A concise introduction to trapped surface formation in general relativity

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Abstract

Trapped surface formation in general relativity can be studied through a coupled set of nonlinear equations, where various terms can be neglected, as was proved by a rigorous mathematical analysis of Christodoulou. This paper is devoted to a pedagogical synthesis of the mathematical formalism employed in this analysis, i.e. the optical structure of general relativity investigated by Christodoulou, Klainerman and other authors after them.

1 Introduction

In the sixties, when the global techniques of differential topology were applied to investigate causal structure and singularities in gravitational collapse and cosmology, the concept of trapped surface was elaborated for the first time. In Ref. [1], a trapped surface $\Sigma$ is defined to be a compact spacelike two-surface with the property that the null geodesics which meet $\Sigma$ orthogonally, locally converge in future directions. More precisely, this means what follows: we consider in a 4-dimensional spacetime $(V,g)$ the compact spacelike 2-surface $\Sigma$ (i.e. a compact Riemannian 2-manifold) which is a submanifold of a 3-dimensional spacelike submanifold $M$. Let $n$ and $\nu$ be the future unit normal to $M$ and $\Sigma$, respectively, with $\nu$ tangent to $M$. One can then define two null vectors $\ell^\pm \equiv n \pm \nu$ which contribute to form the spacetime tensor

$$h = g + n \otimes n - \nu \otimes \nu = g + \frac{1}{2}(\ell^+ \otimes \ell^- + \ell^- \otimes \ell^+).$$

These tensors are then used to define the null mean curvatures (in our paper we always use explicit summations to achieve a clear distinction between 2-, 3- and 4-dimensional concepts)

$$\chi^\pm \equiv \sum_{\alpha,\beta=1}^4 h^{\alpha\beta} \nabla_\alpha \ell^\pm_\beta = \sum_{\alpha=1}^4 \nabla_\alpha \ell^\pm_\alpha.$$  

The surface $\Sigma$ is said to be a trapped surface in the spacetime $(V,g)$ if both $\chi^+$ and $\chi^-$ are negative on $\Sigma$, i.e., if the null geodesic congruences both converge [2, 3, 4].

Penrose asked himself the question whether it is reasonable to expect trapped surfaces to develop at all in our actual universe, and he found that there can be no reason of principle against a trapped surface developing. However, the proof of theorems on the formation of trapped surfaces for solutions of vacuum Einstein equations was a much harder task, and was accomplished only in 2009 by Christodoulou [5], with an appropriate choice of initial conditions, called the short-pulse method. Later on, a simpler proof of trapped-surface formation was obtained in Ref. [6] by enlarging the admissible set of initial conditions and relaxing the corresponding propagation estimates.

The basic tool in such investigations is the geometry of double null foliations, which leads eventually to the so-called optical structure of vacuum Einstein equations. Although the physics-oriented community is by now familiar with Newman-Penrose formalism and solutions of the eikonal equation in curved spacetime, the techniques used in Refs. [5, 6] are not yet widespread, and hence we find it appropriate to summarize here some key concepts and results in such References. We therefore consider a region $D = D(u_*, v_*)$ of a vacuum spacetime $(M, g)$ spanned by a double null foliation generated by the optical functions $(u, v)$ that solve the eikonal equation

$$(\nabla \varphi, \nabla \varphi) = \sum_{\mu, \nu=1}^4 (g^{-1})^{\mu\nu} (\partial_\mu \varphi)(\partial_\nu \varphi) = 0, \quad \varphi = u, v.$$
and are increasing towards the future, so that \( u \) takes values in the closed interval \([0, u^\ast]\) and \( v \) takes values in the closed interval \([0, v^\ast]\). Let \( H_u \) be the outgoing null hypersurfaces generated by the level surfaces of \( u \), and let \( H_v \) be the incoming null hypersurfaces generated by the level surfaces of \( v \). The two-dimensional surfaces obtained by intersection of \( H_u \) and \( H_v \) are \( S_{uv} \equiv H_u \cap H_v \), (1.4)

and we denote by \( H_u(v_1, v_2) \) the portion of \( H_u \) defined by \( v \in [v_1, v_2] \), while \( H_v(u_1, u_2) \) is the portion of \( H_v \) defined by \( u \in [u_1, u_2] \). The two optical functions \( u \) and \( v \) make it possible to define the vector fields with components given by

\[
L^\mu \equiv -2 \sum_{\nu=1}^{4} (g^{-1})^{\mu \nu} \partial_\nu u, \\
M^\rho \equiv -2 \sum_{\lambda=1}^{4} (g^{-1})^{\rho \lambda} \partial_\lambda v.
\]

These are future-directed null geodesic vector fields, in that \( \nabla L L = 0 \), \( \nabla M M = 0 \), (1.7)

the integral curves of \( L \) being the generators of each \( H_u \), and the integral curves of \( M \) being the generators of each \( H_v \). Indeed, one has in arbitrary local coordinates [5]

\[
\sum_{\mu, \nu=1}^{4} g_{\nu \mu} L^\nu \nabla_\nu L^\mu = -2 \sum_{\nu=1}^{4} L^\nu \nabla_\nu u = -2 \sum_{\nu=1}^{4} L^\nu \nabla_\nu \partial_\nu u = 0,
\]

where we have exploited the commutation of covariant derivatives of functions when torsion vanishes, the condition \( \nabla g = 0 \), the Leibniz rule to express

\[
\sum_{\nu, \rho=1}^{4} (g^{-1})^{\nu \rho} \left[ (\partial_\nu u)(\nabla_\rho \partial_\nu u) + (\partial_\rho u)(\nabla_\nu \partial_\nu u) \right] = \nabla_\lambda (\text{grad} u, \text{grad} u) = 0.
\]

and the eikonal equation (1.1) for \( \varphi = u \). The same holds with \( u \) and \( L \) replaced by \( v \) and \( M \), respectively.

Once the geodesic vector fields \( L \) and \( M \) are at our disposal, we can define

\[
\frac{1}{2} \Omega^2 \equiv -\frac{1}{g(L, M)} = -\frac{1}{\sum_{\mu \nu=1}^{4} g_{\mu \nu} L^\mu M^\nu},
\]

For small values of \( u^\ast \) and \( v^\ast \), the spacetime slab \( D(u^\ast, v^\ast) \) is completely determined by data along the null characteristic hypersurfaces \( H_0 \) and \( \bar{H}_0 \) corresponding to \( v = 0 \) and \( u = 0 \), respectively. We assume that \( H_0 \) can be extended [5] to negative values of \( v \), and that the spacetime \( (M, g) \) is Minkowskian for \( v < 0 \) and for all \( u \geq 0 \). The double null foliation can be chosen in such a way that the function \( \Omega \) defined in (1.8) obeys the condition

\[
\Omega(0, v) = 1 \quad \forall v \in [0, v^\ast],
\]

and one defines the pair \( (e_3, e_4) \) of null vector fields such that

\[
e_3 \equiv \Omega M, \quad e_4 \equiv \Omega L.
\]
Given a two-surface \( S(u, v) \) and an arbitrary frame \((e_1, e_2)\) tangent to it, one can define the Ricci coefficients
\[
\Gamma_{(\lambda)(\mu)(\nu)} = g\left(e_{(\lambda)}, D_{e_{(\nu)}} e_{(\mu)}\right), \quad \lambda, \mu, \nu = 1, 2, 3, 4.
\]
With the understanding that lower-case indices \( a, b \) take only the values 1, 2, and that for covariant derivatives with a subscript \( D_4 \equiv D_{e_3}, \ D_3 \equiv D_{e_4} \), the Ricci coefficients are completely determined by the components
\[
\chi_{ab} \equiv g(D_a e_4, e_b), \quad \hat{\chi}_{ab} \equiv g(D_a e_3, e_b),
\]
\[
\eta_a \equiv -\frac{1}{2} g(D_3 e_a, e_4), \quad \tilde{\eta}_a \equiv -\frac{1}{2} g(D_4 e_a, e_3),
\]
\[
\omega \equiv -\frac{1}{4} g(D_4 e_3, e_4), \quad \hat{\omega} \equiv -\frac{1}{4} g(D_3 e_4, e_3),
\]
\[
\zeta_a \equiv \frac{1}{2} g(D_a e_4, e_3).
\]
For example, one has
\[
\chi_{ab} = \sum_{\rho, \sigma, \beta = 1}^4 g_{a\rho} (\Omega L^\rho)_{\beta} (e_a)^\beta (e_b)^\sigma.
\]
In order to display the null structure equations, we have to consider also the trace-free parts of \( \chi_{ab} \) and \( \hat{\chi}_{ab} \), here denoted by \( \psi_{ab}(e_4) \) and \( \hat{\psi}_{ab}(e_3) \), where we exploit the possibility to define, for any vector field \( X \),
\[
\psi_{ab}(X) \equiv g(D_a X, e_b),
\]
so that
\[
\chi_{ab} = \psi_{ab}(e_4), \quad \hat{\chi}_{ab} = \psi_{ab}(e_3).
\]
Moreover, we need \( \nabla \), the induced covariant derivative operator on the surface \( S(u, v) \), the projection \( \nabla_3 \) (respectively \( \nabla_4 \)) to \( S(u, v) \) of the covariant derivative \( D_3 \) with respect to the vector field \( e_3 \) (respectively \( D_4 \) with respect to \( e_4 \)), and the null curvature components
\[
\alpha_{ab} \equiv R(e_a, e_4, e_b, e_4), \quad \hat{\alpha}_{ab} \equiv R(e_a, e_3, e_b, e_3),
\]
\[
\beta_a \equiv \frac{1}{2} R(e_a, e_4, e_3, e_4), \quad \hat{\beta}_a \equiv \frac{1}{2} R(e_a, e_3, e_4, e_4),
\]
\[
\rho \equiv \frac{1}{4} R(L e_4, e_3, e_4, e_4), \quad \sigma \equiv \frac{1}{4} R(e_4, e_3, e_4, e_3),
\]
where \( ^*R \) is the Hodge dual of \( R \). One then arrives at the null structure equations \[6\]
\[
\nabla_4 \chi_{ab} = -\sum_{c=1}^2 \chi_{ac} \chi^c_b - 2\omega \chi_{ab} - \alpha_{ab},
\]
\[
\nabla_3 \hat{\chi}_{ab} = -\sum_{c=1}^2 \hat{\chi}_{ac} \hat{\chi}^c_b - 2\hat{\omega} \hat{\chi}_{ab} - \hat{\alpha}_{ab},
\]
\[
\nabla_4 \eta_a = -\sum_{c=1}^2 \chi^c_a (\eta_c - \tilde{\eta}_c) - \beta_a,
\]
\[
\nabla_3 \hat{\eta}_a = \sum_{c=1}^2 \hat{\chi}^c_a (\eta_c - \tilde{\eta}_c) + \hat{\beta}_a,
\]
\[
\nabla_4 \hat{\omega} = 2\omega \hat{\omega} + \frac{3}{4} (\eta_a - \tilde{\eta}_a)(\eta^a - \tilde{\eta}^a) - \frac{1}{4} (\eta_a - \tilde{\eta}_a)(\eta^a + \tilde{\eta}^a) - \frac{1}{8} (\eta_a + \tilde{\eta}_a)(\eta^a + \tilde{\eta}^a) + \frac{1}{2} \rho,
\]
\( \nabla_3 \omega = 2 \omega \tilde{\omega} + \frac{3}{4} (\eta_a - \bar{\eta}_a)(\eta^a - \bar{\eta}^a) + \frac{1}{4} (\eta_a - \bar{\eta}_a)(\eta^a + \bar{\eta}^a) - \frac{1}{8} (\eta_a + \bar{\eta}_a)(\eta^a + \bar{\eta}^a) + \frac{1}{2} \beta, \) (1.30)

supplemented by the constraint equations

\[
\begin{align*}
\text{div} \tilde{\psi}(e_4) &= \frac{1}{2} \nabla \chi - \frac{1}{2} (\eta - \bar{\eta}) \left( \tilde{\psi}(e_4) - \frac{1}{2} \text{tr} \chi \right) - \beta, \\
\text{div} \tilde{\psi}(e_3) &= \frac{1}{2} \nabla \bar{\chi} + \frac{1}{2} (\eta - \bar{\eta}) \left( \tilde{\psi}(e_3) - \frac{1}{2} \text{tr} \bar{\chi} \right) + \bar{\beta},
\end{align*}
\]

where \( K \) is the Gauss curvature of the 2-surface \( S \). From these equations one gets in particular a pair of equations which play a key role in the formation of trapped surfaces, i.e. [6]

\[
\nabla_4 \text{tr} \chi + \frac{1}{2} (\text{tr} \chi)^2 = - \sum_{a,b=1}^2 \tilde{\psi}_{ab} \tilde{\psi}^{ab} - 2 \omega \text{tr} \chi,
\]

\[
\nabla_a \tilde{\psi} + \frac{1}{2} (\text{tr} \bar{\chi}) \tilde{\psi} = \nabla \tilde{\psi} \eta + 2 \omega \tilde{\psi}(e_4) - \frac{1}{2} (\text{tr} \chi) \tilde{\psi}(e_3) + \eta \tilde{\psi} \eta \equiv F,
\]

where the tensor products in Eq. (1.34) have components obtainable from the general formula for pairs of 1-forms \( C = \sum_{a=1}^2 C_a dx^a \) and \( E = \sum_{b=1}^2 E_b dx^b \) on the 2-surface \( S \) [5]

\[
(C \otimes E)_{ab} = C_a E_b + C_b E_a - g_{ab} \sum_{f=1}^2 C_f E^f.
\]

Note that the above expression can be written also as a symmetric-tracefree part (modulo a factor of 2)

\[
C \otimes E = 2[C \otimes E]^{\text{STF}}.
\]

2 Approximate form of nonlinear equation

Following Ref. [6], it is instructive to outline an approximate treatment of the nonlinear equation responsible for trapped-surface formation. For this purpose, we assume that spacetime is Minkowskian for \( v < 0 \) and all non-negative values of \( u \). The values of \( v \) are restricted to the closed interval \([0, \delta]\), where \( \delta \) is positive and small. The radius of the 2-surface \( S = S(u, v) \) is denoted by \( r = r(u, v) \), i.e. \( |S(u, v)| = 4\pi r^2 \), and \( r(0,0) = r_0 \). Further assumptions are as follows.

(i) For small values of \( \delta, u \) and \( v \) approach their flat-space values \( u \approx \frac{1}{2}(t - r + r_0) \) and \( v \approx \frac{1}{2}(t + r - r_0) \), while \( \Omega \approx 1 \) and \( \frac{dv}{du} \approx -1 \).

(ii) The value of \( \text{tr} \bar{\chi} \) is close to \(-\frac{2}{r}\), corresponding to the imbedding in flat space.

(iii) The right-hand side \( F \) of Eq. (1.36) can be neglected in a first approximation, as well as \(-2\omega \text{tr} \chi \) on the right-hand side of Eq. (1.35).

In light of these assumptions, Eq. (1.35) reduces to

\[
\frac{d}{dv} \text{tr} \chi \leq -|\tilde{\psi}|^2, \tag{2.1}
\]

which, by integration, yields

\[
\text{tr} \chi(u, v) \leq \text{tr} \chi(u, 0) - \int_0^v |\tilde{\psi}|^2(u, v) dv = \frac{2}{r(u, 0)} - \int_0^v |\tilde{\psi}|^2(u, v) dv. \tag{2.2}
\]
Now we can multiply the exact form of Eq. (1.36) by \( \hat{\psi} \), finding
\[
\frac{d}{du} |\hat{\psi}|^2 + (\text{tr}\hat{\chi})|\hat{\psi}|^2 = \hat{\psi}F, \tag{2.3}
\]
while, by application of the Leibniz rule, adding and subtracting terms that make it possible to exploit the assumptions (i) and (ii), we find
\[
\frac{d}{du} (r^2|\hat{\psi}|^2) = r^2\frac{d}{du} |\hat{\psi}|^2 + 2r \frac{dr}{du} |\hat{\psi}|^2 = r^2|\hat{\psi}|^2 \left( -\text{tr}\hat{\chi} + \frac{2 \, dr}{r \, du} \right) + r^2 \hat{\psi}F,
\]
which yields, upon integration,
\[
r^2|\hat{\psi}|^2(u,v) = r^2(0,v)|\hat{\psi}|^2(0,v) + \int_0^u F(u',v)du'. \tag{2.5}
\]
By virtue of the assumptions (i) and (ii) and of Eq. (2.4), the integral \( \int_0^u F(u',v)du' \) is negligible in the slab \( D(u,\delta) \), and hence one obtains the approximate relation
\[
r^2|\hat{\psi}|^2(u,v) \approx r^2(0,v)|\hat{\psi}|^2(0,v). \tag{2.6}
\]
As a next step, one freely prescribes the trace-free part of the extrinsic curvature along the initial hypersurface \([6]\), so that
\[
\hat{\psi}(0,v) = \hat{\psi}_0(v) \tag{2.7}
\]
for some traceless 2-tensor \( \hat{\psi}_0 \). Hence Eq. (2.6) becomes
\[
|\hat{\psi}|^2(u,v) \approx \frac{r^2(0,v)}{r^2(u,v)} |\hat{\psi}_0|^2(v). \tag{2.8}
\]
Furthermore, since \( |v| \leq \delta \) and \( r(u,v) = r_0 + v - u \), Eq. (2.8) reduces to
\[
|\hat{\psi}|^2(u,v) \approx \frac{r_0^2}{(r_0 - u)^2} |\hat{\psi}_0|^2(v). \tag{2.9}
\]
This formula can be now inserted into the right-hand side of Eq. (2.2), and leads to
\[
\text{tr}\chi(u,v) \leq \frac{2}{(r_0-u)^2} - \frac{r_0^2}{(r_0-u)^2} \int_0^u |\hat{\psi}_0|^2(v')dv' + \text{error term}. \tag{2.10}
\]
It is now clear that the trace of the extrinsic curvature is never positive provided that
\[
\frac{2(r_0-u)}{r_0^2} < \int_0^\delta |\hat{\psi}_0|^2(v')dv'. \tag{2.11}
\]
On the other hand, from Eq. (1.35) the condition for the initial hypersurface not to contain trapped surfaces is
\[
\int_0^\delta |\hat{\psi}_0|^2(v')dv' < \frac{2}{r_0}. \tag{2.12}
\]
The joint effect of majorizations (2.11) and (2.12) is that formation of trapped surfaces is expected provided that the condition
\[
\frac{2(r_0-u)}{r_0^2} < \int_0^\delta |\hat{\psi}_0|^2(v')dv' < \frac{2}{r_0}. \tag{2.13}
\]
is fulfilled. Such a condition requires an upper bound of the form $|\hat{\psi}_0| \leq \frac{1}{\sqrt[2]{\delta}}$. (2.14)

In order to control the error term $F$ in (2.4), we need for some positive $c$ $^{[8]}$
\[
\text{tr}\tilde{\chi} + \frac{2}{r} = O(\delta^c), \quad \frac{d}{du} \delta + 1 = O(\delta^c), \quad \eta = O\left(\delta^{-\frac{1}{2}} + c\right), \quad \omega = O\left(\delta^{-\frac{1}{2}} + c\right). \quad (2.15)
\]

Many optical structure equations (see Sect. 3) have curvature components as sources, and hence one has to derive bounds not just for all Ricci coefficients $\chi, \omega, \eta, \tilde{\chi}$ and $\tilde{\eta}$, but also for all null curvature components $\alpha, \beta, \rho, \sigma, \tilde{\alpha}$ and $\tilde{\beta}$. In Ref. $^{[5]}$ Christodoulou obtained such estimates by making the short-pulse ansatz for the initial data. This means that initial data are taken to be trivial and that $\hat{\psi}_0$ satisfies, relative to coordinates $v$ and transported coordinates $\omega$ along $H_0$ (transport being taken with respect to $\frac{dx}{\delta}$), the condition
\[
\hat{\psi}_0(v, \omega) = \frac{1}{\sqrt[2]{\delta}} f_0(\delta^{-1}v, \omega), \quad (2.16)
\]
where $f_0$ denotes a fixed traceless, symmetric $S$-tangent 2-tensor along $H_0$.

### 3 The optical structure equations

The aim of this section is to summarize the conceptual and technical framework leading to the 16 optical structure equations of vacuum Einstein equations, since their knowledge is not widespread, and the notation used in the literature is sometimes a bit cumbersome, so that its potentialities are hidden rather than being fully appreciated.

We consider a spacetime manifold $(M, g)$ with boundary, where the metric $g = \sum_{\mu, \nu=1}^{4} g_{\mu\nu} dx^\mu \otimes dx^\nu$ is taken to be a smooth solution of the vacuum Einstein equations
\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 0 \implies R_{\mu\nu} = 0.
\]

The past boundary of $M$ is the future null geodesic cone $C_p$ of a point $p$, and the initial data are assigned on $C_p$. The future-directed null geodesics issuing from $p$ are the generators of $C_p$ $^{[7]}$, while a timelike geodesic from $p$ with tangent vector $T$ at $p$ is denoted by $\Gamma_p$. With the notation of the Introduction, let us define the vector fields $Z \equiv \Omega e_4$ and $W \equiv \Omega e_3$. If $\xi$ is a 1-form on $\frac{M}{\Gamma_p}$ such that
\[
\xi(Z) = \left(\sum_{m=1}^{2} \xi_m dx^m \right) \left(\sum_{n=1}^{2} Z^n \frac{\partial}{\partial x^n}\right) = \sum_{m=1}^{2} \xi_m Z^m = \xi(W) = \sum_{n=1}^{2} \xi_n W^n = 0, \quad (3.1)
\]
we then say that $\xi$ is a $S$ 1-form, which is therefore the specification of a 1-form intrinsic to $S_{uv}$ for each $(u, v)$. The Lie derivative of $\xi$ with respect to $Z$ can be restricted to the tangent space $T S_{uv}$, and such a restriction is here denoted by $L_Z \xi|_{S_{uv}}$. This is a $S$ 1-form as well as $\xi$. Related geometrical concepts are as follows.

(i) A $S$ vector field is a vector field $X$ defined on $\frac{M}{\Gamma_p}$ such that, at each point $x \in T_p\frac{M}{\Gamma_p}$, $X$ is tangential to the surface $S_{uv}$ through $x$. This is therefore a vector field intrinsic to $S_{uv}$ for each $(u, v)$.

(ii) A type $T^s_x S$ tensor field $\theta$ is a type $T^s_x$ tensor field defined on $\frac{M}{\Gamma_p}$ such that, at each $x \in T_p\frac{M}{\Gamma_p}$ and each $X_1, ..., X_s \in T_xM$, one has
\[
\theta(X_1, ..., X_s) \in \otimes^s T_x S_{uv},
\]
and \( \theta(X_1, \ldots, X_s) = 0 \) if one of \( X_1, \ldots, X_s \) is either \( Z \) or \( W \). One therefore deals with a type \( T^q_s \) tensor field intrinsic to \( S_{uv} \) for each \((u, v)\).

The work in Ref. [5] proves that, for any given \( S \) vector field \( Y \), the Lie derivatives

\[
\mathcal{L}_Z Y = [Z, Y] \quad \text{and} \quad \mathcal{L}_W Y = [W, Y]
\]

are also \( S \) vector fields. One can therefore define the restricted Lie derivatives

\[
\mathcal{L}Z\theta|_{TS_{uv}} \equiv \mathcal{L}_Z \theta, \quad \mathcal{L}W\theta|_{TS_{uv}} \equiv \mathcal{L}_W \theta.
\]

(3.2)

As a next step, for a \( S \) tensor field \( \theta \) of type \( T^q_s \), the Lie derivative \( \mathcal{L}_Z \theta|_{TS_{uv}} \) is defined by considering \( \theta \) on each \( H_u \) extended to the tangent space \( TH_u \) according to the condition that it vanishes if one of the entries is \( Z \), and setting the Lie derivative of \( \theta \) with respect to \( Z \), when restricted to the tangent space \( TS_{uv} \), equal to the restriction to such a tangent space of the usual Lie derivative with respect to \( Z \) of this extension. In analogous fashion, the restriction to the tangent space \( TS_{uv} \) of the Lie derivative of \( \theta \) with respect to \( W \) is defined by considering \( \theta \) on each \( H_v \) extended to the tangent space \( TH_v \) in such a way that it vanishes if one of the entries is \( W \), and setting

\[
\mathcal{L}_W \theta|_{TS_{uv}} = \text{restriction to } TS_{uv} \text{ of the usual Lie derivative with respect to } W \text{ of this extension.}
\]

This method yields Lie derivatives which are, themselves, \( S \) vector fields of type \( T^q_s \). We write hereafter

\[
\mathcal{L}Z\theta|_{TS_{uv}} \equiv \mathcal{D}\theta, \quad \mathcal{L}W\theta|_{TS_{uv}} \equiv \tilde{\mathcal{D}}\theta.
\]

(3.3)

In particular, if \( \theta \) is a 0-form, i.e. a function \( f \), one has

\[
\mathcal{D}f = Zf = \sum_{a=1}^{2} Z^a \frac{\partial f}{\partial x^a}, \quad \tilde{\mathcal{D}}f = Wf = \sum_{b=1}^{2} W^b \frac{\partial f}{\partial x^b}.
\]

(3.4)

For any function \( f \) defined on \( M^p \), we denote by \( d_{uv} f \) the \( S \) 1-form obtained by restriction to each surface \( S_{uv} \) of the differential \( df \), i.e.

\[
d_{uv} f \equiv df|_{S_{uv}} = \sum_{a=1}^{2} \frac{\partial f}{\partial x^a} dx^a.
\]

(3.5)

This operation commutes with the \( D \) and \( \tilde{D} \) derivatives, i.e. [5]

\[
Dd_{uv} f = d_{uv} Df, \quad \tilde{D}d_{uv} f = d_{uv} \tilde{D}f.
\]

Set now (cf. Sect. 1) \( \hat{L} \equiv e_4 = \Omega L \), \( \tilde{L} \equiv e_3 = \Omega M \). The tangent hyperplane \( T_p H_u \) to a given null hypersurface \( H_u \) at a point \( p \in H_u \) is given by all vectors \( X \) at \( p \) which are orthogonal to \( \hat{L}_p \), i.e.

\[
T_p H_u \equiv \left\{ X \in T_p M : g(X, \hat{L}_p) = 0 \right\}, \quad (3.6)
\]

while the tangent hyperplane \( T_p H_v \) is given by

\[
T_p H_v \equiv \left\{ X \in T_p M : g(X, \tilde{L}_p) = 0 \right\}. \quad (3.7)
\]

Since \( H_u \) and \( H_v \) are null hypersurfaces in spacetime, their induced metrics are degenerate, while the induced metric \( h \) on each surface \( S_{uv} \) is Riemannian (i.e. positive-definite), and is a symmetric 2-covariant tensor field

\[
h = \sum_{a,b=1}^{2} h_{ab} dx^a \otimes dx^b, \quad h_{ab} = h_{(ab)}.
\]
Any vector $X \in T_p H_u$ can be uniquely decomposed into a vector collinear to $\hat{L}_p$ and a vector tangent to the surface $S_{uv}$, i.e. ($a$ being a real number)

$$X \in T_p H_u \implies X = a\hat{L}_p + PX, \quad PX \in T_p S_{uv}. \tag{3.8}$$

If $X$ and $Y$ are any two vectors tangent to $H_u$ at $p$, one has a simple relation between spacetime metric $g$ and induced metric $h$, i.e.

$$g(X,Y) = h(PX, PY). \tag{3.9}$$

Similarly, one has

$$X \in T_p H_v \implies X = a\hat{L}_p + \pi X, \quad \pi X \in T_p S_{uv}, \tag{3.10}$$

and

$$g(X,Y) = h(\pi X, \pi Y) \tag{3.11}$$

for any pair of vectors $X$ and $Y$ tangent to $H_v$ at $p$.

The second fundamental form $\chi$ of a null hypersurface $H_u$ is a bilinear form

$$\chi_u : T_p H_u \times T_p H_u \rightarrow \mathbb{R}$$

defined by (see components in (1.15) and (1.19))

$$\chi_u(X,Y) \equiv g(\nabla_X \hat{L}, Y). \tag{3.12}$$

It can be shown to be symmetric, because

$$\chi_u(X,Y) - \chi_u(Y,X) = -g(\hat{L}, [X,Y]) = 0, \tag{3.13}$$

where $X$ and $Y$ are extended to vector fields along $H_u$ which are tangential to $H_u$. It should be stressed that $\chi$ is intrinsic to $H_u$, because the vector field $\hat{L}$ is tangential to $H_u$. One has

$$\chi_u(X,Y) = \chi_u(PX, PY), \tag{3.14}$$

and hence $\chi_u$ is a symmetric 2-covariant $S$ tensor field. Similarly, for the null hypersurfaces $H_v$ one defines

$$\chi_v : T_p H_v \times T_p H_v \rightarrow \mathbb{R}$$

such that

$$\chi_v(X,Y) \equiv \tilde{\chi}(X,Y) \equiv g(\nabla_X \hat{L}, Y) = \chi_v(Y,X) = \chi_v(\pi X, \pi Y). \tag{3.15}$$

If $\theta$ is any 2-covariant $S$ tensor field, we denote by $\theta^\sharp$ the $S$ tensor field of type $(1,1)$ (i.e. once covariant and once contravariant) such that

$$h(\theta^\sharp X, Y) = \theta(X,Y) \quad \forall X, Y \in T_p S_{uv}. \tag{3.16}$$

If $e_1, e_2$ is an arbitrary basis for the tangent space $T_p S_{uv}$, one has (unlike Ref. [5], we do not use block capital letters for tensor components here, so as to avoid confusion with two-component spinors for which $A, B$ are a standard notation for unprimed spinor indices [1])

$$\theta^\sharp e_a = \sum_{b=1}^{2} (\theta^\sharp)^b_a \ e_b, \quad (\theta^\sharp)^b_a = \sum_{c=1}^{2} \theta_{ac}(h^{-1})^c_b, \quad a, b = 1, 2, \tag{3.17}$$

which verify indeed the explicit form of Eq. (3.16), i.e.

$$\sum_{a,b,c=1}^{2} h_{ab}(\theta^\sharp)^a_c X^c Y^b = \sum_{a,b=1}^{2} \theta_{ab} X^a Y^b. \tag{3.18}$$
If \( \theta \) and \( T \) are symmetric 2-covariant \( S \) tensor fields, their product \( \theta \times T \) is defined by

\[
(\theta \times T)(X,Y) \equiv h(\theta^2 X, T^2 Y), \quad \forall X, Y \in T_p S_{uv}.
\]  
(3.19)

In arbitrary local coordinates for \( S_{uv} \), this formula reads as

\[
(\theta \times T)_{ab} = \sum_{r,s=1}^{2} h_{rs}(\theta^2)_r^a (T^2)_s^b = \sum_{r,s=1}^{2} (h^{-1})^r_s \theta_{ar} T_{bs}.
\]  
(3.20)

The optical structure equations will involve (see below) the rescaled tensor fields (see (1.15) and (1.19)-(1.21))

\[
\chi'_{ab} \equiv \Omega^{-1} \chi_{ab}, \quad \tilde{\chi}'_{ab} \equiv \Omega^{-1} \tilde{\chi}_{ab},
\]  
(3.21)

and a hypersurface version of divergence, curl and covariant derivative. More precisely, one defines

the covariant derivative intrinsic to \( S_{uv} \), for any pair \( X, Y \) of \( S \) vector fields, with the help of a projection operator \( \pi \) to the surfaces \( S_{uv} \), as given by \[5\]

\[
(uv \nabla)_X Y = \pi \nabla_X Y, \quad \pi V = V + \frac{1}{2} g(V, e_3) e_4 + \frac{1}{2} g(V, e_4) e_3 \in T_q S_{uv},
\]  
(3.22)

for all \( V \in T_q (\frac{\text{M}}{\mathcal{F}}) \). Furthermore, the intrinsic divergence is defined by the formula

\[
(uv \text{div} \theta)_a = \sum_{c=1}^{2} (uv \nabla)_c \theta^c_a,
\]  
(3.23)

and one denotes by \( uv \varepsilon \) the area 2-form of \( S_{uv} \), with components

\[
(uv \varepsilon)_{ab} = (uv \varepsilon)(e_a, e_b), \quad a, b = 1, 2.
\]  
(3.24)

The latter concept is used to define the intrinsic curl of a \( S \) 1-form \( \xi \) according to

\[
uw \text{curl} \xi \equiv \frac{1}{2} \sum_{a,b,c,d=1}^{2} (uw \varepsilon)_{cd} (h^{-1})^{ac} (h^{-1})^{bd} (uw \nabla_a \xi_b - uw \nabla_b \xi_a).
\]  
(3.25)

Out of the area 2-form of \( S_{uv} \) one can also build the twice sharp \( \varepsilon \), defined as

\[
(uw \varepsilon^{\sharp\sharp})_{ab} = \sum_{c,d=1}^{2} (uw \varepsilon)_{cd} (h^{-1})^{ac} (h^{-1})^{bd},
\]  
(3.26)

and hence the wedge product of symmetric 2-covariant \( S \) tensor fields \( \theta \) and \( T \), i.e. \[5\]

\[
\theta \wedge T \equiv \sum_{a,b,c,d=1}^{2} (uw \varepsilon^{\sharp\sharp})_{ab} (h^{-1})^{cd} \theta_{ac} T_{bd}.
\]  
(3.27)

Last, one considers

\[
(\xi, \xi') \equiv \sum_{a,b=1}^{2} (h^{-1})^{ab} \xi_a \xi'_b,
\]  
(3.28)

\[
|\xi| \equiv \sqrt{(\xi, \xi)}.
\]  
(3.29)

Since we have defined all concepts that are needed, we can now write down from Ref. \[5\], but with our notation, the 16 equations expressing the optical structure of vacuum Einstein equations. They read as follows:

\[
Dh = 2\Omega \chi, \quad \tilde{D}h = 2\Omega \tilde{\chi},
\]  
(3.30)
\[ D\chi' = \Omega^2 \chi' \times \chi' - \alpha, \quad (3.31) \]
\[ \tilde{D}\chi' = \Omega^2 \tilde{\chi} \times \tilde{\chi}' - \tilde{\alpha}, \quad (3.32) \]
\[ D\eta = \Omega (\chi^\sharp \cdot \tilde{\eta} - \beta), \quad (3.33) \]
\[ \tilde{D}\eta = \Omega (\tilde{\chi}^\sharp \cdot \eta + \tilde{\beta}), \quad (3.34) \]
\[ D\tilde{\omega} = \Omega^2 \left[ 2(\eta, \tilde{\eta}) - |\eta|^2 - \rho \right], \quad (3.35) \]
\[ \tilde{D}\omega = \Omega^2 \left[ 2(\eta, \tilde{\eta}) - |\eta|^2 - \rho \right], \quad (3.36) \]
\[ K + \frac{1}{2} (\text{tr} \chi)(\text{tr} \tilde{\chi}) - \frac{1}{2} (\chi, \tilde{\chi}) = -\rho, \quad (3.37) \]
\[ w_e \text{div} \chi' = w_e d (\text{tr} \chi') + \chi'^\sharp \cdot \eta - (\text{tr} \chi') \eta = -\Omega^{-1} \beta, \quad (3.38) \]
\[ w_e \text{div} \tilde{\chi}' = w_e d (\text{tr} \tilde{\chi}') + \tilde{\chi}^\sharp \cdot \tilde{\eta} - (\text{tr} \tilde{\chi}') \tilde{\eta} = \Omega^{-1} \tilde{\beta}, \quad (3.39) \]
\[ w_e \text{curl} \tilde{\eta} = \frac{1}{2} \chi \wedge \tilde{\chi} - \sigma, \quad (3.40) \]
\[ w_e \text{curl} \eta = w_e \text{curl} \zeta = -w_e \text{curl} \tilde{\eta}, \quad (3.41) \]
\[ D(\Omega \chi) = \Omega^2 \left[ w_e \nabla \tilde{\eta} + w_e \tilde{\nabla} \eta + 2 \eta \otimes \tilde{\eta} + \frac{1}{2} (\chi \times \tilde{\chi} + \tilde{\chi} \times \chi) + \rho h \right], \quad (3.42) \]
\[ \tilde{D}(\Omega \chi) = \Omega^2 \left[ w_e \nabla \eta + w_e \tilde{\nabla} \eta + 2 \eta \otimes \eta + \frac{1}{2} (\chi \times \tilde{\chi} + \chi \times \chi) + \rho h \right], \quad (3.43) \]
\[ \tilde{D}\tilde{\eta} = -\Omega \left( \tilde{\chi} \sharp \cdot \eta + \tilde{\eta} \right) + 2 d w_e \tilde{\omega}, \quad (3.44) \]
\[ D\tilde{\eta} = -\Omega \left( \chi \sharp \cdot \eta - \beta \right) + 2 d w_e \omega. \quad (3.45) \]

The first of Eqs. (3.30), and Eqs. (3.31), (3.33), (3.35), are propagation equations along the generators of each \( H_u \); the second of Eqs. (3.30), and Eqs. (3.32), (3.34), (3.36), are propagation equations along the generators of each \( H_v \). Moreover, Eq. (3.37) is the Gauss equation of the embedding of the surfaces \( S_{uv} \), with Gauss curvature \( K \), in the spacetime manifold \((M, g)\), while Eqs. (3.38) and (3.39) are the Codazzi equations of such an embedding.

## 4 Open problems

As far as we can see, at least two outstanding problems deserve careful consideration:

(i) Despite the elegant proof obtained in Refs. [3] and [4] that the non-linear equations leading to trapped surface formation can be studied by discarding some terms, it would be interesting to solve them numerically without discarding any term, no matter how small it can be.

(ii) It would be interesting to apply optical-structure methods to extended theories of classical gravity, and possibly to the quantum theory of black holes.

## Acknowledgments

G. E. is grateful to the Dipartimento di Fisica “Ettore Pancini” of Federico II University for hospitality and support.
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