SHUFFLE AND FAÀ DI BRUNO HOPF ALGEBRAS IN THE CENTER PROBLEM FOR ORDINARY DIFFERENTIAL EQUATIONS

ALEXANDER BRUDNYI

ABSTRACT. In this paper we describe the Hopf algebra approach to the center problem for the differential equation \( \frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x)v^{i+1}, \quad x \in [0, T], \) and study some combinatorial properties of the first return map of this equation. The paper summarizes and extends previously developed approaches to the center problem due to Devlin and the author.

CONTENTS

1. Introduction 1
2. Hopf Algebras 3
3. Shuffle Hopf Algebra 5
3.1. Definition 6
3.2. Set of Characters 6
3.3. Semigroup of Paths 7
4. Faà di Bruno Hopf Algebra 8
5. Center Problem 9
5.1. Displacement Polynomials 9
5.2. First Return Map 12
5.3. Group of Formal Centers 12
5.4. Equations with Finitely Many Terms 13
6. Generalized Displacement Polynomials 15
6.1. Recurrence Relations 15
6.2. Generalized Displacement Polynomials in the Shuffle Hopf Algebra 17
6.3. Augmentation Homomorphism 23
References 26

1. INTRODUCTION

Given the ordinary differential equation

\[
\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x)v^{i+1}, \quad x \in I_T := [0, T],
\]

with coefficients \( a_i \in L^\infty(I_T) \) (the Banach space of bounded measurable complex-valued functions on \( I_T \) equipped with supremum norm) satisfying

\[
\sup_{x \in I_T, i \in \mathbb{N}} \sqrt{|a_i(x)|} < \infty
\]
the *center problem* asks whether (1.1) determines a *center*, i.e. whether every solution \(v\) of (1.1) with a sufficiently small initial value (existing and Lipschitz due to (1.2)) satisfies \(v(T) = v(0)\). The center problem arises naturally in the framework of the geometric theory of ordinary differential equations created by Poincaré. In particular, it is related to the classical Poincaré Center-Focus problem for planar polynomial vector fields

\[
\frac{dx}{dt} = -y + F(x, y), \quad \frac{dy}{dt} = x + G(x, y),
\]

where \(F\) and \(G\) are real polynomials of a given degree without constant and linear terms, asking about conditions on (1.2) and by (1.3) and expanding the right-hand side of the resulting equation as a series in \(r\) (for \(F, G\) with sufficiently small coefficients) one obtains an equation (1.1) with coefficients being trigonometric polynomials depending polynomially on the coefficients of (1.3). This transforms the Center-Focus problem to the center problem for equations (1.1) with coefficients depending polynomially on a parameter. (For recent advances in the area of the center problem for equation (1.1) see [AL, A1, A2, A3, A4, Br2, Br3, Br4, Br5, Br6, BrY, BRY, C1, CGM1, CGM2, CGM3, D, GGL, GGS, P1, P2] and references therein.)

By \(\mathcal{X}\) we denote the vector space of sequences \(a = (a_1, a_2, \ldots)\) of coefficients of equation (1.1) satisfying (1.2) and by \(\mathcal{C} \subset \mathcal{X}\) the set of centers of (1.1). Let \(v(x; r; a), x \in I_T\), be the Lipschitz solution with initial value \(v(0; r; a) = r\) of equation (1.1) with the sequence of coefficients \(a \in \mathcal{X}\). Clearly, for every \(x \in I_T\), \(v(x; r; a) \in G_c[r]\), the set of locally convergent near zero power series of the form \(r + \sum_{i=1}^{\infty} c_i r^{i+1}\) with all \(c_i \in \mathbb{C}\). By definition, \(\mathcal{P}(a) := v(T; \cdot; a)\) is the *first return map* of (1.1). The explicit expression for \(\mathcal{P}(a)\) was obtained by Devlin [D] (for equations with finitely many nonzero coefficients \(a_i\)) and independently and by a different method by the author [Br1] (the general case):

\[
\mathcal{P}(a) := r + \sum_{i=1}^{\infty} \left( \sum_{i_1 + \ldots + i_k = i} p_{i_1,\ldots,i_k}(i) \cdot I_{i_1,\ldots,i_k}(a) \right) r^{i+1}
\]

(in the inner sum \(k\) runs over the set of natural numbers \(1, \ldots, i\)), where for \(t \in \mathbb{C}\),

\[
p_{i_1,\ldots,i_k}(t) = (t-i_1+1)(t-i_1-i_2+1)(t-i_1-i_2-i_3+1) \cdots (t-i) \quad (1.5)
\]

and \(I_{i_1,\ldots,i_k}(a) := \int \cdots \int_{0 \leq s_1 \leq \ldots \leq s_k \leq T} a_{i_1}(s_1) \cdots a_{i_k}(s_k) \, ds_k \cdots ds_1\).

In particular, one obtains (see [Br3, Th.3.1])

\[
a \in \mathcal{C} \iff \sum_{i_1 + \ldots + i_k = i} p_{i_1,\ldots,i_k} \cdot I_{i_1,\ldots,i_k}(a) \equiv 0 \quad \text{for all } i \in \mathbb{N} \quad (1.6)
\]

\[
\iff \sum_{i_1 + \ldots + i_k = i} p_{i_1,\ldots,i_k}(i) \cdot I_{i_1,\ldots,i_k}(a) = 0 \quad \text{for all } i \in \mathbb{N}.
\]

In [D] the center problem for equation (1.1) with finitely many nonzero coefficients \(a_i\) was reformulated using the language of word-problems. In the same vein, in [Br3] the algebraic model for the center problem for equation (1.1) was constructed. In the present paper we continue this line of research and describe the Hopf algebra approach to the center problem. The key point of this approach is that the first return map of equation (1.1) determines the natural monomorphism of the co-opposite of the *Faà di Bruno Hopf algebra* into the *shuffle Hopf algebra* (see Sections 2–5 below for the corresponding definitions and
results) given in terms of polynomials

\begin{equation}
\mathcal{P}_i(X_1, \ldots, X_i) = \sum_{i_1 + \cdots + i_k = i} p_{i_1, \ldots, i_k}(i) X_{i_1} \cdots X_{i_k}, \quad i \in \mathbb{N},
\end{equation}

in free noncommutative variables $X_1, X_2, \ldots$.

Some recurrence relations for such polynomials (in the sequel called the displacement polynomials as they originated from the expression for the displacement map of equation (1.1)) were established by Devlin, see [D, Th.6.5]. In the present paper we study more general polynomials

\begin{equation}
\tilde{\mathcal{P}}_i(X_1, \ldots, X_i; t) = \sum_{i_1 + \cdots + i_k = i} p_{i_1, \ldots, i_k}(t) X_{i_1} \cdots X_{i_k}, \quad t \in \mathbb{C}, \ i \in \mathbb{N},
\end{equation}

which naturally appear in our Hopf algebra approach to the center problem (see Section 6) and are closely related to the classical Bell polynomials [B]. We establish some recurrence relations for such polynomials (referred to as the generalized displacement polynomials) extending those of [D, Th.6.5] and describe certain important combinatorial properties of their coefficients.

The paper is organized as follows.

Section 2 contains the necessary background material from the theory of Hopf algebras. Sections 3 and 4 are intended as an introduction to the areas of the shuffle and Faà di Bruno Hopf algebras.

Sections 5 and 6 comprise our main results and their proofs. Specifically, in Section 5.1.1 we reveal the algebraic nature of the displacement polynomials showing that they appear in the expression for the natural Hopf algebra monomorphism of the co-opposite of the Faà di Bruno Hopf algebra into the shuffle Hopf algebra. As a result, we obtain some important combinatorial relations between the displacement and the Bell polynomials (see (5.3), (5.4)). In Section 5.1.2, using the 'generating function' for the displacement polynomials (5.6), we prove some recurrence relations for them, see Theorem 5.3 partially established earlier by Devlin [D] by a different method. The first return map of equation (1.1) can be factorized through the so-called Chen map (3.7) of $\mathcal{X}$ into the group of characters of the shuffle Hopf algebra; this result is established in Section 5.2. In Section 5.3 we describe the group of formal centers of equation (1.1) introduced earlier in [Br3]. It turns out that it is a subgroup of the group of characters of the shuffle Hopf algebra isomorphic to the group of characters of the quotient of the algebra by the Hopf ideal generated by the displacement polynomials. In the same way, we describe the Lie algebra of the group of formal centers, see (5.12). The Hopf algebra approach to the center problem for equations (1.1) with finitely many terms is described in Section 5.4.

Finally, Section 6 is devoted to the study of the generalized displacement polynomials. We reveal their algebraic nature (showing that their values for $t \in \mathbb{N}$ appear as the ‘matrix entries’ of the composition of the well-known infinite-dimensional faithful representation of group $(G_c[[r]], \circ)$ and the first return map $\mathcal{P} : \mathcal{X} \to G_c[[r]]$, see Proposition 6.5 and Remark 6.6), prove some important recurrence relations for them (Sections 6.1, 6.2.2), establish their connection with the Bell polynomials (Section 6.2.1) and prove some combinatorial identities for their coefficients (Section 6.3).

2. Hopf Algebras

In this section we collect some basic definitions and results in the area of Hopf algebras, cf., e.g., [C], [CK], [GVF], [M], [S]. All objects are considered over a ground field $\mathbb{K}$.
(A) An associative unital algebra is a \( \mathbb{K} \)-vector space \( A \) together with a multiplication \( m : A \otimes A \to A \), \( m(a_1 \otimes a_2) =: a_1 \cdot a_2, a_1, a_2 \in A \), such that
\[
m \circ (m \otimes \text{id}) = m \circ (\text{id} \otimes m)
\]
(i.e. \( (a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3) \) for all \( a_1, a_2, a_3 \in A \)), and a unit \( \eta : \mathbb{K} \to A \) such that
\[
\eta(k_1 \cdot k_2) = \eta(k_1) \cdot \eta(k_2) \quad \text{for all} \quad k_1, k_2 \in \mathbb{K} \quad \text{and} \quad a \cdot 1_A = a = 1_A \cdot a \quad \text{for all} \quad a \in A;
\]
here \( 1_A := \eta(1_\mathbb{K}) \).

(C) A coassociative counital coalgebra is a \( \mathbb{K} \)-vector space \( C \) together with a comultiplication \( \Delta : C \to C \otimes C \), \( \Delta(c) := \sum c(1) \otimes c(2) \), such that
\[
(\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta,
\]
and a counit \( \varepsilon : C \to \mathbb{K} \) such that
\[
(\varepsilon \otimes \text{id}_C) \circ \Delta = (\text{id}_C \otimes \varepsilon) \circ \Delta = \text{id}_C.
\]

(B) A bialgebra is a \( \mathbb{K} \)-vector space \( B \) which is both an associative algebra and a coassociative coalgebra such that comultiplication \( \Delta \) and counit \( \varepsilon \) are algebra morphisms, i.e. for all \( b_1, b_2 \in B \),
\[
\Delta(b_1 \cdot b_2) = \Delta(b_1) \cdot \Delta(b_2), \quad \Delta(1_H) = 1_H \otimes 1_H, \quad \varepsilon(b_1 \cdot b_2) = \varepsilon(b_1) \cdot \varepsilon(b_2), \quad \varepsilon(1) = 1;
\]
here the product on \( B \otimes B \) is given by \( (b_1 \otimes b_1') \cdot (b_2 \otimes b_2') = (b_1 \cdot b_2) \otimes (b_1' \cdot b_2') \) for all \( b_1, b_2, b_1', b_2' \in B \).

(H) A Hopf algebra is a bialgebra \( H \) together with a \( \mathbb{K} \)-linear map \( S : H \to H \), called the antipode, such that
\[
m \circ (S \otimes \text{id}) \circ \Delta = \eta \circ \varepsilon = m \circ (\text{id} \otimes S) \circ \Delta
\]
and \( S \) is both an antimorphism of algebras and an antimorphism of coalgebras, i.e. for all \( h_1, h_2 \in H \),
\[
S(h_1 \cdot h_2) = S(h_2) \cdot S(h_1), \quad S(1_H) = 1_H, \quad \Delta(S(h_1)) = (S \otimes S)(\Delta^{op}(h_1)), \quad \varepsilon(S(h_1)) = \varepsilon(h_1),
\]
where \( \Delta^{op} = \tau \circ \Delta, \quad \tau(u \otimes v) = v \otimes u \).

**Example 2.1.** Recall that the universal enveloping algebra \( U(g) \) of a Lie algebra \( (g, [\cdot, \cdot]) \) is the quotient of the free associative unital algebra on the vector space \( g \) by the two-sided ideal generated by elements of the form \( x \cdot y - y \cdot x - [x, y] \cdot 1, \quad x, y \in g \). It is a Hopf algebra with comultiplication, counit and antipode given on the generators \( x \in g \) by
\[
\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x.
\]
Note that the Hopf algebra \( U(g) \) is cocommutative (i.e. \( \Delta(u) = \Delta^{op}(u) \) for all \( u \in U(g) \)).

A bialgebra \( B \) is called graded if there are \( \mathbb{K} \)-vector spaces \( B_n, n \in \mathbb{Z}_+, \) such that
\[
B = \bigoplus_{n \geq 0} B_n, \quad m(B_n \otimes B_m) \subset B_{n+m}, \quad \Delta(B_n) \subset \bigoplus_{r+s=n} B_r \otimes B_s.
\]

Elements \( b \in B_n \) have degree \( n \) (written, \( \deg(b) = n \)). A graded bialgebra \( B \) is called connected if \( B_0 \) is one-dimensional, i.e. \( B_0 = \mathbb{K} \cdot 1_H \).

Every connected graded bialgebra \( B \) is a Hopf algebra with the antipode \( S \) defined by certain recursive relations and satisfying \( S(B_n) \subset B_n \) for all \( n \in \mathbb{Z}_+ \).

Let \( (B, m, \eta, \Delta, \varepsilon) \) be a bialgebra and \( A \) be an associative unital algebra with multiplication \( m_A \) and unit \( \eta_A : \mathbb{K} \to A \). The vector space \( L(B; A) \) of \( \mathbb{K} \)-linear maps from \( B \).
SHUFFLE AND FAà DI BRUNO HOPF ALGEBRAS IN THE CENTER PROBLEM FOR ODES

5
to $A$ inherits a canonical associative unital algebra structure with multiplication given by convolution:

\begin{equation}
\alpha \ast \beta := m_A \circ (\alpha \otimes \beta) \circ \Delta, \quad \alpha, \beta \in L(B; A),
\end{equation}

and unit $\iota := \eta_A \circ \varepsilon$.

If $(H, m, \eta, \Delta, \varepsilon, S)$ is a commutative Hopf algebra and $(A, m_A, \eta_A)$ is a commutative unital algebra, then the subset $G_H(A) \subset L(H, A)$ of $A$-valued characters of $H$ (i.e. unital algebra morphisms from $H$ to $A$) forms a group with respect to the convolution product $\ast$ with unit $\iota = \eta_A \circ \varepsilon$ and the inverse given by $\alpha^{*-1} := \alpha \circ S, \alpha \in G_H(A)$. Thus $H$ can be regarded as the algebra of $A$-valued functions on group $G_H(A)$ equipped with pointwise multiplication, i.e. each $h \in H$ can be seen as a function on $G_H(A)$ given by $h(\alpha) := \alpha(h), \alpha \in G_H(A)$. Moreover, comultiplication $\Delta$ then coincides with comultiplication on functions determined by the group law in $G_H(A)$, i.e. $\Delta(h)(\alpha, \beta) = h(\alpha \ast \beta), h \in H, \alpha, \beta \in G_H(A)$.

Further, an $A$-valued infinitesimal character of $H$ is a map $\alpha \in L(H, A)$ such that

$$\alpha(h_1 \cdot h_2) = \alpha(h_1) \cdot \iota(h_2) + \iota(h_1) \cdot \alpha(h_2), \quad h_1, h_2 \in H.$$ 

Since, $\iota(1_H) = 1_A, \alpha(1_H) = 0$. It follows that the set $g_H(A)$ of $A$-valued infinitesimal characters of $H$ is a Lie algebra with bracket given by

$$[\alpha, \beta] := \alpha \ast \beta - \beta \ast \alpha, \quad \alpha, \beta \in g_H(A).$$

Now, assume that $H = \oplus_{n \geq 0} H_n$ is a connected graded commutative Hopf algebra and $\mathbb{K}$ is of characteristic zero. Then by the Milnor-Moore theorem, see, e.g., [C] Th.3.8.3, $H$ is a free commutative algebra generated by homogeneous elements. Also, $G_H(A)$ is a pro-unipotent Lie group with the Lie algebra $g_H(A)$ and the exponential map

$$\exp(\alpha) := \iota + \sum_{n=1}^{\infty} \frac{\alpha^n}{n!}, \quad \alpha \in g_H(A),$$

maps $g_H(A)$ bijectively onto $G_H(A)$.

In addition, assume that $\dim_{\mathbb{K}} H_n < \infty$ for all $n$. Let $H^*_n$ be the dual space of $H_n$ over $\mathbb{K}$. Then $H^* = \oplus_{n \geq 0} H^*_n$ is the graded dual Hopf algebra with multiplication $\Delta^*$, comultiplication $m^*$, unit $\varepsilon^*$, counit $\eta^*$ and antipode $S^*$. Since $H$ is commutative, its dual $H^*$ is a cocommutative Hopf algebra. Hence, by the Milnor-Moore theorem, see, e.g., [C] Th.3.8.1, $H^*$ is the universal enveloping algebra of the Lie algebra of its primitive elements

$$\text{Prim } H^* := \{h^* \in H^* : m^*(h^*) = h^* \otimes 1_H^* + 1_H^* \otimes h^* \}.$$ 

Since $H^* \subset L(H; \mathbb{K})$ and $1_H^* = \varepsilon$, $\text{Prim } H^*$ is a Lie subalgebra of the Lie algebra $g_H(\mathbb{K})$. In fact, $g_H(\mathbb{K})$ is the completion of $\text{Prim } H^*$ in $L(H; \mathbb{K})$ equipped with the adic topology induced by the grading of $H$.

Next, the Hopf algebra operations on $H^*$ extend by continuity to similar operations on $L(H; \mathbb{K})$ turning the latter into a topological Hopf algebra. (Here the extension of $\Delta^*$ coincides with the convolution product, cf. [2.3], and of $\varepsilon^*$ with $\iota$.) Then the group of characters $G_H(\mathbb{K}) \subset L(H; \mathbb{K})$ coincides with the set of group-like elements, that is elements $\alpha \in L(H; \mathbb{K})$ such that

$$m^*(\alpha) = \alpha \otimes \alpha \quad \text{and} \quad \eta^*(\alpha) = 1.$$ 

3. Shuffle Hopff Algebra

We refer to [C], [L], [Ra], [Re], [R] and references therein for basic results in the area of shuffle Hopf algebras.
3.1. Definition. Let $\mathcal{A} = \{\alpha_i : i \in \mathbb{N}\}$ be a countable alphabet. By definition, a word is an ordered sequence $\alpha_{i_1} \ldots \alpha_{i_k}$ of (not necessarily distinct) elements from $\mathcal{A}$. The set of all words together with the empty word $\emptyset (=: 1)$ is denoted by $\mathcal{A}^*$. Let $\mathbb{K}(\mathcal{A})$ be the vector space over a field $\mathbb{K}$ of characteristic zero freely generated by elements of $\mathcal{A}^*$. To introduce multiplication on $\mathbb{K}(\mathcal{A})$ we invoke the following definition.

A permutation $\sigma$ of $\{1, 2, \ldots, r + s\}$ is called a shuffle of type $(r, s)$ (denoted $\sigma \in \text{Sh}_{r,s}$) if $\sigma^{-1}(1) < \sigma^{-1}(2) < \cdots < \sigma^{-1}(r)$ and $\sigma^{-1}(r + 1) < \sigma^{-1}(r + 2) < \cdots < \sigma^{-1}(r + s)$.

Now, $\mathbb{K}(\mathcal{A})$ is equipped with the structure of a commutative algebra by defining the shuffle product of words

\begin{equation}
\alpha_{i_1} \ldots \alpha_{i_r} \shuffle \alpha_{i_{r+1}} \ldots \alpha_{i_{r+s}} := \sum_{\sigma \in \text{Sh}_{r,s}} \alpha_{i_{\sigma(1)}} \cdots \alpha_{i_{\sigma(r+s)}}.
\end{equation}

Here the empty word $1 \in \mathcal{A}^*$ is the unit of $\mathbb{K}(\mathcal{A})$, i.e., $1 \shuffle w = w = w \shuffle 1$ for all $w \in \mathcal{A}^*$.

Next, the grading on $\mathbb{K}(\mathcal{A})$ is given by $\deg(\alpha_i) := i$ ($i \in \mathbb{N}$), $\deg(1) := 0$ so that $\mathbb{K}_i(\mathcal{A})$, $i \geq 1$, is the subspace of $\mathbb{K}(\mathcal{A})$ generating by words $w = \alpha_{i_1} \ldots \alpha_{i_p} \in \mathcal{A}^*$ such that $\deg(w) := i_1 + \cdots + i_p = i$. Then the shuffle algebra $(\mathbb{K}(\mathcal{A}), \shuffle) = \bigoplus_{i \geq 0} \mathbb{K}_i(\mathcal{A})$ is a connected graded Hopf algebra with comultiplication defined by
decatenation,

\begin{equation}
\Delta(\alpha_{i_1} \ldots \alpha_{i_p}) := \alpha_{i_1} \ldots \alpha_{i_p} \otimes 1 + \sum_{1 \leq j < p} \alpha_{i_1} \ldots \alpha_{i_{j-1}} \otimes \alpha_{i_j} \alpha_{i_{j+1}} \ldots \alpha_{i_p},
\end{equation}

and with counit $\varepsilon : \mathbb{K}(\mathcal{A}) \to \mathbb{K}$ equal to zero at each nonempty word in $\mathcal{A}^*$ and one at 1. Also, the antipode is given by the formulas

\begin{equation}
S(\alpha_{i_1} \ldots \alpha_{i_p}) = (-1)^p \alpha_{i_p} \otimes \alpha_{i_1}.
\end{equation}

Next, let us introduce the lexicographic order $\prec$ on $\mathcal{A}^*$ assuming initially that $\alpha_i < \alpha_j$ if and only if $i < j$. By cyclic permutations a word $w = \alpha_{i_1} \ldots \alpha_{i_k}$ generates $k$ words $w(1), \ldots, w(k)$ with $w(1) = w$. A Lyndon word is a word $w$ such that $w(1), \ldots, w(k)$ are all distinct and $w \prec w(j)$ for $j = 2, \ldots, k$. The classical result by Radford [Ra] asserts that $\mathbb{K}(\mathcal{A})$ is a graded polynomial algebra over $\mathbb{K}$ in the Lyndon words as generators.

3.2. Set of Characters. The graded dual Hopf algebra of $\mathbb{K}(\mathcal{A})$ is isomorphic to the associative algebra $\mathbb{K}(X)$, $X = \{X_i : i \in \mathbb{N}\}$, with unit $I$ of noncommutative polynomials in $I$ and free noncommutative variables $X_i$ with coefficients in $\mathbb{K}$. The duality between $\mathbb{K}(\mathcal{A})$ and $\mathbb{K}(X)$ is defined by putting the monomial basis $X^*$ of $\mathbb{K}(X)$ in the natural duality with words in $\mathcal{A}^*$. Also, the grading on $\mathbb{K}(X)$ is given by $\deg(X_i) = i$ ($i \in \mathbb{N}$), $\deg(I) = 0$ (by $\mathbb{K}_i(X) \subset \mathbb{K}(X)$ we denote the subspace of elements of degree $i$), and comultiplication $\Delta$ is defined on the generators by

\begin{equation}
\Delta(X_i) = I \otimes X_i + X_i \otimes I, \quad i \in \mathbb{N}.
\end{equation}

The set of primitive elements of $\mathbb{K}(X)$ is the free Lie algebra over $\mathbb{K}$ generated by $X_i$, $i \in \mathbb{N}$ (and $\mathbb{K}(X)$ is its universal enveloping algebra).

Further, the set $L(\mathbb{K}(\mathcal{A}); \mathbb{K})$ of $\mathbb{K}$-linear functionals on $\mathbb{K}(\mathcal{A})$ is the completion of $\mathbb{K}(X)$ with respect to its grading. It is naturally identified with the subalgebra of the associative algebra $\mathbb{K}(X)[[t]]$ of formal power series in $t$ with coefficients in $\mathbb{K}(X)$ consisting of series of the form

\begin{equation}
f = f_0 I + \sum_{i=1}^{\infty} f_i t^i, \quad f_i \in \mathbb{K}_i(X).
\end{equation}

\[1\]The term “shuffle” is used because such permutations arise in riffle shuffling a deck of $r + s$ cards cut into one pile of $r$ cards and a second pile of $s$ cards.
Thus $L(\mathbb{K}(\omega); \mathbb{K})$ has the structure of a topological Hopf algebra with comultiplication extending by linearity and continuity comultiplication $\Delta$ on $\mathbb{K}(X)$. In turn, the group $G_{\mathbb{K}(\omega)}(\mathbb{K})$ of $\mathbb{K}$-valued characters of $\mathbb{K}(\omega)$ is the set of group-like elements of $L(\mathbb{K}(\omega); \mathbb{K})$, and the Lie algebra $g_{\mathbb{K}(\omega)}(\mathbb{K})$ of infinitesimal characters of $\mathbb{K}(\omega)$ is the set of primitive elements of $L(\mathbb{K}(\omega); \mathbb{K})$. It consists of series $g = \sum_{i=1}^{\infty} g_i t^i$ with $g_i \in \mathbb{K}_i(X) \cap g_{\mathbb{K}(\omega)}(\mathbb{K})$

Lie elements of degree $i$ (i.e. having the form

(3.5) \[ g_i = \sum_{i_1 + \ldots + i_k = i} d_{i_1, \ldots, i_k} [X_{i_1}, [X_{i_2}, \ldots, [X_{i_{k-1}}, X_{i_k}] \ldots]]; \]

here $[X, Y] := XY - YX$ and the term with $i_k = i$ is $d_i X_i$.

Hence, the exponential map $\exp(g) := I + \sum_{n=1}^{\infty} \frac{g^n}{n!}$, $g \in L(\mathbb{K}(\omega); \mathbb{K})$, maps $g_{\mathbb{K}(\omega)}(\mathbb{K})$ bijectively onto $G_{\mathbb{K}(\omega)}(\mathbb{K})$.

Remark 3.1. From [M-KO, Th. 3.2] one obtains that

(3.6) \[ \dim(\mathbb{K}_i(X) \cap g_{\mathbb{K}(\omega)}(\mathbb{K})) = \frac{1}{i} \sum_{d|i} (2^{i/d} - 1) \cdot \mu(d), \]

where the sum is taken over all numbers $d \in \mathbb{N}$ that divide $i$, and $\mu : \mathbb{N} \to \{-1, 0, 1\}$ is the Möbius function defined as follows. If $d$ has a prime factorization

\[ d = p_1^{n_1} p_2^{n_2} \cdots p_q^{n_q}, \quad n_i > 0, \]

then

\[ \mu(d) = \begin{cases} 1 & \text{for } d = 1, \\ (-1)^q & \text{if all } n_i = 1, \\ 0 & \text{otherwise.} \end{cases} \]

The basis of $g_{\mathbb{K}(\omega)}(\mathbb{K})$ can be constructed by means of Lyndon words of $X^*$, see [Ly].

3.3. Semigroup of Paths. Let us consider the set $\mathcal{X}$ of coefficients of equation (1.1) as a nonassociative semigroup with the operations given for $a = (a_1, a_2, \ldots)$ and $b = (b_1, b_2, \ldots)$ in $\mathcal{X}$ by

\[ a \ast b = (a_1 \ast b_1, a_2 \ast b_2, \ldots) \in \mathcal{X} \quad \text{and} \quad a^{-1} = (a_1^{-1}, a_2^{-1}, \ldots) \in \mathcal{X}, \]

where for $i \in \mathbb{N}$

\[ (a_i \ast b_i)(x) = \begin{cases} 2b_i(2x) & \text{if } 0 \leq x \leq T/2, \\ 2a_i(2x - T) & \text{if } T/2 < x \leq T \end{cases} \]

and

\[ a_i^{-1}(x) = -a_i(T - x), \quad 0 \leq x \leq T. \]

Let $\mathbb{C}^\infty$ be the vector space of sequences of complex numbers $(c_1, c_2, \ldots)$ equipped with the product topology. For $a = (a_1, a_2, \ldots) \in \mathcal{X}$ by $\bar{a} = (\bar{a}_1, \bar{a}_2, \ldots) : I_T \to \mathbb{C}^\infty$,

\[ \bar{a}_k(x) := \int_{0}^{x} a_k(t) \, dt \]

for all $k \in \mathbb{N}$, we denote a path in $\mathbb{C}^\infty$ starting at 0. The one-to-one map $a \mapsto \bar{a}$ sends the product $a \ast b$ to the product of paths $\bar{a} \ast \bar{b}$, that is the path obtained by translating $\bar{a}$ so that its beginning meets the end of $\bar{b}$ and then forming the composite path. Similarly, $\bar{a}^{-1}$ is the path obtained by translating $\bar{a}$ so that its end meets 0 and then taking it with the opposite orientation.

For $a = (a_1, a_2, \ldots) \in \mathcal{X}$ let us consider the basic iterated integrals

\[ I_{i_1, \ldots, i_k}(a) := \int_{0}^{T} \cdots \int_{0 \leq s_1 \leq \ldots \leq s_k \leq T} a_{i_k}(s_k) \cdots a_{i_1}(s_1) \, ds_k \cdots ds_1 \]

(for $k = 0$ we assume that this equals 1). By the Ree shuffle formula [R] the linear space over $\mathbb{C}$ generated by all such functions on $\mathcal{X}$ is an algebra (i.e. it is closed under the pointwise multiplication of functions on $\mathcal{X}$) denoted by $\mathcal{I}_{\mathbb{C}}(\mathcal{X})$. Also, the correspondence
\( \mathcal{A}^{s} \ni \alpha_{i_{1}} \ldots \alpha_{i_{k}} \mapsto I_{i_{1}, \ldots, i_{k}} \in I_{\mathbb{C}}(\mathcal{X}) \) for all possible indices \( i_{1}, \ldots, i_{k} \) extends (by linearity) to an isomorphism of commutative algebras \( \mathbb{C}(\mathcal{A}) \to I_{\mathbb{C}}(\mathcal{X}) \) whose transpose determines the Chen map \( \varepsilon: \mathcal{X} \to L(I_{\mathbb{C}}(\mathcal{X}); \mathbb{C}) \) (see (3.1)),

\[
E(a) := E(a) = I + \sum_{i=1}^{\infty} \left( \sum_{i_{1}+\ldots+i_{k}=i} I_{i_{1}, \ldots, i_{k}}(a)X_{i_{1}} \cdots X_{i_{k}} \right) t^{i},
\]

that sends \( \mathcal{X} \) into \( G_{\mathbb{C}(\mathcal{A})}(\mathbb{C}) \) and satisfies (cf. [Ch Th. 6.1])

\[
E(a \ast b) = E(a) \cdot E(b), \quad E(a^{-1}) = E(a)^{-1} \quad \text{for all} \quad a, b \in \mathcal{X}.
\]

Group \( G_{\mathbb{C}(\mathcal{A})}(\mathbb{C}) \) has the natural structure of a complete metrizable separable topological group and \( E(\mathcal{X}) \) is its dense subgroup, cf. [Br3 Sect. 2.3.2].

Next, \( \mathcal{X} = E^{-1}(I) \subset \mathcal{C} \) and is called the \textit{set of universal centers} of equation (1.1). Its basic properties are described in [Br3 Sect. 2.2], [BY], [Br5] (see also references therein).

The expression in brackets can be written as

\[
\varepsilon(t_{i}) = \delta_{i0} \quad i \in \mathbb{Z}_{+}; \quad \text{here} \quad t_{0} := 1.
\]

The expression in brackets can be written as

\[
\frac{(j+1)!}{(i+1)!} B_{i+1, j+1}(1, 2! t_{1}, 3! t_{2}, \ldots, (i-j+1)! t_{i-j}),
\]

where

\[
B_{r, s}(t_{1}, \ldots, t_{l}) := \sum_{k_{1}+\ldots+k_{r} = s} \frac{r!}{k_{1}! \cdots k_{r}!} \left( \frac{t_{1}}{1!} \right)^{k_{1}} \cdots \left( \frac{t_{l}}{l!} \right)^{k_{l}}, \quad l = r - s + 1,
\]

are the \textit{Bell polynomials} [B].
The antipode $S_{\text{FdB}}$ of $\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})$ (and of $\mathcal{H}_{\text{FdB}}(\mathbb{K})$ as well) is given by the formulas

\begin{equation}
S_{\text{FdB}}(t_i) = \frac{1}{(i + 1)!} \sum_{j=1}^{i} (-1)^j B_{i+j,j} (0, 2! t_1, 3! t_2, \ldots, (i + 1)! t_i), \quad i \in \mathbb{N}.
\end{equation}

Hopf algebra $\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})$ is connected and graded with generators $t_i$ having degree $i$, $i \in \mathbb{N}$. Let $\mathcal{H}^\text{prim}_{\text{FdB}}(\mathbb{K}) \subset \mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})$ be the subspace of elements of degree $i$. By the Milnor-Moore theorem the dual Hopf algebra $\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})^* = \bigoplus_{i \geq 0} \mathcal{H}^i_{\text{FdB}}(\mathbb{K})^*$ is the universal enveloping algebra (equipped with the convolution product) of the Lie algebra $\text{Prim} \mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})$ of its primitive elements. The latter has a basis $t'_i$, $i \in \mathbb{N}$, with $\deg(t'_i) = i$ such that

\begin{equation}
t'_i(t_j) = \delta_{ij} \quad \text{and} \quad t'_i(t_1 \cdot \ldots t_k) = 0 \quad \text{for} \quad k \geq 2
\end{equation}

which satisfies the following relations

\begin{equation}
[t'_i, t'_j] := t'_i t'_j - t'_j t'_i = (i - j) t'_{i+j} \quad \text{for all} \quad i, j \in \mathbb{N}.
\end{equation}

**Remark 4.1.** It is worth noting that $\text{Prim} \mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{C})$ is isomorphic to the Lie algebra $W_1(1) = \text{span}_C \{ e_n := t^{n+1} \frac{\partial}{\partial t}, \quad n \in \mathbb{N} \}$, the nilpotent part of the Witt algebra of complex formal vector fields on $\mathbb{R}$.

Next, the Lie algebra $\mathfrak{g}_{\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})}$ of $\mathbb{K}$-valued infinitesimal characters of $\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})$ is naturally identified with the set of formal power series of the form $g(t) := \sum_{i=0}^{\infty} (d_i t^i)$, $d_i \in \mathbb{K}$, with the bracket extending by linearity and the adic continuity the bracket on $\text{Prim} \mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})^*$. In turn, the set $L(\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K}); \mathbb{K})$ of $\mathbb{K}$-linear functionals on $\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})$ is the completion of $\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})^*$ with respect to its grading. It is identified with the set of formal power series in $t$ with coefficients in $\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})^*$ of the form

\begin{equation}
f(t) = \sum_{i=0}^{\infty} f_i t^i, \quad f_i \in \mathcal{H}^i_{\text{FdB}}(\mathbb{K})^*, \quad i \in \mathbb{Z}_+,
\end{equation}

with multiplication $*$ extending by linearity and continuity the convolution product on $\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})^*$. By definition, $f(t)(t_i) = f_i(t_i) t^i$ for all $i \in \mathbb{Z}_+$ (here $t_0 := 1$).

Further, the set of characters $G_{\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})}(\mathbb{K}) \subset L(\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K}); \mathbb{K})$ of $\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})$ is the image of $\text{Prim} \mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})^*$ under the exponential map. Each $f \in G_{\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})}(\mathbb{K})$ is uniquely determined by its values on generators. Moreover, map $\Theta : (G_{\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})}, *) \to (G_{\hat{\mathbb{K}}[[r]]}, \circ)$ (the group with respect to the composition of series of formal power series of the form $r + \sum_{i=1}^{\infty} c_i r^{i+1}$ with all $c_i \in \mathbb{K}$) given by

\begin{equation}
\Theta(f)(r) := r + \sum_{i=1}^{\infty} f_i(t_i) r^{i+1}, \quad f \in G_{\mathcal{H}^\text{cop}_{\text{FdB}}(\mathbb{K})},
\end{equation}

is an isomorphism of topological groups.

## 5. Center Problem

In this section we describe the main ingredients of the Hopf algebra approach to the center problem for equation (1.1). Some of our results are already proved in [Br3], so we just adapt them to the new setting.

### 5.1. Displacement Polynomials.
5.1.1. **Embedding of** $\hat{\mathcal{H}}_F^{\text{cop}}(\mathbb{K})$ **into** $\mathbb{K}\langle\mathcal{A}\rangle$. Consider the canonical surjective morphism of Lie algebras $\rho_\mathbb{K} : \text{Prim}\mathbb{K}(X) \to \text{Prim}\hat{\mathcal{H}}_F^{\text{cop}}(\mathbb{K})^*$ defined on the generators by

$$\rho_\mathbb{K}(X_i) = t'_i, \quad i \in \mathbb{N}.$$  

By the universal property of enveloping algebras it extends to a surjective morphism of graded Hopf algebras $\rho_\mathbb{K} : \mathbb{K}\langle X \rangle \to \hat{\mathcal{H}}_F^{\text{cop}}(\mathbb{K})^*$ whose transpose is a monomorphism of graded Hopf algebras $\rho_\mathbb{K}^* : \hat{\mathcal{H}}_F^{\text{cop}}(\mathbb{K}) \to \mathbb{K}\langle\mathcal{A}\rangle$.

**Theorem 5.1.** The values of $\rho_\mathbb{K}^*$ on the generators are the displacement polynomials, i.e.

$$\rho_\mathbb{K}^*(t_i) = \mathcal{P}_i(\alpha_1, \ldots, \alpha_i), \quad i \in \mathbb{N};$$

here the product of words in $\mathcal{A}^*$ is defined by concatenation.

**Proof.** For $k \geq 2$ and $\bar{X} := X_{i_k} \cdots X_{i_1}$, deg $(\bar{X}) := i_1 + \cdots + i_k$, we have

$$\bar{X}(\rho_\mathbb{K}^*(t_i)) = \rho_\mathbb{K}(\bar{X})(t_i) = (t'_{i_k} \cdots t'_{i_1})(t_i) = m_\mathbb{K} \circ ((t'_{i_k} \cdots t'_{i_1}) \odot t'_i) \circ \Delta_\text{FDB}^{\text{cop}}(t_i)$$

$$= \sum_{j=0}^{i} (t'_{i_k} \cdots t'_{i_j})(t_j) \cdot \frac{(j+1)!}{(i+1)!} \cdot t'_i(B_{i+1,j+1}(1, 2!t_1, 3!t_2, \ldots, (i-j+1)!t_{i-j})).$$

The last factor is nonzero if $i_1 \leq i$ and $j = i - i_1$, see (4.1), (4.2). In this case the previous expression implies that

$$(X_{i_k} \cdots X_{i_1})(\rho_\mathbb{K}^*(t_i)) = (i - i_1 + 1) \cdot (X_{i_k} \cdots X_{i_2})(\rho_\mathbb{K}^*(t_{i-1})).$$

From the above recursive relations we obtain

$$\bar{X}(\rho_\mathbb{K}^*(t_i)) = (i - i_1 + 1)(i - i_1 - i_2 + 1) \cdots 1 \cdot \delta_{i\text{deg}(\bar{X})}.$$  

Also, $X_i(\rho_\mathbb{K}^*(t_i)) = t'_i(t_i) = 1$. From here passing to the dual basis $\mathcal{A}^* \subset \mathbb{K}\langle\mathcal{A}\rangle$ we get

$$\rho_\mathbb{K}^*(t_i) = \sum_{\alpha_i, \ldots, \alpha_1 = i_1} p_{i_1, \ldots, i_k}(i) \alpha_i \cdots \alpha_i =: \mathcal{P}_i(\alpha_1, \ldots, \alpha_i).$$

\[\square\]

**Remark 5.2.** (1) Since $\rho_\mathbb{K}^* : \hat{\mathcal{H}}_F^{\text{cop}}(\mathbb{K}) \to \mathbb{K}\langle\mathcal{A}\rangle$ is a morphism of Hopf algebras,

$$\rho_\mathbb{K}^* \circ S_{\text{FDB}} = S_{\mathbb{K}\langle\mathcal{A}\rangle} \circ \rho_\mathbb{K}^* \quad \text{and} \quad (\rho_\mathbb{K} \otimes \rho_\mathbb{K}^*) \circ \Delta_\text{FDB}^{\text{cop}} = \Delta_{\mathbb{K}\langle\mathcal{A}\rangle} \circ \rho_\mathbb{K}.$$  

These and Theorem 5.1 lead to the following combinatorial relations in the shuffle Hopf algebra $\mathbb{K}\langle\mathcal{A}\rangle$ between the displacement and the Bell polynomials:

$$\mathcal{P}_i := \frac{1}{(i+1)!} \sum_{j=1}^{i} (-1)^j B_{i+j,j}(0, 2!\mathcal{P}_1, 3!\mathcal{P}_2, \ldots, (i+1)!\mathcal{P}_i),$$

where

$$\mathcal{P}_i(\alpha_1, \ldots, \alpha_i) := (S_{\mathbb{K}\langle\mathcal{A}\rangle} \circ \rho_\mathbb{K}^*)((t_i) = \sum_{\alpha_i, \ldots, \alpha_1 = i_1} (-1)^k \cdot p_{i_1, \ldots, i_1}(i) \alpha_i \cdots \alpha_i, \quad i \in \mathbb{N};$$

$$\Delta_{\mathbb{K}\langle\mathcal{A}\rangle}((\mathcal{P}_i)) = \sum_{j=0}^{i} \mathcal{P}_j \otimes \frac{(j+1)!}{(i+1)!} B_{i+1,j+1}(1, 2!\mathcal{P}_1, 3!\mathcal{P}_2, \ldots, (i-j+1)!\mathcal{P}_{i-j}),$$

where $\mathcal{P}_0 := 1$ and for $i \in \mathbb{N}$,

$$\Delta_{\mathbb{K}\langle\mathcal{A}\rangle}((\mathcal{P}_i))(\alpha_1, \ldots, \alpha_i)$$

$$:= \sum_{\alpha_i, \ldots, \alpha_i = i_1} p_{i_1, \ldots, i_k}(i) \cdot \left(1 \otimes \alpha_i \cdots \alpha_i + \alpha_i \cdots \alpha_i \otimes 1 + \sum_{j=1}^{k} \alpha_i \cdots \alpha_{i+j-1} \otimes \alpha_{i+j} \cdots \alpha_i \right).$$
(2) It is worth noting that the displacement polynomials \( P_i \in \mathbb{K}(\mathcal{A}) \), \( i \in \mathbb{Z}_+ \), are algebraically independent over \( \mathbb{K} \).

5.1.2. Recurrence Relations for the Displacement Polynomials. We set \( P_0 := I \in \mathbb{K}(X) \).

**Theorem 5.3.** The following relations hold for all \( n \in \mathbb{N} \):

\[
(5.5) \quad P_n(X_1, \ldots, X_n) = \sum_{i=1}^{n} (n-i+1) \cdot P_{n-i}(X_1, \ldots, X_{n-i}) \cdot X_i.
\]

**Proof.** First, we show that

\[
(5.6) \quad P_{K}(1 \otimes I - \sum_{j=1}^{\infty} t_j^{i} \otimes X_j t^j)^{-1} \in \mathcal{H}_{FdB}^{\text{cop}}(\mathbb{K})^{*} \otimes L(\mathbb{K}(\mathcal{A}); \mathbb{K}),
\]

is the ‘generating function’ for the displacement polynomials.

Indeed, by the definition,

\[
P_{K} = 1 \otimes I + \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} t_j^{i} \otimes X_j t^j \right)^i = 1 \otimes I + \sum_{i=1}^{\infty} \left( \sum_{i_1+\cdots+i_k=i} t_{i_k}^{i} \cdots t_{i_1}^{i} \otimes X_{i_k} \cdots X_{i_1} \right) t^i.
\]

In particular, due to the computation of Theorem 5.1,

\[
P_{K}(t_n) := 1(t_n) \cdot I + \sum_{i=1}^{\infty} \left( \sum_{i_1+\cdots+i_k=i} (t_{i_k}^{i} \cdots t_{i_1}^{i})(t_n) \otimes X_{i_k} \cdots X_{i_1} \right) t^i = P_n(X_1, \ldots, X_n) \cdot t^n.
\]

Next, we rewrite the expression for \( P_{K} \) as follows

\[
P_{K} = 1 \otimes I + P_{K} \times \sum_{j=1}^{\infty} t_j^{i} \otimes X_j t^j;
\]

here \( \times \) denotes the product in the algebra \( \mathcal{H}_{FdB}^{\text{cop}}(\mathbb{K})^{*} \otimes L(\mathbb{K}(\mathcal{A}); \mathbb{K}) \).

From this presentation we obtain

\[
P_{K}(t_n) = \left( \sum_{i=1}^{n} \left( \sum_{i_1+\cdots+i_k=n-i} t_{i_k}^{i} \cdots t_{i_1}^{i} \otimes X_{i_k} \cdots X_{i_1} \right) t^{n-i} \times (t_i^{i} \otimes X_i t^i) \right) (t_n)
\]

\[
= \sum_{j=1}^{n} \left( \sum_{i_1+\cdots+i_k=n-i} (t_{i_k}^{i} \cdots t_{i_1}^{i})(t_n) \otimes X_{i_k} \cdots X_{i_1} \cdot X_j \right) t^n
\]

\[
= \sum_{i=1}^{n} \left( \sum_{i_1+\cdots+i_k=n-i} (t_{i_k}^{i} \cdots t_{i_1}^{i})(t_{n-i}) \cdot (n-i+1) \otimes X_{i_k} \cdots X_{i_1} \cdot X_i \right) t^n
\]

\[
= \sum_{i=1}^{n} (n-i+1) \cdot P_{n-i}(X_1, \ldots, X_{n-i}) \cdot X_i \cdot t^n.
\]

Comparing the latter expression for \( P_{K}(t_n) \) with the former one we attain the required recurrence relations. \( \square \)
5.2. First Return Map. Consider equation (1.1) corresponding to \( a = (a_1, a_2, \ldots) \in \mathcal{X} \):

\[
\frac{dv}{dx} = \sum_{i=1}^{\infty} a_i(x) v^{i+1}, \quad x \in I_T.
\]

We relate to this equation the following one:

\[
\frac{dH}{dx} = \left( \sum_{i=1}^{\infty} a_i(x) t_i^t \right) H, \quad x \in I_T.
\]

Solving (5.8) by Picard iteration we get a solution whose coefficients in the series expansion are Lipschitz functions on \( I_T \). One easily checks that (cf. (5.7))

\[
H_a(T) = 1 + \sum_{i=1}^{\infty} \left( \sum_{i_1+\cdots+i_k = i} I_{i_1,\ldots,i_k}(a) t_{i_k}^t \cdots t_{i_1}^t \right) t^i = (\rho_C \circ E)(a).
\]

In particular, \( H_a(T) \in G_{H_{\mathbb{D}}(\mathbb{C})}(\mathbb{C}) \). Thus, the following result for the first return map \( \mathcal{P}(a) \) of (5.7) holds:

**Theorem 5.4.**

\[
\mathcal{P}(a) = \Theta(H_a(T)) = (\Theta \circ \rho_C \circ E)(a).
\]

**Proof.** Using the computation of the proof of Theorem 5.1 we obtain, see (1.4),

\[
\Theta(H_a(T))(r) = r + \sum_{i=1}^{\infty} \left( \sum_{i_1+\cdots+i_k = i} I_{i_1,\ldots,i_k}(a) t_{i_k}^t \cdots t_{i_1}^t \right) r^{i+1}
\]

\[
= r + \sum_{i=1}^{\infty} \left( \sum_{i_1+\cdots+i_k = i} p_{i_1,\ldots,i_k}(i) \cdot I_{i_1,\ldots,i_k}(a) \right) r^{i+1} =: \mathcal{P}(a)(r).
\]

5.3. Group of Formal Centers. Let \( J_{\mathbb{K}} \subset \mathbb{K}(\mathcal{A}) \) be the ideal generated by the displacement polynomials \( \mathcal{P}_i, \quad i \in \mathbb{N} \) (cf. (5.1)). Due to (5.2), \( J_{\mathbb{K}} \) is a graded Hopf ideal of \( \mathbb{K}(\mathcal{A}) \), i.e. \( J_{\mathbb{K}} = \oplus_{i \geq 1} \mathbb{K}_i(\mathcal{A}) \cap J_{\mathbb{K}} \). Thus the quotient \( \mathbb{K}(\mathcal{A})/J_{\mathbb{K}} = \oplus_{i \geq 0} \mathbb{K}_i(\mathcal{A})/(\mathbb{K}_i(\mathcal{A}) \cap J_{\mathbb{K}}) \) is a connected graded Hopf algebra. The dual Hopf algebra \( (\mathbb{K}(\mathcal{A})/J_{\mathbb{K}})^\ast \) is the connected graded Hopf subalgebra of \( \mathbb{K}(X) \) consisting of elements vanishing on \( J_{\mathbb{K}} \). By the definition of \( J_{\mathbb{K}} \),

\[
(\mathbb{K}(\mathcal{A})/J_{\mathbb{K}})^\ast = \mathbb{K}_0(X) \oplus \ker \rho_{\mathbb{K}}.
\]

Hence, \( (\mathbb{K}(\mathcal{A})/J_{\mathbb{K}})^\ast \cap \mathbb{K}_i(X), \quad i \geq 1, \) consists of elements

\[
f_i = \sum_{i_1+\cdots+i_k = i} c_{i_1,\ldots,i_k}X_{i_k} \cdots X_{i_1} \text{ such that } \sum_{i_1+\cdots+i_k = i} p_{i_1,\ldots,i_k}(i) \cdot c_{i_1,\ldots,i_k} = 0.
\]

Next, by the Milnor-Moore theorem \( (\mathbb{K}(\mathcal{A})/J_{\mathbb{K}})^\ast \) is the universal enveloping algebra of the Lie algebra \( \text{Prim}((\mathbb{K}(\mathcal{A})/J_{\mathbb{K}})^\ast) \) of its primitive elements. By definition,

\[
\text{Prim}((\mathbb{K}(\mathcal{A})/J_{\mathbb{K}})^\ast) = \text{Prim} \mathbb{K}(X) \cap \ker \rho_{\mathbb{K}}.
\]

An explicit computation using that \( [t_i^t, t_j^t] = (i-j) t_{i+j}^t \) for all \( i, j \in \mathbb{N} \) (cf. [Br3 Sect. 3.2.2] for similar arguments) shows that \( \text{Prim}((\mathbb{K}(\mathcal{A})/J_{\mathbb{K}})^\ast \cap \mathbb{K}_i(X) \) consists of elements

\[
g_i = \sum_{i_1+\cdots+i_k = i} c_{i_1,\ldots,i_k}[X_{i_k}, [X_{i_{k-1}}, \ldots, [X_{i_2}, X_{i_1}] \cdots]]
\]
such that
\[\rho_K(g_i) = \sum_{i_1+\ldots+i_k = i} c_{i_1\ldots i_k} \cdot \gamma_{i_1\ldots i_k} = 0, \quad \text{where } \gamma_i = 1 \quad \text{and} \]
\[\gamma_{i_1\ldots i_k} = (i_2-i_1)(i_3-i_2-i_1)\cdots(i_k-i_{k-1}-\cdots-i_1) \quad \text{for } k \geq 2.\]

In turn, the Lie algebra \(g_{K\langle A^\omega \rangle}/J_{K}\) of infinitesimal characters of \(K\langle A^\omega \rangle/J_{K}\) is the closure of \(\text{Prim}(K\langle A^\omega \rangle/J_{K})^*\) in \(\text{Prim}(K\langle X \rangle)\), i.e. it consists of series \(\sum_{i=1}^{\infty} g_i t^i\) with \(g_i \in \text{Prim}(K\langle A^\omega \rangle/J_{K})^* \cap K_i(X)\), \(i \in \mathbb{N}\). Also, the group of characters of \(K\langle A^\omega \rangle/J_{K}\) is a subgroup of \(G_{K\langle A^\omega \rangle}(K)\) consisting of characters vanishing on \(J_{K}\), i.e.
\[G_{K\langle A^\omega \rangle}/J_{K} = G_{K\langle A^\omega \rangle}(K) \cap \text{Ker} \rho_K.\]

In particular, if \(K = \mathbb{C}\), then, due to Theorem 5.3, \(G_{C\langle A^\omega \rangle}/J_{C} = F_{\mathbb{C}^1}\), the group of formal centers of equation (1.1).

By definition, \(g_{K\langle A^\omega \rangle}/J_{K}(K)\) is the Lie algebra of the Lie group \(G_{K\langle A^\omega \rangle}/J_{K}(K)\) and the exponential map \(g_{K\langle A^\omega \rangle}/J_{K}(K)\) homeomorphically onto \(g_{K\langle A^\omega \rangle}/J_{K}(K)\).

Note that the linear continuous map \(\pi: g_{K\langle A^\omega \rangle}(K) \rightarrow g_{K\langle A^\omega \rangle}/J_{K}(K)\),
\[\pi \left( \sum_{i=1}^{\infty} g_i t^i \right) := \sum_{i=1}^{\infty} (g_i - \rho_K(g_i) \cdot X_i) t^i, \quad \sum_{i=1}^{\infty} g_i t^i \in g_{K\langle A^\omega \rangle}(K),\]
is a projection so that as a topological vector space \(g_{K\langle A^\omega \rangle}(K)\) is isomorphic to the direct sum of topological vector spaces \(g_{K\langle A^\omega \rangle}/J_{K}(K) \oplus g_{H\langle A^\omega \rangle}/J_{K}(K)\). This implies that as a topological space \(G_{K\langle A^\omega \rangle}(K)\) is homeomorphic to the direct product of topological spaces \(G_{K\langle A^\omega \rangle}/J_{K}(K) \times G_{K\langle A^\omega \rangle}\).

Remark 5.5. According to the Shirshov-Witt theorem (see, e.g., [10]), \(\text{Prim}(K\langle A^\omega \rangle/J_{K})^*\) is a free Lie algebra. It is easily seen that the set of generators of \(\text{Prim}(K\langle A^\omega \rangle/J_{K})^*\) is countable and each generator can be chosen to be homogeneous (i.e. belonging to some \(\text{Prim}(K\langle A^\omega \rangle/J_{K})^* \cap K_i(X)\)). Let \(Y = \{Y_i : Y_i \in K_{n_i}(X), \ i \in \mathbb{N}\}\) be the set of generators of \(\text{Prim}(K\langle A^\omega \rangle/J_{K})^*\) ordered such that \(n_i \leq n_j\) for \(i \leq j\) \((i,j \in \mathbb{N})\). Then the bijective map \(X \rightarrow Y, \ X_i \mapsto Y_i, \ i \in \mathbb{N}\), extends to an isomorphism of (non-graded) Hopf algebras \(\varphi: K\langle X \rangle \rightarrow (K\langle A^\omega \rangle/J_{K})^*\) and induces an isomorphism between topological groups \(G_{K\langle A^\omega \rangle}(K)\) and \(G_{K\langle A^\omega \rangle}/J_{K}(K)\). (Thus the group of formal centers of equation (1.1) is isomorphic to the group of characters of the shuffle Hopf algebra \(C\langle A^\omega \rangle\).) In turn, the transpose map \(\varphi^*\) determines a Hopf algebra isomorphism between \(K\langle A^\omega \rangle/J_{K}\) and the shuffle Hopf algebra \(K\langle A^\omega \rangle\).

5.4. Equations with Finitely Many Terms. The Hopf algebra approach to the center problem for equations (1.1) with finitely many terms is similar to the one already described so we just sketch it leaving details to the readers.

Let \(A^N\) be the subalphabet of \(A\) consisting of letters \(\alpha_1, \ldots, \alpha_N\). Let \(K\langle A^N \rangle\) be the graded Hopf subalgebra of \(K\langle A^\omega \rangle\) generated by words in the letters of \(A^N\). We have the natural projection \((q_N)_K: K\langle A\rangle \rightarrow K\langle A^N \rangle\) sending each word in \(A^*\) containing letters of \(A^N\) to zero and identity on the other words. The dual of \(K\langle A^N \rangle\) is the Hopf subalgebra \(K\langle X_N \rangle, \ X_N := \{X_1, \ldots, X_N\}\), of \(K\langle X \rangle\) generated by polynomials in \(I\) and variables in \(X_N\). We have the natural projection \((Q_N)_K: K\langle X \rangle \rightarrow K\langle X_N \rangle\) defined by equating all variables \(X_i, \ i > N\), to zero. The Hopf map \((Q_N)_K\) extends by continuity to the projection \(L(K\langle A^N \rangle; K) \rightarrow L(K\langle A^\omega \rangle; K)\) denoted by the same symbol.

Next, the Lie algebra Prim \(K\langle A^N \rangle \subset \text{Prim}(K\langle A\rangle)\) is the free Lie algebra generated by \(X_1, \ldots, X_N\) with the grading induced from \(\text{Prim}(K\langle A\rangle)\). The Lie algebra \(g_{K\langle A^N \rangle}(K)\) of infinitesimal characters of \(K\langle A^N \rangle\) is the closure of \(\text{Prim}(K\langle A^N \rangle)\) in \(g_{K\langle A^\omega \rangle}(K)\). Also, it
coincides with \( (Q_N)_\mathbb{K}(\mathfrak{g}_{\mathbb{K}⟨x⟩}(\mathbb{K})) \). Similarly, the Lie group \( G_{\mathbb{K}⟨x⟩}(\mathbb{K}) \) of characters of \( \mathbb{K}⟨x⟩ \) coincides with \( (Q_N)_\mathbb{K}(G_{\mathbb{K}⟨x⟩}(\mathbb{K})) \).

We set (see Section 5.1)

\[
(\rho_N)_\mathbb{K} := \rho_{\mathbb{K}}|_{\mathbb{K}(X_N)} : \mathbb{K}(X_N) \to \mathcal{H}_{\text{FD}B}^\text{cop}(\mathbb{K})^*.
\]

Then the transpose map \((\rho_N)_\mathbb{K}^* : \mathcal{H}_{\text{FD}B}^\text{cop}(\mathbb{K}) \to \mathbb{K}⟨x⟩ \) satisfies

\[
(\rho_N)_\mathbb{K}^* = (q_N)_\mathbb{K} \circ \rho_{\mathbb{K}}^*.
\]

In particular, setting \( P_i^N := P_i \) for \( i \leq N \) and \( P_i^N(X_1, \ldots, X_N) := P_i(X_1, \ldots, X_N, 0, 0, \ldots) \) for \( i > N \), we obtain

**Proposition 5.6.** Map \((\rho_N)_\mathbb{K}^* \) is a monomorphism of graded Hopf algebras given on the generators by the formulas

\[
(\rho_N)_\mathbb{K}^*(t_i) = (q_N)_\mathbb{K}(P_i(\alpha_1, \ldots, \alpha_i)) =: P_i^N(\alpha_1, \ldots, \alpha_N), \quad i \in \mathbb{N}.
\]

Substituting \( X_i = 0 \) for all \( i > N \) in Theorem 5.3 we get the following recurrence relations (originally established in [D] by a different method):

\[
P_i^N(X_1, \ldots, X_N) = \sum_{i=1}^{\min(n,N)} (n-i+1) \cdot P_{i-1}(X_1, \ldots, X_{i-1}) \cdot X_i, \quad n \in \mathbb{N}.
\]

Let \( \mathcal{X}_N \subset \mathcal{X} \) be the subsemigroup of sequences \( a = (a_1, a_2, \ldots) \) with \( a_i = 0 \) for \( i > N \). Each \( a \in \mathcal{X}_N \) determines equation

\[
\frac{dv}{dx} = \sum_{i=1}^N a_i(x)v^i, \quad x \in I_T.
\]

In turn, map \( E : \mathcal{X} \to L(\mathbb{C}⟨x⟩; \mathbb{C}) \), see (5.7), sends \( \mathcal{X}_N \) into the group of characters \( G_{\mathbb{C}⟨x⟩}(\mathbb{C}) \) of \( \mathbb{C}⟨x⟩ \). The subgroup \( \hat{\mathcal{X}}^N := E(\mathcal{X}_N \cap \mathcal{C}) \subset \mathcal{C} \) is called the group of centers of equation (5.16). Its closure in \( G_{\mathbb{C}⟨x⟩}(\mathbb{C}) \) denoted by \( \hat{\mathcal{X}}^N \) is called the group of formal centers of this equation. The Lie algebra of \( \hat{\mathcal{X}}^N \) is then the Lie algebra \( \mathcal{g}_{\mathbb{C}⟨x⟩}/\mathcal{J}^N_{\mathcal{C}} \) of infinitesimal characters of the quotient Hopf algebra \( \mathbb{C}⟨x⟩/\mathcal{J}^N_{\mathcal{C}} \), where \( \mathcal{J}^N_{\mathcal{C}} \) is the Hopf ideal generated by the displacement polynomials \( P_i^N, i \in \mathbb{N} \). One easily checks that \( \mathcal{J}^N_{\mathcal{C}} = (q_N)_{\mathcal{C}}(\mathcal{J}_{\mathcal{C}}) \) and that \( \hat{\mathcal{X}}^N \) is the group of characters of \( \mathbb{C}⟨x⟩/\mathcal{J}^N_{\mathcal{C}} \).

Among other results, let us mention that group \( G_{\mathbb{C}⟨x⟩}(\mathbb{C}) \) is isomorphic to the semidirect product of group \( G_{\mathbb{C}⟨x⟩}(\mathbb{C}) \) (corresponding to the Abel differential equation) and a certain normal subgroup of \( \hat{\mathcal{X}}_f \) (see [Br3, Prop. 3.9]) and that \( G_{\mathbb{C}⟨x⟩}(\mathbb{C}) \) as a topological space is homeomorphic to the product of topological spaces \( \hat{\mathcal{X}}_f^2 \times G_{\mathbb{C}[x]} \) (see [Br3, Prop. 3.12]).

An important class of equations (1.1) with finitely many terms is determined by the subsemigroup of rectangular paths \( \mathcal{X}_{\text{rect}} \subset \mathcal{X} \) of elements \( a \in \mathcal{X} \) whose first integrals \( \tilde{a} : I_T \to \mathbb{C}^\infty \) are paths consisting of segments each going in the direction of some particular coordinate. Every center determined by \( a \in \mathcal{X}_{\text{rect}} \) is universal (i.e. \( \mathcal{C} \cap \mathcal{X}_{\text{rect}} \subset \mathcal{X} \)). This fact was established in [Br4] by means of the deep result of Cohen [Co]. Note that \( G(\mathcal{X}_{\text{rect}}) := E(\mathcal{X}_{\text{rect}}) \subset G_{\mathbb{C}⟨x⟩}(\mathbb{C}) \) is a dense subgroup generated by elements \( e^{c_n x_n t_n} \), \( c_n \in \mathbb{C}, n \in \mathbb{N} \). (In particular, \( G(\mathcal{X}_{\text{rect}}) \) is isomorphic to the free product of countably many copies of \( \mathbb{C} \).) Thus we have \( G(\mathcal{X}_{\text{rect}}) \cap \hat{\mathcal{X}}_f = \{I\} \).
6. Generalized Displacement Polynomials

In this section we reveal the algebraic nature of the generalized displacement polynomials, prove recurrence relations for them and show that, when considered in the shuffle Hopf algebra $\mathbb{K}(\mathcal{A})$, their values in $t$ can be computed by means of the displacement polynomials $\mathcal{P}_i$. Also, we compute the values $\varepsilon_{\mathbb{K}}(\mathcal{P}_i), \ i \in \mathbb{N}$, of the augmentation homomorphism $\varepsilon_{\mathbb{K}} : \mathbb{K}(X) \to \mathbb{K}$.

6.1. Recurrence Relations. Let $\mathbb{K}[[z]]$ be the algebra of formal power series in variable $z$ with coefficients in the field of characteristic zero $\mathbb{K}$. By $D, L : \mathbb{K}[[z]] \to \mathbb{K}[[z]]$ we denote the differentiation and the left translation operators defined on $f(z) = \sum_{k=0}^{\infty} c_k z^k$ by

$$ (Df)(z) := \sum_{k=0}^{\infty} (k+1)c_k z^k \quad \text{and} \quad (Lf)(z) := \sum_{k=0}^{\infty} c_k z^k. $$

Let $\mathcal{A}_\mathbb{K}(D, L)$ be the associative algebra with unit $I := \text{Id}|_{\mathbb{K}[[z]]}$ of polynomials with coefficients in $\mathbb{K}$ in variables $I, D$ and $L$. We define $[A, B] := AB - BA$ for $A, B \in \mathcal{A}_\mathbb{K}(D, L)$. Then we have (see, e.g., [Br3, Lm. 3.2])

$$ [DL^i, DL^j] = (i - j)DL^{i+j+1} \quad i, j \in \mathbb{Z}_+. $$

In particular, the vector space $\mathcal{L}_\mathbb{K} \subset \mathcal{A}_\mathbb{K}(D, L)$ over $\mathbb{K}$ generated by $DL^i, \ i \in \mathbb{Z}_+$, and equipped with the above bracket is a Lie algebra. It is easily seen that the linear map $\sigma_\mathbb{K} : \text{Prim} \mathcal{H}_{\text{FdB}}(\mathbb{K})^* \to \mathcal{L}_\mathbb{K}$ defined on the generators by $\sigma(t'_i) := DL^{i-1}, \ i \in \mathbb{N}$, is an isomorphism of Lie algebras. By the universal property of enveloping algebras it extends to the morphism of associative algebras $\sigma_\mathbb{K} : \mathcal{H}_{\text{FdB}}(\mathbb{K})^* \to \mathcal{A}_\mathbb{K}(D, L)$. The following result exposes the algebraic nature of the generalized displacement polynomials.

**Proposition 6.1. Element**

$$(\sigma_\mathbb{K} \otimes \text{Id})(\mathcal{P}_\mathbb{K}) := (1 \otimes I - \sum_{j=1}^{\infty} DL^{j-1} \otimes X_j t^j)^{-1} \in \mathcal{A}_\mathbb{K}(D, L) \otimes L(\mathcal{A}); \mathbb{K}),$$

see (5.6), satisfies

$$ (\sigma_\mathbb{K} \otimes \text{Id})(\mathcal{P}_\mathbb{K})(z^m) = \sum_{i=0}^{m} \mathcal{P}_i(X_1, \ldots, X_i; m) \cdot z^{m-i} \cdot t^i \quad \text{for all} \quad m \in \mathbb{Z}_+. $$

**Proof.** By definition,

$$ (\sigma_\mathbb{K} \otimes \text{Id})(\mathcal{P}_\mathbb{K})(z^m) = I(z^m) + \sum_{i=1}^{\infty} \left( \sum_{j=1}^{\infty} DL^{j-1} \otimes X_j t^j \right)^i (z^m) $$

$$ = z^m + \sum_{i=1}^{\infty} \left( \sum_{i_1 + \cdots + i_k = i} (DL^{i_k-1} \cdots DL^{i_1-1})(z^m) \otimes X_{i_k} \cdots X_{i_1} \right) t^i $$

$$ = z^m + \sum_{i=1}^{m} \left( \sum_{i_1 + \cdots + i_k = i} (m - i_1 + 1)(m - i_1 - i_2 + 1) \cdots (m - i + 1) z^{m-i} X_{i_k} \cdots X_{i_1} \right) t^i $$

$$ := \sum_{i=0}^{m} \mathcal{P}_i(X_1, \ldots, X_i; m) \cdot z^{m-i} \cdot t^i. $$

□
Using this proposition we establish two basic recurrence relations for the generalized displacement polynomials.

**Theorem 6.2.** For all \( n \in \mathbb{N}, \ t \in \mathbb{C}, \)

\[
\tilde{P}_n(X_1, \ldots, X_n; t) = \sum_{j=1}^{n} \tilde{P}_{n-j}(X_1, \ldots, X_{n-j}; t - j) \cdot X_j
\]

and

\[
\tilde{P}_n(X_1, \ldots, X_n; t) = (t - n + 1) \cdot \sum_{j=1}^{n} X_j \cdot \tilde{P}_{n-j}(X_1, \ldots, X_{n-j}; t).
\]

**Proof.** We set for brevity \( \tilde{\mathcal{P}} := (\sigma_\mathcal{C} \otimes \text{Id})(\mathcal{P}_\mathcal{C}) \). Applying to the identity

\[
\mathcal{P}_\mathcal{C} = 1 \otimes I + \mathcal{P}_\mathcal{C} \times \sum_{j=1}^{\infty} t_j \otimes X_j t^j
\]

operator \( \sigma_\mathcal{C} \otimes \text{Id} \) and using Proposition 6.1 we obtain for all \( m \in \mathbb{Z}_+ \) (below \( \times \) stands for the product on \( \mathcal{A}_\mathcal{C}(D, L) \otimes \mathcal{L}(\mathcal{C}(\mathcal{A}); \mathcal{C}) \)),

\[
\sum_{i=0}^{m} \bar{P}_i(X_1, \ldots, X_i; m) \cdot z^{m-i} \cdot t^i = \tilde{\mathcal{P}}(z^m) = \left( 1 \otimes I + \tilde{\mathcal{P}} \times \sum_{j=1}^{\infty} DL^{j-1} \otimes X_j t^j \right) (z^m)
\]

\[
= z^m + \sum_{j=1}^{m} \tilde{\mathcal{P}}(z^{m-j}) \cdot (m - j + 1) \cdot X_j t^j
\]

\[
= z^m + \sum_{j=1}^{m} (m - j + 1) \left( \sum_{k=0}^{m-j} \bar{P}_k(X_1, \ldots, X_k; m - j) z^{m-j-k} \cdot r^k \right) \cdot X_j t^j
\]

\[
= z^m + \sum_{i=1}^{m} \left( \sum_{j=1}^{i} \bar{P}_{i-j}(X_1, \ldots, X_{i-j}; m - j) \cdot X_j \right) z^{m-i} \cdot t^i.
\]

Choosing here \( m \geq n \) and equating coefficients of \( t^n \) in both sides of the above equation we get

\[
\tilde{P}_n(X_1, \ldots, X_n; m) = \sum_{j=1}^{n} (m - j + 1) \cdot \tilde{P}_{n-j}(X_1, \ldots, X_{n-j}; m - j) \cdot X_j.
\]

Since coefficients of \( \tilde{P}_1, \ldots, \tilde{P}_n \) are polynomials in \( m \) and the above identity is valid for all sufficiently large natural \( m \), it is valid for all complex numbers \( m \) as well. This completes the proof of \( (6.3) \).

To prove the second relation we use a different presentation of \( \mathcal{P}_\mathcal{C} \):

\[
\mathcal{P}_\mathcal{C} = 1 \otimes I + \left( \sum_{j=1}^{\infty} t_j \otimes X_j t^j \right) \times \mathcal{P}_\mathcal{C}.
\]
As before, applying to this equation operator $\sigma_C \otimes \text{Id}$ and using Proposition 6.1 we obtain for all $m \in \mathbb{Z}_+$,

$$
\sum_{i=0}^{m} \mathbb{P}_i(X_1, \ldots, X_i; m) \cdot z^{m-i} \cdot t^i = \mathbb{P}(z^m) = \left(1 \otimes I + \left(\sum_{j=1}^{\infty} DL^{j-1} \otimes X_j t^j\right) \times \mathbb{P}\right)(z^m)
$$

\[= z^m + \sum_{j=1}^{\infty} X_j \cdot DL^{j-1}(\mathbb{P}(z^m)) t^j \]

\[= z^m + \sum_{j=1}^{\infty} X_j \cdot \left(\sum_{k=0}^{m} DL^{j-1}(z^{m-k}) \cdot \mathbb{P}_k(X_1, \ldots, X_k; m) t^k\right) t^j \]

\[= z^m + \sum_{j=1}^{m} X_j \cdot \left(\sum_{k=0}^{m-j} (m - j - k + 1) \cdot \mathbb{P}_k(X_1, \ldots, X_k; m) z^{m-j-k} \cdot t^k\right) t^j \]

\[= z^m + \sum_{i=1}^{m} (m - i + 1) \left(\sum_{j=1}^{i} X_j \cdot \mathbb{P}_{i-j}(X_1, \ldots, X_{i-j}; m)\right) z^{m-i} \cdot t^i.\]

Choosing here $m \geq n$ and equating coefficients of $t^n$ in both sides of the above equation we get

$$
\mathbb{P}_n(X_1, \ldots, X_n; m) = (m - n + 1) \cdot \sum_{j=1}^{n} X_j \cdot \mathbb{P}_{n-j}(X_1, \ldots, X_{n-j}; m).
$$

Again, since coefficients of $\mathbb{P}_1, \ldots, \mathbb{P}_n$ are polynomials in $m$ and the above identity is valid for all sufficiently large natural $m$, it is valid for all complex numbers $m$ as well. This completes the proof of (6.4) and of the theorem. \hfill \Box

6.2. Generalized Displacement Polynomials in the Shuffle Hopf Algebra. In this section we study polynomials $\mathbb{P}_i(\alpha_1, \ldots, \alpha_i; t) \in (K\langle \mathcal{A} \rangle, \cup)$, $i \in \mathbb{Z}_+$, where $\mathbb{P}_0 := 1$ (as before, the product of words in $\mathcal{A}^*$ is defined by concatenation).

6.2.1. Expression via the Bell Polynomials. The main result of this section shows that the generalized displacement polynomials belong to the image of the natural monomorphism $\rho^*_K : \mathcal{H}_{\text{FD}B}(K) \rightarrow K\langle \mathcal{A} \rangle$, see Section 5.1.1. To formulate the result, we introduce polynomials $B_k \in \mathbb{Q}[t_1, \ldots, t_k, t]$, $k \in \mathbb{N}$,

\[B_k(t_1, \ldots, t_k, t) := \sum_{l_1 + 2l_2 + \cdots + kl_k = k} \prod_{l=1}^{0} (t + l) \frac{t_1^{l_1} \cdots t_k^{l_k}}{l_1! \cdots l_k!} \]

Also, we set $B_0 := 1$.

Proposition 6.3. For all $i, j \in \mathbb{Z}_+$, $i - j \geq 1$ polynomials (6.5) satisfy

\[B_{i-j}(t_1, \ldots, t_{i-j}, j + 1) = \frac{(j + 1)!}{(i + 1)!} B_{i+1,j+1}(1, 2!t_1, 3!t_2, \ldots, (i - j + 1)!t_{i-j}).\]
Theorem 6.4. The following identities hold in $(\mathbb{K}(\mathcal{A}), \mu)$ for all $i \in \mathbb{Z}_+$, $t \in \mathbb{K}$:

\begin{equation}
\bar{P}_i(\alpha_1, \ldots, \alpha_i; t) = B_i(\mathcal{P}_1(\alpha_1), \ldots, \mathcal{P}_i(\alpha_1, \ldots, \alpha_i), t - i + 1).
\end{equation}

In particular,

\begin{equation}
\bar{P}_i(\alpha_1, \ldots, \alpha_i; t) \in \rho^*_k(H^\text{cop}_{\text{FdB}}(\mathbb{K})) \cap \mathbb{K}_i(\mathcal{A}) \quad \text{for all} \quad t \in \mathbb{K}, i \in \mathbb{Z}_+.
\end{equation}

**Proof.** Let us consider algebra $\mathcal{A}_C(D, L)$ with the grading defined on the generators by $\deg(D) = \deg(L) = 1$, $\deg(I) = 0$. The closure of $\mathcal{A}_C(D, L)$ in the adic topology induced by the grading is naturally identified with the subalgebra $\mathcal{A}_*$ of the algebra $\mathcal{A}_C(D, L)[[t]]$ of formal power series in $t$ with coefficients in $\mathcal{A}_C(D, L)$ of series of the form

$$f(t) = \sum_{i=0}^{\infty} f_i t^i, \quad f_i \in \mathcal{A}_C(D, L), \quad \deg(f_i) = i, \quad i \in \mathbb{Z}_+.$$ 

Map $\sigma_C$ (see Section 6.1) extends by continuity to the morphism of topological algebras $L(H^\text{cop}_{\text{FdB}}(\mathbb{C}); \mathbb{C}) \to \mathcal{A}_*$ (denoted by $\sigma_C$ as well) which maps the Lie algebra $g_{H^\text{cop}_{\text{FdB}}(\mathbb{C})}^{\text{cop}}(\mathbb{C})$ of infinitesimal characters of $H^\text{cop}_{\text{FdB}}(\mathbb{C})$ isomorphically onto the closure $\text{cl}(\mathcal{L}_C)$ of the Lie algebra $\mathcal{L}_C = \text{span}_\mathbb{C}\{DL^{-1}, i \in \mathbb{N}\}$ in $\mathcal{A}_*$. The latter consists of series of the form

$$g(t) = \sum_{i=1}^{\infty} d_i DL^{-1} t^i, \quad d_i \in \mathbb{C}, \quad i \in \mathbb{N},$$

with the bracket extending by continuity the bracket on $\mathcal{L}_C$. In particular, $\sigma_C$ maps the group $G_{H^\text{cop}_{\text{FdB}}(\mathbb{C})}^{\text{cop}}(\mathbb{C})$ of characters of $H^\text{cop}_{\text{FdB}}(\mathbb{C})$ isomorphically onto the group $\text{exp}(\text{cl}(\mathcal{L}_C)) \subset \mathcal{A}_*$. Further, $\sigma_C$ transfers ordinary differential equation (5.8) with values in $L(H^\text{cop}_{\text{FdB}}(\mathbb{C}); \mathbb{C})$ corresponding to $a \in \mathcal{X}$ to equation

\begin{equation}
\frac{dF}{dx} = \left( \sum_{i=1}^{\infty} a_i(x) DL^{i-1} t^i \right) F, \quad x \in I_T.
\end{equation}

The fundamental solution $H_a$ of (5.8) then goes to the fundamental solution $F_a := \sigma_C \circ H_a$ of (6.9) so that its value at $x = T$ is given by the formula

\begin{equation}
F_a(T; t) = I + \sum_{i=1}^{\infty} \left( \sum_{i_1 + \cdots + i_k = i} I_{i_1, \ldots, i_k}(a) DL^{i_k-1} \cdots DL^{i_1-1} \right) t^i.
\end{equation}

Given $a \in \mathcal{X}$ the expression in the brackets of (6.10) is a linear operator, in what follows denoted by $I_i(a)$, on the algebra of complex polynomials $\mathbb{C}[z]$. The next result computes its values $I_i(a)(z^m)$ on the basis $\{z^m\}_{m \in \mathbb{Z}_+}$ of $\mathbb{C}[z]$. 

Proposition 6.5. (1) If $m = 0, \ldots, i - 1$, then $I_i(a)(z^m) = 0$.

(2) If $m \geq i$, then

$$I_i(a)(z^m) = \left( \sum_{i_1 + \cdots + i_k = i} \frac{p_{i_1, \ldots, i_k}(m)}{(m + 1)!} B_{m+1, m-i+1}(1, 2! p_1(a), \ldots, (i+1)! p_i(a)) \cdot z^{m-i} \right),$$

where $p_k(a)$ is the coefficient of $x^{k+1}$ in the series expansion of the first return map $P(a)$, see (1.4).

Proof. (1) This follows directly from the definition of $I_i(a)$.

(2) Applying $F_a$ to $s(z, r) := r (1 - rz)^{-1} = \sum_{m=0}^{\infty} r^{m+1} \cdot z^m \in \mathbb{C}[z]$ we obtain, see, e.g., [Br2 Sect. 4],

$$(6.11) \quad F_a(T; t)(s(z, r)) = s(z, P(a)(r t)/t) \in \mathbb{C}[t, z, r].$$

For $t$ and $z$ fixed, the latter is the composition of series $s(z, r) \in \mathbb{C}[[r]]$ and $P(a)(r t)/t \in \mathbb{C}[[r]]$. Hence, using the Faà di Bruno formula, cf. (4.1), we obtain

$$s(z, r) + \sum_{i=1}^{\infty} I_i(a)(s(z, r)) t^i =: F_a(T; t)(s(z, r))$$

$$= r + \sum_{m=1}^{\infty} \left( \sum_{k=0}^{m} \frac{(k + 1)!}{(m + 1)!} z^{k \cdot m - k} B_{m+1, m-i+1}(1, 2! p_1(a), \ldots, (m-k+1)! p_{m-k}(a)) \right) r^{m+1}.$$ 

Equating coefficients of $t^i$ in both sides of this identity we get

$$I_i(a)(s(z, r)) = \sum_{m=i}^{\infty} \left( \frac{(m - i + 1)!}{(m + 1)!} B_{m+1, m-i+1}(1, 2! p_1(a), \ldots, (i+1)! p_i(a)) \right) z^{m-i} \cdot r^{m+1}.$$ 

On the other hand, the straightforward application of operators $DL^{i_k-1} \cdots DL^{i_1-1}$ to the basis $\{z^m\}_{m \in \mathbb{Z}_+}$ of $\mathbb{C}[z]$ shows that the left-hand side of the above expression equals

$$\sum_{m=0}^{\infty} I_i(a)(z^m) r^{m+1} = \sum_{m=i}^{\infty} \left( \sum_{i_1 + \cdots + i_k = i} \frac{p_{i_1, \ldots, i_k}(m)}{(m + 1)!} B_{m+1, m-i+1}(1, 2! p_1(a), \ldots, (i+1)! p_i(a)) \right) z^{m-i} \cdot r^{m+1}.$$ 

Comparing coefficients of $r^{m+1}$ in the latter and the former equations we get the required identity. □

Further, recall that there is an isomorphism between the algebra $\mathcal{L}_C(\mathcal{X})$ generated by iterated integrals on $\mathcal{X}$ and the shuffle algebra $(\mathbb{C}(\mathcal{A}), \cup \cup)$ sending functions $I_{i_1, \ldots, i_k}$ on $\mathcal{X}$ to words $\alpha_{i_k} \ldots \alpha_{i_1} \in \mathcal{A}^*$, $i_1, \ldots, i_k, k \in \mathbb{N}$ (see Section 3.3). Applying this algebra isomorphism to the coefficients of $z^{m-i}$ in the identity of Proposition 6.5 and noting that their images belong to $\mathbb{Q}(\mathcal{A}) (\subset \mathbb{K}(\mathcal{A}))$, from Proposition 6.3 we obtain that equation (6.7) is valid for all integers $t \geq i$, $i \in \mathbb{N}$. Since in both sides of (6.7) are polynomials in $t$ and this equation is valid for infinitely many values of $t$, it is valid for all $t \in \mathbb{K}$. This completes the proof of Theorem 6.4 for $i \in \mathbb{N}$. For $i = 0$ the statement of the theorem is trivially true. □

Remark 6.6. Note that representation $\sigma_C \circ \Theta^{-1} : (G_C[[r]], \circ) \to \exp(\mathfrak{cl}(\mathcal{L}_C)) \subset \mathcal{A}_e$, where $\Theta^{-1} : (G_C[[r]], \circ) \to (G_{F_{\mathbb{C}^0}^{\mathbb{C}^0}(\mathbb{C}^0)}, \star)$ is the isomorphism given by (117), is faithful and due to Proposition 6.5 coincides with the well-known ‘smallest faithful representation’ of $(G_C[[r]], \circ)$, see, e.g., [FM] Sect. 3.1; here $\exp(\mathfrak{cl}(\mathcal{L}_C))$ is considered as the subgroup of the

SHUFFLE AND FAÀ DI BRUNO HOPF ALGEBRAS IN THE CENTER PROBLEM FOR ODES
group of invertible linear operators acting on the space of complex polynomials \( \mathbb{C}[z] \). Thus, the values of the generalized displacement polynomials for \( t \in \mathbb{N} \) are the ‘matrix entries’ (written with respect to the basis \( \{z^m\}_{m \in \mathbb{Z}_+} \) of \( \mathbb{C}[z] \)) of the composite homomorphism \( F = (\sigma_C \circ \Theta^{-1}) \circ \mathcal{P} : (\mathcal{X}, *) \to \exp(\text{cl}(\mathcal{L}_C)) \).

For \( a = (a_1, a_2, \ldots) \in \mathcal{X} \), we set
\[
\tilde{P}_i(\int a_1, \ldots, \int a_i; t) := \sum_{i_1 + \cdots + i_k = i} p_{i_1, \ldots, i_k}(t) \cdot I_{i_1, \ldots, i_k}(a).
\]
Note that if we introduce bounded linear operators \( E_{a_i} : L^\infty(I_T) \to L^\infty(I_T) \),
\[
(E_{a_i} f)(x) := \int_0^x a_i(s) f(s) \, ds, \quad x \in I_T, \quad i \in \mathbb{N},
\]
then for each \( t \in \mathbb{C} \),
\[
\tilde{P}_i(\int a_1, \ldots, \int a_i; t) = (\tilde{P}_i(E_{a_1}, \ldots, E_{a_i}; t)(1))(T);
\]
here the product of operators is given by composition.

As the corollary of Theorem \ref{thm:approximation} and Proposition \ref{prop:approximation} we get (cf. Section 3.3):

**Corollary 6.7.** (1) Suppose \( a_k = (a_{1k}, a_{2k}, \ldots) \in \mathcal{X} \), \( k = 1, 2 \). Then
\[ (6.12) \quad \tilde{P}_i(\int a_{11}, \ldots, \int a_{i_1}; t) = \tilde{P}_i(\int a_{12}, \ldots, \int a_{i_2}; t) \quad \text{for all} \quad t \in \mathbb{C}, \; i \in \mathbb{N}, \]
if and only if \( a_1 \ast a_2^{-1} \in \mathcal{E} \).

(2) Let \( g \in G_{C(\mathcal{X})}(\mathbb{C}), \; h \in \widehat{\mathcal{E}_f} \) be some characters of the shuffle Hopf algebra \( \mathbb{C}(\mathcal{X}) \). Then
\[ (6.13) \quad (g \ast h) \left( \tilde{P}_i(a_1, \ldots, a_i; t) \right) = g \left( \tilde{P}_i(a_1, \ldots, a_i; t) \right) \quad \text{for all} \quad t \in \mathbb{C}, \; i \in \mathbb{N}. \]

**Proof.** (1) Since the first return map of equation (1.1) \( \mathcal{P} \) satisfies \( \mathcal{P}(a_1) = \mathcal{P}(a_2) \Leftrightarrow a_1 \ast a_2^{-1} \in \mathcal{E} \), the result follows from Theorem \ref{thm:approximation}.

(2) Since, by Theorem \ref{thm:approximation}, \( \tilde{P}_i(a_1, \ldots, a_i; t) \in \rho_C^*(H_{\text{FdB}}(\mathbb{C})) \) each \( h \in \widehat{\mathcal{E}_f} \) vanishes on it, see Section 5.3. This implies (6.13). \( \square \)

**Remark 6.8.** Suppose \( u_1, u_2, \ldots \) is a sequence of complex-valued Lipschitz functions on \( I_T \) such that \( u_i(0) = 0 \) for all \( i \). We define the sequence \( a_1, a_2, \ldots \) of functions in \( L^\infty(I_T) \) from the equality of formal power series
\[ (6.14) \quad \sum_{i=1}^\infty a_i(x) t^{i+1} = \frac{\sum_{k=1}^\infty u_k'(T - x) t^{k+1}}{1 + \sum_{k=1}^\infty (k+1) u_k(T - x) t^k}, \quad x \in I_T. \]

The differential equation
\[ \frac{dv}{dx} = \sum_{i=1}^\infty a_i(x) v^{i+1} \]
has a solution \( v(x; r), \; x \in I_T, \; v(0; r) = r \), presenting as a formal power series in \( r \) with Lipschitz coefficients. Identity (6.14) implies that
\[ v(x; r) + \sum_{i=1}^\infty u_i(T - x) v(x; r)^{i+1} = r + \sum_{i=1}^\infty u_i(T) r^{i+1} \quad \text{for all} \quad x \in I_T, \]
cf. the second part of the proof of [Br2, Th.3.7] for similar arguments. Since the first return map of the above equation is given by formula (1.4) as well,
\[ (6.15) \quad \mathcal{P}_i(\int a_1, \ldots, \int a_i) = u_i(T) \quad \text{for all} \quad i \in \mathbb{N}. \]
Here \( \mathcal{P}_i(\int a_1, \ldots, \int a_i) := \tilde{P}_i(\int a_1, \ldots, \int a_i; i) \).
For instance, assume that \( u_i(T) := \frac{t_i}{(i+1)!} \) for some \( t_i \in \mathbb{C}, i \in \mathbb{N} \). Then by Theorem 6.4 and (6.15) we obtain for all natural numbers \( i,j \) such that \( i - j \geq 1 \),

\[
\tilde{P}_{i-j}(f a_1, \ldots, f a_{i-j}; i-1) = \frac{j!}{l!} B_{i,j}(1, t_1, \ldots, t_{i-j}).
\]

Thus the Bell polynomials can be expressed in many different ways via the values of the generalized displacement polynomials; e.g., one can choose \( u_i(x) = \frac{t^i x^i}{T(i+1)!}, l, i \in \mathbb{N} \). As an illustration, let us consider in details the case of \( u_i(x) = \frac{t^i x^i}{T(i+1)!}, i \in \mathbb{N} \). Then (6.14) becomes

\[
\sum_{i=1}^{\infty} a_i(x) t^{i+1} = \frac{\sum_{k=1}^{\infty} t_k}{1 + \sum_{k=1}^{\infty} \frac{T-x}{T} t^k} = \left( \sum_{k=1}^{\infty} \frac{t_k}{T} t^{k+1} \right) \cdot \left( \sum_{l=0}^{\infty} \left( \frac{x-T}{T} \right)^l \cdot \left[ \sum_{k=1}^{\infty} \frac{t_k}{k} t^k \right]^l \right).
\]

(Equality of expressions in the square brackets in lines two and three above follows from the Faa di Bruno formula, see, e.g., [Com, page 133]; here \( B_{0,0} := 1, B_{m,0} := 0, m \geq 1 \).)

The latter implies that (6.13) is valid for

\[
a_i(x) = \frac{1}{(i+1)!} \sum_{k=1}^{i} \frac{t_k}{T} \cdot \frac{i+1}{i-k} \cdot \left( \sum_{l=0}^{i-k} \frac{l! \cdot B_{i-k,l}(t_1, \ldots, t_{i-k-l+1}) \cdot \left( \frac{x-T}{T} \right)^l}{T} \right), \quad i \in \mathbb{N}.
\]

### 6.2.2. Recurrence Relations

We apply Theorem 6.4 to establish several ‘recurrence relations’ for polynomials \( P_i(x_1, \ldots, x_i; \cdot) \), \( i \in \mathbb{N} \). Their proofs are based on known recurrence relations for the Bell polynomials, see, e.g., [Com] [Rid] [Rom] [Cv].

**Theorem 6.9.** We have

(A)

\[
\tilde{P}_i(\cdot; t) = \frac{t - i + 1}{t + 1} \cdot \sum_{j=1}^{i+1} \cdot j \cdot \tilde{P}_{j-1} \sqcup \tilde{P}_{i-j+1}(\cdot; t-j);
\]

(B)

\[
\tilde{P}_i(\cdot; t) = \sum_{j=1}^{i} \left( \frac{(t - i + 2)j}{i} - 1 \right) \tilde{P}_j \sqcup \tilde{P}_{i-j}(\cdot; t-j).
\]

(Recall that \( P_k(x_1, \ldots, x_k; \cdot) := \tilde{P}_k(x_1, \ldots, x_k; k) \) and \( \tilde{P}_0 = 1 \).)

**Proof.** (A) First, using the well-known recurrence relation for the Bell polynomials,

\[
B_{n,m}(x_1, \ldots, x_{n-m+1}) = \sum_{j=1}^{n-m+1} \binom{n-1}{j-1} x_j B_{n-j,m-1}(x_1, \ldots, x_{n-j-m+2}),
\]

we establish the following result for polynomials \( B_k \), see (6.5).

**Proposition 6.10.** For all \( k \in \mathbb{N} \),

\[
B_k(t_1, \ldots, t_k, t) = \frac{t}{t+k} \sum_{j=1}^{k+1} j t_{j-1} B_{k-j+1}(t_1, \ldots, t_{k-j+1}, t-1);
\]

here and below \( t_0 := 1 \).
Proof. Applying (6.17) with \( x_j := j! t_{j-1}, 1 \leq j \leq n - m + 1 \), we get

\[
B_{n,m}(1, 2! t_1, \ldots, (n - m + 1)! t_{n-m})
= \sum_{j=1}^{n-m+1} \frac{(n-1)!}{(n-j)! (j-1)!} (j! t_{j-1}) B_{n-j,m-1}(1, 2! t_1, \ldots, (n - m - 2 - j)! t_{n-m+1-j}).
\]

This formula with \( n = i + 1, m = i - k + 1 \) and Proposition 6.3 imply for all integers \( i \geq k \),

\[
B_k(t_1, \ldots, t_k, i - k + 1) = \frac{(i - k + 1)!}{(i + 1)!} B_{i+1,i-k+1}(1, 2! t_1, \ldots, (k + 1)! t_{k+1})
= \frac{(i - k + 1)!}{(i + 1)!} \sum_{j=1}^{k+1} \frac{i!}{(i + 1 - j)!} j t_{j-1} B_{i+1-j,i-k}(1, 2! t_1, \ldots, (k + 2 - j)! t_{k+1-j})
= \frac{(i - k + 1)!}{(i + 1)!} \sum_{j=1}^{k+1} \frac{i!}{(i + 1 - j)!} j t_{j-1} \cdot \frac{(i + 1 - j)!}{(i - k)!} B_{k+1-j}(t_1, \ldots, t_{k+1-j}, i - k)
= \frac{i - k + 1}{i + 1} \sum_{j=1}^{k+1} j t_{j-1} B_{k+1-j}(t_1, \ldots, t_{k+1-j}, i - k).
\]

This shows that (6.18) is valid for all \( t = i - k + 1 \) where \( i \geq k \) is an integer number. Since both parts of the required identity comprise rational functions in \( t \), the latter implies that (6.18) is valid for all \( t \in \mathbb{C} \), as required.

Now, identity (A) follows directly from Proposition 6.10 and Theorem 6.4. Indeed, according to these results for each character \( g \in G_{\mathbb{C}^{(x)}}(\mathbb{C}) \),

\[
\begin{align*}
g \left( \bar{P}_{i}(\alpha_1, \ldots, \alpha_i; t) \right) &= g \left( B_i(P_1(\alpha_1), \ldots, P_i(\alpha_1, \ldots, \alpha_i), t - i + 1) \right) \\
&= \frac{t - i + 1}{t + 1} \sum_{j=1}^{i+1} j g_{j-1} \cdot B_{i-j+1}(g_1, \ldots, g_{i-j+1}, t - i) \\
&= g \left( \frac{t - i + 1}{t + 1} \sum_{j=1}^{i+1} j P_{j-1}(\alpha_1, \ldots, \alpha_{j-1}) \cup \bar{P}_{i-j+1}(\alpha_1, \ldots, \alpha_{i-j+1}; t - j) \right),
\end{align*}
\]

where \( g_k := g \left( P_k(\alpha_1, \ldots, \alpha_j) \right), 0 \leq k \leq i \).

This gives identity (6.18).

(B) Using the recurrence relation for the Bell polynomials established in [Cv],

\[
B_{n,m}(x_1, \ldots, x_{n-m+1})
= \frac{1}{x_1} \cdot \frac{1}{n-m} \sum_{j=1}^{n-m} \binom{n}{j} \left( m + 1 - \frac{n+1}{j+1} \right) x_{j+1} B_{n-j,m}(x_1, \ldots, x_{n-j-m+1}),
\]

we prove the following result for polynomials \( B_k \).

**Proposition 6.11.** For all \( k \in \mathbb{N} \)

\[
B_k(t_1, \ldots, t_k, t) = \sum_{j=1}^{k} \left( \frac{(t+1)j}{k} - 1 \right) t_j B_{k-j}(t_1, \ldots, t_{k-j}, t).
\]
Thus identity (6.20) is valid for all $t = i - k + 1$ where $i \geq k$ is an integer number. Since both parts of the required identity comprise polynomials in $t$, the latter implies that (6.20) is valid for all $t \in \mathbb{C}$, as required. \hfill \Box

As in the proof of (A), identity (B) follows directly from Proposition 6.11 and Theorem 6.4. Indeed, according to these results for each character $g \in G_{\mathcal{C}(\mathcal{A})}(\mathbb{C}),$

$$g\left(\mathcal{P}_i(\alpha_1, \ldots, \alpha_i; t)\right) = g(B_i(\mathcal{P}_1(\alpha_1), \ldots, \mathcal{P}_i(\alpha_1, \ldots, \alpha_i), t - i + 1))$$

$$= \sum_{j=1}^{i} \left(\frac{(t - i + 2)j}{i} - 1\right) g_j B_{i-j}(g_1, \ldots, g_{i-j}, t - i + 1)$$

$$= g\left(\sum_{j=1}^{i} \left(\frac{(t - i + 2)j}{i} - 1\right) \mathcal{P}_j(\alpha_1, \ldots, \alpha_j) \mathcal{P}_{i-j}(\alpha_1, \ldots, \alpha_{i-j}; t - j)\right),$$

where $g_k := g(\mathcal{P}_k(\alpha_1, \ldots, \alpha_j))$, $1 \leq k \leq i$.

This proves identity (6.20). \hfill \Box

6.3. Augmentation Homomorphism. Recall that the augmentation homomorphism $\varepsilon_{\mathbb{C}}: \mathbb{C}(X) \to \mathbb{C}$ is defined on the generators by the formula $\varepsilon_{\mathbb{C}}(X_k) = 1$, $k \in \mathbb{Z}_+$, $X_0 := I$. In this section we compute the values of $\varepsilon_{\mathbb{C}}$ on the generalized Devlin polynomials $\mathcal{P}_i(X_1, \ldots, X_i; t)$, $i \in \mathbb{N}$.

**Proposition 6.12.** For all $i \in \mathbb{N}$, $t, x \in \mathbb{C}$,

$$\varepsilon_{\mathbb{C}}(\mathcal{P}_i(x_1, \ldots, x_i; t)) := \sum_{i_1 + \cdots + i_k = i} (t - i_1 + 1)(t - i_1 - i_2 + 1) \cdots (t - i + 1)x^k$$

$$= (xt + 1)(x(t - 1) + 1) \cdots (x(t - i + 1) + 1)x(t - i + 1).$$

In particular, $\varepsilon_{\mathbb{C}}(\mathcal{P}_i(X_1, \ldots, X_i)) := \sum_{i_1 + \cdots + i_k = i} (i - i_1 + 1)(i - i_1 - i_2 + 1) \cdots 1 = \frac{(i + 1)!}{2}.$
Corollary 6.13. Given $i, k \in \mathbb{N}$, $k \leq i$,

$$S_{i,k}(t) := \sum_{i_1 + \cdots + i_k = i} (t - i_1 + 1)(t - i_1 - i_2 + 1) \cdots (t - i + 1)$$

$$= (t - i + 1) \cdot \sum_{l=0}^{k-1} s(i - 1, l + i - k) \binom{l + i - k}{l} t^l,$$

where $s(\cdot, \cdot)$ are Stirling numbers of the first kind\footnote{In contrast with our previous notation, in the above sum $i$ and $k$ are fixed numbers.}

Note that $S_{i,k}(i - 2) = -|s(i - 1, i - k)|$, $i \geq 2$; $S_{i,k}(0) = (1 - i) \cdot s(i - 1, i - k)$. 

Proof. We give an operatorial proof of this result. For an operator $A$ on $\mathbb{C}[z]$ we assume that $A^0 := I$. We have

$$\left( I - \sum_{j=1}^{\infty} xDL^{j-1}t^j \right)^{-1} = \sum_{i=0}^{\infty} \left( \sum_{j=1}^{\infty} xDL^{j-1}t^j \right)^i$$

$$= I + \sum_{i=1}^{\infty} \left( \sum_{i_1 + \cdots + i_k = i} x^kDL^{i_1-1} \cdots DL^{i_k-1} \right) t^i.$$ 

On the other hand,

$$\left( I - \sum_{j=1}^{\infty} xDL^{j-1}t^j \right)^{-1} = (I - xtD(I - tL)^{-1})^{-1} = (I - tL)(I - tL - xtD)^{-1}$$

$$= (I - tL) \sum_{i=0}^{\infty} (L + xD)^i t^i = I + \sum_{i=1}^{\infty} ((L + xD)^i - L(L + xD)^{i-1}) t^i$$

$$= I + \sum_{i=1}^{\infty} xD(L + xD)^{i-1} t^i.$$ 

Equating the coefficients of $t^i$ on the right-hand sides of (6.21) and (6.22) we get

$$\sum_{i_1 + \cdots + i_k = i} x^kDL^{i_1-1} \cdots DL^{i_k-1} = xD(L + xD)^{i-1}.$$ 

Now, applying both sides of (6.23) to $z^m$ with $m \geq i$ we obtain

$$\sum_{i_1 + \cdots + i_k = i} p_{i_1, \ldots, i_k}(m)x^kz^{m-i} = (xm+1)(x(m-1)+1) \cdots (x(m-i+2)+1)x(m-i+1)z^{m-i}.$$ 

Since in both parts of the required identity are polynomials and the latter shows that this identity is valid for all integers $t \geq i$ and $x \in \mathbb{C}$, it is valid for all $t \in \mathbb{C}$ as well. This completes the proof of the proposition. \qed
Proof. For $x \neq 0$ we have
\[
\sum_{k=1}^{i} S_{i,k}(t) x^k = (xt + 1)(x(t - 1) + 1) \cdots (x(t - i + 2) + 1) x(t - i + 1)
\]
\[
= x^i \left( t + \frac{1}{x} \right) \left( t + \frac{1}{x} - 1 \right) \cdots \left( t + \frac{1}{x} - (i - 2) \right) (t - i + 1)
\]
\[
= \sum_{i=0}^{i-1} s(i-1, l) \left( t + \frac{1}{x} \right)^{l} x^i(t - i + 1) = \sum_{i=0}^{i-1} s(i-1, l) \sum_{m=0}^{l} \left( \frac{l}{m} \right) t^{l-m} x^{i-m} (t - i + 1)
\]
\[
= (t - i + 1) \sum_{k=1}^{i} \left( \sum_{l=0}^{k-1} s(i-1, l + i - k) \left( \frac{l+i-k}{l} \right)^{l} \right) x^k.
\]

\[
\text{Example 6.14.} \text{ Consider a separable equation}
\]
\[
(6.24) \quad \frac{dv}{dx} = \sum_{i=1}^{\infty} v^{i+1}, \quad x \in I_T := [0,T].
\]
For all sufficiently small initial values $v(0) = r \in \mathbb{C}$ its solution $v(\cdot;r)$ satisfies $\sup_{x \in I_T} |v(x,r)| < 1$ and is given by the formula, cf. \[1.4] \text{and Corollary 6.13} with $t = i$,
\[
v(x;r) = r + \sum_{i=1}^{\infty} \left( \sum_{i_1+\cdots+i_k=i} p_{i_1,\ldots,i_k}(i) \cdot \int \cdots \int_{0 \leq s_1 \leq \cdots \leq s_k \leq x} ds_k \cdots ds_1 \right) r^{i+1}
\]
\[
= r + \sum_{i=1}^{\infty} \left( \sum_{k=1}^{i} \frac{S_{i,k}(i)}{k!} x^k \right) r^{i+1}, \quad x \in I_T.
\]
On the other hand, for such $r$,
\[
\sum_{i=1}^{\infty} v(x;r)^{i+1} = \frac{v^2(x;r)}{1 - v(x;r)}, \quad x \in I_T,
\]
so that (6.24) is equivalent to equation
\[
(6.26) \quad \frac{dv}{dx} = \frac{v^2}{1 - v}, \quad x \in I_T := [0,T].
\]
For real initial values $r \neq 0$ it can be solved explicitly producing solutions
\[
(6.27) \quad v(x;r) = \begin{cases} 
\frac{1}{W_0 \left( e^x \cdot W_0^{-1}(\frac{1}{r}) \right)} & \text{if } -\infty < r < 0 \\
\frac{1}{W_{-1} \left( e^x \cdot W_{-1}^{-1}(\frac{1}{r}) \right)} & \text{if } 0 < r \leq -\frac{1}{W_{-1}(-e^{-T-1})}
\end{cases}
\]
such that
\[
\lim_{r \to 0^+} \sup_{x \in I_T} |v(x,r)| = \lim_{r \to 0^+} \sup_{x \in I_T} |v(x,r)| = 0.
\]
Here $W_0^{-1}(s) = W_{-1}(s) = se^s$, $s \in \mathbb{R}$, and $W_0$, $W_{-1}$ are real branches of the Lambert function, see, e.g., [CGHJK] for their properties.

Due to the uniqueness of solutions of initial value problems of (6.24) and (6.26), for all sufficiently small real $r \neq 0$, functions in (6.25) and (6.27) coincide. Thus (6.27) determines
a real analytic function on $I_T \times \left( -\infty, -\frac{1}{W_{-1}(e^{-T-1})} \right)$ satisfying (6.26) with the Taylor series expansion about $(0, 0)$ given by (6.25).

Finally, note that from Theorem 6.4 we obtain for all $i \in \mathbb{N}, t \in \mathbb{C}$,

$$S_i(x, t) = B_i(S_1(x, 1), \ldots, S_{i-1}(x, i), t - i + 1),$$

where

$$S_i(x, t) := \sum_{k=1}^{i} \frac{S_{i,k}(t)}{k!} x^k.$$

**REFERENCES**

[AL] M. A. M. Alwash and N. G. Lloyd, Non-autonomous equations related to polynomial two-dimensional systems, Proc. R. Soc. Edinb. A 105 (1987), 129–152.

[A1] M.A.M. Alwash, On the Composition Conjectures, Electronic Journal of Differential Equations, Vol. 2003 (2003), No. 69, pp. 14.

[A2] M.A.M. Alwash, The composition conjecture for Abel equation, Expo. Math. 27 (2009), 241–250.

[A3] M.A.M. Alwash, Polynomial differential equations with piecewise linear coefficients, Differ. Equ. Dyn. Syst. 19 (2011), no. 3, 267–281.

[A4] M.A.M. Alwash, Composition conditions for two-dimensional polynomial systems, Diff. Eq. and Appl., Volume 5, Number 1, February 2013.

[B] E. T. Bell, Exponential polynomials, Ann. of Math. (2) 35 (1934), 258–277.

[Br1] A. Brudnyi, An explicit expression for the first return map in the center problem, J. Differential Equations 206 (2004), 306–314.

[Br2] A. Brudnyi, On the center problem for ordinary differential equations, Amer. J. Math. 128 no. 2 (2006), 419–451.

[Br3] A. Brudnyi, Formal paths, iterated integrals and the center problem for ordinary differential equations, Bull. Sci. Math. 132 (2008), 455–485.

[Br4] A. Brudnyi, Center problem for the group of rectangular paths, C. R. Math. Rep. Acad. Sci. Canada 31 (2) (2009), 33–44.

[Br5] A. Brudnyi, Composition conditions for classes of analytic functions, Nonlinearity 25 (2012), 3197–3209.

[Br6] A. Brudnyi, Moments finiteness problem and characterization of universal centers of ODEs with analytic coefficients, Nonlinearity 27 (2014), 1611–1631.

[BFY] M. Briskin, J.-P. Francoise and Y. Yomdin, The Bautin ideal of the Abel Equation, Nonlinearity 11 (1998), 41–53.

[BRY] M. Briskin, N. Roytvarf and Y. Yomdin, Center conditions at infinity for Abel differential equations, Annals of Math. 172 (1) (2010), 437–483.

[BY] A. Brudnyi and Y. Yomdin, Tree composition condition and moments vanishing, Nonlinearity 23 (2010), 1651–1673.

[C] P. Cartier, A primer of Hopf algebras, Frontiers in Number Theory, Physics and Geometry II, 537–615, Springer (2007).

[CGHJK] R. Corless, G. Gonnet, D. Hare, D. Jeffrey, D. Knuth, On the Lambert W function. Adv. Comput. Math. 5 (1996), 329–359.

[CGM1] A. Cima, A. Gasull, and F. Mánosas, Centers for trigonometric Abel equations, Qual. Theory Dyn. Syst. 11 (2012), 19–37.

[CGM2] A. Cima, A. Gasull, and F. Mánosas, A simple solution of some composition conjectures for Abel equations, J. Math. Anal. Appl. 398 (2013), 477-486.

[CGM3] A. Cima, A. Gasull, and F. Mánosas, An explicit bound of the number of vanishing double moments forcing composition, J. Diff. Equations 255 (3) (2013), 339–350.

[Ch] K.-T. Chen, Iterated integrals and exponential homomorphisms, Proc. London Math. Soc. 4 (1954) (3), 502–512.

[Co] S. D. Cohen, The group of translations and positive rational powers is free, Q. J. Math. Oxford 46 (2) (1995), 21–93.

[Com] L. Comtet, Advanced combinatorics: The art of finite and infinite expansions, rev. enl. ed. Dordrecht, Netherlands: Reidel, 1974.

[CK] A. Connes, D. Kreimer, Hopf algebras, renormalization and noncommutative geometry, Commun. Math. Phys. 199 (1998) 203–242.
[Cr] C. Christopher, Abel equations: composition conjectures and the model problem, Bull. Lond. Math. Soc. 32 (2000), 332-338.
[Cv] D. Cvijović, New identities for the partial Bell polynomials, Appl. Math. Lett. 24 (2011), 1544–1547.
[D] J. Devlin, Word problems related to periodic solutions of a nonautonomous system, Math. Proc. Cambridge Philos. Soc. 108 (1990) 127–151.
[DbF] F. Faà di Bruno, Sullo sviluppo delle funzioni, Ann. Sci. Mat. Fis., Roma 6 (1855), 479–480.
[FG] H. Figueroa, J. Gracia-Bondía, Combinatorial Hopf algebras in quantum field theory I, Rev. Math. Phys. 17 (2005), 881–976.
[FM] A. Frabetti, D. Manchon, Five interpretations of Faà di Bruno’s formula, Faà di Bruno Hopf Algebras, Dyson-Schwinger Equations, and Lie-Butcher Series, 91–147, IRMA Lect. Math. Theor. Phys. 21, Eur. Math. Soc., Zürich, 2015.
[GGL] J. Giné, M. Grau and J. Llibre, Universal centers and composition conditions,
[GGS] J. Giné, M. Grau and X. Santallusia, Universal centers in the cubic trigonometric Abel equation, Electron. J. Qual. Theory Differ. Equ. No. 1 (2014), 1–7.
[GVF] J. M. Gracia-Bondía, J. C. Várilly, H. Figueroa, Elements of Noncommutative Geometry, Birkhäuser Boston, Boston, MA, 2001.
[JR] S. A. Joni, G.-C. Rota, Coalgebras and bialgebras in combinatorics, Contemp. Math. 6 (1982), 1–47.
[L] M. Lothaire, Combinatorics on words, Encyclopedia of Mathematics and its Applications, vol. 17, Cambridge University Press, 1984.
[Ly] R. C. Lyndon, On Burnside problem I, Trans. Amer. Math. Soc. 77 (1954), 202–215.
[M] D. Manchon, Hopf algebras and renormalisation, Handbook of algebra 5 (M. Hazewinkel ed.) (2008), 365–427.
[M-KO] H. Munthe-Kaas and B. Owren, Computations in a free Lie algebra, Phil. Trans. Royal Soc. A 357 (1999), 957–981.
[P1] F. Pakovich, On the polynomial moment problem, Math. Res. Lett., 10, no. 2-3 (2003), 401–410.
[P2] F. Pakovich, Solution of the parametric center problem for the Abel differential equation, JEMS, in print. arXiv:1407.0150.
[Ra] D. E. Radford, A natural ring basis for the shuffle algebra and an application to group schemes, Journal of Algebra 58 (1979), 432–453.
[Re] C. Reutenauer, Free Lie algebras, Oxford Univ. Press, 1993.
[Rí] J. Riordan, An introduction to combinatorial analysis, Wiley, New York, 1980.
[Rom] S. Roman, The umbral calculus, Academic Press, New York, 1984.
[R] R. Ree, Lie elements and an algebra associated with shuffles, Ann. of Math. (2) 68 (1958), 550–561.
[S] M. E. Sweedler, Hopf algebras, Mathematics Lecture Note Series, W. A. Benjamin, New York, 1969.