Robust hybrid control for sampled-data delayed systems

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Abstract. In this paper, we propose a new idea for dealing with the $H_2$ hybrid control of sampled-data system with state and input delays. Its corresponding digital model is also proposed for the delayed system. We also show that the sampled-data hybrid stabilization design problem is equivalent to discrete-time control stabilization problem. Based on the principle of equivalent areas, a digital control law is developed to deal with sampled-data hybrid stabilization problem of the input and state delayed continuous-time system.

1. Introduction
Most of the practical control systems are described by continuous-time systems mixed sensor noise and actuator error. Some well-developed control design methods are given by Doyle et al.¹, Sinha et al.², and Chen et al.³. Recently, with the rapid advances in digital technologies and computers, more and more hardware designers would like to replace analogue controllers by digital ones for better reliability, lower cost and better performance. The process of converting a continuous-time controller into an equivalent discrete-time one such that the states of the continuous-time system with discrete-time controller will closely match those of the original continuous-time system with continuous-time controller is called the “digital redesign⁴”. It also means that the corresponding digital redesign hybrid system is stable⁵,⁶, if the original closed-loop continuous-time system is stable.

The commonly encountered problem in the analysis and design of control systems is the unavoidable input and/or state delays. About hybrid control problem of sampled-data system without input and/or state delays, many literatures⁷,⁸ have been proposed to discuss it. However, the hybrid control problem for sampled-data systems with both state and input delays has not been fully solved yet. This paper is inspired by Chen et al.⁹ We consider both input and state delays into their problem model and use digital redesign method to solve it, not like them directly construct a digital controller. In other words, to solve the hybrid control problem of sampled-data system with both input and state delays via digital redesign and to prove it is feasible are this paper main researches.

2. Problem formulation
Consider a controllable continuous-time system with input and state delays given by

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\[ \dot{x}(t) = A_0 x(t) + A_1 x(t - \tau_x) + B_1 w(t) + B_2 u(t - \tau_z), \quad (1a) \]

\[ z(t) = C x(t) + D_1 w(t) + D_2 u(t), \quad (1b) \]

where \( u(t) \) is the control input, \( z(t) \) is the signal to be controlled, \( x(t) \) is the state feedback signal, \( \tau_x \) is the state time delay, \( \tau_z \) is the input time delay, \( A_0, A_1, B_1, B_2, C, \) and \( D_1, D_2 \) are the appropriate system nonsingular matrices. The controlled plant is mixed with state delay, input delay and external disturbance. It is highly challenging to construct the corresponding digital controller \( u_c(kT) \) of \( u(t) \) in \((1)\) and to show the closed-loop hybrid system stability. The schematic diagram is shown in Fig. 2.

In another viewpoint, we suppose \( u(t) = K x(t + \tau_x) \) is the predictor continuous-time controller stabilizing overall closed-loop system \((1)\). The schematic block of \((1)\) can be shown as Fig. 1. The result after Laplace Transformation of \((1a) \) and \((1b) \) is

\[ x(s) = (s I - A_0 - A_1 e^{-s \tau_x})^{-1} B_1 w(s) + (s I - A_0 - A_1 e^{-s \tau_x})^{-1} B_2 e^{-s \tau_z} u(s), \quad (2a) \]

\[ z(s) = C (s I - A_0 - A_1 e^{-s \tau_x})^{-1} B_1 w(s) + C (s I - A_0 - A_1 e^{-s \tau_x})^{-1} B_2 e^{-s \tau_z} + D_1 u(s), \quad (2b) \]

where \( I \) is identity matrix. On the basis of \((2)\), we express the controllable continuous plant \( P \) in Fig. 1 as

\[ \begin{bmatrix} x(s) \\ z(s) \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w(s) \\ u(s) \end{bmatrix}, \quad (3a) \]

where

\[ P_{11} = (s I - A_0 - A_1 e^{-s \tau_x})^{-1} B_1, \]
\[ P_{12} = (s I - A_0 - A_1 e^{-s \tau_x})^{-1} B_2 e^{-s \tau_z}, \]
\[ P_{21} = C (s I - A_0 - A_1 e^{-s \tau_x})^{-1} B_1 + D_1, \]
\[ P_{22} = C (s I - A_0 - A_1 e^{-s \tau_x})^{-1} B_2 e^{-s \tau_z} + D_1. \]

The corresponding sampled-data system with input and state delays of \((1)\) is shown as Fig. 2. We induce two operators, Sampler and Holder, into the continuous plant \( P \) to obtain the digital plant

\[ P_{sd} = \text{Sampler} \ast P \ast \text{Holder} \triangleq \Delta S \ast P \ast H \]

shown in Fig. 3. Because the actuator/quantization error or sensor noise may happen due to the finite-precision implementation of digital controller, the exogenous signals \( v_1 \) and \( v_2 \) are added to Fig. 3 to replace those mentioned noise or error. Thus, we can rewrite \((3a)\) as

\[ \begin{bmatrix} x_c(kT) \\ z(t) \end{bmatrix} = \begin{bmatrix} S P_{11} \\ S P_{12} H \end{bmatrix} \begin{bmatrix} w(t) \\ u_c(kT) \end{bmatrix} = \begin{bmatrix} \Delta P_{sd} \\ u_c(kT) \end{bmatrix}, \quad (3b) \]

where \( P_{sd} \) is the sampled-data generalized plant with a compatible operator matrix

\[ P_{sd} = \begin{bmatrix} S P_{11} \\ P_{21} S P_{22} H \end{bmatrix}. \quad (3c) \]
In short, given well-designed continuous control gain $K_c$ which can make the continuous-time system (1) stable in Fig. 1, To convert the controlled delayed plant (1a) into digital one; to construct sampled-data control gain $K_s$ in Fig. 3 and to show the $K_s$ can robustly stabilize the closed-loop hybrid system (to be defined) are this paper main tasks.

3. Digital modelling of continuous-time model with input and state delays

Rewrite the continuous-time model with input and state delays (1a) as the following form:

$$\dot{x}_c(t) = A_c x_c(t) + A_{ct} (t - \tau_r) + B_c w(t) + B_{ct} u(t - \tau_r).$$  \hfill (4)

The approximate solution of (4) for $t = kT + T$, where $T$ is the sampling period, and for the piecewise-constant input $u_i(t) = u_i(kT) \equiv u_i(t)$ with $kT \leq t < kT + T$ is

$$x_c(kT + T) = G x_c(kT) + \int_{t-kT}^{t-kT+T} e^{\lambda(t-kT-i)} A_c x_c(\lambda - \tau_r) d\lambda + w_c(kT) + \int_{t-kT}^{t-kT+T} e^{\lambda(t-kT-i)} B_c u_c(\lambda - \tau_r) d\lambda$$ \hfill (5)

where

$$G = e^{\lambda T}, \quad w_c(kT) = \int_{t-kT}^{t-kT+T} e^{\lambda(t-kT-i)} B_c w(\lambda) d\lambda.$$ \hfill (6)

Let the $\tau_r = h_1T + \tau_1$ and $\tau_r = h_2T + \tau_2$ where $h_1$ and $h_2$ are integers, $0 \leq r_1 < T$ and $0 \leq r_2 < T$, then, we can rewrite (5) as

$$x_c(kT + T) = G x_c(kT) + \int_{t-kT}^{t-kT+T} e^{\lambda(t-kT-i)} A_c x_c(\lambda - \tau_r) d\lambda + w_c(kT) + H_c u_c(kT-h_1T) + H_c u_c(kT-h_2T-T),$$ \hfill (7)

where $H_c = (G^{h_1T} - I) A_c^i B_c$, $H_1 = (G - G^{h_1T}) A_c^i B_c$ and $\sigma_1 = \frac{\tau_1}{T}$. Because the convolution integral in (6) is difficult to be evaluated exactly, the principle of equivalent areas$^{10}$ should be used to obtain an approximate one. So, the integral term in (6) can be approximately expressed as

$$\int_{t-kT}^{t-kT+T} e^{\lambda(t-kT-i)} A_c x_c(\lambda - \tau_r) d\lambda \approx \int_{t-kT}^{t-kT+T} e^{\lambda(t-kT-i)} A_c x_c(\lambda - \tau_r) d\lambda = \frac{1}{T} \int_{t-kT}^{t-kT+T} x_c(\lambda - \tau_r) d\lambda.$$ \hfill (8)

Because the digital noise of $w(t)$ in (1a) has been converted into $w_c(kT)$ in (6) in doing digital modeling from (1a) to (6), it is not necessary to consider again $w(t)$ in (1a) for approximating the integral term in (6). Hence, in order to deal with the integral term of (6), we integrate both sides of (1a) as

$$x_c(kT + T) = A_c x(kT) + A_{ct} (T - \tau_r) x_c(kT) + B_c w_c(kT) + B_{ct} u_c(kT - h_1T) + H_c u_c(kT - h_2T - T).$$ \hfill (9a)

Using the well-known trapezoidal rule$^{11}$ shown as

$$\frac{1}{T} \int_{t-kT}^{t-kT+T} x_c(\lambda) d\lambda \approx \frac{1}{2} \left[ x_c(kT) + x_c(kT + T) \right].$$ \hfill (9b)

to replace the first integral term in (8), then, the second integral term in (8) can be expressed as

$$A_c \int_{t-kT}^{t-kT+T} x_c(\lambda - \tau_r) d\lambda = (I - \frac{A_c T}{2}) x_c(kT) + (I + \frac{A_c T}{2}) x_c(kT) - (T - \tau_r) B_c u_c(kT - h_1T) - r_1 B_c u_c(kT - h_2T - T).$$ \hfill (9c)

The last right integral term in (7) can not be evaluated because the intersampled state $x_c(\lambda - \tau_r)$ in the integral term is not available. However, it can be obtained by (9b). Substituting (9b) into (7) and its result into the (6) yields the desired discrete-time model of the continuous-time model with input and state delays in (1) as

$$x_c(kT + T) = G_c x_c(kT) + F_c u_c(kT - h_1T) + F_c u_c(kT - h_2T - T) + w_c(kT).$$ \hfill (10a)
\[ z(kT) = C_{x}x_{j}(kT) + D_{x}w(kT) + D_{u}u_{j}(kT), \quad (10b) \]

where

\[
G_{i} = \left( I - \frac{1}{T}(G-I)A_{i}^{-1}\left( I - \frac{A_{i}T}{2} \right) \right)^{-1},
\]

\[
H_{\alpha} - \left( I - \frac{B_{i}}{T}(G-I)A_{i}^{-1}B_{i} \right), \quad (10c)
\]

\[
F_{i} = \left( I - \frac{1}{T}(G-I)A_{i}^{-1}\left( I - \frac{A_{i}T}{2} \right) \right)^{-1}
\]

\[
H_{i} - \left( I - \frac{B_{i}}{T}(G-I)A_{i}^{-1}B_{i} \right), \quad (10d)
\]

\[
W_{i} = \left( I - \frac{1}{T}(G-I)A_{i}^{-1}\left( I - \frac{A_{i}T}{2} \right) \right)^{-1}, \quad (10e)
\]

Whenever the smaller sampling period \( T \) in (10) is chosen, the discrete-time model (10) nearly approaches exact digital plant \( P_{d} \) in Fig. 3. Here, we regard the discrete-time model (10) as the digital plant \( P_{d} \) in Fig. 3 and use it to derive a hybrid stability theorem in next Section.

Whenever the inverse in (10) does not exist, the desired discrete-time model needs to be reconstructed using a suitably small sampling period. A bisection search is suggested to find a suitable sampling period.

4. Hybrid stability

Before finding the digital control gain \( K_{d} \) in Fig. 3 to stabilize the closed-loop hybrid delayed system, the hybrid stability needs to be defined clearly. The sampled-data control system shown in Fig. 3 is hybrid stable if the nine operators defined from \( w_{j}, v_{j}, \ldots, z_{j}(kT), u_{j}(kT) \) are all bounded. In other words, the bounded input bounded output stable criterion \(^4\) is satisfied. Here, two Hilbert spaces \( L_{2} \) and \( l_{2} \) for the continuous- and discrete-time signal space, respectively, are aided to account for the following Theorem. Thus, the exogenous input \( w \) lives in \( L_{2} \) and \( v_{j}, v_{2} \) in \( l_{2} \).

**Theorem 1:** Assume (10) is both stabilizable and detectable. If \( K_{d} \) internally stabilizes \( P_{d} \) in Fig. 3 in discrete-time, then a piecewise-constant control gain \( K_{d} \) can hybridly stabilize \( P \).

**Proof:** Let \( w=0 \). Suppose \( K_{d} \) internally stabilizes \( P_{d} \) in Fig. 3. It means that the operators \( v_{j}, v_{2} \rightarrow u_{j}(kT), x_{j}(kT) \) are bounded from \( l_{2} \) to \( l_{2} \). In order to prove that the digital control gain \( K_{d} \) can hybridly stabilize \( P \). The remained five operators \( w \rightarrow x_{j}(kT), u_{j}(kT), z(t) \) and \( v_{j}, v_{2} \rightarrow z(t) \) need to be proved that the bounded input bounded output stable criterion is satisfied.

Firstly, before going the proof procedure of bounded operators, the four preliminary norm inequalities on \( kT \leq t < kT + T \) are shown in (11).
\[
\left| e^{k(t-t_0)} \right| \leq e^{k(t-t_0)\Delta a_0},
\]
(11a)
\[
\int_T e^{k(t-t_0)} d\lambda < \left| \int_T e^{k(t-t_0)} d\lambda \right| \leq \Delta a_i.
\]
(11b)

The result of substituting (10a) with \( w_j(kT) = 0 \) into (9b) produces the following norm inequality shown as
\[
\left| \frac{A_i}{t-kT} \right| \left| x_j(\lambda-\tau_i) d\lambda \right| < \left| \frac{A_i}{t-kT} \right| \left| x_j(\lambda-\tau_i) d\lambda \right|
\]
\[
< a_i \left| x_j(kT) \right| + a_i \left| u_y(kT-h_2T) \right|
\]
\[
+ a_i \left| u_y(kT-h_2T-T) \right|
\]
(11c)

where
\[
a_i = \frac{1}{T} \left| I - \frac{A_i}{t} \right| G_i - \left( I + \frac{A_i}{2t} \right) \right|
\]
(11d)

Now, we want to prove those operators \( v_1, v_2 \rightarrow z \) are bounded. On the principle of equivalent areas and \( u_1(t) = u_2(t) = u_3(t) \) with \( kT \leq t < kT + T \), the approximate solution to \( x_i(t) \) (denoted by \( x_i(t) \)) of (1a) with \( w(t)=0 \) is given by
\[
x_i(t) = e^{k(t-t_0)} x_j(kT)
\]
\[
+ \int_T e^{k(t-t_0)} d\lambda \frac{A_i}{t-kT} \int_T x(\lambda-\tau_i) d\lambda
\]
\[
+ \int_T e^{k(t-t_0)} B u_y(\lambda-\tau_i) d\lambda.
\]
(12)

From (11), the norm inequality of (12) can be written as:
\[
\left| x_i(t) \right| \leq \left| x_i(kT) \right|
\]
\[
+ \left| e^{k(t-t_0)} \right| \left| \int_T x(\lambda-\tau_i) d\lambda \right|
\]
\[
< \beta_i \left| x_j(kT) \right| + \beta_i \left| u_y(kT-h_2T) \right|
\]
\[
+ \beta_i \left| u_y(kT-h_2T-T) \right|
\]
(13)

where \( \beta_i = a_i + a_i \), \( \beta_i = a_i + a_i \), \( \beta_i = a_i + a_i \). Based on the norm inequality shown as
\[
\beta_i \left| x_j(kT) \right| + \beta_i \left| u_y(kT-h_2T) \right|
\]
\[
> 2\beta_i \beta_i \left| x_j(kT) \right| \left| u_y(kT-h_2T) \right|
\]
(14a)
the square of (13) is given by
\[ \|x(t)\|^2 < 3\beta^2 \|x(t)\|^2 + 3\beta^2 \|u_h(t-h,T)\|^2 + 3\beta^2 \|u_h(t-h,T-T)\|^2. \] (15)

According to the \(H_2\) norm definition, \(\|A\|^2 = \|A\|^2\), (15) can be expressed as \(H_2\) norm inequality form shown as
\[ \|x(t)\|^2 < 3\beta^2 \|x(t)\|^2 + 3\beta^2 \|u_h(t-h,T)\|^2 + 3\beta^2 \|u_h(t-h,T-T)\|^2. \] (16)

So, setting \(w(t)=0\) and \(u_i(t)\equiv u_i(t) = u_i(kt)\) with \(kT \leq t < kT + T\) in (1b), it is obviously hold that the operators \(v_1, v_2 \rightarrow z\) are bounded from \(l_1\) to \(l_2\).

The next step is to prove the operators \(w \rightarrow x_i(kt), u_i(kt)\) from \(L_2 \rightarrow l_2\). By (1a) with \(w(t) \neq 0\), the approximate solution (denoted \(x_i(t)\) of (1a) can be expressed as
\[ x_i(t) = e^{A_i(kt)}x_i(kt) + \int_{kt}^{t} e^{A_i(t-s)}d\lambda A_i x_i(\lambda - \tau_i) d\lambda + \int_{kt}^{t} e^{A_i(t-s)}B_i u_i(\lambda - \tau_i) d\lambda. \] (17)

Let \(t = kT + T\), the norm inequality of (17) is shown as
\[ \|x_i(t)\|^2 < \|e^{A_iT}x_i(kt)\|^2 + \int_{kt}^{t+T} e^{A_i(t-s)}d\lambda \left[ \|A_i x_i(\lambda - \tau_i)\| \right] + \int_{kt}^{t+T} e^{A_i(t-s)}B_i u_i(\lambda - \tau_i) d\lambda + \int_{kt}^{t+T} e^{A_i(t-s)}B_i w(\lambda) d\lambda. \] (18)

The norm of last right term in (18) can be shown as
\[ \left\| \int_{kt}^{t+T} e^{A_i(t-s)}B_i w(\lambda) d\lambda \right\| = \int_{kt}^{t+T} e^{A_i(t-s)}B_i w(t+s) d\lambda \]
\[ < \int_{kt}^{t+T} \|e^{A_i(t-s)}\| \|B_i\| \|w(t+s)\| d\lambda \]
\[ < e^{A_iT} \|B_i\| \left( \int_{kt}^{t+T} \|w(t+s)\|^2 d\lambda \right)^{1/2} \]
\[ < \sqrt{T} e^{A_iT} \|B_i\| \left( \int_{kt}^{t+T} \|w(t+s)\|^2 d\lambda \right)^{1/2} \]
\[ = a_i \|w\|_2. \] (19)

where \(a_i = \sqrt{T} e^{A_iT} \|B_i\|\) and \(\|w\|_2 = \left( \int_{kt}^{t+T} \|w(t+s)\|^2 d\lambda \right)^{1/2}\). The result after substituting both (19) and (11) into (18) is
Since the digital model (10) is stabilized by \( u_j(kT) = K_jx_j(kT) \), it is apparently true that if the \( \|w\| \) in (20) is bounded then \( \|x_j(kT)\| \) and \( \|u_j(kT)\| \) are bounded. Therefore, the operators \( w \to x_j(kT), u_j(kT) \) are bounded operators from \( L_2 \) to \( L_2 \). To complete the proof, it remains to show last one operator \( w \to z \). By a similar argument used in (17), \( w \to x_j(t) \) is bounded from \( L_2 \) to \( L_2 \), and so is \( w \to z \).

From above mentioned Theorem 1, we see that finding \( K_j \) to make digital model \( P_d \) in (10) internally stable is the sufficient condition of the closed-loop hybrid stability for block diagram Fig. 3. Shieh et al. have applied digital redesign method to find digital controller for internally stabilizing continuous-time plant. Therefore, it is obviously hold that use digital redesign method to instead of directly constructing the hybrid controller which can stabilize the hybrid sampled-data control system. We will derive it in next Section in detail.

5. Construction of robust hybrid control for delayed systems [13]

Here, we use the digital redesigned of continuous-time predictor controller (1c) to instead of directly constructing digital control gain \( K_j \) in this Section. Rewrite the delayed continuous-time system (1a) as

\[
\dot{x}_j(t) = A_jx_j(t) + A_jx_j(t - \tau_j) + B_ju_j(t - \tau_j) 
\]

The well-designed predictor control law is given by (1c)

\[
u_j(t) = K_jx_j(t + \tau_j). \]

The closed-loop form of (21) becomes

\[
\begin{cases}
\dot{x}_j(t) = A_jx_j(t) + A_jx_j(t - \tau_j); & \text{for } 0 \leq t < \tau_j, \\
\dot{x}_j(t) = A_jx_j(t) + A_jx_j(t - \tau_j); & \text{for } \tau_j \leq t,
\end{cases}
\]

where \( A_j = A_j + B_jK_j \). The stability of (22) is a pure state delay model which has been touched by many articles. So, it is reasonable that the predictor control gain \( K_j \) in (21b) is supposed to stabilize (21a).

Let the state equation (21a) with a piecewise-constant control law \( u_j(t - \tau_j) \) be

\[
\dot{x}_j(t) = A_jx_j(t) + A_jx_j(t - \tau_j) + B_ju_j(t - \tau_j),
\]

where

\[
u_j(t) = u_j(kT) \quad \text{for } kT \leq t < kT + T.
\]

From (21a), the \( x_j(t) \) at \( t = kT + T + \tau_j \) can be evaluated as
\[ x_i(kT + T + \tau_z) = e^{\lambda T} x_i(kT + \tau_z) + \int_{\lambda T}^{\lambda T + \tau_z} e^{\lambda T} A_i x_i(\lambda - \tau_z) d\lambda + \int_{\lambda T}^{\lambda T + \tau_z} e^{\lambda T} B_i u_i(\lambda - \tau_z) d\lambda. \]  

(24)

On the principle of equivalent areas, the last right-hand integral input term in (24) can be written as

\[ \int_{\lambda T}^{\lambda T + \tau_z} e^{\lambda T} B_i u_i(\lambda - \tau_z) d\lambda \]

\[ = \int_{\lambda T}^{\lambda T + \tau_z} e^{\lambda T} B_i u_i(\lambda - \tau_z) d\lambda + \frac{1}{T} \int_{\lambda T}^{\lambda T + \tau_z} u_i(\lambda - \tau_z), \]

\[ \Delta H \bar{u}_i(kT), \]

where

\[ H_i = \int_{\lambda T}^{\lambda T + \tau_z} e^{\lambda T} B_i u_i(\lambda - \tau_z) d\lambda = (G - I) A_i B_i \]

(25)

So, (24) can be rewritten as

\[ x_i(kT + T + \tau_z) = e^{\lambda T} x_i(kT + \tau_z) + \int_{\lambda T}^{\lambda T + \tau_z} e^{\lambda T} A_i x_i(\lambda - \tau_z) d\lambda + H_i \bar{u}_i(kT). \]

(27)

Similarly, the solution \( x_j(t) \) in (23) at \( t = kT + T + \tau_z \) can be evaluated as

\[ x_i(kT + T + \tau_z) = e^{\lambda T} x_i(kT + \tau_z) + \int_{\lambda T}^{\lambda T + \tau_z} e^{\lambda T} A_i x_i(\lambda - \tau_z) d\lambda + H_i u_i(kT). \]

(28)

Comparing (27) with (28), if \( u_j(kT) \) in (28) is same as the \( \bar{u}_j(kT) \) in (27), then \( x_j(kT + \tau_z) = x_j(kT + \tau_z) \) and \( x_j(\lambda - \tau_z) = x_j(\lambda - \tau_z) \) obviously hold. Let \( \bar{u}_j(kT) \) in (27) be the \( u_j(kT) \) in (28), then (26b) becomes

\[ u_j(kT) = \bar{u}_j(kT) = \frac{1}{T} \int_{\lambda T}^{\lambda T + \tau_z} u_j(\lambda - \tau_z) d\lambda \]

\[ = \frac{K}{T} \int_{\lambda T}^{\lambda T + \tau_z} x_j(\lambda) d\lambda. \]

(29)

The last right-hand integral term in (29) can be approximately evaluated using the well-known trapezoidal approximation rule as
\[ u_d(kT) = \frac{K}{T} \int_{\tau_1}^{\tau_2} x_d(\lambda) d\lambda \]
\[ \approx \frac{K}{2} \left[ x_d(kT + \tau_2) + x_d(kT + T + \tau_2) \right] \]
\[ = \frac{K}{2} \left[ x_d(kT + \tau_2) + x_d(kT + T + \tau_2) \right]. \tag{30} \]

Because the intersampled states \( x_d(kT + \tau_2) \) and \( x_d(kT + T + \tau_2) \) in (30) are not available, they need to be approximately evaluated by the Newton's backward extrapolation formula as:

\[ x_d(kT + hT + \tau_2) = \left( 1 + 1.5 \frac{r_2}{T} + 0.5 \left( \frac{r_2}{T} \right)^2 \right) x_d(kT + hT) \]
\[ - \left( 2 \frac{r_2}{T} + \left( \frac{r_2}{T} \right)^2 \right) x_d(kT + hT - T) \]
\[ + \left( 0.5 \frac{r_2}{T} + 0.5 \left( \frac{r_2}{T} \right)^2 \right) x_d(kT + hT - 2T), \tag{31a} \]
\[ x_d(kT + hT + T + \tau_2) = \left( 1 + 1.5 \frac{r_2}{T} + 0.5 \left( \frac{r_2}{T} \right)^2 \right) x_d(kT + hT + T) \]
\[ - \left( 2 \frac{r_2}{T} + \left( \frac{r_2}{T} \right)^2 \right) x_d(kT + hT) \]
\[ + \left( 0.5 \frac{r_2}{T} + 0.5 \left( \frac{r_2}{T} \right)^2 \right) x_d(kT + hT - T). \tag{31b} \]

The result of substituting (31) into (30) yields the digital control law \( u_d(kT) \) in Fig. (3) as:

\[ u_d(kT) = K_{d1} x_d(kT + hT + T) + K_{d2} x_d(kT + hT) \]
\[ + K_{d3} x_d(kT + hT - T) + K_{d4} x_d(kT + hT - 2T), \tag{32a} \]

where

\[ K_{d1} = \frac{K}{2} \left( 1 + 1.5 \frac{r_2}{T} + 0.5 \left( \frac{r_2}{T} \right)^2 \right), \tag{32b} \]
\[ K_{d2} = \frac{K}{2} \left( 1 - 0.5 \frac{r_2}{T} - 0.5 \left( \frac{r_2}{T} \right)^2 \right), \tag{32c} \]
\[ K_{d3} = \frac{K}{2} \left( 1.5 \frac{r_2}{T} + 0.5 \left( \frac{r_2}{T} \right)^2 \right), \tag{32d} \]
\[ K_{d4} = \frac{K}{2} \left( 0.5 \frac{r_2}{T} + 0.5 \left( \frac{r_2}{T} \right)^2 \right). \tag{32e} \]

The \( x_d(kT + hT - 2T) \), \( x_d(kT + hT - T) \), \( x_d(kT + hT) \) and \( x_d(kT + hT + T) \) in (32a) can be obtained via iteration of (10a) as follows.
\[ x_j(kT + h_jT - 2T) = G_j^{h_j}x_j(kT) \]
\[ + \sum_{i=0}^{\frac{h_j-1}{2}} G_j^i(G_i F_i + F_j)u_j(kT - iT - 4T) \]
\[ + G_j^{h_j}F_j u_j(kT - h_jT - T) + F_i u_j(kT - 3T), \]  
\[ (33a) \]

\[ x_j(kT + h_jT - T) = G_j^{h_j}x_j(kT) \]
\[ + \sum_{i=0}^{\frac{h_j-1}{2}} G_j^i(G_i F_i + F_j)u_j(kT - iT - 3T) \]
\[ + G_j^{h_j}F_j u_j(kT - h_jT - T) + F_i u_j(kT - 2T), \]  
\[ (33b) \]

\[ x_j(kT + h_jT) = G_j^{h_j}x_j(kT) \]
\[ + \sum_{i=0}^{\frac{h_j-1}{2}} G_j^i(G_i F_i + F_j)u_j(kT - iT - 2T) \]
\[ + G_j^{h_j}F_j u_j(kT - h_jT - T) + F_i u_j(kT - T), \]  
\[ (33c) \]

\[ x_j(kT + h_jT + T) = G_j^{h_j}x_j(kT) \]
\[ + \sum_{i=0}^{\frac{h_j-1}{2}} G_j^i(G_i F_i + F_j)u_j(kT - iT - T) \]
\[ + G_j^{h_j}F_j u_j(kT - h_jT - T) + F_i u_j(kT). \]  
\[ (33d) \]

6. Illustrative Example
Consider an input-state-delayed system in (1a) with the system matrices\(^{12}\) shown as
\[ A_0 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}, \quad A_i = \begin{bmatrix} 0 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \]
\[ B_0 = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]  
\[ (34) \]

Based on (10a), the corresponding discrete-time system matrices with sampling period
\( T = 0.15 \) sec., state delayed \( \tau = 0.1 \) sec. and input delayed \( \tau_2 = 0.1 \) sec. are
\[ G_i = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F_i = \begin{bmatrix} 0.0089 \\ 0.0711 \end{bmatrix}, \quad F_2 = \begin{bmatrix} -0.0109 \\ 0.0431 \end{bmatrix}. \]  
\[ (35) \]

From (22), it is reasonable that the predictor continuous control gain \( K_\infty \) in (1c) can be
considered as the well-designed \( H_\infty \) control for state delayed system. Then, the \( K_c \) proposed by
Ge, et al.\(^{12}\) is shown as
\[ K_c = \begin{bmatrix} 0.4533 \\ 0.6517 \end{bmatrix}. \]  
\[ (36) \]
Following the proposed digital control law (32), the piecewise constant control law is shown as

\[
\begin{align*}
    u_d(kT) &= K_{d1}x_1(kT) + K_{d2}x_2(kT - T) \\
    &\quad + K_{d3}x_3(kT - 2T) + K_{d4}u_d(kT - T),
\end{align*}
\]

(37a)

where

\[
K_{d1} = \begin{bmatrix} -0.5968 & -0.8579 \end{bmatrix},
\]

(37b)

and

\[
K_{d2} = \begin{bmatrix} 0.3240 & 0.4657 \end{bmatrix},
K_{d3} = \begin{bmatrix} -0.1535 & -0.2206 \end{bmatrix},
K_{d4} = -0.0274.
\]

(37c)
7. Conclusions

Before time, it is necessary to make many assumptions for directly building $H_2$ sampled-data controller to stabilize continuous-time system with time delay, not both state and input delays, and the computing procedure is not easy to comprehend. In this paper, a new method is proposed to solve $H_2$ hybrid control of sampled-data system with state and input delays via digital redesign rather than directly construct digital one. It is also proven that the proposed idea is feasible in this paper Theorem 1.

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