Strong Differential Sandwich Results of $\lambda$-Pseudo-Starlike Functions with Respect to Symmetrical Points

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Abstract. In the present investigation, by considering suitable classes of admissible functions, we establish strong differential subordination and superordination properties for $\lambda$-pseudo-starlike functions with respect to symmetrical points in the open unit disk $U$. These results are applied to obtain strong differential sandwich results.

1. Introduction and Preliminaries

Let $\mathcal{H}(U)$ be the class of analytic functions in the open unit disk $U = \{ z \in \mathbb{C} : |z| < 1 \}$. For a positive integer $n$ and $a \in \mathbb{C}$, let $\mathcal{H}[a,n]$ be the subclass of $\mathcal{H}(U)$ consisting of functions of the form:

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \cdots,$$

with $\mathcal{H} = \mathcal{H}[1,1]$.

Let $\mathcal{A}$ stand for the class of all analytic functions in $U$ and having the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in U),$$

A function $f \in \mathcal{A}$ is called starlike with respect to symmetrical points, if (see [10])

$$\text{Re} \left\{ \frac{zf'(z)}{f(z) - f(-z)} \right\} > 0, \quad z \in U.$$

The set of all such functions is denote by $S_s^*$.

Recently, Babalola [2] defined the class $\mathcal{L}_\lambda$ of $\lambda$-pseudo-starlike functions as follows:

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Let $f \in \mathcal{A}$ and $\lambda \geq 1$. Then $f \in \mathcal{L}_\lambda$ of $\lambda$-pseudo-starlike functions in $U$ if and only if
\[
\Re \left\{ \frac{z (f'(z))^\lambda}{f(z)} \right\} > 0, \quad z \in U.
\]

A function $f \in \mathcal{A}$ is called $\lambda$-pseudo-starlike with respect to symmetrical points, if
\[
\Re \left\{ \frac{z (f'(z))^\lambda}{f(z) - f(-z)} \right\} > 0, \quad z \in U.
\]

We denote by $\mathcal{L}_{\lambda,s}^*$ the family of all $\lambda$-pseudo-starlike functions with respect to symmetrical points.

Let $f$ and $g$ be members of $\mathcal{H}(U)$. The function $f$ is said to be subordinate to $g$, or (equivalently) $g$ is said to be superordinate to $f$, if there exists a Schwarz function $w$ which is analytic in $U$ with $w(0) = 0$ and $|w(z)| < 1(z \in U)$ such that $f(z) = g(w(z))$. In such a case, we write $f \prec g$ or $f(z) \prec g(z)(z \in U)$. Furthermore, if the function $g$ is univalent in $U$, then we have the following equivalent (see [5])
\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

Let $G(z, \zeta)$ be analytic in $U \times \bar{U}$ and let $f(z)$ be analytic and univalent in $U$. Then the function $G(z, \zeta)$ is said to be strongly subordinate to $f(z)$ or $f(z)$ is said to be strongly superordinate to $G(z, \zeta)$, written as $G(z, \zeta) \prec \prec f(z)$, if for $\zeta \in \bar{U} = \{ z \in \mathbb{C} : |z| \leq 1 \}$, $G(z, \zeta)$ as a function of $z$ is subordinate to $f(z)$. We note that
\[
G(z, \zeta) \prec \prec f(z) \iff G(0, \zeta) = f(0) \quad \text{and} \quad G(U \times \bar{U}) \subset f(U).
\]

**Definition 1.1** ([6]). Let $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$ and let $h$ be a univalent function in $U$. If $p$ is analytic in $U$ and satisfies the following (second-order) strong differential subordination:
\[
(1) \quad \phi \left( p(z), zp'(z), z^2p''(z); z, \zeta \right) \prec h(z),
\]
then $p$ is called a solution of the strong differential subordination (1). The univalent function $q$ is called a dominant of the solutions of the strong differential subordination or more simply a dominant if $p(z) \prec q(z)$ for all $p$ satisfying (1). A dominant $\tilde{q}$ that satisfies $\tilde{q}(z) \prec q(z)$ for all dominants $q$ of (1) is said to be the best dominant.

**Definition 1.2** ([7]). Let $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$ and let $h$ be analytic function in $U$. If $p$ and $\phi \left( p(z), zp'(z), z^2p''(z); z, \zeta \right)$ are univalent in $U$ for $\zeta \in \bar{U}$ and satisfy the following (second-order) strong differential superordination:
\[
(2) \quad h(z) \prec \prec \phi \left( p(z), zp'(z), z^2p''(z); z, \zeta \right),
\]
then $p$ is called a solution of the strong differential superordination (2). An analytic function $q$ is called a subordinant of the solutions of the strong differential superordination or more simply a subordinant if $q(z) \prec p(z)$ for
all \( p \) satisfying (2). A univalent subordinant \( \tilde{q} \) that satisfies \( q(z) \prec \tilde{q}(z) \) for all subordinants \( q \) of (2) is said to be the best subordinant.

**Definition 1.3** ([6]). Denote by \( Q \) the set consisting of all functions \( q \) that are analytic and injective on \( \tilde{U} \setminus E(q) \), where

\[
E(q) = \left\{ \xi \in \partial U : \lim_{z \to \xi} q(z) = \infty \right\},
\]

and are such that \( q'(\xi) \neq 0 \) for \( \xi \in \partial U \setminus E(q) \).

Furthermore, let the subclass of \( Q \) for which \( q(0) = a \) be denoted by \( Q(a) \), \( Q(0) \equiv Q_0 \) and \( Q(1) \equiv Q_1 \).

**Definition 1.4** ([8]). Let \( \Omega \) be a set in \( \mathbb{C} \), \( q \in Q \) and \( n \in \mathbb{N} \). The class of admissible functions \( \Psi_n[\Omega,q] \) consists of those functions \( \psi : \mathbb{C}^3 \times U \times \tilde{U} \rightarrow \mathbb{C} \) that satisfy the following admissibility condition: \( \psi(r,s,t;z,\zeta) \notin \Omega \), whenever

\[
r = q(\xi), \quad s = kq'(\xi) \quad \text{and} \quad \Re \left\{ \frac{t}{s} + 1 \right\} \geq k \Re \left\{ \frac{\xi q''(\xi)}{q'(\xi) + 1} \right\},
\]

\( z \in U, \: \xi \in \partial U \setminus E(q), \: \zeta \in \tilde{U} \) and \( k \geq n \).

We simply write \( \Psi_1[\Omega,q] = \Psi[\Omega,q] \).

**Definition 1.5** ([8]). Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in \mathcal{H}[a,n] \) with \( q'(z) \neq 0 \). The class of admissible functions \( \Psi'_n[\Omega,q] \) consists of those functions \( \psi : \mathbb{C}^3 \times U \times \tilde{U} \rightarrow \mathbb{C} \) that satisfy the following admissibility condition: \( \psi(r,s,t;\xi,\zeta) \in \Omega \), whenever

\[
r = q(z), \quad s = \frac{zq'(z)}{m} \quad \text{and} \quad \Re \left\{ \frac{t}{s} + 1 \right\} \leq \frac{1}{m} \Re \left\{ \frac{zq''(z)}{q'(z) + 1} \right\},
\]

\( z \in U, \: \xi \in \partial U, \: \zeta \in \tilde{U} \) and \( m \geq n \geq 1 \).

In particular, we write \( \Psi'_1[\Omega,q] = \Psi'[\Omega,q] \).

In our investigations, we will need the following lemmas:

**Lemma 1.1** ([9]). Let \( \psi \in \Psi_n[\Omega,q] \) with \( q(0) = a \). If \( p \in \mathcal{H}[a,n] \) satisfies

\[
\psi \left( p(z), zp'(z), z^2p''(z); z, \zeta \right) \in \Omega,
\]

then \( p(z) \prec q(z) \).

**Lemma 1.2** ([8]). Let \( \psi \in \Psi'_n[\Omega,q] \) with \( q(0) = a \). If \( p \in Q(a) \) and \( \psi \left( p(z), zp'(z), z^2p''(z); z, \zeta \right) \) is univalent in \( U \) for \( \zeta \in \tilde{U} \), then

\[
\Omega \subset \left\{ \psi \left( p(z), zp'(z), z^2p''(z); z, \zeta \right) : z \in U, \zeta \in \tilde{U} \right\}
\]

implies \( q(z) \prec p(z) \).

In recent years, several authors obtained many interesting results in strong differential subordination and superordination [1, 3, 4, 11, 12, 13]. In this work, by making use of the strong differential subordination results and
strong differential superordination results of Oros and Oros [8, 9], we introduce and study certain suitable classes of admissible functions and derive some strong differential subordination and superordination properties of \( \lambda \)-pseudo-starlike functions with respect to symmetrical points.

2. Strong Subordination Results

**Definition 2.1.** Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in Q_1 \cap \mathcal{H} \). The class of admissible functions \( \Phi_L [\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C} \) that satisfy the admissibility condition: \( \phi(u, v, w; z, \zeta) \notin \Omega \), whenever

\[
u = q(\xi), \quad \nu = \frac{k \xi q'(\xi)}{q(\xi)}, \quad q(\xi) \neq 0 \quad \text{and} \quad \Re \left\{ \frac{w + v^2}{u} \right\} \geq k \Re \left\{ \frac{\xi q''(\xi)}{q'(\xi)} + 1 \right\},
\]

where \( z \in U, \ \zeta \in \bar{U}, \ \zeta \in \partial U \setminus E(q) \) and \( k \geq 1 \).

**Theorem 2.1.** Let \( \phi \in \Phi_L [\Omega, q] \). If \( f \in \mathcal{A} \) satisfies

\[
\begin{align*}
\left\{ \phi \left( \frac{2z (f'(z))^\lambda}{f(z) - f(-z)}, 1 + \frac{\lambda z f''(z)}{f'(z)} - \frac{z (f(z) - f(-z))'}{f(z) - f(-z)}, \frac{\lambda z^2 f'''(z)}{f'(z)} \right) + \frac{\lambda z f''(z)}{f'(z)} \left( 1 - \frac{z f''(z)}{f'(z)} \right) & - \frac{z^2 (f(z) - f(-z))''}{f(z) - f(-z)} + \frac{z (f(z) - f(-z))'}{f(z) - f(-z)} \\
\times \left( \frac{z (f(z) - f(-z))'}{f(z) - f(-z)} - 1 \right) ; z, \zeta \right\} : z \in U, \ \zeta \in \bar{U} \right\} \subset \Omega,
\end{align*}
\]

then

\[
\frac{2z (f'(z))^\lambda}{f(z) - f(-z)} < q(z).
\]

**Proof.** We define the function \( p \) by

\[
p(z) = \frac{2z (f'(z))^\lambda}{f(z) - f(-z)}.
\]

It is clear that \( p \) is analytic in \( U \).

Simple calculations from (4), we obtain

\[
z p'(z) p(z) = 1 + \frac{\lambda z f''(z)}{f'(z)} - \frac{z (f(z) - f(-z))'}{f(z) - f(-z)}.
\]

Further computations show that

\[
\begin{align*}
\frac{z^2 p''(z)}{p(z)} + \frac{z p'(z)}{p(z)} - \left( \frac{z p'(z)}{p(z)} \right)^2 &= z \left[ 1 + \frac{\lambda z f''(z)}{f'(z)} - \frac{z (f(z) - f(-z))'}{f(z) - f(-z)} \right]' \\
&= \frac{\lambda z^2 f'''(z)}{f'(z)} + \frac{\lambda z f''(z)}{f'(z)} \left( 1 - \frac{z f''(z)}{f'(z)} \right) - \frac{z^2 (f(z) - f(-z))''}{f(z) - f(-z)} \\
&+ \frac{z (f(z) - f(-z))'}{f(z) - f(-z)} \left( \frac{z (f(z) - f(-z))'}{f(z) - f(-z)} - 1 \right).
\end{align*}
\]
Define the transforms from $\mathbb{C}^3$ to $\mathbb{C}$ by
\[
u = r, \quad w = \frac{s}{r}, \quad w = \frac{r(t + s) - s^2}{r^2}.
\]
Let
\[
(7) \quad \psi(r, s, t; z, \zeta) = \phi(u, v, w; z, \zeta) = \phi \left( r, \frac{s}{r}, \frac{r(t + s) - s^2}{r^2}; z, \zeta \right).
\]

The proof shall make use of Lemma 1.1. Using equations (4), (5) and (6), it follows from (7) that
\[
\begin{align*}
\psi(p(z), zp'(z), z^2p''(z); z, \zeta) &= \phi \left( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)}, 1 + \frac{\lambda z f'''(z)}{f'(z)} - \frac{z(f(z) - f(-z))'}{f(z) - f(-z)}, \frac{\lambda z^2 f''(z)}{f'(z)} \right) \\
&\quad + \frac{\lambda z f'''(z)}{f'(z)} \left( 1 - \frac{z f''(z)}{f'(z)} \right) - \frac{z^2(f(z) - f(-z))''}{f(z) - f(-z)} + \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} \times \\
&\quad \times \left( \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} - 1 \right); z, \zeta \right).
\end{align*}
\]
Thus (3) becomes $\psi(p(z), zp'(z), z^2p''(z); z, \zeta) \in \Omega$.

To complete the proof, we next show that the admissibility condition for $\phi \in \Phi_L[\Omega, q]$ is equivalent to the admissibility condition for $\psi$ as given in Definition 1.4.

Note that
\[
\frac{t}{s} + 1 = \frac{w + v^2}{v}.
\]

Hence $\psi \in \Psi[\Omega, q]$. By Lemma 1.1, $p(z) \prec q(z)$ or equivalently
\[
\frac{2z(f'(z))^{\lambda}}{f(z) - f(-z)} \prec q(z).
\]

We consider the special situation when $\Omega \neq \mathbb{C}$ is a simply connected domain. In this case $\Omega = h(U)$, for some conformal mapping $h$ of $U$ onto $\Omega$ and the class $\Phi_L[h(U), q]$ is written as $\Phi_L[h, q]$. The following result is an immediate consequence of Theorem 2.1.

**Theorem 2.2.** Let $\phi \in \Phi_L[h, q]$. If $f \in \mathcal{A}$ satisfies
\[
\begin{align*}
\phi \left( \frac{2z(f'(z))^{\lambda}}{f(z) - f(-z)}, 1 + \frac{\lambda z f'''(z)}{f'(z)} - \frac{z(f(z) - f(-z))'}{f(z) - f(-z)}, \frac{\lambda z^2 f''(z)}{f'(z)} \right) &\quad + \frac{\lambda z f'''(z)}{f'(z)} \left( 1 - \frac{z f''(z)}{f'(z)} \right) - \frac{z^2(f(z) - f(-z))''}{f(z) - f(-z)} + \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} \times \\
&\quad \times \left( \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} - 1 \right); z, \zeta \right) \prec h(z),
\end{align*}
\]

\[
\square
\]
then
\[
\frac{2z(f'(z))^\lambda}{f(z) - f(-z)} < q(z).
\]

By taking \( \phi(u, v, w; z, \zeta) = u + \frac{v}{\beta u + \gamma}, (\beta, \gamma \in \mathbb{C}) \) in Theorem 2.2, we state the following corollary:

**Corollary 2.1.** Let \( \beta, \gamma \in \mathbb{C} \) and let \( h \) be convex in \( U \) with \( h(0) = 1 \) and \( \text{Re} \{\beta h(z) + \gamma\} > 0 \). If \( f \in \mathcal{A} \) satisfies
\[
2z(f'(z))^\lambda \left(1 + \frac{zf''(z)}{f'(z)} \right) (f(z) - f(-z)) - z(f(z) - f(-z))' - 2\beta z(f'(z))^\lambda + \gamma (f(z) - f(-z))' < h(z),
\]
then
\[
\frac{2z(f'(z))^\lambda}{f(z) - f(-z)} < q(z).
\]

The next result is an extension of Theorem 2.1 to the case where the behavior of \( q \) on \( \partial U \) is not known.

**Corollary 2.2.** Let \( \Omega \in \mathbb{C} \) and \( q \) be univalent in \( U \) with \( q(0) = 1 \). Let \( \phi \in \Phi_L [h, q_\rho] \) for some \( \rho \in (0, 1) \), where \( q_\rho(z) = q(\rho z) \). If \( f \in \mathcal{A} \) satisfies
\[
\phi \left( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)}, 1 + \frac{zf''(z)}{f'(z)} - z(f(z) - f(-z))' + \frac{\lambda z^2 f''(z)}{f'(z)}, 1 + \frac{zf''(z)}{f'(z)} - z^2(f(z) - f(-z))'' - z(f(z) - f(-z))' \right) \in \Omega,
\]
then
\[
\frac{2z(f'(z))^\lambda}{f(z) - f(-z)} < q(z).
\]

**Proof.** Theorem 2.1 yields \( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} < q_\rho(z) \). The result is now deduced from the fact that \( q_\rho(z) < q(z) \). \( \square \)

**Theorem 2.3.** Let \( h \) and \( q \) be univalent in \( U \) with \( q(0) = 1 \) and set \( q_\rho(z) = q(\rho z) \) and \( h_\rho(z) = h(\rho z) \). Let \( \phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C} \) satisfy one of the following conditions:

1. \( \phi \in \Phi_L [h, q_\rho] \) for some \( \rho \in (0, 1) \),
2. there exists \( \rho_0 \in (0, 1) \) such that \( \phi \in \Phi_L [h_\rho, q_\rho] \) for all \( \rho \in (\rho_0, 1) \).

If \( f \in \mathcal{A} \) satisfies (9), then
\[
\frac{2z(f'(z))^\lambda}{f(z) - f(-z)} < q(z).
\]
Proof. Case (1): By applying Theorem 2.1, we obtain
\[ \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} < q_\rho(z), \]
since \( q_\rho(z) \prec q(z) \), we deduce
\[ \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} < q(z). \]

Case (2): Let \( p(z) = \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} \) and \( q_\rho(z) = p(\rho z) \). Then
\[ \phi(p_\rho(z), zp'_\rho(z), z^2p''_\rho(z); \rho z, \zeta) = \phi(p(\rho z), zp'_\rho(z), z^2p''(\rho z); \rho z, \zeta) \in h_\rho(U). \]

By using Theorem 2.1 and the comment associated with
\[ \phi(p(z), zp'_\rho(z), z^2p''(z); w(z), \zeta) \in \Omega, \]
where \( w \) is any function mapping \( U \) into \( U \), with \( w(z) = \rho z \), we obtain
\[ p_\rho(z) \prec q_\rho(z) \] for \( \rho \in (\rho_0, 1) \). By letting \( \rho \to 1^- \), we get \( p(z) \prec q(z) \).
Therefore
\[ \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} < q(z). \]

The next result gives the best dominant of the strong differential subordination (9):

**Theorem 2.4.** Let \( h \) be univalent in \( U \) and \( \phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C} \). Suppose that the differential equation
\[ (10) \quad \phi \left( q(z), \frac{zq'(z)}{q(z)}, \frac{z^2q''(z)}{q(z)} + \frac{zq'(z)}{q(z)} - \left( \frac{zq'(z)}{q(z)} \right)^2 ; z, \zeta \right) = h(z) \]
has a solution \( q \) with \( q(0) = 1 \) and satisfies one of the following conditions:

1. \( q \in Q_1 \) and \( \phi \in \Phi_L[h, q] \),
2. \( q \) is univalent in \( U \) and \( \phi \in \Phi_L[h, q_\rho] \) for some \( \rho \in (0, 1) \),
3. \( q \) is univalent in \( U \) and there exists \( \rho_0 \in (0, 1) \) such that \( \phi \in \Phi_L[h_\rho, q_\rho] \) for all \( \rho \in (\rho_0, 1) \).

If \( f \in A \) satisfies (9), then
\[ \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} < q(z). \]
and \( q \) is the best dominant.

Proof. By applying Theorem 2.2 and Theorem 2.3, we deduce that \( q \) is a dominant of (9). Since \( q \) satisfies (10), it is also a solution of (9) and therefore \( q \) will be dominated by all dominants.

Hence \( q \) is the best dominant of (9). \(\Box\)

In the particular case \( q(z) = 1 + Mz, M > 0 \) and in view of Definition 2.1, the class of admissible functions \( \Phi_L[\Omega, q] \) denoted by \( \Phi_L[\Omega, M] \) can be expressed in the following form:
Definition 2.2. Let $\Omega$ be a set in $\mathbb{C}$ and $M > 0$. The class of admissible function $\Phi_L[\Omega, M]$ consists of those functions $\phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C}$ such that

\[
\phi \left( 1 + Me^{i\theta}, \frac{kM}{M + e^{-i\theta}}, \frac{kM + Le^{-i\theta}}{M + e^{-i\theta}} - \left( \frac{kM}{M + e^{-i\theta}} \right)^2 ; z, \zeta \right) \notin \Omega,
\]

whenever $z \in U$, $\zeta \in \bar{U}$, $\theta \in \mathbb{R}$, $\Re \{Le^{-i\theta}\} \geq k(k-1)M$ for all $\theta$ and $k \geq 1$.

Corollary 2.3. Let $\phi \in \Phi_L[\Omega, M]$. If $f \in A$ satisfies

\[
\phi \left( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} + \frac{\lambda z f''(z)}{f'(z)} - \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} + \frac{\lambda z^2 f'''(z)}{f'(z)}, \right.
\]

\[
+ \frac{\lambda z f''(z)}{f'(z)} \left( 1 - \frac{zf''(z)}{f'(z)} \right) - \frac{z^2(f(z) - f(-z))''}{f(z) - f(-z)} + \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} \times
\]

\[
\times \left( \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} - 1 \right) ; z, \zeta \right) \in \Omega,
\]

then

\[
\left| \frac{2z(f'(z))\lambda}{f(z) - f(-z)} - 1 \right| < M.
\]

When $\Omega = q(U) = \{ w : |w - 1| < M \}$, the class $\Phi_L[\Omega, M]$ is simply denoted by $\Phi_L[M]$, then corollary 2.3 takes the following form:

Corollary 2.4. Let $\phi \in \Phi_L[M]$. If $f \in A$ satisfies

\[
\phi \left( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} + \frac{\lambda z f''(z)}{f'(z)} - \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} + \frac{\lambda z^2 f'''(z)}{f'(z)} \right.
\]

\[
+ \frac{\lambda z f''(z)}{f'(z)} \left( 1 - \frac{zf''(z)}{f'(z)} \right) - \frac{z^2(f(z) - f(-z))''}{f(z) - f(-z)} + \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} \times
\]

\[
\times \left( \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} - 1 \right) ; z, \zeta \right) - 1 < M,
\]

then

\[
\left| \frac{2z(f'(z))\lambda}{f(z) - f(-z)} - 1 \right| < M.
\]

Example 2.1. If $M > 0$ and $f \in A$ satisfies

\[
\left| \frac{\lambda z^2 f'''(z)}{f'(z)} - \lambda \left( \frac{zf''(z)}{f'(z)} \right)^2 - \frac{z^2(f(z) - f(-z))''}{f(z) - f(-z)} + \right.
\]

\[
+ \left( \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} \right)^2 \right| < M,
\]
then
\[
\left| \frac{2z (f'(z))^\lambda}{f(z) - f(-z)} - 1 \right| < M.
\]
This implication follows from Corollary 2.4 by taking \( \phi(u, v; z, \zeta) = w - v + 2 \).

**Example 2.2.** If \( M > 0 \) and \( f \in A \) satisfies
\[
\left| \frac{\lambda zf''(z)}{f'(z)} - z \frac{(f(z) - f(-z))'}{f(z) - f(-z)} \right| < \frac{M}{M + 1},
\]
then
\[
\left| \frac{2z (f'(z))^\lambda}{f(z) - f(-z)} - 1 \right| < M.
\]
This implication follows from Corollary 2.3 by taking \( \phi(u, v, w; z, \zeta) = v \) and \( \Omega = h(U) \) where \( h(z) = \frac{M}{M + 1}z, M > 0 \). To apply Corollary 2.3, we need to show that \( \phi \in \Phi_L[\Omega, M] \), that is, the admissibility condition (11) is satisfied follows from
\[
\phi \left( 1 + Me^{i\theta}, \frac{kM}{M + e^{-i\theta}}, \frac{kM + Le^{-i\theta}}{M + e^{-i\theta}} - \left( \frac{kM}{M + e^{-i\theta}} \right)^2 ; z, \zeta \right) = \frac{kM}{M + 1} \geq \frac{M}{M + 1},
\]
for \( z \in U, \zeta \in \bar{U}, \theta \in \mathbb{R} \) and \( k \geq 1 \).

3. **Strong Superordination Results**

In this section, we establish strong differential superordination. For this purpose the class of admissible functions given in the following definition will be required.

**Definition 3.1.** Let \( \Omega \) be a set in \( \mathbb{C} \) and \( q \in \mathcal{H} \). The class of admissible functions \( \Phi_L[\Omega, q] \) consists of those functions \( \phi : \mathbb{C}^3 \times U \times \bar{U} \rightarrow \mathbb{C} \) that satisfy the admissibility condition: \( \phi(u, v, w; \xi, \zeta) \in \Omega, \) whenever
\[
u = q(z), \quad v = \frac{zq'(z)}{mq(z)}, \quad q(z) \neq 0 \quad \text{and} \quad \text{Re} \left\{ \frac{w + v^2}{v} \right\} \leq \frac{1}{m} \text{Re} \left\{ \frac{zq''(z)}{q'(z)} + 1 \right\},
\]
where \( z \in U, \zeta \in \bar{U}, \xi \in \partial U \) and \( m \geq 1 \).

**Theorem 3.1.** Let \( \phi \in \Phi_L[\Omega, q] \). If \( f \in A \), \( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} \in Q_1 \) and
\[
\phi \left( \frac{2z (f'(z))^\lambda}{f(z) - f(-z)}, 1 + \frac{\lambda zf''(z)}{f'(z)} - \frac{z (f(z) - f(-z))'}{f(z) - f(-z)}, \frac{\lambda z^2 f'''(z)}{f'(z)} \right) + \frac{\lambda zf''(z)}{f'(z)} \left( 1 - \frac{z f''(z)}{f'(z)} \right) - \frac{z^2 (f(z) - f(-z))''}{f(z) - f(-z)} + \frac{z (f(z) - f(-z))'}{f(z) - f(-z)} \times
\]

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Let \( \psi \) be analytic in \( \Omega \), then

\[
\Omega \subset \left\{ \phi \left( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} \right), 1 + \frac{\lambda zf''(z)}{f'(z)} - \frac{z(f(z) - f(-z))'}{f(z) - f(-z)}, \frac{\lambda z^2 f'''(z)}{f'(z)} \right. \\
\left. + \frac{\lambda zf''(z)}{f'(z)} \left(1 - \frac{zf''(z)}{f'(z)}\right) - \frac{z^2 (f(z) - f(-z))''}{f(z) - f(-z)} + \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} \right\},
\]

is univalent in \( U \), then

\[
q(z) < \frac{2z(f'(z))^\lambda}{f(z) - f(-z)}.
\]

Proof. Let \( p \) defined by (4) and \( \psi(p(z), zp'(z), z^2p''(z); z, \zeta) \) defined by (8). Since \( \phi \in \Phi^*_L[\Omega, q] \), from (8) and (12), we have

\[
\Omega \subset \left\{ \psi(p(z), zp'(z), z^2p''(z); z, \zeta) : z \in U, \zeta \in \bar{U} \right\}.
\]

From (7), we see that the admissibility condition for \( \phi \in \Phi^*_L[\Omega, q] \) is equivalent to the admissibility condition for \( \psi \) as given in Definition 1.5. Hence \( \psi \in \Psi^*[\Omega, q] \) and by Lemma 1.2, \( q(z) < p(z) \) or equivalently

\[
q(z) < \frac{2z(f'(z))^\lambda}{f(z) - f(-z)}.
\]

We consider the special situation when \( \Omega \neq \mathbb{C} \) is a simply connected domain. In this case \( \Omega = h(U) \), for some conformal mapping \( h \) of \( U \) onto \( \Omega \) and the class \( \Phi^*_L[h(U), q] \) is written as \( \Phi^*_L[h, q] \). The following result is an immediate consequence of Theorem 3.1.

**Theorem 3.2.** Let \( \phi \in \Phi^*_L[h, q], q \in \mathcal{H} \) and \( h \) be analytic in \( U \). If \( f \in \mathcal{A}, \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} \in Q_1 \) and

\[
\phi \left( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)}, 1 + \frac{\lambda zf''(z)}{f'(z)} - \frac{z(f(z) - f(-z))'}{f(z) - f(-z)}, \frac{\lambda z^2 f'''(z)}{f'(z)} \right. \\
\left. + \frac{\lambda zf''(z)}{f'(z)} \left(1 - \frac{zf''(z)}{f'(z)}\right) - \frac{z^2 (f(z) - f(-z))''}{f(z) - f(-z)} + \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} \right]\times
\]

\[
\times \left( \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} - 1 \right) ; z, \zeta\)
is univalent in \( U \), then
\[
\begin{align*}
\frac{2z(f'(z))}{f(z) - f(-z)} & \times \left( \frac{\lambda zf''(z)}{f'(z)} \left( 1 - \frac{zf''(z)}{f'(z)} \right) - \frac{z^2(f(z) - f(-z))''}{f(z) - f(-z)} + \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} \times \\
& \times \left( \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} - 1 \right) ; z, \zeta \right),
\end{align*}
\]
implies
\[
q(z) \prec \frac{2z(f'(z))^2}{f(z) - f(-z)}.
\]

By taking \( \phi(u, v, w; z, \zeta) = u + \frac{v}{\beta u + \gamma}, (\beta, \gamma \in \mathbb{C}) \) in Theorem 3.2, we state the following corollary:

**Corollary 3.1.** Let \( \beta, \gamma \in \mathbb{C} \) and let \( h \) be convex in \( U \) with \( h(0) = 1 \). Suppose that the differential equation \( q(z) + \frac{zq'(z)}{\beta q(z) + \gamma} = h(z) \) has a univalent solution \( q \) that satisfies \( q(0) = 1 \) and \( q(z) \prec h(z) \). If \( f \in \mathcal{A} \), \( \frac{2z(f'(z))^2}{f(z) - f(-z)} \in \mathcal{H} \cap Q_1 \) and
\[
\frac{2z(f'(z))^2}{f(z) - f(-z)} + \left( 1 + \frac{\lambda zf''(z)}{f'(z)} \right) \left( f(z) - f(-z) \right) - z(f(z) - f(-z))' \quad \frac{2\beta z(f'(z))^2 + \gamma(f(z) - f(-z))}{2\beta z(f'(z))^2 + \gamma(f(z) - f(-z))},
\]
is univalent in \( U \), then
\[
\frac{2z(f'(z))^2}{f(z) - f(-z)} + \left( 1 + \frac{\lambda zf''(z)}{f'(z)} \right) \left( f(z) - f(-z) \right) - z(f(z) - f(-z))' \quad \frac{2\beta z(f'(z))^2 + \gamma(f(z) - f(-z))}{2\beta z(f'(z))^2 + \gamma(f(z) - f(-z))},
\]
implies
\[
q(z) \prec \frac{2z(f'(z))^2}{f(z) - f(-z)}.
\]

The next result gives the best subordinant of the strong differential superordination (13):

**Theorem 3.3.** Let \( h \) be analytic in \( U \) and \( \phi : \mathbb{C}^3 \times U \times \bar{U} \to \mathbb{C} \). Suppose that the differential equation
\[
\phi \left( q(z), \frac{zq'(z)}{q(z)} ; z, \zeta \right) = h(z)
\]
has a solution \( q \in Q_1 \). If \( \phi \in \Phi_L^\alpha[h, q], f \in \mathcal{A}, \frac{2z(f'(z))^2}{f(z) - f(-z)} \in Q_1 \) and
\[
\phi \left( \frac{2z(f'(z))^2}{f(z) - f(-z)}, 1 + \frac{\lambda zf''(z)}{f'(z)} - \frac{z(f(z) - f(-z))'}{f(z) - f(-z)}, \frac{\lambda^2 f''''(z)}{f'(z)} \right)
\]


implies

Proof. The proof is similar to that of Theorem 2.4 and is omitted.

4. Sandwich Results

By combining Theorem 2.2 and Theorem 3.2, we obtain the following sandwich theorem:

**Theorem 4.1.** Let $h_1$ and $q_1$ be analytic functions in $U$, $h_2$ be univalent in $U$, $q_2 \in Q_1$ with $q_1(0) = q_2(0) = 1$ and $\phi \in \Phi_L[h_2, q_2] \cap \Phi'_L[h_1, q_1]$. If $f \in \mathcal{L}$, \( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} \in \mathcal{H} \cap Q_1 \) and

\[
\phi \left( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)}, 1 + \frac{\lambda z f''(z)}{f'(z)} - \frac{z(f(z) - f(-z))'}{f(z) - f(-z)}, \frac{\lambda z^2 f'''(z)}{f'(z)} \right)
\]

is univalent in $U$, then

\[
h_1(z) \prec \phi \left( \frac{2z(f'(z))^\lambda}{f(z) - f(-z)}, 1 + \frac{\lambda z f''(z)}{f'(z)} - \frac{z(f(z) - f(-z))'}{f(z) - f(-z)}, \frac{\lambda z^2 f'''(z)}{f'(z)} \right)
\]

is univalent in $U$, and $q$ is the best subordinant.

Proof. The proof is similar to that of Theorem 2.4 and is omitted. \(\square\)
\[
\times \left( \frac{z(f(z) - f(-z))'}{f(z) - f(-z)} - 1 \right); z, \zeta \right) \prec \prec h_2(z),
\]
implies
\[
q_1(z) \prec \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} \prec q_2(z).
\]

By combining Corollary 2.1 and Corollary 3.1, we obtain the following sandwich corollary:

**Corollary 4.1.** Let \(\beta, \gamma \in \mathbb{C}\) and let \(h_1, h_2\) be convex in \(U\) with \(h_1(0) = h_2(0) = 1\). Suppose that the differential equations
\[
q_1(z) + \frac{zq_1'(z)}{\beta q_1(z) + \gamma} = h_1(z), \quad q_2(z) + \frac{zq_2'(z)}{\beta q_2(z) + \gamma} = h_2(z)
\]
have a univalent solutions \(q_1\) and \(q_2\), respectively, that satisfy
\[
q_1(0) = q_2(0) = 1 \quad \text{and} \quad q_1(z) \prec h_1(z), q_2(z) \prec h_2(z).
\]
If \(f \in A\), \(\frac{2z(f'(z))^\lambda}{f(z) - f(-z)} \in \mathcal{H} \cap Q_1\) and
\[
\frac{2z(f'(z))^\lambda}{f(z) - f(-z)} + \frac{\left(1 + \frac{\lambda zf''(z)}{f'(z)}\right)(f(z) - f(-z)) - z(f(z) - f(-z))'}{2\beta z(f'(z))^\lambda + \gamma (f(z) - f(-z))}
\]
is univalent in \(U\), then
\[
h_1(z) \prec \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} + \frac{\left(1 + \frac{\lambda zf''(z)}{f'(z)}\right)(f(z) - f(-z)) - z(f(z) - f(-z))'}{2\beta z(f'(z))^\lambda + \gamma (f(z) - f(-z))}
\prec \prec h_2(z)
\]
implies
\[
q_1(z) \prec \frac{2z(f'(z))^\lambda}{f(z) - f(-z)} \prec q_2(z).
\]

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