On the OA(1536,13,2,7) and related orthogonal arrays

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Abstract

With a computer-aided approach based on the connection with equitable partitions, we establish the uniqueness of the orthogonal array OA(1536,13,2,7), constructed in [D.G.Fon-Der-Flaass. Perfect 2-Colorings of a Hypercube, Sib. Math. J. 48 (2007), 740–745] as an equitable partition of the 13-cube with quotient matrix [[0, 13], [3, 10]]. By shortening the OA(1536,13,2,7), we obtain 3 inequivalent orthogonal arrays OA(768,12,2,6), which is a complete classification for these parameters too.

After our computing, the first parameters of unclassified binary orthogonal arrays OA(N,n,2,t) attending the Friedman bound $N \geq 2^n(1 - n/2(t + 1))$ are OA(2048,14,2,7). Such array can be obtained by puncturing any binary 1-perfect code of length 15. We construct orthogonal arrays with these and similar parameters OA($N = 2^n-m+1, n = 2^m-2, 2, t = 2^{m-1} - 1$), $m \geq 4$, that are not punctured 1-perfect codes.

Additionally, we prove that any orthogonal array OA($N,n,2,t$) with even $t$ attending the bound $N \geq 2^n(1 - (n + 1)/2(t + 2))$ induces an equitable 3-partition of the $n$-cube.

Keywords: orthogonal array, equitable partition, correlation-immune Boolean function, hypercube
MSC: 05B15

1. Introduction

Orthogonal arrays are combinatorial structures interesting from both theoretical and practical points of view. In different applications like design of experiments or software testing, orthogonal arrays are important as a good approximation of the Hamming space. The classification of orthogonal arrays with given parameters is a problem that attracts attention of many researchers, see the recent works [3], [4], and the bibliography there.

An orthogonal array OA($N,n,q,t$) is an $N$ by $n$ array $A$ with entries from $\{0, \ldots, q-1\}$ such that, within any $t$ columns of $A$, every ordered $t$-tuple of symbols from $\{0, \ldots, q-1\}$
occurs in exactly $\lambda = N/q^t$ rows of $A$. In this paper, we characterize the orthogonal arrays with parameters $\text{OA}(1536, 13, 2, 7)$ and $\text{OA}(768, 12, 2, 6)$, which are related to each other, lie on the Bierbrauer–Friedman [1, 8] and Bierbrauer–Gopalakrishnan–Stinson [2] bounds, respectively, and are also connected with equitable partitions of the 13-cube and the 12-cube. These parameters were listed as open questions in Table 12.1 of the monograph [10] ($k = 13, t = 7$ and $k = 12, t = 6$). An orthogonal array $\text{OA}(1536, 13, 2, 7)$ was constructed by Fon-Der-Flaass in [3], in terms of equitable partitions, as a special case of a general construction, and $\text{OA}(768, 12, 2, 6)$ is obtained from $\text{OA}(1536, 13, 2, 7)$ by shortening. The orthogonal-array (correlation-immune) properties of equitable partitions were known, see e.g. [6], but not mentioned in [5], and the construction in that paper was unnoticed by specialists in orthogonal arrays for some time, see, e.g., Table 5 in [3], where these parameters are still marked as unsolved.

Utilizing local properties of equitable partitions, we use exhaustive computer search to find all $\text{OA}(1536, 13, 2, 7)$ up to equivalence and establish that there is only one equivalence class of such orthogonal arrays. A similar approach was already used in [19] to characterize the orthogonal arrays $\text{OA}(1024, 12, 2, 7)$. However, the direct generalization of the approach of [19] did not work for $\text{OA}(1536, 13, 2, 7)$, and in this paper we modify it, considering the value of the first coordinate separately from the others. All calculations took about one core-year of computing on a 2GHz processor. By shortening the unique $\text{OA}(1536, 13, 2, 7)$, we find all inequivalent $\text{OA}(768, 12, 2, 6)$.

The small parameters of binary orthogonal arrays attending the Friedman bound $N \geq 2^n(1 - n/2(t + 1))$ [8, Theorem 2.1] and satisfying the Fon-Der-Flaass–Khalyavin bound $t \leq 2n/3 - 1$ [6, 13] are shown in Table 1. The first unclassified case is $\text{OA}(2048, 14, 2, 7)$.

| OA          | quotient matrix | number of equivalence classes |
|-------------|-----------------|-------------------------------|
| $\text{OA}(2, 3, 2, 1)$ | [0, 3], [1, 2]   | 1                             |
| $\text{OA}(16, 6, 2, 3)$ | [0, 6], [2, 4]   | 1                             |
| $\text{OA}(16, 7, 2, 3)$ | [0, 7], [1, 6]   | 1 (the Hamming (7, 16, 3) code) |
| $\text{OA}(128, 9, 2, 5)$ | [0, 9], [3, 6]   | 2 (see [13])                  |
| $\text{OA}(1024, 12, 2, 7)$ | [0, 12], [4, 8]  | 16 (see [19])                 |
| $\text{OA}(1536, 13, 2, 7)$ | [0, 13], [3, 10] | 1 (Theorem 2)                 |
| $\text{OA}(2048, 14, 2, 7)$ | [0, 14], [2, 12] | $> 14960$ (Corollary 1)       |
| $\text{OA}(2048, 15, 2, 7)$ | [0, 15], [1, 14] | 5983 (1-perfect (15, 2^{11}, 3) codes [23]) |
| $\text{OA}(8192, 15, 2, 9)$ | [0, 15], [5, 10] | ?                             |

Table 1: A list of small parameters $\text{OA}(N, n, 2, t)$ of binary orthogonal arrays satisfying $N = 2^n(1 - n/2(t + 1))$, $t \leq 2n/3 - 1$, and the corresponding equitable partitions

It can be shown that puncturing (projecting in one coordinate) any 1-perfect binary code of length 15 gives an orthogonal array with these parameters. Such codes, of parameters $(15, 2^{11}, 3)$, were classified by Östergård and Pottonen [23], and all the punctured codes can be derived from that classification. In Section 7 we show that this is not enough for the complete classification of $\text{OA}(2048, 14, 2, 7)$: there are such orthogonal arrays that are not punctured 1-perfect codes.
The parameters $\text{OA}(768, 12, 2, 6)$ attend the general theoretical bound $N \geq 2^n - 2^{n-2}(n+1)/[(t+1)/2]$ \cite{2} for $\text{OA}(N, n, 2, t)$. As a theoretical contribution in addition to the computational results, in this paper we prove that the orthogonal arrays attending this bound are in one-to-one correspondence with the equitable 3-partitions with a special quotient matrix. Thus, one more family is added to the collection of classes of optimal objects (e.g., perfect and nearly perfect codes, some other classes of codes \cite{15, 16}, orthogonal arrays \cite{26}, correlation-immune functions \cite{6}) whose parameters guarantee that they can be described in terms of equitable partitions.

The paper is organized as follows. In the next section, we define the basic concepts, mainly related with orthogonal arrays and equitable partitions, and mention some basic theoretical facts. In Section 3, we consider known and new (Theorem 1) general theoretical results connecting equitable partitions and orthogonal arrays. In Section 4, we describe the classification approach. The results of the classification of the orthogonal arrays $\text{OA}(1536, 13, 2, 7)$ and $\text{OA}(768, 12, 2, 6)$ can be found in Section 5. In Section 6, we describe the unique $\text{OA}(1536, 13, 2, 7)$ in two ways, by the Fon-Der-Flaass construction and by the Fourier transform. Section 7 is devoted to the orthogonal arrays $\text{OA}(2048, 14, 2, 7)$ and arrays with similar parameters. In the concluding section, we highlight some open research problems.

2. Definitions

Definition 1 (graphs and related concepts). A (simple) graph is a pair $(V, E)$ of a set $V$, whose elements are called vertices, and a set $E$ of 2-subsets of $V$, called edges. Two vertices in the same edge are called neighbor, or adjacent, to each other. The number of neighbors of a vertex is referred to as its degree. A graph whose vertices have the same degree is called regular. An isomorphism between two graphs is a bijection between their vertices that induces a bijection between the edges. Two graphs are isomorphic if there is an isomorphism between them. An automorphism of a graph is an isomorphism to itself. A set of vertices of a graph is called independent if it does not include any edge.

Definition 2 (Hamming graphs and related concepts). The Hamming graph $H(n, q)$ is a graph whose vertex set is the set $\{0, 1, \ldots, q-1\}^n$ of the words of length $n$ over the alphabet $\{0, 1, \ldots, q-1\}$. Two vertices are adjacent if and only if they differ in exactly one coordinate position, which is referred to as the direction of the corresponding edge. The Hamming distance $d(\bar{x}, \bar{y})$ between vertices $\bar{x}$ and $\bar{y}$ is the number of coordinates in which they differ. The weight $\text{wt}(\bar{x})$ of a word $\bar{x}$ is the number of nonzero elements in it. In this paper, we focus on the binary Hamming graph $H(n, 2)$, also known as the $n$-cube $Q_n$. The vertices of $Q_n$ are also considered as vectors over the 2-element field $\text{GF}(2)$, with the coordinate-wise addition and multiplication by a constant.

For two words $\bar{u}$ and $\bar{v}$, we denote by $\bar{u}|\bar{v}$ their concatenation. The all-zero word and the all-one word are denoted by $\bar{0}$ and $\bar{1}$, respectively (the length is usually clear from the context).

Definition 3 (orthogonal arrays and related concepts). An orthogonal array $\text{OA}(N, n, q, t)$ is a multiset $C$ of vertices of $H(n, q)$ of cardinality $N$ such that every sub-
graph isomorphic to \( H(n-t, q) \) contains exactly \( N/q^t \) elements of \( C \). An orthogonal array is \textit{simple} if it is a usual set; that is, if it does not contain elements of multiplicity more than 1. Two orthogonal arrays are \textit{equivalent} if some automorphism of \( H(n, q) \) induces a bijection between their elements. The \textit{automorphism group} \( \text{Aut}(C) \) of an orthogonal array \( C \) (as well as any other set \( C \) of vertices of \( H(n, q) \)) consists of all automorphisms of \( H(n, q) \) that stabilize \( C \) set-wise. The \textit{orbit} of a vertex \( \bar{v} \) under the action of \( \text{Aut}(C) \) is the set of images \( \text{Aut}(\bar{v}) \) over all \( A \) from \( \text{Aut}(C) \). The \textit{kernel} of \( C \subseteq \{0, 1\}^n \) is the set \( \{\bar{k} \in \{0, 1\}^n : \bar{k} + C = C\} \) of all its periods. For any set \( C \) of vertices of \( H(n, q) \), by \( \overline{C} \) we denote its complement \( \{0, 1, \ldots, q-1\}^n \setminus C \). Obviously, the complement of a simple \( \text{OA}(N, n, q, t) \) is a simple \( \text{OA}(q^n - N, n, q, t) \). We say that an orthogonal array \( C' \) is obtained from an orthogonal array \( C \) by \textit{a-shortening}, or simply by \textit{shortening}, in the \( i \)-th position (by default, in the last position) if \( C' \) is obtained from \( C \) by choosing all words with a symbol \( a \in \{0, \ldots, q-1\} \) in the \( i \)-th position and removing this symbol in this position from all chosen words.

\textbf{Remark 1.} In the current paper, motivated by the coding-theory technique used in this research, we consider orthogonal arrays as sets of vertices of the Hamming graph (for generality, they are defined as multisets, but all considered arrays are simple). The definition above is equivalent to the traditional definition mentioned in the introduction, but uses a different language. According to our definition, the elements of the array are words, corresponding to the rows \( \text{(runs)} \) in the traditional definition, and the positions in the words, or the coordinates, numbered from 1 to \( n \), correspond to the columns \( \text{(factors)} \) in the traditional definition. In the classical literature on orthogonal arrays, the parameters \( N, n, q, t, \lambda = N/q^t \) are known as respectively the number of (experimental) runs, the number of factors, the number of levels, the strength, and the index of the orthogonal array.

\textbf{Definition 4 (equitable partitions).} Let \( G = (V, E) \) be a graph. A partition \((C_0, \ldots, C_{k-1})\) of the set \( V \) is an \textit{equitable partition} (in some literature, \textit{regular partition}, \textit{perfect coloring}, or \textit{partition design}), or \textit{equitable k-partition}, with the quotient matrix \( S = (s_{ij}) \) if for all \( i \) and \( j \) from \( \{0, \ldots, k-1\} \) every vertex of \( C_i \) has exactly \( s_{ij} \) neighbors in \( C_j \). Below, for convenience, a \( 2 \times 2 \) quotient matrix will be represented by its row list: 
\[
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix} = [[a, b], [c, d]].
\]

\textbf{3. Connections between orthogonal arrays and equitable partitions}

The following folklore fact establishes the orthogonal-array properties of an equitable partition. For equitable 2-partitions of \( Q_n \) (the case we focus on), it can be found, e.g., in [6].

\textbf{Proposition 1.} Each cell \( C \) of an equitable partition of \( H(n, q) \) with quotient matrix \( S \) is a simple \( \text{OA}(|C|, n, q, t) \), where \( t = \frac{n(q-1) - \theta}{q} - 1 \) and \( \theta \) is the second largest eigenvalue of \( S \). In particular, \( t = \frac{b+c}{q} - 1 \) if \( S = [[a, b], [c, d]] \).
Proof (a sketch). It follows from the general theory of equitable partitions \[9\] that the characteristic \(\{0, 1\}\)-function of every cell of the equitable partition can be represented as the sum of eigenfunctions of the graph with eigenvalues that coincide with eigenvalues of the quotient matrix \(S\). The eigenspaces of the Hamming graph have very convenient bases from so-called characters (see Section 6.2 for the definition in the binary case). It is straightforward to check that for every character corresponding to a non-largest eigenvalue of \(S\), the sum of values over the vertices of a subgraph isomorphic to \(H(n - t, q)\) is 0. For the largest eigenvalue, an eigenfunction is a constant function. This means that \(C\) has a constant number of vertices in every subgraph isomorphic to \(H(n - t, q)\); i.e., it is an OA(|\(C|, n, q, t\)). ▲

In some cases, the parameters of an orthogonal array guarantee that it is a cell of an equitable partition. One of the known bounds on the parameters of orthogonal arrays, proved by Friedman \[8, Theorem 2.1\] for the binary case \(q = 2\) and by Bierbrauer \[1\] for an arbitrary \(q\), says that the size \(N\) of an OA\((N, n, q, t)\) satisfies the inequality

\[
N \geq q^n \left(1 - \frac{(q - 1)n}{q(t + 1)}\right) . \tag{1}
\]

As follows from the proof, see \[1\] p. 181, line 4], the inequality is strict for non-simple arrays (with repeated elements). Moreover, an orthogonal array that attains this bound is an independent set and forms an equitable 2-partition, in the pair with its complement.

**Proposition 2** (\[25\] \(q = 2\), \[26\]). If \(1\) holds with equality for some OA\((N, n, q, t)\) \(C\), then \((C, \overline{C})\) is an equitable partition with quotient matrix

\[
\begin{pmatrix}
0 & (q-1)n \\
(q+1)-(q-1)n & 2(q-1)n-q(t+1)
\end{pmatrix}, \text{ in particular, } \begin{pmatrix}
0 & n \\
2(t+1)-n & 2n-2(t+1)
\end{pmatrix} \text{ if } q = 2.
\]

So, by Propositions \[1\] and \[2\], there is a bijection between the orthogonal arrays attaining the Bierbrauer–Friedman bound and the equitable 2-partitions of the Hamming graph with the first coefficient of the quotient matrix being 0.

Next, we consider a bound for binary orthogonal arrays of even strength, which follows straightforwardly from the Friedman bound and the following fact about lengthening a binary array of even strength (this fact can be considered as a dual analog of the possibility of extending a binary \(n, M, 2e + 1\) code to an \((n + 1, M, 2e + 2)\) code, well known in the theory of error-correcting codes, see e.g. \[20\] 1.9(I))].

**Proposition 3** (\[27\] Proposition 2.3]). If \(t\) is even, then every orthogonal array OA\((N, n, 2, t)\) can be obtained from some OA\((2N, n+1, 2, t+1)\) by shortening. Specifically, if \(C\) is an OA\((N, n, 2, t)\), then \(C|0 \cup C''|1\), where \(C'' = C + \overline{1}\), is an OA\((2N, n+1, 2, t+1)\).

As was noted by V. Levenshtein (cited in \[2\] as a private communication), Proposition 3 with \(1\) imply the inequality

\[
N \geq 2^n \left(1 - \frac{n + 1}{2(t + 2)}\right) . \tag{2}
\]

for the parameters of a binary orthogonal array OA\((N, n, 2, t)\) of even strength \(t\). We can note that every orthogonal array attending bound \(2\) is a cell of an equitable 3-partition.
**Theorem 1.** Assume that \( C \) is an orthogonal array OA\((N, n, 2, t)\) of even strength \( t \) meeting (2) with equality. If \( C' = C + \bar{1} = \{\bar{c} + \bar{1} : \bar{c} \in C\} \) and \( C'' = \{0, 1\}^n \setminus (C \cup C') \), then \((C, C', C'')\) is an equitable partition with quotient matrix

\[
\begin{pmatrix}
0 & 2t-n+2 & 2n-2t-2 \\
2t-n+2 & 0 & 2n-2t-2 \\
2t-n+3 & 2t-n+3 & 3n-4t-6
\end{pmatrix}
= \begin{pmatrix}
0 & a & n-a \\
a & 0 & n-a \\
a+1 & a+1 & n-2a-2
\end{pmatrix}, \quad \text{where } a = 2t - n + 2.
\]

(3)

**Proof.** Denote \( C' = C + \bar{1} \) and \( B = C|0 \cup C'|1 \). By Proposition 3, \( B \) is an orthogonal array OA\((2N, n+1, 2, t+1)\). By Proposition 2, \((B, \overline{B})\) is an equitable partition with quotient matrix

\[
\begin{pmatrix}
0 & n+1 & 2n-2t-2 \\
2t-n+3 & n+1 & 2n-2t-2
\end{pmatrix}
= \begin{pmatrix}
0 & n+1 & 2n-2t-2 \\
0 & a+1 & n-a
\end{pmatrix}, \quad \text{where } a = 2t - n + 2.
\]

(note that \( n \) and \( t \) in Proposition 2 correspond respectively to \( n+1 \) and \( t+1 \) in our case). Denoting \( C'' = \{0, 1\}^n \setminus (C \cup C') \), we observe that \((C, C', C'')\) is a partition of \( \{0, 1\}^n \) (indeed, \( C \) and \( C' \) are disjoint because \( B \) is an independent set). It remains to check that it is an equitable partition.

Consider a vertex \( \bar{v} \) from \( C \). As \( B \) is an independent set, \( C \) is an independent set too, and \( \bar{v} \) has 0 neighbors in \( C \). Moreover, \( \bar{v}|0 \) from \( B \) has no neighbors in \( B \), and \( \bar{v}|1 \) from \( \overline{B} \) has \( a+1 \) neighbors in \( B \); one of them is \( \bar{v}|0 \) and the other are in \( C''|1 \). Hence, \( \bar{v} \) has \( a \) neighbors in \( C' \). The other neighbors of \( \bar{v} \) are in \( C'' \), and the first row of the quotient matrix (3) is confirmed. The second row is similar. For the third row, consider a vertex \( \bar{u} \) from \( C'' \). Both \( \bar{u}|0 \) and \( \bar{u}|1 \) are in \( \overline{B} \). Each of them has \( a+1 \) neighbors in \( B \), but those neighbors of \( \bar{u}|0 \) are in \( C|0 \), while those neighbors of \( \bar{u}|1 \) are in \( C'|1 \). So, \( \bar{u} \) has exactly \( a+1 \) neighbors in \( C \) and exactly \( a+1 \) neighbors in \( C'' \); the third row of the quotient matrix is confirmed. ▲

**Remark 2.** The equitable partitions with quotient matrices (3) are connected (in a one-to-one manner) with a special class of completely regular codes. A completely regular code of covering radius \( \rho \) is the first cell of an equitable \((\rho + 1)\)-partition with a tridiagonal quotient matrix. We start from an equitable partition \((C, C', C'')\) with quotient matrix (3). Divide each of \( C, C' \) into two subsets, respectively \( C_{\text{even}} \) and \( C_{\text{odd}} \), \( C'_{\text{even}} \) and \( C'_{\text{odd}} \), according to the parity of the weight of vertices. It is straightforward to check that \((C_{\text{even}} \cup C'_{\text{odd}}, C''_{\text{even}} \cup C'_{\text{odd}})\) is an equitable partition with quotient matrix

\[
\begin{pmatrix}
a & n-a & 0 \\
a+1 & n-2a-2 & a+1 \\
0 & n-a & a
\end{pmatrix}.
\]

So, \( C_{\text{even}} \cup C'_{\text{odd}} \) (as well as \( C'_{\text{even}} \cup C'_{\text{odd}} \)) is a completely regular code.

The parameters of orthogonal arrays OA\((1536, 13, 2, 7)\) lie on the Bierbrauer–Friedman bound (1). It is straightforward to see that the corresponding quotient matrix is \([\{0, b\}, \{c, d\}] = \{[0, 13], [3, 10]\} \) (indeed, \( 0 + b = c + d = 13 \) and \( c : (b+c) = 1536 : 2^{13} \)); equitable partitions with this quotient matrix are known to exist [5, Proposition 2]. One of the main results of the current research is establishing that the OA\((1536, 13, 2, 7)\) constructed
4. Classification of $\text{OA}(1536, 13, 2, 7)$

For classification by exhaustive search, we use an approach based on the local properties of the equitable partitions. Say that the pair of disjoint sets $P_+, P_-$ of vertices of $Q_{13}$ is an $(r_0, r_1)$-local partition if

(I) $P_+ \cup P_-$ are the all words starting with 0 and having weight at most $r_0$ or starting with 1 and having weight at most $r_1$;

(II) $P_+$ contains the all-zero word $\vec{0}$;

(III) $P_+$ is an independent set;

(IV) the neighborhood of every vertex $\vec{v} = (v_1, \ldots, v_{13})$ of weight less than $r_{v_1}$ satisfies the local condition from the definition of an equitable partition with quotient matrix $[[0, 13], [3, 10]]$ (that is, if $\vec{v} \in P_+$ then the whole neighborhood of $\vec{v}$ is included in $P_-$; if $\vec{v} \in P_-$ then the neighborhood has exactly 3 elements in $P_+$ and 10 in $P_-$).

Two $(r_0, r_1)$-local partitions $(P_+, P_-)$ and $(P'_+, P'_-)$ are equivalent if there is a permutation of coordinates that fixes the first coordinate and sends $P_+$ to $P'_+$.

We classify all inequivalent $(r_0, r_1)$-local partitions subsequently for $(r_0, r_1)$ equal (2, 2), (2, 3), (2, 4), (3, 4), (4, 4), (4, 5), (5, 5), (13, 13), where (13, 13) corresponds to the complete equitable partitions. In an obvious way, every equitable partition $(C, \overline{C})$ such that $\vec{0} \in C$ includes a (5, 5)-local partition $(P_+^{(5,5)}, P_-^{(5,5)})$, $P_+^{(5,5)} \subseteq C$ and $P_-^{(5,5)} \subseteq \overline{C}$, every (5, 5)-local partition $(P_+^{(5,5)}, P_-^{(5,5)})$ includes a (4, 5)-local partition $(P_+^{(4,5)}, P_-^{(4,5)})$, $P_+^{(4,5)} \subseteq P_+^{(5,5)}$ and $P_-^{(4,5)} \subseteq P_-^{(5,5)}$, and so on. So, the strategy is to reconstruct, in all possible ways, a $(r_0, r_1)$-local partition from each of the inequivalent $(r_0 - 1, r_1)$-local or $(r_0, r_1 - 1)$-local partitions, and then to choose and keep only inequivalent solutions, one representative for each equivalence class found. Our classification is divided into the following steps.
1. (Section 4.1) Manually characterizing the \((2, 2)\)-local partitions, up to equivalence.

2. (Section 4.2) Characterizing, up to equivalence, the \((2, 3)\)-local partitions based on the known representatives of \((2, 2)\)-local partitions and using the exact-covering software \([12]\). Similarly, from \((2, 3)\) to \((2, 4)\), from \((2, 4)\) to \((3, 4)\), from \((3, 4)\) to \((4, 4)\), from \((4, 4)\) to \((4, 5)\), from \((4, 5)\) to \((5, 5)\). Equivalence is recognized using the graph-isomorphism software \([21]\). The results (see Table 2) are validated by double-counting using the orbit-stabilizer theorem.

3. (Section 4.3) Reconstructing an equitable partition from a \((5, 5)\)-local partition. It follows from the definition of orthogonal arrays that a complete equitable partition can be reconstructed in a unique way.

Table 2: The number of equivalence classes of \((r_0, r_1)\)-local partitions classified by the type of the included \((2, 2)\)-local partition

| type | \((r_0, r_1) = (2, 3)\) | \((2, 4)\) | \((3, 4)\) | \((4, 4)\) | \((4, 5)\) | \((5, 5)\) |
|------|----------------|--------|--------|--------|--------|--------|
| 4+3+3+3+3 | 266 | 33077 | 912 | 0 |
| 3+4+3+3+3 | 475 | 97550 | 187335 | 0 |
| 7+3+3 | 2315 | 861699 | 97841 | 0 |
| 3+7+3+3 | 2540 | 839273 | 1198056 | 0 |
| 6+4+3+3 | 3492 | 1362844 | 37234 | 0 |
| 4+6+3+3 | 4134 | 748748 | 3724 | 0 |
| 3+6+4 | 2404 | 861732 | 452111 | 0 |
| 5+5+3+3 | 2611 | 1194122 | 69325 | 10 | 20 | 20 |
| 3+5+5+3 | 1156 | 444846 | 330614 | 12 | 12 | 12 |
| 10+3 | 25784 | 11598959 | 699031 | 14 | 20 | 20 |
| 3+10 | 10579 | 4336586 | 3656845 | 19 | 15 | 12 |
| 5+4+4+4 | 1397 | 565938 | 7864 | 0 |
| 4+5+4+4 | 3785 | 701873 | 1192 | 0 |
| 9+4 | 19809 | 9262166 | 186257 | 0 |
| 4+9 | 15802 | 3240956 | 9203 | 0 |
| 8+5 | 15149 | 7843990 | 229791 | 0 |
| 5+8 | 9518 | 5006596 | 147247 | 0 |
| 7+6 | 12777 | 6436913 | 185167 | 0 |
| 6+7 | 10901 | 5446544 | 124577 | 0 |
| 13 | 150346 | 77748861 | 242510 | 0 |
| any | 295240 | 138633273 | 10049836 | 55 | 67 | 64 |

4.1. The \((2, 2)\)-local partitions

A starting point of our classification is the \((2, 2)\)-local partitions, which can be classified manually.
Lemma 1. There are exactly 20 equivalence classes of (2, 2)-local partitions.

Proof. Assume that \((P_+, P_-)\) is a (2, 2)-local partition. By the definition, \(\emptyset \in P_+\). Moreover, all 13 weight-1 words belong to \(P_-\). Each of them has 3 neighbors in \(P_+\), by the definition of an equitable partition. One of these 3 neighbors is \(\emptyset\), while the other two have weight 2. On the 13 weight-1 words, we construct a graph \(\Gamma_{13}\), two vertices being connected if and only if they have a common weight-2 neighbor in \(P_+\). We see that this graph is regular of degree 2 (i.e., a 2-factor, consisting of disjoint cycles), and it completely determines \(P_+\) and hence \(P_-\). There are 10 such graphs, up to isomorphism, with cycle structures \(4+3+3+3, 7+3+3, 6+4+3, 5+5+3, 10+3, 5+4+4, 9+4, 8+5, 7+6,\) and 13. However, two isomorphic graphs correspond to inequivalent (2, 2)-local partitions if and only if the weight-1 word with 1 in the first coordinate belongs to cycles of different length in these two graphs. So, inequivalent (2, 2)-local partitions correspond to non-isomorphic pairs (a 2-factor on 13 vertices, a chosen vertex). There are exactly 20 such non-isomorphic pairs, with the cycle structures \(3+4+3+3, 4+3+3+3, 3+7+3, 7+3+3, 3+6+4, 4+6+3, 6+4+3, 3+5+3, 5+5+3, 3+10, 10+3, 4+5+4, 5+4+4, 4+9, 9+4, 5+8, 8+5, 6+7, 7+6,\) and 13, where the first (dotted) summand corresponds to the length of the cycle that contains the chosen vertex. ▲

4.2. From (2, 2) to (2, 3), (2, 4), \ldots, (5, 5)

We describe these steps by the example of the case \((2, 3) \rightarrow (2, 4)\), as the other cases are completely similar and solved with the same c++ program with different parameters.

4.2.1. Completing to (2, 4)-local partition

Denote by \(W_i^j\) the set of words of weight \(j\) that start with \(i\). As the result of the previous step, we keep representatives of all the equivalence classes of (2, 3)-local partitions. For each representative \((P_+, P_-)\), we need to find a subset \(R\) of \(W_i^4\) such that \((P_+ \cup R, P_- \cup (W_i^4 \setminus R))\) is a (2, 4)-local partition, i.e., satisfies (I)–(IV).

Conditions (I) and (II) are satisfied automatically. To satisfy condition (III), we remove from \(W_i^4\) all the words that have a neighbor from \(P_+\). The set obtained, call it \(U\), is the set of candidates for the role of elements of \(R\). It remains to satisfy condition (IV) for all the vertices from \(W_i^3 \cap P_-\) (for the vertices from \(P_+\), it is satisfied by (III); for the vertices from \(W_0^0, W_0^1, W_1^1,\) and \(W_2^2\), it is satisfied because of the (2, 3)-local property). For each vertex \(\bar{u}\) from \(W_i^3 \cap P_-\), denote \(\alpha(\bar{u}) = 3 - \beta(\bar{u})\), where \(\beta(\bar{u})\) is the number of neighbors of \(\bar{u}\) in \(P_+\). By the definition of a (2, 4)-local partition, \(\bar{u}\) must have exactly \(\alpha(\bar{u})\) neighbors in \(R\). So, to meet (IV), we have to find a collection \(R\) of elements from \(U\) such that every element \(\bar{u}\) from \(W_i^3 \cap P_-\) belongs to exactly \(\alpha(\bar{u})\) neighbors of elements of \(R\). This is an instance of the problem known as exact covering. A convenient package to solve this problem (with different multiplicities \(\alpha(\bar{u})\), which is important in our case) in C and C++ programs is libexact [12]. After finding all the solutions \(R\), we have all the (2, 4)-local partitions that include the given (2, 3)-local partition \((P_+, P_-)\).
4.2.2. Isomorph rejection

As we need to keep only inequivalent (2, 4)-local partitions, it is important to compare such partitions for equivalence. It is done with the help of the well-known graph-isomorphism software [21]. The standard technique, described in [11], consists of constructing for each object (in our case, a (2, 4)-local partition) a graph such that two objects are equivalent if and only if the corresponding graphs are isomorphic. Using the nauty&traces package [21], from each graph we can construct the canonical-labeling graph such that two graphs are isomorphic if and only if the corresponding canonical-labeling graphs are equal. Each time we find a new (2, 4)-local partition, we construct the canonical-labeling graph \( G \) and check whether it is contained in our collection (of inequivalent (2, 4)-local partitions and the corresponding canonical-labeling graphs). If not, we update the collection with the new representative and the corresponding canonical-labeling graph \( G \), and set the value of a special variable \( N(G) \), the number of occurrences, equal to 1 (the final value of \( N(G) \) is utilized in the validation step, see the next subsection). If the graph \( G \) is already in the collection, we only increase \( N(G) \) by 1. When the search is finished, our collection contains representatives of all the equivalence classes of (2, 4)-local partitions.

4.2.3. Validation

We can validate the results of the calculation by double-counting the size of each equivalence class found. Let \((P'+, P'_-)\) be a (2, 4)-local partition, and let it include a (2, 3)-local partition \((P_+, P_-)\). On one hand, there are exactly

\[
\frac{12!}{|\text{Aut}(P', P')|}
\]

(2, 4)-local partitions equivalent to \((P'+, P'_-)\), where \(\text{Aut}(P'+, P'_-)\) is the set of permutations of the last 12 coordinates that stabilize \(P'_+\) and \(P'_-\) set-wise. On the other hand, this number equals

\[
N(P'_+, P'_-) \cdot \frac{12!}{|\text{Aut}(P_+, P_-)|},
\]

where \(N(P'_+, P'_-)\) is the number of (2, 4)-local partitions that are equivalent to \((P'_+, P'_-)\) and include the (2, 3)-local partition \((P_+, P_-)\). As our algorithm finds all (2, 4)-local partitions that include a given (2, 3)-local partition, the number \(N(P'_+, P'_-)\) for each found equivalence class is computed during the isomorph rejection step and equals the final value of \(N(G)\) for the corresponding graph \(G\), see the previous subsection. If the number \(N(P'_+, P'_-)\) is counted correctly, then we know that we did not miss any representative of the equivalence class during the experiment. So, calculating the values (4) and (5) and comparing them for equality prevents many kinds of random and systematical errors. This strategy represents a special case of the general double-counting validation technique described in [12], 10.2. Note that \(|\text{Aut}(P'_+, P'_-)|\) coincides with the order of the automorphism group of the corresponding characteristic graph; it is computed by nauty&traces as a part of finding the canonical-labeling graph.
4.3. Completing to an equitable partition

Completing a $(5, 5)$-local partition $(P_+, P_-)$ to an equitable partition $(C, \overline{C})$ of $Q_{13}$ is the easiest step, and the result is always unique (however, the fact that it always exists is still only empiric). We know that $P_+$ consists of all the vertices of the orthogonal array $C$ of weight 5 or less. Every subgraph of $Q_{13}$ isomorphic to $Q_6$ contains exactly $\lambda = 12$ vertices of $C$. For every vertex $\bar{u}$ of weight 6, there is such subgraph that contains $\bar{u}$ and $2^6 - 1$ vertices of smaller weight. Counting the number of vertices of $P_+$ among them, we can determine whether $\bar{u}$ belongs to $C$ or not. After finding, in this way, all weight-6 elements of $C$, we can repeat the similar procedure for the weight 7, then 8, 9, 10, 11, 12, and 13.

5. Results of the classification

**Theorem 2.** There is only one orthogonal array $OA(1536, 13, 2, 7)$, up to equivalence. Its automorphism group has order 480; the orthogonal array is partitioned into orbits of sizes 240, 240, 240, 240, 240, 48, 48, and the complement is partitioned into 2 orbits of size 48, 4 orbits of size 80, 18 orbits of size 240, and 4 orbits of size 480. The kernel has size 4 and contains words of weight 0, 6, 7, and 13.

By Proposition 3 every OA$(768, 12, 2, 6)$ can be obtained by shortening some OA$(1536, 13, 2, 7)$. Since the OA$(1536, 13, 2, 7)$ is unique up to equivalence, shortening it in different positions we get all the OA$(768, 12, 2, 6)$, also up to equivalence. Under the action of the automorphism group of the OA$(1536, 13, 2, 7)$, the positions are divided into three orbits by 1, 6, and 6, corresponding to the three equivalence classes of OA$(768, 12, 2, 6)$. We should also note that the OA$(1536, 13, 2, 7)$ is invariant under translation by $\bar{1}$ (see Proposition 3 or the claim of Theorem 2 about the kernel); so, the results of 0-shortening and 1-shortening in the same position are equivalent.

**Theorem 3.** There are three orthogonal arrays $OA(768, 12, 2, 6)$, up to equivalence. One of them has the automorphism group of order 240, with orbit sizes 120, 120, 120, 120, 120, 120, 24, 24. Each of the other two arrays has the automorphism group of order 40; two orbits of size 4, 14 orbits of size 20, and 12 orbits of size 40.

6. Representations of OA$(1536, 13, 2, 7)$

6.1. The Fon-Der-Flaass construction

The following is a special case of a construction from [5]. The construction starts from an equitable partition $(C_6, \overline{C}_6)$ with quotient matrix $[[1, 5], [3, 3]]$. The cell $C_6$ is partitioned into edges; we use the notation $i(\bar{c})$ to indicate the direction of the edge that contains a vertex $\bar{c}$ of $C_6$. To be explicit, we list all words $\bar{c}$ of the cell $C_6$ (which is known as
The Fourier transform of a real-valued (or complex-valued) function $f$ on $\{0, 1\}^n$ is the collection of the coefficients $\hat{f}(\bar{y})$, $\bar{y} \in \{0, 1\}^n$, in the expansion

$$f(\bar{x}) = \sum_{\bar{y} \in \{0, 1\}^n} \hat{f}(\bar{y})(-1)^{\bar{y} \cdot \bar{x}}$$

of $f$ in terms of the orthogonal basis from the characters $\psi_\bar{y}(\bar{x}) = (-1)^{\bar{y} \cdot \bar{x}}$, where $\langle(y_1, \ldots, y_n), (x_1, \ldots, x_n)\rangle = y_1x_1 + \ldots + y_nx_n$. The Fourier transform, whose variants are also known as the Walsh–Hadamard transform and the MacWilliams transform, is an important representation of a function or a set of vertices in $\{0, 1\}^n$. In particular, it is well known and straightforward that a multiset of vertices in $\{0, 1\}^n$ is an OA$(N, n, 2, t)$ if and only if the Fourier transform $\hat{f}$ of its multiplicity function satisfies $\hat{f}(\bar{0}) = N/2^n$ and $\hat{f}(\bar{y}) = 0$ for all $\bar{y}$ of weight $1, 2, \ldots, t$. On the other hand, a set of vertices of $Q_n$ is a cell of an equitable 2-partition of $Q_n$ if and only if the nonzero $(\bar{y} \neq \bar{0})$ nonzeros $(\hat{f}(\bar{y}) \neq 0)$ of the Fourier transform $\hat{f}$ of its characteristic function have the same weight. The Fourier transform of $C_{13}$ was found computationally. It can be seen from the construction in the previous subsection that the two-cycle coordinate permutation $(2 3 4 5 6)(8 9 10 11 12)$ is an automorphism or $C_{13}$; it follows that the Fourier transform is also invariant under this coordinate permutation.

**Proposition 4 (a special case of [15, Proposition 2]).** The partition $(C_{13}, \overline{C}_{13})$, where

$$C_{13} = \{(\bar{b}|\bar{c}|b_1+b_2+b_3+b_4+b_5+b_6+b_i(\bar{c})+c_i(\bar{c})) : \bar{b} = (b_1\ldots b_6) \in \{0, 1\}^6, \bar{c} = (c_1\ldots c_6) \in C_6\},$$

is equitable with quotient matrix $[[0, 13], [3, 10]]$, and $C_{13}$ is an OA$(1536, 13, 2, 7)$.

**Remark 3.** The Fon-Der-Flaass construction [5] admits the possibility of switching the resulting equitable partition. In our case, we can choose an edge $\{c', c''\}$ in $C_6$, and change the value of the last coordinate for the $2^7$ vertices of $C_{13}$ corresponding in (7) to $\bar{c} \in \{\bar{c}', \bar{c}''\}$. This operation, switching, results in an equitable partition with the same quotient matrix. Since this can be done with each of the 12 edges, switching gives $2^{12}$ different equitable partitions. By Theorem [2] all these partitions are equivalent in the considered special case, which can be considered as a surprising result of the classification. In general, the construction in combination with switching gives a huge number of inequivalent equitable partitions of $Q_n$ as $n$ grows [30].

### 6.2. The Fourier transform

The Fourier transform of a real-valued (or complex-valued) function $f$ on $\{0, 1\}^n$ is the collection of the coefficients $\hat{f}(\bar{y})$, $\bar{y} \in \{0, 1\}^n$, in the expansion

$$f(\bar{x}) = \sum_{\bar{y} \in \{0, 1\}^n} \hat{f}(\bar{y})(-1)^{\bar{y} \cdot \bar{x}}$$

of $f$ in terms of the orthogonal basis from the characters $\psi_\bar{y}(\bar{x}) = (-1)^{\bar{y} \cdot \bar{x}}$, where $\langle(y_1, \ldots, y_n), (x_1, \ldots, x_n)\rangle = y_1x_1 + \ldots + y_nx_n$. The Fourier transform, whose variants are also known as the Walsh–Hadamard transform and the MacWilliams transform, is an important representation of a function or a set of vertices in $\{0, 1\}^n$. In particular, it is well known and straightforward that a multiset of vertices in $\{0, 1\}^n$ is an OA$(N, n, 2, t)$ if and only if the Fourier transform $\hat{f}$ of its multiplicity function satisfies $\hat{f}(\bar{0}) = N/2^n$ and $\hat{f}(\bar{y}) = 0$ for all $\bar{y}$ of weight $1, 2, \ldots, t$. On the other hand, a set of vertices of $Q_n$ is a cell of an equitable 2-partition of $Q_n$ if and only if the nonzero $(\bar{y} \neq \bar{0})$ nonzeros $(\hat{f}(\bar{y}) \neq 0)$ of the Fourier transform $\hat{f}$ of its characteristic function have the same weight. The Fourier transform of $C_{13}$ was found computationally. It can be seen from the construction in the previous subsection that the two-cycle coordinate permutation $(2 3 4 5 6)(8 9 10 11 12)$ is an automorphism or $C_{13}$; it follows that the Fourier transform is also invariant under this coordinate permutation.
Theorem 4. The Fourier decomposition

\[ \chi_{C_{13}}(\bar{x}) = \sum_{\bar{y} \in \{0,1\}^{13}} \phi(\bar{y})(-1)^{\langle \bar{y}, \bar{x} \rangle} \]

of the characteristic \{0,1\}-function of the orthogonal array \( C_{13} \) defined in (7) has 1 + 111 nonzero coefficients \( \phi(\bar{y}) \). The collection of coefficients \( \phi(\bar{y}) \) is invariant under the coordinate permutation \( \pi = (2 \ 3 \ 4 \ 5 \ 6)(8 \ 9 \ 10 \ 11 \ 12) \). Below is the list of representatives \( \bar{y} \) under \( \pi \) corresponding to the nonzero values of \( \phi(\bar{y}) \).

| value of \( \phi \) | representatives under \( \pi = (2 \ 3 \ 4 \ 5 \ 6)(8 \ 9 \ 10 \ 11 \ 12) \) |
|---------------------|-------------------------------------------------|
| 3/16                | 0 000000|0 000000|0 |
| -1/16               | 1 01011|1 01011|0 |
| 1/16                | 1 00111|1 00111|0, 0 01111|0 01111|0 |
| -1/32               | 0 10100|1 01111|1, 0 01010|1 01111|1, 0 01111|1 00100|1, 0 01111|1 00010|1, 1 00111|0 11100|1 |
| 1/32                | 1 11111|1 00000|1, 1 01110|1 00000|1, 1 11001|1, 1 10101|1 01010|1, 1 00100|1 11001|1, 1 00100|1 01111|1, 1 11111|0 00100|1, 1 10101|0 11110|1 01010|1, 1 01110|0 10101|1, 1 01110|0 10101|1, 0 11110|1 10001|1, 0 01111|1 10001|1, 0 11000|1 01111|1, 0 00011|1 11110|1 |

It can be noted that all \( \bar{y} \) with \( \phi = \pm 1/16 \) are all the 15 words of form \((\bar{u}|\bar{u}|0), \) where \( \bar{u} \in \{0,1\}^6 \) and \( wt(\bar{u}) = 4 \). Further, all \( \bar{y} \) with \( \phi = \pm 1/32 \) are all the 96 weight-8 words of form \((\bar{u}|\bar{w}|1), \) where \( \bar{u}, \bar{w} \in \{0,1\}^6, \) wt(\(\bar{u}\)) is even, and the positions of zeros in \( \bar{u} \) and \( \bar{w} \) are disjoint.

7. On \( OA(2048, 14, 2, 7) \) and similar parameters

In this section, we construct orthogonal arrays \( OA(2^{2m-3}, 2^{m-2}, 2, 2^{m-1}-1) \) that cannot be extended to 1-perfect codes of length \( 2^m - 1 \). In particular, this means that the characterization of the orthogonal arrays \( OA(2048, 14, 2, 7) \) cannot be done by characterizing only punctured 1-perfect codes in \( Q_{14} \).

Definition 5 (1-perfect codes and related concepts). A set \( C \) of vertices of \( H(n, q) \) is called an \( l \)-fold 1-perfect code (in the case \( l = 1 \), simply a 1-perfect code) if \((C, \overline{C})\) is an equitable partition with quotient matrix

\[
\begin{pmatrix}
  l-1 & n(q-1)-l+1 \\
  l & n(q-1)-l
\end{pmatrix}
\]

(in particular, \[
\begin{pmatrix}
  0 & n(q-1) \\
  1 & n(q-1)-1
\end{pmatrix}
\]
if \( l = 1 \)),

that is, if every radius-1 ball in \( H(n, q) \) contains exactly \( l \) words of \( C \). Obviously, the union of disjoint \( l \)- and \( l' \)-fold 1-perfect codes in \( H(n, q) \) is an \((l + l')\)-fold 1-perfect code. A 2-fold 1-perfect code is called splittable (unsplittable) if it can (respectively, cannot) be represented as the union of two 1-perfect codes. A set \( C \) of vertices in \( Q_n \) is called a punctured 1-perfect code if \( C = C' \cup C'' \) where \( C'|0 \cup C''|1 \) is a 1-perfect code.

Theorem 5. For every \( m \geq 4 \), there is an orthogonal array \( OA(2^{2m-3}, 2^{m-2}, 2, 2^{m-1}-1) \) that is not a punctured 1-perfect code.
Proof. Unsplittable 2-fold 1-perfect binary codes were constructed in [18] in every $Q_n$ such that $n = 2^m - 1 \geq 15$. We will construct such set with an additional property such that after shortening it gives a required orthogonal array.

At first, we need a set $M_k \subset \{0, 1, 2, 3\}^k$, $k = 2^{m-2} \geq 4$, of vertices of $H(k, 4)$ with the following properties:

(I) for every word $\bar{x}$ in $\{0, 1, 2, 3\}^k$ and for every position $i$ from $\{1, \ldots, k\}$, exactly two words from $M_k$ have the same values as $\bar{x}$ in all positions may be except the $i$-th position (in terms of [18], $M_k$ is a 2-fold MDS code);

(II) $M_k$ cannot be partitioned into two independent sets (in terms of [18], it is unsplittable);

(III) $(x_1, \ldots, x_{k-1}, 0) \in M_k$ if and only if $(x_1, \ldots, x_{k-1}, 1) \in M_k$;

$(x_1, \ldots, x_{k-1}, 2) \in M_k$ if and only if $(x_1, \ldots, x_{k-1}, 3) \in M_k$.

We construct $M_k$ in three steps.

1. We start with defining $M_2, M'_2 \subset \{0, 1, 2, 3\}^2$ by listing their elements:

$$M_2 = \{00, 01, 10, 12, 22, 23, 31, 33\}, \quad M'_2 = \{00, 01, 11, 12, 22, 23, 30, 33\}.$$  \hfill (8)

2. Define $M_3 = M_2|0 \cup M'_2|1 \cup \overline{M_2}|2 \cup \overline{M'_2}|3$.

3. Recursively define $M_i = M_{i-1}|0 \cup M_{i-1}|1 \cup \overline{M_{i-1}}|2 \cup \overline{M'_{i-1}}|3$, $i = 4, \ldots, k$.

From step 1, we can directly check (I) for $i = 1, 2$. Step 2 guarantees (I) for $i = 3$. Step 3 guarantees (I) for $i = 4, \ldots, k$ and (III). The 7-cycle induced by the vertices

$$010\ldots 0, \ 000\ldots 0, \ 100\ldots 0, \ 120\ldots 0, \ 121\ldots 0, \ 111\ldots 0, \ 011\ldots 0$$

supports (II) because it is impossible to distribute these 7 elements between two independent sets.

Next, we enumerate the words of $\{0, 1\}^3$:

$$\bar{z}_{0,0} = 000, \ \bar{z}_{1,0} = 110, \ \bar{z}_{2,0} = 011, \ \bar{z}_{3,0} = 101,$$

$$\bar{z}_{0,1} = 111, \ \bar{z}_{1,1} = 001, \ \bar{z}_{2,1} = 100, \ \bar{z}_{3,1} = 010,$$

the even-weight words of $\{0, 1\}^4$:

$$\bar{y}_{i,0,b} = \bar{z}_{i,b}|b, \quad i = 0, 1, 2, 3, \quad b = 0, 1,$$

and the odd-weight words of $\{0, 1\}^4$:

$$\bar{y}_{i,1,b} = \bar{z}_{i,b}|(1 - b), \quad i = 0, 1, 2, 3, \quad b = 0, 1.$$

We choose a 1-perfect code in $\{0, 1\}^{k-1}$ and denote it $P_{k-1}$. For example, $P_3$ can be $\{000, 111\}$. Now, we define

$$C_{2^{m-1}} = \{(\bar{y}_{a_1,c_1,b_1}| \cdots |\bar{y}_{a_{k-1},c_{k-1},b_{k-1}}|\bar{z}_{a_k,b_k}) : (a_1, \ldots, a_k) \in M_k, \ (c_1, \ldots, c_{k-1}) \in P_{k-1}, \ (b_1, \ldots, b_k) \in \{0, 1\}^k\}.$$ \hfill (9)

Construction (9) is a variant of the Phelps construction [24] of 1-perfect binary codes adopted in [18] to construct $l$-fold 1-perfect codes. We state the following.
completed by considering only the punctured partition. The Fon-Der-Flaass construction \cite[Proposition 2]{5} allows to construct equitable Friedman bound is not known in infinitely many cases; this is another challenging direction. The existence of binary (and non-binary) orthogonal arrays attending the Bierbrauer–Friedman bound is not known in infinitely many cases; this is another challenging direction. The Bierbrauer–Friedman bound is not known in infinitely many cases; this is another challenging direction.

Theorem 6. There are exactly 14960 equivalence classes of punctured 1-perfect codes in $Q_{14}$. For codes in 14874 of these classes, the set of even-weight codewords is equivalent to the set of odd-weight codewords; for the remaining 86 classes, this is not the case.

Corollary 1. The number of equivalence classes of OA(2048, 14, 2, 7) is strictly greater than 14960.

8. Conclusions and open problems

We classified all the orthogonal arrays OA(1536, 13, 2, 7) and OA(768, 12, 2, 6) and proved that the classification of the orthogonal arrays OA(2048, 14, 2, 7) cannot be completed by considering only the punctured 1-perfect binary codes of length 14. The classification of the OA(2048, 14, 2, 7) remains an open challenging problem. The approach developed in the current paper is probably too hard to complete the case OA(2048, 14, 2, 7), and finishing the classification is expected to require more theoretical results or(and) more computing capacity.

The existence of binary (and non-binary) orthogonal arrays attending the Bierbrauer–Friedman bound is not known in infinitely many cases; this is another challenging direction. The Fon-Der-Flaass construction \cite[Proposition 2]{5} allows to construct equitable 2-partitions of hypercubes for infinite series of quotient matrices with the first coefficient 0. However, there are putative quotient matrices of type $[[0, n], [c, d]]$ that are not covered.
by that construction. In this direction, the first two open questions are about existence of equitable partitions with quotient matrices \([0, 25], [7, 18]\) and \([0, 27], [5, 22]\) (equivalently, orthogonal arrays \(\text{OA}(7 \cdot 2^{20}, 25, 2, 15)\) and \(\text{OA}(5 \cdot 2^{22}, 27, 2, 15)\)).

Summarizing two theoretical results of the current paper, Theorem 1 and Theorem 5 we can conclude that an orthogonal array with the orthogonal-array parameters \(\text{OA}(2^{n-m+1}, n = 2^m - 3, 2, 15)\) of a shortened punctured 1-perfect binary code (a code obtained by puncturing and then shortening a 1-perfect code) induces an equitable 3-partition with the quotient matrix \([0, 1, 2^m - 4], [1, 0, 2^m - 4], [2, 2, 2^m - 7]\), but is not necessarily a shortened punctured 1-perfect code. This is an OA analog of similar results for the error-correcting codes with (code) parameters of doubly or triply shortened 1-perfect binary codes \([15, 16, 17, 22]\). Noting the nice algebraic and combinatorial properties of equitable partitions, it worth to look for more results showing that some classes of (optimal) combinatorial configurations are in one-to-one correspondence with classes of equitable partitions with specially defined quotient matrices.

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