Asymptotics of the number of 2-threshold functions

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August 27, 2020

Abstract

A \( k \)-threshold function on a rectangular grid of size \( m \times n \) is the conjunction of \( k \) threshold functions on the same domain. In this paper, we focus on the case \( k = 2 \) and show that the number of two-dimensional 2-threshold functions is 

\[
\frac{25}{12\pi^4} m^4n^4 + o(m^4n^4).
\]

Keywords: threshold function, \( k \)-threshold function, intersection of halfplanes, integer lattice, rectangular grid, asymptotic formula

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1 Introduction

Let \( G_{m,n} \) denote an integer two-dimensional rectangular grid, that is, \( G_{m,n} = \{0, \ldots, m-1\} \times \{0, \ldots, n-1\} \). In this paper we study \( \{0,1\} \)-valued functions on \( G_{m,n} \) defined by two linear inequalities. In other words, we observe partitions of \( G_{m,n} \) by two straight lines in general position. The problem belongs to the study of different configurations and properties of lattice points (points having integer coordinates) produced by using different classes of lines (in two dimensions) and surfaces (in higher dimensions).
where proposed. Herein, as it has been mentioned above, we consider one of such problems. Position, can be much more difficult, especially for a proper performance analysis of the methods centered at the same point. The situation, once the data space partition surfaces are in general not assumed to be in general position (e.g. parallel hyperplanes or $d$-dimensional spheres centered at the same point). The situation, once the data space partition surfaces are in general position, can be much more difficult, especially for a proper performance analysis of the methods proposed. Herein, as it has been mentioned above, we consider one of such problems.

For a $\{0,1\}$-valued function $f$ defined on $G_{m,n}$ we denote

$$M_{\nu}(f) = \{ x \in G_{m,n} | f(x) = \nu \},$$

where $\nu \in \{0,1\}$. For a given set of points $S$ we denote by $\text{Conv}(S)$ the convex hull of $S$.

We say that a $\{0,1\}$-valued function $f$ defined on the grid $G_{m,n}$ is threshold if its sets of true and false points are separable by a line, i.e.

$$\text{Conv}(M_0(f)) \cap \text{Conv}(M_1(f)) = \emptyset.$$ 

Let $k$ be a natural number, a function $f : G_{m,n} \rightarrow \{0,1\}$ is called $k$-threshold if it can be represented as a conjunction of at most $k$ threshold functions $f_1, \ldots, f_k$, i.e. $f = f_1 \land \cdots \land f_k$. One can say a $k$-threshold function is the intersection of $k$ threshold functions. We also say that the functions $f_1, \ldots, f_k$ define the $k$-threshold function $f$. A $k$-threshold function is called proper $k$-threshold if it is not $(k-1)$-threshold.

In this work we focus on 2-threshold functions, i.e. the conjunctions of two threshold functions. Some previous works on this topic dealt with learning issues ([14, 21, 22]) and the structure ([23]) of 2-threshold functions whereas this paper is devoted to the estimation of their number asymptotically.

We denote by $t_k(m,n)$ the number of $k$-threshold functions on $G_{m,n}$. Throughout the paper we will write $t(m,n)$ instead of $t_1(m,n)$, as the former is a common notation in the literature. The asymptotics of the number of threshold functions for square grids was first obtained in [18]:

$$t(n,n) = \frac{6}{\pi^2}n^4 + O(n^3 \log n),$$

and for arbitrary rectangular grids in [1]:

$$t(m,n) = \frac{6}{\pi^2}m^2n^2 + O(m^2n \log n + mn^2 \log \log n),$$

where $m < n$ is assumed.

The current best known formula was obtained in [12]:

$$t(m,n) = \frac{6}{\pi^2}m^2n^2 + O(mn^2).$$

An important point to note here is that all the above results are based on the relation between non-constant threshold functions and (oriented) prime segments.

The above asymptotics provides a trivial upper bound on the number of $k$-threshold functions for a fixed $k > 1$:
\[
t_k(m,n) \leq \binom{t(m,n)}{k} = \frac{t(m,n)^k}{k!} + O\left(\frac{t(m,n)^{k-1}}{k!}\right) = \frac{6^k}{\pi^{2k}k!} m^{2k}n^{2k} + O\left(m^{2k-1}n^{2k}\right).
\]

However, no asymptotics was known for the number of \(k\)-threshold functions for any \(k > 1\). In this paper we derive an asymptotics for the number of 2-threshold functions on arbitrary rectangular grids. More specifically, the main result of the paper is the following

**Theorem 1.** \(t_2(m,n) = \frac{25}{24m^4n^4} + o(m^4n^4)\).

In order to prove Theorem 1, we will first make use of the structural characterization of 2-threshold functions from [23] to reduce the problem to the enumeration of pairs of prime segments in convex position. Then we will derive the asymptotic formula for the number such pairs.

The organization of the paper is as follows. All preliminary information can be found in Section 2. In Section 3 we recall the results from [23] introducing proper pairs of segments and describing their relation to 2-threshold functions. We also express the number of 2-threshold function through the number of proper pairs of segments. In Section 4 we reveal the relation between proper pairs of segments and pairs of prime segments in convex position and reduce the problem of the estimation of the number of 2-threshold functions to that of pairs of prime segments in convex position. Finally, Section 5 is devoted to the estimation of the number of pairs of prime segments in convex position. In Section 6 we use the obtained formula to improve the trivial upper bound on the number of \(k\)-threshold functions (1) for \(k \geq 3\).

\section{Preliminaries}

In this paper we use capital letters \(A, B, C\) etc. to denote points on the plane. The distance between two points \(A\) and \(B\) is denoted by \(d(A, B)\). The distance between two sets of points \(S_1, S_2\) is the minimum distance between two points \(A \in S_1\) and \(B \in S_2\) and denoted by \(d(S_1, S_2)\). The distance between a point and a set of points is denoted analogously. The line passing through two distinct points \(A\) and \(B\) is denoted by \(\ell(AB)\). For a convex polygon \(\mathcal{P}\) we denote by \(\text{Area}(\mathcal{P})\) the area of \(\mathcal{P}\).

We say that a point \(A = (x, y)\) is integer, if both of its coordinates \(x\) and \(y\) are integer. If two distinct integer points \(A, B\) are the only integer points on the segment \(AB\), we say that the segment \(AB\) is prime.

For a polygon \(P\) denote by \(\text{Vert}(P)\) the set of vertices of \(P\). We say that the points \(A_1, A_2, \ldots, A_n\) are in convex position if \(\{A_1, \ldots, A_n\} = \text{Vert}(\text{Conv}\{A_1, \ldots, A_n\})\). We will say that two segments are in convex position if they are opposite sides of a convex quadrilateral. We also denote by \(P(f)\) the convex hull of true points of \(f\), that is \(P(f) = \text{Conv}(M_1(f))\).

Let \(\mathcal{C}\) be a convex set. A convex polygon \(\mathcal{P}\) is called circumscribed about \(\mathcal{C}\) if for every edge \(AB\) of \(\mathcal{P}\) the line \(\ell(AB)\) is a tangent to \(\mathcal{C}\) and \(AB \cap \mathcal{C} \neq \emptyset\).

\subsection{Segments, triangles, quadrilaterals and their orientation}

We often denote a convex polygon by a sequence of its vertices in either clockwise or counterclockwise order. For instance, we denote by

- \(AB\) the segment with endpoints \(A, B\);
- \(ABC\) the triangle with vertices \(A, B, C\);
- \(ABCD\) the convex quadrilateral with edges \(AB, BC, CD, DA\).

We call a polygon or a segment oriented and add an arrow in the notation if the order of vertices is important. For example, \(\overrightarrow{AB}, \overrightarrow{ABC}, \overrightarrow{ABCDA}\) denote the oriented segment, the oriented triangle, and the oriented convex quadrilateral, respectively. An oriented convex polygon is clockwise or counterclockwise depending on the orientation of the rotation. If \(\overrightarrow{ABC}\) is a clockwise triangle, then \(\overrightarrow{ACB}\) is a counterclockwise triangle and vice versa.
### 2.2 Number theoretic preliminaries

In the subsequent sections we will use the following formulas. For the harmonic number:

\[
\sum_{i=1}^{n} \frac{1}{i} = \log n + \gamma + O \left( \frac{1}{n} \right) = \log n + O(1),
\]

(2)

where \( \gamma \) is the Euler-Mascheroni constant.

For a fixed natural \( k \) the asymptotics of the sum of \( k \)-th powers can be estimated as

\[
\sum_{i=1}^{n} i^k = \frac{n^{k+1}}{k+1} + O(n^k).
\]

(3)

Also, for a fixed natural \( q \) and integer \( k \geq 0 \) we have

\[
\sum_{p \perp q}^{n} p^k = \frac{\phi(q) n^{k+1}}{q (k+1)} + O(n^k 2^{w(q)}),
\]

(4)

where \( \phi(q) \) is the Euler function, \( w(q) \) is the number of different prime divisors of \( q \) and

\[
\sum_{q=1}^{n} O(2^{w(q)}) = O(n \log n).
\]

(5)

For the negative powers of \( p \) we have

\[
\sum_{p=1}^{n} \frac{1}{p^k} = \frac{\phi(q)}{q} \log n + \frac{\phi(q)}{q} \gamma + O \left( \frac{1}{n} \right) - \sum_{d \mid q} \mu(d) \frac{1}{d} \log d.
\]

(6)

Some sums regarding the Euler function are as follows:

\[
\sum_{x=1}^{n} \phi(x) \log(x) = \frac{3}{\pi^2} n^2 \log n - \frac{3}{2\pi^2} n^2 + o(n^2).
\]

(7)

The general formula for the power of \( x \) follows:

\[
\sum_{x=1}^{n} \phi(x) x^k = \frac{6}{\pi^2} \frac{n^{k+2}}{(k+2)} + O(n^{k+1} \log n),
\]

(8)

where \( k \) is integer.

More details about the derivations of the previous sums are provided in [11] and [4].

The following sum is obtained from (4) and (8):

\[
\sum_{p=1}^{m} \sum_{q=1}^{n} \frac{1}{q^k} = \frac{6}{\pi^2} mn + O(m \log n).
\]

(9)

For the Möbius function \( \mu_n \) we have ([4]):

\[
\sum_{n \mid k} \mu_n = \begin{cases} 1 & \text{if } k = 1 \\ 0 & \text{if } k > 1. \end{cases}
\]

(10)

where \( n \mid k \) means that \( n \) is a divisor of \( k \), and

\[
\sum_{n=1}^{\infty} \frac{\mu(n)}{n^2} = \frac{1}{\zeta(2)} = \frac{6}{\pi^2},
\]

(11)

where \( \zeta(n) \) is the Riemann zeta function.
3 Proper pairs of oriented prime segments and the number of 2-threshold functions

The known asymptotic formulas for the number of threshold functions ([18, 1, 12]) are based on their relation to (oriented) prime segments.

**Definition 3.1.** Let $AB$ be a prime segment and $A, B \in \mathcal{G}_{m,n}$. We say that $\overrightarrow{AB}$ defines a $\{0, 1\}$-valued function $f$ on $\mathcal{G}_{m,n}$ if:

1. $f(A) = 1, f(B) = 0$;
2. for any $X \in \mathcal{G}_{m,n} \cap \ell(AB)$ we have $f(X) = 1$ if and only if $d(A, X) < d(B, X)$;
3. for any $X \in \mathcal{G}_{m,n} \setminus \ell(AB)$ we have $f(X) = 1$ if and only if $\overrightarrow{AB}X$ is a counterclockwise triangle.

The function defined by $\overrightarrow{AB}$ is denoted as $f_{\overrightarrow{AB}}$.

It was proved in [18] that the function $f_{\overrightarrow{AB}}$ defined by an oriented prime segment $\overrightarrow{AB}$ is threshold. Moreover, in the same paper, a bijection between all non-constant threshold functions and oriented prime segments in $\mathcal{G}_{m,n}$ was established.

Following a similar approach with 2-threshold functions, we introduced the next definition in [23]:

**Definition 3.2.** We say that a pair of oriented prime segments $\overrightarrow{AB}, \overrightarrow{CD}$ in $\mathcal{G}_{m,n}$ defines a 2-threshold function $f$ on $\mathcal{G}_{m,n}$ if

$$f = f_{\overrightarrow{AB}} \land f_{\overrightarrow{CD}}.$$ 

However, since the same 2-threshold function can be represented as the conjunction of different pairs of threshold functions, it can be defined by different pairs of oriented prime segments. To overcome this difficulty, we imposed an extra restriction on pairs of segments, which resulted in the notion of proper pairs of segments [23].

**Definition 3.3.** We say that a pair of oriented segments $\overrightarrow{AB}, \overrightarrow{CD}$ is proper if the segments are prime and

$$f_{\overrightarrow{CD}}(A) = f_{\overrightarrow{CD}}(B) = f_{\overrightarrow{AB}}(C) = f_{\overrightarrow{AB}}(D) = 1.$$ 

In [23] we proved that any proper pair of segments defines a proper 2-threshold function and that for any proper 2-threshold function there exists a proper pair of segments that defines the function. We also showed that such a pair is unique if the function has a true point on the boundary of the grid. In this way we established a bijection between proper 2-threshold functions having a true point on the boundary of the grid and the proper pairs of segments defining such functions.

In this section we will show that almost all 2-threshold functions satisfy the above condition, and use this fact to derive that the number of 2-threshold functions is asymptotically equal to the number of proper pairs of segments.

**Claim 2.** Let $\{\overrightarrow{AB}, \overrightarrow{CD}\}$ be a proper pair of segments in $\mathcal{G}_{m,n}$, and let $f$ be the 2-threshold function defined by $\{\overrightarrow{AB}, \overrightarrow{CD}\}$. If $f$ does not have true points on the boundary of the grid, i.e. $M_1(f) \subseteq \{1, \ldots, m-2\} \times \{1, \ldots, n-2\}$, then the distances $d(A, \ell(\overrightarrow{CD}))$ and $d(B, \ell(\overrightarrow{CD}))$ do not exceed one.

**Proof.** The statement is obvious for $\ell(\overrightarrow{AB}) = \ell(\overrightarrow{CD})$, so we assume that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are not collinear.

Let us first assume that $\ell(\overrightarrow{AB})$ and $\ell(\overrightarrow{CD})$ are not parallel and denote by $O$ the intersection point of the two lines. We start by showing that there exists a point $X \in \ell(\overrightarrow{AB}) \cap B(\mathcal{G}_{m,n})$ such that $\overrightarrow{AB} \subseteq OX$. Indeed, since $f(A) = 1$, the point $A$ is an interior point of $\text{Conv}(\mathcal{G}_{m,n})$, and hence the line $\ell(\overrightarrow{AB})$ intersects $B(\mathcal{G}_{m,n})$ in exactly two points, which we denote by $X$ and $Y$. Furthermore, as $\ell(\overrightarrow{CD})$ does not separate $A$ and $B$, we have either $\overrightarrow{AB} \subseteq OX$ or $\overrightarrow{AB} \subseteq OY$. Without loss of generality assume $\overrightarrow{AB} \subseteq OX$. Let $Z \in B(\mathcal{G}_{m,n})$ be the closest point to $X$ such that $f_{\overrightarrow{AB}}(Z) = 1$. Clearly, $d(X, Z) \leq 1$. The assumption $M_1(f) \subseteq \{1, \ldots, m-2\} \times \{1, \ldots, n-2\}$ implies that $f(Z) = 0$, and therefore $f_{\overrightarrow{CD}}(Z) = 0$. Hence, either $Z \in \ell(\overrightarrow{CD})$ or the triangle $\overrightarrow{CDZ}$ is clockwise. The former implies...
that \( d(X, \ell(CD)) \leq 1 \). The latter leads to the same conclusion, if we notice that the triangle \( \overline{CDX} \) is counterclockwise as \( X \) and \( A \) lie on the same side of \( \ell(CD) \), and hence \( \ell(CD) \) intersects \( XZ \). Finally, since \( A, B \in OX \), we conclude that \( \max\{d(A, \ell(CD)), d(B, \ell(CD))\} \leq d(X, \ell(CD)) \leq 1 \), as required.

The proof for parallel \( \ell(AB) \) and \( \ell(CD) \) is similar and uses the fact that the distance from any point of \( \ell(AB) \) to \( \ell(CD) \) is the same.

**Claim 3.** There are \( O(m^2n^2(m + n)^2) \) proper pairs of segments \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) in \( \mathcal{G}_{m,n} \) such that the 2-threshold function defined by \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) does not have true points on the boundary of \( \mathcal{G}_{m,n} \).

**Proof.** There are at most \( mn \) ways to choose each of \( C \) and \( D \). Given the segment \( CD \), by Claim 2, each of \( A \) and \( B \) lies at distance at most one from \( \ell(CD) \). Since there are \( O(m + n) \) such points, we conclude that there are \( O(m^2n^2(m + n)^2) \) desired pairs of segments.

**Claim 4.** There are \( O(m^2n^2(m + n)^2) \) 2-threshold functions on \( \mathcal{G}_{m,n} \) that are either threshold or do not have true points on the boundary of \( \mathcal{G}_{m,n} \).

**Proof.** Since the number of threshold functions is \( \Theta(m^2n^2) \), it is enough to prove that the number of proper 2-threshold functions that do not have a true point on the boundary of \( \mathcal{G}_{m,n} \) is \( O(m^2n^2(m + n)^2) \). Indeed, every proper 2-threshold function with no true points on the boundary of \( \mathcal{G}_{m,n} \) is defined by at least one proper pair of segments, and hence the number of such functions can not exceed the number of proper pairs of segments defining these functions, which is estimated in Claim 3 as \( O(m^2n^2(m + n)^2) \).

We are now in a position to state formally the main result of the section. Denote by \( q(m, n) \) the number of proper pairs of segments in \( \mathcal{G}_{m,n} \). Claims 3 and 4 and a bijection between proper 2-threshold functions with true points on the boundary of \( \mathcal{G}_{m,n} \) and proper pairs of segments defining these functions lead to the following corollary.

**Corollary 5.**

\[
t_2(m, n) = q(m, n) + O(m^2n^2(m + n)^2). \tag{12}
\]

Corollary 5 is useful if only the order of the number of 2-threshold functions (and proper pairs of segments) is larger than \( m^2n^2(m + n)^2 \). From (1) it follows that \( t_2(m, n) = O(m^4n^4) \). In the subsequent sections we will prove that \( q(m, n) = \Theta(m^4n^4) \). This result together with Claims 3 and 4 will establish a bijection between almost all 2-threshold functions and almost all proper pairs of segments.

4 Pairs of prime segments in convex position

In this section we will reduce the estimation of the number of proper pairs of oriented segments to that of prime (non-oriented) segments in convex position. First we will show that the number of those proper pairs of segments, which are not in convex position, does not affect the asymptotics. To this end, we will use the structure of proper pairs of segments revealed in [23]:

**Theorem 6 ([23]).** The pair of prime segments \( \overrightarrow{AB}, \overrightarrow{CD} \) is proper if and only if one of the following holds:

1. \( AC \subset BD \);
2. \( A \in BD \) and \( \overrightarrow{CDB} \) is a counterclockwise triangle or \( C \in BD \) and \( \overrightarrow{ABD} \) is a counterclockwise triangle;
3. \( \overrightarrow{ABCD} \) is a counterclockwise quadrilateral.

Let \( p(n, m) \) denote the number of proper pairs of segments, which are in convex position. Then we have the following
Claim 7. 

\[ q(m, n) = p(m, n) + O(m^3n^3(m + n)). \] (13)

Proof. We will show that the number of proper pairs of segments \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) in \( G_{m,n} \) such that Conv\(\{A, B, C, D\}\) is a segment or triangle is \( O(m^3n^3(m + n)) \). If Conv\(\{A, B, C, D\}\) is a segment, then all of the four points \( A, B, C, \) and \( D \) belong to the same line. If Conv\(\{A, B, C, D\}\) is a triangle, then, by Theorem 6, three of the points belong to the same line. In both cases there are three collinear points, say \( A, B, C \). There are \( O(m^2n^2) \) ways to choose two of these three points. Given two fixed points, there are at most \( \max\{m - 2, n - 2\} = O(m + n) \) ways to choose the third one. For the fourth point, whether it lies on the same line with \( A, B, C \) or not, there are \( O(mn) \) choices. Hence, altogether there are \( O(m^3n^3(m + n)) \) proper pairs of segments \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) in \( G_{m,n} \) such that Conv\(\{A, B, C, D\}\) is a segment or triangle, which implies (13).

Corollary 5 and Claim 7 reduce the estimation of the number of 2-threshold functions to that of the number of proper pairs of segments in convex position. Further, we will show that it suffices to consider non-oriented prime segments in convex position. The following claim is a convenient necessary and sufficient condition for a pair of segments to be in convex position.

Claim 8. Segments \( AB \) and \( CD \) are in convex position if and only if

\[
\begin{align*}
\ell(AB) \cap CD &= \emptyset, \\
\ell(CD) \cap AB &= \emptyset.
\end{align*}
\] (14)

Proof. Clearly, if \( AB \) and \( CD \) are in convex position, then (14) holds. To prove the converse, we observe that (14) implies that Conv\(\{A, B, C, D\}\) is not a segment or triangle, hence it is a convex quadrilateral with vertices \( A, B, C, \) and \( D \). Moreover, \( AB \cap CD = \emptyset \), and hence the segments are neither diagonals nor adjacent edges, and consequently they are opposite edges of the quadrilateral Conv\(\{A, B, C, D\}\)\).

The relation between proper pairs of segments in convex position and pairs of non-oriented prime segments in convex position is revealed in the following theorem.

Theorem 9. There is one-to-one correspondence between pairs of (non-oriented) prime segments in convex position and proper pairs of segments in convex position.

Proof. To prove the claim, we establish a bijective mapping between the two sets of pairs of segments. Clearly, if \( \{\overrightarrow{AB}, \overrightarrow{CD}\} \) is a proper pair of segments in convex position, then \( AB \) and \( CD \) are prime and in convex position.

Now, let \( AB \) and \( CD \) be prime segments in convex position, then Conv\(\{A, B, C, D\}\) is a quadrilateral, and \( AB \) and \( CD \) are two of its four edges. Assume, without loss of generality, that Conv\(\{A, B, C, D\}\) has also the edges \( CD \) and \( DA \) (see Fig. 1). There are two oriented quadrilaterals corresponding to Conv\(\{A, B, C, D\}\), namely, \( ABCD \) and \( DCBA \), and these quadrilaterals have opposite orientations. Without loss of generality, we assume that \( ABCD \) is
Figure 2: \{AB, CD\} is a pair of segments in convex position. The grey shape is Conv(\{A, B, C, D\}). The rectangle is circumscribed about Conv(\{A, B, C, D\}) in (a) and (b) and not circumscribed in (c) and (d).

the counterclockwise one, and hence, \{\overrightarrow{AB}, \overrightarrow{CD}\} is a unique proper pair of segments in convex position corresponding to AB, CD.

Because of the bijection established in Theorem 9, \(p(m, n)\) denotes both the number of proper pairs of segments in convex position and the number of pairs of (non-oriented) prime segments in convex position. By Corollary 5, Claim 7, and Theorem 9, we conclude that the number of 2-threshold functions can be expressed via the number of pairs of prime segments in convex position:

**Corollary 10.**

\[ t_2(m, n) = p(m, n) + o(m^4 n^4). \]

As we will show in Theorem 22, the number of pairs of prime segments in convex position is \(\Theta(m^4 n^4)\), and hence, Claim 7 and Theorem 9 establish a bijection between almost all proper pairs of segments and pairs of prime segments in convex position.

## 5 The number of pairs of prime segments in convex position

In what follows we will extensively use rectangles with horizontal and vertical sides circumscribed about the convex quadrilaterals being the convex hulls of pairs of segments in convex position (see Fig. 2).

Denote by \(R_{u,v}\) a \(u \times v\) rectangle Conv(\(G_{u+1,v+1}\)) for natural numbers \(u\) and \(v\). Denote by \(Z(u, v)\) the set of pairs of prime segments \{AB, CD\} in convex position such that \(R_{u,v}\) is circumscribed about Conv(\{A, B, C, D\}).

**Theorem 11.**

\[ p(m, n) = \sum_{u=1}^{m} \sum_{v=1}^{n} (m - u)(n - v)|Z(u, v)|. \]

**Proof.** First for every convex quadrilateral with vertices in \(G_{m,n}\) there exists a unique rectangle with sides parallel to the sides of Conv(\(G_{m,n}\)) circumscribed about it. Hence, the statement follows from the fact that there are exactly \((m - u)(n - v)\) rectangles in \(G_{m,n}\) with sides of length \(u\) and \(v\) that are parallel to the sides of Conv(\(G_{m,n}\)).

\(\square\)

Let \(Z_i(u, v) \subseteq Z(u, v)\) be the set of those pairs of segments AB, CD in Z(u, v), for which exactly \(i\) points in \{A, B, C, D\} are vertices of \(R_{u,v}\). Clearly, \(Z(u, v)\) is the disjoint union of \(Z_i(u, v), i = 0, 1, \ldots, 4\), and therefore

\[ |Z(u, v)| = \sum_{i=0}^{4} |Z_i(u, v)|. \]

Our next step is to estimate the cardinality of \(Z_i(u, v)\) for every \(i \in \{0, 1, \ldots, 4\} \). The cases \(i \in \{4, 3\} \) are easy and we consider them below. The cases \(i \in \{0, 1, 2\} \) are more involved and we treat them independently in Sections 5.1–5.3.
Lemma 12. $|Z_3(u, v)| + |Z_4(u, v)| = O(uv)$.

Proof. By definition, for any pair of segments $\{AB, CD\} \in Z_3(u, v) \cup Z_4(u, v)$ at least three of the endpoints of the segments are vertices of $R_{u,v}$. Therefore, since there is a constant number of ways to map 3 of the endpoints of the segments to the vertices of $R_{u,v}$, and there are $O(uv)$ ways to place the fourth point in $G_{u+1,v+1}$, we conclude the lemma.

5.1 The number of pairs of segments with two corner points

In this section we estimate $|Z_2(u, v)|$, i.e. the number of pairs of segments $\{AB, CD\}$ in $Z(u, v)$, for which exactly 2 points in $\{A, B, C, D\}$ are vertices of $R_{u,v}$.

Let $\{X, Y\} = \{A, B, C, D\} \cap \text{Vert}(R_{u,v})$. We consider the partition of $Z_2(u, v)$ into the following three subsets:

1. $Z_2^a(u, v)$ is the subset of $Z_2(u, v)$ such that $X$ and $Y$ are adjacent vertices of $R_{u,v}$, i.e. $XY$ is a side of $R_{u,v}$.
2. $Z_2^b(u, v)$ is the subset of $Z_2(u, v)$ such that $X$ and $Y$ are opposite vertices of $R_{u,v}$ and belong to the same segment.
3. $Z_2^c(u, v)$ is the subset of $Z_2(u, v)$ such that $X$ and $Y$ are opposite vertices of $R_{u,v}$ and belong to the different segments.

Clearly,

$$|Z_2(u, v)| = |Z_2^a(u, v)| + |Z_2^b(u, v)| + |Z_2^c(u, v)|.$$  

Let us show that the first summand does not affect the asymptotics of the sum which will be proved to be $\Theta(u^2v^2)$.

Lemma 13. $|Z_2^a(u, v)| = O(u^2v + w^2)$.

Proof. Since $R_{u,v}$ is circumscribed about Conv($AB \cup CD$) and two of the points $A, B, C, D$ belong to the same side of $R_{u,v}$, at least one of the other two points belongs to the opposite side of $R_{u,v}$. Therefore there are $O(u + v)$ ways to place this point. Furthermore, there are $O(uv)$ ways to place the fourth point in $R_{u,v}$, which implies the desired estimate.

Lemma 14. Let $AB$ and $CD$ be segments with endpoints in $G_{m,n}$. Then $AB$ and $CD$ are in convex position if and only if $A, B, C, D$ are in general position, the triangle $\overline{ABD}$ has the same orientation as $\overline{ABC}$, and the triangle $\overline{CDB}$ has the same orientation as $\overline{CDB}$.

Proof. We will prove the lemma by showing that its conditions are equivalent to those of Claim 8. First we claim that the equation $\ell(AB) \cap \overline{CD} = \emptyset$ is equivalent to the statement that the points in both sets $\{A, B, C\}$ and $\{A, B, D\}$ are in general position and the orientations of $\overline{ABC}$ and $\overline{ABD}$ are the same. Indeed, $\overline{ABC}$ and $\overline{ABD}$ are triangles if and only if $C, D \notin \ell(AB)$. Moreover, the orientations of $\overline{ABC}$ and $\overline{ABD}$ are the same if and only if $\ell(AB)$ does not separate $C$ and $D$.

Using similar arguments, one can establish the equivalence of the equation $\ell(AB) \cap AB = \emptyset$ and the statement that the points in both sets $\{C, D, A\}$ and $\{C, D, B\}$ are in general position and the orientations of triangles $\overline{CDA}$ and $\overline{CDB}$ are the same.

We will employ Lemma 14 to describe the admissible region for the point $D$ under fixed points $A, B, C$ such that $AB$ and $CD$ are in convex position. Figure 3 illustrates the admissible region for $D$. It follows from Lemma 14 that for a segment $AB$ and a point $C \notin \ell(AB)$ the segments $AB$ and $CD$ are in convex position if and only if $D$ belongs to the interior of $P_1 \cup P_2$.

For a polygon $\mathcal{P}$, denote by $\mathcal{L}(\mathcal{P})$ the number of integer points in $\mathcal{P}$, i.e.

$$\mathcal{L}(\mathcal{P}) = |\mathbb{Z}^2 \cap \mathcal{P}|.$$

For a polygon $\mathcal{P}$ and a point $A$ denote by Prime$(\mathcal{P}, A)$ the number of integer points $X \in \mathcal{P}$ such that $AX$ is a prime segment. If $A$ is the origin $O = (0,0)$ we simply write Prime$(\mathcal{P})$. 9
Figure 3: The stripped region is the area of points $X$ such that the triangle $\overrightarrow{ABX}$ has the same orientation as $\overrightarrow{ABC}$. The region $P_1$ is the set of points $X$ such that $\overrightarrow{CXA}$, $\overrightarrow{CXB}$ are counterclockwise. The region $P_2$ is the set of points $X$ such that $\overrightarrow{CXA}$ and $\overrightarrow{CXB}$ are clockwise. The union $P_1 \cup P_2$ is the admissible region for $D$ under fixed points $A$, $B$, and $C$.

**Lemma 15.** Let $R_{u,v}$ be circumscribed about a triangle $ABC$. Then

$$Prime(ABC, A) = \frac{6}{\pi^2} Area(ABC) + O(u + v). \quad (17)$$

**Proof.** Without loss of generality we assume that $A$ coincides with the origin $O = (0,0)$. Denote by $i \cdot ABC$ the triangle $ABC$ scaled for a given factor $i > 0$, i.e.

$$i \cdot ABC = \{(i \cdot x, i \cdot y) \in \mathbb{Z}^2 | (x, y) \in ABC\}.$$  

We start with the equation

$$L(ABC) = Prime(ABC) + Prime\left(\frac{1}{2} \cdot ABC\right) + Prime\left(\frac{1}{3} \cdot ABC\right) + \ldots$$

$$= \sum_{j=1}^{u+v} Prime\left(\frac{1}{j} \cdot ABC\right)$$

and the consequent one

$$L\left(\frac{1}{p} \cdot ABC\right) = \sum_{q=1}^{\infty} Prime\left(\frac{1}{p \cdot q} \cdot ABC\right)$$

$$= \sum_{q=1}^{(u+v)/p} Prime\left(\frac{1}{p \cdot q} \cdot ABC\right).$$

Using (10) we proceed with

$$Prime(ABC) = \sum_{l=1}^{u+v} Prime\left(\frac{1}{l} \cdot ABC\right) \left(\sum_{k|l} \mu(k)\right)$$

$$= \sum_{h=1}^{u+v} \mu(h) \sum_{i=1}^{(u+v)/h} Prime\left(\frac{1}{hi} \cdot ABC\right)$$

$$= \sum_{h=1}^{u+v} \mu(h) L\left(\frac{1}{h} \cdot ABC\right).$$

From [7] we have

$$L(ABC) = Area(ABC) + O(u + v),$$
Figure 4: $C \in ABR_1$, the grey triangle is the admissible region for $D$

and hence

\[
\sum_{h=1}^{u+v} \mu(h) \mathcal{L} \left( \frac{1}{h} \cdot ABC \right) = \sum_{h=1}^{u+v} \mu(h) \frac{1}{h^2} \left( \text{Area}(ABC) + O(u + v) \right)
\]

\[
= \text{Area}(ABC) \left( \sum_{h=1}^{\infty} \frac{\mu(h)}{h^2} - \sum_{h=u+v+1}^{\infty} \frac{\mu(h)}{h^2} \right) + O(u + v)
\]

\[
(11) \quad \frac{6}{\pi^2} \text{Area}(ABC) + O \left( \frac{\text{Area}(ABC)}{u + v} + u + v \right)
\]

\[
= \frac{6}{\pi^2} \text{Area}(ABC) + O(u + v).
\]

**Corollary 16.** Let $R_{u,v}$ be circumscribed about a triangle $ABC$. Then the number of internal points $X$ of $ABC$ such that $AX$ is a prime segment is

\[
\frac{6}{\pi^2} \text{Area}(ABC) + O(u + v).
\]

The lemma and the corollary above will be applied in the rest of the section and in Section 5.2 in the following way. Lemma 15 implies that the set of points $D$ such that $AB$ and $CD$ are in convex position is contained in some admissible region $P$, which is either a triangle with $C$ being its vertex or a pair of triangles that have a unique common point, which is $C$ and a vertex of each of them. In both cases we use Lemma 15 (or its corollary) to estimate the number of possible points $D$ in $P$ such that $CD$ is a prime segment.

**Lemma 17.**

\[
|Z_2^b(u, v)| = \begin{cases} 
\frac{1}{\pi^2} u^2 v^2 + O(u^2 v + uv^2) & \text{if } u \perp v, \\
0 & \text{otherwise}. 
\end{cases}
\]

**Proof.** First, we notice that for non-coprime $u$ and $v$ the diagonal of $R_{u,v}$ is not a prime segment and the set $Z_2^b(u, v)$ is empty.

Let now $u$ and $v$ be coprime. Let us denote the vertices of $R_{u,v}$ by $R_1, R_2, R_3,$ and $R_4$ as in Fig. 4, and consider a pair $\{AB, CD\}$ from $Z_2^b(u, v)$. Without loss of generality we assume that $AB$ is a diagonal of $R_{u,v}$, i.e. either $AB = R_2R_4$ or $AB = R_1R_3$. Let us assume that $AB = R_2R_4$. Since, by definition, $CD$ does not intersect $AB$, either $CD \in ABR_1$ or $CD \in ABR_3$. Let us assume that $CD \in ABR_1$ and let $C = (c_1, c_2), D = (d_1, d_2)$. Clearly, $c_1 \neq d_1$ as otherwise $\ell(CD)$ would intersect $AB$. Without loss of generality we assume that $c_1 > d_1$. Let us denote

\[
X = \ell(AC) \cap R_1B = \left( \frac{vc_1}{v - c_2}, 0 \right)
\]
and

\[ Y = \ell(BC) \cap R_1A = \left( 0, \frac{uc_2}{u - c_1} \right). \]

It follows from Lemma 14 that \( AB \) and \( CD \) are in convex position if and only if \( D \) is an interior point of \( ACY \). By Lemma 15, the number of choices for point \( D \) such that \( CD \) is a prime segment is

\[ \frac{6}{\pi^2} \text{Area}(ACY) + o(c_1(v - c_2)) = \frac{6}{\pi^2} \text{Area}(ACY) + O(u + v). \]

Hence, summing up over all possible choices for the point \( C \) in \( ABR_1 \) and multiplying by 4 to take into account the cases of \( C \in ABR_3 \) and \( AB = R_1R_3 \) we derive

\[
|Z_2^b(u, v)| = 4 \sum_{c_1=1}^{u-1} \sum_{c_2=1}^{u-1} \left( \frac{6}{\pi^2} \text{Area}(ACY) + O(u + v) \right)
\]

\[
= \frac{24}{\pi^2} \sum_{c_1=1}^{u-1} \sum_{c_2=1}^{u-1} \text{Area}(ACY) + O(u^2v + uv^2),
\]

where

\[
\text{Area}(ACY) = \frac{c_1 \cdot d(A, Y)}{2} = \frac{1}{2} c_1 \left( v - \frac{uc_2}{u - c_1} \right) = \frac{1}{2} \left( vc_1 - \frac{uc_1c_2}{u - c_1} \right).
\]

Therefore, we have

\[
|Z_2^b(u, v)| = \frac{12}{\pi^2} \sum_{c_1=1}^{u-1} \sum_{c_2=1}^{u-1} \left( \frac{6}{\pi^2} \text{Area}(ACY) + O(u^2v + uv^2) \right)
\]

\[
= \frac{3}{\pi^2} \sum_{c_1=1}^{u-1} \left( \frac{v^2c_1(u - c_1)}{u} - \frac{uc_1}{u - c_1} \left( \frac{v^2(u - c_1)^2}{2u^2} + O \left( \frac{v(u - c_1)}{u} \right) \right) \right)
\]

\[
+ O(u^2v + uv^2)
\]

\[
= \frac{12}{\pi^2} \sum_{c_1=1}^{u-1} \left( \frac{v^2c_1(u - c_1)}{2u} \right) + O(u^2v + uv^2)
\]

\[
= \frac{6v^2}{\pi^2} \sum_{c_1=1}^{u-1} \left( c_1 - \frac{c_1^3}{u} \right) + O(u^2v + uv^2)
\]

\[
= \frac{6v^2}{\pi^2} \left( \frac{(u - 1)^2}{2} - \frac{(u - 1)^3}{3u} + O(u) \right) + O(u^2v + uv^2)
\]

\[
= \frac{1}{\pi^2} u^2v^2 + O(u^2v + uv^2).
\]

\[ \square \]

**Lemma 18.** \( |Z_2^b(u, v)| = \frac{42}{\pi^2} u^2v^2 + o(u^2v^2). \)

**Proof.** Consider a pair \( \{AB, CD\} \) from \( Z_2^b(u, v) \). Without loss of generality we assume \( \{A, B, C, D\} \cap \text{Vert}(R_{u,v}) = \{A, C\} \), that is either \( \{A, C\} = \{R_1, R_3\} \) or \( \{A, C\} = \{R_2, R_4\} \). The cases are symmetric, and hence it suffices to consider one of them, say \( \{A, C\} = \{R_1, R_3\} \). Without loss of generality we assume \( A = R_1 \) and \( C = R_3 \) as in Fig. 5. The point \( B = (b_1, b_2) \) belongs to one of the triangles \( ACR_2 \) and \( ACR_4 \). Due to symmetry, we assume without loss of generality \( B \in ACR_2 \), in which case we have \( b_2 > \frac{v b_1}{u} \). Let us denote

\[ X = \ell(AB) \cap CR_2 = \left( \frac{vb_1}{b_2}, v \right). \]

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Figure 5: The point $B$ belongs to $AR_2R_3$, the grey triangles form the admissible region for $D$.

It follows from Lemma 14 that $AB$ and $CD$ are in convex position if and only if $D$ is an interior point of $ACR_4 \cup BCX$ or an interior point of one of the segments $CX$, $AR_4$, or $CR_4$. By Lemma 15 and Corollary 16, the number of possible choices for $D$ such that $CD$ is a prime segment for a fixed $B$ is

$$\frac{6}{\pi^2} (\text{Area}(ACR_4) + \text{Area}(BCX)) + O(u + v),$$

where

$$\text{Area}(ACR_4) = \frac{uv}{2}$$

and

$$\text{Area}(BCX) = \frac{(v - b_2) \cdot d(C,X)}{2} = \frac{1}{2} (v - b_2) \left( u - \frac{vb_1}{b_2} \right) = \frac{1}{2} \left( uv + vb_1 - ub_2 - \frac{v^2 b_1}{b_2} \right).$$

Therefore, summing over all possible choices of $B$ and multiplying by 4 to take into account the cases $B \in ACR_4$ and $\{A, C\} = \{R_2, R_4\}$ we derive:

$$|Z_2^f(u, v)| = 4 \sum_{b_1=1}^{u-1} \sum_{b_2=\left\lfloor \frac{vb_1}{b_2} + 1 \right\rfloor}^{v} \frac{6}{\pi^2} (\text{Area}(ACR_4) + \text{Area}(BCX) + O(u + v))$$

$$= \frac{12}{\pi^2} \sum_{b_1=1}^{u-1} \sum_{b_2=\left\lfloor \frac{vb_1}{b_2} + 1 \right\rfloor}^{v} \left( 2uv + vb_1 - ub_2 - \frac{v^2 b_1}{b_2} \right) + O(u^2 v + uv^2). \quad (19)$$

We will estimate the asymptotics of different summands of (19) separately.

1. **Estimation of** $\sum \sum 2uv$.

Using formulas (4), (8), and (5) we obtain

$$\sum_{b_1=1}^{u-1} \sum_{b_2=\left\lfloor \frac{vb_1}{b_2} + 1 \right\rfloor}^{v} 1 = \sum_{b_1=1}^{u-1} \left( v \frac{\phi(b_1)}{b_1} - \frac{v}{u} \phi(b_1) + O \left( 2^w(b_1) \right) \right)$$

$$= v \left( \frac{6}{\pi^2} u + O(\log u) \right) - \frac{v}{u} \left( \frac{3}{\pi^2} u^2 + O(u \log u) \right) + O(u \log u)$$

$$= \frac{3}{\pi^2} uv + O(v \log u) + O(u \log u). \quad (21)$$
Changing the order of summation in the above sum, we deduce the same result, but with a slightly different error term:

\[
\sum_{b_1=1}^{u-1} \sum_{\substack{b_2=1 \ b_2 \perp b_1 \ b_2=1 \ b_2 \perp b_1 \ b_2+1}}^{v} 1 = \sum_{b_2=1}^{v} \left[ \sum_{b_1=1}^{\left\lfloor \frac{ub_2}{v} \right\rfloor} 1 \right] \\
= \sum_{b_2=1}^{v} \left( \frac{\phi(b_2)}{b_2} \left\lfloor \frac{ub_2}{v} - 1 \right\rfloor + O\left(2^{\omega(b_2)}\right) \right) \\
= \sum_{b_2=1}^{v} \left( \frac{b_2}{v} \phi(b_2) + O\left(\frac{\phi(b_2)}{b_2} \right) + O\left(2^{\omega(b_2)}\right) \right) \\
= \frac{u}{v} \left( \frac{3}{\pi^2} v^2 + O(v \log v) \right) + O(v) + O(v \log v) \\
= \frac{3}{\pi^2} uv + O(u \log v) + O(v \log v). \tag{22}
\]

Finally, denoting \(\alpha = \max(u, v)\) and \(\beta = \min(u, v)\) we derive from (21) and (22)

\[
\sum_{b_1=1}^{u-1} \sum_{\substack{b_2=1 \ b_2 \perp b_1 \ b_2+1}}^{v} 1 = \frac{3}{\pi^2} uv + O(\alpha \log \beta) + O(\beta \log \beta) \\
= \frac{3}{\pi^2} uv + O(u \log v + v \log u), \tag{23}
\]

and hence

\[
\sum_{b_1=1}^{u-1} \sum_{\substack{b_2=1 \ b_2 \perp b_1 \ b_2+1}}^{v} 2uv = \frac{6}{\pi^2} u^2 v^2 + O(u^2 v \log v + u v^2 \log u). \tag{24}
\]

2. Estimation of \(\sum \sum vb_1\).

Using formulas (20) and (8) we obtain

\[
\sum_{b_1=1}^{u-1} \sum_{\substack{b_2=1 \ b_2 \perp b_1 \ b_2+1}}^{v} b_1 = \sum_{b_1=1}^{u-1} b_1 \sum_{\substack{b_2=1 \ b_2 \perp b_1 \ b_2+1}}^{v} 1 \\
= \sum_{b_1=1}^{u-1} \left( v\phi(b_1) - \frac{v}{u} b_1 \phi(b_1) + O\left(b_1 2^{\omega(b_1)}\right) \right) \\
= \frac{3}{\pi^2} u^2 v + O(u v \log u) - \frac{v}{u} \left( \frac{2}{\pi^2} u^3 + O(u^2 \log u) \right) + O(u^2 \log u) \\
= \frac{1}{\pi^2} u^2 v + O(uv \log u) + O(u^2 \log u). \tag{25}
\]

Again, changing the order of summation in the above sum, we deduce the same result with a
Similarly to the previous case, by changing the order of summation, one can show that

different error term:

\[
\sum_{b_1=1}^{u-1} \sum_{b_2=\left[\frac{ub_2}{v}+1\right]}^{v} b_1 = \sum_{b_2=1}^{v} \left[ \sum_{b_1=1}^{\left[\frac{ub_2}{v}\right]} b_1 \right]
\]

\[
= \sum_{b_2=1}^{v} \left( \frac{\phi(b_2)}{2b_2} \left[ \frac{ub_2}{v} - 1 \right]^2 + \frac{ub_2}{v} O \left( 2^w(b_2) \right) \right)
\]

\[
= \sum_{b_2=1}^{v} \left( \frac{u^2}{2v} b_2 \phi(b_2) + \frac{u}{2v} \phi(b_2) + \frac{ub_2}{v} O \left( 2^w(b_2) \right) \right)
\]

\[
= \frac{1}{\pi^2} u^2 v + O(u^2 \log v) + O(u v \log v).
\]

Finally, denoting \( \alpha = \max(u, v) \) and \( \beta = \min(u, v) \) we derive from (25) and (26)

\[
\sum_{b_1=1}^{u-1} \sum_{b_2=\left[\frac{ub_2}{v}+1\right]}^{v} b_1 = \frac{1}{\pi^2} u^2 v + u \left( O(\alpha \log \beta) + O(\beta \log \beta) \right) = \frac{1}{\pi^2} u^2 v + o(u^2 v),
\]

and hence

\[
\sum_{b_1=1}^{u-1} \sum_{b_2=\left[\frac{ub_2}{v}+1\right]}^{v} v b_1 = \frac{1}{\pi^2} u^2 v^2 + o(u^2 v^2)
\]

3. Estimation of \( \sum \sum ub_2 \).

Using formulas (4), (5), and (8) we obtain:

\[
\sum_{b_1=1}^{u-1} \sum_{b_2=\left[\frac{ub_2}{v}+1\right]}^{v} b_2 = \sum_{b_1=1}^{u-1} \left( \frac{\phi(b_1)v^2}{2b_1} - \frac{\phi(b_1)b_1 v^2}{2u^2} + O \left( v^2 w(b_1) \right) \right)
\]

\[
= \frac{v^2}{2} \sum_{b_1=1}^{u-1} \left( \frac{\phi(b_1)}{b_1} - \frac{1}{u^2} b_1 \phi(b_1) \right) + O(u v \log u)
\]

\[
= \frac{v^2}{2} \left( \frac{6}{\pi^2} u + O(\log u) - \frac{1}{u^2} \left( \frac{2}{\pi^2} u^3 + O(u^2 \log u) \right) \right) + O(u v \log u)
\]

\[
= \frac{2}{\pi^2} u v^2 + O(u v \log u) + O(v^2 \log u).
\]

Similarly to the previous case, by changing the order of summation, one can show that

\[
\sum_{b_1=1}^{u-1} \sum_{b_2=\left[\frac{ub_2}{v}+1\right]}^{v} b_2 = \frac{2}{\pi^2} u v^2 + O(u v \log v) + O(v^2 \log v),
\]

which together with (28) imply

\[
\sum_{b_1=1}^{u-1} \sum_{b_2=\left[\frac{ub_2}{v}+1\right]}^{v} b_2 = \frac{2}{\pi^2} u v^2 + o(u v^2),
\]
and hence
\[
\sum_{b_1=1}^{u-1} \sum_{b_2=1}^{v} \sum_{b_2 \perp b_1}^{\lfloor \frac{v b_1}{b_2} \rfloor+1} u b_2 = \frac{2}{\pi^2} u^2 v^2 + o(u^2 v^2).
\] (29)

4. Estimation of \( \sum \sum \frac{v^2 b_1}{b_2} \).

Using formulas (6), (7), (8), and the fact that \( \log |x| = \log x + O \left( \frac{1}{x} \right) \) for \( x \geq 1 \), we obtain
\[
\sum_{b_1=1}^{u-1} \sum_{b_2=1}^{v} \sum_{b_2 \perp b_1}^{\lfloor \frac{v b_1}{b_2} \rfloor+1} b_1 = \sum_{b_1=1}^{u-1} b_1 \left( \sum_{b_2=1}^{\lfloor \frac{v b_1}{b_2} \rfloor} \frac{1}{b_2} - \sum_{b_2=1}^{\lfloor \frac{v b_1}{b_2} \rfloor} \frac{1}{b_2} \right)
\]
\[
= \sum_{b_1=1}^{u-1} b_1 \left( \phi(b_1) \log v + \phi(b_1) O \left( \frac{1}{v} \right) - \phi(b_1) \log \frac{v b_1}{b_1} \right)
\]
\[
= \sum_{b_1=1}^{u-1} \phi(b_1) \left( \log v - \log \frac{v b_1}{b_1} + O \left( \frac{1}{v} \right) + O \left( \frac{u}{v b_1} \right) \right)
\]
\[
= \sum_{b_1=1}^{u-1} \phi(b_1) \left( \log u - \log b_1 + O \left( \frac{1}{v b_1} \right) + O \left( \frac{u^2}{v b_1} \right) \right)
\]
\[
= \frac{3}{2 \pi^2} u^2 \log u + O(u \log^2 u) - \frac{3}{2 \pi^2} u^2 \log u + \frac{3}{2 \pi^2} u^2 + o(u^2) + O \left( \frac{u^2}{v} \right)
\]
\[
= \frac{3}{2 \pi^2} u^2 + o(u^2) + O \left( \frac{u^2}{v} \right),
\] (30)

and hence
\[
\sum_{b_1=1}^{u-1} \sum_{b_2=1}^{v} \sum_{b_2 \perp b_1}^{\lfloor \frac{v b_1}{b_2} \rfloor+1} v^2 b_1 = \frac{3}{2 \pi^2} u^2 v^2 + o(u^2 v^2).
\] (31)

Finally, combining (19), (24), (27), (29), and (31) we derive
\[
|Z_2^U(u, v)| = \frac{12}{\pi^2} \left( \frac{6}{\pi^2} u^2 v^2 + \frac{1}{\pi^2} u^2 v^2 - \frac{2}{\pi^2} u^2 v^2 - \frac{3}{2 \pi^2} u^2 v^2 + o(u^2 v^2) \right)
\]
\[
= \frac{42}{\pi^2} u^2 v^2 + o(u^2 v^2).
\]

5.2 The number of pairs of segments with one corner point

There is no loss of generality in assuming \( \{A, B, C, D\} \cap \text{Vert}(R_{u,v}) = \{A\} \) for every pair \( \{AB, CD\} \in Z_1(u, v) \). We consider the partition of \( Z_1(u, v) \) into the following two subsets:

1. \( Z_2^U(u, v) \) the set of those pairs \( \{AB, CD\} \) in which the point \( B \) is an interior point of \( R_{u,v} \);

2. \( Z_2^C(u, v) \) the set of those pairs \( \{AB, CD\} \) in which the point \( B \) belongs to the boundary of \( R_{u,v} \).

In the rest of the section we estimate the sizes of these sets in separate lemmas.

Lemma 19. \( |Z_2^C(u, v)| = \frac{72}{\pi^2} u^2 v^2 + o(u^2 v^2) \).

Proof. Due to symmetry, for a corner point \( R \) of \( R_{u,v} \) the number of pairs \( \{AB, CD\} \in Z_2^C(u, v) \), where \( A \) coincides with \( R \), is the same for every
Figure 6: The case $B \in AR_2R_3$. The points $C$ and $D$ belong to the segments $R_3R_4$ and $XR_3$ respectively, where $X = \ell(AB) \cap R_2R_3$.

$R \in \{R_1, R_2, R_3, R_4\}$. Therefore, it is enough to estimate the number of pairs where $A$ coincides with a fixed corner point of $R_{u,v}$, and we assume that $A = R_1$.

Since $B$ is an interior point of $R_{u,v}$ and neither $C$ nor $D$ is a corner point of $R_{u,v}$, we conclude that one of $C$ and $D$ belongs to the interior of $R_2R_3$ and the other belongs to the interior of $R_3R_4$. Without loss of generality, we assume that $C$ is an interior point of $R_3R_4$ and $D$ is an interior point of $R_2R_3$.

Under the above assumptions, we will first estimate the number of pairs in $Z^u_1(u,v)$ in which $B = (b_1, b_2)$ belongs to the triangle $AR_2R_3$. Notice that the latter assumption is equivalent to the inequality $\frac{b_1}{b_2} \leq \frac{3}{2}$. Let us denote

$$X = \ell(AB) \cap R_2R_3 = \left(\frac{vb_1}{b_2}, v\right).$$

It follows from Lemma 14 that $AB$ and $CD$ are in convex position if and only if $D$ is an interior point of $XR_3$ (see Fig. 6). Therefore, by denoting $D = (d_1, v)$ and $C = (u, c_2)$, the number of desired prime pairs segments can be expressed as

$$\sum_{b_1=1}^{u-1} \sum_{b_2=1}^{v-1} \sum_{c_2=1}^{u-1} \sum_{d_1=1}^{v-1} 1.$$ (32)

We start by estimating the contribution of the latter two sums.

$$\sum_{c_2=1}^{u-1} \sum_{d_1=1}^{v-1} \sum_{d_1=1}^{v-1} \sum_{c_2=1}^{u-1} 1 + O(u)$$ (33)

$$= \sum_{c_2=1}^{u-1} \sum_{d_1=1}^{v-1} \left( \phi\left(\frac{c_2}{2}\right) \left( u - v \frac{b_1}{b_2} + O(1) \right) + O\left(2^{\phi\left(\frac{c_2}{2}\right)}\right) \right) + O(u)$$

$$= \sum_{c_2=1}^{u-1} \left( \phi\left(\frac{c_2}{2}\right) \left( u - v \frac{b_1}{b_2} + O(1) \right) + O\left(2^{\phi\left(\frac{c_2}{2}\right)}\right) \right) + O(u)$$

$$= \left(\frac{6v}{\pi^2} + O(\log v)\right) \left( u - v \frac{b_1}{b_2} + O(1) \right) + O(v \log v) + O(u)$$

$$= \left(\frac{6v}{\pi^2}\right) \left( u - v \frac{b_1}{b_2} \right) + O(u \log v) + O\left(\frac{b_1}{b_2} v \log v\right) + O(v \log v)$$

$$= \left(\frac{6v}{\pi^2}\right) \left( u - v \frac{b_1}{b_2} \right) + O(v \log v) + O(u \log v).$$
By changing the order of summation in (33), one can show that

$$
\sum_{c_2=1}^{v-1} \sum_{d_1=\left\lfloor \frac{u}{b_2} \right\rfloor +1}^u 1 = \frac{6v}{\pi^2} \left( u - v \frac{b_1}{b_2} \right) + O(u \log v + v \log u).
$$

Now, plugging in the above result to (32) and using formulas (23) and (30) we obtain:

$$
\sum_{b_1=1}^{u-1} \sum_{b_2=\left\lfloor \frac{u}{b_2} \right\rfloor +1}^u \sum_{c_2=1}^{v-1} \sum_{d_1=\left\lfloor \frac{u}{b_2} \right\rfloor +1}^u 1
$$

$$
= \sum_{b_1=1}^{u-1} \sum_{b_2=\left\lfloor \frac{u}{b_2} \right\rfloor +1}^u \left( \frac{6v}{\pi^2} \left( u - v \frac{b_1}{b_2} \right) + O(u \log v + v \log u) \right)
$$

$$
= \frac{6v}{\pi^2} \left( u \left( \frac{3}{\pi^2} uv + O(u \log v + v \log u) \right) - v \left( \frac{3}{2\pi^2} u^2 + o(u^2) + O \left( \frac{u^2}{v} \right) \right) \right)
$$

$$
+ O(u^2v \log v + uv^2 \log u)
$$

$$
= \frac{9}{\pi^4} u^2v^2 + v^2o(u^2) + O(u^2v \log v + uv^2 \log u).
$$

Note that the obtained estimation is symmetric with respect to $u$ and $v$, which implies that the number of pairs in $Z_1^*(u, v)$ in which $B$ belongs to $AR_3R_4$ has the same asymptotics. Therefore, taking into account additionally all symmetric cases corresponding to the location of $A$, we finally conclude that

$$
|Z_1^*(u, v)| = \frac{72}{\pi^4} u^2v^2 + o(u^2v^2).
$$

**Lemma 20.**

$$
|Z_1^*(u, v)| = \frac{6v}{\pi^2} \sum_{b_1=1}^u (2u - b_1) + \frac{6u^2}{\pi^2} \sum_{c_2=1}^v (2v - c_2) + O(u^2v + uv^2).
$$

**Proof.** As in the proof of Lemma 19, due to symmetry, for a corner point $R$ of $\mathcal{R}_{u,v}$ the number of pairs $\{AB, CD\} \in Z_1^*(u, v)$, where $A$ coincides with $R$, is the same for every $R \in \{R_1, R_2, R_3, R_4\}$. Therefore, it is enough to estimate the number of pairs where $A$ coincides with a fixed corner point of $\mathcal{R}_{u,v}$, and we assume that $A = R_1$.

It is easy to see that if $B$ is an internal point of $R_1R_2$ or $R_1R_4$, then one of $C$ and $D$ belongs to the interior of $R_2R_3$ and the other belongs to the interior of $R_3R_4$. Therefore, taking into account primality of $AB$, the number of pairs, in which $B$ is an internal point of $R_1R_2$ or $R_1R_4$, is $O(uv)$. The latter does not affect the asymptotics, and without loss of generality we assume from now on that $B$ is an internal point of one of the sides $R_2R_3$ and $R_3R_4$.

Suppose first that $B$ is an internal point of $R_2R_3$, i.e. $B = (b_1, v)$ for some $0 < b_1 < u$, and $b_1 \perp v$. Then $AB \cap R_3R_4 = \emptyset$, and hence $CD \cap R_3R_4 \neq \emptyset$, which implies that either $C$ or $D$ belongs to the interior of $R_3R_4$. Without loss of generality we assume the former, i.e. $C = (u, c_2)$ for some $0 < c_2 < v$ (see Fig. 7). Under these assumptions, Lemma 14 implies that $AB$ and $CD$ are in convex position if and only if the point $D$ belongs to $BCR_3 \setminus (BC \cup \{R_3\})$ or $ACR_1 \setminus (AC \cup \{R_4\})$. Therefore, using
Lemma 15 and Corollary 16, we conclude that the number of such pairs of prime segments is

\[
\sum_{b_1=1, c_2=1}^{u-1} \sum_{b_1 \perp v}^{v-1} \frac{6}{\pi^2} (\text{Area}(BCR_3) + \text{Area}(ACR_4) + O(u + v))
\]

\[
= \frac{3}{\pi^2} \sum_{b_1=1, c_2=1}^{u} \sum_{b_1 \perp v}^{v} ((u - b_1)(v - c_2) + uc_2) + O(u^2v + uv^2)
\]

\[
= \frac{3}{\pi^2} \sum_{b_1=1, c_2=1}^{u} \sum_{b_1 \perp v}^{v} (uv - vb_1 + b_1c_2) + O(u^2v + uv^2)
\]

\[
= \frac{3}{\pi^2} \sum_{b_1=1, c_2=1}^{u} \left( (uv - vb_1)v + b_1 \left( \frac{v^2}{2} + O(v) \right) \right) + O(u^2v + uv^2)
\]

\[
= \frac{3v^2}{\pi^2} \sum_{b_1=1, c_1=1}^{u} \left( u - \frac{b_1}{2} \right) + O(u^2v + uv^2).
\]

By symmetry, the number of pairs in which \(B\) is an internal point of \(R_3R_4\) is

\[
\frac{3u^2}{\pi^2} \sum_{c_2=1, c_2 \perp u}^{v} \left( v - \frac{c_2}{2} \right) + O(u^2v + uv^2).
\]

Putting all together and taking into account the symmetric cases corresponding to the location of \(A\), we finally conclude that

\[
|Z_1^b(u, v)| = \frac{6v^2}{\pi^2} \sum_{b_1=1, b_1 \perp v}^{u} (2u - b_1) + \frac{6u^2}{\pi^2} \sum_{c_2=1, c_2 \perp u}^{v} (2v - c_2) + O(u^2v + uv^2).
\]

We note that in Lemma 20 we deliberately did not compute a closed-form asymptotic, as the obtained formula will be crucial later to obtain a better error term.

5.3 The number of pairs of segments with no corner points

In this section we estimate the size of \(Z_0(u, v)\), i.e. the number of those pairs of segments in \(Z(u, v)\) none of whose endpoints is a corner of \(R_{u,v}\).
Figure 8: Each of $A$, $B$, $C$, and $D$ belongs to a unique side of $R_{u,v}$. The endpoints of the same segment belong to the adjacent sides of $R_{u,v}$.

Lemma 21. $|Z_0(u,v)| = \frac{72}{\pi^2}u^2v^2 + O(u^2v\log v)$.

Proof. Let $\{AB, CD\}$ be an arbitrary pair in $Z_0(u,v)$. The fact that none of the points $A, B, C,$ and $D$ is a corner of $R_{u,v}$ implies that each of the sides of $R_{u,v}$ contains exactly one of these points. Furthermore, since $AB$ and $CD$ are in convex position, we conclude that the endpoints of the same segment belong to the adjacent sides of $R_{u,v}$. Therefore, without loss of generality we can assume $A \in R_2R_3$ and $C \in R_3R_4$, in which case either $B \in R_1R_2$ and $D \in R_1R_4$, or $B \in R_1R_4$ and $D \in R_2R_3$.

The two cases are symmetric and we assume the former one, i.e. $B \in R_1R_2$ and $D \in R_1R_4$ (see Fig. 8).

Let us denote $A = (0, a_2)$, $C = (u, c_2)$, $B = (b_1, v)$, and $D = (d_1, 0)$. Under the above assumptions the segments $AB$ and $CD$ are prime if and only if $(v - a_2) \perp b_1$ and $(u - d_1) \perp c_2$, and the number of such pairs is

$$\sum_{a_2=1}^{v-1} \sum_{b_1=1}^{u-1} \sum_{c_2=1}^{v-1} \sum_{d_1=1}^{u-1} 1 = \sum_{a_2'=1}^{v-1} \sum_{b_1=1}^{u-1} \sum_{c_2'=1}^{v-1} \sum_{d_1'=1}^{u-1} 1.$$

Using formula (9) we obtain

$$\sum_{a_2'=1}^{v-1} \sum_{b_1=1}^{u-1} \sum_{c_2'=1}^{v-1} \sum_{d_1'=1}^{u-1} \left( \frac{6}{\pi^2}uv + O(u\log v) \right) = \frac{36}{\pi^2}u^2v^2 + O(u^2v\log v).$$

Finally, taking into account the symmetric case of $B \in R_1R_4$ and $D \in R_2R_3$, we derive the desired result

$$|Z_0(u,v)| = \frac{72}{\pi^2}u^2v^2 + O(u^2v\log v).$$

5.4 Summarizing results

In the following theorem we prove the main result of the paper by putting everything together.

Theorem 22. $p(m,n) = \frac{25}{12\pi^4}m^4n^4 + o(m^4n^4)$. 

\(\square\)
Proof. First, using (16) and Lemmas 12 and 13, we expand formula (15) as follows:

\[
p(m,n) = \sum_{u=1}^{m-1} (m-u) \sum_{v=1}^{n-1} (n-v) \cdot |Z(u,v)|
\]

\[
= \sum_{u=1}^{m-1} (m-u) \sum_{v=1}^{n-1} (n-v) \cdot (|Z_4(u,v)| + |Z_3(u,v)| + |Z_2(u,v)| + |Z_1(u,v)| + |Z_0(u,v)|)
\]

\[
= \sum_{u=1}^{m-1} (m-u) \sum_{v=1}^{n-1} (n-v) \cdot \left( |Z^b_2(u,v)| + |Z^a_2(u,v)| + |Z^a_1(u,v)| + |Z^b_1(u,v)| + |Z^b_0(u,v)| \right) + o(m^4n^4).
\]

Next, different parts of the above sum can be estimated separately, as follows

\[
\sum_{u=1}^{m-1} (m-u) \sum_{v=1}^{n-1} (n-v) \cdot |Z^b_2(u,v)| = \frac{1}{24\pi^4}m^4n^4 + o(m^4n^4). \quad (34)
\]

\[
\sum_{u=1}^{m-1} (m-u) \sum_{v=1}^{n-1} (n-v) \left( |Z^b_2(u,v)| + |Z^a_2(u,v)| + |Z^a_1(u,v)| + |Z^b_1(u,v)| + |Z^b_0(u,v)| \right) = \frac{31}{24\pi^4}m^4n^4 + o(m^4n^4). \quad (35)
\]

\[
\sum_{u=1}^{m-1} (m-u) \sum_{v=1}^{n-1} (n-v) \cdot |Z^b_1(u,v)| = \frac{3}{4\pi^4}m^4n^4 + o(m^4n^4). \quad (36)
\]

Finally, plugging in (34), (35), and (36) into the initial formula we obtain

\[
p(m,n) = \frac{1}{24\pi^4}m^4n^4 + \frac{31}{24\pi^4}m^4n^4 + \frac{3}{4\pi^4}m^4n^4 + o(m^4n^4) = \frac{25}{12\pi^4}m^4n^4 + o(m^4n^4).
\]

Corollary 10 and Theorem 22 imply Theorem 1.

The following corollary from Corollary 5, Corollary 10, and Theorem 22, reveal the relations between 2-threshold functions, proper pairs of segments, and pairs of prime segments on convex position.

**Corollary 23.** There is a bijection between almost all 2-threshold functions and almost all proper pairs of segments, and a bijection almost all 2-threshold functions and pairs of prime segments in convex position.

### 6 On the number of k-threshold functions for \( k \geq 3 \)

The obtained asymptotic formula for the number of 2-threshold functions can be used to improve upper bound (1) on the number of k-threshold functions for \( k \geq 3 \). Indeed, since a k-threshold function can be seen as a conjunction of several 2-threshold functions and at most one threshold function, we have:

\[
t_k(m,n) \leq \left( \frac{t_2(m,n)}{\binom{k}{2}} \right) = \frac{t_2(m,n)^k}{k^k} + o\left( m^{2k}n^{2k} \right)
\]

\[
= \frac{5^k}{12^k \pi^{2k} \binom{k}{2}} m^{2k}n^{2k} + o\left( m^{2k}n^{2k} \right) \quad (37)
\]

\(^1\)Due to their simplicity and technicality, the proofs are excluded from the main part of the paper and included in the appendix.
for even $k$ and

$$t_k(m, n) \leq \left(\frac{t_2(m, n)}{\binom{\left\lfloor \frac{k}{2} \right\rfloor}{k}}\right) \frac{t(m, n)}{k} = \frac{t_2(m, n)^{\left\lfloor \frac{k}{2} \right\rfloor}}{\binom{\left\lfloor \frac{k}{2} \right\rfloor}{k}} t_k(m, n) + o\left(m^{2k}n^{2k}\right)$$

(38)

for odd $k$. Since for even $k$

$$\frac{5^k}{12\left\lfloor \frac{k}{2} \right\rfloor \pi^{2k} k!} \leq \frac{6^k}{\pi^{2k} k!}$$

if and only if $k \leq 22$, and for odd $k$

$$\frac{5^{k-1}6}{12\left\lfloor \frac{k}{2} \right\rfloor \pi^{2k} k!} \leq \frac{6^k}{\pi^{2k} k!}$$

if and only if $k \leq 23$, we conclude that the upper bounds in (37) and (38) improve estimation (1) for every $3 \leq k \leq 23$.

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A Estimations from Theorem 22

A.1 Estimation of $\sum(m - u) \sum(n - v) \cdot |Z^b_2(u, v)|$

Using Lemma 17 and formulas (4), (5), and (8), we obtain

$$
\sum_{u=1}^{m-1} (m - u) \sum_{v=1}^{n-1} (n - v) \cdot |Z^b_2(u, v)|
= \sum_{u=1}^{m-1} (m - u) \sum_{v=1}^{n-1} (n - v) \cdot |Z^b_2(u, v)|
= \sum_{u=1}^{m-1} (m - u) \sum_{v=1}^{n-1} (n - v) \left( \frac{1}{\pi^2} u^2 v^2 + O(u^2 v + uv^2) \right)
= \frac{1}{\pi} \sum_{u=1}^{m} (mu^2 - u^3) \sum_{v=1}^{n} (nv^2 - v^3) + O(m^4 n^3 + m^3 n^4)
= \frac{1}{\pi} \sum_{u=1}^{m} (mu^2 - u^3) \left( \frac{\phi(u)}{3u} n^4 - \frac{\phi(u)}{4u^2} n^4 + O \left( n^3 2^{w(u)} \right) \right) + O(m^4 n^3 + m^3 n^4)
= \frac{1}{12\pi^2} \sum_{u=1}^{m} \left( mn^4 u \phi(u) - n^4 u^2 \phi(u) + O \left( mn^3 u^2 2^{w(u)} \right) \right) + O(m^4 n^3 + m^3 n^4)
= \frac{1}{12\pi^4} \left( \frac{2m^4 n^4}{\pi^2} - \frac{3m^4 n^4}{2\pi^2} + O \left( m^4 n^3 \log m \right) \right) + O(m^4 n^3 + m^3 n^4)
= \frac{1}{24\pi^4} m^4 n^4 + O(m^4 n^3 \log m + m^3 n^4).
$$

(39)

Now, symmetry of formula (39) implies also the estimation with a symmetric error term

$$
\sum_{u=1}^{m-1} (m - u) \sum_{v=1}^{n-1} (n - v) \cdot |Z^b_2(u, v)| = \frac{1}{24\pi^4} m^4 n^4 + O \left( m^3 n^4 \log n + m^4 n^3 \right).
$$

(40)

Finally, comparing the two estimations one can derive

$$
\sum_{u=1}^{m-1} (m - u) \sum_{v=1}^{n-1} (n - v) \cdot |Z^b_2(u, v)| = \frac{1}{24\pi^4} m^4 n^4 + o(m^4 n^4).
$$

(40)
A.2 Estimation of \( \sum (m - u) \sum (n - v) \left( |Z^*_2(u,v)| + |Z^*_1(u,v)| + |Z_0(u,v)| \right) \)

Using Lemmas 18, 19, 21, and formula (3), we obtain

\[
\sum_{u=1}^{m-1} (m - u) \sum_{v=1}^{n-1} (n - v) \left( |Z^*_2(u,v)| + |Z^*_1(u,v)| + |Z_0(u,v)| \right)
= \sum_{u=1}^{m-1} (m - u) \sum_{v=1}^{n-1} (n - v) \left( \frac{42}{\pi^4} u^2 v^2 + \frac{72}{\pi^4} u^2 v^2 + \frac{72}{\pi^4} u^2 v^2 + o(u^2 v^2) \right)
= \frac{186}{\pi^4} \sum_{u=1}^{m} (m - u) u^2 \sum_{v=1}^{n} (n - v) v^2 + o(m^4 n^4)
= \frac{186}{\pi^4} \sum_{u=1}^{m} (m - u) u^2 \left( \frac{n^4}{3} - \frac{n^4}{4} + O(n^3) \right) + o(m^4 n^4)
= \frac{31}{2\pi^4} n^4 \sum_{u=1}^{m} (m - u) u^2 + o(m^4 n^4) = \frac{31}{24\pi^4} m^4 n^4 + o(m^4 n^4). \quad (41)
\]

A.3 Estimation of \( \sum (m - u) \sum (n - v) \cdot |Z^*_1(u,v)| \)

Using Lemma 20 we derive

\[
\sum_{u=1}^{m-1} (m - u) \sum_{v=1}^{n-1} (n - v) \cdot |Z^*_1(u,v)|
= \sum_{u=1}^{m-1} (m - u) \sum_{v=1}^{n-1} (n - v) \left( \frac{6v^2}{\pi^2} \sum_{b_1=1}^{u} (2v - b_1) + \frac{6u^2}{\pi^2} \sum_{c_2=1}^{v} (2v - c_2) + O(u^2 v + u v^2) \right)
= \frac{6}{\pi^2} \sum_{u=1}^{m} (m - u) \sum_{v=1}^{n} (n - v) v^2 \sum_{b_1=1}^{u} (2v - b_1)
+ \frac{6}{\pi^2} \sum_{v=1}^{n} (n - v) \sum_{u=1}^{m} (m - u) u^2 \sum_{c_2=1}^{v} (2v - c_2) + o(m^4 n^4). \quad (42)
\]

We notice that the first of the summands in the latter formula is obtained from the second one by swapping \( u \) with \( v \), \( b_1 \) with \( c_2 \), and \( m \) with \( n \), hence it suffices to find a closed-form estimation only for one of them, say for the first one.

Using formula (4) we obtain

\[
\sum_{v=1}^{n} (n - v) v^2 \sum_{b_1=1}^{u} (2v - b_1)
= \sum_{v=1}^{n} (n - v) v^2 \left( 2\phi(v) \frac{u^2}{v} - \phi(v) \frac{u^2}{2v} u^2 + O(u^2 v^2) \right)
= \frac{3}{2} \sum_{v=1}^{n} \left( u^2 n v \phi(v) - u^2 v^2 \phi(v) + O(u^2 v^2) \right)
= \frac{3}{2} \left( u^2 n \frac{2}{\pi^2} n^3 - u^2 \frac{3}{2\pi^2} n^4 \right) + O(u^2 n^3 \log n) + O(u n^4 \log n)
= \frac{3}{4\pi^2} u^2 n^4 + n^2 \left( O(u^2 n \log n) + O(u n^2 \log n) \right). \quad (43)
\]
By changing the order of summation in the above sum, we deduce the same result with a different error term:

\[ \sum_{v=1}^{n} (n - v)v^2 \sum_{b_1=1}^{u} (2u - b_1) = \sum_{b_1=1}^{u} (2u - b_1) \sum_{v=1}^{n} (n - v)v^2 \]

\[ = \sum_{b_1=1}^{u} (2u - b_1) \left( \frac{\phi(b_1)}{3b_1} n^4 - \frac{\phi(b_1)}{4b_1} n^4 + O \left( n^3 w(b_1) \right) \right) \]

\[ = \frac{1}{12} \sum_{b_1=1}^{u} \left( 2un^4 \frac{\phi(b_1)}{b_1} - n^4 \phi(b_1) + O \left( n^3 u^2 w(b_1) \right) \right) \]

\[ = \frac{1}{12} \left( 2un^4 \frac{6}{\pi^2} u + O(un^4 \log u) - n^4 \frac{3}{\pi^2} u^2 + O(un^4 \log u) + O(n^3 u^2 \log u) \right) \]

\[ = \frac{3}{4\pi^2} u^2 n^4 + n^2 \left( O(un^2 \log u) + O(nu^2 \log u) \right). \quad (44) \]

Comparing the error terms in (43) and (44) we obtain

\[ \sum_{v=1}^{n} (n - v)v^2 \sum_{b_1=1}^{u} (2u - b_1) = \frac{3}{4\pi^2} u^2 n^4 + o(u^2 n^4). \]

Using the obtained formula and formula (3) we proceed

\[ \frac{6}{\pi^2} \sum_{u=1}^{m} (m - u) \sum_{v=1}^{n} (n - v)v^2 \sum_{b_1=1}^{u} (2u - b_1) = \frac{6}{\pi^2} \sum_{u=1}^{m} (m - u) \left( \frac{3}{4\pi^2} u^2 n^4 + o(u^2 n^4) \right) \]

\[ = \frac{9n^4}{2\pi^4} \sum_{u=1}^{m} (mu^2 - u^3) + o(m^4 n^4) \]

\[ = \frac{9n^4}{2\pi^4} \left( \frac{m^4}{3} - \frac{m^4}{4} \right) + o(m^4 n^4) \]

\[ = \frac{3}{8\pi^4} m^4 n^4 + o(m^4 n^4). \]

Due to symmetry, the second summand in formula (42) has the same asymptotics, and therefore

\[ \sum_{u=1}^{m-1} (m - u) \sum_{v=1}^{n-1} (n - v) \cdot |Z^b(u, v)| = \frac{3}{4\pi^4} m^4 n^4 + o(m^4 n^4). \]

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