SYZYGIES FOR THE VECTOR INVARIANTS OF THE DIHEDRAL GROUP

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Abstract. The problem of finding generators of the \( GL \)-ideal of the relations between the generators of the algebra of invariants of the dihedral group acting on \( m \)-tuples of vectors from its defining 2-dimensional representation is studied. It is shown that this \( GL \)-ideal is generated by relations depending on no more than 3 vector variables. A minimal \( GL \)-ideal generating system is found for the case when \( m = 2 \), and for the case of the dihedral group of order 8 and arbitrary \( m \).

1. Introduction

Given a group \( G \) of linear transformations on a finite dimensional complex vector space \( V \) and a positive integer \( m \), consider the diagonal action of \( G \) on the space \( V^m \) of \( m \)-tuples of vectors from \( V \). We call the algebra \( R(m) := \mathbb{C}[V^m]^G \) of polynomial functions on \( V^m \) constant along the \( G \)-orbits the algebra of vector invariants of \( G \). Following Weyl [17] who gave a systematic study of the case when \( G \) is one of the classical subgroups of the general linear group \( GL(V) \), a description of the generators of \( R(m) \) is referred to as a First Fundamental Theorem, whereas a description of the generators of the ideal of relations between the generators of \( R(m) \) is referred to as a Second Fundamental Theorem for the vector invariants of \( G \). An indispensable tool to establish such theorems is to take into account a natural right action of the group \( GL_m(\mathbb{C}) \) of invertible \( m \times m \) matrices on \( V^m \) that commutes with the \( G \)-action: for \( g = (g_{ij})_{i,j=1}^m \in GL_m(\mathbb{C}) \) and \( v = (v_1, \ldots, v_m) \in V^m \) we set

\[
v \cdot g = \left( \sum_{i=1}^m g_{i1}v_i, \ldots, \sum_{i=1}^m g_{im}v_i \right).
\]

This induces a left action of \( GL_m(\mathbb{C}) \) on \( \mathbb{C}[V^m] \) via \( \mathbb{C} \)-algebra automorphisms. Namely, for \( f \in \mathbb{C}[V^m] \), \( v \in V^m \) and \( g \in GL_m(\mathbb{C}) \) we have \((g \cdot f)(v) = f(v \cdot g)\). The subalgebra \( R(m) \) is a \( GL_m(\mathbb{C}) \)-invariant subspace. Moreover, \( R(m) \) is a graded subalgebra of \( \mathbb{C}[V^m] \), where the latter is endowed with the standard grading. Write \( R(m)_+ \) for the maximal ideal of \( R(m) \) spanned by its homogeneous elements of positive degree. It is clearly a \( GL_m(\mathbb{C}) \)-submodule, just like \((R(m)_+)^2\). Let \( W(m) \) be a \( GL_m(\mathbb{C}) \)-module direct complement of \((R(m)_+)^2\) in \( R(m) \). Then the subspace \( W(m) \) minimally generates the algebra \( R(m) \). For example, for

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the case when $G = GL_3(\mathbb{C})$ acting on the space $V = \mathbb{C}^{3 \times 3}$ of $3 \times 3$ matrices by conjugation, an explicit description of a minimal homogeneous generating system of $R(m)$ for general $m$ is rather complicated. This was made transparent in [1] by determining the $GL_m(\mathbb{C})$-module structure of $W(m)$. Let us turn next to the relations between the generators of $R(m)$. The identity map $W(m) \to W(m)$ induces a $\mathbb{C}$-algebra surjection

\[ \varphi(m) : S(W(m)) \to R(m) \]

from the symmetric tensor algebra $S(W(m))$ of $W(m)$ onto $R(m)$, and $\ker(\varphi(m))$ is the ideal of relations for the minimal generating subspace $W(m)$ of $R(m)$. Clearly, $\ker(\varphi(m))$ is a $GL$-ideal, i.e. a $GL_m(\mathbb{C})$-stable ideal in $S(W(m))$. A sensible way to describe $\ker(\varphi(m))$ is to find elements in it which generate irreducible $GL_m(\mathbb{C})$-submodules of $S(W(m))$, such that the sum of these $GL_m(\mathbb{C})$-submodules is direct and forms a minimal generating subspace of the ideal $\ker(\varphi(m))$ (so in particular, these elements constitute a minimal generating system of $\ker(\varphi(m))$ as a $GL$-ideal). For example, in the special case $G = GL_3(\mathbb{C})$ and $V = \mathbb{C}^{3 \times 3}$, the $GL_3(\mathbb{C})$-module structure of the minimal degree non-zero homogeneous component of $\ker(\varphi(m))$ was computed in [3].

In this paper we study the problem of finding generators of the $GL$-ideal $\ker(\varphi(m))$ for the defining 2-dimensional representation of the dihedral group $D_{2n}$ of order $2n$ ($n \geq 3$ is a positive integer). For the case $n = 3$ the result can be easily deduced from [10], see Theorem 6.9. For arbitrary $n$ and $m = 2$ the solution will be given in Theorem 5.1. Moreover, for arbitrary $n$ we reduce the problem to the case $m = 3$ (see Theorem 4.8). As an application of this reduction we give a complete solution of the problem when $n = 4$ in Theorem 6.8.

2. First Fundamental Theorem for the dihedral group

Let $n \geq 3$ be a positive integer and $\omega$ a complex primitive $n$th root of 1. We take for $G$ the dihedral group $D_{2n}$, the subgroup of $GL_2(\mathbb{C})$ generated by the matrices

\[
\begin{pmatrix}
\omega & 0 \\
0 & \omega^{-1}
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}.
\]

The given representation of $D_{2n}$ is the complexification of the defining real representation of $D_{2n}$ as the group of isometries of the Euclidean plane mapping a given regular $n$-gon into itself. Denoting by $x, y$ the coordinate functions on $V := \mathbb{C}^2$, the algebra $\mathbb{C}[x, y]^{D_{2n}}$ of polynomial invariants is generated by the algebraically independent invariants

\[ q := xy, \quad p := x^n + y^n. \]

Consider now the diagonal action of $D_{2n}$ on the space $V^m := V \oplus \cdots \oplus V$ ($m$ direct summands) of $m$-tuples of vectors. Denote by $x_i$ (respectively $y_i$) the function mapping an $m$-tuple of vectors to the first (respectively second) coordinate of its $i$th vector component, $i = 1, \ldots, m$. The coordinate ring $\mathbb{C}[V^m]$ is the $2m$-variable polynomial ring $\mathbb{C}[x_1, y_1, \ldots, x_m, y_m]$. For a degree $d$ homogeneous element $g \in \mathbb{C}[x, y]$ and $\alpha = (\alpha_1, \ldots, \alpha_m)$ with $\sum_j \alpha_j = d$ denote by $g_{\alpha}$ the multihomogeneous component of
where $\binom{d}{\alpha}$ stands for the $d$-th symmetric tensor power of the $k$th symmetric tensor power of the $GL(U)$-module $U$.

Write $\mathcal{F}(n, m) := S(W(m)) = \sum_{d=0}^{\infty} S^d(W(m))$ for the symmetric tensor algebra of $W(m)$ (endowed with the natural $GL_m(C)$-module structure induced by the $GL_m(C)$-action on $W(m)$). Although $S^1(W(m)) \subset \mathcal{F}(n, m)$ is just $W(m)$, to avoid confusion later we need to distinguish in the notation the elements of $W(m)$ when they are considered as elements of $\mathcal{F}(n, m)$. We shall write $\rho_\alpha \in \mathcal{F}(n, m)$ for the element corresponding to $q_\alpha$, and $\pi_\beta \in \mathcal{F}(n, m)$ for the element corresponding to $p_\beta$. So $\mathcal{F}(n, m)$ is the polynomial ring

$$\mathcal{F}(n, m) = \mathbb{C}[\rho_\alpha, \pi_\beta \mid \sum \alpha_i = 2, \sum \beta_j = n].$$

**Theorem 2.1.** [12] Theorem 4.1] The polarizations of $q$ and $p$ form a minimal homogeneous generating system of the $\mathbb{C}$-algebra $\mathbb{C}[V^m]^{D_{2n}}$.

**Remark 2.2.** In fact [12] works over the real field, but by well known basic principles [12, Theorem 4.1] implies the variant Theorem 2.1 above.

Therefore in this case $W(m)$ (cf. Section 1) is

$$W(m) = \langle q \rangle_{GL_m(C)} \oplus \langle p \rangle_{GL_m(C)} = \text{Span}_C \{q_\alpha, p_\beta \mid \alpha, \beta \in \mathbb{N}_0^m, \sum_{i=1}^{m} \alpha_i = 2, \sum_{j=1}^{m} \beta_j = n\}.$$

We have the $GL_m(C)$-module isomorphisms

$$\langle q \rangle_{GL_m(C)} \cong S^2(C^m) \quad \text{and} \quad \langle p \rangle_{GL_m(C)} \cong S^n(C^m)$$

where $S^k(U)$ stands for the $k$th symmetric tensor power of the $GL(U)$-module $U$.
Thus in our case $\varphi(m)$ (cf. [1] in Section 1) is the $\mathbb{C}$-algebra (as well as $GL_m(\mathbb{C})$-module) surjection

$$\varphi(n, m) : \mathcal{F}(n, m) \rightarrow \mathbb{C}[V^m]^{D_{2n}}, \quad \rho_\alpha \mapsto q_\alpha, \quad \pi_\beta \mapsto p_\beta.$$ 

Throughout the paper for $l \leq m$ we shall treat $\mathbb{C}[V^l]$ and $\mathcal{F}(n, l)$ as a subalgebra of $\mathbb{C}[V^m]$ and $\mathcal{F}(n, m)$ in the obvious way. Our aim is to describe the kernel $\ker(\varphi(n, m))$ of $\varphi(n, m)$ as a $GL$-ideal.

3. Preliminaries on $GL_m(\mathbb{C})$-modules

As a general reference to the material in this section, see for example the book [14]. Recall that the isomorphism classes of the irreducible polynomial $GL_m(\mathbb{C})$-modules of degree $d$ are labeled by partitions of $d$ with $m$ parts, where by a partition $\lambda$ of $d$ with $m$ parts (notation: $\lambda \in \text{Par}_m(d)$) we mean a sequence $(\lambda_1, \ldots, \lambda_m)$ of non-negative integers with $\lambda_1 \geq \cdots \geq \lambda_m$ and $\lambda_1 + \cdots + \lambda_m = d$. We denote by $\text{ht}(\lambda)$ the number of non-zero elements in the sequence $(\lambda_1, \ldots, \lambda_m)$. Moreover, for $l < m$ and non-negative integers $\lambda_1 \geq \cdots \geq \lambda_l$ with $\sum \lambda_i = d$ we identify the sequence $(\lambda_1, \ldots, \lambda_l)$ with $(\lambda_1, \ldots, \lambda_l, 0, \ldots, 0) \in \text{Par}_m(d)$.

For an $\ell$-dimensional vector space $U$ write $S^\lambda(U)$ for the $GL(U)$-module associated to $U$ by the Schur functor $S^\lambda(-)$. This is an irreducible polynomial $GL(U)$-module if $\text{ht}(\lambda) \leq \ell$ and $S^\lambda(U)$ is the zero module when $\text{ht}(\lambda) > \ell$. For example, for the partition $\lambda = (n, 0, \ldots)$ with only one non-zero part we have $S^\lambda(U) = S^n(U)$, the $n$th symmetric tensor power of $U$. Moreover, for $\lambda \in \text{Par}_m(d)$, $S^\lambda(\mathbb{C}^m)$ is the irreducible polynomial $GL_m(\mathbb{C})$-module labeled by $\lambda$. Note that when $U$ has a $GL_m(\mathbb{C})$-module structure ($m$ may differ from $\ell$), then $S^\lambda(U)$ becomes naturally a (typically not irreducible) $GL_m(\mathbb{C})$-module.

$\mathcal{F}(n, m)$ and $\mathbb{C}[V^m]$ are polynomial $GL_m(\mathbb{C})$-modules. An element $v$ of a polynomial $GL_m(\mathbb{C})$-module generates a submodule isomorphic to $S^\lambda(\mathbb{C}^m)$ if

1. it is fixed by $UT_m(\mathbb{C})$, the subgroup of upper triangular unipotent matrices;
2. for a diagonal element $\text{diag}(z_1, \ldots, z_m) \in GL_m(\mathbb{C})$ we have $\text{diag}(z_1, \ldots, z_m) \cdot v = z_1^{\lambda_1} \cdots z_m^{\lambda_m} \cdot v$.

Such an element is called a highest weight vector. An irreducible polynomial $GL_m(\mathbb{C})$-module contains a unique (up to non-zero scalar multiples) highest weight vector.

The action of $GL_m(\mathbb{C})$ on $\mathcal{F}(n, m)$ and on $\mathbb{C}[V^m]$ induces a representation of its Lie algebra $\mathfrak{gl}_m(\mathbb{C})$ on $\mathcal{F}(n, m)$ and on $\mathbb{C}[V^m]$, such that $\varphi(n, m)$ is a homomorphism of $\mathfrak{gl}_m(\mathbb{C})$-modules. In particular, $\ker(\varphi(n, m))$ is preserved by $\mathfrak{gl}_m(\mathbb{C})$. To detect explicit highest weight vectors it is convenient to pass to the Lie algebra action. Therefore for later use we shall record the formulae determining this $\mathfrak{gl}_m(\mathbb{C})$-representation on $\mathcal{F}(n, m)$ in the case $m = 3$. Denote by $E_{i, j}$ the matrix unit having entry 1 in the $(i, j)$ position and the entry 0 in all other positions. The Lie algebra $\mathfrak{gl}_m(\mathbb{C})$ of $GL_m(\mathbb{C})$ has basis $\{E_{i, j} \mid 1 \leq i, j \leq m\}$. We have

$$E_{1, 2} \pi_{i, j, k} = j \pi_{i+1, j-1, k}, \quad E_{2, 3} \pi_{i, j, k} = k \pi_{i, j+1, k-1}, \quad E_{ss} \pi_{\alpha_1, \alpha_2, \alpha_3} = \alpha_s \pi_\alpha$$

$$E_{2, 1} \pi_{i, j, k} = i \pi_{i-1, j+1, k}, \quad E_{3, 2} \pi_{i, j, k} = j \pi_{i-1, j+1, k}$$

$$E_{1, 2} \rho_{i, j, k} = j \rho_{i+1, j-1, k}, \quad E_{2, 3} \rho_{i, j, k} = k \rho_{i, j+1, k-1}, \quad E_{ss} \rho_{\alpha_1, \alpha_2, \alpha_3} = \alpha_s \rho_\alpha$$
\[ E_{2,1} \cdot \rho_{i,j,k} = i \rho_{i-1,j+1,k}, \quad E_{3,2} \cdot \rho_{i,j,k} = j \rho_{i,j-1,k+1} \]

where in the above formulae \( \pi_{a,b,c} \) or \( \rho_{a,b,c} \) is interpreted as zero unless \( a, b, c \) are all non-negative.

4. Reduction to the case \( m = 3 \)

The algebra \( \mathbb{C}[V^m]^{D_{2n}} \) is a graded subalgebra of \( \mathbb{C}[V^m] \), where the latter is endowed with the standard grading. We introduce a grading on \( F \) by making the homomorphism \( \varphi(n, m) \) degree preserving; that is, the degree of the variables \( \rho_\alpha \) is 2, and the degree of the variables \( \pi_\beta \) is \( n \). We shall denote by \( F(n, m)_d \) and by \( \ker(\varphi(n, m))_d \) the degree \( d \) homogeneous component of \( F(n, m) \) and its graded subspace \( \ker(\varphi(n, m)) \), and we write \( F(n, m)_{\leq d} \) and \( \ker(\varphi(n, m))_{\leq d} \) for the sum of the homogeneous components of degree at most \( d \) in \( F(n, m) \) and \( \ker(\varphi(n, m)) \).

**Proposition 4.1.** The ideal \( \ker(\varphi(n, m)) \) is generated by its elements of degree at most \( 2n + 2 \).

**Proof.** By a general result of Derksen [8, Theorem 2] the ideal \( \ker(\varphi(n, m)) \) is generated by its elements of degree at most \( 2\tau_{D_{2n}}(V^m) \), where \( \tau_{D_{2n}}(V^m) \) denotes the minimal positive integer \( k \) such that all homogeneous polynomials in \( \mathbb{C}[V^m] \) of degree \( k \) are contained in the ideal \( \mathbb{C}[V^m]^{D_{2n}} \cdot \mathbb{C}[V^m] \) of \( \mathbb{C}[V^m] \) generated by the homogeneous \( D_{2n} \)-invariants of positive degree (called the Hilbert ideal). The number \( \tau_{D_{2n}}(V^m) \) is bounded by the Noether number of \( D_{2n} \) by [7, Lemma 1.6]. The Noether number of \( D_{2n} \) is \( n + 1 \) by [15] (see also [6, Corollary 5.6] for a stronger statement). This implies \( \tau_{D_{2n}}(V^m) \leq n + 1 \). Note finally that obviously \( \tau_{D_{2n}}(V^m) \geq \tau_{D_{2n}}(V) \), and the latter number is known to be \( n + 1 \): indeed, \( \mathbb{C}[V]^{D_{2n}} \) is generated by algebraically independent homogeneous invariants of degree 2 and \( n \), and thus the Hilbert series of the corresponding coinvariant algebra \( \mathbb{C}[V^m]/\mathbb{C}[V^m]^{D_{2n}} \cdot \mathbb{C}[V^m] \) equals \( (1 + t)(1 + t + \cdots + t^{n-1}) \) (see [5]).

Note that \( F(n, m) \) has the tensor product decomposition \( F(n, m) = D(m) \otimes E(n, m) \) where

- \( D(m) = S(\langle q \rangle_{\text{GL}_m(\mathbb{C})}) = \mathbb{C}[\rho_\alpha \mid \alpha \in \mathbb{N}_0^m, \sum \alpha_i = 2] \)
- \( E(n, m) = S(\langle p \rangle_{\text{GL}_m(\mathbb{C})}) = \mathbb{C}[\pi_\beta \mid \beta \in \mathbb{N}_0^m, \sum \beta_j = n] \)

For \( m \geq 3 \) set

\[ R_{2,2,2} := \det \begin{pmatrix} \rho_{2,0,0} & \rho_{1,1,0} & \rho_{1,0,1} \\ \rho_{1,1,0} & \rho_{0,2,0} & \rho_{0,1,1} \\ \rho_{1,0,1} & \rho_{0,1,1} & \rho_{0,0,2} \end{pmatrix} \in D(m) \subset F(n, m). \]

**Lemma 4.2.** For \( m \geq 3 \) the element \( R_{2,2,2} \) belongs to \( \ker(\varphi(n, m)) \), and \( \langle R_{2,2,2} \rangle_{\text{GL}_m(\mathbb{C})} \cong S^{(2,2,2)}(\mathbb{C}^m) \) as \( \text{GL}_m(\mathbb{C}) \)-modules.
Proof. The matrix
\[
\begin{pmatrix}
q_{2,0,0} & q_{1,1,0} & q_{1,0,1} \\
q_{1,1,0} & q_{0,2,0} & q_{0,1,1} \\
q_{1,0,1} & q_{0,1,1} & q_{0,0,2}
\end{pmatrix}
= \begin{pmatrix} x_1 & y_1 \\
x_2 & y_2 \\
x_3 & y_3 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1/2 \\
1/2 & 0 \end{pmatrix} \cdot \begin{pmatrix} x_1 & x_2 & x_3 \\
y_1 & y_2 & y_3 \end{pmatrix}
\]
has rank 2, implying that its determinant is zero. On the other hand, this determinant is \(\varphi(n,m)\)(\(R_{2,2,2}\)). Thus \(R_{2,2,2} \in \ker(\varphi(n,m))\). A straightforward calculation yields \(E_{1,2}R_{2,2,2} = 0, E_{2,3}R_{2,2,2} = 0,\) and \(E_{i,j}R_{2,2,2} = 2R_{2,2,2}\) (\(i = 1, 2, 3\)), hence \(R_{2,2,2}\) is a highest weight vector in \(F(n,m)\) with weight \((2,2,2)\).

\[\square\]

Remark 4.3. In fact it is known that \(R_{2,2,2}\) generates as a \(GL\)-ideal the kernel of the restriction of \(\varphi(n,m)\) to \(D(m)\) (see the Second Fundamental Theorem for the orthogonal group in \([17, \text{Theorem} 2.17.A]\)).

Setting \(\bar{D}(m) := D(m)/(R_{2,2,2})GL_m(\mathbb{C})D(m)\), by Lemma 4.2 we conclude that \(\varphi(n,m)\) factors through the natural surjection \(F(n,m) \to \bar{D}(m) \otimes E(n,m)\), so we get the graded \(GL_m(\mathbb{C})\)-module algebra surjection
\[
\bar{\varphi}(n,m) : \bar{D}(m) \otimes E(n,m) \to \mathbb{C}[V^m]^{D_{2n}}.
\]

Lemma 4.4. For \(m \geq 3\) the \(GL\)-ideal \(\ker(\varphi(n,m))\) is generated by \(R_{2,2,2}\) and any subset of \(\ker(\varphi(n,m))\) whose image under the natural surjection \(F(n,m) \to \bar{D}(m) \otimes E(n,m)\) generates \(\ker(\varphi(n,m))\) as a \(GL\)-ideal.

Proof. This is an immediate consequence of the construction of \(\bar{\varphi}(n,m)\).

Our next aim is to compute the \(GL_m(\mathbb{C})\)-module structure of the homogeneous components of \(\bar{D}(m) \otimes E(n,m)\) up to degree \(2n + 2\).

Proposition 4.5. We have the following isomorphisms of \(GL_m(\mathbb{C})\)-modules:

(i) \(\bar{D}(m) \cong \sum_{d=0}^{\infty} \sum_{\lambda \in \text{Par}_m(d)} S^{2\lambda}(\mathbb{C}^m)\) where \(2\lambda\) stands for the partition \((2\lambda_1, \ldots, 2\lambda_m)\) of \(2d\).

(ii) \(\bar{D}(m) \cong \sum_{d=0}^{\infty} \sum_{\lambda \in \text{Par}_m(d), \text{ht}(\lambda) \leq 2} S^{2\lambda}(\mathbb{C}^m)\).

(iii) \(E(n,m) \cong S(S^n(\mathbb{C}^m)) = \bigoplus_{d=0}^{\infty} S^d(S^n(\mathbb{C}^m))\) (the symmetric tensor algebra of \(S^n(\mathbb{C}^m)\)).

(iv) \(S^2(S^n(\mathbb{C}^m)) \cong \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} S^{2(n-2j,2j)}(\mathbb{C}^m)\).

Proof. (i) We have \(\langle q \rangle_{GL_m(\mathbb{C})} \cong S^2(\mathbb{C}^m)\) and therefore (i) follows from the well-known decomposition
\[
S^d(S^2(\mathbb{C}^m)) \cong \sum_{\lambda \in \text{Par}_m(d)} S^{2\lambda}(\mathbb{C}^m)
\]
(see for example \([14, \text{Section} 11.4.5, \text{Theorem}]\)).

(ii) \(\bar{D}(m)\) can be identified with the coordinate ring of the variety of \(m \times m\) symmetric matrices of rank at most 2 endowed with the natural \(GL_m(\mathbb{C})\)-action. For the well-known decomposition of this \(GL_m(\mathbb{C})\)-module see for example \([14, \text{Section} 11.5.1, \text{Second Fundamental Theorem}]\).

(iii) follows from the isomorphism \(\langle p \rangle_{GL_m(\mathbb{C})} \cong S^n(\mathbb{C}^m)\).
Corollary 4.6.\ As a $GL_m(\mathbb{C})$-module, $(\bar{D}(m) \otimes \mathcal{E}(n, m))_{\leq d}$ is isomorphic to

$$
\sum_{d=0}^{n+1} \sum_{\lambda \in \text{Par}_m(n(d), \text{ht}(\lambda) \leq 2)} S^{2\lambda}(\mathbb{C}^m) + \sum_{d=0}^{n+2} \sum_{\lambda \in \text{Par}_m(n(d), \text{ht}(\lambda) \leq 2)} S^m(\mathbb{C}^m) \otimes S^{2\lambda}(\mathbb{C}^m)
$$

$$
+ \sum_{j=0}^{\lfloor \frac{d}{2} \rfloor} (S^{(2n-2j, 2j)}(\mathbb{C}^m) + S^{(2n-2j, 2j)}(\mathbb{C}^m) \otimes S^2(\mathbb{C}^m)).
$$

Proof. As $\mathcal{E}(n, m)_d$ is non-zero only if $d$ is a multiple of $n$, the sum of the homogeneous components of $\bar{D}(m) \otimes \mathcal{E}(n, m)$ of degree at most $2n + 2$ is

$$
\sum_{d=0}^{2n+2} \bar{D}(m)_d \otimes \mathcal{E}(n, m)_0 + \sum_{d=0}^{n+2} \bar{D}(m)_d \otimes \mathcal{E}(n, m)_n + \sum_{d=0}^{2} \bar{D}(m)_d \otimes \mathcal{E}(n, m)_2.
$$

Now the statement follows from the $GL_m(\mathbb{C})$-module isomorphisms given in Proposition 4.5. \qed

Recall that for $l \leq m$ we view $\mathcal{F}(n, l)$ as a subspace of $\mathcal{F}(n, m)$ in the obvious way. Similarly, $\bar{D}(l)$, $\mathcal{D}(l)$, $\mathcal{E}(n, l)$, are viewed as subspaces of $\bar{D}(m)$, $\mathcal{D}(m)$, $\mathcal{E}(n, m)$, and so $\mathcal{D}(l) \otimes \mathcal{E}(n, l)$ is viewed as a subspace of $\bar{D}(m) \otimes \mathcal{E}(n, m)$.

Proposition 4.7. Let $U$ be a $GL_m(\mathbb{C})$-invariant subspace of $(\bar{D}(m) \otimes \mathcal{E}(n, m))_{\leq 2n+2}$. Then

$$
U = (U \cap \bar{D}(3) \otimes \mathcal{E}(n, 3))_{GL_m(\mathbb{C})}.
$$

Proof. By complete reducibility of $GL_m(\mathbb{C})$-modules, it is sufficient to prove the statement in the special case when $U$ is a minimal $GL_m(\mathbb{C})$-invariant subspace of $(\bar{D}(m) \otimes \mathcal{E}(n, m))_{\leq 2n+2}$. So assume that $U$ is a minimal $GL_m(\mathbb{C})$-invariant subspace. Pieri’s Formula (cf. [13, I.5.16]) implies that if $S^\nu(\mathbb{C}^m)$ occurs as a summand in $S^\lambda(\mathbb{C}^m) \otimes S^\mu(\mathbb{C}^m)$ then $\text{ht}(\nu) \leq \text{ht}(\lambda) + \text{ht}(\mu)$. Thus by Corollary 4.6 we conclude that $U \cong S^\lambda(\mathbb{C}^m)$ for some partition $\lambda$ with $\text{ht}(\lambda) \leq 3$. It follows that $U = (w)_{GL_m(\mathbb{C})}$ for some highest weight vector $w$ with weight $\lambda$. That is, $w$ is a non-zero element of $U$ fixed by the subgroup of unipotent upper triangular matrices in $GL_m(\mathbb{C})$, and for a diagonal element $\text{diag}(z_1, \ldots, z_m) \in GL_m(\mathbb{C})$ we have $\text{diag}(z_1, \ldots, z_m) \cdot w = z_1^{\lambda_1} \cdots z_m^{\lambda_m} w$. In particular, $\text{ht}(\lambda) \leq 3$ implies that $w$ belongs to $\bar{D}(3) \otimes \mathcal{E}(n, 3)$. \qed

We arrived at the main result of this Section:

Theorem 4.8. For $m \geq 3$ the ideal $\ker(\varphi(n, m))$ of $\mathcal{F}(n, m)$ is generated by the $GL_m(\mathbb{C})$-submodule $\langle \ker(\varphi(n, 3))_{\leq 2n+2} \rangle_{GL_m(\mathbb{C})}$ of $\mathcal{F}(n, m)$. That is, $\ker(\varphi(n, m))$ is generated as a $GL$-ideal by elements of degree $\leq 2n + 2$ contained in the kernel of $\varphi(n, 3)$. \qed
Proof. The natural surjection $F(n, m) \to \mathcal{D}(m) \otimes \mathcal{E}(n, m)$ maps $\ker(\varphi(n, m))$ onto $\ker(\varphi(n, m))$, therefore by Proposition 4.7, the latter is generated as an ideal by $\ker(\varphi(n, m))_{\leq 2n+2}$. Now $\ker(\varphi(n, m))_{\leq 2n+2}$ is a $GL_m(\mathbb{C})$-invariant subspace of $(\mathcal{D}(m) \otimes \mathcal{E}(n, m))_{\leq 2n+2}$, hence by Proposition 4.7 we have

$$\ker(\varphi(n, m))_{\leq 2n+2} = \langle \ker(\varphi(n, m))_{\leq 2n+2} \rangle_{\leq 2n+2} \cap (\mathcal{D}(3) \otimes \mathcal{E}(n, m))_{\leq 2n+2} = \langle \ker(\varphi(n, m))_{\leq 2n+2} \rangle_{\leq 2n+2} \cap \mathcal{E}(n, m)_{\leq 2n+2}.$$ 

So the image $\ker(\varphi(n, m))_{\leq 2n+2}$ of $\ker(\varphi(n, m))_{\leq 2n+2}$ under the natural surjection $F(n, m) \to \mathcal{D}(m) \otimes \mathcal{E}(n, m)$ generates $\ker(\varphi(n, m))$ as a $GL$-ideal. Since the element $R_{2,2,2}$ belongs to $\ker(\varphi(n, m))_{\leq 2n+2}$, we can conclude by Lemma 4.4 that $\ker(\varphi(n, m), 3)_{\leq 2n+2}$ generates $\ker(\varphi(n, m))$ as a $GL$-ideal. Equivalently, $\ker(\varphi(n, m))$ is generated as an ideal of $F(n, m)$ by its subspace $\langle \ker(\varphi(n, m))_{\leq 2n+2} \rangle_{\leq 2n+2}$.

\[ \square \]

5. The case $m = 2$

Set

\[ \mathcal{R}(n)_{2n-2k, 2k} := \pi_{n,0}\rho_{0,2} - 2\pi_{n-1,1}\rho_{1,1} + \pi_{n-2,2}\rho_{2,0}. \]

For $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ set

\[ \mathcal{R}(n)_{2n-2k, 2k} := (-1)^k \frac{1}{2} \binom{2k}{k} \pi_{n-k,k}^2 + \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} \pi_{n-j,j} \pi_{n-2k+j, 2k-j} - 4^n \rho_{2,0}^2 (\rho_{1,1} - \rho_{2,0} \rho_{0,2})^k. \]

Theorem 5.1. For $\lambda \in \{(n, 2), (2n-2, 2), (2n-4, 4), \ldots, (2n-2 \lfloor \frac{n}{2} \rfloor, 2 \lfloor \frac{n}{2} \rfloor)\}$ consider the element $\mathcal{R}(n)_\lambda$ in $F(n, 2)$ introduced above.

(i) For $m \geq 2$ the element $\mathcal{R}(n)_\lambda$ generates a $GL_m(\mathbb{C})$-submodule in $\ker(\varphi(n, m))$ isomorphic to $S^\lambda(\mathbb{C}^m)$.

(ii) For $m = 2$ the ideal, $\ker(\varphi(n, 2))$ is minimally generated by the above elements $\mathcal{R}(n)_\lambda$ as a $GL_2(\mathbb{C})$-ideal.

Remark 5.2. A minimal presentation of the algebra $\mathbb{C}[V \oplus V^*]^{D_{2n}}$ by generators and relations is given in [2] for $n = 3, 4, 6$ and in [4] Theorem 2.1 for arbitrary $n$. Since $V^* \cong V$ as $D_{2n}$-modules, these results can be translated to an explicit minimal presentation by generators and relations of $\mathbb{C}[V^2]^{D_{2n}}$. So the novel part of our Theorem 5.1 is the nice explicit form of the relations $\mathcal{R}(n)_\lambda$ and the understanding of the $GL_2(\mathbb{C})$-module structure of the minimal syzygies.

Proof of Theorem 5.1(i). We have $E_{1,2}.\mathcal{R}(n)_{n,2} = 0$, $E_{1,1}.\mathcal{R}(n)_{n,2} = n\mathcal{R}(n)_{n,2}$, $E_{2,2}.\mathcal{R}(n)_{n,2} = 2\mathcal{R}(n)_{n,2}$. Thus $\mathcal{R}(n)_{n,2}$ is a highest weight vector with weight $(n, 2)$, and therefore it generates an irreducible $GL_m(\mathbb{C})$-submodule of $F(n, m)$ isomorphic to $S^{(n,2)}(\mathbb{C}^m)$ for $m \geq 2$. The elements $\rho_{2,0}, \rho_{1,1}^2 - \rho_{2,0} \rho_{0,2}$ and $(-1)^k \frac{1}{2} \binom{2k}{k} \pi_{n-k,k}^2 + \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} \pi_{n-j,j} \pi_{n-2k+j, 2k-j}$ are annihilated by $E_{1,2} \in \mathfrak{sl}_2(\mathbb{C})$, so they are highest weight vectors with weights $(2, 0)$, $(2, 2)$ and $(2n-2k, 2k)$. It follows that $\mathcal{R}(n)_{2n-2k, 2k}$ is a highest weights vector of weight.
Note that the equalities (3), (4), (5) show that this gives an explicit formula for the highest weight vector of the summand $S^{(2n-2k,2k)}(C^m)$ for $m \geq 2$. The equality
\[ p_{n,0}q_{0,2} - 2p_{n-1,1}q_{1,1} + p_{n-2,2}q_{2,0} = 0 \]
can be verified by direct computation, showing that $\varphi(n,2)(\mathcal{R}(n)_{2n-2k,2k}) = 0$. To verify that $\mathcal{R}(n)_{2n-2k,2k}$ belongs to $\ker(\varphi(n,2))$ for $k = 1, \ldots, \lfloor \frac{n}{2} \rfloor$ we need to prove the equality
\[ (-1)^k \frac{1}{2} \binom{2k}{k} p_{n-k,k}^2 + \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} p_{n-j,j} p_{n-2k+j,2k-j} - 4^k q_{2,0}^{n-2k} (q_{1,1}^2 - q_{2,0} q_{0,2})^k = 0. \]

Since $\mathcal{R}(n)_{2n-2k,2k}$ is a highest weight vector, it is fixed by the subgroup $UT_2(C)$ of unipotent upper triangular matrices in $GL_2(C)$. Therefore $\varphi(n,2)(\mathcal{R}(n)_{2n-2k,2k})$ (the left hand side of (4)) is a $UT_2(C)$-invariant in $C[V^2]$, and thus it is constant along the $UT_2(C)$-orbits in $V^2$. The $UT_2(C)$-orbit of each point from a Zariski dense open subset in $V^2$ has non-empty intersection with the subset of lower triangular matrices in $C^{2 \times 2} = V^2$. Therefore it is sufficient to show that $\sigma(\varphi(n,2)(\mathcal{R}(n)_{2n-2k,2k})) = 0$, where $\sigma$ is the homomorphism $C[x, y_1, x_2, y_2] \rightarrow C[x, y_1, y_2]$ given by the specialization $x_2 \mapsto 0$. Now $\sigma(p_{n-j,j}) = y_1^{n-j} y_2^j$ for $j > 0$, $\sigma(q_{0,2}) = 0$, $\sigma(q_{1,1}) = \frac{1}{2} x_1 y_2$, hence
\[ \sigma(4^k q_{2,0}^{n-2k} (q_{1,1}^2 - q_{2,0} q_{0,2})^k) = x_1^{n-2k} y_2^k. \]

On the other hand, we have
\[ \sigma((-1)^k \frac{1}{2} \binom{2k}{k} p_{n-k,k}^2 + \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} p_{n-j,j} p_{n-2k+j,2k-j}) \]
\[ = ((-1)^k \frac{1}{2} \binom{2k}{k}) + \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j}) y_1^{2n-2k} y_2^k + x_1^n y_1^{-2k} y_2^k. \]

Note that
\[ (-1)^k \frac{1}{2} \binom{2k}{k} + \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} = \frac{1}{2} \sum_{j=0}^{2k} \binom{2k}{j} = \frac{1}{2} (1 - 1)^{2k} = 0. \]

The equalities (5), (6), (7) show $\sigma(\varphi(n,2)(\mathcal{R}(n)_{2n-2k,2k})) = 0$, implying in turn the equality (4), and thus (i) is proved. \[\Box\]

**Remark 5.3.** The expression $(-1)^k \frac{1}{2} \binom{2k}{k} \pi_{n-k,k}^2 + \sum_{j=0}^{k-1} (-1)^j \binom{2k}{j} \pi_{n-j,j} \pi_{n-2k+j,2k-j}$ appears in the invariant theory of $m$-ary forms of degree $n$, see \[11\] Section 114. Note also that this gives an explicit formula for the highest weight vector of the summand $S^{(2n-2k,2k)}(C^m)$ of $S^2(S^n(C^m))$ in the decomposition given in Proposition 4.3 (iv).
5.1. Hironaka decomposition. The following statement is known, see for example [4] Theorem 2.1. We shall present an alternative proof.

Proposition 5.4. The elements \(p_{n,0}, q_{2,0}, p_{0,n}, q_{0,2}\) form a homogeneous system of parameters in \(\mathbb{C}[V^2]^{D_{2n}}\), and \(\mathbb{C}[V^2]^{D_{2n}}\) is a free module over the subalgebra \(P(n,2) := \mathbb{C}[p_{n,0}, q_{2,0}, p_{0,n}, q_{0,2}]\) generated by

\[
\{q_{1,1}, p_{n-i,i} \mid j = 0, 1, \ldots, n; \ i = 1, \ldots, n-1\}.
\]

Proof. The common zero locus in \(V^2\) of \(p_{n,0}, q_{2,0}, p_{0,n}, q_{0,2}\) is the zero element of \(V^2\), hence these polynomials form a homogeneous system of parameters in \(\mathbb{C}[V^2]^{D_{2n}}\) and in \(\mathbb{C}[V^2]\) (so both of these algebras are finitely generated free modules over their subalgebra \(P(n,2)\), see for example [8, Section 2.3]).

Denote by \(H\) the cyclic subgroup of \(D_{2n}\) generated by \(\begin{pmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{pmatrix}\). Then \(\mathbb{C}[V^2]^H\) is spanned as a \(\mathbb{C}\)-vector space by the monomials \(x_1^{\alpha_1}x_2^{\alpha_2}y_1^{\beta_1}y_2^{\beta_2}\) where \(\alpha_1 + \alpha_2 - \beta_1 - \beta_2\) is divisible by \(n\). One can easily deduce (see for example the method of the proof of Proposition 5.3) that \(\mathbb{C}[V^2]^H\) is a free \(P(n,2)\)-module generated by

\[
\{(x_1y_2)^j, (x_2y_1)^j, x_1^{n-i}x_2^i, y_1^{n-i}y_2^i, x_1^n, x_2^n, x_1^n x_2^n \mid j = 0, 1, \ldots, n-1, \ i = 1, \ldots, n-1\}.
\]

Denote by \((P(n,2)_+)\) the ideal in \(\mathbb{C}[V^2]\) generated by \(p_{n,0}, q_{2,0}, p_{0,n}, q_{0,2}\). The other generator \(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\) of \(D_{2n}\) maps the cosets of \(x_1^n\) and \(x_2^n\) in \(\mathbb{C}[V^2]/(P(n,2)_+)\) to their negative, fixes the coset of \(x_1^n x_2^n\) in \(\mathbb{C}[V^2]/(P(n,2)_+)\), permutes the other elements in the above \(P(n,2)\)-module generating system of \(\mathbb{C}[V^2]^H\), and fixes all elements of \(P(n,2)\). It follows that \(\mathbb{C}[V^2]^{D_{2n}}\) is a free \(P(n,2)\)-module generated by \(\mathbb{C}[V^2]/(P(n,2)_+)\)

\[
\{(x_1y_2)^j + (y_1x_2)^j, x_1^{n-i}x_2^i + y_1^{n-i}y_2^i, x_1^n, x_2^n \mid j = 0, 1, \ldots, n-1, \ i = 1, \ldots, n-1\}.
\]

Note finally that \((x_1y_2)^j + (y_1x_2)^j\) is congruent to \(2^j q_{1,1}^j\) modulo \((P(n,2)_+)\), and \(x_1^n x_2^n\) is congruent to \(-\frac{1}{2}(2q_{1,1})^n\) modulo \((P(n,2)_+)\). This shows that \(\mathbb{C}[V^2]^{D_{2n}}\) is a free module over \(P(n,2)\) generated by [8].

5.2. Further relations. Define \(R(n)_{n,2}^{n-j,2+j}\) for \(j = 0, 1, \ldots, n-2\) recursively by setting \(R(n)_{n,2}^{n,2} := R(n,n,2)\), and set

\[
R(n)_{n,2}^{n-j,2+j} := \frac{1}{n-2-j}E_{2,1}\cdot R(n)_{n,2}^{n-j,2+j}
\]

for \(j = 0, 1, \ldots, n-3\). Then \(\{R(n)_{n,2}^{n-j,2+j} \mid j = 0, 1, \ldots, n-2\}\) is a \(\mathbb{C}\)-vector space basis in \(\langle R(n)_{n,2} \rangle_{GL_2(\mathbb{C})}\). Moreover, one shows by induction on \(j\) that

\[
R(n)_{n,2}^{n-j,2+j} = \pi_{n-j,j}p_{0,2} - 2\pi_{n-j-1,j+1}p_{1,1} + \pi_{n-j-2,j+2}p_{2,0}.
\]

Proof of Theorem 5.1 (ii). The elements \(q_{2,0}, q_{1,1}, q_{0,2}\) in \(\mathbb{C}[V^2]^{D_{2n}}\) are algebraically independent. Therefore any non-zero element in \(\text{ker}(\varphi(n,2))\) involves a variable \(\pi_{n-j,j}\) for some \(j\). Moreover, a non-trivial linear combination of the elements \(p_{n-j,j} \in \mathbb{C}[V^2]^{D_{2n}}\)
Let \( P \) be any monomial of the form \( x_1^{j_1}x_2^{j_2} \). Consider the term \( -\sum_{j=1}^{n-1} \rho_{1,j} \pi_{n-j,j} \) to an element of the ideal of \( \ker(\varphi(n,2)) \). It follows from Proposition \ref{prop} that there must exist \( \lambda \) elements of degree 2n in \( \ker(\varphi(n,2)) \) that allow to rewrite the products \( \pi_{n-j,j} \pi_{n-i,i} \) as an element of \( \sum_{i=0}^{n-1} \mathcal{P} \rho_{1,1}^{i} + \sum_{j=1}^{n-1} \mathcal{P} \pi_{n-j,j} \), and these relations together with the earlier relations of degree 2 are sufficient to generate \( \ker(\varphi(n,2)) \) up to degree 2n. Consider the element \( \rho_{1,1}^{n+1} \) to an element in \( \sum_{i=0}^{n} \mathcal{P} \rho_{1,1}^{i} + \sum_{j=1}^{n-1} \mathcal{P} \pi_{n-j,j} \). When \( n \) is even, consider the element \( \rho_{1,1}^{n+1} \mathcal{R}_{n,n} \). It belongs to \( \mathcal{J} \) and has the term \(-4^{n} \rho_{1,1}^{n+1} \). The other terms of \( \rho_{1,1} \mathcal{R}_{n,n} \) are congruent modulo \( \mathcal{I} \) to an element of the ideal of \( \mathcal{F}(n,2) \) generated by \( \pi_{n,0}, \pi_{0,n}, \rho_{2,0}, \rho_{0,2} \) by \( \mathcal{J} \). It follows that \( \rho_{1,1}^{n+1} \) is congruent modulo \( \mathcal{J} \) to an element of \( \sum_{i=0}^{n} \mathcal{P} \rho_{1,1}^{i} + \sum_{j=1}^{n-1} \mathcal{P} \pi_{n-j,j} \). So we are done when \( n \) is even. For odd \( n \) consider the element \( \rho_{1,1}(E_{2,1} \mathcal{R}_{n+1,n-1}) \) of \( \mathcal{J} \). It has the term \(-2^{n} \rho_{1,1}^{n+1} \), and all the other terms are congruent modulo \( \mathcal{I} \) to elements in \( \sum_{i=0}^{n} \mathcal{P} \rho_{1,1}^{i} + \sum_{j=1}^{n-1} \mathcal{P} \pi_{n-j,j} \). This finishes the proof also for the case of odd \( n \). \( \square \)
6. The case \( n = 4 \)

6.1. Secondary S-generating systems. In this subsection we return to the general setup of Section 6.1. The action of the subgroup of diagonal matrices in \( GL_m(\mathbb{C}) \) induces an \( \mathbb{N}_0^m \)-grading on \( \mathbb{C}[V^m] \) and \( \mathbb{C}[V^m]^G \). Write \( \mathbb{C}[V^m]_\alpha \) and \( \mathbb{C}[V^m]^G_\alpha \) for the multihomogeneous component of \( \mathbb{C}[V^m] \) and \( \mathbb{C}[V^m]^G \) of multidegree \( \alpha = (\alpha_1, \ldots, \alpha_m) \).

Now assume that \( G \) is finite. Take a homogeneous system of parameters \( p_1, \ldots, p_k \) in \( \mathbb{C}[V]^G \) (so \( k = \dim(V) \)). For \( j = 1, \ldots, m \) and \( i = 1, \ldots, k \) denote by \( p_i^{(j)} \) the element of \( \mathbb{C}[V^m]^G \) that maps \( (v_1, \ldots, v_m) \in V^m \) to \( p_i(v_j) \). Then

\[
(10) \quad p_1^{(j)}, \ldots, p_k^{(j)} \quad (j = 1, \ldots, m)
\]

is a homogeneous system of parameters for \( \mathbb{C}[V^m]^G \). This means that denoting by \( P(m) \) the \( \mathbb{C} \)-subalgebra of \( \mathbb{C}[V^m]^G \) generated by the (algebraically independent) elements \( \{10\} \), the space \( \mathbb{C}[V^m]^G \) is a finitely generated free \( P(m) \)-module. The elements \( \{10\} \) are called primary generators of \( \mathbb{C}[V^m]^G \), whereas a finite free \( P(m) \)-module generating system of \( \mathbb{C}[V^m]^G \) is called a system of secondary generators of \( \mathbb{C}[V^m]^G \). By the Graded Nakayama Lemma a set of homogeneous elements in \( \mathbb{C}[V^m]^G \) forms a system of secondary generators if and only if they form a basis in a vector space direct complement in \( \mathbb{C}[V^m]^G \) of \( (P(m))^+ \), the ideal in \( \mathbb{C}[V^m]^G \) generated by the elements \( \{10\} \).

View the symmetric group \( S_m \) as the subgroup of permutation matrices in \( GL_m(\mathbb{C}) \). Then \( s \in S_m \) maps \( \mathbb{C}[V^m]_\alpha \) to \( \mathbb{C}[V^m]_{s^\alpha} \), where \( s \cdot \alpha = (\alpha_{s^{-1}(1)}, \ldots, \alpha_{s^{-1}(m)}) \).

Definition 6.1. A subset \( L \) of \( \mathbb{C}[V^m]^G \) is called a system of secondary S-generators of \( \mathbb{C}[V^m]^G \) if \( L \) consists of multihomogeneous elements of decreasing multidegrees and

\[
L^* := \{ s \cdot f \mid f \in L, \quad s \in S_m/\text{Stab}(\deg(f)) \},
\]

is a system of secondary generators of \( \mathbb{C}[V^m]^G \) where \( \deg(f) \in \mathbb{N}_0^m \) stands for the multidegree of \( f \) and \( S_m/\text{Stab}(\deg(f)) \) stands for a chosen set of left coset representatives in \( S_m \) with respect to the stabilizer subgroup \( \text{Stab}(\deg(f)) \) of the multidegree of \( f \). (Note that this notion depends on the choice \( p_1, \ldots, p_k \) of homogeneous system of parameters in \( \mathbb{C}[V]^G \).)

Proposition 6.2. A system of secondary S-generators of \( \mathbb{C}[V^m]^G \) exists.

Proof. Since the elements in \( \{10\} \) are multihomogeneous, the ideal \( (P(m))^+ \) of \( \mathbb{C}[V^m]^G \) is spanned by multihomogenous elements, and therefore a system of secondary generators consisting of multihomogeneous elements exist. Moreover, a set \( M \) of multihomogeneous elements forms a system of secondary generators if and only if for each multidegree \( \alpha \in \mathbb{N}_0^m \) the subset \( M \cap \mathbb{C}[V^m]_\alpha \) is a \( \mathbb{C} \)-vector space basis in a direct complement in \( \mathbb{C}[V^m]^G_\alpha \) of \( (P(m))^+ \cap \mathbb{C}[V^m]^G_\alpha \).

The action of \( S_m \) preserves the set \( \{10\} \), hence it preserves the algebra \( P(m) \) and the ideal \( (P(m))^+ \) in \( \mathbb{C}[V^m]^G \). An element \( s \in S_m \) gives a vector space isomorphism between \( \mathbb{C}[V^m]_\alpha \) and \( \mathbb{C}[V^m]_{s^\alpha} \), such that the subspace \( \mathbb{C}[V^m]^G_\alpha \) is mapped onto \( \mathbb{C}[V^m]_{s^\alpha}^G \) and \( \mathbb{C}[V^m]^G_\alpha \cap (P(m))^+ \) is mapped onto \( \mathbb{C}[V^m]_{s^\alpha} \cap (P(m))^+ \). Therefore to get a system of secondary S-generators of \( \mathbb{C}[V^m]^G \) we just need to take for each decreasing multidegree \( \alpha \) a
basis in a direct complement in $\mathbb{C}[V^m]^G$ of $\mathbb{C}[V^m]_\alpha \cap (P(m)^+)$, and the union of these sets as $\alpha$ ranges over all decreasing multidegrees will be a system of secondary $S$-generators. □

6.2. Hironaka decomposition for $n = 4, m = 3$. The elements

\[(11)\]

\[p_{4,0,0}, p_{0,4,0}, p_{0,0,4}, q_{2,0,0}, q_{0,2,0}, q_{0,0,2}\]

constitute a homogeneous system of parameters in the algebras $\mathbb{C}[V^3]$ and $\mathbb{C}[V^3]^{D_8}$. We shall refer to the elements (11) as the primary generators of $\mathbb{C}[V^3]^{D_8}$, and denote by $P(4,3)$ the subalgebra of $\mathbb{C}[V^3]$ generated by them. We are looking for an explicit free $P(4,3)$-module generating system (called system of secondary generators) of $\mathbb{C}[V^3]^{D_8}$.

Denote by $H$ the cyclic subgroup of $D_8$ generated by \( \left( \begin{array}{cc} \omega & 0 \\ 0 & \omega^{-1} \end{array} \right) \). The elements (11) constitute a homogeneous system of parameters in $\mathbb{C}[V^3]^H$ as well.

**Proposition 6.3.** The following table gives a secondary $S$-generating system of $\mathbb{C}[V^3]^H$ with respect to $P(4,3)$:

| multidegree | generator |
|-------------|-----------|
| (0, 0, 0)   | \( x_1y_2, y_1x_2 \) |
| (1, 1, 0)   | \( x_1^2x_2x_3, y_1^2y_2y_3, x_1^2y_2y_3, y_1^2x_2x_3 \) |
| (2, 1, 1)   | \( x_1^3y_2y_3, y_1^3x_2^2x_3, x_1^3x_2^2x_3, y_1^3y_2^2x_3 \) |
| (2, 2, 0)   | \( x_1y_2, y_1y_2 \) |
| (3, 1, 0)   | \( x_1^4, y_1^4 \) |
| (3, 2, 1)   | \( x_1^2y_2, y_1^2x_2, x_1^2y_2y_3, y_1^2x_2x_3 \) |
| (3, 3, 0)   | \( x_1^3y_2, y_1^3x_2, x_1^3y_2y_3, y_1^3x_2x_3 \) |
| (4, 1, 1)   | \( x_1^4x_2y_3, x_1^4y_2x_3 \) |
| (4, 2, 2)   | \( x_1^3x_2^2, y_1^3x_2^2, x_1^3x_2^2, y_1^3y_2x_3 \) |
| (4, 3, 1)   | \( x_1^4x_2^2, x_1^4y_2x_3 \) |
| (4, 4, 0)   | \( x_1^4x_2^2, x_1^4y_2x_3 \) |
| (4, 3, 3)   | \( x_1^4y_2^2, x_1^4y_2^2 \) |
| (4, 4, 4)   | \( x_1^4x_2^3, x_1^4x_2^3 \) |

**Proof.** $\mathbb{C}[V^3]^H$ is spanned as a $\mathbb{C}$-vector space by the monomials $x_1^iy_2x_3^jy_1^my_2^n$ such that $i + j + k - m - n$ is divisible by 4. Consider the lexicographic monomial order in $\mathbb{C}[x, y]$ induced by the order $x < y$ of the variables. Then it is easy to check that the ideal generated by $xy$ and $x^4 + y^4$ has the Gröbner basis $xy, x^4 + y^4, x^5 = x(x^4 + y^4) - y^3(xy)$, hence the ideal generated by the initial monomials in $(xy, x^4 + y^4)$ is $(xy, y^4, x^5)$. It follows that the monomials not divisible by any of $x_1y_1, x_2y_2, x_3y_3, y_1^4, y_2^4, y_3^4, x_1^3, x_2^3, x_3^3, y_1^3, y_2^3, y_3^3, x_1^2, x_2^2, x_3^2, y_1^2, y_2^2, y_3^2$ form a basis in a vector space direct complement of the ideal in $\mathbb{C}[V^3]$ generated by the elements in (11). As each monomial spans an $H$-invariant subspace in $\mathbb{C}[V^3]$, the $H$-invariant monomials among them form a system of secondary generators for $\mathbb{C}[V^3]^H$, and the table above contains all those with decreasing multidegree. □
Proposition 6.4. The following table gives a secondary $S$-generating system of $\mathbb{C}[V^3]^{D_8}$:

| multidegree | generator |
|-------------|-----------|
| $(0, 0, 0)$ | 1         |
| $(1, 1, 0)$ | $q_{1,1,0}$ |
| $(2, 1, 1)$ | $p_{2,1,1}$, $q_{1,0,0}q_{1,0,1}$ |
| $(2, 2, 0)$ | $p_{2,2,0}$, $q_{1,2,0}$ |
| $(3, 1, 0)$ | $p_{3,1,0}$ |
| $(3, 2, 1)$ | $p_{3,1,0}q_{0,1,0}$, $q_{1,2,0}q_{1,0,1}$ |
| $(3, 3, 0)$ | $q_{1,2,0}$ |
| $(4, 1, 1)$ | $p_{3,1,0}q_{1,0,1}$ |
| $(3, 3, 2)$ | $p_{2,1,1}p_{1,2,1}$, $p_{3,1,0}q_{0,1,0}^2$ |
| $(4, 2, 2)$ | $p_{3,1,0}q_{1,0,1}q_{0,1,0}$, $q_{1,2,0}q_{1,0,1}$ |
| $(4, 3, 1)$ | $q_{1,2,0}^3$ |
| $(4, 4, 0)$ | $q_{1,2,0}$ |
| $(4, 3, 3)$ | $p_{3,1,0}q_{1,0,1}q_{0,1,0}^2$ |

Proof. Following the notation of Section 6.1, $(P(4, 3)^+)$ stands for the ideal in $\mathbb{C}[V^3]^{D_8}$ generated by the elements $\langle 11 \rangle$. The element $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ of $D_8$ permutes up to sign the cosets modulo $(P(4, 3)^+)$ of the monomials in the table in Proposition 6.3 (note that $x^4$ is congruent to $-y^4$ modulo $(P(4, 3)^+)$). It follows by Proposition 6.3 that a $D_8$-invariant direct complement of $(P(4, 3)^+)$ in the sum of the homogeneous components of $\mathbb{C}[V^3]^H$ with decreasing multidegree has the basis $C_+ \cup C_-$, where

$$C_+ = \{1, x_1 y_2 + y_1 x_2, x_1^2 x_2 x_3 + y_1^2 y_2 y_3, x_1^3 y_2 y_3 + y_1^3 x_2 x_3, x_1^2 x_2^2 + y_1^2 y_2^2, x_1^2 y_2^2 + y_1^2 x_2^2,$$

$$x_1^3 x_2 + y_1^3 y_2 y_3 + y_1^3 x_2 x_3, x_1^3 x_2^2 y_3 + y_1^3 y_2^2 y_3, x_1^3 x_2 y_3 + y_1^3 x_2^2 x_3 + y_1^3 y_2^2 x_3, x_1^3 x_2 y_3 + y_1^3 x_2^2 x_3 + y_1^3 y_2^2 x_3,$$

$$x_1^3 y_2 x_3 + y_1^3 x_2 y_3, x_1^3 x_2^2 y_3 + y_1^3 y_2^2 x_3 - x_1^3 y_2 y_3, x_1^3 x_2^2 x_3 + y_1^3 x_2 y_3 - x_1^3 x_2 x_3 \}$$

and

$$C_- = \{x_1 y_2 - y_1 x_2, x_1^2 x_2 x_3 - y_1^2 y_2 y_3, x_1^2 y_2 y_3 - y_1^2 x_2 x_3, x_1^3 x_2 x_3 - y_1^3 y_2 y_3, x_1^3 y_2 y_3 - y_1^3 x_2 x_3,$$

$$x_1^3 x_2 y_3 - y_1^3 y_2^2 y_3, x_1^3 x_2^2 x_3 + y_1^3 x_2 y_3 - x_1^3 x_2 x_3, x_1^3 x_2 x_3 - y_1^3 y_2^3 x_3 - y_1^3 x_2^2 x_3 + y_1^3 y_2^2 x_3 - y_1^3 x_2^2 x_3 + y_1^3 y_2^2 x_3,$$

$$x_1^3 x_2 y_3 - y_1^3 y_2 y_3, x_1^3 x_2^2 x_3 + y_1^3 x_2 y_3 - x_1^3 x_2 x_3 \}.$$

The elements in $C_+$ are $D_8$-invariant, whereas the elements in $C_-$ span a 1-dimensional $D_8$-invariant subspace on which $D_8$ acts via the determinant representation. It follows that $C_+$ is a system of secondary $S$-generators for $\mathbb{C}[V^3]^{D_8}$. It is easy to see that modulo the ideal $(P(4, 3)^+)$ the elements listed in the table in the statement of our proposition agree with non-zero scalar multiples of the elements in $C_+$. $\square$
6.3. Relations for \( n = 4 \) and \( m = 3 \).

**Proposition 6.5.** The \( GL_3(\mathbb{C}) \)-ideal \( \ker(\varphi(4,3)) \) is minimally generated by \( R_{2,2,2}, R(4)_{4,2}, R(4)_{6,2}, \) and \( R(4)_{4,4} \).

**Remark 6.6.** We have \( \dim(S^{(4,2)}(\mathbb{C}^3)) = 27 \), \( \dim(S^{(6,2)}(\mathbb{C}^3)) = 60 \), \( \dim(S^{(4,4)}(\mathbb{C}^3)) = 15 \), and \( \dim(S^{(2,2,2)}(\mathbb{C}^3)) = 1 \). Therefore Proposition 6.5 implies that a minimal homogeneous generating system of the ideal \( \ker(\varphi(4,3)) \) consists of 103 elements. For comparison we mention that a minimal homogeneous generating system of \( \ker(\varphi(4,2)) \) consists of \( 3 + 5 + 1 = 9 \) elements.

The logic of the proof of Proposition 6.5 is furnished by the following general lemma:

**Lemma 6.7.** Let \( \varphi : F \to R \) be a surjective homomorphism of graded \( \mathbb{C} \)-algebras, where \( R \) is a connected graded Cohen-Macaulay algebra (cf. Section 2.3), so there exist homogeneous elements \( h_1, \ldots, h_k \) and \( t_1, \ldots, t_l \) in \( F \) with the following properties:

1. \( \varphi(h_1), \ldots, \varphi(h_k) \) is a homogeneous system of parameters in \( R \).
2. \( R = \mathbb{P} \varphi(t_1) \oplus \cdots \oplus \mathbb{P} \varphi(t_l) \) where \( \mathbb{P} = \mathbb{C}[\varphi(h_1), \ldots, \varphi(h_k)] \).

Let \( \mathcal{K} \) be a homogeneous ideal in \( F \) contained in \( \ker(\varphi) \), and assume that for some \( d \in \mathbb{N} \) we have

\[
F_{\leq d} = \text{Span}_\mathbb{C} \mathcal{T}_{\leq d} + \mathcal{H}_{\leq d} + \mathcal{K}_{\leq d}
\]

where \( \mathcal{H} \) is the ideal in \( F \) generated by \( h_1, \ldots, h_k \), \( \mathcal{T} := \{ t_1, \ldots, t_l \} \), \( d \in \mathbb{N} \) and for a subset \( A \) of homogeneous elements (respectively graded subspace) in \( F \) we write \( A_{\leq d} \) for the set of elements (respectively sum of homogeneous components) of \( A \) with degree \( \leq d \).

Then we have \( \mathcal{K}_{\leq d} = \ker(\varphi)_{\leq d} \).

**Proof.** One can show by an induction on the degree that our assumptions imply the equality

\[
F_{\leq d} = \mathcal{K}_{\leq d} + \mathcal{P}_{\leq d - \deg(t_1)} t_1 + \cdots + \mathcal{P}_{\leq d - \deg(t_l)} t_l,
\]

where \( \mathcal{P} = \mathbb{C}[h_1, \ldots, h_k] \). Moreover, the restriction of \( \varphi \) to \( \mathcal{P} \) and to \( \mathcal{P} t_1 + \cdots + \mathcal{P} t_l \) is injective. This clearly implies the desired equality \( \mathcal{K}_{\leq d} = \ker(\varphi)_{\leq d} \).

**Proof of Proposition 6.5.** For a decreasing multidegree \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) denote by \( \mathcal{T}_\alpha \) the set of products of the variables \( \pi_{i_1,j_2,k_3}, \rho_{j_1,j_2,j_3} \) having multidegree \( \alpha \) that correspond to the elements of multidegree \( \alpha \) in the secondary \( S \)-generating system of \( \mathbb{C}[V^3]^D_8 \) given in Proposition 6.2. For example, \( \mathcal{T}_{1,1,1} = \emptyset \) and \( \mathcal{T}_{4,2,2} = \{ \pi_{3,1,0}\rho_{1,0,1}, \pi_{0,1,1}, \rho_{1,1,0}^2 \} \). Denote by \( \mathcal{K} \) the \( GL \)-ideal in \( F_3 \) generated by \( R_{2,2,2}, R(4)_{4,2}, R(4)_{6,2}, \) and \( R(4)_{4,4} \), and write \( \mathcal{H} \) for the ideal in \( F(4,3) \) generated by \( \pi_{4,0,0}, \pi_{0,4,0}, \pi_{0,0,4}, \rho_{2,0,0}, \rho_{0,2,0}, \rho_{0,0,2} \). The subspaces \( \mathcal{K} \) and \( \mathcal{H} \) are spanned by multihomogeneous elements of \( F(4,3) \), and we shall write \( \mathcal{K}_\alpha, \mathcal{H}_\alpha, F(4,3)_\alpha \) for their components of multidegree \( \alpha \). We shall show that

\[
(12) \text{Span}_\mathbb{C} \{ \mathcal{T}_\alpha \} + \mathcal{K}_\alpha + \mathcal{H}_\alpha = F(4,3)_\alpha \text{ for all decreasing } \alpha \in \mathbb{N}_0^3 \text{ with } \alpha_1 + \alpha_2 + \alpha_3 \leq 10.
\]

Note that \( \mathcal{H} \) is not a \( GL \)-ideal in \( F(4,3) \), however, it is preserved by the subgroup \( S_3 \) of \( GL_3(\mathbb{C}) \). For \( s \in S_3 \) we have \( s \cdot \mathcal{H}_\alpha = \mathcal{H}_{s \cdot \alpha} \) and \( s \cdot \mathcal{K}_\alpha = \mathcal{K}_{s \cdot \alpha} \). Thus (12) implies...
\( F_{s\alpha} = \text{Span}_C\{s \cdot T_\alpha\} + H_{s\alpha} + K_{s\alpha} \). Set
\[
T := \bigcup_{\alpha \in \mathbb{N}_0^3, \alpha_1 \geq \alpha_2 \geq \alpha_3 \in S_3/\text{Stab}_{S_3}(\alpha)} s \cdot T_\alpha.
\]
Then the generators of \( H \) are mapped to a homogeneous system of parameters in \( R := \mathbb{C}[V^3]^{D_8} \) and \( T \) is mapped by \( \varphi(4, 3) \) to a system of secondary generators of \( R \) (see Definition 6.1). Given (12) the above considerations show that the assumptions of Lemma 6.7 hold for \( F = F(4, 3) \), \( \varphi = \varphi(4, 3) \) and \( d = 10 \), and therefore by Lemma 6.7 we conclude that \( K_{\leq 10} = \ker(\varphi(4, 3))_{\leq 10} \). Now \( \ker(\varphi(4, 3)) \) is generated by its elements of degree at most 10 by Proposition 4.4, implying the equality \( K = \ker(\varphi(4, 3)) \).

It remains to prove (12). For multidegrees \( \alpha \) with \( \alpha_3 = 0 \) this was done in the proof of Theorem 5.1. Note also that \( F_{\alpha} = \{0\} \) if \( \alpha_1 + \alpha_2 + \alpha_3 \) is odd.

Denote by \( M_\alpha \) the products of multidegree \( \alpha \) in \( \{\pi_\beta, \rho_\gamma \mid \beta_i < 4, \gamma_j < 2\} \) (the generators of \( F(4, 3) \) different from \( \pi_4, 0, 0, \pi_4, 4, 0, \pi_0, 0, 4, \rho_2, 0, 0, \rho_0, 2, 0, \rho_0, 0, 2 \) ). In the first table below we collect for all decreasing \( \alpha \in \mathbb{N}_0^3 \) with \( \alpha_3 > 0 \) and \( \sum_{i=1}^3 \alpha_i \leq 6 \) even the elements of \( M_\alpha \).

| \( \alpha \) | \( T_\alpha \) | \( M_\alpha \) | \( \mathcal{M}_\alpha \) |
|---|---|---|---|
| (2, 1, 1) | \( \pi_{2,1,1}, \rho_{1,1,0}\rho_{1,0,1} \) | \( \pi_{3,0,1}\rho_{1,1,0} \) | |
| (4, 1, 1) | \( \pi_{3,1,0}\rho_{1,0,1} \) | \( \pi_{3,0,1}\rho_{1,1,0} \) | |
| (3, 2, 1) | \( \rho_{1,1,0}\rho_{1,0,1}, \pi_{3,1,0}\rho_{0,1,1} \) | \( \pi_{2,1,1}\rho_{1,1,0}, \pi_{2,2,0}\rho_{1,0,1} \) | |
| (2, 2, 2) | \( \rho_{1,1,0}\rho_{1,0,1}\rho_{0,1,1}, \pi_{2,1,1}\rho_{1,1,0}, \pi_{1,2,1}\rho_{1,0,1}, \pi_{1,1,2}\rho_{1,1,0} \) | | |

Now going multidegree by multidegree we shall show that the products \( M_\alpha \) are congruent modulo \( H + K \) to a \( \mathbb{C} \)-linear combination of the elements in \( T_\alpha \). Denote by \( I \) the \( GL \)-ideal of \( F(4, 3) \) generated by \( R_{2,2,2} \) and \( R_{4,2,2} \), and we shall write \( a \equiv_I b \) for some \( a, b \in F(4, 3) \) if \( a - b \in H + I \). We shall write \( a \equiv b \) for \( a, b \in F(4, 3) \) if \( a - b \in H + K \). Note that \( I \subseteq K \) and therefore \( a \equiv_I b \) implies \( a \equiv b \).

There is nothing to do for the multidegree (2, 1, 1). Taking into account Theorem 5.1 this shows also that for \( d < 6 \) we have \( \ker(\varphi(4, 3))_d = \{0\} \).

(4, 1, 1): The relation (2) in the special case \( n = 4, m = 3 \) is
\[
R_{4,2,2} := \pi_{4,0,0}\rho_{0,2,0} - 2\pi_{3,1,0}\rho_{1,1,0} + \pi_{2,2,0}\rho_{2,0,0}. 
\]
Applying \( E_{3,2} \in gl_3(\mathbb{C}) \) to \( R_{4,2,2} \) we get the following element of \( \langle R_{4,2,2} \rangle_{GL_3(\mathbb{C})} \):
\[
R_{4,1,1} := \frac{1}{2} E_{3,2} R_{4,2,2} = \pi_{4,0,0}\rho_{0,1,1} - \pi_{3,1,0}\rho_{1,1,0} + \pi_{2,1,1}\rho_{2,0,0} 
\]
This relation implies that
\[
\pi_{3,0,1}\rho_{1,1,0} \equiv_I -\pi_{3,1,0}\rho_{1,1,0}. 
\]

(3, 2, 1): Consider the following elements of \( \langle R_{4,2,2} \rangle_{GL_3(\mathbb{C})} \):
\[
R_{4,3,1} := \frac{1}{2} E_{3,1} R_{4,2,2} = -\pi_{3,1,0}\rho_{0,1,1} + \pi_{2,2,0}\rho_{1,0,1} 
\]
\[
+ 2\pi_{3,0,1}\rho_{0,2,0} + \pi_{1,2,1}\rho_{2,0,0} 
\]
ideal zero term of any element of $\pi(19)$ and (16) and (18) imply $\pi$

Set, Consequently, ker($\pi$ and taking into account that the ideal $\pi$ Denote by $B$

The relations (9) in the special case $n$ give that

Moreover, for later reference we mention some consequences of the relations of degree 6.

The relations (9) in the special case $n = 4$ give that

Denote by $A$ the union of the $S_3$-orbits of $\pi_{2,1,1}\rho_{1,1,0}$, $\pi_{2,1,1}\rho_{0,1,1}$, $\rho_{1,1,0}\rho_{1,0,1}\rho_{1,0,1}$, $\pi_{3,1,0}\rho_{1,1,0}$ and $\pi_{2,2,0}\rho_{1,1,0}$. Each element of $A$ is congruent to zero modulo $H + I$ by (19), (20), (21) (and taking into account that the ideal $H + I$ is preserved by the action of $S_3$ on $F(4,3)$). Denote by $B$ the $S_3$-orbit of $\pi_{2,2,0}\rho_{1,0,1}$. By (19) and as $I + H$ is $S_3$-stable, we conclude that each element of $B$ is congruent modulo $H + I$ to an element of the $S_3$-orbit of $\pi_{3,1,0}\rho_{0,1,1}$. Set $C := \{\pi_{3,0,1}\rho_{1,1,0}\}$, by (19) the element of $C$ is congruent modulo $H + I$ to $-\pi_{3,1,0}\rho_{0,1,1}$. Summarizing, we have that the factor space

Now we turn to the relations of degree 8. Every non-zero term in any element of the $GL$-ideal $I$ of $F(4,3)$ generated by $R_{2,2,2}$ and $R(4)_{4,2}$ involves a variable $\rho_{i,j,k}$, and every non-zero term of any element of $H$ involves a variable from $\{\pi_{4,0,0}, \pi_{0,4,0}, \pi_{0,0,4}, \rho_{2,0,0}, \rho_{0,2,0}, \rho_{0,0,2}\}$.
Hence none of $\mathcal{R}(4)_{6,2}$ or $\mathcal{R}(4)_{4,4}$ is contained in $I + H$. Consequently, by basic principles about semisimple representations we have

$$\mathcal{R}(4)_{6,2} \cup \mathcal{R}(4)_{4,4} = \emptyset.$$
the number of semi-standard tableaux of shape $\lambda$ and content $\alpha$. The dimensions relevant to us are given in the following table:

| $\alpha$ | $(6,1,1)$ | $(5,2,1)$ | $(4,3,1)$ | $(4,2,2)$ | $(3,3,2)$ |
|----------|-----------|-----------|-----------|-----------|-----------|
| dim($S^{(6,2)}(\mathbb{C})_\alpha$) | 1         | 2         | 2         | 3         | 3         |
| dim($S^{(4,4)}(\mathbb{C})_\alpha$) | 0         | 0         | 1         | 1         | 1         |

In the above table the sum of the dimensions in the column of $\alpha$ agrees with $|N_\alpha|$, hence we showed that

$$\mathcal{K}_{\leq 8} = \langle \mathcal{R}(4,6,2)_{GL_4(\mathbb{C})} \rangle \oplus \langle \mathcal{R}(4,4,4)_{GL_4(\mathbb{C})} \rangle \oplus I_{\leq 8}.$$

Finally we turn to the relations of degree 10. Our study of the degree 8 relations implies that any product $\pi_{i_1,i_2,i_3} \pi_{j_1,j_2,j_3}$ is congruent modulo $\mathcal{K} + \mathcal{H}$ to a linear combination of elements from the $S_3$-orbit of $\pi_{2,1,1} \pi_{1,2,1}$ and products of the variables of $\mathcal{F}(4,3)$ involving at most one factor of the form $\pi_{k_1,k_2,k_3}$. Moreover,

$$\pi_{2,1,1} \pi_{1,2,1} \rho_{i,j,k} \equiv I \ 0$$

by (18) and (21). Taking into account (22) we conclude that

$$(27) \quad \mathcal{F}(4,3)_{10} \text{ is spanned modulo } \mathcal{H} + \mathcal{K} \text{ by products involving at most one variable of the form } \pi_{i_1,i_2,i_3} \text{ and not divisible by any element of } A \cup B \cup C.$$

For a decreasing $\alpha$ with $\alpha_1 + \alpha_2 + \alpha_3 = 10$ denote by $\mathcal{V}_\alpha$ the subset of $\mathcal{M}_\alpha \setminus \mathcal{T}_\alpha$ obtained by removing the products divisible by some $\pi_{i_1,i_2,i_3} \pi_{j_1,j_2,j_3}$ or by any element of $A \cup B \cup C$. By (27) it is sufficient to prove that every element of $\mathcal{V}_\alpha$ is congruent modulo $\mathcal{K} + \mathcal{H}$ to an element in $\text{Span}_{\leq} \mathcal{T}_\alpha$. The table below gives $\mathcal{T}_\alpha$ and $\mathcal{V}_\alpha$ for all decreasing $\alpha$ with $\alpha_3 \neq 0$ and $\sum \alpha_i = 10$.

| $\alpha$ | $\mathcal{T}_\alpha$ | $\mathcal{V}_\alpha$ |
|----------|-----------------|-----------------|
| $(8,1,1)$ |                 |                 |
| $(7,2,1)$ |                 |                 |
| $(6,3,1)$ |                 |                 |
| $(6,2,2)$ |                 |                 |
| $(5,4,1)$ | $\rho_{1,1,0}^1 \rho_{1,0,1}^1$ |                 |
| $(5,3,2)$ | $\rho_{1,1,0}^2 \rho_{1,0,1}^2$ |                 |
| $(4,4,2)$ |                 |                 |
| $(4,3,3)$ | $\pi_{3,1,0} \rho_{1,0,1} \rho_{0,1,1} \rho_{1,0,1}$ | $\pi_{1,3,0} \rho_{1,0,1} \rho_{1,1,0} \pi_{1,0,3} \rho_{1,1,0}$ |

We see that $\mathcal{V}_\alpha$ is non-empty only for $\alpha \in \{(5,4,1), \ (5,3,2), \ (4,3,3)\}$.

$(5,4,1)$: The relation $\mathcal{R}_{2n-4,4,4}$ in the special case $n = 4$ is

$$\mathcal{R}(4,4,4) := \pi_{4,0,0} \pi_{0,4,0} - 4 \pi_{3,1,0} \pi_{1,3,0} + 3 \pi_{2,2,0}^2 - 16 (\rho_{1,1,0}^2 + \rho_{2,0,0} \rho_{0,2,0})^2.$$  

Apply $\frac{1}{4}E_{4,2}$ to $\mathcal{R}(4,4,4)$ we get

$$\mathcal{R}(4,4,4) := \frac{1}{4}E_{3,2} \mathcal{R}(4,4,4) = \pi_{4,0,0} \pi_{0,3,1} - \pi_{3,0,1} \pi_{1,3,0} - 3 \pi_{3,1,0} \pi_{1,2,1} + 3 \pi_{2,2,0} \pi_{2,1,1}.$$
\[-16(\rho_{2,0,0}\rho_{0,2,0} - \rho_{1,1,0}^2)(\rho_{2,0,0}\rho_{0,1,1} - \rho_{1,1,0}\rho_{1,0,1}).\]

We deduce from $R(4)^{4,3,1}_{4,4} \in K$ that

\[(28) \quad \rho_{1,1,0}^3 \rho_{1,1,0} \equiv \frac{1}{16}(-\pi_{3,0,1}\pi_{1,3,0} - 3\pi_{3,1,0}\pi_{1,2,1} + 3\pi_{2,2,0}\pi_{2,1,1}).\]

Multiplying (28) by $\rho_{1,1,0}$ and using

\[\pi_{1,3,0}\rho_{1,1,0} \equiv 0, \quad \pi_{3,1,0}\rho_{1,1,0} \equiv 0, \quad \pi_{2,2,0}\rho_{1,1,0} \equiv 0\]

we get

\[\rho_{1,1,0}^4 \rho_{1,1,0} \equiv \mathcal{I} 0.\]

\[(5,3,2): \quad \text{Multiplying (28) by } \rho_{1,0,1} \text{ and using} \]

\[\pi_{3,0,1}\rho_{1,0,1} \equiv \mathcal{I} 0, \quad \pi_{1,2,1}\rho_{1,0,1} \equiv \mathcal{I} 0, \quad \pi_{2,1,1}\rho_{1,0,1} \equiv \mathcal{I} 0\]

we get

\[\rho_{1,1,0}^3 \rho_{1,0,1}^2 \equiv \mathcal{I} 0.\]

\[(4,3,3): \quad \text{Applying the transposition } (1,2) \in S_3 \text{ to (19) we get} \]

\[\pi_{1,3,0}\rho_{1,0,1} \equiv \mathcal{I} \pi_{2,2,0}\rho_{0,1,1}.\]

Multiplying this by $\rho_{1,0,1}^2$ we get

\[\pi_{1,3,0}\rho_{1,0,1}^3 \equiv \mathcal{I} \pi_{2,2,0}\rho_{1,0,1}\rho_{0,1,1} \equiv \mathcal{I} \pi_{3,1,0}\rho_{1,0,1}^2 \rho_{0,1,1}^2\]

(for the second congruence see (19)). Finally, applying the transposition $(2,3) \in S_3$ to the congruence $\pi_{1,3,0}\rho_{1,0,1}^3 \equiv \mathcal{I} \pi_{3,1,0}\rho_{1,0,1}^2 \rho_{0,1,1}^2$ and using (15) we obtain

\[\pi_{1,0,3}\rho_{1,1,0}^3 \equiv \mathcal{I} -\pi_{3,1,0}\rho_{1,0,1}\rho_{0,1,1}^2.\]

□

\textbf{Theorem 6.8.} For arbitrary $m \geq 3$ the kernel of $\varphi(4, m) : F(4, m) \to \mathbb{C}[V^m]^D_8$ is minimally generated as a $GL$-ideal by $R_{2,2,2}, R(4)_{4,2}, R(4)_{6,2}, R(4)_{4,4}$.

\textit{Proof.} This is an immediate consequence of Proposition 6.5 and Theorem 4.8 □

The symmetric group $S_3$ is isomorphic to the dihedral group $D_6$ of order 6, and the natural 3-dimensional permutation representation of $S_3$ can be identified with the sum of the trivial representation and the defining 2-dimensional representation of $D_6$. Therefore \cite{10} Theorem 3.1] dealing with multisymmetric polynomials can be restated in the notation of the present paper as follows:

\textbf{Theorem 6.9.} For arbitrary $m \geq 3$ the kernel of $\varphi(3, m) : F(3, m) \to \mathbb{C}[V^m]^D_6$ is minimally generated as a $GL$-ideal by $R_{2,2,2}, R(3)_{3,2}, R(3)_{4,2}$.

\textbf{Remark 6.10.} (i) The method of the present paper would yield a shorter proof of Theorem 6.9 than the proof in \cite{10}.
(ii) For \( n = 2 \) the group \( D_4 \) is isomorphic to Klein’s four group, so it is abelian and its action on \( V \) is diagonalizable. In an appropriate basis we have \( \mathbb{C}[V]^{D_4} = \mathbb{C}[x^2, y^2] \), and \( \mathbb{C}[V^m] = \mathbb{C}[x_i x_j, y_i y_j \mid 1 \leq i \leq j \leq m] \). The corresponding \( GL \)-ideal of \( GL \)-induces an action of \( \mathbb{C} \) on \( \mathbb{C}[V^m]^{D_4} \) is generated by the two relations \( x_1 x_2^2 = 0 \) and \( y_1^3 y_2^2 - (y_1 y_2)^2 = 0 \). For some results on presentations of rings of invariants of abelian groups see for example [9] and the references therein.

7. Some computations

Next we determine the \( GL_m(\mathbb{C}) \)-module structure of \( \mathbb{C}[V^m]^{D_{2n}} \). It turns out that the multiplicities of the irreducible summands are conveniently expressed in terms of the coefficients of the Hilbert series of \( \mathbb{C}[V^m]^{D_{2n}} \). Denote by \( h(d) \) the dimension of the degree \( d \) homogeneous component of \( \mathbb{C}[x, y]^{D_{2n}} \). Note that in the formal power series ring \( \mathbb{Z}[t] \) we have the equality

\[
\sum_{d=0}^{\infty} h(d) t^d = \frac{1}{(1 - t^2)(1 - t^4)}.
\]

**Proposition 7.1.** The multiplicity of \( S^\lambda(\mathbb{C}^m) \) as a summand in \( \mathbb{C}[V^m]^{D_{2n}} \) is non-zero only if \( \text{ht}(\lambda) \leq 2 \), and in this case the multiplicity is

\[
\begin{cases} 
  h(\lambda_1 - \lambda_2) & \text{if } 2 \mid \lambda_2 \\
  h(\lambda_1 - \lambda_2 - n) & \text{if } 2 \nmid \lambda_2.
\end{cases}
\]

**Proof.** Identify \( V^m \) with the space \( \mathbb{C}^{2 \times m} \) of \( 2 \times m \) matrices, endowed with the \( GL_2(\mathbb{C}) \times GL_m(\mathbb{C}) \)-action \( (g, h) \cdot A := gAh^{-1} \) for \( g \in GL_2(\mathbb{C}) \), \( h \in GL_m(\mathbb{C}) \), and \( A \in \mathbb{C}^{2 \times m} \). This induces an action of \( GL_2(\mathbb{C}) \times GL_m(\mathbb{C}) \) on \( \mathbb{C}[V^m] \) via \( \mathbb{C} \)-algebra automorphisms in the standard way: \( ((g, h) \cdot f)(A) = f(g^{-1}Ah) \) for \( g \in GL_2(\mathbb{C}) \), \( h \in GL_m(\mathbb{C}) \), \( A \in \mathbb{C}^{2 \times m} \) and \( f \in \mathbb{C}[V^m] \). Note that the restriction of this action to the subgroup \( GL_m(\mathbb{C}) \) agrees with the \( GL_m(\mathbb{C}) \)-action on \( \mathbb{C}[V^m] \) considered in the previous sections. By the Cauchy Formula (see for example [14] Section 9.6.3) we have

\[
\mathbb{C}[V^m]_d \cong \sum_{\lambda \in \text{Par}_m(d), \ \text{ht}(\lambda) \leq 2} S^\lambda(V^*) \otimes S^\lambda(\mathbb{C}^m)
\]

as \( GL_2(\mathbb{C}) \times GL_m(\mathbb{C}) \)-modules. It follows that we have the following isomorphism of \( GL_m(\mathbb{C}) \)-modules:

\[
\mathbb{C}[V^m]^{{D_{2n}}} \cong \sum_{\lambda \in \text{Par}_m(d), \ \text{ht}(\lambda) \leq 2} S^\lambda(V^*)^{{D_{2n}}} \otimes S^\lambda(\mathbb{C}^m).
\]

Note that for a partition \( \lambda = (\lambda_1, \lambda_2) \), we have the \( D_{2n} \)-module isomorphism \( S^\lambda(V^*) \cong S^{\lambda_1 - \lambda_2}(V^*) \) when \( \lambda_2 \) is even, whereas for \( \lambda_2 \) odd we have \( S^\lambda(V^*) \cong \det \otimes S^{\lambda_1 - \lambda_2}(V^*) \), where \( \det \) stands for the 1-dimensional representation of \( D_{2n} \) obtained by composing the determinant with the 2-dimensional defining representation of \( D_{2n} \). Now the result follows from the well-known description of \( \mathbb{C}[V] \cong S(V^*) \) as a \( D_{2n} \)-module. \( \square \)
For a fixed $n$ Proposition 7.1 and Corollary 4.6 allow us to compute the multiplicities of the simple $GL_3(\mathbb{C})$-module summands of the homogeneous components of $\ker(\bar{\varphi}(n, 3))$ up to degree $2n + 2$. We shall do this computation for the case $n = 4$. To simplify notation write $S(\lambda)$ for $S^\lambda(\mathbb{C}^m)$. Corollary 4.6 yields

\begin{equation}
\bar{\mathcal{D}}(m) \otimes \mathcal{E}(4, m)_{\leq 10} \cong S(0) + S(2) + 2S(4) + S(2, 2) + 2S(6) + S(5, 1) + 2S(4, 2) + 3S(8) + S(7, 1) + 4S(6, 2) + S(5, 3) + 3S(4, 4) + 3S(10) + 2S(9, 1) + 5S(8, 2) + 3S(7, 3) + 5S(6, 4) + S(5, 2, 1) + S(4, 2, 2) + 2S(7, 2, 1) + 2S(6, 3, 1) + 2S(5, 4, 1) + 2S(6, 2, 2) + S(5, 3, 2) + 2S(4, 4, 2).
\end{equation}

By Proposition 7.1 and (29) we have

\begin{equation}
\mathbb{C}[V_{nm}]_{D_8} \cong S(0) + S(2) + 2S(4) + S(2, 2) + 2S(6) + S(5, 1) + S(4, 2) + 3S(8) + S(7, 1) + 2S(6, 2) + S(4, 4) + 3S(10) + 2S(9, 1) + 2S(8, 2) + S(7, 3) + S(6, 4).
\end{equation}

Combining (30) and (31) we get

\begin{equation}
\ker(\bar{\varphi}(4, m)_{\leq 10}) \cong S(4, 2) + 2S(6, 2) + 2S(4, 4) + 3S(8, 2) + 4S(6, 4) + S(5, 3) + 2S(7, 3) + S(5, 2, 1) + S(4, 2, 2) + 2S(7, 2, 1) + 2S(6, 3, 1) + 2S(6, 2, 2) + 2S(5, 4, 1) + 2S(4, 4, 2) + S(5, 3, 2).
\end{equation}

An alternative approach to Theorem 6.8 would be to find the highest weight vectors in the $GL$-ideal generated by $R_{2, 2, 2}$, $R(4)_{4, 2}$, $R(4)_{6, 2}$, $R(4)_{4, 4}$ required by the decomposition (32). However, this seems to be more laborious than the approach in Section 6 based on the Hironaka decomposition.

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