Growth of solutions with $L^2(p+2)$-norm for a coupled nonlinear viscoelastic Kirchhoff equation with degenerate damping terms

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Abstract: In this work, we consider a coupled nonlinear viscoelastic Kirchhoff equations with degenerate damping, dispersion and source terms. Under suitable hypothesis, we will prove that when the initial data are large enough (in the energy point of view), the energy grows exponentially and thus so the $L^2(p+2)$-norm.

Keywords: viscoelastic equation; exponential growth; degenerate damping term

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1. Introduction

In this paper, we consider the following problem

$$\begin{align*}
|u|^r u_t - \Delta u + \int_0^t h_1(t-s)\Delta u(s)ds - \Delta u_t + (|u|^k + |v|^l) |u|^{j-1} u_t &= f_1(u, v), \quad (x, t) \in \Omega \times (0, T), \\
|v|^p v_t - \Delta v + \int_0^t h_2(t-s)\Delta v(s)ds - \Delta v_t + (|v|^q + |u|^\theta) |v|^{s-1} v_t &= f_2(u, v), \quad (x, t) \in \Omega \times (0, T), \\
u(x, t) &= v(x, t) = 0, \quad (x, t) \in \partial \Omega \times (0, T), \\
u(x, 0) &= u_0(x), \quad u_t(x, 0) = u_1(x), \\
v(x, 0) &= v_0(x), \quad v_t(x, 0) = v_1(x), \quad x \in \Omega,
\end{align*}$$

(1.1)

where $k, l, \theta, \varrho \geq 0; j, s \geq 1$ for $N = 1, 2$, and $0 \leq j, s \leq \frac{N+2}{N-2}$ for $N \geq 3$; and $\eta \geq 0$ for $N = 1, 2$ and $0 < \eta \leq \frac{2}{N-2}$ for $N \geq 3$, $h_i(.) : R^+ \rightarrow R^+$ ($i = 1, 2$) are positive relaxation functions which will be
specified later. \((|.|)^n + |.|^p \big| (.,.)|^{r-1} (.,.)\) and \(-\Delta (.,.)\) are the degenerate damping term and the dispersion term, respectively.

And

\[
\begin{align*}
  f_1(u, v) &= a_1|u + v|^{2(p+1)}(u + v) + b_1|u|^p . u |v|^{p+2}, \\
  f_2(u, v) &= a_1|u + v|^{2(p+1)}(u + v) + b_1|v|^p . v |u|^{p+2}.
\end{align*}
\]

(1.2)

It is well known that viscous materials are the opposite of elastic materials which have the capacity to store and dissipate mechanical energy. As the mechanical properties of these viscous substances are of great importance when they appear in many applications of other applied sciences.

Physically, the relationship between the stress and strain history in the beam inspired by Boltzmann theory called viscoelastic damping term, where the kernel of the term of memory is the function \(h\) (see [1–9]). If \(\eta \geq 0\), this type of problem has been studied by many authors. For more depth, here are some papers that focused on the study of this damping. See for example [10–15]. The effect of the degenerate damping terms often appear in many applications and piratical problems and turns a lot of systems into different problems worth studying.

The well known “Growth” phenomenon is one of the most important phenomena of asymptotic behavior, where many researches omit from its study especially when it comes from the evolution problems. It gives us very important information to know the behavior of equation when time arrives at infinity, it differs from global existence and blow up in both mathematically and in applications point of view.

Recently, the stability, the asymptotic behavior, blowing up and exponential growth of solutions for evolution systems with time degenerate damping has been studied by many authors. See [16–20].

The great importance of the source term with nonlinear functions \(f_1\) and \(f_2\) satisfying appropriate conditions. In physics is that they appear in several issues and theories. Many researchers also touched on this type of problem in several different issues, where the global existence of solutions, stability, blow up and growth of solutions were studied. For more information, the reader is referred to ( [21–28]). Recently, If \(\gamma = 0, \alpha_1 = 1\) our problem (1.1) has been studied in [27], under some restrictions on the initial datum, standard conditions on relaxation functions, the authors are established the global existence and proved the general decay of solutions.

Based on all of the above, the combination of these terms of damping (Memory term, degenerate damping, dispersion and the source terms ) we believe that it constitutes a new problem worthy of study and research, different from the above that we will try to shed light on.

In fact it will be proved that the \(L^{2(p+2)}\) norm of the solution grows as an exponential function. An essential tool of the proof is an idea used in the literature, which based on an auxiliary function (which is a small perturbation of the total energy), in order to obtain a differential inequality leads to the exponential growth result provided that under suitable hypothesis.

Our paper is divided into several sections: in the next section we lay down the hypotheses, concepts and lemmas we need. In the third section we prove our main result. Finally, a general conclusion has been drawn up.
2. Preliminaries

We prove the exponential growth of solutions under the following suitable assumptions.

(A1) $h_i : \mathbb{R}_+ \to \mathbb{R}_+$ are a differentiable and decreasing functions such that

$$h_i(t) \geq 0, \quad 1 - \int_0^\infty h_i(s) \, ds = l_i > 0, \quad i = 1, 2.$$  \hspace{1cm} (2.1)

(A2) There exists a constants $\xi_1, \xi_2 > 0$ such that

$$h_i'(t) \leq -\xi_i h_i(t), \quad t \geq 0, \quad i = 1, 2.$$  \hspace{1cm} (2.2)

**Theorem 2.1.** Assume (2.1) and (2.2) holds. Let

$$p \in \left(-1, \frac{2}{n-2}\right), \quad n \geq 3; \quad p \geq -1, \quad n = 1, 2.$$  \hspace{1cm} (2.3)

Then for any initial data

$$(u_0, u_1, v_0, v_1) \in \mathcal{H},$$

the problem (1.1) has a unique solution, for some $T > 0$

$$u \in C([0, T]; H^2(\Omega) \cap H_0^1(\Omega)),$$

where

$$\mathcal{H} = H_0^1(\Omega) \times L^2(\Omega) \times H_0^1(\Omega) \times L^2(\Omega).$$

In the next theorem we give the global existence result, its proof based on the potential well depth method in which the concept of so-called stable set appears, where we show that if we restrict our initial data in the stable set, then our local solution obtained is global in time. We will make use of arguments in [15].

**Theorem 2.2.** Suppose that (2.1), (2.2) and (2.3) holds. If $u_0, v_0 \in H_0^1(\Omega), u_1, v_1 \in L^2(\Omega)$

$$p \left( \frac{2(p+2)}{(p+1)} \right)^{p+1} E(0)^{p+1} < 1,$$  \hspace{1cm} (2.4)

where $\rho > 0$ is a constant. Then the local solution $(u, v)$ is global in time.

To achieve our goal, we need the following lemmas.

**Lemma 2.1.** There exists a function $F(u, v)$ such that

$$F(u, v) = \frac{1}{2(\rho + 2)} [uf_1(u, v) + vf_2(u, v)]$$

$$= \frac{1}{2(\rho + 2)} \left[ a_1 |u + v|^{2(p+2)} + 2b_1 |uv|^{p+2} \right] \geq 0,$$

where

$$\frac{\partial F}{\partial u} = f_1(u, v), \quad \frac{\partial F}{\partial v} = f_2(u, v),$$
we take $a_1 = b_1 = 1$ for convenience.

**Lemma 2.2.** [2] There exist two positive constants $c_0$ and $c_1$ such that
\[
\frac{c_0}{2(p + 2)} (|u|^{2(p+2)} + |v|^{2(p+2)}) \leq F(u, v) \leq \frac{c_1}{2(p + 2)} (|u|^{2(p+2)} + |v|^{2(p+2)}).
\] (2.5)

Now, we define the energy functional

**Lemma 2.3.** Assume (2.1), (2.2) and (2.3) hold, let $(u, v)$ be a solution of (1.1), then $E(t)$ is non-increasing, that is
\[
E(t) = \frac{1}{\eta + 2} \left[ \|u\|^{\eta + 2}_{2(p+2)} + \|v\|^{\eta + 2}_{2(p+2)} \right] + \frac{1}{2} \|\nabla u\|_2^2 + \|\nabla v\|_2^2
\]
\[
+ \frac{1}{2} \left( 1 - \int_0^t h_1(s)ds \right) \|\nabla u\|_2^2 - \int_0^t \int_\Omega F(u, v)dx,
\] (2.6)
\[
E'(t) \leq \frac{1}{\eta + 2} \left[ (h'_1 \partial_t \nabla u)(t) + (h'_2 \partial_t \nabla v)(t) \right] - \frac{1}{2} \left[ h_1(t) \|\nabla u\|_2^2 + h_2(t) \|\nabla v\|_2^2 \right]
\]
\[
- \int_\Omega (|u|^k + |v|^k) |u|^{\eta+1} dx - \int_\Omega (|v|^\eta + |u|^\eta) |v|^{\eta+1} dx
\]
\[
\leq 0.
\] (2.7)

**Proof.** By multiplying (1.1)$_1$, (1.1)$_2$ by $u_t, v_t$ and integrating over $\Omega$, we get
\[
\frac{d}{dt} \left( \frac{1}{\eta + 2} \left[ \|u\|^{\eta + 2}_{2(p+2)} + \|v\|^{\eta + 2}_{2(p+2)} \right] + \frac{1}{2} \|\nabla u\|_2^2 + \frac{1}{2} \|\nabla v\|_2^2 \right)
\]
\[
+ \frac{1}{2} \left( 1 - \int_0^t h_1(s)ds \right) \|\nabla u\|_2^2 - \int_\Omega F(u, v)dx
\]
\[
+ \frac{1}{2} \left( h_1 \partial_t \nabla u)(t) + \frac{1}{2} (h'_2 \partial_t \nabla v)(t) - \int_\Omega F(u, v)dx \right)
\]
\[
= - \int_\Omega (|u|^k + |v|^k) |u|^{\eta+1} dx - \int_\Omega (|v|^\eta + |u|^\eta) |v|^{\eta+1} dx
\]
\[
+ \frac{1}{2} (h'_1 \partial_t \nabla u) - \frac{1}{2} h_1(t) \|\nabla u\|_2^2 + \frac{1}{2} (h'_2 \partial_t \nabla v) - \frac{1}{2} h_2(t) \|\nabla v\|_2^2,
\] (2.8)
\[
\begin{align*}
&-\frac{1}{2}\left[\left(1 - \int_0^t h_1(s)ds\right)\|\nabla u\|_2^2 + \left(1 - \int_0^t h_2(s)ds\right)\|\nabla v\|_2^2\right] \\
&-\frac{1}{2}\left[h_1(0)\nabla u(t) + h_2(0)\nabla v(t)\right] \\
&+ \frac{1}{2(p+2)}\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2}. \\
\end{align*}
\] (3.1)

**Theorem 3.1.** Assume (2.1), (2.2), and (2.3) hold, and suppose that \(E(0) < 0\), and
\[
2(p + 2) > \max\left\{k + j + 1; l + j + 1; \theta + s + 1; \varrho + s + 1; \frac{\eta + 2}{\eta + 1}\right\}. \tag{3.2}
\]

Then the solution of problem (1.1) grows exponentially.

**Proof.** From (2.6), we have
\[
E(t) \leq E(0) \leq 0. \tag{3.3}
\]
Therefore
\[
H'(t) = -E'(t) \geq \int_{\Omega} (|u|^k + |v|^l)|u|^j + \frac{1}{2}\int_{\Omega} (|v|^\theta + |u|^\varrho)|v|^s + 1, \tag{3.4}
\]

hence
\[
\begin{align*}
H'(t) &\geq \int_{\Omega} (|u|^k + |v|^l)|u|^j + 1 \geq 0 \\
H'(t) &\geq \int_{\Omega} (|v|^\theta + |u|^\varrho)|v|^s + 1 \geq 0. \tag{3.5}
\end{align*}
\]

By (3.1) and (2.5), we have
\[
0 \leq H(0) \leq H(t) \leq \frac{1}{2(p+2)}\|u + v\|_{2(p+2)}^{2(p+2)} + 2\|uv\|_{p+2}^{p+2} \\
\leq \frac{c_1}{2(p+2)}\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \tag{3.6}
\]

We set
\[
\begin{align*}
K(t) &= H(t) + \frac{\varepsilon}{\eta + 1} \int_{\Omega} [u|u|^\eta u + v|v|^\varphi v]dx \\
&+ \varepsilon \int_{\Omega} [\nabla u \nabla u + \nabla v \nabla v]dx, \tag{3.7}
\end{align*}
\]
where \(\varepsilon > 0\) to be assigned later.

By multiplying (1.1)_1, (1.1)_2 by \(u, v\) and with a derivative of (3.7), we get
\[
\begin{align*}
K'(t) &= H'(t) + \frac{\varepsilon}{\eta + 1} (\|u\|_{\eta+2}^{\eta+2} + \|v\|_{\varphi+2}^{\varphi+2}) + \varepsilon (\|\nabla u\|_2^2 + \|\nabla v\|_2^2)
\end{align*}
\]
At this point, we use Young's inequality, for

\[
\text{From (3.8), we find}
\]

\[
\text{Hence, we have}
\]

\[
\begin{align*}
J_1 &= \varepsilon \int_0^t h_1(t-s)ds \int_\Omega \nabla u.(\nabla u(s) - \nabla u(t))dxd + \varepsilon \int_0^t h_1(s)ds \|\nabla u\|_2^2 \\
&\geq \frac{\varepsilon}{2} \int_0^t h_1(s)ds \|\nabla u\|_2^2 - \frac{\varepsilon}{2}(h_1o\nabla u). \\
J_2 &= \varepsilon \int_0^t h_2(t-s)ds \int_\Omega \nabla v.(\nabla v(s) - \nabla v(t))dxd + \varepsilon \int_0^t h_2(s)ds \|\nabla v\|_2^2 \\
&\geq \frac{\varepsilon}{2} \int_0^t h_2(s)ds \|\nabla v\|_2^2 - \frac{\varepsilon}{2}(h_2o\nabla v).
\end{align*}
\]

From (3.8), we find

\[
\mathcal{K}'(t) \geq \mathcal{H}'(t) + \frac{\varepsilon}{\eta + 1} (\|u\|_q^{p+2} + \|v\|_{q+2}^{p+2}) + \varepsilon(\|\nabla u\|_2^2 + \|\nabla v\|_2^2) \\
- \varepsilon\left( \left(1 - \frac{1}{2} \int_0^t h_1(s)ds \right)\|\nabla u\|_2^2 + \left(1 - \frac{1}{2} \int_0^t h_2(s)ds \right)\|\nabla v\|_2^2 \right) \\
- \frac{\varepsilon}{2}(h_1o\nabla u) - \frac{\varepsilon}{2}(h_2o\nabla v) - J_3 - J_4 + J_5.
\]

At this point, we use Young's inequality, for \(\delta > 0\)

\[
XY \leq \frac{\delta^\alpha X^\alpha}{\alpha} + \frac{\delta^\beta X^\beta}{\beta}, \quad \alpha,\beta > 0, \quad \frac{1}{\alpha} + \frac{1}{\beta} = 1,
\]

we get, for \(\delta_1,\delta_2 > 0\)

\[
\begin{align*}
|u|^{j+1}u &= \frac{\delta_1^{j+1}}{j+1}|u|^{j+1} + \frac{j}{j+1}\delta_1^{-j}\|u\|^{j+1}, \\
|v|^{s+1}v &= \frac{\delta_2^{s+1}}{s+1}|v|^{s+1} + \frac{s}{s+1}\delta_2^{-s}\|v\|^{s+1}.
\end{align*}
\]

Hence, we have

\[
J_3 \leq \frac{\varepsilon \delta_1^{j+1}}{j+1} \int_\Omega (|u|^k + |v|^j)|u|^{j+1}dx + \frac{\varepsilon \delta_1^{-j}}{j+1} \int_\Omega (|u|^k + |v|^j)|u|^{j+1}dx,
\]
By Young’s inequality, we find for
\[
J_4 \leq \frac{\delta^{j+1}}{s+1} \int_{\Omega} (|v|^q + |u|^p)|v|^{j+1} dx + \frac{\varepsilon\delta^{(\frac{q}{p})}}{s+1} \int_{\Omega} (|v|^q + |u|^p)|v|^{j+1} dx.
\]
(3.14)

Therefore, using (3.5) and by setting \(\delta_1, \delta_1\) so that,
\[
\frac{j\delta_1^{(\frac{1}{j+1})}}{j+1} = \frac{\kappa}{2}, \quad \frac{s\delta_2^{(\frac{1}{s})}}{s+1} = \frac{\kappa}{2},
\]
substituting in (3.11), we get
\[
\mathcal{K}'(t) \geq \left[ 1 - \varepsilon \kappa \right] \mathcal{K}(t) + \frac{\varepsilon}{\eta + 1} \left( ||u||_{\eta^2}^q + ||v||_{\eta^2}^q \right)
- \varepsilon \left[ \left( 1 - \frac{1}{2} \int_0^t h_1(s) ds \right) ||\nabla u||^2_2 + \left( 1 - \frac{1}{2} \int_0^t h_2(s) ds \right) ||\nabla v||^2_2 \right]
+ \varepsilon \left[ ||\nabla u||^2_2 + ||\nabla v||^2_2 \right] - \frac{\varepsilon}{2} (h_1 \rho \nabla u - \varepsilon) (h_2 \rho \nabla v)
- \varepsilon C_1(\kappa) \int_{\Omega} (|u|^k + |v|^l)|u|^{j+1} dx
- \varepsilon C_2(\kappa) \int_{\Omega} (|v|^q + |u|^p)|v|^{j+1} dx + J_5,
\]
(3.15)

where
\[
C_1(\kappa) := \left( \frac{2j}{\kappa(j+1)} \right)^{j+1} \frac{1}{j+1}, \quad C_2(\kappa) := \left( \frac{2s}{\kappa(s+1)} \right)^{s+1} \frac{1}{s+1},
\]
(3.16)

we have
\[
\int_{\Omega} (|u|^k + |v|^l)|u|^{j+1} dx = ||u||_{k+j+1}^{k+j+1} + \int_{\Omega} |v|^l|u|^{j+1} dx,
\]
\[
\int_{\Omega} (|v|^q + |u|^p)|v|^{j+1} dx = ||v||_{q+s+1}^{q+s+1} + \int_{\Omega} |u|^p|v|^{j+1} dx.
\]
(3.17)

By Young’s inequality, we find for \(\delta_3, \delta_4 > 0\)
\[
\int_{\Omega} |v|^l|u|^{j+1} dx \leq \frac{l}{l+j+1} \delta_3^{\frac{l}{j+1}} ||v||_{l+j+1}^{l+j+1} + \frac{j+1}{l+j+1} \delta_3^{-\frac{j}{l+j+1}} ||u||_{l+j+1}^{l+j+1},
\]
\[
\int_{\Omega} |u|^p|v|^{j+1} dx \leq \frac{\rho}{\rho + s + 1} \delta_4^{\frac{p}{\rho}} ||u||_{p+s+1}^{p+s+1} + \frac{s+1}{\rho + s + 1} \delta_4^{-\frac{s}{p+s+1}} ||v||_{p+s+1}^{p+s+1}.
\]
(3.18)

Hence
\[
\int_{\Omega} (|u|^k + |v|^l)|u|^{j+1} dx \leq ||u||_{k+j+1}^{k+j+1} + \frac{l}{l+j+1} \delta_3^{\frac{l}{j+1}} ||v||_{l+j+1}^{l+j+1}
+ \frac{j+1}{l+j+1} \delta_3^{-\frac{j}{l+j+1}} ||u||_{l+j+1}^{l+j+1},
\]
Similarly, by (3.2) we get

\[ \int_{\Omega} (|v|^p + |u|^p)|v|^{\rho+1} \, dx \leq |v|^{\frac{(\rho+1)}{n+1}} + \frac{\rho}{\rho + s + 1} \delta_4 \left( |v|^{\frac{(\rho+1)}{n+1}} + |u|^{\frac{(\rho+1)}{n+1}} \right)^{\frac{\rho+1}{\rho+2}} \]

\[ + \frac{(s+1)}{\rho + s + 1} \delta_4 \left( |v|^{\frac{(\rho+1)}{n+1}} + |u|^{\frac{(\rho+1)}{n+1}} \right)^{\frac{\rho+1}{\rho+2}}. \]  

(3.19)

By using (3.6) and (3.2), since \( 2(p+2) > k + j + 1 \), we have from the embedding \( L^{2(p+2)}(\Omega) \hookrightarrow L^{k+j+1}(\Omega) \),

\[ ||u||_{k+j+1}^{k+j+1} \leq C ||u||_{2(p+2)}^{k+j+1} \leq \left( ||u||_{2(p+2)}^{2(p+2)} \right)^{\frac{k+j+1}{2(p+2)}}, \]  

(3.20)
since \( 0 < \frac{k+j+1}{2(p+2)} < 1 \), to find by using the algebraic inequality

\[ B^\varsigma \leq (B + 1) \leq (1 + \frac{1}{b})(B + b), \quad \forall B > 0, \quad 0 < \varsigma < 1, \quad b > 0, \]  

(3.21)

\[ \left( \frac{||u||_{2(p+2)}^{2(p+2)}}{2(p+2)} \right)^{k+j+1} \leq K(||u||_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)), \]  

(3.22)

where \( K = 1 + \frac{1}{b(0)} \).

Similarly, by (3.2) we get

\[ ||v||_{k+j+1}^{k+j+1} \leq \left( ||v||_{2(p+2)}^{2(p+2)} \right)^{\frac{k+j+1}{2(p+2)}} \leq K(||v||_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)), \]

\[ ||v||_{\rho+1}^{\rho+1} \leq \left( ||v||_{2(p+2)}^{2(p+2)} \right)^{\frac{\rho+1}{2(p+2)}} \leq K(||v||_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)), \]

\[ ||u||_{\rho+1}^{\rho+1} \leq \left( ||u||_{2(p+2)}^{2(p+2)} \right)^{\frac{\rho+1}{2(p+2)}} \leq K(||u||_{2(p+2)}^{2(p+2)} + \mathbb{H}(0)). \]  

(3.23)

Hence, by fixed \( \delta_3, \delta_4 > 0 \), and (3.19), gives

\[ \int_{\Omega} (|v|^p + |u|^p)|v|^{\rho+1} \, dx \]

\[ \leq M_1 \left( 1 + \frac{l \delta_3^{\frac{j+1}{n+1}}}{l + j + 1} + \frac{(j+1) \delta_3^{\frac{j+1}{n+1}}}{l + j + 1} \right) \left( ||v||_{2(p+2)}^{2(p+2)} \right) + ||u||_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \]

\[ \int_{\Omega} (|v|^p + |u|^p)|v|^{\rho+1} \, dx \]

\[ \leq M_2 \left( 1 + \frac{\rho \delta_4^{\frac{\rho-1}{\rho+1}}}{\rho + s + 1} + \frac{(s+1) \delta_4^{\frac{\rho-1}{\rho+1}}}{\rho + s + 1} \right) \left( ||v||_{2(p+2)}^{2(p+2)} \right) + ||u||_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)), \]  

(3.24)

for some constants \( M_1, M_2 > 0 \).

Now, for \( 0 < \alpha < 1 \), from (3.1)

\[ J_5 = \varepsilon ||u + v||_{2(p+2)}^{2(p+2)} + 2 ||uv||_{p+2}^{p+2} = \varepsilon a \left[ ||u + v||_{2(p+2)}^{2(p+2)} + 2 ||uv||_{p+2}^{p+2} \right] \]
\[
K'(t) \geq \left\{ 1 - \epsilon \kappa \right\} \mathbb{H}(t) + \epsilon \left\{ (p + 2)(1 - a) + 1 \right\} (\|\nabla u\|_2^2 + \|\nabla v\|_2^2) + \epsilon \left( \frac{2\epsilon(p + 2)(1 - a)}{\eta + 2} + \frac{1}{\eta + 1} \right) (\|u\|_{p+2}^{p+2} + \|v\|_{p+2}^{p+2}) + \epsilon \left( (p + 2)(1 - a) \left( 1 - \int_0^t h_1(s)ds \right) - \left( 1 - \frac{1}{2} \int_0^t h_1(s)ds \right) \right) \|\nabla u\|_2^2 + \epsilon \left( (p + 2)(1 - a) \left( 1 - \int_0^t h_2(s)ds \right) - \left( 1 - \frac{1}{2} \int_0^t h_2(s)ds \right) \right) \|\nabla v\|_2^2 + \epsilon \left( (p + 2)(1 - a) - \frac{1}{2} \right) (h_1 \sigma \nabla u + h_2 \sigma \nabla v) + \epsilon \left( c_0 \alpha - \left( M_3 C_1(\kappa) + M_4 C_2(\kappa) \right) \right) (\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)}) + \epsilon \left( 2(p + 2)(1 - a) - \left( M_3 C_1(\kappa) + M_4 C_2(\kappa) \right) \right) \mathbb{H}(t),
\]

where

\[
M_3 := M_1 \left( 1 + \frac{l i(\frac{i}{i+1})}{l + j + 1} + \frac{(j + 1) i(\frac{i}{i+1})}{l + j + 1} \right) > 0,
\]

\[
M_4 := M_2 \left( 1 + \frac{s \sigma(\frac{i}{i+1})}{\sigma + s + 1} + \frac{(s + 1) \sigma(\frac{i}{i+1})}{\sigma + s + 1} \right) > 0.
\]

In this stage, we take \( a > 0 \) small enough so that

\[
\lambda_1 = (p + 2)(1 - a) - 1 > 0,
\]

and we assume

\[
\text{max} \left\{ \int_0^\infty h_1(s)ds, \int_0^\infty h_2(s)ds \right\} < \frac{(p + 2)(1 - a) - 1}{((p + 2)(1 - a) - \frac{1}{2})} = \frac{2\lambda_1}{2\lambda_1 + 1},
\]

which gives

\[
\lambda_2 = \left\{ (p + 2)(1 - a) - 1 - \int_0^\infty h_1(s)ds \left( (p + 2)(1 - a) - \frac{1}{2} \right) \right\} > 0.
\]
\[ \lambda_3 = \left\{ \left[ (p + 2)(1 - a) - 1 \right] - \int_0^\infty h_2(s)ds \left( (p + 2)(1 - a) - \frac{1}{2} \right) \right\} > 0, \]

then we choose \( \kappa \) so large that

\[ \lambda_4 = ac_0 - \left( M_3 C_1(\kappa) + M_4 C_2(\kappa) \right) > 0, \]

\[ \lambda_5 = 2(p + 2)(1 - a) - \left( M_3 C_1(\kappa) + M_4 C_2(\kappa) \right) > 0. \]

Finally, we fixed \( \kappa, a \), and we appoint \( \varepsilon \) small enough so that

\[ \lambda_6 = 1 - \varepsilon \kappa > 0, \]

and, from (3.7)

\[ \mathcal{K}(t) \leq \frac{1}{2(p + 2)} \left[ \| u + v \|_{L^{2(p+2)}}^2 + 2\| uv \|_{L^{p+2}} \right] \]

\[ \leq \frac{c_1}{2(p + 2)} \left[ \| u \|_{L^{2(p+2)}}^2 + \| v \|_{L^{2(p+2)}}^2 \right]. \]  

(3.28)

Thus, for some \( \beta > 0 \), estimate (3.26) becomes

\[ \mathcal{K}'(t) \geq \beta \left\{ \bar{\Omega}(t) + \| u \|_{L^{q+2}}^2 + \| v \|_{L^{q+2}}^2 + \| \nabla u \|_2^2 + \| \nabla v \|_2^2 + \| \nabla u \|_2^2 + \| \nabla v \|_2^2 \right. \]

\[ + \left( h_1 \partial \nabla u \right) + \left( h_2 \partial \nabla v \right) + \| u \|_{L^{2(p+2)}}^2 + \| u \|_{L^{2(p+2)}}^2 \} \]  

(3.29)

By (2.5), for some \( \beta_1 > 0 \), we obtain

\[ \mathcal{K}'(t) \geq \beta_1 \left\{ \bar{\Omega}(t) + \| u \|_{L^{q+2}}^2 + \| v \|_{L^{q+2}}^2 + \| \nabla u \|_2^2 + \| \nabla v \|_2^2 + \| \nabla u \|_2^2 + \| \nabla v \|_2^2 \right. \]

\[ + \left( h_2 \partial \nabla u \right) + \left( h_2 \partial \nabla v \right) + \| u \|_{L^{2(p+2)}}^2 + 2\| uv \|_{L^{p+2}} \} \]  

(3.30)

and

\[ \mathcal{K}(t) \geq \mathcal{K}(0) > 0, \quad t > 0. \]  

(3.31)

Next, using Holder’s and Young’s inequalities, we have

\[ \left| \int_{\Omega} \left( u|u|^\mu u_t + v|v|^\mu v_t \right) dx \right| \leq C \left[ \| u \|_{L^{2(p+2)}}^\mu + \| u \|_{L^{q+2}}^\mu \right. \]

\[ + \left( \| v \|_{L^{2(p+2)}}^\mu + \| v \|_{L^{q+2}}^\mu \right). \]  

(3.32)

where \( \frac{1}{\mu} + \frac{1}{\theta} = 1 \).

We take \( \mu = (\eta + 2) \), to get

\[ \theta = \frac{(\eta + 2)}{(\eta + 1)} \leq 2(p + 2). \]
Subsequently, by using (3.2) and (3.21), we obtain
\[
\|u\|_{2(p+2)}^{2(p+2)} \leq K\|u\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t)
\]
\[
\|v\|_{2(p+2)}^{2(p+2)} \leq K\|v\|_{2(p+2)}^{2(p+2)} + \mathbb{H}(t), \quad \forall t \geq 0.
\]
Therefore,
\[
\left| \int_{\Omega} (u|u|^p u_t + v|v|^p v_t) \, dx \right| \leq c_{13}\left\{ \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} + \|u_t\|_{2(p+2)}^{p+2} + \|v_t\|_{2(p+2)}^{p+2} + \mathbb{H}(t) \right\}.
\] (3.33)

Hence,
\[
\mathcal{K}(t) = \left( \mathbb{H}(t) + \frac{\epsilon}{\eta + 1} \int_{\Omega} (u|u|^p u_t + v|v|^p v_t) \, dx + \epsilon \int_{\Omega} (\nabla u_t \nabla u + \nabla v_t \nabla v) \, dx \right)
\]
\[
\leq c \left( \mathbb{H}(t) + \|u_t\|_{p+2}^{p+2} + \|v_t\|_{p+2}^{p+2} + \|\nabla u\|_{2}^{2} + \|\nabla v\|_{2}^{2} + \|\nabla u_t\|_{2}^{2} + \|\nabla v_t\|_{2}^{2} + (h_1 o \nabla u) + (h_2 o \nabla v) + \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right).
\] (3.34)

From (3.29) and (3.34), gives
\[
\mathcal{K}'(t) \geq \lambda \mathcal{K}(t),
\] (3.35)
where \( \lambda > 0 \), this depends only on \( \beta \) and \( c \).

By integration of (3.35), we obtain
\[
\mathcal{K}(t) \geq \mathcal{K}(0)e^{\lambda t}, \quad \forall t > 0.
\] (3.36)

From (3.7) and (3.28), we have
\[
\mathcal{K}(t) \leq \frac{c_1}{2(p+2)} \left[ \|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \right].
\] (3.37)

By (3.36) and (3.37), we have
\[
\|u\|_{2(p+2)}^{2(p+2)} + \|v\|_{2(p+2)}^{2(p+2)} \geq Ce^{\lambda t}, \quad \forall t > 0.
\]

Hence, we conclude that the solution in the \( L^{2(p+2)} \) norm is growths exponentially. This completes the proof.

\[\square\]

4. Conclusions

The purpose of this work was to study when the initial data are large enough, the energy grows exponentially with \( L^{2(p+2)} \) norm of solutions for a coupled nonlinear viscoelastic Kirchhoff equations with degenerate damping, dispersion and source terms. This type of problem is frequently found in some mathematical models in applied sciences. Especially in the theory of viscoelasticity. What interests us in this current work is the combination of these terms of damping (degenerate damping, dispersion and source terms), which dictates the emergence of these terms in the problem. In the next work, we will try to using the same method with same problem. But in added of other damping (Balakrishnan-Taylor damping and Logarithmic terms).
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Conflict of interest

This work does not have any conflicts of interest.

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