Syllepsis in Homotopy Type Theory

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In Homotopy Type Theory, the following two properties hold:

- **Eckmann-Hilton** (Favonia, Christensen, Shulman, et al.): any two 2-loops \( p, q : 1 = 1 \) based at reflexivity commute.

- **Syllepsis** (S., Rijke): for any two 3-loops \( p, q : 1_1 = 1_1 \) based at reflexivity on reflexivity, the Eckmann-Hilton proof that \( q \) and \( p \) commute is the inverse of the Eckmann-Hilton proof that \( p \) and \( q \) commute.

The dimensions cannot be lowered: Eckmann-Hilton does not hold for 1-loops (counterexample: non-commuting endofunctions) and syllepsis does not hold for 2-loops (counterexample due to Vicary).
Outline

- Introduction
- Preliminaries
- The Eckmann-Hilton Proof
- Properties of The Eckmann-Hilton Proof
- Syllepsis
- Proof of Syllepsis: The Square, The Triangles, and The Result
- Future Work
Lemma

For any points \( a, b, c : A \), 1-paths \( u : a = b \), \( x, y : b = c \), and 2-path \( q : x = y \), we have a term

\[
\text{whisk-L}(u, q) : u \cdot x = u \cdot y
\]

Pictorially:

\[
\begin{array}{ccc}
\begin{array}{c}
 a \\
 b \\
 c
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
 u \\
 x \\
 y \\
 q \downarrow
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
 a \\
 b \\
 c
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
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 u \\
 y \\
 q \downarrow
\end{array}
\end{array}
\]

\[
\begin{array}{ccc}
\begin{array}{c}
 a \\
 b \\
 c
\end{array}
\end{array}
\]
Lemma
For any points \( a, b, c : A \), 1-paths \( u, v : a = b \), \( x : b = c \), and 2-path \( p : u = v \), we have a term

\[
\text{whisk-R}(p, x) : u \cdot x = v \cdot x
\]

Pictorially:

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {a};
\node (b) at (1,0) {b};
\node (c) at (2,0) {c};
\node (p) at (1,1) {p};
\draw (a) -- (b) node[midway, above] {u};
\draw (b) -- (c) node[midway, above] {x};
p \Downarrow
a \quad b \quad c
\end{tikzpicture}
\end{array}
\]

\[
\begin{array}{c}
\begin{tikzpicture}
\node (a) at (0,0) {a};
\node (b) at (1,0) {b};
\node (c) at (2,0) {c};
\node (p) at (1,1) {v};
\draw (a) -- (b) node[midway, above] {v};
\draw (b) -- (c) node[midway, above] {x};
p \Downarrow
\end{tikzpicture}
\end{array}
\]
Whiskering Exchange Law

Lemma

For any points \( a, b, c : A \), 1-paths \( u, v : a = b, x, y : b = c \), and 2-paths \( p : u = v, q : x = y \), we have a term

\[
\text{whisk-L-R}(p, q)
\]

witnessing the commutativity of the diagram

\[
\begin{array}{ccc}
  & \text{whisk-R}(p, x) & \\
\hline
u \cdot x & \text{whisk-L}(u, q) & v \cdot x \\
\hline
whisk-R(p, y) & \text{whisk-L}(v, q) & v \cdot y
\end{array}
\]
Lemma

Concatenation on the left by reflexivity is natural: for any points \( a, b : A \), 1-paths \( u, v : a = b \), and 2-path \( p : u = v \), we have a term

\[
\Box-1-L-nat(p)
\]

witnessing the commutativity of the diagram

\[
\begin{array}{ccc}
1_a \cdot u & \Box-1-L(u) & u \\
\downarrow \text{whisk-}L(1_a, p) & & \downarrow p \\
1_a \cdot v & \Box-1-L(v) & v \\
\end{array}
\]
Lemma

Concatenation on the right by reflexivity is natural: for any points $a, b : A$, 1-paths $x, y : a = b$, and 2-path $q : x = y$, we have a term

$$\mathsf{■-1-R-nat}(q)$$

witnessing the commutativity of the diagram

$$\begin{array}{ccc}
  x \cdot 1_b & \quad & \mathsf{■-1-R}(x) \\
\downarrow & & \downarrow \phantom{1}q \\
\mathsf{whisk-R}(q, 1_b) & & \mathsf{■-1-R}(y) \\
  y \cdot 1_b & \quad & y
\end{array}$$
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The Eckmann-Hilton Proof

Theorem (Eckmann-Hilton)

For any point $\star : A$ and 2-loops $p, q : 1_\star = 1_\star$, we have a 3-path $EH(p, q)$:

$$p \cdot q$$

$$\text{whisk-L}(1_\star, p) \cdot \text{whisk-R}(q, 1_\star)$$

$$\text{whisk-R}(q, 1_\star) \cdot \text{whisk-L}(1_\star, p)$$

$$q \cdot p$$
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The term $EH(1, q)$ is equal to

$$1 \cdot q \quad -1-L(q) \quad q \quad -1-R(q)^{-1} \quad q \cdot 1$$

The term $EH(p, 1)$ is equal to

$$p \cdot 1 \quad -1-R(p) \quad p \quad -1-L(p)^{-1} \quad 1 \cdot p$$
Naturality of Eckmann-Hilton

Lemma
For any 2-loops $u, v, x : 1 = 1$, and 3-path $q : u = v$, we have a term

$$EH-L-nat(q, x)$$

witnessing the commutativity of the diagram

$$
\begin{array}{ccc}
EH(u, x) & & EH(v, x) \\
\downarrow whisk-R(q, x) & & \downarrow whisk-L(x, q) \\
v \cdot x & & x \cdot v \\
\end{array}
$$
Naturality of Eckmann-Hilton

Lemma
For any 2-loops $u, x, y : 1 = 1$, and 3-path $p : x = y$, we have a term

$$EH-R-nat(u, p)$$

witnessing the commutativity of the diagram

\[
\begin{array}{ccc}
EH(u, x) & & x \cdot u \\
\mathit{EH}(u, x) & & \\
\mathit{whisk-L}(u, p) & & \mathit{whisk-R}(p, u) \\
\mathit{whisk-L}(u, p) & & \mathit{whisk-R}(p, u) \\
\mathit{EH}(u, y) & & y \cdot u \\
\end{array}
\]
The term EH-L-nat(q, 1_1) is equal to

\[ \text{whisk-R}(q, 1_1) \]
The term $\text{EH-R-nat}(1_1, p)$ is equal to

$$\begin{align*}
\text{whisk-}L(1, p) & \\
1_1 \cdot 1_1 & \rightarrow_{\text{■-1-L}(1)} 1_1 \quad & \rightarrow_{\text{■-1-R}(1)^{-1}} 1_1 \cdot 1_1
\end{align*}$$

$p$

$$\begin{align*}
\text{whisk-}R(p, 1) & \\
1_1 \cdot 1_1 & \rightarrow_{\text{■-1-L}(1)} 1_1 \quad & \rightarrow_{\text{■-1-R}(1)^{-1}} 1_1 \cdot 1_1
\end{align*}$$
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Syllepsis

Theorem
For any point \(* : A\) and 3-loops \(p, q : 1_{1*} = 1_{1*}\), we have

\[ EH(q, p) = EH(q, p)^{-1} \]
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Syllepsis: The Square, The Triangles, and The Result

We can split the diagram as follows:

\[
\begin{align*}
p \cdot q \quad & \rightarrow \quad whisk-R(p, 1_1) \cdot whisk-L(1_1, q) \\
whisk-L(1_1, p) \cdot whisk-R(q, 1_1) \quad & \rightarrow \quad whisk-R(p, 1_1) \cdot whisk-L(1_1, q) \\
whisk-R(q, 1_1) \cdot whisk-L(1_1, p) \quad & \rightarrow \quad whisk-L(1_1, q) \cdot whisk-R(p, 1_1) \\
q \cdot p
\end{align*}
\]
Syllepsis: The Square

Generalize to $p : x = y$ and $q : u = v$ for arbitrary 2-loops $x, y, u, v : 1 = 1$:

\[
\begin{align*}
\text{whisk-L}(u, p) \cdot \text{whisk-R}(q, y) & \quad ? \quad \text{whisk-R}(p, u) \cdot \text{whisk-L}(y, q) \\
\text{whisk-R}(q, x) \cdot \text{whisk-L}(v, p) & \quad ? \quad \text{whisk-L}(x, q) \cdot \text{whisk-R}(p, v)
\end{align*}
\]

But: endpoints do not match! We need to insert Eckmann-Hilton.
Syllepsis: The Square

Generalize to $p : x = y$ and $q : u = v$ for arbitrary 2-loops $x, y, u, v : 1 = 1$:

$$\text{whisk-L}(u, p) \cdot \text{whisk-R}(q, y) \quad ? \quad \text{whisk-R}(p, u) \cdot \text{whisk-L}(y, q)$$

$$\text{whisk-R}(q, x) \cdot \text{whisk-L}(v, p) \quad ? \quad \text{whisk-L}(x, q) \cdot \text{whisk-R}(p, v)$$

But: endpoints do not match!
Syllepsis: The Square

Generalize to $p : x = y$ and $q : u = v$ for arbitrary 2-loops $x, y, u, v : 1 = 1$:

\[
\begin{align*}
\text{whisk-L}(u, p) \cdot \text{whisk-R}(q, y) & \quad ? \quad \text{whisk-R}(p, u) \cdot \text{whisk-L}(y, q) \\
\text{whisk-R}(q, x) \cdot \text{whisk-L}(v, p) & \quad ? \quad \text{whisk-L}(x, q) \cdot \text{whisk-R}(p, v)
\end{align*}
\]

But: endpoints do not match! We need to insert Eckmann-Hilton.
Syllepsis: The Square

To construct the first horizontal path, we need to fill the following square:

\[
\begin{array}{c}
\text{whisk-L}(u, p) \\
\text{whisk-R}(q, y) \\
\text{whisk-R}(p, u) \\
\text{whisk-L}(y, q)
\end{array}
\]

\[
\begin{array}{c}
EH(u, x) \\
EH(v, y)
\end{array}
\]
Syllepsis: The Square

We use the naturality of Eckmann-Hilton:

\[
\begin{array}{c}
\begin{array}{c}
whisk-L(u, p) \quad \text{EH}(u, x) \quad whisk-R(p, u) \\
\hline
u \cdot y \quad \text{EH}(u, y) \quad y \cdot u \\
whisk-R(q, y) \quad whisk-L(y, q) \\
v \cdot y \quad \text{EH}(v, y) \quad y \cdot v
\end{array}
\end{array}
\]
To construct the second horizontal path, we need to fill the following square:

\[
\begin{array}{ccc}
  & EH(u, x) & \\
\hline
u \cdot x & x \cdot u & \\
\hline
whisk-R(q, x) & & whisk-L(x, q) \\
\hline
v \cdot x & x \cdot v & \\
\hline
whisk-L(v, p) & & whisk-R(p, v) \\
\hline
v \cdot y & EH(v, y) & y \cdot v
\end{array}
\]
Syllepsis: The Square

We use the naturality of Eckmann-Hilton:

\[
\begin{array}{c}
\begin{array}{ccc}
u \cdot x & \overset{EH(u, x)}{\longrightarrow} & x \cdot u \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
whisk-R(q, x) & & whisk-L(x, q) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
v \cdot x & \overset{EH(v, x)}{\longrightarrow} & x \cdot v \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
whisk-L(v, p) & & whisk-R(p, v) \\
\end{array}
\end{array}
\]

\[
\begin{array}{c}
\begin{array}{ccc}
v \cdot y & \overset{EH(v, y)}{\longrightarrow} & y \cdot v \\
\end{array}
\end{array}
\]
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Future Directions

Where to go next:

▶ Use the syllepsis term to compute the Brunerie number, \( i.e. \), prove that \( \pi_4(S^3) \) is 2.
▶ Adapt the techniques from this proof to further open problems in synthetic homotopy type theory.
▶ Suggestions here: ...