New classical brackets for dissipative systems.

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A set of brackets for classical dissipative systems, subject to external random forces, are derived. The method is inspired to the old procedure found by Peierls, for deriving the canonical brackets of conservative systems, starting from an action principle. It is found that an adaptation of Peierls' method is applicable also to dissipative systems, when the friction term can be described by a linear functional of the coordinates, as is the case in the classical Langevin equation, with an arbitrary memory function. The general expression for the brackets satisfied by the coordinates, as well as by the external random forces, at different times, is determined, and it turns out that they all satisfy the Jacobi identity. Upon quantization, these classical brackets are found to coincide with the commutation rules for the quantum Langevin equation, that have been obtained in the past, by appealing to microscopic conservative quantum models for the friction mechanism.

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I. INTRODUCTION

The study of quantum dissipative systems, has attracted, in the last decades, a lot of interest, in view of its broad spectrum of applications, ranging from quantum optics through statistical mechanics, etc. The standard approach to deal with quantum dissipation, is based on the idea that the physical origin of dissipation is the interaction of the system with a heat bath, consisting of a large number of degrees of freedom. One considers then some microscopic, conservative model for the heat bath (and its interaction with the system), and tries to recover the macroscopic quantum behavior of the dissipative system alone, by eliminating from the description the degrees of freedom describing the bath. In Ref.([1]), it is shown, indeed, that the most general quantum Langevin Equation, which is one of the most popular models for dissipation, can be obtained from a simple microscopic model, where the heat bath is described by a set of independent oscillators, linearly coupled to the system of interest.

One may wonder whether it is possible to find a quantization method for dissipative systems, which is based on the macroscopic description of dissipation only, and makes therefore no use of microscopic models. As is well known, quantization of dissipative systems is by no means straightforward, because in general they admit neither a Lagrangian nor a Hamiltonian formulation. Moreover, even in those special instances where such a formulation can be given, the application of the conventional canonical quantization methods leads to physically incorrect results [2]. In this Letter, we show that new classical brackets can be consistently built for dissipative systems, by generalizing the covariant definition of Poisson Brackets for Lagrangian systems, discovered long ago by Peierls [3] (See also Refs.[4, 5, 6, 7]). Our bracket is defined on the infinite-dimensional functional space consisting of all possible classical trajectories, that are accessible to the system under the influence of the random force. It turns out that, when dissipation is present, the random external force has a non-vanishing bracket with the system coordinates, which implies that it cannot be consistently taken to be zero. This seems to be in agreement with the fluctuation-dissipation theorem, which requires fluctuating forces, in the presence of dissipation.

By the correspondence principle, our classical brackets can be eventually quantized, upon substituting them by $(1/(i\hbar))$ commutators. In this way, we recover the same expressions for the commutators between the system coordinates and the random forces, which were derived from the independent oscillator microscopic model of Ref.[1].

In what follows, we make no attempt at mathematical rigor, and the presentation is totally heuristic. We hope to clarify elsewhere the delicate issues involved in the consideration of Poisson structures in infinite-dimensional functional spaces.

II. THE CLASSICAL BRACKETS

We consider a mechanical system, with coordinates $(q^1, \ldots, q^n)$, described by an action functional $S = \int dt \mathcal{L}(q^i, \dot{q}^i, t)$, where dot denotes a time derivative. We assume that the Lagrangian is a polynomial of second degree in the velocities $\dot{q}^i$, and that its Hessian $\partial^2 \mathcal{L}/\partial \dot{q}^i \partial \dot{q}^j$ is a constant, non-degenerate matrix $M_{ij}$. We imagine that the system is in contact with a heat bath, and we assume that the influence of the heat bath can be de-
scribed, effectively, by a mean force, characterized by a bounded memory function $\mu_{ij}(t-t')$, and a random force $F_i(t)$. The time evolution of the system is then described by the following equation of Langevin type:

$$
- \frac{\delta S}{\delta q^i(t)} + \int_{-\infty}^{t} dt' \mu_{ij}(t-t') \dot{q}^j(t') = F_i(t) .
$$

(1)

Here, $\delta S/\delta q^i(t)$ denotes the functional derivative of the action $S$:

$$
\frac{\delta S}{\delta q^i(t)} \equiv -\frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}^i} \right) + \frac{\partial L}{\partial q^i} .
$$

(2)

The original form of the Langevin Equation results from the singular limit, where $\mu_{ij}(t-t')$ approaches $\gamma_{ij}(t-t')$.

Mimicking the procedure found by Peierls, to compute the Poisson Brackets of a conservative Lagrangian system [3], one can consider the effect, on the system evolution, of a small disturbance, produced by an infinitesimal change $\delta S$ in the form of the action. We consider changes of the form $\delta S = \epsilon A$, where $\epsilon$ is an infinitesimal constant and $A$ is a local functional of the trajectory $q^i(t)$, taken from a finite time interval. The small disturbance causes an infinitesimal shift $\delta A q^j(t)$ in the trajectory $q^j(t)$, and it is easy to see that, to first order in $\epsilon$, $\delta A q^j$ satisfies the following linear integro-differential equation:

$$
(L \delta A q)_i(t) \equiv -\int dt' \frac{\delta^2 S}{\delta q^i(t) \delta q^j(t')} \delta A q^j(t') + \int_{-\infty}^{t} dt' \mu_{ij}(t-t') \delta A \dot{q}^j(t') = \epsilon \frac{\delta A}{\delta q^j(t)} ,
$$

(3)

where it is understood that all functional derivatives are evaluated along the undisturbed trajectory. When writing the above Equation, we have also assumed that the random force does not undergo any variation, to first order in $\epsilon$. We point out that, by virtue of our assumptions on the Lagrangian, the coefficients of Eq. (3) depend only on the coordinates $q^j(t)$ and the velocities $\dot{q}^j(t)$ of the undisturbed trajectory, while they are independent of the accelerations. This is reassuring, because, by virtue of the random external force, the classical trajectories possess, in general, smooth velocities, while the acceleration does not exist, in the ordinary sense of time-derivatives of the velocity $\dot{q}^j(t)$.

Since the disturbance $A$ is localized in a finite time interval, it makes sense to consider the solution $\delta A q^j(t)$ of Eq. (3) satisfying \textit{retarded} boundary conditions:

$$
\delta A q^j(t) = 0 \quad \text{at early times} .
$$

(4)

The non-degeneracy condition for the Hessian $M_{ij}$ of the Lagrangian, ensures that $\delta^+_A q^j(t)$ exists and is unique. We consider also the \textit{advanced} solution $\delta^+_A q^j(t)$:

$$
\delta^+_A q^j(t) = 0 \quad \text{at late times}
$$

(5)

of the adjoint equation of Eq. (3):

$$
(L^T \delta^+_A \dot{q})_i(t) = -\int dt' \frac{\delta^2 S}{\delta \dot{q}^i(t') \delta \dot{q}^j(t')} \delta^+_A \dot{q}^j(t') - \int_{-\infty}^{t} dt' \mu_{ji}(t'-t) \delta^+_A \dot{q}^j(t') = \epsilon \frac{\delta A}{\delta \dot{q}^j(t)} ,
$$

(6)

where the superscript $T$ stands for transpose (the transpose coincides with the adjoint, because we are in the real field). If $B$ is another functional of the trajectory, with support in a finite time interval, we define the bracket $\{ A, B \}$ as the following expression, involving the quantities $\delta^+_A q^j(t)$:

$$
\{ A, B \} := \frac{1}{\epsilon} \int dt \frac{\delta B}{\delta \dot{q}^j(t)} (\delta^+_A \dot{q}^j(t) - \delta^+_A \dot{q}^j(t)) .
$$

(7)

It is immediate to verify that the bracket is bilinear and satisfies the Leibniz rule:

$$
\{ AB, C \} = \{ A, C \} B + A \{ B, C \} ,
$$

(8)

$$
\{ A, BC \} = A \{ B, C \} + B \{ A, C \} .
$$

(9)

To verify that the bracket Eq. (7) is also antisymmetric and that it satisfies the Jacobi identity, it is useful to reexpress it in terms of the Green functions $G^{\pm ij}(t,t';q)$, defined so that:

$$
\delta^+_A q^j(t) = \epsilon \int dt' G^{\pm ij}(t,t';q) \frac{\delta A}{\delta \dot{q}^j(t')} .
$$

(10)
The Green functions $G^{\pm ij}(t,t')$ satisfy the following boundary conditions:

$$G^{-ij}(t,t';q) = 0 \quad \text{for } t \leq t'$$  \hspace{1cm} (11)
$$G^{+ij}(t,t';q) = 0 \quad \text{for } t \geq t'$$  \hspace{1cm} (12)
$$\lim_{t-t' \Rightarrow t'} \frac{\partial G^{\pm ij}(t,t';q)}{\partial t} = \mp (M^{-1})^{ij}.$$  \hspace{1cm} (13)

We define now the commutator function $\tilde{G}^{ij}(t,t';q)$:

$$\tilde{G}^{ij}(t,t';q) := G^{+ij}(t,t';q) - G^{-ij}(t,t';q).$$  \hspace{1cm} (14)

Note that, by virtue of the boundary conditions satisfied by the retarded and the advanced Green functions, $\tilde{G}^{ij}(t,t')$ and $\partial_t \tilde{G}^{ij}(t,t')$ are continuous, in the coincidence time limit, $t \rightarrow t'$. By using $\tilde{G}^{ij}(t,t')$, we can rewrite the bracket Eq. (7) as:

$$\{A,B\} = \int dt \int dt' \frac{\delta B}{\delta q^i(t)} \tilde{G}^{ij}(t,t';q) \frac{\delta A}{\delta q^j(t')}.$$  \hspace{1cm} (15)

The antisymmetry of the bracket Eq. (7) follows from the fact that the commutator function $\tilde{G}^{ij}$ is antisymmetric, as a consequence of the following reciprocity relation, satisfied by the advanced and retarded Green functions:

$$G^{+ij}(t,t';q) = G^{-ji}(t',t;q).$$  \hspace{1cm} (16)

Before turning to the proof of Eq. (16), it is useful to introduce the condensed index notation devised by DeWitt [3]. With this notation, the trajectory $q^i(t)$ is just denoted as $q^i$, with the single latin index $i$ playing the rôle of both the discrete index, and the time variable. Consequently, repeated condensed indices mean a summation on the discrete indices as well as a time integration. For example, Eq. (3), with the condensed notation, is written as:

$$L_{ij} \delta_A q^i \equiv (\delta s_{ij} + \kappa_{ij}) \delta_A q^i = \epsilon_A i,$$  \hspace{1cm} (17)

where commas denote functional differentiation, and $\kappa_{ij} \delta_A q^i$ is a symbolic notation for the integral linear operator, depending on the memory function, in Eq. (3). To prove the reciprocity relation Eq. (16), we point out that the Green functions satisfy by definition the following Equations:

$$L_{ij} G^{-jk} = \delta^i_k, \quad (L^T)_{ij} G^{+jk} = \delta^i_k.$$  \hspace{1cm} (18)

Upon multiplying the second of the above Equations by $G^{-il}$, we obtain:

$$G^{-il} (L^T)_{ij} G^{+jk} = G^{-il} \delta^i_k = G^{-kl}.$$  \hspace{1cm} (19)

However, upon using the first of Eq. (18), we can rewrite the l.h.s. of the above Equation as:

$$G^{-il} (L^T)_{ij} G^{+jk} = G^{-il} (L)_{ji} G^{+jk} = \delta_j^i G^{+jk} = G^{+lk}.$$  \hspace{1cm} (20)

Upon comparing the r.h.s. of Eq. (19) and Eq. (20), the reciprocity relation Eq. (16) follows. We can verify now the Jacobi identity. Direct evaluation of the quantity $\{\{A,B\},C\} + \{(C,A),B\} + \{(B,C),A\}$, using Eq. (16) shows that:

$$\{\{A,B\},C\} + c.p. = A_i B_{ij} C_{ik} T^{ijk},$$  \hspace{1cm} (21)

where c.p. stands for cyclic permutations of the functions $A,B,C$. The terms involving second order functional derivatives of $A, B$ and $C$ cancel exactly, by virtue of the antisymmetry of $\tilde{G}^{ij}$. In the above expression, $T^{ijk}$ denotes the following quantity, constructed out of functional derivatives of the commutator function:

$$T^{ijk} = \tilde{G}^{il} \tilde{G}^{jk}_{kl} + \tilde{G}^{ij} \tilde{G}^{ki}_{kl} + \tilde{G}^{ij} \tilde{G}^{ik}_{kl}.$$  \hspace{1cm} (22)

By using the reciprocity relation, the quantity $T^{ijk}$ can be written solely in terms of the retarded Green function $G^{-ij}$, and its functional derivatives. On the other hand, the functional derivatives $G^{-jk}_{kl}$ can be computed by functionally differentiating the first of Eqs. (18):

$$L_{ij} G^{-jk} + L_{ij} G^{-jk}_{kl} = 0.$$  \hspace{1cm} (23)

Multiplication by $G^{+mi}$ then gives:

$$G^{-mk}_{il} = -G^{+mi} L_{ij} G^{-jk} = -G^{+mi} L_{ij} G^{-jk},$$  \hspace{1cm} (24)

where in the last passage use has been made again of the reciprocity relation. By using this expression into Eq. (22), it is possible to verify that:

$$T^{ijk} = (G^{-li} G^{-mj} G^{-nk} + c.p.)(L_{mn} - L_{nm}),$$  \hspace{1cm} (25)

where c.p. stands for cyclic permutations of the indices $ijk$. It is easy to check now that $T^{ijk}$ vanishes. Indeed, in view of Eq. (17), we see that the quantity between the brackets of the r.h.s. is equal to:

$$S_{mn} - S_{nm} + \kappa_{mn} - \kappa_{nm} = 0.$$  \hspace{1cm} (26)

The terms involving third order functional derivatives of the action functional cancel each other, because functional derivatives commute with each other. On the other hand, the quantities $\kappa_{ij}$ are independent of the trajectories $q^i$, and hence their functional derivatives vanish identically. It follows then that $T^{ijk}$ vanishes, and thus the Jacobi identity holds.

Thus we have shown that it is possible to define a bracket on the space of all trajectories. We can evaluate now the brackets satisfied by the random force $F_i(t)$. To do this, we can use the expression for $F_i(t)$, provided by the Langevin Equation, Eq. (11). In this way, we find:
\{F_i, q^k\} = \{-S_{ij} + \kappa_{ij} q^i q^k\} = L_{ij}(q^i, q^k) = L_{ij}(G^{+jk} - G^{-jk}) = (L - L^T)_{ij} G^{+jk} + (L^T)_{ij} G^{+jk} - L_{ij} G^{-jk} = \\
= (S_{ij} - S_{ji} + \kappa_{ij} - \kappa_{ji}) G^{-kj} = (\kappa - \kappa^T)_{ij} G^{-kj}, \quad (27)

\&

\{F_i, F_j\} = \{-S_{ij} + \kappa_{ik} q^i, -S_{ij} + \kappa_{jl} q^l\} = L_{ik} L_{jl}(q^i, q^j) = L_{ik} L_{jl}(G^{+kl} - G^{-kl}) = \\
= L_{ik} L_{jl}(G^{-ik} - G^{-kl}) = L_{ij} - L_{ji} = S_{ij} - S_{ji} + \kappa_{ij} - \kappa_{ji} = (\kappa - \kappa^T)_{ij}. \quad (28)

It is useful to write the above bracket in plain form:

\[
\{F_i(t), F_j(t')\} = \frac{d\mu_{ij}}{dt}(t - t') + \frac{d\mu_{ji}}{dt}(t' - t).
\quad (29)
\]

We see from these Equations that, when friction is present, the external forces have non-vanishing brackets, which implies that they cannot be set to zero.

Using Eq. 28, it is possible to verify that the equal-time brackets for the coordinates \(q^i(t)\) and the momenta \(p_i(t) = \partial L / \partial \dot{q}^i(t)\) have the familiar canonical form:

\[
\{q^i(t), q^j(t)\} = 0, \quad (30)
\]

\[
\{q^i(t), p_j(t)\} = \delta^i_j, \quad (31)
\]

\[
\{p_i(t), p_j(t)\} = 0. \quad (32)
\]

The verification is similar to the conservative case 4, because the memory function contributes to \(\tilde{G}^{ij}(t, t')\) only to order \((t - t')^3\). This can be seen by inserting the expansions of \(G^{\pm ij}(t, t')\) in powers of \((t - t')\) into Eqs. 13, and exploiting the boundedness of the memory function.

III. CONCLUSIONS

We have constructed a set of brackets for a classical dissipative system, described by a Langevin Equation, with an arbitrary memory function. The brackets satisfy the usual properties enjoyed by Poisson Brackets of Hamiltonian systems. It is worth pointing out the essential rôle played, in our treatment, by external random forces. When dissipation occurs, they have non-vanishing brackets with the system coordinates, and thus cannot be consistently set to zero.

Quantization can be carried out according to the traditional procedure, by replacing the classical brackets with commutators. The resulting commutation rules coincide with those that are obtained in standard treatments of quantum dissipation, by making recourse to a microscopic model for the heat bath, after elimination of the bath degrees of freedom (see for example Ref. 1 and references therein).

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[1] G.W. Ford, J.T. Lewis and R.F. O’Connell, Phys. Rev. A 37, 4419 (1988).
[2] I.R. Senitzky, Phys. Rev. 119, 670 (1960).
[3] R. E. Peierls, Proc. Roy. Soc. (London), A 214, 143
(1952).
[4] B.S. DeWitt, *Dynamical Theory of Groups and Fields*, (Gordon & Breach, New York, 1965).
[5] D. Marolf, Ann. Phys. (N.Y.) **236**, 392 (1994).
[6] M. Dütsch and K. Fredenhagen, The Master Ward Identity and Generalized Schwinger-Dyson Equation in Classical field theory, [hep-th/0211242]
[7] G. Bimonte, G. Esposito, G. Marmo and C. Stornaiolo, Peierls Brackets in Field Theory, [hep-th/0301113] to appear in Int. J. Mod. Phys. **A**.
[8] E. Nelson, Phys. Rev. **150**, 1079 (1966) and references therein.