A NEW UPPER BOUND FOR SPHERICAL CODES

NASER T. SARDARI AND MASOUD ZARGAR

ABSTRACT. We introduce a new linear programming method for bounding the maximum number
M(n, θ) of points on a sphere in n-dimensional Euclidean space at an angular distance of not less
than θ from one another. We give the unique optimal solution to this linear programming problem
and improve the best known upper bound of Kabatyanskii and Levenshtein [KL78]. By well-known
methods, this leads to new upper bounds for δn, the maximum packing density of an n-dimensional
Euclidean space by equal balls.

1. Introduction
1.1. Sphere packings. Packing densities have been studied extensively, for purely mathematical
reasons as well as for their connections to coding theory. The work of Conway and Sloane is a great
reference for this subject [CS99]. We proceed by defining the basics in this subject. Consider \( \mathbb{R}^n \)
equipped with the Euclidean metric \(|.|\) and associated volume \( \text{vol}(.). \). For each real \( r > 0 \) and each
\( x \in \mathbb{R}^n \), we denote by \( B_n(x, r) \) the open ball in \( \mathbb{R}^n \) centered at \( x \) and of radius \( r \). For each discrete
set of points \( S \subset \mathbb{R}^n \) such that any two distinct points \( x, y \in S \) satisfy \( |x - y| \geq 2 \), we can consider
\[ \mathcal{P} := \bigcup_{x \in S} B_n(x, 1), \]
the union of unit open balls centered at the points of \( S \). This is called a sphere packing (\( S \) may
vary), and we may associate to it the function mapping each real \( r > 0 \) to
\[ \delta_{\mathcal{P}}(r) := \frac{\text{vol}(\mathcal{P} \cap B_n(0, r))}{\text{vol}(B_n(0, r))}. \]
The packing density of \( \mathcal{P} \) is by definition
\[ \delta_{\mathcal{P}} := \limsup_{r \to \infty} \delta_{\mathcal{P}}(r). \]
Clearly, this is a finite number. The maximal packing density in \( \mathbb{R}^n \) is
\[ \delta_n := \sup_{\mathcal{P} \subset \mathbb{R}^n} \delta_{\mathcal{P}}, \]
a supremum over all sphere packings \( \mathcal{P} \) of \( \mathbb{R}^n \) by non-overlapping unit balls.

The linear programming method is a powerful tool for giving upper bounds on sphere packing
densities [Del72]. We only know the optimal sphere packing densities for dimensions 1, 2, 3, 8 and 24
[FT43, Hal05, Via17, CKM+17]. In dimension 1, this is trivial with \( \delta_1 = 1 \). In dimension 2, the
best sphere packing is achieved by the usual hexagonal lattice packing with \( \delta_2 = \pi/\sqrt{12} \). A rigorous
proof was provided by L. Fejes Tóth in 1943 [FT43]; however, a proof was also given by A.Thue
in 1910 [Thu10], but it was considered incomplete by some experts in the field. In dimension 3,
this is the subject of the Kepler conjecture, and was resolved in 1998 by T.Hales [Hal05]. As a
result of his work, we know that $\delta_3 = \pi/\sqrt{18}$. The other two known cases of optimal sphere packings were recently resolved in dimensions 8 and 24 by M. Viazovska and her collaborators in 2016. Based on some of the ideas of Cohn and Elkies [CE03], M. Viazovska first resolved the dimension 8 case [Via17] in 2016. Shortly afterwards, she along with Cohn, Kumar, Miller, and Radchenko resolved the case of 24 dimensions [CKM+17]. As a result of her work, we now know that the maximal packing in 8 dimensions is obtained by the $E_8$-lattice with $\delta_8 = \pi^4/384$. In 24 dimensions, it is achieved by the Leech lattice with $\delta_{24} = \pi^{12}/12!$. All these optimal sphere packings are coming from even unimodular lattices. Very recently, the first author proved an optimal upper bound on the sphere packing density of all but a tiny fraction of even unimodular lattices in high dimensions; see [Sar19, Theorem 1.1].

The best known upper bounds on sphere packing densities in low dimensions is based on the linear programming method developed by Cohn and Elkies [CE03]. However, in higher dimensions, the most successful method is due to Kabatyanskii-Levenshtein from 1978 [KL78]. This method is based on first bounding from above the maximal size of spherical codes. We discuss this in the next subsection.

1.2. Spherical codes. A notion closely related to sphere packings of Euclidean spaces is that of spherical codes. Given $S^{n-1}$, the unit sphere in $\mathbb{R}^n$, a spherical code is a finite subset $A \subset S^{n-1}$. For each $0 < \theta \leq \pi$, we define $M(n, \theta)$ to be the largest size of a spherical code $A \subset S^{n-1}$ such that no two distinct $x, y \in A$ are at an angular distance less than $\theta$.

Suppose that $p_m^{\alpha,\beta}(t)$ is the Jacobi polynomial of degree $m$ with parameters $(\alpha, \beta)$. We denote by $t^{\alpha,\beta}_{1,m}$ its largest root. In 1978, Kabatyanskii and Levenshtein proved the following inequality.

**Theorem 1.1** (Kabatyanskii-Levenshtein,[KL78]). If $\cos \theta \leq t^{\alpha,\alpha}_{1,m}$ for some $0 < \theta < \pi/2$, then

$$M(n, \theta) \leq 4 \left( \frac{m + n - 2}{m} \right) / (1 - t^{\alpha,\alpha}_{1,m+1}).$$

The bounds on sphere packing densities follow from this theorem via a geometric interpolation argument that allows them to relate sphere packings in $\mathbb{R}^n$ to spherical codes. Indeed, for any $0 < \theta \leq \pi$,

$$\delta_n \leq \sin^n(\theta/2)M(n + 1, \theta).$$

Cohn and Zhao [CZ14] improved sphere packing density upper bounds by combining the above upper bound of Kabatyanskii-Levenshtein on $M(n, \theta)$ with their [CZ14, Proposition 2.1] stating that for $\pi/3 \leq \theta \leq \pi$,

$$\delta_n \leq \sin^n(\theta/2)M(n, \theta),$$

an improvement to the previously used inequality above. What we do is revisit the inequality on $M(n, \theta)$ itself proved by Kabatyanskii and Levenshtein, and improve it by defining and studying a new linear programming problem. See Corollary 1.4 below. That being said, the best asymptotic exponent still belongs to Kabatyanskii and Levenshtein. The focus of the rest of this paper is on bounding the size of spherical codes. In the following subsections, we discuss our contributions.

1.3. The new Linear programming method. In this section, we describe a new linear programming method for bounding the maximum number $M(n, \theta)$ of points of a sphere in $n$-dimensional Euclidean space at an angular distance of not less than $\theta$ from one another. Next, we state our main theorem which gives the optimal solution to this linear programming problem.

We suppose that the reader is familiar with the Delsarte linear programming upper bound on $M(n, \theta)$; see [DGS77] and [KL78, section 5]. For simplicity, we write $p_m(t)$ for the $L^2$-normalized
Jacobi polynomials $p_{m}^{\alpha,\beta}(t)$, where $\alpha = \beta = \frac{2-3}{2}$. For some of the properties of Jacobi polynomials that we will use, see Section 2.

Let $A(\alpha, d, s)$ be the set of all polynomials $f(t) := \sum_{i=0}^{d} f_{i}p_{i}(t)$ of degree at most $d$ that satisfies the following inequalities:

1. $f_{i} \geq 0$ and $f_{0} > 0$,
2. $f(t) \leq 0$ for $-1 \leq t \leq s$.

Set

$$N(\alpha, d, s) := \inf_{f \in A(\alpha, d, s)} \frac{f(1)}{f_{0}p_{0}},$$

where $p_{0}$ is the value of the $L^{2}$-normalized constant function. Then the Delsarte linear programming upper bound implies that $M(n, \theta) \leq N \left( \frac{2n-3}{2}, d, \cos \theta \right)$; see [DGS77]. Solving the above linear programming problem is very hard. Instead, we introduce $B(\alpha, d, s) \subset A(\alpha, 2d - 1, s)$ a subset of polynomials that satisfies the second condition trivially, and then find the minimum of $\frac{f(1)}{f_{0}p_{0}}$ over $B(\alpha, d, s)$. Next, we introduce $B(\alpha, d, s)$ and one of our main theorems. Let $B(\alpha, d, s)$ be the set of all polynomials $g(t) = \frac{1}{(t-s)^{k}} f(t)^{2} = \sum_{i=0}^{2d-1} g_{i}p_{i}(t)$, where $f(t)$ is some polynomial of degree at most $d$ which vanishes at $s$, and

$$g_{i} \geq 0, \text{ and } g_{0} > 0.$$

Note that $B(\alpha, d, s) \subset A(\alpha, 2d - 1, s)$. For $0 < s < 1$, we define $d_{\alpha}(s) \in \mathbb{N}$ to be the least natural number such that there exists a Jacobi polynomial of weight $(\alpha, \alpha)$ and degree $d_{\alpha}(s)$ with a root bigger than $s$. We state one of our main theorems.

**Theorem 1.2.** Let

$$g_{\min}^{\text{KL}}(t) := \frac{1}{t-s} \left( \sum_{i=0}^{d_{\alpha}(s)-1} \lambda_{i}^{\min} \det \begin{bmatrix} p_{i+1}(t) & p_{i+1}(s) \\ p_{i}(t) & p_{i}(s) \end{bmatrix} \right)^{2},$$

where $\lambda_{i}^{\min} := \frac{1}{a_{i+1}} \left( \frac{p_{i+1}(1)}{p_{i}(s)} - \frac{p_{i+1}(1)}{p_{i+1}(s)} \right)$. Then $g_{\min}^{\text{KL}}(t) \in B(\alpha, d_{\alpha}(s), s)$ and

$$g_{\min}^{\text{KL}}(1) = \inf_{g \in B(\alpha, d_{\alpha}(s), s)} \frac{g(1)}{p_{0}g_{0}^{\min}}.$$

Moreover $g_{\min}^{\text{KL}}$ (up to a positive scalar multiple) is the unique minimum.

**Remark 2.** We note that Kabatyanskii and Levenshtein used the following test function

$$g_{KL}(t) := \frac{1}{t-s} \det \begin{bmatrix} p_{d_{\alpha}(s)}(t) & p_{d_{\alpha}(s)}(s) \\ p_{d_{\alpha}(s)-1}(t) & p_{d_{\alpha}(s)-1}(s) \end{bmatrix}^{2}.$$

It follows from Theorem 1.5 that $g_{KL}(t) \in B(\alpha, d_{\alpha}(s), s)$. Theorem 1.2 implies that our upper bound is better than that of Kabatyanskii and Levenshtein. More precisely, we give an asymptotic for the ratio of our optimal upper bound and that of Kabatyanskii and Levenshtein.

We give an exact formula for $\frac{g_{\min}^{\text{KL}}(1)}{p_{0}g_{0}^{\min}}$ in the following theorem that we prove in Section 5.

**Theorem 1.3.** We have

$$\frac{g_{\min}^{\text{KL}}(1)}{p_{0}g_{0}^{\min}} = \left( m + 2\alpha + 1 \right) \left( m + \alpha + 1 \right) \left( \frac{m + \alpha + 1}{\alpha + 1} + \frac{1}{1-s} \left( 1 - \frac{m + \alpha + 1}{m + 1} \right) \right),$$

where $m := d_{\alpha}(s) - 1$.

Using this, we obtain the following corollary.
Corollary 1.4. If $\cos \theta \leq t_{1,m}^{\alpha,\alpha}$ for some $0 < \theta < \pi/2$, then

$$M(n, \theta) \leq \left( \frac{m + n - 2}{m} \right) \left( 1 + \frac{2m}{n-1} + \frac{1}{1-t_{1,m}^{\alpha,\alpha}} \right).$$

Comparing this bound to that of Kabatyanskii and Levenshtein gives us that their bound is weaker than our bound by a factor of

$$\frac{4}{(1-t_{1,m+1}^{\alpha,\alpha}) \left( 1 + \frac{2m}{n-1} + \frac{1}{1-t_{1,m}^{\alpha,\alpha}} \right)},$$

which asymptotically as the dimension $n \to \infty$ amounts to being worse by a factor of

$$\frac{4}{\frac{1-\cos \theta}{\sin \theta} + 1},$$

a decreasing function of $\theta$ on the interval $[0, \pi/2]$ with maximal value of 4 and minimal value of 2. On the other, if we fix the dimension and let $m \to \infty$, our bound becomes better by a factor of 4.

1.4. Key idea of the paper. Note that the condition (1) for $g(t) = \frac{1}{(t-s)}f(t)^2$ is a system of quadratic inequalities in terms of the coefficients of $f$, and the quadratic forms depends on the multiplicative structure of the Jacobi polynomials. So, checking condition (1) directly for $g(t)$ is very hard. Our main innovation in this paper is to give linear programming criteria in terms of the coefficients of $f$ which implies condition (1) for $g(t) = \frac{1}{(t-s)}f(t)^2$. Note that the linear programming criteria, though sufficient, is not a necessary condition for (1). We show that $g^{\min}(t)$ satisfies these linear programming criteria and hence condition (1).

We proceed by introducing some new notations and then introduce the linear programming criteria. Fix $s \in \mathbb{R}$, and define

$$q_i(s,t) := \det \begin{bmatrix} p_{i+1}(t) & p_{i+1}(s) \\ p_i(t) & p_i(s) \end{bmatrix},$$

which is a polynomial of degree $i + 1$ and vanishes at $t = s$. Let

$$B_{s,m} : \{q_i(s,t) : 0 \leq i \leq m-1\}.$$

Note that $B_{s,m}$ is a basis for polynomials of degree less than or equal to $m$ that vanish at $s$. Suppose that $g(t) = \frac{1}{(t-s)}f(t)^2$ for some $f(t)$ of degree $m$ which vanishes at $s$. We write $f$ uniquely as a linear combination of elements of $B_{s,m}$

$$f(t) := \sum_{i=0}^{m-1} \lambda_i q_i(s,t).$$

We state our theorem relating to our linear criteria, and prove it in Section 3. It is easy to check that $g^{KL}$ satisfies these linear criteria.

**Theorem 1.5.** Let $f(t) := \sum_{i=0}^{m-1} \lambda_i q_i(s,t)$, and $g(t) := \frac{1}{(t-s)}f(t)^2$ with the zeroth Fourier coefficient $g_0 \neq 0$. Suppose that

$$p_j(s) \sum_{l=j}^{m-1} a_{l+1} \lambda_l \geq 0, \quad \text{for every } j \leq m-1,$$

$$(3) \quad -\lambda_j p_{j+1}(s) + \lambda_{j-1} p_{j-1}(s) \geq 0, \quad \text{for every } i \in \mathbb{Z}.$$

Here, we suppose that $\lambda_i = 0$ for $i \geq m$ and $i < 0$. Then $g(t)$ satisfies condition (1) which implies $g(t) \in B(\alpha, m, s)$. 
2. Jacobi polynomials

We record some well-known properties of the Jacobi polynomials (see [Sze39, Chapter IV]) as well as the Christoffel-Darboux formula that will be used repeatedly in this paper. We denote by \( p_n^\alpha,\beta(t) \) the Jacobi polynomial of degree \( n \) with parameters \( \alpha \) and \( \beta \). These are orthogonal polynomials with respect to the measure \( d\mu_{\alpha,\beta} := (1-t)^\alpha(1+t)^\beta dt \) on the interval \([-1,1] \). When \( \alpha = \beta \), we denote this measure simply as \( d\mu_{\alpha} \). For simplicity, we write \( p_n(t) \) for the \( L^2 \)-normalized Jacobi polynomials \( \frac{p_n^\alpha,\beta(t)}{|p_n^\alpha,\beta|_2} \) when \( \alpha = \beta \). We denote the top coefficient of \( p_n(t) \) with \( k_n \). The Christoffel-Darboux formula states the following (see [Sze39, Theorem 3.2.2]):

\[
\frac{1}{(t-s)} \det \begin{bmatrix} p_{n+1}(t) & p_{n+1}(s) \\ p_n(t) & p_n(s) \end{bmatrix} = k_{n+1} \frac{k_n}{k_n} \sum_{j=0}^{n} p_j(s)p_j(t) = a_{n+1} \sum_{j=0}^{n} p_j(s)p_j(t),
\]

where \( a_{n+1} = \frac{k_{n+1}}{k_n} \). In fact, this formula holds more generally for sequences of orthonormal polynomials. The recursive relation for orthonormal polynomials (see [Sze39, Theorem 3.2.1]) gives

\[
p_{n+1}(t) = (a_{n+1}t + b_{n+1})p_n(t) - c_{n+1}p_{n-1}(t),
\]

where

\[
c_{n+1} = a_{n+1} = \frac{k_{n+1}k_{n-1}}{k_n^2} > 0, \text{ and } b_{n+1} = 0 \text{ for } \alpha = \beta.
\]

We also have; see [KL78, Equation (38)]

\[
p_i(t)p_j(t) = \sum_{l=0}^{i+j} a_{i,j}^l p_l(t),
\]

where \( a_{i,j}^l \geq 0 \). The Jacobi polynomials that we use are suitably normalized so that we have the following formulas:

\[
p_n^{\alpha,\alpha}(1) = \binom{n + \alpha}{n},
\]

\[
\omega_n := \int_{-1}^{1} (p_n^{\alpha,\alpha})^2 d\mu_{\alpha}(t) = \frac{2^{2\alpha+1}\Gamma(n + \alpha + 1)^2}{(2n + 2\alpha + 1)n!\Gamma(n + 2\alpha + 1)},
\]

\[
n(n+2\alpha)p_n^{\alpha,\alpha}(t) = (2n + 2\alpha - 1)(n+\alpha)p_{n-1}^{\alpha,\alpha}(t) - (n+\alpha-1)(n+\alpha)p_{n-2}^{\alpha,\alpha}(t),
\]

and

\[
\frac{d}{dt}p_n^{\alpha,\alpha}(t) = \frac{n + 2\alpha + 1}{2} p_{n-1}^{\alpha,\alpha+1}(t).
\]

Another identity that we will use in our computations in this paper is the following.

**Lemma 2.1.** We have

\[
a_n = \sqrt{\frac{4(n + \alpha)^2 - 1}{n(n + 2\alpha)}}.
\]

**Proof.** It follows from (10) that

\[
n(n+2\alpha)\sqrt{\omega_n} p_n(t) = (2n + 2\alpha - 1)(n + \alpha)\sqrt{\omega_{n-1}} p_{n-1}(t) - (n + \alpha - 1)(n + \alpha)\sqrt{\omega_{n-2}} p_{n-2}(t).
\]

Note that by equation (9)

\[
\omega_n = \omega_{n-1} \frac{(n + \alpha)^2(2n + 2\alpha - 1)}{n(n + 2\alpha)(2n + 2\alpha + 1)}.
\]
By using the above and (5), we obtain
\[ a_n = \frac{(2n + 2\alpha - 1)(n + \alpha) \sqrt{\omega_{n-1}}}{n(n + 2\alpha) \sqrt{\omega_n}} \]
\[ = \frac{(2n + 2\alpha - 1)(n + \alpha) \sqrt{n(n + 2\alpha)(2n + 2\alpha + 1)}}{n(n + 2\alpha)(n + \alpha)(2n + 2\alpha - 1)} \]
\[ = \sqrt{\frac{4(n + \alpha)^2 - 1}{n(n + 2\alpha)}}. \]

\[ \square \]

3. Linear Programming Criteria

In this section, we prove Theorem 1.5.

Proof of Theorem 1.5. We write \( \lambda := (\lambda_0, \ldots, \lambda_{m-1}) \). By assumption, \( \lambda_i = 0 \) for \( i \geq m \) and \( i < 0 \), and
\[ p_j(s) \sum_{l=j}^{m-1} a_{l+1} \lambda_l \geq 0, \text{ for every } j \leq m - 1, \]
\[ -\lambda_i p_{i+1}(s) + \lambda_{i-1} p_{i-1}(s) \geq 0, \text{ for every } i \in \mathbb{Z}. \]

Let
\[ g(t; s, \lambda) := \frac{1}{t - s} \left( \sum_{i=0}^{m-1} \lambda_i \det \begin{bmatrix} p_{i+1}(t) & p_{i+1}(s) \\ p_i(t) & p_i(s) \end{bmatrix} \right)^2. \]

We note that \( g(t; s, \lambda) \) satisfies the second condition of the Delsarte linear programming trivially. We show that \( g(t) \) satisfies condition (1). By (4), we have
\[ \frac{1}{t - s} \left( \sum_{i=0}^{m-1} \lambda_i \det \begin{bmatrix} p_{i+1}(t) & p_{i+1}(s) \\ p_i(t) & p_i(s) \end{bmatrix} \right) = \sum_{j=0}^{m-1} p_j(s) \left( \sum_{l=j}^{m-1} a_{l+1} \lambda_l \right) p_j(t). \]

By our assumption for every \( j \leq m - 1 \), we have
\[ p_j(s) \sum_{l=j}^{m-1} a_{l+1} \lambda_l \geq 0. \]

Therefore, all the coefficients of \( p_j(t) \) for every \( j \) on the right hand side of (12) are non-negative. We also have
\[ \sum_{i=0}^{m-1} \lambda_i \det \begin{bmatrix} p_{i+1}(t) & p_{i+1}(s) \\ p_i(t) & p_i(s) \end{bmatrix} = \sum_{i=0}^{m} (-\lambda_i p_{i+1}(s) + \lambda_{i-1} p_{i-1}(s)) p_i(t). \]

By our assumption, for every \( i \geq 0 \)
\[ -\lambda_i p_{i+1}(s) + \lambda_{i-1} p_{i-1}(s) \geq 0. \]

Similarly, all the coefficients of \( p_j(t) \) on the left hand side of (13) are non-negative. Therefore, by multiplying equation (12) with (13) and using (7), it follows that \( g(t; s, \lambda) \) has all Fourier coefficients non-negative. Since we also know that \( g_0 \neq 0 \) by assumption, we have \( g_0 > 0 \). Therefore, \( g(t; s, \lambda) \) satisfies condition (1). This concludes our theorem. \[ \square \]
4. The optimal linear programming function

In this section, we give a proof of Theorem 1.2. Recall from the previous section that

$$g(t; s, \lambda) := \frac{1}{t-s} \left( \sum_{i=0}^{m-1} \lambda_i \det \begin{bmatrix} p_{i+1}(t) & p_{i+1}(s) \\ p_i(t) & p_i(s) \end{bmatrix} \right)^2.$$ 

We write $g(t; s, \lambda) = \sum_{i=0}^{2m-1} g_i(s, \lambda)p_i(t)$.

4.1. Computing $g_0(s, \lambda)$.

**Proposition 4.1.** We have

$$g_0(s, \lambda) = -p_0 \sum_{j=0}^{m-1} a_{j+1}p_j(s)p_{j+1}(s)\lambda_j^2.$$ 

As a result for every $0 \leq i \leq m-1$, we have

$$\frac{\partial g_0}{\partial \lambda_i} = -2p_0a_{i+1}\lambda_ip_i(s)p_{i+1}(s).$$

**Proof.** By equations (12) and (13), we have

$$g_0(s, \lambda) = \int_{-1}^{1} p_0g(t; s, \lambda)d\mu_{\alpha}(t)$$

$$= p_0 \int_{-1}^{1} \sum_{j=0}^{m-1} \left( \sum_{i=j}^{m-1} a_{i+1}\lambda_i \right) p_j(t)p_j(s)$$

$$\times \left( p_{m-1}(s)p_{d_{\alpha}(s)}(t) + \sum_{i=0}^{m-1} (-\lambda_ip_{i+1}(s) + \lambda_{i-1}p_{i-1}(s))p_i(t) \right) d\mu_{\alpha}(t)$$

$$= p_0 \sum_{j=0}^{m-1} \left( \sum_{i=j}^{m-1} a_{i+1}\lambda_i \right) p_j(s)(-\lambda_jp_{j+1}(s) + \lambda_{j-1}p_{j-1}(s))$$

$$= p_0 \sum_{j=0}^{m-1} \lambda_jp_j(s)p_{j+1}(s) \left( \sum_{i=j+1}^{m-1} a_{i+1}\lambda_i - \sum_{i=j}^{m-1} a_{i+1}\lambda_i \right)$$

$$= -p_0 \sum_{j=0}^{m-1} a_{j+1}p_j(s)p_{j+1}(s)\lambda_j^2.$$ 

Hence, we also have

$$\frac{\partial g_0}{\partial \lambda_i} = -2p_0a_{i+1}\lambda_ip_i(s)p_{i+1}(s).$$

This completes the proof of our proposition. \qed

4.2. Lagrange multiplier. Let

$$R(s, \lambda) := \frac{g(1; s, \lambda)}{g_0(s, \lambda)}.$$ 

Note that for fixed $s$, $R(s, \lambda)$ is invariant under multiplying $\lambda$ with a scalar. So, for fixed $s$, we may consider $R(s, \lambda)$ as a function on the projective space $\mathbb{P}^{d_{\alpha}(s)-1}(\mathbb{R})$. We define

$$\lambda^{\min} := (\lambda_{0}^{\min}, \ldots, \lambda_{d_{\alpha}(s)-1}^{\min}),$$

(14)
where

\[ \lambda_i^{\min} = \frac{1}{a_{i+1}} \left( \frac{p_i(1)}{p_i(s)} - \frac{p_{i+1}(1)}{p_{i+1}(s)} \right). \]

We prove a stronger version of Theorem 1.2 in the following proposition.

**Proposition 4.2.** Fix \(0 < s < 1\). \(\lambda^{\min}\) is the unique minimum of \(R(s, \lambda)\) subject to \(g_0 > 0\). Moreover, \(g(t, s, \lambda^{\min})\) satisfies the linear programming criteria in Theorem 1.5, and as a result all the linear inequalities in (1).

**Proof.** We begin by checking the linear programming criteria in Theorem 1.5. We need to show that

\[ p_j(s) \sum_{i=j}^{d_\alpha(s)-1} a_{i+1} \lambda_i^{\min} \geq 0, \]

\[-\lambda_i^{\min} p_{i+1}(s) + \lambda_{i-1}^{\min} p_{i-1}(s) \geq 0.\]

We check the first inequality. We have

\[ p_j(s) \sum_{i=j}^{d_\alpha(s)-1} a_{i+1} \lambda_i^{\min} = p_j(s) \sum_{i=j}^{d_\alpha(s)-1} \left( \frac{p_i(1)}{p_i(s)} - \frac{p_{i+1}(1)}{p_{i+1}(s)} \right). \]

\[ = p_j(1) - \frac{p_{d_\alpha(s)}(1)p_j(s)}{p_{d_\alpha(s)}(s)} \geq 0. \]

By definition of \(d_\alpha(s)\), we have \(p_j(s) > 0\) for \(j < d_\alpha(s)\), and \(p_{d_\alpha(s)}(s) \leq 0\). Note that \(p_j(1) > 0\) for every \(j\). Therefore, the above quantity is non-negative. Next, we check the second inequality. We have for every \(i \leq d_\alpha(s) - 1,\)

\[-\lambda_i^{\min} p_{i+1}(s) + \lambda_{i-1}^{\min} p_{i-1}(s) = \frac{1}{a_{i+1}} \left( p_{i+1}(1) - \frac{p_i(1)p_{i+1}(s)}{p_i(s)} \right) + \frac{1}{a_i} \left( p_{i-1}(1) - \frac{p_i(1)p_{i-1}(s)}{p_i(s)} \right) \]

\[= \frac{1}{p_i(s)} \left( \frac{1}{a_{i+1}} \det \begin{bmatrix} p_i(1) & p_{i+1}(s) \\ p_{i+1}(1) & p_i(s) \end{bmatrix} - \frac{1}{a_i} \det \begin{bmatrix} p_i(1) & p_i(s) \\ p_{i-1}(1) & p_{i-1}(s) \end{bmatrix} \right) \]

\[= \frac{(1-s)}{p_i(s)} \left( \sum_{j=0}^{i} p_j(1)p_j(s) - \sum_{j=0}^{i-1} p_j(1)p_j(s) \right) = (1-s)p_i(1) \geq 0. \]

Hence, \(g(t, s, \lambda^{\min})\) satisfies the linear programming criteria in Theorem 1.5, and as a result all the linear inequalities in (1). Moreover, we have

\[ \sum_{i=0}^{d_\alpha(s)-1} \lambda_i^{\min} \det \begin{bmatrix} p_{i+1}(1) & p_{i+1}(s) \\ p_i(1) & p_i(s) \end{bmatrix} \]

\[= \lambda^{\min} p_{d_\alpha(s)-1}(s) p_{d_\alpha(s)}(1) + \sum_{i=0}^{d_\alpha(s)-1} (-\lambda_i^{\min} p_{i+1}(s) + \lambda_{i-1}^{\min} p_{i-1}(s)) p_i(1) \]

\[= \lambda^{\min} p_{d_\alpha(s)-1}(s) p_{d_\alpha(s)}(1) + \sum_{i=0}^{d_\alpha(s)-1} (1-s)p_i(1)^2 > 0, \]

where the last inequality also uses the fact that from the definition of \(d_\alpha(s), \lambda^{\min}_{d_\alpha(s)-1} p_{d_\alpha(s)-1}(s) p_{d_\alpha(s)}(1) \geq 0.\)

Finally, we apply a version of the Lagrange multiplier method and show that \(\lambda^{\min}\) (up to scalar) is the unique minimum of \(R(s, \lambda)\) subject to \(g_0 > 0\). Since \(R(s, \lambda)\) is a function on the projective
space, without loss of generality we may assume \( g_0(s, \lambda) = 1 \). So, minimizing \( R(s, \lambda) \) subjected to \( g_0 > 0 \) is equivalent to finding the minimum of
\[
g(1; s, \lambda) = \frac{1}{1 - s} \left( \sum_{i=0}^{d_0(s)-1} \lambda_i \det \begin{bmatrix} p_{i+1}(1) & p_{i+1}(s) \\ p_i(1) & p_i(s) \end{bmatrix} \right)^2
\]
on the quadric \( g_0(s, \lambda) = 1 \). First, we show that \( (\lambda_0^{\min}, \ldots, \lambda_{d_0(s)-1}^{\min}) \) is a critical point for the restriction of \( R \) on \( g_0(s, \lambda) = 1 \). We have
\[
\nabla R = \frac{1}{g_0} \nabla g(1; s, \lambda) - \frac{g(1; s, \lambda)}{g_0^2} \nabla g_0.
\]
Therefore, \( \lambda \) is a critical point for the restriction of \( R \) on \( g_0(s, \lambda) = 1 \) if and only if \( \nabla g(1; s, \lambda) = \nabla g_0 \) as points in the projective space \( \mathbb{P}^{d_0(s)-1}(\mathbb{R}) \). In what follows we consider vectors as elements of \( \mathbb{P}^{d_0(s)-1}(\mathbb{R}) \), so we ignore the scalars.
\[
\nabla g(1; s, \lambda) = \left( \frac{\partial g(1; .)}{\partial \lambda_i} \right)_{0 \leq i \leq d_0(s)-1} = \left( \sum_{i=0}^{d_0(s)-1} \lambda_i \det \begin{bmatrix} p_{i+1}(1) & p_{i+1}(s) \\ p_i(1) & p_i(s) \end{bmatrix} \right) / (1 - s) = \left( \det \begin{bmatrix} p_{i+1}(1) & p_{i+1}(s) \\ p_i(1) & p_i(s) \end{bmatrix} \right)_{0 \leq i \leq d_0(s)-1} \in \mathbb{P}^{d_0(s)-1}(\mathbb{R}).
\]
By Proposition 4.1, we have
\[
\nabla g_0(s, \lambda) = (a_{i+1} \lambda_i p_i(s) p_{i+1}(s))_{0 \leq i \leq d_0(s)-1} \in \mathbb{P}^{d_0(s)-1}(\mathbb{R}).
\]
If \( \lambda = (\lambda_i)_{0 \leq i \leq d_0(s)-1} \) is a critical point then
\[
(\lambda_i)_{0 \leq i \leq d_0(s)-1} = \left( \frac{1}{a_{i+1}} (p_i(1)/p_i(s) - p_{i+1}(1)/p_{i+1}(s)) \right)_{0 \leq i \leq d_0(s)-1} = \lambda^{\min} \in \mathbb{P}^{d_0(s)-1}(\mathbb{R}).
\]
This implies that \( \lambda^{\min} \in \mathbb{P}^{d_0(s)-1}(\mathbb{R}) \) is the unique critical point for \( R \) subjected to \( g_0 > 0 \). We note that the quadratic form \( g_0(s, \lambda) \) has signature \((1, n)\). Therefore the set
\[
C := \{ \lambda : g_0(s, \lambda) \geq 1 \}
\]
is a convex set. By (15), the tangent hyperplane of the quadric \( g_0(s, \lambda) = 1 \) at \( \lambda^{\min} \) separates the origin and the quadric. Hence, it follows that \( \lambda^{\min} \) is the unique global minimum of \( R \). This concludes the proof of our proposition.

\[\square\]

5. NEW UPPER BOUNDS

In this section, we give a proof of Theorem 1.3 and that of its Corollary 1.4. Recall that
\[
g^{\min}(t) = \frac{1}{t - s} \left( \sum_{i=0}^{d_0(s)-1} \lambda_i^{\min} \det \begin{bmatrix} p_{i+1}(t) & p_{i+1}(s) \\ p_i(t) & p_i(s) \end{bmatrix} \right)^2.
\]
For the rest of this section, we suppose that \( m := d_0(s) - 1 \). Also note that \( p_0 = \frac{1}{\sqrt{\omega_0}} \). We use the following proposition.
Proposition 5.1.

\[
g_{\min}^{\alpha}(1) = \frac{3/2}{1 - s}g_0^{\min} = \frac{\omega_0}{1 - s} \sum_{i=0}^{m} \lambda_i^{\min} \det \begin{bmatrix} p_{i+1}(1) & p_{i+1}(s) \\ p_i(1) & p_i(s) \end{bmatrix}.
\]

Proof. We have

\[
g_{\min}^{\alpha}(1) := \frac{1}{1 - s} \left( \sum_{i=0}^{m} \lambda_i^{\min} \det \begin{bmatrix} p_{i+1}(1) & p_{i+1}(s) \\ p_i(1) & p_i(s) \end{bmatrix} \right)^2
\]

\[
= \frac{1}{1 - s} \left( \sum_{i=0}^{m} \lambda_i^{\min} p_i(s) p_{i+1}(s) \left( \frac{p_i(1)}{p_i(s)} - \frac{p_{i+1}(1)}{p_{i+1}(s)} \right) \right)^2
\]

\[
= \frac{1}{1 - s} \left( \sum_{i=0}^{m} a_{i+1} p_i(s) p_{i+1}(s) \lambda_i^{\min} \right)^2
\]

\[
= \frac{\omega_0}{1 - s} \left( \sum_{i=0}^{m} a_{i+1} p_i(s) p_{i+1}(s) \lambda_i^{\min} \right)^2
\]

\[
= \frac{\omega_0}{1 - s} (g_0^{\min})^2,
\]

where the last equality follows from Proposition 4.1. The other equality also follows from these computations.

We are now ready to deduce the following theorem.

Theorem 5.2.

\[
\frac{g_{\min}^{\alpha}(1)}{p_0 g_0^{\min}} = \left( m + 2\alpha + 1 \right) \left( \frac{m + \alpha + 1}{\alpha + 1} + \frac{1}{1 - s} \left( 1 - \frac{m + \alpha + 1}{m + 1} \frac{p_{m+1}(s)}{p_{m}(s)} \right) \right).
\]

Proof. Indeed, Proposition 5.1 implies that it suffices to compute

\[
\frac{\omega_0}{1 - s} \sum_{i=0}^{m} \lambda_i^{\min} \det \begin{bmatrix} p_{i+1}(1) & p_{i+1}(s) \\ p_i(1) & p_i(s) \end{bmatrix}.
\]

As in the proof of Proposition 4.2, we can show that

\[
\sum_{i=0}^{m} \lambda_i^{\min} \det \begin{bmatrix} p_{i+1}(1) & p_{i+1}(s) \\ p_i(1) & p_i(s) \end{bmatrix} = \lambda_i^{\min} p_m(s) p_{m+1}(1) + \sum_{i=0}^{m} (1 - s)p_i(1)^2,
\]

from which we obtain

\[
\frac{\omega_0}{1 - s} \sum_{i=0}^{m} \lambda_i^{\min} \det \begin{bmatrix} p_{i+1}(1) & p_{i+1}(s) \\ p_i(1) & p_i(s) \end{bmatrix}
\]

\[
= \omega_0 \left( \sum_{i=0}^{m} p_i(1)^2 + \frac{\lambda_i^{\min} p_m(s) p_{m+1}(1)}{1 - s} \right)
\]

\[
= \omega_0 \left( \sum_{i=0}^{m} p_i(1)^2 + \frac{p_m(s) p_{m+1}(1)}{a_{m+1}} \left( \frac{p_m(1)}{p_m(s)} - \frac{p_{m+1}(1)}{p_{m+1}(s)} \right) \right)
\]

\[
= \omega_0 \sum_{i=0}^{m} p_i(1)^2 + \frac{\omega_0 a_{m+1}(1)^2}{(1 - s)a_{m+1}/\sqrt{\omega_m} \sqrt{\omega_{m+1}}} \left( \frac{m + 1}{m + \alpha + 1} - \frac{\alpha_{m+1}(s)}{p_{m+1}(s)} \right).
\]
By the Christoffel-Darboux formula (4),
\[ \sum_{i=0}^{m} p_i(1)^2 = \frac{1}{a_{m+1}} (p'_{m+1}(1)p_m(1) - p'_m(1)p_{m+1}(1)). \]

On the other hand, we know from equation (11) that for every \( i \)
\[ \frac{d}{dt} \rho_i^{\alpha,i}(t) = \frac{i + 2\alpha + 1}{2} \rho_{i-1}^{\alpha,i+1}(t). \]

Evaluating at \( t = 1 \), we obtain
\[ p'_m(1) = \frac{(p_m^{\alpha,i})'(1)}{\sqrt{\omega_m}} = \frac{m + 2\alpha + 1}{2\sqrt{\omega_m}} \left( \frac{m + \alpha}{\alpha} \right). \]

Using this along with equation (8) that
\[ p_i^{\alpha,i}(1) = \frac{(i + \alpha)_{\alpha}}{\alpha} \]
for every \( i \), we obtain
\[ p'_{m+1}(1)p_m(1) - p'_m(1)p_{m+1}(1) = \frac{m + 2\alpha + 2}{2\sqrt{\omega_m}\sqrt{\omega_{m+1}}} \left( \frac{m + \alpha + 1}{\alpha + 1} \right) \left( \frac{m + \alpha}{\alpha} \right) - \frac{m + 2\alpha + 1}{2\sqrt{\omega_m}\sqrt{\omega_{m+1}}} \left( \frac{m + \alpha}{\alpha} \right) \left( \frac{m + 1 + \alpha}{\alpha} \right) \]
\[ = \frac{(m + \alpha)(m + \alpha + 1)}{2\sqrt{\omega_m}\sqrt{\omega_{m+1}}} \left( \frac{m + 2\alpha + 2}{m + 1} - \frac{(m + 2\alpha + 1)m}{m + 1} \right) \]
\[ = \frac{(m + \alpha + 1)(m + \alpha + 1)}{(m + 1)\sqrt{\omega_m}\sqrt{\omega_{m+1}}} \]
\[ = \frac{(m + \alpha + 1)(m + \alpha + 1)}{\sqrt{\omega_m}\sqrt{\omega_{m+1}}} \]

Consequently,
\[ \sum_{i=0}^{m} p_i(1)^2 = \frac{(m + \alpha + 1)(m + \alpha + 1)}{a_{m+1}\sqrt{\omega_m}\sqrt{\omega_{m+1}}}. \]

Therefore, we have
\[ \frac{g_{\alpha, \alpha}^{\min(1)}}{p_0 a_0^{\min}} = \omega_0 \left( \frac{(m + \alpha + 1)(m + \alpha + 1)}{a_{m+1}\sqrt{\omega_m}\sqrt{\omega_{m+1}}} + \frac{(m + \alpha + 1)^2}{(1 - s)a_{m+1}\sqrt{\omega_m}\sqrt{\omega_{m+1}}} \left( \frac{m + 1}{m + \alpha + 1} - \frac{p_m^{\alpha,i}(s)}{p_i^{\alpha,i}(s)} \right) \right) \]
\[ = \omega_0 \left( \frac{(m + \alpha + 1)(m + \alpha + 1)}{a_{m+1}\sqrt{\omega_m}\sqrt{\omega_{m+1}}} \left( \frac{m + \alpha + 1}{\alpha + 1} + \frac{1}{1 - s} \left( 1 - \frac{m + \alpha + 1}{m + 1} \frac{p_m^{\alpha,i}(s)}{p_i^{\alpha,i}(s)} \right) \right) \right) \]

Note that using formula (9) and Lemma 2.1,
\[ \frac{(m + \alpha + 1)^2}{a_{m+1}^2 \omega_m \omega_{m+1}} = \frac{\Gamma(m + \alpha + 2)^2}{\Gamma(\alpha + 1)^2 \Gamma(m + 2)^2} \]
\[ = \frac{(2m + 2\alpha + 1)(2m + 2\alpha + 3)}{(m + 1)(m + 2\alpha + 1)} \frac{2^{4\alpha + 2} \Gamma(m + \alpha + 1)^2 \Gamma(m + \alpha + 2)^2}{(m + 1)(m + 2\alpha + 1)^2 \Gamma(m + 2) \Gamma(m + 2\alpha + 1) \Gamma(m + 2\alpha + 2)} \]
\[ = \frac{\Gamma(m + 2\alpha + 2)^2}{2^{4\alpha + 2} \Gamma(\alpha + 1)^2 \Gamma(m + \alpha + 1)^2} \frac{(m + 2\alpha + 1)(m + 2\alpha)^2}{\alpha} \].
Taking squareroots, we obtain
\[
\frac{(m+\alpha+1)}{a_{m+1}\sqrt{\omega_{m}}/\omega_{m+1}} = \frac{(m+2\alpha+1)}{2^{2\alpha+1}} \left( \frac{m+2\alpha}{\alpha} \right).
\]
Therefore,
\[
g_{\min}^{\alpha}(1) = \omega_{0} \frac{(m+2\alpha+1)(m+\alpha)(m+2\alpha)}{2^{2\alpha+1}} \left( \frac{m+\alpha+1}{\alpha+1} + \frac{1}{1-s} \left( 1 - \frac{m+\alpha+1}{m+1} \frac{p_{m,\alpha}^{\alpha}(s)}{p_{m+1}^{\alpha}(s)} \right) \right).
\]
However, by formula (9),
\[
\omega_{0} = \frac{2^{2\alpha+1}(\alpha+1)^{2}}{\Gamma(2\alpha+2)},
\]
from which we obtain
\[
g_{\min}^{\alpha}(1) = \frac{(m+2\alpha+1)\Gamma(\alpha+1)^{2}(m+\alpha)(m+2\alpha)}{\Gamma(2\alpha+2)} \left( \frac{m+\alpha+1}{\alpha+1} + \frac{1}{1-s} \left( 1 - \frac{m+\alpha+1}{m+1} \frac{p_{m,\alpha}^{\alpha}(s)}{p_{m+1}^{\alpha}(s)} \right) \right).
\]
as required. \(\square\)

As a result, we have the following corollary.

**Corollary 5.3.** For \(\sigma \leq t_{1,\alpha}^{\alpha}\), we have
\[
N(\alpha,2m+1,\sigma) \leq \left( \frac{m+2\alpha+1}{m} \right) \left( \frac{m+\alpha+1}{\alpha+1} + \frac{1}{1-t_{1,\alpha}^{\alpha}} \right).
\]

**Proof.** Note that, from definition, \(N(\alpha,2m+1,x)\) is a non-decreasing function of \(x\). Therefore, for \(\sigma\) and \(s\) such that \(\sigma \leq t_{1,\alpha}^{\alpha} \leq s < t_{1,\alpha}^{\alpha} + 1\), we have
\[
N(\alpha,2m+1,\sigma) \leq N(\alpha,2m+1,s).
\]
In particular, we can take \(s = t_{1,\alpha}^{\alpha}\) and apply the previous theorem to obtain
\[
N(\alpha,2m+1,\sigma) \leq \left( \frac{m+2\alpha+1}{m} \right) \left( \frac{m+\alpha+1}{\alpha+1} + \frac{1}{1-t_{1,\alpha}^{\alpha}} \right).
\]
It is easy to see that among such \(s, s = t_{1,\alpha}^{\alpha}\) is optimal. Indeed, for the other \(s \in (t_{1,\alpha}^{\alpha}, t_{1,\alpha}^{\alpha} + 1),\)
\[
- \frac{m+\alpha+1}{m+1} \frac{p_{m,\alpha}^{\alpha}(s)}{p_{m+1}^{\alpha}(s)} > 0.
\]
\(\square\)

**Corollary 5.4.** If \(\cos \theta \leq t_{1,\alpha}^{\alpha}\), for some \(0 < \theta < \pi/2\), then
\[
M(n,\theta) \leq \left( \frac{m+n-2}{m} \right) \left( 1 + \frac{2m}{n-1} + \frac{1}{1-t_{1,\alpha}^{\alpha}} \right).
\]
This follows from the inequality \(M(n,\theta) \leq N \left( \frac{n-3}{2}, 2m+1, \cos \theta \right)\) and Corollary 5.3 above.

As in [KL78], by taking \(m,n \rightarrow \infty\), \(\alpha = (n-3)/2\), such that the ratio \(\alpha/m \rightarrow \frac{\sin \theta}{1-\sin \theta}\), we obtain that for fixed \(0 < \theta < \pi/2\),
\[ \frac{1}{n} \log M(n, \theta) \lesssim \frac{1 + \sin \theta}{2 \sin \theta} \log \left( \frac{1 + \sin \theta}{2 \sin \theta} \right) - \frac{1 - \sin \theta}{2 \sin \theta} \log \left( \frac{1 - \sin \theta}{2 \sin \theta} \right), \]

which is the same asymptotic bound as in [KL78]. Therefore, our Corollary 5.4 does not improve upon the exponent for sphere packing densities. That being said, our corollary is an improvement to the Kabatyanskii-Levenshtein bound. Indeed, Kabatyanskii-Levenshtein prove the bound
\[ M(n, \theta) \leq \frac{4 (m + n - 2)}{m} \]
whenever \( 0 < \theta < \pi/2 \) is such that \( \cos \theta \leq t_{1,m}^{a,\alpha} \) [KL78, Equation 52]. As a result, the bound given in Kabatyanskii-Levenshtein for \( M(n, \theta) \) is more than our bound by a factor of \( \frac{4}{(1 - t_{1,m+1}^{a,\alpha}) \left( 1 + \frac{2m}{n-1} + \frac{1}{1 - t_{1,m}^{a,\alpha}} \right)} \).

Note that by [Sze39, Theorem 8.1.2], we know that if we write \( t_{1,m}^{a,\alpha} = \cos \theta_{1,m} \) with \( 0 < \theta_{1,m} < \pi \), then
\[ \lim_{m \to \infty} m \theta_{1,m} = j_1, \]
where \( j_1 \) is the smallest positive zero of the Bessel function \( J_\alpha \). Therefore, this along with Taylor expansion gives us \( t_{1,m}^{a,\alpha} \sim 1 - \frac{j_1^2}{2m^2} \) as \( m \to \infty \). Consequently, as \( m \to \infty \) with \( \alpha \) fixed,
\[ (1 - t_{1,m+1}^{a,\alpha}) \left( 1 + \frac{2m}{n-1} + \frac{1}{1 - t_{1,m}^{a,\alpha}} \right) \sim \frac{j_1^2}{2(m+1)^2} \left( 1 + \frac{2m}{n-1} + \frac{2m^2}{j_1^2} \right) \to 1. \]

Therefore, for large \( m \), the bound of Kabatyanskii-Levenshtein on \( M(n, \theta) \) is 4 times ours.

When \( m, n \to \infty \) such that \( \alpha/m \to \frac{\sin \theta}{1 - \sin \theta} \), then \( t_{1,m}^{a,\alpha} \to \cos \theta \) [KL78, Lemma 4]. Therefore, asymptotically as the dimension \( n \to \infty \) our bound on \( M(n, \theta) \) is better by a factor of
\[ \frac{4}{\frac{1 - \cos \theta}{\sin \theta} + 1}, \]
a decreasing function of \( \theta \) on the interval \([0, \pi/2]\) with maximal value of 4 and minimal value of 2.

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Institute for Advanced Study, Princeton, NJ, USA

*E-mail address*: ntalebiz0@mpim-bonn.mpg.de

Department of Mathematics, University of Regensburg, Regensburg, Germany

*E-mail address*: masoud.zargar@ur.de