PERIODIC SOLUTIONS AND HOMOCLINIC SOLUTIONS FOR A SWIFT-HOHENBERG EQUATION WITH DISPERSION

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Dedicated to Professor Boling Guo on the occasion of his 80th birthday.

ABSTRACT. We investigate the 1D Swift-Hohenberg equation with dispersion

\[ u_t + 2u_{xx} - \sigma u_{xxxx} + u_{xxxxxx} = \alpha u + \beta u^2 - \gamma u^3, \]

where \( \sigma, \alpha, \beta \) and \( \gamma \) are constants. Even if only the stationary solutions of this equation are considered, the dispersion term \( -\sigma u_{xxxx} \) destroys the spatial reversibility which plays an important role for studying localized patterns. In this paper, we focus on its traveling wave solutions and directly apply the dynamical approach to provide the first rigorous proof of existence of the periodic solutions and the homoclinic solutions bifurcating from the origin without the reversibility condition as the parameters are varied.

1. Introduction. This paper concerns the traveling solutions of the 1D Swift-Hohenberg equation with dispersion

\[ u_t + 2u_{xx} - \sigma u_{xxxx} + u_{xxxxxx} = \alpha u + \beta u^2 - \gamma u^3, \]

where \( \alpha, \beta, \sigma \) and \( \gamma \) are constants. This equation was first introduced by Burke, Houghton and Knobloch [5] in order to examine the consequences of breaking spatial reversibility on the snakes-and-ladders structure.

If the dispersion term \( -\sigma u_{xxxx} \) disappears, the equation (1) is also called the generalized Swift-Hohenberg equation or the 23 Swift-Hohenberg equation. It has been used to model different phenomena for different constants in various physical contexts such as hydrodynamics [7, 30], nonlinear optics [20, 23], granular layers [3], plasma [18], chemical reactions [19] and so on. Its complicated dynamics have numerically and analytically been studied extensively such as the long-time behavior, the periodic solutions, the homoclinic solutions, the bifurcations, the pattern selections and so on. For example, see the references [1, 2, 4, 8, 9, 10, 11, 12, 14, 15, 16, 22, 24, 25, 26, 27, 28, 29, 31].

Especially, Burke, Houghton and Knobloch [5] stressed that the equation (1) for \( \sigma = 0 \) admits multiple stationary spatially localized states in the snaking or pinning region and these are organized in the snakes-and-ladders structure. This

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structure appears because the equation (1) has some good properties like the variational structure and in particular the spatial reversibility, i.e., the invariance of the equation (1) under
\[ R: \xi \rightarrow -\xi, \quad u \rightarrow u. \]
Then they consider its traveling wave solutions with \( x = \xi + ct \)
\[ cu_x + 2u_{xx} - \sigma u_{xxx} + u_{xxxx} = \alpha u + \beta u^2 - \gamma u^3. \] (2)
Using the numerical method, they pointed out that the equation (2) loses the reversibility which destroys the pitchfork bifurcations responsible for the rung states such that the snakes-and-ladders structure breaks up into a stack of isolas. After this, the meromorphic exact solutions of (2) are also given in [17] with the aid of the Laurent series. As we know, of particular interest in these studies is the existence of periodic solutions and homoclinic solutions. In this paper, we directly apply the dynamical approach and present the first rigorous proof of existence of the periodic solutions and homoclinic solutions of (2) near the origin if some constants are adjusted. Unlike most of papers listed above, we do not require that the equation has the reversibility and the variational structure. Since we here focus only on the solutions bifurcating from the origin, we mention that the dominant nonlinear term in (2) is \( \beta u^2 \), which means that the nonlinear term \( -\gamma u^3 \) is not crucial.

The plan of this paper is the following. In Section 2, we consider the case that the linear operator of the equation (2) has an eigenvalue 0, a pair of purely imaginary eigenvalues and an eigenvalue with a nonreal part. Applying the center manifold reduction theorem and the averaging theory, we show that the zero Hopf bifurcation will occur such that an unstable periodic solution bifurcates from the origin. Section 3 yields the existence of the homoclinic solution and the periodic solution if the linear operator has a double eigenvalue 0 and two eigenvalues with nonreal parts by means of the well-known Melnikov function. The existence of the periodic solution is also given in Section 4 if the linear operator has a pair of purely imaginary eigenvalues and two eigenvalues with nonreal parts.

2. Zero Hopf bifurcation. Let \( u_1 = u_x, u_2 = u_{xx} \) and \( u_3 = u_{xxx} \). Then the equation (2) is equivalently changed into a system with dimension four
\[ u_x = u_1, \]
\[ u_{1x} = u_2, \]
\[ u_{2x} = u_3, \]
\[ u_{3x} = \alpha u + \beta u^2 - \gamma u^3 - cu_1 - 2u_2 + \sigma u_3. \] (3)
Before we study the zero Hopf bifurcation of the system (3), we need the following result (For example, see [13, 32]), which provides a first order approximation for the periodic solutions of a periodic differential system by the averaging theory.
Consider the differential system
\[ \dot{X}' = \epsilon F_1(x, X) + \epsilon^2 F_2(x, X, \epsilon), \quad X(0) = X_0 \] (4)
for \( X \in D \), where the dot denotes the derivative with respect to \( x \), \( D \) is an open subset of \( \mathbb{R}^n \), \( x \geq 0 \), \( F_1(x, X) \) and \( F_2(x, X, \epsilon) \) are \( T \)-periodic in \( x \), and the constant \( T \) is independent of \( \epsilon \). The averaged differential equation is
\[ \dot{Y} = \epsilon G_1(Y), \quad Y(0) = Y_0, \] (5)
Lemma 2.1. Suppose that

1. \( F_1, \partial F_1/\partial Y, \partial^2 F_1/\partial X^2, F_2 \) and \( \partial F_2/\partial Y \) are defined, continuous and bounded by a constant independent of \( \epsilon \) in \([0, \infty) \times D \) and \( \epsilon \in (0, \epsilon_0] \).
2. \( F_1 \) and \( F_2 \) are \( T \)-periodic in \( x \).

Then the following statements hold.

(a) If \( p \) is an equilibrium of (5) and

\[
\text{Det} \left( \frac{\partial G_1}{\partial Y} \right) \bigg|_{Y=p} \neq 0,
\]

then there exists a \( T \)-periodic solution \( \phi(x, \epsilon) \) of (4) such that \( \phi(0, \epsilon) \to p \) as \( \epsilon \to 0 \).

(b) If the eigenvalues of the equilibrium \( p \) all have negative real part, the corresponding periodic solution \( \phi(x, \epsilon) \) is asymptotically stable for \( \epsilon \) sufficiently small. If one of the eigenvalues has positive real part, then \( \phi(x, \epsilon) \) is unstable.

Apply this lemma together with the center manifold reduction theorem, and we obtain the existence of an unstable periodic solution emanating from the zero Hopf bifurcation of the system (3).

Theorem 2.2. Suppose that

\[
\alpha = \alpha_1 \mu^{3/2}, \quad c = -2\sigma + c_1 \mu^{3/2}, \quad \beta \sigma \neq 0
\]

and

\[
|\alpha_1 \sigma| > 2|c_1|,
\]

where \( \mu > 0 \) is a small parameter and \( \alpha_1, c_1 \) are fixed constants. Then for \( \mu > 0 \) sufficiently small there exists an unstable periodic solution of (2) that shrinks to the origin.

Remark 1. The condition (7) guarantees that the linear operator of (3) for \( \mu = 0 \) has an eigenvalue zero and a pair of purely imaginary eigenvalues. The condition (8) confirms that the averaged differential equation has a nontrivial equilibrium.

Proof of Theorem 2.2. In order to investigate the zero Hopf bifurcation of the system (3) near the origin, we adjust the constants and first assume that

\[
\alpha = \alpha_0 \mu, \quad c = -2\sigma + c_0 \mu, \quad \beta \sigma \neq 0,
\]

where \( \mu > 0 \) is a small parameter, and \( \alpha_0, c_0 \) are constants. Take

\[
(u, u_1, u_2, u_3) = \mu(w, w_1, w_2, w_3)
\]

such that the system (3) is equivalent to

\[
\begin{align*}
w_x &= w_1, \\
w_{1x} &= w_2, \\
w_{2x} &= w_3, \\
w_{3x} &= \alpha_0 \mu w + \beta \mu w^2 - \gamma \mu^2 w^3 + (2\sigma - c_0 \mu) w_1 - 2w_2 + \sigma w_3.
\end{align*}
\]
In order to eliminate the affect of the terms of order which corresponding eigenvectors are

\[ \lambda_1 = 0, \quad \lambda_2 = \sigma, \quad \lambda_3 = i\sqrt{2}, \quad \lambda_4 = -i\sqrt{2}, \]

which corresponding eigenvectors are

\[ U_1 = (1, 0, 0, 0)^T, \quad U_2 = \left( \frac{1}{\sigma^3}, \frac{1}{\sigma^2}, \frac{1}{\sigma}, 1 \right)^T, \]
\[ U_3 = \left( i, -\frac{1}{2i}, -\frac{1}{2}, \frac{1}{\sqrt{2}} \right)^T, \quad U_4 = \bar{U}_3 = \left( -i, \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2} \right)^T. \]

The solution of (10) can be decomposed in terms of the above eigenvectors. Note that the system (10) is real. If let

\[ \begin{bmatrix} w, w_1, w_2, w_3 \end{bmatrix}^T = AU_1 + BU_2 - \frac{i}{2}v_1(U_3 - \bar{U}_3) + \frac{1}{2}v_2(U_3 + \bar{U}_3) \]

and plug it into (10), we obtain its equivalent system

\[ \dot{X} = H_0(\mu, X) + H_1(\mu, X), \quad (11) \]

where \( H_0(\mu, X) = (h_1, h_2, h_3, h_4)^T \) and

\[ h_1 = -\frac{\mu}{16\sigma^7}h_{10}, \quad h_2 = \sigma B + \frac{\mu}{8\sigma^4(\sigma^2 + 2)}h_{10}, \]
\[ h_3 = -\sqrt{2}v_2 + \frac{\mu}{4\sqrt{2}\sigma^5(\sigma^2 + 2)}h_{10}, \quad h_4 = \sqrt{2}v_1 + \frac{\mu}{4\sigma^6(\sigma^2 + 2)}h_{10}, \]
\[ h_{10} = \sigma^6\left( 8\alpha_0A + 2\sqrt{2}\alpha_0v_1 + 4\alpha_0v_2 + 8\beta A^2 + 4\sqrt{2}\beta A v_1 + \beta v_1^2 \right) \]
\[ + 4\sigma^3B\left( 2\alpha_0 - 2\alpha_0\sigma + 4\beta A + \sqrt{2}\beta v_1 \right) + 8\beta B^2. \]

Since \( \sigma \neq 0 \), the center manifold reduction theorem (see [13, 32]) shows that (11) has a smooth reduction function \( B = \Phi(\mu, A, v_1, v_2) \) near the origin, which satisfies

\[ B = \Phi(\mu, A, v_1, v_2) = O(||(A, v_1, v_2)||^2, \mu, A, v_1, v_2)). \quad (12) \]

Before we analyze the zero Hopf bifurcation of (11), we need the leading terms of \( B \) in (12). Assume

\[ \Phi(\mu, A, v_1, v_2) = \mu(b_1 A + b_2 v_1 + b_3 v_2) + a_1 A^2 + a_2 A v_1 + a_3 A v_2 \]
\[ + a_4 v_1^2 + a_5 v_1 v_2 + a_6 v_2^2 + O(||(A, v_1, v_2)||^2, \mu, A, v_1, v_2)^2). \quad (13) \]

Differentiating (12) with respect to \( x \) and using the system (11) together with (13), we have \( a_i = 0 \) for \( i = 1, \ldots, 6 \) and

\[ b_1 = -\frac{\alpha_0\sigma}{\sigma^2 + 2}, \quad b_2 = \frac{\sigma^2(2\alpha_0 + \alpha_0\sigma)}{2\sqrt{2}(\sigma^2 + 2)^2}, \quad b_3 = \frac{\sigma^2(\alpha_0 - \alpha_0\sigma)}{2(\sigma^2 + 2)^2}. \]

In order to eliminate the affect of the terms of order \( O(||(A, v_1, v_2)||^2, \mu, A, v_1, v_2)^2) \) in \( B \), we further assume that

\[ \alpha_0 = \sqrt{\mu}\alpha_1, \quad c_0 = \sqrt{\mu}c_1, \quad (A, v_1, v_2) = \sqrt{\mu}(\tilde{A}, \tilde{v}_1, \tilde{v}_2), \]

where \( \alpha_1 \) and \( c_1 \) are constants, which implies

\[ B = \mu^2O(||(\tilde{A}, \tilde{v}_1, \tilde{v}_2)||^2) + \mu^{3/2}O(||(\tilde{A}, \tilde{v}_1, \tilde{v}_2)||^2). \]
Thus, the conditions in (9) become

\[ \alpha = \alpha_1 \mu^{3/2}, \quad c = -2\sigma + c_1 \mu^{3/2}, \quad \beta \sigma \neq 0, \]

which is given in (7) of Theorem 2.2. The reduced equations of \( A, v_1 \) and \( v_2 \) can be changed into

\[ \dot{X} = F_0(\mu, \bar{X}) + F_1(\mu, \bar{X}), \quad (14) \]

where \( \bar{X} = (\bar{A}, \bar{v}_1, \bar{v}_2)^T \), \( F_1(\mu, \bar{X}) = \mu^2 O((\bar{A}, \bar{v}_1, \bar{v}_2)) \), \( F_0(\mu, \bar{X}) = (f_1, f_2, f_3)^T \), and

\[

t_0 = \frac{-\mu^{3/2}}{16\sigma} f_{10}, \quad f_2 = -\sqrt{2} v_2 + \frac{\mu^{3/2} \sigma}{4\sqrt{2}(\sigma^2 + 2)} f_{10}, \quad f_3 = \sqrt{2} v_1 + \frac{\mu^{3/2}}{4(\sigma^2 + 2)} f_{10},
\]

\[
f_{10} = 8\alpha_1 \bar{A} + 2\sqrt{2} \alpha_1 \bar{v}_1 + 4c_1 \bar{v}_2 + 8\beta \bar{A}^2 + 4\sqrt{2} \beta \bar{A} \bar{v}_1 + \beta \bar{v}_1^2.
\]

Now we write the system (14) in the form of (4) so that we can apply Lemma 2.1. Let \( \bar{v}_1 = r \cos(\theta) \) and \( \bar{v}_2 = r \sin(\theta) \), and we have

\[ \dot{\bar{A}} = \mu^{3/2} g_1(\theta, r, \bar{A}) + g_{10}(\mu, \theta, r, \bar{A}), \]

\[ \dot{\bar{r}} = \mu^{3/2} g_2(\theta, r, \bar{A}) + g_{20}(\mu, \theta, r, \bar{A}), \]

\[ \dot{\theta} = \sqrt{2} + \mu^{3/2} g_3(\theta, r, \bar{A}) + g_{30}(\mu, \theta, r, \bar{A}), \quad (15) \]

where \( g_{10}, g_{20} \) and \( g_{30} \) are of order \( \mu^2 O((\mu, \theta, r, \bar{A})) \), and

\[
g_1(\theta, r, \bar{A}) = -\frac{1}{16\sigma} \left[ r \left( 2\sqrt{2} \alpha_1 \cos(\theta) + 4c_1 \sin(\theta) + 4\sqrt{2} \beta \bar{A} \cos(\theta) + \beta r \cos^2(\theta) \right) \\
+ 8\bar{A}(\alpha_1 + \beta \bar{A}) \right],
\]

\[
g_2(\theta, r, \bar{A}) = \frac{1}{8(\sigma^2 + 2)} \left[ 4\sqrt{2} \cos(\theta) \left( \alpha_1 + c_1 + 2\beta \bar{A} \right) r \sin(\theta) \\
+ 2\sigma(\alpha_1 + \beta \bar{A}) \bar{A} + 8 \sin(\theta) \left( 2\alpha_1 \bar{A} + c_1 r \sin(\theta) + 2\beta \bar{A}^2 \right) \\
+ 2 \left( 2\sigma \alpha_1 + \beta r \sin(\theta) + 4\sigma \beta \bar{A} \right) r \cos^2(\theta) + \sqrt{2} \beta \sigma r^2 \cos^3(\theta) \right],
\]

\[
g_3(\theta, r, \bar{A}) = \frac{1}{8r(\sigma^2 + 2)} \left[ 4\sqrt{2} \sin(\theta) \left( -2(\alpha_1 + \beta \bar{A}) A - rc_1 \sin(\theta) \right) \\
+ 4 \cos(\theta) \left( r \sin(\theta)(2c_1 - \sigma \alpha_1 - 2\sigma \beta \bar{A}) + 4(\alpha_1 + \beta \bar{A}) \bar{A} \right) \\
+ \sqrt{2} r \cos^2(\theta) \left( 4\alpha_1 - \beta \sigma r \sin(\theta) + 8\beta \bar{A} \right) + 2\beta r^2 \cos^3(\theta) \right]. \quad (16)
\]

Note that the above system is only well defined for \( r > 0 \) since \( r \) appears in the denominator of \( g_3 \). Using the fact that \( \theta' = \sqrt{2} + O(\mu^{3/2}) > 0 \) for small \( \mu > 0 \), we rewrite the system (15) into

\[
\frac{dr}{d\theta} = \mu^{3/2} H_1(\theta, r, \bar{A}) + H_{10}(\mu, \theta, r, \bar{A}),
\]

\[
\frac{d\bar{A}}{d\theta} = \mu^{3/2} H_2(\theta, r, \bar{A}) + H_{20}(\mu, \theta, r, \bar{A}), \quad (17)
\]
where $H_{10}(\mu, \theta, r, \tilde{A})$ and $H_{20}(\mu, \theta, r, \tilde{A})$ are of order $\mu^2 O(|(\mu, \theta, r, \tilde{A})|)$, and

$$H_1(\theta, r, \tilde{A}) = -\frac{1}{16\sqrt{2}\sigma} \left[ 2\sqrt{2}r \cos(\theta)(\alpha_1 + 2\beta \tilde{A}) + 4c_1 r \sin(\theta) + \beta r^2 \cos^2(\theta) + 8\tilde{A}(\alpha_1 + \beta \tilde{A}) \right],$$

$$H_2(\theta, r, \tilde{A}) = \frac{1}{8\sqrt{2}(\sigma^2 + 2)} \left[ 4\sqrt{2} \cos(\theta) \left( r \sin(\theta)(\alpha_1 + c_1 \sigma + 2\beta \tilde{A}) + 2\sigma \tilde{A}(\alpha_1 + \beta \tilde{A}) + 8 \sin(\theta) \left( 2\tilde{A}(\alpha_1 + \beta \tilde{A}) + c_1 r \sin(\theta) \right) + 2r \cos^2(\theta) \left( 2\sigma(\alpha_1 + 2\beta \tilde{A}) + \beta r \sin(\theta) \right) + \sqrt{2} \beta r^2 \cos^3(\theta) \right].$$

The system (17) is exactly the form of (4) and $2\pi$-periodic in the variable $\theta$. According to (6), we define the bifurcation function $B(\tilde{\omega})$ by

$$B(\tilde{\omega}) = \frac{1}{2\pi} \int_0^{2\pi} F_1(\theta, r, \tilde{A}) d\theta = \left( \frac{-160(1) \tilde{A} + 16\beta \tilde{A}^2 + \beta r^2}{32\sqrt{2}\sigma}, \frac{r(2c_1 + \sigma(\alpha_1 + 2\beta \tilde{A}))}{4\sqrt{2}(\sigma^2 + 2)} \right)^T,$$

where $\tilde{\omega} = (\tilde{A}, r)^T, F_1(\theta, r, \tilde{A}) = (H_1, H_2)^T(\theta, r, \tilde{A})$. If the condition (8) holds, it is easy to check that $B(\tilde{\omega})$ has a nontrivial solution $w_0 = (A_0, \sigma_0, \mu_0)^T$ where

$$A_0 = \frac{-2c_1 - \alpha_1 \sigma}{2\beta \sigma}, \quad \sigma_0 = \frac{2}{|\beta \sigma|} \sqrt{\alpha_1^2 \sigma^2 - 4c_1^2}.$$

since we need $\sigma_0 > 0$. Moreover, the Jacobian matrix $D_{\tilde{\omega}}B(\tilde{\omega})$ of $B(\tilde{\omega})$ at $w_0$ has two eigenvalues

$$\hat{\lambda}_1 = -\frac{1}{8\sqrt{2}\sigma} \sqrt{\alpha_1^2 \sigma^2 - 4c_1^2}, \quad \hat{\lambda}_2 = \frac{1}{\sqrt{2}(\sigma^2 + 2)} \sqrt{\alpha_1^2 \sigma^2 - 4c_1^2}.$$

Lemma 2.1 implies that there exists a periodic solution $W_\mu(\mu, \theta)$ of the system (17) with period $2\pi$ for sufficiently small $\mu > 0$, which approaches to $w_0$ as $\mu \to 0$. Since $\hat{\lambda}_2 > 0$, the periodic solution $W_\mu(\mu, \theta)$ is unstable.

Now going back through the center manifold reduction theorem, the changes of variables and the rescaling which keep the instability of this periodic solution, we know that the original equation (2) for $\mu > 0$ sufficiently small has one unstable periodic solution. Hence the proof is completed. \[\square\]

3. Double eigenvalue 0. In this section, we assume that

$$\alpha = \mu \alpha_0, \quad c = \mu c_0, \quad \sigma \neq 0. \tag{18}$$

Thus, the linear operator of the system (3) for $\mu = 0$ has a double eigenvalue 0, $\lambda_1 = \frac{\alpha - \sqrt{\alpha^2 - 8}}{2}$ and $\lambda_2 = \frac{\alpha + \sqrt{\alpha^2 - 8}}{2}$. For simplicity, we suppose

$$|\sigma| > 2\sqrt{2}. \tag{19}$$

Then the corresponding eigenvectors and the generalized eigenvectors for the eigenvalues 0, $\lambda_1$ and $\lambda_2$ are

$$U_1 = (1, 0, 0, 0)^T, \quad U_2 = (0, 1, 0, 0)^T,$$

$$U_3 = \left( \frac{\lambda_2}{2\lambda_1}, \frac{\sigma \lambda_2 - 2}{4}, \frac{\lambda_2}{2}, 1 \right)^T, \quad U_4 = \left( \frac{\lambda_1}{2\lambda_2}, \frac{\sigma \lambda_1 - 2}{4}, \frac{\lambda_1}{2}, 1 \right)^T.$$
Since the solution of (3) can be written in terms of the above eigenvectors and generalized eigenvectors, we let

\[(u, u_1, u_2, u_3)^T = AU_1 + BU_2 + CU_3 + DU_4. \tag{20}\]

Plugging (20) into (3) yields its equivalent system given by

\[
\begin{align*}
\dot{A} &= B + h_1(\mu, A, B, C, D), \\
\dot{B} &= h_2(\mu, A, B, C, D), \\
\dot{C} &= \lambda_1 C + h_3(\mu, A, B, C, D), \\
\dot{D} &= \lambda_2 D + h_4(\mu, A, B, C, D),
\end{align*}\tag{21}
\]

where the smooth functions \(h_i(\mu, A, B, C, D) = O(||(A, B, C, D)||(||\mu, A, B, C, D)||)\) for \(i = 1, 2, 3, 4\). The expressions of \(h_i\) are complicated and some terms are not crucial for the study of the bifurcation so that we do not explicitly write them out. Since the real eigenvalues \(\lambda_1\) and \(\lambda_2\) are non zero, we apply the center manifold reduction theorem to solve for \(C\) and \(D\) as smooth functions of \((\mu, A, B)\) such that

\[
C = O(||(A, B)||(||\mu, A, B)||), \quad D = O(||(A, B)||(||\mu, A, B)||). \tag{22}\]

Substituting (22) into the first two equations of (21), we get the reduced system as follows

\[
\begin{align*}
\dot{A} &= B + \tilde{R}_1(\mu, A, B), \\
\dot{B} &= \frac{c_0}{2} \mu A - \frac{c_0}{2} \mu B + \frac{\beta}{2} A^2 + \tilde{R}_2(\mu, A, B),
\end{align*}\tag{23}
\]

where \(\tilde{R}_1(\mu, A, B)\) and \(\tilde{R}_2(\mu, A, B)\) are smooth, and

\[
\begin{align*}
\tilde{R}_1(\mu, A, B) &= \frac{\alpha_0 \sigma}{4} \mu A - \frac{\alpha_0 \sigma}{4} \mu B + \frac{\beta \sigma}{4} A^2 + O(||(A, B)||(||\mu, A, B)||^2), \\
\tilde{R}_2(\mu, A, B) &= O(||(A, B)||(||\mu, A, B)||^2).
\end{align*}
\]

Note that \(\tilde{R}_1(\mu, A, B)\) can be always chosen equal to 0 since we consider \((A, B)\) near the origin. This can be done by a change of coordinates of the type \(\tilde{B} = B + \tilde{R}_1(\mu, A, B)\). Thus, (23) can be changed into (the tilde is dropped)

\[
\begin{align*}
\dot{A} &= B, \\
\dot{B} &= \frac{c_0}{2} \mu A + \frac{\alpha_0 \sigma - 2\alpha_0}{4} \mu B + \frac{\beta \sigma}{4} AB + \frac{\beta}{2} A^2 + R_2(\mu, A, B),
\end{align*}\tag{24}
\]

where \(R_2(\mu, A, B) = O(||(A, B)||(||\mu, A, B)||^2)\). In order to find its dominant system, let \(A = \mu \tilde{A}, B = \mu^{3/2} \tilde{B}\) and \(x = \frac{1}{\sqrt{\mu}} \tilde{x}\) which transform (24) into

\[
\begin{align*}
\dot{\tilde{A}} &= \tilde{B}, \\
\dot{\tilde{B}} &= \frac{c_0}{2} \tilde{A} + \frac{\alpha_0 \sigma - 2\alpha_0}{4} \sqrt{\mu} \tilde{B} + \frac{\beta \sigma}{4} \sqrt{\mu} \tilde{A} \tilde{B} + \frac{\beta}{2} \tilde{A}^2 + \tilde{R}_2(\mu, \tilde{A}, \tilde{B}),
\end{align*}\tag{25}
\]

where the prime denotes the derivative with respect to \(\tilde{x}\) and

\[
\tilde{R}_2(\mu, \tilde{A}, \tilde{B}) = \frac{1}{\mu^{3/2}} R_2(\mu, A, B) = \mu O(||(\tilde{A}, \tilde{B})||).
\]

The dominant system of the system (25) is

\[
\begin{align*}
\dot{\tilde{A}} &= \tilde{B}, \\
\dot{\tilde{B}} &= \frac{\alpha_0}{2} \tilde{A} + \frac{\beta}{2} \tilde{A}^2,
\end{align*}\tag{26}
\]

which is the KdV system. For simplicity, we first assume that

\[
\alpha_0 < 0, \quad \beta \neq 0. \tag{27}
\]
Obviously, the system (26) has a center equilibrium $(0, 0)$ and a saddle equilibrium $(-\frac{\alpha_0}{\beta}, 0)$. Moreover, it has a homoclinic solution connecting the saddle equilibrium and infinite many periodic solutions around the center equilibrium. In what follows, we will use the bifurcation method and in particular the Melnikov function to rigorously prove the persistence of the homoclinic solution and the periodic solutions when the higher order terms are added, that is, the system (25) owns the homoclinic solution and the periodic solution.

3.1. Persistence of homoclinic solutions. In order to move the saddle equilibrium $(-\frac{\alpha_0}{\beta}, 0)$ to the origin, we let

$$\tilde{A} = -\frac{\alpha_0}{\beta} + \hat{A}, \quad \tilde{B} = \hat{B}.$$  

The system (25) is changed into

$$\hat{A}' = \hat{B}, \quad \hat{B}' = -\frac{\alpha_0}{2} \hat{A} - \frac{2c_0 + \alpha_0\sigma}{4}\sqrt{\mu} \hat{B} + \frac{\beta\sigma}{2} \sqrt{\mu} \hat{A} \hat{B} + \frac{\beta}{2} \hat{A}^2 + \hat{R}(\mu, \hat{A}, \hat{B}),$$  

(28)

where the smooth function $\hat{R}(\mu, \hat{A}, \hat{B}) = \tilde{R}_2(\mu, \hat{A}, \hat{B}) = O(\mu)$. Its dominant system is

$$\hat{A}' = \hat{B}, \quad \hat{B}' = -\frac{\alpha_0}{2} \hat{A} + \frac{\beta}{2} \hat{A}^2,$$  

(29)

which is a Hamiltonian system with a Hamiltonian function

$$H(\hat{A}, \hat{B}) = \frac{1}{2} \hat{B}^2 + \frac{\alpha_0}{4} \hat{A}^2 - \frac{\beta}{6} \hat{A}^3.$$  

It is easy to check that the system (29) has a homoclinic solution

$$(H_A, H_B)^T(\tilde{x}) = (a \text{sech}^2(b \tilde{x}), -2ab \text{sech}^2(b \tilde{x}) \tanh(b \tilde{x}))^T,$$  

(30)

which exponentially approaches to the origin where $a = \frac{3\alpha_0}{2\sigma}$ and $b = \sqrt{-\frac{\alpha_0}{2\sqrt{2}}}$.  

**Theorem 3.1.** Suppose that (27) holds and

$$c_0 = \frac{5}{14} \alpha_0 \sigma + O(\sqrt{\mu}).$$  

(31)

Then the system (28) has a homoclinic solution for sufficiently small $\mu > 0$.

Going back to the origin equation (2), we obtain the following theorem.

**Theorem 3.2.** Suppose that

$$\alpha = \mu \alpha_0, \quad c = \mu c_0, \quad c_0 = \frac{5}{14} \alpha_0 \sigma + O(\sqrt{\mu}), \quad \alpha_0 < 0, \quad \beta \neq 0, \quad |\sigma| > 2\sqrt{2}.$$  

(32)

Then, for sufficiently small $\mu > 0$, the equation (2) has a homoclinic solution $u_h(x)$ given by

$$u_h(x) = -\frac{\alpha_0}{\beta} \mu + \frac{3\alpha_0}{2\beta} \mu \text{sech}^2\left(\frac{\sqrt{-\alpha_0}}{2\sqrt{2}} \mu x\right) + R_0(\sqrt{\mu}) + R_1(\sqrt{\mu}, \sqrt{\mu} x)$$  

(33)

for $x \in \mathbb{R}$ where the smooth functions $R_0$ and $R_1$ are of order $\mu^{3/2}$, $R_0$ is independent of $x$, and $R_1$ exponentially tends to 0 as $x \to \pm \infty$.  

Proof of Theorem 3.1. The expressions of the homoclinic solution for the dominant system (29) have been obtained in (30). Consider the system (28) and we use the Melnikov function to prove the persistence of this homoclinic solution under small perturbation, which is not time-dependent and is defined by (For example, see [13, 32])

\[
M_0(\mu, \alpha_0, c_0) = \sqrt{\mu} \int_{-\infty}^{\infty} H_B(t) \left( -\frac{2c_0 + \alpha_0 \sigma}{4} H_B(t) + \frac{\beta \sigma}{2} H_A(t) H_B(t) \right) dt
\]

\[
+ \int_{-\infty}^{\infty} H_B(t) \tilde{R}(\mu, H_A(t), H_B(t)) dt
\]

\[
= \frac{3(-\alpha_0)^{5/2}(5\alpha_0 \sigma - 14c_0)}{70\sqrt{2}\beta^2} \sqrt{\mu} + M_1(\mu, \alpha_0, c_0),
\]

(34)

where the smooth function \( M_1(\mu, \alpha_0, c_0) = O(\mu) \). The bifurcation situation, when the saddle connection is preserved, is given by \( M_0(\mu, \alpha_0, c_0) \equiv 0 \), or, for \( \mu > 0 \) small by the implicit function theorem

\[
c_0 = \frac{5}{14} \alpha_0 \sigma + O(\sqrt{\mu}),
\]

which is given in (31). Thus this implies that the system (28) has a homoclinic solution for small \( \mu > 0 \).

3.2. Persistence of periodic solutions. Now we consider the persistence of periodic solutions of the system (25) and have the following theorem.

Theorem 3.3. Suppose that (27) is valid and

\[
\sigma > 2\sqrt{2}, \quad \frac{\alpha_0 \sigma}{2} < c_0 < \frac{5\alpha_0 \sigma}{14} \quad \text{or} \quad \sigma < -2\sqrt{2} \quad \text{and} \quad \frac{5\alpha_0 \sigma}{14} < c_0 < \frac{\alpha_0 \sigma}{2}.
\]

(35)

Then the system (25) has a periodic solution for sufficiently small \( \mu > 0 \).

This above theorem and its proof imply the following theorem for the equation (2).

Theorem 3.4. Suppose that

\[
\alpha = \mu \alpha_0, \quad c = \mu c_0, \quad \alpha_0 < 0, \quad \beta \neq 0,
\]

and

\[
\sigma > 2\sqrt{2}, \quad \frac{\alpha_0 \sigma}{2} < c_0 < \frac{5\alpha_0 \sigma}{14} \quad \text{or} \quad \sigma < -2\sqrt{2} \quad \text{and} \quad \frac{5\alpha_0 \sigma}{14} < c_0 < \frac{\alpha_0 \sigma}{2}.
\]

(37)

Then, for sufficiently small \( \mu > 0 \), there exists a constant \( k_0 \in (0, 1) \) such that the equation (2) has a periodic solution \( u_p(x) \) given by

\[
u_p(x) = a \mu + b \mu \sqrt{\mu} \Omega(x, k_0) + R_2(\sqrt{\mu} x, k_0),
\]

(38)

where \( \Omega(x, k_0) \) is the Jacobi elliptic function, the smooth function \( R_2 \) is periodic in \( x \) and of order \( \mu^{3/2} \), and

\[
a = -\frac{\alpha_0}{2\beta} + \frac{\alpha_0 (k_0^2 + 1)}{2\beta \sqrt{k_1}}, \quad b = -\frac{3k_0^2 \alpha_0}{2\beta \sqrt{k_1}}, \quad \Omega = \frac{\sqrt{-\alpha_0}}{2\sqrt{2k_1}}, \quad k_1 = k_0^2 - 1.
\]

(39)
Proof of Theorem 3.3. The Hamiltonian function of the dominant system (26) is

\[ H(\tilde{A}, \tilde{B}) = \frac{\tilde{B}^2}{2} - \frac{\alpha_0}{4} \tilde{A}^2 - \frac{\beta}{6} \tilde{A}^3 \]

with

\[ e_0 \triangleq H(0, 0) = 0, \quad e_1 \triangleq H(-\frac{\alpha_0}{\beta}, 0) = -\frac{\alpha_0^3}{12\beta^2}. \]

It is clear that all the periodic solutions \((\tilde{A}, \tilde{B})^T(\tilde{x})\) of (26) in the \((\tilde{A}, \tilde{B})\)-plane must lie on the level curve

\[ H(\tilde{A}, \tilde{B}) = \frac{\tilde{B}^2}{2} - \frac{\alpha_0}{4} \tilde{A}^2 - \frac{\beta}{6} \tilde{A}^3 = e \quad (40) \]

for \(e \in (e_0, e_1)\). In the following, we look for some \(e \in (e_0, e_1)\) such that the corresponding periodic solution persists if the higher order terms are included.

In order to do this, we need the exact expression of the periodic solution. Suppose that the curve \(H(\tilde{A}, \tilde{B}) = e\) intersects the \(\tilde{A}\)-axis with three points \(C_1(\tilde{A}_1, 0), C_2(\tilde{A}_2, 0)\) and \(C_3(\tilde{A}_3, 0)\). Then they satisfy

\[ \beta \frac{\tilde{A}^3}{6} + \frac{\alpha_0}{4} \tilde{A}^2 + e = \frac{\beta}{6}(\tilde{A} - \tilde{A}_1)(\tilde{A} - \tilde{A}_2)(\tilde{A} - \tilde{A}_3). \]

It is easy to check that

\[ \tilde{A}_1 = -\frac{\alpha_0}{2\beta}(1 + 2\cos(\theta)), \quad \tilde{A}_2 = -\frac{\alpha_0}{2\beta}(1 + \sqrt{3}\sin(\theta) - \cos(\theta)), \]

\[ \tilde{A}_3 = -\frac{\alpha_0}{2\beta}(1 - \sqrt{3}\sin(\theta) - \cos(\theta)) \quad (41) \]

with \(\theta = \frac{1}{3} \arccos(1 + 24e\beta^2/\alpha_0^3)\). Thus, (40) yields

\[ \tilde{B} = \pm \sqrt{\frac{\beta}{3}} \tilde{A}^3 + \frac{\alpha_0}{2} \tilde{A}^2 + 2e = \pm \sqrt{\frac{\beta}{3}} (\tilde{A} - \tilde{A}_1)(\tilde{A} - \tilde{A}_2)(\tilde{A} - \tilde{A}_3), \quad e \in (e_0, e_1). \]

Due to the symmetry of the system (26), we take only the upper branch for the periodic solutions, i.e.,

\[ \tilde{B} = \sqrt{\frac{\beta}{3}} (\tilde{A} - \tilde{A}_1)(\tilde{A} - \tilde{A}_2)(\tilde{A} - \tilde{A}_3), \quad e \in (e_0, e_1). \]

The relationship \(\tilde{x} = \dot{\tilde{A}}\) and the properties of the elliptic integrals (see [6]) give for \(\beta > 0\)

\[ \tilde{x} = \int_{\tilde{A}_1}^{\tilde{A}} \frac{1}{\sqrt{\frac{\beta}{3}(z - \tilde{A}_1)(z - \tilde{A}_2)(z - \tilde{A}_3)}} dz = \frac{1}{\Omega} \int_0^\omega \frac{d\rho}{\sqrt{(1 - \rho^2)(1 - k^2\rho^2)}}, \quad \tilde{A} \in (\tilde{A}_3, \tilde{A}_2) \quad (42) \]

and \(\tilde{A}_3 < \tilde{A}_2 < \tilde{A}_1\) where

\[ \omega = \sqrt{\frac{\tilde{A}_2 - \tilde{A}_3}{\tilde{A}_2 - \tilde{A}_1}}, \quad k = \sqrt{\frac{\tilde{A}_2 - \tilde{A}_3}{\tilde{A}_1 - \tilde{A}_3}}, \quad \Omega = \frac{\sqrt{\beta(\tilde{A}_1 - \tilde{A}_3)}}{2\sqrt{3}}, \quad (43) \]
or for $\beta < 0$

$$\ddot{x} = \int_{\tilde{A}_2}^{\tilde{A}_1} \frac{1}{\sqrt{\frac{3}{4}(z - \tilde{A}_1)(z - \tilde{A}_2)(z - \tilde{A}_3)}} \, dz$$

$$= \frac{1}{\Omega} \int_{0}^{\omega} \frac{d\rho}{\sqrt{(1 - \rho^2)(1 - k^2\rho^2)}}$$

$\tilde{A} \in (\tilde{A}_2, \tilde{A}_3)$ (44)

and $\tilde{A}_1 < \tilde{A}_2 < \tilde{A}_3$ where

$$\omega = \sqrt{(\tilde{A}_3 - \tilde{A}_1)(\tilde{A}_3 - \tilde{A}_2)} \quad k = \sqrt{\frac{\tilde{A}_3 - \tilde{A}_1}{\tilde{A}_3 - \tilde{A}_2}} \quad \Omega = \frac{\sqrt{\beta(\tilde{A}_1 - \tilde{A}_3)}}{2\sqrt{3}}.$$ (45)

From (42) and (44) we obtain the periodic solutions of (26) given by

$$P_A(\tilde{x}) = a + b \operatorname{sn}^2(\Omega \tilde{x}, k), \quad P_B(\tilde{x}) = d \operatorname{sn}(\Omega \tilde{x}, k) \operatorname{cn}(\Omega \tilde{x}, k) \operatorname{dn}(\Omega \tilde{x}, k)$$

with a period

$$T(k) = \frac{2}{\Omega} K(k),$$

where $K(k) = F(\frac{\pi}{2}, k)$ is the complete elliptic integral of the first kind, $\operatorname{sn}$, $\operatorname{cn}$ and $\operatorname{dn}$ are the Jacobi elliptic functions, and

$$\begin{align*}
F(\phi, k) &= \int_{0}^{\phi} \frac{1}{\sqrt{1 - k^2 \sin^2(\eta)}} \, d\eta = \int_{0}^{\sin(\phi)} \frac{1}{\sqrt{(1 - t^2)(1 - k^2t^2)}} \, dt, \\
a &= -\frac{\alpha_0}{2\beta} + \frac{\alpha_0(k^2 + 1)}{2\beta \sqrt{k_1}}, \\
b &= \frac{3k^2\alpha_0}{2\beta \sqrt{k_1}}, \\
d &= \frac{3k^2(-\alpha_0)^{3/2}}{2\sqrt{2}\beta k_1^{3/4}}, \\
0 \leq k < 1, \quad \Omega = \frac{\sqrt{-\alpha_0}}{2\sqrt{2k_1^{3/4}}}, \quad k_1 = k^4 - k^2 + 1.
\end{align*}$$

Clearly, when $e$ is increasing from $e_0$ to $e_1$, $\theta$ is strictly increasing from $0$ to $\frac{\pi}{3}$. By (41), (43) and (45), we have

$$k = \sqrt{\frac{2\sqrt{3}\sin(\theta)}{\sqrt{3}\sin(\theta) + 3\cos(\theta)}} = \sqrt{\frac{2}{1 + \sqrt{3}\cot(\theta)}}.$$ 

This shows that $k$ is strictly increasing in $\theta$, which implies that $k$ is strictly increasing in $e$. In the following, we shall use the parameter $k$ rather than $e$. Thus, the problem of the persistence of the periodic solution for some fixed $e \in (e_0, e_1)$ can be changed into one for some $k \in (0, 1)$.

Lemma 3.5. The period $T(k)$ is strictly increasing with respect to $k \in [0, 1)$ and

$$T(0) = \frac{2\sqrt{2}\pi}{\sqrt{-\alpha_0}}, \quad \lim_{k \to 1^-} T(k) = \infty.$$

The proof can easily be obtained by using the fact that $K(k)$ is strictly increasing in $k \in [0, 1)$ and

$$\frac{\pi}{2} \leq K(k) \leq \frac{1}{2} \log \frac{16}{1 - k} \left(1 + \sqrt{\frac{2}{7}(1 - k)}\right), \quad k \in [0, 1)$$

which is given in Lemma 23 of [21].
In order to obtain the persistence of the periodic solution \((P_A(x), P_B(x))^T\) when the higher order terms are added in (25), it is sufficient by the Melnikov theory to verify that for small \(\mu > 0\)

\[
M_p(\mu, k) = \sqrt{\mu} \int_0^{T(k)} P_B(t) \left( \frac{\alpha_0 \sigma - 2c_0}{4} P_B(t) + \frac{\beta \sigma}{2} P_A(t) P_B(t) \right) dt \\
+ \int_0^{T(k)} P_B(t) \tilde{R}_2(\mu, P_A(t), P_B(t)) dt
\]

is zero for some \(k \in (0, 1)\). Thus, we have

\[
M_p(\mu, k) = -\frac{3(-\alpha_0)^{5/2}}{140\sqrt{2} \beta^2 k_1^{7/4}} \left( M_2(k) E(k) + (k^2 - 1) M_3(k) K(k) \right) \sqrt{\mu} + M_4(\mu, k)
\]

\[
\triangleq M_5(\mu, k) \sqrt{\mu} + M_4(\mu, k), 
\]

(46)

where \(E(k)\) is the complete elliptic integral of the second kind, and

\[
M_2(k) = 28c_0k_1^{3/2} + 5\alpha_0 \sigma (k^2 + 1)(k^2 - 2)(2k^2 - 1),
\]

\[
M_3(k) = -14c_0(k^2 - 2)\sqrt{k_1} - 5\alpha_0 \sigma (k^4 + 2k^2 - 2),
\]

\[
M_4(\mu, k) = \int_0^{T(k)} P_B(t) \tilde{R}_2(\mu, P_A(t), P_B(t)) dt = O(\mu).
\]

It is clear to see that by the properties of \(E(k)\) and \(K(k)\) in [6]

\[
M_5(k) = -\frac{9\pi (-\alpha_0)^{5/2}(2c_0 - \alpha_0 \sigma)}{64\sqrt{2} \beta^2} k^4 + O(k^6) \quad \text{near } k = 0
\]

and

\[
\lim_{k \to 1^-} M_5(k) = -\frac{3(-\alpha_0)^{5/2}(14c_0 - 5\alpha_0 \sigma)}{70\sqrt{2} \beta^2}.
\]

If

\[
\sigma > 2\sqrt{2}, \quad \frac{\alpha_0 \sigma}{2} < c_0 < \frac{5\alpha_0 \sigma}{14} \quad (\text{or } \sigma < -2\sqrt{2} \text{ and } \frac{5\alpha_0 \sigma}{14} < c_0 < \frac{\alpha_0 \sigma}{2}),
\]

which is given in (37), we have

\[
M_5(k) < 0 \quad \text{or } M_5(k) > 0 \quad \text{near } k = 0
\]

and

\[
\lim_{k \to 1^-} M_5(k) > 0 \quad \text{or } \lim_{k \to 1^-} M_5(k) < 0.
\]

Thus, from (46), there exists at least one \(k_0 \in (0, 1)\) for sufficiently small \(\mu > 0\) by the intermediate values theorem such that

\[
M_p(\mu, k_0) = 0.
\]

This gives the existence of the periodic solutions of the system (25). The proof is completed.

Using a similar argument as above, we have the following results.

Remark 2. (1) Suppose that

\[
\alpha = \mu \alpha_0, \quad \sigma = \mu c_0, \quad c_0 = -\frac{5}{14} \alpha_0 \sigma + O(\sqrt{\mu}), \quad \alpha_0 > 0, \quad \beta \neq 0, \quad |\sigma| > 2\sqrt{2}.
\]

Then the equation (2) has a homoclinic solution for sufficiently small \(\mu > 0\).
(2) Suppose that
\[ \alpha = \mu \alpha_0, \quad c = \mu c_0, \quad \alpha_0 > 0, \quad \beta \neq 0, \]
and
\[ \sigma > 2\sqrt{2}, \quad -\alpha_0 \sigma < c_0 < -\frac{5\alpha_0 \sigma}{4} \]
(or \( \sigma < -2\sqrt{2} \) and \( -\frac{5\alpha_0 \sigma}{4} < c_0 < -\frac{\alpha_0 \sigma}{2} \)).

Then the equation (2) has at least one periodic solution for sufficiently small \( \mu > 0 \).

(3) If \( 0 < |\sigma| \leq 2\sqrt{2} \), using a similar way, we get the reduced system which is the same as (23) so that Theorem 3.2, Theorem 3.4, and the above two remarks still hold.

4. A pair of purely imaginary eigenvalues. In this section, we consider the case: the system (3) has a pair of purely imaginary eigenvalues and two eigenvalues with nonzero real parts. We have the following theorem.

**Theorem 4.1.** Assume that
\[ \alpha > 0, \quad \sigma = \mu \sigma_1, \tag{47} \]
where \( \sigma_1 \) is an arbitrary constant. For any positive constant \( A_0 \), there exists a smooth function \( \tilde{c}_0(\mu, \sigma_1, A_0) \) such that
\[ c = \mu (c_0 + \tilde{c}_0(\mu, \sigma_1, A_0)), \quad c_0 = -\frac{\alpha \sigma_1}{\sqrt{1+\alpha} - 1}, \tag{48} \]
and \( \tilde{c}_0(\mu, \sigma_1, A_0) = O(\mu) \). Then, for \( \mu > 0 \) sufficiently small, the equation (2) has a periodic solution \( u_p(x) \) given by
\[ u_p(x) = -\frac{A_0}{\lambda_1} \mu \sin(\lambda_1 x) + R_3(\mu, \lambda_1 x), \tag{49} \]
where the smooth function \( R_3 \) is periodic in \( x \) and of order \( \mu^2 \), and
\[ \lambda_1 = \sqrt{1 + \sqrt{1+\alpha}}. \tag{50} \]

**Proof.** Under the condition (47), the linear operator of the system (3) for \( \mu = 0 \) has a pair of purely imaginary eigenvalues \( \pm i \lambda_1 \) and two real eigenvalues \( \pm \lambda_2 \), whose eigenvectors are
\[ U_1 = \begin{pmatrix} i \lambda_1^2, -1, -i \lambda_1^2, 1 \end{pmatrix}^T, \quad U_2 = \begin{pmatrix} -i \lambda_1^2, -1, i \lambda_1^2, 1 \end{pmatrix}^T, \]
\[ U_3 = \begin{pmatrix} 1, 1, 1, 1 \end{pmatrix}^T, \quad U_4 = \begin{pmatrix} -1, 1, -1, 1 \end{pmatrix}^T, \]
where \( \lambda_1 \) is given in (50) and \( \lambda_2 = \sqrt{1 + \sqrt{1+\alpha}} - 1 \). Again, the solution of (3) can be decomposed in terms of the above eigenvectors. Let
\[ (u, u_1, u_2, u_3)^T = AU_1 + BU_2 + CU_3 + DU_4 \]
so that the system (3) is equivalent to the following system
\[ \dot{A} = i\lambda_1 A + F_{10}, \quad \dot{B} = -i\lambda_1 B + F_{10}, \]
\[ \dot{C} = \lambda_2 C + F_{20}, \quad \dot{D} = -\lambda_2 D + F_{20}, \tag{51} \]
where
\[ F_{10} = \frac{\lambda_1^2}{4\sqrt{\alpha + 1}} f_{10}, \quad F_{20} = \frac{\alpha}{4\lambda_2^2 \sqrt{\alpha + 1}} f_{10}, \]
\[ f_{10} = \frac{c}{\lambda_2^2 \lambda_3^2} \left( \lambda_2^2 (A + B) - \lambda_1^2 (C + D) \right) + \beta \left( \frac{1}{\lambda_2^4} (C - D) + \frac{i}{\lambda_2^4} (A - B) \right)^2 \]
\[ - \frac{\gamma}{\lambda_2^4 \lambda_3^2} \left( \lambda_1^2 (C - D) + i \lambda_2^2 (A - B) \right)^3 + \sigma (A + B + C + D). \]

Furthermore, we assume that
\[ w_1 = -i (A - B), \quad w_2 = A + B, \quad w_3 = C - D, \quad w_4 = C + D, \]
which transforms the system (51) into
\[ \dot{w}_1 = \lambda_1 w_2, \quad \dot{w}_2 = -\lambda_1 w_1 + \frac{1}{2\lambda_2^2 \sqrt{1 + \alpha}} H_1, \]
\[ \dot{w}_3 = \lambda_2 w_4, \quad \dot{w}_4 = \lambda_2 w_3 + \frac{\alpha}{2\lambda_2^2 \sqrt{1 + \alpha}} H_2, \]
where
\[ H_1 = c \left( \lambda_2^2 u_2 - \lambda_3^2 u_4 \right) + \frac{\beta}{\lambda_2^4 \lambda_3^2} \left( \lambda_1^2 u_3 - \lambda_2^2 u_1 \right)^2 - \frac{\gamma}{\lambda_2^4 \lambda_3^2} \left( \lambda_1^2 u_3 - \lambda_2^2 u_1 \right)^3 \]
\[ + \alpha \sigma (u_2 + u_4), \]
\[ H_2 = c \left( \frac{u_2}{\lambda_1^2} - \frac{u_4}{\lambda_2^2} \right) + \frac{\beta}{\lambda_2^4 \lambda_3^2} \left( \lambda_1^2 u_3 - \lambda_2^2 u_1 \right)^2 - \frac{\gamma}{\lambda_2^4 \lambda_3^2} \left( \lambda_1^2 u_3 - \lambda_2^2 u_1 \right)^3 \]
\[ + \sigma (u_2 + u_4). \]
The center manifold reduction theorem shows that \( w_3 \) and \( w_4 \) can be solved as smooth functions of \( w_1 \) and \( w_2 \) such that
\[ w_3(\mu, w_1, w_2) = O(||(w_1, w_2)||||\mu, w_1, w_2||), \]
\[ w_4(\mu, w_1, w_2) = O(||(w_1, w_2)||||\mu, w_1, w_2||). \]
Hence the reduced system is by (47)
\[ \dot{w}_1 = \lambda_1 w_2, \]
\[ \dot{w}_2 = -\lambda_1 w_1 + \frac{\lambda_2^2 c_1 + \alpha \sigma_1}{2\lambda_2^2 \sqrt{1 + \alpha}} \mu w_2 + \frac{\beta}{2\lambda_2^2 \sqrt{1 + \alpha}} w_1^2 + G(\mu, w_1, w_2), \quad (52) \]
where \( c = \mu c_1 \) and the smooth function \( G(\mu, w_1, w_2) = O(||(w_1, w_2)||||\mu, w_1, w_2||^2) \).
We rescale as follows, letting
\[ w_1 = \mu y_1, \quad w_2 = \mu y_2, \]
such that the system (52) becomes
\[ \dot{y}_1 = \lambda_1 y_2, \]
\[ \dot{y}_2 = -\lambda_1 y_1 + \frac{\lambda_2^2 c_1 + \alpha \sigma_1}{2\lambda_2^2 \sqrt{1 + \alpha}} \mu y_2 + \frac{\beta}{2\lambda_2^2 \sqrt{1 + \alpha}} y_1^2 + \tilde{G}(\mu, w_1, w_2), \quad (53) \]
where the smooth function \( \tilde{G} \) satisfies
\[ \tilde{G}(\mu, y_1, y_2) = G(\mu, w_1, w_2) / \mu = O(||(y_1, y_2)||||\mu, y_1, y_2||^2) = O(\mu^2). \]
For \( \mu = 0 \), the system (53) has a periodic solution
\[ (\mathcal{X}_1, \mathcal{X}_2)^T(x) = A_0(\sin(\lambda_1 x), \cos(\lambda_1 x))^T(x). \]
Then the Melnikov function is defined by

\[ M_0(\mu, c_1, \sigma_1, A_0) = \int_0^{2\pi/\lambda_1} \mathcal{Y}_2(t) \left( \frac{\lambda_2^2 \mathcal{C}_1 + \alpha \sigma_1}{2\lambda_2^2 \sqrt{1 + \alpha}} \mu \mathcal{V}_2(t) + \frac{\beta}{2\lambda_1^2 \sqrt{1 + \alpha}} \mu \mathcal{V}_1^2(t) \right) dt \]

where

\[ M_1(\mu, c_1, \sigma_1, A_0) = \int_0^{2\pi/\lambda_1} \mathcal{Y}_2(t) \tilde{G}(\mu, \mathcal{Y}_1(t), \mathcal{Y}_2(t)) dt = A_0^2 O(\mu^2). \]

When higher order terms are added, the persistence of the periodic solution \((\mathcal{Y}_1, \mathcal{Y}_2)^T(x)\) is equivalent to that \(M_0(\mu, c_1, \sigma_1, A_0) \equiv 0\). For sufficiently small \(\mu > 0\), the implicit function theorem allows us to solve the equation \(M_0(\mu, c_1, \sigma_1, A_0) \equiv 0\) for \(c_1\) as a smooth function of \((\mu, \sigma_1, A_0)\), that is,

\[ c_1 = -\frac{\alpha \sigma_1}{A_0^2} + \tilde{c}_0(\mu, \sigma_1, A_0), \]

where the smooth function \(\tilde{c}_0(\mu, \sigma_1, A_0) = O(\mu)\), which is given in (47) and (48). Thus this implies that the system (53) or the system (52) has at least one periodic solution for small \(\mu > 0\). This completes the proof of Theorem 4.1.

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