Twisted Weibel’s conjecture for smooth proper connective
differential graded algebras

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Abstract

We prove Weibel’s conjecture for twisted $K$-theory when twisting by a smooth proper connective dg-algebra. Our main contribution is showing we can kill a negative twisted $K$-theory class using a projective birational morphism (in the same twisted setting). We extend the vanishing result to relative twisted $K$-theory of a smooth affine morphism and describe counter examples to some similar extensions.

1 Introduction

The so-called fundamental theorem for $K_1$ and $K_0$ states that for any ring $R$ there is an exact sequence

$$0 \to K_1(R) \to K_1(R[t]) \oplus K_1(R[t^{-1}]) \to K_1(R[t^\pm]) \to K_0(R) \to 0.$$ 

We see $K_0$ can be defined using $K_1$. There is an analogous exact sequence, truncated on the right, for $K_0$. Bass defines $K_{-1}(X)$ as the cokernel of the final morphism. He then iterates the construction to define a theory of negative $K$-groups [Bas68, Sections XII.7 and XII.8].

Weibel’s conjecture, originally posed in [Wei80], asks if $K_{-i}(R) = 0$ for $i > \dim R$ when $R$ has finite Krull dimension. Kerz–Strunk–Tamme [KST18] have proven Weibel’s conjecture for any noetherian scheme of finite Krull dimension (see the introduction for a historical summary of progress) by establishing pro cdh-descent for algebraic $K$-theory. Land–Tamme [LT18] have shown that a general class of localizing invariants satisfy pro cdh-descent. With this improvement, we extend Weibel’s vanishing to some cases of twisted $K$-theory.

Theorem 1.1. Let $X$ be a noetherian quasi-separated $d$-dimensional scheme and $A$ a sheaf of smooth proper connective quasi-coherent differential graded algebras over $X$, then $K_{-i}(\text{Perf}(A))$ vanishes for $i > d$.

The original goal of this paper was to extend Weibel’s conjecture to an Azumaya algebra over a scheme. To an Azumaya algebra $A$ of rank $r^2$ on $X$ we can associate a Severi-Brauer variety $P$ of relative dimension $r - 1$ over $X$. Such a variety is étale-locally isomorphic over $X$ to $\mathbb{P}_X^{r-1}$. In Quillen’s work [Qui73], he generalizes the projective bundle formula to Severi-Brauer varieties showing (for $i \geq 0$)

$$K_i(P) \cong \bigoplus_{n=0}^{r-1} K_i(A^\oplus n).$$

At the root of this computation is a semi-orthogonal decomposition of $\text{Perf}(P)$. Consequently, the computation lifts to the level of nonconnective $K$-theory spectra. Statements about the $K$-theory of Azumaya
algebras can generally be extracted through this decomposition. In our case, the dimension of the Severi-Brauer variety jumps and so Weibel’s conjecture (for our noncommutative dg-algebra) does not follow from the commutative setting.

We could remedy this by characterizing a class of morphisms to \( X \), which should include Severi-Brauer varieties, and then show the relative \( K \)-theory vanishes under \(-d-1\). In Remark 4.4, we show that smooth and proper morphisms (in fact, smooth and projective) are not sufficient. We warn the reader that we will use the overloaded words “smooth and proper” in both the scheme and dg-algebra settings.

For dg-algebras and dg-categories, properness and smoothness are module and algebraic finiteness conditions, see Toën–Vaquié [TV07, Definition 2.4]. Together, the two conditions characterize the dualizable objects in \( \text{Mod}_{\text{Mod}_A}(\text{Pr}^l_{\text{st},\omega}) \), whose objects are \( \omega\)-compactly generated \( R \)-linear stable presentable \( \infty \)-categories.

More surprisingly, the invertible objects of \( \text{Mod}_{\text{Mod}_R}(\text{Pr}^l_{\text{st},\omega}) \) are exactly the module categories over derived Azumaya algebras, see Antieau–Gepner [AG14, Theorem 3.15]. So Theorem 1.1 recovers the discrete Azumaya algebra case.

However, any connective derived Azumaya algebras is discrete. After base-changing to a field \( k \), \( A_k \cong H_\ast A_k \) is a connective graded \( k \)-algebra and \( H_\ast A_k \otimes_k (H_\ast A_k)^{op} \) is Morita equivalent to \( k \). So \( H_\ast A_k \) is discrete.

The scope of Theorem 1.1 is not wasted as smooth proper connective dg-algebras can be nondiscrete, see Raedschelders–Stevenson [RS19, Section 5].

The proof of Theorem 1.1 follows Kerz [Ker18]. In Section 2, we define and study twisted \( K \)-theory. We kill a negative twisted \( K \)-theory class using a projective birational morphism in Section 3. Lastly, Section 4 holds the main theorem and we consider some extensions.

**Conventions:** We make very little use of the language of \( \infty \)-categories. For a commutative ring \( R \), there is an equivalence of \( \infty \)-categories between the \( \mathbb{E}_1 \)-ring spectra over \( HR \) and differential graded algebras over \( R \) localized at the quasi-isomorphisms (see [Lur, 7.1.4.6]). For a dg-algebra (or \( \mathbb{E}_1 \)-ring) \( A \), we can consider the \( \infty \)-category RMod\((A)\) of spectra which have a right \( A \)-module structure. We will refer to this \( \infty \)-category as the derived category of \( A \) and denote it by \( D(A) \). The subcategory Perf\((A)\) consists of all compact objects of RMod\((A)\), or the right \( A \)-modules which corepresent a functor that commutes with filtered colimits. We shall refer to objects of Perf\((A)\) as perfect complexes over \( A \).

We use \( K(-) \) undecorated as non-connective algebraic \( K \)-theory and consider it as a localizing invariant in the sense of Blumberg–Gepner–Tabuada [BGT13]. In particular, it is an \( \infty \)-functor from \( \text{Cat}_{\infty} \), the \( \infty \)-category of idempotent complete small stable infinity categories with exact functors, taking values in Sp, the \( \infty \)-category of spectra. For \( X \) a quasi-compact quasi-separated scheme, \( K(\text{Perf}(X)) \) is equivalent to the non-connective \( K \)-theory spectrum of Thomason–Trobaugh [TT90]. The \( \infty \)-category \( \text{Cat}_{\infty}^{\text{perf}} \) has a symmetric monoidal structure which we will denote by \( \otimes \). For \( R \) an \( \mathbb{E}_\infty \)-ring spectrum, \( \text{Perf}(R) \) is an \( \mathbb{E}_\infty \) algebra in \( \text{Cat}_{\infty}^{\text{perf}} \). We will restrict the domain of algebraic \( K \)-theory to \( \text{Mod}_{\text{Pr}^l(\text{Ring})}(\text{Cat}_{\infty}^{\text{perf}}) \).

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## 2 Twisted \( K \)-theory

In Grothendieck’s original papers [Gro68a] [Gro68b] [Gro68c], he globalizes the notion of a central simple algebra over a field.

**Definition 2.1.** A locally free sheaf of \( \mathcal{O}_X \)-algebras \( A \) is a sheaf of Azumaya algebras if it is étale-locally isomorphic to \( M_n(\mathcal{O}_X) \) for some \( n \).

An Azumaya algebra is then a \( \text{PGL}_n \)-torsor over the étale topos of \( X \) and so, by Giraud, isomorphism

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classes are in bijection with $H^1_{\text{et}}(X, PGL_n)$. The central extension of sheaves of groups in the étale topology

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

leads to an exact sequence of nonabelian cohomology

$$\cdots \to H^1_{\text{et}}(X, \mathbb{G}_m) \to H^1_{\text{et}}(X, GL_n) \to H^1_{\text{et}}(X, PGL_n) \to H^2_{\text{et}}(X, \mathbb{G}_m).$$

For $d \mid n$ we have a morphism of exact sequences

$$1 \to \mathbb{G}_m \to GL_n \to PGL_n \to 1$$

$$1 \to \mathbb{G}_m \to GL_d \to PGL_d \to 1$$

with the two right arrows given by block-summing the matrix along the diagonal $n/d$ times. The Brauer group is the filtered colimit of cofibers

$$Br(X) := \operatorname{colim}_{n \mid m} (\text{cofib}(H^1_{\text{et}}(X, GL_n) \to H^1_{\text{et}}(X, PGL_n)))$$

along the partially-ordered set of the natural numbers under division. This is the group of Azumaya algebras modulo Morita equivalence with group operation given by tensor product (see [Gro68a]). We have an injection $Br(X) \hookrightarrow H^2_{\text{et}}(X, \mathbb{G}_m)$ and when $X$ is quasi-compact this injection factors through the torsion subgroup. We will call $Br'(X) := H^2_{\text{et}}(X, \mathbb{G}_m)_{\text{tor}}$ the cohomological Brauer group. Grothendieck asked if the injection $Br(X) \hookrightarrow Br'(X)$ is an isomorphism.

This map is not generally surjective. Edidin–Hassett–Kresch–Vistoli [EHKV01] give a non-separated counter example by connecting the image of the Brauer group to quotient stacks. There are two ways to proceed in addressing the question. The first is to provide a class of schemes for when this holds. In [dJ], de Jong publishes a proof of O. Gabber that $Br(X) \cong Br'(X)$ when $X$ is equipped with an ample line bundle. Along with reproving Gabber’s result for affines, Lieblich [Lie04] shows that for a regular scheme with dimension less than or equal to 2 there are isomorphisms $Br(X) \cong Br'(X) \cong H^2_{\text{et}}(X, \mathbb{G}_m)$.

The second perspective is to enlarge the class of objects considered. The Morita equivalence classes of $\mathbb{G}_m$-gerbes over the étale topos of a scheme $X$ are in bijection with $H^2_{\text{et}}(X, \mathbb{G}_m)$. In [Lie04], Lieblich associates to any Azumaya algebra $A$ a $\mathbb{G}_m$-gerbe of Morita-theoretic trivializations. Over an étale open $U \to X$, the gerbe gives a groupoid of Morita equivalences from $A$ to $O_X$. The gerbe of trivializations represents the boundary class $\delta([A]) = \alpha \in H^2_{\text{et}}(X, \mathbb{G}_m)$.

Any class $\alpha \in H^2_{\text{et}}(X, \mathbb{G}_m)$ is realizable on a Čech cover. We can use this data to build a well-defined category of sheaves of $O_X$-modules which “glue up to $\alpha$”, see Căldăraru [Căl00, Chapter 1]. Let $\operatorname{Mod}_X^\infty$ denote the corresponding derived $\infty$-category and $\operatorname{Perf}_X^\infty$ the full subcategory of compact objects. $K(\operatorname{Perf}_X^\infty)$ is the classical definition of $\alpha$-twisted algebraic $K$-theory. Determining when the cohomology class $\alpha$ is represented by an Azumaya algebra reduces to finding a twisted locally-free sheaf with trivial determinant on a $\mathbb{G}_m$-gerbe associated to $\alpha$ [Lie04, Section 2.2.2]. The endomorphism algebra of the twisted locally-free sheaf gives the Azumaya algebra and the twisted module represents the tilt $\operatorname{Mod}_X^\infty \simeq \operatorname{Mod}_A$.

Lieblich also compactifies the moduli of Azumaya algebras. This necessarily includes developing a definition of a derived Azumaya algebra.

**Definition 2.2.** A **derived Azumaya algebra** over a commutative ring $R$ is a proper dg-algebra $A$ such that the natural map of $R$-algebras

$$A \otimes^L_R A^{op} \xrightarrow{\sim} \mathbb{R} \operatorname{Hom}_{D(R)}(A, A)$$

is a quasi-isomorphism.
The lemma follows since $K^A$.

**Proof.** We have the following isomorphisms of discrete rings

$$\pi_0(A \otimes_R S) \cong \pi_0(A \otimes_R \pi_0(S)) \cong \pi_0(\pi_0(A) \otimes_R S) \cong \pi_0(\pi_0(A) \otimes_R \pi_0(S)).$$

The lemma follows since $K_i(R) \cong K_i(\pi_0(R))$ for $i \leq 0$ (see Theorem 9.53 of [BGT13]).

The previous proposition suggests we can work discretely and then transfer the results to the derived setting. This is true to some extent. However, taking $\pi_0$ of a connective dg-algebra does not preserve smoothness, which is a necessary property for our proof of Proposition 3.2. We will also need reduction invariance for low dimensional K-groups.
Proposition 2.7. Let $R$ be a commutative ring and $A$ a connective dg-algebra over $R$. Let $S$ be a commutative ring under $R$ and let $I$ be a nilpotent ideal of $S$. Then the induced morphism $K^A_i(S) \xrightarrow{\sim} K^A_i(S/I)$ is an isomorphism for $i \leq 0$.

Proof. By naturality of the fundamental exact sequence of twisted $K$-theory (see (†) and the surrounding discussion at the beginning of Section 3), we can restrict the proof to $K^A_0$. By Proposition 2.6, we can assume $A$ is a discrete algebra. Let $\varphi : S \rightarrow S/I$ be the surjection. After $- \otimes_R A$ we have a surjection $(\ker \varphi) \otimes_R A \twoheadrightarrow \ker(\varphi \otimes_R A)$. The nonunital ring $(\ker \varphi) \otimes_R A$ is nilpotent. So $\ker(\varphi \otimes_R A)$ is nilpotent as well. The proposition follows from nil-invariance of $K^A_0$. 

A Zariski descent spectral sequence argument gives us a global result.

Corollary 2.8. Let $X$ be a noetherian quasi-separated scheme of finite Krull dimension $d$ and $A$ a sheaf of connective quasi-coherent dg-algebras over $X$. The natural morphism $f : X_{\text{red}} \rightarrow X$ induces isomorphisms $K^A_{-i}(X_{\text{red}}) \cong K^A_{-i}(X)$ for $i \geq d$.

Proof. We have descent spectral sequences

$$E^p,q_2 = H^p_{\text{Zar}}(X, (\pi_q K^A)^\sim) \Rightarrow \pi_{q-p}K^A(X)$$

$$E^p,q_2 = H^p_{\text{Zar}}(X, f_* (\pi_q K^f(A))^{\sim}) \Rightarrow \pi_{q-p}K^f(A)(X_{\text{red}})$$

both with differential $d_2 = (2, 1)$. We let $F^{\sim}$ denote the Zariski sheafification of the presheaf $F$. The spectral sequences agree for $q \leq 0$ and vanish for $p > d$. 

In Theorem 4.3, we extend our main theorem across smooth affine morphisms. We will need reduction invariance in this setting.

Definition 2.9. For $f : S \rightarrow X$ a morphism of quasi-compact quasi-separated schemes and $A$ a sheaf of quasi-coherent dg-algebras over $X$, the relative $A$-twisted $K$-theory of $f$ is

$$K^A(f) := \text{fib}(K^A(X) \xrightarrow{f^*} K^A(S))$$

As defined, $K^A(f)$ is a spectrum. There is an associated presheaf of spectra on the base scheme $X$ given by $U \mapsto K^A(f|_U)$. This presheaf sits in a fiber sequence

$$K^A(f) \rightarrow K^A \rightarrow K^A_S$$

where the presheaf $K^A_S$ is also defined by pullback along $f$. Both presheaves $K^A$ and $K^A_S$ satisfy Nisnevich descent and so $K^A(f)$ does as well.

Corollary 2.10. Let $f : S \rightarrow X$ be a smooth affine morphism of noetherian schemes with $X$ a $d$-dimensional scheme. Let $A$ be a sheaf of connective quasi-coherent dg-algebras over $X$. Then the commutative diagram

$$
\begin{array}{ccc}
S_{\text{red}} & \xrightarrow{f_{\text{red}}} & X_{\text{red}} \\
\downarrow \quad \quad \quad & \quad & \quad \quad \downarrow \quad \quad \quad \\
S & \xrightarrow{f} & X
\end{array}
$$

induces an isomorphism of relative twisted $K$-theory groups

$$K^g_{-i}(f_{\text{red}}) \cong K^A_{-i}(f)$$

for $i \geq d + 1$. 

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Proof. We have two descent spectral sequences
\[ E_2^{p,q} = H^p_{Zar}(X, (\pi_q K^A(f))^\sim) \Rightarrow \pi_{q-p} K^A(f)(X) \] and
\[ E_2^{p,q} = H^p_{Zar}(X, g_*(\pi_q K^{g^* A}(fred))^\sim) \Rightarrow \pi_{q-p} K^{g^* A}(fred)(X_{red}) \]
with differential of degree \( d = (2, 1) \) and \( F^\sim \) the sheafification of the presheaf \( F \). For an open affine \( \text{Spec} R \to X \) with pullback \( \text{Spec} A \to S \) we examine the morphism of long exact sequences when \( q \leq 0 \)
\[ \cdots \to \pi_q K^A(R) \to \pi_q K^A(A) \to \pi_{q-1} K^A(f) \to \pi_{q-1} K^A(R) \to \pi_{q-1} K^A(A) \to \cdots \]
\[ \cdots \to \pi_q K^A(R_{red}) \to \pi_q K^A(A_{red}) \to \pi_{q-1} K^A(f_{red}) \to \pi_{q-1} K^A(R_{red}) \to \pi_{q-1} K^A(A_{red}) \to \cdots \]
By the 5-lemma, this induces sheaf isomorphisms \( g_*(\pi_q K^{g^* A}(f_{red}))^\sim \cong (\pi_q K^A(f))^\sim \) for \( q < 0 \) and cohomology vanishes for \( p > d \).

We will need pro-excision for abstract blow-up squares. Recall that an abstract blow-up square is a pullback square
\[ D \to \tilde{X} \]
\[ Y \to X \]
with \( Y \to X \) a closed immersion and \( \tilde{X} \to X \) a proper morphism which restricts to an isomorphism of open subschemes \( \tilde{X} \setminus D \to X \setminus Y \). The theorem is stated using the \( \infty \)-category of pro-spectra \( \text{Pro}(\text{Sp}) \), where an object is a small cofiltered \( \infty \)-category, \( E : \Lambda \to \text{Sp} \), valued in spectra. We write \( \{E_n\} \) for the corresponding pro-spectrum and will always consider \( n \) as the index. If the brackets and index are omitted, then we consider it as a constant pro-spectrum. After adjusting equivalence class representatives, we may assume the cofiltered diagram is fixed when working with a finite set of pro-spectra. Any morphism can then be represented by a natural transformation of diagrams. We will need no knowledge of the \( \infty \)-category beyond the following definition.

**Definition 2.11.** A square of pro-spectra
\[
\{E_n\} \to \{F_n\} \\
\downarrow \quad \downarrow \\
\{X_n\} \to \{Y_n\}
\]
is pro-cartesian if and only if the induced map on the level-wise fiber pro-spectra is a weak equivalence.

The following is Theorem A.8 of Land–Tamme [LT18]. The theorem holds much more generally for any \( k \)-connective localizing invariant (see Definition 2.5 of [LT18]). Twisted \( K \)-theory is 1-connective.

**Theorem 2.12** (Land–Tamme [LT18]). Given an abstract blow-up square (\( \ast \)) of schemes and a sheaf of dg-algebras \( \mathcal{A} \) on \( X \) then the square of pro-spectra
\[
K^\mathcal{A}(X) \to K^\mathcal{A}(\tilde{X}) \\
\downarrow \quad \downarrow \\
\{K^\mathcal{A}(Y_n)\} \to \{K^\mathcal{A}(D_n)\}
\]
is pro-cartesian (where $Y_n$ is the infinitesimal thickening of $Y$).

The pro-cartesian square of pro-spectra gives a long exact sequence of pro-groups

\[ \cdots \to \{ K^A_{n+1}(E_n) \} \to K^A_n(X) \to \tilde{K}^A_n(X) \oplus \{ K^A_{-1}(Y_n) \} \to \{ K^A_{-1}(E_n) \} \to \cdots \]

which is the key to our induction argument.

3 Blowing-up negative twisted $K$-theory classes

We turn to our main contribution of the existence of a projective birational morphism which kills a given negative twisted $K$-theory class (when twisting by a smooth proper connective dg-algebra). Let $X$ be a quasi-separated quasi-compact scheme and $A$ a sheaf of quasi-coherent dg-algebras on $X$. We first construct geometric cycles for negative twisted $K$-theory classes on $X$ using a classical argument of Bass (see XII.7 of [Bas68]) which works for a general additive invariant. We have an open cover

\[ X[t^\pm] \xrightarrow{f} X[t^-] \]

\[ X[t] \xrightarrow{k} \mathbb{P}^1_X. \]

Since twisted $K$-theory satisfies Zariski descent, there is an associated Mayer-Vietoris sequence of homotopy groups

\[ \cdots \to K^A_n(\mathbb{P}^1_X) \xrightarrow{(j^* \kappa')} K^A_n(X[t]) \oplus K^A_n(X[t^-]) \xrightarrow{f^* - g^*} K^A_n(X[t^\pm]) \xrightarrow{\partial} K^A_{n-1}(\mathbb{P}^1_X) \to \cdots. \]

As an additive invariant, $K^A_n(\mathbb{P}^1_X) \simeq K^A_n(X) \oplus K^A_n(X)$ splits as a $K^A_n(X)$-module with generators

\[ [\mathcal{O} \otimes_{\mathcal{O}_X} A] = [A] \quad \text{and} \quad [\mathcal{O}(1) \otimes_{\mathcal{O}_X} A] = [A(1)] \]

corresponding to the Beilinson semiorthogonal decomposition. Adjusting the generators to $[A]$ and $[A] - [A(1)]$, we can identify the map $(j^*, \kappa')$ as it is a map of $K^A_n(X)$-modules. The second generator vanishes under each restriction. This identifies the map as

\[ K^A(\mathbb{P}^1_X) \simeq K^A_n(X) \oplus K^A_n(X) \]

with $\Delta$ the diagonal map corresponding to pulling back along the projections $X[t] \to X$ and $X[t^-] \to X$. As $\Delta$ is an embedding the long exact sequence splits as

\[ 0 \to K^A_n(X) \xrightarrow{\Delta} K^A_n(X[t]) \oplus K^A_n(X[t^-]) \xrightarrow{\pm} K^A_n(X[t^\pm]) \xrightarrow{\partial} K^A_{n-1}(X) \to 0. \quad (\dagger) \]

After iterating the complex

\[ K^A_n(X[t]) \to K^A_n(X[t^\pm]) \to K^A_{n-1}(X), \]

we can piece together a complex

\[ K^A_0(A^{n+1}_X) \to K^A_0(\mathbb{G}^{n+1}_{m,X}) \to K^A_{-1}(X). \]

Negative twisted $K$-theory classes have geometric representations as twisted perfect complexes on $\mathbb{G}^{m,X}_m$. There is even a sufficient geometric criterion implying a given representative is 0; it is the restriction of a twisted perfect complex on $\mathbb{G}^{m,X}_m$. Our proof of the main proposition of this section will use these representatives. We first need a lemma about extending finitely-generated discrete modules in a twisted setting.
Lemma 3.1. Let \( j : U \to X \) be an open immersion of Noetherian schemes. Let \( \mathcal{A} \) be a sheaf of proper connective quasi-coherent dg-algebras on \( X \) and \( j^* \mathcal{A} \) its restriction. Let \( \mathcal{N} \) be a discrete \( j^* \mathcal{A} \)-module which is finitely generated as an \( \mathcal{O}_U \)-module. Then there exists a discrete \( \mathcal{A} \)-module \( \mathcal{M} \), finitely generated over \( \mathcal{O}_X \), such that \( j^* \mathcal{M} \cong \mathcal{N} \).

Proof. Note that \( H_{\geq 1}(j^* \mathcal{A}) \) necessarily acts trivially on \( \mathcal{N} \). So the \( j^* \mathcal{A} \)-module structure on \( \mathcal{N} \) comes from forgetting along the map \( j^* \mathcal{A} \to H_0(j^* \mathcal{A}) \) and the natural \( H_0(j^* \mathcal{A}) \)-module structure. Under restriction, \( j^* H_0(\mathcal{A}) \cong H_0(j^* \mathcal{A}) \).

We reduce to when \( \mathcal{A} \) is a quasi-coherent sheaf of discrete \( \mathcal{O}_X \)-algebras, finite over the structure sheaf. We have an isomorphism \( \mathcal{N} \cong j^* j_* \mathcal{N} \). Write \( j_* \mathcal{N} \) as a filtered colimit of its finitely generated \( \mathcal{A} \)-submodules \( j_* \mathcal{N} \cong \operatorname{colim}_\lambda \mathcal{M}_\lambda \). Each \( \mathcal{M}_\lambda \) is a coherent \( \mathcal{O}_X \)-module by finiteness of \( \mathcal{A} \). After taking the pullback, \( \mathcal{N} \cong j^* (\operatorname{colim}_\lambda \mathcal{M}_\lambda) \cong \operatorname{colim}_\lambda j^* \mathcal{M}_\lambda \).

As \( \mathcal{N} \) is coherent, this isomorphism factors at some stage and we get a retraction \( \mathcal{N} \to j^* \mathcal{M}_\lambda \to \mathcal{N} \).

The right morphism is essentially (up to a natural isomorphism) \( j^* \) applied to the injection \( \mathcal{M}_\lambda \hookrightarrow \mathcal{N} \). Complete the injection to a short exact sequence \( 0 \to \mathcal{M}_\lambda \hookrightarrow \mathcal{N} \to Q \to 0 \).

After application of \( j^* \), we can extend on the left via Tor sheaves. However \( j^* Q \cong 0 \), and so its higher Tor sheaves vanish, implying \( j^* \mathcal{M}_\lambda \to \mathcal{N} \) is an isomorphism.

Proposition 3.2. Let \( X = \text{Spec} \, R \) be a reduced noetherian affine scheme and \( \mathcal{A} \) a smooth proper connective dg-algebra on \( R \). Let \( \gamma \in K^{<1}_A(X) \) for \( i > 0 \). Then there is a projective birational morphism \( \rho : \tilde{X} \to X \) so that \( \rho^* \gamma = 0 \in K^{<1}_A(\tilde{X}) \).

Proof. We fix a diagram of schemes over \( X \)

\[
\begin{array}{ccc}
\mathbb{G}^i_{m,X} & \xrightarrow{j} & \mathbb{A}^i_X \\
\pi_1 \downarrow & & \downarrow \pi_2 \\
X & & 
\end{array}
\]

For any morphism \( f : Y_1 \to Y_2 \), we let \( \tilde{f} : \mathbb{G}^i_{m,Y_1} \to \mathbb{G}^i_{m,Y_2} \) denote the pullback.

Lift \( \gamma \) to a \( K^{<1}_A(\mathbb{G}^i_{m,X}) \)-class \([P_] - [Q_]\), with \( P_\bullet, Q_\bullet \) some choice of \( \pi_1^* \mathcal{A} \)-twisted perfect complexes on \( \mathbb{G}^i_{m,X} \). We can construct projective birational morphisms for both \( P_\bullet \) and \( Q_\bullet \) and then take the pullback. So we work with \( P_\bullet \).

The Induction Step:

We induct on the range of homology of \( P_\bullet \). As \( \pi_1^* \mathcal{A} \) is a sheaf of proper quasi-coherent dg-algebras, \( P_\bullet \) is perfect on \( \mathbb{G}^i_{m,X} \) by Lemma 2.5. Since \( \mathbb{G}^i_{m,X} \) has an ample family of line bundles, we may choose \( P_\bullet \) to be strict perfect without changing the quasi-isomorphism class. After some (de)suspension, we may assume \( P_\bullet \).
is connective as this only alters the $K_0$-class by $\pm 1$. For the lowest nontrivial differential of $P_\bullet$, $d_1$, we utilize part (iv) of Lemma 6.5 of [KST18] (with the morphism $\mathbb{G}^i_m,X \to X$) to construct a projective birational morphism $\rho : X_1 \to X$ so that $\coker(\tilde{\rho}^*d_1) (= H_0(\tilde{\rho}^*P_\bullet))$ has tor-dimension $\le 1$ over $X_1$. Consider the following distinguished triangle of $\tilde{\rho}^*\pi_1^*\mathcal{A}$-complexes on $\mathbb{G}^i_m,X_1$.

$$F_\bullet \to \tilde{\rho}^* P_\bullet \to H_0(\tilde{\rho}^*P_\bullet) \cong \coker \tilde{\rho}^*d_1.$$  

In Lemma 3.3 below, we cover the base induction step, when the homology is concentrated in a single degree. Using this, construct a projective birational morphism $\phi : X_2 \to X_1$ such that $L\tilde{\phi}^*H_0(\tilde{\rho}^*P_\bullet)$ is a perfect complex and is the restriction of a perfect complex from $\mathbb{A}_X$. By two out of three, $L\tilde{\phi}^*F_\bullet$ is perfect and $[\tilde{\phi}^*\tilde{\rho}^*P_\bullet] = [L\tilde{\phi}^*F_\bullet] + [L\tilde{\phi}^*H_0(\tilde{\rho}^*P_\bullet)]$ in $K^b_\mathbb{A}(\mathbb{G}^i_m,X_2)$. We then repeat the entire induction step with $L\tilde{\phi}^*F_\bullet$.

We need to guarantee the induction will terminate, which is the purpose of the first projective birational morphism of each step. Since $\coker(\tilde{\rho}^*d_1)$ has tor-dimension $\le 1$ over $X_1$,

$$L\tilde{\phi}^* \coker(\tilde{\rho}^*d_1) = \tilde{\phi}^* \coker(\tilde{\rho}^*d_1) \cong \coker(\tilde{\phi}^*\tilde{\rho}^*d_1).$$

The first equality guarantees $L\tilde{\phi}^*F_\bullet$ will have no homology outside the original range of homology of $P_\bullet$. Both equivalences guarantee $H_0(L\tilde{\phi}^*F_\bullet) = 0$, so the homology of $L\tilde{\phi}^*F_\bullet$ lies in a strictly smaller range than $\tilde{\phi}^*\tilde{\rho}^*P_\bullet$. Proposition 3.2 follows from the next lemma.

**Lemma 3.3.** Let $X$ be a noetherian quasi-separated scheme which is quasi-projective over an affine scheme. Let $\mathcal{A}$ be a sheaf of smooth proper connective quasi-coherent dg-algebras on $X$. Let $\mathcal{N}$ be a discrete $\pi_1^*\mathcal{A}$-module which is coherent on $\mathbb{G}^i_m,X$. Then there exists a blow-up $\phi : \tilde{X} \to X$ so that $\tilde{\phi}^*\mathcal{N}$ is perfect over $\tilde{\phi}^*\pi_1^*\mathcal{A}$ on $\mathbb{G}^i_{\tilde{X}}$ and is the restriction of a perfect complex over the pullback of $\mathcal{A}$ to $\mathbb{A}^i_{\tilde{X}}$.

**Proof.** Using Lemma 3.1, extend $\mathcal{N}$ from $\mathbb{G}^i_m,X$ to a coherent $\pi_1^*\mathcal{A}$-module $\mathcal{M}$ on $\mathbb{A}^i_{\tilde{X}}$. Using the ample family, choose a resolution in $\mathcal{O}_{\mathbb{A}^i_{\tilde{X}}}$-modules of the form

$$0 \to K \to F \to M \to 0$$

where $F$ is a vector bundle and $K$ is the coherent kernel. As $X$ is reduced, $K$ is flat over some nonempty open set $U$ of $X$. By platification par éclatement (see Theorem 5.2.2 of Raynaud–Gruson [RG71]), there is a $U$-admissible blow-up $\phi : \tilde{X} \to X$ so that the strict transform of $K$ along the pullback morphism $p : \mathbb{A}^i_{\tilde{X}} \to \mathbb{A}^i_X$ is flat over $\tilde{X}$.

We now show the pullback $p^*\mathcal{M}$ is perfect as a $p^*\pi_2\mathcal{A}$-module. Let $j : \mathbb{A}^i_{\tilde{X}} \to \mathbb{A}^i_X$ be the inclusion of the open set and $Z$ the closed complement. For any sheaf of modules $\mathcal{G}$ on $\mathbb{A}^i_X$, we let $\mathcal{G}_Z$ denote the subsheaf of sections supported on $Z$. We have a short exact sequence natural in $\mathcal{G}$

$$0 \to \mathcal{G}_Z \to \mathcal{G} \to j^*\mathcal{G} \to 0.$$  

We also obtain the following exact sequence of sheaves of abelian groups via pullback

$$0 \to \mathcal{F}or_{\mathbb{A}^i_X}^1 (p^{-1}\mathcal{M}, \mathcal{O}_{\mathbb{A}^i_{\tilde{X}}}) \to p^*K \to p^*F \to p^*\mathcal{M} \to 0.$$  

To make our notation clearer, we set $\mathcal{T} = \mathcal{F}or_{\mathbb{A}^i_X}^1 (p^{-1}\mathcal{M}, \mathcal{O}_{\mathbb{A}^i_{\tilde{X}}})$. We flesh both these exact sequences

9
out into a (nonexact) commutative diagram of $p^{-1}\mathcal{O}_{\mathcal{A}_{X}}$-modules

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \\
0 & \mathcal{T}_Z & \mathcal{T} & j^{st}\mathcal{T} \\
\downarrow & \downarrow & \downarrow & \\
0 & (p^{*}\mathcal{K})_Z & p^{*}\mathcal{K} & j^{st}p^{*}\mathcal{K} \\
\downarrow & \downarrow & \downarrow & \\
0 & (p^{*}\mathcal{F})_Z & p^{*}\mathcal{F} & j^{st}p^{*}\mathcal{F} \\
\downarrow & \downarrow & \downarrow & \\
0 & (p^{*}\mathcal{M})_Z & p^{*}\mathcal{M} & j^{st}p^{*}\mathcal{M} \\
\downarrow & \downarrow & \downarrow & \\
0 & 0 & 0 & 0 \\
\end{array}
$$

We observe that every row and the middle column is exact. The first map in the left column is an injection and the last map in the right column is a surjection. Since $p^{*}\mathcal{F}$ is flat, we have $(p^{*}\mathcal{F})_Z = 0$. This induces a lifting of the injection

$$
\mathcal{T}_Z \longrightarrow \mathcal{T} \longleftarrow (p^{*}\mathcal{K})_Z \longrightarrow p^{*}\mathcal{K}.
$$

We finish the proof by showing $j^{*}\mathcal{F}or_{1}^{p^{-1}\mathcal{O}_{\mathcal{A}_{X}}}(p^{-1}\mathcal{M}, \mathcal{O}_{\mathcal{A}_{X}}) = 0$. Since $j : \mathcal{A}_{U} \rightarrow \mathcal{A}_{X}$ is flat, the sheaf is isomorphic to $\mathcal{F}or_{1}^{\mathcal{A}_{U}}(j^{*}p^{-1}\mathcal{M}, j^{*}\mathcal{O}_{\mathcal{A}_{X}})$ and $j^{*}\mathcal{O}_{\mathcal{A}_{X}} \cong \mathcal{O}_{\mathcal{A}_{U}}$. Our big diagram can be rewritten as

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \mathcal{T}_Z & \mathcal{T} & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & (p^{*}\mathcal{K})_Z & p^{*}\mathcal{K} & j^{st}p^{*}\mathcal{K} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & p^{*}\mathcal{F} & j^{st}p^{*}\mathcal{F} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & (p^{*}\mathcal{M})_Z & p^{*}\mathcal{M} & j^{st}p^{*}\mathcal{M} \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & 0 & 0 & 0 \\
\end{array}
$$

and we can glue together to get a flat resolution of $p^{*}\mathcal{M}$ as an $\mathcal{O}_{\mathcal{A}_{X}}$-module

$$
0 \rightarrow j^{st}p^{*}\mathcal{K} \rightarrow p^{*}\mathcal{F} \rightarrow p^{*}\mathcal{M} \rightarrow 0
$$
implying globally finite Tor-amplitude. It remains to show the complex is pseudo-coherent. This follows since $A_{\tilde{X}}$ is Noetherian and $p^*M$ is coherent. Since $p^*\pi_2^*A$ is a sheaf of smooth quasi-coherent dg-algebras over $\mathcal{O}_{k_X}$, the complex $p^*M$ is perfect over $p^*\pi_2^*A$ by Lemma 2.5. By commutativity, $p^*M$ restricts to $\tilde{p}^*N$ on $\mathbb{G}^i_{m,\tilde{X}}$. This completes the proof of Proposition 3.2.

We will need a relative version of Proposition 3.2.

**Corollary 3.4.** Let $f : S \to X$ be a smooth quasi-projective morphism of noetherian quasi-separated schemes with $X$ reduced and quasi-projective over a noetherian base ring. Let $f_{\mathbb{A}}$ be a sheaf of smooth proper connective quasi-coherent dg-algebras over $X$ and consider a negative twisted $K$-theory class $\gamma \in K_{-i}^A(S)$ for $i < 0$. Then there exists a projective birational morphism $\rho : \tilde{X} \to X$ such that, under the pullback of the pullback morphism, $\rho^*_p \gamma = 0$.

**Proof.** We will briefly check that we can run the induction argument in the proof of Proposition 3.2. The assumptions of this corollary are invariant under pullback along projective birational morphisms $\tilde{X} \to X$. We need to ensure we can select projective birational morphisms to our base $X$. Lemma 6.5 of Kerz–Strunk–Tamme [KST18] is stated in a relative setting. The proof also relies on platification par éclatement. This can still be applied in our relative setting as $X$ is reduced (see Proposition 5 of Kerz–Strunk [KS17]).

4 Twisted Weibel’s conjecture

We now prove Theorem 1.1 and an extension across a smooth affine morphism. We begin with the base induction step for both theorems. Kerz–Strunk [KS17] use a sheaf cohomology result of Grothendieck along with a spectral sequence argument to show vanishing for a Zariski sheaf of spectra can be reduced to the setting of local ring.

**Proposition 4.1.** Let $R$ be a regular noetherian ring of Krull dimension $d$ over a local Artinian ring $k$. Let $A$ be a smooth proper connective quasi-coherent dg-algebra over $R$, then $K_i^A(R) = 0$ for $i < 0$.

**Proof.** By Proposition 2.10, we may assume $k$ is a field. Proposition A.4 of [RS19] shows that the t-structure on $D(A)$ restricts to a t-structure on Perf($A$), which is observably bounded. The heart is the category of modules over $H_0(A)$. As $H_0(A)$ is finite-dimensional over $k$, this is a noetherian abelian category. By Theorem 1.2 of Antieau–Gepner–Heller [AGH19]), the negative $K$-theory vanishes.

**Theorem 1.1.** Let $X$ be a noetherian quasi-separated scheme of Krull dimension $d$ and $A$ a sheaf of smooth proper connective quasi-coherent dg-algebras on $X$, then $K_i^A(X)$ vanishes for $i > d$.

**Proof.** Proposition 4.1 covers the base case so assume $d > 0$. By the Kerz–Strunk spectral sequence argument and Proposition 2.8, we may assume $X$ is a noetherian reduced affine scheme.

Choose a negative $K_A$-theory class $\gamma \in K_{-i}^A(X)$ for $i \geq \dim X + 1$. Using Proposition 3.2, construct a projective birational morphism that kills $\gamma$ and extend it to an abstract blow-up square

$$
\begin{array}{ccc}
E & \to & \tilde{X} \\
\downarrow & & \downarrow \\
Y & \to & X
\end{array}
$$

By [LT18, Theorem A.8], there is a Mayer-Vietoris exact sequence of pro-groups

$$
\cdots \to \{K_{i+1}^A(E_n)\} \to K_i^A(X) \to K_i^A(\tilde{X}) \oplus \{K_i^A(Y_n)\} \to \{K_i^A(E_n)\} \to \cdots
$$
When \( i \geq \dim X + 1 \), by induction every nonconstant pro-group vanishes and \( K^A_{-i}(X) \cong K^A_{-i}(\tilde{X}) \) showing \( \gamma = 0 \).

By [AG14, Theorem 3.15], we recover Weibel’s vanishing for discrete Azumaya algebras.

**Corollary 4.2.** For \( X \) a noetherian quasi-separated \( d \)-dimensional scheme and \( A \) a quasi-coherent sheaf of discrete Azumaya algebras, then \( K^A_{-i}(X) = 0 \) for \( i > d \).

The next result nearly covers the K-regularity portion of Weibel’s conjecture, but we are missing the boundary case \( K^A_{-d}(X) \cong K^A_{-d}(\tilde{X}) \).

**Theorem 4.3.** Let \( f : S \to X \) be a smooth affine morphism of noetherian quasi-separated schemes and \( A \) a sheaf of smooth proper connective quasi-coherent dg-algebras on \( X \). Then \( K^A_{-i}(f) = 0 \) for \( i > \dim X + 1 \).

**Proof.** The base case is covered by Proposition 4.1 and our reductions are analogous to those in the proof of Theorem 1.1. So assume \( X \) is a noetherian reduced affine scheme of dimension \( d \). Choose \( \gamma \in K^A_{-i}(S) \) with \( i \geq d \). Using Corollary 3.4, construct a projective birational morphism \( \rho : \tilde{X} \to X \) that kills \( \gamma \). We then build a morphism of abstract blow-up squares

\[
\begin{array}{c}
D \\
\downarrow E \\
V \\
\downarrow Y \\
S \\
\downarrow \\
X
\end{array}
\]

By Theorem 2.12, we again get a long exact sequence of pro-groups corresponding to the back square

\[
\cdots \to \{K^A_{-i+1}(D_n)\} \to K^A_{-i}(S) \to K^A_{-i}(\tilde{S}) \oplus \{K^A_{-i}(V_n)\} \to \{K^A_{-i}(D_n)\} \to \cdots .
\]

When \( i \geq \dim X + 1 \), every nonconstant pro-group vanishes by induction and we have an isomorphism \( K^A_{-i}(S) \cong K^A_{-i}(\tilde{S}) \) implying \( \gamma = 0 \).

**Remark 4.4.** The conditions on the morphism in Corollary 3.4 are more general than those of Theorem 4.3. We might hope to generalize Theorem 4.3 to a smooth quasi-projective or smooth projective map of noetherian schemes. Although the induction step is present, both base cases fail. Consider the descent spectral sequence

\[
E^{p,q}_2 := H^p(X, \tilde{K}_q) \Rightarrow K_{q-p}(X) \text{ with } d_2 = (2,1)
\]

If \( \dim X \leq 3 \), then

\[
E^{2,1}_3 = E^{2,1}_\infty = \text{coker} (H^0(X, \mathbb{Z}) \xrightarrow{d_2} H^2(X, \mathcal{O}_X^*))
\]

contributes to \( K_{-1}(X) \). The differential is zero as the edge morphism

\[
K_0(X) \xrightarrow{\text{rank}} E^{0,0}_\infty
\]

identifies \( E^{0,0}_\infty \) with the rank component of \( K_0 \), implying \( E^{0,0}_2 = E^{0,0}_\infty \). We now construct a family of examples for schemes \( X \) with nontrivial \( H^2(X, \mathcal{O}_X^*) \). Let \( X_{red} \) be quasi-projective smooth over a field \( k \) and form the
cartesian diagram

\[
\begin{array}{ccc}
X & \xrightarrow{f} & X_{\text{red}} \\
\downarrow & & \downarrow \\
\text{Spec}(k[t]/(t^2)) & \longrightarrow & \text{Spec } k
\end{array}
\]

The pullback \( X \) will be our counter-example. We have an isomorphism

\[ \mathcal{O}_X^* \cong g^*(\mathcal{O}_{X_{\text{red}}}^*) \oplus g^*(\mathcal{O}_{X_{\text{red}}}) \]

of sheaves of abelian groups on \( X \) with \( g : X_{\text{red}} \to X \) the pullback of the reduction morphism \( \text{Spec } k \to \text{Spec } k[t]/(t^2) \). Locally, \((R[t]/(t^2))^\times\) consists of all elements of the form \( u + v \cdot t \) where \( u \in R^\times \) and \( v \in R \). Sheaf cohomology commutes with coproducts so this turns into an isomorphism

\[ H^2(X, \mathcal{O}_X^*) \cong H^2(X, g^*(\mathcal{O}_{X_{\text{red}}}^*)) \oplus H^2(X, g^*(\mathcal{O}_{X_{\text{red}}})) \cong H^2(X_{\text{red}}, \mathcal{O}_{X_{\text{red}}}^*) \oplus H^2(X_{\text{red}}, \mathcal{O}_{X_{\text{red}}}). \]

Now the problem reduces to finding a surface or 3-fold \( X_{\text{red}} \) with nontrivial degree 2 sheaf cohomology. Take a smooth quartic in \( \mathbb{P}_k^3 \) for a counter-example which is smooth and proper. Here is a counter-example which is smooth and quasi-affine. Let \((A, m)\) be a 3-dimensional local ring which is smooth over a field \( k \). Take \( X = \text{Spec } A \setminus \{m\} \) to be the punctured spectrum. Then \( H^2(X, \mathcal{O}_X) \cong H^3_m(A) \), which is the injective hull of the residue field \( A/m \).

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