An Improved Determination of the Fermi Coupling Constant, $G_F$

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Over 40 years after the calculation of the 1-loop QED corrections to the muon lifetime, new theoretical developments have made it possible to obtain an analytic expression for the complete 2-loop QED contributions in the Fermi theory. The exact result for the effects of virtual and real photons, virtual electrons, muons and hadrons as well as $e^+e^-$ pair creation is

$$\Delta \Gamma_{\text{QED}}^{(2)} = \Gamma_0 \left(\frac{\alpha}{\pi}\right)^2 \left( \frac{156815}{5184} - \frac{1036}{27} \zeta(2) - \frac{895}{36} \zeta(3) + \frac{67}{8} \zeta(4) + 53 \zeta(2) \ln 2 - (0.042 \pm 0.002) \right)$$

where $\Gamma_0$ is the tree-level width. This eliminates the theoretical error in the extracted value of the Fermi coupling constant, $G_F$, which was previously the source of the dominant uncertainty. The new value is

$$G_F = (1.16637 \pm 0.00001) \times 10^{-5} \text{ GeV}^{-2}.$$

The overall error has been roughly halved and is now entirely experimental. Several experiments are planned for the next generation of muon lifetime measurements and these can proceed unhindered by theoretical uncertainties.

I. INTRODUCTION

The three fundamental input parameters that enter into all calculations of electroweak physics are the electromagnetic coupling constant, $\alpha$, the Fermi coupling constant, $G_F$, and the mass of the $Z^0$ boson, $M_Z$. Their current best values, along with their absolute and relative errors are

$$\alpha = 1/(137.0359895 \pm 0.0000061) \quad (0.045 \text{ ppm})$$
$$G_F = (1.16639 \pm 0.00002) \times 10^{-5} \text{ GeV}^{-2} \quad (17 \text{ ppm})$$
$$M_Z = 91.1867 \pm 0.0021 \text{ GeV} \quad (23 \text{ ppm})$$

In the mid-80’s, just before the turn on of LEP, a CERN report concluded that the error on $M_Z$ would be $\pm 50 \text{ MeV}$ or $550 \text{ ppm}$ and that “A factor of 2–3 improvement can be reached with a determined effort.” It was thus generally believed that the error on $M_Z$ represented the limiting factor in the precision with which theoretical predictions could be made. The situation has changed adiabatically, however, and the relative error on $M_Z$ now approaches that of $G_F$.

Since LEP, as a machine, is not a radically new design, the lesson that we should take is that it is extremely difficult to predict the accuracy with which physical quantities will be measured, even in the relatively short term, and that one should constantly strive to reduce such errors to the minimum level consistent with the available technology. The possibility of precision physics at a muon collider serves to emphasize this point.

With this in mind, and given the great cost and effort that was expended in reducing the error on $M_Z$ to its current value, it is reasonable to look again at $G_F$ and see what is required to reduce its error to a level where it can never become an obstacle limiting the accuracy with which theoretical predictions can be made.

$G_F$ is extracted from the measured value of the muon lifetime, $\tau_\mu = (2.19703 \pm 0.00004) \mu\text{s}$ and on the experimental side this is currently the source of the dominant error. New experiments are planned at the Brookhaven National Laboratory, the Paul Scherrer Institute and the Rutherford-Appleton Laboratory and it is likely that the uncertainty on $G_F$ from this source will be reduced to somewhere in the range 0.5–1 ppm.

Most of the work reported here appears in refs [4,5].
II. THE FERMI COUPLING CONSTANT

As given above the current relative error on the Fermi constant is \( \delta G_F / G_F = 1.7 \times 10^{-5} \). Of this 0.9 \times 10^{-5} is experimental and 1.5 \times 10^{-5} is theoretical being an estimate of unknown 2-loop QED corrections.

\[ G_F \] is related to the measured muon lifetime, \( \tau \), by the formula

\[ \frac{1}{\tau_\mu} \equiv \Gamma_\mu = \Gamma_0 (1 + \Delta q). \] (1)

where

\[ \Gamma_0 = \frac{G_F^2 m_\mu^5}{192\pi^3} \] (2)

as calculated using the Fermi theory in which the weak interactions are described by a contact interaction. \( \Delta q \) encapsulates the higher order QED corrections and may written as a perturbation series in \( \alpha_r = e^2 / (4\pi) \), the renormalized electromagnetic coupling constant. Thus

\[ \Delta q = \sum_{i=0}^{\infty} \Delta q^{(i)} \] (3)

in which the index \( i \) gives the power of, \( \alpha_r \) that appears in \( \Delta q^{(i)} \). Note that Eq.(3) differs from the usual formula in ways that begin to become important at the part-per-million level. It is known that \( \Delta q^{(0)} = -8x - 12x^2 \ln x + 8x^3 - x^4 \) and \( \Delta q^{(1)} = \left( \frac{\alpha_r}{\pi} \right) \left( \frac{25}{8} - 3\zeta(2) \right) + \mathcal{O}(\alpha_r x \ln x) \) (4)

where \( x = m_e^2/m_\mu^2 \) and \( \zeta \) is the Riemann zeta function with \( \zeta(2) = \pi^2/6 \). That the \( \Delta q^{(i)} \) remain finite in the limit \( m_e \to 0 \) is a consequence of the Kinoshita-Lee-Nauenberg theorem whose discovery was largely prompted by this particular observation.

Although the Fermi theory is not renormalizable, the \( \Delta q^{(i)} \) can be shown to be finite for all \( i \). This remarkable feature follows from the fact that the \( V - A \) interaction is invariant under a Fierz rearrangement that interchanges the wavefunctions of the electron and the muon neutrino. Thus Fermi theory is equivalent to an effective theory in which the muon and electron occupy the same fermion current in the weak interaction lagrangian. After fermion mass renormalization is performed the divergences in the vector part of this current are independent of the fermion mass and hence cancel in exactly the way they would for QED. The lagrangian of Fermi theory is invariant under the transformations \( \psi_e \to \gamma_5 \psi_e \) and \( m_e \to -m_e \). The QED corrections to the axial vector of the part can thus be obtained from the those of the vector part by changing the sign of the electron mass and hence are finite as well. Moreover the two sets of corrections are equal in the limit \( m_e \to 0 \) and, in that case, calculations need only be performed using the vector part of the Fermi interaction. This conclusion holds under any regularization prescription and avoids the complications associated with the use of \( \gamma_5 \) in dimensional regularization.

The foregoing discussion does not apply to the \( \beta \)-decay of the neutron where the Fierz rearrangement generates scalar and pseudoscalar terms that bear no resemblance to QED and the radiative corrections are consequently not finite.

III. THE 2-LOOP QED CORRECTIONS TO THE MUON LIFETIME

The complete 2-loop QED corrections to the muon lifetime require the calculation of matrix element for the processes, \( \mu^- \to \nu_\mu e^-\bar{\nu}_e, \mu^- \to \nu_\mu e^-\bar{\nu}_e\gamma, \mu^- \to \nu_\mu e^-\bar{\nu}_e\gamma\gamma \) and \( \mu^- \to \nu_\mu e^-\bar{\nu}_e e^+e^- \) with up to two virtual photons. All processes contain infrared (IR) divergences coming from either virtual photons, soft bremsstrahlung or both.
The cancellation of IR divergences occurs between the various processes but this complication may be avoided by exploiting cutting relations and calculating the 2-loop corrections as imaginary parts of 4-loop diagrams, some of which are shown in Figs 2 and 3. In these Feynman diagrams thick lines represent a muon and the thin lines represent either the electron or the neutrinos all of which are taken to be massless. Since the external muon is on-shell any cut passing through a muon line will vanish and the only cuts contributing to the imaginary part are precisely the ones that generate the diagrams appearing in the calculation of muon decay.

Recursion relations [11] obtained by integration-by-parts were first applied to reduce all dimensionally regularized integrals to a small set of relatively simple integrals. The well-behaved primitive integrals were then calculated by taking the external muon momentum, $q$, off mass shell to obtain expressions as power series in $x = -q^2/m_{\mu}^2$ and logarithms of $x$ using well-established large mass expansion techniques [12]. As the large mass expansion proceeds many terms, such as those that are topologically tadpoles, can be immediately discarded since they do not give rise to imaginary parts. Since the final result is required for $x = 1$ the complete series must be summed which can now be done in closed form in terms of polygamma functions and certain classes of multiple nested sums [13].

All diagrams were calculated in a general covariant gauge for the photon field and exact cancellation in the final result of the dependence on the gauge parameter was demonstrated.

A. Hadronic Contributions

Hadronic effects enter $\tau_\mu$ at the 2-loop level through the diagrams shown in Fig.1. The shaded blob represents the hadronic vacuum polarization of the photon. The hadronic contribution can be calculated in the usual way using dispersion relations but, in contrast to other well-known situations for which such effects have been calculated [14][15], the momenta of the external fermions, to which the virtual photon is attached, is not fixed. Here the electron participates in the phase-space integration which complicates matters somewhat.

Hadronic contributions are always afflicted to some degree by an uncertainty that arises from the experimental error on the measured cross-section $\sigma_{\text{had}} \equiv \sigma(e^+e^- \rightarrow \text{hadrons})$. If this uncertainty turned out to be large there would be little point proceeding with the perturbative calculation of the 2-loop QED contributions to the muon lifetime.

The shift induced in the inverse lifetime, $\Gamma_{\mu}$, of the muon is given as a convolution integral [1]

$$\Delta \Gamma_{\text{had}} = \frac{\alpha_r}{3\pi} \int_{4\rho}^{\infty} \frac{dz}{z} R(m_{\mu}^2z) \Delta \Gamma(z)$$

over the hadronic spectrum, $R(q^2) \equiv \sigma_{\text{had}}/\sigma_{\text{point}}$, and in which $\rho = m_{\mu}^2/m_{\pi}^2 = 1.61395...$ The convolution kernel, $\Delta \Gamma(z)$ is obtained exactly as an analytic function. When the integral is performed using actual hadronic data the result is

$$\Delta \Gamma_{\text{had}} = -\Gamma_0 \left( \frac{\alpha_r}{\pi} \right)^2 (0.042 \pm 0.002)$$

...(6)
which includes a rather conservative estimate of the hadronic uncertainty. Still the latter amounts to only 2 parts in $10^8$ and so is well under control.

The integral \( \int \) can be used to obtain an expression for the contribution from diagrams where the hadronic vacuum polarization has been replaced by muon loop by setting

\[
R(m^2 \mu^z) = \left( 1 + \frac{2}{z} \right) \sqrt{1 - \frac{4}{z}}.
\]

which gives

\[
\Delta \Gamma_{\text{muon}} = \Gamma_0 \left( \frac{\alpha_r}{\pi} \right)^2 \left( \frac{16987}{976} - \frac{85}{36} \zeta(2) - \frac{64}{3} \zeta(3) \right)
\]

\[
= -\Gamma_0 \left( \frac{\alpha_r}{\pi} \right)^2 \cdot 0.0364333.
\]

The result agrees with that subsequently obtained by perturbative methods. The effect of tau loops can be obtained in a similar way and, as expected on the basis of the decoupling theorem, is very small.

### B. Photonic Corrections

Examples of photonic diagrams which when cut give rise to contributions to the muon lifetime at $O(\alpha^2)$ are shown in Fig.2.

![Diagram](image)

**FIG. 2.** Examples of diagrams whose cuts give contributions to $\mu^- \rightarrow \nu_\mu e^- \bar{\nu}_e$, $\mu^- \rightarrow \nu_\mu e^- \bar{\nu}_e \gamma$ or $\mu^- \rightarrow \nu_\mu e^- \bar{\nu}_e \gamma \gamma$.

The result obtained for the complete set of photonic diagrams is

\[
\Delta \Gamma^{(2)}_{\gamma \gamma} = \Gamma_0 \left( \frac{\alpha_r}{\pi} \right)^2 \left( \frac{11047}{2592} - \frac{1030}{27} \zeta(2) - \frac{223}{36} \zeta(3) + \frac{67}{8} \zeta(4) + 53 \zeta(2) \ln(2) \right)
\]

\[
= \Gamma_0 \left( \frac{\alpha_r}{\pi} \right)^2 \cdot 3.55877
\]

where $\zeta(3) = 1.20206...$ and $\zeta(4) = \pi^4/90$.

### C. Electron-Loops and $e^+e^-$ Pair Creation

Diagrams containing an electron loop whose cuts give contributions to muon decay are shown in Fig.3. The result obtained for these diagrams is
\[ \Delta \Gamma^{(2)}_{\text{elec}} = -\Gamma_0 \left( \frac{\alpha_r}{\pi} \right)^2 \left( \frac{1009}{288} - \frac{77}{36} \zeta(2) - \frac{8}{3} \zeta(3) \right) \]

\[ = \Gamma_0 \left( \frac{\alpha_r}{\pi} \right)^2 3.22034. \]  

The value given in Eq. (14) is consistent with a numerical study carried out by Luke et al. [16] in the context of semi-leptonic decays of heavy quarks.

In order to obtain a UV finite answer the a diagrams in which the electron loop is replaced by the photon 2-point counterterm must be included and therefore a decision has to take as to the renormalization scheme that is to be adopted. This will be discussed further in section IV.

IV. THE RENORMALIZED ELECTROMAGNETIC COUPLING CONSTANT, \( \alpha_R \)

The use of dispersion relations to calculate the hadronic and muon loop contributions in the previous section naturally invokes a subtraction of the photon vacuum polarization at \( q^2 = 0 \) and is therefore equivalent to the on-shell renormalization scheme. In cases where there are two or more widely separated scales, such as \( m_e \) and \( m_\mu \), use of the \( \overline{\text{MS}} \) renormalization scheme is indicated since it automatically incorporates the large logarithms that arise into the value of the renormalized coupling constant, \( \alpha_r \), at tree level.

It is therefore appropriate here to adopt the \( \overline{\text{MS}} \) renormalization scheme. The hadronic contributions of section III.A that were obtained via dispersion relations must be corrected to convert them from the on-shell to \( \overline{\text{MS}} \) renormalization scheme. As it turns out the contribution from muon loops is the same in both schemes when the 't Hooft mass is taken set to \( \mu = m_\mu \) as is appropriate here. It can be shown [17] that the \( \overline{\text{MS}} \) renormalization scheme is implemented in a consistent manner by using the results of section III.A as they are given and setting

\[ \alpha_r = \alpha_e(m_\mu) \equiv \frac{\alpha}{1 - \frac{\alpha}{3\pi} \ln \frac{m_\mu^2}{m_e^2}} + \frac{\alpha^3}{4\pi^2} \ln \frac{m_\mu^2}{m_e^2}, \]  

where the logarithm of \( \mathcal{O}(\alpha^3) \) was first calculated by Jost and Luttinger [18]. The substitution (15) correctly resums logarithms of the form \( \alpha^n \ln^{n-1}(m_\mu^2/m_e^2) \) for all \( n > 0 \) and incorporates those of \( \alpha^3 \ln(m_\mu^2/m_e^2) \). Upon evaluation

\[ \alpha_e(m_\mu) = 1/135.90 = 0.0073582. \]
V. CONCLUSIONS

It has been over 40 years since the 1-loop QED corrections to the muon lifetime were calculated. The 2-loop contributions have had to await the development of new theoretical techniques, as well substantial increases in computer speed and storage capacity, but are now available.

The complete 2-loop QED contribution to the muon lifetime in the Fermi model may be encapsulated in the quantity $\Delta q^{(2)}$, as defined in Eq.s (1) and (3). After including the effects of virtual and real photons, virtual electrons, muons, taus and hadrons as well as $e^+e^-$ pair creation is found to be

\[
\Delta q^{(2)} = \left( \frac{\alpha_e(m_{\mu})}{\pi} \right)^2 \left( \frac{156815}{5184} - \frac{1036}{27} \zeta(2) - \frac{895}{36} \zeta(3) + \frac{67}{8} \zeta(4) + 53 \zeta(2) \ln 2 - (0.042 \pm 0.002) \right)
\](16)

\[
\Delta q^{(2)} = \left( \frac{\alpha_e(m_{\mu})}{\pi} \right)^2 (6.700 \pm 0.002)
\](17)

The tiny effect of tau loops has been included in the numerical value given in eq.(17).

This translates into a new value for the Fermi coupling constant of

\[
G_F = (1.16637 \pm 0.00001) \times 10^{-5} \text{ GeV}^{-2}
\](9 ppm)

The error has been halved relative to its previous value and is now entirely experimental.

New measurements of the muon lifetime are planned at the Brookhaven National Laboratory, the Paul Scherrer Institute and the Rutherford-Appleton Laboratory and it is therefore likely that the uncertainty on $G_F$ from this source will be reduced to somewhere in the range 0.5–1 ppm.

In that case the theoretical error should still be negligible but other issues such, as error on the muon mass, $m_{\mu}$, and the upper limit on the muon neutrino mass, $m_{\nu_\mu}$, need to be considered.

Finally many of the results and techniques employed here can be readily taken over and applied to inclusive decays of the $b$-quark.

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