A Realistic Formalism for 4N Bound State in a Three-Dimensional Yakubovsky Scheme

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A spin-isospin dependent Three-Dimensional formalism based on the momentum vectors for the four-nucleon bound state is presented. The four-nucleon Yakubovsky equations with two- and three-nucleon interactions are formulated as a function of the vector Jacobi momenta. Our formalism, according to the number of spin-isospin states that one takes into account, leads to only a strictly finite number of the coupled three dimensional integral equations to be solved. The evaluation of the transition and permutation operators as well as the coordinate transformations due to considering the continuous angle variables instead of the discrete angular momentum quantum numbers are less complicated in comparison with partial wave representation. With respect to partial wave the present formalism with the smaller number of equations leads to higher dimensionality of the integral equations. We have concluded that three dimensional formalism is less cumbersome for considering the three-nucleon forces.

\textsection 1. Introduction

The evaluation of the four-nucleon bound state properties because of the presence of the fourth nucleon is a challenging task both from the formalism aspect as well as the numerical one\textsuperscript{1,19}. The binding energy of the $\alpha$-particle with both $2 + 2$ and $3 + 1$ structures needs both realistic two- and three-nucleon forces in order to get closer to the experimental number\textsuperscript{14}.

In order to introduce the formalism of four-nucleon bound state we choose the three dimensional (3D) approach. Let us seek the answer to this main question which is our motivation to select this approach. Why do we use 3D instead of partial wave (PW) approach? Few-body calculations are traditionally carried out by solving the relevant equations in a PW basis. After truncation they lead to coupled equations on angular momentum quantum numbers. A few PWs often provide qualitative insight, but modern calculations need many different spin, isospin and angular momentum combinations. On the other hand the 3D approach replaces the discrete angular momentum quantum numbers with continuous angle variables and consequently it considers automatically all PWs. It may be useful to mention that in PW approach we should sum all PWs to infinite order, but in practice we truncate the sum to a finite angular momentum number which is dependent to the energy that we are working. It means that in higher energies we will need more PW components to obtain a convergence, whereas in 3D approach continues angle variables sum all PW
components to infinite order. So the number of equations in 3D representation is energy independent, whereas in PW representation it is energy dependent. It appears therefore natural to avoid a PW representation completely and work directly with vector variables.

The motivation for developing this non PW approach is introducing a direct solution of the integral equations avoiding the very involved angular momentum algebra occurring for the permutations, transformations and especially for the three-body forces.

So in contrast to the truncated PW approach, the number of equations in the non truncated 3D representation is energy independent. Therefore this non PW method is more efficient and applicable to the three- and four-nucleon scattering problems which consider higher energies than the corresponding bound state problems. To show this efficiency the leading order\textsuperscript{20} and full formulation\textsuperscript{21} of 3N scattering with the inclusion of 3NFs has been done. Certainly the 4N scattering formulation and calculations are major additional tasks to be carried out.

We have recently applied the spin-isospin dependent 3D approach based on momentum vectors to the 3N bound state. The Faddeev equations with NN interactions are formulated successfully as a function of the vector Jacobi momenta, specifically the magnitudes of the momenta and the angle between them, as well as the spin-isospin quantum numbers.\textsuperscript{22,23} This novel formalism, according to the number of spin-isospin states that one takes into account, leads to only strictly finite number of coupled three dimensional integral equations to be solved. We have shown that this formalism for both $^3\text{H}$ and $^3\text{He}$ bound states yields the same number of coupled equations which for fully charge dependent case leads to only 24 coupled equations. As an application the spin-isospin dependent three dimensional Faddeev integral equations are solved with Bonn-B potential.\textsuperscript{24,25} Our result for $^3\text{H}$ binding energy with the value of $-8.152$ MeV is in good agreement with the achievements of the other partial wave based methods. The calculation of $^3\text{H}$ binding energy with most modern NN potentials, i.e. AV18 and chiral potentials, is currently underway.

It should be mentioned that the calculation of $^3\text{He}$ binding energy, because of the presence of coulomb interaction, is more complicated in comparison to $^3\text{H}$ case and with the recent achievement considering coulomb interaction it can be included in the calculation more easily in future.

We have developed in this article the 3D approach for the four-nucleon bound state studies with the realistic interactions. The present work lies a formal ground for numerical investigations which are planned. This will helps us to reach to the full solution of the four-nucleon bound state in a straightforward manner. We propose the spin and isospin degrees of freedom to be implemented into the three dimensional four-nucleon Yakubovskiy equations. We work directly with the vector variables in the Yakubovskiy scheme in the momentum space, which leads to two coupled sets of a very limited number of equations in the three vector variables for the amplitudes which greatly simplifies the calculations without using a partial wave (PW) decomposition.

In the present article we also show that how three-nucleon forces (3NFs), which we have chosen for example the Tucson-Melbourne two-pion exchange 3NF for con-
sidering the full solution of the four-nucleon bound system, can be presented in very simpler formalism in comparison with the PW.\textsuperscript{12) With regard to this simplifications the study of Tucson-Melburn 3NF effects in the $^3H$ binding energy is also achievable and currently is underway.

This paper is organized as follows. In section 2 we briefly represent the coupled Yakubovsky equations for the four-nucleon bound state with two- and three-nucleon interactions. We also introduce the four-nucleon basis states in a realistic 3D approach. In sections 3 and 4 we derive the coupled Yakubovsky equations in the realistic 3D approach as a function of the momentum vectors as well as the spin-isospin quantum numbers with and without the 3NFs, respectively. We add appendix A to show how would be less cumbersome in compare with the PW approach, if we evaluate the 3NF matrix elements for example for the Tucson-Melbourne $2\pi$-exchange 3NF in the 3D approach. In section 5 we discuss about the number of coupled Yakubovsky equations in both the 3D and the PW approaches. Finally we summarize in section 6.

\section{The 4N Basis States in 3D Representation}

The bound state of the four nucleon (4N) system, in the presence of the three-nucleon forces, is described by the two coupled Yakubovsky equations:

\begin{align}
|\psi_1\rangle &= G_0 tP[(1-P_{34})|\psi_1\rangle + |\psi_2\rangle] + (1 + G_0 t)G_0 V_{123}^{(3)}|\Psi\rangle, \\
|\psi_2\rangle &= G_0 t\tilde{P}[(1-P_{34})|\psi_1\rangle + |\psi_2\rangle],
\end{align}

where the Yakubovsky components $|\psi_1\rangle$ and $|\psi_2\rangle$ belong to "$3 + 1$"(123,4) and "$2 + 2$"(12,34) partitions of the four particles respectively.\textsuperscript{14) Here the free 4N propagator is given by $G_0 = (E - H_0)^{-1}$, and $H_0$ stands for the free Hamiltonian. The operator $t$ is the NN transition matrix determined by a two-body Lippmann-Schwinger equation. $P$, $\tilde{P}$ and $P_{34}$ are permutation operators. $P = P_{12}P_{23} + P_{13}P_{23}$ permutes the particles in the three-body subsystem (123) and $\tilde{P} = P_{13}P_{24}$ interchanges the two two-body subclusters (12) and (34). The quantity $V_{123}^{(3)}$ defines a part of the 3NF in the cluster (123), which is symmetric under the exchange of the particles 1 and 2. This can be related by an interchange of three particles to the two other parts $V_{123}^{(1)}$ and $V_{123}^{(2)}$ that sum up to the total 3NF of particles 1,2 and 3: $V_{123} = V_{123}^{(1)} + V_{123}^{(2)} + V_{123}^{(3)}$. For the 3NFs based on a meson-exchange picture, $V_{123}^{(3)}$ describes the interaction induced by a pion interchanged between the particles 1, 2 and on its path rescattered by the third particle, see Fig.\textsuperscript{11} Applying a combination of the transpositions to the set of the Yakubovsky components, one obtains the total wave function $|\Psi\rangle$ as

\begin{equation}
|\Psi\rangle = [1 - (1 + P)P_{34}] (1 + P)|\psi_1\rangle + (1 + P)(1 + \tilde{P})|\psi_2\rangle.
\end{equation}

The antisymmetry property of the $|\psi_1\rangle$ under the exchange of the particles 1,2 and the $|\psi_2\rangle$ under separate exchanges of particles 1,2 and 3,4 guarantee that the $|\Psi\rangle$ is totally antisymmetric. According to the two types of partitions (123,4) and
(12, 34), as shown in Fig. 2, we introduce two different sets of the 4N basis states in a 3D representation which are suitable to represent the Yakubovsky components |ψ1⟩ and |ψ2⟩ in the coupled equations (2.1):

\[
|\textbf{u} \alpha \rangle \equiv |u_1 u_2 u_3 \alpha_{1234} \rangle, \\
|\textbf{v} \beta \rangle \equiv |v_1 v_2 v_3 \beta_{1234} \rangle.
\] (2.3)

The spin-isospin part of the basis states are:

\[
|\alpha_{1234} \rangle \equiv |\alpha_{1234}^S \alpha_{1234}^T \rangle, \\
|\beta_{1234} \rangle \equiv |\beta_{1234}^S \beta_{1234}^T \rangle,
\] (2.4)

where

\[
|\alpha_{1234}^S \rangle = \langle (s_1 s_2 s_12 s_3 s_{123} s_4) S M_S \rangle \equiv \langle (s_1 s_2 s_12 s_3 s_{123} s_4) S M_S \rangle, \\
|\alpha_{1234}^T \rangle = \langle (s_1 s_2 s_12 s_3 s_4) S M_S \rangle \equiv \langle (s_1 s_2 s_12 s_3 s_4) S M_S \rangle,
\] (2.5)

and the isospin parts of the basis states |α_{1234}^T⟩ and |β_{1234}^T⟩ are similar to the spin parts.

Each one of the basis states involves three standard Jacobi momenta \(u_1, u_2, u_3\) or \(v_1, v_2, v_3\).\(^{14}\) As indicated in the Fig. 2, the angular dependence explicitly appears in the Jacobi vector variables, whereas in a standard PW representation the angular dependence leads to three orbital angular momentum quantum numbers for each kind of the basis states, i.e. \(l_{12}, l_3\) and \(l_4\) or \(l_{12}, l_2\) and \(l_{34}\).\(^{12}\) It indicates that in our 3D formalism there is not any coupling between the orbital angular momenta and corresponding spin quantum numbers. Therefore we couple the spin quantum numbers \(s_{12}, s_3\) and \(s_4\) or \(s_{12}\) and \(s_{34}\) to the total spin \(S\) and its third component \(M_S\), by using only one intermediate quantum number \(s_{123}\) for the first basis states: \(|(s_{12} s_3 s_{123} s_4) S M_S \rangle |(s_{12} s_3) S M_S \rangle \) or \(|(s_{12} s_3) S M_S \rangle \). For the isospin quantum numbers similar coupling schemes to the total isospin \(T M_T\) involve one intermediate quantum number \(t_{123}\): \(|(t_{12} t_3) T M_T \rangle \) or \(|(t_{12} t_3) T M_T \rangle \). In order to be able to evaluate the transition and permutation operators we need the free 4N basis states \(|\textbf{u} \ \gamma \rangle \) and \(|\textbf{v} \ \gamma \rangle \), where

\[
|\gamma \rangle \equiv |\gamma_{1234} \rangle \equiv |\gamma_{1234}^S \gamma_{1234}^T \rangle, \\
|\gamma^S \rangle \equiv |m_{s_1} m_{s_2} m_{s_3} m_{s_4} \rangle.
\] (2.6)
Fig. 2. Definition of the 4N basis states in both the 3D and the PW schemes which are constructed by 3 + 1 and 2 + 2 type of the Jacobi coordinates and the corresponding spin-isospin quantum numbers.

where \( m_{s_i}, i = 1, 2, 3, 4 \) indicates the projection of the spin of each nucleon. The isospin part of the basis states \( |\gamma_{1234}\rangle \) is similar to the spin part. In this respect when we are going from one of the 4N basis states, \( |\alpha\rangle \) and \( |\beta\rangle \), to the free 4N basis states, \( |\gamma\rangle \), we should calculate the following Clebsch-Gordan coefficients;

\[
\begin{align*}
\langle \gamma | \alpha \rangle & = g_{\gamma \alpha} = g_{\gamma_{1234} \alpha_{1234}} = g_{\gamma_{1234} \alpha_{1234}}^S g_{\gamma_{1234} \alpha_{1234}}^T \\
& = \langle m_{s_1} m_{s_2} m_{s_3} m_{s_4} | ( (s_{12} \frac{1}{2}) s_{123} \frac{1}{2} ) S M_S \rangle \\
& \quad \times \langle m_{t_1} m_{t_2} m_{t_3} m_{t_4} | ( (t_{12} \frac{1}{2}) t_{123} \frac{1}{2} ) T M_T \rangle,
\end{align*}
\]

\[
\begin{align*}
\langle \gamma | \beta \rangle & = g_{\gamma \beta} = g_{\gamma_{1234} \beta_{1234}} = g_{\gamma_{1234} \beta_{1234}}^S g_{\gamma_{1234} \beta_{1234}}^T \\
& = \langle m_{s_1} m_{s_2} m_{s_3} m_{s_4} | ( s_{12} s_{34} ) S M_S \rangle \\
& \quad \times \langle m_{t_1} m_{t_2} m_{t_3} m_{t_4} | ( t_{12} t_{34} ) T M_T \rangle.
\end{align*}
\]  

(2.7)
The introduced basis states are complete in the four-nucleon Hilbert space as:

$$\sum_{\xi}^{A} \langle A | \xi \rangle \langle \xi | A \rangle = 1, \quad \sum_{\xi}^{A} \equiv \sum_{\xi}^{A} \int D^{3}A \equiv \sum_{\xi}^{A} \int d^{3}A_{1} \int d^{3}A_{2} \int d^{3}A_{3},$$

(2.8)

where $A$ indicates each one of the $u$ and $v$ vector sets and $\xi$ indicates $\alpha, \beta$ and $\gamma$ quantum number sets. They are also normalized according to:

$$\langle A | \xi \rangle \langle \xi | A^{' \neq \xi} \rangle = \delta^{3}(A - A^{'}) \delta_{\xi \xi^{'}}.$$

(2.9)

Clearly the basis states $|u \alpha \rangle$ are adequate to expand the first Yakubovsky component $|\psi_{1}\rangle$ and correspondingly the basis states $|v \beta \rangle$ are adequate for the second one $|\psi_{2}\rangle$

§3. Yakubovsky Equations in 3D Representation Without 3NFs

Let us now represent the coupled equations (2.1), without the 3NFs, with respect to the basis states which have been introduced in Eq. (2.3):

$$\langle u \alpha | \psi_{1} \rangle = \sum_{\alpha''}^{u''} \langle u \alpha | G_{0}tP(1 - P_{34})| u'' \alpha'' \rangle \langle u'' \alpha'' | \psi_{1} \rangle + \sum_{\beta}^{v'} \langle u \alpha | G_{0}tP | v' \beta' \rangle \langle v' \beta' | \psi_{2} \rangle,$$

$$\langle v \beta | \psi_{2} \rangle = \sum_{\alpha}^{u} \langle v \beta | G_{0}t\tilde{P}(1 - P_{34})| u' \alpha' \rangle \langle u' \alpha' | \psi_{1} \rangle + \sum_{\alpha'}^{v} \langle v \beta | G_{0}t\tilde{P} | v' \beta' \rangle \langle v' \beta' | \psi_{2} \rangle.$$

(3.1)

It is convenient to insert the free 4N completeness relations between the permutation operators, it results:

$$\langle u \alpha | \psi_{1} \rangle = \sum_{\gamma}^{u''} \sum_{\alpha''}^{u'''} \langle u \alpha | G_{0}tP | u' \gamma' \rangle \langle u' \gamma' | (1 - P_{34})| u'' \alpha'' \rangle \langle u'' \alpha'' | \psi_{1} \rangle + \sum_{\gamma'} \sum_{\beta}^{v'} \langle u \alpha | G_{0}tP | u' \gamma' \rangle \langle u' \gamma' | v' \beta' \rangle \langle v' \beta' | \psi_{2} \rangle,$$

$$\langle v \beta | \psi_{2} \rangle = \sum_{\gamma}^{v'} \sum_{\alpha'}^{u'} \langle v \beta | G_{0}t\tilde{P} | v' \gamma' \rangle \langle v' \gamma' | (1 - P_{34})| u' \alpha' \rangle \langle u' \alpha' | \psi_{1} \rangle + \sum_{\alpha'}^{v} \langle v \beta | G_{0}t\tilde{P} | v' \beta' \rangle \langle v' \beta' | \psi_{2} \rangle.$$

(3.2)

For evaluating the above coupled equations, Eq. (3.2), we need to evaluate the following matrix elements:

$$\langle u \alpha | G_{0}tP | u' \gamma' \rangle,$$

(3.3)
\begin{align}
\langle v \beta | G_0 \hat{P} | v' \beta' \rangle, \quad & \langle v \beta | G_0 \hat{P} | v' \gamma' \rangle, \tag{3.4} \\
\langle u' \gamma' | 1 - P_{34} | u'' \alpha'' \rangle, \quad & \langle v' \gamma' | 1 - P_{34} | u' \alpha' \rangle, \tag{3.5} \\
\langle u' \gamma' | v' \beta' \rangle. \quad & \tag{3.6} \\
\langle u' \gamma' | 1 - P_{34} | u' \alpha' \rangle, \quad & \tag{3.7}
\end{align}

For evaluating the first term, Eq. (3.3), we should insert again a free 4N completeness relation between the NN $t$-matrix operator and the permutation operator $P$ as:

\begin{align}
\langle u \alpha | G_0 t P | u' \gamma' \rangle &= \frac{1}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m} - \frac{2u_4^2}{3m}} \sum_{\alpha''} \sum_{\gamma''} \langle \alpha \gamma'' | t \rangle | u'' \gamma'' \rangle | u' \gamma' \rangle | P | u' \gamma' \rangle, \tag{3.8}
\end{align}

where the matrix elements of the NN $t$-matrix and the permutation operator $P$ are evaluated separately as:

\begin{align}
\langle u' \gamma'' | t | u'' \gamma'' \rangle &= \delta^3(u_2 - u''_1) \delta^3(u_3 - u''_2) \delta^3(u_4 - u''_3) \delta^3(u''_2 - u''_1 + 1/2 u''_2) \\
&\quad \times \langle u_1 m''_{s_1} m''_{t_{s_2}} m''_{t_{s_3}} m''_{t_{s_4}} | t(e) | u''_1 m''_{s_1} m''_{t_{s_2}} m''_{t_{s_3}} m''_{t_{s_4}} \rangle, \tag{3.9}
\end{align}

\begin{align}
\langle u'' \gamma'' | P | u' \gamma' \rangle &= \delta^3(u_3' - u_3) \delta^3(u''_{s_4} m''_{t_{s_4}} m''_{t_{s_4}} m''_{t_{s_4}} m''_{t_{s_4}} \tag{3.10}
\end{align}

For evaluation of the matrix elements of the permutation operator $P$ we have used the relation between the Jacobi momenta in different three-body subsystems (312, 4), (231, 4) and (123, 4). Inserting Eqs. (3.9) and (3.10) into Eq. (3.8) leads to:

\begin{align}
\langle u \alpha | G_0 t P | u' \gamma' \rangle &= \frac{\delta^3(u_3 - u''_1)}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m} - \frac{2u_4^2}{3m}} \sum_{\alpha''} \sum_{\gamma''} g_{\alpha' \gamma''} \delta^3(u''_{s_4} m''_{t_{s_4}} m''_{t_{s_4}} m''_{t_{s_4}} \tag{3.11}
\end{align}
where $E_1 = \frac{v_1}{m} - \frac{v_2}{2m} - \frac{v_2}{m}$.

The matrix elements of the NN $t$-matrix and the permutation operator $\hat{P}$ are evaluated as:

$$\langle v \gamma'' | t | v'' \gamma'' \rangle = \delta^3(v_2 - v'_2) \delta^3(v_3 - v'_3) \delta m''_2 \delta m''_4 \delta m''_1 \delta m''_3 \delta m''_4$$

$$\times \langle v_1 m''_{s_1} m''_{s_2} m''_{t_1} m''_{t_2} | t(e^*) | v''_{s_1} m''_{s_2} m''_{t_1} m''_{t_2} \rangle, \quad (3.14)$$

$$\langle v'' \gamma'' | \hat{P} | v' \gamma' \rangle = \delta^3(v''_2 - v'_2) \delta^3(v''_3 + v'_3) \delta^3(v''_1 - v'_1)$$

$$\times \delta m''_3 \delta m''_4 \delta m''_3 \delta m''_1 \delta m''_3 \delta m''_1$$

$$\times \delta m''_3 \delta m''_1 \delta m''_3 \delta m''_1 \delta m''_1 \delta m''_1. \quad (3.15)$$

Inserting Eqs. (3.14) and (3.15) into Eq. (3.13) leads to:

$$\langle v \beta | G_0 t \hat{P} | v' \gamma' \rangle = \frac{\delta^3(v_2 + v'_2) \delta^3(v_3 - v'_1)}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_2^2}{m}}$$

$$\times \sum_{\gamma''} g_{\gamma'' m''} \delta m''_2 m''_1 \delta m''_3 m''_2 \delta m''_1 m''_4$$

$$\times \langle v_1 m''_{s_1} m''_{s_2} m''_{t_1} m''_{t_2} | t(e^*) | v''_{s_1} m''_{s_2} m''_{t_1} m''_{t_2} \rangle. \quad (3.16)$$

For evaluation of the third term, Eq. (3.16), we should use the relation between the Jacobi momenta in different chains $(123, 4)$ and $(124, 3)$, which results:

$$\langle u' \gamma' | 1 - P_{34} | u'' \alpha'' \rangle = \sum_{\gamma''} g_{\gamma'' \alpha''} \delta^3(u'_1 - u''_1)$$
ering Eqs. (3.17) and (3.18) as:

\[
\times \left\{ \delta^3 (u''_2 - u''_1) \delta^3 (u'_3 - u''_3) \delta_{1243} \gamma_{1234}'' \right.
\]

\[
\left. - \delta^3 (u'_2 - \frac{1}{3} u''_2 - \frac{8}{9} u''_3) \delta^3 (u'_3 - u''_2 + \frac{1}{3} u''_3) \delta_{1243} \gamma_{1243}'' \right\}.
\]

(3.17)

And finally for the evaluation of the fourth and fifth terms, Eqs. (3.16) and (3.17), we should use the relation between the Jacobi momenta in two naturally different chains (123, 4) and (12, 34). the result is:

\[
\langle u' \gamma' | v' \beta' \rangle = g_{\gamma' \beta'} \delta^3 (v'_1 - u'_1) \delta^3 (v'_2 + u'_2 + \frac{2}{3} u'_3) \delta^3 (v'_3 - \frac{1}{2} u'_2 + \frac{2}{3} u'_3),
\]

(3.18)

\[
\langle v' \gamma' | 1 - P_{34} \rangle u' \alpha' \rangle = \sum_{\gamma''} g_{\gamma'' \alpha''} \delta^3 (u'_1 - v'_1)
\]

\[
\times \left\{ \delta^3 (u'_2 + \frac{2}{3} v'_2 - \frac{2}{3} v'_3) \delta^3 (u'_3 + \frac{1}{2} v'_2 + v'_3) \delta_{1243} \gamma_{1243}'' \right.
\]

\[
\left. + \delta^3 (u'_2 + \frac{2}{3} v'_2 + \frac{2}{3} v'_3) \delta^3 (u'_3 + \frac{1}{2} v'_2 - v'_3) \delta_{1243} \gamma_{1243}'' \right\}.
\]

(3.19)

By these considerations, in the following we evaluate the first and second Yakubovsky components separately. For the first component we can rewrite Eq. (3.2) by considering Eqs. (3.17) and (3.18) as:

\[
\langle u \alpha | \psi_1 \rangle = \sum_{\gamma} \sum_{\alpha''} \langle u \alpha | G_0 t P | u' \gamma' \rangle \sum_{\gamma''} g_{\gamma'' \alpha''} \delta^3 (u'_1 - u''_1)
\]

\[
\times \left\{ \delta^3 (u'_2 - u''_2) \delta^3 (u'_3 - u''_3) \delta_{1243} \gamma_{1243}'' \right.
\]

\[
\left. - \delta^3 (u'_2 - \frac{1}{3} u''_2 - \frac{8}{9} u''_3) \delta^3 (u'_3 - u''_2 + \frac{1}{3} u''_3) \delta_{1243} \gamma_{1243}'' \right\}
\]

\[
\times \langle u'' \alpha'' | \psi_1 \rangle
\]

\[
+ \sum_{\gamma} \sum_{\beta'} \langle u \alpha | G_0 t P | u' \gamma' \rangle
\]

\[
\times g_{\gamma' \beta'} \delta^3 (v'_1 - u'_1) \delta^3 (v'_2 + u'_2 + \frac{2}{3} u'_3) \delta^3 (v'_3 - \frac{1}{2} u'_2 + \frac{2}{3} u'_3)
\]

\[
\times \langle v' \beta' | \psi_2 \rangle,
\]

(3.20)

by integrating over \( u'' \) and \( v' \) vector sets in the first and the second terms respectively,
and considering Eq. (3.11), we obtain:

\[
\langle u \alpha | \psi_1 \rangle = \sum_{\gamma'} \frac{\delta^3(u_3 - u'_4)}{E - \frac{\delta}{m} - \frac{3u_2}{4m} - \frac{2u_3}{3m}} \sum g_{\alpha''} \delta_{m''_{s}m'_{t}l_{t}} \delta_{m''_{s}m'_{t}l_{t}}
\times \left[ \delta^3(u_2 - u'_1 + \frac{1}{2}u_2') \delta_{m''_{s}m'_{t}l_{t}} \delta_{m''_{s}m'_{t}l_{t}} \right.
\times \left( u_1 m''_{s}m''_{s}m''_{t}m''_{t}l(t(e)) - \frac{1}{2}u_2 - u'_2 m''_{s}m'_{t}l_{t} \right)
\right]
\times \left\{ \sum_{\alpha''} g_{\alpha''} \langle u' \alpha'' | \psi_1 \rangle 
- \sum g_{\alpha''} \delta_{\gamma_{1243}^{\gamma_{1243}}} \langle u'_1 1/3 u_2' + 8/9 u_3' - 1/3 u'_3 \alpha'' | \psi_1 \rangle 
+ \sum g_{\alpha''} \langle u'_2 - 2/3 u'_3 + 1/2 u'_2 - 2/3 u_3' | \psi_2 \rangle \right\}, \quad (3.21)
\]

and by integrating over \(u'_1\) and \(u'_3\) vectors:

\[
\langle u \alpha | \psi_1 \rangle = \int d^3u'_2 \sum_{\gamma', \gamma''} g_{\alpha''} \delta_{m''_{s}m'_{t}l_{t}} \delta_{m''_{s}m'_{t}l_{t}}
\times \left[ \delta_{m''_{s}m'_{t}l_{t}} \delta_{m''_{s}m'_{t}l_{t}} \langle u_1 m''_{s}m''_{s}m''_{t}m''_{t}l(t(e)) - \frac{1}{2}u_2 - u'_2 m''_{s}m'_{t}l_{t} \rangle \right.
\times \left\{ \sum_{\alpha''} g_{\alpha''} \langle u_2 + \frac{1}{2}u_2' u_3' u_3 \alpha' | \psi_1 \rangle 
- \sum g_{\alpha''} \delta_{\gamma_{1243}^{\gamma_{1243}}} \langle u_2 + \frac{1}{2}u_2' - \frac{2}{3} u_3' + \frac{1}{2} u'_2 - \frac{2}{3} u_3' | \psi_2 \rangle \right\}
\right.
\times \left\{ \sum_{\alpha''} g_{\alpha''} \langle -u_2 - \frac{1}{2}u_2' u_3' u_3 \alpha'' | \psi_1 \rangle 
- \sum g_{\alpha''} \delta_{\gamma_{1243}^{\gamma_{1243}}} \langle -u_2 - \frac{1}{2}u_2' u_3' u_3 | \psi_1 \rangle \right\}, \quad (3.21)
\]
\[
+ \sum_{\beta'} g_{\gamma'\beta'} \left\langle \mathbf{u}_2 - \frac{1}{2} \mathbf{u}'_2 - \mathbf{u}_3' - \frac{2}{3} \mathbf{u}_3 - \frac{2}{3} \mathbf{u}_2' - \frac{1}{3} \mathbf{u}_3' \beta' | \psi_2 \right\rangle \right]. \quad (3.22)
\]

In order to evaluate the Eq. (3.22) we consider the result of the permutation operators \(P_{12}\) and \(P_{34}\) action on the Yakubovsky components, the space and also the spin-isospin parts of the basis states as:

\[
P_{12} | \psi_1 \rangle = - | \psi_1 \rangle,
P_{12} | \psi_2 \rangle = - | \psi_2 \rangle,
P_{34} | \psi_2 \rangle = - | \psi_2 \rangle,
P_{12} | \mathbf{u} \rangle = - | \mathbf{u}_1 \mathbf{u}_2 \mathbf{u}_3 \rangle,
P_{12} | \mathbf{v} \rangle = - | \mathbf{v}_1 \mathbf{v}_2 \mathbf{v}_3 \rangle,
P_{34} | \mathbf{v} \rangle = | \mathbf{v}_1 \mathbf{v}_2 - \mathbf{v}_3 \rangle,
P_{12} | \alpha \rangle = (-)^{s_1 + s_2 - s_12} (-)^{t_1 + t_2 - t_12} | \alpha \rangle = (-)^{s_12 + t_12} | \alpha \rangle,
P_{12} | \beta \rangle = (-)^{s_1 + s_2 - s_12} (-)^{t_1 + t_2 - t_12} | \beta \rangle = (-)^{s_12 + t_12} | \beta \rangle,
P_{34} | \beta \rangle = (-)^{s_3 + s_4 - s_34} (-)^{t_3 + t_4 - t_34} | \beta \rangle = (-)^{s_34 + t_34} | \beta \rangle,
P_{12} | \gamma \rangle = | m_{s_1} m_{s_2} m_{s_3} m_{t_1} m_{t_2} m_{t_3} \rangle \equiv | \gamma_{1234} \rangle,
P_{34} | \gamma \rangle = | m_{s_1} m_{s_2} m_{s_3} m_{t_1} m_{t_2} m_{t_3} \rangle \equiv | \gamma_{1243} \rangle. \quad (3.23)
\]

By taking the Eq. (3.23) into consideration the matrix elements of the NN \(t\)-matrix as well as the Yakubovsky components in the second term can be rewritten as:

\[
\langle \mathbf{u}_1 m''_1 m''_2 m''_3 m''_4 m''_5 \mathbf{t}(\epsilon) \frac{1}{2} \mathbf{u}_2 + \mathbf{u}'_2 m' m' m' m' m' \rangle
\equiv \langle \mathbf{u}_1 m''_1 m''_2 m''_3 m''_4 m''_5 \mathbf{t}(\epsilon) P_{12} P_{12} \frac{1}{2} \mathbf{u}_2 + \mathbf{u}'_2 m' m' m' m' m' \rangle
= \langle \mathbf{u}_1 m''_1 m''_2 m''_3 m''_4 m''_5 \mathbf{t}(\epsilon) P_{12} \rangle - \frac{1}{2} \mathbf{u}_2 - \mathbf{u}'_2 m' m' m' m' m' , \quad (3.24)
\]

\[
\sum_{\alpha''} g_{\gamma'\alpha''} \left\langle - \mathbf{u}_2 - \frac{1}{2} \mathbf{u}'_2 \mathbf{u}_3 \alpha'' | \psi_1 \right\rangle
\equiv \sum_{\alpha''} g_{\gamma'\alpha''} \left\langle - \mathbf{u}_2 - \frac{1}{2} \mathbf{u}'_2 \mathbf{u}_3 \alpha'' | P_{12} P_{12} | \psi_1 \right\rangle
= -(-)^{s''_1 + t''_1} \sum_{\alpha''} g_{\gamma'\alpha''} \left\langle \mathbf{u}_2 + \frac{1}{2} \mathbf{u}'_2 \mathbf{u}_3 \alpha'' | \psi_1 \right\rangle, \quad (3.25)
\]

\[
\sum_{\alpha'' \gamma''} g_{\gamma'' \alpha''} \delta_{1234 \gamma''_{1243}} \left\langle - \mathbf{u}_2 - \frac{1}{2} \mathbf{u}'_2 \mathbf{u}_3 \alpha'' | P_{12} P_{12} | \psi_1 \right\rangle
\equiv \sum_{\alpha'' \gamma''} g_{\gamma'' \alpha''} \delta_{1234 \gamma''_{1243}} \left\langle - \mathbf{u}_2 - \frac{1}{2} \mathbf{u}'_2 \mathbf{u}_3 \alpha'' | P_{12} P_{12} | \psi_1 \right\rangle
\]

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\[ (-1)^{s_1 t_2 + s_1 t_2} \sum_{\alpha''} g_{\gamma'' \alpha''} \delta_{t_{1245} \gamma_{1243}} \]
\[ \times (u_2 + \frac{1}{2} u'_2 - \frac{1}{3} u'_2 + \frac{8}{9} u_3 u'_2 - \frac{1}{3} u_3 \alpha'' | P_{12} P_{12} | \psi_1), \]  
\[ (3.26) \]
\[ \sum_{\beta''} g_{\gamma'' \beta''} \langle -u_2 - \frac{1}{2} u'_2 - u'_2 - \frac{2}{3} u_3 + \frac{1}{2} u'_2 - \frac{2}{3} u_3 \beta | \psi_2 \rangle \]
\[ \sum_{\beta''} g_{\gamma'' \beta''} \langle -u_2 - \frac{1}{2} u'_2 - u'_2 - \frac{2}{3} u_3 + \frac{1}{2} u'_2 - \frac{2}{3} u_3 \beta | P_{12} P_{12} | \psi_2 \rangle \]
\[ = (-1)^{s_1 t_2 + s_1 t_2} \sum_{\beta''} g_{\gamma'' \beta''} \langle u_2 + \frac{1}{2} u'_2 - u'_2 - \frac{2}{3} u_3 + \frac{1}{2} u'_2 - \frac{2}{3} u_3 \beta | P_{12} P_{12} | \psi_2 \rangle. \]  
\[ (3.27) \]

After inserting Eqs. (3.24)–(3.27) in Eq. (3.22) we obtain:
\[ \langle u_\alpha | \psi_1 \rangle = \int d^3 u'_2 \sum_{\gamma'' \gamma''} g_{\gamma'' \alpha''} \frac{\delta m_{m_4} m_{m_4} \delta m_{m_4} m_{m_4}}{E - \frac{\alpha_1}{m} - \frac{\beta_2}{m} - \frac{\alpha_3}{m}} \]
\[ \times \left\{ \sum_{\alpha''} g_{\gamma'' \alpha''} \langle u_2 + \frac{1}{2} u'_2 - u'_2 - \frac{2}{3} u_3 + \frac{1}{2} u'_2 - \frac{2}{3} u_3 \beta | \psi_1 \rangle \right\} \]
\[ + \langle u_1 m_{m_1} m_{m_2} m_{m_1} m_{m_2} | t(e) | u_2 - u'_2 m_{s_2} m_{s_2} m_{t_3} m_{t_3} \rangle \delta_{m_{s_3} m_{s_3}} \delta_{m_{t_1} m_{t_1}} \delta_{m_{t_3} m_{t_3}} \]
\[ \times \left\{ \sum_{\alpha''} g_{\gamma'' \alpha''} \langle u_2 + \frac{1}{2} u'_2 - u'_2 - \frac{2}{3} u_3 + \frac{1}{2} u'_2 - \frac{2}{3} u_3 \beta | \psi_1 \rangle \right\} \]
\[ + \langle u_1 m_{m_1} m_{m_2} m_{m_1} m_{m_2} | t(e) P_{12} | -\frac{1}{2} u_2 - u'_2 m_{s_2} m_{s_2} m_{t_3} m_{t_3} \rangle \delta_{m_{s_3} m_{s_3}} \delta_{m_{t_1} m_{t_1}} \delta_{m_{t_3} m_{t_3}} \]
\[ \times \left\{ \sum_{\alpha''} g_{\gamma'' \alpha''} \langle -(-1)^{t_{12} + t_{12}} u_2 + \frac{1}{2} u'_2 - u'_2 - \frac{2}{3} u_3 + \frac{1}{2} u'_2 - \frac{2}{3} u_3 \beta | \psi_1 \rangle \right\} \]
\[ + \langle u_1 m_{m_1} m_{m_2} m_{m_1} m_{m_2} | t(e) P_{12} | -\frac{1}{2} u_2 - u'_2 m_{s_2} m_{s_2} m_{t_3} m_{t_3} \rangle \delta_{m_{s_3} m_{s_3}} \delta_{m_{t_1} m_{t_1}} \delta_{m_{t_3} m_{t_3}} \]
\[ \times \left\{ \sum_{\alpha''} g_{\gamma'' \alpha''} \langle -(-1)^{t_{12} + t_{12}} u_2 + \frac{1}{2} u'_2 - u'_2 - \frac{2}{3} u_3 + \frac{1}{2} u'_2 - \frac{2}{3} u_3 \beta | \psi_1 \rangle \right\} \right\}, \]  
\[ (3.28) \]

The exchange of the labels \( m_{s_1} m_{t_1} \) to \( m_{s_2} m_{t_2} \) and reverse of it in the second term and consequently the following relations:
\[ g_{\gamma'' \alpha''} \rightarrow (-1)^{s_1 t_2 + t_1 t_2} g_{\gamma'' \alpha''} \]
\[ g_{\gamma'_{1243} \alpha''} \rightarrow (-)^{s_2 + \frac{1}{3}} g_{\gamma'_{1243} \alpha''} \]
\[ g_{\gamma' \beta'} \rightarrow (-)^{s_2 + \frac{1}{3}} g_{\gamma' \beta'} \]  
\[ \text{(3.29)} \]

lead to:

\[ \langle \mathbf{u} \alpha | \psi_1 \rangle = \int d^3 u_2 \sum_{\gamma', \gamma''} g_{\alpha \gamma''} \frac{\delta_{m_{s_4}^m m_{t_4}^t} \delta_{m_{s_4}^m m_{t_4}^t}}{E - \frac{u_2^2}{m} - \frac{3u_2^3}{4m} - \frac{2u_2^4}{3m}} \delta_{m_{s_3}^m m_{t_3}^t} \delta_{m_{s_3}^m m_{t_3}^t} \]
\[ \times \langle \mathbf{u}_1 m_{s_1}^m m_{s_2}^m m_{t_1}^t m_{t_2}^t | t(\epsilon) (1 - P_{12}) | \frac{-1}{2} u_2 - u_2' m_{s_2}^m m_{s_3}^m m_{t_2}^t m_{t_3}^t \rangle \]
\[ \times \left\{ \sum_{\alpha''} g_{\gamma' \alpha''} \langle \mathbf{u}_2 + \frac{1}{2} u_2' u_2 \mathbf{u}_3 \alpha'' | \psi_1 \rangle \right. \]
\[ - \sum_{\alpha'' \gamma''} g_{\gamma'' \alpha''} \delta_{\gamma'_{1234} \gamma''_{1234}} \langle \mathbf{u}_2 + \frac{1}{2} u_2' u_2 \frac{1}{3} u_2' + \frac{8}{9} u_3 u_2' - \frac{1}{3} u_3 \alpha'' | \psi_1 \rangle \]
\[ + \sum_{\beta'} g_{\gamma' \beta'} \langle \mathbf{u}_2 + \frac{1}{2} u_2' \mathbf{u}_2' - \frac{2}{3} u_3 + \frac{1}{2} u_2' - \frac{2}{3} u_3 \beta' | \psi_2 \rangle \right\}, \quad \text{(3.30)} \]

For the second component we can rewrite Eq. (3.22) by considering Eq. (3.19) as:

\[ \langle \mathbf{v} \beta | \psi_2 \rangle = \sum_{\gamma} \frac{v^\prime}{\gamma} \frac{v^\prime}{\gamma} \sum_{\gamma''} \langle \mathbf{v} \beta | G_0 t \tilde{P} | \mathbf{v'} \beta' \rangle \sum_{\gamma''} \sum_{\alpha''} \delta^3(\mathbf{u}_1' - \mathbf{v}_1') \]
\[ \times \left\{ \delta^3\left(\mathbf{u}_2' + \frac{2}{3} \mathbf{v}_2' + \frac{2}{3} \mathbf{v}_3' \right) \delta^3\left(\mathbf{u}_3' + \frac{1}{2} \mathbf{v}_2' + \mathbf{v}_3' \right) \delta_{\gamma'_{1234} \gamma''_{1234}} \right. \]
\[ + \delta^3\left(\mathbf{u}_2' + \frac{2}{3} \mathbf{v}_3' + \frac{2}{3} \mathbf{v}_3' \right) \delta^3\left(\mathbf{u}_3' + \frac{1}{2} \mathbf{v}_2' - \mathbf{v}_3' \right) \delta_{\gamma'_{1234} \gamma''_{1234}} \left. \right\} \times \langle \mathbf{u'} \alpha' | \psi_1 \rangle \]
\[ + \sum_{\beta'} \langle \mathbf{v} \beta | G_0 t \tilde{P} | \mathbf{v'} \beta' \rangle \langle \mathbf{v'} \beta' | \psi_2 \rangle. \quad \text{(3.31)} \]

In this stage by integrating over \( \mathbf{u'} \) vector set and considering Eq. (3.16) we obtain:

\[ \langle \mathbf{v} \beta | \psi_2 \rangle = \sum_{\gamma'} \frac{\delta^3(\mathbf{v}_2 + \mathbf{v}_3') \delta^3(\mathbf{v}_3 - \mathbf{v}_1')}{E - \frac{v_2^2}{m} - \frac{v_3'^2}{2m} - \frac{v_3'^2}{3m}} \]
\[ \times \sum_{\alpha'} g_{\gamma' \alpha'} \langle \mathbf{v}_1' - \frac{2}{3} \mathbf{v}_2' + \frac{2}{3} \mathbf{v}_3' - \frac{1}{2} \mathbf{v}_2' - \mathbf{v}_3' \alpha' | \psi_1 \rangle \]
\[ - \sum_{\alpha'' \gamma''} g_{\gamma'' \alpha''} \delta_{\gamma'_{1234} \gamma''_{1234}} \langle \mathbf{v}_1' - \frac{2}{3} \mathbf{v}_2' - \frac{2}{3} \mathbf{v}_3' - \frac{1}{2} \mathbf{v}_2' + \mathbf{v}_3' \alpha' | \psi_1 \rangle \]
\[
+ \sum_{\beta''} g_{\gamma' \beta''} \langle \mathbf{v'} \beta' | \psi_2 \rangle \bigg\},
\]

and in the next stage by integrating over \( \mathbf{v}'_1 \) and \( \mathbf{v}'_2 \) vectors we arrive to the following equation:

\[
\langle \mathbf{v} \beta | \psi_2 \rangle = \int d^3 v'_3 \sum_{\gamma',\gamma''} g_{\gamma' \gamma''} \frac{\delta m''_{32} m_{21} \delta m''_{m_3} m'_{t_2}}{E - \frac{v^2_1}{m} - \frac{v^2_2}{2m} - \frac{v^2_3}{m}} \times \langle \mathbf{v}_1 m''_{m_1} m''_{m_2} m''_{t_1} m''_{t_2} t(\epsilon^*)| \mathbf{v}_3 m'_{m_3} m'_{s_4} m'_{t_3} m'_{t_4} \rangle \\
x \left\{ \begin{array}{l}
\sum_{\alpha'} g_{\gamma' \alpha'} \langle \mathbf{v}_3 \frac{2}{3} \mathbf{v}_2 + \frac{2}{3} \mathbf{v}'_3 \frac{1}{2} \mathbf{v}_2 - \mathbf{v}'_3 \alpha' | \psi_1 \rangle \\
- \sum_{\alpha'} g_{\gamma' \alpha'} \langle \mathbf{v}_3 \frac{2}{3} \mathbf{v}_2 - \frac{2}{3} \mathbf{v}'_3 \frac{1}{2} \mathbf{v}_2 + \mathbf{v}'_3 \alpha' | \psi_1 \rangle \\
+ \sum_{\beta''} g_{\gamma' \beta''} \langle \mathbf{v}_3 - \mathbf{v}_2, \mathbf{v}'_3 \beta' | \psi_2 \rangle \bigg\}.
\]

(3.33)

Therefore by considering the result of the permutation operators \( P_{12} \) and \( P_{34} \) action, Eq. (3.23), the matrix elements of the NN \( t \)-matrix as well as the Yakubovsky components, under the exchange of the labels \( m'_{s_3}, m'_{t_3} \) to \( m'_{s_4}, m'_{t_4} \) and reverse of it with changing \( \mathbf{v}'_3 \) to \( -\mathbf{v}'_3 \), can be obtained as:

\[
\langle \mathbf{v}_1 m''_{m_1} m''_{m_2} m''_{t_1} m''_{t_2} t(\epsilon^*)| \mathbf{v}_3 m'_{m_3} m'_{s_4} m'_{t_3} m'_{t_4} \rangle \\
\rightarrow \langle \mathbf{v}_1 m''_{m_1} m''_{m_2} m''_{t_1} m''_{t_2} t(\epsilon^*)| - \mathbf{v}_3 m'_{s_3} m'_{s_4} m'_{t_3} m'_{t_4} \rangle \\
= \langle \mathbf{v}_1 m''_{m_1} m''_{m_2} m''_{t_1} m''_{t_2} t(\epsilon^*) P_{12} P_{12} | - \mathbf{v}_3 m'_{s_4} m'_{s_3} m'_{t_4} m'_{t_3} \rangle \\
= \langle \mathbf{v}_1 m''_{m_1} m''_{m_2} m''_{t_1} m''_{t_2} t(\epsilon^*) P_{12} | \mathbf{v}_3 m'_{s_3} m'_{s_4} m'_{t_3} m'_{t_4} \rangle \\
(3.34)
\]

\[
\sum_{\alpha'} g_{\gamma' \alpha'} \langle \mathbf{v}_3 \frac{2}{3} \mathbf{v}_2 + \frac{2}{3} \mathbf{v}'_3 \frac{1}{2} \mathbf{v}_2 - \mathbf{v}'_3 \alpha' | \psi_1 \rangle \\
\rightarrow \sum_{\alpha'} g_{\gamma' \alpha'} \langle \mathbf{v}_3 \frac{2}{3} \mathbf{v}_2 - \frac{2}{3} \mathbf{v}'_3 \frac{1}{2} \mathbf{v}_2 + \mathbf{v}'_3 \alpha' | \psi_1 \rangle, \quad (3.35)
\]

\[
\sum_{\alpha'} g_{\gamma' \alpha'} \langle \mathbf{v}_3 \frac{2}{3} \mathbf{v}_2 - \frac{2}{3} \mathbf{v}'_3 \frac{1}{2} \mathbf{v}_2 + \mathbf{v}'_3 \alpha' | \psi_1 \rangle \\
\rightarrow \sum_{\alpha'} g_{\gamma' \alpha'} \langle \mathbf{v}_3 \frac{2}{3} \mathbf{v}_2 + \frac{2}{3} \mathbf{v}'_3 \frac{1}{2} \mathbf{v}_2 - \mathbf{v}'_3 \alpha' | \psi_1 \rangle, \quad (3.36)
\]

\[
\sum_{\beta''} g_{\gamma' \beta''} \langle \mathbf{v}_3 - \mathbf{v}_2, \mathbf{v}'_3 \beta' | \psi_2 \rangle \rightarrow \sum_{\beta''} g_{\gamma' \beta''} \langle \mathbf{v}_3 - \mathbf{v}_2, - \mathbf{v}'_3 \beta' | \psi_2 \rangle \\
\equiv \sum_{\beta''} (-)^{\epsilon_{34} + \epsilon_{34}} g_{\gamma' \beta''} \langle \mathbf{v}_3 - \mathbf{v}_2, - \mathbf{v}'_3 \beta' | P_{34} P_{34} | \psi_2 \rangle
\]
be obtained by rewriting Eqs. (3.30) and (3.39): the final representation of the three dimensional Yakubovsky integral equations can (B) of Ref. 23)

With these considerations we can rewrite the Eq. (3.33) as:

\[
\langle v | p \gamma \beta | \psi_2 \rangle = \int d^3v_3 \sum_{\gamma' \gamma''} g_{\beta \gamma''} \frac{\delta_{m_5' m_5} \delta_{m_4' m_4} \delta_{m_3' m_3} \delta_{m_2' m_2}}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_3^2}{m}} \times \langle v_1 m_5' m_4' m_3' m_2' | t(\epsilon^*) P_{12} | v_3 m_5' m_4' m_3' m_2' \rangle \\
\times (-) \left\{ \sum_{\alpha'} g_{\gamma' \alpha'} \langle v_3 \frac{2}{3} v_2 + \frac{2}{3} v_3 - \frac{1}{2} v_2 - v_3 \alpha' | \psi_1 \rangle \\
- \sum_{\alpha'} g_{\gamma'' \alpha''} \langle v_3 \frac{2}{3} v_2 - \frac{2}{3} v_3 - \frac{1}{2} v_2 + v_3 \alpha' | \psi_1 \rangle \\
+ \sum_{\beta'} g_{\gamma' \beta'} \langle v_3 - v_2 v_3 \beta' | \psi_2 \rangle \right\},
\]

(3.38)

and finally considering the Eqs. (3.33) and (3.38) together leads to:

\[
\langle v | p \gamma \beta | \psi_2 \rangle = \frac{1}{2} \int d^3v_3 \sum_{\gamma' \gamma''} g_{\beta \gamma''} \frac{\delta_{m_5' m_5} \delta_{m_4' m_4} \delta_{m_3' m_3} \delta_{m_2' m_2}}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_3^2}{m}} \times \langle v_1 m_5' m_4' m_3' m_2' | t(\epsilon^*) (1 - P_{12}) | v_3 m_5' m_4' m_3' m_2' \rangle \\
\times \left\{ \sum_{\alpha'} g_{\gamma' \alpha'} \langle v_3 \frac{2}{3} v_2 + \frac{2}{3} v_3 - \frac{1}{2} v_2 - v_3 \alpha' | \psi_1 \rangle \\
- \sum_{\alpha'} g_{\gamma'' \alpha''} \langle v_3 \frac{2}{3} v_2 - \frac{2}{3} v_3 - \frac{1}{2} v_2 + v_3 \alpha' | \psi_1 \rangle \\
+ \sum_{\beta'} g_{\gamma' \beta'} \langle v_3 - v_2 v_3 \beta' | \psi_2 \rangle \right\}.
\]

(3.39)

By introducing the physical representation of the NN $t$-matrix (see appendix (B) of Ref. 23):

\[
a \langle p m_{s_1} m_{s_2} m_{t_1} m_{t_2} | t(\epsilon) | p' m'_{s_1} m'_{s_2} m'_{t_1} m'_{t_2} \rangle_a \\
= \langle p m_{s_1} m_{s_2} m_{t_1} m_{t_2} | t(\epsilon) (1 - P_{12}) | p' m'_{s_1} m'_{s_2} m'_{t_1} m'_{t_2} \rangle,
\]

(3.40)

the final representation of the three dimensional Yakubovsky integral equations can be obtained by rewriting Eqs. (3.30) and (3.39):

\[
\langle u | p \alpha | \psi_1 \rangle = \frac{1}{E - \frac{u_1^2}{m} - \frac{u_2^2}{4m} - \frac{u_3^2}{3m}}
\]
\[ \langle v | v' \rangle = \frac{1}{E - \frac{u_1^2}{m} - \frac{u_2^2}{m} - \frac{u_3^2}{m}} \]

\[ \times \int d^3u' \sum_{\gamma', \gamma''} g_{\gamma'' \gamma'} \delta_m \delta_{m_{s_1}, m_{s_2}, m_{t_1}, m_{t_2}} |t(e)| \frac{1}{2} u_2 - u_2' m_{s_2}, m_{s_3}, m_{t_2}, m_{t_3} a \]

\[ \times \left\{ \sum_{\alpha''} g_{\gamma'' \alpha''} \langle u_2 + \frac{1}{2} u_2' u_2', u_3 \alpha'' | \psi_1 \rangle - \sum_{\alpha'} g_{\alpha'' \alpha'} \langle u_2 + \frac{1}{2} u_2' \frac{1}{3} u_2' + \frac{8}{9} u_3 | \psi_1 \rangle \right. \]

\[ + \sum_{\beta'} g_{\gamma' \beta'} \langle u_2 + \frac{1}{2} u_2' - u_2' - \frac{2}{3} u_3 \frac{1}{2} u_2' - \frac{2}{3} u_3 \beta' | \psi_2 \rangle \right\} \]
The matrix elements of the second term can be evaluated by inserting the suitable completeness relations as:

\[
\langle \mathbf{u} \alpha | t G_0 V_{123}^{(3)} | \Psi \rangle = \sum_{\gamma'} \sum_{\gamma''} \sum_{\alpha''} u''^m \\
\times \langle \mathbf{u} \gamma' | t G_0 | u'' \gamma'' \rangle \langle \gamma'' | \alpha'' \rangle \langle \alpha'' | V_{123}^{(3)} | \Psi \rangle \\
\equiv \sum_{\gamma'} \sum_{\gamma''} \sum_{\alpha''} u''^m \frac{g_{\gamma' \alpha} g_{\gamma'' \alpha''}}{E - \frac{u''_1^2}{m} - \frac{3u''_2^2}{4m} - \frac{2u''_3^2}{3m}} \\
\times \langle \mathbf{u} \gamma' | t | u'' \gamma'' \rangle \langle \alpha'' | V_{123}^{(3)} | \Psi \rangle,
\]

after evaluating the matrix elements of the NN \( t \)-matrix, Eq. (3.9), and integrating over \( u''^m \) and \( u_3''^m \) vectors, we obtain:

\[
\langle \mathbf{u} \alpha | t G_0 V_{123}^{(3)} | \Psi \rangle = \sum_{\gamma', \gamma'', \alpha''} \int d^3 \mathbf{u}_1'' \frac{g_{\gamma' \alpha} g_{\gamma'' \alpha''}}{E - \frac{u''_1^2}{m} - \frac{3u''_2^2}{4m} - \frac{2u''_3^2}{3m}} \\
\times \delta_{m'_3 m''_3} \delta_{m'_4 m''_4} \delta_{m'_5 m''_5} \delta_{m'_6 m''_6} \\
\times \langle \mathbf{u}_1 m'_s m''_s m'_t m''_t | t(\epsilon) P_{12} | \mathbf{u}_1'' m''_s m''_s m''_t m''_t \rangle \\
\times \langle \mathbf{u}_1'' \mathbf{u}_2 \mathbf{u}_3 \alpha'' | V_{123}^{(3)} | \Psi \rangle,
\]

By using Eq. (5.9), the symmetry property of the 3NF and the anti-symmetry property of the total wave function under exchange of nucleons 1 and 2, one can write:

\[
\langle \mathbf{u} \alpha | t G_0 V_{123}^{(3)} | \Psi \rangle = \sum_{\gamma', \gamma'', \alpha''} \int d^3 \mathbf{u}_1'' \frac{g_{\gamma' \alpha} g_{\gamma'' \alpha''}}{E - \frac{u''_1^2}{m} - \frac{3u''_2^2}{4m} - \frac{2u''_3^2}{3m}} \\
\times \delta_{m'_3 m''_3} \delta_{m'_4 m''_4} \delta_{m'_5 m''_5} \delta_{m'_6 m''_6} \\
\times \langle \mathbf{u}_1 m'_s m''_s m'_t m''_t | t(\epsilon) P_{12} | \mathbf{u}_1'' m''_s m''_s m''_t m''_t \rangle \\
\times \left( -(-)^{s'_3 + s''_3} \right) \langle \mathbf{u}_1'' \mathbf{u}_2 \mathbf{u}_3 \alpha'' | V_{123}^{(3)} | \Psi \rangle,
\]

under the exchange of the labels \( m''_s, m''_t \), \( m''_s, m''_t \). Reverse of it and changing \( \mathbf{u}_1'' \) to \( -\mathbf{u}_1'' \) we find:

\[
\langle \mathbf{u} \alpha | t G_0 V_{123}^{(3)} | \Psi \rangle = \sum_{\gamma', \gamma'', \alpha''} \int d^3 \mathbf{u}_1'' \frac{g_{\gamma' \alpha} g_{\gamma'' \alpha''}}{E - \frac{u''_1^2}{m} - \frac{3u''_2^2}{4m} - \frac{2u''_3^2}{3m}} \\
\times \delta_{m'_3 m''_3} \delta_{m'_4 m''_4} \delta_{m'_5 m''_5} \delta_{m'_6 m''_6} \\
\times \langle \mathbf{u}_1 m'_s m''_s m'_t m''_t | -t(\epsilon) P_{12} | \mathbf{u}_1'' m''_s m''_s m''_t m''_t \rangle \\
\times \langle \mathbf{u}_1'' \mathbf{u}_2 \mathbf{u}_3 \alpha'' | V_{123}^{(3)} | \Psi \rangle.
\]
Now we consider Eqs. (4.3) and (4.5) together to achieve:

\[
\langle \mathbf{u} \alpha | tG_0 V_{123}^{(3)} | \Psi \rangle = \frac{1}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}} \times \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \\
\times \langle \mathbf{u}_1 m_s^t m_s^t m_t^t | t(\epsilon)(1 - P_{12}) | \mathbf{u}_1 m_s^t m_s^t m_t^t \rangle \\
\times \langle \mathbf{u}_1 m_s^t m_s^t m_t^t | V_{123}^{(3)} | \Psi \rangle,
\]  

(4.6)

by applying the introduction of the anti-symmetrized NN t-matrix, Eq. (3.40), we can rewrite the final representation of the three dimensional Yakubovsky integral equations, Eq. (3.41), as:

\[
\langle \mathbf{u} \alpha | \psi_1 \rangle = \frac{1}{E - \frac{u_1^2}{m} - \frac{3u_2^2}{4m} - \frac{2u_3^2}{3m}} \times \left[ \int d^3 u_2' \sum_{\gamma',\gamma''} g_{\gamma' \gamma''} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \\
\times \left\{ \sum_{\alpha''} g_{\gamma' \gamma''} \langle \mathbf{u}_2 + \frac{1}{2} \mathbf{u}_2' \mathbf{u}_2 \mathbf{u}_3 \alpha'' | \psi_1 \rangle \\
- \sum_{\alpha''} g_{\gamma' \gamma''} \langle \mathbf{u}_2 + \frac{1}{2} \mathbf{u}_2' - \frac{1}{2} \mathbf{u}_3 | \psi_1 \rangle \right\} \\
+ \left\{ \langle \mathbf{u} \alpha | V_{123}^{(3)} | \Psi \rangle \\
+ \frac{1}{2} \sum_{\gamma',\gamma'',\alpha''} g_{\gamma' \gamma''} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \\
\times \langle \mathbf{u}_1 m_s^t m_s^t m_t^t | t(\epsilon)(1 - P_{12}) | \mathbf{u}_1 m_s^t m_s^t m_t^t \rangle_a \langle \mathbf{u}_1' \mathbf{u}_2 \mathbf{u}_3 \alpha'' | V_{123}^{(3)} | \Psi \rangle \right\} \right],
\]

\[
\langle \mathbf{v} \beta | \psi_2 \rangle = \frac{1}{E - \frac{v_1^2}{m} - \frac{v_2^2}{2m} - \frac{v_3^2}{3m}} \times \left[ \int d^3 v_3' \sum_{\gamma',\gamma''} g_{\beta' \gamma''} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \delta_{m_s^t m_s^t} \\
\times \langle \mathbf{v}_1 m_s^t m_s^t m_t^t | t(\epsilon^*) | \mathbf{v}_3' m_s^t m_s^t m_t^t \rangle_a \langle \mathbf{v}_1 m_s^t m_s^t m_t^t | V_{123}^{(3)} | \Psi \rangle \right].
\]
\[ \sum_{\alpha'} g_{\gamma'\alpha'} \langle v_3 \frac{2}{3} v_2 + \frac{2}{3} v_3' \frac{1}{2} v_2 - v_3' \alpha' | \psi_1 \rangle - \sum_{\alpha'} g_{\gamma'1243\alpha'} \langle v_3 \frac{2}{3} v_2 - \frac{2}{3} v_3' \frac{1}{2} v_2 + v_3' \alpha' | \psi_1 \rangle + \sum_{\beta'} g_{\gamma'\beta'} \langle v_3 - v_2 v_3' \beta' | \psi_2 \rangle \] 

To represent the generality of our 3D formalism we can simplify the Eq. (4.7) to the bosonic case by switching off the spin-isospin quantum numbers, see Ref. 19.)

In order to show the efficiency of our formalism we have also chosen a realistic 2π-exchange 3NF, i.e. TM, to evaluate the matrix elements of \( \langle u_\alpha | V^{(3)}_{123} | \Psi \rangle \). As a matter of reference we present this formalism in appendix (A) to indicate the simplicity of 3D representation.

§5. The Number of Coupled Yakubovsky Equations in both the PW and the 3D Formalisms

In this section we discuss about the number of coupled Yakubovsky equations in both the 3D and the PW approaches. Already in the PW representation of the 3N bound state equations the number of channels must be high, e.g. of the order of 34, in order to achieve reasonably well enough converged energy eigenvalues.24) In contrast to the 3N system the number of channels for the 4N bound state \( N = N_\alpha + N_\beta \), where \( N_\alpha \) and \( N_\beta \) are the numbers of \( \alpha \) and \( \beta \) quantum number combinations respectively, is in principle unlimited even if the 2N interaction is assumed to act only in a certain 2N states. Nevertheless the approach to that unlimited number can be classified for instance in the following possibly useful manner. For fixed \( l_4 \) and fixed total quantum numbers \( J \) and \( T \) the numbers of \( \alpha \)-states is strictly finite once the 2N interaction is assumed to act only up to \( j_{12}^{\text{max}} \). Correspondingly, the number of \( \beta \)-states is strictly finite once \( l_2 \) is fixed and again the 2N interaction is assumed to be zero in states for \( j_{12}, j_{34} < j_{13}^{\text{max}} \). We display examples for those maximum \( N_\alpha \) and \( N_\beta \) values in Table 5. Thus even if assuming only \( j_{13}^{\text{max}} = 1 \), and restricting \( l_2 \) and \( l_4 \) to be at most 1, one leads to \( N = 60 \) channels, while the total isospin is restricted to be zero.9)

In Tables II and III we present the number of the spin-isospin states for both kind of the Jacobi coordinates, \( \alpha \) and \( \beta \), as well as the number of coupled Yakubovsky equations in our realistic 3D formalism. Since the angular momentum quantum numbers, i.e. \( l_{12}, l_3, l_4 \) and \( l_{12}, l_2, l_{34} \), do not appear explicitly in our formalism, therefore the number of coupled equations which are fixed according to the spin-isospin states are strongly reduced. This is an indication that the present formalism automatically consider all partial waves without any truncation on the space part. Considering the spin-isospin degrees of freedom for \( ^4\text{He} \) one leads to \( 8, 38, 58, 18, 87, 132, 22, 106 \) and \( 161 \) coupled equations for different combinations of total spin-isospin states \( (S - T): (0 - 0), (1^* - 0), (2^* - 0), (0 - 1^*), (1^* - 1^*), (2^* - 1^*) \),
Table I. The number of partial wave channels contributing to both kind of the 4N Jacobi coordinates for $J^\pi = 0^+$ dependent on maximal values of intercluster orbital angular momenta $l_i$ and total two-body angular momenta $j_{ij}$. For (a) and (b) parts the total isospin is restricted to $T = 0$ and for (c) part $T = 0, 1, 2$. $N$ is the total number of channels, where $N_\alpha$ and $N_\beta$ are the number of channels corresponding to $3 + 1$ and $2 + 2$ partitions. The results of part (a) are given according to the notation of Ref.,\cite{9} where $j_{12}, j_{34} \leq j_{ij}^{\text{max}}$ and $l_2, l_4 \leq l_i^{\text{max}}$, and the results of parts (b) and (c) are given according to the notation of Ref.,\cite{14} where $j_{12}, j_{34} \leq j_{ij}^{\text{max}}$, $l_2, l_3, l_4 \leq l_i^{\text{max}}$ and $l_1 + l_3 + l_4 + l_{12} + l_{34} + l_2 \leq l_{\text{sum}}^{\text{max}}$. It should be mentioned that for (c) part, $N_\alpha = 4200$ and $N_\beta = 2000$.

| $l_i^{\text{max}}$ | $j_{ij}^{\text{max}}$ | $N_\alpha$ | $N_\beta$ | $N$ |
|-------------------|------------------------|-----------|-----------|-----|
| $l_i^{\text{max}} = 0$ | 1 | 10 | 10 | 20 |
|                   | 2 | 18 | 18 | 36 |
|                   | 3 | 26 | 26 | 52 |
| $l_i^{\text{max}} = 1$ | 1 | 34 | 26 | 60 |
|                   | 2 | 66 | 58 | 124 |
|                   | 3 | 98 | 90 | 188 |
| $l_i^{\text{max}} = 2$ | 1 | 62 | 34 | 96 |
|                   | 2 | 130 | 98 | 228 |
|                   | 3 | 202 | 170 | 372 |
| $l_i^{\text{max}} = 3$ | 1 | 90 | 34 | 124 |
|                   | 2 | 198 | 122 | 320 |
|                   | 3 | 322 | 242 | 564 |
| $l_i^{\text{max}} = 4$ | 1 | 118 | 34 | 152 |
|                   | 2 | 266 | 130 | 396 |
|                   | 3 | 446 | 290 | 736 |
| (b) Ref.\cite{13} | $j_{ij}^{\text{max}} = 6$ | $l_i^{\text{max}} = 8$ | $l_{\text{sum}}^{\text{max}} = 12$ | $T = 0$ | $N = 1572$ |
| (c) Ref.\cite{14} | $j_{ij}^{\text{max}} = 6$ | $l_i^{\text{max}} = 8$ | $l_{\text{sum}}^{\text{max}} = 14$ | $T = 0, 1, 2$ | $N = 6200$ |

$(0 - 2^*), (1^* - 2^*)$ and $(2^* - 2^*)$ respectively. The star superscript indicates all spin or isospin states that we have taken into account up to a specific value. It is clear that in the 3D formalism for a fully charge dependent calculation there is only 161 coupled equations, whereas in PW approach after truncation of the Hilbert space to $T = 0$ there is 1572 coupled equation, see part (b) of Table 5 and 6200 coupled equations for a fully charge dependent, see part (c) of Table 5. So our 3D formalism leads to a very small number of coupled equations in comparison with the very large number of coupled equations in the truncated PW formalism. However, it should be mentioned that our formulation leads to coupled integral equations which depend on three vector variables for the amplitudes, whereas the
Table II. The number of the spin-isospin states for both kind of the 4N Jacobi coordinates in a realistic 3D formalism. \( N_{\alpha S} (N_{\beta S}) \) and \( N_{\alpha T} (N_{\beta T}) \) are the number of the spin (isospin) states for 3 + 1 and 2 + 2 partitions correspondingly.

(a) 3 + 1 partitions

| \((s_{12} \frac{1}{2}) s_{123} \frac{1}{2}) SM_S\) | \(S = 0\) | \(S = 0, 1\) | \(S = 0, 1, 2\) |
|---|---|---|---|
| \((0 \frac{1}{2} \frac{1}{2} \frac{1}{2}) 0\) | 1 | 1+0 | 1+0+0 |
| \((0 \frac{1}{2} \frac{1}{2} \frac{1}{2}) 1\) | 0 | 0+3 | 0+3+0 |
| \((1 \frac{1}{2} \frac{1}{2} \frac{1}{2}) 0\) | 1 | 1+0 | 1+0+0 |
| \((1 \frac{1}{2} \frac{1}{2} \frac{1}{2}) 1\) | 0 | 0+3 | 0+3+0 |
| \((1 \frac{1}{2} \frac{1}{2} \frac{1}{2}) 1\) | 0 | 0+3 | 0+3+0 |
| \((1 \frac{1}{2} \frac{1}{2} \frac{1}{2}) 2\) | 0 | 0+0 | 0+0+5 |

| \((t_{12} \frac{1}{2}) t_{123} \frac{1}{2}) T0\) | \(T = 0\) | \(T = 0, 1\) | \(T = 0, 1, 2\) |
|---|---|---|---|
| \((0 \frac{1}{2} \frac{1}{2}) 0\) | 1 | 1+0 | 1+0+0 |
| \((0 \frac{1}{2} \frac{1}{2}) 1\) | 0 | 0+1 | 0+1+0 |
| \((1 \frac{1}{2} \frac{1}{2}) 0\) | 1 | 1+0 | 1+0+0 |
| \((1 \frac{1}{2} \frac{1}{2}) 1\) | 0 | 0+1 | 0+1+0 |
| \((1 \frac{1}{2} \frac{1}{2}) 1\) | 0 | 0+1 | 0+1+0 |
| \((1 \frac{1}{2} \frac{1}{2}) 2\) | 0 | 0+0 | 0+0+1 |

\(N_{\alpha S} = 2\) \(N_{\alpha T} = 11\) \(N_{\alpha T} = 16\)

(b) 2 + 2 partitions

| \((s_{12} s_{34}) SM_S\) | \(S = 0\) | \(S = 0, 1\) | \(S = 0, 1, 2\) |
|---|---|---|---|
| \((0 0) 0\) | 1 | 1+0 | 1+0+0 |
| \((0 1) 1\) | 0 | 0+3 | 0+3+0 |
| \((1 0) 1\) | 0 | 0+3 | 0+3+0 |
| \((1 1) 0\) | 1 | 1+0 | 1+0+0 |
| \((1 1) 2\) | 0 | 0+0 | 0+0+5 |

\(N_{\beta S} = 2\) \(N_{\beta S} = 8\) \(N_{\beta S} = 13\)

| \((t_{12} t_{34}) T0\) | \(T = 0\) | \(T = 0, 1\) | \(T = 0, 1, 2\) |
|---|---|---|---|
| \((0 0) 0\) | 1 | 1+0 | 1+0+0 |
| \((0 1) 1\) | 0 | 0+1 | 0+1+0 |
| \((1 0) 1\) | 0 | 0+1 | 0+1+0 |
| \((1 1) 0\) | 1 | 1+0 | 1+0+0 |
| \((1 1) 2\) | 0 | 0+0 | 0+0+1 |

\(N_{\beta T} = 2\) \(N_{\beta T} = 4\) \(N_{\beta T} = 5\)

PW formulation after truncation leads to a finite number of coupled equations in three variables for the amplitudes.

The coupled Yakubovsky equations, Eq. (4.7), represent a set of three dimensional homogenous integral equations, which after discretization turns into a huge matrix eigenvalue equation. In order to solve these coupled integral equations directly without employing the PW projections, one has to define a coordinate system.
Table III. The number of coupled Yakubovsky equations for $\alpha$ particle in a realistic 3D formalism according to the spin-isospin states $(S-T)$ that we have taken into account. $N = N_\alpha + N_\beta$ is the total number of coupled Yakubovsky equations, where $N_\alpha = N_\alpha^S \times N_\alpha^T$ and $N_\beta = N_\beta^S \times N_\beta^T$ are the number of $3+1$ and $2+2$ states correspondingly. The star superscript indicates all spin or isospin states that we have taken into account up to a specific value.

| $(S-T)$   | $N_\alpha$ | $N_\beta$ | $N$   |
|-----------|------------|------------|-------|
| $(0-0)$   | 4          | 4          | 8     |
| $(1^*-0)$ | 22         | 16         | 38    |
| $(2^*-0)$ | 32         | 26         | 58    |
| $(0-1^*)$ | 10         | 8          | 18    |
| $(1^*-1^*)$ | 55   | 32         | 87    |
| $(2^*-1^*)$ | 80   | 52         | 132   |
| $(0-2^*)$ | 12         | 10         | 22    |
| $(1^*-2^*)$ | 66   | 40         | 106   |
| $(2^*-2^*)$ | 96   | 65         | 161   |

According to experience of $^3$H binding energy calculations, Ref., $^{23}$ it is convenient to choose the spin polarization direction parallel to the $z$-axis and express the momentum vectors in this coordinate system. As indicated in Ref.,$^{19}$ generally one needs six independent variables, three magnitudes of the Jacobi momentum vectors and three angle variables, to uniquely specify the geometry of the three vectors. But in contrast to the bosonic case, Ref.,$^{18}$ one does not have this freedom to choose the third vectors, i.e. $\mathbf{u}_3$ and $\mathbf{v}_3$, parallel to the $z$-axis and second vectors, i.e. $\mathbf{u}_2$ and $\mathbf{v}_2$, in the $x-z$ plane. So the angular dependencies of spin-isospin-dependent Yakubovsky components will be more complicated rather than bosonic ones. The dependence on the continuous momentum and angle variables should be replaced in the numerical treatment by a dependence on certain discrete values. Let the number of these discrete points be denoted by $N_{\text{jacobi}}$ and $N_{\text{angle}}$ corresponding to momentum and angle variables, the dimension of the eigenvalue problem in both PW and 3D approaches is:

$$N^{PW} = N_{\text{jacobi}}^3 \times (N_\alpha^{PW} + N_\beta^{PW})$$
$$N^{3D} = N_{\text{jacobi}}^3 \times N_{\text{angle}}^3 \times (N_\alpha^{3D} + N_\beta^{3D})$$

(5.1)

It should be indicated that in the formulation of the 4N bound state we can consider the parity and the time reversal invariance to cut down the dimension of the Yakubovsky components, see appendix (B). So according to our experience in four-bosonic calculations and also by considering the parity and the time reversal invariance one needs in 4N realistic calculations thirty grid points for Jacobi momentum variables and ten grid points for angle variables. By these number of mesh grids the dimension of the eigenvalue problem for a fully charge dependent calculation, i.e. for total spin-isospin states $(S^n - T^n) = (2^* - 2^*)$, will be:

$$N^{PW} = 30^3 \times (4200 + 2000) \approx 1.7 \times 10^8$$
$$N^{3D} = 30^3 \times 10^3 \times (96 + 65) \approx 4.4 \times 10^9$$

(5.2)

Thus, the price for the smaller number of equations in 3D representation is the higher
dimensionality of the integral equations. In other words, algebraic simplification is achieved by a more involved numerical scheme. As indicated in ref18 for bosonic case these dimensions reduce to the values $N_{PW}^{3D} \simeq 2.7 \times 10^6$ and $N_{3D}^{3N} \simeq 2.7 \times 10^7$. As it is clear from the close similarity of dimension values and also based on the experience of simpler bosonic calculations, we expect that for more rigorous nucleonic case the numerical calculation could be achievable.

§6. Summary and Outlook

We propose a new representation of three dimensional Yakubovsky equations for the four-nucleon bound state including the spin and isospin degrees of freedom in the momentum space. This formalism stays closely to the bosonic structure where the spin and isospin degrees of freedom are ignored. This is an important step forward since the formulation based on partial wave decomposition, which includes the spin and isospin degrees of freedom, after truncation leads to two coupled sets of a finite number of coupled equations in three variables for the amplitudes. In contrast our 3D formulation leads to two coupled sets of a strictly finite number of equations in three vector variables for the amplitudes. The comparison of 3D and PW formalisms shows that our 3D formalism avoids the very involved angular momentum algebra occurring for the permutations and transformations and it is more efficient and less cumbersome for considering the three-nucleon forces. This formalism enables us to handle realistic 2N and 3N potentials with all their complexity in four-nucleon bound state calculations. This work provides the necessary formalism for the calculation of four-nucleon binding energy which is under preparation.

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Appendix A

The Evaluation of $\langle u_{\alpha} | V_{123}^{(3)} | \Psi \rangle$ for the TM 2$\pi$-exchange 3NF

A.1. Preparation of the TM 3NF for the 3D Representation

For the evaluation of the coupled Yakubovsky equations, Eq. [4-7], matrix elements of the form $\langle u_{\alpha} | V_{123}^{(3)} | \Psi \rangle$ need to be calculated. In this section we evaluate these matrix elements in our 3D approach for the TM 2$\pi$-exchange 3NF. In the notation from Fig. [H] this force is given by[26]

$$V_{123}^{(3)} = V_0 \sigma_2 \cdot Q' \sigma_1 \cdot Q \frac{F(Q^2)}{Q^2 + m_\pi^2} \frac{F(Q'^2)}{Q'^2 + m_\pi^2} \times \left[ \tau_1 \cdot \tau_2 \left( A + B \cdot Q \cdot Q' + C \left( Q^2 + Q'^2 \right) \right) + D \tau_3 \cdot \tau_1 \times \tau_2 \sigma_3 \cdot Q \times Q' \right],$$

(A.1)

with the momentum transfers $Q$ and $Q'$. The $\sigma_i$'s are Pauli spin matrices and the form factors are chosen to be $F(Q^2) = \left( \frac{Q^2 - m_\pi^2}{\Lambda^2} \right)^2$ with the cut-off parameter $\Lambda$. One
The constants \( A \) and \( B \) are the scalar triple product of the spin and momentum transfer vectors, \( v \), \( e \), respectively. The momentum transfers are given by operators in the form that can be evaluated easily in our 3D representation as:

\[
Q = k_1 - k'_1 \equiv \left\{ (+u_1 - \frac{1}{2}u_2) - (+u'_1 - \frac{1}{2}u'_2) \right\},
\]

\[
Q' = k_2 - k_2' \equiv \left\{ (-u'_1 - \frac{1}{2}u'_2) - (-u_1 - \frac{1}{2}u_2) \right\},
\]

as indicated in Fig. In order to be able to evaluate the TM 3NF matrix elements we should evaluate the scalar product of the spin and momentum transfer vectors, \( \sigma \cdot Q \) and \( \sigma \cdot Q' \), the scalar product of both the momentum transfer vectors, \( Q \cdot Q' \), the scalar triple product of the spin and momentum transfer vectors, \( \sigma \cdot Q \times Q' \), as well as the isospin operators. Toward this aim we can rewrite the spin-space operators in the form that can be evaluated easily in our 3D representation as:

\[
\begin{align*}
\sigma_1 \cdot Q &= Q \sigma_1 \cdot \hat{Q}, \\
\sigma_2 \cdot Q' &= Q' \sigma_2 \cdot \hat{Q}, \\
Q \cdot Q' &= QQ' \gamma, \\
\sigma_3 \cdot Q \times Q' &= |Q \times Q'| \sigma_3 \cdot \hat{Q} \times \hat{Q}' = QQ' a \left\{ \frac{3ia}{4\gamma} - \frac{i}{2\gamma a} (\sigma_3 \cdot \hat{Q})² + \frac{i}{2a} \sigma_3 \cdot \hat{Q}' \sigma_3 \cdot \hat{Q} \right. \\
&\left. + \frac{i}{8\gamma a} (\sigma_3 \cdot \hat{Q})² (\sigma_3 \cdot \hat{Q})² - \frac{i}{2\gamma a} (\sigma_3 \cdot \hat{Q})² \right\},
\end{align*}
\]

where

\[
\gamma = \hat{Q} \cdot \hat{Q}' \quad a = \sqrt{1 - \gamma²}.
\]

The scalar product of the spin-momentum vectors can be evaluated as:

\[
\langle \hat{z} m_s' | \sigma \cdot \hat{Q} | \hat{z} m_s'' \rangle = \sum_{m_s} m_s D^{\frac{1}{2}}_{m_s' m_s} (\hat{Q}) D^{\frac{1}{2}+*}_{m_s' m_s} (\hat{Q}) = \delta_{m_s' m_s},
\]

where \( D^{\frac{1}{2}}_{m_s' m_s} \) is Wigner D-function which is defined generally as \( D^{\frac{1}{2}}_{m_s' m_s} (\hat{q}) = \langle \hat{z} m_s' | \hat{q} m_s'' \rangle \). The application of the TM 3NF to the total wave function \( |\Psi\rangle \) can be considered as sum of the the four independent terms:

\[
|\psi\rangle = V^{(3)}_{123} |\psi\rangle = \sum_{i=A}^{D} V^{(i)}_{31} I^{(i)} V^{(i)}_{23} |\psi\rangle = \sum_{i=A}^{D} |\psi^{(i)}\rangle,
\]

where we have abbreviated the isospin operators by \( I^{(i)} \):

\[
I^{(i)} = \begin{cases} 
\tau_1 \cdot \tau_2 & i = A, B, C \\
\tau_3 \cdot \tau_1 \times \tau_2 & i = D.
\end{cases}
\]
A2. Explicit 3D Evaluation of the A-, B-, C- and D-Terms

In the following we evaluate the matrix elements of \( \langle u\alpha |\psi^i \rangle \) terms. From Fig. 1 we see that \( V^{(3)}_{123} \) can be split into two parts and each part contains a meson exchange. The first meson is exchanged in the subsystem (31), where it is called for convenience subsystem 2, and the second is exchanged in subsystem (23), is called 1. Since the structure of the 3BF is specified by two momentum transfers of consecutive meson exchanges, it is convenient to insert a complete set of states of the type 2 between \( V^{(3)}_{123} \) and \( |\Psi\rangle \) and another complete set of states of type 1 between the two meson exchanges. Then the matrix element of \( V^{(3)}_{123} \) is rewritten as

\[
3\langle u\alpha |\psi^i \rangle = \sum_{\alpha'} 3\langle u\alpha |u'\alpha' \rangle_1 \times \sum_{\alpha''} \langle u'\alpha' |V^{(i)}_{31}|u''\alpha'' \rangle_1 \times \sum_{\alpha'''} \langle u''\alpha'' |I^{(i)}_{23}|u'''\alpha''' \rangle_2 \times 2\langle u'''\alpha''' |\psi \rangle,
\]

(A.8)

Here the subscripts 1, 2, 3 of the bra and ket vectors stand for the different types of three-nucleon coordinate systems of (3 + 1)-type fragmentation \((ijk, 4)\). The coordinate transformation from the system of type 1 to one of type 3 can be evaluated explicitly as:

\[
3\langle u\alpha |u'\alpha' \rangle_1 = g_{\alpha_3 \alpha_1} \delta^3(u'_1 + \frac{1}{2}u_1 + \frac{3}{4}u_2)\delta^3(u'_2 - u_1 + \frac{1}{2}u_2)\delta^3(u'_3 - u_3).\]

(A.9)

The matrix elements of the isospin coordinate transformations can be rewritten as:

\[
1\langle u''\alpha'' |I^{(i)}|u'''\alpha''' \rangle_2 = 1\langle u''\alpha''_S |u'''\alpha'''_S \rangle_2 \times 1\langle \alpha''_T |I^{(i)}|\alpha'''_T \rangle_2,
\]

(A.10)

where

\[
1\langle u''\alpha''_S |u'''\alpha'''_S \rangle_2 = g_{\alpha''_1 \alpha''_2}^S \delta^3(u''_1 + \frac{1}{2}u_1 + \frac{3}{4}u_2)\delta^3(u''_2 - u_1 + \frac{1}{2}u_2)\delta^3(u''_3 - u_3).
\]

(A.11)

The matrix elements of the coordinate transformations \( 1\langle \alpha''_T |I^{(i)}|\alpha'''_T \rangle_2 \) are given in Ref.\textsuperscript{12} The isospin matrix elements have been derived in Ref.\textsuperscript{29} for the 3N system.
and are given below for the 4N system for completeness:

\[
1\langle \alpha''_T | I^{(i)} | \alpha''_T \rangle_2 = \delta_{T''T'} \delta_{M''_{1,2}M_{12}''} \delta_{\nu''_{12}, \nu_{12}''} (-6) (-)^{2\nu_{23}''} \sqrt{i^{\mu}_{23}'' i^{\rho}_{31}''}
\times \left\{ \begin{array}{c|c|c}
\frac{1}{2} & \frac{1}{2} & \frac{\mu''_{12}}{i^{\mu}_{23}''} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array} \right\}
\]

\[
i = A, B, C
\]

\[
1\langle \alpha''_T | V^{(i)}_{31} | \alpha''_T \rangle_1 = \delta^3 (u''_1 - u''_3) \delta^3 (u''_3 - u''_4) \delta_{\alpha''_1 \alpha''_2} 1\langle \alpha''_S | \alpha''_S \rangle, \quad (A.15)
\]

and the spin-space parts can be more simplified for the \(A\)-, \(B\)-, \(C\)- and \(D\)-terms separately as:

\[
1\langle \alpha''_S | V^{(A,B,C)}_{31} | \alpha''_S \rangle_1 = \sum_{\gamma''_S} \sum_{\gamma''_S} g^{1}_S \gamma''_S \delta_{\alpha''_1 \alpha''_2} \delta_{m''_{23} m''_{31}} \delta_{m''_{s2} m''_{s3}} \delta_{m''_{s3} m''_{s4}} \times
1\langle \alpha''_S | \alpha''_S \rangle_1, \quad (A.14)
\]

\[
1\langle \alpha''_S | V^{(D)}_{31} | \alpha''_S \rangle_1 = \sum_{\gamma''_S} \sum_{\gamma''_S} g^{1}_S \gamma''_S \delta_{\alpha''_1 \alpha''_2} \delta_{m''_{23} m''_{31}} \delta_{m''_{s2} m''_{s3}} \delta_{m''_{s3} m''_{s4}} \times
1\langle \alpha''_S | \alpha''_S \rangle_1, \quad (A.15)
\]

where the \(A\)-, \(B\)- and \(C\)-terms can be evaluated as:

\[
1\langle \alpha''_1 m''_{s1} | V^{(A)}_{31} | \alpha''_2 m''_{s1} \rangle_1 = 1\langle \alpha''_1 m''_{s1} | F(Q^2) \frac{Q_1 \cdot \vec{Q}}{Q^2 + m^2_{\pi}} Q | \alpha''_2 m''_{s1} \rangle_1
\]

\[
= \frac{F((\vec{u''_1} - \vec{u''_2}))^2}{(\vec{u''_2} \cdot \vec{u''_2}) + m^2_{\pi}} |\vec{u''_1} - \vec{u''_2}| O^{(\vec{u''_1} - \vec{u''_2})}_{m''_{s1} m''_{s1}}, \quad (A.16)
\]

\[
1\langle \alpha''_1 m''_{s1} | V^{(B)}_{31} | \alpha''_2 m''_{s1} \rangle_1 = 1\langle \alpha''_1 m''_{s1} | F(Q^2) \frac{Q_1 \cdot \vec{Q}}{Q^2 + m^2_{\pi}} Q | \alpha''_2 m''_{s1} \rangle_1
\]

\[
= \frac{F((\vec{u''_1} - \vec{u''_2}))^2}{(\vec{u''_2} \cdot \vec{u''_2}) + m^2_{\pi}} |\vec{u''_1} - \vec{u''_2}| O^{(\vec{u''_1} - \vec{u''_2})}_{m''_{s1} m''_{s1}}, \quad (A.17)
\]
and for the different parts of the D-term, see Eq. (A.3), we obtain:

\[
1 \langle \mathbf{u}_2^{m_1} | V_{31}^{(D)} | \mathbf{u}_2^{m_1} m_{s_3} \rangle_{1} = 1 \langle \mathbf{u}_2^{m_1} m_{s_3} | V_{31}^{(D)} | \mathbf{u}_2^{m_1} m_{s_3} \rangle_{1}
\]

(A.18)

The matrix elements of \( V_{23} \) can be evaluated by following the same algorithm as above. So we obtain:

\[
2 \langle \mathbf{u}_2^{m_1} m_{s_2} | V_{23}^{(A,B,C)} | \mathbf{u}_2^{m_1} m_{s_2} \rangle_2 = \sum_{\gamma_3} g_{\gamma_3}^{m_1} \sum_{\gamma_3} g_{\gamma_3}^{m_1} \delta_{m_1, m_{s_2}} \delta_{m_{s_1}, m_{s_2}} \delta_{m_{s_3}, m_{s_3}} \delta_{m_{s_4}, m_{s_4}}
\]

(A.23)

where

\[
2 \langle \mathbf{u}_2^{m_1} m_{s_2} | V_{23}^{(A,B,C)} | \mathbf{u}_2^{m_1} m_{s_2} \rangle_2 = \sum_{\gamma_3} g_{\gamma_3}^{m_1} \sum_{\gamma_3} g_{\gamma_3}^{m_1} \delta_{m_1, m_{s_2}} \delta_{m_{s_1}, m_{s_2}} \delta_{m_{s_3}, m_{s_3}} \delta_{m_{s_4}, m_{s_4}}
\]

(A.24)
\[
2 \langle \mathbf{u}_2^{m} \mid V_{23}^{(B)} \mid \mathbf{u}_2^{m} \rangle = 2 \langle \mathbf{u}_2^{m} \mid F(Q^2) \cdot \tilde{Q} \mid \mathbf{u}_2^{m} \rangle \frac{\sigma_2 \cdot \tilde{Q}}{Q^2 + m_\pi^2} \langle \mathbf{u}_2^{m} \rangle \quad \text{(A.26)}
\]

\[
2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} \rangle_2 = 2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} \mid F(Q^2) \cdot \tilde{Q} \mid \mathbf{u}_2^{m} m_{s_2}^{m} \rangle_2 \frac{\sigma_2 \cdot \tilde{Q}}{Q^2 + m_\pi^2} \langle \mathbf{u}_2^{m} \rangle \quad \text{(A.27)}
\]

also the different parts of the D-term which have been considered in the right side of Eq. \[(A.21)\] can be evaluated in the following:

\[
2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2 = 2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2 = 2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2
\]

\[
2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2 = 2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2 = 2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2
\]

\[
2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2 = 2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2 = 2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2
\]

\[
2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2 = 2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2 = 2 \langle \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \mid V_{23}^{(D)} \mid \mathbf{u}_2^{m} m_{s_2}^{m} m_{s_3}^{m} \rangle_2
\]

**Appendix B**

**Parity and Time Reversal Properties of the Total 4N Wave Function**

In our formulation of 4N bound state we have not yet used the properties of total wave function under the parity and time reversal invariance. In this section we discuss about these properties.
The parity invariance would mean:

\[
\langle u \alpha | \Psi \rangle = \langle u \alpha | PP | \Psi \rangle = \langle -u \alpha | PP | \Psi \rangle = \langle -u \alpha | P | \Psi \rangle = \langle -u \alpha | \Psi \rangle
\]

(B.1)

The parity invariance or equivalently the symmetry property of the total wave function \( | \Psi \rangle \) under the exchange of the vector sets \( | u \rangle \) to \( | -u \rangle \) can be used in the numerical solution of the coupled equations (4.7) to reduce the dimension of the 4N problem. On the other hand this symmetry property can be used explicitly to cut down the size of the Yakubovsky components and thus save the time and memory when computing the coupled three dimensional integral equations.

The time reversal invariance might be more interesting. The total wave function can be written as:

\[
\langle u \alpha | \Psi \rangle = \sum_\gamma \langle \alpha | \gamma \rangle \langle u \gamma | \Psi \rangle = \sum_\gamma g_{\alpha \gamma} \langle u \gamma | \Psi \rangle,
\]

(B.2)

where

\[
\langle u \gamma | \Psi \rangle = \langle u \gamma | TT | \Psi \rangle = \prod_{i=1}^{4} (-)^{(1/2-m_{s_i})} (-)^{(1/2-m_{t_i})} \langle -u - \gamma | T | \Psi \rangle = \prod_{i=1}^{4} (-)^{(1-m_{s_i}-m_{t_i})} \langle -u - \gamma | T | \Psi \rangle.
\]

(B.3)

By using the time reversal invariance of the total wave function, which is applicable for the ground state of \(^4\text{He}\) with the total angular momentum \(J = 0\), i.e. \(T | \Psi \rangle = | \Psi \rangle\), as well as the parity invariance, Eq. (B.1), we can rewrite Eq. (B.3) as:

\[
\langle u \gamma | \Psi \rangle = \prod_{i=1}^{4} (-)^{(1-m_{s_i}-m_{t_i})} \langle -u - \gamma | \Psi \rangle
\]

(B.4)

The time reversal, and the parity invariance, as indicated in Eq. (B.4) can be also used to cut down the size of the total wave function, by reducing the dimension of the spin-isospin parts of total wave function, which will be valuable in the calculations of 3NFs.

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