RESONANT OSCILLATIONS AND TIDAL HEATING
IN COALESCING BINARY NEUTRON STARS

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ABSTRACT

Tidal interaction in a coalescing neutron star binary can resonantly excite the g-mode oscillations of the neutron star when the frequency of the tidal driving force equals the intrinsic g-mode frequencies. We study the g-mode oscillations of cold neutron stars using recent microscopic nuclear equations of state, where we determine self-consistently the sound speed and Brunt-Väisälä frequency in the nuclear liquid core. The properties of the g-modes associated with the stable stratification of the core depend sensitively on the pressure-density relation as well as the symmetry energy of the dense nuclear matter. The frequencies of the first ten g-modes lie approximately in the range of $10^{-3} - 10^0$ Hz. Resonant excitations of these g-modes during the last few minutes of the binary coalescence result in energy transfer and angular momentum transfer from the binary orbit to the neutron star. The angular momentum transfer is possible because a dynamical tidal lag develops even in the absence of fluid viscosity. However, since the coupling between the g-mode and the tidal potential is rather weak, the amount of energy transfer during a resonance and the induced orbital phase error are very small.

Resonant excitations of the g-modes play an important role in tidal heating of binary neutron stars. Without the resonances, viscous dissipation is effective only when the stars are close to contact. The resonant oscillations result in dissipation at much larger orbital separation. The actual amount of tidal heating depends on the viscosity of the neutron star. Using the microscopic viscosity, we find that the binary neutron stars are heated to a temperature $\sim 10^8$ K before they come into contact.

Key Words: binaries: close — radiation mechanisms: gravitational — stars: neutron — oscillations — hydrodynamics — equation of state — dense matter

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1. INTRODUCTION

Coalescing neutron star-neutron star (NS-NS) and neutron star-black hole (NS-BH) binaries are believed to be one of the most promising sources of gravitational radiation (e.g., Schutz 1986, Thorne 1987) that could be detected by the new generation of instruments such as the Laser Interferometer Gravitational-Wave Observatory (LIGO, see Abramovici et al 1992) and its French-Italian counterpart VIRGO (Bradaschia et al 1990). Extrapolation of the estimated galactic formation rates of compact binaries based on the observed local population of binary radio pulsars leads to an estimate of the rate of coalescence in the Universe of about $10^{-7} \, \text{yr}^{-1} \, \text{Mpc}^{-3}$ (Narayan, Piran & Shemi 1991, Phinney 1991). Extracting gravity wave signals from noise requires accurate theoretical waveforms in the frequency range $10 - 1000 \, \text{Hz}$, corresponding to the last few minutes of the binaries’ life (e.g., Cutler et al. 1993). Effects of tidal interaction on the orbital decay and gravitational radiation waveforms have been found to be small except during the final stage of coalescence (Kochanek 1992, Bildsten & Cutler 1992, Lai, Rasio & Shapiro 1993, 1994a, 1994b). However, all these studies implicitly assume equilibrium or quasi-equilibrium tides, while dynamical tidal responses of the stars are completely neglected. Such dynamical tides are possible when the interaction potential between the two components of the binary resonantly excites the intrinsic oscillations of the neutron stars.

Recent studies of neutron star oscillations have demonstrated the existence of low frequency g-modes of various kinds. McDermott, Van Horn & Scholl (1983) and McDermott, Van Horn & Hansen (1988) calculated the g-modes associated with thermally induced buoyancy for warm (hot) neutron stars. Finn (1987) considered crustal g-modes due to the composition discontinuities in the outer envelope of cold neutron stars. Reisenegger & Goldreich (1992, hereafter RG1) studied g-modes associated with the buoyancy induced by proton-neutron composition gradient in the cores of neutron stars. Regardless of the nature of the g-mode oscillations and various theoretical uncertainties, the existence of such g-modes indicates that a neutron star in a binary system can be resonantly excited when the frequency of the tidal driving force equals the g-mode frequencies. These resonant oscillations and their consequences to the binary orbital decay and the neutron stars themselves are the subjects of this paper.

For a star in a binary system, the tidal potential due to the companion acts like an external perturbing force, with a driving frequency $2\Delta \Omega = 2(\Omega - \Omega_s)$, where $\Omega$ and $\Omega_s$ are the orbital angular frequency and the spin angular frequency of the star (the factor of 2 results from the quadrupole nature of the tidal potential, and will become apparent in §2). The star behaves as a collection of harmonic oscillators with different eigen-frequencies. Let us consider just one such oscillation mode with frequency $\omega$. Schematically, the equation governing the tidal distortion $\xi$ can be written as

$$\ddot{\xi} + \omega^2 \xi - \frac{1}{t_{\text{visc}}} \dot{\xi} \propto e^{2i\Delta \Omega t},$$

(1.1)

where $t_{\text{visc}}$ is the timescale for viscous dissipation (or other dissipations) in the star. For stationary state, we have

$$\xi \propto \frac{1}{\omega^2 - 4\Delta \Omega^2 - 2i\Delta \Omega/t_{\text{visc}}} e^{2i\Delta \Omega t}.$$  

(1.2)
Since in general the viscous dissipation timescale is much larger than the dynamical timescale of the star, we see that when $2\Delta\Omega = \omega$, the amplitude of $\xi$ can be very large.

Such resonant oscillations in close binaries were first considered by Cowling (1941) in his seminal paper on nonradial stellar oscillations. He noted that the coupling between the tidal potential and the high-order g-mode is weak, therefore a close resonance is needed to greatly enhance the equilibrium tide (see also Zahn 1970). He also suggested that such close resonance will be destroyed by second order effects due to the large horizontal displacement of the high-order g-mode. Clearly, for long-lived binary systems, such as binaries consisting of main sequence stars, what exactly happens to the stars at resonance is a complicated issue (e.g., Zahn 1977, Nicholson 1978, Goldreich & Nicholson 1989).

For an inspiraling neutron star binary, the problem becomes “cleaner”. The rapid decay of the orbit induces a dynamical tidal lag, and implies that the orbital frequency does not stay at a particular resonant value very long. Thus the rapid decay of the orbit “kills” the infinity inherent in equation (1.2) even without dissipative or other higher-order effects. Moreover, we shall see that the resonant tidal distortion is small and a linear approximation for the excited oscillations is adequate.

In this paper, we shall concentrate on the excitations of core g-modes. These modes involve the bulk region of a neutron star, while the g-modes driven by entropy gradient and density discontinuities are confined to its outer envelope (Finn 1987, McDermott, Van Horn & Hansen 1988). Previous study (RG1) has yielded a qualitative understanding of the basic properties of the core g-modes. However, the detailed properties of these g-modes, especially the strengths of their coupling to the tidal field, which directly affect the amplitudes of the excitations, depend sensitively on the values of the Brunt-Väisälä frequency in different regions of the star. The calculations by RG1 were based on an approximate, inconsistent ansatz for the Brunt-Väisälä frequency in the core. We examine some recent microscopic equations of state of nuclear matter, and use them to obtain the self-consistent Brunt-Väisälä frequency, upon which our calculations of g-modes are based (see §4 and §5).

A related question of interest concerns how much energy is dissipated into heat in coalescing binary neutron stars. Mészáros & Rees (1992) considered the consequence of such tidal heating in the context of cosmological Gamma ray burst models. Significant tidal heating of the neutron stars can induce mass loss from the stars even before merging takes place. Such mass loss can lead to baryon contamination of the possible Gamma ray burst emission. Clearly, this result depends on the viscosity of the neutron star matter. We show that the resonant excitations of g-modes are important in the viscous heating of binary neutron stars (§8).

Some aspects of the problem of resonant excitations of g-modes have also recently been considered by Reisenegger & Goldreich (1994, hereafter RG2). They calculated the total energy transfer from the binary orbit to the star in a resonance using the formalism of Press & Teukolsky (1977, hereafter PT), who considered a similar problem in the context of parabolic encounter of two stars. We present here a different, more complete analysis of oscillations in coalescing binary neutron stars, including dynamical tidal
lag, energy transfer and angular momentum transfer due to resonances, etc. Our method enables us to study the time-dependence of the resonant energy transfer process (rather than the total energy transferred), as well as the non-resonant, quasi-equilibrium tidal effects. We include comparisons of our results with those of RG2.

In §2 we develop the formalism to study tidal excitations of normal modes in inspiraling binary stars, where we show that the problem can naturally be divided into two parts. The dynamical part is discussed in §3, while the physics of core g-modes and the calculations of the mode properties are discussed in §4 and §5. In §6 energy transfer and angular momentum transfer from the binary orbit to the star are considered. The effects of such transfer on the orbital decay and the induced orbital phase error are discussed in §7. Tidal heating of binary neutron stars are considered in §8. We present our conclusions in §9, where we also discuss the importance of g-mode resonances in coalescing white dwarf binaries.

2. DYNAMICAL TIDAL INTERACTION IN COALESCING BINARIES: THE FORMALISM

Consider a star with mass $M$ and radius $R$ in circular orbit with its companion, which we treat as a point object with mass $M'$. When $M'$ also has a finite size, there is additional quadrupole-quadrupole coupling between $M$ and $M'$; but this is a higher-order effect which will be neglected. Also, general relativistic effects introduce a small correction to our analysis; these effects are neglected.

The interaction potential due to $M'$ is

$$U = -\frac{GM'}{|r - D|}, \tag{2.1}$$

where $r$ is the position vector of a fluid element in star $M$, and $D = D(t)$ specifies the position of the point mass. In a comoving coordinate system $(r, \theta, \phi)$ with its origin at the center of $M$ and the orbital plane at $\theta = \pi/2$, we have $D(t) = (D(t), \pi/2, \Phi(t))$, and the potential can be expanded into spherical harmonics

$$U = -GM'\sum_{lm} \frac{4\pi}{2l+1} \frac{r^l}{D(t)^{l+1}} Y^*_lm\left(\frac{\pi}{2}, \Phi\right) Y_l^m(\theta, \phi), \tag{2.2}$$

(using the notation of PT), where the numerical coefficient $W_{lm}$ is

$$W_{lm} = (-)^{(l+m)/2} \left[\frac{4\pi}{2l+1}(l+m)!(l-m)\right]^{1/2} \left[2^l \left(\frac{l+m}{2}\right) \left(\frac{l-m}{2}\right)\right]^{-1}. \tag{2.3}$$

Here the symbol $(-)^k$ is to be interpreted as zero when $k$ is not an integer. In equation (2.2), the $l = 0$ and $l = 1$ terms can be be dropped, since they are not relevant for tidal deformation. Although our equations are valid for general $l$, throughout this paper we shall only consider the leading quadrupole term, $l = 2$, for which the relevant coefficients $W_{lm}$ are

$$W_{20} = -\left(\frac{\pi}{5}\right)^{1/2}, \quad W_{2\pm 1} = 0, \quad W_{2\pm 2} = \frac{1}{2} \left(\frac{6\pi}{5}\right)^{1/2}. \tag{2.4}$$
In the linear approximation, the effect of the tidal potential on star $M$ is specified by the Lagrangian
\[ \left( \rho \frac{\partial^2}{\partial t^2} + \mathcal{L} \right) \tilde{\xi} = -\rho \nabla U, \]
where $\rho$ is the density, and $\mathcal{L}$ is an operator which specifies the internal restoring forces of the star. We now analyze $\tilde{\xi}(\mathbf{r}, t)$ in to the normal modes of the star
\[ \tilde{\xi}(\mathbf{r}, t) = \sum_{\alpha} a_{\alpha}(t) \hat{\xi}_{\alpha}(\mathbf{r}), \]
where $\alpha$ specifies the “quantum number” of a normal mode. The mode eigenfunction $\hat{\xi}_{\alpha}$ satisfies
\[ (\mathcal{L} - \rho \omega_{\alpha}^2) \hat{\xi}_{\alpha}(\mathbf{r}) = 0, \]
where $\omega_{\alpha}$ is the angular frequency of the eigenmode. For a spherical star, the normal modes are labeled by the spherical harmonic indices, $l$, $m$, and by a “radial quantum number” $n$, i.e., $\{\alpha\} = \{n, l, m\}$. For convenience, the indices $\alpha$ and $\{n, l, m\}$ will be used interchangeably throughout the paper. For a particular spheroidal mode, the eigenfunction can be written as the sum of the radial and tangential pieces
\[ \hat{\xi}_{\alpha}(\mathbf{r}) = \tilde{\xi}_{nlm}(r) = \left[ \xi_r n_{l}(r) \mathbf{e}_r + r \xi^\perp n_{l}(r) \nabla Y_{lm}(\theta, \phi) \right], \]
where $\mathbf{e}_r$ is the unit radial vector. Note that toroidal modes, which are characterized by identically vanishing Eulerian variation of density (e.g., Cox 1980, p222), are not excited by potential forces such as the tidal interaction considered here. Using the orthogonal relations for $\hat{\xi}_{\alpha}$ and the normalization
\[ \int d^3x \rho \hat{\xi}_{\alpha}^* \cdot \hat{\xi}_{\alpha'} = \delta_{\alpha\alpha'}, \]
equation (2.5) can be reduced to a set of equations for $a_{\alpha}(t)$:
\[ \ddot{a}_{\alpha} + \omega_{\alpha}^2 a_{\alpha} = \frac{GM' W_{lm} Q_{nl}}{D^{l+1}} e^{-im\Phi(t)}, \]
where we have defined a tidal coupling coefficient
\[ Q_{nl} = \int d^3x \rho \xi_{nlm}^* \cdot \nabla \left[ r Y_{lm}(\theta, \phi) \right], \]
\[ = \int_0^R \rho r^{l+1} dr \left[ \xi_r n_{l}(r) + (l + 1) \xi^\perp n_{l}(r) \right], \]
(see Zahn 1970, PT). Since $\tilde{\xi}$ is real, we have $a_{m=2}^* = a_{m=-2}$.

Equation (2.10) then essentially determines the response of the star to the tidal potential, provided we know the functions $D(t)$ and $\Phi(t)$ which specify the decaying orbit. Using the quadrupole formula for the gravitational radiation of two point masses, the rate of change of the orbital separation is given by
\[ \dot{D} = -\frac{64G^3}{5c^5} \frac{MM'(M + M')}{D^3}, \]
(2.12)
The use of this point-mass formula is adequate, since the effect of tidal interaction on the orbit is small (see §6; a fully self-consistent formalism, incorporating the feedback of the tidal interaction on the orbital trajectory, is discussed in Lai 1994). It is convenient to define an orbital decay time \( t_D \) as

\[
t_D \equiv \frac{D}{|\dot{D}|} = \frac{56^5}{64G^3} \frac{D^4}{M M'(M + M')},
\]

(2.13)

The orbital phase function \( \Phi(t) \) is

\[
\Phi(t) = \int^t dt \Delta \Omega = \int^t dt (\Omega - \Omega_s),
\]

(2.14)

where \( \Omega_s \) is the spin of the star, and \( \Omega \) is the orbital angular frequency, for which we have

\[
\Omega^2 = \frac{G(M + M')}{D^3}, \quad \dot{\Omega} = -\frac{3 \dot{D}}{2D} \Omega.
\]

(2.15)

When \( \Omega_s \neq 0 \), the normal modes of the star become much more complicated, especially when \( \Omega_s \) becomes comparable to the mode frequencies. Throughout our paper, we shall assume \( \Omega >> \Omega_s \), so that the eigen-modes can be adequately approximated by those of a non-rotating spherical star, for which \( \omega_\alpha = \omega_{nl} \) does not depend on \( m \). Kochanek (1992), and Bildsten and Cutler (1992) have shown that the viscosity is too small for significant spin-up of the neutron star during the inspiral. We point out that even in the absence of fluid viscosity, angular momentum can be transferred to the star due to dynamical tidal lag. In §6, we will show that such dynamical spin-up of the neutron star is also negligible.

It is convenient to define a function \( b_\alpha(t) \) such that

\[
a_\alpha(t) = (GM'W_{lm}Q_{nl})b_\alpha(t)e^{-im\Phi(t)}.
\]

(2.16)

Equation (2.10) then becomes

\[
\ddot{b}_\alpha - 2i m \Omega \dot{b}_\alpha + (\omega_\alpha^2 - m^2 \Omega^2 - i m \dot{\Omega}) b_\alpha = \frac{1}{D^{l+1}}.
\]

(2.17)

Note that apart from the mode frequency, \( b_\alpha(t) \) does not depend on the properties of the mode and the structure of the star. The mode properties enter through the tidal coupling coefficient \( Q_{nl} \). Thus the whole problem is divided into two parts: Find \( b_\alpha(t) \) by solving equation (2.17) for a given \( \omega_\alpha \), and calculate the frequencies and tidal coupling coefficients \( Q_{nl} \) for different modes.

The natural units of mass, length, time, and energy in this problem are \( M, R, (R^3/GM)^{1/2} \), and \( GM^2/R \) respectively. We shall use these units as well as real physical units throughout the paper, whichever is more convenient. Using the natural units to nondimensionalize \( Q_{nl}, \xi_\alpha \) and \( b_\alpha \), we have

\[
\ddot{\xi}(r, t) = h_0 \sum_{\alpha} W_{lm}Q_{nl} \frac{b_\alpha(t)}{(R/D)^{l+1}} e^{-im\Phi(t)} \tilde{\xi}_\alpha(r),
\]

(2.18)

(only \( l = 2 \) terms are included in the sum), where \( h_0 \) is simply the typical equilibrium tide height

\[
h_0 \equiv R \left( \frac{M'}{M} \right) \left( \frac{R}{D} \right)^3.
\]

(2.19)
The equilibrium tide corresponds mainly to the f-mode distortion, for which $b_\alpha \sim (R/D)^3$ and $Q_{nl} \sim 1$, as we shall see later.

### 3. Dimensionless Dynamical Tidal Amplitude

In this section, we consider the dimensionless tidal amplitude $b_\alpha(t)$ for a given mode frequency $\omega_\alpha$. This determines the dynamical aspects of the tidal excitation.

#### 3.1 Solution Prior to The Resonance

From equation (2.4), we see that $b(t)$ is nonzero only for $m = 0$ or $m = \pm 2$. For $m = 0$, the tidal potential is quasi-static, and equation (2.17) becomes

$$\ddot{b}_\alpha + \omega_\alpha^2 b_\alpha = \frac{1}{D^{l+1}}, \quad (m = 0). \quad (3.1)$$

When the orbital decay timescale $t_D$ is much longer than the oscillation period $1/\omega_\alpha$, the function $b_\alpha(t)$ is approximately equal to the quasi-equilibrium value $b_{\alpha,eq}$:

$$b_\alpha(t) \simeq b_{\alpha,eq}(t) = \frac{1}{\omega_\alpha^2 D(t)^{l+1}}, \quad (m = 0). \quad (3.2)$$

Now consider the dynamical tide ($m = \pm 2$). Resonance occurs when $\omega_\alpha = 2\Delta \Omega \simeq 2\Omega$, at a resonance radius given by

$$\frac{D_\alpha}{R} = \left[ \frac{4(1 + q)GM}{\omega_\alpha^3 R^3} \right]^{1/3}, \quad (3.3)$$

where $q = M'/M$ is the mass ratio. Before reaching this resonance, when $\omega_\alpha^2 - (2\Omega)^2 \gg |\dot{\Omega}|$, an approximate solution for $b_\alpha(t)$ can be found. Dropping the $\ddot{b}_\alpha$ and $\dot{b}_\alpha$ terms in equation (2.17), we obtain a zero-th order solution

$$b_\alpha^{(0)} = \frac{1}{D^{l+1}(\omega_\alpha^2 - m^2\Omega^2)}. \quad (3.4)$$

Thus the zero-th order term in $\dot{b}_\alpha$ is

$$\dot{b}_\alpha \simeq \left[ -(l + 1) \frac{\dot{D}}{D} + \frac{2m^2\Omega \dot{\Omega}}{\omega_\alpha^2 - m^2\Omega^2} \right] b_\alpha. \quad (3.5)$$

Substituting (3.5) into equation (2.17), and dropping the term $\ddot{b} \sim b/t_D^2$, we obtain an approximate solution for $b_\alpha(t)$, valid for $\omega_\alpha^2 - 4\Omega^2 \gg |\dot{\Omega}|$:

$$b_\alpha(t) \simeq \frac{1}{D^{l+1}} \left[ \omega_\alpha^2 - m^2\Omega^2 + i \left( 2(l + 1)m\Omega \frac{\dot{D}}{D} - m\dot{\Omega} - \frac{4m^3\Omega^2 \dot{\Omega}}{\omega_\alpha^2 - m^2\Omega^2} \right) \right]^{-1}. \quad (3.6)$$

It is important to note that even without viscous dissipation, the orbital decay naturally provides a dynamical tidal lag. This is because the star does not respond instantaneously to the changing tidal potential. For $\omega_\alpha \gg \Omega$, this lag angle is

$$\delta_\alpha \sim \frac{\Delta \Omega}{\omega_\alpha t_D}, \quad (3.7)$$
(recall $\Delta \Omega \simeq \Omega$, since the spin is assumed to be small). This should be compared with the standard viscosity-induced tidal lag (cf. eq. [1.2])

$$\delta_{\text{visc}} \sim \frac{\Delta \Omega}{\omega_{\alpha}^{2/3} t_{\text{visc}}}.$$  \hfill (3.8)

For most likely viscosities (see §8), we have $t_D << t_{\text{visc}}$, thus the viscous tidal lag is negligible compared to the dynamical lag induced by the changing tidal potential.

### 3.2 Resonance Amplitude

As the orbital separation decreases, $b_{\alpha}(t)$ increases according to equation (3.6). We can estimate how close to resonance this equation can still be valid, from which we can obtain a scaling relation for the maximum amplitude during a resonance.

Suppose equation (3.6) starts to break down when $2\Omega \sim (1 - \epsilon)\omega_\alpha$. Note that $\epsilon$ is not determined by $\omega_\alpha^2 - 4 \Omega^2 \sim \dot{\Omega}$, which gives $\epsilon \sim t_{\text{orb}}/t_D$ (evaluated at the resonance radius), where $t_{\text{orb}} = 2\pi/\Omega$ is the orbital period; rather, equation (3.6) breaks down at a larger orbital radius, when the second term (the absolute value) in equation (2.17) becomes comparable to the third term, or, when the real part and the imaginary part in equation (3.6) becomes comparable, i.e.,

$$\frac{4m^3 \Omega^2 \dot{\Omega}}{\omega_\alpha^2 - m^2 \Omega^2} \sim \omega_\alpha^2 - m^2 \Omega^2.$$  \hfill (3.9)

Therefore, the critical “closeness” to the resonance, before which equation (3.6) can be applied, is given by

$$\epsilon \sim \left(\frac{\dot{\Omega}}{\omega_\alpha^2}\right)_{\alpha}^{1/2} \sim \left(\frac{t_{\text{orb}}}{t_D}\right)_{\alpha}^{1/2}.$$  \hfill (3.10)

where the subscript “$\alpha$” implies values at the resonance radius $D_\alpha$. Evaluating equation (3.6) at $2\Omega \sim (1 - \epsilon)\omega_\alpha$, we obtain the amplitude at this near-resonance radius

$$b_\alpha \sim \frac{1}{\epsilon L_{\alpha}^{1/2}} \sim \frac{1}{\omega_\alpha D_{\alpha}^{1/2}} \left(\frac{1}{\Omega}\right)^{1/2}.$$  \hfill (3.11)

The maximum amplitude $|b_{\alpha_{\max}}|$ during a resonance is of the same order. From equation (3.2) for the equilibrium tide, we have

$$\frac{|b_{\alpha_{\max}}|}{b_{\alpha_{\eq}}}_{\alpha} \sim \frac{1}{\epsilon} \sim \left(\frac{t_D}{t_{\text{orb}}}\right)_{\alpha}^{1/2}.$$  \hfill (3.12)

The maximum amplitude attained during a resonance can also be calculated as follows. Before the resonance is reached, $a_{\alpha}(t)$ is essentially forced to oscillate at frequency $2\Delta \Omega$ (cf. eqs. [2.10], [2.14], [3.6]). But after the resonance, we expect $a_{\alpha}(t)$ to oscillate at its own eigenfrequency $\omega_\alpha$. Thus, for post-resonance oscillation, instead of equation (2.16), it is more appropriate to define a function $c_\alpha(t)$ such such

$$a_{\alpha}(t) = (GM^W_i m Q_{\alpha}) c_\alpha(t)e^{-i\omega_\alpha t},$$  \hfill (3.13)
where \( s_m = 1 \) for \( m = +2 \) and \( s_m = -1 \) for \( m = -2 \). Thus we have

\[
b_\alpha(t) = c_\alpha(t)e^{im\Phi - is_m\omega_m t}.
\] (3.14)

Substitution of (3.13) into equation (2.10) yields

\[
\ddot{c}_\alpha - 2is_m\omega_m \dot{c}_\alpha = \frac{1}{D^{l+1}}e^{-im\Phi + is_m\omega_m t}.
\] (3.15)

Integrating (3.15) with respect to time, and neglecting the \( \dot{c}_\alpha \) term, we obtain the post-resonance tidal amplitude

\[
c_\alpha \simeq -\frac{1}{2i s_m \omega_m} \int dt \frac{1}{D(t)^{l+1}}e^{-im\Phi + is_m\omega_m t}.
\] (3.16)

When \( t_D \gg 1/\omega_m \), we can approximately set the upper (lower) limit of the integral to \( \infty \) (\( -\infty \)), and the integral can be evaluated using the stationary phase approximation. The maximum amplitude is then given by

\[
|c_{\alpha,\text{max}}| \simeq \frac{1}{2\omega_m D_{l+1}} \left( \frac{\pi}{\Omega} \right)^{1/2}_\alpha , \quad (m = \pm 2).
\] (3.17)

This has the same scaling as equation (3.11). Using equations (2.12), (2.15), (3.3) and (3.17), we have

\[
|c_{\alpha,\text{max}}| \simeq \frac{1}{16} \left( \frac{5\pi}{6} \right)^{1/2} \omega_m^{-5/6} \left( \frac{R_c^2}{GM} \right)^{5/4} q^{-1/2} \left( \frac{2}{1 + q} \right)^{5/6} \]
\[
= \frac{1}{2^{21/4}} \left( \frac{5\pi}{6} \right)^{1/2} \left( \frac{D_\alpha}{R} \right)^{5/4} \left( \frac{R_c^2}{GM} \right)^{5/4} q^{-1/2} \left( \frac{2}{1 + q} \right)^{5/4} \]
\[
= 5.42 \left( D_{\alpha,10} R_{10} M_{1.4} \right)^{5/4} q^{-1/2} \left( \frac{2}{1 + q} \right)^{5/4},
\] (3.18)

where \( D_{\alpha,10} = D_\alpha/(10R) \), \( R_{10} = R/(10 \text{ km}) \), \( M_{1.4} = M/(1.4M_\odot) \), and \( M_\odot \) is the solar mass. We see that the dimensionless resonance amplitude is larger for larger values of \( D_\alpha/R \) and \( R_c^2/(GM) \). This is because the orbital decay timescale is larger for larger \( R_c^2/(GM) \) (cf. eq.[2.13]), and therefore the binary spends a longer time at a particular resonance radius. For neutron stars, the ratio \( R_c^2/(GM) \) ranges from 4 to 8 (e.g., Arnett & Bowers 1977); for white dwarfs, \( R_c^2/(GM) \sim 10^3 - 10^4 \).

### 3.3 Numerical Solutions

The dimensionless tidal amplitude \( b_\alpha(t) \) can be determined more accurately by numerical integration. Define

\[
b_\alpha(t) = b^{(r)}_\alpha(t) + ib^{(i)}_\alpha(t),
\] (3.19)

where \( b^{(r)}_\alpha \) and \( b^{(i)}_\alpha \) are real. Equation (2.17) then gives

\[
\ddot{b}^{(r)}_\alpha + 2m\Omega b^{(i)}_\alpha + (\omega^2 - m^2\Omega^2)b^{(r)}_\alpha + m\dot{\Omega}b^{(i)}_\alpha = \frac{1}{D^{l+1}},
\] (3.20)

\[
\ddot{b}^{(i)}_\alpha - 2m\Omega b^{(r)}_\alpha - m\dot{\Omega}b^{(r)}_\alpha + (\omega^2 - m^2\Omega^2)b^{(i)}_\alpha = 0.
\]
Numerical integration can be started at a distance larger than the resonance radius, where \((D - D_\alpha)/D_\alpha >> \epsilon\) (cf. eq.[3.10]). The initial condition for \(b^{(r)}_\alpha\) and \(b^{(i)}_\alpha\) can be obtained from equation (3.6), i.e.,

\[
\begin{align*}
    b^{(r)}_\alpha &= \frac{1}{D^{l+1}(\omega^2_\alpha - m^2\Omega^2)}, \\
    \dot{b}^{(r)}_\alpha &= -\left[(l + 1)\frac{\dot{D}}{D} + \frac{2m^2\Omega\dot{\Omega}}{\omega^2_\alpha - m^2\Omega^2}\right]b^{(r)}_\alpha, \\
    b^{(i)}_\alpha &= \frac{1}{\omega^2_\alpha - m^2\Omega^2}(2m\Omega\dot{b}^{(r)}_\alpha + \dot{\Omega}b^{(r)}_\alpha), \\
    \dot{b}^{(i)}_\alpha &\simeq 0.
\end{align*}
\]

(3.21)

Note that solutions for \(m = 2\) and \(m = -2\) are related by

\[
\begin{align*}
    b^{(r)}_{m=2} &= b^{(r)}_{m=-2}, \\
    b^{(i)}_{m=2} &= -b^{(i)}_{m=-2}.
\end{align*}
\]

(3.22)

Thus both \(m = -2\) and \(m = 2\) modes are equally excited.

In Figure 1 we show a typical example of the numerical result. We see that for larger orbital separation, \(D > D_\alpha\), the tidal amplitude indeed evolves according \(a_\alpha(t) \propto e^{-im\Phi}\) (cf. eqs.[2.16], [3.6]). Past the resonance, the tidal amplitude approximately evolves according to \(a_\alpha(t) \propto e^{-ism\omega_\alpha t}\). The absolute amplitude \(|c_\alpha(t)|\) or \(|b_\alpha(t)|\) does not exactly stay at a constant value after the mode is excited. This is because the tidal force continues to act upon the oscillation mode, although the net energy transfer is negligible after the resonance. Indeed, from our numerical results, we find that the average post-resonance amplitude \(|c_\alpha(t)|\) agree very well with equation (3.18), except when \(D_\alpha/R < \sim 3 - 4\).

Also note that the maximum amplitude given by equation (3.18) is not attained instantaneously at \(D = D_\alpha\). From equation (3.16), the “duration of the resonance” \(\delta t_\alpha\), during which the resonance is effective, is given by

\[
\delta t_\alpha \sim \left|\int dt e^{-im\Phi + ism\omega_\alpha t}\right| \simeq \left(\frac{\pi}{\Omega}\right)_{\alpha}^{1/2} = \left(\frac{1}{3}t_{orb} D\right)_{\alpha}^{1/2},
\]

(3.23)

(see also RG2). The corresponding change in the orbital separation \(\delta D_\alpha\) is

\[
\delta D_\alpha \sim |\dot{D}\delta t|_{\alpha} \simeq \left(\frac{t_{orb}}{3t_D}\right)_{\alpha}^{1/2} D_\alpha
\]

\[
\simeq 0.05D_\alpha \left[ q \left(\frac{1+q}{2}\right)^{1/2} \left(D_{\alpha,10} R_{10}^{-1} M_{1.4}\right)^{5/2}\right]^{1/2},
\]

(3.24)

This expression agrees with the numerical result shown in Figure 1.

4. EQUATION OF STATE AND CONVECTIVE STABILITY OF NEUTRON STARS

Having understood the dynamical aspect of the problem of tidal excitations in §3, we now proceed to discuss the properties of the g-mode oscillations of neutron stars.
The restoring force for g-mode oscillations in the core of a cold neutron star is the buoyancy due to the chemical composition gradient, as first identified by RG1. However, the calculations of RG1 were based on an approximate (and inconsistent) ansatz for the Brunt-Väisälä frequency. In this section, we discuss how g-mode oscillations depend on the properties of nuclear matter, and how the Brunt-Väisälä frequency can be obtained self-consistently from some recent microscopic equations of state (EOS). Our calculations of g-modes will be presented in §5.

4.1 Composition Gradient and Convective Stability

The existence of stellar g-modes is closely related to the convective stability (stable stratification) of the star. For a sufficiently old neutron star, we can assume zero temperature and zero entropy throughout the star. Thus buoyancy can only arise from a composition gradient. Consider a blob of matter in equilibrium with its surrounding, with pressure $P$, the number of electrons per baryon $Y_e$, and density $\rho(P, Y_e)$. Imagine the blob is displaced upwards (against the gravity) adiabatically by a distance $dr$, where the surrounding matter has pressure $P'$, composition $Y'_e$, and density $\rho(P', Y'_e)$. The blob is in pressure equilibrium with the surroundings, but its composition is still $Y_e$ during the adiabatic process, since the timescale for the weak interaction, which is responsible for the change of $Y_e$, is much longer than the dynamical timescale. The displaced blob density is therefore $\rho(P', Y_e)$. Convective stability requires

$$\rho(P', Y_e) > \rho(P', Y'_e),$$  \hspace{1cm} (4.1)

which gives the stability criterion (the Ledoux criterion)

$$\left( \frac{\partial \rho}{\partial Y_e} \right)_{P} \left( \frac{dY_e}{dr} \right) < 0.$$  \hspace{1cm} (4.2)

The gravity mode oscillation frequencies are closely related to the Brunt-Väisälä frequency $N$, defined as

$$N^2 = g^2 \left( \frac{1}{c_s^2} - \frac{1}{c_e^2} \right),$$  \hspace{1cm} (4.3)

(e.g., Cox 1980). Here $g = |\mathbf{g}|$ is the local gravitational acceleration, $c_s$ is the *adiabatic sound speed*, as given by

$$c_s^2 = \left( \frac{\partial P}{\partial \rho} \right)_{Y_e},$$  \hspace{1cm} (4.4)

(the subscript “s” means “adiabatic”, which in this case implies constant composition), and $c_e$ is given by

$$c_e^2 = \frac{dP}{d\rho},$$  \hspace{1cm} (4.5)

(the subscript “e” stands for “equilibrium”). We can easily rewrite the Brunt-Väisälä frequency (eq. [4.3]) as

$$N^2 = g^2 \left( \frac{\partial \rho}{\partial Y_e} \right)_{P} \left( \frac{dY_e}{dP} \right) = -\frac{g}{\rho} \left( \frac{\partial \rho}{\partial Y_e} \right)_{P} \left( \frac{dY_e}{dr} \right),$$  \hspace{1cm} (4.6)
where we have used the hydrostatic equilibrium condition \( dP/dr = -\rho g \). Thus we see that the stability criterion (4.2) is equivalent to \( N^2 > 0 \), or

\[
c^2_s - c^2_e = -\left( \frac{\partial P}{\partial Y_e} \right)_\rho \left( \frac{dY_e}{d\rho} \right) > 0, \quad \text{convective stability.} \tag{4.7}
\]

We can now easily understand the existence of crustal g-modes (Finn 1987) and core g-modes (RG1) in a cold neutron star. In the outer crust, the matter consists of an electron gas embedded in a lattice of nuclei. As \( \rho \) increases, the nuclei become more neutron-rich (Baym, Pethick & Sutherland 1971), so that \( dY_e/d\rho < 0 \). On the other hand, since the pressure mainly comes from the electron gas, for a given total density, the pressure increases as the electron fraction increases, i.e., \( (\partial P/\partial Y_e)_\rho > 0 \). Therefore we have \( c^2_s - c^2_e > 0 \), and crustal g-modes are implied.

In the core of a neutron star, the signs of \( dY_e/d\rho \) and \( (\partial P/\partial Y_e)_\rho \) are opposite to those in the outer crust. Consider a simple free npe (neutron, proton and electron) gas model for the core (e.g., Shapiro & Teukolsky 1983, RG1). The electron fraction \( Y_e \) increases with density, \( dY_e/d\rho > 0 \). But since the pressure is mainly provided by the neutrons, we have \( (\partial P/\partial Y_e)_\rho < 0 \). Again we see \( N^2 > 0 \). More detailed discussions of these points based on general EOS of nuclear matter are given in §4.2.

### 4.2 Nuclear Equation of State and Brunt-Väisälä Frequency

We see from §4.1 that calculations of the core g-modes of a neutron star requires knowledge of both \( c_s \) and \( c_e \), in addition to the density-pressure relation. Most of the early EOS’s (e.g., Arnett & Bowers 1977) do not provide enough information to calculate the sound speed and to determine the Brunt-Väisälä frequency. Here we discuss how to obtain these information from a general parametrization of nuclear matter. Such parametrization has been used in several recent microscopic nuclear EOS’s.

Consider the high density liquid core of a neutron star consisting of neutrons, protons and electrons in \( \beta \)-equilibrium. In general, as a function of the baryon number density \( n \) and the proton fraction \( x = n_p/n \), the energy per baryon of the nuclear matter can be written as (Lagaris & Pandharipande 1981; Prakash, Ainsworth & Lattimer 1988; Wiringa, Fiks & Fabrocini 1988, hereafter WFF)

\[
E_n(n, x) = T_n(n, x) + V_0(n) + V_2(n)(1 - 2x)^2, \tag{4.8}
\]

where

\[
T_n(n, x) = \frac{3}{5} \frac{\hbar^2}{2m_n} (3\pi^2 n)^{2/3} \left[ x^{5/3} + (1 - x)^{5/3} \right], \tag{4.9}
\]

is the Fermi kinetic energy of the nucleons, \( m_n \) is the nucleon mass. The last two terms in equation (4.8) represent contributions due to nucleon interactions: \( V_0 \) mainly specifies the bulk compressibility of the matter (hence the pressure-density relation), and \( V_2 \) (the “symmetry potential”) is related to the symmetry energy

\footnote{Note that the change of \( Y_e \) in the crust is actually discontinuous.}
of nuclear matter \(^3\), which plays an important role in determining the equilibrium proton fraction. The parametrization of dense nuclear matter by equation (4.8) is expected to be generic, and it can simulate many recent microscopic calculations.

The energy per baryon of relativistic electrons is given by

\[
T_e(n, x_e) = \frac{3}{4}\hbar c(3\pi^2 n)^{1/3} x_e^{4/3},
\]

where \(x_e = n_e/n = Y_e\). Charge neutrality requires \(x = x_e\). The total energy per baryon in the liquid core is then

\[
E(n, x) = E_n(n, x) + T_e(n, x_e).
\]

Minimization of \(E(n, x)\) with respect to \(x\), i.e., \(\partial E/\partial x = 0\), yields the condition for \(\beta\)-equilibrium

\[
\mu_n - \mu_p = \mu_e,
\]

where \(\mu_i\)'s are the chemical potentials of the three species of particles, with

\[
\mu_n - \mu_p = -\frac{\partial E_n}{\partial x} = 4(1 - 2x)V_2 + \frac{\hbar^2}{2m_n} (3\pi^2 n)^{2/3} \left[ (1 - x)^{2/3} - x^{2/3} \right],
\]

and

\[
\mu_e = \hbar c(3\pi^2 n)^{1/3} x_e^{1/3}.
\]

The equilibrium proton fraction \(x(n) = x_e(n)\) can then be obtained by solving equations (4.12)-(4.14). The mass density and pressure are determined via

\[
\rho(n, x) = n[m_n + E(n, x)/c^2],
\]

\[
P(n, x) = n\frac{\partial E(n, x)}{\partial n} = \frac{2n}{3}T_n + \frac{n}{3}T_e + n^2 \left[ V'_0 + V'_2(1 - 2x)^2 \right],
\]

where \(V'_0 = dV_0/dn\), etc. The equilibrium EOS is then obtained when \(\rho\) and \(P\) are evaluated at the equilibrium composition \(x = x(n)\). The adiabatic sound speed \(c_s\) is given by

\[
c_s^2 = \frac{\partial P}{\partial \rho} = \frac{n}{\rho + P/c^2} \frac{\partial P}{\partial n} = \frac{n}{\rho + P/c^2} \left\{ \frac{10}{9}T_n + \frac{4}{9}T_e + 2n \left[ V'_0 + V'_2(1 - 2x)^2 \right] + n^2 \left[ V''_0 + V''_2(1 - 2x)^2 \right] \right\}.
\]

The difference between \(c_s^2\) and \(c_e^2\) is given by

\[
c_s^2 - c_e^2 = \frac{n}{\rho + P/c^2} \left( \frac{\partial P}{\partial n} - \frac{dP}{dn} \right) = -\frac{n}{\rho + P/c^2} \left( \frac{\partial P}{\partial x} \right) \frac{dx}{dn}
\]

\[
= -\frac{n^3}{\rho + P/c^2} \left[ \frac{\partial}{\partial n} (\mu_e + \mu_p - \mu_n) \right] \frac{dx}{dn}.
\]

\(^3\) The symmetry energy \(E_s\) is defined such that \(E_n(n, x) = E_n(n, 1/2) + E_s(n)(1 - 2x)^2 + \cdots\). Therefore, \(E_s(n) = 5T_n(n, 1/2)/9 + V_2(n)\). The experimental value is \(E_s \sim 30\) MeV at the saturation density \(n = 0.16\) fm\(^{-3}\).
From equation (4.12), we have
\[
\frac{dx}{dn} = -\left[ \frac{\partial}{\partial n} (\mu_e + \mu_p - \mu_n) \right] \left[ \frac{\partial}{\partial x} (\mu_e + \mu_p - \mu_n) \right]^{-1}.
\] (4.18)

Equation (4.17) then becomes
\[
c_s^2 - c_e^2 = \frac{n^3}{\rho + P/c^2} \left[ \frac{\partial}{\partial n} (\mu_e + \mu_p - \mu_n) \right]^2 \left[ \frac{\partial}{\partial x} (\mu_e + \mu_p - \mu_n) \right]^{-1}.
\] (4.19)

Clearly, convective stability (cf. eq.[4.7]) requires \( \partial (\mu_e + \mu_p - \mu_n) / \partial x > 0 \), i.e.,
\[
\frac{1}{3x} \mu_e + 8V^2 + \frac{2}{3} \frac{\hbar^2}{2m_n} \left( 3\pi^2 n \right)^2 / x^{1/3} \left( 1 - x \right)^{-1/3} + x^{-1/3} > 0.
\] (4.20)

Thus unless \( V^2 \) is extremely negative, core g-modes always exist. On the other hand, zero-temperature neutron stars must always be convectively stable, since for the cold nuclear matter in the ground state, there is no energy available to sustain convective motion \(^4\). Therefore equation (4.20) provides a constraint on the nuclear equilibrium state.

Finally, we note that just above the nuclear density, when \( \mu_e \) exceeds the muon rest mass \( m_\mu \), muons are energetically allowed via \( n \leftrightarrow p + \mu \). The \( \beta \)-equilibrium condition becomes
\[
\mu_n - \mu_p = \mu_e = \mu_\mu = [m_\mu^2 c^4 + \hbar^2 c^2 (3\pi^2 n x_\mu) ^{2/3}]^{1/2},
\] (4.21)

with \( x = x_e + x_\mu \). The expressions derived in this section can similarly be modified. However, there are greater uncertainties in our knowledge of nuclear EOS, especially \( V^2(n) \). For definiteness, we will not include muons in our calculations.

### 4.3 Neutron Star Core EOS Models

In this paper, four nuclear EOS’s will be considered. The first three are based on recent microscopic calculations of WFF. For \( n \geq 0.07 \text{ fm}^{-3} \), the functions \( V_0(n) \) and \( V_2(n) \) obtained from different nucleon Hamiltonians have been tabulated in Table IV of WFF. However, to facilitate smooth derivatives of \( V_0 \) and \( V_2 \) with respect to \( n \) (which are needed to determine \( c_s \) and especially \( c_s^2 - c_e^2 \)), we find it more convenient to use approximate fitting formulae, which we give below:

(1) **Model AU:** This is the EOS based on nuclear potential AV14+UVII of WFF. \( V_0 \) and \( V_2 \) (in MeV) are fitted as
\[
V_0 = -43 + 330 (n - 0.34)^2, \quad V_2 = 21 n^{0.25},
\] (4.22)

where \( n \) is the baryon number density in \( \text{fm}^{-3} \).

(2) **Model UU:** This is based on potential UV14+UVII of WFF. The fitting formulae are (in the same units as in eq.[4.22])
\[
V_0 = -40 + 400 (n - 0.3)^2, \quad V_2 = 42 n^{0.55}.
\] (4.23)

\(^4\) I thank Ed Salpeter and Andreas Reisenegger for pointing this out to me.
(3) Model UT: This is based on potential UV14+TNI of WFF. The fitting formulae are (in the same units as in eq.[4.22])

\[ V_0 = -42 + 350 (n - 0.28)^2, \]
\[ V_2 = 18 - 130 (n - 0.29)^2. \]  

(4.24)

Once the functions \( V_0(n) \) and \( V_2(n) \) are given, all the relevant EOS parameters needed for the calculations of g-modes can be obtained using the expressions in §4.2. Some of the results are shown in Figure 2.

The above fitting formulae are valid only for \( n \lesssim 1 \text{ fm}^{-3} \). This is adequate for our purpose, since we will be mainly concerned with \( M = 1.4M_\odot \) NS’s, for which all three EOS’s yield a central density that is less than \( 1 \text{ fm}^{-3} \). For higher density, both AU and UU violate causality, indicating considerable theoretical uncertainties. We note that for AU, the fitted function \( V_2(n) \) does not reflect the exact behavior of \( V_2(n) \) in Table IV of WFF (e.g., \( V_2(n) \) in that table shows non-monotonic behavior for \( 0.07 \text{ fm}^{-3} < n < 0.7 \text{ fm}^{-3} \), while our fitted formula is monotonic in \( n \)). However, there is much greater uncertainty in the three-body nucleon interaction potential (especially its isospin dependence) that determines \( V_2(n) \) itself. Also note that for model UT, the quantity \( c_s^2 - c_e^2 \) becomes negative at very high density. This implies that the model is not reliable in such a high density regime (see discussion following eq. [4.20]).

(4) Model UU2: Our fourth EOS has the same \( P(n), \rho(n), \) and \( c_e(n) \) as UU. However, we shall adopt the ansatz similar to that of RG1 for \( c_s^2 - c_e^2 \) in order to determine to what extent the g-mode properties are affected. This ansatz is based on the following consideration.

For free npe gas (i.e., set \( V_0 = V_2 = 0 \) for the equations in §4.2), \( \beta \)-equilibrium gives (see eqs.[4.12]-[4.14])

\[ x = 3 \pi^2 n \left( \frac{\hbar}{2m_n c} \right)^3 \left[ (1 - x)^{2/3} - x^{2/3} \right] \approx \frac{3 \pi^2 n}{2} \left( \frac{\hbar}{2m_n c} \right)^3 \]
\[ \approx 5.6 \times 10^{-3} \frac{\rho}{2.8 \times 10^{14} \text{ g/cm}^3}. \]  

(4.25)

where the second equality follows from \( x << 1 \). The sound speed is given by (eq.[4.16] with \( V_0 = V_2 = 0 \))

\[ c_s^2 \approx \frac{1}{m_n} \frac{\partial P}{\partial n} \approx \frac{2}{3m_n} \frac{\hbar^2}{2m_n} (3\pi^2 n)^{2/3} \left( 1 - \frac{x}{2} \right). \]  

(4.26)

From equations (4.13)-(4.14) and (4.17), we have

\[ c_s^2 - c_e^2 = -\frac{1}{m_n} \frac{dx}{dn} \left[ \frac{n}{3} \mu_e + \frac{2n}{3} (\mu_p - \mu_n) \right]. \]  

(4.27)

Using equations (4.12), (4.26)-(4.27), we then have

\[ c_s^2 - c_e^2 \approx \frac{1}{3m_n} \frac{\hbar^2}{2m_n} (3\pi^2 n)^{2/3} x \approx \frac{x}{2} c_e^2 \approx \frac{x}{2} c_s^2. \]  

(4.28)

This agrees with the expressions given in RG1 (see their eqs. [28]-[29]).

\[ \text{The results tabulated in Table IV of WFF contain slight errors (Wiringa 1993, private communication). We have used the updated results provided by Bob Wiringa.} \]
Now the ansatz for UU2 is to use equation (4.28) to obtain $c_s$ from $c_e$ and to use $x(n)$ from equation (4.25), although both equations are only valid for free npe gas EOS model. This ansatz is clearly not a consistent procedure to obtain $c_s^2 - c_e^2$ and hence the Brunt-Väisälä frequency. In a realistic description of nuclear matter, the sound speed $c_s$ is determined by the compressibility of the matter, while the quantity $c_s^2 - c_e^2$ is mainly related to the symmetry energy. Therefore, equation (4.28) cannot hold in general. Indeed, from Figure 2 we see that the ratio $(c_s^2 - c_e^2)/c_s^2$ in UU2 is qualitatively different from that in UU, although the other quantities are the same. Nevertheless, in view of the great uncertainties in our knowledge of $V_2(n)$, it is worth exploring the g-mode properties based on this ansatz.

5. CORE g-MODE OSCILLATIONS AND TIDAL COUPLING COEFFICIENTS

We now discuss our calculations of the g-mode oscillations of an isolated cold neutron star based on the EOS models of §4. The resulting mode frequencies $\omega_\alpha$ and the tidal coupling coefficients $Q_{nl}$ (eq. [2.11]) will be used in §6 to determine the energy transfer and angular momentum transfer due to resonant excitations in coalescing neutron star binaries.

5.1 Basic Equations

In this paper, we use Newtonian linearized equations to calculate the oscillation modes. The use of Newtonian equations is consistent with our Newtonian description of tidal interactions (§2). For f-mode, general relativistic effects are expected to modify our results of oscillation frequencies by not more than $GM/(Rc^2) \sim 20\%$. Linear oscillations of nonrotating stars in Newtonian theory have been discussed extensively in the literature (e.g., Ledoux 1974, Cox 1980). Here, for completeness and to clarify the notations, we give some of the relevant equations below.

The basic equations that govern the oscillations of stars are the equations of hydrodynamics, i.e., the continuity equation, the equation of motion, energy equation (which specifies the adiabatic nature of the oscillations), and the Poisson equation for the gravitational potential. The linearized equations can be written in terms of the radial component of the fluid displacement $\xi^r$ (cf. eq.[2.8]), the Eulerian perturbations of density, pressure and gravitational potential, $\delta\rho$, $\delta P$ and $\delta \phi$. Separating out the angular dependence $Y_{lm}(\theta, \phi)$, these linearized equations are (here and throughout this section, the mode index $\alpha = \{n, l, m\}$ is suppressed)

$$\frac{\partial}{\partial r} (r^2 \xi^r) = \frac{\rho g}{\Gamma_1 P} (r^2 \xi^r) + \left[ \frac{l(l+1)}{\omega^2} - \frac{r^2 \rho}{\Gamma_1 P} \right] \left( \frac{\delta P}{\rho} \right) + \frac{l(l+1)}{\omega^2} \delta \phi,$$

$$\frac{\partial}{\partial r} \left( \frac{\delta P}{\rho} \right) = \omega^2 + A g \left( r^2 \xi^r \right) - A \left( \frac{\delta P}{\rho} \right) - \frac{\partial}{\partial r}(\delta \phi),$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( \frac{r^2}{r^2} \delta \phi \right) = \frac{l(l+1)}{r^2} \delta \phi + 4\pi G \delta \rho,$$

RG1 have also considered the ansatz using more realistic $x(n)$ in equation (4.28). This is also not a consistent procedure.
where $\omega$ is the eigenfrequency of the mode, and

$$\Gamma_1 = \frac{\rho c_s^2}{P}, \quad A = -\frac{N^2}{g}, \quad \delta \rho = \frac{\rho^2}{\Gamma_1 P} \left( \frac{\delta P}{\rho} \right) - \frac{\rho A}{r^2} (r^2 \xi^r). \quad (5.2)$$

Here, $c_s^2$, $N^2$ can be calculated using equations (4.3), (4.16)-(4.17). The transverse component of the displacement vector (see eq. [2.8]) is given by

$$\xi^\perp(r) = \frac{1}{\omega^2 r} \left[ \frac{\delta P(r)}{\rho} + \delta \phi(r) \right]. \quad (5.3)$$

The eigenmodes can be determined by solving equation (5.1) with appropriate boundary conditions. For $r \to 0$, regularity requires

$$r^2 \xi^r = \frac{l}{\omega^2} (Y_o + \Phi_o) r^{l+1}, \quad \frac{\delta P}{\rho} = Y_o r^l, \quad \delta \phi = \Phi_o r^l, \quad (r \to 0), \quad (5.4)$$

where $Y_o$ and $\Phi_o$ are constants. At the stellar surface, the Lagrangian perturbation of the pressure, $\Delta P$, must vanish, i.e.,

$$\Delta P = \delta P - \rho g \xi_r = 0, \quad (r = R). \quad (5.5)$$

Also, the gravitational potential and force must be continuous across the surface. This gives

$$\frac{\partial}{\partial r} \delta \phi + \frac{l+1}{r} \delta \phi = -4\pi G \rho \xi_r, \quad (r = R). \quad (5.6)$$

Numerically, we fix $Y_o = 1$, which simply sets a scale for the eigenfunctions. Equations (5.1) are integrated outward from $r \to 0$ with boundary conditions (5.4), using a fifth-order Runge-Kutta scheme. The outer boundary conditions at $r = R$ (eqs. [5.5]-[5.6]) can be matched using a standard shooting routine (Press et al. 1992), therefore the eigenvalues $\omega^2$ and $\Phi_o$ can be obtained after an initial guess. The number of nodes in the functions is counted to determine the orders of the modes. After $\omega^2$ is obtained for a specific mode, the eigenfunctions are normalized according to equation (2.9), i.e.,

$$\int_0^R \rho r^2 dr \left[ \xi^r(r)^2 + l(l+1) \xi^\perp(r)^2 \right] = 1. \quad (5.7)$$

Then the tidal coupling coefficient $Q_{nl}$ is calculated using equation (2.11). To test our numerical code, we have done calculations for polytropic stellar models, with $P = K \rho^\Gamma$, where the polytropic index $\Gamma$ and the adiabatic index $\Gamma_1 > \Gamma$ are constant throughout the star. For a few specific values of $\Gamma$ and $\Gamma_1$, the mode frequencies and $Q_{nl}$ have already been tabulated in the literature (e.g., Lee & Ostriker 1986). Our results for the polytropic models agree with these tabulated values.

We construct equilibrium neutron star models by solving Newtonian hydrostatic equations. We use Newtonian equations for consistency with our use of Newtonian equations in calculating oscillation modes and in describing the tidal interaction. The EOS’s we have used are those discussed in §4.2-4.3 for $n \geq 0.07$ fm$^{-3}$, where the pressure, sound speed, and $c_s^2$ (hence the Brunt-Väisälä frequency) are all calculated
self-consistently (except model UU2). For lower density crustal regions, we use the Baym, Bethe & Pethick (1971) EOS above neutron drip \( \rho \gtrsim 4 \times 10^{11} \text{ g/cm}^3 \), and then join onto the Baym, Pethick & Sutherland (1971) EOS below neutron drip. In all of our calculations, we set \( c_s^2 = c_e^2 \) in the crustal region, therefore effectively suppressing the crustal g-modes while concentrating on the core g-modes. Since the density in the crustal region is much smaller, from equation (2.11) we see that the tidal coupling coefficients for these crustal modes are likely to be very small (as found by Shibata 1993, who used highly simplified neutron star models).

5.2 Numerical Results and Discussions

Table 1 gives our numerical results for the oscillation frequencies and tidal coupling coefficients \( Q_{nl} \) of the f-mode and the first two g-modes for neutron stars based on the four EOS models discussed in §4.3. Note that since the gravitational wave frequency is twice that of the orbital frequency \( \Omega \), and resonances occur when \( \omega_\alpha = 2\Omega \), the listed mode frequencies also correspond to the gravitational wave frequencies at resonances. We see that the four neutron star models considered have very similar stellar radii for the given canonical mass \( M = 1.4M_\odot \), and their f-mode properties are also very similar. This is because the four EOS’s of §4.3 predict very similar bulk properties (e.g., pressure, compressibility) for the nuclear matter. However, the g-mode properties are very different for the four models. In particular, although UU and UU2 yield the identical neutron star mass-radius relations, the g-mode frequencies predicted by them differ by large factors. These differences reflect the sensitive dependence of g-modes on the symmetry energy of the nuclear matter.

From Table 1 we see that the frequencies of g-modes are much smaller than that of f-mode. This is a direct consequence of the small difference between \( c_s \) and \( c_e \) (cf. Fig.2). Also, \( |Q_{nl}| \) becomes smaller as the radial order of modes increases. This is because of the strong cancellation in the integral for \( Q_{nl} \) (eq. [2.11]), as already noted by Cowling (1941). Indeed, using the equation of continuity,

\[
\delta \rho_\alpha = -\nabla \cdot (\rho \xi_\alpha),
\]

the expression for \( Q_{nl} \) (eq. [2.11]) can be rewritten as

\[
Q_{nl} = \int_0^R r^{l+2} \delta \rho_{nl}(r) dr.
\]

Since the order of a mode corresponds to the number of nodes in \( \delta \rho_{nl}(r) \), we clearly see that for the higher-order mode \( |Q_{nl}| \) is smaller. Based on our numerical values of \( Q_{nl} \) for the first few g-modes, we find that for all NS models considered, \( |Q_{nl}| \) decreases as \( \omega_{nl} \) decreases faster than \( |Q_{nl}| \propto \omega_{nl}^{3.3} \).

The frequencies of the first ten quadrupole g-modes are given in Figure 3. They generally lie in the range of 10 Hz to 100 Hz. An approximate asymptotic expression for the frequencies can be easily obtained. For high-order g-modes \( n \gg 1 \), the perturbation of the gravitational potential \( \delta \phi \) can be neglected (“Cowling
approximation”). Eliminating \( \delta P/\rho \) from the first two equations in (5.1), and noticing that \( \omega \to 0 \), we have

\[
\frac{\partial^2}{\partial r^2} (r^2 \xi^r) \simeq \frac{l(l+1)}{\omega^2} \frac{\partial}{\partial r} \left( \frac{\delta P}{\rho} \right) \simeq -\frac{l(l+1)}{\omega^2 r^2} N^2 \left( r^2 \xi^r \right). \tag{5.10}
\]

Assuming \( r^2 \xi^r \sim e^{ikr} \), we obtain the WKB local dispersion relation for high-order g-modes

\[
k(r)^2 \simeq \frac{l(l+1)}{\omega^2 r^2} N(r)^2. \tag{5.11}
\]

The eigenvalues for \( \omega^2 \) are then determined by the stationarity condition

\[
\int_0^R k(r) dr = (n + C)\pi, \tag{5.12}
\]

where \( C \) is a constant of order unity. Thus, asymptotically, the high-order g-mode angular frequencies are given by

\[
\omega_{nl} \simeq \sqrt{l(l+1)} \frac{N(r)}{(n+C)\pi} \int_0^R \frac{N(r)}{r} dr. \tag{5.13}
\]

The values of \( Q_{nl} \) for the higher-order g-modes are harder to calculate numerically as they become much smaller. Indeed, in evaluating the values of \( Q_{nl} \), it is necessary to know the eigenfunctions of the g-modes very precisely, since a small f-mode contamination can significantly change the obtained values (Reisenegger 1994). To obtain meaningful numbers of \( Q_{nl} \) for the g-modes, it is also important to implement self-consistent hydrodynamical equations in the numerical calculations, so that the Hermiticity of the oscillation operator \( L \) is maintained. For example, we have found that significant systematic errors in \( Q_{nl} \) can be introduced if \( c_e^2 = dP/d\rho \) is not satisfied to a high accuracy when interpolating tabulated EOS. For the g-modes given in Table 1, we have checked that the orthogonality relation (2.9) is satisfied to sufficient accuracy, therefore our values of \( Q_{nl} \) for these g-modes are not contaminated by the f-mode.

We have also done calculations of the oscillation modes using the Cowling approximation. We find, as expected, that the g-mode frequencies are nearly unaffected. However, for a given stellar model, the Cowling approximation leads to a larger f-mode frequency (by \( \sim 50\% \)) and larger values of \( |Q_{nl}| \) for the first few g-modes (by as large as 80%).

6. TIDAL ENERGY AND ANGULAR MOMENTUM

In this section, we combine our results of §3 and §5 to calculate the amplitude of distortion, energy transfer and angular momentum transfer to the neutron star due to tidal interaction. The effects on the orbital phase will be considered in §7.

6.1 Tidal Distortion Amplitude

A measure of the tidal distortion of the neutron star is the quantity \( a_\alpha \) (cf. eq.[2.6]). For equilibrium, quasi-static tide \( (m = 0) \), equation (3.2) gives

\[
|a_{\alpha,eq}| \simeq q \omega_{nl}^2 |W_{20} Q_{nl}| \left( \frac{R}{D} \right)^3. \tag{6.1}
\]
where $\omega_{nl}$ is the mode angular frequency in units of $(GM/R^3)^{1/2}$. The dominant tidal distortion comes from the f-mode, for which $\omega_{nl} \sim 1$ and $Q_{nl} \sim 1$ (see Table 1), thus

$$|a_{0,eq}| \sim \frac{h_0}{R} = q \left(\frac{R}{D}\right)^3,$$

where $h_0$ is the typical height of the tidal bulge (see eq. [2.19]).

Now consider the dynamical tides ($m = \pm 2$). Before the resonance, the distortion amplitude associated with a particular mode increases as the orbit decays (see eq. [3.6]). After the resonance, the amplitude stays nearly constant. From equations (2.16) and (3.18), the tidal distortion amplitude resulting from the resonant excitation is given by

$$|a_{\alpha,\text{max}}| \approx \frac{\pi}{32} \omega_{nl}^{-5/6} |Q_{nl}| \left(\frac{Rc^2}{GM}\right)^{5/4} q^{1/2} \left(\frac{2}{1+q}\right)^{5/6},$$

where $f_{nl} = \omega_{nl}/2\pi$ is the mode frequency, which is also equal to the frequency of the gravitational wave when the resonance occurs. The reference values $f_{nl} = 100$ Hz and $|Q_{nl}| = 0.0003$ have been chosen to approximately agree with the lowest order g-mode in model UU (see Table 1). Note that equation (6.3) is valid for both $m = 2$ and $m = -2$, and it is half of the result given by RG2 (see their eq. [14]) \footnote{In the PT formalism used by RG2, the contributions from $m = 2$ and $m = -2$ terms are not equal, while it is clear from our analysis (see eqs. [2.10], [3.16] or [3.22]) that the $m = 2$ and $m = -2$ modes contribute to tidal distortion and energy equally. The difference comes from the fact that contributions from $m = 2$ and $m = -2$ modes have been re-grouped to derive the final expressions of PT. Thus, although explicit evaluations in RG2 are made only for $l = m = 2$, RG2’s results actually include contributions from both the $m = 2$ and $m = -2$ modes, since the other term in PT’s formula are much smaller.}

### 6.2 Tidal Energy Transfer

Next we look at tidal energy transferred to the neutron star. The kinetic energy of the star is given by

$$E_k(t) = \frac{1}{2} \int d^3x \rho \frac{d\xi^*}{dt} \cdot \frac{d\xi}{dt} = \frac{1}{2} \sum_\alpha |\dot{a}_\alpha|^2,$$

where we have used equations (2.6) and (2.9). Similarly, using equation (2.7), the potential energy (including the internal energy) of the oscillations is given by

$$E_p(t) = \frac{1}{2} \int d^3x \xi^* \cdot L \cdot \xi = \frac{1}{2} \sum_\alpha \omega_\alpha^2 |a_\alpha|^2,$$
(e.g., Shapiro & Teukolsky 1983, p141-143). The total tidal energy of the neutron star is $E = E_k + E_p$. Substituting equations (2.16) and (3.13), and using the natural units introduced at the end of §2, we have

$$E_k(t) = \frac{1}{2} \sum_\alpha (M'W_{lm}Q_{nl})^2 |b_\alpha - im\Omega b_\alpha|^2 = \frac{1}{2} \sum_\alpha (M'W_{lm}Q_{nl})^2 |\hat{c}_\alpha - i s_m\omega c_\alpha|^2,$$

$$E_p(t) = \frac{1}{2} \sum_\alpha (M'W_{lm}Q_{nl})^2 \omega^2 |b_\alpha(t)|^2 = \frac{1}{2} \sum_\alpha (M'W_{lm}Q_{nl})^2 \omega^2 |c_\alpha(t)|^2.$$

(6.6)

Another way to consider the tidal energy of the neutron star is to look at the rate of energy transfer:

$$\dot{E} = - \int d^3x \rho \frac{\partial \xi}{\partial t} \cdot \nabla U^*.$$

(6.7)

Using equations (2.2), (2.6), (2.11) and (2.16), we have

$$\dot{E} = \sum_\alpha (M'W_{lm}Q_{nl}) \frac{1}{Dl+1} e^{im\Phi} a_\alpha = \sum_\alpha (M'W_{lm}Q_{nl})^2 \frac{1}{Dl+1} (b_\alpha - im\Omega b_\alpha).$$

(6.8)

It is easy to check that $\dot{E}$ in this expression is real (as it should be), since the contributions to the imaginary part from the $m = +2$ term and that from the $m = -2$ term cancel. Moreover, using equation (2.10), we can show that this expression for $\dot{E}$ is consistent with equations (6.4)-(6.5) for $E_k$ and $E_p$.

For equilibrium, quasi-static tide ($m = 0$), when orbital decay time $t_D >> 1/\omega_{nl}$, the kinetic tidal energy is small compared to the potential energy (thus “quasi-static”). The equilibrium tidal energy associated with a particular mode is then

$$E_{\alpha,eq} = \frac{1}{2} \frac{GM^2}{R} q^2 \lambda_{nl}^{-2} (W_{20} Q_{nl})^2 \left( \frac{R}{D} \right)^6. $$

(6.9)

Again, the dominant equilibrium tidal energy comes from the f-mode

$$E_{0,eq} \sim \frac{GM^2}{R} q^2 \left( \frac{R}{D} \right)^6,$$

(6.10)

which is the expected result (e.g., Lai et al 1993).

Now consider the energy transfer $\Delta E_{nl}$ due to the resonant excitation of a specific g-mode. Since $c_\alpha(t)$ is approximately constant after the resonance, the kinetic energy and the potential energy are equal. Equations (3.18) and (6.6) then yield

$$\Delta E_{nl} \approx 2\omega_{nl}^2 |a_{\alpha,max}|^2,$$

$$\approx \frac{\pi^2}{512} \left( \frac{GM^2}{R} \right) \omega_{nl}^{1/3} Q_{nl}^{2} \left( \frac{Rc^2}{GM} \right)^{5/2} q \left( \frac{2}{1 + q} \right)^{5/3}. $$

(6.11)

Note that the factor of 2 in the first equality comes about because the $m = 2$ and $m = -2$ terms contribute equally to the excitation energy, and we have included both of them in (6.11). The ratio of $\Delta E_{nl}$ to the orbital energy

$$E_{\text{orb}} \simeq -\frac{GM M'}{2D},$$

(6.12)
is given by
\[
\frac{\Delta E_{nl}}{|E_{orb}|} \approx \frac{\pi^2}{128} \omega^{-1/3} Q_{nl}^2 \left( \frac{Rc^2}{GM} \right)^{5/2} q^2 \left( \frac{2}{1+q} \right)^{4/3} = 1.0 \times 10^{-6} \left( \frac{f_{nl}}{100 \text{ Hz}} \right)^{-1/3} \left( \frac{|Q_{nl}|}{0.0003} \right)^2 M^{-7/3}_{1.4} R^{2} q^2 \left( \frac{2}{1+q} \right)^{4/3},
\]
(6.13)
where \(|E_{orb}|\) is evaluated at \(D = D_{nl}\). This ratio plays an important role in determining the effect of resonant energy transfer on the orbital decay rate (see §7). Since \(\Delta E_{nl} \ll |E_{orb}|\), the resonance has only a small effect on the orbit, and the use of the trajectory of ideal two point masses (cf. eq.[2.12]) is a good approximation. Our result for \(\Delta E_{nl}\) agree with that of RG2 (see their eqs. [15] and [17]), but note that our equations (6.11) and (6.13) include both \(m = 2\) and \(m = -2\) modes (see footnote 7).

### 6.3 Angular Momentum Transfer and Dynamical Spin-up of the Neutron Star

Associated with the tidal energy transfer is an angular momentum transfer to the neutron star. This is possible because of the dynamical tidal lag induced by the decaying orbit, as already noted in §3.1 (see eqs. [3.6]-[3.7]). The \(z\)–component of the tidal torque on the neutron star per unit mass is given by
\[
\tau_z = -e_z \cdot (r \times \nabla)U = -\frac{\partial}{\partial \phi} U.
\]
(6.14)
Thus the total torque is
\[
N_z = \int d^3x (\rho + \delta \rho) \tau_z^* = \int d^3x \delta \rho \tau_z^*.
\]
(6.15)
Using the continuity equation (5.8), we have
\[
N_z = -\sum_{\alpha} a_\alpha(t) \int d^3x \cdot (\rho \vec{\kappa}_\alpha) \tau_z^* = \sum_{\alpha} (M' W_{lm} Q_{nl})^2 \frac{1}{D^{i+1}} (-i m b_{\alpha}) = \sum_{\alpha} (M' W_{lm} Q_{nl})^2 \frac{1}{D^{i+1}} m b_{\alpha}^{(i)}.
\]
(6.16)
Note that this is non-zero since the contribution from the \(m = +2\) term and that from the \(m = -2\) term add up (cf. eq.[3.22]). Clearly, the equilibrium tide \((m = 0)\) does not contribute to angular momentum transfer.

Prior to the resonance, the tidal torque can be evaluated using equation (3.6). In order of magnitude, the torque associated with a specific mode is
\[
N_{z, \alpha} \sim \frac{GM^2}{R} q^2 \omega_{nl}^{-2} Q_{nl}^2 \left( \frac{R}{D} \right)^6 \delta_{\alpha},
\]
(6.17)
where the dynamical tidal lag angle \(\delta_{\alpha}\) is given by equation (3.7). The dominant f-mode contribution is
\[
N_{z, 0} \sim \frac{GM^2}{R} q^2 \left( \frac{R}{D} \right)^6 \delta_0 \sim \frac{GM^2}{R} q^2 \left( \frac{R}{D} \right)^6 \frac{\Omega}{\omega_0^2 t_D}.
\]
(6.18)
It is easy to see that this non-resonant dynamical tidal torque does not appreciably affect the neutron star spin. In particular, this torque is not sufficient to synchronize the spin with the orbital angular velocity: the timescale for synchronization is given by
\[
t_{\text{sym}} \sim \frac{MR^2 \Omega}{N_{z, 0}} \sim q^{-2} \left( \frac{D}{R} \right)^6 t_D.
\]
(6.19)
which is much larger than the orbital decay time $t_D$ unless $D$ becomes comparable to $R$.

Now consider the total angular momentum transferred to the star, $\Delta J_\alpha$, due to a particular resonance. From equation (6.16), this is given by

$$\Delta J_\alpha = (M'W_{lm}Q_{nl})^2 \int \frac{1}{D+1}(-im\beta_\alpha), \quad (6.20)$$

where only the real part should be included (recall that the imaginary part from the $m = 2$ term and that from the $m = -2$ term cancel). Using equations (3.14)-(3.15), we have

$$\Delta J_\alpha = (M'W_{lm}Q_{nl})^2(-im) \int \frac{1}{D+1} c_\alpha e^{im\Phi - i\omega_\alpha s_m t} \quad (6.21)$$

Integrate by parts, and notice that the term proportional to $c_\alpha \dot{c}_\alpha^* \alpha$ can be neglected, since after the resonance, $c_\alpha$ is nearly constant. Adding up contributions from both $m = 2$ and $m = -2$ terms, the total angular momentum transfer $\Delta J_{nl}$ due to the resonance is given by

$$\Delta J_{nl} \simeq 2\pi^2 (GM_3 R) \omega^{-2/3} Q_{nl}^2 \left( \frac{Rc_{\alpha, max}}{GM} \right)^{5/2} q \left( \frac{2}{1 + q} \right)^{5/3} (6.22)$$

Using equation (6.3), we have

$$\Delta J_{nl} \simeq \frac{\pi^2}{256} (GM_3 R)^{1/2} \omega_{nl}^{-2/3} Q_{nl}^2 \left( \frac{Rc_{\alpha, max}}{GM} \right)^{5/2} q \left( \frac{2}{1 + q} \right)^{5/3} \quad (6.23)$$

Note that the energy transfer and angular momentum transfer are related by

$$\Delta J_{nl} = \Delta E_{nl} \Omega_{nl} = 2\pi^2 (GM_3 R) \omega_{nl}^{-2/3} Q_{nl}^2 \left( \frac{Rc_{\alpha, max}}{GM} \right)^{5/2} q \left( \frac{2}{1 + q} \right)^{5/3} \quad (6.24)$$

where $\Omega_{nl} = \omega_{nl}/2$ is the orbital angular velocity at the resonance.

The effect of this resonant angular momentum transfer on the spin of the neutron star can be considered if we assume that the angular momentum goes into uniform rotation of the star. The resulting change of the spin rate due to the resonance is then given by

$$\Delta \Omega_{s,nl} = \frac{\Delta J_{nl}}{I} \quad (6.25)$$

where $I = 2\kappa M R^2/5$ is the moment of inertia of the star, $\kappa < 1$ is of order unity. The ratio of $\Delta \Omega_{s,nl}$ and $\Omega_{nl}$ is

$$\Delta \Omega_{s,nl} \Omega_{nl} = \frac{5\pi^2}{256\kappa} \omega_{nl}^{-5/3} Q_{nl}^2 \left( \frac{Rc_{\alpha, max}}{GM} \right)^{5/2} q \left( \frac{2}{1 + q} \right)^{5/3} \quad (6.26)$$

where $I = 2\kappa M R^2/5$ is the moment of inertia of the star, $\kappa < 1$ is of order unity. The ratio of $\Delta \Omega_{s,nl}$ and $\Omega_{nl}$ is

$$\Delta \Omega_{s,nl} \Omega_{nl} = \frac{5\pi^2}{256\kappa} \omega_{nl}^{-5/3} Q_{nl}^2 \left( \frac{Rc_{\alpha, max}}{GM} \right)^{5/2} q \left( \frac{2}{1 + q} \right)^{5/3} \quad (6.26)$$

Therefore, spin-up due to resonant angular momentum transfer is negligible. In reality, the angular momentum transferred does not manifest itself as uniform spin of the star, since the viscous timescale to achieve
uniform rotation is long. Thus the actual change of spin rate is even smaller than that given by the above expression.

7. EFFECTS ON ORBITAL DECAY RATE: PHASE ERROR

We now calculate the orbital phase error due to the resonant energy transfer and angular momentum transfer from the binary orbit to the neutron star. This is important because, as mentioned in the introduction, a small phase error in the theoretical gravitational wave template can destroy a possible detection using the matched filter technique.

Because of the relation (6.24), a circular orbit will remain circular after the resonance. Therefore it is sufficient to consider the energy of the system only. Excluding a constant, the total energy of the system, $E_{\text{tot}}$, can be written as a sum of the orbital energy $E_{\text{orb}}$ and the stellar energy $E$, i.e., $E_{\text{tot}} = E_{\text{orb}} + E$. Thus

$$
\frac{dE_{\text{tot}}}{dD} \dot{D} = \frac{dE_{\text{orb}}}{dD} + \frac{dE}{dD} \dot{D}.
$$

(7.1)

The number of orbital cycles, $N_{\text{orb}}$, can be obtained from

$$
dN_{\text{orb}} = \frac{\Omega}{2\pi} \frac{dD}{\dot{D}} = \frac{\Omega}{2\pi} dD \left( \frac{dE_{\text{orb}}}{dD} + \frac{dE}{dD} \right) / \dot{E}_{\text{tot}}. 
$$

(7.2)

The first term in (7.2) is simply the point mass result. The second term represents the error induced by the change of stellar energy:

$$
d(\Delta N_{\text{orb}}) = \frac{\Omega}{2\pi} dD \left( \frac{dE}{dD} \right) / \dot{E}_{\text{tot}}.
$$

(7.3)

This is a general expression, which applies to dynamical tidal effects as well as equilibrium tidal effects. Now consider the total accumulated error on the orbital cycles $\Delta N_{\text{orb},nl}$ due to resonant excitation of a particular mode with angular frequency $\omega_{nl}$. Since $dE/dD$ is large only near the resonance radius $D_{nl}$ (see eq. [3.3]), we can integrate equation (7.3) to obtain

$$
\Delta N_{\text{orb},nl} \simeq \frac{\Omega_{nl} \Delta E_{nl}}{2\pi E_{\text{tot}}} = -\frac{t_D}{t_{\text{orb}}} \frac{\Delta E_{nl}}{|E_{\text{orb}}|},
$$

(7.4)

where $t_D = |E_{\text{orb}}/\dot{E}_{\text{tot}}| = |D/\dot{D}|$ (see eq. [2.13]), and $t_{\text{orb}} = 2\pi/\Omega$ is the orbital period. Note that all the quantities in equation (7.4) are evaluated at the resonance radius $D_{nl}$. Here the negative sign implies that the effect of energy transfer is to make the binary coalesce slightly faster (i.e., slightly larger $|\dot{D}|$) compared with the ideal point mass case. Substituting equations (2.13) and (6.13) into (7.4), we have

$$
\Delta N_{\text{orb},nl} \simeq -\frac{5\pi}{4096} \omega_{nl}^{-2} Q_{nl}^{2} \left( \frac{R_c}{GM} \right)^{5} q \left( \frac{2}{1+q} \right)
$$

$$
\simeq -4.3 \times 10^{-4} \left( \frac{f_{nl}}{100 \text{ Hz}} \right)^{-2} \left( \frac{|Q_{nl}|}{0.0003} \right)^{2} M_{1.4}^{-4} R_{10}^{2} q \left( \frac{2}{1+q} \right). 
$$

(7.5)

Note that this phase error does not accumulate instantaneously at $D = D_{\alpha}$ (see §3.3). From equation (3.23), the number of orbital cycles that unfold during the period of effective resonance is given by

$$
\delta N_{\text{orb},nl} \sim \frac{\Omega_{\alpha}}{2\pi} \delta t_{\alpha} \simeq 12 \left( \frac{f_{nl}}{100 \text{ Hz}} \right)^{-5/6} M_{1.4}^{-5/6} q^{-1/2} \left( \frac{2}{1+q} \right)^{1/6}.
$$

(7.6)
The small size of $\Delta N_{\text{orb,nl}}$ in equation (7.5) implies that the phase errors due to resonant excitations of g-modes are negligible for constructing gravitational wave templates. Therefore, to a very good approximation, binary neutron stars can be treated as point masses during the inspiraling phase. Hydrodynamical tidal effects are dominated by the f-mode distortion, and they are important only when the neutron stars are close to contact (see eqs. [6.2], [6.10]).

8. TIDAL HEATING OF BINARY NEUTRON STARS

Having determined in §6 the tidal energy in the decaying binary neutron stars, a natural question concerns where this energy goes. Without damping mechanisms, this energy is stored in the form of potential energy and kinetic energy of various oscillation modes. There are basically three processes that can lead to damping of g-modes (RG1): gravitational radiation, relaxation toward chemical equilibrium by neutrino emission, and viscous dissipation. The damping timescales for all of these mechanisms are likely to be much larger than the orbital decay time of interest here, i.e., the last few minutes prior to the binary merger. However, viscous dissipation leads to heating of the neutron stars, thus changing the physical properties of the stars even before merging takes place. Therefore it is of interest of consider the heating processes in some detail.

The standard estimation of viscous tidal dissipation goes as follows. The rate of energy dissipation is

$$\dot{E}_{\text{visc}} \sim \nu M \left| \frac{\partial v_i}{\partial x_k} \right|^2,$$

where $\nu$ is the kinematic shear viscosity (in units of cm$^2$/s), and $\partial v_i/\partial x_k$ is the strain tensor. The typical tidal bulge height is $h_0$, and the bulge rises and falls at angular velocity $2\Delta \Omega$ (see eq. [1.2]). If the tidal distortion of the star varies over a length-scale comparable to the stellar radius, then the shear strain tensor is of order $\partial v_i/\partial x_k \sim \Delta \Omega h_0/R$. Thus the viscous dissipation rate is

$$\dot{E}_{\text{visc}} \sim \nu M \left( \Delta \Omega \frac{h_0}{R} \right)^2 = \frac{M(h_0 \Delta \Omega)^2}{R^2/\nu} = \nu M(\Delta \Omega)^2 q^2 \left( \frac{R}{D} \right)^6.$$

This is simply the tidal kinetic energy $E_k \sim M(h_0 \Delta \Omega)^2$ divided by the viscous time $t_{\text{visc}} \sim R^2/\nu$. For a synchronized system ($\Delta \Omega = 0$), there is no viscous dissipation.

It is clear from the above derivation that this simple relation is mainly for the f-mode tide, i.e., higher-order modes and resonances have been neglected. However, higher-order mode oscillations involve more shear motion of the fluid, thus the length-scale and timescale for viscous dissipation are smaller for these modes. More importantly, when g-mode resonances occur, the viscous dissipation can start at a larger separation, at which the orbit decay is still slow. As a result, more heat can be generated during the whole inspiral. By contrast, dissipation due to the non-resonant f-mode oscillation is mainly effective for small orbital separation (see eq. [8.2]), at which binary decay is fast.

Of course, the amount of viscous dissipation ultimately depends on the viscosity of the fluid, which is not at all well understood. Our main intention in this section is to consider the effect of g-mode resonances
on the viscous dissipation rate. Although the stellar spin can be incorporated in our formalism without much
difficulty, we shall focus on the non-spinning case, so that \( \Delta \Omega = \Omega \). In this case, equation (8.2) indicates
that for non-resonant viscous dissipation \( \dot{E}_{\text{visc}} \propto 1/D^9 \).

### 8.1 Viscous Damping Rate of Normal Oscillation Modes

Here we consider the viscous damping rates of individual modes. To leading order, viscous stress does
not change the oscillation frequency, but it induces mode damping. This damping effect is equivalent to
adding an imaginary part to the oscillation frequency, i.e., \( \omega'_\alpha = \omega_\alpha + i\gamma_\alpha \). The mode energy damping rate
is given by

\[
2\gamma_\alpha = \frac{\dot{E}_{\text{visc},\alpha}}{E_\alpha}.
\]  

(8.3)

Here \( E_\alpha \) is the energy of the eigenmode:

\[
E_\alpha = 2E_{\alpha,k} = \omega^2_\alpha \int d^3x \bar{\xi}_\alpha \cdot \bar{\xi}_\alpha = \omega^2_\alpha.
\]  

(8.4)

where we have used the normalization (2.9). The viscous dissipation rate is given by (suppressing the mode
index \( \alpha \))

\[
\dot{E}_{\text{visc}} = - \int d^3x \sigma_{ik} v^*_i k = - \int d^3x \left[ \frac{1}{2} \eta \left| v_{i,k} + v_{k,i} - \frac{2}{3} \delta_{ik} \nabla \cdot \mathbf{v} \right|^2 + \zeta |\nabla \cdot \mathbf{v}|^2 \right],
\]  

(8.5)

(Landau & Lifshitz 1987) where \( \sigma_{ik} \) is the viscous stress tensor

\[
\sigma_{ik} = \eta \left( v_{i,k} + v_{k,i} - \frac{2}{3} \delta_{ik} \nabla \cdot \mathbf{v} \right) + \zeta \delta_{ik} \nabla \cdot \mathbf{v},
\]  

(8.6)

\( \eta = \rho \nu \) is the dynamical shear viscosity, and \( \zeta \) is the bulk viscosity.

Substituting \( \mathbf{v} = -i\omega \bar{\xi} \) and equation (2.8) into equation (8.5), the damping rate can be evaluated. In
the absence of bulk viscosity, the result has been obtained by Higgins & Kopal (1968, see also Kopal 1978).

For \( m = 0 \) modes,

\[
\gamma^{(\text{shear})}_{m=0} = \frac{1}{2} \int_0^R r^2 dr \eta F(r),
\]  

(8.7)

where

\[
F(r) = 2 \left( \frac{\partial \xi^r}{\partial r} \right)^2 + \left( \frac{2 \xi^r}{r} - l(l+1) \frac{\xi^r}{r} \right)^2
+ l(l+1) \left( \frac{\partial \xi^\perp}{\partial r} + \frac{\xi^r}{r} - \frac{\xi^\perp}{r} \right)^2
+ (l-1)l(l+1)(l+2) \left( \frac{\xi^\perp}{r} \right)^2
- \frac{2}{3} G(r)^2,
\]  

(8.8)

and

\[
G(r) = \frac{\partial \xi^r}{\partial r} + \frac{2 \xi^r}{r} - l(l+1) \frac{\xi^\perp}{r}.
\]  

(8.9)

For modes with \( m \neq 0 \)

\[
\gamma^{(\text{shear})}_{m \neq 0} = \frac{(l+|m|)!}{(l-|m|)!} \gamma^{(\text{shear})}_{m=0}.
\]  

(8.10)

\( ^8 \) Our definition of \( Y_{lm} \) is that of Jackson (1975), which is different from that used in Higgins & Kopal
(1968). Therefore the results are different by a numerical factor. We have checked that for the f-mode of
incompressible sphere (the Kelvin mode), our result using equations (8.7)-(8.9) agrees with the standard
one, as first derived by H. Lamb; see Cutler & Lindblom (1987).
When bulk viscosity is present, we can add the bulk viscous dissipation to the damping rate, i.e.,

$$\gamma = \gamma^{(\text{shear})} + \gamma^{(\text{bulk})},$$

(8.11)

where

$$\gamma^{(\text{bulk})}_m = \frac{1}{2} \int_0^R r^2 d r \zeta G(r)^2, \quad \gamma^{(\text{bulk})}_m = \frac{(l+|m|)!}{(l-|m|)!} \gamma^{(\text{bulk})}_m.$$  

(8.12)

To specify the dependence of $\gamma$ on the mode structure, independent of the actual values of viscosities, let us consider viscosities of the form

$$\eta = \eta_N \left( \frac{\rho}{\rho_N} \right)^2, \quad \zeta = \zeta_N \left( \frac{\rho}{\rho_N} \right)^2,$$

(8.13)

where $\rho_N$ is some fiducial density. Such a density dependence is appropriate for the microscopic viscosity in the core of neutron star (§8.3). The damping coefficient can be written as (for the $m = 0$ mode)

$$\gamma^{(\text{shear})} = \frac{\eta_N R}{M} \left( \frac{M}{R^3 \rho_N} \right)^2 \Sigma^{(\text{shear})}, \quad \gamma^{(\text{bulk})} = \frac{\zeta_N R}{M} \left( \frac{M}{R^3 \rho_N} \right)^2 \Sigma^{(\text{bulk})},$$

(8.14)

where $\Sigma$'s are dimensionless coefficients depending only on the mode eigenfunctions:

$$\Sigma^{(\text{shear})} = \frac{1}{2} \int_0^R r^2 d r \rho^2 F(r), \quad \Sigma^{(\text{bulk})} = \frac{1}{2} \int_0^R r^2 d r \rho^2 G(r)^2.$$  

(8.15)

Here all quantities in the integrands are in units such that $M = R = 1$.

The values of $\Sigma^{(\text{shear})}$ and $\Sigma^{(\text{bulk})}$ are given in Table 1 for various modes. Clearly, $\Sigma^{(\text{shear})}$ increases as the order of the mode increases. For higher-order g-modes, our numerical results yield an approximate scaling relation, $\Sigma^{(\text{shear})}_{nl} \propto \omega_{nl}^{-2}$.

### 8.2 Viscous Dissipation in Coalescing Binaries

With the results of §8.1, we can now write down the expressions for the viscous energy dissipation rate in the coalescing binary neutron stars. Using equations (2.6), (2.16) and (8.3)-(8.5), we have

$$\dot{E}_{\text{visc}} = \sum_{a} (M' W_{lm} Q_{nl})^2 |b_a - i m \Omega \omega_{a}^2 2 \omega_{l0} | \gamma_{a} = \sum_{a} (M' W_{lm} Q_{nl})^2 |\dot{c}_a - i s m \omega_{a} c_a | 2 \omega_{l0}.$$  

(8.16)

It is clear that the contribution from each mode in this expression is simply twice of the tidal kinetic energy (see eq. [6.6]) of the mode divided by energy damping time of that mode, $1/(2 \gamma_{a})$.

Consider first the contribution from the f-mode. In this case, $b_a$ is given by equation (3.6). Since $|b_a| << \Omega |b_a|$, we have, after adding contributions from both $m = 2$ and $m = -2$ terms,

$$\dot{E}_{\text{visc},0} \simeq 2 (M' W_{l+2} Q_0)^2 |2 \Omega b_0|^2 2 \gamma_{l0} \simeq 24 \pi \left( \frac{G M^2}{R} \right) \omega_{0}^{-4} Q_0^2 \left( \frac{R}{D} \right)^9 2 \gamma_{l0},$$  

(8.17)

where we have assumed $\omega_0 >> \Omega$ and have used equation (2.4). Here and below, $\gamma$ refers to $m = \pm 2$ modes, thus we have, e.g., $\gamma^{(\text{shear})} = 4 \gamma_{m=0}^{(\text{shear})} = 4 \eta (R/M) (M/\rho_N R^3)^2 \Sigma^{(\text{shear})}$ (see eqs. [8.10], [8.14]). Clearly, equation (8.17) agrees with our simple dimensional analysis in (8.2).
Now consider the dissipation associated with a specific g-mode. Prior to the resonance, one can still use equation (3.6) in (8.16) to obtain an expression similar to (8.17). After the resonance, the tidal energy stored in the mode stays nearly constant. The viscous dissipation rate is then given by

\[
\dot{E}_{\text{visc},\alpha} \simeq (\Delta E_{\alpha})^2 \dot{\gamma}_\alpha
\]

\[
\simeq 3.2 \times 10^{-8} \left( \frac{GM^2}{R} \right) \left( \frac{f_{nl}}{100 \, \text{Hz}} \right)^{1/3} \left( \frac{Q_{nl}}{0.0003} \right)^2 R_{10}^3 M_{1.4}^{-8/3} q \left( \frac{2}{1 + q} \right)^{5/3} 2^{\dot{\gamma}_\alpha},
\]

where we have used equation (6.11) for \( \Delta E_{\alpha} = \Delta E_{nl} \), and we have included both \( m = 2 \) and \( m = -2 \) contributions.

The relative importance of the non-resonant viscous dissipation (8.17) and the resonant dissipation (8.18) can be compared by calculating the total viscous dissipation accumulated during the inspiral from \( D = \infty \) to the orbital radius before merging \( D = D_m \):

\[
E_{\text{visc}} = \int dt \dot{E}_{\text{visc}} = - \int_{\infty}^{D_m} \frac{dD}{D} t_D \dot{E}_{\text{visc}},
\]

where \( t_D \) is given by equation (2.13). For the f-mode non-resonant dissipation, we have

\[
E_{\text{visc},0} \simeq \frac{3\pi}{80} \left( \frac{GM^2}{R} \right)^{2/3} \left( \frac{GM}{c^3} \right) \left( \frac{R}{D_m} \right)^5 \left( \frac{Re^2}{GM} \right)^4 \frac{Q_0^2}{\omega_0^4} \omega_0^4, \tag{8.20}
\]

where in the second equality we have assumed \( D_m/R = 3 \) and have used \( \omega_0^2 \simeq 1.5, \ Q_0 \simeq 0.56 \) (see Table 1). For the g-mode, we can neglect its contribution prior to the resonance, i.e., we assume \( \dot{E}_{\text{visc},\alpha} \simeq 0 \) for \( D > D_\alpha \). The total accumulated viscous dissipation due to the resonant g-mode oscillation is then

\[
E_{\text{visc},\alpha} \simeq \frac{5\pi^2}{128\gamma_0} Q_{nl}^{-7/3} \left( \frac{f_{nl}}{100 \, \text{Hz}} \right)^{13/2} \left( \frac{Q_{nl}}{0.0003} \right) R_{10}^3 M_{1.4}^{-16/3} \left( \frac{2}{1 + q} \right)^{4/3} \left( \frac{GM^2}{R} \right)^{2\gamma_\alpha} \left( \frac{GM}{c^3} \right)^{2\gamma_\alpha},
\]

\[
\simeq 0.010 \left( \frac{f_{nl}}{100 \, \text{Hz}} \right)^{-7/3} \left( \frac{Q_{nl}}{0.0003} \right)^2 R_{10}^3 M_{1.4}^{-16/3} \left( \frac{2}{1 + q} \right)^{4/3} \left( \frac{GM^2}{R} \right)^{2\gamma_\alpha} \left( \frac{GM}{c^3} \right)^{2\gamma_\alpha}. \tag{8.21}
\]

Because \(|Q_{nl}| \) decreases rapidly as \( \omega_{nl} \) decreases, only the resonance with the lowest order g-mode is important. In obtaining equations (8.20) and (8.21), we have assumed that \( \gamma_\alpha \) remains constant during the inspiral. In reality, the viscosities depend on temperature, which increases as the neutron star is heated up. Nevertheless, since \( \gamma_\alpha^{(\text{shear})} > \gamma_0^{(\text{shear})} \), equations (8.20) and (8.21) already indicated that g-mode resonances can be important in heating the neutron star compared to the standard f-mode non-resonant heating.

### 8.3 Neutron Star Viscosity

The expressions derived in §8.1 and §8.2 are quite general, and can be used with different viscosities. To actually determine the amount of heating of the coalescing binary neutron stars before merging, we need to know the viscosity of the nuclear matter. Calculations by Flowers & Itoh (1976, 1979) yield an estimate
for the microscopic shear viscosity in liquid core of neutron star. Approximate fitting formulae have been given by Cutler and Lindblom (1987). For normal liquid core, the viscosity is dominated by neutron-neutron scattering, with

\[ \eta \simeq 1.1 \times 10^{19} T_8^{-2} \left( \frac{\rho}{\rho_N} \right)^{9/4} \text{g cm}^{-1}\text{s}^{-1}, \quad (8.22) \]

where \( \rho_N = 2.8 \times 10^{14} \text{g/cm}^3 \), and \( T_8 = T/(10^8 \text{K}) \). When the temperature falls below a critical temperature \( \sim 10^9 \text{K} \), both neutron and proton are likely to undergo a phase transition to become a superfluid (see, e.g., Wambach, Ainsworth & Pines 1991 for review). In this low temperature regime, the viscosity is dominated by electron-electron scattering, and it is given by

\[ \eta \simeq 4.7 \times 10^{19} T_8^{-2} \left( \frac{\rho}{\rho_N} \right)^2 \text{g cm}^{-1}\text{s}^{-1}. \quad (8.23) \]

Since coalescing binary neutron stars are extremely old, their internal heat must have radiated away. All cooling calculations indicate that neutron stars older than \( 10^8 \) years have core temperature less than \( \sim 10^6 \text{K} \) (and surface temperature less than \( 10^5 \text{K} \) (e.g., Tsuruta 1992). Therefore, without tidal heating, the cores of the neutron stars in the coalescing binary systems are likely to be in the superfluid state.

The bulk viscosity of neutron star arises from the delay in achieving beta equilibrium as the density is changed. Calculation of Sawyer (1989) yields

\[ \zeta \simeq 4.8 \times 10^{18} T_8^{-6} \left( \frac{\rho}{\rho_N} \right)^2 \left( \frac{\omega}{1 \text{s}^{-1}} \right)^{-2} \text{g cm}^{-1}\text{s}^{-1}, \quad (8.24) \]

where \( \omega \) is the angular frequency of the perturbation \(^9\). Because of the strong temperature dependence, bulk viscosity is important only when \( T >> 10^8 \text{K} \). We shall neglect it in our following discussions.

Of course, one should be cautious in using these microscopic viscosities. The stable stratification indicates that turbulent viscosity can not be present in a cold neutron star. Also, tidal distortion is not likely to induce turbulence and convection in the binary stars, at least in the linear regime (Seguin 1976). However, it is not completely clear whether any other “anomalous” viscosities, such as the mutual friction between the electrons and the superfluid neutron vortices (e.g., Sauls 1989, Mendell 1991), can be present in a cold neutron star.

### 8.4 Heating of the Neutron Stars

With the microscopic viscosity given by equation (8.23), the mode damping constant is (for \( m = \pm 2 \)):

\[ \gamma_\alpha = 4 \times 10^{-5} T_8^{-2} \Sigma_\alpha M_{1.4} R_{10}^{-5} \text{ s}^{-1}. \quad (8.25) \]

\(^9\) This result is based on modified Urca process. When the proton fraction is sufficiently large to allow for direct Urca process (Lattimer et al. 1991), the value of \( \zeta \) is larger than equation (8.24) by a factor of \( (\mu_n/kT)^2 \sim 5 \times 10^7 T_8^{-2} \) (where \( \mu_n \) is the Fermi energy of the neutron). Also, equation (8.24) is valid only when the deviation from chemical equilibrium \( \delta \mu = \mu_n - \mu_p - \mu_e \) is much smaller than \( kT \). More general expression can be found in Haensel (1992).
where we have suppressed the superscript “shear”.

Below we shall focus on neutron stars with $M = 1.4M_\odot$ and $R = 10$ km. Equations (8.17) and (8.25) then give the non-resonant heating rate

$$\dot{E}_{\text{visc},0} \simeq 2.6 \times 10^{50} \left( \frac{R}{D} \right)^9 T_8^{-2} \text{ erg s}^{-1},$$

(8.26)

where we have used $\Sigma_0 \simeq 3$ (see Table 1). Similarly, equations (8.18) and (8.25) yield the resonant heating rate

$$\dot{E}_{\text{visc},\alpha} \simeq 1.3 \times 10^{42} \left( \frac{f_{nl}}{100 \text{ Hz}} \right)^{1/3} \left( \frac{Q_{nl}}{0.005} \right)^2 T_8^{-2} \Sigma_\alpha \text{ erg s}^{-1}.$$

(8.27)

The heat content of the neutron star is mainly given by the thermal energy of the nonrelativistic degenerate free neutron gas in the core:

$$U \simeq \frac{\pi^2}{4} N k_B T k_B T \mu_n \simeq 4.5 \times 10^{45} T_8^2 \text{ erg},$$

(8.28)

where $N$ is the total number of nucleons and $k_B$ is the Boltzmann constant. The thermal evolution equation of the neutron star during the orbital decay can be simply written as

$$\frac{dU}{dt} = \dot{E}_{\text{visc}} + \dot{E}_{\text{cool}}.$$

(8.29)

The cooling term $\dot{E}_{\text{cool}}$ due to neutrino emission and surface photon emission can be shown to be small at this low temperature and will be neglected. If we include only the f-mode non-resonant heating $\dot{E}_{\text{visc},0}$, and use equation (8.28), then integrating equation (8.29) yields:

$$T_8 \simeq 0.36 \left( \frac{3R}{D} \right)^{5/4}.$$

(8.30)

On the other hand, if we include only the resonant dissipation, we obtain, for $D << D_{nl}$,

$$T_8 \simeq 0.42 \left( \frac{f_{nl}}{100 \text{ Hz}} \right)^{-7/12} \left( \frac{|Q_{nl}|}{0.0003} \right)^{1/2} \left( \frac{\Sigma_{nl}}{20} \right)^{1/4}.$$

(8.31)

Thus the resonant excitation of the g-mode can be as important as the standard f-mode tidal dissipation in heating the neutron star. Moreover, this resonant excitation leads to significant heating at larger orbital separation, where the f-mode tidal heating is small. Our value of the temperature is much smaller than that estimated by Mészáros & Rees (1992). To obtain their value of $\sim 10^{11}$ K, we would require a viscosity that is larger than the microscopic value by a factor of order $10^{10} - 10^{12}$, corresponding to a kinematic viscosity

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10 When both neutron and proton become superfluid, their heat capacity is greatly reduced by a factor of order $e^{\Delta/kT}$, where $\Delta \sim 0.1 - 1$ MeV is the superfluid gap energy. In this regime, the main thermal content of the neutron star is due to that of the relativistic degenerate electrons, and $U$ is smaller by a factor $2x$. Using equation (4.25) as a simple estimate of $x$, we have $U \simeq 1.2 \times 10^{14} T_8^2$ (erg). If we use this expression instead of eq. (8.28), the resulting temperature is higher than eqs. (8.30)-(8.31) by a factor of 2.4.
comparable to the maximum possible value in a neutron star $\nu_{\text{max}} \sim (GMR)^{1/2} \sim 10^{16}$ cm$^2$/s. We consider this rather unlikely. We therefore expect that before the binary neutron stars merge, they remain relatively cold and degenerate. Tidal dissipation heats up the neutron stars from less than $10^6$ K to $\sim 10^8$ K before the stars come into contact.

9. CONCLUSIONS

We have studied in this paper various consequences of tidal excitations of oscillations in coalescing binary neutron stars. The energy transfer and angular momentum transfer from the orbit to the star due to the resonant excitations of low-frequency g-modes are found to be small because of the weak coupling between the g-mode and the tidal potential. The induced orbital phase errors are negligible for the detection of gravitational wave from such binary systems by LIGO and VIRGO. Therefore, only the f-mode, quasi-equilibrium tide needs to be considered. Binary models based on ellipsoidal figures (e.g., Lai, Rasio & Shapiro 1994a, 1994b) are thus expected to give a good description of the effects of tidal interaction on the orbits of coalescing neutron star binaries. However, we find that the resonant excitation of the lowest-order g-mode can play a significant role in the tidal heating of the neutron stars prior to merger.

Our calculations of the core g-mode oscillations of neutron stars indicate that the g-mode frequencies depend not only on the pressure-density relation, but also on the symmetry properties of the nuclear matter. Since the resonant amplitude of the tidal distortion associated with the lowest-order g-mode can be as large as $\sim 0.5\%$, if some electromagnetic signature of this induced oscillation can be detected, as suggested by RG2, then we may be able to constrain the properties of nuclear matter in the neutron star. We caution that the actual values of the g-mode frequencies obtained in this paper are by no means conclusive. In our calculations, we have made several assumptions, e.g., we have used Newtonian theory, we have neglected the crustal density discontinuities and shear modulus, the thermal effects, as well as possible superfluid effects. Also, our calculations are based on a single type of nuclear equations of state (WFF), since these are the only EOS available to us from which we can extract enough information to calculate the g-modes self-consistently (see §4). It is desirable that future microscopic calculations of high density nuclear matter not only provide the pressure-density relation, but also provide sufficient details which would allow one to obtain the sound speed $c_s$ and the small difference $c_s^2 - c_e^2$, so that g-mode properties based on a wider variety of EOS can be calculated.

Finally, we note that although the focus of this paper has been neutron star binaries, the expressions we have established in §2-3 and §6-8 are generally valid and can be applied to some other binary systems as well, as long as the tidal distortion remains small and the linear approximation is valid. For example, coalescing white dwarf binaries have been considered as important sources of low-frequency gravitational waves that are potentially detectable by future space-based interferometers (Evans, Iben & Smarr 1987). The restoring force for the g-modes of white dwarfs is provided by the thermal energy of degenerate electrons (Baglin & Heyvaerts 1969, Chanmugam 1972). Since the orbital decay rate is much smaller for a white dwarf binary,
the effects of g-mode resonances are much more pronounced. Indeed, using the typical value of the ratio \( R_c^2/(GM) \sim 10^3 - 10^4 \) for white dwarfs, and using similar dimensionless values for \( \omega_{nl} \) and \( Q_{nl} \) (e.g., \( \omega_{nl} \sim 0.1, Q_{nl} \sim 10^{-4} \)), we find that the energy transferred to the white dwarf during a resonance can be significant compared to the orbital energy (see eq. [6.13]). Thus g-mode resonances are important for the orbital evolution of coalescing binary white dwarfs.

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REFERENCES

Abramovici, A., et al. 1992, Science, 256, 325
Arnett, W.D., & Bowers, R.L. 1977, ApJS, 33, 415
Baglin, A., & Heyvaerts, J. 1969, Nature, 222, 1258
Baym, G., Bethe, H.A., & Pethick, C.J. 1971, Nucl. Phys. A175, 225
Baym, G., Pethick, C.J., & Sutherland, P. 1971, ApJ, 170, 299
Bildsten, L., & Cutler, C. 1992, ApJ, 400, 175
Bradaschia, C., et al. 1990, Nucl. Instrum. & Methods, A289, 518
Chanmugam, G. 1972, Nature, 236, 83
Cowling, T.G. 1941, MNRAS, 101, 367
Cox, J.P. 1980, Theory of Stellar Pulsation (Princeton University Press)
Cutler, C., & Lindblom, L. 1987, ApJ, 314, 234
Cutler, C. et al. 1993, Phys. Rev. Lett., 70, 2984
Evans, C. R., Iben, I., & Smarr, L. 1987, ApJ, 323, 129
Finn, L.S. 1987, MNRAS, 227, 265
Flowers, E., & Itoh, N. 1976, ApJ, 206, 218
Flowers, E., & Itoh, N. 1979, ApJ, 230, 847
Goldreich, P., & Nicholson, P. 1989, 342, 1079
Haensel, P. 1992, A&A, 262, 131
Thorne, K. S. 1987, in 300 Years of Gravitation, eds. S. W. Hawking & W. Israel (Cambridge: Cambridge University Press), 330

Tsuruta, S. 1992, in Physics of Isolated Pulsars, eds. K. Van Riper, R. Epstein & C. Ho (Cambridge University Press)

Wambach, J., Ainsworth, T. L., & Pines 1991, in Neutron Stars: Theory and Observations, eds. J. Ventura & D. Pines (Kluwer: Dordrecht)

Wiringa, R. B., Fiks, V., & Fabrocini, A. 1988, Phys. Rev. C, 38, 1010

Zahn, J.P. 1970, A&A, 4, 452

Zahn, J.P. 1977, A&A, 57, 383
Figure 1.— Dimensionless tidal amplitude associated with a particular g-mode of coalescing binary neutron stars as a function of the binary separation $D$. The resonance radius is assumed to be $D_\alpha/R = 8$, and $Rc^2/(GM) = 5$, $q = M'/M = 1$, where $R$ is the stellar radius, $M$ is the neutron star mass. (a) shows the real part of the function $b_\alpha(t)$ (light solid line) and $|b_\alpha(t)|$ (dark solid line) for the $m = 2$ mode; the dashed line corresponds to the quasi-equilibrium solution (eq. [3.2]). (b) shows the real part (solid line) and the imaginary part (dotted line) of the function $c_\alpha(t)$ for $m = 2$. The orbital phase function is chosen such that $\Phi(D = 0) = 0$.

Figure 2.— Equations of state for four neutron star models considered in this paper: (a) pressure $P$; (b) adiabatic sound speed $c_s$ (where $c$ is the speed of light); (c) the proton fraction $x = n_p/n$; (d) the fractional difference between $c_s^2$ and $c_e^2$. The dotted lines are for model AU, the solid lines for model UU, the short-dashed lines for model UT, and the long-dashed lines for model UU2. Note that model UU2 has the same pressure and $c_e$ as model UU.

Figure 3.— The frequencies of the first ten quadrupole ($l = 2$) g-modes based on four different neutron star models. $n$ specifies the radial order of the g-mode. The dotted line and round circles are for model AU, the solid line and squares for model UU, the short-dashed line and triangles for model UT, and the long-dashed line for model UU2.
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