Local unitary equivalence of quantum states and simultaneous orthogonal equivalence

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Abstract

The correspondence between local unitary equivalence of bipartite quantum states and simultaneous orthogonal equivalence is thoroughly investigated and strengthened. It is proved that local unitary equivalence can be studied through simultaneous similarity under projective orthogonal transformations, and four parametrization independent algorithms are proposed to judge when two density matrices on $\mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2}$ are locally unitary equivalent in connection with trace identities, Weierstrass pencils, Albert determinants and Smith normal forms.

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1 INTRODUCTION

As one of the interesting and non-classic properties of quantum theory and information science \cite{1,2}, quantum entanglement has played an important role in quantum computing \cite{3}, quantum dense coding \cite{4}, quantum cryptography \cite{5} and quantum teleportation \cite{6}. It is necessary to determine and classify entanglement status of quantum states in quantum information theory. One step to solve this question is to determine the local unitary (LU) equivalence, as quantum entanglement is invariant under LU transformation.

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In [7], a set of 18 tensor invariants of local unitary equivalence is constructed for the 2-qubit mixed quantum state. Recently a refined set of 12 polynomial invariants [8] for generic 2-qubits and 90 polynomial invariants for generic 3-qubits have been found by using matrix elements of the Bloch representation. Nonlocal properties of multiqubits have been studied in [9] long ago and a necessary and sufficient condition has been set up for the local unitary equivalence problem in multipartite pure qubits [10, 11]. In the case of bipartite qubits, a parametrization dependent criterion for LU equivalence was given in [12]. While for multipartite quantum states, certain properties of LU equivalence are also considered in some special situations [13, 14]. In an indirect approach, generating sets of local SL-equivalent classes are found for multipartite entanglements [15] and abelian symmetry of the LU equivalence has been studied in [16]. It is also known that LU equivalence of density operators can be classified using a finite set of polynomials [17, 18] and spectrum-dependent bounds are given in [19]. Very recently a method to judge LU equivalence for multi-qubits [20] was also proposed and more generally SLOCC invariants for multi-partite states are found [21]. Despite all these developments, it remains a challenging problem to effectively determine the LU equivalence by an operational procedure using invariant polynomials. It is also noted that almost all previous methods do not work for two particles with different dimensions.

In this paper, we strengthen the correspondence between the local unitary equivalence of bipartite quantum states and simultaneous orthogonal equivalence of associated matrix triples and prove that the local unitary equivalence can be transformed to the classical problem of simultaneous projective orthogonality. We then introduce the concept of quasi-LU equivalence for bipartite states using the latter matrix identities, while the quasi-LU equivalence becomes LU equivalence in the case of qubits. Since the correspondence between simultaneous orthogonality and similarity has been known in linear algebra [24], our new characterization simplifies and emphasizes the connection with LU equivalence. This enables us to give four algorithms to judge the local unitary equivalence of mixed bipartite quantum states on any tensor product of two Hilbert spaces with dimensions not necessarily the same. In particular, we define a new canonical form called Smith normal form for any bipartite quantum state, which provides a set of invariant polynomials for LU equivalent quantum
states. Moreover, our correspondence is completely general as it can treat LU equivalence for any two particles over different dimensions.

One example is given to show how the algorithms uncovered in this work are applied and the pros and cons of these algorithms are analyzed. It is shown that through a finite procedure of checking trace identities the problem of classifying LU equivalence can be completely settled for qubits. We then demonstrate that the set of invariant polynomials given by the Smith normal form also provides effective necessary conditions for two qubit states being LU equivalent.

## 2 LU Equivalence of Bipartite Quantum States

Let $\rho$ be the density matrix of a bipartite state on $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$, and let $\{\lambda_i^{(k)}, 0 \leq i \leq d_k^2 - 1, k = 1, 2\}$ be the Gell-Mann bases for each partite, then $\rho$ can be expressed in the following form:

$$\rho = \frac{1}{d_1d_2} I_{d_1d_2} + \sum_{i=1}^{N_1} u_i \lambda_i^{(1)} \otimes \lambda_0^{(2)} + \sum_{j=1}^{N_2} v_j \lambda_0^{(1)} \otimes \lambda_j^{(2)}$$

$$+ \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} w_{ij} \lambda_i^{(1)} \otimes \lambda_j^{(2)}, \quad N_k = d_k^2 - 1, k = 1, 2 \quad (1)$$

where $u_i = \langle \rho, \lambda_i^{(1)} \otimes \lambda_0^{(2)} \rangle = \text{tr} \rho (\lambda_i^{(1)} \otimes \lambda_0^{(2)})$, $v_j = \langle \rho, \lambda_0^{(1)} \otimes \lambda_j^{(2)} \rangle = \text{tr} \rho (\lambda_0^{(1)} \otimes \lambda_j^{(2)})$, $w_{ij} = \langle \rho, \lambda_i^{(1)} \otimes \lambda_j^{(2)} \rangle = \text{tr} \rho (\lambda_i^{(1)} \otimes \lambda_j^{(2)})$. We associate three matrices for $\rho$:

$$u(\rho) = [u_1, u_2, \cdots, u_{N_1}]^t, \quad v(\rho) = [v_1, v_2, \cdots, v_{N_2}]^t, \quad W(\rho) = [w_{ij}]_{N_1 \times N_2} \quad (2)$$

and call them a matrix representation of the density matrix $\rho$. For convenience we denote $UMU^\dagger$ by $M^U$, where $M \in M(n), U \in U(n)$.

Suppose $\rho' = \rho^{U_1 \otimes U_2}$ is another mixed bipartite state on $\mathcal{H}_{d_1} \otimes \mathcal{H}_{d_2}$ for two unitary matrices $U_i \in U(d_i)$. Therefore we can write that $(\lambda_i^{(1)})^{U_1} = \sum_{j=1}^{N_1} a_{ij} \lambda_j^{(1)}$, $(\lambda_i^{(2)})^{U_2} = \sum_{j=1}^{N_2} b_{ij} \lambda_j^{(2)}$ for two complex matrices $A$ and $B$, and one sees that

$$\sum_{i=1}^{N_1} u_i (\lambda_i^{(1)} U_1) \otimes \lambda_0^{(2)} = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} u_i a_{ij} \lambda_j^{(1)} \otimes \lambda_0^{(2)} = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} u_i a_{ji} \lambda_i^{(1)} \otimes \lambda_0^{(2)} \quad (3)$$
i.e. \( u(\rho^{U_1 \otimes U_2}) = A^t u(\rho) \). Similarly \( v(\rho^{U_1 \otimes U_2}) = B^t v(\rho) \), and \( W(\rho^{U_1 \otimes U_2}) = A^t W(\rho) B \).

**Lemma 2.1** Let \( \rho \) and \( \rho' \) be two locally unitary equivalent density matrices, then there exist two real orthogonal matrices \( A \in O(N_1) \) and \( B \in O(N_2) \) such that \( u(\rho') = A^t u(\rho) \), \( v(\rho') = B^t v(\rho) \), and \( W(\rho') = A^t W(\rho) B \).

**Proof.** Let \( \{\lambda_i\} \) be an orthonormal hermitian basis in \( \text{End}(V) \) under the trace form, and let \( U \) be a unitary matrix \( \in \text{End}(V) \). Write \( \lambda_i^U = U \lambda_i U^\dagger = \sum_{ij} m_{ij} \lambda_j \). As \( (\lambda_i^U)^\dagger = U \lambda_i^\dagger U^\dagger = U \lambda_i U^\dagger \), the coefficients \( m_{ij} \) are real numbers. The orthogonality of \( \{\lambda_i^U\} \) is an easy consequence of the following computation:

\[
tr(\lambda_i^U \lambda_j^U) = tr(U \lambda_i \lambda_j U^\dagger) = tr(\lambda_i \lambda_j U^\dagger U) = tr(\lambda_i \lambda_j) = \delta_{ij}.
\]

By a general result of linear algebra, any two orthonormal bases are transformed by an orthogonal matrix, therefore \( M M^T = M^T M = I \). The matrix equations have already been verified above. \( \square \)

Two bipartite density matrices \( \rho_1 \) and \( \rho_2 \) over \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \) are then called *quasi local unitary equivalent* if there exist two orthogonal matrices \( O_1, O_2, O_t \in O(d_i^2 - 1) \) such that

\[
u(\rho_2) = O_1 u(\rho_1), \quad v(\rho_2) = O_2 v(\rho_1), \quad W(\rho_2) = O_1 W(\rho_1) O_2^t.
\]

By Lemma 2.1, two LU equivalent bipartite mixed states are quasi-LU equivalent. In the case of two qubits, it is well-known that quasi-LU equivalence is also a sufficient condition for LU equivalence (see for example, [8]).

### 3 Criteria of Simultaneous Orthogonal Equivalence

Suppose \( \{W_i, u_i, v_i\} \) is a matrix representation of the density matrix \( \rho_i \) on \( \mathbb{C}^{d_1} \otimes \mathbb{C}^{d_2} \), where \( W_i \) is an \( m \times n \) matrix and \( u_i, v_i \) are column vectors of dimension \( m \) and \( n \) respectively (here \( m = d_1^2 - 1, n = d_2^2 - 1 \)).

By the remark after Lemma 2.1, two qubits \( \rho_1 \) and \( \rho_2 \) are LU equivalent if and only if there are orthogonal matrices \( O_i \) such that \( O_1 W_1 O_2^t = W_2, O_1 u_1 = u_2, \) and \( O_2 v_1 = v_2 \), it
follows that the set \( \{W_1^tW_1, v_1u_1^t\} \) is simultaneously orthogonally equivalent to \( \{W_2^tW_2, v_2u_2^t\} \). However, the converse direction is not true in general.

We first give a simplified correspondence between the quasi-LU equivalence and the projective orthogonal equivalence of two real matrices. In particular, it implies that under a norm condition if \( \{W_1^tW_1, v_1u_1^t\} \) is simultaneously orthogonal equivalent to \( \{W_2^tW_2, v_2u_2^t\} \), then the two mixed states \( \rho_1 \) and \( \rho_2 \) are quasi-LU equivalent.

**Theorem 3.1 (Correspondence between quasi-LU and simultaneous orthogonal equivalence).** Let \( W_i \in \mathbb{R}_{m \times n} \), \( u \in \mathbb{R}_m \) and \( v \in \mathbb{R}_n \). There exist orthogonal matrices \( O_1 \in O(m) \) and \( O_2 \in O(n) \) such that \( O_1W_1O_2^t = W_2, O_1u_1 = u_2, O_2v_1 = v_2 \) if and only if \( \{W_1^tW_1, v_1u_1^t\} \) is simultaneously orthogonal equivalent to \( \{W_2^tW_2, v_2u_2^t\} \) and \( |u_1| = |u_2| \) or \( |v_1| = |v_2| \).

**Proof.** The necessity has already been checked. Suppose there exist two orthogonal matrices \( O_i \) such that \( O_1W_1O_2^t = W_2, O_1u_1v_1O_2^t = u_2v_2 \). Without loss of generality we can assume that both \( u_2, v_2 \neq 0 \). Note that \( u_2O_1u_1v_1O_2^t v_2 = u_2u_2v_2 \neq 0 \), then \( \alpha = \frac{v_1v_2}{v_1O_2^tv_2} = \frac{u_2O_1u_1}{u_2v_2} \neq 0 \), which implies that \( O_1u_1 = \alpha u_2 \), \( O_2v_1 = \alpha^{-1}v_2 \). Let \( O_1 = \alpha^{-1}O_1, O_2 = \alpha O_2 \). As \( |u_1| = |u_2| \) or \( |v_1| = |v_2| \), we see that \( \alpha = \pm 1 \). Then \( O_1W_1O_2^t = W_2, O_1u_1 = u_2, O_2v_1 = v_2 \), where \( O_i \in O(n) \).

Through this correspondence, we have transformed the LU problem to that of simultaneous orthogonal equivalence between two pairs of matrices plus the norm condition. To solve this problem, we first look at several algorithms to judge when two sets of real matrices simultaneously orthogonal similar, and then reduce the problem of simultaneous orthogonal equivalence to that of simultaneous (orthogonal) similarity, which is one of the classical problems in linear algebra.

Recall that two square matrices \( A \) and \( B \) are similar if there exists an orthogonal matrix \( O \) such that \( A = OBO^t \). The fundamental Specht’s criterion [22] says that a square matrix \( A \) is similar to \( B \) if and only if

\[ \text{trw}(A, A^t) = \text{trw}(B, B^t) \] (5)

for any word \( w(x, y) = x^{m_i}y^{n_i} \cdots x^{m_k}y^{n_k} \), where \( m_i, n_i \in \mathbb{Z}_+ \) and \( k \in \mathbb{N} \). Specht’s criterion has been generalized to two sets of normal matrices [23, 24], where the trace identities are
for all words in the alphabet of the matrix set and the transpose. We can give our second result to judge when two density matrices are quasi-LU equivalent.

**Theorem 3.2** (Simultaneous orthogonal equivalence). Let $\rho_i$ be two bipartite density matrices over the same Hilbert space and suppose $\{W_i, u_i, v_i\}$ are the associated matrix triples. Let 

$$
\{A_1, A_2, A_3\} = \{W_1W_1^t, W_1v_1u_1^t, u_1u_1^t\}, \text{ and } \{B_1, B_2, B_3\} = \{W_2W_2^t, W_2v_2u_2^t, u_2u_2^t\}.
$$

Then $\rho_1$ and $\rho_2$ are quasi-LU equivalent if and only if the trace identities hold:

$$
\text{tr}(A_{i_1}A_{j_1}^t \cdots A_{i_k}A_{j_k}^t) = \text{tr}(B_{i_1}B_{j_1}^t \cdots B_{i_k}B_{j_k}^t)
$$

for any compositions $i_1, \cdots, i_k$ and $j_1, \cdots, j_k$ of $\{1, 2, 3\}$ such that $1 \leq i_1 \leq j_1 \leq 3, \cdots, 1 \leq i_k \leq j_k \leq 3$. Moreover, Eq. (6) are sufficient conditions for LU equivalence in the case of two qubits.

**Proof.** First of all, from our previous discussion it follows that if two density matrices $\rho_1$ and $\rho_2$ are quasi-LU equivalent, then $\{W_i, u_i, v_i\}$ are simultaneous orthogonal equivalent and $|u_1| = |u_2|$. Subsequently the sets $\{W_1W_1^t, W_1v_1u_1^t, u_1u_1^t\}$ are simultaneously orthogonal similar. According to [24, Th. 3.3] two sets of rectangular matrices $\{A_1, \cdots, A_l\}$ and $\{B_1, \cdots, B_l\}$ of the same size are orthogonally equivalent if and only if the equations in (6) hold for any compositions $i_1, \cdots, i_k$ of the integers $\{1, \cdots, k\}$ such that $1 \leq i_t \leq j_t \leq k$, $t = 1, \cdots, k$. So the theorem is proved. \(\Box\)

There is a simpler necessary condition arising from the connection with Jordan algebras [25], which can be proved directly.

**Theorem 3.3** (Albert’s criterion) Suppose that $(W, u, v)$ is a matrix representation of the density matrix $\rho$, then

$$
\det(xI - x_1WW^t - x_2uu^t - x_3Wv\bar{u}^t)
$$

is an invariant polynomial in the $x_i$ under the LU equivalence. This partly generalizes Makhlin’s invariants.

We remark that Albert’s criterion is a natural generalization of the characteristic polynomial of a square matrix. It is also easy to see that the generalized characteristic polynomial given by Albert contains several invariants considered by Makhlin [7].
Gerasimova, Horn and Sergeichuk [26] gave an algorithm to judge simultaneous orthogonal similarity using block matrices to reduce the problem to Specht’s criterion. We reformulate it as follows.

**Theorem 3.4 (GHS algorithm).** Suppose \( m \leq n \), and consider the nilpotent matrices

\[
\begin{pmatrix}
0 & I_m & u_1^t W_1^t W_1 \\
0 & I_m & W_1 v_1 u_1^t \\
0 & I_m & 0
\end{pmatrix}, \quad
\begin{pmatrix}
0 & I_m & u_2^t W_2^t W_2 \\
0 & I_m & W_2 v_2 u_2^t \\
0 & I_m & 0
\end{pmatrix}
\] (8)

Then these two \( 4m \times 4m \) matrices are orthogonal similar if and only if the set \( \{ W_1 W_1^t, u_1^t u_1^t, W_1 v_1^t u_1^t \} \) is simultaneous orthogonal equivalent to the set \( \{ W_2 W_2^t, u_2^t u_2^t, W_2 v_2^t u_2^t \} \) or \( \rho_1 \) and \( \rho_2 \) are quasi-LU equivalent.

**Proof.** The criterion is directly checked by working out the matrix product and see that the equations of entries imply the simultaneous orthogonality. \( \square \)

The GHS algorithm transforms the LU problem into that of orthogonality similarity between two block matrices.

**4 Smith Normal Forms of Kronecker Pencils**

We now introduce an effective criterion for simultaneous similarity of the triple matrices. Let \( \rho \) be a density matrix on \( H_{d_1} \otimes H_{d_2} \) associated with \( (W(\rho), u(\rho), v(\rho)) \), we consider the auxiliary \( \lambda \)-matrix \( \lambda W(\rho) + u(\rho)v(\rho)^t \) known as the Kronecker pencil [27]. As an element of the ring \( \mathbb{C}[\lambda] \) of matrix polynomials in \( \lambda \), the \( \lambda \)-matrix \( \lambda W(\rho) + u(\rho)v(\rho)^t \) is equivalent to the Smith normal form [27, 28] under elementary row/column operations. The Smith normal form is defined by the property that it is a diagonal matrix over \( \mathbb{C}[\lambda] \) and each non-zero main diagonal entry divides its next diagonal entry. It is normalized such that the diagonal
entries $d_i(\lambda)$ are monic polynomials, i.e.

$$S(\lambda) = \begin{bmatrix}
  d_1(\lambda) & 0 & \cdots & 0 & 0 & \cdots \\
  0 & d_2(\lambda) & \cdots & 0 & 0 & \cdots \\
  \vdots & \vdots & \ddots & \vdots & \vdots & \ddots \\
  0 & 0 & \cdots & d_m(\lambda) & 0 & \cdots \\
  0 & 0 & \cdots & 0 & 0 & \cdots \\
  \cdots & \cdots & \cdots & \cdots & \cdots & \cdots 
\end{bmatrix}_{N \times M},$$

(9)

where $d_i(\lambda) \in \mathbb{C}[\lambda]$ such that $d_i(\lambda) | d_{i+1}(\lambda), \forall 1 \leq i \leq m$ and $m \leq N$. As $d_i(\lambda)$ are successively given by the principal minors of $\lambda$-matrix $\lambda W(\rho) + u(\rho)v(\rho)^t$, the Smith normal form is uniquely determined and invariant under elementary operations. Furthermore it is well-known [28] that there are $P(\lambda) \in \text{GL}_N(\mathbb{C}[\lambda]), Q(\lambda) \in \text{GL}_M(\mathbb{C}[\lambda])$ such that $P(\lambda)(\lambda W(\rho) + u(\rho)v(\rho)^t)Q(\lambda) = S(\lambda)$.

Let $\rho$ be a density matrix for bipartite system over $H_{d_1} \otimes H_{d_2}$ with the matrix representation $(W(\rho), u(\rho), v(\rho))$, we will simply call the Smith normal form of $\lambda W(\rho) + u(\rho)v(\rho)^t$ as the Smith normal form of the triple $(W(\rho), u(\rho), v(\rho))$. We can now state our fourth criterion.

**Theorem 4.1** (Smith Normal Form). For any two bipartite quantum states $\rho, \rho'$ associated with $(W(\rho), u(\rho), v(\rho))$ and $(W(\rho'), u(\rho'), v(\rho'))$ respectively. If $\rho$ is local unitary equivalent to $\rho'$, then the Smith normal forms of the triple systems $(W(\rho), u(\rho), v(\rho))$ and $(W(\rho'), u(\rho'), v(\rho'))$ are the same.

We remark that the normal form of a $\lambda$-matrix was introduced years ago by Weierstrass for regular cases, by Kronecker for singular cases [27], and in general by Smith [28]. It should not be confused with the much younger term of the canonical form given by the Schmidt decomposition in quantum computation.

**Proposition 4.2** For any matrices $X, X', Y$ and $Y' \in M_N(\mathbb{C})$, there exists $U_1, U_2 \in U(N)$ (or $O(N)$), such that $U_1XU_2^\dagger = X', U_1YU_2^\dagger = Y'$ if and only if there exists $U_1, U_2 \in U(N)$ (or $O(N)$), such that $U_1(X + \lambda Y)U_2^\dagger = (X' + \lambda Y')$.
Proof. This sufficient direction can be easily seen as follows. Suppose there exist \( U_1, U_2 \in U(N) \) such that \( U_1 (X + \lambda Y) U_2^\dagger = (X' + \lambda Y') \). Let \( \lambda = 0, 1 \), we obtain that \( U_1 X U_2^\dagger = X' \) and \( U_1 (X + Y) U_2^\dagger = X' + Y' \). Taking difference, the other equation is also obtained. \( \square \)

**Proof of Smith normal form.** We know that if \( \rho \) is equivalent to \( \rho' \) there exist two orthogonal matrices \( U_i \) in \( O(N_i) \) such that \( u(\rho') = U_1 u(\rho), v(\rho') = U_2 v(\rho), W(\rho') = U_1 W(\rho) U_2^\dagger \). It follows that \( U_1 u(\rho) v(\rho') U_2^\dagger = u(\rho') v(\rho') \dagger \). Subsequently

\[
U_1 (\lambda W(\rho) + u(\rho) v(\rho') U_2^\dagger = \lambda W(\rho') + u(\rho') v(\rho') \dagger, \tag{10}
\]

thus they have the same normal form. \( \square \)

Suppose \( \lambda W_1(\rho) + u_1(\rho) v_1(\rho)^t \) and \( \lambda W_2(\rho) + u_2(\rho) v_2(\rho)^t \) have the same Smith normal form. Then there are invertible matrices \( P(\lambda) \) and \( Q(\lambda) \) such that

\[
P(\lambda)(\lambda W_1(\rho) + u_1(\rho) v_1(\rho)^t) Q(\lambda) = \lambda W_2(\rho) + u_2(\rho) v_2(\rho)^t. \tag{11}
\]

Since \( P(\lambda) \) and \( Q(\lambda) \) are obtained by Gauss elimination, \( P(\lambda) \) and \( Q(\lambda) \) are polynomial functions of \( \lambda \) with non-zero constants. In fact the constants must be invertible matrices. Therefore one obtains that

\[
P(\lambda W_1(\rho) + u_1(\rho) v_1(\rho)^t) Q = \lambda W_2(\rho) + u_2(\rho) v_2(\rho)^t. \tag{12}
\]

for two invertible matrices \( P, Q \). i.e. They are strictly equivalent in the sense of Gantmacher [27].

**Example.** Consider the following quantum state \( \rho \) in \( \mathcal{H}_2 \otimes \mathcal{H}_3 \).

\[
\rho = \frac{1}{6} I_6 + \frac{1 - p}{3} \lambda_1^{(1)} \otimes \lambda_2^{(2)} - \frac{1 - p}{2} \lambda_0^{(1)} \otimes \lambda_3^{(2)} + \frac{1}{2 \sqrt{3}} \lambda_0^{(1)} \otimes \lambda_8^{(2)}
\]

\[
+ \frac{2p - 1}{2} \lambda_1^{(1)} \otimes \lambda_3^{(2)} + \frac{1 - p}{2 \sqrt{3}} \lambda_1^{(1)} \otimes \lambda_8^{(2)} + \frac{p}{2} \lambda_2^{(1)} \otimes \lambda_1^{(2)} + \frac{p}{2} \lambda_3^{(1)} \otimes \lambda_2^{(2)}, \tag{13}
\]

where \( p \in [0, 1] \) and \( \lambda_0^{(1)} = I_2/\sqrt{2}, \lambda_1^{(1)} = (|0\rangle \langle 0| - |1\rangle \langle 1|)/\sqrt{2}, \lambda_2^{(1)} = (|0\rangle \langle 1| + |1\rangle \langle 0|)/\sqrt{2}, \lambda_3^{(1)} = (i|0\rangle \langle 1| - i|1\rangle \langle 0|)/\sqrt{2}, \lambda_0^{(2)} = I_3/\sqrt{2}, \lambda_1^{(2)} = \frac{1}{\sqrt{2}} (|0\rangle \langle 1| + |1\rangle \langle 0|), \lambda_2^{(2)} = -\frac{i}{\sqrt{2}} (|0\rangle \langle 1| - |1\rangle \langle 0|), \lambda_3^{(2)} = \frac{1}{\sqrt{2}} (|0\rangle \langle 0| - |1\rangle \langle 1|), \lambda_8^{(2)} = \frac{1}{\sqrt{6}} (|0\rangle \langle 0| + |1\rangle \langle 1| - 2|2\rangle \langle 2|). \) Three matrices for \( \rho \)
are \( \mu(\rho) = (\frac{1-p}{3}, 0, 0)^T \), \( \nu(\rho) = (0, 0, -\frac{1-p}{2}, 0, 0, 0, \frac{1}{2\sqrt{3}})^T \), and

\[
W(\rho) = \begin{pmatrix}
\frac{p}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{p}{2} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Suppose \( \rho \) is local unitary equivalent to \( \rho' \) under

\[
U_1 \otimes U_2 = \begin{pmatrix}
\frac{1}{\sqrt{2}} & i \frac{1}{\sqrt{2}} \\
-\frac{1}{\sqrt{2}} & -i \frac{1}{\sqrt{2}}
\end{pmatrix} \otimes \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix},
\]

three associated matrices for \( \rho' \) are \( \mu(\rho') = (0, 0, \frac{1-p}{3})^T \), \( \nu(\rho') = (0, 0, \frac{1-p}{2}, 0, 0, 0, \frac{1}{2\sqrt{3}})^T \), and

\[
W(\rho') = \begin{pmatrix}
0 & -\frac{p}{2} & 0 & 0 & 0 & 0 & 0 \\
-\frac{p}{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\frac{2p-1}{2} & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.
\]

Then there exist orthogonal matrices

\[
A = \begin{pmatrix}
0 & 0 & -1 \\
0 & -1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \quad B = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

such that \( \mu(\rho') = A^T \mu(\rho), \nu(\rho') = B^T \nu(\rho), W(\rho') = A^T W(\rho) B \). And \( \rho \) and \( \rho' \) have the same Smith normal forms

\[
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \frac{p}{2} \lambda & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}
\]

Moreover the polynomial

\[
det(xI - x_1 W(\rho) W(\rho)^t - x_2 u(\rho) u(\rho)^t - x_3 W(\rho) v(\rho) u(\rho)^t)
\]

\[
= \det(xI - x_1 W(\rho') W(\rho')^t - x_2 u(\rho') u(\rho')^t - x_3 W(\rho') v(\rho') u(\rho')^t)
\]

\[
= (x - \frac{p^2}{4} x_1)^2 (x - \frac{3(2p - 1)^2 + (1 - p)^2}{12} x_1 - \frac{(1 - p)^2}{9} x_2 - \frac{2 - 3p)(1 - p)^2}{18} x_3)
\]

is an invariant polynomial of the LU equivalence. It can be directly checked that

\[
\begin{bmatrix}
0 & I_3 & u(\rho_1) u(\rho_1)^T & W(\rho_1) W(\rho_1)^T \\
0 & I_3 & W(\rho_1) v(\rho_1) u(\rho_1)^T \\
0 & I_3 & 0
\end{bmatrix}
\]
\[
= I_{12}^T \begin{bmatrix}
0 & I_3 & u(\rho'_1)u(\rho'_1)^T & W(\rho'_1)W(\rho'_1)^T \\
0 & I_3 & W(\rho'_1)v(\rho'_1)u(\rho'_1)^T & I_3 \\
0 & 0 & W(\rho'_1)W(\rho'_1)^T & I_3
\end{bmatrix} I_{12},
\]

According to the GHS algorithm, two $12 \times 12$ nilpotent matrices are orthogonal similar under a matrix. We find that this matrix is given by

\[
\begin{bmatrix}
P & 0 & 0 & 0 \\
P & 0 & 0 & 0 \\
P & 0 & 0 & 0 \\
P & 0 & 0 & 0
\end{bmatrix}, \quad P = \begin{bmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}.
\]

Then \{\(WW^t, uu^t, Wvu^t\}\} is simultaneous orthogonal similar to \{\(W'W'^T, u'u'^T, W'v'u'^T\}\}.

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