Renormalizability of $\mathcal{N} = \frac{1}{2}$ Wess-Zumino model in superspace

Alberto Romagnoni*

*Dipartimento di Fisica dell’Università degli studi di Milano-Bicocca, and INFN, Sezione di Milano, piazza della Scienza 3, I-20126 Milano, Italy

ABSTRACT

In this letter we use the spurion field approach adopted in [hep-th/0307099] in order to show that by adding $F$ and $F^2$ terms to the original lagrangian, the $\mathcal{N} = \frac{1}{2}$ Wess-Zumino model is renormalizable to all orders in perturbation theory. We reformulate in superspace language the proof given in the recent work [hep-th/0307165] in terms of component fields.

PACS: 03.70.+k, 11.15.-q, 11.10.-z, 11.30.Pb, 11.30.Rd
Keywords: Noncommutative geometry, $\mathcal{N} = \frac{1}{2}$ Supersymmetry, Wess-Zumino model.

*alberto.romagnoni@mib.infn.it
It has been recently shown [1, 2] that the IIB superstring in the presence of a graviphoton background defines a superspace geometry with nonanticommutative spinorial coordinates. This deformation of superspace was previously considered in [3]. Field theories defined over $\mathcal{N} = 1/2$ superspace (i.e. $\mathcal{N} = 1$ euclidean superspace deformed by a nonanticommutativity parameter $\{\theta^\alpha, \bar{\theta}^\beta\} = 2C_{\alpha\beta}$ with $C$ a nonzero constant) have been considered in [1, 4, 5, 6, 7].

In this non(anti)commutative superspace we study the Wess-Zumino model

$$S = \int d^8z \bar{\Phi} \Phi - \frac{m}{2} \int d^6z \Phi^2 - \frac{\bar{m}}{2} \int d^6\bar{z} \bar{\Phi}^2 - \frac{g}{3} \int d^6z \bar{\Phi} \Phi \Phi - \frac{\bar{g}}{3} \int d^6\bar{z} \bar{\Phi} \Phi \bar{\Phi}$$

(1)

that in [1] was shown to reduce to the usual WZ augmented by a nonsupersymmetric component term $\frac{g}{6} \int d^4x C^2 F^3$ (with $C^2 = C^\alpha\beta C_{\alpha\beta}$).

In [6], by introducing a spurion field [8], $U = C^2 \theta^2 \bar{\theta}^2$, to represent the supersymmetry breaking term $F^3$, the divergence structure and renormalizability of the $\mathcal{N} = 1/2$ WZ model have been studied systematically in superspace through two loops.

In this approach the classical action reads

$$S = \int d^8z \bar{\Phi} \Phi - \frac{m}{2} \int d^6z \Phi^2 - \frac{\bar{m}}{2} \int d^6\bar{z} \bar{\Phi}^2 - \frac{g}{3} \int d^6z \bar{\Phi} \Phi \Phi - \frac{\bar{g}}{3} \int d^6\bar{z} \bar{\Phi} \Phi \bar{\Phi} + \frac{g}{6} \int d^8z U(D^2\Phi)^3.$$  

(2)

It has been proven [6] that, up to this order, divergences are at most logarithmic, that divergent terms have at most one $U$-insertion (i.e. there is at most one power of $C^2$) and they are of the form $F^\alpha \bar{G}^k$, with $\bar{G} = \bar{m} \bar{\phi} + \bar{g} \phi^2$ and $\alpha \geq 1$, $\alpha + k \leq 3$ (here $\Phi = \phi$, $D_\alpha \Phi = \psi_\alpha$, $D^2 \Phi = F$ and analogous relations for the antichiral superfield).

Finally, a counterterm of the form $F^\alpha \bar{G}^k$ has been shown to be completely equivalent to a counterterm of the form $F^{\alpha+k}$. After adding by hand the terms $\int d^8z U(D^2\Phi)^2$ and $\int d^8z U(D^2\Phi)$, the model is renormalizable up to two loop order.

In the recent paper [7] it has been shown that the same results hold to all orders in perturbation theory: in particular the authors of [7], working in terms of component fields, constrain the form of divergent terms in the effective action using the two global $U(1)$ (pseudo)symmetries of the theory [4] and making general considerations on the structure and combinatorics of the Feynman diagrams.

In this short letter we reformulate in superspace formalism the discussion of [7], since this approach is usually more suitable when some supersymmetry is left. We use the conventions of [9].

We parametrize the terms $F$ and $F^2$ in the classical lagrangian as

$$\lambda_1 g^2 \bar{m}^4 \int d^8z U(D^2\Phi)^2 + \lambda_2 g^2 \bar{m}^2 \int d^8z U(D^2\Phi)^2.$$  

(3)

We consider the two global $U(1)$ (pseudo)symmetries of the theory, the $U(1)_\Phi$ flavor symmetry and $U(1)_R$ R-symmetry [4]. In superspace language we have the charge assignment given in table [11].

In particular with the parametrization [3] the coefficients $\lambda_1$ and $\lambda_2$ are charge neu-
The most general divergent term in the effective action has the form
\[
\int d^4 x \Gamma_\mathcal{O} = \lambda \int d^4 x d^4 \theta (D^2)^{\gamma} (\bar{D}^2)^{\delta} (D_\alpha \partial^{\alpha \dot{\alpha}} \bar{D}_{\dot{\alpha}})^{\eta} \Box^{\zeta} U^\rho \Phi^\alpha \bar{\Phi}^\beta
\] (4)

with \(\gamma, \delta, \eta, \zeta, \rho, \alpha, \beta\) non-negative integers. It is understood that every \(D^2, \bar{D}^2, D_\alpha, \bar{D}_{\dot{\alpha}}\) is acting on \(U, \Phi, \bar{\Phi}\) superfields, taking into account that
\[
[D_\alpha, \bar{D}_{\dot{\alpha}}] = i \partial^{\alpha \dot{\alpha}}, \quad [D_\alpha^2, \bar{D}_{\dot{\alpha}}^2] = i \partial^{\alpha \dot{\alpha}} D_\alpha, \quad D_\alpha^2 \bar{D}_{\dot{\alpha}}^2 = \Box D^2, \quad \bar{D}_{\dot{\alpha}}^2 D_\alpha^2 = \Box \bar{D}^2.
\] (5)

In our notation the coefficient \(\lambda\), with dimension \(d\) and charges \(q_R = R\) and \(q_\Phi = S\), is
\[
\lambda \sim \Lambda^d g^{x-R} \bar{g}^{x} \left( \frac{m_1}{\Lambda} \right)^y \left( \frac{\bar{m}_1}{\Lambda} \right)^{y+\frac{S-3R}{2}} \lambda_2^{\omega_2}
\] (6)

where \(\Lambda\) is an ultraviolet momentum cutoff. \(\lambda\) cannot be a function of \(\lambda_1\) since we cannot form a 1PI connected diagram with a \(\int U (D^2 \Phi)\) term. Moreover \(\omega_2 \leq \rho\) since \(\lambda_2\) appears only in terms with a \(U\) insertion (see (3)).

Since the term \(\Gamma_\mathcal{O}\) has dimension 4 and zero charge, we have
\[
d = 2 + 4 \rho - \alpha - \beta - \gamma - \delta - 2 \eta - 2 \zeta
R = \beta - \alpha + 2 \gamma - 2 \delta - 4 \rho
S = \beta - \alpha.
\] (7)

The overall power of \(\Lambda\) in \(\Gamma_\mathcal{O}\) is
\[
P = d - 2y - \frac{S - 3R}{2}
\] (8)

and using eq. (7)
\[
P = 2 + 2 \gamma - 2 \rho - 2 \alpha - 4 \delta - 2y - 2 \eta - 2 \zeta.
\] (9)

Obviously we have a divergent contribution iff \(P \geq 0\). We consider the different cases:
• \( \rho = 0 \)
  It is the ordinary Wess-Zumino case.

• \( \rho = 1 \)
  We have
  \[
  \gamma - \alpha - 2\delta - y - \eta - \zeta \geq 0 .
  \]
  (10)
  Since the \( U \) superfield has only the \( \theta^2\bar{\theta}^2 \) component, the \( d^4\theta \) integration acts on it. Moreover the covariant \( D^2 \) derivatives can act only on \( \Phi \) superfields (in fact \( D^3 = 0, D^2\bar{D}\alpha\Phi = 0 \) and \( D^2\bar{D}^2\Phi = \Box\Phi \)), so we have
  \[
  \gamma \leq \alpha .
  \]
  (11)
  Therefore the only possibility to satisfy (10) is for
  \[
  \gamma = \alpha , \quad \delta = y = \eta = \zeta = 0
  \]
  (12)
  and we find the general divergent term
  \[
  \int d^8zU(D^2\Phi)^{\alpha}\bar{\Phi}^{\beta}.
  \]
  (13)
  With the assignment (12) we have \( P = 0 \) showing that there is at most a logarithmic divergence.

• \( \rho = 1 + n, \ n > 0 \)
  Since the \( U \) superfield has only the \( \theta^2\bar{\theta}^2 \) component, we need at least \( n \) \( D^2 \) and \( n \) \( \bar{D}^2 \). Therefore
  \[
  \gamma = n + \gamma_1, \quad \delta = n + \delta_1
  \]
  (14)
  and then
  \[
  \gamma_1 - \alpha - 2n - 2\delta_1 - y - \eta - \zeta \geq 0 .
  \]
  (15)
  Since \( \gamma_1 \leq \alpha \) (as in the previous case), and \( n > 0 \), we see that eq. (15) cannot be satisfied.

In conclusion, we have only (logarithmic) divergent terms of the form (13). Now we look for constraints on the coefficients \( \alpha \) and \( \beta \) in order to show that there are only finitely many divergent terms.

As seen in the discussion above, we have
  \[
  \rho = 1 , \quad \gamma = \alpha , \quad \delta = y = \eta = \zeta = 0
  \]
  (16)
  therefore \( \lambda \) takes the form
  \[
  \lambda \sim g^{x-R} \bar{g}^x \tilde{m}^{S-3R} \lambda^{\omega_2}
  \]
  (17)
where
\[ \frac{S - 3R}{2} = -\beta - 2\alpha + 6. \quad (18) \]

If we look only at the UV divergent part of a diagram, the evaluation of the integral cannot depend on the mass parameter (in fact in dimensional regularization the divergences appear just as poles in $1/\epsilon$). Therefore powers of $\bar{m}$ in the coupling constant $\lambda$ can appear:

- from the vertex $\lambda_2 g^2 \bar{m}^2 \int d^8 z U(D^2 \Phi)^2$
- from the propagators $\langle \Phi \Phi \rangle = -\frac{\bar{m} D^2}{\mu^2 (p^2 + mm)} \delta^{(4)}(\theta - \theta')$.

Then, if we consider that the number of propagators $\langle \Phi \Phi \rangle$ is always nonnegative and that $\omega_2 \leq \rho$, we have

- $\omega_2 = 0 \rightarrow -\beta - 2\alpha + 6 \geq 0 \rightarrow \beta + 2\alpha \leq 6$
- $\omega_2 = 1 \rightarrow -\beta - 2\alpha + 4 \geq 0 \rightarrow \beta + 2\alpha \leq 4$

We have also the condition $\alpha \geq 1$: in fact, after $D$-algebra at least one $D^2$ survives and then, using (12), there must be at least one chiral $\Phi$ superfield.

To summarize, we have found that at any loop order the logarithmic divergent terms have the form
\[ \int d^8 z U(D^2 \Phi)^\alpha \bar{\Phi}^\beta, \quad \alpha \geq 1, \quad \beta + 2\alpha \leq 6 - 2\omega_2, \quad \omega_2 = 0, 1. \quad (19) \]

Now we show that we can repackage them into the form
\[ \int d^8 z U(D^2 \Phi)^\alpha \bar{\mathcal{G}}^k \quad (20) \]
with $\bar{\mathcal{G}} = \bar{m} \bar{\Phi} + g \bar{\Phi}^2$ and $0 \leq k \leq 3 - \omega_2 - \alpha$.

The condition $y = 0$ implies that in a divergent diagram the coupling constant does not contain $m$ factors and so that there are not propagators $\langle \Phi \Phi \rangle$ (by the same observations done for $\bar{m}$). Therefore we have divergent contributions only from diagrams without adjacent $\bar{\Phi}^3$ vertices. Then a divergent diagram with $\bar{\Phi}$ external legs is analogous to a yet divergent diagram with the insertion of $\bar{\Phi}^3$ vertices on $\langle \Phi \Phi \rangle$ propagators. In fact, this operation does not modify the divergence of the diagram, since $\langle \Phi \Phi \rangle \sim \Lambda^{-4} \sim (\langle \Phi \bar{\Phi} \rangle)^2$ and since the $D$-algebra is not modified if we look only at divergent contributions. The only differences are the substitution $\bar{m} \rightarrow g \bar{\Phi}$ for every insertion and a combinatorial factor $\binom{q}{k}^2$ that takes into account the $\binom{q}{k}$ ways to insert $k$ vertices in $q$ $\langle \Phi \Phi \rangle$ propagators and a symmetry factor $2 = 3 \cdot \frac{1}{3} \cdot 2$ for every vertex.

Therefore, with this operation, it is possible to start with divergent diagrams that give
contributions to terms $\int U(D^2\Phi)^\alpha$ at a given loop order, and build all possible diagrams that give contributions to terms $\int U(D^2\Phi)^\alpha\bar{\Phi}^\beta$ at the same order.

If we start with a divergent base diagram with fixed $\omega_2$ and $\alpha$, and with symmetry factor $S$ (that we can understand to include also the poles in $1/\epsilon$), the sum of all the divergent contributions with $k \geq 1$ is

$$Sg^{x-\alpha+4\bar{g}^x\lambda_2^{\alpha}m^{6-2\alpha}} \sum_{k=1}^{\infty} 2^k \left( \frac{q}{k} \right) \left( \frac{\bar{g}}{m} \right)^k \int d^8z U(D^2\Phi)^\alpha\bar{\Phi}^k.$$  \hspace{1cm} (21)

Since

$$\sum_{k=1}^{\infty} 2^k \left( \frac{q}{k} \right) \left( \frac{\bar{g}}{m} \right)^k \bar{\Phi}^k = \left( 1 + 2\frac{\bar{g}}{m} \bar{\Phi} \right)^q - 1$$

$$= \left( 1 + 4\frac{\bar{g}}{m} \bar{\Phi} + 4\frac{\bar{g}^2}{m^2} \bar{\Phi}^2 \right)^{\frac{q}{2}} - 1$$

$$= \left( 1 + \frac{4\bar{g}}{m^2} \bar{G} \right)^{\frac{q}{2}} - 1$$  \hspace{1cm} (22)

and observing that for a diagram without $\bar{\Phi}$ external legs ($\beta = 0$) $q = 6 - 2\omega_2 - 2\alpha$, we can finally rewrite eq. (21) as

$$Sg^{x-\alpha+4\bar{g}^x\lambda_2^{\alpha}m^{6-2\alpha}} \sum_{k=1}^{3-\omega_2-\alpha} 4^k \left( \frac{3 - \omega_2 - \alpha}{k} \right) \left( \frac{\bar{g}}{m^2} \right)^k \int d^8z U(D^2\Phi)^\alpha\bar{\Phi}^k$$  \hspace{1cm} (23)

which agrees with the two loop results of \[6\].

Taking into account that $\alpha = 1, 2, 3$ we can conclude, in agreement with \[7\], that to all orders in perturbation theory, the divergent terms generated are (in component fields, with $\bar{G} = \bar{G}$)

$$\omega_2 = 0 \quad \rightarrow \quad F, F^2, F^3, F\bar{G}, F^2\bar{G}, F\bar{G}^2$$

$$\omega_2 = 1 \quad \rightarrow \quad F, F^2, F\bar{G}$$  \hspace{1cm} (24)

Now we show that the counterterms $F, F^2, F^3$ are sufficient to renormalize the theory. We can follow the argument of \[4\] to claim that a contraction of any field with $\bar{G}$ is equivalent to its contraction with $F$. This is possible also in completely superspace language and translates into the equivalence $\bar{G} \rightarrow D^2\Phi$. In fact, let us consider for example the effect of a superfield factor $U(D^2\Phi_b)^2[D^2\Phi - \bar{m}\bar{\Phi}]$ as compared to $\bar{g}U(D^2\Phi_b)^2\bar{\Phi}^2$ (here $\Phi_b$ is the background superfield). The superfield propagators are

$$\langle \Phi\bar{\Phi} \rangle = \frac{1}{p^2 + \bar{m}\bar{m}} \delta^{(4)}(\theta - \theta')$$

$$\langle \Phi\Phi \rangle = -\frac{\bar{m}D^2}{p^2(p^2 + \bar{m}\bar{m})} \delta^{(4)}(\theta - \theta')$$

$$\langle \bar{\Phi}\bar{\Phi} \rangle = -\frac{m\bar{D}^2}{p^2(p^2 + \bar{m}\bar{m})} \delta^{(4)}(\theta - \theta')$$  \hspace{1cm} (25)
and from the Feynman rules for each chiral (antichiral) field there is an extra $\bar{D}^2 (D^2)$ derivative on each line leaving a vertex (except for one of the lines at a (anti)chiral vertex).

In the Wick expansion, the operator $U(D^2\Phi_b)^2[D^2\Phi - \bar{m}\Phi]$ can be contracted either with a $\Phi^3$ vertex, or with a $\bar{\Phi}^3$ vertex. Taking in account the $D$-algebra (and in particular $D^2\bar{D}^2D^2 = -p^2D^2$), and given the form of the propagators, in the first case the result is zero. In the second case, the $D$-algebra is analogous and, given the form of the propagators, the result is $\bar{g}U(D^2\Phi_b)^2\bar{\Phi}^2$. In this last case there is a little subtlety: when we contract $D^2\Phi$ with $\bar{\Phi}$, after using $D^2\bar{D}^2D^2 = -p^2D^2$, we remain with a $D^2$ on the propagator $\langle \Phi\bar{\Phi}\rangle$; this $D^2$ can be integrated by parts onto the $\bar{\Phi}^3$ vertex, to give the exact Feynman rules for a vertex $U(D^2\Phi_b)^2\bar{\Phi}^2$. We can treat in a similar way the operators $U(D^2\Phi_b)\bar{G}$ and $U(D^2\Phi_b)\bar{G}^2$, thus showing the equivalence of the two forms of counterterms when inserted into diagrams.

Therefore the counterterms

$$\int U(D^2\Phi), \int U(D^2\Phi)^2, \int U(D^2\Phi)^3$$

are sufficient to renormalize the theory at any order of perturbation theory.

Acknowledgments I thank R. Britto, B. Feng, M. Grisaru and S. Penati for helpful discussions. This work has been supported by INFN, MURST and the European Commission RTN program HPRN-CT-2000-00131, in which I am associated to the University of Padova.
References

[1] N. Seiberg, *Noncommutative superspace, N = 1/2 supersymmetry, field theory and string theory*, JHEP 06 (2003) 010, hep-th/0305248

[2] J. de Boer, P.A. Grassi and P. van Nieuwenhuizen, *Non-commutative superspace from string theory*, hep-th/0302078
H. Ooguri and C. Vafa, *The c-deformation of gluino and non-planar diagrams*, hep-th/0302109
H. Ooguri and C. Vafa, *Gravity induced c-deformation*, hep-th/0303063
N. Berkovits and N. Seiberg, *Superstrings in graviphoton background and N = 1/2 + 3/2 supersymmetry*, JHEP 07 (2003) 010, hep-th/0306226

[3] S. Ferrara and M.A. Lledó, *Some aspects of deformations of supersymmetric field theories*, JHEP 05 (2000) 008, hep-th/0002084
D. Klemm, S. Penati and L. Tamassia, *Non(anti)commutative superspace*, Class.Quant.Grav. 20 (2003) 2905-2916, hep-th/0104190

[4] R. Britto, B. Feng and S.-J. Rey, *Deformed superspace, N = 1/2 supersymmetry and (non)renormalization theorems*, JHEP 07 (2003) 067, hep-th/0306215

[5] S. Terashima and J.-T. Yee, *Comments on noncommutative superspace*, hep-th/0306237
S. Ferrara, M.A. Lledó and O. Maciá, *Supersymmetry in noncommutative superspaces*, JHEP 09 (2003) 068, hep-th/0307039
J.-H. Park, *Superfield theories and dual supermatrix models*, JHEP 09 (2003) 046, hep-th/0307060
T. Araki, K. Ito and A. Ohtsuka, *Supersymmetric gauge theories on noncommutative superspace*, hep-th/0307076
R. Britto, B. Feng and S.-J. Rey, *Non(anti)commutative superspace, UV/IR mixing and open Wilson lines*, JHEP 08 (2003) 001, hep-th/0307091

[6] M.T. Grisaru, S. Penati and A. Romagnoni, *Two-loop renormalization for nonanticommutative N = 1/2 supersymmetric WZ model*, JHEP 08 (2003) 003, hep-th/0307099

[7] R. Britto and B. Feng, *N = 1/2 Wess-Zumino model is renormalizable*, hep-th/0307165

[8] L. Girardello and M.T. Grisaru, Nucl. Phys. B194 (1982) 65.

[9] S.J. Gates Jr., M.T. Grisaru, M. Roček and W. Siegel, *Superspace*, Benjamin Cummings, Reading 1983.