FINITE-DIMENSIONAL QUASI-HOPF ALGEBRAS OF CARTAN TYPE

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Abstract. In this paper, we present a general method for constructing finite-dimensional quasi-Hopf algebras from finite abelian groups and braided vector spaces of Cartan type. The study of such quasi-Hopf algebras leads to the classification of finite-dimensional radic ally graded basic quasi-Hopf algebras over abelian groups with dimensions not divisible by 2, 3, 5, 7 and associators given by abelian 3-cocycles. As special cases, the small quasi-quantum groups are introduced and studied. Many new explicit examples of finite-dimensional genuine quasi-Hopf algebras are obtained.

1. Introduction

Quasi-Hopf algebras are generalizations of Hopf algebras, and are fundamental in the study of finite integral tensor categories [14]. Recall that a tensor category is called integral if the Frobenius-Perron dimension of each object is an integer. According to [15], any finite integral tensor category over an algebraically closed field is equivalent to the representation category of some finite-dimensional quasi-Hopf algebra. Pointed tensor categories are special examples of integral tensor categories, and the corresponding quasi-Hopf algebras are called basic quasi-Hopf algebras.

In the past and half decades, the classification of finite-dimensional basic quasi-Hopf algebras have attracted lots of attention. Since the dual of a finite-dimensional pointed Hopf algebra is a basic Hopf algebra, the duals of the finite dimensional pointed Hopf algebras over abelian groups classified in [1, 4, 17, 6] provide a big family of finite-dimensional basic quasi-Hopf algebras. Since our ultimate goal is to classify the tensor categories, we are only interested in those quasi-Hopf algebras whose representation categories do not arise from any Hopf algebra. Such quasi-Hopf algebras are said to be genuine. In [10, 11, 12], Etingof and Gelaki gave a method for constructing basic genuine quasi-Hopf algebras from known basic Hopf algebras, and classified the finite-dimensional radically graded basic quasi-Hopf algebras over cyclic groups of prime order. In [16], Gelaki constructed the finite-dimensional basic quasi-Hopf algebras of dimension $N^3$ over cyclic groups of order $N$. Utilizing the classification result of [4], Angiono [5] classified the finite-dimensional radically graded basic quasi-Hopf algebras over cyclic groups with dimensions not divisible by small prime divisors. In [19, 20, 21], the quasi-commutative finite-dimensional graded pointed Majid algebras of low ranks (dual basic quasi-Hopf algebras) are classified by the first author and his cooperators. Although the aforementioned classification work covered a lot of new finite dimensional quasi-Hopf algebras, the most important family of finite dimensional pointed Hopf algebras of Cartan type is not yet covered by the above classifications of quasi-Hopf algebras. In particular, we have not found a natural quasi-version of the (generalized) small quantum groups although a very close quasi-version of the Frobenius-Lusztig kernel was constructed by means of the
quasi-quantum double in [25]. This is because the classic construction of a small quantum group as a particular quotient of the quantum double works not for the quasi-Hopf algebra case. So we have to look for an alternative way to define the notion of a small quasi-quantum group. The fact that the classical small quantum groups form a special class of the finite dimensional pointed Hopf algebras of finite Cartan type, see [22, 23, 1, 4], inspires us: if we could construct finite dimensional quasi-Hopf algebras from Cartan matrices, then the small quasi-quantum groups must be the special cases of those quasi-Hopf algebras of Cartan type. This motivates us to study the finite-dimensional quasi-Hopf algebras of Cartan type. The main work of this paper are threefold.

First of all, we present a general method for constructing finite-dimensional quasi-Hopf algebras from finite Cartan matrices. Such a quasi-Hopf algebra is generated by an abelian group and a braided vector space of Cartan type. In more detail, let $G$ be a finite abelian group and $G$ a bigger abelian group uniquely determined by $G$, see (3.4). Let $u(\mathcal{D}, \lambda, \mu)$ ([4], or see Theorem 2.14) be the generalized small quantum group generated by grouplike elements $\mathcal{G}$ and skew-primitive elements $\{X_1, \ldots, X_n\}$. We then determine the 2-cochains $J$ on $\mathcal{G}$ such that the subalgebra of the twist quasi-Hopf algebra $u(\mathcal{D}, \lambda, \mu)^J$ generated by $G$ and $\{X_1, \ldots, X_n\}$ is a quasi-Hopf subalgebra, denoted $u(\mathcal{D}, \lambda, \mu, \Phi_J)$, see Theorem 3.4. Note that if $\lambda = 0$ and $\mu = 0$, then $u(\mathcal{D}, 0, 0)$ is a radically graded basic Hopf algebra. Moreover, when $G$ is a cyclic group, the quasi-Hopf algebra $u(\mathcal{D}, 0, 0, \Phi_J)$ is the same as those constructed in [5, 10, 11, 12]. However, if $G$ is not cyclic, or $u(\mathcal{D}, \lambda, \mu)$ is not radically graded and basic, then the construction and the study of $u(\mathcal{D}, \lambda, \mu, \Phi_J)$ are much more complicated. One of the difficulties is to compute suitable 2-cochain $J$'s on $\mathcal{G}$ such that $u(\mathcal{D}, \lambda, \mu, \Phi_J)$ is a quasi-Hopf algebra. Even if we can compute such a suitable cochain $J$, we still have no standard method to determine whether $u(\mathcal{D}, \lambda, \mu, \Phi_J)$ is genuine or not. While in the case of $\lambda = 0, \mu = 0$ and $G$ is a cyclic group, this problem is trivial.

Secondly, the obtained quasi-Hopf algebras of Cartan type deliver the classification of finite-dimensional radically graded basic quasi-Hopf algebras over abelian groups. Let $H$ be a finite-dimensional basic quasi-Hopf algebra, and rad$(H)$ the Jacobson radical of $H$. Then we have $H/\text{rad}(H) \cong [kG]^*$, where $G$ is the Grothendieck group of the representation category of $H$. We say that the basic quasi-Hopf algebra $H$ is over the group $G$. When $G$ is abelian, it is obvious that $H/\text{rad}(H) \cong kG$. If $H$ is a radically graded and basic quasi-Hopf algebra over $G$, then the associator of $H$ is determined by a normalized 3-cocycle on $G$, see [5, 19, 21]. We show that a finite-dimensional radically graded and basic quasi-Hopf algebra $H$ over an abelian group $G$ with dimension not divisible by 2, 3, 5, 7, and the associator is given by an abelian 3-cocycle of $G$, must be isomorphic to a quasi-Hopf algebra of Cartan type $u(\mathcal{D}, \lambda, \mu, \Phi_J)$, where $\lambda = 0, \mu = 0$, see Theorem 4.4. Since each normalized 3-cocycle of a cyclic group or an abelian group of the form $\mathbb{Z}_m \times \mathbb{Z}_n$ is abelian, our classification extends the corresponding classification results of [5, 19] to more general cases.

Thirdly, we introduce the quasi-version of the small quantum groups, which form a class of finite dimensional quasi-Hopf algebras of Cartan type, namely, those $H_{\mathcal{L}}$, where $\mathcal{L}$ is a family of parameters. When $\mathcal{L}$ approaches 0, the small quasi-quantum group $H_{\mathcal{L}}$ will be the usual small quantum group, see Proposition 5.3. As mentioned before, the small quasi-quantum group defined in this paper is substantially different from the one defined in [25], where a small quasi-quantum group is defined as the quantum double of a quasi-Hopf algebra $A_q(\mathfrak{g})$ constructed in [11], where $\mathfrak{g}$ is a simple Lie algebra. Note that the quantum double $D(A_q(\mathfrak{g}))$ is a quasi-Hopf algebra of Cartan type as well. Unlike the Hopf algebra case, the small quasi-quantum group $H_{\mathcal{L}}$ is not a quotient of the double $D(A_q(\mathfrak{g}))$ in general. For example, if the
order of $q$ is odd and not divisible by 3 in case $g$ is of type $G_2$, then the double $D(A_q(g))$ is not a
genuine quasi-Hopf algebra, see [13]. Under the same conditions for $q$, we can show that there
are many genuine small quasi-quantum groups. This means that those small quasi-quantum
groups can not be the quotients of $D(A_q(g))$.

Beside the study of the small quasi-quantum groups, we will provide lots of other genuine
quasi-Hopf algebras associated to finite Cartan matrices in Section 6. As a matter of fact,
our method will not only systematically produce many nonsemisimple, nonlinearly graded
genuine quasi-Hopf algebras, but also yield many new classes of finite integral and non-pointed
tensor categories.

The paper is organized as follows. In Section 2, we introduce some concepts and known results
about quasi-Hopf algebras, generalized small quantum groups and 3-cocycle of abelian groups.
In Section 3, the quasi-Hopf algebras of Cartan type are constructed, and some low rank
nonradically graded examples are provided. In Section 4, we classify the finite-dimensional
radically graded quasi-Hopf algebras which are basic over abelian groups, and show that
all the radically graded quasi-Hopf algebras of Cartan type are genuine. In Section 5, we
introduce the small quasi-quantum groups, which are special nonradically graded quasi-Hopf
algebras of Cartan type, and present explicitly examples of genuine quasi-Hopf algebras
associated to connected finite Cartan matrices.

Throughout this paper, $\mathbb{k}$ denotes an algebraically closed field of characteristic zero. All the
algebras, tensor categories and the unadorned tensor product $\otimes$ are over $\mathbb{k}$.

2. Preliminaries

In this section, we introduce some notations and basic facts about Quasi-Hopf algebras, tensor
categories and some important results [4] about pointed Hopf algebras.

2.1. Quasi-Hopf algebras. A quasi-bialgebra $H = (H, \triangle, \varepsilon, \Phi)$ is an unital associative al-
gebra with two algebra maps $\triangle : H \to H \otimes H$ (the comultiplication) and $\varepsilon : H \to \mathbb{k}$ (the
counit), and an invertible element $\Phi \in H \otimes^3$ (the associator), subject to:

$$(\varepsilon \otimes \text{id}) \triangle (h) = (\text{id} \otimes \varepsilon) \triangle (h),$$
$$(\text{id} \otimes \triangle) \varepsilon (h) = \Phi \cdot (\triangle \otimes \text{id}) \varepsilon (h) \cdot \Phi^{-1},$$
$$(\text{id} \otimes \text{id} \otimes \triangle)(\Phi) \cdot (\triangle \otimes \text{id} \otimes \text{id})(\Phi) = (1 \otimes \Phi) \cdot (\text{id} \otimes \varepsilon \otimes \text{id})(\Phi) \cdot (\Phi \otimes 1),$$
$$(\text{id} \otimes \varepsilon \otimes \text{id})(\Phi) = 1$$

for all $h \in H$. Write $\Phi = \Phi^1 \otimes \Phi^2 \otimes \Phi^3$ and $\Phi^{-1} = \Phi^1 \otimes \Phi^2 \otimes \Phi^3$. A quasi-Hopf algebra
$H = (H, \triangle, \varepsilon, \Phi, S, \alpha, \beta)$ is a quasi-bialgebra $(H, \triangle, \varepsilon, \Phi)$ with an antipode $(S, \alpha, \beta)$, where
$\alpha, \beta \in H$ and $S : H \to H$ is an algebra anti-homomorphism satisfying

$$\sum S(a_1) \alpha a_2 = \varepsilon(a) \alpha, \quad \sum a_1 \beta S(a_2) = \varepsilon(a) \beta,$$
$$\Phi^1 \beta S(\Phi^2) \alpha \Phi^3 = 1, \quad S(\Phi^1) \alpha \Phi^2 \beta S(\Phi^3) = 1$$

for all $a \in H$. Here we use Sweedler’s notation $\triangle(a) = \sum a_1 \otimes a_2$.

Definition 2.1. A twist for a quasi-Hopf algebra $H$ is an invertible element $J \in H \otimes H$
satisfying

$$(\varepsilon \otimes \text{id})(J) = (\text{id} \otimes \varepsilon)(J) = 1.$$
Suppose that \( J = \sum_i f_i \otimes h_i \) is a twist of \( H \) with inverse \( J^{-1} = \sum_i \overline{f}_i \otimes \overline{h}_i \). Write
\[
\alpha_J = \sum_i S(\overline{f}_i) \alpha_{\overline{h}_i}, \quad \beta_J = \sum_i f_i S(g_i).
\]

According to [8], if \( \beta_J \) is invertible then one can define a new quasi-Hopf algebra structure \( H^J = (H, \triangle, \varepsilon, \Phi_J, \delta, \beta_J, \alpha_J, 1) \) on the algebra \( H \), where
\begin{align*}
\triangle_J(h) &= J \triangle(h)J^{-1}, \quad h \in H, \\
\Phi_J &= (1 \otimes J)(\text{id} \otimes \triangle)(J)(\Phi \otimes \text{id})(J^{-1})(J \otimes 1)^{-1}, \\
S_J(h) &= \beta_J S(h)\beta_J^{-1}, \quad h \in H.
\end{align*}

Two quasi-Hopf algebras \( H \) and \( H' \) are said to be twist equivalent if \( H' \cong H^J \) for some twist \( J \) of \( H \).

**Definition 2.2.** A quasi-Hopf algebra \( H \) is genuine if \( H \) is not twist (or gauge) equivalent to any Hopf algebra.

The following theorem is useful in Section 5.

**Theorem 2.3.** [27, Theorem 2.2] Let \( H \) and \( B \) be two finite-dimensional quasi-Hopf algebras. Then the two module categories \( H\text{-mod} \) and \( B\text{-mod} \) are tensor equivalent if and only if \( H \) is equal to \( B^J \) for some twist \( J \) of \( B \).

Let \( H = (H, \triangle, \varepsilon, \Phi, S, \alpha, \beta) \) be a quasi-Hopf algebra. If \( H = \oplus_{i \geq 0} H[i] \) is a graded algebra such that \( (H[0], \varepsilon, \Phi, S, \alpha, \beta) \) is a quasi-Hopf subalgebra, and \( I = \oplus_{i \geq 1} H[i] \) is the Jacobson radical of \( H \) and \( I^k = \oplus_{i \geq k} H[i] \) for each \( k \geq 1 \), then we call \( H \) a radically graded quasi-Hopf algebra. Suppose that \( H = (H, \triangle, \varepsilon, \Phi, S, \alpha, \beta) \) is a quasi-Hopf algebra, \( I \) is the Jacobson radical of \( H \). If \( I \) is a quasi-Hopf ideal of \( H \), i.e., \( \triangle(I) \subseteq H \otimes I + I \otimes H, S(I) = I \) and \( \varepsilon(I) = 0 \), then we can construct a radically graded quasi-Hopf algebra associated to \( H \). Let \( H[0] = H/I \) and \( \pi : H \to H[0] \) is the canonical projection. Define \( H[k] = I^k/I^{k+1} \) for \( k \geq 1 \), then the graded algebra \( gr(H) = \oplus_{i \geq 0} H[i] \) has a natural quasi-Hopf algebra structure, with the associator \( \pi \otimes \pi \otimes \pi(\Phi) \), and the antipode \( (\pi \circ S, \pi(\alpha), \pi(\beta)) \). For radically graded quasi-Hopf algebras, we have the following useful lemma.

**Lemma 2.4.** [10, Lemma 2.1] Let \( H = \oplus_{i \geq 0} H_i \) be a radically graded quasi-Hopf algebra. Then \( H \) is generated by \( H[0] \) and \( H[1] \).

### 2.2. Datum of Cartan type, root system and Wyle group.

For a finite group \( G \), by \( \hat{G} \) we mean the character group of \( G \). We give the definition of a datum of Cartan type according to [4].

**Definition 2.5.** A datum of Cartan type
\[
\mathfrak{D} = \mathfrak{D}(G, (h_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, A)
\]
consists of an abelian group \( G \), elements \( h_i \in G \), characters \( \chi_i \in \hat{G}, 1 \leq i \leq \theta \), and a generalized Cartan matrix \( A = (a_{ij})_{1 \leq i, j \leq \theta} \) of size \( \theta \) satisfying
\[
q_{ij}q_{ji} = q_{ij}^{a_{ij}}, \quad \text{where } q_{ij} = \chi_j(h_i), \text{ for all } 1 \leq i, j \leq \theta.
\]

We call \( \theta \) the rank of \( \mathfrak{D} \). A datum of Cartan type \( \mathfrak{D} \) is called finite Cartan type if the associated Cartan matrix \( A \) is finite; \( \mathfrak{D} \) is said to be connected if \( A \) is a connected Cartan matrix.
Fix a datum \( \mathcal{D} = \mathcal{D}(G, (h_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq \theta}, A) \) of finite Cartan type. Let \( \{\alpha_i|1 \leq i \leq \theta\} \) be the set of free generators of \( \mathbb{Z}^\theta \) and \( s_i : \mathbb{Z}^\theta \to \mathbb{Z}^\theta \) the reflection \( s_i(\alpha_j) = \alpha_j - \alpha_i \alpha_j \) for \( 1 \leq i, j \leq \theta \). The Weyl group \( W \) of \( A \) is generated by \( \{s_i|1 \leq i \leq \theta\} \) and the root system \( R = \bigcup_{i=1}^\theta W(\alpha_i) \). Let \( R^+ \) be the set of positive roots with respect to the simple roots \( \{\alpha_i|1 \leq i \leq \theta\} \). For each \( \alpha = \sum_{i=1}^\theta k_i \alpha_i \in \mathbb{Z}^\theta \), denote by \( ht(\alpha) = \sum_{i=1}^\theta k_i \), the height of \( \alpha \), and

\[
\begin{align*}
(2.7) & \quad h_\alpha = h_1^{k_1} h_2^{k_2} \cdots h_\theta^{k_\theta}, \\
(2.8) & \quad \chi_\alpha = \chi_1^{k_1} \chi_2^{k_2} \cdots \chi_\theta^{k_\theta}.
\end{align*}
\]

If \( \alpha = \sum_{i=1}^\theta k_i \alpha_i \in R^+ \), it is obvious that \( k_i \geq 0 \), \( 1 \leq i \leq \theta \), and \( ht(\alpha) > 0 \). For a datum of Cartan type \( \mathcal{D} \), we can define a Yetter-Drinfeld module \( V(\mathcal{D}) \) in \( YD^\mathcal{D} \) by

\[
(2.9) \quad V(\mathcal{D}) = \bigoplus_{i=1}^\theta V^X_{hi},
\]

where \( V^X_{hi} = k \{X_i\} \) is the 1-dimensional Yetter-Drinfeld module such that the module and the comodule structures are given by

\[
(2.10) \quad \delta(X_i) = X_i \otimes h_1, \quad g \triangleright X_i = \chi_i(h) X_i
\]

for all \( g \in G \). A basis \( \{X_1, \cdots, X_n\} \) of Yetter-Drinfeld module \( V(\mathcal{D}) \) satisfying (2.10) is called a canonical basis. It is well-known that \( YD^\mathcal{D} \) is a braided tensor category. The natural braiding on \( V(\mathcal{D}) \) is given by

\[
(2.11) \quad c_{V,V} : V \otimes V \to V \otimes V, \quad X_i \otimes X_j \to q_{ij} X_j \otimes X_i, \quad 1 \leq i, j \leq \theta.
\]

2.3. Braided Hopf algebras. Let \( (V, c) \) be a braided vector space with a basis \( \{X_1, \cdots, X_n\} \) such that

\[
 c(X_i \otimes X_j) = q_{ij} X_j \otimes X_i, \quad q_{ij} \in k, \quad 1 \leq i, j \leq n.
\]

Then we call \( (V, c) \) a braided vector space of diagonal type, \( \{X_1, \cdots, X_n\} \) a canonical basis of \( V \), and \( (q_{ij})_{1 \leq i, j \leq n} \) the braiding constants of \( V \). Moreover, if

\[
 q_{ij} q_{ji} = q_{ij}^{a_{ji}}, \quad 1 \leq i, j \leq n,
\]

where \( A = (a_{ij})_{1 \leq i, j \leq n} \) is a Cartan matrix, then \( (V, c) \) is called a \textbf{braided vector space of Cartan type}. For a datum of Cartan type \( \mathcal{D} \), it is obvious that \( V(\mathcal{D}) \) is a braided vector space of Cartan type.

Note that the braiding matrix \((q_{ij})_{1 \leq i, j \leq \theta}\) of \( (V, c) \) defines a braided commutator on \( T(V) \) as follows:

\[
(2.12) \quad [X, Y]_c = XY - \left( \prod_{1 \leq s \leq \theta} q_{i_s j_s}^{a_{i_s j_s}} \right) YX,
\]

where \( X = X_{i_1}^{x_1} X_{i_2}^{x_2} \cdots X_{i_s}^{x_s} \) and \( Y = Y_{j_1}^{y_1} Y_{j_2}^{y_2} \cdots Y_{j_s}^{y_s} \). The braided adjoint action of an element \( X \in T(V) \) is defined by

\[
(2.13) \quad ad_c(X)(Y) = [X, Y]_c
\]

for any \( Y \in T(V) \).

In the rest of this subsection, we let \( \mathcal{D} = \mathcal{D}(G, (h_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq \theta}, A) \) be a connected datum of finite Cartan type. In addition, we assume for \( 1 \leq i \leq \theta \),

\[
(2.14) \quad q_{ii} \text{ has odd order,}
\]

\[
(2.15) \quad \text{the order of } q_{ii} \text{ is prime to } 3, \text{ if } A \text{ is of type } G_2,
\]
where \( q_{ij} = \chi_j(g_i) \) for \( 1 \leq i, j \leq \theta \). With these assumptions, we have the following:

**Lemma 2.6.** [4, Lemma 2.3] There exists a root of unit \( q \) of odd order and integers \( d_i \in \{1, 2, 3\}, 1 \leq i \leq \theta \), such that for \( 1 \leq i, j \leq \theta \),

\[
q_{ij} = q^{d_i d_j} \quad ; \quad d_i a_{ij} = d_j a_{ji}.
\]

Moreover, if \( A \) is of type \( G_2 \). Then the order of \( q \) is prime to 3.

An immediate consequence of Lemma 2.6 is that the elements \( q_{ij}, \ 1 \leq i \leq \theta \), have the same order, hence we define

\[
N = |q_{ii}|, \ 1 \leq i \leq \theta.
\]

Let \( V = V(\mathfrak{D}) \) and \( \{X_1, \cdots, X_{\theta}\} \) a canonical basis of \( V \). Then the tensor algebra \( T(V) \) is a braided Hopf algebra in \( G(\mathfrak{D}) \) with comultiplication determined by

\[
\Delta(X_i) = X_i \otimes 1 + 1 \otimes X_i, \ 1 \leq i \leq \theta.
\]

Since \( (ad_c X_i)^{1-a_{ij}}(X_j) \), \( 1 \leq i \neq j \leq \theta \), are primitive elements in \( T(V) \), they generate a braided Hopf idea of \( T(V) \), denoted \( I \). So we have a quotient braided Hopf algebra:

\[
\mathcal{R}(\mathfrak{D}) = T(V)/I
\]

in \( G(\mathfrak{D}) \). For convenience, we still denote by \( X_i, \ 1 \leq i \leq \theta \), the image of the element \( X_i \) in \( \mathcal{R}(\mathfrak{D}) \).

Now let \( w_0 = s_{i_1}s_{i_2}\cdots s_{i_P} \) be a fixed reduced presentation of the longest element of \( W \) in terms of simple reflections. Then

\[
\beta_l = s_{i_1}\cdots s_{i_{l-1}}(\alpha_{i_l}) | 1 \leq l \leq P
\]

is a convex order of positive roots. The root vectors \( \{X_\alpha | \alpha \in \mathbb{R}^+\} \) can be defined as iterated braided commutators of the elements \( X_1, \cdots, X_{\theta} \) with respect to the braiding given by \( (g_{ij})_{1 \leq i, j \leq n} \) such that \( X_{g_i} = X_i, 1 \leq i \leq \theta \), see [3, 4, 23] for detailed definition. Denote by \( K(\mathfrak{D}) \) the subalgebra of \( \mathcal{R}(\mathfrak{D}) \) generated by the elements \( X_l = X_{\beta_l}^N, 1 \leq l \leq P \). The following description of \( K(\mathfrak{D}) \) comes from [4].

**Theorem 2.7.** [4, Theorem 2.6]

1. The elements

\[
X_{\beta_1}^{a_1}X_{\beta_2}^{a_2}\cdots X_{\beta_P}^{a_P}, \ a_1, \cdots, a_P \geq 0,
\]

form a basis of \( \mathcal{R}(\mathfrak{D}) \).

2. \( K(\mathfrak{D}) \) is a braided Hopf subalgebra of \( \mathcal{R}(\mathfrak{D}) \).

3. For all \( \alpha, \beta \in \mathbb{R}^+ \), \( [X_\alpha, X_{\beta}^N]_C = 0 \), that is, \( X_\alpha X_{\beta}^N = X_{\beta}^N (g_\alpha) X_\alpha^N X_\alpha = 0 \).

Let \( \epsilon_l = (\delta_{kl})_{1 \leq k, l \leq P} \in \mathbb{N}^P \), where \( \delta_{kl} \) is the Kronecker sign. For each \( a = (a_1, a_2, \cdots, a_P) \in \mathbb{N}^P \), define

\[
Y^a = Y_1^{a_1}Y_2^{a_2}\cdots Y_P^{a_P},
\]

\[
h^a = h_1^{N_{\beta_1}^{a_1}}h_2^{N_{\beta_2}^{a_2}}\cdots h_P^{N_{\beta_P}^{a_P}},
\]

\[
a = a_1\beta_1 + a_2\beta_2 + \cdots + a_P\beta_P.
\]

Let \( \Delta_{\mathcal{R}(\mathfrak{D})} \) be the comultiplication of \( \mathcal{R}(\mathfrak{D}) \), then we have the following lemma.
Lemma 2.8. [4, Lemma 2.8] For any nonzero \( a \in \mathbb{N}^P \), there are uniquely determined scalars \( t_{b,c}^a \in \mathbb{k}, 0 \neq b, c \in \mathbb{N}^P \), such that
\[
\Delta_{\mathcal{R}(G)}(Y^a) = Y^a \otimes 1 + 1 \otimes Y^a + \sum_{b,c \not= 0, \sum_{b,c} = a} t_{b,c}^a Y^b \otimes Y^c.
\]

Definition 2.9. Let \((\mu_a)_{a \in \mathbb{N}^P}\) be a family of elements in \( \mathbb{k} \) such that for all \( a, h^a = 1 \) implies \( \mu_a = 0 \). Then we can define \( u^a \in \mathbb{k}G \) inductively on \( ht(a) \) by
\[
u^a = \mu_a(1 - h^a) + \sum_{b,c \not= 0, \sum_{b,c} = a} t_{b,c}^a \mu_b u^c.
\]

Proposition 2.10. Let \((\mu_1)_{1 \leq i \leq P}\) be a family of elements in \( \mathbb{k} \) such that: \( g^1_{bij} = 1 \) or \( \chi_{bi}^1 \not= \varepsilon \) implies \( \mu_i = 0 \). Then there exists a unique family \((\mu_a)_{a \in \mathbb{N}^P}\) satisfying \( \mu_{ei} = \mu_i \) for \( 1 \leq l \leq P \) such that
\[
u^a = u^{a-e_i} u^e, \text{ if } a = (a_1, \ldots, a_i, 0, \ldots, 0), \quad a_l \geq 1, \quad 1 \leq l \leq P, \quad \text{and } a \not= e_i.
\]

Proof. It follows from [4, Lemma 2.10, Theorem 2.13] \( \square \)

Definition 2.11. Suppose that \( \mu = (\mu_1)_{1 \leq i \leq P} \) is a family of elements in \( \mathbb{k} \) satisfying the condition of Proposition 2.10. For a root \( \alpha \in R^+ \), then there exists \( 1 \leq l \leq P \) such that \( \alpha = \beta_i \) and define
\[
u_i(\mu) = u^{\varepsilon_i}.
\]

2.4. Andruskiewitsch-Schneider’s Hopf algebras. Fix a datum of finite Cartan type \( \mathcal{D} = \mathcal{D}(G, (h_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, A) \), where \( A \) may not be a connected Cartan matrix. For \( 1 \leq i, j \leq \theta \) we define \( i \sim j \) if \( i \) and \( j \) are in the same connected component of the Dynkin diagram of Cartan matrix \( A \), and \( i, j \) are said to be connected if \( i \sim j \). Let \( \Omega = \{I_1, \ldots, I_{l}\} \) be the set of the connected components of \( I' = \{1, 2, \ldots, \theta\} \). Here we also assume that the conditions (2.14)-(2.15) hold for each connected component of \( I \). For \( J \in \Omega \), let \( R_J \) be the root system of \( A_J = (\alpha_{ij})_{i,j \in J} \) and \( N_J \) the corresponding number defined by (2.16). Let \( R_J^+ \) be the set of positive roots of \( A_J \) with respect to the simple roots \( \{\alpha_i | i \in J\} \). The following partitions are obvious:
\[
R = \bigcup_{J \in \Omega} R_J, \quad R^+ = \bigcup_{J \in \Omega} R_J^+.
\]

Definition 12. A family \( \lambda = (\lambda_{ij})_{1 \leq i, j \leq \theta, i \sim j} \) of elements in \( \mathbb{k} \) is called a family of linking parameters for \( \mathcal{D} \) if \( h_i h_j = 1 \) or \( \chi_i \chi_j \not= \varepsilon \) implies \( \lambda_{ij} = 0 \) for all \( 1 \leq i, j \leq \theta, i \sim j \). Vertices \( 1 \leq i, j \leq \theta \) are called linkable if \( i \sim j, h_i h_j \not= 1 \) and \( \chi_i \chi_j = \varepsilon \).

Definition 13. A family \( \mu = (\mu_a)_{a \in \mathbb{R}^+} \) of elements in \( \mathbb{k} \) is called a family of root vector parameters for \( \mathcal{D} \) if \( h_{\alpha_J}^a = 1 \) or \( \chi_{\alpha_J}^a \not= \varepsilon \) implies \( \mu_{\alpha} = 0 \) for all \( \alpha \in R_J^+, J \in \Omega \).

With these definitions and notations, we can give one of the main results of [4].

Theorem 2.14. [4, Theorem 4.5] Let \( \mathcal{D} = \mathcal{D}(G, (h_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, A) \) be a datum of finite Cartan type such that each connected component of \( I' = \{1, 2, \ldots, \theta\} \) satisfies the conditions (2.14)-(2.15). Let \( \lambda \) and \( \mu \) be a family of linking parameters and a family of root vector parameters for \( \mathcal{D} \) respectively. Then we have a finite-dimensional pointed Hopf algebra \( H(\mathcal{D}, \lambda, \mu) \).
generated by the group $G$ and the skew-primitive elements $\{X_i|1 \leq i \leq \theta\}$ subject to the following relations:

(1.24) $gX_ig^{-1} = \chi_i(g)X_i$, for all $1 \leq i \leq \theta, g \in G$,

(1.25) $ad_e(X_i)^{-\omega_j}(X_j) = 0$, for all $i \neq j, i \sim j$,

(1.26) $ad_e(X_i)(X_j) = \lambda_{ij}(1 - h_{ij})$, for all $i < j, i \sim j$,

(1.27) $X^{N_J}_\alpha = u_\alpha(\mu)$, for all $\alpha \in R^+_J, J \in \Omega$.

The coalgebra structure is determined by

$$\nabla(X_i) = X_i \otimes 1 + h_i \otimes X_i, \quad \nabla(g) = g \otimes g, \quad \text{for all } 1 \leq i \leq \theta, g \in G.$$ 

The Hopf algebras constructed in Theorem 2.14 can be viewed as an axiomatic description of generalized the small quantum groups, and Lusztig’s small quantum groups are special examples of such Hopf algebras. Another main result of [4] says that any finite-dimensional pointed Hopf algebra over an abelian group $G$ is of the form $\mathcal{u}(D, \lambda, \mu)$ for some $D, \lambda, \mu$. In the sequel, we call such a Hopf algebra $\mathcal{u}(D, \lambda, \mu)$ an Andruskiewitsch-Schneider Hopf algebra (or AS-Hopf algebra for short) following [12].

2.5. Normalized 3-cocycles on finite groups. Let $G$ be an arbitrary abelian group. So $G \cong \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_n}$ with $m_j \in \mathbb{N}$ for $1 \leq j \leq n$. A function $\phi: G \times G \times G \mapsto \mathbb{k}^*$ is called a 3-cocycle on $G$ if

$$\phi(ef,g,h)\phi(e,f,gh) = \phi(e,f,g)\phi(e,fg,h)\phi(f,g,h)$$

for all $e, f, g, h \in G$. A 3-cocycle is called normalized if $\phi(f,1,g) = 1$. Denote by $\mathcal{A}$ the set of all sequences

$$\{c_{1}, \ldots, c_{l}, \ldots, c_{m}, c_{l2}, \ldots, c_{ij}, \ldots, c_{n-1,n}, c_{l23}, \ldots, c_{rst}, \ldots, c_{n-2,n-1,n}\}$$

such that $0 \leq c_{l} < m_{l}, 0 \leq c_{ij} < (m_{i}, m_{j}), 0 \leq c_{rst} < (m_{r}, m_{s}, m_{t})$ for $1 \leq l \leq n, 1 \leq i < j \leq n, 1 \leq r < s < t \leq n$, where $c_{ij}$ and $c_{rst}$ are ordered in the lexicographic order. We denote by $\omega_{\mathcal{A}}$ the sequence (2.25) in the following.

Let $g_i$ be a generator of $\mathbb{Z}_{m_i}, 1 \leq i \leq n$. For any $\omega_{\mathcal{A}} \in \mathcal{A}$, define

$$\omega_{\mathcal{A}}(G, \mathbb{k}) : G \times G \times G \mapsto \mathbb{k}^*$$

$$[g_{i_1}^{l_1} \cdots g_{n}^{l_n}, g_{i_1}^{m_1} \cdots g_{n}^{m_n}] \mapsto \prod_{l=1}^{n} \xi_{m_{l}}^{c_{ij}^{l_1} [\frac{1}{m_{j}}]} \prod_{1 \leq s < t \leq n} \zeta_{m_{l}}^{c_{rst}^{l_1} [\frac{1}{m_{r}}, \frac{1}{m_{s}}, \frac{1}{m_{t}}]} \zeta_{m_{l}}^{c_{rst}^{l_1} [\frac{1}{m_{r}}, \frac{1}{m_{s}}, \frac{1}{m_{t}}]}.$$ 

Here and below $\zeta_{m}$ stands for an $m$-th primitive root of unity.

Proposition 2.15. [18, Proposition 3.1] $\{\omega_{\mathcal{A}}|\mathcal{A} \in \mathcal{A}\}$ forms a complete set of representatives of the normalized 3-cocycles on $G$ up to 3-cohomology.

For the purpose of this paper, we need another class of representatives of the normalized 3-cocycles on $G$. Let $Z^3(G, \mathbb{k}^*)$ be the set of the normalized 3-cocycles on $G$. Define a map

$$\sigma : Z^3(G, \mathbb{k}^*) \mapsto Z^3(G, \mathbb{k}^*), \quad \sigma(\phi)(f,g,h) = \phi(h,g,f), \quad \text{for all } f, g, h \in G.$$
To see if the map is well-defined, we just need to show that \( \sigma(\phi) \) is a normalized 3-cocycle on \( G \) for each \( \phi \in Z^3(G, k^*) \). Indeed, for any \( e, f, g, h \in G \) and \( \phi \in Z^3(G, k^*) \), we have

\[
\partial(\sigma(\phi))(e, f, g, h) = \frac{\sigma(\phi)(e, f, g)\sigma(\phi)(e, f, gh)\sigma(\phi)(f, g, h)}{\sigma(\phi)(ef, gh)\sigma(\phi)(f, g, h)} - \frac{\phi(g, f, e)\phi(h, fg, e)\phi(h, g, f)}{\phi(h, g, ef)\phi(gh, f, e)} = \partial(\phi)(h, g, f, e) = 1.
\]

This implies that \( \sigma(\phi) \) is a 3-cocycle on \( G \). The fact that \( \sigma(\phi) \) is normalized follows from the equation:

\[
\sigma(\phi)(f, 1, g) = \phi(g, 1, f) = 1, \text{ for all } f, g \in G.
\]

It is obvious that \( \sigma \) is bijective since \( \sigma^2 = id \). Moreover, we have the following.

**Lemma 2.16.** The map \( \sigma \) induces an involution of \( H^3(G, k^*) \).

**Proof.** It suffices show that \( \sigma \) preserves 3-coboundaries. Suppose that \( \phi \) is a 3-coboundary. There exists a 2-cochain \( J : G \times G \to k^* \) such that \( \phi = \partial(J) \). Define \( J' : G \times G \to k^* \) by

\[
J'(f, g) = J^{-1}(g, f), \text{ for all } f, g \in G.
\]

Then we have:

\[
\sigma(\phi)(f, g, h) = \frac{J(h, g)J(gh, f)}{J(g, f)J(h, fg)} = \frac{J^{-1}(g, h)J^{-1}(f, gh)}{J^{-1}(f, g)J^{-1}(fg, h)} = \frac{J'(f, g)J'(fg, h)}{J'(g, h)J'(f, gh)} = \partial(J')(f, g, h)
\]

for all \( f, g, h \in G \). This implies that \( \sigma \) preserves 3-coboundaries. Thus, we have completed the proof. \( \square \)

For each \( \underline{c} \in \mathcal{A} \), define

\[
\phi_{\underline{c}} : G \times G \times G \to k^*, \quad [g^1 \cdots g^n, g'^1 \cdots g'^n, g''^1 \cdots g''^n] \mapsto \prod_{i=1}^n \zeta_{c_{1i}} \prod_{1 \leq s < t \leq n} \zeta_{c_{st}} \prod_{1 \leq r < s \leq t \leq n} \zeta_{c_{rst}}.
\]

It is obvious that \( \sigma(\omega_{\underline{c}}) = \phi_{\underline{c}} \) for each \( \underline{c} \in \mathcal{A} \). It follows from Proposition 2.15 and Lemma 2.16 that we have the following:

**Proposition 2.17.** \( \{\phi_{\underline{c}} | \underline{c} \in \mathcal{A}\} \) forms a complete set of representatives of the normalized 3-cocycles on \( G \) up to 3-cocohomology.

The original definition of an abelian cocycle was given in [9], and an equivalent definition via the twisted quantum double appeared in [26]. Let \( \phi \) be a 3-cocycle on \( G \), and \( D^\phi(G) \) the twisted quantum double of \( (kG, \phi) \) (see [19] for the detail). \( \phi \) is called an abelian 3-cocycle if \( D^\phi(G) \) is commutative. Using Proposition 2.17, one can easily determine all the abelian 3-cocycles on \( G \). A straightforward computation shows that \( \phi_{\underline{c}} \) is an abelian 3-cocycle if and only if \( c_{rst} = 0 \) for all \( 1 \leq r < s < t \leq n \). We point out that the twisted Yetter-Drinfeld category \( \tilde{D}YD^\phi_k \) is a pointed fusion category in case the 3-cocycle \( \phi_{\underline{c}} \) is abelian.
Denote by Vec\(_G\) the category of \(G\)-graded vector spaces. Let \(\omega\) be a 3-cocycle on \(G\). We define a tensor category Vec\(_{G}^{\omega}\). As a category, Vec\(_{G}^{\omega}\) = Vec\(_G\). The tensor product \(V \otimes W\) of two graded modules is endowed with the canonical grading:
\[(V \otimes W)_g = \oplus_{e,f,g} V_e \otimes V_f, \forall g \in G.\]
The associator \(\alpha\) is given by
\[\alpha_{U,V,W} : (U \otimes V) \otimes W \rightarrow U \otimes (V \otimes W)\]
\[(x \otimes y) \otimes z \mapsto \omega^{-1}(e,f,g)x \otimes (y \otimes z),\]
where \(x \in U, y \in V_f, z \in W_g\). According to [14, Proposition 2.6.1], Vec\(_{G}^{\omega}\) is tensor equivalent to the representation category of some Hopf algebra if and only if \(\phi\) is a 3-coboundary on \(G\).

3. Finite-dimensional Quasi-Hopf algebras

3.1. General setup. In this subsection, we fix some notations on abelian groups, which will be used throughout this paper. Suppose that \(G\) is a finite abelian group, say, \(G = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle\) such that \(|g_i| = m_i\) for \(1 \leq i \leq n\). Let \(\hat{G}\) be the character group of \(G\) over \(\mathbb{k}\). For each \(g = \prod_i g_i^{a_i}\), define a character \(\chi_g : G \rightarrow \mathbb{k}^*\) by
\[(3.1) \quad \chi_g(h) = \prod_i c_{m_i}^{a_i \beta_i},\]
where \(h = \prod_i g_i^{a_i} \in G\). From the definition of \(\chi_g\), it is obvious that \(\chi_g^{-1}(h) = \chi_{g^{-1}}(h) = \chi_g(h^{-1})\). So \(\chi : G \rightarrow \hat{G}, g \mapsto \chi_g\) is an group isomorphism. Let \(k[G]\) be the group algebra of \(G\) over field \(\mathbb{k}\). One can verify that
\[(3.2) \quad \{1_g = \frac{1}{|G|} \sum_{h \in G} \chi_g(h)\}
forms a complete set of the orthogonal primitive idempotents of the algebra \(k[G]\).

Lemma 3.1. For \(g,h \in G\), we have \(1_g h = h 1_g = \chi_g^{-1}(h) 1_g\).

Proof. \(1_g h = \frac{1}{|G|} \sum_{f \in G} \chi_g(f)fh = \frac{1}{|G|} \sum_{f \in G} \chi_g(fh)\chi_g(h^{-1})fh = \chi_g^{-1}(h) 1_g.\) \(\square\)

Now let \(G\) be an abelian group. We can define a bigger abelian group \(\mathbb{G}\) associated to \(G\) in the following way: assume
\[(3.3) \quad G = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle, \quad |g_i| = m_i, 1 \leq i \leq n;\]
define the group \(\mathbb{G}\) as follows:
\[(3.4) \quad \mathbb{G} = \langle \mathbb{g}_1 \rangle \times \cdots \times \langle \mathbb{g}_n \rangle, \quad |\mathbb{g}_i| = m_i^2, 1 \leq i \leq n.\]
It is obvious that there is a group injection:
\[(3.5) \quad \iota : G \rightarrow \mathbb{G}, \ i(g_i) = \mathbb{g}_i^{m_i}, 1 \leq i \leq n.\]
Let \(\zeta_{m_i}\) be an \(m_i^2\)-th primitive root of unity such that \(\zeta_{m_i}^{m_i} = \zeta_{m_i}\) for \(1 \leq i \leq n\). For each \(g = \prod_i g_i^{a_i} \in \mathbb{G}\), define \(\chi_g : \mathbb{G} \rightarrow \mathbb{k}^*\) by
\[\chi_g(h) = \prod_{i=1}^n \zeta_{m_i}^{a_i \beta_i}, \quad h = \prod_{i=1}^n \mathbb{g}_i^{a_i}.\]
Lemma 3.2. The following holds for all $0 \leq s_i \leq m_i - 1$, $1 \leq i \leq n$:

$$
\sum_{0 \leq k_j \leq m_j - 1, 1 \leq j \leq n} 1_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(h) = 1_{\prod_{i=1}^{n} g_i^{r_i}}.
$$

Proof. By definition we have

$$
\sum_{0 \leq k_j \leq m_j - 1, 1 \leq j \leq n} 1_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(h) = \frac{1}{|G|} \sum_{0 \leq k_j \leq m_j - 1, 1 \leq j \leq n} \sum_{h \in G} \chi_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(h).
$$

Suppose $h = g_1^{r_1} g_2^{r_2} \cdots g_n^{r_n}$. Then we have the equation:

$$
\sum_{0 \leq k_j \leq m_j - 1, 1 \leq j \leq n} \chi_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(h) = \prod_{i=1}^{n} \sum_{0 \leq k_j \leq m_j - 1, 1 \leq j \leq n} \epsilon_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(r_i).
$$

Note that $\sum_{0 \leq k_j \leq m_j - 1} \epsilon_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(r_i) = 0$ if $r_i \neq t_{m_i}$ for some integer $0 \leq t \leq m_i - 1$. Hence

$$
\sum_{0 \leq k_j \leq m_j - 1, 1 \leq j \leq n} \chi_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(h) \neq 0
$$

if and only if $r_i = t_{m_i}$ for $0 \leq t_i \leq m_i - 1, 1 \leq i \leq n$, i.e., $h$ is contained in the subgroup $G$. If $r_i = t_{m_i}$ for $1 \leq i \leq n$, then $h = g_1^{t_{m_1}} g_2^{t_{m_2}} \cdots g_n^{t_{m_n}}$ and we have:

$$
\sum_{0 \leq k_j \leq m_j - 1, 1 \leq j \leq n} \chi_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(h) = \prod_{i=1}^{n} \left( \sum_{0 \leq k_j \leq m_j - 1} \epsilon_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(t_{m_i}) \right) = \prod_{i=1}^{n} \left( \sum_{0 \leq k_j \leq m_j - 1} \epsilon_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(t_{m_i}) \right) = \prod_{i=1}^{n} (m_i \epsilon_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(t_{m_i})).
$$

It follows that

$$
\sum_{0 \leq k_j \leq m_j - 1, 1 \leq j \leq n} 1_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(h) = \frac{1}{|G|} \sum_{0 \leq k_j \leq m_j - 1, 1 \leq j \leq n} \sum_{h \in G} \chi_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(h).
$$

$$
= \frac{|G|}{|G|} \sum_{0 \leq t_j \leq m_j - 1, 1 \leq j \leq n} \prod_{i=1}^{n} \epsilon_{\prod_{i=1}^{n} E_i^{m_i k_i + r_i}}(t_{m_i}) = \frac{1}{|G|} \sum_{0 \leq t_j \leq m_j - 1, 1 \leq j \leq n} \chi_{\prod_{i=1}^{n} g_i^{r_i}}(g_i^{t_{m_i}}) \prod_{i=1}^{n} g_i^{t_{m_i}}.
$$
For each $t$ions of the last subsection. Let $D$ be the set of the connected components of $I = \{1, 2, \ldots, \theta\}$. Denote by $(s_{ij})_{1 \leq i, j \leq \theta}$ and $(r_{ij})_{1 \leq i, j \leq \theta}$ the two families of root vector parameters.

Thus, the claimed equality holds. □

3.2. Finite dimensional quasi-Hopf algebras and Cartan matrices. Keep the notations of the last subsection. Let $D = \mathcal{D}(\mathbb{G}, (h_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, A)$ be a datum of finite Cartan type, where $A = (a_{ij})_{1 \leq i, j \leq \theta}$ is a finite Cartan matrix. Let $\Omega$ be the subset of the connected components of $I = \{1, 2, \ldots, \theta\}$. Here $A$ family of linking parameters

\[
\lambda = (\lambda_{ij})_{1 \leq i, j \leq \theta}
\]

such that for each $\mathcal{D}(\mathbb{G}, (h_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, A)$ is a finite Cartan matrix. Let $\Omega$ be the set of the con-

\[
\lambda_{ij} \equiv c_j r_{ij} \mod m_j, 1 \leq i \leq \theta, 1 \leq j \leq \theta.
\]

It is obvious that $(s_{ij})_{1 \leq i, j \leq \theta}$ and $(r_{ij})_{1 \leq i, j \leq \theta}$ are uniquely determined by $D$. Let $\Gamma(\mathcal{D})$ be the subset of $\mathcal{A}$ such that for each $\mathcal{L} \in \Gamma(\mathcal{D})$, $e_{r,s} = 0$ for all $0 \leq r < s < t \leq \theta$ and $s_{ij} \equiv c_j r_{ij} \mod m_j, 1 \leq i \leq \theta, 1 \leq j \leq \theta$.

For each $\mathcal{L} \in \Gamma(\mathcal{D})$, define on $G$ the functions $\Theta, \Psi, \Upsilon$ and $\mathcal{F}$ as follows:

\[
\Theta_l(g) = \prod_{i=1}^{n} \zeta_{m_i}^{-l_i s_{li}}, \quad g = \prod_{i=1}^{n} g_i^{l_i} \in G, 1 \leq l \leq \theta.
\]

\[
\Psi_l(f, h) = \prod_{i=1}^{n} \zeta_{m_i}^{c_i g_i \varphi_i(f)}, \quad f = \prod_{i=1}^{n} f_i^{p_i}, \quad h = \prod_{i=1}^{n} h_i^{q_i} \in G, 1 \leq l \leq \theta.
\]

\[
\Upsilon(g) = \prod_{i=1}^{n} \zeta_{m_i}^{-c_i l_i m_i}, \quad g = \prod_{i=1}^{n} g_i^{l_i}.
\]

\[
\mathcal{F}_l(g) = \prod_{j=1}^{n} \zeta_{m_j}^{-c_j (k_j - r_{ij}) \varphi_j(g)}, \quad g = \prod_{i=1}^{n} g_i^{k_j}.
\]

Here

\[
\varphi_i(g) = (k_i - r_{il})' - (k_i - r_{il}), \quad g = \prod_{i=1}^{n} g_i^{k_i},
\]

and $(p_i - r_{il})'$ is the remainder of $p_i - r_{il}$ divided by $m_i$.

In order to construct quasi-Hopf algebras, we need the notions of modified linking parameters and modified root vector parameters.

\[
\varphi_i(g) = (k_i - r_{il})' - (k_i - r_{il}), \quad g = \prod_{i=1}^{n} g_i^{k_i},
\]

and $(p_i - r_{il})'$ is the remainder of $p_i - r_{il}$ divided by $m_i$.

\[
\mathcal{F}_l(g) = \prod_{j=1}^{n} \zeta_{m_j}^{-c_j (k_j - r_{ij}) \varphi_j(g)}, \quad g = \prod_{i=1}^{n} g_i^{k_j}.
\]

Here

\[
\varphi_i(g) = (k_i - r_{il})' - (k_i - r_{il}), \quad g = \prod_{i=1}^{n} g_i^{k_i},
\]

and $(p_i - r_{il})'$ is the remainder of $p_i - r_{il}$ divided by $m_i$.

In order to construct quasi-Hopf algebras, we need the notions of modified linking parameters and modified root vector parameters.

**Definition 3.3.** A family of linking parameters $\lambda = (\lambda_{ij})_{1 \leq i, j \leq \theta}$ for $D$ is said to be modified if

\[
h_i h_j \notin G \text{ implies } \lambda_{ij} = 0 \text{ for all } 1 \leq i < j \leq \theta, i \sim j.
\]

A family of root vector parameters $\mu = (\mu_{\alpha})_{\alpha \in \mathcal{R}}$ for $D$ is said to be modified if

\[
\mu_{\alpha}^N \notin G \text{ implies } \mu_{\alpha} = 0 \text{ for all } \alpha \in \mathcal{R}_j^+, J \in \Omega.
\]
Now we can give the main result of this paper.

**Theorem 3.4.** Let \( \lambda = (\lambda_{ij})_{1 \leq i,j \leq \theta} \) and \( \mu = (\mu_{i})_{i \in \mathbb{R}} \) be two families of modified linking parameters and root vector parameters respectively for a datum of Cartan type \( \mathfrak{D} = \mathfrak{D}(\mathbb{G}, (\mathfrak{h}_{i})_{1 \leq i \leq \theta}, (\chi_{i})_{1 \leq i \leq \theta}, A) \), and \( c \) a nonzero element in \( \Gamma(\mathfrak{D}) \). Then we have a finite-dimensional quasi-Hopf algebra \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \) generated by \( G \) and \( \{X_{1}, \ldots, X_{\theta}\} \) subject to the relations:

\[
\begin{align*}
(3.19) & \quad gX_{ig}^{-1} = \chi_{i}(g)X_{i}, \text{ for all } 1 \leq i \leq \theta, g \in G; \\
(3.20) & \quad ad_{c}(X_{ij}) = 0, \text{ for all } i \neq j, i \sim j; \\
(3.21) & \quad ad_{c}(X_{ij})(X_{ij}) = \lambda_{ij}(1-h_{ij}), \text{ for all } i \neq j, i \sim j; \\
(3.22) & \quad X_{\alpha}^{N_{\gamma}} = u_{\alpha}(\mu), \text{ for all } \alpha \in R_{\gamma}^{+}, J \in \Omega.
\end{align*}
\]

The coalgebra structure of \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \) is given by

\[
\Delta(X_{i}) = \sum_{f,g \in G} \Psi_{\mathfrak{M}}(f,g)X_{i}1 \otimes 1_{g} + \sum_{f \in G} \Theta_{\mathfrak{M}}(f)1_{f} \otimes X_{i}, \quad \Delta(g) = g \otimes g, 1 \leq \theta, g \in G.
\]

The associator of \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \) is determined by

\[
\Phi_{\mathfrak{M}} = \sum_{f,g,h \in G} \phi_{\mathfrak{M}}(f,g,h)(1_{f} \otimes 1_{g} \otimes 1_{h}).
\]

The antipode \((S, \alpha, 1)\) is defined by

\[
\alpha = \sum_{g \in G} T(g)1_{g}, \quad S(X_{i}) = \sum_{g \in G} F_{i}(g)X_{i}1_{g}.
\]

The proof of Theorem 3.4 will be delivered in the next subsection. Since the quasi-Hopf algebra \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \) is generated by the abelian group \( G \) and the braided vector space of Cartan type \( V = k[X_{1}, \ldots, X_{\theta}] \), we shall call it a quasi-Hopf algebra of Cartan type in the sequel. We will say that \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \) is associated to the Cartan matrix \( A \), and call \( \theta \) the rank of \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \), as well as the rank of \( A \).

**Remark 3.5.** The relations (3.20)-(3.23) for \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \) are similar to those of AS-Hopf algebras \( u(\mathfrak{D}, \lambda, \mu) \), but the generators of the two algebras are different. In fact, \( u(\mathfrak{D}, \lambda, \mu) \) is generated by \( G \) and \( \{X_{1}, \ldots, X_{\theta}\} \), and \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \) is the subalgebra of \( u(\mathfrak{D}, \lambda, \mu) \) generated by subgroup \( G \) and \( \{X_{1}, \ldots, X_{\theta}\} \). Moreover, we will prove that \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \) is a quasi-Hopf subalgebra of \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}})^{J} \) for some twist \( J \) of \( u(\mathfrak{D}, \lambda, \mu) \).

It is obvious that \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \) is radically graded if and only if \( u_{\alpha}(\mu) = 0, \lambda_{ij}(1-h_{ij}) = 0 \), for all \( \alpha \in R_{\gamma}^{+}, 1 \leq i,j \leq \theta \), and the radical is the ideal generated by \( X_{1}, \ldots, X_{\theta} \). Note that if \( \mu = 0 \), then \( u_{\alpha}(\mu) = 0 \) by (2.19). It follows the definitions of a family of modified linking parameters and a family of modified root vector parameters thta the quasi-hopf algebra \( u(\mathfrak{D}, \lambda, \mu, \Phi_{\mathfrak{M}}) \) is radically graded if and only if both \( \lambda = 0 \) and \( \mu = 0 \).

**Remark 3.6.**

1. Let \( G \) be a cyclic group, \( H \) the AS-Hopf algebra \( u(\mathfrak{D}, 0, 0) \). The quasi-Hopf algebra \( u(\mathfrak{D}, 0, 0, \Phi_{\mathfrak{M}}) \) is nothing but the basic quasi-Hopf algebra \( \Lambda(H, c) \) (over the cyclic group \( G \)) classified in [5].

2. Suppose that \( A \) is a connected Cartan Matrix of rank \( n \), and \( \mathfrak{g} \) is the simple Lie algebra associated to \( A \). Let \( G = \mathbb{Z}_{m}^{n} \) for some positive odd integer \( m \), which is not divisible by 3 if \( A \) is of type \( G_{2} \). Let \( h_{i} = \prod_{j=1}^{n} \xi_{ij}^{a_{ij}}, \chi_{i}(\mathfrak{g}) = \zeta_{m}^{a_{ij}} \) for \( 1 \leq i, l \leq n \). Let \( c_{i} = a_{ii}, c_{j,k} = a_{jk} \) for \( 1 \leq i \leq n, 1 \leq j < k \leq n \). Then \( u(\mathfrak{D}, 0, 0, \Phi_{\mathfrak{M}}) \) is the half small quasi-quantum groups \( A_{q}(\mathfrak{g}) \) given in [11], where \( q = \zeta_{m}^{a_{ij}} \).
2. Suppose that $A$ is the diagonal Cartan matrix $A_1 \times A_1 \times \cdots \times A_1$. Then the dual of the quasi-Hopf algebras $u(D, 0, \Phi)$ is a quasi-quantum linear space, see [21].

3.3. The proof of Theorem 3.4. Let $H = u(D, \lambda, \mu)$ be the AS-Hopf algebra given in Theorem 2.14, which is generated by $G$ and $\{X_1, \cdots, X_n\}$. By [4, Theorem 4.5], the group of group-like elements of $H$ is $G$. According to Subsection 3.1, we know that $\chi : G \rightarrow \hat{G}$ is a group isomorphism. Let $\{\eta_1, \cdots, \eta_n\}$ be the set of elements in $G$ such that $\chi_{\eta_i} = \chi_i$ for $1 \leq i \leq \theta$. By (3.7), it is obvious that

$$\eta_i = \prod_{j=1}^{n} g_j^{-1}, 1 \leq i \leq \theta.$$ 

Moreover we have the following.

Lemma 3.7.

$$(3.27)$$

Moreover we have the following.

$$\mathbb{I}_g X_i = X_i \mathbb{I}_g \eta_i,$$ for all $g \in G$, $1 \leq i \leq \theta$.

Proof. Follows from the following equations:

$$\mathbb{I}_g X_i = \frac{1}{|G|} \sum_{f \in G} \chi_g(f) f X_i = \frac{1}{|G|} \sum_{f \in G} \chi_g(f) \chi_{\eta_i}(f) X_i f = \frac{1}{|G|} X_i \sum_{f \in G} \chi_g(f) \chi_{\eta_i}(f) f = X_i \mathbb{I}_g \eta_i.$$ 

\[G \in \Gamma(D), \text{ we define:}\]

$$(3.28) J_{\iota} : G \times G \rightarrow k^*; \quad (g_1 x_1 \cdots x_n, g_1 y_1 \cdots y_m) \rightarrow \prod_{i=1}^{n} c_{\iota_i \eta_i(x_i-x_i)} \prod_{1 \leq s < t \leq n} c_{\iota_i \eta_i(x_s-x_t)},$$

where the element $x'_i$ stands for the remainder of $x_i$ divided by $m_i$, for $1 \leq i \leq n$. One can easily verify that $J_g$ is a 2-cochain of $G$. We will see that this 2-cochain $J_g$ induces a twist of the Hopf algebra $H$. Define $J_{\iota} = \sum_{f, g \in G} J_{\iota}(f, g) \mathbb{I}_f \otimes \mathbb{I}_g \in H \otimes H$.

Lemma 3.8. $J_{\iota}$ is a twist of $H$.

Proof. It is obvious that $J_{\iota}$ is invertible with inverse $J_{\iota}^{-1} = \sum_{f, g \in G} J_{\iota}^{-1}(f, g) \mathbb{I}_f \otimes \mathbb{I}_g$. Next we verify that $(\varepsilon \otimes id)(J_{\iota}) = (id \otimes \varepsilon)(J_{\iota}) = 1$ holds. Suppose $g = \prod_{i=1}^{n} g_i^{k_i}$ for some $0 \leq k_i \leq m_i, 1 \leq i \leq n$. Then the following equations hold:

$$\varepsilon(\mathbb{I}_g) = \frac{1}{|G|} \varepsilon \left( \sum_{h \in G} \chi_g(h) \mathbb{I}_h \right) = \frac{1}{|G|} \sum_{h \in G} \chi_g(h)$$

$$= \frac{1}{|G|} \prod_{0 \leq i \leq n} \sum_{l_i \leq m_i - 1} c_{\iota_i \eta_i(l_i)}$$

$$= \frac{1}{|G|} \prod_{i=1}^{n} \left( \sum_{0 \leq l_i \leq m_i - 1} c_{\iota_i \eta_i(l_i)} \right).$$

It follows that

$$\varepsilon(\mathbb{I}_h) = \begin{cases} 0, & \text{if } h \neq 1; \\ 1, & \text{if } h = 1. \end{cases}$$

Hence, $(\varepsilon \otimes id)(J_{\iota}) = \sum_{g \in G} J_{\iota}(1, g) \mathbb{I}_g = \sum_{g \in G} \mathbb{I}_g = 1$. Similarly, the equation: $(id \otimes \varepsilon)(J_{\iota}) = 1$ holds. \[\square\]
Since $H$ is a Hopf algebra, we can view it as a quasi-Hopf algebras with the trivial associator $\Phi = 1 \otimes 1 \otimes 1$ and the usual antipode $(S, 1, 1)$. According to Subsection 2.1, we can construct a quasi-Hopf algebra $H^{\mathbb{L}} = (H^{\mathbb{L}}, \Delta^{\mathbb{L}}, \varepsilon, \Phi^{\mathbb{L}}, S^{\mathbb{L}}, \beta^{\mathbb{L}})$ of Cartan type $1$. The associator of $H^{\mathbb{L}}$ can be explicitly described as follows:

**Lemma 3.9.**

\[
\Phi^{\mathbb{L}} = \sum_{f, g, h \in G} \phi_2(f, g, h) 1_f \otimes 1_g \otimes 1_h
\]

**Proof.** First of all, we need to verify the comultiplication of the element $1_g$ for every $g \in G$:

\[
\triangle (1_g) = \sum_{f, h : f h = g} 1_f \otimes 1_h.
\]

Indeed, we have:

\[
\sum_{f, h : f h = g} 1_f \otimes 1_h = \frac{1}{|G|^2} \sum_{\mathfrak{f} = g, \mathfrak{x} \in \mathfrak{G}} \chi_\mathfrak{f} (\mathfrak{x}) \mathfrak{x} \otimes \chi_\mathfrak{h} (\mathfrak{y}) \mathfrak{y} = \frac{1}{|G|^2} \sum_{\mathfrak{x}, \mathfrak{y} \in \mathfrak{G}} \left[ \sum_{\mathfrak{f} : \mathfrak{f} h = g} \chi_\mathfrak{f} (\mathfrak{x}) \chi_\mathfrak{h} (\mathfrak{y}) \mathfrak{x} \otimes \mathfrak{y} \right] = \frac{1}{|G|^2} \sum_{\mathfrak{x} \in \mathfrak{G}} \chi_\mathfrak{g} (\mathfrak{x}) \mathfrak{x} \otimes \mathfrak{x} = \triangle (1_\mathfrak{g}),
\]

where the third identity follows from the equation:

\[
\sum_{\mathfrak{f} : \mathfrak{f} h = g} \chi_\mathfrak{f} (\mathfrak{x}) \chi_\mathfrak{h} (\mathfrak{y}) = \begin{cases} 
0, & \text{if } \mathfrak{x} \neq \mathfrak{y}; \\
|G| \chi_\mathfrak{g} (\mathfrak{x}), & \text{if } \mathfrak{x} = \mathfrak{y}.
\end{cases}
\]

Hence, it yields:

\[
\Phi^{\mathbb{L}} = (1 \otimes 1_\mathfrak{g})(i d \otimes \triangle)(1_\mathfrak{g})(\triangle \otimes i d)(1_\mathfrak{g} \otimes 1)^{-1} = \sum_{f, g, h : f \otimes g h = g} J_{\mathfrak{g}}(f, g) J_{\mathfrak{g}}(h, g) 1_f \otimes 1_g \otimes 1_h
\]

Now suppose $f = \prod_{i=1}^n g_i^{x_i y_i + z_i}$, $g = \prod_{i=1}^n g_i^{y_i m_i + z_i}$, $h = \prod_{i=1}^n g_i^{z_i m_i + t_i}$ for $0 \leq y_i, z_i, r_i, \leq n$. Let $f = \prod_{i=1}^n g_i^{x_i}$, $g = \prod_{i=1}^n g_i^{y_i}$, $h = \prod_{i=1}^n g_i^{z_i}$. We compute the element $\partial(J_{\mathfrak{g}})(f, g, h)$:

\[
\partial(J_{\mathfrak{g}})(f, g, h) = \prod_{i=1}^n \zeta_{c_{i, t_i}^{x_i}}^{x_i \zeta_{m_i}} \prod_{1 \leq j \leq n} \zeta_{c_{i, t_i}^{y_i}}^{y_i \zeta_{m_i}} = \phi_2(f, g, h).
\]
Applying Lemma 3.2, we obtain:

\[
\Phi_{\mathbf{1}} = \sum_{0 \leq r_1, t_1, s_1 \leq m_1 - 1, 0 \leq r_2, t_2, s_2 \leq m_2 - 1, 0 \leq r_3, t_3, s_3 \leq m_3 - 1} \prod_{i=1}^{n} c_{i}^{t_i} \frac{r_i + s_i}{r_i + s_i} \prod_{1 \leq j < k \leq n} \zeta_{m_j m_k} c_{j}^{t_j} \frac{r_i + s_i}{r_i + s_i} \]

\[
= \sum_{0 \leq r_1, t_1, s_1 \leq m_1 - 1, 0 \leq r_2, t_2, s_2 \leq m_2 - 1, 0 \leq r_3, t_3, s_3 \leq m_3 - 1} \prod_{i=1}^{n} c_{i}^{t_i} \frac{r_i + s_i}{r_i + s_i} \prod_{1 \leq j < k \leq n} \zeta_{m_j m_k} c_{j}^{t_j} \frac{r_i + s_i}{r_i + s_i} \times 1^{(\prod_{i=1}^{n} \zeta_{s_i}^{t_i})} \prod_{i=1}^{n} \zeta_{s_i}^{t_i} \prod_{i=1}^{n} \zeta_{s_i}^{t_i}
\]

as desired. □

**Lemma 3.10.** \(u_{\alpha}(\mu) \in \mathbb{k}G\) for all \(\alpha \in R^+\).

**Proof.** Since \(R^+ = \cup_{J \in \Omega} R^+_J\), it suffices to show \(u_{\alpha}(\mu) \in \mathbb{k}G\) for any \(\alpha \in R^+_J\) with a fixed \(J \in \Omega\). Suppose that \(J = \{\alpha_1, \cdots, \alpha_n\} \subset I\), and \(\{\alpha_1, \cdots, \alpha_n\}\) is the set of the simple roots corresponding to the vertexes of \(J\). Let \(w_J = s_{j_1} s_{j_2} \cdots s_{j_p}\) be the reduced presentation of the longest element of the Weyl group \(W_{J}\) in terms of simple reflections. Define

\[(3.31) \quad \beta_{j_i} = s_{j_1} s_{j_2} \cdots s_{j_{i-1}}(\alpha_{j_i}) \text{ for } 1 \leq i \leq p_J\]

and

\[(3.32) \quad \mathbf{a} = a_1 \beta_{j_1} + a_2 \beta_{j_2} + \cdots + a_{p_J} \beta_{j_{p_J}}, a \in N^{P_J}.
\]

We show that \(u^a \in \mathbb{k}G\) for each \(a \in N^{P_J}\). Consequently, it leads to \(u_{\alpha}(\mu) \in \mathbb{k}G\) for all \(\alpha \in R^+_J\). We will prove it by induction on \(ht(a)\).

In case \(ht(a) = 1\), then \(a\) is a simple root contained in \(\{\alpha_1, \cdots, \alpha_n\}\), say, \(\alpha_{i_k}\). By (2.22), we have \(u^a = \mu_{i_k}(1 - h^a) = h^{N_{i_k}}.\) It follows from 3.18 that \(u^a = \mu_{i_k}(1 - h^a) \in \mathbb{k}G\).

Now assume that \(u^a \in \mathbb{k}G\) holds for all \(a \in N^{P_J}\) such that \(ht(a) < l\). Let \(a \in N^{P_J}\) such that \(ht(a) = l\). If \(a = \beta_{j_s}\) for some \(1 \leq s \leq P_J\), then

\[u^a = \mu_a(1 - h^a) + \sum_{b, c \neq 0, b + c = a} t_{b, c}^a \mu_b u^c,
\]

and \(h^a = h^{N_{j_s}}.\) From 3.18 we see that the part \(\mu_a(1 - h^a) \in \mathbb{k}G\). The fact that second part \(\sum_{b, c \neq 0, b + c = a} t_{b, c}^a \mu_b u^c\) belongs to \(\mathbb{k}G\) follows from the induction assumption. If \(a \neq \beta_{j_s}\) for any \(1 \leq s \leq P_J\), then \(a = (a_1, a_2, \cdots, a_s, 0, \cdots, 0)\) with \(a_s > 0\) for some \(1 \leq s \leq P_J\). Let

\[e_s = (0, \cdots, 1, 0, \cdots, 0).
\]

By Proposition 2.10, we have \(u^a = u^{a-e_s} u^{e_s}.\) Since both heights of \(a - e_s\) and \(e_s\) are less than \(l\), the elements \(u^{a-e_s}, u^{e_s}\) belong to \(\mathbb{k}G\) by the induction assumption. This implies that \(u^a \in \mathbb{k}G\). □

Now we denote by \(A(H, \mathcal{L})\) the subalgebra of \(H_{\mathbb{F}_\mathbb{K}}\) generated by \(G\) and \(\{X_1, \cdots, X_p\}\). We are going to show that \(A(H, \mathcal{L})\) is the desired quasi-Hopf algebra if we choose an appropriate element \(\mathcal{L}\). We first describe the defining relations of the generators of \(A(H, \mathcal{L})\).
Proposition 3.11. The algebra $A(H, \zeta)$ can be presented by the generators $G$ and $\{X_1, \cdots, X_\theta\}$ and the following relations:

\[(3.33) \quad gX_ig^{-1} = \chi_i(g)X_i, \text{ for all } 1 \leq i \leq \theta, g \in G, \]
\[(3.34) \quad ad_C(X_i)^{1-a_{ij}}(X_j) = 0, \text{ for all } i \neq j, i \sim j, \]
\[(3.35) \quad ad_C(X_i)(X_j) = \lambda_{ij}(1-h_ih_j), \text{ for all } i < j, i \sim j, \]
\[(3.36) \quad X_{\alpha_J}^N = u_\alpha(\mu), \text{ for all } \alpha \in R_J^+, J \in \Omega. \]

Moreover, $A(H, \zeta)$ has a basis of the form

$X_{\beta_1}^{x_1}X_{\beta_2}^{x_2} \cdots X_{\beta_p}^{x_p}g, \; g \in G, \; 0 \leq x_i \leq N_J, \; \beta_i \in R_J^+.$

Proof. Since the relations (3.34)-(3.35) hold in $H$ for the group $\zeta$ and the generators $X_1, \cdots, X_\theta$, they hold as well for the subgroup $G$ and $X_1, \cdots, X_\theta$. Relation (3.37) follows from Lemma 3.10. For the relation (3.36), it is enough to show that the elements $\lambda_{ij}(1-h_ih_j)$ fall in $kG$. But this is true because of (3.17). The last part of the proposition follows from [4, Theorem 3.3] and Relation (3.29).

The algebra $A(H, \zeta)$ is apparently not a Hopf subalgebra of $H$. However, it is a quasi-Hopf subalgebra of some twist of $H$.

Proposition 3.12. $A(H, \zeta)$ is a quasi-Hopf subalgebra of $H^V$ if and only if $\zeta \in \Gamma(D)$. 

Proof. $\Leftarrow$. First of all, we show that $A(H, \zeta)$ is closed under the comultiplication $\Delta_\zeta$ of $H^V$. It is obvious that $\Delta_\zeta(g) = J_\zeta(g \otimes g)J_\zeta^{-1} = g \otimes g$ for any $g \in G \subset \zeta$ since $G$ is abelian. It remains to show that $\Delta_\zeta(X_i) \in A(H, \zeta) \otimes A(H, \zeta)$ for $1 \leq i \leq \theta$. By Lemma 3.1 and 3.7, we have

\[(3.37) \quad \psi_{ij} = \left\{ \begin{array}{ll}
(k_j - r_{ij})' - (x_jm_j + k_j - r_{ij}), & \text{if } x_jm_j + k_j - r_{ij} \geq 0; \\
(k_j - r_{ij})' - m_j - (x_jm_j + k_j - r_{ij}), & \text{if } x_jm_j + k_j - r_{ij} < 0.
\end{array} \right. \]

Then by (3.11) we have

\[(3.38) \quad J_\zeta(\zeta^{-1}g, \zeta) = \prod_{s \leq t \leq \eta} c_{m_s} c_{m_t} \prod_{j=1}^{n} c_{r_{ij}(k_j - r_{ij})} \prod_{1 \leq s \leq t \leq \eta} c_{m_s m_t}. \]
Similarly, by (3.9) and (3.10), we obtain

\begin{equation}
\frac{J_{\chi}(f, g)}{J_{\chi}(f, g)} \chi_{\mathcal{L}^{-1}}(h_i) = J_{\chi}(f, g) \chi_{\mathcal{L}^{-1}}(h_i)
\end{equation}

\begin{equation}
= \prod_{j=1}^{n} \left[ \frac{c_{ij}}{m_{ij}} \right] \prod_{1 \leq s < t \leq n} \frac{c_{is}}{m_{is}} \frac{c_{it}}{m_{it}}
\end{equation}

\begin{equation}
= \prod_{j=1}^{n} \left( c_{ij} - s_{ij} \right) x_{j} m_{ij}
\end{equation}

\begin{equation}
= \prod_{j=1}^{n} \left( c_{ij} - s_{ij} \right).
\end{equation}

Hence,

\begin{equation}
\Delta_{\mathcal{L}}(X_i) = \sum_{0 \leq x_{ij}, y_{ij}, \leq m_{ij} - 1, 1 \leq r \leq n} \prod_{j=1}^{n} \frac{c_{ij}}{m_{ij}} \prod_{1 \leq s < t \leq n} \frac{c_{is}}{m_{is}} \frac{c_{it}}{m_{it}}
\end{equation}

\begin{equation}
\times X_i \left[ \prod_{j=1}^{n} y_{ij}^{m_{ij}+1} \right] X_i \left[ \prod_{j=1}^{n} x_{ij}^{m_{ij}+1} \right]
\end{equation}

\begin{equation}
= \sum_{0 \leq k_{ij}, l_{ij}, \leq m_{ij} - 1, 1 \leq s < t \leq n} \prod_{j=1}^{n} \frac{c_{ij}}{m_{ij}} \prod_{1 \leq s < t \leq n} \frac{c_{is}}{m_{is}} \frac{c_{it}}{m_{it}}
\end{equation}

\begin{equation}
\times X_i^{1} \left[ \prod_{j=1}^{n} g_{ij}^{k_{ij}+1} \right] X_i^{1} \left[ \prod_{j=1}^{n} g_{ij}^{l_{ij}+1} \right]
\end{equation}

\begin{equation}
+ \sum_{f, g \in \mathbf{G}} \Psi(f, g) X_i^{1} f_{1} \otimes 1_{g} + \sum_{f \in \mathbf{G}} \Theta(f) 1_{f} \otimes X_i.
\end{equation}

The second equality follows from Lemma 3.2. So we have proved that $\Delta_{\mathcal{L}}(X_i) \in A(H, \chi) \otimes A(H, \chi)$, hence $A(H, \chi)$ is closed under the comultiplication $\Delta_{\mathcal{L}}$ of $H^*_{\mathcal{L}}$.

Next we will show that $(S_{A(H, \chi)}, \alpha_{\mathcal{L}}, \beta_{\mathcal{L}})$ is a antipode of $A(H, \chi)$. For all $g \in \mathbf{G}$, we have

\begin{equation}
S(1_{g}) = S(1_{|\mathbf{G}|} \sum_{h \in \mathbf{G}} \chi_{g}(h) h) = 1_{|\mathbf{G}|} \sum_{h \in \mathbf{G}} \chi_{g}(h) h^{-1} = 1_{g^{-1}}.
\end{equation}

So we obtain

\begin{equation}
\alpha_{\mathcal{L}} = \sum_{f, g \in \mathbf{G}} J_{\chi}(f, g) 1_{f^{-1}} 1_{g} = \sum_{g \in \mathbf{G}} J_{\chi}(g^{-1}, g) 1_{g},
\end{equation}

\begin{equation}
\beta_{\mathcal{L}} = \sum_{f, g \in \mathbf{G}} J_{\chi}(f, g) 1_{g} 1_{g^{-1}} = \sum_{g \in \mathbf{G}} J_{\chi}(g, g^{-1}) 1_{g}.
\end{equation}
It is obvious that $\beta_{\underline{\beta}}$ is invertible with inverse $\sum_{g \in G} J_{\underline{L}}^{-1}(g, g^{-1}) 1_g$, and we have:

$$
(3.43) \quad \beta_{\underline{\beta}} \alpha_{\underline{\beta}} = \sum_{g \in G} J_{\underline{L}}(g, g^{-1}) 1_g \quad \text{if} \quad \alpha_{\underline{\beta}} = \sum_{g \in G} J_{\underline{L}}(g, g^{-1}) 1_g
$$

$$
= \sum_{0 \leq l_i, l_i - m_i - 1, -1 \leq l_i \leq n} \prod_{i=1}^{n} \zeta_{m_i}^{c_{l_i} l_i m_i} \prod_{1 \leq s < t \leq n} \zeta_{m_s m_t}^{c_{l_s} l_s m_t} \prod_{i=1}^{n} \zeta_{m_i}^{h_i}
$$

$$
= \sum_{0 \leq l_i, l_i - m_i - 1, -1 \leq l_i \leq n} \prod_{i=1}^{n} \zeta_{m_i}^{c_{l_i} l_i m_i} \prod_{1 \leq s < t \leq n} \zeta_{m_s m_t}^{c_{l_s} l_s m_t} \prod_{i=1}^{n} \zeta_{m_i}^{h_i}
$$

$$
= \sum_{g \in G} \gamma(g) 1_g.
$$

Here the second identity follows from (3.11), and the fourth identity follows from Lemma 3.2. Hence we have showed $\beta_{\underline{\beta}} \alpha_{\underline{\beta}} \in A(H, \underline{\beta})$. Next we will show that $S_{\underline{\beta}}$ preserves $A(H, \underline{\beta})$. It is obvious that $S_{\underline{\beta}}(g) = g^{-1}$ for all $g \in G$. For each $X, 1 \leq i \leq \theta$ we have:

$$
(3.44) \quad S_{\underline{\beta}}(X_i) = -\beta_{\underline{\beta}}(h_i^{-1} X_i) \beta_{\underline{\beta}}^{-1}
$$

$$
= - \sum_{g \in G} J_{\underline{L}}(g, g^{-1}) 1_g \chi(g) \prod_{1 \leq s < t \leq n} \zeta_{m_s m_t}^{c_{l_s} l_s m_t} \prod_{i=1}^{n} \zeta_{m_i}^{h_i}
$$

where the third identity follows from Lemma 3.1 and 3.7; the fourth identity follows from (3.9)-(3.11); and the fifth identity follows from Lemma 3.2. So $S_{\underline{\beta}}(X_i) \in A(H, \underline{\beta})$, and $S_{\underline{\beta}}$ preserves $A(H, \underline{\beta})$ because $\beta_{\underline{\beta}}$ is an anti-algebra morphism and $A(H, \underline{\beta})$ is generated by $G$ and $\{X_1, \cdots, X_\theta\}$.

$\Rightarrow$. We omit the detailed computation, and point out that $\triangle_{\underline{\beta}}(X_i) \in A(H, \underline{\beta}) \oplus A(H, \underline{\beta})$ implies (3.9)-(3.11), and $S_{\underline{\beta}}(X_i) \in A(H, \underline{\beta})$ implies (3.9)-(3.11). Hence $\underline{\beta}$ must be contained in $\Gamma(\mathcal{D})$. □
Proof of Theorem 3.4. Let $\mathfrak{g} \in \Gamma(\mathfrak{D})$. By Proposition 3.11, we know that $A(H, \mathfrak{g})$ is identical to $u(\mathfrak{D}, \lambda, \mu, \Phi)$ as algebras. By Proposition 3.12, $A(H, \mathfrak{g})$ is a quasi-Hopf algebra. Thus, $u(\mathfrak{D}, \lambda, \mu, \Phi)$ is also a quasi-Hopf algebra with the comultiplication $\Delta$ satisfying $\Delta(g) = g \otimes g$ and \eqref{3.40}. The antipode $(S, \alpha, 1)$ is determined by \eqref{3.43} and \eqref{3.44}, and the associator $\Phi$ is given by \eqref{3.29}. Therefore, we have proved the theorem.

3.4. Examples of quasi-Hopf algebras of Cartan type. In this subsection, we will give some examples of quasi-Hopf algebras of Cartan type. We make a convention that the comultiplications, the associators, and the antipodes of the quasi-Hopf algebras in those examples below can be written in the forms as listed in \eqref{3.23}-\eqref{3.25}, and hence will be omitted.

Example 3.13. Basic quasi-Hopf algebras over cyclic groups. Let $G = \mathbb{Z}_m = \langle g \rangle$. In this case, $\mathfrak{G} = \mathbb{Z}_m^2 = \langle g \rangle$, and $G$ is identical to the subgroup $\mathfrak{G}^m$ of $G$, see \eqref{3.5}. Let $\mathfrak{D} = \mathfrak{D}(G, (h_i)_{1 \leq i \leq \theta}, (\chi_i)_{1 \leq i \leq \theta}, A)$ be a datum of finite Cartan type, where

$$h_i = g^{s_i}, \chi_1(g) = \zeta_{m^2}^{r_i}$$

for some $0 < s_i, r_i < m^2, 1 \leq i \leq \theta$. Denote by $H$ the AS-Hopf algebra $u(\mathfrak{D}, 0, 0)$, and let $s$ be a number satisfying $0 < s < m$ and $sr_i \equiv s_1 \mod m$ for all $1 \leq i \leq \theta$, and $\xi = \{s\}$. Then we get a quasi-Hopf algebra $u(\mathfrak{D}, 0, 0, \Phi) = A(H, s)$, see \cite[Proposition 3.1.1]{5} for the definition of $A(H, s)$. According to \cite[Theorem 3.4.1]{5}, any nonsesimply, genuine basic graded quasi-Hopf algebra over a cyclic group with dimension not divisible by 2 and 3 must be twist equivalent to some $u(\mathfrak{D}, 0, 0, \Phi)$. Let $m = p$ and $\mathfrak{D} = \mathfrak{D}(G, g, \chi, A_1)$ such that $\chi(g) = \zeta_p^2$. Then it is obvious that $s = 1$, $\xi = \{1\}$, and we get a quasi-Hopf algebra $u(\mathfrak{D}, 0, 0, \Phi)$ generated by $G$ and $X$ with the relations

$$gXg^{-1} = \zeta_p^2 X, \quad X^{p^2} = 0.$$

According to the classification of pointed Hopf algebras of dimension $p^3$ in \cite{2}, we know that $u(\mathfrak{D}, 0, 0, \Phi)$ does not admit a pointed Hopf algebra structure.

Next we will construct a few more non-radically graded quasi-Hopf algebras of rank 2. First we give an example of a quasi-Hopf algebra associated to $A_1 \times A_1$, and then present some examples of quasi-Hopf algebras associated to $A_2, B_2, G_2$.

Example 3.14. The quasi-version of $u_q(s_l)$. Let $N > 2$ and $d$ be two positive odd numbers, and $G = \mathbb{Z}_m = \langle g \rangle$, $\mathfrak{G} = \mathbb{Z}_m^2 = \langle g \rangle$, where $m = Nd$. As usual, $G$ is viewed as the subgroup of $G$. Let $\mathfrak{D} = \mathfrak{D}(G, (h_1, h_2), (\chi_1, \chi_2), A_1 \times A_1)$, where

$$h_1 = h_2 = g^m, \chi_1(g) = \zeta_m^{2d}, \chi_2(g) = \zeta_m^{-2d}.$$

It is easy to verify that $\mathfrak{D}$ is a datum of Cartan type. Since $\Gamma(\mathfrak{D})$ is the set of numbers $0 \leq c \leq m - 1$ satisfying

$$m \equiv 2cd \mod m,$$

$$m \equiv -2cd \mod m.$$

Both equations are equivalent with $N|c$ since $m = Nd$ and $N$ is odd.

In this case, it is clear that $\Gamma(\mathfrak{D}) = \{c = kN|0 \leq k < d\}$. Let $q = \zeta_m^{nd}$. By $H_c$ we denote the algebra generated by $g, X_1, X_2$ subject to the relations as follows:

$$gX_1g^{-1} = q^2 X_1, gX_2g^{-1} = q^{-2} X_2,$$

$$X_1X_2 - q^{-2} X_2X_1 = \lambda(1 - g^2),$$

$$X_1^N = X_2^N = 0.$$
Let $E = X_1, F = X_2 g^{-1}$, and $\lambda = q^{-1} - q$, then we can see that $H_c$ is generated by $g, E, F$ satisfying the relations:

$$
gEg^{-1} = q^2 X_1, qFg^{-1} = q^{-2} X_2, $$
$$
EF - FE = \frac{q - g^{-1}}{q - q^{-1}}, $$
$$
E^N = F^N = 0.
$$

When $d = 1$, we have $c = 0$ and $H_c \cong u_q(sl_2)$. When $d > 1$, we know that $\Gamma(\mathcal{D})$ has nonzero elements. If $c = 0$, we have $H_c \cong u(\mathcal{D}', \lambda, 0)$, where $\mathcal{D}' = \mathcal{D}(G, (g, y), (\chi_1', \chi_2'), A_1 \times A_1)$ and $\chi_1'(g) = q^2, \chi_2'(g) = q^{-2}$. If $c \neq 0$, then we have $H_c = u(\mathcal{D}, \lambda, 0, \Phi_c)$. Because of this fact, $u(\mathcal{D}, \lambda, 0, \Phi_c)$ can be viewed as the quasi-version of $u_q(sl_2)$, and is called a small quasi-quantum group. More small quasi-quantum groups will be studied in Section 5.

The above example provides us some non-radically graded quasi-Hopf algebras associated to $A_1 \times A_1$. We point out that there is a similar notion of a quasi-version of $u_q(sl_2)$ in [24], where Liu defined a quasi-Hopf analogue of $u_q(sl_2)$ as a quantum double of a quasi-Hopf algebra associated to $A_1$. It is obvious that these two definitions are different, since the dimension of a quasi-version of $u_q(sl_2)$ is not a square in general. Hence, it should not be a quantum double. In order to construct non-radically graded examples of type $A_2, B_2, G_2$, we need the following well-known proposition from number theory.

**Proposition 3.15.** Let $a, b, n$ be nonzero integers. Then the equation $ax \equiv b \mod n$ has solutions if and only if $(a, n)$ divides $b$. Moreover, if there exists a solution, then it is unique up to modulo $\frac{n}{(a, n)}$.

**Example 3.16.** Quasi-Hopf algebras associated to $A_2, B_2, G_2$. Let $A$ be a Cartan matrix of type $A_2, B_2$ or $G_2$. Suppose that $m, n, d$ are positive odd numbers such that $(m, n) = (m, d) = (n, d) = 1$. In addition, in case $A$ is of type $G_2$, we will assume that the three numbers $m, n$ and $d$ are not divisible by 3. Let $G = \langle g_1 \rangle \times \langle g_2 \rangle \times \langle g_3 \rangle \times \langle g_4 \rangle$, where $|g_1| = md, |g_2| = nd, |g_3| = md^2$, and $|g_4| = nd^2$. The group $G$ and the generators $g_i, 1 \leq i \leq 4$ are defined in a similar manner as before. Let $a, b, k$ be the numbers listed in the Table 1. Define $\mathcal{D} = \mathcal{D}(G, (h_1, h_2), (\chi_1, \chi_2), A)$, where

$$
h_1 = (g_1 g_2)^{mn}, \quad h_2 = (g_1 g_2 g_3 g_4)^{mn}, $$
$$
\chi_1(g_1) = \zeta_d^a, \quad \chi_1(g_2) = \zeta_d^a, \quad \chi_1(g_3) = \zeta_d^a, \quad \chi_1(g_4) = \zeta_d^a, $$
$$
\chi_2(g_1) = \zeta_d^b, \quad \chi_2(g_2) = \zeta_d^b, \quad \chi_2(g_3) = \zeta_d^b, \quad \chi_2(g_4) = \zeta_d^{2k^2 - 1}.
$$

Let $A = (a_{ij})_{1 \leq i, j \leq 2}$ such that $a_{12} \leq a_{21}$. Then one can easily show that

$$
\chi_1(h_2)\chi_2(h_1) = \chi_1(h_1)^{a_{12}} = \chi_2(h_2)^{a_{21}}.
$$

Hence $\mathcal{D}$ is a datum of Cartan type.

Next we will show that $\Gamma(\mathcal{D})$ contains nonzero elements. By definition, $\Gamma(\mathcal{D})$ is the set of families $\mathcal{C} = (c_1, c_2, c_3, c_4)_{1 \leq i, j \leq 4}$ satisfying (3.9)-(3.11). So we only need to show that Equations (3.9) have nonzero solutions $\{c_1, c_2, c_3, c_4\}$, since (3.10)-(3.11) always have solutions. Equations (3.9) are equivalent to:

\begin{align}
(3.45) \quad & mn \equiv c_1 m^2 a \mod md; \\
(3.46) \quad & mn \equiv c_2 n^2 a \mod nd; \\
(3.47) \quad & 0 \equiv c_3 (bm^2 d^2) \mod md^2, mn \equiv c_3 m^2 \mod md^2; \\
(3.48) \quad & 0 \equiv c_4 (bm^2 d^2) \mod nd^2, mn \equiv c_4 n^2 (kd^2 - 1) \mod nd^2.
\end{align}
Remark 3.17. Consider the subalgebra of a Hopf algebra over a cyclic group, then the root vector relation must be trivial, i.e., all the radically graded quasi-Hopf algebras given in Theorem 3.16. In this section, we study radically graded quasi-Hopf algebras of Cartan type. We show that noncyclic groups.

Let \( H = \bigoplus_{i \geq 0} H[i] \) be a finite-dimensional radically graded quasi-Hopf algebra. Then \( H \) is genuine if and only if \( H[0] \) is a genuine quasi-Hopf algebra.

Proof. “\( \Rightarrow \)”: Suppose \( H = (H, \Delta, \varepsilon, \Phi, S, \alpha, \beta) \). By the definition of a radically graded quasi-Hopf algebra, \( H[0] = (H[0], \Delta, \varepsilon, \Phi, S, \alpha, \beta) \) is a quasi-Hopf subalgebra of \( H \). If \( H \) is not genuine, then there is a twist \( J \) of \( H \), such that \( H^J \) is a Hopf algebra, i.e.,

\[
\Phi_J = 1 \otimes 1 \otimes 1, \quad \alpha_J \beta_J = 1.
\]

Let \( \pi : H \to H[0] \) be the natural projection, where \( I = \bigoplus_{i \geq 1} H[i] \) is the Jacobson radical of \( H \). Define \( J_0 = (\pi \otimes \varepsilon)(J) \). It is clear that \( J = J_0 + J_{\geq 1} \), where \( J_{\geq 1} \in H \otimes I + I \otimes H \).

Since \( \varepsilon(I) = 0 \), we have \((\text{id} \otimes \varepsilon)(J_0) = (\pi \otimes \varepsilon)(\text{id} \otimes \varepsilon)(J) = 1 \). Similarly, \((\varepsilon \otimes \text{id})(J_0) = 1 \). It

| Table 1. \( a, b, k \) associated to \( A_2, B_2, G_2 \). |
|---------------------------------|----------------|
| \( a, b, k \) associated to \( A \) | Cartan matrix \( A \) |
| 1. \( a = 1, b = -3, k = 0 \) | \( A_2 \) |
| 2. \( a = 1, b = -3, k = -1 \) | \( B_2 \) |
| 3. \( a = 3, b = -6, k = -4 \) | \( G_2 \) |

By Proposition 3.15, these equations have solutions. It is obvious that any solution \((c_1, c_2, c_3, c_4)\) of Equations (3.45)-(3.48) should not be zero since \((n, d) = 1\). Hence, \( \Gamma(\mathfrak{D}) \) contains nonzero elements.

At last, we will show that there exists a family of nonzero modified root vector parameters \( \mu \) for \( \mathfrak{D} \). Note that \( A \) is connected, so \( \lambda \) must be zero. Since \( N = |\chi_i(h_i)| \) for \( i = 1, 2 \), (see (2.16) for definition), it is obvious that \( N = |\zeta_{2\alpha i mn}| = d^2 \). Let \( \alpha_i \) be the simple root corresponding to \( X_i \), \( 1 \leq i \leq 2 \). We have \( h_i^N = (g_1 g_2)^d^e = g_1^a g_2^b \in G \), \( h_i^N \neq 1 \), and \( \chi_i^N = \chi_i^a = 1 \). So \( \mu_\alpha \) is a nonzero parameter. Thus, there exists a family \( \mu \) of nonzero modified root vector parameters for \( \mathfrak{D} \). The quasi-Hopf algebra \( u(\mathfrak{D}, 0, \mu, A) \) is a nonradically graded quasi-Hopf algebra associated to \( A \). In Section 6, we will show that these quasi-Hopf algebras are genuine.

4. Radically graded quasi-Hopf algebras of Cartan type

In this section, we study radically graded quasi-Hopf algebras of Cartan type. We show that all the radically graded quasi-Hopf algebras given in Theorem 3.4 are genuine quasi-Hopf algebras, which leads to some interesting classification results.

4.1. Radically graded quasi-Hopf algebras of Cartan type are genuine. In general, it is very difficult to determine whether a nonradically graded quasi-Hopf algebra is genuine or not. However, for radically graded quasi-Hopf algebras, we have the following proposition.

Proposition 4.1. Suppose that \( H = \bigoplus_{i \geq 0} H[i] \) is a finite-dimensional radically graded quasi-Hopf algebra. Then \( H \) is genuine if and only if \( H[0] \) is a genuine quasi-Hopf algebra.
is obvious that \( J_0 \) has the inverse \((\pi \otimes \pi)(J^{-1})\) because \( \pi \) is an algebra morphism. It follows that \( J_0 \) is a twist for \( H[0] \). Now \( \Phi \in H[0] \#^3 \) implies that

\[
\begin{cases}
\Phi_{J_0} \in H[0] \#^3, \\
\Phi_{J_2} \in H \otimes H \otimes I + H \otimes I \otimes H + I \otimes H \otimes H.
\end{cases}
\]

Combining \( \Phi_J = 1 \otimes 1 \otimes 1 \), we obtain \( \Phi_{J_0} = \Phi_{J} = 1 \otimes 1 \otimes 1 \). Similarly, we have \( \alpha_{J_1} \beta_{J_0} = 1 \). Hence, \( H[0] \#^3 \) is a Hopf algebra, a contradiction to the fact that \( H[0] \) is genuine.

\( \Rightarrow \): If \( H[0] \) is not genuine, then there is a twist \( J \) of \( H[0] \) such that \( H[0] \#^J \) is a Hopf algebra. It is easy to see that \( J \) is also a twist of \( H \), and \( H^J \) is a Hopf algebra. \( \square \)

**Theorem 4.2.** Suppose that \( u(\mathfrak{D}, \lambda, \mu, \Phi_\mathfrak{D}) \) is a quasi-Hopf algebra of Cartan type with \( \lambda = 0, \mu = 0 \). Then \( u(\mathfrak{D}, \lambda, \mu, \Phi_\mathfrak{D}) \) is a genuine quasi-Hopf algebra.

**Proof.** Keep the same notations as those in Theorem 3.4. Let \( H = u(\mathfrak{D}, 0, 0, \Phi_\mathfrak{D}) \) and the radically graded structure is given by \( H = \oplus_{i \geq 0} H[i] \). Then \( H[0] = \mathbb{k}G \), and the associator

\[
\Phi_e = \sum_{e,f,g \in G} \phi_{\mathfrak{D}}(e,f,g)1_e \otimes 1_f \otimes 1_g
\]

for a nonzero \( \mathfrak{D} \in \Gamma(\mathfrak{D}) \). By Proposition 2.17, \( \phi_{\mathfrak{D}} \) is a 3-cocycle on \( G \), but not a coboundary. So \( (H[0], \Phi_\mathfrak{D}) \) is a genuine quasi-Hopf algebra. It follows from Proposition 4.1 that \( H = u(\mathfrak{D}, 0, 0, \Phi_\mathfrak{D}) \) is a genuine quasi-Hopf algebra. \( \square \)

### 4.2. Some classification results

Let \( H = \oplus_{i \geq 0} H[i] \) be a finite-dimensional radically graded quasi-Hopf algebra. The ideal \( I = \oplus_{i \geq 1} H[i] \) is the radical of \( H \). Note that \( H[i] = I^i/I^{i+1} \). In fact, we have the following relations:

\[
H[i] = H[1]^i, i \geq 1.
\]

Assume that \( H_0 = \mathbb{k}[G] \) and \( G \) an abelian group. Then \((\mathbb{k}[G], \Phi)\) is a quasi-Hopf subalgebra of \( H \) with the inherited associator \( \Phi \) and the restricted antipode \((S|_{H_0}, \alpha, \beta)\). Now we construct a new quasi-Hopf algebra \( \hat{H} \). By \( \triangleright \) we denote the inner action of \( H[0] \) on \( H[1] \):

\[
g \triangleright X = g \cdot X \cdot g^{-1}, \ g \in G, \ X \in H[1],
\]

where \( \cdot \) stands for the multiplication of \( H \). Extend the action of \( H[0] \) on \( H[1] \) to the tensor algebra \( T(H[1]) \) naturally.

Let \( \hat{H} \) be the smash product algebra \( T(H[1]) \rtimes H[0] \). The algebra \( \hat{H} \) has a natural comultiplication given by

\[
\Delta H(X) = \Delta H(X), \Delta H(g) = g \otimes g, \ X \in H[1], \ g \in G.
\]

Let \( S' : \hat{H} \to \hat{H} \otimes \hat{H} \) be an algebra anticomorphism such that \( S'|_{H[0] \otimes H[1]} = S|_{H[0] \otimes H[1]} \). One may verify straightforwardly that \((\hat{H}, \Delta H, \Phi, S', \alpha, \beta)\) forms a quasi-Hopf algebra. It is obvious that we have a canonical surjective homomorphism \( P : \hat{H} \to H \) such that \( P \) restricts to the identity on \( H[0] \otimes H[1] \).

In what follows, the elements in \( H[n] \) will be said to be of degree \( n \). In order to classify the finite-dimensional radically graded quasi-Hopf algebras over abelian groups, we need the following proposition, whose proof is parallel to [5, Proposition 3.3.2, Proposition 3.3.3], hence will be omitted.
Proposition 4.3. Let $H$ be a finite-dimensional radically graded quasi-Hopf algebra, and $\pi : H \to u(\mathcal{D}, 0, 0, \Phi_c)$ a quasi-Hopf algebra epimorphism such that the restriction of $\pi$ to the parts of degree 0 and 1 is the identity. Then $\text{ad}_c(X_i)^{1-\alpha}(X_j) = 0, i \neq j$ and $X_i^{N_i} = 0, \alpha \in R_i^+$. Now we are able to give one of the main results of the paper. The notations of $G, \mathcal{G}, g_i, \mathcal{g}_i, 1 \leq i \leq n$ are the same as those in Subsection 3.1.

Theorem 4.4. Suppose that $H$ is a radically graded finite-dimensional genuine quasi-Hopf algebra over an abelian group $G$ with an associator $\Phi = \sum e, f, g \in G \phi(e, f, g)1_e \otimes 1_f \otimes 1_g$, where $\phi$ is a 3-cocycle on $G$ satisfying $2, 3, 5, 7 \mid \dim(H)$. Then $H \cong u(\mathcal{D}, 0, 0, \Phi_c)$ for some datum of finite Cartan type $\mathcal{D}$ and some nonzero $c \in \Gamma(\mathcal{D})$.

Proof. According to Subsection 2.5, we know that $\{ \phi_{e, f, g} : e, f, g \in A, c, r, s, t \geq 0 \}$ is a complete set of representatives of abelian 3-cocycles on $G$. Thus there exists a 2-cochain $J$ on $G$ such that $\phi J = \phi_0$ for some $c \in A$ satisfying $c, r, s, t = 0$ for all $1 \leq r < s < t \leq n$. Let $\mathcal{J} = \sum_{f, g}^J f, g, 1_r \otimes 1_s \otimes 1_t$. It is clear that the associator of $H^J$ is $\Phi_c$. Thus, without loss of generality, we may assume that the associator of $H$ is $\Phi_c$ for some $c \in \{ c \in A | c, r, s, t \geq 0, 1 \leq r < s < t < n \}$.

Since $H = \bigoplus_{i \geq 2} H[i]$ is a radically graded quasi-Hopf algebra, we can construct a new quasi-Hopf algebra $\tilde{H}$, so that there is an epimorphism $P : \tilde{H} \to H$ such that $P$ restricts to the identity on $H[0] \oplus H[1]$. Denote by $L$ the sum of all quasi-Hopf ideals of $\tilde{H}$ contained in $\sum_{i \geq 2} H[i]$. It is easy to see that $ker P \subset L$. Let $\overline{H}$ be the quotient $\tilde{H}/L$, and $\eta : \tilde{H} \to \overline{H}$ the canonical projection. Thus, there exists an epimorphism $\pi : H \to \overline{H}$ such that $\pi = \eta P$. Note that $\pi$ restricts to the identity as well on $H[0] \oplus H[1]$.

Next we show that $\overline{H} \cong u(\mathcal{D}, 0, 0, \Phi_c)$ for some $\mathcal{D}$ and some $c \in \Gamma(\mathcal{D})$. Then, it follows from Proposition 4.3 that $\pi$ must be an isomorphism, and the proof will be done. Decompose $H[1] = \bigoplus_{c \in \mathcal{G}} H_c[1]$, where

$$H_c[1] = \{ X \in H[1] | gXg^{-1} = \chi(g)X, \forall g \in \mathcal{G} \}.$$

(4.2)

For each $c \in \mathcal{G}$, define a $\tilde{\chi} \in \hat{\mathcal{G}}$ by $\tilde{\chi}(g_i) = \chi(g_i)\chi^{-1}, 1 \leq i \leq n$. Denote by $\tilde{H}$ the quasi-Hopf algebra generated by $\overline{H}$ and $\mathcal{g}_i, 1 \leq i \leq n$, where $\mathcal{g}_i^{-1} = g_i$, and $gXg^{-1} = \tilde{\chi}(g)X$ for all $g \in \mathcal{G}$ and $X \in H_c[1]$. It is obvious that $\tilde{H}$ is a radically graded quasi-Hopf algebra over $\mathcal{G}$, and $\overline{H}$ is the quasi-Hopf subalgebra of $\tilde{H}$ generated by $\tilde{H}[1]$ and $\mathcal{g}_i, 1 \leq i \leq n$.

Consider $A = \tilde{H}^{-1}$. Since $A$ is a finite-dimensional radically graded Hopf algebra over $\mathcal{G}$, and $A$ is of the form $R \# \mathcal{G}$ for some braided graded Hopf algebra in the category of Yeter-Drinfeld modules over $\mathcal{G}$. So $A$ is also a finite-dimensional pointed Hopf algebra over $\mathcal{G}$. By the classification result of $[4]$, there exists a datum of finite Cartan type $\mathcal{D} = \mathcal{D}(\mathcal{G}, (h_1), 1 \leq i \leq \mathcal{G}, (\chi_j), 1 \leq j \leq \mathcal{G})$ such that $A = u(\mathcal{D}, 0, 0)$. Since $\overline{H}$ is generated by $A[1]$ and $\mathcal{g}_i, 1 \leq i \leq n$, there is a nonzero $c \in \Gamma(\mathcal{D})$ such that $\overline{H} \cong u(\mathcal{D}, 0, 0, \Phi_c)$ by Proposition 3.12. Now we are able to give one of the main results of the paper. The notations of $G, \mathcal{G}, g_i, \mathcal{g}_i, 1 \leq i \leq n$ are the same as those in Subsection 3.1.

Theorem 4.4. Suppose that $H$ is a radically graded finite-dimensional genuine quasi-Hopf algebra over an abelian group $G = \mathbb{Z}_m \otimes \mathbb{Z}_n$ such that $2, 3, 5, 7 \mid \dim(H)$. Then $H \cong u(\mathcal{D}, 0, 0, \Phi_c)$ for some datum of finite Cartan type $\mathcal{D}$ and some $c \in \Gamma(\mathcal{D})$. From Proposition 2.17 we know that every 3-cocycle on a cyclic group or on an abelian group of the form $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ is abelian. So we have the following.

Corollary 4.5. Suppose that $H$ is a radically graded finite-dimensional genuine quasi-Hopf algebra over an abelian group $G = \mathbb{Z}_m \otimes \mathbb{Z}_n$ such that $2, 3, 5, 7 \mid \dim(H)$. Then $H \cong u(\mathcal{D}, 0, 0, \Phi_c)$ for some datum of finite Cartan type $\mathcal{D}$ and some $c \in \Gamma(\mathcal{D})$. From Proposition 2.17 we know that every 3-cocycle on a cyclic group or on an abelian group of the form $\mathbb{Z}_{m_1} \times \mathbb{Z}_{m_2}$ is abelian. So we have the following.
5. Small quasi-quantum groups

In this section, we will introduce small quasi-quantum groups. These algebras can be viewed as natural generalization of small quantum groups. We will present several examples of small quasi-quantum groups which are genuine quasi-Hopf algebras. We fix a finite abelian group \( G \) with free generators \( \{ g_i \mid 1 \leq i \leq n \} \). The notations \( \mathcal{G} \) and \( \{ g_i \mid 1 \leq i \leq n \} \) are defined in the the same way as those in Subsection 3.1.

5.1. Small quasi-quantum groups. Suppose that \( A = (a_{ij})_{1 \leq i,j \leq n} \) is a finite Cartan matrix and that \( D(A) \) is the Cartan matrix \( \begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix} \). Let \( G = \mathbb{Z}_m^n = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle \).

**Definition 5.1.** Let \( \mathcal{D}(\mathcal{G}, (h_i)_{1 \leq i \leq 2n}, (\chi_i)_{1 \leq i \leq 2n}, D(A)) \) be a datum of finite Cartan type such that \( h_i = h_{i+1} \in \mathcal{G}, \chi_i = \chi_{i+1}, 1 \leq i \leq n \). Suppose that \( \mathcal{G} \in \Gamma(\mathcal{D}) \) is nonzero. We call the quasi-Hopf algebra \( Qu(\mathcal{D}, \lambda, \phi) = u(\mathcal{D}, \lambda, \mu, \phi) \) a quasi-small quantum group if \( \mu = 0 \) and \( \lambda_{ij} \neq 0 \) if and only if \( j = i + n \) for \( 1 \leq i \leq n \).

The quasi-Hopf algebras in Example 3.14 are examples of quasi-small quantum groups. More examples will be given before we show that small quasi-quantum groups are natural generalization of small quantum groups. For \( 1 \leq i \leq n \), define \( E_i = X_i, F_i = X_{i+n}h_i^{-1} \) and \( X'_i = X_{i+n} \). Let \( V = k \{ E_1, \cdots, E_n \}, V' = k \{ X'_1, \cdots, X'_n \} \) and \( U = k \{ F_1, \cdots, F_n \} \). It is obvious that \( V \) and \( V' \) are braided vector spaces of Cartan type with the braiding matrices \((q_{ij})_{1 \leq i,j \leq n}\) and \((q_{ij}^{-1})_{1 \leq i,j \leq n}\) respectively, where \( q_{ij} = \chi_j(h_i), 1 \leq i,j \leq n \). Note that By definition de braided vector spaces of Cartan type (see Subsection 2.3), the associated Cartan matrices of \( V \) and \( V' \) are both equal to \( A \). Let \( q_{ij} = q_{ij}^{-1}, 1 \leq i,j \leq n \). We define a braiding \( c \) on \( U \) as follows:

\[
c(F_i \otimes F_j) = q_{ij}^2 F_j \otimes F_i, 1 \leq i,j \leq n.
\]

For all \( 1 \leq i,j \leq n \), we have \( q_{ij}q_{ji} = q_{ij}^{-1}q_{ji}^{-1} = q_{ij}^{-1} = q_{ji}^{-1}q_{ij}^{-1} \). So \( (U, c) \) is also a braided vector space of Cartan type, and the associated Cartan matrix is \( A \) as well. So we can define braided commutators, the braided adjoint action and the root vectors over \( T(U) \), see Subsection 2.3 for details.

Now let \( R^+ \) be the positive root system corresponding to the Cartan matrix \( A \) with respect to the simple roots \( \alpha_1, \cdots, \alpha_n \), and \( F_\alpha, \alpha \in R^+ \) the root vectors such that \( F_\alpha = F_i \) for \( 1 \leq i \leq n \). Let \( \Omega \) be the set of the connected components of \( I = \{ 1, \cdots, n \} \), and \( R^+_J \) be the positive root system corresponding to \( J \in \Omega \).

**Proposition 5.2.** \( Qu(\mathcal{D}, \lambda, \phi) \) is generated by \( g_i, E_i, F_i, 1 \leq i \leq n \) subject to the relations:

\[
 g_iE_j g_i^{-1} = \chi_j(g_i)E_j, \quad g_iF_j g_i^{-1} = \chi_j^{-1}(g_i)F_j, \quad 1 \leq i,j \leq n, \\
 E_iF_j - F_jE_i = \delta_{ij} \lambda_{i+n}h_i^{-1} - h_i, \quad \lambda_{i+n} \neq 0, \quad 1 \leq i,j \leq n, \\
 ad_c(E_i)^{-1} - a_{ii}(E_j) = 0, \quad ad_c(F_i)^{-1} - a_{ji}(F_j) = 0, \quad 1 \leq i \neq j \leq n, \\
 E_i^N_{\alpha} = 0, \quad F_\alpha^N = 0, \quad \alpha \in R^+_J, \quad J \in \Omega.
\]

The comultiplication is determined by

\[
 \Delta(E_i) = \sum_{f,g \in \mathcal{G}} \Psi_i(f,g) E_i 1_f \otimes 1_g + h_i \otimes E_i, \\
 \Delta(F_i) = \sum_{f,g \in \mathcal{G}} \Psi_{i+n}(f,g) \chi_{i}(h_{i}^{-1}) F_i 1_f \otimes 1_g + 1 \otimes F_i.
\]
The associator is \( \Phi \) and the antipode \((S, \alpha, 1)\) is given by
\begin{align}
\alpha &= \sum_{g \in G} Y(g)1_g, \\
S(F_i) &= \chi_i^{-1}(h_i) \sum_{g \in G} \chi_g(h_i) F_{i+1}(g) F_i 1_g, \\
S(E_i) &= \sum_{g \in G} F_i(g) E_i 1_g,
\end{align}
for \( 1 \leq i \leq n \).

**Proof.** By Theorem 3.4, \( Qu(\mathcal{D}, \lambda, \Phi) \) is generated by \( G \) and \( X_i, 1 \leq i \leq n, 1 \leq j \leq 2n \) subject to the relations (3.19)-(3.22). Since \( F_i h_i = X_i \), the algebra \( Qu(\mathcal{D}, \lambda, \Phi) \) is also generated by \( g_i, E_i, F_i, 1 \leq i \leq n \). Now we show that the relations (5.1)-(5.4) are equivalent to the relations (3.19)-(3.22). It is easy to see that (5.1) equals (3.19). For all \( 1 \leq i, j \leq n \), we have:
\[
E_i F_j - F_j E_i = X_i X_{j+n} h_j^{-1} - X_{j+n} h_j^{-1} X_i
\]
Next we compute the comultiplication and the antipode of \( Qu(\mathcal{D}, \lambda, \Phi) \) for the generators. Formula (5.5) follows from the fact \( h_i = \sum_{f \in G} \Theta_i(f) 1_f \) for \( h_i \in G \). Formula (5.6) holds because of the following equations:
\[
\Delta(F_i) = \Delta(X_{i+n} h_i^{-1})
= \left( \sum_{f, g \in G} \Psi_{i+n}(f, g) X_{i+n} 1_f \otimes 1_g + \sum_{f \in G} \Theta_{i+n}(f) 1_f \otimes X_{i+n}(h_i^{-1} \otimes h_i^{-1}) \right)
= \sum_{f, g \in G} \Psi_{i+n}(f, g) \chi_g(h_i^{-1}) F_i 1_f \otimes 1_g + \sum_{f \in G} \Theta_{i+n}(f) \chi_f(h_i^{-1}) 1_f \otimes F_i
= \sum_{f, g \in G} \Psi_{i+n}(f, g) \chi_g(h_i^{-1}) F_i 1_f \otimes 1_g + 1 \otimes F_i.
\]
Formula (5.7) is obvious. Formula (5.8) follows from
\[ S(F_i) = S(h_i^{-1})S(X_{i+n}) = h_i \sum_{g \in G} F_{i+n}(g)X_{i+n}1_g \]
\[ = \chi_{i+n}(h_i) \sum_{g \in G} \chi_g(h_i^2)F_{i+n}(g)F_i1_g, \]
where \( \chi_g, g \in G, \) is defined by (3.1).

Now let \( D(G, (h_i)_{1 \leq i \leq 2n}, (\chi_i)_{1 \leq i \leq 2n}, D(A)) \) be a datum of finite Cartan type such that \( h_i = h_{n+i} \in G, \chi_i = \chi_{i+n}^{-1}, 1 \leq i \leq n. \) Note that \( \Gamma(D) \) is not empty since \( 0 \in \Gamma(D). \)

Take an element \( \varepsilon \in \Gamma(D). \) We define an algebra \( H_{\varepsilon} \) generated by \( G \) and \( E_i, F_i, 1 \leq i \leq n \) subject to the Relations (5.1)-(5.4). Define an algebra morphism \( \triangle : H_{\varepsilon} \to H_{\varepsilon} \otimes H_{\varepsilon} \) by (5.5)-(5.6), and an algebra antimorphism \( S : H_{\varepsilon} \to H_{\varepsilon} \) by (5.8). Let \( \alpha \) be an element of \( H_{\varepsilon} \) defined by (5.7). Define an algebra morphism \( \varepsilon : H_{\varepsilon} \to \mathbb{k} \) such that \( \varepsilon(g) = 1, \varepsilon(E_i) = \varepsilon(F_i) = 0 \) for \( g \in G, 1 \leq i \leq n. \) We have the following identification of the algebra \( H_{\varepsilon}. \)

**Proposition 5.3.**

1. If \( \varepsilon = 0, \) then \( (H_{\varepsilon}, \triangle, \varepsilon, S) \) is isomorphic to the AS-Hopf algebra \( u(D', \lambda, 0), \) where \( D' = D(G, (h_i)_{1 \leq i \leq 2n}, (\chi_i')_{1 \leq i \leq 2n}, D(A)) \), and \( \chi_i' = \chi_i|_{\mathbb{G}}, 1 \leq i \leq n. \) Each small quantum group is isomorphic to a Hopf algebra in the following way.
2. If \( \varepsilon \neq 0, \) then \( (H_{\varepsilon}, \triangle, \varepsilon, S, \alpha, 1) \) is isomorphic to the small quasi-quantum group \( Qu(D, \lambda, \Phi_\varepsilon). \)

**Proof.** Observe that the functions defined by (3.13)-(3.15) are equivalent to the constant function 1 if \( \varepsilon = 0. \) Hence, the algebra morphism \( \Upsilon : H_{\varepsilon} \to u(D', \lambda, 0) \) given by
\[ \Upsilon(g) = g, \quad \Upsilon(E_i) = X_i, \quad \Upsilon(F_i) = X_{i+n}h_i, \]
for all \( g \in G, 1 \leq i \leq n, \) is a Hopf algebra isomorphism. The second part of the proposition follows from Proposition 5.2. \( \square \)

From this proposition, we can see that small quasi-quantum groups are natural generalizations of small quantum groups.

### 5.2. Triangular decomposition and half small quasi-quantum groups.

In this subsection, we study the triangular decomposition of small quasi-quantum groups. Fix an abelian group \( G = \mathbb{Z}^n = (g_1) \times \cdots \times (g_n) \) and a finite Cartan matrix \( A = (a_{ij})_{1 \leq i, j \leq n}. \) Assume that \( D = D(G, (h_i)_{1 \leq i \leq 2n}, (\chi_i)_{1 \leq i \leq 2n}, D(A)) \) is a datum of finite Cartan type and \( Qu(D, \lambda, \Phi_\varepsilon) \) is a small quantum group. Denote by \( Qu^+(D, \Phi_\varepsilon) \) (resp. \( Qu^-(D, \Phi_\varepsilon) \)) the subalgebra generated by \( E_i, 1 \leq i \leq n \) (resp. \( F_i, 1 \leq i \leq n \)), and \( Qu^0 = \mathbb{k} G. \) So we have a natural linear isomorphism
\[ \varphi : Qu^+(D, \Phi_\varepsilon) \otimes Qu^0 \otimes Qu^-(D, \Phi_\varepsilon) \to Qu(D, \lambda, \Phi_\varepsilon) \]
\[ x \otimes y \otimes z \to xyz, \]
for all \( x \in Qu^+(D, \Phi_\varepsilon), \ y \in Qu^0, \ z \in Qu^-(D, \Phi_\varepsilon). \) This decomposition \( Qu(D, \lambda, \Phi_\varepsilon) = Qu^+(D, \Phi_\varepsilon) \otimes Qu^0 \otimes Qu^-(D, \Phi_\varepsilon) \) is called the triangula decomposition of \( Qu(D, \lambda, \Phi_\varepsilon). \)

Denote by \( Qu^{\geq 0}(D, \Phi_\varepsilon) \) (resp. \( Qu^{< 0}(D, \Phi_\varepsilon) \)) the subalgebra of \( Qu(D, \lambda, \Phi_\varepsilon) \) generated by \( G \) and \( E_i, 1 \leq i \leq n. \) The following isomorphisms are obvious.

(5.10) \[ Qu^{\geq 0}(D, \Phi_\varepsilon) \cong u(D', 0, 0, \Phi_\varepsilon); \]
(5.11) \[ Qu^{< 0}(D, \Phi_\varepsilon) \cong u(D'', 0, 0, \Phi_\varepsilon). \]
where \( D' = D(G, (h_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, A) \), \( D'' = D(G, (h_i)_{n+1 \leq i \leq 2n}, (\chi_i)_{n+1 \leq i \leq n}, A) \). The two radically graded quasi-Hopf algebras are called the half small quasi-quantum groups. Keep the assumption of \( G \) and \( A \) as above, we have the following.

**Proposition 5.4.** Let \( D = D(g, (h_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, A) \) be a datum of finite Cartan type such that \( h_i \in G, 1 \leq i \leq n \), and \( \chi(h_j) = \chi(h_i), 1 \leq i, j \leq n \). Then for any nonzero \( \xi \in \Gamma(D) \), \( u(0, 0, \Phi_\xi) \) is a half small quasi-quantum group.

**Proof.** We need to show that there exists a small quasi-quantum group \( Qu(D', \lambda, \Phi_\xi) \) such that \( u(D', 0, 0, \Phi_\xi) \cong Qu(D', \Phi_\xi) \). Define \( h_{i+n} = h_i \) and \( \chi_{i+n} = \chi_i^{-1} \) for \( 1 \leq i \leq n \). For \( 1 \leq i, j \leq n \), we have the following:

\[
\chi_i(h_{j+n}) \chi_{j+n}(h_i) = \chi_i(h_j) \chi_j^{-1}(h_i) = 1.
\]

It is clear that \( D' = D(G, (h_i)_{1 \leq i \leq 2n}, (\chi_i)_{1 \leq i \leq 2n}, D(A)) \) is a datum of finite Cartan type. For each \( 1 \leq i \leq n \), let \( h_i = \prod_{j=1}^n \gamma^j_{i} \). Since \( h_i \in G \), we have \( \gamma^j_{i} \equiv 0 \mod m \). It follows that \( \Gamma(D) = \Gamma(D') \). Moreover, \( h_i h_{i+j} = h_i^2 \in G \) and \( \chi(h_i) = \chi_i \chi_i^{-1} = 1 \). Thus, we obtain a family of linking parameters \( \lambda = (\lambda_{ij})_{1 \leq i \leq 2n, i \neq j} \) for \( D' \) such that \( \lambda_{ij} \neq 0 \) if and only if \( j = i + n \).

Since each connected component of \( I = \{1, \ldots, n\} \) satisfies (2.14), we have \( h_i h_{i+n} = h_i^2 \neq 1 \). This implies that \( \lambda \) is a family of modified linking parameters. Therefore, we obtain a small quasi-quantum group \( Qu(D', \lambda, \Phi_\xi) \) such that \( u(D', 0, 0, \Phi_\xi) \cong Qu(D', \Phi_\xi) \).

5.3. **Examples of genuine small quasi-quantum groups.** Let \( A = (a_{ij})_{1 \leq i, j \leq n} \) be a finite Cartan matrix, \( (d_1, \ldots, d_n) \) a vector with elements in \( \{1, 2, 3\} \) such that \( (d, a_{ij})_{1 \leq i, j \leq n} \) is symmetric. Let \( q \) be an \( N \)-th primitive root of unity, where \( N \) is an odd positive integer. Moreover, in case \( A \) has a connected component of type \( G_2 \), we will add one more assumption that \( N \) is prime to 3. Let \( p \) be a positive odd integer and \( m = pN \). Choose \( \zeta_m \) and \( \zeta_m \) such that \( \zeta_m^m = \zeta_m \). Let \( G = \mathbb{Z}_m = \{g_1, \ldots, g_m\} \), and \( l, 1 \leq l < m \), be an integer such that \( lp \neq 0 \mod m \). Define characters \( \{\chi_i|1 \leq i \leq n\} \) on \( G \) as follows:

\[
\chi_i(g_j) = \begin{cases} 
\zeta_m^{2pd_i}, & \text{if } i = j; \\
\zeta_m^{pd_i a_{ij}}, & \text{if } i \neq j \text{ and } a_{ij} \neq 0; \\
1, & \text{if } a_{ij} = 0.
\end{cases}
\]

With the above notations, it is easy to verify that \( D = D(G, (h_i)_{1 \leq i \leq 2n}, (\chi_i)_{1 \leq i \leq 2n}, D(A)) \) is a datum of finite Cartan type, where \( h_i = h_{i+n} = g_{i}^{nl} \) and \( \chi_{i+n} = \chi_i^{-1} \) for \( 1 \leq i \leq n \).

**Lemma 5.5.** There are nonzero elements \( \xi = (c_i, c_{jk})_{1 \leq i \leq n, 1 \leq j < k \leq n} \) in \( \Gamma(D) \) such that \( c_i = k_iN \) for some \( 1 \leq k_i < p \), \( 1 \leq i \leq n \).

**Proof.** It is obvious that (3.10)-(3.11) are solvable for \( (c_{jk})_{1 \leq j < k \leq n} \). Thus, it suffices to prove that Equation (3.9) has solutions \( c_i = k_iN \) for some \( 1 \leq k_i < p \), \( 1 \leq i \leq n \). Now the equations in variable \( c_i \) are

\[
ml \equiv 2c_ipd_i \mod m,
\]

\[
ml \equiv -2c_ipd_i \mod m,
\]

\[
0 \equiv c_ipd_ia_{ij} \mod m, \quad i \neq j, a_{ij} \neq 0,
\]

\[
0 \equiv -c_ipd_ia_{ij} \mod m, \quad i \neq j, a_{ij} \neq 0.
\]

It is not difficult to see that Equations (5.13)-(5.16) have a set of solutions

\[
\{c_i = k_iN|0 \leq k < p\}.
\]

We have thus proved the lemma.

\[\square\]
We need the following lemma.

**Lemma 5.6.** There exists a family of modified linking parameters $\lambda$ for $\mathcal{D}$ satisfying condition: $\lambda_{ij} \neq 0$ if and only if $i + n = j$.

**Proof.** Note that for each $1 \leq i \leq n$, we have $\chi_i \chi_{i+n} = \varepsilon$ because $\chi_{i+n} = \chi_i^{-1}$. The fact that $lp \neq 0 \mod m$, implies that $l \neq 0 \mod N$. Hence $2l \neq 0 \mod N$ because $N$ is odd. It follows that $h_i h_{i+n} = g_{-n}^{2m} = g_{i+1}^{2l} \in G$ and $h_i h_{i+n} \neq 1$. By definition, there exists a family of modified linking parameters $\lambda$ for $\mathcal{D}$ such that $\lambda_{ij} \neq 0$ if and only if $i + n = j$. \qed

From Lemmas 5.5-5.6 and the definition of small quasi-quantum groups, we obtain the following.

**Proposition 5.7.** Let $\xi$ be a nonzero parameter in $\Gamma(\mathcal{D})$, and $\lambda$ a a family of modified linking parameters such that $\lambda_{ij} \neq 0$ if and only if $i + n = j$. Then $u(\mathcal{D}, \lambda, 0, \Phi_\xi)$ is a small quasi-quantum group.

In what follows, we let $Qu(\mathcal{D}, \lambda, \Phi_\xi) = u(\mathcal{D}, \lambda, 0, \Phi_\xi)$, $\xi \neq 0$, be a small quasi-quantum group given in Proposition 5.7. We conjecture that $Qu(\mathcal{D}, \lambda, \Phi_\xi)$ is genuine. At this moment, we are not able to prove it in general. But we have the following partial result.

**Proposition 5.8.** Suppose $l > 1$, $l|m$ and $l \nmid c_i$ for $1 \leq i \leq n$. Then $Qu(\mathcal{D}, \lambda, \Phi_\xi)$ is genuine.

**Proof.** Let $I$ be the quasi-Hopf ideal of $Qu(\mathcal{D}, \lambda, \Phi_\xi)$ generated by $\{X_i \mid 1 \leq i \leq n\}$. Set $\tilde{u} = Qu(\mathcal{D}, \lambda, \Phi_\xi)/I$. It is evident that $\tilde{u} \cong kG'$, where $G' = G/(g_i^{2l} - 1 \mid 1 \leq i \leq n)$. Since $\gcd(2l, m) = l$, we have $G' = G/(g_i^{2l} - 1 \mid 1 \leq i \leq n)$. For an element $g \in G$, we denote by $\overline{g}$ the corresponding element in the quotient group $G'$. Let $G''$ be the subgroup of $G$ generated by $\{g_i^{2l} \mid 1 \leq i \leq n\}$. Define a group isomorphism $\varphi: G'' \to G'$ by $\varphi(g_i^{2l}) = \overline{g_i}$, $1 \leq i \leq n$.

Let $f = \prod_i g_i^{a_i}$. We have the following expression of the element $\overline{f}$ in $\mathbf{k}G'$:

$$
\overline{f} = \frac{1}{|G|} \sum_{g \in G} \chi_f(g) \overline{g}
= \frac{1}{|G|} \prod_{0 \leq i_j < l, 0 \leq k_j < m} \sum_{1 \leq j \leq n} [\prod_n c_{m}^{n_{i_j+k_j}}(\prod_j g_{j}^{a_j})]
= \frac{1}{|G|} \prod_{j} \sum_{0 \leq i_j < l, 0 \leq k_j < m} c_{m}^{n_{i_j+k_j}}(\prod_j g_{j}^{a_j})
$$

It follows that $\overline{f} \neq 0$ if and only if $f \in G''$. Now we assume that $f = \prod_j g_j^{a_j} \in G''$. Then we have:

$$
\overline{f} = \frac{1}{|G|} \prod_{j} \sum_{0 \leq i_j < l, 0 \leq k_j < m} c_{m}^{a_j+k_j}(\prod_j g_{j}^{a_j})
= \frac{m}{|G|} \prod_{j} \sum_{0 \leq i_j < l} c_{m}^{a_j+k_j}(\prod_j g_{j}^{a_j})
= \frac{1}{|G|} \prod_{0 \leq i_j < l, 1 \leq j \leq n} [\prod_n c_{m}^{a_j}(\prod_j g_{j}^{a_j})]
= 1_{\varphi(f)}.
$$

This completes the proof. \qed
Define a 3-cocycle $\phi'_c$ on $G'$ as follows:

\[
\phi'_c(e, f, g) = \prod_{1 \leq i \leq n} \zeta_{l}^{c_i l_i + 1} \prod_{1 \leq j < k \leq n} \zeta_{l}^{c_j k + c_k}\]

for $e = \prod_{i=1}^{n} \prod_{i}^{\gamma_i}$, $f = \prod_{i=1}^{n} \prod_{j}^{\gamma_j}$, $g = \prod_{i=1}^{n} \prod_{i}^{\gamma_i}$.

Now we compute the associator $\Psi_\omega$ of $\mathfrak{k}G'$:

\[
\Psi_\omega = \sum_{e, f, g \in G} \phi(e, f, g) 1_e \otimes 1_f \otimes 1_g
\]

The second identity follows from (5.18), and $\prod_{f} = 0$ if $f \notin G^\circ$. The third identity follows from the definition of $\varphi$. Since $l \nmid c_i$ for $1 \leq i \leq \theta$, we know that $\phi'_c$ is a 3-cocycle on $G'$ according to Proposition 2.17. Moreover, $\phi'_c$ is not a 3coboundary. Hence $\hat{u} \cong (\mathfrak{k}G', \overline{\Phi})$ is a genuine quasi-Hopf algebra.

Suppose that $Qu(\mathfrak{D}, \lambda, \Phi_\omega)$ is not genuine. Then $Qu(\mathfrak{D}, \lambda, \Phi_\omega)$ is twist equivalent to a Hopf algebra, or equivalently the tensor category $\mathcal{M}$ of representations of $Qu(\mathfrak{D}, \lambda, \Phi_\omega)$ has a fiber functor. Let $Rep_{\mathcal{M}}(\mathfrak{k}G')$ be the representation category of the quasi-Hopf algebra $\hat{u} \cong \mathfrak{k}G'$.

It is obvious that $Rep_{\mathcal{M}}(\mathfrak{k}G')$ is a full tensor subcategory of $\mathcal{M}$. Hence $Rep_{\mathcal{M}}(\mathfrak{k}G')$ has a fiber functor as well. Thus $\hat{u}$ is twist equivalent to a Hopf algebra, a contradiction since $\hat{u}$ is genuine.

It is easy to see that there are (infinitely) many numbers $l, p, N$ to choose such that $l, m, c_i$, $1 \leq i \leq n$ satisfy the conditions in Proposition 5.8. Therefore, we obtain many examples of genuine small quasi-quantum groups associated to each finite Cartan matrix.

6. **Nonradically graded genuine Quasi-Hopf algebras associated to connected Cartan matrices**

In this section, we construct many new examples of nonradically graded genuine quasi-Hopf algebras associated to each finite connected Cartan matrix. Explicitly, for each finite connected Cartan matrix $A$, we will show that there exists a Cartan datum $\mathfrak{D}$ associated to $A$, nontrivial modified root vector parameters $\mu$ for $\mathfrak{D}$ and a nonzero $\rho \in \Gamma(\mathfrak{D})$ such that $u(\mathfrak{D}, 0, \mu, \Phi_\omega)$ is a finite-dimensional genuine quasi-Hopf algebra. All these quasi-Hopf algebras have only trivial linking relations since the Cartan matrices are assumed to be connected.

In the following, the groups $G = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle$ and $G = \langle g_1 \rangle \times \cdots \times \langle g_n \rangle$ are defined to be the same as those in Subsection 3.1. For each fixed Cartan matrix $A$, the corresponding root system $R$, positive root system $R^+$ with simple roots $\{\alpha_i| 1 \leq i \leq n\}$ are defined to be the same as those in Subsection 2.2.

Throughout this section, $p, q, d$ are positive odd numbers satisfying $(p, q) = (p, d) = (q, d) = 1$.

6.1. **Quasi-Hopf algebras of Cartan type $A_n$, $B_n$ and $C_n$**. Notice that examples of non-radically graded quasi-Hopf algebras associated to $A_n$, $B_n$, and $C_n$ have been given in Subsection 3.5, so in this subsection we always assume $n \geq 3$. Let $A = (a_{ij})_{1 \leq i, j \leq n}$ be the
Cartan matrix of type $A_n, B_n$ or $C_n$. Let $G = \langle g_1 \rangle \times \langle g_2 \rangle \times \cdots \times \langle g_{2n} \rangle$ be the abelian group determined by

$$|g_i| = \begin{cases} 
pd, & \text{if } i = 4k + 1 \text{ for some } k; \\
qd, & \text{if } i = 4k + 2 \text{ for some } k; \\
pd^2, & \text{if } i = 4k + 3 \text{ for some } k; \\
qd^2, & \text{if } i = 4k \text{ for some } k.
\end{cases}$$

(6.1)

For each $1 \leq i \leq n$, define an element in $\mathbb{G}$ as follows:

$$h_i = \begin{cases} 
(g_{2i-1}g_{2i})^{pq}, & \text{if } i \text{ is odd}; \\
(g_{2i-3}g_{2i-2}g_{2i-1}g_{2i+1}g_{2i+2})^{pq}, & \text{if } i \text{ is even and } i \neq n; \\
(g_{2i-3}g_{2i-2}g_{2i-1}g_{2i})^{pq}, & \text{if } i \text{ is even and } i = n.
\end{cases}$$

(6.2)

In order to give a datum of Cartan type associated to Cartan matrix $A$, we need to introduce some characters on $\mathbb{G}$. Let $a, b, r$ be the numbers given in Table 2. Define

$$\chi_1(g_i) = \begin{cases} 
\zeta_{3}, & j = 1,2; \\
\zeta_{4}, & j = 3,4; \\
1, & j \geq 5;
\end{cases}$$

(6.3)

$$\chi_2(g_i) = \begin{cases} 
\zeta_{3}^2, & j = 1,2; \\
\zeta_{4}^2, & j = 3; \\
\zeta_{4}^{2ad^2-1}, & j = 4; \\
\zeta_{4}^2, & j = 5,6; \\
1, & j \geq 7.
\end{cases}$$

(6.4)

For $3 \leq i < n$, define

$$\chi_i(g_i) = \begin{cases} 
\zeta_{3}^{-3b}, & \text{if } i \text{ is odd and } j = 2i-2,2i+1,2i+2; \\
\zeta_{4}^{2}, & \text{if } i \text{ is odd and } j = 2i-1,2i; \\
\zeta_{4}^{2}, & \text{if } i \text{ is even and } j = 2i-2,2i+1,2i+2; \\
\zeta_{4}^{2}, & \text{if } i \text{ is even and } j = 2i-1; \\
\zeta_{4}^{2-2ad^2-1}, & \text{if } i \text{ is even and } j = 2i; \\
1, & \text{otherwise}.
\end{cases}$$

(6.5)

When $n$ is odd, define

$$\chi_n(g_i) = \begin{cases} 
\zeta_{3}^{-3b}, & j = 2n-3,2n-2; \\
\zeta_{4}^{2}, & j = 2n-1,2n; \\
1, & \text{otherwise}.
\end{cases}$$

(6.6)

When $n$ is even, define

$$\chi_n(g_i) = \begin{cases} 
\zeta_{4}^{2}, & j = 2n-3,2n-2; \\
\zeta_{4}^{-1}, & j = 2n-1; \\
1, & \text{otherwise}.
\end{cases}$$

(6.7)

With these notations, we can give the following lemma.

**Lemma 6.1.** $\mathcal{D} = \mathcal{D}(\mathbb{G}, (h_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, A)$ is a datum of Cartan type.

**Proof.** Let $q_{ij} = \chi_j(h_i)$ for all $1 \leq i \neq j \leq n$. By definition of datum of Cartan type, we only need to show

$$q_{ij}q_{ji} = q_{ji}^{a_{ij}}, \quad 1 \leq i \neq j \leq n,$$

(6.8)

and it follows a direct verification. □
Lemma 6.2. \( \Gamma(\mathcal{D}) \) is not empty. For each family \( \mathcal{E} = (c_i, c_{jk})_{1 \leq i \leq 2n, 1 \leq j < k \leq 2n} \) in \( \Gamma(\mathcal{D}) \), we have \( c_i \neq 0 \) for \( 1 \leq i \leq 2n \).

Proof. Notice that Equations (3.10)-(3.11) always have solutions, for examples \( c_{ij} = 0 \) for all \( 1 \leq i < j \leq 2n \). So we only need to prove that Equations (3.9) have solutions and \( c_i \neq 0 \) for \( 1 \leq i \leq 2n \). Let \( m_i = |g_i| \) for each \( 1 \leq i \leq 2n \). It is clear that Equations (3.9) on variables \( c_1, c_2, c_3, c_4 \) are given by

\[
\begin{align*}
(6.9) & \quad c_1 ap^2 \equiv pq \mod pd, \\
(6.10) & \quad c_2 aq^2 \equiv pq \mod qd, \\
(6.11) & \quad c_3 p^2 \equiv pq \mod pd^2, \\
(6.12) & \quad c_3 rp^2d^2 \equiv 0 \mod pd^2, \\
(6.13) & \quad c_4(-2ad^2 - 1)q^2 \equiv pq \mod qd^2, \\
(6.14) & \quad c_4rq^2d^2 \equiv 0 \mod qd^2.
\end{align*}
\]

When \( 3 \leq i \leq n \) and \( i \) is odd, Equations (3.9) are

\[
\begin{align*}
(6.15) & \quad c_{2i-1} bp^2 \equiv pq \mod pd, \\
(6.16) & \quad c_{2i} bq^2 \equiv pq \mod qd.
\end{align*}
\]

When \( 3 < i < n \) and \( i \) is even, Equations (3.9) become

\[
\begin{align*}
(6.17) & \quad c_{2i-1} p^2 \equiv pq \mod pd^2, \\
(6.18) & \quad c_{2i-1}(-3b)p^2d^2 \equiv 0 \mod pd^2 \\
(6.19) & \quad c_{2i}(-2bd^2 - 1)q^2 \equiv pq \mod qd^2, \\
(6.20) & \quad c_{2i}(-3b)q^2d^2 \equiv 0 \mod qd^2.
\end{align*}
\]

When \( i = n \) is even, the equations (3.9) are given by

\[
\begin{align*}
(6.21) & \quad c_{2n-1} p^2 \equiv pq \mod pd^2, \\
(6.22) & \quad c_{2n-1}(-3b)p^2d^2 \equiv 0 \mod pd^2, \\
(6.23) & \quad c_{2n}(-1)q^2 \equiv pq \mod qd^2, \\
(6.24) & \quad c_{2n}(-3b)q^2d^2 \equiv 0 \mod qd^2.
\end{align*}
\]

It is obvious that any integers \( c_{2i-1}, c_{2i} \) are solution of (6.12), (6.14), (6.18), (6.20), (6.22) and (6.24). Since \( (ap^2, pd) = p, (aq^2, qd) = q, (p^2, pd^2) = p, ((-2ad^2 - 1)q^2, qd^2) = q \), so (6.9)-(6.14) have solutions by Proposition (3.15). Any solution \( c_1, c_2, c_3 \) or \( c_4 \) of (6.9)-(6.14) should not be zero since \( (p, d) = (q, d) = 1 \). Similarly, one can show that (6.15)-(6.17), (6.19)-(6.21) and (6.23) have nonzero solutions. \( \square \)

Lemma 6.3. There exists modified root vector parameters \( \mu \) for \( \mathcal{D} \) satisfying the condition:

\[
\begin{align*}
(6.25) & \quad \mu_\alpha \text{ is a nonzero parameter if and only if } \alpha = \alpha_i \text{ for some odd number } 1 \leq i \leq n.
\end{align*}
\]
Proof. Firstly, one can verify that \(|\chi_i(h_i)| = d_i^2| for 1 \leq i \leq n\). So for all \(1 \leq i \leq n\) and \(i\) is odd, we have \(h_i^{d_i} = (g_{2i-1}g_{2i}^{d_i})^{d_i} = g_{2i-1}g_{2i}^{d_i} \in G, h_i^{d_i} \neq 1\) and \(\chi_i^{d_i} = \varepsilon\), hence \(\mu_{\alpha_i} \neq 0\).

We proved the lemma. \(\square\)

**Proposition 6.4.** Let \(\zeta \in \Gamma(D)\) and \(\mu\) a family of modified root vector parameters for \(D\) satisfying condition (6.25). Then \(u(D, 0, \mu, \Phi_{k+})\) is a finite-dimensional nonradically graded genuine quasi-Hopf algebra associated to \(A\).

**Proof.** By Lemma 6.2-6.3 and Theorem 3.4, \(u(D, 0, \mu, \Phi_{k+})\) is a finite-dimensional nonradically graded quasi-Hopf algebra. So we only need to prove that \(u(D, 0, \mu, \Phi_{k+})\) is a genuine quasi-Hopf algebra.

Let \(\alpha\) be a positive root in \(R^+\), then we have \(\mu_{\alpha} = 0\) if \(\alpha \neq \alpha_i\) some odd number \(1 \leq i \leq n\).

Let \(I\) be the ideal of \(u(D, 0, \mu, \Phi_{k+})\) generated by

\[\{X_i, 1 - g_j|1 \leq i \leq n, j = 1 \text{ or } 2 \mod 4\} \]

It is obvious that \(I\) is a quasi-Hopf ideal of \(u(D, 0, \mu, \Phi_{k+})\). Denote by

\[G' = \langle g_i | i = 0 \text{ or } 3 \mod 4 \rangle \]

Then it is obvious that

\[(6.26)\quad kG' = u(D, 0, \mu, \Phi_{k+})/I.\]

Similar to the proof of Proposition 5.8, one can show that the associator of the \(kG'\) is

\[\Phi_{kG'} = \sum_{e, f, g \in G'} \Phi_{kG'}(e, f, g)1_e \otimes 1_f \otimes 1_g.\]

By lemma 6.2, \(\Phi_{kG'}\) is a not 3-coboundary on \(G'\). Hence \((kG', \Phi_{k})\) is a genuine quasi-Hopf algebra. This implies \(u(D, 0, \mu, \Phi_{k+})\) is genuine, since otherwise \((kG', \Phi_{k}) = u(D, 0, \mu, \Phi_{k+})/I\) should not be genuine, which is a contradiction. \(\square\)

### 6.2. Quasi-Hopf algebras of Cartan type \(D_n\)

In this subsection, we always assume that \(n \geq 4\). Let \(G = \langle g_1 \rangle \times \cdots \times \langle g_{2n} \rangle\) such that

\[(6.27)\quad |g_i| = \begin{cases} pd_i, & i = 1 \text{ or } 4k - 1 \text{ for } k \geq 1; \\ qd_i, & i = 2 \text{ or } 4k \text{ for } k \geq 1; \\ pd_i, & i = 4k + 1 \text{ for } k \geq 1; \\ qd_i, & i = 4k + 2 \text{ for } k \geq 1. \end{cases}\]

Let \((h_i)_{1 \leq i \leq n}\) be a family of elements in \(G\) given by

\[(6.28)\quad h_i = \begin{cases} (g_{2i-1}g_{2i})^{pq}, & \text{if } i = 1 \text{ or } 2k \text{ for } k \geq 1; \\ (\prod_{j=1}^{2} \mathbb{R})^{pq}, & \text{if } i = 3; \\ (g_{2i-3}g_{2i-2}g_{2i-1}g_{2i}g_{2i+1}g_{2i+2})^{pq}, & \text{if } i = 2k + 1 \neq n \text{ for } k \geq 2; \\ (g_{2n-3}g_{2n-2}g_{2n-1}g_{2n})^{pq}, & \text{if } i = n \text{ is odd}. \end{cases}\]

Now define a family \((\chi_i)_{1 \leq i \leq n}\) of characters on \(G\) as following.

When \(i = 1, 2\),

\[(6.29)\quad \chi_i(g_j) = \begin{cases} \zeta_{d_j^2}, & j = 2i - 1, 2i; \\ \zeta_{d_j^3}, & j = 5, 6; \\ 1, & \text{otherwise}. \end{cases}\]


With these definitions, one can verify that \( \mathcal{D} = \mathcal{D}(\mathfrak{g}, (h_i)_{1 \leq i \leq n}, (\chi_i)_{1 \leq i \leq n}, D_n) \) is a datum of Cartan type. Moreover, we have the following two lemmas, and the proofs, omitted, are similar as the proofs of Lemma 6.2 and 6.3.

**Lemma 6.5.** \( \Gamma(\mathcal{D}) \) is a nonempty set, and for each family \( \mathcal{E} = (c_i, c_{jk})_{1 \leq i \leq 2n, 1 \leq j < k \leq 2n} \) in \( \Gamma(\mathcal{D}) \), we have \( c_i \neq 0 \) for \( 1 \leq i \leq 2n \).

**Lemma 6.6.** There exists a family of modified root vector parameters \( \mu \) for \( \mathcal{D} \) satisfying the condition:

\[
\mu_{\alpha} \text{ is a nonzero if and only if } \alpha = \alpha_1 \text{ or } \alpha_i \text{ for some even number } 1 \leq i \leq n.
\]

**Proposition 6.7.** Let \( \mathcal{E} \in \Gamma(\mathcal{D}) \), and \( \mu \) a family of modified root vector parameters for \( \mathcal{D} \) satisfying the condition (6.34). Then \( u(\mathcal{D}, 0, \mu, \Phi_\mathcal{E}) \) is a genuine quasi-Hopf algebra.

**Proof.** Similar to the proof of Proposition 6.4. \( \square \)

### 6.3. Quasi-Hopf algebras of Cartan type \( E_6 \), \( E_7 \) and \( E_8 \)

In this subsection, we always assume \( n = 6, 7 \) or \( 8 \). Define an abelian group \( G_8 = \langle g_1 \rangle \times \langle g_2 \rangle \times \cdots \times \langle g_{16} \rangle \) such that

\[
|g_{2i-1}| = pd^2, \quad |g_{2i}| = qd^2 \quad \text{for } i = 1, 3, 6, 8
\]

\[
|g_{2i-1}| = pd, \quad |g_{2i}| = qd \quad \text{for } i = 2, 4, 5, 7.
\]
Let $G_6 = \langle g_1 \rangle \times \cdots \times \langle g_{12} \rangle$ and $G_7 = \langle g_1 \rangle \times \cdots \times \langle g_{14} \rangle$. Define

\begin{align*}
h_i &= (g_{2i-1}g_{2i})^{pq} \text{ for } i = 2, 4, 5, 7, \\
h_1 &= (g_1g_2g_4g_9)^{pq}, \\
h_3 &= \prod_{i=3}^{10} (g_i)^{pq}. \\
h_6 &= (g_8g_9g_{10}g_{11}g_{12})^{pq}, \\
h_0' &= (g_9g_{10}g_{11}g_{12}g_{13}g_{14})^{pq}, \\
h_8 &= (g_{13}g_{14}g_{15}g_{16})^{pq}. \end{align*}

Let $\chi_i, 1 \leq i \leq 8$ and $\chi_i'$ be the characters of $G_8$ given in Table 3, then one can verify that

\begin{align*}
\mathcal{D}_6 &= \mathcal{D}(G_6, (h_1, \cdots, h_8), (\chi_1, \cdots, \chi_8), E_6), \\
\mathcal{D}_7 &= \mathcal{D}(G_7, (h_1, \cdots, h_5, h_6', h_7), (\chi_1, \cdots, \chi_5, \chi_6', \chi_7), E_7), \\
\mathcal{D}_8 &= \mathcal{D}(G_8, (h_1, \cdots, h_5, h_6', h_7, h_8), (\chi_1, \cdots, \chi_5, \chi_6', \chi_7, \chi_8), E_8). \end{align*}

are datums of Cartan type. And similar as Lemma 6.2 and 6.3 we have the following:

**Lemma 6.8.** $\Gamma(\mathcal{D}_n)$ is a nonempty set. For any $\mathcal{L} = (c_i, c_{jk})_{1 \leq i \leq n, 1 \leq j < k \leq n}$ in $\Gamma(\mathcal{D}_n)$, we have $c_i \neq 0$ for each $1 \leq i \leq 2n$.

**Lemma 6.9.** There exists nonzero modified root vector parameters $\mu$ for $\mathcal{D}_n$ satisfying the conditions: if $n = 6$, $\mu_\alpha \neq 0$ if and only if $\alpha = \alpha_i$ for $i = 2, 4$ or $5$; if $n = 7$ or $8$, $\mu_\alpha \neq 0$ if and only if $\alpha = \alpha_i$ for $i = 2, 4, 5$ or $7$.

**Proposition 6.10.** Let $\mathcal{L} \in \Gamma(\mathcal{D}_n)$, and $\mu$ a family of modified root vector parameters for $\Gamma(\mathcal{D}_n)$ satisfying the conditions of Lemma 6.9. Then $\mathfrak{u}(\mathcal{D}_n, 0, \mu, \Phi_\mathcal{L})$ is a nonradically graded genuine quasi-Hopf algebra.
Table 4. Characters of $G = (g_1) \times \cdots \times (g_8)$

|   | $g_1$ | $g_2$ | $g_3$ | $g_4$ | $g_5$ | $g_6$ | $g_7$ | $g_8$ |
|---|-------|-------|-------|-------|-------|-------|-------|-------|
| $\chi_1$ | $c_1^g$ | $c_2^g$ | $c_3^g$ | $c_4^g$ | $\zeta_{2^d}$ | $1$ | $1$ | $1$ |
| $\chi_2$ | $c_1^g$ | $c_2^g$ | $c_3^g$ | $c_4^g$ | $\zeta_{2^d}^{4d^2-1}$ | $\zeta_d$ | $\zeta_d$ | $1$ |
| $\chi_3$ | $1$ | $1$ | $c_2^g$ | $c_4^g$ | $c_3^g$ | $c_6^g$ | $c_5^g$ | $c_2^g$ |
| $\chi_4$ | $1$ | $1$ | $1$ | $1$ | $c_2^g$ | $c_5^g$ | $c_6^g$ | $\zeta_{2^d}$ |

Proof. Similar to the proof of Proposition 6.4. □

6.4. Quasi-Hopf algebras of Cartan type $F_4$. Let $G = (g_1) \times \cdots \times (g_8)$ such that

\[(6.43) \quad |g_i| = \begin{cases} 
pd, & i = 1, 5; 
qd, & i = 2, 6; 
pd^2, & i = 3, 7; 
qd^2, & i = 4, 8. 
\end{cases}\]

Denote by $(h_i)_{1 \leq i \leq 4}$ a family of elements in $G$ through

\[(6.44) \quad h_1 = (g_1 g_2)^{pq}, \quad h_2 = (g_1 g_3 g_4 g_5 g_6)^{pq}, \quad h_3 = (g_5 g_6)^{pq}, \quad h_4 = (g_5 g_7 g_8)^{pq}.\]

Let $(\chi_i)_{1 \leq i \leq 4}$ be the characters of $G$ given in Table 4, then one can easily verify that

$\mathcal{D} = \mathcal{D}(G, (h_i)_{1 \leq i \leq 4}, (\chi_i)_{1 \leq i \leq 4}, F_4)$

is a datum of Cartan type. Similar to Lemma 6.2-6.3, we have the following two lemmas.

**Lemma 6.11.** $\Gamma(\mathcal{D})$ is a nonempty set. Let $\underline{c} = (c_i, c_{jk})_{1 \leq i \leq 8, 1 \leq j < k \leq 8}$ be an element in $\Gamma(\mathcal{D})$. Then we have $c_i \neq 0$ for $1 \leq i \leq 8$.

**Lemma 6.12.** There exists a family of modified root vector parameter $\mu$ for $\Gamma(\mathcal{D})$ satisfying the condition: $\mu_\alpha$ is nonzero if and only if $\alpha = \alpha_1$ or $\alpha_3$.

**Proposition 6.13.** Let $\underline{c} \in \Gamma(\mathcal{D})$, and $\mu$ a family of modified root vector parameters satisfying the condition of Lemma 6.12. Then $u(\mathcal{D}, 0, \mu, \Phi_\underline{c})$ is a nonradically graded genuine quasi-Hopf algebra.

Proof. Similar to the proof of Proposition 6.4. □

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