REGULARITY PROPERTIES FOR EVOLUTION FAMILY GOVERNED BY NON-AUTONOMOUS FORMS

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ABSTRACT. This paper gives further regularity properties of the evolution family associated with a non-autonomous evolution equation

\[ \dot{u}(t) + A(t)u(t) = f(t), \quad t \in [0, T], \quad u(0) = u_0, \]

where \( A(t) \), \( t \in [0, T] \), arise from non-autonomous sesquilinear forms \( a(t, \cdot, \cdot) \) on a Hilbert space \( H \) with constant domain \( V \subset H \). Results on norm continuity, compactness and results on the Gibbs character of the evolution family are established. The abstract results are applied to the Laplacian operator with time dependent Robin boundary conditions.

INTRODUCTION

Let \( H, V \) be two Hilbert spaces such that \( V \) is continuously and densely embedded into \( H \) and consider a non-autonomous sesquilinear form \( a : [0, T] \times V \times V \to \mathbb{K} \) with

\[ |a(t, u, v)| \leq M\|u\|_V\|v\|_V \text{ and } \Re a(t, u, u) \geq \alpha\|u\|_V^2, \]

for all \( t \in [0, T] \), \( v \in V \) and for some \( \alpha, M > 0 \). For each \( t \in [0, T] \) we associate with \( a(t, \cdot, \cdot) \) a unique operator \( A(t) \in \mathcal{L}(V, V') \) such that

\[ a(t, u, v) = \langle A(t)u, v \rangle \quad \text{for all } u, v \in V. \]

The non-autonomous Cauchy problem

(1) \[ \dot{u}(t) + A(t)u(t) = f(t), \quad u(0) = u_0 \]

is said to have \( L^2 \)-maximal regularity in \( H \) if for every \( f \in L^2(0, T; H) \) and \( u_0 \in V \) there exists a unique function \( u \) belonging to \( L^2(0, T; V') \cap H^1(0, T; H) \) such that \( u \) satisfies (1). Maximal regularity is a very important aspect in ongoing research in theory of non-autonomous evolution equations and addressed by many authors. This is due to its applicability in the prove of existence and regularity of solutions. More interestingly, maximal regularity can be used to solve nonlinear evolution equations by means of fixed point theorems. Considering (1) on \( V' \), in 1961 J. L. Lions proved the following well known result regarding \( L^2 \)-maximal regularity in \( V' \):

**Theorem 0.1** (Lions). Given \( f \in L^2(0, T; V') \) and \( u_0 \in H \), the problem (1) has a unique solution \( u \in MR(V, V') := L^2(0, T; V') \cap H^1(0, T; V') \).

Theorem 0.1 has been proved by Lions in [27] using a representation theorem of linear functionals, known in the literature as Lions’ representation Theorem. Other proofs of Theorem 0.1 can be found in [15, XVIII Chapter 3, p. 620] where a Galerkin’s method is used or in [39, Section 5.5] where a fundamental solution has been constructed. Furthermore, the author gave an alternative proof in [34, Section 2] using the approach of the integral product of semigroups developed in [20, 21, 22].

Lions’ theorem requires only measurability of \( t \mapsto a(t, u, v) \) for all \( u, v \in V \). However, in applications to boundary problems maximal regularity in \( V' \) is not sufficient because it is only the part \( A(t) \) of \( A(t) \) in \( H \) that realizes the boundary conditions in question. Precisely one is more interested on \( L^2 \)-maximal regularity in \( H \), i.e., the solution \( u \) of (1) belong to \( H^1(0, T; H) \) if \( f \in L^2(0, T; H) \) and \( u_0 \in V \). The problem of \( L^2 \)-maximal regularity in \( H \) was initiated by Lions in [27, p. 68] for \( u_0 = 0 \) and \( a \) is symmetric. In general, we have to impose more regularity on the form \( a \) then measurability of the form is not sufficient [16, 8]. However, under additional regularity assumptions on the form \( a \), the initial value \( u_0 \) and the inhomogeneity \( f \), some positive results were already done by Lions in [27, p. 68, p. 94, ], [27,
Theorem 1.1, p. 129] and [27, Theorem 5.1, p. 138] and by Bardos [13]. More recently, this problem has been studied with some progress and different approaches [9, 10, 29, 17, 31, 32, 22, 18, 12, 24]. Results on multiplicative perturbation are established in [9, 17, 11]. See also the recent review paper [8] for more details and references.

In this paper we are mainly interested by further regularity of the evolution family generated by $\mathcal{A}(t), t \in [0, T]$. Recall that it is well known that, under suitable conditions, the solution of a non-autonomous linear evolution equation may be given by a strongly continuous evolution family

$$\{U(t, s) : 0 \leq s \leq t \leq T\} \subset \mathcal{L}(H)$$

i.e., a family that has the following properties:

(i) $U(t, t) = I$ and $U(t, s) = U(t, r)U(r, s)$ for every $0 \leq r \leq s \leq t \leq T$,

(ii) for every $x \in H$ the function $(t, s) \mapsto U(t, s)x$ is continuous into $H$ for $0 \leq s \leq t \leq T$.

In the autonomous case, i.e., if $a(t, \cdot, \cdot) = a_0(\cdot, \cdot)$ for all $t \in [0, T]$, then one knows that $-A_0$, the operator associated with $a_0$ in $H$, generates a holomorphic $C_0$-semigroup $(T(t))_{t \geq 0}$ in $H$. In this case $U(t, s) := T(t - s)$ yields a strongly continuous evolution family on $H$.

The study of regularity properties of the evolution family with respect to $(t, s)$ in general Banach spaces has been investigated in the Literature in the case of constant domains by Komatsu [26] and Lunardi [28] and by Acquistapace [2] for the case of time-dependent domains.

Consider a non-autonomous sesquilinear form $a(t, \cdot, \cdot) : V \times V \to \mathbb{C}$ form such that there exists $0 \leq \gamma < 1$ and a continuous function $\omega : [0, T] \to [0, +\infty)$ with

$$\sup_{t \in [0, T]} \frac{\omega(t)}{t^{\gamma/2}} < \infty \quad \text{and} \quad \int_0^T \frac{\omega(t)}{t^{1+\gamma/2}} < \infty$$

such that

$$|a(t, u, v) - a(s, u, v)| \leq \omega(|t - s|)||u||V||v||_{V'} \quad (t, s \in [0, T], u, v \in V),$$

where $V_{\gamma} := [H, V]$ is the complex interpolation space. Under this assumptions, (1) has $L^2$-maximal regularity in $H$ provided $a(0; \cdot, \cdot)$ has the square root property by a recent result of Arendt and Monniaux [10]. In this paper we push their analysis forward by discussing additional regularity properties of the solution.

In Section 2 we establish that in this case the solution of the non-autonomous Cauchy problem (1) is governed by a strongly continuous evolution family $\mathcal{U}(\cdot, \cdot)$ in $H$ such that the mapping $(t, s) \mapsto \mathcal{U}(t, s)$ is norm continuous with value in $\mathcal{L}(V)$ for $0 \leq s < t \leq T$. If in addition the embedding $V \subset H$ is compact then we show in Section 3 that the operator $U(t, s)$ is compact with value in $\mathcal{L}(V)$, respectively in $\mathcal{L}(H)$, for each $0 \leq s < t \leq T$. This implies, in particular, that $(t, s) \mapsto \mathcal{U}(t, s)$ is also norm continuous with value in $\mathcal{L}(H)$ for $0 \leq s < t \leq T$ provided that $V \subset H$ is compact. An other important result of Section 3 concerns the Gibbs character of the evolution family $\mathcal{U}(\cdot, \cdot)$. Recall that the trace class operators is a special case of $p$-Schatten class operators $S_p(H)$ when $p = 1$. Then we call that an evolution family is Gibbs if $U(t, s)$ is a trace operator for each $0 \leq s < t \leq T$. In the autonomous situation, the concept of Gibbs semigroups has been introduced by Dietrich A. Uhlenbrock [40], and it is known that this class play an important role in spectral theory and mathematics physics. In Section 3 we show that the evolution family associated with (1) is a Gibbs evolution family provided that there exists some $p \in (1, \infty)$ such that the embedding $V \subset H$ is a $p$-Schatten operator. We apply our abstract results in Section 4 to parabolic equations governed by the Laplacian operator with time dependent Robin boundary conditions.

1. Preliminary results

Throughout this paper $H, V$ are two Hilbert spaces over $\mathbb{C}$ such that $V \hookrightarrow V'$; i.e., $V$ is densely embedded into $H$ and

$$||u|| \leq c_H ||u||_V \quad (u \in V)$$

for some constant $c_H > 0$. Let $V'$ denote the antidual of $V$. The duality between $V'$ and $V$ is denoted by $\langle \cdot, \cdot \rangle$. As usual, by identifying $H$ with $H'$, we have $V \hookrightarrow H \cong H' \hookrightarrow V'$. These embeddings are continuous
and
\[ \|f\|_{V'} \leq c_H \|f\| \quad (f \in V') \]
see e.g., [14]. We denote by \( \langle \cdot | \cdot \rangle_V \) the scalar product and \( \| \cdot \|_V \) the norm on \( V \) and by \( (\cdot, \cdot) \) the corresponding quantities in \( H \). Let \( a : [0, T] \times V \times V \to \mathbb{C} \) be a continuous and coercive non-autonomous sesquilinear form, i.e., \( a(\cdot, u, v) \) is measurable,
\[ |a(t, u, v)| \leq M \|u\|_V \|v\|_V \]
and
\[ \Re a(t, u, u) \geq \alpha \|u\|_V^2 \]
for some constants \( \alpha, M > 0 \) and for all \( t \in [0, T], u, v \in V \). By the Lax-Milgram theorem, for each \( t \in [0, T] \) there exists an isomorphism \( A(t) : V \to V' \) such that \( (A(t)u, v) = a(t, u, v) \) for all \( u, v \in V \).

It is well known that \( -A(t) \), regarded as an unbounded operator with domain \( V \), generates a bounded holomorphic semigroup \( e^{-A(t)} \) of angle \( \theta := \frac{\pi}{2} - \arctan(\frac{M}{\alpha}) \) on \( V' \). We call \( A(t) \) the operator associated with \( a(t, \cdot, \cdot) \) on \( V' \). We have also to consider the operator \( A(t) \) associated with \( a(t, \cdot, \cdot) \) on \( H \):
\[ D(A(t)) := \{ u \in V : A(t)u \in H \} \]
\[ A(t)u = A(t)u. \]
i.e., \( A(t) \) is the part of \( A(t) \) in \( H \). Then \( -A(t) \) generates a holomorphic \( C_0 \)-semigroup (of angle \( \theta \)) \( e^{-A(t)} \) on \( H \) which is the restriction to \( H \) of \( e^{-A(t)} \), and we have
\[ e^{-A(t)} = \frac{1}{2\pi i} \int_{\Gamma} e^{\mu} (\mu + A(t))^{-1} d\mu \]
where \( \Gamma := \{ re^{i\varphi} : r > 0 \} \) for some fixed \( \varphi \in (\theta, \frac{\pi}{2}) \) (see e.g. [4, Lecture 7],[25],[30, Chapter 1] or [39, Chap. 2]).

We assume in addition, that there exists \( 0 \leq \gamma < 1 \) and a continuous function \( \omega : [0, T] \to [0, +\infty) \) with
\[ \sup_{t \in [0, T]} \frac{\omega(t)}{t^{1/2}} < \infty, \]
(5)
\[ \int_0^T \frac{\omega(t)}{t^{1+\gamma/2}} dt < \infty \]
(6)
and
\[ |a(t, u, v) - a(s, u, v)| \leq \omega(|t - s|) \|u\|_V \|v\|_V \]
(7)
for all \( t, s \in [0, T] \) and for all \( u, v \in V \) where \( V_\gamma := [H, V]_\gamma \) is the complex interpolation space. Note that
\[ V \hookrightarrow V_\gamma \hookrightarrow H \hookrightarrow V'_\gamma \hookrightarrow V' \]
with continuous embeddings.

**Remark 1.1.** Remark that conditions (5) and (6) implies that
\[ \int_0^T \frac{\omega(t)^2}{t^{1+\gamma}} dt < \infty \]
(8)
and for each \( \varepsilon > 0 \) there exists \( \delta_0 > 0 \) such that
\[ \int_0^{\delta_0} \frac{\omega(t)}{t^{1+\gamma/2}} dt < \varepsilon. \]
(9)

The following proposition is of great interest for this paper.

**Proposition 1.2.** [10, Section 2] Let \( b \) be any sesquilinear form that satisfies assumptions (2)-(3) with the same constants \( M \) and \( \alpha \) and let \( \gamma \in [0, 1] \). Let \( B \) and \( B' \) be the associated operators on \( V' \) and \( H \), respectively. Then the following estimates holds
\[ (i) \quad \| (\lambda - B)^{-1} \|_{\mathcal{L}(V'_\gamma, H)} \leq \frac{e}{(1 + |\lambda|)^{1-\frac{\gamma}{2}}} \]
(1)
(2) \( \| (\lambda - \mathcal{B})^{-1} \|_{\mathcal{L}(V)} \leq \frac{c}{1 + |\lambda|} \),

(3) \( \| (\lambda - \mathcal{B})^{-1} \|_{\mathcal{L}(H,V)} \leq \frac{c}{(1 + |\lambda|)^{1/2}} \),

(4) \( \| (\lambda - \mathcal{B})^{-1} \|_{\mathcal{L}(V^*,H)} \leq \frac{c}{(1 + |\lambda|)^{1/2}} \),

(5) \( \| (\lambda - \mathcal{B})^{-1} \|_{\mathcal{L}(V^*,V)} \leq c \),

(6) \( \| (\lambda - \mathcal{B})^{-1} \|_{\mathcal{L}(V^*_s,V)} \leq \frac{c}{(1 + |\lambda|)^{1/2}} \),

(7) \( \| e^{-sB} \|_{\mathcal{L}(V^*,V)} \leq \frac{c}{s} \),

(8) \( \| e^{-sB} \|_{\mathcal{L}(V^*_s,H)} \leq \frac{c}{s^{1/2}} \),

(9) \( \| e^{-sB} \|_{\mathcal{L}(V^*_s,V)} \leq \frac{c}{s^{1/2}} \),

(10) \( \| B e^{-sB} \|_{\mathcal{L}(H)} \leq \frac{c}{s} \),

(11) \( \| e^{-sB} \|_{\mathcal{L}(V)} \leq c \)

for each \( s \geq 0, \lambda \notin \Sigma_0 := \{ re^{i\varphi} : r > 0, |\varphi| < \theta \} \) and for some constant \( c > 0 \) which depend only on \( M, \alpha, \gamma \) and \( c_H \).

**Remark 1.3.** All estimates in Proposition 1.2 holds for \( A(t) \) with constant independent of \( t \in [0,T] \), since \( A \) satisfies (2)-(3) with the same constants \( M \) and \( \alpha \), also \( \gamma \) and \( c_H \) does not depend on \( t \in [0,T] \).

If \( b : V \times V \to \mathbb{C} \) is a coercive and bounded form and \( \mathcal{B} \in \mathcal{L}(V,V') \) the associated operator on \( V' \), then one defines \( \mathcal{B}^{-1/2} \in \mathcal{L}(V') \) by

\[
\mathcal{B}^{-1/2} u := \frac{1}{\pi} \int_0^\infty t^{-1/2} (t - \mathcal{B})^{-1} dt, \quad \text{for all } u \in V',
\]

see [33, (6.4), Section 2.6] or [5, (3.52), Section 3.8]. The operator \( \mathcal{B}^{-1/2} \) is also given by the following formula

\[
\mathcal{B}^{-1/2} u = \frac{1}{\sqrt{\pi}} \int_0^\infty t^{-1/2} e^{-tB} u dt, \quad \text{for all } u \in V'
\]

see [33, (6.9), Section 2.6]. Moreover, \( \mathcal{B}^{-1/2} \) is one-to-one and one defines \( \mathcal{B}^{1/2} \) by

\[
D(\mathcal{B}^{1/2}) := \text{Ran}(\mathcal{B}^{-1/2}) \quad \text{and} \quad \mathcal{B}^{1/2} u = (\mathcal{B}^{-1/2})^{-1} u \quad \text{for all } u \in D(\mathcal{B}^{1/2}).
\]

Let \( B \) be the part of \( \mathcal{B} \) in \( H \). Then \( \mathcal{B}^{-1/2} u = \mathcal{B}^{-1/2} u \) for each \( u \in H \) and \( \mathcal{B}^{1/2} = (\mathcal{B}^{-1/2})^{-1} \) with domain \( D(\mathcal{B}^{1/2}) = \text{Ran}(\mathcal{B}^{-1/2}) \). We recall that \( b : V \times V \to \mathbb{C} \) has the square root property if the following equivalent conditions are satisfied

(i): \( D(\mathcal{B}^{1/2}) = V \),

(ii): \( D(\mathcal{B}^{1/2}) = H \).

Conditions (i) and (ii) above are indeed equivalent since \( \mathcal{B}^{-1/2} \mathcal{B}^{-1/2} = \mathcal{B}^{-1} \) is an isomorphism from \( V' \) into \( V \). If a coercive and bounded sesquilinear form \( b : V \times V \to \mathbb{C} \) has the square root property, then \( \mathcal{B}^{-1/2} \) is an isomorphism from \( V' \) into \( H \) with inverse \( \mathcal{B}^{1/2} \), and \( \mathcal{B}^{-1/2} \) is an isomorphism from \( H \) into \( V \) with inverse \( \mathcal{B}^{1/2} \). Further, \( \mathcal{B}^{1/2} \) is the part of \( \mathcal{B}^{1/2} \) in \( H \).

**Lemma 1.4.** Assume that \( a : [0,T] \times V \times V \to \mathbb{C} \) satisfies (2)-(7) and \( D(A^{1/2}(t)) = V \) for every \( t \in [0,T] \). Then there exists a constant \( c_0 > 0 \) such that

\[
\| A^{1/2}(t) - A^{1/2}(s) \|_{\mathcal{L}(V,H)} \leq c_0 w(|t - s|),
\]

(13) \( \| A^{-1/2}(t) - A^{-1/2}(s) \|_{\mathcal{L}(V,V)} \leq c_0 w(|t - s|) \)

and

(14) \( \| A^{1/2}(t) - A^{1/2}(s) \|_{\mathcal{L}(H,H)} \leq c_0 w(|t - s|) \)

for all \( t, s \in [0,T] \).
Proof. Let $u \in V'$ and $t, s \in [0, T]$. Then using (11) we obtain

$$A^{-1/2}(t)u - A^{-1/2}(s)u = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{\sqrt{\tau}} (e^{-\tau A(t)}u - e^{-\tau A(s)}u) d\tau = \frac{1}{\sqrt{\pi}} \int_0^\infty \frac{1}{2\pi i \sqrt{\tau}} \int e^{-\tau \lambda (A(t) - A(s))} (A(t) - A(s))^{-1} u d\lambda d\tau$$

By Proposition 1.2-(1), (5) and assumption (7)

$$\|(\lambda - A(t))^{-1}(A(t) - A(s))\|(\lambda - A(t))^{-1} u\|_H \leq \|(\lambda - A(t))^{-1}\|_{L(V', H)} \|A(t) - A(s)\|_{L(V, V')} \|(\lambda - A(t))^{-1}\|_{L(V', V)} |u|_{V'} \leq c\omega(|t - s|) \frac{1}{(1 + |\lambda|)^{1/2}} |u|_{V'}$$

for each $\lambda \in \Gamma$ and some constant $c > 0$ depending only on $M, \alpha, \gamma$ and $c_H$. This proves inequality (12), the second inequality also follows easily by a similar way. Let now prove the last one. First remark that (7) implies, in particular, that

$$\|A(t) - A(s)\|_{L(V', V')} \leq c\omega(|t - s|).$$

On the other hand, $A^{-1/2}(\cdot) : [0, T] \to L(V', H)$ is continuously (12) since $\omega(|t - s|) \to 0$ as $t \to s$ by assumption. Thus

$$\kappa := \sup_{r \in [0, T]} \|A^{-1/2}(r)\|_{L(V', H)} < \infty.$$

Since $A^{1/2}(\cdot)u = A^{-1/2}(\cdot)A(\cdot)u$ we deduce from (2) and (12) that

$$\|A^{1/2}(t)u - A^{1/2}(s)u\|_H = \|A^{-1/2}(t)A(t)u - A^{-1/2}(s)A(s)u\|_H \leq \|A^{-1/2}(t)\|_{L(V', H)} \|A(t) - A(s)\|_{L(V', V')} + \|(A^{-1/2}(t) - A^{-1/2}(s))\|_{L(V', H)} \|A(s)u\|_{V'} \leq c_0\omega(|t - s|) |u|_{V'}$$

where the constant $c_0 > 0$ depends only on $M, \alpha, \gamma, c_H$ and $\kappa$, as thus the claim is proved.

The following lemma is a direct consequence of Lemma 1.4.

**Lemma 1.5.** Assume that $a : [0, T] \times V \times V \to \mathbb{C}$ satisfies (2)-(7) and $D(A^{1/2}(t)) = V$ for every $t \in [0, T]$. Then there exists a constant $\sigma > 0$ such that

$$\frac{1}{\sigma} |u|_{V} \leq \|A^{1/2}(t)u\|_H \leq \sigma |u|_{V}$$

for all $u \in V$ and every $t \in [0, T]$.

**Notation 1.6.** To keep notations simple as possible we will in the sequel denote all positive constants depending on $M, \alpha, \gamma, c_H, T$ and $\sigma$ that appear in proofs and theorems uniformly as $c > 0$.  

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**REGULARITY PROPERTIES FOR EVOLUTION FAMILY GOVERNED BY NON-AUTONOMOUS FORMS**

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2. Norm continuous evolution family

Let \( a : [0, T] \times V \times V \to \mathbb{C} \) be a bounded and coercive non-autonomous sesquilinear form. Then we know from Theorem 0.1 that for each given \( s \in [0, T) \) and \( x \in H \), the Cauchy problem

\[
\dot{u}(t) + \mathcal{A}(t)u(t) = 0 \quad \text{a.e. on } [s, T], \quad u(s) = x,
\]

has a unique solution \( u \in \MR(V, V') := \MR(s, T; V, V') := L^2(s, T; V) \cap H^1(s, T; V') \). Recall that the maximal regularity space \( \MR(V, V') \) is continuously embedded in \( C([s, T], H) \). Therefore, for every \( (t, s) \in \overline{\Delta} \) and every \( x \in H \) we can define

\[
\MR(s, T)x := u(t),
\]

where \( u \) is the unique solution in \( \MR(V, V') \) of (16) and \( \Delta := \{(t, s) \in [0, T]^2 : t < s\} \).

**Proposition 2.1.** The family \( \{\MR(t, s) : (t, s) \in \overline{\Delta}\} \) yields a contractive, strongly continuous evolution family on \( H \). Moreover, for each \( f \in L^2(0, T, H) \)

\[
v(t) = \int_0^t \MR(t, r)f(r)dr
\]

is the unique solution in \( \MR(V, V') \) of the inhomogeneous problem

\[
\dot{v}(t) + \mathcal{A}(t)v(t) = f(t) \quad \text{a.e. on } [0, T], \quad u(0) = 0,
\]

**Proof.** The last assertion and the fact that \( \{\MR(t, s) : (t, s) \in \overline{\Delta}\} \) yields a bounded, strongly continuous evolution family on \( H \) follows immediately from [7, Proposition 2.3, Proposition 2.4]. It remains to show that

\[
\|\MR(t, s)x\| \leq \|x\|, \quad \text{for all } x \in H \text{ and } (t, s) \in \overline{\Delta}.
\]

Let \( x \in H \) and \( s \in [0, T) \). Since \( \MR(\cdot, s)x \in \MR(V, V') \) then the function \( \|\MR(\cdot, s)\|^2 \) is absolutely continuous on \([s, T]\) and

\[
\frac{d}{dt}\|\MR(t, s)\|^2 = 2 \text{Re} \langle \MR(t, s), \MR(t, s) \rangle,
\]

see e.g., [37, Chapter III, Proposition 1.2] or [39, Lemma 5.5.1]. It follows

\[
\frac{d}{dt}\|\MR(t, s)x\|^2 = 2 \text{Re} \langle -\mathcal{A}(t)\MR(t, s)x, \MR(t, s)x \rangle
\]

\[
= -2 \text{Re} a(t, \MR(t, s)x, \MR(t, s)x) \leq -2\alpha \|\MR(t, s)x\|^2 \leq 0
\]

for almost every \( t \in [s, T] \). Integrating this inequality on \([s, t]\) and using (3) we obtain

\[
\|\MR(t, s)x\|^2 - \|\MR(s, s)x\|^2 \leq -2\alpha \int_s^t \|\MR(r, s)x\|^2 dr \leq 0,
\]

and the claim follows. \( \square \)

For the remainder of this section, we assume that \( a : [0, T] \times V \times V \to \mathbb{C} \) satisfies (2)-(7) and \( D(A(0)^{1/2}) = V \) and we set

\[
\MR_{a,s} := \{u \in H^1(s, T, H) \cap L^2(s, T, V) : u(s) \in V \text{ and } \mathcal{A}(\cdot)u(\cdot) \in L^2(s, T, H)\},
\]

see [10, Theorems 4.1, 4.2]. The main result of this section states that \( \{\MR(t, s) : (t, s) \in \overline{\Delta}\} \subset \mathcal{L}(V) \) define a norm continuous evolution family on \( V \), that is \( \MR(\cdot, \cdot) \in C(\Delta, \mathcal{L}(V)) \). To this end, we recall that the solution \( u = \MR(\cdot, \cdot) \) of (16) satisfies the following key formula

\[
\MR(t, s)x = e^{-(t-s)A(t)}x + \int_s^t e^{-(t-r)A(t)}(A(t) - A(\tau))\MR(r, s)x dr
\]
for all \((s, t) \in \Delta\) and \(x \in V\). This formula is due to Acquistapace and Terreni [3] and was proved in a more general setting in [10, Proposition 3.5]. First we introduce the following notations:

\[
U_1(t, s)x := e^{-(t-s)A(t)}x,
\]

and

\[
U_2(t, s)x := \int_s^t e^{-(t-r)A(t)}(A(t) - A(r))U(r, s)xdr.
\]

Moreover, consider the operators \(P_s : C(s, T, V) \to C(s, T, V), s \in [0, T]\), defined for each \(h \in C(s, T, V)\) and \(\tau \in [s, T] \) by

\[
(P_s h)(\tau) := \int_s^\tau e^{-(\tau-r)A(\tau)}(A(\tau) - A(r))h(r)dr.
\]

Let \(c_0 > 0\) be arbitrary and fixed. Then replacing \(A(t)\) with \(A(t) + \mu\) for \(\mu \in \mathbb{R}\) we may assume without loss of generality that

\[
\|A(t)^{-\frac{1}{2}}\|_{L(V)} \leq c_0, \quad \text{for all } t \in [0, T].
\]

With this notations we have the following first result.

**Lemma 2.2.** The operator \(P_s\) is well defined and \(\|P_s\|_{L(C(s, T, V))} < \frac{1}{T}\) by choosing \(c_0 > 0\) in (22) to be small enough.

This Lemma has been proved in [10, Step.3 of the proof of Theorem 4.4]. For completeness we will carry out the proof.

**Proof.** (of Lemma 2.2) For each \(s \in [0, T]\), \(P_s\) is well defined thanks to [10, Lemma 4.5] and the Lebesgue domination theorem. Moreover, since (7), Proposition 1.2-(9) imply

\[
\|A^{1/2}(t_2)e^{-(t_2-t_1)A(t_2)}(A(t_2) - A(t_1))h(t_1)x\|_V \\
=\|A^{1/2}(t_2)e^{-(t_2-t_1)A(t_2)}e^{-(t_2-t_1)A(\tau)}(A(t_2) - A(t_1))h(\tau)x\|_V \\
\leq c\frac{\omega(t_2-t_1)}{(t_2-t_1)^{1+\frac{s}{2}}}\|x\|_V
\]

for all \(t_1, t_2 \in [0, T]\) with \(t_1 < t_2\), it follows

\[
\|P_s h\|_\infty \leq c\|A^{-1/2}(\tau)\|_{L(V)}\left(\int_0^T \frac{\omega(s)}{s^{1+\frac{s}{2}}}ds\right)\|h\|_\infty \leq cc_0\|h\|_\infty
\]

The last inequality above holds thanks to (6). Choosing then \(c_0 > 0\) small enough we obtain that \(\|P_s\|_{L(C(s, T, V))} < \frac{1}{T}\). \(\square\)

**Lemma 2.3.** Assume \(a : [0, T] \times V \times V \to \mathbb{C}\) satisfies (2)-(7) and \(D(A(0)^{1/2}) = V\). Then

\[
\|U_1(t, s) - U_1(t', s)\|_{L(V)} \leq c\left[\log \left(\frac{t'-s}{t-s}\right) + \omega(|t-t'|)\right]
\]

holds for each \(s \in [0, T]\) and all \(0 \leq s < t \leq t' \leq T\).
Proof. Let $x \in V$ and $0 \leq s < t \leq t' \leq T$. Because of assumptions (2)-(7) we have $D(\mathcal{A}(t)^{1/2}) = V$ for all $t \in [0, T]$ [10, Proposition 2.5]. Then we deduce from (7), Lemma 1.5 and Proposition 1.2.(10) that

$$\|e^{-(t-s)\mathcal{A}(t)}x - e^{-(t'-s)\mathcal{A}(t')}x\|_V \leq \int_{t-s}^{t'-s} \|\mathcal{A}(t)e^{-r\mathcal{A}(t)}x\|_V \leq \sigma^{-1} \int_{t-s}^{t'-s} \|\mathcal{A}^{1/2}(t)e^{-r\mathcal{A}(t)}x\|_H dr \leq \sigma^{-1} \int_{t-s}^{t'-s} \|\mathcal{A}(t)e^{-r\mathcal{A}(t)\mathcal{A}^{1/2}(t)}x\|_H dr \leq c \sigma^{-1} \int_{t-s}^{t'-s} \frac{1}{r} \|\mathcal{A}^{1/2}(t)x\|_H dr \leq c \sigma^{-1} \int_{t-s}^{t'-s} \frac{1}{r} dr \|x\|_V = c \log \left(\frac{t'-s}{t-s}\right) \|x\|_V.$$

On the other hand, since

$$e^{-(t'-s)\mathcal{A}(t)}x - e^{-(t'-s)\mathcal{A}(t')}x = \frac{1}{2\pi i} \int_{\gamma} e^{-(t'-s)\lambda} (\mathcal{A}(t) - \mathcal{A}(t'))^{-1} (\mathcal{A}(t) - \mathcal{A}(t')) (\mathcal{A}(t) - \mathcal{A}(t'))^{-1} x d\lambda.$$ 

by (4), we deduce from assumption (7) and Proposition 1.2.(6) that

$$\|e^{-(t'-s)\mathcal{A}(t)}x - e^{-(t'-s)\mathcal{A}(t')}x\|_V \leq c \int_{\gamma} e^{-(t'-s)\lambda} \mathcal{R}e \frac{\omega([t - t'])}{(1 + \lambda)^{\frac{3}{2}}} d\lambda \|x\|_V = c \omega([t - t']) \int_{0}^{\infty} \frac{1}{(1 + r)^{\frac{3}{2}}} dr \|x\|_V \leq c \omega([t - t']) \int_{0}^{\infty} \frac{1}{(1 + r)^{\frac{3}{2}}} dr \|x\|_V.$$ 

Now (24) follows by the triangle inequality. \hfill \Box

Now we arrive to our crucial technical Lemma.

**Lemma 2.4.** For all $0 \leq s < t \leq t' \leq T$, $h \in C(s, T, V)$ and $0 < \delta_0 < t - s$ we have

$$\|(P_{s\delta})(t') - (P_{s\delta})(t)\|_V \leq c \left[\kappa_s(t' - t) + \int_{0}^{\delta_0} \frac{\omega(\sigma)}{\sigma^{1/2}} d\sigma\right] \|h\|_{\infty}, \tag{25}$$

where $\kappa_s : [0, T] \to \mathbb{R}_+$ with $\kappa_s(r) \to 0$ as $r \to 0$.

**Proof.** We write $(P_{s\delta})(t') - (P_{s\delta})(t)$ as follows

$$(P_{s\delta})(t') - (P_{s\delta})(t) = \int_{0}^{t'-s} e^{-(t'-s-r)\mathcal{A}(t')} (\mathcal{A}(t') - \mathcal{A}(r + s)) h(r + s) dr - \int_{0}^{t'-s} e^{-(t'-s-r)\mathcal{A}(t)} (\mathcal{A}(t) - \mathcal{A}(r + s)) h(r + s) dr \tag{26}$$

$$= \int_{0}^{t'-s-\delta_0} e^{-(t'-s-r)\mathcal{A}(t')} (\mathcal{A}(t') - \mathcal{A}(r + s)) h(r + s) dr - \int_{0}^{t'-s-\delta_0} e^{-(t-r)\mathcal{A}(t)} (\mathcal{A}(t) - \mathcal{A}(r + s)) h(r + s) dr + \int_{t'-s-\delta_0}^{t'-s} e^{-(t'-s-r)\mathcal{A}(t')} (\mathcal{A}(t') - \mathcal{A}(r + s)) h(r + s) dr - \int_{t'-s-\delta_0}^{t'-s} e^{-(t-s-r)\mathcal{A}(t)} (\mathcal{A}(t) - \mathcal{A}(r + s)) h(r + s) dr. \tag{27}$$
Then a similar argument as in the proof of Lemma 2.2 one obtain

\[
\| \int_{t'-s}^{t'-s-r} e^{-(t'-s-r)A(t')}(A(t') - A(r+s))h(r+s)dr \|_V \leq c \int_{t'-s}^{t'-s-r} \frac{\omega(t' - s - r)}{(t' - s - r)^{1+\frac{\gamma}{2}}} dr \\
= \|h\|_\infty \int_0^{\delta_0} \frac{\omega(r)}{r^{1+\frac{\gamma}{2}}} dr,
\]

and

\[
\| \int_{t'-s}^{t'-s-r} e^{-(t'-s-r)A(t)}(A(t) - A(r+s))h(r+s)dr \|_V \leq c \|h\|_\infty \int_0^{\delta_0} \frac{\omega(r)}{r^{1+\frac{\gamma}{2}}} dr.
\]

It remains to treat integral terms in (26)

\[
I_{3,\delta_0} := \int_{0}^{t'-s-\delta_0} e^{-(t'-r-s)A(t')}(A(t') - A(r+s))h(r+s)dr \\
- \int_{0}^{t'-s-\delta_0} e^{-(t'-r-s)A(t)}(A(t) - A(r+s))h(r+s)dr \\
= \int_{0}^{t'-s-\delta_0} \left[ e^{-(t'-r-s)A(t')} - e^{-(t'-r-s)A(t)} \right] (A(t') - A(r+s))h(r+s)dr \\
\quad + \int_{0}^{t'-s-\delta_0} e^{-(t'-r-s)A(t')} (A(t') - A(t))h(r+s)dr \\
\quad + \int_{t'-s-\delta_0}^{t'-s-r} e^{-(t'-r-s)A(t')}(A(t') - A(r+s))h(r+s)dr.
\]

We denote the integral terms in (30), (31), (32) and (33) by $I_{3,\delta_0,1}, I_{3,\delta_0,2}, I_{3,\delta_0,3}$ and $I_{3,\delta_0,4}$ respectively. Then, Lemma 1.5 and the Proposition 1.2(7) imply that

\[
\|A(t')e^{-rA(t')}z\|_V \leq \|A(t')A(t')e^{-rA(t')}z\|_H = \sigma \|A(t')e^{-\frac{\omega}{2}A(t')}A(t')e^{-\frac{\omega}{2}A(t')}z\|_H \\
\leq \delta \frac{1}{r} \|A(t')e^{-\frac{\omega}{2}A(t')}z\|_H \\
\leq \delta \frac{1}{r} \|e^{-\frac{\omega}{2}A(t')}z\|_V \leq \frac{c}{r^{1+\frac{\gamma}{2}}} \|z\|_V,
\]

for each $z \in V'$ and $r > 0$. Therefore,

\[
\|I_{3,\delta_0,1}\|_V = \| \int_{0}^{t'-s-\delta_0} \int_{t'-s}^{t'-r-s} \|A(t')e^{-\sigma A(t')} (A(t') - A(t))h(r+s)dr\|_V \\
\leq 2M\|h\|_\infty \int_0^{t'-s-\delta_0} \int_{t'-s}^{t'-r-s} \|A(t')e^{-\sigma A(t')}\|_{L(V',V')} d\sigma dr \\
\leq \|h\|_\infty \int_0^{t'-s-\delta_0} \int_{t'-s}^{t'-r-s} \frac{1}{\sigma^2} d\sigma dr \\
\leq c\|h\|_\infty \left[ \log \left( \frac{t'-t}{\delta_0} + 1 \right) + \log \left( \frac{t'-s}{t'-s} \right) \right]
\]

(34)
Next, by (7) and Proposition 1.2-(9) and since $\omega : [0, T] \to [0, \infty)$ is bounded
\[\begin{align*}
\|I_{3, \delta_0, 3}\| V &\leq c\omega(t' - t)\|h\|_\infty \int_0^{t-s-\delta_0} (t' - r - s) - \frac{\delta_0}{2} \, dr \\
&= c\omega(t' - t)\|h\|_\infty \left[ (t - s)^{1/2} - \delta_0 \right] \\
&\leq c\omega(t' - t)\|h\|_\infty \left[ T^{1/2} \right] \leq c\omega(t' - t)\|h\|_\infty.
\end{align*}\]
(36)

and
\[\begin{align*}
\|I_{3, \delta_0, 4}\| V &\leq c\|h\|_\infty \int_{t-s-\delta_0}^{t-s-\delta_0} \frac{\omega(t' - r - s)}{(t' - r - s)^{1/2}} \, dr \\
&\leq c\|h\|_\infty \sup_{t \in [0, T]} \frac{\omega(t)}{\tau^{1/2}} \int_0^{t-s-\delta_0} \frac{1}{(t' - r - s)^{1/2}} \, dr \\
&\leq c\|h\|_\infty \left[ (t' - t + \delta_0)^{1/2} - \delta_0^{1/2} \right].
\end{align*}\]
(37)

In the last inequality we have used that $\sup_{t \in [0, T]} \frac{\omega(t)}{\tau^{1/2}} < \infty$. Thus we conclude that
\[\begin{align*}
\|I_{3, \delta_0}\| V &\leq c\left[ \omega(t' - t) + (t' - t + \delta_0)^{1/2} - \delta_0^{1/2} + \log \left( \frac{t' - t}{\delta_0} + 1 \right) + \log \left( \frac{t - s}{\delta_0} + 1 \right) \right].
\end{align*}\]
(38)

This estimates together with (28) and (29) proof the desired inequality. \hfill \Box

**Proposition 2.5.** Let $\{ U(t, s) \mid (t, s) \in \Delta \} \subset \mathcal{L}(V)$ be defined by (17). Then for each fixed $s_0 \in [0, T)$ the function
\[ t \mapsto U(t, s_0) \]
is norm continuous on $(s_0, T]$ into $\mathcal{L}(V)$.

*Proof.* Let $s_0 \in [0, T)$. Lemma 2.3 implies that $I - P_{s_0}$ is invertible on $\mathcal{L}(C(s_0, T, V))$ by the Neumann series. Therefore, $R_n \to 0$ as $n \to \infty$ in $\mathcal{L}(C(s_0, T, V))$ where
\[ R_n = \sum_{k=n+1}^{\infty} P_{s_0}^k, \quad n \in \mathbb{N} \]
and, thanks to representation formula (21),
\[ U(\cdot, s_0)x = (I - P_{s_0})^{-1}U_1(\cdot, s_0)x = \sum_{k=0}^{\infty} P_{s_0}^k U_1(\cdot, s_0)x \]
(39)

Next, proceeding by induction, we see that (25) in Lemma 2.4 holds if we replace $P_{s_0}$ by $P_{s_0}^k$ for each $k \in \mathbb{N} \setminus \{0\}$ since $\|P_{s_0}\|_{\mathcal{L}(C(s_0, T, V))} < 1/4$. The case $k = 0$ is treated in Lemma 2.3.

Finally, since
\[ \|U_1(t, s_0)x\|_V = \|e^{-(t-s_0)A(t)}x\|_V \leq c\|x\|_V, \]
(40)
and \(\kappa_{s_0}(t-t')\) in (25) converges to 0 as \(t \to t'\) and taking into account Remark 1.1, the proof follows by a 3-\(\varepsilon\)-argument. \(\square\)

In the following proposition, we will proves that the mapping \(U(\cdot, \cdot)\) is also norm continuous on the second variable.

**Proposition 2.6.** Let \(\{U(t, s) \mid (t, s) \in \Delta\} \subset \mathcal{L}(V)\) be defined by (17). Then for each fixed \(t_0 \in (0, T]\) the function
\[
s \mapsto U(t_0, s)
\]
is norm continuous on \([0, t_0]\) into \(\mathcal{L}(V)\).

**Proof.** The proof is an easy consequence of the representation formula (39). Indeed, let \(0 \leq s_0 \leq s < t \leq T\) and \(x \in V\). Then (5), (7) and Proposition 1.2-(9) imply
\[
\|P_s U_1(\cdot, s)x - P_{s_0} U_1(\cdot, s_0)x\|_V \leq \int_{s_0}^{s} \|e^{-(t-r)\mathcal{A}(t)}(\mathcal{A}(t) - \mathcal{A}(r))U_1(r, s_0)x\|_V \, dr
\]
\[
\leq c \left[(t-s)^{1/2} - (t-s_0)^{1/2}\right]\|x\|_V.
\]
On the other hand, similarly as in the first part of the proof of Lemma 2.3 we obtain
\[
\|U_1(t, s) - U_1(t, s_0)\|_{\mathcal{L}(V)} \leq c \log \left(\frac{t-s}{t-s_0}\right).
\]
Finally (39) implies
\[
U(\cdot, s) - U(\cdot, s_0) = (I - P_s)^{-1}U_1(\cdot, s) - U_1(\cdot, s_0)\ + [(I - P_s)^{-1}(P_s - P_{s_0})(I - P_{s_0})^{-1}U_1(\cdot, s_0)
\]
from which the claim follows. \(\square\)

Combining Proposition 2.5 and Proposition 2.6, we conclude that \(\{U(t, s) \mid (t, s) \in \Delta\}\) is norm continuous with value in \(\mathcal{L}(V)\). In Section 3 below we will see that \(\mathcal{U}\) is also norm continuous with value in \(\mathcal{L}(H)\) provided that \(V\) is compactly embedded in \(H\).

**Theorem 2.7.** Assume that a satisfies (2)-(7) and \(D(\mathcal{A}(0))^{1/2} = V\). Let \(\{U(t, s) : (t, s) \in \Delta\}\) given by (20). Then the function
\[
(t, s) \mapsto U(t, s)
\]
is a norm continuous on \(\Delta\) into \(\mathcal{L}(V)\).

**Proof.** Due to the evolution law \(U(t, s) = U(t, r)U(r, s)\) it suffices to proof that \(\{U(t, s) : (t, s) \in \Delta\}\) is bounded on \(\mathcal{L}(V)\), the norm continuity follows then from Proposition 2.5 and Proposition 2.6. In the proof of Proposition 2.5 we have seen that \(U(t, s)\) is given by the Neumann series (39). This representation together with (40) and the fact that \(\|P_s\|_{\mathcal{L}(C(s, T, V))} \leq 1/4\) for all \(0 \leq s < T\) imply the claim. \(\square\)

### 3. Compact and Gibbs Evolution Family

Throughout this section we adopt the notations and assumptions of Section 2. In this section we provide a further regularity properties of the evolution family \(\{\mathcal{U}(t, s) \mid (t, s) \in \Delta\}\). To this end we will introduce some definitions.

**Definition 3.1.** The family \(\{\mathcal{U}(t, s) \mid (t, s) \in \Delta\}\) is said to be a **compact evolution family** on a Hilbert space \(X \subseteq H\) if the operator \(\mathcal{U}(t, s) \in \mathcal{K}(X)\) for every \((t, s) \in \Delta\) where \(\mathcal{K}(X)\) denotes the space of compact operators on \(X\).

For strongly continuous semigroups, it is well known that compactness is a sufficient condition for norm continuity [23, Lemma 4.22]. It is easy to verify that this is also true for strongly continuous evolution families. In the following we show that \(\mathcal{U}(t, s) \in \mathcal{K}(H)\) and \(\mathcal{U}(t, s) \in \mathcal{K}(V)\) for every \((t, s) \in \Delta\) whenever \(V\) is compactly embedded in \(H\). This would implies, in particular, that \((t, s) \mapsto \mathcal{U}(t, s)\) is norm continuous on \(\Delta\) with value in \(\mathcal{L}(H)\).

The following Lemma is essential for the results of this section.
Lemma 3.2. Under the assumptions of Theorem 2.7, we have \( U(t, s) \in \mathcal{L}(H, V) \) for every \((t, s) \in \Delta\).

Proof. We first proof that \( U(t, s) \) maps \( H \) into \( V \) for every \((t, s) \in \Delta\). Since \( U(\cdot) \) provides the solution of (16), there exists a null set \( N \subset [0, T] \) such that \( U(t, s)x \in V \) for every \( s \in [0, T] \) and each \( t \in [s, T] \setminus N \). Let \( x \in H \) and \((t, s) \in \Delta \) be arbitrary and fixed. Choose \( t_0 \in [s, T] \setminus N \) with \( t_0 < t \). Then we obtain that

\[
U(t, s)x = U(t, t_0)U(t_0, s)x
\]

belongs to \( V \) since \( U(t_0, s)x \in V \) and \( U(\cdot, \cdot)V \subset V \). The boundedness of \( U \) from \( H \) to \( V \) follows by Banach Steinhaus Theorem using (44) and the strong continuity of \( U(\cdot, s) : [s, T] \rightarrow V \).

\[\square\]

Theorem 3.3. Assume that a satisfies (2)-(7) and \( D(A(0)^{1/2}) = V \). If \( V \) is compactly embedded in \( H \) then the evolution family \( \{U(t, s) \mid (t, s) \in \Delta\} \) is compact on \( H \), respectively on \( V \). In particular, \( \{U(t, s) \mid (t, s) \in \Delta\} \) is then norm continuous on \( H \).

Proof. The first assertion follows from the evolution family law \((ii)\) and Lemma 3.2. The last one follows from the remarks above.

Finally, we introduce the concept of Gibbs evolution family. For a separable Hilbert space \( X \) and \( p \in [1, \infty) \) the \( p \)-Schatten class operators space \( S_p(X) \) is given by

\[
S_p(X) := \left\{ T \in \mathcal{K}(X) \mid \|T\|_{S_p} := \|(s_n)_{n \in \mathbb{N}}\|_{\mathcal{P}(\mathbb{N})} < \infty \right\},
\]

where \((s_n)_{n \in \mathbb{N}}\) is the sequence of singular values of \( T \), that is the sequence of eigenvalue of \( |T| := (TT^*)^{1/2} \). This classes of operators have been introduced by Robert Schatten and John von Neumann [35] and are also known in the literature as von Neumann-Schatten classes. The function \( \| \cdot \|_{S_p} \) is a norm on \( S_p(X) \) called \( p \)-Schatten norm and, \( (S_p, \| \cdot \|_p) \) is a Banach space. For \( p = 1 \) we obtain the well known trace class and the Hilbert-Schmidt operators for \( p = 2 \). The \( S_p(X) \) is a \(*\)-ideal in \( \mathcal{L}(X) \), i.e., if \( T \in S_p(X) \) and \( S \in \mathcal{L}(X) \) then \( T^* \in S_p(X), TS \in S_p(X) \) and \( ST \in S_p(X) \). Moreover,

\[
S_1(X) \subseteq S_p(X) \subseteq S_q(X) \subseteq \mathcal{K}(X)
\]

for every \( 1 \leq p \leq q < \infty \) since \( p \mapsto \|T\|_{S_p} \) is non-increasing. Further, applying Hölder inequality to \( \| \cdot \|_{S_p} \), one obtain that for \( T \in S_p(X) \) and \( S \in \mathcal{S}(X)_q \), we have \( TS \in S_r(X) \) for \( r^{-1} = p^{-1} + q^{-1} \). For all this and more details on \( p \)-Schatten class we refer e.g. to [36, 41].

The following definition generalizes the concept of Gibbs semigroups introduced by Dietrich A. Uhlenbrock [40] (see also [41]) to the non-autonomous situation.

Definition 3.4. Let \( X \) be separable Hilbert space. An evolution family \( \{U(t, s) \mid (t, s) \in \Delta\} \) is said to be a Gibbs evolution family on \( X \) if each \((t, s) \in \Delta \) the operator \( U(t, s) \) is of trace class.

In view of the properties of \( S_p(X) \) and the evolution family law, any evolution family \( U(t, s) \) for which there exists \( p \in (1, \infty) \) such that \( U(t, s) \in S_p(X) \) for every \((t, s) \in \Delta\) is a Gibbs evolution family.

Theorem 3.5. Assume that a satisfies (2)-(7) and \( D(A(0)^{1/2}) = V \). If the embedding \( V \subset H \) is of \( p \)-Schatten class for some \( p \in (1, \infty) \) then \( \{U(t, s) \mid (t, s) \in \Delta\} \) is a Gibbs evolution family on \( H \) and \( V \), respectively.

Proof. From Lemma 3.2 we have that \( U(t, s) \in \mathcal{L}(H, V) \) for each \((t, s) \in \Delta\). The assertion follows then thanks to the ideal property of \( S_p(X) \).

\[\square\]

4. THE LAPLACIAN WITH TIMES DEPENDENT ROBIN BOUNDARY CONDITIONS

Let \( \Omega \) be a bounded domain of \( \mathbb{R}^N \) with Lipschitz boundary \( \Gamma \). Denote by \( \sigma \) the \((d-1)\)-dimensional Hausdorff measure on \( \Gamma \). Let \( T > 0 \) and \( \alpha > 1/4 \). Let

\[
\beta : [0, T] \times \Gamma \rightarrow \mathbb{R}
\]

be a bounded measurable function such that

\[
|\beta(t, x) - \beta(t, x)| \leq c|t - s|^\alpha
\]
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for some constant \( c > 0 \) and every \( t, s \in [0, T], x \in \Gamma \). We consider the from \( a : [0, T] \times V \times V \rightarrow \mathbb{C} \) defined by

\[
a(t; u, v) := \int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Gamma} \beta(t, \cdot) u \nu \cdot v \, d\sigma
\]

where \( u \rightarrow u_{1\Gamma} : H^1(\Omega) \rightarrow L^2(\Gamma, \sigma) \) is the trace operator. Fix \( r_0 \in (0, 1/2) \) such that \( r_0 + 1/2 < 2\alpha \).

Then the form \( a \) satisfies (7) with \( \gamma := r_0 + 1/2 \) and \( \omega(t) = t^\gamma \) since the trace operators is bounded from \( H^\gamma(\Omega) \) with value in \( H^{r_0}(\Gamma) \). The operator \( A(t) \) associated with \( a(t; \cdot, \cdot) \) on \( H := L^2(\Omega) \) is minus the Laplacian with time dependent Robin boundary conditions

\[
\partial_{\nu} u(t) + \beta(t, \cdot) u = 0 \quad \text{on} \ \Gamma.
\]

Here \( \partial_{\nu} \) is the weak normal derivative: let \( v \in H^1(\Omega) \) such that \( \Delta u \in L^2(\Omega) \) then for \( h \in L^2(\Gamma, \sigma) \) we have \( \partial_{\nu} u := h \) if and only if \( \int_{\Omega} \nabla v \cdot \nabla w \, dx + \int_{\Gamma} \Delta v \cdot w \, d\sigma = \int_{\Gamma} hw \, d\sigma \) for all \( w \in H^1(\Omega) \). Thus the domain of \( A(t) \) is the set

\[
D(\Lambda(t)) = \left\{ u \in H^1(\Omega) \mid \Delta u \in L^2(\Omega), \partial_{\nu} u(t) + \beta(t, \cdot) u_{1\Gamma} = 0 \right\}
\]

and for \( u \in D(\Lambda(t)) \), \( \Lambda(t) u := -\Delta u \). By Theorem 3.3 the non-autonomous Cauchy problem

\[
\begin{aligned}
\dot{u}(t) - \Delta u(t) &= f(t), \ u(0) \in H^1(\Omega) \\
\partial_{\nu} u(t) + \beta(t, \cdot) u &= 0 \quad \text{on} \ \Gamma
\end{aligned}
\]

has \( L^2 \)-maximal regularity in \( H \) and is governed by a compact and thus norm continuous evolution family \( U_{\Omega}(\cdot, \cdot) \) since the embedding of \( H^1(\Omega) \subset L^2(\Omega) \) is compact. Moreover, this embedding is also of \( p \)-Schatten class for all \( p > N \). We deduce then from Theorem 3.5 that \( U_{\Omega}(\cdot, \cdot) \) is in addition a Gibbs evolution family.

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