Renormalization group flow for $SU(2)$ Yang-Mills theory and gauge invariance

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Abstract

We study the formulation of the Wilson renormalization group (RG) method for a non-Abelian gauge theory. We analyze the simple case of $SU(2)$ and show that the local gauge symmetry can be implemented by suitable boundary conditions for the RG flow. Namely we require that the relevant couplings present in the physical effective action, i.e. the coefficients of the field monomials with dimension not larger than four, are fixed to satisfy the Slavnov-Taylor identities. The full action obtained from the RG equation should then satisfy the same identities. This procedure is similar to the one we used in QED. In this way we avoid the conspicuous fine tuning problem which arises if one gives instead the couplings of the bare Lagrangian. To show the practical character of this formulation we deduce the perturbative expansion for the vertex functions in terms of the physical coupling $g$ at the subtraction point $\mu$ and perform one loop calculations. In particular we analyze to this order some ST identities and compute the nine bare couplings. We give a schematic proof of perturbative renormalizability.

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1 Introduction

An elegant treatment of local gauge theories is provided by dimensional regularization\[1\] which preserves BRST invariance. However, this method is necessarily bounded to the perturbative regime. Moreover, in the case of chiral gauge theories, even this regularization conflicts\[2\] with the local symmetry. The reason for this is that the usual definition of $\gamma_5$ is not consistent in complex dimension. Therefore it is useful to study and exploit alternative formulations which work in four dimensions and are in principle nonperturbative.

The exact renormalization group (RG) formulation\[3\] works in the physical space-time and provides a deep physical meaning of the ultraviolet divergences. The original motivation for this method was the study of the flow of Lagrangians at various scales. Once the action $S_{\Lambda_0}$ at the ultraviolet (UV) scale $\Lambda_0$ is given, one computes the Wilsonian action $S_{\Lambda}$ at a lower scale $\Lambda$, by integrating the fields with frequencies $p$ between $\Lambda$ and $\Lambda_0$ ($\Lambda^2 < p^2 < \Lambda_0^2$). A very simple way of constructing the RG equations has been written few years ago by Polchinski\[4\]. See also\[5\]-\[10\]. This method not only allows the study of the evolution in $\Lambda$ of $S_{\Lambda}$, but provides a practical way of computing Green’s functions. For instance the iterative solution of RG equations gives the usual loop expansion.

The RG formulation was not considered suitable for local gauge theories since the presence of a momentum scale $\Lambda$ implies that the BRST invariance is completely lost. As a consequence the bare couplings of the UV action $S_{\Lambda_0}$ are not constrained by BRST invariance. Obviously the various bare couplings in $S_{\Lambda_0}$ are anyway related by the fact that the physical Green’s functions must satisfy Slavnov-Taylor (ST) identities. Even assuming that this is possible, the bare couplings would be constrained by a tremendous fine tuning process\[11\]. Recently Becchi\[6\], by studying the “quantum action principle” for the UV action in $SU(2)$ Yang-Mills (YM) case was able to extend the BRST transformations and obtain a functional fine tuning equation for the bare couplings. See also\[12\].

In this paper we present an attempt of giving a RG formulation for gauge theories which avoids the fine tuning problem. In a previous paper\[9\] we discussed the Abelian case while here we analyze the non-Abelian case of $SU(2)$ Yang-Mills theory. The method is quite general, uses some well known procedures but organized in a rather unusual way. Therefore it is useful to summarize the main steps.

1) Definition of the cutoff effective action. Consider the physical effective action $\Gamma[\Phi]$ where $\Phi(x)$ denotes all the fields of the theory. For QED $\Phi = (A_{\mu}, \bar{\psi}, \psi)$, i.e. the photon and electron fields; for SU(2) YM case $\Phi = (A^a_\mu, c^a, \bar{c}^a, u^a_\mu, v^a)$, i.e. the gauge, ghost, antighost fields and the sources for the BRST transformations of $A^a_\mu$ and $c_a$ respectively. $\Gamma[\Phi]$ must satisfy Ward or ST identities. We consider now the “cutoff effective action” $\Gamma[\Phi; \Lambda]$, obtained by putting an infrared (IR) cutoff $\Lambda$ and an UV cutoff $\Lambda_0$ for all propagators in the vertices, i.e. we set to zero each propagator if its frequency is lower than $\Lambda$ and larger than $\Lambda_0$. In the $\Lambda_0 \to \infty$ limit $\Gamma[\Phi; \Lambda = 0]$ is just the physical effective action, while for $\Lambda \neq 0$ it does not satisfy Ward or ST identities. The functional $\Gamma[\Phi; \Lambda]$ is related to the Wilsonian action $S_\Lambda$ and we introduced it only because of technical reasons. Actually $\Gamma[\Phi; \Lambda = \Lambda_0]$ is the UV action $S_{\Lambda_0}$.

2) RG equation for $\Gamma[\Phi; \Lambda]$. The RG equation for $S_\Lambda$ can be formulated directly for $\Gamma[\Phi; \Lambda]$
and, as we shall discuss, takes the form (see [4])

$$\Lambda \partial_\Lambda \Gamma[\Phi; \Lambda] = I[\Phi; \Lambda]$$

(1)

where the functional $I[\Phi; \Lambda]$ is given (non linearly) in terms of $\Gamma[\Phi; \Lambda]$. Once the boundary conditions are given, the RG equation (1) allows one to obtain $\Gamma[\Phi; \Lambda]$ for any $\Lambda$ and in particular, for $\Lambda = 0$, the physical effective action. As one expects, the boundary conditions for the various vertices depend on dimensional counting.

3) Relevant and irrelevant parts of $\Gamma[\Phi; \Lambda]$. One distinguishes irrelevant vertices, which have negative mass dimension, and relevant couplings with non-negative dimension. One can therefore write

$$\Gamma[\Phi; \Lambda] = \Gamma_{\text{rel}}[\Phi; \sigma_i(\Lambda)] + \Gamma_{\text{irr}}[\Phi; \Lambda],$$

(2)

where $\Gamma_{\text{irr}}[\Phi; \Lambda]$ involves only the irrelevant vertices and $\sigma_i(\Lambda)$ are the relevant couplings: seven for QED and nine for the $SU(2)$ case. Therefore $\Gamma_{\text{rel}}$ is given by a polynomial in the fields. The relevant couplings are defined as the values of some vertex functions or their derivatives at a given normalization point. We are ready now to discuss the boundary conditions for the evolution equation in (1).

4) Boundary conditions for $\Gamma_{\text{irr}}[\Phi; \Lambda]$. For $\Lambda = \Lambda_0 \to \infty$ one assumes

$$\Gamma[\Phi; \Lambda_0] = \Gamma_{\text{rel}}[\Phi; \sigma_i(\Lambda_0)],$$

(3)

i.e. the irrelevant vertices vanish in the UV region. Notice that this hypothesis is essential to have perturbative renormalizability. The parameters $\sigma_i(\Lambda_0)$ are the bare parameters of the UV action. In the usual procedure one gives these bare parameters and, by using [4], evaluates $\Gamma[\Phi; \Lambda = 0]$. The requirement that this resulting effective action satisfies the ST identities gives rise to the fine tuning problem, which we want to avoid.

5) Boundary conditions for $\Gamma_{\text{rel}}[\Phi; \sigma_i(\Lambda)]$. Instead of fixing the boundary conditions for the relevant parameters $\sigma_i(\Lambda)$ at $\Lambda = \Lambda_0 \to \infty$, we set them at the physical point $\Lambda = 0$. The values $\sigma_i(0)$ are then the physical masses, wave function constants and couplings. Moreover, for a gauge theory we must fix these couplings in such a way that $\Gamma_{\text{rel}}[\Phi; \sigma_i(0)]$ satisfies Ward or ST identities. This is then the place where we implement the symmetry of the theory. For the Abelian case, studied in the previous paper, we showed that one can simply assume $\Gamma_{\text{rel}}[\Phi; \sigma_i(0)] = S_{\text{cl}}[\Phi]$, where $S_{\text{cl}}[\Phi]$ is the QED classical action including the gauge fixing term. Then $\Gamma_{\text{rel}}[\Phi; \sigma_i(0)]$ satisfies Ward identities. For the non-Abelian theory, as we shall illustrate, the situation is more complex. In this case one has two features which are new with respect to QED. The first difference is that in the YM case one can construct relevant monomials, such as for instance $(A^a_\mu A^a_{\mu})^2$, which are not present in the classical Lagrangian. The second important difference between QED and YM theories is that Ward identities are linear while ST identities are not linear. In QED, Ward identities do not couple relevant and irrelevant couplings, so that the relevant part of the effective action satisfies Ward identities by itself. In the YM case instead the ST identities couple the relevant and irrelevant part of the effective action. Therefore the requirement that $\Gamma_{\text{rel}}[\Phi; \sigma_i(0)]$ satisfies ST identities implies that some of the nine relevant couplings are fixed in terms of irrelevant vertices. This will be discussed in detail in sect. 3.

6) Analysis of the ST identities. These boundary conditions and the RG equation (1) completely determine $\Gamma[\Phi; \Lambda]$, at least perturbatively. The fundamental point is whether the physical effective action $\Gamma[\Phi; \Lambda = 0]$ does satisfy Ward or ST identities. Recall that these identities have been implemented only for the relevant part of $\Gamma[\Phi; \Lambda = 0]$. In the
Abelian case we have been able to show [9] that the renormalized Green’s functions indeed satisfy Ward identities to all order in perturbation theory. The method was based on the properties of RG flow and the use of the above boundary conditions. However the proof still required the analysis of graphs. The same method could be extended to the non-Abelian case but it would be quite more complicated. In this paper we limit ourself to the analysis of some ST identities to one loop. In this way we illustrate the key points needed to obtain these identities. A general proof to all loops should use more synthetic techniques.

The paper is organized as follows. In sect. 2 we define the cutoff effective action for the $SU(2)$ YM theory for which we write the RG equation. In sect. 3 we define the relevant couplings and deduce in detail their values at $\Lambda = 0$ to satisfy ST identities. In sect. 4 we convert the differential functional equation into an integral equation which embodies the boundary conditions. In sect. 5 we show that the iterative solution gives the loop expansion for the vertex functions. This perturbative expansion involves only the physical parameter $g$ defined at the subtraction point $\mu$. To one-loop order we explicitly derive: the various vertex functions, the bare parameters in the UV Lagrangian, i.e. the relevant couplings $\sigma_i(\Lambda)$ at $\Lambda = \Lambda_0$, and the usual one-loop beta function. We check the ST identities at one loop for the vector propagator and for the ghost-vector vertex in the physical limit ($\Lambda = 0$ and $\Lambda_0 \to \infty$). Finally we give a schematic proof of perturbative renormalizability. In sect. 6 we show that the cutoff effective action depends, as the classical Lagrangian, on the combination $w_a = u_a^\mu / g + \partial_\mu \bar{c}^a$. This fact simplify greatly the analysis of relevant field monomials. Sect. 8 contains some conclusions.

## 2 Renormalization group flow and effective action

In this section we introduce the cutoff effective action and deduce the RG group flow equations. We follow the same steps [8, 9] as in the scalar and QED theory. This allows us to set up the necessary notations.

In the $SU(2)$ gauge theory the classical Lagrangian in the Feynman gauge is

$$S_{cl}[A_\mu, c, \bar{c}] = \int d^4x \left\{ -\frac{1}{4} (F_{\mu\nu} \cdot F_{\mu\nu}) - \frac{1}{2} (\partial_\mu A_\mu)^2 - \bar{c} \cdot \partial_\mu D_\mu c \right\},$$

where

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g (A_\mu \land A_\nu)^a, \quad (A_\mu \land A_\nu)^a = \epsilon^{abc} A_\mu^b A_\nu^c,$$

$$D_\mu c = \partial_\mu c + g A_\mu \land c.$$

This action is invariant under the BRS transformations [13]

$$\delta A_\mu = \frac{1}{g} \eta D_\mu c, \quad \delta c = -\frac{1}{2} \eta c \land c, \quad \delta \bar{c} = -\frac{1}{g} \eta \partial_\mu A_\mu,$$

with $\eta$ a Grassmann parameter. Introducing the sources $u_\mu$ and $v$ associated to the variations of $A_\mu$ and $c$ one has the BRS action

$$S_{BRS}[A_\mu, c, \bar{c}, u_\mu, v] = S_{cl} + \int d^4x \left\{ \frac{1}{g} u_\mu \cdot D_\mu c - \frac{1}{2} v \cdot c \land c \right\} = S_2 + S_{int},$$

(5)
where $K$ and $\Lambda$ are present in the classical action (4). The remaining two seven monomials $\text{SU}$ which are obtained by considering the limit $\Lambda_0 \rightarrow \infty$. We regularize these divergences by assuming that in the path integral one integrates only the fields with frequencies smaller than a given UV cut off $\Lambda$. We now define for this theory the cutoff effective action discussed in the introduction.

### 2.1 Cutoff effective action

We now define for this theory the cutoff effective action discussed in the introduction. In order to compute the vertices one needs a regularization procedure of the ultraviolet divergences. We regularize these divergences by assuming that in the path integral one integrates only the fields with frequencies smaller than a given UV cut off $\Lambda$. This procedure is equivalent to assume that the free propagators vanish for $p^2 > \Lambda_0^2$. The physical theory is obtained by considering the limit $\Lambda_0 \rightarrow \infty$. In order to study the Wilson renormalization group flow (3), one introduces in the free propagators also an infrared cutoff $\Lambda$. The quadratic part of the action (3) becomes, in momentum space,

$$ S^2_{\Lambda, \Lambda_0} = \int_p \left\{ -\frac{1}{2} A_\mu(-p) \cdot A_\mu(p) p^2 + \frac{p^2}{2} c(-p) \cdot c(p) \right\} K^{-1}_{\Lambda\Lambda_0}(p) - \frac{i}{2} \int_p (\partial_\mu A_\mu(p)) \cdot c(p) - \frac{i}{2} \int_q v(p) \cdot c(q) \wedge c(-p - q), $$

where $K_{\Lambda\Lambda_0}(p) = 1$ in the region $\Lambda^2 \ll p^2 \ll \Lambda_0^2$ and rapidly vanishing outside and

$$ \int_p \equiv \int \frac{d^4 p}{(2\pi)^4}. $$

The introduction of a cutoff in the propagators breaks the gauge invariance properties of the action. Therefore at the UV scale the interaction $S^2_{\text{int}}$ must contain all the nine monomials which are $SU(2)$ singlets and Lorentz scalar with dimension not higher than four. The first seven monomials

$$ A_\mu \cdot A_\mu, \quad (\partial_\nu A_\mu) \cdot (\partial_\nu A_\mu), \quad (\partial_\mu A_\mu) \cdot (\partial_\nu A_\nu), \quad w_\mu \cdot \partial_\mu c, \quad (A_\mu \wedge A_\nu) \cdot (\partial_\mu A_\nu), \quad w_\mu \cdot c \wedge A_\mu, \quad (A_\mu \wedge A_\nu) \cdot (A_\mu \wedge A_\nu) $$

are present in the classical action (3). The remaining two

$$ 2(A_\mu \cdot A_\nu) (A_\mu \cdot A_\nu) + (A_\mu \cdot A_\mu) (A_\nu \cdot A_\nu), \quad v \cdot c \wedge c $$

are not present in (3) and are generated by the interaction.

In general we consider two functionals $\Pi_{1,\text{rel}}$ and $\Pi_{2,\text{rel}}$, containing the two groups of monomials in (3) and (4) respectively. The first one depends on the seven couplings $\sigma_i$

$$ \Pi_{1,\text{rel}}[\Phi; \sigma_i] = \int d^4 x \left\{ \frac{1}{2} A_\mu \cdot \left[ g_{\mu\nu}(\sigma_{m\alpha} - \sigma_\alpha \partial^2) - \sigma_A (g_{\mu\nu} \partial^2 - \partial_\mu \partial_\nu) \right] A_\nu + i \sigma_{w_\mu} w_\mu \cdot \partial_\mu c + i \sigma_{3A} (\partial_\mu A_\mu) \cdot A_\nu \wedge A_\mu + \frac{\sigma_{4A}}{4} (A_\mu \wedge A_\nu) \cdot (A_\mu \wedge A_\nu) + \sigma_{w_\mu} w_\mu \cdot c \wedge A_\mu \right\}. $$

(9)
The second functional depends on two additional couplings. Since they will play a particular rôle, we denote them with the different letter \( \rho \). The functional \( \Pi_{2,\text{rel}} \) is then
\[
\Pi_{2,\text{rel}}[\Phi; \rho_A] = \int d^4 x \left\{ \frac{\rho_{\text{vec}}}{2} v \cdot c \wedge c + \frac{\rho_{\text{el}} A}{8} \left[ 2 (A_\mu \cdot A_\nu) (A_\mu \cdot A_\nu) + (A_\mu \cdot A_\mu) (A_\nu \cdot A_\nu) \right] \right\}
\]
and contains the contributions from field monomials in \( S \), which do not appear in the classical action. By \( \Phi \) we indicate the classical fields and sources \( A_\mu, c, w_\mu \) and \( v \). In conclusion the interacting part of the UV action can be written as
\[
S^{\Lambda_0}_{\text{int}} = \Pi_{1,\text{rel}}[\Phi; \sigma_i^B] + \Pi_{2,\text{rel}}[\Phi; \rho_i^B],
\]
where \( \sigma_i^B \) and \( \rho_i^B \) are the bare couplings. The mass parameter \( \sigma_{m_A} \) has positive dimension, while the other parameters are dimensionless. \( \sigma_B^P \) and \( \sigma_{wc}^B \) are related to the vector and ghost wave function renormalization, \( \sigma_a^B \) to the gauge fixing parameter renormalization, \( \sigma_{vca}^B \), \( \sigma_{3A}^B \) and \( \sigma_{4A}^B \) are related to the bare interaction couplings. The additional coupling \( \rho_{\text{vec}}^B \) corresponds to the interaction among four vectors with a group structure not present in the classical action \( S_{\text{el}} \). Similarly \( \rho_{\text{vec}}^B \) is an interaction term not present in \( S_{\text{el}} \).

The generating functional of the Green’s functions is
\[
Z[j, u, v; \Lambda] = \exp i W[j, u, v; \Lambda] = \int D[\phi_A] \exp i \left\{ S_{\text{int}}^A + S_{\text{int}}^{\Lambda_0} + (j, \phi) \right\},
\]
where we introduced the compact notation for the fields and corresponding sources
\[
\phi_A = (A_\mu, c^a, \bar{c}^a), \quad j_A = (j_\mu^a, \bar{\chi}^a, -\chi^a),
\]
\[
(j, \phi) \equiv \int_p j_\mu (-p) \cdot A_\mu (p) + \bar{\chi} (-p) \cdot c(p) + \bar{c} (-p) \cdot \chi (p).
\]
The free cutoff propagators are described by the matrix \( D_{AB} \) defined by
\[
\int_p \frac{i}{2} \phi_A (-p) D_{AB}^{-1} (p; \Lambda) \phi_B (p) \equiv \int_p \left\{ -\frac{1}{2} A_\mu (-p) \cdot A_\mu (p) p^2 + p^2 c (-p) \cdot c (p) \right\} K_{\Lambda A}^{-1} (p).
\]
In \( Z[j, u, v; \Lambda] \) and \( D_{AB} (p; \Lambda) \) we have explicitly written only the cutoff \( \Lambda \) since we will consider in any case the limit \( \Lambda_0 \rightarrow 0 \) and \( \Lambda \rightarrow \infty \). The physical functional \( Z[j, u, v] \) is obtained by taking the limits \( \Lambda \rightarrow 0 \) and \( \Lambda_0 \rightarrow \infty \).

Finally, the cutoff effective action is defined by taking the Legendre transform
\[
\Gamma[\phi, u, v; \Lambda] = W[j, u, v; \Lambda] - W[0, 0, 0; \Lambda] - (j, \phi), \quad \phi_A (p) = (2\pi)^4 \frac{\delta W[j, u, v; \Lambda]}{\delta j_A (-p)}.
\]

### 2.2 Exact renormalization group equation

Usually the RG equations are obtained by requiring that the physical quantities are independent of \( \Lambda \). In the present formulation the same flow equation can be simply obtained by observing that all the \( \Lambda \) dependence of \( Z[j, u, v; \Lambda] \) is contained in the cutoffs in the propagators. Thus one easily derives \( \Pi \) the equation
\[
\Lambda \partial_\Lambda Z[j, u, v; \Lambda] = -i \frac{(2\pi)^8}{2} \int_q \Lambda \partial_\Lambda D_{BA}^{-1} (-q; \Lambda) \frac{\delta^2 Z[j, u, v; \Lambda]}{\delta j_B (q) \delta j_A (-q)}.
\]
In the same way one finds the corresponding equation for the cutoff effective action. The equation is schematically given in fig. 1 and is obtained by observing that the operator $\Lambda \partial_\Lambda$ acts on each internal propagator. The first term involves only one vertex; the second term involves two vertices in such a way to reproduce, upon $q$-integration, a one-particle irreducible contribution. The other terms will involve any possible number of vertices. Of course one can deduce formally this equation. By following the same steps of refs. \[8, 9\] we obtain

$$\Lambda \partial_\Lambda \Pi[\Phi; \Lambda] = -\frac{i}{2} \int_q M_{BA}(q; \Lambda) \bar{\Gamma}_{AB}[-q, q; \Phi; \Lambda] = I[\Phi; \Lambda],$$

(12)

where

$$\Gamma[\phi, u, v; \Lambda] = S_{2}^{\Lambda, \Lambda_0} + \Pi[\Phi; \Lambda],$$

$$M_{BA} = \Delta_{BC}(q; \Lambda) \Lambda \partial_\Lambda D_{CD}^{-1}(q; \Lambda) \Delta_{DA}(q; \Lambda)$$

and $\Delta_{AB}(q; \Lambda)$ is the full propagator. The auxiliary functional $\bar{\Gamma}_{AB}[q, q'; \Phi]$ is the inverse of $\delta^2 \Gamma[\phi, u, v]/\delta \phi_A(q) \delta \phi_B(q')$. To obtain this we isolate the contribution of the two point function

$$(2\pi)^8 \frac{\delta^2 \Gamma[\phi, u, v]}{\delta \phi_A(q) \delta \phi_B(q')} = (2\pi)^4 \delta^4(q + q') \Delta_{BA}(q; \Lambda) + \Gamma_{BA}^{int}[q', q; \Phi].$$

The auxiliary functional $\bar{\Gamma}_{AB}[q, q'; \Phi]$ is then given by the integral equation

$$\bar{\Gamma}_{AB}[q, q'; \Phi] = (-\delta_B) \Gamma_{AB}^{int}[q, q'; \Phi] - \int_{q''} \bar{\Gamma}_{CB}[-q'', q'; \Phi] \Delta_{DC}(q''; \Lambda) \Gamma_{CD}^{int}[q, q''; \Phi],$$

(14)

where $\delta_B$ is the ghost number and the indices $A$ and $B$ run over the indices of the fields $A^a$, $c^a$ and $c^a$. In terms of the proper vertices of $\Pi[\Phi; \Lambda]$ the evolution equations are

$$\Lambda \partial_\Lambda \Pi_{C_1\cdots C_n}(p_1, \cdots p_n; \Lambda) = -\frac{i}{2} \int_q M_{BA}(q; \Lambda) \Gamma_{AB,C_1\cdots C_n}(-q, q; p_1, \cdots p_n; \Lambda),$$

where $\Gamma_{AB,C_1\cdots C_n}(q, q'; p_1, \cdots p_n; \Lambda)$ are the vertices of the auxiliary functional $\bar{\Gamma}[\Phi; \Lambda]$ obtained by differentiating with respect to the fields $\Phi$. The vertices of this auxiliary functional are obtained in terms of the proper vertices by expanding (14) and one finds

$$\bar{\Gamma}_{AB,C_1\cdots C_n}(q, q'; p_1, \cdots p_n; \Lambda) = \Gamma_{AB,C_1\cdots C_n}(q, q', p_1, \cdots p_n; \Lambda)$$

$$- \sum' \Gamma_{AC_1\cdots C_{i_k}}(q, p_{i_1}, \cdots, p_{i_k}, Q; \Lambda) \Delta_{C\prime C''}(Q; \Lambda) \bar{\Gamma}_{C\prime B,C_{i_{k+1}}\cdots C_{i_n}}(-Q, q', p_{i_{k+1}}, \cdots p_{i_n}; \Lambda),$$

(15)

where $Q = q + p_{i_1} + \cdots + p_{i_k}$, and $\sum'$ is the sum over the combinations of photon and ghost indices $(i_1 \cdots i_n$) taking properly into account the symmetrization and anti-symmetrization. Notice that this construction involves vertices with $\bar{c}$ external fields, which can be obtained from the relation $\Gamma_{\bar{c} \cdots}(p, \cdots) = -i p_\mu \Gamma_{\bar{c} \mu \cdots}(p, \cdots)$. 

### 3 Relevant parameters and physical conditions

In order to define the boundary conditions for the evolution equation of the cutoff effective action we have to distinguish the relevant and irrelevant vertices. The interaction part $\Pi[\Phi; \Lambda]$ in (13) can be written as

$$\Pi[\Phi; \Lambda] = \Pi_{1,\text{rel}}[\Phi; \sigma_1(\Lambda)] + \Pi_{2,\text{rel}}[\Phi; \rho_1(\Lambda)] + \Gamma_{\text{irrel}}[\Phi; \Lambda],$$

where $\Pi_{1,\text{rel}}$ and $\Pi_{2,\text{rel}}$ are the relevant parts of the interaction, and $\Gamma_{\text{irrel}}$ is the irrelevant part.
where the functional \( \Gamma_{\text{irrel}} \) contains all irrelevant vertices. The relevant part of \( \Pi[\Phi; \Lambda] \) involves only field monomials with dimension not larger than four. Therefore the structure of \( \Pi_{1,2;\text{rel}} \) is the same as the one defined in (9) and (10) for the UV action. Now the couplings \( \sigma_i \) and \( \rho_i \) depend on \( \Lambda \). These nine relevant couplings are defined as the value of some vertex functions or their derivatives at a given normalization point. In order to split the functional \( \Pi[\Phi; \Lambda] \) in its relevant and irrelevant parts, it is then necessary to specify the procedure of extracting the relevant parameters. This is done in full detail in the next subsection.

### 3.1 Relevant couplings and subtraction conditions

In this subsection we define the couplings \( \sigma_i(\Lambda) \) and \( \rho_i(\Lambda) \) as the values at the subtraction points of the vertices with non-negative dimension. We work in the momentum space. The subtraction point for the two point functions is assumed at \( p^2 = \mu^2 \), while for the \( N \) point vertex functions is assumed at the symmetric point (NSP) defined by

\[
\bar{p}_i \bar{p}_j = \frac{\mu^2}{N - 1} (N \delta_{ij} - 1), \quad N = 3, 4, \ldots
\]

The relevant couplings are involved in the following six \( SU(2) \) singlets vertices with

\[
n_A + n_c + 2n_w + 2n_v \leq 4,
\]

where \( n_i \) is the number of fields of the type \( i \).

1) The vector propagator is the coefficient of \( \frac{1}{2} A_\mu(-p) \cdot A_\nu(p) \) and has the form

\[
\Gamma_{\mu\nu}(p; \Lambda) = -g_{\mu\nu} \frac{p^2}{N - 1} K_{\Lambda_\alpha}^{-1}(p) + g_{\mu\nu} \Pi_L(p; \Lambda) + t_{\mu\nu}(p) \Pi_T(p; \Lambda),
\]

with

\[
t_{\mu\nu}(p) \equiv g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}.
\]

The three relevant couplings are defined by

\[
\Pi_L(p; \Lambda) = \sigma_{m_A}(\Lambda) + p^2 \sigma_A(\Lambda) + \Sigma_L(p; \Lambda), \quad \Sigma_L(0; \Lambda) = 0, \quad \frac{\partial \Sigma_L(p; \Lambda)}{\partial p^2} |_{p^2 = \mu^2} = 0,
\]

and

\[
\Pi_T(p; \Lambda) = p^2 \sigma_A(\Lambda) + \Sigma_T(p; \Lambda), \quad \Sigma_T(0; \Lambda) = 0, \quad \frac{\partial \Sigma_T(p; \Lambda)}{\partial p^2} |_{p^2 = \mu^2} = 0.
\]

From these conditions we can factorize in the vertices \( \Sigma_{L,T} \) a dimensional function of \( p \). Thus \( \Sigma_{L,T} \) are “irrelevant” and contribute to \( \Gamma_{\text{irrel}}[\Lambda] \).

2) The contribution to \( \Gamma[\Lambda] \) from three vectors can be written as

\[
A_\mu(p) \cdot A_\nu(q) \wedge A_\rho(k) \Gamma^{(3A)}_{\mu\nu\rho}(p, q, k; \Lambda),
\]

\[
\Gamma^{(3A)}_{\mu\nu\rho}(p, q, k; \Lambda) = p_\mu g_{\mu\nu} \Gamma^{(3A)}(p, q, k; \Lambda) + \tilde{\Gamma}^{(3A)}_{\mu\nu\rho}(p, q, k; \Lambda).
\]
In the last term all three Lorentz indices are carried by external momenta. Thus, after factorizing these momenta, one remains with a function of dimension $-2$. Then the vertex $\tilde{\Gamma}^{(3A)}_{\mu \nu \rho} (p, q, k; \Lambda)$ is irrelevant and contributes to $\Gamma_{irrel}$. The relevant coupling $\sigma_{3A}(\Lambda)$ is defined by

$$\Gamma^{(3A)}(p, q, k; \Lambda) = \sigma_{3A}(\Lambda) + \Sigma^{(3A)}(p, q, k; \Lambda), \quad \Sigma^{(3A)}(p, q, k; \Lambda)|_{3SP} = 0,$$

so that $\Sigma^{(3A)}(p, q, k; \Lambda)$ is irrelevant.

3) The contribution to $\Gamma[\Lambda]$ from four vectors is given by two different $SU(2)$ scalars. The first one involves the field combination which appears in the classical action:

$$\frac{1}{4} [A_\mu(p) \wedge A_\nu(q)] \cdot [A_\rho(k) \wedge A_\sigma(h)] \Gamma^{(4A)}_{1, \mu \nu \rho \sigma}(p, q, k, h; \Lambda),$$

$$\Gamma^{(4A)}_{1, \mu \nu \rho \sigma}(p, q, k, h; \Lambda) = g_{\mu \rho} g_{\nu \sigma} \Gamma^{(4A)}_1(p, q, k, h; \Lambda) + \tilde{\Gamma}^{(4A)}_{1, \mu \nu \rho \sigma}(p, q, k, h; \Lambda). \quad (20)$$

The other singlet structure is given by

$$\frac{1}{8} [A_\mu(p) \cdot A_\nu(q)] [A_\rho(k) \cdot A_\sigma(h)] \Gamma^{(4A)}_{2, \mu \nu \rho \sigma}(p, q, k, h; \Lambda),$$

$$\Gamma^{(4A)}_{2, \mu \nu \rho \sigma}(p, q, k, h; \Lambda) = (2 g_{\mu \rho} g_{\nu \sigma} + g_{\mu \sigma} g_{\nu \rho}) \Gamma^{(4A)}_2(p, q, k, h; \Lambda) + \tilde{\Gamma}^{(4A)}_{2, \mu \nu \rho \sigma}(p, q, k, h; \Lambda). \quad (21)$$

For the two vertices $\bar{\Gamma}^{(4A)}_{1, 2, \mu \nu \rho \sigma}$ at least two Lorentz indices are carried by external momenta. Thus these vertices are irrelevant. The two relevant couplings $\sigma_{4A}(\Lambda)$ and $\rho_{4A}(\Lambda)$ are defined by

$$\Gamma^{(4A)}_1(p, q, k, h; \Lambda) = \sigma_{4A}(\Lambda) + \Sigma^{(4A)}_1(p, q, k, h; \Lambda), \quad \Sigma^{(4A)}_1(p, q, k, h; \Lambda)|_{4SP} = 0,$$

$$\Gamma^{(4A)}_2(p, q, k, h; \Lambda) = \rho_{4A}(\Lambda) + \Sigma^{(4A)}_2(p, q, k, h; \Lambda), \quad \Sigma^{(4A)}_2(p, q, k, h; \Lambda)|_{4SP} = 0.$$

4) The interaction part of the ghost propagator and $u-c$ vertex is given by the same function. The ghost propagator is the coefficient of $\bar{c}(-p) \cdot c(p)$

$$\Gamma^{(uc)}(p; \Lambda) = p^2 K_{\Lambda \Lambda_0}^{-1}(p) + ip^2 \Pi^{(uc)}(p; \Lambda). \quad (22)$$

The coefficient of $u_\mu(-p) \cdot c(p)$ has the form

$$\Gamma^{(uc)}_\mu(p; \Lambda) = -\frac{ip_\mu}{g} + \frac{p_\mu}{g} \Pi^{(uc)}(p; \Lambda). \quad (23)$$

The function $\Pi^{(uc)}$ contains the relevant coupling $\sigma_{uc}(\Lambda)$

$$\Pi^{(uc)}(p; \Lambda) = \sigma_{uc}(\Lambda) + \Sigma^{(uc)}(p; \Lambda), \quad \partial_{p^2} \left(p^2 \Sigma^{(uc)}(p; \Lambda)\right)|_{p^2 = \mu^2} = 0. \quad (24)$$

5) The contribution to $\Gamma[\Lambda]$ from $w-c-A$ is given by the $SU(2)$ scalar

$$w_\mu(p) \cdot c(q) \wedge A_\nu(k) \Gamma^{(wCA)}_{\mu \nu}(p, q, k; \Lambda),$$

$$\Gamma^{(wCA)}_{\mu \nu}(p, q, k; \Lambda) = g_{\mu \nu} \Gamma^{(wCA)}(p, q, k; \Lambda) + \tilde{\Gamma}^{(wCA)}_{\mu \nu}(p, q, k; \Lambda). \quad (25)$$
The two Lorentz indices in $\tilde{\Gamma}^{(wcA)}(p, q, k; \Lambda)$ are carried by external momenta. Thus this vertex is irrelevant. The coupling $\sigma_{wcA}(\Lambda)$ is defined by

$$\Gamma^{(wcA)}(p, q, k; \Lambda) = \sigma_{wcA}(\Lambda) + \Sigma^{(wcA)}(p, q, k; \Lambda), \quad \Sigma^{(wcA)}(p, q, k; \Lambda)|_{3SP} = 0.$$ (26)

Due to the fact that $\Gamma$ depends on $\bar{c}$ and $u$ only through the combination $w_\mu$ one has that the coefficient of $\bar{c}-c-A$ is

$$\Gamma^{(\bar{c}cA)}(\bar{c}, c, A) = \frac{i p_\mu}{\Gamma^{(wcA)}(p, q, k; \Lambda)}.$$ (27)

6) The only vertex involving the source $v$ which contains a relevant coupling is the coefficient of $\frac{1}{2}v(p) \cdot c(q) \wedge c(k)$. The coupling $\rho_{vcc}$ is given by

$$\Gamma^{(vcc)}(p, q, k; \Lambda) = -1 + \rho^{(vcc)}(\Lambda) + \Sigma^{(vcc)}(p, q, k; \Lambda), \quad \Sigma^{(vcc)}(p, q, k; \Lambda)|_{3SP} = 0.$$ (28)

All the remaining vertices are irrelevant, since they are coefficients of monomials with dimension higher than four.

### 3.2 Boundary conditions

As discussed in the introduction for the irrelevant vertices we assume the following boundary condition

$$\Gamma_{irr}[\Phi, \Lambda_0] = 0,$$ (29)

since, due to dimensional reasons, they vanish at the UV scale. For $\Lambda = \Lambda_0$ the cutoff effective action becomes then local and corresponds to the bare action (11), where the nine bare parameters $\sigma^B_i$ and $\rho^B_i$ are given by $\sigma_i(\Lambda_0)$ and $\rho_i(\Lambda_0)$. However, in our procedure these couplings are fixed at $\Lambda = 0$ and their value at $\Lambda = \Lambda_0$ can be perturbatively computed.

For the relevant part, we first give the boundary conditions for the terms which appear also in the classical action. For them we impose

$$\Pi_{1rel}[\Phi; \sigma_i(\Lambda = 0)] = S_{int},$$

which gives

$$\sigma_{mA}(0) = \sigma_A(0) = \sigma_{wc}(0) = 0,$$

$$\sigma_{3A}(0) = ig, \quad \sigma_{4A}(0) = -g^2, \quad \sigma_{wcA}(0) = -g.$$ (30)

The remaining two couplings $\rho_{4A}(0)$ and $\rho_{vcc}(0)$ are fixed by the ST identities and given in terms of some irrelevant vertices as follows.

The physical effective action, namely $\Gamma[\Phi; \Lambda = 0]$, must satisfy the ST identities (in the rest of this subsection it will be understood that $\Lambda = 0$ and $\Lambda_0 \to \infty$). Since, as discussed, the effective action depends on $\bar{c}$ and $u_\mu$ fields only through the combination $w_\mu$, the Slavnov-Taylor identities are

$$\int_s \left\{ \frac{\delta \Gamma'}{\delta u_\mu(s)} \frac{\delta \Gamma'}{\delta A_\mu^a(-s)} + \frac{\delta \Gamma'}{\delta v^a(s)} \frac{\delta \Gamma'}{\delta c^a(-s)} \right\} = 0,$$ (31)
where $\Gamma'$ is obtained from the effective action $\Gamma$ by subtracting the gauge fixing contribution, i.e. the second term in (4).

These relations, due to their non-linearity, couple the relevant and irrelevant vertices. Since the total action $S_{BRS}$ (without the gauge fixing term) satisfies these identities, only the relevant couplings $\rho_{4A}(0)$ and $\rho_{vcc}(0)$ are coupled to irrelevant vertices. We now determine the relations giving the couplings $\rho_{4A}(0)$ and $\rho_{vcc}(0)$.

To find the relation for $\rho_{vcc}(0)$ we use the simplest identity involving the $v-c-c$ vertex. This is obtained by differentiating (30) with respect to the fields $c^a(p)$, $c^b(q)$ and $\bar{c}^c(k)$ and setting the fields to zero. We obtain

$$\Gamma^{(wc)}(p) \Gamma^{(\bar{c}A)}(k, q, p) + \Gamma^{(wA)}(q) \Gamma^{(\bar{c}A)}(k, p, q) + \Gamma^{(\bar{c}c)}(k) \Gamma^{(vcc)}(k, p, q) = 0.$$  \hspace{1cm} (31)

By taking $p$, $q$ and $k$ at the symmetric point and using (22), (23) and (27) we obtain

$$\rho_{vcc}(0) = -\frac{2}{g} \left[ \frac{k_\mu p_\nu}{p^2} \tilde{\Gamma}^{(wA)}(k, q, p) \right]_{3SP},$$  \hspace{1cm} (32)

where we used $\Gamma^{(vcc)}(p, q, k)|_{3SP} = -1 + \rho_{vcc}(0)$ and $\Gamma^{(wA)}(p, q, k)|_{3SP} = \sigma_{wA}(0) = -g$.

We discuss now the determination of value of $\rho_{4A}(0)$. From the ST identities for three and four vector vertices, one deduces

$$\left[ p_\mu q_\nu k_\rho h_\sigma \Gamma^{(4A)}_{2,\mu\nu\rho\sigma}(p, q, k, h) \right]_{4SP} = 0,$$

which gives

$$\rho_{4A}(0) = -\left[ \frac{3}{p^4} p_\mu q_\nu k_\rho h_\sigma \tilde{\Gamma}^{(4A)}_{2,\mu\nu\rho\sigma}(p, q, k, h) \right]_{4SP}.$$  \hspace{1cm} (33)

## 4 Integral equation

Once the boundary conditions (28), (29), (32) and (33) are given, the RG equation (12) allows one to compute all vertices, at least perturbatively. The nine relevant parameters are fixed to satisfy the ST identities by definition and this avoids the fine tuning problem. The couplings $\sigma_i(\Lambda_0)$ and $\rho_i(\Lambda_0)$ can be computed as a function of $g$ and $\Lambda_0/\mu$ and they will be already fine tuned.

The boundary conditions we use are of the mixed type: the irrelevant vertices are fixed at $\Lambda = \Lambda_0$, the relevant couplings at $\Lambda = 0$. Moreover, two of the relevant couplings are given in terms of irrelevant vertices at the physical point $\Lambda = 0$. This kind of boundary conditions does not allow to solve the RG equation as an evolution equation, as in the usual case in which one fixes both irrelevant and relevant vertices at $\Lambda = \Lambda_0$. In our case the best way to formulate the problem is to convert the RG equation (12) into a Volterra integral equation which embodies the boundary conditions. As we shall show this formulation is also quite suitable for generating the iterative solution, i.e. loop expansion, and for proving renormalizability. Moreover, it could be used in principle also for nonperturbative studies.
Finally, by using the form of $\Pi_{\text{rel}}$, we consider the following vertices of $I_{\Phi; \Lambda}$:

$$I_{C_1\cdots C_n}(p_1, \cdots, p_n; \Lambda) = -\frac{i}{2} \int_q MBA(q; \Lambda) \tilde{\Gamma}_{AB,C_1\cdots C_n}(-q, q; p_1, \cdots, p_n; \Lambda).$$

(34)

The functional $I[\Phi; \Lambda]$ can be separated into its relevant and irrelevant part

$$I[\Phi; \Lambda] = I_{\text{rel}}[\Phi; \Lambda] + I_{\text{irr}}[\Phi; \Lambda].$$

To extract the relevant parameters in $I_{\text{rel}}[\Phi; \Lambda]$ we proceed as in subsect. 3.1. For example from the following vertices of $I_{\Phi; \Lambda}$

$$I_{\mu \nu}(p; \Lambda) = g_{\mu \nu} I_L(p; \Lambda) + \tau_{\mu \nu} I_T(p; \Lambda),$$

$$I_{\text{wcA}}(p, q, k; \Lambda) = g_{\mu \nu} I_{\text{wcA}}(p, q, k; \Lambda) + \tilde{I}_{\text{wcA}}(p, q, k; \Lambda),$$

$$I_{2,\mu \nu \rho \sigma}(p, q, k, h; \Lambda) = (2g_{\mu \rho}g_{\nu \sigma} + g_{\mu \sigma}g_{\nu \rho}) I_{2,\text{wcA}}(p, q, k, h; \Lambda) + \tilde{I}_{2,\mu \nu \rho \sigma}(p, q, k, h; \Lambda),$$

we extract the relevant parameters

$$I_L(p; \Lambda) = I_L(0; \Lambda) + p^2 \partial_{\rho} I_L(p; \Lambda) + \cdots \equiv S_m(\Lambda) + p^2 S_A(\Lambda) + \cdots,$$

$$I_T(p; \Lambda) = p^2 \partial_{\rho} I_T(p; \Lambda) + \cdots \equiv p^2 S_A(\Lambda) + \cdots,$$

$$I_{\text{wcA}}(p, q, k; \Lambda) = I_{\text{wcA}}(p, q, k; \Lambda)|_{3SP} + \cdots \equiv S_{\text{wcA}}(\Lambda) + \cdots,$$

$$I_{2,\text{wcA}}(p, q, k, h; \Lambda) = I_{2,\text{wcA}}(p, q, k, h; \Lambda)|_{4SP} + \cdots \equiv R_4(\Lambda) + \cdots,$$

where the dots stand for irrelevant parts. The relevant part of the functional $I[\Phi; \Lambda]$ can be again written in terms of $\Pi_{1,2,\text{rel}}$

$$I_{\text{rel}}[\Phi; \Lambda] = \Pi_{1,\text{rel}}[\Phi; S_i(\Lambda)] + \Pi_{2,\text{rel}}[\Phi; R_i(\Lambda)].$$

The integral equations for the relevant and irrelevant parts are then

$$\Pi_{\text{rel}}[\Phi; \Lambda] = \Pi_{\text{rel}}[\Phi; 0] + \int_0^\Lambda \frac{d\lambda}{\lambda} I_{\text{rel}}[\Phi; \lambda],$$

$$\Pi_{\text{irr}}[\Phi; \Lambda] = -\int_\Lambda^{\Lambda_0} \frac{d\lambda}{\lambda} I_{\text{irr}}[\Phi; \lambda].$$

Finally, by using the form of $\Pi_{\text{rel}}[\Phi; 0]$, we have

$$\Pi[\Phi; \Lambda] = S_{\text{int}}[\Phi] + \Pi_{2,\text{rel}}[\Phi; \rho_i(0)] + \int_0^\Lambda \frac{d\lambda}{\lambda} I[\Phi; \lambda] - \int_0^\infty \frac{d\lambda}{\lambda} \{I[\Phi; \lambda] - I_{\text{rel}}[\Phi; \lambda]\},$$

(38)

where $S_{\text{int}}[\Phi]$ is the interacting part of the BRS action (3). $\Pi_{2,\text{rel}}[\Phi; \rho_i(0)]$ contains the two additional relevant couplings fixed by the ST constraint and given by

$$\rho_{\text{exc}}(0) = \frac{2}{g} \int_0^\infty \frac{d\lambda}{\lambda} \left[ \frac{k^\mu p^\nu}{p^2} \tilde{I}_{\text{wcA}}(k, q; p; \lambda) \right]_{3SP},$$

$$\rho_{1A}(0) = \frac{3}{g} \int_0^\infty \frac{d\lambda}{\lambda} \left[ \frac{p^\mu q^\nu k^\rho h_{\sigma}}{p^4} \tilde{I}_{2,\mu \nu \rho \sigma}(p, q, k, h; \lambda) \right]_{4SP},$$

(39)
where $\tilde{I}_{\mu\nu}^{(wCA)}$ and $\tilde{I}_{2\mu\nu\rho\sigma}^{(4A)}$ are given in (34).

In the last $\lambda$ integration in (38) we have sent $\Lambda_0 \to \infty$. The proof that the subtractions make all integrals finite in the $\Lambda_0 \to \infty$ limit is sketched in subsect. 5.4. In the next section, where we compute at one loop some vertex functions, we will see how the subtractions make all integrals finite in the $\Lambda_0 \to \infty$ limit. Only for $\Lambda = 0$ the total effective action $\Gamma[\Phi] = S_2 + \Pi[\Phi; 0]$ must satisfy the ST identities. This will be analyzed in the next section at one loop level.

5 Loop expansion

The loop expansion is obtained by solving iteratively (38). The starting point is the zero loop order $\Pi^{(0)}[\Phi; \sigma_i(0)] = S_{int}[\Phi]$. This corresponds to give the relevant vertices in the tree approximation, i.e.

$$\sigma^{(0)}_{m_A}(\Lambda) = \sigma^{(0)}_{\alpha}(\Lambda) = \sigma^{(0)}_{wC}(\Lambda) = 0,$$

$$\sigma^{(0)}_{3A}(\Lambda) = ig, \quad \sigma^{(0)}_{4A}(\Lambda) = -g^2, \quad \sigma^{(0)}_{wCA}(\Lambda) = -g,$$

$$\rho^{(0)}_{4A}(\Lambda) = \rho^{(0)}_{wCC}(\Lambda) = 0.$$

At zero loop we have also the auxiliary vertices in $\bar{\Gamma}^{(0)}[\Phi; \Lambda]$. For example we have the vector-vector contributions to order $g^2$ given in fig. 2. They correspond to the usual Feynman diagrams at tree level with cutoff propagators.

Assume we have computed the cutoff effective action $\Pi^{(\ell)}[\Phi; \Lambda]$ up to loop $\ell$. At the next loop this functional can be computed iteratively by the following steps. First from eq. (34) we compute the vertices $I_{C_1 \cdots C_n}^{(\ell+1)}(p_1, \cdots p_n; \lambda)$. From eq. (35) and eq. (39) we obtain the vertices $\tilde{I}_{\mu\nu}^{(wCA)(\ell+1)}(\Lambda)$ and $\tilde{I}_{2\mu\nu\rho\sigma}^{(4A)(\ell+1)}(\Lambda)$ and the couplings $\rho^{(\ell+1)}_{wC}(0)$ and $\rho^{(\ell+1)}_{4A}(0)$. We then obtain the functionals $I^{(\ell+1)}[\Phi; \Lambda]$, $I^{(\ell+1)}_{rel}[\Phi; \Lambda]$ and $\Pi^{(\ell+1)}_{2,rel}[\Phi; \rho_i(0)]$. Finally from eq. (38) we compute the functional $\Pi^{(\ell+1)}[\Phi; \Lambda].$

5.1 One loop vertex functions

In this subsection we compute to one loop the six vertices with $n_A + n_c + 2n_w + 2n_v \leq 4$ which contain the relevant couplings. We also evaluate to one loop the couplings $\sigma_i B$ and $\rho_i B$ of the bare action (11) given by $\sigma_i(\Lambda_0)$ and $\rho_i(\Lambda_0)$.

1) Vector propagator.

We have three contributions to the integrand, corresponding to the three usual Feynman amplitudes

$$I_{\mu\nu}(p; \Lambda) = -\frac{ig^2}{3} \Lambda \partial_\Lambda \int_{q} \frac{K_{\Lambda\Lambda_0}(q)}{q^2} j^{abba}_{\mu\alpha\beta\gamma}$$

$$- ig^2 \Lambda \partial_\Lambda \int_{q} \frac{K_{\Lambda\Lambda_0}(q,p,q)}{q^2(p+q)^2} \left[ V_{\mu\alpha\beta}(p,q,-p-q) V_{\alpha\beta\gamma}(-q,p+q,-p) + q_\mu(p+q)_\nu \right],$$

(40)
where \( K_{\Lambda_0}(q, p + q) = K_{\Lambda_0}(p + q) \),

\[
V_{\mu \nu \rho}(p, q, k) = g_{\mu \nu}(p - q) \rho + g_{\nu \rho}(q - k) \mu + g_{\mu \rho}(k - p) \nu ,
\]

\[
i_{\mu_1 \cdots \mu_4}^{a_1 \cdots a_4} = \left( \epsilon_{a_1 a_2 c} \epsilon_{a_3 a_4} - \epsilon_{a_1 a_4 c} \epsilon_{a_2 a_3} \right) g_{\mu_1 \mu_3} g_{\mu_2 \mu_4} + \left( \epsilon_{a_1 a_2 c} \epsilon_{a_3 a_4} - \epsilon_{a_1 a_4 c} \epsilon_{a_2 a_3} \right) g_{\mu_1 \mu_4} g_{\mu_2 \mu_3} ,
\]

which are the three and four vector elementary vertices, respectively. Notice that there is a factor \( i \) coming from the \( q \)-integration. Thus we get

\[
I_L(p; \Lambda) = -ig^2 \Lambda \partial_\Lambda \left[ \int_q \frac{K_{\Lambda_0}(q, p + q)}{q^2(p + q)^2} \left( 2q^2 + 10pq + 3p^2 + 8 \frac{(pq)^2}{p^2} \right) - 6 \int_q \frac{K_{\Lambda_0}(q)}{q^2} \right],
\]

\[
I_T(p; \Lambda) = -\frac{ig^2}{3} \Lambda \partial_\Lambda \left[ \int_q \frac{K_{\Lambda_0}(q, p + q)}{q^2(p + q)^2} \left( 8q^2 - 24pq + 6p^2 - 32 \frac{(pq)^2}{p^2} \right) \right].
\]

The relevant couplings for large \( \Lambda \) are

\[
\sigma^{(1)}_{m_\Lambda}(\Lambda) = \int_0^\Lambda \frac{d\lambda}{\lambda} I_L(0; \lambda) = ig^2 \int_q \frac{K_{\Lambda_0}(q)}{q^4} \left( 8 \frac{(pq)^2}{p^2} - 4q^2 \right) = \frac{g^2}{8\pi^2} \Lambda^2 + O(1) .
\]

\[
\sigma^{(1)}_\Lambda(\Lambda) = \int_0^\Lambda \frac{d\lambda}{\lambda} \frac{\partial}{\partial p^2} I_L(\bar{p}; \lambda) = O(1) ,
\]

\[
\sigma^{(1)}_A(\Lambda) = \int_0^\Lambda \frac{d\lambda}{\lambda} \frac{\partial}{\partial p^2} I_T(\bar{p}; \lambda) = -\frac{10}{3} \frac{g^2}{16\pi^2} \log(\frac{\Lambda^2}{\mu^2}) + O(1) .
\]

From (37) the irrelevant parts of the vector propagator are

\[
\Sigma_{L,T}(p; \Lambda) = - \int_\Lambda \frac{d\lambda}{\lambda} \left[ I_{L,T}(p; \lambda) - I_{L,T}(0; \lambda) - p^2 \frac{\partial}{\partial p^2} I_{L,T}(\bar{p}; \lambda) \right].
\]

Because of the subtractions, we can take the \( \Lambda_0 \to \infty \) limit. The physical value is obtained setting \( \Lambda = 0 \) and we have

\[
\Sigma_L(p; 0) = -ig^2 \int_q \left[ \frac{2q^2 + 10pq + 3p^2 + 8 \frac{(pq)^2}{p^2}}{q^2(p + q)^2} - \frac{4}{q^4} - p^2 \frac{\partial}{\partial p^2} \left( 2q^2 + 10pq + 3p^2 + 8 \frac{(pq)^2}{p^2} \right) \right] ,
\]

\[
\Sigma_T(p; 0) = -\frac{ig^2}{3} \int_q \left[ \frac{8q^2 - 24pq + 6p^2 - 32 \frac{(pq)^2}{p^2}}{q^2(p + q)^2} - p^2 \frac{\partial}{\partial p^2} \left( 8q^2 - 24pq + 6p^2 - 32 \frac{(pq)^2}{p^2} \right) \right] .
\]

In subsect. 5.3 we will show that \( \Sigma_L(p; 0) = 0 \), namely the vector propagator is transverse, as required by the ST identities. For \( \Sigma_T(p; 0) \), for instance, we can write

\[
\frac{\partial}{\partial p^2} \Sigma_T(p; 0) = -\frac{10}{3} \frac{g^2}{16\pi^2} \log(\frac{p^2}{\mu^2}) .
\]

2) Ghost propagator.

The ghost propagator is given in terms of \( \Pi^{(w)} \). We have one contribution

\[
p_\mu I^{(w)}(p; \Lambda) = -2g^2 \Lambda \partial_\Lambda \left[ \int_q \frac{K_{\Lambda_0}(q, p + q)}{q^2(p + q)^2} q_\mu \right].
\]
We obtain then the ghost wave function coupling

\[ \sigma_{wc}(\Lambda) = \int_{\bar{p}} \delta \left( \frac{\Lambda}{\partial \bar{p}} \right) \left( \bar{p}^2 I^{(wc)}(\bar{p}; \lambda) \right) = -i \frac{g^2}{16\pi^2} \log(\frac{\Lambda^2}{\mu^2}) + \mathcal{O}(1) , \] (45)

for large \( \Lambda \). From (37) the irrelevant part of the \( w-c \) vertex is, at \( \Lambda = 0 \)

\[ p^2 \Sigma^{(wc)}(p; 0) = -2g^2 \int_q \left[ \frac{pq}{q^2(p+q)^2} - p^2 \frac{\partial}{\partial p^2} \frac{\bar{p}q}{q^2(p+q)^2} \right] , \]

where again we have removed the UV cutoff since the momentum integration is convergent.

3) \( w-c-A \) vertex.

In this case the integrand has two contributions

\[ I^{(wcA)}_{\mu\nu}(p, k, -p - k; \Lambda) = \Lambda \partial_\Lambda \left\{ i g^3 \int_q \frac{K_{\Lambda\Lambda\Lambda\lambda}(q, q + k, p - q)}{q^2(q + k)^2(p - q)^2} \right. \]

\[ \left. \times [g_{\mu\nu}(q^2 - qk - 2qp) - q_\mu k_\nu + q_\nu (3k_\mu + p_\mu) - p_\nu k_\mu] \right\} . \] (46)

The coupling \( \sigma_{wcA}(\Lambda) \) for large \( \Lambda \) is

\[ \sigma_{wcA}^{(1)}(\Lambda) = \int_{\bar{p}} \delta \left( \frac{\Lambda}{\partial \bar{p}} \right) I^{(wcA)}(\bar{p}, k; \lambda) \big|_{3\text{SP}} = \frac{g^3}{16\pi^2} \log(\frac{\Lambda^2}{\mu^2}) + \mathcal{O}(1) . \] (47)

This vertex allows one to obtain the coupling \( \rho_{vcc}(0) \). Extracting from (46) the vertex \( \bar{I}_{\mu\nu}^{(wcA)} \) and inserting it in (39) we find

\[ \rho_{vcc}^{(1)}(0) = -i g^2 \int_q \left[ \frac{pk + 2qp + qk}{q^2(q + k)^2(p - q)^2} \right] \big|_{3\text{SP}} \simeq 4.6878 \frac{g^2}{16\pi^2} , \] (48)

where the value is found by numerical integration over the Feynman parameters.

4) \( v-c-c \) vertex.

In this case the integrand has one contribution and we have

\[ I^{(vec)}(p, k, -p - k; \Lambda) = -i g^2 \Lambda \partial_\lambda \left[ \int_q \frac{K_{\Lambda\lambda\lambda\lambda}(q, q + k, p - q)}{q^2(q + k)^2(p - q)^2} (pq - q^2) \right] . \] (49)

The relevant coupling for large \( \Lambda \) is

\[ \rho_{vcc}^{(1)}(\Lambda) = \int_{\bar{p}} \delta \left( \frac{\Lambda}{\partial \bar{p}} \right) I^{(vec)}(\bar{p}; \lambda) = \frac{g^2}{16\pi^2} \log(\frac{\Lambda^2}{\mu^2}) + \mathcal{O}(1) . \] (50)

5) Three vector vertex.

The integrand for the three vector vertex is too long and we give only the result for the coupling for large \( \Lambda \)

\[ \sigma_{3A}^{(1)}(\Lambda) = i \frac{4}{3} \frac{g^3}{16\pi^2} \log(\frac{\Lambda^2}{\mu^2}) + \mathcal{O}(1) . \] (51)
6) Four vector vertex.

Also for this vertex the integrand is too long and we give only the values for the two couplings for large \( \Lambda \)

\[
\sigma_{4A}^{(1)}(\Lambda) = \frac{2}{3} \frac{g^4}{16\pi^2} \log(\frac{\Lambda^2}{\mu^2}) + O(1),
\]

\[
\rho_{4A}^{(1)}(\Lambda) = -10 \frac{g^4}{16\pi^2} \log(\frac{\Lambda^2}{\mu^2}) + O(1).
\]

From this vertex we obtain the coupling \( \rho_{4A}(0) \) given in (39). From a numerical integration over the Feynman parameters we find

\[
\rho_{4A}^{(1)}(0) \simeq 1.2329 \frac{g^2}{16\pi^2}.
\]

### 5.2 One loop beta function

We discuss in the present formulation the role of the subtraction point \( \mu \) and deduce the beta function (see also ref. [14] for the scalar case). In this subsection we take the physical values \( \Lambda = 0 \) and \( \Lambda_0 \to \infty \). Denote by \( \Gamma_{nA,mwc}(g,\mu) \) the vertices with \( n \) vector fields and \( m \) pairs \( w_\mu c \), which satisfy the physical conditions [29] with couplings \( \sigma_i \) and \( \rho_i \) defined as in subsect. 3.1 at the scale \( \mu \). Therefore \( \mu \) is the only dimensional parameter, apart for the external momenta. However physical measurements should not depend on the specific value of \( \mu \). Consider the vertices \( \Gamma_{nA,mwc}(g',\mu') \) with coupling \( g' \) at a new scale \( \mu' \). The request that the two sets of vertices \( \Gamma_{nA,mwc}(g,\mu) \) and \( \Gamma_{nA,mwc}(g',\mu') \) describe the same theory implies that the corresponding effective actions \( \Gamma[\Phi; g, \mu] \) and \( \Gamma[\Phi'; g', \mu'] \) are equal, where the fields in \( \Phi \) and \( \Phi' \) are related by a rescaling, \( A_\mu'(p) = \sqrt{Z_A} A_\mu(p) \) and \( w'_\mu(q)c'(p) = Zcw_\mu(q)c(p) \). This implies that the two sets of vertices are related by

\[
\Gamma_{nA,mwc}(g',\mu') = Z_A^{-n/2} Z_c^{-m} \Gamma_{nA,mwc}(g,\mu).
\]

To obtain \( Z_A \) we use (16)-(18), which give

\[
Z_A = 1 - \frac{\partial}{\partial p^2} \Sigma_T(p; g, \mu)|_{p^2=\mu^2} \equiv 1 - f_A(\mu'; g, \mu).
\]

Similarly from (24) we find

\[
Z_c = 1 + i \frac{\partial}{\partial p^2} \left[ p^2 \Sigma^{(wc)}(p; g, \mu) \right]|_{p^2=\mu^2} \equiv 1 + f_c(\mu'; g, \mu).
\]

We can now give the expression of the beta function. By using the physical condition (29) for the \( w-c-A \) vertex and the relation \( \Gamma_{\alpha\beta}^{(wcA)}(g',\mu') = Z_A^{-1/2} Z_c^{-1} \Gamma_{\alpha\beta}^{(wcA)}(g,\mu) \) we obtain the renormalization group relation

\[
g' = \frac{g - f_g(\mu'; g, \mu)}{(1 + f_c(\mu'; g, \mu))(1 - f_A(\mu'; g, \mu))^{1/2}},
\]
where \( f_g(\mu'; g, \mu) = \Sigma^{(\text{uca})}(p, q, k; g, \mu)_{3SP'} \). The beta function is obtained by considering an infinitesimal scale change and is given by

\[
\beta(g) = \mu' \frac{\partial}{\partial \mu'} \left\{ -f_g(\mu'; g, \mu) - g f_c(\mu'; g, \mu) + \frac{1}{2} g f_A(\mu'; g, \mu) \right\} |_{\mu' = \mu}.
\]

The three dimensionless quantities \( f_i(\mu'; g, \mu) \) are functions of \( g \) and the ratio \( \mu'/\mu \), thus the beta function depends only on \( g \). From the above results we find, at one loop order,

\[
f_A = -\frac{10}{3} \frac{g^2}{16\pi^2} \log\left(\frac{\mu'^2}{\mu^2}\right), \quad f_c = \frac{g^2}{16\pi^2} \log\left(\frac{\mu'^2}{\mu^2}\right), \quad f_g = \frac{g^3}{16\pi^2} \log\left(\frac{\mu'^2}{\mu^2}\right).
\]

Thus

\[
\beta^{(1)}(g) = -\frac{11}{3} \frac{g^2}{16\pi^2} g^3 \mu' \frac{\partial}{\partial \mu'} \ln \frac{\mu'^2}{\mu^2} = -\frac{22}{3} \frac{g^3}{16\pi^2},
\]

which is the usual one loop result.

Since there is only one coupling, the beta function can be defined also from the three and four vector vertex. One finds

\[
\beta(g) = \mu' \frac{\partial}{\partial \mu'} \left\{ -if_{3A}(\mu'; g, \mu) + \frac{3}{2} g f_A(\mu'; g, \mu) \right\} |_{\mu' = \mu},
\]

and

\[
\beta(g) = \mu' \frac{\partial}{\partial \mu'} \left\{ -\frac{1}{2} g f_{4A}(\mu'; g, \mu) + g f_A(\mu'; g, \mu) \right\} |_{\mu' = \mu},
\]

where \( f_{3A}(\mu'; g, \mu) = \Gamma^{(3A)}(p; g, \mu)_{3SP'} \) and \( f_{4A}(\mu'; g, \mu) = \Gamma^{(4A)}_{1}(p; g, \mu)_{4SP'} \). At one loop they are given by

\[
f_{3A} = i \frac{4}{3} \frac{g^3}{16\pi^2} \log\left(\frac{\mu'^2}{\mu^2}\right), \quad f_{4A} = \frac{2}{3} \frac{g^4}{16\pi^2} \log\left(\frac{\mu'^2}{\mu^2}\right).
\]

We have then that these definitions give the same one loop value for the beta function.

### 5.3 One loop ST identities

In this subsection we show how two simple ST identities are satisfied at one loop order in the physical limit. Thus in this subsection we set \( \Lambda = 0 \) and we will take \( \Lambda_0 \to \infty \). The first one is the vector propagator transversality and is obtained from (30) by derivating with respect to \( c^a(p) \) and \( A^b_\mu(q) \). We get

\[
\Gamma^{(\text{uc})}_{\mu}(p) [p_\mu p_\nu + \Gamma_{\mu\nu}(p)] = 0,
\]

which implies \( \Sigma_L(p) = 0 \). To show this at one loop, we consider first the unsubtracted integral

\[
\Sigma'_L(p) = -\int_0^{\Lambda_0} \frac{d\lambda}{\lambda} I_L(p; \lambda).
\]

From (11), after some algebra, we find

\[
\Sigma'_L(p) = -i \int_q \frac{K_{0\Lambda_0}(q)}{q^2} \left[ \frac{4pq}{p^2} (K_{0\Lambda_0}(q + p) - K_{0\Lambda_0}(q - p)) + 6 (K_{0\Lambda_0}(q + p) - K_{0\Lambda_0}(q)) \right].
\]
Due to the difference of the two cutoff functions we have that $q^2$ is forced into the region $q^2 \sim \Lambda_0^2$. By taking for instance an exponential UV cutoff one has

$$K_{0\Lambda_0}(q + p) - K_{0\Lambda_0}(q - p) = -4\frac{p \cdot q}{\Lambda_0^2} \left[ 1 - \frac{p^2}{\Lambda_0^2} + \frac{2(\hat{p} \cdot q)^2}{3\Lambda_0^4} + \cdots \right] e^{-q^2/\Lambda_0^2}$$

and obtains

$$\Sigma'_L(p) = a\Lambda_0^2 + bp^2 + O\left(\frac{p^2}{\Lambda_0^2}\right),$$

where $a$ and $b$ are numerical constants. In this calculation the effect of the non invariant regularization is clear. The divergent integral with the cutoff functions gives a surface term which destroys the transversality of the propagator. However, the longitudinal contributions are of relevant type, thus they are cancelled by imposing the boundary conditions and taking the limit $\Lambda_0 \to \infty$. One finds

$$\Sigma^{(1)}_L(p) = \Sigma'_L(p) - \Sigma'_L(0) - p^2\partial_{\hat{p}^2}\Sigma'_L(\hat{p})|_{\hat{p}^2 = \mu^2} = O\left(\frac{p^2, \mu^2}{\Lambda_0^2}\right) \to 0.$$

Notice that for $\Lambda \neq 0$ the longitudinal part of the photon propagator is different from zero both in the relevant ($\sigma_m (\Lambda)$ and $\sigma_\alpha (\Lambda)$) and irrelevant ($\Sigma_L(p; \Lambda)$) parts.

We now verify the identity (31). At one loop order it becomes

$$\int_{0}^{\Lambda_0} \frac{d\lambda}{\lambda} \left[ p_\mu k_\mu \left( I^{(\text{wc})}_{\mu\nu}(k, h; \lambda, \lambda) - g_{\mu\nu} I^{(\text{wc})}_{\text{rel}}(\mu^2; \lambda) \right) + h \leftrightarrow p \right]$$

$$-i \int_{0}^{\Lambda_0} \frac{d\lambda}{\lambda} \left[ k p \left( I^{(\text{wc})}(p, \lambda) - I^{(\text{wc})}_{\text{rel}}(\mu^2; \lambda) \right) + p \to h + p \to k \right]$$

$$+ k^2 \int_{0}^{\Lambda_0} \frac{d\lambda}{\lambda} \left( I^{(\text{vcc})}(k, p, h; \lambda) - I^{(\text{vcc})}_{\text{rel}}(\mu^2; \lambda) \right) - k^2 \rho_{\text{vcc}}(0) \to 0 \quad \text{for} \ \Lambda_0 \to \infty.$$

As above we keep the UV cutoff $\Lambda_0$, so that we can consider separately the integrands $I$ and their subtractions $I_{\text{rel}}$. From (14), (16) and (13) the unsubtracted contribution can be cast in the form

$$ig^2 k_\mu \left\{ \int_{0} k_\mu \frac{K_{0\Lambda_0}(q, q + k)}{q^2(q + k)^2} [K_{0\Lambda_0}(q - h) - 1] + (k \to h, h \to k) + (k \to p, h \to h) \right\}.$$

As in the previous case, the factor $(K_{0\Lambda_0} - 1)$ forces the integration momentum to be of the order $q^2 \sim \Lambda_0^2$ so that the integral vanishes for $\Lambda_0 \to \infty$. It is easy to verify that also the remaining terms vanish for $\Lambda_0 \to \infty$.

### 5.4 Perturbative renormalizability

Perturbative renormalization is essentially based on power counting. To prove that the theory is perturbatively renormalizable one has to show that the integral equations give a finite result in the limit $\Lambda_0 \to \infty$. This can be done perturbatively by iterating eqs. (36) and (37). In order to see that the integrations over $\lambda$ are convergent, we have to estimate
the behaviour of the integrands for large $\lambda$. The analysis can be simplified, following Polchinski, by introducing the norm

$$|\Gamma(n_A, n_{\bar{c}c}, n_u, n_v)|_\Lambda \equiv \text{Max}_{p_i^2 \leq \epsilon x^2} |\Gamma_{C_1 \cdots C_n}(p_1, \cdots p_n; \Lambda)|,$$

which allows us to ignore the momentum dependence. Since the $\Lambda$-dependence is fixed only by the number of fields, to simplify the notation, we have indicated in the vertices only the numbers $n_A$ of vectors, $n_{\bar{c}c}$ of ghost-antighost pairs, $n_u$ and $n_v$ of $u$ and $v$ sources. We deduce perturbative renormalizability by proving that the $\Lambda$ dependence of this norm is given by power counting. We show this by induction on the number of loops. Namely, at loop $\ell$, we assume for large $\Lambda$ the following behaviour

$$|\Gamma^{(\ell)}(n_A, n_{\bar{c}c}, n_u, n_v)|_\Lambda = \mathcal{O}(\Lambda^{4-n_A-2n_{\bar{c}c}-2n_u-2n_v}), \quad (53)$$

which is satisfied at $\ell = 0$. We neglect for simplicity all possible $\ell$-dependent powers of $\log \frac{\Lambda}{\mu}$.

We now proceed by iteration and prove that the behaviours (53) are reproduced at the loop $\ell + 1$. We follow the method of ref. [8, 9]. First of all one notices that the norm of the auxiliary vertices have the same behaviours as the corresponding vertices, as can be seen from their definition. We have then

$$|\Gamma_A^{(\ell)}(-q, q; n_A, n_{\bar{c}c}, n_u, n_v; \Lambda)|_\Lambda = \mathcal{O}(\Lambda^{2-n_A-2n_{\bar{c}c}-2n_u-2n_v}). \quad (54)$$

From this behaviour one obtains the bound for the integrands (54) of the vertices at loop $\ell + 1$

$$|I^{(\ell+1)}(n_A, n_{\bar{c}c}, n_u, n_v)|_\Lambda = \mathcal{O}(\Lambda^{4-n_A-2n_{\bar{c}c}-2n_u-2n_v}), \quad (55)$$

where one uses the fact that $M_{AB}$ gives a factor $\Lambda^{-2}$. Moreover, the derivative of the integrands $I^{(\ell+1)}$ with respect to the external momenta reduces the powers of $\Lambda$

$$|\partial^m p I^{(\ell+1)}(n_A, n_{\bar{c}c}, n_u, n_v)|_\Lambda = \mathcal{O}(\Lambda^{4-n_A-2n_{\bar{c}c}-2n_u-2n_v-m}). \quad (56)$$

The proof of this is given for instance in ref. [8].

Finally, from these behaviours to obtain the vertices at loop $\ell + 1$. For the relevant couplings the integrand in (56) grows with $\lambda$ and the result is dominated by the upper limit $\Lambda$. This reproduces immediately (53).

For the irrelevant vertices, which are given by (57), the $\lambda$-integration goes up to infinity. We treat separately the irrelevant vertices with negative and non-negative dimension. For the first ones the integration over $\lambda$ is convergent and the integral is dominated by the lower limit, thus reproducing at loop $\ell + 1$ the assumption (53). For the second ones due to the subtractions the integrand can be expressed as derivative with respect to the external momenta for which we can use (50). Consider for instance the case of $\Sigma^{(\ell+1)}_L(p; \Lambda)$ We have

$$|I^{(\ell+1)}_L(p; \lambda) - I^{(\ell+1)}_{rel L}(p; \lambda)|_\Lambda \sim p^4 |\partial^4 I^{(\ell+1)}_L(p; \lambda)|_\Lambda = p^4 \mathcal{O}(\lambda^{-4}) \quad (57)$$

for $p^2 < \lambda^2$. Then by inserting the behaviour (50) in (57) we have that the $\lambda$-integration is convergent and dominated by the lower limit $\Lambda$. Thus the assumption (53) is recovered.
In this paper we have assumed that the interaction part of the cutoff effective action depends only on the combination of fields \( w_\mu(p) \), namely we have the following functional identity
\[
-ig p_\mu \frac{\delta \Pi[\Phi; \Lambda]}{\delta u_\mu^a(p)} = \frac{\delta \Pi[\Phi; \Lambda]}{\delta \bar{c}^a(p)}.
\]  
(58)

Since \( w_\mu \) has mass dimension 2 we have reduced the number of vertices with non-negative dimension and consequently the number of relevant couplings. Moreover, this identity simplifies the ST identities.

In this section we will prove that, by a proper choice of the boundary conditions of the evolution equation, this identity is maintained at all loops.

If we do not assume (58) the fields \( \bar{c} \) and \( u_\mu \) enter independently in the effective action. Then instead of the two monomials containing \( w_\mu \) in (7) we would have four relevant couplings associated to the monomials
\[
\bar{c} \cdot c, \quad u_\mu \cdot c, \quad (\partial_\mu \bar{c}) \cdot (c \wedge A_\mu), \quad u_\mu \cdot (c \wedge A_\mu).
\]

The four couplings associated with these monomials are defined by
\[
\Pi^{(\bar{c}c)}(p; \Lambda) = ip^2(\sigma_{\bar{c}c}(\Lambda) + \Sigma^{(\bar{c}c)}(p; \Lambda)),
\]
\[
\Pi^{(uc)}_\mu(p; \Lambda) = p_\mu(\sigma_{uc}(\Lambda) + \Sigma^{(uc)}(p; \Lambda)),
\]
\[
\Gamma^{(\bar{c}cA)}_\mu(p, q; \Lambda) = p_\mu[\sigma_{\bar{c}cA} + \Sigma^{(\bar{c}cA)}(p, q; \Lambda)] + q_\mu\tilde{\Gamma}^{(\bar{c}cA)}(p, q; \Lambda),
\]
\[
\Gamma^{(ucA)}_{\mu\nu}(p, q; \Lambda) = g_{\mu\nu}[\sigma_{ucA}(\Lambda) + \Sigma^{(ucA)}(p, q; \Lambda)] + \tilde{\Gamma}^{(ucA)}_{\mu\nu}(p, q; \Lambda),
\]
where the normalization conditions are
\[
\partial_{p^2} \left( p^2 \Sigma^{(\bar{c}c)}(p; \Lambda) \right) |_{p^2=\mu^2} = \partial_{p^2} \left( p^2 \Sigma^{(uc)}(p; \Lambda) \right) |_{p^2=\mu^2} = 0,
\]
\[
\Sigma^{(\bar{c}cA)}(p, q; \Lambda)_{3SP} = \Sigma^{(ucA)}(p, q; \Lambda)_{3SP} = 0.
\]
The evolution equations and their integral solutions are completely analogous to the previous case.

The field \( \bar{c} \) enters in four other monomials of dimension four
\[
(\partial_\mu c) \cdot (\bar{c} \wedge A_\mu), \quad (\bar{c} \wedge c) \cdot (A_\mu \wedge A_\mu), \quad (\bar{c} \cdot A_\mu)(c \cdot A_\mu), \quad (\bar{c} \wedge c) \cdot (\bar{c} \wedge c).
\]

If the functional identity holds, whenever there is a \( \bar{c} \) field in a vertex, we can extract a power of its momentum and we are left with an irrelevant vertex. We can therefore assume that there are no relevant couplings associated with these monomials (notice that for this reason we have assumed that there is only one relevant coupling in \( \Gamma^{(\bar{c}cA)}_\mu \)). The corresponding vertices are then fixed to vanish at \( \Lambda = \Lambda_0 \).

Since the Lorentz decomposition and the normalization conditions for the vertices \( w-c \) and \( w-c-A \) in (23) and (24) are analogous to the one for the vertices \( u-c \) and \( u-c-A \) given above, we select the boundary conditions
\[
\sigma_{uc}(0) = \frac{1}{g}\sigma_{wc}(0) = 0, \quad \sigma_{ucA}(0) = \frac{1}{g}\sigma_{wcA}(0) = -1.
\]
This corresponds to fix these two couplings as they appear in the total action (3). We now show that it is possible to fix the boundary conditions on the parameters \( \sigma_{\bar{c}c} \) and \( \sigma_{\bar{c}cA} \) in such a way that the identity (58) is valid at any loop and for any \( \Lambda \). At the tree level this requirement gives

\[
\sigma^{(0)}_{\bar{c}c}(0) = 0, \quad \sigma^{(0)}_{\bar{c}cA}(0) = ig.
\]

Let us suppose that the identity is true for the cutoff effective action \( \Pi^{(\ell)} \) at loop \( \ell \). The identity then holds for the auxiliary vertices \( \bar{\Gamma}^{(\ell)} \) and, from (34), also for the integrands \( I^{(\ell+1)} \). Then the validity of (58) at loop \( \ell + 1 \) depends only on the boundary conditions for \( \sigma^{(\ell+1)}_{\bar{c}c} \) and \( \sigma^{(\ell+1)}_{\bar{c}cA} \). In the following, in order to simplify the notation, in all the integrands and vertices we do not write the loop order which is understood to be \( \ell + 1 \).

1) Ghost propagator.

The integral equations give \((\bar{p}^2 = \mu^2)\)

\[
\Pi^{(\bar{c}c)}(p; \Lambda) = ip^2 \sigma^{(\ell+1)}_{\bar{c}c}(0) + \int_0^{\Lambda_0} \frac{d\lambda}{\lambda} \partial_{\bar{p}^2} I^{(\bar{c}c)}(\bar{p}; \lambda) - ip^2 \int_\Lambda^{\Lambda_0} \frac{d\lambda}{\lambda} I^{(\bar{c}c)}(p; \lambda)
\]

and

\[
p_{\mu} \Pi^{(\bar{c}cA)}_{\mu}(p; \Lambda) = p^2 \int_0^{\Lambda_0} \frac{d\lambda}{\lambda} \partial_{\bar{p}^2} (\bar{p}^2 I^{(\bar{c}cA)}(\bar{p}; \lambda)) - p^2 \int_\Lambda^{\Lambda_0} \frac{d\lambda}{\lambda} I^{(\bar{c}cA)}(p; \lambda).
\]

From the inductive hypothesis

\[
gI^{(\bar{c}cA)}(p; \lambda) = I^{(\bar{c}c)}(p; \lambda),
\]

we have that the identity is satisfied at any loop and for any \( \Lambda \), by imposing \( \sigma^{(\ell+1)}_{\bar{c}c}(0) = 0 \), which implies to all loop order

\[
\sigma_{\bar{c}c}(0) = 0.
\]

2) Vector-ghost vertex.

The integral equations give

\[
\Gamma^{(\bar{c}cA)}_{\mu}(p, q; \Lambda) = p_{\mu} \sigma^{(\ell+1)}_{\bar{c}cA}(0) + p_{\mu} \int_0^{\Lambda_0} \frac{d\lambda}{\lambda} I^{(\bar{c}cA)}(p, q; \lambda)|_{3SP} - \int_\Lambda^{\Lambda_0} \frac{d\lambda}{\lambda} I^{(\bar{c}cA)}_{\mu}(p, q; \lambda)
\]

and

\[
\Gamma^{(\bar{c}cA)}_{\nu\mu}(p, q; \Lambda) = -g_{\mu\nu} + g_{\mu\nu} \int_0^{\Lambda_0} \frac{d\lambda}{\lambda} I^{(\bar{c}cA)}(p, q; \lambda)|_{3SP} - \int_\Lambda^{\Lambda_0} \frac{d\lambda}{\lambda} I^{(\bar{c}cA)}_{\nu\mu}(p, q; \lambda).
\]

The inductive hypothesis

\[
-ig p_{\nu} I^{(\bar{c}cA)}_{\nu\mu}(p, q; \lambda) = I^{(\bar{c}cA)}_{\mu}(p, q; \lambda),
\]

at 3SP gives

\[
-ig \left[ I^{(\bar{c}cA)}(p, q; \lambda) + \frac{2}{3p^2} p_{\mu} Q_{\bar{c}c} I^{(\bar{c}cA)}_{\mu}(p, q; \lambda) \right]_{3SP} = I^{(\bar{c}cA)}(p, q; \lambda)|_{3SP},
\]

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where \( Q_\nu = 2p_\nu + q_\nu \) is vector orthogonal to \( q_\nu \) at \( 3SP \). Using these two results one finds that in this case the identity (58) is valid for any \( \Lambda \) if we fix

\[
\sigma_{ecA}^{(\ell+1)}(0) = ig \left[ \frac{2}{3p^2} Q_\alpha p_\beta \int_0^{\Lambda_0} \frac{d\lambda}{\lambda} \tilde{f}_{\beta \alpha}^{(ucA)}(p, q; \lambda) \right]_{3SP}.
\]

To all loops we have then

\[
\sigma_{ecA}(0) = ig \left[ 1 - \frac{2}{3p^2} p_\beta \tilde{f}_{\beta \alpha}^{(ucA)}(p, q; 0) Q_\alpha \right]_{3SP}.
\]

Notice that this condition implies

\[-ig \sigma_{ucA}(\Lambda_0) = \sigma_{ecA}(\Lambda_0),\]

namely, in the usual language, no counterterms violating (58) are generated by the evolution equation.

7 Conclusions

In this paper we have analyzed a RG formulation of a non-Abelian gauge theory which avoids the usual fine tuning problem [6, 11]. This problem arises when one treats the RG equations as evolution equations and starts by giving the relevant couplings at the UV scale \( \Lambda_0 \) (bare parameters). These various parameters have to be fine tuned to depend on a single parameter, the coupling \( g \) at the normalization point \( \mu \), in such a way that the ST identities are satisfied by the computed physical effective action. Our procedure avoids the fine tuning problem simply by fixing the relevant parameters at the physical point \( \Lambda = 0 \). We have shown that the formulation so obtained is quite practical and deduced the usual loop expansion of the renormalized vertex functions in terms of the physical coupling \( g \) at \( \mu \). The calculations do not seem more complex than the ones done in dimensional regularization. The cancellations of UV divergences in the Feynman graphs is provided by subtractions which are systematically generated by implementing the boundary conditions. These cancellations are very similar to the ones obtained by the Bogoliubov \( R \)-operation [15]. We described very briefly the proof of perturbative renormalizability. This proof is simply based on power counting and does not involve any new feature with respect to the one for the scalar case and QED.

We have performed detailed calculations only to one loop. We computed the six vertices with non-negative dimension which contain the relevant couplings. From them we obtained for large \( \Lambda \) the nine relevant couplings \( \sigma_i(\Lambda) \) and \( \rho_i(\Lambda) \), which are used to compute the usual one loop beta function and are the coefficients of the UV bare action (11).

One then has to show that the complete (physical) effective action indeed satisfies ST identities. In the QED case we have shown [6] to all perturbative orders that the complete effective action computed by this method does indeed satisfy the Ward identities. This implies that, in the RG flow, all couplings and vertices are constrained for every \( \Lambda \) by the symmetry. In the YM case we have verified the ST identities only to one loop and for two
vertices. A general proof of ST identities, based on the results of the recent work [6] by Becchi, is under study.

Our analysis has been perturbative. However, contrary to dimensional regularization, the basic integral equations in (38) and (39) are in principle nonperturbative. The only possible perturbative element at the basis of (38) is simply the separation of \( \Gamma[\Phi; \Lambda] \) in relevant and irrelevant parts, which is based on naïve power counting. This should be correct for an asymptotically free theory. A next natural step in this formulation is the analysis of anomalies and chiral gauge theories. Since we work in four dimensions, the presence of \( \gamma_5 \) does not lead to any direct complication. This analysis is left to a separate publication [10].

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Figure Captions

Figure 1: The circles represent vertex functions with the IR cutoff \( \Lambda \). Internal lines involve only the fields \( \phi_A = (A_\mu, c, \bar{c}) \), while the external lines could involve also the sources \( u_\mu \) and \( v \). The crosses represent the derivative with respect to \( \Lambda \) of the internal propagators. Integration over \( q \) in the loop is understood.

Figure 2: a) Graphical representation of the contribution to \( \bar{\Gamma}^{(0)b_1b_2 aa'}_{\nu_1\nu_2,\mu\mu'}(q_1, q_2; p, p'; \Lambda) \) obtained by expanding (14) to second order in \( \Gamma^{int} \).

b) Graphical representation of the contribution to \( \bar{\Gamma}^{(0)b_1b_2 aa'}_{\nu_1\nu_2,\mu\mu'}(q_1, q_2; p, p'; \Lambda) \) obtained by expanding (14) to first order in \( \Gamma^{int} \).

c) Graphical representation of the contribution to \( \bar{\Gamma}^{(0)b_1b_2 aa'}_{\mu\mu',\nu_1\nu_2}(q_1, q_2; p, p'; \Lambda) \) obtained by expanding (14) to second order in \( \Gamma^{int} \).