THE BASIC SUBALGEBRA STRUCTURE OF THE CAYLEY-DICKSON ALGEBRA OF DIMENSION 32 (TRIGINTADUONIONS)

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ABSTRACT. The Cayley-Dickson algebras \( \mathbb{R} \) (real numbers), \( \mathbb{C} \) (complex numbers), \( \mathbb{H} \) (quaternions), \( \mathbb{O} \) (octonions), \( \mathbb{S} \) (sedenions) and \( T \) (trigintaduonions) have attracted the attention of several mathematicians and theoretical physicists because of their important applications in both pure mathematics and theoretical physics. This paper deals with the determination of the basic subalgebra structure of the algebra \( T \) by analyzing the loop \( T_L \) of order 64 generated by its 32 basis elements. The analysis shows that \( T_L \) is a non-associative finite invertible loop (NAFIL) with 373 non-trivial subloops of orders 32, 16, 8, 4, and 2 all of which are normal. These subloops generate subalgebras of \( T \) of dimensions 16, 8, 4, 2, and 1 which form the elements of its structure.

1. INTRODUCTION

The Cayley-Dickson algebras \( \mathbb{C} \) (complex numbers 2-D), \( \mathbb{H} \) (quaternions 4-D), \( \mathbb{O} \) (octonions 8-D), \( \mathbb{S} \) (sedenions 16-D), and \( T \) (trigintaduonions 32-D) are real algebras obtained from the real numbers \( \mathbb{R} \) (1-D) by a doubling procedure called the Cayley-Dickson (C-D) process [1, 7]. Thus we have the following C-D doubling chain:

\[
\mathbb{R} \subset \mathbb{C} \subset \mathbb{H} \subset \mathbb{O} \subset \mathbb{S} \subset T \subset ...
\]

This shows that the trigintaduonions \( T \) contains \( \mathbb{S} \), \( \mathbb{O} \), \( \mathbb{H} \), \( \mathbb{C} \), and \( \mathbb{R} \) as subalgebras. These, however, are not the only subalgebras of \( T \): any subalgebra of \( \mathbb{S} \), \( \mathbb{O} \), \( \mathbb{H} \), and \( \mathbb{C} \) is also a subalgebra of \( T \) as well as others generated by its basis elements.

In a previous study [2, 3] we determined (using the software \textit{FINITAS} [4]) the basic subalgebra structure of the C-D sedenion algebra \( S \) of dimension 16. We showed that it contains an embedded loop \( S_L \) of order 32 (called the standard sedenion loop) generated by its 16 basis elements. Analysis of \( S_L \) showed that it contains 15 maximal subloops of order 16 all of which are non-abelian NAFILs (non-associative finite invertible loops) [5]. Of these, 8 are isomorphic to the standard octonion loop \( O_L \) of order 16 generated by the 8 basis elements of \( \mathbb{O} \) while 7 are isomorphic to a newly identified loop \( \tilde{O}_L \) which we called the quasi-octonion loop. Up to isomorphisms, these subloops generate 8-dimensional subalgebras, \( \mathbb{O} \) and \( \tilde{O} \), of the sedenions \( S \).

\textit{Key words and phrases.} Cayley-Dickson process, loop, octonions, quasi-octonions, sedenions, trigintaduonions, NAFILs. \textit{FINITAS}.

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1The 32-dimensional Cayley-Dickson algebra known as the trigintaduonions (from the Latin word trigintaduo, meaning 32) is also called the \( 2^5 \)-ions.
The next C-D algebra in the chain after $S$ is the trigintaduonion algebra $T$ of dimension 32. The captivating thing about $T$ is that all of the well-known real division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$, and $\mathbb{O}$ fit nicely inside it as subalgebras. Hence any object involving these algebras can be dealt with in $T$. Moreover, it also contains the sedenions $S$ and several sedenion-type algebras that have potential applications in pure mathematics and theoretical physics. This is our motivation for studying these algebras.

2. Methodology

The important question related to the study of any algebra is to know its subalgebra composition and lattice which determine its structure.

This paper presents an attempt to determine the basic subalgebra structure of $T$ by determining the subloop composition and lattice of the loop $T_L$ of order 64 generated by the 32 basis elements of $T$. The subloops of $T_L$ are of orders 32, 16, 8, 4, and 2. These generate subalgebras of $T$ of dimensions 16, 8, 4, 2, and 1, respectively. Thus there is a one-to-one correspondence between the subloops of $T_L$ and the basic subalgebras of $T$. Because of this the basic subalgebra composition of $T$ is directly related to that of $T_L$. To complete the description of the structure of $T_L$ we must show how its subloops fit inside it and how they are related to each other by determining its subloop lattice; and hence, by the correspondence, also the basic subalgebra lattice of $T$.

2.1. Definitions. In the standard literature the term algebra [1] is taken to mean a finite dimensional vector space over a field $F$ (like $\mathbb{R}$) with a bilinear multiplication (not necessarily associative) and with a unit element. Being a vector space, we can choose a basis in terms of which each element of the algebra can be written as a linear combination of the basis elements. Such an algebra is completely defined by the multiplication rules of its basis by means of a multiplication table. A loop [5], on the other hand, is a binary system satisfying all group axioms but not necessarily the associative property. For finite loops of small order, multiplication is usually defined by a Cayley table.

The Cayley-Dickson algebras [1] is a sequence of algebras over the real numbers $\mathbb{R}$, each with twice the dimension of the previous one as indicated by the doubling chain in Section 1. Thus, these algebras have dimensions of the form $2^n$, where $n \geq 0$ is an integer. Any algebra in the chain contains all of the algebras (and their subalgebras) before it as subalgebras. For $n \geq 3$ such an algebra is not associative.

2.2. Notation. In what follows, let the $32 = 2^5$ basis elements of the trigintaduonion algebra $T$ be represented by the set $T_E = \{e_0, e_1, e_2, e_3, \ldots, e_{31}\}$, where $e_0$ is the unit element. If $e_i, e_j, e_k \in T_E$, then $e_i \cdot e_j = \pm e_k$, where $-e_k \notin T_E$. Because of this the set $T_E$ is not closed under the operation $\cdot$ of trigintaduonion multiplication since it contains only positive elements. Thus the 32 elements of $T_E$ must be extended to include 32 negative elements $\in T$ to form the trigintaduonion loop $T_L$ of order $64 = 2^6$. Accordingly, these 64 loop elements will be represented by the set $T_L = \{\pm e_0, e_1, e_2, e_3, \ldots, e_{31}\}$. If $i = 0$, then $e_0 = 1$ (the unit) while if $i \geq 1$, then $\pm e_i$ is an imaginary such that $(\pm e_i)^2 = -1$. Moreover, $(-e_i) \cdot (e_j) = e_i \cdot (-e_j) = -(e_i \cdot e_j)$.\footnote{This correspondence can be considered as a form of duality.}
2.3. Computer Use. In this study we relied heavily on the use of the computer software FINITAS [4], the LOOPS package for GAP [9], and other programs for the construction and analysis of the various algebraic structures involved.

First, the Cayley table of the trigintaduonion loop $T_L$ was computer generated by means of a special computer program based on the Cayley-Dickson process. The result is shown in Table 1 which only shows its main portion that corresponds to the multiplication table of the 32 basis elements of $T$.

Next, using the software FINITAS we determined the subloop composition of the trigintaduonion loop $T_L$ by generating the Cayley tables of its subloops. We also analyzed $T_L$ to determine its basic structural properties (Section 3.5). Then we identified its subloops and classified them into isomorphy classes by subjecting them to isomorphism and other tests. For this, we used both FINITAS and LOOPS.

The data obtained from the various tests conducted were then analyzed.

3. Results and Discussions

The trigintaduonion algebra $T$ contains an embedded loop $T_L$ of order 64 generated by its 32 basis elements. Analysis (using FINITAS) shows that this loop is a NAFIL (non-associative finite invertible loop) with 373 non-trivial subloops of orders $m = 32$ (31 loops), $m = 16$ (155 loops), $m = 8$ (155 loops), $m = 4$ (31 loops), and $m = 2$ (1 loop). The subloops of orders 32 and 16 are NAFILs while those of orders 8, 4, and 2 are groups. Moreover, all of these 373 subloops of $T_L$ are normal. Such a loop, called a Hamiltonian loop, is known to have a modular lattice [6].

| R | C | H | O | S |
|---|---|---|---|---|
| 1 | 2 | 3 | 4 | 5 |

Table 1. Main portion of the Cayley table of the trigintaduonion loop $T_L = \pm\{e_0, e_1, e_2, e_3, ..., e_{31}\}$ of order $m = 64$. This portion corresponds to the multiplication table of the basis $T_E = \{e_0, e_1, e_2, e_3, ..., e_{31}\}$ of $T$. Note how $S$, $O$, $H$, and $R$ are contained in $T$. For convenience of notation, we represent each loop element $e_i$ by its subscript $i$, that is, we set $i = e_i$.  

\[3\text{Prof. Michael Kinyon (Denver University, USA) used the LOOPS package to assist us in determining the isomorphy classes of the 31 sedenion-type subloops of } T_L.\]
3.1. Subloops of Order 32. The maximal subloops of \( T_L \) are the 31 subloops of order 32 (called sedenion-type loops). One of these is the “standard” sedenion loop \( S_L \) generated by the basis of the sedenion algebra \( S \). In addition to \( S_L \) three more of these 31 sedenion-type loops of order 32 have been identified as distinct (non-isomorphic). In terms of the basis elements of \( T \), we have:

- \( S_L(#2) = \pm\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\} \)
  (std sedenion loop)
- \( S_\alpha L(#7) = \pm\{0, 1, 2, 3, 12, 13, 14, 15, 20, 21, 22, 23, 28, 29, 30, 31\} \)
  (\( \alpha \)-sedenion loop)
- \( S_\beta L(#10) = \pm\{0, 1, 2, 3, 12, 13, 14, 15, 20, 21, 22, 23, 24, 25, 26, 27\} \)
  (\( \beta \)-sedenion loop)
- \( S_\gamma L(#4) = \pm\{0, 1, 2, 3, 4, 5, 6, 7, 24, 25, 26, 27, 28, 29, 30, 31\} \)
  (\( \gamma \)-sedenion loop)

3.1.1. Isomorphy Classes of Sedenion-type Subloops. Further analysis shows that these four distinct subloops represent exactly four isomorphy classes:

- \( C_{\sim} = \{S_L(#2)\} \rightarrow 16 \) subloops
- \( C_{\sim} = \{S_\alpha L(#7)\} \rightarrow 7 \) subloops
- \( C_{\sim} = \{S_\beta L(#10)\} \rightarrow 7 \) subloops
- \( C_{\sim} = \{S_\gamma L(#4)\} \rightarrow 1 \) subloop

This means that each of the 31 sedenion-type subloops of \( T_L \) belongs to just one, and only one, of these classes. Up to isomorphism, it can be shown that the subloops \( S_L, S_\alpha L, S_\beta L, S_\gamma L \) of \( T_L \) generate subalgebras \( S, S^\alpha, S^\beta, S^\gamma \) of \( T \).

The Cayley tables of \( S_L, S_\alpha L, S_\beta L, \) and \( S_\gamma L \) are shown in Tables 2, 3, 4, and 5. Like Table 1, these Cayley tables only show their main portions; these correspond to the multiplication tables of the subalgebras \( S, S^\alpha, S^\beta, \) and \( S^\gamma \) of \( T \).

| \( n \) | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|-----|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0   | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
| 1   | 1 | -1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 2   | 2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 3   | 3 | -3 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 4   | 4 | -4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 5   | 5 | -5 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 6   | 6 | -6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 7   | 7 | -7 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 8   | 8 | -8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 9   | 9 | -9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10  | 10 | -10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11  | 11 | -11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 12  | 12 | -12 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 13  | 13 | -13 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 14  | 14 | -14 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 15  | 15 | -15 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

Table 2. Main portion of the Cayley table of the standard sedenion loop \( S_L(#2) = \pm\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15\} \) of order \( n = 32 \).

\( ^4 \)FINITAS decomposes the loop \( T_L \) into its 373 non-trivial subloops numbered #2 - #374; subloop #1 is the trivial subloop of order 1.
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Table 3. Main portion of the Cayley table of the $\alpha$-sedenion loop $S^\alpha_L(\#7) = \pm\{0, 1, 2, 3, 8, 9, 10, 11, 20, 21, 22, 23, 28, 29, 30, 31\}$ of order $n = 32$.

| * | 0 | 1 | 2 | 3 | 8 | 9 | 10 | 11 | 20 | 21 | 22 | 23 | 28 | 29 | 30 | 31 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 1 | 2 | 3 | 8 | 9 | 10 | 11 | 20 | 21 | 22 | 23 | 28 | 29 | 30 | 31 |
| 1 | 1 | 0 | 3 | 2 | 9 | 8 | 11 | 10 | 21 | 20 | 23 | 22 | 29 | 28 | 31 | 30 |
| 2 | 2 | 0 | 0 | 1 | 11 | 10 | 10 | 11 | 31 | 30 | 9 | 8 | 20 | 21 | 22 | 23 |
| 3 | 3 | 3 | 0 | 0 | 9 | 8 | 11 | 10 | 21 | 20 | 23 | 22 | 29 | 28 | 31 | 30 |
| 8 | 8 | 8 | 8 | 0 | 0 | 9 | 8 | 11 | 10 | 21 | 20 | 23 | 22 | 29 | 28 | 31 |
| 9 | 9 | 9 | 9 | 9 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 10 | 10 | 10 | 10 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 11 | 11 | 11 | 11 | 11 | 11 | 11 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 10 | 10 | 10 | 10 | 10 | 10 | 10 | 10 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 20 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 21 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 22 | 0 | 0 | 0 | 0 | 0 | 0 |

Remark 1. The Cayley table of the loop $T_L$ (Table 1) of order 64 has 64 rows and 64 columns consisting of four portions (partitions) each with 32 rows and 32 columns. Such a table is somewhat large and we only show its main portion: the multiplication table of the basis $T_E$ of $T$. Similarly, Tables 2, 3, 4, and 5 only show their main portions.
3.2. **Subloops of order 16.** Analysis shows that $T_L$ has 155 subloops of order 16. These are octonion-type NAFIL loops that form exactly two isomorphy classes:

- $C_{\sim\{O_L\}}$ of octonion loops [Class representative: #5 = $\pm\{0, 1, 2, 3, 4, 5, 6, 7\}$] = 50 loops
- $C_{\sim\{\tilde{O}_L\}}$ of quasi-octonion loops [Class representative: #11 = $\pm\{0, 1, 2, 3, 12, 13, 14\}$] = 105 loops

Thus, every octonion-type loop either belongs to the class $C_{\sim\{O_L\}}$ or to the class $C_{\sim\{\tilde{O}_L\}}$. These are the maximal subloops of the sedenion-type loops.

3.3. **Subloops of Order 8, 4, and 2.** The subloops of order 8, 4, and 2 are all groups. Analysis shows that all of the 155 subloops of order $m = 8$ are groups isomorphic to the quaternion group $Q_8$. On the other hand, the 31 subloops of order $m = 4$ are groups isomorphic to the cyclic group $C_4$; the lone subloop of order $m = 2$ is a group isomorphic to the cyclic group $C_2$. Thus they form the following isomorphy classes:

- $C_{\sim\{Q_8\}}$ of quaternion groups $Q_8$: $\longrightarrow$ 155 subloops
- $C_{\sim\{C_4\}}$ of cyclic groups $C_4$: $\longrightarrow$ 31 subloops
- $C_{\sim\{C_2\}}$ of cyclic group $C_2$: $\longrightarrow$ 1 subloop

3.4. **The Isomorphy Classes of the Subloops of $T_L$.** We have now identified all of the 9 isomorphy classes of the 373 non-trivial subloops of $T_L$:

- $C_{\sim\{S_L\}}, C_{\sim\{S^\alpha_L\}}, C_{\sim\{S^\beta_L\}}, C_{\sim\{S^\gamma_L\}}, C_{\sim\{O_L\}}, C_{\sim\{\tilde{O}_L\}}, C_{\sim\{Q_8\}}, C_{\sim\{C_4\}},$ and $C_{\sim\{C_2\}}$.

where $S_L, S^\alpha_L, S^\beta_L, S^\gamma_L, O_L, \tilde{O}_L, Q_8, C_4,$ and $C_2$ are class representatives. Here, $S_L$ is the standard sedenion loop of order 32 while $S^\alpha_L, S^\beta_L, S^\gamma_L$ are newly identified loops of order 32. The loop $O_L$ of order 16 is the standard octonion loop while $\tilde{O}_L$ is called the quasi-octonion loop of order 16. On the other hand, $Q_8$ is the quaternion group of order 8, $C_4$ is the cyclic group of order 4, and $C_2$ is the cyclic group of order 2.

It can be shown that each of the 373 subloops under these classes generate subalgebras of T. Thus the basic subalgebra composition of the algebra $\tilde{T}$ corresponds to the subloop composition of the loop $T_L$. This also means that each of the subloop isomorphy classes of $T_L$ determines a class of isomorphic subalgebras of $\tilde{T}$.

3.5. **Maximal Subloop Compositions of Sedenion-Type Loops.** Analysis also shows that each of the 31 sedenion-type subloops of $T_L$ belong to exactly one of the following three maximal subloop composition types:

- $[8 O_L + 7 \tilde{O}_L] = [8 + 7] \longrightarrow$ 17 subloops
- $[2 O_L + 13 \tilde{O}_L] = [2 + 13] \longrightarrow$ 7 subloops
- $[0 O_L + 15 \tilde{O}_L] = [0 + 15] \longrightarrow$ 7 subloops

Thus we find that any sedenion-type loop contains at least 7 quasi-octonion loops $\tilde{O}_L$ as subloops. Table 6 shows the four identified sedenion-type loops in Section 3.1 and their subloop composition types.
From this we find that the loops $S_L$, $S^2_L$, and $S^3_L$ differ in their maximal subloop compositions. Hence they are not isomorphic. Note that the loop $S^4_L$ has the same subloop composition as $S_L$ but analysis shows that they are not isomorphic. Therefore all of these four subloops are distinct.

The sedenion-type loops generate 16-dimensional non-associative real subalgebras of $T$ with zero divisors [2]. This is due to the fact that (up to isomorphism) each of them contains $\tilde{O}_L$, and hence $\tilde{O}$ as a subalgebra. Such algebras (and their subalgebras) have potential applications in both pure and applied mathematics and in theoretical physics [7, 8].

### 3.6. Properties of $T_L$ and its Subloops

All Cayley-Dickson algebras of dimension $D \geq 8$ are non-associative and non-commutative. Analysis of the loop $T_L$ shows that it is a NAFIL that satisfies the following identities (universally quantified equations) listed in Table 7. Naturally, these identities are also satisfied by all subloops of $T_L$ (both associative and non-associative).

| Special Loop Property     | Acronym | Defining Equation |
|---------------------------|---------|------------------|
| Inverse Property          | IP      | $\ell^{-1}(\ell q) = (q \ell)^{-1} = q$ |
| Alternative Property      | AP      | $\ell (\ell q) = \ell^2 q$ and $(\ell q) q = \ell q^2$ |
| Flexible Law              | FL      | $\ell_i (\ell_k \ell_j) = (\ell_i \ell_k) \ell_j$ |
| C Loop Property           | CP      | $\ell_i [\ell_j (\ell_k \ell_j)] = [\ell_i \ell_j] \ell_k$ |
| Power Associative Property| PAP     | $\ell^a \cdot \ell^b = \ell^{a+b}$ |
| Weak Inverse Property     | WIP     | $\ell (q \ell)^{-1} = q^{-1}$ |
| Anti-Automorphic Inverse Property | AAIP | $\ell (q \ell)^{-1} = q^{-1} \ell^{-1}$ |

Table 7. Some special loop properties (identities) satisfied by all Cayley-Dickson loops of order $m \geq 16$. The trigintaduonion loop $T_L$ and its subloops satisfy these identities. The algebra $T$, however, satisfies only the flexible and power-associative identities.

Not all of the identities satisfied by the loop $T_L$ listed in Table 7 are satisfied by the algebra $T$ as a whole. Moreover there are identities satisfied only by its subloops. For instance, the octonion loop and the quaternion and cyclic groups satisfy the Moufang identity.

### 3.7. Subloop Lattice of $T_L$

The subloop lattice of $T_L$ (and hence the basic subalgebra lattice of $T$) is determined by its subloop composition. This consists of its 373 subloops classified into 9 isomorphy classes: $C \subset \{S_L\}$, $C \subset \{S^2_L\}$, $C \subset \{S^3_L\}$, $C \subset \{S^4_L\}$, $C \subset \{O_L\}$, $C \subset \{\tilde{O}_L\}$, $C \subset \{Q_8\}$, $C \subset \{C_4\}$, and $C \subset \{C_2\}$.

The diagram shown in Figure 1 indicates the general form of the subloop lattice of $T_L$ in terms of the isomorphy classes of its subloops. The determination of the complete lattice is a complicated problem that we are now trying to address. So
far, we know that all subloops of $T_L$ are normal. Such a loop has been shown to have a modular lattice [6].

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{lattice_diagram.png}
\caption{Lattice diagram (in block form) of the 9 isomorphy classes of the subloops of the loop $T_L$.}
\end{figure}

4. Summary

The Cayley-Dickson algebra $\mathbb{T}$ of dimension $D = 32$ (called the trigintaduonions) contains an embedded NAFIL loop $T_L$ of order 64. This loop contains 31 subloops of order 32, 155 of order 16, 155 of order 8, 31 of order 4, and 1 of order 2, all of which are normal. Such a loop is known to have a modular lattice.

The subloops of orders 32 (sedenion-type) and 16 (octonion-type) are NAFILs, while those of orders 8, 4, and 2 are groups. These non-trivial subloops of $T_L$ are classified into nine isomorphy classes: $C_\approx \{S_L\}, C_\approx \{S_{L}^\beta\}, C_\approx \{S_{L}^{\alpha}\}, C_\approx \{S_{L}^{\gamma}\}, C_\approx \{O_L\}, C_\approx \{\tilde{O}_L\}, C_\approx \{Q_8\}, C_\approx \{C_4\}$, and $C_\approx \{C_2\}$.

It can be shown that these subloops of $T_L$ of orders 32, 16, 8, 4, and 2 generate subalgebras of $\mathbb{T}$ of dimensions 16, 8, 4, 2, and 1, respectively. Hence the basic subalgebra structure of $\mathbb{T}$ is directly related to the subloop structure of $T_L$.

Up to isomorphism, the octonion-type subloops of $T_L$ have the same subloop composition: 7 quaternion groups $Q_8$, 7 cyclic groups $C_4$, and 1 cyclic group $C_2$. Nevertheless, they differ in many fundamental aspects (e.g. $O_L$ satisfies the Moufang identity while $\tilde{O}_L$ does not).

The 31 maximal subloops of order 32 form 4 isomorphy classes and 3 distinct maximal subloop composition types: $[8 O_L + 7 \tilde{O}_L], [2 O_L + 13 \tilde{O}_L]$, and $[0 O_L +$
15 $\tilde{O}_L$]. This shows that (up to isomorphism) every sedenion-type loop contains at least 7 quasi-octonion loops as subloops.

All 16-dimensional subalgebras of the trigintaduonions $T$ (like $S$, $S^\alpha$, $S^\beta$, and $S^\gamma$) have zero divisors. Except for $S$, these interesting algebraic structures have not been previously identified. To date not much is known about these algebras and it is therefore a challenge to determine their structural features and their possible applications.

The lattice of the isomorphy classes of the subloops of $T_L$ is now known (Figure 1). However, the complete subloop lattice of $T_L$ has not yet been determined. This aspect of the problem is now being studied.

Remark 2. A recent review of the literature has shown that S. Catto and D. Chesley (Twisted octonions and their symmetry groups, Nuclear Physics Proceedings Supplements, Volume 6, p. 428-432) have independently identified the quasi-octonions which they have called the "twisted octonions." Their approach to this problem is different from our method: it is based on the analysis of quaternion triples. However, their results agree with our own findings. Robert de Marrais (The 42 Assessors and the Box-Kites they Fly: Diagonal axis-pairs systems of zero-divisors in the Sedenions, http://arXiv.org/abs/math.GM/0011260) has also been studying the Cayley-Dickson algebras using a different method which he calls the Box-Kites approach.

References

[1] Baez, John, The Octonions, Bul. Amer. Math. Soc. 30 (2001), No. 2, 145-205.[See also: Springer Encyclopedia of Mathematics, http://eom.springer.de]
[2] Cawagas, R. E. & S. A. G. Gutierrez, Subloop Structure of the Cayley-Dickson Sedenion Loop, Matimyas Mathematica, Vol. 28, Nos. 1-3 (2005)
[3] Cawagas, R. E., Loops Embedded in Generalized Cayley Algebras of Dimension $2^r$, Int. J. Math. Math. Sci., 28:3 (2001) 181-187
[4] Cawagas, R. E., FINITAS - A Software for the Construction and Analysis of Finite Algebraic Structures, PUP Jour. Res. Expo, Vol. 1, No. 1 (1997)
[5] Cawagas, R. E., Introduction to: Non-Associative Finite Invertible Loops, PUP Journal of Science and Technology, Vol. 1, No. 2 (2007) [See also: H. O. Pflugfelder, Quasigroups and Loops: Introduction, Sigma Series in Pure Mathematics, Helderman Verlag, Berlin (1990)]
[6] Lausch, H., On the Lattice Formed by all Normal Subloops of a Finite Loop, Letters in Mathematical Physics 56: 121-127, 1995
[7] Okubo, S., Introduction to Octonions and Other Non-Associative Algebras in Physics, Cambridge University Press
[8] Zihua Weng., Compounding Fields and Their Quantum Equations in the Trigintaduonion Space, arXiv: physics/0704.0136
[9] Nagy, G. and Vojtechovsky, P., LOOPS: Computing with quasigroups and loops in GAP
http://www.math.du.edu/loops/

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