Model-independent Mass-Radius Constraint for Neutron Stars†

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Abstract

While model-independent limits are always interesting, a limit on neutron star radius as a function of mass attains special interest in light of recent interpretations of the periodic as well as quasi-periodic oscillations (QPOs) in brightness of X-rays emitted from neutron stars that are accreting matter from a low-mass companion. Here we derive such a limit based only on well accepted principles. We discuss our limit in connection with a recent interpretation of X-ray pulsations from SAX J1808.4-3658 as indicating a strange-star candidate, and show that this object can also be a normal neutron star, though one whose central core has very high density. The most plausible high-density phase of hadronic matter, which is also expected to be very compressible, is quark matter. So an alternative to the strange star interpretation of SAX J1808.4-3658 is that it is a hybrid neutron star.

1 Motivation

An accreting X-ray binary having millisecond pulses was recently discovered ([23, 3]). Presumably it is one of a class of objects that represent the missing link between canonical and millisecond pulsars. If so, it confirms a long-standing conjecture that millisecond pulsars are formed by accretion onto canonical pulsars ([2, 3, 18]). Many other accreting X-ray sources have been discovered consisting of a neutron star and a low-mass companion from which material is gathered, presumably, from an accretion disk ([21, 20, 22, 23]). An interpretation of quasi-periodic oscillations in X-ray brightness characteristic of these sources suggests that upper limits on mass and radius of the neutron star may possibly be deduced ([20, 17]). However, the interpretation in such terms is the subject of some controversy and it remains to be seen how the models of the observations finally play out ([15]). Certainly there is much uncertainty concerning the magnetic field and accretion disk interaction which play an important role in the modeling of X-ray pulsations.

Tentative limits on mass and radius deduced from models of quasi periodic oscillations in X-ray brightness (QPOs) have been employed recently to discriminate among models of the equation of state of dense nuclear matter ([19]). The X-ray pulsar, Sax J1808.4-3658, is a particularly interesting object; it produces coherent X-ray emission with a 2.5 ms period as well as X-ray bursts. Based on an analysis of radiation from...
this object, a limiting mass-radius relationship was derived which is difficult to reconcile with existing neutron star models ([16]). The mass-radius relationship derived would be consistent with an interpretation of Sax J1808.4 as a strange star candidate as found by the above authors.

Against this background our purpose is to derive a model-independent mass-radius constraint for neutron stars that depends only on minimal and well accepted principles. The limiting relation is analogous to a previously obtained lower limit on the Kepler period of a rotating star as a function of its mass ([10, 13]), and to an even earlier analysis of limits on the gravitational redshift from neutron stars ([4]).

The most conservative minimal principles and constraints are:

1. Einstein’s general relativistic equations for stellar structure hold.
2. The matter of the star satisfies $dp/dρ \geq 0$ which is a necessary condition that a body is stable, both as a whole and also with respect to the spontaneous expansion or contraction of elementary regions away from equilibrium (Le Chatelier’s principle).
3. The equation of state satisfies the causal constraint for a perfect fluid; a sound signal cannot propagate faster than the speed of light, $v(ε) \equiv \sqrt{dp/dε} \leq 1$, which is also the appropriate expression for sound signals in General Relativity ([7, 9]).
4. The high-density equation of state matches continuously in energy and pressure to the low-density equation of state of [4] and has no bound state at any density.

The last condition assures that the $M − R$ relation obtained is for a neutron star and not some sort of exotic. We mean “neutron star” in the generic sense: it is made of charge neutral nuclear matter at low density, while at higher density in the interior, matter may be in a mixed or pure quark-matter or other high-density phase of nuclear matter. The last condition also implies that the star is bound by gravity as is a neutron star and is not a self-bound star such as a strange star. As we will see, a self-bound star can lie in a region of the $M − R$ plane that is forbidden to neutron stars. In referring to the constraints as conservative, we mean that we make no assumption about dense matter aside from the constraints mentioned and we specifically allow for a phase transition above a baryon density of $0.1625 \text{ fm}^{-3}$. We discuss this further in the Section, Caveats.

We can adapt the results of our earlier search for a model-independent minimum Kepler period by searching for the radius at fixed mass that minimizes $P \sim (R^3/M)^{1/2}$ ([10]). Several researchers found that the above classical result applies to relativistic stars to within a few percent accuracy with a suitable constant of proportionality ([12, 8]). We use variational equations of state subject to the above constraints and techniques as described in the above reference. Our earlier results for the Kepler period agree to six percent with the results of [13], who performed a numerical solution for rotating stars in place of the above approximation formula for the Kepler period in terms of mass and radius of the non-rotating counterparts.

Our results are shown in Fig. 1. Neutron stars at the mass limit can have radii as small as those shown by the line, and otherwise must lie in the shaded region marked for neutron stars. The region can be approximated in the interval illustrated by

$$R \geq (3.1125 − 0.44192x + 2.3089x^2 − 0.38698x^3) \text{ km},$$
Of course, for neutron stars (unlike white dwarfs), there is only one equation of state in nature; all neutron stars form a single family, and whatever the trajectory of the mass-radius relationship is for that family, the limiting mass star has the smallest radius, and it is greater or equal to the limit derived. For example, if the most massive neutron star that could exist in nature, independent of formation mechanism, is $2M_\odot$, the radius of all neutron stars would have to exceed 8.37 km. (Recall that measured masses tell us nothing about the maximum possible mass that can be supported by nature’s equation of state.)

\[
1 \leq x = \frac{M}{M_\odot} \leq 2.5.
\] (1)

Another limit of interest follows from the properties of General Relativity. Schartzchild’s limit $R > 2M$ is actually less stringent than $R > 9M/4$, which must be obeyed by any relativistic star \cite{5,24}. The latter is also plotted. If a star’s mass and radius placed it in the region between the above described regions, it could be made of matter that is self-bound at high density, matter that would be bound in microscopic to stellar like-objects even in the absence of gravity (see Eq. 3). Strange stars, if the strange matter hypothesis is true, are examples.

![Graph](image-url)
2 Application to X-ray Emitters

In the approximations and hypotheses that have been used to interpret the oscillations in X-ray luminosity, there appears a Keplerian radius. Such a radius expresses the balance of gravitational and centrifugal forces. In classical physics as well as in General Relativity for a non-rotating star (units are $G = c = 1$):

$$\Omega = \sqrt{\frac{M}{R^3}}$$

where $M$ is the mass of the star and $\Omega$ is the angular velocity of a particle in circular orbit at $R_K$. This relation has nothing to do with the nature of the interior of the star, whatever that may be. It only relates gravity in the exterior region to the centrifugal force on a particle at $R_K$. The expression is exact for a non-rotating star in General Relativity, and only approximate for a rotating star, but there is a multiplicative factor on the right side, found to be accurate for many models that have been tested, which is $\zeta \approx 0.65$ ([12, 8]).

The same relationship $M/R^3 = \text{const}$ holds also for a self-bound object, such as a strange star as can be seen as follows: The average energy density $\bar{\epsilon}$ satisfies the identity

$$\epsilon_{\text{equil.}} \leq \bar{\epsilon} \equiv M\left(\frac{4\pi}{3}R^3\right)$$

where $\epsilon_{\text{equil.}}$ denotes the equilibrium density at which hypothetical strange matter is bound. The equality would hold for spherical objects of mass such that gravity is unimportant. Thus in either case, $M/R^3 = \text{const}$ characterizes both a Keplerian orbit and a strange star, though they have nothing to do with each other a priori. We make this point since superficially the constraint derived from X-ray luminosity oscillations looks like the $M - R$ relationship for a strange star [cf. Fig. 3 in Ref. (1)].

In fact, strange stars have been considered as candidates that satisfy the constraints of the QPO model ([19]) and of a model of of periodic pulsations from the X-ray pulsar ([16]). The suggestions seem especially appealing because of the above coincidence. They also have some appeal in as much as many explicit neutron star models cannot satisfy the constraints imposed by the theoretical analyses. Figure 2 illustrates the limit obtained for the 2.5 ms X-ray pulsar, Sax J1808.4, by Li et al. (1999), together with the limit obtained in our model independent way. Neutron stars must lie to the right of our limit and the X-ray object must lie on or to the left of the line so marked. Li et al. (1999) have proposed the object as a strange-star candidate because the neutron star models they tested did not meet their constraint, whereas the strange star models did. However, from our model independent constraint, it is clear that neutron stars cannot be ruled out, even if many explicit models can, always provided the X-ray phenomena are modeled correctly.
Figure 2: According to the QPO model of [16] the X-ray star must lie on or to the left of the curved line $R \propto M^3$ and by our model independent determination, neutron stars must lie to the right of the line marked ‘Ns allowed’.

3 Caveats

We have derived a mass-radius relationship for “neutron stars” employing the minimal conservative constraints enumerated above. In doing so we are recognizing that there is no empirical knowledge of the properties of nuclear matter above saturation density of $\sim 0.15 \text{ fm}^{-3}$. In our search for the minimum radius as a function of mass, we have allowed a constant pressure region to develop above the fiducial density of 0.1625 $\text{fm}^{-3}$, the density closest to saturation in the BPS tables of the low-density equation of state. The minimum radius is increased by $\sim 0.1 \text{ km}$ if the equation of state is merely very soft just above the fiducial density. These features are permitted by our ignorance above saturation density, and in this sense provide a conservative estimate of the minimum radius as a function of mass. However, we may be permitted some prejudice: If a low-density phase transition of any kind were not plausible, then the minimum radius that we have derived, would be increased.

The central density of the minimum radius star, in either of the above two cases is about 26 times nuclear density for a canonical $1.44 M_\odot$ star. Almost certainly, stars of such high central density must contain a deconfined quark matter core. However, these stars, with the constraints that we have imposed, are gravitationally bound, rather than self-bound, as a strange star would be. So, if the analyses of these X-ray objects really does imply an extraordinarily small radius, that fact would be consistent with the star having a quark matter core, the quarks being liberated from hadrons by the high pressure. This is in distinction with the strange matter hypothesis, according to which the entire star would be made of self-bound quark matter, a so-far undiscovered state, the actual ground state of hadronic matter, if the hypothesis...
is true.

4 Comments

Two or three properties of pulsars can be measured with great accuracy—the period of rotation, sometimes the time rate of change of period, and the masses involved in close binaries. The first two are directly observed, and the third deduced from measurement of orbital parameters. With sufficient observation time, these can be determined accurately, and little doubt surrounds orbital mechanics. It is possible that no other properties gained from any other phenomena will rival these types of measurements either in accuracy or in clarity of interpretation.

The detection of a pulsar with a rotational period smaller for its mass than that obtained as a model independent limit for neutron stars would be decisive in distinguishing between the neutron star interpretation of pulsars as compared to an exotic star—a star that is self-bound at very high equilibrium density (see Eq. (8) in Ref. ([10])). However, nature may never provide a mechanism for approaching the limiting period, which for a neutron star of mass $1.44M_\odot$ is about 0.3 ms.

On the other hand, the pulsation phenomenon in X-ray stars may involve a relation between mass and radius which could also be decisive. However, the interpretation is subject to some uncertainty, both as to the origin of the pulsations and most certainly as to the accuracy of the mass-radius connection, Eq. [2]. This relationship holds only for classical and for non-rotating relativistic stars. There is no formula for the Kepler frequency of a particle orbiting rotating relativistic stars because of the position dependent frame-dragging frequency; rather the Kepler frequency can be determined only as a self-consistency condition on the solution of Einstein’s equations and therefore only for specific model assumptions. There is no possibility of evading this model dependence; however it becomes weaker at further distance outside the star. [See Eq. (8) in ([11])].

Therefore, it is of general interest to have a model-independent limit on radius as a function of mass of neutron stars, such as we have provided here, and it is of particular interest in connection with the oscillations in X-ray brightness of neutron star accreters.

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