Light deflection in Weyl gravity: critical distances for photon paths.

Sophie Pireaux ‡
UMR 5562, Dynamique Terrestre et Planétaire (DTP), B105
Observatoire Midi-Pyrénées, 
14 Avenue Edouard Belin, 
31400 Toulouse, 
FRANCE
E-mail: sophie.pireaux@cnes.fr

Abstract. The Weyl gravity appears to be a very peculiar theory. The contribution of the Weyl linear parameter to the effective geodesic potential is opposite for massive and nonmassive geodesics. However, photon geodesics do not depend on the unknown conformal factor, unlike massive geodesics. Hence light deflection offers an interesting test of the Weyl theory. In order to investigate light deflection in the setting of Weyl gravity, we first distinguish between a weak field and a strong field approximation. Indeed, the Weyl gravity does not turn off asymptotically and becomes even stronger at larger distances. We then take full advantage of the conformal invariance of the photon effective potential to provide the key radial distances in Weyl gravity. According to those, we analyze the weak and strong field regime for light deflection. We further show some amazing features of the Weyl theory in the strong regime.

PACS numbers: 04.05.+h,04.80.Cc,04.90.+e,95.30Sf,95.35.+d,96.40.Cd,98.80.Es

Submitted to: Class. Quantum Grav.

‡ Previously working in
Unité de Physique Théorique et Mathématique (FYMA),
Université catholique de Louvain (UCL), BELGIUM.
1. Introduction

General Relativity, this monumental theory elaborated by Albert Einstein, is a pure tensor theory which simply reduces to Newton’s theory in the weak field limit. It agrees, so far, with all the observations made in our solar system and has to be taken into account in modern technologies like the Global Positioning System (GPS) or when computing orbits. Nevertheless, General Relativity cannot be the final theory of gravitation. Indeed, solely from a theoretical point of view, several questions are pending. First of all, the minimal choice of the Hilbert-Einstein action is not based on any fundamental principle. Additionally, Einstein’s theory of gravitation cannot be properly described by quantum field perturbation theory; and this makes it impossible to unify it with other fundamental interactions. It is also not invariant under conformal transformations. However, if we wish to achieve the unification of gravitation with particle physics, in which conformal invariance plays a crucial role, we might consider a theory of gravitation that incorporates this property. Furthermore, nothing guarantees that the Newtonian potential is valid on very short, or very long distances...

From the point of view of experiments, even though Einstein’s theory passed so far all the relevant solar system tests (light bending, time delay, perihelion shift of Mercury) within the experimental errors [Wi2001], we recall that General Relativity cannot reproduce the flat velocity distributions in the vicinity of galaxies without using copious amounts of dark matter. The Newtonian potential would indeed predict a decreasing distribution. We are thus confronted with the following dilemma: either we assume the existence of a huge amount of dark matter, or we modify the potential for galactic distances. This second solution would immediately invalidate General Relativity with a null cosmological constant. Note that the solution to the “dark matter dilemma” could also be a combination of the two solutions mentioned here above, thus requiring a more acceptable amount of dark matter.

Hence, an alternative theory of gravitation is highly desirable. Regarding this demand, and among many other candidates, Weyl theory is an interesting prototype. Not only does it present an interesting conformal invariance property, but it also contains an additional linear contribution to the Newtonian potential. This latter feature is encoded in the key parameter of the theory, $\gamma_W$, and General Relativity is recovered for $\gamma_W = 0$.

Like any alternative theory of gravitation, Weyl theory needs to be tested. But if as for various alternative theories, light deflection is an interesting probe [Pi2002], it proves to be even more relevant when Weyl gravity is tackled. Indeed, similarly to other theories, the sole Weyl gravitational action (no prescription for the matter action needed) can be used to derive photon paths. It allows to deduce important constraints on the theory free parameter. The crucial advantage of light deflection in Weyl theory resides in the fact that such a study is free from considerations on the unknown Weyl conformal factor. Whereas this factor, to be provided by some symmetry breaking mechanism, must be specified to analyze predictions involving the motion of massive particles, like galactic rotation curves.
In the approach developed in the present article, we show how some critical radii in Weyl theory structure the space-time with respect to photon paths. However, we still leave open the question of the sign of the parameter $\gamma_W$. Indeed, we do not make any assumption on the conformal scale factor, unlike in the approach used by Mannheim and Kazanas, who deduced a positive value for $\gamma_W$ by fitting galactic rotation curves. Nevertheless, light deflection provides us with some useful hints on the physics implied by one or the other sign.

We give here the orders of magnitude for the critical distances as implied by the Mannheim-Kazanas parametrization ($\gamma_W > 0$). In another article “Light deflection in Weyl gravity: constraints on the linear parameter” [Pi2003], we provide some less stringent constraints on $\gamma_W$ than Mannheim and Kazanas, but unbiased by the scale factor for our constraints are based solely on light deflection experiments. The orders of magnitude of critical radii, weak and strong field limits can be recalculated accordingly.

2. The Weyl theory

2.1. The gravitational action

The Weyl theory (W) [Ma1994a, Ma1996, MaKa1989, MaKa1991] is a purely tensorial theory with a metric coupling of gravitation to matter like in General Relativity (GR), but with higher order derivatives of the metric in the dynamical sector. A particularity of this theory is to be conformally invariant (under transformations $g_{\mu\nu} \mapsto \chi^2(x) g_{\mu\nu}$ with $\chi^2(x)$ a finite, positive, nonvanishing, continuous real function), which makes it attractive since renormalizable perturbatively [FrTs1982, St1977] and asymptotically free [FrTs1981, JuTo1978] in a way similar to the theory of strong interactions.

The Weyl gravitational action given by

$$I_W^{\text{gravitation}} = \int dx^4 \sqrt{-g} \, W^{\mu\nu\rho\sigma} \, W_{\mu\nu\rho\sigma}$$

where $g$ is the determinant of the metric $g_{\mu\nu}$; $W^{\mu\nu\rho\sigma}$, $R^{\mu\nu\rho\sigma}$, $R_{\mu\nu}$ are the Weyl, Riemann and Ricci tensors respectively; and $R$, is the scalar of curvature associated with the metric. The latter action is conformally invariant since

$$W^{\mu\nu\rho\sigma} \mapsto \chi^2(x) \, W^{\mu\nu\rho\sigma}.$$ 

The geodesics for massive particles are obviously not conformally invariant, but the way to implement this symmetry breaking is still obscure. However, light geodesics are conformally invariant, see Subsection 3.1 so that any gravitational test based on the trajectories of photons (like the light deflection and the radar echo delay) should not carry this ambiguity.

In vacuum, the variational principle applied to action (1) with respect to the metric leads to the Bach equations:

$$B^{\mu\nu} \equiv R_{\alpha\beta} \, W^{\mu\alpha\nu\beta} + 2 \, W^{\alpha\beta\nu}_{\, \, \, \, \, |\alpha|\beta} = 0$$

(2)
where $\alpha$ is the covariant derivative. The vacuum solutions of GR ($R_{\alpha\beta} = 0$) are also vacuum solutions of the Weyl theory thanks to the well-known Bianchi identities. Thus, the Schwarzschild metric is a particular solution of the spherically symmetric Bach equations.

The generalized Birkhoff theorem proves that there exists a three parameter family ($\beta_W$, $\gamma_W$, $k_W$, all constants) of static and spherically symmetric solutions to those Bach equations. Any of these solutions can be recast, thanks to conformal and coordinate transformations into a canonical line element [MaKa1989], for an efficient derivation using Killing vectors see [Pi1997]. It provides the general solution to the Bach equations,

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

$$= \chi^2(r) \left\{ \left[ 1 + \frac{2V_W}{c^2} \right] c^2 dt^2 - \left[ 1 + \frac{2V_W}{c^2} \right]^{-1} dr^2 - r^2 \left( d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right\}$$  (3)

where

$$V_W(r) = -\frac{\beta_W}{2} \left( 2 - 3\beta_W \gamma_W \right) c^2 - \frac{3}{2} \beta_W \gamma_W c^2 + \frac{\gamma_W}{2} r^2 c^2 - \frac{k_W}{2} r^2 c^2.$$  (4)

The Weyl metric contains, as expected, the Schwarzschild line element as a particular case ($\gamma_W = k_W = 0$ with $\beta_M = \frac{GM}{c^2}$, where $G$ is the Newtonian constant, and $M$, the gravitational mass). The conformal factor $\chi^2(r)$ is arbitrary unless a conformal symmetry breaking mechanism is specified.

2.2. The gravitational potential

The gravitational potential corresponding to the Weyl static and spherically symmetric metric is not Newtonian anymore. If choosing $\chi^2(r) \equiv 1$ (or choosing a constant $\chi^2$, in which case $r$ and $t$ are rescaled by a constant factor), this is the so-called Weyl gravitational potential [4]. It presents a constant ($\beta_W \gamma_W$)-term in addition to the traditional Newtonian potential (1st term), so that the solution cannot, with any conformal transformation, be brought to a Minkowskian space, even asymptotically. It contains also a linear term that might dominate at galactic distance scales; as well as a quadratic term that should become significant only at cosmological distances from the gravitational source. When $\gamma_W$ is null (classical Schwarzschild solution), $k_W$ is proportional to the cosmological scalar curvature of a de Sitter background [MaKa1989].

2.3. Mannheim-Kazanas parametrization

When fitting the experimental galactic rotation curves with the gravitational Weyl potential (Newtonian plus galactic term only), without the assumption of any dark matter, Mannheim and Kazanas [Ma1994b, Ma1995, MaKa1989] provided the following constraints on the parameters $\beta_W$ and $\gamma_W$ of the theory:

$$\gamma_W \sim +10^{-26} \text{ m}^{-1},$$

$$\beta_W (M_{\text{Galaxy}}) \sim +10^{+14} \text{ m}^{-1}.$$  (5)
On radial distances \((r)\) smaller than the parsec, the \((\beta_W \gamma_W)\)- and \(k_W\)-contributions can be neglected in the effective potential \(\text{[H]}\).

The explanation of the flattening of rotation curves on galactic distance scales given by Mannheim and Kazanas requires \(\gamma_W\) to be positive. The order of magnitude of \(\gamma_W\), close to the inverse of the Hubble length, \(c/H_0\), was noted as an interesting coincidence. However, we have to keep in mind that the Mannheim-Kazanas parametrization \(\text{[5]}\) is based on the specific choice\(\text{§}\) \(\chi^2(r) \equiv 1\), while any gravitational test or measurement requiring a distance/time scale depends on the conformal factor.

2.4. The weak field versus the strong field limit

If we neglect the \(k_W\)-term, to which photons are blind as we shall see, the radial distance scale at which the Weyl gravitational potential in \(\text{[H]}\) can be considered to be a weak field \(\left(2 V_W(r) / c^2 << 1\right)\) is given by

\[
\begin{align*}
    r_{\text{weak field}} &\ll \\
    \left(\beta_W \gamma_W\right)\text{-term neglected} &\sim \\
    \pm \frac{1 + 3 \beta_W \gamma_W \pm \sqrt{1 \pm \left\{ \begin{array}{c} +14 \\ -2 \end{array} \right\} \beta_W \gamma_W - 3 (\beta_W \gamma_W)^2}}{2 \gamma_W} \\
    \text{if } \chi^2(r) = 1 &\sim \\
    \text{and } &\sim +10^{26} \text{ m ,}
\end{align*}
\]

where \(\beta_W \sim G_N M / c^2\) with gravitational mass \(M\); with the upper sign and number for \(\gamma_W > 0\) and the lower sign and number for \(\gamma_W < 0\). When neglecting the \((\beta_W \gamma_W)\)-term, the weak field limit is independent of the gravitational mass creating the field.

\(\text{§}\) In a private communication \(\text{[Ma2002]}\), Mannheim advocated that “all mass is dynamical” in his theory. In other words, both the test particle and the gravitational source would get their mass from the same Higgs field, such that all massive particles possess the same conformal factor, \(\chi^2(r)\), which makes it unobservable.

However, it is known that there should exist at least two different symmetry breaking mechanisms generating the ordinary matter mass. For example, the electron gets its mass from the Higgs mechanism of the Standard Electroweak Model, whereas the mass of the proton is mainly due to the Quantum Chromodynamics. Hence, we claim that it is not allowed to attribute a common conformal factor \(\chi^2(r)\) to all types of particles.
On the other hand, if distances are sufficiently large with respect to the order of magnitude of \(1/|\gamma_W|\) and of the deflector mass \(M\), that is to say,

\[
\begin{align*}
    r_{\text{strong field}} &\gg \sqrt{\frac{2\beta_W}{|\gamma_W|}} \\
    &\gg \sqrt{3 \cdot 10^{10^{-3}} \frac{1}{|\gamma_W|}} \text{ m for } M = 10^2 M_{\text{Sun}}, \gamma_W \text{ in } [m^{-1}] \\
\end{align*}
\]

if \(\chi^2(r) = 1\) and \(\beta_W = 0\), the radial variable \(r\) must belong to certain allowed ranges to insure a proper definition of the line element in Schwarzschild coordinates,

\[
ds^2 = A^2(r) c^2 dt^2 - B^2(r) dr^2 - r^2 [d\theta^2 + \sin^2 \theta d\varphi^2].
\]

In other words, the diagonal metric terms in (3) must be strictly positive. The conditions are listed in discussion (22), section 6. The relevance of the radial distances \(r_{\text{min}}\) and \(r_{\text{null}}\) introduced there will become evident in the next section.

Of course, one would need to analyze further the intermediate regime between the weak field and strong field limits.

3. Light deflection in Weyl theory

3.1. The geodesic equation

In the Weyl theory, the shape of the effective geodesic potential for light crucially changes according to the sign chosen for the parameter \(\gamma_W\), and to whether the contribution of some parameters (\(\gamma_W, \beta_W, k_W\)) is considered as negligible or not. There might exist no black-hole solution. Moreover, the effective geodesic potential for photons versus that for massive particles presents different features that might enlighten us on the role of the effective sign of the parameter \(\gamma_W\), measured through light deflection or galactic rotation curves. We can check those statements by comparing the Weyl effective geodesic potential for light (\(F \equiv 0\)) with that for matter (\(F > 0\)) in Schwarzschild coordinates using the metric given in equation (3) with definition (4). The geodesic equations can be expressed as

\[
\begin{align*}
    \left(\frac{dr}{d\lambda}\right)^2 &+ \left\{\frac{1}{r^2} + F \frac{\chi^2(r)}{J^2}\right\} \left\{1 + 2 \frac{V_W(r)}{c^2}\right\} = \frac{E^2}{J^2}
\end{align*}
\]

where

- \(\left(\frac{dr}{d\lambda}\right)^2\) is the "Kinetic Energy",
- \(V_{\text{geodesic}} = \frac{\chi^2(r)}{J^2}\) is the "Effective potential",
- \(1 + 2 \frac{V_W(r)}{c^2}\) is the "Total Energy".
Light deflection in Weyl gravity: critical distances for photon paths.

where $\frac{dr}{d\lambda} \equiv \frac{1}{r^2} \frac{dr}{d\varphi}$, while $E$ and $J$ are integration constants for the total energy and the total angular momentum of the particle respectively.

First note that, as stated earlier, photon geodesics, unlike massive ones, are independent of the conformal factor $\chi^2(r)$, to be specified by the symmetry breaking mechanism. This is why light deflection is a powerful test for Weyl theory.

To illustrate further the different behavior of massive versus massless geodesics, let us fix arbitrarily $\chi^2(r)$ to 1 and consider the effective gravitational force. The derivative of the effective geodesic potential is given by [EdPa1998]

$$- F_{\text{geodesic}}(r) \propto \frac{dV_{\text{geodesic}}}{dr} = -\frac{2}{r^3} + \beta_W (2 - 3\beta_W \gamma_W) \left\{ \frac{2}{2r^4} + \frac{1}{r^2} \right\} + 3\beta_W \gamma_W \left\{ \frac{2}{r^4} \right\} + \gamma_W \left\{ -\frac{1}{r^2} + \frac{J^2}{r^2} \right\} + k_W \left\{ 0 - 2\frac{J^2}{r^2} \right\}.$$ (10)

The interpretation of the first three terms contributing to the “geodesic force” is unambiguous: the first term is asymptotically convergent with respect to the variable $r$ and negative (repulsive) for any type of particle. Whereas the factor corresponding to the Newtonian term (multiplying $\beta_W (2 - 3\beta_W \gamma_W)$) is always positive (attractive), regardless of the type of particle. The same conclusion holds for the factor in front of the $(\beta_W \gamma_W)$-term, which is always positive (thus attractive) if $\gamma_W$ is positive, or alternatively negative (thus repulsive) for a negative $\gamma_W$. Both the Newtonian term and this last term are asymptotically convergent.

On the contrary, the fifth term clearly distinguishes between massive and nonmassive particles. The $k_W$-term has a null contribution for photons, unlike for massive particles; and for nonrelativistic particles, it is repulsive (negative) for a positive $k_W$ or conversely for a negative $k_W$. Moreover, the corresponding term in the potential, $V_{\text{geodesic}}(r)$, is asymptotically divergent for massive particles at large $r$. However, in any case, we can neglect the $k_W$-contribution (set $k_W = 0$) on non-cosmological distances.

Let us therefore focus on the original feature of the Weyl gravity encoded in the key parameter $\gamma_W$, namely the fourth term in (10). This term cannot be neglected at intermediate distances. We realize that the sign of the factor corresponding to the parameter $\gamma_W$ depends on the type of particle: it is always negative for photons ($F = 0$) and for sufficiently relativistic particles, i.e. for particles verifying

$$\left\{ -\frac{1}{r^2} + \frac{J^2}{r^2} \right\} < 0$$

$\Downarrow$ $u \frac{dr}{d\lambda} \sim 0 \text{ in } r \text{ small.}$

In contrast, it is positive for massive, not too relativistic particles. For photons, moreover, the contribution of the $\gamma_W$-term in the potential $V_{\text{geodesic}}(r)$ is convergent on large radial distances; whereas it diverges linearly for massive, not too relativistic
particles. Again, for photons, this term is attractive for $\gamma_W < 0$ or repulsive for $\gamma_W > 0$; while it is just the opposite for nonrelativistic particles.

3.2. Critical radii from the effective geodesic potential for photons

We now analyze thoroughly the case of null geodesics ($F \equiv 0$ and we set $k_W$ to zero), that is, the trajectory of photons and ultra relativistic particles. We find the following characteristics for the effective geodesic light potential: a radius at which the effective geodesic light potential is minimal, one at which it shows a local maximum, one being an inflection point, and finally, a radius at which the effective geodesic light potential cancels. We can approximate those critical radii assuming that $\beta_W \gamma_W$ is small in comparison with $\beta_W$ and $\gamma_W$ alone. Their expressions are respectively,

$$r_{\text{min}} = \frac{3\beta_W \gamma_W}{2} \sim -\frac{2}{\gamma_W},$$

$$r_{\text{max}} = 3\beta_W,$$

$$r_{\text{inflection}} = -3+9\beta_W \gamma_W - \sqrt{9-54\beta_W \gamma_W + 81(\beta_W \gamma_W)^2 + 48\gamma_W - 72\beta_W \gamma_W^2} \sim -\frac{3}{\gamma_W},$$

$$r_{\text{null}} = -1+3\beta_W \gamma_W - \sqrt{1+2\beta_W \gamma_W - 3(\beta_W \gamma_W)^2} \sim -\frac{1}{\gamma_W}.$$

Analyzing the shape of the effective geodesic potential curve for light in some limiting cases will provide some insight on the key radial distances introduced here.

**Case A. $\gamma_W = 0$**

When $\gamma_W$ is null, we essentially recover General Relativity: the effective geodesic light potential is that of a Schwarzschild black hole with a generalized Schwarzschild radius $\mathcal{R}$ (Figure 2). There thus exists a critical value of the closest approach distance, $r_{\text{max}}$, under which no deflection can take place because photons are captured. Hence, the condition for a photon on a trajectory $r(t)$ to be deflected with a closest approach distance $r_0$ to the gravitational mass is:

$$r \geq r_0 > r_{\text{max}} > \mathcal{R} \equiv 2\beta_W .$$

**Case B. $\beta_W = 0$**

When $\beta_W$ is assumed to be zero, the shape of the potential for light depends crucially on the sign of $\gamma_W$.

**B1. $\gamma_W > 0$**

For positive values of $\gamma_W$, the effective geodesic potential for light is a wall (Figure 3), and thus there exists no critical value of the closest approach distance. Indeed, the characteristic points of the curve (minimum, inflection point) are at negative radii and thus are non physical. Light deflection can occur at any radial distance and is characterized by

$$r_{\text{null}} = -1+3\beta_W \gamma_W - \sqrt{1+2\beta_W \gamma_W - 3(\beta_W \gamma_W)^2} \sim -\frac{1}{\gamma_W}.$$

for $\gamma_W > 0$ : $ r \geq r_0$.
B2. $\gamma_\text{W}<0$

In case of a negative $\gamma_\text{W}$, we have, again, a potential wall with no critical value of the closest approach distance, but it admits a minimum and negative energies (Figure 4). This $r_{\text{min}}$ constitutes the radius of a stable circular orbit for photons, and together with other orbits of negative effective total energy ($E^2_{\text{photon}} < 0$) it belongs to the class of bound states. There thus exists a maximum closest approach distance, $r_{\text{null}}$, allowed for asymptotically free trajectories. Consequently, the condition for light deflection to take place is:

$$\text{for } \gamma_\text{W} < 0 : \quad r \geq r_0 \quad \text{and} \quad r_0 \leq r_{\text{null}}.$$  \hspace{1cm} (14)

The role of $r_{\text{null}}$ will be clarified in the next paragraph, particularly with expression (20).

Case C. Nonzero values of $\gamma_\text{W}$ and $\beta_\text{W}$

On short distance scales ($r << 1/|\gamma_\text{W}|$), the potential is Schwarzschild-like and analogous to case A discussed above. On intermediate distances ($r \sim 1/|\gamma_\text{W}|$), if $\gamma_\text{W} < 0$, the effective geodesic potential for light presents a minimum as described in case B2. Otherwise, for $\gamma_\text{W} > 0$, the comments given in case B1 above apply.

3.3. Conditions for light deflection

From the geodesic equations for photons, one derives the photon path, parametrized as the angle $\varphi$ as a function of the radial distance $r$ to the gravitational source. For a generic metric in Schwarzschild coordinates ($r$, $\theta$, $\varphi$):

$$\varphi(r) = \int_{0}^{\sin \Phi=r_0/r} \frac{\cos \Phi}{\sqrt{A^2(r_0) - \sin^2 \Phi A^2(r)}} d\Phi.$$  \hspace{1cm} (15)

The asymptotic light deflection angle, $\hat{\alpha}$, is usually defined as the angle between the inner and outer asymptotes to the photon trajectory, with $r_0$ the closest approach distance to the gravitational body (Figure 4):

$$\hat{\alpha}_{\text{exact}}(r_0) = 2 |\varphi(r_0)| - \pi.$$  \hspace{1cm} (16)

3.3.1. The weak field regime

The light deflection angle

Let us apply the general expression for the photon path (15) and the asymptotic deflection angle (16) to the Weyl metric (3) in the weak field regime (6). We obtain:

$$\hat{\alpha}_{\text{weak field}}(r_0) \simeq \frac{2\beta_\text{W} \left(2 - 3\beta_\text{W} \gamma_\text{W} \right)}{r_0} + \frac{3}{2} \beta_\text{W} \gamma_\text{W} \pi - \gamma_\text{W} r_0,$$  \hspace{1cm} (17)

where $r_0$ verifies (6) with (12), if $\gamma_\text{W} \geq 0$, (12) and (14), if $\gamma_\text{W} < 0$. 
We see that the \((\beta_W, \gamma_W)\)-term contributes to a small constant deflection, independent of the radial distance, while the linear term tends to decrease or increase the usual Newtonian contribution (first term in (17)), according to the sign of the parameter \(\gamma_W\).

Critical radius from the weak field deflection angle
If \(\gamma_W\) is negative, the asymptotic deflection angle in (17) is always convergent (positive); whereas if \(\gamma_W\) is positive, the deflection angle cancels at \(r_0\), which is a function of the deflector mass, and becomes divergent (negative) beyond that distance:

\[
\begin{align*}
r_0 &= 3\pi \frac{\beta_W \gamma_W + \sqrt{9 (\beta_W \gamma_W)^2 + 32 \beta_W \gamma_W (2 - 3 \beta_W \gamma_W)}}{4 \gamma_W} \\
&\sim \sqrt{6} \cdot \frac{10^{4.5+\frac{1}{2}}}{|\gamma_W|} \text{ m for } M = 10^7 M_{\odot}, \gamma_W \text{ in } [\text{m}^{-1}]
\end{align*}
\]

If \(\chi^2(r) = 1\) and (4)

3.3.2. The strong field regime
Open versus closed orbits in the strong field regime
If we apply the general expression for the photon path (15) to the Weyl metric (3) in the strong field regime (7, 8), we see that we need additional conditions on the distances \(r\) and \(r_0\), so that the term in the square-root of the integrand of equation (15), which we call \(\Delta(r)\), be positive \(\parallel\) when \(\beta_W = 0\). Those conditions, together with the definition of \(\Delta_{\beta_W=0}(r)\), are listed in discussion (23), section 6.

Bearing discussions (22) and (23) from section 6 in mind, the potential in the strong field regime (8) permits an exact integration of the photon trajectory (15) given by

\[
\varphi(r)_{\beta_W=0} = \left[ \pm \arctan \left( \frac{r_0 (\gamma_W r + 2)}{2 \sqrt{\Delta(r)}} \right) \right]_{r_{initial}}^r
\]

with \(r_{initial} = +\infty\) for an unbound orbit \(r_{initial} = r_0\) for a bound orbit

\[
\Rightarrow \pm \varphi(r)_{\beta_W=0} \pm \varphi_{initial} = \arcsin \left( \frac{2 r_0/r + \gamma_W r_0}{2 + \gamma_W r_0} \right)
\]

\(\parallel\) It should be further restricted to a strict inequality for the integration variable \(r\) in expression (15).
\[ r_{\beta W=0} = \frac{-2/\gamma_W}{1 - \frac{2+\gamma_W r_0}{\gamma_W r_0} \cdot \sin (\pm \varphi \pm \varphi_{\text{initial}})} \] (19)

where

\[ \varphi_{\text{initial}} \equiv \begin{cases} + \arcsin \left( \frac{\gamma_W r_0}{2+\gamma_W r_0} \right) & \text{for } r_0 > r_{\text{null}} \text{ if } \gamma_W > 0; \\ \pm \frac{\pi}{2} & \text{for } r_0 = r_{\text{null}} \text{ or } r_0 > r_{\text{min}} \text{ if } \gamma_W < 0; \\ \pm \frac{\pi}{2} & \text{for } r_{\text{null}} < r_0 < r_{\text{min}} \text{ if } \gamma_W < 0. \end{cases} \]

\[ e \equiv \left| \frac{2+\gamma_W r_0}{\gamma_W r_0} \right| \text{ the eccentricity.} \]

The types of allowed orbits can thus be classified \[EdPa1998\] according to (20), with respect to \( r_{\text{null}}, r_{\text{min}} \) and \( r_\ast \) given in (11) and (23). Figures 5 and 6 illustrate the different photon orbits.

**types of orbits in Schwarzschild coordinates:**

for \( r \) verifying (22) and (23):
- if \( \gamma_W > 0 \): \( \forall r_0 \), hyperbolic
- if \( \gamma_W < 0 \):
  - \( r_0 < r_{\text{null}} \), hyperbolic \((e > 1)\)
  - \( r_0 = r_{\text{null}} \), parabolic \((e = 1)\)
  - \( r_0 > r_{\text{null}} \), elliptic \((e < 1)\)
  - with perihelion/aphelion given by \( r_0 \) or \( r_\ast \)
  - particular case if \( r_0 = r_{\text{min}} \), circular \((e = 0)\).

We see that, in fact, conditions (22) only reflect the different orbits allowed for photons.

Note that, strictly speaking, expression (19) cannot be used for the circular orbit. Instead, one needs to come back to the geodesic equation for photons \((F = 0)\) (9) and to its derivative with respect to \( r \). Then, constant radius orbits are found when imposing \( \partial r / \partial \lambda = \partial^2 r / \partial^2 \lambda = 0 \). The circular orbit radii obtained correspond to \( r_{\text{min}} \) and \( r_{\text{max}} \) given in (11), of which \( r_{\text{max}} \) is unstable.

The light deflection angle

Consequently, in Weyl theory, light deflection in the strong field regime is described by the following exact equation

\[ \hat{\alpha}_{\beta W=0}(r_0) = -2 \arcsin \left( \frac{\gamma_W r_0}{2 + \gamma_W r_0} \right), \] (21)

where \( r_0 \) verifies (7) and (13), if \( \gamma_W > 0 \)

(14), if \( \gamma_W < 0 \).

---

**Light deflection in Weyl gravity: critical distances for photon paths.**
It corresponds to asymptotically free orbits (hyperbolic or parabolic) described in discussion (20).

4. **Amazing features of the strong field limit for a negative value of $\gamma_W$**

Let us now get a visual overview on how the various critical radii introduced here structure the space-time with respect to light paths, and hence structure information transfer.

4.1. **An accumulation point**

To start with, we realize from the graph of photon trajectories (Figure 6) that the semi-lattice rectum, located at $r = -2/\gamma_W$ for any orbit in the strong field limit, is an accumulation point. Indeed, if we interpolate between the weak field regime, in which the description of light deflection is qualitatively analogous to the description in General Relativity, and the strong field regime, we might possibly find the picture illustrated in Figure 7. Consequently, the surface of a sphere of radius $r = -2/\gamma_W$, centered on a given lens, is a very particular region of space.

4.2. **Peculiar alignment configuration**

Another remark is the existence of a particular Observer-Lens-Source alignment configuration in the strict strong field regime ($\beta_W = 0$), in which the observer and the source are located on the semi-lattice rectum points. There would then be an infinite number of photon trajectories coming from the source to the observer (see Figure 8)!

4.3. **Observable regions of space**

It would also be interesting to investigate further which regions of space are observable (connected by light geodesics) in the strong field regime. For example, in the presence of one single lens (L), when we wish to understand if an observer, in the strict strong field limit ($\beta_W = 0$), will be able to see some given source, we realize that it depends on the position of the source with respect to the concentric spheres of radius $r_{null}$ and $r_{min}$ surrounding the lens. The equation of orbits described in (19) and (20) are useful for such a discussion. Their consequences are summarized in Figure 9.

Indeed, if the light source ($S_1$) is inside the smallest sphere of radius $r_{null}$, then, the closest approach distance of any photon originating from the source is necessarily smaller than $r_{null}$. This means that all these photons have unbound orbits (hyperbolas centered on L).

Also, if the source ($S_2$) is on the sphere of radius $r_{null}$, the closest approach distance of its photons can be either smaller or equal to $r_{null}$, which means that all the orbits are again unbound (hyperbolas or a parabola). In those two cases ($S_1$ and $S_2$), the source can be seen from any region of space (except just behind the lens if it is not transparent).
Whereas if the source (S₃) is located in between the two spheres, there are in addition photons originating from this source that have a closest approach distance to the lens larger than \( r_{\text{null}} \). Accordingly, those photons are captured on elliptical orbits. Hence, in this case, there will be some regions of space that cannot be reached by any photon of S₃.

We have already evoked the particular case of a source (S₄) situated on the sphere of radius \( r_{\text{min}} \), which allows all four types of orbits: hyperbolic, parabolic, circular and elliptic.

For a source outside the larger sphere (S₅) there will also be some regions of space that cannot be reached by any photon originating from the source.

One could now complicate the game by considering many lenses and relax the strong field regime!

5. Conclusions

We have shown how in Weyl theory different critical radial distances structure the space-time surrounding a gravitational mass with respect to photon paths. This structure and the physical relevance of the different radii (relevant when positive!) strongly depend on the sign of the key parameter of the theory, \( \gamma_W \).

When \( \gamma_W \) is negative, it is important to notice the coincidence between the weak field limiting radius (15) and the critical radius \( r_{\text{null}} \) (11) that separates between open and closed orbits; in other words, between the possibility and the impossibility of light deflection. This means that in the weak field limit, that is when the Newtonian contribution dominates, light deflection always takes place. On the contrary, in the strong field regime, when the \( \gamma_W \)-term is nonnegligible, light deflection is not always possible.

If we consider the order of magnitude given for \( \gamma_W \) by Mannheim and Kazanas, resting on a particular scale factor, then \( r_{\text{null}} \sim 10^{26} \) m is a cosmological distance. On the other hand, if we consider the more conservative estimate of \( \gamma_W \) derived in article [P2003], then \( r_{\text{null}} \gtrsim 10^{18} \) m might be within galactic distances. It is important to note that this later estimate does not imply any restriction on the conformal factor.

When \( \gamma_W \) is positive, light deflection always takes place, whether the weak field regime be valid or not. However, a new feature of Weyl gravity, by comparison to General Relativity, appears at closest approach distances larger than the critical value \( r_0 \) (18): light deflection from being a convergent phenomenon becomes divergent. In the strong field regime, light deflection is always divergent because the order of magnitude of the strong field limiting radius (7) is the same as that of the critical closest approach distance (18).
6. TABLES AND FIGURES

Figure 1. Definition of the asymptotic light deflection angle. I= Image; S= Source; O= Observer; L= Lens (gravitational potential); \( \hat{\alpha} \)= asymptotic light deflection angle; \( r_0 \)= closest approach distance, written here in Schwarzschild coordinates \((r, \theta, \varphi)\); \( b \)= impact parameter. The solid line shows the light path while the dash line shows the inner and outer asymptotes to the geodesics.
Light deflection in Weyl gravity: critical distances for photon paths.

Proper definition of the Weyl line element in Schwarzschild coordinates

$$A_{\beta W=0}^2(r) = B_{\beta W=0}^{-2}(r) = \left[1 + \frac{2\gamma_W}{c} r\right] > 0 \quad \text{with } [\mathbb{N}]$$

Let $K \equiv -k_W - \frac{3\gamma_W}{4}$,

if $K > 0 \iff k_W < -\frac{3\gamma_W}{4} < 0$ : $r \in [0, +\infty[$

if $K = 0 \iff k_W = -\frac{3\gamma_W}{4}$:

$$\begin{cases} 
\gamma_W > 0 \Rightarrow r_{\min} < 0 : r \in [0, +\infty[ \\
\gamma_W < 0 \Rightarrow r_{\min} > 0 : r \in [0, r_{\min}] 
\end{cases}$$

if $K < 0$:

$$\begin{cases} 
k_W > 0 \Rightarrow r_- < 0 \text{ and } r_+ > 0 : r \in [0, r_+] \\
k_W = 0 : 
\begin{cases} 
\gamma_W > 0 \Rightarrow r_{null} < 0 : r \in [0, +\infty[ \\
\gamma_W < 0 \Rightarrow r_{null} > 0 : r \in [0, r_{null}] 
\end{cases} \\
k_W < 0 : 
\begin{cases} 
\gamma_W > 0 \Rightarrow r_- < 0 \text{ and } r_+ < 0 : r \in [0, +\infty[ \\
\gamma_W < 0 \Rightarrow r_- > r_+ > 0 : r \in [0, r_+ \cup r_-, +\infty[ 
\end{cases} 
\end{cases}$$

where $r_{\min}$ is the root of $A_{\beta W=0}^2(r)$ when $K = 0$,

$r_\pm \equiv \frac{\gamma_W \pm \sqrt{-4K}}{2k_W}$ are the roots of $A_{\beta W=0}^2(r)$ when $K < 0$,

$r_{null}$ is the root of $A_{\beta W=0}^2(r)$ when $k_W = 0$.

$$\ldots \notin \text{ roots of } A_{\beta W=0}^2(r) \text{ when } K > 0.$$ 

Conditions for $\Delta_{\beta W=0}(r) \equiv (1 + \gamma_W r_0) r^2 - \gamma_W r_0^2 r - r_0^4 \geq 0$ in (15):

if $\gamma_W > 0$:

$$r \in [r_0, +\infty[$$

$$\begin{cases} 
0 < r_0 < r_{null} \Rightarrow r_+ < 0 : r \in [r_0, +\infty[ \\
r_0 = r_{null} : r \in [r_0, +\infty[ 
\end{cases}$$

if $\gamma_W < 0$:

$$\begin{cases} 
r_{null} < r_0 < r_{\min} \Rightarrow 0 < r_0 < r_+ : r \in [r_0, r_+] \\
r_0 = r_{\min} \Rightarrow r_+ = r_0 \\
r_0 > r_{\min} \Rightarrow 0 < r_+ < r_0 : r \in [r_+, r_0] 
\end{cases}$$

where

$$r_+ \equiv \frac{-r_0}{1+\gamma_W r_0} \bigg\{ \text{are the roots of } \Delta_{\beta W=0}(r) \text{ when } r_0 < r_{\min} \text{ and } r_0 \neq r_{null} \bigg\}$$

$r_{null}$ is the root of $\Delta_{\beta W=0}(r)$ when $r_0 = r_{null}$

$$\notin \text{ roots of } \Delta_{\beta W=0}(r) \text{ when } r_0 > r_{\min} \text{ and } \gamma_W < 0.$$
Light deflection in Weyl gravity: critical distances for photon paths.

Figure 2. Effective geodesic potential for light when $\gamma_W$ is null.

Figure 3. Effective geodesic potential for light when $\gamma_W$ is positive and $\beta_W$ is null.

Figure 4. Effective geodesic potential for light when $\gamma_W$ is negative and $\beta_W$ is null.
Figure 5. The strong field regime. Asymptotic photon trajectories for a positive value of the Weyl parameter $\gamma_W$ and the deflector located at the origin of the coordinate system. All the orbits are hyperbolic and divergent. In this plot, $r_0 = \frac{1}{2\gamma_W}$, $r_0 = \frac{1}{\gamma_W}$, $r_0 = \frac{3}{2\gamma_W}$, $r_0 = \frac{2}{\gamma_W}$, and $r_0 = \frac{5}{2\gamma_W}$ respectively for the dash-dot curve, solid line, dash curve, bold curve and dotted line.

Figure 6. The strong field regime. Asymptotic photon trajectories for a negative value of the Weyl parameter $\gamma_W$ and the deflector located at the origin of the coordinate system. With respect to the characteristic distances $r_{null} \equiv -\frac{2}{\gamma_W}$ and $r_{min} \equiv -\frac{1}{\gamma_W}$, the orbits are convergent and elliptic for $r_0 > r_{null}$ (in particular, circular for $r_0 = r_{min}$), parabolic for $r_0 = r_{null}$ and hyperbolic for $r_0 < r_{null}$. Hence, light deflection is possible only when $r_0 \leq r_{null}$. Note that all the orbits have the same semi-lattice rectum (position located at an angle $\pi/2$ from the closest approach distance position) value of $\frac{2}{|\gamma_W|}$. In this plot, $r_0 = \frac{-1}{2\gamma_W}$, $r_0 = \frac{-1}{\gamma_W}$, $r_0 = \frac{-3}{2\gamma_W}$, $r_0 = \frac{-2}{\gamma_W}$, and $r_0 = \frac{-5}{2\gamma_W}$ respectively for the dash-dot curve, solid line, dash curve, bold curve and dotted line.
Light deflection in Weyl gravity: critical distances for photon paths.

Figure 7. The intermediate regime. The semi-lattice rectum, located at \( r_{\text{min}} = -2/\gamma W \) for any orbit in the strong field limit, is an accumulation point. In the weak field limit, light deflection for a point mass is similar to the general relativistic predictions: light converges on short distances when it passes close to the gravitational mass, while it converges on farther distances when it has a larger closest approach distance from the lens. We may guess the intermediate regime.

Figure 8. In the strict strong field regime \((\beta_W = 0)\), there exists a particular O-L-S alignment configuration in which the observer and the source aligned with the lens are located on the semi-lattice rectum points. There would then be an infinite number of photon trajectories coming from the source to the observer.
Figure 9. In the strong field regime, for a light source, $(S_1)$ located inside the sphere of radius $r_{\text{null}}$, the closest approach distance of any photon originating from the source is necessarily smaller than $r_{\text{null}}$. This means that all these photons have unbound orbits: hyperbolas centered on $L$.

When the source $(S_2)$ is on the sphere of radius $r_{\text{null}}$, the closest approach distance of its photons can be either smaller or equal to $r_{\text{null}}$, which means that all the orbits are again unbound: hyperbolas or a parabola.

Whereas for a source $(S_3)$ located in between the two spheres of radius $r_{\text{null}}$ and $r_{\text{min}}$, there are additionally photons originating from this source that have a closest approach distance to the lens larger than $r_{\text{null}}$. Accordingly, those photons are captured on elliptic orbits. The particular case of a source $(S_4)$ situated on the sphere of radius $r_{\text{min}}$ allows all four types of orbits: hyperbolic, parabolic, circular and elliptic. The photon trajectories will be again hyperbolic, parabolic or elliptic for a source $(S_5)$ outside the larger sphere of radius $r_{\text{min}}$. 
Acknowledgments

The research work presented here was carried out in the FYMA Institute at the University of Louvain la Neuve, during a Ph.D. thesis financed by the I.I.S.N research assistantship. We are thankful to Professor J-M. Gérard (FYMA, UCL) for pertinent advice and carefully proofreading the manuscript.

[EdPa1998] A. Edery and M. B. Paranjape. Causal structure of vacuum solutions to conformal (Weyl) gravity. General Relativity and Gravitation, 31, 1031 (1999) [astro-ph/9808345] v2.

[FrTs1981] E. S. Fradkin and A. A. Tseytlin. Renormalizable asymptotically free quantum theory of gravity. Physics Letters B, 104, 377-381 (1981).

[FrTs1982] E. S. Fradkin and A. A. Tseytlin. Renormalizable asymptotically free quantum theory of gravity. Nuclear Physics B, 201, 5, 469-491 (1982).

[JuTo1978] J. Julve and M. Tonin. Quantum gravity with higher derivative terms. Nuovo Cimento B, 46, 1, 137-152 (1978)

[Ma1994a] P. D. Mannheim. Open Questions in Classical Gravity. Foundations of Physics, 24, 4, 487-511 (1994).

[Ma1994b] P. D. Mannheim. Microlensing, Newton-Einstein gravity, and conformal gravity. UCONN-94-10, astro-ph/9412007 (1994).

[Ma1995] P. D. Mannheim. Conformal Cosmology and the Age of the Universe. UCONN 95-08 (1995).

[Ma1996] P. D. Mannheim. Local and Global Gravity. Foundations of Physics, 26, 12, 1683-1709 (1996).

[Ma2002] P. D. Mannheim. Private Communication. University of Connecticut, August 2002.

[MaKa1989] P. D. Mannheim and D. Kazanas. Exact Vacuum Solution to Conformal Weyl Gravity and Galactic Rotation Curves. Astrophysical Journal, 342, 635-638 (1989).

[MaKa1991] P. D. Mannheim and D. Kazanas. General Structure of the Gravitational Equations of Motion in Conformal Weyl Gravity. Astrophysical Journal, Supplement Series, 76, 431-453 (1991).

[Pi1997] S. Pireaux. Etude de Solutions Particulières d’une Théorie Invariante Conforme de la Gravitation. Graduate thesis, Université catholique de Louvain (UCL), June 1997.

[Pi2002] S. Pireaux. Light deflection experiments as a test of relativistic theories of gravitation. Ph.D. thesis, Université catholique de Louvain (UCL), September 2002.

[Pi2003] S. Pireaux. Light deflection in Weyl gravity: constraints on the linear parameter. Submitted to Classical and Quantum Gravity.

[St1977] K. S. Stelle. Renormalization of higher order derivative quantum gravity. Physical Review D, 16, 4, 953-969 (1977).

[Wi2001] C. M. Will. The confrontation between General Relativity and experiments. Living Reviews, 4 (2001). \www.livingreviews.org/Articles/Volume4/2001-4will