THE UNIVERSALITY THEOREM FOR NEIGHBORLY POLYTOPES

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ABSTRACT. In this note, we prove that every open primary basic semialgebraic set is stably equivalent to the realization space of an even-dimensional neighborly polytope. This in particular provides the final step for Mnëv’s proof of the universality theorem for simplicial polytopes.

1. INTRODUCTION

Mnëv’s Universality Theorem was a fundamental breakthrough in the theory of oriented matroids and convex polytopes. It states that the realization spaces of oriented matroids and polytopes, i.e. the spaces of point configurations with fixed oriented matroid/face lattice, can be arbitrarily complex. It comes in four flavours:

Theorem 1 (Universality Theorem [Mnë88]). Let \( V \) be a primary basic semialgebraic set defined over \( \mathbb{Z} \), then

(i) there is an oriented matroid of rank 3 whose realization space is stably equivalent to \( V \), and
(ii) there is a polytope whose realization space is stably equivalent to \( V \);

if moreover \( V \) is open, then

(iii) there is a uniform oriented matroid of rank 3 whose realization space is stably equivalent to \( V \), and
(iv) there is a simplicial polytope whose realization space is stably equivalent to \( V \).

Mnëv announced this theorem in 1985 [Mnë85] and published a sketch of the proof in 1988 [Mnë88]. A more detailed reasoning can be found in his thesis [Mnë86] (in Russian). Shor [Sho91] simplified a key step in Mnëv’s line of reasoning for part (i) and (iii).

Moreover, part (i) of Theorem 1 was later elaborated upon by Richter-Gebert [RG95] and Günzel [Gün96], who proved the stronger Universal Partition Theorem for oriented matroids. Using Lawrence extensions to rigidify the face lattices, it is easy to prove part (ii) from part (i) [Mnë88][RG98]. Additionally, Theorem 1(ii) was generalized greatly by Richter-Gebert, who proved that already 4-dimensional polytopes are universal [RGZ95][RG96].

In contrast, part (iv), for which the elaborations in [Mnë86][Mnë88] were especially concise, remained open (although widely believed to be true) except for some preliminary results of Sturmfels [Stu88b] and Bokowski–Guedes de Oliveira [BGdO90]. It is important to stress that, despite the wrong common belief, Lawrence extensions cannot be used to deduce the universality theorem for simplicial polytopes. We use a different approach to rigidify matroids, namely one based on a result of Shemer proving rigidity of neighborly polytopes [She82].

In particular, we establish here that neighborly polytopes, i.e. \( d \)-polytopes with a complete \( \lfloor \frac{d}{2} \rfloor \)-skeleton, are universal. Even more, we obtain this result with even-dimensional polytopes. Since all neighborly polytopes of even dimension are simplicial, this provides a proof of Theorem 1(iv).

Theorem 2. Every open primary basic semialgebraic set defined over \( \mathbb{Z} \) is stably equivalent to the realization space of some neighborly \((2n - 4)\)-dimensional polytope on \( 2n \) vertices.
This is in contrast to the case of cyclic polytopes, who have trivial realization spaces. This holds more generally for all (Gale sewn) polytopes [She82][Pad13], of which cyclic polytopes are a particular case.

**Theorem 3.** The realization space of

- any simplicial, d-dimensional seen polytope on n vertices and of
- any Gale sewn, neighborly d-polytope on n vertices, d even, is a PL \( d(n - d - 1) \)-ball.

Encouraged by the work of Richter-Gebert, we make the daring conjecture that universality holds even when we restrict to neighborly d-dimensional polytopes.

**Conjecture 4.** Every open primary basic semialgebraic set defined over \( \mathbb{Z} \) is stably equivalent to the realization space of some neighborly (and hence simplicial) \( d \)-polytope.

A universality conjecture for simplicial 4-polytopes is supported by the existence of simplicial 4-polytopes without the isotopy property [BGdO90].

The following idea provides further motivation for our conjecture. Altshuler and Steinberg proved that vertex figures of neighborly 4-polytopes are always 3-dimensional stacked polytopes [AS73]. Despite the fact that realization spaces of stacked polytopes are trivial, their matroids can be complicated: Notice that if \( M \) is any point configuration the plane, then there exists a stacked 3-polytope \( P \) with a distinguished vertex \( v \) such that the contraction of \( v \) in \( P \) coincides with \( M \).

Now, realization spaces of planar point configurations are universal, and it is conceivable that these realization spaces can be captured by constructing neighborly 4-polytopes having them as edge contractions, combined with the fact that all neighborly 4-polytopes are rigid.

2. **Universality for simplicial neighborly polytopes**

The realization space \( \mathcal{R}(M) \) of an oriented matroid \( M \) is the set of vector configurations that share the same oriented matroid, i.e. if \( M \) is of rank \( d \), and \( E = E(M) \) is the ground set of \( M \), then

\[
\mathcal{R}(M) = \{ X \in \mathbb{R}^{E \times d} : X \text{ realizes } M \}/\text{GL}(\mathbb{R}^d)
\]

Similarly, if \( P \) is a polytope in \( \mathbb{R}^d \), and \( V = V(P) \) is its vertex set, then we define its realization space as

\[
\mathcal{R}(P) = \{ X \in \mathbb{R}^{V \times d} : \text{conv}X \text{ realizes } P \}/\text{Aff}(\mathbb{R}^d)
\]

A primary basic semialgebraic set in \( \mathbb{R}^d \) is the set of solutions to a finite number of rational polynomial equalities and strict inequalities. A basic semialgebraic set \( S \subset \mathbb{R}^d \) is a stable projection of a basic semialgebraic set \( T \subset \mathbb{R}^{d+d} \) if, for the projection \( \pi : \mathbb{R}^{d+d} \to \mathbb{R}^d \), we have that \( \pi(T) = S \) and that for every \( x \in S \), the fiber \( \pi^{-1}(x) \) is the relative interior of a non-empty polyhedron defined by equalities and strict inequalities that depend polynomially on \( x \). Two basic semialgebraic sets \( S \) and \( T \) are rationally equivalent if there is a homeomorphism \( f : S \to T \) such that \( f \) and \( f^{-1} \) are rational functions. Two basic semialgebraic sets \( S \) and \( T \) are stably equivalent if they belong to the same equivalence class generated by stable projections and rational equivalences. We denote the stable equivalence of \( S \) and \( T \) as \( S \approx T \). We refer to [RG96][RG98] for more detailed definitions of these concepts.

A technique that pervades our proofs is the use of lexicographic extensions (see [BLSWZ93, Section 7.2]). When \( M \) is realized by a vector configuration \( V \), then its lexicographic extension by \( [a_1^{\sigma_1}, \ldots, a_k^{\sigma_k}] \), where \( a_i \) are elements of \( V \) and \( \sigma_i \) are signs, is realized by adjoining to \( V \) the vector \( \sigma_1 a_1 + \varepsilon \sigma_2 a_2 + \cdots + \varepsilon^{k-1} \sigma_k a_k \) for any \( \varepsilon > 0 \) small enough. The following lemma is straightforward, see also [BLSWZ93, Lemma 8.2.1 and Proposition 8.2.2].

**Lemma 5.** Let \( M \) denote any matroid, and let \( M[a_1^{\sigma_1}, \ldots, a_k^{\sigma_k}] \), \( (a_i) \) independent, denote a lexicographic extension of \( M \). Then the projection

\[
\mathcal{R}(M[a_1^{\sigma_1}, \ldots, a_k^{\sigma_k}]) \to \mathcal{R}(M)
\]

\[
X \mapsto X|_{E(M)}
\]

induced by deletion is surjective, and its fibers are (polynomially parametrized) polyhedra of dimension \( k \).
We start with an observation of Mnëv/Shor concerning universality of uniform matroids \([\text{Mnë85}] [\text{Sho91}]\), which is proven using constructible oriented matroids and a substitution technique from \([\text{Mnë88}] [\text{JMLSW89}]\).

**Lemma 6.** For every open primary basic semialgebraic set \(S\) defined over \(\mathbb{Z}\), there exists a uniform matroid \(M\) such that
\[
S \approx R(M).
\]

The key step of the proof of Theorem 2 is to use a construction of Kortenkamp, who proved that every \(d\)-dimensional point configuration of at most \(d + 4\) points appears as a face figure of a neighborly polytope. For larger point configurations, this is still an open problem, first asked by Perles (compare \([\text{Stu88a}]\)).

For the proof, recall that in oriented matroid theory a polytope is called rigid if the face lattice of \(P\) determines the oriented matroid \(M\) spanned by the vertices of \(P\) (see \([\text{Zie95, Section 6.6}]\)). In particular, for all rigid polytopes, we have \(R(M) \cong R(P) \times \mathbb{R}_{>0}^n \approx R(P)\).

**Lemma 7.** For every uniform matroid \(M\) of rank 3 on \(n\) elements, there exists a neighborly polytope \(P\) with \(2n\) vertices in dimension \(2n - 4\) such that
\[
R(M) \cong R(P).
\]

**Proof.** By a theorem of Kortenkamp \([\text{Kor97, Theorem 1.2}]\), every matroid of rank 3 can be extended to the Gale dual of an even-dimensional neighborly polytope by performing \(n\) lexicographic extensions, obtaining a rank 3 matroid \(\tilde{M}\) on \(2n\) elements. By Lemma 5 we obtain
\[
R(M) \cong R(\tilde{M}).
\]

Let \(P\) be the face lattice of the Gale dual of \(\tilde{M}\). Now, Gale duality preserves realization spaces and, by \([\text{She82, Theorem 2.10}]\) and \([\text{Stu88a, Theorem 4.2}]\), every neighborly polytope of even dimension is rigid, i.e. the face lattice determines the whole oriented matroid. Hence,
\[
R(P) \cong R(\tilde{M}).
\]

Together with Lemma 6, this finishes the proof of Theorem 2. It remains to characterize the realization spaces of Gale sewn polytopes.

**Proof of Theorem 3.** If \(P\) is a \(d\)-polytope obtained by (general position) sewing a vertex \(v\) onto a \(d\)-polytope \(Q\), then the fibers of the natural projection map \(R(P) \rightarrow R(Q)\) are relative interiors of polyhedra (by Grünbaum’s beneath-beyond \([\text{Gri03, Section 5.2}]\)). Moreover, since sewing can be always realized by a specific lexicographic extension, these polyhedra are \(d\)-dimensional by Lemma 5 (and in particular nonempty). The claim follows by induction.

Gale sewing produces even-dimensional neighborly polytopes, and hence rigid, through a sequence of lexicographic extensions of the Gale dual. The claim follows by Lemma 5 and induction.

**Remark 8.** Unlike the first part of Theorem 3, the second part requires the polytopes to be neighborly. To explain this, notice that the (classical) sewing operation consists of lexicographic extensions which only depend on the positive cocircuits of the associated matroid, and hence is fundamentally a polytope operation.
This is unlike the case of Gale sewing, for which we apply lexicographic extensions to the Gale dual of $P$. Hence it is a matroid operation, so that we rely on rigidity to relate the realization spaces of matroid and polytope. Indeed, observe that if the polytope is not rigid then the deletion map might not be surjective: If $P$ is realized by different matroids $M$ and $M'$, lexicographic extensions with the same signature of the Gale duals of $M$ and $M'$ might give rise to different polytopes.

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