Constraining $CP$ violation in a softly broken $A_4$ symmetric Model

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Abstract

To understand the mass spectra of charged lepton and neutrino $A_4$ symmetry has been proposed in addition with the Standard $SU(2)_L \times U(1)_Y$ model. We break $A_4$ symmetry softly and the deviation from the tri-bimaximal mixing arises due to Zee mechanism. In the present work, we express two mixing angles $\theta_{13}$ and $\theta_{23}$ in terms of a single model parameter and experimental observables, such as, mixing angle $\theta_{12}$, mass squared differences $\Delta m^2_{21}$ and $\Delta m^2_{32}$. Using the experimental values of $\theta_{23}$, $\theta_{12}$, $\Delta m^2_{21}$ and $\Delta m^2_{32}$ we restrict the model parameter and we predict $\theta_{13}$. This model gives rise to $\theta_{13} \simeq 11^\circ$ if we allow $1\sigma$ deviation of $\theta_{23}$ and $2^\circ$ deviation of $\theta_{12}$ from their best fit values. Utilizing all those constraints, we explore the extent of CP violation parameter $J_{CP}$ in the present model and found a value of $J_{CP} \approx 2.65 \times 10^{-3}$ (for $1\sigma$ deviation of $\theta_{23}$ and $2^\circ$ deviation of $\theta_{12}$) consistent with the other neutrino experimental results. We have studied the mass pattern of neutrino and neutrinoless double beta decay ($\beta\beta_{0\nu}$) parameter $|(M_\nu)_{ee}|$ in this model.

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An interesting way to obtain hierarchical charged lepton mass matrix along with appropriate texture of neutrino mass matrix which can accommodate present neutrino experimental results, namely, solar, atmospheric and CHOOZ is through the introduction of non-abelian discrete $A_4$ symmetry in a model [1]. The interplay of $A_4$ symmetry predicts diagonal and hierarchical charged lepton mass matrix in addition with the neutrino masses which could be quasi-degenerate or hierarchical. Altarelli and Feruglio have proposed a version of $A_4$ symmetric model (AF model) [2] to obtain hierarchical charged lepton mass matrix along with tri-bimaximal neutrino mixing ($\sin \theta_{12} = 1/\sqrt{3}$, $\sin \theta_{23} = -1/\sqrt{2}$, $\sin \theta_{13} = 0$) [3]. Although the model gives rise to $\theta_{13} = 0$ ($|U_{e3}| = 0$) which is consistent with the CHOOZ-Palo Verde experimental upper bound ($\theta_{13} < 12^\circ$ at 3$\sigma$), however, the non-zero value of $|U_{e3}|$ opens up a possibility to explore $CP$ violation in leptonic sector which is the main goal of many future short and long baseline experiments.

A prediction for non-zero $U_{e3}$ has been realized in a recently proposed modified AF model through the inclusion of three gauge singlet charged scalars due to radiative correction of the off diagonal elements of the neutrino mass matrix [4]. The model successfully predicts solar and atmospheric neutrino mixing and mass-squared differences along with small but non-zero value of $\theta_{13}$ well below the present experimental upper bound for a reasonable choice of model parameters. A relationship between different mixing angles is an outcome of the model and the predictability of the model is also testable in future neutrino experiments.

Aim of this paper is to generalize the assumptions made in [4]. In the present work we bring down the value of $\theta_{12}$ to its best fit value $\theta_{12} = 34.0^\circ$ from the tri-bimaximal value of the above $\theta_{12} = 35.26^\circ$ which is at the 1$\sigma$ edge of experimental value. We investigate how much $\theta_{13}$ can be within 1$\sigma$ variation of $\theta_{23}$ and at the best fit value as well as 2$^\circ$ variation about the best fit value of $\theta_{12}$. Then we shift our concentration to $CP$ violating parameter $J_{CP}$ and Dirac phase $\delta_D$ and figure out their values. We also see the mass pattern of the neutrinos and also see the variation of $\beta\beta_{0\nu}$ experimental parameter in this model. For our analysis we have used the best fit value of the mass squared differences, $\Delta m^2_{21} = 8.0 \times 10^{-5} \text{eV}^2$ and $\Delta m^2_{32} = 2.1 \times 10^{-3} \text{eV}^2$.

For completeness, we briefly summarize here the model proposed in [4]. The lepton content with their representation and the Higgs content with their vevs and representation under $SU(2)_L \times U(1)_Y \times A_4$ symmetry is presented in Table 1, where all fields except the charged scalar $\chi^+_i$ have been used in the original AF model [2]. The Yukawa interaction in the leptonic sector is given by

$$L_{AF}^l = y_e e^c (\phi_T l) h_d/\Lambda + y_{\mu} \mu^c (\phi_T l)' h_d/\Lambda + y_{\tau} \tau^c (\phi_T l)'' h_d/\Lambda$$
\[ x_a \xi (l_h l_{h_a})/\Lambda^2 + x_b (\phi_s l_{h_a} l_{h_a})/\Lambda^2 \]  

(1)

where \( x_a, x_b, y_e, y_\mu, y_\tau \) are Yukawa couplings and \( \Lambda \) is the new mass scale. After spontaneous breaking of the symmetry of the model, the above Lagrangian gives rise to the following mass terms

\[ \mathcal{L}_{AF} = v_d v_\tau/\Lambda(y_e e^c e + y_\mu \mu^c \mu + y_\tau \tau^c \tau) \]
\[ + x_a v_u^2 (u/\Lambda^2)(\nu_e \nu_e + 2\nu_\mu \nu_\tau) \]
\[ + x_b v_u^2 v_S/3\Lambda^2(\nu_e \nu_e + \nu_\mu \nu_\mu + \nu_\tau \nu_\tau - \nu_e \nu_\mu - \nu_\mu \nu_\tau - \nu_\tau \nu_e) + \text{h.c.} \]  

(2)

The charged lepton mass matrices come out as

\[
m_l = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix}, m_{\nu}^{AF} = m_0 \begin{pmatrix} a + 2d/3 & -d/3 & -d/3 \\ -d/3 & 2d/3 & a - d/3 \\ -d/3 & a - d/3 & 2d/3 \end{pmatrix},
\]  

(3)

where

\[
m_e = y_e v_d v_\tau/\Lambda, m_\mu = y_\mu v_d v_\tau/\Lambda, m_\tau = y_\tau v_d v_\tau/\Lambda, \]
\[ a = 2x_a u/\Lambda, d = 2x_b v_S/\Lambda, m_0 = v_u^2/\Lambda. \]  

(4)

The charged lepton mass matrix is diagonal so the leptonic mixing solely occurs from the neutrino sector and diagonalising \( m_\nu \) by the way \( U^\dagger m_\nu U^* \) we get the three mass eigenvalues as

\[
m_1 = a + d, m_2 = a, m_3 = d - a
\]  

(5)

with the exact tri-bimaximal mixing pattern. Thus, in the AF model, tri-bimaximal mixing occurs naturally. In [4], a modified version of the above model has been investigated to generate non-zero \( \theta_{13} \) which at the leading order gives tri-bimaximal mixing form. In order to do that, an \( A_4 \) triplet \( SU(2)_L \) singlet charged scalar \( \chi_i^+ (= \chi_1^+, \chi_2^+, \chi_3^+) \) has been incorporated and the leptonic part of the Lagrangian becomes

\[ L = L_{AF}^I + L_{MAF}^I \]  

(6)

where \( L_{MAF}^I \) has two parts as

\[ L_{MAF}^I = \mathcal{L}_1 + \mathcal{L}_2 \]  

(7)
**L**$_1$ is $A_4$ symmetry preserving part and is given by

\[
L_1 = f \left( L \ L \chi_i \right) \subset \left( 3 \times 3 \times 3 \right)
\]

\[
= f(\nu_{\tau} \chi_1^+ + \nu_{e} \chi_2^+ + \nu_{e} \mu \chi_3^+ - \nu_{e} \tau \chi_1^+ - \nu_{\tau} \mu \chi_2^+ - \nu_{\tau} \chi_3^+).
\]

(8)

however, $L_2$ contains Zee-type term which is explicit soft $A_4$ symmetry breaking and is given by

\[
L_2 = c_{12} \ h^T \ u \ \tau \ h \ ( \chi_1^+ + \chi_2^+ + \chi_3^+).
\]

(9)

The charged lepton mass matrix is still diagonal in the present model and the neutrino mass matrix comes out as

\[
m_\nu = \begin{pmatrix}
a + 2d/3 & -d/3 & -d/3 - \epsilon \\
-d/3 & 2d/3 & a - d/3 + \epsilon \\
-d/3 - \epsilon & a - d/3 + \epsilon & 2d/3
\end{pmatrix}
\]

(10)

where the $a$ and $d$ parameters defined earlier and are obtained due to higher dimension operators in the same way as AF model. The $\epsilon$ term is the additional contribution arises at the one-loop level due to well known Zee mechanism and is shown in Fig.1. The parameter $\epsilon$ is given by

\[
\epsilon = f m^2 r c_{12} v_u \ F(m^2_\chi, m^2_\nu) \]

(11)

with the definition,

\[
F(M_1^2, M_2^2) = \frac{1}{16 \pi^2 (M_1^2 - M_2^2)} \ln \frac{M_1^2}{M_2^2}.
\]

(12)

Although the Lagrangian given in Eq.(8) can generate corrections to all off-diagonal entries of the mass matrix given in Eq. (10), however, dominant terms proportional to $m^2_\tau$ are retained.

In this model neutrinos are Majorana-type in nature. In general, the parameters $a$, $d$, $\epsilon$ are all complex, however, it is possible to rotate out one of the phase. For our analysis, we consider only the parameter $d$ is complex and parameters $a, \epsilon$ are real. The neutrino mass matrix in this case comes out as

\[
M_\nu = \begin{pmatrix}
a + 2d e^{i\phi}/3 & -de^{i\phi}/3 & -de^{i\phi}/3 - \epsilon \\
-de^{i\phi}/3 & 2de^{i\phi}/3 & a - de^{i\phi}/3 + \epsilon \\
-de^{i\phi}/3 - \epsilon & a - de^{i\phi}/3 + \epsilon & 2de^{i\phi}/3
\end{pmatrix}
\]

(13)

A relationship between the parameters $a$ and $d$ has been considered as

\[
d = \kappa a \cos \phi
\]

(14)
Figure 1: One-loop radiative $\nu_{e,\mu} - \nu_{\tau}$ mass due to charged Higgs exchange.

| Lepton | $SU(2)_L$ | $A_4$ |
|--------|-----------|-------|
| ($\nu_i, l_i$) | 2 | 3 |
| $l_i^c$ | 1 | 1 |

| Scalar | | VEV |
|--------| |------|
| $h_u$ | 2 | $< h_u^0 > = v_u$ |
| $h_d$ | 2 | $< h_d^0 > = v_d$ |
| $\xi$ | 1 | $< \xi^0 > = u$ |
| $\phi_S$ | 1 | 3 | $< \phi_S^0 > = (v_S, v_S, v_S)$ |
| $\phi_T$ | 1 | 3 | $< \phi_T > = (v_T, 0, 0)$ |
| $\chi_i^+$ | 1 | 3 |

Table 1: List of fermion and scalar fields used in this model.
where $\kappa$, $d$ and $a$ are the real parameters. The analysis of Ref. [4] has been done for a specific value of $\kappa$ and in the present work, we have generalized the whole analysis. We constrain the value of $\kappa$ from the existing bounds of two mixing angles $\theta_{12}$ and $\theta_{23}$. Then we have utilized the result to calculate $\theta_{13}$ and the CP violation parameter $J_{\text{CP}}$ [5] and explore the extent at which $J_{\text{CP}}$ is allowed in the present model.

With only assumption $\epsilon$ is small, we diagonalize mass matrix Eq. (13) as

$$U^\dagger M_\nu U^* = \text{diag} \left( de^{i\phi} + a + \epsilon, \ a, \ de^{i\phi} - a - \epsilon \right)$$

upto first order in $\epsilon$ where diagonalizing matrix

$$U = \begin{pmatrix}
\sqrt{\frac{2}{3}} & \sqrt{\frac{1}{3}} & 0 \\
-\sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & -\sqrt{\frac{1}{2}} \\
-\sqrt{\frac{1}{6}} & \sqrt{\frac{2}{3}} & \sqrt{\frac{1}{2}} \\
\end{pmatrix} + \epsilon \begin{pmatrix}
\frac{-2a+de^{-i\phi}}{\sqrt{6(d+2a\cos \phi)}} & \frac{2a+de^{i\phi}}{\sqrt{3d(d+2a\cos \phi)}} & \frac{1}{3\sqrt{2}} \left( \frac{1}{a} - \frac{e^{-i\phi}}{d-2a\cos \phi} \right) \\
\frac{d^2-4a^2+2ad\sin \phi}{2\sqrt{6(d+2a\cos \phi)}} & \frac{2a^2\cos \phi - d(a+de^{i\phi})}{\sqrt{3(d^2-4a^2d\cos^2 \phi)}} & -\frac{1}{6\sqrt{2}} \left( \frac{1}{a} + \frac{2e^{-i\phi}}{d-2a\cos \phi} \right) \\
\frac{4a^2+d^2+4ad \cos \phi - 2ad \sin \phi}{2\sqrt{6(d+2a\cos \phi)}} & \frac{a(2a^2\cos \phi + de^{2i\phi})}{\sqrt{3(d^2-4a^2d\cos^2 \phi)}} & -\frac{1}{6\sqrt{2}} \left( \frac{1}{a} + \frac{2e^{-i\phi}}{d-2a\cos \phi} \right) \\
\end{pmatrix}.$$  

(16)

It is to be noted that for vanishing value of $\epsilon$ the matrix $U$ leads to tri-bimaximal form. Three approximate mass eigenvalues in Eq. (15) take the following forms

$$m_1^2 = \left| de^{i\phi} + a + \epsilon \right|^2$$

$$m_2^2 = a^2$$

$$m_3^2 = \left| de^{i\phi} - a - \epsilon \right|^2$$

$$\simeq a^2 \left[ 1 + 2\epsilon' - 2\kappa \cos^2 \phi - 2\epsilon' \kappa \cos^2 \phi + \kappa^2 \cos^2 \phi \right]$$

(17)

where $\epsilon' = \epsilon/a$ and we use the relation given in Eq. (14). The solar and atmospheric neutrino mass squared differences are coming out as

$$\Delta m_{\odot}^2 = \Delta m_{21}^2 = m_2^2 - m_1^2 = a^2 \left[ -\kappa \cos^2 \phi (\kappa + 2) - 2\epsilon' (1 + \kappa \cos^2 \phi) \right]$$

$$\Delta m_{\text{atm}}^2 = \Delta m_{32}^2 = m_3^2 - m_2^2 = a^2 \left[ -\kappa \cos^2 \phi (2 - \kappa) + 2\epsilon' (1 - \kappa \cos^2 \phi) \right].$$

(18)

From mixing matrix $U$ in Eq. (16) we obtain the mixing angles:

$$\sin \theta_{12} = |U_{12}| = \frac{1}{\sqrt{3}} + \frac{\epsilon' (2 + \kappa \cos^2 \phi)}{\sqrt{3 \kappa \cos^2 \phi (\kappa + 2)}}$$
Figure 2: Plot of $\cos^2 \phi$ with respect to $\kappa$ and $\theta_{12}$. We keep $\Delta m^2_{32}$ and $\Delta m^2_{21}$ to their best fit values.

$$\sin \theta_{13} = |U_{13}| = \left| \frac{\epsilon'}{3\sqrt{2}(\kappa - 2)} \right| \left[ 1 + \frac{\cos^2 \phi(\kappa^2 + 8 - 6\kappa)}{\cos^2 \phi} \right]^{1/2}$$

$$\tan^2 \theta_{23} = \frac{|U_{23}|^2}{1 - |U_{23}|^2} = 1 + \frac{2\epsilon' \kappa}{3(\kappa - 2)}.$$  \hspace{1cm} (19)

We define the ratio $R$ as

$$R = \frac{\Delta m^2_{21}}{\Delta m^2_{\text{atm}}} = \frac{\kappa \cos^2 \phi(\kappa + 2) + 2\epsilon'(1 + \kappa \cos^2 \phi)}{\kappa \cos^2 \phi(2 - \kappa) - 2\epsilon'(1 - \kappa \cos^2 \phi)}.$$  \hspace{1cm} (20)

Using the first relation of Eq. (19) we have

$$\epsilon' = \frac{\sqrt{3} \kappa \cos^2 \phi(\kappa + 2)}{2 + \kappa \cos^2 \phi} \times \left( \sin \theta_{12} - 1/\sqrt{3} \right).$$  \hspace{1cm} (21)

Using Eq. (21) in Eq. (20) we get

$$\cos^2 \phi = -\frac{2}{\kappa} \left[ \frac{2(1 - R) + \kappa(1 + R) + \sqrt{3}(\kappa + 2)(\sin \theta_{12} - 1/\sqrt{3})(1 + R)}{2(1 - R) + \kappa(1 + R) + 2\sqrt{3}(\kappa + 2)(\sin \theta_{12} - 1/\sqrt{3})(1 - R)} \right].$$  \hspace{1cm} (22)

Again using expression of $\epsilon'$ from Eq. (21) to the expression of $\Delta m^2_{32}$ in Eq. (18) we get the dependence of the $a^2$ on $\kappa$ and $\cos^2 \phi$:

$$a^2 = \frac{\Delta m^2_{32}(2 + \kappa \cos^2 \phi)}{\kappa \cos^2 \phi[(\kappa - 2)(2 + \kappa \cos^2 \phi) + 2\sqrt{3}(\kappa + 2)(1 - \kappa \cos^2 \phi)(\sin \theta_{12} - 1/\sqrt{3})]}.$$  \hspace{1cm} (23)
Thus from Eq. (22) and Eq. (23), we see that $a^2$ only depends on single model parameter $\kappa$. From the above expression of $\cos^2 \phi$ we can put bound on $\kappa$ for the given values of $R$ and $\theta_{12}$. For the best fit value of $\Delta m_{32}^2$ and $\Delta m_{21}^2$ we have the approximate bound:

$$\kappa < -2$$ \hfill (24)

in the course of variation $32^\circ \leq \theta_{12} \leq 36^\circ$ for $\cos^2 \phi < 1$ and it is shown in Fig. 2. It also ensures $\cos^2 \phi > 0$. Again compatibility of the above bound of $\kappa$ with the restriction $a^2 > 0$ demands that $\Delta m_{32}^2 > 0$. This leads to normal ordering of neutrino masses.

Using the expression of $\epsilon'$ from Eq. (21) into the expression of $\sin \theta_{13}$ and $\tan^2 \theta_{23}$ in Eq. (19) we get the two mixing angles in terms of $\kappa$, $\cos^2 \phi$:

$$\tan^2 \theta_{23} = 1 + \frac{2\sqrt{3}\kappa^2 \cos^2 \phi (\kappa + 2)}{3(\kappa - 2)(2 + \kappa \cos^2 \phi)} \times (\sin \theta_{12} - 1/\sqrt{3})$$ \hfill (25)
and

$$\sin \theta_{13} = \left| \frac{\kappa \cos^2 \phi (\kappa + 2)}{\sqrt{6}(\kappa - 2)(2 + \kappa \cos^2 \phi)} \right| \left| \frac{1 + \cos^2 \phi (\kappa^2 + 8\kappa)}{\cos^2 \phi} \right|^{1/2} \left| \sin \theta_{12} - 1/\sqrt{3} \right| \quad (26)$$

Here also $\tan^2 \theta_{23}$ and $\sin \theta_{13}$ only depend on $\kappa$ as $\cos^2 \phi$ is only function of $\kappa$. In addition to this dependence on parameter $\kappa$, they also depend on the well measured quantities, mixing angle $\theta_{12}$, and the ratio of solar and atmospheric mass-squared differences $R$. We keep the solar and atmospheric mass-squared differences on their best fit values. We study the variation of $\theta_{23}$ with mixing angle $\theta_{12}$ and the parameter $\kappa$ in Fig.3. We have varied $\kappa$ from its analytical upper bound $-2.0$ to $-2.5$ and also have varied $\theta_{12}$ from $32^\circ$ to $36^\circ$ ($2^\circ$ deviation about best fit value $34^\circ$ of $\theta_{12}$). For a fixed value of $\kappa$ variation of $\theta_{23}$ with $\theta_{12}$ is small and is smaller in higher value of $\kappa$, e.g. for the variation of $\theta_{12}$ from $32^\circ$ to $36^\circ$ for $\kappa = -2.0$, $\theta_{23}$ remains almost at $45.8^\circ$ and for $\kappa = -2.5$, $\theta_{23}$ changes from $48.4^\circ$ to $48.6^\circ$. If we allow $\theta_{23}$ upto its $1^\circ$ deviated value $46^\circ$, range of $\kappa$ shrinks to $-2.04 \leq \kappa \leq -2.0$ which is insensitive to the variation of $\theta_{12}$. To keep $\theta_{23}$ within $1\sigma$ deviated value $48^\circ$ for the whole range of variation of $\theta_{12}$ from $32^\circ$ to $36^\circ$, we have to keep $\kappa \geq -2.39$. From Fig.3 we can also study the variation of $\theta_{23}$ with $\kappa$ for a fixed value of $\theta_{12}$, e.g. for the best fit value of $\theta_{12} = 34^\circ$, $\theta_{23}$ changes from $45.8^\circ$ to $48.5^\circ$ for the variation of $\kappa$ from $-2.0$ to $-2.5$. So, from Fig.3 we have gathered the information where we should keep the value of $\kappa$ in the light of experimental values of $\theta_{23}$ and $\theta_{12}$. With this information we study other observables.

Fig.4 we have plotted $\theta_{13}$ with $\theta_{12}$ and $\kappa$ in their allowed region using the best fit value of solar and atmospheric mass squared differences. We have promised to generate nonzero $\theta_{13}$ with changing $\theta_{12}$ from its tri-bimaximal value $35.26^\circ$. Thus, we obtain the prediction on $\theta_{13}$ as $2.91^\circ \leq \theta_{13} \leq 3.7^\circ$ for $-2.04 \leq \kappa \leq -2.0$ (alternatively for $45.8^\circ \leq \theta_{23} \leq 46.0^\circ$) for the whole range of variation of $\theta_{12}$ $32^\circ$ to $36^\circ$. If we allow $\theta_{23}$ upto its $1\sigma$ deviated value $45.8^\circ \leq \theta_{23} \leq 48.0^\circ$ (equivalently $-2.39 \leq \kappa \leq -2.0$), we have the prediction of $\theta_{13}$ as $2.91^\circ \leq \theta_{13} \leq 10.7^\circ$ for the whole range of variation of $\theta_{12}$ from $32^\circ$ to $36^\circ$. So, the upper bound is near the largest possible allowed value from the experiment (CHOOZ $\theta_{13} < 12^\circ$ at $3\sigma$). From the plot of Fig.4 this is to be noted that the upper bound of $\theta_{13}$ is mildly sensitive to $\theta_{12}$, it varies from $9.93^\circ$ to $10.7^\circ$ at $\kappa = -2.39$. But the lower bound of $\theta_{13}$ is $2.91^\circ$ which is insensitive to the variation of $\theta_{12}$ at $\kappa = -2.0$.

Keeping all those constraints in view next we explore the parameter space of CP violation parameter $J_{CP}$. The parameter $J_{CP}$ defined as

$$J_{CP} = \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta_D = \frac{Im[h_{12}h_{23}h_{31}]}{\Delta m^2_{21}\Delta m^2_{31}\Delta m^2_{32}} \quad (27)$$
Figure 4: Plot of $\theta_{13}$ with respect to $\kappa$ and $\theta_{12}$. We keep $\Delta m^2_{32}$ and $\Delta m^2_{21}$ to their best fit values.
where \( h = M_\nu M_\nu^T \), \( \delta_D \) is Dirac phase. This \( J_{\text{CP}} \) is associated with CP violation in neutrino oscillation and is directly related to Dirac phase of mixing matrix. Using Eq. (13), Eq. (14) in Eq. (27) \( CP \) violating parameter \( J_{\text{CP}} \) takes the following form

\[
J_{\text{CP}} = \frac{\alpha^6}{\Delta m_{21}^2 \Delta m_{32}^2 \Delta m_{31}^2} \times \frac{2e'}{9} \kappa^3 (\kappa + 2) \cos^4 \phi \sqrt{\cos^2 \phi - \cos^4 \phi}. \tag{28}
\]

upto first order term in \( e' \). Using the expression of \( e' \) from Eq. (21) and \( a^2 \) from Eq. (23) into the Eq. (28) we have

\[
J_{\text{CP}} = \frac{2\sqrt{3}}{9} \frac{\kappa (\kappa + 2)^2 (2 + \kappa \cos^2 \phi) \kappa (\sin \theta_{12} - 1/\sqrt{3}) \sqrt{\cos^2 \phi - \cos^4 \phi}}{R[(\kappa - 2)(2 + \kappa \cos^2 \phi) + 2\sqrt{3}(\kappa + 2)(1 - \kappa \cos^2 \phi)(\sin \theta_{12} - 1/\sqrt{3})]^3}. \tag{29}
\]

\( J_{\text{CP}} \) is also only function of \( \kappa \) because \( \cos^2 \phi \) in Eq. (22) is function of \( \kappa \) only. It is clear from the above expression if \( \kappa = -2 \) the value of \( J_{\text{CP}} \) becomes zero. This \( \kappa + 2 \) factor appear in \( J_{\text{CP}} \) in Eq. (28) is purely from \( A_4 \) symmetric structure of the neutrino mass matrix. In our analysis we have seen that \( \kappa \) and \( \cos^2 \phi \) are not independent. \( \kappa = -2 \) approximately corresponds to \( \cos^2 \phi = 1 \) where \( J_{\text{CP}} \) also vanishes. Using the expression of \( \cos^2 \phi \) from Eq. (22) into above Eq. (29) we can make \( J_{\text{CP}} \) only \( \kappa \) dependent. It also depends on the observables quantities \( \theta_{12}, \Delta m_{32}^2 \) and \( \Delta m_{21}^2 \). We have plotted in Fig.5 \( J_{\text{CP}} \) with respect \( \kappa \) and \( \theta_{12} \) in their allowed region keeping \( \Delta m_{32}^2 \) and \( \Delta m_{21}^2 \) to their best fit values. We have the values of \( J_{\text{CP}} \) in the range \( 3.27 \times 10^{-5} \leq J_{\text{CP}} \leq 7.6 \times 10^{-5} \) for \( -2.04 \leq \kappa \leq -2.0 \) (alternatively for \( 45.8^\circ \leq \theta_{23} \leq 46.0^\circ \)) for the whole range of variation of \( \theta_{12} \) from \( 32^\circ \) to \( 36^\circ \). If we allow \( \theta_{23} \) upto its 1\( \sigma \) deviated value \( 45.8^\circ \leq \theta_{23} \leq 48.0^\circ \) (equivalently \( -2.39 \leq \kappa \leq -2.0 \)), we have the prediction of \( J_{\text{CP}} \) as \( 3.27 \times 10^{-5} \leq J_{\text{CP}} \leq 2.65 \times 10^{-3} \) for the whole range of variation of \( \theta_{12} \) from \( 32^\circ \) to \( 36^\circ \). Upper bound of \( J_{\text{CP}} \) is varying from \( 2.12 \times 10^{-3} \) to \( 2.65 \times 10^{-3} \) for the the variation of \( \theta_{12} \) from \( 32^\circ \) to \( 36^\circ \). For the best fit value of \( \theta_{12} \) (\( \approx 34^\circ \)), upper bound is \( 2.42 \times 10^{-3} \). So, larger value of \( J_{\text{CP}} \approx 2.65 \times 10^{-3} \) is possible and it can be probed through upcoming base-line experiments. From the Eq. (21) we can find the expression for \( \sin \delta_D \). Using expressions for \( \theta_{13} \) from Eq. (26), \( \theta_{23} \) from Eq. (25), \( J_{\text{CP}} \) from Eq. (29) and \( \cos^2 \phi \) from Eq. (22) into Eq. (27) we can have \( \kappa \) dependent function for \( \sin \delta_D \). We have plotted \( \delta_D \) with respect to \( \kappa \) in Fig.6 for the best fit values of \( \theta_{12}, \Delta m_{32}^2 \) and \( \Delta m_{21}^2 \). The figure reflects the prediction of \( \delta_D \) as \( \delta_D = 3.6^\circ \) for \( \theta_{23} = 48^\circ \) at \( \kappa = -2.39 \).

Now we are going to see the behavior of mass eigenvalues and their sum with respect \( \kappa \) and hence the other observables. Using the the expressions for \( a^2 \) from Eq. (23), \( e' \) from Eq. (21) and \( \cos^2 \phi \) from Eq. (22) into Eq. (17) we get the \( \kappa \) dependent functions for mass eigenvalues. We have plotted \( m_1, m_2, m_3 \) and their sum in Fig.7 with respect to \( \kappa \) for the best
Figure 5: Plot of $J_{CP}$ with respect to $\kappa$ and $\theta_{12}$. We keep $\Delta m_{32}^2$ and $\Delta m_{21}^2$ to their best fit values.
Figure 6: Plot of Dirac phase $\delta_D$ with respect to $\kappa$ for the best fit values of $\theta_{12}$, $\Delta m_{32}^2$ and $\Delta m_{21}^2$. 

$\theta_{12} = 34.0$

$\Delta m_{32}^2 = 8.0 \times 10^{-5}$

$\Delta m_{21}^2 = 2.1 \times 10^{-3}$
fit values of $\theta_{12}$, $\Delta m^2_{32}$ and $\Delta m^2_{21}$. The observations of the plots in Fig.7 suggest that mass pattern is normal-hierarchical. We also have seen that $0.07 \, eV < m_1 + m_2 + m_3 < 0.076 \, eV$ in the course of variation of $\kappa$ $-2.39 < \kappa < -2 \ (48^\circ > \theta_{23} > 45.8^\circ)$. It also satisfy the cosmological bound $m_1 + m_2 + m_3 < 0.7 \, eV$ [6].

Again parameter responsible for the $\beta\beta_{0\nu}$ experiment is also studied in the present model and the relevant quantity:

$$|(M_\nu)_{ee}| = \left|a + \frac{2d\exp^{i\phi}}{3}\right| = a \left[1 + \frac{4\kappa(\kappa + 3)\cos^2\phi}{9}\right]^{1/2}.$$ (30)

Using the expressions for $a$ and $\cos^2\phi$ from Eq. (23) and Eq. (22) respectively, we get $|(M_\nu)_{ee}|$ in terms of model parameter $\kappa$ and other physical observables. Keeping experimental values of the $\theta_{12}$, $\Delta m^2_{32}$ and $\Delta m^2_{21}$ to their best fit value, we have plotted $|(M_\nu)_{ee}|$ with respect to $\kappa$. For the physical region of $\kappa$ Fig.7 shows that $|(M_\nu)_{ee}|$ is well below the experimental bound 0.89 eV.

A point to be noted as our analysis of mixing angles are matching with [4] in the real limit($\phi = 0^\circ$). However, the expressions for the mixing angles for complex case in [4] are not in exact correspondence with our result in the present work for $\kappa = -2$ limit. This is because in the complex analysis in [4], we have assumed that eigenvectors of $M_\nu$ construct $U$. This $U$ has ability to diagonalize $M_\nu$ to its true eigenvalues but this $U$ may not be unitary. It is better to diagonalize $h (M_\nu M_\nu^\dagger)$ which is hermitian and its diagonalizing matrix is unitary. But it is always not easy to do the same. In this paper we have solved the 18 equation from Eq. (15) and find out $U$ in Eq. (16) which also is unitary keeping terms upto the order $\epsilon$: $U^\dagger U = UU^\dagger = 1 + O(\epsilon^2)$.

In summary, we explore the parameter space of a softly broken $A_4$ symmetric model for different mixing angles and a model parameter $\kappa$. We expressed the two mixing angles $\theta_{12}$ and $\theta_{23}$ in terms of a single parameter $\kappa$, and constrained the parameter space for the best fit values of $\Delta m^2_{32}$ and $\Delta m^2_{21}$. With the allowed parameter value we predict $\theta_{13}, \theta_{13} \simeq 11^\circ \ (for \ 1\sigma \ deviation \ of \ \theta_{23} \ and \ 2^\circ \ deviation \ of \ \theta_{12} \ about \ their \ best \ fit \ value)$. Utilising the above result, we expressed the CP violation parameter $J_{CP}$ in terms of $\kappa$ and $\sin \theta_{12}$ and explore the extent of $J_{CP}$ allowed in the present model. A comparatively larger value of $J_{CP}$ is allowed by the present model ($J_{CP} = 2.65 \times 10^{-3}$ for $1\sigma$ variation of the angle $\theta_{23}$) and consistent with other neutrino experimental results.
Figure 7: Plot of neutrino masses $m_1$, $m_2$, $m_3$ and their sum, and also $|\langle M_e e\rangle|$ with respect to $\kappa$ for the best fit values of $\theta_{12}$, $\Delta m^2_{32}$ and $\Delta m^2_{21}$.
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