The algebra of Feistel-Toffoli schemes

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Abstract

The process of replacing an arbitrary Boolean function by a bijective one, a fundamental tool in reversible computing and in cryptography, is interpreted algebraically as a particular instance of a certain group homomorphism from the $X$-fold cartesian power of a group $G$ into the automorphism group of the free $G$-set over the set $X$. It is shown that this construction not only can be generalized from groups to monoids but, more generally, to internal categories in arbitrary finitely complete categories where it becomes a cartesian isomorphism between certain discrete fibrations.

MSC 2020: 18D40, 18D30, 68Q09

Keywords: Internal category, discrete fibration, G-set, convolution monoid, Feistel scheme.

1 Introduction

The basic idea behind the so-called Toffoli gate, a fundamental tool in reversible computing and, hence, one of the ingredients to prove that every classical program may be executed by a quantum computer, and called the Fundamental Theorem in [19] is the following fact: every function $f: 2^m \rightarrow 2^n$ can be turned into some bijective function $E_m(f): 2^m \times 2^n \rightarrow 2^{m+n} = 2^m \times 2^n$ in such a way that $f$ may be retrieved from it. Thinking of $2 = \{0, 1\}$ as $\mathbb{Z}_2$ and using the notations $\bar{x} := (x_1, \ldots , x_m) \in \mathbb{Z}_2^m$ and $f(\bar{x}) =: (f_1(\bar{x}), \ldots , f_n(\bar{x}))$ the map $E_m(f): \mathbb{Z}_2^{m+n} \rightarrow \mathbb{Z}_2^{m+n}$ is given by

$$(\bar{x}, \bar{y}) \mapsto (\bar{x}, f_1(\bar{x}) + y_1, \ldots , f_n(\bar{x}) + y_n).$$

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About a decade older is the consideration of its special instance with \( n = m \); this is of interest in cryptography, since it provides a method for developing the general architecture of some symmetric encryption schemes commonly known as Feistel ciphers \cite{15}. Definition 7.81.

Hence, the natural question arises whether this map \( E_m \) has any conceptual algebraic meaning and, if so, whether this might make sense in a more general context. In order to find an answer to this question let us observe first that in the definition of \( E_m(f) \) neither the algebraic structure of the domain \( \mathbb{Z}_2^m \) of \( f \) is used nor the fact that the codomain of \( f \), that is, the additive monoid \( \mathbb{Z}_2^n \) is a group. This leads us to start simply with a map \( X \xrightarrow{f} M \) where \( X \) is a set and \( M = (\mathbb{M}, \cdot) \) is a monoid such that we obtain by the construction above a map \( X \times M \xrightarrow{E_X(f)} X \times M, (x, m) \mapsto (x, f(x) \cdot m) \). Noting that \( X \times M \) is the (underlying set of the) free object \( F_M X \) over \( X \) in the category \( \text{Set}_M \) of right \( M \)-acts one immediately sees that the map \( E_X(f) \) is nothing but the homomorphic extension \( \langle \text{id}_X, f \rangle^\sharp \) of the map \( X \xrightarrow{(\text{id}_X, f)} X \times M \) with \( x \mapsto (x, f(x)) \), that is, the unique map making the following diagram commute, where \( \eta_X(x) = (x, 1) \).

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X} & X \times M \\
\downarrow{(\text{id}_X, f)} & & \downarrow{E_X(f) = \langle \text{id}_X, f \rangle^\sharp} \\
X \times M & & \\
\end{array}
\]

This not only answers the question at the beginning of the previous paragraph but also describes the reconstruction of \( f \) from \( E_X(f) \) structurally:

\[
f = X \xrightarrow{\eta_X} X \times M \xrightarrow{E_X(f)} X \times M \xrightarrow{\pi_M} M =: L_X(E_X(f)). \tag{1}
\]

Considering the map \( f: X \to M \) as an element of the \( X \)-fold cartesian power of \( M \) the map \( E_X \) becomes a map \( M^X \to \text{Set}_M(F_M X, F_M X) \). Denoting by \( s \ast t \) the multiplication in \( M^X \) one gets by a straightforward calculation the equations \( \langle \text{id}_X, s \ast t \rangle^\sharp = \langle \text{id}_X, s \rangle^\sharp \circ \langle \text{id}_X, t \rangle^\sharp \) and \( \langle \text{id}_X, 1_M^X \rangle^\sharp = \text{id}_{F_M X} \), for any pair \( (s, t) \in M^X \). Thus, \( E_X \) even is a monoid homomorphism from the \( X \)-fold power \( M^X \) of \( M \) in the category \( \text{Mon} \) of (small) monoids into the endomorphism monoid \( \text{Set}_M(F_M X, F_M X) \). Since \( E_X \) is injective by Equation (\ref{1}) one concludes that \( M^X \) is isomorphic to a submonoid of this endomorphism monoid.

In view of the starting examples, where \( E_X(f) \) was bijective, we note that this homomorphism property implies that the monoid \( E_X(M^X) \) is a subgroup of the automorphism group of the free \( M \)-act over \( X \), provided that the monoid \( M \) and, hence, the monoid \( M^X \) even is a group.

This construction will be shown to be functorial in the following sense: the assignments \( X \mapsto M^X \) and \( X \mapsto E_X(M^X) \) define contravariant functors \( P^*_M \) and \( Q^*_M \) from \( \text{Set} \) to \( \text{Set} \) which are naturally isomorphic by means of the family \( (E_X)_X \). This is in other words: the natural transformation \( (E_X)_X \) determines a cartesian isomorphism between
the discrete fibrations $P_M$ and $Q_M$ over $Set$ whose base change functors are the functors $P_M^*$ and $Q_M^*$.

All of this generalizes quite obviously to any category $C$ with finite products instead of the category $Set$ of (small) sets as will be shown in Section 2. Having in mind that a monoid is a one-object internal category in $Set$ one may wonder whether even a generalization to arbitrary internal categories in arbitrary finitely complete categories $C$ is possible; and this the more so, since an internal category in $C$, which has $O$ as its object of objects, is the same as a monoid in a certain monoidal category $C_O$; since $C_O$ contains an isomorphic copy of $C/O$ as a full monoidal subcategory, one is lead to the following conjecture.

1. All of the above can be generalized to pairs $(X,M)$ where $M$ is an internal category in a finitely complete category $C$ and $X$ is an object of $C/O$. The proof of this conjecture then is the purpose of this note, where particular emphasis is given to the question in which sense the occurring constructions are functorial. The main part of this note, Section 3, contains the proof of this conjecture, where particular emphasis is given to the question in which sense the occurring constructions are functorial.

2 The case of internal monoids

Let $C$ be a cartesian category, that is, a category with finite products and $M = (M, M \times M \overset{m}{\to} M, 1 \overset{i}{\to} M)$ a monoid in $C$. Then the following hold for any $C$-object $C$:

1. $(C(C, M), *, e_C)$ is a monoid, for each $C$-object $C$ with $e_C := C \overset{i}{\to} 1 \overset{c}{\to} M$ where $1$ is the terminal object of $C$, and $\phi * \psi := C \overset{\phi \times \psi}{\to} C \times C \overset{\phi \times \psi}{\to} M \times M \overset{m}{\to} M$. This is a special instance of the so-called convolution monoid given by a comonoid $C$ and a monoid $M$ in a monoidal category (see also Section 3.2.1), when considering $(C, \times, 1)$ as a cartesian monoidal category, that is, a monoidal category whose monoidal structure is given by binary products and a terminal object. Note, that in case $C = Set$, this monoid is simply the $C$-fold power of the monoid $M$.

If $M = (M, M \times M \overset{m}{\to} M, 1 \overset{i}{\to} M, M \overset{1}{\to} M)$ even is a group in $C$ then $(C(C, M), e_C, *)$ is a group, since $\phi * (i \circ \phi) = e_C = (i \circ \phi) * \phi$, for each $\phi \in C(C, M)$.

For a different argument using the Yoneda functor see e.g. [14, III.6].

2. There is the category $C_M$ of $M$-acts $(C, C \times M \overset{c}{\to} C)$ in $C$ defined in the obvious way and an obvious forgetful functor $|-| : C_M \to C$. This again is a special instance of the category of right $M$-modules with respect to a monoid in a monoidal category.

3. The forgetful functor $|-|$ of the category of right $M$-modules in a monoidal category $C$ has a left adjoint $F_M$ (see e.g. [16]). This specializes to a category with finite products as follows.

$C_O$ is the monoidal category $C_O := \text{Span}(C)(O, O)$, where $\text{Span}(C)$ denotes the bicategory of spans in $C$ (see below). Note that the categories $\text{Span}(Set)(1, 1) = Set_1$, $Set/1$ and $Set$ are isomorphic.
(a) $\mid - \mid : \mathcal{C}_M \to \mathcal{C}$ has a left adjoint $F_M$ given by

$$F_M(C) = C \times M \times M \overset{id_C \times m}{\longrightarrow} C \times M$$

with units $C \overset{\eta_C}{\longrightarrow} C \times M = C \simeq C \times 1 \overset{id_C \times e}{\longrightarrow} \|(C \times M, \alpha)\|$.

(b) The underlying $\mathcal{C}$-morphism of the homomorphic extension $f^\sharp$ of a $\mathcal{C}$-morphism $C \overset{f}{\to} Y \times M$ is given by $C \times M \overset{f \times id_M}{\longrightarrow} Y \times M \times M \overset{id_Y \times m}{\longrightarrow} Y \times M$.

Note that Items (a) and (b), when specialized to $\text{Set}$, are precisely the descriptions of the units and homomorphic extensions in the introduction.

Consequently, the arguments used to prove the properties of the map $E_X$ in the introduction apply literally in the case where the category under consideration is not $\text{Set}$ but an arbitrary category with finite products. We so obtain the

**Proposition 1** Let $M = (M, M \times M \overset{m}{\longrightarrow} M, 1 \overset{e}{\longrightarrow} M)$ be a monoid in a category $\mathcal{C}$ with finite products. Then the assignment $(C \overset{f}{\to} M) \mapsto (id_C, f)^\sharp$ defines a map $E_C : \mathcal{C}(C, M) \to \mathcal{C}_M(F_M C, F_M C)$ such that the following hold.

1. $E_C$ is monoid morphism from the convolution monoid into the $\mathcal{C}_M$-endomorphism monoid.
2. $E_C$ is a section.
3. If $M$ even is a group in $\mathcal{C}$ then, for each $C \overset{f}{\to} M$, the endomorphism $E_C(f)$ is an automorphism.

**Remark 2** One certainly could have shown the above without explicitly referring to the theory of monoidal categories. We prefer our approach for the following reasons: (a) the monoidal methods will have to be used below in any case. (b) When interpreting varieties (in the sense of universal algebra) as those of monoids, groups and $M$-acts in a category with finite products one cannot prove the existence of free algebras in general. Hence, the existence of free $M$-acts used above is rather untypical in this context — it becomes natural, however, when considering a category with finite products as a cartesian monoidal category.

3 The case of internal categories

As from now the category $\mathcal{C}$ is supposed to be finitely complete.
3.1 Prerequisites

3.1.1 Some notation

Given morphisms \( M \xrightarrow{c} O \) and \( N \xrightarrow{d} O \) in \( C \), the pullback of \( c \) along \( d \) over \( O \) will be denoted by

\[
\begin{array}{ccc}
M_d \times_c N & \xrightarrow{\pi_c^M} & N \\
\downarrow{\pi_d^N} & & \downarrow{c} \\
M & \xrightarrow{d} & O
\end{array}
\]

If possible, we will write only \( \pi_c \) and \( \pi_d \) instead of \( \pi_c^M \) and \( \pi_d^N \), respectively; occasionally we may simply write \( p \) instead of \( \pi_c \) and \( q \) instead of \( \pi_d \).

Note that by the standing assumption on \( C \) the object \( M \times_c N \) is a (regular) subobject of \( M \times N \). Given a pair of morphisms \( M' \xrightarrow{f} M \) and \( N' \xrightarrow{g} N \) such that the morphism \( M' \times N' \xrightarrow{f \times g} M \times N \) factors over \( M_d \times_c N \) we will by slight abuse of notation its corestriction \( M' \xrightarrow{f} M \times N \) simply denote by \( f \times g \) as well; more generally, any restriction of this morphism to a subobject \( X \) of \( M' \times N' \) which factors over \( M_d \times_c N \) will be denoted simply by \( X \xrightarrow{f \times g} M_d \times_c N \) as well.

3.1.2 The monoidal category \( C_O \)

The categories described in this section, to which we referred already in the introduction, are of crucial importance when working with internal categories.

Let \( O \) be a fixed object in the finitely complete category \( C \). Then the category \( C_O \) has as its objects all pairs \( (x \xrightarrow{f} O, x' \xrightarrow{g} O) \) of morphisms in \( C \), called *spans* and usually denoted by \( O \leftarrow x \xrightarrow{f} O \), and as its morphisms, also referred to as 2-*cells* (see Remark 3 below), \( O \xleftarrow{f'} x' \xrightarrow{g'} O \Rightarrow O \xleftarrow{f} x' \xrightarrow{g} O \) all \( C \)-morphisms \( x \xrightarrow{f} x' \) making the following diagram commute.

\[
\begin{array}{ccc}
O & \xleftarrow{f} & x \\
\downarrow{t} & & \downarrow{g} \\
O & \xrightarrow{f'} & x'
\end{array}
\]

The composition in \( C_O \) is the composition in \( C \) and the identity on \( O \xleftarrow{f} x \xrightarrow{g} O \) is the identity \( id_x \) in \( C \). The functor \( C_O \xrightarrow{\Rightarrow} C \) denotes the forgetful functor given by the assignment \((O \xleftarrow{f} x \xrightarrow{g} O) \mapsto (O \xleftarrow{f'} x' \xrightarrow{g'} O) \mapsto (x \xrightarrow{f} x') \). Note that when \( C \) is locally small then \( C_O \) is locally small as well. The category \( C_O \) is equipped with a monoidal structure defined as follows.
1. \((O \xleftarrow{h} y \rightarrow O) \otimes (O \xleftarrow{f} x \rightarrow O) := O \xrightarrow{f \circ \pi_g} x_g \times_h y \xrightarrow{k \circ \pi_h} O\) is given by pullbacks as visualized by the following diagram, where we occasionally denote the \(\mathcal{C}\)-object \(x_g \times_h y\) by \(x \times_O y\).

\[
\begin{array}{ccc}
O & \xleftarrow{\pi_g} & x \times_O y \\
& \searrow f & \downarrow g \\
& & O \\
\end{array}
\begin{array}{ccc}
& & \pi_h \\
& \nearrow h & \uparrow k \\
& & O
\end{array}
\]

2. Given \(\mathcal{C}_O\)-morphisms \(s: O \xleftarrow{k} y \rightarrow O \Rightarrow O \xleftarrow{k'} y' \rightarrow O\) and \(t: O \xleftarrow{f} x \rightarrow O \Rightarrow O \xleftarrow{f'} x' \rightarrow O\) then \(t \otimes s\) is the \(\mathcal{C}\)-morphism from \(x \times_O y\) to \(x' \times_O y'\) induced by the pullback property of \(x' \times_O y'\) from the \(\mathcal{C}\)-morphisms \(x \times_O y \xrightarrow{p} x \xrightarrow{f} x'\) and \(x \times_O y \xrightarrow{q} y \xrightarrow{k} y'\) as visualized by the commutative diagram below. In other words, \(t \otimes s\) is the unique morphism with \(p' \circ (t \otimes s) = t \circ p\) and \(q' \circ (t \otimes s) = s \circ q\).

\[
\begin{array}{ccc}
O & \xleftarrow{t \otimes s} & x \times_O y \\
& \searrow t & \downarrow s \\
& & O \\
\end{array}
\begin{array}{ccc}
& & \pi_g \\
& \nearrow f & \uparrow p \\
& & O \\
\end{array}
\begin{array}{ccc}
& & \pi_h \\
& \nearrow g & \uparrow k \\
& & O \\
\end{array}
\begin{array}{ccc}
& & \pi_h \\
& \nearrow f' & \uparrow p' \\
& & O \\
\end{array}
\begin{array}{ccc}
& & \pi_h \\
& \nearrow f' & \uparrow p' \\
& & O
\end{array}
\]

3. The monoidal unit is the span \(\bar{O}: = O \xleftarrow{id} O \xrightarrow{id} O\).

The associativity and units constraints are easily obtained from the pullback property.

**Remark 3** The fact above is a consequence of the well known results due to Bénabou [4], that the spans of a finitely complete category \(\mathcal{C}\) form a bicategory, and that the categories of 1-cells of a bicategory are monoidal (see also [9]).

**The cartesian subcategory \(\widehat{\mathcal{C}}_O\)**

In the sequel we denote \(\mathcal{C}_O\)-objects of the form \(O \xleftarrow{f} A \xrightarrow{g} O\) simply by \(f_A\) and by \(\widehat{\mathcal{C}}_O\) the full subcategory of \(\mathcal{C}_O\) spanned by all the spans \(f_A\). This category is obviously isomorphic to the slice category \(\mathcal{C}/O\) by means of the functor \(D: \mathcal{C}/O \rightarrow \mathcal{C}_O\) with \(((X \xrightarrow{f} O) \xrightarrow{\phi} (Y \xrightarrow{g} O)) \mapsto (f_X \xrightarrow{\phi} g_Y)\). We, hence, may not distinguish notationally between (objects of) \(\mathcal{C}/O\) and \(\widehat{\mathcal{C}}_O\).
For $\widehat{C}_O$-objects $f_A$ and $g_B$ one has $f_A \otimes g_B = O \xleftarrow{\dagger} A \xleftarrow{\pi_f} A_f \times g_B \xrightarrow{\pi_g} B \xrightarrow{\dagger} O$, where $f \circ \pi_f = g \circ \pi_g$. Hence, the $\widehat{C}_O$-object $f_A \otimes g_B = (f \circ \pi_f)_{A_f \times g_B}$ belongs to $\widehat{C}_O$. Since the $\mathcal{C}$-morphisms $(A_f \times g_B \xrightarrow{\pi_f} A)$ and $(A_f \times g_B \xrightarrow{\pi_g} B)$ are the product projections of $(A \xleftarrow{\dagger} O) \times (B \xrightarrow{\dagger} O)$ in $\mathcal{C}/O$ one obtains

**Lemma 4**: $\widehat{C}_O$ is a full monoidal subcategory of $C_O$ and $D: \mathcal{C}/O \rightarrow C_O$ is a monoidal equivalence between the cartesian category $\mathcal{C}/O$ and the full monoidal subcategory $\widehat{C}_O$ of $C_O$. In more detail: given $\widehat{C}_O$-objects $f_A$ and $g_B$ the following hold.

1. $(f_A \otimes g_B, \pi_f, \pi_g)$ is a product of $f_A$ and $g_B$ in $\widehat{C}_O$.

2. Since for each $f_A = O \xleftarrow{\dagger} A \xrightarrow{\dagger} O$ in $\widehat{C}_O$ the $\mathcal{C}$-morphism $f$ is the unique $\widehat{C}_O$-morphism $f_A \xrightarrow{\dagger} (O \xleftarrow{\dagger} A \xrightarrow{id} O)$, the $\widehat{C}_O$-object $O = id_O$ is terminal in $\widehat{C}_O$.

**Some calculation rules**

Given spans $\tilde{X} = (O \xleftarrow{\pi} X \xrightarrow{\eta} O)$ and $\tilde{M} = (O \xleftarrow{\pi} M \xrightarrow{\eta} O)$ there is, for any pair of $\mathcal{C}$-morphisms $A \xrightarrow{\alpha} M$ and $A \xrightarrow{\xi} X$ satisfying the condition $d \circ \alpha = y \circ \xi$, the unique $\mathcal{C}$-morphism $A \xrightarrow{\langle\xi, \alpha\rangle \xi}$ with $\pi_d \circ \langle\xi, \alpha\rangle = \alpha$ and $\pi_y \circ \langle\xi, \alpha\rangle = \xi$ (see Section 3.1.1). The following facts concerning the interplay of the monoidal structure of $C_O$ and the cartesian structure of its subcategory $\widehat{C}_O$ are easy to prove and will be of use later.

1. If $\alpha$ and $\xi$ are 2-cells $f_A \Rightarrow \tilde{M}$ and $f_A \Rightarrow \tilde{X}$, respectively, then $\langle\xi, \alpha\rangle$ exists and is a 2-cell $f_A \Rightarrow \tilde{X} \otimes \tilde{M}$.

2. Denoting for $f_A$ in $\widehat{C}_O$ by $f_A \xrightarrow{\Delta} f_A \otimes f_A$ or $\Delta_A$, if necessary, its diagonal (w.r.t. the cartesian structure of $\widehat{C}_O$), for any pair of $\mathcal{C}_O$-morphisms $f_A \xrightarrow{\xi} \tilde{X}$, $f_A \xrightarrow{\alpha} \tilde{M}$ one has $\langle\xi, \alpha\rangle = (\xi \otimes \alpha) \circ \Delta$.

3. For any pair of $\mathcal{C}_O$-morphisms $\alpha, \beta: f_A \rightarrow \tilde{M}$ the following diagram commutes.

4. Concerning the construction $f_A \otimes \tilde{M}$ we note moreover:

   (a) the pullback projection $f_A \otimes \tilde{M} \xrightarrow{\pi_f} \tilde{M}$ is a 2-cell;
   
   (b) for each 2-cell $\alpha: f_A \Rightarrow f_A \otimes \tilde{M}$ the composite $\pi_f \circ \alpha$ is a 2-cell $f_A \Rightarrow f_A$ (though the pullback projection $\pi_f$ will fail to be a 2-cell in general).
(c) for any 2-cell $\phi: f_A \Rightarrow g_B$ the following diagrams commute

\[
\begin{array}{ccc}
  f_A \otimes \bar{M} & \xrightarrow{\pi_d} & \bar{M} \\
  \phi \otimes \text{id} \downarrow & & \downarrow \text{id} \\
  g_B \otimes \bar{M} & \xrightarrow{\pi_d} & \bar{M}
\end{array}
\quad
\begin{array}{ccc}
  |f_A| \otimes \bar{M} & \xrightarrow{|\pi_f|} & |f_A| \\
  |\phi \otimes \text{id}| \downarrow & & \downarrow |\phi| \\
  |g_B| \otimes \bar{M} & \xrightarrow{|\pi_g|} & |g_B|
\end{array}
\]

### 3.1.3 Internal categories and groupoids

**Internal categories**

Recall (see e.g. [3, Ex. 3C]) that a small category can be seen as a sixtuple

\[ M = (O, M, d, c: M \to O, \eta: M \to M \times O M \to M) \]

of sets and maps, where $O$ and $M$ are the sets of objects and morphisms, respectively, $c$ and $d$ are the (co)domain maps, $\eta$ is thought of as the family of identities, and $\mu$ is the composition map. These data must satisfy the category axioms, that is, they make the obvious diagrams commute. This all makes sense if the terms *sets* and *maps* are replaced by *objects* and *morphisms* of a finitely complete category $\mathcal{C}$. In this case then $M$ is called an *internal category* in $\mathcal{C}$. Obviously an internal category with $O = 1$, the terminal object of $\mathcal{C}$, is essentially the same thing as an internal monoid $(M, \mu, \eta)$ in $\mathcal{C}$.

The following generalizes a familiar fact about internal monoids (see e.g. [14, III.6]) and is easy to see. The Yoneda functor $Y: \mathcal{C} \to \text{Set}^{\mathcal{C}^{\text{op}}}$ maps an internal category $M = (O, M, d, c, \mu, \eta)$ in $\mathcal{C}$ (here assumed locally small) to an internal category in $\text{Set}^{\mathcal{C}^{\text{op}}}$ and each evaluation functor $\text{Set}^{\mathcal{C}^{\text{op}}} \xrightarrow{\text{ev}_C} \text{Set}$ maps this to the small category $M_C$ which has

- $\mathcal{C}(O, O)$ as its set of objects and $\mathcal{C}(C, M)$ as its set of morphisms,
- the maps $d_C := \mathcal{C}(C, M) \xrightarrow{\mathcal{C}(C,d)} \mathcal{C}(C, O)$ and $c_C := \mathcal{C}(C, M) \xrightarrow{\mathcal{C}(C,c)} \mathcal{C}(C, O)$ as domain and codomain assignments and the map $\mathcal{C}(C, O) \xrightarrow{\mathcal{C}(C,\eta)} \mathcal{C}(C, M)$ as its map assigning units.
- the map $\mathcal{C}(C, M) \times_{\mathcal{C}(C, O)} \mathcal{C}(C, M) = \mathcal{C}(C, M \times_O M) \xrightarrow{\mathcal{C}(C,\mu)} \mathcal{C}(C, M)$ as its composition, where $\mathcal{C}(C, M) \times_{\mathcal{C}(C, O)} \mathcal{C}(C, M)$ is the pullback of $c_C$ along $d_C$ in $\text{Set}$.

In other words, the category $M_C$ has

- the $\mathcal{C}$-morphisms $C \xrightarrow{f} O$ as its objects,
- the sets $\{ C \xrightarrow{\alpha} M \mid d_C(\alpha) = d \circ \alpha = f \text{ and } c_C(\alpha) = c \circ \alpha = g \}$ as its hom-sets $\text{hom}((C, f), (C, g))$, 

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the $\mathcal{C}$-morphisms $\text{id}_{(C,f)} := C \xrightarrow{f} O \xrightarrow{\eta} M$ as its units,

- the composition $((C, g) \xrightarrow{\beta} (C, h)) \circ ((C, f) \xrightarrow{\alpha} (C, g)) = C \xrightarrow{(\beta, \alpha)} M \times_O M \xrightarrow{\mu} M$.

**Remark 5** Given an internal category $\mathcal{M} = (O, M, d, c, \mu, \eta)$ in $\mathcal{C}$, the assignment $C \mapsto \mathcal{M}_C$ can obviously be extended to a functor $\mathcal{C}^{\text{op}} \to \text{Cat}$ (where $\text{Cat}$ denotes the category of (small) categories); consequently, there exists a split fibration $U : \mathcal{C}(O, M) \to \mathcal{C}$ called the **externalization of** $\mathcal{M}$ (see e.g. [12]), whose fibres then are the categories $\mathcal{M}_C$ just defined.

Recall that an **internal functor** $(O, M, c, d, \eta, \mu) \to (O', M', c', d', \eta', \mu')$ between internal categories in $\mathcal{C}$ is a pair of $\mathcal{C}$-morphisms $F = (O \xrightarrow{F^d} O', M \xrightarrow{F^m} M')$ such that the following diagrams commute.

![Diagram](image.png)

One so obtains $\text{Cat}_{\text{int}}(\mathcal{C})$, the category of internal categories and internal functors in $\mathcal{C}$.

**Remark 6** Obviously, every finite limit preserving functor $\mathcal{C} \xrightarrow{K} \mathcal{C}'$ between finitely complete categories induces a functor $\text{Cat}_{\text{int}}(K) : \text{Cat}_{\text{int}}(\mathcal{C}) \to \text{Cat}_{\text{int}}(\mathcal{C}')$.

**Internal categories in the language of spans**

In the language of spans the description of an internal category in a finitely complete category $\mathcal{C}$ reads as follows: an internal category $\mathcal{M} = (O, M, d, c, \eta, \mu)$ is a triple $(\bar{\mathcal{M}}, \eta, \mu)$ where

- $\bar{\mathcal{M}} = O \xleftarrow{d} M \xrightarrow{c} O$ is an object of $\mathcal{C}_O$,
- $\eta : \bar{\mathcal{O}} = O \xleftarrow{id} O \xrightarrow{id} O \Rightarrow \bar{\mathcal{M}}$ and $\mu : \bar{\mathcal{M}} \otimes \bar{\mathcal{M}} \Rightarrow \bar{\mathcal{M}}$ are morphisms in $\mathcal{C}_O$,
- the following diagrams commute.

![Diagram](image.png)

Since these equations are nothing but the axioms for the triple $(\bar{\mathcal{M}}, \mu, \eta)$ with $\bar{\mathcal{M}} = (O \xleftarrow{d} M \xrightarrow{c} O)$ to be a monoid in the monoidal category $\mathcal{C}_O$, we have got the following fact.
**Fact 7** If $\mathcal{C}$ is a finitely complete category then internal categories in $\mathcal{C}$ with $O$ its object of objects are precisely the monoids in the monoidal category $\mathcal{C}_O$.²

**Internal groupoids**

As groups and groupoids can equivalently be considered as monoids and categories, respectively, with an additional property (every element has an inverse) or an additional structure (the map mapping an element to its inverse), the same holds for internal groupoids. Here, however, neither the additional property nor the respective equivalence is as obvious as in the cases just mentioned (see e.g. [11]). We, hence, refrain from describing internal groupoids by a particular property of an internal category and only use the following classical definition (see [6]) which suits our needs best.

**Definition 8** Let $\mathcal{C}$ be a finitely complete category. An internal groupoid in $\mathcal{C}$ is a pair $(M, \iota)$ where $M = (O, M, d, c, \eta, \mu)$ is an internal category in $\mathcal{C}$ and $\iota: M \to M$ is a $\mathcal{C}$-morphism such that

1. $c \circ \iota = d$ and $d \circ \iota = c$ and
2. $\mu \circ (\iota, id) = \eta \circ c$ and $\mu \circ (id, \iota) = \eta \circ d$.

If now $(M, \iota) = (O, M, d, c, \eta, \mu)$ is an internal groupoid then, for each $\mathcal{C}$-object $C$, the category $M_C$ is equipped with a map $\iota_C = \mathcal{C}(C, \iota): \mathcal{C}(C, M) \to \mathcal{C}(C, M)$ satisfying the conditions

1. $c_C \circ \iota_C = d_C$ and $d_C \circ \iota_C = c_C$ and
2. the following diagram commutes

$$
\begin{array}{ccc}
\mathcal{C}(C, M) & \xrightarrow{\mathcal{C}(C, \eta_C)} & \mathcal{C}(C, M) \\
\mathcal{C}(C, (id, \iota)) & & \mathcal{C}(C, \mu) \\
\mathcal{C}(C, M \times_O M) & \xleftarrow{\mathcal{C}(\iota, id)} & \end{array}
$$

We, hence, can conclude that every morphism $\alpha$ in $M_C$ is an isomorphism with inverse $\alpha^{-1} = \iota_C(\alpha)$, such that $M_C$ in fact is a groupoid and that $\iota_C \circ \iota_C = id_{\mathcal{C}(C, M)}$, for each $\mathcal{C}$-object $C$. Since the family $(\iota_C)_C$ is the image of the full and faithful Yoneda embedding $Y$ the $\mathcal{C}$-morphism $\iota$ is an idempotent isomorphism as well. We so have got the following results.

**Facts 9** If $(M, \iota)$ is an internal groupoid in $\mathcal{C}$ then all categories $M_C$ are groupoids and $\iota \circ \iota = id_M$.²

²Alternatively one may say that these internal categories are monads in the bicategory $\text{Span}(\mathcal{C})$ on the object $O$ in the sense of [13]; this view however is not the appropriate one for our purpose.
3.1.4 Discrete fibrations

We recall the following well known concepts and facts (see e.g. [13]).

1. A functor $P: \mathcal{C} \to \mathcal{A}$ with small fibres, that is, such that for each $\mathcal{A}$-object $A$ the fibre $P^*(A)$ over $A$ is a small set, is a discrete fibration over $\mathcal{A}$ if for each $\mathcal{A}$-morphism $f: A' \to PC$ there exists a unique $\mathcal{C}$-morphism $g: C' \to C$ such that $Pg = f$.

The full subcategory of the slice category $\text{CAT/}\mathcal{A}$ spanned by all discrete fibrations over $\mathcal{A}$, where $\text{CAT}$ stands for the category of all locally small categories, is called the category of discrete fibration over $\mathcal{A}$ and denoted by $\text{DFib(}\mathcal{A})$. An equivalence in $\text{CAT/}\mathcal{A}$ is called cartesian equivalence.

2. Given a discrete fibration $P: \mathcal{C} \to \mathcal{A}$ its change of base functor $P^*: \mathcal{A}^{\text{op}} \to \text{Set}$ is defined by the assignment $(A, a) \mapsto (\mathcal{A} \xrightarrow{\phi} B) \mapsto (P^*(B) \xrightarrow{P^*(\phi)} P^*(A))$ with as above $P^*(A)$ the fibre over the $\mathcal{A}$-object $A$ and $P^*(\phi)$ maps any $b \in P^*(B)$ to the domain of the unique $\mathcal{C}$-morphism $f$ with codomain $b$ such that $P(f) = \phi$.

3. The category $\text{El}(P^*)$ of elements of a functor $P^*: \mathcal{A}^{\text{op}} \to \text{Set}$, has as

- objects all pairs $(A, a)$ with $A \in \text{ob}(\mathcal{A})$ and $a \in P^*(A)$,
- morphisms $(A, a) \xrightarrow{\phi} (B, b)$ all $\mathcal{A}$-morphisms $A \xrightarrow{\phi} B$ with $P^*(\phi)(b) = a$ with composition and identities as in $\mathcal{A}$,
- a functor $P: \text{El}(P^*) \to \mathcal{A}$ given by $((A, a) \xrightarrow{\phi} (B, b)) \mapsto (A \xrightarrow{\phi} B)$.

4. For any functor $\mathcal{A}^{\text{op}} \xrightarrow{P^*} \text{Set}$ the functor $\text{El}(P^*) \xrightarrow{P} \mathcal{A}$ is a discrete fibration and the assignment $P^* \mapsto P$ defines an equivalence of the categories $\text{Set}^{\mathcal{A}^{\text{op}}}$ and $\text{DFib(}\mathcal{A})$.

3.1.5 Kleisli categories

As shown in the introduction we will be concerned with the endomorphism monoids of free algebras. An alternative way to describe, for any monad $\mathcal{T}$ on a category $\mathcal{A}$, the category of all free algebras $\mathcal{A_T}$ of $\mathcal{T}$ (see e.g. [14, VI.5]). The equivalence of these categories is given by the fact that homomorphisms $F^T X \to F^T Y$ between free algebras are nothing but the homomorphic extensions of $\mathcal{A}$-morphisms $X \to F^TY$. Consequently, the Kleisli category $\mathcal{A_T}$ of a monad $\mathcal{T} = (T, \mu, \eta)$ has as

1. its objects the objects of $\mathcal{A}$ and
2. as its hom-sets $\mathcal{A_T}(X, Y)$ the sets $\mathcal{A}(X, TY)$; its identities $1_X$ are the $\mathcal{A}$-morphisms $\eta_X$ and composition is defined by

\[(Y \xrightarrow{g} TZ) \bullet (X \xrightarrow{f} TY) := X \xrightarrow{f} TY \xrightarrow{Tg} TTZ \xrightarrow{\mu_Z} TZ.\]
The naturality of this concept in the context of this note is evident: if we would have used in the introduction, instead of $\text{Set}$, the Kleisli category $\mathbb{T}_M$ of the free (right) $M$-act monad, the description of the map $E_X$ would have simply read as $f \mapsto \langle id_X, f \rangle$, while the reconstruction map $L_X$ could simply be described as $(X \xrightarrow{(id_X, f)} X \times M) \mapsto (X \xrightarrow{(id_X, f)} X \times M \xrightarrow{\pi_M} M)$

### 3.2 Two discrete fibrations of an internal category

From now on one assumes that the finitely complete category $C$ is locally small. Given an internal category in $C$, that is, a monoid $M = (\bar{M}, \eta, \mu)$ in the monoidal category $C_O$, we denote by $\mathbb{M} = (T_M, \mu_M, \eta_M)$ the monad of its right modules, that is in detail, $T_M(\bar{A}) = \bar{A} \otimes \bar{M}$, $(\mu_M)_{\bar{A}} = \bar{A} \otimes \bar{M} \otimes \bar{M} \xrightarrow{id \otimes \mu} \bar{A} \otimes \bar{M}$ and $(\eta_M)_{\bar{A}} = \bar{A} \simeq \bar{A} \otimes \bar{O} \xrightarrow{id \otimes \eta} \bar{A} \otimes \bar{M}$. $\hat{T}_M$ denotes the composite $T_M \circ D$, where $D : C/O \to C_O$ is the embedding.

In view of the introduction we should be interested in

1. the category $\text{Conv}_M = D \downarrow \hat{\bar{M}}$ of $\bar{M}$-valued morphisms on $\hat{\bar{C}}$-objects. (Note that $\text{Conv}_M$ is locally small as so is $\bar{C}$.) This category is equipped with the forgetful functor $P_M : \text{Conv}_M \to C/O$ acting as

   $$\left( (f_A \xrightarrow{\alpha} \bar{M}) \xrightarrow{\phi} (g_B \xrightarrow{\beta} \bar{M}) \right) \mapsto \left( A \xrightarrow{f} O \xrightarrow{\phi} (B \xrightarrow{g} O) \right)$$

   (of course, $P_M$ has small fibres since $C$ is locally small)

2. the endomorphism monoids of free $\mathbb{M}$-modules over $\hat{\bar{C}}$-objects $f_A$, i.e., of the monoids

   $$C_O(f_A, f_A \otimes \bar{M}) = (C_O)_{\mathbb{M}}(f_A, f_A)$$

   where $(C_O)_{\mathbb{M}}$ denotes the Kleisli category of the monad $\mathbb{M}$ and the monoid structure is given by the Kleisli composition $\bullet$ as

   $$(f_A \xrightarrow{\beta} f_A \otimes \bar{M}) \bullet (f_A \xrightarrow{\alpha} f_A \otimes \bar{M}) = f_A \xrightarrow{\alpha} f_A \otimes \bar{M} \xrightarrow{\beta \otimes \bar{M}} f_A \otimes \bar{M} \otimes \bar{M} \xrightarrow{f_A \otimes \mu} f_A \otimes \bar{M}.$$ 

   These endomorphisms are objects of the comma category $D \downarrow \hat{T}_M$ and, hence, generate a full subcategory $\text{End}_{\hat{T}_M}$ of $D \downarrow \hat{T}_M$. In more detail $\text{End}_{\hat{T}_M}$ has as objects all 2-cells $(f_A \xrightarrow{\alpha} f_A \otimes \bar{M})$ and as morphisms $(f_A \xrightarrow{\alpha} f_A \otimes \bar{M}) \to (g_B \xrightarrow{\beta} g_B \otimes \bar{M})$ pairs of 2-cells $(f_A \xrightarrow{\alpha} g_B, f_A \xrightarrow{\tau \otimes id} g_B)$ making the following diagram commute. (Note that $\text{End}_{\hat{T}_M}$ is locally small as so is $\bar{C}$.)

\[
\begin{array}{ccc}
\hat{f}_A & \xrightarrow{\alpha} & \hat{f}_A \otimes \bar{M} \\
\downarrow_{\sigma} & & \downarrow_{\tau \otimes id} \\
\hat{g}_B & \xrightarrow{\beta} & \hat{g}_B \otimes \bar{M}
\end{array}
\]
Given an internal category in $\mathcal{C}$ considered as a monoid $\mathcal{M} = (\mathcal{M}, \mu, \eta)$ in $\mathcal{C}_O$ then, for each $\mathcal{C}_O$-objects $f_A$, the hom-set $\mathcal{C}_O(f_A, \mathcal{M})$ carries the structure of a monoid defined by $\phi * \psi := f_A \xrightarrow{\Delta} f_A \otimes f_A \xrightarrow{\phi \otimes \psi} \mathcal{M} \otimes \mathcal{M} \xrightarrow{\mu} \mathcal{M}$ and $\epsilon_f := f_A \xrightarrow{f} O \xrightarrow{n} \mathcal{M}$ and the functor $P^* := \mathcal{C}_O^{op} \hookrightarrow \mathcal{C}_O^{op} \xrightarrow{\mathcal{C}_O(-, \mathcal{M})} \text{Set}$ factors over the category $\text{Mon}$. As one easily sees $El(P^*) = \text{Conv}_\mathcal{M}$. This explains the notation chosen for this category: its objects are the elements of the respective convolution monoids.

Proposition 11 For every internal category $\mathcal{M}$ in $\mathcal{C}$ and every $f_A$ in $\mathcal{C}/\mathcal{O}$ the endomorphism monoid $M_A(f_A, f_A)$ coincides with the convolution monoid $\mathcal{C}_O(f_A, \mathcal{M})$.

Proof By definition of the category $M_A$ (see Section 3.1.3) the endomorphism set $M_A(f_A, f_A)$
Lemma 12 The functor $P_M: \text{Conv}_M \to \mathcal{C}/O$ is a discrete fibration whose change of base functor is $P_M^*: \overset{\sim}{\mathcal{C}}^\text{op}/O \hookrightarrow \mathcal{C}^\text{op}/(\mathcal{O},M)$ Set, and this functor factors over the category Mon of monoids.

3.2.2 The free-module-endomorphism fibration

Recall that for every 2-cell $f_A \overset{\alpha}{\to} f_A \otimes \tilde{M}$ the 2-cells $\tilde{\alpha} = f_A \overset{\pi^A \circ \alpha}{\to} \tilde{M}$ and $\alpha' := f_A \overset{\pi^A \circ \alpha}{\to} f_A$ form the only pair of 2-cells $(f_A \overset{\xi}{\to} f_A, f_A \overset{\nu}{\to} \tilde{M})$ presenting $\alpha$, that is, such that $f_A \overset{\xi, \nu}{\to} f_A \otimes \tilde{M}$ exists and equals $\alpha$.

Definition 13 A 2-cell $f_A \overset{\alpha}{\to} f_A \otimes \tilde{M}$ is simply presented if it is presented by $(id_A, \tilde{\alpha})$. $\text{spEnd}_{T_M}$ denotes the full subcategory of $\text{End}_{T_M}$ spanned by all simply presented endomorphisms while $\text{spAut}_{T_M}$ is its full subcategory spanned by the automorphisms.

Remark 14 A 2-cell $f_A \overset{\alpha}{\to} f_A \otimes \tilde{M}$ is simply presented iff any of the following holds.

1. $\alpha = f_A \overset{\langle id, \nu \rangle}{\to} f_A \otimes \tilde{M} = f_A \overset{\Delta}{\to} f_A \otimes f_A \overset{id \otimes \nu}{\to} f_A \otimes \tilde{M}$ for some 2-cell $f_A \overset{\nu}{\to} \tilde{M}$

2. $\alpha$ belongs to the image $\mathcal{E}_{1_A}$ of the map $E_{1_A}$.

3. $E_{1_A}(L_{1_A}(\alpha)) = \tilde{\alpha} = \alpha$.

Lemma 15 If $(\sigma, \tau): (f_A \overset{\langle id, \alpha \rangle}{\to} f_A \otimes \tilde{M}) \to (g_B \overset{\langle id, \beta \rangle}{\to} g_B \otimes \tilde{M})$ is a morphism in $\text{spEnd}_{T_M}$, then $\sigma = \tau$.

Proof If $(\sigma, \tau): (f_A \overset{\alpha}{\to} f_A \otimes \tilde{M}) \to (g_B \overset{\beta}{\to} g_B \otimes \tilde{M})$ is a morphism in $\text{spEnd}_{T_M}$, then $\tau = \sigma \circ \pi_f \circ \alpha = \pi_f \circ \alpha \circ (id \otimes \tau) \circ \alpha = \pi_g \circ \beta \circ \sigma = \sigma$. □

Correspondingly, we may denote morphisms in this category simply by $\sigma$.

Fact 16 For each $f_A \overset{\sigma}{\to} g_B$ in $\mathcal{C}/O$ there is the map $\mathcal{E}_{g_B} \overset{\sigma^*}{\to} \mathcal{E}_{1_A}$ given by the assignment $(g_B \overset{\beta}{\to} g_B \otimes \tilde{M}) \mapsto (f_A \overset{\langle id_A, \pi_A \circ \beta \circ \sigma \rangle}{\to} f_A \otimes \tilde{M})$ and the assignment $(f_A \overset{\sigma}{\to} g_B) \mapsto Q^*_M(g_B) := \mathcal{E}_{g_B} \overset{\sigma^*}{\to} \mathcal{E}_{1_A} = Q^*_M(f_A)$ defines a functor $(\mathcal{C}/O)^\text{op} \overset{Q^*_M}{\to} \text{Set}$. 

14
El(Q_M^*) has as objects all pairs \((f_A, (id_A, \alpha))\) with a 2-cell \(f_A \overset{\alpha}{\rightarrow} \bar{M}\), while a morphism \((f_A \overset{(id, \alpha)}{\rightarrow} f_A \otimes \bar{M}) \overset{\sigma}{\rightarrow} (g_B \overset{(id, \beta)}{\rightarrow} g_B \otimes \bar{M})\) in El(Q_M^*) is a 2-cell \(f_A \overset{\sigma}{\rightarrow} g_B\) satisfying the condition \(\beta \circ \sigma = \alpha\). The obvious commutativity of the following diagram shows that this condition is equivalent to the fact that \(\sigma\) is a morphism in \(^{sp}End_{T_M}\).

Thus, El(Q_M^*) is isomorphic to the category \(^{sp}End_{T_M}\). Q_M maps \((f_A \overset{(id, \alpha)}{\rightarrow} f_A \otimes \bar{M}) \overset{\sigma}{\rightarrow} (g_B \overset{(id, \beta)}{\rightarrow} g_B \otimes \bar{M})\) to \((A \overset{f}{\rightarrow} O) \overset{\sigma}{\rightarrow} (B \overset{g}{\rightarrow} O)\). (Q_M has small fibres since \(\mathcal{C}\) is locally small.)

**Lemma 17** The map \(E_{f_A}\) is a homomorphism of monoids from the convolution monoid \(\mathcal{C}_O(f_A, \bar{M})\) into the endomorphism monoid \((\mathcal{C}_O)_{T_M}(f_A, f_A)\). In particular, \(E_{f_A}\) is a sub-monoid of the endomorphism monoid \((\mathcal{C}_O)_{T_M}(f_A, f_A)\).

**Proof** First, as easily seen, one has \(E_{f_A}(f_A \overset{f}{\rightarrow} \bar{O} \overset{\eta}{\rightarrow} \bar{M}) = f_A \simeq f_A \otimes \bar{O} \overset{id \otimes \eta}{\rightarrow} f_A \otimes \bar{M}\), that is \(E_{f_A}\) preserves units. \(E_{f_A}\) preserves multiplication iff \(E_{f_A}(\alpha \ast \beta) = E_{f_A}(\alpha) \cdot E_{f_A}(\beta)\), for any pair of \(\mathcal{C}_O\)-morphisms \(\alpha, \beta: f_A \rightarrow \bar{M}\), that is, iff the following diagram commutes in \(\mathcal{C}_O\). But this is the case by Item 3 of Section 3.1.2

By Section 3.1.4 we so obtain

**Proposition 18** The functor \(Q_M: ^{sp}End_{T_M} \rightarrow \mathcal{C}/O\) is a discrete fibration whose fibres are the sets \(\mathcal{E}_{f_A}\) of simply presented elements and whose change of base functor factors over Mon.
The final and somewhat surprising result of this section requires the following generalization of Lemma 15:

**Lemma 19** For each \( \text{End}_{\hat{T}} \)-morphism \((g_B \xrightarrow{\beta} g_B \otimes \tilde{M})\) \((f_A \xrightarrow{\alpha} f_A \otimes \tilde{M})\) with domain in \(^{sp}\text{End}_{\hat{T}}\), one has \(\psi = \phi_\alpha := g_B \xrightarrow{\phi} f_A \xrightarrow{\alpha} f_A \otimes \tilde{M} \xrightarrow{\pi_f} f_A\).

**Proof** The claim follows by the following calculation where the second equality comes from the definition of \(\text{End}_{\hat{T}}\)-morphisms and the third one holds by definition of \(\otimes\).

\[
\begin{align*}
\phi_\alpha &= g_B \xrightarrow{\phi} f_A \xrightarrow{\alpha} f_A \otimes \tilde{M} \xrightarrow{\pi_f} f_A = g_B \xrightarrow{(id,\beta)} g_B \otimes \tilde{M} \xrightarrow{\psi \otimes id} f_A \otimes \tilde{M} \xrightarrow{\pi_f} f_A \\
&= g_B \xrightarrow{(id,\beta)} g_B \otimes \tilde{M} \xrightarrow{\pi_f} g_B \xrightarrow{\psi} f_A = g_B \xrightarrow{id} g_B \xrightarrow{\psi} f_A = \psi.
\end{align*}
\]

\(\square\)

**Proposition 20** The category \(^{sp}\text{End}_{\hat{T}}\) is a full coreflective subcategory of \(\text{End}_{\hat{T}}\).

**Proof** Observe that the 2-cell \(C(\alpha) := (id, \pi_f \circ \alpha) : f_A \xrightarrow{(id,\alpha)} f_A \otimes \tilde{M}\) belongs to \(^{sp}\text{End}_{\hat{T}}\), for each \(\text{End}_{\hat{T}}\)-object \((f_A \xrightarrow{\alpha} f_A \otimes \tilde{M})\).

Since by Item 4 (b) of Section 3.1.2 if \(f_A \xrightarrow{\alpha} f_A \otimes \tilde{M} \xrightarrow{\pi_f} f_A\) is a 2-cell, the pair \((id, \pi_f \circ \alpha)\) will be a \(\text{End}_{\hat{T}}\)-morphism \(C(\alpha) \xrightarrow{\pi_\alpha} \alpha\) if the following diagram commutes.

\[
\begin{array}{ccc}
f_A & \xrightarrow{(id,\alpha)} & f_A \otimes \tilde{M} \\
\downarrow{id} & & \downarrow{(\pi_f \circ \alpha) \otimes id} \\
f_A & \xrightarrow{\alpha} & f_A \otimes \tilde{M}
\end{array}
\]

But this is easily seen to be true.

We claim that \(c_\alpha\) in fact is a coreflection. We thus have to show that, for any morphism \(\text{End}_{\hat{T}}\)-morphism \((g_B \xrightarrow{\beta} g_B \otimes \tilde{M})\) \((f_A \xrightarrow{\alpha} f_A \otimes \tilde{M})\) with an \(^{sp}\text{End}_{\hat{T}}\)-object \((g_B \xrightarrow{\beta} g_B \otimes \tilde{M})\), there exists a unique \(^{sp}\text{End}_{\hat{T}}\)-morphism \((g_B \xrightarrow{\beta} g_B \otimes \tilde{M}) \xrightarrow{(\sigma,\sigma)} C(\alpha)\) such that \(c_\alpha \circ (\sigma, \sigma) = (\phi, \psi)\). By the preceding Lemma this condition is equivalent to

\[c_\alpha \circ (\sigma, \sigma) = (id, \pi_f \circ \alpha) \circ (\sigma, \sigma) = (\sigma, \pi_f \circ \alpha \circ \sigma) = (\phi, \phi_\alpha) = (\phi, \pi_f \circ \alpha \circ \phi)\]

and, hence, satisfied obviously. \(\square\)
3.2.3 Comparing the fibrations

As seen above we have \( Q_M^*(f_A) = E_{f_A}(P_M^*(f_A)) \) and (see Remark 14 above) a 2-cell \( f_A \overset{\alpha}{\to} f_A \otimes \tilde{M} \) is simply presented iff \( E_{f_A}(L_{f_A}(\alpha)) = \tilde{\alpha} = \alpha \).

Moreover, we can restrict the map \( L_{f_A} \) to a map \( Q_M^*(f_A) \to P_M^*(f_A) \) and observe that, for each 2-cell \( f_A \overset{\alpha}{\to} \tilde{M} \), one has \( L_{f_A}(E_{f_A}(\alpha)) = \tilde{\alpha} = \langle \text{id}, \alpha \rangle = \alpha \).

One so obtains the first of the following statements; the second one follows by a straightforward calculation, while the third one is Lemma 17. The final item then follows from Lemma 12 by Items 2 and 3.

**Lemma 21**

1. \( L_{f_A} \) is the inverse of \( E_{f_A} \).

2. The family \( (E_{f_A})_{f_A} : P_M^* \Rightarrow Q_M^* \) is a natural isomorphism with \( (L_{f_A})_{f_A} \) as its inverse.

3. Each map \( E_{f_A} \) is an isomorphism of monoids \( P_M^*(f_A) \to Q_M^*(f_A) \) with \( L_{f_A} \) as its inverse.

4. For each morphism \( f_A \overset{\sigma}{\to} g_B \) in \( C/O \) the map \( Q_M^*(\sigma) \) is a homomorphism of monoids \( Q_M^*(g_B) \to Q_M^*(f_A) \).

In view of Section 3.1.4 we can summarize the above alternatively as follows.

**Proposition 22** For any internal category \( M = (O, M, d, c, \eta, \mu) \) in a finitely complete category \( C \) the following hold.

1. The functors \( P_M : \text{Conv}_M \to C/O \) and \( Q_M : \text{spEnd}_{\hat{T}_M} \to C/O \) are discrete fibrations whose change of base functors \( P_M^* \) and \( Q_M^* \) are naturally isomorphic and factor over the category \( \text{Mon} \) of monoids.

2. The respective cartesian isomorphism \( E : \text{Conv}_M \to \text{spEnd}_{\hat{T}_M} \) and its inverse \( L : \text{spEnd}_{\hat{T}_M} \to \text{Conv}_M \) act as follows:

\[
E((f_A \overset{\alpha}{\to} \tilde{M}) \overset{\phi}{\to} (g_B \overset{\beta}{\to} \tilde{M})) = (g_B \overset{\beta}{\to} g_B \otimes \tilde{M}) \overset{\phi}{\to} (f_A \overset{\tilde{\alpha}}{\to} f_A \otimes \tilde{M})
\]

\[
L((f_A \overset{\alpha}{\to} f_A \otimes \tilde{M}) \overset{(\phi, \psi)}{\to} (g_B \overset{\beta}{\to} g_B \otimes \tilde{M})) = (f_A \overset{\tilde{\alpha}}{\to} \tilde{M}) \overset{\phi}{\to} (g_B \overset{\beta}{\to} \tilde{M})
\]

**Remark 23** By Proposition 11 the convolution monoids \( P_M^*(f_A) \) are groups if the categories \( M_A \) are groupoids, and this clearly is the case if the internal category under consideration is an internal groupoid (see Fact 9). We so obtain: if \( (O, M, d, c, \eta, \mu, \iota) \) is an internal groupoid in \( C \) then the functors \( P_M^* \) and \( Q_M^* \) even factor over the category \( \text{Grp} \) of groups.

With \( E \) and \( L \) the isomorphisms of Proposition 22 and \( C \) the coreflector of Proposition 20 one obtains
Corollary 24  Let \( M = (O, M, d, c, \eta, \mu) \) be an internal category in the finitely complete category \( C \). Then the following hold:

1. The functor \( \hat{E} = \text{Conv}_M \overset{E}{\rightarrow} \text{spEnd}_{\hat{T}_M} \hookrightarrow \text{End}_{\hat{T}_M} \) is left adjoint to the functor \( \hat{L} : \text{End}_{\hat{T}_M} \overset{L}{\rightarrow} \text{Conv}_M \).

2. If \((M, \iota)\) even is an internal groupoid in \( C \) then this adjunction induces, an adjunction \( E' \dashv L' \) with \( L' = \text{Aut}_{\hat{T}_M} \hookrightarrow \text{End}_{\hat{T}_M} \overset{L}{\rightarrow} \text{Conv}_M \), the restriction of the functor \( \hat{L} \), and \( E' : \text{Conv}_M \rightarrow \text{Aut}_{\hat{T}_M} \) the corestriction of \( \hat{E} \).

3.3 Functoriality

We finally investigate to what extent the assignments \( M \mapsto P_M \) and \( M \mapsto Q_M \) are functorial. For doing so it seems appropriate to define the category \( \text{IntCat} \) of all internal categories in analogy to the category of all modules and also the category of all discrete fibrations \( \text{DFib} \). We will need the observation that the construction of Remark \( \text{R}\) defines a functor \( \text{Cat} \subset \text{Lex} \rightarrow \text{CAT} \) where \( \text{Lex} \) denotes the category of all finitely complete locally small categories and finite limit preserving functors and \( \text{CAT} \) that of all locally small categories.

Since these constructions involve categories of categories let us clarify our conventions concerning the sizes of the categories that are considered below: we will here use the term “category” also for structures which are roughly like “metacategories” in \( \text{[14]} \) or could be called “very large categories”, such as \( \text{Lex} \), \( \text{CAT} \), \( \text{IntCat} \) or \( \text{DFib} \) (see below). In particular such categories are not assumed locally small\( ^3 \).

The category \( \text{IntCat} \) of internal categories has as objects the pairs \((C, M)\) formed by a finitely complete category \( C \) and an internal category \( M \) in \( C \), and as morphisms the pairs \((C, M) \xrightarrow{(K, (F_o, F_m))} (C', M')\) consisting of a finite limit preserving functor \( K \in \text{Lex}(C, C') \) and an internal functor \( \text{Cat}_{in}(K)(M) \xrightarrow{(F_o, F_m)} M' \) (see Remark \( \text{R}\)). In other words, \( \text{IntCat} \) is the category obtained from the functor \( \text{Cat}_{in} : \text{Lex} \rightarrow \text{CAT} \) using the Grothendieck construction (see e.g. [12] Theorem 1.10.7, p. 111)).

\(^3\)More formally, for the sake of the reader interested in the foundations of the theory of categories: let us choose e.g. as our foundational system ZFC + Grothendieck universes (that is, for every set there exists a Grothendieck universe containing it), and let us called “category” any set-theoretic model of the axioms of categories. Let \( U_1 \in U_2 \in U_3 \) be Grothendieck universes. Let us fix the terminology, as already but informally used in this note: a set which belongs to \( U_1 \) (resp. \( U_2 \), resp. \( U_3 \)) is referred to as small (resp. large, resp. very large). A small (resp. large, resp. very large) category is then a category whose sets of objects and arrows are small (resp. large, resp. very large). A large category is locally small when its hom-sets are small. E.g. the categories of small sets \( \text{Set} \) and of small categories \( \text{Cat} \) are locally small. The categories \( \text{Lex} \), \( \text{CAT} \), \( \text{IntCat} \) and \( \text{DFib} \) are very large, while only “locally large”, that is, their hom-sets are large.
Given a morphism \((C,M) \xrightarrow{(K,(F_o,F_m))} (C',M')\) in \(\text{IntCat}\) we denote by \(K/F_o : C/O \to C'/O'\) the functor mapping \(((C \xrightarrow{f} O) \xrightarrow{\alpha} (D \xrightarrow{g} O))\) to \(((K(C) \xrightarrow{K(f)} K(O) \xrightarrow{F_o} O') \xrightarrow{K(\alpha)} (K(D) \xrightarrow{K(g)} K(O) \xrightarrow{F_o} O')).\)

The category \(\text{DFib}\) of discrete fibrations is the full subcategory of the category of arrows in \(\text{CAT}\) spanned by all discrete fibrations; in other words, if \(A \xrightarrow{P} C\) and \(B \xrightarrow{Q} D\) are discrete fibrations, then a morphism \(P \to Q\) is a pair of functors \((A \xrightarrow{F} B, C \xrightarrow{G} D)\) such that \(Q \circ F = G \circ P\).

For any pair \((C,M)\) in \(\text{IntCat}\) we denote by

1. Conv the full subcategory of \(\text{DFib}\) spanned by the discrete fibrations \(\text{Conv}_{M} \xrightarrow{P_M} C/O\)
2. \(\text{spEnd}\) be the full subcategory of \(\text{DFib}\) spanned by the discrete fibrations \(\text{spEnd}_{\hat{T}} \xrightarrow{Q_M} C/O\).

One now can define functors \(\text{IntCat} \to \text{DFib}\) as follows:

\(\text{P}\) For any \((C,M) \xrightarrow{(K,(F_o,F_m))} (C',M')\) in \(\text{IntCat}\) let

\([K,(F_o,F_m)] : \text{Conv}_{M} \to \text{Conv}_{M'}\)

be the functor acting

- on objects by \((f_A \xrightarrow{\alpha} \bar{M}) \mapsto ((F_o \circ K(f)) \xrightarrow{F_m \circ K(\alpha)} \bar{M}')\)
- and on morphisms by \(\sigma \mapsto K(\sigma)\).

It is now easy to see that the following diagram commutes and, hence, the pair \([K,(F_o,F_m)], K/F_o\) is a morphism \(P_M \to P_{M'}\) in Conv.

\[
\begin{array}{ccc}
\text{Conv}_{M} & \xrightarrow{[K,(F_o,F_m)]} & \text{Conv}_{M'} \\
\downarrow{P_M} & & \downarrow{P_{M'}} \\
C/O & \xrightarrow{K/F_o} & C'/O'
\end{array}
\]

In fact, commutativity of this diagram is equivalent to saying that for any morphism \((f_A \xrightarrow{\alpha} \bar{M}) \xrightarrow{\sigma} (g_B \xrightarrow{\beta} \bar{M})\) in \(\text{Conv}_{M}\), i.e., for any commutative diagram

\[
\begin{array}{ccc}
& A & \\
& \downarrow{f} & \downarrow{f} \\
O & \xrightarrow{g} B & \xrightarrow{g} O \\
& \downarrow{\sigma} & \downarrow{\sigma} \\
& B & \xrightarrow{\beta} O \\
& \downarrow{\beta} & \downarrow{\beta} \\
& M & \xrightarrow{d} O
\end{array}
\]
with $\alpha = \beta \circ \sigma$ the following diagram commutes

$$
\begin{array}{ccc}
K(A) & \xrightarrow{K(f)} & K(B) \\
\downarrow & & \downarrow \\
K(O) & \xrightarrow{K(g)} & K(O)
\end{array}
\quad
\begin{array}{ccc}
K(f) & \xrightarrow{K(\sigma)} & K(f) \\
\downarrow & & \downarrow \\
K(g) & \xrightarrow{K(\sigma)} & K(g)
\end{array}
$$

But this is obvious.

A further straightforward calculation shows that the assignment

$$
((C, M) \xrightarrow{(K, (F_o, F_m))} (C', M')) \rightarrow \left( P_M \xrightarrow{([K, (F_o, F_m)], K/F_o)} P_{M'} \right)
$$

defines a functor $\text{IntCat} \xrightarrow{P} \text{Conv} \hookrightarrow \text{DFib}$.

Q For any $(C, M) \xrightarrow{(K, (F_o, F_m))} (C', M')$ in $\text{IntCat}$ let

$$
\{K, (F_o, F_m)\} : \text{spEnd}_{\tilde{T}_M} \rightarrow \text{spEnd}_{\tilde{T}_{M'}}
$$

be the functor acting

- on objects by

$$
\left( (f_A \xrightarrow{id, \alpha} f_A \otimes \tilde{M}) \mapsto ((F_o \circ K(f))_{K(A)} \xrightarrow{id, F_m \circ K(\alpha)} (F_o \circ K(f))_{K(A)} \otimes \tilde{M}') \right)
$$

- and on morphisms by $\sigma \mapsto K(\sigma)$.

Again by a simple calculation one obtains that the following diagram commutes and, hence, the pair $(\{K, (F_o, F_m)\}, K/F_o)$ is a morphism $Q_M \rightarrow Q_{M'}$ in $\text{spEnd}$,

$$
\begin{array}{ccc}
\text{spEnd}_{\tilde{T}_M} & \xrightarrow{\{K, (F_o, F_m)\}} & \text{spEnd}_{\tilde{T}_{M'}} \\
\downarrow & & \downarrow \\
Q_M & \rightarrow & Q_{M'}
\end{array}
\quad
\begin{array}{ccc}
C/O & \xrightarrow{K/F_o} & C'/O' \\
\downarrow & & \downarrow \\
\text{C'/O'} & \rightarrow & \text{C'/O'}
\end{array}
$$

while a further calculation shows that the assignment

$$
((C, M) \xrightarrow{(K, (F_o, F_m))} (C', M')) \rightarrow \left( Q_M \xrightarrow{([K, (F_o, F_m)], K/F_o)} Q_{M'} \right)
$$

defines a functor $\text{IntCat} \xrightarrow{Q} \text{spEnd} \hookrightarrow \text{DFib}$.  

20
Conclusion The functors $P$ and $Q$, considered as functors $\text{IntCat} \to \text{DFib}$, are naturally isomorphic.

4 Possible applications

4.1 A “quantum” Feistel scheme

The purpose of this section is a generalization of the cryptographic construction of Feistel schemes referred to in the introduction, which is suitable for quantum computing. We therefore briefly recall the original construction as follows, where plaintexts and ciphertexts are members of $2^{2m}$: choose, for some $N \in \mathbb{N}$ (called number of rounds) “key”-maps $f_i: 2^m \to 2^m$ for each $1 \leq i \leq N$ and form the maps $E_m(f_i) \circ \sigma: 2^m \times 2^m \to 2^m \times 2^m$, where $\sigma: (\vec{x}, \vec{y}) \in 2^m \times 2^m \mapsto (\vec{y}, \vec{x}) \in 2^m \times 2^m$ and their composition $E := (E_m(f_N) \circ \sigma) \circ \cdots \circ (E_m(f_1) \circ \sigma)$. Then the encoded ciphertext $c_t$ corresponding to some plaintext $t$ is computed as $E(t)$.

Making this construction implementable on a quantum computer requires to find a category $C$ with finite products such that, for any internal group $(H, \mu, \eta, \iota)$ in $C$ and each morphism $H \xrightarrow{f} H$ in $C$, the morphism $E_H(f): H \times H \to H \times H$ generalizing the fundamental step of constructing Feistel ciphers is a unitary operator on a finite dimensional Hilbert space (see e.g. [5]).

In order to define a suitable category $C$ recall first the following facts about the category $\text{Hilb}_{fd}$ of finite dimensional complex Hilbert spaces and linear maps (these are automatically bounded): (a) $(\text{Hilb}_{fd}, \otimes, \mathbb{C}, \sigma)$ with $\otimes$ the usual tensor product and $\sigma$ the usual twist is a symmetric monoidal category; (b) the usual Hilbert adjoint functor $\dagger: \text{Hilb}_{fd}^{\text{op}} \to \text{Hilb}_{fd}$ makes it a dagger compact closed category (see e.g. [17]); (c) $\text{Hilb}_{fd}$ has finite biproducts. By $\lvert - \rvert$ we denote its underlying space functor into the category of complex vector spaces.

As for any monoidal category we, hence, can consider its category $\text{coCComon}(\text{Hilb}_{fd})$ of cocommutative comonoids. Its objects are triples $(H, \delta, \epsilon)$ where $H$ is a finite dimensional Hilbert space, and where $(\lvert H \rvert, \lvert H \rvert \xrightarrow{\delta} \lvert H \rvert \otimes \mathbb{C} \lvert H \rvert, \lvert H \rvert \xrightarrow{\epsilon} \mathbb{C})$ is an ordinary cocommutative (complex) coalgebra with a counit. As is well-known the tensor product of $\text{Hilb}_{fd}$ lifts to $\text{coCComon}(\text{Hilb}_{fd})$ and here is the binary product; in particular, this category has all finite products (the Hilbert space $\mathbb{C}$ with its trivial coalgebra structure is a terminal object) [10].

A cocommutative comonoid $(H, \delta, \epsilon)$ is a $\dagger$-Frobenius coalgebra when $(\delta \dagger \otimes \text{id})(\delta \otimes \text{id}) = \delta \circ \delta \dagger = (\text{id} \otimes \delta \dagger)(\delta \otimes \text{id})$. It is called special when $\delta$ is an isometry, that is, $\delta \dagger \circ \delta = \text{id}$. One immediately sees that $(H, \delta \dagger, \epsilon \dagger)$ then is a special commutative $\dagger$-Frobenius algebra in $\text{Hilb}_{fd}$ (see [7]).

The full subcategory $C_{Hilb}$ of $\text{coCComon}(\text{Hilb}_{fd})$ generated by the special $\dagger$-Frobenius coalgebras turns out to be suitable for our purpose. It first is closed under binary products and contains $\mathbb{C}$ (a direct consequence of [1 Proposition 2.2.6, p. 18] for instance) and, thus, has all finite products. Moreover, an internal group in this category is a tuple.
\((H, \delta, \epsilon), \mu, \eta, S)\) where \((H, \delta, \epsilon)\) is a special cocommutative \(\dagger\)-Frobenius coalgebra and \((\mathbb{H}, \delta, \epsilon, \mu, \eta, S)\) is an ordinary (finite dimensional) Hopf algebra (see e.g. [8, p. 377] concerning the relation between cocommutative Hopf algebras and group objects in the category of cocommutative coalgebras). In addition one observes the following: the set \(G(H, \delta, \epsilon)\) of group-like elements, that is, the elements \(x \in H\) such that \(\delta(x) = x \otimes x\) and \(\epsilon(x) = 1\), of a special cocommutative \(\dagger\)-Frobenius coalgebra \((H, \delta, \epsilon)\) forms an orthonormal basis of the Hilbert space \(H\) (see [7, Corollary 4.7] with [2, Propositions 14 and 15]), and for a morphism of coalgebras \((H, \delta, \epsilon) \xrightarrow{f} (H', \delta', \epsilon')\) between special \(\dagger\)-Frobenius coalgebras, \(f(G(H, \delta, \epsilon)) \subseteq G(H', \delta', \epsilon')\).

Let now \((H, \delta, \epsilon, \mu, \eta, S)\) be an internal group in \(C_{Hilb}\) and let \(f: H \to H\), with \(H := (H, \delta, \epsilon)\), be an endomorphism. Then, using the notations from the Introduction, \(E_H(f): H \otimes H \to H \otimes H\), given in details by \(u \otimes v \mapsto \sum_{(u)} u_1 \otimes \mu(f(u_2) \otimes v)\) (with \(\delta(u) = \sum_{(u)} u_1 \otimes u_2\)), is an automorphism. In particular, since \(E_H(f)(G(H \otimes H)) = G(H \otimes H)\), \(E_H(f)\) is in fact a unitary operator on \(H \otimes H\) as aspired.

Admittedly, we don’t know whether the execution of these generalized ‘ciphers’ in the context of quantum computing is of great importance.

### 4.2 Higher-order Boolean functions

Let \(\bigcup_{x \in X}(G_x, \ast_x, 1_x)\) be the coproduct, in the category of groupoids, of an \(X\)-indexed family \((G_x, \ast_x, 1_x)_{x \in X}\) of groups, each of them being considered as a one-object groupoid. The domain \(d\) and codomain \(c\) of this groupoid are both equal to the canonical projection \(\pi: \bigcup_{x \in X} G_x \to X, (x, g) \mapsto x\). Let \(\alpha: X \to \bigcup_{x \in X} G_x\) be a section of \(\pi\), that is, \(\alpha(x) = (x, \alpha^g(x))\) where \(\alpha^g(x) \in G_x, x \in X\). By Remarks 23 and 24 the map \(E(\alpha): \bigcup_{x \in X} G_x \to \bigcup_{x \in X} G_x\) given by \(E(\alpha)(x, g) = (x, \alpha^g(x) \ast_x g)\) is a permutation.

In the case where \(G_x = \text{Set}(2^{m_x}, 2^{n_x})\) is the group \(\mathbb{Z}_2^{n_x} \times 2^{m_x}\), \(\alpha: X \to \bigcup_{x \in X} \text{Set}(2^{m_x}, 2^{n_x})\) is, in the language of functional programming, a higher-order Boolean function. Hence our construction turns such a higher-order Boolean function into a higher-order Boolean permutation. This construction might be of use in functional programming.

### Acknowledgements

The first author thanks Prof. Sami Harari (Toulon) for fruitful discussions about the Feistel scheme which were the germ of the work presented here.

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