ABEL MAPS AND LIMIT LINEAR SERIES

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ABSTRACT. We explore the relationship between limit linear series and fibers of Abel maps in the case of curves with two smooth components glued at a single node. To an $r$-dimensional limit linear series satisfying a certain exactness property (weaker than the refinedness property of Eisenbud and Harris) we associate a closed subscheme of the appropriate fiber of the Abel map. We then describe this closed subscheme explicitly, computing its Hilbert polynomial and showing that it is Cohen–Macaulay of pure dimension $r$. We show that this construction is also compatible with one-parameter smoothings.

1. INTRODUCTION

The classical theory of linear series on smooth curves is closely related to that of Abel maps and their fibers, which consist precisely of complete linear series. This relationship also amplifies the relationship between linear series and (families of) effective divisors. For (singular) curves of compact type, Eisenbud and Harris [5] developed the theory of limit linear series as an analogue of linear series, while Coelho and Pacini [3] have studied Abel maps. However, the relationship between these two concepts is far murkier than in the smooth case. On the side of limit linear series, there is no obvious concept of a complete limit linear series, nor of families of divisors associated to a limit linear series. On the other hand, fibers of Abel maps are not very well behaved: for instance, they are in general not even equidimensional.

Our aim is to relate limit linear series to fibers of Abel maps via the definition of limit linear series and construction of their moduli space in [7]. For the sake of simplicity, we restrict our attention to the case treated in loc. cit., which is that of a curve $X$ with two smooth components glued together at a single node.

There is an open subset of the moduli space of limit linear series on $X$ consisting of “exact” limit linear series (see Definition 2.2 below). These contain in particular all limits of linear series on the generic fiber in a regular smoothing family; see Section 5. If $g$ is an exact limit linear series of dimension $r$ with underlying line bundle $L$, we construct a closed subscheme $P(g)$ of $A_d^{-1}(L)$, the corresponding fiber of the $d$th Abel map. This subscheme is by definition reduced, and we show in Theorem 4.3 that $P(g)$ is connected and Cohen–Macaulay, of dimension $r$, with the same Hilbert polynomial as $P^r$. We also show in Theorem 5.2 that if $g$ is the limit of a $g^r_d$ on the generic fiber of a one-parameter regular smoothing of $X$, then $P(g)$ is the flat limit of the corresponding $P^r$ in the fiber of the classical $d$th Abel map on the generic (smooth) curve. Finally, we observe in Proposition 6.1 that if a fiber

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of the $d$th Abel map for $X$ has any component of dimension less than $r$, then there
is no limit linear series of dimension $r$ for the corresponding line bundle.

Finally, we mention that there is a somewhat parallel construction of Eisenbud
and Harris in Section 5 of [5], where they describe how the target projective space
of the morphism associated to a linear series degenerates in the special case that
the limit is a refined limit series. The space they describe is quite similar to the
special case of ours (see Remark 4.9 below). However, the constructions ought to be viewed as dual to one another: if $g$ is a linear series on the generic fiber, we construct degenerations of $P(g)$, while the natural target of the associated
morphism is not $P(g)$, but rather its dual space $P(g^*)$.

2. Limit linear series

Throughout this article, $X$ will denote the union of two smooth curves $Y$ and
$Z$, meeting transversally at a point $P$.

Let $\mathcal{L}$ be an invertible sheaf on $X$. It is determined by its restrictions $\mathcal{L}|_Y$ and
$\mathcal{L}|_Z$. Also, there are natural short exact sequences,

\begin{equation}
0 \to \mathcal{L}|_Z(-P) \to \mathcal{L} \to \mathcal{L}|_Y \to 0,
\end{equation}

\begin{equation}
0 \to \mathcal{L}|_Y(-P) \to \mathcal{L} \to \mathcal{L}|_Z \to 0.
\end{equation}

For each integer $i$, let $\mathcal{L}^i$ be the invertible sheaf on $X$ with restrictions $\mathcal{L}|_Y(-iP)$
and $\mathcal{L}|_Z(iP)$. There are natural maps $\varphi^i: \mathcal{L}^i \to \mathcal{L}^{i+1}$ and $\varphi_1: \mathcal{L}^{i+1} \to \mathcal{L}^i$, defined
as the compositions:

$\varphi^i: \mathcal{L}^i \to \mathcal{L}^i|_Z = \mathcal{L}|_Z(iP) = \mathcal{L}|_Z((i+1)P)(-P) = \mathcal{L}_{i+1}|_Z(-P) \to \mathcal{L}^{i+1}$,

$\varphi_1: \mathcal{L}^{i+1} \to \mathcal{L}^{i+1}|_Y = \mathcal{L}|_Y(-(i+1)P) = \mathcal{L}|_Y(-iP)(-P) = \mathcal{L}^i|_Y(-P) \to \mathcal{L}^i$,

where the first map in each composition is the restriction map, and the
last maps are the inclusions in (2.1) and (2.2) for $\mathcal{L}^{i+1}$ and $\mathcal{L}^i$ instead of $\mathcal{L}$.
Notice that the compositions $\varphi^i \varphi_1$ and $\varphi_1 \varphi^i$ are zero.

**Definition 2.1.** Fix integers $d$ and $r$. A limit (linear) series on $X$ of degree $d$ and
dimension $r$ is a collection consisting of an invertible sheaf $\mathcal{L}$ on $X$ of degree $d$ on
$Y$ and degree $0$ on $Z$, and vector subspaces $V_i \subseteq \Gamma(X, \mathcal{L}^i)$ of dimension $r + 1$, for
each $i = 0, \ldots, d$, such that $\varphi^i(V_i) \subseteq V_{i+1}$ and $\varphi_1(V_{i+1}) \subseteq V_i$ for each $i$.

Given a limit series $(\mathcal{L}, V_0, \ldots, V_d)$, we denote by $V_{i}^{{Y_0},0}$ the subspace of $V_i$ of
sections that vanish on $Y$, and by $V_{i}^{{Z_0},0}$ the subspace of $V_i$ of sections that vanish
on $Z$. Also, let $V_i|_Y$ denote the subspace of $\Gamma(Y, \mathcal{L}^i|_Y)$ generated by $V_i$ and $V_i|_Z$
that of $\Gamma(Z, \mathcal{L}^i|_Z)$ generated by the same $V_i$. Of course, $V_i^{{Y_0},0}$ is the kernel of the
surjection $V_i \to V_i|_Y$, and $V_i^{{Z_0},0}$ is the kernel of the surjection $V_i \to V_i|_Z$. Also,
the map $\varphi^i: V_i \to V_{i+1}$ has kernel $V_i^{{Z_0},0}$ and image contained in $V_{i+1}^{{Y_0},0}$, whereas
$\varphi_1: V_{i+1} \to V_i$ has kernel $V_{i+1}^{{Y_0},0}$ and image contained in $V_i^{{Z_0},0}$.

**Definition 2.2.** A limit linear series $(\mathcal{L}, V_0, \ldots, V_d)$ is called exact if, for each $i$,

$\text{Im}(\varphi^i: V_i \to V_{i+1}) = V_{i+1}^{{Y_0},0} = \text{Ker}(\varphi_1: V_{i+1} \to V_i)$,

$\text{Im}(\varphi_1: V_{i+1} \to V_i) = V_i^{{Z_0},0} = \text{Ker}(\varphi^i: V_i \to V_{i+1})$. 
It is a theorem of Liu [6] that if X is general (i.e., if both (Y, P) and (Z, P) are general 1-marked curves) the exact limit linear series are dense in the space of all limit linear series.

One of the key properties of exact limit series is the following, which may be thought of as a simultaneous diagonalization lemma.

**Lemma 2.3.** If \((L, V_0, \ldots, V_d)\) is an exact limit series, then there exist nonnegative integers \(i_0 \leq i_1 \leq \cdots \leq i_r \leq d\) and sections \(s_0, \ldots, s_r\) with \(s_j \in V_{i_j}\) such that for each \(i = 0, \ldots, d\), the \(s_{i_j}\) with \(i_j = i\) form a basis of \(V_i/(V_i^{Y,0} \oplus V_i^{Z,0})\), and the iterated images of all the \(s_j\) form a basis for \(V_i\).

For the argument, see the proof of Lemma A.12 (ii) of [7].

### 3. Abel maps

To our knowledge, higher-degree Abel maps for curves of compact type appeared first in [3], though they are the natural offspring of the construction of degree-1 Abel maps for stable curves in [1] or [2].

We will need them in a very special situation, where they are easy to describe. Recall that X is the union of two smooth curves Y and Z, meeting transversally at a point P. Let \(S^d(X)\) denote the symmetric product of \(X\), thus parameterizing 0-cycles, or Weil divisors, on \(X\) of degree \(d\). The degree-\(d\) Abel map is a map

\[
A_d: S^d(X) \longrightarrow \text{Pic}^d(X),
\]

where \(\text{Pic}^d(X)\) is the Picard scheme of \(X\), parameterizing line bundles of a fixed multidegree \((d_1, d_2)\) with total degree \(d_1 + d_2 = d\). The specific multidegree varies according to choices of components and polarizations; see [3].

For our purposes, it is better to think of \(\text{Pic}^d(X)\) as parameterizing equivalence classes of line bundles of total degree \(d\), where two line bundles \(L_1\) and \(L_2\) are said to be **equivalent** if there exists an integer \(j\) such that \(L_1|_Y \cong L_2|_Y(-jP)\) and \(L_1|_Z \cong L_2|_Z(jP)\). The map \(A_d\) is then given as follows: Given a 0-cycle \(D\) on \(X\) of degree \(d\), write it as \(D = D_Y + D_Z\), where \(D_Y\) and \(D_Z\) are 0-cycles, the first supported on \(Y\) and the second on \(Z\); then the image of \(D\) under \(A_d\) is the (class of the) line bundle on \(X\) whose restrictions to \(Y\) and \(Z\) are \(O_Y(D_Y)\) and \(O_Z(D_Z)\).

(Note the abuse of notation, where we view a 0-cycle of \(X\) supported on \(Y\) or \(Z\) as a 0-cycle on \(Y\) or \(Z\), and vice versa.) This description of \(A_d(D)\) does not depend on how \(D\) is decomposed as \(D_Y + D_Z\), as the line bundles resulting from different decompositions are all equivalent to each other.

The fibers of \(A_d\) are also easy to describe, at least set-theoretically. A point of \(\text{Pic}^d(X)\) has a unique representative \(L\) of degree \(d\) on \(Y\) and 0 on \(Z\). Define the bundles \(\mathcal{L}'\) as in the last section. Set \(\Gamma_Y^i := \Gamma(Y, \mathcal{L}'^{d-i}|_Y)\) and \(\Gamma_Z^i := \Gamma(Z, \mathcal{L}'|_Z)\) for each \(i = 0, \ldots, d\). There are natural closed embeddings

\[
\mathbb{P}(\Gamma_Y^i) \longrightarrow S^i(Y) \quad \text{and} \quad \mathbb{P}(\Gamma_Z^i) \longrightarrow S^i(Z),
\]

sending the class of a nonzero section \(s\) to the 0-cycle \(\text{div}(s)\) associated to its zero scheme. Taking products of these embeddings, and composing with the natural embeddings

\[
S^{d-i}(Y) \times S^i(Z) \longrightarrow S^d(X),
\]

which send a pair of 0-cycles to their sum, we obtain as images subsets of \(S^d(X)\), whose union is the fiber of \(A_d\) over \(\mathcal{L}\). Abusing notation, by not keeping record of
the embeddings, we have:

\[ A_d^{-1}(\mathcal{L}) = \mathbb{P}(\Gamma^d_X) \times \mathbb{P}(\Gamma^d_Y) \cup \mathbb{P}(\Gamma^d_z) \times \mathbb{P}(\Gamma^{d-1}_Z) \cup \cdots \cup \mathbb{P}(\Gamma^d_y) \times \mathbb{P}(\Gamma^{d-1}_z). \]

The natural inclusions

\[ S^i(Y) \rightarrow S^{i+1}(Y) \quad \text{and} \quad S^i(Z) \rightarrow S^{i+1}(Z), \]

sending a 0-cycle \( D \) to \( D + P \) in both cases, take \( \mathbb{P}(\Gamma^d_Y) \) to \( \mathbb{P}(\Gamma^{d+1}_Y) \) and \( \mathbb{P}(\Gamma^d_z) \) to \( \mathbb{P}(\Gamma^{d+1}_z) \), respectively. Abusing notation again, inside \( S^d(Y) \) and \( S^d(Z) \) we have chains of subschemes:

\[ \mathbb{P}(\Gamma^0_Y) \subseteq \mathbb{P}(\Gamma^1_Y) \subseteq \cdots \subseteq \mathbb{P}(\Gamma^d_Y) \subseteq \mathbb{P}(\Gamma^0_z), \]

\[ \mathbb{P}(\Gamma^0_z) \subseteq \mathbb{P}(\Gamma^1_z) \subseteq \cdots \subseteq \mathbb{P}(\Gamma^d_z) \subseteq \mathbb{P}(\Gamma^0_z). \]

So we may consider the union on the right-hand side of (3.1) inside the product \( \mathbb{P}(\Gamma^d_Y) \times \mathbb{P}(\Gamma^d_z) \) instead of \( S^d(X) \). It is equal to \( A_d^{-1}(\mathcal{L}) \) nonetheless. Indeed, the product \( S^d(Y) \times S^d(Z) \) can be viewed naturally inside \( S^{2d}(X) \), by sending a pair of 0-cycles to their sum. Also, \( S^d(X) \) can be viewed inside \( S^{2d}(X) \), by sending a 0-cycle \( D \) to \( D + dP \). Under these inclusions, each \( \mathbb{P}(\Gamma^d_Y) \times \mathbb{P}(\Gamma^d_z) \), whether viewed as a subset of \( S^d(X) \) or of \( \mathbb{P}(\Gamma^d_Y) \times \mathbb{P}(\Gamma^d_z) \), gives the same subset of \( S^{2d}(X) \).

Given \( R \in X - P \), let

\[ R + S^{d-1}(X) := \{ D \in S^d(X) \mid D \geq R \}. \]

Fixing points \( R_1 \in Y - P \) and \( R_2 \in Z - P \), consider the divisors \( H_{s,t} \) on \( S^d(X) \) given by

\[ H_{s,t} := s(R_1 + S^{d-1}(X)) + t(R_2 + S^{d-1}(X)) \]

for each \( s, t \in \mathbb{Z} \). If \( s, t > 0 \), the restriction of \( H_{s,t} \) to any fiber of \( A_d \) is ample.

Given a subscheme \( W \) of a fiber \( A_d^{-1}(\mathcal{L}) \), we may compute its bivariate Hilbert polynomial

\[ P_W(s, t) := \dim \Gamma(W, \mathcal{O}_W(s, t)) = \dim \Gamma(W, \mathcal{O}_{S^{2d}(X)}(H_{s,t}))(W) \]

for \( s, t > 0 \). Notice that, viewing \( W \) inside \( \mathbb{P}(\Gamma^d_Y) \times \mathbb{P}(\Gamma^d_z) \) under the natural embedding \( A_d^{-1}(\mathcal{L}) \rightarrow \mathbb{P}(\Gamma^d_Y) \times \mathbb{P}(\Gamma^d_z) \), the Hilbert polynomial \( P_W \) is the bivariate Hilbert polynomial of a subscheme in the product of two projective spaces, and is thus independent of the choices of \( R_1 \) and \( R_2 \).

4. Limit linear series and fibers of the Abel map

**Definition 4.1.** Let \( \mathfrak{g} := (\mathcal{L}, V_0, \ldots, V_d) \) be an exact limit linear series on \( X \) of degree \( d \) and dimension \( r \). Let \( \mathbb{P}(\mathfrak{g}) \) denote the closure of the subset of \( A_d^{-1}(\mathcal{L}) \) consisting of points of the form

\[ \text{div}(s|_Y) + \text{div}(s|_Z) \in S^d(X), \]

for \( s \in V_i - (V_i^{Y,0} \cup V_i^{Z,0}) \) for some \( i \). We give \( \mathbb{P}(\mathfrak{g}) \) the reduced induced subscheme structure.

The definition makes sense, because, for each \( i \), the line bundle associated to \( \text{div}(s|_Y) + \text{div}(s|_Z) \) under \( A_d \) is clearly \( \mathcal{L}^i \), which is equivalent to \( \mathcal{L} \).
Remark 4.2. Note that the definition of \( \mathbb{P}(g) \) will only yield divisors from \( V_i \) if \( V_i \neq V_i^{Y,0} \cup V_i^{Z,0} \). However, this condition is weak and easy to control: it is violated if and only if \( V_i = V_i^{Y,0} \) or \( V_i = V_i^{Z,0} \), or equivalently, if and only if \( V_i|_Z = 0 \) or \( V_i|_Y = 0 \). Moreover, since the \( V_i \) give filtrations of \( V_0|_Y \) and \( V_d|_Z \), violation occurs only in “extremal” degrees. More precisely, there exist certain \( t_{low} \leq t_{high} \) between 0 and \( d \) such that \( V_i \neq V_i^{Y,0} \cup V_i^{Z,0} \) if and only if \( t_{low} \leq i \leq t_{high} \).

Furthermore, exactness yields

\[
V_{t_{low}} \neq V_{t_{low}}^{Y,0} \oplus V_{t_{low}}^{Z,0} \quad \text{and} \quad V_{t_{high}} \neq V_{t_{high}}^{Y,0} \oplus V_{t_{high}}^{Z,0}.
\]

Indeed, if \( t_{low} = 0 \), then \( V_0^{Y,0} = 0 \). Otherwise, since \( V_{t_{low} - 1}^{Z,0} = V_{t_{low} - 1} \), the image of \( V_{t_{low}}^{Z,0} \) in \( V_{t_{low}} \) is 0, whence by exactness \( V_{t_{low}}^{Y,0} = 0 \). In either case, since \( V_{t_{low}}^{Z,0} \neq V_{t_{low}}^{Y,0} \) by definition of \( t_{low} \), we get the desired assertion for \( V_{t_{low}} \). The same argument works for \( V_{t_{high}} \), switching the role of \( Y \) and \( Z \).

Theorem 4.3. If \( g = (L, V_0, \ldots, V_d) \) is an exact limit linear series on \( X \) of degree \( d \) and dimension \( r \), then \( \mathbb{P}(g) \) is reduced, connected, and Cohen–Macaulay of pure dimension \( r \), has bivariate Hilbert polynomial \( P(s,t) = (s + t + r) \) and is a flat degeneration of \( \mathbb{P}^r \).

The proof of the theorem is lengthy, so we break it into a number of steps. The first three lemmas are independent of limit linear series, and will ultimately be used to describe the geometry of \( \mathbb{P}(g) \), starting with its irreducible components.

Lemma 4.4. Given \( p, q, m \) nonnegative integers, let \( Q_{p,q,m} \subseteq \mathbb{P}^{m+q} \times \mathbb{P}^{m+p} \) be the subscheme given by the equations

\[
x_i y_j - x_j y_i = 0 \quad \text{for } i, j = p, \ldots, m + p,
\]

where \( \mathbb{P}^{m+q} \) is given coordinates \( x_p, \ldots, x_{m+p+q} \) and \( \mathbb{P}^{m+p} \) is given coordinates \( y_0, \ldots, y_{m+p} \). Then \( Q_{p,q,m} \) is integral of dimension \( m + p + q \), determinantal and hence Cohen–Macaulay, with bivariate Hilbert polynomial \( P_{Q_{p,q,m}}(s,t) \) given by

\[
\sum_{\ell=0}^{m} \binom{s + q + \ell}{q + \ell} \binom{t + p + m - \ell}{p + m - \ell} - \sum_{\ell=0}^{m-1} \binom{s + q + \ell}{q + \ell} \binom{t + p + m - 1 - \ell}{p + m - 1 - \ell}.
\]

(The variable indexing in the lemma may appear ad hoc, but it will be useful when we consider certain unions of these varieties inside larger products of projective spaces.)

Proof. Denote by \( S \) the bigraded ring \( k[x_p, \ldots, x_{m+p}, y_p, \ldots, y_{m+p}]/I \), where \( I \) is the ideal generated by

\[
x_i y_j - x_j y_i = 0 \quad \text{for } i, j = p, \ldots, m + p.
\]

Then \( S \) is the homogeneous coordinate ring of the diagonal in \( \mathbb{P}^m \times \mathbb{P}^m \), and its bivariate Hilbert polynomial is \( \binom{s+t+m}{m} \). Moreover, since the cohomology of projective space vanishes in positive degree for \( O(s+t) \) with \( s + t \geq 0 \), we have that the Hilbert polynomial agrees with the Hilbert function.

The homogeneous coordinate ring of \( Q_{p,q,m} \) is then

\[
S_Q := S[x_{m+p+1}, \ldots, x_{m+p+q}, y_0, \ldots, y_{p-1}],
\]

from which it follows that \( Q_{p,q,m} \) is integral. Furthermore, a monomial of bidegree \( (s,t) \) in \( S_Q \) may be written uniquely as a monomial of bidegree \( (i,j) \) in \( S \) times
a monomial of degree $s - i$ in the $x_n$ for $n > m + p$, and a monomial of degree $t - j$ in the $y_n$ for $n < p$, where $i$ and $j$ are nonnegative integers less than or equal to $s$ and $t$ respectively. But we know that there are $(i+j+n)$ monomials of bidegree $(i,j)$ in $S$, and this is equal to the number of monomials of degree $i + j$ in $m + 1$ variables, say $z_0, \ldots, z_m$. We thus conclude that the number of monomials of bidegree $(s,t)$ in $S_Q$ is equal to the number of monomials of total degree $s + t$ in \( x_{m+p+1}, \ldots, x_{m+p+q}, y_0, \ldots, y_{p-1}, z_0, \ldots, z_m \), with the degrees in the $x_n$ and $y_n$ bounded by $s$ and $t$, respectively. Denote the latter set of monomials by $S_Q(s,t)$. In the definition of $S_Q(s,t)$ and hereafter the $x_n$ are assumed to have $n > m + p$ and the $y_n$ are assumed to have $n < p$.

On the other hand, each summand in the first sum of the desired formula for the Hilbert polynomial is equal to the product of the number of monomials of degree $s$ in the $x_n$ and $z_0, \ldots, z_t$ with the number of monomials of degree $t$ in the $y_n$ and $z_t, \ldots, z_m$. Taking the product of such a pair of monomials trivially gives an element of $S_Q(s,t)$. For a given $\ell$, the resulting map from pairs of monomials to $S_Q(s,t)$ is clearly injective. And if we take the union of all these maps over all $\ell$, it is clear that we get a surjective map, as for each monomial in $S_Q(s,t)$ we may choose $\ell$ minimal so that its total degree in the $x_n$ and $z_0, \ldots, z_t$ is at least $s$.

However, a given monomial in $S_Q(s,t)$ may arise in this way from more than one pair of monomials. The number of times it arises is equal to the number of $\ell$ such that the total degree in the $x_n$ and $z_0, \ldots, z_t$ is at least $s$ and the total degree in the $y_n$ and $z_t, \ldots, z_m$ is at least $t$. If this holds for $\ell', \ell' + 1, \ldots, \ell''$, with $\ell' < \ell''$, it immediately follows that the total degree of the monomial in the $x_n$ and $z_0, \ldots, z_{\ell'}$ is $s$, its total degree in $z_{\ell'+1}, \ldots, z_{\ell''-1}$ is 0, and its total degree in the $y_n$ and $z_{\ell''}, \ldots, z_m$ is $t$. Each summand in the second sum of the desired formula may be interpreted as counting pairs of monomials of degree $s$ in the $x_n$ and $z_0, \ldots, z_t$ and of degree $t$ in the $y_n$ and $z_{t+1}, \ldots, z_m$. These likewise map to $S_Q(s,t)$, and we see that the number mapping to a given monomial is precisely $\ell'' - \ell'$, with notation as above. We thus conclude the desired formula for the Hilbert polynomial. And it follows from the formula that the dimension of $Q_{p,q,m}$ is the one stated.

Now, the description of $Q_{p,q,m}$ is visibly determinantal, coming from the condition that the matrix

\[
\begin{bmatrix}
x_p & x_{p+1} & \ldots & x_{m+p-1} & x_{m+p} \\
y_p & y_{p+1} & \ldots & y_{m+p-1} & y_{m+p}
\end{bmatrix}
\]

have rank at most 1. Moreover,

\[
\text{codim}(Q_{p,q,m}, \mathbb{P}^{m+q} \times \mathbb{P}^{m+p}) = (m + q) + (m + p) - (m + p + q) = m,
\]

and the expected codimension of $Q_{p,q,m}$ is $(2 - 1)((m + 1) - 1) = m$, so we conclude that $Q_{p,q,m}$ is determinantal of the expected codimension, and consequently Cohen–Macaulay.

\begin{lemma}
\label{lemma:4.5}
Given nonnegative integers $r$ and $m_0, \ldots, m_n$ with $\sum_i (m_i + 1) \leq r + 1$, define sequences $p_0, \ldots, p_n$ and $q_0, \ldots, q_n$ as follows:

1. $p_0 := 0$ and $p_i := p_{i-1} + m_{i-1} + 1$ for each $i = 1, \ldots, n$;
2. $q_i := r - m_i - p_i$ for each $i = 0, \ldots, n$.

Give $\mathbb{P}^r \times \mathbb{P}^r$ bihomogeneous coordinates $x_j$ and $y_j$ for $j = 0, \ldots, r$, and for each $i = 0, \ldots, n$, view the $Q_{p_i,q_i,m_i}$ of Lemma 4.4 as a subvariety $Q_i$ of $\mathbb{P}^r \times \mathbb{P}^r$ in the
\end{lemma}
natural way, by considering
\[ \mathbb{P}^{m_i+q_i} \times \mathbb{P}^{m_i+p_i} = V(x_0, \ldots, x_{p_i-1}, y_{m_i+p_i+1}, \ldots, y_r) \subseteq \mathbb{P}^r \times \mathbb{P}^r. \]

Set
\[ Q = \bigcup_{i=0}^{n} Q_i \subseteq \mathbb{P}^r \times \mathbb{P}^r. \]

Then \( Q \) is connected, reduced and Cohen–Macaulay of dimension \( r \), with bivariate Hilbert polynomial \( P_Q(s,t) \) given by
\[
\sum_{\ell=0}^{p_n+m_n} \binom{s + r - \ell}{r - \ell} \binom{t + \ell}{\ell} - \sum_{\ell=0}^{p_n+m_n-1} \binom{s + r - 1 - \ell}{r - 1 - \ell} \binom{t + \ell}{\ell}.
\]

If further \( p_n + m_n = r \), then
\[
P_Q(s,t) = \binom{s + t + r}{r}.
\]

(Note that \( p_n + m_n + 1 = \sum_i (m_i + 1) \), so by hypothesis the case \( p_n + m_n = r \) is maximal.)

**Proof.** We begin by observing that the final assertion, for the case \( p_n + m_n = r \), follows immediately from the rest of the lemma. Indeed, by the general formula, our Hilbert polynomial in this case agrees with that in the case \( n = 0, m_0 = r \), in which \( Q = Q_0 \) is simply the diagonal in \( \mathbb{P}^r \times \mathbb{P}^r \), and therefore has the desired Hilbert polynomial.

For the main statement of the lemma, the case \( n = 0 \) follows immediately from Lemma 4.4. (The stated form of the Hilbert polynomial follows from that in Lemma 4.4 by replacing \( \ell \) by \( m - \ell \) in the first sum and \( \ell \) by \( m - 1 - \ell \) in the second.) Now, suppose \( n > 0 \). Denote by \( Q' \) the union of the \( Q_i \) for \( i = 0, \ldots, n - 1 \), so that \( Q = Q' \cup Q_n \). Set \( Q'' := Q' \cap Q_n \). We claim that (scheme-theoretically)
\[ Q'' = V(x_0, \ldots, x_{p_n-1}, y_{p_n}, \ldots, y_r) \supseteq \mathbb{P}^r - p_n \times \mathbb{P}^{p_n-1}. \]

The lemma follows from the claim and induction on \( n \). Indeed, \( Q_n \) is reduced of pure dimension \( r \) by Lemma 4.4. So is \( Q' \) by induction, and thus so is their union \( Q \). Also, \( Q' \) and \( Q_n \) are connected, and thus so is \( Q \) because \( Q'' \) is nonempty by the claim. Furthermore, \( Q' \) and \( Q_n \) are Cohen–Macaulay, and thus so is \( Q \) by the claim and [4], Ex. 18.13, p. 467. Finally, we obtain the desired form for the Hilbert polynomial simply by applying induction, the claim, and the identity
\[
P_Q(s,t) = P_{Q'}(s,t) + P_{Q_n}(s,t) - P_{Q''}(s,t).
\]

We now prove the claim. Denote by \( J_i \) the ideal defining \( Q_i \), so that
\[
J_i = (x_0, \ldots, x_{p_i-1}, y_{m_i+p_i+1}, \ldots, y_r) + (x_j y_{\ell} - x_j y_{\ell}' \mid p_i \leq j, \ell \leq m_i + p_i).
\]

We need only check that
\[
J_n + (J_{n-1} \cap \cdots \cap J_0) = J_n + J_{n-1} = (x_0, \ldots, x_{p_n-1}, y_{m_n+p_n+1}, \ldots, y_r),
\]
from which the claim follows by noting that \( p_n = m_n + 1 + 1 \). The second equality above is trivial. As for the first, we clearly have \( J_n + (J_{n-1} \cap \cdots \cap J_0) \subseteq J_n + J_{n-1} \). To show the reverse inclusion we observe that \( x_0, \ldots, x_{p_n-1} \in J_n \), while \( y_{m_n+p_n+1}, \ldots, y_r \in J_{n-1} \cap \cdots \cap J_0 \). Thus we have proved the claim and the lemma. 

\[ \square \]
Lemma 4.6. In the situation of Lemma 4.5, if \( p_n + m_n = r \) then \( Q \) is a flat degeneration of the diagonal in \( \mathbb{P}^r \times \mathbb{P}^r \).

Proof. Consider the subscheme \( \hat{W} \) of \( \mathbb{P}^r \times \mathbb{P}^r \times (\mathbb{A}_z^1 \times \{0\}) \) given by the equations \( z^{c_{i,j}}x_i y_j - z^{c_{i,j}}x_j y_i = 0 \) for \( 0 \leq i \leq j \leq r \), where \( c_{j,i} = n - i \) if \( p_i \leq j \leq p_i + m_i \). At \( z = 1 \), we recover the equations of the diagonal in \( \mathbb{P}^r \times \mathbb{P}^r \), and we see moreover that for any \( z \neq 0 \), the scheme we obtain is simply a change of coordinates of the diagonal, given by \( y_j \mapsto z^{c_{i,j}}y_j \). Let \( W \) be the closure of \( \hat{W} \) in \( \mathbb{P}^r \times \mathbb{P}^r \times \mathbb{A}_z^1 \), and \( W_0 \) the fiber over \( z = 0 \). Then \( W_0 \) is a flat degeneration of the diagonal. Thus, by Lemma 4.5 we have that \( W_0 \) has the same Hilbert polynomial as \( Q \). Since, by the same lemma, \( Q \) is reduced, if we show that \( Q \subseteq W_0 \) set-theoretically, we conclude scheme-theoretic equality, and the lemma is proved.

Now, by definition a point \( ((a_0, \ldots, a_r), (b_0, \ldots, b_r)) \) of \( Q \) lies in \( Q_i \) for some \( i \). We then have

\[
a_0 = \cdots = a_{p_i - 1} = b_{m_i + p_i + 1} = \cdots = b_r = 0;
\]

also, there exist \( \lambda, \mu \) not both zero with \( \lambda a_i = \mu b_i \) for all \( i = p_i, \ldots, m_i + p_i \). Since \( Q_i \) is irreducible by Lemma 4.4, it is enough to show that a nonempty open subset of \( Q_i \) lies in \( W_0 \), so we may further assume that neither \((a_{p_i}, \ldots, a_{m_i + p_i})\) nor \((b_{p_i}, \ldots, b_{m_i + p_i})\) is identically zero, so that \( \lambda \) and \( \mu \) are both nonzero. Then we can renormalize the \( a_i \) and \( b_i \) so that \( \lambda = \mu = 1 \). In order to show that our point is in \( W_0 \), we define \( e_j = -\epsilon_j \) for \( j \geq p_i \), and \( e_j = -\epsilon_{p_i} \) for \( j < p_i \). Then set

\[
c_j = \begin{cases} a_j z^{c_{j,i}} : & j \geq p_i \\ b_j z^{c_{j,i}} : & \text{otherwise.} \end{cases}
\]

Then we evidently have the section

\[
((c_0 z^{a_0}, \ldots, c_r z^{a_r}), (c_0, \ldots, c_r))
\]

in \( W \) for all \( z \neq 0 \), and hence its limit at \( z = 0 \) is in \( W_0 \) by definition. To obtain the limit at \( z = 0 \) we have to divide through on the left and right by the minimum powers of \( z \) occurring on each side. On the right, since the \( c_j \) are nonincreasing in \( j \), this minimum is \( -\epsilon_{p_i} \), achieved precisely for the \( j \in \{0, \ldots, p_i + m_i\} \) such that \( b_j \neq 0 \). We thus see that the limit on the right is

\[
(b_0, \ldots, b_{p_i - 1}, a_{p_i}, \ldots, a_{p_i + m_i}, 0 \ldots 0) = (b_0, \ldots, b_r).
\]

Similarly, the minimum on the left is 0, achieved precisely for the \( j \in \{p_i, \ldots, r\} \) such that \( a_j \neq 0 \), so the limit is

\[
(0, \ldots, 0, a_{p_i}, \ldots, a_r) = (a_0, \ldots, a_r).
\]

We have thus shown that our chosen point is in \( W_0 \), and we conclude that \( Q_i \subseteq W_0 \) for each \( i \), and thus that \( Q \subseteq W_0 \), as desired. \( \square \)

We now relate the previous lemmas to \( \mathbb{P}(g) \). Given \( g \), we observe that for any \( i \) we have a closed embedding

\[
(4.1) \quad \mathbb{P}(V_i|_Y) \times \mathbb{P}(V_i|_Z) \longrightarrow A_{d-1}(\mathcal{L});
\]

see (3.1).
Definition 4.7. Let \( g = (L, V_0, \ldots, V_d) \) be an exact limit linear series. For each \( i \) such that \( V_i \neq V_i^{Y,0} \cup V_i^{Z,0} \), denote by \( \mathbb{P}(g_i) \) the closure of the subset of \( A_d^{-1}(L) \) consisting of points of the form

\[
\text{div}(s|y) + \text{div}(s|z) \in S^d(X),
\]

for \( s \in V_i - (V_i^{Y,0} \cup V_i^{Z,0}) \).

Observe that the construction of \( \mathbb{P}(g_i) \) visibly factors through (4.1), so we have

\[
\mathbb{P}(g_i) \subseteq \mathbb{P}(V_i|y) \times \mathbb{P}(V_i|z).
\]

Lemma 4.8. Let \( g = (L, V_0, \ldots, V_d) \) be an exact limit linear series. Let \( i \) be such that \( V_i \neq V_i^{Y,0} \cup V_i^{Z,0} \). Set \( p := \dim V_i^{Y,0} \), \( q := \dim V_i^{Z,0} \), and \( m := r - p - q \), and let \( s_0, \ldots, s_r \in V_i \) be any basis such that \( s_0, \ldots, s_p \) form a basis for \( V_i^{Y,0} \), and \( s_{r+1-q}, \ldots, s_r \) form a basis for \( V_i^{Z,0} \). Let the \( s_i \) induce coordinates \( x_p, \ldots, x_r \) on \( \mathbb{P}(V_i|y) \) and \( y_0, \ldots, y_{r-q} \) on \( \mathbb{P}(V_i|z) \).

Then if \( m = -1 \), we have

\[
\mathbb{P}(g_i) = \mathbb{P}(V_i|y) \times \mathbb{P}(V_i|z).
\]

If \( m \geq 0 \), we have

\[
\mathbb{P}(g_i) = Q_{p,q,m} \subseteq \mathbb{P}(V_i|y) \times \mathbb{P}(V_i|z),
\]

with notation as in Lemma 4.4.

(Note that since \( V_i^{Y,0} \cap V_i^{Z,0} = (0) \), we necessarily have \( p + q \leq r + 1 \), with equality occurring when \( \mathbb{P}(V_i|y) \times \mathbb{P}(V_i|z) \) has dimension \( r - 1 \).)

Proof. Since the kernel of \( V_i \to V_i|y \) is by definition \( V_i^{Y,0} \), we may view the points of \( \mathbb{P}(V_i|y) \) as the subspaces of \( V_i \) containing \( V_i^{Y,0} \) with codimension 1. An analogous interpretation holds for \( \mathbb{P}(V_i|z) \). With this point of view, consider the subset

\[
Q_i \subseteq \mathbb{P}(V_i|y) \times \mathbb{P}(V_i|z)
\]

parameterizing pairs \( (W_1, W_2) \) of subspaces of \( V_i \), with \( V_i^{Y,0} \subseteq W_1 \) and \( V_i^{Z,0} \subseteq W_2 \) of codimension 1, such that \( W_1 \cap W_2 \) is nontrivial. Then \( Q_i \) is a closed subset. If \( p + q = r + 1 \), it is clear that \( Q_i = \mathbb{P}(V_i|y) \times \mathbb{P}(V_i|z) \); otherwise we claim that \( Q_i \) agrees with the \( Q_{p,q,m} \) of Lemma 4.4.

Indeed, in the coordinates \( x_j \), we may represent a point of \( \mathbb{P}(V_i|y) \) by \( (a_r, \ldots, a_r) \). The corresponding subspace of \( V_i \) is then the span of \( s_0, \ldots, s_p, \sum_{i \geq j} a_i s_i \). Similarly, the point \( (b_0, \ldots, b_{r-q}) \in \mathbb{P}(V_i|z) \) corresponds to the subspace of \( V_i \) spanned by \( \sum_{i \leq r-q} b_i s_i, s_{r-q+1}, \ldots, s_r \). The intersection of these subspaces is nontrivial if and only if the span of all the above sections generate a subspace of dimension at most \( p + q + 1 \), thus if and only if the matrix

\[
\begin{bmatrix}
a_p & a_{p+1} & \ldots & a_{r-q-1} & a_{r-q} \\
b_p & b_{p+1} & \ldots & b_{r-q-1} & b_{r-q}
\end{bmatrix}
\]

has rank at most 1. Thus, in the coordinates \( x_j, y_m \), we see that \( Q_i \) is the closed subset given by the equations

\[
x_j y_m - x_m y_j = 0 \quad \text{for } j, m = p, \ldots, r - q.
\]

We thus conclude that \( Q_i = Q_{p,q,m} \), as claimed. In particular, by Lemma 4.4, we have that \( Q_i \) is integral.

It thus suffices to show (set-theoretically) that \( \mathbb{P}(g_i) = Q_i \). One containment is clear: let \( v \in V_i - (V_i^{Y,0} \cup V_i^{Z,0}) \). Then \( W_1 := kv + V_i^{Y,0} \) and \( W_2 := kv + V_i^{Z,0} \) are
subspaces of $V_i$ with nontrivial intersection, and thus the pair $(W_1, W_2)$ defines a point of $Q_i$. So $\mathbb{P}(g_i) \subseteq Q_i$. Since $\mathbb{P}(g_i)$ is defined via closure, it suffices to prove that it contains a nonempty open subset of $Q_i$. First, if $p + q = r + 1$, we have $V_i = V_i^{Y, 0} \oplus V_i^{Z, 0}$, so any $(W_1, W_2) \in Q_i = \mathbb{P}(V_i|Y) \times \mathbb{P}(V_i|Z)$ may be represented by $(w_1, w_2) \in (V_i^{Y, 0} \setminus \{0\}) \times (V_i^{Z, 0} \setminus \{0\})$, and then $(W_1, W_2)$ is the image of $w_1 + w_2$. Thus, we see in this case that $Q_i$ coincides with the set whose closure defines $\mathbb{P}(g_i)$, so closure is unnecessary in this case, and in particular $Q_i = \mathbb{P}(g_i)$. On the other hand, if $p + q < r + 1$, then there is a nonempty open subset of $Q_i$ consisting of $(W_1, W_2)$ such that $W_1 \cap V_i^{Z, 0} = 0$ and $W_2 \cap V_i^{Y, 0} = 0$. On this subset, any nonzero $v \in W_1 \cap W_2$ is neither in $V_i^{Y, 0}$ nor in $V_i^{Z, 0}$, and thus $W_1 = kv + V_i^{Y, 0}$ and $W_2 = kv + V_i^{Z, 0}$. So $(W_1, W_2) \in \mathbb{P}(g_i)$. We thus conclude $\mathbb{P}(g_i) = Q_i$, as desired.

We are now ready to complete the proof of our main theorem.

**Proof of Theorem 4.3.** Let $i_0, \ldots, i_r$ and $s_0, \ldots, s_r$ be as in Lemma 2.3. So the $s_j$ yield bases for each $V_i$, and in particular for $V_0$ and $V_q$. Since the restriction maps $V_0 \to V_0|Y$ and $V_q \to V_q|Z$ are isomorphisms, the $s_j$ induce coordinates on $\mathbb{P}(V_0|Y)$ and $\mathbb{P}(V_q|Z)$, to be denoted $x_0, \ldots, x_r$ and $y_0, \ldots, y_r$ respectively.

Let $m_0$ be the number of $j > 0$ with $i_j = i_0$, let $m_1$ be the number of $j > m_0 + 1$ with $i_j = i_{m_0 + 1}$, and so forth. Suppose that $n + 1$ is the number of distinct values of the $i_j$, so that we obtain a sequence of nonnegative numbers $m_0, \ldots, m_n$. In light of Lemmas 4.5 and 4.6, the theorem will follow if we show that under the coordinates $x_i, y_i$, we have that $\mathbb{P}(g)$ coincides with the $Q$ of Lemma 4.5. Following through the definitions and applying Lemma 4.8, we see that $Q$ is precisely the union of the $\mathbb{P}(g_i)$ for those $V_i$ with $i = i_j$ for some $j$. It thus suffices to show that if $i \neq i_j$ for any $j$, we get nothing new from $V_i$, that is, $\mathbb{P}(g_i)$ is already contained in some $\mathbb{P}(g_i)$.

According to Remark 4.2, if we either increase or decrease $i$ we will eventually get to some $i' \neq V_{i'}^{Y, 0} \oplus V_{i'}^{Z, 0}$, or equivalently $i' = i_j$ for some $j$. Suppose we chose to decrease $i$. Thus $i_j < i$, and there is no $i_j'$ with $i_j < i_j' \leq i$. It follows from exactness that for $i' > i_j$, with $i' \leq i$, we have $V_{i'}^{Y, 0}$ identified with $V_i^{Y, 0}$ under the map $V_{i'} \to V_i$, and $V_{i'}^{Z, 0}$ identified with $V_i^{Z, 0}$ under the map $V_{i'} \to V_i$. Thus, each $\mathbb{P}(g_{i'})$ is equal to $\mathbb{P}(g_i)$. Furthermore, we still have $V_i^{Z, 0}$ identified with $V_i^{Z, 0}$, and the codimension of the image of $V_i^{Y, 0}$ in $V_i^{Y, 0}$ is equal to $\dim V_{i_j}/(V_{i_j}^{Y, 0} \oplus V_{i_j}^{Z, 0}) = m_{i_j}$, if $i_j$ is the $(\ell + 1)$st value taken on by the $i_{j'}$. It then follows from the explicit descriptions given in Lemma 4.8 that $\mathbb{P}(g_i) \subseteq \mathbb{P}(g_{i_j})$. We conclude that $\mathbb{P}(g_i) \subseteq Q_i$, and thus the theorem. \hfill $\square$

**Remark 4.9.** It follows from the above proof that

$$\mathbb{P}(g) = \bigcup_{\ell=0}^{m} \mathbb{P}(g_i),$$

where $i_0, \ldots, i_m$ is the sequence of integers $i = 0, \ldots, d$ such that $V_i \neq V_i^{Y, 0} \oplus V_i^{Z, 0}$.

In the special case that $V_i^{Y, 0} \oplus V_i^{Z, 0}$ has codimension 1 in $V_i$, it follows from Lemma 4.8 that $\mathbb{P}(g_i)$ is isomorphic to an $r$-dimensional product of two projective spaces.
Such a situation arises frequently: Indeed, for a refined limit linear series (see Definition 6.5 of [7]) the codimension of $V_i^{Y,0} \oplus V_i^{Z,0}$ in $V_i$ is either 0 or 1 for every $i$. Thus, in this case, $\mathbb{P}(g)$ consists of a union of $r+1$ irreducible components, each isomorphic to an $r$-dimensional product of two projective spaces.

5. Limits of linear series

Let $B$ be the spectrum of a discrete valuation ring with algebraically closed residue field, and let $\eta$ denote its generic point.

**Definition 5.1.** Let $\pi: \mathcal{X} \to B$ be a flat, projective map, where $\mathcal{X}$ is regular, the generic fiber of $\pi$ is smooth, and the special fiber is isomorphic to $X$, the union of two smooth curves $Y$ and $Z$ meeting transversally at a point $P$. We call $\pi$ or $\mathcal{X}/B$ a regular smoothing of $X$.

Let $L_\eta$ be an invertible sheaf of degree $d$ on the generic fiber $\mathcal{X}_\eta$. Since $\mathcal{X}$ is regular, $L_\eta$ extends to an invertible sheaf $L$ on $\mathcal{X}$. Since $\mathcal{X}$ is regular, $Y$ and $Z$ are Cartier divisors on $\mathcal{X}$, and thus $L(-iZ) := L \otimes \mathcal{O}_\mathcal{X}(-iZ)$ are also extensions of $L_\eta$ for all $i$. Thus there is an extension $L$ such that $L|_X$ has degree $d$ on $Y$ and 0 on $Z$. Fix this extension $L$, and set $L'_i := L(-iZ)|_X$ for $i = 0, \ldots, d$.

Let $V_\eta$ be a vector subspace of $\Gamma(\mathcal{X}_\eta, L_\eta)$ of dimension $r+1$. Viewing $V_\eta$ as a subspace of $\Gamma(\mathcal{X}_\eta, L(-iZ)|_{X_\eta})$ for each $i = 0, \ldots, d$, set

$$\tilde{V}_i := \Gamma(\mathcal{X}, L(-iZ)) \cap V_\eta,$$

and denote by $V_i \subseteq \Gamma(\mathcal{X}, L')$ the image of the restriction of $\tilde{V}_i$ to the special fiber.

Notice that

$$\tilde{V}_i \cap \Gamma(\mathcal{X}, L(-(i+1)Z)) = \tilde{V}_{i+1}.$$

Thus, not only is the image of $V_{i+1}$ under the map on global sections induced by the natural map $L^{i+1} \to L'$ contained in $V_i$, but it is also equal to the subspace of $\tilde{V}_i$ of sections that vanish on $Z$. An analogous statement can be made with regard to the natural map $L' \to L^{i+1}$ in the reverse direction. So

$$g := (L^0, V_0, V_1, \ldots, V_d)$$

is an exact limit linear series. We say that $g$ is the limit of the linear series $(L_\eta, V_\eta)$.

**Theorem 5.2.** Let $\mathcal{X}/B$ be a regular smoothing of $X$ and $(L_\eta, V_\eta)$ a linear series of dimension $r$ and degree $d$ on the generic fiber. Let $g$ be the linear series that is limit of $(L_\eta, V_\eta)$. Then $\mathbb{P}(V_\eta)$, viewed as a subscheme of the fiber of the relative symmetric product $S^d(\mathcal{X}/B)$ over $\eta$, has closure intersecting $S^d(\mathcal{X})$ in $\mathbb{P}(g)$.

**Proof.** Up to making an étale base change, we may assume that $\mathcal{X}/B$ has two sections $\Sigma_1$ and $\Sigma_2$, the first intersecting $Y$ away from $P$, the second intersecting $Z$ away from $P$. Let

$$H_{s,t} := s(\Sigma_1 + S^{d-1}(\mathcal{X}/B)) + t(\Sigma_2 + S^{d-1}(\mathcal{X}/B))$$

for each $s, t \in \mathbb{Z}$. The restriction of $H_{s,t}$ to $\mathbb{P}(V_\eta)$ is equal to $O_{\mathbb{P}(V_\eta)}(s + t)$, thus

$$P_{\mathbb{P}(V_\eta)}(s, t) := \dim \Gamma(\mathbb{P}(V_\eta), O_{S^d(\mathcal{X}/B)}(H_{s,t}))|_{\mathbb{P}(V_\eta)} = \binom{s + t + r}{r}$$

for $s, t >> 0$. On the other hand, also

$$P_{\mathbb{P}(g)}(s, t) := \dim \Gamma(\mathbb{P}(g), O_{S^d(\mathcal{X}/B)}(H_{s,t}))|_{\mathbb{P}(g)} = \binom{s + t + r}{r}$$

are the natural maps.
for $s, t >>> 0$ by Theorem 4.3. Thus, we need only show that the closure of $\mathbb{P}(V_\eta)$ contains $\mathbb{P}(g)$.

Consider now on the product $X \times_B \mathbb{P}(\tilde{V}_i)$ the composition

$$O_{\mathbb{P}(\tilde{V}_i)}(-1) \rightarrow \tilde{V}_i \rightarrow L(-iZ)$$

where the first map is the tautological map of $\mathbb{P}(\tilde{V}_i)$ and the second is the evaluation map, all sheaves and maps being viewed on the product under the appropriate pullback. Let $F$ denote the degeneracy scheme of this composition. Then $F$ is a relative Cartier divisor of degree $d$ pullbacks. Let $\eta$ denote the degeneracy scheme of this composition. Then $\eta$ consists of a pair $((\mathcal{L}', V'), (\mathcal{L}^Z, V^Z))$ of linear series, and every Eisenbud–Harris limit linear series arises in this way. As to Proposition 6.6 of [7], the resulting pair is in fact an Eisenbud–Harris limit linear series, and every Eisenbud–Harris limit linear series arises in this way. As a consequence, the statement of Proposition 6.1 is equally valid in either context. Note also that the converse of the statement does not hold: even if the fiber of the Abel maps has every component of dimension at least $r$, there may be no limit linear series of dimension $r$; see Example 7.3 below.

We will also find the following terminology convenient:

**Definition 6.3.** Suppose $\mathcal{L}$ is a line bundle on $X$ of degree $d$ on $Y$ and degree 0 on $Z$. Then the **vanishing sequence** of $\mathcal{L}$ on $Y$ at $P$ is the vanishing sequence at $P$...
of the complete linear series for $\mathcal{L}|_Y$. Similarly, the vanishing sequence of $\mathcal{L}$ on $Z$ at $P$ is the vanishing sequence at $P$ of the complete linear series for $(\mathcal{L}|_Z)(dP)$.

**Lemma 6.4.** Let $\mathcal{L}$ be a line bundle on $X$ of degree $d$ on $Y$ and degree $0$ on $Z$, with vanishing sequences $a_1^Y, \ldots, a_s^Y$ and $a_1^Z, \ldots, a_t^Z$ on $Y$ and $Z$ respectively at $P$. For $\ell$ between $0$ and $d$, the subset $\mathbb{P}(\Gamma^\ell_Y) \times \mathbb{P}(\Gamma^{d-\ell}_Z)$ inside $\mathcal{A}^{-1}_d(\mathcal{L})$ constitutes an irreducible component of the fiber if and only if there exist $\ell'$ such that $a_i^Y = d - \ell'$, $a_j^Z = \ell''$, $a_{i-1}^Y < d - \ell''$, and $a_{j-1}^Z < \ell'$. In this case, the dimension of $\mathbb{P}(\Gamma^\ell_Y) \times \mathbb{P}(\Gamma^{d-\ell}_Z)$ is equal to $p + q - i - j$.

**Proof.** Because of (3.1), since each subset $\mathbb{P}(\Gamma^\ell_Y) \times \mathbb{P}(\Gamma^{d-\ell}_Z)$ is irreducible, one such subset is an irreducible component of the fiber if and only if it is not strictly contained in any other. There are only two ways such a strict containment could occur: as $\ell$ decreases, we could have $\mathbb{P}(\Gamma^\ell_Y)$ remaining unchanged until after $\mathbb{P}(\Gamma^{d-\ell}_Z)$ has strictly increased, or as $\ell$ increases, we could have $\mathbb{P}(\Gamma^{d-\ell}_Z)$ remaining unchanged until after $\mathbb{P}(\Gamma^\ell_Y)$ has strictly increased. These two conditions are precisely what is ruled out by the conditions in the statement of the lemma.

**Proof of Proposition 6.1.** According to Lemma 6.4, any irreducible component of the fiber is of dimension $p + q - i - j$, where $i, j$ satisfy $a_{i-1}^Y + a_j^Z < d$. We prove that if there is an Eisenbud–Harris limit linear series with underlying line bundle $\mathcal{L}$ and dimension $r$, then we must have $p + q - i - j \geq r$. Given $\mathcal{L}$, such a limit series is determined by the spaces $V^Y, V^Z$, which must have their vanishing sequences at $P$ being subsequences of length $r + 1$ of the $a_i^Y$ and $a_j^Z$; suppose these are given by indices $i_0 < i_1 < \cdots < i_r$ and $j_0 < j_1 < \cdots < j_r$. Then by the definition of an Eisenbud–Harris limit series, we must have $a_{i_s}^Y + a_{j_{r-s}}^Z \geq d$ for $s = 0, \ldots, r$. Choose $s$ maximal with $i_s \leq i - 1$. Then $i_{s+1}, \ldots, i_r \geq i$, so in order to have enough remaining choices of indices, we conclude that $r - s \leq p + 1 - i$. On the other hand, since $a_{i_s}^Y + a_{j_{r-s}}^Z \geq d$, we have

$$a_{j_{r-s}}^Z \geq d - a_{i_s}^Y \geq d - a_{i-1}^Y > a_j^Z,$$

so as before we conclude $s + 1 = r + 1 - (r - s) < q + 1 - j$. Putting the two inequalities together we conclude that $r \leq p + q - i - j$, as desired.

7. Examples and Further Discussion

In general, the map from exact limit linear series to subschemes of fibers of Abel maps need not be injective. The reason is essentially that if $V_i$ is obtained by gluing together sections of $\mathbb{P}(\Gamma^{d-i}_Y)$ and $\mathbb{P}(\Gamma^i_Z)$ which vanish at the node $P$, but we do not have $V_i = V_i^{Y,0} \oplus V_i^{Z,0}$, then scaling sections on $Z$ while holding them fixed on $Y$ will yield different choices for the subspaces $V_i$, but will not change the associated subsets of $S^d(X)$. For instance, if $r = 0$ an exact limit linear series is uniquely determined by a single choice of section of some $\mathcal{L}^i$ which does not vanish on either $Y$ or $Z$. If this section vanishes at $P$, we may obtain different choices of $V_i$ by scaling the section on $Z$ while holding its value on $Y$ fixed. In this case, we see that the map from the moduli space of exact limit linear series to that of subschemes of fibers of the Abel map actually factors through the space of Eisenbud–Harris limit linear series. However, this is not the case in general.

**Example 7.1.** Suppose $X$ has genus $0$, and set $d = 2$, $r = 1$. In this case we have a 2-dimensional family of exact limit series which all correspond to the same
Eisenbud–Harris limit series. Explicitly, if we choose coordinates \( y, z \) on \( Y \) and \( Z \) such that \( P \) is \( y = 0 \) and \( z = 0 \), we may represent sections by pairs of polynomials in \( y \) and \( z \) of degree at most 2. Then \((y, y^2), (z, z^2)\) gives a (crude) Eisenbud–Harris limit linear series. The corresponding choices of limit linear series in our sense are (as always) uniquely determined for \( V_0, V_2 \), but \( V_1 \) may be any 2-dimensional subspace of the vector space of pairs \((a_1 y + a_2 y^2, a_1 z + a_3 z^2)\).

This vector space is 3-dimensional, so we have a 2-dimensional projective space of choices for \( V_1 \). The exact limit series are an open subset (specifically, those spaces which remain 2-dimensional after restriction to either \( Y \) or \( Z \)), and we see that depending on our choice of 2-dimensional subspace, we will obtain different pencils of corresponding divisors, and hence different 1-dimensional subschemes of the fiber of the Abel map.

Thus, we cannot define our map using Eisenbud–Harris limit series instead of our limit series, and in particular this is not a viable approach to resolving the lack of injectivity. However, it appears that the failure of injectivity may be resolved if instead of associating subschemes of \( S^d(X) \) to a given limit series, we associate subschemes of \( \text{Hilb}^d(X) \).

One may also wonder to what extent the map we have constructed is surjective. There is more than one way to interpret this question, and we consider more specifically whether each fiber of the Abel map is the union of the closed subschemes we construct. For this question, it is necessary to specify further which \( r \) we consider. If we take \( r = 0 \), we have very few constraints on constructing limit linear series, but since every exact limit series yields a point of the fiber of the Abel map, and since we do not associate subschemes of the fiber to a non-exact limit linear series, we should only expect to get an open subset of the fiber in this way, and this is indeed what happens.

**Example 7.2.** Suppose \( Y \) and \( Z \) are both elliptic curves. Consider the case \( d = 1 \), and \( \mathcal{L} \) the line bundle of degree 1 such that the degree-1 restrictions to \( Y \) and \( Z \) are both isomorphic to \( O(P) \). Then each restriction has a unique nonzero section up to scalar, which vanishes to order 1 at \( P \). The fiber \( A_{1}^{-1}(\mathcal{L}) \) is the single point \( P \), which we may view either as \( \mathbb{P}(\Gamma_0^Y) \times \mathbb{P}(\Gamma_1^Z) \) or as \( \mathbb{P}(\Gamma_1^Y) \times \mathbb{P}(\Gamma_2^Z) \). However, while there is a non-exact limit series of dimension 0 with underlying line bundle \( \mathcal{L} \), there is no exact one, so we are unable to associate any subschemes to this fiber.

The lack of surjectivity exhibited in the previous example could in principle be addressed by extending our construction to non-exact limit linear series. However, we see that for \( r > 0 \), there may not be any limit linear series even when the fiber of the Abel map is pure of dimension \( r \).

**Example 7.3.** With \( Y \) and \( Z \) still both elliptic curves, suppose \( d = 2 \), and the degree-2 restrictions of \( \mathcal{L} \) to \( Y \) and to \( Z \) are \( O_Y(2P) \) and \( O_Z(P + Q) \) for some \( Q \neq P \), respectively. Then \( \mathcal{L} \) has vanishing sequences at \( P \) given by \( a_0^Y = 0, a_1^Y = 2 \) and \( a_0^Z = 0, a_1^Z = 1 \). We thus see that the corresponding fiber of \( A_2 \) is equal to \( \mathbb{P}(\Gamma_0^Y) \times \mathbb{P}(\Gamma_2^Z) \cong \mathbb{P}^1 \). However, we see immediately that there is no Eisenbud–Harris limit linear series of dimension 1 with underlying line bundle \( \mathcal{L} \), and thus no limit linear series in our sense, either.

Thus, to address the surjectivity question one also has to determine what range of \( r \) is appropriate to consider for a given fiber of the Abel map.
Finally, we give an example for which the fiber of the Abel map is not equidimensional.

**Example 7.4.** With \(Y\) and \(Z\) still both elliptic curves, suppose \(d = 3\), and the degree-3 restrictions of \(\mathcal{L}\) to \(Y\) and \(Z\) are \(\mathcal{O}_Y(3P)\) and \(\mathcal{O}_Z(2P+Q)\) for some \(Q \neq P\), respectively. Then \(\mathcal{L}\) has vanishing sequences at \(P\) given by \(a_0^Y = 0, a_1^Y = 1, a_2^Y = 3,\) and \(a_0^Z = 0, a_1^Z = 1, a_2^Z = 2\). In this case, we see that \(A_{3}^{-1}(\mathcal{L})\) is the union of \(\mathbb{P}(\Gamma_0^Y) \times \mathbb{P}(\Gamma_3^Z)\) and \(\mathbb{P}(\Gamma_2^Y) \times \mathbb{P}(\Gamma_1^Z)\), which have dimension 2 and 1 respectively, and intersect only at a point.

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