Chiral fermions on a finite lattice

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We discuss how to formulate Dirac fermion operator on a finite lattice such that it can provide a nonperturbative regularization for massless fermion interacting with a background gauge field.

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I. Introduction

Consider a massless Dirac fermion interacting with a background gauge field. Our present goal is to formulate a nonperturbatively regularized quantum field theory which at least satisfies the following physical constraints:

(A) In the classical limit, it reproduces the classical physics of the action,
\[ \mathcal{A} = \int_x \bar{\psi}(x) \gamma_\mu [\partial_\mu + igA_\mu(x)] \psi(x). \]

(B) For topologically trivial gauge backgrounds, and in the weak coupling limit, it agrees with the predictions of weak coupling perturbation theory of the action.

(C) For topologically nontrivial gauge backgrounds, it possesses exact zero modes satisfying the Atiyah-Singer index theorem.

Although Wilson’s idea [1] of formulating gauge theories on the spacetime lattice is the most successful nonperturbative regularization for pure gauge fields, putting massless Dirac fermions [2] on the lattice has been a notorious problem for more than twenty years. The resolution of the lattice fermion problem first appeared in the context of the Domain-Wall fermion [3], and it motivated the Overlap formalism [4] which led to the construction of Overlap-Dirac operator [5] in 1997. We refer to ref. [6] for a recent review of the Domain-Wall fermions, and to ref. [7] for a recent review of the Overlap.

However, if we consider a Weyl fermion interacting with a background gauge field, then a completely satisfactory nonperturbative regularization for chiral gauge theories (e.g., the standard model) has not yet been presented up to this moment.

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In the following, we will concentrate our discussions on the general principles to construct chiral Dirac fermion operators on a finite lattice, in particular, for vector-like gauge theories such as QCD. With the constraints imposed by the Nielson-Ninomiya no-go theorem\[8\], one can construct a gauge covariant Dirac operator $D$ on a finite lattice such that:

(i) $D$ breaks the chiral symmetry (i.e., $D\gamma_5 + \gamma_5 D = 0$) at finite lattice spacing but recovers the chiral symmetry in the continuum limit $a \to 0$.

(ii) $D$ is local.

$|D(x, y)| \sim \exp(-|x - y|/l)$ with $l \sim a$; or $D(x, y) = 0$ for $|x - y| > z$, where $z$ is much less than the size of the lattice.

(iii) In the free fermion limit, $D$ is free of species doublings.

The free fermion propagator $D^{-1}(p)$ has only one simple pole at the origin $p = 0$ in the Brillouin zone.

(iv) In the free fermion limit, $D$ has correct continuum behavior.

In the limit $a \to 0$, $D(p) \sim i\gamma_\mu p_\mu$ around $p = 0$.

However, one cannot push the property (i) any further, while maintaining properties (ii)-(iv). For example, if $D$ is chirally symmetric at finite lattice spacing, then it must violate at least one of the three properties (i)-(iv). We note that these four properties (i)-(iv) form the necessary conditions to meet the requirements (A)-(C), however, they are not sufficient to guarantee that (C) will be satisfied.

An example satisfying (i)-(iv) is the standard Wilson-Dirac fermion operator\[2\]

$$D_w = \gamma_\mu t_\mu + w$$

where

$$t_\mu(x, y) = \frac{1}{2}[U_\mu(x)\delta_{x+\bar{\mu}, y} - U_\mu^\dagger(y)\delta_{x-\bar{\mu}, y}] ,$$

$$\sigma_\mu \sigma_\nu + \sigma_\nu \sigma_\mu = 2\delta_{\mu\nu} ,$$

$$\gamma_\mu = \begin{pmatrix} 0 & \sigma_\mu \\ \sigma_\mu^\dagger & 0 \end{pmatrix}$$

and $w$ is the Wilson term

$$w(x, y) = \frac{a}{2} \sum_\mu \left[2\delta_{x,y} - U_\mu(x)\delta_{x+\bar{\mu}, y} - U_\mu^\dagger(y)\delta_{x-\bar{\mu}, y} \right] .$$
The color, flavor and spinor indices have been suppressed in (1). The first term on the r.h.s. of (1) is the naive fermion operator which is chirally symmetric at any lattice spacing and satisfies properties (ii) and (iv) but violates (iii) since it has $2^d - 1$ fermion doubled modes. The purpose of the Wilson term $w$ is to give each doubled mode a mass of $\sim 1/a$ such that in the continuum limit ($a \to 0$), each doubled mode becomes infinitely heavy and decouples from the fermion propagator. However, the introduction of the Wilson term has serious drawbacks. It causes $O(a)$ artifacts and also leads to the notorious problems such as vector current renormalization, additive fermion mass renormalization, and mixings between operators in different chiral representations.

During the last two years, it has become clear that the proper way to break the chiral symmetry of $D$ at finite lattice spacing is to conform with the Ginsparg-Wilson relation [9]

$$D\gamma_5 + \gamma_5 D = 2a DR\gamma_5 D$$  \hspace{1cm} (6)

where $R$ is a positive definite hermitian operator which is local in the position space and trivial in the Dirac space. Then the generalized chiral symmetry (6) can ensure that the theory is free of above mentioned problems of the Wilson-Dirac operator [10].

The general solution to the Ginsparg-Wilson relation can be written as [11]

$$D = D_c(\mathbb{1} + aRD_c)^{-1} = (\mathbb{1} + aD_cR)^{-1}D_c$$  \hspace{1cm} (7)

where $D_c$ is any chirally symmetric ($D_c\gamma_5 + \gamma_5 D_c = 0$) Dirac operator which must violate at least one of the three properties (ii)-(iv) above. Now we must require $D_c$ to satisfy (iii) and (iv), but violate (ii), i.e, $D_c$ is nonlocal), since (7) can transform the nonlocal $D_c$ into a local $D$ on a finite lattice for $R = r\mathbb{1}$ with $r$ in the proper range [12, 16], while the properties (iii) and (iv) are preserved. Moreover, the zero modes and the index of $D_c$ are invariant under the transformation [11, 12]. That is, a zero mode of $D_c$ is also a zero mode of $D$ and vice versa, hence,

$$n_+(D_c) = n_+(D), \quad n_-(D_c) = n_-(D),$$  \hspace{1cm} (8)

$$\text{index}(D_c) = n_-(D_c) - n_+(D_c) = n_-(D) - n_+(D) = \text{index}(D).$$  \hspace{1cm} (9)

Since the massless Dirac fermion operator in continuum is antihermitian, we also require that $D_c$ is antihermitian ($D_c^\dagger = -D_c$) even at finite lattice spacing. Then the chiral symmetry of $D_c$ together with its antihermiticity implies that $D_c$ satisfies the $\gamma_5$-hermiticity

$$D_c^\dagger = \gamma_5 D_c \gamma_5.$$  \hspace{1cm} (10)
This implies that $D$ in the general solution (7) also satisfies the $\gamma_5$-hermiticity
\[ D^\dagger = \gamma_5 D \gamma_5. \tag{11} \]
Then the eigenvalues of $D$ are either real or come in complex conjugate pairs. Furthermore, from (7), since $R$ is a positive definite hermitian operator, the lower bound of real eigenvalues of $D$ is zero, thus $\det(D)$ is real and nonnegative, and is amenable to Hybrid Monte Carlo simulation for dynamical fermions with any number of fermion flavors.

In particular, for $R = r I$ with $r > 0$, the analysis [Eqs. (24)-(41)] in ref. [13] goes through with trivial modifications. The main results are:

(a) The eigenvalues of $D$ fall on a circle with center at $1/2r$, and radius $1/2r$, and have the reflection symmetry with respect to the real axis.

(b) The real eigenmodes (if any) at 0 and $1/r$ have definite chirality $+1$ or $-1$.

(c) The chirality of any complex eigenmodes is zero.

(d) Total chirality of all eigenmodes must vanish.
\[ \text{Tr}(\gamma_5) = \sum_s \phi_s^\dagger \gamma_5 \phi_s = n_+ - n_- + N_+ - N_- = 0 \]
where $n_+(n_-)$ denotes the number of zero modes of positive (negative) chirality, and $N_+(N_-)$ the number of $1/r$ modes of positive (negative) chirality. From (d), we immediately see that any zero mode must be accompanied by a real $1/r$ mode with opposite chirality, and the index of $D$ is
\[ \text{index}(D) \equiv n_- - n_+ = -(N_- - N_+). \tag{12} \]

Now the central problem is to construct the chirally symmetric $D_c$ which is nonlocal, and satisfies (iii), (iv), and (10). Furthermore we also require that $D_c$ is topologically proper (i.e., satisfying the Atiyah-Singer index theorem) for any prescribed smooth gauge background. These constitute the necessary requirements [12] for $D_c$ to enter (7) such that $D$ could provide a nonperturbative regularization for a massless Dirac fermion interacting with a background gauge field. Explicitly, these necessary requirements are:

(a) $D_c$ is antihermitian (hence $\gamma_5$-hermitian) and it agrees with $\gamma_\mu (\partial_\mu + igA_\mu)$ in the classical continuum limit.

(b) $D_c$ is free of species doubling.

(c) $D_c$ is nonlocal.
(d) $D_c$ is well defined in topologically trivial background gauge field.

(e) $D_c$ has zero modes as well as simple poles in topologically non-trivial background gauge fields (each zero mode of $D_c$ must be accompanied by a simple pole of $D_c$). Furthermore, the zero modes of $D_c$ satisfy the Atiyah-Singer index theorem for any prescribed smooth gauge background.

The general solution of $D_c$ satisfying these requirements had been investigated in ref. [18] and was reported by the author at Chiral’99. However, in general, given any lattice Dirac operator $D$, there exists a transformation $\mathcal{T}(R_c)$ for $D$ such that the transformed Dirac operator $D_c = \mathcal{T}(R_c)[D]$ is chirally symmetric [14, 15]. Therefore, for the sake of completeness, the transformation $\mathcal{T}(R_c)$ is outlined in section 2. Then the construction in ref. [18] is reviewed in section 3. Concluding remarks are briefly outlined in section 4.

II. A transformation for lattice Dirac operators

Given any lattice Dirac operator $D$, in general, there are many different ways to construct a chirally symmetric $D_c$ out of $D$. For example, we can construct

$$D_s = \frac{1}{2}(D - \gamma_5 D \gamma_5)$$

which is chirally symmetric ($D_s \gamma_5 + \gamma_5 D_s = 0$). However, this transformation does not necessarily preserve the property (iii). For example, if we apply this transformation to the Wilson-Dirac operator [1], we obtain $D_s = \gamma_\mu t_\mu$, the naive fermion operator which suffers from the species doublings. Although $D_w$ is free of species doublings in the continuum limit, the transformation (13) cannot preserve this property since $D_s$ is local. Therefore, we need a transformation which preserves the properties (iii), (iv), (11) and (8), but exchanges the locality of $D$ for its chiral symmetry at finite lattice spacing. The transformation [14, 15]

$$\mathcal{T}(R) : \quad D \rightarrow D' = \mathcal{T}(R)[D] \equiv D(\mathbb{I} + RD)^{-1} = (\mathbb{I} + DR)^{-1}D$$

which generalizes (13) can serve our purposes. Evidently, the set of these transformations, $\{\mathcal{T}(R)\}$, form an abelian group with group parameter space $\{R\}$ [14, 15].

For any $D$ satisfying $\gamma_5$-hermiticity, (11), there exists a hermitian $R_c$,

$$R_c = -\frac{1}{2}(D^{-1} + \gamma_5 D^{-1} \gamma_5)$$

such that

$$D_c = \mathcal{T}(R_c)[D] = 2\gamma_5 D(\gamma_5 D - D \gamma_5)^{-1}D$$

(16)
is chirally symmetric ($\gamma_5 D_c + D_c \gamma_5 = 0$) and antihermitian ($D_c^\dagger = -D_c$).

It is evident that for any two lattice Dirac operators $D^{(1)}$ and $D^{(2)}$ satisfying (11), their corresponding chiral limits obtained from (16), say, $D_c^{(1)}$ and $D_c^{(2)}$, are in general different. However, they are related by the transformation

$$D_c^{(1)} = \sum_i T_i D_c^{(2)} T_i^\dagger$$

where each $T_i$ commutes with $\gamma_5$. In general, given any two $D_c^{(1)}$ and $D_c^{(2)}$, it is a nontrivial task to obtain all $T_i$ in (17).

For the Wilson-Dirac operator (1), the transformation (16) gives

$$D_c = \gamma_\mu t_\mu - w (\gamma_\mu t_\mu)^{-1} w .$$

It is antihermitian, nonlocal (due to the second term), and satisfies (i), (iii) and (iv). Then we can substitute (18) into (11) with $R = r I$ to obtain a GW Dirac operator

$$D = \begin{bmatrix} r C^\dagger C (1 + r^2 C^\dagger C)^{-1} & -C^\dagger (1 + r^2 C^\dagger C)^{-1} \\ C (1 + r^2 C^\dagger C)^{-1} & r C C^\dagger (1 + r^2 C^\dagger C)^{-1} \end{bmatrix}$$

where

$$C = (\sigma_\mu^i t_\mu) - w (\sigma_\mu^i t_\mu)^{-1} w .$$

On a finite lattice, the GW Dirac operator (19) can be constructed to be local but not highly peaked in the diagonal elements if the value of $r$ is within a proper range [16]. In other words, if $r$ is too small, then $D$ must be nonlocal since $D$ is close to $D_c$. On the other hand, if $r$ is too large, then $D$ becomes highly peaked in the diagonal elements. In these two extreme cases, $D$ does not respond properly to the background gauge field (e.g., the anomaly function of $D$ does not agree with the Chern-Pontryagin density of the prescribed gauge background). In the limit the number of sites in each dimension goes to infinity, the proper range of $r$ extends to $(0, \infty)$.

The anomaly function of $D$ (19) can be written

$$A_D(x) = 2 \text{tr}[(1 + r^2 CC^\dagger)^{-1} - (1 + r^2 C^\dagger C)^{-1}](x,x)$$

where $\text{tr}$ denotes the trace over the color, flavor and spinor space.

It is instructive to compare the anomaly function of the Wilson-Dirac operator $D_w$ (1) to that of the GW Wilson-Dirac operator $D$ (19), in a topologically trivial background gauge field, on a finite lattice, as shown in Fig. 1 in ref. [17] and Fig. 1 in ref. [15], respectively. The anomaly function of $D_w$ is very different from the Chern-Pontryagin density $\rho(x)$ on a $12 \times 12$ torus, while that of the GW Dirac-Wilson operator $D$ is in good
agreement with $\rho(x)$ at each site. This demonstrates that the transformation $\mathcal{T}(r + R_c)$ plays the important role in converting $D_w$ into a GW Dirac operator $D = \mathcal{T}(r + R_c)[D_w]$ which is free of $O(a)$ lattice artifacts, thus $D$ can reproduce the correct anomaly function even on a finite lattice provided that the local fluctuations of the background gauge field are not too violent.

However, a lattice Dirac operator satisfying properties (i)-(iv), (11) and the GW relation (6) does not guarantee that it has the correct anomaly function in topologically nontrivial gauge backgrounds. This can be seen as follows. It is well known that the sum of the axial anomaly over all sites of a finite lattice is equal to the index of the lattice Dirac operator. Therefore the necessary condition for a lattice Dirac operator to have the correct anomaly function is that it possesses exact zero modes satisfying the Atiyah-Singer index theorem i.e., $n_- - n_+ = Q$. However, not every lattice Dirac operator has exact zero modes in topologically nontrivial gauge backgrounds. For example, the Wilson-Dirac operator, $D_w$ in (1), it does not have exact zero modes in topologically nontrivial sectors. Then, according to (6), $D = \mathcal{T}(r + R_c)[D_w]$ also does not have any exact zero modes, even though $D$ satisfies the GW relation (6), and properties (i)-(iv) and (11). This implies that the anomaly function of $D_w$ or $D = \mathcal{T}(r + R_c)[D_w]$ does not agree with the Chern-Pontryagin density for topologically nontrivial gauge fields on any finite lattices. Consequently, the disagreement must persist in the continuum limit $a \to 0$.

This provides an example to illustrate that any lattice Dirac operator $D$ must possess a nonperturbative attribute, the topological characteristics, $c[D]$. In general, $c[D]$ is a rational number, a functional of $D$ and the gauge configuration. In a gauge background with nonzero topological charge $Q$, the index of a lattice Dirac operator is

$$n_- - n_+ = c[D] Q .$$

In the case of $D_w$ and $D = \mathcal{T}(r + R_c)[D_w]$, $n_- = n_+ = 0$, thus the topological characteristics $c[D_w] = c[D] = 0$. For the vacuum sector with nonzero topological charge density (i.e., $Q = 0$ but $\rho(x) \neq 0$), $c[D]$ cannot be defined by (22), however, it can be defined unambiguously in the context of the anomaly function.

We note in passing that for the Overlap-Dirac operator $D_o = \mathds{1} + V$, the transformation (16) gives $D_o = 2(\mathds{1} + V)(\mathds{1} - V)^{-1}$, as expected. Substituting $D_o$ into (11) with $R = r\mathds{1}$, we obtain the generalized Overlap-Dirac operator

$$D = 2(\mathds{1} + V)[\mathds{1} - V + 2r(\mathds{1} + V)]^{-1} .$$

III. A construction of $D_c$

Suppose that we do not have any lattice Dirac operator to begin with. It is still possible for us to construct $D_c$ according to the necessary requirements (a)-(e) listed
in section 1. Since $D_c$ is antihermitian, there exists one to one correspondence between $D_c$ and a unitary operator $V$,

$$D_c = (I+V)(I-V)^{-1}, \quad V = (D_c - I)(D_c + I)^{-1}.$$  \hfill (24)

where $V$ also satisfies the $\gamma_5$-hermiticity, $\gamma_5 V \gamma_5 = V^\dagger$. Then the unitary operator $V$ can be expressed in terms of a hermitian operator $h$,

$$V = \gamma_5 h = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \begin{pmatrix} h_1 & h_2 \\ h_2^\dagger & h_3 \end{pmatrix} = \begin{pmatrix} h_1 & h_2 \\ -h_2^\dagger & -h_3 \end{pmatrix}$$  \hfill (25)

where $h_1^\dagger = h_1$ and $h_3^\dagger = h_3$. Using the unitarity condition $V^\dagger V = I$, we have $h^2 = I$,

$$h^2 = \begin{pmatrix} h_1^2 + h_2 h_2^\dagger & h_1 h_2 + h_2 h_3 \\ h_2 h_1 + h_3 h_2^\dagger & h_2^\dagger h_2 + h_3^2 \end{pmatrix} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}$$  \hfill (26)

Then we obtain

$$D_c \equiv \begin{bmatrix} 0 & D_R \\ D_L & 0 \end{bmatrix} = \begin{bmatrix} 0 & (I - h_1)^{-1} h_2 \\ -h_2^{-1}(I + h_1) & 0 \end{bmatrix}$$  \hfill (27)

where $D_L = -D_R^\dagger$. The general solution to Eq. (24) can be written as

$$h_1 = \pm \frac{1}{\sqrt{1 + bb^\dagger}}$$  \hfill (28)

$$h_2 = \frac{1}{\sqrt{1 + bb^\dagger}} b e^{i\theta}$$  \hfill (29)

where $e^{i\theta}$ is an arbitrary phase, and $b$ is any operator. In the following we will restrict $\theta$ to zero, and also pick the minus sign for $h_1$. Then the general solution for $D_c$ can be written in the following form

$$D_L = b^{-1} \left[ I - \sqrt{I + bb^\dagger} \right]$$  \hfill (30)

$$D_R = \left[ I + \sqrt{I + bb^\dagger} \right]^{-1} b$$  \hfill (31)

Due to the presence of the square root in (30) and (31), $D_c$ is nonlocal for nontrivial $b$, thus the requirement (b) is satisfied. In the following, we outline a construction of $b$ which satisfies constraints (a)-(d) but not (e).

In order to have $D_c$ satisfy the constraint (a), we first try

$$b = w^{-1} \sum \sigma_{\mu} t_{\mu}$$  \hfill (32)
where \( t_\mu \) and \( \sigma_\mu \) are defined in (2) and (3) respectively, and \( w \) is a non-singular hermitian operator which is trivial in the Dirac space and goes to a constant in the classical continuum limit. Then Eqs. (30) and (31) become

\[
D_L = (\sigma \cdot t)^{-1} \left[ w - \sqrt{w^2} \sqrt{1 + w^{-1} t^2} w^{-1} \right] \tag{33}
\]

\[
D_R = -D_L^\dagger \tag{34}
\]

where

\[
\sigma \cdot t = \sum_\mu \sigma_\mu t_\mu \tag{35}
\]

\[
t^2 = -(\sigma \cdot t)(\sigma^\dagger \cdot t) \tag{36}
\]

However, \( t_\mu \) suffers from species doublings. Now our task is to construct a hermitian operator \( w \) such that the doubled modes are completely decoupled from the fermion propagator \( D_c^{-1} \), even at finite lattice spacing.

As discussed in ref. [18], we want to construct a hermitian \( w \) such that in the free fermion limit, it satisfies the following condition,

\[
w(p) = \begin{cases} 
> 0 & \text{for the primary mode at } p = 0 \\
< 0 & \text{for the doubled modes at } p \in \otimes_\mu \{0, \pi/a\} \setminus \{p = 0\}. 
\end{cases} \tag{37}
\]

A simplest solution to (37) is

\[
w(p) = c - 2 \sum_\mu \sin^2(p_\mu a/2), \quad c \in (0, 2) \tag{38}
\]

Note that the role of \( w \) in the general solution of \( D_c \) is quite different from the Wilson term (5) in the Wilson-Dirac operator (1). In the general solution of \( D_c \), the chiral symmetry is always preserved, and the role of \( w \) is to suppress the doubled modes completely at finite lattice spacing, while in the Wilson-Dirac operator, the Wilson term breaks the chiral symmetry explicitly and gives a mass of order \( a^{-1} \) to the doubled modes such that they can be decoupled in the continuum limit (\( a \to 0 \)). After the gauge links are restored, \( w \) in the position space becomes

\[
w(x, y) = c \delta_{x,y} - \frac{1}{2} \sum_\mu \left[ 2\delta_{x,y} - U_\mu(x)\delta_{x+\hat{\mu},y} - U_\mu^\dagger(y)\delta_{x-\hat{\mu},y} \right] \tag{39}
\]

This is one of the simplest solution of \( w \) satisfying the requirement (37) in the free fermion limit. Certainly, there exists other solutions to (37).

\[\dagger\] The \( w \) operator in this section is different from the Wilson term in Eq. (5).
Substituting the $D_c$ [Eqs. (33)-(34) with $t_\mu$ in (2), and $w$ in (39) with $c = 1$, and the normalization constant $1/2$] into (7) with $R = r I$, we obtain the GW Dirac operator

$$D = \begin{bmatrix}
    r(BB^\dagger + r^2)^{-1} & -B(B^\dagger B + r^2)^{-1} \\
    B^\dagger(BB^\dagger + r^2)^{-1} & r(B^\dagger B + r^2)^{-1}
\end{bmatrix}$$

(40)

where

$$B = \frac{1}{2} \left( w - \sqrt{w^2} \sqrt{1 + w^{-1} t^2} \right)^{-1} (\sigma \cdot t)$$

(41)

The anomaly function of $D$ is

$$A_D(x) = 2r^2 \text{tr} \left[ (BB^\dagger + r^2)^{-1} - (B^\dagger B + r^2)^{-1} \right] (x, x)$$

(42)

where tr stands for the trace over the color, flavor and spinor space.

Since the $D_c$ of (40) in the free fermion limit is free of species doubling and has the correct continuum behavior, the perturbation calculation in ref. [19] showed that $D = D_c(1 + rD_c)^{-1}$ has the correct chiral anomaly in topologically trivial gauge backgrounds. This has been verified explicitly on a finite lattice. An example is shown in Fig. 2 of ref. [17].

However, for topologically nontrivial sectors, the anomaly function of $D$ also depends on the topological characteristics, $c[D]$, which is a non-perturbative attribute of $D$. Since $D$ does not have any exact zero modes in nontrivial sectors, the anomaly function of $D$ does not agree with the Chern-Pontryagin density for nontrivial gauge backgrounds. Now we have two examples of GW Dirac operators, (19) and (40), which have the correct axial anomaly in the vacuum sector but not in the nontrivial sectors.

In general, if $D_c$ is well defined in nontrivial sectors (i.e., not satisfying (e)), then index$[D_c] = \text{index}[D] = 0$ [11, 12] (i.e., $c[D] = c[D_c] = 0$, $D$ and $D_c$ are topologically trivial). On the other hand, if $D_c$ has zeros and poles, then $c[D] \neq 0$, however, it does not necessarily imply that $c[D] = 1$. Sometimes $c[D]$ may even become a fraction (see Table 4 in ref. [14]). The important point is that in general we cannot predict the value of $c[D]$ in the topologically nontrivial sectors solely based on the properties of $D$ in the free fermion limit or in the vacuum sector.

IV. Summary and discussions

The necessary conditions for a lattice Dirac operator $D$ to provide a nonperturbative regularization for massless fermion interacting with a background gauge field are (A)-(C) listed in section 1. The conditions (A) and (B) can be satisfied if $D$ fulfills (i)-(iv) and (11), and the chiral symmetry is broken according to the Ginsparg-Wilson
relation (6). Since any GW Dirac operator can be represented by the transformation (7) [preserving (iii), (iv), (11) and (8)] on a chirally symmetric $D_c$, then the conditions for $D$ become the necessary requirements (a)-(e) for $D_c$. It should be emphasized that the requirement (C) for $D$ (or the requirement (e) for $D_c$) cannot be expressed in terms of any conditions in the free fermion limit or in the vacuum sector. This naturally leads to the concept of topological characteristics [12, 17] associated with each lattice Dirac operator $D$. Although we have no problems to construct $D$ to satisfy (i)-(iv), (11) and (6), it remains an interesting question whether one can construct a topologically proper $D$ on a four dimensional lattice without square root operations.

If one has a lattice Dirac operator $D$ satisfying (i)-(iv), (11) and (C), then one can transform $D$ into a GW Dirac operator through the transformation $D' = T(r + R_c)[D]$ which not only preserves all essential physics of $D$ but also can restrict the quantum corrections to behave properly such that $D$ is free of $O(a)$ lattice artifacts, additive mass renormalization, etc.

Recently it has been shown that the effective four-dimensional action ($N_s \to \infty$) of the Domain Wall fermion is local and satisfying the GW relation, for gauge fields with small field strength [20]. However, it is possible to transform the Domain-Wall fermion operator into a local GW Dirac operator even at finite $N_s$ [15], then the anomalous effects (e.g., the residual pion mass in Domain-Wall QCD) due to chiral symmetry violations at finite $N_s$ could be suppressed even at moderate $N_s$. Further investigations are required before one can tell whether this scenario can be realized or not.

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