Slow relaxation dynamics and aging in random walks on activity driven temporal networks

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Received 17 November 2014 / Received in final form 15 December 2014
Published online 4 February 2015 – © EDP Sciences, Società Italiana di Fisica, Springer-Verlag 2015

Abstract. We investigate the dynamic relaxation of random walks on temporal networks by focusing in the recently proposed activity driven model [N. Perra, B. Gonçalves, R. Pastor-Satorras, A. Vespignani, Sci. Rep. 2, 469 (2012)]. For realistic activity distributions with a power-law form, we observe the presence of a very slow relaxation dynamics compatible with aging effects. A theoretical description of this processes in achieved by means of a mapping to Bouchaud’s trap model. The mapping highlights the profound difference in the dynamics of the random walks according to the value of the exponent $\gamma$ in the activity distribution.

1 Introduction

The heterogeneous topology of a complex network [1] can have a very relevant impact on the properties of dynamical systems running on top of it [2,3]. Already classical studies in network science have thus shown that a heterogeneous connectivity pattern can lead to a null percolation threshold [4,5], set a strong resilience against random failures [6], as well as to induce a vanishing epidemic threshold for disease propagation [7], indicative of a strong weakness against infective agents. Similar and additional remarkable effects have been observed in a wide variety of dynamical processes, both in and out of equilibrium (see Refs. [2,3] for extensive reviews on this subject).

Such dynamical effects, originally reported for static networks [1], in which nodes and edges are fixed and do not change over time, can take a different, more complex turn when one considers the intrinsic time-varying, temporal nature of many real networks [8]. Indeed, networked systems are often not static, but show connections which appear and disappear with some characteristic time scales that can be of the same order of magnitude of those ruling a dynamical process on top of the network. Social networks [9] represent the prototypical example of this behavior, being defined in terms of a sequence of social contacts that are continuously established and broken. This mixing of time scales can induce new phenomenology on dynamics on temporal networks, in stark contrast with what is observed in static networks. Moreover, the bursty nature [10–12] of the time evolution of temporal network contacts, characterized by long stretches of inactivity, interspersed by bursts of intense activity, can complicate the picture, inducing for example a noticeable dynamical slowing down in dynamical processes as varied as epidemic spreading, diffusion or synchronization [13–17].

The random walk [18] is one of the simplest dynamical processes, although still underlying many practical realistic applications such as diffusion, searching, community detection and spreading dynamics. Even in this simplest of cases, a time-varying substrate can induce very noticeable differences with respect to the behavior expected in static networks [19–23]. Particularly relevant in this sense is the analysis of the random walk behavior in activity driven networks [24], a class of social temporal network models based on the observation that the establishment of social contacts is driven by the activity of individuals, prompting them to interact with their peers with different levels of intensity, and on the empirical measurement of heterogeneous levels of activity $a$ across different datasets, activity which is found to be distributed according to a power law form, $F(a) \sim a^{-\gamma}$ [24]. The analytic study of the random walk in this class of models, focusing on steady state, large time properties, points out the striking differences imposed by a time-varying topology, with respect to a static one [20]. In the present paper we extend the study of these differences, by considering the time evolution of the dynamics towards the steady state. Using a combination of analytic arguments and numerical simulations, we show that random walks on activity driven networks exhibit a slow relaxation to their steady state. The time scale of the relaxation is inversely proportional to a parameter $\varepsilon$, measuring the smallest activity in the network. In the limit $\varepsilon \to 0$, we found evidence of aging behavior in the random walk relaxation [25], characterized by a breaking of time translation invariances for time scales smaller than $\varepsilon^{-1}$. By means of a mapping to Bouchaud’s trap model [26,27] we show that, for $\gamma < 1$,
random walks on activity driven networks exhibit simple aging, characterized by a unique relevant time scale. On the other hand, the case \( \gamma > 1 \) corresponds to a more complex picture, with several competing characteristic time scales.

We have organized our manuscript as follows: in Section 2 we recall the definition of the activity driven model. Section 3 defines the continuous time implementation of directed random walks on activity driven networks and derives the analytic steady state solution for the occupation probability in terms of a generalized Montroll-Weiss equation. In Section 4 we present numerical evidence for aging behavior of the system, which are numerically analyzed. The mapping suggests additional quantities characterizing the aging behavior of the system, which are numerically analyzed. Our conclusions are finally presented in Section 6.

2 The activity driven network model

The activity driven network model \[24,28\] is defined as follows: \( N \) nodes (individuals) in the network are endowed with an activity \( a_i \in [\varepsilon, 1] \), extracted randomly from an activity distribution \( F(a) \). Every time step \( \Delta t = 1/N \), an agent \( i \) is chosen uniformly at random. With probability \( a_i \), the agent becomes active and generates \( m \) links that are connected to \( m \) other agents, chosen uniformly at random. These links last for a period of time \( \Delta t \) (i.e. are erased at the next time step). Time is updated by \( t \rightarrow t + \Delta t \). For simplicity, we will consider in the following \( m = 1 \). The topological properties of the integrated network at time \( t \) (i.e. the network in which nodes \( i \) and \( j \) are connected if there has ever been a connection between them at any time \( t' \leq t \)) have been studied in reference \[28\], obtaining as a main result that the integrated degree distribution at time \( t \), \( P_t(k) \), scales in the large \( t \) limit as the activity distribution, i.e.

\[
P_t(k) \sim t^{-1}F \left( \frac{k}{t} - \langle a \rangle \right).\tag{1}
\]

Empirical measurements report activity distributions in real temporal networks exhibiting long tails of the form \( F(a) \sim a^{-\gamma} \) \[24\]. This expression thus relates in a simple way the functional form of the activity distribution and the degree distribution of the integrated network at time \( t \), and allows to explain the scale-free form of the latter observed in social networks \[29,30\].

In this paper we will consider activity distributions with this power-law form. The range of values of the \( a \) is restricted to \( a \in [\varepsilon, 1] \), where a minimum activity \( \varepsilon \) is set to avoid divergencies close to zero. The normalized form of the distribution thus depends on the value of \( \gamma \):

\[
F(a) = \frac{1 - \varepsilon}{1 - \varepsilon^{1-\gamma}} a^{-\gamma}.\tag{2}
\]

As we will see later, the parameter \( \varepsilon \) will play a significant role in the analysis of random walks on activity driven networks.

3 Random walks on activity driven networks

The dynamics of a random walk on activity driven networks is defined as follows \[20-22\]: a walker arriving at a node \( j \) at time \( t \) remains on it until an edge is created joining \( i \) and other node \( j \) at a subsequent time \( t' > t \). The walker then jumps instantaneously to node \( j \) and waits there until an edge departing from it is created. To simplify calculations, here we will focus on activated random walks: a walker can leave node \( i \) only when \( i \) becomes active and creates an edge pointing at another node \( \psi_i(n) = a_i(1 - a_i)^{n-1} \), independently of the time of the last activation of \( i \). In the limit of large \( N \) we can take the continuous time limit and define a waiting time \( \tau = n/N \), which is given by a local waiting time distribution

\[
\rho_\tau(a_i) = a_i e^{-a_i \tau}.\tag{3}
\]

That is, the dynamics of hopping from one node to another follows in time a Poisson process with a rate \( a_i \) that depends on node \( i \).

The dynamics of activated random walks under this restriction is particularly easy to implement in continuous time: considering that the walker arrives at vertex \( i \) with activity \( a_i \) at time \( t \), it hops to a randomly selected node \( j \) and time is updated as \( t \rightarrow t + \tau \), where \( \tau \) is a random variable extracted from the distribution equation (3). This continuous time implementation has the additional benefit of not restricting the maximum possible value of the activity \( a \), which can be now considered as a probability rate. With this definition, a directed random walk on an activity driven network can be directly mapped to a continuous time random walk (CTRW) on a fully connected network in which each node has a different distribution of waiting times \( \rho_\tau(\tau) \) \[18\].

Steady state solution

The time evolution of the activated random walk on activity driven networks can be studied by means of the generalized Montroll-Weiss equation approach \[18,22\]. For a Poissonian waiting time distribution, as in equation (3), the occupation probability \( P(i,t) \) of finding the walker in node \( i \) at time \( t \) fulfills the exact equation \[22\]

\[
\frac{dP(i,t)}{dt} = -A_i P(i,t) + \sum_j \lambda_{ij} P(j,t),\tag{4}
\]

where \( \lambda_{ij} \) is the probability per unit time that the walker jumps from node \( j \) to node \( i \), and \( A_i = \sum_j \lambda_{ij} \) is the
probability per unit time that the walker at \( j \) leaves this node. For the activated random walk on activity driven networks, we obviously have \( \lambda_j = a_j/N \) and \( A_j = a_j \). Equation (4) then reads

\[
dP(i,t) \over dt = -a_i P(i,t) + \frac{1}{N} \sum_j a_j P(j,t).
\]

We can obtain an effective equation for the probability \( P(a,t) \) that the walker is in a node of activity \( a \) at time \( t \) by performing a coarse-graining of equation (5), in which we define \( P(a,t) = \frac{1}{w} \sum_{i \in a(t)} P(i,t), \) where \( V(a) \) is the set of nodes with activity \( a \), with an average size \( N_a = N F(a). \) In performing this coarse-graining, vertices with the same activity are treated as equivalent, reasonably assuming that all dynamical properties of a node depend exclusively of its activity. Applying this definition on equation (5), and rearranging the summation over \( j \), we obtain

\[
dP(a,t) \over dt = -a P(a,t) + F(a) \sum_{a'} a' P(a',t). \tag{6}
\]

From equation (6) it is straightforward to obtain the steady state solution \( \lim_{t \to \infty} P(a,t) \equiv P_\infty(a) \) by imposing \( \frac{dP(a,t)}{dt} = 0 \), obtaining

\[
P_\infty(a) = \frac{F(a)}{a} \sum_{a'} a' P_\infty(a') \equiv \frac{1}{\langle a^{-1} \rangle} F(a) \tag{7}
\]

where in the last term we have applied the normalization condition \( \sum_a P_\infty(a) = 1 \), thus recovering the result obtained in reference [20].

### 4 Slow relaxation dynamics

Equation (6) yields information about the occupation probability of nodes with activity \( a \) at large times, equation (7), expression whose accuracy has been checked numerically [20]. From it, however, it is hard to extract information about the time evolution of the process, and in particular, about the time scales of the relaxation to the steady state. We explore this issue by means of numerical simulations. Thus, in Figure 1 we plot the occupation probability \( P(a,t_w) \) of nodes of activity \( a \), measured after letting the walker evolve for a time \( t_w \). As Figure 1 shows, the occupation probability exhibits a very slow relaxation from a state \( P(a,t_w \to 0) \sim F(a) \) at short times to the equilibrium state, \( P_\infty(a) \sim F(a)/a \), at large times (see Eq. (7)). As a function of \( a \) for fixed \( t_w \), this relaxation translates in a crossover between both scaling regimes at a crossover activity \( a_c(t_w) \) which is a decreasing function of \( t_w \).

We can understand the origin of this crossover by the following argument [31]: the average time \( \tau_a \) to exit from a node with activity \( a \) is

\[
\tau_a = \int_0^\infty \tau \psi_a(\tau) = \frac{1}{a} \tag{8}
\]

A walker initially at a node of activity \( a \), is expected to remain there for any time smaller than \( \tau_a \). Since the smallest activity in the network is \( \varepsilon \), for \( t_w > \varepsilon^{-1} \) the walker has had the chance to explore (and scape from) all nodes in the network, and therefore we expect to find it in the steady state. For any arbitrary time \( t_w < \varepsilon^{-1} \), one can thus consider that all nodes with activity \( a \) such that \( \tau_a < t_w \) (large \( a \)) will have had time to relax and reach the steady state, while nodes with activity fulfilling \( \tau_a > t_w \) (small \( a \)) will not have relaxed. We thus see that the crossover activity fulfills \( \tau_a \sim t_w \) or, from equation (8), \( a_c(t_w) \sim t_w^{-1} \).

The previous argument suggests therefore the following scaling form for the whole occupation probability:

\[
P(a,t_w) = t_w P(a, t_w), \tag{9}
\]

where \( P(z) \) is a scaling function satisfying

\[
P(z) \sim \begin{cases} z^{-\gamma} & \text{for } z \ll 1 \\ z^{-\gamma_1} & \text{for } z \gg 1. \end{cases} \tag{10}
\]

This scaling regime is expected to hold for times \( t_w < \varepsilon^{-1} \), i.e. before the full relaxation to the steady state.

In Figure 2 we check the scaling form in equation (9) for activated random walks in activity driven networks with power-law activity distribution. For values of \( \gamma < 1 \), Figure 2a, we observe that the scaling form of the occupation probability is perfectly fulfilled for all times \( t_w < \varepsilon^{-1} \). Surprisingly, however, the scaling form fares quite badly for \( \gamma > 1 \), performing increasingly worst for larger values of \( \gamma \).

### 5 Mapping to Bouchaud’s trap model and aging behavior

The radical difference in behavior of the random walk for \( \gamma \) larger or smaller than 1 can be understood in terms of a
with rate $\tau$ random time $\tau$ temperature, the system remains in a trap of depth $T$ traps, ruled by the temperature of the system, $T$. The trap model proceeds by a succession of jumps between the $\rho$ from the probability distribution $N$ networks with a fined in terms of a phase space consisting on $i$ deep traps, each one with a depth $E_i$, $i = 1, \ldots, N$, extracted randomly from the probability distribution $\rho(E)$. The dynamics of the model proceeds by a succession of jumps between the traps, ruled by the temperature of the system, $T$. At this temperature, the system remains in a trap of depth $E$ a random time $\tau$ distributed according to a Poisson process with rate $\tau E^{-1} \equiv \exp(-E/T)/\tau_0$, where $\tau_0$ is a microscopic time scale that can be arbitrarily set equal to 1. After this time, the system jumps to a randomly chosen trap. Since all traps are equivalent, the probability that the system lands on a trap of depth $E$ after a jump is $\rho(E)$. The average time spent in any trap is thus

$$\langle \tau \rangle = \int \rho(E) \tau E \, dE.$$  \hspace{1cm} (11)

The trap model exhibits a phase transition between a high temperature phase, where $\tau$ is finite, to a low temperature, glassy state characterized by very slow relaxation dynamics and aging behavior [25], when $\langle \tau \rangle$ diverges. This transition takes place at a finite temperature $T_0$ for a depth probability distribution of the form.

$$\rho(E) = \exp(-E/T_0) [26,27],$$  \hspace{1cm} (12)

in which case the distribution of trapping times takes the form:

$$P(\tau) \sim \tau^{1-T/T_0}.$$  \hspace{1cm} (13)

For $T < T_0$, the exponent in the power-law in equation (12) is smaller than 2 and thus leads to an infinite average trapping time.

Activated random walkers can be exactly mapped to the trap model by noticing the mapping between the respective jumping rates, that is:

$$a = \tau_E^{-1} = \exp(-E/T),$$  \hspace{1cm} (13)

from where we obtain the equivalence between depth and activity

$$E = -T \ln(a),$$  \hspace{1cm} (14)

with a range of variation $E \in [0, T \ln(\varepsilon^{-1})]$. The relation between the activity distribution and the depth distribution is given by $\rho(E) dE = F(a) \, da$, leading to

$$\rho(E) = \frac{e^{-E/T}}{T} F\left(\frac{e^{-E/T}}{T}\right).$$  \hspace{1cm} (15)

Assuming now an activity with a power-law distribution as is equation (2), we obtain

$$\rho(E) \sim \frac{1}{T} \exp\left(-\gamma E/T\right).$$  \hspace{1cm} (16)

Let us consider separately the different possible values of $\gamma$.

5.1 Case $\gamma < 1$

From equation (16), the mapping to the trap model makes perfect sense for $\gamma < 1$. In this case, $\rho(E)$ is a decreasing function of the depth $E$, corresponding to the presence of many shallow traps and a few deep ones. Deep traps represent rare long trapping events that eventually dominate the dynamics below the glass transition and induce a very slow relaxation and aging behavior [26,27]. In this case, however, temperature plays no role in the equivalent dynamics of the activated random walkers, as it can be absorbed in a change of variables $\tilde{E} = E/T$, with a range of variation $\tilde{E} \in [0, \ln(\varepsilon^{-1})]$. From the point of view of the mapping to the trap model, the activated random walker is a system in a fully glass state, corresponding to an infinite glass transition temperature. This can easily be seen by looking at the average trapping time distribution, i.e.

$$\psi(\tau) = \sum_a F(a) a e^{-a \tau} \simeq \frac{1}{1 - \varepsilon^{-1}} \int_{\varepsilon}^{\infty} a^{1-\gamma} e^{-a \tau} da \sim \tau^{-2+\gamma} e^{-\varepsilon \tau},$$  \hspace{1cm} (17)

where we have made an expansion for $\tau \ll \varepsilon^{-1}$. The exponent in equation (17) is always smaller than 2, indicative of an infinite glass transition temperature. The average

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig2}
\caption{Occasion probability $P(a,t_w)$ as a function of the activity $a$ at different times $t_w$. Data refer to activity-driven networks with $N = 5 \times 10^3$, $\varepsilon = 10^3$, and (a) $\gamma = 0.25$, (b) $\gamma = 2.00$. Insets: data collapse according to the scaling form equation (9).}
\end{figure}
trapping time is thus modulated by the exponential factor, diverging when $\varepsilon \to 0$, i.e. when the upper cut-off of the associated energy tends to infinity.

This analogy allows to explore in the random walk problem other features of the glassy dynamics of the trap model, in particular aging effects [25]. Aging effects are here usually measured by looking at the two-time correlation function $C(t; t_w)$, between the states of the system at times $t_w$ and $t + t_w$. This correlation function, which is defined as the average probability that the system in a given trap at time $t_w$ has not performed a jump at time $t + t_w$, fulfills in the trap model the scaling relation

$$C(t; t_w) = C\left(\frac{t}{t_w}\right),$$  \hspace{1cm} (18)

corresponding to the so-called “simple” aging [26,27]. This scaling can be simply deduced for the random walk on activity driven networks: since the jumping dynamics is Poissonian in every node, the probability of not leaving a node with activity $a$ in a time interval $t$ is $e^{-at}$. We thus can write

$$C(t; t_w) = \int_z^1 P(a, t_w) e^{-at} da = \int_z^1 t_w P(a, t_w) e^{-at} da$$

$$= \int_{t_w}^{t_w z} P(z) e^{-z t/t_w} dz,$$  \hspace{1cm} (19)

where we have used the scaling relation equation (9) for $P(a, t_w)$ and performed a change of variable. For large $z$, the upper limit of the integral can be safely set equal to infinity, due to the exponential cut-off. In the limit of small $z$, we have $P(z) \sim z^{-\gamma}$, and its integral also converges for $t_w\varepsilon$ small. Thus, in the double limit $1 \ll t_w \ll \varepsilon^{-1}$, we have

$$C(t; t_w) \simeq \int_0^{\infty} P(z) e^{-z t/t_w} dz,$$  \hspace{1cm} (20)

recovering the scaling relation for the correlation equation (18), which is expected to hold for waiting times $t_w \ll \varepsilon^{-1}$. In Figure 3 we show that, for $\gamma < 1$, the scaling of the correlations is very well fulfilled in the random walk process.

Additional information on the aging properties of the system can be gathered by looking at the average escape time $t_{esc}(t_w)$ that a walker at a given node at time $t_w$ requires to escape from it [31]. In Figure 4 (inset) we plot the curves of the escape time as a function of $t_w$, for different values of $\varepsilon$. As we can see, the escape time is a increasing function of $t_w$ for small values of $t_w$, indicating another of the typical features of aging systems, namely a breaking of scale invariance translation. Due to its Poissonian nature, the average time to leave a node with activity $a$ is $\tau_a = 1/a$. Therefore, we can write $t_{esc}(t_w) = \int da P(a, t_w)/a$. Applying the scaling relation equation (9) and performing a change of variables, we have

$$t_{esc}(t_w) = t_w \int_{t_w}^{t_w z} \frac{P(z)}{z} dz.$$  \hspace{1cm} (21)

For large $t_w$, the upper limit of the integral is not singular, and we can set it to infinity. We are therefore led to the scaling form

$$t_{esc}(t_w) \simeq t_w f(t_w \varepsilon),$$  \hspace{1cm} (22)

which is valid for $t_w \ll \varepsilon^{-1}$.

We check this theoretical predictions by means of a data collapse analysis, plotting $t_{esc}(t_w)/t_w$ as a function of $t_w\varepsilon$. In Figure 4 (main) we plot the result, showing an excellent agreement with the prediction.

**5.2 Case $\gamma > 1$**

For the case $\gamma > 1$, equation (16) indicates that the mapping to the trap model is not physical. For this range of $\gamma$ values, the density of traps increases with depth, meaning that very deep traps are much more probable than shallow ones. Long time trapping is thus not a rare event, but the norm of the system. This implies a qualitatively different,
much slower dynamics than for the case $\gamma < 1$. This effect can be simply seen by looking at the coverage $S(t)$ of the random walk as a function of time, defined as the average fraction of different nodes that a walker has visited up to time $t$ (see Fig. 5). For very large $t > \varepsilon^{-1}$, the system has reached the steady state, and the walker jumps from any node at an average increment time $\Delta t = \langle \tau \rangle$, where the average trapping time $\langle \tau \rangle$ is defined as:

$$\langle \tau \rangle = \int_{\varepsilon}^{1} F(a) \tau_a \, da = \langle a^{-1} \rangle = \frac{1 - \gamma \varepsilon^{-\gamma} - 1}{\gamma (1 - \varepsilon^{1-\gamma})}, \quad (23)$$

where we have used equations (23) and (2). In this limit of large $t$, and since walker jumps to randomly chosen new nodes, we expect

$$S(t) \sim \langle a^{-1} \rangle t, \quad t \gg \varepsilon^{-1}. \quad (24)$$

For $\gamma < 1$ and short times, we observe a power law increase of the coverage, $S(t) \sim t^\alpha$. In the case $\gamma > 1$, on the other hand the initial growth is extremely slow, slower numerically than logarithmical, in the region $t < \varepsilon^{-1}$, where the dynamics is dominated by an exceedingly large number of deep traps. After this very slow regime, the linear, steady state behavior is recovered. We can explain the exponent of the initial growth of $S(t)$ for $\gamma < 1$ by the following argument: At any time $t \ll \varepsilon^{-1}$, the random walker will have explored in average nodes with activity restricted to $a > t^{-1}$, since it will not have had time to escape deeper traps. In this case, the average inverse activity of those nodes explored will depend on time as:

$$\langle a^{-1} \rangle_t \sim \int_{t^{-1}}^{1} \frac{F(a)}{a} \, da \sim t^\gamma. \quad (25)$$

We therefore estimate a coverage

$$S(t) \sim \frac{t}{\langle a^{-1} \rangle_t} \sim t^{1-\gamma}, \quad (26)$$

with an exponent $\alpha = 1 - \gamma$ which is a decreasing function of $\gamma$, in agreement with the results in Figure 5.

The anomalous density of deep traps for $\gamma > 1$ has a strong effect in the scaling of the occupation probability. In Figure 2b we have plotted the function $P(a, t_w)$ for $\gamma = 2.0$. From the rescaled plot, it is evident that a simple scaling behavior such as the one represented in equation (9) is doomed to fail. In particular, as we can observe, the occupation probability for small values of $a$ is apparently independent of $t_w$. This fact is in stark contrast with the behavior for $\gamma < 1$, Figure 2a, in which the value of $P(a, t_w)$ for small $a$ increases for increasing $t_w$, as expected from equation (9). This phenomenology is again due to the anomalous abundance of deep traps in the region $\gamma > 1$: thus, while at $t_w$ the walker has had the opportunity to explore and escape from nodes with activity $a > t_w^{-1}$, the fraction of such nodes is very small for large values of $\gamma$. Indeed, denoting by $\phi(t_w)$ the fraction of such nodes, we have

$$\phi(t_w) = \int_{t_w}^{1} F(a) = \frac{1 - t_w^{-1}}{1 - \varepsilon^{1-\gamma}}. \quad (27)$$

Denoting by $\phi^>(t_w)$ (resp. $\phi^<(t_w)$) the fraction corresponding to $\gamma > 1$ (resp. $\gamma < 1$), we have, in the limit of small $\varepsilon$,

$$\phi^>(t_w) \sim (t_w \varepsilon)^{\gamma-1}, \quad \phi^<(t_w) \sim 1 - t_w^{-1}. \quad (28)$$

I.e. the fraction of visited nodes up to time $t_w$ grows much faster for $\gamma < 1$ than in the opposite case.

While the scaling relation equation (9) is invalid for $\gamma > 1$, we can still understand the functional form of the occupation probability by means of the following argument: let us consider a walker at time $t_w$. Initially, the walker starts at a randomly chosen node with activity $a_0$. With probability $e^{\alpha t_w}$, the walker does not move from its initial position, and therefore it contributes to the initial activity $a_0$. With the complementary probability $1 - e^{\alpha t_w}$, the walker has performed one or more jumps. Assuming that after its first jump it already reaches the steady state $F(a)/[a(a-\alpha)]$, we have the average occupation probability

$$P(a, t_w) = F(a) e^{-\alpha t_w} + F(a) \int_{\varepsilon}^{1} e^{-\alpha t_w} F(a_0) \, da_0 \int_{\varepsilon}^{1} e^{-\alpha t_w} F(a_0) \, da. \quad (29)$$

For the activity distribution given by equation (2), we have

$$P(a, t_w) = F(a) e^{-\alpha t_w} + \frac{F(a)}{a(a-\alpha)} \left[ 1 - \int_{\varepsilon}^{1} e^{-\alpha t_w} F(a_0) \, da_0 \right]. \quad (29)$$

where in the integral we have extended the upper limit up to infinity, and $F(a, z)$ is the incomplete Gamma function [33]. In Figure 6 we plot the numerical data for $\gamma = 2$, compared with the theoretical prediction in equation (29). As we can see, except for very small values of...
the activity $a$, the fit of the theoretical prediction with the numerical results is excellent. The mixing of time scales in equation (29), namely $a^{-1}$ and $\varepsilon^{-1}$, allows to understand the failure of the simple scaling ansatz performed to derive equation (9).

Finally, let us consider the average escape time $t_{esc}(t_w)$ for $\gamma > 1$. The lack of simple scaling evidenced in equation (29) indicates that the simple collapse predicted in equation (22) cannot be correct (see Fig. 7 (inset)). We can recover however some form of scaling law for this quantity. Considering a fixed value of $\varepsilon$, the plateau at large $t_w$ is given by the escape time at the stationary state, $t_{esc}^a$, which is independent of $t_w$ and, from equation (7), given by $t_{esc}^a = \int P_\infty(a)/a \, da = (a^{-2})/(a^{-1}) \sim \varepsilon^{-1}$, for all $\gamma$. On the other hand, for small $t_w$, $t_{esc}$ is dominated by the deepest traps, and it starts only to increase when the walker has had time to explore a finite fraction of the network, that is, when time is larger than the average trapping time $\langle \tau \rangle$, which, from equation (23) is proportional to $\varepsilon^{-1}$ for $\gamma > 1$. This reasoning suggests a scaling behavior for the escape time of the form

$$t_{esc}(t_w) = \varepsilon^{-1} \mathcal{G}(t_w, \varepsilon).$$  

(30)

This scaling form in checked in Figure 7 (main), where we can see that it is quite well satisfied. Incidentally, this scaling form can also be cast in the form valid for $\gamma < 1$, equation (22), by simply defining $F(z) \equiv \mathcal{G}(z)/z$.

6 Conclusions

In this paper we have investigated the temporal relaxation of the simplest dynamical process, namely the random walk, on the class of activity driven temporal networks. We have focused in particular in the case of activated random walks, in which a walker can only leave a node when the latter becomes active. By means of a combination of analytic calculations and numerical experiments, we have shown that, for networks with a power law distribution of activity, the random walk experiences a very slow relaxation towards its steady state. The speed of this relaxation is mainly controlled by the parameter $\varepsilon$, bounding the smallest activity of any node. In the limit of small $\varepsilon \to 0$, the dynamics exhibits aging behavior, characterized additionally by a breaking of time translation symmetry. The aging properties of the random walk are studied by examining different quantities usually applied to characterize aging in glassy systems. Crucially, the aging properties of the random walk depend on the exponent $\gamma$ in the activity distribution. For $\gamma < 1$, the random walk exhibits a relaxation compatible with “simple” aging, with a unique characteristic time scale given by the average escape time from the least active node, $\varepsilon^{-1}$. In this regime, simple scaling forms for the two-time correlation function and the average escape time can be worked out, starting from a scaling ansatz for the occupation probability $P(a, t_w)$. For $\gamma > 1$, on the other hand, the picture is more complex, with a scaling ruled by several characteristic time scales. This different behavior according to $\gamma$ can be understood by means of a mapping to Bouchaud’s trap model: the case $\gamma > 1$ corresponds in this case to an unphysical representation of the trap model, in which there is a majority of deep traps that induce an extraordinary slow relaxation dynamics.

The results obtained here indicate that, apart form the slowing down already reported in dynamical systems on temporal networks, more complex effects, such as aging, can also be observed. While here we have focused on the simplest case of a random walk, those effects should be relevant, and must be taken into account, in the study of general dynamical processes on temporal networks. In the present case, in which the temporal network substrate is the activity driven model, aging emerges as the result of the mixing of Poisson activation processes with widely different time scales. We remark that this mixing, which we have made evident in a power-law activity distribution (the empirically observed one), can arise for any functional form $F(a)$ leading to a diverging average trapping time, as given by $\langle \tau \rangle = \int F(a)/a$. Obviously, a more complex
phenomenology is to be expected in real temporal networks with non Poissonian, bursty activation rates.

We thank A. Barrat for very helpful comments and discussions. This work has been supported by the CAPES under project No. 5511-13-5. RP-S acknowledges financial support from the Spanish MINECO, under projects Nos. FIS2010-21781-C02-01 and FIS2013-47282-C2-2, and EC FET-Proactive Project MULTIPLEX (Grant No. 317532).

References

1. M.E.J. Newman, Networks: An introduction (Oxford University Press, Oxford, 2010)
2. S.N. Dorogovtsev, A.V. Goltsev, J.F.F. Mendes, Rev. Mod. Phys. 80, 1275 (2008)
3. A. Barrat, M. Barthélemy, A. Vespignani, Dynamical Processes on Complex Networks (Cambridge University Press, Cambridge, 2008)
4. R. Cohen, K. Erez, D. ben Avraham, S. Havlin, Phys. Rev. Lett. 85, 4626 (2000)
5. D.S. Callaway, M.E.J. Newman, S.H. Strogatz, D.J. Watts, Phys. Rev. Lett. 85, 5468 (2000)
6. R. Albert, H. Jeong, A. Barabasi, Nature 406, 378 (2000)
7. R. Pastor-Satorras, C. Castellano, P. Van Mieghem, A. Vespignani, arXiv:1408.2701 (2014)
8. P. Holme, J. Saramäki, Phys. Rep. 519, 97 (2012)
9. M. Jackson, Social and Economic Networks (Princeton University Press, Princeton, 2010)
10. J.G. Oliveira, A.L. Barabási, Nature 437, 1251 (2005)
11. J.P. Onnela, J. Saramäki, J. Hyvönen, G. Szabó, D. Lazer, K. Kaski, J. Kertész, A.L. Barabási, Proc. Natl. Acad. Sci. 104, 7332 (2007)
12. C. Cattuto, W. Van den Broeck, A. Barrat, V. Colizza, J.F. Pinton, A. Vespignani, PLoS One 5, e11596 (2010)
13. A. Vazquez, B. Rácz, A. Lukács, A.L. Barabási, Phys. Rev. Lett. 98, 158702 (2007)
14. H.H. Jo, J.I. Perotti, K. Kaski, J. Kertész, Phys. Rev. X 4, 011041 (2014)
15. M. Kivelä, R. Kumar Pan, K. Kaski, J. Kertesz, J. Saramaki, M. Karsai, J. Stat. Mech. 2012, P03005 (2012)
16. J. Stehle et al., BMC Medicine 9, 87 (2011)
17. N. Fujisawa, J. Kurths, A. Díaz-Guilera, Phys. Rev. E 83, 025101 (2011)
18. G.H. Weiss, Aspects and Applications of the Random Walk (North-Holland Publishing Co., Amsterdam, 1994)
19. M. Starnini, A. Baronchelli, A. Barrat, R. Pastor-Satorras, Phys. Rev. E 85, 056115 (2012)
20. N. Perra, A. Baronchelli, D. Mocanu, B. Gonçalves, R. Pastor-Satorras, A. Vespignani, Phys. Rev. Lett. 109, 238701 (2012)
21. B. Ribeiro, N. Perra, A. Baronchelli, Sci. Rep. 3, 3006 (2013)
22. T. Hoffmann, M.A. Porter, R. Lambiotte, Phys. Rev. E 86, 046102 (2012)
23. L. Speidel, R. Lambiotte, K. Aihara, N. Masuda, Phys. Rev. E 91, 012806 (2015)
24. N. Perra, B. Gonçalves, R. Pastor-Satorras, A. Vespignani, Sci. Rep. 2, 469 (2012)
25. M. Henkel, M. Pleimling, Non-equilibrium Phase Transition: Ageing and Dynamical Scaling far from Equilibrium (Springer-Verlag, Dordrecht, 2010)
26. J. P. Bouchaud, J. Phys. I France 2, 1705 (1992)
27. C. Monthus, J.P. Bouchaud, J. Phys. A 29, 3847 (1996)
28. M. Starnini, R. Pastor-Satorras, Phys. Rev. E 87, 062807 (2013)
29. M.E.J. Newman, Proc. Natl. Acad. Sci. USA 98, 404 (2001)
30. M.E.J. Newman, Phys. Rev. E 64, 016132 (2001)
31. A. Baronchelli, A. Barrat, R. Pastor-Satorras, Phys. Rev. E 80, 020102 (2009)
32. P. Moretti, A. Baronchelli, A. Barrat, R. Pastor-Satorras, J. Stat. Mech. 2011, P03032 (2011)
33. M. Abramowitz, I.A. Stegun, Handbook of Mathematical Functions (Dover Publications, New York, 1972)