A mean identity for longest increasing subsequence problems

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Abstract

We show that a wide variety of generalized increasing subsequence problems admit a one parameter family of extensions for which we can exactly compute the mean length of the longest increasing subsequence. By the nature of the extension, this gives upper bounds on the mean in the unextended model, which turn out to be asymptotically tight for all of the models that have so far been analyzed. A heuristic analysis based on this fact gives not just the asymptotic mean but also the asymptotic scale factor, again agreeing with all known cases.

1 Introduction

In [14], Prähofer and Spohn consider a certain polynuclear growth (PNG) model with stationary initial conditions, and show that it maps to the following increasing subsequence problem:

Let \( t \) be a positive real number. Pick a random set of points in the unit square \([0, 1] \times [0, 1]\) as follows. On the left and bottom edges, take a Poisson process of mean \( t \); inside the square, take a Poisson process of mean \( t^2 \). (Thus our total mean is \( t^2 + 2t \).) A sequence of these points is “increasing” if we have \( x \leq x', y \leq y' \) whenever \((x, y)\) and \((x', y')\) are consecutive points in the sequence; the length of the sequence is defined to be the number of points. The problem is then to determine the asymptotic distribution of the length of the longest increasing subsequence. (Note that without the extra points on the left and bottom edge, this is just the standard Poisson model for increasing subsequences of random permutations [1].)

Prähofer and Spohn then observe [13] that the stationarity of the initial conditions can be used to show that the length of the longest increasing subsequence has mean exactly \( 2t \). This fact is striking for two reasons. The first is that the mean in the standard model is rather complicated; it is thus surprising that a fairly simple extension gives rise to an exact formula for the extended mean. The second is that since adding points can only help the longest increasing subsequence, we conclude that \( 2t \) is an upper bound on the mean in the standard model. This bound is quite tight; indeed, in the standard model, the mean takes the form \( 2t + O(t^{1/3}) \). (We could also derive this upper bound from the (strictly stronger) result of [13] that the expected length of the longest increasing subsequence of a permutation of length \( n \) is at most \( 2\sqrt{n} \); the present method is more generally applicable, however.)

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The first object of the present paper is to generalize this fact in a number of ways. It turns out, for instance, that the standard Poisson model admits a one-parameter family of extensions with explicit means; this further extends to give explicit information about the moment generating function in a neighborhood of this family. Also, we can replace the Poisson model by the generalized model considered in \cite{1} (based in turn on a model of Johansson \cite{11}).

Our other object is to explore the asymptotic relations between the extended models and the unextended models. It turns out that by a careful (if heuristic) analysis, we can use the moment generating function identities to determine not just the asymptotic mean of the longest increasing subsequence length, but also the asymptotic scale factor. This gives a uniform prescription for the scaling information, agreeing with the results of all of the cases that have so far been analyzed.

Section 1 defines the models of interest, as well as a certain continuous limiting case. Section 2 gives a short, algebraic proof of the moment generating function and mean identities; this is followed by a somewhat more complicated, but also more enlightening combinatorial proof in Section 3. Finally, Section 4 considers the asymptotic consequences, giving explicit conjectures for the asymptotics of the general case.

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## 2 Extended growth models

The model we will be considering is a generalization of a model considered by Johansson \cite{11}, combining the generalizations of \cite{1}, \cite{14}, and \cite{7}. We define a “parameter set” $p$ to be a triple $(t, q, r)$, where $t$ is a nonnegative number, and $q$ and $r$ are sequences of nonnegative numbers with

$$\sum_i q_i + \sum_i r_i < \infty; \quad (2.1)$$

we will write such a parameter set as $t:q/r$, omitting $t$ if $t = 0$ and omitting any trailing 0’s from $q$ and $r$. From the existence of the sum, we conclude that $Q(p) := \sup_i q_i$ and $R(p) := \sup_i r_i$ (2.2) are well-defined and attained. Given a parameter set $p$, we define two functions

$$H(z;p) = e^{t z} \prod_i (1 - z q_i)^{-1} (1 + z r_i) \quad (2.3)$$

$$E(z;p) = e^{t z} \prod_i (1 + z q_i) (1 - z r_i)^{-1}; \quad (2.4)$$

$H(z;p)$ converges for $|z| < Q(p)^{-1}$, while $E(z;p)$ converges for $|z| < R(p)^{-1}$. We will say that two parameter sets $p_+ = t^+q^+/r^+$ and $p_- = t^-q^-/r^-$ are **compatible** if $Q(p_+)<Q(p_-)<1$, $R(p_+)<R(p_-)<1$. Note that we then have

$$H(p_+;p_-) := e^{t^+ t^-} \prod_i e^{t^- (q_i^+ r_i^+ + q_i^- q_j^-)} e^{t^+ (q_i^+ r_i^- + q_j^+ q_j^-)} \prod_{i,j} (1 - q_i^+ q_j^-)^{-1} (1 - r_i^+ r_j^-)^{-1} (1 + q_i^+ r_j^-) (1 + r_i^+ q_j^-) < \infty. \quad (2.5)$$
Let \((\mathbb{Z}^+)\)' be a disjoint copy of \(\mathbb{Z}^+\), and consider the set \(\Omega := [0, 1] \cup \mathbb{Z}^+ \cup (\mathbb{Z}^+)\)' . Then we associate to a pair \(p_+, p_\cdot\) of compatible parameter sets a random multiset \(M(p_+, p_\cdot) \subseteq \Omega \times \Omega\) as follows. We let \(P(t)\) denote a Poisson random variable of parameter \(t\), \(g(q)\) denote a geometric random variable with parameter \(q\), and \(b(q)\) denote a random variable which is 0 with probability \(\frac{q}{1+q}\) and 1 with probability \(\frac{1}{1+q}\).

\[\begin{align*}
\text{• On } [0, 1] \times [0, 1], \text{ we choose } P(t^+t^-) \text{ i.i.d. uniform points,} \\
\text{• On } [0, 1] \times i, \text{ we choose } P(t^+q^-_i) \text{ i.i.d. uniform points,} \\
\text{• On } i \times j, \text{ we have multiplicity } g(q^+_i q^+_j), \\
\text{• On } i \times j', \text{ we have multiplicity } b(q^+_i r^-_j), \\
\text{• On } i' \times j', \text{ we have multiplicity } g(r^+_i r^-_j), \\
\end{align*}\]

and so on, with all of the multiplicities chosen independently.

Choose a pair of total orderings (denoted \(<_+\) and \(<_-\)) on \(\Omega\) compatible with the usual ordering on \([0, 1]\). A subsequence of a multiset \(M\) in \(\Omega \times \Omega\) (a sequence of points \((x_i, y_i)\) from \(M\) with no point occurring more often than its multiplicity) is “increasing” if we always have \(x_i \leq_+ x_{i+1}, y_i \leq_- y_{i+1}\), subject to the further conditions

\[\begin{align*}
x_i = x_{i+1} &\implies x_i \notin \mathbb{Z}^+ \quad (2.6) \\
y_i = y_{i+1} &\implies y_i \notin \mathbb{Z}^+ \quad (2.7)
\end{align*}\]

In other words, the sequence must be strictly increasing along rows and columns from \(\mathbb{Z}^+\). We then define a sequence \(\lambda_i(M)\) by setting

\[\sum_{1 \leq i \leq l} \lambda_i(M) \quad (2.8)\]

equal to the size of the longest subsequence of \(M\) which is a union of \(l\) increasing subsequences. We then define \(\lambda(p_+, p_\cdot) := \lambda(M(p_+, p_\cdot))\) (with the latter notation preferred when particular multisets are being compared).

We recall the following results from [4]:

**Theorem 2.1.** For any pair \(p_+, p_\cdot\) of compatible parameter sets, \(\lambda(p_+, p_\cdot)\) is a random partition, finite with probability 1. The distribution of \(\lambda(p_+, p_\cdot)\) is independent of the choice of total orderings on \(\Omega\).

**Theorem 2.2.** For any pair \(p_+, p_\cdot\) of compatible parameter sets, we have the identity

\[\Pr(\lambda_1(p_+, p_\cdot) \leq l) = H(p_+; p_-)^{-1} \mathbb{E}_{U \in U(l)} \det(H(U; p_+)H(U^\dagger; p_-)). \quad (2.9)\]

**Remark.** As in [4], this is a formal integral, defined by analytic continuation from the region \(Q(p_\pm) < 1\).

For our purposes, we need to extend the model slightly, by adding a special row and column to the random multiset. Extend \(\Omega\) to \(\Omega^+\) by adding a new element, denoted \(\Sigma\), and extend the total orderings so that \(\Sigma\) is smallest in both orderings. Then we define a new random multiset \(M(p_+, p_\cdot; \alpha_+, \alpha_-)\) as follows.
On $\Omega \times \Omega$, we take $M(p_+, p_-)$,

On $[0, 1] \times \{\Sigma\}$, we choose $P(\alpha_- t^+)$ i.i.d. uniform points,

On $(i, \Sigma)$, we have multiplicity $b(\alpha_- q_i^+)$,

On $(i', \Sigma)$, we have multiplicity $q(\alpha_- r_i^+)$,

and so on, and allow increasing subsequences to be weakly increasing in the new row and column. Note that the point $(\Sigma, \Sigma)$ has multiplicity fixed at 0; otherwise, the new model would simply be a special case of the old model. By the argument in [7], we have:

**Theorem 2.3.** For any compatible pair $p_+, p_-$ of parameter sets and any $\alpha_+, \alpha_-$ with $\alpha_+ R(p_-) < 1$ and $\alpha_- R(p_+) < 1$, we have

$$\Pr(\lambda_1(p_+, p_-; \alpha_+, \alpha_-) \leq l) = E(\alpha_+; p_-)^{-1} E(\alpha_-; p_+)^{-1} \{D_1(p_+, p_-; \alpha_+, \alpha_-) - \alpha_+ \alpha_- D_{1-1}(p_+, p_-; \alpha_+, \alpha_-)\}, \quad (2.10)$$

where

$$D_1(p_+, p_-; \alpha_+, \alpha_-) = E_{U \in U(l)} \det((1 + \alpha_+ U)(1 + \alpha_- U^t) H(U; p_+) H(U^t; p_-)). \quad (2.11)$$

Aside from intrinsic interest ([7]), this new model in principle can give us some information about the original model $\lambda_1(p_+, p_-)$, since we have the coupling

$$\lambda_1(M(p_+, p_-)) \leq \lambda_1(M(p_+, p_-; \alpha_+, \alpha_-)). \quad (2.12)$$

Thus it is particularly interesting that, as we shall see, there is an exact formula for

$$E(\lambda_1(p_+, p_-; \alpha, \alpha^{-1})) \quad (2.13)$$

whenever $R(p_+) < \alpha < R(p_-)^{-1}$.

We will also consider a continuous limiting case of the above model, combining the exponential limit of [11] and the heavy-traffic limit of queueing theory [9] (studied in the present context in [18] and [8]).

The parameters for the continuous model consist of a pair of sequences $\rho_i^\pm \in \mathbb{R} \cup \{\infty\}$, a nonnegative real $u$, and two numbers $a_\pm \in \mathbb{R} \cup \{\infty\}$, subject to the convergence constraint

$$\sum_{i} \frac{1}{\rho_i^+ - z} < \infty, \text{ for } z < \inf(\rho^\pm), \quad (2.14)$$

and the compatibility constraints

$$\inf(\rho^+) + \inf(\rho^-) > 0, \quad \inf(\rho^+) + a_- > 0, \quad a_+ + \inf(\rho^-) > 0. \quad (2.15)$$

We omit any trailing $\infty$’s from $\rho^\pm$. Define an infinite matrix $\Lambda_{ij}$, $0 \leq i, j$ by

$$\Lambda_{ij} = \begin{cases} 0, & (i, j) = (0, 0) \\ \rho_i^+ \rho_j^-, & (i, j) \neq (0, 0), \end{cases} \quad (2.16)$$
with the convention that \( \rho^+_0 = \alpha_\pm \), and let \( M \) be a random matrix filled with independent exponential random variables, such that \( M_{ij} \) has mean \( \Lambda_{ij} \). Also, for \( 0 \leq i \) such that \( \rho^+_i < \infty \), let \( B_i \) be a Brownian motion on \([0,1] \) with

\[
E(B_i(x)) = -u\rho^+_ix \quad \text{Var}(B_i(x)) = ux;
\]

by convention, \( \rho^+_0 = a_\pm \). Then we define a random sequence \( \chi(\rho^+, u;\rho^-; a_+; a_-) \) in terms of increasing paths in \( \mathbb{N} \times (\mathbb{N} \cup [0,1]) \), where the contribution of an interval \( i \times [x_i,y_i] \) is \( B_i(y_i) - B_i(x_i) \) and the contribution of a point \( i \times j \) is \( M_{ij} \). When \( u > 0 \), we require that the union of \( k \) increasing paths used to define \( \chi_k \) have total Lebesgue measure \( k \) in \( \mathbb{N} \times [0,1] \), and that the paths contain no point in \( i \times [0,1] \) when \( \rho^+_i = \infty \). If no such path exists, we set \( \chi_k = -\infty \).

**Theorem 2.4.** For any valid choice of \( \rho^+, u;\rho^-; a_+; a_- \), of compatible parameter sets, \( \chi(\rho^+, u;\rho^-; a_+; a_-) \) is a random nonincreasing sequence in \( \mathbb{R} \cup \{-\infty\} \), and is nonnegative if \( u = 0 \). The distribution is invariant under reordering of the sequences \( \rho^\pm \).

**Proof.** That the sequence \( \chi \) is nonincreasing, invariant under reordering, and nonnegative when \( u = 0 \) follows from the fact that it is a scaled limit of \( \lambda \) (see below); it suffices therefore to show that \( \chi_1 < \infty \) with probability 1. We first observe that

\[
\chi_1(\rho^+, u;\rho^-; a_+; a_-) \leq \chi_1(\rho^+, u;\rho^-; a_+; \infty) + \chi_1(\rho^+, u;\rho^-; \infty, a_-).
\]

Since decreasing \( a_\pm \) cannot decrease \( \chi_1 \), we conclude that it suffices to prove finiteness when \(-\inf(\rho^\pm) < a_\pm < \inf(\rho^\pm) \) and \( a_+ + a_- > 0 \). But then Corollary 3.4 below applies, expressing \( E(e^{(a_+ + a_-)\chi_1}) \) as an infinite product. Moreover, the conditions on \( \rho^\pm \) suffice to force convergence of this product, and thus \( E(e^{(a_+ + a_-)\chi_1}) < \infty \). But this immediately implies \( \chi_1 < \infty \), as desired. \( \square \)

As we alluded to above, this is a limiting case of the discrete model:

\[
\chi(\rho^+, u;\rho^-; a_+; a_-) = \lim_{t \to \infty} \frac{\lambda(e^{-\rho^+/t}; ut^2; e^{-\rho^-/t}; e^{-a_+/t}; e^{-a_-/t}) - ut^2}{t},
\]

with the limit taken in the sense of distribution. This corresponds to the facts that if \( x \) is an exponential random variable of mean \( 1/m \), then \( [\frac{x}{t}] \) is a geometric random variable of parameter \( e^{-m/t} \), and that as the parameter tends to infinity, Poisson processes converge (with proper scaling) to Brownian motion.

The main significance of the continuous model is that it contains two classical matrix ensembles as special cases. Let \( 0_n \) denote the finite sequence consisting of \( n \) copies of 0. We obtain the Gaussian Unitary Ensemble from \( \chi(0_n, 1; \infty, \infty) \) that is, \( \chi(0_n, 1; \infty, \infty) \) is distributed as the (ordered) eigenvalues of an \( n \times n \) Hermitian Gaussian matrix (extended to an infinite sequence by adding \(-\infty\)). (This is essentially proved in [18]; to be precise, they show that one obtains traceless GUE under a constraint equivalent to \( \sum_i B_i(1) = 0 \); it follows easily that without this constraint, one obtains ordinary GUE.) Similarly, we obtain the Laguerre Unitary Ensemble from \( \chi(0_{n_+}, 0_{n_-}; \infty, \infty) \); that is, the distribution of the singular values of a \( n_+ \times n_- \) complex Gaussian matrix.
3 An algebraic proof

Let $p_+, p_-$ be a pair of compatible parameter sets.

**Lemma 3.1.** $D_l(\alpha_+, \alpha_-) := D_l(p_+, p_-; \alpha_+, \alpha_-)$ is a polynomial, satisfying the identity

$$D_l(\alpha_+, \alpha_-) = (\alpha_+ \alpha_-)^l D_l(\alpha_-^1, \alpha_+^1). \tag{3.1}$$

**Proof.** This follows immediately from the corresponding fact for $\det((1 + \alpha U)(1 + \alpha_+ U^\dagger))$. \hfill \Box

For $\alpha_+ < R(p_-)^{-1}$, $\alpha_- < R(p_+)^{-1}$, define $L(\alpha_+, \alpha_-) = \lambda_1(p_+, p_-; \alpha_+, \alpha_-)$; then

**Theorem 3.2.** For $R(p_+) < \alpha_+ < R(p_-)^{-1}$ and $R(p_-) < \alpha_- < R(p_+)^{-1}$,

$$E((\alpha_+ \alpha_-)^{-L(\alpha_+, \alpha_-)}) = E(\alpha_+; p_-)^{-1} E(\alpha_-; p_+)^{-1} E(\alpha_-^1; p_+ E(\alpha_-^1; p_-) \tag{3.2}$$

**Proof.** By Theorem 2.3 above,

$$\Pr(L(\alpha_+, \alpha_-) \leq l) = E(\alpha_+; p_-)^{-1} E(\alpha_-; p_+)^{-1} H(p_+; p_-)^{-1}[D_l(\alpha_+, \alpha_-) - \alpha_+ \alpha_- D_l-1(\alpha_+, \alpha_-)] \tag{3.3}$$

$$= (\alpha_+ \alpha_-)^{-L(\alpha_+, \alpha_-)} E(\alpha_-; p_-)^{-1} E(\alpha_-; p_+)^{-1} H(p_+; p_-)^{-1}[D_l(\alpha_-^1, \alpha_+^1) - D_l-1(\alpha_-^1, \alpha_+^1)]. \tag{3.4}$$

Then

$$\sum_{0 \leq l \leq k} (\alpha_+ \alpha_-)^{-l} \Pr(L(\alpha_+, \alpha_-) = l)$$

$$= \sum_{0 \leq l \leq k} (\alpha_+ \alpha_-)^{-l} [\Pr(L(\alpha_+, \alpha_-) \leq l) - \Pr(L(\alpha_+, \alpha_-) \leq l - 1)] \tag{3.5}$$

$$= (\alpha_+ \alpha_-)^{-k} \Pr(L(\alpha_+, \alpha_-) \leq k) + \frac{\alpha_+ \alpha_- - 1}{\alpha_+ \alpha_-} \sum_{0 \leq l \leq k} (\alpha_+ \alpha_-)^{-l} \Pr(L(\alpha_+, \alpha_-) \leq l) \tag{3.6}$$

$$= E(\alpha_+; p_-)^{-1} E(\alpha_-; p_+)^{-1} H(p_+; p_-)^{-1} \left[ \frac{1}{\alpha_+ \alpha_-} \left( D_k(\alpha_-^1, \alpha_+^1) - D_{k-1}(\alpha_-^1, \alpha_+^1) \right) + \frac{\alpha_+ \alpha_- - 1}{\alpha_+ \alpha_-} D_k(\alpha_-^1, \alpha_+^1) \right] \tag{3.7}$$

$$= E(\alpha_+; p_-)^{-1} E(\alpha_-; p_+)^{-1} H(p_+; p_-)^{-1} [D_k(\alpha_-^1, \alpha_+^1) - (\alpha_+ \alpha_-)^{-1} D_{k-1}(\alpha_-^1, \alpha_+^1)] \tag{3.8}$$

$$= E(\alpha_+; p_-)^{-1} E(\alpha_-; p_+)^{-1} E(\alpha_-^1; p_+ E(\alpha_-^1; p_-) \Pr(L(\alpha_-^1, \alpha_+^1) \leq k), \tag{3.9}$$

where the last step is valid since $\alpha_-^1 < R(p_-)^{-1}$ and $\alpha_+^1 < R(p_+)^{-1}$.

The theorem then follows by taking the limit $k \to \infty$. \hfill \Box

Taking a limit as $\alpha_+ \alpha_- \to 1$, we obtain:

**Corollary 3.3.** Whenever $R(p_+) < \alpha < R(p_-)^{-1}$,

$$E(L(\alpha, \alpha^1)) = \frac{\alpha E'(\alpha; p_-)}{E(\alpha; p_-)} + \frac{\alpha^{-1} E'(\alpha^{-1}; p_+)}{E(\alpha^{-1}; p_+)} \tag{3.10}$$

$$E(\alpha; p_-) + E(\alpha^{-1}; p_+)$$
Remark. If \( p = t:q/r \), then
\[
\frac{\alpha E'(\alpha; p)}{E(\alpha; p)} = \alpha t + \sum_i \frac{\alpha q_i}{1 + \alpha q_i} + \sum_i \frac{\alpha r_i}{1 - \alpha r_i}
\]  
(3.11)

We also observe that
\[
\frac{\alpha E'(\alpha; p_-)}{E(\alpha; p_-)} + \frac{\alpha^{-1} E'(\alpha^{-1}; p_+)}{E(\alpha^{-1}; p_+)} = \frac{\alpha}{d\alpha} \log \frac{E(\alpha; p_-)}{E(\alpha^{-1}; p_+)}
\]  
(3.12)

In the continuous limit, we write \( X(a_+, a_-) := \chi(\rho^+, w_0; \rho^-; a_+, a_-) \). Taking the appropriate limit gives:

**Corollary 3.4.** Let \( \rho^+ \) and \( w: \rho^- \) be compatible continuous parameter sets. Then whenever \( -\inf(\rho^+) < a_+ < \inf(\rho^+) \) and \( -\inf(\rho^+) < a_- < \inf(\rho^-) \),
\[
E(e^{(a_+ + a_-) X(a_+, a_-)}) = e^{u(a^2 - a_i^2)/2} \prod_i \frac{\rho_i^1 + a_-}{\rho_i^1 - a_+} \prod_i \frac{\rho_i^{-1} + a_+}{\rho_i^{-1} - a_-}
\]  
(3.13)

Whenever \( -\inf(\rho^-) < a < \inf(\rho^+) \),
\[
E(X(a, -a)) = -ua + \sum_i \frac{1}{\rho_i^1 + a} + \sum_i \frac{1}{\rho_i^{-1} - a}.
\]  
(3.14)

**4 A combinatorial proof**

Fix parameters as in the previous section, and set
\[
N_+ = |M(p_+, p_-; \alpha_+, \alpha_-) \cap (\Sigma \times \Omega)|
\]  
(4.1)
\[
N_- = |M(p_+, p_-; \alpha_+, \alpha_-) \cap (\Omega \times \Sigma)|
\]  
(4.2)

Then we observe
\[
E((\alpha_+ \alpha_-)^{-N_+}) = \frac{E(\alpha_+^{-1}; p_-)}{E(\alpha_+; p_-)} \quad E((\alpha_+ \alpha_-)^{-N_-}) = \frac{E(\alpha_+^{-1}; p_+)}{E(\alpha_-; p_+)}.
\]  
(4.3)

and for \( \alpha_+ = \alpha_-^{-1} = \alpha \),
\[
E(N_+) = \frac{E'(\alpha; p_-)}{E(\alpha; p_-)} \quad E(N_-) = \frac{E'(\alpha^{-1}; p_+)}{E(\alpha^{-1}; p_+)}.
\]  
(4.4)

So we can restate Theorem 3.2 and Corollary 3.3 as
\[
E((\alpha_+ \alpha_-)^{-L(\alpha_+ \alpha_-)}) = E((\alpha_+ \alpha_-)^{-N_+ - N_-})
\]  
(4.5)
\[
E(L(\alpha_+ \alpha_-)) = E(N_+ + N_-)
\]  
(4.6)

We give a direct proof of this fact, for \( \alpha_+ \alpha_- \leq 1 \):

**Proof.** Let \( \alpha' < \alpha_- \). Then \( \alpha_+ \alpha' < 1 \), so we can extend \( M(p_+, p_-; \alpha_+, \alpha') \) by adjoining \( (\Sigma, \Sigma) \) with multiplicity \( N_0 \) of distribution \( g(\alpha_+ \alpha') \); denote the resulting random multiset by \( M' \). But then by Theorem 2.1, we can change the total ordering \( <_\Sigma \) so that \( \Sigma \) becomes maximal instead of minimal. We then find that
\[
\Sigma \times \Omega, (\Sigma, \Sigma), \Omega \times \Sigma
\]  
(4.7)
induces an increasing subsequence of $M'$ with respect to this new ordering; thus

$$N_0 + N_+ + N_- \leq \lambda_1(M').$$  \hfill (4.8)

On the other hand, this is the only maximal increasing subsequence that passes through $(\Sigma, \Sigma)$; any other maximal increasing subsequence can have size at most $N_+ + N_- + \lambda_1(M(p_+, p_-))$. We thus find

$$\lambda_1(M') \leq N_+ + N_- + \max(\lambda_1(M(p_+, p_-)), N_0).$$  \hfill (4.9)

But $\lambda_1(M')$ is distributed as $N_0 + \lambda_1(M(p_+, p_+; \alpha_+, \alpha'))$ (since before the reordering every maximal increasing subsequence passes through $(\Sigma, \Sigma)$, and $N_0$ is independent of $\lambda_1(M(p_+, p_-; \alpha_+, \alpha'))$). So if we take the expectations and subtract/divide by the contribution of $N_0$, we find that we need only show

$$\lim_{\alpha' \to (1/\alpha)} \frac{\lambda_1(M_{\alpha_+})(\alpha_+)}{\lambda_1(M_{\alpha_+})(\alpha_+)} = 0.$$  \hfill (4.10)

**Lemma 4.1.** Let $X$ be a nonnegative-integer-valued random variable with finite first moment, and let $Y$ be an independent geometric random variable of parameter $t$. Then

$$\lim_{t \to 1-} E(\max(X, Y)) - E(Y) = 0.$$  \hfill (4.12)

Similarly, for $s < 1$, if $E(s^{-X})$ is finite, then

$$\lim_{t \to s-} E(s^{-Y})^{-1} E(s^{-\max(X, Y)}) = 1.$$  \hfill (4.13)

**Proof.**

$$E(\max(X, Y) - Y) = E((X - Y) \Pr(Y < X))$$

$$= E(X) + \frac{t}{1-t} (E(t^X) - 1)$$

$$\to 0.$$  \hfill (4.16)

Similarly,

$$E(s^{-\max(X, Y)}) = E(s^{-X} \Pr(Y < X) + \Pr(Y \geq X) E(s^{-Y} | Y \geq X))$$

$$= E(s^{-X}) + \frac{t(s-1)}{t-s} E((t/s)^X).$$  \hfill (4.18)

and thus

$$\lim_{t \to s-} E(s^{-Y})^{-1} E(s^{-\max(X, Y)}) = \lim_{t \to s-} \frac{t-s}{s(t-1)} E(s^{-X}) + \frac{t(s-1)}{s(t-1)} E((t/s)^X)$$

$$= 1.$$  \hfill (4.20)
The theorem then follows from the following lemma, since
\[ \alpha_+ \alpha_- > R(p_+) R(p_-) \] by assumption. \[\square\]

**Lemma 4.2.** For all \( z > R(p_+) R(p_-) \), \( E(z^{-\lambda_1(p_+,p_-)}) \) is finite. In particular, since \( 1 > R(p_+) R(p_-) \), \( \lambda_1(p_+,p_-) \) has moments of all orders.

**Proof.** An increasing subsequence in \( M(p_+,p_-) \) can pass through a point on a strict row or column at most once; thus \( \lambda_1(M(p_+,p_-)) \) is unchanged if we remove any excess multiplicity in those rows and columns. Let \( M^o \) be the resulting multiset, then
\[ \lambda_1(M(p_+,p_-)) = \lambda_1(M^o) \leq |M^o|. \] It will thus suffice to prove that \( E(z^{-|M^o|}) < \infty \). But the moment generating function of \( |M^o| \) is \( f(z)/f(1) \), where
\[
f(z) = e^{-\epsilon^+ z} \prod_i e^{\epsilon^+ q_i^+ z} e^{\epsilon^- r_i^- z} e^{\epsilon^+ t_i^+ z} e^{\epsilon^- t_i^- z} \prod_{i,j} (1 - r_i^+ r_j^- z)^{-1} (1 + q_i^+ q_j^- z) (1 + r_i^+ q_j^- z) (1 - q_i^+ q_j^- z) \]
This product converges to an analytic function with no pole inside the open disc \( |z| < (R(p_+) R(p_-))^{-1} \), and thus the result follows. \[\square\]

5 \hspace{1cm} Asymptotic consequences

Since \( \lambda_1(p_+,p_-;\alpha_+,\alpha_-) \) is nondecreasing in \( \alpha_+ \) and \( \alpha_- \), we obtain the following bound:

**Theorem 5.1.** For any compatible parameters \( p_+, p_- \),
\[
E(\lambda_1(p_+,p_-)) \leq \inf_{R(p_+)<\alpha<R(p_-)^{-1}} E(\lambda_1(p_+,p_-;\alpha,\alpha^{-1})).
\] (5.1)

For instance, in the purely Poisson case, \( p_+ = p_- = t/\epsilon \), we find
\[
E(\lambda_1(p_+,p_-)) \leq \inf_{\alpha>0} (\alpha + \alpha^{-1}) t = 2t.
\] (5.2)

This bound is remarkably tight; indeed, we have \[\[\]\]
\[
\lambda_1(p_+,p_-) = 2t - O(t^{1/3+\epsilon}).
\] (5.3)

This suggests the following conjecture:

**Conjecture 5.2.** Fix parameters \( p_+, p_- \), and define
\[
m(\alpha;p_+,p_-) = E(\lambda_1(p_+,p_-;\alpha,\alpha^{-1})).
\] (5.4)
Then

\[
\lim_{n \to \infty} n^{-1} \lambda_1(p^n_+, p^n_-) = \inf_{R(p_+)<\alpha<R(p_-)^{-1}} m(\alpha; p_+, p_-),
\]

with probability 1, where

\[
(t:q/r)^n := (nt):q^n/r^n
\]

\[
q^n := q_1, q_1, \ldots q_1, q_2, q_2, \ldots q_2, \ldots
\]

Remark 1. Roughly speaking, this is an analogue of the law of large numbers. As such, it can most likely be strengthened considerably (considering different sequences of parameter sets than just \(p^n_+, p^n_-\)). See, for instance, the result of [16].

Remark 2. The existence of the limit (5.5) follows from superadditivity and the bound (5.1). This has been verified in a number of special cases (see below). In each case, we in fact find that

\[
\lambda_1(p^n_+, p^n_-) - \mu n
\]

converges to a limit distribution.

Fix parameters \(p_+, p_-\). An increasing subsequence of \(M(p_+, p_-; \alpha, \alpha^{-1})\) cannot include points from both \(\{\Sigma\} \times \Omega\) and \(\Omega \times \{\Sigma\}\). We would thus expect that for \(\alpha\) “large”, the typical longest increasing subsequence will avoid \(\Omega \times \{\Sigma\}\) entirely. In particular, we would expect

\[
E(\lambda_1(p_+, p_-; \alpha, \alpha^{-1})) \sim E(\lambda_1(p_+, p_-; \alpha, 0))
\]

whenever \(N_+ \gg N_-\). For asymptotic purposes, this condition is simply that \(\alpha > \tilde{\alpha} \), where \(\tilde{\alpha} \) minimizes \(m(\alpha; p_+, p_-)\). (We also define \(\tilde{\alpha} = (\tilde{\alpha}^{-1})^{-1}\), which of course minimizes \(m(\alpha; p_-, p_+)\)).

In particular, this tells us that \(\tilde{\alpha} \) is a critical point; if \(\alpha_+ < \tilde{\alpha}\) and \(\alpha_- < \tilde{\alpha}\), we have

\[
E(\lambda_1(p_+, p_-; \alpha_+, \alpha_-)) \sim E(\lambda_1(p_+, p_-; \tilde{\alpha}_+, \tilde{\alpha}_-)),
\]

while if either is greater, the mean is determined by the dominant parameter.

This behaviour is, of course, confirmed by the analysis of [7], in which the asymptotics for general \(\alpha_\pm\) are determined for the Poisson case \(p = p' = t\)/ and the Johansson case \(p = p' = /\sqrt{q}\) (where \(\sqrt{q}\) is the finite sequence consisting of \(n\) copies of \(\sqrt{q}\)). In both cases, we obtain the same behaviour near the critical point. This suggests that for general parameters there should exist constants \(\mu\), \(\sigma\), and \(\sigma_\pm\) so that the following holds:

If we fix \(w_\pm\), \(w_-\), and define

\[
\alpha_\pm = \tilde{\alpha}_\pm \exp(- \frac{2w_\pm}{\sigma_\pm n^{1/3}}),
\]

then as \(n \to \infty\),

\[
\frac{\lambda_1(p^n_+, p^n_-; \alpha_+, \alpha_-) - \mu n}{\sigma n^{1/3}}
\]

converges to the distribution \(H(w_+, w_-)\) (5.5).

We recall the following information about the distribution \(H(w_+, w_-)\):
Lemma 5.3. Let $X$ be distributed as $H(w_+, w_-)$. Then $E(\exp(2(w_+ + w_-)X)) = \exp(\frac{8}{3}(w_+^3 + w_-^3))$. If $w_+ = -w_- = w$, then $E(X) = 4w^2$.

Only the latter equation was actually shown in [7], but essentially the same calculation gives the first equation as well. Furthermore, in the cases that have been fully analyzed, this is precisely the analogue for $\tilde{w}$.

Assume that $\tilde{w}$ is distributed according to $\mathcal{H}$. Then we have the following limiting distributions as $n \to \infty$:

$$E(\exp(2(w_+ + w_-)X)) = \exp(\frac{8}{3}(w_+^3 + w_-^3)),$$

with $X$ distributed according to $H(w_+, w_-)$ in the limit. Thus to retain the analogy, we must have $\sigma_+ = \sigma_- = \sigma$.

On the other hand, we have:

$$\log(E((\alpha_+ + \alpha_-)^{-\lambda_1(p^p_{\alpha_+}; \alpha_+; \alpha_-)})) = 2\mu n^{2/3}(\frac{w_+}{\sigma_+} + \frac{w_-}{\sigma_-}) + \log(E(\exp(2w_+ X/\sigma_+ \exp(2w_- X/\sigma_-)))$$

Comparing the asymptotics, we find

$$\sigma = ((\theta^3 g) / (\tilde{\alpha}^2)/2)^{1/3} = (\tilde{\alpha}_+^2 m''(\tilde{\alpha}_+; p_+, p_-)/2)^{1/3}.$$

Similar considerations (based on part (iv) below) give us the scale factors for $\alpha_+ > \tilde{\alpha}_+$, thus giving us the following conjecture (see [8] for the definitions of the limiting distributions):

Conjecture 5.4. Fix parameters $p_+, p_-$, define $\tilde{\alpha}_\pm$ as above, and further define

$$\mu = m(\tilde{\alpha}_+; p_+, p_-) \quad \sigma = (\tilde{\alpha}_+^2 m''(\tilde{\alpha}_+; p_+, p_-)/2)^{1/3}$$

$$\mu_+(z) = m(z; p_+, p_-) \quad \sigma_+(z) = (zm'(z; p_+, p_-))^{1/2}$$

$$\mu_-(z) = m(z; p_-, p_+) \quad \sigma_-(z) = (zm'(z; p_-, p_+))^{1/2}$$

Assume that $\tilde{\alpha}_0 \notin \{0, \infty\}$, $\sigma > 0$, $\alpha_+(\alpha)^2 > 0$ for $\tilde{\alpha}_+ < \alpha_+ < R(p_-)^{-1}$, and $\alpha_-(\alpha)^2 > 0$ for $\tilde{\alpha}_- < \alpha_- < R(p_+)^{-1}$. Then we have the following limiting distributions as $n \to \infty$:

- (i) If $0 \leq \alpha_+ < \tilde{\alpha}_+$ and $0 \leq \alpha_- < \tilde{\alpha}_-$ are fixed, then

$$\frac{\lambda_1(p^p_{\alpha_+}; \alpha_+; \alpha_-) - \mu n}{\sigma n^{1/3}} \to F_{\text{GUE}}$$

Near the critical point, set $w_\pm$ by $\alpha_\pm = \tilde{\alpha}_\pm \exp(-2w_\pm / \sigma n^{1/3})$.

- (ii) If $w_\pm$ and $0 \leq \alpha_+ < \tilde{\alpha}_+$ are fixed,

$$\frac{\lambda_1(p^p_{\alpha_+}; \alpha_+; \alpha_-) - \mu n}{\sigma n^{1/3}} \to G(w_\pm).$$
• (iii) If \( w_+ \) and \( w_- \) are fixed,
\[
\frac{\lambda_1(p^n_+, p^n_-; \alpha_+, \alpha_-) - mn}{\sigma n^{1/3}} \rightarrow H(w_+, w_-).
\] (5.21)

Finally (the Gaussian regime), let \( \alpha^0_+ \) and \( \alpha^0_- \) be fixed such that \( \alpha^0_+ > \tilde{\alpha}_+ \) and \( \mu_+(\alpha^0_+) = \mu_-(\alpha^0_-) \), and set
\[
\alpha_\pm = \alpha^0_\pm \exp(x_\pm / \sigma_\pm (\alpha^0_\pm) n^{1/2}).
\]

• (iv) If \( x_\pm \) and \( 0 \leq \alpha_\mp < \alpha^0_\pm \) are fixed,
\[
\frac{\lambda_1(p^n_+, p^n_-; \alpha_+, \alpha_-) - \mu_\pm(\alpha^0_\pm)n}{\sigma n^{1/2}} \rightarrow N(x_\pm, \sigma_\pm(\alpha^0_\pm)^2). \] (5.22)

• (v) If \( x_+ \) and \( x_- \) are fixed,
\[
\frac{\lambda_1(p^n_+, p^n_-; \alpha_+, \alpha_-) - \mu_\pm(\alpha^0_\pm)n}{\sigma n^{1/2}} \rightarrow \max(N(x_+, \sigma_+(\alpha^0_+)^2), N(x_-, \sigma_-(\alpha^0_-)^2)). \] (5.23)

Remark 1. This, in turn, is analogous to the central limit theorem, so again can probably be strengthened considerably (although not nearly to the same extent as Conjecture 5.2 most likely can). In particular, it is presumably sufficient for the parameters \( \alpha_\pm, w_\pm, x_\pm \) to tend to limits as appropriate, rather than simply be fixed.

Remark 2. We recall that
\[
\mu_+(\alpha) = \mathbf{E}(N_+) + \mathbf{E}(N_-) \] (5.24)
at \( \alpha_+ = \alpha, \alpha_- = \alpha^{-1} \). Similarly,
\[
\sigma_+(\alpha) = \operatorname{Var}(N_+) - \operatorname{Var}(N_-) \] (5.25)

Remark 3. The analogous conjectures for models of the other symmetry types (\[4\], \[5\], \[6\]) are straightforward. We note in particular that when \( p_+ = p_- = p \), we find \( \alpha m'(\alpha) = -\alpha^{-1} m'(\alpha^{-1}) \), and \( \alpha m'(\alpha) > 0 \) whenever \( \alpha > 1 \). So the hypotheses of the above conjectures hold in such cases, with \( \tilde{\alpha}_\pm = 1 \).

In the continuous limit, we make a similar conjecture; the main difference is that the model is nonincreasing in the parameters, not nondecreasing, so the \( F_{\text{GUE}} \) region is now \( a_\pm > \tilde{a}_\pm \). The scale factors are:
\[
\mu = m_c(\tilde{a}_+; \rho^+, u; \rho^-) \quad \sigma = (m_c''(\tilde{a}_+; \rho^+, u; \rho^-)/2)^{1/3}
\] (5.26)
\[
\mu_+(z) = m_c(z; \rho^+, u; \rho^-) \quad \sigma_+(z)^2 = -m_c'(z; \rho^+, u; \rho^-), \quad \mu_-(z) = \mu_+(-z) \quad \sigma_-(z)^2 = -\sigma_+(z)^2
\] (5.27)

where we define
\[
m_c(z; \rho^+, u; \rho^-) = -uz + \sum_i \frac{1}{\rho_i^+ + a} + \sum_i \frac{1}{\rho_i^- - a} \] (5.29)
Near the critical point, we take \( a^\pm = \tilde{a}^\pm + 2w^\pm/\sigma n^{1/3} \), while in the Gaussian regime, we take \( a^\pm = a_0^\pm - x^\pm/\sigma_\pm(a_0^\pm) n^{1/2} \).

As remarked above, Conjecture 5.4 was proved in [8] (with the exception of parts (iv) and (v), which are straightforward using the argument in section 7 of [5]) for the cases \( p_\pm = t/\) and \( p_\pm = 1/\sqrt{q} \). The only other known results are for the case \( \alpha_\pm = 0 \); the references in the following examples refer to this case alone.

**Example 1.** If we take \( p_\pm = t/\), we have
\[
\mu_\pm(z) = (z + z^{-1})t, \quad \sigma_\pm(z) = ((z - z^{-1})t)^{1/2}, \quad \tilde{\alpha} = 1, \quad \mu = 2t, \quad \sigma = t^{1/3}.
\] (5.30)

This corresponds to the classical case of increasing subsequences of random permutations, studied in [8].

**Example 2.** If we take \( p_\pm = 1/(\sqrt{q} n^\pm) \), with \( n_+ / n_- \) tending to a constant, we have
\[
\mu_\pm(z) = \frac{\sqrt{q} n_+}{1 - z/\sqrt{q}} + \frac{\sqrt{q} n_-}{z - \sqrt{q}}, \quad \sigma_\pm(z)^2 = \frac{\sqrt{q} n_+}{(1 - z/\sqrt{q})^2} - \frac{\sqrt{q} n_-}{(z - \sqrt{q})^2}
\] (5.31)

and thus
\[
\tilde{\alpha}_+ = \frac{\sqrt{q} n_+ + \sqrt{q} n_-}{\sqrt{q} n_+ + \sqrt{q} n_-}, \quad \mu = \frac{q(n_+ + n_-) + 2\sqrt{q} n_+ n_-}{1 - q}, \quad \sigma = \frac{(q n_+ n_-)^{1/6}(1 + \sqrt{\sigma_{n_-}^2}/2)(1 + \sqrt{\sigma_{n_+}^2}/2)}{1 - q}.
\] (5.32)

This model was analyzed in [11], along with the continuous (Laguerre) limit, in which case we have
\[
\mu_\pm(z) = \frac{1}{2} \left( \frac{n_+}{1 + 2z} + \frac{n_-}{1 - 2z} \right), \quad \sigma_\pm(z)^2 = \frac{4n_+}{(1 + 2z)^2} - \frac{4n_-}{(1 - 2z)^2}
\] (5.33)

and thus
\[
\tilde{\alpha}_+ = \frac{\sqrt{n_+} - \sqrt{n_-}}{2(\sqrt{n_+} + \sqrt{n_-})}, \quad \mu = (\sqrt{n_+} + \sqrt{n_-})^2, \quad \sigma = (n_+ n_-)^{1/6}(\sqrt{n_+} + \sqrt{n_-})^{1/3}.
\] (5.34)

**Example 3.** If we take \( p_\pm = 1^n, p_- = t/\), we have
\[
\mu_\pm(z) = zt + \frac{n}{z - 1}, \quad \sigma_\pm(z)^2 = zt - \frac{zn}{(z - 1)^2},
\] (5.35)

and thus
\[
\tilde{\alpha}_+ = 1 + \sqrt{n/t}, \quad \mu = t + 2\sqrt{n}, \quad \sigma = \sqrt{n^{-1/6}(1 + \sqrt{n/t})^2/3}.
\] (5.36)

This model, corresponding to weakly increasing subsequences of random words, was studied in [10] and [18]. In the continuous limit (corresponding to the \( n \times n \) GUE [14]), we have:
\[
\mu = 2\sqrt{n}, \quad \sigma = n^{-1/6}
\] (5.37)

These are precisely the scale factors required to make the largest eigenvalue of an \( n \times n \) Gaussian Hermitian matrix tend to the limit \( F_{\text{GUE}} \).
Example 4. If we take \( p_+ = (\sqrt{q})^{n_+} / \), \( p_- = (\sqrt{q})^{n_-} / \), with \( n_+/n_- \) tending to a constant, we have

\[
\mu_+(z) = \frac{\sqrt{q}z n_+}{1 + z \sqrt{q}} + \frac{\sqrt{q} n_-}{z + \sqrt{q}} \quad \sigma_+(z)^2 = \frac{\sqrt{q}z n_+}{(1 + z \sqrt{q})^2} - \frac{\sqrt{q} z n_-}{(z + \sqrt{q})^2}.
\] (5.38)

Here we have three cases. If \( qn_+ \geq n_- \), then \( \tilde{\alpha}_+ = 0 \), and if \( qn_- \geq n_+ \), then \( \tilde{\alpha}_+ = \infty \); in either case, the above conjectures do not apply (indeed, in those cases one expects the limiting distribution to be atomic, \( \lambda_1(p_+, p_-) = \min(n_+, n_-) \)). Otherwise,

\[
\tilde{\alpha}_+ = \frac{\sqrt{n_-} - \sqrt{q n_+}}{\sqrt{n_+} - \sqrt{q n_-}} \quad \mu = \frac{2\sqrt{q n_+} n_- - q(n_+ + n_-)}{1 - q} \quad \sigma = \frac{(qn_+ n_-)^{1/6}(1 - \sqrt{q n_-})^{2/3}(1 - \sqrt{2n_-})^{2/3}}{1 - q}.
\] (5.39)

The mean in this model was derived in [15]; the refined asymptotics of a symmetrized version was studied in [2].

Example 5. If we take \( p_+ = (\sqrt{q})^{n_+} / \), \( p_- = (\sqrt{q})^{n_-} / \), with \( n_+/n_- \) tending to a constant, we have

\[
\mu_+(z) = \frac{\sqrt{q}z n_+}{1 - z \sqrt{q}} + \frac{\sqrt{q} n_-}{z + \sqrt{q}} \quad \sigma_+(z)^2 = \frac{\sqrt{q}z n_+}{(1 - z \sqrt{q})^2} - \frac{\sqrt{q} z n_-}{(z + \sqrt{q})^2}.
\] (5.40)

There are two cases. If \( qn_+ \geq n_- \), then \( \tilde{\alpha}_+ = 0 \), and the above remark applies. Otherwise

\[
\tilde{\alpha}_+ = \frac{\sqrt{n_-} - \sqrt{q n_+}}{\sqrt{n_+} + \sqrt{q n_-}} \quad \mu = \frac{2\sqrt{q n_+} n_- + q(n_- - n_+)}{1 + q} \quad \sigma = \frac{(qn_+ n_-)^{1/6}(1 - \sqrt{q n_-})^{2/3}(1 + \sqrt{2n_-})^{2/3}}{1 + q}.
\] (5.41)

The mean in this model was first derived in [17]; the fluctuations have been analyzed in section 5 of [10].

Example 6. If we take \( p_+ = n_+/n_- \), \( p_- = t/n_- \), we have

\[
\mu_+(z) = zt + \frac{n}{z + 1} \quad \sigma_+(z)^2 = zt - \frac{zn}{(z + 1)^2}
\] (5.42)

If \( n \leq t \), then \( \tilde{\alpha}_+ = 0 \); otherwise we have

\[
\tilde{\alpha}_+ = \sqrt{n/t - 1} \quad \mu = 2\sqrt{n} - t \quad \sigma = (tn)^{1/6}(1 - \sqrt{t/n})^{2/3}.
\] (5.43)

This corresponds to strictly increasing subsequences of random words, and was studied to a small extent in [18].

Note that the pathological case \( \tilde{\alpha}_+ = 0 \) (resp. \( \tilde{\alpha}_+ = \infty \)) can only occur if all the rows (resp. columns) are strict.

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