The K-group of Substitutional Systems

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Abstract. In another article we associated a dynamical system to a non-properly ordered Bratteli diagram. In this article we describe how to compute the $K$-group $K_0$ of the dynamical system in terms of the Bratteli diagram. In the case of properly ordered Bratteli diagrams this description coincides with what is already known, namely the so-called dimension group of the Bratteli diagram. The new ordered group defined here is more relevant for non-properly ordered Bratteli diagrams. We use our main result to describe $K_0$ of a substitutional system.

0. Introduction

An important tool in the study of Cantor minimal dynamical systems $(X, T)$ is its $K$-theory; in particular the $K_0$–group $K_0(X, T)$, which is an ordered group, is an important invariant. After the celebrated Vershik-Herman-Putnam-Skau approach of codifying minimal Cantor dynamical systems by using the so-called ordered Bratteli diagrams, it became relevant to understand the group $K_0$ directly through diagrams. This is achieved in [HPS, Thm.5.4 and Cor.6.3] when properly ordered Bratteli diagrams are employed. Recently, we showed how to associate dynamical systems to non-properly ordered Bratteli diagrams. We generalise the above result of [HPS] by a careful modification (see 3.1) of the notion of dimension group of an ordered Bratteli diagram. In doing this we have employed the “tripling” construction that was first introduced in [EP]. The result which describes the group $K_0$ in the case of a substitutional system arising from a primitive aperiodic non-proper substitution is described in theorem 3.12. It may be remarked that a method of computing $K_0$ even in the case of non-proper substitutions is indicated in [DHS, sections 5,6,7]; it relies on showing that the substitutional dynamical system is isomorphic to another one arising from a proper substitution. The proof of [DHS, proposition 20] and [DHS, proposition 23] relies heavily on ‘return words’ and ‘derivative sequences’ (loc.cit). But this method seems to us to be quite indirect and not entirely transparent; ‘return words’ are essentially of an existential nature and hence do not afford a feasible method by which to compute effectively the dimension groups or even the Bratteli diagrams of the preferred proper substitutional system. In contrast we feel that our description in theorem 3.12 for non-proper substitutions is direct and closer in its approach and simplicity to the above cited Herman-Putnam-Skau description for proper substitutions. It eliminates the handicap of having to first work out details in the properly ordered case and then do the job of reducing to one such. Our methods have the advantage of standing up equally well for the task of computing the dimension group of the dynamical system associated to any simple non-stationary non-properly ordered Bratteli diagram (theorem 3.9).
1. Preliminaries

Since our description of the modification of dimension group in the case of non-properly ordered Bratteli diagrams depends heavily on the key constructions that were first introduced in [EP] we summarize the same for the benefit of the reader following closely the text of the first chapter of [EP]. Some of the basic definitions and concepts in the study of Cantor dynamical systems are also recalled in this section.

A topological dynamical system is a pair \((X, \varphi)\) where \(X\) is a compact metric space and \(\varphi\) is a homeomorphism in \(X\). We say that \(\varphi\) is minimal if for any \(x \in X\), the \(\varphi\)-orbit of \(x := \{\varphi^n(x) | n \in \mathbb{Z}\}\) is dense in \(X\). We say that \((X, \varphi)\) is a Cantor dynamical system if \(X\) is a Cantor set, i.e. \(X\) is totally disconnected without isolated points. \((X, \varphi)\) is a Cantor minimal dynamical system if, in addition, \(\varphi\) is minimal. Some of the basic concepts of the theory are recalled below, mostly from the more detailed sources [DHS] and [HPS].

1.1. Bratteli diagram. A Bratteli diagram is an infinite directed graph \((V, E)\), where \(V\) is the vertex set and \(E\) is the edge set. Both \(V\) and \(E\) are partitioned into non-empty disjoint finite sets

\[
V = V_0 \cup V_1 \cup V_2 \cdots \quad \text{and} \quad E = E_1 \cup E_2 \cup \cdots
\]

There are two maps \(r, s : E \rightarrow V\) the range and source maps. The following properties hold:

(i) \(V_0 = \{v_0\}\) consists of a single point, referred to as the ‘top vertex’ of the Bratteli diagram

(ii) \(r(E_n) \subseteq V_n, s(E_n) \subseteq V_{n-1}, n = 1, 2, \cdots\). Also \(s^{-1}(v) \neq \emptyset \forall v \in V\) and \(r^{-1}(v) \neq \emptyset \forall v \in V_1, V_2, \cdots\).

Maps between Bratteli diagrams are assumed to preserve gradings and intertwine the range and source maps. If \(v \in V_n\) and \(w \in V_m\), where \(m > n\), then a path from \(v\) to \(w\) is a sequence of edges \((e_{n+1}, \cdots, e_m)\) such that \(s(e_{n+1}) = v, r(e_m) = w\) and \(s(e_{j+1}) = r(e_j)\). Infinite paths from \(v_0 \in V_0\) are defined similarly. The Bratteli diagram is called simple if for any \(n = 0, 1, 2, \cdots\) there exists \(m > n\) such that every vertex of \(V_n\) can be joined to every vertex of \(V_m\) by a path.

1.2. Ordered Bratteli diagram. An ordered Bratteli diagram \((V, E, \geq)\) is a Bratteli diagram \((V, E)\) together with a linear order on \(r^{-1}(v), \forall v \in V = \{v_0\} = V_1 \cup V_2 \cup V_3 \cdots\). We say that an edge \(e \in E_n\) is a maximal edge (resp. minimal edge) if \(e\) is maximal (resp. minimal) with respect to the linear order in \(r^{-1}(r(e))\).

Given \(v \in V_n\), it is easy to see that there exists a unique path \((e_1, e_2, \cdots, e_n)\) from \(v_0\) to \(v\) such that each \(e_i\) is maximal (resp. minimal).

Note that if \(m > n\), then for any \(w \in V_m\), the set of paths starting from \(V_n\) and ending at \(w\) obtains an induced (lexicographic) linear order:

\[
(e_{n+1}, e_{n+2}, \cdots, e_m) > (f_{n+1}, f_{n+2}, \cdots, f_m)
\]

if for some \(i\) with \(n + 1 \leq i \leq m, e_i = f_j\) for \(1 < j \leq m\) and \(e_i > f_i\).

1.3. Proper order. A properly ordered Bratteli diagram is a simple ordered Bratteli diagram \((V, E, \geq)\) which possesses a unique infinite path \(x_{\text{max}} = (e_1, e_2, \cdots)\) such that each \(e_i\) is a maximal edge and a unique infinite path \(x_{\text{min}} = (f_1, f_2, \cdots)\) such that each \(f_i\) is a minimal edge.

Given a properly ordered Bratteli diagram \(B = (V, E, \geq)\) we denote by \(X_B\) its infinite path space. So

\[
X_B = \{(e_1, e_2, \cdots) | e_i \in E_i, r(e_i) = s(e_{i+1}), i = 1, 2, \cdots\}
\]

For an initial segment \((e_1, e_2, \cdots, e_n)\) we define the cylinder sets

\[
U(e_1, e_2, \cdots, e_n) = \{(f_1, f_2, \cdots) \in X_B | f_i = e_i, 1 \leq i \leq n\}.
\]
By taking cylinder sets to be a basis for open sets $X_B$ becomes a topological space. We exclude trivial cases (where $X_B$ is finite, or has isolated points). Thus, $X_B$ is a Cantor set. $X_B$ is a metric space, where for two paths $x, y$ whose initial segments to level $m$ agree but not to level $m+1, d(x,y) = 1/m + 1$.

1.4. Vershik map for a properly ordered Bratteli diagram. If $x = (e_1,e_2,\cdots, e_n,\cdots) \in X_B$ and if at least one $e_i$ is not maximal define

$$V_B(x) = y = (f_1, f_2, \cdots, f_j, e_{j+1}, e_{j+2}, \cdots) \in X_B$$

where $e_1, e_2, \cdots, e_j$ are maximal, $e_j$ is not maximal and has $f_j$ as successor in the linearly ordered set $r^{-1}(r(e_j))$ and $(f_1, f_2, \cdots, f_{j-1})$ is the minimal path from $v_0$ to $s_0(f_j)$. Extend the above $V_B$ to all of $X_B$ by setting $V_B(x_{\text{max}}) = x_{\text{min}}$. Then $(X_B, V_B)$ is a Cantor minimal dynamical system.

Next, we describe the construction of a dynamical system associated to a non-properly ordered Bratteli diagram. The Bratteli diagram need not be simple. To motivate this construction, it is perhaps worthwhile to begin by indicating how it works in the case of an ordered Bratteli diagram associated to a nested sequence of Kakutani-Rohlin partitions of a Cantor dynamical system $(X, T)$.

1.5. K-R partition. A Kakutani-Rohlin partition of the Cantor minimal system $(X, T)$ is a clopen partition $\mathcal{P}$ of the kind

$$\mathcal{P} = \{T^jZ_k \mid k \in A \text{ and } 0 \leq j < h_k\}$$

where $A$ is a finite set and $h_k$ is a positive integer. The $k^{th}$ tower $S_k$ of $\mathcal{P}$ is $\{T^jZ_k \mid 0 \leq j < h_k\}$; its floors are $T^jZ_k, (0 \leq j < h_k)$. The base of $\mathcal{P}$ is the set $Z = \bigcup_{k \in A} Z_k$.

Let $\{\mathcal{P}_n\}, (n \in \mathbb{N})$ be a sequence of Kakutani-Rohlin partitions

$$\mathcal{P}_n = \{T^jZ_{n,k} \mid k \in A_n \text{ and } 0 \leq j < h_{n,k}\},$$

with $\mathcal{P}_0 = \{X\}$ and with base $Z_n = \bigcup_{k \in A_n} Z_{n,k}$. We say that this sequence is nested if, for each $n$,

(i) $Z_{n+1} \subseteq Z_n$

(ii) $\mathcal{P}_{n+1}$ refines the partition $\mathcal{P}_n$.

For the Bratteli-Vershik system $(X_B, V_B)$ of sections 1.3-1.4, one obtains a Kakutani-Rohlin partition $\mathcal{P}_n$ for each $n$ by taking the sets in the partition to be the cylinder sets $U(e_1, e_2, \cdots, e_n)$ of section 1.3 and taking as the base of the partition the union $\bigcup U(e_1, e_2, \cdots, e_n)$ over minimal paths (i.e., each $e_i$ is a minimal edge). This is a nested sequence.

1.6. To any nested sequence $\{\mathcal{P}_n\}, (n \in \mathbb{N})$ of Kakutani-Rohlin partitions we associate an ordered Bratteli diagram $B = (V, E, \geq)$ as follows (see [DHS, section 2.3]): the $|A_n|$ towers in $\mathcal{P}_n$ are in $1-1$ correspondence with $V_n$, the set of vertices at level $n$. Let $v_{n,k} \in V_n$ correspond to the tower $S_{n,k} = \{T^jZ_{n,k} \mid 0 \leq j < h_{n,k}\}$ in $\mathcal{P}_n$. We refer to $T^jZ_{n,k}, 0 \leq j < h_{n,k}$ as floors of the tower $S_{n,k}$ and to $h_{n,k}$ as the height of the tower. We will exclude nested sequences of K-R partitions where the infimum (over $k$ for fixed $n$) of the height $h_{n,k}$ does not go to infinity with $n$.

Let us view the tower $S_{n,k}$ against the partition $\mathcal{P}_{n-1} = \{T^jZ_{n-1,k} \mid k \in A_{n-1}, \text{ and } 0 \leq j < h_{n-1,k}\}$. As the floors of $S_{n,k}$ rise from level $j = 0$ to level $j = h_{n,k} - 1$, $S_{n,k}$ will start traversing a tower $S_{n-1,i_1}$ from the bottom to the top floor, then another tower $S_{n-1,i_2}$ from the bottom to the top floor, then another tower $S_{n-1,i_3}$ likewise and so on till a final segment of $S_{n,k}$ traverses a tower $S_{n-1,i_m}$ from the bottom to the top. Note that in this final step the top floor $T^jZ_{n,k}$ for $j = h_{n,k} - 1$ of $S_{n,k}$ reaches the top floor $T^qZ_{n-1,i_m}$ for $q = h_{n-1,i_m} - 1$ of $S_{n-1,i_m}$ as a consequence of the assumption $Z_n \subseteq Z_{n-1}$ and the fact that $T^{-1}$ (union of bottom floors) = union of top floors. Bearing in mind this order in which $S_{n,k}$ traverses $S_{n-1,i_1}, S_{n-1,i_2}, \cdots, S_{n-1,i_m}$ we associate $m$ edges, ordered
as \( e_{1,k} < e_{2,k} < \cdots < e_{m,k} \) and we set the range and source maps for edges by 

\[ r(e_{j,k}) = r_{n,k} \quad \text{and} \quad s(e_{j,k}) = v_{n-1,j}. \]

Note that \( m \) depends on the index \( k \in A_n \)
and (that by convention the indexing sets \( A_n \) are disjoint). \( E_n \) is the disjoint union over \( k \in A_n \) of the edges having range in \( V_n \).

**1.7.** For \( x \in X \), we define \( x_n \in \mathcal{P}_n^Z \), \( n \in \mathbb{N} \) as follows: \( x_n = (x_{n,i})_{i \in \mathbb{Z}} \), where \( x_{n,i} \in \mathcal{P}_n \) is the unique floor in \( \mathcal{P}_n \) to which \( T'(x) \) belongs. If \( m > n \), let \( j_{m,n} : P_m \to P_n \) be the unique map defined by \( j_{m,n}(F) = F' \) if \( F \subseteq F' \). (By abuse of notation, we use the same symbol \( F \) to denote a point of the finite set \( P_m \) and also to denote the subset of \( X \), in the partition \( \mathcal{P}_n \), which \( F \) represents). An important property of the map

\[ X \to \prod_n (\mathcal{P}_n^Z), \quad x \mapsto (x_1, x_2, \cdots), x_n = (x_{n,i})_{i \in \mathbb{Z}}, \]

defined above is the following:

**1.8.** If \( F \) and \( TF \) are two successive floors of a \( \mathcal{P}_n \)-tower and if \( x_{n,i} = F \) then \( x_{n+1,i} = TF \). If \( x_{n,i} \) is the top floor of a \( \mathcal{P}_n \)-tower, then \( x_{n,i+1} \) is the bottom floor of a \( \mathcal{P}_n \)-tower. More importantly, given integers \( K \) and \( n \), there exist \( m > n \) and a single tower \( S_{m,k} \) of level \( m \) such that the finite sequence \( (x_{n,i})_{K \leq i \leq K} \) is an interval segment contained in

\[ \{j_{m,n}(T^\ell(Z_{m,k})) \mid 0 \leq \ell < h_{m,k}\}. \]

This is a consequence of the assumption that the infimum of the heights of level-

\( n \) towers goes to infinity. It is true that \( x_{n,i} = j_{m,n}(x_{m,i}) \), but the sequence \( (x_{m,i})_{K \leq i \leq K} \) need not be an interval segment of \( \{T^\ell(Z_{m,k}) \mid 0 \leq \ell < h_{m,k}\} \).

The foregoing observations in the case of an ordered Bratteli diagram associated to a nested sequence of Kakutani-Rohlin partitions gives us the hint to define a dynamical system \((X_B, T_B)\) of a non properly ordered Bratteli diagram \( B = (V, E, \geq) \) as follows:

**1.9. Definition.** For each \( n \) define \( \varpi_n = \) the set of paths from \( V_0 \) to \( V_n \). There is an obvious truncation map \( j_{m,n} : \varpi_m \to \varpi_n \) which truncates paths from \( V_0 \) to \( V_m \) to the initial segment ending in \( V_n \). For each \( v \in V_n \), the set \( \varpi(v) \) of paths from \( \{\ast\} \in V_0 \) ending at \( v \) will be called a \textit{\`\`tower parametrised by} \( v \). Each tower is a linearly ordered set (whose elements may be referred to as floors of the tower) since paths from \( v_0 \) to \( v \) acquire a linear order (cf. 1.2). \textit{We will exclude unusual examples of ordered Bratteli diagram where the infimum of the height of level-} \( n \) \textit{towers does not go to infinity, with} \( n \) (for example like [HPS, Example 3.2]). Now, we define

**1.10. Definition.** \( X_B = \{x = (x_1, x_2, \cdots, x_n, \cdots)\} \) where

(i) \( x_n = (x_{n,i})_{i \in \mathbb{Z}} \in \varpi_n \),

(ii) \( j_{m,n}(x_{m,i}) = x_{n,i} \) for \( m > n \) and \( i \in \mathbb{Z} \) and

(iii) given \( n \) and \( K \) there exists \( m \) such that \( m > n \) and a vertex \( v \in V_m \),

such that the interval segment \( x_n[-K, K] := (x_{n,K}, x_{n-K+1}, \cdots, x_{n,K}) \)

is obtained by applying \( j_{m,n} \) to an interval segment of the linearly ordered set of paths from \( v_0 \) to \( v \).

The condition (iii) is the crucial part of the definition. Without it what one gets is an inverse system.

The condition (iii) implies that a property similar to (1.8) holds. Since each \( \varpi_n \) is a finite set \( \varpi_n^Z \) has a product topology which makes it a compact set - in fact a Cantor set. Likewise, \( \prod_n (\varpi_n^Z) \) is again a Cantor set. Thus, \( X_B \subseteq \prod_n (\varpi_n^Z) \) has an induced topology. The lemma below and the following proposition are analogous to corresponding facts for the Vershik model associated to properly ordered Bratteli diagrams.

The following results (1.11) and (1.12) are proved in [EP].
1.11. Lemma. The topological space \( X_B \) is compact.

Denote by \( T_B \) the restriction of the shift operator to \( X_B \). So, if \( x = (x_1, x_2, \ldots, x_n, \ldots) \), where \( x_n = (x_{n,i})_{i \in \mathbb{Z}} \in \omega^n \), then \( T_B(x) = (x'_1, x'_2, \ldots, x'_n, \ldots) \), where \( x'_n = (x'_{n,i})_{i \in \mathbb{Z}} \in \omega^n \) and \( x'_{n,i} = x_{n,i+1} \).

\((X_B, T_B)\) will be called the dynamical system associated to \( B = (V, E, \geq) \).

1.12. Proposition. If \( B = (V, E, \geq) \) is a simple ordered Bratteli diagram, then \((X_B, T_B)\) is a Cantor minimal dynamical system.

1.13. In (1.7), given a nested sequence of Kakutani-Rohlin partitions \((X, T)\), we defined a map from \((X, T)\) to the dynamical system \((X_B, T_B)\) of the associated ordered Bratteli diagram. It follows that if \((X, T)\) is minimal, and if the Bratteli diagram of the nested sequence of K-R partitions is a simple Bratteli diagram, then \((X, T) \to (X_B, T_B)\) is onto. If the topology of \((X, T)\) is spanned by the collection of the clopen sets belonging to the K-R partitions then clearly the map \((X, T) \to (X_B, T_B)\) is injective. In particular, if the Bratteli diagram is properly ordered then the Bratteli-Vershik system is naturally isomorphic to the system given by our construction in 1.10.

1.14. Note that the same term ‘towers’ has been used to denote two different but related objects [in (1.5) and (1.9)]. For \( v \in V_n \), let \( y \) be a path from \( \{x\} \) to \( v \) in \((V, E, \geq)\). So, \( y \) is a ‘tower’ (consisting of the single element \( y \)) belonging to the \( \omega^v \)-tower \( \omega(y) \) (a finite set) parametrized by \( v \in V_n \) - all in the sense of (1.9). Here, \( \omega(v) = \) all paths from \( \{x\} \) to \( v \). Put \( F_y = \{x = (x_1, x_2, \ldots, x_n, \ldots) \in X_B \mid x_{n,0} = y\} \). \( F_y \) is a clopen set of the Cantor set \( X_B \). Put \( P_n = \{F_y \mid y \in \omega(v), v \in V_n\} \). Then, in the sense of (1.5) \( P_n \) is a K-R partition of \( X_B \) whose base is the union of \( \bigcup F_y \) (\( y \) minimal \( \in \omega(v) \), \( v \in V_n \)). Its towers \( S_v \) are parametrized by \( v \in V_n \); \( S_v = \{F_y \mid y \in \omega(v)\} \). Then for each \( v \in V_n \), \( F_y, (y \in \omega(v)) \) are the floors of the tower \( S_v \). (We encountered this K-R partition earlier in the case of the Bratteli-Vershik system at the end of 1.5.) The ordered Bratteli diagram obtained from \( \{F_y \mid y \in \omega(v), v \in V_n\} \) is \((V, E, \geq)\).

2. The Bratteli diagram \((V^Q, E^Q, \geq)\)

2.1. We will now define two nested sequences of K-R partitions of \( X \). For \( v \in V_n \), let \( y \) be a path from \( \{x\} \) to \( v \) in \((V, E, \geq)\). So, \( y \) is a ‘tower’ belonging to the \( \omega^v \)-tower \( \omega(y) \) parametrized by \( v \). Put \( F_y = \{x = (x_1, x_2, \ldots, x_n, \ldots) \in X \mid x_{n,0} = y\} \)

\[ P_n = \{F_y \mid y \in \omega(v), v \in V_n\}. \]

Then \( \{P_n\}_n \) is a nested sequence of K-R partitions of \( X \). But, the topology of \( X \) need not be spanned by the collection of clopen sets \( \{F_y\}_y \), \( (y \in \omega(v), v \in V_n, n \in \mathbb{N}) \). In contrast, the topology of \( X \) is indeed spanned by the collection of clopen sets in another nested sequence \( \{Q_n\}_n \) of K-R partitions, defined below. Let \( \omega = \omega(u), \omega' = \omega(v), \omega'' = \omega(w) \) be three \( \omega^v \)-towers and \( y \) a floor of \( \omega' \). For any \( x \in X \) and for any \( n \) if \( x_{n,i} \) is a floor of a \( \omega^n \)-tower \( \omega \), then for some \( a, b \in \mathbb{Z} \) such that \( a \leq i \leq b \), the segment \( x_{n,a:b} \) is just the sequence of floors in \( \omega \). We define

\[ F(\omega, \omega', \omega''; y) = \text{the clopen subset of } F_y \text{ consisting of the elements } x = (x_1, x_2, \ldots, x_n, \ldots) \text{ with the property that for some } a_1 < a_2 \leq 0 < a_3 < a_4 \in \mathbb{Z}, \text{ the segment } x_{n[a_1, a_2 - 1]} \text{ is the sequence of floors of } \omega, \text{ the segment } x_{n[a_2, a_3 - 1]} \text{ is the sequence of floors of } \omega' \text{ and the segment } x_{n[a_3, a_4]} \text{ is the sequence of floors of } \omega'' \text{. Some of the sets } F(\omega, \omega', \omega''; y) \text{ may be empty, but the non-empty sets } F(\omega, \omega', \omega''; y) \text{ form a K-R partition which we denote by } Q_n \text{. For fixed } \omega, \omega', \omega'' \text{ the subcollection } \{F(\omega, \omega', \omega''; y) \text{ as } y \text{ varies through the floors of } \omega' \text{, is a } Q_n \text{-tower parametrized by } [u, v, w] \}. \]

We denote this \( Q_n \)-tower by \( S(\omega, \omega', \omega'') \). The floors of the tower \( S(\omega, \omega', \omega'') \) are \( \{F(\omega, \omega', \omega''; y) \text{ as } y \text{ runs through the sequence of floors of } \omega' \}. \)
2.2. The tripling of \((V, E, \geq)\). Let \((V, E, \geq)\) be an arbitrary simple, ordered Bratteli diagram. Define \((V^Q, E^Q, \geq)\) as follows: \(V^Q_0 = \{\varepsilon\}\), a single point.

\(V^Q_n\) consists of triples \((u, v, w)\in V_n \times V_n \times V_n\) such that for some \(y\in V_m\) where \(m > n\), the level-\(m\) tower \(\varpi(y)\) passes successively through the level-\(n\) tower \(\varpi(u)\), then \(\varpi(v)\) and then \(\varpi(w)\). An edge \(e\in E^Q_n\) is a triple \((u, e, w)\) such that \(e\) is an edge of \((V, E)\) and \((u, r(e), w)\in V^Q_n\). Let

\[
\{e_1, e_2, \cdots, e_k\} \text{ all the edges in } r^{-1}(r(e)),
\{f_1, f_2, \cdots, f_l\} \text{ all the edges in } r^{-1}(u) \text{ and}
\{g_1, g_2, \cdots, g_m\} \text{ all the edges in } r^{-1}(w).
\]

The sources of \((u, e_1, w), (u, e_2, w), \cdots, (u, e_k, w)\) are defined to be \((s(f_l), s(e_1), s(e_2)), (s(e_1), s(e_2), s(e_3)), \cdots, (s(e_{k-1}), s(e_k), s(g_1))\) respectively. The range of \((u, e, w)\) is of course \((u, r(e), w)\). If \(r^{-1}(r(e))\) is ordered as \(\{e_1, e_2, \cdots, e_k\}\), we declare the ordering of \(r^{-1}(u, e, w))\) to be \(\{(u, e_1, w), (u, e_2, w), \cdots, (u, e_k, w)\}\). The ordered Bratteli diagram \((V^Q, E^Q, \geq)\) thus defined will be called the tripling of \((V, E, \geq)\).

The map \(\pi : (V^Q, E^Q, \geq) \rightarrow (V, E, \geq)\) given by \((u, v, w)\mapsto v, (u, e, w)\mapsto e\) enjoys the ‘unique path lifting’ property in the following sense. If \(m > n \geq 1\), and \((e_n, e_{n+1}, \cdots, e_m)\) is a path in \((V, E)\) from \(V_{n-1}\) to \(V_m\) with \(r(e_m) = v\) then for any \((u, v, w)\in V^Q_m\), there is a unique path \((e_n, e_{n+1}, \cdots, e_m)\) in \((V^Q, E^Q)\) which maps onto \((e_n, e_{n+1}, \cdots, e_m)\) under \(\pi\) and such that \(r(e_m) = (u, v, w)\). It is quite elementary to check that the map \(\pi : (V^Q, E^Q, \geq) \rightarrow (V, E, \geq)\) induces an isomorphism between the corresponding dynamical systems given by 1.10 ([EP, 2.20]).

Two different edges on the left with the same source may map into the same edge on the right. Two different edges on the left with the same range cannot map to the same edge on the right.

Let \(\{n_k\}_{k=0}^{\infty}\) be a subsequence of \(\{0, 1, 2, \cdots\}\) where we assume \(n_0 = 0\). A Bratteli diagram \((V', E')\) is called a ‘telescoping’ of \((V, E)\) if \(V'_k = V_n\) and \(E'_k\) consists of paths \((e_{n_{k-1}+1}, \cdots, e_{n_k})\) from \(V_{n_{k-1}}\) to \(V_{n_k}\) in \((V, E)\), the range and source maps being the obvious ones. It is easy to see that tripling is compatible with telescoping.

2.3. Stationary Bratteli diagrams. A Bratteli diagram is stationary if the diagram repeats itself after level 1. (One may relax by allowing a period from some level onwards; but, a telescoping will be stationary in the above restricted sense.) If \((V, E, \geq)\) is an ordered Bratteli diagram and the diagram together with the order repeats itself after level 1, then \((V, E, \geq)\) will be called a stationary ordered Bratteli diagram. We refer the reader to [DHS, section (3.3)] for the usual definition of a substitutional system and how they give rise to stationary Bratteli diagrams. Some details are recalled below. Let \((V, E, \geq)\) be as above and suppose moreover that it is a simple Bratteli diagram. We have

1. an enumeration \(\{v_{n,1}, v_{n,2}, \cdots, v_{n,L}\}\) of \(V_n, \forall n \geq 1\),
2. for \(n > 1\) and \(1 \leq j \leq L\) an enumeration \(\{e_{n,j,1}, e_{n,j,2}, \cdots, e_{n,j,a_j}\}\) of \(r^{-1}(e_{n,j})\) which is assumed to be listed in the linear order in \(r^{-1}(v_{n,j})\),
3. in the enumerations above, \(L\) does not depend on \(n\) and \(a_j\) depends only on \(j\) and not on \(n\). Moreover, the ordering in \(r^{-1}(e_{n,j})\) is stationary, i.e., if \(n, m > 1\), if \(1 \leq j \leq L, 1 \leq k \leq L, 1 \leq i \leq a_j\), then \(s(e_{n,j,i}) = v_{m-1,k}\)

\[\Rightarrow \; s(e_{m,j,i}) = v_{m-1,k}.\]

2.4. Substitutional systems. Let \(A\) be an alphabet set. Write \(A^+\) for the set of words of finite length in the alphabets of \(A\). Let \(\sigma : A \rightarrow A^+\) be a primitive aperiodic non-proper substitution, written, \(\sigma(a) = \alpha \beta \gamma \cdots\). The stationary ordered Bratteli diagram \(B = (V, E, \geq)\) associated to \((A, \sigma)\) (cf. [DHS, section 3.3]) can be
described as
\[ V_n = A, \forall n \geq 1, V_0 = \{\ast\} \]
\[ E_n = \{(a, k, b) \mid a, b \in A, k \in \mathbb{N}, a \text{ is the } k^{th} \text{ alphabet in the word } \sigma(b)\}. \]

(The reader who prefers a more carefully evolved notation can consider introducing an extra factor ‘×{n}’ so that vertices and edges at different levels are seen to be disjoint). The source and range maps \(s \) and \(r\) are defined by \(s(a, k, b) = a, r(a, k, b) = b\).

In the linear order in \(r^{-1}(b), (a, k, b)\) is the \(k^{th}\) edge.

To the stationary ordered Bratteli diagram \(B\) of \((A, \sigma)\) (which may not be properly ordered unless \(\sigma\) is a primitive, aperiodic, proper substitution, – see [DHS, section 3]) we can associate a dynamical system \(X_B\) following the construction of 1.10; this is naturally isomorphic to the substitutional dynamical system \((X_\sigma, T_\sigma)\) associated to \((A, \sigma)\) defined for example in [DHS, section 3.3.1]. (See [EP, section 2.5].)

2.5. Tripling for a substitutional system \((A, \sigma)\). Let \((A, \sigma)\) be a substitutional system and suppose \(B = (V, E, \geq)\) is the stationary ordered Bratteli diagram associated to \((A, \sigma)\). Define \(A^Q = \{(a, b, c) \in A \times A \times A \mid abc \text{ occurs as a subword of } \sigma^n(d) \text{ for some } d \in A \text{ and some } n\}\). Define
\[ \sigma^Q : A^Q \to (A^Q)^+ \]
by \(\sigma^Q(a, b, c) = (a_m, b_1, b_2) \cdot (b_1, b_2, b_3) \cdots (b_{n-2}, b_{n-1}, b_n) \cdot (b_{n-1}, b_n, c_1)\), where \(\sigma(b) = b_1 \cdot b_2 \cdots b_n\) and \(a_m\) is the last alphabet in \(\sigma(a)\), while \(c_1\) is the first alphabet in \(\sigma(c)\). Then \((V^Q, E^Q, \geq)\) is the stationary ordered Bratteli diagram associated to \((A^Q, \sigma^Q)\).

3. The groups \(K^0(X, T), K_{-0}(V, E, \geq)\) and \(K_0(V, E)\).

3.1. Definition. Let \((X, T)\) be a Cantor minimal system. Let \(C(X, Z)\) be the space of integer valued continuous functions on \(X\). Let
\[ K^0(X, T) = C(X, Z)/\partial_T C(X, Z) \]
where \(\partial_T : C(X, Z) \to C(X, Z)\) denotes the coboundary operator \(\partial_T(f) = f - f \circ T\).

A function of the form \(f - f \circ T\) is called a coboundary. Define the positive cone
\[ K^0(X, T)^+ = \{[f] \mid f \in C(X, Z)^+\} \]
where \([f]\) denotes the projection modulo coboundaries. The ordered group \((K^0(X, T), K^0(X, T)^+)\) has a distinguished order unit, namely \([1]\), the projection of the constant function 1.

Let \((V, E)\) be a Bratteli diagram and \((V, E, \geq)\) the same thing equipped with a linear order on edges which makes it an ordered Bratteli diagram. As usual the dimension group \(K_0(V, E)\) is defined to be the inductive limit of the system of ordered groups
\[ Z^{V_0} \xrightarrow{A_0} Z^{V_1} \xrightarrow{A_1} Z^{V_2} \xrightarrow{A_2} Z^{V_3} \xrightarrow{A_3} \ldots \]
where the positive homomorphism \(A_n\) is given by matrix multiplication with the incidence matrix between levels \(n - 1\) and \(n\). The inductive limit \(K_0(V, E)\) is endowed with the induced order, the positive cone being denoted by \(K_0(V, E)^+\).

The image of \(1 \in Z^{V_0}\) in \((K_0(V, E), K_0(V, E)^+)\) is an order unit.

On the other hand we define the group \(K_{-0}(V, E, \geq)\) in the following way. Whenever we have \(m \geq n\) and two paths \(\varpi_1\) and \(\varpi_2, (\varpi_1 \leq \varpi_2)\) from \(V_n\) to \(V_m\) with the same range \(u \in V_m\) define \([\varpi_1, \varpi_2]\) to be the set consisting of all paths from \(V_n\) to \(V_m\) lying between \(\varpi_1\) (included) and \(\varpi_2\) (excluded) ranging at \(u\). Put \(BZ^{V_0} = \{[\varpi]\} \subseteq Z^{V_0} \mid \Sigma_{u \in V_{m+1}} m_u = 0\) for all \(m, n\) and all \(\varpi_1, \varpi_2\) as above with the same source and same range. Observe that \(A_n(BZ^{V_0}) \subseteq BZ^{V_{n+1}}\). Moreover, suppose \(\overline{\varpi}, \overline{\varphi} \in Z^{V_0,+} \), \(\overline{\varpi} = BZ^{V_0}\) and \(\overline{\varphi} = -\overline{\varphi} = \overline{\varpi} + \overline{\varphi}\). Then,
\[\overline{m} = \overline{n} + \overline{q} \in \mathbb{Z}^{[V_\alpha]}\mathbb{Z},\] which forces \(\overline{m}\) to be zero because of the defining conditions of \(B\mathbb{Z}^{[V_\alpha]}\). Thus, the natural order in \(\mathbb{Z}^{[V_\alpha]}\) induces an order in the quotient group \(\mathbb{Z}^{[V_\alpha]}/B\mathbb{Z}^{[V_\alpha]}\) making it an ordered group. Define \((K_{-0}(V, E, \geq), K_{-0}(V, E, \geq)^+\) to be the inductive limit of the system of ordered groups

\[
\mathbb{Z}^{[V_0]} \xrightarrow{A_0} \mathbb{Z}^{[V_1]} \xrightarrow{A_1} \mathbb{Z}^{[V_2]} \xrightarrow{A_2} \mathbb{Z}^{[V_3]} \xrightarrow{A_3} \cdots
\]

Observe that \(B\mathbb{Z}^{[V_0]} = 0\). The image of 1 in \(\mathbb{Z}^{[V_0]}\) in \((K_{-0}(V, E, \geq), K_{-0}(V, E, \geq)^+)\) is an isomorphism.

### 3.2. Theorem. For \(B = (V, E, \geq)\) let \((X_B, T_B)\) be defined as in (1.10). Write \((X, T) = (X_B, T_B)\). Define the tripling \(B^Q = (V^Q, E^Q, \geq)\) as in 2.2. Then \(K^Q(X, T)\) is naturally order isomorphic to \(K_{-0}(V^Q, E^Q, \geq)\), preserving distinguished order units.

**Proof.** We recall the notation introduced in 2.1. Given \(f \in C(X, \mathbb{Z})\), choose \(n\) sufficiently large such that \(f, \partial_f(f)\) are both constant on the sets of the partition \(Q_n\). The vertices of the Bratteli diagram \((V^Q, E^Q, \geq)\) correspond to towers \(S_{(\varphi', \varphi'', \varphi')}\) of a K-R partition which in turn are partitioned into floors \(\varphi(\varphi', \varphi'', \varphi')\) as \(y\) varies through the floors of \(\varphi'\). For \(h\) as above, define \(\gamma_n(h) \in \mathbb{Z}^{[V_{n+1}]}\) by \(\gamma_n(h)(\varphi', \varphi'', \varphi') = f(x) + f(Tx) + f(T^2x) + \cdots + f(T^{h-1}x)\), where \(x\) belongs to the lowest floor of \(S_{(\varphi', \varphi'', \varphi')}\) and \(h\) is the height of the tower \(S_{(\varphi', \varphi'', \varphi')}\). Then \(A_n^\circ(\gamma_n(f)) = \gamma_{n+1}(f)\) and \(\gamma_n(\partial_f(f)) \in B\mathbb{Z}^{[V_{n+1}]}\). This gives rise to a map

\[\gamma : K^Q(X_B, T_B) \longrightarrow K_{-0}(V^Q, E^Q, \geq)\].

### 3.3. Lemma. Let \(f \in C(X, \mathbb{Z})\) and suppose that \(f\) is constant on the sets of the partition \(Q_n\). Suppose that \(\gamma_n(f) \in B\mathbb{Z}^{[V_{n+1}]}\). Then, \(f = \partial_f(g)\), for some \(g \in C(X, \mathbb{Z})\).

### 3.4. Lemma. With the same assumptions as in 3.3 suppose that \(x, y \in X\) both lie in the same floor of a \(Q_n\)-tower \(S_{(\varphi', \varphi', \varphi')}\). Furthermore, suppose that for some positive integers \(k, \ell\) both \(T^kx\) and \(T^\ell y\) lie in the same floor of a \(Q_n\)-tower \(S_{(\varphi', \varphi', \varphi')}\). Then,

\[f(x) + f(Tx) + \cdots + f(T^kx) = f(y) + f(Ty) + \cdots + f(T^\ell y)\].

**Proof of 3.4.** Let \(m \geq n\). Let \(U_y\) be a neighborhood of \(y\) such that \(\forall z \in U_y\) and for \(i \in [0, \ell]\), \(T^i y\) belong to the same floor of the \(Q_n\) partition. Since the orbit of \(T^kx\) by iterations of \(T\) is dense \(\exists j\) such that \(T^{j+k}x \in U_y\). For sufficiently large \(m \geq n, \exists a Q_m\)-tower \(S\) such that \(x, T^kx, T^{k+j}x, T^{k+j+\ell}x\) belong to different floors of \(S\).

Let \(u\) be the vertex of \(V_m^Q\) represented by \(S\). The floors of the \(Q_m\)-tower \(S\) are linearly ordered reflecting the linear order in the set of paths in \((V^Q, E^Q, \geq)\) from the top vertex to \(u\). Similarly, the paths from \(V_n^Q\) to \(u \in V_n^Q\) are linearly ordered reflecting the order in which \(S\) traverses the level-\(n\) towers of \((V^Q, E^Q, \geq)\). Denote by \(\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_L\) the paths from \(V_n^Q\) to \(u \in V_m^Q\) in their linear order. Write \(s(\varphi_1), s(\varphi_2), s(\varphi_3), \ldots, s(\varphi_L) \in V_n^Q\) for their sources and \(S_{s(\varphi_1)}, S_{s(\varphi_2)}, S_{s(\varphi_3)}, \ldots, S_{s(\varphi_L)}\) for the \(Q_n\)-towers represented by these sources. Thus, \(S\) traverses \(Q_n\)-towers in the order \(S_{s(\varphi_1)}, S_{s(\varphi_2)}, S_{s(\varphi_3)}, \ldots, S_{s(\varphi_L)}\). Choose \(1 \leq a < b < c < d \leq L\) such that \(x, (\text{resp.} T^kx, \text{resp.} T^{k+j}x, \text{resp.} T^{k+j+\ell}x)\), is picked up by \(S\) at the \(a^{th}\) (resp. \(b^{th}\), resp. \(c^{th}\), resp. \(d^{th}\)) instance of \(S\) traversing through a \(Q_n\)-tower, namely, \(S_{s(\varphi_a)}, S_{s(\varphi_b)}, S_{s(\varphi_c)}, S_{s(\varphi_d)}\). In particular, observe that \(s(\varphi_a) = s(\varphi_c)\) and \(s(\varphi_b) = s(\varphi_d)\).
Since $\gamma_n(f) \in BZ^{V_n^Q}$, we have

$$\gamma_n(f)(s(\varpi_a)) + \gamma_n(f)(s(\varpi_{a+1})) + \cdots + \gamma_n(f)(s(\varpi_c)) = 0$$

and similarly,

$$\gamma_n(f)(s(\varpi_b)) + \gamma_n(f)(s(\varpi_{b+1})) + \cdots + \gamma_n(f)(s(\varpi_d)) = 0.$$ 

These two equations imply that

$$f(x) + f(Tx) + \cdots + f(T^{k+j-1}x) = 0$$

and

$$f(T^{k+1}x) + f(T^{k+2}x) + \cdots + f(T^{k+j}x) = 0.$$ 

Hence,

$$f(x) + f(Tx) + \cdots + f(T^kx) = f(y) + f(Ty) + \cdots + f(T^\ell y).$$

This ends the proof of 3.4. \qed

**Proof of 3.3.** Choose $x_0 \in X$. Now, for any $z \in X$ choose $k \in \mathbb{Z}^+$ such that $T^kx_0$ and $z$ belong to the same $Q_n$-floor. Define $g \in C(X, \mathbb{Z})$ by $g(z) = f(x_0) + f(Tx_0) + \cdots + f(T^kx_0)$. Then, 3.4 implies that $g$ is well defined and $\partial_T(g) = -f \circ T$. So, $\partial_T(-g \circ T^{-1}) = f$.

From 3.3 one can immediately deduce that the map $\gamma : K^0(X_B, T_B) \to K_0(V^Q, E^Q, \geq)$ defined just before the statement of lemma 3.3 is an isomorphism.

This completes the proof of Theorem 3.2. \qed

**3.5. A subgroup of $BZ^{V_n^Q}$.** In practice it is quite tedious to determine whether a given element $\overline{v}$ of $Z^{V_n^Q}$ lies in $BZ^{V_n^Q}$. We now begin to describe a subgroup $\Delta Z^{V_n^Q} \subseteq BZ^{V_n^Q}$, which is more easily identifiable than $BZ^{V_n^Q}$. Eventhough, in general, this inclusion is proper we will later see that the distinction disappears when one takes inductive limits. As a consequence, we are able to obtain theorem 3.9, which yields a feasible method to compute $K_0$ effectively. Clearly, $Z^{V_n^Q}$ is the space of integral valued functions on the set $V_n^Q$. For a function $\varphi : V_n \times V_n \to \mathbb{Z}$ define $\delta(\varphi) \in Z^{V_n^Q}$ by $\delta(\varphi)(a, b, c) = \varphi(b, c) - \varphi(a, b)$.

**Lemma 3.6.** $\delta(\varphi) \in BZ^{V_n^Q}$.

**Proof.** Let $\overline{v} \in Z^{V_n^Q}$. Write $\overline{v} = \{p(u, v, w)\}_{(u, v, w) \in V_n^Q}$. Take two paths from $V_n^m$ to $V_m^Q(m > n)$ with the same source in $V_n^{Q}$ and same range in $V_m^{Q}$. The sequence of sources of paths lying between the above two paths is of the form \{(u_1, v_1, w_1), (u_2, v_2, w_2), \ldots, (u_j, v_j, w_j)\} where

(i) \(u_1, v_1, w_1) = (u_j, v_j, w_j)\),

(ii) \(u_{i+1} = v_i\) and

(iii) \(v_{i+1} = w_i\), for \(i = 1, 2, \ldots, j - 1\).

If $\overline{p} = \delta(\varphi)$, the sum $p(u_1, v_1, w_1) + p(u_2, v_2, w_2) + \cdots + p(u_{j-1}, v_{j-1}, w_{j-1})$ equals

$$\{\varphi(v_1, w_1) - \varphi(u_1, v_1)\} + \{\varphi(v_2, w_2) - \varphi(u_2, v_2)\} + \cdots + \{\varphi(v_{j-1}, w_{j-1}) - \varphi(u_{j-1}, v_{j-1})\}$$

$$= -\varphi(u_1, v_1) + \varphi(v_1, w_1)$$

(in view of (ii) and (iii) above)

$$= -\varphi(u_1, v_1) + \varphi(u_1, v_1)$$

(in view of (ii) and (iii) above)

$$= 0$$

(in view of (i) above).

The condition for $\overline{p}$ to belong to $BZ^{V_n^Q}$ is precisely that the sums of the type $p(u_1, v_1, w_1) + p(u_2, v_2, w_2) + \cdots + p(u_{j-1}, v_{j-1}, w_{j-1})$ as above should all vanish. As the foregoing calculation shows this holds whenever $\overline{p} = \delta(\varphi)$ for some $\varphi : V_n \times V_n \to \mathbb{Z}$. \qed
We define $\Delta Z^{[V_n^\mathcal{Q}]}$ to be the subgroup $\delta(Z^{[V_n \times V_n]})$ of $Z^{[V_n^\mathcal{Q}]}$.

Recall the map $A_n^\mathcal{Q}: Z^{[V_n^\mathcal{Q}]} \to Z^{[V_{n+1}^\mathcal{Q}]}$ given by matrix multiplication by the incidence matrix between the levels $V_n^\mathcal{Q}$ and $V_{n+1}^\mathcal{Q}$. For a function $\phi: V_n \times V_n \to Z$ define $\phi': V_{n+1} \times V_{n+1} \to Z$ by $\phi'(u, v) = \phi(u, v)$ where $u$ (resp. $v$) is the source of the last (resp. first) edge ranging at $u$ (resp. $v$).

**Lemma 3.7.** With notation as above, $A_n^\mathcal{Q}(\delta(\phi)) = \delta(\phi')$. In particular the identity endomorphism of $Z^{[V_n^\mathcal{Q}]}$ induces a map

$$\frac{Z^{[V_n^\mathcal{Q}]} \Delta Z^{[V_n^\mathcal{Q}]}}{\Delta Z^{[V_n^\mathcal{Q}]}} \xrightarrow{\rho_n} \frac{Z^{[V_n^\mathcal{Q}]} \Delta Z^{[V_n^\mathcal{Q}]}}{\Delta Z^{[V_{n+1}^\mathcal{Q}]}}$$

and

$$\frac{Z^{[V_n^\mathcal{Q}]}}{\Delta Z^{[V_n^\mathcal{Q}]}} \xrightarrow{A_n^\mathcal{Q}} \frac{Z^{[V_{n+1}^\mathcal{Q}]}}{\Delta Z^{[V_{n+1}^\mathcal{Q}]}}$$

is commutative.

**Proof.** The proof of the assertion $A_n^\mathcal{Q}(\delta(\phi)) = \delta(\phi')$ is a straightforward calculation using definitions and notation. The rest follows immediately.

We might wish to ask whether every element $\overline{p}$ of $BZ^{[V_n^\mathcal{Q}]}$ is of the form $\delta(\phi)$ for some function $\phi: V_n \times V_n \to Z$. The proposition 3.8 below shows that after applying a finite iteration $A_{n+1}^\mathcal{Q} \cdots A_{n+1}^\mathcal{Q} \circ A_n^\mathcal{Q}$ to $\overline{p}$ it will indeed be so.

Let $\overline{p} \in BZ^{[V_n^\mathcal{Q}]}$. Let $g \in C(X, Z)$ be chosen as in Lemma 3.3 so that

1. $\partial_T(g)$ is constant on the sets of the partition $\mathcal{Q}_n$ and moreover, $\overline{p} = \gamma_n(\partial_T(g))$.

2. $g$ itself is constant on the sets of the partition $\mathcal{Q}_{n+i}$ for some positive integer $i$.

Choose a positive integer $j$ such that any $\mathcal{Q}_{n+i+j}$ -tower traverses through at least two $\mathcal{Q}_{n+i}$ -towers. Then, of course, any $\mathcal{P}_{n+i+j}$ -tower traverses through at least two $\mathcal{P}_{n+i}$ -towers.

For $u', \ v' \in V_{n+i+j}$ let $u_n$ (resp. $u_{n-1}$, resp. $v_1$, resp. $v_2$) be the source of the last (resp. last but one, resp. first, resp. second) path from $V_{n+i}$ to $V_{n+i+j}$ ranging at $u'$ (resp. $u_{n-1}$, resp. $v_1$, resp. $v_2$). If there exists $x \in X$ such that

1. $x \in$ bottom floor of the $\mathcal{Q}_{n+i}$ -tower represented by $(u_n, v_1, v_2)$,

2. $T^{-1}x \in$ top floor of the $\mathcal{Q}_{n+i+j}$ -tower represented by $(u_{n-1}, u_n, v_1)$

define $\varphi'(u', v') = g(x)$; then, $\varphi'(u', v')$ is independent of $x$. For given $u', v' \in V_{n+i+j}$ if no such $x$ exists, define $\varphi'(u', v')$ arbitrarily.

**Proposition 3.8.** With notation as above $A_{n+i+j-1}^\mathcal{Q} \cdots A_{n+1}^\mathcal{Q} \circ A_n^\mathcal{Q}(\overline{p}) = \delta(\varphi')$.

**Proof.** $\delta(\varphi')(u', v', w') = \varphi'(u', w') - \varphi'(u', v') = g(T^k y) - g(y)$, if $y$ lies in the lowest floor of the $\mathcal{Q}_{n+i+j}$ -tower of height $h$ represented by $(u', v', w')$. Also, for $(u, v, w) \in V_n^\mathcal{Q}$, $\overline{p}(u, v, w) = \gamma_n(\partial_T(g))(u, v, w) = (g \circ T - g)(T^2 z) + \cdots + (g \circ T - g)(T^{k-1} z)$, where $k$ is the height of the $\mathcal{Q}_n$ -tower represented by $(u, v, w)$ and $z$ lies in its lowest floor. Thus, $A_{n+i+j-1}^\mathcal{Q} \cdots A_{n+1}^\mathcal{Q} \circ A_n^\mathcal{Q}(\overline{p})(u', v', w')$ is the sum of $g \circ T - g$ taken over all the floors of the $\mathcal{Q}_{n+i+j}$ -tower represented by $(u', v', w')$. This sum also equals $g \circ T^k(y) - g(y)$. 


We can therefore give an alternate description of $K^0(X_B, T_B)$ which is more
elegant than the description in Theorem 3.2. For the same reasons as in the case
of $\frac{\mathbb{Z}^{|V_n^Q|}}{\mathbb{B}Z^{|V_n^Q|}}$, we see that the natural order in $\frac{\mathbb{Z}^{|V_n^Q|}}{\Delta Z^{|V_n^Q|}}$
induces an order in $\frac{\mathbb{Z}^{|V_n^Q|}}{\Delta Z^{|V_n^Q|}}$
As we already observed, $A_n^Q[\delta(Z_{n_0, n_1})] \subseteq \delta(Z_{n_0, n_1})$.

3.9. Theorem. For $B = (V, E, \geq)$ let $(X_B, T_B)$ be defined as in (1.10). Write $(X, T) = (X_B, T_B)$. Define the tripling $B^Q = (V^Q, E^Q, \geq)$ as in 2.2. Then the
map induced between the inductive limits of the two (horizontal) systems of ordered
groups in the following diagram is an isomorphism.

\[
\begin{array}{ccccccccc}
\frac{\mathbb{Z}^{|V_n^Q|}}{\mathbb{B}Z^{|V_n^Q|}} & \overset{A_n^Q}{\longrightarrow} & \frac{\mathbb{Z}^{|V_n^Q|}}{\mathbb{B}Z^{|V_n^Q|}} & \overset{A_n^Q}{\longrightarrow} & \frac{\mathbb{Z}^{|V_n^Q|}}{\mathbb{B}Z^{|V_n^Q|}} & \overset{A_n^Q}{\longrightarrow} & \frac{\mathbb{Z}^{|V_n^Q|}}{\mathbb{B}Z^{|V_n^Q|}} & \overset{A_n^Q}{\longrightarrow} & \cdots \\
\rho_0 \uparrow & & \rho_1 \uparrow & & \rho_2 \uparrow & & \rho_3 \uparrow \\
\frac{\mathbb{Z}^{|V_n^Q|}}{\Delta Z^{|V_n^Q|}} & \overset{A_n^Q}{\longrightarrow} & \frac{\mathbb{Z}^{|V_n^Q|}}{\Delta Z^{|V_n^Q|}} & \overset{A_n^Q}{\longrightarrow} & \frac{\mathbb{Z}^{|V_n^Q|}}{\Delta Z^{|V_n^Q|}} & \overset{A_n^Q}{\longrightarrow} & \frac{\mathbb{Z}^{|V_n^Q|}}{\Delta Z^{|V_n^Q|}} & \overset{A_n^Q}{\longrightarrow} & \cdots \\
\end{array}
\]

Furthermore, the two inductive limits are both isomorphic to $K^0(X, T)$.

[To avoid messy notation and display, we have hidden $G^+$ while referring to
the ordered group $(G, G^+)$]

Proof. By lemma 3.6, $\Delta Z^{|V_n^Q|} \subseteq \mathbb{B}Z^{|V_n^Q|}$. By lemma 3.8, for sufficiently large
$K$, $A_{n+K-1}^Q \circ \cdots \circ A_{n+1}^Q \circ A_n^Q (\mathbb{B}Z^{|V_n^Q|}) \subseteq \Delta Z^{|V_n^Q|}$. Hence, the induced map bet-
ween the inductive limits is an isomorphism. That $K^0(X, T)$ is isomorphic to
the inductive limit of the top horizontal was already proved in theorem 3.2. Observe
that $\mathbb{Z}^{|V_n^Q|} = \mathbb{Z}$ and $\mathbb{B}Z^{|V_n^Q|} = \Delta Z^{|V_n^Q|} = 0$. The image of $1 \in \mathbb{Z}^{|V_n^Q|}$ in the inductive
limit maps to the order unit $u$ in $K^0(X, T)$ corresponding to the image of the
constant function $1 \in C(X, \mathbb{Z})$.

\[\blacksquare\]

Remark. Since it is known that $K^0(X, T)$ is isomorphic to the $K_0$-group of the
associated $C^*$-crossed product $C(X) \rtimes_T \mathbb{Z}$, we see that as a corollary to theore-
m 3.9, we can effectively compute $K_0(C(X_B) \rtimes_{T_B} \mathbb{Z})$.

Finally, we should point out how these descriptions simplify further for properly
ordered Bratteli diagrams and yield the isomorphism $K^0(X, T) \simeq K_0(V, E)$, (see
3.1), proved by Hermann, Putnam and Skau [HPS, Theorem 5.4 and Corollary 6.3].

3.10. Let $(V, E, \geq)$ be a properly ordered Bratteli diagram. Telescoping if nec-
essary, assume that every level $n + 1$-tower traverses through at least two level
$n$-towers. Telescoping further if necessary, (see [HPS, Proposition 2.8]), we can
assume that any two maximal edges of $E_n$ have the same source. Similarly, we can
assume that any two minimal edges of $E_n$ have the same source. For the rest of
the paper we assume that these properties hold. Then for any $Q_{n+2}$-tower $S(u, v, w)$
the first $Q_n$-tower traversed by $S(u, v, w)$ is independent of $(u, v, w) \in V_{n+2}$. Thus
one sees from the definition of $\mathbb{B}Z^{|V_n^Q|}$ that $A_{n+1}^Q \circ A_n^Q (\mathbb{B}Z^{|V_n^Q|}) = 0$. As a conse-
quence, the map induced between the inductive limits of the top two horizontals in
the following diagram is an isomorphism.
The map \( \pi^* : \mathbb{Z}[V_n] \to \mathbb{Z}[V_n] \) is induced by the map \( \pi : V_n^Q \to V_n \) given by \((u,v,w) \mapsto v\). Thus the map induced between the inductive limits of the two bottom horizontals still denoted by \( \pi \) (resp. \( \beta \)). Define \( \delta(\varphi(a,b,c)) = \varphi(b,c) - \varphi(a,b) \). Let \( \mathbf{\Delta}Z^{|A^Q|} \) be the subgroup \( \delta(Z^{|(A \times A)|}) \) of \( Z^{|A^Q|} \). Suppose \( \overline{a}, \overline{b} \in \mathbf{Z}^{|A^Q|} \) and take values in \( \mathbb{Z}^+ \) and further that \( \overline{a} = -\overline{b} \) mod \( \mathbf{\Delta}Z^{|A^Q|} \). Then \( \overline{b} = \overline{0} \). The natural order in \( \mathbf{Z}^{|A^Q|} \) induces an order in \( \mathbf{Z}^{|A^Q|}/\mathbf{\Delta}Z^{|A^Q|} \) making it an ordered group.

Let \( \beta^Q : \mathbf{Z}^{|A^Q|} \to \mathbf{Z}^{|A^Q|} \) be given by matrix multiplication by the incidence matrix of the substitution \( \sigma^Q \). For a function \( \varphi : A \times A \to \mathbf{Z} \) define \( \varphi' : A \times A \to \mathbf{Z} \) by \( \varphi'(u,v) = \varphi(u,v) \) where \( u,v \) is the last alphabet in the substitution \( \sigma(u') \). Then \( \beta^Q(\delta(\varphi)) = \delta(\varphi') \); thus, \( \beta^Q[\delta(Z^{|A \times A|})] \subseteq \delta(Z^{|A \times A|}) \). Hence, \( \beta^Q \) induces a homomorphism of ordered groups

\[
\frac{Z^{|A^Q|}}{\mathbf{\Delta}Z^{|A^Q|}} \xrightarrow{\beta^Q} \frac{Z^{|A^Q|}}{\mathbf{\Delta}Z^{|A^Q|}} \xrightarrow{\beta^Q} \frac{Z^{|A^Q|}}{\mathbf{\Delta}Z^{|A^Q|}} \xrightarrow{\beta^Q} \frac{Z^{|A^Q|}}{\mathbf{\Delta}Z^{|A^Q|}} \xrightarrow{\beta^Q} \cdots
\]
is isomorphic to the dimension group $K^0(X_\sigma, T_\sigma)$ of the substitution system associated to $(A, \sigma)$.

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