A classical solution in SU(2) Yang-Mills gauge theory

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Abstract
A solution of euclidean Yang-Mills gauge theory, which is governed by 
\( \pi_4(SU(2)) \), is given.

1 Introduction
A new classical solution of the SU(2) Yang-Mills gauge theory is given. It is based on the homotopy group 
\( \pi_4(SU(2)) = \pi_4(S^3) = \mathbb{Z}_2 \). Various ansätze are discussed, either selfdual as in the case of instantons \([17]\) with finite action but zero energy or in Minkowski space with finite energy as in the meron case \([12]\). We will also investigate whether the Corrigan-Fairlie ansatz works, i.e., finding a \( \phi^4 \) solution and plugging it into the general ansatz given by them \([16]\).

2 Main part
In four dimensional euclidean Yang-Mills gauge theory, the base space (refering to the language of principal fiber bundles) \( R^4 \) can be compactified to the fourdimensional sphere \( S^4 \) because pure Yang-Mills gauge theory is conformally invariant and the metric of the fourdimensional sphere is conformally flat:

\[
g_{\mu\nu} = \Omega(x)\eta_{\mu\nu}
\]

where \( g \) is the metric of the sphere \( S^4 \), \( \eta \) the metric of \( R^4 \) (flat metric) and \( \Omega \) the conformal transformation. \([18]\) Since the local gauge transformations \( g \) (or gauge transformations of the second
kind) define mappings from the base space of the principal fiber bundle into its structure group SU(N) (say), they are classified by $\pi_4(SU(N))$ which is equal to $\mathbb{Z}_2$ in the case $N = 2$, otherwise 0. [15] In the case of instantons [13] and merons one employs boundary conditions, that enforce mappings from the equator of the base space $S^4$, that is $S^3$, to SU(N). These are classified by $\pi_3(SU(N)) = \pi_3(SU(2)) = \pi_3(S^3) = \mathbb{Z}$ because of Bott periodicity (for instance D.Husemoller, fiber bundles). The solutions are then classified by elements of $\pi_3(S^3) = \mathbb{Z}$, which is called their topological charge $k$. In the case of instantons, selfduality of the curvature tensor $F_{\mu\nu}$ is demanded, to have finite action solutions (and zero energy), because it is believed, that these dominate the euclidean path integral in the case of semiclassical calculations. [14] The instanton solution with $k = 1$ is given by

$$A_\mu = \frac{i\sigma_{\mu\nu}x_\mu x_\nu}{x^2 + \lambda^2} g^{-1} \partial_\mu g$$

where $\lambda$ is the scaling factor (pure Yang-Mills gauge theory is scale invariant and for merons with $k = 1$

$$A_\mu = \frac{1}{2}g^{-1} \partial_\mu g$$

(due to the factor $1/2$ this is not a pure gauge, hence $F_{\mu\nu}$ is not zero, and gives a solution with finite energy [12]. (We can generalize the factor $1/2$ to $m/n$ with $m \neq n$) The gauge transformations $g$ are representants of $\pi_3(S^3)$, and hence mappings from $S^3$ to $S^3$ with mapping degree 1 while $g^{-1} \partial_\mu g$ is the pullback (see Bott, Differential forms in Algebraic Topology) Explicit formulas for the representants shall be given in a forthcoming paper. To find first solutions one can replace the $[g]$ of $\pi_3(S^3)$ with $[g]$ out of $\pi_4(S^3)$ in the instanton and meron solution ($[g]$ means equivalence class of the mapping $g$). These have then topological charge out of $\mathbb{Z}_2$. Further solutions can be found by employing the theorem of Corrigan and Fairlie [16]: $A_\mu = i\sigma_{\mu\nu} \partial_\nu \ln \phi$ is a solution of the Yang-Mills gauge theory, if $\phi$ is a solution of the $\phi^4$ theory. There is a relationship between $\pi_4(S^3)$ and braid groups. [21] Also there is a lot to be said about $\pi_4(SU(2))$ itself. [15] These two items and further solutions shall be investigated in a forthcoming paper. An explicit, nontrivial representant of the homotopy group $\pi_4(SU(2))$ can be found by suspending the Hopf map, i.e., a representant of $\pi_3(SU(2))$. [10] Although $\pi_4(SU(N))$ is zero for $N$ larger than 2 [20], the author wonders whether there is any connection to the confinement problem and triality ($Z_3$ vortices). An explicit representation of the nontrivial element of $\pi_4(S^3)$ was given by D.Friedan [10] based on a paper written by T.Puettmann and Rigas. [15]. In what follows, we will describe this
mapping and find an interesting reformulation of it, linking it to Skyrmes Hedgehog solution [3]. For the ansatz of the vector potential $A^a_\mu$ we had chosen the generalized meron ansatz

$$A^a_\mu = \frac{m}{n} \partial_\mu gg^{-1}$$

The pre factor $\frac{m}{n}$ guarantees a non vanishing field strength $F_{\mu\nu}$ For convenience and without breaking generality we choose this pre factor as $\frac{1}{\sqrt{8}}$. The Yang-Mills Lagrangian then reads as

$$L = \frac{1}{32} \int dx^4 [\partial_\mu gg^{-1}, \partial_\nu gg^{-1}]^2$$

which is exactly the term Skyrme added to the nonlinear $\sigma$ model to stabilize soliton solutions. The mapping $g$ will be given now $g(x, z)$

$$= \cos \varphi(x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i \sin \varphi(x) \begin{pmatrix} |z_1|^2 - |z_2|^2 & 2z_1 \bar{z_2} \\ 2z_1 \bar{z_2} & -|z_1|^2 + |z_2|^2 \end{pmatrix} \cos \varphi(x)$$

$$= \frac{1}{2} i \begin{pmatrix} \bar{z}_1 \bar{z}_2 & \bar{z}_1 \bar{z}_2 \\ \bar{z}_1 \bar{z}_2 & \bar{z}_1 \bar{z}_2 \end{pmatrix}$$

where $z = (z_1, z_2) \epsilon C^2$ and $x \in \mathbb{R}$ and $z_1 = x_1 + i x_2$ and $z_2 = x_3 + i x_4$. Evaluating the matrix in the exponent according to the Lie algebra of $SU(2)$, i.e., the Pauli matrices $\sigma^a$, $a = 1, 2, 3$, $\phi(x, z)$ can be written as follows:

$$\phi(x, z) = \exp[i \sigma^a \zeta^a \varphi(x)]$$

where the $\zeta^a$ constitute the Hopf map

$$f : S^3 \to S^2$$

with Hopf invariant 1 (Heinz Hopf, Collected papers, Springer, Berlin) with

$$\zeta_1 = 2(x_1 x_3 + x_2 x_4) = \sin \theta \cos \varphi$$

$$\zeta_2 = 2(x_1 x_4 - x_2 x_3) = \sin \theta \sin \varphi$$

$$\zeta_3 = x_1^2 + x_2^2 - x_3^2 - x_4^2 = \cos \varphi$$

The $x_i$, $i = 1, 2, 3, 4$, are the coordinates of the Euclidean space, in which $S^3$ is embedded. $\varphi(x)$ obeys the following boundary conditions:

$$\varphi(-\infty) = \pi$$

$$\varphi(\infty) = 0$$

($\varphi$ is often called the chiral angle.) This representation reflects better the non-abelian character of the mapping $\phi$. We then have

$$\phi(x, z) = \cos(\varphi(x)) + i(\sigma^1 \sin \theta \cos \varphi$$

$$+ \sigma^2 \sin \theta \sin \varphi + \sigma^3 \cos \varphi) \sin(\varphi(x))$$

This is Skyrmes Hedgehog field [3]. The function $\varphi(x)$ and the angle $\varphi$ are different objects. It is Skyrmes hedgehog solution on a second level, because the $\zeta^a$ coordinates have to be replaced by the $x^i$, $i = 1, 2, 3, 4$, coordinates. Hence, in contrast to the Skyrme hedgehog solution, our new solution is explicitly (Euclidean) time dependent. The solution for the vector potential $A^a_\mu$ with $A^a_\mu = -i Tr(\sigma^a A)$ is then

$$A^a_\mu = \frac{1}{2}(\zeta^a(x_i) \partial_\mu \varphi(x) + (\sin(\varphi(x)) \partial_\mu \zeta^a(x)$$

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\[
+ \sin^2(\phi(x)) \epsilon_{abc} \partial_\mu \zeta^b(x_i) \zeta^c(x_i)
\]

with \(x_i\) as above and \(x \in \mathbb{R}\).

\(\phi(x)\) is determined by solving the equation of motion

\[
\partial_\mu [A, A] + [[A, A], A] = 0
\]

or by minimizing the energy with respect to \(F(x)\):

\[
M = \frac{s^2}{2} (2(F')^2 + \frac{s^2}{x})
\]

with \(s = \sin(F(x))\). Minimizing \(M\) with respect to \(F(x)\) gives

\[
2s^2F'' + 2sc(F')^2 - \frac{2s^3c}{x^2} = 0
\]

and \(c = \cos(F(x))\). This equation will give a profile function \(F(x)\) similar to the Skyrme model, but the exact form has to be determined by computer. It will be given in the following paper. The new solution could also play a role in the Electro-Weak theory (GSW-Theory), since its gauge group is 

\[
G_{GSW} = SU(2) \times U(1).
\]

Hence we have

\[
\pi_4(G_{GSW}) = \pi_4(SU(2) \times U(1))
\]

\[
= \pi_4(SU(2))
\]

and so

\[
\pi_4(G_{GSW}) = Z_2
\]

Furthermore, because of the identity

\[
\pi_n(S^2) = \pi_n(S^3)
\]

for \(n \geq 3\) [21] But

\[
S^2 = SU(2)/U(1)
\]

which is isomorphic to the vacuum configurations of the isotriplet Higgs scalars of the GSW theory.

### 2.1 Spin structure and \(\pi_4(SU(2))\)

The difference of the two representant mappings of the two equivalence classes of \(\pi_4(SU(2))\) can be described by the spin structures on the circles of \(S^4\) which are the inverse images of our representant mappings above. [19]

Spin structures are classified by the first Stiefel Whitney class \(H^1(M, Z_2)\) where \(M\) is the manifold under consideration. In our case these are the circles \(S^1\), which are the inverse images back from \(SU(2) = S^3\), covering \(S^4\) (similar to the Hopf bundle case, where the Hopf invariant counts the linking number of the circles covering \(S^3\). There is an analog in our \(S^4\) case to be described below by cobordism theory.) Now

\[
H^1(S^1, Z_2) = Hom(H_1(S^1, Z_2)
\]

\[
= Hom(\pi_1(S^1), Z_2)
\]

The second isomorphism comes about by the Hurwitz isomorphism [20] But this sequence links the classification of the spin structures on the collection of circles on \(S^4\) to the inverse images of

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the generic mappings \( \phi : S^4 \to S^3 \)
which make up the collection of circles covering \( S^4 \). But these are the representatives of the equivalence classes of \( \pi_4(SU(3)) = \pi_4(S^3) \). Hence the mappings \( \phi : S^4 \to S^3 \) are classified by the spin structures on the generic inverse images, the circles covering \( S^4 \).

2.2 Braids, Knots and \( \pi_4(SU(2)) \)

According to the work of the South-Korean mathematician Jie Wu [21] there is a link between a combinatoric group \( G(n) \) to be defined in the appendix and homotopy groups \( \pi_n(S^2) \) for \( n > 2 \) and hence by the Hopf mapping a link with \( \pi_n(S^3) \) and \( G(n) \) for \( n > 2 \). What he shows is, that \( Z(G(n)) \), the center of \( G(n) \), is isomorphic to the \( n \)-th homotopy group of the 2-sphere and so \( Z(G(n)) \) is isomorphic to the \( n \)-th homotopy group of the 3-sphere for \( n \leq 2 \) by the Hopf fibration. From our knowledge about especially \( \pi_4(S^3) \) we know now, that \( Z(G(n)) \) is isomorphic to \( Z_2 \). The connection with Artins braid and pure braid group [7] comes into the game by the further results, Wu found: Let \( B_n \) be Artins braid group for \( n \) strands and \( P_n \) the pure Artin braid group (see appendix for definitions). Then:

1. The set of fixed points of \( B_n \) action on \( G_n \), as a group, is isomorphic to the subgroup of \( \pi_n(S^2) \) consisting of elements of order two.

2. The set of fixed points of \( P_n \) on \( G(n) \), as a group, is isomorphic to \( \pi_n(S^2) \).

They should also play a prominent role in the two dimensional non-linear sigma model, since the instanton solutions are directly linked with the the Hopf bundle (see above at the end of the section about the sigma model) Braid groups and knots played already a longer time in Yang-Mills theory, Quantum gravity and generally in quantum field theory a role (see [22]), but it seems that the new classical solution characterized by \( \pi_4(SU(2)) \) gives a more natural link with this subject, while before it seemed to be introduced artificially. Closer examination of this link shall be the subject of further investigation later on.

Another aspect, how knot theory might enter is the augmentation of the standard Yang-Mills action

\[
S = \frac{1}{4} \int d^4 x F F
\]

by a term

\[
\frac{\theta}{32(\pi)^2} \int d^4 x F^* F
\]

where \( F \) is the Yang-Mills curvature and the augmentation term is the well-known expression for the second Chern
class times the factor $\frac{2}{7}$, which can be written as a total differential

$$\int F^* F = \int d^4 x (\partial (A \wedge dA + A \wedge A \wedge A))$$

But this is nothing but equal to

$$\int_{S^3} d\sigma (A \wedge dA + A \wedge A \wedge A)$$

which is the Chern-Simon Lagrangian [?] of which Witten [?] showed that it gives a derivation of the Jones Polynomials [?], an invariant characterizing knots. Mind the first term in the Chern Simons Lagrangian (a so called secondary characteristic class [?])

$$l = \int_{S^3} d\sigma (A \wedge dA)$$

which is nothing but the Hopf linking number for the Hopf fiber bundle giving the linking number of the inverse images of the Hopf mapping $\pi : S^3 \rightarrow S^2$. the inverse images are the fibers of the Hopf bundle, being circles, here on $S^3$, the bundle space of the Hopf bundle. So l gives us the linking number of the inverse images of the Hopf mapping. That means taking a U(1) gauge theory and hence the gauge potential $A$, being an element of the Lie algebra of the gauge group, abelian, the second term in the Chern Simons Lagrangian vanishes and we are left the expression for the Hopf linking number as the Lagrangian. (plus the convetional term $\int F F d^4 x$).

### 2.3 Skyrme model and nonlinear $\sigma$ model

Houghtom et al. [4] made the suggestion to take the similarities of magnetic monopoles and skyrmions literally and adapt an idea of Donaldson [5] to classify and determine t’Hooft magnetic monopole solutions by introducing rational maps $R : S^2 \rightarrow S^2$. In a seminal paper Arafune et al. [7] showed, that in a certain gauge, the t'Hooft-Polyakov monopoles [8] are topologically classified by the Brouwer degree, the mapping being the Higgs field

$$n = \frac{1}{8\pi e} \int_{S^3} \epsilon_{ijk} \epsilon_{abc} \phi^a \partial_j \phi^b \partial_k \phi^c (d^2 \sigma)_i$$

where the $\phi^a$ are a triple of unit length Higgs fields because we look at a SO(3) Yang-Mills Higgs theory. The field theory is defined on $R^3$ but the origin can be deleted to avoid singularities. But $R^3$-0 is a deformation retract of $S^2$. On the other side, the Higgs-field fulfills the condition $\phi^a \phi^a = \text{const}$ which defines a $S^2$ in field space. Hence the Higgs field defines a mapping

$$\phi^a : S^2 \rightarrow S^2$$

The Higgs field transforms under SO(3) as a vector and is left invariant by the subgroup SO(2). Hence all points reached by $\phi^a$ are equivalent to $\frac{SO(3)}{SO(2)} = S^2$. All mappings $\phi^a$ are classified by $\pi_2(SO(3)/SO(2))$. All this carries over...
in a one to one fashion to the two dimensional sigma model (or Heisenberg Ferromagnet) [9] All classical solutions with nontrivial Brouwer degree are given by rational functions, i.e., algebraic mappings

$$\phi^a = \frac{P(z)}{Q(z)}$$

where $z$ is the coordinate parametrizing the Riemann sphere $= S^2$. Hence, as in the case of monopoles due to the description of Donaldson, here, we have rational mappings. The Skyrmions are produced by suspending the mappings $\phi^a : S^2 \rightarrow S^2$ to mappings $U: S^3 \rightarrow S^3$, $U$ being the Skyrme function. [3]. The author feels, that this is a more natural junction than with magnetic monopoles, also since the Skyrme model is in fact a non-linear sigma model. So, skyrmions should be gotten by suspension of spin wave solutions [9]. Also, according to the article of Arafune et al., depending on the gauge, the magnetic charge of the t’Hooft monopole is carried either by the Higgs or by the gauge field. This makes t’Hooft monopoles to the opinion of the author interesting objects in a mathematical sense but it seems to him that they are unrealistic.

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