ELLIPTIC SOLUTIONS TO NONSYMMETRIC MONGE-AMПÈRE TYPE EQUATIONS I. THE $d$-CONCAVITY AND THE COMPARISON PRINCIPLE

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Abstract. We introduce the so-called $d$-concavity, $d \geq 0$, and prove that the nonsymmetric Monge-Ampère type function of matrix variable is concave in an appropriate unbounded and convex set. We prove also the comparison principle for nonsymmetric Monge-Ampère type equations in the case when they are so-called $\delta$-elliptic with respect to compared functions with $0 \leq \delta < 1$.

1. Introduction

In this paper we consider the following nonsymmetric Monge-Ampère type equations:

\[ (1.1) \quad \det [D^2u - A(x, u, Du) - B(x, u, Du)] = f(x, u, Du), \quad x \in \Omega, \]

where $\Omega$ is a bounded domain in $n$ dimensional Euclidean space $\mathbb{R}^n$ with smooth boundary, $Du$ and $D^2u$ denote the gradient vector and the Hessian matrix of the second order derivatives of the function $u : \Omega \to \mathbb{R}$, respectively, $A$ is a given $n \times n$ symmetric matrix function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, $B$ is a given $n \times n$ skew-symmetric matrix function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$, $f$ is a positive scalar valued function defined on $\Omega \times \mathbb{R} \times \mathbb{R}^n$. As usual, we use $x, z, p, r$ to denote points in $\Omega, \mathbb{R}, \mathbb{R}^n, \mathbb{R}^{n \times n}$, respectively.

In the case that $B(x, z, p) \equiv 0$, equation (1.1) becomes

\[ (1.2) \quad \det [D^2u - A(x, u, Du)] = f(x, u, Du), \quad x \in \Omega. \]

For functions $u(x) \in C^2(\Omega)$, we set

\[ (1.3) \quad \omega(x, u) \equiv D^2u(x) - A(x, u(x), Du(x)). \]

We recall that the equation (1.1) or (1.2) is elliptic with respect to function $u(x) \in C^2(\Omega)$ whenever

\[ \lambda_{\min}(\omega(x, u)) > 0, \quad \forall x \in \Omega. \]

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Here and in what follows, we denote by $\lambda_{\min}(M)$ the smallest eigenvalue of a symmetric matrix $M \in \mathbb{R}^{n \times n}$.

For the Dirichlet problem for equation (1.2), the existence of elliptic solutions was settled in [4], [5], [6] by the method of continuity. In this method, the solvability of the Dirichlet problem is reduced to the establishment of $C^{2,\alpha}(\Omega)$ estimates for its elliptic solutions. It is well-known that the concavity of the following function

$$F(\omega) = \log(\det \omega),$$

considered as a function on the set of symmetric positive definite matrices $\omega = [\omega_{ij}]_{n \times n}$, has one of essential roles in establishing these a priori estimates.

As it had been remarked in [6], the question on the solvability of Dirichlet problem for equations (1.1) when $B(x, z, p) \not\equiv 0$ is an open one. To investigate this problem, instead of the function $F(\omega)$, we will consider in the below the following function of matrix variable,

$$(1.4) \quad F(R) = \log(\det R),$$

where $R = [R_{ij}] \in \mathbb{R}^{n \times n}$, which is represented by the form

$$R = \omega + \beta, \quad \omega^T = \omega, \quad \omega > 0, \quad \beta^T = -\beta.$$

We will show in the below that $\det \beta \geq 0$ and will prove that

$$\det R = \det(\omega + \beta) \geq \det \omega + \det \beta > 0.$$

Thus the matrix $R = \omega + \beta$ is always non-singular, the function $F(R)$ is well-defined and infinitely differentiable. The function $F(R)$ is called the Monge-Ampère type function, associated to equation (1.1).

Suppose that $\delta, \mu$ are fixed nonnegative numbers, where $\delta \in [0, 1)$. For the function $F(R)$, we consider the following set of matrices:

$$(1.5) \quad D_{\delta,\mu} \equiv \left\{ R \mid R = \omega + \beta, \omega^T = \omega, \beta^T = -\beta, \lambda_{\min}(\omega) > 0, \delta \lambda_{\min}(\omega) \geq \mu, \|\beta\| \leq \mu \right\}.$$

Here and in what follows, $\| \cdot \|$ denotes the operator norm on $\mathbb{R}^{n \times n}$. It is easy to verify that $D_{\delta,\mu}$ is an unbounded and convex set in $\mathbb{R}^{n \times n}$. If $\delta = 0$ then $\mu = 0$, $\beta = 0$ and the set $D_{0,0}$ consists of symmetric positive definite matrices. In order to generalise the notion of usual concavity for the function $\log(\det \omega)$, we introduce the so-called $d$-concavity for the function $F(R)$.

**Definition 1.** Suppose that $d \geq 0$ is a nonnegative number. The function $F(R)$ is said to be $d$-concave in the set $D_{\delta,\mu}$ if for any matrices $R^{(0)} = [R^{(0)}_{ij}]_{n \times n}$ and
\[ R^{(1)} = \begin{bmatrix} R^{(1)}_{ij} \end{bmatrix}_{n \times n} \] from \( D_{\delta,\mu} \), we have
\[
F(R^{(1)}) - F(R^{(0)}) \leq \sum_{i,j=1}^{n} \frac{\partial F(R^{(0)})}{\partial R_{ij}} (R^{(1)}_{ij} - R^{(0)}_{ij}) + d.
\]

When \( d = 0 \), the 0-concavity is indeed the usual concavity. One of our main results in this paper is the Theorem in which we prove that the function \( F(R) \) is \( d \)-concave in the set \( D_{\delta,\mu} \) with some \( d \geq 0 \), which depends only on \( \delta \) and \( n \).

Another aspect of our studying in this paper is the comparison principle for nonsymmetric Monge-Ampère type equations (1.1). It is well-known that when \( B(x, z, p) \equiv 0 \), the comparison principle holds for elliptic solutions to the equation (1.2). In [2], this principle had been considered for fully nonlinear second-order elliptic equations. However, in applying to the equation (1.1) to compare functions \( u(x), v(x) \in C^{2}(\Omega) \), the following condition needs to be satisfied: for any \( t \in [0, 1] \), the matrix \( \omega(x, (1-t)u(x) + tv(x)) \) must be positive definite for all \( x \in \Omega \). But, in general, the equation (1.1) do not satisfy this condition. The new point of this paper is that we can prove in the Theorem the comparison principle to the equation (1.1) in the case when it is \( \delta \)-elliptic with respect to compared functions.

**Definition 2.** Suppose that \( \delta \in [0, 1) \) is a fixed number. We say that the equation (1.1) is \( \delta \)-elliptic with respect to function \( u(x) \in C^{2}(\Omega) \) if it is elliptic with respect to \( u(x) \) and
\[
\delta \lambda_{\min}(\omega(x, u)) \geq \mu(B), \quad \forall x \in \Omega,
\]
where \( \omega(x, u) \) is defined by (1.3) and
\[
(1.6) \quad \mu(B) \equiv \sup_{\Omega \times \mathbb{R} \times \mathbb{R}^n} \|B(x, z, p)\|,
\]
which is assumed to be finite.

Based on the two results mentioned above, in our incoming paper [3], we will get a priori estimates in \( C^{2,\alpha}(\overline{\Omega}) \) for \( \delta \)-elliptic solutions to the Dirichlet problem for (1.1). Moreover, by the method of continuity we will prove in that paper that when \( A(x, z, p), f(x, z, p) \) satisfy some conditions which are like those for the Dirichlet problem for (1.2) ([4], [5], [6]), there exists a unique \( \delta \)-elliptic solution to the Dirichlet problem for (1.1) in the space \( C^{2,\alpha}(\overline{\Omega}) \) for some \( 0 < \alpha < 1 \), provided that the matrix \( B(x, z, p) \) is sufficiently small.

The structure of the paper is as follows. In Section §2, we recall the notion of the 2nd compound and its properties for square matrices. In Section §3, we will study the second differentials for the Monge-Ampère type function \( F(R) \)
in the set $D_{\delta,\mu}$ and prove its $d$-concavity. In the last section, we will prove the comparison principle for the equations (1.1), which are $\delta$–elliptic with respect to compared functions.

2. THE 2ND COMPOUNDS OF SQUARE MATRICES

Definition 3. (II) Let $M = [M_{ij}]$ be an $n \times n$ matrix with entries in $\mathbb{R}$ or $\mathbb{C}$. Suppose that $i < k$ and $j < \ell$. We denote by $M_{ik,j\ell}^{(2)}$ the minor, which is the determinant at the intersection of rows $i, k$ and columns $j, \ell$ of the matrix $M$, that is,

$$M_{ik,j\ell}^{(2)} = \begin{vmatrix} M_{ij} & M_{i\ell} \\ M_{kj} & M_{k\ell} \end{vmatrix}.$$

When the pairs $(ik), (j\ell)$ with $i < k$ and $j < \ell$ are arranged in the lexical order, the resulting $\left( \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right) \times \left( \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right)$ matrix, consisting of corresponding minors is called the 2nd compound of the matrix $M$ and written as $M^{(2)}$. In symbols, we write

$$M^{(2)} = \left[ M_{ik,j\ell}^{(2)} \right]_{\left( \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right) \times \left( \begin{smallmatrix} n \\ 2 \end{smallmatrix} \right)}.$$

Some principal properties of the 2nd compound matrices are listed in the following proposition.

Proposition 1. (II) Let $M$ and $N$ be matrices in $\mathbb{C}^{n \times n}$. Then we have the following assertions:

(i) Binet-Cauchy Theorem:

$$(MN)^{(2)} = M^{(2)}N^{(2)}.$$

(ii) $(M^{(2)})^T = (M^T)^{(2)}$, where $M^T$ is the transpose of $M$.

(iii) $M^{(2)} = \overline{M}^{(2)}$, where $\overline{M}$ is the complex conjugate of $M$.

(iv) $(M^{(2)})^* = (M^*)^{(2)}$, where $M^*$ is the Hermitian adjoint of $M$, $M^* = \overline{M}^T$.

(v) $M$ is non-singular if and only if $M^{(2)}$ is non-singular, and

$$(M^{(2)})^{-1} = (M^{-1})^{(2)}.$$

(vi) Suppose that $M \in \mathbb{R}^{n \times n}$ and $M$ is symmetric or skew-symmetric, then $M^{(2)}$ is symmetric.

(vii) $(kM)^{(2)} = k^2 M^{(2)}$, $\forall k \in \mathbb{C}$.

(viii) If $M = \text{diag}(\lambda_1, \ldots, \lambda_n)$, then

$$M^{(2)} = \text{diag}(\lambda_j \lambda_k, j < k).$$

To investigate the $d$-concavity of function $F(R)$ in the next section, we need the following proposition.
Proposition 2. Let $M = [M_{ij}]$ be a square matrix of order $n$. Then
\begin{equation}
(2.1) 
M^{(2)} + (M^T)^{(2)} = \frac{1}{2} (M + M^T)^{(2)} + \frac{1}{2} (M - M^T)^{(2)}.
\end{equation}

Proof. For all $i, j, k, \ell = 1, \ldots, n$ such that $i < k, j < \ell$, we have
\begin{align*}
(M + M^T)_{ik,j\ell}^{(2)} &= \left| \begin{array}{ccc}
M_{ij} + M_{ji} & M_{i\ell} + M_{i\ell} \\
M_{kj} + M_{jk} & M_{k\ell} + M_{k\ell}
\end{array} \right| \\
&= 2 \left( \begin{array}{c}
M_{ij} \\
M_{kj}
\end{array} \right) + M_{ij} M_{i\ell} + M_{ik} M_{k\ell} \\
&- \left( \begin{array}{c}
M_{ij} \\
M_{kj}
\end{array} \right) + M_{ij} M_{i\ell} - M_{ik} M_{k\ell} - M_{ij} M_{i\ell} - M_{ijk} M_{k\ell} \\
&= 2 \left( M_{ik,j\ell}^{(2)} + (M^T)_{ik,j\ell}^{(2)} - (M - M^T)_{ik,j\ell}^{(2)} \right).
\end{align*}
This implies the desired equality (2.1). \qed

3. The $d$-concavity of the nonsymmetric Monge-Ampère type functions

3.1. Some properties of matrices $R$ belonging the set $D_{\delta, \mu}$. Let $D_{\delta, \mu}$ is the set given in [15]. We shall introduce some properties of matrices $R = \omega + \beta$ from $D_{\delta, \mu}$.

Proposition 3. Suppose that $R = \omega + \beta \in \mathbb{R}^{n \times n}$, where $\omega$ is symmetric positive definite, $\beta$ is skew-symmetric. Then
(i) $\det R \geq \det \omega + \det \beta \geq \det \omega > 0$.
(ii) Particularly, when $n = 2$,
\[
\det R = \det \omega + \det \beta \geq \det \omega > 0.
\]
Consequently, $\det R > 0$ and $R$ is always non-singular when $\omega > 0$.

Proof. (i) Set
\begin{equation}
(3.1) 
\sigma = \omega^{-\frac{1}{2}} \beta \omega^{-\frac{1}{2}}.
\end{equation}
Then $\sigma$ is skew-symmetric and
\begin{equation}
(3.2) 
R = \omega + \beta = \omega^{\frac{1}{2}} \left( E + \omega^{-\frac{1}{2}} \beta \omega^{-\frac{1}{2}} \right) \omega^{\frac{1}{2}} = \omega^{\frac{1}{2}} (E + \sigma) \omega^{\frac{1}{2}}.
\end{equation}
Set
\begin{equation}
(3.3) 
D_1 = \text{diag} (i\sigma_1, \ldots, i\sigma_n),
\end{equation}
where $i\sigma_1, \ldots, i\sigma_n$ are the eigenvalues of $\sigma$, $i$ is the imaginary unit, $\sigma_j \in \mathbb{R}, j = 1, \ldots, n$ and

\[(3.4) \quad \sigma_{2j-1} = -\sigma_{2j}, j = 1, 2, \ldots, \left[\frac{n}{2}\right] \text{ and } \sigma_n = 0 \text{ if } n \text{ is odd.}\]

Then we can write for some unitary matrix $C_1 \in \mathbb{C}^{n \times n}$,

\[(3.5) \quad \sigma = C_1D_1C_1^*,\]

where $C_1^*$ is the Hermitian adjoint of $C_1$, $C_1^* = C_1^{-1}$. It follows from (3.2) and (3.5) that

\[(3.3) \quad R = \omega^\frac{1}{2}C_1(E + D_1)C_1^*\omega^\frac{1}{2},\]

which together with (3.3) and (3.4) yields

\[(3.6) \quad \det R = (\det \omega)(1 + \sigma_1^2)(1 + \sigma_3^2) \cdots (1 + \sigma_{2\left[\frac{n}{2}\right]-1}^2).\]

Also from (3.4), we have

\[(3.7) \quad \det \sigma = 0 \text{ if } n \text{ is odd, } \det \sigma = \sigma_1^2\sigma_3^2 \cdots \sigma_{n-1}^2 \text{ if } n \text{ is even.}\]

It follows that

\[(3.8) \quad 0 \leq \det \sigma \leq \sigma_1^2\sigma_3^2 \cdots \sigma_{2\left[\frac{n}{2}\right]-1}^2.\]

This together with (3.1) gives

\[(3.9) \quad \det \beta = (\det \omega)(\det \sigma) \geq 0.\]

Combining (3.3) - (3.8), we obtain

\[(3.10) \quad \det R \geq (\det \omega)(1 + \det \sigma) = \det \omega + \det \beta \geq \det \omega > 0.\]

(ii) When $n = 2$, $\det \sigma = \sigma_1^2$. We infer from this, (3.6) and (3.8) that

\[(3.11) \quad \det R = (\det \omega)(1 + \sigma_1^2) = \det \omega + (\det \omega)(\det \sigma) = \det \omega + \det \beta \geq \det \omega > 0.\]

The proof is completed. \(\square\)

**Proposition 4.** Suppose that $R = \omega + \beta \in D_{\delta, \mu}$ and the matrix $\sigma$ is given in (3.1). Then the following assertions hold:

(i) $\|\sigma\| \leq \delta < 1$.

(ii) All eigenvalues $i\sigma_j$ of $\sigma$ satisfy: $|\sigma_j| \leq \delta < 1, j = 1, \ldots, n$.

Proof. (i) Since $R = \omega + \beta \in D_{\delta, \mu}$, we have $\delta \lambda_{\min}(\omega) \geq \mu$ and $\|\beta\| \leq \mu$. From these estimates and (3.1), we obtain

\[(3.12) \quad \|\sigma\| \leq \|\omega^{-\frac{1}{2}}\|\|\beta\| \leq \frac{1}{\lambda_{\min}(\omega)}\mu \leq \delta < 1.\]

(ii) The estimate (ii) follows directly from (i) and the fact that $|\sigma_j| \leq \|\sigma\|, j = 1, \ldots, n$. \(\square\)
Proposition 5. Suppose that $R = \omega + \beta \in D_{\delta,\mu}$. Then
\begin{align*}
\frac{1}{\delta^n} \|\beta\|^n + (2^{\frac{n}{2}} - 1) \det \beta \leq \det \omega + (2^{\frac{n}{2}} - 1) \det \beta \leq \det R \leq (1 + \delta^2)^{\frac{n}{2}} \det \omega,
\end{align*}
where, when $\delta = 0$ we have $\beta = 0$ and $\frac{0}{0} = 0$.

Proof. By (3.6),
\begin{align*}
\det R = (\det \omega) (1 + \sigma^2_1) (1 + \sigma^2_2) \cdots (1 + \sigma^2_{\frac{n}{2} - 1}).
\end{align*}
By Proposition 4, $|\sigma_j| \leq \delta < 1, j = 1, \ldots, n$. Thus
\begin{align*}
(1 + \delta^2)\sigma^2_j & \geq (1 + \sigma^2_1) (1 + \sigma^2_2) \cdots (1 + \sigma^2_{\frac{n}{2} - 1}) \\
& \geq 1 + (2^{\frac{n}{2}} - 1)\sigma^2_1\sigma^2_2 \cdots \sigma^2_{\frac{n}{2} - 1} \geq 1 + (2^{\frac{n}{2}} - 1) \det \sigma,
\end{align*}
where the last inequality is by (3.7). Moreover, we have
\begin{align*}
\det \omega \geq (\lambda_{\min}(\omega))^n \geq \frac{1}{\delta^n} \mu^n \geq \frac{1}{\delta^n} \|\beta\|^n.
\end{align*}
From these estimates and (3.8), we obtain the conclusion of Proposition 5. □

Proposition 6. Suppose that $R = \omega + \beta \in D_{\delta,\mu}$ and the matrix $\sigma$ is given in (3.1). Then
\begin{align*}
\frac{R^{-1} + (R^{-1})^T}{2} & = \omega^{-\frac{1}{2}} (E - \sigma^2)^{-1} \omega^{-\frac{1}{2}}, \\
\frac{R^{-1} - (R^{-1})^T}{2} & = \omega^{-\frac{1}{2}} (-\sigma) (E - \sigma^2)^{-1} \omega^{-\frac{1}{2}}.
\end{align*}
Proof. It follows from (3.2) that
\begin{align*}
R^{-1} = \omega^{-\frac{1}{2}} (E + \sigma)^{-1} \omega^{-\frac{1}{2}}, \\
(R^{-1})^T = \omega^{-\frac{1}{2}} ((E + \sigma)^{-1})^T \omega^{-\frac{1}{2}} = \omega^{-\frac{1}{2}} (E - \sigma)^{-1} \omega^{-\frac{1}{2}}.
\end{align*}
Thus
\begin{align*}
\frac{R^{-1} + (R^{-1})^T}{2} & = \omega^{-\frac{1}{2}} (E + \sigma)^{-1} + (E - \sigma)^{-1} \omega^{-\frac{1}{2}}, \\
\frac{R^{-1} - (R^{-1})^T}{2} & = \omega^{-\frac{1}{2}} (E + \sigma)^{-1} - (E - \sigma)^{-1} \omega^{-\frac{1}{2}}.
\end{align*}
Note that $E - \sigma^2 = (E - \sigma)(E + \sigma)$, so we have
\begin{align*}
\frac{(E + \sigma)^{-1} + (E - \sigma)^{-1}}{2} & = \frac{(E - \sigma) + (E + \sigma)}{2} = E, \\
\frac{(E + \sigma)^{-1} - (E - \sigma)^{-1}}{2} & = \frac{(E - \sigma) - (E + \sigma)}{2} = -\sigma.
\end{align*}
Therefore,
\[
\frac{(E + \sigma)^{-1} + (E - \sigma)^{-1}}{2} = (E - \sigma^2)^{-1}, \\
\frac{(E + \sigma)^{-1} - (E - \sigma)^{-1}}{2} = (-\sigma)(E - \sigma^2)^{-1}.
\]

From these equalities and (3.11), we obtain the desired equalities in (3.10). □

**Corollary 1.** Suppose that \( R = \omega + \beta \in D_{\delta,\mu} \) and suppose that the matrix \( \sigma = \omega^{-\frac{1}{2}}\beta\omega^{-\frac{1}{2}} \) is diagonalised by a unitary matrix \( C_1 \in \mathbb{C}^{n \times n} \) as in (3.5),

\[
\sigma = C_1 D_1 C_1^*,
\]

where \( D_1 \) is the diagonal matrix given by (3.3).

Then
\[
\frac{R^{-1} + (R^{-1})^T}{2} = \omega^{-\frac{1}{2}}C_1 D_2 C_1^* \omega^{-\frac{1}{2}},
\]

\[
\frac{R^{-1} - (R^{-1})^T}{2} = \omega^{-\frac{1}{2}}C_1 D_3 C_1^* \omega^{-\frac{1}{2}},
\]

where
\[
D_2 = (E - D_1^2)^{-1} = \text{diag} \left( \frac{1}{1 + \sigma_1^2}, \ldots, \frac{1}{1 + \sigma_n^2} \right),
\]

\[
D_3 = (-D_1) (E - D_1^2)^{-1} = \text{diag} \left( \frac{-i\sigma_1}{1 + \sigma_1^2}, \ldots, \frac{-i\sigma_n}{1 + \sigma_n^2} \right).
\]

**Proof.** All equalities in (3.12), (3.13) are followed easily from (3.3), (3.5) and (3.10). □

**Corollary 2.** Suppose that \( R = \omega + \beta \in D_{\delta,\mu} \). Then
\[
\frac{1}{1 + \delta^2} \text{Tr} \omega^{-1} \leq \text{Tr} R^{-1} \leq \text{Tr} \omega^{-1},
\]

here and in the below, \( \text{Tr} \) stands for the trace operator of square matrices.

**Proof.** From (3.12) and (3.13), we have
\[
\text{Tr} R^{-1} = \text{Tr} \left( \frac{R^{-1} + (R^{-1})^T}{2} \right) = \text{Tr} \left( \omega^{-\frac{1}{2}}C_1 D_2 C_1^* \omega^{-\frac{1}{2}} \right) = \text{Tr} D_2 C_1^* \omega^{-1} C_1 = \sum_j (D_2)_{jj} (C_1^* \omega^{-1} C_1)_{jj}.
\]

Note that \( \omega^{-1} \) is positive definite, \( C_1 \) is unitary and, by Proposition 4, \( \frac{1}{1 + \delta^2} \leq (D_2)_{jj} \leq 1, j = 1, \ldots, n \). We then obtain from (3.15) that
\[
\frac{1}{1 + \delta^2} \text{Tr} \omega^{-1} \leq \frac{1}{1 + \delta^2} \sum_j (C_1^* \omega^{-1} C_1)_{jj} \leq \text{Tr} R^{-1} \leq \sum_j (C_1^* \omega^{-1} C_1)_{jj} = \text{Tr} \omega^{-1}.
\]
This completes the proof. □

3.2. The second differentials of the nonsymmetric Monge-Ampère type functions.

**Proposition 7.** Let $F(R)$ be the function given by (1.4), where $\det R > 0$. Let $R^{-1} = [R^{ij}]$ denote the inverse of $R = [R_{ij}]$. Then for all $i, j, k, \ell = 1, \ldots, n$, we have that

\begin{align}
F^{ij} &\equiv \frac{\partial F(R)}{\partial R_{ij}} = R^{ji}, \\
F^{ij,k\ell} &\equiv \frac{\partial^2 F(R)}{\partial R_{ij}\partial R_{k\ell}} = -R^{\ell i}R^{jk}.
\end{align}

**Proof.** Let $U = [U_{ij}]$ denote the cofactor matrix of $R$, i.e., $U^T = (\det R)R^{-1}$. For a fixed $i, i = 1, \ldots, n$, we expand the determinant $\det R$ according to the $i$-th row,

$$
\det R = R_{i1}U_{i1} + \cdots + R_{in}U_{in}.
$$

Then

$$
\frac{\partial F(R)}{\partial R_{ij}} = \frac{1}{\det R} \frac{\partial (\det R)}{\partial R_{ij}} = \frac{1}{\det R} U_{ij} = R^{ji}, \text{ for } i, j = 1, \ldots, n.
$$

Thus (3.16) is proved.

It follows from (3.16) that

$$
\sum_p R_{sp}F^{jp} = \sum_p R_{sp}R_{pi} = \delta_{is}, \text{ for } i, s = 1, \ldots, n.
$$

Differentiating this equation with respect to $R_{k\ell}$, we get

$$
\sum_p \frac{\partial R_{sp}}{\partial R_{k\ell}}F^{jp} + \sum_p R_{sp}F^{ip,k\ell} = 0,
$$

and thus

$$
\delta_{sk}F^{i\ell} + \sum_p R_{sp}F^{ip,k\ell} = 0, \text{ for } i, s, k, \ell = 1, \ldots, n.
$$

Multiplying this equality by $R^{js}$ and summing over $s$, we have for $i, j, k, \ell = 1, \ldots, n$,

$$
\sum_s \delta_{sk}F^{i\ell}R^{js} + \sum_{p,s} R^{js}R_{sp}F^{ip,k\ell} = 0,
$$

or

$$
R^{\ell i}R^{jk} + F^{ij,k\ell} = 0,
$$

which gives the required result (3.17). The proof is now completed. □
Now we consider the second order differentials of the function \( F(R) \) given by (1.4), where \( R \in D_{\delta,\mu} \), \( D_{\delta,\mu} \) is the unbounded and convex set given in (1.5). Let \( M = [M_{ij}] \in \mathbb{R}^{n \times n} \). We consider the function \( F \) defined as follows:

\[
F(R, M) : D_{\delta,\mu} \times \mathbb{R}^{n \times n} \to \mathbb{R},
\]

\[
F(R, M) = \sum_{i,j,k,\ell} \frac{\partial^2 F}{\partial R_{ij} \partial R_{k\ell}} M_{ij} M_{k\ell} = -\sum_{i,j,k,\ell} R_{\ell i} R_{jk} M_{ij} M_{k\ell}.
\]

Proposition 8. Suppose \( R \in D_{\delta,\mu} \). Then for any matrix \( M = P + Q \in \mathbb{R}^{n \times n} \), the following equality holds

\[
F(R, M) = F(R, P) + F(R, Q) + 2L(R, P, Q),
\]

where

\[
L(R, P, Q) = -\sum_{i,j,k,\ell} R_{\ell i} R_{jk} P_{ij} Q_{k\ell}.
\]

Proof. It follows from (3.18) that

\[
F(R, M) = -\sum_{i,j,k,\ell} R_{\ell i} R_{jk} P_{ij} P_{k\ell} - \sum_{i,j,k,\ell} R_{\ell i} R_{jk} Q_{ij} Q_{k\ell} - \sum_{i,j,k,\ell} R_{\ell i} R_{jk} P_{ij} Q_{k\ell} - \sum_{i,j,k,\ell} R_{\ell j} R_{ik} P_{ij} Q_{k\ell}.
\]

Note that

\[
\sum_{i,j,k,\ell} R_{\ell i} R_{jk} P_{k\ell} Q_{ij} \leftrightarrow \sum_{i,j,k,\ell} R_{\ell i} R_{jk} P_{ij} Q_{k\ell}.
\]

Combining these equalities, we obtain the conclusion of Proposition 8 \( \square \)

Proposition 9. Suppose that \( R \in D_{\delta,\mu} \). Then for any symmetric matrix \( P \in \mathbb{R}^{n \times n} \), the following equality holds

\[
F(R, P) = -[G(R, P)]^2 + \mathcal{H}(R, P),
\]

where

\[
G(R, P) = \text{Tr}(R^{-1}P),
\]

\[
\mathcal{H}(R, P) = 2 \text{Tr} \left( (R^{-1})^{(2)} P^{(2)} \right).
\]

Proof. From (3.18) and the fact that \( P^T = P \), we have

\[
F(R, P) = -\sum_{i,j,k,\ell} R_{\ell i} R_{jk} P_{ij} P_{k\ell} - \sum_{i,j,k,\ell} R_{\ell i} R_{jk} P_{ij} P_{k\ell} + \sum_{i,j,k,\ell} R_{\ell j} R_{ik} P_{ij} P_{k\ell} - \sum_{i,j,k,\ell} R_{\ell j} R_{ik} P_{ij} P_{k\ell}.
\]
It follows that

\[ F(R, P) = -\frac{1}{2} \left( \sum_{i,j} R^{ij} P_{ij} \right) \left( \sum_{k,\ell} R^{k\ell} P_{k\ell} \right) - \frac{1}{2} \left( \sum_{i,\ell} R^{i\ell} P_{i\ell} \right) \left( \sum_{j,k} R^{kj} P_{kj} \right) \]

\[ + \frac{1}{2} \left[ \left( \sum_{i,j} R^{ij} P_{ij} \right) \left( \sum_{k,\ell} R^{k\ell} P_{k\ell} \right) + \left( \sum_{i,\ell} R^{i\ell} P_{i\ell} \right) \left( \sum_{j,k} R^{kj} P_{kj} \right) \right. \]

\[ - \sum_{i,j,k,\ell} R^{i\ell} R^{kj} P_{ij} P_{k\ell} - \sum_{i,j,k,\ell} R^{ij} R^{k\ell} P_{ij} P_{k\ell} \]

\[ = - \left( \sum_{i,j} R^{ij} P_{ij} \right)^2 + 2 \sum_{i<k, j<\ell} (R^{ij} R^{k\ell} - R^{i\ell} R^{kj}) (P_{ij} P_{k\ell} - P_{i\ell} P_{kj}) \]

\[ = - \left( \sum_{i,j} R^{ij} P_{ij} \right)^2 + 2 \sum_{i<k, j<\ell} \left( (R^{-1})_{ik,j\ell} P_{j\ell,ik} \right) \]

\[ = - \left[ \text{Tr} (R^{-1} P) \right]^2 + 2 \text{Tr} \left[ (R^{-1})_{ik,j\ell} P_{j\ell,ik} \right]. \]

This completes the proof. □

**Proposition 10.** Suppose that \( R \in D_{\delta,\mu} \). Then for any skew-symmetric matrix \( Q \in \mathbb{R}^{n \times n} \), the following equality holds

\[ (3.23) \quad F(R, Q) = - [G(R, Q)]^2 + H(R, Q), \]

where the functions \( G \) and \( H \) are defined as in (3.22).

*Proof.* The proof is similar to that of Proposition 9. □

**Proposition 11.** Suppose that \( R \in D_{\delta,\mu} \). Then for any symmetric matrix \( P \in \mathbb{R}^{n \times n} \) and any skew-symmetric matrix \( Q \in \mathbb{R}^{n \times n} \), the following equality holds

\[ (3.24) \quad L(R, P, Q) = - \frac{1}{2} \text{Tr} \left[ \left( R^{-1} - (R^{-1})^T \right) P \left( R^{-1} + (R^{-1})^T \right) Q \right], \]

where \( L(R, P, Q) \) is defined by (3.20).

*Proof.* Note that

\[ L(R, P, Q) = - \sum_{i,j,k,\ell} R^{ij} R^{jk} P_{ij} Q_{k\ell} R^{i\ell} P_{i\ell} - \sum_{i,j,k,\ell} R^{kj} R^{i\ell} P_{ji} Q_{\ell k}. \]
From this and the fact that $P^T = PQ^T = -Q$, we get
\[
\mathcal{L}(R, P, Q) = -\frac{1}{2} \sum_{i,j,k,l} R^{ij} R^{jk} P_{ij} Q_{kl} - \frac{1}{2} \sum_{i,j,k,l} R^{kj} R^{il} P_{ji} Q_{lk} \\
= \frac{1}{2} \sum_{i,j,k,l} R^{jk} P_{ji} R^{il} Q_{lk} - \frac{1}{2} \sum_{i,j,k,l} R^{kj} P_{ji} R^{il} Q_{lk} \\
= \frac{1}{2} \text{Tr} \left[ (R^{-1})^T P (R^{-1})^T Q \right] - \frac{1}{2} \text{Tr} \left[ R^{-1} PR^{-1} Q \right] \\
= -\frac{1}{2} \text{Tr} \left[ (R^{-1} - (R^{-1})^T) P (R^{-1})^T Q \right] - \frac{1}{2} \text{Tr} \left[ (R^{-1} - (R^{-1})^T) PR^{-1} Q \right] \\
= -\frac{1}{2} \text{Tr} \left[ (R^{-1} - (R^{-1})^T) P (R^{-1})^T + R^{-1} Q \right],
\]
where in the fourth step, we have used the equality
\[
\text{Tr} \left[ R^{-1} P (R^{-1})^T Q \right] = \text{Tr} \left[ (R^{-1})^T P R^{-1} Q \right] = 0,
\]
which holds due to the skew-symmetry of $Q$ and the symmetry of matrices $R^{-1} P (R^{-1})^T$, $(R^{-1})^T P R^{-1}$. The proof is completed.

For $R = \omega + \beta \in D_{\delta, \mu}$ fixed and for matrix $M \in \mathbb{R}^{n \times n}$, we set
\[
\tilde{M} \equiv \omega^{-\frac{1}{2}} M \omega^{-\frac{1}{2}} = [\tilde{M}_{jk}], \quad \tilde{M} \equiv C_1^* \tilde{M} C_1 = [\tilde{M}_{jk}],
\]
where $C_1$ is the unitary matrix defined in (3.5). It is obvious that
\[
|\tilde{M}| = |\tilde{M}|, \quad \|\tilde{M}\| = \|\tilde{M}\|
\]
where $| \cdot |$ and $\| \cdot \|$ denote, respectively, the Frobenius norm and the operator norm on $\mathbb{C}^{n \times n}$, which are defined as follows: for any matrix $K = [K_{ij}] \in \mathbb{C}^{n \times n}$,
\[
|K| = \left( \sum_{i,j} |K_{ij}|^2 \right)^{1/2}, \quad \|K\| = \sup_{\xi \in \mathbb{C}^n, \|\xi\| = 1} |K\xi|.
\]

**Proposition 12.** For any matrix $M \in \mathbb{R}^{n \times n}$, we have the following estimate
\[
(\lambda_{\text{max}}(\omega))^{-2} |M|^2 \leq |\tilde{M}|^2 \leq (\lambda_{\text{min}}(\omega))^{-2} |M|^2,
\]
where $\lambda_{\text{max}}(\omega)$ and $\lambda_{\text{min}}(\omega)$ denote, respectively, the largest and smallest eigenvalues of $\omega$.

**Proof.** Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of $\omega$, where $\lambda_1 \geq \cdots \geq \lambda_n > 0$. Write
\[
\omega = CDC^{-1},
\]
where $C$ is orthogonal and $D = \text{diag}(\lambda_1, \ldots, \lambda_n)$. Then
\[
\omega^{-\frac{1}{2}} = CD^{-\frac{1}{2}} C^{-1}, \quad D^{-\frac{1}{2}} = \text{diag} \left( \lambda_1^{-\frac{1}{2}}, \ldots, \lambda_n^{-\frac{1}{2}} \right).
\]
Therefore,

\[
|\tilde{M}|^2 = \left| \omega^{-\frac{1}{2}} M \omega^{-\frac{1}{2}} \right|^2 = \left| \left( CD^{-\frac{1}{2}} C^{-1} \right) M \left( CD^{-\frac{1}{2}} C^{-1} \right) \right|^2
\]

\[
= \left| D^{-\frac{1}{2}} (C^{-1} M C) D^{-\frac{1}{2}} \right|^2 = \sum_{i,j} \lambda_i^{-1} \lambda_j^{-1} ( (C^{-1} M C)_{ij} )^2.
\]

From this and the fact that \(0 < \lambda_i^{-1} \leq \lambda_i^{-1} \leq \lambda_i^{-1} (i = 1, \ldots, n)\), we obtain

\[
\lambda_i^{-2} |M|^2 = \lambda_i^{-2} \sum_{i,j} ( (C^{-1} M C)_{ij} )^2 \leq |\tilde{M}|^2 \leq \lambda_i^{-2} \sum_{i,j} ( (C^{-1} M C)_{ij} )^2 = \lambda_i^{-2} |M|^2.
\]

The proof is completed. \(\square\)

**Proposition 13.** Suppose that \( R = \omega + \beta \in D_{\delta,\mu} \). Then for any symmetric matrix \( P \in \mathbb{R}^{n \times n} \), we have

(3.28) \( \mathcal{F}(R, P) = - \sum_{j,k=1}^{n} \frac{1 - \sigma_j \sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \left| \tilde{p}_{jk} \right|^2, \)

where \( \sigma_1, \ldots, \sigma_n \) are the eigenvalues of the matrix \( \sigma \), defined by (3.1).

**Proof.** Since \( P \) is symmetric, by Proposition 1, \( P^{(2)} \) is also symmetric. Hence from (2.1), (3.21) and (3.22), we have

(3.29) \[
\mathcal{F}(R, P) = - [\mathcal{G}(R, P)]^2 + \mathcal{H}(R, P) = - [\mathcal{G}(R, P)]^2 + 2 \text{Tr} \left[ \left( R^{-1} \right)^{(2)} P^{(2)} \right]
\]

\[
= - [\mathcal{G}(R, P)]^2 + 2 \text{Tr} \left[ \left( R^{-1} \right)^{(2)} + \left( R^{-1} \right)^{(2)} \right] P^{(2)} \]

\[
= - [\mathcal{G}(R, P)]^2 + 2 \text{Tr} \left[ \left( R^{-1} \right)^{(2)} \right] P^{(2)} + 2 \text{Tr} \left[ \left( R^{-1} - \left( R^{-1} \right)^{(2)} \right)^{(2)} \right] P^{(2)} \].
\]

It follows from (3.12) and (3.13) that

\[
\mathcal{G}(R, P) = \text{Tr} \left( \frac{R^{-1} + (R^{-1})^T}{2} P \right) = \text{Tr} \left( \omega^{-\frac{1}{2}} C_1 D_2 C_1^* \omega^{-\frac{1}{2}} P \right)
\]

\[
= \text{Tr} \left( D_2 C_1^* \omega^{-\frac{1}{2}} P \omega^{-\frac{1}{2}} C_1 \right) = \text{Tr} \left( D_2 \tilde{P} \right) = \sum_{j} \tilde{p}_{jj} \frac{1}{1 + \sigma_j^2}.
\]

(3.30)
Since $P$ is symmetric, $\tilde{P}$ is Hermitian. Hence $\tilde{P}_{jk}\tilde{P}_{kj} = |\tilde{P}_{jk}|^2$, $j, k = 1, \ldots, n$. From these equalities, (3.12), (3.13) and Proposition 1 we obtain

\begin{equation}
(3.31)
2 \text{Tr} \left[ \left( R^{-1} + \left( R^{-1} \right)^T \right)^{(2)} \overline{P}^{(2)} \right] = 2 \text{Tr} \left[ (\omega^{-\frac{1}{2}} C_1 D_2 C_1^* \omega^{-\frac{1}{2}})^{(2)} \overline{P}^{(2)} \right] \\
= 2 \text{Tr} \left[ D_2^{(2)} (C_1^*)^{(2)} (\omega^{-\frac{1}{2}})^{(2)} \overline{P}^{(2)} (\omega^{-\frac{1}{2}})^{(2)} C_1^{(2)} \right] \\
= 2 \text{Tr} \left[ D_2^{(2)} \tilde{P}^{(2)} \right] = 2 \sum_{j<k} (D_2^{(2)})_{jk,jk} \tilde{P}_{jk,jk} \\
= 2 \sum_{j<k} \frac{\tilde{P}_{jj}\tilde{P}_{kk} - |\tilde{P}_{jk}|^2}{(1 + \sigma_j^2)(1 + \sigma_k^2)} = \sum_{j\neq k} \frac{\tilde{P}_{jj}\tilde{P}_{kk}}{(1 + \sigma_j^2)(1 + \sigma_k^2)} - \sum_{j\neq k} \frac{|\tilde{P}_{jk}|^2}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \\
= \left( \sum_j \frac{\tilde{P}_{jj}}{1 + \sigma_j^2} \right)^2 - \sum_{j,k} \frac{|\tilde{P}_{jk}|^2}{(1 + \sigma_j^2)(1 + \sigma_k^2)}.
\end{equation}

Combining (3.30), (3.31) yields

\begin{equation}
(3.32) \quad 2 \text{Tr} \left[ \left( R^{-1} + \left( R^{-1} \right)^T \right)^{(2)} \overline{P}^{(2)} \right] = |G(R,P)|^2 - \sum_{j,k} \frac{|\tilde{P}_{jk}|^2}{(1 + \sigma_j^2)(1 + \sigma_k^2)}.
\end{equation}

From (3.12), (3.13) and Proposition 1 we also get

\begin{equation}
(3.33)
2 \text{Tr} \left[ \left( R^{-1} - \left( R^{-1} \right)^T \right)^{(2)} \overline{P}^{(2)} \right] = 2 \text{Tr} \left[ (\omega^{-\frac{1}{2}} C_1 D_3 C_1^* \omega^{-\frac{1}{2}})^{(2)} \overline{P}^{(2)} \right] \\
= 2 \text{Tr} \left[ D_3^{(2)} \tilde{P}^{(2)} \right] = 2 \sum_{j<k} (D_3^{(2)})_{jk,jk} \tilde{P}_{jk,jk} \\
= 2 \sum_{j<k} \frac{-\sigma_j\sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \left( \tilde{P}_{jj}\tilde{P}_{kk} - |\tilde{P}_{jk}|^2 \right) \\
= - \sum_{j\neq k} \frac{\sigma_j\sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \tilde{P}_{jj}\tilde{P}_{kk} + \sum_{j\neq k} \frac{\sigma_j\sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} |\tilde{P}_{jk}|^2.
\end{equation}
Obviously, $\text{Tr}\left(\frac{R^{-1} - (R^{-1})^T}{2} P\right) = 0$. It then follows from (3.12) and (3.13) that

$$\text{Tr}(\omega^{-\frac{1}{2}} C_1 D_3 C_1^* \omega^{-\frac{1}{2}} P) = \text{Tr}\left(D_3 \tilde{P}\right) = \sum_j \frac{-i\sigma_j}{1 + \sigma_j^2} \tilde{P}_{jj} = 0,$$

or equivalently,

$$\left(\sum_j \frac{\sigma_j}{1 + \sigma_j^2} \tilde{P}_{jj}\right)^2 = 0.$$

Hence

$$\sum_j \frac{\sigma_j^2}{(1 + \sigma_j^2)^2} \left(\tilde{P}_{jj}\right)^2 = -\sum_{j \neq k} \frac{\sigma_j \sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \tilde{P}_{jj} \tilde{P}_{kk}.$$

This together with (3.33) gives

$$2 \text{Tr}\left(\left(\frac{R^{-1} - (R^{-1})^T}{2}\right) P^{(2)}\right) = \sum_{j,k} \frac{\sigma_j \sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \left|\tilde{P}_{jk}\right|^2.$$

The proof is straightforward from (3.29), (3.32) and (3.34).

Corollary 3. Suppose that $R = \omega + \beta \in D_{\delta, \mu}$. Then for any symmetric matrix $P \in \mathbb{R}^{n \times n}$, we have

$$\mathcal{F}(R, P) \leq -\frac{1 - \delta^2}{(1 + \delta^2)^2} |\tilde{P}|^2 \leq -\frac{1 - \delta^2}{(1 + \delta^2)^2} (\lambda_{\text{max}}(\omega))^{-2} |P|^2.$$

Proof. By Proposition 4, we have $|\sigma_j| \leq \delta < 1$, $j = 1, \ldots, n$. Hence from (3.26) and (3.28), we obtain

$$\mathcal{F}(R, P) = -\sum_{j,k} \frac{1 - \sigma_j \sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \left|\tilde{P}_{jk}\right|^2 \leq -\sum_{j,k} \frac{1 - |\sigma_j| |\sigma_k|}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \left|\tilde{P}_{jk}\right|^2 \leq -\frac{1 - \delta^2}{(1 + \delta^2)^2} \sum_{j,k} \left|\tilde{P}_{jk}\right|^2 = -\frac{1 - \delta^2}{(1 + \delta^2)^2} |\tilde{P}|^2.$$

Thus we get the first inequality in (3.35). Combining this with Proposition 12, we can easily obtain the second inequality in (3.35). □

Proposition 14. Suppose that $R = \omega + \beta \in D_{\delta, \mu}$. Then for any skew-symmetric matrix $Q \in \mathbb{R}^{n \times n}$, we have

$$\mathcal{F}(R, Q) = \sum_{j,k=1}^n \frac{1 - \sigma_j \sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \left|\tilde{Q}_{jk}\right|^2.$$
Proof. Since $Q$ is skew-symmetric, by Proposition 1, $Q^{(2)}$ is symmetric. By arguing as in (3.29), we obtain from (3.23),

\[ \mathcal{F}(R, Q) = - [\mathcal{G}(R, Q)]^2 + \mathcal{H}(R, Q) = - [\mathcal{G}(R, Q)]^2 \]

\[ + 2 \text{Tr} \left[ \frac{(R^{-1} + (R^{-1})^T)}{2} Q^{(2)} \right] + 2 \text{Tr} \left[ \frac{(R^{-1} - (R^{-1})^T)}{2} Q^{(2)} \right]. \]

It follows from (3.12) and (3.13) that

\[ \mathcal{G}(R, Q) = \text{Tr} \left( \frac{R^{-1} - (R^{-1})^T}{2} Q \right) = \text{Tr} \left( \omega^{-\frac{1}{2}} C_1 D_3 C_1^* \omega^{-\frac{1}{2}} Q \right) \]

\[ = \text{Tr} \left( D_3 C_1^* \omega^{-\frac{1}{2}} Q \omega^{-\frac{1}{2}} C_1 \right) = \text{Tr} \left( D_3 \tilde{Q} \right) = \sum_{j} -\sigma_j \tilde{Q}_{jj}. \]

Since $Q$ is skew-symmetric, $\tilde{Q}$ is skew-Hermitian. Hence $\tilde{Q}_{jk} \tilde{Q}_{kj} = -|\tilde{Q}_{jk}|^2$, $j, k = 1, \ldots, n$. From these equalities, (3.12), (3.13) and Proposition 1, we obtain

\[ 2 \text{Tr} \left[ \frac{(R^{-1} - (R^{-1})^T)}{2} Q^{(2)} \right] = 2 \text{Tr} \left[ \left( \omega^{-\frac{1}{2}} C_1 D_3 C_1^* \omega^{-\frac{1}{2}} \right)^{(2)} Q^{(2)} \right] \]

\[ = 2 \sum_{j<k} \frac{-\sigma_j \sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \left( \tilde{Q}_{jj} \tilde{Q}_{kk} + |\tilde{Q}_{jk}|^2 \right) \]

\[ = \sum_{j\neq k} \frac{-\sigma_j \sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} |\tilde{Q}_{jk}|^2 + \sum_{j<k} \frac{-\sigma_j \sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} |\tilde{Q}_{jk}|^2. \]

Combining (3.38), (3.39) gives

\[ 2 \text{Tr} \left[ \frac{(R^{-1} - (R^{-1})^T)}{2} Q^{(2)} \right] = [\mathcal{G}(R, Q)]^2 + \sum_{j<k} \frac{-\sigma_j \sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} |\tilde{Q}_{jk}|^2. \]
From the equalities $\tilde{Q}_{jk}\tilde{Q}_{kj} = -|\tilde{Q}_{jk}|^2$ ($j, k = 1, \ldots, n$), (3.12), (3.13) and Proposition 1, we also get

$$2 \text{Tr} \left[ \left( \frac{R^{-1} + (R^{-1})^T}{2} \right)^{(2)} Q^{(2)} \right] = 2 \text{Tr} \left[ \left( \frac{\omega^{-\frac{1}{2}}C_1D_2C_1^*\omega^{-\frac{1}{2}}}{2} \right)^{(2)} Q^{(2)} \right]$$

$$= 2 \text{Tr} \left[ D_2^{(2)}\tilde{Q}^{(2)} \right] = 2 \sum_{j<k} \frac{1}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \left( \tilde{Q}_{jj}\tilde{Q}_{kk} + \tilde{Q}_{jk} \right)^2$$

$$= \sum_{j,k} \frac{1}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \tilde{Q}_{jj}\tilde{Q}_{kk} + \sum_{j,k} \frac{1}{(1 + \sigma_j^2)(1 + \sigma_k^2)} |\tilde{Q}_{jk}|^2$$

$$= \left( \sum_{j} \left( \frac{\tilde{Q}_{jj}}{1 + \sigma_j^2} \right)^2 + \sum_{j,k} \frac{1}{(1 + \sigma_j^2)(1 + \sigma_k^2)} |\tilde{Q}_{jk}|^2 \right).$$

(3.41)

Obviously, $\text{Tr} \left( \frac{R^{-1} + (R^{-1})^T}{2} Q \right) = 0$. It follows from this, (3.12) and (3.13) that

$$\text{Tr} \left( \frac{\omega^{-\frac{1}{2}}C_1D_2C_1^*\omega^{-\frac{1}{2}}}{2} Q \right) = \text{Tr} \left( D_2\tilde{Q} \right) = \sum_{j} \frac{\tilde{Q}_{jj}}{1 + \sigma_j^2} = 0.$$

Combining this and (3.41) gives

$$2 \text{Tr} \left[ \left( \frac{R^{-1} + (R^{-1})^T}{2} \right)^{(2)} Q^{(2)} \right] = \sum_{j,k} \frac{1}{(1 + \sigma_j^2)(1 + \sigma_k^2)} |\tilde{Q}_{jk}|^2.$$ 

(3.42)

The proof is straightforward from (3.37), (3.40) and (3.42).

□

**Corollary 4.** Suppose that $R = \omega + \beta \in D_{\delta,\mu}$. Then for any skew-symmetric matrix $Q \in \mathbb{R}^{n \times n}$, we have

$$\mathcal{F}(R, Q) \leq |\tilde{Q}|^2.$$ 

(3.43)

Proof. Note that

$$\frac{1 - \sigma_j\sigma_k}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \leq \frac{1 + |\sigma_j||\sigma_k|}{(1 + \sigma_j^2)(1 + \sigma_k^2)} \leq 1, \ j, k = 1, \ldots, n.$$

From this, (3.26) and (3.36), we obtain that

$$\mathcal{F}(R, Q) \leq \sum_{j,k} |\tilde{Q}_{jk}|^2 = |\tilde{Q}|^2 = |\tilde{Q}|^2.$$

This completes the proof.
Proposition 15. Suppose that \( R = \omega + \beta \in D_{\delta, \mu} \). Then for any symmetric matrix \( P \in \mathbb{R}^{n \times n} \) and any skew-symmetric matrix \( Q \in \mathbb{R}^{n \times n} \), we have
\[
|\mathcal{L}(R, P, Q)| \leq \frac{2n\delta}{1 + \delta^2} |\tilde{P}| |\tilde{Q}|.
\]

Proof. By \( \text{(3.10)} \) and \( \text{(3.24)} \), we have
\[
\mathcal{L}(R, P, Q) = -2\text{Tr}
\left[
\frac{(R^{-1} - (R^{-1})^T)P(R^{-1}) + (R^{-1})^TQ}{2}
\right]
\]
\[
= -2\text{Tr}
\left[
\left(\omega^{-\frac{1}{2}}(-\sigma)(E - \sigma^2)^{-1}\omega^{-\frac{1}{2}}\right)P\left(\omega^{-\frac{1}{2}}(E - \sigma^2)^{-1}\omega^{-\frac{1}{2}}\right)Q
\right]
\]
\[
= 2\text{Tr}
\left[
\sigma (E - \sigma^2)^{-1} \left(\omega^{-\frac{1}{2}}P\omega^{-\frac{1}{2}}\right)(E - \sigma^2)^{-1} \left(\omega^{-\frac{1}{2}}Q\omega^{-\frac{1}{2}}\right)
\right]
\]
\[
= 2\text{Tr}
\left[
\sigma (E - \sigma^2)^{-1} \tilde{P} (E - \sigma^2)^{-1} \tilde{Q}
\right].
\]

Hence
\[
|\mathcal{L}(R, P, Q)| \leq 2 \left| \sigma (E - \sigma^2)^{-1} \right| \left| (E - \sigma^2)^{-1} \right| |\tilde{P}| |\tilde{Q}|.
\]

From \( \text{(3.3)} \), \( \text{(3.5)} \) and Proposition 4, we can easily obtain
\[
\left| \sigma (E - \sigma^2)^{-1} \right| = \left| D_1 (E - D_1^2)^{-1} \right| = \left( \sum_j \frac{\sigma_j^2}{(1 + \sigma_j^2)^2} \right)^{1/2} \leq \frac{\sqrt{n}\delta}{1 + \delta^2},
\]
\[
\left| (E - \sigma^2)^{-1} \right| = \left| (E - D_1^2)^{-1} \right| = \left( \sum_j \frac{1}{(1 + \sigma_j^2)^2} \right)^{1/2} \leq \sqrt{n}.
\]

Combining these estimates with \( \text{(3.3)} \), we get the desired estimate \( \text{(3.44)} \). \( \square \)

In the next theorem we will give an upper estimate for second-order differentials of the function \( F(R) \).

Theorem 1. Suppose that \( R = \omega + \beta \in D_{\delta, \mu} \). Then for any matrix \( M = P + Q \), where \( P \in \mathbb{R}^{n \times n} \) is symmetric and \( Q \in \mathbb{R}^{n \times n} \) is skew-symmetric, we have
\[
\mathcal{F}(R, M) \leq -(1 - \eta) \frac{1 - \delta^2}{(1 + \delta^2)^2} |\tilde{P}|^2 + \left( 1 + \frac{4n^2\delta^2}{\eta(1 - \delta^2)} \right) |\tilde{Q}|^2,
\]
for any constant \( \eta \in (0, 1] \), where \( \tilde{P} = \omega^{-\frac{1}{2}}P\omega^{-\frac{1}{2}}, \tilde{Q} = \omega^{-\frac{1}{2}}Q\omega^{-\frac{1}{2}} \).

Proof. From \( \text{(3.11)} \), \( \text{(3.22)} \), \( \text{(3.15)} \) and \( \text{(3.44)} \), we have
\[
\mathcal{F}(R, M) = \mathcal{F}(R, P) + \mathcal{F}(R, Q) + 2\mathcal{L}(R, P, Q)
\]
\[
\leq -\frac{1 - \delta^2}{(1 + \delta^2)^2} |\tilde{P}|^2 + |\tilde{Q}|^2 + \frac{4n\delta}{1 + \delta^2} |\tilde{P}| |\tilde{Q}|.
\]
By using Cauchy’s inequality, we have for any positive constant $\eta \in (0, 1]$, 
\[
\frac{4n\delta}{1 + \delta^2} |\tilde{P}| |\tilde{Q}| \leq \frac{\eta(1 - \delta^2)}{(1 + \delta^2)^2} |\tilde{P}|^2 + \frac{4n^2\delta^2}{\eta(1 - \delta^2)} |\tilde{Q}|^2.
\]
Combining these estimates, we obtain the estimate (3.46). The proof is completed. □

3.3. The $d$–concavity of the function $F(R)$.

**Theorem 2.** For any matrices $R^{(0)} = \omega^{(0)} + \beta^{(0)} = \begin{bmatrix} R_{ij}^{(0)} \end{bmatrix}$, $R^{(1)} = \omega^{(1)} + \beta^{(1)} = \begin{bmatrix} R_{ij}^{(1)} \end{bmatrix}$ from the set $D_{\varnothing, \mu}$, we have
\[
F(R^{(1)}) - F(R^{(0)}) \leq \sum_{i,j=1}^{n} \frac{\partial F(R^{(0)})}{\partial R_{ij}} (R_{ij}^{(1)} - R_{ij}^{(0)})
\]
\[
+ \frac{1}{2} \left(1 + \frac{4n^2\delta^2}{1 - \delta^2}\right) (\lambda_{\min}(\omega^{(s)}))^2 |\beta^{(1)} - \beta^{(0)}|^2,
\]
where $\omega^{(s)} \equiv (1 - s)\omega^{(0)} + s\omega^{(1)}$ for some constant $s \in (0, 1)$.

**Proof.** We set for all $t \in [0, 1]$,
\[
g(t) := F((1 - t)R^{(0)} + tR^{(1)}) = F(R^{(t)}),
\]
where $R^{(t)} \equiv (1 - t)R^{(0)} + tR^{(1)} = \omega^{(t)} + \beta^{(t)}$, $\omega^{(t)} = (1 - t)\omega^{(0)} + t\omega^{(1)}$, $\beta^{(t)} = (1 - t)\beta^{(0)} + t\beta^{(1)}$. Since $D_{\varnothing, \mu}$ is convex, we infer that $R^{(t)} \in D_{\varnothing, \mu}$.

By the Taylor expansion, we have for some constant $s \in (0, 1)$,
\[
F(R^{(1)}) - F(R^{(0)}) = g(1) - g(0) = g'(0) + \frac{1}{2}g''(s).
\]
By computation, we have for all $t \in (0, 1)$,
\[
g'(t) = \sum_{i,j} \frac{\partial F(R^{(t)})}{\partial R_{ij}} (R_{ij}^{(1)} - R_{ij}^{(0)}),
\]
\[
g''(t) = \sum_{i,j,k,l} \frac{\partial^2 F(R^{(t)})}{\partial R_{ij} \partial R_{kl}} (R_{ij}^{(1)} - R_{ij}^{(0)}) (R_{kl}^{(1)} - R_{kl}^{(0)}) = F(R^{(t)}; R^{(1)} - R^{(0)}),
\]
where the function $F$ is defined by (3.18). Hence
\[
g'(0) = \sum_{i,j} \frac{\partial F(R^{(0)})}{\partial R_{ij}} (R_{ij}^{(1)} - R_{ij}^{(0)}),
\]
Moreover, by applying Theorem 1 with \( R = R^{(s)} = \omega^{(s)} + \beta^{(s)}, M = R^{(1)} - R^{(0)} = (\omega^{(1)} - \omega^{(0)}) + (\beta^{(1)} - \beta^{(0)}) \equiv P + Q \) and \( \eta = 1 \), we obtain

\[
g''(s) = \mathcal{F}(R^{(s)}, R^{(1)} - R^{(0)}) \leq \left(1 + \frac{4n^2\delta^2}{1 - \delta^2}\right) \left| (\omega^{(s)})^{-\frac{1}{2}} (\beta^{(1)} - \beta^{(0)}) (\omega^{(s)})^{-\frac{1}{2}} \right|^2 
\leq \left(1 + \frac{4n^2\delta^2}{1 - \delta^2}\right) \left(\lambda_{\min}(\omega^{(s)})\right)^{-2} |\beta^{(1)} - \beta^{(0)}|^2,
\]

where the last inequality is by Proposition 12. Combining this estimate with (3.48) and (3.49), we arrive at the estimate (3.47).

Now, we obtain the following theorem on \( d \)-concavity in the set \( D_{\delta,\mu} \) for the Monge-Ampère type function \( F(R) \).

**Theorem 3.** The function \( F(R) = \log(\det R) \) is \( d \)-concave in the set \( D_{\delta,\mu} \), where

\[ d = 2n\delta^2 \left(1 + \frac{4n^2\delta^2}{1 - \delta^2}\right) \]

depends only on \( \delta \) and \( n \). That means, for any matrices

\[ R^{(0)} = \omega^{(0)} + \beta^{(0)} = \begin{bmatrix} R^{(0)}_{ij} \end{bmatrix}, R^{(1)} = \omega^{(1)} + \beta^{(1)} = \begin{bmatrix} R^{(1)}_{ij} \end{bmatrix} \] from \( D_{\delta,\mu} \), we have

\[
F(R^{(1)}) - F(R^{(0)}) \leq \sum_{i,j=1}^{n} \frac{\partial F(R^{(0)})}{\partial R^{(0)}_{ij}} \left( R^{(1)}_{ij} - R^{(0)}_{ij} \right) + d. 
\]

**Proof.** By the assumptions and the definition of \( D_{\delta,\mu} \) in (1.5), we have

\[
|\beta^{(1)} - \beta^{(0)}|^2 \leq n \|\beta^{(1)} - \beta^{(0)}\|^2 \leq 2n \left( \|\beta^{(0)}\|^2 + \|\beta^{(1)}\|^2 \right) \leq 4n\mu^2,
\]

and

\[
\delta\lambda_{\min}\left((1-s)\omega^{(0)} + s\omega^{(1)}\right) \geq \mu, \forall s \in [0, 1].
\]

From these estimates and (3.47), we can easily obtain the desired estimate (3.50).

\[ \square \]

4. **Comparison principle for nonsymmetric Monge-Ampère type equations**

In this section, we shall establish the comparison principle for the Monge-Ampère type equation (1.1) in the case that it is \( \delta \)-elliptic, \( 0 \leq \delta < 1 \) with respect to compared functions. Consider the following operator associated to the equation (1.1),

\[
G[u](x) \equiv \log \det \left[ D^2u - A(x, u, Du) - B(x, u, Du) \right] - \log f(x, u, Du), x \in \Omega.
\]

**Theorem 4.** Let \( u(x), v(x) \in C^2(\Omega) \) satisfying \( G[u](x) \leq G[v](x) \) in \( \Omega \), \( u \geq v \) on \( \partial\Omega \), where \( A, B, f \) are in \( C^{1}(\Omega \times \mathbb{R} \times \mathbb{R}^n) \) and \( f > 0 \) on \( \Omega \times \mathbb{R} \times \mathbb{R}^n \). Suppose that the following conditions are satisfied for some nonnegative constants \( \delta, \alpha_1, \beta_1, \beta_2 \),
$0 \leq \delta < 1$ and for all $x \in \Omega$, $z \in \mathbb{R}$, $p \in \mathbb{R}^n$,

(i) $\lambda_{\min}(\omega(x, u)) > 0$, $\lambda_{\min}(\omega(x, v)) > 0$;
(ii) $\delta \min\{\lambda_{\min}(\omega(x, u)), \lambda_{\min}(\omega(x, v))\} \geq \mu(B)$;
(iii) $\lambda_{\min}(D_zA(x, z, p)) \geq (-\alpha_1) \min\{\lambda_{\min}(\omega(x, u)), \lambda_{\min}(\omega(x, v))\}$;
(iv) $\beta_1 \min\{\lambda_{\min}(\omega(x, u)), \lambda_{\min}(\omega(x, v))\} \geq \mu(D_zB)$;
(v) $\inf_{\Omega \times \mathbb{R} \times \mathbb{R}^n} \left( \frac{D_{u^l}}{\mu(B)} \right) \geq n \left( \alpha_1 + \frac{\delta}{1 + \delta^2 \beta_1} \right)$,

where the quantities $\mu(B)$, $\mu(D_zB)$ are defined as in [1.0].

Then we have that either $u > v$ or $u \equiv v$ in $\Omega$.

**Proof.** For all $x \in \Omega$ and for all $t \in [0, 1]$, we set

$$w(x) = v(x) - u(x),$$

$$u^l(x) = (1 - t)u(x) + tv(x),$$

and

$$R^{(0)}(x) = D^2u(x) - A(x, u(x), Du(x)) - B(x, u(x), Du(x)),$$
$$R^{(1)}(x) = D^2v(x) - A(x, v(x), Dv(x)) - B(x, v(x), Dv(x)),$$
$$R^l(x) = (1 - t)R^{(0)}(x) + tR^{(1)}(x),$$
$$\omega^{(0)}(x) = D^2u(x) - A(x, u(x), Du(x)),$$
$$\omega^{(1)}(x) = D^2v(x) - A(x, v(x), Dv(x)),$$
$$\omega^l(x) = (1 - t)\omega^{(0)}(x) + t\omega^{(1)}(x).$$

Set

$$g(t, x) \equiv \log \det \left( (1 - t)R^{(0)}(x) + tR^{(1)}(x) \right) = \log \det \left( R^l(x) \right).$$

Then by the mean value Theorem and [3.10], we have

$$\log \det \left( R^{(1)}(x) \right) - \log \det \left( R^{(0)}(x) \right) = g(1, x) - g(0, x)$$

$$= g_t(s, x) = \sum_{i,j=1}^n \left( R^{(s)}(x) \right)_{ji}^{-1} \left( R^{(1)}(x) - R^{(0)}(x) \right)_{ji},$$

where $s \in (0, 1)$ is the constant depending on $x$.

Set

$$h(t, x) = \sum_{i,j=1}^n \left( R^{(s)}(x) \right)_{ji}^{-1} \left[ D_{ij}u^l(x) - A_{ij}(x, u^l(x), Du^l(x)) \right.$$

$$- B_{ij}(x, u^l(x), Du^l(x))] - \log f(x, u^l(x), Du^l(x)).$$
Then by the mean value Theorem and (4.1), we obtain
\[ G[v](x) - G[u](x) = h(1, x) - h(0, x) = h'(\tau, x) \]
\[ = \sum_{i,j=1}^{n} (R^{(s)}(x))^{-1}_{ij} \left[ (D_{ij}v(x) - D_{ij}u(x)) \right. \]
\[ - \sum_{i=1}^{n} \left. \left. \left( D_{pk}A_{ij} + D_{pk}B_{ij} \right) \left( x, u^{(\tau)}(x), D_{u^{(\tau)}}(x) \right) \right) \left( D_{k}v(x) - D_{k}u(x) \right) \right. \]
\[ - (D_{z}A_{ij} + D_{z}B_{ij}) \left( x, u^{(\tau)}(x), D_{u^{(\tau)}}(x) \right) (v(x) - u(x)) \]
\[ - \frac{1}{f(x, u^{(\tau)}(x), D_{u^{(\tau)}}(x))} \sum_{k=1}^{n} D_{pk} f(x, u^{(\tau)}(x), D_{u^{(\tau)}}(x)) (D_{k}v(x) - D_{k}u(x)) \]
\[ \left. - \frac{1}{f(x, u^{(\tau)}(x), D_{u^{(\tau)}}(x))} D_{z} f(x, u^{(\tau)}(x), D_{u^{(\tau)}}(x)) (v(x) - u(x)) \right), \]
where \( \tau \in (0, 1) \) is the constant depending on \( x \) and \( s \).
Consequently,
\begin{equation}
G[v](x) - G[u](x) = a^{ij}(x) D_{ij} w(x) + b^{k}(x) D_{k} w(x) + c(x) w(x),
\end{equation}
where
\[ a^{ij}(x) = \frac{(R^{(s)}(x))^{-1}_{ij} + (R^{(s)}(x))^{-1}_{ij}}{2}, \]
\[ b^{k}(x) = - \sum_{i,j=1}^{n} \left( R^{(s)}(x) \right)^{-1}_{ij} \left[ (D_{pk}A_{ij} + D_{pk}B_{ij}) \left( x, u^{(\tau)}(x), D_{u^{(\tau)}}(x) \right) \right] \]
\begin{equation}
- \frac{1}{f(x, u^{(\tau)}(x), D_{u^{(\tau)}}(x))} D_{pk} f(x, u^{(\tau)}(x), D_{u^{(\tau)}}(x)),
\end{equation}
\[ c(x) = - \sum_{i,j=1}^{n} \left( R^{(s)}(x) \right)^{-1}_{ij} \left[ (D_{z}A_{ij} + D_{z}B_{ij}) \left( x, u^{(\tau)}(x), D_{u^{(\tau)}}(x) \right) \right] \]
\begin{equation}
- \frac{1}{f(x, u^{(\tau)}(x), D_{u^{(\tau)}}(x))} D_{z} f(x, u^{(\tau)}(x), D_{u^{(\tau)}}(x)).
\end{equation}
Consider the second order linear partial differential operator \( L \) given by
\begin{equation}
L := a^{ij}(x) D_{ij} + b^{k}(x) D_{k} + c(x),
\end{equation}
where the coefficients \( a^{ij}, b^{k}, c \) are defined by (4.3). We have the following claims.

Claim 1. The operator \( L \) is uniformly elliptic; that is, there exists positive constants \( \lambda, \Lambda \) such that
\begin{equation}
\lambda |\xi|^2 \leq a^{ij}(x) \xi_i \xi_i \leq \Lambda |\xi|^2, \quad \forall x \in \overline{\Omega}, \forall \xi \in \mathbb{R}^n.
\end{equation}
Indeed, it follows from conditions (i), (ii) that \( R^{(0)}(x), R^{(1)}(x) \) are in the set \( D_{\delta, \mu(B)} \), so is \( R^{(s)}(x) \). Also from (i) and our regularity assumptions for \( A, B \) and
u, we infer that there exists positive constants $\lambda_0, \Lambda_0$ such that

$$
\lambda_0 E \leq \omega(x, u) \leq \Lambda_0 E, \quad \lambda_0 E \leq \omega(x, v) \leq \Lambda_0 E, \quad \forall x \in \overline{\Omega},
$$

where $E$ is the unit matrix of order $n$. It follows that

$$(4.6) \quad \lambda_0 E \leq \omega(s) (x) \leq \Lambda_0 E, \quad \forall x \in \overline{\Omega}.
$$

Therefore

$$
\frac{1}{\Lambda_0} E \leq \omega(s) (x) \leq \frac{1}{\lambda_0} E, \quad \forall x \in \overline{\Omega}.
$$

Moreover, by Proposition 4 and Corollary 1, one can easily show that

$$
\frac{1}{1 + \delta^2} \left( (\omega(s)(x))^{-1} \xi, \xi \right) \leq (H(x) \xi, \xi) \leq \left( (\omega(s)(x))^{-1} \xi, \xi \right), \quad \forall x \in \overline{\Omega},
$$

where

$$
H(x) := \frac{(R(s)(x))^{-1} + ((R(s)(x))^{-1})^T}{2}.
$$

Then (4.5) follows from the above estimates by taking $\lambda = \frac{1}{\lambda_0}$ and $\Lambda = \frac{1}{\Lambda_0}$.

**Claim 2.** The coefficients $b^k(x), c(x)$ are bounded in $\overline{\Omega}$.

This claim easily follows from the fact that the set $\{(x, u^{(\tau)}(x), Du^{(\tau)}(x))\}$ is bounded in $\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^n$ and

$$
det R(s)(x) \geq det \omega(s)(x) \geq \lambda_0^n > 0, \quad \forall x \in \overline{\Omega},
$$

which holds by Proposition 3 and (4.6).

**Claim 3.** The coefficient $c(x) \leq 0$ for all $x \in \overline{\Omega}$.

**Claim 3** follows from (4.3), condition (v) and the two following inequalities

$$
(4.7) \quad - \sum_{i,j=1}^{n} (R(s)(x))^{-1}_{ji} Dz A_{ij}(x, u^{(\tau)}(x), Du^{(\tau)}(x)) \leq n \alpha_1, \quad \forall x \in \overline{\Omega},
$$

$$
(4.8) \quad - \sum_{i,j=1}^{n} (R(s)(x))^{-1}_{ji} Dz B_{ij}(x, u^{(\tau)}(x), Du^{(\tau)}(x)) \leq n \frac{\delta}{1 + \delta^2} \beta_1, \quad \forall x \in \overline{\Omega}.
$$

So it remains to prove (4.7) and (4.8).

Since $Dz A$ is symmetric and $H(x)$ is positive definite, we have

$$
(4.9) \quad \sum_{i,j=1}^{n} (R(s)(x))^{-1}_{ji} Dz A_{ij} = \text{Tr}[(Dz A)(H(x))] \geq \lambda_{\text{min}}(Dz A) \text{Tr}H(x).
$$

Given any point $x \in \overline{\Omega}$. Assume that $\lambda_{\text{min}}(Dz A(x, u^{(\tau)}(x), Du^{(\tau)}(x))) \geq 0$ at this point. Then by (4.9), the left hand side of (4.7) is nonpositive and thus (4.7)
follows. Assume the contrary, that $\lambda_{\min}(DzA(x, u(\tau)(x), Du(\tau)(x))) < 0$. Then (4.7) follows from (4.9) and the following estimates

$$
\text{Tr}H(x) \leq \text{Tr}\left[\left(\omega^{(s)}(x)\right)^{-1}\right] \leq \frac{n}{\lambda_{\min}(\omega^{(s)}(x))},
$$

$$
(-\alpha_1)\lambda_{\min}(\omega^{(s)}(x)) \leq (-\alpha_1)\min\{\lambda_{\min}(\omega(x, u)), \lambda_{\min}(\omega(x, v))\}
$$

$$
\leq \lambda_{\min}(DzA(x, u(\tau)(x), Du(\tau)(x))),
$$

which are inferred from Corollary 2 and condition (iii), respectively. Thus (4.7) is proved.

We now prove (4.8). Set

$$
K(x) := \frac{(R^{(s)}(x))^{-1} - (R^{(s)}(x))^{-1}}{2}.
$$

By Proposition 4 and Corollary 1, one can easily show that

$$
\|K(x)\| \leq \frac{\delta}{1 + \delta^2 \lambda_{\min}(\omega^{(s)}(x))}, \quad \forall x \in \Omega,
$$

and, by condition (iv),

$$
\mu(DzB) \leq \beta_1 \min\{\lambda_{\min}(\omega(x, u)), \lambda_{\min}(\omega(x, v))\} \leq \beta_1 \lambda_{\min}(\omega^{(s)}(x)), \quad \forall x \in \Omega.
$$

From these estimates and the following inequality:

$$
\text{Tr}(MN) \leq |M||N| \leq n\|M\||N\|, \quad \text{for all } M, N \in \mathbb{R}^{n \times n},
$$

we obtain for all $x \in \Omega$,

$$
- \sum_{i,j=1}^{n} (R^{(s)}(x))_{ij}^{-1} DzB_{ij}(x, u(\tau)(x), Du(\tau)(x))
$$

$$
= \text{Tr}\left[\left(K(x)\right)\left(-DzB(x, u(\tau)(x), Du(\tau)(x))\right)\right]
$$

$$
\leq n\frac{\delta}{1 + \delta^2 \lambda_{\min}(\omega^{(s)}(x))} \leq n\frac{\delta}{1 + \delta^2 \beta_1}.
$$

Thus (4.8) is proved.

To complete the proof of this theorem, we note that, if $G[u] \leq G[v]$ in $\Omega$, $u \geq v$ on $\partial \Omega$ then, by (4.2) and (4.4), $Lw \geq 0$ in $\Omega$, $w \leq 0$ on $\partial \Omega$. By Claims 1, 2, 3, we can apply the strong maximum principle of E. Hopf (Theorem 3.5, [2]) to obtain the conclusion of Theorem 4.

**Corollary 5.** Under the assumptions of Theorem 4, where $G[u] < G[v]$ in $\Omega$, $u = v$ on $\partial \Omega$, $\partial \Omega \in C^2$, we have the following strict inequalities

$$
u > v, \quad \text{in } \Omega,
$$

$$
\frac{\partial u}{\partial \nu} > \frac{\partial v}{\partial \nu}, \quad \text{on } \partial \Omega.
$$
where $\nu$ is the unit inner normal to $\partial \Omega$.

Proof. Set $w = v - u$. Following the proof of Theorem 4, we have

$$Lw > 0 \text{ in } \Omega, \quad w = 0 \text{ on } \partial \Omega,$$

where $L$ is the uniformly elliptic operator defined by (4.4). Further, by Theorem 4, we have that $w < 0$ in $\Omega$. Hence, we can apply the Hopf’s lemma (Theorem 3.4, [2]) to obtain

$$\frac{\partial w}{\partial \nu} < 0, \text{ on } \partial \Omega.$$

The proof is completed. $\square$

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