Robust RHC for wheeled vehicles with bounded disturbances

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Summary
The robust receding horizon control (RHC) synthesis approach is developed in this paper, for the simultaneous tracking and regulation problem (STRP) of wheeled vehicles with bounded disturbances. Considering the bounded disturbances, we firstly provide a robust positively invariant (RPI) set and associated feedback controller for the perturbed vehicles, which contribute to the foundation of the robust RHC synthesis approach. Then, by extending the tube-based approach introduced in the article of Mayne et al (robust model predictive control of constrained linear systems with bounded disturbances in Automatica, 2005, vol. 41) to the STRP of wheeled vehicles, we employ the designed RPI set to determine the robust tube and terminal state region, and further construct a nominal optimal control problem. The actual control input is implemented by correcting the solved nominal input with the designed feedback controller. Following the contributed properties of the developed RPI set and extended tube-based approach, a robust RHC algorithm is finally proposed with the guarantees of recursive feasibility and robust convergence, which can also be adapted for real-time implementation. Additionally, due to the elaborate control design, the effect of disturbances can be completely nullified to achieve better tracking performance. The effectiveness and advantage of the proposed approach are illustrated by two simulation examples.

KEYWORDS
robust positively invariant (RPI) set, robust receding horizon control (RHC), simultaneous tracking and regulation problem (STRP), tube-based, wheeled vehicles

1 | INTRODUCTION

Due to the advantages in handling physical constraints and optimizing control performance, the receding horizon control (RHC), also referred to as model predictive control (MPC), has been widely employed in various applications for constrained systems, eg, chemicals, food processing, automotive, and aerospace applications, surveyed by Qin and Badgwell. Examples of RHC application for unmanned vehicles include the air/ground traffic control management, the motion/trajectory planning for the vehicles, and the tracking and/or regulation of vehicles. In these problems, the motions and/or trajectories of vehicles need to be planned to achieve the desired objective in some optimal senses, and some given physical constraints (eg, the collision avoidance constraint, the velocity constraint, and the input saturation constraint) need to be satisfied. Therefore, the RHC strategy is preferred by the researchers to others (eg, sliding mode control, dynamic feedback linearization, backstepping technique, etc), and more and more approaches emerge to be

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effective solutions for these problems. Some of these emerged approaches are proposed in synthesis framework (ie, both the recursive feasibility and stability are guaranteed) for the control of unmanned vehicles. Following the common practice of RHC synthesis approach in the work of Mayne et al, choosing a proper terminal (penalty) cost, developing a terminal-state region (or terminal constraint set) with (robustly) positive invariance, and designing an associated controller are the three key ingredients.

In many general control problems of unmanned vehicles, the dynamics of vehicles are simply modeled or linearized as discrete-time linear systems, then the effectiveness of the proposed RHC approaches can be shown clearly. However, the simplification of dynamics omits some practical characteristics of vehicles, such as the nonlinearity and mechanical dynamics of wheeled vehicles, eg, the unicycle-modeled vehicle is constrained by its nonholonomic constraint \( \dot{y} \cos \theta - \dot{x} \sin \theta = 0 \). More details for nonholonomic mechanics and control of complex vehicles can refer to the work of Bloch.

Focused on the regulation case of wheeled vehicle, many researchers attempt to present RHC approach, eg, the controllers designed for nominal vehicle and for perturbed vehicle. With directly borrowing a linear formed terminal-state region or simply choosing a nonlinear formed terminal-state region, the controllers are designed following the common practice of RHC synthesis approach, whereas neither of these can achieve synthesis approach due to the failure of guaranteeing (robustly) positive invariance for the terminal-state region. As the (robust) positively invariant terminal constraint set and associated control law are successfully developed, the approaches proposed by Fontes can fall in synthesis category. To avoid the challenge of developing positively invariant terminal constraint set, an alternative RHC synthesis approach can be presented without stabilizing constraints and costs, where the asymptotic stability depends much on the prediction horizon. Unfortunately, the aforementioned synthesis approaches cannot be applied for tracking case because their associated control laws (or control input candidates) are designed only for stationary reference. In fact, it is hard to develop a (robust) positively invariant set and associated control law for the simultaneous tracking and regulation problem (STRP), ie, tracking a dynamic reference with its speeds approaching to zeros (eg, the parallel parking). Hence, the STRP arises to be a challenging issue of TPSC, especially for perturbed vehicle.

Aiming at the STRP of nominal vehicle, a terminal constraint set is defined by Gu and Hu, whereas its positive invariance cannot be ensured. In the work of Wang and Ding, a positively invariant terminal constraint set with associated controller is successfully developed for the STRP of nominal multiple vehicles. However, its positive invariance is no longer guaranteed for perturbed vehicle, which hinders its adaption to robust case. Motivated by this situation, this paper intends to propose a robust RHC synthesis approach for the STRP of perturbed wheeled vehicle, where an RPI set and associated feedback control law designed in our conference work serve as the fundamental contribution here. To clearly distinguish the contributions of the aforementioned works, a brief comparison is provided in Table 1, with more details concretely clarified in the following.

By explicitly taking bounded uncertainties into account, various approaches are proposed for the robust RHC and can be classified into two general categories: min-max approach and tube-based approach. The former considers the possibly worst-case conditions of bounded uncertainties over the prediction horizon and constructs the optimal control problem in tree structured form, where the numbers of decision variables and constraints scale exponentially with the prediction horizon. In contrast, the latter directly handles bounded uncertainties by tightening the constraint and constructs the optimal control problem in nominal form, where the optimal control problem includes its initial state as a decision variable and adopts the nominal dynamics for prediction. Due to the nominal optimization problem solved for implementation, the tube-based RHC is of similar complexity to the nominal (conventional) RHC and is now widely extended, eg, the stochastic case, the nonlinear discrete-time case, the linear continuous-time case, and the nonlinear continuous-time case. However, the aforementioned extensions for nonlinear continuous-time case require linear controllability within the neighborhood of equilibrium, hence cannot be applied for the STRP of perturbed wheeled vehicle. In this paper, we successfully extend the tube-based approach for the aforementioned problem by adapting the designed RPI set to develop the initial, tightened, and terminal constraint sets, and then present a robust RHC algorithm with recursive feasibility and robust convergence. Moreover, the presented algorithm can be modified for implementation.
without on-line optimization, which can meet the requirement of real-time application. This is the main contribution of this work.

Under the robust RHC frame some auxiliary techniques aiming at handling the uncertainties and/or disturbances, such as integral action and disturbance compensation, can be introduced to improve the robust control performance. By augmenting the original dynamics with integral term of tracking error\textsuperscript{19} or representing the state and/or input in the integrator form,\textsuperscript{37,38} the integral action can be included to suppress the effect of uncertainties and/or disturbances (even to achieve zero steady-state error\textsuperscript{19}). Through incorporating the disturbance estimation scheme into controller design,\textsuperscript{20,39} the perturbed effect can be estimated and compensated to enhance the disturbance resistance. Without adopting any of the aforementioned auxiliary techniques, this paper directly utilize the deviation between actual and nominal states (which can be seen as the perturbed effect) to correct the solved control input for better disturbance rejection. Due to the feedback control law elaborately designed for correction, the null steady-state tracking error can be achieved by applying the presented robust RHC algorithm. This contributes to the performance advantage of the proposed approach in this work.

A partial and preliminary version of this paper has been presented in our conference work.\textsuperscript{40} Here, further potential of the presented robust RHC approach is exploited, with handling the state constraint and providing more technical details and simulation examples for better demonstration. The rest of this paper is organized as follows. In Section 2, we introduce the perturbed wheeled vehicle system, describe the desired control objective, and formulate the cost function accordingly. According to the framework of tube-based approach, Section 3 presents the robust RHC synthesis approach for the STRP of perturbed wheeled vehicle, by designing an RPI set and associated feedback control law in Section 3.1, developing the terminal constraint and choosing the terminal cost in Section 3.2, building the terminal constraint and choosing the terminal cost in Section 3.3, and providing the implementation algorithm in Section 3.4. Section 4 provides the simulation examples to demonstrate the effectiveness and advantage of the proposed approach. Finally, a conclusion summarizes this paper in Section 5.

**Notation.** For column vectors/scalars $x$ and $y$, $[x; y] = [x^T, y^T]^T$; $||x||$ ($||x||_P$) denotes the two-norm (P-weight two-norm) of $x$, ie, $||x|| = \sqrt{x^T x}$ ($||x||_P = \sqrt{x^T P x}$); $x(\tau|\tau_k)$ denotes the value of $x$ at time $\tau$, predicted at the previous time instant $\tau_k$; and $x^\tau (x^\tau)$ denotes the value of $x$ corresponding to the feasible (optimal) solution of optimization. Denote $T$ as the prediction horizon with $0 < T < \infty$, sgn(·) a sign function, and $\lambda^M_P$ the maximum eigenvalue of a positive definite matrix $P > 0$. The operators “$\ominus$” and “$\oplus$” denote the Minkowski sum and Pontryagin difference, respectively, ie, $A \ominus B = \{a + b|a \in A, b \in B\}$ and $A \oplus B = \{a|a \ominus B \subseteq A\}$.

## 2 Problem Statement

Consider a wheeled vehicle with external disturbances, which is formulated by the following perturbed unicycle-modeled dynamics:

$$\dot{z}(t) = f(z(t), u(t), d(t)) = \begin{bmatrix} (\varphi(t) + \omega_d(t)) \cos \theta(t) \\ (\varphi(t) + \omega_d(t)) \sin \theta(t) \\ \omega(t) + \omega_d(t) \end{bmatrix}, \quad (1)$$

### TABLE 1  Brief comparison of contributions between this paper and the existing relevant works

| Type of Vehicle | Type of Problem | Synthesis Approach? | Contribution Robust Approach? | Null perturbed Tracking Error? | Relevant Work |
|-----------------|-----------------|---------------------|-------------------------------|--------------------------------|--------------|
| Linearly modeled or handled | Tracking | √ | × | — | works\textsuperscript{14,16,17} |
| Nonlinearly modeled and nonholonomic | Tracking case (ie, with persistently dynamic reference) | √ | × | — | works\textsuperscript{22,23} |
| | Regulation case (ie, with stationary reference) | √ | × | × | work\textsuperscript{24} |
| | Simultaneous tracking and regulation | × | × | — | work\textsuperscript{7} |
| | | √ | × | — | work\textsuperscript{8} |
| | | √ | √ | √ | This paper |
where \( z(t) = [p(t); \theta(t)], u(t) = [v(t); \omega(t)], \) and \( d(t) = [d_r(t); d_n(t)] \) represent the state, control input, and external disturbance of vehicle, respectively; \( p(t) = [x(t); y(t)] \) and \( \theta(t) \) the position (of the midpoint of the rear axis) in the Cartesian coordinate frame and the orientation of vehicle, respectively; \( v(t) \) and \( \omega(t) \) the linear and angular speeds of vehicle, respectively; and \( d_r(t) \) and \( d_n(t) \) the disturbances affecting on the linear and angular speeds, respectively. Here, we assume that the actual state \( z(t) \) is measurable in real time. For a wheeled vehicle, the linear and angular speeds are generally derived according to the velocities of the left and right driving wheels (say \( v_L(t) \) and \( v_R(t) \), respectively) as \( v(t) = \frac{v_L(t) + v_R(t)}{2} \) and \( \omega(t) = \frac{v_L(t) - v_R(t)}{L} \), with \( L \) being the distance between the left and right driving wheels. Due to the unexpected forces and/or torques perturbed on the vehicle, the velocities \( v_L(t) \) and \( v_R(t) \) are inevitably affected by additive disturbances (say \( d_{v_L}(t) \) and \( d_{v_R}(t) \), respectively), which cause the speed disturbances \( d_v(t) = \frac{d_{v_L}(t) + d_{v_R}(t)}{2} \) and \( d_\omega(t) = \frac{d_{v_L}(t) - d_{v_R}(t)}{L} \).

In practice, the state is required to satisfy some specific constraints with a general form

\[
z(t) \in Z \triangleq \{ [x; y; \theta] | c_i(x, y, \theta) \leq c^M_i, i = 1, 2, \ldots, s \} , \quad \forall t \geq 0 ,
\]

(2)

where \( c_i(\cdot, \cdot, \cdot) \) denotes the \( i \)-th convex constraint function, \( c^M_i \) the \( i \)-th prespecified bound, and \( s \) the number of state constraints. The control input \( u(t) \) is required to satisfy the saturation constraint

\[
u(t) \in U \triangleq \{ [v; \omega] | v \leq v^M, |\omega| \leq \omega^M \} , \quad \forall t \geq 0 ,
\]

(3)

with the prespecified bounds \( v^M > 0 \) and \( \omega^M > 0 \). The external disturbance is unknown but bounded, i.e., \( d(t) \in D \triangleq \{ [d_r; d_n] | d_r \leq d^M_r, |d_n| \leq d^M_n \} \) holds for all \( t \geq 0 \) with the bounds \( d^M_r > 0 \) and \( d^M_n > 0 \) known a priori. By neglecting the disturbance \( d(t) \), the nominal dynamics can be obtained as \( \dot{z}(t) = f(z(t), u(t), 0), \) with the nominal state \( \dot{z}(t) = [\dot{x}(t); \dot{y}(t); \dot{\theta}(t)] \) and control input \( \dot{u}(t) = [\dot{v}(t); \dot{\omega}(t)] \).

In this paper, the control objective is to steer the perturbed vehicle (1) to track a prespecified reference, without violating the constraints (2) and (3), i.e., to solve a robust TPSC. For realizability of the control objective, the reference trajectory is generated by a virtual vehicle of nominal dynamics

\[
\dot{z}_r(t) = f (z_r(t), u_r(t), 0),
\]

(4)

with the reference state \( z_r(t) = [x_r(t); y_r(t); \theta_r(t)] \) and reference input \( u_r(t) = [v_r(t); \omega_r(t)] \) satisfying

\[
c_i (x_r(t), y_r(t), \theta_r(t)) \leq c^M_i, i = 1, 2, \ldots, s, \forall t \geq 0,
\]

(5)

\[
|v_r(t)| \leq v^M, |\omega_r(t)| \leq \omega^M, \forall t \geq 0,
\]

(6)

where \( c^M_i < v^M, v^M < v^M, \) and \( \omega^M < \omega^M \) are the bounds of reference known a priori. Note that, when the reference input \( u_r(t) \) approaches to zero (i.e., \( \lim_{t \to \infty} (v_r^2(t) + \omega_r^2(t)) = 0 \)), the robust TPSC becomes robust STRP.

For ease of control design, the aforementioned tracking problem is always formulated in the vehicle-based coordinate frame, then the tracking error state and control input are defined as\(^{7,11}\)

\[
z_e(t) \triangleq \begin{bmatrix} \chi_e(t) \\ y_e(t) \\ \theta_e(t) \end{bmatrix} = \Phi(\theta(t)) \begin{bmatrix} x(t) - x_r(t) \\ y(t) - y_r(t) \\ \theta(t) - \theta_r(t) \end{bmatrix}
\]

(7)

and \( u_e(t) = [v(t); \omega(t)] = [v(t) - v_r(t) \cos \theta(t); \omega(t) - \omega_r(t)], \) respectively, where \( \Phi(\theta(t)) \) is referred to as the coordinate transformation matrix at time \( t \). Accordingly, the nominal tracking errors can be defined as \( z_e(t) = [\dot{x}_e(t); \dot{y}_e(t); \dot{\theta}_e(t)] = \Phi(\dot{\theta}(t)) (\dot{z}_r(t) - z_e(t)) \) and \( \dot{u}_e(t) = [\dot{v}_e(t); \dot{\omega}_e(t)] = [\dot{v}(t) - v_r(t) \cos \theta(t); \dot{\omega}(t) - \omega_r(t)]. \) Similarly, the error between actual and nominal states, say perturbed state error, is defined as \( z(t) = [\dot{x}(t); \dot{y}(t); \dot{\theta}(t)] = \Phi(\dot{\theta}(t)) (z_r(t) - \dot{z}(t)). \)

Given the weighting matrices \( Q > 0 \) and \( R > 0 \), the tracking cost function is formulated as

\[
J(t_k, z_e, u_e) = \int_{t_k}^{t_k + T} L(t) |z_e(t)|^2 dt + g(z_e(t_k + T)|t_k)),
\]

where the stage cost \( L(t)|z_e(t), u_e(t)| = ||z_e(t)|^2| + ||u_e(t)|^2 \) and the terminal cost \( g(z_e(t_k + T)|t_k)) \) is a continuous and differentiable function (will be chosen later), satisfying \( g(0) = 0 \) and \( g(z_e(t)) > 0 \) for any \( z_e(t) \neq 0 \).
3 | ROBUST RECEDING HORIZON CONTROLLER DESIGN

In the common framework of RHC, the predictive state and input trajectories are partitioned into two parts according to time: (i) the trajectories planned over the prediction horizon (ie, \( \tau \in [t_k, t_k + T] \)); and (ii) the trajectories after the prediction horizon (ie, \( \tau \geq t_k + T \)). For both parts of trajectories, the desired constraints need to be handled (against the disturbances) to guarantee recursive feasibility. Based on recursive feasibility, some additional properties are further required to achieve (robust) stability. With considering the disturbances, the latter part of trajectories is commonly handled by the min-max approach or the tube-based approach. Due to the complexity advantage (ie, similar complexity to the conventional one) and easy extensibility for nonlinear continuous-time case, we prefer the tube-based approach here.

Let \( \mathcal{O} \) and \( \kappa(\cdot, \cdot) \) denote an RPI set and associated feedback control law (both designed later) for robust TPSC of vehicle (1), respectively; \( \mathcal{O}[\mathcal{O}] \) the set mapped from set \( \mathcal{O} \) by function \( \kappa(\cdot, \cdot) \); and \( \mathcal{O}_T(\cdot) \) the desired terminal constraint set. According to the tube-based RHC explored for linear discrete-time system by Mayne et al, we extend its fundamental measures to nonlinear continuous-time case via four steps.

i. Adopt the nominal cost \( J(t_k, \bar{z}_e, \bar{u}_e) \) for minimization and nominal dynamics \( f(\bar{z}, \bar{u}, 0) \) for prediction, hence the value of nominal cost is zero in an RPI set \( \mathcal{O} \) that can serve as the “origin” for TPSC of perturbed vehicle (1).

ii. Include the initial nominal state \( \bar{z}(t_k|t_k) \) as a decision variable and impose the current actual state lying in an RPI set around it (ie, \( \bar{z}(t_k) \in \bar{z}(t_k|t_k) \oplus \mathcal{O} \)), hence the feasible domain can be enlarged (see proposition 2 in the work of Mayne et al).

iii. Tighten the original constraints based on a mapped set of \( \mathcal{O} \) (ie, \( \bar{z}(\tau|t_k) \in \bar{z}(\tau|t_k) \oplus \mathcal{O} \) and \( \bar{u}(\tau|t_k) \in \bar{u}(\tau|t_k) \oplus \mathcal{O}[\mathcal{O}] \)), hence the recursive feasibility of the optimization problem and the admissibility of actual control input can be guaranteed.

iv. Implement the solved optimal control input \( \bar{u}^*(\tau|t_k) \) corrected by the feedback control law \( \kappa(\cdot, \cdot) \), hence the external disturbances can be resisted.

Following these steps, the optimization problem of tube-based RHC is firstly presented in conceptual form as

\[
\min_{\bar{z}(t_k|t_k), \bar{u}(\tau|t_k)} J(t_k, \bar{z}_e, \bar{u}_e) \quad \text{(8a)}
\]

\[
\text{s.t. } \bar{z}(t_k) \in \bar{z}(t_k|t_k) \oplus \mathcal{O}, \quad \text{(8b)}
\]

\[
\text{for all } \tau \in [t_k, t_k + T], \quad \bar{z}(\tau|t_k) = f(\bar{z}(\tau|t_k), \bar{u}(\tau|t_k), 0), \quad \text{(8c)}
\]

\[
\bar{z}(\tau|t_k) \in \bar{z}(\tau|t_k) \oplus \mathcal{O}, \quad \text{(8d)}
\]

\[
\bar{u}(\tau|t_k) \in \bar{u}(\tau|t_k) \oplus \mathcal{O}[\mathcal{O}], \quad \text{(8e)}
\]

\[
\bar{z}(t_k + T|t_k) \in \mathcal{O}_T(z_e(t_k + T)). \quad \text{(8f)}
\]

In the optimization problem (8), the constraint (8b) is referred to as initial constraint (similar to the terminal constraint (8f)), the constraints (8d) and (8e) are referred to as tightened state and input constraints, respectively. After optimization, the solved nominal control input \( \bar{u}^*(\tau|t_k) \) is corrected for actual implementation as

\[
u(\tau) = \kappa(\bar{z}^*(\tau|t_k), \bar{u}^*(\tau|t_k)), \quad \tau \in [t_k, t_{k+1}]. \quad \text{(9)}
\]

with \( \bar{z}^*(\tau|t_k) = \Phi(\theta(\tau))(z(\tau) - \bar{z}^*(\tau|t_k)) \), where the predictive nominal state \( \bar{z}^*(\tau|t_k) \) is evolved along the nominal dynamics (8c) from \( \bar{z}^*(t_k|t_k) \) under the solved control input \( \bar{u}^*(\tau|t_k) \).

Generally, the development of the (robust) positively invariant set for nonlinear system is based on linearization near the equilibrium point. However, this method is invalid for the regulation problem of unicycle-modeled vehicle due to controllability loss of its linearized dynamics. For the STRP of unicycle-modeled vehicle, especially of perturbed vehicle, designing a (robust) positively invariant set and associated control law is still a challenging issue. Hence, in the following, we firstly focus on tackling this issue for the STRP of perturbed vehicle, then develop the actual initial and tightened constraints based on the designed RPI set and associated control law. The terminal constraint set will also be built with choosing a proper terminal cost.
A robust positively invariant set $\mathcal{S}$ for the simultaneous tracking and regulation problem of perturbed vehicle in the forward tracking case (i.e., $v_r(t) \geq 0$), which is described in the vehicle-based coordinate frame (i.e., the $x-y$ coordinate frame) by the constraint (12) [Colour figure can be viewed at wileyonlinelibrary.com]

### 3.1 Design of an RPI set and associated feedback control law

According to the common practice of tube-based RHC, the set $\mathcal{O}$ in problem (8) is desired to be RPI for the perturbed state error $\tilde{z}(t|t_k)$. The terminal constraint set $\mathcal{O}_T(\cdot)$ is desired to be positively invariant for the nominal tracking state error $\bar{z}_e(t|t)$.

Note that the reference and nominal trajectories are generated by the same nominal dynamics $f(\cdot, \cdot, 0)$, hence the set $\bar{O}$ is a more general version of the set $\mathcal{O}_T(\cdot)$ that possesses robust property and can be developed of a general structure.

**Definition 1.** For STRP of the perturbed vehicle (1) subject to constraints (2) and (3) and the prespecified reference (4) satisfying (5) and (6), an RPI set $\mathcal{S}$ is such that, if $z_e(t) \in \mathcal{S}$, then the following conditions hold for any $d(t) \in \mathcal{D}$ and $t \geq \tau$,

RPI: $z_e(t) \in \mathcal{S},$  
Admissibility: $z(t) \in \mathcal{Z}$ and $u(t) = \kappa(z_e(t), u_r(t)) \in \mathcal{U},$

where $\kappa(z_e(t), u_r(t))$ is a designed feedback control law.

Here, we can immediately describe the designed RPI set (see the region enclosed by dotted lines in Figure 1) as

$$\mathcal{S} \triangleq \{ z_e(t) | (12) \text{ holds} \},$$

with the following constraints:

$$|y_e(t)| \leq \alpha_1 |x_e(t)| \theta_e(t),$$

$$|\theta_e(t)| \leq \alpha_2 |x_e(t)|,$$

$$|x_e(t)| \leq \alpha_3,$$

$$|\theta_e(t)| \leq \alpha_4 \leq \frac{\pi}{2},$$

$$\begin{cases} x_e(t) \leq 0, \ y_e(t) \theta_e(t) \geq 0, \text{ for } v_r(t) \geq 0, \\
 x_e(t) \geq 0, \ y_e(t) \theta_e(t) \leq 0, \text{ for } v_r(t) \leq 0, \end{cases}$$

and present the designed feedback control law as

$$\kappa(z_e(t), u_r(t)) = \begin{bmatrix} v^\xi(t) \\ \omega^\xi(t) \end{bmatrix} = \begin{bmatrix} v_r(t) \cos \theta_e(t) + v^\xi_r(t) \\ \omega_r(t) + \omega^\xi_r(t) \end{bmatrix},$$

with $v^\xi_r(t) = -\beta_1 x_e(t) - \beta_2 \text{sgn}\{x_e(t)\}$ and $\omega^\xi_r(t) = \frac{\beta x_e(t) \theta_e(t)}{x^2_e(t) + v^2_e(t)} - \beta_4 \text{sgn}\{\theta_e(t)\}$, where $\alpha_i \geq 0$ and $\beta_i > 0$, $i = 1, 2, 3, 4$, are nonnegative real scalars to be chosen.
Remark 1. Although a positive invariant set with auxiliary control law has been designed in our previous work\(^8\) (see equations (16) and (17) therein) for STRP of wheeled vehicles without considering the state constraint, it cannot guarantee robust positive invariance for perturbed vehicle and may lead to infeasibility of optimization problem (8). Hence, an RPI set (11) with associated feedback control law (13) is designed specially for the robust STRP of perturbed vehicles subject to both state constraint (2) and input constraint (3).

To meet the requirements in Definition 1, the real scalars \(a_i\) and \(\beta_i\), \(i = 1, 2, 3, 4\), need to be properly chosen to satisfy the following conditions.

**Theorem 1.** For the given bounds \(\omega^M, i = 1, 2, \ldots, s, v^M, \omega^M, d^M, d^M, i = 1, 2, \ldots, s, v^M,\) and \(\omega^M, i = 1, 2, \ldots, s, v^M,\) and \(\omega^M,\) if the chosen scalars \(a_i\) and \(\beta_i\), \(i = 1, 2, 3, 4,\) satisfy

\[
\beta_1 \geq \alpha_1 a_4 \left( \omega^M + d^M \right),
\]
\[
\beta_2 \geq d^M,
\]
\[
\beta_3 + \beta_3 \leq v^M - v^M,
\]
\[
\beta_3 \geq 1 + a_1^2 a_3^2,
\]
\[
\beta_4 \geq \max \{ \eta_1, \eta_2 \},
\]
\[
\beta_3 \alpha_3 v^M + \beta_4 \leq \omega^M - \omega^M,
\]
\[
\max_{\mu_x, \mu_y, \mu_\theta \in [-1, 1]} \left\{ c_i(\mu_x \eta_3, \mu_y \eta_3, \mu_\theta a_4) \right\} \leq c_i^M, i = 1, 2, \ldots, s,
\]

where \(\eta_1 = \omega^M + d^M + a_1 a_5 (\omega^M + \beta_3 (\beta_3 + \beta_3) + a_1 (\omega^M - \omega^M + d^M), \eta_2 = d^M + \beta_4 a_4 + \beta_2 (\beta_3 + d^M), \) and \(\eta_3 = \alpha_3 \sqrt{1 + a_1^2 a_3^2} \), then the set \(S\) described in (11) and the control law presented in (13) satisfy the requirements in Definition 1.

**Proof.** The proof follows that, under the designed feedback control law (13), the constraint (12) is always guaranteed by itself and the condition (14) for any \(z(r) \in S\) and \(t \geq r\). The satisfaction of the admissibility requirements (10b) and (10c) is also guaranteed by the conditions (12) and (14).

For the satisfaction of conditions (10a) and (10c), the detailed proof and a simulation demonstration have been provided in our conference work.\(^2\) Here, the proof for the satisfaction of requirement (10b) is provided as follows. According to the constraints (12a), (12c), and (12d), we have \(\sqrt{x^2(t) + y(t)} \leq \sqrt{1 + a_1^2 a_3^2} |x(t)| \leq \eta_3\) and \(|\theta(t)| \leq \alpha_4\), which, following the definition (7), yields

\[
|x(t) - x(t)| \leq \eta_3, |y(t) - y(t)| \leq \eta_3, |\theta(t) - \theta(t)| \leq \alpha_4.
\]

Due to the convexity of the constraint function \(c_i(\cdot, \cdot, \cdot), i = 1, 2, \ldots, s,\) the following condition holds for any \(x(t), y(t)\) and \(\theta(t)\) satisfying (15):

\[
c_i(x(t), y(t), \theta(t)) \leq \max_{\mu_x, \mu_y, \mu_\theta \in [-1, 1], |x(t)| \leq \eta_3, |y(t)| \leq \eta_3, |\theta(t)| \leq \alpha_4} \left\{ c_i(x(t) + \mu_x \eta_3, y(t) + \mu_\theta \eta_3, \theta(t) + \mu_\theta \eta_3) \right\}
\]
\[
\leq \max_{\mu_x, \mu_y, \mu_\theta \in [-1, 1], \eta_3 \leq \theta(t) \leq \alpha_4} \left\{ c_i(\eta_3 x(t) + \mu_x \eta_3, \eta_3 y(t) + \mu_\theta \eta_3, \theta(t) + \mu_\theta \eta_3) \right\}
\]
\[
\leq \max_{\mu_x, \mu_y, \mu_\theta \in [-1, 1], \eta_3 \leq \theta(t) \leq \alpha_4} \left\{ c_i(\mu_x \eta_3, \mu_y \eta_3, \mu_\theta a_4) \right\} + \max_{\mu_x, \mu_y, \mu_\theta \in [-1, 1]} \left\{ c_i(\mu_x \eta_3, \mu_y \eta_3, \mu_\theta a_4) \right\}
\]
\[
\leq \max_{\mu_x, \mu_y, \mu_\theta \in [-1, 1]} \left\{ c_i(\mu_x \eta_3, \mu_y \eta_3, \mu_\theta a_4) \right\} + c_i^M, i = 1, 2, \ldots, s.
\]

Then, the condition (14g) leads to \(c_i(x(t), y(t), \theta(t)) \leq c_i^M, i = 1, 2, \ldots, s.\) Hence, the robustly positive invariance of the set \(S\) defined in (11) guarantees the satisfaction of requirement (10b).

This completes the proof. \(\square\)

The scalars \(a_i\) and \(\beta_i\), \(i = 1, 2, 3, 4\), can be chosen off-line to determine the RPI set \(S\) and the control law \(\kappa(\cdot, \cdot)\). The larger the scalars \(a_i, i = 1, 2, 3, 4\), are chosen, the larger the RPI set \(S\) can be determined, however, the less margin is left to choose the parameters \(\beta_i, i = 1, 2, 3, 4\), for \(\kappa(\cdot, \cdot)\). Hence, a large RPI set \(S\) should be determined properly to guarantee the existence of the control law \(\kappa(\cdot, \cdot)\). Note that the conditions in (14) are more easily satisfied for some smaller scalars \(a_i,\)
i = 1, 2, 3, 4. Hence, the minimal RPI set can be determined as $\mathbb{S}^m = \{ z_e(t) = 0 \}$ by choosing the minimal scalars $\alpha_i = 0$, $i = 1, 2, 3, 4$. This property is achieved by introducing the sliding mode control terms (e.g., $\beta_s \text{sgn} x_i(t)$ and $\beta_s \text{sgn} \theta_i(t)$) into the control law (13) because the matched disturbance $d(t)$ can be completely nullified in the sliding mode.\(^{22}\) The detailed proof and a simulation illustration for the convergence of the tracking error $z_e(t)$ to zero by implementing the designed control law (13) can refer to our conference work.\(^{28}\) Note that, if the external disturbance $d(t)$ is not directly acting on input channel, ie, $d(t)$ is additive unmatched disturbance, then the aforementioned property cannot be guaranteed, which is because the additive time-varying unmatched disturbance cannot be completely eliminated by sliding mode control or any other control. To additionally reduce the effect of unmatched disturbance, sliding mode control with compensating the disturbance by estimation is recommended.\(^{39}\)

**Remark 2.** Other than determining a fixed RPI set, we design the RPI set $\mathbb{S}$ of a general structure, ie, its shape and size can be tuned by the scalars $\alpha_i$, $i = 1, 2, 3, 4$. This provides much convenience to specify an RPI set for miscellaneous requirements (e.g, a smaller set for better robust convergence, or a bigger set for larger feasible domain).

### 3.2 Development of actual initial and tightened constraints

Note in the optimization problem (8) that both the initial and tightened constraints are dependent on the set $\mathbb{O}$, which is desired to be RPI for the perturbed state error $\tilde{z}(t|t_k)$. Due to the similar evolution of the errors $z_e(\cdot)$ and $\tilde{z}(\cdot)$, we can design the desired RPI set $\mathbb{O}$ as

$$\mathbb{O} \triangleq \{ \tilde{z}(t) \}$$

with proper nonnegative real scalars $\alpha_{ij}$, $i = 1, 2, 3, 4$, and adopt the associated feedback control law $\kappa(\tilde{z}(t), \tilde{v}(t|t_k))$ with proper positive scalars $\beta_{ij}$, $i = 1, 2, 3, 4$. Then, following the definition of Minkowski sum and formulation (17), $z(t_k) \in \mathbb{Z}(t_k|t_k) \oplus \mathbb{O}$ is equal to $\mathbb{Z}(t_k|t_k) \in \mathbb{O}$.

As the actual state $z(t_k)$ can be measured in real time, we can reformulate the initial constraint to constrain the decision variable $\tilde{z}(t_k|t_k)$ as

$$\tilde{z}(t_k|t_k) \in \mathbb{O}_I(z(t_k)) \triangleq \{ \tilde{z}(t_k|t_k) \}$$

with

\[
\begin{align*}
|\tilde{z}(t_k|t_k)| & \leq \alpha_{1,3} |\tilde{x}(t_k|t_k)\tilde{\theta}(t_k|t_k)|, \\
|\tilde{\theta}(t_k|t_k)| & \leq \alpha_{2,3} |\tilde{x}(t_k|t_k)|, \\
|\tilde{x}(t_k|t_k)| & \leq \alpha_{3,3}, \\
|\tilde{\theta}(t_k|t_k)| & \leq \alpha_{4,3} \leq \frac{\pi}{2}, \\
\end{align*}
\]

\[
\begin{align*}
\tilde{x}(t_k|t_k) & \leq 0, \quad \tilde{y}(t_k|t_k)\tilde{\theta}(t_k|t_k) \geq 0, \quad \text{for} \quad v_f(t) \geq 0, \\
\tilde{y}(t_k|t_k) & \geq 0, \quad \tilde{x}(t_k|t_k)\tilde{\theta}(t_k|t_k) \leq 0, \quad \text{for} \quad v_f(t) \leq 0.
\end{align*}
\]

It deserves to note that, in both cases of constraint (19e), the reference linear speed $v_f(t)$ should also be replaced by the nominal predictive one $\tilde{v}(t|t_k)$ for robustly positive invariance. Here, this substitution can be omitted with the consideration that the input constraint is generally imposed as $0 \leq v(t) \leq v^M$ ($-v^M \leq v(t) \leq 0$) when the forward (backward) STRP is considered,\(^{7,16}\) which requires $\tilde{v}(t|t_k) \geq 0$ for $v_f(t) \geq 0$ ($\bar{v}(t|t_k) \leq 0$ for $v_f(t) \leq 0$).

Because the actual initial constraint (18) is of the similar form to the terminal one, we refer to $\mathbb{O}_T(\cdot)$ as the initial constraint set and distinguish the initial and terminal constraint sets with subscripts “I” and “T”, respectively.

Now, based on the designed RPI set $\mathbb{O}$ in (17) with $\alpha_{ij}$, $i = 1, 2, 3, 4$, we can develop the actual tightened constraints as follows. Similar to the procedure in proof of Theorem 1, it can be easily derived that, if $\mathbb{Z}(t) \in \mathbb{O}$ in (17), then the constraint $c_i(x(t), y(t), \theta(t)) \leq c^M_i$ is guaranteed for any $\tilde{z}(t|t_k)$, satisfying

$$c_i(\tilde{x}(t|t_k), \tilde{y}(t|t_k), \tilde{\theta}(t|t_k)) \leq c^M_i \triangleq \max_{\mu_s, \mu_i, \mu_r \in [-1, 1]} \{ c_i(\mu_s \eta_{1,3}, \mu_r \eta_{3,3}, \mu_r \alpha_{4,4}) \}$$

(20)

where $\eta_{1,3} = \alpha_{1,3} \sqrt{1 + a_{i,3}^2 a_{i,4}^2}$. Hence, the actual tightened state constraint can be developed as

$$\tilde{z}(t|t_k) \in \tilde{\mathbb{Z}} \triangleq \{ \tilde{x}; \tilde{y}; \tilde{\theta} | c_i(\tilde{x}, \tilde{y}, \tilde{\theta}) \leq c^M_i, i = 1, 2, \ldots, s \}$$

(21)
where \( \tilde{Z} \) is the actual tightened state constraint set. According to formulation (13), conditions (14c) and (14f), the mapped set \( \kappa[\mathcal{O}] \) can be determined as

\[
\kappa[\mathcal{O}] \triangleq \left\{ \begin{array}{c}
-\beta_{1,1} \tilde{x} - \beta_{1,2} \text{sgn}(\tilde{x}) \\
\beta_{1,3} \hat{\dot{\theta}} - \beta_{1,4} \text{sgn}\hat{\theta}
\end{array} \right\} \bigg| \begin{array}{c}
[\tilde{x}; \{\hat{\theta}\}] \in \mathcal{O} \text{ in (17)}
\end{array}
\]

\[
= \{ [\tilde{v}; \bar{\omega}] | [\bar{v}] \leq \beta_{1,1} \alpha_{i,3} + \beta_{1,2}, |\bar{\omega}| \leq \beta_{1,3} \alpha_{i,2} (v^M - \beta_{1,3} \alpha_{i,3} - \beta_{1,2}) + \beta_{1,4} \}. \tag{22}
\]

Then, following the definition of Pontryagin difference, we can develop the actual tightened input constraint as

\[
\tilde{u}(r|t_k) \in \tilde{U} \triangleq \left\{ [\tilde{v}; \bar{\omega}] | [\tilde{v}] \leq \tilde{v}^M \triangleq v^M - \beta_{1,1} \alpha_{i,3} - \beta_{1,2}, |\bar{\omega}| \leq \bar{\omega}^M \triangleq \omega^M - \beta_{1,3} \alpha_{i,2} (v^M - \beta_{1,3} \alpha_{i,3} - \beta_{1,2}) - \beta_{1,4} \right\}, \tag{23}
\]

where \( \tilde{U} \) is the actual tightened input constraint set.

Note that the RPI property of the set \( \mathcal{O} \) in (17) depends much on the chosen scalars \( \alpha_{i,1} \) and \( \beta_{1,i}, i = 1, 2, 3, 4 \), and so does the determination of the tightened constraint bounds \( \bar{c}^M_i, i = 1, 2, \ldots, s, v^M, \) and \( \bar{\omega}^M \). Here, we adapt the condition (14) to provide a guidance for choosing the scalars \( \alpha_{i,1} \) and \( \beta_{1,i}, i = 1, 2, 3, 4 \), as

\[
\beta_{1,1} \geq \alpha_{i,1} \alpha_{i,4} (\omega^M + d^M_{\omega}), \tag{24a}
\]

\[
\beta_{1,2} \geq d^M_v, \tag{24b}
\]

\[
\beta_{1,1} \alpha_{i,3} + \beta_{1,2} \leq v^M - v^M, \tag{24c}
\]

\[
\beta_{1,3} \geq 1 + \alpha_{i,1}^2 \alpha_{i,4}^2, \tag{24d}
\]

\[
\beta_{1,4} \geq \max \{\eta_{1,1}, \eta_{1,2}\}, \tag{24e}
\]

\[
\beta_{1,3} \alpha_{i,2} (v^M - \beta_{1,3} \alpha_{i,3} - \beta_{1,2}) + \beta_{1,4} \leq \omega^M - \omega^M, \tag{24f}
\]

\[
\max_{\mu_e, \mu_i, \mu_f \in \{-1, 1\}} \left\{ c_i (\mu_e \eta_{1,3}, \mu_i \eta_{1,3}, \mu_f \alpha_{i,4}) \right\} \leq c^M_i, i = 1, 2, \ldots, s, \tag{24g}
\]

with \( \eta_{i,1} = \frac{\omega^M + (\alpha_{i,1} - 1) \beta_{1,3} \alpha_{i,2} (v^M - \beta_{1,3} \alpha_{i,3} - \beta_{1,2}) + (\alpha_{i,1} + 1) \beta^M_{1,3} + \alpha_{i,1} \beta_{1,3} (d^M_{\omega} + \beta_{1,3} \alpha_{i,1} + \beta_{1,2})}{2 - \alpha_{i,1}}, \eta_{i,2} = d^M_{\omega} + \beta_{1,1} \alpha_{i,4} + \alpha_{i,2} (\beta_{1,2} + d^M_v). \]

### 3.3 Building of terminal constraint with choosing terminal cost

By slightly adapting the usual axioms in the work of Mayne et al\(^{15}\) (see p. 797) for the STRP in this paper, we can directly give the following conditions to build the terminal constraint set \( \mathcal{O}_T(\cdot) \) and to choose the terminal cost \( g(\cdot) \):

\[
\tilde{z}(t|t_k) \in \mathcal{O}_T(z_e(t)) \subseteq \tilde{Z}, \tag{25}
\]

\[
\tilde{u}(t|t_k) \in \tilde{U}, \tag{26}
\]

\[
\tilde{g}(\tilde{z}_e(t|t_k)) + L(t|t_k, \tilde{z}_e, \tilde{u}_e) \leq 0, \tag{27}
\]

for any \( \tilde{z}(r|t_k) \in \mathcal{O}_T(z_e(\tau)) \) and \( t \geq r \). According to the procedure in Section 3.1, we can build the terminal constraint set to guarantee conditions (25) and (26) as

\[
\mathcal{O}_T(z_e(t_k + T)) \triangleq \{ \tilde{z}(t_k + T|t_k)[(29) \text{ holds}] \}, \tag{28}
\]

with

\[
|\tilde{y}_e(t_k + T|t_k)| \leq \alpha_{T,1} |\tilde{x}_e(t_k + T|t_k)| \tag{29a},
\]

\[
|\tilde{\theta}_e(t_k + T|t_k)| \leq \alpha_{T,2} |\tilde{x}_e(t_k + T|t_k)|, \tag{29b}
\]

\[
|\tilde{x}_e(t_k + T|t_k)| \leq \alpha_{T,3}, \tag{29c}
\]

\[
|\tilde{\theta}_e(t_k + T|t_k)| \leq \alpha_{T,4} \leq \frac{\pi}{2}, \tag{29d}
\]

\[
\begin{cases}
\tilde{x}_e(t_k + T|t_k) \leq 0, & \tilde{y}_e(t_k + T|t_k) \tilde{\theta}_e(t_k + T|t_k) \geq 0, \text{ for } v_e(t) \geq 0, \\
\tilde{x}_e(t_k + T|t_k) \geq 0, & \tilde{y}_e(t_k + T|t_k) \tilde{\theta}_e(t_k + T|t_k) \leq 0, \text{ for } v_e(t) \leq 0,
\end{cases} \tag{29e}
\]
and adopt the terminal controller $\bar{u}(t|t_k) = \kappa(\bar{z}_e(t|t_k), u_c(t))$ with scalars $\beta_{T,i}$, $i = 1, 2, 3, 4$. Here, the scalars $\alpha_{T,i}$ and $\beta_{T,i}$, $i = 1, 2, 3, 4$, should satisfy the following conditions to guarantee the positive invariance of the built set $\mathcal{O}_T(z_e(t_k + T))$:

\begin{align}
\beta_{T,1} & \geq \max \left\{ \alpha_{T,1} \alpha_{T,4} \overline{M}, \alpha_{T,1} \alpha_{T,2} \sin \alpha_{T,4} \nu_T \right\}, \\
\beta_{T,1} \alpha_{T,3} + \beta_{T,2} & \leq \overline{v}_T - \nu_T, \\
\beta_{T,3} & \geq 1 + \alpha_{T,1}^2 \alpha_{T,4}^2, \\
\beta_{T,4} & \geq \max \{ \eta_{T,1}, \eta_{T,2} \}, \\
\beta_{T,3} \alpha_{T,2} \phi_T^M + \beta_{T,4} & \leq \alpha_T - \alpha_T^M, \\
\max_{\nu_T, \nu_T, \nu_T \in [-1, 1]} \left\{ c(\mu, \eta_T, \mu_T, \eta_T, \mu_T, \mu_T) \right\} & \leq \bar{c}_T - c_T^M, \quad i = 1, 2, \ldots, s,
\end{align}

(30a) (30b) (30c) (30d) (30e) (30f)

where $\eta_{T,1} = \alpha_T^M + \alpha_{T,1} \alpha_{T,4} (\beta_{T,1} \alpha_{T,3} + \beta_{T,2}) + \alpha_{T,1} (\alpha_T^M - \alpha_T^M), \eta_{T,2} = \beta_{T,1} \alpha_{T,4} + \alpha_{T,2} \beta_{T,2}, \eta_{T,3} = \alpha_{T,1} \alpha_{T,4} \sqrt{1 + \alpha_{T,1}^2 \alpha_{T,4}^2}$. Because both states $z(t)$ and $\bar{z}(t)$ are evolved along the nominal dynamics $f(\cdot, \cdot, 0)$, condition (30) is an adaption of (14) with setting $\alpha_T = \alpha_T^M = 0$, then one can choose $\beta_{T,2} = 0$ for the adopted terminal controller.

By choosing the terminal cost

$$\mathcal{L}(z_e(t)) = \frac{\gamma}{2} (\bar{x}_e^2(t) + \bar{y}_e^2(t) + \gamma |\bar{\theta}_e(t)|$$

(31)

with the positive weighting scalar satisfying

$$\gamma \geq \max \left\{ \frac{\lambda^M (1 + \alpha_{T,1} \alpha_{T,4}^2 + \alpha_{T,2}^2)}{\beta_{T,1} - \alpha_{T,1} \alpha_{T,2} \sin \alpha_{T,4} \nu_T^M}, \frac{\lambda^M (\beta_{T,1} \alpha_{T,2} \phi_T^M + \beta_{T,4})}{\lambda^M (\beta_{T,1} \alpha_{T,2} \phi_T^M + \beta_{T,4})} \right\}.$$  

(32)

we can guarantee condition (27) as follows (time indices “(t|t_k)” and “(t)” are omitted):

$$\mathcal{L}(z_e(t|t_k)) + L(t|t_k, z_e, u_c) \leq \gamma \left( \bar{x}_e \nu_T + \bar{y}_e v_T \sin \bar{\theta}_e + \text{sgn}(\bar{\theta}_e) \bar{\omega}_e \right) + \lambda^M \left( \bar{x}_e^2 - \bar{y}_e^2 + \bar{\theta}_e^2 \right)$$

$$\leq - \gamma \left( \beta_{T,1} \bar{x}_e^2 - |\bar{y}_e v_T \sin \bar{\theta}_e| + \left( \frac{\beta_{T,1} \bar{x}_e v_T \bar{\theta}_e}{\bar{x}_e^2 + \bar{y}_e^2} + \beta_{T,4} \right) |\text{sgn}(\bar{\theta}_e)| \right)$$

$$+ \lambda^M \left( \bar{x}_e^2 - \bar{y}_e^2 + \bar{\theta}_e^2 \right) + \lambda^M \left( \frac{\beta_{T,1} \bar{x}_e v_T \bar{\theta}_e}{\bar{x}_e^2 + \bar{y}_e^2} + \beta_{T,4} \right)^2 |\text{sgn}(\bar{\theta}_e)|$$

$$\leq - \left( \gamma \left( \beta_{T,1} - \alpha_{T,1} \alpha_{T,2} \sin \alpha_{T,4} \nu_T^M \right) - \lambda^M \beta_{T,1} - \lambda^M (1 + \alpha_{T,1}^2 \alpha_{T,4}^2 + \alpha_{T,2}^2) \right) \bar{x}_e^2$$

$$- \left( \beta_{T,1} \alpha_{T,2} \phi_T^M + \beta_{T,4} \right) \gamma - \lambda^M \left( \beta_{T,1} \alpha_{T,2} \phi_T^M + \beta_{T,4} \right) |\text{sgn}(\bar{\theta}_e)|$$

$$\leq 0.$$  

(33)

### 3.4 Implementation of robust RHC

Now, by substituting the actual initial and tightened constraints, the built terminal constraint set, and the chosen terminal cost into the conceptual optimization problem (8), an actual optimization problem can be formulated as

$$\min_{z_e(t_k | t_k), u_c(t_k)} J(t_k, z_e, u_c)$$  

(34a)

s.t. (19), (29), for all $\tau \in [t_k, t_k + T], (8c), (21)$ and (23),

(34b)

and the robust RHC algorithm is provided as follows.

According to the common practice of RHC, the implementation of RHC algorithm assumes trivial on-line optimization time, which may hinder the practical application of Algorithm 1. To meet the requirement of real-time control in
practice, we can adapt Algorithm 1 by avoiding the on-line optimization procedure. For expression convenience, we firstly define a nominal control input candidate at time instant $t_k$ as

$$\bar{u}^\ast(\tau|t_k) = \begin{cases} \bar{u}^\ast(\tau|t_k), & \tau \in [t_k, t_k + T), \\ \left( \tilde{z}^\ast(\tau|t_k) - z_r(\tau), \bar{u}^\ast(\tau|t_k) \right), & \tau \geq t_k + T, \end{cases}$$

(35)

with the chosen scalars $\beta^{T, i}, i = 1, 2, 3, 4$, where $\tilde{z}^\ast(\tau|t_k) = \Phi(\bar{u}^\ast(\tau|t_k))$.

Then, the adapted algorithm is presented as follows.

**Algorithm 2** Robust RHC algorithm without on-line optimization

**Off-line stage:** For the given bounds $c_i^M, i = 1, 2, \ldots, s, \nu^M, \omega^M, \nu^M, \omega^M, \alpha_i^M, i = 1, 2, \ldots, s, \nu^M$ and $\omega^M$, choose some proper real scalars $\alpha_{i,j} \geq 0$ and $\beta_{i,j} > 0$, $i = 1, 2, 3, 4$ satisfying the condition (24), then determine the tightened constraint bounds $z_i^M, i = 1, 2, \ldots, s, \nu^M$ and $\omega^M$ according to (20) and (23); choose some proper real scalars $\alpha_{T,i} \geq 0$ and $\beta_{T,i}, i = 1, 2, 3, 4$ satisfying the condition (30), then choose the weighting scalar $\gamma$ satisfying the condition (32).

**On-line stage:** For any time $\tau \geq t_0$, the vehicle measures the real-time state $z(\tau)$ and implements the control input (9) with the chosen scalars $\beta^{T, i}, i = 1, 2, 3, 4$.

In both Algorithm 1 and Algorithm 2, the initial constraint set $\mathcal{O}_I(\cdot)$ and the terminal constraint set $\mathcal{O}_T(\cdot)$ are adopted with different controllers. It deserves to note that the controller associated with set $\mathcal{O}_T(\cdot)$ is never used in Algorithm 1, and the scalars $\beta^{T, i}, i = 1, 2, 3, 4$, are only used to determine the weighting scalar $\gamma$. However, the controller associated with set $\mathcal{O}_I(\cdot)$ is utilized in Algorithm 1 (Algorithm 2) to correct the solved control input $\bar{u}^\ast(\tau|t_0)$ (off-line determined control input $\bar{u}^\ast(\tau|t_0)$) for actual implementation. The relationships between the both constraint sets and between different trajectories are illustrated in Figure 2.

Note that the implementation of Algorithm 1 is a recursive application of the first part of (36), and the implementation of Algorithm 2 is an application of (36) calculated at initial time instant $t_0$. Hence, if the recursive feasibility and robust convergence (ie, the trajectory of perturbed vehicle (1) converges robustly to that of the desired reference (4)) can be guaranteed by applying (36) calculated at any time instant $t_k$, they can also be guaranteed by both Algorithm 1 and Algorithm 2. These properties are illustrated as follows.

**Theorem 2.** For a perturbed vehicle (1) with disturbance $d(t) \in \mathbb{D}$, and a desired reference (4) satisfying (5) and (6), if the optimization problem (34) is feasible at any time instant $t_0$, then, by implementing the control input $u(\tau) = u^\ast(\tau|t_0)$ in (36), we have the following.

a) The problem (34) is always feasible for $\tau \geq t_k$.

b) The actual state and the implemented control input always satisfy the desired constraints (2) and (3), respectively.
c) The perturbed vehicle (1) will be steered to track the desired reference (4) robustly, i.e., the tracking error state $z_e(t)$ will converge into the RPI set $\mathcal{S}$ described in (11) with $a_{I,i}$, $i = 1, 2, 3, 4$, and finally converges to zero.

**Proof.** At the time instant $t_k$, the feasibility of problem (34) yields the optimal solution $\bar{z}^*(t_k|t_k)$ and $\bar{u}^*(\tau|t_k)$, $\tau \in [t_k, t_k + T]$, then the nominal control input candidate $\bar{u}^*(\tau|t_k)$ in (35) and associated nominal state $\bar{z}^*(\tau|t_k)$ can be determined. By applying $u(t) = u^*(t|t_k)$ in (36), we have the following derivations.

a) According to the procedures in Section 3.2 and Section 3.3, the satisfactions of (19) and (29) are, respectively, equivalent to $\bar{z}^*(t_k|t_k) = \bar{z}^*(t_k|t_k) \in \mathcal{O}_T(z(t_k))$ with $a_{I,i}$ and $\bar{z}^*(t_k + T|t_k) = \bar{z}^*(t_k + T|t_k) \in \mathcal{O}_T(z(t_k + T))$ with $a_{T,i}$, $i = 1, 2, 3, 4$. Due to the robust positive invariance of $\mathcal{O}_T(\cdot)$, $\bar{z}^*(t|t_k) \in \mathcal{O}_T(z(t))$ can be guaranteed for all $t \geq t_k$. The implementation of the control input $u(t) = u^*(t|t_k)$ in (36) guarantees $\bar{u}^*(t|t_k) \in \mathcal{U}$ in (23) for $t \in [t_k, t_k + T]$ and yields $\bar{z}^*(t|t_k) = \bar{z}^*(t|t_k) \in \mathcal{Z}$ in (21) for $t \in [t_k, t_k + T]$. The positive invariance of $\mathcal{O}_T(\cdot)$ can guarantee $\bar{z}^*(t|t_k) \in \mathcal{O}_T(z(t_k)) \subseteq \mathcal{Z}$ and $\bar{u}^*(t|t_k) \in \mathcal{U}$ for all $t \geq t_k + T$. Hence, for any time $t \geq t_k$, the optimization problem (34) can always find a feasible solution as $\bar{z}^*(t|t_k)$ and $\bar{u}^*(\tau|t_k) = \bar{u}^*(\tau|t_k)$, $\tau \in [t, t + T]$, then conclusion a) holds.

b) According to the procedures in Section 3.2, for any $\bar{z}(t|t_k) = \bar{z}^*(t|t_k) \in \mathcal{O}_T(z(t))$ and $t \geq t_k$, the satisfaction of constraints (21) and (23) guarantees $z(t) \in \mathcal{Z}$ and $u^*(t|t_k) \in \mathcal{U}$. Then, following conclusion a), the desired constraints (2) and (3) are always satisfied by implementing the control input $u(t) = u^*(t|t_k)$ in (36), hence conclusion b) holds.

c) By choosing the Lyapunov function as $V(t_k) = J(t_k, \bar{z}_e, \bar{u}_e)$, condition (27) guarantees

$$V(t_{k+1}) - V(t_k) = J(t_{k+1}, \bar{z}_e^*, \bar{u}_e^*) - J(t_k, \bar{z}_e^*, \bar{u}_e^*)$$

$$= \int_{t_k}^{t_{k+1}+T} L(\tau|t_k, \bar{z}_e^*, \bar{u}_e^*) \, d\tau + g(\bar{z}_e^*(t_{k+1} + T|t_k)) - \int_{t_k}^{t_{k+1}+T} L(\tau|t_k, \bar{z}_e^*, \bar{u}_e^*) \, d\tau - g(\bar{z}_e^*(t_k + T|t_k))$$

$$= - \int_{t_k}^{t_{k+1}+T} L(\tau|t_k, \bar{z}_e^*, \bar{u}_e^*) \, d\tau + \int_{t_k}^{t_{k+1}+T} (L(\tau|t_k, \bar{z}_e^*, \bar{u}_e^*) - L(\tau|t_k, \bar{z}_e^*, \bar{u}_e^*)) \, d\tau$$

$$+ \int_{t_k}^{t_{k+1}+T} L(\tau|t_k, \bar{z}_e^*, \bar{u}_e^*) \, d\tau + g(\bar{z}_e^*(t_{k+1} + T|t_k)) - g(\bar{z}_e^*(t_k + T|t_k))$$

$$\leq - \int_{t_k}^{t_{k+1}+T} L(\tau|t_k, \bar{z}_e^*, \bar{u}_e^*) \, d\tau,$$

for any $s \geq k$, which implies $\lim_{t \to \infty}(\bar{z}_e^*(t|t_k) - z_e(t)) = 0$. Additionally, the recursive satisfaction of constraint (19) in problem (34) guarantees $\bar{z}^*(t|t_k) \in \mathcal{O}_T(z(t))$ in (18), and the implementation of the control input (36) guarantees $\lim_{t \to \infty} \bar{z}^*(t|t_k) = 0$ (more details can refer to the proof of theorem 1 in our conference work[28]). Hence, the tracking error state $z_e(t)$ will converge into the RPI set $\mathcal{S}$ described in (11) with $a_{I,i}$, $i = 1, 2, 3, 4$, and finally converges to zero, i.e., conclusion c) holds.

These complete the proof.
Following the proof of Theorem 2, we can conclude the main result as follows. For the perturbed nonholonomic vehicle (1) with \(d(t) \in \mathbb{D}\) and the reference trajectory (4) satisfying (5) and (6), if the optimization control problem (34) is feasible at the initial time instant \(t_0\), then, by implementing either Algorithm 1 or Algorithm 2, the perturbed vehicle can be steered to robustly track the desired reference trajectory with null steady-state tracking error and always satisfies the desired constraints (2) and (3). It deserves to note that the control implementation depends much on the initial feasibility of problem (34), and the robust convergence depends much on the size of the RPI set \(S\) with \(\alpha_{1,i}, i = 1, 2, 3, 4\). We prefer to choose smaller scalars \(\alpha_{1,i}, i = 1, 2, 3, 4\), for better robust convergence (ie, smaller deviation between the actual state \(z(t)\) and the optimal state \(\tilde{z}^*(t|t_k)\)), whereas choose larger scalars \(\beta_{1,i}, i = 1, 2, 3, 4\), for faster convergence. However, it can result in a smaller feasible domain. To offset this effect and enlarge the feasible domain, we suggest to choose the scalars \(\alpha_{1,i}, i = 1, 2, 3, 4\), large enough.

Remark 3. If the initial constraint set is chosen as the minimal one, ie, \(O^m_i(z(t_k)) = \{\tilde{z}(t_k|t_k) | \tilde{z}(t_k|t_k) = z(t_k)\}\), then the original state constraint (2) needs not to be tightened, and the minimal parameters \(\beta_{1,1} = 0, \beta_{1,2} = d^m_0, \beta_{1,3} = 1\), and \(\beta_{1,4} = \max(\frac{\omega^M + \omega^H}{2}, d^M_0)\) can be chosen to reduce the conservatism. With this choice, the actual state \(z(t)\) is evolved the same as the solved optimal state \(\tilde{z}^*(t|t_k)\) because \(\tilde{z}^*(t|t_k) \in O^m_i(z(t))\) is guaranteed for any \(t \geq t_0\) by implementing the control input \(u(t) = u^*(t|t_k)\) in (36).

4 | SIMULATION RESULTS

To verify the validity of the proposed approach, we give a simulation example to illustrate the effectiveness of the main results for robust STRP and a simulation example to provide performance comparison with other approaches. All the simulation examples are implemented in MATLAB on a PC, with a dual-core 3.20 GHz Intel i5 CPU and a 3.47 GB RAM. The optimization problem of the vehicle is described and transcribed by the Imperial College London Optimal Control Software (ICLOCS)\(^{13}\) and solved by the open-source code Interior Point OPTimizer (IPOPT).\(^{44}\)

4.1 | Illustrative example: simultaneous tracking and regulation problem

Consider a perturbed wheeled vehicle with the state constraint functions \(c_1(x, y, \theta) = |x|\) and \(c_2(x, y, \theta) = |y|\), the state constraint bounds \(c^M_1 = c^M_2 = 15\) m, and the speed limits \(v^M = 3\) m/s and \(\omega^M = 2.5\) rad/s. The disturbance is composed of the constant term \(d_1(t) = [0.1; 0.05]\) and the time-varying term \(d_2(t) = [0.4 \sin 0.5t; 0.25 \cos 0.5t]\), hence the bounds of disturbance \(d(t) = d_1(t) + d_2(t)\) can be calculated as \(d^M_0 = 0.5\) m/s and \(d^M_0 = 0.3\) rad/s. The reference trajectory (a parallel parking problem is chosen for testing forward STRP) is described as

\[
x_r(t) = \begin{cases} 
10 \cos \varphi(t), & 0 \leq t < 5\pi, \\
-5\sqrt{2}, & t \geq 5\pi,
\end{cases}
\]

\[
y_r(t) = \begin{cases} 
5 \sin 2\varphi(t), & 0 \leq t < 5\pi, \\
-5, & t \geq 5\pi,
\end{cases}
\]

with \(\varphi(t) = 0.1t + \frac{\pi}{4}\). This trajectory is evolved along the dynamics (4) from initial state \(z_r(0) = [5\sqrt{2}\text{ m}; 5\text{ m}; \pi\text{ rad}]\) by implementing control input\(^{13}\)

\[
v_r(t) = \begin{cases} 
\sqrt{\sin^2 \varphi(t) + \cos^2 2\varphi(t)}, & 0 \leq t \leq 5\pi, \\
0, & t \geq 5\pi,
\end{cases}
\]

\[
\omega_r(t) = \begin{cases} 
\frac{0.1(\cos \varphi(t) \cos 2\varphi(t) + 2 \sin \varphi(t) \sin 2\varphi(t))}{\sin^2 \varphi(t) + \cos^2 2\varphi(t)}, & 0 \leq t \leq 5\pi, \\
0, & t \geq 5\pi.
\end{cases}
\]

Then, the state and input constraint bounds of reference can be calculated as \(c^M_1 = 5\sqrt{2}\text{ m}, c^M_2 = 5\text{ m}, v^M = \sqrt{2}\text{ m/s, and }\omega^M = \frac{\sqrt{2}}{5}\text{ rad/s.}\) To clearly show the effectiveness of the proposed algorithms, four simulation cases are provided by choosing different initial states for the perturbed wheeled vehicles as \(z_1(0) = [0\text{ m}; 15\text{ m}; \frac{3\pi}{2}\text{ rad}], z_2(0) = [15\text{ m}; 14\text{ m}; \pi\text{ rad}], z_3(0) = [15\text{ m}; -10\text{ m}; \pi\text{ rad}],\) and \(z_4(0) = [0\text{ m}; -9\text{ m}; \frac{\pi}{2}\text{ rad}].\)
In this simulation, we choose the weighting matrices \( Q = \text{diag}(0.5, 0.5, 0.1) \) and \( R = \text{diag}(0.1, 0.1) \), the prediction horizon \( T = 6 \) s, the sampling period (ie, time interval between two adjacent sampling instants \( t_k \) and \( t_{k+1} \) ) \( \delta = 2 \) s, and the simulation step size (ie, time interval between two adjacent simulation steps) \( \Delta t = 0.02 \) s. According to the guidance condition (24), a proper initial constraint set \( \mathcal{O}_c(\cdot) \) is chosen with \( \alpha_{T,1} = 0.03, \alpha_{T,2} = 0.05, \alpha_{T,3} = 1, \) and \( \alpha_{T,4} = 0.05, \) and the associated controller \( \kappa(\cdot, \cdot) \) is chosen with \( \beta_{I,1} = 0.1, \beta_{I,2} = 0.5, \beta_{I,3} = 1.1, \) and \( \beta_{I,4} = 1.362. \) Then, the tightened state constraint set \( \mathcal{Z} \) can be determined with \( \mathcal{Z}_T = \mathcal{Z}_I \approx 14 \) m according to (20), and the tightened input constraint set \( \mathcal{U} \) can be determined with \( \mathcal{U}_T = 2.4 \) m/s and \( \mathcal{U}_I = 1.006 \) rad/s according to (23). For better robust convergence, the minimal initial constraint set \( \mathcal{O}_c(\cdot) \) and the associated controller can be chosen according to Remark 3. Then, the tightened state constraint set \( \mathcal{Z} = \mathcal{Z}_I \) and the tightened input constraint set \( \mathcal{U} \) can be determined with \( \mathcal{U}_T = 2.5 \) m/s and \( \mathcal{U}_I = 1.1 \) rad/s. According to condition (30), a large terminal constraint set \( \mathcal{O}_T(\cdot) \) is chosen with \( \alpha_{T,1} = 0.01, \alpha_{T,2} = 0.2, \alpha_{T,3} = 10, \) and \( \alpha_{T,4} = 0.7854, \) and the associated controller \( \kappa(\cdot, \cdot) \) is chosen with \( \beta_{T,1} = 0.1, \beta_{T,2} = 0, \beta_{T,3} = 1.01, \) and \( \beta_{T,4} = 0.293. \) Following requirement (32), the terminal weighting scalar is chosen as \( \gamma = 8. \)

By applying Algorithm 1 with either \( \mathcal{O}_c(\cdot) \) or \( \mathcal{O}_c(\cdot) \) for each simulation case, the optimization problem (34) is always feasible over the whole simulation horizon [0 s, 16 s], and the associated optimal solutions are obtained. This verifies the recursive feasibility of Algorithm 1. Under the corrected control input (9), the solved nominal and actual motion trajectories of the perturbed vehicle (see \( \mathbf{p}_i(t_k) \) and \( \mathbf{p}_i(t_k) \) for Case \( i = 1, 2, 3, 4 \) with choosing \( \mathcal{O}_c(\cdot) \) and \( \mathcal{O}_c(\cdot) \) are shown in Figure 3A and Figure 3B, respectively. By applying Algorithm 2 for each simulation case, the optimization problem (34) is solved only at the initial time instant \( t_0. \) Then, under the control input \( u(t) = u^* (t|t_0) \) in (36), the motion trajectories of perturbed vehicle are shown in Figure 3C. In Figure 3, the vehicles are denoted by circles with orientation arrows, and the actual (solved nominal) start poses by thick (thin) orientation arrows. It can be seen that, although the wheeled vehicle is perturbed by uncertain disturbance \( d(t) \), it can be steered to robustly achieve the simultaneous tracking and regulation objectives. Note in Figure 3 that choosing a large initial constraint set \( \mathcal{O}_c(\cdot) \) allows large deviation between the solved nominal and actual motion trajectories (see Figure 3B and Figure 3C).

To provide a clear illustration for the tracking and regulation process, we plot the solved nominal and actual state trajectories of all simulation cases in Figure 4, Figure 5, and Figure 6. These figures show that, by applying either Algorithm 1 or Algorithm 2, the actual evolved states (see \( \mathbf{x}_i(t), y_i(t), \) and \( \mathbf{e}_i(t) \) for Case \( i = 1, 2, 3, 4 \) can track the solved nominal trajectories (see \( \tilde{X}_i(t_k), \tilde{Y}_i(t_k), \) and \( \tilde{D}_i(t_k) \) for Case \( i = 1, 2, 3, 4 \) closely and converge to the final reference state \( \mathbf{z}_i(t_k) = [-5\sqrt{2}; -5; \pi] \) without steady-state tracking error. This verifies conclusion c) in Theorem 2. Note in these figures (especially in their magnified plots) that the solved nominal and actual state trajectories are allowed to be different by choosing the proper initial constraint set \( \mathcal{O}_c(\cdot) \) and are enforced to be identical by choosing the minimal initial constraint set \( \mathcal{O}_c(\cdot). \) This allowed difference can possibly induce sudden changes of the solved nominal state trajectory at each sampling instant \( t_k \), and then leads to less smooth of the actual state trajectory (see Figure 4A, Figure 5A, and Figure 6A).
The linear (angular) speeds of solved nominal and actually implemented control inputs (see $\dot{v}_i(t|t_k)$ and $v_i(t)$, $\dot{\omega}_i(t|t_k)$, and $\omega_i(t)$) for Case $i$, $i = 1, 2, 3, 4$) are plotted in Figure 7 (Figure 8) for all simulation cases. It can be easily seen that both the solved nominal and actually implemented control inputs converge to the reference input. Note in Figure 4, Figure 5, Figure 7, and Figure 8 that, as the solved nominal state and input satisfying the actual tightened constraints (21) and (23), respectively, the admissibility of the actually evolved state and implemented control input is guaranteed, ie, conclusion b) in Theorem 2 is verified. Hence, the effectiveness of the proposed robust RHC algorithms is demonstrated.

### 4.2 Comparative example: tracking problem

Consider a tracking problem adopted in the work of Sun et al,\textsuperscript{24} where the desired reference trajectory can be described as

$$
\begin{align*}
    x_r(t) &= 0.475 \left( -\frac{\sqrt{3}}{2} + \sin \left( 0.04t + \frac{\pi}{3} \right) \right), \\
    y_r(t) &= 0.475 \left( \frac{1}{2} - \cos \left( 0.04t + \frac{\pi}{3} \right) \right),
\end{align*}
$$

\text{\textsuperscript{(40)} (41)}
FIGURE 6  The orientation trajectories for all simulation cases. A, Applying Algorithm 1 with $\mathbb{O}_1^p(\cdot)$; B, Applying Algorithm 1 with $\mathbb{O}_1^m(\cdot)$; C, Applying Algorithm 2 with $\mathbb{O}_1^m(\cdot)$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 7  The linear speeds for all simulation cases. A, Applying Algorithm 1 with $\mathbb{O}_1^p(\cdot)$; B, Applying Algorithm 1 with $\mathbb{O}_1^m(\cdot)$; C, Applying Algorithm 2 with $\mathbb{O}_1^m(\cdot)$ [Colour figure can be viewed at wileyonlinelibrary.com]

which is evolved along dynamics (4) from initial state $z_r(0) = [0 \text{ m}; 0 \text{ m}; \frac{\pi}{3} \text{ rad}]$ by implementing control input $u_r(t) = [0.019 \text{ m/s}; 0.04 \text{ rad/s}]$. In this simulation, we choose the initial state of the perturbed vehicle $z(0) = [0.4 \text{ m}; -0.2 \text{ m}; \frac{\pi}{2} \text{ rad}]$, the weighting matrices $Q = \text{diag}[0.2, 0.2, 0]$ and $R = \text{diag}[0.4, 0.4]$, the prediction horizon $T = 10$ s, the sampling period $\delta = 5$ s, and the simulation step size $\Delta t = 0.05$ s.

For algorithm 1 proposed in the work of Sun et al.\textsuperscript{24} (referred to as algorithm Sun1 hereafter), the input constraint is desired as

$$u(t) \in \mathbb{U} = \left\{ [v(t); \omega(t)] \left| \frac{|v(t)|}{0.13} + \frac{|\omega(t)|}{4.8598} \leq \lambda = 1 \right\},$$

the disturbance is introduced only on the linear speed and bounded by $\eta = 0.015$, the terminal feedback gain $\tilde{k}_1 = \tilde{k}_2 = 1.2$, and the feedback gain $K = \text{diag}[-2.0, -2.0]$ are chosen the same as in the work of Sun et al.\textsuperscript{24} According to the procedures in the work of Sun et al.\textsuperscript{24} the aforementioned parameters induce the tightened input constraint set $\mathbb{U}_{\text{tube}} = \lambda_{\text{tube}} \mathbb{U}$ with $\lambda_{\text{tube}} = 0.5439$, and the terminal region $\Omega_{\text{tube}} = \{ \tilde{p}_e \mid |\tilde{x}_e| + |\tilde{y}_e| \leq 0.0422 \}$. For Algorithm 1 in this paper, the speed limits are set as $v^M = 0.1 \text{ m/s}$ and $\omega^M = 0.8 \text{ rad/s}$ to satisfy (42), the external disturbances acting on both linear and angular speeds are considered and bounded as $d^v_{\text{ext}} = d^\omega_{\text{ext}} = 0.015$, the minimal initial constraint set $\mathbb{O}_1^m(\cdot)$ and
FIGURE 8  The angular speeds for all simulation cases. A, Applying Algorithm 1 with $\mathbb{O}_i^P(\cdot)$; B, Applying Algorithm 1 with $\mathbb{O}_i^M(\cdot)$; C, Applying Algorithm 2 with $\mathbb{O}_i^M(\cdot)$ [Colour figure can be viewed at wileyonlinelibrary.com]

FIGURE 9  The control performance comparison between the approaches proposed in this paper and in the work of Sun et al.24 A, The motion trajectories; B, The x- and y-direction trajectories; C, The position tracking errors [Colour figure can be viewed at wileyonlinelibrary.com]

its associated controller are adopted according to Remark 3. Then, the tightened input constraint set $\bar{U}$ can be determined with $\bar{v}_M = 0.085\text{m/s}$ and $\bar{\omega}_M = 0.392\text{rad/s}$. According to condition (30), the terminal constraint set $\mathbb{O}_T(\cdot)$ can be chosen with $\alpha_{T,1} = 0.01$, $\alpha_{T,2} = 0.2$, $\alpha_{T,3} = 2$, and $\alpha_{T,4} = 0.4$, and the associated controller $\kappa(\cdot, \cdot)$ can be chosen with $\beta_{T,1} = 0.04$, $\beta_{T,2} = 0$, $\beta_{T,3} = 1.01$, and $\beta_{T,4} = 0.044$. Following requirement (32), the terminal weighting scalar is chosen as $\gamma = 6$.

By applying algorithm Sun1 and Algorithm 1 in this paper with $\mathbb{O}_i^M(\cdot)$, the perturbed vehicle can be steered to robustly track the desired reference trajectory, with its motion trajectories, position trajectories, and position tracking errors shown in Figure 9. It can be easily seen from the magnified plots in Figure 9 that, although both the linear and angular speeds are perturbed, Algorithm 1 proposed in this paper can achieve null steady-state tracking error, hence provides better control performance than Algorithm Sun1. This demonstrates the advantage of our proposed approach.

5 | CONCLUSION

In this paper, a synthesis approach of robust RHC is studied for the STRP of perturbed wheeled vehicle. Based on a designed RPI set, a tube-based RHC algorithm is proposed for the STRP of perturbed wheeled vehicle, by developing the
initial constraint, the tightened state and input constraints and, a nominal terminal constraint. With the RPI set given in a general structure, much convenience can be provided to specify the set to meet miscellaneous requirements, eg, specify a smaller initial constraint set to achieve better robust convergence and a larger terminal constraint set to enlarge the feasible domain. Due to the adopted common measures of tube-based approach, all the advanced properties of the robust RHC (eg, the recursive feasibility and the robust convergence) are inherited naturally, and the proposed algorithm is enabled for real-time application without on-line optimization.

As a tube-based approach for the perturbed wheeled vehicle, the proposed robust RHC algorithm has comparable complexity with the nominal one. Additionally, the imposed initial constraint and its associated control law can guarantee that the actual evolved trajectory always lies in a specified set around the reference trajectory and finally converges to the reference trajectory with null steady-state tracking error. These factors facilitate the extension of the proposed robust RHC approach for distributed case, which will be considered in our future research works.

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