BRACKET STRUCTURES FOR ADJOINT-SYMMETRIES AND SYMMETRIES, AND THEIR APPLICATIONS

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Abstract. Infinitesimal symmetries of a partial differential equation (PDE) can be defined as the solutions of the linearization (Frechet derivative) equation holding on the space of solutions to the PDE, and they are well-known to comprise a linear space having the structure of a Lie algebra. Solutions of the adjoint linearization equation holding on the space of solutions to the PDE are called adjoint-symmetries. Their algebraic structure for general PDE systems is studied herein. This is motivated by the correspondence between variational symmetries and conservation laws arising from Noether’s theorem, which has a well-known generalization to non-variational PDEs, where infinitesimal symmetries are replaced by adjoint-symmetries, and variational symmetries are replaced by multipliers (adjoint-symmetries satisfying a certain Euler-Lagrange condition). Several main results are obtained. Symmetries are shown to have three different linear actions on the linear space of adjoint-symmetries. These linear actions are used to construct bilinear adjoint-symmetry brackets, one of which is a pull-back of the symmetry commutator bracket and has the properties of a Lie bracket. The brackets do not use or require the existence of any local variational structure (Hamiltonian or Lagrangian) and thus apply to general PDE systems. In the case of variational PDEs, adjoint-symmetries coincide with symmetries, and the linear actions themselves constitute new bilinear symmetry brackets which differ from the commutator bracket when acting on non-variational symmetries. In addition, one of the symmetry actions is shown to encode a pre-sympletic (Noether) operator and a symplectic 2-form, which lead to the construction of a Hamiltonian structure. Several examples of physically interesting nonlinear PDEs are used to illustrate all of the results.

1. Introduction

In the study of partial differential equations (PDEs), symmetries are a fundamental intrinsic (coordinate-free) structure of a PDE and have numerous important uses [1, 2, 3], such as finding exact solutions, mapping known solutions into new solutions, detecting integrability, and finding linearizing transformations. In addition, when a PDE has a variational principle, then through Noether’s theorem [2, 3] the infinitesimal symmetries of the PDE under which the variational principle is invariant — namely, variational symmetries — yield conservation laws.

Like symmetries, conservation laws [2, 3, 7] are another important intrinsic (coordinate-free) structure of a PDE. They provide conserved quantities and conserved norms, which are used in the analysis of solutions; they detect integrability and can be used to find linearizing
transformations; they also can be used to check the accuracy of numerical solution methods and give rise to discretizations with good properties.

A modern form of the Noether correspondence between variational symmetries and conservation laws has been developed in the past few decades [4, 5, 2, 6, 8, 9, 7] and generalized to non-variational PDEs. From a purely algebraic viewpoint, infinitesimal symmetries of a PDE are the solutions of the linearization (Frechet derivative) equation holding on the space of solutions to the PDE. Solutions of the adjoint linearization equation, holding on the space of solutions to the PDE, are called adjoint-symmetries [10, 11, 8]. In the generalization of the Noether correspondence, infinitesimal symmetries are replaced by adjoint-symmetries, and variational symmetries are replaced by multipliers which are adjoint-symmetries satisfying an Euler-Lagrange condition [4, 8, 9, 7].

As an important consequence, the problem of finding the conservation laws for a PDE is reduced to a kind of adjoint of the problem of finding the symmetries of the PDE. In particular, for any PDE system, conservation laws can be explicitly derived in a similar algorithmic way to the standard way that symmetries are derived (see Ref. [7] for a review).

These developments motivate studying the basic mathematical properties of adjoint-symmetries and their connections to infinitesimal symmetries. As is well known, the set of infinitesimal symmetries of a PDE has the structure of a Lie algebra, in which the subset of variational symmetries is a Lie subalgebra, and the set of conservation laws of a PDE is mapped into itself under the symmetries of the PDE. This leads to several interesting basic questions:

- How do symmetries act on adjoint-symmetries and multipliers?
- Does the set of adjoint-symmetries have any kind of algebraic structure, such as a generalized Lie bracket or Poisson bracket, with the set of multipliers inheriting a corresponding structure?
- Do there exist generalized analogs of Hamiltonian and (Noether) symplectic operators for general PDE systems?

In Ref. [12, 13], the explicit action of infinitesimal symmetries on multipliers is derived for general PDE systems and used to study invariance of conservation laws under symmetries. Recently in Ref. [14], for scalar PDEs, a linear mapping from infinitesimal symmetries into adjoint-symmetries is constructed in terms of any fixed adjoint-symmetry that is not a multiplier. This mapping can be viewed as a (Noether) pre-symplectic operator, in analogy with symplectic operators that map symmetries into adjoint-symmetries for Hamiltonian systems [15, 2]. The inverse mapping thus can be viewed as a pre-Hamiltonian operator.

The present paper is addressed to expanding substantially on this work and will give answers to the basic questions just posed for general PDE systems.

Firstly, it will be shown that there are two basic different actions of infinitesimal symmetries on adjoint-symmetries. One action represents a Lie derivative, and the other action comes from the adjoint relationship between the determining equation for infinitesimal symmetries and adjoint-symmetries. For adjoint-symmetries that are multipliers, these two actions coincide with the known action given in Ref. [12, 13]. Furthermore, the difference of the two actions produces a third action that vanishes on multipliers. This third action yields a generalization of the pre-symplectic operator for scalar PDEs, and its inverse provides a general pre-Hamiltonian operator. For evolution PDEs and Euler-Lagrange PDEs, this structure further yields a symplectic 2-form and an associated Poisson bracket, which can be
used to derive a corresponding Hamiltonian structure. In particular, Hamiltonian structures are naturally encoded in a simple way in the adjoint-symmetry structure of non-dissipative PDE systems.

Secondly, these three actions of infinitesimal symmetries on adjoint-symmetries will be used to construct associated bracket structures on the subset of adjoint-symmetries given by the range of each action. Two different constructions will be given: the first bracket is antisymmetric and can be viewed as a pull-back of the symmetry commutator (Lie bracket) to adjoint-symmetries; the second bracket is non-symmetric and does not utilize the commutator structure of symmetries. Most significantly, one of the antisymmetric brackets will be shown to satisfy the Jacobi identity, and thus it gives a Lie algebra structure to a natural subset of adjoint-symmetries. In certain situations, this subset will coincide with the whole set of adjoint-symmetries. More generally, a correspondence (homomorphism) will exist between Lie subalgebras of symmetries and adjoint-symmetries, which will hold even for dissipative PDEs that lack any local variational (Hamiltonian or Lagrangian) structure.

Thirdly, for Euler-Lagrange PDEs, adjoint-symmetries coincide with infinitesimal symmetries and thus each of the three symmetry actions themselves represents a bracket on the set of infinitesimal symmetries. One of these brackets reduces to the standard commutator (Lie bracket) when acting on variational symmetries. The other two brackets are new: one of them yields variational symmetries from symmetries, and the other one vanishes on variational symmetries.

All of these main results are new and provide important steps in understanding the basic algebraic structure of adjoint-symmetries and its application to pre-Hamiltonian operators, (Noether) pre-symplectic operators, and symplectic 2-forms for general PDE systems.

Apart from the intrinsic mathematical interest in developing and exploring such structures, a more applied utilization of the results is that symmetry actions on adjoint-symmetries can be used to produce a new adjoint-symmetry (and hence possibly a multiplier) from a known adjoint-symmetry and a known symmetry, while brackets on adjoint-symmetries allow a pair of known adjoint-symmetries to generate a new adjoint-symmetry (and hence possibly a multiplier), just as a pair of known symmetries can generate a new symmetry from their Lie bracket. Additional adjoint-symmetries can be obtained through the interplay of these structures.

The main results will be illustrated for six physical examples of PDE systems:
(1) a nonlinear reaction-diffusion system
(2) the Navier-Stokes equations for compressible viscous fluid flow
(3) coupled solitary wave equations
(4) a generalized nonlinear Schrodinger (NLS) equation
(5) a coupled Boussinesq system
(6) the free-space Maxwell’s equations

The first five PDE systems will be considered in one spatial dimension; the last PDE system will be considered in two spatial dimensions.

The rest of the paper is organized as follows. Section 2 gives a short review of infinitesimal symmetries, adjoint-symmetries, and multipliers, from an algebraic viewpoint. Section 3 presents the actions of infinitesimal symmetries on adjoint-symmetries and multipliers, and explains the construction of general pre-Hamiltonian and pre-symplectic (Noether) operators from these actions. Section 5 derives the bracket structures for adjoint-symmetries,
and discusses their properties. In particular, the conditions under which a Lie algebra structure arises for adjoint-symmetries from a commutator bracket is explained. Sections 7 and 8 specialize the results to Euler-Lagrange PDEs and evolution PDEs. Construction of a pre-symplectic operator and an associated symplectic 2-form and Poisson bracket is also explained. Section 8 contains the examples. Finally, section 9 provides some concluding remarks.

Throughout, the mathematical setting will be calculus in jet space [2], which is summarized in an Appendix. Partial derivatives and total derivatives will be denoted using a standard (multi-) index notation. The Frechet derivative will be denoted by \( ' \). Adjoints of total derivatives and linear operators will be denoted by \( * \). Prolongations will be denoted as \( \text{pr} \).

Hereafter, a “symmetry” will refer to an infinitesimal symmetry in evolutionary form.

2. Symmetries and adjoint-symmetries

An algebraic perspective will be utilized to allow symmetries and adjoint-symmetries to be defined and handled in a unified way (following Ref. [7]).

Consider a general PDE system of order \( N \) consisting of \( M \) equations

\[
G^A(x, u^{(N)}) = 0, \quad A = 1, \ldots, M
\]

where \( x^i, i = 1, \ldots, n \), are the independent variables, and \( u^\alpha, \alpha = 1, \ldots, m \), are the dependent variables. The space of formal solutions \( u^\alpha(x) \) of the PDE system will be denoted \( \mathcal{E} \). As is usual in symmetry theory [1, 2, 3], the PDE system will be assumed to be well posed in the sense that the standard tools of variational calculus in jet space can be applied. In particular, no integrability conditions are assumed to arise from the equations and their differential consequences; namely, the PDE system and its differential consequences are involutive. (A more precise formulation can be found in [2, 6, 16] from a geometric/algebraic point of view, and in [7] from a computational point of view.)

An underlying technical condition will be that a PDE system admits a solved-form for a set of leading derivatives, and likewise all differential consequences of the PDE system admit a solved-form in terms of differential consequences of the leading derivatives. This condition allows Hadamard’s lemma to hold in the setting of jet space [7].

**Lemma 2.1.** If a function \( f(x, u^{(k)}) \) vanishes on \( \mathcal{E} \) then \( f = R_f(G) \) holds identically, where \( R_f \) is some linear differential operator in total derivatives whose coefficients are functions that are non-singular on \( \mathcal{E} \).

When the preceding technical conditions hold, a PDE system will be called **regular**. Essentially all PDE systems of interest in physical applications are regular systems. (See Ref. [7, 17] for examples and further discussion.) Hereafter, only regular PDE systems are considered.

An additional technical condition, which is not needed for the main results, will be useful for certain developments. The proof is similar to that of the previous lemma [7].

**Lemma 2.2.** Suppose \( R(G) = 0 \) holds identically for a linear differential operator \( R \) in total derivatives whose coefficients are functions that are non-singular on \( \mathcal{E} \). If the PDE system \( G^A = 0 \) does not obey any differential identities, then \( R \) vanishes on \( \mathcal{E} \).
2.1. Determining equations and identities. An infinitesimal symmetry of a PDE system (2.1) is a set of functions $P^\alpha(x,u^{(k)})$ that are non-singular on $\mathcal{E}$ and satisfy
\[ G^\prime(P)^A|_\mathcal{E} = 0. \]  
(2.2)
This is the determining equation for $P^\alpha$, called the characteristic functions of the symmetry.

Off of the solution space $\mathcal{E}$, the symmetry determining equation is given by
\[ G^\prime(P)^A = R_P(G)^A \]  
(2.3)
(due to Lemma 2.1) where $R_P = (R_P)^A_B D_I$ is some linear differential operator in total derivatives whose coefficients $(R_P)^A_B$ are functions that are non-singular on $\mathcal{E}$.

The determining equation for adjoint-symmetries is the adjoint of the symmetry determining equation (2.2). It is obtained by using the Frechet derivative identity
\[ Q_A G^\prime(P)^A = P^\alpha G'^*(Q)_\alpha + D_I \Psi^i(P,Q). \]  
(2.4)
An adjoint-symmetry of a PDE system (2.1) is a set of functions $Q_A(x,u^{(k)})$ that are non-singular on $\mathcal{E}$ and satisfy
\[ G'^*(Q)_\alpha|_\mathcal{E} = 0. \]  
(2.5)

Off of the solution space $\mathcal{E}$, this determining equation is given by
\[ G'^*(Q)_\alpha = R_Q(G)_\alpha \]  
(2.6)
(again due to Lemma 2.1) where $R_Q = (R_Q)^I_B D_I$ is some linear differential operator in total derivatives whose coefficients $(R_Q)^I_B$ are functions that are non-singular on $\mathcal{E}$.

The geometrical meaning of symmetries is well known. From the algebraic viewpoint, it comes from the relation $G^\prime(P)^A = (\text{pr}P^\alpha \partial_{u^\alpha})G^A$ whereby the symmetry determining equation (2.2) can be expressed as
\[ ((\text{pr}P^\alpha \partial_{u^\alpha})G^A)|_\mathcal{E} = 0. \]  
(2.7)
This is usually the starting point for defining symmetries, since it indicates that $X_P = P^\alpha \partial_{u^\alpha}$ is a vector field that is tangent to surfaces $G^A = 0$ (and their prolongations $D^k G^A = 0, k = 0,1,2,\ldots$) in jet space. A geometrical meaning for adjoint-symmetries has recently been developed in Ref. [18], based on evolutionary 1-forms that functionally vanish on the solution space $\mathcal{E}$.

Recall that a multiplier is a set of functions $\Lambda_A(x,u^{(k)})$ that are non-singular on $\mathcal{E}$ and satisfy $\Lambda_A G^A = D_I \Psi^i$ off of $\mathcal{E}$, for some vector function $\Psi^i$ in jet space. This total divergence condition is equivalent to
\[ E_{u^\alpha}(\Lambda_A G^A) = 0. \]  
(2.8)
It can be further reformulated through the product rule of the Euler operator, which yields the equivalent condition $\Lambda'^*(G)_\alpha + G'^*(\Lambda)_\alpha = 0$. Consequently, on $\mathcal{E}$,
\[ G'^*(\Lambda)_\alpha|_\mathcal{E} = 0 \]  
(2.9)
whereby $\Lambda_A$ is an adjoint-symmetry. Off of $\mathcal{E}$, the adjoint-symmetry determining equation (2.6) yields
\[ G'^*(\Lambda)_\alpha = R_A(G)_\alpha \]  
(2.10)
where $R_A$ is a linear differential operator in total derivatives. Hence, one sees that $\Lambda'^*(G)_\alpha = -G'^*(\Lambda)_\alpha = -R_A(G)_\alpha$. Now suppose that $G^A = 0$ does not obey any differential identities. Then one can conclude (from Lemma 2.2) that $\Lambda'^* = -R_A + S^{I,J}(D_I G) D_J$ where $S^{I,J} = -S^{J,I}$ holds off of $\mathcal{E}$ and $S^{I,J}$ is non-singular on $\mathcal{E}$. Furthermore, suppose that $\Lambda_A$ contains no
Then one can assume without loss of generality that $S^{IJ} = 0$. Therefore, in this situation, $\Lambda^* = -R_\Lambda$ holds identically. The adjoint of this equation yields the relation

$$\Lambda' = -R'_\Lambda.$$  \hfill (2.11)

3. Action of symmetries on adjoint-symmetries

Symmetries of any given PDE system are well-known to form a Lie algebra via their commutators. From the algebraic viewpoint, if $P_1^\alpha$, $P_2^\alpha$ are symmetries, then so is the commutator defined by

$$[P_1, P_2]^\alpha = P_2'(P_1)^\alpha - P_1'(P_2)^\alpha.$$  \hfill (3.1)

The geometrical formulation is the same:

$$[\text{pr}X_{P_1}, \text{pr}X_{P_2}] = \text{pr}X_{[P_1, P_2]}.$$  \hfill (3.2)

Stated precisely, the set of symmetries comprises a linear space on which the commutator defines a bilinear antisymmetric bracket that obeys the Jacobi identity. This bracket is called the Lie bracket of the symmetry vector fields. Any symmetry has a natural action on the linear space of all symmetries via the algebraic commutator [3.1]. This action is commonly denoted by $\text{ad}(P_1)P_2 = [P_1, P_2]$.

Symmetries also have a natural action on the set of adjoint-symmetries, since this set is a linear space that is determined by the given PDE system whose solution set $\mathcal{E}$ is mapped into itself by a symmetry. Actually, there are two distinct actions of symmetries on the linear space of adjoint-symmetries, as shown next.

The first symmetry action arises directly from the prolonged action of a symmetry $P^\alpha$ applied to the adjoint-symmetry determining equation (2.6). To begin, from the lefthand side of this equation, one gets

$$\text{pr}X_P(G'^*(Q)_\alpha) = G'^*(\text{pr}X_P(Q))_\alpha + \text{pr}X_P(G'^*(Q))_\alpha.$$  \hfill (3.3)

The last term can be simplified by the following steps. First, one has $\text{pr}X_P(G'^*) = (\text{pr}X_P(G))' - P'^*G'^*$ (by identity [A.13]), whence $\text{pr}X_P(G'^*)(Q)_\alpha = (\text{pr}X_P(G))'*(Q)_\alpha - P'^*(G'^*(Q))_\alpha$. Second, through the symmetry equation (2.3), one can simplify $(\text{pr}X_P(G))'|_\mathcal{E} = (R_P(G))'|_\mathcal{E} = (R_PG')|^*_\mathcal{E} = G'^*R'_P|^*_\mathcal{E}$, where $R'_P$ is the adjoint of the linear differential operator $R_P$ (in total derivatives). Thus, expression (3.3) on $\mathcal{E}$ becomes

$$\text{pr}X_P(G'^*(Q))_\alpha|_\mathcal{E} = G'^*(Q'_P + R'_P(Q))_\alpha|_\mathcal{E}.$$  \hfill (3.4)

Next, from the righthand side of equation (2.6), one has

$$\text{pr}X_P(R_Q(G))_\alpha = (\text{pr}X_P R_Q)(G)_\alpha + R_Q(\text{pr}X_P(G))_\alpha.$$  \hfill (3.5)

On $\mathcal{E}$, this yields

$$\text{pr}X_P(R_Q(G))_\alpha|_\mathcal{E} = 0.$$  \hfill (3.6)

Finally, from equating expressions (3.6) and (3.4), one gets

$$G'^*(Q'_P + R'_P(Q))_\alpha|_\mathcal{E} = 0$$  \hfill (3.7)

which shows that $Q'_P + R'_P(Q)$ is an adjoint-symmetry. Therefore, this yields a linear mapping

$$Q_A \xrightarrow{X_P} Q'_P + R'_P(Q)_A$$  \hfill (3.8)

acting on the linear space of adjoint-symmetries.
This action (3.8) can be interpreted geometrically as a Lie derivative [18] and is a generalization of a better known action of symmetries on conservation law multipliers, which is found in Ref. [12, 13]. Further discussion is given in section 3.1.

The second symmetry action arises from the adjoint relation between the respective determining equations (2.2) and (2.5) for symmetries and adjoint-symmetries.

As is well known [10, 20, 8, 21], when \( P^\alpha \) is a symmetry and \( Q_A \) is an adjoint-symmetry, the adjoint relation (2.4) yields a conservation law since

\[
D_i \Psi^i (P, Q) |_{\mathcal{E}} = Q_A G^i (P) A |_{\mathcal{E}} - P^\alpha G^\alpha (Q) A |_{\mathcal{E}} = 0 \tag{3.9}
\]

from the determining equations (2.2) and (2.5). Off of \( \mathcal{E} \), this formula is given by

\[
D_i \Psi^i (P, Q) = Q_A R_P (G) A - P^\alpha R_Q (G) A \tag{3.10}
\]

and hence \( (R_P^* (Q) A - R_Q^* (P) A) G^A \) is a total divergence in jet space. This implies that the set of functions \( R_P^* (Q) A - R_Q^* (P) A \) constitute a conservation law multiplier. Since every multiplier is an adjoint-symmetry, there is a linear mapping

\[
Q_A \xrightarrow{X} R_P^* (Q) A - R_Q^* (P) A := \Lambda_A \tag{3.11}
\]

which acts on the linear space of adjoint-symmetries.

The preceding results are a full and complete generalization of the symmetry actions derived for scalar PDEs in Ref. [14]. They will now be summarized, and then some of their consequences will be developed.

**Theorem 3.1.** For any (regular) PDE system (2.1), there are two actions (3.8) and (3.11) of symmetries on the linear space of adjoint-symmetries. The second symmetry action (3.11) maps adjoint-symmetries into conservation law multipliers. The difference of the first and second actions yields the linear mapping

\[
Q_A \xrightarrow{X} Q' (P) A + R_Q^* (P) A. \tag{3.12}
\]

The action (3.12) will be trivial when the adjoint-symmetry is a conservation law multiplier, as follows from the relation (2.11) which holds under certain mild conditions on the form of the PDE system \( G^A = 0 \) (Lemma 2.2) and the functions \( Q_A \).

**Proposition 3.2.** For a (regular) PDE system \( G^A = 0 \) with no differential identities, the symmetry action (3.12) on adjoint-symmetries \( Q_A \) that contain no leading derivatives (and their differential consequences) in the PDE system is trivial iff \( Q_A \) is a conservation law multiplier.

The conditions in Proposition 3.2 are satisfied by evolution PDEs, as shown in section 6.
Thus,
\[ \text{pr}X_P(A_\Lambda G^A) = (\Lambda'(P)_A + R^*_p(A)_A)G^A \quad \text{modulo total derivatives.} \quad (3.15) \]

Now, from equating expressions (3.15) and (3.13), one concludes that \((\Lambda'(P)_A + R^*_p(A)_A)G^A\) is a total derivative. Therefore, \(\Lambda'(P)_A + R^*_p(A)_A\) is a multiplier.

This yields the following well-known action [12, 13]:
\[ \Lambda_A \xrightarrow{X_p} \Lambda'(P)_A + R^*_p(A)_A. \quad (3.16) \]

Theorem 3.3 shows that this action extends from conservation law multipliers to adjoint-symmetries through the symmetry action (3.8) on adjoint-symmetries.

3.2. **Action of Lie point symmetries.** An explicit expression for the first symmetry action (3.8) in Theorem 3.1 can be derived in the case of Lie point symmetries.

A **Lie point symmetry** vector field has the form [2, 3]
\[ X_p = P^\alpha_p \partial_{u^\alpha}, \quad P^\alpha_p = \eta^\alpha(x, u) - \xi^i(x, u)u^\alpha_i, \quad (3.17) \]

which generates a point transformation group acting on the space \((x, u)\), as given by exponentiation of the corresponding canonical vector field
\[ Y_p = \xi^i \partial_{x^i} + \eta^\alpha \partial_{u^\alpha}. \quad (3.18) \]

The prolongations of these vector fields are related by [2, 3]
\[ \text{pr}Y_p = \xi^i D_i + \text{pr}X. \quad (3.19) \]

A function \(F(x, u^{(k)})\) is **symmetry invariant** iff \(\text{pr}Y_p F\) vanishes identically. More generally, a function \(F(x, u^{(k)})\) is **symmetry homogeneous** iff \(\text{pr}Y_p F = \sigma F\) holds identically for some function \(\sigma F(x, u)\).

The symmetry determining equation (2.3) for Lie point symmetries can be expressed as
\[ \text{pr}Y_p(G) = R_p(G) \quad (3.20) \]

where \(R_p = (R_p)_B^A D_I\) is some linear differential operator in total derivatives whose coefficients \((R_p)_B^A\) are functions that are non-singular on \(E\). When every PDE in the system \(G^A = 0\) has the same differential order, and the system has no differential identities, then \(R_p\) will be purely algebraic, namely \((R_p)_B^A\) vanishes for \(I \neq \emptyset\).

**Proposition 3.3.** The first symmetry action (3.8) for a Lie point symmetry (3.17) on an adjoint-symmetry is given by
\[ Q_A \xrightarrow{X_p} Y_p(Q)_A + R^*_p(Q)_A + (D_i \xi^i)Q_A \quad (3.21) \]

where \(R^*_p\) is the adjoint of \(R_p\).

The proof is a straightforward computation of the terms \(Q'(P)_A + R^*_p(Q)_A\) in the action (3.8). One has \(Q'(P)_A = \text{pr}Y_p(Q)_A - \xi^i D_i Q_A\) and \(R^*_p(G)_A = R_p(G) - \xi^i D_i G^A\) from identity (3.19). Hence, \(R^*_p(Q)_A = R^*_p(Q)_A + D_i (\xi^i Q_A)\), and thus after cancellation of terms, one obtains the action (3.21).

Similar explicit expressions can be obtained for the other two symmetry actions (3.11), (3.12) in Theorem 3.1 in the case of adjoint-symmetries with a first-order linear form
\[ Q_A = \kappa_A(x, u) + \rho_{\Lambda \alpha}^I (x, u)u^\alpha_i. \quad (3.22) \]
This form is a counterpart of Lie point symmetries (more generally, first-order linear symmetries). The adjoint-symmetry determining equation (2.6) implies that

\[
G'^\alpha(Q)_\alpha = \rho^\alpha_{A\alpha} D_i G^A + K_{A\alpha} G^A
\]  

(3.23)

for some functions \(K_{A\alpha}\) that are non-singular on \(\mathcal{E}\), when every PDE in the system \(G^A = 0\) has the same differential order, and the system has no differential identities.

This leads to the following result.

**Proposition 3.4.** For a Lie point symmetry (3.17), the second and third symmetry actions (3.11) and (3.12) on a first-order linear adjoint-symmetry (3.22) are given by

\[
Q_A \xrightarrow{X_p} R_p^*(Q)_A + u^\alpha_j D_i (2\xi^i \rho^j_{A\alpha}) + D_i (\xi^i \kappa_A + \rho^i_{A\alpha} \eta^\alpha) - K_{A\alpha} (\eta^\alpha - \xi^i u^\alpha_i),
\]  

(3.24)

\[
Q_A \xrightarrow{X_p} Y_p(Q)_A + (D_i \xi^i) Q_A - u^\alpha_j D_i (2\xi^i \rho^j_{A\alpha}) - D_i (\xi^i \kappa_A + \rho^i_{A\alpha} \eta^\alpha) + K_{A\alpha} (\eta^\alpha - \xi^i u^\alpha_i),
\]  

(3.25)

where \(R_p^*\) is the adjoint of \(R_p\).

The proof is similar to that for the action (3.5). One has \(R_p^*(Q)_A = R_p^*(Q)_A + D_i (\xi^i Q)\), where \(D_i (\xi^i Q) = D_i (\xi^i \kappa_A) + D_i (\xi^i \rho^j_{A\alpha}) u^\alpha_j + \xi^i \rho^j_{A\alpha} u^\alpha_j\). Next, from relation (3.23), one obtains \(R_p^*(P_p)_A = K_{A\alpha} P_{\alpha} - D_i (\rho^i_{A\alpha} P^\alpha_p)\), where \(D_i (\rho^i_{A\alpha} P^\alpha_p) = D_i (\rho^i_{A\alpha} \eta^\alpha) - D_i (\rho^i_{A\alpha} \xi^j) u^\alpha_j - \rho^i_{A\alpha} \xi^j u^\alpha_j\) and \(K_{A\alpha} P^\alpha_p = K_{A\alpha} (\eta^\alpha - \xi^i u^\alpha_i)\). Then, combining the terms \(R_p^*(Q)_A - R_p^*(P_p)_A\), one gets expression (3.24). Likewise, combining the terms \(Q'(P_p)_A + R_p^*(P_p)_A\) yields expression (3.25).

Two basic types of Lie point symmetries which appear in numerous applications are translations \(Y_{\text{trans.}} = a^i \partial_{x^i}\) and scalings \(Y_{\text{scal}} = w_{(a)} x^i + w_{(i)} a^i \partial_{a^i}\). Here the vector \(a^i\) represents the direction of the translation; the scalars \(w_{(a)}, w_{(i)}\) represent the scaling weights of \(a^i\) and \(x^i\). The corresponding evolutionary form of these symmetries is given by

\[
P^\alpha_{\text{trans.}} = -a^i u^\alpha_i
\]  

(3.26)

and

\[
P_{\text{scal.}} = w_{(a)} u^\alpha_a - w_{(i)} x^i u^\alpha_i.
\]  

(3.27)

Their action on adjoint-symmetries has a very simple form, which is an immediate consequence of Propositions 3.3 and 3.4.

**Corollary 3.5.** (i) Suppose \(Q_A\) and \(G^A\) are translation invariant: \(Y_{\text{trans.}}(Q)_A = 0\) and \(Y_{\text{trans.}}(G)^A = 0\). Then the three symmetry actions respectively consist of

\[
Q_A \xrightarrow{X_p} 0,
\]  

(3.28)

\[
Q_A \xrightarrow{X_p} 2u^\alpha_j a^i [D_i \rho^j_{A\alpha}],
\]  

(3.29)

\[
Q_A \xrightarrow{X_p} -2u^\alpha_j a^i [D_i \rho^j_{A\alpha} - a^i D_i \kappa_A - a^i u^\alpha_i K_{A\alpha}],
\]  

(3.30)
(ii) Suppose $Q_A$ and $G^A$ are scaling homogeneous: $Y_{\text{scal}}(Q)_A = w(A)Q_A$ and $Y_{\text{scal}}(G)^A = \omega(A)G^A$. Then the three symmetry actions respectively consist of

\begin{align}
Q_A \xrightarrow{X_p} (\omega(A) + w(A) + \sum_i w(i))Q_A, \\
Q_A \xrightarrow{X_p} \omega(A)Q_A + u^\alpha w(i)D_i(2x^i\rho_{Ax}^j) + w(i)D_i(x^i\kappa_A) + w(\alpha)D_i(\rho_{Ax}^j u^\alpha) - K_{Ax}(w(\alpha)u^\alpha - w(i)x^i u^\alpha), \\
Q_A \xrightarrow{X_p} (w(A) + \sum_i w(i))Q_A - u_j^\alpha w(i)D_i(2x^i\rho_{Ax}^j) - w(i)D_i(x^i\kappa_A) + w(\alpha)D_i(\rho_{Ax}^j u^\alpha) + K_{Ax}(w(\alpha)u^\alpha - w(i)x^i u^\alpha).
\end{align}

For both translations and scalings, the second and third symmetry actions here are considered only for first-order linear adjoint-symmetries (3.22) - (3.23).

The first symmetry actions (3.28) and (3.31) are a generalization of the same actions derived on multipliers in Ref. [12, 21]. The other results are new.

4. Generalized pre-symplectic and pre-Hamiltonian structures (Noether operators) from symmetry actions

It will be useful to begin with a general discussion. Let

\begin{align}
\text{Symm}_G := \{P^\alpha(x,u^{(k)}), k \geq 0, \text{ s.t. } G'(P)^A|_E = 0\} \\
\text{AdjSymm}_G := \{Q_A(x,u^{(k)}), k \geq 0, \text{ s.t. } G^*(Q)_\alpha|_E = 0\}
\end{align}

denote the linear spaces of symmetries and adjoint-symmetries for a given PDE system $G^A(x,u^{(N)}) = 0$. Also, let

\begin{align}
\text{Multr}_G := \{\Lambda_A(x,u^{(k)}), k \geq 0, \text{ s.t. } G^*(\Lambda)_\alpha + \Lambda^*(G)_\alpha = 0\}
\end{align}

denote the linear space of multipliers, which is a subspace of the linear space of adjoint-symmetries (4.2).

Suppose that the PDE system possesses the extra structure

\begin{align}
DG' = G^*J
\end{align}

where $D$ and $J$ are linear differential operators in total derivatives whose coefficients are non-singular on $E$. Then, for any symmetry $P^\alpha$, $G^*(J(P))|_E = DG'(P)|_E = 0$ shows that

\begin{align}
J(P)_A := Q_A
\end{align}

is an adjoint-symmetry. If $J(P)_A$ is a multiplier, then $J$ represents a pre-symplectic operator for the PDE system, in the sense that it is a mapping from Symm$_G$ into Multr$_G$, analogous to a symplectic operator in the case of Hamiltonian systems. When $J(P)_A$ is an adjoint-symmetry but not a multiplier, it will be called a Noether operator [15].

Similarly, suppose that a PDE system (2.1) possesses the extra structure

\begin{align}
DG'^* = G'H
\end{align}

where $D$ and $H$ are linear differential operators in total derivatives whose coefficients are non-singular on $E$. For any adjoint-symmetry $Q_A$, $G'(H(Q))|_E = DG'^*(Q)|_E = 0$ whereby

\begin{align}
H(Q)_\alpha := P^\alpha
\end{align}
is a symmetry. Since $H$ is a mapping from $\text{AdjSymm}_G \supseteq \text{Mult}_{G}$ into $\text{Symm}_G$, it represents a pre-Hamiltonian operator (or inverse Noether operator) for the PDE system, analogous to a Hamiltonian operator in the case of Hamiltonian systems [15].

When the inverses of $J$ and $H$ are well defined, then $J^{-1} := H$ defines a pre-Hamiltonian (inverse Noether) operator, and $H^{-1} := J$ defines a Noether operator.

These definitions can be generalized to allow $J$, $H$, and $D$ to be linear operators in partial derivatives with respect to jet space variables in addition to total derivatives. In this case, $J$ and $H$ will be respectively called a generalized pre-symplectic (Noether) structure and a generalized pre-Hamiltonian (inverse Noether) structure.

**Remark 4.1.** For $H$ to be a Hamiltonian structure, there must exist a non-degenerate integral pairing $\langle Q, P \rangle$ (modulo total derivatives) between symmetries and adjoint-symmetries such that $\{Q_1, Q_2\}_H := \langle Q_1, H(Q_2) \rangle$ is a Poisson bracket, namely it must be skew-symmetric and satisfy the Jacobi identity. Similarly, for $J$ to be a symplectic structure, the analogous bilinear-form $\omega_J(P_1, P_2) := \langle J(P_1), P_2 \rangle$ must be skew-symmetric and closed.

**Proposition 4.2.** The Frechet derivative identity (2.4) provides a natural integral pairing
\[
\langle Q, P \rangle = \int_{\Omega} \Psi^i(P, Q) \hat{n}_i \, dV
\] (4.8)
defined on a suitable domain $\Omega$ of codimension 1 in $\mathbb{R}^n$, with $\hat{n}_i$ denoting a unit normal 1-form of $\Omega$.

Now, it will be shown how an action of symmetries on adjoint-symmetries can be used itself to define a generalized pre-symplectic (Noether) structure and, when its inverse exists, a generalized pre-Hamiltonian (inverse Noether) structure.

Consider, in general, any symmetry action
\[
Q_A \xrightarrow{X_P} S_P(Q)_A
\] (4.9)
on $\text{AdjSymm}_G$, where $S_P$ is a linear operator which is also linear in $P^\alpha$. Note that $S_P$ may be constructed from both total derivatives $D_I$ and partial derivatives $\partial_{u_I}$. The action $S_P(Q)_A$ also defines a dual linear operator
\[
S_Q(P)_A := S_P(Q)_A
\] (4.10)
from $\text{Symm}_G$ into $\text{AdjSymm}_G$, which constitutes a generalized pre-symplectic (Noether) structure. For a fixed adjoint-symmetry $Q_A$, $S_Q$ will have an inverse $S_Q^{-1}$ which is defined modulo its kernel, $\ker(S_Q) \subset \text{Symm}_G$, and which acts on the linear subspace given by its range, $S_Q(\text{Symm}_G) \subset \text{AdjSymm}_G$. This inverse $S_Q^{-1}$ constitutes a generalized pre-Hamiltonian (inverse Noether) structure when $S_Q(\text{Symm}_G) = \text{AdjSymm}_G$, and otherwise it is a restricted type of that structure.

From the three symmetry actions in Theorem 3.1, the following structures are obtained.

**Theorem 4.3.** For a general PDE system (2.1), let $Q_A$ be any fixed adjoint-symmetry. Then, a generalized Noether structure is given by the first symmetry action (3.8),
\[
J_1(P)_A := S_1 Q(P)_A = Q(P)_A + R^*_P(Q)_A;
\] (4.11)
a generalized pre-symplectic structure is given by and the second symmetry action (3.11),
\[
J_2(P)_A := S_2 Q(P)_A = R^*_P(Q)_A - R_Q(P)_A;
\] (4.12)
a Noether operator is given by the third symmetry action \((3.12)\),
\[
J_3 := S_3 Q = Q' + R^*_Q.
\] (4.13)
The formal inverse of each structure \((4.11)\) and \((4.12)\) gives a generalized pre-Hamiltonian (inverse Noether) structure, while the formal inverse of the operator \((4.13)\) gives a pre-Hamiltonian (inverse Noether) operator.

The statement about the inverse of \(J_3\) is proven as follows, relying on a direct derivation of the symmetry action \(S_3 P(Q)_A = Q'(P)_A + R^*_Q(P)_A\). Similar proofs hold for the inverse of \(J_1\) and \(J_2\), using the derivations that were given in establishing Theorem \(3.11\).

For any set of differential functions \(P^\alpha\), one has \(\text{pr} X_P (G^{**}(Q)_\alpha - R_Q(G)_\alpha) = 0\) from the determining equation \((2.6)\), where \(Q_A\) is any fixed adjoint-symmetry. One also has \(E^{\nu\alpha} (P^\beta G^{**}(Q)_\beta - Q_A G'(P)^A) = 0\) from the adjoint relation \((2.4)\). These two expressions can be simplified, on \(\mathcal{E}\), by the following steps with \(H^A := G'(P)^A - R_P(G)^A\):
\[
\begin{align*}
E^{\nu\alpha} (P^\beta G^{**}(Q)_\beta - Q_A G'(P)^A)|_\mathcal{E} &= E^{\nu\alpha} (P^\beta R_Q(G)_\beta - Q_A R_P(G)^A)|_\mathcal{E} - E^{\nu\alpha} (Q_A H^A)|_\mathcal{E} \\
&= E^{\nu\alpha} (G^A (R^*_Q(P) - R^*_P(Q))^A)|_\mathcal{E} - E^{\nu\alpha} (Q_A H^A)|_\mathcal{E} \\
&= G^{**} (R^*_Q(P) - R^*_P(Q))|_\mathcal{E} - Q^*(H)|_\mathcal{E} - H^*(Q)|_\mathcal{E},
\end{align*}
\] (4.14)
which has used the product rule for the Euler operator and integration by parts; and
\[
\begin{align*}
\langle \text{pr} X_P (G^{**}(Q) - R_Q(G)) \rangle|_\mathcal{E} &= G''(Q'(P))|_\mathcal{E} + (G'(P))^{*\alpha}(Q)|_\mathcal{E} - R_Q(G'(P))|_\mathcal{E} \\
&= (\text{pr} X_F f^*) = (\text{pr} X_F f^*) - F^* f^* \\
&= G^{**} (Q'(P))|_\mathcal{E} + (R_P(G))^{*\alpha}(Q)|_\mathcal{E} + H^*(Q)|_\mathcal{E} - R_Q(H)|_\mathcal{E} \\
&= G^{**} (Q'(P) + R^*_P(Q))|_\mathcal{E} + H^*(Q)|_\mathcal{E} - R_Q(H)|_\mathcal{E},
\end{align*}
\] (4.15)
which has used the identity \((A.13)\), combined with the adjoint-symmetry determining equation \((2.5)\), in addition to \((R_P(G))^{*\alpha}|_\mathcal{E} = (R_P G^*)|_\mathcal{E} = G^{**} R^*_P|_\mathcal{E}\). Then, combining the two expressions \((4.14)\) and \((4.15)\), both of which vanish, one obtains
\[
0 = E^{\nu\alpha} (P^\beta G^{**}(Q)_\beta - Q_A G'(P)^A)|_\mathcal{E} + (\text{pr} X_P (G^{**}(Q) - R_Q(G))|_\mathcal{E}
\]
\[
= G^{**} (Q'(P) + R^*_P(Q))|_\mathcal{E} - (Q^*(H) + R_Q(H))|_\mathcal{E}. (4.16)
\]
This equation shows that \(G^{**}(S_3 Q(P))|_\mathcal{E} = 0\) iff \((Q^* + R_Q) H|_\mathcal{E} = 0\). Assuming that \(Q^* + R_Q\) is formally invertible, one can conclude that \(H^A|_\mathcal{E} = 0\) whenever \(S_3 Q(P)_A\) is an adjoint-symmetry, thereby showing that \(P^\alpha\) is a symmetry. Hence, \(S_3^{-1} = J_3^{-1}\) maps adjoint-symmetries into symmetries. This completes the proof.

It is worth noting that this proof gives the relation \(G^{**}(J_3(P)) = J_3^{**}(G'(P))\) on \(\mathcal{E}\), which is stronger than the structure \((4.13)\).

Further development of the structures in Theorem \(4.3\) will given for evolution PDEs in section \(6\) and Euler-Lagrange PDEs in section \(7\).

5. Bracket structures for adjoint-symmetries

The commutator \((3.1)\) of symmetries defines a Lie bracket on the linear space of symmetries \((4.1)\). An interesting fundamental question is whether there exists any bilinear bracket on the linear space of adjoint-symmetries \((4.2)\). Such a structure would allow the possibility for
a pair of known adjoint-symmetries to generate a new adjoint-symmetry, just as a pair of
known symmetries can generate a new symmetry.

Every action of symmetries on adjoint-symmetries will now be shown to give rise to two
different bilinear bracket structures on adjoint-symmetries. The first bracket is a Lie bracket
constructed from the pull-back of the symmetry commutator \((3.1)\) under an inverse of the
symmetry action on adjoint-symmetries. This yields a homomorphism from the Lie algebra
of symmetries into a Lie algebra of adjoint-symmetries. The second bracket does not involve
the symmetry commutator \((3.1)\) and instead uses the symmetry action composed with an
inverse action to construct a recursion operator on adjoint-symmetries.

These constructions will be carried out in terms of the dual linear operator \((4.10)\) associated
to a general symmetry action \((1.9)\). Afterward, the properties of the resulting brackets
will be discussed for each of the three actions \((3.8), (3.11), (3.12)\).

5.1. Adjoint-symmetry commutator brackets from symmetry actions. The con-
struction of the first bracket goes as follows.

**Proposition 5.1.** Fix an adjoint-symmetry \(Q_A\) in \(\text{AdjSymm}_G\), and let \(S_Q\) be the dual linear
operator \((4.10)\) associated to a symmetry action \(S_P\) on \(\text{AdjSymm}_G\). If the kernel of \(S_Q\) is
an ideal in \(\text{Symm}_G\), then

\[
Q[Q_1, Q_2]_A := S_Q([S_Q^{-1} Q_1, S_Q^{-1} Q_2])_A
\]

defines a bilinear bracket on the linear space \(S_Q(\text{Symm}_G) \subseteq \text{AdjSymm}_G\). This bracket can
be expressed as

\[
Q[Q_1, Q_2]_A = Q'_2(S_Q^{-1} Q_1) - Q'_1(S_Q^{-1} Q_2) - S'_Q(S_Q^{-1} Q_2)(S_Q^{-1} Q_1) + S'_Q(S_Q^{-1} Q_1)(S_Q^{-1} Q_2)
\]

where \(S'_Q\) denotes the Frechet derivative of \(S_Q\).

Any one of the symmetry actions \((3.8), (3.11), (3.12)\) can be used to write down formally
a corresponding bracket \((5.1)\). However, \(S_Q^{-1}\) is well-defined only modulo \(\ker(S_Q)\), and so
in the absence of any extra structure to fix this arbitrariness, the condition that \(\ker(S_Q)\)
is an ideal is necessary and sufficient for the bracket to be well defined (namely, invariant
under \(S_Q^{-1} \rightarrow S_Q^{-1} + \ker(S_Q)\)). This condition will select a set of adjoint-symmetries \(Q_A\)
that can be used in constructing the bracket. When \(\ker(S_Q)\) is an ideal, so is \(\ker(S_Q) = \lambda \ker(S_Q)\),
for any constant \(\lambda\). Hence, the set of adjoint-symmetries \(Q_A\) for which \(\ker(S_Q)\)
is an ideal comprises a projective subspace in \(\text{AdjSymm}_G\). In the case when the dimension
of this subspace is larger than 1, it is natural to select \(Q_A\) such that \(\text{ran}(S_Q)\) is maximal in
\(\text{AdjSymm}_G\).

For the space \(\ker(S_Q) \subseteq \text{Symm}_G\) to be an ideal, it must be a subalgebra that is preserved
by the action of \(\text{Symm}_G\) given by the Lie bracket \((3.1)\). The subalgebra condition

\[
[\ker(S_Q), \ker(S_Q)] \subseteq \ker(S_Q)
\]

states that \(S_Q([P_1, P_2]) = 0\) is required to hold for all pairs of symmetries \(X_{P_1} = P_1^\alpha \partial_{\alpha}\) and
\(X_{P_2} = P_2^\alpha \partial_{\alpha}\) such that \(S_Q(P_1)_A = S_P(Q)_A = 0\) and \(S_Q(P_2)_A = S_P(Q)_A = 0\). The question
of whether this condition \((5.3)\) is satisfied by each of the three symmetry actions will now
be addressed.

For the first symmetry action \((3.8)\), consider

\[
0 = S_1 Q(P_1)_A = Q'(P_1)_A + R_{P_1}^*(Q)_A, \quad 0 = S_1 Q(P_2)_A = Q'(P_2)_A + R_{P_2}^*(Q)_A.
\]
Applying the symmetries $X_{P_2}$ and $X_{P_1}$ respectively to these two equations and subtracting them yields

$$0 = \text{pr}X_{P_2}(Q'(P_1)_A + R_{P_1}^*(Q)_A) - \text{pr}X_{P_1}(Q'(P_2)_A + R_{P_2}^*(Q)_A)$$

$$= Q'(\text{pr}X_{P_2}(P_1) - \text{pr}X_{P_1}(P_2)) + \text{pr}X_{P_2}(R_{P_1}^*)(Q)_A - \text{pr}X_{P_1}(R_{P_2}^*)(Q)_A$$

$$+ R_{P_1}(\text{pr}X_{P_2}(Q))_A - R_{P_2}(\text{pr}X_{P_1}(Q))_A$$

(5.5)

using $\text{pr}X_{P_2}Q'(P_1) - \text{pr}X_{P_1}Q'(P_2) = Q''(P_2, P_1) - Q''(P_1, P_2) = 0$. The first term in equation (5.3) reduces to a commutator expression

$$Q'(\text{pr}X_{P_2}(P_1) - \text{pr}X_{P_1}(P_2))_A = Q'([P_2, P_1])_A.$$  

(5.6)

The middle two terms can be expressed as

$$\text{pr}X_{P_2}(R_{P_1}^*)(Q)_A - \text{pr}X_{P_1}(R_{P_2}^*)(Q)_A = R_{[P_2, P_1]}^*(Q)_A - [R_{P_2}^*, R_{P_1}^*](Q)_A$$

(5.7)

by use of the identity

$$R_{[P_2, P_1]}^* = [R_{P_2}^*, R_{P_1}^*] + \text{pr}X_{P_2}(R_{P_1}^*) - \text{pr}X_{P_1}(R_{P_2}^*)$$

(5.8)

which can derived straightforwardly from the symmetry determining equation (2.3) off of $E$. Next, the last term in equation (5.5) can be combined with the last two terms in equation (5.3), yielding

$$R_{P_1}^*(Q'(P_2)_A + R_{P_2}^*(Q)_A) - R_{P_2}^*(Q'(P_1)_A + R_{P_1}^*(Q)_A) = 0$$

(5.9)

de due to equations (5.4). Hence, after these simplifications, equation (5.5) becomes $0 = Q'([P_2, P_1])_A + R_{[P_2, P_1]}^*(Q)_A = S_1 Q([P_2, P_1])_A$. This establishes the following result.

**Lemma 5.2.** For the first symmetry action (3.8), $\ker(S_Q)$ is a subalgebra in $\text{Symm}_G$.

To continue, consider the third symmetry action (3.12). Similar steps will now be carried out, starting from

$$0 = S_3 Q([P_2, P_1])_A = Q'(P_1)_A + R_{Q}^*(P_1)_A.$$

$$0 = S_3 Q(P_2)_A = Q'(P_2)_A + R_{Q}^*(P_2)_A.$$  

(5.10)

Respectively applying the symmetries $X_{P_2}$ and $X_{P_1}$ to these two equations and subtracting, one obtains

$$0 = Q'([P_2, P_1])_A + R_{Q}^*([P_2, P_1])_A + \text{pr}X_{P_2}(R_{Q}^*)(P_1)_A - \text{pr}X_{P_1}(R_{Q}^*)(P_2)_A.$$  

(5.11)

Hence, one sees that $S_3 Q([P_2, P_1]) = Q'([P_2, P_1])_A + R_{Q}^*([P_2, P_1])_A = \text{pr}X_{P_2}(R_{Q}^*)(P_2)_A - \text{pr}X_{P_2}(R_{Q}^*)(P_1)_A$ does not vanish in general. This represents an obstruction to the bracket being well-defined. A useful remark is that if $Q = \Lambda$ is a conservation law multiplier for a PDE system with no differential identities (Lemma 2.2), then the relation (2.11) shows that

$$\text{pr}X_{P_2}(R_{Q}^*)(P_2)_A - \text{pr}X_{P_2}(R_{Q}^*)(P_1)_A = \text{pr}X_{P_2}(Q')(P_1)_A - \text{pr}X_{P_1}(Q')(P_2)_A$$

$$= Q''(P_1, P_2) - Q''(P_2, P_1) = 0$$

(5.12)

whereby the obstruction vanishes.

A similar obstruction arises for the bracket given by the second symmetry action (3.11). Specifically, by the same steps used for the first and third symmetry actions, one obtains $S_2 Q([P_2, P_1]) = R_{[P_2, P_1]}^*(Q)_A - R_{Q}^*([P_2, P_1])_A = \text{pr}X_{P_2}(R_{Q}^*)(P_1)_A - \text{pr}X_{P_1}(R_{Q}^*)(P_2)_A + R_{P_2}(S_1 Q(P_1)_A) - R_{P_1}(S_1 Q(P_2))_A$. This expression contains the same obstruction terms as for the third symmetry action, as well as terms that involve the first symmetry action itself.
If \( Q = \Lambda \) is a conservation law multiplier for a PDE system with no differential identities (Lemma 2.2), then this obstruction vanishes.

Consequently, the following two results have been established.

**Lemma 5.3.** For the third symmetry action (3.12), \( \ker(S_{3Q}) \) is a subalgebra in Symm\(_G\) iff the condition

\[
\text{pr}X_{P_2}(R^*_{Q})(P_1)_A - \text{pr}X_{P_1}(R^*_{Q})(P_2)_A = 0
\]

(5.13)

holds for all symmetries \( X_{P_1} = P_1^a \partial_{\alpha} \) and \( X_{P_2} = P_2^a \partial_{\alpha} \) in \( \ker(S_{3Q}) \).

**Lemma 5.4.** For the second symmetry action (3.11), \( \ker(S_{2Q}) \) is a subalgebra in Symm\(_G\) iff the condition

\[
\text{pr}X_{P_2}(R^*_{Q})(P_1)_A - \text{pr}X_{P_1}(R^*_{Q})(P_2)_A + R^*_P(S_1Q(P_1))_A - R^*_P(S_1Q(P_2))_A = 0
\]

(5.14)

holds for all symmetries \( X_{P_1} = P_1^a \partial_{\alpha} \) and \( X_{P_2} = P_2^a \partial_{\alpha} \) in \( \ker(S_{2Q}) \).

The preceding developments can be summarized as follows.

**Proposition 5.5.** The adjoint-symmetry commutator bracket (5.1) associated to each of the symmetry actions (3.8), (3.11), (3.12) is well-defined on \( S_Q(\text{Symm}_G) \subseteq \text{AdjSymm}_G \) if \( \text{ad}(\text{Symm}_G)\ker(S_Q) \subseteq \ker(S_Q) \) and, for the actions (3.11) and (3.12), if the respective conditions (5.14) and (5.13) hold when \( \dim \ker(S_Q) > 1 \). These latter conditions are identically satisfied when \( Q \) is a conservation law multiplier for a PDE system with no differential identities.

An alternative way to have the bracket be well defined is if the quotient \( \text{Symm}_G/\ker(S_Q) \) can be naturally identified with a subspace in \( \text{Symm}_G \). This is equivalent to requiring that the symmetry Lie algebra admits an extra structure of a direct-sum decomposition as a linear space

\[
\text{Symm}_G = \ker(S_Q) \oplus \text{coker}(S_Q)
\]

(5.15)

such that the decomposition is independent of a choice of basis. Then \( S_Q^{-1} \) can be defined as belonging to the subspace \( \text{coker}(S_Q) \), and hence the bracket will be well defined.

It will now be shown that the extra structure (5.15) will typically exist for a symmetry Lie algebra that contains a scaling symmetry (3.27).

Every symmetry in \( \text{Symm}_G \) can be decomposed into a sum of symmetries that are scaling homogeneous. Consequently, there will exist a basis for \( \text{Symm}_G \) consisting of \( P_{\text{scal}} \) and \( \{P_k\}_{k=1, \ldots, \dim(\text{Symm}_G)-1} \), such that \( [P_{\text{scal}}, P_k] = r^{(k)}P_k \) where the constant \( r^{(k)} \) is the scaling weight of the symmetry \( P_k \). Then there exists a direct-sum decomposition

\[
\text{Symm}_G = \text{span}(P_{\text{scal}}) \oplus \bigoplus_k \text{span}(P_k).
\]

(5.16)

which is basis independent. This will provide the extra structure (5.15) if the subspaces \( \ker(S_Q) \) and \( \text{coker}(S_Q) \) can be uniquely characterized in terms of their scaling weights.

**Proposition 5.6.** Suppose \( \text{Symm}_G \) contains a scaling symmetry (3.27). For each of the symmetry actions (3.8), (3.11), (3.12), if \( \ker(S_Q) \) is a scaling-homogeneous subspace in \( \text{Symm}_G \), then the adjoint-symmetry commutator bracket (5.1) is well-defined on the linear space \( S_Q(\text{Symm}_G) \subseteq \text{AdjSymm}_G \) by taking \( S_Q^{-1} \) to belong to a sum of scaling-homogeneous subspaces with scaling weights that are different than that of \( \ker(S_Q) \).
This result can be generalized if \( \ker(S_Q) \) is a direct sum of scaling homogeneous subspaces that have no scaling weights in common with any scaling homogeneous subspace in \( \coker(S_Q) \).

Now, the basic properties of the general adjoint-symmetry commutator bracket \((5.1)\) will be studied. Recall that the underlying symmetry commutator bracket is antisymmetric and obeys the Jacobi identity. This implies that the same properties are inherited by the bracket \((5.1)\).

**Theorem 5.7.** The adjoint-symmetry commutator bracket \((5.1)\) is a Lie bracket, namely it is antisymmetric
\[
Q[Q_1, Q_2]_A + Q[Q_2, Q_1]_A = 0
\]
and obeys the Jacobi identity
\[
Q[Q_1, [Q_2, Q_3]]_A + Q[Q_2, [Q_3, Q_1]]_A + Q[Q_3, [Q_1, Q_2]]_A = 0.
\]
Hence, the linear subspace \( S_Q(\text{Symm}_G) \subseteq \text{AdjSymm}_G \) of adjoint-symmetries acquires a Lie algebra structure which is isomorphic to the symmetry Lie algebra. If there exists an adjoint-symmetry \( Q_A \) such that \( S_Q(\text{Symm}_G) = \text{AdjSymm}_G \) where \( \ker(S_Q) \) satisfies the conditions in either of Propositions 5.5 and 5.6 then the whole space \( \text{AdjSymm}_G \) will be a Lie algebra.

Since \( S_Q \) is a linear mapping, the condition \( S_Q(\text{Symm}_G) = \text{AdjSymm}_G \) can be expressed equivalently as
\[
\dim \text{AdjSymm}_G + \dim \ker(S_Q) = \dim \text{Symm}_G.
\]
Hence, \( \dim \text{Symm}_G \geq \dim \text{AdjSymm}_G \) is a necessary condition. This version is most useful when the dimensions are finite.

### 5.2. Adjoint-symmetry commutators associated to symmetry subalgebras.

The Lie algebra structure identified in Theorem 5.7 motivates a related construction of adjoint-symmetry commutator brackets given by a pull-back of Lie subalgebras in \( \text{Symm}_G \) under \( S_Q^{-1} \).

As the starting point, the linear subspace \( S_Q(\text{Symm}_G) \) will be replaced by \( S_Q(\mathcal{A}) \) where \( \mathcal{A} \) is any Lie subalgebra in \( \text{Symm}_G \) and where \( Q_A \) is chosen such that \( \ker(S_Q) \cap \mathcal{A} \) is empty. The set of such adjoint-symmetries will, as before, be a projective subspace in \( \text{AdjSymm}_G \).

Then the construction of the commutator bracket given in Proposition 5.1 is modified as follows.

**Proposition 5.8.** Given a Lie subalgebra \( \mathcal{A} \) in \( \text{Symm}_G \) and a symmetry action \( S_P \) on \( \text{AdjSymm}_G \), fix an adjoint-symmetry \( Q_A \) in \( \text{AdjSymm}_G \) such that the kernel of \( S_Q \) restricted to \( \mathcal{A} \) is empty, where \( S_Q \) is the dual linear operator \((4.10)\) of the symmetry action. Then the commutator bracket \((5.1)\) is well-defined on the linear space \( S_Q(\text{Symm}_G) \subseteq \text{AdjSymm}_G \), and this structure is isomorphic to the Lie subalgebra \( \mathcal{A} \).

In particular, \( S_Q^{-1} \) provides an isomorphism under which the commutator bracket \((5.1)\) on \( S_Q(\text{Symm}_G) \) is the pull-back of the Lie bracket on \( \mathcal{A} \). The condition
\[
\ker(S_Q) \cap \mathcal{A} = \emptyset
\]
will select the adjoint-symmetries \( Q_A \) that can be used in constructing this bracket. If this condition fails to be satisfied by all adjoint-symmetries, then it implies that there is no subspace in \( \text{AdjSymm}_G \) on which the bracket produces a Lie algebra isomorphic to \( \mathcal{A} \).
The question of which Lie subalgebras $\mathcal{A}$ in $\text{Symm}_G$ have counterparts in $\text{AdjSymm}_G$ for a PDE system $G^A = 0$ thereby becomes an interesting algebraic classification problem.

5.3. Adjoint-symmetry non-commutator brackets from symmetry actions. The construction of the second bracket disregards the symmetry commutator but lacks the attendant properties.

Proposition 5.9. Fix an adjoint-symmetry $Q_A$ in $\text{AdjSymm}_G$, and let $S_Q$ be the dual linear operator (4.10) associated to a symmetry action $S_P$ on $\text{AdjSymm}_G$. If the kernel of $S_Q$ satisfies

$$S_P = 0 \text{ for all } P \in \ker(S_Q),$$ (5.21)

then a bilinear bracket from $\text{AdjSymm}_G \times S_Q(\text{Symm}_G)$ into $S_Q(\text{Symm}_G) \subseteq \text{AdjSymm}_G$ is defined by

$$Q(Q_1,Q_2)_A := S_{Q_1}(S_Q^{-1}Q_2)_A.$$ (5.22)

Any one of the symmetry actions (3.11), (3.8), (3.12) can be used to write down formally a corresponding bracket (5.22). Note that, unlike the situation for the commutator bracket (5.1), the condition (5.21) only involves the properties of the symmetry action $S_Q$ and does not depend on the Lie algebra structure of $\text{Symm}_G$. This condition can be by-passed when a scaling symmetry (3.27) is contained in the symmetry Lie algebra.

Proposition 5.10. Suppose $\text{Symm}_G$ contains a scaling symmetry (3.27). For any symmetry action, if $\ker(S_Q)$ is a scaling-homogeneous subspace in $\text{Symm}_G$, then the adjoint-symmetry bracket (5.22) is well-defined on $S_Q(\text{Symm}_G) \subseteq \text{AdjSymm}_G$ by taking $S_Q^{-1}$ to belong to a sum of scaling-homogeneous subspaces.

In contrast to the commutator bracket (5.1), the bracket (5.22) is non-symmetric. Its only general property is that

$$Q(Q,Q_2) = Q_2$$ (5.23)

for all $Q_2$ in the linear subspace $S_Q(\text{Symm}_G) \subseteq \text{AdjSymm}_G$.

There are two worthwhile remarks that can be made.

Remark 5.11. (i) The bracket (5.22) can be viewed as arising from the property that $S_{Q_1}S_Q^{-1}$ is a recursion operator on adjoint-symmetries in $S_Q(\text{Symm}_G)$. (ii) A symmetric version and a skew-symmetric version of the bracket (5.22) can be defined by respectively symmetrizing and antisymmetrizing on $Q_1$ and $Q_2$:

$$Q[Q_1,Q_2]_A := S_{Q_1}(S_Q^{-1}Q_2)_A - S_{Q_2}(S_Q^{-1}Q_1)_A$$ (5.24)

and

$$Q\{Q_1,Q_2\}_A := S_{Q_1}(S_Q^{-1}Q_2)_A + S_{Q_2}(S_Q^{-1}Q_1)_A.$$ (5.25)

The recursion operator $S_{Q_1}S_Q^{-1}$ was derived originally for scalar PDE systems in Ref. [14].

5.4. Computational aspects. Both brackets (5.1) and (5.22) are constructed in terms of the inverse $S_Q^{-1}$ of the dual linear operator $S_Q$ defined by a symmetry action (4.9). In the case of the two symmetry actions (3.8) and (3.11) (cf Theorem 3.1), the operator $S_Q$ involves total derivatives $D_I$ and partial derivatives $\partial_\omega^I$. This means that $S_Q^{-1}$ will involve an integral (operator) with respect to the variables $u_I^\omega$ in jet space, whereby the brackets are essentially nonlocal in jet space. Nevertheless, as an alternative, $S_Q$ can be represented.
in terms of structure constants that are defined with respect to any fixed basis of the linear spaces Symm\(_G\) and AdjSymm\(_G\). With such a representation, the pre-image of any given adjoint symmetry can be found directly in terms of these structure constants. The resulting brackets thus should be viewed as an a posteriori structure on the linear space AdjSymm\(_G\).

In contrast, for the third symmetry action \((3.12)\), \(S^{-1}_3 Q = Q' + R_Q^*\) is a linear operator in total derivatives, and hence \(S^{-1}_3\) only involves the inverse total derivatives \(D_t^{-1}\), so that the brackets are local in jet space and thereby constitute an a priori structure, just like the symmetry commutator.

The same considerations pertain to the corresponding pre-symplectic and pre-Hamiltonian (Noether) structures shown in Theorem 4.3.

6. Results for evolution PDEs

The preceding general results will next be specialized to evolution PDEs.

Consider a general system of evolution PDEs for \(u^\alpha (t, x)\),

\[
\frac{d}{dt} u^\alpha = g^\alpha(x, u, \partial_x u, \ldots, \partial_x^N u) \tag{6.1}
\]

where \(x\) now denotes the spatial independent variables \(x^i, i = 1, \ldots, n\), while \(t\) is the time variable. In this setting, the number of PDEs and the number of dependent variables in the system are equal, \(M = m\). Hence, we can identify the corresponding indices \(\alpha = \alpha\). Thus, we now have

\[
G^\alpha(t, x, u^{(N)}) = u^\alpha_t - g^\alpha(x, u, \partial_x u, \ldots, \partial_x^N u). \tag{6.2}
\]

It will be useful to note that, on the solution space \(E\) of the evolution system \((6.1)\), all \(t\)-derivatives of \(u^\alpha\) can be eliminated in any expression through substituting the equation \((6.1)\) and its spatial derivatives. This demonstrates, in particular, that any evolution system satisfies Lemma 2.1 and cannot obey any differential identities [2, 7]. In particular, all of the technical conditions assumed in section 2 for general PDE systems hold automatically for evolution systems \((6.1)\).

The determining equation \((2.2)\) for symmetries takes the form \((D_t P^\alpha - g'(P)^\alpha)|_E = 0\) for a set of functions \(P^\alpha(t, x, u, \partial_x u, \ldots, \partial_x^k u)\) containing no \(t\)-derivatives of \(u^\alpha\). The first term can be expressed as \(D_t P^\alpha = \partial_t P^\alpha + P^\alpha(u_t)^\alpha = \partial_t P^\alpha + P'(g)^\alpha + P'(G)^\alpha\), whence

\[
\partial_t P^\alpha + P'(g)^\alpha - g'(P)^\alpha = \partial_t P^\alpha + [g, P]^\alpha = 0 \tag{6.3}
\]

is the symmetry determining equation in simplified form. This equation implies that \(G'(P)^\alpha = P'(G)^\alpha\) holds off of \(E\). Consequently, one has

\[
R_P = P'. \tag{6.4}
\]

Likewise, the determining equation \((2.5)\) for adjoint-symmetries is given by \((-D_t Q_\alpha - g'^*(Q)_\alpha)|_E = 0\) for a set of functions \(Q_\alpha(t, x, u, \partial_x u, \ldots, \partial_x^k u)\) containing no \(t\)-derivatives of \(u^\alpha\). This equation simplifies to the form

\[
-(\partial_t Q_\alpha + Q'(g)_\alpha + g'^*(Q)_\alpha) = 0. \tag{6.5}
\]

Hence, off of \(E\), one has \(G'^*(Q)_\alpha = -Q'(G)_\alpha\), which yields

\[
R_Q = -Q'. \tag{6.6}
\]
A useful remark is that the adjoint-symmetry determining equation (6.5) can be expressed in the form

\[ \partial_t Q_\alpha + \{Q, g\}^*_\alpha = 0 \]  

(6.7)

in terms of the anti-commutator \( \{A, B\} = A'(B) + B'(A) \), where \( \{A, B\}^* = A'^*(B) + B'^*(A) \). This formulation emphasizes the adjoint relationship between the determining equations for adjoint-symmetries and symmetries.

The necessary and sufficient condition for an adjoint-symmetry to be a conservation law multiplier is that its Frechet derivative is self-adjoint [5, 2, 8, 9, 3, 7]

\[ Q' = Q'^* \]  

(6.8)

This well-known condition can be expressed more explicitly as the system of Helmholtz equations

\[ \partial u^I Q_\alpha = (-1)^{|I|} E^I u_\alpha(Q_\beta), \quad |I| = 0, 1, \ldots \]  

(6.9)

in terms of the higher Euler operators \( E^I u_\alpha \). (cf equation (A.7)). The determining system for multipliers thereby consists of equations (6.9) and (6.5).

Self-adjointness (6.8) is also necessary and sufficient for \( Q_\alpha \) to be a variational derivative (gradient)

\[ \Lambda_\alpha = E u_\alpha(\Phi) \]  

(6.10)

for some function \( \Phi(x, u^{(k)}) \), \( k \geq 0 \). Consequently, as is well-known, multipliers are variational (gradient) adjoint-symmetries.

6.1. **Symmetry actions on adjoint-symmetries.** The symmetry actions in Theorem 3.1 can be simplified by use of the relations (6.4) and (6.6). Combined with the condition (6.8) characterizing multipliers, this yields the following result.

**Theorem 6.1.** The actions (3.8) and (3.11) of symmetries on the linear space of adjoint-symmetries are respectively given by

\[ Q_\alpha \xrightarrow{X_P} Q'(P)_\alpha + P'^*(Q)_\alpha, \]  

(6.11)

\[ Q_\alpha \xrightarrow{X_P} Q'^*(P)_\alpha + P'^*(Q)_\alpha = E u_\alpha(P^\beta Q_\beta), \]  

(6.12)

which coincide if \( Q_\alpha \) is a conservation law multiplier. The action (3.12) given by the difference of these two actions consists of

\[ Q_\alpha \xrightarrow{X_P} Q'(P)_\alpha - Q'^*(P)_\alpha \]  

(6.13)

which is trivial if \( Q_\alpha \) is a conservation law multiplier.

For the sequel, indices will be omitted for simplicity of notation wherever it is convenient.

6.2. **Adjoint-symmetry brackets.** For evolution PDEs (6.1), the dual linear map \( S_Q \) in the form of the adjoint-symmetry commutator bracket (5.1) and the non-commutator bracket (5.22) is given by any of the three symmetry actions in Theorem 6.1.

Recall that the commutator bracket is well defined when \( \ker(S_Q) \) satisfies the conditions in either of Propositions 5.5 and 5.6. The conditions in the first Proposition can be expressed entirely in terms of \( Q \) and a pair of symmetries \( P_1, P_2 \), by means of the relations (6.6) and (6.4). In particular, condition (5.13) takes the form

\[ \text{pr} X_{P_1}(Q'^*)(P_2) - \text{pr} X_{P_2}(Q'^*)(P_1) = 0 \]  

(6.14)
which must hold for all functions $f$ (anti-self adjoint), adjoint-symmetries, as discussed in the preceding theorem. Clearly, this operator is skew the relation (6.6). It can be viewed as a Noether operator, namely it maps symmetries into $X$ Then, in the cyclic sum $\text{pr} X f_1 Q''(P_1, P_2) = Q''(P_1, P_2)$. It is worth emphasizing that the existence of these adjoint-symmetry brackets does not rely on a PDE system having any variational structure. Indeed, examples of non-trivial brackets for dissipative PDE systems will be given in section 8.

6.3. A Noether operator and a symplectic 2-form. The third symmetry action (6.13) yields the operator

$$J = Q' - Q'^*$$

(6.16)

which is the form of the third operator in Theorem 4.3 specialized to evolution PDEs through the relation (6.6). It can be viewed as a Noether operator, namely it maps symmetries into adjoint-symmetries, as discussed in the preceding theorem. Clearly, this operator is skew (anti-self adjoint), $J^* = -J$. In general, however, it fails to be a pre-symplectic operator, since for all symmetries $P$, $\tilde{Q} = J(P)$ is not necessarily a variational (gradient) adjoint-symmetry

Nevertheless, the operator (6.16) can be used as follows to construct a symplectic 2-form, without any conditions on $Q$ other than it being a non-variational (non-gradient) adjoint-symmetry so that $J \neq 0$.

The construction uses a natural non-degenerate pairing between adjoint-symmetries and symmetries, as given by the inner product $\langle Q, P \rangle = \int Q_\alpha P^\alpha \, dx$. Then the skew bilinear expression

$$\omega_Q(P_1, P_2) := \langle J(P_1), P_2 \rangle = \int (P_1^\alpha Q'(P_2)_\alpha - P_2^\alpha Q'(P_1)_\alpha) \, dx$$

(6.17)

defines a 2-form on the linear space of symmetries $P^\alpha \partial_\alpha$. As discussed in Remark 4.1, a 2-form is symplectic if it is closed. This condition, $d\omega_Q = 0$, can be formulated as

$$\text{pr} X f_3 \omega_Q(f_1, f_2) + \text{cyclic} = 0$$

(6.18)

which must hold for all functions $f_1^\alpha(t, x)$, $f_2^\alpha(t, x)$, $f_3^\alpha(t, x)$.

**Theorem 6.2.** For any evolution system (6.1), the 2-form (6.17) is symplectic. Hence, whenever an evolution system admits a non-variational (non-gradient) adjoint-symmetry, the system possesses a non-trivial associated symplectic structure.

**Proof.** Consider

$$\text{pr} X f_1 \omega_Q(f_1, f_2) = \int (f_1^\alpha \text{pr} X f_3 Q'(f_2)_\alpha - f_2^\alpha \text{pr} X f_5 Q'(f_1)_\alpha) \, dx$$

(6.19)

Then, in the cyclic sum $\text{pr} X f_2 \omega_Q(f_1, f_2) + \text{pr} X f_3 \omega_Q(f_2, f_1) + \text{pr} X f_1 \omega_Q(f_2, f_3)$, all terms cancel pairwise, due to the symmetry of $Q''$ in its two arguments. Hence the condition (6.18) is satisfied. □
The proof can be straightforwardly generalized to show that
\[ \text{pr} X_{P_1} \omega_Q(P_1, P_2) + \text{cyclic} = 0 \] (6.20)
holds for all symmetries \( P_1^\alpha \partial_u^\alpha, P_2^\alpha \partial_u^\alpha, P_3^\alpha \partial_u^\alpha \).
Examples of the symplectic structure in Theorem 6.2 will be given in section 8.

7. Results for Euler-Lagrange PDEs

The general results in Theorem 3.1 and Proposition 5.5 will now be developed more specifically for Euler-Lagrange PDEs.

Consider a general system of Euler-Lagrange PDEs for \( u^\alpha(x) \), with a Lagrangian \( L(x, u^{(N/2)}) \):
\[ E_{u^\alpha}(L) = G_\alpha(x, u^{(N)}) = 0. \] (7.1)
Here the number of PDEs and the number of dependent variables in the system are equal, \( M = m \), which allows the corresponding indices to be identified, \( A = \alpha \), with a transpose in their up/down position. In the case when time \( t \) is one of the independent variables, then it will be denoted \( x^0 \), so that \( x^i, i = 0, 1, \ldots, n \), is the set of all independent variables.

The same technical conditions stated in section 2 for general PDE systems will be assumed here for Euler-Lagrange systems.

As is well known \([2, 3, 7]\), a PDE system has an Euler-Lagrange form iff the Frechet derivative of the system is self-adjoint,
\[ G' = G'^*. \] (7.2)
Consequently, adjoint-symmetries coincide with symmetries,
\[ Q^\alpha = P^\alpha, \] (7.3)
and thereby \( \text{AdjSymm}_G = \text{Symm}_G \).

The necessary and sufficient condition for the components of a symmetry \( P^\alpha \) to be a conservation law multiplier is that they satisfy
\[ E_{u^\alpha}(P^\beta G_\beta) = 0 \] (7.4)
off of \( \mathcal{E} \). This condition is the same as \( X_P = P^\alpha \partial_u^\alpha \) being a variational symmetry, namely \( \text{pr} X_P(L) = L'(P) = D_i \Phi^i(x, u^{(k)}) \) for some vector function \( \Phi^i(x, u^{(k)}) \), since \( E_{u^\alpha}(\text{pr} X_P(L)) = 0 \) where \( \text{pr} X_P(L) = L'(P) = P^\alpha E_{u^\alpha}(L) + D_i \Phi^i(x, u^{(k)}) \).

Through the product rule for the Euler operator combined with the self-adjoint property (7.2), one sees that the variational symmetry condition (7.4) can be expressed as
\[ G'(P)^\alpha = -P'^*(G)^\alpha = R_P(G)^\alpha. \] (7.5)

For the sequel, indices will be omitted for simplicity of notation.

7.1. Symmetry brackets. From the preceding preliminaries, the three symmetry actions in Theorem 3.1 can be formulated as follows for Euler-Lagrange PDEs.
Proposition 7.1. For any Euler-Lagrange PDE system (7.1), the symmetry actions (3.8), (3.11), (3.12) yield corresponding bilinear brackets on Symm$_G$:

\[
P'_1(P_2) + R_{P_2}(P_1) := (P_1, P_2)_{1}, \quad (7.6)
\]

\[
R_{P_2}(P_1) - R_{P_1}(P_2) := (P_1, P_2)_{2}, \quad (7.7)
\]

\[
P'_1(P_2) + R_{P_1}(P_2) := (P_1, P_2)_{3}. \quad (7.8)
\]

They are related to the commutator bracket by

\[
[P_1, P_2] = (P_2, P_1)_{1} - (P_1, P_2)_{3}. \quad (7.9)
\]

If an Euler-Lagrange PDE system (7.1) has no differential identities, then for variational symmetries that contain no leading derivatives (and their differential consequences) in the PDE system, the bracket (7.8) is trivial while the brackets (7.6) and (7.7) reduce to the commutator.

It is natural to decompose these three brackets into symmetric and skew-symmetric parts. The first and third brackets have the same symmetric part: $\mathcal{P}'_{1}(P_2) + \mathcal{P}'_{2}(P_1) + R_{P_1}(P_2) + R_{P_2}(P_1)$. Their skew-symmetric parts are a linear combination of the second bracket (7.7) and the commutator bracket (3.1). Hence, the following result is obtained.

Theorem 7.2. On Symm$_G$ for any Euler-Lagrange PDE system (7.1), there are two natural bilinear antisymmetric brackets

\[
[P_1, P_2]_{\mathcal{P}'} = \mathcal{P}'_{2}(P_1) - \mathcal{P}'_{1}(P_2), \quad (7.10)
\]

\[
R[P_1, P_2] := R_{P_1}(P_2) - R_{P_2}(P_1), \quad (7.11)
\]

and a natural bilinear symmetric bracket

\[
R\{P_1, P_2\} = \mathcal{P}'_{1}(P_2) + \mathcal{P}'_{2}(P_1) + R_{P_1}(P_2) + R_{P_2}(P_1). \quad (7.12)
\]

The second antisymmetric bracket gives a mapping of a pair of symmetries into a variational symmetry. Under the same conditions stated in Proposition 7.1, the symmetric bracket vanishes and the antisymmetric bracket (7.11) reduces to the commutator bracket (7.10).

The statement about the second antisymmetric bracket (7.11) follows from the property that the corresponding second symmetry action (3.11) in Theorem 3.1 maps symmetries into multipliers. In contrast, the commutator bracket (7.10) does not have this property.

In general, the two new brackets (7.11) and (7.10) have no further properties, whereas the commutator bracket is a Lie bracket. These new brackets will be illustrated by examples in section 8.

In addition to these natural symmetry brackets, more brackets on symmetries are provided by the results in Propositions 5.1, 5.5, and 5.6. Specifically, those results have the following counterparts.

Theorem 7.3. Fix a symmetry $P$ and consider the linear operator

\[
P_1 \rightarrow S_{P}(P_1) \quad (7.13)
\]

defined by any one of the brackets (7.6)–(7.12). (i) If the kernel of $S_{P}$ either is an ideal in Symm$_G$, or is uniquely characterized by its homogeneity with respect to a scaling symmetry in Symm$_G$, then

\[
P[P_1, P_2] := S_{P}([S_{P}^{-1}P_1, S_{P}^{-1}P_2]) \quad (7.14)
\]
defines a Lie bracket on $S_P(\text{Symm}_G) \subseteq \text{Symm}_G$. This bracket is a deformation of the standard commutator bracket \eqref{eq:commutator}. (iii) Given a Lie subalgebra $\mathcal{A}$ in $\text{Symm}_G$, if the kernel of $S_P$ restricted to $\mathcal{A}$ is empty, then the bracket \eqref{eq:bracket} restricted to the linear subspace $S_P(\mathcal{A}) \subseteq \text{Symm}_G$ defines a Lie bracket whose structure is isomorphic to the Lie subalgebra $\mathcal{A}$.

The deformed Lie bracket \eqref{eq:bracket} has a simpler form when $S_P = \text{ad}(P)$ is given by the commutator \eqref{eq:commutator}, with $\text{ad}$ denoting the action defined in the Lie algebra $\text{Symm}_G$. In this case:

$$F[P_1, P_2] = \text{ad}(P)((\text{ad}(P)^{-1}P_1, \text{ad}(P)^{-1}P_2)) = [P_1, \text{ad}(P)^{-1}P_2] + [\text{ad}(P)^{-1}P_1, P_2]$$ \tag{7.15}

where $P_1$ and $P_2$ belong to the range of $\text{ad}(P)$ in $\text{Symm}_G$. This bracket \eqref{eq:bracket} will, in general, differ from the commutator \eqref{eq:commutator}. They will be closely related if $X_P$ is scaling symmetry \eqref{eq:scaling_symmetry}, since then both $[P_1, \text{ad}(P)^{-1}P_2]$ and $[\text{ad}(P)^{-1}P_1, P_2]$ will be constant multiples of $[P_1, P_2]$. Examples will be given in Section 8.

Additional symmetry brackets arise from Propositions \ref{prop:3.9} and \ref{prop:4.10}. Specifically, for a fixed symmetry $P^\alpha$, if the kernel of $S_P$ either satisfies $S_{\ker(S_P)} = 0$ or is uniquely characterized by its homogeneity with respect to a scaling symmetry in $\text{Symm}_G$, then

$$F(P_1, P_2) := S_{P_1}(S_{P}^{-1}P_2)$$ \tag{7.16}

defines a bilinear bracket from $\text{Symm}_G \times S_P(\text{Symm}_G)$ into $S_P(\text{Symm}_G) \subseteq \text{Symm}_G$. This bracket \eqref{eq:bracket} is non-symmetric and obeys $F(P, P_2) = P_2$ for all symmetries $P_2$ in $S_P(\text{Symm}_G)$. It is constructed from $S_{P_1}S_{P}^{-1}$ which is a recursion operator on symmetries in the linear subspace $S_P(\text{Symm}_G) \subseteq \text{Symm}_G$.

### 7.2. Recursion operators and a symplectic 2-form

Since adjoint-symmetries coincide with symmetries for Euler-Lagrange PDEs, the general notions of Noether operators, pre-symplectic and pre-Hamiltonian operators, also coincide and can be viewed simply as recursion operators on symmetries. A pre-symplectic operator in this setting is a recursion operator under which symmetries are mapped into variational symmetries (multipliers).

The linear operator \eqref{eq:operator} associated to any given bilinear bracket on symmetries can be viewed as a symmetry (recursion) operator, namely it can be iterated on $\text{Symm}_G$. To distinguish this property, the notation $R_P$ will be used instead of $S_P$. From Theorem 7.2 there are three natural symmetry operators:

$$R_P = [P, \cdot]$$ \tag{7.17}

$$R_P = R[P, \cdot]$$ \tag{7.18}

$$R_P = R\{P, \cdot\}. \tag{7.19}$$

All of the resulting linear mappings $R_P(P_1)$ involve partial derivatives $\partial_{u^2}$ of $P_1$ in addition to total derivatives $D_t$ on $P_1$, and consequently their formal inverses will be essentially nonlocal in jet space, similarly to the situation for brackets discussed in Section 5.4.

A particular linear combination

$$R_P = [P, \cdot] - R[P, \cdot] + R\{P, \cdot\} = P' + R_P'$$ \tag{7.20}

corresponding to the third bracket \eqref{eq:bracket} in Proposition 7.1 involves only total derivatives. Note that this operator coincides with the operator $J_2$ in Theorem 4.3, while $R_P = R[P, \cdot]$ coincides with $J_2$. 

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Corollary 7.4. The symmetry operator (7.18) is a pre-symplectic structure, namely, it maps symmetries into variational symmetries.

The action of this symmetry operator \( R(P_1) = R[P, P_1] = R^r_P(P_1) - R^r_{P_1}(P) \) involves partial derivatives \( \partial_{u_i} \) of \( P_1 \) and thus is not an operator in total derivatives, in contrast to the operator (7.20).

An interesting question is whether any of the symmetry operators give rise to a symplectic 2-form. Two different constructions can be considered.

Firstly, a symmetry operator \( R_P \) can be combined with a non-degenerate pairing given by the Euclidean inner product \( \langle Q, P \rangle = \int Q^\alpha P^\beta \delta_{\alpha \beta} \, dx \) where \( \delta_{\alpha \beta} \) is the Kronecker symbol (components of the algebraic inner product). This allows one to define a bilinear-form \( \omega_R \) for evolution PDEs. However, for each of the four symmetry operators (7.17), (7.18), (7.19), (7.20), the resulting bilinear-form fails to be closed, \( \text{d} \omega_R \neq 0 \), where the closure condition [15, 2] is explicitly given by

\[
\text{d} \omega_R(f_1, f_2, f_3) = \text{pr} X_f \omega_R(f_1, f_2) + \text{cyclic} = 0
\]  

(7.21)

for all functions \( f^a(x), f^b(x), f^c(x) \).

Secondly, if a symmetry operator is pre-symplectic, then it produces a variational symmetry which yields a conservation law through Noether’s theorem. The conservation law implicitly defines a pairing that can be used to construct a bilinear-form, as indicated in Proposition 4.2.

From Corollary 7.4 and Theorem 7.2 consider the variational symmetry

\[
\bar{P} = R^r_{P_1}(P_2) - R^r_{P_2}(P_1) = R[P_1, P_2]
\]

(7.22)

produced from any pair of symmetries \( P^a_1 \partial_{u^a} \) and \( P^b_2 \partial_{u^b} \). Rather than use Noether’s theorem directly, there is a simpler alternative to obtain the resulting conservation law.

Lemma 7.5. The conservation law produced by the variational symmetry (7.22) is locally equivalent to the conservation law \( D_t \Psi^1(P_1, P_2)|_E = 0 \) given by the identity

\[
D_t \Psi(P_1, P_2) = P^a_2 G^a(P_1) - P^\alpha_1 G^\alpha(P_2),
\]

(7.23)

with

\[
\Psi(P_1, P_2) = (D JP^a_1)(D JP^\beta_2)(-1)^{|I|} E^I_{\alpha \beta}(G^\alpha) .
\]

(7.24)

Proof. Consider the bilinear expression \( P^a_2 G^a(P_1) - P^\alpha_1 G^\alpha(P_2) \). On one hand, the symmetry determining equation (2.3) off of \( E \) shows that \( P^a_2 G^a(P_1) - P^\alpha_1 G^\alpha(P_2) = P^a_2 R^r_{P_1}(G^a) - P^\alpha_1 R^r_{P_2}(G_\alpha) = R[P_1, P_2]^a G^a + D_t \Gamma^a(P_1, P_2; G) \) where \( \Gamma^a(P_1, P_2; G) E = 0 \). Since \( \bar{P} = R[P_1, P_2]^a \) is a variational symmetry, it is a conservation law multiplier (by Noether’s theorem), and hence \( R[P_1, P_2]^a G^a = D_t \Phi^a(P_1, P_2) \) yields a conservation law \( D_t \Phi^a(P_1, P_2)|_E = 0 \). On the other hand, \( G^\alpha = G^{* \alpha} \) shows that \( P^a_2 G^a(P_1) - P^\alpha_1 G^\alpha(P_2) = D_t \Psi^a(P_1, P_2) \) via integration by parts, where \( \Psi^a(P_1, P_2) \) is the expression (7.24). Thus, \( D_t \Phi^a(P_1, P_2) + \Gamma^a(P_1, P_2; G) = D_t \Psi^a(P_1, P_2) \), whereby \( D_t \Phi^a(P_1, P_2)|_E = D_t \Psi^a(P_1, P_2)|_E = 0 \) are locally equivalent conservation laws. \( \square \)
Now, in terms of the conservation law (7.24), define the 2-form

\[ \omega_\Psi(P_1, P_2) := \int_\Omega \Psi^i(P_1, P_2) \hat{n}_i \, dV \]  

(7.25)

where \( \Omega \) is any fixed hypersurface in the space coordinatized by \( x^i \), \( \hat{n}_i \) is a normal covector of \( \Omega \), and \( dV \) is the volume element of \( \Omega \). In the case when time \( t = x^0 \) is one of the independent variables, a natural choice for \( \Omega \) is a spatial hyperplane \( t = \text{const.} \), whereby the 2-form is given by

\[ \omega_\Psi(P_1, P_2) := \int_{\mathbb{R}^n} \Psi^0(P_1, P_2) \, d^n x \]  

(7.26)

where \( x \) denotes the \( n \) spatial variables \( x^i, i = 1, \ldots, n \). This 2-form (7.26) is the simply the conserved integral arising from the conservation law determined by the variational symmetry (7.22). The following interesting result will now be proven.

**Theorem 7.6.** For any Euler-Lagrange system (7.1), \( \omega_\Psi(P_1, P_2) \) is a symplectic 2-form which can be expressed as \( \int_{\mathbb{R}^n} P_i J(P_2) \, d^n x \), modulo boundary terms, where \( J \) is a symplectic operator (in total derivatives).

**Proof.** To be a symplectic 2-form, the bilinear-form \( \omega_\Psi(P_1, P_2) \) must be skew and closed. Lemma 7.5 shows that \( \Psi^i(P_1, P_2) \) is antisymmetric in the symmetries \( P_1 \) and \( P_2 \), whence \( \omega_\Psi(P_1, P_2) \) is skew. The closure condition is the same as equation (7.21). Thus, consider

\[ \text{pr} X^i \omega_\Psi(f_1, f_2) = \int_\Omega \text{pr} X^i \Psi^i(f_1, f_2) \hat{n}_i \, dV \]

for arbitrary functions \( f_1(t, x), f_2(t, x), f_3(t, x) \). One has

\[ \text{pr} X^i D_i \Psi^i(f_1, f_2) = f_2^a \text{pr} X^i f_3 \Psi^i(f_1, f_2) \hat{n}_i - f_1^a \text{pr} X^i f_3 \Psi^i(f_2, f_1) \hat{n}_i \]

(7.27)

and hence \( \text{pr} X^i D_i \Psi^i(f_1, f_2) + \text{cyclic} = 0 \) because all terms in the cyclic sum cancel pairwise. Since \( \text{pr} X^i \) commutes with \( D_i \), one then gets

\[ D_i (\text{pr} X^i \Psi^i(f_1, f_2) + \text{cyclic}) = 0. \]  

(7.28)

Now, a well known result in variational calculus shows that any total divergence that vanishes identically must be a total curl. Thus, \( \text{pr} X^i \Psi^i(f_1, f_2) + \text{cyclic} = D_j \Theta^{ij}(f_1, f_2, f_3) \) holds where \( \Theta^{ij} \) is some skew tensor function in jet space, which is totally antisymmetric in \( f_1^a(x), f_2^a(x), f_3^a(x) \). Therefore,

\[ \text{pr} X^i \omega_\Psi(f_1, f_2) + \text{cyclic} = \int_\Omega (D_j \Theta^{ij}) \hat{n}_i \, dV = \int_\Omega D_j (\Theta^{ij} \hat{n}_i) \, dV = \int_{\partial \Omega} \Theta^{ij} \hat{n}_i \, dA_j \]  

(7.29)

reduces to an integral over the boundary \( \partial \Omega \) of the hypersurface \( \Omega \), where \( dA_i \) denotes the corresponding area element. This boundary integral will vanish by taking the functions \( f_i \) to have either compact support in \( x \) or sufficient asymptotic decay for large \( |x| \). Thus, \( \omega_\Psi(f_1, f_2) \) is a closed 2-form.

These steps can be straightforwardly generalized to show that

\[ \text{pr} X^i \omega_\Psi(P_1, P_2) + \text{cyclic} = 0 \]  

(7.30)

holds for all symmetries \( P_1^\alpha \partial_\nu, P_2^\alpha \partial_\nu, P_3^\alpha \partial_\nu \). Since \( \Psi^i(P_1, P_2) \) is constructed from total derivatives of \( P_1 \) and \( P_2 \), the density \( \Psi^i(P_1, P_2) \hat{n}_i \) thereby can be expressed as \( P_1 J P_2 \) modulo
a total divergence, where $J = -J^*$ is a skew operator in total derivatives. The closure (7.30) of $\omega_\Psi(P_1, P_2)$ then shows that $J$ is symplectic.

As a consequence, the symmetry operator (7.18) encodes a symplectic structure given in terms of a symplectic operator $J$ in total derivatives. The formal inverse $J^{-1}$ then will be a Hamiltonian operator. An example will be shown in section 8.

8. Examples

The main results will now be applied to six examples of physically interesting nonlinear PDE systems: (1) a nonlinear reaction-diffusion system; (2) the Navier-Stokes equations for compressible viscous fluid flow; (3) coupled solitary wave equations; (4) a generalized nonlinear Schrodinger (NLS) equation; (5) a coupled Boussinesq system; (6) Maxwell’s equations in free space.

Examples (1) to (5) will be treated in one spatial dimension; example (6) will be covered in two spatial dimensions. In each example, firstly, the Lie point symmetries (in evolutionary form) and the low-order adjoint-symmetries will be described. Secondly, the three actions of the Lie point symmetries on the adjoint-symmetries will be presented, and the corresponding adjoint-symmetry commutator brackets will be obtained. Thirdly, in examples (1) and (2), a correspondence between symmetries and adjoint-symmetries will shown to exist in the absence of any local variational structure (Hamiltonian or Lagrangian) for dissipative PDE systems. Fourthly, a Noether (pre-symplectic) operator and a symplectic 2-form will be shown to arise directly from one of the symmetry actions in examples (3) to (6) and also will be used to derive a corresponding Hamiltonian structure. This will illustrate how Hamiltonian structures are naturally encoded in the adjoint-symmetry structure of non-dissipative PDE systems. Finally, in examples (4) and (5) where Euler-Lagrange equations arise, the novel symmetry brackets will also be illustrated.

All of the symmetries and adjoint-symmetries in the examples are obtained by solving the determining equations (2.2) and (2.5) through a standard method (see Ref. [2, 7]).

8.1. Reaction-diffusion system. Consider a system of coupled nonlinear reaction-diffusion equations

\[
\begin{align*}
    u_t & = \kappa_1 u_{xx} + \alpha u^{p+1} - \beta v^{p+1}, \\
    v_t & = \kappa_2 v_{xx} + \beta v^{p+1} - \alpha u^{p+1},
\end{align*}
\]

where $\kappa_1 > 0$, $\kappa_2 > 0$ are the diffusivity coefficients; $p > 0$ is the nonlinearity power of the reaction terms; $\alpha$, $\beta$ are the reaction coefficients. This evolution system is a model for two interacting reactive chemicals or ions that are diffusing in a solute, with densities $u(t, x)$ and $v(t, x)$. All of the parameters will be treated as being arbitrary.

Symmetries (in evolutionary form) $P^u \partial_u + P^v \partial_v$ are determined by the equations

\[
\begin{align*}
    (D_t P^u - \kappa_1 D_x^2 P^u - \alpha (p + 1) u^p P^u + \beta (p + 1) v^p P^v)|_\mathcal{E} & = 0, \\
    (D_t P^v - \kappa_2 D_x^2 P^v - \beta (p + 1) v^p P^v + \alpha (p + 1) u^p P^u)|_\mathcal{E} & = 0,
\end{align*}
\]

where $\mathcal{E}$ denotes the solution space of the reaction-diffusion system (8.1). Adjoint-symmetries $(Q^u, Q^v)$ are determined by the adjoint equations

\[
\begin{align*}
    (-D_t Q^u - \kappa_1 D_x^2 Q^u - \alpha (p + 1) u^p (Q^u - Q^v))|_\mathcal{E} & = 0, \\
    (-D_t Q^v - \kappa_2 D_x^2 Q^v - \beta (p + 1) v^p (Q^v - Q^u))|_\mathcal{E} & = 0.
\end{align*}
\]
A basis for the linear space of Lie point symmetries, with \( P = (P^u, P^v) \), consists of
\[
P_1 = (u_t, v_t), \quad P_2 = (u_x, v_x), \quad P_3 = \left( \frac{2}{p} u + 2tu_x + xu_x, \frac{1}{p} v + 2tv_x + xv_x \right), + tu_t, \frac{1}{p} v + tv_t, \quad (8.4)
\]
which represent generators for a time-translation, a space-translation, and a scaling. Their algebra is given by the non-zero commutators
\[
\begin{align*}
[P_1, P_3] &= -2P_1, \quad [P_2, P_3] = -P_2.
\end{align*}
\]
(8.5)

A basis of the linear space of adjoint-symmetries, \( Q = (Q^u, Q^v) \), is given by
\[
Q_1 = (1, 1), \quad Q_2 = (x, x),
\]
(8.6)

which are also multipliers for conservation laws of mass \( M = \int_R (u + v) \, dx \) and center of mass \( X = \int_R x(u + v) \, dx \).

Consequently, from Theorem 6.1, the third symmetry action (6.13) is trivial, while the other two symmetry actions (6.12), (6.11) are given by the linear operator
\[
S_P(Q) = \left( E_u(PQ_t), E_v(PQ_t) \right).
\]
(8.7)

(Here \( t \) denotes the transpose.) This action is summarized in Table 1. Note that, for evaluating the symmetry actions, all \( t \)-derivatives of \( u \) and \( v \) are replaced through equations (8.1).

Table 1. Reaction-diffusion system: symmetry action (8.7) on adjoint-symmetries

|       | \( P_1 \) | \( P_2 \) | \( P_3 \) |
|-------|-----------|-----------|-----------|
| \( Q_1 \) | 0         | 0         | \( (\frac{2}{p} - 1)Q_1 \) |
| \( Q_2 \) | 0         | \(-Q_1\)  | \( 2(\frac{1}{p} - 1)Q_2 \) |

8.1.1. Adjoint-symmetry bracket. The adjoint-symmetry commutator bracket (5.1) arising from this symmetry action is defined via the dual operator
\[
S_Q(P) = \left( E_u(PQ^t), E_v(PQ^t) \right).
\]
(8.8)

To obtain the maximal domain, namely the whole linear space \( \text{span}(Q_1, Q_2) \), one can choose \( Q = Q_2 \), whence
\[
S_{Q_2}(P) = \left( E_u(x(P^u + P^v)), E_v(x(P^u + P^v)) \right).
\]
(8.9)

Thereby, one has \( \ker(S_{Q_2}) = \text{span}(P_1) \), which is an ideal, and \( \text{ran}(S_{Q_2}^{-1}) = \text{span}(P_2, P_3) \) modulo \( \ker(S_{Q_2}) \). From Table 1, one then obtains
\[
S_{Q_2}^{-1}(Q_1) = -P_2, \quad S_{Q_2}^{-1}(Q_2) = \frac{p}{2(1-p)} P_3,
\]
(8.10)

and thus the adjoint-symmetry commutator bracket can be directly computed by
\[
\begin{align*}
Q^2[Q_1, Q_2] &= S_{Q_2}([-P_2, \frac{p}{2(1-p)} P_3]) = -\frac{p}{2(1-p)} S_{Q_2}(P_2) = \frac{p}{2(1-p)} Q_1
\end{align*}
\]
(8.11)

through the symmetry commutator (8.3).

This bracket (8.11) is a non-trivial Lie bracket. It is isomorphic to the symmetry subalgebra \( \mathcal{A} = \text{span}(P_2, P_3) \), which is generated by space translation and scaling. This correspondence between symmetries and adjoint-symmetries (cf Proposition 5.8) exists in the absence of any local variational structure (Hamiltonian or Lagrangian) for the reaction-diffusion equations (8.1).
Theorem 6.1 shows that the third symmetry action (6.13) is trivial, while the other two symmetry actions (6.12), (6.11) are given by the linear operator (8.7) with the center of mass \( X \) and Galilean momentum generators for a time-translation, a space-translation, a Galilean boost, and a scaling:

They are multipliers for conservation laws of mass \( M \) and symplectic 2-form (6.17) are trivial. This is expected, as reaction-diffusion systems are inherently dissipative (namely, the linearized system is parabolic).

8.2. **Navier-Stokes equations.** Consider the Navier-Stokes equations for compressible fluid flow \([22, 23]\) in one spatial dimension

\[
\frac{\partial u}{\partial t} + uu_x = (1/\rho)(-p_x + \mu u_{xx}), \quad \rho_t + (u\rho)_x = 0
\]

for the fluid velocity \( u(t, x) \) and the density \( \rho(t, x) \), where \( \mu \neq 0 \) is the viscosity coefficient. Here the pressure will be given by a polytropic equation of state

\[
p(\rho) = k\rho^q, \quad q \neq 0
\]

with coefficient \( k > 0 \). All of the parameters will be treated as being arbitrary.

The determining equations for symmetries \( P^n\partial_u + P^\rho\partial_\rho \) (in evolutionary form) are given by

\[
(D_t P^n + D_x(uP^n) + qD_x((p/\rho^2)P^n) + \mu(1/\rho)^2 u_{xx}P^n - \mu(1/\rho)D_x^2 P^n)|_E = 0,
\]

\[
(D_t P^\rho + D_x(uP^\rho + \rho P^n))|_E = 0,
\]

where \( E \) denotes the solution space of system (8.12). The adjoint of these equations yields the determining equations for adjoint-symmetries \((Q^n, Q^\rho)\):

\[
(-D_t Q^n - uD_x Q^n - \rho D_x Q^\rho - \mu D_x^2 ((1/\rho)Q^n))|_E = 0,
\]

\[
(-Q_t P^n - uD_x Q^\rho - q(p/\rho^2)D_x Q^n + \mu(1/\rho)^2 u_{xx}Q^n)|_E = 0.
\]

A basis for the linear space of Lie point symmetries, with \( P = (P^n, P^\rho) \), consists of generators for a time-translation, a space-translation, a Galilean boost, and a scaling:

\[
P_1 = (u_t, \rho_t), \quad P_2 = (u_x, \rho_x), \quad P_3 = (1 - tu_x, -t\rho_x),
\]

\[
P_4 = (\frac{1+q}{1+q}u - \frac{2q}{1+q}tu_t - xu_x, -\frac{2}{1+q}\rho - \frac{2q}{1+q}t\rho_t - x\rho_x).
\]

Their algebra is given by the non-zero commutators

\[
[P_1, P_3] = P_2, \quad [P_1, P_4] = \frac{2q}{1+q}P_1, \quad [P_2, P_4] = P_2, \quad [P_3, P_4] = \frac{1-q}{1+q}P_3.
\]

A basis of the linear space of adjoint-symmetries, \( Q = (Q^n, Q^\rho) \), is given by

\[
Q_1 = (0, 1), \quad Q_2 = (\rho, u), \quad Q_3 = (t\rho, tu - x).
\]

They are multipliers for conservation laws of mass \( \mathcal{M} = \int_{\mathbb{R}} \rho \, dx \), momentum \( \mathcal{P} = \int_{\mathbb{R}} \rho u \, dx \), and Galilean momentum \( \mathcal{G} = \int_{\mathbb{R}} \rho(tu - x) \, dx = t\mathcal{P} - \mathcal{X}(t) \) which is related to the motion of the center of mass \( \mathcal{X}(t) = \int_{\mathbb{R}} x\rho \, dx \).

Table 2. Reaction-diffusion system: adjoint-symmetry commutator bracket

| Q_1 | Q_2 |
|-----|-----|
| 0   | \frac{p}{2(1-p)} Q_1 |

| Q_2 |
|-----|
| 0   |

Since the third symmetry action is trivial, both the corresponding Noether operator (6.16) and symplectic 2-form (6.17) are trivial. This is expected, as reaction-diffusion systems are inherently dissipative (namely, the linearized system is parabolic).
This action is summarized in Table 3. For evaluating the symmetry actions, all \( t \)-derivatives of \( u \) and \( \rho \) are replaced through equations (8.12).

**Table 3. Navier-Stokes equations: symmetry action (8.19) on adjoint-symmetries**

|     | \( P_1 \) | \( P_2 \) | \( P_3 \) | \( P_4 \) |
|-----|---------|---------|---------|---------|
| \( Q_1 \) | 0       | 0       | 0       | \( \frac{q+1}{q+1} Q_1 \) |
| \( Q_2 \) | 0       | 0       | \( Q_1 \) | 0       |
| \( Q_3 \) | \(-Q_2\) | \( Q_1 \) | 0       | \( \frac{q}{q+1} Q_3 \) |

8.2.1. **Adjoint-symmetry bracket.** To obtain a maximal domain on which an adjoint-symmetry commutator bracket (5.1) can be defined via the symmetry action (8.19), one seeks a maximal range for the dual operator

\[
S_Q(P) = (E_u(PQ^t), E_\rho(PQ^t)).
\] (8.20)

From Table 3, it is clear that the maximal range will be the whole linear space of adjoint-symmetries, which is obtained for the choice \( Q = Q_3 + c_2 Q_2 + c_1 Q_1 \), where one has

\[
\ker(S_{Q_3+c_2 Q_2+c_1 Q_1}) = \text{span}(P_3 - c_2 P_2).
\] (8.21)

The subalgebra (8.21) is not an ideal, since \([P_1, P_3 - c_2 P_2] = P_2\), and consequently the resulting adjoint-symmetry bracket will depend on how \( \ker(S_{Q_3+c_2 Q_2+c_1 Q_1}) \) is chosen in \( \text{span}(P_1, P_2, P_3, P_4) \). However, the scaling symmetry \( P_4 \) can be utilized in accordance with Proposition 5.6 to fix a canonical choice of \( \ker(S_{Q_3+c_2 Q_2+c_1 Q_1}) \) as follows. From the symmetry commutators (8.17), observe that \( \text{span}(P_3) \) has a different scaling weight compared to \( \text{span}(P_1) \) and \( \text{span}(P_2) \). Also observe that \( \text{span}(Q_1) \), \( \text{span}(Q_2) \), and \( \text{span}(Q_3) \) have different scaling weights. Hence, one can take \( c_2 = c_1 = 0 \), whereby \( \ker(S_{Q_3}) = \text{span}(P_3) \) and \( \ker(S_{Q_3}) = \text{span}(P_3) \oplus \text{span}(P_1) \oplus \text{span}(P_2) \) provides a well-defined decomposition (5.16) of the symmetry Lie algebra under scaling. This determines \( \text{ran}(S_{Q_3}^{-1}) = \text{span}(P_1, P_2, P_4) \).

From Table 3 one now obtains

\[
S_{Q_3}^{-1}(Q_1) = P_2, \quad S_{Q_3}^{-1}(Q_2) = -P_1, \quad S_{Q_3}^{-1}(Q_3) = \frac{q+1}{2q} P_4.
\] (8.22)

Hence, the adjoint-symmetry commutator bracket (5.1) can be directly computed by

\[
Q_3[Q_1, Q_2] = S_{Q_3}([P_2, -P_1]) = S_{Q_3}(0) = 0,
\] (8.23a)

\[
Q_3[Q_1, Q_3] = S_{Q_3}([P_2, \frac{q+1}{2q} P_4]) = \frac{q+1}{2q} S_{Q_3}(P_2) = \frac{q+1}{2q} Q_1,
\] (8.23b)

\[
Q_3[Q_2, Q_3] = S_{Q_3}([-P_1, \frac{q+1}{2q} P_4]) = -S_{Q_3}(P_1) = Q_2,
\] (8.23c)

through the symmetry commutator (8.17).

This bracket (8.23) is a non-trivial Lie bracket on the whole linear space \( \text{span}(Q_1, Q_2, Q_3) \) of adjoint-symmetries. In particular, from the inverse of the dual operator (8.22), one sees that this Lie bracket structure is isomorphic to the symmetry subalgebra \( A = \text{span}(P_1, P_2, P_4) \), which is generated by time translation, space translation, and scaling.

Remarkably, this correspondence between symmetries and adjoint-symmetries (cf Proposition 5.8) exists despite the lack of any local variational structure (Hamiltonian or Lagrangian) for the Navier-Stokes equations (8.12).
Finally, since the third symmetry action is trivial, both the corresponding Noether operator (6.16) and symplectic 2-form (6.17) are trivial. This is expected, as viscous fluid flow is inherently dissipative.

8.3. **Coupled solitary wave equations.** The near-resonant interaction of weakly nonlinear solitary waves can be described by a coupled system of KdV equations [24]. Consider, in particular, the nonlinearly-coupled equations

\begin{align}
  u_t + \alpha (uv)_x + \kappa u_{xxx} &= 0, \quad v_t + uu_x + v_{xxx} = 0,
\end{align}

where, after scaling of the variables, $\alpha \neq 0$ is a coupling parameter, and $\kappa \neq 0$ is a relative dispersion parameter. Here $u(t, x)$ and $v(t, x)$ are the amplitudes of the two waves. This system models [25, 26] the energy transfer between Rossby waves in equatorial latitudes and mid latitudes in the atmosphere.

The linear space of Lie point symmetries of system (8.24), with $P = (P^u, P^v)$, is generated by a time-translation, a space-translation, and a scaling:

\begin{align}
  P_1 &= (u_t, v_t), \quad P_2 = (u_x, v_x), \quad P_3 = (2u + xu_x + 3tu_t, 2v + xv_x + 3tv_t).
\end{align}

Their algebra has the non-zero commutators

\begin{align}
  [P_1, P_3] &= -3P_1, \quad [P_2, P_3] = -P_2.
\end{align}

The linear space of low-order adjoint-symmetries, $Q = (Q^u, Q^v)$, is given by the basis

\begin{align}
  Q_1 &= (1, 0), \quad Q_2 = (0, 1), \quad Q_3 = (\frac{1}{\alpha}u, v), \quad Q_4 = (uv + \frac{\kappa}{\alpha}u_{xx}, \frac{1}{2}u^2 + v_{xx}).
\end{align}

These adjoint-symmetries are also multipliers for conservation laws of (up to an overall factor) the mass $M^u = \int_R u \, dx$ and $M^v = \int_R v \, dx$ for each wave, the combined momentum of the waves $P = \int_R (u^2 + \alpha v^2) \, dx$, and the net energy of the waves $E = \int_R \frac{1}{2}(ku_x^2 + \alpha(v_x^2 - vu^2)) \, dx$.

From Theorem (6.13) one sees that the third symmetry action (6.13) is trivial, while the other two symmetry actions (6.12), (6.11) are given by the linear operator (8.7). This action is summarized in Table 5. For evaluating the symmetry actions, all $t$-derivatives of $u$ and $v$ are replaced through equations (8.24).

---

### Table 4. Navier-Stokes equations: adjoint-symmetry commutator bracket

|   | $Q_1$ | $Q_2$ | $Q_3$ |
|---|-------|-------|-------|
| $Q_1$ | 0     | 0     | $\frac{q+1}{2q}Q_1$ |
| $Q_2$ | 0     | $Q_2$ |       |
| $Q_3$ |       | 0     |       |
Table 5. Coupled solitary wave equations: symmetry action (8.7) on adjoint-symmetries

| P | P | P |
|---|---|---|
| Q | 0 | Q1 |
| Q2 | 0 | Q2 |
| Q3 | 0 | 3Q5 |
| Q4 | 0 | 5Q4 |

8.3.1. A nonlocal adjoint-symmetry and associated adjoint-symmetry brackets. The symmetry action shown in Table 5 has \( \ker S_P(Q) = \text{span}(P_1, P_2) \), as explained by Corollary 3.5. Hence, the cokernel, which is given by \( \text{span}(P_3) \), is only of dimension 1. Since the cokernel is the maximal domain on which a corresponding adjoint-symmetry commutator bracket can be defined, the resulting bracket is trivial.

However, the situation changes when one considers the possibility of nonlocal adjoint-symmetries arising from potentials given by the mass conservation laws. These potentials are obtained via \( u = U_x \) and \( v = V_x \), and they satisfy the coupled system

\[
U_t + \alpha U_x V_x + \kappa U_{xxx} = 0, \quad V_t + \frac{1}{2} U^2_x + V_{xxx} = 0. \tag{8.30}
\]

Adjacent-symmetries \((Q^U, Q^V)\) of this system are determined by the equations

\[
(-D_t Q^U - D_x (\alpha V_x Q^U + U_x Q^V) - \kappa D^3_x Q^U)|_E = 0, \tag{8.31a}
\]

\[
(-D_t Q^V - \alpha D_x (U_x Q^U) - D^3_x Q^V)|_E = 0. \tag{8.31b}
\]

The linear space of low-order adjacent-symmetries is generated by three adjacent-symmetries, two of which are inherited from the adjacent-symmetries \( Q_3 \) and \( Q_4 \) of the original system (8.24) for \( u, v \), through the relation \( (Q^U, Q^V) = -D_x (Q^u, Q^v) \). The other low-order adjacent-symmetry is given by

\[
(Q^U, Q^V) = -\left(\frac{1}{\alpha}(2U_x + xU_{xx} + 3tU_{tx}), 2V_x + xV_{xx} + 3tV_{tx}\right). \tag{8.32}
\]

Applying the inverse relation

\[
(Q^u, Q^v) = -D_x^{-1}(Q^U, Q^V), \tag{8.33}
\]

one obtains the nonlocal adjacent-symmetry

\[
Q_5 = \left(\frac{1}{\alpha}(U + xU_x + 3tU_t), V + xV_x + 3tV_t\right) = \left(\frac{1}{\alpha}(U + xu) - 3t(uv + \frac{2}{\alpha} u_{xx}), V + xv - 3t(\frac{1}{2} u^2 + v_{xx})\right). \tag{8.34}
\]

admitted by the system (8.24) for \( u, v \). One can straightforwardly show that this adjacent-symmetry is not a multiplier.

When the first symmetry action, as given by the linear operator (8.7), is applied to \( Q_5 \), one obtains

\[
S_P(Q_5) = 5Q_4, \quad S_{P_2}(Q_5) = -3Q_3, \quad S_{P_3}(Q_5) = 0, \tag{8.35}
\]

by using the variational derivative relations \( E_u = -D_x^{-1} E_U \) and \( E_v = -D_x^{-1} E_V \). Consequently, if one chooses

\[
Q = Q_5 + c_2 Q_2 + c_1 Q_1 := Q_5', \tag{8.36}
\]

with at least one of \( c_1, c_2 \) being non-zero, then \( \ker S_P(Q) \) is empty, and so the cokernel consists of the whole linear space of Lie point symmetries, \( \text{span}(P_1, P_2, P_3) \). This choice produces a maximal domain for defining the adjacent-symmetry commutator bracket (5.1).

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From equation (8.35) and Table 5, one obtains

\[ S_{P_1}(Q_5') = 5Q_4, \quad S_{P_2}(Q_5') = -3Q_3, \quad S_{P_3}(Q_5') = c_1Q_1 + c_2Q_2, \]  

(8.37)

which yields

\[ S_{Q_5'}^{-1}(c_1Q_1 + c_2Q_2) = P_3, \quad S_{Q_5'}^{-1}(Q_3) = -\frac{1}{3}P_2, \quad S_{Q_5'}^{-1}(Q_4) = \frac{1}{5}P_1, \]  

(8.38)

for the inverse of the dual linear operator. The resulting adjoint-symmetry commutator bracket (5.1) on the linear subspace \( \text{span}(c_1Q_1 + c_2Q_2, Q_3, Q_4) \) is shown in Table 6. This bracket is a Lie bracket which is isomorphic to the symmetry algebra (8.28).

**Table 6.** Coupled solitary wave equations: adjoint-symmetry commutator bracket from symmetry action (8.7) with \( Q = Q_5 + c_2Q_2 + c_1Q_1 \)

|       | \( c_1Q_1 + c_2Q_2 \) | \( Q_3 \) | \( Q_4 \) |
|-------|------------------------|----------|----------|
| \( c_1Q_1 + c_2Q_2 \) | 0                      | -3Q_3    | -3Q_4    |
| \( Q_3 \) |                       | 0        | 0        |
| \( Q_4 \) |                       |          | 0        |

Since \( Q_5 \) is not a multiplier, the third symmetry action (6.13) is now non-trivial. In terms of components \( Q = (Q^u, Q^v) \) and \( P = (P^u, P^v) \), the form of this symmetry action is given by the linear operator

\[ S_P(Q) = (\text{pr}X_f(Q^u) - E_u(Qf^t), \text{pr}X_f(Q^v) - E_v(Qf^t)) \bigg|_{f=P} \]  

(8.39)

with \( X_f = f^u(t, x)\partial_u + f^v(t, x)\partial_v \), where \( \text{pr}X, E_u, E_v \) are regarded as operators in total derivatives when \( f = (f^u(t, x), f^v(t, x)) \) is replaced by \( P = (P^u, P^v) \). For \( Q = Q_5 \), the resulting action is summarized in Table 7.

**Table 7.** Coupled solitary wave equations: symmetry action (8.39) on the nonlocal adjoint-symmetry (8.34)

|       | \( P_1 \) | \( P_2 \) | \( P_3 \) |
|-------|----------|----------|----------|
| \( Q_5 \) | -2Q_4    | 2Q_3     | 2Q_5     |

The range of this action is the linear subspace of adjoint-symmetries \( \text{span}(Q_3, Q_4, Q_5) \), which provides a maximal domain for the adjoint-symmetry commutator bracket (5.1) with \( Q = Q_5 \), as shown in Table 8. The resulting bracket is a Lie bracket which is isomorphic to the symmetry algebra (8.28).

**Table 8.** Coupled solitary wave equations: adjoint-symmetry commutator bracket from symmetry action (8.39) with \( Q = Q_5 \)

|       | \( Q_3 \) | \( Q_4 \) | \( Q_5 \) |
|-------|----------|----------|----------|
| \( Q_3 \) | 0        | 0        | -\frac{1}{5}Q_3 |
| \( Q_4 \) | 0        | 0        | -\frac{4}{5}Q_4   |
| \( Q_5 \) |          |          | 0         |
8.3.2. Symplectic 2-form and Hamiltonian operator. Theorem 6.2 shows that the symmetry action (8.39) constructed in terms of the nonlocal adjoint-symmetry (8.34) encodes a Noether operator \( J \) for the coupled KdV equations (8.24). Specifically, one has \( J(P) = S_P(Q_5) = (\pr X_f Q_5^u - E_u(Q_5 f^v), \pr X_f Q_5^v - E_v(Q_5 f^u)) |_{f=pr} \), where

\[
(\pr X_f Q_5^u, \pr X_f Q_5^v)|_{f=pr} = \left( \frac{1}{\alpha} (D_x^{-1} P^u + x P^u) - 3t (v P^u + u P^v + \frac{\kappa}{\alpha} D_x^2 P^u), D_x^{-1} P^v + x P^v - 3t (u P^u + D_x^2 P^v) \right)
\]

and

\[
(E_u(Q_5 f^v), E_v(Q_5 f^u))|_{f=pr} = \left( \frac{1}{\alpha} (-D_x^{-1} P^u + x P^u) - 3t (v P^u + u P^v + \frac{\kappa}{\alpha} D_x^2 P^u), -D_x^{-1} P^v + x P^v - 3t (u P^u + D_x^2 P^v) \right).
\]

This yields

\[
J = 2 \left( \frac{1}{\alpha} D_x^{-1} \begin{pmatrix} 0 & D_x^{-1} \\ D_x^{-1} & 0 \end{pmatrix} \right)
\]

which actually is a symplectic operator. In particular, there is a bilinear form (6.17) associated to this operator,

\[
\omega_{Q_5}(P, \tilde{P}) = \int \tilde{P} J(P)^t \, dx = 2 \int (\frac{1}{\alpha} \tilde{P}^u D_x^{-1} P^u + \tilde{P}^v D_x^{-1} P^v) \, dx,
\]

where \( P^u \partial_u + P^v \partial_v \) and \( \tilde{P}^u \partial_u + \tilde{P}^v \partial_v \) are any pair of symmetries, and \( t \) denotes the transpose. Modulo boundary terms, this bilinear form is skew and closed (6.18), whence it defines a symplectic 2-form.

The inverse of \( J \) defines a Hamiltonian operator

\[
\mathcal{H} = J^{-1} = \left( \frac{\alpha}{2} D_x \begin{pmatrix} 0 & D_x^{-1} \\ D_x^{-1} & 0 \end{pmatrix} \right).
\]

Consequently, the coupled KdV equations (8.24) have the Hamiltonian formulation

\[
(\dot{u}_t, \dot{v}_t)^t = \frac{1}{\alpha} \mathcal{H}(\delta \mathcal{E}/\delta u, \delta \mathcal{E}/\delta v)^t
\]

in terms of the conserved energy, \( \mathcal{E} \). From this formulation, one has \( u_t = D_x (\delta \mathcal{E}/\delta u) \) and \( v_t = \frac{1}{\alpha} D_x (\delta \mathcal{E}/\delta v) \), which are analogous to the well-known first Hamiltonian structure of the KdV equation.

The symmetry action (8.39) involving the nonlocal adjoint-symmetry (8.34) thereby directly encodes the Hamiltonian structure of the system (8.24).

8.4. Nonlinear Schrodinger system. The generalized NLS equation \( i \psi_t + \psi_{xx} - k |\psi|^{2p} \psi = 0 \), with a nonlinearity power \( p > 0 \) and a real coefficient \( k \neq 0 \), is an important equation in physics and in analysis. When \( p = 1 \), it is an integrable system [27, 28] which is a model for light waves in nonlinear optical fibers and planar wave-guides, in addition to surface gravity waves in deep water, Langmuir waves in plasmas, and energy transport in alpha-helix molecules. Optical fibers that have a highly nonlinear refractive index (non-Kerr media) are modelled by \( p > 1 \) [29]. The case \( p = 2 \) has an extra conformal-type symmetry and is the critical power for global well-posedness of the Cauchy problem in \( L^2 \) [30].

This equation will be considered as an evolution system

\[
u_t = -v_{xx} + k(u^2 + v^2)^p v, \quad u_t = u_{xx} - k(u^2 + v^2)^p u,
\]

(8.46)
where $u(t, x)$ and $v(t, x)$ represent the real and imaginary parts of $\psi(t, x)$. Both parameters $p$ and $k$ will be treated as being arbitrary.

The system (8.46) possesses a Lagrangian formulation after multiplication by a $2 \times 2$ matrix

$$J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix},$$  
(8.47)

satisfying $J^2 = -I$ where $I$ is the identity matrix. In particular, the equation for $u$ is given by the variational derivative of the Lagrangian

$$L = \frac{1}{2}(v_u t - u v_t - u_x^2 - v_x^2) - \frac{k}{2(p+1)}(u^2 + v^2)^{p+1}$$  
(8.48)

with respect to $v$, while the variational derivative of Lagrangian with respect to $u$ yields the negative of the equation for $v$.

The determining equations for symmetries (in evolutionary form) $P^u \partial_u + P^v \partial_v$ are given by

$$(D_t P^u + D^2 P^v - k(u^2 + v^2)^{p-1}(2pvuP^u + (2p+1)v^2)P^v)|_E = 0,$$  
(8.49a)

$$(D_t P^v + D^2 P^u - k(u^2 + v^2)^{p-1}(2pvuP^u + (2p+1)v^2)P^v)|_E = 0,$$  
(8.49b)

where $E$ denotes the solution space of the Euler-Lagrange system (8.46).

The Lie point symmetries of this system (8.46), where $P = (P^u, P^v)$, are well known to be generated by

$$P_1 = (-v, u), \quad P_2 = (u_t, v_t), \quad P_3 = (u_x, v_x), \quad P_4 = (\frac{1}{2}xv + tu_x, -\frac{1}{2}xu + tv_x), \quad P_5 = (\frac{1}{p}u + 2tu_t + xu_x, \frac{1}{p}v + 2tv_t + xv_x),$$  
(8.50)

which represent an internal (phase) rotation, a time-translation, a space-translation, a combined Galilean transformation and phase shift, and a scaling. There are no first-order linear symmetries (as shown by a straightforward computation). The non-zero commutators in the symmetry algebra are given by

$$[P_2, P_3] = -P_3, \quad [P_2, P_5] = -2P_2, \quad [P_3, P_4] = \frac{1}{2}P_1, \quad [P_3, P_5] = -P_3, \quad [P_4, P_5] = P_4.$$  
(8.51)

From the Lagrangian structure, symmetries are in one-to-one correspondence with adjoint-symmetries via the relation

$$Q = (Q^u, Q^v) = P J = (P^v, -P^u).$$  
(8.52)

This yields

$$Q_1 = (u, v), \quad Q_2 = (v_t, -u_t), \quad Q_3 = (v_x, -u_x), \quad Q_4 = (-\frac{1}{2}xu + tv_x, -\frac{1}{2}xv - tu_x), \quad Q_5 = (\frac{1}{p}v + 2tv_t + xv_x, -\frac{1}{p}u - 2tu_t - xu_x),$$  
(8.53)

which constitutes a basis for the linear space of first-order linear adjoint-symmetries.

For evaluating the symmetry actions, all $t$ derivatives of $(u, v)$ are replaced through equations (8.46).

The first symmetry action (6.12) is given by the linear operator (8.51), which is summarized in Table 9. Note the entries have the form of an antisymmetric matrix, which originates from the underlying Lagrangian structure.
Consider the first symmetry action, which is energy; 
\[ C = \int_{\mathbb{R}} (u^2 + v^2) \, dx = \int_{\mathbb{R}} |\psi|^2 \, dx, \] (8.54)
which is the \( L^2 \) norm (representing the net charge of \( \psi(t,x) \));
\[ E = \int_{\mathbb{R}} (u_x^2 + v_x^2 + \frac{k}{p+1}(u^2 + v^2)^{p+1}) \, dx = \int_{\mathbb{R}} (|\psi_x|^2 + \frac{k}{p+1}|\psi|^{2p+2}) \, dx, \] (8.55)
which is energy;
\[ P = \int_{\mathbb{R}} (uv_x - vu_x) \, dx = \int_{\mathbb{R}} \frac{i}{2}(\bar{\psi}_x \psi - \psi_x \bar{\psi}) \, dx, \] (8.56)
which is momentum; and
\[ G = \int_{\mathbb{R}} (t(uv_x - vu_x) - \frac{1}{2}x(u^2 + v^2)) \, dx = \int_{\mathbb{R}} (\frac{1}{2}it(\bar{\psi}_x \psi - \psi_x \bar{\psi}) - \frac{x}{2}|\psi|^2) \, dx, \] (8.57)
which is Galilean momentum. This quantity is related to the motion of the center of charge \( \mathcal{X}(t) = \int_{\mathbb{R}} x|\psi|^2 \, dx \), due to \( G = tP - \mathcal{X}(t) \).

The adjoint-symmetry \( Q_5 = P_5J \), which corresponds to the scaling symmetry \( P_5 \), is not a multiplier. In this situation, the three symmetry actions (8.12), (8.11), and (8.13) will be distinct from each other. The most interesting ones are the first and third.

8.4.1. Adjoint-symmetry brackets and deformations. Consider the first symmetry action, which is given by the linear operator (8.7). A maximal domain for the resulting adjoint-symmetry commutator bracket occurs when one chooses \( Q \) so that the range of the dual operator (8.8) is maximal in the linear space of adjoint-symmetries. Correspondingly, one wants the kernel of this operator to be of minimal dimension in the linear space of Lie point symmetries. It is clear that this happens for the choice \( Q = Q_5 \), where the range has dimension 4. From Table 9, one has \( \ker(S_{Q_5}) = \text{span}(P_5) \), which is a subalgebra but not an ideal. As a consequence, \( \text{coker}(S_{Q_5}) \) does not have a unique representation in \( \text{span}(P_1, P_2, P_3, P_4, P_5) \). The same situation can be shown to occur for all other choices of \( Q \) that yield a one-dimensional kernel: \( Q = Q_5 + c_4 Q_4 + c_3 Q_3 + c_2 Q_2 + c_1 Q_1 \), where \( \ker(S_Q) = \text{span}(P_1 + c_4 P_4 + c_3 P_3 + c_2 P_2 + c_1 P_1) \). Nevertheless, a canonical choice of \( \text{coker}(S_{Q_5}) \) can be fixed by decomposing the symmetry Lie algebra under the scaling symmetry \( P_5 \), which gives \( \text{coker}(S_{Q_5}) = \text{span}(P_1) \oplus \text{span}(P_2) \oplus \text{span}(P_3) \oplus \text{span}(P_4) \) where each subalgebra has a distinct non-zero scaling weight while \( \text{span}(P_5) \) itself has zero scaling weight. This determines \( \text{ran}(S_{Q_5}^{-1}) = \text{span}(P_1, P_2, P_3, P_4) \).
Hence, one obtains
\[ S_{Q_5}^{-1}(Q_1) = \frac{p}{p-2} P_1, \quad S_{Q_5}^{-1}(Q_2) = -\frac{p}{p+2} P_2, \quad S_{Q_5}^{-1}(Q_3) = -\frac{p}{2} P_3, \quad S_{Q_5}^{-1}(Q_4) = \frac{p}{2p-2} P_4. \tag{8.58} \]

Then the adjoint-symmetry commutator bracket \((5.1)\) can be directly computed by
\[ Q_5 [Q_1, Q_2] = S_{Q_5}([\frac{p}{p-2} P_1, -\frac{p}{p+2} P_2]) = 0, \tag{8.59a} \]
\[ Q_5 [Q_1, Q_3] = S_{Q_5}([\frac{p}{p-2} P_1, -\frac{p}{2} P_3]) = 0, \tag{8.59b} \]
\[ Q_5 [Q_1, Q_4] = S_{Q_5}([\frac{p}{p-2} P_1, \frac{p}{2p-2} P_4]) = 0, \tag{8.59c} \]
\[ Q_5 [Q_2, Q_3] = S_{Q_5}([-\frac{p}{p+2} P_2, -\frac{p}{2} P_3]) = 0, \tag{8.59d} \]
\[ Q_5 [Q_2, Q_4] = S_{Q_5}([-\frac{p}{p+2} P_2, \frac{p}{2p-2} P_4]) = \frac{p^2}{2(p^2+p-2)} S_{Q_5}(P_3) = -\frac{p}{p^2+p-2} Q_3, \tag{8.59e} \]
\[ Q_5 [Q_3, Q_4] = S_{Q_5}([-\frac{p}{2} P_3, \frac{p}{2p-2} P_4]) = -\frac{p^2}{8(p-8)} S_{Q_5}(P_1) = -\frac{p(p-2)}{8(p-8)} Q_1. \tag{8.59f} \]

This result is summarized in Table 10. The bracket constitutes a non-abelian Lie bracket on the linear subspace \(\text{span}(Q_1, Q_2, Q_3, Q_4)\) (with codimension 1) of adjoint-symmetries \((8.53)\). In particular, this is the subspace of multipliers, and in accordance with Proposition 5.8, the bracket structure is isomorphic to the Lie algebra of variational symmetries, \(\mathcal{A} = \text{span}(P_1, P_2, P_3, P_4)\), which is generated by a phase rotation, a time-translation, a space-translation, and a Galilean transformation combined with a phase shift.

**Table 10. Nonlinear Schrodinger system: adjoint-symmetry commutator bracket with \(Q = Q_5\)**

|     | \(Q_1\) | \(Q_2\) | \(Q_3\) | \(Q_4\) |
|-----|---------|---------|---------|---------|
| \(Q_1\) | 0       | 0       | 0       | 0       |
| \(Q_2\) | 0       | 0       | \(-\frac{p}{p+2} P_3\) | \(-\frac{p(p-2)}{8(p-8)} Q_1\) |
| \(Q_3\) | 0       | -\frac{p}{2} P_3 | -\frac{p(p-2)}{8(p-8)} Q_1 | 0 |
| \(Q_4\) | 0       | 0       | 0       | 0       |

Next, consider the third symmetry action \((6.13)\), which is given by the linear operator \((8.39)\) using the non-multiplier \(Q_5\). One obtains
\[ S_P(Q_5) = (\frac{2-p}{p} P^v, -\frac{2-p}{p} P^u). \tag{8.60} \]

The resulting symmetry action of the Lie symmetries \((8.50)\) is summarized in Table 11. It consists of simply a multiple of the adjoint-symmetry that corresponds to each symmetry through the Noether relation \((8.52)\).

**Table 11. Nonlinear Schrodinger system: symmetry action \((8.39)\) on a non-gradient adjoint-symmetry**

|     | \(P_1\) | \(P_2\) | \(P_3\) | \(P_4\) | \(P_5\) |
|-----|---------|---------|---------|---------|---------|
| \(Q_5\) | \(\frac{2-p}{p} Q_1\) | \(\frac{2-p}{p} Q_2\) | \(\frac{2-p}{p} Q_3\) | \(\frac{2-p}{p} Q_4\) | \(\frac{2-p}{p} Q_5\) |
Consequently, the adjoint-symmetry bracket produced from this symmetry action is isomorphic to the Lie symmetry commutator bracket (8.51):

\[
\begin{align*}
[Q_2, Q_4] &= -\frac{p}{p-2}Q_3, \\
[Q_2, Q_5] &= -\frac{2p}{p-2}Q_2, \\
[Q_3, Q_4] &= \frac{p}{2(p-2)}Q_1, \\
[Q_3, Q_5] &= -\frac{p}{p-2}Q_3, \\
[Q_4, Q_5] &= \frac{p}{p-2}Q_4.
\end{align*}
\tag{8.61}
\]

Finally, Proposition 5.8 indicates that an adjoint-symmetry commutator bracket (5.1) also exists whenever \( \text{coker}(S_Q) \) coincides with a Lie subalgebra of symmetries. This condition (5.20) can be formulated as determining system for \( Q \), which can be solved straightforwardly.

The full symmetry algebra (8.50)–(8.51) possesses three four-dimensional subalgebras:

\[
\begin{align*}
\mathcal{A}_1 &= \text{span}(P_1, P_2, P_3, P_4); \\
\mathcal{A}_2 &= \text{span}(P_1, P_2, P_3, P_5); \\
\mathcal{A}_3 &= \text{span}(P_1, P_3, P_4, P_5).
\end{align*}
\tag{8.62}
\]

The first subalgebra is the same as \( \mathcal{A} \). The second subalgebra is generated by a phase rotation, a time-translation, a space-translation, and a scaling, while the third subalgebra is generated by a phase rotation, a space-translation, a combined Galilean transformation and phase shift, and a scaling.

Each of these subalgebras is found to concise with \( \text{coker}(S_Q) = \text{span}(Q_1, Q_2, Q_3, Q_4) \) for, respectively,

\[
\begin{align*}
Q &= Q_5 + c_4 Q_4; & Q &= Q_5 + c_4 Q_4, c_4 \neq 0; & Q &= Q_5 + c_4 Q_4 + c_3 Q_3 + c_2 Q_2, c_2 \neq 0.
\end{align*}
\tag{8.63}
\]

The resulting three commutator brackets are shown in Tables 12 to 14. These brackets are deformations of the commutator bracket in Table 10. Specifically, the first bracket is a one-parameter \( (c_4) \) continuous deformation; the second bracket is a finite deformation with one parameter \( (c_4 \neq 0) \); the third bracket is a finite deformation with three parameters \( (c_2 \neq 0, c_3, c_4) \).

**Table 12.** Nonlinear Schrodinger system: adjoint-symmetry commutator bracket deformation with \( Q = Q_5 + c_4 Q_4 \)

| \( Q_1 \) | \( Q_2 \) | \( Q_3 \) | \( Q_4 \) |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | \( \frac{p}{(p+2)(p-1)}(\frac{p}{8}c_4 Q_1 - Q_3) \) | \( -\frac{p(p-2)}{8(p-1)}Q_1 \) |
| 0 | \( \frac{p^2}{(p+2)(p-1)}Q_1 \) | 0 | 0 |

**Table 13.** Nonlinear Schrodinger system: adjoint-symmetry commutator bracket deformation with \( Q = Q_5 + c_4 Q_4, c_4 \neq 0 \)

| \( Q_1 \) | \( \tilde{Q}_2 \) | \( \tilde{Q}_3 \) | \( Q_4 \) |
|---|---|---|---|
| 0 | 0 | 0 | 0 |
| 0 | 0 | \( \frac{p}{p-1}(\tilde{Q}_2 + \frac{p}{8(p+2)}Q_1 - \frac{p}{2(p+2)}Q_3) \) | \( \frac{p}{p-1}(\frac{1}{2}Q_3 - \frac{p}{8}Q_1) \) |
| 0 | \( \frac{p}{p-1}(\frac{1}{2}Q_3 - \frac{p}{8}Q_1) \) | 0 | 0 |

\[ Q_2 = \frac{1}{c_4^2}Q_2, \quad \tilde{Q}_3 = \frac{1}{c_4^2}Q_3 \]
Table 14. Nonlinear Schrödinger system: adjoint-symmetry commutator bracket deformation with $Q = Q_5 + c_4 Q_4 + c_3 Q_3 + c_2 Q_2$, $c_2 \neq 0$

| $\tilde{Q}_1$ | $\tilde{Q}_2$ | $\tilde{Q}_3$ | $\tilde{Q}_4$ |
|--------------|--------------|--------------|--------------|
| $Q_1$        | 0            | 0            | 0            |
| $Q_2$        | 0            | $-\frac{p}{p+2} (Q_4 + \frac{p}{p-1} Q_3 + \frac{p(p-4)}{8(p-1)} c_4 Q_1 - \frac{1}{2(p-1)} c_3 Q_1)$ | 0            | 0            |
| $Q_3$        | 0            | $\frac{-\frac{p(p-2)}{8(p-1)} c_4}{Q_4}$ | $\frac{1}{c_2} Q_1$, | $\tilde{Q}_4 = c_2 Q_2$ |
| $Q_4$        | $\frac{1}{c_2} Q_1$, | $\tilde{Q}_4 = c_2 Q_2$ | $\tilde{c}_4 = c_4 c_2$ |

8.4.2. Symplectic 2-form and Hamiltonian operator. From Theorem 6.2, the third symmetry action (8.60) defines a Noether operator $S_P(Q_5) = J(P)$. Its explicit form is given by

$$J(P) = \frac{2-p}{p} P J. \quad (8.64)$$

The matrix operator $J$ is well-known to be a symplectic operator. Therefore, it yields an associated symplectic 2-form (6.17):

$$\omega_{Q_5}(\tilde{P}, P) = \frac{2-p}{p} \int \tilde{P} J^P dx = \frac{2-p}{p} \int (\tilde{P}^u P^v - P^u \tilde{P}^v) dx, \quad (8.65)$$

where $P^u \partial_u + P^v \partial_v$ and $\tilde{P}^u \partial_u + \tilde{P}^v \partial_v$ are any pair of symmetries. The inverse of $J$ defines a Hamiltonian operator $\mathcal{H} = J^{-1} = \frac{p}{p-2} J$. In particular, the evolution system (8.46) has the Hamiltonian formulation

$$(u_t, v_t) = (\delta \mathcal{E} / \delta u, \delta \mathcal{E} / \delta v) J. \quad (8.66)$$

These structures are a 2-component representation of the well-known Hamiltonian structure of the gNLS equation, $\psi_t = -i (\psi_{xx} - k |\psi|^2 \psi) = i \delta \mathcal{E} / \delta \bar{\psi}$. Hence, the symmetry action (8.60) directly encodes the Hamiltonian structure of the evolution equations for $u(t, x)$ and $v(t, x)$.

8.4.3. Novel symmetry brackets. Using the Euler-Lagrange form of equations (8.46),

$$\begin{pmatrix} \delta L/\delta u \\ \delta L/\delta v \end{pmatrix} = \begin{pmatrix} -v_t + u_{xx} - k(u^2 + v^2)^{p} u \\ u_t + v_{xx} - k(u^2 + v^2)^{p} v \end{pmatrix}, \quad (8.67)$$

one can examine the two novel symmetry brackets (7.11) and (7.12), which are respectively antisymmetric and symmetric. These brackets are constructed in terms of the symmetry generators $X_P = P^u \partial_u + P^v \partial_v$ and the associated linear operators $R_P$ given by

$$\text{pr} X_P \left( \begin{pmatrix} \delta L/\delta u \\ \delta L/\delta v \end{pmatrix} \right) = R_P \left( \begin{pmatrix} \delta L/\delta u \\ \delta L/\delta v \end{pmatrix} \right). \quad (8.68)$$
One has, by direct computation from the Lie point symmetries \((8.50)\),

\[
R_{P_1} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = J, \quad R_{P_2} = \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix} = ID_t, \quad R_{P_3} = \begin{pmatrix} D_x & 0 \\ 0 & D_x \end{pmatrix} = ID_x, \\
R_{P_4} = \begin{pmatrix} tD_x & \frac{1}{2}x \\ \frac{x}{2} & tD_x \end{pmatrix} = ItD_x - J \frac{1}{2}x, \\
R_{P_5} = \begin{pmatrix} 2p+1 \\ \frac{1}{p} 2tD_t + xD_x \\ 0 \\ 2p+1 \\ \frac{1}{p} 2tD_t + xD_x \end{pmatrix} = I(2p+1) + 2tD_t + xD_x,
\]

where \(J\) is the complex-structure matrix \((8.47)\) and \(I\) is the identity matrix. Hence, \(R_{P_i}^*\) is given by

\[
R_{P_1}^* = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = -J, \quad R_{P_2}^* = \begin{pmatrix} -D_t & 0 \\ 0 & -D_t \end{pmatrix} = -ID_t, \quad R_{P_3}^* = \begin{pmatrix} -D_x & 0 \\ 0 & -D_x \end{pmatrix} = -ID_x, \\
R_{P_4}^* = \begin{pmatrix} -tD_x & -\frac{x}{2} \\ \frac{x}{2} & -tD_x \end{pmatrix} = -ItD_x + J \frac{1}{2}x, \\
R_{P_5}^* = \begin{pmatrix} \frac{1-p}{p} & 2tD_t - xD_x \\ 0 & \frac{1-p}{p} & 2tD_t - xD_x \end{pmatrix} = -I(\frac{1-p}{p} + 2tD_t + xD_x).
\]

Evaluation of the two brackets \((7.11)\) and \((7.12)\) is then straightforward. The results are given in Tables \((15)\) and \((16)\).

Through the relation \((8.52)\), the antisymmetric bracket \((7.11)\) can be identified with the symmetry action on adjoint-symmetries in Table \((9)\).

The symmetric bracket \((7.12)\) in Table \((16)\) is a new structure.

**Table 15.** Nonlinear Schrödinger system: antisymmetric symmetry bracket \((7.11)\)

| \(P_1\) | \(P_2\) | \(P_3\) | \(P_4\) | \(P_5\) |
|---|---|---|---|---|
| \(P_1\) | 0 | 0 | 0 | 0 | \(\frac{p-2}{p} P_1\) |
| \(P_2\) | 0 | 0 | \(P_3\) | \(\frac{-p+2}{p} P_2\) |
| \(P_3\) | 0 | \(\frac{1}{p} P_1\) | 0 | \(\frac{-p}{p} P_3\) |
| \(P_4\) | 0 | 0 | 0 | \(\frac{2(p-1)}{p} P_4\) |
| \(P_5\) | 0 | 0 | 0 | 0 |

**Table 16.** Nonlinear Schrödinger system: symmetric symmetry bracket \((7.12)\)

| \(P_1\) | \(P_2\) | \(P_3\) | \(P_4\) | \(P_5\) |
|---|---|---|---|---|
| \(P_1\) | 0 | 0 | 0 | 0 | \(\frac{p-2}{p} P_1\) |
| \(P_2\) | 0 | 0 | 0 | \(\frac{p}{p} P_2\) |
| \(P_3\) | 0 | 0 | \(\frac{p}{p} P_3\) |
| \(P_4\) | 0 | \(\frac{p}{p} P_4\) |
| \(P_5\) | \(\frac{2(p-1)}{p} P_5\) |
In addition, one can also consider the deformed Lie bracket (7.15), which intertwines the commutator with \( \text{ad}(P) \) for any fixed symmetry \( P \). The domain for this bracket will be maximal in the linear space of symmetries (8.50) if the kernel of \( \text{ad}(P) \) is of minimum dimension. For the choice \( P = P_5 \) using the scaling symmetry, ker(\( \text{ad}(P_5) \)) = span(\( P_1, P_3 \)) is two dimensional, and coker(\( \text{ad}(P_5) \)) is three dimensional. However, ker(\( \text{ad}(P_5) \)) is not an ideal since it contains the scaling symmetry \( P_5 \), and so \( \text{ad}(P_5)^{-1} \) does not have a unique representation in span(\( P_1, P_2, P_3, P_4, P_5 \)). Nevertheless, the scaling induces a canonical decomposition span(\( P_1, P_2, P_3, P_4, P_5 \)) = span(\( P_1, P_3 \)) \( \oplus \) span(\( P_2 \)) \( \oplus \) span(\( P_3 \)) \( \oplus \) span(\( P_4 \)) where each subalgebra has a distinct scaling weight, as shown by the commutators (8.51). One thereby has

\[
\text{ad}(P_5)^{-1}P_2 = \frac{1}{2}P_2, \quad \text{ad}(P_5)^{-1}P_3 = P_3, \quad \text{ad}(P_5)^{-1}P_4 = -P_4. \tag{8.71}
\]

Consequently, the deformed Lie bracket (7.15) on span(\( P_2, P_3, P_4 \)) can be computed by

\[
P_5[P_2, P_3] = \text{ad}(P_5)[\text{ad}(P_5)^{-1}P_2, \text{ad}(P_5)^{-1}P_3] = \frac{1}{2} \text{ad}(P_5)[P_2, P_3] = \frac{1}{2} \text{ad}(P_5)(0) = 0, \tag{8.72a}
\]

\[
P_5[P_2, P_4] = \text{ad}(P_5)[\text{ad}(P_5)^{-1}P_2, \text{ad}(P_5)^{-1}P_4] = -\frac{1}{2} \text{ad}(P_5)[P_2, P_4] = -\frac{1}{2} \text{ad}(P_5)(-P_3) = \frac{1}{2}P_3, \tag{8.72b}
\]

\[
P_5[P_3, P_4] = \text{ad}(P_5)[\text{ad}(P_5)^{-1}P_3, \text{ad}(P_5)^{-1}P_4] = -\text{ad}(P_5)[P_3, P_4] = -\text{ad}(P_5)(\frac{1}{2}P_1) = 0. \tag{8.72c}
\]

Note that this linear subspace span(\( P_2, P_3, P_4 \)) is not itself a subalgebra in the symmetry algebra (8.50)–(8.51). Thus, the deformation (8.72) is non-trivial.

### 8.5. Coupled Boussinesq system

Consider a nonlinear system of coupled Boussinesq-type equations (31)

\[
v_t + u_x + \alpha uv_x - v_{txx} = 0, \quad u_t + v_x + \alpha(uv)_x - u_{txx} = 0, \quad \tag{8.73}
\]

where the parameter \( \alpha \neq 0 \) is the convection coefficient. This system models a long-wavelength regime of inviscid fluid flow in shallow water with a free surface, where \( v \) is the horizontal velocity at a certain depth, and \( u \) is the deviation of the surface from equilibrium.

Equations (8.73) are an Euler-Lagrange system in terms of potentials \( v = \psi_x \) and \( u = \phi_x \),

\[
\phi_{tx} + \psi_{tx} + \alpha(\phi_x \psi_x)_x - \phi_{txxx} = -\delta L/\delta \psi = 0,
\]

\[
\psi_{tx} + \phi_{xx} + \alpha \frac{1}{2}(\psi_x^2)_x - \psi_{txxx} = -\delta L/\delta \phi = 0, \tag{8.74}
\]

with a Lagrangian

\[
L = \phi_t \psi_x + \frac{1}{2}(\phi_x^2 + \psi_x^2) + \phi_{tx} \psi_{xx} + \frac{1}{2} \alpha \psi_x^2 \phi_x. \tag{8.75}
\]

Through this variational structure, adjoint-symmetries (\( Q^\psi, Q^\phi \)) of the Euler-Lagrange equations coincide with symmetries \( P^\psi \partial_\psi + P^\phi \partial_\phi \) (in evolutionary form) up to a sign:

\[
Q^\psi = -P^\psi, \quad Q^\phi = -P^\phi. \tag{8.76}
\]

The determining equations for symmetries (and, hence, adjoint-symmetries) are given by

\[
(D_t D_x P^\phi + D_x^2 P^\psi_P + \alpha D_x(\phi_x D_x P^\phi + \psi_x D_x P^\phi) - D_x D_x^3 P^\phi)|_E = 0, \tag{8.77a}
\]

\[
(D_t D_x P^\psi + D_x^2 P^\phi + \alpha D_x(\psi_x D_x P^\psi) - D_x D_x^3 P^\psi)|_E = 0, \tag{8.77b}
\]

where \( E \) denotes the solution space of the system (8.74).
In the original variables, symmetries $P^v \partial_v + P^u \partial_u$ are given by the prolongation
\[ (P^v, P^u) = D_x (P^\psi, P^\phi), \] (8.78)
and adjoint-symmetries are given by the transpose relation
\[ (Q^v, Q^u) = (Q^\phi, Q^\psi). \] (8.79)
A basis of the linear space of first-order linear symmetries $P^\psi \partial_\psi + P^\phi \partial_\phi$ can be shown to consist of the Lie point symmetry generators, with $P = (P^\psi, P^\phi)$,
\[ P_1(g, f) = (g(t), f(t)), \quad P_2 = (\psi_t, \phi_t), \quad P_3 = (\psi_x, \phi_x), \quad P_4 = (\psi + t\psi_t, 2\phi + \frac{1}{\alpha}x + t\phi_t), \] (8.80)
which represent a time-dependent shift, a time-translation, a space-translation, and a generalized scaling. The non-zero commutators in the symmetry algebra consist of
\[ [P_1(g, f), P_2] = P_1(g', f'), \quad [P_1(g, f), P_4] = P_1(g + tg', 2f + tf'), \quad [P_2, P_4] = -P_2, \quad [P_3, P_4] = -P_1(0, \frac{2}{\alpha}). \] (8.81)
A straightforward computation shows that the symmetries $P_1$, $P_2$, and $P_3$ (but not $P_4$) are variational. The resulting conservation laws, in terms of $u$ and $v$, are respectively given by
\[ \mathcal{M} = f(t) \int_R u \, dx + g(t) \int_R v \, dx, \] (8.82)
which is linear combination of the mass of $u$ and $v$;
\[ \mathcal{E} = \int_R \frac{1}{2}(u^2 + v^2 + \alpha uv^2) \, dx, \] (8.83)
which is energy;
\[ \mathcal{P} = \int_R (uv + u_x v_x) \, dx, \] (8.84)
which is momentum.

8.5.1. Novel symmetry brackets. For the Euler-Lagrange equations (8.74), in addition to the symmetry commutator (8.81), there exist two new symmetry brackets. The first bracket (7.11) is antisymmetric, and the second bracket (7.12) is symmetric. These brackets are constructed in terms of the symmetry generators $X_P = P^\phi \partial_\phi + P^\psi \partial_\psi$ and the associated linear operators $R_P$ which are given by the righthand side of the determining equations (8.77) off of the solution space $\mathcal{E}$: namely, $R_P(G)$ with $G = (G^\psi, G^\phi)$ denoting the Euler-Lagrange equations (8.74). This yields
\[ R_{P_1} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad R_{P_2} = \begin{pmatrix} D_t & 0 \\ 0 & D_t \end{pmatrix}, \quad R_{P_3} = \begin{pmatrix} D_x & 0 \\ 0 & D_x \end{pmatrix}, \quad R_{P_4} = \begin{pmatrix} 3 + tD_t & 0 \\ 0 & 2 + tD_t \end{pmatrix}, \] (8.85)
from the symmetries (8.80). One then has
\[ R^*_{P_1} = R_{P_1}, \quad R^*_{P_2} = -R_{P_2}, \quad R^*_{P_3} = -R_{P_3}, \quad R^*_{P_4} = \begin{pmatrix} 2 - tD_t & 0 \\ 0 & 1 - tD_t \end{pmatrix}. \] (8.86)
Evaluation of the brackets (7.11) and (7.12) straightforwardly yields the results in Tables 17 and 18.
using the scaling variational point symmetries. The metric bracket given in Table 17 defines a Noether structure through the symmetry operator kernel is given by span(\(g\)), \(f\)) \(\alpha\). This subspace is a Lie subalgebra, as shown by the first commutator in Theorem 7.3. The preceding symmetry brackets are defined on the whole linear space of Lie symmetries, \(\text{span}(P_1(g, f), P_2, P_3, P_4)\). There are also exist brackets defined on linear subspaces, as follows from Theorem 7.3.

Consider the action of the scaling symmetry, \(\text{ad}(P_4)\), on \(\text{span}(P_1(g, f), P_2, P_3, P_4)\). Its kernel is given by \(\text{span}(P_1(g_{-1}/t, f_{-2}/t^2), P_2, P_3 + P_1(0, \frac{1}{\alpha}), P_4)\), where \(g_{-1}, f_{-2}\) are arbitrary constants, while its range consists of the linear subspace \(\text{span}(P_1(\tilde{g}, \tilde{f}), P_2)\), for arbitrary functions \(\tilde{g}(t), \tilde{f}(t)\). This subspace is a Lie subalgebra, as shown by the first commutator in the symmetry algebra (8.81). Hence, \(\text{ad}(P_4)\) is one-to-one on the Lie subalgebra

\[
A = \text{span}(P_1(\tilde{g}, \tilde{f}), P_2), \quad \tilde{g}(t) = \sum_{j=0}^{d_2} g_j t^j, \quad \tilde{f}(t) = \sum_{i=0}^{d_1} f_i t^i, \quad (8.87)
\]

involving polynomials with arbitrary coefficients \(g_j, f_i\). A deformed Lie bracket on \(A\) is defined by the pull-back (7.15) of the commutator bracket under the inverse action of \(\text{ad}(P_4)\):

\[
\text{ad}(P_4)^{-1} P_1(t^j, t^i) = P_1(\frac{1}{j+1} t^j, \frac{1}{i+1} t^i), \quad \text{ad}(P_4)^{-1} P_2 = -P_2. \quad (8.88)
\]

The non-zero deformed commutators are thus given by

\[
P_k[P_1(t^j, t^i), P_2] = [P_1(t^j, t^i), \text{ad}(P_4)^{-1} P_2] + [\text{ad}(P_4)^{-1} P_1(t^j, t^i), P_2]
\]

\[
= -[P_1(t^j, t^i), P_2] + [P_1(\frac{1}{j+2} t^j, \frac{1}{i+1} t^i), P_2]
\]

\[
= -P_1(j t^{j-1}, i t^{i-1}) + P_1(\frac{j+1}{j+2} t^{j-1}, \frac{i+1}{i+1} t^{i-1}) \quad (8.89)
\]

which differ in comparison to the symmetry commutators \([P_1(t^j, t^i), P_2] = P_1(j t^{j-1}, i t^{i-1})\).

### 8.5.2. Symplectic 2-form and Hamiltonian operator

Corollary 7.4 shows that the antisymmetric bracket given in Table 17 defines a Noether structure through the symmetry operator \(\mathcal{R}_P = R[P, \cdot]\) given in terms of any chosen symmetry \(P\). A maximal range is obtained by using the scaling \(P = P_4\), whereby \(\text{ran}(\mathcal{R}_P_4) = \text{span}(P_1(g, f), P_2, P_3)\) is the subalgebra of variational point symmetries.

### Table 17. Coupled Boussinesq system: antisymmetric symmetry bracket (7.11)

|         | \(P_1(\tilde{g}, \tilde{f})\) | \(P_2\) | \(P_3\) | \(P_4\) |
|---------|-------------------------------|---------|---------|---------|
| \(P_1(g, f)\) | 0                              | 0       | \(P_1(tg' - 2g, tf' - f)\) | \(-4P_2\) |
| \(P_2\) | 0                              | 0       | 0       | \(3P_3\) |
| \(P_3\) | 0                              | 0       | 0       | 6\(P_4\) |
| \(P_4\) |                               |         |         |          |

### Table 18. Coupled Boussinesq system: symmetric symmetry bracket (7.12)

|         | \(P_1(\tilde{g}, \tilde{f})\) | \(P_2\) | \(P_3\) | \(P_4\) |
|---------|-------------------------------|---------|---------|---------|
| \(P_1(g, f)\) | 0                              | 0       | 0       | 3\(P_1(g, f)\) |
| \(P_2\) | 0                              | 0       | 0       | 3\(P_2\) |
| \(P_3\) | 0                              | 0       | 3\(P_3\) |         |
| \(P_4\) |                               |         |         |          |
From Theorem [7.6] this Noether structure yields an associated symplectic 2-form [7.26], which is constructed in terms of the conservation law arising from pairs of variational symmetries, as described explicitly in Lemma [7.5]. Furthermore, the symplectic 2-form will give rise to an associated Hamiltonian structure $H = J^{-1}$ through the inverse of a symplectic operator $J$ in total derivatives.

To carry out this construction, one starts from the symmetry determining equations (8.77) expressed in the operator form

$$
\left( D_t D_x - D_t D_x^3 + \alpha D_x \psi_x D_x \right) \frac{D^2}{D_x} + \alpha D_x \phi_x D_x
\right) \left( P^\phi \right) \right|_\xi = 0. \tag{8.90}
$$

Then, for any pair of symmetries $P^\psi \partial_\psi + P^\phi \partial_\phi$ and $\tilde{P}^\phi \partial_\psi + \tilde{P}^\phi \partial_\phi$, one has the identity (7.23) which takes the explicit form

$$
\left( \tilde{P}^\phi \right)^t \left( D_t D_x \right) - D_t D_x^3 + \alpha D_x \psi_x D_x \right) \frac{D^2}{D_x} + \alpha D_x \phi_x D_x
\right) \left( P^\phi \right)
- \left( P^\psi \right)^t \left( D_t D_x \right) - D_t D_x^3 + \alpha D_x \psi_x D_x \right) \frac{D^2}{D_x} + \alpha D_x \phi_x D_x
\right) \left( \tilde{P}^\phi \right)
= D_t \Psi^t (P, \tilde{P}) + D_x \Psi^x (P, \tilde{P}),
$$

where

$$
\Psi^t (P, \tilde{P}) = \frac{1}{2} \left( D_x \tilde{P}^\psi \right)^t \left( 1 - D_x^2 \right) \left( P^\phi \right)
- \frac{1}{2} \left( D_x P^\psi \right)^t \left( 1 - D_x^2 \right) \left( \tilde{P}^\phi \right). \tag{8.92}
$$

Through the prolongation (8.78) from $(\psi, \phi)$ to $(u, v)$, this expression has the simpler form

$$
\Psi^t (P, \tilde{P}) = \frac{1}{2} \left( \tilde{P}^u \nu \right)^t \left( 0 \ D \right) \left( P^u \nu \right) - \frac{1}{2} \left( P^u \nu \right)^t \left( 0 \ D \right) \left( \tilde{P}^u \nu \right)
= \frac{1}{2} \left( \tilde{P}^u \nu D P^u - P^u \nu D \tilde{P}^u + \tilde{P}^u \nu D P^u - P^u \nu D \tilde{P}^u \right) \tag{8.93}
$$

in terms of the skew operator

$$
D = D_x^{-1} - D_x = -D^*. \tag{8.94}
$$

Finally, the symplectic 2-form (7.26) is given by

$$
\omega (P, \tilde{P}) = \int \Psi^t (P, \tilde{P}) \, dx = \int \left( \tilde{P}^u \nu \right)^t \left( 0 \ D \right) \left( P^u \nu \right) \, dx = \int \tilde{P}^t J P \, dx \tag{8.95}
$$

after integration by parts and use of the skew property of $D$, where

$$
J = \left( \begin{array}{cc} 0 & D \\ D & 0 \end{array} \right). \tag{8.96}
$$

This operator is symplectic. In particular, $J^* = -J$, where the adjoint involves the adjoint of $D$ combined with the transpose of the matrix.

The inverse of the symplectic operator (8.96) defines a Hamiltonian operator

$$
H = J^{-1} = \left( \begin{array}{cc} 0 & D^{-1} \\ D^{-1} & 0 \end{array} \right) \tag{8.97}
$$
with respect to the variables \((v, u)\). As a consequence, the coupled Boussinesq system \((8.73)\) possesses the Hamiltonian formulation

\[
\begin{pmatrix} v_t \\ u_t \end{pmatrix} = H \begin{pmatrix} \delta \mathcal{E} / \delta v \\ \delta \mathcal{E} / \delta u \end{pmatrix}
\]  
(8.98)

where the Hamiltonian functional is the energy \((8.83)\). The corresponding Hamiltonian form of the Euler-Lagrange system \((8.74)\) is given by

\[
\begin{pmatrix} \psi_t \\ \phi_t \end{pmatrix} = D \begin{pmatrix} \delta \mathcal{E} / \delta \psi \\ \delta \mathcal{E} / \delta \phi \end{pmatrix}, \quad D = D_x^{-1} \mathcal{H} D_x^{-1}
\]  
(8.99)

with \(D\) being a Hamiltonian operator with respect to the potentials \((\psi, \phi)\). This formulation is equivalent to the system

\[
(1 - D_x^2)(\phi_t, \psi_t) = D_x^{-1}(\delta \mathcal{E} / \delta \psi, \delta \mathcal{E} / \delta \phi).
\]  
(8.100)

Hence, the new antisymmetric symmetry bracket in Table \([17]\) directly encodes a symplectic structure and a corresponding Hamiltonian structure for the coupled Boussinesq system \((8.73)\).

8.6. **Two-dimensional Maxwell equations.** The free-space Maxwell equations in two space dimensions are given by the linear system \([8.32]\)

\[
\begin{align*}
E_1^1 &= c B_y, & E_2^2 &= -c B_x, \\
B_t &= c(E_1^1 - E_2^2), \\
E_x^1 + E_y^2 &= 0,
\end{align*}
\]  
(8.101a-101c)

for the electric field \(\vec{E} = (E^1(t, x, y), E^2(t, x, y))\) and the magnetic field \(B(t, x, y)\), where \(c\) is the speed of light. In this system, the spatial equation \((8.101c)\) has the role of a constraint (absence of electric charge) that is preserved by the evolution equations \((8.101a)-(8.101b)\).

The determining equations for symmetries (in evolutionary form) \(P^{E_1} \partial_{E_1} + P^{E_2} \partial_{E_2} + P^B \partial_B\) are given by

\[
(D_t P^{E_i} - c D_y P^B)|_\mathcal{E} = 0, \quad (D_t P^{E_2} + c D_x P^B)|_\mathcal{E} = 0, \\
(D_t P^B - c D_y P^{E_1} - D_x P^{E_1})|_\mathcal{E} = 0, \\
(D_x P^{E_1} + D_y P^{E_2})|_\mathcal{E} = 0
\]  
(8.102a-102c)

holding on the solution space \(\mathcal{E}\) of Maxwell’s equations \((8.101)\). A basis for the linear space of Lie point symmetries, with \(P = (P^{E_1}, P^{E_2}, P^B)\), consists of

\[
\begin{align*}
P_1 &= (E^1, E^2, B), & P_2 &= (E_1^1, E_2^1, B_t), & P_3 &= (E_1^x, E_2^x, B_x), & P_4 &= (E_1^y, E_2^y, B_y), \\
P_5 &= (E^2 + x E_1^y - y E_1^x, -E^1 + x E_2^y - y E_2^x, x B_y - y B_x), \\
P_6 &= (c^2 t E_x^y + x E_1^1, c^2 t E_x^2 + x E_1^2 - c B, c^2 t B_x + x B_t - c E^2), \\
P_7 &= (c^2 t E_y^1 + y E_1^1 + c B, c^2 t E_y^2 + y E_1^2, c^2 t B_y + y B_t + c E^1), \\
P_8 &= (t E_1^1 + x E_1^y + y E_1^x, t E_1^2 + x E_2^y + y E_2^x, t B_t + x B_x + y B_y),
\end{align*}
\]  
(8.103)

in addition to \(P_0 = (f^{E_1}(t, x, y), f^{E_2}(t, x, y), f^{B}(t, x, y))\), where these functions satisfy Maxwell’s equations \((8.101)\). The symmetries \((8.103)\) are generators for, respectively, a
scaling, a time-translation, two space translations, a rotation, two Lorentz boosts, and a space-time dilation. Their algebra is given by the non-zero commutators

\begin{equation}
[P_2, P_6] = -c^2 P_3, \quad [P_2, P_7] = -c^2 P_4, \quad [P_2, P_8] = -P_2,
\end{equation}

\begin{equation}
[P_3, P_5] = P_4, \quad [P_3, P_6] = -P_2, \quad [P_3, P_8] = -P_3,
\end{equation}

\begin{equation}
[P_4, P_5] = -P_3, \quad [P_4, P_7] = -P_2, \quad [P_4, P_8] = -P_4,
\end{equation}

\begin{equation}
[P_5, P_6] = -P_7, \quad [P_5, P_7] = P_6, \quad [P_5, P_8] = c^2 P_5.
\end{equation}

The determining equations for adjoint-symmetries \((Q^{E_1}, Q^{E_2}, Q^B, Q^{div})\) are given by

\begin{equation}
(-D_t Q^{E_1} + cD_y Q^B - D_x Q^{div})|_\mathcal{E} = 0, \quad (-D_t Q^{E_2} - cD_x Q^B - D_y Q^{div})|_\mathcal{E} = 0,
\end{equation}

\begin{equation}
(-D_t Q^B + c(D_y Q^{E_1} - D_x Q^{E_2})|_\mathcal{E} = 0.
\end{equation}

A basis of the linear space of zeroth-order adjoint-symmetries, with \(Q = (Q^{E_1}, Q^{E_2}, Q^B, Q^{div})\), is comprised by

\begin{equation}
Q_1 = (E^2, -E^1, 0, cB), \quad Q_2 = (E^1, E^2, B, 0),
\end{equation}

\begin{equation}
Q_3 = (0, B, E^2, -cE^1), \quad Q_4 = (B, 0, E^1, cE^2),
\end{equation}

\begin{equation}
Q_5 = (x B, y B, x E^1 + y E^2, c(x E^2 - y E^1)),
\end{equation}

\begin{equation}
Q_6 = (x E^1, x E^2 - ct B, x B - ct E^2, c^2 t E^1), \quad Q_7 = (y E^1 + ct B, y E^2, y B + ct E^1, c^2 t E^2),
\end{equation}

in addition to \(Q_0 = (g^{E_1}(t, x, y), g^{E_2}(t, x, y), g^B(t, x, y), g^{div}(t, x, y))\), where these functions satisfy the adjoint system associated to Maxwell’s equations \((8.101)\):

\begin{equation}
g^{E_1}_t = c g^B_y - g^{div}_x, \quad g^{E_2}_t = -c g^B_x - g^{div}_y, \quad g^B_t = c(g^{E_1}_x - g^{E_2}_y).
\end{equation}

The linear adjoint-symmetries \((8.106)\) apart from \(Q_1\) are multipliers for conservation laws of energy

\begin{equation}
\mathcal{E} = \int_{\mathbb{R}^2} \frac{1}{2}(|\vec{E}|^2 + B^2) \, dx \, dy,
\end{equation}

momenta

\begin{equation}
\mathcal{P}^x = \int_{\mathbb{R}^2} E^1 B \, dx \, dy, \quad \mathcal{P}^y = \int_{\mathbb{R}^2} E^2 B \, dx \, dy,
\end{equation}

angular momentum

\begin{equation}
\mathcal{J} = \int_{\mathbb{R}^2} (x E^1 + y E^2) B \, dx \, dy,
\end{equation}

and boost momenta

\begin{equation}
\mathcal{G}^x = \int_{\mathbb{R}^2} \left(\frac{1}{2}x(|\vec{E}|^2 + B^2) - ct E^2 B\right) \, dx \, dy, \quad \mathcal{G}^y = \int_{\mathbb{R}^2} \left(\frac{1}{2}y(|\vec{E}|^2 + B^2) + ct E^1 B\right) \, dx \, dy,
\end{equation}

respectively.

Since Maxwell’s equations are linear, they possess symmetry recursion operators and adjoint-symmetry recursion operators, which generate infinite hierarchies of higher-order symmetries and adjoint-symmetries \([33, 34]\). These hierarchies will not be considered in what follows.

Maxwell’s equations also possess gauge adjoint-symmetries

\begin{equation}
Q_\chi = (D_x \chi, D_y \chi, 0, -D_t \chi)
\end{equation}
where $\chi$ is an arbitrary function on jet space. These adjoint-symmetries arise from the differential identity that expresses the compatibility between the spatial divergence equation \((8.101c)\) and the evolution equations \((8.101a)-(8.101b)\). Another way to look at them is that there is one less determining equation than the number of components for adjoint-symmetries, and hence the solution will contain an arbitrary function of all variables.

8.6.1. **Symmetry action on adjoint-symmetries.** The adjoint-symmetry $Q_1$ is not a multiplier, as shown by the simple observation that $E^2(E^1_i - cB_y) - E^1(E^2_i - cB_x) + cB(E^1_i + E^2_y)$ is not a total divergence. (In particular, this expression is not annihilated by the Euler operators with respect to $E^1, E^2, B$.) As a consequence, the first symmetry action \((3.8)\) differs from the second symmetry action \((3.11)\), while the third symmetry action \((3.12)\) is non-trivial.

In terms of components $(P^{E_1}, P^{E_2}, P^B)$ and $(Q^{E_1}, Q^{E_2}, Q^B, Q^{\text{div}})$, the first symmetry action is given by the linear operator

$$S_F(Q) = (\text{pr}X_F(Q^{E_1}) + R^*_F(Q)E^1, \text{pr}X_F(Q^{E_2}) + R^*_F(Q)E^2, \text{pr}X_F(Q^B) + R^*_F(Q)B, \text{pr}X_F(Q^{\text{div}}) + R^*_F(Q)^{\text{div}})$$ \((8.113)\)

where $R^*_F$ is the adjoint of the linear operator $R_F$ associated to the action of the symmetry generator $\text{pr}X_F = P^{E_1}\partial_{E_1} + P^{E_2}\partial_{E_2} + P^B\partial_B$ on Maxwell’s equations, $G = 0$. Specifically, $R_F(G)$ is the righthand side of the determining equations \((8.102)\) off of the solution space $\mathcal{E}$, where

$$G = (E^1_i - cB_y, E^2_i + cB_x, B_t - c(E^1_y - E^2_x), E^1_x + E^2_y)^t$$ \((8.114)\)

is a column vector assembled from Maxwell’s equations.

From the symmetries \((8.103)\), one obtains

$$R_{P_1} = I, \quad R_{P_2} = ID_t, \quad R_{P_3} = ID_x, \quad R_{P_4} = ID_y, \quad R_{P_5} = J + I(yD_x - xD_y), \quad R_{P_6} = c^2K_1 + I(xD_t + c^2tD_x), \quad R_{P_7} = c^2K_2 + I(yD_t + c^2tD_y), \quad R_{P_8} = I(1 + tD_t + xD_x + yD_y).$$ \((8.115)\)

where

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad K_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$ \((8.116)\)

One then has

$$R^*_{P_1} = I, \quad R^*_{P_2} = -ID_t, \quad R^*_{P_3} = -ID_x, \quad R^*_{P_4} = -ID_y, \quad R^*_{P_5} = -J - I(yD_x - xD_y), \quad R^*_{P_6} = c^2K_1 - I(xD_t + c^2tD_x), \quad R^*_{P_7} = c^2K_2 - I(yD_t + c^2tD_y), \quad R^*_{P_8} = -I(2 + tD_t + xD_x + yD_y).$$ \((8.117)\)

It is straightforward now to evaluate the action \((8.113)\). The result is summarized in Table 19.

The third symmetry action \((3.12)\) is given by the linear operator

$$S_F(Q) = (\text{pr}X_F(Q^{E_1}) + R^*_Q(P)^E_1, \text{pr}X_F(Q^{E_2}) + R^*_Q(P)^E_2, \text{pr}X_F(Q^B) + R^*_Q(P)^B, \text{pr}X_F(Q^{\text{div}}) + R^*_Q(P)^{\text{div}})$$ \((8.118)\)
where \( R_Q^* \) is the adjoint of the linear operator \( R_Q \) arising from the righthand side of the adjoint-symmetry determining equations (8.105) off of the solution space \( E \). For evaluating the symmetry action (8.118), only \( Q = Q_1 \) needs to be considered, since the action will be trivial for all of the other adjoint-symmetries (8.106), which are multipliers. One has

\[
R_{Q_1} = \begin{pmatrix}
0 & -1 & 0 & 0 \\
1 & 0 & 0 & 0 \\
0 & 0 & -c & 0
\end{pmatrix},
\] (8.119)

and thus

\[
R_{Q_1}^* = R_{Q_1}^t = \begin{pmatrix}
0 & 1 & 0 \\
-1 & 0 & 0 \\
0 & 0 & -c
\end{pmatrix}.
\] (8.120)

The resulting action by the Lie point symmetries (8.103) is shown in Table 20. In particular, the space-time dilation symmetry \( P_8 \) yields

\[
S_{P_8}(Q_1) = (2t E_2^2 + 2x E_x^2 + 2y E_y^2, -2t E_1^2 - 2x E_x^1 - 2y E_y^1, 0, c(2t B_2 + 2x B_x + 2y B_y))
\]

\[
= (2E^2, -2E^1, 0, cB) + (D_x, D_y, 0, -D_t)(2x E^2 - 2y E^1 - 2tB)
\] (8.121)

on solutions of Maxwell’s equations, where the second term is a gauge adjoint-symmetry.

**Table 20.** Maxwell’s equations: symmetry action (8.118) on the non-gradient adjoint-symmetry (modulo gauge)

| \( Q_1 \) | \( 2Q_1 \) | \( P_1 \) | \( P_2 \) | \( P_3 \) | \( P_4 \) | \( P_5 \) | \( P_6 \) | \( P_7 \) | \( P_8 \) |
|---|---|---|---|---|---|---|---|---|---|
| \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( 0 \) | \( -2Q_1 \) | \( -Q_3 \) | \( cQ_4 \) | \( -2Q_2 \) |

8.6.2. **Nonlocal adjoint-symmetries.** Maxwell’s equations in three space dimensions have a well-known formulation using a scalar potential and a vector potential, arising from the form of the evolution equation and divergence equation for the magnetic field vector. In free-space, there is a dual potential formulation [35] coming from the corresponding equations for the electric field vector. This latter formulation carries over to Maxwell’s equations in two space dimensions (8.101).
The divergence equation (8.101c) for the electric field will become an identity with the introduction of a scalar potential \( \phi(t, x, y) \) given by

\[
E^1 = \phi_y, \quad E^2 = -\phi_x. \tag{8.122a}
\]

Then the electric field evolution equation (8.101a) is identically satisfied by

\[
B = \frac{1}{c} \phi_t. \tag{8.122b}
\]

Finally, the magnetic field evolution equation (8.101b) yields

\[
\frac{1}{c^2} \phi_{tt} = \phi_{xx} + \phi_{yy}. \tag{8.123}
\]

which is a scalar wave equation.

Symmetries \( P^\phi \partial_\phi \) (in evolutionary form) of the wave equation (8.123) give rise to symmetries and adjoint-symmetries of Maxwell’s equations via the following relations [32]. The potential system (8.122) directly implies that

\[
P^E_1 = D_y P^\phi, \quad P^E_2 = -D_x P^\phi, \quad P^B = \frac{1}{c} D_t P^\phi. \tag{8.124}
\]

As the wave equation (8.123) is self-adjoint, its adjoint-symmetries coincide with its symmetries. Then the identity

\[
B_t - c(E^1_y - E^2_x) = \frac{1}{c} \phi_{tt} - c(\phi_{xx} + \phi_{yy}) \implies Q^B = c P^\phi. \tag{8.125}
\]

From this relation, the determining equations for adjoint-symmetries now can be solved to obtain \( Q^E_1, Q^E_2, Q^{\text{div}} \) in terms of \( P^\phi \). The first step is to observe that the gauge adjoint-symmetry freedom allows one to impose the divergence equation

\[
D_x \tilde{Q}^E_1 + D_y \tilde{Q}^E_2 = 0,
\]

by putting

\[
\chi = -\Delta^{-1} (D_x \tilde{Q}^E_1 + D_y \tilde{Q}^E_2). \tag{8.126}
\]

Next, this divergence equation implies that \( \tilde{Q}^E_1 = D_y \Phi \) and \( \tilde{Q}^E_2 = -D_x \Phi \) holds for some function \( \Phi \). Substitution of these two curl expressions and expression (8.125) into the determining equation (8.105b) yields \( D_t P^\phi = \Delta \Phi \), which gives \( \Phi = \Delta^{-1} D_t P^\phi \). Hence, one obtains \( \tilde{Q}^E_1 = D_y \Delta^{-1} D_t P^\phi \) and \( \tilde{Q}^E_2 = -D_x \Delta^{-1} D_t P^\phi \). This yields the relations

\[
Q^E_1 = D_y \Delta^{-1} D_t P^\phi + D_x \chi, \quad Q^E_2 = -D_x \Delta^{-1} D_t P^\phi + D_y \chi. \tag{8.127}
\]

Finally, the remaining determining equations (8.105a) then give

\[
Q^{\text{div}} = -D_t \chi. \tag{8.128}
\]

A useful observation from these relations (8.125), (8.127), (8.128) is that \( Q^B \) represents the gauge-invariant part of an adjoint-symmetry arising from the potential system.

The Lie point symmetries of the wave equation are well known. In two space dimensions, they are generated by [232] a scaling, a time-translation, two space translations, a rotation, two Lorentz boosts, and a space-time dilation. Through the prolongation relation (8.124), these symmetries correspond to the Lie point symmetries (8.103) of Maxwell’s equations. The situation is more interesting for adjoint-symmetries. One can straightforwardly show that the time-translation, two space translations, rotation, and two Lorentz boosts respectively yield \( Q_2, Q_3, Q_4, Q_5, Q_6, Q_7 \). Both the scaling symmetry, \( P^\phi = \phi \), and the space-time dilation symmetry, \( P^\phi = t \phi_t + x \phi_x + y \phi_y \), correspond to nonlocal adjoint-symmetries. In particular, the linear combination

\[
P^\phi_{(\lambda)} = \frac{1}{c} (\lambda \phi + t \phi_t + x \phi_x + y \phi_y), \tag{8.129}
\]

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with \( \lambda \) being an arbitrary constant, yields a one-parameter family of nonlocal adjoint-symmetries:

\[
Q^{E_1} = ctE^1 + yB + (\lambda - 1)\Delta^{-1}B_y, \quad Q^{E_2} = ctE^2 - xB - (\lambda - 1)\Delta^{-1}B_x,
\]
\[
Q^B = ctB + yE^1 - xE^2 + \lambda \phi, \quad Q^{\text{div}} = xE^1 + yE^2.
\]

The symmetry action \( Q^{(\lambda)} \) on this adjoint-symmetry family \( Q \) is found to be given by

\[
S_{\phi}(Q^{(\lambda)}) = (2\lambda - 1)Q_{\phi}
\]

modulo gauge, with

\[
Q_{\phi} = (D_y\Delta^{-1}D_tP^\phi, -D_x\Delta^{-1}D_tP^\phi, cP^\phi, 0)
\]

which is the adjoint-symmetry \( Q \) determined by any symmetry of Maxwell’s equations under projection to a symmetry \( \frac{\partial}{\partial \phi} \) of the wave equation \( (8.123) \) through inversion of the prolongation relations \( (8.124) \). In explicit form, the inversion is given by

\[
P^\phi = \Delta^{-1}(D_yP^{E_1} - D_xP^{E_2}) = \frac{1}{c}\Delta^{-1}D_tP^B,
\]

whereby

\[
Q_{\phi} = (Q^{E_1}, Q^{E_2}, Q^B, Q^{\text{div}}) = (cD_y\Delta^{-1}P^B, -cD_x\Delta^{-1}P^B, D_t\Delta^{-1}P^B, 0)
\]

\[
= (D_t\Delta^{-1}P^E, D_t\Delta^{-1}P^E, c\Delta^{-1}(D_yP^{E_1} - D_xP^{E_2}), 0)
\]

with the use of the relation \( (D_t\Delta^{-1})^2P^\phi = c^2\Delta^{-1}P^\phi \) from the wave equation \( (8.123) \). The derivation of the symmetry action \( (8.131) \) is straightforward. Consider the adjoint-symmetry \( \partial_\lambda Q^{(\lambda)} = (\Delta^{-1}B_y, -\Delta^{-1}B_x, \Delta^{-1}(E^1 - E^2), 0) \). One has

\[
\text{pr}X_P(\partial_\lambda Q^{(\lambda)}) = c(\Delta^{-1}D_yP^B, -\Delta^{-1}D_xP^B, \Delta^{-1}(D_yP^{E_1} - D_xP^{E_2}), 0)
\]

and

\[
R_{\partial_\lambda Q^{(\lambda)}}(G) = c(-\Delta^{-1}D_yG^B, \Delta^{-1}D_xG^B, \Delta^{-1}(-D_yG^{E_1} + D_xG^{E_2}))
\]

after use of relation \( (8.133) \). The adjoint of the linear operator \( R_{\partial_\lambda Q^{(\lambda)}} \) applied to \( P \) yields

\[
R_{\partial_\lambda Q^{(\lambda)}}(P) = c(D_y\Delta^{-1}P^B, -D_x\Delta^{-1}P^B, \Delta^{-1}(D_yP^{E_1} - D_xP^{E_2}), 0).
\]

Hence, one obtains \( \text{pr}X_P(\partial_\lambda Q^{(\lambda)}) + R_{\partial_\lambda Q^{(\lambda)}}(P) = 2Q_{\phi} \), in accordance with the \( \lambda \) term in expression \( (8.131) \). The term independent of \( \lambda \) in expression \( (8.131) \) can be derived in a similar way.

Now, the dual of the linear operator given by the symmetry action \( (8.131) \) produces an adjoint-symmetry commutator bracket \( (5.1) \) on the linear space span\((Q_2, Q_3, Q_4, Q_5, Q_6, Q_7)\). This bracket will be isomorphic to the Lie bracket of the subalgebra of Lie point symmetries generated by the time-translation, two space translations, rotation, and two Lorentz boosts: span\((P_2, P_3, P_4, P_5, P_6, P_7)\). Through the relation \( (8.132) \), one can extend these brackets to the linear spaces span\((P_2, P_3, P_4, P_5, P_6, P_7, P^{(\lambda)}\)) and span\((Q_2, Q_3, Q_4, Q_5, Q_6, Q_7, Q^{(\lambda)}\)). The resulting adjoint-symmetry commutator bracket is shown in Table 21.
Table 21. Maxwell’s equations: adjoint-symmetry commutator bracket from symmetry action (8.131) with $\lambda = 1$.

| $Q_2$ | $Q_3$ | $Q_4$ | $Q_5$ | $Q_6$ | $Q_7$ | $Q_{(\lambda)}$ |
|-------|-------|-------|-------|-------|-------|---------------|
| 0     | 0     | 0     | 0     | $-c^2 Q_3$ | $-c^2 Q_4$ | $-Q_2$        |
| 0     | 0     | $Q_4$ | $-Q_2$ | 0     | $-Q_3$ |               |
| 0     | $Q_3$ | 0     | $Q_2$ | $Q_7$ | 0     |               |
| 0     | $-Q_7$| $Q_6$ | 0     |       |       |               |
| 0     | 0     | $c^2 Q_5$ | 0    |       |       |               |
| $Q_{(\lambda)}$ | 0 | 0 | 0 | 0 | 0 |

8.6.3. Symplectic 2-form and Hamiltonian operator. From Theorem 4.3, the symmetry action (8.131) and (8.134) encodes a Noether operator (4.13) in total derivatives for Maxwell’s equations (8.101) as follows. For simplicity, put $\lambda = 1$, and consider $J = \frac{1}{c^2} S Q_{(1)}$.

The explicit form of this operator is given by

$$J = \begin{pmatrix}
0 & 0 & \Delta^{-1} D_y \\
\Delta^{-1} D_y & 0 & -\Delta^{-1} D_x \\
0 & -\Delta^{-1} D_x & 0
\end{pmatrix}$$

(8.138)

where a spatial representation for the components has been chosen via the two expressions (8.134).

An associated symplectic 2-form can be obtained if one considers the integral pairing (4.8) given in Proposition 4.2, with the domain taken to be $\mathbb{R}^2$. This pairing is given by

$$\langle P, Q \rangle = \int_{\mathbb{R}^2} (P E^1 Q^E_1 + P E^2 Q^E_2 + P B Q^B) \, dx \, dy.$$  

(8.139)

It is non-degenerate because if $Q$ is a gauge adjoint-symmetry (8.112) then

$$\langle P, Q \rangle = \int_{\mathbb{R}^2} (P E^1 D_x \chi + P E^2 D_y \chi) \, dx \, dy = -\int_{\mathbb{R}^2} (D_x P E^1 + D_y P E^2) \, dx \, dy = 0$$  

(8.140)

vanishes modulo boundary terms, after use of the symmetry determining equation (8.102c). Hence,

$$\omega_{\psi}(\tilde{P}, P) = \int_{\mathbb{R}^2} (\tilde{P} E^1 D_y \Delta^{-1} P B - \tilde{P} E^2 \Delta^{-1} D_x \Delta^{-1} P B + \tilde{P} B \Delta^{-1}(D_y P E^1 - D_x P E^2)) \, dx \, dy$$  

(8.141)

is the 2-form associated to the operator (8.138), where $\tilde{P} E^1 \partial E^1 + \tilde{P} E^2 \partial E^2 + \tilde{P} B \partial B$ and $P E^1 \partial E^1 + P E^2 \partial E^2 + \partial B$ are any symmetries of Maxwell’s equations. Modulo boundary terms, this 2-form (8.141) is readily seen to be skew and closed (in the sense of condition (6.18)). Hence, it defines a symplectic 2-form.

As a consequence, the operator (8.138) turns out to be a symplectic operator for Maxwell’s equations. In addition, the formal inverse of this operator thereby defines a Hamiltonian
operator

\[ H = J^{-1} = \begin{pmatrix} 0 & 0 & D_y & 0 \\ 0 & 0 & -D_x & 0 \\ D_y & -D_x & 0 & 0 \end{pmatrix} \]  

(8.142)

whose kernel contains all gauge adjoint-symmetries (8.112). Note that the range of this operator includes all solutions of the divergence equation (8.102c).

A Hamiltonian formulation of Maxwell’s equations is then given by

\[ E^1_t = cH(\delta E / \delta E^1), \quad E^2_t = cH(\delta E / \delta E^2), \quad B_t = cH(\delta E / \delta B), \]  

(8.143)

in terms of the conserved energy (8.108), with \((E^1, E^2)\) taken to be divergence free. This spatial constraint is compatible with the structure of the Hamiltonian operator, since variational derivatives will satisfy the relation \(D_y \delta / \delta E^1 = D_x \delta / \delta E^2\). Specifically, if \((\delta E^1, \delta E^2) = (D_y \chi, -D_x \chi)\) which preserves \(E^1_x + E^2_y = 0\), then for any functional \(F\),

\[ 0 = \delta F = (\delta F / \delta E^1)D_y \chi - (\delta F / \delta E^2)D_x \chi = (-D_y \delta F / \delta E^1 + D_x \delta F / \delta E^2)\chi \]  

(8.144)

is required to hold modulo total derivatives.

9. Concluding remarks

The work in sections 2 to 6 has initiated a mathematical study of the algebraic structure of adjoint-symmetries for general PDE systems, \(G^A(x, u^{(N)}) = 0\). Several main results have been obtained.

Three linear actions of symmetries on adjoint-symmetries have been derived. The first action \(S_1 P : \text{AdjSymm}_G^+ P \rightarrow \text{AdjSymm}_G^+\) comes from applying a symmetry to the determining equation for adjoint-symmetries. It yields a generalization of a better known action of symmetries on conservation law multipliers, \(\text{Multr}_G^+ \rightarrow \text{Multr}_G^+\). The second action arises from a well-known formula that yields a conservation law multiplier, \(\Lambda_A \in \text{Multr}_G^+\), from a pair consisting of a symmetry, \(P^\alpha \in \text{Symm}_G\), and an adjoint-symmetry, \(Q_A \in \text{AdjSymm}_G^+\). Since multipliers are adjoint-symmetries that satisfy certain extra (Helmholtz-type) conditions, the formula gives an action \(S_3 P : \text{AdjSymm}_G^+ \rightarrow \text{Multr}_G^+ \subseteq \text{AdjSymm}_G^+\). A third action \(S_3 P := S_1 P - S_2 P\) has the feature that it is non-trivial only on adjoint-symmetries that are not multipliers.

For each of these linear actions, two different bilinear brackets on adjoint-symmetries have been constructed by use of the dual linear action \(S_Q(P) := S_P(Q)\) for a fixed adjoint-symmetry. The first bracket is a pull-back of the symmetry commutator bracket and has the properties of a Lie bracket, whereas the second bracket does not involve the commutator structure of symmetries and is non-symmetric. Under certain algebraic conditions on \(S_Q\), the brackets are well-defined on the entire space of adjoint-symmetries, \(\text{AdjSymm}_G^+\).

In the case of Euler-Lagrange PDEs, the three symmetry actions themselves define bilinear brackets on symmetries, due to the identification \(\text{AdjSymm}_G = \text{Symm}_G\). On variational symmetries, two of these brackets reduce to the commutator bracket, while the third bracket vanishes. Otherwise, the three brackets are different than the commutator bracket. Thus, symmetries can possess multiple (distinct) bracket structures.

The third symmetry action is able to produce a Noether (pre-symplectic) operator whenever a PDE system possesses an adjoint-symmetry that is not a multiplier. In additional, for evolution PDEs and Euler-Lagrange PDEs, this Noether operator gives rise to an associated
symplectic 2-form which defines a Poisson bracket structure. For Hamiltonian systems, the
Poisson bracket yields an explicit Hamiltonian operator.

In general, the adjoint-symmetry brackets give a correspondence between symmetries and
adjoint-symmetries, which can exist in the absence of any local variational structure (Hamil-
tonian or Lagrangian) for a PDE system. For the adjoint-symmetry commutator bracket,
the correspondence constitutes a homomorphism of a Lie (sub) algebra of symmetries into
a Lie algebra of adjoint-symmetries.

As shown by several examples of physically interesting PDE systems, all of these struc-
tures are non-trivial, which indicates a very rich interplay among conservation laws, adjoint-
symmetries and symmetries, going beyond the connection provided by Noether’s theorem
and its modern generalization. Exploring this interplay more deeply will be an interesting
problem for future work.

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fruitful work that we started together on multipliers and adjoint-symmetries more than two
decades ago [8].

APPENDIX A. CALCULUS IN JET SPACE

General references are provided by Ref. [2, 7].

The following notation is used:

\( x^i, i = 1, \ldots, n \) are independent variables;
\( u^\alpha, \alpha = 1, \ldots, m \) are dependent variables;
\( u^\alpha_i = \frac{\partial u^\alpha}{\partial x^i} \) are partial derivatives;
\( \partial^k u \) is the set of all partial derivatives of \( u \) of order \( k \geq 0 \);
\( u^{(k)} \) is set of all partial derivatives of \( u \) with all orders up to \( k \geq 0 \);

**Multi-indices**

\[
I = \emptyset, \quad u^\alpha_I = u^\alpha, \quad |I| = 0
\]

\[
I = \{i_1, \ldots, i_N\}, \quad u^\alpha_I = u^\alpha_{i_1 \ldots i_N}, \quad |I| = N \geq 1
\]

**Summation convention:** sum over any repeated (multi-) index in an expression.

Jet space is the coordinate space \( J = (x^i, u^\alpha, u^\alpha_i, \ldots) \), and \( J^{(k)} = (x, u^{(k)}) \) is the finite
subspace of order \( k \geq 0 \).

Total derivatives in jet space are defined by

\[
D_i = \partial_{x^i} + u^\alpha_i \partial_{u^\alpha} + \cdots, \quad i = 1, \ldots, n
\]  
\[ \text{(A.1)} \]

The Frechet derivative of a function \( f \) on jet space is defined by

\[
(f')^\alpha_\alpha = f_{u^\alpha_i} D^I
\]  
\[ \text{(A.2)} \]

which acts on functions \( F^\alpha \). The Frechet second-derivative is given by the expression

\[
f''(F_1, F_2) = f_{u^\alpha_i u^\beta_j} (D^I F^\alpha_1)(D^J F^\beta_2)
\]  
\[ \text{(A.3)} \]
which is symmetric in the pair of functions \((F_1^a, F_2^a)\). The adjoint of the Frechet derivative of \(f\) is defined by

\[
(f^{**})_\alpha = D^*_i f_{u_i}^\gamma = (-1)^{|I|} D_I f_{u_i}^\gamma
\]  

(A.4)

which acts on functions \(F\), where the righthand side is a composition of operators.

The Euler operator (variational derivative) is defined by

\[
E_{u^\alpha} = (-1)^{|I|} D_I \partial_{u_i}^\gamma
\]  

(A.5)

It has the property that \(E_{u^\alpha}(f) = 0\) holds identically iff \(f = D_i F^i\) for some vector function \(F^i(x, u^{(k)})\). The product rule for the Euler operator is given by

\[
E_{u^\alpha} (f_1 f_2) = f_1^{**} (f_2)_\alpha + f_2^{**} (f_1)_\alpha
\]  

(A.6)

The higher Euler operators are defined similarly

\[
E^I_{u^\alpha} = (i) (-1)^{|I|/|J|} D_{I/J} \partial_{u_i}^\gamma
\]  

(A.7)

See Ref. [2, 7] for their properties.

Some useful relations:

\[
f' (F) = F^\alpha E_{u^\alpha} (f) + D_i \Gamma^i (F; f), \quad \Gamma^i (F; f) = (D_I F^\alpha) E_{u_i}^\gamma (f);
\]  

(A.8)

\[
H f' (F) = F f'^* (H) = D_i \Psi^i (H, F), \quad \Psi^i (H, F) = (D_I H) (D_J F^\alpha) (-1)^{|I|} E^I_{u_i} (f);
\]  

(A.9)

\[
f' (F) = \text{pr} X_F f, \quad X_F = F^\alpha \partial_{u^\alpha}, \quad \text{pr} X_F = (D_I F^\alpha) \partial_{u_i}^\gamma;
\]  

(A.10)

\[
[F_1, F_2] = \text{pr} X_{F_1} F_2 - \text{pr} X_{F_2} F_1 = F_2' (F_1) - F_1' (F_2);
\]  

(A.11)

\[
[\text{pr} X_{F_1}, \text{pr} X_{F_2}] = \text{pr} X_{[F_1, F_2]};
\]  

(A.12)

and

\[
(\text{pr} X_F f') = (\text{pr} X_F f)' - f' F';
\]  

(A.13)

\[
(\text{pr} X_F f^{**}) = (\text{pr} X_F f)^{**} - F^{**} f^{**}.
\]  

(A.14)

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