Cosmic ray-modified shocks: appearance of an isothermal jump

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Abstract

We point out that for sufficiently strong shocks, with Mach number $M_1 > \sqrt{\frac{3\gamma-1}{(3-\gamma)\gamma}} = 1.34$ ($\gamma = 5/3$), the solutions for cosmic ray-modified shocks experiences a bifurcation. As a result, for super-critical flows an isothermal jump forms (which is not a shock). The isothermal jump forms due to the energy diffusion of fast, but energetically subdominant cosmic rays. For super-critical flows the isothermal jump appears regardless of a particular feed-back mechanism from the CRs. The compression ratio at the isothermal jump is $\frac{2}{(\gamma - 1)} = 3$, so that in the test particle regime the expected spectrum of low energy CRs experiencing first-order Fermi process is $p = \frac{2\gamma}{(3 - \gamma)} = \frac{5}{2}$, steeper than conventional $p = 2$.

1. Cosmic rays’ feedback on shock structure

Cosmic rays (CRs) modify internal structure of astrophysical shocks (e.g. Blandford 1980, Axford et al. 1982, Drury et al. 1982, Blandford & Eichler 1987, Malkov & Drury 2001, Amato & Blasi 2006). Blandford (1980) calculated perturbative effects of the CR on the shock structure expanding in powers of small CR pressure (see also Drury 1983).

The simplest way to calculate the CR ray feedback is within two-fluid model, whereby CR form a separate light, highly diffusive fluid. Here, experice with radiative shocks comes handy. It is well known in the theory of radiative shocks (e.g. Landau & Lifshitz 1959 parag. 95), see also Zel'dovich & Raizer (2003), that for sufficiently strong shocks the internal structure of the solution changes qualitatively - in some limits regardless of the strength of the feed-back an isothermal jump forms within the flow. Similar effect should occur in CR-modified shocks: an extended precursor is followed by an isothermal jump, not a sub-shock, as we argue below. Mathematically, addition of CR diffusion leads to a higher order differential equation for the velocity and, thus, cannot be treated as a perturbation.
2. Non-perturbative CR feedback

2.1. The iso-thermal jump

For sufficiently strong shocks the CR feedback is non-perturbative, as we discuss next. The first most important effect on the shock structure from cosmic rays (similar to effects of radiation in atmospheric explosions, Zeldovich & Raizer 2003) is the diffusive spreading of energy of CRs. This can be seen from the following argument. Strong (initial pressure equals zero) CR-modified shocks in the hydrodynamic approximation obey the following equations

\[
\begin{align*}
\beta_1 \rho_1 &= \beta \rho \\
\rho_1 \beta_1^2 &= p_{\text{tot}} + \rho_{\text{tot}} \beta^2 \\
\rho_1 \beta_1^3 / 2 &= (w_{\text{tot}} + \rho_{\text{tot}} \beta^2 / 2) \beta + F_{\text{CR}} \\
p_{\text{tot}} &= \frac{\rho}{m_p} T + \frac{u_{\text{CR}}}{3} \\
w_{\text{tot}} &= \frac{\gamma}{\gamma - 1} \frac{\rho}{m_p} T + \frac{4}{3} u_{\text{CR}} 
\end{align*}
\]

where \( p_{\text{tot}} \) and \( w_{\text{tot}} \) are total pressure and enthalpy, composed of plasma and CR contribution, and \( F_{\text{CR}} \) is the energy flux carried by CRs. In the diffusive approximation \( F_{\text{CR}} \propto \partial_z u_{\text{CR}} \). Values on the left refer to the far upstream. Thus, cosmic rays contribute to pressure and energy flux. Importantly, CR contribution to pressure is an addition - and thus is small for \( u_{\text{CR}} \ll p_{\text{gas}} \). On the other hand, the term with energy flux \( F_{\text{CR}} \) changes the order of the differential equation, and hence the structure of the solutions. This is the most important effect.

Thus, the first effects of CRs on the shock is the redistribution energy due to CR diffusion, leaving only \( F_{\text{CR}} \) term in (1). Then at each point a 1D stationary non-relativistic flow is described by the following set of equations (mass, momentum and energy flux conservation)

\[
\begin{align*}
\rho_1 v_1 &= \rho v \\
\rho_1 v_1^2 &= p + \rho v^2 \\
\frac{1}{2} \rho_1 v_1^3 &= F_{\text{CR}} + v \left( \frac{\rho v^2}{2} + w \right)
\end{align*}
\]

(cf. Zeldovich & Raizer 2003, Sec. VII.3 and Eqns. (7.10), (7.40)).

Both far upstream and far downstream the CR flux is zero. Introducing (the inverse of the) compression ratio \( \eta = \rho_1 / \rho \), the shock jump conditions give

\[
\begin{align*}
\eta_2 &= \frac{\gamma - 1}{\gamma + 1} \\
T_2 &= 2 \left( \frac{\gamma - 1}{\gamma + 1} \right) m_p v_1^2
\end{align*}
\]

where subscript 2 denotes values far downstream.
Within the shock, independently of the energy flux equation, the momentum conservation can be written as

\[ T = (1 - \eta) m_p v_1^2 \] (4)

\[ \eta(T) = \frac{1}{2} \pm \sqrt{\frac{1}{4} - \frac{T}{m_p v_1^2}} = \frac{1}{2} \left( 1 \pm \sqrt{1 - \frac{T}{T_{\text{max}}}} \right). \] (5)

where \( T_{\text{max}} = m_p v_1^2 / 4 \). Thus, there are two branches of \( \eta(T) \), see Fig. 1. It is the upper branch that connects to the pre-shock state with \( \eta = 1 \). Importantly, for super-critical shocks the final state (3) is located at the lower branch.

Note, that

\[ \frac{T_2}{T_{\text{max}}} = 8 \frac{\gamma - 1}{(\gamma + 1)^2} = \frac{3}{4} < 1 \] (6)

Thus, as the state evolves along the upper branch, the terminal temperature is reached before the terminal compression. It is required that temperature increase monotonically (e.g. Landau & Lifshitz 1959, Eq. (95.3)). Thus, since \( T_2 < T_{\text{max}} \), the final state cannot be reached continuously. There should be an isothermal jump at \( T = T_2 \), Fig. 1.

Note that we have derived the behavior of the compression ratio (and thus velocity of the flow) as a function of temperature without specifying a particular CR feedback mechanism! How the system evolves toward the iso-thermal jump depends on the particular form of \( F_{\text{CR}} \), but the existence of the iso-thermal jump is a consequence of the momentum conservation and total jump conditions, which are independent of \( F_{\text{CR}} \).

Qualitatively, shock jump conditions without diffusive effects may be written as continuous algebraic relations. Addition of diffusive terms modifies the structure of these relations - instead of algebraic, the energy evolution becomes a differential equation. There are spatial points in the equation - e.g., stationary solutions correspond to the shock jump conditions. A continuous solution cannot pass through some of the special points, e.g. \( T = T_{\text{max}} \) - this determines the formation of the iso-thermal jump.

### 2.2. Appearance of the iso-thermal jump

The above derivation assumed that the upstream medium is cold, so that the shock is infinitely strong. If the upstream plasma has temperature \( T_1 \) (so that Mach number is \( M_1 = \sqrt{\frac{m_p}{\gamma v_1^2}} v_1 \)), the compression ratio is

\[ \eta_\pm = \frac{1}{2} \left( \frac{T_1}{v_1^2 m_p} - \sqrt{\left( \frac{T_1}{v_1^2 m_p} + 1 \right)^2 - \frac{4T}{v_1^2 m_p} + 1} \right) \] (7)

Thus, the maximal temperature is

\[ T_{\text{max}} \left( \frac{v_1^2 m_p + T_1}{4v_1^2 m_p} \right)^2 = \frac{v_1^2 m_p (\gamma M_1^2 + 1)^2}{4\gamma^2 M_1^4} \] (8)
Fig. 1.— Compression ratio as function of temperature for very strong shocks. Two highlighted points correspond to the jump solutions; only the lower point is physical. To reach the physical solution it is necessary to pass through two special points: unphysical solution corresponding to the same final temperature and a special point at $\theta_{T,max}$.

Post-shock temperature and compression ratios are

$$T_2 = \frac{-2(\gamma - 1)\gamma T_1^2}{(\gamma + 1)^2 v_1^2 m_p} + \frac{2(\gamma - 1)v_1^2 m_p}{(\gamma + 1)^2} - \frac{(\gamma^2 - 6\gamma + 1) T_1}{(\gamma + 1)^2}$$

$$\eta_2 = \frac{(\gamma - 1)v_1^2 m_p + 2\gamma T_1}{(\gamma + 1)v_1^2 m_p}$$

Equating $T_{max}$ to $T_2$ we find that isothermal jump forms for

$$T_1 < \frac{(3 - \gamma)v_1^2 m_p}{3\gamma - 1}, \quad M_1 > M_{crit} = \sqrt{\frac{3\gamma - 1}{(3 - \gamma)\gamma}} = \frac{3}{\sqrt{5}} = 1.34$$

At this point $p_2/p_1 = (\gamma + 1)/(3 - \gamma)$, cf. Landau & Lifshitz [1959] Eq. (95.7).

We also point out that the transition through critical Mach number can be viewed as a bifur-
cation problem. For $M_1 < M_{\text{crit}}$ we have one branch,
\[
\eta = \frac{(\gamma - 1)M_1^2 + 2}{(\gamma + 1)M_1^2},
\]
while for $M_1 > M_{\text{crit}}$ there is another branch
\[
\eta = \frac{\gamma(2M_1^2 - 1) + 1}{\gamma(\gamma + 1)M_1^2},
\]
see Fig. 2.

Fig. 2.— Transition to supercritical shocks as a bifurcation problem. The final compression ratio $\eta_2$ (solid line) as a function of upstream Mach number $M_1$. For $M_1 < M_{\text{crit}} = 3/\sqrt{5}$ the final solution is reached continuously, while for $M_1 > M_{\text{crit}}$ there is bifurcation of solutions (dashed line), so that the final state is reached though an isothermal jump.

Evolution of quantities in the flow are depicted in Fig. 3. For $M < M_{\text{crit}} = 3/\sqrt{5}$ the final solution is reaches in a continuous way. There is a bifurcation point \( \{M_1 = 3/\sqrt{5}, \eta = 2/3\} \). For larger $M_1$ the final state is reached through an isothermal jump.

Note that the ratio of the final temperature $T_2$ to maximal temperature $T_{\text{max}}$ never exceeds
Fig. 3.— Evolution of compression ratio $\eta$ versus temperature (normalized by $m_p v_1^2$) for different Mach numbers $M = 1.1, 1.2...2$). The flow starts at $\eta = 1$. For $M_1 < M_{\text{crit}} = 3/\sqrt{5}$ the final solution is reached in a continuous way. There is a bifurcation point at $M = M_{\text{crit}}$: for larger $M_1$ the final state is reached through an isothermal jump.

unity:

$$\frac{T_2}{T_{\text{max}}} = \frac{-8(\gamma - 3)^2(\gamma - 1)\gamma + 4(3\gamma^4 - 28\gamma^3 + 66\gamma^2 - 28\gamma + 3) m_1^2 + 8(1 - 3\gamma)^2(\gamma - 1) m_1^4}{(\gamma + 1)^2 (-\gamma + (3\gamma - 1)m_1^2 + 3)^2} \leq 1$$

$$m_1 = \frac{M_1}{M_{\text{crit}}}$$

This ratio reaches unity only at $m_1 = 1$. In this case

$$\eta_{\text{crit}} = \frac{1 + \gamma}{3\gamma - 1} = 2/3$$
2.3. Structure of the precursor

The appearance of the isothermal jump is independent of the particular form of $F$, but the evolution towards the iso-thermal jump depends on it. To resolve the structure of the precursor one needs to relate the CR pressure to the fluid parameters. The energy conservation gives

$$\frac{F_{CR}}{n_1 m_p v_1^2} = \frac{(1 - \eta)(-\gamma(1 - \eta) + \eta + 1)}{2(1 - \gamma)} = -\frac{(1 - \eta)(\eta - \eta_2)}{2\eta_2} \rightarrow \frac{1}{2}(1 - \eta)(1 - 4\eta)$$  \hspace{1cm} (15)

For finite $F_{CR}$ (15) and (5) - with a proper form of $F(\eta, T)$ - determine the structure of the shock.

As a qualitative example, let us assume that density of CRs follows plasma density, so that $F_{CR} = -\kappa \partial_z \rho$ (16)

where $\kappa$ absorbs the diffusion coefficient of CR and the scaling with plasma density. We find

$$\frac{\kappa}{v_1^3} \partial_z \eta = -\frac{(1 - \eta)\eta^2 (\eta - \eta_2)}{2\eta_2} \rightarrow \frac{1}{2}(1 - \eta)(1 - 4\eta)\eta^2$$  \hspace{1cm} (17)

where $\eta_2 = (\gamma - 1)/(\gamma + 1)$.

Dimensinalizing distance by $\kappa/v_1^3$, Eq. (17) can be integrated

$$z = \log \left( (1 - \eta)^{-\frac{2\eta_2}{\eta_2 - 1}} \eta^{\frac{2}{\eta_2 - 1}} + 2 \right) \frac{\eta^2}{\eta_2 - 1} (1 - \eta_0) \frac{2(\eta_2 + 1)}{\eta_2} \eta_2^{2\eta_2} \frac{2\eta_2}{\eta_2 - 1} (\frac{\eta_2 - 1}{\eta_2 - 1})^{-\frac{2\eta_2}{(\eta_2 - 1)\eta_2}} - \frac{2}{\eta} - \frac{2}{\eta_2 - 1}$$

$$\rightarrow \log \left( \frac{1024 \cdot 2^{2/3} (1 - \eta)^{2/3} \eta^{10}}{59049 (\eta - \frac{1}{4})^{32/3}} \right) - \frac{2}{\eta} + \frac{8}{3}$$  \hspace{1cm} (18)

where the integration constant as been chose so that the thermal jump is located at $z = 0$, see Fig. 4.

3. Cosmic rays acceleration at the iso-thermal jump

Let us first give relations for the strong shock limit $M_1 \rightarrow \infty$. At the iso-thermal jump the sound speed is

$$c_s = \sqrt{\frac{\gamma T}{m_p}} = \frac{\sqrt[3]{2} \sqrt{\gamma - 1} \sqrt[3]{\gamma}}{\gamma + 1}$$  \hspace{1cm} (19)

At this point, on the upper branch the parameters of the flow are

$$\eta_+ = \frac{2}{(\gamma + 1)}$$
$$v_+ = \frac{2}{\gamma + 1} v_1 = \frac{3}{4} v_1$$
$$M_+ = \frac{\sqrt[3]{2}}{\sqrt[3]{\gamma - 1} \sqrt[3]{\gamma}} = \frac{3}{\sqrt[3]{5}}$$  \hspace{1cm} (20)
Fig. 4.— Evolution of the compression ration and the velocity in CR-modified shock with $\gamma = 5/3$. Iso-thermal jump is at $z = 0$, dot-dashed line corresponds to the absence of the isothermal jump, Eq. (18). The iso-thermal jump connects states with $\eta_+ = 3/4$ and $\eta_2 = 1/4$ (compare, e.g., with Axford et al. (1982) Fig. 5 and Drury & Voelk (1981), Fig. 2).

While in the post-jump flow

\[
\begin{align*}
\eta_2 &= \frac{\gamma - 1}{\gamma + 1} \\
v_2 &= \frac{\gamma + 1}{\gamma - 1} v_1 = \frac{1}{4} v_1 \\
M_2 &= \frac{\sqrt{\gamma - 1}}{\sqrt{2\gamma}} = \frac{1}{\sqrt{5}}
\end{align*}
\]  

(21)
The compression ratio at the isothermal jump is \((\gamma - 1)/2 = 1/3\), so that the expected spectrum of CRs is \(p = 2\gamma/(3 - \gamma) = 5/2\) (e.g. Blandford & Eichler [1987]), steeper than conventional 2 for \(\gamma = 5/3\).

For finite upstream Mach number the compression jump \(r_{IJ}\) at the isothermal shock is

\[
r_{IJ} = \frac{n_+}{n_-} = \frac{(3\gamma^2 - 4\gamma + 1) m_1^2 - 2(\gamma - 3)\gamma}{\gamma^2 - 4\gamma + (6\gamma - 2)m_1^2 + 3} \to 1/3,
\]

where the last limit assumes \(m_1 \gg 1\) and \(\gamma = 5/3\). see Fig. 5.

![Graph](image)

Fig. 5.— Inverse of the compression ration at the isothermal jump as a function of the ratio of upstream Mach number to the critical one.

4. Discussion

In this work we point out that for sufficiently strong shocks, with Mach number \(M_1 > \sqrt{3\gamma - 1}/(3 - \gamma)^{\gamma} = 3/\sqrt{5} = 1.34\), the energy diffusion induced by CRs modifies the global structure, creating a special kind of a discontinuity - an isothermal jump. At the isothermal jump temperature
remains constant - hence, it is not a shock. On the other hand the flow does change from supersonic to subsonic, with a compression ratio of 3. Thus, many models of diffusive shock acceleration that rely on the shock compression ratio are likely to remain valid for the isothermal jump as well.

We stress that the appearance of the isothermal jump is, generally, independent on the particular form of CR feedback - it is the evolution of the flow towards the isothermal jump that is affected by a particular feed-back mechanism. It is not that large conductivity makes the subshock isothermal - even the minimal CR-diffusion leads to the formation of the iso-thermal jump (for super-critical shocks).

The density compression ratio at the iso-thermal jump of 3 will lead to the spectral index of accelerated particles of $p = 2.5$ (in the limit of test-particle experiencing first-order Fermi process Krymskii [1977], Blandford & Ostriker [1978]), somewhat steeper than the conventional values of $p = 2$ for strong shocks with compression ratio of 4. In fact, in many settings the inferred spectra are closer to $p = 2.5$ (e.g., as discussed by Caprioli [2012]). At the nonlinear stage, when CRs start strongly affect the thermodynamic properties of the flow, the adiabatic index can decrease to $\gamma = 4/3$, which would give compression ratio at the isothermal transition of $2/(\gamma - 1) = 6$.

Our hydrodynamic approach naturally has severe limitations. Kinetic effects are likely to be important, especially for the very highest energy particles. But since the predicted spectrum is soft, the CR escape from the shock will not be important.

Finally, let us comment on the effects of magnetic field. Magnetic fields slightly modify the appearance of the isothermal jump. For strong perpendicular shocks instead of (4) we have

$$T = (1 - \eta)\eta m_p v_i^2 \left( 1 - \frac{1 + \eta}{2\eta^2} \frac{1}{M_A^2} \right)$$

where $M_A = v_1/v_A$ and $v_A$ is Alfvén velocity. Thus, correction is small for highly super-Alfvenic flows. Generally, for perpendicular shocks the effect of magnetic field on the fluid flow can be completely absorbed into the definition of sound speed - which becomes the fast magnetosonic speed.

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