A note on Lusin-type approximation of Sobolev functions on Gaussian spaces

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Abstract

We extend Shigekawa’s Meyer-type inequality in $L^1$ to more general Ornstein–Uhlenbeck operators and establish new approximation results in the sense of Lusin for Sobolev functions $f$ with $|\nabla f| \in L \log L$ on infinite-dimensional spaces equipped with Gaussian measures. The proof relies on some new pointwise estimate for the approximations based on the corresponding semigroup which can be of independent interest.

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1. Introduction

We say that a function $f : X \to \mathbb{R}$ on a metric measure space $(X, d, m)$ is approximable in the sense of Lusin by Lipschitz functions if for any given $\varepsilon > 0$ there exists a Lipschitz function $g : X \to \mathbb{R}$ and a Borel set $S \subset X$ such that $m(X \setminus S) < \varepsilon$ and $f \equiv g$ on $S$. A quantitative version of this property can be formulated as follows:

$$|f(x) - f(y)| \leq d(x, y)(g(x) + g(y)) \quad (1.1)$$

for some measurable nonnegative function $g$ and $x, y \in X \setminus N$, where $N$ is a Borel set of $m$-measure zero. For Sobolev or BV functions on $\mathbb{R}^d$ equipped with the standard Lebesgue measure $\lambda$, F.C. Liu (see [10]) obtained the following important result:

$$|f(x) - f(y)| \leq |x - y|(M(x) + M(y)),$$

$$M(x) := C_d \sup_{r > 0} \frac{1}{\lambda(B(x, r))} \int_{B(x, r)} |\nabla f| \, d\lambda,$$

see also the book [20] for a detailed discussion of this problem. In particular, for $p > 1$ and a function $f$ from the Sobolev class $W^{1,p}$ in the inequality (1.1) one can choose $g \in L^p$. Moreover, it is well-known that in the class of metric measure spaces $(X, d, m)$ satisfying the doubling and 1-Poincaré inequality, the property (1.1) with $g \in L^p$ characterizes

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the class $W^{1,p}$, while for general metric measure structures it is the basis of the definition of the so-called Hajlasz Sobolev functions, see e.g. [9]. For metric measure spaces without the doubling property the classical finite dimensional arguments are not available anymore. In the recent paper [2] L. Ambrosio, E. Brue, D. Trevisan put forward an alternative approach based on some estimates for heat semigroups which applies to Gaussian and $RCD(K, \infty)$ spaces. For Sobolev functions $f \in W^{1,p}$, $p > 1$ on the Wiener space $(W, \mathcal{H}, m)$ the result from [2] reads as follows: there exists a version of the function $f$, such that
\[ |f(x) - f(y)| \leq |x - y|\mathcal{H}(M(x) + M(y)), \]
where $M(x) := C\left( \sup_{t>0} T_t |\nabla \mathcal{H} f|(x) + \sup_{t>0} T_t |\sqrt{1-L} f|(x) \right)$, $\{T_t\}$ is the standard Ornstein-Uhlenbeck semigroup, $L$ is its generator and $C$ is some universal constant. In the Gaussian setting for $p > 1$ this theorem provides a natural counterpart of the classical finite-dimensional results, however, for $p = 1$ the arguments break down for multiple reasons. First, for the Sobolev space $W^{1,1}$ Meyer’s equivalence is not available anymore, in particular, $f$ from $W^{1,1}$ does not need to belong to $D_1(\sqrt{1-L})$ in $L^1$. Second, even for functions $f$ with $\sqrt{1-L} f \in L^1$ it is unknown if the maximal function
\[ \sup_{t>0} T_t |\sqrt{1-L} f| \]
is finite $m$-a.e. or not. In particular, whether the maximal operator
\[ \sup_{t>0} T_t g, \ g \in L^1(X, m) \]
is of weak $(1, 1)$ type in the infinite-dimensional case has been an open problem for a long time. In this paper we construct a Lusin-type approximation for Sobolev functions $f$ with $|\nabla f| \in L \log L$ on Gaussian spaces based on a modification of the approach from [2]. Our starting point is the result by I. Shigekawa [15] which gives a weaker form of Meyer’s equivalence for $L^1$:
\[ \|\sqrt{-L} f\|_1 \leq C \|\nabla f\|_{L \log L}, \ \|\nabla f\|_1 \leq C \|\sqrt{-L} f\|_{L \log L} \]
To overcome the lack of the weak bound for the maximal operator in $L^1$ we modify the smoothing procedure. This enables us to obtain a dimension-independent bound using the classical Hopf–Dunford–Schwartz maximal inequality that might be of independent interest. The paper is organized as follows. Section 2 contains the main abstract semigroup-theoretic tools. In Section 3 we recall the proof of Shigekawa’s bound and extend it to more general Ornstein-Uhlenbeck semigroups. Section 4 contains the main results.
2. Abstract semigroup-theoretic results

Let \((X, \mathcal{F}, m)\) be an abstract measure space, where \(m\) is a probability measure. Let \(\{T_t\}\) be a symmetric Markov semigroup acting on \(L^2(X, \mathcal{F}, m)\) and let \(L\) be its generator. A semigroup of this class has a canonical extension to a contraction semigroup on all \(L^p(X, \mathcal{F}, m)\) spaces. For \(p \in [1, \infty)\) and \(f \in L^p(X, \mathcal{F}, m)\) we write \(f \in D_p(\sqrt{-L})\) if there exists a sequence \((f_n) \subset D(\sqrt{-L}) \cap L^p(X, \mathcal{F}, m)\) converging to \(f\) in \(L^p(X, \mathcal{F}, m)\) with \((\sqrt{-L}f_n)\) converging to some function \(g\) in \(L^p(X, \mathcal{F}, m)\). Using the symmetry of \(\sqrt{-L}\) it is easy to see that if the set of functions \(h \in D(\sqrt{-L}) \cap L^p(X, \mathcal{F}, m)\) is dense in \(L^p(X, \mathcal{F}, m)\) in the weak-\(\ast\) topology (in duality with \(L^p(X, \mathcal{F}, m)\)) then \(g = \sqrt{-L}f\) is uniquely determined. In our cases of interest, where \(\{T_t\}\) is an Ornstein-Uhlenbeck semigroup, the required density can be easily verified explicitly.

The next proposition is the classical Hopf–Dunford–Schwartz maximal inequality which will play the crucial role in the proof of the main results.

**Proposition 2.1.** Let \(f \in L^1(X, \mathcal{F}, m)\). Then for any \(\lambda > 0\)

\[
m\left( x : \sup_{t>0} \frac{1}{t} \int_{[0,t]} T_s f \, ds \geq \lambda \right) \leq \frac{\|f\|_1}{\lambda}.
\]

*Proof.* See [8], chapter VIII (6,7). \(\square\)

Now let us introduce the “smoothing” operators \(\{A_t\}\) as follows:

\[A_t := \frac{1}{t} \int_{[t,2t]} T_s \, ds\]

**Theorem 2.2.** There exists a universal constant \(C > 0\) such that for any \(f \in D(\sqrt{-L})\)

\[|A_t f(x) - f(x)| \leq C \sqrt{t} \sup_{s>0} A_s |\sqrt{-L}f|(x) \text{ for } m\text{-a.e. } x \in X.
\]

*Proof.* Using simple density arguments one can see that it is sufficient to prove this estimate just for functions from \(D(\sqrt{-L})\). In this case for the difference \(T_r f - f\) we have the following classical representation:

\[T_r f - f = \int_{[0,\infty)} K(s, r) T_s \sqrt{-L} f \, ds, \quad r \geq 0,
\]

where

\[K(s, r) := \frac{1}{\sqrt{\pi}} \left( \frac{\chi_{s>r}}{(s-r)^{1/2}} - \frac{\chi_{s>0}}{s^{1/2}} \right),\]
e.g. see Proposition 2.1 in [2] and also [16]. Then:

\[ A_t f - f = \int_{[0, \infty)} U(s, t)T_s \sqrt{-L} f \, ds, \tag{2.1} \]

where

\[ U(s, t) := \frac{1}{t} \int_{[t, 2t]} K(s, r) \, dr. \]

The equality here holds on some set of full measure \( \Omega_{f,t} \) which depends on \( f \) and \( t \). However, using Fubini’s theorem one can easily show that there exists a set \( \Omega'_f \) of full measure which depends only on \( f \) such that the mapping

\[ t \mapsto \int_{[0, t]} T_s f(x) \, ds \]

is absolutely continuous on \((0, \infty)\) whenever \( x \in \Omega'_f \). Consequently, for \( x \in \Omega'_f \) the mapping \( t \mapsto A_t f(x) \) is continuous. Analogously applying Fubini’s theorem one can show that the function

\[ t \mapsto \frac{1}{t} \int_{[0, t]} \int_{[0, \infty)} K(s, r)T_s \sqrt{-L} f \, ds \, dr = \int_{[0, \infty)} U(s, t) T_s \sqrt{-L} f \, ds \]

continuously depends on \( t \) for \( x \in \Omega''_f \), \( \mu(\Omega''_f) = 1 \). Therefore, the equality 2.1 holds simultaneously for all \( t > 0 \) on the set

\[ \Omega_f := \Omega''_f \cap \Omega'_f \cap \bigcap_{t_i \in Q \cap [0, \infty)} \Omega_{f,t_i}, \]

it is clear that \( m(\Omega_f) = 1 \).

One can see that \( K(s, r) \) is not smooth with respect to \( s \) at \( r \). However, it turns out that the “averaged” over \( r \) version of \( K \) is already absolutely continuous with respect to \( s \) for all \( s > 0 \). Below we will show that there exists a universal constant \( C > 0 \) such that for the function

\[ Q(s, t) := s \frac{\partial U}{\partial s}(s, t) \]

the following inequality holds:

\[ \int_{[0, \infty)} |Q(s, t)| \, ds \leq C \sqrt{t}. \tag{2.2} \]

Now let us prove the inequality (2.2). It is easy to see that the functions \( K, U \) and \( Q \) have the following homogeneity property: for any \( a > 0 \)

\[ K(as, ar) = \frac{1}{\sqrt{a}} K(s, r), \quad U(as, at) = \frac{1}{\sqrt{a}} U(s, t), \]

\[ Q(as, at) = \frac{1}{\sqrt{a}} Q(s, t). \]
Then:

\[ \int_{[0, \infty)} |Q(s, t)| \, ds = \int_{[0, \infty)} |Q(ts', t)| \, ds' = \sqrt{t} \int_{[0, 1)} |Q(s', 1)| \, ds'. \]

Consequently, to establish the bound (2.2) it is sufficient to prove that

\[ \int_{[0, \infty)} |Q(s, 1)| \, ds < \infty. \]

For \( s \in (0, 1) \)

\[ U(s, 1) = -\frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{s}}. \]

For \( s \in (1, 2) \)

\[ U(s, 1) = -\frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{s}} + \frac{2}{\sqrt{\pi}} \sqrt{s - 1}. \]

For \( s \in (2, \infty) \)

\[
U(s, 1) = -\frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{s}} + \frac{2}{\sqrt{\pi}} (\sqrt{s - 1} - \sqrt{s - 2}) \\
= -\frac{1}{\sqrt{\pi}} \frac{1}{\sqrt{s}} + \frac{2}{\sqrt{\pi}} \frac{1}{\sqrt{s - 1} + \sqrt{s - 2}} \\
= \frac{1}{\sqrt{\pi}} \frac{\sqrt{s - \sqrt{s - 1} + \sqrt{s - s - 2}}}{\sqrt{s} \sqrt{s - 1} + \sqrt{s - 2}}.
\]

Now it is easy to verify that

\[ Q(s, 1) = s \frac{\partial U}{\partial s}(s, 1) \]

is integrable on \([0, \infty)\). Excluding if necessary a set of measure zero from \( \Omega_f \) by Proposition 2.1 we can assume that for any \( x \in \Omega_f \)

\[ |\sup_{t>0} A_t \sqrt{-Lf}| < \infty. \]

Applying integration by parts we obtain the following equality:

\[
\int_{[1/n, n]} U(s, t) T_s \sqrt{-L} f \, ds = -\int_{[1/n, n]} \frac{\partial U(s, t)}{\partial s} s A_s \sqrt{-L} f \, ds \\
+ U(n, t) n A_n \sqrt{-L} f - U(1/n, t) \frac{1}{n} A_{1/n} \sqrt{-L} f.
\]

Since

\[ |U(1/n, t)| = O(\sqrt{n}), \quad |U(n, t)| = O\left(\frac{1}{n \sqrt{n}}\right), \quad n \to \infty, \]
then
\[ U(1/n, t)\frac{1}{n}A_{1/n}\sqrt{-L}f \to 0, \quad U(n, t)nA_n\sqrt{-L}f \to 0, \quad \text{as } n \to \infty. \]
Consequently,
\[ A_t f - f = -\int_{[0,\infty)} Q(s, t) [A_s \sqrt{-L}f] \, ds, \]
and the bound (2.2) yields the required estimate
\[ |A_t f(x) - f(x)| \leq C \sqrt{t} \sup_{s>0} A_s |\sqrt{-L}f|(x) \text{ for } m\text{-a.e. } x \in X. \]
□

3. Meyer-type inequality in \( L^1 \)

Let us define the Gaussian measure \( \mu \) on \( \mathbb{R}^d \) by the formula
\[ \mu(dx) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2} \, dx \]
and let \( L \) be the standard Ornstein–Uhlenbeck operator:
\[ L := \Delta - \langle x, \nabla \rangle. \quad (3.1) \]
The Ornstein–Uhlenbeck semigroup \( \{T_t\} \) generated by \( L \) is given by
\[ T_t f(x) = \int f(e^{-t}x + \sqrt{1 - e^{-2t}}y) \, d\mu(y) \]
In the \( L^p \) setting Meyer’s inequalities establish the equivalence of two kinds of norms on the Sobolev space in Malliavin calculus, one is defined by means of the gradient and the other by means of the square root of the Ornstein–Uhlenbeck operator, see [12], [13]: for any fixed \( p \in (1, \infty) \) there exist positive constants \( C_1(p), C_2(p) \) such that for any \( f \in C^\infty_0(\mathbb{R}^d, \mathbb{R}) \)
\[ C_1(p)\|\sqrt{1 - L}f\|_p \leq \|\nabla f\|_p + \|f\|_p \leq C_2(p)\|\sqrt{1 - L}f\|_p \]
Later this result was extended to more general Ornstein–Uhlenbeck semigroups by I. Shigekawa in [14], see also [6]. For \( p = 1 \) these inequalities do not hold anymore, however, in the work [15] a weaker replacement was proposed: for a fixed \( \alpha \geq 0 \) there exist \( C_1(\alpha), C_2(\alpha) > 0 \) such that for any \( f \in C^\infty_0(\mathbb{R}^d, \mathbb{R}) \)
\[ C_1(\alpha)\|\sqrt{\alpha - L}f\|_1 \leq \|\nabla f\|_{L \log L} + \|f\|_1, \]
\[ \|\nabla f\|_1 \leq C_2(\alpha)\|\sqrt{\alpha - L}f\|_{L \log L}. \]
Here $L \log L$ is the set of all elements of $L^1$ such that
\[ \int |f| \log(1 + |f|) \, d\mu < \infty, \]
equipped with the norm
\[ \|f\|_{L \log L} := \inf \left\{ \lambda > 0 : \int \Phi(|f|/\lambda) \, d\mu \leq 1 \right\}, \quad (3.2) \]
where
\[ \Phi(a) := \int_{[0,a]} \log(1 + t) \, dt. \quad (3.3) \]
In the paper [15] only the case of the standard Ornstein–Uhlenbeck operator (3.1) was considered. For the sake of completeness we present a proof of these inequalities below for more general Ornstein–Uhlenbeck operators and fill in some details omitted in [15] and add some missing arguments. This extension will enable us to prove the main results for Sobolev classes on an abstract Wiener space as well as for Da Prato’s Sobolev classes on a Hilbert space equipped with a Gaussian measure.

Let $m = N_Q$ be a centered nondegenerate Gaussian measure on $\mathbb{R}^d$ with the covariance operator $Q$. We will be concerned with the semigroup given by Mehler’s formula (see e.g. [3])
\[ P_t f(x) := \int_H f(e^{At}x + \sqrt{1 - e^{2At}}y) \, dm(y) \]
\[ = \int_H f(e^{At}x + y) \, dN_Q(t)(y) = \int_H f(y) \, dN_{e^{At}x,Q_t}(y), \]
where we have set
\[ A := -\frac{1}{2} Q^{-1}, \quad Q_t := \int_{[0,t]} e^{2As} \, ds = Q(1 - e^{2At}) \]
and $N_{e^{At}x,Q_t}$ denotes the the unique Gaussian measure with mean $e^{At}x$ and covariance $Q_t$. Although in this section we assume that the underlying space is finite-dimensional, the final inequalities do not include any dimension-dependent constants and are valid for the infinite-dimensional case as well, this can be justified by the standard approximation arguments. The generator of $\{P_t\}$ is the Ornstein–Uhlenbeck operator $L$, where
\[ L := \frac{1}{2} \Delta + \langle Ax, \nabla \rangle \]
and $N_Q$ is the unique invariant measure of $\{P_t\}$. We will also assume that the operator $-A$ is diagonal with eigenvalues $(\lambda_1, \ldots, \lambda_d)$, where
\[ \beta \leq \lambda_1 \leq \ldots \leq \lambda_d \quad (3.4) \]
for some constant $\beta > 0$. The semigroup $\{P_t\}$ corresponds to the unique strong solution of the stochastic differential equation

$$dX_t = AX_t \, dt + dW_t,$$

where $W_t$ is the standard Brownian motion on $\mathbb{R}^d$.

Following [15], let us introduce another one-dimensional Brownian motion $(B_t)$ that is independent from $(W_t)$, for convenience we will assume that the generator of $(B_t)$ is $\frac{d^2}{da^2}$ instead of the standard one $\frac{1}{2} \frac{d^2}{da^2}$. The starting point of $(B_t)$ is $N$ and to indicate this we denote by $\mathbb{E}_N$ the expectation with respect to the joint law of $(B_t), (W_t)$. Let $\tau$ be the hitting time of $(B_t)$ to the point 0. For a given $\alpha > 0$ let $R^\alpha_t$ be the semigroup generated by $-\sqrt{\alpha - L}$, it is well-known that $\{R^\alpha_t\}$ is a subordination of $\{P_t\}$:

$$R^\alpha_t = \int_{[0, \infty)} e^{-\alpha s} P_s \, d\nu_t(s), \quad (3.5)$$

where $\nu_t$ is a probability measure on $[0, \infty)$ with the Laplace transform

$$\int_{[0, \infty)} e^{-\gamma s} \, d\nu_t(s) = e^{-\sqrt{\gamma t}}, \quad \gamma \geq 0.$$

Let us recall the commutation relation between $\nabla$ and $L$:

$$\nabla L = L \nabla + A \nabla$$

For the semigroup $\{P_t\}$ this implies the following identities:

$$\nabla P_t = e^{At} P_t \nabla,$$

or, equivalently,

$$\nabla_i P_t = e^{-\lambda_i t} P_t \nabla_i, \quad i = 1, \ldots, d.$$

For $f \in C_0^\infty(\mathbb{R}^d, \mathbb{R})$ one can see that the function

$$u(a, x) := R^\alpha_a f(x)$$

satisfies the equation

$$u(0, x) = f(x),$$

$$\frac{\partial u}{\partial a}(a, \cdot) = -\sqrt{\alpha - L} u(a, \cdot).$$

Consequently,

$$\frac{\partial^2 u}{\partial a^2}(a, x) = (\alpha - L) u(a, x),$$

$$\frac{\partial^2 u}{\partial a^2}(a, x) + Lu(x, a) - \alpha u(a, x) = 0.$$
Set
\[ M^>_f(t) := \int_{[0,t\wedge\tau]} \frac{\partial u}{\partial a}(B_s, X_s) dB_s = -\int_{[0,t\wedge\tau]} \sqrt{\alpha - LR^\alpha_{B_s} f(X_s)} dB_s, \]
\[ M^<_f(t) := \int_{[0,t\wedge\tau]} \nabla u(B_s, X_s) dW_s = \int_{[0,t\wedge\tau]} \nabla R^\alpha_{B_s} f(X_s) dW_s. \]

By Ito’s formula
\[ u(B_{t\wedge\tau}, X_{t\wedge\tau}) - u(B_0, X_0) = M^>_f(t) + M^<_f(t) + \alpha \int_{[0,t\wedge\tau]} u(B_s, X_s) ds. \] (3.6)

Using the commutation identities for \( \nabla \) and \( P_t \) one can observe that
\[ \nabla_i R^\alpha_t = R^{\alpha+\lambda_i}_t \nabla_i, \quad i = 1, \ldots, d. \] (3.7)

Indeed, for any \( \varphi \in C^\infty_0(\mathbb{R}^d, \mathbb{R}) \) by the subordination formula 3.5
\[ \nabla_i R^\alpha_t \varphi = \int_{[0,\infty)} e^{-\alpha s} \nabla_i P_s \varphi \, dw_t(s) = \int_{[0,\infty)} e^{-\alpha s} e^{-\lambda_i s} \nabla P_s \varphi \, dw_t(s) = R^{\alpha+\lambda_i}_t \nabla_i \varphi. \]

Let \( g = (g_1, g_2, \ldots, g_d) \) be a vector-valued \( C^\infty_0 \) function on \( \mathbb{R}^d \). Let us define \( v(x, a) \) as follows:
\[ v = (v_1, \ldots, v_d), \]
\[ v_i(x, a) := R^{\alpha+\lambda_i}_t g_i, \quad i = 1, \ldots, d. \]

The function \( v \) satisfies the equation
\[ v(0, x) := g, \]
\[ \frac{\partial^2 v}{\partial^2 a}(a, x) + L v(a, x) + Av(a, x) - \alpha v(a, x) = 0. \]

Hence, by Ito’s formula
\[ v(B_{t\wedge\tau}, X_{t\wedge\tau}) - v(B_0, X_0) \]
\[ = \int_{[0,t\wedge\tau]} \frac{\partial v}{\partial a}(B_s, X_s) dB_s + \int_{[0,t\wedge\tau]} \nabla v(B_s, X_s) dW_s \]
\[ + \int_{[0,t\wedge\tau]} (\alpha - A) v(B_s, X_s) ds. \] (3.8)

Let us define the vector-valued martingales \( N^>_g \) and \( N^<_g \) by the formula
\[ N^>_g(t) := -\int_{[0,t\wedge\tau]} \frac{\partial v}{\partial a}(B_s, X_s) dB_s, \]
\[ N^<_g(t) := \int_{[0,t\wedge\tau]} \nabla v(B_s, X_s) dW_s. \]
\[ N_g^\dagger(t) := \int_{[0,t\wedge\tau]} \nabla v(B_s, X_s) \, dW_s. \]

Set
\[ U_t := |u(B_{t\wedge\tau}, X_{t\wedge\tau})|, \quad V_t := |v(B_{t\wedge\tau}, X_{t\wedge\tau})|. \]

**Remark 3.1.** Due to the commutation relations \ref{eq:commutation_relations} it is easy to see that for \( g = \nabla f \) the constructed above function \( v(t, x) \) coincides with \( \nabla R_f \). The necessity to consider this additional helper function \( v \) is the main difference with the case \( A = \text{Id} \) which was studied in \cite{15}.

**Lemma 3.2.** The processes \((U_t), (U_t^2), (V_t), (V_t^2)\) are submartingales. Moreover, for the processes \((A_{U,t}), (A_{V,t})\) from the Doob–Meyer decompositions of the submartingales \((U_t^2), (V_t^2)\) the following inequalities hold:

\[ A_{U,t} \geq \int_{[0,t\wedge\tau]} |\nabla u(B_s, X_s)|^2 \, ds, \]

\[ A_{V,t} \geq \int_{[0,t\wedge\tau]} \left| \frac{\partial v}{\partial a}(B_s, X_s) \right|^2 \, ds. \]

**Proof.** We will consider the process \((V_t)\), the case of \((U_t)\) is handled analogously, see also Proposition 2.1 from \cite{15}. The identity \ref{eq:id3.8} can be written as follows:
\[ v(B_{t\wedge\tau}, X_{t\wedge\tau}) = v(B_0, X_{t\wedge\tau}) + N_t + \int_{[0,t\wedge\tau]} (\alpha - A)v(B_s, X_s) \, ds, \]
where the martingale \((N_t)\) is given by the formula
\[ N_t = \int_{[0,t\wedge\tau]} \frac{\partial v}{\partial a}(B_s, X_s) \, dB_s + \int_{[0,t\wedge\tau]} \nabla v(B_s, X_s) \, dW_s \]

Set
\[ v_t := v(B_{t\wedge\tau}, X_{t\wedge\tau}). \]

Then:
\[ dV_t^2 = 2\langle v_t, dN_t \rangle + 2\langle (\alpha - A)v_t, v_t \rangle + d\langle N \rangle_t. \]  \hfill (3.9)

Taking into account the assumption \ref{eq:assumption3.4} now one can see that \((V_t^2)\) is a submartingale. Moreover, for the Doob–Meyer decomposition of the process \((V_t^2)\)
\[ V_t^2 = V_0^2 + \text{martingale} + A_{V,t} \]  \hfill (3.10)
we have the inequality
\[ A_{V,t} \geq \int_{[0,t\wedge\tau]} \left| \frac{\partial v}{\partial a}(B_s, X_s) \right|^2 \, ds. \]  \hfill (3.11)
In order to prove that \((V_t)\) is a submartingale it is sufficient to show that for every \(\epsilon > 0\) the process \(\sqrt{\epsilon + V_t^2}\) has this property.

\[
d\sqrt{\epsilon + V_t^2} = \frac{1}{2}(V_t^2 + \epsilon)^{-1/2}(v_t, dN_t)
+ \frac{1}{2}(V_t^2 + \epsilon)^{-1/2}[\langle (\alpha - A)v_t, v_t \rangle dt + d\langle N \rangle_t]
- \frac{1}{222}(V_t^2 + \epsilon)^{-3/2}(V_t^2 + \epsilon)^{-3/2}(2v_t dN_t, 2v_t dN_t)
= \frac{1}{2}(V_t^2 + \epsilon)^{-1/2}2\langle v_t, dN_t \rangle
+ \frac{1}{2}(V_t^2 + \epsilon)^{-1/2}\langle (\alpha - A)v_t, v_t \rangle dt
+ \frac{1}{2}(V_t^2 + \epsilon)^{-3/2}(V_t^2 + \epsilon)^{-3/2}\left[(V_t^2 + \epsilon)d\langle N \rangle_t - \langle v_t dN_t, v_t dN_t \rangle\right].
\]

To finish the proof it remains to refer to the well-known inequality (see e.g. Proposition 2.1 in [15])
\[
\langle v_t dN_t, v_t dN_t \rangle \leq |v_t|^2d\langle N \rangle_t = V_t^2d\langle N \rangle_t
\]
to conclude that the term
\[
(V_t^2 + \epsilon)d\langle N \rangle_t - \langle v_t dN_t, v_t dN_t \rangle
\]
is nonnegative.

\(\square\)

**Lemma 3.3.** For any \(f \in C_0^\infty(\mathbb{R}^d, \mathbb{R}), g \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)\)

\[
\|M_f\|_{BMO} \leq \|f\|_\infty, \quad \|M^\uparrow_f\|_{BMO} \leq \|f\|_\infty, \\
\|N^\downarrow_g\|_{BMO} \leq \|g\|_\infty, \quad \|N^\uparrow_g\|_{BMO} \leq \|g\|_\infty.
\]

**Proof.** By Ito’s formula we have the following identity:

\[
V_t^2 - V_0^2 = \text{martingale} + A_{V,t},
\]

where

\[
A_{V,t} := \int_{[0,t\wedge \tau]} 2\langle (\alpha - A)v_t, v_t \rangle ds + \langle N^\downarrow_g \rangle_t + \langle N^\uparrow_g \rangle_t
\]
is a nondecreasing process with \(A_{V,0} = 0\). Hence, for any stopping time \(T\) we have

\[
\mathbb{E}_N [\|N^\downarrow_g(\infty) - N^\downarrow_g(T)\|^2|\mathcal{F}_T] = \mathbb{E}_N [\langle N^\downarrow_g \rangle_\infty - \langle N^\downarrow_g \rangle_T|\mathcal{F}_T]
\leq \mathbb{E}_N [A_{V,\infty} - A_{V,T}|\mathcal{F}_T] = \mathbb{E}_N [V_\infty^2 - V_T^2|\mathcal{F}_T]
\leq \mathbb{E}_N [g^2(X_T)|\mathcal{F}_T] \leq \|g\|^2_\infty.
\]

The cases of \(N^\uparrow_g, M_f^\uparrow, M_f^\downarrow\) are handled completely analogously. \(\square\)
Lemma 3.4. For any $\zeta \in C_0^\infty(\mathbb{R}^{d+1}, \mathbb{R})$
\[
\mathbb{E}_N \int_{[0,\tau]} \zeta(B_s, X_s) \, ds = \int_{[0,\infty)} (N \wedge a) \int_{\mathbb{R}^d} \zeta(a, x) \, dm(x) \, da.
\]

Proof. Without loss of generality we can assume that $\zeta$ is nonnegative. Let us set
\[
\eta(a) := \int_{\mathbb{R}^d} \zeta(a, x) \, dm(x).
\]
Since $B = (B_t)$ and $X = (X_t)$ are independent it is easy to see that
\[
\mathbb{E}_N \int_{[0,\tau]} \zeta(B_s, X_s) \, ds
= \mathbb{E}_N \mathbb{E} \left[ \int_{[0,\tau]} \zeta(B_s, X_s) \, ds \mid B \right]
= \mathbb{E}_N \int_{[0,\tau]} \eta(B_s) \, ds.
\]
Let $\theta$ be a smooth function such that
\[
\theta'' = \eta, \quad \theta'(N) = 0, \quad \theta(N) = 0.
\]
Let us apply Ito’s formula to $\theta$ and $(B_t)$:
\[
\theta(B_t) - \theta(B_0) = \int_{[0,t]} \theta'(B_s) \, dB_s + \int_{[0,t]} \theta''(B_s) \, ds
= \int_{[0,t]} \theta'(B_s) \, dB_s + \int_{[0,t]} \eta(B_s) \, ds.
\]
Then for any $t > 0$
\[
\theta(B_{t \wedge \tau}) - \theta(B_0) = \int_{[0,t \wedge \tau]} \theta'(s) \, dB_s + \int_{[0,t \wedge \tau]} \eta(B_s) \, ds
= \mathbb{E}_N \left[ \theta(B_{t \wedge \tau}) - \theta(N) \right]
= \mathbb{E}_N \int_{[0,t \wedge \tau]} \eta(B_s) \, ds.
\]
By passing to the limit $t \to \infty$ we obtain the equality
\[
\theta(0) - \theta(N) = \mathbb{E}_N \int_{[0,\tau]} \eta(B_s) \, ds.
\]
Then:
\[
\mathbb{E}_N \int_{[0,\tau]} \eta(B_s) \, ds = \theta(0) - \theta(N) = - \int_{[0,N]} \theta'(a) \, da
= - \int_{[0,\infty)} \frac{d}{da}[(N \wedge a)\theta'(a) \, da = \int_{[0,\infty)} (N \wedge a)\eta(a) \, da
= \int_{[0,\infty)} (N \wedge a) \int_{\mathbb{R}^d} \zeta(a, x) \, dm(x) \, da.
\]
Lemma 3.5. For $\varphi, \psi \in C^\infty_0(\mathbb{R}^d, \mathbb{R})$

$$\lim_{N \to \infty} \mathbb{E}_N \left[ M^\rightarrow_\varphi(\infty) M^\rightarrow_\psi(\infty) \right] = \begin{cases} \frac{1}{2} \langle \varphi, \psi \rangle_{L^2(m)}; & \alpha > 0 \\ \frac{1}{2} \langle \bar{\varphi}, \bar{\psi} \rangle_{L^2(m)}; & \alpha = 0 \end{cases} (3.12)$$

$$\lim_{N \to \infty} \mathbb{E}_N \left[ M^\uparrow_\varphi(\infty) M^\uparrow_\psi(\infty) \right] = \begin{cases} \frac{1}{4} \langle \varphi, \psi \rangle_{L^2(m)} - \frac{1}{4} \alpha \langle (\alpha - L)^{-1} \varphi, \psi \rangle; & \alpha > 0 \\ \frac{1}{4} \langle \bar{\varphi}, \bar{\psi} \rangle_{L^2(m)}; & \alpha = 0 \end{cases} (3.13)$$

where $\bar{\varphi} = \varphi - \int_{\mathbb{R}^d} \varphi \, dm$, $\bar{\psi} = \psi - \int_{\mathbb{R}^d} \psi \, dm$.

Proof. Applying the standard polarization decomposition one can see that it is sufficient to prove these equalities just for $\varphi = \psi$. For the martingale $M^\rightarrow_\varphi$ we have the chain of equalities

$$\lim_{N \to \infty} \mathbb{E}_N \left[ M^\rightarrow_\varphi(\infty) \right]^2 = 2 \lim_{N \to \infty} \mathbb{E}_N \int_{[0, \tau]} \left| \frac{\partial u}{\partial a} \left( B_s, X_s \right) \right|^2 \, ds$$

$$= 2 \lim_{N \to \infty} \int_{[0, \infty)} (N \wedge a) \int_{\mathbb{R}^d} \left| \frac{\partial u}{\partial a} (a, x) \right|^2 \, dm(x) \, da$$

$$= 2 \int_{[0, \infty)} a \int_{\mathbb{R}^d} \left| \frac{\partial u}{\partial a} (a, x) \right|^2 \, dm(x) \, da$$

$$= 2 \int_{[0, \infty)} a \int_{\mathbb{R}^d} \left| \sqrt{\alpha - Lu(a, x)} \right|^2 \, dm(x) \, da$$

$$= 2 \int_{[0, \infty)} a \int_{[0, \infty)} (\alpha + \lambda) e^{-2a\sqrt{\alpha + \lambda}} \, d\langle E_{\lambda} \varphi, \varphi \rangle \, da$$

$$= \int_{[0, \infty)} \int_{[0, \infty)} 2a(\alpha + \lambda) e^{-2a\sqrt{\alpha + \lambda}} \, d\langle E_{\lambda} \varphi, \varphi \rangle$$

$$= \begin{cases} \frac{1}{2} \langle \varphi, \varphi \rangle_{L^2(m)}; & \alpha > 0 \\ \frac{1}{2} \langle \bar{\varphi}, \bar{\varphi} \rangle_{L^2(m)}; & \alpha = 0 \end{cases}$$
At the same time
\[ \mathbb{E}_N \left[ M_{\varphi}^{\top} (\infty) \right]^2 = 2 \mathbb{E}_N \int_{[0, \tau]} \left| \frac{\partial u}{\partial a} (B_s, X_s) \right|^2 ds \]
\[ = 2 \int_{[0, \infty)} (N \wedge a) \int_{\mathbb{R}^d} \left| \frac{\partial u}{\partial a} (a, x) \right|^2 dm(x) da \]
\[ = 2 \int_{[0, \infty)} (N \wedge a) \int_{\mathbb{R}^d} \left| \sqrt{\alpha - Lu} (a, x) \right|^2 dm(x) da \]
\[ = 2 \int_{[0, \infty)} (N \wedge a) \int_{\mathbb{R}^d} \sqrt{\alpha} \left| \nabla u (a, x) \right|^2 dm(x) da \]
\[ = 2 \int_{[0, \infty)} (N \wedge a) \int_{\mathbb{R}^d} \alpha \left| \nabla u (a, x) \right|^2 dm(x) da + 2 \mathbb{E}_N \sum_{s=0}^{\tau} |\nabla u (B_s, X_s)|^2 ds \]
\[ = 2 \int_{[0, \infty)} (N \wedge a) \int_{\mathbb{R}^d} \alpha \left| \nabla u (a, x) \right|^2 dm(x) da + 2 \mathbb{E}_N \left[ M_{\psi}^{\top} (\infty) \right]^2. \]

Consequently,
\[ \lim_{N \to \infty} \mathbb{E}_N \left[ M_{\varphi}^{\top} (\infty) \right]^2 = \frac{1}{2} \lim_{N \to \infty} \mathbb{E}_N \left[ M_{\varphi}^{\top} (\infty) \right]^2 - \int_{[0, \infty)} a \int_{\mathbb{R}^d} \alpha \left| \nabla u (a, x) \right|^2 dm(x) da, \]
where
\[ \lim_{N \to \infty} \int_{[0, \infty)} a \int_{\mathbb{R}^d} \alpha \left| \nabla u (a, x) \right|^2 dm(x) da = \int_{[0, \infty)} \alpha \int_{[0, \infty)} \alpha e^{-2a\sqrt{\alpha + \lambda}} d\langle E_{\lambda} \varphi, \varphi \rangle = \alpha \int_{[0, \infty)} \frac{\alpha}{4(\alpha + \lambda)} d\langle E_{\lambda} \varphi, \varphi \rangle = \frac{\alpha}{4} \langle (\alpha - L)^{-1} \varphi, \varphi \rangle. \]

**Lemma 3.6.** For \( \varphi, \psi \) be \( C_0^{\infty}(\mathbb{R}^d, \mathbb{R}^d) \)
\[ \lim_{N \to \infty} \mathbb{E}_N \left[ N_{\varphi} \cdot N_{\psi} (\infty) \right] = \frac{1}{2} \langle \varphi, \psi \rangle_{L^2(m)}. \]

**Proof.** Applying the previous lemma to every individual coordinate of \( N_{\varphi} \) we obtain
\[ \lim_{N \to \infty} \mathbb{E}_N \left[ N_{\varphi} \cdot N_{\psi} (\infty) \right] = \frac{1}{2} \langle \varphi, \psi \rangle_{L^2(m)}, \]
where we have used the assumption 3.4. Now it remains to sum up these equalities for \( i = 1, \ldots, d \). \( \square \)

Below we will be extensively using the classical inequalities for submartingales which are summarized in the propositions 3.7, 3.8 and 3.9. Let \( Z_t \) be a nonnegative continuous submartingale with the Doob–Meyer decomposition

\[
Z_t = Z_0 + M_t + A_t,
\]

where \((M_t)\) is a continuous martingale with \(M_0 = 0\) and \((A_t)\) is a nondecreasing continuous process with \(A_0 = 0\).

**Proposition 3.7.** *(A version of Doob’s inequality)* There exists \( C > 0 \) such that

\[
E \sup_{t > 0} Z_t \leq C \|Z_\infty\|_{L\log L}.
\]

**Proposition 3.8.** *(Langlart–Lepingle–Pratelli inequality)* For every \( p \in (0, \infty) \) there exists \( C_p > 0 \) such that

\[
E A_p^\infty \leq C_p E \sup_{t > 0} Z_t^p.
\]

**Proposition 3.9.** *(Burkholder–Davis–Gundy inequality)* Let \((Y_t)\) be a continuous martingale such that \( Y_0 = 0 \). For every \( 0 < p < \infty \) there exist \( c_p, C_p > 0 \) such that

\[
c_p E \langle Y \rangle_\infty^p \leq E \sup_{t > 0} |Y_t|^p \leq C_p E \langle Y \rangle_\infty^p.
\]

Now we are ready to prove the main theorem of this section.

**Theorem 3.10.** For any \( \alpha \geq 0 \) there exists a positive constant \( C(\alpha) \) such that for any \( f \in C^\infty_0(\mathbb{R}^d, \mathbb{R}) \)

\[
C(\alpha) \|\sqrt{\alpha - L} f\|_1 \leq \|\nabla f\|_{L\log L} + \sqrt{\alpha} \|f\|_1
\]

(3.14)

**Proof.** For \( \varphi = \sqrt{\alpha - L} f \) and \( \psi \in C^\infty_0(\mathbb{R}^d, \mathbb{R}) \) with \( \|\psi\|_\infty \leq 1 \) we have the following chain of inequalities:

\[
\langle \varphi, \psi \rangle_{L^2(m)} = 4 \lim_{N \to \infty} \mathbb{E} N M_\varphi^N(\infty) \cdot M_\psi^N(\infty) + 4\alpha \langle (\alpha - L)^{-1} \varphi, \psi \rangle
\]

\[
\leq C \left[ \lim_{N \to \infty} \|M_\varphi^N\|_{H^1} \|M_\psi^N\|_{BMO} + \alpha \|\sqrt{\alpha - L}^{-1} f\|_1 \right],
\]

where the first equality is provided by Lemma 3.5. Since \( \{T_t\} \) is a contraction semigroup in \( L^1(m) \) and

\[
\sqrt{\alpha - L}^{-1} = \frac{1}{\Gamma(1/2)} \int_{[0,\infty)} t^{-1/2} e^{-\alpha t} dT_t dt
\]
it is readily seen that
\[ \| \alpha \sqrt{\alpha - L}^{-1} f \|_1 \leq \sqrt{\alpha} \| f \|_1. \]
Applying Lemma 3.3 we obtain the bound
\[ \| M_\varphi^\dagger \|_{BMO} \leq \| \psi \|_\infty \leq 1. \]
Hence, to finish the proof it is sufficient to establish the inequality
\[ \lim_{N \to \infty} \| M_\varphi^\dagger \|_{H^1} \leq C \| \nabla f \|_{L \log L}, \varphi = \sqrt{\alpha - L} f. \tag{3.15} \]
One can observe that
\[ \langle M_\varphi^\dagger \rangle_t = \int_{[0, t \land \tau]} \left| \nabla R_{B_s}^\alpha \varphi(X_s) \right|^2 ds = \int_{[0, t \land \tau]} \left| \nabla R_{B_s}^\alpha \sqrt{\alpha - L} f(X_s) \right|^2 ds \]
\[ = \int_{[0, t \land \tau]} \left| \nabla \frac{\partial}{\partial \alpha} R_{B_s}^\alpha f(X_s) \right|^2 ds = \int_{[0, t \land \tau]} \left| \frac{\partial}{\partial \alpha} \nabla R_{B_s}^\alpha f(X_s) \right|^2 ds. \]
Combining this identity with the bound 3.11 yields the inequality
\[ \langle M_\varphi^\dagger \rangle_t \leq A_{V,t}, \ t \geq 0, \tag{3.16} \]
where \((A_{V,t})\) is the increasing process from the Doob–Meyer decomposition (3.10) of the submartingale \((V_t^2)\) constructed for \(g := \nabla f\). Now the required estimate (3.15) follows by the standard inequalities for submartingales:
\[ \| M_\varphi^\dagger \|_{H^1} = \mathbb{E}_N \sup_{t > 0} | M_\varphi^\dagger (t) | \leq C \mathbb{E}_N \langle M_\varphi^\dagger \rangle_{t}^{1/2} \]
\[ \leq C \mathbb{E}_N A_{V,\infty}^{1/2} \leq C \mathbb{E}_N \sup_{t > 0} V_t \leq C \| V_\infty \|_{L \log L(P_N)} \]
\[ \lim_{N \to \infty} \| M_\varphi^\dagger \|_{H^1} \]
\[ \leq C \lim_{N \to \infty} \| V_\infty \|_{L \log L(P_N)} = C \lim_{N \to \infty} \| v(B_{\tau}, X_\tau) \|_{L \log L(P_N)} \]
\[ = C \lim_{N \to \infty} \| v(0, X_\tau) \|_{L \log L(P_N)} = C \lim_{N \to \infty} \| \nabla f(X_\tau) \|_{L \log L(P_N)} \]
\[ = C \| \nabla f \|_{L \log L}, \]
where we have used the fact that by construction the stopping time \(\tau\) is independent from the process \((X_t)\). \(\square\)

**Theorem 3.11.** For any \(\alpha \geq 0\) there exists a positive constant \(C(\alpha)\) such that for any \(f \in C_0^\infty(\mathbb{R}^d, \mathbb{R})\)
\[ \| \nabla f \|_1 \leq C(\alpha) \| \sqrt{\alpha - L} f \|_{L \log L}. \]
Proof. For $\varphi := \nabla f$ and $\psi \in C_0^\infty(\mathbb{R}^d, \mathbb{R}^d)$ with $\|\psi\|_\infty \leq 1$ we have the following chain of inequalities:

$$\langle \varphi, h \rangle_{L^2(m)} = 2 \lim_{N \to \infty} \mathbb{E}_N N_{\varphi}^+(\infty) \cdot N_{\psi}^+(\infty) \leq C \lim_{N \to \infty} \|N_{\varphi}^+\|_{H^1} \|N_{\psi}^-\|_{BMO},$$

where the first equality is provided by Lemma 3.6. Applying Lemma 3.3 we obtain the bound

$$\|N_{\psi}^-\|_{BMO} \leq \|\psi\|_\infty \leq 1.$$

Hence, it remains to estimate $\mathbb{E}_N \|N_{\varphi}^+\|_{H^1}$. Let us notice that

$$\langle N_{\varphi}^+ \rangle_t = \int_{[0,t\wedge \tau]} \left| \frac{\partial \psi}{\partial a}(B_s, X_s) \right|^2 ds = \int_{[0,t\wedge \tau]} \sum_{i=1}^d \left| \frac{\partial}{\partial a} R_{B_s}^{\alpha, \lambda} \nabla_i f \right|^2 ds = \int_{[0,t\wedge \tau]} \sum_{i=1}^d \left| \nabla_i R_{B_s}^\alpha \sqrt{\alpha - Lf} \right|^2 ds = \int_{[0,t\wedge \tau]} \left| \nabla R_{B_s}^\alpha \sqrt{\alpha - Lf} \right|^2 ds = \langle M\uparrow_{\sqrt{\alpha - Lf}} \rangle_t.$$

Then:

$$\|N_{\varphi}^+\|_{H^1} = \mathbb{E}_N \sup_t |N_{\varphi}^+(t)| \leq C \mathbb{E}_N \langle N_{\varphi}^+ \rangle_{\infty}^\frac{1}{2} \mathbb{E}_N \langle M\uparrow_{\sqrt{\alpha - Lf}} \rangle_{\infty}^\frac{1}{2} \leq C \|R_{B_s}^\alpha \sqrt{\alpha - Lf}\|_{L\log L(P_N)} = C \|\sqrt{\alpha - Lf}(X_\tau)\|_{L\log L(P_N)} = C \|\sqrt{\alpha - Lf}\|_{L\log L(m)},$$

where again we have used the fact that the stopping time $\tau$ is independent from the process $(X_t)$.

4. Main results

4.1. Sobolev functions on the Wiener space. In this section we consider a Wiener space $(\mathcal{W}, \mathcal{H}, \mu)$, i.e. $\mathcal{W}$ is a separable Banach space equipped with a nondegenerate centered Gaussian measure $\mu$ and $\mathcal{H}$ is its Cameron–Martin space, see e.g. [3], [18]. Let us recall that a Borel probability measure $\mu$ on $\mathcal{W}$ is called centered Gaussian if every continuous linear functional $l \in \mathcal{W}^*$ is a centered Gaussian random variable on $(\mathcal{W}, \mu)$, i.e.

$$\int_{\mathcal{W}} \exp(il) d\mu = \exp\left(-\frac{1}{2} \int_{\mathcal{W}} l^2 d\mu\right).$$
The Cameron–Martin space $H = \mathcal{H}(\mu)$ of this measure is the set of all vectors $h \in W$ with $|h|_H < \infty$, where
\[ |h|_H = \sup \{ l(h) : l \in W^*, \|l\|_{L^2(\mu)} \leq 1 \}. \]
This is also the set of all vectors the shifts along which give measures equivalent to $\mu$. The nondegeneracy of $\mu$ means that it is not concentrated on any proper linear subspace of $W$. It is known (see e.g. [3]) that in this case $(H, |\cdot|_H)$ is a separable Hilbert space densely embedded into $W$. Let $\mathcal{FC}_b^\infty$ denote the class of all functions on $W$ of the form
\[ f(x) = f_0(l_1(x), \ldots, l_n(x)), \quad f_0 \in C_0^\infty(\mathbb{R}^n), l_i \in W^*. \]
The gradient $\nabla_H f$ of $f \in \mathcal{FC}_b^\infty$ along the subspace $H$ is defined by the equality
\[ (\nabla_H f(x), h)_H = \partial_h f(x). \]
The Sobolev space $W^{1,p}$ (see [3], [4]) is defined as the completion of $\mathcal{FC}_b^\infty$ in the norm
\[ \|f\|_{1,p} := \|\nabla_H f\|_p + \|f\|_p, \]
where $\|\cdot\|_p$ denotes the norm in $L^p(\mu)$. In this context the Ornstein–Uhlenbeck semigroup $\{T_t\}$ is given by Mehler’s formula
\[ T_t f(x) := \int_W f(e^{-t}x + \sqrt{1-e^{-2t}}y) d\mu(y) \]
and its generator is the standard Ornstein–Uhlenbeck operator $L$
\[ L := \Delta - \langle x, \nabla_H \rangle. \]
Similarly to $W^{1,p}$ one can define the space $W^{1,L\log L}$, where the norm from $L^p(\mu)$ is replaced with the Orlicz norm $L \log L$ (3.2):
\[ \|f\|_{1,L\log L} := \|\nabla_H f\|_{L\log L} + \|f\|_{L\log L}. \]
Alternatively, one can describe the class $W^{1,L\log L}$ as a linear subspace of $W^{1,1}$ consisting of the functions for which the norm of the gradient $\|\nabla f\|_{L\log L}$ is finite. Indeed, to see this it is sufficient to show the validity of $L \log L$-Poincaré-type inequality.

**Proposition 4.1.** For any $\varphi \in \mathcal{FC}_b^\infty$ with $\int \varphi d\mu = 0$
\[ \|\varphi\|_{L\log L} \leq C\|\nabla \varphi\|_{L\log L}. \]

**Proof.** By the standard approximation arguments one can see that it would be sufficient to establish this inequality just for smooth cylindrical functions. Consequently, without loss of generality one may assume
that $\mathcal{W} = \mathbb{R}^d$, $\nabla = \nabla_{\mathcal{H}}$. Since $\Phi$ (defined by 3.3) is convex then by Jensen’s inequality for any $t > 0$

$$\|\nabla T_t \varphi\|_{L \log L} = \|e^{-t} T_t \nabla \varphi\|_{L \log L} \leq e^{-t} \|\nabla \varphi\|_{L \log L}.$$ 

Let us prove the inequality

$$\|T_1 \varphi - \varphi\|_{L \log L} \leq C \|\nabla \varphi\|_{L \log L}.$$ 

Without loss of generality we may assume that $\|\nabla \varphi\|_{L \log L} = 1$, i.e.

$$\int_X \Phi(\|\nabla \varphi(x)\|) \, d\mu = 1.$$ 

In this case it is sufficient to prove that for $\psi := T_1 \varphi - \varphi$

$$\int \Phi(\|\psi(x)\|) \, d\mu(x) \leq C,$$

where $C$ is some positive constant. For any $x, y$

$$\varphi(e^{-1} x - \sqrt{1 - e^{-2} y}) - \varphi(x)$$

$$= \int_{[0,1]} c_t \langle \nabla \varphi(e^{-t} x + \sqrt{1 - e^{-2t}} y), -\sqrt{1 - e^{-2t}} x + e^{-t} y \rangle \, dt,$$

where

$$c_t := \frac{e^{-t}}{\sqrt{1 - e^{-2t}}}.$$ 

It is easy to see that for any $a, b \geq 0$

$$\Phi(ab) = \int_{[0,ab]} \log(1 + t) \, dt = \int_{[0,b]} \log(1 + at) \, adt$$

$$\leq a \Phi(b) + a \log(1 + a)b.$$
Then:

\[
\int_X \Phi(|\psi(x)|) \, d\mu(x)
\]

\[
= \int_X \Phi\left(\left|\int_X \varphi(x) - \varphi\left(e^{-1}x + \sqrt{1-e^{-2y}}\right) \, d\mu(y)\right|\right) \, d\mu(x)
\]

\[
= \int_X \Phi\left(\int_X \int_{[0,1]} c_t |\nabla \varphi| \langle \nabla \varphi / |\nabla \varphi| \rangle \left(e^{-t}x + \sqrt{1-e^{-2ty}}\right),
- \sqrt{1-e^{-2tx}+e^{-ty}} \right) \, d\mu(y) \, d\mu(x)
\]

\[
\leq \int_X \int_{[0,1]} \int_X \Phi\left(c_t |\nabla \varphi| \langle \nabla \varphi / |\nabla \varphi| \rangle \left(e^{-t}x + \sqrt{1-e^{-2ty}}\right),
- \sqrt{1-e^{-2tx}+e^{-ty}} \right) \, d\mu(y) \, d\mu(x)
\]

\[
\leq \int_{[0,1]} \int_X \int_X c_t |\nabla \varphi| \Phi\left(\langle \nabla \varphi / |\nabla \varphi| \rangle \left(e^{-t}x + \sqrt{1-e^{-2ty}}\right),
- \sqrt{1-e^{-2tx}+e^{-ty}} \right) \, d\mu(y) \, d\mu(x) \, dt
\]

\[
+ \int_{[0,1]} \int_X \int_X c_t |\nabla \varphi| \log(1 + c_t |\nabla \varphi|) \langle \nabla \varphi / |\nabla \varphi| \rangle \left(e^{-t}x + \sqrt{1-e^{-2ty}}\right),
- \sqrt{1-e^{-2tx}+e^{-ty}} \right) \, d\mu(y) \, d\mu(x) \, dt.
\]

Taking into account that the rotation

\[
(x, y) \mapsto (e^{-t}x + \sqrt{1-e^{-2ty}}, -\sqrt{1-e^{-2tx}+e^{-ty}})
\]

preserves the Gaussian measure \(\mu \otimes \mu\) on \(X \times X\) we can continue the previous chain of inequalities with

\[
\int_{[0,1]} \int_X \int_X c_t |\nabla \varphi| \Phi\left(\langle \nabla \varphi / |\nabla \varphi| \rangle \left(x, y\right)\right) \, d\mu(y) \, d\mu(x) \, dt
\]

\[
+ \int_{[0,1]} \int_X \int_X c_t |\nabla \varphi| \log(1 + c_t |\nabla \varphi|) \langle \nabla \varphi / |\nabla \varphi| \rangle \left(x, y\right) \, d\mu(y) \, d\mu(x) \, dt
\]

\[
\leq C_1 \int_{[0,1]} \int_X c_t |\nabla \varphi| \, d\mu(x) \, dt
\]

\[
+ C_2 \int_{[0,1]} \int_X c_t \log(1 + c_t |\nabla \varphi|) \log(1 + |\nabla \varphi|) \, d\mu(x) \, dt
\]

\[
\leq C \int_X \Phi(|\nabla \varphi(x)|) \, d\mu(x) \leq C'.
\]
Then the required estimate follows by the triangle inequality:

$$\|\varphi\|_{L^\log L} \leq \sum_{n \geq 0} \|T_{n+1}\varphi - T_n\varphi\|_{L^\log L} \leq \sum_{n \geq 0} e^{-nC}\|\nabla \varphi\|_{L^\log L} \leq C\|\nabla \varphi\|_{L^\log L}.$$

□

Now let us recall the log-convexity property of the semigroup \(\{T_t\}\) which will play an important role below.

**Lemma 4.2.** For every nonnegative Borel function \(g \in L^1\) and every \(t > 0\) the map \(\log T_t g\), where \(T_t g\) is defined by Mehler’s formula, is \(\frac{1}{t}\)-convex with respect to the Cameron–Martin distance, i.e.

$$T_t g((1-s)x_0 + sx_1) \leq \exp\left\{\frac{s(1-s)}{2t} |x_1 - x_0|^2_H\right\}(T_t g(x_0))^{1-s}(T_t g(x_1))^s.$$

for every \(x_0, x_1 \in \mathcal{W}\) with \(x_0 - x_1 \in \mathcal{H}\) and \(s \in [0, 1]\).

**Proof.** See e.g. Lemma 3.4 in [2] or Lemma 5.14 in [5]. □

It is worth noting that in Lemma 4.2 it is important that we consider the version of \(T_t g\) given by Mehler’s formula in the pointwise sense. This is possible since \(g\) is nonnegative and Borel.

Let us define the “smoothing” operators \(\{A_t\}\) by the formula

$$A_t := \frac{1}{t} \int_{[t,2t]} T_s \, ds.$$

**Lemma 4.3.** Let \(g\) be a nonnegative Borel function in \(L^1\) and \(t > 0\). Then for the function

$$A_t g = \frac{1}{t} \int_{[t,2t]} T_s g \, ds,$$

where \(T_s g\) is defined by Mehler’s formula, and every \(x_0, x_1 \in \mathcal{W}\) with \(x_0 - x_1 \in \mathcal{H}\), \(s \in [0, 1]\) the following inequality holds:

$$A_t g((1-s)x_0 + sx_1) \leq \exp\left\{\frac{s(1-s)}{2t} |x_1 - x_0|^2_H\right\}(A_t g(x_0))^{1-s}(A_t g(x_1))^s.$$
Proof. Indeed, the required bound follows from Lemma 4.2 and the standard Hölder’s inequality:

\[ A_t g((1 - s)x_0 + sx_1) = \frac{1}{t} \int_{[t, 2t]} T_u g((1 - s)x_0 + sx_1) \, du \]

\[ \leq \frac{1}{t} \int_{[t, 2t]} \exp \left\{ \frac{s(1 - s)}{2t} |x_1 - x_0|_H^2 \right\} (T_u g(x_0))^{1-s}(T_u g(x_1))^s \, du \]

\[ \leq \exp \left\{ \frac{s(1 - s)}{2t} |x_1 - x_0|_H^2 \right\} \left[ \frac{1}{t} \int_{[t, 2t]} (T_u g(x_0))^{1-s}(T_u g(x_1))^s \, du \right] \]

\[ \leq \exp \left\{ \frac{s(1 - s)}{2t} |x_1 - x_0|_H^2 \right\} \left[ \frac{1}{t} \int_{[t, 2t]} T_u g(x_0) \, du \right]^{1-s} \left[ \frac{1}{t} \int_{[t, 2t]} T_u g(x_1) \, du \right]^s \]

\[ \square \]

Lemma 4.4. Let \( f \in W^{1,1} \). Then there exists a Borel set \( \Omega_f \) with \( \mu(\Omega_f) = 1 \) such that for any \( t > 0, x_0, x_1 \in \Omega_f \) with \( x_0 - x_1 \in \mathcal{H} \):

\[ |A_t f(x_1) - A_t f(x_0)| \leq |x_1 - x_0|_H^2 \left( A_t |\nabla f|((1 - s)x_0 + sx_1) \right) \]

\[ \leq |x_1 - x_0|_H^2 \left( A_t |\nabla f|((1 - s)x_0 + sx_1) \right) \]

\[ \leq |x_1 - x_0|_H^2 \left( A_t |\nabla f|((1 - s)x_0 + sx_1) \right) \]

Proof. Let \( h := x_1 - x_0, h \in \mathcal{H} \). We first assume that \( t > 0 \) is fixed and \( f \) is a smooth cylindrical function. Then

\[ |A_t f(x_1) - A_t f(x_0)| = \left| \int_{[0,1]} \langle \nabla \mathcal{H} A_t f((1 - s)x_0 + sx_1), h \rangle \, ds \right| \]

\[ \leq |h| \int_{[0,1]} |\nabla \mathcal{H} A_t f((1 - s)x_0 + sx_1)| \, ds, \]
Lemma 4.4 is not claimed to be of the form 

Now for a given functions \((f, \Omega)\) there exists a set \(\Omega\) to a subsequence we may assume that \(\Omega\) converges to \(\Omega\) of full measure. 

Therefore, 

\[
\left| A_t f(x) - A_t f(x_0) \right| \leq |x_1 - x_0| \|A_t f_n(x_1) - A_t f_n(x_0)\|_H \\
\leq \lim_{n \to \infty} |x_1 - x_0| \|e^{\frac{|x_1 - x_0|^2}{4t}}(A_t \|\nabla f_n\|(x_1) + A_t \|\nabla f_n\|(x_0))\|_H \\
= |x_1 - x_0| \|e^{\frac{|x_1 - x_0|^2}{4t}}(A_t \|\nabla f\|(x_1) + A_t \|\nabla f\|(x_0))\|_H.
\]

Now similarly to the proof of Theorem 2.2 we can notice that there exists a set \(\Omega_f\) of full measure such that for every \(x \in \Omega_f\) the mappings 

\[
t \mapsto A_t(x), \quad t \mapsto A_t \|\nabla f\|(x)
\]

are continuous on \((0, \infty)\). It easy to see that for any \(x_0, x_1 \in \Omega_f\) with \(x_1 - x_0 \in \mathcal{H}\) and any \(t > 0\) 

\[
\left| A_t f(x_1) - A_t f(x_0) \right| \leq |x_1 - x_0| \|e^{\frac{|x_1 - x_0|^2}{4t}}(A_t \|\nabla f\|(x_1) + A_t \|\nabla f\|(x_0))\|_H,
\]

where 

\[
\Omega_f := \Omega_f' \cap \bigcap_{t_i \in \mathbb{Q} \cap (0, \infty)} \Omega_{f,t_i}.
\]

**Remark 4.5.** Note that unlike Proposition 3.3 from [2] the set \(\Omega_{f,t}\) in Lemma 4.4 is not claimed to be of the form \(\Omega'_{f,t} + H\) for some \(\Omega'_{f,t}\) of full
measure. The reason is that for functions which are merely integrable
Theorem 2.5 from [2] is not directly applicable anymore.

**Lemma 4.6.** Let \( f \in W^{1,L \log L} \), i.e.

\[
\int_{\mathbb{R}^d} |\nabla_H f(x)| \log(1 + |\nabla_H f(x)|) \, d\mu(x) < \infty.
\]

Then \( f \in D_1(\sqrt{-L}) \) and

\[
\|\sqrt{-L} f\|_1 \leq C \|\nabla_H f\|_{L \log L},
\]

where \( C \) is some positive constant which does not depend on \( f \).

**Proof.** For a function \( f \in FC_\infty \) this statement easily follows by Theo-
rem 1.1 from [15], this is also a particular case of Theorem 3.10 from
Section 3. The general case follows from the standard approximation
arguments since smooth cylindrical functions are dense in \( W^{1,L \log L} \).

**Lemma 4.7.** Let \( f \in W^{1,L \log L} \). There exist a universal constant \( C > 0 \)
and a set \( \Omega_f \) with \( \mu(\Omega_f) = 1 \) such that for any \( t > 0 \)

\[
|A_t f(x) - f(x)| \leq C \sqrt{t} \sup_{s > 0} A_s |\nabla_H f|(x), \quad x \in \Omega_f.
\]

**Proof.** This follows immediately by Theorem 2.2 and Lemma 4.6.

The next theorem is our main result for Sobolev functions on the
Wiener space.

**Theorem 4.8.** Let \( f \in W^{1,L \log L} \). There exist a set \( \Omega_f \) with \( \mu(\Omega_f) = 1 \)
and a universal constant \( C > 0 \) such that for any \( x_0, x_1 \in \Omega_f \) with
\( x_1 - x_0 \in \mathcal{H} \)

\[
|f(x_1) - f(x_0)| \leq C|y|_\mathcal{H}(M(x_0) + M(x_1)),
\]

where

\[
M(x) := \sup_{t > 0} \frac{1}{t} \int_{[t,2t]} T_s |\sqrt{-L} f|(x) \, ds + \sup_{t > 0} \frac{1}{t} \int_{[t,2t]} T_s |\nabla_H f|(x).
\]

**Proof.** Let \( \Omega_f \) be the intersection of the sets of full measure provided
by Lemma 4.4 and Lemma 4.7. For any \( x_0, x_1 \in \Omega_f \) and any \( t > 0 \)

\[
|f(x_1) - f(x_0)|
\leq |f(x_1) - A_t f(x_1)| + |A_t f(x_1) - A_t f(x_0)| + |f(x_0) - A_t f(x_0)|
\leq C \sqrt{t} \sup_{s > 0} A_s f(x_1) + C \sqrt{t} \sup_{s > 0} A_s f(x_0)
\]

\[
+ |x_1 - x_0|_\mathcal{H} e^{\frac{|x_1 - x_0|^2}{4t}} (A_t |\nabla_H f|(x_1) + A_t |\nabla_H f|(x_0))
\]

and it is easy to see that picking \( t := |x_1 - x_0|^2 \) yields the required estimate. \( \square \)
Theorem 4.9. Let $f \in W^{1, \log L}$. Then for every $\varepsilon > 0$ there exists an $\mathcal{H}$-Lipschitz $\mu$-measurable function $g_\varepsilon$, i.e.

$$|g_\varepsilon(x_1) - g_\varepsilon(x_0)| \leq C_\varepsilon |x_1 - x_0|_H, \quad x_0, x_1 \in \mathcal{W}, \ x_1 - x_0 \in \mathcal{H}$$

such that

$$\mu(x : g_\varepsilon(x) \neq f(x)) \leq \varepsilon.$$

Proof. Applying the Hopf–Dunford–Schwartz maximal inequality (see Proposition 2.1) to the semigroup $\{T_t\}$ and the integrable functions $|\sqrt{-L}f|$ and $|\nabla_H f|$ yields that for every $\lambda > 0$

$$\mu(x : CM(x) \geq \lambda) \leq C'' \frac{\|\sqrt{-L}f\|_1 + \|\nabla_H f\|_1}{\lambda} \leq C'' \frac{\|\nabla_H f\|_{\log L}}{\lambda},$$

where $C$ is the constant from Theorem 4.8. Let us choose

$$\lambda := \frac{1}{\varepsilon C'' \|\nabla_H f\|_{\log L}}$$

and set

$$\Omega_{f,\varepsilon} := \{x : CM(x) \leq \lambda\}.$$ 

Then:

$$\mu(W \setminus \Omega_{f,\varepsilon}) \leq \varepsilon$$

and for any $x_0, x_1 \in \Omega_{f,\varepsilon}$

$$|f(x_0) - f(x_1)| \leq \lambda |x_0 - x_1|_H.$$ 

Now we can apply the result from [19] to the function $f|_{\Omega_{f,\varepsilon}}$ and obtain a measurable $\mathcal{H}$-Lipschitz function $g_\varepsilon$ defined on the whole space $\mathcal{W}$ such that

$$g|_{\Omega_{f,\varepsilon}} = f|_{\Omega_{f,\varepsilon}}.$$ 

It is clear that by construction

$$\mu(x : g_\varepsilon(x) \neq f(x)) \leq \varepsilon.$$

\[\Box\]

Remark 4.10. As it is clear from the proofs, the statements of Theorem 4.8 and Theorem 4.9 remain valid for any function

$$f \in W^{1,1} \cap D_1(\sqrt{-L}).$$

However, the case of $f \in W^{1,1}$ or $f \in BV$ when the underlying space $W$ is infinite-dimensional is still open, see also the discussion of this problem in [2].
Remark 4.11. In the paper [1] G. Alberti proved that any Borel vector field on \( \mathbb{R}^d \) coincides with the gradient of some \( C^1 \) function outside of a set of arbitrarily small Lebesgue measure. This result was extended to the Wiener space setting in [17], where the following theorem was obtained:

**Theorem 4.12.** Let \( v : W \to \mathcal{H} \) be a Borel vector field with values in the Cameron–Martin space \( \mathcal{H} \). The for any \( \varepsilon > 0 \) and \( \theta > 0 \) there exists an \( \mathcal{H} \)-Lipschitz function \( f \) such that

\[
\mu(x: \ v(x) \neq f(x)) \leq \varepsilon, \\
\|f\|_\infty \leq \theta, \\
\|\nabla_H f\|_p \leq C \varepsilon^{1/p-1} \|v\|_p,
\]

where \( C \) is some universal constant.

4.2. Da Prato’s Sobolev spaces. We refer to the book [7] (see also [3], [4]) for a detailed introduction into this topic. In this setting the underlying space \( W = \mathcal{H} \) is a separable Hilbert space, \( m \) is a centered nondegenerate Gaussian measure. We denote by \( Q \) the covariance operator associated with \( m \). It is well-known (see [7], [3]) that in this case \( Q \) is a nonnegative symmetric operator with finite trace. The Cameron–Martin space of \( m \) will be denoted by \( \mathcal{H} \). In fact, \( \mathcal{H} \) coincides with the range of \( Q^{1/2} \) and moreover \( \|x\|_\mathcal{H} = |Q^{-1/2}x| \). Using the Hilbertian structure of the underlying space \( \mathcal{H} \) we can introduce the Sobolev spaces \( W^{1,p}(H,m) \) obtained as the closure of smooth cylindrical functions with respect to the norm

\[
\|f\|_{1,p} := \|\nabla f\|_p + \|f\|_p.
\]

The difference with the Sobolev classes on the Wiener space which were considered in the previous subsection is that here the gradient with respect to the Hilbertian structure of the underlying space \( \mathcal{H} \) is involved rather than with respect to the structure of the Cameron–Martin space \( \mathcal{H} \). In this context the natural semigroup is given by a Mehler-type formula

\[
P_t f(x) := \int_{\mathcal{H}} f(e^{At}x + \sqrt{1-e^{2At}}y) \, dm(y) \\
= \int_{\mathcal{H}} f(e^{At}x + y) \, dN_{Q_t}(y) = \int_{\mathcal{H}} f(y) \, dN_{e^{At}x, Q_t}(y),
\]

where we have set

\[
A := -\frac{1}{2}Q^{-1}, \quad Q_t := \int_{[0,t]} e^{2As} \, ds = Q(1-e^{2At})
\]
and \( N_{e^{At}x,Q_t} \) denotes the unique Gaussian measure with mean \( e^{At}x \) and covariance \( Q_t \). In this section we denote by \( L \) the generator of the semigroup \( \{ P_t \} \):
\[
L := \frac{1}{2} \Delta + \langle Ax, \nabla \rangle.
\]
We will also assume that for the operator \( A \) the following bound holds:
\[
A \leq -\beta,
\]
where \( \beta \) is a positive constant. It is easy to see that in the infinite-dimensional setting we still have the commutation identity
\[
\nabla P_tf = e^{-At}P_t\nabla f.
\]
Consequently, in this case
\[
|\nabla P_tf| \leq e^{-\beta t}P_t|\nabla f|.
\]
Similarly to the case of the abstract Wiener space we can introduce the Sobolev class \( W^{1,L\log L}(H,m) \) with the norm
\[
\|f\|_{1,L\log L} = \|\nabla f\|_{L\log L} + \|f\|_{L\log L}.
\]
Now we can make use of the extension of Shigekawa’s bound established in Section 3 and obtain the natural counterpart of Lemma 4.6 for Da Prato’s spaces.

**Lemma 4.13.** Let \( f \in W^{1,L\log L}(H,m) \), i.e.
\[
\int_{W} |\nabla f(x)| \log(1 + |\nabla f(x)|) \, d\mu(x) < \infty.
\]
Then \( f \in D_1(\sqrt{-L}) \) and
\[
\|\sqrt{-L}f\|_1 \leq C\|\nabla f\|_{L\log L}.
\]

**Proof.** For a function \( f \in \mathcal{F}\mathcal{C}_b^\infty \) this statement follows by Theorem 3.10 from Section 3. The general Sobolev case is again handled by the standard approximation arguments using the density of smooth cylindrical functions in \( W^{1,L\log L}(H,m) \). \qed

The rest of our intermediate steps work the same as in the Wiener space setting. Therefore, let us conclude this section with the formulation of the final results.

**Theorem 4.14.** Let \( f \in W^{1,L\log L}(H,m) \). There exist a set \( \Omega_f \) with \( m(\Omega_f) = 1 \) and a universal positive constant \( C \) such that for any \( x_0, x_1 \in \Omega_f \)
\[
|f(x_1) - f(x_0)| \leq C|x - y|(M(x_0) + M(x_1)),
\]
where
\[ M(x) := \sup_{t > 0} \frac{1}{t} \int_{[t,2t]} P_s|\sqrt{-L}f|(x) \, ds + \sup_{t > 0} \frac{1}{t} \int_{[t,2t]} P_s|\nabla f|(x). \]

As a by-product we obtain a Lusin-type approximation for functions from Da-Prato’s Sobolev class analogous to Theorem 4.9. The key difference with the case of the Wiener space is that here the Lipschitz functions with respect to the norm of the underlying Hilbert space are involved rather than the functions which are Lipschitz-continuous along the Cameron–Martin space $\mathcal{H}$.

**Theorem 4.15.** Let $f \in W^{1,1,\log^L}(H, m)$. Then for every $\varepsilon > 0$ there exists a Lipschitz $m$-measurable function $g_{\varepsilon}$ such that
\[ \mu(\{x : g_{\varepsilon}(x) \neq f(x)\}) \leq \varepsilon. \]

**Remark 4.16.** The statements of Theorem 4.14 and Theorem 4.15 remain valid for any function $f \in W^{1,1}(H, m) \cap D_1(\sqrt{-L})$.

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