Detecting unstable periodic spatio-temporal states of spatial extended chaotic systems

A. E. Hramov and A. A. Koronovskii

Faculty of Nonlinear Processes, Saratov State University - Astrakhanskaya, 83, Saratov, 410012, Russia

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Abstract – A method of detection of the unstable periodic spatio-temporal states of spatial extended chaotic systems has been proposed. The application of this method is illustrated by the consideration of two different systems: i) the fluid model of Pierce diode being one of the fundamental system of the physics of plasmas and microwave electronics and ii) the complex one-dimensional Ginzburg-Landau equation demonstrating different regimes of spatio-temporal chaos.

Abstract

It is well known that unstable periodic orbits (UPOs) embedded in chaotic attractors play an important role in the dynamics of the systems with a small number of degrees of freedom [1–3]. The chaotic regime of the system may be characterized by means of the set of UPOs [4]. As a universal and powerful tool for exploration of chaotic dynamics [5], unstable periodic orbits proved to be especially efficient in the context of chaotic synchronization [6–9]. The different types of chaotic synchronization (such as phase synchronization [8], lag [10] and complete synchronization [11]) may be explained in terms of unstable periodic orbits [12]. Eventually, UPOs play the key role for the chaos-controlling problem [13], since unstable periodic orbits may be stabilized by means of the weak influence on the system dynamics, e.g., with the help of small variations of the control parameter [14] or with the feedback of different types [15–17].

In spatial extended systems unstable periodic spatio-temporal states (UPSTSs) exist [18] which are similar to the unstable periodic orbits in the chaotic systems with a small number of degrees of freedom. In particular, the chaotic dynamics of spatial extended systems may be controlled by stabilizing such unstable periodic spatio-temporal states [19]. Therefore, one of the important problems connected with the study of spatial extended chaotic systems is finding these unstable periodic states. It is appropriate to suggest that the methods aimed at the search of UPOs of discrete maps (and the flow dynamical systems with small dimension of phase space, too) may be adapted to the spatial extended systems. The method proposed by Lathrop and Kostelich [2], as an example, had been used to pick out UPSTSs for the fluid model of the Pierce diode [20]. This method is based on obtaining the histograms describing the frequency of the system returning to the vicinity of UPOs (in the systems with a small number of degrees of freedom) or UPSTSs (in the spatial extended systems), respectively. Nevertheless, this method applied to spatial extended systems is rather imprecise and time-consuming. Let us also note the work of Zoldi and Greenside [21], where the numerical analysis of UPOs for a high-fractal-dimension chaotic solution of the partial differential equation is carried out with the help of the innovative damped-Newton method.

In this letter we describe the modification of the method of P. Schmelcher and F. Diakonos (SD-method) [22,23] allowing precise detection of UPSTSs in the spatial extended chaotic systems. As analyzed samples of spatially extended chaotic systems we consider here the fluid model of the Pierce diode and the complex one-dimensional Ginzburg-Landau equation.

As primary system under study we have used the fluid model of Pierce diode [24–29] being one of the simplest beam-plasma systems demonstrating complex chaotic dynamics. It consists of two plane-parallel infinite grids pierced by the monoenergetic (at the entrance) electron beam. The grids are grounded and the distance between them is $L$. The entrance space charge density $\rho_0$ and velocity $v_0$ are maintained constant. The space between the grids is evenly filled by neutralizing ions with density $|\rho_i/\rho_0| = 1$. The dynamics of this system is defined
by only one parameter, the so-called Pierce parameter $\alpha = \omega_p L / v_0$, where $\omega_p$ is the plasma frequency of the electron beam. With $\alpha > \pi$ in the system, the so-called Pierce instability [27,30] develops, which leads to the appearance of the virtual cathode. At the same time, with $\alpha \sim 3\pi$, the instability is limited by non-linearity and the regime of complete passing of the electron beam through the diode space can be observed. In this case the system can be described by the partial differential equations:

$$\frac{\partial \varphi}{\partial t} + v \frac{\partial \varphi}{\partial x} = \frac{\partial^2 \varphi}{\partial x^2}, \quad \frac{\partial \rho}{\partial t} + v \frac{\partial \rho}{\partial x} + \rho \frac{\partial v}{\partial x} = 0,$$

with the boundary conditions

$$v(0,t) = 1, \quad \rho(0,t) = 1, \quad \varphi(0,t) = \varphi(1,t) = 0.$$  

In eqs. (1) non-dimensional variables (space charge potential $\varphi$, density $\rho$, velocity $v$, space coordinate $x$ and time $t$) are used. They are related to the corresponding dimensional variables as follows:

$$\varphi' = \frac{v_0^2}{\eta} \varphi, \quad E' = \frac{v_0^2}{L \eta} E, \quad \rho' = \rho_0 \rho, \quad v' = v_0 v, \quad x' = L x, \quad t' = (L/v_0) t,$$

where the symbols denoted by a prime correspond to the dimensional values, $\eta$ is the specific electron charge, $v_0$ and $\rho_0$ are the non-perturbed velocity and density of the electron beam, $L$ is the length of the diode space. Equations (1) are integrated numerically with the help of the one-step explicit two-level scheme with upstream differences and the Poisson equation (2) is solved by the method of the error vector propagation. The time and space integration steps have been taken as $\Delta t = 0.003$ and $\Delta x = 0.005$, respectively.

One of the core problems related to the consideration of the spatial extended system is the infinite dimension of the “phase space” $W^\infty$. As a consequence, the state $U(x,t)$ of the system of investigation should be considered instead of the vector $x(t)$ in $\mathbb{R}^n$ as in the case of the flow systems [1]. After the transient finished ($i.e., t > t_r$) the set of the states $U(x,t)$, $\forall t > t_r$ may be considered as attracting the subspace $W^s$ of the infinite-dimensional “phase space” $W^\infty$ of the spatial extended system under study. If the dimension of this subspace is finite, the finite-dimensional space $\mathbb{R}^m$ of variables may be used to describe the dynamics of the spatial extended system.

It is well known that SD-method was developed for the UPOs detection in the systems with discrete time, although it may be also applied to the flow systems [23] by means of reducing them to maps with the help of Poincaré secant. In order to apply the SD method in an extended system, we assume that its infinite-dimensional phase space possesses the low-dimensional attracting invariant subspace $W^s$, and the desired solution lies in this subspace. Further, we construct the auxiliary system $y(t)$ in which the vector field $y$ is in one-to-one correspondence with $W^s$.

The stationary states $U^0(x,t) = U^0(x)$ of the spatial extended system correspond to the fixed points in the phase space of the auxiliary system, while the periodic spatio-temporal states of (1), (2) are in one-to-one correspondence with the periodic orbits of the finite-dimensional system $y(t)$. Therefore, the UPSTs of spatial extended system may be found by means of the detection of the UPOs of the auxiliary finite-dimensional system.

There are many well-known methods for applying the low-dimensional variable space to describe the behavior of a spatial extended system, among which a typical one is the mode expansion method. In particular, in ref. [28] the low-dimensional model has been constructed for the Pierce diode (1), (2) by means of the extraction of several principal modes with the help of Galerkin method. In the present work we propose the use of variables taken from several points $x_i$ of the extended system space to construct the finite-dimensional system

$$y(t) = (\rho(x_1,t), \ldots, \rho(x_m, t))^T,$$

where $m$ is the dimension of the auxiliary system, $x_i = i L / (m + 1)$, $i = 1, m$. In comparison with the other known methods, such approach allows us to go easily from the spatial extended system state $U(x,t)$ to the low-dimensional vector $y(t)$ without any additional calculations or measurements.

For the system under study (1), (2) we have estimated the dimension of the auxiliary vector $y(t)$ as $m = 3$. This assumption is based on the previous results of the consideration of the finite-dimensional model of the Pierce diode dynamics obtained with the help of Galerkin method [28].

To confirm meeting of the requirements of the one-to-one correspondence between the state $U(x,t)$ of the spatial extended system and the vector $y(t)$ of the constructed auxiliary system with a small number of degrees of freedom we have used the neighbour method [31]. We have examined that the distance $d(y_1, y_2) = ||y_1 - y_2||$ between two vectors $y_1 = y(t_1)$ and $y_2 = y(t_2)$ taken in arbitrary moments of time $t_1$ and $t_2$ is close to zero, if and only if the distance $S(U_1, U_2)$ between the two different states $U(x, t_1)$ and $U(x, t_2)$ of the spatial extended system taken in the same moments of time $t_1$ and $t_2$ is also small. The distance $S(U_1, U_2)$ has been defined as

$$S(U_1, U_2) = \left( \int_0^1 \|U_1(x,t) - U_2(x,t)\|^2 \, dx \right)^{1/2},$$

where $||\cdot||$ is the Euclidian norm.

According to the neighbour method it means that there is a one-to-one correspondence between $U(x,t)$ and $y(t)$, therefore we can use the constructed auxiliary low-dimensional system $y(t)$ to find UPTSTs by means of the SD-method.
Having constructed the auxiliary flow system (5), we can use the SD-method to detect UPOs in it and UPSTs in the initial spatial extended chaotic system (1), (2), respectively. In $\mathbb{R}^3$ space a plane $\rho(x = 0.25, t) = 1.0$ has been selected as Poincaré secant. Let us denote the vectors $y(t_n) = (1, \rho(0.5, t_n), \rho(0.75, t_n))^T$ corresponding to the $n$-th crossing the selected secant surface by the trajectory $y(t)$ as $y_n$. Then the description of the system dynamics can be made with the help of the discrete map

$$y_{n+1} = G(y_n),$$

where $G(\cdot)$ is the evolution operator. Obviously, it is impossible to find the analytical form for the operator $G$, but numerical integration of the initial system of partial differential equations (1), (2) can give us a sequence of values $\{y\}_n$, generated by the map (7).

The SD-method for picking out unstable periodic orbits in the map (7) supposes consideration of the following map [23]:

$$y_{n+1} = y_n + \lambda C [G(y_n) - y_n],$$

where $\lambda = 0.1$ is the method constant and $C$ is a certain matrix of the set $C_k$. Each of matrices $C_k$ should have only one non-vanishing entry +1 or −1 in row and column, i.e., they are orthogonal. In two dimensions the complete set of matrices consists of eight ones.

In works [22,23] it was shown that map (8) under the appropriate choice of the matrix $C$ allows to stabilize effectively the unstable saddle periodic orbits of systems (7) and (5). The positions of the UPOs in phase space are the same for the original chaotic system (7) and the transformed dynamical system (8) but their stability properties have changed: unstable fixed points turned into stable ones. A trajectory of transformed system (8) starting in the domain of attraction of a stabilized fixed point converges to it. Therefore, the UPOs of a chaotic dynamical system (7) can be obtained by iterating the transformed systems (8) using a robust set of initial conditions. In our calculation the matrix

$$C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is suitable to find UPOs in (7). Having obtained UPOs for the auxiliary systems (5) and (7) we can also obtain UPSTs corresponding to them in the original spatial extended system (1), (2).

The transformed system (8) allows to find only the unstable periodic orbits of length 1. To consider UPOs of length $p$ the map

$$y_{n+1} = y_n + \lambda C [G^p(y_n) - y_n],$$

should be considered instead of (8), where $G^{(\cdot)}(\cdot)$ is the $p$-times–iterated map (7). As far as the spatial extended system and the auxiliary flow system are considered, only the $p$-th crossing of the Poincaré secant by the trajectory $y(t)$ should be taken into account.

So, by numerical iteration of the map (10) with different values of $p$ one can find the set of the unstable periodic spatio-temporal states of the extended system (1), (2). However, there is a problem related with searching the state $U(x,t_{n+1})$ at the moment $t_{n+1}$ based on the known vector $y_{n+1}$. Indeed, we know only the coordinates of the state $y(t_{n+1})$ in the Poincaré secant but we do not know the corresponding distribution of space charge density $\rho(x,t_{n+1})$, velocity $v(x,t_{n+1})$ of the electron beam and the potential $\varphi(x,t_{n+1})$, and, correspondingly, we do not know the state $U(x,t_{n+1})$ of the extended system (1), (2). However, as we have determined above with the help of the nearest-neighbours method, the state $y(t_{n+1})$ in the Poincaré secant uniquely defines the corresponding spatial state $U(x,t_{n+1})$ belongs to the attracting finite-dimensional subspace $W^s$ of the infinite-dimensional phase space $W^\infty$. To obtain this spatial state $U(x,t_{n+1})$ mentioned above, we have used the following procedure. The system of partial differential equations (1), (2) describing the fluid model of the Pierce diode is integrated (and vector $y(t)$ is calculated) until some vector $y(t_s)$ is close to the required one $y_{n+1}$ with some demanded precision: $||y_{n+1} - y(t_s)|| < \delta$, where $\delta$ is taken as $\delta = 10^{-3}$. When this condition is satisfied, the space state $U(x,t_s)$ corresponding to the found vector $y(s)$ are considered as the required one $U(x,t_{n+1})$ and then the next iteration according to (10) should be done.

The spatio-temporal chaotic dynamics of the charge density $\rho(x,t)$ of the electron beam of the Pierce diode is shown in fig. 1 for the Pierce parameter value $\alpha = 2.858\pi$.

Fig. 1: The spatio-temporal dynamics of the charge density $\rho(x,t)$ of the electron beam of the Pierce diode. The oscillations for the selected control parameter value $\alpha = 2.858\pi$ are chaotic both in space and time.

The convergence of the iteration procedure (10) is illustrated by fig. 2, which shows the dependence of the space charge density $\rho_0(x = 0.75)$ in the moments of time when the trajectory $y(t)$ in $\mathbb{R}^3$ space crosses the Poincaré secant upon the number of iterations $n$ of the SD-method when the unstable periodic spatio-temporal state of the length $p = 1$ is studied. One can
Fig. 2: The dependence of the space charge density $\rho_n(x = 0.75)$ taken in the moments of time when the trajectory $y(t)$ in the $\mathbb{R}^3$ space crosses the Poincaré secant upon the number of iterations of the SD-method for the UPSTS of the length 1 ($T = 4.2$). The Pierce parameter has been selected as $\alpha = 2.858\pi$.

see clearly that the iteration process of the SD-method converges to a value corresponding to the unstable time-periodical spatio-temporal state of the system. Figure 3 shows the distribution of the space charge density $\rho(x, t)$ corresponding to the unstable spatio-temporal states with different periods $T$ detected by means of the SD-method.

To verify both the correctness of the chosen value of the dimension $m = 3$ of the auxiliary system and the obtained results, we have repeated the SD-method procedure for the value of the auxiliary system dimension being equal to $m = 4$. In this case the fourth-dimensional vector of the auxiliary is

$$y(t) = (\rho(0.2, t), \rho(0.4, t), \rho(0.6, t), \rho(0.8, t))^T.$$ (11)

For $m = 4$ all UPSTSs founded coincide with the ones obtained above for the dimension of the auxiliary system $m = 3$, although the time of calculations increases in this case sufficiently.

To show the universality of the proposed approach, we also report the results of detecting the unstable periodic spatio-temporal states for the one-dimensional complex Ginzburg-Landau equation (CGLE) [32]. The CGLE is a fundamental model for the pattern formation and turbulence description. This equations is used frequently to describe many different nonlinear phenomena in laser physics [33], chemical turbulence [34], fluid dynamics [35], bluff body wakes [36], coupled spatial extended systems [37,38].

We have considered one-dimensional CGLE

$$\frac{\partial u}{\partial t} = u - (1 - i\alpha)|u|^2 u + (1 + i\beta)\frac{\partial^2 u}{\partial x^2}$$ (12)

with periodical boundary conditions $u(L, t) = u(0, t)$. All calculations were performed for the fixed system parameters $\alpha = \beta = 4$ and random initial conditions. The numerical code was based on a semi-implicit scheme in time with finite differences in space. In all simulations we used a time step $\Delta t = 0.0002$ for the integration and a space discretization $\Delta x = 0.04$.

The system length $L$ has been chosen as the control parameter. In our study we examined two values of the control parameter: $L_1 = 12.63$ and $L_2 = 13.25$. For both these values of the control parameter $L$ CGLE demonstrates the spatiotemporal chaotic regime. The corresponding spatio-temporal chaotic dynamics of CGLE are shown in fig. 4 for the system lengths $L = 12.63$ and $L = 13.25$. One can see easily that the second case is characterized by more complex irregular spatio-temporal chaotic dynamics. Indeed, in the first case ($L = 12.63$) the chaotic dynamics is characterized by only one positive Lyapunov exponent $\Lambda_1 = 0.04$, while the second chaotic regime ($L = 12.63$) is characterized by two positive Lyapunov exponents $\Lambda_1 = 0.10$ and $\Lambda_2 = 0.07$.

Applying the modified SD-method to the spatial extended CGLE, we can find the demanded unstable periodical spatio-temporal states as well as for the fluid model of the Pierce diode. We have constructed the
vector (5) of the auxiliary low-dimensional system as
\[ y(t) = (u(x_1, t), \ldots, u(x_m, t))^T, \quad (13) \]
where \( m \) is the dimension of the auxiliary system vector,
\( x_i = iL/m, \ i = 1, m. \)

In contrast to the fluid model of the Pierce diode (1), (2) the dimension \( m \) of the auxiliary vector \( y(t) \) is unknown for CGLE (12). Therefore, we have to try to find UPSTSSs by means of the SD-method (10) for different values of the auxiliary system dimension \( m \) starting from the minimal dimension value \( m = 3. \) If the required UPSTSS is not found for the selected value of the auxiliary vector dimension \( m^* \), the SD-method procedure should be repeated for the greater dimension value \( m = m^* + 1. \)

For the system length \( L = 12.63 \) the dimension of the auxiliary system \( m = 3 \) is found to be adequate for the correct UPSTSS detection. As was mentioned above the system behavior is characterized by one positive Lyapunov exponent. For the more complicated case \( L = 13.25 \) (when the behavior of CGLE is characterized by two positive Lyapunov exponents) the dimension of the auxiliary vector should be taken as \( m = 4 \) for UPSTSS to be detected successfully. The matrices
\[ C = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad (14) \]
are found to be suitable to find UPSTSSs for the system lengths \( L = 12.63 \) and \( L = 13.25, \) respectively.

The convergence of the iteration procedure (10) is illustrated in fig. 5. One can see clearly that the iteration process of the SD-method converges to a value corresponding to the unstable periodic spatio-temporal state of the system. Figure 6 shows the evolution of the profiles \( |u(x, t)| \) corresponding to the unstable periodic spatio-temporal states with the different periods \( T \) detected by means of the SD-method for the system length \( L = 12.63, \) when the dimension of the auxiliary vector (13) has been chosen as \( n = 3. \) The analogous evolution of the profiles \( |u(x, t)| \) corresponding to the unstable periodic spatio-temporal states with the different lengths \( p \) and periods \( T \) is shown in fig. 7 for \( L = 13.25 \) and \( m = 4. \)

In conclusion, we have proposed a method for the detection of the unstable periodic spatio-temporal states of spatial extended chaotic systems, that is an extension of the well-known SD-method. The effectiveness of this method is illustrated by the consideration of the fluid
model of the Pierce diode and the complex Ginzburg-Landau equation.

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