Noncommutative scalar fields in compact spaces: quantization and implications

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In this paper we consider a two-component scalar field theory, with noncommutativity in its conjugate momentum space. We quantize such a theory in a compact space with the help of dressing transformations and we reveal a significant effect of introducing such noncommutativity as the splitting of the energy levels of each individual mode that constitutes the whole system. We further compute the thermal partition function exactly with predicted deformed dispersion relations from noncommutative theories and compare the results with others’. It is found that thermodynamic quantities in noncommutative models, irrespective of whether the model is more deformed in the infrared/UV region, show deviation from standard results in the high temperature region.

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1. Introduction

It is a common concept that the usual picture of spacetime as a smooth pseudo-Riemannian manifold would break down due to quantum gravity effects at very short distances of the order of the Planck length. The deviation from the flat-space concept at the order of the Planck length is actually motivated by new concepts such as quantum groups [1], quantum loop gravity [2], deformation theories [3], noncommutative geometry [4], string theory [5], etc. Besides, the idea of noncommutative spacetime was also discovered in string theory and in the matrix model of M theory, where in a certain limit due to the presence of a background field $B$, noncommutative gauge theory appears [5]. Recently, increasing interest in noncommutative theories has been triggered by the results in string theory [6]. A vast number of papers dealing with the problem of formulating noncommutative field theory [7–18] has appeared in the literature. Some implications of such noncommutativity in field theory, including connections between Lorentz invariance violation and noncommutativity of fields [14], deformed energy eigenvalue of the Hamiltonian [14,16], replusive Casimir force [17], deformed Kac–Moody algebra [16], Bose–Einstein condensation [19], noncommutative field gas driven inflation [20], UV/IR mixing [21], GZK cutoff [22], path integral [23], matter–antimatter asymmetry [14],
etc. have appeared in the literature. Due to the huge potential of noncommutative field theories to produce interesting results, such theories need extensive attention.

Several groups have reported that quantum gravity relics could be seen from the Lorentz-violating dispersion relations [24]. Lorentz invariance is then considered as a good low-energy symmetry which may be violated at very high energies. As our low-energy theories are quantum field theories (QFTs), it is interesting to explore possible generalizations of the QFT framework which could produce departures from exact Lorentz symmetry. The assumption of noncommutativity in the field space of a QFT produces Lorentz-violating dispersion relations in both compact [16] and non-compact space [17]. We expect that signatures of noncommutativity will appear in future experiments involving ultra-high-energy cosmic rays, the cosmic microwave background (CMB), or neutrino experiments [15]. As the blackbody spectrum coincides with CMB radiation with great accuracy, at least for low to medium frequencies, the deformed blackbody radiation (with deformed dispersion relation) needs to be studied well. Blackbody radiation with deformed dispersion coming from other theories such as doubly special relativity [25], generalized uncertainty principle [26], phenomenological quantum gravity theories [27,28], etc. are already well studied, but the blackbody radiation with deformed dispersion relation due to noncommutativity in field theory is yet to be studied in detail. Although Balachandran et al. [16] have visited some of the properties of deformed blackbody radiation, they did not solve the partition function, which we set out to do in this paper.

Noncommutativity in QFT has been heavily studied by several groups, as any deviation from the usual free massless boson theory may have some influence on the modeling of experimental observables. In the usual two-component scalar field theory, the field \( \phi^i (i = 1, 2) \) and the canonical conjugate momentum \( \pi^i \) are assumed to be operators satisfying the canonical commutation relations

\[
[\phi^i(\vec{x}, t), \phi^j(\vec{y}, t)] = 0, \\
[\pi_i(\vec{x}, t), \pi_j(\vec{y}, t)] = 0, \\
[\phi^i(\vec{x}, t), \pi_j(\vec{y}, t)] = i\delta^i_j \delta(\vec{x} - \vec{y}).
\]

Here, \((\vec{x}, t)\) are elements of base space. In their work, Balachandran et al. [16] and Khelili [17] considered noncommutative massless scalar fields with commutative base space and noncommutative target space. Therefore, the above commutation relations take the form\(^1\)

\[
[\hat{\phi}^i(\vec{x}, t), \hat{\phi}^j(\vec{y}, t)] = ie^{ij} \theta \delta(\vec{x} - \vec{y}), \\
[\hat{\pi}_i(\vec{x}, t), \hat{\pi}_j(\vec{y}, t)] = 0, \\
[\hat{\phi}^i(\vec{x}, t), \hat{\pi}_j(\vec{y}, t)] = i\delta^i_j \delta(\vec{x} - \vec{y}),
\]

where \(e^{ij}\) is an antisymmetric constant matrix and \(\theta\) is a parameter with the dimension of length. After constructing the Hamiltonian formulation of this theory and quantizing it in a compact space, Balachandran et al. obtained a splitting of the energy levels of each individual mode that constitutes the whole system. The resemblance of this effect to the well-known Zeeman effect in a quantum system in the presence of a magnetic field was noticed [16]. Balachandran et al. considered an \(S^1 \times S^1 \times S^1 \times \mathbb{R}\)-type geometry, where all the compactified spatial coordinates, i.e., \(S^1\), are of

\(^1\) We have used hat in field space noncommutativity and tilde for momentum space noncommutativity.
the same radius. In this manuscript we investigate a different type of noncommutativity in scalars. Here, we explore the case where the fields are commutative but the conjugate momentum space is noncommutative. Therefore the commutation relations are of the form

\[ \begin{align*}
[\tilde{\phi}(\vec{x}, t), \tilde{\phi}(\vec{y}, t)] &= 0, \\
[\tilde{\pi}(\vec{x}, t), \tilde{\pi}(\vec{y}, t)] &= i\epsilon^{ij}\delta(\vec{x} - \vec{y}), \\
[\tilde{\phi}(\vec{x}, t), \tilde{\pi}(\vec{y}, t)] &= i\delta_i^j\delta(\vec{x} - \vec{y}).
\end{align*} \tag{3a,b,c} \]

At first, we canonically quantize a free massless boson theory with the commutation relation in Eq. (3) in \((1 + 1)\) dimensions following the regularization procedure shown in the seminal paper of Balachandran et al. \[16\]. We then construct a Fock space, since the Hamiltonian can be diagonalized using the Schwinger representation of \(SU(2)\). Afterwards, we generalize the results in arbitrary dimensions. Finally, we compute the thermal partition function for the deformed Hamiltonian due to noncommutativity in Eqs. (2) and (3) and compare them.

2. Review of commutative scalar field theory in a compact space

Let us consider a theory in a \((d + 1)\)-dimensional base space with the target space set in a commutative plane \(\mathbb{R}^2\). Here, we present a free massless boson theory with commutative base space and noncommutative target space. The target space is the space where the field takes its values. The spatial part of the base space is a \(d\)-dimensional torus. Now, if the field components are denoted by \(\phi^i\) where \(i = 1, 2\), then we can write

\[ \phi : S^1_1 \times S^1_2 \times \cdots \times S^1_d \times \mathbb{R} \to \mathbb{R}^2, \]

\[ (\vec{x}, t) \mapsto \phi(\vec{x}, t). \tag{4a,b} \]

We invoke that each spatial direction is compactified in \(S^1_j\) with radius \(R_j\), which causes the field components to be periodic in the spatial coordinates,

\[ \phi(\vec{x} + \vec{R}, t) = \phi(\vec{x}, t). \tag{5} \]

Let us write the components of the field \(\phi^i(\vec{x}, t)\) as a Fourier series,

\[ \phi^i(\vec{x}, t) = \sum_{\vec{n}, j} e^{-\frac{2\pi i \vec{n} \cdot \vec{x}}{R_j}} \varphi^i_{\vec{n}}(t), \tag{6} \]

where \(\vec{n} = (n_1, n_2, \ldots, n_d)\), and thus the Fourier components of the field are

\[ \varphi^i_{\vec{n}}(t) = \frac{1}{M} \sum_j \int d^d x \ e^{\frac{2\pi i \vec{n} \cdot \vec{x}}{R_j}} \phi^i(\vec{x}, t). \tag{7} \]

Here,

\[ V = \prod_{j=1}^d R_j. \tag{8} \]

One can notice a real condition \(\phi^*_{-\vec{n}}(t) = \phi_{-\vec{n}}(t)\) from Eq. (7). Now the Lagrangian is

\[ L = \frac{1}{2} \sum_j \int d^d x [ (\partial_t \phi^j)^2 - (\nabla \phi^j)^2 ]. \tag{9} \]
The above Lagrangian in terms of the Fourier modes for \(d\)-dimensional target space is

\[
L = \frac{V}{2} \sum_{i, \vec{n}} \left\{ \dot{\phi}_i^{\vec{n}} \dot{\phi}_i^{\vec{n}} - \omega_{\vec{n}}^2 \phi_i^{\vec{n}} \phi_i^{\vec{n}} \right\}.
\]  

(10)

Here, \(\omega_{\vec{n}}\) is defined as

\[
\omega_{\vec{n}}^2 = (2\pi)^2 \sum_{j=1}^{d} \left( \frac{n_j^2}{R_j^2} \right).
\]  

(11)

From the Lagrangian, we can now evaluate the expression for the momentum:

\[
\pi_i^{\vec{n}} = \frac{\partial L}{\partial \dot{\phi}_i^{\vec{n}}} = V \dot{\phi}_i^{\vec{n}}.
\]  

(12b)

Armed with the Lagrangian and the momentum, we can finally write the Hamiltonian of the system:

\[
H = \sum_{i, \vec{n}} \pi_i^{\vec{n}} \phi_i^{\vec{n}} - L
\]  

(13a)

\[
= \sum_{i, \vec{n}} \left[ \frac{1}{2V} \pi_i^{\vec{n}} \pi_i^{\vec{n}} + \frac{V}{2} \omega_{\vec{n}}^2 \phi_i^{\vec{n}} \phi_i^{\vec{n}} \right].
\]  

(13b)

So, \(\omega_{\vec{n}}\) corresponds to the frequencies of the set of harmonic oscillators describing the system as defined in Eq. (11).

3. Noncommutative field theory

It is well known that the phase space of a single particle in \(\mathbb{R}^2\) has a natural group structure which is the semidirect product of \(\mathbb{R}^2\) with \(\mathbb{R}^2\). The generators of its Lie algebra can be taken to be coordinates \(x^a\), with \(a = 1, 2\), and momenta \(p_a\):

\[
[x^a, x^b] = 0, \\
[x^a, p_b] = i\delta^a_b, \\
[p_a, p_b] = 0.
\]  

(14)

One can now twist/deform the generators of the above algebra into \(\tilde{x}^a, \tilde{p}^b\) and thus obtain new algebras [16].

3.1. Model 1: Noncommutativity in momentum space, quantization in \((1+1)\) dimensions

In this model, the algebra of derivatives is deformed, but the function algebra is not. Here we twist (or deform) the generators of the above algebra into \(\tilde{x}^a, \tilde{p}^b\) and thus obtain a new algebra:

\[
[\tilde{x}^a, \tilde{x}^b] = 0, \\
[\tilde{x}^a, \tilde{p}^b] = i\delta^a_b, \\
[\tilde{p}^a, \tilde{p}^b] = i\delta^a_b.
\]  

(15)
Here, $\theta$ is a parameter and $\epsilon^{ab}$ is an antisymmetric constant matrix. We can relate the noncommutative coordinates with their commutative counterparts in terms of the deformation parameter with the help of a dressing transformation [16,30–32],

$$\tilde{p}^a = p^b + \frac{1}{2} \tilde{\theta}^{ab} x_b,$$  (16a)

$$\tilde{x}^b = x^b.$$  (16b)

The above dressing transformation map (16) can be easily generalized to scalar field theory. We start with a free real massless bosonic field. Its base space is a cylinder with circumference $R$, and its target space is $\mathbb{R}^2$:

$$\phi : S^1 \times \mathbb{R} \rightarrow \mathbb{R}^2,$$  (17)

$$(x, t) \rightarrow \phi(x, t).$$  (18)

The compactification of the space coordinate makes each field component periodic, i.e.

$$\phi^i(x + R, t) = \phi^i(x, t).$$  (19)

As a result, $\phi^i(x, t)$ can be written as a Fourier expansion as in Eq. (6), where the Fourier components can be rewritten as Eq. (7). Now, following the spirit of Balachandran et al. [16], we rewrite Eq. (3):

$$[\tilde{\phi}^i(x, t), \tilde{\phi}^j(y, t)] = \delta^{ij} \delta(\sigma; x - y),$$  (20b)

$$[\tilde{\phi}^i(x, t), \tilde{\pi}^j(y, t)] = i\epsilon^{ij} \delta(\sigma; x - y),$$  (20c)

where

$$\theta(\sigma; x - y) = \frac{\theta}{\sigma \sqrt{2\pi}} e^{-\frac{(x-y)^2}{2\sigma^2}}.$$  (21)

The redefinition of the term $\theta \delta(x - y)$ is to be seen as a regularization procedure (see Ref. [16]). It should be noted that these new commutation relations reduce to those in Eq. (3) in the limit $\sigma \rightarrow 0$. Also, in the limit $\theta \rightarrow 0$ we obtain the usual commutation relation. The parameter $\sigma$ indicates a new distance scale in the equal-time commutation relations for the fields [16]. Here, $\sigma$ has dimension of length, and $\theta$ has dimension$^2$ of (length)$^1$. The novelty of Balachandran et al.’s work was to regularize the delta function in the commutation relation, which prevented the energy density diverging with respect to frequency. We will use their method in our work as well. Following Eq. (16), the dressing transformation for field theory in this model is

$$\tilde{\pi}^i(x, t) = \pi^i(x, t) + \frac{1}{2} \epsilon^{ij} \int dy \theta(\sigma; x - y) \phi^j(y, t),$$  (22)

$$\tilde{\phi}^i(y, t) = \phi^i(y, t).$$  (23)

$^2$ In $d + 1$ dimensions $\theta$ has dimensions of $(\text{length})^{d+1}$, but $\sigma$ always has dimension of length.
Here, $i = 1, 2$ and $\tilde{\pi}(\vec{x}, t)$ is the canonical conjugate momentum of the field $\tilde{\phi}(\vec{x}, t)$. The map defined above reads, in Fourier modes,

$$\tilde{\pi}^i_n = \pi^i_n + \frac{1}{2R} e^{ij} \psi^j_{-n} \theta(n),$$

$$(24)$$

$$\tilde{\phi}^i_n = \psi^i_n.$$ 

$$(25)$$

The original commutation relations are

$$[\psi^i_m, \psi^j_n] = [\pi^i_m, \phi^j_n] = 0,$$

$$(26a)$$

$$[\psi^i_m, \pi^j_n] = i \delta_{mn} \delta^{ij},$$

$$(26b)$$

and the modified commutation relations in Fourier space are

$$[\tilde{\psi}^i_m, \tilde{\psi}^j_n] = 0,$$

$$(27a)$$

$$[\tilde{\pi}^i_n, \tilde{\pi}^j_m] = \frac{i}{R} e^{ij} \theta(n) \delta_{n+m,0},$$

$$(27b)$$

$$[\tilde{\psi}^i_n, \tilde{\pi}^j_m] = i \delta_{nm} \delta^{ij}.$$ 

$$(27c)$$

Considering free massless noncommutative scalar fields, the Lagrangian can be written as

$$L = \frac{1}{2} \sum_i \int dx \left[ (\partial^2_t \tilde{\phi}^i)^2 - (\partial^2_x \tilde{\psi}^i)^2 \right].$$

$$(28)$$

Now, the Hamiltonian in Fourier space is

$$\tilde{H} = \frac{1}{2R} \sum_i \langle \tilde{\pi}^i_n \tilde{\pi}^i_n \rangle + \frac{1}{2R} \sum_{i,n \neq 0} \left\{ \tilde{\pi}^i_n \tilde{\pi}^i_{-n} + (2\pi |n| g)^2 \tilde{\psi}^i_n \tilde{\psi}^i_{-n} \right\}$$

$$= \frac{1}{2R} \sum_i \left( \pi^0_0 \pi^0_0 + \frac{1}{R} \theta(0) e^{ij} \psi^j_0 \pi^i_0 + \frac{1}{4R^2} \theta^2(0) \psi^0_0 \psi^0_0 \right)$$

$$+ \frac{1}{2R} \sum_{i,n \neq 0} \left\{ \pi^i_0 \pi^i_{-n} + \left[ (2\pi |n|)^2 + \frac{\theta^2(n)}{4R^2} \right] \psi^i_n \psi^i_{-n} + \frac{1}{R} \theta(n) e^{ij} \psi^j_n \pi^i_n \right\}. \quad (29)$$

Now, the standard harmonic oscillator Hamiltonian can be written as

$$H_n = \sum_i \left( \frac{1}{2M} \pi^i_n \pi^i_n + \frac{1}{2} \tilde{\omega}_n^2 \psi^i_n \psi^i_{-n} \right).$$

$$(30)$$

Comparing with the above equation, we can write

$$\tilde{\omega}_n = \frac{1}{M} \sqrt{(2\pi |n|)^2 + \frac{\theta^2(n)}{4R^2}}.$$ 

$$(31)$$

Now, we can define the creation and annihilation operators as

$$a^i_n = \sqrt{\frac{\Delta_n}{2}} \left( \psi^i_n + \frac{\pi^i_n}{\Delta_n} \right),$$

$$(32)$$

$$a^i_n \dagger = \sqrt{\frac{\Delta_n}{2}} \left( \psi^i_{-n} - \frac{\pi^i_n}{\Delta_n} \right).$$ 

$$(33)$$
Here,

\[ \Delta_n = R\bar{\omega}_n = \sqrt{(2\pi |n| g)^2 + \frac{\theta^2(n)}{4R^2}}. \]  

(34)

Now,

\[ [a^\dagger_m, a^\dagger_n] = [a^\dagger_m, a^\dagger_n] = 0, \]  

(35)

\[ [a^\dagger_m, a^\dagger_n] = \delta_{mn}\delta_{ij}. \]  

(36)

The original Hamiltonian can be written in terms of the creation and annihilation operators defined above:

\[ H_n = \sum_i \left( \frac{1}{2R} \bar{\omega}_n \pi^i_n \pi^i_n + \frac{R}{2} \bar{\omega}_n \varphi^i_n \varphi^i_{-n} \right) \]  

(37)

\[ = \sum_i \frac{1}{2} \bar{\omega}_n \left( a^\dagger_n a^\dagger_n + a^\dagger_{-n} a^\dagger_{-n} \right). \]  

(38)

Now, after normal ordering the Hamiltonian looks like

\[ H_n = \sum_i \bar{\omega}_n a^\dagger_n a^\dagger_n. \]  

(39)

Now, the \( \varphi^i_n \pi^i_n \) term of the Hamiltonian can be written as

\[ \epsilon^{ij} \varphi^i_n \pi^j_n = -\frac{i}{2} \epsilon^{ij} [a^\dagger_{-n} a^\dagger_{-n} - a^\dagger_n a^\dagger_n] \]  

\[ = i \epsilon^{ij} a^\dagger_n a^\dagger_n. \]  

Normal ordering has been used to derive this expression. The complete Hamiltonian can be written as

\[ \tilde{H} = H_0 + \sum_{i,n \neq 0} \bar{\omega}_n a^\dagger_n a^\dagger_n + \frac{i}{2} \frac{\theta(n)}{M^2} \sum_{i,j,n \neq 0} \epsilon^{ij} a^\dagger_n a^\dagger_n. \]  

(40)

Now lets define new creation and annihilation operators,

\[ A^1_n = \frac{1}{\sqrt{2}} (a^\dagger_n - ia^2_n), \]  

(41)

\[ A^2_n = \frac{1}{\sqrt{2}} (a^\dagger_n + ia^2_n). \]  

(42)
If we write the Hamiltonian in terms of these new creation and annihilation operators, it looks like\(^3\)

\[
\tilde{H} = H_0 + \sum_n \tilde{\omega}_n [A_n^{\dagger} A_n^1 + A_n^{\dagger} A_n^2] - \frac{g}{2M^2} \sum_n \theta(n) [A_n^{\dagger} A_n^1 - A_n^{\dagger} A_n^2] 
= H_0 + \sum_n (\tilde{\omega}_n - \frac{1}{2R^2} \theta(n)) A_n^{\dagger} A_n^1 + \sum_n (\tilde{\omega}_n + \frac{1}{2R^2} \theta(n)) A_n^{\dagger} A_n^2. \tag{44}
\]

This is how energy splitting occurs due to noncommutativity in momentum space. A consequence of such noncommutativity in momentum space is the appearance of a term proportional to a component of angular momentum in the Hamiltonian of the theory. It affects the splitting of the energy levels. Splitting is also noticed if noncommutativity is introduced in field space [16]. But the functional form of the two types of splitting are quite different.

### 3.1.1. The deformed conformal generators

Now, we will have a look at the deformed conformal generators. The deformed Hamiltonian written with hatted operators is

\[
\tilde{H} = \sum_i \left( \frac{\tilde{\pi}_i^2}{2R} \right)^2 + \frac{1}{2R} \sum_{i,n \neq 0} \left\{ \tilde{\pi}_i \tilde{\pi}_{-n} + (2\pi |n|)^2 \tilde{\varphi}_n \tilde{\varphi}_{-n} \right\}. \tag{45}
\]

The deformed creation and annihilation operators can be written as

\[
\tilde{a}_n^i = \frac{1}{\sqrt{4\pi |n|}} (2\pi |n| \tilde{\varphi}_n^i + i \tilde{\pi}_n^i), \tag{46}
\]

\[
\tilde{a}_n^{i\dagger} = \frac{1}{\sqrt{4\pi |n|}} (2\pi |n| \tilde{\varphi}_{-n}^i - i \tilde{\pi}_{-n}^i). \tag{47}
\]

So,

\[
[\tilde{a}_m^i, \tilde{a}_n^j] = \frac{1}{4\pi |n|} \frac{-i}{R} \epsilon^{ij} \theta(n) \delta_{n+m,0}, \tag{48}
\]

\[
[\tilde{a}_m^{i\dagger}, \tilde{a}_n^{j\dagger}] = \delta^{ij} \delta_{mn} + \frac{1}{4\pi |n|} \frac{i}{R} \epsilon^{ij} \theta(n) \delta_{mn}, \tag{49}
\]

\[
[\tilde{a}_m^{i\dagger}, \tilde{a}_n^j] = \frac{1}{4\pi |n|} \frac{-i}{R} \epsilon^{ij} \theta(n) \delta_{n+m,0}. \tag{50}
\]

It should be noted that if we make \(\theta \to 0\), the creation–annihilation operators of noncommutative theories coincide with the usual theory. The generators of the modified \(U(1)\) Kac–Moody algebra would be:

For \(n > 0\),

\[
J_n^i = -i \sqrt{n} \tilde{a}_n^i, \tag{51}
\]

\[
\tilde{J}_n^i = -i \sqrt{n} \tilde{a}_n^{i\dagger}. \tag{52}
\]

\(^3\) Following Ref. [16] we ignore the zero mode. It is not relevant, since it is associated with the overall translation of the system.
For $n < 0$,

$$J^i_n = i\sqrt{-n}\hat{a}^i_{-n},$$  \hspace{2cm} (53)  

$$\bar{J}^i_n = i\sqrt{-n}\hat{a}^i_{-n}.$$  \hspace{2cm} (54)  

The commutators between the generators can be written as

$$[J^j_{m}, J^i_{n}] = m\delta^{ij}\delta_{n+m,0} + \frac{i}{4\pi R} \epsilon^{ij}(n)\delta_{n+m,0},$$  \hspace{2cm} (55)  

$$[\bar{J}^j_{m}, \bar{J}^i_{n}] = m\delta^{ij}\delta_{n+m,0} + \frac{1}{4\pi R} \epsilon^{ij}(n)\delta_{n+m,0},$$  \hspace{2cm} (56)  

$$[J^j_{m}, \bar{J}^i_{n}] = \frac{i}{4\pi R} \epsilon^{ij}(n)\delta_{n+m}.$$  \hspace{2cm} (57)  

It can be easily observed that a term dependent upon noncommutative parameter $\theta$ has appeared in the commutation relations of the $U(1)$ Kac–Moody algebra. This deformed $U(1)$ Kac–Moody due to momentum space noncommutativity is quite different compared to field space noncommutativity deformed $U(1)$ Kac–Moody[16]. Now, the non-zero mode terms of the Hamiltonian can be written as

$$\frac{2\pi}{R} \sum_{i,n > 0} (J^i_{-n}J^j_n + \bar{J}^i_{-n}\bar{J}^j_n).$$  \hspace{2cm} (58)  

So,

$$[\tilde{H}, J^k_{-m}] = \frac{2\pi}{R} \sum_{i,m > 0} \{2mJ^k_{-m} + \frac{1}{2\pi g R} \epsilon^{ik}(J^j_{-m} + \bar{J}^j_{-m})\}.$$  \hspace{2cm} (59)  

Now we can write the conformal generators:

$$\hat{L}_0 = \frac{1}{2} \sum_i J^2_0 + \sum_{i,n > 0} J^i_{-n}J^i_n,$$  \hspace{2cm} (60)  

$$\hat{L}_n = \frac{1}{2} \sum_{i,m,n \neq 0} J^i_{n-m}J^i_m,$$  \hspace{2cm} (61)  

$$\hat{\bar{L}}_0 = \frac{1}{2} \sum_i \bar{J}^2_0 + \sum_{i,n > 0} \bar{J}^i_{-n}\bar{J}^i_n,$$  \hspace{2cm} (62)  

$$\hat{\bar{L}}_n = \frac{1}{2} \sum_{i,m} \bar{J}^i_{n-m}\bar{J}^i_m.$$  \hspace{2cm} (63)  

Here,

$$J^i_0 = J^i_0 = \frac{1}{\sqrt{4\pi}} \sqrt{\frac{\theta_0}{\theta(0)} + \frac{1}{R} \theta(0)\phi^i_0\phi^i_0 + \frac{1}{4R^2} \theta^2(0)\phi^i_0\phi^i_0}.$$  \hspace{2cm} (64)  

So, the Hamiltonian can be written as,

$$\tilde{H} = \frac{2\pi}{R} (\hat{L}_0 + \hat{\bar{L}}_0).$$  \hspace{2cm} (65)  

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3.2. Momentum noncommutativity in (d+1) dimensions

We will now generalize the results of the noncommutativity in (d+1) dimensions. The spatial part of the base space is a d-dimensional torus, just like Eq. (4) but with a noncommutative scalar field. Therefore, the field can be written as

\[ \tilde{\phi}^i(\vec{x}, t) = \sum_{\vec{n}} \exp \left[ 2\pi i \left( \sum_j \frac{n_j x_j}{R_j} \right) \right] \tilde{\phi}^i_{\vec{n}}(t) \] (66)

The Fourier components can be written as

\[ \tilde{\phi}^i_{\vec{n}}(t) = \frac{1}{R_1 R_2 \cdots R_d} \int d^d x \exp \left[ -2\pi i \left( \sum_j \frac{n_j x_j}{R_j} \right) \right] \tilde{\phi}^i(\vec{x}, t). \] (68)

So, the Lagrangian looks like

\[ L = \frac{1}{2} \sum_i \int d^d x \left[ (\partial_t \tilde{\phi}^i)^2 - (\nabla \tilde{\phi}^i)^2 \right]. \]

If we write Lagrangian in terms of Fourier modes, then

\[ L = \frac{1}{2} R_1 R_2 \cdots R_d \sum_{i, \vec{n}} \left\{ \tilde{\phi}^i_{\vec{n}} \tilde{\phi}^i_{-\vec{n}} - 4\pi^2 \left[ \sum_j \frac{n_j^2}{R_j^2} \right] \tilde{\phi}^i_{\vec{n}} \tilde{\phi}^i_{-\vec{n}} \right\}. \] (69)

The canonical momentum is defined by

\[ \tilde{\pi}^i_{\vec{n}} = \frac{\partial L}{\partial \dot{\tilde{\phi}}^i_{\vec{n}}} = R_1 R_2 \cdots R_d \tilde{\phi}^i_{-\vec{n}}. \] (70)

Now, the deformation map can be written as:

\[ \tilde{\phi}^i_{\vec{n}} = \phi^i_{\vec{n}}, \] (71)

\[ \tilde{\pi}^i_{\vec{n}} = \pi^i_{\vec{n}} + \frac{1}{2 R_1 R_2 \cdots R_d} \epsilon^{ab} \phi^j_{-\vec{n}} \theta(\vec{n}). \] (72)

The commutation relationships between the deformed field modes are:

\[ [\tilde{\phi}^i_{\vec{m}}, \tilde{\phi}^j_{\vec{n}}] = 0, \] (73)

\[ [\tilde{\pi}^i_{\vec{m}}, \tilde{\pi}^j_{\vec{n}}] = \frac{i}{R_1 R_2 \cdots R_d} \delta^{ij} \theta(\vec{n}) \delta_{n+m,0}. \] (74)

\[ [\tilde{\phi}^a_{\vec{n}}, \tilde{\pi}^b_{\vec{m}}] = i \delta_{mn} \delta^{ab}. \] (75)

Here, the term \( \theta(\vec{n}) \) is defined as

\[ \theta(\vec{n}) = \theta \exp \left[ -2\pi^2 \sigma^2 \sum_j \frac{n_j^2}{R_j^2} \right]. \] (76)
If we write the Hamiltonian in terms of the hatted operators, then
\[
H = \frac{1}{2R_1R_2\ldots R_d} \sum_i \hat{\pi}_i^i \hat{\pi}_i^i + \frac{1}{2R_1R_2\ldots R_d} \sum_{i,j \neq 0} \left\{ \hat{\pi}_i^i \hat{\pi}_j^j + (2\pi R_1R_2\ldots R_d)^2 \left[ \sum_j n_j^2 R_j^2 \right] \hat{\psi}_i^i \hat{\psi}_j^j \right\}.
\]

So, using the dressing transformation,
\[
H = H_0 + \frac{1}{2R_1R_2\ldots R_d} \sum_i \pi_i^i \pi_i^i + \frac{1}{2R_1R_2\ldots R_d} \sum_{i,j \neq 0} \left\{ (2\pi R_1R_2\ldots R_d)^2 \left[ \sum_j n_j^2 R_j^2 \right] \right\} + \theta^2(\vec{n}) \frac{1}{4R_1^2R_2^2\ldots R_d^2} \hat{\psi}_i^i \hat{\psi}_j^j.
\]

Now, let us define
\[
H_{\vec{n}} = \sum_i \frac{1}{2R_1R_2\ldots R_d} \pi_i^i \pi_i^i + \sum_i \frac{1}{2R_1R_2\ldots R_d} \left[ (2\pi R_1R_2\ldots R_d)^2 \left[ \sum_j n_j^2 R_j^2 \right] \right] + \theta^2(\vec{n}) \frac{1}{4R_1^2R_2^2\ldots R_d^2} \hat{\psi}_i^i \hat{\psi}_j^j.
\]

The standard harmonic oscillator Hamiltonian can be written as
\[
H_{\vec{n}} = \sum_i \left( \frac{1}{2V} \pi_i^i \pi_i^i + \frac{V}{2} \hat{\omega}_n^2 \hat{\psi}_i^i \hat{\psi}_i^i \right).
\]

Comparing these two equations, we can write:
\[
V = R_1R_2\ldots R_d, \hat{\omega}_n = \frac{1}{M} \left[ (2\pi R_1R_2\ldots R_d)^2 \left[ \sum_j n_j^2 R_j^2 \right] \right] + \frac{\theta^2(\vec{n})}{4R_1^2R_2^2\ldots R_d^2}.
\]

Let us now define creation and annihilation operators:
\[
d_{\vec{n}}^i = \sqrt{\frac{\Delta_{\vec{n}}}{2}} \left( \hat{\psi}_i^i + \frac{\pi_i^n}{\Delta_{\vec{n}}} \right),
\]
\[
da_{\vec{n}}^{i\dagger} = \sqrt{\frac{\Delta_{\vec{n}}}{2}} \left( \hat{\psi}_i^i - \frac{\pi_i^n}{\Delta_{\vec{n}}} \right).
\]

Here,
\[
\Delta_{\vec{n}} = V \hat{\omega}_n = \left[ (2\pi R_1R_2\ldots R_d)^2 \left[ \sum_j n_j^2 R_j^2 \right] \right] + \frac{\theta^2(\vec{n})}{4R_1^2R_2^2\ldots R_d^2}.
\]

Using these operators, the Hamiltonian \( H_{\vec{n}} \) can be written as
\[
H_{\vec{n}} = \sum_i \frac{\omega_n}{2} (d_{\vec{n}}^i a_{\vec{n}}^{\dagger i} + a_{\vec{n}}^{i\dagger} d_{\vec{n}}^i).
\]
The time-ordered form of this Hamiltonian is

$$\sum_{\vec{n} \neq 0} H_{\vec{n}} = \sum_{i, \vec{n} \neq 0} \omega_{\vec{n}} (a_{\vec{n}}^i \hat{a}_{\vec{n}}^i).$$

(86)

The last term of the Hamiltonian can be written as

$$\frac{\theta(\vec{n})}{2R_1R_2 \cdots R_d} \sum_{i, \vec{n} \neq 0} \frac{i}{R_1R_2 \cdots R_d} \epsilon_{ij} a_{\vec{n}}^i \hat{a}_{\vec{n}}^j.$$  

(87)

So, the total Hamiltonian can be written as

$$H = H_0 + \sum_{\vec{n} \neq 0} \omega_{\vec{n}} {\vec{a}}^\dagger {\vec{a}} + \frac{i \theta(\vec{n})}{2V^2} \sum_{i, j \neq 0} \epsilon_{ij} a_{\vec{n}}^i \hat{a}_{\vec{n}}^j.$$ 

(88)

We can repeat the Schwinger process done previously and define new creation and annihilation operators, $A_{\vec{n}}^1$ and $A_{\vec{n}}^2$. Using these operators, the Hamiltonian is

$$H = H_0 + \sum_{\vec{n} \neq 0} \left( (\omega_{\vec{n}} - \frac{1}{2V^2} \theta(\vec{n})) A_{\vec{n}}^1 A_{\vec{n}}^\dagger + (\omega_{\vec{n}} + \frac{1}{2V^2} \theta(\vec{n})) A_{\vec{n}}^2 A_{\vec{n}}^\dagger \right).$$

(93)

3.3. Model 2: Noncommutativity in field space, quantization in (d+1) dimensions

One can also consider noncommutativity in field space instead of the momentum space. Canonical quantization of such theories in compact space was already considered by Balachandran et al. The twisted algebra reads, in this model,

$$[\hat{x}^a, \hat{x}^b] = i\epsilon^{ab} \theta, \quad \bar{\theta} = i\theta^{ab},$$

(91a)

$$[\hat{p}^a, \hat{p}^b] = 0,$$

(91b)

$$[\hat{x}^a, \hat{p}^b] = i\delta_a^b.$$  

(91c)

Also, the corresponding equal-time commutation relations in field theory are

$$[\hat{\phi}^i(\vec{x}, t), \hat{\phi}^j(\vec{y}, t)] = i\epsilon_i^j \theta(\sigma; \vec{x} - \vec{y}),$$

(92a)

$$[\hat{\pi}_i(\vec{x}, t), \hat{\pi}_j(\vec{y}, t)] = 0,$$

(92b)

$$[\hat{\phi}^i(\vec{x}, t), \hat{\pi}_j(\vec{y}, t)] = i\delta^i_j \delta(\vec{x} - \vec{y}).$$

(92c)

To see the the canonical quantization procedure in compact space for this type of model see Ref. [16]. The spatial part of the base space is a d-dimensional torus. But in their paper, they considered compactified spatial coordinates, i.e., $S^1$, all of them with the same radius $R$. Here, however, we present the results invoking that each spatial direction is compactified in $S^1$ with radius $R_j$. Mimicking the calculation of Ref. [16], we find the quantized Hamiltonian (normal ordered):

$$H = H_0 + \sum_{\vec{n} \neq 0} \omega_{\vec{n}} (\Gamma_{\vec{n}}^1 A_{\vec{n}}^1 A_{\vec{n}}^1 + \Gamma_{\vec{n}}^2 A_{\vec{n}}^2 A_{\vec{n}}^2),$$

(93)
where

\[ \Gamma_1^\mathbf{n} = \Omega_\mathbf{n} - \frac{\omega_\mathbf{n}\theta(n)}{2}, \]  
\[ \Gamma_2^\mathbf{n} = \Omega_\mathbf{n} + \frac{\omega_\mathbf{n}\theta(n)}{2}, \]  
\[ \theta(\mathbf{n}) = \theta \exp \left[ -2\pi^2 \sigma^2 \left( \frac{n_1^2}{R_1^2} + \frac{n_2^2}{R_2^2} + \cdots + \frac{n_d^2}{R_d^2} \right) \right], \]  
\[ \omega_\mathbf{n}^2 = 4\pi^2 \left( \frac{n_1^2}{R_1^2} + \frac{n_2^2}{R_2^2} + \cdots + \frac{n_d^2}{R_d^2} \right), \]  
\[ \Omega_\mathbf{n}^2 = 1 + \pi^2 \theta(\mathbf{n})^2 \left( \frac{n_1^2}{R_1^2} + \frac{n_2^2}{R_2^2} + \cdots + \frac{n_d^2}{R_d^2} \right). \]

In the limit \( R_1 = R_2 = \cdots = R_d \) the above equations coincide with the result of Balachandran et al. [16]. Therefore, the split deformed dispersion relations take the form:

\[ \Lambda_1^\mathbf{n} = \omega_\mathbf{n} \left( \Omega_\mathbf{n} - \frac{\omega_\mathbf{n}\theta(n)}{2} \right), \]  
\[ \Lambda_2^\mathbf{n} = \omega_\mathbf{n} \left( \Omega_\mathbf{n} + \frac{\omega_\mathbf{n}\theta(n)}{2} \right). \]

### 4. Deformed blackbody radiation

In this section, we analyze the blackbody radiation due to the deformed energy momentum relation coming from field space [Eqs. (101) and (102)] and momentum space noncommutativity [Eqs. (91) and (92)]. Although Balachandran et al. briefly discussed it, they did not calculate the thermodynamic quantities. However, we have numerically evaluated the thermodynamic quantities and we will present them in this section. We start from the partition function of quantum gases in the grand canonical ensemble [33],

\[ Z = \text{Tr}(e^{-\beta H}). \]  

As there is a split in energy eigenvalues due to both types of noncommutativity, we find from the above equation that [16,33]

\[ \ln Z = -\sum_{\mathbf{k} \neq 0} \ln(1 - e^{-\beta \Lambda_1^\mathbf{k}}) + \ln(1 - e^{-\beta \Lambda_2^\mathbf{k}}). \]

Here, \( \Lambda_1 \) and \( \Lambda_2 \) refer to two distinct classes of modes due to noncommutativity conditions, \( \mathbf{k} \) is the momentum vector, and \( \beta = \frac{1}{T} \). Therefore, the internal energy is

\[ U = -\frac{\partial}{\partial \beta} \ln Z. \]  

Now, the entropy \( S \) can be obtained from the partition function,

\[ S = -\frac{\partial F}{\partial T}. \]

---

\footnote{We have chosen the Boltzmann constant \( k_B = 1 \), \( c = 1 \).}
Fig. 1. Plot of internal energy $U$ (a) and entropy $S$ (b) of blackbody radiation against temperature $T$ for the special relativity theory and the noncommutative models. Following Refs. [14,16,20] we have chosen $\theta = 1.7 \times 10^{-14}$ and $\sigma = 10^{-15}$.

where $F = \frac{1}{\beta} \ln Z$ is the free energy.

In the thermodynamic limit we consider all $R_i \to \infty$, which allows us to convert the sum in Eq. (104) to an integral. We performed the integrals numerically using Mathematica [34], choosing specific values for $\theta$ and $\sigma$ following Balachandran et al. [16]. Finally, using Eqs. (105) and (106) we evaluated the internal energy and entropy for both types of noncommutative models and for special relativity (SR). We have compared the results in Fig. 1. We found that all three models agree at lower temperatures, but the noncommutativity effects surely modify the Stefan–Boltzmann law ($U \propto T^4$), which is clearly visible in the high-temperature regime. But at high temperatures a significant difference is noticed between them—see Fig. 1(a). In the high-temperature regime it is seen that at any temperature $T = T_0$ the internal energy coming from these models maintains the relation $U_{\text{SR}}(T_0) < U_{\text{model 1}}(T_0) < U_{\text{model 2}}(T_0)$. This trend is also noticed for other thermodynamic quantities such as entropy, specific heat, etc. The dispersion relation predicted from both types of noncommutative field theories are clearly Lorentz violating. This trend of faster rate of growth (with respect to temperature) of thermodynamic quantities at high temperatures compared to SR is also noticed in other Lorentz-violating studies on the thermodynamics of blackbody radiation [24]. As one can see, in model 1 the modifications of the dispersion relations (91) and (92) occur for small wave number $n$ and become the usual ones in the large-$n$ limit (more deformed in the infrared region), whereas in model 2 the dispersion relations (101) and (102) have large modifications for large wave number $n$ and become the usual one in the small-$n$ limit (more deformed in the UV region). But interestingly, thermodynamic quantities in both the models show deviation from standard special relativity results in the high-temperature regime.

In the case of noncommutative models, due to Lorentz-violating dispersion relations, the number of available states grows. As a result, when we do the integration over all the modes in Eq. (104) we find the modified internal energy. The Planck distribution function picks up a smaller value in the low-temperature region compared to the high-temperature region. The noncommutative parameter makes some modification in the Planck distribution but it is not extremely drastic. As a result, when we do integration over it no such significant change is noticed at low temperatures due to the noncommutative parameter. Now as the temperature rises abruptly the Planck distribution attains higher values and a small change due to the noncommutative parameter makes the change big.
enough that when we do integration over all the modes a difference is noticed. The effect of the noncommutative parameter in internal energy is less clear in the low-temperature region, as here the (modified) Planck distribution picks up a very small value. As a result, thermodynamic quantities in both of these models, irrespective of whether the model is more deformed in infrared/UV region, show deviation from standard results in the high-temperature regime.

5. Conclusion

In this paper we have canonically quantized a noncommutative scalar field theory in a compact space with noncommutativity in momentum space, following the seminal work of Balachandran et al. [16]. As a result of this noncommutativity, we have noticed the splitting of the energy levels of each individual mode that constitutes the whole system. This type of splitting in energy eigenvalue was also noticed in noncommutative scalars, where the noncommutativity is in the field space [16]. But the functional forms of deformed dispersion relations due to the two types of noncommutativity are quite different, and as a result their predictions are also quite different in blackbody radiation. We have paid special attention to the special case of (1+1)-dimensional theory and found the deformed conformal generators. We are in the process of evaluating the deformed Virasoro algebra for noncommutative theories and determining the status of the central charge in such field theories. The central charge is a very significant concept in conformal theories as the theories are characterized by this number. A different central charge would imply a new interpretation of the central extension. Such noncommutativity would be even more interesting for gauge fields as the cancellation of degrees of freedom with Gupta–Bleuler quantization or the Faddev–Popov method by the appearance of ghost fields, which can lead to new physics. The central charge of the ghost fields plays a significant role in the critical dimension of string theory. Furthermore, it should be noticed that we have considered that the spatial part of the base space is a \( d \)-dimensional torus where we invoked that each spatial direction is compactified in \( S^1_j \) with radius \( R_j \). The reason behind keeping the result more general is we are in the process of computing the Casimir force for noncommutative theories in compact space. The general result will help us to make any particular direction, say \( R_1 \), finite and other directions to put in the bulk limit. In the future, we would also like to investigate the finite-temperature status of these types of theories. As a result we notice that thermodynamic quantities in both of these models show deviation from standard results.

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