Hunt’s hypothesis (H) and Getoor’s conjecture for Lévy processes

Ze-Chun Hu\textsuperscript{a,},* Wei Sun\textsuperscript{b}

\textsuperscript{a}Department of Mathematics, Nanjing University, Nanjing 210093, China
\textsuperscript{b}Department of Mathematics and Statistics, Concordia University, Montreal, H3G 1M8, Canada

Received 12 February 2011; received in revised form 5 January 2012; accepted 19 March 2012
Available online 29 March 2012

Abstract

In this paper, Hunt’s hypothesis (H) and Getoor’s conjecture for Lévy processes are revisited. Let $X$ be a Lévy process on $\mathbb{R}^n$ with Lévy–Khintchine exponent $(a, A, \mu)$. First, we show that if $A$ is non-degenerate then $X$ satisfies (H). Second, under the assumption that $\mu(\mathbb{R}^n \setminus \sqrt{A}\mathbb{R}^n) < \infty$, we show that $X$ satisfies (H) if and only if the equation

$$\sqrt{A}y = -a - \int_{\{x \in \mathbb{R}^n \setminus \sqrt{A}\mathbb{R}^n: |x| < 1\}} x\mu(dx), \quad y \in \mathbb{R}^n,$$

has at least one solution. Finally, we show that if $X$ is a subordinator and satisfies (H) then its drift coefficient must be 0.

MSC: primary 60J45; secondary 60G51

Keywords: Hunt’s hypothesis; Getoor’s conjecture; Lévy processes

1. Introduction and main results

Let $X$ be a nice Markov process. Hunt’s hypothesis (H) says that “every semipolar set of $X$ is polar”. (H) plays a crucial role in the potential theory of (dual) Markov processes. We refer the reader to Blumenthal and Getoor [1, Chapter VI], [2] for details. In spite of its importance, (H)
has been verified only in some special situations. Let $X$ and $\hat{X}$ be a pair of dual Markov processes as in [1, Chapter VI]. Then, (H) holds if and only if the fine and cofine topologies differ by polar sets, see [1, VI.4.10] and [8, Theorem (2.2)]. Some forty years ago, Getoor conjectured that essentially all Lévy processes satisfy (H).

Throughout this paper, we let $(\Omega, F, P)$ be a probability space and $X = (X_t)_{t \geq 0}$ be an $\mathbb{R}^n$-valued Lévy process on $(\Omega, F, P)$ with Lévy–Khintchine exponent $\psi$, i.e.,

$$E[\exp(i\langle z, X_t \rangle)] = \exp(-t\psi(z)), \quad z \in \mathbb{R}^n, \ t \geq 0,$$

where $E$ denotes the expectation with respect to $P$. For $\psi$, we have the following famous Lévy–Khintchine formula:

$$\psi(z) = i\langle a, z \rangle + \frac{1}{2} \langle z, Az \rangle + \int_{\mathbb{R}^n} \left(1 - e^{ix z} + i\langle z, x \rangle 1_{|x|<1} \right) \mu(dx),$$

where $a \in \mathbb{R}^n$, $A$ is a symmetric nonnegative definite $n \times n$ matrix, and $\mu$ is a measure (called the Lévy measure) on $\mathbb{R}^n \setminus \{0\}$ satisfying $\int_{\mathbb{R}^n \setminus \{0\}} (1 \wedge |x|^2) \mu(dx) < \infty$. Hereafter, we use $\text{Re}(\psi)$ and $\text{Im}(\psi)$ to denote the real part and imaginary part of $\psi$, respectively, and use $(a, A, \mu)$ to denote $\psi$ sometimes. For every $x \in \mathbb{R}^n$, we denote by $P^x$ the law of $x + X$ under $P$. In particular, $P^0 = P$.

Let $B \subset \mathbb{R}^n$. We define the first hitting time of $B$ by

$$\sigma_B := \inf\{t > 0 : X_t \in B\}.$$

Denote by $\mathcal{B}^*$ the family of all nearly Borel sets relative to $X$ (cf. [1, I.10.21]). A set $B \subset \mathbb{R}^n$ is called polar (resp. essentially polar) if there exists a set $C \in \mathcal{B}^*$ such that $B \subset C$ and $P^x(\sigma_C < \infty) = 0$ for every $x \in \mathbb{R}^n$ (resp. $dx$-almost every $x \in \mathbb{R}^n$). Hereafter $dx$ denotes the Lebesgue measure on $\mathbb{R}^n$. $B$ is called a thin set if there exists a set $C \in \mathcal{B}^*$ such that $B \subset C$ and $P^x(\sigma_C = 0) = 0$ for every $x \in \mathbb{R}^n$. $B$ is called semipolar if $B \subset \cup_{n=1}^{\infty} B_n$ for some thin sets $\{B_n\}_{n=1}^{\infty}$.

Before introducing our results, we first recall some important results obtained so far for Getoor’s conjecture. When $n = 1$, Kesten [15] (cf. also [3]) showed that if $X$ is not a compound Poisson process, then every $\{x\}$ is non-polar if and only if

$$\int_0^{\infty} \text{Re}([1 + \psi(z)]^{-1})dz < \infty.$$

(If $X$ is a compound Poisson process, then it is easy to see that every $x$ is regular for $\{x\}$, i.e., $P^x(\sigma_{\{x\}} > 0) = 0$.) Port and Stone [17] proved that for the asymmetric Cauchy process on the line every $x$ is regular for $\{x\}$. Hence only the empty set is a semipolar set and therefore (H) holds in this case. Further, Blumenthal and Getoor [2] showed that all stable processes with index $\alpha \in (0, 2)$ on the line satisfy (H).

Kanda [13] and Forst [5] proved that (H) holds if $X$ has bounded continuous transition densities (with respect to $dx$) and the Lévy–Khintchine exponent $\psi$ satisfies $|\text{Im}(\psi)| \leq M(1 + \text{Re}(\psi))$ for some positive constant $M$. Rao [18] gave a short proof of the Kanda–Forst theorem under the weaker condition that $X$ has resolvent densities. In particular, for $n > 1$ all stable processes of index $\alpha \neq 1$ satisfy (H). Kanda [14] settled this problem for the case $\alpha = 1$ assuming the linear term vanishes. Silverstein [20] extended the Kanda–Forst condition to the non-symmetric Dirichlet forms setting, and Fitzsimmons [4] extended it to the semi-Dirichlet forms setting. Glover and Rao [9] proved that $\alpha$-subordinates of general Hunt processes satisfy (H). Rao [19] proved that if all 1-excessive functions of $X$ are lower semicontinuous and
\[ |\text{Im}(\psi)| \leq (1 + \text{Re}(\psi)) f (1 + \text{Re}(\psi)), \]
where \( f \) is an increasing function on \([1, \infty)\) such that \( \int_{N}^{\infty} z f(z)^{-1} \, dz = \infty \) for any \( N \geq 1 \), then \( X \) satisfies (H).

Now we introduce the main results of this paper. To state the first result, we let \( \bar{X} \) be an independent copy of \( X \). Define the symmetrization \( \tilde{X} \) of \( X \) by \( \tilde{X} := X - \bar{X} \).

**Theorem 1.1.** Suppose that \( A \) is non-degenerate, i.e., \( A \) is of full rank. Then:

(i) \( X \) satisfies (H);

(ii) The Kanda–Forst condition \( |\text{Im}(\psi)| \leq M (1 + \text{Re}(\psi)) \) holds for some positive constant \( M \);

(iii) \( X \) and \( \tilde{X} \) have the same polar sets.

Denote \( b := -a \) and \( \mu_{1} := \mu|_{\mathbb{R}^{n} \setminus \sqrt{A} \mathbb{R}^{n}} \). If \( \int_{|x|<1} |x| \mu_{1}(dx) < \infty \), we set \( b' := b - \int_{|x|<1} x \mu_{1}(dx) \). To state the second result, we define the following solution condition:

(S) The equation \( \sqrt{A} y = b', \ y \in \mathbb{R}^{n} \), has at least one solution.

**Theorem 1.2.** Suppose that \( \mu(\mathbb{R}^{n} \setminus \sqrt{A} \mathbb{R}^{n}) < \infty \). Then, the following three claims are equivalent:

(i) \( X \) satisfies (H);

(ii) (S) holds;

(iii) The Kanda–Forst condition \( |\text{Im}(\psi)| \leq M (1 + \text{Re}(\psi)) \) holds for some positive constant \( M \).

**Remark 1.3.** (i) Theorem 1.1 tells us that if a Lévy process on \( \mathbb{R}^{n} \) is perturbed by an independent (small) \( n \)-dimensional Brownian motion, then the perturbed Lévy process must satisfy (H).

(ii) By Theorem 1.2 and Jacob [12, Example 4.7.32], one finds that if \( X \) satisfies (H) and \( \mu(\mathbb{R}^{n} \setminus \sqrt{A} \mathbb{R}^{n}) < \infty \), then \( X \) must be associated with a Dirichlet form on \( L^{2}(\mathbb{R}^{n}; dx) \).

**Proposition 1.4.** Suppose that \( \mu(\mathbb{R}^{n} \setminus \sqrt{A} \mathbb{R}^{n}) < \infty \). Then:

(i) \( X \) has transition densities implies that all the claims of Theorem 1.2 are fulfilled.

(ii) If one of the claims in Theorem 1.2 is fulfilled, then the following four claims are equivalent:

(a) Every essentially polar set of \( X \) is polar;

(b) \( X \) has resolvent densities;

(c) \( X \) has transition densities;

(d) \( A \) is of full rank.

**Proposition 1.5.** Suppose that \( X \) has bounded continuous transition densities, and \( X \) and \( \tilde{X} \) have the same polar sets. Then \( X \) satisfies (H).

Suppose that \( X \) is a subordinator. Then \( \psi \) can be expressed by

\[ \psi(z) = -idz + \int_{(0,\infty)} \left( 1 - e^{izx} \right) \mu(dx), \quad z \in \mathbb{R}, \]

where \( d \geq 0 \) (called the drift coefficient) and \( \mu \) satisfies \( \int_{(0,\infty)} (1 \wedge x) \mu(dx) < \infty \).

**Proposition 1.6.** If \( X \) is a subordinator and satisfies (H), then \( d = 0 \).

The rest of this paper is organized as follows. In Section 2, we recall the Lévy–Itô decomposition of Lévy processes and discuss the orthogonal transformation of Lévy processes. In Section 3, we present the proofs of our results.
2. Lévy–Itô decomposition and orthogonal transformation of Lévy processes

2.1. Lévy–Itô decomposition

**Theorem 2.1** (Lévy–Itô). Let \( X \) be a Lévy process on \( \mathbb{R}^n \) with exponent \((a, A, \mu)\). Then there exist a Brownian motion \( B_A \) on \( \mathbb{R}^n \) with covariance matrix \( A \) and an independent Poisson random measure \( N \) on \( \mathbb{R}^+ \times (\mathbb{R}^n \setminus \{0\}) \) such that, for each \( t \geq 0 \),

\[
X_t = bt + B_A(t) + \int_{|x| \geq 1} xN(t, dx) + \int_{|x| < 1} x\tilde{N}(t, dx),
\]

(2.1)

where \( b = -a, \tilde{N}(t, F) = N(t, F) - t\mu(F) \).

Define

\[
X_t^{(I)} := bt + B_A(t), \quad X_t^{(II)} := \int_{|x| \geq 1} xN(t, dx), \quad X_t^{(III)} := \int_{|x| < 1} x\tilde{N}(t, dx),
\]

\( t \geq 0 \).

Then \( X^{(I)} \), \( X^{(II)} \) and \( X^{(III)} \) are mutually independent, \( X^{(II)} \) is a compound Poisson process, and \( X^{(III)} \) is a square integrable martingale. For convenience, we write \( X_t^{(I)} = bt + \sqrt{A}B_t \), where \( B = (B_t)_{t \geq 0} \) is a standard \( n \)-dimensional Brownian motion.

2.2. Orthogonal transformation

Let \( X \) be a Lévy process on \( \mathbb{R}^n \) with exponent \((a, A, \mu)\). Since \( A \) is a symmetric nonnegative definite matrix, there exists an orthogonal matrix \( O \) such that

\[
OAO^T = \text{diag}(\lambda_1, \ldots, \lambda_n) := D,
\]

where \( \lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0 \) and \( O^T \) denotes the transpose of \( O \). We fix such an orthogonal matrix \( O \) and define \( Y_t := OX_t, t \geq 0 \). Then \( Y = (Y_t)_{t \geq 0} \) is a Lévy process on \( \mathbb{R}^n \) and \( X \) satisfies (H) if and only if \( Y \) satisfies (H). We will see that sometimes it is more convenient to work with \( Y \). By the expression of the exponent of \( X \) and simple computation, we get that the exponent of \( Y \) is \((Oa, D, \mu^{-1})\), where \( \mu^{-1}(B) = \mu(\{x \in \mathbb{R}^n : Ox \in B\}) \) for any Borel set \( B \) of \( \mathbb{R}^n \).

From now on, we denote by \( k \) the rank of \( A \). Then, the orthogonal transformation satisfies the following properties:

1. \( \mu(\mathbb{R}^n \setminus \sqrt{A}\mathbb{R}^n) < \infty \) if and only if \( \mu^{-1} \) is a finite measure on \( \mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\}) \). (When \( k = n, \mathbb{R}^n \setminus \sqrt{A}\mathbb{R}^n \) and \( \mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\}) \) are the empty set.)
2. If \( \int_{|x| < 1} |x|\mu_1(dx) < \infty \), then

\[
\int_{\{y \in \mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\}) : |y| < 1\}} |y|\mu O^{-1}(dy) = \int_{|y| < 1} |y|\mu_1 O^{-1}(dy) = \int_{|x| < 1} |x|\mu_1(dx) < \infty.
\]

Recall that \( b' = b - \int_{|x| < 1} x\mu_1(dx) \). Define \( \tilde{b} := Ob' \). Then

\[
\tilde{b} = Ob - \int_{\{y \in \mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\}) : |y| < 1\}} y\mu O^{-1}(dy).
\]
Note that $\sqrt{A} = O^T \sqrt{DO}$. Then, the equation $\sqrt{A}y = b'$ is equivalent to $\sqrt{DO}y = Ob'$. Therefore, the equation $\sqrt{A}y = b'$, $y \in \mathbb{R}^n$, has a solution if and only if the equation $\sqrt{DO}y = \bar{b}$, $y \in \mathbb{R}^n$, has a solution.

(3) Suppose that $\int_{|x| < 1} |x| \mu_1(dx) < \infty$. Then, by the Lévy–Itô decomposition (2.1), $Y$ can be expressed by

$$Y_t = Obt + \sqrt{D} \tilde{B}_t + \int_{|y| \geq 1} y\tilde{N}(t, dy) + \int_{|y| < 1} \tilde{N}(t, dy),$$

where $\tilde{B} = Ob$ is a standard Brownian motion on $\mathbb{R}^n$, $\tilde{N}$ is a Poisson random measure on $\mathbb{R}^+ \times (\mathbb{R}^n \setminus \{0\})$ with $\mu O^{-1}$ being its intensity measure, $\tilde{N}(t, F) = \tilde{N}(t, F) - t \mu O^{-1}(F)$. $\tilde{B}$ and $\tilde{N}$ are independent. We rewrite (2.2) as

$$Y_t = Y_t^{(1)} + Y_t^{(2)},$$

where

$$Y_t^{(1)} := \tilde{b}t + \sqrt{D} \tilde{B}_t + \int_{|y| \geq 1} y\tilde{N}(t, dy) + \int_{|y| < 1} \tilde{N}(t, dy),$$

$$Y_t^{(2)} := \int_{\mathbb{R}^k \times (\mathbb{R}^n \setminus \{0\})} y\tilde{N}(t, dy),$$

$Y^{(1)}$ and $Y^{(2)}$ are independent.

By (2), we can see that (S) holds if and only if $\tilde{b} \in \mathbb{R}^k \times \{0\}$. In this case, $Y^{(1)}$ can be regarded as a $k$-dimensional Lévy process on $\mathbb{R}^k \times \{0\}$, which has a non-degenerate Gaussian component. If $\mu(\mathbb{R}^n \setminus \sqrt{AR}^n) < \infty$, then $Y^{(2)}$ is a compound Poisson process.

3. Proofs of the main results

3.1. Proof of Theorem 1.1

First, we prove (ii). Since $A$ is of full rank, there exists a constant $c > 0$ such that $\langle z, Az \rangle \geq c \langle z, z \rangle$, $\forall z \in \mathbb{R}^n$. Then

$$\Re \psi(z) = \frac{1}{2} \langle z, Az \rangle + \int_{\mathbb{R}^n} (1 - \cos(z, x)) \mu(dx) \geq \frac{1}{2} \langle z, Az \rangle \geq \frac{c}{2} \langle z, z \rangle. \quad (3.1)$$

By the Cauchy–Schwarz inequality, one finds that $|\langle a, z \rangle|$ is controlled by $1 + \Re \psi(z)$. To establish the Kanda–Forst condition, we need only show that $|\Im[\int_{|x| < 1} (1 - e^{i\langle z, x \rangle} + i \langle z, x \rangle) \mu(dx)]|$ is controlled by $\langle z, z \rangle$. Note that $|t - \sin t| \leq t^2/2$ for any $t \in \mathbb{R}$. Then,

$$\left| \Im \left\{ \int_{|x| < 1} (1 - e^{i\langle z, x \rangle} + i \langle z, x \rangle) \mu(dx) \right\} \right| = \left| \int_{|x| < 1} (\langle z, x \rangle - \sin \langle z, x \rangle) \mu(dx) \right|$$

$$\leq \frac{1}{2} \int_{|x| < 1} |\langle z, x \rangle|^2 \mu(dx)$$

$$\leq \left( \frac{1}{2} \int_{|x| < 1} |x|^2 \mu(dx) \right) |z|^2.$$

Therefore (ii) holds.
Second, we prove (i). By \((3.1)\), we get
\[
\lim_{|z| \to \infty} \frac{\text{Re} \psi(z)}{\ln(1 + |z|)} = \infty. \tag{3.2}
\]
By Hartman and Wintner [10] (cf. also [16]) and \((3.2)\), we find that \(X\) has bounded continuous transition densities. Then, by (ii) and the Kanda–Forst theorem, we obtain (i).

Finally, we prove (iii). Denote by \(\tilde{\psi}\) the Lévy–Khintchine exponent of \(\tilde{X}\). Note that \(\tilde{\psi} = 2 \text{Re} \psi\). Then, for any \(\lambda \geq 1\), by (ii) we get (cf. [13, Page 163])
\[
2 \text{Re} \left( \frac{1}{\lambda + \tilde{\psi}(\xi)} \right) = \frac{1}{\lambda + \text{Re} \psi(\xi)} \geq \frac{1}{\lambda + \text{Re} \psi(\xi)} \geq \frac{1}{\lambda + \text{Re} \psi(\xi)} \left[ 1 + \left( \frac{\text{Im} \psi(\xi)}{\lambda + \text{Re} \psi(\xi)} \right)^2 \right]^{-1} \geq \frac{1}{\lambda + \text{Re} \psi(\xi)} \left[ 1 + \left( \frac{M(1 + \text{Re} \psi(\xi))}{\lambda + \text{Re} \psi(\xi)} \right)^2 \right]^{-1} \geq \frac{1}{1 + M^2} \left( \frac{1}{\lambda + \frac{1}{2} \tilde{\psi}(\xi)} \right) \geq \frac{1}{1 + M^2} \text{Re} \left( \frac{1}{\lambda + \tilde{\psi}(\xi)} \right). \tag{3.3}
\]
By (3.3), the above proved fact that \(X\) has bounded continuous transition densities, and Kanda [13, Theorem 1] (or Hawkes [11, Theorems 2.1 and 3.3]), we obtain (iii). \(\square\)

3.2. Proof of Theorem 1.2

By the discussion of Section 2.2, we know that \(X\) satisfies (H) if and only if \(Y\) satisfies (H), and (S) holds if and only if \(\tilde{b} \in \mathbb{R}^k \times \{0\}\). By the expression of the exponent of \(Y\), it is easy to see that the Kanda–Forst condition holds for \(X\) if and only if it holds for \(Y\). Hence, to prove Theorem 1.2, we may and do assume without loss of generality that \(A = \text{diag}(\lambda_1, \ldots, \lambda_n) \equiv D\), where \(\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0, \lambda_{k+1} = \lambda_{k+2} = \cdots = \lambda_n = 0\) \((k \geq 0)\), and \(X\) has the expression
\[
X_t = X_t^{(1)} + X_t^{(2)}, \quad t \geq 0,
\]
where
\[
X_t^{(1)} := b't + \sqrt{D}B_t + \int_{\{x \in \mathbb{R}^k \times \{0\}: |x| \geq 1\}} xN(t, dx) + \int_{\{x \in \mathbb{R}^k \times \{0\}: |x| < 1\}} x\tilde{N}(t, dx), \tag{3.4}
\]
\[
X_t^{(2)} := \int_{\mathbb{R}^k \times (\mathbb{R}^n - \mathbb{R}^k) \setminus \{0\}} xN(t, dx),
\]
\(b'\) is the same as in Section 1, and \(B, N\) and \(\tilde{N}\) are the same as in Section 2.1.
If \( k = 0 \), then \( X_t = b't + X_t^{(2)} \). Since \( X^{(2)} \) is a compound Poisson process, it is easy to see that (i)–(iii) are equivalent in this case. Below we assume that \( k \geq 1 \).

(ii) \( \Rightarrow \) (iii): Suppose that (S) holds, i.e., \( b' \in \mathbb{R}^k \times \{0\} \). Then \( X^{(1)} \) stays in \( \mathbb{R}^k \times \{0\} \) if it starts there. By Theorem 1.1, the Kanda–Forst condition holds for \( X^{(1)} \). Since \( X^{(2)} \) is a compound Poisson process, its Lévy–Khintchine exponent is bounded. Hence the Kanda–Forst condition holds for \( X \), i.e., (iii) holds.

(iii) \( \Rightarrow \) (ii): Suppose that the Kanda–Forst condition holds for \( X \). Since the Lévy–Khintchine exponent of \( X^{(2)} \) is bounded, we get that the Kanda–Forst condition holds for \( X^{(1)} \). Assume that \( b' \notin \mathbb{R}^k \times \{0\} \). We will reach a contradiction. Denote \( b' = (b'_1, \ldots, b'_n) \). Without loss of generality, we assume that \( b''_n \neq 0 \). Let \( \psi_1 \) be the Lévy–Khintchine exponent of \( X^{(1)} \). Then

\[
\psi_1(z) = i \langle b', z \rangle + \frac{1}{2} (z, \sqrt{D} z) + \int_{\mathbb{R}^k \times \{0\}} \left( 1 - e^{i \langle z, x \rangle} + i \langle z, x \rangle 1_{\{\|x\| < 1\}} \right) \mu(dx).
\]

It follows that if \( z = (z_1, \ldots, z_n) \) with \( z_i = 0, \ i = 1, \ldots, n-1 \) and \( z_n \neq 0 \), then \( \psi_1(z) = b''_n z_n i \) and thus the Kanda–Forst condition cannot hold for \( X^{(1)} \). Hence \( b' \notin \mathbb{R}^k \times \{0\} \) and therefore (S) holds.

(i) \( \Rightarrow \) (ii): We will show \( b' \notin \mathbb{R}^k \times \{0\} \) implies that \( X \) does not satisfy (H). We first consider the case that \( \mu_1 \neq 0 \).

Suppose that \( b' \notin \mathbb{R}^k \times \{0\} \). First, we show that \( \mathbb{R}^k \times \{0\} \) is a thin set of \( X \). Let \( T_1^{(2)} \) be the first jumping time of \( X^{(2)} \). Since \( X^{(2)} \) is a compound Poisson process, \( T_1^{(2)} \) has an exponential distribution, in particular,

\[
P(T_1^{(2)} > 0) = 1.
\]  

(3.5)

For any \( x \in \mathbb{R}^k \times \{0\} \) and any \( t > 0 \), we know that \( x + X_t^{(1)} \notin \mathbb{R}^k \times \{0\} \) since \( b' \notin \mathbb{R}^k \times \{0\} \), which together with (3.5) implies that

\[
P^x(\sigma_{\mathbb{R}^k \times \{0\}} = 0) \leq P^0 \left( \exists t \in (0, T_1^{(2)}) \text{ s.t. } x + X_t \in \mathbb{R}^k \times \{0\} \right) = P^0 \left( \exists t \in (0, T_1^{(2)}) \text{ s.t. } x + X_t^{(1)} \in \mathbb{R}^k \times \{0\} \right) = 0.
\]  

(3.6)

For any \( x \notin \mathbb{R}^k \times \{0\} \), the distance between \( x \) and the subspace \( \mathbb{R}^k \times \{0\} \) is strictly positive. By (3.5) and the right continuity of the sample path of \( X^{(1)} \), we get

\[
P^x(\sigma_{\mathbb{R}^k \times \{0\}} = 0) = P^0 \left( \exists [t_n, n \geq 1] \subset (0, T_1^{(2)}) \text{ s.t. } x + X_{t_n} \in \mathbb{R}^k \times \{0\}, t_n \downarrow 0 \right) = P^0 \left( \exists [t_n, n \geq 1] \subset (0, T_1^{(2)}) \text{ s.t. } x + X_{t_n}^{(1)} \in \mathbb{R}^k \times \{0\}, t_n \downarrow 0 \right) = 0.
\]  

(3.7)

It follows from (3.6) and (3.7) that \( \mathbb{R}^k \times \{0\} \) is a thin set and thus a semipolar set of \( X \).

Next, we show that \( \mathbb{R}^k \times \{0\} \) is not a polar set of \( X \). Note that \( P^0(T_1^{(2)} > s) > 0 \) for any \( s > 0 \). Then

\[
P^{-b's}(\sigma_{\mathbb{R}^k \times \{0\}} < \infty) = P^{-b's} \left( \exists t > 0 \text{ s.t. } X_t \in \mathbb{R}^k \times \{0\} \right) \geq P^{-b's} \left( X_s \in \mathbb{R}^k \times \{0\} \right)
\]
Hence $\mathbb{R}^k \times \{0\}$ is not a polar set of $X$. Therefore $X$ does not satisfy (H).

The case that $\mu_1 = 0$ can be proved similarly by $T_1^{(2)} \equiv \infty$.

(ii) $\Rightarrow$ (i): Suppose that (S) holds, i.e., $b' \in \mathbb{R}^k \times \{0\}$. Let $F$ be a semipolar set of $X$. We will show that $F$ is a polar set of $X$. Without loss of generality, we assume that $F$ is a nearly Borel set. For $y \in \mathbb{R}^{n-k}$, we define

$$F_y := F \cap (\mathbb{R}^k \times \{y\}).$$

Since $X^{(2)}$ is a compound Poisson process, one finds that $F_y$ is semipolar for the $k$-dimensional Lévy process $(X^{(1)}, P^{(x,y)})_{x \in \mathbb{R}^k}$ on $\mathbb{R}^k \times \{y\}$. Hence $F_y$ is polar for $(X^{(1)}, P^{(x,y)})_{x \in \mathbb{R}^k}$ by Theorem 1.1. Therefore,

$$P \left( \exists t > 0 \text{ s.t. } (x, y) + X_t^{(1)} \in F_y \right) = 0, \quad \forall x \in \mathbb{R}^k, \forall y \in \mathbb{R}^{n-k}. \quad (3.8)$$

Denote by $\eta$ the distribution of $T_1^{(2)}$ under $P$. Let $\xi$ be a random variable taking values on $\mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\})$, which has distribution $\mu_1$ and is independent of $X^{(1)}$ and $T_1^{(2)}$. Then, for any $x_0 = (u, v) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$, we obtain by (3.8) that

$$P \left( x_0 + X_t^{(1)} + \xi \in F \right) = \int_{\mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\})} \int_{(0, \infty)} P((u, v) + X_t^{(1)} + (x, y) \in F) \eta(dt) \mu_1(dx, dy)$$

$$= \int_{\mathbb{R}^k \times (\mathbb{R}^{n-k} \setminus \{0\})} \int_{(0, \infty)} P((u + x, v + y) + X_t^{(1)} \in F_{v+y}) \eta(dt) \mu_1(dx, dy)$$

$$= 0.$$

Since $x_0$ is arbitrary, by the strong Markov property of Lévy process, $F$ is a polar set of $X$. Therefore, $X$ satisfies (H).

3.3. Proof of Proposition 1.4

(i) Suppose that $X$ has transition densities. We will show that $A$ is of full rank. We adopt the setting of Section 3.2. Assume that $k < n$. Set $X = (X^1, \ldots, X^n)$ and $b' = (b'^1, \ldots, b'^n)$. Without loss of generality, we suppose $\mu_1 \neq 0$. Let $T_1^{(2)}$ be the first jumping time of $X^{(2)}$. Then $T_1^{(2)}$ has an exponential distribution and thus $P(T_1^{(2)} > 1) > 0$. It follows from (3.4) that $P(X^n_t = b'^n) > 0$. This contradicts with the assumption that $X$ has transition densities. Hence $A$ is of full rank. Therefore, the proof is completed by Theorem 1.1.

(ii) (a) $\Leftrightarrow$ (b) follows from [6, (viii)]. (d) $\Rightarrow$ (c) $\Rightarrow$ (b) is easy. (b) $\Rightarrow$ (c) follows from the Kanda–Forst condition, Silverstein [21, Theorem 3.2] and the spatial homogeneity of Lévy processes. (Note that if the Lévy process $X$ is associated with a Dirichlet form on $L^2(\mathbb{R}^n; dx)$, then the Dirichlet form is regular by Silverstein [20, Lemma 1.5].) (c) $\Rightarrow$ (d) follows from the above proof of (i).
3.4. Proof of Proposition 1.5

The main idea has been used in the proof of Kanda [14, Theorem 2]. Denote by $\tilde{\psi}$ the Lévy–Khintchine exponent of $\tilde{X}$. Then, for any $\lambda > 0$, we have

$$\text{Re} \left( \frac{1}{\lambda + \psi(\xi)} \right) \leq \frac{1}{\lambda + \text{Re} \psi(\xi)} = \frac{1}{\lambda + \frac{1}{2} \tilde{\psi}(\xi)} \leq 2 \text{Re} \left( \frac{1}{\lambda + \psi(\xi)} \right).$$

By Kanda [13, Remark 2.1] or Hawkes [11, Theorem 3.3], we find that there exists a positive constant $M$ such that for every $\lambda > 0$ and every compact $K$,

$$C^{\lambda}(K) \geq M \tilde{C}^{\lambda}(K), \quad (3.9)$$

where $C^{\lambda}(K)$ (resp. $\tilde{C}^{\lambda}(K)$) is $\lambda$-capacity of $K$ relative to $X$ (resp. $\tilde{X}$). Since $\tilde{X}$ is a symmetric Lévy process with bounded continuous transition densities, it satisfies (H), i.e., every semipolar set of $\tilde{X}$ is a polar set of $\tilde{X}$. By Kanda [14, Theorem 1], we get

$$\lim_{\lambda \uparrow \infty} \tilde{C}^{\lambda}(K) = \infty$$

for every non-polar compact set $K$ of $\tilde{X}$. (We remark that, more generally, (H) implies (3.10) under the weaker condition that $\tilde{X}$ has resolvent densities, see [7, Theorem (11.21)].) By the assumption, we find that every non-polar compact set $K$ of $X$ is a non-polar compact set of $\tilde{X}$. Thus, by (3.9) and (3.10), we get

$$\lim_{\lambda \uparrow \infty} C^{\lambda}(K) = \infty$$

for every non-polar compact set $K$ of $X$. Then, by Kanda [14, Theorem 1] again, we obtain that every semipolar set of $X$ is a polar set of $X$. □

3.5. Proof of Proposition 1.6

Suppose that $d > 0$. Then $X$ is strictly increasing, which together with the right continuity of sample paths implies that singletons are thin and thus semipolar. By Kesten [15] or Bretagnolle [3], we know that $X$ hits points with positive probability. Hence (H) cannot hold. Therefore we must have $d = 0$. □

Acknowledgments

We thank an Associate Editor and a referee for their valuable suggestions, which improved the presentation of this paper. We are grateful to the support of NNSFC (Grant No. 10801072) and NSERC (Grant No. 311945-2008).

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