Asymmetric double well system as effective model for the kicked one

V.I. Kuvshinov, A.V. Kuzmin and V.A. Piatrou
Joint Institute for Power and Nuclear Research,
Krasina str. 99, Minsk, 220109, Belarus

Abstract

Effective Hamiltonian for the kicked double well system was derived using the Campbell-Baker-Hausdorff expansion formula. Asymmetric model for the kicked system was constructed. Analytical description of the quasienergy levels splittings for the low laying doublets was given in the framework of the model. Numerical calculations confirm applicability of the proposed effective asymmetric approach for the double well system with the kick-type perturbation.

1 Introduction

The connection between the semiclassical properties of chaotic systems and purely quantum processes such as tunneling is a reach rapidly developing field of research nowadays. Our insight in some novel phenomena in this field was extended in the last decades. The most intriguing among them are the chaos assisted tunneling (CAT) and the closely related coherent destruction of tunneling (CDT).

The first one is an enhancement of tunneling in perturbed low-dimensional systems at relatively high external field strengths and high driving frequencies (in order the singlet-doublet crossings to occur) [1, 2, 3]. This phenomenon takes place when levels of the regular doublet undergo an avoided crossing with the chaotic state [4, 5]. At the semiclassical level of description one considers tunneling between KAM-tori embedded into the ”chaotic sea”. The region of chaotic motion affects tunneling rate because compared to direct tunneling between tori it is easier for the system to penetrate primarily into the chaotic region, to travel then along some classically allowed path and finally to tunnel onto another KAM-torus [6, 7].

CDT phenomenon is a suppression of tunneling when values of amplitude and frequency of driving force belong to some one-dimensional manifold in the perturbation parameters’ space [8]. This phenomenon occurs due to the exact crossing of two states with different symmetries from the tunneling doublet. In this parameter region tunneling time diverges which means the total localization of quantum state on the initial torus.

*E-mail:V.Kuvshinov@sosny.bas-net.by
†E-mail:avkuzmin@sosny.bas-net.by
‡E-mail:vadzim.piatrou@gmail.com
CAT phenomenon as well as CDT were experimentally observed in a number of real physical systems. The CAT observation between whispering gallery-type modes of microwave cavity having the form of the annular billiard was reported in the Ref. [9]. The same phenomenon for ultracold atoms was experimentally investigated in Refs. [10, 11]. The study of the dielectric microcavities provided evidences for CAT as well [12]. Both phenomena were observed in two coupled optical waveguides [13, 14]. Recently experimental evidence of coherent control of single particle tunneling in strongly driven double well potential was reported in Ref. [15].

The most common methods which are used to investigate the CAT are numerical methods based on the Floquet theory [6, 16, 17]. Among other approaches to CAT we would like to mention the scattering approach for billiard systems [18, 19] and quantum mechanical amplitudes in the complex configuration space [20, 21, 22]. There is an analytical approach based on instanton technique, which was proposed in [23, 24, 25] and independently used in [26]. Alternative approach based on quantum instantons which are defined using an introduced notion of quantum action was suggested in [27].

We will investigate a quasienergy spectrum of paradigmatic model for different physical systems, namely double well potential. We consider this system with perturbation of the kicked type. One of the most attractive features of kicked systems is the well investigated simple quantum map which stroboscopically evolves the system from kick $n$ to kick $n + 1$ and greatly facilitates theoretical analysis.

The main idea of this paper is to study the possibility to construct an effective autonomous model for the non-autonomous perturbed system using Campbell-Baker-Hausdorff expansion formula and to test it in numerical calculations of the quasienergy spectrum. Both CAT and CDT are connected with the behavior of the quasienergy spectrum (avoided or exact crossing of the levels). Thus the development of the methods for calculation of this spectrum is important for extending one’s knowledge in CAT and CDT. We regard the kick-type perturbation which is proportional to $x$. This perturbation, in contrast to perturbation proportional to $x^2$, destroys the spacial symmetry in the system which is important for presence of the CAT phenomenon [28]. The main role in quantum dynamics of our system is played by the classical asymmetry. There is no chaos induced processes in it but in our future work we will use this approach to system with CAT and CDT.

In this paper we propose the effective model for the kicked double well system which gives a possibility to simplify the numerical calculations of the quasienergy spectrum and allows to determine both analytically and numerically the quasienergy splitting dependence on both the perturbation strength and frequency.

2 Effective Hamiltonian for the kicked system

Now lets construct the effective Hamiltonian for the double well system with the perturbation of the kick-type. Hamiltonian of the particle in the double-well potential can be written in the following form:

$$H_0 = \frac{p^2}{2m} + a_0 x^4 - a_2 x^2,$$

where $m$ - mass of the particle, $a_0, a_2$ - parameters of the potential.
We consider the perturbation of the kick-type which is proportional to $x$

$$V_{\text{per}} = \epsilon x \sum_{n=-\infty}^{+\infty} \delta(t - nT),$$

(2)

where $\epsilon$ - perturbation strength, $T$ - perturbation period, $t$ - time.

Full Hamiltonian of the system is the following:

$$H = H_0 + V_{\text{per}}.$$  

(3)

Now we will construct an effective Hamiltonian for the system under investigation using the following definition:

$$\exp(-iH_{\text{eff}}T) = \exp(-i\epsilon x)\exp(-iH_0T),$$

(4)

where RHS is a one-period evolution operator. We restrict our consideration by sufficiently small values of both the perturbation strength and period. Using the Campbell-Baker-Hausdorff expansion formula for the kicked dynamical systems we can rewrite the last expression (4) in the following way [29]:

$$H_{\text{eff}} = H_0 + \frac{\epsilon \nu}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \int_0^1 ds \left[ \exp(-i\epsilon s\hat{x})\exp(-i\hat{H}_0T) \right]^k \hat{x},$$

(5)

where

$$g(z) = \frac{\ln|z|}{z-1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} (z-1)^n \quad \text{and} \quad \nu = \frac{2\pi}{T}.$$

With the definition of $g(z)$ formula (5) can be expanded in the following form:

$$H_{\text{eff}} = H_0 + \frac{\epsilon \nu}{2\pi} \sum_{n=0}^{\infty} \frac{1}{n+1} \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!} \int_0^1 ds \left[ \exp(-i\epsilon s\hat{x})\exp(-i\hat{H}_0T) \right]^k \hat{x},$$

(6)

where the expression under the integral is a map in the power $k$ which for sufficiently small values of both the perturbation strength and period can be rewritten as follows:

$$\left[ \exp(-i\epsilon s\hat{x})\exp(-i\hat{H}_0T) \right]^k \hat{x} = \hat{x} - kT \hat{p} + O(\epsilon^2, \epsilon T, T^2).$$

(7)

Substituting (7) in expression (6) one obtains the following form of the effective Hamiltonian:

$$H_{\text{eff}} = \frac{p^2}{2m} + a_0x^4 - a_2x^2 + \frac{\epsilon \nu}{2\pi} x + \frac{\epsilon}{2m} p + O(\epsilon^2, \epsilon T, T^2).$$

(8)

The fourth term in RHS of the last expression has the same order in perturbation parameters as first three main terms. The fifth term is proportional to small parameter, namely perturbation strength. We restrict our consideration by terms without small parameters and neglect all terms with order higher than zero in the perturbations parameters. As a result we have the following effective Hamiltonian for the kicked double well system:

$$H_{\text{eff}} = \frac{p^2}{2m} + a_0x^4 - a_2x^2 + \frac{\epsilon \nu}{2\pi} x.$$  

(8)

This is the Hamiltonian for the asymmetric double well potential without perturbation. In contrast to the kicked system it is autonomous. In the next section we will consider the properties for this system and construct the effective asymmetric model for the kicked system. In section 4 the correspondence between kicked and asymmetric effective double well systems will be tested numerically.
3 Effective asymmetric model for the kicked system

Hamiltonian of the classical particle in the asymmetric double-well potential is the following:

\[ H_{as} = \frac{p^2}{2m} + a_0 x^4 - a_2 x^2 + \sigma \sqrt{\frac{a_0}{2a_2}} x, \]  

(9)

where \( a_0 \) and \( a_2 \) are parameters of the potential, \( \sigma \) - asymmetric parameter. The asymmetric potential with parameters \( m = 1, a_0 = 1/128, a_2 = 1/4 \) and \( \sigma = 0.15 \) is shown in the figure 1 (thick solid line). Eight lowest energy levels are shown in the same figure with thin solid lines. The form of the last term in the RHS of the Hamiltonian (9) is more handy due to special choice of the asymmetric parameter \( \sigma \). Parameter \( \sigma \) for this form of the Hamiltonian is equal to shift between bottoms of the wells (dashed lines in the figure 1).

![Asymmetric double well potential](image)

Figure 1: Asymmetric double well potential (thick solid line) with eight lowest energy levels (thin solid lines). Minima shift (\( \sigma = 0.15 \)) is shown by dashed lines. The model parameters are \( m = 1, a_0 = 1/128, a_2 = 1/4 \).

In order to construct the effective model for the kicked system we should to introduce an effective value of the shift in the kicked system. Comparing obtained effective Hamiltonian \( H_{as} \) and Hamiltonian for the asymmetric double well system \( H_{as} \) we define effective asymmetric parameter in the following way:

\[ \sigma_{eff} = \sqrt{\frac{a_2}{2a_0}} \varepsilon \nu. \]  

(10)

Now we can give definition of the proposed effective model: asymmetric double well system with the effective parameter \( \sigma_{eff} \) defining by expression (10) is an effective model for the kicked system with the perturbation parameters \( \varepsilon \) and \( \nu \). The parameters \( a_0, a_2 \) and \( m \) are the same for both systems.

The definition of the effective asymmetric parameter (10) shows that perturbation strength and frequency appears in it as a product \( \varepsilon \nu \). This is the first advantage of the effective approach. We have effectively only one asymmetric parameter (\( \sigma \)) instead of two perturbation parameters (\( \varepsilon \) and \( \nu \)). The second advantage is the more simple way of the numerical calculations which will be discussed in the next section.
The third advantage of the proposed approach is that splitting for the doublets laying below the barrier hump in the asymmetric double well can be described analytically. The asymmetric model can be considered as a pair of shifted harmonic oscillators. The shift is equal to asymmetric parameter $\sigma$. It is obvious that in this constructed system non-degenerate energy doublets have splitting $\Delta = \sigma$. In asymmetric system splitting between levels remains close to $\sigma$ for all doublets lying below the barrier top as for case shown in the figure. This correspondence can be used in order to give analytical description of the quasienergy spectrum in the kicked system. Using expression (10) we can write down the formula for low lying quasienergy doublets’ splittings of the time-dependent system (3)

$$\Delta = \sqrt{\frac{a_2^2 \nu}{2a_0 \pi}}$$

It worth to mention the linear dependence of the levels splitting on both the perturbation strength and frequency. The applicability of this analytical description will be tested in the numerical calculations in the next section.

4 Numerical calculations

For the computational purposes it is convenient to choose the eigenvectors of harmonic oscillator as basis vectors. In this representation matrix elements of the Hamiltonian (1) and the perturbation (2) are real and symmetric. They have the following forms ($n \geq m$):

$$
H^0_{m,n} = \delta_{m,n} \left[ \hbar \omega \left( n + \frac{1}{2} \right) + \frac{g}{2} \left( \frac{3}{2} g a_0 (2m^2 + 2m + 1) - a'_2(2m + 1) \right) \right] + \delta_{m+2,n} \frac{g}{2} (g a_0 (2m + 3) - a'_2) \sqrt{(m + 1)(m + 2)} + \delta_{m+4,n} \frac{a_0 g^2}{4} \sqrt{(m + 1)(m + 2)(m + 3)(m + 4)},
$$

$$
x_{m,n} = \delta_{m+1,n} \sqrt{\frac{g}{2} \sqrt{m + 1}},
$$

where $g = \hbar/m\omega$ and $a'_2 = a_2 + m \omega^2/2$, $\hbar$ is Planck constant which we put equal to 1, $\omega$ - frequency of harmonic oscillator which is arbitrary, and so may be adjusted to optimize the computation. We use the value $\omega = 0.2$ with parameters $m = 1$, $a_0 = 1/128$, $a_2 = 1/4$. The matrix size is chosen to be equal to 200 $\times$ 200. Calculations with larger matrices give the same results. System of computer algebra Mathematica was used for numerical calculations.

In order to obtain quasienergy levels ($\eta_k$) in the kicked double well system directly we calculate eigenvalues ($\lambda_k$) of the one-period evolution operator $e^{-iHT}e^{-iV}$ and express quasienergy levels through the definition $\eta_k = i \ln \lambda_k/T$. Then we get ten levels with the lowest one-period average energy which is calculated using the formula $\langle v_i | H_0 + V/T | v_i \rangle$ ($| v_i \rangle$ are the eigenvectors of the one-period evolution operator). The dependence of the quasienergies of these ten levels on the strength of the perturbation is shown in the figure. Quasienergies of the two doublets with the minimal average energy (thick lines in the figure) has a linear dependence on the strength of the perturbation in the considered parameter region. They are strongly influenced by the perturbation while some of the quasienergy states are not.
Figure 2: Quasienergy spectrum for the ten lowest average energy levels. All levels are numbered in order of the average energy values. Solid lines - quasienergy levels for the kicked system. Thick lines - two doublets with the minimal average energy. Empty squares - shifted energy levels of the asymmetric model. The model parameters are \( m = 1, a_0 = 1/128, a_2 = 1/4 \) and \( \nu = 0.5 \).

Figure 3: Quasienergy splitting as a function of the strength (a) and frequency (b) of the perturbation. Filled circles - results of the numerical calculations for the kicked system. Empty squares - shifted energies for the asymmetric model. The model parameters are \( m = 1, a_0 = 1/128, a_2 = 1/4 \).

To obtain the same levels in the framework of the effective model approach we have to calculate the eigenvalues of the Hamiltonian of the asymmetric double well potential (9) with the asymmetric parameter defined by the expression (10). Then we shift obtained eigenvalues to zone \((-\frac{\nu}{2}, \frac{\nu}{2})\) in order to compare results with ones in the kicked system. The result of calculations on the base of this procedure is shown in the figure (2) by empty squares for two lowest doublets. Comparing results of direct and model calculations we make the conclusion that the levels of the doublets laying below the potential hump are correctly described by the effective model.

Performed numerical calculations for the kicked and the effective system give the depen-
dence of the ground quasienergy splitting both on the strength (fig.3(a)) and the frequency (fig.3(b)) of the perturbation. Filled circles in figures correspond to kicked double well system, empty squares to numerical results obtained in the framework of the effective model. There is good agreement between splitting’s dependencies for these two systems. They are linear as it was predicted by expression (11). It should be mentioned that all dependencies for the effective model was obtained from one series of the numerical calculations. We fix model parameters $a_0, a_2, m$ and calculate numerically one set of numerical points for the dependence on the asymmetric parameter. In kicked system we should to perform one series of numerical calculations for every dependency. This is the first advantage which was discussed in the previous section.

The second advantage which we should discuss after description of the used numerical methods is a more simple algorithm of the calculations. In the kicked system we should to calculate eigenvalues of the matrix exponents. This is more difficult task than in asymmetric model where we calculate the eigenvalues of the system Hamiltonian.

Analytical result (11) which was put forward as third advantage of the method is plotted in the figures 3(a) and 3(b) by straight solid lines. Numerical points lie close to these lines. The agreement between numerical calculations and analytical expression (11) is good (near 6%) in the parametric region considered.

5 Conclusions

Effective Hamiltonian for the kicked double well system was obtained using the Campbell-Baker-Hausdorff expansion formula. Effective autonomous asymmetric model for this system was constructed. This model is more convenient in numerical calculations than kicked one. Results of numerical calculations show that model correctly describes quasienergy spectrum of the kicked system for low laying levels.

The analytical formula for the ground quasienergy splitting dependence on both the perturbation strength and frequency was obtained in the framework of the effective asymmetric model. This formula predicts linear dependence of the ground quasienergy splitting on these parameters for the small perturbation strength and period values. Numerical results for the quasienergy splitting as a function of the perturbation frequency and strength demonstrate linear dependence as well. They are in a good agreement with the formula (11). Proposed approach will be used in future for investigation of the CAT and CDT phenomena.

References

[1] W. A. Lin and L. E. Ballentine, Phys. Rev. Lett. 65, 2927 (1990).
[2] J. Plata and J. M. G. Llorente, J. Phys. A: Math. Gen. 25, L303 (1992).
[3] M. Holthaus, Phys. Rev. Lett. 69, 1596 (1992).
[4] O. Bohigas, S. Tomsovic, and D. Ullmo, Phys. Rep. 223, 43 (1993).
[5] M. Latka, P. Grigolini, and B. J. West, Phys. Rev. A 50, 1071 (1994).
[6] R. Utermann, T. Dittrich, and P. Hänggi, Phys. Rev. E 49, 273 (1994), arXiv:chao-dyn/9310006v1.
[7] A. Mouchet, C. Miniatura, R. Kaiser, B. Gremaud, and D. Delande, Phys. Rev. E 64, 016221 (2001), arXiv:nlin/0012013v1.

[8] F. Grossmann, T. Dittrich, P. Jung, and P. Hänggi, Phys. Rev. Lett. 67, 516 (1991).

[9] C. Dembowski et al., Phys. Rev. Lett. 84, 867 (2000), arXiv:chao-dyn/9911023v2.

[10] D. A. Steck, W. H. Oskay, and M. G. Raizen, Science 293, 274 (2001).

[11] W. K. Hensinger et al., Nature 412, 52 (2001).

[12] V. A. Podolskiy and E. E. Narimanov, Optics Letters 30, 474 (2005).

[13] I. Vorobeichik, E. Narevicius, G. Rosenblum, M. Orenstein, and N. Moiseyev, Phys. Rev. Lett. 90, 176806 (2003).

[14] G. Della Valle et al., Phys. Rev. Lett. 98, 263601 (2007), arXiv:quant-ph/0701121v1.

[15] E. Kierig, U. Schnorrberger, A. Schietinger, J. Tomkovic, and M. K. Oberthaler, Phys. Rev. Lett. 100, 190405 (2008), arXiv:0803.1406v1 (quant-ph).

[16] J. H. Shirley, Phys. Rev. 138, B979 (1965).

[17] M. Grifoni and P. Hanggi, Phys. Rep. 304, 229 (1998).

[18] S. D. Frischat and E. Doron, Phys. Rev. E 57, 1421 (1998), arXiv:chao-dyn/9707005.

[19] E. Doron and S. D. Frischat, Phys. Rev. Lett. 75, 3661 (1995), arXiv:cond-mat/9505010v3.

[20] A. Shudo and K. S. Ikeda, Phys. Rev. Lett. 76, 4151 (1996).

[21] A. Shudo and S. Kensuke, Physica D: Nonlinear Phenomena 115, 234 (1998).

[22] T. Onishi, A. Shudo, K. S. Ikeda, and K. Takahashi, Phys. Rev. E 64, 025201(R) (2001), arXiv:nlin/0105067v1.

[23] V. I. Kuvshinov and A. V. Kuzmin, PEPAN 36, 100 (2005).

[24] V. I. Kuvshinov, A. V. Kuzmin, and R. G. Shulyakovsky, Phys. Rev. E 67, 015201(R) (2003), arXiv:nlin/0305028.

[25] V. I. Kuvshinov, A. V. Kuzmin, and R. G. Shulyakovsky, Acta Phys.Polon. B33, 1721 (2002), arXiv:hep-ph/0209292.

[26] A. Igarashi and H. S. Yamada, Physica D: Nonlinear Phenomena 221, 146 (2006), arXiv:cond-mat/0508483.

[27] H. Jirari, H. Kroger, X. Q. Luo, K. J. M. Moriarty, and S. G. Rubin, Phys. Lett. A 281, 1 (2001), arXiv:quant-ph/9910116v3.

[28] A. Peres, Phys. Rev. Lett. 67, 158 (1991).

[29] R. Scharf, J. Phys. A: Math. Gen. 21, 2007 (1988).