Many-body-localization: strong disorder perturbative approach for the local integrals of motion

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Abstract
For random quantum spin models, the strong disorder perturbative expansion of the local integrals of motion around the real-spin operators is revisited. The emphasis is on the links with other properties of the many-body-localized phase, in particular the memory in the dynamics of the local magnetizations and the statistics of matrix elements of local operators in the eigenstate basis. Finally, this approach is applied to analyze the many-body-localization transition in a toy model studied previously from the point of view of the entanglement entropy.

Keywords: many-body-localization, local integrals of motion, strong disorder perturbation

1. Introduction
In the field of many-body-localization (MBL) (see the recent reviews [1–8] and references therein), the notion of local integrals of motion (LIOMs) has emerged as an essential notion to understand the various properties of the many-body-localized phase [9–21]. However, the general definitions of LIOMs may remain somewhat abstract and elusive for the newcomers in the field, so that it seems useful to have a more concrete picture in the simplest limit, namely in the strong disorder limit deep in the MBL-phase, where the LIOMs remain perturbatively close to the real-space degrees of freedom defining the model. In addition, the notion of LIOMs is often used at a qualitative level to explain the behavior of various observables, so it is important to discuss the quantitative link in this strong disorder limit with other signatures of the MBL-phase. The goal of this paper is thus to revisit the strong disorder perturbative approach for the local integrals of motion from this perspective and to give explicit calculations up to second order for various quantum spin models.
The paper is organized as follows. In section 2, we recall how the pseudo-spins can be constructed from the true spins by the unitary transformation that diagonalizes the Hamiltonian. In section 3, we describe how the expansion of the true spins in the Pauli basis of the pseudo-spins is related to the matrix elements of a single spin operator in the eigenstate basis and to the dynamics of the local magnetizations. In section 4, we mention the reciprocal expansion of the pseudo-spins in terms of the true spins. In section 5, the lowest order of the strong disorder perturbative expansion for the pseudo-spins is described in detail. The application to the random field XXZ chain is given in section 6. The application to the toy model considered in [22, 23] is studied in section 7 to analyze the stability of the MBL-phase. Our conclusions are summarized in section 8. Finally, the appendix A describes the non-perturbative notion of LIOMs for the trivial case involving only two spins, while the appendix B contains some useful results on Lévy sums of correlated variables.

2. Definition of LIOMS in the strong disorder limit

2.1. Random quantum spin models

Let us consider a model of $N = L^d$ quantum spins $\sigma_i$ described by the hermitian Pauli matrices at each site

$$\sigma^{(0)} = \hat{I}d = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma^x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma^y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma^z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(1)

with an Hilbert space of size

$$\mathcal{N} = 2^N = 2^{L^d}.$$  

(2)

The Hamiltonian can be decomposed into a diagonal part and an off-diagonal part in the $\sigma^z$ basis

$$H = H^{\text{diag}} + H^{\text{off}}.$$  

(3)

The diagonal part $H^{\text{diag}}$ contains disorder variables such as random fields $h_j$ that are drawn with some continuous distribution to avoid any exact degeneracy in the spectrum.

2.2. Unitary transformation diagonalizing the Hamiltonian

When the off-diagonal part vanishes $H^{\text{off}} = 0$, the $\mathcal{N} = 2^N$ eigenstates of $H^{\text{diag}}$ are simply labelled by the eigenvalues of the $\sigma^z$

$$|\psi^{(0)}_{S_1,\ldots,S_N}\rangle \equiv |\sigma^z_1 = S_1, \sigma^z_2 = S_2, \ldots, \sigma^z_N = S_N\rangle$$

(4)

where the random energies

$$E^{(0)}_{S_1,\ldots,S_N} = \langle S_1S_2\ldots S_N|H^{\text{diag}}|S_1S_2\ldots S_N\rangle$$

(5)

are non-degenerate as a consequence of the random fields. Then each eigenstate can be followed via the non-degenerate perturbation theory in $H^{\text{off}}$, and this defines a unitary transformation between the basis of unperturbed eigenstates and the basis of perturbed eigenstates

$$|\psi_{S_1,\ldots,S_N}\rangle = U|S_1S_2\ldots S_N\rangle = \sum_{S'_1,\ldots,S'_N} |S'_1S'_2\ldots S'_N\rangle \langle S'_1S'_2\ldots S'_N |U|S_1S_2\ldots S_N\rangle.$$  

(6)
It is then interesting to consider the action of this unitary transformation of the spin operators with $a = x, y, z$ [15, 20, 24]

$$\tau^a_n = U \sigma^a_n U^\dagger$$

(7)
because these pseudo-spins $\tau_n$ inherit the commutation relations of the true spins $\sigma_n$.

2.3. Interpretation of the eigenstates in terms of the pseudo-spins

The unperturbed eigenstate of equation (4) is associated to the projector

$$|S_1 S_2 \ldots S_N \rangle \langle S_1 S_2 \ldots S_N| = \prod_{i=1}^{N} \left( \frac{1 + S_i \sigma^z_i}{2} \right).$$

(8)

Its transformation via the unitary transformation $U$ reads using the definition of pseudo-spins in equation (7)

$$U|S_1 S_2 \ldots S_N \rangle \langle S_1 S_2 \ldots S_N| U^\dagger = \prod_{i=1}^{N} \left( \frac{1 + S_i \tau^z_i}{2} \right) = |\tau^1_1 = S_1, \ldots, \tau^N_N = S_N \rangle \langle \tau^1_1 = S_1, \ldots, \tau^N_N = S_N|$$

(9)

and thus corresponds to the projector on the state $\tau^1_1 = S_1$. On the other hand, by definition of the unitary transformation of equation (6), this coincides with the projector onto the perturbed eigenstate $|\psi_{S_1 \ldots S_N}\rangle$

$$U|S_1 S_2 \ldots S_N \rangle \langle S_1 S_2 \ldots S_N| U^\dagger = |\psi_{S_1 \ldots S_N}\rangle \langle \psi_{S_1 \ldots S_N}|.$$  

(10)

By identification, one obtains that the perturbed eigenstates $|\psi_{S_1 \ldots S_N}\rangle$ corresponds to the eigenstate of the pseudo-spin operators $\tau^1_1 = S_1$

$$|\psi_{S_1 \ldots S_N}\rangle = |\tau^1_1 = S_1, \tau^2_2 = S_2, \ldots, \tau^N_N = S_N \rangle.$$

(11)

The $N$ pseudo-spins $\tau^j_1$ are thus a very convenient way to label the $2^N$ eigenstates.

2.4. Hamiltonian in terms of the pseudo-spins

The Hamiltonian is diagonal in the basis of the eigenstates labelled by the pseudo-spins (equation (12))

$$H = \sum_{T_1 = \pm \ldots T_N = \pm} E_{T_1 \ldots T_N} |\tau^1_1 = T_1, \tau^2_2 = T_2, \ldots, \tau^N_N = T_N \rangle \langle \tau^1_1 = T_1, \tau^2_2 = T_2, \ldots, \tau^N_N = T_N|$$

(12)

and can be thus rewritten at the operator level as

$$H = \sum_{a_1 = 0, z} \ldots \sum_{a_N = 0, z} \mathcal{H}_{a_1 \ldots a_N} \tau^{(a_1)}_1 \tau^{(a_2)}_2 \ldots \tau^{(a_N)}_N$$

(13)

where the $N' = 2^N$ coefficients $\mathcal{H}_{a_1 \ldots a_N}$ can be computed to reproduce the $2^N$ energies $E_{T_1 \ldots T_N}$

2.5. Pauli basis of the pseudo-spins

More generally, the notion of the basis of Pauli matrices for operators is very useful [15, 25, 26]. Any operator $O$ can be thus expanded in the basis of Pauli matrices of the $N$ pseudo-spins as [15]
\[ O = \sum_{a_1=0,0,\ldots} \sum_{a_N=0,0,\ldots} O(a_1 \ldots a_N) \tau_1^{(a_1)} \tau_2^{(a_2)} \ldots \tau_N^{(a_N)} = \sum_{\bar{a}} O(\bar{a}) \tau_1^{(a_1)} \tau_2^{(a_2)} \ldots \tau_N^{(a_N)} \]  
\tag{14}

where the \(4^N\) coefficients labelled by \(\bar{a} = (a_1, \ldots, a_N)\) read

\[ O(\bar{a}) = \frac{1}{2^N} \text{Tr} (O \tau_1^{(a_1)} \tau_2^{(a_2)} \ldots \tau_N^{(a_N)}). \]  
\tag{15}

### 3. Expansion of the real spins in terms of the pseudo-spins

It is interesting to consider the expansion of equation (14) for the real-spin-operator \(O = \sigma_z^n\)

\[ \sigma_z^n = \sum_{\bar{a}} S_n(\bar{a}) \tau_1^{(a_1)} \tau_2^{(a_2)} \ldots \tau_N^{(a_N)}. \]  
\tag{16}

#### 3.1. Properties of the coefficients

The coefficients

\[ S_n(\bar{a}) = \frac{1}{2^N} \text{Tr} (\sigma_z^n \tau_1^{(a_1)} \tau_2^{(a_2)} \ldots \tau_N^{(a_N)}) \]  
\tag{17}

are real

\[ (S_n^*(\bar{a}))^* = S_n(\bar{a}) \]  
\tag{18}

as a consequence of hermiticity of Pauli matrices, while the zero-coefficient vanishes as a consequence of the vanishing trace of Pauli matrices

\[ S_n(\bar{0}) = \frac{1}{2^N} \text{Tr} (\sigma_z^n) = 0. \]  
\tag{19}

The condition of identity for the square

\[ I = (\sigma_z^n)^2 = \sum_{\bar{a}} S_n(\bar{a}) \sum_{\bar{a}'} S_n^*(\bar{a}') \tau_1^{(a_1)} \tau_2^{(a_2)} \ldots \tau_N^{(a_N)} \tau_1^{(a_1')} \tau_2^{(a_2')} \ldots \tau_N^{(a_N')}. \]  
\tag{20}

yields in particular by taking the trace

\[ 1 = \frac{1}{2^N} \text{Tr} ((\sigma_z^n)^2) = \sum_{\bar{a}} S_n^2(\bar{a}) \]  
\tag{21}

that the sum of the square of the \(4^N\) coefficients \(S_n(\bar{a})\) is normalized to unity.

#### 3.2. Matrix elements of a single real spin operator in the eigenstate basis

The behavior of matrix elements of local operators in the eigenstate basis has been proposed as an important criterion to distinguish the many-body-localized phase from the delocalized phase [24, 27–29]. It is thus interesting to consider the consequence of the expansion equation (16) on the matrix element of the single real-spin operator \(\sigma_z^n\) between two eigenstates labelled as \(|\bar{T}\rangle = |\tau_1^T = T_1, \ldots, \tau_N^T = T_N\rangle\) and \(|\bar{F}\rangle = |\tau_1^F = T_1', \ldots, \tau_N^F = T_N'\rangle\)

\[ \langle \bar{T} | \sigma_z^n | \bar{F} \rangle = \sum_{\bar{a}} S_n(\bar{a}) \tau_1^{(a_1)} \tau_2^{(a_2)} \ldots \tau_N^{(a_N)}. \]
\begin{equation}
\langle \vec{T} | \sigma_n^z | \vec{T}' \rangle = \langle T_1, \ldots, T_N \rangle \sum_{\vec{a}} S_n(\vec{a}) \tau_1^{(a_1)} \cdots \tau_N^{(a_N)} | T_1', \ldots, T_N' \rangle = \sum_{\vec{a}} S_n(\vec{a}) \prod_{i=1}^N \langle T_i | \tau_i^{(a_i)} | T_i' \rangle.
\end{equation}

To characterize the amplitude of these matrix elements, it is convenient to introduce the Edwards–Anderson matrix \([28, 29]\)

\begin{equation}
Q_{\vec{T}, \vec{T}'} \equiv | \langle \vec{T} | \sigma_n^z | \vec{T}' \rangle |^2 = \langle \vec{T} | \sigma_n^z | \vec{T} \rangle \langle \vec{T} | \sigma_n^z | \vec{T}' \rangle
\end{equation}

that has the nice property to be doubly stochastic, i.e. it is a square matrix of size \(N \times N\) of non-negative real numbers, where the sums over any row or any column is unity

\begin{equation}
\sum_{\vec{p}} Q_{\vec{T}, \vec{p}} = 1 = \sum_{\vec{p}} Q_{\vec{p}, \vec{T}}.
\end{equation}

as a consequence of the completeness identity for the eigenstate basis and the Pauli matrix identity \((\sigma_i^z)^2 = I\).

The normalization of equation (24) means that for a fixed eigenstate \(| \vec{T} \rangle\), the \(N\) numbers \(Q_{\vec{T}, \vec{p}}\) can be interpreted as \(N\) weights normalized to unity. Their statistics can be characterized by the multifractal formalism as follows (see more details in [29]) : the number of weights of order \(Q_{\vec{T}, \vec{p}} \propto N^{-\alpha}\) among the \(N\) weights scales as

\begin{equation}
\text{Number}(Q_{\vec{T}, \vec{p}} \propto N^{-\alpha}) \propto N^{f(\alpha)}.
\end{equation}

Equivalently, the generalized moments can be computed via a saddle-point analysis in the exponent \(\alpha\)

\begin{equation}
Y_q \equiv \sum_{\vec{p}} | \langle \vec{T} | \sigma_n^z | \vec{T}' \rangle |^{2q} = \int d\alpha N^{f(\alpha) - q \alpha} = N^{-\tau(q)}
\end{equation}

with the usual Legendre transformation between the multifractal spectrum \(f(\alpha)\) and the exponents \(\tau(q)\)

\begin{align}
f(\alpha) - q \alpha &= -\tau(q) \\
f'(\alpha) - q &= 0
\end{align}

When the state \(| \vec{T} \rangle\) is in the middle of the spectrum, the the Ergodic phase where the eigenstate thermalization hypothesis (ETH) [30–34] holds is characterized by the monofractal

\begin{equation}
f^{\text{ETH}}(\alpha) = \delta(\alpha - 1)
\end{equation}

i.e. there is an extensive number \(O(N)\) of weights that are of order \(Q_{\vec{T}, \vec{p}} \propto 1/N\), so that the generalized moments scale linearly in \(q\) as

\begin{equation}
Y_q^{\text{ETH}} \propto N^{1-q}.
\end{equation}

On the contrary in the limit where \(H_{\text{eff}} = 0\) where the pseudo spins \(\tau^z\) coincides with the true spins \(\sigma^z\), only one weight is non-zero \(Q_{\vec{T}, \vec{p}} = \delta_{\vec{T}, \vec{p}}\). More generally in the many-body-localized phase, this weight remains finite

\begin{equation}
Q_{\vec{T}, \vec{T}'}^{\text{MBL}} = O(1)
\end{equation}

i.e. the multifractal spectrum begins at \(\alpha = 0\) with \(f^{\text{MBL}}(\alpha = 0) = 0\). Introducing the linear slope around the origin

\begin{equation}
f^{\text{MBL}}(\alpha) = q \alpha + o(\alpha)
\end{equation}
one obtains that the saddle-point calculation of equation (26) is dominated by this boundary
\[ \alpha = 0 \] for \( q > q_c \), yielding finite generalized moments
\[ Y_{\text{MBL}}^{\text{q}} = O(1). \] (32)

3.3. Dynamics of the local magnetizations
The dynamics of the local magnetizations
\[ \langle \sigma^z_n \rangle_t = \text{Tr} (\sigma^z_n e^{-iHt} \rho(t=0) e^{iHt}) \] (33)
can be considered as the most important criterion to distinguish the many-body-localized phase from the delocalized phase. To make the link with the above framework, it is convenient to focus on the simple initial density matrix [15]
\[ \rho(t=0) = \frac{1 + \sigma^z_{n_0}}{2N} \] (34)
corresponding to magnetization unity on the site \( n_0 \) and zero magnetization on all the other sites
\[ \langle \sigma^z_n \rangle_t = \delta_{n,n_0}. \] (35)

The expansion upon the basis of eigenstates \( |\vec{T}\rangle = |T_1, \ldots, T_N\rangle \)
\[ \langle \sigma^z_n \rangle_t = \sum_{\vec{T}} \sum_{\vec{T}'} \langle \vec{T} | \sigma^z_n | \vec{T}' \rangle \langle \vec{T}' | \rho(0) | \vec{T} \rangle e^{i(E_{\vec{T}} - E_{\vec{T}'})t} \] (36)
yields that the time-average eliminates the off-diagonal terms \( \vec{T}' \neq \vec{T} \)
\[ \frac{1}{t_{\text{max}}} \int_0^{t_{\text{max}}} dt \langle \sigma^z_n \rangle_t = \frac{1}{t_{\text{max}}} \int_{t_{\text{max}} \rightarrow +\infty} dt \langle \vec{T} | \sigma^z_n | \vec{T} \rangle \langle \vec{T} | \rho(0) | \vec{T} \rangle \equiv m^\infty_n. \] (37)

For the initial condition of equation (34), one obtains that these magnetization \( m^\infty_n \) are directly related to the matrix elements discussed above
\[ m^\infty_n = \frac{1}{2N} \sum_{\vec{T}} \langle \vec{T} | \sigma^z_n | \vec{T} \rangle \langle \vec{T} | \sigma^z_{n_0} | \vec{T} \rangle. \] (38)

In particular, an important property of the many-body-localized phase is the presence of some memory of the initial magnetization unity on the site \( n = n_0 \)
\[ m^\infty_{n=n_0} = \frac{1}{2N} \sum_{\vec{T}} (\langle \vec{T} | \sigma^z_n | \vec{T} \rangle)^2 = \frac{1}{2N} \sum_{\vec{T}} Q_{\vec{T},\vec{T}} \] (39)
which involves an average over the eigenstate \( \vec{T} \) of the diagonal terms \( Q_{\vec{T},\vec{T}} \) of equation (23).

4. Expansion of the LIOMs in terms of the real spins
4.1. Expansion of the pseudo-spins in the Pauli basis of the real spins
Reciprocally, it is interesting to consider the expansion of the LIOMs \( \tau^z_n \) in the Pauli basis of the true spins \( \sigma_i \)
\[ \tau_n^z = \sum_{\vec{a}} T_n(\vec{a}) \sigma_1^{(a_1)} \sigma_2^{(a_2)} ... \sigma_N^{(a_N)} \]  
(40)

where the coefficients
\[ T_n(\vec{a}) = \frac{1}{2N} \text{Tr} (\tau_n^z \sigma_1^{(a_1)} \sigma_2^{(a_2)} ... \sigma_N^{(a_N)}) \]  
(41)

are real, \( T_n^*(\vec{a}) = T_n(\vec{a}) \), satisfy \( T_n(\vec{0}) = 0 \) and the normalization similar to equation (21)
\[ 1 = \frac{1}{2N} \text{Tr} (\tau_n^z)^2 = \sum_{\vec{a}} T_n^2(\vec{a}). \]  
(42)

4.2. Overlap between one pseudo-spin and the corresponding real spin

The overlap \( O_n \) between the real spin \( \sigma_n^z \) and the corresponding pseudo-spin \( \tau_n^z = U \sigma_n U^\dagger \) (equation (7)) appears as the coefficient associated to \( (a_n = z; a_{i \neq n} = 0) \) either in the expansion of equation (16) or in the expansion of equation (40)
\[ O_n \equiv \frac{1}{2N} \text{Tr} (\tau_n^z \sigma_n^z) = S_n(\sigma_n^z) = T_n(\sigma_n^z) = T_n(\sigma_n^z = z; a_{i \neq n} = 0). \]  
(43)

It is the direct measure of the locality of the pseudo-spin \( \tau_n^z \); it remains finite in the many-body-localized phase
\[ O_n^{\text{MBL}} = O(1) \]  
(44)

while it vanishes in the thermodynamic limit in the delocalized phase.

4.3. Dynamics from a fixed initial condition

The \( N \) operators \( \tau_n^z \) commuting with each other and with the Hamiltonian represent \( N \) elementary integrals of motion, from which one can generate all the other ones by linear combination of products \( \tau_n^z \tau_m^z ... \) (up to the \( 2^N \) projectors on the eigenstates of equation (9)). Within the strong disorder perturbative expansion, the \( N \) pseudospins \( \tau_n^z \) are thus clearly the extensive set of the most ‘local’ integrals of motion that one can construct.

If one starts from a given initial condition in the physical \( \sigma \) basis \( |\psi(t=0)\rangle = |S_1,..,S_N\rangle \) [20], the values of these integrals of motion read
\[ \langle \tau_n^z(t) \rangle_t = \langle \tau_n^z(0) \rangle_{t=0} = \langle S_1,..,S_N | \tau_n^z | S_1,..,S_N \rangle. \]  
(45)

5. Self-consistent first-order perturbative expansion

5.1. Standard first order perturbation theory

For \( H_{\text{eff}} = 0 \), the \( N = 2^N \) eigenstates of \( H_{\text{diag}} \) are simply given by the tensor products of equation (4). The usual non-degenerate first order perturbation theory yields the eigenstates using the simplified notation \( |\vec{S}\rangle = |S_1,..,S_N\rangle \)
\[ |\psi_\vec{S}^{(0+1)} \rangle = |\vec{S}\rangle + \sum_{\vec{S'} \neq \vec{S}} |\vec{S}\rangle \frac{\langle \vec{S} | H_{\text{eff}} | \vec{S'} \rangle}{E_{\vec{S'}} - E_{\vec{S}}} |\vec{S'}\rangle \]  
(46)
This result can be directly translated for the perturbative expansion of the unitary transformation $U$

$$U = 1 + U_1$$ \hfill (47)

describing this change of basis (equation (6))

$$|\psi_S^{(0+1)}\rangle = (1 + U_1)|\tilde{S}\rangle.$$ \hfill (48)

The identification with equation (46) yields the matrix elements of the first order $U_1$

$$\langle \tilde{S}| U_1 |\tilde{S}\rangle = \frac{\langle \tilde{S}| H_{\text{off}} |\tilde{S}\rangle}{E_\tilde{S}^{(0)} - E_{\tilde{S}}^{(0)}}.$$ \hfill (49)

At the operator level, this can be rewritten as the usual commutator equation for $U_1$

$$[U_1, H_{\text{diag}}] = H_{\text{off}}.$$ \hfill (50)

The transformation of any operator $O$ then reads at this order

$$\tilde{O} = UOU^\dagger = (1 + U_1 + ..)O(1 - U_1 + ..) = O + [U_1, O] + ....$$ \hfill (51)

In particular, the pseudo-spins $\tau_n^{z(0+1)} = U\sigma_n^{z}U^\dagger$ (equation (7)) display the following perturbative expansion around the real spins

$$\tau_n^{z(0+1)} = \sigma_n^{z} + [U_1, \sigma_n^{z}].$$ \hfill (52)

Reciprocally, the real spins $\sigma_n^{z} = U^\dagger\tau_n^{z}U$ can be expanded around the pseudo-spins

$$\sigma_n^{z(0+1)} = \tau_n^{z} - [U_1, \tau_n^{z}].$$ \hfill (53)

### 5.2. Self-consistent first order perturbation theory for the eigenstates

In random systems, resonances may appear in the first-order eigenstate of equation (46) when the amplitude $|\langle \tilde{S}| H_{\text{off}} |\tilde{S}\rangle|^{2}$ is not small. The simplest way to take into account these rare possible resonances is to introduce the normalized version of equation (46)

$$|\psi_S^{(0+1)}\rangle_{\text{norm}} \equiv \frac{|\psi_S^{(0+1)}\rangle}{\sqrt{\langle \psi_S^{(0+1)}|\psi_S^{(0+1)}\rangle}} = |\tilde{S}\rangle + \sum_{S' \neq \tilde{S}} \frac{|\tilde{S}'| H_{\text{off}} |\tilde{S}'\rangle}{E_\tilde{S}'^{(0)} - E_{\tilde{S}}^{(0)}}$$ \hfill (54)

as explained in detail in [22, 35] for Anderson-localization and many-body-localization respectively. What happens at the technical level is that the term $|\langle \tilde{S}| H_{\text{off}} |\tilde{S}\rangle|^{2}$ of the denominator, although formally of second order, is actually of first order in the off-diagonal couplings, as a consequence of resonances [22, 35]. Equation (54) can be thus considered as the self-consistent first order perturbative expression for the eigenstate.

### 5.3. Self-consistent first order perturbation theory for the LIOMs and for the spins operators

Here we wish to apply the same idea to LIOMs that are operators, so that the appropriate norm is the Frobenius or Hilbert–Schmidt norm, based on the inner product of two operators
\[(A, B) = \frac{1}{2N} \text{Tr}(A^\dagger B).\]  

So the normalized version of equation (52) reads
\[
\tau_{n(0+1)}^{z,\text{norm}} \equiv \frac{\tau_{n(0+1)}^z}{\sqrt{\langle \tau_{n(0+1)}^z \tau_{n(0+1)}^z \rangle}} = \frac{\sigma_n^z + [U_1, \sigma_n^z]}{\sqrt{1 + \frac{1}{4N} \text{Tr}([U_1, \sigma_n^z]^2)}}. \tag{56}
\]

Similarly, the normalized version of equation (53) reads
\[
\sigma_{n(0+1)}^{z,\text{norm}} \equiv \frac{\sigma_{n(0+1)}^z}{\sqrt{\langle \sigma_{n(0+1)}^z, \sigma_{n(0+1)}^z \rangle}} = \frac{\tau_{n}^z - [U_1, \tau_n^z]}{\sqrt{1 + \frac{1}{4N} \sum_i \langle \mathcal{T} | [U_1, \tau_i^z]^2 | \mathcal{I} \rangle}}. \tag{57}
\]

so that the memory of the initial magnetization of equation (39) becomes
\[
m_{\infty = n_0} = \frac{1}{2N} \sum_j \langle \langle \mathcal{T} | \sigma_{n(0+1)}^z | \mathcal{I} \rangle \rangle^2 = \frac{1}{1 + \frac{1}{4N} \text{Tr}([U_1, \tau_0^z]^2)} \equiv \frac{1}{1 + \frac{1}{4N} \sum_i \langle \mathcal{T} | [U_1, \tau_i^z]^2 | \mathcal{I} \rangle}. \tag{58}
\]

In the two following sections, this self-consistent first order perturbation theory is applied to specific models.

6. Application to the random field XXZ chain

In this section, we consider the nearest-neighbor XXZ chain with random fields \(h_j\), corresponding to the diagonal part
\[
H^{\text{diag}} = \sum_j h_j \sigma_j^z + \sum_j \Delta \sigma_j^x \sigma_{j+1}^x \tag{59}
\]
and the off-diagonal part
\[
H^{\text{off}} = \sum_j J(\sigma_j^x \sigma_{j+1}^x + \sigma_j^y \sigma_{j+1}^y) = \sum_j 2J(\sigma_j^+ \sigma_{j+1}^- + \sigma_j^- \sigma_{j+1}^+). \tag{60}
\]
The isotropic case \(J = 1 = \Delta\) is the most studied many-body-localization model where many numerical results are available [36–49].

6.1. Perturbation in the off-diagonal coupling \(J\)

When applied to the XXZ Hamiltonian of equations (59) and (60), the first order perturbation theory described in the previous section yields the following first order contribution \(U_1\) for the unitary transformation \(U = 1 + U_1\)
\[
U_1 = \sum_{j_0} \theta_{j_0,j_0+1}(\tau_{j_0-1}^z, \tau_{j_0+2}^z)(\tau_{j_0}^+ \tau_{j_0+1}^- - \tau_{j_0}^- \tau_{j_0+1}^+) \tag{61}
\]
with the notation
\[
\theta_{j_0,j_0+1}(\tau_{j_0}^z, \tau_{j_0+2}^z) \equiv \frac{J}{(h_{j_0+1} + \Delta \tau_{j_0+2}^z) - (h_{j_0} + \Delta \tau_{j_0}^z)}. \tag{62}
\]

The commutator appearing in equation (57) reads...
\[ [U, \tau_n^z] = 2\theta_{n-1,n}(\tau_{n-2}^z; \tau_{n+1}^z)(\tau_n^z \tau_{n-1}^- + \tau_n^- \tau_{n+1}^-) - 2\theta_{n,n+1}(\tau_n^z; \tau_{n+2}^z)(\tau_n^z \tau_{n+1}^- + \tau_n^- \tau_{n+1}^-). \]  

(63)

The diagonal part of its square

\[ \text{Diag}([U, \tau_n^z]^2) = 2\theta_{n-1,n}(\tau_{n-2}^z; \tau_{n+1}^z)(1 - \tau_{n-1}^z \tau_n^z) + 2\theta_{n,n+1}(\tau_{n-1}^z; \tau_{n+2}^z)(1 - \tau_{n}^z \tau_{n+1}^-) \]

(64)

yields that the trace needed for the magnetization memory of equation (58) simplifies into the following sum

\[ T = \frac{1}{2\mathcal{N}} \text{Tr}([U, \tau_n^z]^2) = \frac{1}{2\mathcal{N}} \sum \bar{T}([U, \tau_n^z]^2) = \frac{1}{2\mathcal{N}} \sum \theta_{n,n-1}(\tau_{n-2}^z; \tau_{n+1}^z) + \theta_{n,n+1}^2(\tau_{n-1}^z; \tau_{n+2}^z) \]

\[ = \frac{1}{2} \sum_{n=1}^{\mathcal{N}} \theta_{n,n-1}(T_{n-2}; T_{n+1}) + \theta_{n,n+1}^2(T_{n-1}; T_{n+2}) \]

\[ = \frac{1}{2} \left( \frac{2}{(h_n - h_{n-1})^2} + \frac{1}{(h_n - h_{n+1} + 2\Delta)^2} + \frac{1}{(h_n - h_{n-1} - 2\Delta)^2} \right) \]

\[ + \frac{1}{2} \left( \frac{2}{(h_{n+1} - h_n)^2} + \frac{1}{(h_{n+1} - h_{n+2} + 2\Delta)^2} + \frac{1}{(h_{n+1} - h_n - 2\Delta)^2} \right) \]

(65)

that involves only the three consecutive random fields \( (h_{n-1}, h_n, h_{n+1}) \).

6.2. Magnetization memory as lowest order in the off-diagonal coupling \( J \)

The magnetization memory of equation (58) is directly related to the positive sum \( T \) introduced in equation (65)

\[ m_{\mathcal{N}=\infty} = \frac{1}{1 + T}. \]

(66)

As detailed in appendix B, its disorder-averaged value is not of order \( J^2 \) but of order \( |J| \) as a consequence of the Lévy distribution of \( T \) induced by the resonances (equation (B.10))

\[ m_{\mathcal{N}=\infty} = 1 - |J|\mathcal{A} + o(|J|) \]

(67)

where the amplitude \( \mathcal{A} \) can be computed (equation (B.10)) from the properties of the individual terms appearing in equation (65). This behavior in \( |J| \) was already found in [20] and can be also computed exactly (i.e. without perturbation theory) in the trivial two-spins example described in the appendix A without perturbation theory (equations (A.13), (A.14) and (A.15)).

7. Application to the toy model [22, 23]

In this section, we consider the toy model [22, 23] where the diagonal part

\[ H_{\text{diag}} = \sum_{i=1}^{N} h_i \sigma_i^z \]

contains only random fields that are distributed with the Gaussian distribution of variance \( W^2 \)

\[ P(h) = \frac{1}{\sqrt{2\pi W^2}} e^{-\frac{h^2}{2W^2}} \]

(69)
while the off-diagonal part contains hopping to all other configurations of the Hilbert space

$$H_{\text{off}} = \sum_{k=1}^{N} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} J_{i_1 \ldots i_k} \sigma_{i_1}^x \sigma_{i_2}^x \ldots \sigma_{i_k}^x. \quad (70)$$

The question is how weak the behavior of the off-diagonal couplings $J_{i_1 \ldots i_k}$ should be in order to allow the existence of the many-body-localized phase. In [22], the analysis was based on the entanglement entropy for the one-dimensional case. Here we wish instead to analyze this type of model in dimension $d$ from the point of view of the pseudo-spins.

7.1. Perturbation in the off-diagonal couplings

The unitary transformation $U = 1 + U_1$ reads at first order in the off-diagonal couplings

$$U_1 = \sum_{k=1}^{N} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} J_{i_1 \ldots i_k} \sigma_{i_1}^x \sigma_{i_2}^x \ldots \sigma_{i_k}^x. \quad (71)$$

To simplify the notations, let us focus on the first spin $\sigma_{i_1}^z$ and compute the commutator of equation (57)

$$[U_1, \tau_{i_1}^z] = \sum_{k=1}^{N} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} J_{i_1 \ldots i_k} \tau_{i_1}^z \tau_{i_2}^x \ldots \tau_{i_k}^x. \quad (72)$$

The diagonal part of its square

$$\text{Diag}([U_1, \tau_{i_1}^z]^2) = \sum_{k=1}^{N} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} \left( \frac{J_{i_1 \ldots i_k}}{\sum_{p=1}^{k} h_p \tau_{i_p}^z} \right)^2 \quad (73)$$

yields that the trace needed for the magnetization memory of equation (58) simplifies into the following sum

$$T \equiv \frac{1}{2N} \text{Tr}([U_1, \tau_1^z]^2) = \frac{1}{2N} \sum_{T} \langle \bar{T} | [U_1, \tau_1^z]^2 | \bar{T} \rangle = \frac{1}{2N} \sum_{T_1 \ldots T_k} \sum_{1 \leq i_1 < i_2 < \ldots < i_k \leq N} \left( \frac{J_{i_1 \ldots i_k}}{\sum_{p=1}^{k} h_p T_p} \right)^2 \quad (74)$$

7.2. Memory of the local magnetization

The disorder-averaged value of the magnetization memory of equation (58) reads in terms of equation (74)
\[
\begin{align*}
\mathcal{M}_n^{-\infty} &= \int_{-\infty}^{+\infty} \mathrm{d}h_1 P(h_1) \ldots \int_{-\infty}^{+\infty} \mathrm{d}h_N P(h_N) \frac{1}{1 + \frac{1}{N} \sum_{k=1}^{N} \sum_{1 \leq i_1 < \ldots < i_k \leq N} \frac{1}{T_{i_1}T_{i_2} \ldots T_{i_k} \sum_{p=1}^{k} h_{ip}} \left( \frac{J_{i_1 \ldots i_k}}{\sum_{p=1}^{k} h_{ip}} \right)^2}.
\end{align*}
\]

(75)

It is thus convenient to make the change of variables \( \tilde{h}_i = h_i T_i \) and to use the Gaussian distribution of equation (69) to rewrite
\[
\begin{align*}
\mathcal{M}_n^{-\infty} &= \int_{-\infty}^{+\infty} \mathrm{d}\tilde{h}_1 P(\tilde{h}_1) \ldots \int_{-\infty}^{+\infty} \mathrm{d}\tilde{h}_N P(\tilde{h}_N) \frac{1}{1 + \frac{1}{N} \sum_{k=1}^{N} \sum_{1 \leq i_1 < \ldots < i_k \leq N} \frac{1}{T_{i_1}T_{i_2} \ldots T_{i_k} \sum_{p=1}^{k} \tilde{h}_{ip}} \left( \frac{J_{i_1 \ldots i_k}}{\sum_{p=1}^{k} \tilde{h}_{ip}} \right)^2}.
\end{align*}
\]

(76)

where
\[
\Sigma = \sum_{k=1}^{N} \sum_{1 \leq i_1 < \ldots < i_k \leq N} \left( \frac{J_{i_1 \ldots i_k}}{\sum_{p=1}^{k} \tilde{h}_{ip}} \right)^2
\]

(77)

is a Lévy sum of correlated variables, whose statistical properties are discussed in appendix B.

Since the sum of \( k \) Gaussian variables (equation (69))
\[
E = \sum_{p=1}^{k} \tilde{h}_p
\]

(78)

is distributed with the Gaussian distribution of variance \((kW^2)\)
\[
\rho_{\tilde{h}_1 \ldots \tilde{h}_k}(E) = \frac{1}{\sqrt{2\pi kW^2}} e^{-\frac{E^2}{2kW^2}}
\]

(79)

the application of equation (B.10)–(76) yields
\[
\begin{align*}
\mathcal{M}_n^{-\infty} &= 1 - \pi \sum_{k=1}^{N} \sum_{1 \leq i_1 < \ldots < i_k \leq N} |J_{i_1 \ldots i_k}| \rho_{\tilde{h}_1 \ldots \tilde{h}_k}(0) + o(|J_{i_1 \ldots i_k}|) \\
&= 1 - \frac{\sqrt{\pi}}{W \sqrt{2}} \sum_{k=1}^{N} k^{-\frac{1}{2}} \sum_{1 \leq i_1 < \ldots < i_k \leq N} |J_{i_1 \ldots i_k}| + o(|J_{i_1 \ldots i_k}|).
\end{align*}
\]

(80)

7.3. Criterion for the stability of the many-body-localized phase

The result of equation (80) for the magnetization memory yields that the criterion for the existence of the many-body-localized phase is the convergence in the thermodynamic limit \( N \to +\infty \) of the following sum involving the averaged values \( |J_{i_1 \ldots i_k}| \)
\[
\text{MBL phase criterion} : \sum_{k=1}^{N} k^{-\frac{1}{2}} \sum_{1 \leq i_1 < \ldots < i_k \leq N} |J_{i_1 \ldots i_k}| < +\infty.
\]

(81)
In [22, 23], we have considered only the one-dimensional model with the following specific form of the couplings $J_{i_1, \ldots, i_k}$: they were assumed to depend only on the spatial range $r = (i_k - i_1)$ via some exponential decay governed by the control parameter $b$ and possibly some power-law prefactor governed by the parameter $a$.

\[ J_{i_1, \ldots, i_k} = V e^{-b|i_k-i_1|} v_{i_1, \ldots, i_k} \]  \hspace{1cm} (82)

with $O(1)$ random variables $v_{i_1, \ldots, i_k}$, and the prefactor $V$ being the global vanishing amplitude to perform the perturbative expansion. The properties of the entanglement entropy at the critical point $b_c = 1$ could be then studied explicitly [22].

Here it is thus interesting to discuss more generally the criterion of equation (81) in dimension $d$ with $N = L^d$ spins. It is clear that within the sum of equation (81) containing a number of terms equal to the number of configurations

\[ \sum_{k=1}^{N} \sum_{1=i_1<t_2<\ldots<i_k\leq N} = \sum_{k=1}^{N} \binom{N}{k} = 2^N - 1 \]  \hspace{1cm} (83)

the most dangerous terms for the convergence are the terms corresponding to $k \approx N^2$ that maximizes the binomial coefficient $\binom{N}{k}$ counting the number of choices for the $k$ positions $(i_1, \ldots, i_k)$ within $N$

\[ \binom{N}{N^2} \propto N \rightarrow +\infty 2^{N/2} N^{-1/4} \]  \hspace{1cm} (84)

that displays the same exponential behavior as the whole number of terms (equation (83)). As a consequence, the important property of the couplings $J_{i_1, \ldots, i_k}$ for the MBL-transition is the behavior for extensive resonances involving $k \approx N^2$ spins $(i_1, \ldots, i_{N^2})$ in a volume $N = L^d$. Using the change of variable $k = \frac{N}{2} \left(1 + \frac{x}{\sqrt{N}}\right)$ and the behavior of the binomial coefficient

\[ \binom{N}{N^2} \left(1 + \frac{x}{\sqrt{N}}\right)^{\infty} \propto N \rightarrow +\infty 2^{N/2} N^{-1/4} e^{-x^2} \]  \hspace{1cm} (85)

one obtains

\[ \sum_{k=1}^{N} k^{-\frac{1}{2}} \binom{N}{k} \approx \frac{2^N}{\sqrt{N}} \]  \hspace{1cm} (86)

so that the criterion of equation (81) becomes a bound for $|J_{i_1, \ldots, i_{N^2}}|$ in the thermodynamic limit

\[ \left|J_{i_1, \ldots, i_{N^2}}\right| < \frac{\sqrt{N}}{2^N}. \]  \hspace{1cm} (87)

Since the right-handside represents the scaling of the energy-level-spacing in the middle of the spectrum, the physical interpretation of this bound is thus very clear and natural: the MBL-phase is stable only if the coupling between two configurations having an extensive number $N^2$ of different spins is smaller than the level spacing $\Delta_N \propto \sqrt{N} 2^{-N}$.
8. Conclusion

In summary, the strong disorder perturbative expansion of the local integrals of motion (LIOMS) around the real-spin operators has been described to make the link with other properties of the many-body-localized phase, in particular the statistics of matrix elements of local operators in the eigenstate basis, and the memory in the dynamics of the local magnetizations. The lowest-order contribution to the magnetization memory was discussed for the random field XXZ chain, which is the most studied MBL model. Finally, this approach has been used to analyze the MBL-transition in the toy model considered previously in [22, 23] via other points of view.

As a final remark, it is interesting to mention the recent work on strong zero modes in the non-random XYZ chain [50]: although it is not directly related to the problem of many-body-localization in random models, the construction of an exact non-trivial operator that squares to the identity and whose commutator with the Hamiltonian is exponentially small in the system size [50] is nevertheless somewhat reminiscent of the notion of LIOMS in random models.

Appendix A. Non-perturbative LIOMS for the simplest two-spin Hamiltonian

Besides the perturbative framework described in the text for systems with a large number of spins, it is useful to see on the trivial case involving only two spins

\[ H = h_1 \sigma_1^+ \sigma_1^- + h_2 \sigma_2^+ \sigma_2^- + \Delta \sigma_1^+ \sigma_1^- + 2J (\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) \]  

(A.1)

how the finite unitary transformation \( U = e^\eta \) based on the antihermitian generator \( \eta \)

\[ U = e^\eta = \cos \theta \sigma_1^+ \sigma_2^+ + \sin \theta (\sigma_1^- \sigma_2^- - \sigma_1^+ \sigma_2^+) \]  

(A.2)

allows to obtain the pseudo-spins and the other observables described in the text.

A.1. LIOMS in terms of spins

Equation (A.2) yields the pseudo-spins

\[ \tau_1^+ = U \sigma_1^+ U^\dagger = \cos^2 \theta \sigma_1^+ + \sin^2 \theta \sigma_2^+ - 2 \cos \theta \sin \theta (\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) \]

\[ \tau_1^- = U \sigma_1^- U^\dagger = \sin^2 \theta \sigma_1^+ + \cos^2 \theta \sigma_2^+ + 2 \cos \theta \sin \theta (\sigma_1^+ \sigma_2^- + \sigma_1^- \sigma_2^+) \]  

(A.3)

and the ladder operators

\[ \tau_1^\pm = U \sigma_1^\pm U^\dagger = \cos \theta \sigma_1^\pm + \sin \theta \sigma_1^\dagger \sigma_2^\pm \]

\[ \tau_2^\pm = U \sigma_2^\pm U^\dagger = - \sin \theta \sigma_1^\pm \sigma_2^\dagger + \cos \theta \sigma_1^\dagger \sigma_2^\pm \]  

(A.4)

so that the hopping terms read for instance

\[ \tau_1^+ \tau_2^- = \cos^2 \theta \sigma_1^+ \sigma_2^- - \sin^2 \theta \sigma_1^- \sigma_2^+ + \cos \theta \sin \theta \frac{\sigma_1^- - \sigma_2^+}{2} \]

\[ \tau_1^- \tau_2^+ = \cos^2 \theta \sigma_1^- \sigma_2^+ - \sin^2 \theta \sigma_1^+ \sigma_2^- + \cos \theta \sin \theta \frac{\sigma_1^+ - \sigma_2^-}{2} \]  

(A.5)

A.2. Spins in terms of LIOMS

In terms of the pseudo-spins, the generator keeps the same form as a consequence of equation (A.5)
\[ \eta = \theta (\sigma_1^+ \sigma_2^- - \sigma_1^- \sigma_2^+) = \theta (\tau_1^+ \tau_2^- - \tau_1^- \tau_2^+) \] (A.6)

and leads to the expression of the real spins in terms of the pseudo-spins
\[ \begin{align*}
\sigma_1^+ &= U^\dagger \tau_1^+ U = \cos^2 \theta \tau_1^+ + \sin^2 \theta \tau_2^+ + 2 \cos \theta \sin \theta (\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+) \\
\sigma_2^- &= U^\dagger \tau_2^- U = \sin^2 \theta \tau_1^+ + \cos^2 \theta \tau_2^+ - 2 \cos \theta \sin \theta (\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+). 
\end{align*} \] (A.7)

A.3. Choice of the angle \( \theta \) to diagonalize the Hamiltonian

The Hamiltonian of equation (A.1) reads in terms of the pseudo-spins
\[ \begin{align*}
H &= h_1 \left( \cos^2 \theta \tau_1^+ + \sin^2 \theta \tau_2^+ + \sin(2\theta)(\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+) \right) \\
&\quad + h_2 \left( \sin^2 \theta \tau_1^+ + \cos^2 \theta \tau_2^+ - \sin(2\theta)(\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+) \right) \\
&\quad + \Delta \tau_1^+ \tau_2^- + 2J \left( \cos(2\theta)(\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+) - \cos \theta \sin \theta (\sigma_1^+ - \sigma_2^-) \right) \\
&\quad = (h_1 \cos^2 \theta + h_2 \sin^2 \theta - J \sin(2\theta)) \tau_1^+ + (h_1 \sin^2 \theta + h_2 \cos^2 \theta + J \sin(2\theta)) \tau_2^+ + \Delta \tau_1^+ \tau_2^- \\
&\quad + ((h_1 - h_2) \sin(2\theta) + 2J \cos(2\theta))(\tau_1^+ \tau_2^- + \tau_1^- \tau_2^+). \tag{A.8}
\end{align*} \]

It is thus diagonal in the \( \tau^z \) basis when the hopping term vanishes, i.e. for the choice of the angle \( \theta \in \left[ -\frac{\pi}{4}, \frac{\pi}{4} \right] \) satisfying
\[ \begin{align*}
\cos(2\theta) &= \frac{1}{\sqrt{1 + \left( \frac{2J}{-h_1 + h_2} \right)^2}} \\
\sin(2\theta) &= \frac{2J}{\sqrt{1 + \left( \frac{2J}{-h_1 + h_2} \right)^2}} \tag{A.9}
\end{align*} \]

leading to
\[ \begin{align*}
H &= \left( \frac{h_1 + h_2}{2} + \frac{h_1 - h_2}{2} \sqrt{1 + \left( \frac{2J}{-h_1 + h_2} \right)^2} \right) \tau_1^+ \\
&\quad + \left( \frac{h_1 + h_2}{2} - \frac{h_1 - h_2}{2} \sqrt{1 + \left( \frac{2J}{-h_1 + h_2} \right)^2} \right) \tau_2^+ + \Delta \tau_1^+ \tau_2^-.
\end{align*} \]

A.4. Matrix elements of a single real spin operator in the eigenstate basis

The matrix element of equation (22)
\[ \langle T_1 T_2 | \sigma_1^z | T_1' T_2' \rangle = \langle T_1 \cos^2 \theta + T_2 \sin^2 \theta \rangle \delta_{T_1 T_1'} \delta_{T_2 T_2'} + \sin(2\theta) \delta_{T_1 - T_1'} \delta_{T_2 - T_2'} \] (A.10)

yields the doubly stochastic matrix of equation (23)
\[ \mathcal{Q}_{T_1 T_2, T_1' T_2'} \equiv | \langle T_1 T_2 | \sigma_1^z | T_1' T_2' \rangle |^2 \\
= (\cos^4 \theta + \sin^4 \theta + 2T_1 T_2 \cos^2 \theta \sin^2 \theta) \delta_{T_1 - T_1'} \delta_{T_2 - T_2'} + \sin^2(2\theta) \delta_{T_1 - T_1'} \delta_{T_2 - T_2'}. \] (A.11)
A.5. Dynamics of the local magnetizations

The magnetization of equation (39) reads

\[ m_\infty n = n_0 = 1 = \frac{1}{4} \sum_{T_1 = \pm 1, T_2 = \pm 1} Q_{T_1 T_2 T_1} = \cos^2 \theta + \sin^4 \theta = (\cos^2 \theta + \sin^2 \theta)^2 - 2 \cos^2 \theta \sin^2 \theta \]

\[ = 1 - \frac{1}{2} \sin^2(2\theta) \]  

(A.12)

or more explicitly using equation (A.9)

\[ m_\infty n = n_0 = 1 = 1 - \frac{1}{2} J_2 (-h_1 + h_2)^2 + O(J^2) \]  

(A.13)

When the two random fields \((h_1, h_2)\) are given, the correction is of second order in the coupling \(J\)

\[ m_\infty n = n_0 = 1 = 1 - \frac{1}{2} J_2 (-h_1 + h_2)^2 + O(J^2) \]  

(A.14)

but the averaged value over the two random fields gives a correction of order \(|J|\) via the change of variable \(h_2 = h_1 + 2|J|x\)

\[ \overline{m_\infty n} = 1 - \int \text{d}h_1 P(h_1) \int \text{d}h_2 P(h_2) \frac{J^2}{(-h_1 + h_2)^2 + 4J^2} \]

\[ = 1 - |J| \int \text{d}h_1 P(h_1) \int \text{d}x \frac{1}{(x^2 + 1)} \]

\[ = 1 - \pi |J| \int \text{d}h_1 P^2(h_1) + o(|J|). \]  

(A.15)

Appendix B. Magnetization memory in terms of the Lévy sum of correlated variables

The strong disorder perturbative approach described in the text yields that the magnetization memory of equation (58) reads

\[ m_\infty n_0 = \frac{1}{1 + \sum} \]  

(B.1)

in terms of the sum of \(M\) positive terms

\[ \sum = \sum_{i=1}^{M} x_i \]  

(B.2)

of the form

\[ x_i = \frac{J_i^2}{y_i} \]  

(B.3)

where \(J_i\) are the fixed small perturbative off-diagonal couplings, while the variables \(y_i\) depending on the random fields are correlated random variables, such that the partial law \(\rho_i(y_i)\) of \(y_i\) has some finite weight \(\rho_i(y_i = 0) > 0\) at the origin \(y_i = 0\). As a consequence, the partial law \(P_i(x_i)\) of \(x_i\) of equation (B.3) displays the following Lévy power-law tail for large \(x_i\).
\[ P_i(x_i) = \int_{-\infty}^{+\infty} dy_i \rho_i(y_i) \delta \left( x_i - \frac{y_i^2}{4} \right) = \int_{-\infty}^{+\infty} dy_i \rho_i(y_i) \frac{\delta \left( y_i - \frac{J_i}{\sqrt{\pi}} \right) + \delta \left( y_i + \frac{J_i}{\sqrt{\pi}} \right)}{2x_i^{+\infty} - x_i^{-\infty}}. \tag{B.4} \]

and its Laplace transform displays the following singular behavior for small \( p \)

\[
e^{-p\Sigma} = \int_0^{+\infty} dx_i P_i(x_i) e^{-px_i} = 1 - \int_0^{+\infty} dx_i P_i(x_i) (1 - e^{-px_i}) = 1 - \int_0^{+\infty} \frac{du}{p} \rho_i \left( \frac{u}{p} \right) \left( 1 - e^{-u} \right) = 1 - \frac{p^2}{2} |J_i| \rho_i(0) + o(p^\frac{3}{2}). \tag{B.5} \]

Since the average value of \( x_i \) diverges
\[
\overline{x_i} = \infty \tag{B.6} \]

the average value of the sum of equation (B.2) also diverges
\[
\overline{\Sigma} = \infty. \tag{B.7} \]

To characterize the statistical properties of the sum \( \Sigma \), one thus needs to evaluate the singular behavior of the Laplace transform of its probability distribution for small \( p \) using equation (B.5)

\[
1 - e^{-p\Sigma} = 1 - \prod_{i=1}^{M} (1 - (1 - e^{-p|J_i|})) = \sum_{i=1}^{M} (1 - e^{-p|J_i|}) + o(p^\frac{3}{2})
= p^\frac{1}{2} \left( \sum_{i=1}^{M} |J_i| \rho_i(0) \right) 2\sqrt{\pi} + o(p^\frac{3}{2}). \tag{B.8} \]

The disorder-averaged value of the magnetization memory of equation (B.1) can be computed from the Laplace transform of the probability distribution of \( \Sigma \) via

\[
\overline{m_{\Sigma}}_{n=0} = \frac{1}{1 + \overline{\Sigma}} = \int_0^{+\infty} dp e^{-p} e^{-p\Sigma}. \tag{B.9} \]

Equation (B.8) then yields that the lowest order in the off-diagonal couplings \( J_i \) reads

\[
\overline{m_{\Sigma}}_{n=0} = 1 - \int_0^{+\infty} dp e^{-p} (1 - e^{-p\Sigma})
= 1 - \left( \sum_{i=1}^{M} |J_i| \rho_i(0) \right) 2\sqrt{\pi} \int_0^{+\infty} dp e^{-p} p^{\frac{1}{2}} + o(|J_i|)
= 1 - \left( \sum_{i=1}^{M} |J_i| \rho_i(0) \right) \pi + o(|J_i|). \tag{B.10} \]

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