Dirac operator and a twisted cyclic cocycle on the standard Podleś quantum sphere

Konrad Schmüdgen and Elmar Wagner

Fakultät für Mathematik und Informatik
Universität Leipzig, Augustusplatz 10, 04109 Leipzig, Germany
E-mail: schmuedg@mathematik.uni-leipzig.de / wagner@mathematik.uni-leipzig.de

Abstract

A Dirac operator $D$ on the standard Podleś sphere $S^2_q$ is defined and investigated. It yields a real spectral triple such that $|D|^{-z}$ is of trace class for $\Re z > 0$. Commutators with the Dirac operator give the distinguished 2-dimensional covariant differential calculus on $S^2_q$. The twisted cyclic cocycle associated with the volume form of the differential calculus is expressed by means of the Dirac operator.

Keywords: Quantum spheres, differential calculus, spectral triples, twisted cyclic cocycles

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0 Introduction

Spectral triples, cyclic cocycles and Dirac operators are basic concepts of Alain Connes’ non-commutative geometry [C1]. In general, covariant differential calculi on quantum groups cannot be described by spectral triples. For instance, the commonly used covariant differential calculus on the quantum group $SU_q(2)$ cannot even be given by bounded commutators [S]. For the quantum group $SU_q(2)$, an equivariant spectral triple has been constructed in [CP] and the corresponding non-commutative geometry has been studied by A. Connes in [C3]. In this paper we show (among other things) that the distinguished 2-dimensional covariant
differential calculus of the standard Podleś quantum sphere $S^2_q$ can be given by a spectral triple.

The starting point of our construction is the representation of the left crossed product algebra $\mathcal{O}(S^2_q) \rtimes U_q(su_2)$ of the Hopf algebra $U_q(su_2)$ and the coordinate algebra $\mathcal{O}(S^2_q)$ of the standard Podleś sphere on the subspace $V^+ \oplus V^-$ of $\mathcal{O}(SU_q(2))$, where $V^+ = \mathcal{O}(S^2_q)c + \mathcal{O}(S^2_q)d$ and $V^- = \mathcal{O}(S^2_q)a + \mathcal{O}(S^2_q)b$. Then the right actions $R_E$ and $R_F$ of the generators $E, F \in U_q(su_2)$ on $\mathcal{O}(SU_q(2))$ map $V^+$ into $V^-$ and $V^-$ into $V^+$, respectively. The Dirac operator

$$D = \begin{pmatrix} 0 & R_F \\ R_E & 0 \end{pmatrix}$$

commutes with the left action of $U_q(su_2)$ on $V^+ \oplus V^-$. Since $D$ has discrete spectrum of eigenvalues $[l + 1]_q$, $l \in \mathbb{N}_0$, with multiplicities $2l + 1$, and since commutators $[D, x]$, $x \in \mathcal{O}(S^2_q)$, are bounded, the Dirac operator $D$ leads to a spectral triple such that $|D|^{-z}$ is of trace class for all $z \in \mathbb{C}$, $\text{Re} \, z > 0$.

Note that our Dirac operator $D$ is unitarily equivalent to the Dirac operator constructed in [DS], where also a real structure was obtained. However, we show that the Dirac operator fits into the framework of covariant differential calculus. That is, we prove that the Dirac operator gives a commutator representation $dx \cong i[D, x]$ of the distinguished 2-dimensional covariant first order differential calculus on $\mathcal{O}(S^2_q)$. The associated higher order calculus has up to multiples a unique invariant volume 2-form $w$. Let $h$ denote the $U_q(su_2)$-invariant state on $\mathcal{O}(S^2_q)$. We show that the associated twisted cyclic 2-cocycle $\tau_{w,h}$ represents a non-trivial cohomology class in the twisted cyclic cohomology and prove that

$$\tau_{w,h}(x_0, x_1, x_2) = h(x_0(R_F(x_1)R_E(x_2) - q^2R_E(x_1)R_F(x_2)))$$
$$= (q - q^{-1})^{-1}(\log q) \text{Res}_{z=2} \text{Tr}_K \gamma_q K^2 |D|^{-z} x_0[D, x_1][D, x_2]$$

for $x_0, x_1, x_2 \in \mathcal{O}(S^2_q)$, where

$$\gamma_q = \begin{pmatrix} 1 & 0 \\ 0 & -q^2 \end{pmatrix}.$$

This paper is organized as follows. In Section 1 twisted cyclic cocycles associated with volume forms of differential calculi are defined. A number of preliminary facts on the Podleś sphere $S^2_q$ and the quantum group $SU_q(2)$ are collected in Section 2. The Dirac operator and its properties are developed in Section 3 while the twisted cyclic cocycle is studied in Section 4. The form of the cocycle $\tau_{w,h}$ was
kindly communicated to us by I. Heckenberger. His proof of the corresponding result will be given in the appendix.

Throughout this paper, \( q \) is a real number such that \( 0 < q < 1 \) and \( i \) denotes the complex unit. We abbreviate \( \lambda := q - q^{-1} \) and \([n]_q := (q^n - q^{-n})/(q - q^{-1})\).

## 1 Twisted cyclic cocycles

Let \( \mathcal{X} \) be a unital algebra and \( \sigma \) an algebra automorphism of \( \mathcal{X} \). We shall use the following notations on the twisted cyclic cohomology resp. twisted cyclic homology (see e.g. [KMT] and [KR]) of the pair \((\mathcal{X}, \sigma)\).

Let \( \varphi \) be an \((n + 1)\)-linear form on \( \mathcal{X} \) and let \( \eta = \sum_k x_0^k \otimes \cdots \otimes x_n^k \in \mathcal{X}^{\otimes n+1} \), where \( x_j^k \in \mathcal{X} \). The \( \sigma \)-twisted coboundary resp. boundary operator \( b_\sigma \) and the \( \sigma \)-twisted cyclicity operator \( \lambda_\sigma \) are defined by

\[
(b_\sigma \varphi)(x_0, \ldots, x_{n+1}) = \sum_{j=0}^{n} (-1)^j \varphi(x_0, \ldots, x_j, x_{j+1}, \ldots, x_{n+1}) + (-1)^{n+1} \varphi(\sigma(x_{n+1})x_0, x_1, \ldots, x_n),
\]

(2)

\[
(\lambda_\sigma \varphi)(x_0, \ldots, x_n) = (-1)^n \varphi(\sigma(x_n), x_0, \ldots, x_{n-1}),
\]

(3)

\[
b_\sigma \eta = \sum_{k,j=0}^{n-1} (-1)^j x_0^k \otimes \cdots \otimes x_j^k \otimes x_{j+1}^k \otimes \cdots \otimes x_n^k + (-1)^n \sigma(x_n^k) x_0^k \otimes x_1^k \otimes \cdots \otimes x_{n-1}^k,
\]

(4)

\[
\lambda_\sigma \eta = \sum_k (-1)^n \sigma(x_n^k) x_0^k \otimes \cdots \otimes x_{n-1}^k.
\]

(5)

An \((n + 1)\)-form \( \varphi \) is called a \( \sigma \)-twisted cyclic \( n \)-cocycle if \( b_\sigma \varphi = 0 \) and \( \lambda_\sigma \varphi = \varphi \). A \( \sigma \)-twisted cyclic \( n \)-cocycle is called non-trivial if there is no \( n \)-form \( \psi \) such that \( \varphi = b_\sigma \psi \) and \( \lambda_\sigma \psi = \psi \). Similarly, \( \eta \) is a \( \sigma \)-twisted cyclic \( n \)-cycle if \( b_\sigma \eta = 0 \) and \( \lambda_\sigma \eta = \eta \). Cycles \( \eta = \sum_k x_0^k \otimes \cdots \otimes x_n^k \in \mathcal{X}^{\otimes n+1} \) can be paired with \((n + 1)\)-forms \( \varphi \) by setting \( \varphi(\eta) := \sum_k \varphi(x_0^k, \ldots, x_n^k) \).

Let \( \Gamma = \bigoplus_{k=0}^{\infty} \Gamma^{\wedge k} \) be a differential calculus on \( \mathcal{X} \) with differentiation \( d : \Gamma^{\wedge k} \to \Gamma^{\wedge (k+1)} \) (see e.g. [KS, Part IV]). Suppose that \( \Gamma^{\wedge n} \) is a free \( \mathcal{X} \)-module generated by \( \omega \) such that \( \omega x = x \omega \) for all \( x \in \mathcal{X} \). For \( \eta \in \Gamma^{\wedge n} \), let \( \pi(\eta) \in \mathcal{X} \) denote the unique element such that \( \eta = \pi(\eta) \omega \). Assume that \( \sigma \) is an algebra automorphism of \( \mathcal{X} \) and \( h \) is a linear functional on \( \mathcal{X} \) such that, for all \( x, y \in \mathcal{X} \) and \( \omega_{n-1} \in \Gamma^{\wedge (n-1)} \),

\[
h(xy) = h(\sigma(y)x), \quad h(\pi(d\omega_{n-1})) = 0.
\]
Then it is easy to check by using repeatedly the Leibniz rule that
\[ \tau_{\omega,h}(x_0, \ldots, x_n) := h(\pi(x_0 dx_1 \wedge \ldots \wedge dx_n)), \quad x_0, \ldots, x_n \in X, \]
defines a \( \sigma \)-twisted cyclic \( n \)-cocycle \( \tau_{\omega,h} \) on \( X \).

\section{Preliminaries on the quantum group \( SU_q(2) \) and the Podleś sphere \( S^2_q \)}

First we collect some definitions, facts and notations used in what follows (see e.g. [KS]). For brevity, we employ the Sweedler notation \( \Delta(x) = x(1) \otimes x(2) \) for the comultiplication \( \Delta(x) \).

Let \( \mathcal{O}(SU_q(2)) \) denote the coordinate Hopf \( \ast \)-algebra of the quantum \( SU(2) \) group (see e.g. [KS, Chapter 4]). As usual, the generators of \( \mathcal{O}(SU_q(2)) \) are denoted by \( a, b, c, d \). The Hopf \( \ast \)-algebra \( U_q(\mathfrak{su}_2) \) has four generators \( E, F, K, K^{-1} \) with defining relations
\[ KK^{-1} = K^{-1}K = 1, \quad KE = qEK, \quad FK = qKF, \quad EF - FE = \lambda^{-1}(K^2 - K^{-2}), \]
involution \( E^\ast = F, \quad K^\ast = K \), comultiplication
\[ \Delta(E) = E \otimes K + K^{-1} \otimes E, \quad \Delta(F) = F \otimes K + K^{-1} \otimes F, \quad \Delta(K) = K \otimes K, \]
counit \( \epsilon(E) = \epsilon(F) = \epsilon(K - 1) = 0 \) and antipode \( S(K) = K^{-1}, S(E) = -qE, S(F) = -q^{-1}F \). There is a dual pairing \( \langle \cdot, \cdot \rangle \) of the Hopf \( \ast \)-algebras \( U_q(\mathfrak{su}_2) \) and \( \mathcal{O}(SU_q(2)) \) given on the generators by the values
\[ \langle K^{\pm 1}, a \rangle = q^{\pm 1/2}, \quad \langle E, c \rangle = \langle F, b \rangle = 1 \quad (6) \]
and zero otherwise. Therefore, \( \mathcal{O}(SU_q(2)) \) is a left and right \( U_q(\mathfrak{su}_2) \)-module \( \ast \)-algebra with left action \( \triangleright \) and right action \( \triangleleft \) defined by
\[ f \triangleright x = \langle f, x(2) \rangle x(1), \quad x \triangleleft f = \langle f, x(1) \rangle x(2), \quad x \in \mathcal{O}(SU_q(2)), \quad f \in U_q(\mathfrak{su}_2). \quad (7) \]
The actions satisfy
\[ (f \triangleright x)^\ast = S(f)^\ast \triangleright x^\ast, \quad (x \triangleleft f)^\ast = x^\ast \triangleleft S(f)^\ast, \quad (8) \]
\[ f \triangleright xy = (f \triangleright x)(f \triangleright y), \quad xy \triangleleft f = (x \triangleleft f_{(1)})(y \triangleleft f_{(2)}). \quad (9) \]
On generators, we have
\[
E_\alpha a = b, \quad E_\alpha c = d, \quad E_\beta b = E_\beta d = 0, \quad F_\alpha b = a, \quad F_\alpha d = c, \quad F_\alpha a = F_\alpha c = 0,
\]
\[
K_\alpha a = q^{-1/2} a, \quad K_\alpha b = q^{1/2} b, \quad K_\alpha c = q^{-1/2} c, \quad K_\alpha d = q^{1/2} d;
\]
\[
c_\alpha E = a, \quad d_\alpha E = b, \quad c_\alpha E = b_\alpha E = 0, \quad c_\alpha F = c, \quad b_\alpha F = d, \quad c_\alpha F = d_\alpha F = 0,
\]
\[
\alpha_\alpha K = q^{-1/2} a, \quad b_\alpha K = q^{-1/2} b, \quad c_\alpha K = q^{1/2} c, \quad d_\alpha K = q^{1/2} d.
\]

Let \( h \) denote the Haar state of \( \mathcal{O}(SU_q(2)) \) and let \( \mathcal{L}^2(SU_q(2)) \) be the Hilbert space completion of \( \mathcal{O}(SU_q(2)) \) equipped with the scalar product \( (x, y) = h(y^* x) \), \( x, y \in \mathcal{O}(SU_q(2)) \). For \( x \in \mathcal{O}(SU_q(2)) \) and \( f \in \mathcal{U}_q(su_2) \), set
\[
R_f(x) = x_s S^{-1}(f).
\]

Using (8) and the \( \mathcal{U}_q(su_2) \)-invariance of \( h \), we compute
\[
(x, R_f(y)) = h((y^*_s S^{-1}(f)^* x) = h((y^* f^*_s) x) = \varepsilon(f_{(1)}^*) h((y^* f^*_s) x)
\]
\[
= h((y^* f^*_s S^{-1}(f_{(2)}^*)) (x_s S^{-1}(f_{(1)}^*))) = (R_f^*(x), y).
\]

Hence the mapping \( R \) given by \( f \mapsto R_f \) is a \(*\)-representation of \( \mathcal{U}_q(su_2) \). Likewise, the left action \( \cdot \) defines a \(*\)-representation of \( \mathcal{U}_q(su_2) \) on \( \mathcal{O}(SU_q(2)) \).

Let \( t_{jk}^l, \ l \in \frac{1}{2} \mathbb{N}_0, \ j, k = -l, -l + 1, \ldots, l \), denote the matrix elements of finite dimensional unitary corepresentations of \( \mathcal{O}(SU_q(2)) \). By the Peter-Weyl theorem, the elements \( v_{jk}^l := [2l+1]^{1/2} q^l t_{jk}^l \) form an orthonormal vector space basis of \( \mathcal{O}(SU_q(2)) \). For our considerations below, we shall need the elements \( t_{\pm 1/2, j}^l \)
\( l \in \mathbb{N}_0, \ j = -(l+1/2), \ldots, l + 1/2 \). They can be expressed explicitly by the following formulas (see e.g. [KS, p. 109]):
\[
t_{\pm 1/2, j}^l = M_j^l p_{l-j+1/2}(\zeta; q^{2(1/2-j)}, q^{-2(1/2+j)}) b_j^{-1/2} d_j^{1/2}, \quad j > 0,
\]
\[
t_{\pm 1/2, j}^l = N_j^l a_{j+1/2}^{-1/2} c_{j-1/2}^{1/2} p_{l+j+1/2}(\zeta; q^{-2(1/2-j)}, q^{2(1/2+j)}), \quad j < 0,
\]
\[
t_{-1/2, j}^l = N_j^l p_{l-j+1/2}(\zeta; q^{2(1/2-j)}, q^{-2(1/2+j)}) b_j^{1/2} d_j^{-1/2}, \quad j > 0,
\]
\[
t_{-1/2, j}^l = M_j^l a_{j+1/2}^{-1/2} c_{j-1/2}^{1/2} p_{l+j+1/2}(\zeta; q^{-2(1/2+j)}, q^{2(1/2-j)}), \quad j < 0,
\]
where \( p_k(\cdots, \cdots) \) are the little \( q \)-Jacobi polynomials, \( \zeta = -qbc \) and \( M_j^l, N_j^l \) are positive constants. On the vector space \( W_k^l := \text{Lin}\{v_{jk}^l ; j = -l, \ldots, l\} \), the \(*\)-representation \( R \) becomes a spin \( l \) representation. Let \( \alpha_j^l := ([l-j]_q[l+j+1]_q)^{1/2} \).

Then
\[
R_E(v_{jk}^l) = -\alpha_{j-1}^l v_{j-1,k}^l, \quad R_F(v_{jk}^l) = -\alpha_j^l v_{j+1,k}^l, \quad R_K(v_{jk}^l) = q^{-j} v_{jk}^l.
\]
For the left action $\triangleright$ of $\mathcal{U}_q(\mathfrak{su}_2)$ on $\mathcal{O}(\mathfrak{SU}_q(2))$, one obtains

$$E \triangleright v_{jk}^l = \alpha_k^l v_{j,k+1}^l, \quad F \triangleright v_{jk}^l = \alpha_{k-1}^l v_{j,k-1}^l, \quad K \triangleright v_{jk}^l = q^k v_{jk}^l. \quad (20)$$

The coordinate $*$-algebra $\mathcal{O}(S_q^2)$ of the standard Podleś sphere is the unital $*$-subalgebra of $\mathcal{O}(\mathfrak{SU}_q(2))$ generated by the elements $A := -q^{-1}bc, B := ac, B^* := -db$. One can also define $\mathcal{O}(S_q^2)$ as the abstract unital $*$-algebra with three generators $A = A^*, B, B^*$ and defining relations

$$BA = q^2 AB, \quad AB^* = q^2 B^* A, \quad B^* B = A - A^2, \quad BB^* = q^2 A - q^4 A^2. \quad (21)$$

It is a well-known fact and easy to verify that $\mathcal{O}(S_q^2)$ is the set of all elements of $\mathcal{O}(\mathfrak{SU}_q(2))$ which are invariant under the right action of the generator $K$ of $\mathcal{U}_q(\mathfrak{su}_2)$, that is,

$$\mathcal{O}(S_q^2) = \{ x \in \mathcal{O}(\mathfrak{SU}_q(2)) ; x = x \triangleright K \}. \quad (22)$$

From (19) and (22), it follows that $\mathcal{O}(S_q^2)$ is the linear span of elements $t_{0k}^l$, where $k = -l, \ldots, l$ and $l \in \mathbb{N}_0$.

The importance of the Podleś sphere $S_q^2$ stems from the fact that it is a quantum homogeneous space for the compact quantum group $\mathfrak{SU}_q(2)$. In fact, $\mathcal{O}(S_q^2)$ is a right $*$-coideal of the Hopf $*$-algebra $\mathcal{O}(\mathfrak{SU}_q(2))$ and hence a left $\mathcal{U}_q(\mathfrak{su}_2)$-module $*$-algebra with respect to the left action (7). The generators

$$x_{-1} := (1 + q^{-2})^{-1/2} B, \quad x_0 := 1 - (1 + q^2) A, \quad x_1 := -(1 + q^2)^{-1/2} B^*$$

transform by the spin 1 matrix corepresentation of $\mathcal{O}(\mathfrak{SU}_q(2))$, see [KS, p. 124] or [P1]. We have

$$\Delta(B) = B \otimes a^2 + (1 - (1 + q^2)A) \otimes ac - qB^* \otimes c^2, \quad (23)$$

$$\Delta(A) = -q^{-2}B \otimes ab - q^{-1}(1 - (1 + q^2)A) \otimes bc + A \otimes 1 - B^* \otimes dc, \quad (24)$$

$$\Delta(B^*) = -q^{-1}B \otimes b^2 - (1 - (1 + q^2)A) \otimes db + B^* \otimes d^2. \quad (25)$$

The left crossed product algebra $\mathcal{O}(S_q^2) \times \mathcal{U}_q(\mathfrak{su}_2)$ is the $*$-algebra generated by the $*$-subalgebras $\mathcal{O}(S_q^2)$ and $\mathcal{U}_q(\mathfrak{su}_2)$ with respect to the cross relations

$$f \triangleright x = (f_{(1)} \triangleright x)f_{(2)} \equiv (f_{(1)}, x_{(2)}; x_{(1)}f_{(2)}), \quad f \in \mathcal{U}_q(\mathfrak{su}_2), \quad x \in \mathcal{O}(S_q^2).$$

From (6) and (23)–(24), we obtain the following cross relations for the generators

$$KA = AK, \quad EA = AE + q^{-1/2}B^* K, \quad FA = AF - q^{-3/2} BK, \quad (26)$$

$$KB = q^{-1}BK, \quad EB = qBE + q^{1/2}(1 - (1 + q^2)A)K, \quad FB = qBF, \quad (27)$$

$$KB^* = qB^* K, \quad EB^* = q^{-1}B^* E, \quad FB^* = q^{-1}B^* F - q^{-1/2}(1 - (1 + q^2)A)K. \quad (28)$$
For the Haar state $h$ on $\mathcal{O}(SU_q(2))$, it is known that $h(xy) = h((K^{-2}v_yK^{-2})x)$ for $x, y \in \mathcal{O}(SU_q(2))$. By a slight abuse of notation, we denote the restriction of $h$ to $\mathcal{O}(S_q^2)$ also by $h$. Then $h$ is the unique $\mathcal{U}_q(su_2)$-invariant state on the left $\mathcal{U}_q(su_2)$-module $\ast$-algebra $\mathcal{O}(S_q^2)$. Since $\mathcal{O}(S_q^2)$ is right $K$-invariant, we have

$$h(xy) = h(\sigma(y)x) \text{ with } \sigma(y) = K^{-2}v_y, \quad x, y \in \mathcal{O}(S_q^2).$$

(29)

3 Dirac operator and differential calculus on $S_q^2$

First we introduce linear subspaces $V^+$, $V^-$ and $V$ of the $\ast$-algebra $\mathcal{O}(SU_q(2))$ which are left $\mathcal{O}(S_q^2)$-modules and left $\mathcal{U}_q(su_2)$-modules. Set

$$V^+ = \text{Lin}\{v_{l+1/2,k}^{l+1/2} ; k = -(l+1/2), \ldots, l+1/2, l \in \mathbb{N}_0\},$$

$$V^- = \text{Lin}\{v_{l-1/2,k}^{l+1/2} ; k = -(l+1/2), \ldots, l+1/2, l \in \mathbb{N}_0\}$$

and $V = V^+ \oplus V^-$, where $v_{\pm 1/2,k}^{l+1/2} = [2l+1]_q q^{1/2} t_{\pm 1/2,k}^{l+1/2}$ are the matrix elements from Section 2. For brevity, we shall frequently write $fv^\pm$ for $f \circ v^\pm$, where $v^\pm \in V^\pm$.

Lemma 3.1

(i) $V^+ = \mathcal{O}(S_q^2)c + \mathcal{O}(S_q^2)d = c \mathcal{O}(S_q^2) + d \mathcal{O}(S_q^2)$,

$V^- = \mathcal{O}(S_q^2)a + \mathcal{O}(S_q^2)b = a \mathcal{O}(S_q^2) + b \mathcal{O}(S_q^2)$.

(ii) $f \circ V^\pm \subseteq V^\pm$ for $f \in \mathcal{U}_q(su_2)$.

(iii) $R_E(\mathcal{O}(S_q^2)) \subseteq \mathcal{O}(S_q^2)a^2 + \mathcal{O}(S_q^2)ab + \mathcal{O}(S_q^2)b^2$

$= a^2 \mathcal{O}(S_q^2) + ab \mathcal{O}(S_q^2) + b^2 \mathcal{O}(S_q^2)$,

$R_F(\mathcal{O}(S_q^2)) \subseteq \mathcal{O}(S_q^2)c^2 + \mathcal{O}(S_q^2)cd + \mathcal{O}(S_q^2)d^2$

$= c^2 \mathcal{O}(S_q^2) + cd \mathcal{O}(S_q^2) + d^2 \mathcal{O}(S_q^2)$.

(iv) $R_E(x)R_F(y) \in \mathcal{O}(S_q^2)$, $R_F(x)R_E(y) \in \mathcal{O}(S_q^2)$ for $x, y \in \mathcal{O}(S_q^2)$.

Proof. (i) The above formulas for the matrix elements imply that $t_{1/2,k}^{l+1/2} \subseteq \mathcal{O}(S_q^2)d$ and $t_{-1/2,-k}^{l+1/2} \subseteq \mathcal{O}(S_q^2)c$ for $k > 0$, so $V^+ \subseteq \mathcal{O}(S_q^2)c + \mathcal{O}(S_q^2)d$. Conversely, from the form of the Clebsch-Gordan coefficients (see e.g. [KS, Section 3.4]) and the definition of $t_{1/2,k}^{l+1/2}$, it follows that $t_{0,k}^{l}c \subseteq V^+$ and $t_{0,k}^{l}d \subseteq V^+$ for all
l ∈ \mathbb{N}_0$. Thus, $V^+ = \mathcal{O}(S_2^q)c + \mathcal{O}(S_2^q)d$. The other relations are proved by a similar argumentation.

(ii) follows from (i) and Equations (11) and (11), or from the formulas (20).

(iii) From the definition of the dual pairing (6) and the formulas (23)–(25), we obtain for the right action of $E$ and $F$ on the generators $A, B, B^*$

$$R_E(B) = -q^{1/2}a^2, \quad R_E(B^*) = q^{-3/2}b^2, \quad R_E(A) = q^{-3/2}ba,$$

$$R_F(B) = -q^{3/2}c^2, \quad R_F(B^*) = q^{1/2}d^2, \quad R_F(A) = q^{1/2}dc.$$

Using Equation (9) and the $K$-invariance (22) of $\mathcal{O}(S_2^q)$, assertion (iii) follows easily by induction on powers of generators.

(iv) follows at once from (iii) because pairwise products of the elements $a^2, ab, b^2$ with $c^2, cd, d^2$ are contained in $\mathcal{O}(S_2^q)$.

From Lemma 3.1(i) and (ii), we conclude that $V^+$ and $V^-$ are invariant under the action of the left crossed product algebra $\mathcal{O}(S_2^q) \rtimes \mathcal{U}_q(su_2)$. It can be shown that the corresponding modules are irreducible $*$-representations of the cross product $*$-algebra $\mathcal{O}(S_2^q) \rtimes \mathcal{U}_q(su_2)$.

Now we define the Dirac operator $D$ on $S_2^q$. In order to do so, we essentially use the action of $\mathcal{U}_q(su_2)$ on $\mathcal{O}(SU_q(2))$. Since $\alpha^{l+1/2} = [l + 1]_q$, it follows from (19) that

$$R_E(v_{l+1/2,k}) = -[l+1]_q v_{-1/2,k}, \quad R_F(v_{l+1/2,k}) = -[l+1]_q v_{1/2,k}$$

for $k = -(l + \frac{1}{2}), \ldots, l + \frac{1}{2}$. Hence we have $R_E : V^+ \to V^-$ and $R_F = (R_E)^* : V^- \to V^+$. Therefore, the operator $D$, given by the matrix

$$D := \begin{pmatrix} 0 & R_F \\ R_E & 0 \end{pmatrix}$$

on $V = V^+ \oplus V^-$, maps $V$ into itself and is hermitian. Moreover, by (30), $\psi_{l+1,k}^+ := \frac{1}{\sqrt{2}}(v_{l+1/2,k}, \epsilon v_{l+1/2,k})^t, k = -(l + \frac{1}{2}), \ldots, l + \frac{1}{2}$, is an orthonormal sequence of eigenvectors of the operator $D$ with respect to the eigenvalues $-\epsilon[l + 1]_q$, where $\epsilon = \pm 1$ and $l \in \mathbb{N}_0$. Let $\mathcal{K}$ denote the closure of the subspace $V$ in the Hilbert space $L^2(SU_q(2))$. By the preceding, the closure of $D$, denoted again by $D$, is a self-adjoint operator on $\mathcal{K}$, has a bounded inverse and $|D|^{-1/2}$ is of trace class for all $z \in \mathbb{C}, \text{Re } z > 0$. The vectors $\varphi_{l+1,k}^+ := (v_{l+1/2,k}, 0)^t, \varphi_{l+1,k}^- := (0, v_{l+1/2,k})^t, \ q = -(l + \frac{1}{2}), \ldots, l + \frac{1}{2}, l \in \mathbb{N}_0$, form an orthonormal basis of the Hilbert space $\mathcal{K}$ consisting of eigenvectors of the self-adjoint operator $|D| = (D^*D)^{1/2}$ with respect to the eigenvalues $|l + 1]_q.$
Let \( x \in \mathcal{O}(S_q^2) \). Using (9), (13) and (22), we compute

\[
R_F(xv^-) = -q xv^-sF = -q(xsF)(v^-sK) - q(xsK^{-1})(v^-sF) = q^{-1/2} R_F(x)v^- + xR_F(v^-),
\]

so \( [R_F, x]v^- = q^{-1/2} R_F(x)v^- \) for \( v^- \in V^- \). Notice that \( R_F(x)v^- \in V^+ \) by Lemma 3.1 and Equations (15)–(18). Similarly, we have for all \( v^+ \in V^+ \) the identity \( [R_E, x]v^+ = q^{1/2} R_E(x)v^+ \in V^- \). Thus, for all \( (v^+, v^-) \in V^+ \oplus V^- \), we obtain \( [D, x](v^+, v^-) = (q^{-1/2} R_F(x)v^-, q^{1/2} R_E(x)v^+) \), that is,

\[
[D, x] = \begin{pmatrix} 0 & q^{-1/2} R_F(x) \\ q^{1/2} R_E(x) & 0 \end{pmatrix},
\]

where the elements \( R_E(x) \) and \( R_F(x) \) of \( \mathcal{O}(SU_q(2)) \) act by left multiplication on \( V^- \) and \( V^+ \), respectively. In particular, \([D, x]\) is a bounded operator on the Hilbert space \( \mathcal{K} \) for all \( x \in \mathcal{O}(S_q^2) \). Since \( D \) is a self-adjoint operator with compact resolvent as noted above, \((D, \mathcal{O}(S_q^2), \mathcal{K})\) is a spectral triple [1,FGV].

We turn now to the reality structure. Let \( J \) denote the involution of the \(*\)-algebra \( \mathcal{O}(SU_q(2)) \). By Lemma 3.1, \( J \) maps \( V \) onto \( V^\perp \). More precisely, from the formulas for the elements \( t^{\pm 1/2}_{\pm 1/2,j} \) listed in Section 2 and the fact that the polynomials \( p_k(\cdot;\cdot,\cdot) \) have real coefficients it follows that

\[
(t^{\pm 1/2}_{\pm 1/2,j})^* = (-q)^{i\pm 1/2} t^{\mp 1/2}_{\mp 1/2,-j}.
\]

Define an anti-linear operator \( J_0 := iKR_{K^{-1}}J \) on \( V \), that is,

\[
J_0(v) = iKsv^sK, \quad v \in V.
\]

By (8), \( J_0^2(v) = iKsv^sK = Ksv^sK = v \) for \( v \in V \), and so \( J_0^2 = I \). The same arguments imply

\[
J_0y^* J_0^{-1}v = v(Ksv^sK), \quad y \in \mathcal{O}(S_q^2), \quad v \in V,
\]

so that \( xJ_0y^* J_0^{-1}v = J_0y^* J_0^{-1}xv = xv(Ksv^sK) \) for all \( x, y \in \mathcal{O}(S_q^2) \). Hence \([x, J_0y^* J_0^{-1}] = 0\). Also, by (31) and (33), \([D, x], J_0y^* J_0^{-1}\) = 0 which means that \( J_0 \) satisfies the order-one condition.

From (19), (20) and (32), we get

\[
J_0(t^{\mp 1/2}_{\pm 1/2,j}) = i[2l + 1]q^{\mp 1/2}(-q)^{j\mp 1/2}Kv^{l+1/2}_{\mp 1/2,-j}K = \pm i^{2j}q^{l+1/2}Kv^{l+1/2}_{\mp 1/2,-j}.
\]
for \( j = -(l + 1/2), \ldots, l + 1/2 \) and \( l \in \mathbb{N}_0 \). Remind that \((-1)^{j+1/2}\) is real and \(2j\) is an odd integer. Since \( \{v_{\pm 1/2,j}\} \) forms an orthonormal basis of the Hilbert space \( \mathcal{K} \), the operator \( J_0 \) is anti-unitary on \( \mathcal{K} \). It is easy to check that \( J_0 \) anti-commutes with the Dirac operator \( D \).

Recall that the spectral triple \((D, \mathcal{O}(S^2_q), \mathcal{K})\) is called even [C2, Definition 1], if there is a grading operator \( \gamma \) on \( \mathcal{K} \) such that \( \gamma^2 = I \) and \( \gamma x = x \gamma \) for all \( x \in \mathcal{O}(S^2_q) \). With \( \gamma \) defined by \( \gamma v^\pm = (\pm 1)v^\pm \) for all \( v^\pm \in V^\pm \), the spectral triple \((D, \mathcal{O}(S^2_q), \mathcal{K})\) is even. To obtain a real structure on \((D, \mathcal{O}(S^2_q), \mathcal{K})\), we change \( J_0 \) slightly and set \( J := \gamma J_0 \). Then \( J \) is an anti-unitary operator, \( J^2 = -I \) and \( J \mathcal{J} \) is even. Moreover, \([D, J] = 0\) and \([D, \mathcal{J}] = 0\). This means that \( \mathcal{J} \) is a real structure on the even spectral triple \((D, \mathcal{O}(S^2_q), \mathcal{K})\) according to [C2, Definition 3].

Next we construct a first order differential calculus on \( S^2_q \). Let \( \Gamma = \mathcal{L}(V) \) be the algebra of linear operators on \( V \). By Lemma [5.1], \( \Gamma \) is an \( \mathcal{O}(S^2_q)\)-bimodule with respect to the left and right multiplication of the algebra \( \mathcal{O}(S^2_q) \). Defining \( d : \mathcal{O}(S^2_q) \to \Gamma = \mathcal{L}(V) \) by \( dx := i[D, x] \), we obtain a first order differential \( * \)-calculus \((\Gamma, d)\) on the algebra \( \mathcal{O}(S^2_q) \). Since the right actions \( R_E \) and \( R_F \) commute with the left action of \( f \in \mathcal{U}_q(su_2) \) on \( \mathcal{O}(SU_q(2)) \), the operator \( D \) and so the differentiation \( d \) commute with the \( \mathcal{U}_q(su_2) \)-action. The latter means that the differential calculus \((\Gamma, d)\) is covariant with respect to the left \( \mathcal{U}_q(su_2) \)-action.

We show that \((\Gamma, d)\) has the quantum tangent space \( \mathcal{T} = \text{Lin}\{E, F\} \). Indeed, let \( x_j, y_j, j = 1, \ldots, n \), be such that \( \sum_j dx_j y_j = 0 \). Then 

\[
\sum_j dx_j y_j (v^+, v^-)^t = \sum_j i(q^{-1/2} R_F(x_j)y_j v^-, q^{1/2} R_E(x_j)y_j v^+)^t = 0
\]

for all \( v^\pm \in V^\pm \) or, equivalently, \( \sum_j R_E(x_j)y_j = \sum_j R_F(x_j)y_j = 0 \). Hence \( \sum_j x_j(1) \otimes x_j(2)y_j \in \mathcal{T}^\perp \otimes \mathcal{O}(SU_q(2)) \), where \( \mathcal{T}^\perp \) is the \( \mathcal{O}(S^2_q) \)-module generated by \( \{x \in \mathcal{O}(S^2_q) ; \langle E, x \rangle = \langle F, x \rangle = 0\} \). This shows that \( \mathcal{T} = \text{Lin}\{E, F\} \) is the quantum tangent space of \((\Gamma, d)\). Since \( \dim \mathcal{T} = 2 \), \((\Gamma, d)\) is 2-dimensional.

The universal differential calculus \((\Gamma_u^\wedge, d)\) associated with \((\Gamma, d)\) is defined by \( \Gamma_u^\wedge = \bigoplus_{k=0}^{\infty} \Gamma^{\otimes k}/\mathcal{I} \), where \( \Gamma^{\otimes k} = \Gamma \otimes \mathcal{O}(S^2_q) \otimes \cdots \otimes \mathcal{O}(S^2_q) \) \( (k\)-fold tensor product), \( \Gamma^{\otimes 0} = \mathcal{O}(S^2_q) \) and \( \mathcal{I} \) denotes the 2-sided ideal of the tensor algebra \( \bigoplus_{k=0}^{\infty} \Gamma^{\otimes k} \) generated by the subset \( \{ \sum_i dx_i \otimes dy_i ; \sum_i dx_i y_i = 0 \} \). The product of the algebra \( \Gamma^\wedge \) is denoted by \( \wedge \). Let \((\Gamma^\wedge, d)\) be the higher order calculus on \( \mathcal{O}(S^2_q) \) obtained from \((\Gamma_u^\wedge, d)\) by setting \( \Gamma_u^{\wedge k} = 0 \) for \( k \geq 3 \). (It can be shown that \( \Gamma_u^{\wedge k} = 0 \) for \( k \geq 3 \) but we do not need this here.) A covariant differential calculus \((\Gamma_p^\wedge, d_p)\) on \( \mathcal{O}(S^2_q) \) was constructed by P. Podleś in [P2]. The following lemma is
proved in the appendix.

Lemma 3.2  
(i) \((\Gamma, d)\) is the unique 2-dimensional covariant first order differential calculus on \(O(S^2_q)\) such that \(\{dA, dB, dB^*\}\) generate the right (left) \(O(S^2_q)\)-module \(\Gamma\).

(ii) The differential calculi \((\Gamma^\wedge, d)\) and \((\Gamma^\flat, d_P)\) are isomorphic.

The next theorem collects some properties of the Dirac operator \(D\).

Theorem 3.3  
(i) \(D\) is a self-adjoint operator with discrete spectrum consisting of eigenvalues \(\epsilon[n]_q\) with multiplicities \(2n\), where \(\epsilon = \pm 1\) and \(n \in \mathbb{N}\). In particular, \(|D|^{-1}\) is of trace class for \(z \in \mathbb{C}, \text{Re } z > 0\).

(ii) \(D\) commutes with the left action of \(U_q(su_2)\) on \(V\).

(iii) Equipped with the grading operator \(\gamma\), the spectral triple \((D, O(S^2_q), K)\) is even. The operator \(\mathcal{J}\) defines a real structure on \((D, O(S^2_q), K)\) according to [C2, Definition 3].

(iv) There is a unique 2-dimensional covariant first order differential calculus \((\Gamma, d)\) on \(O(S^2_q)\) such that \(\{dA, dB, dB^*\}\) generate the left \(O(S^2_q)\)-module \(\Gamma\). This calculus is given by given by \(dx := i[D, x], x \in O(S^2_q)\).

We close this section by giving a purely algebraic construction of the differential calculus \((\Gamma, d)\). Since the subset \(S = \{b^k c^n \mid k, n \in \mathbb{N}_0\}\) is a left and right Ore set and the algebra \(O(SU_q(2))\) has no zero divisors, the localization algebra \(\hat{O}(SU_q(2))\) of \(O(SU_q(2))\) at \(S\) exists. Then \(O(SU_q(2))\) is a *-subalgebra of \(\hat{O}(SU_q(2))\) and \(b\) and \(c\) are invertible in the larger algebra \(\hat{O}(SU_q(2))\). The crucial observation is Equation (34) in the following lemma.

Lemma 3.4  
(i) The mappings \(R_F, R_E : O(S^2_q) \to O(SU_q(2))\) are derivations, that is, \(R_f(xy) = xR_f(y) + R_f(x)y\) for \(f = F, E\) and \(x, y \in O(S^2_q)\).

(ii) For all \(x \in O(S^2_q)\),

\[
R_F(x) = [q^{1/2} \lambda^{-1} db^{-1}, x], \quad R_E(x) = -[q^{-1/2} \lambda^{-1} ac^{-1}, x],
\tag{34}
\]

where the commutators are taken in the algebra \(\hat{O}(SU_q(2))\).
Lemma 4.2

3.1(iv).
Proof. We begin with a couple of preliminary lemmas.

Note that the products of (34) are also derivations. Hence it suffices to check that Equation (34) holds for the generators $x = A, B, B^*$. We omit the details of this easy computation. □

From relations (31) and (34), it follows that, for all $x \in O(S^2_q)$,

$$dx = i[D, x] = i\lambda^{-1} \begin{pmatrix} 0 & [db^{-1}, x] \\ -[ac^{-1}, x] & 0 \end{pmatrix}. $$

The reason for the somehow surprising identity (34) is the following fact [SW2]: The elements $q^{1/2}K^{-1}E + K^{-2}c^{-1}a$ and $q^{1/2}KF^{-1} - qdb^{-1}K^{-2}$ of the right crossed product algebra $U_q(su_2) \ltimes O(SU_q(2))$ commute with all elements of $O(SU_q(2))$. By the $K$-invariance (22), this implies (34).

4 The twisted cyclic cocycle on $S^2_q$

We begin with a couple of preliminary lemmas.

Lemma 4.1 For all $x, y \in O(S^2_q)$, we have

$$h(R_F(x)R_E(y)) = q^2h(R_E(x)R_F(y)). \quad (35)$$

Proof. Let $x, y \in O(S^2_q)$. By (8) and (14), $R_F(x^*) = x^*E = -qR_E(x^*)$. The $K$-invariance (22) of $O(S^2_q)$ implies $(R_F, R_E - R_E, R_F)(y) = 0$. Hence

$$h(R_F(x)R_E(y)) = (R_E(y), R_F(x)^*) = -q(R_E(y), R_E(x^*))$$
$$= -q(R_F, R_E(y), x^*) = -q(R_F(y), R_F(x^*))$$
$$= q^2(R_F(y), R_E(x)^*) = q^2h(R_E(x)R_F(y)). \quad \square$$

For $x_0, x_1, x_2 \in O(S^2_q)$, we define our cocycle $\tau$ by

$$\tau(x_0, x_1, x_2) = h(x_0(R_F(x_1)R_E(x_2) - q^2R_E(x_1)R_F(x_2))). \quad (36)$$

Note that the products $R_F(x_1)R_E(x_2)$ and $R_E(x_1)R_F(x_2)$ are in $O(S^2_q)$ by Lemma 3.1(iv).

Lemma 4.2 Let $\sigma$ denote the algebra automorphism of $O(S^2_q)$ given by $\sigma(x) = K^{-2p}x, x \in O(S^2_q)$. Then $\tau$ is a non-trivial $\sigma$-twisted cyclic 2-cocycle on $O(S^2_q)$.
Proof. First, consider \( \tau_1(x_0, x_1, x_2) := h(x_0 R_F(x_1) R_E(x_2)) \) and \( \tau_2(x_0, x_1, x_2) := h(x_0 R_E(x_1) R_F(x_2)) \). Recall that \( R_F \) and \( R_E \) act as derivations on \( O(S_q^2) \) by Lemma 3.4 and \( h \) satisfies condition (29). Applying the boundary operator \( b_\sigma \) to \( \tau_1 \) and \( \tau_2 \) and using the Leibniz rule for the derivations \( R_E \) and \( R_F \), one sees that the sum in (2) telescopes to zero. Hence \( b_\sigma \tau = 0 \). Next,
\[
\tau(x_0, x_1, x_2) = h(R_F(x_0 x_1) R_E(x_2)) - q^2 R_E(x_0 x_1) R_F(x_2)) \\
- h(R_F(x_0) R_E(x_1 x_2) - q^2 R_E(x_0) R_F(x_1 x_2)) \\
+ h(R_F(x_0) R_E(x_1) x_2 - q^2 R_E(x_0) R_F(x_1) x_2) = \tau(\sigma(x_2), x_0, x_1),
\]
where the first equality is an application of the Leibniz rule and the second equation follows from Lemma 4.1 and condition (29). Thus \( \tau = \lambda_\sigma \tau \).

We prove that \( \tau \) is non-trivial. Consider the element
\[
\eta := B^* \otimes A \otimes B + q^2 B \otimes B^* \otimes A + q^2 A \otimes B \otimes B^* - q^{-2} B^* \otimes B \otimes A \\
- q^{-2} A \otimes B^* \otimes B - B \otimes A \otimes B^* + (q^6 - q^{-2}) A \otimes A \otimes A
\]
of the tensor product \( O(S_q^2)^{\otimes 3} \). By (11) and the definition of \( \sigma \), we have \( \sigma(A) = A \), \( \sigma(B) = q^2 B \) and \( \sigma(B^*) = q^{-2} B^* \). Since \( \tau = \lambda_\sigma \tau \), it follows that
\[
\tau(\eta) = 3 \tau(B^*, A, B) - 3q^{-2} \tau(B^*, B, A) + (q^6 - q^{-2}) \tau(A, A, A).
\]

From the definition of \( \tau \), we obtain
\[
\tau(B^*, A, B) = (q^2 - q^{-4}) h(A^3 - A^2) + q^{-2} h(A^2 - A), \\
\tau(B^*, B, A) = (q^4 - q^{-2}) h(A^3 - A^2) - q^2 h(A^2 - A), \\
\tau(A, A, A) = (q^{-2} - q^4) h(A^3) - (q^{-2} - q^2) h(A^2).
\]
Inserting the values \( h(A^j) = (1-q^2)/(1-q^{2j+2}) \) and summing up gives \( \tau(\eta) = -1 \).

On the other hand, one computes \( b_\sigma(\eta) = 2(1-q^2) A \otimes A \) by using algebra relations (21). Note that Equation (3) and \( \sigma(A) = A \) imply \( \tau'(A, A) = 0 \) for any \( \sigma \)-twisted cyclic 1-cocycle \( \tau' \). If there were a \( \sigma \)-twisted cyclic 1-cocycle \( \tau' \) such that \( \tau = b_\sigma(\tau') \), we would get
\[
\tau(\eta) = (b_\sigma \tau')(\eta) = \tau'(b_\sigma \eta) = 2(q^4 - q^{-2}) \tau'(A, A) = 0,
\]
a contradiction. Thus \( \tau \) is non-trivial. \( \Box \)

The left action of \( f \in U_q(\mathfrak{su}_2) \) on cycles \( \eta = \sum_k x_0 \otimes \cdots \otimes x_n \in O(S_q^2)^{\otimes n+1} \) is defined by
\[
f \circ \eta = \sum_k f(1)^k x_0 \otimes \cdots \otimes f(n+1)^k x_n.
\]
Pairing a 2-cycle \( \eta \) with \( \tau \) gives \( \tau(f \cdot \eta) = \varepsilon(f) \tau(\eta) \) since the right actions \( R_E \) and \( R_F \) commute with the left action of \( \mathcal{U}_q(\mathfrak{su}_2) \) on \( \mathcal{O}(\mathfrak{su}_q(2)) \) and \( h \) is \( \mathcal{U}_q(\mathfrak{su}_2) \)-invariant. Hence \( \tau \) is \( \mathcal{U}_q(\mathfrak{su}_2) \)-invariant.

We next describe \( \tau \) analytically. Let \( \mathcal{K}^\pm \) be the closure of \( V^\pm \) in the Hilbert space \( \mathcal{L}^2(\mathfrak{su}_q(2)) \) and, for \( z \in \mathbb{C}, \Re z > 2 \), let \( \zeta(z) \) denote the holomorphic function given by

\[
\zeta(z) = \sum_{n=1}^{\infty} [n]^{-z} [2n]_q.
\]

The following lemma is a slight modification of Theorem 5.7 in [SW1].

**Lemma 4.3** Let \( z \in \mathbb{C} \) and \( \Re z > 2 \). For any \( x \in \mathcal{O}(S^2_q) \), the closure of the operator \( K^2|D|^{-z}x \) restricted to the Hilbert spaces \( \mathcal{K}^\pm \) is of trace class and

\[
h(x) = \zeta(z)^{-1} \text{Tr}_{\mathcal{K}^\pm} K^2|D|^{-z}x.
\]

**Proof.** The operators \( |D|^{-z} \) and \( K \) act on the orthonormal basis \( \{v^{l+1/2}_{\pm 1/2,k}\} \) of the Hilbert spaces \( \mathcal{K}^\pm \) by \( |D|^{-z}v^{l+1/2}_{\pm 1/2,k} = [l + 1]q^{-z}v^{l+1/2}_{\pm 1/2,k} \) and \( Kv^{l+1/2}_{\pm 1/2,k} = q^kv^{l+1/2}_{\pm 1/2,k} \), respectively. Since \( x \in \mathcal{O}(S^2_q) \) acts as a bounded operator on \( \mathcal{K}^\pm \), \( K^2|D|^{-z}x \) is of trace class. Thus, \( h_z(x) := \text{Tr}_{\mathcal{K}^\pm} K^2|D|^{-z}x \) is well defined for \( x \in \mathcal{O}(S^2_q) \).

We show that \( h_{2z} \) is a \( \mathcal{U}_q(\mathfrak{su}_2) \)-invariant linear functional on \( \mathcal{O}(S^2_q) \), that is, \( h_{2z}(f \cdot x) = \varepsilon(f) h_{2z}(x) \) for \( f \in \mathcal{U}_q(\mathfrak{su}_2) \) and \( x \in \mathcal{O}(S^2_q) \). In order to do so, we essentially use the fact that \( V^\pm \) are left modules of the left crossed product algebra \( \mathcal{O}(S^2_q) \rtimes \mathcal{U}_q(\mathfrak{su}_2) \). It suffices to verify the invariance for the generators \( f = E, F, K, K^{-1} \). We carry out the proof for \( f = E \) and show that \( h_{2z}(E \cdot x) = 0 \).

From the relations (26)–(28) of the cross product algebra \( \mathcal{O}(S^2_q) \rtimes \mathcal{U}_q(\mathfrak{su}_2) \), it follows that for any \( x \in \mathcal{O}(S^2_q) \) there are elements \( x, x', x'' \in \mathcal{O}(S^2_q) \) such that \( Kx = yK \) and \( xe = Ex' + Kx'' \). Thus, since \( Ev^{l+1/2}_{\pm 1/2,k} = \alpha_k^{l+1/2}v^{l+1/2}_{\pm 1/2,k+1} \) and \( |\alpha_k^{l+1/2}| \leq \text{const} q^l \), the operators \( K^2|D|^{-2z}ExK^{-1}, |D|^{-z}KxE \) and \( KE|D|^{-z} \) are of trace class. Since \( |D|^{-z} \) commutes with \( E \) and \( K \), we compute

\[
\text{Tr}_{\mathcal{K}^\pm}(K^2|D|^{-2z}ExK^{-1}) = \text{Tr}_{\mathcal{K}^\pm}(|D|^{-2z}K^2ExK^{-1}) = q^{2} \text{Tr}_{\mathcal{K}^\pm}(KE|D|^{-z})|D|^{-zy}y
=q^{2} \text{Tr}_{\mathcal{K}^\pm}(|D|^{-z}yKE|D|^{-z})=q^{2} \text{Tr}_{\mathcal{K}^\pm}(|D|^{-z}y^2KE)|D|^{-z}y
=q^{2} \text{Tr}_{\mathcal{K}^\pm}(|D|^{-z}KKE)|D|^{-z}y
=q^{2} \text{Tr}_{\mathcal{K}^\pm}(|D|^{-2z}ExE)=q^{2} \text{Tr}_{\mathcal{K}^\pm}(K^2|D|^{-2z}K^{-1}x),
\]

where all operators in parentheses are of trace class. (Strictly speaking, one has to take the closures of these operators.) In the left crossed product algebra we have
\[ f^v x = f(1) x S(f(2)) \] and so \( E^v x = E x K^{-1} - q K^{-1} x E \). Hence \( h_{2z}(E^v x) = 0 \) by (38).

The functional \( h_{2z} \) on \( \mathcal{O}(S_q^2) \) is \( \mathcal{U}_q(\mathfrak{su}_2) \)-invariant. Since

\[
\begin{align*}
\gamma_{x_0} &= \left( \begin{array}{cc} 1 & 0 \\ 0 & -q^2 \end{array} \right) \\
\end{align*}
\]

it follows that \( (2z)^{-1} h_{2z} \) is the invariant state \( h \) on \( \mathcal{O}(S_q^2) \). Finally, \( \zeta(z) h(x) \) and \( h_z(x) \) are holomorphic functions for \( z \in \mathbb{C}, \text{Re} \, z > 2 \). Since they are equal for \( \text{Re} \, z > 4 \) as just shown, they coincide also for \( \text{Re} \, z > 2 \).

Using Lemma 4.3, we now express the cocycle \( \tau \) defined by (36) in terms of our Dirac operator \( D \). Let

\[
\gamma_q = \left( \begin{array}{cc} 1 & 0 \\ 0 & -q^2 \end{array} \right)
\]

be the “grading” operator on \( K = K^+ \oplus K^- \). Indeed, using formulas (31), (36) and (37), we obtain

\[
\begin{align*}
\text{Tr}_K \gamma_q K^2 [D]^{-z} x_0 [D, x_1] [D, x_2] &= \text{Tr}_K K^2 [D]^{-z} x_0 R_F(x_1) R_E(x_2) - q^2 \text{Tr}_K K^{-2} [D]^{-z} x_0 R_E(x_1) R_F(x_2) \\
&= \zeta(z) h(x_0 R_F(x_1) R_E(x_2) - q^2 x_0 R_E(x_1) R_F(x_2)) = \zeta(z) \tau(x_0, x_1, x_2) \tag{39}
\end{align*}
\]

for \( z \in \mathbb{C}, \text{Re} \, z > 2, \) and \( x_0, x_1, x_2 \in \mathcal{O}(S_q^2) \). Using the binomial series, one can show that

\[
\zeta(z) = (q^{-1} - q)^{-1} \sum_{k=0}^{\infty} \left( \begin{array}{c} 1 - z \\ k \end{array} \right) \left( 1 - q^{2z - 2k} \right)^{-1} + \left( 1 - q^{2(z - 1)k} \right)^{-1}
\]

The right-hand side is a meromorphic function which is denoted again by \( \zeta(z) \). It has a simple pole at \( z = 2 \) with residue \( \lambda(\log q)^{-1} \). Therefore, by (39),

\[
\text{res}_{z=2} \text{Tr}_K \gamma_q K^2 [D]^{-z} x_0 [D, x_1] [D, x_2] = \lambda(\log q)^{-1} \tau(x_0, x_1, x_2).
\]

On the other hand, the cocycle \( \tau \) can also be obtained from the differential calculus \((\Gamma^\wedge, d) \cong (\Gamma_P^\wedge, d_P)\). As shown in [P2], there is a left-invariant 2-form \( \omega \neq 0 \) such that \( \Gamma^\wedge 2 = \mathcal{O}(S_q^2) \omega \) and \( x \omega = \omega x \) for \( x \in \mathcal{O}(S_q^2) \). The following remarkable result is due to I. Heckenberger. His proof will be given in the appendix.

**Lemma 4.4** The volume form \( \omega \) can be chosen such that \( \tau \) is equal to the \( \sigma \)-twisted cyclic cocycle \( \tau_{\omega, h} \).
Retaining the preceding notation, we now summarize the main results obtained in this section.

**Theorem 4.5** There is a non-trivial $\sigma$-twisted cyclic 2-cocycle $\tau$ on the algebra $\mathcal{O}(S^2_q)$ such that

$$
\tau(x_0, x_1, x_2) = h(x_0(R_F(x_1)R_E(x_2) - q^2 R_E(x_1)R_F(x_2)))
$$

$$
= \zeta(z)^{-1}\text{Tr}_K \gamma_q K^2 |D|^{-z} x_0[D,x_1][D,x_2]
$$

$$
= \lambda^{-1}(\log q) \text{res}_{z=2} \text{Tr}_K \gamma_q K^2 |D|^{-z} x_0[D,x_1][D,x_2]
$$

for $z \in \mathbb{C}$, $\text{Re} \, z > 2$, and $x_0, x_1, x_2 \in \mathcal{O}(S^2_q)$. The cocycle $\tau$ is $\mathcal{U}_q(\mathfrak{su}_2)$-invariant and coincides with the $\sigma$-twisted cyclic cocycle $\tau_{\omega, h}$ associated with the volume form $\omega$ of the differential calculus $(\Gamma, d)$.

In ”ordinary” non-commutative geometry the grading operator anti-commutes with the Dirac operator $D$ and with the anti-unitary operator implementing the real structure. This is not true for our grading operator $\gamma_q$, but $\gamma = q^{-1}\gamma_q R_{K^{-2}}$ does anti-commute with $D$ and $J$.

## 5 Appendix

In this appendix we present the proofs of Lemmas 3.2 and 4.4 by I. Heckenberger. Let us first state some general facts on covariant differential calculi (see e.g. [H]). To an (arbitrary) covariant first order differential calculus $(\Gamma, d)$ over $\mathcal{O}(S^2_q)$, one associates the left ideal $\mathcal{L} := \{b \in \mathcal{O}(S^2_q); \varepsilon(b) = \omega_R(b) = 0\}$, where the map $\omega_R : \mathcal{O}(S^2_q) \to \Gamma \otimes_{\mathcal{O}(S^2_q)} \mathcal{O}(\mathfrak{su}_2(2))$ is given by $\omega_R(b) = db(1) \otimes S(b(2))$. The left ideal $\mathcal{L}$ determines $(\Gamma, d)$ uniquely. The (right) quantum tangent space associated with $\Gamma$ is the linear subspace $\mathcal{T} := \{f \in \mathcal{O}(S^2_q)'; \langle f, 1 \rangle = \langle f, \mathcal{L} \rangle = 0\}$ of the dual vector space $\mathcal{O}(S^2_q)'$ of $\mathcal{O}(S^2_q)$. Set $\mathcal{O}(S^2_q)^+ = \{x \in \mathcal{O}(S^2_q); \varepsilon(x) = 0\}$. The cardinal number $\dim \Gamma := \dim_{\mathbb{C}} \Gamma / \Gamma \mathcal{O}(S^2_q)^+ = \dim_{\mathbb{C}} \mathcal{O}(S^2_q)^+ / \mathcal{L}$ is called the (right) dimension of $\Gamma$. If $\dim \Gamma < \infty$, then the quantum tangent space $\mathcal{T}$ also determines $(\Gamma, d)$ uniquely and $\dim \Gamma = \dim_{\mathbb{C}} \mathcal{T}$.

From now on, $(\Gamma, d)$ stands for the first order differential calculus over $\mathcal{O}(S^2_q)$ with quantum tangent space $\mathcal{T} = \text{Lin}\{E_q, F\}$ constructed in Section 3.

**Proof of Lemma 3.2** (i) Let $(\tilde{\Gamma}, d)$ be a finite dimensional covariant first order differential calculus over $\mathcal{O}(S^2_q)$ with quantum tangent space $\tilde{\mathcal{T}}$. By the right-handed version of [HK2, Corollary 5], we have $\tilde{\mathcal{T}} \mathbb{C}[K, K^{-1}] \subset \tilde{T}$. Since $\dim \tilde{\Gamma} < \infty$,
we deduce from [HK1, Theorem 6.5.1] that $\tilde{T} \subset U_q(\mathfrak{sl}_2)|_{O_q(S^2)}$. Using once more [HK2, Corollary 5], one checks that only the linear spaces Lin$\{E, F\}$, Lin$\{E, E^2\}$ and Lin$\{F, F^2\}$ are quantum tangent spaces of 2-dimensional covariant first order differential calculi over $O(S^2_q)$. However, only one of them, namely $T = \text{Lin} \{E, F\}$, is separated by the right $O(SU_q(2))$-comodule Lin$\{B, B^*, 1 - (1 + q^2)A\}$. Hence, by the right-handed version of [HK2, Lemma 7], $(\Gamma, d)$ is the unique 2-dimensional covariant first order differential calculus generated by $G := \text{Lin}\{dB, dB^*, dA\}$ as a right $O(S^2_q)$-module.

(ii) We first show that the first order differential calculi $(\Gamma_P, d_P)$ and $(\Gamma, d)$ are isomorphic. By the definition in [P2], the right $O(S^2_q)$-module $\Gamma_P$ is also generated by $G$. Since $\Gamma_P$ is the quotient of $\text{Lin} G \otimes O(S^2_q)$ by a submodule generated by a non-zero element [P2], we deduce $\dim \Gamma_P = 2$. Let $T_P$ denote the right quantum tangent space of $(\Gamma_P, d_P)$. Again by [HK2, Lemma 7], Lin$\{B, B^*, 1 - (1 + q^2)A\}$ separates $T_P$. Applying the uniqueness result from the proof of Lemma 3.2(i) shows that $(\Gamma_P, d_P)$ and $(\Gamma, d)$ are isomorphic.

By definition, $\Gamma^\wedge k = 0$ for $k \geq 3$. To complete the proof of Lemma 3.2(ii), it remains to verify that $\Gamma_P^\wedge 2$ and $\Gamma^\wedge 2$ are isomorphic as right $O(S^2_q)$-modules and right $O(SU_q(2))$-comodules. Since $\Gamma^\wedge 2$ is obtained from the universal differential calculus, $\Gamma_P^\wedge 2$ is a quotient of $\Gamma^\wedge 2$. From the definition of $\Gamma_P^\wedge 2$ in [P2], it follows that $\dim \Gamma^\wedge 2/\Gamma_P^\wedge 2 O(S^2_q)^+ = 1$. Thus it suffices to show that $\dim \Gamma^\wedge 2/\Gamma_P^\wedge 2 O(S^2_q)^+ = 1$ which follows from the right-handed version of [HK3, Proposition 3.11(iv)].

Proof of Lemma 4.4. To apply the results of [HK3], the pairing in [HK3, Subsection 2.3.5] has to be replaced by

\[ \langle \cdot, \cdot \rangle : (\mathcal{T} \otimes \mathcal{T}_0) \times (\Gamma \otimes_{O(S^2_q)} \Gamma \otimes_{O(S^2_q)} O(SU_q(2))) \rightarrow \mathbb{C}, \]
\[ \langle t \otimes s, dx \otimes dy \otimes z \rangle := \langle t \otimes s, x_{(1)} \otimes x_{(2)}y^+\rangle \varepsilon(z) = \langle ts_{(1)} \otimes s_{(2)}^+, x \otimes y \rangle \varepsilon(z), \]
\[ (40) \]

where $t \in \mathcal{T}$, $s \in \mathcal{T}_0 := \text{Lin}\{K^{-1}E, K^{-1}F\}$, $x, y \in O(S^2_q)$, $z \in O(SU_q(2))$ and $f^+ = f - \varepsilon(f)$. Set $\mathcal{T}_2 := \{\sum_i t_i \otimes s_i \in \mathcal{T} \otimes \mathcal{T}_0 ; \sum_i t_is_i \in \mathcal{T} \}$. Similarly to [HK3, Corollary 2.10], one shows that (40) induces a non-degenerate pairing

\[ \langle \cdot, \cdot \rangle : \mathcal{T}_2 \times \Gamma^\wedge 2/\Gamma_P^\wedge 2 O(S^2_q)^+ \rightarrow \mathbb{C}. \]

Observe that $\mathcal{T}_2 = \mathbb{C}t_2$ with $t_2 := q^2 F \otimes K^{-1}E - E \otimes K^{-1}F$. By (40) and the
right $K$-invariance (22) of $O(S^2_q)$,
\[
\langle \langle t_2, dx \wedge dy \rangle \rangle = (q^2 FK^{-2} \otimes K^{-1} E - EK^{-2} \otimes K^{-1} F, x \otimes y) \\
= (F \otimes E - q^2 E \otimes F, x \otimes y)
\]
for all $x, y \in O(S^2_q)$. For notational convenience, we rename the generators $a, b, c$ and $d$ of $O(SU_q(2))$ by $u_{11}, u_{12}, u_{21}$ and $u_{22}$, respectively. Set
\[
(p_{ij})_{i,j=1,2} := (S(u^i_i)u^j_j)_{i,j=1,2} = \begin{pmatrix} A & B^* \\ B & 1-q^2 A \end{pmatrix}, \quad \omega := \sum_{i,j,k} q^{2-2i} dp_{ij} \wedge dp_{jk} p_{ki}.
\]
As $\Delta(p_{ij}) = \sum_k p_{kl} \otimes S(u^k_l)u^j_j$ and the calculus $\Gamma(\delta d)$ is covariant, the 2-form $\omega$ is right-coinvariant. Using Equation (41) and $\varepsilon(p_{ij}) = \delta_i \delta_j$, one computes
\[
\langle \langle t_2, \omega \rangle \rangle = \sum_j \langle \langle t_2, q^{-2} dp_{2j} \wedge dp_{j2} \rangle \rangle = \sum_j \langle F \otimes E - q^2 E \otimes F, q^{-2} p_{2j} \otimes p_{j2} \rangle = 1.
\]
In particular, $\omega \neq 0$. By the definition in [P2], the right $O(S^2_q)$-module $\Gamma^{\wedge 2}_p \cong \Gamma^{\wedge 2}$ is free and generated by a non-zero right-coinvariant central element. Consequently, $\Gamma^{\wedge 2} = \omega O(S^2_q)$. Moreover, $\langle \langle t_2, \omega u_{(1)} \rangle \rangle u_{(2)} = \langle \langle t_2, \omega \rangle \rangle \varepsilon(u_{(1)}) u_{(2)} = u$ for all $u \in O(S^2_q)$. Let $\rho = \omega u \in \Gamma^{\wedge 2}$. Then, by the last relation and the coinvariance of $\omega$, $\rho = \omega \langle \langle t_2, \omega u_{(1)} \rangle \rangle u_{(2)} = \omega \langle \langle t_2, \rho_{(1)} \rangle \rangle \rho_{(2)}$. This identity and Equation (41) imply
\[
xdy \wedge dz = x\omega \langle \langle t_2, dy_{(1)} \wedge dz_{(1)} \rangle \rangle y_{(2)} z_{(2)} \\
= x\omega (F \otimes E - q^2 E \otimes F, y_{(1)} \otimes z_{(1)}) y_{(2)} z_{(2)} \\
= x\left(R_F(y) R_E(z) - q^2 R_E(y) R_F(z)\right) \omega,
\]
which proves Lemma 4.4.

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