Multi-scale fluctuations near a Kondo Breakdown Quantum Critical Point

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We study the Kondo-Heisenberg model using a fermionic representation for the localized spins. The mean-field phase diagram exhibits a zero temperature quantum critical point separating a spin liquid phase where the f-conduction hybridization vanishes, and a Kondo phase where it does not. Two solutions can be stabilized in the Kondo phase, namely a uniform hybridization when the band masses of the conduction electrons and the f spinons have the same sign, and a modulated one when they have opposite sign. For the uniform case, we show that above a very small Fermi liquid temperature scale ($\sim 1$ mK), the critical fluctuations associated with the vanishing hybridization have dynamical exponent $z = 3$, giving rise to a specific heat coefficient that diverges logarithmically in temperature, as well as a conduction electron inverse lifetime that has a $T \log T$ behavior. Because the f spinons do not carry current, but act as an effective bath for the relaxation of the current carried by the conduction electrons, the latter result also gives rise to a $T \log T$ behavior in the resistivity. This behavior is consistent with observations in a number of heavy fermion metals.

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I. INTRODUCTION

A large number of experiments have been performed on metallic heavy fermion compounds close to a zero temperature phase transition (a quantum critical point (QCP)) driven by applied magnetic field, chemical doping or pressure. In the quantum critical regime, the thermodynamics and transport properties indicate a breakdown of the Fermi liquid. In many cases, the resistivity is quasi-linear in temperature over several decades, and the specific heat coefficient diverges logarithmically. The spin susceptibility typically exhibits an anomalous exponent in temperature. Neutron scattering experiments on some of these materials have revealed that the anomalous exponent in the dynamical susceptibility is identical for all points in the Brillouin zone, suggesting a local character for the fluctuations. deHaas-vanAlphen experiments also find a divergence of the effective mass when approaching the critical point, along with a change in the Fermi surface topology when going through it.

These unusual observations have motivated many theoretical studies that have attempted to capture these effects. Most theories\cite{2,6,7,8,9,10} are based on the assumption that at the QCP, a spin density wave forms, and therefore the critical fluctuations that destabilize the Fermi liquid are magnetic in nature\cite{2,6,7,8,9}. In three dimensions, these theories fail to capture simultaneously the linear temperature dependence of the resistivity, the logarithmic divergence of the specific heat coefficient\cite{10,11,12}, and the anomalous exponent of the spin susceptibility\cite{10,11,12}. For an antiferromagnetic spin density wave transition, a central problem is that the critical fluctuations are confined to an inverse coherence length about the spin density ordering vector, and consequently, only parts of the Fermi surface couple effectively with the critical bosonic modes.

More recently, the problem has been approached from another perspective which takes the point of view that at the QCP, magnetic fluctuations suppress the formation of the heavy Fermi liquid, driving the effective Kondo temperature of the lattice ($T_K$) to zero\cite{10,12,13,14}. In this picture, the QCP is a fractionalized critical point at which the heavy quasiparticle deconfines into a spinon and holon. One feature that distinguishes between these two classes of theories is that the first predicts the Fermi surface to change smoothly across the QCP, while the second predicts an abrupt change\cite{10,12}. Recent results of the Hall effect for YbRh$_2$Si$_2$\cite{15}, as well as the earlier mentioned dHvA data\cite{15}, have lent support to theories of the second type.

Here, we explore the possibility that in the quantum critical regime, the magnetic fluctuations are not the dominant ones at the QCP, and that the unusual behavior in thermodynamics and transport is due to critical fluctuations of a non-magnetic order parameter associated with the vanishing energy scale $T_K$. One motivation for this point of view is the fact that in some compounds like YbRh$_2$Si$_2$, the gain in entropy inside the magnetically ordered phase represents only a few percent of the total entropy per localized spin\cite{16}. The order parameter we advocate is the field $\sigma$ associated with the hybridization between the localized spins and the conduction electrons\cite{17,18}. At the QCP, the effective Kondo temperature for the lattice goes to zero, leading to a ‘Kondo breakdown’ of the heavy Fermi liquid. The critical fluctuations of $\sigma$ are gapless excitations, and we study how these fluctuations influence the properties of the metal using the formalism of the large $N$ Kondo-Heisenberg model.

There have been several earlier studies of this model\cite{13,14,19}. Beyond the mean-field level, the Kondo-Heisenberg model can be treated as a lattice gauge theory. Senthil et. al\cite{20} have examined the effect of the gauge fluctuations in this model, while Coleman et. al\cite{21}
studied the zero temperature transport anomalies. In our work, we find a number of novel effects associated with the fluctuations of the $\sigma$ field which were not discovered in these earlier studies.

At the Kondo breakdown QCP, the metal passes from a magnetic phase (which we approximate, as in earlier work\textsuperscript{13}, as a uniform spin liquid) to a Kondo phase. In the spin liquid phase, the $f$ spinons are characterized by a ‘Fermi surface’ which generically differs in size from the conduction electron Fermi surface. In the Kondo phase, these two surfaces become coupled due to the non-zero expectation value of $\sigma$. In our study, we observe two new phenomena associated with this. First, for the case where the spinon and conduction electron masses have opposite sign, $\sigma$ can order at a finite wavevector, leading to spatial modulations of the Kondo hybridization analogous to the LOFF state of superconductivity\textsuperscript{20,21}. Second, we find the presence of multiple energy scales, spread over a very large range in energy, due to the mismatch between the two Fermi surfaces. The lowest scale, below which Fermi liquid behavior is restored, is extremely small (of order 1 mK), above which, up to an ultraviolet cutoff of order the single ion Kondo temperature, the critical fluctuations of $\sigma$ exhibit a dynamical exponent $z = 3$. This gives rise to a marginal Fermi liquid like behavior in $d = 3$ for the conduction electrons along the entire Fermi surface, due to scattering with the critical fluctuations. This property is to be contrasted with antiferromagnetic spin density wave models, where only on parts of the Fermi surface the scattering of the electrons with the critical mode is effective. Next, since the $f$ spinons do not carry current, but act as an effective bath for the transport lifetime (i.e., in the latter models, forward scattering does not degrade the current), or that of antiferromagnetic spin density wave models in which the “cold” parts of the Fermi surface dominate the transport properties\textsuperscript{22}. Moreover, a logarithmic dependence is found for the specific heat coefficient from both the gauge\textsuperscript{13} and $\sigma$ fluctuations. The latter also give rise to an anomalous temperature exponent of 4/3 in the uniform spin susceptibility. A summary of our results have been presented in a shorter paper\textsuperscript{23}.

The phenomenon of the breakdown of the Kondo effect at a QCP can also be studied in the more general context of a periodic Anderson model. This generalization is discussed in other works\textsuperscript{23,25}.

II. MODEL AND FORMALISM

The starting point of our theory is the microscopic Kondo-Heisenberg model in three dimensions, which describes a broad band of conduction electrons interacting with a periodic array of localized spins through antiferromagnetic Kondo coupling $J_K > 0$. Additionally, the localized spins interact with one another via nearest neighbour exchange $J_H > 0$. The Hamiltonian for the large $N$ version of this model, where $N$ denotes the enlarged spin symmetry group $SU(N)$, is given by

$$
\mathcal{H} = -t \sum_{\langle ij \rangle, \alpha} c_{i\alpha}^\dagger c_{j\alpha} + \frac{J_K}{N} \sum_{i, \alpha, \beta} c_{i\alpha}^\dagger c_{i\beta} f_{i\beta}^\dagger f_{i\alpha} + \frac{J_H}{N} \sum_{\langle ij \rangle, \alpha, \beta} f_{i\alpha}^\dagger f_{j\beta} f_{i\beta} f_{j\alpha}.
$$

Here $c_{i\alpha}$ ($c_{i\alpha}$) are creation (annihilation) operators for the conduction electrons with spin index $\alpha = (1, N)$ at site $i$, and $\langle ij \rangle$ refers to nearest neighbour sites. $t$ is the hopping matrix element between neighbouring sites for the conduction electrons. The $SU(N)$ generalization of the localized spins $S^a_i$ with $a = (1, ..., N^2 - 1)$ at each site $i$ are expressed in terms of Abrikosov pseudofermions (or spinons) by $S^a_i = \sum_{\alpha, \beta} f_{i\alpha}^\dagger \Gamma^a_{\alpha\beta} f_{i\beta}$, where $\Gamma^a$ are the generators of the $SU(N)$ group in the fundamental representation. This fermionic representation of the spin operator gives rise to a local constraint at each site, $i$

$$
\sum_{\alpha} f_{i\alpha}^\dagger f_{i\alpha} = \frac{N}{2}, \quad \forall i.
$$

We note that in the context of the heavy fermion systems, the Heisenberg exchange term is often equated to the RKKY interaction between the localized spins which is mediated by the mobile conduction electrons. In such a scenario, the Heisenberg coupling $J_H \propto \rho_0 J_K^2$, where $\rho_0$ is the density of states of the conduction electrons at the Fermi level. However, for the purpose of the present study, it is convenient to consider $J_H$ as a parameter independent of $J_K$. Microscopically this can be justified by noting that, in principle, there can be other sources which generate this coupling, such as superexchange within the narrow band of $f$-electrons.

In order to perform a systematic large $N$ study of the system defined by Eqs. (1) and (2), the first step is to decouple the interaction terms which are quartic in fermionic operators using a Hubbard-Stratonovich transformation. The Heisenberg exchange term is decoupled using a bosonic link variable $\phi_{ij} \rightarrow \sum_{\alpha} f_{i\alpha}^\dagger f_{j\alpha}$, while the Kondo interaction is decoupled by introducing a complex bosonic field $\sigma^\dagger_i \rightarrow \sum_{\alpha} f_{i\alpha}^\dagger c_{i\alpha}$. In the next step, following Ref.\textsuperscript{12} we assume that in three dimensions, $\phi_{ij}$ condenses in a uniform spin liquid phase, i.e., $\langle \phi_{ij} \rangle = \phi_0$ at the mean field level. This provides a dispersion to the spinon band which, as we show later, is an essential ingredient to obtain the breakdown of the Kondo effect. We note that there is no clear evidence of a spin liquid phase in any heavy fermion system near its quantum critical point. Rather, the typical phase diagram exhibits a QCP that separates a magnetic ground state (typically an antiferromagnet) from a paramagnetic heavy Fermi
liquid. Consequently, it is useful to discuss the motivation for our choice of a uniform spin liquid phase for the Heisenberg link variable $\phi_{ij}$. This choice is partly guided by convenience; since our main purpose is to study the consequences of the breakdown of the Kondo effect, the choice of a uniform spin liquid can be viewed as the simplest device which allows the vanishing of the Kondo energy scale (indicating the breakdown of the Kondo effect) at the mean field level. More physically, one can view the uniform spin liquid as a mean field description of the short range magnetic correlations that persist when a magnetic ground state is destroyed by quantum fluctuations. However, to demonstrate this point concretely is not simple, and beyond the scope of the present study. The key point is that the spin liquid provides a bandwidth for the $f$ electrons. Other approaches, for instance, the single-ion Kondo effect of imaginary time) under a local vector potentials can also be understood by noting that the physical subspace. Consequently, for the kinematics of the spinons, only simultaneous opposite hops between two neighbouring sites is a physically allowed process. This implies that the local spinon current operator $\tilde{J}_f = 0$ at every site $i$. The gauge fields $a_{ij}$ (vector potential), associated with the phase of $\phi_{ij}$, ensure that this constraint is satisfied. The appearance of the scalar and vector potentials can also be understood by noting that $L$ is invariant (up to a term which is a total derivative of imaginary time) under a local $U(1)$ gauge transformation $f_{ia} \rightarrow f_{ia} e^{i\theta_i}$, $\sigma_i \rightarrow e^{-i\theta_i}$, $\lambda_i \rightarrow \lambda_i + i\partial_\tau \theta_i$, $a_{ij} \rightarrow a_{ij} - \theta_i + \theta_j$, a consequence of the fermionic representation of the spin and the constraint Eq. (2).32.

In the following we examine the above Lagrangian, first in a mean field approximation and then consider Gaussian fluctuations of the action around the mean field solution. This involves studying the possibility of hybridization between the conduction and the spinon bands (for $\langle \sigma_i \rangle \neq 0$) as well as calculating the hybridization fluctuation which is an interband particle-hole excitation. As such, one needs to characterize the dispersions of the conduction and the spinon bands. We do this by assuming that the bands have a parabolic dispersion (to facilitate calculations), and we introduce the following two important parameters. First, $\alpha \equiv \phi_0/D$, is the ratio of the spinon bandwidth $\phi_0$ and the conduction bandwidth $D$.

As we will see in the next section, at the Kondo breakdown QCP $\phi_0 \sim J_H \sim T_K^0$, where $T_K^0 \equiv D e^{-1/(\mu_0 J_K)}$ is the single-ion Kondo energy scale of the system, which is typically of order 10 K in heavy fermion systems. Assuming $D \sim 10^4$ K, we get $\alpha \sim 10^{-3}$. Second, while the spinon band is half filled due to the constraint (for $N = 2$), the conduction band filling is generic. Without any loss of generality, we take the conduction band to be less than half filled. This implies that the Fermi wave vector of the conduction band $k_F$ is different from that of the spinon band $k_{F0}$. We denote this mismatch by $q^* \equiv k_{F0} - k_F$, and assume that the fraction $(q^*/k_F)$ is of the order $0.1$.27 This would mean that while $k_F$ and $k_{F0}$ are of the order of the Brillouin zone dimension, the mismatch wave vector $q^*$ is one order of magnitude smaller. The parameters $\alpha$ and $(q^*/k_F)$ affect the important energy scales of the system. This is illustrated in Fig. IIII where we show the conduction and spinon dispersions.

**III. MEAN FIELD TREATMENT**

At the level of the mean field approximation, we replace the bosonic Hubbard-Stratonovich fields and the Lagrange multipliers by their expectation values, and we study the approximate Lagrangian given by

$$L_{MF} = \sum_{(ij)\alpha} \left[ c^\dagger_{ia} (\partial_\tau \delta_{ij} - t) c_{ja} + f^\dagger_{ia} ((\partial_\tau - \langle \lambda_i \rangle) \delta_{ij} - \phi_0) f_{ja} \right] + \frac{N}{2} \sum_i \langle \lambda_i \rangle + \frac{N}{J_K} \sum_i |\langle \sigma_i \rangle|^2 + \frac{N\phi_0^2}{J_H} + \sum_{ia} \left( c^\dagger_{ia} f_{ia} \langle \sigma_i \rangle + \text{H.c.} \right).$$

![FIG. 1: (Color online) Dispersion of conduction and spinon bands, with the mismatch wavevector, $q^*$, and the mismatch energy, $E_x \equiv \alpha \phi_0 q^*$, indicated, where $\alpha$ is the ratio of the spinon and conduction bandwidths. An artificially large value of $\alpha$ was used in this plot (0.5) so as to better illustrate the origin of $E_x$.](image-url)
In the following we write the dispersion ($\epsilon_k$) of the conduction band as

$$\epsilon_k = \epsilon + \frac{\epsilon^2}{D}, \quad (5)$$

where $\epsilon = v_F(k - k_F)$, $k$ is the magnitude of $k$, and $v_F$ is the Fermi velocity of the conduction electrons. The dispersion ($\epsilon_k^0$) of the spinon band is similarly written as

$$\epsilon_k^0 = \alpha \left[ (\epsilon - v_F q^*) + \frac{(\epsilon - v_F q^*)^2}{D} \right]. \quad (6)$$

We note that, in the above, both the bands are taken as electron-like, for which we find that the mean field equations yield a spatially uniform solution, namely, $\langle \sigma_i \rangle = \sigma_0$ and $\langle \lambda_i \rangle = \lambda_0$. In the case where one of the bands is chosen to be hole-like, we find a spatially modulated solution which we discuss in appendix A. The free energy corresponding to Eq. (4) is given by

$$F_{MF}^{MF} = -\frac{1}{\beta} \text{Tr} \left[ \ln \left(-G_{a}^{-1}(i\omega_n, k)\right) + \ln \left(-G_{b}^{-1}(i\omega_n, k)\right) \right]$$

$$+ \frac{\sigma_0^2}{J_K} + \frac{\phi_0}{J_H} + \frac{\lambda_0}{2}, \quad (7)$$

where $\beta$ is the inverse temperature, $\omega_n$ is the fermionic Matsubara frequency, and Tr corresponds to a trace over space-time co-ordinates. In the above

$$G_{a}^{-1}(i\omega_n, k) = i\omega_n - \epsilon_{k}^{a,b}, \quad (8)$$

where

$$\epsilon_{k}^{a,b} = \frac{1}{2} \left[ \epsilon_k + \epsilon_k^0 \mp \sqrt{(\epsilon_k - \epsilon_k^0)^2 + 4\sigma_0^2} \right]. \quad (9)$$

We evaluate the free energy given by Eq. (7) at zero temperature ($T = 0$) in the limit $(q^*/k_F) \to 0$. The details of this evaluation is given in appendix A.2. As a function of $\alpha$ and $\sigma_0$, and to $O(\sigma_0^4)$ accuracy, we find ($\alpha \ll 1$)

$$F_{MF}^{MF} = \frac{\rho_0 D^2}{2} \left[ \frac{\alpha^2}{2\rho_0 J_H} - \frac{\alpha}{3} \right] + \rho_0 \sigma_0^2 \left[ \frac{1}{\rho_0 J_K} \right.$$

$$\left. - \ln \left(\frac{1}{\alpha}\right) \right] + \frac{\rho_0 \sigma_0^4}{\alpha^2 D^2} + \text{const}, \quad (10)$$

where the constant part has explicit $\lambda_0$ dependence. Since the precise value of $\lambda_0$ is of no importance for our results, in the following we ignore the mean field equation for $\lambda_0$. Minimizing $F_{MF}$ with respect to $\alpha$ and $\sigma_0$ we get

$$\rho_0 D^2 \left[ \left( \frac{\alpha}{\rho_0 J_H} - \frac{1}{3} \right) + \frac{2\sigma_0^2}{\alpha D^2} - \frac{4\sigma_0^4}{\alpha^3 D^4} \right] = 0, \quad (11)$$

$$2\rho_0 \sigma_0 \left[ \left( \frac{1}{\rho_0 J_K} - \ln \left(\frac{1}{\alpha}\right) \right) + \frac{2\sigma_0^2}{\alpha^2 D^2} \right] = 0, \quad (12)$$

respectively. We study these equations by keeping the Heisenberg parameter $J_H$ fixed, while varying the Kondo parameter $J_K$, and find two solutions corresponding to two mean field ground states. (i) First, a uniform spin liquid phase where $\sigma_0 = 0$, which implies that in this phase, the Kondo effect fails to occur and the localized spins remain unscreened in a uniform spin liquid state. In this phase, $\alpha = \alpha_0 \equiv (\rho_0 J_H)/\beta$, which implies that the Heisenberg coupling sets the scale for the spinon dispersion, since $\phi_0 = (\rho_0 D J_H)/6 \sim J_H$. It is simple to check that this solution is stable for $J_K < J_{K_2}$, where

$$\frac{1}{\rho_0 J_{K_2}} = \ln \left(\frac{1}{\alpha}\right). \quad (13)$$

(ii) For $J_K > J_{K_2}$ the stable mean field solution corresponds to $\sigma_0 \neq 0$, indicating a ground state where the local moments are screened by the Kondo effect and a heavy Fermi liquid is established below an energy scale $T_k \approx \pi \rho_0 \sigma_0^2$. The growth of the Kondo order parameter in this phase is given by

$$\sigma_0 \propto J_H \ln \left(\frac{1}{\alpha_0} \frac{[J_K - J_{K_2}]}{D}\right)^{\beta}, \quad (14)$$

where $\beta = 1/2$ is the typical mean field exponent. We also find that the spin liquid order parameter decreases in this phase, and is given by

$$\alpha = \alpha_0 - \frac{6\sigma_0^2}{D^2} + O(\sigma_0^4). \quad (15)$$

Thus, from the above mean field study, we find that, in the presence of a finite bandwidth of the spinons, the Kondo effect takes place only when the Kondo coupling $J_K$ is larger than a finite value $J_{K_2}$. This establishes the Kondo breakdown QCP where the lattice Kondo energy scale $T_K$ vanishes. In the current formulation of the mean field theory, the Kondo breakdown QCP separates a uniform spin liquid ground state ($J_K < J_{K_2}$) from a heavy Fermi liquid ground state ($J_K > J_{K_2}$). It is important to note that if we define a single-ion Kondo scale ($T^0_K$) as a function of $J_K$ for the system by

$$T^0_K(J_K) \equiv D e^{-1/(\rho_0 J_K)}, \quad (16)$$

using Eq. (13) we conclude that at the QCP

$$J_H \sim T^0_K(J_{K_2}). \quad (17)$$

This shows that the Kondo breakdown QCP is established as a result of a competition between the Kondo energy scale and the magnetic energy scale, even though there is no long range magnetic order in the present study. The reduction of the spin liquid order parameter, given by Eq. (13), provides further evidence for this competition. Therefore, this mean field study can be viewed as a microscopic realization of the energetic argument that Doniach had proposed several decades ago for the existence of a QCP in heavy fermion systems.28.
IV. FLUCTUATIONS

In this section, we study the massless fluctuations in the quantum critical regime. There are two such modes: (a) one associated with the phase of $\phi_{ij}$ which are the gauge fluctuations, and (b) the fluctuations of the complex order parameter $\langle \sigma_i^x, \sigma_i \rangle$ which are gapless due to the vanishing of the Kondo energy scale $T_K$ at the Kondo breakdown QCP.

A. Gauge Fluctuations

Since the gauge fluctuations of the system have been studied earlier, here we just summarize the main points for the sake of completeness. It is convenient to work in the Coulomb gauge $\nabla \cdot \vec{a} = 0$, where the vector gauge fields $a_\mu (\mu = x, y, z)$ are purely transverse. In this gauge the fluctuations of the scalar potential $\lambda$ decouple from $a_\mu$, and give rise to a screened Coulomb interaction between the spinons which can be neglected. Next, since the fields $a_\mu$ enter the theory as vectorial Lagrange multipliers to satisfy the constraint that the local spinon current is zero, they behave as ‘artificial photons’ without any intrinsic dynamics of their own. Their dynamics is only generated by their coupling to the matter field, namely the spinon band, and therefore these bosonic modes are overdamped. The propagator for the transverse gauge fields is defined as $D_{\mu
u}(x, \tau) = \langle \partial_\mu(x, \tau) a_\nu(0, 0) \rangle$, which in frequency-momentum space has the standard form $D_{\mu\nu}(q, i\Omega_n) = (\delta_{\mu\nu} - q_\mu q_\nu/q^2)\Pi^{-1}(q, i\Omega_n)$, with $\Pi(q, i\Omega_n) \propto [(q/2k_F)^2 + |\Omega_n|/(\alpha D q^2)]$. Here $\Omega_n$ is a bosonic Matsubara frequency, and the above expression for the gauge propagator $D_{\mu\nu}(q, i\Omega_n)$ is valid for frequencies smaller than the spinon bandwidth $\alpha D$. As a result, the gauge excitations are characterized by a dynamical exponent $z = 3$, which in $d = 3$ are known to give a contribution to the specific heat coefficient $\gamma \equiv -\partial^2 F/\partial T^2 \propto \ln(\alpha D/T)$ and to the static spin susceptibility $\delta \chi_s \propto T^2 \ln(\alpha D/T)$. Finally, it has been argued in the literature that the gauge fluctuations convert the finite temperature mean field phase transition line into a crossover line.

B. Fluctuations of the Kondo Boson

At the QCP, where the Kondo coupling is tuned to its critical value $J_{K_*}$, the critical fluctuations of the continuous phase transition are given by those of the complex order parameter fields $\langle \sigma_i^x, \sigma_i \rangle$. The propagator for these fluctuations is defined by $D_{\sigma}(x, \tau) = \langle \sigma^x(x, \tau) \sigma(0, 0) \rangle$. We get $D_{\sigma}^{-1}(q, i\Omega_n) = 1/\lambda + \Pi_{fc}(q, i\Omega_n)$.

The interband polarization bubble between the conduction and the spinon bands. In the above $G_{\sigma}^{-1}(q, i\omega_n) = (i\omega_n - \epsilon_k)$ is the propagator for the conduction electrons, while $G_f^{-1}(k, i\omega_n) = (i\omega_n - \epsilon_k^f)$ is the propagator for the dispersive spinons. We write $\Pi_{fc}(q, i\Omega_n) = \Pi_{fc}(q, 0) + \Delta \Pi_{fc}(q, i\Omega_n)$, where $\Pi_{fc}(q, 0)$ is the static part of the fluctuations and $\Delta \Pi_{fc}(q, i\Omega_n)$ is the dynamic part. We first compute the static part which can be written as

$$\Pi_{fc}(q, 0) = \sum_k \frac{n_F(\epsilon_k) - n_F(\epsilon_k^0)}{\epsilon_k - \epsilon_{k+q}^0},$$

where $n_F(\epsilon)$ is the Fermi function. We find that $\Pi_{fc}(q, 0)$ is independent of momentum if the dispersions are linearized in Eq. (19). This implies that the momentum dependence is due to $k \sim k_F$ in the $k$-integral of Eq. (19), for which it is important to retain the quadratic dispersions of the bands. Furthermore, since the main contribution is for $k \sim k_F$, the small momentum scale $q^*$ is unimportant and can be set to zero to facilitate the calculation, and we write $\epsilon_k = (k^2 - k_F^2)/(2m)$ and $\epsilon_k^0 = (k^2 - k_F^2)/(2m_0)$. Then, in terms of $\alpha$, the ratio of the two bandwidths that we introduced earlier, we have $\alpha = m/m_0$. Using $k \leftrightarrow k + q$ inside the $k$-summation we
get
\[ \Pi_{fc}(q,0) = \sum_{k \leq k_F} \left\{ \frac{1}{\epsilon_k - \epsilon_{k+q}^0} - \frac{1}{\epsilon_{k+q} - \epsilon_k^0} \right\} \]
\[ = \int_0^{k_F} \frac{dkk^2}{4\pi^2} \int_{-1}^{1} dz \left\{ \frac{1}{B - \frac{qk}{m}} - \frac{1}{C + \frac{qk}{m}} \right\} \]
\[ = \frac{m_0}{2} \ln \left[ \frac{C + gk/m_0}{C - gk/m_0} \right] , \]

where \( B = A(k_F^2 - k^2) - q^2/(2m) \), \( C = A(k_F^2 - k^2) - q^2/(2m_0) \), and \( A = [1/(2m) - 1/(2m_0)] \). After performing the momentum integration, we expand the resulting expression in powers of \((q/k_F)\), and we use \( \rho_0 = mk_F/(2\pi^2) \), the density of states per spin of the conduction electrons at the Fermi level. To leading order in \((q/k_F)\) we get
\[ \Pi_{fc}(q,0) \approx \rho_0 \left[ \ln \left( \frac{1}{1 - \alpha} \right) + \frac{1}{4(1 - \alpha)^3} \left( \frac{q}{k_F} \right)^2 \right] \]

Note that the \( \rho_0 \ln(\alpha) \) term in the above equation has been derived in appendix A2 using a slightly different method for the calculation of the mean field free energy in Eq. (10). This term, along with \( 1/J_K \), define the mass \((1/J_K + \rho_0 \ln(\alpha))\) of the Kondo boson, which goes to zero at the QCP.

Next we calculate the dynamic part of the fluctuations which can be written as
\[ \Delta \Pi_{fc}(q,i\Omega_n) = \Pi_{fc}(q,i\Omega_n) - \Pi_{fc}(q,0) = \frac{1}{\beta} \sum_{k,\omega_n} G_k(k,\omega_n) \left( G_f(k+q,\omega_n-i\Omega_n) - G_f(k+q,\omega_n) \right) \]

Unlike in the case of the static part, here the dominant contribution is from the interband particle-hole excitations around the two Fermi surfaces, for which the spectra can be linearized. We write \( \epsilon_k = \epsilon \) for the dispersion of the conduction electrons, and \( \epsilon_{k+q}^0 = \alpha(\epsilon - v_F q^* + v_F qz) \) for the dispersion of the spinons, where \( z \) is the cosine of the angle between wavevectors \( k \) and \( q \). Approximating the \( k \)-summation by
\[ \frac{\rho_0}{2} \int_{-\infty}^{\infty} d\epsilon \int_{-1}^{1} dz , \]
at zero temperature we get
\[ \Delta \Pi_{fc}(q,i\Omega_n) = \frac{\rho_0}{2(1 - \alpha)} \left[ Y_1 + Y_2 + Y_3 + Y_4 \right] , \]

where
\[ Y_{1,2} = \left( 1 + \frac{E_1}{v_F q^*} \right) \ln(\epsilon_1 + v_F q^*) \]

with \( E_1 = v_F q^* - i\Omega_n/\alpha \), and
\[ Y_{3,4} = - \left( 1 + \frac{E_2}{v_F q^*} \right) \ln(\epsilon_2 + v_F q^*) \]

with \( E_2 = v_F q^* - i\Omega_n \). From the above expression of the dynamic part given by Eqs. (22a)–(22c), we next extract the leading behaviour in different regimes of frequency and momentum. For this we need to compare the momentum \( q \) with \( q^* \), and the frequency \( \Omega_n \) (a continuous variable at \( T = 0 \)) with the energy scales \( E_x \equiv \alpha v_F q^* \) and \( \alpha v_F q \). Note that \( v_F q^* \sim 10^3 \) K is an energy scale much larger than the ultraviolet cut-off of the theory \( \alpha D \sim 10 \) K (the spinon bandwidth), and therefore we need to consider only \(|\Omega_n| \ll v_F q^*\). We find five distinct regimes which are as follows:

(i) \(|\Omega_n| < E_x \) and \( q < q^* \), where
\[ \sum_{i} Y_i \approx -2(1 - \alpha) \frac{i\Omega_n}{E_x} \left[ \frac{1}{3} \left( \frac{q}{q^*} \right)^2 + \frac{1}{2} \frac{(1 + \alpha) \Omega_n}{E_x} \right] . \]

Note that in the above, we retained two sub-leading terms because there are regimes where the sub-leading terms are larger than the static \((q/k_F)^2\) term.

(ii) \(|\Omega_n| < E_x \) and \( q > q^* \), where
\[ \sum_{i} Y_i \approx -2(1 - \alpha) \frac{i\Omega_n}{\alpha v_F q} \left[ \frac{\pi}{2} \text{sgn}(\Omega_n) + q^*/q \right] . \]

(iii) \(|\Omega_n| > E_x \) and \( q < q^* \), where
\[ \sum_{i} Y_i \approx 2 \left[ \ln \left( \frac{-i\Omega_n}{E_x} \right) + \frac{1}{6} \left( \frac{q}{q^*} \right)^2 - \frac{E_x}{i\Omega_n} \right] . \]

(iv) \(|\Omega_n| > \alpha v_F q > E_x \) and \( q > q^* \), where
\[ \sum_{i} Y_i \approx 2 \left[ \ln \left( \frac{-i\Omega_n}{\alpha v_F q} \right) + 1 + \frac{i\pi}{2} \text{sgn}(\Omega_n) \right] . \]

(v) \( \alpha v_F q > |\Omega_n| > E_x \) and \( q > q^* \), where
\[ \sum_{i} Y_i \approx (1 - \alpha) \frac{i\Omega_n}{\alpha v_F q} \left[ -i\pi \text{sgn}(\Omega_n) + (1 + \alpha) \frac{i\Omega_n}{\alpha v_F q} \right] . \]

At the quantum critical point, the mass of the Kondo boson goes to zero due to Eq. (13). The leading frequency and momentum dependences of \( D_{fc}(q,\Omega_n) \) are determined using Eqs. (20), (22) and (23). The details of the various asymptotic structures of \( D_{fc}(q,\Omega_n) \) in different regimes of frequency and momentum are discussed in appendix B.1. Among the forms of \( D_{fc}(q,\Omega_n) \) given in
Eqs. (111)–(113), only the following two asymptotic structures are important for obtaining the leading contribution of the Kondo boson to thermodynamic and transport properties.

First, for $|\Omega_n| < [\alpha D/(2\pi)](q^*/k_F)^3$ and $q < q^*$, we get

$$D_{\sigma}^{-1}(q, i\Omega_n) \approx \rho_0 \left[ \frac{1}{4} \left( \frac{q}{k_F} \right)^2 - \frac{i\Omega_n}{E_x} \right],$$

(24)

which gives rise to an undamped propagating mode with dynamical exponent $z = 2$ (the dispersion of which is given by setting Eq. (24) to zero). The existence of this mode is a direct consequence of the mismatch between the Fermi surfaces of the conduction and the spinon bands. Due to this mismatch, a minimum momentum of $q^*$ is necessary to excite an interband particle-hole pair. Consequently, for momentum $q < q^*$, the spectrum of the Kondo boson lies outside the continuum of the interband particle-hole excitations and thereby remains undamped. Note that this massless mode corresponds to hybridization fluctuations about the QCP, and becomes massive for $J_K < J_{KE}$ (this is realized by adding a constant term $\delta$ to Eq. (24)). Since $\Pi_f$, at $q = 0$ diverges logarithmically at $E_x$, the mode energy never exceeds $E_x$. The mode dispersion, which is quadratic about $q = 0$, is more complicated as $q$ approaches $q^*$ due to logarithmic corrections to $\Pi_f$, and is described in greater detail in appendix B.2.2.

Second, for most of the phase space, the spectrum for the fluctuations of $\sigma$ lies within the interband particle-hole continuum, and we get

$$D_{\sigma}^{-1}(q, i\Omega_n) \approx \rho_0 \left[ \frac{1}{4} \left( \frac{q}{k_F} \right)^2 + \pi \frac{|\Omega_n|}{2 \alpha v F q} \right],$$

(25)

e.i., an overdamped critical mode with dynamical exponent $z = 3$. Next we note that, since we assume $q^* \ll k_F$, the overdamped $z = 3$ critical mode occupies most of the momentum space and therefore almost always it provides the leading contribution to thermodynamic and transport properties. In this regime, the scaling of frequency is given by $\Omega_n \sim [(\alpha D)/(2\pi)](q/k_F)^3$, and since this regime ends for $q < q^*$, one obtains the infrared energy scale

$$E^* \approx c\alpha D \left( \frac{q^*}{k_F} \right)^3,$$

(26)

where $c$ is $1/(2\pi)$. The true value of $c$ is slightly smaller ($\sim 0.1$) since there are logarithmic corrections to $\Pi_f$, as $q$ approaches $q^*$. A more detailed account is given in appendix B.2.2. We note that $E^*$, which can be estimated to be $\sim 1$ mK, appears as an infrared crossover scale for any physical property that is affected by the excitations of $\sigma$. On the other hand, the ultraviolet cutoff is provided by $\alpha D \sim 10$ K, which is the bandwidth of the spinons, or equivalently the single ion Kondo scale by Eq. (17).

V. THERMODYNAMICS OF THE KONDO BOSON

In this section, we study the effect of the fluctuations of the Kondo boson $\sigma$ on the thermodynamics of the system in the quantum critical regime. In particular, we compute (a) the contribution to the free energy, (b) the temperature dependence of the static spin susceptibility, and (c) the crossover lines in temperature which demarcate the quantum critical regime.

A. Free energy

The contribution of the fluctuations of $\sigma$ to the free energy (per unit volume) is given by

$$F = \sum_q \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \Im \ln \left[ D_{\sigma}^{-1}(q, \Omega + i\eta) \right],$$

(27)

depicting the retarded propagator for the Kondo bosons. We find that, for all temperatures $T < \alpha D$, the leading $T$ dependence of the free energy $F$ is given by that part of phase space where the mode is overdamped (with dynamical exponent $z = 3$) and for which the expression for the propagator is approximately given by Eq. (25). The details of this demonstration, as well as the evaluation of the sub-leading contribution from the other regimes, is given in appendix B.3. For $T > E^*$, the leading $T$ dependence of $F$ is given by

$$F \approx \frac{k_F^3\alpha D}{2\pi^3} \int_0^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_c}^1 dq q^2 \times \tan^{-1} \left( \frac{2\pi \Omega}{q^2} \right),$$

(28)

Here $q$ and $q_c$ are dimensionless momenta in units of $k_F$, and $\Omega$ and $T$ are dimensionless energies in units of $\alpha D$. Since the $\alpha$-integral is ultraviolet divergent, we use the Fermi momentum as an upper cutoff. The infrared cutoff, $q_c$, for the $z=3$ regime is dependent on the particular temperature range considered, since the leading $T$ dependence comes from frequencies $\Omega \sim T$. For $T > E^*$, $q \gtrsim \Omega^{1/3}$, for which we can approximate $\tan^{-1}(x) \approx x$ and replace the cutoff $q_c$ by $\Omega^{1/3}$. Performing the integrals, we find

$$F(T) \approx -\left( \frac{k_F^3}{9} \right) \ln \left( \frac{\alpha D}{T} \right) \frac{T^3}{\alpha D}, \ T > E^*. \quad (29)$$

We note that this contribution adds to a similar $T^2 \ln(T)$ contribution from the transverse gauge fluctuations (which are massless $z = 3$ excitations). They give rise to a $\ln(T)$ behavior for the specific heat coefficient.

For $T < E^*$, the leading contribution to the free energy is again given by Eq. (28) with $q_c = q^*/k_F$ for the infrared cutoff of the $q$-integral. This is because for $\Omega \sim T < E^*$, the $z = 3$ regime exists for $q > q^*$. As a result, because
\( \Omega^{1/3} < q^*/k_F \) in this temperature regime, the lower cutoff remains at \( q^*/k_F \). This gives,

\[
F(T) \approx - \left( \frac{k_F^3}{3} \right) \ln \left( \frac{k_F}{q^*} \right) \frac{T^2}{\alpha D}, \quad T < E^*.
\] (30)

This \( T^2 \)-dependence cannot be distinguished from ordinary Fermi liquid corrections, and in this temperature regime the free energy is dominated by the \( T^2 \ln(T) \) contribution from the transverse gauge fluctuations.\(^{29}\)

The collective mode gives a magnon-like contribution to the free energy (\( F \sim T^{5/2} \)), and is sub-leading relative to the \( z=3 \) contribution (see appendix 13). We illustrate this by showing in Fig. 3 a numerical determination of the contribution of the specific heat coefficient, C/T, coming from the Kondo boson, using the \( \sigma \) polarization bubble of Eq. (22a). In this plot, one sees the sub-leading contribution arising from the \( z=3 \) region, the logarithmic contribution from the \( z=3 \) region which saturates for \( T < E^* \), and the small difference between the positive and negative \( \Omega \) contributions from the \( z=3 \) region due to the chirality of the \( \sigma \) polarization bubble.

### B. Static spin susceptibility

At the mean field level, where the critical fluctuations of \( \sigma \) are ignored, the temperature dependence of the static spin susceptibility \( \chi_s(T) \) is entirely analytic, namely a constant (Pauli susceptibility) plus a \( T^2 \) term, which is usual for band fermions. Next, when we take the critical fluctuations into account, we expect the correction to \( \chi_s(T) \) to be non-singular (since the transition is non-magnetic and the excitations of \( \sigma \) are in the singlet channel), but non-analytic (due to the massless excitations). In order to evaluate this temperature dependence, we first need to compute \( D_\sigma(q, i\Omega) \) in the presence of a magnetic field (\( B \)). For a finite \( B \), the effect of the Zeeman term is to shift the Fermi wave vectors \( k_F \) and \( k_F^0 \) of the conduction and the spinon bands, respectively. We get, \( k_F^0 \rightarrow k_F^0 \pm (\mu_B g_f B)/(\alpha v_F) \), and \( k_F \rightarrow k_F \pm (\mu_B g_s B)/v_F \), where \( g_f \) and \( g_s \) are effective Lande \( g \)-factors of the spinons and the conduction electrons, respectively, \( \mu_B \) is the Bohr magneton, and \( \pm \) refers to the up and down spins, respectively. Since \( \alpha \ll 1 \), and in general \( g_f > g_s \), we can ignore the coupling of \( B \) to the \( c \)-electrons and consider the effect of the Zeeman term as a renormalization of the mismatch wave vector \( q^* \), which is given by

\[
q^* \rightarrow q^* + \frac{\mu_B g_f B}{\alpha v_F}.
\]

Next, we note that, in the presence of a finite \( q^* \), one expects \( \Pi_{bc}(0,0) \) to have corrections of the type \( q^*/k_F \) and \( (q^*/k_F)^2 \) (which are not calculated in Eq. (21) since the evaluation was performed in the limit \( q^* \rightarrow 0 \)). This implies that, in the presence of a magnetic field, we expect a correction to \( \Pi_{bc}(0,0) \) which is proportional to \( [(\mu_B g_f B)/(\alpha D)]^2 \) (since the excitation of \( \sigma \) is in the singlet channel, we do not expect a linear term in \( B \)). Adding such a term to \( D_\sigma(q, i\Omega) \), and noting that the leading temperature dependence is due to the overdamped \( z = 3 \) mode, we can generalize Eq. (28) to obtain the \( B \) dependence of the free energy as

\[
F(B, T) \approx - \frac{k_F^3 \alpha D}{2\pi^4} \int_0^\infty d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_c}^1 dq q^2 \times \tan^{-1} \left( \frac{2\pi\Omega}{q^3 + h_2^2 q} \right),
\]

(31)

Here energy and momenta are in dimensionless units (as in Eq. (28)) and \( h = (\mu_B g_f B)/(\alpha D) \) is the dimensionless magnetic field. Writing the correction to the static spin susceptibility due to the fluctuations of \( \sigma \) as \( \delta\chi_s(T) \equiv -[\partial^2 F/(\partial B)^2]_{B=0} \), we get for \( T > E^* \)

\[
\delta\chi_s(T) \approx -(\mu_B g_f)^2 \left[ \frac{2^{4/3} \Gamma(4/3) \zeta(4/3)}{\pi^5/3^{3/2}} \right] \left( \frac{k_F^3}{\alpha D} \right)^{4/3},
\]

(32)

while for \( T < E^* \), the lower cut-off is at \( q^* \), making the mode effectively massive, and we get

\[
\delta\chi_s(T) \approx -(\mu_B g_f)^2 \left[ \frac{1}{3} \left( \frac{k_F^3}{(q^*)^2} \right)^2 \right] \left( \frac{T^2}{\alpha D} \right)^7.
\]

(33)

As in the case of the free energy, the non-analyticity in the leading temperature dependence is cutoff below \( E^* \) due to the mismatch wave vector \( q^* \). As noted before, the gauge bosons give rise to a \( T^2 \ln(T) \) contribution to \( \chi_s \).
\[ \delta m(T) = u_0 \sum_{q} \int_{-\infty}^{\infty} \frac{d\Omega}{2\pi} \coth \left( \frac{\Omega}{2T} \right) \text{Im} D_x(q, \Omega + \eta). \]  

Denoting the contribution of the \( z = 2 \) mode as \( \delta m_1(T) \), we get Eq. (24) as 

\[ \delta m_1(T) = \left( \frac{u_0E_x}{2\pi^2\rho_0} \right) \int_0^{\infty} dq q^2 n_B \left( \frac{E_xq^2}{4k_F^2} \right), \]

where \( n_B(x) = (x/e^x - 1)^{-1} \) is the Bose function. For the leading \( T \)-dependence, we write \( u_B(x) \approx 1/x \), with an appropriate ultraviolet cutoff for the \( q \)-integral. For \( T < E^* \), this cutoff is \( k_F(T/E_x)^{1/2} \), and for \( T > E^* \), this cutoff remains at \( q^* \). We get

\[ \delta m_1(T) \approx \left( \frac{4u_0k_F^3}{\pi^2\rho_0} \right) \frac{T^{3/2}}{E_x^{1/2}}, \quad T < E^*, \]

\[ \approx \left( \frac{2u_0k_F^2q^*}{\pi^2\rho_0} \right) T, \quad T > E^*. \]  

Next, denoting the contribution of the \( z = 3 \) mode as \( \delta m_2(T) \), we get

\[ \delta m_2(T) = \left( \frac{4u_0\alpha Dk_F^3}{\pi^2\rho_0} \right) \int_0^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \Omega \quad \times \quad \int_{q_c}^\infty dq q^2 \left( \frac{q^2}{4\pi^2\Omega^2} \right), \]

where \( q \) and \( q_c \) are dimensionless in units of \( k_F \), and \( \Omega \) and \( T \) are dimensionless in units of \( \alpha D \). For \( T > E^* \), we can put \( q_c \approx \Omega^{1/3} \) for the leading term, while for \( T < E^* \), we have \( q_c = q^* \). This gives,

\[ \delta m_2(T) \approx \left[ \frac{2u_0k_F^2}{3\rho_0} \right] \left( \frac{k_F}{q^*} \right)^2 \frac{T^2}{\alpha D}, \quad T < E^*, \]

\[ \approx \left[ \frac{2^{7/3}T^2(4/3)(4/3)u_0k_F^3}{3^{2/3}\pi^{5/3}\rho_0} \right] \frac{T^{4/3}}{(\alpha D)^{1/3}}, \quad T > E^*. \]  

Comparing Eqs. (35) and (36) we find that, for \( T < E^* \), the leading \( T \)-dependence is given by the \( z = 2 \) mode and \( \delta m(T) \approx \delta m_1(T) \), while for \( T > E^* \), the leading term is from the \( z = 3 \) damped mode and \( \delta m(T) \approx \delta m_2(T) \). Consequently, the crossover lines in temperature which define the quantum critical regime are given by

\[ T \propto |\delta - \delta_c|^{2/3}, \quad T < E^*, \]

\[ \propto |\delta - \delta_c|^{3/4}, \quad T > E^*. \]  

VI. QUASIPARTICLE LIFETIME AND TRANSPORT

In this section, we first evaluate the quasiparticle lifetime (\( \tau_\sigma \)) of the conduction electrons due to scattering from the critical excitations of \( \sigma \) whose dynamical exponent is \( z = 3 \). In three dimensions, this has a marginal Fermi liquid form.

\[ \text{Im} \Sigma^R(k, \omega) = \sum_q \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \left[ n_B(\Omega) + n_F(\Omega - \omega) \right] \times \text{Im} G^R_\sigma(k - q, \omega - \Omega), \]

where \( n_F(x) = (e^{x/T} + 1)^{-1} \) is the Fermi function, and \( R \) denotes retarded functions. At zero temperature, this gives

\[ \text{Im} \Sigma^R(k, \omega) = \sum_q \int_0^\omega \frac{d\Omega}{\pi} \text{Im} D^R_\sigma(k, \Omega) \times \text{Im} G^R_f(k - q, \omega - \Omega). \]
We evaluate the above expression for a conduction electron on the Fermi surface, i.e., for $|k| = k_F$, and we find that the leading frequency dependence is always due to the overdamped $z = 3$ mode whose expression is given by Eq. (24). The $z = 2$ mode does not contribute since it cannot kinematically connect the $f$ and $c$ electrons. We write

$$\sum_q \to \frac{1}{4\pi^2} \int_0^\infty dq q^2 \int_{-1}^1 dz,$$

where $z$ is the cosine of the angle between $k$ and $q$. After linearizing the spectrum for the spinons, we have

$$\text{Im}G_f^R(k_F - q, \omega - \Omega) = -\pi \delta(\omega - \Omega + E_c + \alpha v_F q z).$$

Since, $\Omega \sim \omega$, and $q > q^*$ for the overdamped mode, the constraint from the $\delta$-function is always satisfied. After the angular integral, we get

$$\text{Im} \Sigma_c^R(k_F, \omega) = -\left(\frac{2k_F^3}{\pi \rho_0 \alpha D}\right) \int_0^\omega d\Omega \int_0^{\infty} dq q^2 \int_0^{\infty} dq q^2 \int_{-1}^1 dz,$$

where momenta and frequencies are dimensionless in units of $k_F$ and $\alpha D$, respectively. For $\Omega \sim \omega > E^*$, the leading contribution of the $q$-integral comes from $q \sim (\Omega/\alpha D)^{3/2}$ and therefore the infrared cutoff $q^*$ can be set to zero. But for $\Omega \sim \omega < E^*$, $q^* > (\Omega/\alpha D)^{3/2}$, and the lower cut-off at $q^*$ comes into play. We finally get

$$\text{Im} \Sigma_c^R(k_F, \omega) \approx -\left(\frac{k_F^3}{6\pi \rho_0 \alpha D}\right) \omega^2, \quad |\omega| > E^*, \quad (40)$$

Thus we find that above the infrared cutoff scale $E^*$, the Kondo breakdown scenario, in which the conduction electrons interact with the critical hybridization fluctuations, provides a microscopic mechanism to obtain a marginal Fermi liquid in three dimensions.

Next we evaluate the temperature dependence of the imaginary part of the self-energy at $\omega = 0$ on the Fermi surface. Denoting this as $\text{Im} \Sigma_c(T)$, we get from Eq. (38)

$$\text{Im} \Sigma_c^R(T) = \sum_q \int_{-\infty}^{\infty} \frac{d\Omega}{\pi} \frac{1}{\sinh(\Omega/T)} \text{Im}D^R_q(q, \Omega) \times \text{Im}G_f^R(k_F - q, -\Omega). \quad (41)$$

The evaluation of the above expression is very similar to the finite frequency case, except for $T > E^*$, the thermal factors $n_B(x) + n_F(x) = 1/\sinh(x)$ gives an additional logarithm which is cut off by $E^*$. We get

$$\text{Im} \Sigma_c^R(T) \approx -\left(\frac{k_F^3}{3\pi \rho_0 \alpha D}\right) T \ln \left(\frac{2T}{E^*}\right), \quad T > E^*,$$

$$\approx -\left(\frac{k_F^3}{6\pi \rho_0 \alpha D E^*}\right) T^2, \quad T < E^*. \quad (42)$$

In Fig. 5 we show a plot of this quantity from a numerical evaluation of Eq. (11) using Eq. (25). One can see the approximate linear $T$ behavior except at very low temperatures, where one crosses over to a $T^2$ behavior.

Next we evaluate the temperature dependence of the resistivity $\delta\rho(T) \equiv \rho(T) - \rho(0)$. In order to proceed, we first need to address whether the transport lifetime $\tau_r$ can be identified with the quasiparticle lifetime $\tau_c(\omega, T) \propto [\text{Im} \Sigma_c(\omega, T)]^{-1}$, whose frequency and temperature dependences are given by Eqs. (40) and (41). For this, it is useful to compare the Kondo-Heisenberg model with a single band model. In the latter case, the two lifetimes have a different temperature dependence because the leading contribution to the self-energy comes from forward scattering processes with momentum transfer $q \simeq 0$, which are not effective in relaxing the current. As such, when vertex corrections are taken into account, $\tau_r \simeq \alpha D$ acquires an additional temperature dependence proportional to $T^2 \sim T^2/\alpha$. However, this is not the case for the Kondo-Heisenberg model which has two bands, one of light conduction electrons and the other of heavy spinons. Due to the constraint of half filling [Eq. (2)], the spinon current operator $J_f = 0$ at every site $i$. Therefore, it is guaranteed by gauge invariance that a vertex correction involving the exchange of a single $\sigma$ boson [Fig. (6a)], which involves an external spinon current operator, is identically zero. The first non-zero vertex correction involves the exchange of two $\sigma$ bosons, and we expect such a correction to be small by a factor of $\alpha$. This can be understood as well in a Boltzmann approach, where the transport vertex correction $1 - \cos(\theta)$ gets replaced by $1 - \alpha \cos(\theta)$, which is essentially unity since $\alpha \ll 1$.

Consequently, in the present theory, the transport lifetime is proportional to the quasiparticle lifetime. The physical picture that emerges from the above discussion is that, when scattered from a $\sigma$ boson ($c \Rightarrow f + \sigma$), a
conduction electron transmutes into a spinon and relaxes its current in the bath of the spinons. More formally, the expression for the conductivity \( \sigma_c \) obtained from the current-current correlator in the Kubo formalism is given by

\[
\sigma_c = \frac{v_F^2}{3} \sum_k \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left[ \frac{\partial}{\partial \omega} \tanh\left(\frac{\omega}{2T}\right) \right] \left[ \text{Im} G^R(k,\omega) \right]^2.
\]

(43)

We write

\[
\left[ G^R_c(k,\omega) \right]^{-1} = \omega - \epsilon_k - \text{Re} \Sigma(\omega) + \frac{i}{2\tau_c(\omega,T)}. \tag{44}
\]

where for \( (\omega, T) > E^* \)

\[
[\tau_c(\omega, T)]^{-1} = \tau^{-1} + \left( \frac{2k_F^2}{3\pi\rho_0\alpha D} \right) T \ln \left( \frac{2T}{E^*} \right) + \frac{\omega}{2}. \tag{45}
\]

Here \( \tau \) is an elastic scattering lifetime of the conduction electrons due to impurities, and sets the scale of the temperature independent part of \( \sigma_c \). We linearize the dispersion of the conduction electrons and replace the momentum sum by an energy integral, and we finally obtain \( \sigma_c(T) = \frac{\rho_0 v_F}{3} \alpha D, \) \( E^* < T < \alpha D, \)

\[
\delta \rho(T) \propto T \ln \left( \frac{2T}{E^*} \right). \tag{46}
\]

Therefore, the scenario of the breakdown of the Kondo effect captures one of the most enigmatic features of heavy fermion systems close to quantum criticality, namely the quasi-linear temperature dependence of the resistivity observed for most compounds over a large range of temperature. For \( T < E^* \), the usual Fermi liquid result is recovered and \( \delta \rho(T) \propto T^2 \). It is interesting to note that, the recovery of the Fermi liquid \( T^2 \) behavior of resistivity below a finite temperature scale in the quantum critical regime of YbRh\(_2\)Si\(_2\) has recently been reported\(^{22}\). Finally, we note that for the same reason that equates the single particle and transport lifetimes, the electrical and thermal transport lifetimes are the same.

VII. CONCLUSION

To summarize, we studied the Kondo-Heisenberg model in three dimensions using a fermionic representation for the localized spins. The mean-field phase diagram in the \( T - J_K \) plane, where \( J_K \) is the Kondo coupling, exhibits a quantum critical point that separates a uniform spin liquid phase from a heavy Fermi liquid phase. In the uniform spin liquid phase, the Kondo hybridization between the conduction band and the band of fermionic spinons that constitute the local moments vanishes, thereby indicating that in this phase, the Kondo effect fails to occur. For a Kondo coupling larger than the critical value \( (J_K > J_K^*) \), a heavy Fermi liquid ground state is established with finite hybridization between the bands. This implies that at the quantum critical point \( (J_K = J_K^*) \), the lattice Kondo energy scale \( T_K \) vanishes, indicating the breakdown of the Kondo effect for couplings smaller than the finite value \( J_K^* \).

In general, the size of the (hot) “Fermi surface” of the spinon band is different from that of the conduction electrons, and we characterized their mismatch by a wavevector \( q^* \equiv k_{F0} - k_F \), where \( k_{F0} \) is the spinon Fermi wavevector and \( k_F \) is the conduction Fermi wavevector. As a consequence of this mismatch, we found that two mean field solutions are possible in the Fermi liquid phase. First, one with a uniform hybridization, which is stabilized when the two band masses have the same sign. This is the standard Kondo phase which appears in the mean-field pseudofermion description of the Kondo lattice. Second, a novel Kondo phase with the hybridization modulated in space with wavevector \( q_0 \approx 1.2q^* \), which appears when the two band masses have opposite signs (i.e., one band is electron-like and the other hole-like). Conceptually, this phase is analogous to the LOFF state of superconductivity, and is characterized by nodes in space where \( T_K \) is zero. In this paper, we did not examine the physical consequences of the modulated hybridization, which will be the topic for future work. For the uniform case, we showed that at the quantum critical point the single ion Kondo scale \( T_K^0 \) is approximately equal to the Heisenberg coupling \( J_H \). This demonstrates that the Kondo breakdown is a consequence of the competition between the Kondo energy scale and the magnetic energy scale, even though there is no long range magnetic order in the present formulation.

Then, we studied the effect of the critical hybridization fluctuations (excitations of the order parameter \( \sigma \)) associated with the vanishing energy scale \( T_K \) on the thermodynamic and transport properties of the system. We found that, due to the mismatch \( q^* \), the critical fluctuations are affected by energy scales \( E^{**} \sim [\alpha D/(2\pi)](q^*/k_F)^3 \) and \( E_x \sim \alpha v_F q^* \), where \( \alpha \sim J_H/D \) is the ratio of the spinon
bandwidth $J_H$ and the conduction bandwidth $D$. The propagator for the critical modes has several asymptotic structures in different regimes of frequency-momentum space, out of which the following two are important and readily understood. (i) For momentum $q < q^\ast$, the spectrum of the critical fluctuations lies outside the interband particle-hole continuum and therefore their dynamics is undamped and is characterized by a dynamical exponent $z = 2$. (ii) For most of momentum space ($q > q^\ast$), the spectrum of the critical modes lies within the particle-hole continuum, and therefore have overdamped dynamics with exponent $z = 3$ (Landau damping). The leading contribution to thermodynamics and transport is almost always governed by the latter asymptotic structure, in contrast to most Ginzburg-Landau approaches, where only the critical modes within $1/\xi(T)$ of the ordering vector ($q = 0$) are important. Above the temperature scale $E^\ast \sim 1$ mK, this rises to anomalous metallic behavior, such as a specific heat coefficient that diverges logarithmically with temperature, and the inverse lifetime of the conduction electrons which has a $T \ln T$ temperature dependence. The latter is a consequence of the conduction electrons scattering with the critical bosons with the dynamical exponent $z = 3$, which in three dimensions provide a microscopic mechanism to obtain marginal Fermi liquid behavior. Since the spinons do not carry current, but are effective in relaxing the current carried by the conduction electrons, the $T$-dependence of the inverse particle lifetime also gives rise to a $T \ln T$ behavior of the resistivity. From a scaling point of view, in this regime the frequency $\Omega$ of the critical fluctuations scale as $\Omega \sim q^3$, where $q$ is their momentum. For $T < E^\ast$, however, the infrared cutoff $q^\ast$ prohibits the $z = 3$ scaling, and the leading $T$ dependence of the specific heat coefficient and the resistivity are Fermi liquid like.

The Kondo breakdown scenario is promising in that it can explain one of the least understood features of the heavy fermions near quantum criticality, namely the quasi-linear temperature dependence of the resistivity, and the existence of multiple energy scales, over decades of temperature.

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APPENDIX A

1. Spatially modulated mean field solution

Here we demonstrate that when the conduction electron and spinon masses have opposite signs, i.e., when one band is electron-like and the other hole-like, the mean field theory admits a solution where the Kondo hybridization is modulated in space. This is a consequence of the mismatch between the two Fermi surfaces, and conceptually is analogous to the LOFF state of superconductivity. In the following, we choose the conduction band to be electron-like and the spinon band hole-like, and linearize their dispersions. This gives $\epsilon_k = \epsilon$ for the dispersion of the conduction band, where $\epsilon = v_F(k - k_F)$, and $\epsilon_k^\ast = -\alpha(\epsilon - v_Fq^\ast)$ for the dispersion of the spinon band. For this case, we evaluate the static interband polarization $\Pi_{fc}(q,0)$, whose general expression is given by Eq. (19). Approximating the momentum summation by

$$\sum_{\bf k} \rightarrow \frac{\rho_0}{2} \int_D^d \rho \, dz,$$

where the conduction bandwidth $D$ enters as an ultraviolet cutoff for the energy integral, we get,

$$\Pi_{fc}(q,0) = \frac{\rho_0}{1 + \alpha} \left\{ \ln \left[ \frac{\alpha v_F^2 (q^\ast)^2 - q^2}{(1 + \alpha)^2 D^2} \right] - 2 \right. $$

$$+ \frac{q^\ast}{q} \ln \left[ \frac{q + q^\ast}{q^\ast - q} \right] \right\}. \quad (A1)$$

It is easy to see that the maximum of $-\Pi_{fc}(q,0)$ is at a finite wavevector $q_0$ where

$$q_0 \approx 1.2q^\ast. \quad (A2)$$

Therefore for $J_K > J_{K^\ast}$, where the critical value of the Kondo coupling is given by

$$\frac{1}{J_{K^\ast}} + \Pi_{fc}(q_0,0) = 0,$$

the Kondo boson condenses in the Fermi liquid phase at a finite wavevector $q_0$ (i.e., $\langle \sigma_{q_0} \rangle \neq 0$). This implies that the Kondo hybridization is modulated, with nodes in space where $T_K$ vanishes.

2. Calculation of the mean field free energy

In this part, we give the technical details for the evaluation of the mean field free energy at $T = 0$. This can be written as

$$\frac{F_{MF}}{N} = \sum_{\bf k, i=a,b} \epsilon_k^i \theta(-\epsilon_k^i) + \frac{\sigma_0^2}{J_K} + \frac{\phi_0^2}{D_H} + \frac{\lambda_0}{2}. \quad (A3)$$

where $\theta(x)$ is the Heaviside step function, and $\epsilon_k^{a,b}$ are given by Eq. (11). We replace $\sum_{\bf k} \rightarrow \rho_0 \int d\epsilon$, and from the solution of the equations $\epsilon_k^a = 0$, we get
\[
\sum_{k,i=a,b} \epsilon_k^i \theta(-\epsilon_k^i) = \frac{\rho_0}{2} \int_{-D}^{s-s_1} d\varepsilon \left[ (1+\alpha) \left( \varepsilon + \frac{\varepsilon^2}{D} \right) - \left( 1-\alpha \right)^2 \left( \varepsilon + \frac{\varepsilon^2}{D} \right)^2 + 4\sigma_0^2 \right]^{1/2} + \frac{\rho_0}{2} \int_{-D}^{-s-s_1} d\varepsilon \left[ (1+\alpha) \left( \varepsilon + \frac{\varepsilon^2}{D} \right) + \left( 1-\alpha \right)^2 \left( \varepsilon + \frac{\varepsilon^2}{D} \right)^2 + 4\sigma_0^2 \right]^{1/2},
\]

where \(s = \sigma_0/\alpha^{1/2}\) and \(s_1 = \sigma_0^2/(2D) + \mathcal{O}(1/D^2)\). We expand the expression under the square root in powers of \((1/D)\) and keep terms up to \(\mathcal{O}(1/D^2)\), since higher orders contribute to \(\mathcal{O}(\sigma_0^2)\) and beyond which we neglect. Performing the \(\varepsilon\)-integral, to \(\mathcal{O}(\sigma_0^2)\) accuracy we get

\[
\frac{F_{MF}}{N} = \frac{\rho_0 D^2}{2} \left[ \frac{\alpha^2}{2\rho_0 J_H} - \frac{\alpha}{3} \right] + \rho_0 \sigma_0 \left[ \frac{1}{\rho_0 J_K} \right] - \frac{1}{1-\alpha} \ln \left( \frac{1}{\alpha} \right) + \frac{\rho_0 \sigma_0}{\alpha^2 D^2} (1 - 4\alpha + \alpha^2) \left( 1 + \alpha \right),
\]

where a constant part has been ignored. Since \(\alpha \ll 1\), in the terms proportional to \(\sigma_0^2\) and \(\sigma_0^4\), we retain only the dominant \(\alpha\)-dependence, and get Eq. \((10)\).

**APPENDIX B**

1. Asymptotic structure of the Kondo boson

In this appendix, we determine the leading frequency and momentum dependences of the propogator for the Kondo boson \(D_\sigma(q, i\Omega_n)\) in the quantum critical regime using Eqs. \((20)\), \((22)\) and \((23)\). Its leading frequency dependence is given by the first terms in Eqs. \((23a)\)–\((23c)\), while the next term is determined by comparing the static \((q/k_F)^2\) term in Eq. \((20)\) with the sub-leading terms of Eqs. \((23a)\)–\((23c)\). The asymptotic structure of \(D_\sigma(q, i\Omega_n)\) in different regimes of frequency and momentum are as follows:

1. \(\Omega_n \ll E^* \approx (\alpha D)/(2\pi) [q^*/k_F]^3\). In this frequency interval there are three sub-regimes depending on the magnitude of the momentum \(q\) we get (a) \(q \ll q_{01} \equiv |\Omega_n/E_x| k_F\) (where \(E_x \equiv |\alpha v_F q^*|\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx -\rho_0 \left( \frac{i\Omega_n}{E_x} \right) \left[ 1 + \frac{1}{2} (1 + \alpha) \frac{i\Omega_n}{E_x} \right],
\]

(b) \(q_{01} \ll q \ll q^*\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx \rho_0 \left[ \frac{1}{4} \left( \frac{q}{k_F} \right)^2 - \frac{i\Omega_n}{E_x} \right],
\]

(c) \(q^* \ll q \ll k_F\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx \rho_0 \left[ \frac{1}{4} \left( \frac{q}{k_F} \right)^2 + \frac{\pi |\Omega_n|}{2\alpha v_F q} \right].
\]

2. \(E^* \ll \Omega_n \ll E_x\). In this frequency interval there are four sub-regimes given by

(a) \(q \ll q_{02} \equiv (\Omega_n/E_x)^{1/2} q^*\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx -\rho_0 \left( \frac{i\Omega_n}{E_x} \right) \left[ 1 + \frac{1}{2} (1 + \alpha) \frac{i\Omega_n}{E_x} \right],
\]

(b) \(q_{02} \ll q \ll q^*\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx -\rho_0 \left( \frac{i\Omega_n}{E_x} \right) \left[ 1 + \frac{1}{3} \left( \frac{q}{q^*} \right)^2 \right],
\]

(c) \(q^* \ll q \ll q_{03} \equiv k_F [(q^*/\Omega_n)/(\alpha k_F D)]^{1/4}\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx -\rho_0 \left( \frac{i\Omega_n}{\alpha v_F q} \right) \left[ \frac{\pi}{2} \text{sgn}(\Omega_n) + \frac{q^*}{q} \right],
\]

(d) \(q_{03} \ll q \ll k_F\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx \rho_0 \left[ \frac{1}{4} \left( \frac{q}{k_F} \right)^2 + \frac{\pi |\Omega_n|}{2\alpha v_F q} \right].
\]

3. \(E_x \ll \Omega_n \ll \alpha D\). In this frequency range there are five sub-regimes given by

(a) \(q \ll q_{04} \equiv (E_x/\Omega_n)^{1/2}\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx \rho_0 \left( \frac{1}{1-\alpha} \right) \ln \left( \frac{-i\Omega_n}{E_x} \right) \left[ \frac{E_x}{i\Omega_n} \right],
\]

(b) \(q_{04} \ll q \ll q^*\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx \rho_0 \left( \frac{1}{1-\alpha} \right) \ln \left( \frac{-i\Omega_n}{E_x} \right) + \frac{1}{6} \left( \frac{q}{q^*} \right)^2,
\]

(c) \(q^* \ll q \ll q_{05} \equiv k_F [\Omega_n/(\alpha D)]\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx \rho_0 \left( \frac{1}{1-\alpha} \right) \ln \left( \frac{-i\Omega_n}{\alpha v_F q} \right) + 1 + \frac{\pi}{2} \text{sgn}(\Omega_n),
\]

(d) \(q_{05} \ll q \ll q_{06} \equiv k_F [\Omega_n/(\alpha D)]^{1/2}\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx \rho_0 \left( \frac{\pi |\Omega_n|}{2\alpha v_F q} - \frac{1}{2} (1 + \alpha) \frac{\Omega_n^2}{(\alpha v_F q)^2} \right),
\]

(e) \(q_{06} \ll q \ll k_F\),

\[
D^{-1}_\sigma(q, i\Omega_n) \approx \rho_0 \left[ \frac{1}{4} \left( \frac{q}{k_F} \right)^2 + \frac{\pi |\Omega_n|}{2\alpha v_F q} \right].
\]
2. Spectral response of the Kondo boson

In the paper, several simplified expressions were used for the spectral response of the Kondo boson. Here, we give a more complete account. The fc polarization bubble has some similarities to the Lindhard function\(^{22}\), but also differs from it in important respects. In particular, the particle-hole continuum of the Lindhard function exists for all momenta, while this is not the case for the fc polarization as a result of the mismatch between the conduction and spinon Fermi surfaces. We have performed numerical calculations including the full quadratic dispersion of the fermions, but they are very similar to results we present here that are based on Eq. (22a) plus the static curvature correction (last term in Eq. (20)). The advantage of using Eq. (22a) is that it is valid for arbitrarily small \(\alpha\). All results here are for the retarded response function at \(T=0\). We confine our discussion to the case where both conduction and spinon bands have the same sign for the mass.

We begin with the fc bubble at \(q=0\)

\[
Re\Pi_{fc}(0, \Omega) = \frac{\rho_0}{1 - \alpha} \ln \left(\frac{\Omega - \alpha v_F q^*}{\Omega - |v_F q^*|}\right) \quad (B4)
\]

This expression contains two logarithmic singularities at the energies \(E_x \equiv \alpha v_F q^*\) and \(v_F q^*\), where \(q^* \equiv k_{F0} - k_F\) is the mismatch vector between the conduction and spinon Fermi surfaces. The imaginary part of \(\Pi_{fc}\) is non-zero. It guarantees that the Kondo boson propagator, \(D \equiv J_K/(1 + J_K \Pi_{fc})\), always has a pole between zero and \(E_x\). This pole is undamped since \(\Im \Pi_{fc}\) is zero before \(E_x\).

The general structure of \(\Pi_{fc}\) can be appreciated from Fig. 2, where the various domains for the imaginary part are shown. Note that the imaginary part vanishes in the regime we label as \(z=2\). For the positive frequency side, this is a triangle in \((q, \Omega)\) space bounded by \((0, E_x)\) and \((q^*, 0)\). For low frequencies appropriate for the dispersive peaks of \(\Im D\), it will be sufficient to expand Eq. (22a) for small \(\Omega\). When we do this, we find

\[
ReD^{-1} = \delta - \frac{\rho_0 \Omega}{2 \alpha v_F q} \ln |q + q^*| + \frac{\rho_0 \Omega^2}{4 k_F^2} \quad (B5)
\]

where \(\delta\) is the deviation from the quantum critical point (QCP) and the last term is the static curvature correction. Below the kinematic boundary, \(\Im D^{-1}\) is zero, so the zeros of Eq. (B5) in this regime give the collective mode dispersion, which for \(\delta = 0\) is

\[
\Omega_{coll}/E_x = 0.5(q^*/k_F)^2(q/q^*)^{1/2} \ln \left(\frac{|q + q^*|}{|q - q^*|}\right) \quad (B6)
\]

We compare this in Fig. 8 to the expression where the log in Eq. (B5) is expanded for small \(q/q^*\), the latter being

\[
\Im D^{-1} = -\frac{\rho_0 \pi \Omega}{2 \alpha v_F q} \ln |q + q^*| \quad (B7)
\]

Eq. (24). Note that formally, Eq. (B6) vanishes as \(q\) goes to \(q^*\), but this is of no concern, since the mode intersects the kinematic boundary before this occurs, and thus it terminates at a finite energy, corresponding to \(c \sim 0.1\) in Eq. (20).

Above the kinematic boundary, \(\Im D^{-1}\) is non-zero. For \(q > q^*\), it is

\[
\Im D^{-1} = -\frac{\rho_0 \pi \Omega}{2 \alpha v_F q} \left|q + q^*\right| \quad (B7)
\]

This leads to a pseudo-Lorentzian behavior for \(\Im D\). The location of the maximum of \(\Im D\), denoted as \(\Gamma\), can be found upon differentiation with respect to \(\Omega\), leading
FIG. 9: (Color online) Plots of $I_mD$ for positive (left) and negative (right) $\Omega$. The quantum critical point ($\delta = 0$) is shown on the top, away from this ($\delta = 1$) is shown on the bottom. The $z = 2$ dispersion is not visible on the scale of this plot. Note the approximate (anti)symmetry of the damped ($z = 3$) response at the QCP as compared to away. This damped dispersion at the QCP closely follows the analytic expression of Eq. (B8). The intensity scale for the bottom plots are a factor of ten smaller than the top ones.

FIG. 10: (Color online) Dispersion of the $I_mD$ maxima for $\delta$ ranging from zero (bottom curve) to 1 (top curve). The undamped modes are to the left of the kinematic boundary (dashed line), the damped modes to the right. Note the reversed magnon-like dispersion of the undamped modes and the approximate linear $q$ behavior of the damped modes for non-zero $\delta$.

3. Free energy

Here we compute the free energy due to the excitations of the Kondo boson, whose expression is given by Eq. (27), and take into account all the different asymptotic structures of the propagator $D_\sigma(q, i\Omega_n)$ which are given in Eqs. (B1)–(B3). The goal of this exercise is to prove that for all temperatures $T < \alpha D$, the leading contribution comes from that part of the phase space where the boson is overdamped with dynamical exponent $z = 3$, and whose propagator is given by Eq. (25).

(1) $T < E^*$. Since for the leading $T$ dependence we expect $\Omega \sim T$, in this temperature regime $D_\sigma(q, i\Omega_n)$ has three asymptotic forms which are given in Eq. (B1). Accordingly, we split the $q$-integral into three parts, namely $q < q_{01}, q_{01} < q < q^*$, and $q^* < q < k_F$, and denote their contributions as $F_{1a}, F_{1b}$ and $F_{1c}$, respectively. Keeping only the leading terms for each sub-regime, we get

$$\Gamma / E_x = 0.5(q^*/k_F)^2(q/q^*)^3 / \sqrt{\pi^2 + \ln^2 \left( \frac{|q + q^*|}{|q - q^*|} \right)}$$  (B8)

which is also plotted in Fig. 8. If instead, we ignore the $\Omega$ term in Eq. (B5), we get Eq. (25) instead. The latter is a true Lorentzian, and its dispersion is plotted as well in Fig. 8. Although formally Eq. (B8) vanishes as $q$ goes to $q^*$, the actual results based on Eq. (22a) do not, and we again find $c \sim 0.1$ in Eq. (26).

We finish this discussion by showing in Fig. 9 $I_mD$ based on Eq. (22a) for both positive and negative $\Omega$ for two cases, the quantum critical point ($\delta = 0$) and somewhat away ($\delta = 1$). The collective mode is not visible on the scale of this plot, but we note that it is only present on the positive frequency side. The damped response is approximately (anti)symmetric in $\Omega$ for $\delta = 0$ but becomes highly asymmetric for non-zero $\delta$. As $\delta$ increases, the most intense part of the damped response moves up the kinematic boundary $\Omega/E_x = 1 - q/q^*$ and approaches the log singularity at $q = 0, \Omega = E_x$. In Fig. 10, the dispersion of the $I_mD$ maxima is plotted for various $\delta$. Note the reversed magnon-like dispersion of the undamped modes and the approximate linear $q$ behavior of the damped modes for non-zero $\delta$. 

Here we compute the free energy due to the excitations of the Kondo boson, whose expression is given by Eq. (27), and take into account all the different asymptotic structures of the propagator $D_\sigma(q, i\Omega_n)$ which are given in Eqs. (B1)–(B3). The goal of this exercise is to prove that for all temperatures $T < \alpha D$, the leading contribution comes from that part of the phase space where the boson is overdamped with dynamical exponent $z = 3$, and whose propagator is given by Eq. (25).
\[ F_{1a} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{0}^{q_{101}} dq q^2 \ln \left[ -\frac{\Omega}{E_x} - i\eta \right] = -\left( \frac{\pi^2}{90k_F^2} \right) \frac{T^4}{E_x^3} \]  
(\text{B9a})

\[ F_{1b} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_{102}}^{q_*} dq q^2 \ln \left[ \frac{q^2}{4k_F^2} - \frac{\Omega}{E_x} - i\eta \right] = -\left( \frac{\zeta(5/2)}{\pi^{3/2}k_F} \right) \frac{T^{5/2}}{E_x^{3/2}} \]  
(\text{B9b})

\[ F_{1c} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_*}^{k_F} dq q^2 \ln \left[ \frac{q^2}{4k_F^2} - \frac{\pi}{2} \frac{\Omega}{\alpha v_F q} \right] = -\left( \frac{k_F^3}{3} \right) \ln \left( \frac{k_F}{q} \right) \frac{T^2}{\alpha D} \]  
(\text{B9c})

We note that, since \( T < E^* \), the leading temperature dependence is due to the \( z = 3 \) mode whose contribution is given by Eq. (B9c), and thus \( F \approx F_{1c} \).

(2) \( E^* < T < E_x \). In this temperature regime, \( D_\alpha(q, i\Omega_n) \) has four asymptotic forms which are given in Eq. (B3).

Now we split the \( q \)-integral into four parts, namely \( q < q_{102}, \ q_{102} < q < q^*, \ q^* < q < q_{313} \) and \( q_{313} < q < k_F \), and denote their contributions as \( F_{2a}, \ F_{2b}, \ F_{2c}, \ F_{2d} \) respectively. Once again, keeping only the leading terms for each sub-regime, we get

\[ F_{2a} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{0}^{q_{102}} dq q^2 \ln \left[ -\frac{\Omega}{E_x} - i\eta \right] = -\left( \frac{\zeta(5/2)}{4\pi^{1/2}k_F^3} \right) \left( \frac{E^*}{\alpha D E_x^3} \right)^{5/2} \]  
(\text{B10a})

\[ F_{2b} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_{102}}^{q_*} dq q^2 \ln \left[ -\frac{\Omega}{E_x} - i\eta \right] = -\left( \frac{q^*}{6T} \right) T ln \left( \frac{T}{E^*} \right) \]  
(\text{B10b})

\[ F_{2c} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_*}^{q_{313}} dq q^2 \ln \left[ -\left( \frac{q}{q^*} \right) \frac{\Omega}{\alpha v_F q} - i\frac{\Omega}{2\alpha v_F q} \right] = -\left( \frac{k_F^3}{3} \right) \frac{T^{7/4}}{(\alpha D)^{3/4}} \]  
(\text{B10c})

\[ F_{2d} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_{313}}^{k_F} dq q^2 \ln \left[ \frac{q^2}{4k_F^2} - \frac{\pi}{2} \frac{\Omega}{\alpha v_F q} \right] = -\left( \frac{k_F^3}{9} \right) \ln \left( \frac{\alpha D}{T} \right) \frac{T^2}{\alpha D} \]  
(\text{B10d})

After comparing the various contributions above, once again we find that the leading temperature dependence is due to the \( z = 3 \) mode, whose contribution is given by Eq. (B10c), and we have \( F \approx F_{2d} \).

(3) \( E_x < T < \alpha D \). In this temperature regime \( D_\alpha(q, i\Omega_n) \) has five asymptotic forms which are given in Eq. (B3).

Now we split the \( q \)-integral into five parts, namely \( q < q_{314}, \ q_{314} < q < q^*, \ q^* < q < q_{316} \) and \( q_{316} < q < k_F \), and denote their contributions as \( F_{3a}, \ F_{3b}, \ F_{3c}, \ F_{3d} \) and \( F_{3e} \) respectively. Once again, keeping only the leading terms for each sub-regime, we get

\[ F_{3a} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{0}^{q_{314}} dq q^2 \ln \left[ -\left( \frac{\Omega}{E_x} - i\eta \right) - \frac{E_x}{\Omega} \right] = -\left( \frac{q^*}{18\pi^2} \right) T \]  
(\text{B11a})

\[ F_{3b} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_{314}}^{q_*} dq q^2 \ln \left[ -\left( \frac{\Omega}{E_x} - i\eta \right) + \frac{1}{6} \left( \frac{q}{q^*} \right)^2 \right] = -\left( \frac{q^*}{12\pi^2} \right) T \ln \left( \frac{T}{E_x} \right) \]  
(\text{B11b})

\[ F_{3c} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_*}^{q_{315}} dq q^2 \ln \left[ -\left( \frac{\Omega}{\alpha v_F q} \right) + 1 - \frac{\pi}{2} \text{sgn}(\Omega) \right] = -\left( \frac{\pi}{90k_F^2} \right) T^4 \]  
(\text{B11c})

\[ F_{3d} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_{316}}^{q_{315}} dq q^2 \ln \left[ \frac{\Omega^2}{2(\alpha v_F q)} - \frac{\pi}{2} \frac{\Omega}{\alpha v_F q} \right] = -\left( \frac{\zeta(5/2)}{8\pi^{3/2}k_F^3} \right) T^{5/2} \]  
(\text{B11d})

\[ F_{3e} = \frac{1}{4\pi^3} \int_{-\infty}^{\infty} d\Omega \coth \left( \frac{\Omega}{2T} \right) \int_{q_{316}}^{k_F} dq q^2 \ln \left[ \frac{q^2}{4k_F^2} - i\frac{\pi}{2} \frac{\Omega}{\alpha v_F q} \right] = -\left( \frac{k_F^3}{9} \right) \ln \left( \frac{\alpha D}{T} \right) \frac{T^2}{\alpha D} \]  
(\text{B11e})
As before, we find that the leading temperature dependence is given by the \( z = 3 \) mode, whose contribution is given by Eq. (1511e), and we have \( F \approx F_{3e} \).

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27. Heavy fermion metals have complex Fermi surfaces where both conduction and f surfaces are large. As a consequence, large values of \( q^* \) are unlikely. Very small values of \( q^* \) are possible due to degeneracies. Within our simple model, a value of \( q^*/k_F \approx 0.1 \) is a reasonable estimate.
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31. As we showed in Section IV.B1, gaussian fluctuations about the mean field solution generate a dispersion for the \( \sigma \) field. As a consequence of gauge invariance, this also implies a coupling between the \( \sigma \) and the gauge fields which leads to a back-flow current. From the point of view of an effective field theory, where the \( \sigma \) fields are dressed, the constraint \( J_{F_1} = 0 \) is generalized to \( J_{F_1} + J_{\sigma_1} = 0 \), and leads to an additional contribution to the conductivity via the Ioffe-Larkin composition rules. This contribution, though, is of order \( \alpha \), and therefore does not change our findings, since \( \alpha \ll q_0^* \).
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34. Note that the \( z = 2 \) modes make sub-leading contributions to all quantities but the crossover lines. They also make no contribution to the conduction electron self-energy and transport due to kinematic constraints.
35. For spherical Fermi surfaces, as many ordering wave vectors as allowed by lattice symmetry will condense, each with a modulus of \( q_0 \).
36. see e.g., A. L. Fetter, and J. D. Walecka, Quantum Theory of Many-Particle Systems, (Dover Publications Inc., New York, 2003), p. 158-163.