Fluctuations of $1/f$ noise and the low frequency cutoff paradox

Markus Niemann,$^1$ Holger Kantz,$^2$ and Eli Barkai$^3$

$^1$Institut für Physik, Carl von Ossietzky Universität Oldenburg, 26111 Oldenburg, Germany
$^2$Max-Planck-Institut für Physik komplexer Systeme, Nöthnitzer Straße 38, 01187 Dresden, Germany
$^3$Department of Physics, Bar Ilan University, Ramat Gan 52900, Israel

Recent experiments on blinking quantum dots and weak turbulence in liquid crystals reveal the fundamental connection between $1/f$ noise and power law intermittency. The non-stationarity of the process implies that the power spectrum is random – a manifestation of weak ergodicity breaking. Here we obtain the universal distribution of the power spectrum, which can be used to identify intermittency as the source of the noise. We solve an outstanding paradox on the non integrability of $1/f$ noise and the violation of Parseval’s theorem. We explain why there is no physical low frequency cutoff and therefore cannot be found in experiments.

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The power spectrum $S(f)$ of a wide variety of physical systems exhibits enigmatic $1/f$ noise $^1$ $^2$ at low frequencies,

$$S(f) \sim \frac{\text{constant}}{f^\gamma} \quad \text{where} \quad 0 < \gamma < 2. \quad (1)$$

Starting with Bernamont $^3$, in the context of resistance fluctuations of thin films, many models of these widely observed phenomena were put forward. Indeed, $1/f$ noise is practically universal, ranging from voltages and currents in vacuum tubes, diodes and transistors, to annual amounts of rainfall, to name only a few examples. A closer look at the phenomenon reveals several themes which demand further explanation. The first is that $1/f$ noise is not integrable: $\int_{f_0}^\infty S(f) \, df = \infty$, due to the low frequency behavior, when $\gamma \geq 1$. This violates the Parseval theorem from which one may deduce that the spectrum of a random process is integrable (see details below). So how can we find in laboratory $1/f$ noise if a mathematical theorem forbids it? One simple explanation would be that the phenomenon has a cutoff at some low frequency, namely that below $f < f_0$ Eq. (1) is not valid. Experimentalists have therefore carefully searched for this cutoff, increasing the measurement time as far as reasonable: three weeks for noise in MOSFET $^4$, and 300 years for weather data $^5$. No cutoff frequency is observed even after these long measurement times. This is one of the outstanding features of $1/f$ noise. A second old controversy, related to the first, is the suggestion of Mandelbrot $^6$ that models of $1/f$ noise for $\gamma \geq 1$ should be related to non-stationarity processes, though the nature of this non-stationarity is still an open question. Further, experiments find that at least in some cases the amplitude of the power spectrum varies among identical systems measured at different times, but the shape and in particular the value of the exponent $\gamma$ is quite consistent $^1$ $^2$. This means that a $1/f$ spectrum is a non self averaging observable, at least in some systems.

While these observations where made long ago, the verdict on them is not yet out. However, recent measurements of blinking quantum dots $^7$ $^8$ and liquid crystals in the electrohydrodynamic convection regime $^9$ shed new light on the nature of $1/f$ noise. Both systems, while very different in their nature, reveal a power law intermittency route to $1/f$ noise. This means that power law waiting times in a micro-state of the system are responsible for the observed spectrum. This approach was suggested as a fundamental mechanism for $1/f$ noise in the context of intermittency of chaos and turbulence, with the work of Manneville $^{10}$. Subsequently, it has been found in many intermittent chaotic systems $^{11}$ $^{14}$, and has been used successfully as a model for transport in geological formations $^{15}$. For a quantum dot driven by a continuous wave laser, this mechanism means that the dot switches from a dark state to a bright state where photons are emitted, and that sojourn times in both states exhibit power law statistics which is scale free $^{16}$ $^{17}$. Waiting times probability density functions (PDF) in these states follow $\psi(t) \sim t^{-\alpha}$ and $0 < \alpha < 1$ (for bright state times this behavior is found for low laser intensity and low temperature). The dynamics is scale free, because the average sojourn times diverge, and we expect weak ergodicity breaking $^{18}$ $^{19}$. This means that the power spectrum remains a random variable even in the long time limit $^{20}$ $^{21}$.

Here we investigate the non self averaging power spectrum, and show that indeed this observable exhibits large but universal fluctuations, while the estimation of $\gamma$ is rather robust. Our work gives experimentalists a way to verify whether a data set exhibiting $1/f$ noise belongs to the intermittency class, and this we believe will help unravel the origin of an old mystery of statistical physics. We also remove the paradox based on Parseval’s identity, showing that as $t \rightarrow \infty$ the integrability remains, and that there is no cutoff frequency $f_0$. So experimentally searching for this “lost” low frequency might be in vain.

Parseval’s identity and $1/f$ noise. We consider a measurement of a random signal $I(t)$ in the time interval $(0, t)$, so that its Fourier transform is $I(\omega) = \int_0^t I(t) e^{-i \omega t} dt$. If $I(t)$ is integrable, then Parseval’s identity states that the power spectrum $S(\omega) = \left| \mathcal{F}[I(t)](\omega) \right|^2$ satisfies

$$\int_0^\infty S(\omega) \, d\omega = \int_0^t I^2(t) \, dt.$$
\[ I'(t') \exp(-i\omega t')dt'. \] The power spectrum \( S_\ell(\omega) = \langle \dot{I}_\ell(\omega) \dot{I}_\ell^*(\omega) \rangle / t, \) is considered in the long measurement time limit. The ensemble average power spectrum is \( \langle S_\ell(\omega) \rangle. \) Note that in an experiment with one realization of the time series, for example a measurement of the intensity of a single molecule or a quantum dot, the ensemble average is not performed, though in experiments one introduces smoothing methods which reduce the noise level of the reported power spectrum \[22\]. More importantly note that the integral over the power spectrum is

\[
\int_{-\infty}^{\infty} S_\ell(\omega) d\omega = \frac{1}{t} \int_{-\infty}^{\infty} d\omega \int_{0}^{t} dt_1 \exp(-i\omega t_1) I(t_1) \int_{0}^{t} dt_2 \exp(i\omega t_2) I(t_2) = \frac{2\pi}{t} \int_{0}^{t} I^2(t_1) dt_1, \tag{2}
\]

where we used a well known identity of the delta function \( \int_{-\infty}^{\infty} d\omega \exp[-i\omega(t_1-t_2)] = 2\pi \delta(t_2-t_1) \) and \( S_\ell(w) = S_\ell(-\omega) \) by definition. For any bounded process be it ergodic or non-ergodic, stationary or non-stationary, \( I_\ell^2(t) \leq \langle I_{\text{max}}^2 \rangle^2 \) and hence \( \int_{-\infty}^{\infty} S_\ell(\omega) d\omega \leq 2\pi \langle I_{\text{max}}^2 \rangle^2. \) So the integral is finite for a wide class of processes. As a consequence, non-integrable \( 1/f \) noise is strictly prohibited. The classical way out was to assume a violation of the \( 1/f \)-behaviour in the limit of \( f \to 0. \)

The ensemble-average of Eq. \[2\], under the additional assumption that the process reaches a stationary state, reads \( \lim_{t \to \infty} \int_{-\infty}^{\infty} S_\ell(\omega) d\omega = 2\pi \langle I^2 \rangle. \) If the system is ergodic, i.e., \( \overline{I_\ell^2} = \int_{0}^{t} I^2(t') dt' / t \to \langle I^2 \rangle, \) we have for a single trajectory \( I(t) \)

\[
\lim_{t \to \infty} \int_{-\infty}^{\infty} S_\ell(\omega) d\omega = 2\pi \langle I^2 \rangle. \tag{3}
\]

Thus fluctuations of the total area under the power spectrum are an indication for ergodicity breaking.

**Models.** For simplicity we consider a two state model. The generalization to \( N \) states will be discussed elsewhere. The two states of our model are \( \text{up} \) or \( \text{down} \) where \( I(t) = I_0 \) and \( \text{down} \) or \( \text{up} \) with \( I(t) = -I_0. \) The sojourn times in these states are independently identically distributed random variables with PDFs \( \psi(\tau). \) Thus after waiting a random time in state \( \text{up} \), the particle jumps to state \( \text{down} \) and vice versa. The waiting times PDFs have long tails \( \psi(\tau) \propto \tau^{-(1+\alpha)} \) with \( 0 < \alpha < 1, \) hence the averages of \( \text{up} \) and \( \text{down} \) times are infinite. The Laplace transform \( t \to \lambda \) of these PDFs is for small \( \lambda: \overline{\psi(\lambda)} \simeq 1 - (\lambda \overline{\tau})^{\alpha} \) where \( \overline{\tau} \) is a scaling constant. This is a simple stochastic model of a blinking quantum dot, for which typically \( \alpha = 1/2 \) though \( 1/2 < \alpha < 1 \) was also reported.

More general models where the random process \( I(t) \) has \( N \) internal states with possible different waiting time distributions can describe annealed trap models used for glass phenomenology \[10\] or for continuous time random walks describing motion of single molecules in live cells \[29\]. Our results for \( N > 2 \) are structurally similar to those for \( N = 2 \) but deserve their own discussion, which will be presented in a longer paper. For these models, we have derived detailed expressions for the power spectrum of the process \( I(t) \) and its statistical properties.

**Statement of the main results.** For \( \alpha < 1 \) the expectation value of the spectrum is not constant, but decreases with measurement time \( \langle S_\ell(\omega) \rangle \simeq t^{-\alpha}/\sigma_\ell(\omega). \) Expanding the \( t \)-independent function \( \sigma_\ell(\omega) \) for small frequencies \( \omega, \) one finds a typical non-integrable \( 1/f \)-noise

\[
\langle S_\ell(\omega) \rangle \simeq C \frac{t^{-\alpha-1}}{\omega^{2-\alpha}}. \tag{4}
\]

In general, the value \( S_\ell(\omega) \) of the spectrum is a fluctuating quantity even in the \( t \to \infty \) limit. The statistical behavior of the general class of processes for large \( t \) (for pairwise disjoint \( \omega_i \neq 0 \) is fully described by

\[
\left( \frac{S_\ell(\omega_1)}{\langle S_\ell(\omega_1) \rangle}, \ldots, \frac{S_\ell(\omega_n)}{\langle S_\ell(\omega_n) \rangle} \right) \to Y_\alpha \cdot (\xi_1, \ldots, \xi_n), \tag{5}
\]

where \( Y_\alpha \) is a random variable of normalized Mittag-Leffler distribution with exponent \( \alpha \) whose moments are \( \langle Y_\alpha^n \rangle = n! \Gamma(1+\alpha)^n / \Gamma(1+n\alpha). \) \[24\] The \( \xi_i \) are independent exponential random variables with unit mean. For \( \alpha = 1 \) the Mittag-Leffler random variable becomes \( Y_1 = 1, \) so that the powers \( S_\ell(\omega_i) \) of different frequencies become independent exponentially distributed random variables - a result known for several ergodic random processes \[25\]. In the case of weak ergodicity breaking \( (\alpha < 1), \) the whole spectrum obtains a common random prefactor \( Y_\alpha \) which shifts the complete observed spectrum.

Many procedures for the estimation of the spectrum from one finite time realization are designed to suppress the statistical fluctuations due to the uncorrelated random variables \( \xi_i. \) \[22, 23\]. These cannot account for the fluctuations of \( Y_\alpha, \) common to all estimators of a given realization. For these procedures the prefactor affects all estimated values for the spectrum. However, being a common prefactor, it does not affect the shape of the estimated spectrum so that features as \( 1/f \)-noise can be detected independently of the realization.
Motivation of the results. Whereas an exact derivation for the general N-state model is based on \cite{26} and will be presented in a longer paper, we report here on the 2-state model introduced above. To simplify our arguments further, we assume that after a waiting time the next state will be chosen randomly (i.e., a + state with probability 1/2).

Let $\tau_i$ be the $i$th waiting time and $\chi_i = \pm I_0$ the value taken during this waiting time. We denote by $T_j = \sum_{i=1}^{j-1} \tau_i$ the times at which one waiting time ends. If $n(t)$ is the number of completed waiting times up to time $t$, we can approximate by ignoring the waiting time in progress at $t$

\[
\int_0^t d\tau \exp(i\omega\tau) I(\tau) \simeq \sum_{j=1}^{n(t)} d_j(\omega)
\]

with $d_j(\omega) = i\chi_j \exp(i\omega T_j) \frac{1 - \exp(i\omega \tau_j)}{\omega}$. \hfill (6)

With this approximation and $\langle \chi_i \chi_j \rangle = \delta_{ij} I_0^2$ one obtains

\[
S_t(\omega) \simeq \frac{1}{t} \sum_{k,l=1}^{n(t)} d_k(\omega) d_l(-\omega).
\]

Assuming that $n(t)$ is for large $t$ independent of the waiting time of a single step, $\tau_i$, we get for the ensemble average

\[
\langle S_t(\omega) \rangle \simeq \frac{1}{t} \langle n(t) \rangle \langle d_1(\omega) d_1(-\omega) \rangle
\]

\[
\simeq I_0^2 \frac{1}{t} \langle n(t) \rangle 2 - \hat{\psi}(i\omega) - \hat{\psi}(-i\omega).
\]

It has been shown that $n(t) \simeq Y_0 t^\alpha / (\Gamma(1 + \alpha) t^\gamma)$ \cite{27}. Therefore

\[
\langle S_t(\omega) \rangle \simeq \frac{I_0^2 t^{\alpha-1}}{\Gamma(1 + \alpha) / \Gamma(1 + \alpha) t^\gamma} 2 - \hat{\psi}(i\omega) - \hat{\psi}(-i\omega)
\]

\[
\simeq \frac{2I_0^2 \cos(\alpha \pi/2)}{\Gamma(1 + \alpha)/\Gamma(1 + \alpha)} \frac{t^{\alpha-1}}{\omega^{2-\alpha}} \text{ as } \omega \to 0.
\]

The last line shows the typical $1/f$ noise \cite{21}. It is important that the observation limit $t \to \infty$ is taken before the frequency limit $\omega \to 0$.

We motivate the main result Eq. (7) with help of a random phase approximation. The random phase approximation assumes that terms of the form $\exp(i\omega T_j)$ are just random phases and any average over them vanishes. Especially, $\langle d_{j_1}(\nu_1) \cdots d_{j_n}(\nu_n) \rangle = 0$ if $\nu_1 T_{j_1} + \cdots + \nu_n T_{j_n} \neq 0$ for some $T_{j}$s (the $\nu$ being $\pm \omega$). Looking at the second moment of Eq. (7) and using (3):

\[
\langle S_t^2(\omega) \rangle \simeq \frac{1}{t^2} \langle \sum_{k,l,p,q=1}^{n(t)} d_k(\omega) d_l(-\omega) d_p(\omega) d_q(-\omega) \rangle
\]

\[
\simeq \frac{2}{t^2} \langle n(t)^2 \rangle \langle d_1(\omega) d_1(-\omega) \rangle^2
\]

\[
\simeq 2 \langle Y_0^2 \rangle \langle S_t(\omega) \rangle^2
\]

where we ignored terms with $k = l = p = q$ as there are only $\langle n(t) \rangle$ of them. The factor 2 stems from the fact that the sum in the first line of Eq. (10) has contributions for $k = l$, $p = q$ and for $k = q$, $l = p$. In contrast to this, for the term $\langle S_t(\omega_1) S_t(\omega_2) \rangle$ with $\omega_1 \neq \omega_2$ this symmetry factor will not be present. Following the same steps as in Eq. (10) gives

\[
\langle S_t(\omega_1) S_t(\omega_2) \rangle \simeq \langle Y_0^2 \rangle \langle S_t(\omega_1) \rangle \langle S_t(\omega_2) \rangle.
\]

This shows the equality of the second moments of Eq. (5).

The equality of the higher moments follows similarly by using combinatorial methods to determine these symmetry factors. We see that the random number of jumps $n(t)$ is responsible for the Mittag-Leffler fluctuations while the random phases generate the exponential noise. See Supplemental Material at [URL will be inserted by publisher] for a more detailed version of this approximation. The exact proofs for the general model will be published in a longer paper.

Numerical results. We simulated the two state model with $I_0 = 1$ for different length of time series and different $\alpha$. The waiting times were generated by using a uniformly distributed random number $0 < X \leq 1$ and setting $\tau = c_\alpha X^{-1/\alpha}$. The constant $c_\alpha$ was chosen such that $\langle n(1) \rangle \simeq 10000$. The ensemble consists of 10000
realizations of the time series.

In Fig. 1(a) we have plotted different realizations of \( S_t(f) \). The stochastic fluctuations inside and between the realizations are clearly observable. In Fig. 1(b) the ensemble average of the power spectrum for different lengths is plotted. The \( 1/f \) spectrum and its decay with observation time is clearly visible. Note that at very low frequencies we find \( S_t(\omega) \approx \text{const} \) independent of frequency - an effect we soon explain.

In a second step we want to check the statistical properties described by Eq. (11). To isolate the Mittag-Leffler fluctuations, we have calculated the spectrum for a fixed set of \( N \) frequencies \( \omega_i \) and determined the values

\[
M = \frac{1}{N} \sum_{i=1}^{N} \frac{S_i(\omega_i)}{\langle S_i(\omega_i) \rangle}.
\]  

(12)

As the exponential distributions are uncorrelated, they average out for sufficiently large \( N \) and the value taken by \( M \) should be distributed as the Mittag-Leffler distribution \( \mathcal{M} \). We have compared this for different \( \alpha \) values. The histogram of \( M \) values with the Mittag-Leffler density is shown in Fig. 2 for \( \alpha = 0.2, \alpha = 0.5 \) and \( \alpha = 0.8 \). A good agreement with the theory is apparent.

Removing the non integrability paradox of \( 1/f \) noise.

As mentioned in the introduction, the \( 1/\omega^{2-\alpha} \) noise is non integrable, \( \int_0^{\infty} \langle S(\omega) \rangle d\omega = \infty \), due to the low frequency behavior. This in turn violates the simple bound we have found. To start understanding this behavior notice that the random phase approximation breaks down when \( \omega = 0 \), as the phase \( \omega T_m \) is clearly non random. Hence the distribution of the power spectrum in Eq. (15) is not valid for \( \omega = 0 \) and this case must be treated separately.

For \( \omega = 0 \) we have for a single realization \( S_t(0) = T^2 t \) with the time average \( T = \int_0^t T(t') dt'/t \). For ergodic processes the time average \( \langle T \rangle \) is equal to the ensemble average \( \langle I \rangle \). However, for non ergodic processes under investigation, the time average \( T \) remains a random variable even in the infinite time limit [20–22]. For the two state process introduced above, we have \( T = I_0(T^+ - T^-)/t \) where \( T^\pm \) is the total time spent in state up or down. The value of \( T \) follows an arcsine like distribution [20–22, 28]. This simply means that for a given realization, the system will spend most of the time either in state \( u \) or in state \( d \), and hence \( T \) is random, which would not be the case for an ergodic process.

This has a consequence for the non integrability of the power spectrum. As \( S_t(0) = T^2 t \), the spectrum at zero frequency tends to infinity, but for any finite measurement time it is finite. For the two state model we have on average \( \langle S_t(0) \rangle = I_0^2 (1 - \alpha) t \). So indeed theoretically there is a low-frequency cut-off of the divergence of the \( 1/f \) spectrum, and we now define a crossover frequency \( \omega_c \) for the transition between the zero frequency limit, where arcsine statistics takes control (failure of random phase approximation), and higher frequencies where the Mittag-Leffler statistics takes control. This frequency is defined by merging the two behaviours,

\[
\langle S_t(\omega_c) \rangle \approx \frac{C}{\omega_c^{\alpha-1}} = \langle S_t(0) \rangle.
\]  

(13)

We see that

\[
\omega_c = \left( \frac{C}{\langle T \rangle} \right)^{1/(2-\alpha)} \frac{1}{t}.
\]  

(14)

The values of \( \langle T \rangle \) and \( C \) can be obtained from measurements, or from theory, for example for the two state model \( \langle T \rangle = I_0^2 (1 - \alpha) \) and \( C = 2I_0^2 \cos(\alpha \pi/2)/\Gamma(1 + \alpha) \). More importantly, we see that the crossover frequency depends on the measurement time as \( 1/t \). Although at first sight surprising, this is the only way how such a cross-over can take place in the absence of a characteristic time scale for dynamics: measurement time itself sets the time scale for crossover. Additionally, \( 1/t \) appears as the frequency resolution of the discrete Fourier transform typically used in spectral analysis. Importantly, experiments report a lowest frequency at \( f = 1/t \).

We see that increasing measurement time merely stretches the domain of frequency where the \( 1/f \) noise is observed, which is clearly seen in the numerical simulations (see Fig. 1(b)). There is no point in increasing measurement time in order to identify better the crossover, since a time independent crossover frequency does not exist. Thus the \( 1/f \) noise stretches to the lowest frequencies compatible with measurement time (of the order of \( 1/t \)). This resolves the non integrability paradox. The
amplitude of the power spectrum itself is also decreasing in time, in such a way that integrability is maintained. Namely
\[
\int_0^\infty \langle S_t(\omega) \rangle d\omega \approx (T^2)t\omega_c + \int_{\omega_c}^\infty C t^{\alpha-1}/\omega^{2-\alpha} d\omega \\
= \frac{2-\alpha}{1-\alpha} \left( \frac{T^2}{1-\alpha} C \right)^{1/(\alpha-2)}
\]
(15)
is indeed finite and time independent.

Thus we conclude that the power spectrum is integrable as it should. This seems to indicate the generality of our results, since a crossover frequency is only found in few experiments. From a different angle, assuming that the natural frequency is also the limit of measurement \(\omega_c \sim 1/t\), we must demand the decrease of the amplitude of power spectrum with time to maintain integrability as required for bounded signals. This together with the universal fluctuations of \(1/f\) noise (Eq. (5)) are strong fingerprints of power law intermittency. The tools developed here can be tested in a vast number of physical systems.

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* Electronic address: markus.niemann@uni-oldenburg.de

[1] M. S. Keshner, Proceedings of the IEEE 70 212 (1982).
[2] F. N. Hooge, Physica 60, 130 (1972).
[3] J. Bernamont, Proc. Phys. Soc. 49 (extra part) 138 (1937).
[4] M. A. Caloyannides, J. Appl. Phys. 45, 307 (1974).
[5] B. B. Mandelbrot, J. R. Wallis, Water Resources Res. 5, 321 (1969).
[6] B. B. Mandelbrot, IEEE Trans. Infor. Theory IT-13 289 (1967).
[7] M. Pelton, D. G. Grier, and P. Guyot-Sionnest Appl. Phys. Lett. 85 819 (2004).

[8] M. Pelton, G. Smith, N. F. Scherer and R. A. Marcus Proc. National Academy of Science 104 14249 (2007).
[9] L. Silvestri, L. Fronzoni, P. Grigolini, and P. Allegrini Phys. Rev. Lett. 102, 014502 (2009).
[10] P. Manneville, J. Physique 41, 1235 (1980).
[11] T. Geisel, A. Zacherl, G. Radons, Phys. Rev. Lett. 59, 2503 (1987).
[12] G. Zumofen, J. Klafter, Physica D 69, 436 (1993).
[13] A. Ben-Mizrachi, I. Procaccia, N. Rosenberg, A. Schmidt, H. G. Schuster, Phys. Rev. A 31, 1830 (1985).
[14] T. Geisel, J. Nierwetberg, A. Zacherl, Phys. Rev. Lett. 54, 616 (1985).
[15] H. Scher, G. Margolin, R. Metzler, J. Klafter, B. Berkowitz, Geophys. Res. Lett. 2002, 1061 (2002).
[16] P. Frantsuzov, M. Kuno, B. Jánko, R. A. Marcus, Nature Physics 4, 519 (2008).
[17] F. D. Stefani, J. P. Hoogenboom, E. Barkai, Physics today 62, 35 (2009).
[18] X. Brokmann, J.-P. Hermier, G. Messin, P. Desbiolles, J.-P. Bouchaud, M. Dahan, Phys. Rev. Lett. 90, 126001 (2003).
[19] J. P. Bouchaud, Journal de Physique I 2, 1705 (1992).
[20] G. Margolin, E. Barkai, Phys. Rev. Lett. 94, 080601 (2005).
[21] G. Margolin, and E. Barkai J. of Statistical Physics 122 137 (2006).
[22] W. H. Press, S. A. Teukolsky, W. T. Vetterling, B. P. Flannery, Numerical Recipes in FORTRAN - The Art of Scientific Computing, 2nd ed., Cambridge University Press, Cambridge, 1992
[23] E. Barkai, Y. Garini, R. Metzler, Physics today 65, 29 (2012).
[24] W. Feller, An Introduction to Probability Theory and Its Applications - Volume II, John Wiley & Sons, New York (1971).
[25] M. B. Priestley, Spectral Analysis and Time Series - Volume 1: Univariate Series, Academic Press, London, 1981.
[26] M. Niemann, I. G. Szendro, H. Kantz, Chem. Phys. 375 370 (2010).
[27] J.-P. Bouchaud, A. Georges, Phys. Rep. 195, 127 (1990).
[28] A. Rebenshtok, E. Barkai, Phys. Rev. Lett. 99, 210601 (2007).
[29] C. Godrèche, J. M. Luck, J. Stat. Phys. 104, 489 (2001).
[30] A. Rebenshtok, E. Barkai, J. Stat. Phys. 133, 565 (2008).
Supplementary material to:
Fluctuations of $1/f$ noise and the low frequency cutoff paradox

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This supplementary material contains a more detailed derivation of the main results. We will still work with the approximations made in the main text – an exact proof will be published in a longer paper.

Using the same notations as in the main text we see

\[ d_j(\omega) = \int_{T_j}^{T_{j+1}} d\tau \chi_j \exp(i\omega\tau) \]
\[ = i\chi_j \exp(i\omega T_j) \frac{1 - \exp(i\omega T_j)}{\omega}. \]  

(S1)

With this, we get for the Fourier transform of the signal by neglecting the last waiting time in progress at $t$:

\[
\int_0^t d\tau \exp(i\omega\tau) I(\tau) = \sum_{j=1}^{n(t)} \int_{T_j}^{T_{j+1}} d\tau \chi_j \exp(i\omega\tau) + \int_{T_{n(t)+1}}^{t} d\tau \chi_j \exp(i\omega\tau) \\
\approx \sum_{j=1}^{n(t)} d_j(\omega),
\]

(S2)

where $n(t)$ denotes the number of finished waiting times up to time $t$ also called the number of renewals in $(0,t)$.

With this approximation we get for the spectrum

\[ S_t(\omega) \approx \frac{1}{t} \sum_{k,l=1}^{n(t)} d_k(\omega) d_l(-\omega). \]  

(S3)

We now assume that the $T_i$ are independent from the $\tau_j$ and in turn the average of $n(t)$ can be performed independently from the average over the $\tau_j$. Using $\langle \chi_i \chi_j \rangle = \delta_{ij} I_0^2$ one obtains (where a distinction is necessary, we write the variables whose average is taken as an index)

\[
\langle S_t(\omega) \rangle \approx \frac{1}{t} \left\langle \sum_{k,l=1}^{n(t)} d_k(\omega) d_l(-\omega) \right\rangle \\
\approx \frac{1}{t} \left\langle \sum_{k,l=1}^{n(t)} \chi_k \chi_l \exp \left( i\omega(T_k - T_l) \right) \frac{1 - \exp(i\omega\tau_k)}{\omega} \frac{1 - \exp(-i\omega\tau_l)}{\omega} \right\rangle \\
= I_0^2 \frac{1}{t} \left\langle \sum_{k=1}^{n(t)} \frac{2 - \exp(i\omega\tau_k) - \exp(-i\omega\tau_k)}{\omega^2} \right\rangle \\
\approx I_0^2 \left\langle n(t) \right\rangle \frac{2 - \hat{\psi}(i\omega) - \hat{\psi}(-i\omega)}{\omega^2} \\
\approx \frac{\left\langle n(t) \right\rangle}{t} (d_1(\omega) d_1(-\omega)).
\]

(S4)

It has been shown that $n(t) \approx Y_\alpha t^\alpha / (\Gamma(1 + \alpha) \tau^\alpha)$ from which we get $\langle n(t) \rangle \approx t^\alpha / (\Gamma(1 + \alpha) \tau^\alpha)$

\[
\langle S_t(\omega) \rangle \approx \frac{I_0^2 t^{\alpha - 1}}{\Gamma(1 + \alpha) \tau^\alpha} \frac{2 - \hat{\psi}(i\omega) - \hat{\psi}(-i\omega)}{\omega^2}.
\]  

(S5)

Here $Y_\alpha$ is a random variable of Mittag-Leffler distribution with unit mean which is defined by its moments $\langle Y_\alpha^n \rangle = n!\Gamma(1 + \alpha)^2/\Gamma(1 + na\alpha)$. Using a random variable $L_\alpha$ of totally asymmetric Lévy stable distribution with exponent $\alpha$ and distribution $l_\alpha(\xi)$

\[
\langle \exp(-\lambda L_\alpha) \rangle = \int_0^\infty d\xi \exp(-\lambda\xi) l_\alpha(\xi) = \exp(-\lambda^\alpha),
\]

(S6)
a representation of \(Y_\alpha\) can be obtained by
\[
Y_\alpha = \Gamma(1 + \alpha)L_{\alpha-\alpha}^{-}\alpha
\] (S7)
which is equivalent of \(Y_\alpha\) having the probability density function
\[
y_\alpha(\xi) = \frac{\Gamma^{1/\alpha}(1 + \alpha)}{\alpha\xi^{1+1/\alpha}} I_\alpha \left[ \frac{\Gamma^{1/\alpha}(1 + \alpha)}{\xi^{1/\alpha}} \right].
\] (S8)

Plots of \(y_\alpha(\xi)\) are used in Fig. 2 for various \(\alpha\).

For small \(\lambda\) we have the expansion in the Laplace variable \(\hat{\psi}(\lambda) \simeq 1 - (\alpha\lambda)^{\alpha}\). By analytic continuation (putting the branch cut to the negative real axis such that it is not crossed) we have
\[
(i\omega)^{\alpha} = |\omega|^\alpha \left( \cos \left( \frac{\pi}{2\alpha} \right) + i \text{sgn}(\omega) \sin \left( \frac{\pi}{2\alpha} \right) \right)
\] (S9)
such that for small \(\omega\)
\[
1 - \hat{\psi}(i\omega) \simeq \Gamma |\omega|^\alpha \left( \cos \left( \frac{\pi}{2\alpha} \right) + i \text{sgn}(\omega) \sin \left( \frac{\pi}{2\alpha} \right) \right)
\] (S10)
and
\[
\langle S_t(\omega) \rangle \simeq \frac{2T^2_0 \cos(\alpha\pi/2)}{\Gamma(1 + \alpha)} \frac{t^{\alpha-1}}{|\omega|^{2-\alpha}} \text{ as } \omega \to 0.
\] (S11)

It is important, that the observation limit \(t \to \infty\) is taken before the frequency limit \(\omega \to 0\).

The Eq. (5) is shown by checking the equality of the moments on both sides. Assuming that we have some natural numbers \(p_1, \ldots, p_q\) (\(P = p_1 + \cdots + p_q\)), we need to show
\[
\left\langle \prod_{j=1}^{q} \left( \frac{S_t(\omega_j)}{\langle S_t(\omega_j) \rangle} \right)^{p_j} \right\rangle \overset{!}{=} \left\langle \prod_{j=1}^{q} (Y_\alpha \xi_j)^{p_j} \right\rangle
\] (S12)
\[
= \langle Y_\alpha^P \rangle \prod_{j=1}^{q} \langle \xi_j^{p_j} \rangle
\]
\[
= \langle Y_\alpha^P \rangle \prod_{j=1}^{q} p_j!.
\]

Here, we denote by \(\overset{!}{=}\) an equality which needs to be proven. We have used the fact that the \(\xi_i\) are independent random variables which are exponentially distributed with unit mean, i.e., \(\langle \xi_i^n \rangle = n!\). Especially, we have to show for the second moments \((\omega_1 \neq \omega_2)\)
\[
\left\langle \left( \frac{S_t(\omega)}{\langle S_t(\omega) \rangle} \right)^2 \right\rangle \overset{!}{=} 2\langle Y_\alpha^2 \rangle
\] (S13)
and
\[
\left\langle \frac{S_t(\omega_1)}{\langle S_t(\omega_1) \rangle} \frac{S_t(\omega_2)}{\langle S_t(\omega_2) \rangle} \right\rangle \overset{!}{=} \langle Y_\alpha^2 \rangle.
\] (S14)

We motivate this with help of a random phase approximation. The random phase approximation assumes that terms of the form \(\exp(i\omega T_j)\) are just random phases and any average over them vanishes. Especially, the phase factor of \(d_{j_1}(\nu_1) \cdots d_{j_n}(\nu_n)\) is \(\exp(i[\nu_1 T_{j_1} + \cdots + \nu_n T_{j_n}])\) (the \(\nu_i\) take values of \(\pm \omega\)). Therefore, we assume
\begin{align}
\langle d_{j_1}(\nu_1) \cdots d_{j_n}(\nu_n) \rangle &= 0 \text{ if } \nu_1 T_{j_1} + \cdots + \nu_n T_{j_n} \neq 0 \text{ for some } T_j. \text{ Looking at } \\
\langle S_t^2(\omega) \rangle &\approx \frac{1}{t^2} \left\langle \sum_{k,l,p,q=1}^{n(t)} d_k(\omega)d_l(\omega)d_p(\omega)d_q(\omega) \right\rangle \\
&\approx \frac{1}{t^2} \left\langle \sum_{k,l,p,q=1}^{n(t)} (d_k(\omega)d_l(\omega)d_p(\omega)d_q(\omega))_{\chi_1,T_1,\tau_1} \right\rangle_{n(t)} \\
&\approx \frac{1}{t^2} \left\langle \sum_{k,l,p,q=1}^{n(t)} (\chi_k \chi_l \chi_p \chi_q)_{\chi_1} \left\langle \frac{1 - \exp(i\omega \tau_k)}{\omega} - \frac{1 - \exp(-i\omega \tau_l)}{\omega} - \frac{1 - \exp(i\omega \tau_p)}{\omega} - \frac{1 - \exp(-i\omega \tau_q)}{\omega} \right\rangle_{\tau_1} \right. \\
&\left. \langle \exp(i\omega (T_k - T_l + T_p - T_q)) \rangle_{T_1} \right\rangle_{n(t)}. 
\end{align}

In our approximation, the non vanishing terms in the sum must have the property \(\omega(T_k - T_l + T_p - T_q) = 0\) for any realization. This is possible either for \(k = l\) and \(p = q\), or for \(k = q\) and \(p = l\). In other words, the random phase approximation boils down to

\begin{equation}
\langle \exp(i\omega (T_k - T_l + T_p - T_q)) \rangle_{T_1} \simeq \delta_{k\ell} \delta_{pq} + \delta_{kq} \delta_{pl}. 
\end{equation}

Plugging this back into Eq. (S15) and using (S4) gives

\begin{align}
\langle S_t^2(\omega) \rangle &\approx \frac{2}{t^2} \left\langle \sum_{k,p=1}^{n(t)} (d_k(\omega)d_k(\omega)) (d_p(\omega)d_p(\omega)) \right\rangle + \frac{1}{t^2} \left\langle \sum_{k=1}^{n(t)} (d_k(\omega)d_k(\omega)d_k(\omega)d_k(\omega)) \right\rangle \\
&\approx \frac{2}{t^2} \langle n(t)(n(t) - 1) \rangle (d_1(\omega)d_1(\omega))^2 + \frac{1}{t^2} \langle n(t) \rangle (d_1(\omega)d_1(\omega)d_1(\omega)d_1(\omega)) \\
&\approx \frac{2}{t^2} \langle n(t)^2 \rangle (d_1(\omega)d_1(\omega))^2 \\
&\approx 2 \langle Y^2 \rangle \langle S_t(\omega) \rangle^2.
\end{align}

where we ignored terms of the order \(n(t)\) as for large \(t\) the leading order is \(n(t)^2\). This shows Eq. (S16).

In contrast to this, for the term \(\langle S_t(\omega_1) S_t(\omega_2) \rangle\) with \(\omega_1 \neq \omega_2\) we get

\begin{align}
\langle S_t(\omega_1) S_t(\omega_2) \rangle &\approx \frac{1}{t^2} \left\langle \sum_{k,l,p,q=1}^{n(t)} d_k(\omega_1)d_l(\omega_1)d_p(\omega_2)d_q(\omega_2) \right\rangle \\
&\approx \frac{1}{t^2} \left\langle \sum_{k,l,p,q=1}^{n(t)} (d_k(\omega_1)d_l(\omega_1)d_p(\omega_2)d_q(\omega_2))_{\chi_1,T_1,\tau_1} \right\rangle_{n(t)} \\
&\approx \frac{1}{t^2} \left\langle \sum_{k,l,p,q=1}^{n(t)} (\chi_k \chi_l \chi_p \chi_q)_{\chi_1} \left\langle \frac{1 - \exp(i\omega_1 \tau_k)}{\omega_1} - \frac{1 - \exp(-i\omega_1 \tau_l)}{\omega_1} - \frac{1 - \exp(i\omega_2 \tau_p)}{\omega_2} - \frac{1 - \exp(-i\omega_2 \tau_q)}{\omega_2} \right\rangle_{\tau_1} \right. \\
&\left. \langle \exp(i\omega_1 (T_k - T_l) + i\omega_2 (T_p - T_q)) \rangle_{T_1} \right\rangle_{n(t)}. 
\end{align}

This time, the non vanishing terms in the sum must have the property \(\omega_1(T_k - T_l) + \omega_2(T_p - T_q) = 0\) which is possible only for \(k = l\) and \(p = q\):

\begin{equation}
\langle \exp(i\omega_1 (T_k - T_l) + i\omega_2 (T_p - T_q)) \rangle_{T_1} \simeq \delta_{k\ell} \delta_{pq}. 
\end{equation}
We get with this approximation

\[
(S_t(\omega_1)S_t(\omega_2)) \approx \frac{1}{t^2} \left( \sum_{k,p=1}^{n(t)} \langle dk_1(\omega_1)dk_2(\omega_1)dp_1(\omega_2)dp_2(\omega_2) \rangle \right) + \frac{1}{t^2} \left( \sum_{k=1}^{n(t)} \langle dk_1(\omega_1)dk_2(\omega_2)dp_1(\omega_2)dp_2(\omega_2) \rangle \right)
\]

\[
\approx \frac{1}{t^2} \langle n(t)(n(t) - 1) \rangle \langle d_1(\omega_1) d_1(-\omega_1) \rangle \langle d_1(\omega_2) d_1(-\omega_2) \rangle + \frac{1}{t^2} \langle n(t) \rangle \langle d_1(\omega_1) d_1(-\omega_1) d_1(\omega_2) d_1(-\omega_2) \rangle
\]

\[
\approx \langle Y_2^2 \rangle \langle S_t(\omega_1) \rangle \langle S_t(\omega_2) \rangle.
\]

(S20)

This shows Eq. (S14) and therefore with Eq. (S13) the equality of the second moments of Eq. (S5). Similar calculations can be performed in general for higher order moments giving rise to Eq. (S5).

[1] G. Margolin, and E. Barkai, J. of Statistical Physics 122, 137 (2006).
[2] C. Godrèche, J. M. Luck, J. Stat. Phys. 104, 489 (2001).
[3] J.-P. Bouchaud, A. Georges, Phys. Rep. 195, 127 (1990).
[4] Y. He, S. Burov, R. Metzler, E. Barkai, Phys. Rev. Lett. 101, 058101 (2008).