A characterisation of elementary fibrations

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Abstract

We present a characterisation of elementary fibrations, also known as fibrations with equality, that generalises the one for faithful fibrations, and employ it for a comparison with the structures used in the semantics of the identity type of Martin-Löf type theory.

1 Introduction

Fibrations provide an algebraic framework that underlies the treatment of syntax and semantics of (fragments of) first and higher order logics, as well as of dependent type theories. The former approach dates back to Lawvere’s hyperdoctrines [Law69, Law70] where, in the spirit of functorial semantics, equality is specified requiring left adjoints to certain reindexing functors, giving rise to what is called an elementary fibration. On the other hand, models of dependent type theory that do not collapse the (whole) hierarchy of identity types do not treat equality as an adjunction. Rather, they often rely on weak factorisation systems or related structures.

We present a characterisation of elementary fibrations that contributes to shed light on the relation between the two approaches to equality. As it will become clear, the relation is based on a structure which, in type-theoretic terms, can be understood as a transport structure. The complete statement of our main result lists other equivalent characterisations of an elementary fibration and the proof builds on the well-known observation that existence of left adjoints to reindexing is equivalent to existence of cocartesian lifts. In the case of faithful fibrations, the characterisation reduces to the well-known characterisation of first-order equality as a reflexive and substitutive relation stable under products.

We use our characterisation to discuss the relation between elementary fibrations and fibrations coming from the homotopical semantics of identity types. Not surprisingly, the latter fibrations are rarely elementary. But they are all part of comprehension categories or related structures. On the other hand, the richer structure of algebraic weak factorisation systems, as compared to other structures to model identity types, provides us with more structured fibrations. In particular, consider the algebraic weak factorisation system on \( \mathbf{Cat} \) (and \( \mathbf{Gpd} \)) whose underlying weak factorisation system is the one of acyclic cofibrations and fibrations from the canonical, or “folk”, model structure on \( \mathbf{Cat} \) (and \( \mathbf{Gpd} \)). We shall use the characterisation to prove that the fibration of algebras for the monad on the right functor is elementary. In the case of the algebraic weak factorisation system on \( \mathbf{Gpd} \), the associated full comprehension category is the Hofmann–Striecher groupoid model from [HS98].

In Section 2 we recall notations and results from the theory of fibrations necessary for the following sections. In Section 3 we introduce the structure of transporters in a

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fibration and prove some elementary properties of these. These are put to use in Section 4 which contains the characterisation theorem. Finally Section 5 contains applications to algebraic weak factorisation systems.

2 Preliminaries

Let \( p: \mathcal{E} \to \mathcal{B} \) be a functor. An arrow \( \varphi \) in \( \mathcal{E} \) is said to be **over** an arrow \( f \) in \( \mathcal{B} \) when \( p(\varphi) = f \). For \( X \) in \( \mathcal{B} \), the fibre \( \mathcal{E}_X \) is the subcategory of \( \mathcal{E} \) of arrows over \( \text{id}_X \). In particular, an object \( A \) in \( \mathcal{E} \) is said to be **over** \( X \) when \( p(A) = X \).

Recall that an arrow \( \varphi: A \to B \) is **cartesian** if, for every \( \chi: A' \to B \) such that \( p(\chi) \) factors through \( p(\varphi) \) via an arrow \( g: X' \to X \), there is a unique \( \psi: A' \to A \) over \( g \) such that \( \varphi \psi = \chi \), as in the left-hand diagram below. And an arrow \( \theta: A \to B \) is **cocartesian** if it satisfies the dual universal property of cartesian arrows depicted in the right-hand diagram below.

![Diagram](image)

Once we fix an arrow \( f: X \to Y \) in \( \mathcal{B} \) and an object \( B \) in \( \mathcal{E}_Y \), a cartesian arrow \( \varphi: A \to B \) over \( f \) is uniquely determined up to isomorphism, i.e. if \( \varphi': A' \to B \) is cartesian over \( f \), then there is a unique iso \( \psi: A' \to A \) such that \( \varphi \psi = \varphi' \).

Clearly, every property of cartesian arrows applies dually to cocartesian arrows. So for an arrow \( f: X \to Y \) in \( \mathcal{B} \) and an object \( A \) in \( \mathcal{E}_X \), a cocartesian arrow \( \theta: A \to B \) over \( f \) is uniquely determined up to isomorphism.

In the following, we write cartesian arrows as \( \to \) and cocartesian arrows as \( \twoheadrightarrow \).

A functor \( p: \mathcal{E} \to \mathcal{B} \) is a **fibration** if, for every arrow \( f: X \to Y \) in \( \mathcal{B} \) and for every object \( A \) in \( \mathcal{E}_Y \), there is a **cartesian lift** of \( f \) into \( A \), that is, an object \( f^* A \) and a cartesian arrow \( f^* A \twoheadrightarrow A \) over \( f \). A **cleavage** for the fibration \( p \) is a choice of a cartesian lift for each arrow \( f: X \to Y \) in \( \mathcal{B} \) and object \( B \) in \( \mathcal{E}_Y \), and a **cloven fibration** is a fibration equipped with a cleavage. In a cloven fibration, for every \( f: X \to Y \) in \( \mathcal{B} \), there is a functor \( f^*: \mathcal{E}_Y \to \mathcal{E}_X \) called **reindexing** along \( f \). Henceforth we assume that fibrations can be endowed with a cleavage.

2.1 Remark. It is well-known that, for the fibration \( p: \mathcal{E} \to \mathcal{B} \), an arrow \( f: X \to Y \) in \( \mathcal{B} \) has cocartesian lifts if and only if the reindexing functor \( f^*: \mathcal{E}_Y \to \mathcal{E}_X \) has a left adjoint. The value of the left adjoint at an object \( A \) over \( X \) can be chosen as the codomain \( A' \) of a cocartesian lift \( A \to A' \) of \( f: X \to Y \) at \( A \). Conversely, the cocartesian lift is given by the composition

\[
A \xrightarrow{\eta_A} f^*(\exists_f(A)) \xrightarrow{f^*\exists_f(A)} \exists_f(A)
\]

of the unit \( \eta_A: A \to f^*(\exists_f(A)) \) of the adjunction \( \exists_f \dashv f^* \) and the cartesian lift of \( f \).

Fibrations are ubiquitous in mathematics and the list of examples is endless. Since our aim is to characterise those fibrations which encode a proof-relevant notion of equality, we choose the following classes of examples.

2.2 Examples. (a) Given a category \( \mathcal{C} \), let \( \text{Fam}(\mathcal{C}) \) be the category whose objects are set-indexed families of objects in \( \mathcal{C} \), i.e. pairs \( (I_i(A_i))_{i \in I} \) where \( I \) is a set and \( A_i \) is an
object in \( C \), for \( i \in I \), and an arrow from \((I, (A_i)_{i \in I})\) to \((I', (A'_j)_{j \in I'})\) is a pair \((f, \varphi)\) where \( f: I \to I' \) is a function and \( \varphi = (\varphi_i: A_i \to A_{f(i)})_{i \in I} \) is a family of arrows in \( C \), see [Jac99, 1.2.1].

Equivalently, an object of \( \text{Fam}(C) \) is a functor \( A: I \to C \) where \( I \) is a set seen as a discrete category, and an arrow from \( A \) to \( B: J \to C \) is a pair \((f, \varphi)\) where \( f: I \to J \) is a function and \( \varphi: A \to Bf \) is a natural transformation as in the diagram

\[
\begin{array}{ccc}
I & \xrightarrow{f} & A \\
\downarrow & & \downarrow \varphi \\
J & \xrightarrow{\cdot} & C.
\end{array}
\]

Since \( I \) is discrete, all commutative diagrams for naturality are trivial.

The functor \( \text{Pr}_1: \text{Fam}(C) \to \text{Set} \) that sends \((f, \varphi): A \to B \) to \( f: I \to J \) is a fibration. An arrow \((f, \text{id}): Bf \to B \) is cartesian into \( B \) over \( f: I \to J \). Note that \( \text{Fam}(I) \equiv \text{Set} \) and that the fibration \( \text{Pr}_1: \text{Fam}(C) \to \text{Set} \) is isomorphic to \( \text{Fam}(!): \text{Fam}(C) \to \text{Fam}(I) \), where \(!: C \to I\) is the unique functor.

(b) Let \( \mathcal{F} \) be a full subcategory of \( C^2 \), so that an arrow \( f: a \to b \) in \( \mathcal{F} \), where \( a \) and \( b \) are arrows in \( C \), is a commutative square

\[
\begin{array}{ccc}
A & \xrightarrow{f_1} & B \\
\downarrow a & & \downarrow b \\
X & \xrightarrow{f_0} & Y.
\end{array}
\]

Suppose that, for every \( f: X \to Y \) in \( C \) and \( g: B \to Y \) in \( \mathcal{F} \), there is a pullback square

\[
\begin{array}{ccc}
P & \xrightarrow{g'} & B \\
\downarrow & & \downarrow g \\
X & \xrightarrow{f} & Y
\end{array}
\]

and \( g' \in \mathcal{F} \). Then the composite

\[
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\text{pr}_1} & C^2 \\
\downarrow & & \downarrow \text{cod} \uparrow \mathcal{F} \\
C & \xrightarrow{\text{cod}} & \text{C}
\end{array}
\]

is a fibration. Given \( f: X \to Y \) in \( C \), a cartesian lift into the object \( g: B \to Y \), where \( g \in \mathcal{F} \), is a pullback square as the one above. Since \( \mathcal{F} \) is full, in the following we shall often confuse it with its collection of objects, i.e. a collection of arrows of \( C \).

If \( C \) has pullbacks we can choose \( \mathcal{F} \) as \( C^2 \) itself. In the particular case of \( C := \text{Set} \) the example in (a) come to be the same as the example in (b) since there is an equivalence

\[
\begin{array}{ccc}
\text{Fam}(\text{Set}) & \xrightarrow{\text{pr}_1} & \text{Set}^2 \\
\downarrow & & \downarrow \text{cod} \\
\text{Set} & \xrightarrow{\text{cod} \uparrow \mathcal{F}} & \text{C}
\end{array}
\]

Recall that a fibration \( p: \mathcal{E} \to \mathcal{B} \) has finite products if the base \( \mathcal{B} \) has finite products as well as each fibre \( \mathcal{E}_X \), and each reindexing functor preserves products. Equivalently, both \( \mathcal{B} \) and \( \mathcal{E} \) have finite products and \( p \) preserves them.
2.3 Notation. We do not require a functorial denotation for products; when we write 1 we refer to any terminal object in \( \mathcal{B} \) and, similarly for objects \( X \) and \( Y \) in \( \mathcal{B} \), when we write \( X \times Y \), \( \text{pr}_1: X \times Y \to X \) and \( \text{pr}_2: X \times Y \to Y \), we refer to any diagram of binary products in \( \mathcal{B} \). Universal arrows into a product induced by lists of arrows shall be denoted as \( (f_1, \ldots, f_n) \), but lists of projections \( (\text{pr}_{i_1}, \ldots, \text{pr}_{i_k}) \) will always be abbreviated as \( \text{pr}_{i_1}, \ldots, \text{pr}_{i_k} \). In particular, as an object \( X \) is a product of length 1, sometimes we find it convenient to denote the identity on \( X \) as \( \text{pr}_1 \), the diagonal \( X \to X \times X \) as \( \text{pr}_{1,1} \) and the unique \( X \to 1 \) as \( \text{pr}_0 \). As the notation is ambiguous, we shall always indicate domain and codomain of lists of projections and sometimes we may distinguish projections decorating some of them with a prime symbol.

We shall employ a similar notation for binary products and projections in a fibre \( \mathcal{E}_X \), as \( \bigtriangleup_X \), \( A \times_X B \), \( \pi_1: A \times_X B \to A \) and \( \pi_2: A \times_X B \to B \), dropping the subscript in \( \bigtriangleup_X \) and \( \times_X \) when it is clear from the context. Moreover, given objects \( A \) in \( \mathcal{E}_X \) and \( B \) in \( \mathcal{E}_Y \), write

\[
A \boxtimes B := (\text{pr}_1^* A) \times_X (\text{pr}_2^* B)
\]

for the product of \( A \) and \( B \) in the total category \( \mathcal{E} \). Given a third object \( C \) and two arrows \( \varphi_1: C \to A \) and \( \varphi_2: C \to B \), we denote the induced arrow into \( A \boxtimes B \) also as \( \langle \varphi_1, \varphi_2 \rangle \).

2.4 Examples. (a) Consider the fibration \( \text{Pr}_1: \text{Fam}(C) \to \text{Set} \) defined in Example 2.2(a) and suppose that \( C \) has finite products. Then the fibration \( \text{Pr}_1 \) has finite products. Indeed a product of the two families \( A: I \to C \) and \( B: I \to C \) in the fibre \( \text{Fam}(C)_I \) is the family \( \bigtriangleup_X A \times_X B \) where \( (A \times B)_i \) is \( A_i \times B_i \) with projections \( (\text{id}_I, \text{pr}_1) \) and \( (\text{id}_I, \text{pr}_2) \) where \( (\text{pr}_1)_i \) is the first projection \( A_i \times B_i \to A_i \). A terminal object in \( \text{Fam}(C)_I \) is given by the family \( 1: I \to C \) which is constantly a chosen terminal object of \( C \).

(b) Assume that the base \( C \) of the fibration \( \text{cod}^\circ \gamma \) defined in Example 2.2(b) has finite products. Then the fibration \( \text{cod}^\circ \gamma \) has finite products, and in the fibres these are given by pullback of arrows in \( \gamma \).

3 Transporters

This section presents a structure that will be useful in the characterisation in Theorem 4.8 providing some examples and some instrumental results.

3.1 Definition. Let \( p: \mathcal{E} \to \mathcal{B} \) be a fibration with finite products and consider an object \( X \) in \( \mathcal{B} \). A transporter on \( X \) consists of

(i) an object \( I_X \) over \( X \times X \);

(ii) an arrow \( \partial_X: \bigtriangleup_X \to I_X \) over \( \text{pr}_{1,1}: X \to X \times X \);

(iii) for every \( A \) over \( X \), an arrow \( t_A: (\text{pr}_1^* A) \times_I X \to A \) over \( \text{pr}_2: X \times X \to X \).

We refer to the object \( I_X \) as the hinge and to the arrows \( \partial_X \) and \( t_A \) respectively as the loop and the carrier for the transporter on \( X \). We say that a transporter is strict if \( t_A((\text{pr}_1^* A), \partial_X! A) = \text{id}_A \) for every \( A \). A fibration \( p: \mathcal{E} \to \mathcal{B} \) has transporters if it has a transporter on each \( X \) in \( \mathcal{B} \).

3.2 Notation. We often write \( \delta_A \) for the arrow \( \langle (\text{pr}_{1,1}^* A), \partial_X! A \rangle: A \to (\text{pr}_1^* A) \times_I X \). In Definition 3.9 we shall also need a parametric version of it. We write \( \delta^Y_A \) for the arrow \( \langle (\text{pr}_{1,2,2}^* A), \theta \rangle: A \to (\text{pr}_{1,2}^* A) \times (\text{pr}_{2,3}^* I_X) \), where \( \theta: A \to \text{pr}_{2,3}^* I_X \) is the unique arrow.
over \(\text{pr}_{1,2,2}\) obtained by cartesianness of \(\text{pr}_{2,3}^*I_X \rightarrow I_X\) as shown in the diagram

![Diagram](image)

**3.3 Examples.** (a) Consider the fibration \(\text{Pr}_1: \text{Fam}(C) \rightarrow \text{Set}\) from Example 2.2(a).

And suppose that \(C\) has a stable initial object, i.e., an initial object \(0\) such that \(0 \times A \rightarrow 0\) for all \(A\). Let \(1\) be a terminal object of \(C\), and consider the family \(I_X: X \times X \rightarrow C\) as the function that maps \((a, b)\) to \(1\) if \(a = b\) and to \(0\) otherwise. There are two natural transformations \(i: I_X \rightarrow I_X\text{pr}_{1,1}\), whose component on \(x \in X\) is the identity, and \(r_A: (\text{Ap}_1) \wedge I_X \rightarrow \text{Ap}_2\) whose component on \((x_1, x_2)\) is the identity on \(A(x_1)\) if \(x_1 = x_2\), and the unique arrow \(0 \rightarrow A(x_2)\) otherwise. The object \(I_X\) and arrows \((\text{pr}_{1,1}, i), (\text{pr}_2, r_A)\) for \(A\) over \(X\), form a strict transporter for the set \(X\).

(b) Let \(C\) be a category with finite products and suppose that \(C\) has a weak factorisation system \((\mathcal{L}, \mathcal{R})\) such that \(C\) has pullbacks of arrows in \(\mathcal{R}\) along any arrow. It follows that arrows in the right class \(\mathcal{R}\) satisfy the hypothesis of Example 2.2(b), so \(\text{cod}|\mathcal{R}: \mathcal{R} \rightarrow C\) is a fibration with products. If arrows in the left class \(\mathcal{L}\) are stable under pullback along arrows in the right class \(\mathcal{R}\), then every object \(X\) of \(C\) has a strict transporter defined as follows. A loop \(\partial_X := (r_X, \text{pr}_{1,1})\) is obtained factoring the diagonal \(\text{pr}_{1,1}: X \rightarrow X \times X\) as

![Diagram](image)

where \(r_X\) is in \(\mathcal{L}\). For an arrow \(a \in \mathcal{R}\), consider the following commutative diagram.

![Diagram](image)

Since the arrow \(r_a\) is a pullback of \(r_X\) along an arrow in \(\mathcal{R}\), it is in \(\mathcal{L}\). It follows by weak orthogonality that there is \(t_a: A \times_X PX \rightarrow A\) filling in the previous diagram.

![Diagram](image)
A carrier at \( a \) is then \((pr_2, t_a)\). Instances of this situation can be found in any Quillen model category where acyclic cofibrations are stable under pullback along fibrations, but also in Shulman’s type-theoretic fibration categories \cite{Shulman:2015} and Joyal’s tribes \cite{Joyal:2017}.

(c) Let \((C, \mathcal{W}, \mathcal{F})\) be a path category. It consists of two full subcategories \( \mathcal{W} \) and \( \mathcal{F} \) of \( C^2 \) closed under isomorphisms and satisfying some additional conditions, see \cite{BM18}. In particular, the category \( \mathcal{F} \) satisfies the hypothesis of Example 3.3(b), so \( \text{cod}_{\mathcal{F}}: \mathcal{F} \rightarrow C \) is a fibration with products. In the notation of \cite{BM18}, the arrow \((s, t): PX \rightarrow X \times X\) together with the arrow \( r: X \rightarrow PX \) and, for every \( f \in \mathcal{F} \), a transport structure in the sense of \cite[Def. 2.24]{BM18}, provides a (not necessarily strict) transporter for \( C \).

In the following, when dealing with a path category, we shall always try to stick to the notation in \cite{BM18}; however we prefer to denote the arrow \( r: X \rightarrow PX \) as \( r_X \).

3.4 Remark. Example 3.3(c) fits only momentarily in the framework that we are developing. This will become clear after Theorem 4.8, as our aim is to characterise elementary fibrations. This suggests the relevance of a weaker notion than elementary fibration, which we shall consider in future work.

3.5 Definition. Let \( p: \mathcal{E} \rightarrow \mathcal{B} \) be a fibration with finite products. We say that \( p \) has **productive transporters** if

(i) for every object \( X \) in \( \mathcal{B} \), there is a transporter for \( X \);

(ii) for every \( X \) and \( Y \) in \( \mathcal{B} \), there is a vertical arrow \( \chi_{X,Y}: I_X \otimes I_Y \rightarrow I_{X \times Y} \);

and \( p \) has **strict productive transporters** if it has productive transporters, transporters are strict, and for every \( X \) and \( Y \) in \( \mathcal{B} \) the following diagram commutes

\[
\begin{array}{ccc}
\partial_X \otimes \partial_Y & \xrightarrow{\chi_{X,Y}} & \partial_{X \times Y} \\
I_X \otimes I_Y & \downarrow & I_{X \times Y} \\
\end{array}
\]

3.6 Remark. Strict productive transporters give a commutative diagram

\[
\begin{array}{ccc}
\partial_X \otimes \partial_Y & \xrightarrow{\chi_{X,Y}} & \partial_{X \times Y} \\
I_X \otimes I_Y & \downarrow \omega_{X,Y} & I_{X \times Y} \\
\end{array}
\]

for a canonical arrow \( \omega_{X,Y}: I_{X \times Y} \rightarrow I_X \otimes I_Y \). The reader will see this in the proof of (iv)\(\Rightarrow\)(v) in Theorem 4.8.

3.7 Examples. (a) The fibration \( \text{Pr}_1: \text{Fam}(C) \rightarrow \text{Set} \) from Example 3.3(a) has strict productive transporters when \( C \) has a stable initial object as in Example 3.3(a). Indeed, the components on \((x_1, y_1, x_2, y_2)\) of \( I_X \otimes I_Y \) and \( I_{X \times Y} \) are both initial or both terminal. Hence we may take the canonical iso as the component of \( \chi_{X,Y} \) on \((x_1, y_1, x_2, y_2)\).

(b) Consider the category \( C \) equipped with a w.f.s. \((\mathcal{L}, \mathcal{R})\) satisfying the same assumptions in Example 3.3(b), so that the fibration \( \text{cod}_{\mathcal{R}}: \mathcal{R} \rightarrow C \) has strict transporters. Suppose further that the class \( \mathcal{L} \) is stable under products in the sense that, for every object \( X \) in \( C \), the functor \((-) \times X: C \rightarrow C \) maps \( \mathcal{L} \) into \( \mathcal{L} \). Then \( \text{cod}_{\mathcal{R}} \) has strict productive transporters. Indeed in this case \( \partial_X \otimes \partial_Y = (pr_{1,2,1,2}, r_X \times r_Y) \) and the arrow \( r_X \times r_Y \) is in \( \mathcal{L} \) as it factors as shown below.

\[
\begin{array}{ccc}
X \times Y & \xrightarrow{r_X \times Y} & PX \times Y \\
& \xrightarrow{PX \times r_Y} & PX \times PY \\
\end{array}
\]
And the rest of the argument is similar to the one in (c) below.

The assumption that \( \mathcal{L} \) is stable under products is satisfied in particular when terminal arrows are in \( \mathcal{R} \). Hence type-theoretic fibration categories and tribes provide examples of fibrations with strict productive transporters. It is also satisfied when \( \mathcal{L} \) consists of the monos in \( \mathcal{C} \). It follows that the weak factorisation system given by the acyclic cofibrations and fibrations of any right-proper Čisinski model structure \([\text{Cis}02, \text{Cis}06]\) yields a fibration with strict productive transporters.

(c) In a path category \((\mathcal{C}, \mathcal{W}, \mathcal{F})\) as in Example 3.2.3 one can prove that the arrows in \( \mathcal{W} \) are stable under pullback along arrows in \( \mathcal{F} \), see \([\text{BM18}, \text{Prop. 2.7}]\). It follows that the fibration \( \text{cod}|_C \) has productive transporters. Note that these are not necessarily strict as the lower filler need not make the upper triangle commute.

3.8 Remark. For \( X, Y \) in \( \mathcal{B} \), we can rewrite \( \partial_X \otimes \partial_Y : \top_X \to I_X \otimes I_Y \) as the composite \( \langle \iota, \partial_Y \rangle \partial_X \) as shown in the diagram below.

3.9 Definition. Let \( p : \mathcal{E} \to \mathcal{B} \) be a fibration with transporters. Let \( X, Y \) be in \( \mathcal{B} \) and let \( A \) be over \( Y \times X \) in \( \mathcal{E} \). A **parametrised carrier** at \( A \) for the transporter on \( X \) is an arrow

\[ t_A^Y : (\text{pr}_{1,2} A) \wedge (\text{pr}_{2,3} I_X) \to A \]

over \( \text{pr}_{1,3} Y \times X \times X \to Y \times X \). We say that the parametrised carrier \( t_A^Y \) is **strict** if \( t_A^Y \delta_X^Y = \text{id}_A \), where \( \delta_X^Y : A \to (\text{pr}_{1,2} A) \wedge (\text{pr}_{2,3} I_X) \) is the arrow defined in Notation 4.3.

3.10 Proposition. Let \( p : \mathcal{E} \to \mathcal{B} \) be a fibration.
(i) If \( p \) has productive transporters, then for every \( X,Y \) in \( \mathcal{B} \), there is a parametrised carrier at every \( A \) over \( Y \times X \).

(ii) If the productive transporters are strict, then so are the parametrised carriers.

**Proof.** (i) The arrow \( t^Y_A \) can be obtained as the composite

\[
\begin{array}{ccc}
(pr_{1,2}^* A) \land (pr_{2,3}^* I_X) & \xrightarrow{\alpha \land (\partial'_Y \!, \iota)} & (pr_{1,2}^* A) \land (I_Y \otimes I_X) \\
\downarrow \chi_{X,Y} & & \downarrow \chi_{X,Y} \\
(pr_{1,2}^* A) \land I_{Y \times X} & \xrightarrow{t_A} & A
\end{array}
\]

where \( \alpha : pr_{1,2}^* A \rightarrow pr_{1,2}^* A \) and \( \iota : pr_{2,3}^* (I_X) \rightarrow pr_{2,4}^* (I_X) \) are cartesian over \( pr_{1,2,1,3} \) and \( \partial'_Y : pr_{1,2,1,3} \rightarrow pr_{1,3}^* I_Y \) is the unique arrow over \( pr_{1,2,1,3} \) obtained by cartesianness from the composite

\[
pr_{1,2,1,3} \xrightarrow{pr_{1,2}^* \gamma_Y} Y \times X \times X \xrightarrow{pr_{1,3}^* I_Y} pr_{1,2}^* I_Y.
\]

(ii) Let \( \partial'_X : pr_{2,3}^* I_X \rightarrow pr_{2,3}^* I_X \) be the unique arrow over \( pr_{2,3} \) obtained by cartesianness from \( \partial_X (pr_{2,3}^* I_X) \). By Remark [3.8] it is \( \langle \partial'_Y \!, \iota \rangle \partial'_X = \partial_Y \otimes \partial_X \). It follows that \( t^Y_A \delta_Y = t_A \delta_A = id_A \). \( \square \)

### 4 Elementary fibrations

Recall from [Jac99, 3.4.1 (ii)] the following definition.

**4.1 Definition.** A fibration with products \( p : E \rightarrow B \) is **elementary** if, for every pair of objects \( Y \) and \( X \) in \( \mathcal{B} \), reindexing along the parametrised diagonal \( pr_{1,2} : Y \times X \rightarrow Y \times X \times X \) has a left adjoint \( \exists_{Y,X} : E_{Y \times X \times X} \rightarrow E_{Y \times X} \), and these satisfy:

**Frobenius Reciprocity:** for every \( A \) over \( Y \times X \) and \( B \) over \( Y \times X \times X \), the canonical arrow

\[
\exists_{Y,X} (pr_{1,2}^* B \land A) \rightarrow B \land \exists_{Y,X} A
\]

is iso, and

**Beck-Chevalley Condition:** for every pullback square

\[
\begin{array}{ccc}
Y \times X & \xrightarrow{f \times X} & Z \times X \\
pr_{1,2} \downarrow & & \downarrow pr_{1,2} \\
Y \times X \times X & \xrightarrow{f \times X} & Z \times X \times X
\end{array}
\]

and every \( A \) over \( Z \times X \), the canonical arrow

\[
\exists_{Y,X} (f \times X)^* A \rightarrow (f \times X \times X)^* \exists_{Z,X} A
\]

is iso.
Since the collection of parametrised diagonals will be often referred to in the following, it is useful to introduce a notation for it.

**4.2 Notation.** We write $\Delta$ for the class of arrows of the form $\text{pr}_{1,2,2}: Y \times X \rightarrow Y \times X \times X$ in $\mathcal{B}$.

**4.3 Examples.** (a) The fibration $\text{Pr}_1: \text{Fam}(\mathcal{C}) \rightarrow \text{Set}$ of the Example 2.2 is elementary when $\mathcal{C}$ has finite products and a stable initial object. Indeed, let $I_X: X \times X \rightarrow \mathcal{C}$ be the family defined in Example 3.3. Then, for every $A: Y \times X \rightarrow \mathcal{C}$, the family

$$Y \times X \times X \xrightarrow{3_{Y,X}(A)} \mathcal{C}$$

$$(x,a,b) \xrightarrow{A(x,a) \times I_X(a,b)}$$

determines the required left adjoint, see [Jac99, 3.4.3 (iii)].

(b) When the right class of a weak factorisation system $(\mathcal{L}, \mathcal{R})$ on a category $\mathcal{C}$ with finite limits consists of all arrows in $\mathcal{C}$, the associated fibration $\text{cod}|_\mathcal{R} = \text{cod}$ is elementary.

Let $p: \mathcal{E} \rightarrow \mathcal{B}$ be a functor. For a class of arrows $\Theta$ in $\mathcal{B}$, say that an arrow $\varphi$ in $\mathcal{E}$ is over $\Theta$ if $p(\varphi) \in \Theta$. Recall from [Str20] that an arrow $\varphi: A \rightarrow B$ is locally epic with respect to $p$ if, for every pair $\psi, \psi': B \rightarrow B'$ such that $p(\psi) = p(\psi')$, whenever $\psi \varphi = \psi' \varphi$ it is already $\psi = \psi'$.

**4.4 Remark.** (i) When $p$ is a fibration, $\varphi$ is locally epic with respect to $p$ if and only if $\psi \varphi = \psi' \varphi$ implies $\psi = \psi'$ just for vertical arrows $\psi$ and $\psi'$.

(ii) Every cocartesian arrow is locally epic with respect to $p$.

(iii) An arrow $\varphi: A \rightarrow B$ that factors as a cocartesian arrow followed by a vertical $\psi$ is locally epic with respect to $p$ if and only if $\psi$ is locally epic with respect to $p$.

Moreover, if $p$ is a fibration, this happens if and only if $\varphi$ is epic in the fibre $\mathcal{E}_p(B)$.

Assume from now on that $p: \mathcal{E} \rightarrow \mathcal{B}$ is a fibration. We need to introduce a few definitions that will be instrumental in formulating the main Theorem 4.8.

It is well-known that, whenever there is a commuting square in $\mathcal{E}$ with two opposite arrows cartesian and sitting over a pullback square in $\mathcal{B}$

\[
\begin{array}{ccc}
A' & \xrightarrow{\varphi'} & A \\
\downarrow^{\varphi} & & \downarrow^{p} \\
B' & \xrightarrow{\psi} & B \\
\end{array}
\]

then the left-hand square in (1) is a pullback too.

We say that a class $\Phi$ of arrows in $\mathcal{E}$ is **product-stable** when, in every diagram (1) where the right-hand pullback is of the form

\[
\begin{array}{ccc}
U \times Y & \xrightarrow{g \times Y} & V \times Y \\
\downarrow^{U \times f} & & \downarrow^{V \times f} \\
U \times X & \xrightarrow{g \times X} & V \times X \\
\end{array}
\]

and $\varphi$ is in $\Phi$, also $\varphi'$ is in $\Phi$. In such a situation, we may say that $\varphi'$ is a **parametrised reindexing of $\varphi$ along $g$**.
4.5 Lemma. Suppose $\mathcal{B}$ has binary products and $p: \mathcal{E} \to \mathcal{B}$ is a fibration with left adjoints to reindexing along arrows in $\Delta$. Let $\Phi$ be the class of cocartesian lifts of arrows in $\Delta$, which exist thanks to Remark 2.1. Then the following are equivalent:

(i) The class $\Phi$ is product-stable.

(ii) The left adjoints satisfy the Beck-Chevalley Condition.

Proof. $(i) \implies (ii)$ Given $f: Y \to Z$ and an object $A$ over $Z \times X$, in the commutative diagram

$$(f \times X)^* A \longrightarrow A$$

the dotted arrow is cocartesian by $(i)$. The statement follows by Remark 2.1.

$(ii) \implies (i)$ Consider a diagram like in $(1)$ for $f$ in $\Delta$ and $\varphi$ cocartesian over it:

So, by Remark 2.1 it is $B \cong \mathcal{Y}_{U,X}(A)$. Hence $B' \cong \mathcal{Y}_{U,X}(A')$ by $(ii)$ which yields that also $\varphi'$ is cocartesian, again by Remark 2.1.

We say that $\Phi$ is **pairable** if, for every $\varphi: A \to B$ and every cartesian arrow $\psi: C' \to C$ over $p(\varphi)$, the arrow $\varphi \wedge \psi := (\varphi \pi_1, \psi \pi_2): A \wedge C' \to B \wedge C$ is in $\Phi$ whenever $\varphi$ is.

4.6 Lemma. Let $p: \mathcal{E} \to \mathcal{B}$ be a fibration with finite products and suppose that it has left adjoints to reindexing along arrows in $\Delta$. Let $\Phi$ be the class of cocartesian lifts of arrows in $\Delta$, which exist thanks to Remark 2.1. Then the following are equivalent:

(i) The class $\Phi$ is pairable.

(ii) The left adjoints satisfy the Frobenius Reciprocity.

Proof. For a parametrised diagonal $\text{pr}_{1,2,2}: Y \times X \to Y \times X \times X$ and an object $A$ over $Y \times X$, consider a commutative diagram

$A \longleftarrow A \wedge \text{pr}_{1,2,2}^*(B) \longrightarrow \text{pr}_{1,2,2}^*(B)$

By Remark 2.1 the middle arrow is cocartesian if and only if $\mathcal{Y}_{Y,X}(A) \wedge B \cong \mathcal{Y}_{Y,X}(A \wedge \text{pr}_{1,2,2}^*(B))$. Hence the statement follows.
Let \( r: Y \to X \) and \( s: X \to Y \) be a retraction pair in \( B \) with \( rs = \text{id}_X \) and let \( \rho: C' \to C \) and \( \sigma: C \to C' \) be cartesian over \( r \) and \( s \) respectively so that \( \rho \sigma = \text{id}_C \). Given \( \varphi: A \to B \) over \( s \), we say that \( \sigma \land \varphi: C \land A \to C' \land B \) is a **split pairing of \( \varphi \) with \( C \)**.

Given a retraction pair \( r: Y \to X \) and \( s: X \to Y \) in \( B \), and an arrow \( \varphi: A \to B \) over \( V \times s: V \times X \to V \times Y \), let \( \Xi^s(\varphi) \) be the class of all arrows obtained from \( \varphi \) by first a parametrised reindexing and then a split pairing. More specifically, the class \( \Xi^s(\varphi) \) consists of those arrows \( \psi \) fitting in a commutative diagram

![Diagram](image)

**4.7 Remark.** Note that the class \( \Xi^s(\varphi) \) is closed under isomorphism.

We are at last in a position to state the main result of the paper.

**4.8 Theorem.** Let \( p: E \to B \) be a fibration with products. The following are equivalent:

(i) The fibration \( p: E \to B \) is elementary.

(ii) a. Every arrow in \( \Delta \) has all cocartesian lifts.

b. The cocartesian arrows over \( \Delta \) are product-stable and pairable.

(iii) a. For every object \( X \) in \( B \) there is an object \( I_X \) over \( X \times X \) and a cocartesian arrow \( \partial_X: \top_X \to I_X \) over \( \text{pr}_{1,1}: X \to X \times X \).

b. The cocartesian arrows over \( \Delta \) are product-stable and pairable.

(iv) a. The fibration \( p \) has strict productive transporters.

b. Let \( \partial_X: \top_X \to I_X \) denote the loop for the transporter on \( X \). Every arrow in \( \Xi^{\text{pr}_{1,1}}(\partial_X) \) is locally epic with respect to \( p \) over \( \Delta \).

(v) a. For every \( X \) in \( B \) there are an object \( I_X \) over \( X \times X \) and an arrow \( \partial_X: \top_X \to I_X \) over \( \text{pr}_{1,1}: X \to X \times X \).

b. For every object \( X \) in \( B \), the arrows in \( \Xi^{\text{pr}_{1,1}}(\partial_X) \) are cocartesian over \( \Delta \).
(vi) For every $X$ in $\mathcal{B}$ there is an object $I_X$ over $X \times X$ such that, for every $Y, X \in \mathcal{B}$ and every $A$ over $Y \times X$, the assignment

$$A \mapsto (\text{pr}_{1,2}^* A) \land (\text{pr}_{2,3}^* I_{X \times X})$$

gives rise to a left adjoint to the reindexing functor $\text{pr}_{1,2,2}^*: \mathcal{E}_{Y \times X \times X} \to \mathcal{E}_{Y \times X}$. 

Proof. (i) By Remark 2.1, the equivalence follows from Lemma 4.5 and Lemma 4.6.

(ii) The object $I_X$ over $X \times X$ and the arrow $\delta_X$ are obtained taking a cocartesian lift of $\text{pr}_{1,1}^*: X \to X \times X$ from $\tau_X$.

(iii) We begin constructing a strict transporter on an object $X$. To this aim, it is enough to construct a carrier $t_A$ for $A$ over $X$.

To prove condition (iv).b, recall that arrows in $\mathcal{E}_{\text{pr}_{1,1}^*} (\partial_X)$ are cocartesian in particular locally epic with respect to $\partial_X$. Hence, each is cocartesian, thus so is $\partial_X \cong \partial_Y$. Its universal property applied to $\partial_X \times Y$ then yields the required $\chi_{X,Y}$.

To prove condition (iv).b recall that arrows in $\mathcal{E}_{\text{pr}_{1,1}^*} (\partial_X)$ are obtained from $\partial_X$ first by parametrised reindexing and then with a split pairing, as in diagram (2). Since $\partial_X$ is a cocartesian arrow over $\Delta$ and these are product-stable and pairable, arrows in $\mathcal{E}_{\text{pr}_{1,1}^*} (\partial_X)$ are cocartesian, in particular locally epic with respect to $p$.

(iv) There is only condition (v) to prove; we need to show that, given $X$ in $\mathcal{B}$, arrows in $\mathcal{E}_{\text{pr}_{1,1}^*} (\partial_X)$ are cocartesian. These are obtained by parametrised reindexing of $\partial_X$ along any projection $\text{pr}_{2,3}: Y \times X \times X \to X \times X$, and then by split pairing. Hence, for every $A$ over $Y \times X$, there is exactly one arrow $A \mapsto (\text{pr}_{1,2,2}^* A) \land (\text{pr}_{2,3}^* I_X)$ in $\mathcal{E}_{\text{pr}_{1,1}^*} (\partial_X)$ which is $\delta_A := (\text{pr}_{1,2,2}^* A, \delta)$ introduced in Notation 4.2. Let $\varphi: A \to B$ be an arrow over $\text{pr}_{1,2,2}: Y \times X \to Y \times X \times X$ and consider the following diagram.
where $\text{pr}_{1,2} = \text{pr}_{1,2} \circ \text{pr}_{1,2} : Y \times X \times X \rightarrow Y \times X \rightarrow Y \times X \times X$, the arrow $t_{Y \times X}$ is a strict parametrised carrier at $B$ obtained by Proposition 3.10 and $\beta$ and $\gamma$ are cartesian. Commutativity of the upper square with vertex $B$ follows from $t_{B \times Y} \delta_{B \times Y} = \text{id}_B$ and commutativity of the diagram below.

Therefore diagram (3) commutes. Since $\delta_{X}^\prime$ is locally epic with respect to $p$, it follows that there is exactly one vertical arrow $\varphi : (\text{pr}_{1,2} \circ A) \wedge (\text{pr}_{2,3} \circ I_X) \rightarrow B$ such that $\varphi \delta_{X}^\prime = \varphi$. Hence $\delta_{X}^\prime$ is cocartesian.

This is just an instance of the equivalence in Remark 2.1. It is straightforward to verify that the left adjoints specified in (vi) satisfy the Beck-Chevalley condition and Frobenius reciprocity. For the latter, a useful remark is that the left adjoint to reindexing along $\text{pr}_{1,2,1,2} : X \times X \rightarrow X \times X \times X \times X$

\[ A \rightarrow \text{pr}_{1,2}^\ast A \wedge I_{X \times X} \]

is isomorphic to $A \rightarrow \text{pr}_{1,2}^\ast A \wedge (I_X \otimes I_X)$ as it follows from the fact that the diagram

\[
\begin{array}{c}
X \times X \\
\text{pr}_{1,2,2} \\
\downarrow \\
X \times X \\
\text{pr}_{1,2,1,2}
\end{array}
\quad \sim 
\begin{array}{c}
X \times X \\
\text{pr}_{2,3} \\
\downarrow \\
X \times X \\
\text{pr}_{2,3,1}
\end{array}
\quad \sim 
\begin{array}{c}
X \times X \\
\text{pr}_{1,2,3,3} \\
\downarrow \\
X \times X \\
\text{pr}_{3,1,4,2}
\end{array}
\]

commutes.

4.9 Remark. In an elementary fibration $p$, the canonical arrow $\omega_{X,Y} : I_{X \times Y} \rightarrow I_X \otimes I_Y$ of Remark 3.6 is the inverse of $\chi_{X \times Y}$. 

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4.10 Remark. Since faithful fibrations are equivalent to indexed posets, the equivalence between condition [i] and condition [iv] in Theorem 4.8 gives Proposition 2.4 of [EPR20].

4.11 Proposition. If \( p: E \to B \) is an elementary fibration, \( A \) is a category with finite products, and \( F: A \to B \) is a functor which preserves finite products, then the fibration \( F^*p: F^*E \to A \) is also elementary.

Proof. Since \( F \) preserves finite products, the fibration \( F^*p \) has finite products. To see that \( F^*p \) is elementary, we apply Theorem 4.8(iv). For \( V \) an object in \( C \), a transporter on \( F(V) \) for the fibration \( p \) is also a transporter on \( V \) for the fibration \( F^*p \) since \( \langle F(pr_1), F(pr_2) \rangle: F(V \times V) \to FV \times FV \). Condition [iv] for \( F^*p \) follows directly from the same condition for \( p \) since the arrows in \( \Xi_{pr_1,1}^p(\partial_Y) \) are the arrows in \( \Xi_{F(pr_1)}^{F^*p}(\partial_{FV}) \). \( \square \)

5 Applications

Consider the fibration \( \text{cod}|_{\mathcal{R}}: \mathcal{R} \to \mathcal{C} \) associated to the subcategory \( \mathcal{R} \) from a weak factorisation system \((L, \mathcal{R})\) in Example 2.2(b). As we saw in Examples 3.3(b) and 3.7(b), the fibration \( \text{cod}|_{\mathcal{R}} \) has strict productive transporters when the base category \( C \) has finite products, \( L \) is closed under products and closed under pullbacks along arrows in \( \mathcal{R} \). In this case, for every \( (f_0, f_1) \in \Xi_{pr_1,1}^p(\partial_X) \), the arrow \( f_1 \) is in \( L \). Indeed, \( r_X \) is in \( L \) by construction and, for every \( \delta_A^X = (pr_{1,2}, (id_A, r_X pr_2 a)) \), the arrow \( (id_A, r_X pr_2 a) \) is a pullback along an arrow in \( \mathcal{R} \) of a product of \( r_X \) as in the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\langle id_A, r_X pr_2 a \rangle} & A \times_X PX \\
\downarrow{a} & & \downarrow{a} \\
Y \times X & \xrightarrow{id_{Y \times X}} & Y \times PX \\
\end{array}
\]

where the right-hand square is a pullback.

5.1 Lemma. Let \( \mathcal{F} \) be a full subcategory of \( \mathcal{C}^2 \) closed under pullbacks. Given an arrow in \( \mathcal{F} \)

\[
\begin{array}{ccc}
A & \xrightarrow{f_1} & B \\
\downarrow{a} & & \downarrow{b} \\
X & \xrightarrow{f_0} & Y, \\
\end{array}
\]

the following are equivalent:

(i) The arrow \((f_0, f_1)\) is locally epic with respect to \( \text{cod}|_{\mathcal{F}} \).

(ii) Every left lifting problem for \( f_1 \) against arrows in \( \mathcal{F} \) has at most one solution.

Proof. (i)⇒(ii) It is enough to show that a lifting problem of the form

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
f_1 \downarrow & & \downarrow{c} \\
B & \xrightarrow{id_B} & B \\
\end{array}
\]
for \( c \in \mathcal{F} \) has at most one solution. Let then \( g, g': B \to C \) be two diagonal fillers. They fit in the diagram below which commutes except for the two parallel arrows.

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow a & & \downarrow b_c \\
X & \xrightarrow{b} & Y \\
\end{array}
\quad \begin{array}{ccc}
B & \xrightarrow{g} & C \\
\downarrow & & \downarrow \\
Y & \xrightarrow{id_Y} & Y \\
\end{array}
\quad \begin{array}{ccc}
A & \xrightarrow{f_1} & C \\
\downarrow f_0 & & \downarrow \\
X & \xrightarrow{id_X} & X \\
\end{array}
\]

But, since by hypothesis \( (f_0, f_1) \) is locally epic with respect to \( p \), also \( g = g' \).

(ii)\( \Rightarrow \) (i) Let \( (k, g), (k, g'): b \to c \) be such that \( (f_0, f_1)(k, g) = (f_0, f_1)(k, g') \). Then the commutative diagram

\[
\begin{array}{ccc}
A & \xrightarrow{h} & C \\
\downarrow f_1 & & \downarrow c \\
B & \xrightarrow{g} & Z \\
\downarrow & & \downarrow \\
X & \xrightarrow{kb} & Y \\
\end{array}
\]

exhibits \( g \) and \( g' \) as solutions to a lifting problem for \( f_1 \). It follows that \( g = g' \). \( \square \)

Combining Lemma 5.1 with Theorem 4.8 we immediately obtain a sufficient condition for the fibration \( \text{cod}_{\mathcal{R}} \) to be elementary.

5.2 Corollary. Let \((\mathcal{L}, \mathcal{R})\) be a weak factorisation system on a category \( \mathcal{C} \) with finite products such that \( \mathcal{L} \) is closed under products and under pullbacks along arrows in \( \mathcal{R} \). If \((\mathcal{L}, \mathcal{R})\) is a strong factorisation system i.e. diagonal fillers are unique, then the fibration \( \text{cod}_{\mathcal{R}} \) is elementary.

5.3 Remark. Lemma 5.1 provides also a way to find examples of non-elementary fibrations among those of the form \( \text{cod}_{\mathcal{R}} \) for a weak factorisation system \((\mathcal{L}, \mathcal{R})\) as in Corollary 5.2 and Example 3.7(b). Indeed, a cocartesian arrow \((\text{pr}_{1,1}, r): \text{id}_X \to I_X \) would give rise to a factorisation of the diagonal \( \text{pr}_{1,1}: X \to X \times X \) as the arrow \( r \) followed by \( I_X \in \mathcal{R} \). It follows that \( r \) is a retract of \( r_X \), and thus it is in \( \mathcal{L} \). As \((\text{pr}_{1,1}, r)\) is in particular locally epic with respect to \( \text{cod}_{\mathcal{R}} \), the arrow \( r \) would have unique solutions to lifting problems. Hence, as soon as there is an object \( X \) such that no \( L \)-map in a factorisation of \( \text{pr}_{1,1}: X \to X \times X \) has unique solutions to lifting problems, \( \text{cod}_{\mathcal{R}} \) cannot be elementary.

The situation is however different if, instead of looking at the fibration associated to the full subcategory \( \mathcal{R} \) of \( \mathcal{C}^2 \), one looks at the fibration associated to a non-full subcategory of \( \mathcal{R} \). We present one such situation.

Recall from \([GT06]\), see also \([BGT16]\), that an algebraic weak factorisation system \((L, M, R)\) on a category \( \mathcal{C} \) (an awfs, for short) consists of functors \( M: \mathcal{C}^2 \to \mathcal{C} \), \( R: \mathcal{C}^2 \to \mathcal{C}^2 \), and \( L: \mathcal{C}^2 \to \mathcal{C}^2 \) giving rise to a functorial factorisation

\[
\begin{array}{ccc}
A & \xrightarrow{Lf} & Mf & \xrightarrow{Rf} & B, \\
\downarrow f & & & & \downarrow f \\
& & & & \\
\end{array}
\]

and suitable monad and comonad structures on \( R \) and \( L \) respectively, together with a distributivity law between them. Let \( R\text{-Alg} \) be the category of algebras for the monad on \( R \) and let \( R\text{-Map} \) be the category of algebras for the pointed endofunctor on \( R \).
Similarly, let \( L\text{-Coalg} \) and \( L\text{-Map} \) be the categories of coalgebras for the comonad on \( L \) and the pointed endofunctor on \( L \), respectively. When \( C \) has finite limits, the two forgetful functors

\[
\begin{array}{ccc}
R\text{-Alg} & \xrightarrow{U'} & R\text{-Map} \\
S & \downarrow N & \downarrow \text{cod} \\
& C \end{array}
\]

are homomorphisms of fibrations with finite products.

The category \( Gpd \) of groupoids admits an awfs \((L, M, R)\) such that the fibrations of \( R\text{-Alg} \) and \( R\text{-Map} \) over \( Gpd \) are equivalent to the fibrations of split cloven isofibrations and of normal cloven isofibrations, respectively, with arrows the commutative squares that preserve the cleavage strictly, see \([GL19, \text{Section 3}]\). Similar calculations show that also the category of small categories \( \text{Cat} \) admits an awfs \((L, M, R)\) such that the fibrations of \( R\text{-Alg} \) and \( R\text{-Map} \) over \( \text{Cat} \) are equivalent to the categories of split cloven isofibrations and of normal cloven isofibrations, respectively, each with arrows the homomorphisms of fibrations that preserve the cleavage on the nose.

The underlying weak factorisation systems \((L, R)\) and \((L', R')\) of the two awfs are the (acyclic cofibrations, fibrations) weak factorisation systems of the canonical, aka “folk”, Quillen model structures on \( \text{Cat} \) and \( Gpd \), respectively. As discussed in Remark 5.3, the associated fibrations \( \text{cod}|_R \) and \( \text{cod}|_{R'} \) are not elementary. On the other hand, the fibration \( S: R\text{-Alg} \rightarrow \text{Cat} \) is elementary as we shall see promptly. From this it will also follow that the fibration \( R\text{-Alg} \rightarrow Gpd \) is elementary, and we shall be able to comment about the other two obtained with the pointed endofunctor.

For a functor \( F: A \rightarrow B \) between small categories, \( MF \) is the category whose objects are pairs \((A, x: B \xrightarrow{\sim} FA)\) where \( A \) is an object in \( A \) and \( x \) is an iso in \( B \), and whose arrows \((b, a): (A, x: B \xrightarrow{\sim} FA) \rightarrow (A', x': B' \xrightarrow{\sim} FA')\) are pairs of an arrow \( b: B \rightarrow B' \) in \( B \) and an arrow \( a: A \rightarrow A' \) in \( A \) such that the square

\[
\begin{array}{ccc}
B & \xrightarrow{x} & FA \\
\downarrow b & & \downarrow F a \\
B' & \xrightarrow{x'} & FA'
\end{array}
\]

commutes. Denote \( i_B: \text{Iso}(B) \rightarrow B^2 \) the embedding of the full subcategory \( \text{Iso}(B) \) of \( B^2 \) on the isos. Write

\[
\begin{array}{ccc}
\text{Iso}(B) & \xrightarrow{i_B} & B^2 \\
\downarrow c_B & & \downarrow \text{id} \\
\downarrow d_B & & \downarrow \text{cod} \\
\downarrow r_B & & \downarrow \text{dom} \\
B & \xrightarrow{\text{id}} & B
\end{array}
\]

the restrictions to \( \text{Iso}(B) \) of the three structural functors. Note that \( MF \) appears in the pullback of categories and functors
and the functorial factorisation is obtained directly from it:

\[
\begin{array}{ccc}
A & \xrightarrow{F} & B \\
\downarrow{LF} & & \downarrow{F} \\
Id_A & \xrightarrow{MF} & Iso(B) \\
\downarrow{c_B} & & \downarrow{c_B} \\
A & \xrightarrow{F} & B
\end{array}
\]

The factorisation extends to an awfs on \(\text{Cat}\): for the comonad the component at \(F\) of the counit is

\[
\Delta_F(A, x: B \twoheadrightarrow FA) = (A, (x, id_A)): (A, (A, x: B \twoheadrightarrow FA)) \twoheadrightarrow (LF)A)
\]

---from here onward we leave out the definition of a functor on arrows when it is obvious. The component at \(F\) of the unit of the monad is

\[
\begin{array}{ccc}
A & \xrightarrow{LF} & MF \\
\downarrow{F} & & \downarrow{RF} \\
B & \xrightarrow{Id_B} & B
\end{array}
\]

and (the top component of) that of the multiplication \(RR \twoheadrightarrow R\) is

\[
\mu_F((A, x: B \twoheadrightarrow FA), x': B' \twoheadrightarrow B) = (A, xx': B' \twoheadrightarrow FA).
\]

The required distributivity law follows from the identities

\[(RLF) \circ \Delta_F = id_{MF} = \mu_F \circ (LRF),\]

see [BG16, 2.2].

5.4 Proposition. The fibration \(S: R-\text{Alg} \rightarrow \text{Cat}\) is elementary.

Proof. We shall make good use Theorem [BG16, 4.8] checking that the fibration \(S\) verifies condition (iv) To construct transporters consider a small category \(B\) in \(\text{Cat}\). The hinge on \(B\) is the functor \(\langle c_B, d_B\rangle: \text{Iso}(B) \twoheadrightarrow B \times B\) together with the structure map \(s_B\) defined by

\[
\begin{array}{ccc}
M(c_B, d_B) & \xrightarrow{s_B} & \text{Iso}(B) \\
R(c_B, d_B) & \downarrow{Id_{B \times B}} & \downarrow{(c_B, d_B)} \\
B \times B & & B \times B.
\end{array}
\]

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To provide a loop on \( \mathcal{B} \) it is then enough to show that the pair \((pr_{1,1}, r_B)\) is a morphism from the algebra \((\text{Id}_B, R\text{Id}_B)\) to the algebra \(((c_B, d_B), s_B)\), which is an easy diagram chase in

\[
\begin{array}{ccc}
\text{MId}_B & \xrightarrow{M(pr_{1,1}, r_B)} & \text{M}(c_B, d_B) \\
R\text{Id}_B & \downarrow & \downarrow s_B \\
\mathcal{B} & \xrightarrow{r_B} & \text{Iso}(\mathcal{B}).
\end{array}
\]

The construction of carriers is postponed to Lemma 5.6. But note that, once carriers are determined, transporters will be strictly productive as the iso

\[(\text{Iso}(pr_1), \text{Iso}(pr_2)): \text{Iso}(\mathcal{B} \times C) \cong \text{Iso}(\mathcal{B}) \times \text{Iso}(C)\]

is clearly a morphism of algebras.

Finally, to see that morphisms in \( \mathcal{E}_{pr_{1,1}}(pr_{1,1}, r_B) \) are locally epic with respect to \( S \), consider an algebra \((A, F, \mathcal{B}, S)\); write \( D: \mathcal{A} \times \mathcal{B} \text{Iso}(\mathcal{B}) \rightarrow \mathcal{I} \times \mathcal{B} \times \mathcal{B} \) for the underlying functor of \((pr_{1,2}^*F) \wedge (pr_{2,3}^*(c_B, d_B))\) and let \( T: MD \rightarrow \mathcal{A} \times \mathcal{B} \text{Iso}(\mathcal{B}) \) be its structure map, which maps an object \(((A, x), (i, b_1, b_2), (I, B_2, B_1)) \sim (FA, B)) \rightarrow (S(A, i, b_2), b_2^{-1}xb_1)\). Note that there is a functor \( K: \mathcal{A} \times \mathcal{B} \text{Iso}(\mathcal{B}) \rightarrow \text{M}(pr_{1,2}^*F)\) mapping an iso \( x: B \rightarrow \text{pr}_2 FA\) to

\[(A, (\text{id}_{FA}, x)): (FA, B) \sim \text{pr}_{1,2}^*FA)\]

and that the composite \(M(\text{Id}_{\mathcal{I} \times \mathcal{B} \times \mathcal{B}}, (\text{Id}_A, r_B\text{pr}_2F))K: \mathcal{A} \times \mathcal{B} \text{Iso}(\mathcal{B}) \rightarrow \text{MD},\) is a section of the algebra structure map. Then for every vertical morphism \( G: (pr_{1,2}^*F) \wedge (pr_{2,3}^*(c_B, d_B)) \rightarrow (F, S),\) it is

\[G = GTM(\text{Id}_{\mathcal{I} \times \mathcal{B} \times \mathcal{B}}, (\text{Id}_A, r_B\text{pr}_2F))K = SM(\text{Id}_{\mathcal{I} \times \mathcal{B} \times \mathcal{B}}, G)(\text{Id}_A, r_B\text{pr}_2F))K.\]

As \( \delta_F^I = (pr_{1,2,3}, (\text{Id}_A, r_B\text{pr}_2F)),\) algebra morphisms out of \(((c_B, d_B), s_B)\) are determined by their precomposition with \( \delta_F^I.\)

5.5 Corollary. The fibration \( \text{R-Alg} \rightarrow \mathcal{Gpd} \) is elementary.

Proof. The algebraic weak factorisation system structure on \( \mathcal{Gpd} \) is obtained pulling back that on \( \text{Cat} \) along the embedding \( \mathcal{Gpd} \rightarrow \text{Cat} \). It follows that \( \text{R-Alg} \rightarrow \mathcal{Gpd} \) is a change of base of \( \text{R-Alg} \rightarrow \text{Cat} \) along the embedding \( \mathcal{Gpd} \rightarrow \text{Cat} \). Hence \( \text{R-Alg} \rightarrow \mathcal{Gpd} \) is elementary by Proposition 4.11.

5.6 Lemma. Given an algebra \((F: \mathcal{A} \rightarrow \mathcal{B}, S: MF \rightarrow \mathcal{A})\) in \( \text{R-Alg} \), there is exactly one carrier for the loop \((pr_{1,1}, r_B): (\text{Id}_B, R\text{Id}_B) \rightarrow ((c_B, d_B), s_B)\) and it is \((pr_2, S): pr_1^*(F, S) \wedge ((c_B, d_B), s_B) \rightarrow F).\)

Proof. We may assume, without loss of generality, that the underlying functor of the algebra \( pr_1^*(F, S) \wedge ((c_B, d_B), s_B) \) is the diagonal \( D: MF \rightarrow \mathcal{B} \times \mathcal{B} \) in the pullback of categories and functors

\[
\begin{array}{ccc}
MF & \xrightarrow{F'} & \text{Iso}(\mathcal{B}) \\
\langle c'_B, RF \rangle & \downarrow D & \downarrow \langle c_B, d_B \rangle \\
A \times \mathcal{B} & \xrightarrow{F \times \text{Id}_B} & \mathcal{B} \times \mathcal{B}
\end{array}
\]

with the notation of diagram \( \square \). The structure map \( S_D: MD \rightarrow MF \) is induced by those on \( F \) and \( (c_B, d_B) \) and maps a pair \((A, x: B \sim FA), (b_1, b_2): (B_1, B_2) \sim (FA, B)\) to
the pair $S(A,b_1), b_1^{-1}xb_2: B_2 \xrightarrow{\sim} B_1$. A functor $T: MF \rightarrow A$ in the second component of the carrier has to fit in the commutative diagram

\[
\begin{array}{ccc}
MF & \xrightarrow{T} & A \\
\downarrow D & & \downarrow F \\
B \times B & \xrightarrow{pr_2} & B
\end{array}
\]

and, since it has to be a homomorphism of algebras, the following diagram must commute

\[
\begin{array}{ccc}
MD & \xrightarrow{M(pr_2,T)} & MF \\
\downarrow S_D & & \downarrow S \\
MF & \xrightarrow{T} & A.
\end{array}
\]

Moreover, the strictness condition imposes that the diagram

\[
\begin{array}{ccc}
A & \xrightarrow{\text{Id}_A} & A \\
\downarrow L_F & & \downarrow T \\
MF & \xrightarrow{T} & A
\end{array}
\]

(6)

commutes. Note also that there is an arrow $H: M(RF) \rightarrow MD$ such that $S_DH = \mu_F$ and $M(pr_2,T)H = M(\text{Id}_B, T)$. Precomposing diagram (5) with $H$ and using (6) together with a triangular identity for the monad, the commutative diagram

\[
\begin{array}{ccc}
MF & \xrightarrow{\text{Id}_MF} & MF \\
\downarrow M(RF) & & \downarrow S \\
MF & \xrightarrow{\mu_F} & S
\end{array}
\]

shows that the only possible choice for $T$ is the structural functor $S: MF \rightarrow A$, and it is straightforward to see that that choice makes diagrams (5) and (6) commute. \qed

5.7 Corollary. The fibrations $R \text{-Map} \rightarrow \text{Cat}$ and $R \text{-Map} \rightarrow \text{Gpd}$ are not elementary.

Proof. The forgetful fibred functor $U': R \text{-Alg} \rightarrow R \text{-Map}$ takes loops to loops. But, by Lemma \ref{lemma:transporter}, a transporter for an algebra in $R \text{-Alg}$ exists if and only if the algebra underlies an algebra for the monad. The same argument applies to the fibration $R \text{-Map} \rightarrow \text{Cat}$. \qed

5.8 Remark. Note that the argument in the proof of Proposition 5.4 which shows that arrows in $\Xi^{pr}_{pr_1,pr_2}$ locally epic with respect to $S: R \text{-Alg} \rightarrow \text{Cat}$ can be repeated to show that the same arrows are locally epic with respect to $N: R \text{-Map} \rightarrow \text{Cat}$. Also, since the forgetful fibred functor $U': R \text{-Alg} \rightarrow R \text{-Map}$ preserves finite products, $N: R \text{-Map} \rightarrow \text{Cat}$ has the underlying arrows to ensure strictly productive transporters.

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