^*_m M$ are related to each other by the equalities
\[
(\text{Span } dP^L)^\perp = \text{Span } dP^R, \quad (\text{Span } dP^R)^\perp = \text{Span } dP^L. \tag{3.19}
\]

**Proof.** Let us fix a system of coordinates $x$ in a neighborhood of $g \in G$, and let $(x, p)$ be the local coordinates induced from $x$ on $T^*G$. According to (3.7), we have $V^L_{ij} = \partial P^L_{ij} / \partial p$. Since at any point $g$ of $G$ the $N$ vector fields $V^L_{ij}$, $1 \leq i < j \leq n$, are linearly independent, we have that the differentials $d_p P^L_{ij} \in T^*_p G$ with respect to variables $p$ of the $N$ functions $P^L_{ij}$ are linearly independent. This implies in particular the independence of their differentials $dP^L_{ij}$ in $T^*_m M$ with respect to variables $(x, p)$, so that rank $P^L = N$. We obtain in a similar way that rank $P^R = N$. Equalities (3.18) are thus proved.

Formula (3.10) implies $\text{Span } dP^R \subseteq (\text{Span } dP^L)^\perp$. Furthermore, formula (3.18) is equivalent to $\dim (\text{Span } dP^L) = \dim (\text{Span } dP^R) = N$. Therefore, applying lemma 3.1 to the set $P^L$ we obtain $\dim (\text{Span } dP^L)^\perp = N = \dim (\text{Span } dP^R)$. These facts imply the former of equalities (3.19). The latter is obtained in a similar way. 

Let $B$ denote the set $B := (P^L, P^R)$ of $2N = n(n-1)$ functions on $T^*G$.

**Corollary 3.3.** On all $M = T^*G$ we have
\[
(\text{Span } dB)^\perp = \text{Span } dP^L \cap \text{Span } dP^R. \tag{3.20}
\]

**Proof.** Recalling equalities (3.19), from the relation $B = P^L \cup P^R$ we obtain $(\text{Span } dB)^\perp = (\text{Span } dP^R)^\perp \cap (\text{Span } dP^L)^\perp = \text{Span } dP^L \cap \text{Span } dP^R$. 

In order to establish the linear dimension of subspace \([3.20]\), it is first necessary to recall a basic property of skew-symmetric matrices.

**Lemma 3.4.** Let us consider a skew-symmetric operator in an \(n\)-dimensional euclidean space \(L\). Then there is a cartesian system of coordinates in which the operator is defined by a matrix which has the “normal block-diagonal” form:

\[
\bar{A} = \begin{pmatrix}
\begin{array}{cccccc}
0 & \alpha_1 & 0 & 0 & \cdots & 0 \\
-\alpha_1 & 0 & 0 & 0 & \cdots & 0 \\
0 & 0 & \alpha_2 & 0 & \cdots & 0 \\
0 & 0 & -\alpha_2 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & \alpha_s \\
0 & 0 & 0 & 0 & \cdots & -\alpha_s \\
0 & 0 & 0 & 0 & \cdots & 0
\end{array}
\end{pmatrix}.
\]

(3.21)

where \(s = \left\lfloor \frac{n}{2} \right\rfloor\) is the integer part of \(\frac{n}{2}\), \(\alpha_k \geq 0\) for \(k = 1, \ldots, s\), and the last vanishing row and column are present only for odd \(n\). Let us suppose that all the eigenvalues \(\pm i\alpha_1, \ldots, \pm i\alpha_s, (0)\) of \(\bar{A}\) (where \(i = \sqrt{-1}\)) are pairwise different.

(This means that \(\alpha_k > 0\), \(\alpha_h \neq \alpha_k \forall h, k = 1, \ldots, s, h \neq k\). Then a skew-symmetric matrix \(B\) commutes with \(\bar{A}\), i.e., \([\bar{A}, B] = 0\), if and only if \(B\) has the same block-diagonal form, without any condition on its eigenvalues.

An equivalent intrinsic formulation is the following. For any skew-symmetric operator \(\bar{A}\) with pairwise different eigenvalues on the euclidean space \(L\), there exists a unique decomposition \(L = L_1^2 \oplus L_2^2 \oplus \cdots \oplus L_s^2 \oplus L_0^2\) of \(L\) into \(s\) 2-dimensional invariant subspaces \(L_i^2\) and (for odd \(n\)) a 1-dimensional subspace \(L_0^2\). Any skew-symmetric operator \(B\) commutes with \(\bar{A}\) if and only if all subspaces \(L_i^2\) and \(L_0^2\) of this decomposition are invariant under the action of \(B\).

We shall denote with \(\mathfrak{g}_{\text{typ}} \subset \mathfrak{g} = \text{so}(n)\) the set of all skew-symmetric matrices whose eigenvalues are pairwise different. Obviously, “almost all” elements of \(\mathfrak{g}\) also belongs to \(\mathfrak{g}_{\text{typ}}\). More exactly, \(\mathfrak{g}' \setminus \mathfrak{g}_{\text{typ}}\) is a closed subset of \(\mathfrak{g}\) having vanishing measure. It follows that, if some statement is true in \(\mathfrak{g}_{\text{typ}}\), then it is true (at least) almost everywhere in \(\mathfrak{g}\).

Let us fix an element \(a\) of the Lie algebra \(\mathfrak{g}\), and consider the linear operator \(\text{ad}_a : \mathfrak{g} \to \mathfrak{g}\) in this algebra, \(\text{ad}_a : b \mapsto [a, b]\). The kernel of \(\text{ad}_a\) is the subalgebra of all elements of \(\mathfrak{g}\) which commute with \(a\), i.e., \(\text{Ker} \text{ad}_a = \{b \in \mathfrak{g} : [a, b] = 0\}\).

**Corollary 3.5.** If \(a \in \mathfrak{g}_{\text{typ}}\), then

\[
\dim \text{Ker} \text{ad}_a = \left\lfloor \frac{n}{2} \right\rfloor.
\]

(3.22)

Let \(\text{ad}_a(\mathfrak{g}) \subset \mathfrak{g}\) denote the image of \(\mathfrak{g}\) with respect to the operator \(\text{ad}_a\). Then

\[
\dim \text{ad}_a(\mathfrak{g}) = N - \left\lfloor \frac{n}{2} \right\rfloor.
\]

(3.23)

Let \(a\) be represented by a matrix \(A\) having the normal block-diagonal form \((3.21)\), with \(\alpha_k > 0\), \(\alpha_k \neq \alpha_k \forall h, k = 1, \ldots, s, h \neq k\). Then \(\text{ad}_a(\mathfrak{g})\) is the linear space of the matrices \(B \in \mathfrak{g}\) such that \(B_{12} = B_{34} = \cdots = B_{2s-1,2s} = 0\), with \(s = \left\lceil \frac{n}{2} \right\rceil\), i.e.,

\[
\text{ad}_a(\mathfrak{g}) = \{B \in \mathfrak{g} : B_{2i-1,2i} = 0 \ \forall \ i = 1, \ldots, \left\lfloor \frac{n}{2} \right\rfloor\}.
\]

(3.24)
Lemma 3.6. For all \(m\) \(\in\) \(M\), \(\sigma\) is the linear dimension of the subalgebra of all matrices \(A\) \(\in\) \(so(n)\) such that \([P_L, A] = 0\).

Equality (3.22) follows from lemma 3.4, and from the observation that the linear space of the matrices of the form (3.21) has dimension \(s = \left\lfloor \frac{n}{2} \right\rfloor\).

Equality (3.23) follows directly from (3.22), since \(\dim g = N = n(n-1)/2\). A direct computation shows that, for any \(C \in g\), one has \(B_{2i-1,2i} = 0\) \(\forall i = 1, \ldots , \left\lfloor \frac{n}{2} \right\rfloor\), where \(B := [A, C]\). Therefore the linear space on the right-hand side of (3.24) contains \(\text{ad}_A(g)\). Furthermore, it is obvious that the right-hand side of (3.24) has linear dimension \(N = \left\lfloor \frac{n}{2} \right\rfloor\). Taking into account (3.23), these facts imply (3.24).

Proof. According to formula (3.15), \(\forall A \in g = so(n)\) we have

\[[P_L, A] = [X^{-1}P_R X , A] = X^{-1}[P_R , \text{Ad}_X(A)]X ,\]

where \(\text{Ad}_X(A) = XAX^{-1}\) is the image of \(A\) with respect to adjoint action \(\text{Ad}_X : g \rightarrow g\) of \(X\) on \(g\). Hence, \(A \in \text{Ker ad}_{P_L}\) if and only if \(\text{Ad}_X(A) \in \text{Ker ad}_{P_R}\).

This implies (3.26), since the adjoint action \(\text{Ad}_X\) is an algebra automorphism of \(g\). Equality (3.26) follows from formula (3.22) of corollary 3.5.

Proposition 3.7. Let us consider the set \(B = (P_L, P_R)\) of \(2N\) functions defined by formula (3.7). At any point \(m \in M = T^*G\), where \(G = SO(n)\), we have

\[(\text{Span}\ dB) = \left\{ a \in T_m^*M : a = \sum_{i<j} A_{ij} dP^L_{ij} , \tilde{A} = -A , [P_L, A] = 0 \right\} = \left\{ a \in T_m^*M : a = \sum_{i<j} A_{ij} dP^R_{ij} , \tilde{A} = -A , [P_R, A] = 0 \right\} (3.28)\]

and\[\text{rank } B = 2N - \sigma , (3.29)\]

where \(\sigma\) is defined by formula (3.25).
Proof. Equalities (3.19) and (3.20) imply that
\[(\text{Span } dB)^\perp = \text{Span } dP^L \cap (\text{Span } dP^L)^\perp .\]
Therefore \((\text{Span } dB)^\perp\) is the set of all those elements \(a \in \text{Span } dP^L\) such that \(\Pi(a, dP^L_{hk}) = 0\ \forall \ h, k\) with \(1 \leq h < k \leq n\). According to (3.18), \(dP^L\) is a set of linearly independent elements of \(T^*_mG\). Hence any covector \(a \in \text{Span } dP^L\) can be expressed as \(a = \sum_{i<j} A_{ij} dP^L_{ij}\), where \(A_{ij}\) are elements of a univocally determined skew-symmetric matrix \(A\). We have
\[
\Pi(a, dP^L_{hk}) = \sum_{i<j} A_{ij} \Pi(dP^L_{ij}, dP^L_{hk}) = \{P^L_A, P^L_{hk}\},
\]
where for the last equality use has been made of formula (3.11). The therefore \(a \in (\text{Span } dB)^\perp\) if and only if \(\{P^L_A, P^L_{hk}\} = 0\). This implies the first equality of formula (3.28); the second one can be obtained in a similar way.

Formula (3.28) shows that \((\text{Span } dB)^\perp\) is in one-to-one correspondence with the subalgebra of matrices \(A \in \text{so}(n)\) such that \(\{P^L_A, A\} = 0\). Hence
\[
\dim(\text{Span } dB)^\perp = \dim \ker dP^L = \sigma ,
\]
so that (3.29) is obtained by applying lemma 3.1 to the set \(B\). □

From formulas (3.29) and (3.27) we immediately obtain the following:

**Corollary 3.8.** At all points of \(M_{\text{typ}}\) (hence, almost everywhere on \(M = T^*G\)) we have
\[
\text{rank } B = 2N - \left\lceil \frac{n}{2} \right\rceil .
\]

The previous corollary, together with proposition 2.1 implies that there exist at most \(\left\lceil \frac{n}{2} \right\rceil\) functionally independent functions on \(M\) which are in involutions with the whole set \(B\). In order to explicitly construct a set of \(\left\lceil \frac{n}{2} \right\rceil\) such functions, it is useful to extend to Lie algebras the notion of Casimir function which has already been introduced for Lie co-algebras in [2].

**Definition 3.2.** A *Casimir function* on a Lie algebra \(\mathfrak{g}\) is an invariant of the adjoint action of the local Lie group \(G\), i.e., a function which is constant on the orbits of this action. In other words, a Casimir function \(K : \mathfrak{g} \to \mathbb{R}\) is a first integral which is common to all differential equations in \(\mathfrak{g}\) of type \(\dot{b} = [a, b]\), where \(b \in \mathfrak{g}\) and \(a\) is a fixed element of algebra \(\mathfrak{g}\).

Let us consider the case \(\mathfrak{g} = \text{so}(n)\), i.e., \(\mathfrak{g}\) is the algebra of skew-symmetric matrices. Fix \(A \in \mathfrak{g}\) and consider the characteristic polynomial \(\det(A - \lambda E)\), where \(E\) is the identity matrix. The relation \(\hat{A} = -A\) implies \(\det(\hat{A} + \lambda E) = \det(A + \lambda E) = (-1)^n(A - \lambda E)\), so that the characteristic polynomial has the form \(\det(\lambda E - A) = \lambda^n + C_1(A)\lambda^{n-2} + C_2(A)\lambda^{n-4} + \cdots\). The coefficients \(C_1(A), \ldots, C_s(A)\) of this polynomial, where \(s = \left\lceil \frac{n}{2} \right\rceil\), are clearly polynomial functions of the \(N = n(n-1)/2\) independent elements \(A_{ij}, 1 \leq i < j \leq n\), of the skew-symmetric matrix \(A\). Such elements are the coefficients of \(A\) with respect to the basis \(D^ij\) of \(\mathfrak{g}\) given by formula (2.24). We will call \(C = (C_1, \ldots, C_s)\) the standard set of Casimir functions on \(\text{so}(n)\).
Lemma 3.9. The functions of the standard set $C$ of Casimir functions on $g = \text{so}(n)$ are really Casimir functions in the sense of definition 3.2. Moreover, the functions of set $C$ form a basis in the space of all Casimir functions in the following “functional” sense. The differentials of the functions of set $C$ are linearly independent at almost all points of $g_{\text{typ}}$; therefore, they are linearly independent at almost all points of $g$. Moreover, every Casimir function $K$ on $g$ can be locally expressed as a function of the elements of this set: $K = K(C)$.

Proof. These are actually well-known facts, but we give the proof for completeness. The adjoint action of the group $G = \text{SO}(n)$ on the Lie algebra $g = \text{so}(n)$ takes in the matrix representation the form $A \mapsto \text{Ad}_X(A) = XAX^{-1}$, where $X \in G$, $A \in g$. Therefore $\det(A - \lambda E) = \det(\text{Ad}_X(A) - \lambda E)$, i.e., the polynomial $\det(A - \lambda E)$ in the variable $\lambda$ is invariant under the adjoint action of $G$. This means that the coefficients of this polynomial, i.e., the functions of set $C$, are also invariant, and are thus Casimir functions in the sense of definition 3.2.

Let us now examine the functional independence of the $\mathcal{S}$ functions of set $C$, where $\mathcal{S} = \left[\begin{smallmatrix} i \\ j \end{smallmatrix}\right]$, at a given point $\tilde{B} \in g$. To this end, we shall evaluate the rank of the $\mathcal{S} \times N$ matrix

$$J = J(\tilde{B}) := \left( \begin{array}{cccc} \frac{\partial C_h(B)}{\partial B_{ij}} & , & 1 \leq h \leq s, & 1 \leq i < j \leq n \end{array} \right),$$

whose rows are labelled by the index $h$ and the columns by the double index $ij$. In the above formula, we have denoted with $\frac{\partial C_h(B)}{\partial B_{ij}}$ the partial derivatives of the Casimir function $C_h$ with respect to the independent elements $B_{ij}$ of the skew symmetric matrix $B$. We know from lemma 3.4 that there exists $X = X(\tilde{B}) \in G$ such that $\text{Ad}_X(\tilde{B}) = \tilde{A}$, where $\tilde{A}$ is a skew-symmetric matrix in the normal form (3.21), such that $\alpha_k \geq 0$ for $k = 1, \ldots, s$, and $\pm i\alpha_1, \ldots, \pm i\alpha_s, 0)$ are the eigenvalues of $\tilde{B}$. Since $C$ is a set of Casimir functions, we have that $C(\tilde{B}) = C(\tilde{A})$. Taking into account that $\text{Ad}_X$ is an invertible linear operator in $g$, it follows that $\text{rank } J(\tilde{B}) = \text{rank } J(\tilde{A})$.

Let $P(B, \lambda) := \det(\lambda E - B)$ denote the characteristic polynomial for the matrix $B \in g$. For any $\lambda \in \mathbb{R}$ it is easy to see that

$$\left. \frac{\partial P(B, \lambda)}{\partial B_{ij}} \right|_{B = A} = 0 \quad \forall i, j = 1, \ldots, n \text{ such that } j > i + 1. \quad (3.33)$$

Recalling the definition of the set $C$, equality (3.33) implies that all the columns of matrix $J(\tilde{A})$, corresponding to indexes $ij$ with $j > i + 1$, are zero. We can thus write $\text{rank } J(\tilde{A}) = \text{rank } J(\tilde{A})$, where we have introduced the $\mathcal{S} \times \mathcal{S}$ matrix

$$\tilde{J} = \tilde{J}(\tilde{A}) := \left( \begin{array}{cccc} \frac{\partial C_h(B)}{\partial B_{2i-1, 2j-1}} & , & 1 \leq h \leq s, & 1 \leq i \leq s \end{array} \right). \quad (3.34)$$

It is easy to see that, for a matrix $\tilde{A}$ of the form (3.21), the characteristic polynomial has the form $\det(\lambda E - \tilde{A}) = (\lambda)^s \prod_{i=1}^{s}(\tilde{\alpha}_1^2 + \tilde{\alpha}_2^2) = \lambda^n + \sum \alpha_i^2 \lambda^{n-2} + \sum \alpha_i^2 \alpha_j^2 \lambda^{n-4} + \ldots$. Therefore the standard set of Casimir functions can be written as $C(\tilde{A}) = D(\beta(\alpha))$, where $\alpha = (\alpha_1, \ldots, \alpha_s)$, $\beta(\alpha) = (\beta_1(\alpha), \ldots, \beta_s(\alpha)) = (\alpha_1^2, \ldots, \alpha_s^2)$, $D(\beta) = (D_1(\beta), \ldots, D_s(\beta))$, and

$$D_h(\beta) := \sum_{i_1 < i_2 \cdots < i_h} \beta_{i_1} \beta_{i_2} \cdots \beta_{i_h} \quad \text{for } h = 1, \ldots, s.$$
Note that in particular $D_1(\beta) = \sum_{i=1}^s \beta_i$ and $D_s(\beta) = \beta_1 \beta_2 \cdots \beta_s$. Since $\bar{A}_{2i-1, 2i} = \alpha_i$, we have

$$\frac{\partial C_h(B)}{\partial B_{2i-1, 2i}} \bigg|_{B=A} = 2\alpha_i \frac{\partial D_h(\beta)}{\partial \beta_i} \bigg|_{\beta=\beta(\alpha)}.$$  

Therefore, recalling (3.34), we have $\det \tilde{J}(\bar{A}) = 2^s \alpha_1 \alpha_2 \cdots \alpha_s K_s(\beta(\alpha))$, where $K_s(\beta)$ is the determinant of the $s \times s$ jacobian matrix $\frac{\partial D}{\partial \beta}(\beta)$. It is easy to see that $K_s(\beta)$ is a symmetric polynomial of order $s(s-1)/2$ in the variables $\beta$, which vanishes whenever $\beta_i = \beta_j$ for some $i \neq j$. Therefore we can write $K_s(\beta) = c_s \prod_{i<j} (\beta_i - \beta_j)$. In order to evaluate the constant coefficient $c_s$, we note that, when $\beta_s = 0$, one has $\partial D_s/\partial \beta_i = 0$ for $i = 1, \ldots, s-1$, and $\partial D_s/\partial \beta_s = \beta_1 \beta_2 \cdots \beta_{s-1}$. It easily follows that $K_s(\beta) = \beta_1 \beta_2 \cdots \beta_{s-1} K_{s-1}(\beta')$ for $\beta_s = 0$, where $\beta' = (\beta_1, \ldots, \beta_{s-1})$. Using this fact, it can be easily shown, by induction with respect to $s$, that $c_s = 1 \forall s \in \mathbb{N}$. We thus conclude that

$$\det \tilde{J}(\bar{A}) = 2^s \alpha_1 \alpha_2 \cdots \alpha_s \prod_{i<j} (\alpha_i^2 - \alpha_j^2). \quad (3.35)$$

When all eigenvalues of matrix $\bar{B}$ are pairwise different, we have $\alpha_k > 0$, $\alpha_k \neq \alpha_h \forall h, k = 1, \ldots, s, h \neq k$. In such cases we see therefore from (3.35) that $\det \tilde{J}(\bar{A}) \neq 0$, so that rank $\tilde{J}(\bar{B}) = \text{rank} \tilde{J}(\bar{A}) = s$. This means that the differentials of the $s$ Casimir functions of set $C$ are linearly independent at $\bar{B}$.

In order to complete the proof of the lemma, we note that, according to corollary 3.3, at almost each point $B \in g$ the set of vectors $[A, B]$, where $A$ varies on all $g$, forms a subspace of codimension $s = \left[\frac{n}{2}\right]$ in $g$. Since the differential of any Casimir function must be zero when acting on this subspace, we deduce that the differentials of any $r$ Casimir functions, where $r \geq s$, are linearly dependent at any point $B \in g$. Therefore the differential of any Casimir function $K$ at any point of algebra $so(n)$ is a linear combination of the differentials of the functions of the standard set, i.e., we have locally $K = K(C)$.

Note that the set of all Casimir functions on $g$ defines the orbits $O$ of the adjoint representation of the corresponding local Lie group $G$ by their common level surfaces: $C^{-1}(c) = O$, where $C^{-1}(c)$ is the pre-image of a point $c \in \mathbb{R}^s$, and $s = \left[\frac{n}{2}\right]$. More exactly, this is true in the domain of regularity of the map $C : g \rightarrow \mathbb{R}^s$, defined by the set $C$. From this it is easy to obtain that the typical orbit $O$ has codimension $s$ in $g$, i.e., $\dim O = \dim g - s$.

For $g = so(n)$, we can associate with every element $b \in g^*$ a unique skew symmetric matrix $B$ such that $(A, b) = \frac{1}{2} \text{Tr} (AB) = \sum_{i<j} A_{ij} B_{ij} \forall A \in g$. Clearly, the elements $B_{ij}$, $1 \leq i < j \leq n$, of matrix $B$ are just the coefficients of $b$ with respect to the dual basis of the basis $D^\vee$ of $so(n)$ defined by formula (2.2). According to this one-to-one correspondence, in the following we will often identify $g^*$ with the co-algebra of skew-symmetric matrices, and we shall denote with $\mathfrak{g}^*_{\text{typ}}$ the set of all elements of $g^*$ whose eigenvalues are pairwise different. Obviously, almost all elements of $g^*$ also belongs to $\mathfrak{g}^*_{\text{typ}}$. It is easy to see that, in the matrix representation, the co-adjoint action of the group $G = SO(n)$ on an element $A \in g^*$ has the same form as the adjoint action on the corresponding element of $g$, i.e., $A \mapsto XAX^{-1}$, with $X \in G$. It follows that the standard set of Casimir functions on the Lie algebra $so(n)$, introduced in
Definition 3.3. Let $\mathfrak{g}^*$ be the dual co-algebra of $\mathfrak{g} = so(n)$. Fix a skew-symmetric matrix $A \in \mathfrak{g}^*$ and consider the characteristic polynomial $\det(\lambda E - A) = \lambda^n + C_1(A)\lambda^{n-2} + C_2(A)\lambda^{n-4} + \cdots$. The coefficients $C_1(A), \ldots, C_s(A)$ of this polynomial, where $s = \left\lfloor \frac{n}{2} \right\rfloor$ is the integer part of $\frac{n}{2}$, are polynomial functions of the set $\alpha$ of the matrix elements of $A$. We will call $C = (C_1, \ldots, C_s)$ the standard set of Casimir functions on $so(n)^*$. 

Lemma 3.10. The functions of the standard set $C$ of Casimir functions on the coalgebra $\mathfrak{g}^*$, where $\mathfrak{g} = so(n)$, are really Casimir functions in the sense of the definition given in [2]. Moreover, the functions of set $C$ form a basis in the space of all Casimir functions in the following “functional” sense. The differentials of the functions of set $C$ are linearly independent at almost each point of $\mathfrak{g}^*$. Therefore, they are linearly independent at almost each point of $\mathfrak{g}^*$. Moreover, every Casimir function $K$ on $\mathfrak{g}^*$ can be locally expressed as a function of the elements of this set: $K = K(C)$.

Proposition 3.11. Let $C$ be the standard set of Casimir functions on the Lie algebra $so(n)$. Then $C(P^L) = C(P^R)$ on all symplectic manifold $M = T^*G$.

Proof. From relation (3.15), taking into account that $\tilde{X} = X^{-1} \forall X \in SO(n)$, we obtain that $\det(\lambda E - P^R) = \det(\lambda E - P^L)$ on all $M$ for any $\lambda \in \mathbb{R}$. The thesis then follows from definition 3.9 of the set $C$.

In the following we shall denote with $\tilde{C}$ the set $\tilde{C} := C(P^L) = C(P^R)$ of $s = \left\lfloor \frac{n}{2} \right\rfloor$ real functions on $M$.

Proposition 3.12. On all $M = T^*G$, where $G = SO(n)$, we have

$$\{\tilde{C}, B\}_M = 0.$$  \hspace{1cm} (3.36)

Moreover, on all $M_{\text{typ}}$ (hence, almost everywhere on $M$) we have

$$\text{rank } \tilde{C} = \left\lfloor \frac{n}{2} \right\rfloor,$$  \hspace{1cm} (3.37)

$$\text{Span } d\tilde{C} = (\text{Span } dB)\mathcal{C}.$$  \hspace{1cm} (3.38)

Proof. Since $\tilde{C} = C(P^L)$, formula (3.11) implies $\{\tilde{C}, P^R\} = 0$. On the other hand, according to proposition 3.11 we also have $\tilde{C} = C(P^R)$, so that using again (3.11) we obtain $\{\tilde{C}, P^L\} = 0$. Equality (3.37) is thus proved.

Since both $\mathfrak{g}$ and $\mathfrak{g}^*$ have been identified with the set of $n \times n$ skew-symmetric matrices, we have $M_{\text{typ}} = \{m \in M : P^L(m) \in \mathfrak{g}_{\text{typ}}\}$. Lemma 3.9 implies that $\text{rank } C = \left\lfloor \frac{n}{2} \right\rfloor$ at all points of $\mathfrak{g}_{\text{typ}}$. Moreover, according to (3.13), $P^L : M \rightarrow \mathfrak{g}$ is a regular map. Therefore, it follows from the definition $\tilde{C} = C \circ P^L$ that equality (3.37) holds at all points $m \in M_{\text{typ}}$.

Formula (3.8) implies that $\text{Span } d\tilde{C} \subseteq (\text{Span } dB)\mathcal{C}$. Moreover, from (3.37), (3.31), and (3.24), we obtain that $\dim \text{Span } d\tilde{C} = \left\lfloor \frac{n}{2} \right\rfloor = \dim (\text{Span } dB)\mathcal{C}$ at all points of $M_{\text{typ}}$. These relations imply (3.38).
Taking into account (3.19) and (3.20), equality (3.38) also implies
\[
\text{Span } d\bar{C} = \text{Span } dP^L \cap \text{Span } dP^R = \text{Span } dP^L \cap (\text{Span } dP^L)^c. \tag{3.39}
\]

Remark 3.2. For any function \( f : M \to \mathbb{R} \), the condition \( \{ f, B \} = 0 \) is equivalent to \( df \in (\text{Span } dB)^c \). According to formula (3.38), the latter condition implies that locally \( f = g(\bar{C}) \), where \( g : \mathbb{R}^{[n/2]} \to \mathbb{R} \) is an arbitrary function. Recalling the definition of the set \( \bar{C} \), this means that \( f = g(C(P^L)) = c(P^L) \), where \( C \) is the standard set of Casimir functions, and therefore \( c := g \circ C \) is also a Casimir function. Taking into account proposition 3.11 we also have \( f = c(P^R) \). We conclude that the functions on \( M \) which are in involution with the whole set \( B \) are all and only the Casimir functions of the set \( P^L \) (or equivalently \( P^R \)).

3.2 Free rotation of a classical rigid body

Let us consider the system with hamiltonian function of the form
\[
H = \frac{1}{2} \langle P^L, J P^L \rangle, \tag{3.40}
\]
where \( J \) is an operator in the \( N \)-dimensional linear space \( \mathbb{R}^N \) of impulses \( P^L \), with \( N = n(n - 1)/2 \). The map \( P^L : T^*G \to \mathbb{R}^N = \mathfrak{g} \) is linear with respect to impulses \( p \), see formula (3.7). Therefore the function \( H = H(P^L(X)) \), \( H : T^*G \to \mathbb{R} \), at any point \( X \in G \) defines a quadratic form on cotangent space \( T^*_X G \) to \( G \) at \( X \), and this quadratic form \( H|_X := H|_{T^*_X G} \) is invariant with respect to the action of \( G \) on itself by left shifts. This fact is the motivation of the following definition.

Definition 3.4. We say that a system with hamiltonian function (3.40) describes the free motion of a point on a Lie group \( G \) provided with left-invariant metric.

Note that if the restriction \( H|_{X=e} \) of \( H \) on \( T^*_e G \) is positively defined, then \( H \) defines on \( G \) a left-invariant riemannian metric, and the system with hamiltonian \( H \) defines a geodesic flow on cotangent bundle \( T^*G \) which corresponds to this metric. (\( H \) can be considered as the kinetic energy \( T \) of the point.) From a mathematical point of view, the positivity of the 2-form \( H|_{X=e} \), which is transferred on all \( G \) by left shifts, is not important (see [19]).

Definition 3.5. Let the operator \( J \) in formula (3.40) be diagonal, and let its element be such that
\[
H = H^\lambda = \frac{1}{2} \sum_{i<j} \frac{(P^L_{ij})^2}{\lambda_i + \lambda_j}, \tag{3.41}
\]
with \( \lambda_i > 0 \) for \( i = 1, \ldots, n \). In this particular case we also say that the system with hamiltonian function (3.40) describes the rotation of a rigid body in \( n \)-dimensional space, with generalized moments of inertia \( \lambda_1, \ldots, \lambda_n \). In the following of this section we will always consider systems of this type.

Lemma 3.13. If \( H = H(P^L) \), then \( \{ H, P^R \} = 0 \).

This result follows from relation (3.10).
Lemma 3.14. Let $H = H^\lambda$ have the form (3.41), and suppose that $\lambda_i = \lambda_j$ for some $i, j$. Then $\{H, P^L_{ij}\} = 0$.

Proof. From (3.38) and (3.41) we get for all $i, j$, $1 \leq i < j \leq n$,

$$\{H, P^L_{ij}\} = (\lambda_i - \lambda_j) \sum_{k=1}^{n} \frac{P^L_{ik}P^L_{kj}}{\lambda_i + \lambda_k}(\lambda_k + \lambda_j).$$

The right-hand side obviously vanishes when $\lambda_i = \lambda_j$. Note that the above formula is equivalent to the well-known Euler equations for the free rotation of a rigid body.

Denoting $\lambda = (\lambda_1, \ldots, \lambda_n)$, let us group together the elements of the set $\lambda$ which are equal to each other. More precisely, we suppose that there exist $u$ distinct positive real numbers $\mu_1, \ldots, \mu_u$, with $1 \leq u \leq n$ and $\mu_h \neq \mu_k$ for $h \neq k$, such that

$$\lambda_1 = \lambda_2 = \cdots = \lambda_{p_1} = \mu_1,$$

$$\lambda_{p_1+1} = \lambda_{p_1+2} = \cdots = \lambda_{p_2} = \mu_2,$$

$$\vdots$$

$$\lambda_{p_{u-1}+1} = \lambda_{p_{u-1}+2} = \cdots = \lambda_{p_u} = \mu_u,$$

with $p_u = n$. Let us consider the set $q = q(\lambda) = (q_1, \ldots, q_u)$, where $q_1 = p_1$, $q_2 = p_2 - p_1$, $\ldots$, $q_u = p_u - p_{u-1}$. The set of all possible $\lambda$ can be divided into classes $l(q)$ each characterized by a given set $q = (q_1, \ldots, q_u)$, with $q_j \in \mathbb{N}$ for $j = 1, \ldots, u$, and $\sum_{j=1}^{u} q_j = n$. The order of the different generalized moments of inertia $\mu_h$ is not important, so one may always arrange them in such a way that $1 \leq q_1 \leq q_2 \leq \cdots \leq q_u$.

For a given $\lambda$, consider the decomposition of $\mathbb{R}^n$ which corresponds to this $\lambda$:

$$\mathbb{R}^n = L_1 \oplus L_2 \oplus \cdots \oplus L_u,$$

where $\dim L_j = q_j$, $\lambda \in l(q)$, $q = (q_1, \ldots, q_u)$. Let us consider the subalgebra $g(\lambda) \subseteq g = so(n)$ of all skew-symmetric matrices $A$ such that the subspaces $L_j$ are invariant with respect to the action of the operators defined by these matrices for all $j = 1, \ldots, q$:

$$g^\lambda = \{A \in g : A(L_j) \subseteq L_j \quad \forall j = 1, \ldots, u\}.$$ 

Let us consider a cartesian basis in $\mathbb{R}^n$ corresponding to the decomposition (3.43), and the set of skew-symmetric matrices which represent the elements of Lie algebra $g$ in this basis. A basis of $g$ is then given by the set of $N$ matrices $D^{ij}$ of the form (2.2), with $1 \leq i < j \leq n$. This basis is orthonormal with respect to the euclidean structure induced in $g$ by the bilinear form $(A, B) := \text{Tr}(AB)/2$, for $A, B \in g$. Let $I^\lambda$ denote the set of the pairs of indexes which correspond to equal generalized moments of inertia. Recalling equalities (3.42), we have $I^\lambda = (I^1, \ldots, I^u)$, where

$$I^k := \{(i, j) \in \mathbb{N} \times \mathbb{N} : p_{k-1} < i < j \leq p_k\} \quad \text{for} \quad k = 1, \ldots, u.$$ 

Then the set $D^\lambda := \{D^{ij} : (i, j) \in I^\lambda\}$ of matrices of the form (2.2) forms a basis of $g^\lambda$. Let us consider the set $P^L\lambda := \{P^L_{ij} : (i, j) \in I^\lambda\} \subseteq P^L$ of impulses of the form (3.7). The following proposition is an obvious consequence of lemmas 3.13 and 3.14.
**Proposition 3.15.** If \( H = H^\lambda \) has the form (3.44), then \( \{ H, B^\lambda \} = 0 \), where \( B^\lambda := (P^L, P^R) \).

In other words, \( B^\lambda \) contains the elements of \( B \) which are in involution with the hamiltonian \( H^\lambda \). We are now going to investigate whether it is possible to construct an integrable set using suitable functions of the set \( B^\lambda \) and possibly \( H^\lambda \). We will find that this depends on the properties of the set \( \lambda \) or, more exactly, of the set of integers \( q(\lambda) \).

Let us consider a set of functions \( F = (F_1, \ldots, F_p) \) on the 2N-dimensional symplectic manifold \( M^{2N} \). We suppose that \( \text{rank} \ F \) is the same almost everywhere. Let \( k = k(F) \) be the maximal number of functionally independent functions, defined on all \( M^{2N} \), which are functions of \( F \) and are in involution with all functions of set \( F \). More exactly, there exists a set \( Z = (Z_1, \ldots, Z_k) \), such that these functions are functionally independent almost everywhere on \( M \) and \( \{ Z, F \} = 0 \) on \( M \), and \( k \) is the maximum integer for which such a set \( Z \) exists.

**Definition 3.6.** We will say that the number \( k \) is the centrality of the set \( F \), while \( r = r(F) := 2N - \text{rank} \ F - k(F) \) is the defect of integrability of the set \( F \).

**Lemma 3.16.** For any set \( F \) (such that \( \text{rank} \ F \) is the same almost everywhere) we have \( k(F) \leq \dim W \), where \( W := \text{Span} \ dF \cap (\text{Span} \ dF)^\perp \) at a typical point \( m \in M \). Moreover, \( r(F) \geq 0 \).

**Proof.** For any function \( Z = Z(F) \), such that \( \{ Z, F \} = 0 \) at \( m \), we must obviously have \( z \in W \). From this it easily follows that \( k(F) \leq \dim W \). The inequality \( r(F) \geq 0 \) follows instead from proposition 2.4.

**Proposition 3.17.** The centrality and defect of integrability of the set of functions \( B = (P^L, P^R) \) are respectively

\[
k(B) = \left\lceil \frac{m}{2} \right\rceil, \tag{3.45}
\]

\[
r(B) = 0. \tag{3.46}
\]

**Proof.** Equality (3.45) follows from proposition 3.12 and remark 3.2. Then equality (3.46) follows from (3.32).

Let \( P^\lambda : \mathfrak{g} \to \mathfrak{g}^\lambda \) denote the projector onto the subalgebra \( \mathfrak{g}^\lambda \) with respect to the euclidean structure in \( \mathfrak{g} \) introduced above. For any linear operator \( L : \mathfrak{g} \to \mathfrak{g} \), we denote by \( L|_{\mathfrak{g}^\lambda} \) its restriction to the subalgebra \( \mathfrak{g}^\lambda \). For any \( m \in M \) we can then consider the operators

\[
P^\lambda \circ \text{ad}_{PL} : \mathfrak{g} \to \mathfrak{g}^\lambda,
\]

\[
P^\lambda \circ \text{ad}_{PL}|_{\mathfrak{g}^\lambda} : \mathfrak{g}^\lambda \to \mathfrak{g}^\lambda,
\]

\[
\text{ad}_{PL}|_{\mathfrak{g}^\lambda} : \mathfrak{g}^\lambda \to \mathfrak{g},
\]

where \( \circ \) denotes the composition of linear operators, and \( P^L = P^L(m) \). We define the three integers \( \sigma_1^\lambda, \sigma_2^\lambda, \) and \( \sigma_3^\lambda \), dependent on the point \( m \), as the linear dimension of the kernels of the three above operators:

\[
\sigma_1^\lambda := \dim \text{Ker} \left( P^\lambda \circ \text{ad}_{PL} \right), \tag{3.47}
\]

\[
\sigma_2^\lambda := \dim \text{Ker} \left( P^\lambda \circ \text{ad}_{PL}|_{\mathfrak{g}^\lambda} \right), \tag{3.48}
\]

\[
\sigma_3^\lambda := \dim \text{Ker} \left( \text{ad}_{PL}|_{\mathfrak{g}^\lambda} \right). \tag{3.49}
\]
From these definitions and from (3.25) it is immediate to see that $\sigma_1^\lambda \geq \sigma \geq \sigma_3^\lambda$ and $\sigma_1^\lambda \geq \sigma_2^\lambda \geq \sigma_3^\lambda$.

**Proposition 3.18.** The rank of the set $B^\lambda = (P^L, P^R)$ at any point $m \in M$ is given by

$$\text{rank } B^\lambda = 2N - \sigma_1^\lambda.$$  

(3.50)

In addition we have

$$\dim W = \sigma + \sigma_2^\lambda - \sigma_3^\lambda,$$  

(3.51)

where

$$W := \text{Span } dB^\lambda \cap (\text{Span } dB^\lambda)^\perp.$$  

(3.52)

**Proof.**

Since $B^\lambda = P^L \cup P^R$, we have

$$(\text{Span } dB^\lambda)^\perp = (\text{Span } dP^R)^\perp \cap (\text{Span } dP^L)^\perp,$$

(3.53)

where the last equality follows from (3.19). Therefore $(\text{Span } dB^\lambda)^\perp$ is the set of all those elements $a \in \text{Span } dP^L$ such that $\Pi(a, dP^L_{hk}) = 0 \forall (h, k) \in I^\lambda$. Any covector $a \in \text{Span } dP^L$ can be expressed as $a = \sum_{i<j} A_{ij} dP^L_{ij}$, where $A_{ij}$ are elements of a univocally determined skew-symmetric matrix $A$. Hence $a = dP^L_A$ and

$$\Pi(a, dP^L_{hk}) = \{ P^L_A, P^L_{hk} \} = [P^L, A]_{hk},$$

(3.54)

where for the last equality use has been made of formula (3.11). The matrix elements on the right-hand side of the above formula, for $(h, k) \in I^\lambda$, are just the components, with respect to the basis $D^\lambda$, of the projection $P^\lambda([P^L, A])$ of the element $[P^L, A] \in \mathfrak{g}$ onto the subalgebra $g^\lambda$. We see therefore that $(\text{Span } dB^\lambda)^\perp$ is in one-to-one correspondence with the linear space of matrices $A \in so(n)$ such that $P^\lambda([P^L, A]) = 0$. Hence

$$\text{dim}(\text{Span } dB^\lambda)^\perp = \dim \ker (P^\lambda \circ \text{ad}_{P^L}) = \sigma_1^\lambda,$$

(3.55)

so that (3.50) is obtained by applying lemma 3.1 to the set $B^\lambda$. From (3.53) and (3.52) it follows that

$$W = \text{Span } dB^\lambda \cap (\text{Span } dP^R)^\perp \cap (\text{Span } dP^L)^\perp.$$  

(3.56)

Let us then consider an arbitrary element $w \in W$. The condition $w \in \text{Span } dB^\lambda$ implies that there exist $z_1 \in \text{Span } dP^R$ and $z_2 \in \text{Span } dP^L$ such that

$$w = z_1 + z_2.$$  

Since $P^L \subseteq P^L$, from (3.19) it follows that

$$\Pi(z_1, dP^L) = \Pi(z_2, dP^R) = 0.$$  

(3.57)

Hence the condition $w \in (\text{Span } dP^R)^\perp$ implies $0 = \Pi(w, dP^R) = \Pi(z_1, dP^R)$. Therefore, recalling (3.19) and (3.20), we have

$$z_1 \in \text{Span } dP^R \cap (\text{Span } dP^R)^\perp = (\text{Span } dB)^\perp.$$  

Using proposition 3.7 we thus conclude that $z_1 = \sum_{i<j} C_{ij} dP^L_{ij}$, where $C$ is a skew-symmetric matrix such that $[P^L, C] = 0$, i.e., $C \in \ker \text{ad}_{P^L}$. Finally, the
condition \(w \in \text{Span } dP^{L\lambda}\), using again (3.57), implies \(0 = \Pi(w, dP^{L\lambda}) = \Pi(z_2, dP^{L\lambda})\). Therefore

\[
z_2 \in \text{Span } dP^{L\lambda} \cap \text{Span } dP^{L\lambda}\).\]

Using formula (3.54), we see that a covector \(a \in \text{Span } dP\) belongs to \(\text{Span } dP^{L\lambda} \cap \text{Span } dP^{L\lambda}\) if and only if \(a = \sum_{i<j} A_{ij} dP_{ij}^L\), with \(A \in g^\lambda\) and \(P^\lambda([P^L, A]) = 0\). These two conditions on \(A\) can be simultaneously expressed as \(A \in S^\lambda\), where \(S^\lambda := \text{Ker } (P^\lambda \circ \text{ad}_P^L|_{g^\lambda})\).

We have thus shown that \(w = \sum_{i<j} (C_{ij} + A_{ij}) dP_{ij}^L\), where \(C \in S := \text{Ker } \text{ad}_{P^L}^\lambda\) and \(A \in S^\lambda\). It follows that

\[
W = \left\{w \in T^*_m M : w = \sum_{i<j} K_{ij} dP_{ij}^L, K \in \text{Span } (S, S^\lambda)\right\}.
\] (3.58)

Since covectors \(dP_{ij}^L\), \(1 \leq i < j \leq n\), are linearly independent (see proposition 3.2), according to (3.57), there exists a linear isomorphism between \(W\) and \(\text{Span } (S, S^\lambda)\). Therefore

\[
\dim W = \dim \text{Span } (S, S^\lambda) = \dim S + \dim S^\lambda - \dim (S \cap S^\lambda).
\] (3.59)

Recalling (3.25) and (3.48), we have \(\dim S = \sigma^2\) and \(\dim S^\lambda = \sigma_2^\lambda\). Furthermore, it is easy to see that \(S \cap S^\lambda = \text{Ker } \text{ad}_{P^L}^\lambda \cap g^\lambda = \text{Ker } \text{ad}_{P^L}^\lambda|_{g^\lambda}\), so that recalling (3.39) we can write \(\dim (S \cap S^\lambda) = \sigma^2_2\). Hence (3.59) is equivalent to (3.51).

**Corollary 3.19.** At a typical point \(m \in M\), then the centrality \(k(B^\lambda)\) and the defect of integrability \(r(B^\lambda)\) (see definition 3.7) of the set \(B^\lambda\) satisfy the inequalities

\[
k(B^\lambda) \leq \sigma + \sigma_2^\lambda - \sigma_3^\lambda,\]

\[
r(B^\lambda) \geq \sigma^2_2 + \sigma_3^\lambda - \sigma - \sigma_2^\lambda.
\] (3.60) (3.61)

**Proof.** Relation (3.60) follows from lemma 3.16 and formula (3.51). Then relation (3.61) follows from (3.50) and (3.60).

Note that formula (3.52) implies \(\dim W \leq \dim (\text{Span } dB^\lambda)^\perp\). Therefore using (3.55) and (3.51) we obtain the inequality \(\sigma^2_2 \geq \sigma + \sigma_2^\lambda - \sigma_3^\lambda\).

Let us consider again the set \(q = (q_1, \ldots, q_n)\) introduced after formula (3.82). If \(u = 1\), i.e., \(q = (n)\), \(0 < \lambda_1 = \lambda_2 = \cdots = \lambda_n\), then \(g^\lambda = g\), so that \(\sigma^2_2 = \sigma_2^\lambda = \sigma\) and \(B^\lambda = B\). We recall that, according to lemma 3.6, we have \(\sigma = \left[\frac{n}{4}\right]\) for a typical \(m \in M\). Therefore, in this particular case formulas (3.50) and (3.60)–(3.61) agree with the results \(B = 2N - \left[\frac{n}{4}\right]\), \(k(B) = \left[\frac{n}{4}\right]\), and \(r(B) = 0\), see formula (3.32) and proposition 3.17.

**Lemma 3.20.** Suppose that \(u > 1\), i.e., there exist at least two generalized moments of inertia which are different from each other. Then at a typical point \(m \in M\) we have

\[
\sigma^1_3 = \sum_{i<j} q_i q_j = \frac{1}{2} \left(n^2 - \sum_{i=1}^u q_i^2\right),
\] (3.62)

\[
\sigma^2_3 = \sum_{i=1}^u \left[\frac{q_i}{2}\right] = \frac{u - d(q)}{2},
\] (3.63)

\[
\sigma^3_3 = 0.
\] (3.64)

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On the right-hand side of formula (3.63), the function $d(q)$ is defined as the number of odd $q_i$, $1 \leq i \leq u$ or, equivalently:

$$d(q) = \frac{1}{2} \left( u - \sum_{i=1}^{u} (-1)^{q_i} \right).$$

Proof. We shall outline the scheme of the proof for the case $u = 2$, i.e., $q = (q_1, q_2)$, with $q_1 + q_2 = n$. The generalization to the case of arbitrary $u$ should be obvious.

The decomposition (3.43) of $\mathbb{R}^n$ for $u = 2$ becomes $\mathbb{R}^n = L_1 \oplus L_2$, with $\dim L_1 = q_1$, $\dim L_2 = q_2$. According to this decomposition, the matrix $P^L(m) \in \mathfrak{so}(n)$ can be represented in a blockwise form as

$$P^L = \begin{pmatrix} A_1 & B \\ -\tilde{B} & A_2 \end{pmatrix},$$

where $A_1 \in \mathfrak{so}(q_1)$, $A_2 \in \mathfrak{so}(q_2)$, whereas $B$ is a generic $q_1 \times q_2$ matrix. According to remark 3.1, for a generic $m \in M$ both matrices $A_1$ and $A_2$ will have pairwise different eigenvalues. By applying lemma 3.4 to the two subspaces $L_1$ and $L_2$, it is possible to find a basis in each of them such that both matrices $A_1$ and $A_2$ have the normal block-diagonal form (3.21). In these coordinates, any other arbitrary matrix $Z \in \mathfrak{so}(n)$ can be represented as

$$Z = \begin{pmatrix} V_1 & U \\ -\tilde{U} & V_2 \end{pmatrix},$$

(3.65)

where $V_1 \in \mathfrak{so}(q_1)$, $V_2 \in \mathfrak{so}(q_2)$, whereas $U$ is a generic $q_1 \times q_2$ matrix. We have $Z \in g^\lambda$ if and only if $U = 0$. For the commutator of $P^L$ and $Z$ we obtain

$$[P^L, Z] = \begin{pmatrix} C_1 & D \\ -\tilde{D} & C_2 \end{pmatrix},$$

(3.66)

where

$$C_1 = [A_1, V_1] + U\tilde{B} - B\tilde{U},$$

(3.67)

$$C_2 = [A_2, V_2] + U\tilde{B} - \tilde{B}U,$$

(3.68)

$$D = A_1 U - U A_2 - V_1 B + B V_2.$$  

(3.69)

We will have $\mathcal{P}^\lambda([P^L, Z]) = 0$ if and only if $C_1 = 0$ and $C_2 = 0$.

Let us now consider the number $\sigma_2^\lambda$ defined by formula (3.43). The kernel of the operator $\mathcal{P}^\lambda \circ \text{ad}_{P^L} |_{g^\lambda}$ is made by the matrices $Z$ of the form (3.65), with $U = 0$, such that $C_1 = 0$ and $C_2 = 0$ in (3.66). Using formulas (3.67)–(3.68), we find that $V_1$ and $V_2$ must satisfy the conditions $[A_1, V_1] = 0$ and $[A_2, V_2] = 0$ respectively. According to formula (3.22) we have that $\dim \text{Ker ad}_{A_1} = [q_1/2]$ and $\dim \text{Ker ad}_{A_2} = [q_2/2]$, whence

$$\sigma_2^\lambda = \left[ \frac{q_1}{2} \right] + \left[ \frac{q_2}{2} \right],$$

which corresponds to (3.63).
In a similar way, in order to evaluate the number $\sigma_1^3$ defined by formula (3.49), we observe that the kernel of the operator $\text{ad}_{P^L}P$ is made by the matrices $Z$ of the form (3.65), with $U = 0$, such that $[P^L, Z] = 0$. Using formulas (3.66)–(3.69), we find that $V_1$ and $V_2$ must simultaneously satisfy the conditions

$$[A_1, V_1] = 0, \quad [A_2, V_2] = 0,$$

and

$$V_1B - BV_2 = 0.$$  \hfill (3.70)

We recall that $A_1$ and $A_2$ are skew-symmetric matrices in the normal block-diagonal form (3.21), with pairwise different eigenvalues. Hence, according to lemma 3.4 conditions (3.70) imply that also $V_1$ and $V_2$ must have the normal block-diagonal form (3.21), with arbitrary eigenvalues. But then it is easy to verify that, for a generic $B$, condition (3.71) necessarily implies that all eigenvalues of $V_1$ and $V_2$ must be zero. From this we conclude that $Z = 0$, so that (3.74) is proved.

Finally, in order to evaluate the number $\sigma_1^3$ defined by formula (3.47), we note that the kernel of the operator $P^L \circ \text{ad}_{P^L}$ is made by the matrices $Z$ of the form (3.65), such that $C_1 = 0$ and $C_2 = 0$ in (3.69). Introducing the matrices $T_1 := BU - UB$ and $T_2 := BU - UB$, for such $Z$ formulas (3.64)–(3.69) can be rewritten as

$$[A_1, V_1] = T_1, \quad [A_2, V_2] = T_2.$$  \hfill (3.72)

Hence $U$ must be such that

$$T_1 \in \text{ad}_{A_1}(g_1), \quad T_2 \in \text{ad}_{A_2}(g_2),$$

where $g_1 = \text{so}(q_1)$ and $g_2 = \text{so}(q_2)$. According to formula (3.21), this is equivalent to

$$(T_1)_{12} = (T_1)_{34} = \cdots = (T_1)_{2s_1 - 1, 2s_1} = 0, \quad (T_2)_{12} = (T_2)_{34} = \cdots = (T_2)_{2s_2 - 1, 2s_2} = 0,$$

where $s_1 = [q_1/2]$, $s_2 = [q_2/2]$. It is easy to check that, for a typical matrix $B$, the linear system of equations (3.74) can be solved with respect to $s_1 + s_2$ appropriately chosen elements of the matrix $U$. Therefore matrices $U$ satisfying conditions (3.73) form a linear space having dimension $q_1q_2 - s_1 - s_2$. After choosing $U$ in this space, in order to obtain an element $Z \in \text{Ker} (P^L \circ \text{ad}_{P^L})$ one has to take $V_1$ and $V_2$ satisfying equations (3.72). These are equivalent to

$$V_1 \in \text{ad}^{-1}_{A_1}(T_1), \quad V_2 \in \text{ad}^{-1}_{A_2}(T_2).$$

According to corollary 3.5, we have

$$\dim \text{ad}^{-1}_{A_1}(T_1) = \dim \text{Ker} \text{ad}_{A_1} = s_1, \quad \dim \text{ad}^{-1}_{A_2}(T_2) = \dim \text{Ker} \text{ad}_{A_2} = s_2.$$

More exactly, equations (3.72) determine all elements of $V_1$ and $V_2$ except $(V_1)_{12}$, $(V_1)_{34}, \ldots, (V_1)_{2s_1 - 1, 2s_1}$, and $(V_2)_{12}, (V_2)_{34}, \ldots, (V_2)_{2s_2 - 1, 2s_2}$. These are $s_1 + s_2$ elements which can be arbitrarily chosen, in addition to $q_1q_2 - s_1 - s_2$ elements of the matrix $U$ previously considered. We thus conclude that

$$\sigma_1^3 = q_1q_2,$$

which corresponds to (3.62). \hfill $\square$
Let $C^j$ denote the standard set of Casimir functions for the adjoint action of the group $SO(q_j)$ of orthogonal transformations on euclidean subspace $L_j \subseteq \mathbb{R}^n$, $j = 1, \ldots, u$, see definition 3.2. This set contains $[q_j/2]$ functions on the algebra $so(q_j)$ which corresponds to the subspace $L_j \subseteq \mathbb{R}^n$. We can consider these functions as functions on the algebra $\mathfrak{g} = so(n)$. Let us consider the set $C^\lambda = (C^1, \ldots, C^u)$ of functions on $\mathfrak{g}$, obtained by collection of sets $C^j$. Clearly $C^\lambda$ contains
\[ s^\lambda := \sum_{j=1}^{u} \left[ \frac{q_j}{2} \right] = \frac{n - d(q)}{2}, \quad (3.75) \]
elements. Let $C^{L\lambda} = (C^{L1}, \ldots, C^{Lu})$ denote the set $C^{L\lambda} := C^\lambda \circ P^L$ of functions on $T^*G$, obtained by making the composition of the functions of set $C^\lambda$ with the map $P^L$. We have already introduced the set $\mathcal{C} = C \circ P^L = C \circ P^R$ (see proposition 3.11). Clearly $C^{L\lambda} = \mathcal{C}$ if $u = 1$. We will denote with $Z^\lambda$ the set of functions $Z^\lambda := (\mathcal{C}, C^{L\lambda})$, if $u > 1$, or $Z^\lambda := \mathcal{C}$, if $u = 1$. The set $Z^\lambda$ contains $z^\lambda$ elements, where
\[ z^\lambda = \begin{cases} \left\lfloor \frac{n}{2} \right\rfloor & \text{if } u = 1, \\ \left\lfloor \frac{n}{2} \right\rfloor + s^\lambda. & \text{if } u > 1. \end{cases} \quad (3.76) \]

**Proposition 3.21.** For any $\lambda = (\lambda_1, \ldots, \lambda_n)$, each function of the set $Z^\lambda$ is in involution with each function of the set $B^\lambda$:
\[ \{Z^\lambda, B^\lambda\} = 0. \quad (3.77) \]

Furthermore, almost everywhere in $M = T^*G$, where $G = SO(n)$, the set $Z^\lambda$ is functionally independent, i.e., rank $Z^\lambda = z^\lambda$.

**Proof.** The equality $\{\mathcal{C}, B^\lambda\} = 0$ follows from (3.36). From the definition of Casimir function, and the fact that $P^L$ is a Poisson map, it follows that $\{C^{L^i}, P^{L^j}\} = 0 \forall i = 1, \ldots, u$, where $P^{L^i} := \{P^{L_h}_k : (h, k) \in I^i\}$, see formula (3.44). Relation (3.8) implies $\{P^{L_i}, P^{L_j}\} = 0 \forall i \neq j$, whence $\{C^{L_i}, P^{L_j}\} = 0$. Furthermore, (3.10) implies $\{C^{L\lambda}, P^R\} = 0$. Hence $\{C^{L\lambda}, B^\lambda\} = 0$, and equality (3.77) is proved.

According to proposition 3.12, applied to the group $SO(n)$ and its subgroups $SO(q_j)$, $j = 1, \ldots, u$, both sets $\mathcal{C}$ and $C^{L\lambda}$ are almost everywhere functionally independent, so that
\[ \text{rank } \mathcal{C} = \left\lfloor \frac{n}{2} \right\rfloor, \quad \text{rank } C^{L\lambda} = s^\lambda. \quad (3.78) \]

It follows that
\[ \text{rank } Z^\lambda = \dim \text{Span } (d\mathcal{C}, dC^{L\lambda}) = \left\lfloor \frac{n}{2} \right\rfloor + s^\lambda - \dim (\text{Span } d\mathcal{C} \cap \text{Span } dC^{L\lambda}). \quad (3.79) \]

Let us then consider the set
\[ G = \left\{ w \in T^*_m M : w = \sum_{i<j} K_{ij} dP^L_{ij}, K \in \text{Ker } (\text{ad}_{P^L} \mid g^\lambda) \right\}. \quad (3.80) \]

Using (3.28) and (3.38) we see that
\[ G = \text{Span } d\mathcal{C} \cap \text{Span } dP^{L\lambda}. \quad (3.81) \]
Since \( \text{Span } dC^L \subseteq \text{Span } dP^L \), we obtain
\[
G \supseteq \text{Span } dC \cap \text{Span } dC^L.
\] (3.82)

On the other hand, any \( w \in G \) can be expressed as \( w = \sum_{i=1}^w w_i \), with \( w_i \in \text{Span } dP^L \). For any \( z \in \text{Span } dP^L \) we have \( 0 = \Pi(w, z) = \Pi(w_i, z) \). Hence \( w_i \in \text{Span } dP^L \cap (\text{Span } dP^L)^\perp = \text{Span } dC^L \), where the last equality follows from the application of (3.39) to the subspace \( L_i \). It follows that \( w \in \text{Span } dC^L \).

Hence, using (3.81) and (3.82), we conclude that
\[
G = \text{Span } dC \cap \text{Span } dC^L,
\]
so that
\[
\dim (\text{Span } dC \cap \text{Span } dC^L) = \dim G = \dim (\text{Ker } \text{ad}_{PL}|_{g^\lambda}) = \sigma_3^\lambda.
\] (3.83)

Recalling formula (3.64), and the equalities \( s^\lambda = \sigma_3^\lambda = [n/2] \) for \( u = 1 \), we then obtain from (3.79) that \( \text{rank } Z^\lambda = z^\lambda \) for any \( u \), so that the set \( Z^\lambda \) is functionally independent.

\textbf{Lemma 3.22}. Almost everywhere in \( T^*G \) we have
\[
\text{Span } dZ^\lambda = \text{Span } dB^\lambda \cap (\text{Span } dB^\lambda)^\perp.
\] (3.84)

Moreover, if \( u > 1 \) we have
\[
\text{rank } B^\lambda = 2N - \sum_{i<j} q_i q_j = \frac{1}{2} \left( n^2 - 2n + \sum_{i=1}^u q_i^2 \right),
\] (3.85)
\[
k(B^\lambda) = z^\lambda = \left[ \frac{n}{2} \right] + \sum_{i=1}^u \left[ \frac{q_i}{2} \right] = n - \left[ \frac{d(q) + 1}{2} \right],
\] (3.86)
\[
r(B^\lambda) = \sum_{i<j} q_i q_j - z^\lambda = \frac{1}{2} \left( n^2 - 2n - \sum_{i=1}^u q_i^2 \right) + \left[ \frac{d(q) + 1}{2} \right],
\] (3.87)
where \( k(B^\lambda) \) and \( r(B^\lambda) \) are respectively the centrality and the defect of integrability (see definition 3.6) of the set \( B^\lambda \).

\textbf{Proof}. Since the elements of \( Z^\lambda \) are functions of \( B^\lambda \), from equality (3.77) it follows that \( \text{Span } dZ^\lambda \subseteq W := \text{Span } dB^\lambda \cap (\text{Span } dB^\lambda)^\perp \). From (3.51), taking into account (3.27), (3.63) and (3.64), we obtain that \( \dim W = z^\lambda \). Moreover, we know from proposition 3.21 that the set \( Z^\lambda \) is functionally independent, so that \( \dim(\text{Span } dZ^\lambda) = z^\lambda \). Hence we conclude that \( \text{Span } dZ^\lambda = W \), so that (3.84) is proved.

Equality (3.85) follows from (3.50) and (3.62).

According to definition 3.6, the centrality of \( B^\lambda \) is at least as great as the number of elements of \( Z^\lambda \), i.e., we have \( k(B^\lambda) \geq z^\lambda \). On the other hand, using (3.27), (3.63) and (3.64), we obtain from (3.60) that \( k(B^\lambda) \leq z^\lambda \). Therefore (3.86) is proved.

Finally, (3.87) follows from definition 3.6 and from equalities (3.85)–(3.86).
Note that one can easily prove, by induction on \( n \), that \( r(B^\lambda) \) given by formula (3.87) is always an even integer.

The complete integrability of the system describing the free rotation of an \( n \)-dimensional rigid body can be proved by introducing the so-called Manakov’s integrals [20]. It is not difficult to check that the function \( \mathcal{P}(\rho) \equiv (1/2k)\operatorname{Tr}(P^B + J^2 \rho)^k : T^*G \to \mathbb{R} \), where \( J \) is the diagonal \( n \times n \) matrix with \( \lambda_1, \lambda_2, \ldots, \lambda_n \) as diagonal elements, is in involution with \( H^\lambda \) for any value of the parameter \( \rho \). Hence the coefficients of the polynomial \( \mathcal{P}(\rho) \) in the variable \( \rho \) are also in involution with \( H^\lambda \), i.e. we have \( \{c_{ij}, H^\lambda\} = 0 \), where

\[
\mathcal{P}(\rho) \equiv \frac{1}{2k} \operatorname{Tr}(P^B + J^2 \rho)^k = \sum_{j=0}^{k} c_{kj} \rho^j.
\]

These coefficients are not all functionally independent. One immediately sees that \( c_{kk} \) is just a constant, and that \( c_{kj} = 0 \) whenever \( k-j \) is odd. Furthermore it can be proved that, if one is only interested to functionally independent elements, then one need only consider coefficients \( c_{kj} \) with \( k = 2, 3, \ldots, n \). It is easy to see that \( \mathcal{P}(\rho) \) is in involution with all functions of the set \( B^\lambda \), so that \( \{c_{ij}, B^\lambda\} = 0 \). Moreover, one can prove that all coefficients \( c_{ij} \) are mutually in involution, \( \{c_{ij}, c_{ij'}\} = 0 \). This result provides in particular another proof of the fact that all these coefficients are integrals of the system with hamiltonian \( H^\lambda \), for it can be shown that \( H^\lambda \) can be expressed as a linear combination of the functions \( c_{k,k-2} \) for \( k = 2, \ldots, n \).

It has been proved in general that the system with hamiltonian \( H^\lambda \) is integrable [21] [16] [17]. In the general case in which all generalized moments of inertia are pairwise different, \( \lambda_i \neq \lambda_j \) for \( i \neq j \), an integrable set of functions is given by \( (M; P^R) \), where \( M = (c_{k,k-2}, k = 2, \ldots, n, i = 1, 2, \ldots, [k/2]) \) is the complete set of \( \sum_{k=2}^{n}[k/2] = (1/2)(n(n-1)/2 + [n/2]) \) functionally independent coefficients \( c_{ij} \), and \( P^R \) is a set of \( n(n-1)/2 - [n/2] \) elements of \( P^R \), such that \( (C, P^R) \) is a functionally independent set. Hence this system is integrable with \( (1/2)(n(n-1)/2 + [n/2]) \) central functions. The elements \( c_{ij} \) of \( M \) such that \( j > 0 \) are called Manakov’s integrals. The remaining \([n/2]\) elements of \( M \), i.e. \( c_{2k,0} = (1/2k)\operatorname{Tr}(P^B)^k \) for \( k = 1, 2, \ldots, [n/2], \) are independent of the moments of inertia \( \lambda \) and form a set of Casimir functions equivalent to the set \( C \) introduced in section 3.3. Hence, an equivalent integral set of functions for the free \( n \)-dimensional rigid body with pairwise different moments of inertia is \( (C, M; P^R) \), where \( M \) is the set of \( (1/2)(n(n-1)/2 - [n/2]) \) Manakov’s integrals.

When the moments of inertia are not all pairwise different, the set \( M \) is no longer functionally independent. However the integrability of the system is preserved, which means that one can construct an integrable set of functions whose central subset is made of the elements of \( Z^\lambda \) and of a suitable subset of \( M \). According to proposition 2.1 such a subset of \( M \) must contain just \( r/2 \) elements, where \( r = r(B^\lambda) \) is the defect of integrability of the set \( B^\lambda \) and is given by formula (3.87). Therefore, the central subset will contain \( k(B^\lambda) + r(B^\lambda)/2 \) elements. This result is expressed by the following proposition.

**Proposition 3.23.** The system with hamiltonian \( H = H^\lambda \) given by formula (3.41) is integrable with

\[
\tilde{k}(q) = \frac{1}{4} \left( n^2 + 2n - \sum_{i=1}^{n} q_i^2 \right) - \frac{1}{2} \left[ \frac{d(q) + 1}{2} \right]
\]

(3.88)
central integrals.

Manakov’s integrals can be explicitly represented in the following form:

\[ c_{k,k-2l} = \frac{1}{4l} \sum_{i_1,i_2,\ldots,i_{2l}} a_{k,k-2l}^{i_1i_2\ldots i_{2l}} P_{i_1i_2}^L P_{i_3i_4}^L \cdots P_{i_{2l-1}i_{2l}}^L P_{i_{2l+1}}^L, \]

(3.89)

with \( 0 < l < k/2 \), where

\[ a_{k,k-2l}^{i_1i_2\ldots i_{2l}} = \sum_{b_1 \geq 0, b_2 \geq 0, \ldots, b_{2l} \geq 0} \lambda_{i_1}^{2b_1} \lambda_{i_2}^{2b_2} \cdots \lambda_{i_{2l}}^{2b_{2l}} \delta_{b_1+b_2+\cdots+b_{2l},k-2l}. \]

(3.90)

We see that \( c_{k,k-2l} \) is a homogeneous polynomial of degree \( 2l \) in the left-invariant momenta, while its coefficients \( a_{k,k-2l}^{i_1i_2\ldots i_{2l}} \) are homogeneous polynomials of degree \( 2(k-2l) \) in the generalized moments of inertia, completely symmetrical with respect to permutations of the indexes \( i_1, \ldots, i_{2l} \).

In Table 3 we give the number \( \bar{k}(q) \) of central integrals resulting from the above proposition for free \( n \)-dimensional rigid bodies with \( n \leq 6 \). We also give the quantities \( k(B^\lambda) \) and \( r(B^\lambda) \) resulting from lemma 3.22.

| \( n \) | \( q \) | \( k(B^\lambda) \) | \( r(B^\lambda) \) | \( \bar{k}(q) \) |
|---|---|---|---|---|
| 3 | (3) | 1 | 0 | 1 |
| 3 | (1,2) | 2 | 0 | 2 |
| 3 | (1,1,1) | 1 | 2 | 2 |
| 4 | (4) | 2 | 0 | 2 |
| 4 | (1,3) | 3 | 0 | 3 |
| 4 | (2,2) | 4 | 0 | 4 |
| 4 | (1,1,1) | 2 | 6 | 5 |
| 5 | (5) | 2 | 0 | 2 |
| 5 | (1,4) | 4 | 0 | 4 |
| 5 | (2,3) | 4 | 2 | 5 |
| 5 | (1,1,3) | 3 | 4 | 5 |
| 5 | (1,2,2) | 4 | 4 | 6 |
| 5 | (1,1,1,2) | 3 | 6 | 6 |
| 5 | (1,1,1,1) | 2 | 8 | 6 |
| 6 | (6) | 3 | 0 | 3 |
| 6 | (1,5) | 5 | 0 | 5 |
| 6 | (2,4) | 6 | 2 | 7 |
| 6 | (3,3) | 6 | 2 | 7 |
| 6 | (1,1,4) | 5 | 4 | 7 |
| 6 | (1,2,3) | 5 | 6 | 8 |
| 6 | (1,1,3) | 4 | 8 | 8 |
| 6 | (1,2,2) | 5 | 8 | 9 |
| 6 | (1,1,1,2) | 4 | 10 | 9 |
| 6 | (1,1,1,1,1) | 3 | 12 | 9 |

Table 3: Number of central integrals for free rigid bodies.
3.3 Free rotation of a quantum rigid body

In order to quantize a free rigid body we have to consider the quantum impulses \( \hat{P}_{ij}^L, \hat{P}_{ij}^R \), which are constructed according to formula (2.16) in correspondence with vector fields \( V_{ij}^L, V_{ij}^R \) respectively, \( 1 \leq i < j \leq n \). However, if we want to apply to this system the concept of integral quantum system introduced in \( \text{[1]} \), we are apparently faced by the problem that here the configuration space \( K = G = SO(n) \) is not a domain of the linear space \( \mathbb{R}^N \). This problem is solved by the consideration of local coordinates on \( G \). Note that the main part \( M\mathcal{F} \) of a linear differential operator \( \mathcal{F} \) (see definition in \( \text{[1]} \)) is not defined intrinsically. However the symbol \( (M\mathcal{F})_{\text{sym}} \) can be considered as intrinsically defined, according to the following proposition.

**Proposition 3.24.** The symbol \( S := (M\mathcal{F})_{\text{sym}} \) of the main part of a linear operator of class \( \mathcal{O} \), expressed via local coordinates \( x \) on configuration space \( K \), has the form of a homogeneous polynomial of \( p \). This polynomial \( S = S(x, p) \) behaves under a change of local coordinates \( x \) on \( K \) as a function on the cotangent bundle \( T^*K \) to the manifold \( K \). It follows from this fact that the definition of quasi-independence of a set of operators does not depend on the choice of local coordinates on configuration space \( K \). The same is true for the definition of quasi-integrability of either a set of operators or an individual operator.

The proof of this proposition is obvious. The first part of the proposition, about the representation of \( S \) as a function on \( T^*K \), is actually the reformulation of well-known facts.

Let us consider the quantum system with hamiltonian operator

\[
\hat{H} = \hat{H}^\lambda = \frac{1}{2} \sum_{i<j} \left( \frac{\hat{P}_{ij}^L}{\lambda_i + \lambda_j} \right)^2
\]

(3.91)
on \( C^\infty(SO(n)) \). We consider this system as the system describing the free rotation of a quantum \( n \)-dimensional rigid body.

**Proposition 3.25.** For \( n \leq 6 \) this quantum system is quasi-integrable for any \( \lambda \), with the same number \( k \) of central operators as the number \( \bar{k}(u) \) of central integrals of the corresponding classical system, see proposition 3.23 and Table 3.

Moreover, if \( q = (n) \), this quantum system is quasi-integrable for any \( n \) with \( \lfloor n/2 \rfloor \) central operators. If \( q = (1, n-1) \), this quantum system is quasi-integrable for any \( n \) with \( n-1 \) central operators.

**Proof.** Let \( \hat{P}^L \) and \( \hat{P}^R \) denote the sets of operators \( \hat{P}^L = (\hat{P}_{ij}^L, 1 \leq i < j \leq n) \) and \( \hat{P}^R = (\hat{P}_{ij}^R, 1 \leq i < j \leq n) \). Let us consider the set \( \hat{B} := (\hat{P}^L, \hat{P}^R) \) containing \( 2N \) operators. Let \( \hat{C}^R \) denote the set of operators which are obtained by symmetrization with respect to \( \hat{P}^R \) from the functions of set \( \hat{C}^R = C(\hat{P}^R) \), i.e., \( \hat{C}^R := (\hat{C}^R)_{\text{sym}} \). Analogously we define \( \hat{C}^L, \hat{P}^{\lambda L}, \hat{B}^\lambda, \hat{C}^{\lambda L}, \hat{Z}^\lambda \), i.e., \( \hat{C}^L := (\hat{C}^L)_{\text{sym}} \) etc. We first show that the commutation relations \( [\hat{B}^\lambda, \hat{H}^\lambda] = [\hat{B}^\lambda, \hat{Z}^\lambda] = 0 \) follow from the analogous classical relations \( \{\hat{B}^\lambda, \hat{H}^\lambda\} = \{\hat{B}^\lambda, \hat{Z}^\lambda\} = 0 \) and from the propositions about quantization of [2] and section 2.2 of the present paper. The commutators of the operators of set
\( \hat{B} \) have the same form as the Poisson brackets of the corresponding classical functions, since all these functions are linearly dependent on canonical impulses \( p \), see formulas (3.7)–(3.10) and proposition 2.6

\[
\begin{align*}
[\hat{P}_{ij}^{L}, \hat{P}_{ik}^{L}] &= -\delta_{ik} \hat{P}_{jk}^{L} - \delta_{jk} \hat{P}_{ik}^{L} + \delta_{ik} \hat{P}_{jk}^{R} + \delta_{jk} \hat{P}_{ik}^{R}, \\
[\hat{P}_{ij}^{R}, \hat{P}_{ik}^{R}] &= \delta_{ik} \hat{P}_{jk}^{R} + \delta_{jk} \hat{P}_{ik}^{R} - \delta_{ik} \hat{P}_{jk}^{L} - \delta_{jk} \hat{P}_{ik}^{L}, \\
[\hat{P}_{ij}^{L}, \hat{P}_{ik}^{R}] &= 0.
\end{align*}
\]

The relation \( [\hat{H}^{\lambda}, \hat{B}^{\lambda}] = 0 \) then follows from \( \{ H^{\lambda}, B^{\lambda} \} = 0 \) using proposition 4.2, case b, of [2]. Similarly, since

\[
\{ C^{L}(P^{L}), B^{\lambda} \} = 0, \quad \hat{C}^{L} = (C^{L})^{\text{sym}},
\]

the equality \( \hat{C}^{L}, B^{\lambda} \) is obtained by applying corollary 4.4 of [2]. Analogously, we obtain the equality \( \{ \hat{C}^{L}, B^{\lambda} \} = 0 \) from the corresponding classical relation \( \{ C^{L}(P^{L}), B^{\lambda} \} = 0 \). We have thus proved that \( [\hat{B}^{\lambda}, \hat{Z}^{\lambda}] = 0 \).

If \( q = (n) \) or \( q = (1, n-1) \), according to lemma 5.22, the defect of integrability of the set \( B^{\lambda} \) in the classical case is \( r(B^{\lambda}) = 0 \). Moreover, we have \( k(B^{\lambda}) = [n/2] \) if \( q = (n) \), and \( k(B^{\lambda}) = n - 1 \) if \( q = (1, n-1) \). This implies that in these two cases, for any \( n \), there exists a classical integrable set of functions of the form \( F = (Z^{\lambda}; B') \), where \( B' \subset B^{\lambda} \), and the central subset \( Z^{\lambda} \) contains \( k(B^{\lambda}) \) elements. Let us then consider the corresponding set of operators \( \hat{F} = (\hat{Z}^{\lambda}; \hat{B}') \).

From what we have seen above, it follows that \( [\hat{Z}^{\lambda}, \hat{F}] = 0 \). Moreover, since all functions of \( \hat{F} \) are homogeneous with respect to \( p \), these functions coincide with their main parts with respect to \( p \), i.e., \( M(\hat{F}) = \hat{F} \). It is also easy to see that the elements of \( \hat{F} \) are the symbols of the main parts with respect to \( p \) of the elements of the corresponding set of operators \( \hat{F} \), i.e., \( \hat{F} = (\hat{M}(\hat{F}))^{\text{amb}} \). Hence, the quasi-independence of the sets of operators \( \hat{F} \) follows immediately from the functional independence of set of functions \( F \). One thus concludes that \( \hat{F} \) is an integrable set of operators with \( k(B^{\lambda}) \) central elements. One can also easily show that in these two cases, \( \hat{H}^{\lambda} \) is a linear combination of the elements of \( \hat{Z}^{\lambda} \). The integrability of the system describing the free quantum rigid-body is thus proved for any \( n \) in the two cases \( q = (n) \) and \( q = (1, n-1) \).

In the remaining cases, the classical integrable sets of functions generally include also one or more Manakov’s integrals among their central elements. We define Manakov’s operators \( \hat{c}_{k,k-2l} \) as the symmetrization of the classical functions (3.89) with respect to the left-invariant momenta:

\[
\hat{c}_{k,k-2l} = \frac{1}{4l} \sum_{i_1,i_2,...,i_2l} a_{k,k-2l}^{i_1i_2...i_2l} \text{Sym}_{2l}(\hat{P}_{i_1i_2}, \hat{P}_{i_2i_3}, ..., \hat{P}_{i_{2l-1}i_{2l}}, \hat{P}_{i_{2l}i_1}),
\]

with \( 0 < l < k/2 \). Note that, also in the quantum case, the hamiltonian operator \( \hat{H}^{\lambda} \) can be expressed as a linear combination of the operators \( \hat{c}_{k,k-2l} \) for \( k = 2, \ldots, n \). By applying again the results of [2] we easily see that \( [\hat{c}_{k,k-2l}, \hat{B}^{\lambda}] = 0 \). However, no general theorem ensures that Manakov’s operators commute with \( \hat{H}^{\lambda} \) or among themselves. We will here limit ourselves to studying commutators between Manakov’s operators of degree lower that 6 in the momenta. This will be sufficient to establish the quasi-integrability of the free quantum rigid body in spatial dimensions \( n \leq 6 \).
The commutator between two Manakov’s operators, when one of them is of second degree, can be evaluated by making use of proposition 2.3 of [2], and of the algebra (3.92) of left-invariant momenta. In this way, with some computation we find for any \( l, h \):

\[
[\hat{c}_{l,l-2}, \hat{c}_{h,h-2}] = 0, \quad (3.96)
\]

\[
[\hat{c}_{l,l-2}, \hat{c}_{h,h-4}] = \frac{1}{6} \sum_{i,j,k} b_{i,j,k}^{ijk} \text{Sym}_3(\hat{P}_{L_{ij}}, \hat{P}_{L_{jk}}, \hat{P}_{L_{ki}}), \quad (3.97)
\]

where

\[
b_{i,j,k}^{ijk} = a_{i,j}^{l,l-2} \left( 2a_{i,j,k}^{h,h-4} - 3a_{i,j,k}^{h,h-4} - \sum_{p \neq i,j,k} a_{i,j,k}^{k,p} \right) + \sum_{p \neq i,j,k} a_{i,j,k}^{k,p} \left( a_{i,j,k}^{i,j,p} - a_{i,j,k}^{i,j,p} \right). \quad (3.98)
\]

Note that the operator \( \text{Sym}_3(\hat{P}_{L_{ij}}, \hat{P}_{L_{jk}}, \hat{P}_{L_{ki}}) \) is completely antisymmetrical with respect to permutations of indexes \( i, j, k \), so that formula (3.97) can be rewritten as

\[
[\hat{c}_{l,l-2}, \hat{c}_{h,h-4}] = \sum_{i<j<k} b_{i,j,k}^{ijk} \text{Sym}_3(\hat{P}_{L_{ij}}, \hat{P}_{L_{jk}}, \hat{P}_{L_{ki}}), \quad (3.99)
\]

where \( b_{i,j,k}^{ijk} \) denotes the complete antisymmetrization of coefficient \( b_{i,j,k}^{ijk} \).

In formula (3.98) the coefficients \( a_{i,j}^{l,l-2} \) and \( a_{i,j,k}^{h,h-4} \) have to be replaced by their explicit expressions given by (3.90). We have

\[
a_{i,j}^{l,l-2} = a_{i,j}^{l,l-2} = \sum_{k=0}^{l-2} \lambda_i^{2(l-2-k)} \lambda_j^{2k} = \frac{\lambda_i^{2(l-1)} - \lambda_j^{2(l-1)}}{\lambda_i^2 - \lambda_j^2}. \quad (3.99)
\]

Moreover, for \( h = 5 \) we have

\[
a_{i,j,k}^{h,h-4} = a_{i,j,k}^{5,1} = \lambda_i^2 + \lambda_j^2 + \lambda_k^2 + \lambda_p^2,
\]

so that from (3.98) we easily obtain \( b_{i,j,k}^{ijk} = 0 \), which implies

\[
[\hat{c}_{l,l-2}, \hat{c}_{5,1}] = 0 \quad (3.100)
\]

and consequently \( [\hat{H}_\lambda, \hat{c}_{5,1}] = 0 \). From these results it follows that the free quantum rigid body is a quasi-integrable system for spatial dimensions \( n \leq 5 \). The integrable set of functions of the classical system can in fact be quantized by replacing Manakov’s integrals with the corresponding operators. Since these operators are homogeneous in the momenta, the symbols of their main parts with respect to \( \hat{p} \) coincide with the corresponding classical functions. Hence the quasi-independence of the sets of operators is a consequence of the functional independence of the classical sets of functions.

For \( h = 6 \) the coefficient \( a_{i,j,k}^{h,h-4} \) becomes

\[
a_{i,j,k}^{i,j,k} = \lambda_i^2 + \lambda_j^2 + \lambda_k^2 + \lambda_p^2 + \lambda_i^2 \lambda_j^2 + \lambda_i^2 \lambda_k^2 + \lambda_i^2 \lambda_p^2 + \lambda_j^2 \lambda_k^2 + \lambda_j^2 \lambda_p^2 + \lambda_k^2 \lambda_p^2,
\]

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and from (3.98) we obtain
\[ b_{i,j,l}^{ijk} = \frac{5}{6} \left[ \lambda_i^2 (\lambda_j^2 - \lambda_k^2) a_{ij,l-2}^{ij} + \lambda_j^2 (\lambda_k^2 - \lambda_i^2) a_{ij,l-2}^{ij} + \lambda_k^2 (\lambda_i^2 - \lambda_j^2) a_{ij,l-2}^{ij} \right] \] (3.101)

\[ = \frac{5}{6} \left[ \lambda_i^{2(l-1)} (\lambda_j^2 - \lambda_k^2) + \lambda_j^{2(l-1)} (\lambda_k^2 - \lambda_i^2) + \lambda_k^{2(l-1)} (\lambda_i^2 - \lambda_j^2) \right]. \]

Since for a generic set \( \lambda \) of generalized moments of inertia the above expression is different from 0, we have that in general \([\hat{c}_{i,l}, \hat{c}_{6,2}] \neq 0\). Note also that, putting \( l = 3/2 \) in (3.99), we get \( a_{3/2, -1/2}^{ij} = 1/(\lambda_i + \lambda_j) \), so that we can formally write
\[ \hat{H}^\lambda = -\hat{c}_{3/2, -1/2}. \]

From the above formulas we thus directly obtain
\[ [\hat{H}^\lambda, \hat{c}_{6,2}] = \sum_{i,j<k} b_{i,j,k}^{ijk} \text{Sym}_3(\hat{P}_{ij}, \hat{P}_{jk}, \hat{P}_{ki}), \]
with
\[ b_{i,j,k}^{ijk} = -\frac{5}{6} \left[ \lambda_i (\lambda_j^2 - \lambda_k^2) + \lambda_j (\lambda_k^2 - \lambda_i^2) + \lambda_k (\lambda_i^2 - \lambda_j^2) \right]. \]

Hence \( \hat{c}_{6,2} \) is not in general an integral operator of the quantum system with hamiltonian \( \hat{H}^\lambda \).

By using again proposition 2.3 of [2] and commutation relations \(\hat{c}_{6,2}^{\lambda} \), one finds however that, for arbitrary symmetrical coefficients \( \alpha^{ij} = \alpha^{ji} \),
\[ -\frac{1}{2} \sum_{ij} \alpha^{ij} [\hat{c}_{i,l-2}, (\hat{P}_{ij}^L)^2] = \sum_{i,j<k} b_{i,j,k}^{ijk} \text{Sym}_3(\hat{P}_{ij}, \hat{P}_{jk}, \hat{P}_{ki}), \]
with
\[ b_{i,j,k}^{ijk} = (\alpha^{jk} - \alpha^{ik}) a_{i,j,l-2}^{ij} + (\alpha^{ki} - \alpha^{ij}) a_{i,l-2}^{ij} + (\alpha^{ij} - \alpha^{kj}) a_{i,j,l-2}^{ij}. \] (3.102)

One sees immediately that the above expression becomes identical with (3.101) if one chooses \( \alpha^{ij} = (5/6)\lambda_j^2 \lambda_j^2 \). Hence, if we define the modified Manakov’s operator
\[ \hat{C}_{6,2} \equiv \hat{c}_{6,2} + \frac{5}{12} \sum_{i,j} \lambda_j^2 \lambda_j^2 (\hat{P}_{ij}^L)^2, \]
then we get
\[ [\hat{c}_{i,l-2}, \hat{C}_{6,2}] = 0, \quad [\hat{H}^\lambda, \hat{C}_{6,2}] = 0. \]

For \( n = 6 \), let us then consider the set of operators \( F \) which is obtained from the classical integrable set of functions \( F \) given by proposition 5.28 by replacing Manakov’s integrals \( c_{i,l-2} \) \((l = 3, \ldots, 6)\), \( c_{5,1} \), and \( c_{6,2} \), with the operators \( \hat{c}_{i,l-2} \), \( \hat{c}_{5,1} \), and \( \hat{C}_{6,2} \) respectively. In order to prove that this set of operators satisfies the required commutation relations, we still have to show that
\[ [\hat{c}_{5,1}, \hat{C}_{6,2}] = 0. \] (3.103)

With the techniques already employed one obtains that
\[ \frac{5}{12} \sum_{i,j} \lambda_j^2 \lambda_j^2 [\hat{c}_{5,1}, (\hat{P}_{ij}^L)^2] = -\frac{5}{6} \sum_{hlm} \lambda_i^4 \lambda_j^2 \left( \frac{5}{3} \text{Sym}_3(\hat{P}_{kl}^L, \hat{P}_{lm}^L, \hat{P}_{nh}^L) \right. \]
\[ + \left. \sum_{ij} \text{Sym}_5(\hat{P}_{ij}^L, \hat{P}_{jh}^L, \hat{P}_{hl}^L, \hat{P}_{il}^L, \hat{P}_{mi}^L) \right). \] (3.104)
On the other hand, the commutator $[\hat{c}_{5,1}, \hat{c}_{6,2}]$ between two fourth-order symmetrized polynomials cannot be worked out with the tools provided in [2]. By means of a straightforward and quite heavy calculation, we have verified that this commutator is indeed just the opposite of the expression (3.103), so that equality (3.103) actually holds. Of course, the symbol of the main part of $\hat{C}_{6,2}$ coincides with the classical function $c_{6,2}$, so that the quasi-independence of the set $\hat{F}$ again follows from the functional independence of the classical set $F$. We conclude that $\hat{F}$ is a quasi-integrable set of operators. The proposition is thus completely proved.

We have seen that, for $n \geq 6$, the correct quantization of Manakov’s integral $c_{6,2}$ does not coincide with the symmetrization of the classical function with respect to the left-invariant momenta. We have nevertheless provided a recipe to achieve the quasi-integrability of the free quantum rigid body for $n = 6$ for arbitrary moments of inertia $\lambda$. We can conjecture that analogous procedures can lead to the quasi-integrability of this quantum system also for $n > 6$. However, we are unable at the moment to prove that this conjecture is true.

References

[1] M. Marino and N. N. Nekhoroshev, Quantization of classical integrable systems. Part I: quasi-integrable quantum systems, arXiv:1001.4685 [math-ph] (2010).

[2] M. Marino and N. N. Nekhoroshev, Quantization of classical integrable systems. Part II: quantization of functions on Poisson manifolds, arXiv:1001.4701 [math-ph] (2010).

[3] N. N. Nekhoroshev, Action-angle variables and their generalizations, Trans. Moscow Math. Soc. 26, 180–198 (1972).

[4] A. T. Fomenko, Differential Geometry and Topology, Consultants Bureau (New York), 1987.

[5] F. Fassò, Superintegrable Hamiltonian Systems: Geometry and Perturbations, Acta Appl. Math. 87, 93–121 (2005).

[6] A. M. Olshanetsky and M. A. Perelomov, Classical integrable finite-dimensional systems related to Lie algebras, Phys. Rep. 71, no. 5, 314–400 (1981).

[7] A. M. Olshanetsky and M. A. Perelomov, Quantum integrable systems related to Lie algebras, Phys. Rep. 94, no. 6, 313–404 (1983).

[8] N. W. Evans, Superintegrability in classical mechanics, Phys. Rev. A 41, 5666-5676 (1990).

[9] P. I. Etingof, Quantum integrable systems and representations of Lie algebras, J. Math. Phys. 36, no. 6, 2636–2651 (1995).

[10] P. Tempesta, A. V. Turbiner, and P. Winternitz, Exact solvability of superintegrable systems, J. Math. Phys. 42, 4248-4257 (2001).
[11] S. Gravel and P. Winternitz, Superintegrability with third-order integrals in quantum and classical mechanics, *J. Math. Phys.* 43, 5902–5912 (2002).

[12] I. Marquette and P. Winternitz, Polynomial Poisson algebras for classical superintegrable systems with a third-order integral of motion, *J. Math. Phys.* 48, 012902 (2007).

[13] M. A. Rodriguez and P. Winternitz, Quantum superintegrability and exact solvability in $n$ dimensions, *J. Math. Phys.* 43, 1309-1322 (2002).

[14] L. D. Landau and E. M. Lifshitz, *Mechanics*, Pergamon (London), 1960.

[15] L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, Pergamon (London), 1960.

[16] A. S. Mishchenko and A. T. Fomenko, Euler equations on finite dimensional Lie groups, *Math. URSS, Izvestija* 12, 371–389 (1978).

[17] T. Ratiu, The motion of the free $n$-dimensional rigid body, *Indiana Univ. Math. J.* 29, 609–629 (1980).

[18] M. Adler and P. van Moerbeke, Completely integrable systems, Kac–Moody Lie algebras, and curves, *Adv. in Math.* 38, 267–317 (1980).

[19] V. I. Arnold, *Mathematical methods of classical mechanics*, Springer-Verlag (New York), 1978.

[20] S. V. Manakov, Note on the integration of Euler’s equations of the dynamics of an $n$-dimensional rigid body, *Funcional Anal. Appl.* 4, 328–329 (1976).

[21] A. S. Mishchenko and A. T. Fomenko, On the integration of Euler equations on semisimple Lie algebras, *Soviet Math. Dokl.* 17, 1591–1593 (1976).