An external approach to set theory

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1 Introduction

1.1 Abstract

This approach is ‘external’ in the sense that it begins with a context much larger than sets, and locates “relaxed” set theory inside this. Traditional set theory is ‘internal’ in the sense that the axioms mainly concern sets. The “gold standard” Zermillo-Fraenkel-Choice (ZFC) axioms only concern sets. This paper gives development necessary to establish properties of relaxed sets, but the primary concern is to establish the relationship with traditional set theory. The companion paper, [Quinn], explores applications of the theory. These replaces an earlier version, [Quinn old]

This approach has a number of advantages. First, it seems to provide a better formal foundation for mainstream mathematics than does traditional set theory. It begins at a lower level; the axioms are fewer and simpler; and, most usefully, it resolves a discrepancy between standard set theory and standard practice in mainstream mathematics. This is explained in Section 7.

Next, it gives a good view of the object just outside set theory (the class of all sets), which should be helpful in studying large categories and large universal constructions. This is explored in [Quinn].

Finally, the larger context provides a good setting for category theory, and shares some of its features. The primitive objects are closely related to the ‘object’ and ‘functor’ primitives of category theory. The approach also explains some of the awkwardness of trying to fit category theory into traditional set theory: a weaker logical structure is more appropriate.

To make precise the the relationship between this and ZFC set theories, we describe a universal well-founded pairing. This turns out (modulo a regularity hypothesis) to satisfy the ZFC axioms. Universality implies that if a collection of axioms includes the axiom of Foundation, then models appear as subobjects of this universal theory. This may provide a way to organize the profound work on ZFC models done in the last century. It may also have consequences for the theory of large cardinals.

Some structural issues concerning the upper end of the theory remain unresolved.

1.2 Some ideas

The key to this development is explicit use of logical functions: functions to \{yes, no\}. In the traditional approach these are usually implicit, and rely on the “Law of Excluded Middle”. To illustrate this, suppose we have a collection of objects, \(A\), and a sub-collection \(B \subseteq A\). Given \(a \in A\) we might ask: “is \(a\) in \(B\)?” Excluded-middle asserts that this question always has an answer, either ‘yes’ or ‘no’. In other words, there is a logical function on \(A\) that detects \(B\). As a result the traditional focus is on subcollections, and functions that detect them are implicit. Here excluded-middle fails in that there are subcollections that are not logical. As a result it is often necessary to work more directly with
logical functions.

The logic in the basic setting here is weaker than the traditional logic of set theory. Roughly, excluded-middle, quantification, and a meaning for equality of elements, are all missing. These methods are necessary for mainstream mathematics, so we identify subcontexts in which they are available. A logical domain is a collection with a pairing \( A \times A \to \text{yes/no} \) that returns ‘yes’ if and only if the two inputs are the same (in a sense to be made precise later). These are settings for basic binary logic. Quantification also concerns identifications, but up a level. The powerset of a logical domain, denoted \( \mathcal{P}[A] \), is the collection of all logical functions on \( A \). We say that a logical domain supports quantification if there is a logical function \( \mathcal{P}[A] \to \text{yes/no} \) that detects the empty (always ‘no’) function. This is called ‘quantification’ because the standard expression for this function uses quantification: \((h = \emptyset) \iff (\forall x \in A, h[x] = \text{no})\). Conversely, if this one quantification expression is implemented, other quantification expressions over \( A \) also work.

‘Supports quantification’ is equivalent to ‘\( \mathcal{P}[A] \) is a logical domain’. We say \( A \) “supports \( n \)th order quantification” if \( \mathcal{P}^j[A] \) is defined and is a logical domain for \( j \leq n \). “Infinite order quantification” means \( n \)th order for all positive integers \( n \).

Logical domains that support infinite-order quantification are called ‘relaxed sets’. They have the properties expected of sets, but are ‘free-range’ in the sense that they are identified individually rather than caged in a model for an elaborate axiom system. They are much easier to use with full precision. This is explored in [Quinn]; the focus here is more on the relationship to traditional theories. There is a universal well-founded pairing that—modulo a regularity assumption—satisfies the ZFC axioms; and a domain is a relaxed set if and only if there is a bijection to a set in this ZFC theory. Universality implies any ZFC set theory is isomorphic to a transitive subset of this one.

1.3 Main question

The “Quantification Hypothesis” (QH) asserts that if a logical domain supports quantification then so does its powerset. If QH fails then there is a maximal strong-limit cardinal, and the “Regularity Hypothesis” (RH) is that this is a regular cardinal. If either QH or RH holds then the universal well-founded pairing defined in Section 6 is the universal ZFC set theory. ‘QH False’ leads to a host of secondary questions. For instance, in this case is there a maximal cardinal? QH and RH effect only the largest-scale structures in the theory, so have few consequences for mainstream mathematical practice.

1.4 Outline

Section 2 describes the primitives of descriptor theory (undefined objects, core logic, and assumed hypotheses). Primitive objects are essentially the ‘object’ and ‘morphism’ primitives of category theory. These emerged from a great deal of trial and error with set theory as the goal: their suitability for categories
is a bonus rather than a design objective. The primitive logic is weaker than standard binary logic, and in particular does not include the “law of excluded middle”. Most of it uses assertions rather than binary (yes/no) logic. The primitive hypotheses are mostly standard, including the axiom of Choice. The success of set theory as a foundation provides strong experimental evidence that these are consistent, see Section 7.

Section 3 describes logical functions, logical domains, and quantification. It also includes a discussion of logical pairings (functions $A \times B \to \text{yes/no}$).

The next two sections concern traditional topics (well-orders and cardinals), but reverse some of the logic. Usually they are developed within a set theory. Here we develop them in a more general context, and then use them to deduce information about the set theory. We briefly repeat arguments that have been well-known for a century to be sure the logic reversal does not cause problems, and often to squeeze out a bit more information.

The development of well-orders is recalled in Section 4. $\mathcal{W}$ denotes the domain with elements the isomorphism classes of well-orders. These are often referred to as ‘ordinals’. The Burali-Forti paradox, formulated in the late 1800s, shows that $\mathcal{W}$ cannot be well-ordered. It comes close: it is a logical domain with a linear order such that every bounded subdomain is well-ordered. The resolution of the paradox is that $\mathcal{W}$ does not support quantification. Therefore, we cannot logically distinguish between functions on it. We can, for instance, show that a subdomain is cofinal if and only if it is order-isomorphic to $\mathcal{W}$, but there is no logical function that distinguishes cofinal subdomains from bounded ones.

Section 5 recalls the development of cardinals and cardinality. The outcome is essentially the same as in classical set theory, though some of the arguments are slightly different. Cardinals are used to show relaxed sets have the expected properties, and are needed in the next section.

Section 6 gives the description of the universal well-founded pairing. This uses Cantor’s Beth function (§5.4) as a template, and the proof of universality is a version of Mostowski collapsing. This satisfies all the Zermillo-Fraenkel-Choice axioms if either QH holds or the maximal strong-limit cardinal is regular. If neither of these is true then size constraints are necessary in some theorem statements.

Finally, section 7 discusses differences between ZFC and the theory developed here, and suggests that the differences make relaxed sets a better foundation for mathematics. It also describes uncertainty about structure at the upper end of set theory, particularly the status of the Quantification Hypothesis.
2 Primitives

There are three types of irreducible ingredients: primitive objects, primitive logic, and primitive hypotheses. We particularly focus on the logic since it is weaker than the logic embedded in our language, and mistakes are easy.

2.1 Primitive objects

Standard practice in mathematics is to define new things in terms of old, and use the definition to infer properties from properties of the old things. The old things are typically defined in terms of yet more basic things. But to get started there must be some objects that are not defined. Properties and usage of primitive objects must be specified directly, since they cannot be inferred from a definition.

The primitive objects here are essentially the “object” and “morphism” primitives of category theory. This was not a deliberate choice: set theory was the goal, but a great deal of experimentation led to category theory anyway.

Object descriptors, usually shortened to just “descriptors”, are indicated by the symbol $\in$. Usage takes the form $x \in A$, which we read as “$x$ is an output of the descriptor $\in A$”, or “$x$ is an object in $A$”.

The term “descriptor” is supposed to suggest that these describe things but, unlike the “element of” primitive in set theory, they have no logical ability to identify outputs. In more detail, if $x$ is already specified then in standard set theory the expression “$x \in A$” may be expected (by excluded middle) to be ‘true’ or ‘false’. Here, for previously specified $x$, $x \in A$ is usually a usage error that invalidates arguments. However, see the assertion forms below in §2.2.

Syntax for defining descriptors takes the form “$x \in A$ means ‘...’”. For example, the descriptor whose objects are themselves descriptors is defined by:

$$A \in \text{OD} \text{ means } \text{“} A \text{ is an object descriptor.}\text{”}$$

A more standard example is the descriptor for groups. “$G$ is a group”, or $G \in (\text{groups})$ means “$G$ is a set together with a binary operation that is associative, and has a unit and inverses”.

Morphisms of descriptors are essentially the primitives behind functors of categories. “$f : \in A \rightarrow \in B$ is a morphism” means that every object $x \in A$ specifies an object $f[x] \in B$.

“Specifies” can be made more precise, but this form seems to work well enough in practice that we forego the complication. The logical-function overlay also imposes more discipline. Morphisms have some of the structure expected of functors: for instance morphisms $A \xrightarrow{f} B$ and $B \xrightarrow{g} C$ can be composed to get a morphism $A \xrightarrow{g \circ f} C$.

This composition is associative, for the usual reason, but stating this requires use of the assertion form of $=$ (see ‘Primitive logic’, below).
The definition of ‘morphism’ is implicitly a descriptor. Given descriptors \( \in A, \in B \), the morphism descriptor is defined by: \( f \in \text{morph}[A, B] \) means “\( f \) is a morphism \( A \rightarrow B \)”.

2.2 Primitive logic

Logic provides methods of reasoning with primitive objects and hypotheses. The core logic here is weaker than that of set theory. We describe primitive logical terms, and provide examples to illustrate usage.

In traditional set theory, logic is presented as a formal language, cf. [Manin]. The role of first-order predicate calculus in some of the axioms seems to make this formality necessary. The lack of an analogous restriction here enables us to be less formal, but deeper investigations may require more precision.

Questions and assertions

An assertion is a statement that is known to be correct. These are often indicated by ‘!’; for example \( a \in B \) means “\( a \) is known to be an output of \( B \)”. Similarly, \( a \notin B \) means “\( a \) is known not to be an output of \( B \)”.

A question is a statement implemented by a logical function. These are often indicated by ‘?’; for example, suppose \( a \in A \) and \( A \supset B \) is a subdescriptor. Then \( a ? \in B \) means “there is a logical function \( (# ? \in B) : A \rightarrow y/n \), and \( a ? \in B \) is the value of that function on \( a \)”. This value is either ‘yes’ or ‘no’: the statement itself does not include an assertion about its value.

Discussion

Traditional binary logic presumes that the “law of excluded middle” makes every appropriate statement into a logical function. “Is (statement) true?” should reliably return either ‘yes’ or ‘no’. This is not the case here: to use traditional logic we must first know that the logical function exists. Including ‘?’ in the notation is supposed to indicate this has been done. One goal of this paper is to identify a context in which traditional binary logic works reliably.

Negations work differently in this logic. Logical functions can be negated formally by interchanging ‘yes’ and ‘no’. But the formal negation of “it is known that (statement) is true” is “it is not known that (statement) is true”, which has no logical force. Negation assertions can make sense, and this is the basis for arguments by contradiction. For example we might suppose \( a \in B \), and show that this leads to a contradiction. \( a \in B \) is then known to be false, and therefore \( a \) is known not to be an output of \( B \).

Examples

1. “\( a \neq b \)” is read as “\( a \) is known to be identical to \( b \)” (see below for “identical”);
2. "!∃a | . . ." is read as “it is known that there exist an element a such that (. . . )”.
3. "!∀a | . . ." is read as “it is known that for all a, (. . . ) holds”.
4. "!a∈∈A" is read as “it is known that a is an output of the descriptor ∈∈A”. Similarly a !∈∈A is “it is known that a is not an output of ∈∈A”.
5. "!not[assertion]" is read as “it is known that (assertion) is false”.

We also use the common notation := for “defined by”. Note (:=) ⇒ (!=).

More about ‘!=’
Officially, a != b means that a, b are symbols representing a single output of a descriptor.

Example 1: Suppose ∈∈A and ∈∈B are descriptors, and b0∈∈B. Then there are projections and inclusions

\[ p: A \times B \to A, \quad \text{by} \quad (a, b) \mapsto a \]
\[ j: A \to A \times B, \quad \text{by} \quad a \mapsto (a, b_0) \]

Then p[j[a]] != p[(a, b0)] != a.

Example 2: Sometimes there are more-explicit formulations for identity. For example, suppose f, g: ∈∈A → ∈∈B are morphisms of descriptors. Then (f != g) is equivalent to (∀a∈∈A, f[a] != g[a]). This formulation of “identical” makes the usual proof of associativity of composition work. Explicitly, suppose

\[ A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} A_4 \]

are morphisms of descriptors. Then \( f_3 \circ (f_2 \circ f_1) \) != \( (f_3 \circ f_2) \circ f_1 \).

Example: Russell’s paradox
This gives a statement that cannot be a logical function.

Suppose ∈∈A is a descriptor, and consider the statement (∈∈A)∈∈A. This makes sense as an assertion, ie. “consider A such that A is known to be an object of itself”. Suppose for a moment that it is implemented by a logical function defined on descriptors. In that case we could define a descriptor ∈∈Notln by: A∈∈Notln means “the logical function #?∈∈# has value ‘no’ on A”. This leads to a contradiction: it is easily seen that Notln ?∈∈Notln cannot have value either ‘yes’ or ‘no’. The argument is valid except possibly for the assumption about a logical function, Since it leads to a contradiction, such a logical function cannot exist.
Here are some other views: In “Primitive objects” we emphasized that descriptors cannot recognize their outputs, so any interpretation of \((\in A)\in A\) other than an assertion (eg. as a defining characteristic) is a usage error. In contrast, the logic of traditional set theory implies \(A \in A\) must define a logical function on sets. The traditional conclusion from the contradiction is that the descriptor \(\text{NotIn}\) cannot be a set. Alternatively, ZFC theories are well-founded so \(A \in A\) always has value ‘no’. Negating gives that if \(\text{NotIn}\) is a set then \(\text{NotIn} \in \text{NotIn}\) must be ‘yes’. Since this is not possible, again we conclude \(\text{NotIn}\) cannot be a set.

Example: Images and the set-builder notation

The traditional set-builder notation can be used to describe images of descriptor morphisms, but does not encode whether or not the image is logical. In detail:

Suppose \(f: B \to A\) is a morphism of descriptors. The \textbf{image} of \(f\) is the descriptor 
\[ a \in \text{im}[f] \text{ means } \exists! b \in B \mid f[b] = a \]
together with the projection to \(a \in A\). This is almost the standard set-builder notation for images. Writing it that way, and writing the element of \(B\) explicitly as a dummy variable gives 
\[ a \in \text{im}[f] := \left\{ \exists! \# \in B \mid f[\#] = a \right\}. \]

Note that the use of ‘\(\exists!\)’ means this expression does \textbf{not} define a logical function of \(a\), and an attempt to use it that way is a usage error.

Next, suppose \(p: A \to \text{y/n}\) is a logical function. We say \(p\) \textbf{detects the image} of \(f\) if 
\[ (y \in A \& p[y] = \text{yes}) \iff (\exists! \# \in B \mid f[\#] = y). \]

If such \(p\) exists we say “\(f\) has logical image.”

Finally, a logical function \(p: A \to \text{y/n}\) defines a descriptor with an injective morphism to \(A\) whose image is detected by \(p\). We do \textbf{not} want to use the set-builder notation for the image because it loses the information that it is logical. Instead, we usually denote the descriptor by the same name as the logical function: define \(x \in p := (x \in A \& p[x] = \text{yes})\).

2.3 Primitive hypotheses

Primitive \textbf{hypotheses} are assertions that we believe are consistent, but cannot justify by reasoning with other primitives. Instead we regard these as experimental hypotheses. Extremely heavy use suggests that the ones given should be completely reliable. The “Quantification Hypothesis” (see below) may eventually be added to these, but is still undergoing testing.
Hypotheses

Two: There is a descriptor $\in \{\text{yes}, \text{no}\}$ such that $\text{yes} \in \{\text{yes}, \text{no}\}$, $\text{no} \in \{\text{yes}, \text{no}\}$ and if $a \in \{\text{yes}, \text{no}\}$ then either $a \not= \text{yes}$ or $a \not= \text{no}$.

Choice: Suppose $f : \in A \to \in B$ is a morphism of object descriptors and $f$ is known to be onto. Then there is a morphism $g : \in B \to \in A$ so that $g \circ f$ is (known to be) the identity. We refer to such $g$ as sections of $f$.

Infinity: The natural numbers support quantification of all (finite) orders.

Discussion:

About Two: The force of this hypothesis is that, unlike general descriptors, we can tell the objects apart. This is an excluded-middle property that needs to be made explicit because we do not require the general principle.

The names ‘yes’, ‘no’ are chosen to make it easy to remember operations (‘and’, ‘or’, etc.). One might prefer ‘1’ and ‘0’ for indexing or connections to Boolean algebra. We avoid ‘true’ and ‘false’ because over the last 3,000 years philosophers have attached a great deal of baggage to these terms.

About Choice: The term “choice” comes from the idea that if a morphism is onto, then we can “choose” an element in each preimage to get a morphism $g$. Note that in general there is no logical-function way to determine if a morphism is onto, or if the composition is the identity. These must be assertions, as above.

The axiom of choice in the traditional setting has strong consequences that have been extensively tested for more than a century. No contradictions have been found, and it is now generally accepted. The above form extends the well-established version to contexts without quantification. This extension has implicitly been used in category theory, again without difficulty.

About Infinity: Using primitive objects and hypotheses other than Infinity, we can construct the natural numbers $\mathbb{N}$ as a logical domain. However we cannot show that it supports quantification. The Infinity hypothesis is that this is the case, and further that the iterated powersets $\mathcal{P}^n[\mathbb{N}]$ all support quantification. This is essentially the same as the ZFC axiom “there is an infinite set”.

2.4 The Quantification hypothesis

At present there is uncertainty about structure at the upper end of set theory. The Quantification hypothesis (QH) is that if a logical domain supports quantification then so does its powerset. Alternatives are discussed in Section 7. The author’s guess is that we will be unable to settle it one way or another, but will eventually have enough confidence in its consistency to include it as a primitive hypothesis.
3 Logical domains, and quantification

We provide formal definitions and basic properties of logical domains and quantification. Only first-order quantification is used in this section.

3.1 Logical domains

A descriptor $\in A$ is a logical domain, or simply “domain”, if there is a logical function of two variables that detects equality. Explicitly, there is $\neq : A \times A$ such that $(a \neq b) \Rightarrow (a = b)$. We reserve the notation ‘$\neq$’ for this use, i.e. for equality-detecting pairings on domains.

Logical domain are first-order approximations to traditional sets, so we use traditional terms. Objects in a domain are referred to as elements, and we use $a \in D$ instead of $a \in D$.

Domains of equivalence classes

A pre-domain consists of $(\in D, \equiv)$ where

1. $\in D$ is an object descriptor,
2. $\equiv[x, y]$ is a logical pairing $D \times D \rightarrow y/n$;
3. $\equiv$ satisfies the standard requirements for an equivalence relation:
   - reflexive: $\equiv[x, x] = yes$
   - symmetric: $\equiv[x, y] \Rightarrow \equiv[y, x]$
   - and transitive: $\equiv[x, y] \& \equiv[y, z] \Rightarrow \equiv[x, z]$.

Given this data we define a quotient descriptor by: $h \in D/\equiv$ means “$h$ is a logical function $D \rightarrow y/n$ and $h \neq \equiv[x, #]$ for some $x \in D$”. Define $\neq : D/\equiv \times D/\equiv \rightarrow y/n$ by: suppose $h \neq \equiv[x, #]$ and $g \neq \equiv[y, #]$, then $(h \neq g) := \equiv[x, y]$. The hypothesis that ‘$\equiv$’ is an equivalence relation implies that ‘$\neq$’ is a logical pairing, and does not depend on the representatives $x, y$. Similarly $\neq \Rightarrow ! =$. Thus $(D/\equiv, \neq)$ is a logical domain.

Finally, note that if $\neq \Rightarrow ! =$, as in the definition of domain, then $\neq$ is automatically an equivalence relation. Therefore domains are also pre-domains; exactly those for which the quotient morphism $D \rightarrow D/\equiv$ is a bijection.

3.2 Cantor-Bernstein theorem

This is the first of a number of classical results that we review to clarify quantification and binary-logic requirements. It is put here to emphasize it does not require quantification.

Theorem. (Cantor-Bernstein) Suppose $A, B$ are logical domains and $A \rightarrow B \rightarrow A$ are injections with logical image. Then there is a bijection $A \simeq B$. 
Proof: Composing the injections reduces the hypotheses to the following. Suppose \( J : A \to A \) is an injection, \( j : A \to \{\text{y/n}\} \) detects the image of \( J \), and \( k \) is a logical function with \( j \subset k \subset A \) (\( k \) detects \( B \)). Then there is a bijection \( A \simeq k \).

Composing with iterates of \( J \) gives a sequence

\[
\cdots k \circ J^{n+1} \subset j \circ J^n \subset k \circ J^n \cdots \subset j \circ J^0 \subset k \circ J^0 \subset A.
\]

Define a new function \( \hat{J} : A \to A \) by:

\[
\hat{J}[a] := \begin{cases} 
  a & \text{if } \exists n \in \mathbb{N} \mid a \in k \circ J^n - j \circ J^n, \\
  J[a] & \text{if not}
\end{cases}
\]

Then it is straightforward to see that \( \hat{J} \) is a bijection \( A \to k \simeq B \), as required.

The first case in the definition of \( \hat{J} \) uses quantification over the natural numbers, but the Axiom of Infinity asserts that this is valid. In the second case “if not” makes sense because we can apply \( \text{not}[*] \) to a logical function.

### 3.3 Quantification

The empty function on \( A \) is \( \emptyset[a] := \text{no} \), for all \( a \in A \).

**Definition.** A logical domain \((A, ?=)\) **supports quantification** if there is a logical function \( P[A] \to \{\text{y/n}\} \) that detects the empty function.

We use “\( A \) is \( Q^1 \)” as shorthand for “\((A, ?=) \) supports quantification”.

The traditional quantification notation for the empty-set detecting function is \((\forall a \in A, h[a] ?= \text{no}) \). The logic here does not imply that such expressions define logical functions. However if there is a logical function that implements the *intent* of this particular expression, then all expressions using quantification over \( A \) will define logical functions. This is illustrated by:

**Lemma.** A domain supports quantification if and only if \( P[A] \) is a domain.

Suppose \( ?= \) is a logical pairing on \( P[A] \) such that \( ?= \Rightarrow \text{!=} \). Then \( \# ?= \emptyset \) is a logical function that detects the empty function. Conversely, suppose \( \phi : P[A] \to \{\text{y/n}\} \) detects \( \emptyset \), ie \( (\phi[h] = \text{yes}) \Leftrightarrow (h \neq \emptyset) \). Define ‘\( ?= \)’ by \((h ?= g) := \phi[\# \mapsto (h[\#] = g[\#])])\). Then

\[
(h ?= g) \implies (\forall \# \in A, h[\#] = g[\#]) \implies (h \neq g).
\]

**Operations on \( P[A] \)**

The logical operations on \( y/n \) induce operations on logical functions \( A \to y/n \), and these correspond to traditional operations on subsets. Explicitly:
1. intersection: \( h_1 \cap h_2 := h_1 \& h_2 \)
2. union: \( h_1 \cup h_2 := h_1 \lor h_2 \)
3. complement: \( D - h_1 := \text{not}[h_1] \)

We caution that these operations may not be defined for subdomains that are not logical; see the next section.

### 3.4 Subdomains

Logical functions correspond to subsets in traditional set theory: the function \( h: A \rightarrow y/n \) determines a subset by \( \{ x \in A \mid h[x] = \text{yes} \} \). In the traditional theory “excluded middle” is assumed to hold, which implies that the function can be recovered from the subset. Here, failure of excluded middle means there may be subdomains that are not logical; see ‘Example: images and the set-builder notation’ in 2.2 for explanation and notation. However we have:

**Lemma.** Suppose \( B \subset A \) is a subdomain. If \( B \) is \( Q^1 \) then it is logical. Conversely, if \( A \) is \( Q^1 \) and \( B \) is logical, then \( B \) is \( Q^1 \).

We will eventually see (final corollary in 5.1) that in most cases subdomains of \( Q^1 \) domains must be logical, so “\( B \) is logical” can be dropped from the converse statement.

**Proof:** The traditional expression for a logical function that detects \( B \) is \( b[x] := (x \in A \& \exists y \in B \mid y \neq x) \). If \( B \) is \( Q^1 \) then it supports quantification and ‘\( \exists \)’ defines a logical function. For the converse suppose \( b[\#] \) detects \( B \). A logical function \( h \) on \( B \) extends to one on \( A \) by \( \hat{h}[x] := (b[x] \& h[x]) \), and the extension is empty if and only if \( h \) is empty. But since \( A \) is \( Q^1 \) we can logically detect the empty function on it. Applying this to \( \hat{h} \) detects the empty function on \( B \). \( \square \)

### Functions with \( Q^1 \) support

There is a useful blend of the above two lemmas. Suppose \( A \) is a logical domain, and define \( Q^1 P[A] \) to be the logical functions whose supports are \( Q^1 \).

**Lemma.** \( Q^1 P[A] \) is a logical domain, for any \( A \).

**Proof:** Suppose \( B, C \subset A \) are \( Q^1 \). Then \( (B \subset C) := (\forall y \in B \mid C[y] = \text{yes}) \) is a logical pairing (because \( B \) is \( Q^1 \)) that detects inclusion. Define \( B ?= C \) by \( (B \subset C) \& (C \subset B) \), then \( ?= \implies !\) = , as required. \( \square \)

### 3.5 Logical pairings

A **logical pairing** is a logical function of two variables, or equivalently, a function on a product \( \lambda: A \times B \rightarrow y/n. \)

This section briefly mentions basic structure. In traditional set theory, topology, etc. logical pairings are often called ‘relations’, but this term is not used here.
Examples

Equivalence relations, isomorphisms, orderings, and set theories are all defined in terms of pairings. The ‘?≡’ and ‘eqv’ functions required in the definition of domain and pre-domain are pairings.

Functions \( f : A \to B \) are primitive objects here. On domains, functions determine pairings by \( f[a, b] := (f[a] ?= b) \), and this puts them in a larger context where they can be logically manipulated.

If \( A \) is \( Q^1 \) then \( P[A] \) is a domain and the evaluation pairing

\[
\text{Ev}: P[A] \times A \to \{y/n,\}
\]

is defined by \( \text{Ev}[h, x] := h[x] \). Note the convention that the \( P[A] \) variable comes first in the evaluation pairing. This order seems to be the most convenient, but it is not consistent with some traditional notations. For instance if we think of \( h \) as a subset then \( h[x] \) is traditionally denoted by ‘\( x \in h \)’, with the \( A \) variable first.

Standard structure

Suppose \( \lambda : A \times B \to \{y/n,\} \) is a pairing of \( Q^1 \) domains. Then:

1. the domain of \( \lambda \) is the logical function on \( A \) defined by: \( \text{dom}[\lambda, a] := (\exists b \in B \mid \lambda[a, b]) \). The image is defined similarly.

2. The opposite is defined by formally changing the order of the variables: \( \lambda^{op}[b, a] := \lambda[a, b] \). Note that the domain of \( \lambda^{op} \) is the image of \( \lambda \), and vice versa.

3. The adjoint is the function \( \lambda^{adj}: A \to P[B] \) defined by:

\[
a \mapsto \lambda[a, #].
\]

For example, the evaluation pairing \( \text{Ev}: P[B] \times B \) has adjoint the identity function \( P[B] \to P[B] \).

Composition

Suppose \( \alpha: A \times B \to \{y/n\} \) and \( \beta: B \times C \to \{y/n\} \) are pairings of \( Q^1 \) domains. The composition is defined by

\[
(\alpha * \beta)[a, c] := (\exists b \in B \mid \alpha[a, b] \& \beta[b, c]).
\]

If \( \alpha, \beta \) are both single-valued then pairing-composition corresponds to composition of the associated functions, but the notations have reversed order. If we denote the function associated to a single-valued \( \lambda \) by \( \hat{\lambda}: A \to B \) then

\[
\hat{\alpha} * \hat{\beta} = \hat{\beta} \circ \hat{\alpha}.
\]
4 Well-orders

We briefly recall the properties of well orders, and show that there is a universal almost well-ordered domain. Basic properties of this domain are described. Only first-order quantification is used in this section.

4.1 Definitions

Suppose \((A, =)\) is a logical domain.

1. A linear order is a pairing \((\#1 \geq \#2) : A \times A \rightarrow \text{y/n}\) that is transitive; any two elements are related: \((a \geq b)\) or \((b \geq a)\); and elements related both ways are the same: \((a \geq b) \& (b \geq a) \iff (a = b)\).

2. Suppose \((A, \geq)\) is a linear order. A subdomain \(B \subset A\) is said to be transitive if \((a \notin B) \& (a \geq b) \implies b \notin B\).

3. \((A, \geq)\) is a well-order if it is a linear order, \(A\) is \(Q^1\), and if \(A \supset B\) is transitive then it is logical and either \(B = A\) or the complement \(A - B\) has a least element.

4. \((A, \geq)\) is an almost well-order if subdomains of the form \((x > \#)\) are well-ordered.

Notes

1. In (3), note that ‘logical’ is a conclusion, not a hypothesis, but it is only required for transitive subdomains. Among other things, this ensures that ‘complement’ is defined. The quantification \((Q^1)\) requirement is needed for “\(B = A\)” to have value ‘yes’ or ‘no’, and to define “least element”.

2. In a linear order “logical and least element in the complement” is equivalent to “is of the form \((x > \#)\) for some \(x\)”. The latter is easier to use, the former generalizes to well-founded pairings.

3. The definition of ‘well-order’ is different from the usual one (cf. Jech [Jech] definition 2.3), but it is essentially equivalent and is better for the development here.

4. As usual, well-orders are hereditary in the sense that if \((A, \geq)\) is well-ordered, and \(B \subset A\) is a logical subdomain, then the induced order on \(B\) is a well-order. Similarly for almost well-orders. If \(B\) is nonempty then it has a unique minimal element denoted by \(\text{min}[B]\).

5. We will see that an almost well-order fails to be a well-order if and only if the domain is not \(Q^1\), \S 4.3. It will turn out that there is only one of these, up to order-isomorphism.
Well-ordered equivalence classes

This is a variation on ‘Domains of equivalence classes’ in §3.1. We will use this construction in the definition of the universal almost well-order.

A linear pre-order consists of an object descriptor $\in A$ and a logical pairing $\text{geq}[\#1, \#2]$ defined on pairs of outputs from the object descriptor. The pairing satisfies:

1. (transitive) $\text{geq}[a, b] \& \text{geq}[b, c] \Rightarrow \text{geq}[a, c]$;
2. (reflexive) $\text{geq}[a, a]$; and
3. (pre-linear) for all $a, b \in A$, either $\text{geq}[a, b]$ or $\text{geq}[b, a]$ (or both).

Given this structure, define $\text{eqv}[a, b] := (\text{geq}[a, b] \& \text{geq}[b, a])$. The quotient $A/\text{eqv}$ is a logical domain, and $\text{geq}$ induces a linear order in the ordinary sense on elements (ie. on equivalence classes of objects). The additional conditions used to define well-orders in this context are the same as those on the element level.

4.2 Recursion

Our formulation is slightly different from the standard one (cf. [Jech], Theorem 2.15) in part because we do not find the standard one to be completely clear. There is a version for well-founded pairings in §6.2.

First we need a notation for restrictions. Suppose $f : A \rightarrow B$ is a partially-defined function and $D \subset A$ is $Q^1$. Then $f \upharpoonright D$ is the restriction to $\text{dom}[f] \cap D$.

Now, suppose $(A, \geq)$ is an almost well-order, $B$ is a domain, and $R$ is a partially-defined function $R : \text{pfn}[A, B] \times A \rightarrow B$.

Here ‘\text{pfn}’ denotes partially-defined functions whose domains are $Q^1$. According to ‘Functions with $Q^1$ support’ in §3.4, this is a logical domain and it is reasonable to think about functions defined on it. We refer to such an $R$ as a recursion condition.

A (partially-defined) function $f : A \rightarrow B$ is said to be $R$-recursive if:

1. $\text{dom}[f]$ is transitive; and
2. for every $c \in \text{dom}[f]$, $f[c] = R[f \upharpoonright (c > \#), c]$.

Note that this is hereditary in the sense that if $D \subset \text{dom}[f]$ is transitive then the restriction $f \upharpoonright D$ is also recursive.

**Proposition.** (Recursion) If $(A, \geq), B, R$ are as above, then there is a unique maximal $R$-recursive (partially-defined) function $r : A \rightarrow B$. 

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“Maximal” refers to domains: \( r \) is maximal if there is no recursive function with larger domain. We describe a criterion for maximality below. Note that the domain of \( r \) may not be \( Q^1 \) (i.e., may be all of \( A \)); the recursion condition applies to restrictions to subdomains of the form \((a > \#)\), and according to the definition of ‘almost’ well-order these are all \( Q^1 \).

The proof is essentially the same as the classical one; we sketch it to illustrate the slightly non-standard definitions. First, if \( f, g \) are recursive and have the same domain, then they are equal. Suppose not and let \( a \) be the least element on which they differ. Minimality of \( a \) implies the restrictions to \( \# < a \) are equal. But then

\[
\begin{align*}
  f[a] &= R[f \upharpoonright (\# < a)] = R[g \upharpoonright (\# < a)] = g[a]
\end{align*}
\]

a contradiction.

The maximal \( r \) has domain the union \( \cup (\text{dom}[f] \mid f \text{ is } R \text{- recursive}) \). If \( x \) is in this union then \( x \in \text{dom}[f] \) for some recursive \( f \). Define \( r[x] := f[x] \). Uniqueness implies this is well-defined, and is recursive.

A recursive \( r \) is maximal if \( \text{dom}[r] = A \) or, if \( \text{dom}[r] \neq A \) and \( a \) is the least element not in \( \text{dom}[r] \), then \((r, a)\) is not in the domain of \( R \).

### Order isomorphisms

The first application of recursion is a key result in traditional set theory. It is extended to well-founded pairings in §6.2.

**Proposition.** Suppose \((A, \geq), (B, \geq)\) are almost well-orders. Then there is a unique maximal order-isomorphism, from a transitive subdomain of \( A \) to a transitive subdomain of \( B \). Maximality is characterized by either full domain or full image.

This results from the recursion condition \( R[f, a] := \min[B - \text{im}[f]] \).

### 4.3 Universal almost well-order

In this section the domain \( W \) is defined, and shown to be universal for almost well-orders. \( W \) corresponds to the “ordinal numbers” of classical set theory, cf. Jech [Jech], §2.

**Definition of \( W \)**

\( W \) is the quotient of an equivalence relation on a descriptor \( W\mathcal{Q} \).

1. the object descriptor is defined by: \((A, \geq) \in W\mathcal{Q} \) means “\((A, \geq)\) is a well-order”;

2. \( \text{geq}[(A, \geq), (B, \geq)] := (\text{im}[r] \subseteq B) \), where \( r: A \to B \) denotes the maximal order-isomorphism of transitive subdomains described just above.
We expand on (2). Since $A$ is $Q^1$, the image $\text{im}[r]$ is a logical function on $B$. Since $B$ is $Q^1$ the comparison of logical functions $\text{im}[r] \not= B$ is a logical function. Finally, since $r$ is uniquely determined by $A, B$ and their well-orders, $\text{im}[r] \not= A$ is a logical function of $A, B$.

$(\mathbb{W}, \geq)$ is a linear pre-order; let $(\mathbb{W}, \geq)$ denote the quotient domain with its induced linear order. The elements of the domain $\mathbb{W}$ are order-isomorphism classes, and we denote the class represented by $(A, \geq)$ by $\langle A, \geq \rangle$ (angle brackets).

**Canonical embeddings**

Recall that if $(A, \geq)$ is almost well-ordered and $x \in A$ then $(x > \#) \subset A$, with the induced order, is well-ordered. It therefore determines an equivalence class $\langle (x > \#), \geq \rangle \in \mathbb{W}$. This defines the canonical embedding $\omega: A \rightarrow \mathbb{W}$. Explicitly, $\omega[x] = \langle (x > \#), \geq \rangle$.

**Theorem.** *(universality of $\mathbb{W}$)*

1. $(\mathbb{W}, \geq)$ is almost well-ordered, but not well-ordered because it does not support quantification;

2. If $(A, \geq)$ is almost well-ordered then the canonical embedding $\omega: A \rightarrow \mathbb{W}$ defined above is an order-isomorphism to a transitive subdomain, and $\omega$ is uniquely determined by this property;

3. If $A$ is $Q^1$ then the image of $\omega$ is $(\# < \langle A, \geq \rangle)$; if $A$ is not $Q^1$ then the image is all of $\mathbb{W}$.

**Proof of the theorem**

First, the properties of maximal order-isomorphisms described in §4.2 imply that $(\mathbb{W}, \geq)$ is a linear pre-order. We next see that it is almost well-ordered. If $a \in \mathbb{W}$ then $a$ is an equivalence class of well-ordered domains $\langle A, \geq \rangle$. As explained in “canonical embeddings”, there is an order-preserving bijection, from $(A, \geq)$ to the subdomain $\langle (A, \geq) > \# \rangle \subset \mathbb{W}$.

Since $A$ is well-ordered, so is the indicated subdomain of $\mathbb{W}$. But this is the definition of “almost” well-order.

This proves all but the second halves of (1) and (3). We take these up next.

**Quantification fails in $\mathbb{W}$**

To begin, we make explicit that quantification is the issue.

First, an almost well-order is a well-order if and only if the domain supports quantification. One direction is clear: a well-order is $Q^1$ by definition. For the converse suppose $(A, \geq)$ is an almost well-order with $A Q^1$, and suppose $h$ is a transitive logical subdomain of $A$. Since $A$ is $Q^1$, not $[h] \not= \emptyset$ is a logical function that returns either yes, in which case $h = A$, or no, in which case
∃x | h[x] = no. Transitive implies that h[y] = no for y ≥ x, so h has domain contained in (x > #). But (x > #) is well-ordered, so either h?=(x > #) or there is x > z so that h?=(z > #). In either case h has the form required to show A is well-ordered.

Next we use a form of the Burali-Forti paradox to show that (W, ≥) cannot be a well-order. If it were, then by definition of W there would be x ∈ W and an isomorphism ω: W ≃ (x > #). But then, ω² gives an isomorphism with (ω[x] > #). Since ω[x] < x, this contradicts the fact that isomorphism classes of well-orders correspond uniquely to elements of W. Thus (W, ≥) cannot be a well-order, and therefore does not support quantification.

The final part of the theorem is to see that if A is an almost well-order and not Q¹ then the canonical embedding is an isomorphism to W. Suppose there is x not in the image. The image is transitive, so must be contained in (x > #). But this is well-ordered, so a transitive subdomain of it must be Q¹. This contradicts the hypothesis that A is not Q¹. We conclude that there cannot be any such x, and therefore ω is onto. It follows that it is an order-isomorphism.

This completes the proof of the theorem.

4.4 Existence of well-orders

We review a crucial classical theorem, and get some information about W.

**Proposition.** Suppose A is a logical domain.

1. If A is Q¹ then it has a well-order.
2. if A is not Q¹ then there is an injection W → A.
3. There is no logical function on domains that detects which of these alternatives holds.

Proof: As above, Q¹P[A] denotes the logical functions on A whose supports are Q¹, and Q¹P[A] & (# ≠ A) the ones whose support is not all of A. Comment on the logic: if A is Q¹ then Q¹P[A] = P[A] is a logical domain and # ≠ A is a logical function on it. If A is not Q¹ then P[A] is not a logical domain. But (# ≠ A) is still a logical function because it is always ‘yes’.

Define a **choice function** for A to be ch: (Q¹P[A] & (# ≠ A)) → A, satisfying h[ch[h]] = no (ie. ch[h] is in the complement of h). The axiom of Choice implies that any A has a choice function, as follows: Define c: (Q¹P[A] & (# ≠ A)) × A → y/n by c[h, a] := not[h[a]]. The projection of (the support of) c to Q¹P[A] is known to be onto, due to the “not all of A” condition. According to Choice, there is a section of this, and sections are exactly choice functions.

We now set up for recursion. Fix a choice function and define a condition R: pfn[W, A] → A, where ’pfn[W, #]’ denotes partially-defined functions with bounded domain, by:

\[ R[f, a] := ch[im[f \upharpoonright (a > #))]]. \]
In words, \( f[a] \) is the chosen element in the complement of the image of the restriction \( f \upharpoonright (a > \#) \).

This is clearly a recursive condition. We conclude that partially-defined functions \( r: \mathbb{W} \to A \) satisfying:

1. \( \text{dom}[r] \subset \mathbb{W} \) is transitive;
2. \( f[a] = R[r \upharpoonright (a > \#), a] \) holds for all \( a \in \text{dom}[r] \).

form a linearly-ordered domain with a maximal element, and \( r \) is maximal if and only if either \( \text{im}[r] = A \) or \( \text{dom}[r] = \mathbb{W} \).

If \( \text{dom}[r] = \mathbb{W} \) then \( r \) gives an injective function \( \mathbb{W} \to A \) and \( A \) cannot be \( Q^1 \). If \( \text{dom}[r] \neq \mathbb{W} \) then \( r \) gives a bijection from a transitive proper subdomain of \( \mathbb{W} \) to \( A \). But a transitive proper subdomain has a well-order, so \( A \) has one also.

4.5 Cofinality in \( \mathbb{W} \)

This clarifies the structure of \( \mathbb{W} \) “near infinity”. Suppose \((A, \geq)\) is almost well-ordered and \( A \) does not have a maximal element. A logical function \( h: A \to y/n \) is said to be cofinal if for every \( a \in A \) there is \( b \in h \) such that \( b \geq a \). See Jech [Jech] §3.6. Note: if \( A \) is well-ordered then “is \( h \) cofinal?” is a logical function on \( P[A] \). If \( A \) is not well-ordered (ie. is \( \mathbb{W} \)) then this is not a logical function and “for every \( a \in A \)” has to be interpreted as an assertion.

**Proposition.** A logical function \( h \) on \( \mathbb{W} \) is (known to be) cofinal if and only if \( h \) with the induced order is (known to be) order-isomorphic to \( \mathbb{W} \). There is no logical function that distinguishes between bounded and cofinal \( h \).

Note that this is essentially the traditional definition of “regular cardinal”.

Proof: we show that \( h \) is \( Q^1 \) if and only if it is not cofinal. If it is not cofinal then there is \( m \) larger than any element of \( h \). This means \( h \) is a logical function on \((m > \#)\). But this is \( Q^1 \) and a logical subdomain of a \( Q^1 \) domain is \( Q^1 \).

For the converse, define a function \( p: \mathbb{W} \to h \) by \( p[a] := \min[h[\#] \& (a > \#)] \). If \( b \in h \) then (since \( h \) is cofinal) the preimage \( p^{-1}[b] \) is bounded and therefore \( Q^1 \). According to the Surjection proposition at the end of §3.4, this implies \( \mathbb{W} \) is known to be \( Q^1 \) if and only if \( h \) is known to be \( Q^1 \). But \( \mathbb{W} \) is not \( Q^1 \) so neither is \( h \). The induced order on \( h \) is an almost well-order, so by the last part of the Universal almost well-order Theorem, it is order-isomorphic to \( \mathbb{W} \).

4.6 Higher-order quantification

We say a domain \( A \) is \( Q^j \) if \( P^j[A] \) is defined and is a logical domain. Let \( Q^j \mathbb{W} \) denote the well-orders whose underlying domain is \( A^j \). It is easy to see that \( Q^j \mathbb{W} \) is transitive in \( \mathbb{W} \) (ie. the \( Q^j \) property is hereditary), and \( Q^j \subset Q^{j-1} \).

**Proposition.** There is \( \tau \geq 1 \) (the quantification threshold) such that \( Q^{\tau-1} \neq Q^\tau = Q^\infty \). For \( \tau > j \geq 1 \), the domain \( Q^j \mathbb{W} \) is \( Q^{j-1} \) but not \( Q^j \).
Proof: First, the sequence $Q^1\mathbb{W} \supset \cdots Q^j\mathbb{W} \supset Q^{j+1}\mathbb{W} \supset \cdots$ is non-increasing. This means it must be eventually constant; otherwise we would get a sequence with no minimal element in a well-ordered domain. This implies that there is a quantification threshold $\tau$, as claimed.

The Quantification Hypothesis, $Q^1 = Q^\infty$, corresponds to $\tau = 1$. If this is the case then the proof is complete, because we know $\mathbb{W} = Q^1\mathbb{W}$ is $Q^0$ and not $Q^1$.

Suppose, then, that $\tau > 1$. For $j > 1$ $Q^j\mathbb{W}$ is bounded in $\mathbb{W}$, so the induced order is a well-order. Let $q_j \in \mathbb{W}$ denote its equivalence class. Since $Q^j\mathbb{W}$ is transitive, $q_j$ is the minimal element in the complement. But $Q^j\mathbb{W}$ is defined to be equivalence classes whose underlying domain is $Q^j$. Since $q_j$ is in the complement, its underlying domain $Q^j\mathbb{W}$ is not $Q^j$.

It remains to show that for $\tau \geq j > 1$, $Q^j\mathbb{W}$ is $Q^{j-1}$. For this we need $Q^{j-1}\mathbb{W} \neq Q^j\mathbb{W}$. But it is easy to see that if $Q^{j-1}\mathbb{W} = Q^j\mathbb{W}$ then the sequence stabilizes there: $j - 1 \geq \tau$. This contradicts the constraints on $j$. Alternatively, we can exhibit an element in the complement, $Q^{j-1}\mathbb{W} - Q^j\mathbb{W}$, namely $P^n[Q^\tau\mathbb{W}]$ for $n = \tau - j$. ∎
5 Cardinals

Cardinals in set theories have been studied for well over a century. Here we turn this around a bit by developing cardinality for $Q^1$ domains, and then using it to study set theories.

5.1 Definitions

An element $a \in W$ is a cardinal element if $a > b$ implies that there is no injective function $(a > #) \rightarrow (b > #)$. A cardinal well-order is one whose equivalence class is a cardinal element. In particular if $a$ is a cardinal then $(a > #)$ with its induced order is a cardinal well-order.

For this definition to be legitimate in the logic used here we need:

Lemma. There is a logical function $?\text{inj}: W \times W \rightarrow y/n$ such that $?\text{inj}[a, b] = \text{yes}$ if and only if there is an injection $(a > #) \rightarrow (b > #)$.

Fix $b \in W$ and define a subdomain of $W$ by $
\{ y \in W \mid \text{there is an injective function } (y > #) \rightarrow (b > #) \}.
$

If $y \geq z$ then the inclusion $(z > #) \rightarrow (y > #)$ is an injective function. It follows that the subdomain is transitive. Since $W$ is almost well-ordered the subdomain is logical, and given by either the logical function ‘yes’ (ie. all of $W$) or $(c > #)$ for some $c$.

Definition of $\text{card}[A]$  
Suppose $A$ is a $Q^1$ domain. Choose a well-order $(A, \geq)$, then $\text{card}[A]$ is defined to the the smallest element in $?\text{inj}[(A, \geq), #]$.

Note $(A, \geq) \in ?\text{inj}[(A, \geq), #]$, so it is nonempty, and a nonempty logical subdomain of $W$ has a least element. Minimality implies that $\text{card}[A]$ is a cardinal element of $W$. It also implies that $\text{card}[A]$ is well-defined (ie. doesn’t depend on the choice of well-order on $A$).

The global perspective is that this defines a morphism of descriptors $\text{card}: Q^1 \rightarrow W$,

where $Q^1$ is the descriptor whose outputs are $Q^1$ logical domains.

5.2 Bijections, logicality, and Cantor’s theorem

Proposition. Suppose $A$ is $Q^1$, and $(B, \geq)$ is a representative of the equivalence class $\text{card}[A]$. Then there is a bijection $A \simeq B$.

Proof: Choose a well-order on $A$. By minimality, $(A, \geq) \geq (B, \geq)$, so there is an inclusion $B \rightarrow A$. By definition of $\text{card}[A]$, there is an injection $A \rightarrow B$. But the Cantor-Bernstein theorem 3.2 then asserts that there is a bijection $A \simeq B$.  

Corollary. There is an injection \( A \to B \) if and only if \( \text{card}[A] \leq \text{card}[B] \), and a bijection \( A \simeq B \) if and only if \( \text{card}[A] = \text{card}[B] \).

Higher-order quantification

If the Quantification hypothesis is false \( (Q^1 \neq Q^2) \) then it is easy to see that the elements denoted by \( q_j \in \mathbb{W} \) in 4.6 are cardinals for \( j \geq 2 \). They have the property that a \( Q^1 \) domain \( A \) is \( Q^j \) for some \( j \geq 2 \), if and only if \( \text{card}[A] < q_j \). In particular \( A \) is a relaxed set \( (Q^{\infty}) \) if and only if \( \text{card}[A] < q_\tau \), where \( \tau \) is the quantification threshold of 4.6. This is a version of von Neumann’s “axiom of size”, which postulates that sets can be identified by their cardinalities.

Note that in the above we had to know that \( A \) is \( Q^1 \) before we could use cardinality to identify higher-order properties. This does not extend to give a way to identify \( Q^1 \) domains. We know, for instance, that a subdomain of \( \mathbb{W} \) is \( Q^1 \) if and only if it is bounded. However, there is no logical function that detects whether or not a subdomain is bounded.

Logicality of subdomains

Proposition. If \( A \) is \( Q^1 \) and \( \text{card}[A] \) is not a maximal cardinal, then any subdomain of \( A \) is logical.

Proof: Suppose \( X \subset A \) is a subdomain that is not logical. According to the Lemma in 3.4, this implies \( X \) is not \( Q^1 \). According to the Proposition in 4.4, this implies there is an injection \( \mathbb{W} \to X \). Since \( \text{card}[A] \) is not maximal, there is a \( Q^1 \) domain \( B \subset \mathbb{W} \) with \( \text{card}[B] > \text{card}[A] \). Composing the inclusion and injection gives an injection \( B \to A \). But this contradicts \( \text{card}[B] > \text{card}[A] \). Therefore \( X \) must be logical.

It is currently unknown whether or not there is a maximal cardinal; this will be discussed in §7.2. Cantor’s theorem (below) implies that if \( \text{card}[A] \) is maximal then \( A \) is not \( Q^2 \).

Cantor’s theorem

Theorem. (Cantor) Suppose \( A \) is a \( Q^2 \) domain, so \( \mathcal{P}[A] \) is \( Q^1 \) and \( \text{card}[\mathcal{P}[A]] \) is defined. If \( A \) is nonempty then \( \text{card}[\mathcal{P}[A]] > \text{card}[A] \).

Proof: suppose \( \text{card}[\mathcal{P}[A]] \leq \text{card}[A] \), then there is an injection \( \mathcal{P}[A] \to A \). Since \( A \) is nonempty, this and the axiom of choice imply there is a surjection \( p: A \to \mathcal{P}[A] \). Define a logical function \( h: A \to \{y/n\} \) by \( h[a] := \text{not}((p[a])[a]) \). Then \( h \) is not in the image of \( p \), a contradiction.

5.3 Hessenberg’s theorem

In the classical development this is a key fact about cardinality of sets. Here it is a key ingredient in showing ‘relaxed sets’ have the properties expected of sets. We go through the proof to check the use of quantification, and because
the traditional proof is somewhat muddled by the identification of sets and elements.

The canonical order

Suppose \((A, \geq)\) is a linear order. The **canonical** order on \(A \times A\) is a partially-symmetrized version of lexicographic order.

First define the maximum function \(\max : A \times A \rightarrow A\) by \((a, b) \mapsto \max[a, b]\). This induces a pre-linear order on \(A \times A\), namely
\[
(a_2, b_2) \succ (a_1, b_1) := \max[a_2, b_2] > \max[a_1, b_1].
\]

The canonical order refines this to a linear order as follows: Fix \(c \in A\), then the elements of \(A \times A\) with \(\max\) equal to \(c\) have the form \((c > \#), c\) or \((c, c \geq \#)\). Each of these is given the order induced from \(A\), and pairs of the first form are defined to be smaller than pairs of the second form. The canonical order is denoted by \(\geq_{can}\).

More explicitly, \((a_2, b_2) >_{can} (a_1, b_1)\) means:
\[
(\max[a_2, b_2] > \max[a_1, b_1]) \text{ or } \\
((\max[a_2, b_2] = \max[a_1, b_1]) \& ((a_2 > a_1) \text{ or } (a_2 = a_1 \& b_2 > b_1))
\]

The following is straightforward:

**Lemma.** If \((A, \geq)\) is a well-order then the canonical order \((A \times A, \geq_{can})\) is a well-order. If \(\geq\) is an almost well-order then so is \(\geq_{can}\).

This implies that the classifying function gives a bijection to the image of an order-preserving function
\[
\omega : (\mathbb{W} \times \mathbb{W}, \geq_{can}) \rightarrow (\mathbb{W}, \geq).
\]

The main result is a century-old theorem of Hessenberg (see [Jech], Th. 3.5.)

**Proposition.** If \(c\) is an infinite cardinal then

1. \(\omega[(0, c)] = c\), or equivalently
2. \(\omega[(c > \#) \times (c > \#)] = (c > \#)\)

Version (1) concerns images of elements, while (2) concerns images of transitive subdomains. They are equivalent because
\[
(c > \#1) \times (c > \#2) = ((0, c) >_{can} (#1, #2)).
\]

The first step is that \(\omega[0, c] \geq c\). Note that \((c > \#) \times (c > \#)\) contains a copy of \((c > \#), so \text{card}[(c > \#) \times (c > \#)] \geq \text{card}[(c > \#)] = c. But \(\omega\) is a bijection, so the image has cardinality \(\geq c\). The image is \((\omega[0, c] > \#)\), so by minimality of cardinals we have \(\omega[0, c] \geq c\).
Next we work out the consequences of strict inequality in this last: suppose \( \omega[0, c] > c \). Since \( \omega \) is order-preserving, this implies there is \((0, c) >_{\text{can}} (x, y)\) with \( \omega[x, y] = c \). Denote the maximum of \((x, y)\) by \( m \), then \( m < c \). Since cardinals are limits, the successor \( m + 1 \) is also less than \( c \). Turning to the subdomain form, this means \((m + 1 > #) \times (m + 1 > #)\) contains \((x, y) >_{\text{can}} (#1, #2)\), which is the preimage of \((c > #)\). Thus \( \text{card}[(m + 1 > #) \times (m + 1 > #)] \geq c \). Finally, define \( d := \text{card}[m + 1] \). This gives a cardinal \( c \geq d \) so that \( \text{card}[(d > #) \times (d > #)] \geq c \).

We proceed by induction on \( c \) to show this cannot happen. To begin, suppose \( c = \text{card}[\mathbb{N}] \), the first infinite cardinal. If \( \omega[0, c] \neq c \) then we get \( d \) with \( c < d \), as above. But the domain \((d > #)\) is finite, so the product with itself is finite, and it is false that the cardinality of the product is \( \geq \text{card}[\mathbb{N}] \). Therefore \( \omega[0, \text{card}[\mathbb{N}]] = \text{card}[\mathbb{N}] \).

Now suppose the proposition is false, and let \( c \) be the least cardinal for which \( \omega[0, c] > c \). We get \( c < d \) as above, with \( \text{card}[(d > #) \times (d > #)] \geq c \). But the induction hypothesis implies that \( \text{card}[(d > #) \times (d > #)] = d \), and \( d < c \). This gives a contradiction, so the proposition must be true. \( \square \)

**Products and unions**

For *finite* sets, the cardinality corresponds to the number of elements. Cardinality of a disjoint union is therefore the sum of the cardinals, and cardinality of a product is the product. The preceding section implies that the situation is much simpler for infinite sets.

**Proposition.** Suppose \( A, B \) are \( Q^1 \) domains, and at least one is infinite. Then

1. \( \text{card}[A \times B] = \max[\text{card}[A], \text{card}[B]] \), and
2. if \( A, B \subseteq D \) then \( \text{card}[A \cup B] = \max[\text{card}[A], \text{card}[B]] \).

Proof: suppose \( \text{card}[A] \geq \text{card}[B] \), so \( \max[\text{card}[A], \text{card}[B]] = \text{card}[A] \). Then \( \text{card}[A] \leq \text{card}[A \times B] \leq \text{card}[A \times A] = \text{card}[A] \)

The last step being the Proposition above. This gives (1).

For (2), \( \text{card}[A] \leq \text{card}[A \cup B] \leq \text{card}[A \times B] = \text{card}[A] \), by (1). \( \square \)

One consequence is an extension of the final Proposition in §3.4, from \( Q^1 \) to higher orders. This is new information only if \( \text{QH} \) fails.

**Corollary.** Suppose \( f: A \to B \) is a function, the image of \( f \) is \( Q^j \) and either \( \text{QH} \) holds, or \( j = 1 \), or the cardinalities of the point preimage of \( f \) are bounded below \( q_j \). Then \( A \) is \( Q^j \).

In the non-\( \text{QH} \) case, \( q_j \) is the cardinality of \( Q^j \mathbb{N} \); see the example in 5.2.

Proof: The first two cases follow from the statement for \( Q^1 \). In the third case suppose there is a cardinal \( k < q_j \) and \( \text{card}[f^{-1}[b]] \leq k \) for all \( b \in B \), and suppose \( K \) is a domain with cardinality \( k \). Then there is an injection \( f^{-1}[b] \to K \) for every \( b \in B \). Putting these together gives an injection \( A \to K \times \text{im}[f] \),

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so $\text{card}[A] \leq \text{card}[K \times \text{im}[f]]$. Applying the above we get $\text{card}[K \times \text{im}[f]] = \max[\text{card}[K], \text{card}[\text{im}[f]]] < q_j$, and therefore $\text{card}[A] < q_j$. But this implies that $A$ is $Q^j$. \hfill \Box

5.4 The Cantor Beth function

Cantor introduced several functions related to cardinals. Only one is needed here.

Definition

‘Beth’ (beth) is the second character in the Hebrew alphabet, and was used by Cantor to denote the “iterated powerset function”. \beth is briefly mentioned in Jech §5, p. 55. This, and the associated rank function, play major roles in the construction of the ZFC set theory in §6.

Proposition. There is a unique maximal function \beth: D → W satisfying:

1. $D$ is a transitive subdomain of \W;
2. if $a = 0$ then \beth[a] = \text{card}[N];
3. if $a > 0$ is not a limit and \beth[a - 1] is defined and is $Q^2$, then $a \in D$ and \beth[a] = \text{card}[\text{P}[\beth[a - 1]]]; and
4. if $a$ is a limit of $D$ then $a \in D$ and \beth[a] = \text{sup}[\beth[\# < a]].$

‘sup’ in (3) is the ‘supremum’, which is a convenient shorthand for the minimum of elements greater or equal to the image: \text{sup}[h] := \text{min}[\# \geq (\forall a \mid h[a] = \text{yes})].

It is straightforward to formulate a recursive condition on partially defined functions so that conditions (1)–(4) correspond to fully recursive. Existence of a maximal such function therefore follows from recursion.

If QH holds then the domain is $D = \W$ and the image is cofinal in \W. If QH does not hold then there is a maximal strong limit cardinal, and there are only finitely many images of \beth larger than this. The termination criterion comes from (3): if \beth[\mu] is not $Q^2$ then $\mu$ is the maximal element of the domain.

Limits and strong limits

Recall that $a \in \W$ is a limit if $(a > \#)$ does not have a maximal element. It is a strong limit if $a > b$ implies $a > \text{card}[\text{P}[b > \#]].$

Proposition. 1. $b \leq \beth[b]$;
2. $a \in \W$ is a strong limit if and only if $a = \beth[x]$ for either $x = 0$ or $x$ a limit; and
3. if $b$ is a limit then the image \beth[b > \#] is cofinal in $(\beth[b] > \#)$.
Proof: (1) is standard, and easily proved by considering the least element that fails.

For (2), suppose $a$ is a strong limit. $\beth$ is increasing, so $a > \beth^{-1}[\#]$ is transitive and therefore of the form $x > \#$. Since $b \notin (\beth^{-1}[a > \#]), \beth[b] \geq a$.

We show $b$ is a limit. If not then $(b > \#)$ has a maximal element, $b - 1$. $a > \beth[b - 1]$, so by definition of strong limit $a > \beth[b - 1] > \text{card}[P[\#]$, but the left side is the definition of $\beth[b]$, so this contradicts the choice of $b$. We conclude $(b > \#)$ does not have a maximal element, so $b$ is a limit.

Next, $\beth[b > \#]$ is bounded by $a$, so $\sup[\beth[b > \#]]$ is defined, is $\leq a$, and is the definition of $\beth[b]$. But the choice of $b$ requires $\beth[b] \geq a$, so $\beth[b] = a$, as required.

For the other direction of ‘if and only if’, suppose $b$ is a limit. We want to show that $a = \beth[b] := \sup[\beth[b > \#]]$ is a strong limit. Suppose $a > x$. Then there is $b > y$ with $\beth[y] \geq x$. Thus $\text{card}[P[x > \#]] \leq \text{card}[P[\beth[y] > \#]] \geq \text{card}[P[x > \#]]$. Next, since $b$ is a limit, $b > y + 1$. But the definition of $\beth$ gives $\text{card}[P[# < \beth[y]]] = \beth[y + 1]$. Since $\beth[y + 1] < \sup[\beth[b > \#]] = a$, we get $a > \text{card}[P[x > \#]]$. This verifies the definition of strong limit.

The cofinality conclusion follows from the definition of ‘sup’. □

A corollary of (3) in the Proposition is that the cofinalities are the same: $cf[\beth[b]] = cf[b]$. Therefore if $b < \beth[b]$, or $b$ is not regular, then the strong limit $\beth[b]$ is not regular.

**Beth rank**

Rank plays a central role in the construction in Section 6. This rank follows the $\beth$ function closely, and is a bit different from the rank function defined in [Jech] §6.2: it starts with $\text{rank} = 0 \iff \text{‘finite’}$, rather than $\text{rank} = 0 \iff \text{‘empty’}$, and otherwise differs by $+1$.

The (Beth) **rank** is the function $\mathcal{W} \rightarrow \mathcal{W}$ given by

$$\text{rank}[x] := \min[(\# | \beth[\#] > x)],$$

provided $(\beth[\#] > x)$ is nonempty. This case happens only if the Quantification hypothesis is false. In this case there is a maximal element $\mu$ in the domain of $\beth$, and we define the rank of $x \geq \beth[\mu]$ to be $\mu + 1$.

Note that if $a$ is a limit then there are no elements of rank $a$. The reason is that $\beth[a] = \sup[\beth[a > \#]]$, so if $\beth[a] > x$ then there is some $a > b$ with $\beth[b] > x$.

These hidden values come into play in ranks of logical functions, defined next.

If $h: \mathcal{W} \rightarrow y/n$ has bounded ($\iff Q^1$) support then define

$$\text{rank}[h] := \sup[\text{rank}[(\# | h[\#])]].$$

Examples and special cases: If $h[\#] = (\# = a)$, the function that detects $a$, then the ranks of $a$ and $h$ are the same. More generally, $h[a] \Rightarrow (\text{rank}[a] \leq \text{rank}[h])$.

If $h$ is cofinal in $g$ then $\text{rank}[h] = \text{rank}[g]$.

Finally, consider the logical function $(a > \#)$. $\text{rank}[a > \#] = \text{rank}[a]$ unless $a = \beth[b]$ for some $b$, in which case $\text{rank}[a > \#] = b$ and $\text{rank}[a] = b + 1$. 

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6 The universal well-founded pairing

This section provides an interface between the present theory and traditional
axiomatic set theory. The result is roughly that the universal well-founded
pairing is a ZFC set theory, and a domain is a relaxed set (ie. $Q^\infty$) if and only
if there is a bijection with a set in this ZFC theory.

6.1 Main result

Definitions

Most of these are standard, but included for precision. Suppose \((A,\lambda)\) is a
logical pairing, and \(A \supset B\) is a subdomain.

1. As with well-orders, ‘transitive’ means \(b \in B\) and \(\lambda[b,c] = \text{yes}\) implies \(c \in B\).

2. An element \(a \in A - B\) is minimal if \(\lambda[a,b] \Rightarrow b \in B\).

3. A pairing \((A,\lambda)\) is well-founded if \(A\) is \(Q^\infty\), and if \(B \subset A\) is transitive
with \(A - B\) nonempty, then there is a minimal element in \(A - B\).

4. For the “almost” version we need the transitive closure of \(B \subset A\). This
is the smallest transitive subdomain containing \(\lambda[b,\#]\) for all \(b \in B\). More
explicitly, it consists of \(a\) such that there is a finite sequence \(c_0, c_1, \ldots, c_n,\)
\(n \geq 1\), with \(c_0 \in B\), \(c_n = a\), and \(\lambda[c_i, c_{i+1}] = \text{yes}\). We denote this by
\(tcl[B]\).

5. Now we define a pairing \((A,\lambda)\) to be almost well-founded if the restriction
to the transitive closure of any point, \((tcl[a],\lambda)\), is well-founded.

We note that \(Q^\infty\) is not necessary for the definition in (2) to make sense. How-
ever we have a clean result only for \(Q^\infty\), and putting it in the definition means
we can avoid cluttering theorem statements with it.

Main result

Some of the terms are defined after the statement.

**Theorem.** 1. (Existence) There is a pairing \(\exists: Q^\infty \times Q^\infty \rightarrow y/n\) whose
adjoint gives a bijection \(\exists^{adj}: Q^\infty \rightarrow bP(Q^\infty)\), such that

(a) the restriction \(\mathbb{N} \rightarrow bP[\mathbb{N}]\) is the canonical order isomorphism; and

(b) if \(x \notin \mathbb{N}\) then \(\text{rank}[\exists^{adj}[x]] = \text{rank}[x] - 1\).

2. (Universality) Suppose \((A,\lambda)\) is an almost well-founded pairing, and the
adjoint \(\lambda^{adj}: A \rightarrow P[A]\) is injective. Then there is a unique function
\(\omega: A \rightarrow Q^\infty\) with transitive image and giving an isomorphism of pair-
ings from \(\lambda\) to the restriction of \(\exists\) to \(\text{im}[\omega]\).
3. (Set theories) If QH or a regularity hypothesis hold then \((Q^\infty W, \exists')\) satisfies all the ZFC axioms. If neither holds then size constraints may be needed in the Union and Replacement axioms; see § 7.2.

Clarifications: the adjoint of \(\exists'\) takes elements of rank 1 to the cofinal functions on \(N\). Technically, all functions on \(N\) have rank 0, but the bounded ones are accounted for by condition (1a). In (1b) recall that ranks of elements cannot be limits, so \(\text{rank}[x] - 1\) is defined. Finally, the notation \(\exists'\) is a reminder that this is the opposite of the standard \(\in\)' pairings.

Two such pairings \(\exists'\) differ by a bijection \(Q^\infty W \to Q^\infty W\) that is the identity on \(N\) and preserves the rank function.

The universality property is a version of Mostowski collapsing ([Jech], § 6.15). Consequences are considered in the next section. The injective-adjoint condition is traditionally called “extensional”, and corresponds to the set-theory axiom that sets are determined by their elements.

**Proof of Existence**

We will show that for every \(a\) there is a bijection, from elements of rank \(a + 1\) to logical functions of rank \(a\). The rest of the proof is short: According to the axiom of choice we can make a simultaneous choice of bijections for all \(a\). These fit together to give a bijection \(Q^\infty W \to bP[Q^\infty W]\). This is the adjoint of a pairing with the properties with properties (1a) and (1b).

Now we see that there are bijections as claimed. Suppose \(a \in W\). There is a bijection from elements of rank \(a + 1\) to functions of rank \(a\) if these collections have the same cardinality.

The elements of rank \(a + 1\) are \((\beth[a] \leq \# < \beth[a + 1])\). This is the complement of \((\# < \beth[a])\) in \((\# < \beth[a + 1])\). Since \(\beth[a + 1] = \text{card}[P[\# < \beth[a]]]\), we are removing a set of smaller cardinality. According to §5.3, this does not change cardinality. The cardinality of the collection of elements is therefore \(\text{card}[P[\# < \beth[a]]]\).

There are two cases for functions. Suppose \(a\) is not a limit. Then functions of rank \(a\) are the complement of \(P[\# < \beth[a - 1]]\) in \(P[\# < \beth[a]]\). Again the smaller subdomain has smaller cardinality, so removing it does not change cardinality. Cardinality of the functions is therefore \(\text{card}[P[\# < \beth[a]]]\), same as the elements.

Now suppose \(a\) is a limit. The functions on \(\# < \beth[a]\) of rank \(a\) are the cofinal ones. These are the complement of the bounded functions: \(\text{cfP}[\# < \beth[a]] = P[\# < \beth[a]] - bP[\# < \beth[a]]\). We want to show that removing the bounded functions does not change the cardinality. For this it is sufficient to show that the cardinality of the complement is at least as large as that of the collection being removed. This is so because there is an injection \(bP[\# < \beth[a]] \to \text{cfP}[\# < \beth[a]]\) defined by \(h \mapsto \text{not}[h]\). The conclusion is that the cardinality of the functions is \(\text{card}[P[\# < \beth[a]]]\), again same as the elements.

This completes the existence part of the Theorem. \(\square\)
6.2 Well-founded recursion, and universality

See Jech, [Jech] p. 66. Suppose \((A, \lambda)\) is an almost well-founded pairing. As in §4.2, a recursion condition is a partially-defined function \(R: \text{pfn}[A, B] \times A \rightarrow B\). As before, ‘pfn’ denotes partially-defined functions.

Again as in §4.2, a partially-defined function \(f: A \rightarrow B\) is \(R\)-recursive if:

1. \(\text{dom}[f]\) is \(\lambda\)-transitive; and
2. for every \(c \in \text{dom}[f]\), \(f[c] = R[(f \upharpoonright \lambda[c, \#]), c]\).

Then, exactly as in §4.2, there is a unique maximal \(R\)-recursive partially-defined function \(A \rightarrow B\).

Recall that \(\lambda[c, \#]\) defines a logical function on \(A\), and \(f \upharpoonright \lambda[c, \#]\) is the restriction of \(f\) to the support of this function. To prove universality, suppose \((A, \lambda)\) is an almost well-founded pairing. Define \(R: \text{pfn}[A, W] \times A \rightarrow Q^\infty W\) by:

1. \((f, a) \in \text{dom}[R]\) if \(\lambda[a, \#] \subset \text{dom}[f]\), and in this case
2. \(R(f, a) := (\geq^\text{adj})^{-1}[\lambda[a, \#] \ast f]\).

Note that this definition uses the injectivity of the adjoint \(\exists^\text{adj}: Q^\infty W \rightarrow bP(Q^\infty W)\). Unwinding definitions gives \((R(f, a) \ni x) \iff (\exists b \in A | \lambda[a, b] & f[b] = x)\).

The maximal \(R\)-recursive function \(f: A \rightarrow Q^\infty W\) has domain \(A\), and the recursion condition implies that \(f\) is a morphism \((A, \lambda) \rightarrow (Q^\infty W, \exists)\).

There are a few loose ends to clean up. A function \(f: (A, \lambda) \rightarrow (B, \tau)\) is a morphism of pairings if \(\tau[f[a], x] \iff (\exists b \in A | (f[b] = x) & \lambda[a, b] = \text{yes})\). Note that this implies the image of \(f\) is \(\tau\)-transitive in \(B\).

In general, morphisms of pairings need not be injective. It is straightforward to see:

1. if the adjoint \(\lambda^\text{adj}: A \rightarrow P[A]\) is injective then \(f\) is injective; and
2. if \(\tau^\text{adj}: B \rightarrow P[B]\) is injective then \(f\) is unique.

6.3 Reformulation of the ZFC axioms

A traditional set theory is a pool (class, logical domain) of potential elements, \(\Sigma\), and a binary logical operator ‘\(\in\)’. We change notation to prefix form and, to be consistent with previous material, reverse the order: define \(N: \Sigma \times \Sigma \rightarrow \text{y/n}\) by \(N[a, b] := b \in a\).

A set in the theory is a logical function on \(\Sigma\) of the form \(# \mapsto N[a, \#]\), for some \(a \in \Sigma\). Assume these sets are relaxed sets, by restricting the theory if necessary. See ‘Quantification in ZFC theories’ below.
Axioms

Given all this, we translate the ZFC axioms described in [Jech] Ch. 1.

1. (Well-founded, or Regular) \( N \) is almost well-founded.
2. (Extensionality) \( N[a, \#] = N[b, \#] \) implies \( a = b \) (see note 3);
3. (Union) \( \exists (\bigcup a) \in \Sigma \) such that \( N[\bigcup a, \#] = (N \ast N)[a, \#] \). \( N \ast N \) is the composition of pairings, defined by \( (N \ast N)[a, \#] := (\exists b \mid N[a, b] \& N[b, \#]) \).
4. (Powerset) \( \exists P[a] \in \Sigma \) such that \( N[P[a], b] = (N[b, \#] \subset N[a, \#]) \);
5. (Infinity) There is \( a \) with \( N[a, \#] \) infinite;
6. (Choice) There is a partially-defined function \( ch: \Sigma \to \Sigma \) with domain \( a \in \Sigma \mid N[a, \#] \neq \Sigma \) satisfying \( N[a, ch[a]] = \text{no} \) (see note 4);
7. (Separation) If \( P \) is an appropriate logical function on \( \Sigma \) (a “property”) then there is \( a \& P \in \Sigma \) such that \( N[a \& P, \#] = N[a, \#] \& P[\#] \) (see note 5);
8. (Replacement) Suppose \( f: \Sigma \to \Sigma \) is an appropriate function. Then the \( f \) image of a set is a set (see note 6).

Notes

1. ‘Well-founded’ is usually put near the end of axiom lists. The universality theorem indicates that it is a key ingredient, so we put it first.

2. External quantification requires that there be a logical function, defined on all possible functions \( A \to y/n \), that detects the empty function. The Internal quantification of ZFC requires only that \( \emptyset \) be detectable among functions of the form \( N[a, \#] \). In principle there could be many fewer of these so, again in principle, there might be theories with sets that support internal quantification but not external. If so we discard such sets.

3. The domain \( \Sigma \) may not support quantification, so we may not be able to identify functions \( N[a, \#] \) among all possible logical functions on \( \Sigma \), but we can identify them within functions of the form \( N[x, \#] \) as follows. \( N[a, \#] = N[b, \#] \) means: \((\forall x \in N[a, \#])(N[b, x] = \text{yes})\), and similarly \((\forall x \in N[b, \#])(N[a, x] = \text{yes})\).

4. In this formulation the choice function provides an element not in the given set, rather than (as more usual) one in the set. This is the form used to show \( \Sigma \) has a well-order, which implies any other form one might want. Note that if \( \Sigma \) does not support quantification then “\( N[a, \#] \neq \Sigma \)” does not make good sense. In this case omit this condition: it is redundant anyway because \( N[a, \#] \) cannot be all of \( \Sigma \) since it is assumed to be a set.
5. The point in “Separation” is that $a \& P$ is a set, even if $P$ is not (ie. is not of the form $N[p, #]$).

6. The intent of “Replacement” is that functions should take sets to sets. However, the usual definition of “function” uses set theory, so using it here would make the definition logically circular. The developers of set theory eventually fell back on an earlier view of functions as “given by formulas”; see [Manin] and the historical notes in [Jech]. Thus, in standard ZFC, “appropriate” in Separation and Replacement means “given by an expression in the first-order logic of sets”. In the theory here, functions (in the guise of morphisms of descriptors) are primitive objects, and need not be defined. Roughly speaking, the theory here is universal because “appropriate” is a subset of “all”.

Verifying the axioms

We verify that $(\mathcal{W}, \ni)$ satisfies the translated axioms.

1. ‘Well-founded’ follows from the fact that the pairing reduces rank, and rank takes values in a well-ordered domain.

2. The domains $\ni [a, \#]$ are relaxed sets because they are bounded subdomains of $Q^\infty \mathcal{W}$.

3. ‘Extension’ is equivalent to injectivity of the adjoint of $\ni$, and this is a design requirement in the construction.

4. ‘Infinity’ is a primitive axiom in the system used here.

5. Similarly, ‘Choice’ is a primitive axiom.

6. For Separation, note that $\ni [a, \#]$ is bounded, so $\ni [a, \#] \& P$ is bounded.

7. We interpret ‘Replacement’ in a strong way, namely that any function $Q^\infty \mathcal{W} \to Q^\infty \mathcal{W}$ should take bounded logical functions to bounded logical functions. If QH holds, this is included in §4.5. If QH does not hold, it is equivalent to the hypothesis that the cardinality of $Q^\infty \mathcal{W}$ is a regular cardinal. If this regularity does not hold either, then some theorem statements in set theory need an additional condition that ensures an associated function $Q^\infty \mathcal{W} \to Q^\infty \mathcal{W}$ is bounded.

This completes the ZFC axioms, and the proof of the universality theorem.
7 Comments and questions

A relaxed set is a $Q^\infty$ logical domain. The developments above show that these have the properties commonly used in mainstream mathematics, but they differ in some ways from ZFC sets. We discuss the suitability for use as a foundation, then describe the possible failure of quantification at the upper limit.

7.1 Foundations

A foundation is material assumed without proof, and from which everything else is derived. The need for reliability seems to have driven the evolution of a foundation: if the foundation is consistent then lawful arguments produce reliable results. The evolutionary aspect results from the experience that long delicate arguments that magnify small virtues into theorems, also magnify small flaws in foundation into clear contradictions. These flaws have been tracked down and eliminated. The current great body of successful work thus provides powerful evidence that the current foundations are consistent.

Set theory has been considered the foundation of mathematics for more than a century. However there is a problem: downstream success gives empirical support for the set theory actually used, and this is not quite the theories studied by the set-theory community. In practice the only serious difference between ZFC and relaxed set theory is the use of first-order logic. For instance in the Replacement axiom of ZFC, “function” is interpreted as “given by a first-order logical expression”, cf. [Jech], p. 13.

The historical story is that in the early 1900s it became accepted that the fully-precise definition of functions $A \rightarrow B$ should be in terms of subsets of the product $A \times B$. The intent of the Replacement axiom is that the image of a set under a function should also be a set. However the subset-of-the-product definition of function cannot be used in an axiom of set theory because it causes a circularity. In mainstream practice the intuitions developed from the subset-of-the-product definition seemed to work quite well. For study of set theory itself, however, more precision was required and eventually the community settled on first-order formulas. This led to a divergence between mainstream practice and technical set theory: since there were no mainstream problems that needed fixing, the first-order aspects were ignored. Put another way, if first-order logic was actually necessary then we would expect ignoring it would have lead to problems by now. Downstream success therefore actually gives empirical support for relaxed set theory as a foundation.

Other advantages of relaxed set theory include:

1. It begins at a more primitive level and deduces set theory in that context. The axioms for the primitive context are fewer and simpler than ZFC (and in particular do not involve first-order logic);

2. When working with objects already known to be sets, this theory is essentially the same as naive set theory with “don’t say ‘set of all sets’ ”;
3. It provides ways to recognize sets ab initio (a weak point of both naive set
theory and ZFC).

4. It provides detailed information about the ‘class of sets’ object; and

5. The primitive context is a convenient and effective setting for category
theory.

7.2 The quantification gap

The picture is that \( Q^\infty \) encodes set theory, while \( W \) encodes \( Q^1 \) domains. The region between these is the “quantification gap”, and encodes domains with finite-order quantification but not infinite order. These behave strangely. There is a quantification threshold \( \tau \) so that \( Q^{\tau-1} \neq Q^\tau = Q^\infty \) (§4.6). There seem to be a number of possibilities.

\textbf{QH}  

The Quantification Hypothesis asserts \( Q^\infty = Q^1 \). If this holds the there is no quantification gap, no strangely-behaved domains, and the threshold is 1.

For later comparison we note that QH implies there are arbitrarily large strong-limit cardinals. For a quick proof note that QH implies the Cantor Beth function has domain \( W \) and cofinal image in \( W \). There are arbitrarily large limits, and Beth takes limits to strong limits.

\textbf{Regularity}  

If QH is false then there is a maximal strong-limit cardinal, namely the cardinality of \( Q^\infty \) itself. If this cardinal is regular (equal to its cofinality) then none of the usual operations on sets can produce a non-set. The universal well-founded pairing is a full ZFC set theory, and the quantification gap is irrelevant to standard practice.

\textbf{Regularity fails}  

If QH and Regularity are both false then there is a function \( A \to Q^\infty \) with \( A \) and cofinal image. This gives a family of sets, indexed by a set, whose union is \( Q^1 \) but not a set. The product of this family may not even be \( Q^1 \).

In this situation some theorem statements must be amended to include a size restriction. Suppose we have a family of sets indexed by a set \( A \). Applying the function \texttt{card} to the family gives a function \( A \to Q^\infty \). The union of the family is a set if and only if this function is not cofinal, i.e. is bounded in \( Q^\infty \). The necessary size restriction is therefore “the sets in the family have bounded cardinality”. This causes minor difficulty in category theory, but seems to have no impact on other areas.
Maximal cardinal

There is no maximal cardinal in standard set theory, due to Cantor’s observation that $\text{card}[P[A]] > \text{card}[A]$. However if $A$ is $Q^1$ and not $Q^2$ then $P[A]$ is a domain but not $Q^1$. As a result $\text{card}[P[A]]$ is not defined and the argument fails. If QH fails then we cannot at present rule out the possibility of a maximal cardinal.

Existence of a maximal cardinal means that all sufficiently large $Q^1$ domains are bijective. Further, the argument that subdomains are logical fails for these. This means that if $A$ is a domain of maximal cardinality, then there might be an injection $\mathbb{W} \rightarrow A$ with (necessarily) non-logical image. Conversely, if there is an injection $\mathbb{W} \rightarrow A$ then $A$ has maximal cardinality.
8 Bibliography

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