The reachable regions construction for the inelastic anisotropic billiards

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Abstract. Modern methods of the ballistic design of space flights (entry of a spacecraft with non-zero aerodynamic quality into the planet's atmosphere, carrying out gravity assist maneuvers around planets) are associated with the need to calculate a lot of trajectories (i.e. of the phase beams). For their effective use and determination of dynamically admissible reachability regions, it is necessary to identify and study the structure of accompanying singularities of the phase parameters and to construct the corresponding adequate and adaptive models. In this sense, the analogy of the motion of these systems with the singular motion of billiard systems turns out to be very fruitful. First of all, we are talking about anisotropic billiards with dissipation taken into account. In this work, we set and described the model problems about the flat stones skipping on the surface of the water (the ricochets) and an inelastic ball analogue which is bouncing or rolling on an imperfect surface in the presence of a side deviation effect. The main indicatrices types classification for the singular reflection from the plane surface is carried out. Model problems of the ricocheting pebble on the surface of the water surface and the inelastic ball bouncing or rolling on an imperfect surface in the presence of a side bouncing effect also the attainability region problem of the atmospheric entry are posed and described. The configurations of their maximum reachable regions are described

1. Introduction
Modern methods of the ballistic design of space flights (entry of a spacecraft with non-zero aerodynamic quality into the planet's atmosphere, carrying out gravity assist maneuvers around planets [1, 2, 3, 4] are associated with the need to calculate a lot of trajectories (i.e. of the phase beams). For their effective use and determination of dynamically admissible reachability regions, it is necessary to identify and study the structure of accompanying singularities of the phase parameters and to construct the corresponding adequate and adaptive models. In this sense, the analogy of the motion of these systems with the singular motion of billiard systems [1, 2, 5, 6] turns out to be very fruitful. First of all, we are talking about anisotropic billiards with dissipation taken into account. In this work, we also are present the analogical model problems about the flat stones skipping on the surface of the water (the ricochets) and the inelastic ball which is bouncing or rolling on an imperfect surface in the presence of a side deviation effect. The indicatrices approach for this the model problems of the ricocheting pebble on the surface of the water and the inelastic ball bouncing or rolling on an imperfect surface in the presence of a side bouncing effect also the ricochets in the spacecraft motion

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during entry in the planetary atmosphere or during its flybys is presented. The main indicatrices types classification for the singular reflection from the plane surface are explained.

2. Variational principles and the study of the extreme dynamic capabilities of mechanical systems
The study of the extreme dynamic capabilities of mechanical systems is a priority direction in the development of the modern mechanics. On the one hand, solving problems of this class makes it possible to use existing mechanical objects more widely and efficiently and manage them more economically. On the other hand, optimal control is much closer to a broad class of unmanaged mechanical systems than it might seem at first glance. In particular, the classical calculus of variations can be considered as a subdivision of optimal control theory [5]. The work is presenting the according approach to the dynamic estimation of the ballistic phase beams and the identifying extreme structural properties of the ballistically ricocheting mechanical systems, where the traditional studying of conventional methods of classical mechanics (for example - using the theory of impact and the presence of dissipation) can be very complicated by virtue of its quasi-regularity.

3. The universal indicatrices approach
The representation of the trajectory of a mechanical system as an extremal of the variational principle in its extended phase space has been known since the time of Hamilton. Accordingly, it becomes possible to draw an analogy between the movement of an object of Newtonian mechanics and the propagation of a light beam in geometric optics in an optical medium with a“suitably” constructed refractive index [1, 2, 5]. In the latter case, the role of the variational principle is played by the principle of the smallest optical path (Fermat's principle). At the same time, we can rely on well-studied problems of geometric optics (as well as geometric acoustics and geometric electrodynamics). Using the indicatrices technique for the phase beams representation is well known and is the one of interesting approach for the physical systems obtaining with quasi-singularities [1, 2]. The indicatrices technique was appearing initially in the geometric optic during the light wave fronts studying [5, 6]. Taking into account the fact that the geometry of the gravity assist maneuver is in essence the same as the billiard geometry [1, 2, 5], we can generalize the optic–mechanical analogy, which is valid for billiard trajectories [2, 5], for the case of investigating the spatial localization of the spacecraft on various trajectories emanating from the gravity assist maneuver domain. The optic–mechanical analogy relates the trajectories of a mechanical system with the light trajectories in an anisotropic medium [5, 6,]. For that reason, we may consider the indicatrix as the surface of all possible velocities with which the spacecraft leaves the reflection “point” [1, 2, 5].

4. Main positions of the quasi-singular ballistic indicatrix approach (QSI approach)
The toolkit of the quasi-singular indicatrix approach (QSI approach) is briefly as follows. Consider the motion of a ricochet mechanical system in a gravity field described by a system of differential equations, where \( \dot{x} = f(x, t) \) is the phase state vector of the system and \( t \) is the unidimensional time. Let's introduce the reflection hypersurface \( x = \bar{x} \) (usually it is determined by some characteristic value of the force function \( U : U = U_0 \); for example, the conditional height of the planet's atmosphere \( h_0 \), or the radius of the planet's sphere of influence \( r_d \)).

1. The movement of the ricocheting mechanical system (RM-system) (for example, a spacecraft during its ricochet entry in the planet atmosphere) can be described as a multiple chain of two-element couplings: the smooth ballistic trajectory sections BTA (ballistic trajectory arcs) and the local distortion zones, DZ (distortion zone).
2. The smooth ballistic trajectory sections correspond to conservative motion (for example, when a spacecraft moves near a planet far from the atmosphere), while break zones DZ correspond to strong non-conservative effects (for example, aerodynamic drag when the spacecraft enters the atmosphere).
The change in speed at each current BTA occurs according to the ballistic law of motion, so that the vector of the "entry" velocity in the BTA $v_{\text{in},k}, k = 1, \ldots, N_{\text{in}}$ (in ballistics - the vector of the throwing velocity, or the launch velocity) is equal magnitude to the vector of the "exit" from the BTA (in ballistics - the spacecraft entry angle into the atmosphere), and their trajectory angles $\theta_{\text{in}}, \theta_{\text{out}}$ will coincide up to a sign (Fig. 1).

4. The change in the velocity vector in the distortion zone DZ occurs almost singularly.

5. As a consequence of the previous point, the length of the BTA sections significantly exceeds the length of the break zones, and only these forms the areas of reachability of the ricochet path.

6. To describe the law of the singular change in the spacecraft velocity vector in the distortion zone DZ, one can introduce an operator (by analogy with the light-front propagation indicatrix) an autonomous DZ indicatrix $\hat{I}_{\text{DZ}}(x, u_{\text{DZ}})$ for converting the input velocity vector $v_{\text{in}}$ into the output velocity vector $v_{\text{out}}$, where there are all kinds of virtual control functions in the DZ section, so that

$$v_{\text{out}} = (\hat{I}_{\text{DZ}}) v_{\text{in}} \quad (1)$$

7. It is possible to calculate “virtual” $\hat{I}_{\text{DZ}}$ by precise previously solving in the DZ a set of optimization problems on the, varying $u_{\text{DZ}}$ (cumbersome), or by using advanced modifications of the Monte Carlo method [7] with exact numerical integration (also cumbersome).

8. With the help of the choice of the previously calculated “virtual” $u_{\text{DZ}}$, one can change the initial data for calculating the next element of the chain (the current Cauchy problem generation) in order to maximize the final reachable region or some other special functionals. The cumbersomely mass calculations are remaining “out of the board” during the quick ballistic design.

9. To identify the main structural properties of the RM-systems, we can use also the phenomenological types of indicatrices $\hat{I}_{\text{DZ}}(x, u_{\text{DZ}})$ (Figure 2,3) with the carrying out mass modelling.

The synthesis of trajectory rays is carried out by choosing a large set of fixed vectors $\hat{I}_{\text{DZ}}(x, u_{\text{DZ}}), j = 1, \ldots, N_{\text{mod}}$ and performing the corresponding calculations of the Cauchy problems using the advanced Monte Carlo methods [1, 7].

**Figure 1.** The "entry" velocity vector of in the BTA (altitude $h = h_{0}$) is equal magnitude to the vector of the "exit", their trajectory angles will coincide up to a sign

**Figure 2.** The indicatrix of the isotropic bouncing elastic ball on a horizontal ideal surface
5. Examples
Examples of the ballistic RM-systems: interplanetary spacecraft flights with gravity assist maneuvers (the method of conical sections implies small flyby areas in comparison with heliocentric loops); ricocheting spacecraft entry into the planet's atmosphere; ricochet of a pebble on the horizontal surface of the water; a bouncing inelastic ball on a horizontal rough surface, etc.

Let's consider cuts of the indicatrix borders and the borders of the attainability region in projection onto the reflecting plane in coordinates $D, B$ (longitudinal range and lateral deviation).

6. A bouncing inelastic ball on a horizontal rough surface
The cuts of the exemplary indicatrix borders and the borders of the attainability region in projection onto the reflecting plane bouncing inelastic ball on a horizontal rough surface are presented at fig. 3,4. We can see that the attainability region isn’t simply connected in this case.

![Figure 3](image1.png) ![Figure 4](image2.png)

**Figure 3.** The bouncing inelastic ball indicatrix  
**Figure 4.** The bouncing inelastic ball attainability region onto the reflecting plane flyby indicatrixes

7. The problem of the ricocheting pebble on the surface of the water
Consider a pebble ricocheting with attenuation along the surface of the water, and set the model problem of determining all points on this surface that it can hit at a fixed vector of the initial velocity $V_{in}$. For simplicity, the surface will be considered a horizontal plane. Obviously, the type of the sought-for area will be determined by the nature of the reflection of the pebble at each contact with water, which, in turn, depends on the current speed of the pebble, its rotation and orientation in space. A cumbersome optimization problem in a complex environment - the construction of the region of attainability of the specified motion, becomes solvable if we use the following model assumptions.

1. The increments of the functional of the longitudinal range $D$ and lateral deviation $B$ in the areas of direct reflection from the water are small compared to the areas of free flight.
2. Air resistance in free flight areas can be neglected. We can see that QSI approach includes these positions.

We can see that the attainability region become simply connected. The purpose is in possibilities of pebble dipping every ricochet.
8. Spacecraft entry into the planet’s atmosphere
The main points of the presented QSI-approach and its tools were nurtured precisely for this interesting and urgent task. This approach made it possible to explain the structure of the spacecraft entry trajectories into the planet's atmosphere and to describe its main features. One of the most interesting nuances is the non-smooth beake-like border of the reverse side of the reachable area. Main details are presented in [1, 2, 3]. The structure of the indicatrices and the attainability region are presented below (Figure 7, 8).

9. Interplanetary gravity assist maneuvers
We point out that the heuristics of the presented approach lies in the fact that its applications can be much wider than those announced. Indeed, no one obliges us to optimize the spatial variables B and D, that is, the reachable region of the RM-system. With the help of virtual controls at the indicatrices in DZ-regions, we would also be able to directly optimize the values of the spacecraft velocity at the exit from the "sinking" in order to form them as such, since they can determine, for example, the required heliocentric orbit of the spacecraft for performing GAM. In other words, we can perform the optimization process using virtual controls, previously was been solved in DZ also and in the space of output speeds. In the context of modern interplanetary ballistics, this means that we can optimize the
"phase motion" of the spacecraft into the Tisserand-Poincaré phase diagram using gravity assists maneuvers [1]. Indeed, even in this example, we also can represent the indicatrix and the reachability region (in the phase space of the Tisserand diagram [1, 3, 4]), Figure 9, Figure 10.

![Figure 9. The flyby indicatrices](image1)

![Figure 10. Attainability regions of the interplanetary mission using gravity assist maneuvers on the Tisserand diagram (in format apocenter-pericentre, planets radii)](image2)

**Conclusion**

The study of the extreme dynamic capabilities of mechanical systems is a priority direction in the development of the modern mechanics. On the one hand, solving problems of this class makes it possible to use existing mechanical objects more widely and efficiently and manage them more economically. On the other hand, optimal control is much closer to a broad class of unmanaged mechanical systems than it might seem at first glance. In particular, the classical calculus of variations can be considered as a subdivision of optimal control theory [5]. The work is presenting the identical quasi-singular ballistic indicatrix approach (QSI-approach) to the identifying extremal structural properties of the ballistically ricocheting mechanical systems, where the traditional of conventional methods of classical mechanics (for example - using the theory of impact and the presence of dissipation) can be very complicated by virtue of systems quasi-regularity. The next problems using the toolkits of the QSI-approach are presented and described: interplanetary spacecraft flights with gravity assist maneuvers (the method of conical sections implies small flyby areas in comparison with heliocentric loops); ricocheting spacecraft entry into the planet's atmosphere; ricochet of a pebble on the horizontal surface of the water; a bouncing inelastic ball on a horizontal rough surface. It is shown that the main details of these problems can be described using the QSI approach.

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