THE EXISTENCE OF SOLUTIONS OF 2-DIMENSIONAL INCOMPRESSIBLE NAVIER-STOKES EQUATIONS ON A MOVING DOMAIN IN AN OPTIMAL SOBOLEV SPACE

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Abstract. We establish the existence of a solution to the Navier-Stokes equations on a moving domain with surface tension in an optimal Sobolev space for the case of two space dimension. No compatibility conditions are required to guarantee the existence of a solution.

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1. INTRODUCTION

1.1. The equations. We are concerned with the 2-dimensional Navier-Stokes equations on a moving domain \( \Omega(t) \) with surface tension on the moving boundary \( \partial \Omega(t) \). Let \( \Omega \) be a bounded domain of \( \mathbb{R}^2 \) (the regularity of \( \Omega \) will be specified later) which denotes the initial fluid domain, and \( u \) and \( p \) denote the fluid velocity and pressure, respectively. We consider

\[
\begin{align*}
    u_t + (u \cdot \nabla)u + \nabla p &= \Delta u & \text{in} & \Omega(t), \\
    \text{div} u &= 0 & \text{in} & \Omega(t), \\
    (\text{Def} u - p \text{Id})n &= \sigma H n & \text{on} & \partial \Omega(t), \\
    u &= u_0 & \text{on} & \Omega \times \{t = 0\}, \\
    \mathcal{V}(\partial \Omega(t)) &= u \cdot n & \text{on} & \partial \Omega(t),
\end{align*}
\]

where the viscosity of the fluid is assumed to be 1, \( n \) is the outward-pointing unit normal of \( \Omega(t) \), \( H \) is the mean curvature of the boundary of \( \Omega(t) \), and \( \sigma > 0 \) is the

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surface tension, \( e \) the identity map defined by \( e(x) = x \), and \( \mathcal{V}(\partial \Omega(t)) \) denotes the normal velocity of the moving boundary \( \partial \Omega(t) \).

1.2. Some prior results. Free boundary problems are one of the main sources of highly nonlinear PDEs that require special treatment to establish well-posedness. Due to high nonlinearity, even the local-in-time solution are expected to exist only in spaces with high regularity, and such kind of results usually accompany certain orders of compatibility conditions if the governing equations are of parabolic type. For example, Xinfu Chen & Fernando Reitich [1] established the well-posedness of the Stefan problem with surface tensions provided that the initial data satisfies two compatibility conditions. In the study of the interaction between incompressible viscous fluids and elastic shells, two compatibility conditions also have to be imposed for the purpose of the existence and uniqueness of the solution (see [2] and [3] for the detail).

When considering compressible or incompressible Navier-Stokes with surface tensions, one compatibility condition has to be imposed in order to guarantee the well-posedness in previous literatures. In the compressible case, if \( \rho_0 \) and \( u_0 \) denote the initial fluid density and velocity respectively, by imposing the compatibility condition

\[
\{\mu \text{Def} u_0 + (\lambda \text{div} u_0 - p(\rho_0)) \text{Id}\} n(0) - \sigma H(0) n(0) = -p_x n(0) \quad \text{on} \quad \partial \Omega,
\]

Solonnikov & Tani [7] showed the existence of a unique solution. In the incompressible case, if \( P_{\text{tan}} \) denotes the projection map onto the tangent bundle of \( \partial \Omega \) and \( u_0 \) is the initial velocity, the first order compatibility condition reads

\[
P_{\text{tan}}(\text{Def} u_0 n(0)) = 0 \quad \text{on} \quad \partial \Omega,
\]

and under the assumption that \( u_0 \in H^2(\Omega) \) satisfies the first order compatibility condition Shkoller & Coutand [4] established the well-posedness of the equation. We remark that compatibility condition [4] does not involve the initial pressure \( p_0 \).

To illustrate the importance of our work in this paper, we emphasize that the compatibility conditions put a lot of constraint on the initial data, especially when considering the numerical simulation in which case the initial data can be given in almost arbitrary fashions. For example, for the incompressible case, if \( \Omega = B(0, 1) \) and \( u_0 \) is given by

\[
u_0(x, y) = (F(y), G(x)) .
\]

Then [4] holds only when

\[
F'(\sin \theta) + G'(\cos \theta) = 0 \quad \forall \theta \in (0, 2\pi),
\]

while we know that it is easy to find \( F \) and \( G \) such that [4] does not hold.

1.3. The difficulties. When looking for the solution of the velocity possessing only \( H^2 \) spatial regularity (which is the optimal Sobolev space for a strong solution to exist), it is not clear how the moving boundary \( \Gamma(t) \) is defined since the Lagrangian flow map \( \eta \) satisfying the ODE

\[
\eta_t(x, t) = u(\eta(x, t), t) \quad \forall x \in \Omega, t > 0 ,
\]

\[
\eta(x, 0) = x \quad \forall x \in \Omega ,
\]

is in general not solvable (uniquely) due to the lack of Lipschitz continuity (on the other hand, \( u \in C^{0, \alpha}(\Omega(t)) \) for all \( \alpha \in (0, 1) \) and \( t > 0 \) because of the Sobolev embedding \( C^{0, \alpha}(\Omega(t)) \hookrightarrow H^2(\Omega(t)) \) if \( n = 2 \)). Therefore, it is not adequate to describe
the moving boundary using the Lagrangian flow map $\eta$. Moreover, due to the lack of regularity of the solution, the nonlinearity appears to be much stronger, and some standard ways of constructing solutions fail to work (see Remark 5.6 for the detail), even though the a priori estimates can be easily derived.

1.4. Outlines. In Section 2 we introduce the ALE map which can describe the time-dependent domain $\Omega(t)$ with free boundary moving along with an $H^2$-velocity. However, reasoning in Remark 5.6, the ALE formulation is still not good enough for the purpose of constructing solutions with optimal regularity, so we slightly modify the ALE formulation in the last part of this section (for the purpose of constructing an approximated solution). The functional framework are then introduced in Section 3 and some preliminary results are established in this section as well. The main theorem is stated in Section 4, and we prove the main theorem from Section 5 to Section 8, including the introduction of an approximated regularized problem (with a smooth parameter $\varepsilon$) as well as the construction of a solution to this particular approximation in Section 5, the $\varepsilon$-independent estimates in Section 6, and the argument of the continuation of time in Section 7.

2. The ALE formulation

2.1. A map $\psi$ that maps from a fixed reference domain to $\Omega(t)$. Let $\Gamma$ be a smooth closed curve in the tubular neighborhood of $\partial \Omega$, $O$ be the region enclosed by $\Gamma$, and $N$ is the outward-pointing unit normal of $O$ such that each point $y \in \partial \Omega$ corresponds to a unique $x \in \Gamma$ such that $y = x + h_0(x)N(x)$ for some $h_0$; that is, $\partial \Omega$ is the graph of $h_0$ over $\Gamma$. Let $h : \Gamma \to \mathbb{R}$ denote the signed distance function which measure the signed distance from $\partial \Omega$ to $\Gamma$. In other words, if $x \in \Gamma$, then the point $x + h(x,t)N(x)$ belongs to the curve $\partial \Omega(t)$. Let $\psi$ be the harmonic extension of the map $e + hN$ on $\Gamma$; that is, $\psi$ solves

\[ \Delta \psi = 0 \quad \text{in } O, \quad (4a) \]
\[ \psi = e + hN \quad \text{on } \Gamma. \quad (4b) \]

We remark that if $\|h\|_{L^\infty(\Gamma)} \ll 1$, $\psi : O \to \Omega(t)$ is a diffeomorphism.

2.2. The representation of some geometric quantities. Let $\ell$ be the length of $\Gamma$, and $\Gamma$ be parametrized by the map $X : [-\ell/2, \ell/2] \to \mathbb{R}^2$ with arc-length $s$. Then the map

$\psi(X(s), t) = X(s) + h(X(s), t)N(X(s)) \quad \text{on } [-\ell/2, \ell/2]$ is a parametrization of the moving curve $\partial \Omega(t)$.

2.2.1. The metric. For any function $G$ defined on $\Gamma$, we use the notation $' \cdot ' s$ to denote the derivative with respect to the arc-length $s$; that is,

$G'(X(s), t) = \frac{\partial}{\partial s} G(X(s), t).$

Then $\psi' = X' \circ X^{-1} + h'N + hN'$. As a consequence, the metric $g$ induced by the map $\psi$ is given by

$g = \psi' \cdot \psi' = 1 + 2h_0h + h'^2 + h^2b_0^2 = (1 + b_0h)^2 + h'^2 \quad \text{on } \Gamma, \quad (5)$

where $b_0 = -(X'' \circ X^{-1}) \cdot N = (X' \circ X^{-1}) \cdot N'$ is the curvature of $\Gamma$ (since $|X'| = 1$).

We remark that since $\Gamma$ is assumed to be smooth, $b_0$ is a smooth function (of $s$).
2.2.2. The curvature. Since \( \psi' \) is tangent to \( \partial \Omega(t) \), we find that the normal vector \( n \) is given by

\[
n \circ \psi = \frac{-h'(X' \circ X^{-1}) + (1 + b_0 h)N}{\sqrt{(1 + b_0 h)^2 + h'^2}} = \frac{-h'(X' \circ X^{-1}) + (1 + b_0 h)N}{\sqrt{g}} \quad \text{on } \Gamma. \tag{6}
\]

Therefore, by the formula \( H \circ \psi = g^{-1} \psi'' \cdot (n \circ \psi) \) we obtain that

\[
H \circ \psi = \frac{(1 + b_0 h)h'' - b_0(1 + 2b_0 h + b_0^2 h^2 + 2h'^2) - hh'h'}{g^{3/2}}. \tag{7}
\]

**Remark 2.1.** In the methodology we employ, we need \( H \circ \psi \in H^{0.5}(\Gamma) \). If \( \partial \Omega \) is an \( H^{3.5} \)-surface, then we can simply let \( O = \Omega \) to proceed. However, since we will only assume that \( \partial \Omega \) is an \( H^2 \)-surface, the use of \( O = \Omega \) will result in that \( b_0 \in L^2(\Gamma) \), which implies that the curvature \( H \circ \psi \) at best belongs to \( H^{-1}(\Gamma) \) due to the presence of \( b_0 \) in (4). This is the reason why we choose a smooth \( O \) to start with.

2.3. Some basic identities concerning the map \( \psi \). Let \( J = \det(\nabla \psi) \), and \( A = (\nabla \psi)^{-1} \). Writing \( x = \psi(y) \), then the divergence theorem and the Piola identity suggest that

\[
\int_{\psi(O)} w \cdot ndS_y = \int_{\psi(O)} \text{div}w dx = \int_{\partial \psi(O)} JA^j_i (w^i \circ \psi)_j dy = \int_{\partial \psi(O)} JA^j_i (w^i \circ \psi)N_j dS_y.
\]

Since \( \psi : \partial O \to \partial \psi(O) \) is also a diffeomorphism, the change of variable formula implies that

\[
\int_{\partial \psi(O)} (w^i \circ \psi)(n^i \circ \psi)\sqrt{g}dS_y = \int_{\partial \psi(O)} w \cdot ndS_x = \int_{\partial \psi(O)} JA^j_i (w^i \circ \psi)N_j dS_y.
\]

The identity above holds for all smooth \( w \); thus we obtain that

\[
JA^TN = \sqrt{g}(n \circ \psi) \quad \text{on } \Gamma. \tag{8}
\]

In other words, the direction of the exterior normal \( n \) is parallel to the vector \( A^TN \), and the length of \( JA^TN \) is \( \sqrt{g} \), the square root of the metric.

2.4. The equations in ALE coordinate. Let \( v = u \circ \psi \) and \( q = p \circ \psi \) be the velocity and pressure in ALE coordinate. Taking the composition of (1) and the map \( \psi \), we find that the equation (1) is transformed to

\[
v^i_t + A^k_i v^k - \psi^i \psi^k v^k = A^k_i (A^j_k v^j + A^k_j v^j) \quad \text{in } \Omega \times (0, T), \tag{9a}
\]

\[
A^k_i v^j = 0 \quad \text{in } \Omega \times (0, T), \tag{9b}
\]

\[
[A^j_k v^i + A^k_j v^i] = \sigma(H \circ \psi)A^k_i N_k \quad \text{on } \Gamma \times (0, T), \tag{9c}
\]

\[
\Delta \psi = 0 \quad \text{in } \Omega \times (0, T), \tag{9d}
\]

\[
\psi = (e + hN) \quad \text{on } \Gamma \times (0, T), \tag{9e}
\]

\[
\psi_t \cdot (n \circ \psi) = (u \cdot n) \circ \psi \quad \text{on } \Gamma \times (0, T), \tag{9f}
\]

\[
v = v_0 \equiv u_0 \circ \psi_0 \quad \text{on } \Omega \times \{t = 0\}, \tag{9g}
\]

\[
h = h_0 \quad \text{on } \Gamma \times \{t = 0\}, \tag{9h}
\]

where \( J, A \) are defined in previous sub-section, and \( \psi_0 = \psi(0) \). The boundary condition (5) is obtained by taking the composition of (1) and the map \( \psi \), then applying identity (8). Furthermore, the curvature \( H \circ \psi \) in boundary condition (5) will be represented by (7).
We also remark that boundary condition (13f) reads that the speed of \( \partial \Omega(t) \) in the direction of exterior normal is the same as \( u \cdot n \). In other words, the boundary of \( \Omega(t) \) moves along with the fluid velocity.

2.5. The evolution equation of \( h \). The evolution equation of \( h \) is a direct consequence of the boundary condition (13, f). Differentiating (13) in time, then taking the inner product of the resulting equation and \( n \circ \psi \), by (13) we find that

\[
h_t(N \cdot (n \circ \psi)) = (u \cdot n) \circ \psi \quad \text{on } \Gamma \times (0, T).
\]

By (13) and (8), the equation above reads

\[
h_t = \frac{v \cdot (n \circ \psi)}{N \cdot (n \circ \psi)} = \frac{J \nu^i A^k L_k}{\sqrt{\xi} N \cdot (n \circ \psi)} = \frac{J A^T N}{1 + b_0 h} \cdot v \quad \text{on } \Gamma \times (0, T).
\]

We remark here that the denominator does not vanish for a short period of time if \( |h| \ll 1 \). Equation (10) is the evolution equation of \( h \).

2.6. A modification of the ALE formulation. For the purpose of constructing solutions, we modify the ALE formulation such that the new formulation keeps the structure of the divergence-free “velocity” field and the “pure” pressure gradient.

Let \( w^i = J A^i J w^i \) or equivalently \( v^i = J^{-1} \psi^i r w^r \). Then (13) together with the Piola identity implies that \( w \) is divergence-free. Moreover, since

\[
\psi^i A^k (A^j v^j + A^j \psi^j)_{,k} = \psi^i A^k [A^j (J^{-1} \psi^i r w^r)_{,j} + A^j (J^{-1} \psi^i r w^r)_{,j}]_{,k}
\]

\[
= \left[ \psi^i A^k (J^{-1} \psi^i r w^r)_{,j} + A^k (J^{-1} \psi^i r w^r)_{,s} \right]_{,k}
\]

\[
- (\psi^i A^k)_{,k} \left[ A^j (J^{-1} \psi^i r w^r)_{,j} + A^j (J^{-1} \psi^i r w^r)_{,j} \right]
\]

and

\[
\psi^i A^k \left[ A^j v^j + A^j \psi^j - q \delta^i_k \right] A^k N_k
\]

\[
= \left[ \psi^i A^k A^j (J^{-1} \psi^i r w^r)_{,j} + A^k (J^{-1} \psi^i r w^r)_{,s} - q \delta^i_k \right] N_k,
\]

if \( L \psi \) denotes the second order differential operator given by

\[
[L \psi(w)]^a = \left[ \psi^i A^k A^j (J^{-1} \psi^i r w^r)_{,j} + A^k (J^{-1} \psi^i r w^r)_{,s} - q \delta^i_k \right] N_k,
\]

and \( \ell \psi(w, q) \) denotes the boundary operator given by

\[
[\ell \psi(w, q)]^a = \left[ \psi^i A^k A^j (J^{-1} \psi^i r w^r)_{,j} + A^k (J^{-1} \psi^i r w^r)_{,s} - q \delta^i_k \right] N_k,
\]

we find that \( (w, q) \) satisfies the following equation

\[
J^{-1} \psi^i A^j \psi^j w^r - [L \psi(w)]^a + q_s = F^a \quad \text{in } O \times (0, T), \quad (13a)
\]

\[
\text{div} w = 0 \quad \text{in } O \times (0, T), \quad (13b)
\]

\[
\ell \psi(w, q) = \sigma(H \circ \psi) N \quad \text{on } \Gamma \times (0, T), \quad (13c)
\]

\[
\Delta \psi = 0 \quad \text{in } O \times (0, T), \quad (13d)
\]

\[
\psi = e + h N \quad \text{on } \Gamma \times (0, T), \quad (13e)
\]

\[
h_t = \frac{w \cdot N}{1 + b_0 h} \quad \text{on } \Gamma \times (0, T), \quad (13f)
\]

\[
w = w_0 \quad \text{on } O \times \{ t = 0 \}, \quad (13g)
\]

\[
h = h_0 \quad \text{on } \Gamma \times \{ t = 0 \}, \quad (13h)
\]
where
\[ F^s = - \psi^s A^s_p (J^{-1} \psi^p, w^r - \psi^p) + \psi^s (J^{-1} \psi^p, w^s)_j - \psi^s (J^{-1} \psi^p)_j. \]
(14)

Letting \( F = F - [J^{-1} (\nabla \psi)^T (\nabla \psi) - \text{Id}] w_t \), (13) can be rewritten as
\[ w_t - L_p(w) + \nabla q = F \quad \text{in} \quad O \times (0, T). \] (13′)

3. Notation and preliminary results

3.1. The energy spaces \( \mathcal{V}(T), \mathcal{H}(T), \mathcal{W}(T), \mathcal{H}_1(T) \). Let \( \mathcal{V}(T) \) denote the space (of solutions \( v \))
\[ \mathcal{V}(T) = \left\{ v \in L^2(0, T; H^2(\Omega)) \mid v_t \in L^2(0, T; L^2(\Omega)) \right\}, \]
equipped with norm
\[ \|v\|_{\mathcal{V}(T)} = \|v\|_{L^2(0, T; H^2(\Omega))} + \|v_t\|_{L^2(0, T; L^2(\Omega))}, \]
\( \mathcal{Q}(T) \) denote the space (of solutions \( q \)) \( L^2(0, T; H^1(\Omega)) \), and \( \mathcal{H}(T) \) denote the space (of solutions \( h \))
\[ \mathcal{H}(T) = \left\{ h \in L^2(0, T; H^{2.5}(\Gamma)) \mid h_t \in L^2(0, T; H^{1.4}(\Gamma)) \right\}, \]
(in which the number 1.4 can be replaced by any number closed to but less than 1.5) equipped with norm
\[ \|h\|_{\mathcal{H}(T)} = \|h\|_{L^2(0, T; H^{2.5}(\Gamma))} + \|h_t\|_{L^2(0, T; H^{1.4}(\Gamma))}. \]
We also define two spaces \( \mathcal{W}(T) \) and \( \mathcal{H}_1(T) \) for the purpose of constructing approximated solutions. The space \( \mathcal{W}(T) \) is the collection of all \( w \in \mathcal{V}(T) \) such that \( w \in L^2(0, T; H^2(\Gamma)) \); that is
\[ \mathcal{W}(T) = \left\{ w \in \mathcal{V}(T) \mid w \in L^2(0, T; H^2(\Gamma)) \right\}, \]
and the norm \( \|\cdot\|_{\mathcal{W}(T)} \) is given by
\[ \|w\|_{\mathcal{W}(T)} = \|w\|_{\mathcal{V}(T)} + \|w_t\|_{L^2(0, T; H^2(\Gamma))}. \]
The space \( \mathcal{H}_1(T) \), on the other hand, is not a subspace of \( \mathcal{H}(T) \). It is given by
\[ \mathcal{H}_1(T) = \left\{ h \in L^2(0, T; H^2(\Gamma)) \mid h_t \in L^2(0, T; H^2(\Gamma)) \right\}, \]
equipped with norm
\[ \|h\|_{\mathcal{H}_1(T)} = \|h\|_{L^2(0, T; H^2(\Gamma))} + \|h_t\|_{L^2(0, T; H^{1.4}(\Gamma))}. \]
By the fundamental theorem of Calculus,
\[ \sup_{t \in [0, T]} \|v(t)\|_{H^1(\Omega)} \leq \|v(0)\|_{H^1(\Omega)} + C\|v\|_{\mathcal{V}(T)}, \quad (15a) \]
\[ \sup_{t \in [0, T]} \|h(t)\|_{H^2(\Gamma)} \leq \|h(0)\|_{H^2(\Gamma)} + \|h_t\|_{\mathcal{H}_1(T)}. \quad (15b) \]

3.2. A useful lemma. By interpolation, we can derive the following useful

Lemma 3.1. Suppose that \( f \in H^s(\Gamma) \) for some \( s > 1/2 \), and \( g \in H^{0.5}(\Gamma) \). Then \( fg \in H^{0.5}(\Gamma) \)
and
\[ \|fg\|_{H^{0.5}(\Gamma)} \leq C_s \|f\|_{H^{s}(\Gamma)} \|g\|_{H^{0.5}(\Gamma)} \quad (16) \]
for some generic constant \( C_s > 0 \).
3.3. The horizontal convolution-by-layers operator and a commutator type estimate. Let \( \eta : \mathbb{R} \to \mathbb{R} \) be a non-negative smooth function supported in the interval \( (-\frac{\ell}{4}, \frac{\ell}{4}) \) and satisfying

\[
\int_{-\ell/4}^{\ell/4} \eta(s) \, ds = 1.
\]

Let \( \eta_\varepsilon(s) = \frac{1}{\varepsilon} \eta\left(\frac{s}{\varepsilon}\right) \) for all \( \varepsilon > 0 \). Given a function \( f \) defined on \( \Gamma \), we define the horizontal convolution \( \eta_\varepsilon \ast f \) by

\[
f_\varepsilon(x) = (\eta_\varepsilon \ast f)(x) = (\eta_\varepsilon \ast (f \circ X))(X^{-1}(x)) = \int_{-\ell/2}^{\ell/2} \eta_\varepsilon(X^{-1}(x) - \tilde{s}) f(X(\tilde{s})) \, d\tilde{s} \quad \forall x \in \Gamma.
\]

Since every point \( x \) can be identified as \( X(s) \) for a unique \( s \in [-\ell/2, \ell/2] \), we also write the equation above as

\[
f_\varepsilon(s) = (\eta_\varepsilon \ast f)(s) = \int_{-\ell/2}^{\ell/2} \eta_\varepsilon(s - \tilde{s}) f(\tilde{s}) \, d\tilde{s} \quad \text{if} \quad x = X(s).
\]

We note that in the equation above, \( f(\tilde{s}) \) is understood as \( f(X(\tilde{s})) \). Moreover, given two parameters \( \varepsilon \) and \( \eta \), we have

\[
[\eta_\varepsilon \ast (\eta_\varepsilon \ast f)](s) = \int_{-\ell/2}^{\ell/2} \frac{1}{\varepsilon} \eta\left(\frac{s'}{\varepsilon}\right) \int_{-\ell/2}^{\ell/2} \frac{1}{\varepsilon} \eta\left(\frac{s''}{\varepsilon}\right) f(s'' + s' - s) ds'' ds' = [\eta_\varepsilon \ast (\eta_\varepsilon \ast f)](s);
\]

thus two horizontal convolution commute.

Having the convolution on \( \Gamma \) defined, we define the commutator of the horizontal convolution operator \( \eta_\varepsilon \ast \) and a function \( f \) by

\[
[[\eta_\varepsilon \ast, f] g](s) = (\eta_\varepsilon \ast (fg) - f(\eta_\varepsilon \ast g))(s) = \int_{-\ell/2}^{\ell/2} \eta_\varepsilon(s - \tilde{s}) [f(\tilde{s}) - f(s)] g(\tilde{s}) \, d\tilde{s}.
\]

Then the mean value theorem and Young’s inequality imply that

\[
\|[\eta_\varepsilon \ast, f] g\|_{L^2(\Gamma)} \leq \varepsilon \|f'\|_{L^\infty(\Gamma)} \|g\|_{L^2(\Gamma)}.
\]

Moreover,

\[
([\eta_\varepsilon \ast, f] g)'(s) = \int_{-\ell/2}^{\ell/2} \eta_\varepsilon(s - s') [f(s') - f(s)] g(s') \, ds' - \int_{-\ell/2}^{\ell/2} \eta_\varepsilon(x - y) f'(s) g(s') \, ds';
\]

thus

\[
\|[\eta_\varepsilon \ast, f] g'\|_{L^2(\Gamma)} \leq \|[\eta_\varepsilon \ast, f'\|_{L^\infty(\Gamma)} \|g\|_{L^2(\Gamma)} + \|[f']\|_{L^\infty(\Gamma)} \|g\|_{L^2(\Gamma)}
\]

\[
\leq C \|[f']\|_{L^\infty(\Gamma)} \|g\|_{L^2(\Gamma)}.
\]

For any given \( f \) defined on \( \mathbb{R}^2_+ \), we can also define the horizontal convolution-by-layers operator \( \Lambda_\varepsilon \) by

\[
\Lambda_\varepsilon f(y_1, y_2) = \int_{\mathbb{R}} \eta_\varepsilon(y_1 - z_1) f(z_1, y_1) \, dz_1 \quad \forall f(\cdot, y_2) \in L^1(\mathbb{R}).
\]

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It should be clear that $\Lambda_\varepsilon$ smooths functions defined on $\mathbb{R}^2$ along horizontal $y_1$-direction, but does not smooth these functions in the vertical $y_2$-direction. In addition, we can restrict the operator $\Lambda_\varepsilon$ to act on functions $f : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ as well, in which case $\Lambda_\varepsilon$ becomes the usual mollification operator $\eta_\varepsilon$.

By standard properties of convolution, there exists a constant $C$ which is independent of $\varepsilon$, such that for $s \geq 0$,

$$\|\Lambda_\varepsilon F\|_{H^s(\mathbb{R}^2_+)} \lesssim C \|F\|_{H^s(\mathbb{R}^2_+)} \quad \forall F \in H^s(\mathbb{R}^2_+),$$

and

$$\|\Lambda_\varepsilon F\|_{H^s(\mathbb{R}^2_+)} \lesssim C \|F\|_{H^s(\mathbb{R}^2_+)} \quad \forall F \in H^s(\mathbb{R}^2_+).$$

Furthermore,

$$\epsilon \|\Lambda_\varepsilon F,1\|_{L^2(\mathbb{R}^2_+)} \leq C \|F\|_{L^2(\mathbb{R}^2_+)} \quad \forall F \in L^2(\mathbb{R}^2_+). \tag{20}$$

Similar to (18) and (19), we also have

$$\left\|\left(\Lambda_\varepsilon, f\right)\right\|_{L^2(\mathbb{R}^2_+)} \leq C \epsilon \|f,1\|_{L^2(\mathbb{R}^2_+)} \|g\|_{L^2(\mathbb{R}^2_+)}, \tag{21a}$$

$$\left\|\left(\Lambda_\varepsilon, f\right)\right\|_{L^2(\mathbb{R}^2_+)} \leq C \epsilon \|f,1\|_{L^2(\mathbb{R}^2_+)} \|g\|_{L^2(\mathbb{R}^2_+)}. \tag{21b}$$

### 3.4. The generalized Gronwall inequality

In the process of performing the nonlinear estimates, we need the following Gronwall type inequality.

**Theorem 3.2.** Let $X$ be a non-negative continuous function of $t$, and satisfy that for some positive constants $C$, $M$, $T_1$ and polynomial $P$,

$$X(t) \leq M + C t P(X(t)) \quad \forall t \in [0,T_1].$$

Then there is a $T \in (0,T_1]$ such that $X(t) \leq 2M$ for all $t \in [0,T]$.

### 3.5. The Lagrangian multiplier lemma

Let $V \equiv \{u \in H^1(\Omega) \mid u \in H^1(\Gamma)\}$ equipped with norm

$$\|u\|_V = \left[\|u\|_{L^2(\Omega)}^2 + \|\text{Def} u\|_{L^2(\Omega)}^2 + \|u'\|_{L^2(\Gamma)}^2\right]^{1/2} \tag{22}$$

which is induced by the inner product

$$(u, v)_V = (u, v)_{L^2(\Omega)} + (\text{Def} u, \text{Def} v)_{L^2(\Omega)} + (u', v')_{L^2(\Gamma)}.$$ We note that by Korn’s inequality, the norm defined by (22) is equivalent to the norm

$$\|u\| = \|u\|_{H^1(\Omega)} + \|u\|_{H^1(\Gamma)}.$$

**Lemma 3.3.** Let $T : V \rightarrow \mathbb{R}$ be a bounded linear functional satisfying that $T(\varphi) = 0$ whenever $\text{div} \varphi = 0$. Then there exists a unique $q \in L^2(\Omega)$ such that

$$T(\varphi) = (q, \text{div} \varphi)_{L^2(\Omega)} \quad \forall \varphi \in V.$$ Moreover, for some constant $c > 0$,

$$\frac{1}{c} \|q\|_{L^2(\Omega)} \leq \|T\|_{B(V,\mathbb{R})} = \sup_{\|\varphi\|_V = 1} T(\varphi). \tag{23}$$
Proof. For any given $p \in L^2(O)$, define $L_p(\varphi) = \langle p, \text{div}\varphi \rangle_{L^2(O)}$. Then $L_p : V \to \mathbb{R}$ is a bounded linear functional. By the Riesz representation theorem, there exists $Qp \in V$ such that

$$L_p(\varphi) = \langle Qp, \varphi \rangle \quad \forall \varphi \in V$$

and $\|Qp\|_V = \|L_p\|_{B(V, \mathbb{R})} \leq c_1 \|p\|_{L^2(O)}$. On the other hand, for any $p \in L^2(O)$, there exists $u_p \in V$ such that

$$\text{div} u_p = p \quad \text{in } O \quad (24)$$

and satisfies $\|u_p\|_V \leq c_2 \|p\|_{L^2(O)}$. In fact, with $\bar{p}$ denoting the average of $p$ over $O$; that is, $\bar{p} = \frac{1}{|O|} \int_O p \, dx$, if $v$ solves

$$\text{div} v = p - \bar{p} \quad \text{in } O,$$

$$v = 0 \quad \text{on } \Gamma,$$

and satisfies $\|v\|_{H^1(O)} \leq C\|p - \bar{p}\|_{L^2(O)}$, then $u_p(x, y) \equiv v(x, y) + \frac{\bar{p}}{2}(x, y)$ belongs to $V$ and satisfies (24) and

$$\|u_p\|_{H^1(O)} \leq \|v\|_{H^1(O)} + C\|\bar{p}\|_{L^2(O)}.$$

Therefore,

$$\|p\|_{L^2(O)}^2 = \langle p, \text{div} u_p \rangle_{L^2(O)} = \langle Qp, u_p \rangle_V \leq \|Qp\|_V \|u_p\|_V \leq c_2 \|Qp\|_V \|p\|_{L^2(O)}$$

which implies that $\|p\|_{L^2(O)} \leq c_2 \|Qp\|_V$ for all $p \in L^2(O)$. As a consequence,

$$\frac{1}{c_2} \|p\|_{L^2(O)} \leq \|Qp\|_V \leq c_1 \|p\|_{L^2(O)} \quad \forall p \in L^2(O),$$

so $Q : L^2(O) \to V$ is one-to-one, and $R(Q)$, the range of $Q$, is closed. Therefore, $V = R(Q) \oplus_V R(Q) \perp$. Let $P : V \to R(Q)$ be the projection. Then for any $u \in V$,

$$u = Pu + (\text{Id} - P)u,$$

where $(\text{Id} - P)u \in R(Q) \perp$. Moreover,

$$v \in R(Q) \perp \Leftrightarrow \langle Qq, v \rangle_V = 0 \quad \forall q \in L^2(O) \Leftrightarrow \langle q, \text{div} v \rangle_{L^2(O)} = 0 \quad \forall q \in L^2(O)$$

$$\Leftrightarrow \text{div} v = 0.$$

Let $w \in V$ be the representation of $T$; that is, $T(\varphi) = \langle w, \varphi \rangle_V$ for all $\varphi \in V$. Then $Pw = Qq$ for some $q \in L^2(O)$, and for any given $\varphi \in V$,

$$T(\varphi) = T(P\varphi) = \langle w, P\varphi \rangle_V = \langle Pw, P\varphi \rangle_V = \langle Qq, P\varphi \rangle_V$$

$$= \langle Qq, \varphi \rangle_V = L_q(\varphi) = \langle q, \text{div} \varphi \rangle_{L^2(O)}.$$

Finally, by the solvability of (24),

$$\|q\|_{L^2(O)} = \sup_{\|p\|_{L^2(O)} = 1} \|q, p\|_{L^2(O)} \leq \sup_{\|\varphi\|_V \leq c_2} \|q, \text{div} \varphi\|_{L^2(O)} = \sup_{\|\varphi\|_V \leq c_2} T(\varphi),$$

and (23) follows immediately.
3.6. Elliptic regularity. In this sub-section we study an important regularity theory which is the foundation of our result and is very similar to the regularity of the steady Stokes problem. The proof is provided in Appendix A.

**Theorem 3.4.** Let \( a_{rs}^{jk} \) be a \((2, 2)\)-tensor such that \( a_{rs}^{jk} = a_{sr}^{jk} = a_{sr}^{jk} \), and satisfy
\[
\| a_{rs}^{jk} - \lambda_1 \delta_k^r \delta_s^j - \lambda_2 \delta_k^s \delta_r^j \|_{L^\infty(O)} \ll 1
\]  
for some positive constants \( \lambda_1 \) and \( \lambda_2 \).

1. Suppose that \((w, q) \in V \times L^2(O)\) is a weak solution to the following elliptic equation
\[
-\left[ a_{rs}^{jk} w_{rj}^s \right]_k + q_s = f^s \quad \text{in} \quad O, \\
\text{div}w = 0 \quad \text{in} \quad O, \\
a_{rs}^{jk} w_{rj}^s N_k - q N_s = \varepsilon \Delta_0 w^s + g^s \quad \text{on} \quad \Gamma;
\]
that is, \((w, q)\) satisfies the variational formulation
\[
\left( a_{rs}^{jk} w_{rj}^s, \varphi^s \right)_{L^2(O)} - (q, \text{div} \varphi)_{L^2(O)} + \varepsilon (w', \varphi')_{L^2(\Gamma)} = (f, \varphi)_{L^2(O)} + (g, \varphi)_{L^2(\Gamma)} \quad \forall \varphi \in V.
\]
Then \((w, q) \in H^2(O) \times H^1(\Gamma)\), and there are constants \( C \) and \( C_\varepsilon \) such that
\[
\| w \|^2_{H^2(O)} + \| q \|^2_{H^2(\Gamma)} + \| q \|^2_{H^1(O)} + \| f \|^2_{L^2(O)} + \| g \|^2_{L^2(\Gamma)} \leq C \left( 1 + \| a \|^2_{L^\infty(O)} \right) \times
\]
\[
\left[ \varepsilon \Delta_0 w^s + g^s \right]_{H^2(O)} + \left[ f \right]_{L^2(O)} + \left[ g \right]_{L^2(\Gamma)}.
\]

2. Suppose that \( a \in W^{1, 4}(O) \), and \((w, q) \in H^1_0(O) \times L^2(O)\) is a weak solution to the following elliptic equation
\[
-\left[ a_{rs}^{jk} w_{rj}^s \right]_k + q_s = f^s \quad \text{in} \quad O, \\
\text{div}w = 0 \quad \text{in} \quad O, \\
w = 0 \quad \text{on} \quad \Gamma;
\]
that is,
\[
\left( a_{rs}^{jk} w_{rj}^s, \varphi^s \right)_{L^2(O)} - (q, \text{div} \varphi)_{L^2(O)} = (f, \varphi)_{L^2(O)} \quad \forall \varphi \in H^1_0(O). 
\]
Then \((w, q) \in H^2(O) \times H^1(\Gamma)\) satisfies
\[
\| w \|^2_{H^2(O)} + \| q \|^2_{H^2(\Gamma)} \leq C \left[ 1 + \| f \|^2_{L^2(O)} + \| \nabla a \|^2_{L^2(O)} \right].
\]

The following corollary is a direct consequence of the second part of the theorem above.

**Corollary 3.5.** Suppose that \( a \in W^{1, 4}(O) \), and \((w, q) \in H^2(O) \times H^1(\Gamma)\) is a solution to the following elliptic equation
\[
-\left[ a_{rs}^{jk} w_{rj}^s \right]_k + q_s = f^s \quad \text{in} \quad O, \\
\text{div}w = g \quad \text{in} \quad O, \\
w = h \quad \text{on} \quad \Gamma;
\]
for some \( f \in L^2(\Omega), \ g \in H^1(\Omega) \) and \( h \in H^{1.5}(\Omega) \) with \( \int_{\Gamma} h \cdot N dS = \int_{\Omega} g dx \). Then \( (w, q) \in H^2(\Omega) \times H^1(\Omega) \), and satisfies

\[
\|w\|_{H^2(\Omega)}^2 + \|q\|_{H^1(\Omega)}^2 \leq C\left[1 + \|f\|_{L^2(\Omega)}^2 + \|g\|_{H^1(\Omega)}^2 + \|\nabla a\|_{L^1(\Omega)}^4 \|\nabla w\|_{L^2(\Omega)}^2 + \|h\|_{H^{1.5}(\Omega)}^2 + \|\nabla a\|_{L^1(\Omega)}^2 \|h\|_{H^1(\Omega)}^2\right].
\]

(32)

**Remark 3.6.** When \( a^{jk} = \delta^i_j \delta^k_s \), Part (2) of Theorem 3.5 is the (lowest order) regularity result for the steady Stokes equations. We also note that condition (25) can also be replaced by the usual ellipticity assumption

\[ a^{ik}\xi^k_j \xi^i_l \geq \lambda_1 \xi^i_j \xi^i_l + \lambda_2 \xi^i_j \xi^i_l \]

without any modifications of the proof. Moreover, the condition \( a \in W^{1,\infty}(\Omega) \) in part (1) can be relaxed to \( a \in W^{1,4}(\Omega) \); however, it is not used in our paper, so we omit the detail.

4. **Main Theorem**

Now we state the main result that we establish in this paper.

**Theorem 4.1.** For any bounded smooth domain \( \Omega \subseteq \mathbb{R}^2 \), there is a number \( \varsigma > 0 \) such that for all \( u_0 \in H^1(\Omega) \) and \( h_0 \in H^2(\Gamma) \) satisfying \( \|h_0\|_{H^{1.5}(\Gamma)} < \varsigma \), there exists a unique solution \((v, h, q) \in \mathcal{V}(T) \times \mathcal{H}(T) \times \mathcal{Q}(T)\) to equation (11) for some \( T > 0 \). Moreover, \((v, h, q)\) satisfies

\[
\|v\|_{\mathcal{V}(T)} + \|q\|_{\mathcal{Q}(T)} + \|h\|_{\mathcal{H}(T)} \leq C\left[1 + \|u_0\|_{H^1(\Omega)} + \|h_0\|_{H^2(\Gamma)}\right].
\]

(33)

**Remark 4.2.** If the initial domain \( \Omega \) has smooth boundary, we simply let \( \Omega = \Omega_0 \) and choose \( h_0 = 0 \) which certainly satisfies the condition \( \|h_0\|_{H^{1.5}(\Gamma)} < \varsigma \).

**Remark 4.3.** The fluid velocity in Eulerian coordinate is given by \( u = v \circ \psi^{-1} \). Since \((v, \psi) \in L^2(0, T; H^2(\Omega)) \times L^\infty(0, T; H^{2.5}(\Omega))\), the fluid velocity \( u \) belongs to \( H^2(\Omega(t)) \) for almost all \( t \in [0, T] \), where \( \Omega(t) \) is given by \( \Omega(t) = \Omega(\Omega, t) \). However, as mentioned in Section 3.3 the existence of the flow map \( \eta \) is not guaranteed since \( u \) is not Lipschitz in spatial variable. Therefore, our result does not implies the existence of flow map, but implies that the existence of fluid velocity \( u \) and fluid domain \( \Omega(t) = \psi(\Omega, t) \) such that (11–f) holds.

**Remark 4.4.** Our theorem can also by applied to the case that \( \Omega \) is separated by an interface \( \Gamma(t) \) such that \( \Omega = \Omega^+(t) \cup \Gamma(t) \cup \Omega^-(t) \), where \( \Gamma(t) = \partial \Omega^-(t) \).

5. **The construction of solutions**

5.1. **The regularized problem.** Without loss of generality, we assume that the surface tension \( \sigma = 1 \). For any fixed \( \varepsilon > 0 \), we consider an approximation of (13)
The linearization of the regularized problem.

Let

\[ p \rightarrow \text{smooth}, \]

following linearization of (34):

\[ \begin{align*}
J^{-1} \psi_t \psi_t^i w_t^r - [L_\psi(w)]^s + q_s &= F^s \\
\text{div} w &= 0 \\
\ell_\psi(w, q) &= \mathcal{L}_\varepsilon(h)N + \varepsilon^2 \Delta_0 w \\
\Delta_\psi &= 0 \\
\psi &= e + h_{\varepsilon\varepsilon}N \\
h_t &= \frac{w \cdot N}{1 + b_0 h_{\varepsilon\varepsilon}} \\
w &= w_{0\varepsilon} \\
h &= h_{0\varepsilon}
\end{align*} \]

where as before \( J = \det(\nabla \psi) \) and \( A = (\nabla \psi)^{-1} \), \( F \) is given by (14), \( e \) is the identity map, \( h_{\varepsilon\varepsilon} \equiv \eta_\varepsilon \ast (\eta_\varepsilon \ast h) \) is the double horizontal convolution of \( h \), \( \Delta_0 \) is the Laplace-Beltrami operator defined by

\[ \Delta_0 w = (w \circ X)^\prime \circ X^{-1} \quad \text{on} \; \Gamma, \]

and \( \mathcal{L}_\varepsilon \) is a differential operator defined by

\[ \mathcal{L}_\varepsilon(h) \equiv \frac{(1 + b_0 h_{\varepsilon\varepsilon}) h'' - b_0 [(1 + b_0 h_{\varepsilon\varepsilon})^2 + 2 h_{\varepsilon\varepsilon} h_{\varepsilon\varepsilon} h'' + h_{\varepsilon\varepsilon}^2] - h_{\varepsilon\varepsilon} h_{\varepsilon\varepsilon}^2 h''}{[(1 + b_0 h_{\varepsilon\varepsilon})^2 + h_{\varepsilon\varepsilon}^2]^{3/2}}. \]

We note that the solution \( (w, q, \psi, h) \) depends on \( \varepsilon \).

### 5.2. The linearization of the regularized problem

Define

\[ C_T(M) = \left\{ (w, h) \in \mathcal{W}(T) \times \mathcal{H}_1(T) \left| \|w\|_{\mathcal{W}(T)} + \|h\|_{\mathcal{H}_1(T)} \leq M, \; w(0) = w_{0\varepsilon}, \; h(0) = h_{0\varepsilon}, \; h_t(0) = \frac{w_{0\varepsilon} \cdot N}{1 + b_0 h_{0\varepsilon\varepsilon}} \right. \right\} \]

for some \( M > 1 \) and \( T < 1 \) to be determined later. We note that if \( M \gg 1 \) and \( T > 0 \) small enough, then \( C_T(M) \) is non-empty since

\[ \left( w_{0\varepsilon}, h_{0\varepsilon} + \frac{t w_{0\varepsilon} \cdot N}{1 + b_0 h_{0\varepsilon\varepsilon}} \right) \in C_T(M). \]

Let \((\bar{w}, \bar{h}) \in C_T(M)\) be given. For a fixed \( \varepsilon > 0 \), let \( \bar{\psi} \) be the solution to

\[ \begin{align*}
\Delta \bar{\psi} &= 0 \quad \text{in} \; \Omega, \\
\bar{\psi} &= e + \bar{h}_{\varepsilon\varepsilon}N \quad \text{on} \; \Gamma,
\end{align*} \]

where we recall that \( \bar{h}_{\varepsilon\varepsilon} \equiv \eta_\varepsilon \ast (\eta_\varepsilon \ast \bar{h}) \). We note that since \( \Omega \) is smooth and \( \bar{h}_{\varepsilon\varepsilon} \) is smooth, \( \bar{\psi} \) is a smooth diffeomorphism if \( h \ll 1 \).

Let \( \bar{A} = (\nabla \bar{\psi})^{-1} \) and \( \bar{J} = \det(\nabla \bar{\psi}) \) be defined according. We consider the following linearization of (34):

\[ \begin{align*}
w_t - L_{\bar{\psi}}(w) + \nabla q &= \bar{F} \\
\text{div} w &= 0 \\
\ell_{\bar{\psi}}(w, q) &= \varepsilon^2 \Delta_0 w + \bar{G} \quad \text{on} \; \Gamma \times (0, T),
\end{align*} \]
where
\[
\hat{F}^s = (\delta_s' - \bar{J}^{-1}\bar{\psi}_{i,s}^1 \hat{w}_t^i - \overline{\bar{\psi}}_s^i \bar{A}_x^i (\bar{J}^{-1}\bar{\psi}_{i,r}^1 \hat{w}_r^i - \bar{\psi}_s^i (\bar{J}^{-1}\bar{\psi}_{r,s}^1 \hat{w}_s^i),j \\
- \bar{\psi}_s^i (\bar{J}^{-1}\bar{\psi}_{r,s}^1 \hat{w}_r^i) w_i^r + (\bar{\psi}_s^i \bar{A}_x^k)^o k [\bar{A}_x^i (\bar{J}^{-1}\bar{\psi}_{r,s}^1 \hat{w}_r^i),j + \bar{A}_x^i (\bar{J}^{-1}\bar{\psi}_{s,r}^1 \hat{w}_s^i),j]),
\]
\[
\hat{G} = \frac{1+b_0\overline{\bar{h}}_{zz}}{(1+b_0\overline{\bar{h}}_{zz})^2 + \overline{\bar{h}}_{zz}^2} \frac{b_0}{b_0} - \frac{b_0}{b_0^2}
\]

**Definition 5.1.** A vector-valued function \(w \in L^2(0,T;H^1(O)) \cap L^2(0,T;H^1(\Gamma))\) satisfying \(w_t \in L^2(0,T;L^2(O))\) is said to be a weak solution to (38) if

\[
(w_t, \varphi)_{L^2(O)} + B_{\hat{G}}(w, \varphi) + \varepsilon^2(w', \varphi')_{L^2(\Gamma)} = (\hat{F}, \varphi)_{L^2(O)} + (\hat{G}, \varphi)_{L^2(\Gamma)} \quad \forall \varphi \in H^1_{\text{div}}(O), \quad a.e. \ t \in [0,T],
\]

and

\[
w = w_0 \quad \text{on} \quad O \times \{t = 0\},
\]

where \(B_{\hat{G}} : H^1(O) \times H^1(O) \to \mathbb{R}\) is a bilinear form given by

\[
B_{\hat{G}}(w, \varphi) = \int_0^T \left[ \bar{A}_x^i \overline{\bar{\psi}}_{i,s}^1 \bar{A}_x^k (\bar{J}^{-1}\bar{\psi}_{r,s}^1 \hat{w}_r^i,w_i^r) + \bar{A}_x^k (\bar{J}^{-1}\bar{\psi}_{s,r}^1 \hat{w}_s^i,\hat{w}_s^i) \right] \varphi_k \, dx,
\]

and \(H^1_{\text{div}}(O) \equiv \{v \in H^1(O) \mid \text{div}v = 0\}\) is the collection of divergence-free \(H^1\) vector-valued functions.

5.3. **Some a priori estimates.** In this sub-section, we establish some estimates concerning the smallness of certain important quantities which will be used throughout the paper. We remark that even though we have better regularity for the input \(\overline{\bar{h}}\), we perform the following estimates under the assumption that \((\overline{\bar{h}}, \overline{\bar{h}}_t) \in L^\infty(0,T;H^2(\Gamma)) \times L^2(0,T;H^2(\Gamma)).

**Proposition 5.2.** For any \(\zeta > 0\) (which is an upper bound of \(\|h_0\|_{H^1(\Gamma)}\)), there exists \(T_M > 0\) such that

\[
\|\overline{\bar{h}}(t)\|_{H^1(\Gamma)} < \zeta \quad \forall \ t \in [0,T_M],
\]

**Proof.** By \(H^{0.25} - H^{-0.25}\) duality,

\[
\frac{1}{2} \frac{d}{dt} \|\overline{\bar{h}}\|^2_{H^1(\Gamma)} \leq \|\overline{\bar{h}}\|_{H^2(\Gamma)} \|\overline{\bar{h}}_t\|_{H^{1.5}(\Gamma)} \leq M \|\overline{\bar{h}}\|_{H^{1.5}(\Gamma)}
\]

which suggests that

\[
\|\overline{\bar{h}}(t)\|^2_{H^1(\Gamma)} \leq \|h_0\|^2_{H^{1.5}(\Gamma)} + \int_0^t M \|\overline{\bar{h}}_t\|_{H^{1.5}(\Gamma)} d\tilde{t}
\]

\[
\leq \|h_0\|^2_{H^1(\Gamma)} + \sqrt{T} M \|\overline{\bar{h}}_t\|_{L^2(0,T; H^{1.5}(\Gamma))} \leq \|h_0\|^2_{H^{1.5}(\Gamma)} + \sqrt{T} M^2.
\]

Since \(\|h_0\|_{H^{1.5}(\Gamma)} < \zeta\), the inequality above suggests that

\[
\|\overline{\bar{h}}(t)\|^2_{H^1(\Gamma)} < \zeta^2 \quad \forall \ t \in [0,T_M].
\]

provided that \(T_M > 0\) is chosen small enough.

**Corollary 5.3.** There exists \(\zeta < 1\) (with corresponding \(T_M\) given in Proposition 5.3) such that

\[
\|\nabla \overline{\bar{\psi}}(t) - \text{Id}\|_{L^\infty(O)} + \|\bar{A}(t) - \text{Id}\|_{L^\infty(O)} + \|\bar{J}(t) - 1\|_{L^\infty(O)} \leq C\zeta \quad \forall \ t \in [0,T_M].
\]
for some generic constant $C$ independent of $\varsigma$. In particular,
\[ \frac{1}{2} \leq \| \mathbf{J}(t) \|_{L^\infty(\Omega)} \leq \frac{3}{2} \quad \forall t \in [0, T_M]. \] (43)

**Proof.** By the Sobolev embedding and elliptic regularity, for $t \in [0, T_M]$,
\[ \| \nabla \tilde{\psi}(t) - \text{Id} \|_{L^\infty(\Omega)} \leq C \| \nabla \tilde{\psi} - \text{Id} \|_{H^{1/2}(\Omega)} \leq C \| \tilde{h} \|_{H^{1/2}(\Gamma)} \leq C \varsigma. \]
Therefore, since $\tilde{A} - \text{Id} = \tilde{A} (\text{Id} - \nabla \tilde{\psi})$,
\[ \| \tilde{A}(t) - \text{Id} \|_{L^\infty(\Omega)} \leq C \| \tilde{A} \|_{L^\infty(\Omega)} \varsigma \quad \forall t \in [0, T_M]. \] (44)
On the other hand, $\| \tilde{A} \|_{L^\infty(\Omega)} - 1 \leq \| \tilde{A} - \text{Id} \|_{L^\infty(\Omega)}$; thus
\[ \| \tilde{A} \|_{L^\infty(\Omega)} \leq 1 + C \| \tilde{A} \|_{L^\infty(\Omega)}. \]
By choosing $\varsigma$ small enough,
\[ \| \tilde{A}(t) \|_{L^\infty(\Omega)} \leq \frac{3}{2}. \]
The estimate above, together with (44), in turn implies that
\[ \| \tilde{A}(t) - \text{Id} \|_{L^\infty(\Omega)} \leq C \varsigma \quad \forall t \in [0, T_M]. \] (45)
The estimate for $\| \mathbf{J}(t) - 1 \|_{L^\infty(\Omega)}$ is similar since $\mathbf{J} = \det(\nabla \tilde{\psi})$. \hfill \qed

5.4. The construction of solutions to the regularized problem. In this subsection, our goal is to establish a map $\Phi : C_T(M) \rightarrow C_T(M)$ by choosing $M \gg 1$ and $T \ll 1$, and then show the existence of a fixed-point of the map. This fixed-point then is a strong solution to the regularized problem (34).

Since the velocity $w$ we are looking for is divergence-free, there are lots of ways of constructing the solution to the linear problem (38). Two typical ways are:

1. The Galerkin method: let $\{e_k\}_{k=1}^\infty$ be an orthonomal divergence-free basis in $L^2(\Omega)$ which is orthogonal in $H^1(\Omega)$, and approximate (38) by projecting (38a) onto the subspace span$\{e_1, \cdots, e_n\}$. The projection of (38a) then becomes an ODE so the projected problem is solvable. The remaining thing to do is to guarantee the solution $w_n$ to the projected problem has an $n$-independent estimate in appropriate spaces.

2. The penalty method: approximate (38) by introducing a penalized parameter $\theta > 0$ and approximate the pressure $q$ in (38) by $\frac{1}{\theta} \text{div} w_\theta$. Once a $\theta$-independent estimate is obtained, the limit of $w_\theta$ (as $\theta \rightarrow 0$) satisfies the divergence-free constraint automatically.

Both methods mentioned above can be used to construct a solution to (38); however, it should be clear to the readers that the standard Galerkin method fails to work in constructing solutions to (9) since there is no basis in $V$ which preserves condition (9b) for all $t > 0$. In order to explain why the penalty method does not work for the purpose of constructing a solution to (11) in the space $\mathcal{X}(T) \times \mathcal{H}(T)$ either, in the following we construct a solution to (38) using the penalty method. We remark here that it is much easier to use the Galerkin method to construct a weak solution to (38).
5.4.1. The penalized problem. Let \( \theta > 0 \) be a given positive constant. We consider

\[
\begin{align*}
  w_{\theta t} - L_{\bar{\psi}}(w_\theta) + \nabla q_\theta &= \bar{F} \\
  \ell_{\bar{\psi}}(w_\theta, q_\theta) &= \varepsilon^2 \Delta q_\theta + \bar{G} \\
  w_\theta &= w_0
\end{align*}
\]

in \( O \times (0, T) \),

\[
\ell_{\bar{\psi}}(w_\theta, q_\theta) = \varepsilon^2 \Delta q_\theta + \bar{G} \\
\theta_\varepsilon(t) &= \theta_\varepsilon(0)
\]

on \( \Gamma \times (0, T) \),

\[
w_\theta = w_0
\]

on \( O \times \{ t = 0 \} \), \hspace{1cm} (46a, 46b, 46c)

where \( q_\theta = \frac{1}{\theta} \) \text{div} \( w_\theta \) is the penalized pressure. Similar to Definition 5.1, we have

**Definition 5.4.** A function \( w_\theta \in L^2(0, T; V) \) satisfying \( w_{\theta t} \in L^2(0, T; L^2(O)) \) is said to be a weak solution to (46) if

\[
(w_{\theta t}, \varphi)_{L^2(O)} + B_{\bar{\psi}}(w_\theta, \varphi) - (q_\theta, \text{div} \varphi) + \varepsilon^2 (w_\theta', \varphi')_{L^2(\Gamma)}
\]

\[
= (\bar{F}, \varphi)_{L^2(O)} + (\bar{G}, \varphi)_{L^2(\Gamma)} \hspace{1cm} \forall \varphi \in V, \hspace{0.5cm} \text{a.e. } t \in [0, T],
\]

and

\[
w_\theta = w_0 \hspace{1cm} \text{on} \hspace{0.5cm} O \times \{ t = 0 \}.
\]

We recall that \( V = \{ u \in H^1(O) \mid u \in H^1(\Gamma) \} \), and the space \( L^2(0, T; V) \) consists of \( u : [0, T] \to V \) such that

\[
\int_0^T \| u(t) \|^2_V \, dt < \infty.
\]

Our goal is to obtain a weak solution to (48) by passing the penalized parameter \( \theta \) to the weak limit for the weak solution \( v_\theta \) to (40).

5.4.2. The existence of the unique weak solution to the penalized problem. The construction of a weak solution to (40) can be done using the Galerkin method. Let \( \{ e_k \}_{k=1}^\infty \) be an orthonormal basis in \( L^2(O) \) which is orthogonal in \( V \), and let

\[
w_n = \sum_{k=1}^n d_k(t) e_k(x)
\]

solve

\[
(w_{nt}, \varphi)_{L^2(O)} + B_{\bar{\psi}}(w_n, \varphi) - (q_n, \text{div} \varphi)_{L^2(O)} + \varepsilon^2 (w_n', \varphi')_{L^2(\Gamma)}
\]

\[
= (\bar{F}, \varphi)_{L^2(O)} + (\bar{G}, \varphi)_{L^2(\Gamma)} \hspace{1cm} \forall \varphi \in \text{span}(e_1, \ldots, e_n)
\]

(48)

with the initial condition

\[
w_n(0) = \sum_{k=1}^n (w_0, e_k)_{L^2(O)} e_k,
\]

(49)

where \( q_n = -\frac{1}{\theta} \) \text{div} \( w_n \). We note that (48) is an ODE; thus the fundamental theorem of ODE implies that there exists \( T_n > 0 \) such that \( w_n \) exists in the time interval \( [0, T_n) \).

**Remark 5.5.** A basis of \( V \) can be obtained by the following eigenvalue problem

\[
-u - \Delta u = \lambda u \hspace{1cm} \text{in} \hspace{0.5cm} O,
\]

\[
\frac{\partial u}{\partial N} = \Delta_0 u \hspace{1cm} \text{on} \hspace{0.5cm} \Gamma.
\]

The study of this eigenvalue problem relies on the solvability of the elliptic problem

\[
u - \Delta u = f \hspace{1cm} \text{in} \hspace{0.5cm} O,
\]

\[
\frac{\partial v}{\partial N} = \Delta_0 u \hspace{1cm} \text{on} \hspace{0.5cm} \Gamma,
\]

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while the solvability of the equation above is trivial because of the Lax-Milgram theorem.

The next step is to obtain \( n \)-independent estimates for the finite dimensional approximation \( w_n \) in certain Banach spaces. Before doing so, we note that if assuming that \( \|h_0\|_{H^1(\Gamma)} < \varsigma \), by (12), for all \( w \in H^1(\Omega) \) we find that the bilinear form \( B_{\psi} \) satisfies

\[
B_{\psi}(w, w) = \int_{\Omega} (w^s w^s_k + w^k w^s_k) \, dx - \int_{\Omega} \left[ J^{-1} \bar{A}_l^j \bar{A}_k^l \bar{\psi}_r^j \bar{\psi}_r^l \right] w^r \, w^s_k \, dx \\
+ \int_{\Omega} \left[ \bar{\psi}_r^j \bar{A}_l^n \bar{A}_k^n (J^{-1} \bar{\psi}_r^j)_{,r} \right] w^r \, w^s_k \, dx + \int_{\Omega} (J^{-1} - 1) w^s_k w^r \, dx \\
\geq \frac{1}{2} \|\text{Def} w\|_{L^2(\Omega)}^2 - C \varsigma \|w\|_{H^1(\Omega)}^2 - C \|\bar{\mathbf{h}}\|_{H^2(\Gamma)} \|w\|_{L^2(\Omega)}^{1/2} \|w\|_{H^1(\Omega)}^{3/2} \\
\geq \frac{1}{2} \|\text{Def} w\|_{L^2(\Omega)}^2 - (C \varsigma + \delta_1) \|w\|_{H^2(\Omega)}^2 - C \delta_1 \left( \|h_0\|_{H^2(\Gamma)}^2 + M^2 \right) \|w\|_{L^2(\Omega)}^2.
\]

Now let \( \varphi = w_n \in \text{span}(e_1, \ldots, e_n) \) in (48), by the inequality above we have

\[
\frac{1}{2} \frac{d}{dt} \|w_n(t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\text{Def} w_n\|_{L^2(\Omega)}^2 + \theta \|q_n\|_{L^2(\Omega)}^2 + \varepsilon^2 \|w_n\|_{L^2(\Gamma)}^2 \leq \left( C \varsigma + \delta_1 \right) \|w_n\|_{H^1(\Omega)}^2 + C \delta_1 \left( \|h_0\|_{H^2(\Gamma)}^2 + M^2 \right) \|w_n\|_{L^2(\Omega)}^2 + C \left( \|\mathbf{F}\|_{L^2(\Omega)}^2 + \|\mathbf{G}\|_{L^2(\Gamma)}^2 \right).
\]

Integrating the inequality above in time over the time interval \((0, t)\), by Korn's inequality we further obtain that

\[
\|w_n(t)\|_{L^2(\Omega)}^2 + \int_0^t \left[ \|w_n\|_{H^1(\Omega)}^2 + \frac{1}{\theta} (\text{div} w_n)_{L^2(\Omega)}^2 + \varepsilon^2 \|w_n\|_{L^2(\Gamma)}^2 \right] \, dt \leq C \left[ \|w_0\|_{L^2(\Omega)}^2 + \|\mathbf{F}\|_{L^2(\Omega)}^2 + \|\mathbf{G}\|_{L^2(\Gamma)}^2 \right] \left( \frac{1}{\theta} \right) \left( \frac{1}{\theta} \right) + C \delta_1 \left( \|h_0\|_{H^2(\Gamma)}^2 + M^2 \right) \int_0^t \|w_n\|_{L^2(\Omega)}^2 \, dt + C \int_0^t \left( \|\mathbf{F}\|_{L^2(\Omega)}^2 + \|\mathbf{G}\|_{L^2(\Gamma)}^2 \right) \, dt.
\]

Since

\[
\|\mathbf{F}\|_{L^2(\Omega)} \leq C \varsigma \|\bar{\mathbf{w}}\|_{L^2(\Omega)} + C \|D^2 \bar{\mathbf{u}}\|_{L^2(\Omega)} \|D \bar{\mathbf{u}}\|_{L^2(\Omega)} + C \|D^2 \bar{\mathbf{u}}\|_{L^2(\Omega)} \|\bar{\mathbf{w}}\|_{L^2(\Omega)} \\
+ C \left( \|\bar{\mathbf{w}}\|_{L^2(\Omega)} + \|\bar{\mathbf{u}}\|_{L^2(\Omega)} \right) \left( \|D^2 \bar{\mathbf{u}}\|_{L^2(\Omega)} \|\bar{\mathbf{u}}\|_{L^2(\Omega)} + \|\bar{\mathbf{w}}\|_{W^{1,4}(\Omega)} \right) \\
+ C \left( \|\mathbf{G}\|_{L^2(\Gamma)} \right) \|\bar{\mathbf{u}}\|_{L^2(\Omega)}.
\]

\[
\|\mathbf{G}\|_{L^2(\Gamma)} \leq C \left( \|\bar{\mathbf{h}}\|_{L^2(\Gamma)} + \|\bar{\mathbf{h}}\|_{L^2(\Gamma)}^2 \right) \leq C \left[ \|\bar{\mathbf{h}}\|_{H^2(\Gamma)} + 1 \right],
\]

by (15) and Young's inequality we obtain that

\[
\|\mathbf{F}\|_{L^2(\Omega)}^2 + \|\mathbf{G}\|_{L^2(\Gamma)}^2 \leq C \varsigma^2 \|\bar{\mathbf{w}}\|^2_{L^2(\Omega)} + C \|\bar{\mathbf{h}}\|^2_{H^2(\Gamma)} \\
+ C \left( \|w_0\|^2_{H^1(\Omega)} + \|h_0\|^2_{H^2(\Gamma)} + M^2 \right) \left( \frac{1}{M^2} \right) \|\bar{\mathbf{w}}\|^2_{H^2(\Omega)}.
\]
Therefore, by choosing $\zeta > 0$, $\delta_1 > 0$ small enough, and $0 < T^*_M < T_M$ small enough so that $CM^2(\|w_0\|_{H^{1}(\Omega)}^2 + \|h_0\|_{H^2(\Gamma)}^2 + M^2)^6T^*_M < 1$,

$$\left\|w_n(t)\right\|_{L^2(\Omega)}^2 + \int_0^t \left[ \left\|w_n\right\|_{H^1(\Omega)}^2 + \theta \|q_n\|_{L^2(\Omega)}^2 + \varepsilon^2 \|w'_{n}\|_{L^2(\Gamma)}^2 \right] dt \leq C\left[ 1 + \left\|w_0\right\|_{L^2(\Omega)}^2 \right] + C(\|h_0\|_{H^2(\Gamma)}^2 + M^2)^6 \int_0^t \|w_n\|_{L^2(\Omega)}^2 dt + C\zeta^2 \int_0^t \|\bar{w}_t\|_{L^2(\Omega)}^2 dt$$

for all $t \in [0, T^*_M]$. The Gronwall inequality further suggests that

$$\left\|w_n(t)\right\|_{L^2(\Omega)}^2 + \int_0^t \left[ \left\|w_n\right\|_{H^1(\Omega)}^2 + \theta \|q_n\|_{L^2(\Omega)}^2 + \varepsilon^2 \|w'_{n}\|_{L^2(\Gamma)}^2 \right] dt \leq C\left[ 1 + \left\|w_0\right\|_{L^2(\Omega)}^2 \right] + C\zeta^2 \int_0^t \|\bar{w}_t\|_{L^2(\Omega)}^2 dt$$

(50)

if $T^*_M$ is chosen even smaller.

We also need an estimate of $w_{nt}$ in order to pass $n$ to the limit. Since $w_{nt}$ belongs to the span of $e_1, \ldots, e_n$, it can be used as a test function in (48). By doing so we obtain that

$$\left\|w_{nt}\right\|_{L^2(\Omega)}^2 + \int_\Omega \left[ \bar{\psi}_t \overline{A_k^i} \overline{A_t^k} (J^{-1} \bar{\psi}_t^i w_{n,t})_j + \overline{A_t^k} (J^{-1} \bar{\psi}_t^i w_{n,t}),_j \right] w_{nt,k} dx + \frac{1}{2} \frac{d}{dt} \left[ \theta \|q_n\|_{L^2(\Omega)}^2 + \varepsilon^2 \|w'_{n}\|_{L^2(\Gamma)}^2 \right] = \left\langle F_{nt}, w_{nt} \right\rangle_{L^2(\Omega)} + \int_{\Gamma} \tilde{G} : w_{nt} dS.$$  

(51)

Since

$$\int_\Omega \left[ \bar{\psi}_t \overline{A_k^i} \overline{A_t^k} (J^{-1} \bar{\psi}_t^i w_{n,t})_j + \overline{A_t^k} (J^{-1} \bar{\psi}_t^i w_{n,t}),_j \right] w_{nt,k} dx = \int_\Omega \overline{A_t^k} (J^{-1} \bar{\psi}_t^i w_{n,t}),_j - \overline{A_t^k} (J^{-1} \bar{\psi}_t^i)_{,k} w_{nt,k} - \overline{A_t^k} (J^{-1} \bar{\psi}_t^i)_{,k} w_{nt,k} dx,$$

by the symmetry of $\overline{A_t^k} (J^{-1} \bar{\psi}_t^i w_{n,t}),_j + \overline{A_t^k} (J^{-1} \bar{\psi}_t^i)_{,k} w_{nt,k}$ in $(i, \ell)$ we find that

$$\int_\Omega \overline{A_t^k} (J^{-1} \bar{\psi}_t^i w_{n,t}),_j + \overline{A_t^k} (J^{-1} \bar{\psi}_t^i)_{,k} w_{nt,k} dx \geq \frac{1}{2} \frac{d}{dt} \int_\Omega \left[ \overline{A_t^k} (J^{-1} \bar{\psi}_t^i w_{n,t}),_j + \overline{A_t^k} (J^{-1} \bar{\psi}_t^i)_{,k} w_{nt,k} \right] dx$$

(52)
thus
\[
\int_0^t \left[ \bar{v}_s^j \bar{A}_s^j (\bar{J}^{-1} \bar{v}_r^j w_n^r)_{,j} + \bar{A}_s^j (\bar{J}^{-1} \bar{v}_r^j w_n^r)_{,s} \right] w_{nt,k} \, dx \\
\geq \frac{1}{2} \frac{d}{dt} \int_0^t \left[ \bar{A}_s^j (\bar{J}^{-1} \bar{v}_r^j w_n^r)_{,j} + \bar{A}_s^j (\bar{J}^{-1} \bar{v}_r^j w_n^r)_{,s} \right] \bar{A}_s^j (\bar{J}^{-1} \bar{v}_r^j w_n^r)_{,k} \, dx \\
- C_\delta \left( \| \bar{h}_\varepsilon \|_{H^2(\Gamma)}^2 + \| \bar{h}_t \|_{H^2(\Gamma)}^2 \right) \| w_n \|_{H^1(\Omega)}^2 - \delta \| w_{nt} \|_{L^2(\Omega)}^2.
\]

Moreover, by the embedding \( H^1(\Omega) \to H^{0.5}(\Gamma) \),
\[
\int_0^t \bar{G} \cdot w_{nt} dS \, d\bar{t} = \int_\Gamma \bar{G} \cdot w_{nt} dS + \int_0^t \int_\Gamma \bar{G}_t \cdot w_n dS \, d\bar{t} \\
\leq C_\delta_1 \| \bar{G}(t) \|_{H^{0.5}(\Gamma)}^2 + \delta_1 \| w_n(t) \|_{H^1(\Omega)}^2 + C \| w_n(0) \|_{H^1(\Omega)}^2 \\
+ \frac{1}{M^2} \int_0^t \| \bar{G}_t \|_{H^{0.5}(\Gamma)}^2 \, d\bar{t} + CM^2 \int_0^t \| w_n \|_{H^1(\Omega)}^2 \, d\bar{t}.
\]

As a consequence, integrating (51) in time over the time interval \((0, t)\), by (12) and choosing \( \delta > 0 \) small enough we conclude that
\[
\| \text{Def} w_n(t) \|_{L^2(\Omega)}^2 + \theta \| q_n(t) \|_{L^2(\Omega)}^2 + \varepsilon^2 \| w_n^r(t) \|_{L^2(\Gamma)}^2 + \int_0^t \| w_{nt} \|_{L^2(\Omega)}^2 \, d\bar{t} \leq C \left( \| b_0 \|_{H^1(\Gamma)} + \sqrt{\epsilon} \| h_0 \|_{H^{1.5}(\Gamma)} + \sqrt{\epsilon} \| h_t \|_{H^{1.5}(\Omega, \cdot ; L^2(0, t ; H^{1.5}(\Gamma)))} \right)
\]

for some constant \( C \) independent of the initial data. Since
\[
\| \bar{h}_\varepsilon \|_{H^2(\Gamma)}^2 \leq C \varepsilon^{-3/2} \| \bar{h} \|_{H^2(\Gamma)}^2,
\]
the Gronwall inequality further implies that we may choose \( T_\varepsilon < T_M^* \) such that for all \( t \in [0, T_\varepsilon] \),
\[
\| w_n(t) \|_{H^1(\Omega)}^2 + \theta \| q_n(t) \|_{L^2(\Omega)}^2 + \varepsilon^2 \| w_n^r(t) \|_{L^2(\Gamma)}^2 + \int_0^t \| w_{nt} \|_{L^2(\Omega)}^2 \, d\bar{t} \leq C \left[ 1 + \| w_0 \|_{H^1(\Omega)}^2 + \| h_0 \|_{H^{1.5}(\Gamma)}^2 \right] + C \varepsilon^2 M^2.
\]
Estimate [55] provides an $n$-independent upper bound for $w_{nt} \in L^2(0, T_\varepsilon; L^2(O))$ and $w_n \in L^\infty(0, T_\varepsilon; V)$. Therefore, there exists a subsequence $n_k$ of $n$ such that

$$w_{n_k} \rightharpoonup w_0 \quad \text{in} \quad L^p(0, T_\varepsilon; V) \quad \forall \ p \in (1, \infty),$$

$$w_{nt_k} \rightharpoonup w_{0t} \quad \text{in} \quad L^2(0, T_\varepsilon; L^2(O)).$$

The weak limit $w_0$ satisfies

$$\left[ \int_0^d \left( \|w_0(t)\|_{H^1(O)}^{2p} + \|\sqrt{q_0(t)}(t)\|_{L^2(O)}^{2p} + \|\varepsilon w_{0t}(t)\|_{L^2(\Gamma)}^{2p} \right) dt \right]^{1/p}$$

$$+ \int_0^d \|w_{0t}(t)\|_{L^2(O)}^2 dt \leq C \left[ \|w_0\|_{H^1(O)}^2 + \|h_0\|_{H^{1.5}(\Gamma)}^2 + 1 \right] + C\varepsilon^2M^2$$

and the variational form

$$\int_0^{T_\varepsilon} \left[ (w_{0t}, \varphi)_{L^2(O)} + B_\varphi(w_0, \varphi) - (q_0, \text{div}\varphi) + \varepsilon^2(w_{0t}, \varphi')_{L^2(\Gamma)} \right] dt$$

$$= \int_0^{T_\varepsilon} \left[ (\bar{\varphi}, \varphi)_{L^2(O)} + (\bar{G}, \varphi)_{L^2(\Gamma)} \right] dt \quad \forall \ \varphi \in L^2(0, T_\varepsilon; V).$$

Since the right-hand side of (56) is independent of the exponent $p$, we let $p \to \infty$ and obtain that

$$\sup_{t \in [0, T_\varepsilon]} \left[ \|w_0(t)\|_{H^1(O)}^2 + \theta\|q_0(t)\|_{L^2(O)}^2 + \varepsilon^2\|w_{0t}(t)\|_{L^2(\Gamma)}^2 \right]$$

$$+ \int_0^{T_\varepsilon} \|w_{0t}(t)\|_{L^2(O)}^2 dt \leq C \left[ \|w_0\|_{H^1(O)}^2 + \|h_0\|_{H^{1.5}(\Gamma)}^2 + 1 \right] + C\varepsilon^2M^2.$$\hspace{1cm} (58)

The same argument suggests that there exists $\theta_k \to 0$ such that

$$w_{\theta_k} \rightharpoonup w \quad \text{in} \quad L^p(0, T_\varepsilon; V) \quad \forall \ p \in (1, \infty),$$

$$w_{\theta kt} \rightharpoonup w_t \quad \text{in} \quad L^2(0, T_\varepsilon; L^2(O)),$$

and the weak limit $w$ satisfies

$$\sup_{t \in [0, T_\varepsilon]} \left[ \|w(t)\|_{H^1(O)}^2 + \varepsilon^2\|w'(t)\|_{L^2(\Gamma)}^2 \right] + \int_0^{T_\varepsilon} \|w_t(t)\|_{L^2(O)}^2 dt$$

$$\leq C_1 \left[ \|w_0\|_{H^1(O)}^2 + \|h_0\|_{H^{1.5}(\Gamma)}^2 + 1 \right] + C_2\varepsilon^2M^2,$$\hspace{1cm} (59)

and $\text{div}w = 0$ since $\sqrt{\theta}q_0 = \frac{1}{\sqrt{\theta}}\text{div}w_0$ is uniformly bounded in $L^\infty(0, T_\varepsilon; L^2(O))$. Moreover, [57] implies that for all $a, b \in (0, T_\varepsilon),$

$$\int_a^b \left[ (w_t, \varphi)_{L^2(O)} + B_\varphi(w, \varphi) + \varepsilon^2(w', \varphi')_{L^2(\Gamma)} \right] dt$$

$$= \int_a^b \left[ (\bar{F}, \varphi)_{L^2(O)} + (\bar{G}, \varphi)_{L^2(\Gamma)} \right] dt \quad \forall \ \varphi \in V \cap H^1_{\text{div}}(O),$$

and Lebesgue’s differentiation theorem further suggests that $w$ satisfy [39].

To finish the process of construction a weak solution to (8), it remains to show that $w(0) = w_0$. Let $\zeta : [0, T_\varepsilon] \to \mathbb{R}$ be a non-negative smooth function (of $t$) such that $\zeta(0) = 1$ and $\zeta(T_\varepsilon) = 0$, and $\varphi \in \text{span}(e_1, \cdots, e_n)$. The use of $\zeta \varphi$ as
a test function in (38) and then integrating in time over the time interval \((0, T_\varepsilon)\),
integrating by parts in time we obtain that
\[
(w_n(0), \varphi)_{L^2(O)} - \int_0^{T_\varepsilon} (w_n, \zeta')_{L^2(O)} dt + \int_0^{T_\varepsilon} \left[ B_{\zeta}(w_n, \zeta \varphi) - (q_n, \zeta \div \varphi)_{L^2(O)} \right. \\
+ \left. \varepsilon^2 (w'_n, \zeta \varphi')_{L^2(\Gamma)} \right] dt = \int_0^{T_\varepsilon} \left[ (\bar{F}, \zeta \varphi)_{L^2(O)} + (\bar{G}, \zeta \varphi)_{L^2(\Gamma)} \right] dt.
\]
By (49), passing \(n \to \infty\) we find that
\[
(w_0, \varphi)_{L^2(O)} - \int_0^{T_\varepsilon} (w_0, \zeta')_{L^2(O)} dt + \int_0^{T_\varepsilon} \left[ B_{\zeta}(w_0, \zeta \varphi) - (q_0, \zeta \div \varphi)_{L^2(O)} \right. \\
+ \left. \varepsilon^2 (w'_0, \zeta \varphi')_{L^2(\Gamma)} \right] dt = \int_0^{T_\varepsilon} \left[ (\bar{F}, \zeta \varphi)_{L^2(O)} + (\bar{G}, \zeta \varphi)_{L^2(\Gamma)} \right] dt \quad \forall \varphi \in H^1(O).
\]
On the other hand, the use of \(\zeta \varphi\) as a test function in (57) suggests that
\[
(w_0(0), \varphi)_{L^2(O)} - \int_0^{T_\varepsilon} (w_0, \zeta')_{L^2(O)} dt + \int_0^{T_\varepsilon} \left[ B_{\zeta}(w_0, \zeta \varphi) - (q_0, \zeta \div \varphi)_{L^2(O)} \right. \\
+ \left. \varepsilon^2 (w'_0, \zeta \varphi')_{L^2(\Gamma)} \right] dt = \int_0^{T_\varepsilon} \left[ (\bar{F}, \zeta \varphi)_{L^2(O)} + (\bar{G}, \zeta \varphi)_{L^2(\Gamma)} \right] dt \quad \forall \varphi \in H^1(O).
\]
The comparison between the two identities above enables us to conclude the identity \(w_0(0) = w_0\). Similar argument can be used to conclude that \(w(0) = w_0\), and is left to the readers. The uniqueness of the weak solution should also be clear to the readers.

Finally, let \(T : V \to \mathbb{R}\) be given by
\[
T(\varphi) = (w, \varphi)_{L^2(O)} + B_{\zeta}(w, \varphi) + \varepsilon^2 (w', \varphi')_{L^2(\Gamma)} - (\bar{F}, \varphi)_{L^2(O)} - (\bar{G}, \varphi)_{L^2(\Gamma)}.
\]
where \(w\) is the weak solution to (38). Since \(T : V \cap H^1_{\text{div}}(O) \to \{0\}\), the Lagrange multiplier lemma implies that there exists a unique \(q \in L^2(O)\) such that
\[
T(\varphi) = (q, \div \varphi)_{L^2(O)} \quad \forall \varphi \in V,
\]
and by (12) we also conclude that \(q\) satisfies
\[
\|q\|_{L^2(O)} \leq C \left[ \|w\|_{L^2(O)} + \|\nabla w\|_{L^2(O)} + \varepsilon^2 \|\varphi\|_{H^1(\Gamma)} + \|\bar{F}\|_{L^2(O)} + \|\bar{G}\|_{H^{-\frac{1}{2}}(\Gamma)} \right].
\]

**Remark 5.6.** Now we explain briefly why the usual ALE formulation is not a good choice in obtaining a solution \((v, q) \in V(T) \times Q(T)\). Similar to (34) and (35), we introduce the linear penalized problem of (38) as the following equation
\[
v_{0t} + \bar{A}_k^i v_{0,j} + \bar{A}_k^i v_{0,k} = \bar{F}_1 \quad \text{in} \quad O \times (0, T), \quad (61a)
\]
\[
(\bar{A}_k^i v_{0,j} + \bar{A}_k^i v_{0,k} - q_0 \delta^i_k) \bar{A}_k^i N_k = \varepsilon^2 \Delta_0 v_0 + \bar{G}_1 \quad \text{on} \quad \Gamma \times (0, T), \quad (61b)
\]
\[
v_0 = v_0 \equiv v_0 \circ \psi_0 \quad \text{on} \quad O \times \{t = 0\}, \quad (61c)
\]
for some functions \(\bar{F}_1\) and \(\bar{G}_1\), where \(\bar{A}\) could be obtained from the ALE map \(\bar{\psi}\) or Lagrangian coordinate \(\bar{\eta}\) and \(q_0 = -\frac{1}{\bar{\eta}} \bar{A}_i^i v_{0,j}\). The same as before, let \(v_n(x, t) =
\[ \sum_{k=1}^{n} d_k(t) e_k(x) \] is the finite dimensional approximation of the solution to (61) satisfying the ODE
\[ (v_{nt}, \varphi)_{L^2(O)} + \frac{1}{2} v_{nt}^2 + \frac{1}{2} \left( (\bar{A}_1^k v_{n,j} + \bar{A}_1^k v_{n,j}) + (\bar{A}_1^k \varphi_{j,k} + \bar{A}_1^k \varphi_{j,k}) \right)_{L^2(O)} + \varepsilon^2 (v_n', \varphi')_{L^2(\Gamma)} \]
\[ - (q_n, \bar{A}_1^k v_{n,j})_{L^2(O)} = (\bar{F}_1, \varphi)_{L^2(O)} + (\bar{G}_1, \varphi)_{L^2(\Gamma)} \quad \forall \varphi \in \text{span}(e_1, \ldots, e_n) \]
and the initial condition
\[ v_n(x, 0) = \sum_{k=1}^{n} (v_0, e_k)_{L^2(O)} e_k(x). \]
Then for all fixed \( \theta > 0 \),
\[ v_n \rightarrow v_\theta \quad \text{in} \quad L^2(0, T; H^1(O)), \]
\[ v_{n,j} \rightarrow v_{\theta,j} \quad \text{in} \quad L^2(0, T; H^1(O)). \]
However, the difficulties here is due to a \( \theta \)-independent estimate of \( v_{nt} \) (which is required to pass \( \theta \rightarrow 0 \)). We remind the readers that the way to obtain an estimate of \( v_{nt} \) is to use \( v_{nt} \) as a test function in (62), while in this case, for the last term on the left-hand side of (62) we have
\[ - (q_n, \bar{A}_1^k v_{n,j})_{L^2(O)} = \frac{\theta}{2} \frac{d}{dt} \| q_n \|_{L^2(O)}^2 + (q_n, \bar{A}_1^k v_{n,j})_{L^2(O)}, \]
where \( q_n = -\frac{1}{\theta} \bar{A}_1^k v_{n,j} \). The appearance of the second term suggests that we are not able to obtain \( \theta \)-independent estimate of \( v_{nt} \) and the penalty method is not applicable.

On the other hand, with the satisfaction of the first order compatibility condition (2), we use the Lagrangian formulation and look for a solution \((v, \varphi)\) in the space \( L^2(0, T; H^1(O)) \times L^2(0, T; H^2(O)) \) with \( v \in L^2(0, T; H^1(O)) \). In this case, we first use the Galerkin method to obtain a solution to the following integral equality (which is the time derivative of the weak formulation (63) of the solution to (62))
\[ \langle w_{\theta t}, \varphi \rangle + (\bar{A}_1^k \bar{A}_1^k v_{\theta,j})_{L^2(O)} + \varepsilon^2 (w_{\theta}', \varphi')_{L^2(\Gamma)} \]
\[ - (\bar{A}_1^k q_{\theta t}, \varphi_{j,k})_{L^2(O)} = (\bar{F}_1, \varphi)_{L^2(O)} + (\bar{G}_1, \varphi)_{L^2(\Gamma)} \quad \forall \varphi \in H^1(O) \]
with \( w_\theta(0) = \bar{F}_1(0) - \nabla q_0 + \Delta u_0 \), where \( \theta = \frac{\int_0^\ell w_\theta ds}{\int_0^\ell w_\theta ds} \), and
\[ q_\theta = \frac{\int_0^\ell q_{\theta} ds}{\int_0^\ell w_\theta ds}. \]
Integrating (64) in time we find that \( v_\theta \) satisfies
\[ \langle \varphi_t, \varphi \rangle + (\bar{A}_1^k \bar{A}_1^k v_{\theta,j})_{L^2(O)} + \varepsilon^2 (\varphi_t', \varphi')_{L^2(\Gamma)} \]
\[ - (q_{\theta t}, \bar{A}_1^k \varphi_{j,k})_{L^2(O)} = (\bar{F}_1, \varphi)_{L^2(O)} + (\bar{G}_1, \varphi)_{L^2(\Gamma)} \quad \forall \varphi \in V. \]
We remark that to obtain (65) by integrating (64) in time, the first order compatibility condition (2) is required.

Since the solution \( w_\theta = v_{\theta t} \) to (64) belongs to \( L^2(0, T; H^1(O)) \), we may use it as a test function in (65). In this case, for the last term on the left-hand side of (65) we have
\[ - (q_{\theta t}, \bar{A}_1^k v_{\theta,j})_{L^2(O)} = \frac{\theta}{2} \frac{d}{dt} \| q_{\theta t} \|_{L^2(O)}^2 + (q_{\theta t}, \bar{A}_1^k v_{\theta,j})_{L^2(O)}. \]
Even though the equality above looks similar to the finite dimensional approximation \( (59) \), the situation now is different since \( q_\theta \) is the Lagrange multiplier for the functional \( T : V \to \mathbb{R} \) given by

\[
T(\varphi) = (v_{\theta t}, \varphi)_{L^2(O)} + \frac{1}{2}((\bar{A}_t^j v_{0,j}^t + \bar{A}_t^j v_{0,j}'^t)(\bar{A}_t^k \varphi_{k}^t + \bar{A}_t^k \varphi_{k}'^t))_{L^2(O)} + \epsilon^2(v_{0}^t \cdot \varphi')_{L^2(\Gamma)} - (\bar{F}_1, \varphi)_{L^2(O)} - (\bar{G}_1, \varphi)_{L^2(\Gamma)}.
\]

Therefore, another version of the Lagrange multiplier lemma provides an \( L^2 \)-estimate for \( q_\theta \):

\[
\|q_\theta \|_{L^2(O)} \leq C\left[\|v_{\theta t} \|_{L^2(O)} + \|v_{\theta} \|_{H^1(O)} + \epsilon^2\|v_{\theta} \|_{H^1(\Gamma)} + \|\bar{F}_1 \|_{L^2(O)} + \|\bar{G}_1 \|_{L^2(\Gamma)}\right],
\]

and the estimate above can be used to deduce an \( \theta \)-independent estimate for \( v_{\theta t} \) in \( L^2(0,T;L^2(O)) \) (using Young’s inequality).

5.4.3. The existence of a solution to the regularized problem \( (34) \). In the previous sub-section we have already established the existence of a unique \( (w,q) \in V(T) \times Q(T) \) satisfying

\[
\begin{align*}
B_\psi(w, \varphi) - (q, \text{div} \varphi)_{L^2(O)} + \epsilon^2(w', \varphi')_{L^2(\Gamma)} = (\bar{F} - w_{t}, \varphi)_{L^2(O)} + (\bar{G}, \varphi)_{L^2(\Gamma)} & \quad \forall \varphi \in V, \ a.e. \ t \in [0,T].
\end{align*}
\]

In other words, \( w(t) \) is the weak solution to the elliptic equation

\[
-a_{r,s}^{jk} w_{s}^{r} N_{k} - q N_{s} = \epsilon^2 \Delta w + f_s & \quad \text{on } \Gamma
\]

where

\[
a_{r,s}^{jk} = \bar{J}^{-1} \bar{A}_t^j \bar{A}_t^k \bar{A}_t^p \psi_{r}^p + \bar{J}^{-1} \delta_{s}^{k} \delta_{r}^{j}
\]

and

\[
f_s = \bar{F} - w_s + \left[ \bar{A}_t^j \bar{A}_t^k \bar{A}_t^p \psi_{r}^p (J^{-1} \bar{\psi}_{r}^p)_{,s} w^{r} + \bar{A}_t^k (J^{-1} \bar{\psi}_{r}^p)_{,s} w^{r} \right]_{,k},
\]

\[
g_s = \bar{G} - \left[ \bar{A}_t^j \bar{A}_t^k \bar{A}_t^p \psi_{r}^p (J^{-1} \bar{\psi}_{r}^p)_{,s} w^{r} + \bar{A}_t^k (J^{-1} \bar{\psi}_{r}^p)_{,s} w^{r} \right] N_{k}.
\]

Thanks to the convolution, \( \bar{\psi}, \bar{\psi}_t \in L^\infty(0,T;H^k(O)) \) for all \( k \in \mathbb{N} \). Therefore,

\[
\|a\|_{W^{1,\infty}(O)} \leq C\left[1 + \|\nabla^2 \bar{\psi}\|_{L^\infty(O)}\right] \leq C\left[1 + \|\nabla^2 \bar{\psi}\|_{H^1(\Gamma)}\right] \leq C\left[1 + \|\bar{\psi}_{\infty}\|_{H^{2,75}(\Gamma)}\right] \leq C\left[1 + \epsilon^{-1}\|\bar{h}\|_{H^{1,75}(\Gamma)}\right].
\]

Moreover, by \( (59) \) we find that

\[
\|g\|_{L^2(\Gamma)} \leq C\left[1 + \|\bar{h}\|_{H^2(\Gamma)} + \|\nabla^2 \bar{\psi}\|_{L^4(\Gamma)}\|w\|_{L^4(\Gamma)}\right] \leq C\left[1 + \|\bar{h}\|_{H^2(\Gamma)} + \epsilon^{-1}\|\bar{h}\|_{H^{2,75}(\Gamma)}\|w\|_{H^1(\Gamma)}\right] \leq C\left[1 + \|\bar{h}_{0}\|_{H^2(\Gamma)} + M^{\sqrt{T}} + \epsilon^{-1}(\|w_{0}\|_{H^1(\Gamma)} + \|\bar{h}_{0}\|_{H^{1,75}(\Gamma)} + 1)\right],
\]
and by (13k) we obtain that
\[
\|f\|_{L^2(\Omega)} \leq C \left[ \|\text{Id} - \mathcal{J}^{-1}(\nabla \tilde{\psi})^T(\nabla \tilde{\psi})\|_{L^8(\Omega)} \|\tilde{w}\|_{L^2(\Omega)} + \|w_t\|_{L^2(\Omega)} 
\right.
\]
\[
\left. + (\|\tilde{w}\|_{L^4(\Omega)} + \|\tilde{w}_t\|_{L^4(\Omega)}) (\|\nabla \tilde{\psi}\|_{L^8(\Omega)} + \|\nabla^2 \tilde{\psi}\|_{L^4(\Omega)} \|\tilde{w}\|_{L^2(\Omega)}) + \|\nabla \tilde{\psi}\|_{L^8(\Omega)} \|\tilde{w}\|_{L^2(\Omega)} + \|\nabla^2 \tilde{\psi}\|_{L^4(\Omega)} \|\tilde{w}\|_{L^2(\Omega)} \right] 
\]
\[
\leq C \left[ \|\tilde{w}\|_{L^2(\Omega)} + (1 + \|w_0\|_{H^1(\Omega)} + M \sqrt{T}) \|\tilde{w}\|_{H^1(\Gamma)} \right]
\]
\[+ C\|w_t\|_{L^2(\Omega)} + C_\varepsilon \|w\|_{H^1(\Omega)}.
\]

Therefore, Theorem 3.4 together with (59) implies that
\[
\int_0^{T_\varepsilon} \left[ \|w\|_{H^2(\Omega)}^2 + \varepsilon^2 \|w\|_{H^2(\Gamma)}^2 + \|q\|_{H^1(\Omega)}^2 \right] dt \leq C_\varepsilon \int_0^{T_\varepsilon} \|\tilde{g}\|_{L^2(\Gamma)}^2 dt
\]
\[
+ C \int_0^{T_\varepsilon} \left[ (1 + \|a\|_{W^{1,\infty}(\Omega)}) \|w\|_{H^1(\Omega)}^2 + \|f\|_{L^2(\Omega)}^2 + \|g\|_{H^{-\delta,5}(\Gamma)}^2 \right] dt
\]
\[
\leq C_\varepsilon \left( 1 + \|w_0\|_{H^1(\Omega)}^2 + \|h_0\|_{H^2(\Gamma)}^2 + M^2 T_\varepsilon \right) + C_\varepsilon \int_0^{T_\varepsilon} \|w\|_{H^1(\Omega)}^2 dt + C_\varepsilon^2 M^2
\]
\[
+ C \left( 1 + \|w_0\|_{H^1(\Omega)}^2 + M^2 T_\varepsilon \right) \int_0^{T_\varepsilon} \|w\|_{H^1(\Gamma)}^2 dt + C \int_0^{T_\varepsilon} \|w_t\|_{L^2(\Omega)}^2 dt
\]
\[
\leq C_3 \left( \|w_0\|_{H^1(\Omega)}^2 + \|h_0\|_{H^2(\Gamma)}^2 + 1 \right) + C_4 \varepsilon^2 M^2
\]
(67)

if \( T_\varepsilon > 0 \) is chosen small enough, where \( C_3 \) and \( C_4 \) are generic constants.

Once \((w, q)\) is obtained, we define
\[
h(t) = h_0 + \int_0^t \frac{w \cdot N}{1 + b_0 h} dt.
\]
(68)

Then Hölder’s inequality and the normal trace estimate imply that
\[
\|h\|_{H^1(T_\varepsilon)}^2 = \int_0^{T_\varepsilon} \left[ \|h\|_{H^2(\Gamma)}^2 + \|h_t\|_{H^2(\Gamma)}^2 + \|h_{tt}\|_{H^{-\delta,5}(\Gamma)}^2 \right] dt
\]
\[
\leq \int_0^{T_\varepsilon} \|w_t\|_{L^2(\Omega)}^2 dt + CT_\varepsilon \left[ \|h_0\|_{H^2(\Gamma)}^2 + \int_0^{T_\varepsilon} \left( \|w\|_{H^2(\Gamma)}^2 + \|w\|_{L^\infty(\Gamma)} \|h\|_{H^2(\Gamma)}^2 \right) dt \right]
\]
\[
\leq C_1 \left[ \|w_0\|_{H^1(\Omega)}^2 + \|h_0\|_{H^1(\Gamma)}^2 + 1 \right] + C_2 \varepsilon^2 M^2
\]
\[
+ CT_\varepsilon \left[ \|h_0\|_{H^2(\Gamma)}^2 + \frac{M^2 + 1}{\varepsilon^2} \left( C_3 \left[ \|w_0\|_{H^1(\Omega)}^2 + \|h_0\|_{H^2(\Gamma)}^2 + 1 \right] + C_4 \varepsilon^2 M^2 \right) \right].
\]

As a consequence, the combination of (55), (69) and an upper bound for \( \|h\|_{H^1(T_\varepsilon)} \) (stated in the estimate above) suggests that
\[
\|w\|_{W(T_\varepsilon)} + \|h\|_{H^1(T_\varepsilon)}^2 \leq C_5 \left[ \|w_0\|_{H^1(\Omega)}^2 + \|h_0\|_{H^1(\Gamma)}^2 + 1 \right] + C_6 \varepsilon^2 M^2
\]
(69)

for some constants \( C_5 \geq \max\{C_1, C_3\} \) and \( C_6 \geq \max\{C_2, C_4\} \), if \( T_\varepsilon > 0 \) is chosen even smaller. Let \( \zeta \) be small enough so that \( C_6 \varepsilon^2 \leq \frac{1}{2} \) and
\[
M^2 = 2C_5 \left[ \|w_0\|_{H^1(\Omega)}^2 + \|h_0\|_{H^2(\Gamma)}^2 + 1 \right].
Then
\[ \|w\|_{W(T_c)}^2 + \|h\|_{H^1(T_c)}^2 \leq M^2 \left[ \|w_0\|_{H^1(O)}^2 + \|h_0\|_{H^2(\Gamma)}^2 + 1 \right]. \]  

(70)

At this point we have established a map
\[ \Phi : \begin{cases} 
C_{T_c}(M) \to C_{T_c}(M) \\
(w, h) \mapsto (w, h) 
\end{cases} \]

Suppose that \((w_n, h_n) \in C_{T_c}(M)\) and \((w_n, h_n) \to (w, h)\) in \(W(T_c) \times H_1(T_c)\). Let \((\psi_n, \psi)\) be the corresponding ALE map with \(J_n, A_n, J, A\) defined accordingly, and \((w^n, h^n) = \Phi(w_n, h_n) \in C_{T_c}(M)\). Moreover, let \((\tilde{F}_n, G_n)\) and \((\tilde{F}, \tilde{G})\) be the corresponding forcing. Due to the boundedness of \((w^n, h^n)\) in \(W(T_c) \times H_1(T_c)\) and the convolution we must have
\[(w^{n_t}, h^{n_t}) \to (\tilde{w}, \tilde{h}) \quad \text{in} \quad W(T_c) \times H_1(T_c), \]
\[\psi_{n_t} \to \psi_t \quad \text{in} \quad L^2(0, T_c; H^2(O)), \]
\[(w_{n_t}, h_{n_t}) \to (w, h) \quad \text{in} \quad L^2(0, T_c; H^{1,75}(O)) \times C([0, T_c]; H^{1,75}(\Gamma)), \]
\[(\psi_{n_t}, \psi_{n_t}) \to (\psi, \psi) \quad \text{in} \quad L^\infty(0, T_c; W^{2,\infty}(O)) \times L^\infty(0, T_c; H^1(O)), \]

(71a, 71b, 71c, 71d)

for some subsequence \(n_t\) of \(n\). This implies that \(\Phi : (w^{n_t}, h^{n_t}) \in C_{T_c}(M)\) satisfy the variational identity
\[(w_{t}^{n_t}, \varphi)_{L^2(O)} + B_{\psi_{n_t}}(w^{n_t}, \varphi) + \varepsilon^2 (w^{n_t}, \varphi')_{L^2(\Gamma)} = (\tilde{F}_{n_t}, \varphi)_{L^2(O)} + (\tilde{G}_{n_t}, \varphi)_{L^2(\Gamma)} \quad \forall \varphi \in V \cap H^1_{\text{div}}(O), \]

by \((71a)\) and \((71b)\) we find that \((\tilde{w}, \tilde{h})\) satisfies that
\[(\tilde{w}_{t}, \varphi)_{L^2(O)} + B_{\psi}(\tilde{w}, \varphi) + \varepsilon^2 (\tilde{w}', \varphi')_{L^2(\Gamma)} = (\tilde{F}, \varphi)_{L^2(O)} + (\tilde{G}, \varphi)_{L^2(\Gamma)} \quad \forall \varphi \in V \cap H^1_{\text{div}}(O). \]

However, by the uniqueness of the weak solution (to the linearized problem) we find that \((\tilde{w}, \tilde{h}) = (w, h)\). This implies that \(\Phi : C_{T_c}(M) \to C_{T_c}(M)\) is weakly continuous. Therefore, the Tychonoff fixed-point theorem suggests that there exists a fixed-point of \(\Phi\) in \(C_{T_c}(M)\). Every fixed-point \((w^\varepsilon, h^\varepsilon)\) with associated Lagrange multiplies \(q^\varepsilon\) then is a solution to the regularized equation \((34)\).

6. The \(\varepsilon\)-independent estimates

To avoid confusion and simplify the notation, throughout this section we omit the super-script \(\varepsilon\) and denote the strong solution \((w^\varepsilon, w^{\varepsilon'}, q^\varepsilon, h^\varepsilon)\) by \((v, w, q, h)\). Let \(\Psi\) be the solution to
\[
\Delta \Psi = 0 \quad \text{in} \quad \Omega, \\
\Psi = e + (\eta_e \ast \eta_e \ast h)N \quad \text{on} \quad \Gamma, 
\]

(34)
and $J$, $A$ be defined accordingly. If $v = J^{-1}(\nabla \Psi)\omega$, then $(v, q, h)$ satisfies

$$
v_t^i - A^k_t\left([A^j_t v^j_i + A^j_t v^k_j + A^j_t q_{,j}] + A^j_t \Psi - v^j \right)v^i_t \quad \text{in } O \times (0, T), \quad (73a)
$$

$$
A^j_t v^j_j = 0 \quad \text{in } O \times (0, T), \quad (73b)
$$

$$
\left[A^j_t v^j_j + A^j_t v^k_j - q \delta^j \right]A^j_t N_{,k} = L(h)A^j_t N_i + \varepsilon^2 A^j_t \Delta_0 w^n \quad \text{on } \Gamma \times (0, T), \quad (73c)
$$

$$
h_t = \frac{1}{1 + b_0 h_{,\varepsilon}} \cdot v \quad \text{on } \Gamma \times (0, T), \quad (73d)
$$

$$
(v, h) = (u_0, h_0) \quad \text{on } O \times \{t = 0\}, \quad (73e)
$$

here we recall that $L(h) = \left[(1 + b_0 h_{,\varepsilon})h'' - b_0 (1 + 2b_0 h_{,\varepsilon} + b_0^2 h_{,\varepsilon}^2 + 2h_{,\varepsilon}^2) - h_{,\varepsilon} h_{,\varepsilon} h_{,\varepsilon}^2 \right].$

If $\varphi \in V$ is a test function, then

$$
(Jv_t, \varphi)_{L^2(O)} + \int_O J(A_t^j v^j_j + A_t^j v^k_j) A_t^j \varphi^i_j dx + \varepsilon^2 \int_O (J(A_t^j \varphi^j_j)(JA_t^j \varphi^k_j))^j dS = \int_\Gamma L(h)JA_t^j N_j \varphi^i dS + \int_\Gamma \left[(1 + b_0 h_{,\varepsilon})h'' - b_0 (1 + 2b_0 h_{,\varepsilon} + b_0^2 h_{,\varepsilon}^2 + 2h_{,\varepsilon}^2) - h_{,\varepsilon} h_{,\varepsilon} h_{,\varepsilon}^2 \right] \varphi \quad (74)
$$

In the following, we let $M_1$ denote a positive constant (to be determined later) such that

$$
M_1^2 \geq M^2 = 2C_5 \left[\|\omega_0\|_{H^1(\Omega)}^2 + \|h_0\|_{H^2(\Gamma)}^2 + 1\right],
$$

and $[0, T]$ be the (maximal) time interval in which a solution exists, and will be determined later as well.

6.1. **Key elliptic estimate.** The fundamental reason for that the time of existence $T_\varepsilon$ depends on the smoothing parameter $\varepsilon$ is the requirement for extra regularity for $\psi$. For example, it requires that $\nabla^2 \tilde{\psi} \in L^\infty(0, T; L^2(O))$ in the process of estimating $w_{ut}$ (see estimate (52)), and we need $a_{jk} \in W^{1, \infty}(O)$ and $f \in L^2(0, T; L^2(O))$ to apply Theorem 3.4, while the boundedness of $\nabla^3 \tilde{\psi}$ (in some Sobolev spaces) is required in both cases. However, our functional framework (for the linearized problem) only provides us $\tilde{h} \in L^\infty(0, T; H^2(\Gamma))$ which suggests that $\tilde{\psi} \in L^\infty(0, T; H^{2, \varepsilon}(O))$, so the requirement of extra regularity has to be provided by the convolution. Therefore, an improvement of the regularity of $h$ is important for obtaining $\varepsilon$-independent estimates.

Before proceeding to the derivation of $\varepsilon$-independent estimates, we state the following lemma, while the proof of this Lemma will be provided in Appendix B.

**Lemma 6.1.** Let $(v, q, h) \in W(\Omega) \times Q(\Omega) \times H^1(\Omega)$ be a strong solution to (70). Then for some generic constant $C$ (independent of $\varepsilon$)

$$
\int_0^T \|h(t)\|_{H^2(\Gamma)}^2 dt \leq C\left[1 + \|h_0\|_{H^1(\Gamma)}^2 + \|v\|_{V(\Omega)}^2 + \|q\|_{Q(\Omega)}^2\right]. \quad (75)
$$

In particular, the corresponding $J$, $A$ and $\Psi$ satisfy

$$
\int_0^T \left[\|A\|_{H^2(\Omega)}^2 + \|J\|_{H^2(\Omega)}^2 + \|\nabla \psi\|_{H^2(\Omega)}^2\right] dt \leq C\left[1 + \|h_0\|_{H^1(\Gamma)}^2 + \|v\|_{V(\Omega)}^2 + \|q\|_{Q(\Omega)}^2\right]. \quad (76)
$$
Remark 6.2. We emphasize that with the regularity of $\Psi$ given by \[76\], $w$ and $v$ cannot belong to $L^2(0, T; H^2(O))$ simultaneously: if $v \in L^2(0, T; H^2(O))$, the identity

$$D^2w = D^2(JA)v + 2D(JA)Dv + Jad^2v$$

suggests that $w \notin L^2(0, T; H^2(O))$ since $D^2(JA)v \notin L^2(0, T; L^2(O))$ (for it would require that $v \in L^2(0, T; L^2(O))$). This is the main reason that we switch back to the ALE formulation \[73\] instead of directly looking for $\varepsilon$-independent estimates using \[43\], even though we have improved regularity for $h$.

6.2. The estimate for $v_t$ in $L^2(0, T; L^2(O))$. As suggested in Section 6.2.2 estimating $v_t \in L^2(0, T; L^2(O))$ requires that we use $v_t$ as a test function in \[74\]. Since $v_t$ does not belong to $H^1(O)$, it cannot be used as a test function in \[74\]. To resolve this issue, we adopt the approach of difference quotient. We note that since $v \in C([0, T]; H^1(O))$, $D_{\Delta t}v(t) = \frac{v(t + \Delta t) - v(t)}{\Delta t}$ can be used as a test function in \[74\] for all $t \in [0, T]$. By doing so, for $\tilde{t} = t$ and $\tilde{t} = t + \Delta t$ for almost all $t \in (0, T)$ and $\Delta t > 0$,

\[
\left( J(i)v_t(i), D_{\Delta t}v(t) \right)_{L^2(O)} + \int_0^t J(i)\left[ A^i_k(i)v^i_j(\tilde{t}) + A^i_k(i)v^i_j(\tilde{\tilde{t}}) \right]A^j_k(\tilde{t})\left[ D_{\Delta t}v(t) \right]^i_k dx \\
+ \varepsilon^2 \int_\Gamma J(i)A^i_k(i)v^i_j(\tilde{t})\left[ J(i)A^j_k(\tilde{t})D_{\Delta t}v^k(t) \right]^i_k dS
\]

\[77\]

Summing \[74\] over $\tilde{t} = t$ and $\tilde{t} = t + \Delta t$, integrating the sum over the time interval $(a, b) \subseteq [0, T]$, and then passing $\Delta t \to 0$ will then result in the desired estimates.

To be more precise, we note that

\[
\int_0^t J(t + \Delta t)\left[ A^i_k(t + \Delta t)v^i_j(t + \Delta t) + A^i_k(t)v^i_j(t) \right]A^j_k(t + \Delta t)\left[ D_{\Delta t}v(t) \right]^i_k dx \\
+ \int_0^t J(t)\left[ A^i_k(t)v^i_j(t) + A^i_k(t)v^i_j(t) \right]A^j_k(t)\left[ D_{\Delta t}v(t) \right]^i_k dx \\
= \frac{1}{\Delta t} \int_0^t J(i)\left[ A^i_k(i)v^i_j(i) + A^i_k(i)v^i_j(i) \right]A^j_k(i)v^k(i) dx \bigg|_{\tilde{t} = t + \Delta t}^{\tilde{t} = t} \\
- \int_0^t \frac{(JA^i_kA^j_k)(t + \Delta t) - (JA^i_kA^j_k)(t)}{\Delta t}v^i_j(t)v^k(t + \Delta t) dx
\]

and due to the fact that $A^i_kA^j_k = 0$,

\[
(q(t + \Delta t), J(t + \Delta t)A^i_k(t + \Delta t)D_{\Delta t}v(t))_{L^2(O)} + (q(t), J(t)A^i_k(t)\left[ D_{\Delta t}v(t) \right]^i_j)_{L^2(O)}
\]

\[ = -\left( q(t + \Delta t), \frac{J(t + \Delta t)A^i_k(t + \Delta t) - J(t)A^i_k(t)}{\Delta t}v^k(t) \right)_{L^2(O)} \\
- \left( q(t), \frac{J(t + \Delta t)A^i_k(t + \Delta t) - J(t)A^i_k(t)}{\Delta t}v^k(t + \Delta t) \right)_{L^2(O)}.
\]
By the fact that

\[ f(t + \Delta t) \to f(t) \quad \text{in} \quad L^2(a, b; X) \quad \text{if} \quad f \in L^2(0, T; X), \]

\[ \frac{f(t + \Delta t) - f(t)}{\Delta t} \to f_t(t) \quad \text{in} \quad L^2(a, b; X) \quad \text{if} \quad f_t \in L^2(0, T; X), \]

if \([a, b] \subseteq (0, T),\) we obtain that

\[
\lim_{\Delta t \to 0} \int_a^b \left[ \int_0^1 J(t + \Delta t) (A^l_j(t + \Delta t)v^j(t + \Delta t) + A^l_k(t + \Delta t)v^j(t + \Delta t)) \times A^l_k(t + \Delta t)[D_{\Delta t}\mathbf{v}(t)]^i_{jk} \\
+ J(t)(A^l_j(t)v^j(t) + A^l_k(t)v^j(t))A^l_k(t)[D_{\Delta t}\mathbf{v}(t)]^i_{jk} \right] dt \\
= \int_a^b \frac{d}{dt} \| \nabla J(\mathbf{A} + A^T(\nabla \mathbf{v})^T) \|_{L^2(\Omega)}^2 dt - \int_a^b \int_0^1 (JA^l_k A^l_j)_{t} v^j v^i dx dt
\]

and

\[
\lim_{\Delta t \to 0} \int_a^b \left[ \left( q(t + \Delta t), J(t + \Delta t)A^l_j(t + \Delta t)[D_{\Delta t}\mathbf{v}(t)]^i_{jk} \right)^2 \right]_{L^2(\Omega)} = -2 \int_a^b (q, (JA^l_j v^j))_{L^2(\Omega)} dt.
\]

Moreover,

\[
\lim_{\Delta t \to 0} \int_a^b \left[ \left( \mathcal{L}_e(h)J_{A^l_k} \right)(t + \Delta t) + \left( \mathcal{L}_e(h)J_{A^l_k} \right)(t) \right] \mathbb{N}_d D_{\Delta t}\mathbf{v}(t) dx dt \\
= 2 \int_a^b \left( \mathcal{L}_e(h), (JA^l_k \mathbb{N}_d v)_t \right) dt - 2 \int_a^b \int_0^1 \mathcal{L}_e(h)(JA^l_k \mathbb{N}_d v) \mathbb{N}_d S dt \\
\leq 2 \int_a^b \left( \mathcal{L}_e(h), (JA^l_k \mathbb{N}_d v)_t \right) dt + C \int_a^b \left( \| h \|_{H^2(\Gamma)} + 1 \right) \| h_t \|_{W^{1,4}(\Gamma)} \| \mathbf{v} \|_{L^4(\Gamma)} dt
\]

and by Young’s inequality,

\[
\lim_{\Delta t \to 0} \int_a^b \int_\Gamma \left[ (J\mathcal{A}^l_k \Delta_0(JA^l_j v^j)) (t + \Delta t) + (J\mathcal{A}^l_k \Delta_0(JA^l_j v^j)) (t) \right] D_{\Delta t}\mathbf{v}(t) dS dt \\
= 2 \int_a^b \int_\Gamma (J\mathcal{A}^l_j v^j) \Delta_0(JA^l_j v^j) dS - 2 \int_a^b \int_\Gamma \left[ (J\mathcal{A}^l_j v^j)_t - (J\mathcal{A}^l_j v^j) \Delta_0(JA^l_j v^j) \right] dS \\
\leq 2 \int_a^b \int_\Gamma \left[ (J\mathcal{A}^l_j v^j)_t \Delta_0(JA^l_j v^j) \right] dS dt + \frac{C_b_1}{\varepsilon^2} \int_a^b \| \nabla \mathbb{P}_t \|_{L^2(\Gamma)}^2 \| \mathbf{v} \|_{L^4(\Gamma)}^2 dt \\
+ \delta_1 \varepsilon^2 \int_a^b \| \Delta_0(JA^l_j v^j) \|_{L^2(\Gamma)}^2 dt.
\]
As a consequence, summing (77) over $i = t$ and $i = t + \Delta t$ and then integrating in $t$ over the time interval $(a, b)$ and then passing $\Delta t \to 0$, we find that

$$
\frac{1}{2} \int_a^b \langle \nabla v A + A^T (\nabla v)^T \rangle \|_{L^2(O)}^2 dt + \int_a^b \| \nabla v_i \|_{L^2(O)}^2 dt \\
\leq 2 \int_a^b \langle L_c(h), (JA_i^t N_j v) \rangle dt + 2 \int_a^b \left( \int \left( JA_i^t v^i \right) \Delta_0(JA_i^t v^i) dS dt \right) \\
+ C \int_a^b \left( \| h \|_{H^2(\Gamma)} + 1 \right) \| h \|_{W^{1,4}(\Gamma)} \| v \|_{L^4(\Gamma)} dt \\
+ C \int_a^b \| \nabla v_i \|_{L^2(O)} \left( \| \nabla v \|_{L^2(\Gamma)}^2 + \| q \|_{L^4(\Gamma)} \| \nabla v \|_{L^4(\Gamma)} \right) dt \\
+ C \int_a^b \| \nabla v_i \|_{L^4(\Gamma)}^2 \| v \|_{L^4(\Gamma)}^2 dt + \delta_1 e^4 \int_a^b \Delta_0(JA v) \| \nabla v \|_{L^2(\Gamma)}^2 dt.
$$

We note that

$$
\| \nabla v_i \|_{L^4(\Gamma)} \leq C \| h \|_{H^{2.5}(\Gamma)} \leq C \left[ \| JA \|_{H^{2.5}(\Gamma)} \| v \|_{H^{2.5}(\Gamma)} + \| JA \|_{H^{2.5}(\Gamma)} \| v \|_{H^{2.5}(\Gamma)} \right] \\
\leq C \left[ \| h \|_{H^{2.5}(\Gamma)} \| v \|_{H^{1.5}(\Gamma)} + \| h \|_{H^{2.5}(\Gamma)} \right].
$$

by Young’s inequality we obtain that

$$
\| \nabla v_i \|_{L^4(\Gamma)}^2 \leq C \left[ \| h \|_{H^{2.5}(\Gamma)}^2 \| v \|_{H^{1.5}(\Gamma)}^2 + \| h \|_{H^{2.5}(\Gamma)}^2 \| v \|_{H^{1.5}(\Gamma)}^2 \right] \\
\leq C \left[ \| h \|_{H^{2.5}(\Gamma)}^2 \| v \|_{H^{1.5}(\Gamma)}^2 + \| h \|_{H^{2.5}(\Gamma)}^2 \| v \|_{H^{1.5}(\Gamma)}^2 \right] \\
\leq C \| \nabla v \|_{H^{2}(\Gamma)}^2 + \| h \|_{H^{2.5}(\Gamma)}^2.
$$

Since (78) holds for all $0 < a < b < T$ and $v \in C([0, T]; H^1(\Omega))$, passing $a \to 0$ and $b \to t$ for some $t \leq T$, by (75) estimate (78) implies that

$$
\frac{1}{2} \int_0^t \langle \nabla v A + A^T (\nabla v)^T \rangle \| \nabla v \|_{L^2(O)}^2 dt + \int_0^t \| \nabla v_i \|_{L^2(O)}^2 dt \\
\leq C \left[ 1 + \| u_0 \|_{H^1(\Omega)} + \| h_0 \|_{H^{1.5}(\Gamma)} \right] \\
+ 2 \int_0^t \langle L_c(h), (JA_i^t N_j v) \rangle dt + 2 \int_0^t \left( \int \left( JA_i^t v^i \right) \Delta_0(JA_i^t v^i) dS dt \right) \\
+ C \delta_1 \int_0^t \left( \| h \|_{H^2(\Gamma)}^2 \| v \|_{H^1(\Gamma)}^2 + \| h \|_{H^2(\Gamma)}^2 \| v \|_{H^1(\Gamma)}^2 \right) dt \\
+ \delta_1 e^4 \int_0^t \Delta_0(JA v) \| \nabla v \|_{L^2(\Gamma)}^2 dt.
$$

Integrating by parts in time,

$$
\int_0^t \langle L_c(h), (JA_i^t N_j v) \rangle dt = \int_0^t \langle (L_c(h))_t, JA_i^t N_j v \rangle dt \\
\leq C \delta_2 \left[ 1 + \int_0^t \| \nabla v_i \|_{L^2(O)}^2 dt + \int_0^t \| v \|_{H^2(\Gamma)}^2 dt \right] + \delta_2 \| v \|_{H^1(\Gamma)}^2,
$$

while on the other hand,

$$
\int_0^t \left( \int \left( JA_i^t v^i \right) \Delta_0(JA_i^t v^i) dS dt \right) = - \frac{1}{2} \int_0^t \frac{d}{dt} \| JA v \|_{L^2(\Gamma)}^2 dt = - \frac{1}{2} \| JA v \|_{L^2(\Gamma)}^2 |_{i=0}^{i=t}.
$$
Moreover,
\[
\|\Delta_0(J Av)\|_{L^2(\Gamma)} \leq C \left[ \|h_{\varepsilon}\|_{H^3(\Gamma)} \|v\|_{L^\infty(\Gamma)} + \|h_{\varepsilon}\|_{H^2.25(\Gamma)} \|v\|_{H^1.25(\Gamma)} + \|v\|_{H^2(\Gamma)} \right]
\]
\[
\leq C \left[ \frac{1}{\varepsilon} \|h_{H^2(\Gamma)}\|_{L^2(\Omega)} \|v\|_{L^2(\Omega)}^{1/2} \|v\|_{H^2(\Omega)}^{1/2} + \frac{1}{\varepsilon^{1/2}} \|v\|_{H^2(\Omega)} + \|v\|_{H^2(\Gamma)} \right]
\]
and
\[
\varepsilon^2 \|\Delta_0(J Av)\|_{L^2(\Gamma)}^2 \geq \frac{\varepsilon^2}{C} \|v\|_{L^2(\Gamma)}^2 - C \varepsilon^2 \|h_{\varepsilon}\|_{H^{2.25}(\Gamma)}^2 \|v\|_{H^{0.25}(\Gamma)}^2 \]
\[
\geq \frac{\varepsilon^2}{C} \|v\|_{L^2(\Gamma)}^2 - C \delta_2 \|v\|_{L^2(\Omega)}^2 - \delta_2 \|v\|_{H^1(\Omega)}^2.
\]
Therefore, by \(\varepsilon^2 \|u_0\|_{H^1(\Omega)}^2 \leq C \|u_0\|_{H^1(\Omega)}^2\), if \(\varepsilon < 1\),
\[
\|\text{Def}v(t)\|_{L^2(\Omega)}^2 + \varepsilon^2 \|v(t)\|_{L^2(\Gamma)}^2 + \int_0^t \|v_t\|_{L^2(\Omega)}^2 \, dt \]
\[
\leq C \left[ 1 + \|u_0\|_{H^1(\Omega)}^2 + \|h_0\|_{H^{1.5}(\Gamma)}^2 \right] + C \delta_2 \varepsilon \int_0^t \left( 1 + \|h\|_{H^2(\Gamma)} \right) \|v\|_{H^1(\Omega)}^2 \, dt \]
\[
+ C \delta_2 \varepsilon \int_0^t \left[ \|v\|_{L^2(\Gamma)}^2 + \|q\|_{H^1(\Omega)}^2 \right] \, dt + \delta_1 \left[ \|v(t)\|_{H^2(\Gamma)}^2 + \varepsilon^4 \int_0^t \|v\|_{H^2(\Gamma)}^2 \, dt \right].
\]
(79)

On the other hand, the use of \(v\) as a test function in (74) implies that
\[
\frac{1}{2} \frac{d}{dt} \|\sqrt{\gamma} v\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\sqrt{\gamma} \Delta v + A^T (\nabla v)^T \|_{L^2(\Omega)}^2 \leq C \left( \|h\|_{H^2(\Gamma)} + 1 \right) \|v\|_{L^2(\Omega)}^2
\]
\[
\leq C \delta_3 \left( 1 + \|v\|_{L^2(\Omega)}^2 + |\nabla v||_{L^2(\Omega)}^2 + \|v\|_{H^2(\Gamma)}^2 \right),
\]
where we use the trace estimates and Young’s inequality to conclude the last inequality. Integrating the equality above in time over the time interval \((0, t)\) and choosing \(\delta_3 > 0\) small enough, by the Gronwall inequality we obtain that
\[
\|v(t)\|_{L^2(\Omega)}^2 + \int_0^t \|\nabla v\|_{L^2(\Omega)}^2 \, dt \leq C \left[ \|u_0\|_{L^2(\Omega)}^2 + \int_0^t \|h\|_{H^2(\Gamma)}^2 \, dt \right].
\]
(80)

Combining estimates (74) and (75), by Korn’s inequality and choosing \(\delta_2\) small enough we conclude that
\[
\|v(t)\|_{H^1(\Omega)}^2 + \varepsilon^2 \|v(t)\|_{H^1(\Gamma)}^2 + \int_0^t \|v_t\|_{L^2(\Omega)}^2 \, dt \leq C \delta_3 \left[ 1 + \|u_0\|_{H^1(\Omega)}^2 + \|h_0\|_{H^{1.5}(\Gamma)}^2 \right]
\]
\[
+ C \delta_3 \left( \frac{1}{h_{H^2(\Gamma)}} + 1 \right) \left( 1 + \|v\|_{H^1(\Omega)}^2 + \|v\|_{H^2(\Gamma)}^2 \right) \int_0^t \|v\|_{H^2(\Gamma)}^2 \, dt \]
\[
+ \delta \int_0^t \left[ \|v\|_{L^2(\Gamma)}^2 + \|q\|_{H^1(\Omega)}^2 \right] \, dt + \delta_1 \left[ \|v(t)\|_{H^2(\Gamma)}^2 + \varepsilon^4 \int_0^t \|v\|_{H^2(\Gamma)}^2 \, dt \right].
\]
(81)

6.3. The estimate for \(v \in L^2(0, T; H^{1.5}(\Gamma))\). Let \(\{\chi_m\}_{m=1}^k\) be cut-off functions supported near the boundary \(\Gamma\) so that there exists \(\theta_m : B(0, r_m) \to \text{spt}(\chi_m)\) satisfying

1. \(\theta_m(B_+(0, r_m)) = \text{spt}(\chi_m) \cap \Omega\), where \(B_+(0, r_m) \equiv B(0, r_m) \cap \{y_2 > 0\}\),
2. \(\theta_m(B(0, r_m) \cap \{y_2 = 0\}) = \text{spt}(\chi_m) \cap \Gamma\);
3. \(\text{det}(\nabla \theta_m) = 1\).
For a fixed $m$, define $\tilde{\alpha} = (\nabla \theta_m)^{-1}$, $\tilde{\chi} = \chi_m \circ \theta_m$ in $\text{spt}(\tilde{\chi}_m)$, $(\tilde{w}, \tilde{q}) = (w, q) \circ \theta_m$ in $B_+(0, r_m)$. Our goal is to use

$$\varphi = \left[ \Lambda_e (\tilde{\chi}^2 \Lambda_0 \tilde{v}_1), 1 \right] \circ \theta_m^{-1},$$

as a test function in (82) (which is legitimate since $\varphi \in L^2(0, T; H^1(O))$, and then pass $\epsilon \to 0$.

6.3.1. The estimate of $(J_{\tilde{v}_t}, \varphi)_{L^2(O)}$. Let $\tilde{J} = J \circ \theta_m$. Then

$$(J_{\tilde{v}_t}, \varphi)_{L^2(O)} = (\tilde{J}\tilde{v}_t, \Lambda_e (\tilde{\chi}^2 \Lambda_0 \tilde{v}_1), 1)_{L^2(B_+(0, r_m))}$$

$$= (\Lambda_e (\tilde{J}\tilde{v}_t), (\tilde{\chi}^2 \Lambda_0 \tilde{v}_1), 1)_{L^2(B_+(0, r_m))}$$

$$= (\tilde{J}\Lambda_0 \tilde{v}_t, (\tilde{\chi}^2 \Lambda_0 \tilde{v}_1), 1)_{L^2(B_+(0, r_m))} + \left[ \Lambda_e, \tilde{J} \right] \tilde{v}_t, (\tilde{\chi}^2 \Lambda_0 \tilde{v}_1), 1 \right]_{L^2(B_+(0, r_m))}$$

$$= -(\tilde{J}\Lambda_0 \tilde{v}_t, \Lambda_0 \tilde{v}_1)_{L^2(B_+(0, r_m))} - (\tilde{J} \Lambda_0 \tilde{v}_t, \tilde{\chi}^2 \Lambda_0 \tilde{v}_1)_{L^2(B_+(0, r_m))} + \left[ \Lambda_e, \tilde{J} \right] \tilde{v}_t, (\tilde{\chi}^2 \Lambda_0 \tilde{v}_1), 1 \right]_{L^2(B_+(0, r_m))}.$$

By (16) and the properties of convolution,

$$\left[ \Lambda_e, \tilde{J} \right] \tilde{v}_t, (\tilde{\chi}^2 \Lambda_0 \tilde{v}_1), 1 \right]_{L^2(B_+(0, r_m))} \geq -C\epsilon \|\nabla j\|_{L^\infty(O)} \|v_t\|_{L^2(O)} \|\varphi\|_{H^2(O)} \geq -\frac{C\epsilon}{\epsilon} \|v_t\|_{L^2(O)} \|\varphi\|_{H^2(O)};$$

thus

$$\left( \Lambda_e (\tilde{J}\tilde{v}_t), (\tilde{\chi}^2 \Lambda_0 \tilde{v}_1), 1 \right)_{L^2(B_+(0, r_m))} \geq -\frac{1}{2} \frac{d}{dt} \|\tilde{J}\tilde{\chi}^2 \Lambda_0 \tilde{v}_1\|_{L^2(B_+(0, r_m))}^2$$

$$- \frac{1}{2} \left( \frac{d}{dt} \|\tilde{\chi}^2 \Lambda_0 \tilde{v}_1\|_{L^2(B_+(0, r_m))}^2 - \frac{C\epsilon}{\epsilon} \|v_t\|_{L^2(O)} \|\varphi\|_{H^2(O)}$$

$$\geq -\frac{1}{2} \frac{d}{dt} \|\tilde{J}\tilde{\chi}^2 \Lambda_0 \tilde{v}_1\|_{L^2(B_+(0, r_m))}^2 - C\epsilon \|v_t\|_{L^2(O)} \|\nabla \varphi\|_{L^2(O)}^2 - \frac{C\epsilon}{\epsilon} \|v_t\|_{L^2(O)} \|\varphi\|_{H^2(O)}.$$

Therefore, since $\Lambda_0 v \to v$ in $C([0, T]; H^1(O))$, by interpolation we conclude that

$$\lim_{\epsilon \to 0} \int_0^t \left( \Lambda_e (\tilde{J}\tilde{v}_t), (\tilde{\chi}^2 \Lambda_0 \tilde{v}_1), 1 \right)_{L^2(B_+(0, r_m))} \, dt \geq -\frac{1}{2} \|\tilde{J}\tilde{\chi}^2 \Lambda_0 \tilde{v}_1\|_{L^2(B_+(0, r_m))}^2$$

$$- C\epsilon \left( \int_0^t \|\varphi\|_{H^1(O)}^2 \, dt \right) - C\delta \int_0^t \|v_t\|_{H^2(O)}^2 \, dt - \int_0^t \int_{B_+(0, r_m)} \frac{\nabla^2 \tilde{\chi}^2 \Lambda_0 \tilde{v}_1}{x} \, dx \, dt.$$

(83)

6.3.2. The estimate of $\int_O J(A^1_i \tilde{v}_j + A^1_i \tilde{\chi}^j \Lambda^k \varphi^k \, dx$. Let $\tilde{A} = A \circ \theta_m$. Then

$$\int_O J(A^1_i \tilde{v}_j + A^1_i \tilde{\chi}^j \Lambda^k \varphi^k \, dx \rightarrow \int_{B_+(0, r_m)} J(A^1_i \tilde{v}_j + A^1_i \tilde{\chi}^j \Lambda^k \varphi^k \, dx \rightarrow \int_{B_+(0, r_m)} \tilde{J}(\tilde{A}^k_1 \tilde{A}^k_1 \tilde{\chi}^j \Lambda^k \varphi^k \Lambda_0 \tilde{v}_1), \tilde{\chi}^2 \Lambda_0 \tilde{v}_1)_{L^2} \, dy$$

$$= -\int_{B_+(0, r_m)} \tilde{J}(\tilde{A}^k_1 \tilde{A}^k_1 \tilde{\chi}^j \Lambda^k \varphi^k \Lambda_0 \tilde{v}_1), \tilde{\chi}^2 \Lambda_0 \tilde{v}_1)_{L^2} \, dy + R_1$$

with some error term $R_1$ satisfying

$$R_1 \geq -C \left( \|\nabla^2 \Psi\|_{L^2(O)} \|\nabla v\|_{L^\infty(O)} + \|\nabla v\|_{L^2(O)} \right) \|\nabla v\|_{H^1(O)},$$

where $2^+$ is a number close to but greater than 2 so that the Sobolev embedding $H^{0,2}(O) \hookrightarrow L^{2^+}(O)$ holds, and $\infty^-$ is the corresponding Hölder’s conjugate satisfying...
\[ \frac{1}{2^+} + \frac{1}{2^-} = \frac{1}{2} \] Using the notation of commutators,
\[
\int_{B_+(0, r_m)} \tilde{J}(\tilde{A}_j^j \tilde{A}_{j}^j \tilde{v}_{1r}^j + \tilde{A}_j^j \tilde{A}_{j}^j \tilde{v}_{1r}^j) \tilde{A}_k^k \Lambda_r (\tilde{\chi}^2 \Lambda \tilde{v}_{1s}^j) \, dy
\]
\[ = \int_{B_+(0, r_m)} \tilde{J}[\tilde{A}_j^j \tilde{A}_{j}^j \Lambda_r (\tilde{\chi} \tilde{v}_{1r}^j) + \tilde{A}_j^j \tilde{A}_{j}^j (\tilde{\chi} \Lambda \tilde{v}_{1r}^j)] \tilde{A}_k^k (\tilde{\chi} \Lambda \tilde{v}_{1s}^j) \, dy
\]
\[ + \int_{B_+(0, r_m)} \left( [\Lambda_r, \tilde{J} \tilde{A}_j^j \tilde{A}_{j}^j \tilde{A}_k^k \tilde{v}_{1r}^j] + [\Lambda_r, \tilde{J} \tilde{A}_j^j \tilde{A}_{j}^j \tilde{A}_k^k \tilde{v}_{1r}^j] \right) \tilde{\chi}^2 \Lambda \tilde{v}_{1s}^j \, dy; \]
thus by (18).

\[
\int_{B_+(0, r_m)} \tilde{J} \tilde{A}_j^j (\tilde{A}_j^j \tilde{A}_{j}^j \tilde{A}_k^k \Lambda_r (\tilde{\chi}^2 \Lambda \tilde{v}_{1s}^j) dy
\]
\[ \leq \int_{B_+(0, r_m)} \tilde{J}[\tilde{A}_j^j \tilde{A}_{j}^j \Lambda_r (\tilde{\chi} \tilde{v}_{1r}^j) + \tilde{A}_j^j \tilde{A}_{j}^j (\tilde{\chi} \Lambda \tilde{v}_{1r}^j)] \tilde{A}_k^k (\tilde{\chi} \Lambda \tilde{v}_{1s}^j) dy
\]
\[ + C\varepsilon \| \tilde{J} \Lambda \tilde{A}_j^j \|_{W^{1, \infty}(O)} \| \nabla^2 \tilde{v} \|_{L^2(O)}^2
\]
\[ \leq \int_{B_+(0, r_m)} \tilde{J}[\tilde{A}_j^j \tilde{A}_{j}^j \Lambda_r (\tilde{\chi} \tilde{v}_{1r}^j) + \tilde{A}_j^j \tilde{A}_{j}^j (\tilde{\chi} \Lambda \tilde{v}_{1r}^j)] \tilde{A}_k^k (\tilde{\chi} \Lambda \tilde{v}_{1s}^j) dy + \frac{C\varepsilon}{\varepsilon} \| \nabla^2 \tilde{v} \|_{L^2(O)}^2.
\]

Therefore, since \( \Lambda_r (\tilde{\chi} \tilde{v}_{1r}^j) \to \tilde{\chi} \tilde{v}_{1r}^j \) in \( L^2(0, T; L^2(B_+(0, r_m))) \), by interpolation and Young’s inequality we conclude that

\[
\lim_{\varepsilon \to 0} \int_0^t \int_O \tilde{J}(\tilde{A}_j^j \tilde{v}_{1j}^j + \tilde{A}_j^j \tilde{v}_{1j}^j) \tilde{A}_k^j \varphi_k^j \, dx \, dt
\]
\[ \geq \int_0^t \int_{B_+(0, r_m)} \tilde{J}[\tilde{A}_j^j \tilde{A}_{j}^j \Lambda_r (\tilde{\chi} \tilde{v}_{1r}^j) + \tilde{A}_j^j \tilde{A}_{j}^j (\tilde{\chi} \Lambda \tilde{v}_{1r}^j)] \tilde{A}_k^k (\tilde{\chi} \Lambda \tilde{v}_{1s}^j) dy \, dt
\]
\[ - C \int_0^t \left[ \| \nabla^2 \tilde{v} \|_{L^2(O)} \left( \| \nabla \tilde{v} \|_{L^2(O)} + \| \nabla^2 \tilde{v} \|_{L^2(O)} \right) \right] \| \nabla \tilde{v} \|_{H^1(O)} \, dt
\]
\[ \geq \frac{1}{2} \int_0^t \| \text{Def}(\tilde{\chi} \tilde{v}_{1}) \|^2_{B_+(0, r_m)} \, dt - C\delta \int_0^t \| \tilde{v} \|^2_{H^1(O)} \, dt - (C\varepsilon + \delta) \int_0^t \| \tilde{v} \|^2_{L^2(O)} \, dt. \] (84)

6.3.3. The estimate of \( (q, \tilde{J} \tilde{A}_j^j \varphi_j^j)_{L^2(O)} \). Let \( \tilde{q} = q \circ \theta_m \). Making a change of variable and integrating by parts in \( y_1 \), we obtain that

\[
(q, \tilde{J} \tilde{A}_j^j \varphi_j^j)_{L^2(O)} = (\tilde{q}, \tilde{J} \tilde{A}_j^j \tilde{A}_j^j \Lambda_r (\tilde{\chi}^2 \Lambda \tilde{v}_{1s}^j), \tilde{\Lambda})_{L^2(B_+(0, r_m))}
\]
\[ = - (\tilde{q}, \tilde{J} \tilde{A}_j^j (\tilde{A}_j^j \Lambda_r (\tilde{\chi}^2 \Lambda \tilde{v}_{1s}^j)), \tilde{\Lambda})_{L^2(B_+(0, r_m))}
\]
\[ - (\tilde{q}, \tilde{J} \tilde{A}_j^j (\tilde{A}_j^j \Lambda_r (\tilde{\chi}^2 \Lambda \tilde{v}_{1s}^j)), \tilde{\Lambda})_{L^2(B_+(0, r_m))}
\]
\[ - (\tilde{q}, \tilde{J} \tilde{A}_j^j (\tilde{A}_j^j \Lambda_r (\tilde{\chi}^2 \Lambda \tilde{v}_{1s}^j)), \tilde{\Lambda})_{L^2(B_+(0, r_m))}.
\]

For the first term, we can integrate by parts in \( y_1 \) again and obtain that

\[
- (\tilde{q}, \tilde{J} \tilde{A}_j^j (\tilde{A}_j^j \Lambda_r (\tilde{\chi}^2 \Lambda \tilde{v}_{1s}^j)), \tilde{\Lambda})_{L^2(B_+(0, r_m))}
\]
\[ \leq C \left[ \| q \|_{L^\infty(O)} \| \nabla^2 \tilde{v} \|_{L^2(O)} + \| q \|_{H^1(O)} \right] \| \nabla \tilde{v} \|_{H^1(O)} \leq C \| q \|_{H^1(O)} \| \nabla \tilde{v} \|_{H^1(O)}.
\]
Similarly, it is easy to see that the second term satisfy that

$$- (\tilde{q}, (\tilde{J}A^i_j)_1 \tilde{a}^j_k r (\tilde{\chi}^2 \tilde{\Lambda} \tilde{\psi}_j, \tilde{\lambda}))_{L^2(B_+(0, r_m))}$$

$$\leq C \| \tilde{q} \|_{L^2(0, r_m)} \| \nabla^2 \tilde{\psi} \|_{L^2(0)} \| \tilde{\psi} \|_{H^2(0)} \leq C \varsigma \| \tilde{q} \|_{H^1(0)} \| \tilde{\psi} \|_{H^2(0)}.$$

For the last term, since $0 = (A^i_j \psi_j, \tilde{\theta}_m) = \tilde{J}A^i_j \tilde{\psi}_j$, we find that

$$\tilde{J}A^i_j \tilde{a}^j_k r \psi_j, \tilde{\theta}_m = (\tilde{J}A^i_j \tilde{a}^j_k r \psi_j), \tilde{\psi}_j = - (\tilde{J}A^i_j \tilde{a}^j_k r \psi_j), \psi_j;$$

thus by the convergence of $\Lambda, (\tilde{\chi}^2 \tilde{\psi}_j, \tilde{\lambda}) \rightarrow (\tilde{\chi}^2 \tilde{\psi}_j, \tilde{\lambda})$ in $L^2(0, T; L^2(B_+(0, r_m))$,

$$- \lim_{\varepsilon \to 0} \int_0^t (\tilde{q}, (\tilde{J}A^i_j)_1 \tilde{a}^j_k r (\tilde{\chi}^2 \tilde{\Lambda} \tilde{\psi}_j, \tilde{\lambda}))_{L^2(B_+(0, r_m))} dt$$

$$= - \int_0^t (\tilde{q}, (\tilde{J}A^i_j)_1 \tilde{a}^j_k r (\tilde{\chi}^2 \tilde{\psi}_j, \tilde{\lambda}))_{L^2(B_+(0, r_m))} dt$$

$$\leq C \left( \| \tilde{q} \|_{L^2(0)} \| \nabla^2 \tilde{\psi} \|_{L^2(0)} + \| \tilde{\psi} \|_{H^2(0)} \right) \| \tilde{\psi} \|_{H^1(0)} \leq C \| \tilde{q} \|_{H^1(0)} \| \tilde{\psi} \|_{H^2(0)}.$$

As a consequence, by Young’s inequality we conclude that

$$\lim_{\varepsilon \to 0} \int_0^t (\tilde{q}, (\tilde{J}A^i_j \tilde{\psi}_j)_{L^2(0)} dt$$

$$\leq C \delta \int_0^t \| \tilde{\psi} \|_{H^1(0)} dt + C (C \delta + \delta) \int_0^t \left( \| \tilde{\psi} \|_{H^2(0)} + \| \tilde{\psi} \|_{H^1(0)} \right) dt.$$

6.3.4. The estimate of $(\tilde{J}^i \Lambda^i_j (\tilde{\psi}_j - \tilde{\psi}_j), \tilde{\psi}_j, \tilde{\psi}_j')_{L^2(0)}$. Since

$$\| \tilde{\psi}_j \|_{L^2(0)} \leq C \| \tilde{h} \|_{H^{2,6}(\Gamma)} \leq C \| \tilde{\psi} \|_{H^{1,6}(\Gamma)} \leq C \| \psi \|_{H^{2,3}(\Gamma)},$$

by interpolation and Young’s inequality it is easy to see that

$$(\tilde{J}^i \Lambda^i_j (\tilde{\psi}_j - \tilde{\psi}_j), \tilde{\psi}_j, \tilde{\psi}_j')_{L^2(0)}$$

$$\leq C \left( \| \tilde{\psi}_j \|_{L^2(0)} \| \tilde{\psi} \|_{H^2(0)} \right) \| \tilde{\psi} \|_{H^1(0)} \leq C \delta \| \tilde{\psi} \|_{L^2(0)}^2 \| \tilde{\psi} \|_{H^2(0)}^2 \| \tilde{\psi} \|_{H^1(0)}.$$

(86)

6.3.5. The estimate of $\int_\Gamma \mathcal{L}_r(h)(J\Lambda^i_j \tilde{\psi}_j, N_j) dS$. Let $g = (1 + b_0 h_{\varepsilon})^2 + h_{\varepsilon}^2$. We note that since $h$ satisfies

$$|g(t) - 1|_{H^2(\Gamma)} \leq C \varsigma \quad \forall t \in [0, T_{M_1}],$$

By writing $\mathcal{L}_r(h) = \frac{h^\prime}{1 + b_0 h_{\varepsilon}} + \mathcal{R}$, where by $\mathcal{R} \equiv \mathcal{L}_r(h) - \frac{h^\prime}{1 + b_0 h_{\varepsilon}}$ satisfies

$$\| \mathcal{R} \|_{H^{2,5}(\Gamma)} \leq C \left[ \left( \frac{1 + b_0 h_{\varepsilon}^2}{g^\prime} - 1 \right)_{H^{0,625}(\Gamma)} \| h \|_{H^2(\Gamma)} + 1 \right] \leq C \left( \sqrt{\varsigma} \| h \|_{H^2(\Gamma)} + 1 \right),$$

we find that

$$\int_\Gamma \mathcal{L}_r(h)(J\Lambda^i_j \tilde{\psi}_j, N_j) dS$$

$$= \int_\Gamma \frac{h^\prime}{1 + b_0 h_{\varepsilon}} J^T N \tilde{\psi} dS + \int_\Gamma \left( \mathcal{L}_r(h) - \frac{h^\prime}{1 + b_0 h_{\varepsilon}} \right)(J\Lambda^T N) \cdot \tilde{\psi} dS$$

$$\geq \int_\Gamma \frac{h^\prime}{1 + b_0 h_{\varepsilon}} J^T N \tilde{\psi} dS - C \left( \sqrt{\varsigma} \| h \|_{H^2(\Gamma)} + 1 \right) \| \tilde{\psi} \|_{H^1(\Gamma)}.$$
On the other hand, since \( \varphi \rightarrow (\chi^2 \tilde{\nu}_1) \) in \( L^2(0, T; L^2(B(0, r_m) \cap \{ y_2 = 0 \})) \),

\[
\lim_{\epsilon \to 0} \int_0^t \int_{\Gamma} \frac{\tilde{h}''}{1 + \tilde{b}_h \epsilon} (J \mathcal{A}_h \varphi^i N_j) \, dS d\tilde{t} \\
\geq \int_0^t \int_{\Gamma} \chi^2 \tilde{h}'' \left[ h_t'' - \left( \frac{-h''_{\epsilon\epsilon} (X' \circ X^{-1})}{1 + \tilde{b}_h \epsilon} + N \right) v - 2 \left( \frac{-h''_{\epsilon\epsilon} (X' \circ X^{-1})}{1 + \tilde{b}_h \epsilon} + N \right) v' \right] dS d\tilde{t} \\
- C \int_0^t \| \tilde{h}'' \|_{L^2(\Gamma)} \| v \|_{L^2(\Gamma)} d\tilde{t} \\
\geq \frac{1}{2} \left( \| \tilde{h}'' \|_{L^2(\Gamma)}^2 - \| h_0 \|_{H^2(\Gamma)}^2 \right) - \int_0^t \int_{\Gamma} \frac{(X' \circ X^{-1}) \cdot \tilde{h}''}{1 + \tilde{b}_h \epsilon} \tilde{h}'' \, dS d\tilde{t} \\
- C \int_0^t \left( \| \tilde{h}'' \|_{L^2(\Gamma)}^2 + \| \tilde{h}'' \|_{L^2(\Gamma)} \| v \|_{H^1(\Gamma)} \right) d\tilde{t}.
\]

By commutator estimate \( \mathcal{L}_h \),

\[
\int_{\Gamma} \frac{(X' \circ X^{-1}) \cdot \tilde{h}''}{1 + \tilde{b}_h \epsilon} \tilde{h}'' \, dS \\
= - \int_{\Gamma} \left( \left[ \left( \frac{(X' \circ X^{-1}) \cdot \tilde{h}''}{1 + \tilde{b}_h \epsilon} \right) \right] \right) h'' \, dS - \int_{\Gamma} \left( \frac{(X' \circ X^{-1}) \cdot \tilde{h}''}{1 + \tilde{b}_h \epsilon} \right) h'' \, dS \\
\leq \frac{1}{2} \int_{\Gamma} \left( \left| \left( \frac{(X' \circ X^{-1}) \cdot \tilde{h}''}{1 + \tilde{b}_h \epsilon} \right) \right| \right) \left| h'' \right|^2 \, dS + C \| \tilde{h}'' \|_{H^1(\Gamma)} \| v \|_{L^2(\Gamma)} \| h'' \|_{L^2(\Gamma)} ;
\]

thus by interpolation and Young's inequality,

\[
\int_{\Gamma} \frac{(X' \circ X^{-1}) \cdot \tilde{h}''}{1 + \tilde{b}_h \epsilon} \tilde{h}'' \, dS \\
\leq C \| \tilde{h}'' \|_{L^1(\Gamma)} \| h'' \|_{L^2(\Gamma)} + C \sqrt{\epsilon} \| \tilde{h}'' \|_{H^1(\Gamma)} \| v \|_{H^1(\Gamma)} \| h'' \|_{H^2(\Gamma)} \\
\leq C \| \tilde{h}'' \|_{H^1(\Gamma)} \| h \|_{H^2(\Gamma)} + C_\delta \| \tilde{h}'' \|_{H^1(\Gamma)} \| h \|_{H^2(\Gamma)} \| h'' \|_{H^2(\Gamma)} + \delta_1 \epsilon^2 \| v \|_{H^2(\Gamma)} \\
\leq C_\delta \| h \|_{H^2(\Gamma)} \| h'' \|_{H^1(\Gamma)} + \delta_2 \left( \| h \|_{H^2(\Gamma)} \right) \| h'' \|_{H^1(\Gamma)} \| h'' \|_{H^2(\Gamma)} + \delta_2 \epsilon^2 \| v \|_{H^2(\Gamma)} \\
\leq C_\delta, \delta_2 \left( 1 + \| h \|_{H^2(\Gamma)} \right) \| v \|_{H^2(\Gamma)} + \delta_2 \left( \| h \|_{H^2(\Gamma)} \right) \| h'' \|_{H^1(\Gamma)} + \delta_2 \epsilon^2 \| v \|_{H^2(\Gamma)} .
\]

Therefore, we obtain that

\[
\lim_{\epsilon \to 0} \int_0^t \int_{\Gamma} \mathcal{L}_h (\tilde{h} (J \mathcal{A}_h \varphi^i)) \, dS d\tilde{t} \\
\geq \frac{1}{2} \left( \| \tilde{h}'' \|_{L^2(\Gamma)} - C \| h_0 \|_{H^2(\Gamma)} - C_\delta, \delta_2 \int_0^t \left( 1 + \| h \|_{H^2(\Gamma)} \right) \| v \|_{H^2(\Gamma)} d\tilde{t} \right. \tag{87} \\
- \left. \left( C \sqrt{\epsilon} + \delta \right) \int_0^t \left( \| h \|_{H^2(\Gamma)} \right) \| v \|_{H^2(\Gamma)} d\tilde{t} - \delta_2 \epsilon^2 \int_0^t \| v \|_{H^2(\Gamma)} d\tilde{t} .
\]

6.3.6. The estimate of \( \varepsilon^2 (\Delta_0 (J \mathcal{A}_h \varphi^i) \cdot J \mathcal{A}_h \varphi^i) \) \( L^2(\Gamma) \). Let \( |\theta_{m,1}|^2 = \tilde{g} \). Then

\[
\Delta_0 (J \mathcal{A}_h \varphi^i) \circ \theta_m = \frac{1}{\sqrt{\tilde{g}}} \left( \frac{1}{\sqrt{\tilde{g}}} (\tilde{J A}_h \varphi^i) \right) \cdot 1 \quad \text{on} \quad B(0, r_m) \cap \{ y_2 = 0 \} .
\]

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Therefore, by \( \varphi \to (\hat{\chi}^2 \hat{\varphi}, 1) \) in \( L^2(0, T; L^2(\Gamma)) \),

\[
\lim_{\varepsilon \to 0} \varepsilon^{2} \int_{0}^{t} (\Delta_0 (JA^{\varepsilon}_x \varphi^\varepsilon), JA^{\varepsilon}_x \varphi^\varepsilon)_{L^2(\Gamma)} d\tilde{t} \\
= \varepsilon^{2} \int_{0}^{t} \int_{B(0,r_{m}) \cap (y_{2} = 0)} \left( \frac{1}{\sqrt{g}} \hat{J} \hat{A}^{\varepsilon}_x \varphi^\varepsilon, _1 \right) \hat{J} \hat{A}^{\varepsilon}_x (\hat{\chi}^2 \hat{\varphi}, 1)_{1} d\chi_1 d\tilde{t} \\
\geq \varepsilon^{2} \int_{0}^{t} \int_{B(0,r_{m}) \cap (y_{2} = 0)} \frac{\hat{J}^2 \hat{A}^{\varepsilon}_x \hat{A}^{\varepsilon}_y}{\sqrt{g}} (\hat{\chi}^2 \hat{\varphi}, 1) \hat{\chi}_{11} d\chi_1 d\tilde{t} \\
- C \varepsilon^{2} \int_{0}^{t} \left( \| JA \|_{H^2(\Gamma)} \| v \|_{L^2(\Gamma)} + \| JA \|_{W^{1,\infty}(\Gamma)} \| v \|_{H^1(\Gamma)} \right) \| v \|_{H^2(\Gamma)} d\tilde{t}.
\]

By interpolation and the properties of convolution,

\[
\| JA \|_{H^2(\Gamma)} \leq C \| h \|_{H^2(\Gamma)}, \quad \| JA \|_{W^{1,\infty}(\Gamma)} \leq C \| JA \|_{H^1(\Gamma)}^{1/2} \| JA \|_{H^1(\Gamma)}^{1/2} \leq C \sqrt{\varepsilon} \| h \|_{H^2(\Gamma)};
\]

thus by interpolation, the trace estimate, and Young’s inequality we conclude that

\[
\lim_{\varepsilon \to 0} \varepsilon^{2} \int_{0}^{t} (\Delta_0 (JA^{\varepsilon}_x \varphi^\varepsilon), JA^{\varepsilon}_x \varphi^\varepsilon)_{L^2(\Gamma)} d\tilde{t} \\
\geq \varepsilon^{2} \int_{0}^{t} \int_{B(0,r_{m}) \cap (y_{2} = 0)} \frac{1}{\sqrt{g}} \| \hat{J} \hat{A}^{\varepsilon}_x \hat{\chi}_{11} \|_{L^2(\Gamma)} d\chi_1 d\tilde{t} \\
- C \varepsilon \int_{0}^{t} \left( \| h \|_{H^2(\Gamma)} \| v \|_{L^2(\Gamma)}^{6/7} \| v \|_{H^2(\Gamma)}^{8/7} \right) d\tilde{t} - C \varepsilon^{3/2} \int_{0}^{t} \left( \| h \|_{H^2(\Gamma)} \| v \|_{H^1(\Gamma)}^{2/3} \| v \|_{H^2(\Gamma)}^{4/3} \right) d\tilde{t} \\
\geq \varepsilon^{2} (1 - C \delta) \int_{0}^{t} \int_{0}^{t} \left( \| \hat{g}^{-1/4} \hat{\chi}_{11} \|_{L^2(\Gamma)} \right) d\tilde{t} - \varepsilon \int_{0}^{t} \left( \| v \|_{H^2(\Gamma)} \right) d\tilde{t} \\
- C \delta \int_{0}^{t} \int_{0}^{t} \left( \| \hat{g}^{-1/4} \hat{\chi}_{11} \|_{L^2(\Gamma)} \right) d\tilde{t}.
\]

6.3.7. The combination of (SS) - (SS). Summing (SS) - (SS) over \( m = 1, \ldots, K \), by the Korn’s inequality, the trace estimate, estimate (75) and choosing \( \delta > 0 \) small enough we find that

\[
\| \sqrt{\hat{J}} \hat{\chi}_{11} \|_{L^2(\Gamma)}^2 + \| h \|_{H^2(\Gamma)}^2 + \| v \|_{H^2(\Gamma)}^2 d\tilde{t} \\
\leq C \int_{0}^{t} \left( \| v \|_{H^2(\Gamma)}^2 + \| \hat{g}^{-1/4} \hat{\chi}_{11} \|_{L^2(\Gamma)} \right) d\tilde{t} + C \delta \int_{0}^{t} \| v \|_{H^2(\Gamma)}^2 + \| \hat{g}^{-1/4} \hat{\chi}_{11} \|_{L^2(\Gamma)} \right) d\tilde{t}.
\]

6.4. The implication of the Stokes regularity. First, we combine (SS) and (SS) as well as choose \( \delta > 0 \) small enough to conclude that

\[
\| v(t) \|_{H^1(\Gamma)}^2 + \| h(t) \|_{H^2(\Gamma)}^2 + \int_{0}^{t} \left( \| v \|_{H^2(\Gamma)}^2 + \| \hat{g}^{-1/4} \hat{\chi}_{11} \|_{L^2(\Gamma)} \right) d\tilde{t} \\
\leq C \delta \left( 1 + \| v \|_{H^2(\Gamma)}^2 + \| h(t) \|_{H^2(\Gamma)}^2 \right) + \delta \int_{0}^{t} \left( \| v \|_{H^2(\Gamma)}^2 + \| \hat{g}^{-1/4} \hat{\chi}_{11} \|_{L^2(\Gamma)} \right) d\tilde{t}.
\]
Next, we rewrite (73a, b) as

\[-\Delta v + \nabla q = f \quad \text{in} \quad O,\]
\[\text{div} v = g \quad \text{in} \quad O,\]

where \(f\) and \(g\) are given by

\[f^i = A^j_k (\Psi^k_i - v^i)^j - v^i + (\delta^i_k - A^j_k A^i_j)\nu^i_j + (\delta^i_k A^j_k - A^j_k A^i_j)\nu^i_j - A^j_k (A^i_k \nu^i_j + A^j_k \nu^i_j) + (\delta^i_k - A^j_k)q_j ,\]
\[g = (\delta^i_k - JA^i)\nu^i_j .\]

Applying the regularity theory for the Stokes equation, by interpolation and Young’s inequality we obtain that

\[\|v\|_{H^2(O)}^2 + \|q\|_{H^1(O)}^2 \leq C \left[ \|f\|_{L^2(O)}^2 + \|g\|_{H^1(O)}^2 + \|v\|_{H^1,\gamma(\Gamma)}^2 \right] \]
\[\leq C \left[ \|v\|_{L^2(O)}^2 + \|v\|_{H^1,\gamma(\Gamma)}^2 + \|\nabla A\|_{L^2(O)}^2 \|\nabla v\|_{L^2(O)}^2 \right] \]
\[+ C (\|\Psi\|_{L^4(O)} + \|\nu\|_{L^4(O)}) \|\nabla v\|_{L^4(O)} + C_\delta \left[ \|v\|_{H^2(O)}^2 + \|q\|_{H^1(O)}^2 \right] \]
\[\leq C \left[ \|v\|_{L^2(O)}^2 + \|v\|_{H^2,\gamma(\Gamma)}^2 \right] + C_\delta (\|v\|_{H^2(O)}^2 + \|v\|_{H^2,\gamma(\Gamma)}^2) \]
\[+ (C_\delta + \delta) \left[ \|v\|_{H^2(O)}^2 + \|q\|_{H^1(O)}^2 + \|h\|_{H^2,\gamma(\Gamma)}^2 \right] ;\]

thus by (73) and (90) with \(\zeta < 1\) and \(\delta > 0\) small enough,

\[\int_0^t \left[ \|v\|_{H^2(O)}^2 + \|q\|_{H^1(O)}^2 \right] dt \leq C \left[ 1 + \|u_0\|_{H^1(O)}^2 + \|h_0\|_{H^2(\Gamma)}^2 \right] \]
\[+ C \int_0^t \left[ \|h\|_{H^2(\Gamma)} + 1 \right] \left[ 1 + \|v\|_{H^1(O)}^2 + \|v\|_{H^2,\gamma(\Gamma)}^2 \right] dt .\]  

Let \(E(t)\) be the energy function defined by

\[E(t) = \sup_{s \in [0, t]} \left[ \|v(s)\|_{H^1(O)}^2 + \|h(s)\|_{H^2(\Gamma)}^2 \right] + \int_0^t \|v\|_{H^2(O)}^2 dt .\]

Then \(E(t)\) is continuous in \(t\), and (90) and (91) together imply that

\[E(t) \leq C \left[ 1 + \|u_0\|_{H^1(O)}^2 + \|h_0\|_{H^2(\Gamma)}^2 \right] + tP(E(t)) \]

for some polynomial \(P\). Therefore, there exists \(T_* > 0\) such that

\[E(t) \leq 2C \left[ 1 + \|u_0\|_{H^1(O)}^2 + \|h_0\|_{H^2(\Gamma)}^2 \right] \quad \forall \, t \in [0, T_*] \]  

whenever \(E(t)\) exists.

7. The existence of a solution to the problem

7.1. The continuation argument. Let \(M_1 > 0\) be given by

\[M_1^2 = \max \left\{ 2C \left[ 1 + \|u_0\|_{H^1(O)}^2 + \|h_0\|_{H^2(\Gamma)}^2 \right], M^2 \right\} ,\]

and \(T \equiv \min\{T_M, T_*\}\), where we recall that \(T_M\) is chosen (in Proposition 5.3) so that

\[\|h(t)\|_{H^1,\gamma(\Gamma)} < \zeta \quad \forall \, t \in [0, T_M] .\]

In Section 6, we have shown that for any given \((u_0, h_0) \in H^1_{\text{div}}(O) \times H^2(\Gamma)\) with \(\|h_0\|_{H^1,\gamma(\Gamma)} < \zeta\), there exists a solution \((w^\varepsilon, q^\varepsilon, h^\varepsilon)\) to equation (33) in the time
interval \([0, T_\varepsilon]\) for some \(T_\varepsilon\) depending on the smoothing parameter \(\varepsilon\). In principle, \(T_\varepsilon \to 0\) as \(\varepsilon \to 0\). Nevertheless, estimate (\ref{52}) implies that if \(T_\varepsilon \leq T\),
\[
 w^\varepsilon(T_\varepsilon) \in H^1_{\text{div}}(O) \quad \text{and} \quad h^\varepsilon(T_\varepsilon) \in H^2(\Gamma);
\]
thus if \(T_\varepsilon < T\), we can use \((w^\varepsilon(T_\varepsilon), h^\varepsilon(T_\varepsilon))\) as a new set of initial data and extend the time interval of existence (using the updated initial data) to \([0, T'_\varepsilon]\) for some \(T'_\varepsilon > T_\varepsilon\). Estimate (\ref{52}) still holds in the time interval \([0, T'_\varepsilon]\); thus if \(T'_\varepsilon < T\), we can keep the process of solving (\ref{53}) using new initial data. This implies that solution \((w^\varepsilon, q^\varepsilon, h^\varepsilon)\) exists in the time interval \([0, T]\).

7.2. The existence of a solution to equation (\ref{13}). Since \((w^\varepsilon, q^\varepsilon, h^\varepsilon)\) exists in a time interval independent of \(\varepsilon\), and satisfies estimate (\ref{52}) with \(\varepsilon\)-independent upper bound, we can pass \(\varepsilon \to 0\) and obtain that
\[
(w^\varepsilon, w_\varepsilon') \to w \quad \text{in} \quad L^2(0, T; H^2(O)) \times L^2(0, T; L^2(O)),
\]
\[
q^\varepsilon \to q \quad \text{in} \quad L^2(0, T; H^1(O)),
\]
\[
(h^\varepsilon, h_\varepsilon') \to (h, h_1) \quad \text{in} \quad L^2(0, T; H^{2.5}(\Gamma)) \times L^2(0, T; H^{1.5}(\Gamma))
\]
for some \((w, q, h) \in V(T) \times Q(T) \times H(T)\). In particular, there exists \(\varepsilon_j\) such that
\[
h^{\varepsilon_j} \to h \quad \text{in} \quad C([0, T]; H^{1.75}(\Gamma))
\]
which further implies that
\[
\psi^{\varepsilon_j} \to \psi \quad \text{in} \quad C([0, T]; H^{2.25}(O)),
\]
\[
A^{\varepsilon_j} \to A \quad \text{in} \quad C([0, T]; H^{1.25}(O)) \to C([0, T]; C(O)),
\]
where \(\psi^{\varepsilon_j}\) is the ALE map corresponding to \(h^{\varepsilon_j}\) and \(A^{\varepsilon_j} = (\nabla \psi^{\varepsilon_j})^{-1}\). Since \((w^{\varepsilon_j}, q^{\varepsilon_j}, h^{\varepsilon_j})\) satisfies
\[
(w^{\varepsilon_j}, \varphi)_L^2(O) + B_{w^{\varepsilon_j}}(w^{\varepsilon_j}, \varphi) + \varepsilon_j (w^{\varepsilon_j}, \varphi')_L^2(\Gamma) + (q^{\varepsilon_j}, \text{div} \varphi)_L^2(O)
\]
\[
= (F, \varphi)_L^2(O) + \int_\Gamma L(h^{\varepsilon_j})(\varphi \cdot N) dS \quad \forall \varphi \in H^1(O), \text{ a.e. } t \in [0, T],
\]
integrating the equality above in \(t\) over the time interval \((a, b) \subseteq (0, T)\) and then passing \(j \to \infty\), we conclude that \((w, q, h)\) satisfies
\[
\int_a^b \left[ (w_t, \varphi)_L^2(O) + B_{w}(w, \varphi) + (q, \text{div} \varphi)_L^2(O) \right] dt
\]
\[
= \int_a^b (F, \varphi)_L^2(O) dt + \int_a^b \int_\Gamma L(h)(\varphi \cdot N) dS dt \quad \forall \varphi \in H^1(O),
\]
and the Lebesgue differentiation theorem further implies that \((w, q, \varphi)\) satisfies
\[
(w_t, \varphi)_L^2(O) + B_{w}(w, \varphi) + (q, \text{div} \varphi)_L^2(O)
\]
\[
= (F, \varphi)_L^2(O) + \int_\Gamma L(h)(\varphi \cdot N) dS \quad \forall \varphi \in H^1(O), \text{ a.e. } t \in [0, T].
\]
Moreover, since \((w^\varepsilon(0), h^\varepsilon(0)) = (w_0, h_0)\) for all \(\varepsilon > 0\), we must have \((w(0), h(0)) = (w_0, h_0)\). Therefore, we establish the existence of a solution \((w, q, h)\) to equation (\ref{13}) which is the main result of this paper.
Appendix A. Proof of Theorem 3.4

Theorem 3.4 Let $a^{jk}_{rs}$ be a $(2,2)$-tensor such that $a^{ik}_{rs} = a^{kj}_{rs} = a^{jk}_{sr}$, and satisfy
\[
\|a^{jk}_{rs} - \lambda_1 \delta^j_k \delta^r_s - \lambda_2 \delta^j_k \delta^s_r\|_{L^\infty(O)} \ll 1
\]
for some positive constants $\lambda_1$ and $\lambda_2$.

(1) Suppose that $(u, q) \in V \times L^2(O)$ is a weak solution to the following elliptic equation
\[
-\left[ a^{jk}_{rs} w^r_j \right]_k + q_s = f^s \quad \text{in } O,
\]
\[
\text{div } w = 0 \quad \text{in } O,
\]
\[
a^{jk}_{rs} w^r_j \nu_k - q u_s = \varepsilon \Delta_0 w^s + g^s \quad \text{on } \Gamma;
\]
that is, $(w, q)$ satisfies the variational formulation
\[
(a^{jk}_{rs} w^r_j, \varphi^s_j)_L^2(O) - (q, \text{div } \varphi)_L^2(O) + \varepsilon ((w', \varphi')_L^2(\Gamma) = (f, \varphi)_L^2(O) + (g, \varphi)_L^2(O) \quad \forall \varphi \in V.
\]

Then $(w, q) \in H^2(O) \times H^1(O)$, and there are constants $C$ and $C_e$ such that
\[
\|w\|_{H^2(O)}^2 + \varepsilon \|w\|_{H^2(\Gamma)}^2 + \|q\|_{H^1(O)}^2 \leq C_e \|g\|_{L^2(\Gamma)}^2 + C(1 + \|\nu\|_{L^\infty(O)}) \times
\]
\[
\left[ (1 + \|\nu\|_{H^{1,\infty}(O)}) \|w\|_{H^1(O)}^2 + \|f\|_{L^2(O)}^2 + \|g\|_{H^{-\alpha,\infty}(\Gamma)}^2 \right].
\]

(2) Suppose that $a \in W^{1,4}(O)$, and $(u, q) \in H^1_0(O) \times L^2(O)$ is a weak solution to the following elliptic equation
\[
-\left[ a^{jk}_{rs} w^r_j \right]_k + q_s = f^s \quad \text{in } O,
\]
\[
\text{div } w = 0 \quad \text{in } O,
\]
\[
w = 0 \quad \text{on } \Gamma;
\]
that is,
\[
(a^{jk}_{rs} w^r_j, \varphi^s_j)_L^2(O) - (q, \text{div } \varphi)_L^2(O) = (f, \varphi)_L^2(O) \quad \forall \varphi \in H^1_0(O).
\]

Then $(w, q) \in H^2(O) \times H^1(O)$ satisfies
\[
\|w\|_{H^2(O)}^2 + \|q\|_{H^1(O)}^2 \leq C \left[ 1 + \|f\|_{L^2(O)}^2 + \|\nu\|_{L^\infty(O)}^2 \|w\|_{H^1(O)}^2 \right].
\]

Proof. For simplicity, we assume $\lambda_1 = 1$ and $\lambda_2 = 0$, while the additional term $\lambda_2 \delta^j_k \delta^s_r$ with $\lambda_2 > 0$ can be treated in the same way and with the help of the Korn inequality
\[
c_1 \|w\|_{H^1(O)} \lesssim \|u\|_{L^2(O)} + \|\text{Def } u\|_{L^2(O)} \leq C_1 \|u\|_{H^1(O)}.
\]

Part 1: We prove (1) first.

Basic energy estimates: Letting $\varphi = w$ in (27) we obtain that
\[
\|\nabla w\|_{L^2(O)}^2 + \varepsilon \|w\|_{L^2(O)}^2 \lesssim C \left[ \|w\|_{L^2(O)}^2 + \|f\|_{L^2(O)}^2 + \|g\|_{H^{-\alpha,\infty}(\Gamma)}^2 \right]
\]
and the Lagrange multiplier lemma (with $a^{ij}_{kl} = \delta^i_j$) suggests that
\[
\|g\|_{L^2(O)} \lesssim C \left[ \|\nabla w\|_{L^2(O)} + \varepsilon \|w\|_{L^2(O)} + \|f\|_{L^2(O)} + \|g\|_{H^{-\alpha,\infty}(O)} \right]
\]
\[
\lesssim C \left[ \|w\|_{L^2(O)} + \|f\|_{L^2(O)} + \|g\|_{H^{-\alpha,\infty}(O)} \right].
\]
**Estimates in the interior:** Let $\chi$ be a smooth cut-off function with $\text{spt}(\chi) \subseteq O$, and $\eta_\epsilon$ be a family of mollifiers. Since

\[
((a_{rs}^j \chi w^r_j), \eta_* [x^2(\eta_* w^s), \ell, \ell])_{L^2(O)} = (\eta_* (a_{rs}^j \chi w^r_j), \ell, [x^2(\eta_* w^s), \ell])_{L^2(O)}
\]

\[
= (a_{rs}^j (\eta_* w^r_j), \ell, [x^2(\eta_* w^s), \ell] + 2[\chi \chi, \ell (\eta_* w^s)])_{L^2(O)}
\]

\[
+ ((\eta_* a_{rs}^j \chi w^r_j), \ell, [x^2(\eta_* w^s), \ell] + 2[\chi \chi, \ell (\eta_* w^s)])_{L^2(O)}
\]

\[
\geq (1 - \|a - \text{Id} \otimes \text{Id}\|_{L^\infty(O)}) \|\chi \nabla (\eta_* \nabla w)\|_{L^2(O)}^2
\]

\[
- C\|a\|_{L^\infty(O)} \|\nabla w\|_{L^2(O)} \|\chi \nabla (\eta_* \nabla w)\|_{L^2(O)}
\]

\[
- C\|((\eta_* a_{rs}^j \chi w^r_j), \ell)_{L^2(O)} [\|\chi \nabla (\eta_* \nabla w)\|_{L^2(O)} + \|\nabla w\|_{L^2(O)}]
\]

and by $\text{div} w = 0$,

\[
- (q, \eta_* [x^2(\eta_* w^s), \ell])_{L^2(O)} = -2(q, \eta_* [\chi \chi, \ell (\eta_* w^s)])_{L^2(O)}
\]

\[
\leq C\|q\|_{L^2(O)} \left[ \|\nabla w\|_{L^2(O)} + \|\chi \nabla (\eta_* \nabla w)\|_{L^2(O)} \right],
\]

(96)

the use of $\eta_* [x^2(\eta_* w), \ell]$ as a test function in (27) suggests that

\[
(1 - \|a - \text{Id} \otimes \text{Id}\|_{L^\infty(O)}) \|\chi \nabla (\eta_* \nabla w)\|_{L^2(O)}^2
\]

\[
\leq C\|a\|_{L^\infty(O)} \|\nabla w\|_{L^2(O)} \|\chi \nabla (\eta_* \nabla w)\|_{L^2(O)}
\]

\[
+ C\|((\eta_* a_{rs}^j \chi w^r_j), \ell)_{L^2(O)} [\|\chi \nabla (\eta_* \nabla w)\|_{L^2(O)} + \|\nabla w\|_{L^2(O)}]
\]

\[
+ C\|q\|_{L^2(O)} \left[ \|\nabla w\|_{L^2(O)} + \|\chi \nabla (\eta_* \nabla w)\|_{L^2(O)} \right]
\]

\[
+ C\|f\|_{L^2(O)} \|\chi \nabla (\eta_* \nabla w)\|_{L^2(O)}.
\]

Similar to (14),

\[
\|((\eta_* a_{rs}^j \chi w^r_j), \ell)_{L^2(O)} \leq C\|\nabla a\|_{L^\infty(O)} \|\nabla w\|_{L^2(O)};
\]

thus applying Young's inequality to inequality (97) we find that

\[
\|\chi \nabla (\eta_* \nabla w)\|_{L^2(O)}^2 \leq C\|a\|_{W^{1, \infty}(O)}^2 + C\|\nabla w\|_{L^2(O)}^2 + C\|f\|_{L^2(O)}^2
\]

\[
\leq C\left(1 + \|a\|_{W^{1, \infty}(O)}^2 \|\nabla w\|_{L^2(O)}^2 + \|\nabla w\|_{L^2(O)}^2 + \|f\|_{L^2(O)}^2 + \|g\|_{H^{-0.5}(\Gamma)}^2 \right).
\]

By (94), the right-hand side of the estimate above is independent of $\epsilon$, we can pass $\epsilon \to 0$ and find that $w \in H^2_{\text{loc}}(O)$ satisfying

\[
\|\chi \nabla w\|_{L^2(O)}^2 \leq C\left(1 + \|a\|_{W^{1, \infty}(O)}^2 \|\nabla w\|_{L^2(O)}^2 + \|\nabla w\|_{L^2(O)}^2 + \|f\|_{L^2(O)}^2 + \|g\|_{H^{-0.5}(\Gamma)}^2 \right)
\]

(98)

for some constant $C$ depending on $\nabla \chi$.

Let $\psi \in H^2(O)$ be given, and $\varphi = \chi \nabla \psi$ be a test function in (27). Then

\[
(q, \Delta \psi)_{L^2(O)} = -(a_{rs}^j \chi w^r_j, (\chi \psi, \ell))_{L^2(O)} + (\chi f - q \nabla \chi, \nabla \psi)_{L^2(O)}
\]

\[
= ((a_{rs}^j \chi w^r_j), \ell, \psi)_{L^2(O)} - (a_{rs}^j \chi w^r_j)_{L^2(O)} + (\chi f - q \nabla \chi, \nabla \psi)_{L^2(O)}.
\]
thus \( \chi q \) is a distributional solution to
\[
\Delta (\chi q) = f_1 \equiv -[a^{jk}_{rs} w^r_j \nabla \chi],_s + \left( [a^{jk}_{rs} \chi w^r_j],_s \right) - \text{div}(\chi f - q \nabla \chi) \quad \text{in} \quad O,
\]
\[
\chi q = 0 \quad \text{on} \quad \Gamma.
\]
Since \( f_1 \in H^{-1}(O) \) satisfies that
\[
\| f_1 \|_{H^{-1}(O)} \leq C \left[ \| a^{jk}_{rs} w^r_j \nabla \chi \|_{L^2(O)} + \| [a^{jk}_{rs} \chi w^r_j],_s \|_{L^2(O)} + \| \chi f - q \nabla \chi \|_{L^2(O)} \right]
\]
\[
\leq C \left[ \| a \|_{H^1, \infty(O)} \| w \|_{L^2(O)} + \| a \|_{L^\infty(O)} \| \nabla^2 w \|_{L^2(O)} + \| f \|_{L^2(O)} + C \| q \|_{L^2(O)} \right],
\]
we find that \( q \in H^1_{\text{loc}}(O) \), and
\[
\| \chi q \|_{H^1(O)} \leq C \left[ \| a \|_{L^\infty(O)} \left( 1 + \| a \|_{W^{1, \infty}(O)} \right) \right] \| \nabla w \|_{L^2(O)}
\]
\[
+ C \left( 1 + \| a \|_{L^\infty(O)} \right) \left[ \| w \|_{L^2(O)} + \| f \|_{L^2(O)} + \| q \|_{H^{-0.5}(\Gamma)} \right]
\]
(99)
for some constant \( C \) depending on \( \nabla \chi \).

**Estimates near the boundary:** Now we focus on the boundary regularity. Let \( \{ \chi_m \}_{m=1}^K \) be cut-off functions supported near the boundary \( \Gamma \) introduced in Section 6.3. For a fixed \( m \), define \( \tilde{a} = (\nabla \theta_m)^{-1}, \tilde{\chi} = \chi_m \circ \theta_m \) in \( \text{spt}(\tilde{\chi}_m) \), \( (\tilde{w}, \tilde{q}) = (w, q) \circ \theta_m \) in \( B_+(0, r_m) \), and let
\[
\varphi^s = \left[ \theta_m, \ell (\Lambda_\ell (\chi^2 (\Lambda_\ell (\tilde{\chi}^2 (\tilde{a}^2 (\tilde{w}^2)),1)),1),1) \right] \circ \theta_m^{-1}
\]
be a test function in (27), where if \( x = \theta_m(y) \), then \( ,_1 \) denotes the partial derivative with respect to \( y_1 \). Since
\[
\varphi^s \circ \theta_m = \theta_m, \ell \left( \Lambda_\ell (\chi^2 (\Lambda_\ell (\tilde{\chi}^2 (\tilde{a}^2 (\tilde{w}^2)),1)),1),1 \right) + 2 \theta_m, \ell \left( \Lambda_\ell (\chi^2 (\Lambda_\ell (\tilde{a}^2 (\tilde{w}^2)),1)),1 \right)
\]
\[
+ \theta_m, \ell \left( \Lambda_\ell (\chi^2 (\Lambda_\ell (\tilde{a}^2 (\tilde{w}^2)),1)),1 \right) + \theta_m, \ell \left( \Lambda_\ell (\chi^2 (\tilde{a}^2 (\tilde{w}^2)),1),1 \right),
\]
similar to (95) we have
\[
(a^{jk}_{rs} w^r_j, \varphi^s_k)_{L^2(O)}
\]
\[
= \left( (a^{jk}_{rs} \circ \theta_m) \tilde{a}^k \tilde{w}^j, \varphi^s_k \left[ \theta_m, \ell \left( \Lambda_\ell (\chi^2 (\Lambda_\ell (\tilde{a}^2 (\tilde{w}^2)),1)),1 \right) \right] \right)_{L^2(B_+(0, r_m))}
\]
\[
\geq \left( \lambda - \| a \|_{L^\infty(B_+(0, r_m))} \| a - \text{Id} \otimes \text{Id} \|_{L^\infty(O)} \right) \| \chi_1 \|_{L^2(B_+(0, r_m))}^2
\]
\[
- C \| a \|_{W^{1, \infty}(O)} \| \nabla w \|_{L^2(O)} \left[ \| \chi \|_{L^2(B_+(0, r_m))} + \| q \|_{H^1(O)} \right],
\]
in which we use the property that \( \tilde{a}^k \tilde{a}^j \xi_i \xi_j \geq \lambda |\xi|^2 \) for some positive constant \( \lambda \). Similar to (96), by \( \tilde{a}^k \theta_m, \ell = \delta^k_\ell \) and \( \tilde{a}^k \tilde{w}^j = (\text{div} w) \circ \theta_m = 0 \) in \( B_+(0, r_m) \),
\[
-(q, \text{div} \varphi)_{L^2(O)} = -(\tilde{q}, \tilde{a}^k \theta_m, \ell \left( \Lambda_\ell (\chi^2 (\Lambda_\ell (\tilde{a}^2 (\tilde{w}^2)),1)),1 \right) \right)_{L^2(B_+(0, r_m))}
\]
\[
= -(\tilde{q}, [\Lambda_\ell (\chi^2 (\tilde{a}^2 (\tilde{w}^2)),1)),1 \right)_{L^2(B_+(0, r_m))}
\]
\[
\geq -C \| q \|_{L^2(O)} \left[ \| w \|_{H^1(O)} + \| \chi \|_{L^2(B_+(0, r_m))} \right].
\]
Moreover, for the two integrals over the boundary, we have
\[
(w^i, \varphi^s)_{L^2(\Gamma)} \geq \| \chi \|_{(\tilde{w},1,1)} \| \chi \|_{L^2(B_+(0, r_m) \cap \{ y_2 = 0 \})}
\]
\[
- C \| \chi \|_{(\tilde{w},1,1)} \| \chi \|_{L^2(B_+(0, r_m) \cap \{ y_2 = 0 \})} + \| w \|_{L^2(\Gamma)}
\]
Therefore, if we define a bounded linear function $T \colon \mathcal{V} \to \mathbb{R}$ by

$$
T(\varphi) = \langle \Lambda_{m,t} \tilde{\chi}_{\mbox{,}L}(\tilde{\varphi}^{\psi,j}), \tilde{\varphi}^{\psi,j}, \rangle_{L^2(B_+)} - \langle \Lambda_{m,t} \tilde{\chi}_{\mbox{,}L}(f \circ \theta_m), \tilde{\varphi}^{\psi,j}, \rangle_{L^2(B_+)} - \langle \Lambda_{m,t} \tilde{\chi}_{\mbox{,}L}(g \circ \theta_m), \tilde{\varphi}^{\psi,j}, \rangle_{L^2(B_+)}
$$

we can define a bounded linear function $T : \mathcal{V} \to \mathbb{R}$ by

$$
T(\varphi) = \langle \Lambda_{m,t} \tilde{\chi}_{\mbox{,}L}(\tilde{\varphi}^{\psi,j}), \tilde{\varphi}^{\psi,j}, \rangle_{L^2(B_+)} - \langle \Lambda_{m,t} \tilde{\chi}_{\mbox{,}L}(f \circ \theta_m), \tilde{\varphi}^{\psi,j}, \rangle_{L^2(B_+)} - \langle \Lambda_{m,t} \tilde{\chi}_{\mbox{,}L}(g \circ \theta_m), \tilde{\varphi}^{\psi,j}, \rangle_{L^2(B_+)}
$$

where

$$
\tilde{\varphi}^{\psi,j} = \langle \theta_m^* (\Lambda_{m,t} \tilde{\chi}_{\mbox{,}L}(\tilde{\varphi}^{\psi,j},)), \tilde{\varphi}^{\psi,j}, \rangle_{L^2(B_+)}
$$
by the Lagrange multiplier lemma (with $a_i^j = \tilde{a}_i^j$) we find that $\Lambda_e(\tilde{q}^\eta)$, is the Lagrange multiplier associated to $T$, and

$$
\left\| \Lambda_e(\tilde{q}^\eta) \right\|_{L^2(B_+(0,r_m))} \lesssim C \left[ \| q \|_{L^2(O)} + \| a \|_{L^\infty(O)} \| \chi \nabla^2 w \|_{L^2(O)} + \| \nabla a \|_{L^\infty(O)} \| \nabla w \|_{L^2(O)} + \varepsilon \| w \|_{H^2(G)} + \| f \|_{L^2(O)} + \| g \|_{L^2(G)} \right]
$$

$$
\lesssim C \left( 1 + \| a \|_{L^\infty(O)} \right) \left( 1 + \| a \|_{W^{1,\infty}(O)} \right) \| w \|_{H^1(O)}
$$

$$
+ C \left( 1 + \| a \|_{L^\infty(O)} \right) \left[ \| f \|_{L^2(O)} + \| g \|_{H^{-\alpha,\alpha}(G)} \right] + C \| g \|_{L^2(G)}.
$$

Passing $\varepsilon$ to zero, we obtain that

$$
\left\| \Lambda_e(\tilde{q}^\eta) \right\|_{L^2(B_+(0,r_m))} \lesssim C \left( 1 + \| a \|_{L^\infty(O)} \right) \left( 1 + \| a \|_{W^{1,\infty}(O)} \right) \| w \|_{H^1(O)}
$$

$$
+ C \left( 1 + \| a \|_{L^\infty(O)} \right) \left[ \| f \|_{L^2(O)} + \| g \|_{H^{-\alpha,\alpha}(G)} \right] + C \| g \|_{L^2(G)}.
$$

Estimates of normal derivatives of $w$ and $q$: Since $(w, q) \in H^2_{loc}(O) \times H^1_{loc}(O)$, [20] holds almost everywhere. Making the change of variable $\varphi = \theta_m(y)$, (20a,b) are transformed to

$$
- \left[ \tilde{a}_k^j(a_{k}^{j'} \circ \theta_m)\tilde{a}_j^i\tilde{w}_i^{j'} \right]_x + \tilde{a}_k^j\tilde{q}_k = (f \circ \theta_m) \quad \text{in} \quad B_+(0, r_m), \quad (102a)
$$

$$
- \tilde{a}_k^j\tilde{w}_i^{j'} = 0 \quad \text{in} \quad B_+(0, r_m). \quad (102b)
$$

Differentiating (102a) with respect to $y_2$, after rearrangement we find that

$$
- (a_{r_s}^{j'} \circ \theta_m)\tilde{a}_s^2\tilde{a}_k^j\tilde{w}_s^{j'} + \tilde{a}_k^j\tilde{q}_k = f_2^s \quad \text{in} \quad B_+(0, r_m), \quad (103a)
$$

$$
\tilde{a}_k^j\tilde{w}_s^{j'} = f_3 \quad \text{in} \quad B_+(0, r_m), \quad (103b)
$$

where $f_2$ and $f_3$ are given by

$$
f_2 = (f \circ \theta_m) + \left[ \tilde{a}_k^j(a_{k}^{j'} \circ \theta_m)\tilde{a}_j^i\tilde{w}_i^{j'} \right]_x \tilde{w}_x^{j'} + (a_{r_s}^{j'} \circ \theta_m)\tilde{a}_s^2\tilde{a}_k^j\tilde{w}_s^{j'} + 2(a_{k}^{j'} \circ \theta_m)\tilde{a}_k^j\tilde{w}_s^{j'} - \tilde{a}_k^j\tilde{q}_k,
$$

$$
f_3 = -\tilde{a}_s^2\tilde{w}_s^{j'} - \tilde{a}_s^1\tilde{w}_s^{j'} - \tilde{a}_s^2\tilde{w}_s^{j'},
$$

and satisfy

$$
\left\| \tilde{w}_x f_2 \right\|_{L^2(B_+(0,r_m))} + \left\| \tilde{w}_x f_3 \right\|_{L^2(B_+(0,r_m))} \lesssim C \left[ \| f \|_{L^2(O)} + \| \tilde{w}_x \|_{L^2(B_+(0,r_m))} \right]
$$

$$
+ (1 + \| a \|_{W^{1,\infty}(O)} \| w \|_{H^1(O)} + (1 + \| a \|_{L^\infty(O)} \| \tilde{w}_x \|_{H^1(B_+(0,r_m))})
$$

$$
\lesssim C \left( 1 + \| a \|_{L^\infty(O)} \right) \left( 1 + \| a \|_{W^{1,\infty}(O)} \right) \| w \|_{H^1(O)}
$$

$$
+ C \left( 1 + \| a \|_{L^\infty(O)} \right) \left[ \| f \|_{L^2(O)} + \| g \|_{H^{-\alpha,\alpha}(G)} \right] + C \| g \|_{L^2(G)}.
$$

Write (103) in the matrix form

$$
\begin{bmatrix}
- (a_{k}^{j'} \circ \theta_m)\tilde{a}_s^2\tilde{a}_k^j & \tilde{a}_k^j & \tilde{a}_k^j \\
- (a_{k}^{j'} \circ \theta_m)\tilde{a}_s^2\tilde{a}_k^j & \tilde{a}_k^j & \tilde{a}_k^j \\
\tilde{a}_s^1 & \tilde{a}_s^1 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{w}_2^{j'} \\
\tilde{w}_2^{j'} \\
\tilde{q}_2
\end{bmatrix} =
\begin{bmatrix}
\tilde{w}_2^{j'} \\
\tilde{w}_2^{j'} \\
\tilde{q}_2
\end{bmatrix}.
Since \( a_{ijk}^\epsilon \approx \delta_i^j \delta_k^l \) and \( \det(\tilde{\alpha}) = \det(\nabla \theta_m)^{-1} = 1 \),

\[
\begin{bmatrix}
-\alpha_{21}^j (\theta_m) \hat{\alpha}_2^2 \hat{\alpha}_k^2 & -\alpha_{22}^j (\theta_m) \hat{\alpha}_2^2 \hat{\alpha}_k^2 & \hat{\alpha}_1^2 \\
-\alpha_{21}^j (\theta_m) \hat{\alpha}_2^2 \hat{\alpha}_k^2 & -\alpha_{22}^j (\theta_m) \hat{\alpha}_2^2 \hat{\alpha}_k^2 & \hat{\alpha}_2^2 \\
\hat{\alpha}_1^2 & \hat{\alpha}_2^2 & 0
\end{bmatrix}
\approx \left[ (\hat{\alpha}_1^2)^2 + (\hat{\alpha}_2^2)^2 \right] \neq 0.
\]

Therefore, \([103]\) is solvable, and

\[
\| \tilde{\chi} \|_{L^2(\Omega)} \leq C \left[ \tilde{\chi} f_2 \right]_{L^2(\Omega)} = C (1 + \| a \|_{L^\infty(\Gamma)} + \| w \|_{H^1(\Omega)}^{1/2})
\]

Estimate [28] then follows from the combination of \([98], [99], [100], [101]\), and \([104]\).

**Part 2:** Next, we assume that \( a \in W^{1,4}(\Omega) \) and \( (w, q) \in H^2(\Omega) \times L^2(\Omega) \) is a weak solution to \([29]\). Let \( \epsilon \) be a smoothing parameter and \( a^\epsilon \in W^{1,\infty}(\Omega) \) be a sequence converging to \( a \) in \( W^{1,4}(\Omega) \), and \( a^\epsilon \) still satisfies the requirement \([24]\) by the following construction: let \( E : W^{1,4}(\Omega) \to W^{1,4}(\mathbb{R}^2) \) be an extension map, and \( a^\epsilon \) is defined by

\[
a^\epsilon = \eta_\epsilon \ast (\epsilon a).
\]

Let \( w^\epsilon \) be the weak solution to

\[
-\left[ (a^\epsilon)^{jk}_{rs} w_{ij}^{\epsilon \iota} \right]_k + q^\epsilon_s = f^\epsilon \quad \text{in} \quad \Omega, \quad \text{(105a)}
\]

\[
\text{div} w^\epsilon = 0 \quad \text{in} \quad \Omega, \quad \text{(105b)}
\]

\[
w^\epsilon = 0 \quad \text{on} \quad \Gamma; \quad \text{(105c)}
\]

that is,

\[
\left[ (a^\epsilon)^{jk}_{rs} w_{ij}^{\epsilon \iota}, \varphi^\epsilon_k \right]_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in H^1(\Omega),
\]

The existence of a weak solution is guaranteed by the Lax-Milgram theorem, and \( w^\epsilon \) satisfies the basic energy estimate

\[
\| w^\epsilon \|_{H^1(\Omega)} \leq C \| f \|_{L^2(\Omega)}.
\]

(A different version of) the Lagrange multiplier lemma then suggests that there exists a unique \( q \in L^2(\Omega) \) such that

\[
\left[ (a^\epsilon)^{jk}_{rs} w_{ij}^{\epsilon \iota}, \varphi^\epsilon_k \right]_{L^2(\Omega)} + (q, \text{div} \varphi)_{L^2(\Omega)} = (f, \varphi)_{L^2(\Omega)} \quad \forall \varphi \in H^1(\Omega)
\]

and

\[
\| q^\epsilon \|_{L^2(\Omega)} \leq C \left[ \| \nabla w^\epsilon \|_{L^2(\Omega)} + \| f \|_{L^2(\Omega)} \right] \leq C \| f \|_{L^2(\Omega)}.
\]

The argument in part 1 then can applied again (in the case \( \epsilon = 0 \) and \( g = 0 \)) to show that \( (w^\epsilon, q^\epsilon) \in H^2(\Omega) \times H^1(\Omega) \).

Since \((w^\epsilon, q^\epsilon)\) is a strong solution, we can perform the estimates as those in Part 1 in a slightly different fashion. To illustrate the idea, we focus on the interior estimates. The same as part 1, we use \( \eta_\epsilon \ast [\chi^2(\eta_\epsilon \ast w^\epsilon_{\iota})]_t \) as a test function in \([50]\), here we emphasize that here the convolution parameter is \( \epsilon \) instead of \( \epsilon \). This time we integrate by parts in \( x_t \) first and then move the convolution around next (in
part 1 we move the convolution around first then integrate by parts for the next step since the solution does not belong to $H^2(O)$ yet) to obtain that

$$
- \left( (a^\varepsilon)^{jk}_r w^{\varepsilon, r}_{,j}, \eta_\varepsilon \ast \left[ \chi^2(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right], \partial_r \right)_{L^2(O)}
\leq\left( (a^\varepsilon)^{jk}_r \chi(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}), \chi(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right)_{L^2(O)}
+ \left( (\eta_\varepsilon \ast \chi(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}), \chi(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right)_{L^2(O)}
+ 2 \left( (a^\varepsilon)^{jk}_r w^{\varepsilon, r}_{,j}, \eta_\varepsilon \ast \left[ \chi^2(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right] \right)_{L^2(O)}
+ \left( (a^\varepsilon)^{jk}_r \chi(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}), \chi(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right)_{L^2(O)}.
$$

By $(15)$ and the basic energy estimate $(16)$,  

$$
\| \chi(\eta_\varepsilon \ast \nabla^2 w^\varepsilon) \|_{L^2(O)} \leq \left( (a^\varepsilon)^{jk}_r w^{\varepsilon, r}_{,j}, \eta_\varepsilon \ast \left[ \chi^2(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right], \partial_r \right)_{L^2(O)}
+ \| a - \text{Id} \|_{L^\infty(O)} \| \chi(\eta_\varepsilon \ast \nabla^2 w^\varepsilon) \|_{L^2(O)}^2
+ C \| a^\varepsilon \|_{L^\infty(O)} \| \chi \nabla^2 w^\varepsilon \|_{L^2(O)}^2
+ 2 \| \chi(\eta_\varepsilon \ast \nabla^2 w^\varepsilon) \|_{L^2(O)}^2
+ C \| a^\varepsilon \|_{L^\infty(O)} \| \chi \nabla^2 w^\varepsilon \|_{L^2(O)} + \| \chi(\eta_\varepsilon \ast \nabla^2 w^\varepsilon) \|_{L^2(O)}^2.
$$

Since $w^\varepsilon \in H^2(O)$ independent of $\varepsilon$, by Young’s inequality and passing $\varepsilon \to 0$ we find that

$$
\| \chi \nabla^2 w^\varepsilon \|_{L^2(O)}^2 \leq \left( (a^\varepsilon)^{jk}_r w^{\varepsilon, r}_{,j}, \eta_\varepsilon \ast \left[ \chi^2(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right], \partial_r \right)_{L^2(O)}
+ \| a - \text{Id} \|_{L^\infty(O)} \| \chi \nabla^2 w^\varepsilon \|_{L^2(O)}^2
+ C \| a^\varepsilon \|_{L^\infty(O)} \| \chi \nabla^2 w^\varepsilon \|_{L^2(O)}^2 + \| \chi (\eta_\varepsilon \ast \nabla^2 w^\varepsilon) \|_{L^2(O)}^2.
$$

By interpolation,

$$
C \| \chi \nabla^2 w^\varepsilon \|_{L^2(O)} \| \chi \nabla^2 w^\varepsilon \|_{L^4(O)}^2 \leq C \| \chi \nabla^2 w^\varepsilon \|_{L^2(O)} \| \chi \nabla^2 w^\varepsilon \|_{L^2(O)} \| \chi \nabla^2 w^\varepsilon \|_{H^2(O)}
\leq C \| \chi \nabla^2 w^\varepsilon \|_{L^2(O)} \| \nabla w^\varepsilon \|_{L^2(O)} \| \chi \nabla w^\varepsilon \|_{H^2(O)}
\leq C \| \chi \nabla^2 w^\varepsilon \|_{L^2(O)} \| \nabla w^\varepsilon \|_{L^2(O)}^2 + \| \chi \nabla^2 w^\varepsilon \|_{L^2(O)}^2.
$$

thus by choosing $\delta > 0$ small enough, with the smallness assumption $(19)$ we find that $(109)$ suggests that

$$
\| \chi \nabla w^\varepsilon \|_{L^2(O)}^2 \leq \left( (a^\varepsilon)^{jk}_r w^{\varepsilon, r}_{,j}, \eta_\varepsilon \ast \left[ \chi^2(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right], \partial_r \right)_{L^2(O)}
+ C \left[ 1 + \| \nabla^2 w^\varepsilon \|_{L^2(O)}^2 \right] \| \chi \nabla w^\varepsilon \|_{L^2(O)}^2.
$$

On the other hand, we have

$$
- \left( q^\varepsilon, \text{div} \left[ \eta_\varepsilon \ast \left[ \chi^2(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right] \right] \right)_{L^2(O)}
\leq C \| q^\varepsilon \|_{L^2(O)} \| \chi \eta_\varepsilon \ast \left[ \chi^2(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right] \|_{L^2(O)}
\leq C \| q^\varepsilon \|_{L^2(O)} \| \chi \eta_\varepsilon \ast \left[ \chi^2(\eta_\varepsilon \ast w^{\varepsilon, r}_{,j}) \right] \|_{L^2(O)}.
$$
which implies (by passing $\epsilon \to 0$) that
\[
(q^\varepsilon, (\chi^2 w^\varepsilon), t)_{L^2(\Omega)} \leq C_8 \|f\|_{L^2(\Omega)}^2 + \delta \|\chi \nabla^2 w^\varepsilon\|_{L^2(\Omega)}^2.
\] (111)

Moreover,
\[
-(f, (\chi^2 w^\varepsilon), t)_{L^2(\Omega)} \leq C_8 \|f\|_{L^2(\Omega)}^2 + \delta \|\chi \nabla^2 w^\varepsilon\|_{L^2(\Omega)}^2.
\] (112)

By choosing $\delta > 0$ small enough, the combination of (110), (111) and (112) together with interpolation then suggests that
\[
\|\chi \nabla^2 w^\varepsilon\|_{L^2(\Omega)}^2 \leq C \left[1 + \|f\|_{L^2(\Omega)}^2 + \|\nabla a^\varepsilon\|_{L^1(\Omega)}^4 \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 \right].
\]

The interior $H^1$-estimate of $q^\varepsilon$ is done in the same way as part 1 (via a different version of the Lagrange multiplier lemma), and the estimates near the boundary can be done in the same fashion (by integrating by parts first then moving the convolution around) as the interior estimates. Moreover, the estimates of the normal derivatives of $w^\varepsilon$ and $q^\varepsilon$ are obtained in the same way as part 1, so we conclude that
\[
\|w^\varepsilon\|_{H^2(\Omega)}^2 + \|q^\varepsilon\|_{H^2(\Omega)}^2 \leq C \left[1 + \|f\|_{L^2(\Omega)}^2 + \|\nabla a^\varepsilon\|_{L^1(\Omega)}^4 \|\nabla w^\varepsilon\|_{L^2(\Omega)}^2 \right].
\] (113)

Since $a \in W^{1,4}(\Omega)$, the right-hand side of the estimate above is independent of $\varepsilon$. Therefore, there is a sequence $\varepsilon_j$ such that $(w^\varepsilon_j, q^\varepsilon_j)$ converges weakly to some function $(w, q) \in H^2(\Omega) \times H^1(\Omega)$. Moreover, since $a^\varepsilon \rightarrow a$ in $W^{1,4}(\Omega)$, the variational identity (107) converges to (106); thus $(w, q)$ must be the weak solution to (29). Finally, estimate (31) is a direct consequence of (113) by passing $\varepsilon$ to the limit.

\[\square\]

**Appendix B. Proof of Lemma 6.1**

Before proceeding to the proof of Lemma 6.1, we state the following simple proposition which can be proved easily by interpolation.

**Proposition B.1.** Let $f \in L^2(0, T; H^{0.5}(\Gamma))$, $h_0 \in H^{2.5}(\Gamma)$, and $h \in H_1(\Omega)$ be a strong solution to
\[
\begin{align*}
h'' + \varepsilon^2 h''_t &= f & \text{on } \Gamma \times (0, T), \\
h &= g & \text{on } \Gamma \times \{t = 0\}.
\end{align*}
\] (114a)

Then $h \in L^2(0, T; H^{2.5}(\Gamma))$ and satisfies
\[
\|h\|_{L^2(0, T; H^{2.5}(\Gamma))} \leq C \left[\varepsilon^2 \|g\|_{H^{2.5}(\Gamma)}^2 + \|f\|_{L^2(0, T; H^{0.5}(\Gamma))}^2 \right].
\] (115)

**Lemma 6.1** Let $(w, q, h) \in \mathcal{W}(\Gamma) \times Q(T) \times H_1(\Omega)$ be a strong solution to (73). Then
\[
\int_0^T \|h(t)\|^2_{H^{2.5}(\Gamma)} \, dt \leq C \left[1 + \|h_0\|^2_{H^{1.5}(\Gamma)} + \|\psi\|^2_{\mathcal{V}(T)} + \|\psi\|^2_{{\mathcal{Q}(T)}} \right].
\] (165)

In particular, the corresponding $J$, $A$ and $\Psi$ satisfy
\[
\int_0^T \left[\|A\|^2_{H^2(\Omega)} + \|J\|^2_{H^2(\Omega)} + \|\nabla \psi\|^2_{H^2(\Omega)} \right] \, dt \leq C \left[1 + \|h_0\|^2_{H^2(\Gamma)} + \|\psi\|^2_{\mathcal{V}(T)} + \|\psi\|^2_{{\mathcal{Q}(T)}} \right].
\] (76)
Proof. We note that (73) can be rewritten as
\[
\frac{1 + b_0 h_{\varepsilon\varepsilon}}{g^{3/2}} N + \varepsilon^2 w'' = f \quad \text{on} \quad \Gamma \times (0, T),
\]
where \(f\) is given by
\[
f^s = \psi^i \left[ A^i_j \varphi^i_j + A^i_k \varphi^i_k - \varphi^i_l \right] K_k \frac{1 + b_0 h_{\varepsilon\varepsilon}}{g^{3/2}} N_N + \frac{(1 + b_0 h_{\varepsilon\varepsilon})^2 + 2 h_{\varepsilon\varepsilon} + h_{\varepsilon\varepsilon} H_{\varepsilon\varepsilon} b_0}{g^{3/2}} N_s,
\]
and satisfies
\[
\|f\|_{L^2(0, T; H^{0.5}(\Gamma))} \leq C \left[ \|\psi\|_{C(\Gamma)} + \|\varphi\|_{C(\Gamma)} + 1 \right].
\]
We first show that \(h\) indeed belongs to \(L^\infty(0, T; H^{2.5}(\Gamma))\) (with an \(\varepsilon\)-dependent estimate), and then use this fact to obtain estimate (75).

Taking the inner product of (116) and \(N\), by the Leibniz rule we obtain that
\[
\left[ \frac{1 + b_0 h_{\varepsilon\varepsilon}}{g^{3/2}} h + \varepsilon^2 (w \cdot N) \right]' = \tilde{f}
\]
where \(\tilde{f}\) is given by
\[
\tilde{f} = f \cdot N + \varepsilon^2 w \cdot N'' + 2 \varepsilon^2 w' \cdot N' + \left[ \frac{1 + b_0 h_{\varepsilon\varepsilon}}{g^{3/2}} \right]'' + h + 2 \left( \frac{1 + b_0 h_{\varepsilon\varepsilon}}{g^{3/2}} \right)' h'.
\]
We note that due to the convolution, \(\tilde{f} \in L^2(0, T; H^{0.5}(\Gamma))\). Therefore, by elliptic regularity,
\[
\frac{1 + b_0 h_{\varepsilon\varepsilon}}{g^{3/2}} h + \varepsilon^2 (w \cdot N) = g \in L^2(0, T; H^{2.5}(\Gamma)).
\]
Since \(w\) satisfies \(h_t = \frac{w \cdot N}{1 + b_0 h_{\varepsilon\varepsilon}}\), we can further rewrite the equation above as
\[
g^{-3/2} h + \varepsilon^2 h_t = \frac{g}{1 + b_0 h_{\varepsilon\varepsilon}}.
\]
Solving the ODE above using the method of integrating factor, we find that \(h \in L^\infty(0, T; H^{2.5}(\Gamma))\).

Now we rewrite (116) as
\[
h'' + \varepsilon^2 \frac{w''}{1 + b_0 h_{\varepsilon\varepsilon}} = \frac{f}{1 + b_0 h_{\varepsilon\varepsilon}} + (1 - g^{-3/2}) h'' N
\]
which, by projecting to the normal direction, further suggests that
\[
h'' + \varepsilon^2 h''_t = \tilde{f}
\]
with \(\tilde{f}\) satisfying
\[
\|\tilde{f}\|_{H^{0.5}(\Gamma)} \leq C \left[ \|f\|_{H^{0.5}(\Gamma)} + \left( \|h\|_{H^{1.5}(\Gamma)} + \|g - 1\|_{H^{0.5}(\Gamma)} \right) \|h\|_{H^{2.5}(\Gamma)} + \varepsilon^2 \|w\|_{H^{1.5}(\Gamma)} \right]
\]
\[
\leq C \left[ \|f\|_{H^{0.5}(\Gamma)} + \zeta \|h\|_{H^{2.5}(\Gamma)} + \|g\|_{H^{1.5}(\Gamma)} \|w\|_{H^{1.5}(\Gamma)} \right]
\]
\[
\leq C \left[ \|f\|_{H^{0.5}(\Gamma)} + \|w\|_{H^{2}(\Omega)} + \zeta \|h\|_{H^{2.5}(\Gamma)} \right].
\]
where we use $w = JAv$ and (111) to derive the estimate above. Therefore, by Proposition 111, 
\[
\int_0^T \| h' \|^2_{H^1(\Gamma)} d\tilde{t} \leq C \left[ \varepsilon^2 \| h_{0} \|^2_{H^2(\Gamma)} + \int_0^T \left[ \| f \|^2_{H^0(\Gamma)} + \| \nabla \|^2_{H^2(\Omega)} + \varepsilon \| h \|^2_{H^2(\Gamma)} \right] d\tilde{t} \right] 
\leq C \left[ \| h_{0} \|^2_{H^1(\Gamma)} + \| \nabla \|^2_{H^2(\Gamma)} + 1 \right] + C\varepsilon \int_0^T \| h \|^2_{H^2(\Gamma)} d\tilde{t}.
\] (118)

On the other hand, the evolution equation (73d) implies that 
\[
\| h(t) \|_{L^2(\Gamma)} \leq \| h_0 \|_{L^2(\Gamma)} + \int_0^t \left( \frac{(J^T A_N) \cdot \nabla}{1 + \alpha h_{0,\tau}} \right) d\tilde{t} \leq \| h_0 \|_{L^2(\Gamma)} + \sqrt{t} \| \nabla \|_{L^2(0,t;H^2(\Omega))},
\]
and (75) following from the combination of (118) and the estimate above since $\varepsilon \ll 1$.}

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