New Near Horizon Limit in Kerr/CFT

Yoshinori Matsuo\textsuperscript{a} and Tatsuma Nishioka\textsuperscript{b}

\textsuperscript{a} Harish-Chandra Research Institute, Chhatnag Road, Jhusi, Allahabad 211 019, India
\textsuperscript{b} Department of Physics, Princeton University, Princeton, NJ 08544, USA

Abstract

The extremal Kerr black hole with the angular momentum $J$ is conjectured to be dual to CFT with central charges $c_L = c_R = 12J$. However, the central charge in the right sector remains to be explicitly derived so far. In order to investigate this issue, we introduce new near horizon limits of (near) extremal Kerr and five-dimensional Myers-Perry black holes. We obtain Virasoro algebras as asymptotic symmetries and calculate the central charges associated with them. One of them is equivalent to that of the previous studies, and the other is non-zero, but still the order of near extremal parameter. Redefining the algebras to take the standard form, we obtain a finite value as expected by the Kerr/CFT correspondence.

\textsuperscript{†}ymatsuo@hri.res.in
\textsuperscript{‡}nishioka@princeton.edu
1 Introduction and overview

The origin of the black hole entropy has been of great interest in the past few decades and remains to be fully understood. In string theory, some of black holes and branes can be described in terms of the microscopic degrees of freedom, and the entropy is reproduced by counting their microstates. In the case of anti-de Sitter (AdS) spacetime, these microstates are described by the conformal field theory (CFT). In this case, a lot of information can be obtained by using symmetries. In the case of AdS$_3$, the Bekenstein-Hawking entropy of the BTZ black hole was accounted without specifying details of CFT. The asymptotic symmetry of the geometry is identified with the conformal symmetry of the dual field theory and the entropy is calculated by using the Cardy formula [1, 2].

Recently, it was conjectured that the extremal Kerr black hole in four dimensions corresponds to a two-dimensional CFT [3]. They investigated its near horizon geometry which has the $SL(2,\mathbb{R}) \times U(1)$ isometry [4]. The $U(1)$ symmetry is enhanced to an asymptotic symmetry and identified with the Virasoro algebra for the chiral half of CFT. The Bekenstein-Hawking entropy is reproduced by using the Cardy formula in the extremal case. This is called as the Kerr/CFT correspondence and this Virasoro algebra is regarded as that of left mover. This analysis is generalized to many cases [5] and it is known that another Virasoro algebra can be obtained from extending the $SL(2,\mathbb{R})$ part of the isometry [8]. This symmetry is considered as that of right mover which describes the non-extremal excitation from the extremality. Therefore, the extremal Kerr black hole with an angular momentum $J$ is expected to be dual to a non-chiral CFT with $c_L = c_R = 12J$. The recent discussion about the hidden conformal symmetry of the non-extremal Kerr black hole also provides another evidence for this correspondence [14].

However, the asymptotic Virasoro algebras can be found only in the near horizon limit of (near) extremal black holes. Moreover, the central charge for right mover is zero when it is evaluated by the conventional method. Therefore, a cut-off was introduced into the spacetime to take the near-extremal correction into account, and it turned out to be not finite but infinitesimally small [8]. Although an indirect argument is presented in [9], it is fair to say that the central charge $c_R = 12J$ has not been explicitly derived so far.

In this paper, we investigate this issue in more detail. First, we study the five-dimensional extremal Myers-Perry black hole whose near horizon geometry includes

\footnote{See also [8, 9] for related works.}
AdS$_3$ if it has only a single spin $[^4]$. In this case, we can find dual two-dimensional CFT with central charges $c_L = c_R$ similar to the one given by Brown and Henneaux $[^1]$. The CFT lives on light-cone coordinates which consist of the time and angular directions. On the other hand, we know the Kerr/CFT description where dual CFT is defined on the time and angular directions with central charges $c_L \neq c_R$. The coordinates in Kerr/CFT description are almost equivalent to the light-cone coordinates near the horizon up to a scaling factor. This scaling factor makes central charges different from each other.

To resolve this discrepancy, we reconsider the definitions of the algebras, central charges and temperatures. Usually, the Frolov-Thorne temperature is changed under the rescaling of the coordinates, while the central charges are fixed because it represents a number of degrees of freedom. If we use these definitions of the central charges and temperatures, the Cardy formula gives scale dependent quantity. However, the entropy should be independent of the scale since it counts a number of states. Therefore we need to introduce “covariant” definition of the central charge to make the Cardy formula invariant under the scaling. The covariant central charge $c^{(\text{cov})}$ is related to the scale-invariant central charge $c$ as $c^{(\text{cov})} = \beta c$, where $\beta$ is the period of the coordinate. Then, the covariant central charge depends on the scaling factor via the period $\beta$. In the Kerr/CFT correspondence, the Virasoro algebra for right mover has been studied by using the quasi-local charges $[^{13,14}]$. In these studies generators are defined in the scale covariant form, and the central charge depends on the definition of the time coordinates in the near horizon geometry. In the near horizon limit, these covariant central charge for left and right movers are not generally equal to each other, $c_L^{(\text{cov})} \neq c_R^{(\text{cov})}$. In this paper, we show that the central charges satisfy the relation $c_L = c_R$ even in Kerr/CFT, if we use the scale-invariant definition of $c$.

We also consider general five-dimensional Myers-Perry black hole with two rotations, where the near horizon geometry is AdS$_2$, not AdS$_3$. Thus there is no natural light-cone coordinates, but we introduce new coordinates which agree with the light-cone coordinates in the single rotation limit. The resulting near horizon geometry is the same as the usual one once replacing the time and angular directions with the light-cone directions. We evaluate the central charge for our new near horizon geometry on the timeslice with respect to the original time. By using this coordinates, we can evaluate the central charge associated with the Virasoro algebra for the right mover using the conventional covariant phase space method $[^{15,16}]$ and obtain the scale-invariant central charges satisfying $c_L = c_R$. We apply this analysis to the (four-dimensional) Kerr black

\footnote{The structure of AdS$_3$ sometimes appears in certain limits of extremal rotating black holes. It has been recently studied to investigate the Kerr/CFT correspondence in $[^{11,12}]$.}
hole and obtain the expected value of $c_L = c_R = 12J$.

It is worth noting that the combinations of coordinates in our new near horizon limits are the same as those in the analyses of the hidden conformal symmetry [10]. In these studies our covariant central charges are the equivalent to the invariant one. Then, the Cardy formula can be used with the Frolov-Thorne temperature and these central charges to calculate the entropy. Therefore the central charges $c_L = c_R = 12J$ is naturally obtained in the original coordinates before taking the near horizon limit.

This paper is organized as follows. In Section 2, we introduce the five-dimensional Myers-Perry black hole and its near horizon limit. Then, we review on the Kerr/CFT correspondence for left mover. In Section 3, we show the calculation of the central charge for right mover following the previous study [8]. We introduce a cut-off to evaluate the central charge. In Section 4, we consider the special case in which the near horizon geometry has the structure of AdS$_3$. Then, we discuss the relation between this special case and the general case described in Section 2 and 3. In Section 5, we introduce a new definition of the near horizon limit. By using this definition, we calculate the central charge for right mover without introducing the cut-off into the spacetime. In Section 6, we apply the new definition of the near horizon limit to the Kerr black hole, and calculate the central charge. Section 7 is devoted to the conclusion.

2 Kerr/CFT correspondence in 5D Myers-Perry black hole

In this section, we describe the five-dimensional Myers-Perry black hole and briefly review on the Kerr/CFT correspondence for it. The metric is expressed by using the Boyer-Lindquist coordinates as

$$ds^2 = \frac{\Delta}{r^2 \rho^2} \left( dt - a \sin^2 \theta \, d\phi - b \cos^2 \theta \, d\psi \right)^2$$

$$+ \frac{\sin^2 \theta}{\rho^2} \left[ (r^2 + a^2) d\phi - a \, dt \right]^2 + \frac{\cos^2 \theta}{\rho^2} \left[ (r^2 + b^2) d\psi - b \, dt \right]^2$$

$$+ \frac{1}{r^2 \rho^2} \left[ b(r^2 + a^2) \sin^2 \theta \, d\phi + a(r^2 + b^2) \cos^2 \theta \, d\psi - ab \, dt \right]^2$$

$$+ \frac{r^2 \rho^2}{\Delta^2} dr^2 + \rho^2 d\theta^2 ,$$

(2.1)

where $\Delta$ and $\rho^2$ are given by

$$\Delta = (r^2 + a^2)(r^2 + b^2) - \mu r^2 , \quad \rho^2 = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta .$$

(2.2)

In five-dimensional spacetime, we have two independent angular momenta in $\phi$- and $\psi$-directions, respectively. Hence, this geometry has three parameters $\mu, a, b$, which are
related to the ADM mass $M_{\text{ADM}}$ and two angular momenta $J_\phi$ and $J_\psi$:

$$M_{\text{ADM}} = \frac{3\pi \mu}{8G_N}, \quad J_\phi = \frac{\pi \mu a}{4G_N}, \quad J_\psi = \frac{\pi \mu b}{4G_N},$$

(2.3)

where $G_N$ is the Newton constant. The outer and inner horizons are located at the radii $r_+$ and $r_-$, which are expressed as

$$r_{\pm}^2 = \frac{1}{2} \left( \mu - a^2 - b^2 \pm \sqrt{(\mu - (a + b)^2)(\mu - (a - b)^2)} \right),$$

(2.4)

and the angular velocities on the (outer) horizon are

$$\Omega_\phi = \frac{a}{r_+^2 + a^2}, \quad \Omega_\psi = \frac{b}{r_+^2 + b^2}.$$  

(2.5)

The Hawking temperature $T_H$ and the Bekenstein-Hawking entropy $S$ are given by

$$T_H = \frac{r_+^2 - r_-^2}{2\pi \mu r_+}, \quad S = \frac{\pi^2}{2G_N \mu r_+},$$

(2.6)

respectively.

Now, we consider the near horizon limit of this geometry [4]. We focus on the near-extremal case in which the non-extremality is infinitesimally small. We define this non-extremality parameter by

$$\mu = \mu_0(1 + \epsilon^2 \hat{\mu}), \quad \mu_0 = (a + b)^2,$$

(2.7)

where the extremal condition is given by $\mu = \mu_0$ and $\hat{\mu}$ parametrizes the non-extremality. We introduce the near horizon coordinates as

$$t = \epsilon^{-1} \frac{\sqrt{\mu_0}}{2} \hat{t}, \quad r = r_0 + \epsilon \frac{\sqrt{\mu_0}}{2} \hat{r},$$

(2.8a)

$$\phi = \hat{\phi} + \frac{a}{r_0^2 + a^2} \hat{t}, \quad \psi = \hat{\psi} + \frac{b}{r_0^2 + b^2} \hat{t},$$

(2.8b)

where $r_0$ is the horizon radius in the extremal case and given by

$$r_0^2 = ab.$$  

(2.9)

Then, the near horizon limit is obtained as the $\epsilon \to 0$ limit. In this limit, the metric becomes

$$ds^2 = -\frac{\rho_0^2 \hat{\Delta}}{4} \hat{t}^2 + \frac{\rho_0^2}{4\Delta} d\hat{r}^2 + \rho_0^2 d\theta^2 + \frac{a^2 \mu_0 \sin^2 \theta}{\rho_0^2} \left( d\hat{\phi} + k_\phi \hat{r} d\hat{t} \right)^2 + \frac{b^2 \mu_0 \cos^2 \theta}{\rho_0^2} \left( d\hat{\psi} + k_\psi \hat{r} d\hat{t} \right)^2 + \frac{\mu_0^2 \rho_0^2}{\rho_0^2} \left[ \sin^2 \theta \left( d\hat{\phi} + k_\phi \hat{r} d\hat{t} \right) + \cos^2 \theta \left( d\hat{\psi} + k_\psi \hat{r} d\hat{t} \right) \right]^2,$$

(2.10)
where

$$\hat{\Delta} = (\hat{r}^2 - \hat{\mu}), \quad \rho_0 = r_0^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta, \quad (2.11)$$

$$k_{\hat{\phi}} = \frac{1}{2} \sqrt{\frac{b}{a}}, \quad k_{\hat{\psi}} = \frac{1}{2} \sqrt{\frac{a}{b}}. \quad (2.12)$$

A correspondence between this near horizon geometry and its dual CFT is studied in [18]. It was shown that the asymptotic symmetry gives the Virasoro algebra for left mover if we take an appropriate boundary condition. The asymptotic symmetry is defined as a symmetry which preserves a boundary condition:

$$\mathcal{L}_\xi (g_{\mu\nu} + \mathcal{O}(\chi_{\mu\nu})) = \mathcal{O}(\chi_{\mu\nu}), \quad (2.13)$$

where $\chi_{\mu\nu}$ defines the boundary condition. If the geometry takes the following form:

$$ds^2 = f_0(\theta) \left( -\hat{r}^2 dt^2 + \frac{d\hat{r}^2}{\hat{r}^2} \right) + \gamma_{ij}(\theta) \left( dx^i + k^i r dt \right) \left( dx^j + k^j r dt \right) + f_\theta(\theta)d\theta^2, \quad (2.14)$$

where $x^1 = \hat{\phi}$ and $x^2 = \hat{\psi}$, we can obtain the following asymptotic symmetry groups

$$\xi^{(\hat{\phi})} = \epsilon^{\hat{\phi}}(\hat{\phi}) \partial_{\hat{\phi}} - \hat{r} \epsilon^{\hat{\phi}}(\hat{\phi}) \partial_r, \quad (2.15a)$$

$$\xi^{(\hat{\psi})} = \epsilon^{\hat{\psi}}(\hat{\psi}) \partial_{\hat{\psi}} - \hat{r} \epsilon^{\hat{\psi}}(\hat{\psi}) \partial_r, \quad (2.15b)$$

by taking appropriate boundary conditions, respectively. These vectors form the Virasoro algebra as we will see below. We define $\xi^{(\hat{\phi})}_n$ and $\xi^{(\hat{\psi})}_m$ by $\xi^{(\hat{\phi})}$ and $\xi^{(\hat{\psi})}$ with

$$\epsilon^{\hat{\phi}}(\hat{\phi}) = e^{in\hat{\phi}}, \quad \epsilon^{\hat{\psi}}(\hat{\psi}) = e^{in\hat{\psi}}, \quad (2.16)$$

respectively. Then these vectors obey

$$[\xi^{(\hat{\phi})}_n, \xi^{(\hat{\phi})}_m] = -i(n - m)\xi^{(\hat{\phi})}_{n+m}, \quad [\xi^{(\hat{\psi})}_n, \xi^{(\hat{\psi})}_m] = -i(n - m)\xi^{(\hat{\psi})}_{n+m}, \quad (2.17)$$

respectively.

In order to calculate the central charge, we consider the conserved charge associated with the Virasoro algebras. A definition of the conserved charge is given by [15, 16]. The conserved charge is expressed in terms of the background metric $\bar{g}_{\mu\nu}$ and its small perturbation $h_{\mu\nu}$, and given by

$$Q_\xi[h] = \frac{1}{8\pi G_N} \int_{\partial \Sigma} k_\xi [h, \bar{g}], \quad (2.18)$$
where $\Sigma$ is a timeslice and the integration is taken over its boundary $\partial \Sigma$. The three-form $k_\xi$ is defined by

$$ k_\xi[h, \bar{g}] = \tilde{k}_\xi^{\mu\nu}[h, \bar{g}] (d^3x)_{\mu\nu}, $$

(2.19)

where $d^3x$ is the Hodge dual of the two-form $dx^\mu \wedge dx^\nu$, and the two-form $\tilde{k}_\xi[h, \bar{g}]$ is given by

$$ \tilde{k}_\xi^{\mu\nu}[h, \bar{g}] = \frac{1}{2} \left[ \xi^\mu D^\nu h - \xi^\mu D^\nu h + (D^\mu h^{\nu\lambda}) \xi_\lambda + \frac{1}{2} h D^\mu \xi^\nu - h^{\mu\lambda} D_\lambda \xi^\nu + \frac{1}{2} h^{\mu\lambda} (D^\nu \xi_\lambda + D_\lambda \xi^\nu) - (\mu \leftrightarrow \nu) \right]. $$

(2.20)

The central charge $c$ can be read off from the anomalous transformation of the charge:

$$ \frac{1}{8\pi G_N} \int_{\partial \Sigma} k_\xi \left[ \mathcal{L}_m \bar{g}, \bar{g} \right] = \delta_n + m, 0 \quad n^3 \frac{c}{12}. $$

(2.21)

For the metric (2.14), and asymptotic symmetry groups (2.15), we obtain

$$ c_\phi = \frac{6\pi k_\phi}{G_N} \int d\theta \sqrt{\gamma(\theta)} f_\phi(\theta), \quad c_\psi = \frac{6\pi k_\psi}{G_N} \int d\theta \sqrt{\gamma(\theta)} f_\psi(\theta). $$

(2.22)

By using the explicit form of the metric (2.10), we obtain

$$ c_\phi = \frac{3\pi b \mu_0}{2G_N}, \quad c_\psi = \frac{3\pi a \mu_0}{2G_N}. $$

(2.23)

Since the Frolov-Thorne temperatures are given by

$$ T_\phi = \frac{(a + b)(r_+ + r_-)}{2\pi(r_+^2 + b^2)} \to \frac{r_0}{\pi b}, $$

(2.24)

$$ T_\psi = \frac{(a + b)(r_+ + r_-)}{2\pi(r_+^2 + a^2)} \to \frac{r_0}{\pi a}, $$

(2.25)

the Cardy formula reproduces the Bekenstein-Hawking entropy at the extremality:

$$ S = \frac{\pi^2}{3} c_\phi T_\phi = \frac{\pi^2}{3} c_\psi T_\psi = \frac{\pi^2}{2} \mu_0 r_0. $$

(2.26)

It should be noted that both CFTs corresponding to each asymptotic symmetry can reproduce the entropy.

### 3 Near-extremal correction for 5D Myers-Perry black hole

In this section, we consider the Kerr/CFT correspondence for the right mover in the five-dimensional Myers-Perry black hole. For the Kerr black hole, the asymptotic
symmetry for the right mover is obtained by introducing a different boundary condition to that for the left mover [8]. For the five-dimensional Myers-Perry black hole, we can obtain a similar asymptotic symmetry group with an analogous boundary condition. For metrics which has the form of (2.14), we impose the following boundary condition:

\[
O(\chi_{\mu\nu}) = \begin{pmatrix}
\mathcal{O}(r^0) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-2}) & \mathcal{O}(r^{-3}) \\
\mathcal{O}(r^{-4}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-4}) \\
\mathcal{O}(r^{-2}) & \mathcal{O}(r^{-2}) & \mathcal{O}(r^{-3}) \\
\mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) & \mathcal{O}(r^{-3}) \\
\end{pmatrix}.
\]  

(3.1)

Then, the asymptotic symmetry group

\[
\xi = \left( \varepsilon_{\xi}(\hat{t}) + \frac{\varepsilon'_{\xi}(\hat{t})}{2\hat{r}^2} \right) \partial_{\hat{t}} + \left( -\hat{r} \varepsilon'_{\xi}(\hat{t}) + \frac{\varepsilon''_{\xi}(\hat{t})}{2\hat{r}} \right) \partial_{\hat{r}} \\
+ \left( C_{\phi} - \frac{k_{\phi} \varepsilon'_{\xi}(\hat{t})}{\hat{r}} \right) \partial_{\hat{\phi}} + \left( C_{\psi} - \frac{k_{\psi} \varepsilon'_{\xi}(\hat{t})}{\hat{r}} \right) \partial_{\hat{\psi}} + \mathcal{O}(\hat{r}^{-3}) ,
\]  

(3.2)

preserves the boundary condition for the metric

\[
g_{\mu\nu} + \mathcal{O}(\chi_{\mu\nu}) \rightarrow g_{\mu\nu} + \mathcal{O}(\chi_{\mu\nu}) .
\]  

(3.3)

For the asymptotic symmetry group (3.2) the central charge vanishes, and hence, we have to introduce a cut-off. Here, we calculate the central charge by using the quasi-local charge [13,14]. This charge is defined as an integration of the surface energy-momentum tensor on the boundary. The surface energy-momentum tensor is given by the conjugate momentum of the induced metric on the surface, and can be expressed in terms of the extrinsic curvature as

\[
T^{\mu\nu} = \frac{2}{\sqrt{-\gamma}} \pi^{\mu\nu} = \frac{1}{8\pi G_N} \left( K^{\mu\nu} - \gamma^{\mu\nu} K \right) ,
\]  

(3.4)

where \( \gamma_{\mu\nu} \) is the induced metric on the boundary and \( \pi_{\mu\nu} \) is its conjugate momentum, and \( K_{\mu\nu} \) is the extrinsic curvature. Here, we consider the metric with small perturbation \( h_{\mu\nu} \), and take the difference of the surface energy-momentum tensor from that of the background \( g_{\mu\nu} \):

\[
\tau^{\mu\nu}[h] = T^{\mu\nu}_{\mid_{g=g+\hat{h}}} - T^{\mu\nu}_{\mid_{g=g}} .
\]  

(3.5)

\(^3\) Instead of taking difference from the background, we can introduce a counter term such that the charge \( Q_\xi \) becomes finite (see Appendix A).
Then, the quasi-local charge is defined as
\[ Q_{\xi}^{QL} = \int_{\partial\Sigma} d^2x \sqrt{\sigma} u^\mu \tau_{\mu\nu} \xi^\nu . \]  
(3.6)

where \( u^\mu \) is a timelike unit normal to a timeslice \( \Sigma \) and \( \sigma \) is an induced metric on the timeslice at the boundary \( \partial\Sigma \). This quasi-local charge corresponds to the energy-momentum tensor for the right mover in CFT as,
\[ Q_{\xi}^{QL} \sim \bar{T}(\bar{z})\bar{\epsilon}(\bar{z}) . \]  
(3.7)

The central extension can be read off from the anomalous transformation of this charge:
\[ \delta\xi Q_{\xi}^{QL} = \int_{\partial\Sigma} d^2x \sqrt{\sigma} u^\mu \tau_{\mu\nu} [L_{\xi}\bar{g}]^\nu . \]  
(3.8)

For two-dimensional CFT, the central extension of the Virasoro algebra can be read off from the anomalous transformation of the energy-momentum tensor. In an analogous fashion to this, we estimate the central extension from the anomalous transformation of the ADM mass, which is \( \delta\xi Q_{\xi}^{QL} \) with \( \epsilon_\xi(t) = 1 \). For the metric (2.14) and the asymptotic symmetry group (3.2), we obtain
\[ \delta\xi Q_{\partial\hat{t}}^{QL} = \frac{1}{8\pi G_N} \int d\phi d\psi d\theta \frac{k_1 k_2 \gamma_{ij}(\theta) \sqrt{\gamma(\theta)} f_0(\theta)}{2\Lambda f_0(\theta)} \epsilon^{\prime\prime\prime}_\xi(\hat{t}) , \]  
(3.9)

where we have introduced a cut-off by putting the boundary at \( \hat{r} = \Lambda \). By using the explicit form of the near horizon metric (2.10), it turns out that
\[ \delta\xi Q_{\partial\hat{t}}^{QL} = \frac{\pi r_0 \mu_0}{4G_N \Lambda} \epsilon^{\prime\prime\prime}_\xi(\hat{t}) . \]  
(3.10)

From the definition of the near horizon coordinate of \( \hat{r} \), it must satisfy
\[ \hat{r} \ll \frac{2r_0}{\sqrt{\mu_0}} \epsilon^{-1} , \]  
(3.11)

in order for the expansion in \( \epsilon \) to be valid. Therefore, we put the boundary of the near horizon geometry at
\[ \Lambda = \frac{2r_0}{\sqrt{\mu_0}} \epsilon^{-1} . \]  
(3.12)

The central charge is related to the anomalous transformation of the charge as
\[ \delta\xi \frac{Q^{(QL)}}{12} \epsilon^{\prime\prime\prime}_\xi(\hat{t}) + (\text{non-anomalous terms}) . \]  
(3.13)
Then, the central charge can be evaluated as

\[ c^{(QL)} = \frac{3\pi \mu_0^{3/2}}{2G_N} \epsilon . \]  

(3.14)

The Frolov-Thorne temperature for right mover can be read off from the Boltzmann factor with respect to the charge \( Q \), and given by

\[ T = \frac{\sqrt{\mu}}{2\pi} . \]  

(3.15)

Using the Cardy formula, we obtain

\[ S = \frac{\pi^2}{3} c^{(QL)} T = \frac{\pi^2 \mu_0^{3/2} \sqrt{\mu}}{4G_N} \epsilon . \]  

(3.16)

Since the Bekenstein-Hawking entropy is expanded in the near-extremal case as

\[ S = \frac{\pi^2}{2G_N} \mu r_+ = \frac{\pi^2}{2G_N} \mu_0 \left( r_0 + \frac{1}{2} \epsilon \sqrt{\mu_0 \bar{\mu}} + O(\epsilon^2) \right) , \]  

(3.17)

the expression (3.16) agrees with the leading non-extremal correction.

Even though the Bekenstein-Hawking entropy is correctly reproduced, the identification of the cut-off (3.12) is just a rough estimation. In order to justify this choice of the cut-off, we compare this result with the AdS\(_3\)/CFT\(_2\) correspondence in the next section.

4 **Comparison with the AdS\(_3\)/CFT\(_2\) correspondence**

When one of the angular momenta vanishes, the near horizon geometry of the five-dimensional Myers-Perry black hole has the structure of AdS\(_3\). If this momentum is not exactly zero but infinitesimally small in the near-extremal case, this AdS\(_3\) part becomes the BTZ black hole. In this case, we can simply apply the ordinary AdS/CFT correspondence for the Myers-Perry black hole. In this section, we consider such a case and compare it with the previous two results.

In the previous sections, we have considered the near-extremal case in which the non-extremality is infinitesimally small. Here, we assume that one of the angular momenta is also infinitesimally small and of the same order to the non-extremality. We define the parameters \( \bar{\mu} \) and \( \bar{b} \) by the following relations,

\[ \mu = a^2 + b^2 + 2\epsilon \bar{a} \bar{m} , \quad b = \epsilon \bar{b} , \]  

(4.1)
and redefine the near horizon coordinates as

\begin{align}
    t &= e^{-1}\tilde{t}, \\
    r &= e\tilde{r}, \\
    \phi &= \tilde{\phi} + \frac{a}{r_0^2 + a^2 t}, \\
    \psi &= e^{-1}\psi. 
\end{align}

(4.2)

By taking the near horizon limit \( \epsilon \to 0 \), the metric becomes

\begin{align}
    ds^2 &= -\frac{\cos^2 \theta}{a^2} \frac{\tilde{\Delta}}{\tilde{r}^2} d\tilde{t}^2 + \frac{a^2 \cos^2 \theta}{\tilde{\Delta}} d\tilde{r}^2 + \cos^2 \theta \tilde{r}^2 \left( d\tilde{\psi} - \frac{b}{\tilde{r}^2} d\tilde{t} \right)^2 \\
    &+ a^2 \frac{\sin^2 \theta}{\cos^2 \theta} d\tilde{\phi}^2 + a^2 \cos^2 \theta d\theta,
\end{align}

(4.4)

where \( \tilde{\Delta} \) is given by

\begin{align}
    \tilde{\Delta} &= \tilde{r}^4 - 2a\tilde{m}\tilde{r}^2 + a^2\tilde{b}^2 = (\tilde{r}^2 - \tilde{r}_+^2)(\tilde{r}^2 - \tilde{r}_-^2),
\end{align}

(4.5)

and \( \tilde{r}_\pm \) is the positions of the outer and inner horizons in terms of \( \tilde{r} \), which is expressed as

\begin{align}
    \tilde{r}_\pm^2 &= e^{-2\gamma}_\pm^2 = a \left( \tilde{m} \pm \sqrt{\tilde{m}^2 - \tilde{b}^2} \right).
\end{align}

(4.6)

Then, this geometry has the structure of the BTZ black hole. Strictly speaking, \( \tilde{\psi} \) has an infinitesimal period \( 2\pi\epsilon \).

The analysis of the asymptotic symmetry can be applied to this geometry straight-forwardly. For simplicity, we introduce the light-corn coordinates

\begin{align}
    x^\pm &= \tilde{\psi} \pm \frac{\tilde{t}}{a}.
\end{align}

(4.7)

By imposing the boundary condition:

\begin{align}
    x^+ &\quad \tilde{r} &\quad x^- &\quad \tilde{\phi} &\quad \theta \\
    O(x^+) &= \begin{pmatrix}
    O(r^0) & O(r^{-2}) & O(r^0) & O(r^{-2}) & O(r^{-3}) \\
    O(r^{-4}) & O(r^{-2}) & O(r^0) & O(r^{-3}) & O(r^{-4})
    \end{pmatrix}, \\
    O(\chi_{\mu\nu}) &= \begin{pmatrix}
    O(r^0) & O(r^{-2}) & O(r^0) & O(r^{-3}) \\
    O(r^0) & O(r^{-2}) & O(r^0) & O(r^{-3})
    \end{pmatrix},
\end{align}

(4.8)

which is the same as the original work by Brown and Henneaux \footnote{1} for AdS\(_3\) part, we obtain the following asymptotic symmetry groups:

\begin{align}
    \xi^{(+)} &= \epsilon_+(x^+) \partial_+ - \frac{1}{2} \tilde{r} \epsilon'_+(x^+) \partial_\tilde{r} - \frac{a^2}{2\tilde{r}^2} \epsilon''_+(x^+) \partial_-, \\
    \xi^{(-)} &= \epsilon_-(x^-) \partial_- - \frac{1}{2} \tilde{r} \epsilon'_-(x^-) \partial_\tilde{r} - \frac{a^2}{2\tilde{r}^2} \epsilon''_-(x^-) \partial_+.
\end{align}

(4.9a)

(4.9b)
The coordinate $x^+$ and $x^-$ parametrize the almost same directions to $\hat{t}$ and $\hat{\phi}$, and hence, these asymptotic symmetry groups are almost equivalent to those studied in the previous sections. In order to see this, we take the limit of $b \to 0$ of the near horizon geometry (2.10). Comparing the definition of the coordinates $(\hat{t}, \hat{\psi})$ with $(x^+, x^-)$ we obtain the following relations:

$$x^+ = \hat{t} + \mathcal{O}(\epsilon), \quad x^- = \epsilon \hat{\psi}. \quad (4.10)$$

Then, by taking $\epsilon \to 0$ limit with these coordinates and $b = \epsilon^2 \tilde{b}$, the near horizon metric (2.10) becomes

$$ds^2 = \frac{a^2 \cos^2 \theta}{4} \hat{\mu}(dx^+)^2 + \sqrt{a^3 \tilde{b}} \hat{r} dx^+ dx^- + a \tilde{b} \cos^2 \theta (dx^-)^2$$

$$+ \frac{a^2 \cos^2 \theta}{4\Delta} d\hat{r}^2 + a^2 \frac{\sin^2 \theta}{\cos^2 \theta} d\hat{\psi}^2 + a^2 \cos^2 \theta d\theta^2 . \quad (4.11)$$

Since the parameters $\hat{\mu}$ and $\tilde{m}$ are related as $\hat{\mu}/2 = a(\tilde{m} - \tilde{b})$, this expression agrees with (4.4) in the extremal limit of $\tilde{m} \to \tilde{b}$ if we identify

$$r^2 \sim a \tilde{b} + \sqrt{a^3 \tilde{b}} \hat{r}. \quad (4.12)$$

Then, excluding the last terms in (4.9), which are asymptotically subleading contributions, we obtain

$$\xi^{(+)} \sim \epsilon_+ (\hat{t}) \partial_\hat{t} - \hat{r} \epsilon'_+(\hat{t}) \partial_{\hat{r}} + \mathcal{O}(\epsilon), \quad (4.13)$$

$$\xi^{(-)} \sim \epsilon^{-1} \left( \hat{\epsilon}_-(\hat{\psi}) \partial_{\hat{\psi}} - \hat{r} \hat{\epsilon}'_- (\hat{\psi}) \partial_{\hat{r}} \right) + \mathcal{O}(\epsilon^0), \quad (4.14)$$

where $\hat{\epsilon}_-(\hat{\psi}) = \epsilon_-(\epsilon \hat{\psi})$. Therefore, $\xi^{(+)}$ and $\xi^{(-)}$ correspond to the asymptotic symmetry groups for the right and left movers, respectively.

These two sets of vectors form the Virasoro algebras as in the ordinary AdS$_3$ case, but have slightly different structures. Since the coordinate $\hat{\psi}$ has an infinitesimal period of $2\pi \epsilon$, light-cone coordinates $x^+$ and $x^-$ must satisfy the following periodicities

$$x^+ \sim x^+ + 2\pi n \epsilon, \quad x^- \sim x^- + 2\pi n \epsilon. \quad (4.15)$$

Then, the functions $\epsilon_+(x^+)$ and $\epsilon_-(x^-)$ can be expanded with the following forms:

$$\epsilon_+(x^+) = e^{inx^+/\epsilon}, \quad \epsilon_-(x^-) = e^{inx^-/\epsilon}. \quad (4.16a)$$

---

4 Strictly speaking, here we take the near-extremal limit and the small $b$ limit separately, and hence, the parameter $\epsilon$ here and that in (2.7), (2.8) should be distinguished. The metric (4.11) agrees with (4.4) only in the near-extremal case, because we first take the near-extremal limit for (4.14).

5 This relation is consistent with the definitions of $\hat{r}$ and $\tilde{r}$, since the definition of $\hat{r}$ can be rewritten as $r^2 = r_0^2 + \epsilon r_0 \sqrt{\mu_0} \hat{r}$ in $\epsilon \to 0$ limit.
with arbitrary integers \( n \). Now we define \( \xi_{n}^{(+)} \) and \( \xi_{n}^{(-)} \) by \( \xi^{(+)} \) and \( \xi^{(-)} \) with (4.16). Then, these vectors form the following algebras:

\[
[\xi_{n}^{(+)}, \xi_{m}^{(+)}] = -i \frac{n-m}{\epsilon} \xi_{n+m}^{(+)} , \tag{4.17}
\]

\[
[\xi_{n}^{(-)}, \xi_{m}^{(-)}] = -i \frac{n-m}{\epsilon} \xi_{n+m}^{(-)} , \tag{4.18}
\]

respectively. Here we have an additional factor \( \epsilon^{-1} \) which comes from the period of \( \tilde{\psi} \).

Before calculating the central charge, we discuss the general property of the Virasoro algebra with an additional factor. In general, the following vector forms the Virasoro algebra:

\[
\xi = f(x) \partial_x - r f'(x) \partial_r . \tag{4.19}
\]

If the coordinate \( x \) has the period of \( 2\pi \beta \), the function \( f(x) \) must respect this periodicity. We define \( \xi_{n} \) by \( \xi \) with

\[
f_{n}(x) = e^{inx/\beta} . \tag{4.20}
\]

Then, this vector forms the following algebra

\[
[\xi_{n}, \xi_{m}] = \frac{n-m}{\beta} \xi_{n+m} . \tag{4.21}
\]

This is the Virasoro algebra but has an additional factor of \( 1/\beta \). This factor appears because the vector \( \xi \) is not dimensionless and hence the algebra depends on the choice of the coordinate \( x \). The factor \( \beta \) can be absorbed by taking the coordinate \( x \) to have the period of \( 2\pi \), or equivalently, redefining \( \xi \to \xi' = \beta \xi \).

The Noether charges \( L_{n} \) associated with these vector can have the central extension, and obeys the following algebra:

\[
[L_{n}, L_{m}] = -i(n-m)L_{n+m} - i\delta_{n+m,0} n^{3} \frac{c}{12} , \tag{4.22}
\]

where we have rescaled the vector such that the algebra takes the standard form. In the original definition of the asymptotic symmetry groups, we have chosen the normalization such that \( \xi_{0} \) gives a conjugate momentum of \( x \), namely \( \xi_{0} = \partial_{x} \). Then, the algebra becomes

\[
[L_{n}, L_{m}] = -i \frac{n-m}{\beta} L_{n+m} - i\delta_{n+m,0} n^{3} \frac{c^{(\text{cov})}}{12\beta^{3}} . \tag{4.23}
\]
Here, we have chosen the “covariant” definition of the central charge such that the central extension is related to the anomalous transformation of the charge as

$$\delta Q_\xi \sim \frac{c(QL)}{12} f''(x),$$

(4.24)

where $Q_\xi \sim \sum L_n f_n(x)$. Therefore, the central charge we have derived in the previous section is not $c$ but $c^{(cov)}$. By using this covariant definition, the generators $L_n$, central charge $c^{(cov)}$ and the Frolov-Thorne temperature $T$ have the following scaling properties:

$$x \to \lambda x, \quad L_n \to \lambda^{-1} L_n, \quad c^{(cov)} \to \lambda c^{(cov)}, \quad T \to \lambda^{-1} T,$$

(4.25)

where the Frolov-Thorne temperature $T$ is the weight for the charge $L_0$. These behaviors imply that we can also apply the Cardy formula in these definitions. These two definitions, the standard scale-invariant and covariant one, are related with each other by

$$L_n = \beta L_n, \quad c = \frac{c^{(cov)}}{\beta}, \quad T = \beta T,$$

(4.26)

where $T$ is the scale-invariant Frolov-Thorne temperature which is the weight for $L_0$. Obviously the Cardy formula takes the same form for both of these two definitions. These two sets of definitions are exactly same when the period of the coordinate $x$ is $2\pi$.

Now, we evaluate the central charge. By using the definition of $[13, 16]$, the central extensions of the algebras for $\xi^+$ and $\xi^-$ are obtained as

$$\frac{1}{8\pi G_N} \int k_{\xi^+} [L_{\xi^+} \bar{g}, \bar{g}] = -i\frac{\pi}{8} \delta_{n+m,0} a^3 \frac{n^3}{\epsilon^2}.\quad (4.27)$$

Since the period of $x^+$ and $x^-$ are $2\pi \epsilon$, the scale-covariant central charges for $\xi^+$ and $\xi^-$ are

$$c_{\pm}^{(cov)} = \epsilon \frac{3\pi a^3}{2G_N}.\quad (4.28)$$

The Frolov-Thorne temperatures associated with $\partial_+$ and $\partial_-$ are given by

$$T_{\pm} = \frac{\tilde{\varphi}_{\pm} \mp \tilde{\varphi}_-}{2\pi a}.\quad (4.29)$$

Then, the Cardy formula reproduces the Bekenstein-Hawking entropy of (4.4):

$$S_{CFT} = \frac{\pi^2}{3} c_+^{(cov)} T_+ + \frac{\pi^2}{3} c_-^{(cov)} T_-$$

$$= \frac{\pi^2}{2G_N} a^2 \tilde{\varphi}_+ \epsilon = S_{BH}.\quad (4.30)$$
The central charges (4.28) are actually equivalent to those derived in the previous sections. Since the coordinates are rescaled as
\[ \hat{\psi} \to x^- = \epsilon \hat{\psi}, \]
the central charges before and after the rescaling are related with each other as
\[ c_{-}^{(\text{cov})} = \epsilon c_{\hat{\psi}}. \] (4.32)

From (2.23) and (4.28), it is clear that \( c_{\hat{\psi}} \) and \( c_{-}^{(\text{cov})} \) satisfy this relation in \( b \to 0 \) limit.

For the right mover, the coordinates \( \hat{t} \) and \( x^+ \) are equivalent in \( \epsilon \to 0 \) limit. In fact, the central charge \( c_{+}^{(QL)} \) in (3.14) equals to \( c_{+}^{(\text{cov})} \) in \( b \to 0 \) limit. This justifies the identification of the cut-off (3.12).

Even though the asymptotic symmetry groups (2.13) and (3.2) agree with those in AdS3 at the leading order of \( \epsilon \), there are higher order corrections. The near horizon coordinate \( \hat{t} \) is the time coordinate in AdS3 and not exactly equivalent to the light-cone coordinate \( x^+ \). We can also introduce an asymptotic symmetry group of the time direction for AdS3 by imposing a suitable boundary condition. However it is rather natural to define a new near horizon limit to obtain asymptotic symmetry groups for the general extremal Myers-Perry black holes. We will discuss it in the next section.

5 New near horizon limit of 5D Myers-Perry black hole

In this section, we consider another definition of the near horizon limit. We have defined the near horizon coordinates by (2.8). However, the central charge of right mover cannot be calculated by the covariant phase space method given by [13, 14], and hence we used the quasi-local charge [13, 14]. In the new coordinates, the asymptotic Virasoro symmetries are realized along the light-cone coordinates \( (x^+, x^-) \) similarly to the AdS3 spacetime, while those in the usual coordinates are associated with \( (\hat{t}, \hat{\psi}) \) directions. We will show that the definition of [13, 16] also gives the central charge and reproduces the entropy via the Cardy formula.

We define the new coordinates \( x^+ \) and \( x^- \), and redefine \( \hat{r} \) as follows:
\[ x^+ = \epsilon \left( \psi + \frac{a - b}{\mu} \hat{t} \right), \quad x^- = \psi - \frac{a + b}{\mu} \hat{t}, \quad \hat{r} = r_0 + \epsilon \frac{a}{2} \hat{r}. \] (5.1)

In the near-extremal case of (2.7), the near horizon geometry takes the same form as (2.10), but the coordinates \( \hat{t} \) and \( \hat{\psi} \) are replaced with \( x^+ \) and \( x^- \). Namely, taking \( \epsilon \to 0 \)
limit, we obtain

\[ ds^2 = -\frac{\rho_0^2}{4} \hat{\Delta} (dx^+)^2 + \frac{\rho_0^2}{4\hat{\Delta}} d\hat{r}^2 + \rho_0^2 d\theta^2 \]
\[ + \frac{a^2 \mu_0 \sin^2 \theta}{\rho_0^2} \left( d\hat{\phi} + k_\phi \hat{r} dx^+ \right)^2 + \frac{b^2 \mu_0 \cos^2 \theta}{\rho_0^2} \left( dx^- + k_\psi \hat{r} dx^+ \right)^2 \]
\[ + \frac{\mu_0 r_0^2}{\rho_0^2} \left[ \sin^2 \theta \left( d\hat{\phi} + k_\phi \hat{r} dx^+ \right) + \cos^2 \theta \left( dx^- + k_\psi \hat{r} dx^+ \right) \right]^2 , \tag{5.2} \]

where we have also redefined \( \hat{\Delta} \) by

\[ \hat{\Delta} = \hat{r}^2 - \frac{\mu_0}{a^2} \hat{\mu}. \tag{5.3} \]

Then, we can obtain the asymptotic symmetry group in the same fashion as the previous section (but the coordinate \( \hat{t} \) and \( \hat{\psi} \) are replaced with \( x^+ \) and \( x^- \)). By using the same boundary condition as (3.1), we obtain the asymptotic symmetry groups,

\[ \xi = (\epsilon_+(x^+) + \frac{\epsilon''_\phi(x^+)}{2\hat{r}^2}) \partial_+ + \left( -\hat{r} \epsilon'_\phi(x^+) + \frac{\epsilon''_\phi(x^+)}{2\hat{r}} \right) \partial_\hat{r} \]
\[ + \left( C_\phi - \frac{k_\phi \epsilon''_\phi(x^+)}{\hat{r}} \right) \partial_\phi + \left( C_\psi - \frac{k_\psi \epsilon''_\phi(x^+)}{\hat{r}} \right) \partial_- + \mathcal{O}(\hat{r}^{-3}). \tag{5.4} \]

Since \( \psi \sim \psi + 2\pi \), the coordinates \( x^+ \) and \( x^- \) have the periodicity of

\[ x^+ \sim x^+ + 2\pi n \epsilon, \quad x^- \sim x^- + 2\pi n. \tag{5.5} \]

Then, the function \( \epsilon_\phi(x^+) \) must have the following form:

\[ \epsilon_\phi(x^+) = e^{i n x^+ / \epsilon}. \tag{5.6} \]

We define \( \xi_n \) by \( \xi \) with (5.4). Then, \( \xi_n \) forms the following algebra:

\[ [\xi_n, \xi_m] = -i \frac{n - m}{\epsilon} \xi_{n+m}. \tag{5.7} \]

Now let us consider the conserved charge defined by (2.18) in our new coordinates. In the previous definition of the near horizon coordinates, \( \hat{t} \) is equivalent to the time of the original coordinates up to the scaling factor. Hence, the timeslice in the near horizon geometry is also defined on \( \hat{t} = \text{const.} \) plane. However, in the new definition, the coordinate \( x^+ \) is not equivalent to the original time. We should perform the integration on the original timeslice, hence the timeslice \( \Sigma \) (and its boundary \( \partial \Sigma \)) is not \( x^+ = \text{const.} \) plane.
For the asymptotic symmetry groups (5.4), we obtain
\[ \tilde{k}_\xi^{\tau\nu} [\xi g, \bar{g}] = 0, \quad \tilde{k}_\xi^{\nu\tau} [\xi g, \bar{g}] = -\frac{k_\psi}{f_0(\theta)} \epsilon_\xi(x^+) \epsilon^{\nu\tau}(x^+), \] (5.8)
where \( f_0(\theta), k_\psi, \) etc. are the same as those in the previous section. Since the central extension becomes
\[ \frac{1}{8\pi G_N} \int_{\partial \Sigma} k_m [\xi g, \bar{g}] = \frac{\pi k_\psi}{2G_N \epsilon^2} \int d\theta \sqrt{\gamma(\theta)} f_0(\theta), \] (5.9)
the scale-covariant central charge is evaluated as
\[ c_+^{(cov)} = \frac{6\pi k_\psi \epsilon}{G_N} \int d\theta \sqrt{\gamma(\theta)} f_0(\theta). \] (5.10)
Using the explicit form of the metric (5.2), we obtain
\[ c_+^{(cov)} = \frac{3\pi a \mu_0}{2G_N \epsilon}. \] (5.11)
The central charge for left mover can be calculated straightforwardly. Since the additional term gives only the \( O(\epsilon) \) corrections, the central charge for left mover equals to (2.23). It should be noted that the scale-invariant central charge is given by \( c_- = c_+^{(cov)} / \epsilon \), and equals to the value for left mover \( c_- = c_\psi \).

The Frolov-Thorne temperatures associated with \( \partial_+ \) and \( \partial_- \) are given by
\[ T_+ = \epsilon^{-1} r_+ - \frac{r_-}{2\pi a} \rightarrow \frac{\sqrt{\mu_0 \mu}}{2\pi a}, \] (5.12)
\[ T_- = \frac{r_+ + r_-}{2\pi a} \rightarrow \frac{b}{\pi}. \] (5.13)
Then the Cardy formula reproduces the Bekenstein-Hawking entropy up to \( O(\epsilon^2) \):
\[ S = \frac{\pi^2}{3} c_+^{(cov)} T_+ + \frac{\pi^2}{3} c_-^{(cov)} T_- = \frac{\pi^2}{2G_N \mu_0} \left( r_0 + \frac{1}{2} \epsilon \sqrt{\mu_0 \mu} + O(\epsilon^2) \right). \] (5.14)

Before closing this section, we would like to comment on the definition of the new near horizon coordinates. First, the definitions of \( x^\pm \) is the same as those of the hidden conformal symmetry [17] in the near-extremal limit. When one of the angular velocities \( (b) \) is very small, these coordinates, \( x^\pm \), are equivalent to the light-cone coordinates \( x^+ \) and \( x^- \) defined in the previous section up to the factor of \( \epsilon \) for \( x^- \). In this limit, the central charges \( c_+^{(cov)} \) and the Frolov-Thorne temperatures agree with those in Section [2] when we take into account the factor of \( \epsilon \) or use the scale-invariant definitions. Since the coordinate \( x^+ \) and \( \hat{t} \) are related as \( x^+ = \frac{a}{a+\hat{t}} + O(\epsilon) \), the central charge (3.14) can be reproduced from \( c_+^{(cov)} \) by using (4.23).
6 Kerr/CFT revisited

In the previous section, we have defined the new near horizon limit of the five-dimensional Myers-Perry black hole. By using this, the central charges for right mover can be calculated by using the definition of\cite{13,14}. In this section, we consider such a new near horizon limit for the four-dimensional Kerr black hole.

By using the Boyer-Lindquist coordinates, the Kerr geometry can be expressed as

\[
\begin{align*}
\frac{ds^2}{\rho^2} = -\frac{\Delta}{\rho^2} (dt - a \sin^2 \theta d\phi)^2 + \frac{\sin^2 \theta}{\rho^2} \left[(r^2 + a^2)d\phi - adt\right]^2 + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 ,
\end{align*}
\]

where \(\Delta\) and \(\rho^2\) are given by

\[
\begin{align*}
\Delta &= r^2 - 2Mr + a^2 , \\
\rho^2 &= r^2 + a^2 \cos^2 \theta .
\end{align*}
\]

The Kerr geometry are characterized by two parameters \(M\) and \(a\) which are related to the ADM mass and angular momentum as

\[
M_{\text{ADM}} = \frac{M}{G_N} , \quad J = \frac{aM}{G_N} .
\]

The inner and outer horizons are given by

\[
r_{\pm} = M \pm \sqrt{M^2 - a^2} ,
\]

and the angular velocity at the outer horizon \(r_+\) is

\[
\Omega_H = \frac{a}{r_+^2 + a^2} .
\]

The Hawking temperature and the Bekenstein-Hawking entropy are given by

\[
T_H = \frac{r_+ - r_-}{4\pi Mr_+} , \quad S = \frac{2\pi Mr_+}{G_N} .
\]

We consider the near-extremal case, and define a non-extremality parameter \(\hat{r}_H\) as

\[
M = a(1 + \epsilon \hat{r}_H^2) .
\]

Then, the geometry is parametrized by \(a\) and \(\hat{r}_H\). New near horizon coordinates \(x^\pm\) and \(\hat{x}\) are defined by the following relations,

\[
\begin{align*}
x^+ &= \epsilon \phi , \\
x^- &= \phi - \frac{at}{2M^2} , \\
r &= a(1 + \epsilon \hat{r}) .
\end{align*}
\]
Here, the combination of $t$ and $\phi$ in the definition of $x^+$ and $x^-$ are the same as those appeared in analysis of the hidden conformal symmetry \[10\] (in the definition of $w^\pm$). The geometry is expressed in the near horizon limit of $\epsilon \to 0$ as

$$ds^2 = -(\hat{r}^2 - \hat{r}_H^2)f_0(\theta)(dx^+)^2 + f_\phi(\theta)(dx^- + \hat{r}dx^+)^2 + f_0(\theta)\frac{d\hat{r}^2}{\hat{r}^2 - \hat{r}_H^2} + f_0(\theta)d\theta^2 ,$$  \tag{6.9}$$

where $f_0(\theta)$ and $f_\phi(\theta)$ are given by

$$f_0(\theta) = a^2(1 + \cos^2 \theta) , \quad f_\phi(\theta) = \frac{4a^2 \sin^2 \theta}{1 + \cos^2 \theta} . \tag{6.10}$$

This geometry has the same form to the ordinary so-called NHEK geometry, but $t$ and $\phi$ of the near horizon coordinates are replaced with $x^+$ and $x^-$, respectively.

The boundary condition for right mover is obtained in \[8\]. We use this boundary condition by replacing $\hat{t}$ and $\hat{\phi}$ with $x^+$ and $x^-$:

$$O(\chi_{\mu\nu}) = \begin{pmatrix} x^+ & \hat{r} & x^- & \theta \\ x^+ & O(r^0) & O(r^{-3}) & O(r^{-2}) & O(r^{-3}) \\ \hat{r} & O(r^{-4}) & O(r^{-3}) & O(r^{-4}) \\ x^- & O(r^{-2}) & O(r^{-3}) \\ \theta & O(r^{-3}) \end{pmatrix} \tag{6.11}$$

Then, the following asymptotic symmetry group satisfies this boundary condition:

$$\xi = \left( \epsilon_\xi(x^+) + \frac{\epsilon''_\xi(x^+)}{2\hat{r}^2} \right) \partial_+ + \left( -\hat{r}\epsilon'_\xi(x^+) + \frac{\epsilon''_\xi(x^+)}{2\hat{r}} \right) \partial_{\hat{r}} + \left( C - \frac{\epsilon''_\xi(x^+)}{\hat{r}} \right) \partial_- + O(\hat{r}^{-3}). \tag{6.12}$$

Since the period of $\phi$ is $2\pi$, the coordinates $x^+$ and $x^-$ have the following periodicity:

$$x^+ \sim x^+ + 2\pi n \epsilon , \quad x^- \sim x^- + 2\pi n . \tag{6.13}$$

The central extension is given by an integration of two-from $k_\xi [L_\xi \hat{g}, \hat{g}]$ on a timeslice. As we have discussed in the previous section, the charge should be defined by the integration on the timeslice of the original coordinates. As in the case of the five-dimensional Myers-Perry black hole, the near horizon geometry is usually taken such that the time direction of the near horizon coordinates is equivalent to that of the original one, up to a constant factor. In this case, only $k^{tr}_\xi$ component contributes to the charge and the central charge
becomes zero $\delta$. However, in the new near horizon coordinate (6.8), the coordinate $x^+$ is not the time direction of original coordinates. Then, $\tilde{k}^{-r}_{\xi}$ also contributes to the central extension and we obtain

$$\frac{1}{8\pi G_N} \int_{\partial \Sigma} k^{-r}_{\xi_m} [\xi, \tilde{g}, \tilde{g}] = \delta_{n+m,0} n^3 a^2 \epsilon^2.$$  \hspace{1cm} (6.14)

Then, the scale-covariant central charge is

$$c^{(\text{cov})}_+ = \frac{12a^2}{G_N} \epsilon.$$  \hspace{1cm} (6.15)

This result agrees with that obtained in $\delta$. It should be noted that this is the expected value of $c_R \equiv c_+ = 12J$ if we use the scale-invariant definition.

The Frolov-Thorne temperatures associated to $\partial^+$ and $\partial^-$ are given by

$$T^+ = \epsilon^{-1} \frac{r^+ - r^-}{4\pi a} \to \frac{\hat{r} H}{2\pi},$$  \hspace{1cm} (6.16)

$$T^- = \frac{r^+ + r^-}{4\pi a} \to \frac{1}{2\pi}.$$  \hspace{1cm} (6.17)

Then the Cardy formula gives the entropy

$$S = \frac{2\pi a^2}{G_N} (a + \epsilon \hat{r} H),$$  \hspace{1cm} (6.18)

which agrees with the Bekenstein-Hawking entropy up to $O(\epsilon^2)$,

$$S = \frac{2\pi M r^+}{G_N} \frac{2\pi a^2}{G_N} (a + \epsilon \hat{r} H + O(\epsilon^2)).$$  \hspace{1cm} (6.19)

## Conclusion and discussions

In this paper, we introduced a new near horizon limit. In this limit, structure of the near horizon geometry is the same as that introduced in $\delta$, while the original time direction is embedded in a different way. Then, conserved charges are slightly modified since the timeslice is different in our limit and that in $\delta$. This limit is useful to describe the right mover in the Kerr/CFT correspondence, and the central charge can be calculated explicitly.

By using our new definition of the near horizon limit, the charge density depends on the angular coordinates and hence we can define generators simply integrating on the timeslice. The central charge can be calculated by using the definition of $\delta$, and does not have ambiguities of the cut-off. It turns out that the Virasoro algebra does not have the standard form and depends on the definition of the coordinate. By redefining
the generators to have the standard algebraic relation, the central charge becomes finite and satisfies the expected relation of $c_L = c_R = 12J$.

In our new near horizon coordinates, the combinations of coordinates are equivalent to those in analyses of the hidden conformal symmetry. The presence of the hidden conformal symmetry implies the decoupling of the right and left movers. In order to apply the Cardy formula separately, the right and left movers should be decoupled. Hence it is natural that the appropriate choice of the coordinates is equivalent to those for the hidden conformal symmetry.

Even though the scale-invariant central charge is finite, it is natural to use the covariant definition for the temperature. Then, the (covariant) central charge for right mover takes an infinitesimally small value, and hence, the right mover gives subleading contributions in the near-extremal limit. We took the near-extremal limit and considered its leading corrections. However, we did not include all the next-to-leading contributions in this limit. Hence, we cannot exclude the possibility that these contributions affect the near-extremal corrections. In order to see this, we have to study subleading corrections, or consider more general non-extremal cases.

We computed the central charges for each sector separately by giving each boundary condition. Although we reproduce the Bekenstein-Hawking entropy by summing up the entropy of each sector, it is desirable to find a boundary condition that admit two Virasoro algebras as asymptotic symmetry groups. One example is given by [19], but it cannot fix higher order corrections of the asymptotic symmetries for right mover which contribute to the central charges. Investigation in this direction will give further evidence for the Kerr/CFT correspondence.

**Acknowledgements**

We are grateful to T. Hartman and N. Matsumiya for valuable discussions, and N. Matsumiya for collaboration on a new near horizon limit at an earlier stage. TN would like to thank all members of the High Energy Physics Theory Group of the University of Tokyo for hospitality during his stay. The work of TN was supported in part by the US NSF under Grants No. PHY-0844827 and PHY-0756966.

---

*6 If we use the scale-invariant definition, the Frolov-Thorne temperature becomes infinitesimally small. Therefore, the right mover contributes to the subleading corrections, independent to the definition of the central charge.*
Appendix A: Counter terms for the quasi-local charge

In this appendix, we consider counter terms for the quasi-local charges. The quasi-local charge is defined by (3.6), and we have defined the regularized surface energy-momentum tensor $\tau_{\mu\nu}$ by (3.5). Instead of taking difference from the background in (3.5), a counter term can be introduced to regularize the surface energy-momentum tensor. In this case, any covariant counter terms cannot terminate all of divergent terms in the surface energy-momentum tensor. Here, we take the counter term such that $Q_{\xi}^{QL}$ becomes finite. Then, the surface energy-momentum tensor is given by

$$\tau_{\mu\nu} = T_{\mu\nu} + \lambda g_{\mu\nu}, \quad (A.1)$$

where $g_{\mu\nu}$ is the induced metric on the boundary and $\lambda$ is a constant. Using this definition of $\tau_{\mu\nu}$, the quasi-local charge for (2.14) has the following divergent terms:

$$Q_{\xi}^{QL} = -\frac{1}{8\pi G_N} \int d\phi d\psi d\theta \frac{\Lambda k_i k_j \gamma_{ij}(\theta) \sqrt{\gamma(\theta)f_0(\theta)}}{2f_0(\theta)} + \frac{\lambda}{8\pi G_N} \int d\phi d\psi d\theta \frac{\Lambda f_0(\theta)\gamma(\theta)f_0(\theta)}{2f_0(\theta)} + O(\Lambda^0). \quad (A.2)$$

The constant $\lambda$ is chosen such that these two terms cancel each other. For the near horizon geometry of Myers-Perry black hole (2.10), it turns out that

$$\lambda = \frac{(a^{3/2} - b^{3/2})}{3(a-b)}. \quad (A.3)$$

By using this condition, the central extension becomes

$$\delta_\xi Q_{\theta_i}^{QL} = \frac{\lambda}{8\pi G_N} \int d\phi d\psi d\theta \frac{\sqrt{f_0(\theta)\gamma(\theta)f_0(\theta)}}{\Lambda} \epsilon_\xi' (\hat{t})$$

$$= \frac{1}{8\pi G_N} \int d\phi d\psi d\theta \frac{k_i k_j \gamma_{ij}(\theta) \sqrt{\gamma(\theta)f_0(\theta)}}{2\Lambda f_0(\theta)} \epsilon_\xi' (\hat{t}), \quad (A.4)$$

and then, we obtain the same result to (3.9). If we allow the coefficient $\lambda$ to have $\theta$-dependence, we can make the charge density to be finite. In this case, the coefficient of the counter term becomes

$$\lambda(\theta) = k_i k_j \gamma_{ij}(\theta) \left( \frac{1}{f_0(\theta)} \right)^{3/2}, \quad (A.5)$$

where $f_0(\theta)$, $\gamma_{ij}(\theta)$ and $k_i$ just specify the $\theta$-dependence but do not respond to the variation with respect to the metric. In the limit of $b \to 0$, this coefficient can be expanded as

$$\lambda(\theta) \to \frac{1}{a \cos \theta}. \quad (A.6)$$
This agrees with the counter term introduced in [14]. For small $b$, the near horizon geometry has structure of $\text{AdS}_3$, whose effective radius at each point of $\theta$ is given by

$$R = a \cos \theta.$$  \hfill (A.7)

then, the coefficient of the counter term for this $\text{AdS}_3$ is

$$\lambda = \frac{1}{R} = \frac{1}{a \cos \theta},$$  \hfill (A.8)

which is equals to (A.6).

References

[1] J.D. Brown and M. Henneaux, Commun. Math. Phys. 104 (1986) 207.

[2] A. Strominger, JHEP 9802 (1998) 009 [arXiv:hep-th/9712251].

[3] M. Guica, T. Hartman, W. Song and A. Strominger, Phys. Rev. D80 (2009) 124008, [arXiv:0809.4266[hep-th]].

[4] J.M. Bardeen and G.T. Horowitz, Phys. Rev. D60 (1999) 104030, [arXiv:hep-th/9905099].

[5] F. Loran and H. Soltanpanahi, JHEP 0903 (2009) 035, [arXiv:0810.2620[hep-th]].
K. Hotta, Y. Hyakutake, T. Kubota, T. Nishinaka and H. Tanida, JHEP 0901 (2009) 010, [arXiv:0811.0910[hep-th]].
T. Azeyanagi, N. Ogawa and S. Terashima, JHEP 0904 (2009) 061, [arXiv:0811.4177[hep-th]].
T. Hartman, K. Murata, T. Nishioka and A. Strominger, JHEP 0904 (2009) 019, [arXiv:0811.4393[hep-th]].
Y. Nakayama, Phys. Lett. B673 (2009) 272, [arXiv:0812.2234[hep-th]].
D.D.K. Chow, M. Cvetic, H. Lu and C.N. Pope, Phys. Rev. D79 (2009) 084018, [arXiv:0812.2918[hep-th]].
H. Isono, T.-S. Tai and W.-Y. Wen, Int. J. Mod. Phys. A24 (2009) 5659, [arXiv:0812.4440[hep-th]].
T. Azeyanagi, N. Ogawa and S. Terashima, Phys. Rev. D79 (2009) 106009, [arXiv:0812.4883[hep-th]].
J.-J. Peng and S.-Q. Wu, Phys. Lett. B673 (2009) 216,
[arXiv:0901.0311[hep-th]].
C.-M. Chen and J.E. Wang, Class. Quant. Grav. 27 (2010) 075004,
[arXiv:0901.0538[hep-th]].
F. Loran and H. Soltanpanahi, Class. Quant. Grav. 26 (2009) 155019,
[arXiv:0901.1595[hep-th]].
A.M. Ghezelbash, JHEP 0908 (2009) 045, [arXiv:0901.1670[hep-th]].
H. Lu, J.-W. Mei, C.N. Pope and J.F. Vazquez-Poritz, Phys. Lett. B673 (2009) 77,
[arXiv:0901.1677[hep-th]].
G. Compere, K. Murata and T. Nishioka, JHEP 0905 (2009) 077,
[arXiv:0902.1001[hep-th]].
K. Hotta, Phys. Rev. D79 (2009) 104018, [arXiv:0902.3529[hep-th]].
D. Astefanesei and Y.K. Srivastava, Nucl. Phys. B822 (2009) 283,
[arXiv:0902.4033[hep-th]].
M.R. Garousi and A. Ghodsi, Phys. Lett. B687 (2010) 79,
[arXiv:0902.4387[hep-th]].
A.M. Ghezelbash, [arXiv:0902.4662[hep-th]].
C. Krishnan and S. Kuperstein, Phys. Lett. B677 (2009) 326,
[arXiv:0903.2169[hep-th]].
T. Azeyanagi, G. Compere, N. Ogawa, Y. Tachikawa and S. Terashima, Prog. Theor. Phys. 122 (2009) 355, [arXiv:0903.4176[hep-th]].
X.-N. Wu and Y. Tian, Phys. Rev. D80 (2009) 024014,
[arXiv:0904.1554[hep-th]].
D. Anninos, M. Esole and M. Guica, JHEP 0910 (2009) 083,
[arXiv:0905.2612[hep-th]].
L.-M. Cao, Y. Matsuo, T. Tsukioka and C.-M. Yoo, Phys. Lett. B679 (2009) 390,
[arXiv:0906.2267[hep-th]].
A.J. Amsel, G.T. Horowitz, D. Marolf and M.M. Roberts, JHEP 0909 (2009) 044,
[arXiv:0906.2376[hep-th]].
O.J.C. Dias, H.S. Reall and J.E. Santos, JHEP 0908 (2009) 101,
[arXiv:0906.2380[hep-th]].
V. Balasubramanian, J. de Boer, M.M. Sheikh-Jabbari and J. Simon, JHEP 1002 (2010) 017, [arXiv:0906.3272[hep-th]].
M. Becker, P. Bruillard and S. Downes, JHEP 0910 (2009) 004,
[arXiv:0906.4822[hep-th]].
J. Rasmussen, Int. J. Mod. Phys. A 25 (2010) 1597 [arXiv:0908.0184 [hep-th]].
I. Bredberg, T. Hartman, W. Song and A. Strominger, JHEP 1004 (2010) 019, [arXiv:0907.3477[hep-th]].
M. Cvetic and F. Larsen, JHEP 0909 (2009) 088, [arXiv:0908.1136[hep-th]].
B. Chen and C. S. Chu, JHEP 1005 (2010) 004 [arXiv:1001.3208 [hep-th]].
C.-M. Chen and J.-R. Sun, JHEP 1008 (2010) 034, [arXiv:1004.3963[hep-th]].
Y.-Q. Wang and Y.-X. Liu, JHEP 1008 (2010) 087, [arXiv:1004.4661[hep-th]].
J. Rasmussen, arXiv:1004.4773 [hep-th]; arXiv:1005.2255 [hep-th].
C. Krishnan, [arXiv:1005.1629[hep-th]].
B. Chen and J. Long, JHEP 1006 (2010) 018, [arXiv:1004.5039[hep-th]]; JHEP 1008 (2010) 065, [arXiv:1006.0157[hep-th]].
R. Li, M.-F. Li and J.-R. Ren, Phys. Lett. B691 (2010) 249, [arXiv:1004.5335[hep-th]]; [arXiv:1007.1357[hep-th]].
D. Chen, P. Wang and H. Wu, [arXiv:1005.1404[gr-qc]].
M. Becker, S. Cremonini and W. Schulgin, JHEP 1009 (2010) 022, [arXiv:1005.3571[hep-th]].
H. Wang, D. Chen, B. Mu and H. Wu, [arXiv:1006.0439[gr-qc]].
R. Fareghbal, [arXiv:1006.4034[hep-th]].
C.-M. Chen, Y.-M. Huang, J.-R. Sun, M.-F. Wu and S.-J. Zou, Phys. Rev. D82 (2010) 066003, [arXiv:1006.4092[hep-th]]; Phys. Rev. D82 (2010) 066004, [arXiv:1006.4097[hep-th]].
Y. Matsuo, T. Tsukioka and C. M. Yoo, arXiv:1007.3634 [hep-th].
B. Chen, J. Long and J. j. Zhang, arXiv:1007.4269 [hep-th].
K. N. Shao and Z. Zhang, arXiv:1008.0585 [hep-th].
M. R. Setare and V. Kamali, arXiv:1008.1123 [hep-th].
A. M. Ghezelbash, V. Kamali and M. R. Setare, arXiv:1008.2189 [hep-th].
B. Chen, A. M. Ghezelbash, V. Kamali and M. R. Setare, arXiv:1009.1497 [hep-th].
B. K. Button, L. Rodriguez and C. A. Whiting, arXiv:1009.1661 [hep-th].
R. Li and J. R. Ren, JHEP 1009 (2010) 039 [arXiv:1009.3139[hep-th]].
J. Rasmussen, arXiv:1009.4388 [gr-qc].
B. Chen, C. M. Chen and B. Ning, arXiv:1010.1379 [hep-th].
J. Gegenberg, H. Liu, S. S. Seahra and B. K. Tippett, arXiv:1010.2803 [hep-th].

[6] G. T. Horowitz and M. M. Roberts, Phys. Rev. Lett. 99 (2007) 221601 [arXiv:0708.1346 [hep-th]].
[7] O. J. C. Dias, R. Emparan and A. Maccarrone, Phys. Rev. D 77 (2008) 064018 [arXiv:0712.0791 [hep-th]].

24
[8] Y. Matsuo, T. Tsukioka and C.-M. Yoo, Nucl. Phys. B825 (2010) 231, [arXiv:0907.0303[hep-th]].

[9] A. Castro and F. Larsen, JHEP 0912 (2009) 037, [arXiv:0908.1121[hep-th]].

[10] A. Castro, A. Maloney and A. Strominger, Phys. Rev. D82 (2010) 024008, [arXiv:1004.0996[hep-th]].

[11] M. Guica and A. Strominger, arXiv:1009.5039 [hep-th].

[12] T. Azeyanagi, N. Ogawa and S. Terashima, arXiv:1010.4291 [hep-th].

[13] J.D. Brown and J.W. York, Phys. Rev. D47 (1993) 1407, [arXiv:gr-qc/9209012].

[14] V. Balasubramanian and P. Kraus, Commun. Math. Phys. 208 (1999) 413, [arXiv:hep-th/9902121].

[15] G. Barnich and F. Brandt, Nucl. Phys. B633 (2002) 3, [arXiv:hep-th/0111246].

[16] G. Barnich and G. Compere, J. Math. Phys. 49 (2008) 042901, [arXiv:0708.2378[gr-qc]].

[17] C. Krishnan, JHEP 1007 (2010) 039, [arXiv:1004.3537[hep-th]].

[18] H. Lu, J. Mei and C.N. Pope, JHEP 0904 (2009) 054, [arXiv:0811.2225[hep-th]].

[19] Y. Matsuo, T. Tsukioka and C. M. Yoo, Europhys. Lett. 89 (2010) 60001, [arXiv:0907.4272 [hep-th]].