Weights for Objects of Monoids and Actions of Monoids

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Abstract

The main objective of the paper is to define the object of monoids over a monoidal category object, in any 2-category with finite products, as a weighted limit. We also define the object of actions of monoids along the action of a monoidal category object as a weighted limit.

1 Introduction

Overview

Weighted limits and colimits provide a uniform way to define many interesting operations on 2-categories. It has been known for more than 40 years [Law] that the Eilenberg-Moore object for a monad \( T \) in a 2-category \( \mathcal{K} \) is a weighted limit on a diagram defined by the monad \( T \) on the suspension of the simplicial category \( \Delta \) with a weight defined by the ordinal sum (cf. [L], [Z] for more accessible treatments).

This is an example of a ‘2-algebraic set’ (EM-object) over a 2-dimensional algebraic structure (monad) that can be defined in any sufficiently complete 2-category. One can think that there should be a similar definition of a ‘2-algebraic set of monoids’ over any monoidal category object, another 2-dimensional algebraic structure that can be defined in any sufficiently complete 2-category with finite product. That was a question Bob Pare asked in a personal communication with the second author many years ago, with a further comment ‘After all, a monoid is a bunch of objects and morphisms satisfying some identities’. The main purpose of this paper is to provide a positive answer to this question.

As a byproduct, we also enter the debate how much of a metatheory is needed to develop the theory inside. This taken into the categorical context is referred to as a microcosm principle (cf. [BD], [DS]) saying (at least in the strongest form) that one can generalize an algebraic structure inside its categorified version. It is clear from the considerations below that this is not always the case. For example, to define an object of monoids we need to categorify the notion of a bi-monoid rather than just a monoid, as in the definition of a monoid we need to consider ‘two copies’ of a single universe \( M \) to be able to consider \( M \otimes M \) and the object \( I \), i.e., we need ‘diagonals’ and ‘projections’ which are taken for granted in case the tensor is the usual product but which requires the comonoid structure in general.

There is yet another point that we want to emphasize. Lax monoidal functors between monoidal categories induce morphisms between categories of monoids. In the setting internal to a 2-category \( \mathcal{K} \) this is also true (see Section 5) when we keep our monoidal category objects defined with respect to the product in \( \mathcal{K} \). This is due to the fact that a composition of strict cones with the lax monoidal functors gives rise to the so-called strict-lax cones (that commutes strictly with the co-algebraic morphisms and in a lax way only with the algebraic ones) that form a category having the subcategory of strict cones as a co-reflective subcategory. Then the induced morphism of monoid objects is defined by the strict cone that is the co-reflection of the original strict-lax cone obtained by composition. None of this seems to be true when we move from the products to other monoidal structures in the ambient 2-category \( \mathcal{K} \) and then we are bound to consider the strict monoidal functors only.

Algebra needs coalgebra

Affine algebraic sets over (set-based) algebraic structures (like rings, fields, groups, module etc.) can be defined as limits on the diagrams that involve finite power of the universe and some definable (polynomial)
functions between them. Typically, these limits come from finite sets of equations

\[ f(x_1, \ldots, x_n) = g(x_1, \ldots, x_n) \]

but we do not make any restrictions on the variables that occur on both sides of the equations, so that we can consider equations like

\[ f(x, x, y) = g(x, y, y, y) \]

that use the same variable more than once, not necessarily the same number of times on each side (thus using diagonals), and we can also have equations

\[ m(x, x) = e \]

that have different variables occurring on different sides of the equation (thus using projections).

The limits giving rise to such algebraic sets can be chosen canonically, if we allow weights in their definitions. We shall ‘prove it’ by an example. Let \( A \) be a commutative ring in a complete category \( \mathcal{E} \). Then the equation \( x^2 = y^3 \) defines a subobject \( Z \) of the square of the universe of \( A \) (also denoted by \( A \)). In the internal language it can be expressed as

\[ Z = \{(a, b) \in A^2 | a^2 = b^3\} \]

Let \( B = \mathcal{F}[x, y]/x^2 - y^3 \) be the free commutative ring (in \( \text{Set} \)) on two generators \( x \) and \( y \) divided by the equation \( x^2 = y^3 \), and \( \mathbb{L}_{\text{cring}} \) be the Lawvere theory for commutative rings. We have finite products preserving functors

\[ \bar{A} : \mathbb{L}_{\text{cring}} \to \mathcal{E}, \quad \bar{B} : \mathbb{L}_{\text{cring}} \to \text{Set} \]

corresponding to the rings \( A \) and \( B \). Then, the set \( Z \) is the weighted limit \( \operatorname{Lim}_{\bar{B}} \bar{A} \). We show that it is the case if \( \mathcal{E} \) is the category of sets \( \text{Set} \). We have a sequence of isomorphisms

\[ Z = \{(a, b) \in A^2 | a^2 = b^3\} \cong \]

\[ \operatorname{Hom}(B, A) \cong \]

\[ \operatorname{Nat}(\bar{B}, \bar{A}) \cong \]

\[ \operatorname{Nat}(\bar{B}, \text{Set}(1, \bar{A}(-))) \cong \]

\[ \text{Set}(1, \operatorname{Lim}_{\bar{B}} \bar{A}) \cong \]

\[ \operatorname{Lim}_{\bar{B}} \bar{A} \]

where \( \operatorname{Hom} \) is the hom-set in the category of commutative rings.

Note that we also have a PROP\(^1\) for commutative rings \( \mathbb{P}_{\text{cring}} \) and hence symmetric monoidal functors

\[ \bar{A} : \mathbb{P}_{\text{cring}} \to \mathcal{E}, \quad \bar{B} : \mathbb{P}_{\text{cring}} \to \text{Set} \]

corresponding to rings \( A \) and \( B \). The monoidal structure considered on both \( \mathcal{E} \) and \( \text{Set} \) is the finite product structure. However, it is not the case that \( \operatorname{Lim}_{\bar{B}} \bar{A} \) is isomorphic to the object \( Z \) (even if \( \mathcal{E} \) is \( \text{Set} \)), as natural transformations from \( \bar{B} \) to \( \bar{A} \) do not correspond to homomorphisms from \( B \) to \( A \) in this case. The reason for this is that \( \mathbb{P}_{\text{cring}} \) does not have projections and diagonals, a piece of coalgebra which was vital in the former argument.

In other words, to define the usual algebraic sets we use the coalgebra structure on this set with respect to tensor being the usual cartesian product. This comonoid structure is usually not mentioned for good reasons: it is unique, if our tensor is the binary product. However, if we replace the product by some other tensor, we need to specify the comonoid structure separately, if we want to use it. In this sense to do algebra we need to use a bit of coalgebra. This must be taken into account when we define 2-algebraic structures.

\(^1\)PROP is a strict symmetric monoidal category whose objects are natural numbers such that \( I = 0 \) and \( n \otimes m = n + m \).
Generalizing from 2-category \textbf{Cat} to arbitrary 2-categories

One question to ask is how one knows that an algebraic concept from \textbf{Cat} was successfully generalized to arbitrary 2-categories. The simplest way is to check that a representable functor \( K(A, -) : K \to \textbf{Cat} \) sends the generalized concept from an arbitrary 2-category back to the original one in \textbf{Cat}. For example, a 0-cell \( T \), together with two 2-cells \( \eta : 1 \Rightarrow T \) and \( \mu : T^2 \Rightarrow T \), is a monad in a 2-category \( K \) iff it is sent by any representable functor to a monad in \textbf{Cat} or, equivalently, iff its image under the 2-dimensional Yoneda embedding into \( \textbf{CAT}(K, \textbf{Cat}) \) is a monad (the algebraic structure on \( \textbf{CAT}(K, \textbf{Cat}) \) is inherited from \textbf{Cat}). Similarly, we could define Eilenberg-Moore objects for any monad in any 2-category. However, there are better means to do it without going out of the 2-category in question. In that case, we can say that a monad in a 2-category is a 2-functor \( T : s\Delta \to K \) from the 2-category being the suspension of \( \Delta \) to \( K \) and an Eilenberg-Moore object is a weighted limit with the weight defined by the ordinal sum; see \cite{Law}. To make sure that the notion was correctly internalized, one can check that the latter internal definition agrees with the former external one in case \( K \) is locally small. This can be easily verified in case of the ‘internalized’ notion of a monad.

2-algebra needs 2-coalgebra

As we already learned from the previous discussion, it is not necessarily true that if we can identify internally an algebraic concept (ring), then we will be able to derive all the ‘algebraic sets’ related to it (set of solutions of equations). In fact, now we need to talk about ‘2-algebraic sets’ as the derived concepts will be categories or even 0-cells in a 2-category. The category \( s\Delta \) is not even a 2–\textit{PROP}, (i.e., a strict symmetric monoidal 2-category whose 0-cells are natural numbers such that \( I = 0 \) and \( n \otimes m = n + m \)) but the Eilenberg-Moore object has sufficiently simple structure that we are able to get it as a weighted limit from the functor with domain \( s\Delta \). Thus in this case no coalgebra is needed.

If we want to internalize the notion of a monoidal category, we have to have, in our ambient 2-category \( K \), finite products of 0-cells or at least a 2-monoidal structure. Then we can easily define a 2-category \( \text{PM} \) which is a 2–\textit{PROP} for monoidal categories, i.e., with the property that if \( K \) is a 2-category with finite products, then the 2-monoidal functors from \( \text{PM} \) to \( K \) correspond to monoidal categories in \( K \). However, if we want to derive a ‘2-algebraic set’ of monoids from a monoidal category, the weighted limit of a monoidal 2-functor from \( \text{PM} \) is not enough. This is because when we look at the structure maps of monoids

\[
\mu : m \otimes m \to m \quad e : I \to m
\]

as ‘some kind of equalities’, they are not linear-regular (cf. \cite{SZ1}) as in the left ‘equation’ a variable is repeated and on the right a variable is dropped. Thus this uses full force of the equational logic, not just the linear-regular part. Therefore to define internally the object of monoids either we need the internal version of the notion of a bi-monoidal category (by this we mean the categorification of the notion of a bi-monoid) or we need to define the internal notion of a monoidal category on the basis of finite product, i.e., not using 2–\textit{PROP}’s but Lawvere 2-theories.

In this paper, we shall follow the latter approach and we will work in 2-categories with finite products. We shall describe the weights on Lawvere 2-theories for monoidal categories and symmetric monoidal categories that define many ‘2-algebraic sets’ of interest in any sufficiently complete 2-category.

Organization of the paper

The paper is organized as follows. In Section 2 we give a precise (external) definition of an object of monoids and we show that if a 2-functor \( W : \text{M} \to \textbf{Cat} \) defines the object of monoids in \textbf{Cat}, as \( W \)-weighted limits, and if a 2-category \( K \) with finite products has suitable limits, then \( W \) defines the object of monoids in \( K \) as well. In Section 3 we present the weight functor for object monoids and in Section 4 we present the weight functor for object of actions of monoids. The paper ends with Appendix containing definitions of monoidal category object, action of a monoidal category object, the category of monoids and the category of actions of monoids with the former two phased in the language of an arbitrary 2-category with finite products.
Notation

$\omega$ denotes the set of natural numbers, $n = \{0, \ldots, n - 1\}$ for $n \in \omega$.

In this paper weighted limits in 2-categories are always meant to be pseudo-limits (i.e., unique up to an iso) and we call them simply (weighted) limits. $\textbf{Cat}$ is a 2-category of small 2-categories, functors, and natural transformations. $2\textbf{CAT}$ is the 3-category of 2-categories, i.e., with 2-categories as 0-cells, 2-functors as 1-cells, 2-natural transformations as 2-cells, and 2-modifications as 3-cells. Thus $\textbf{Cat}$ is a 0-cell of $2\textbf{CAT}$.

By a 2-category with finite products we will always mean a 2-category with finite products of 0-cells. Let $2\textbf{CAT}_x$ be the sub-3-category of $2\textbf{CAT}$ full on 2-transformations and 2-modifications, whose 0-cells are 2-categories with finite products, and 1-cells are 2-functors preserving finite products.

2 Basic notions

Some 3-categories, 3-functors and 3-transformations

We have 3-functor $\textbf{Mon}_{st}$ associating to a 2-category $K$ with finite products the 2-category of monoidal category objects in $K$, with strict monoidal functors and monoidal natural transformations. There is an obvious 3-transformation from $\textbf{Mon}_{st}$ to the identity functor denoted $| - |$

$$
\begin{array}{ccc}
2\text{CAT}_x & \xrightarrow{\textbf{Mon}_{st}} & 2\text{CAT}_x \\
| - | & \downarrow & | - | \\
\text{Id} & \xrightarrow{\lambda} & 2\text{CAT}_x
\end{array}
$$

See Appendix for the details

Elements of 2-category theory

Let $K$ be a small 2-category. We have a representable 3-functor

$$
2\text{CAT}_x(\text{K}^{op}, -) : 2\text{CAT}_x \rightarrow 2\text{CAT}_x
$$

The 2-category $2\text{CAT}(\text{K}^{op}, \textbf{Cat})$ inherits the algebraic structure from $\textbf{Cat}$. We have the 2-dimensional Yoneda embedding

$$
Y_K : K \rightarrow 2\text{CAT}(\text{K}^{op}, \textbf{Cat})
$$

This 2-functor induces isomorphisms on hom-categories and preserves finite (weighted) limits. So it reflects algebraic structures from $2\text{CAT}(\text{K}^{op}, \textbf{Cat})$ to $K$, if $K$ has suitable limits. Let $M$ be the 2-Lawvere theory for monoidal categories (we will describe it in detail later). The 2-category $M$ represents the functor $\textbf{Mon}_{st}$. Thus we have a natural 3-isomorphism

$$
\zeta : \textbf{Mon}_{st} \rightarrow 2\text{CAT}_x(M, -)
$$

It can be easily checked that we have the following two squares of 3-categories and 3-functors commuting up to a natural 3-isomorphism:

$$
\begin{array}{ccc}
2\text{CAT}_x & \xrightarrow{\textbf{Mon}_{st}} & 2\text{CAT}_x \\
2\text{CAT}(\text{K}^{op}, -) & \xrightarrow{\lambda} & 2\text{CAT}(\text{K}^{op}, -) \\
\end{array}
$$

and

Note that how ‘small’ are our categories is up to us. The (weighted) limits that we are considering in the paper are always countable.
As finite products in \(2\text{CAT}(K^{op}, \text{Cat})\) are computed pointwise, we have a commuting square of natural 3-isomorphisms:

\[
\begin{array}{ccc}
2\text{CAT}_x & \rightarrow & 2\text{CAT}_x(K^{op}, -) \\
\downarrow \theta & & \downarrow \theta \\
2\text{CAT}_x & \rightarrow & 2\text{CAT}_x(K^{op}, -)
\end{array}
\]

\[
\begin{array}{ccc}
2\text{CAT} & \rightarrow & 2\text{CAT}_x(M, 2\text{CAT}(K^{op}, -)) \\
\downarrow \lambda & & \downarrow \lambda \\
2\text{CAT}(K^{op}, \text{Mon}(\text{Cat})) & \rightarrow & 2\text{CAT}(K^{op}, 2\text{CAT}_x(M, -))
\end{array}
\]

The objects of monoids

Having a monoidal category, we can always construct the category of monoids and this construction is 2-functorial in the sense that we have a 2-functor \(\text{mon} : \text{Mon}_{st}(\text{Cat}) \rightarrow \text{Cat}\) and a natural 2-transformation

\[u : \text{mon} \rightarrow | - |
\]

This motivates the following definitions for arbitrary (small) 2-category \(K\) with finite products.

We say that \(K\) admit objects of monoids iff there is a 2-functor \(\text{mon}_K : \text{Mon}_{st}(K) \rightarrow K\), a natural 2-transformation

\[u_K : \text{mon}_K \rightarrow | - |_K
\]

and a natural 2-isomorphism

\[\sigma_K : Y_K \circ \text{mon}_K \Rightarrow \text{mon}_K^{K^{op}} \circ \lambda_K \circ \text{Mon}_{st}(Y_K)
\]

such that in the square

\[
\begin{array}{ccc}
\text{Mon}_{st}(K) & \rightarrow & \text{Mon}_{st}(Y_K) \\
\downarrow \text{mon}_K & & \downarrow \text{mon}_K^{K^{op}} \\
K & \rightarrow & \text{Cat}_{K^{op}}
\end{array}
\]

we have

\[Y_K(u_K) = (u_K^{K^{op}} \circ \text{Mon}_{st}(Y_K)) \circ \sigma_K
\]

Note that the above equation of 2-cells includes equation of 1-cells

\[Y_K \circ | - |_K = | - |_{K^{op}} \circ \lambda_K \circ \text{Mon}_{st}(Y_K)
\]

Below we show that in order to express the object of monoids in arbitrary 2-category with finite products, it is enough to do it in \(\text{Cat}\).
Lemma 2.1. Let the 2-functor \( \mathcal{W} : \mathcal{M} \to \mathbf{Cat} \) be the weight for objects of monoids in \( \mathbf{Cat} \), i.e., for any monoidal category \( \mathcal{C} \) (in \( \mathbf{Cat} \)), the limit \( \text{Lim}_\mathcal{W} \zeta(\mathcal{C}) \) of the corresponding functor \( \zeta(\mathcal{C}) : \mathcal{M} \to \mathbf{Cat} \) with the weight \( \mathcal{W} \) is naturally isomorphic to the category of monoids \( \text{mon}(\mathcal{C}) \). In other words, the following triangle

\[
\begin{array}{ccc}
\text{Mon}_{st}(\mathcal{C}) & \xrightarrow{\zeta}\ & 2\mathbf{CAT}_x(\mathcal{M}, \mathbf{Cat}) \\
\text{mon} & \searrow \ & \text{Lim}_\mathcal{W}^{\mathbf{Cat}} \\
\mathbf{Cat} & \nearrow \ & \text{Cat}
\end{array}
\]

commutes. Then for any 2-category \( \mathcal{K} \) with finite limits, the limit 2-functor

\[
\text{Lim}_\mathcal{W} : \text{Mon}_{st}(\mathcal{K}) \to \mathcal{K}
\]

of \( \mathcal{W} \)-weighted limit, if it exists, is the object of monoids 2-functor.

Proof. We need to show that if \( \mathcal{W} : \mathcal{M} \to \mathbf{Cat} \) is the weight for the category on monoids in \( \mathbf{Cat} \), then in the following diagram

\[
\begin{array}{c}
\text{Mon}_{st}(\mathcal{K}) \longrightarrow \text{Mon}_{st}(Y_{\mathcal{K}}) \longrightarrow \text{Mon}_{st}(2\mathbf{CAT}(\mathcal{K}^{\text{op}}, \mathbf{Cat})) \longrightarrow 2\mathbf{CAT}(\mathcal{K}^{\text{op}}, \text{Mon}_{st}(\mathbf{Cat})) \\
\downarrow \zeta_{\mathcal{K}} \downarrow \zeta_{2\mathbf{CAT}(\mathcal{K}^{\text{op}}, \mathbf{Cat})} \downarrow \lambda_{\mathcal{K}} \downarrow 2\mathbf{CAT}(\mathcal{K}^{\text{op}}, \text{Mon}_{st}(\mathbf{Cat})) \\
\text{Lim}_\mathcal{W}^{\mathbf{Cat}}(\mathcal{K}) \longrightarrow \text{Lim}_\mathcal{W}^{\mathbf{Cat}}(\mathcal{K}^{\text{op}}) \longrightarrow \text{Lim}_\mathcal{W}^{\mathbf{Cat}}(\mathcal{K}^{\text{op}}, \mathbf{Cat}) \longrightarrow 2\mathbf{CAT}(\mathcal{K}^{\text{op}}, \text{Lim}_\mathcal{W}^{\mathbf{Cat}}(\mathcal{K}^{\text{op}})) \\
\downarrow \downarrow \downarrow \downarrow \\
\mathcal{K} \longrightarrow Y_{\mathcal{K}} \longrightarrow \text{Lim}_\mathcal{W}^{\mathbf{Cat}}(\mathcal{K}^{\text{op}}) \longrightarrow 2\mathbf{CAT}(\mathcal{K}^{\text{op}}, \text{mon})
\end{array}
\]

the outer hexagon commutes. As \( \zeta \) is natural, the left top square commutes. The left bottom square commutes as Yoneda preserves weighted limits. The right top square commutes as it is an instance of the commuting square of 3-transformations involving \( \zeta \), \( \lambda \), and \( \theta \). The triangle below it commutes as weighted limits are computed pointwise in \( 2\mathbf{CAT}(\mathcal{K}^{\text{op}}, \mathbf{Cat}) \). Finally, as Yoneda is faithful, the rightmost triangle commutes iff the triangle in Lemma above commutes. The verification of equality (1) is left for the reader. \( \Box \)

3 The weights for monoids

Let \( M_n \) denote the free monoid on \( n \)-generators, for \( n \in \omega + 1 \). The set \( BW_n \) of binary words on letters \( n \) is a free magma on two operations of arity 0 and 2, i.e., it is the least set such that

1. \( \iota \in BW_n \), where \( \iota \) is a distinguished element,
2. \( k \in BW_n \), if \( k \in n \),
3. \( (u \circ v) \in BW_n \), if \( u, v \in BW_n \).

We have a type function \( ty : BW_\omega \to \text{Lin} \) associating to any tree \( t \) the linear order of occurrences of the numbers in \( t \), i.e., it is the composition of functions

\[
ty : BW_\omega \to M_\omega \to \text{Lin}
\]
where the first function is the obvious homomorphism (of magmas) and the second one associates the linear order of occurrences of numbers in the words. For \( o \in ty(t) \), we denote by \(|o|\) the number corresponding to the occurrence \( o \).

The 2-Lawvere theory \( \mathcal{M} \) is a 2-category defined as follows. The objects of \( \mathcal{M} \) are natural numbers. The morphism in \( \mathcal{M} \)

\[
\vec{u} = (u_0, \ldots, u_{n-1}) : m \to n
\]

is an \( n \)-tuple of binary trees in \( BW_m \), i.e., \( u_i \in BW_m \), for \( i \in n \). The identity is given by

\[
\vec{u} : n \to n
\]

where \( u_i = i \), for \( i \in n \). The composition of morphisms is defined by simultaneous substitution. Given two morphisms

\[
(u_0, \ldots, u_{m-1}) : k \to m, \quad (v_0, \ldots, v_{n-1}) : m \to n
\]

their composition is given by

\[
(v_0(0 \setminus u_0, \ldots, m-1 \setminus u_{m-1}), \ldots, v_{n-1}(0 \setminus u_0, \ldots, n-1 \setminus u_{n-1})) : k \to n.
\]

and will be shortened to

\[
(v_i(j \setminus u_j)_{j \in m}, i \in n : k \to n.
\]

A 2-cell

\[
\alpha : \vec{u} \Rightarrow \vec{v} : m \to n
\]

exists iff \( ty(u_i) = ty(v_i) \), for \( i \in n \).

**Examples.** There are morphisms in \( \mathcal{M}(3,1) \), i.e., 2-cells in \( \mathcal{M} \), between \(((1 \circ 2) \circ (1 \circ 0)) \circ 1) \) and \(((1 \circ 2) \circ (1 \circ 0) \circ 1)) \circ 1) \) but not between \((0 \circ (2 \circ 0)) \) and \((0 \circ (0 \circ 2)) \).

The category of words \( CW \) has binary words in \( BW_1 \) as objects, and a morphism \( f : a \to b \) between two binary words \( a, b \in BW_1 \) is a monotone function \( f : ty(a) \to ty(b) \), i.e., between occurrences of \( o \)'s in \( a \) and \( b \).

The weight 2-functor

\[
W : \mathcal{M} \to \text{Cat}
\]

is defined as follows. For \( m \in \omega \)

\[
W(m) = CW^m.
\]

For \( \vec{u} : m \to n \in \mathcal{M} \), the functor

\[
W(\vec{u}) : CW^m \to CW^n,
\]

is given on object \( \vec{a} = (a_0, \ldots, a_{m-1}) \in CW^m \) as follows

\[
W(\vec{u})(\vec{a}) = (u_i(j \setminus a_j)_{j \in m}, i \in n.
\]

and on morphism \( \vec{f} : \vec{a} \to \vec{b} \in CW^m \)

\[
W(\vec{u})(\vec{f}) = (\prod_{o \in ty(a_i)} f_{|o|}, i \in n,
\]

where \( \prod_{x \in X} Y_x \) is the ordered sum of posets \( Y_x \) indexed by a poset \( X \). In particular, if \( X \) and \( Y_x \) are linear orders, the sum is a linear order as well. \( X \cup Y \) is an ordered sum of two posets.

To a 2-cell

\[
\alpha : \vec{u} \Rightarrow \vec{v} : m \to n
\]

the 2-functor \( W \) associates a natural transformation

\[
W(\alpha) : W(\vec{u}) \to W(\vec{v}),
\]

so that its component at \( \vec{a} \in CW^m \) is

\[
W(\alpha)_{\vec{a}} : \prod_{o \in ty(u_i)} id_{ty(a_{|o|})}(u_i(j \setminus a_j)_{j \in m}, v_i(j \setminus a_j)_{j \in m},
\]

for \( i \in n \).

The fact that \( W(\alpha) \) is indeed a natural transformation follows from the commutativity of the following diagram:
\[
W(\vec{u})(\vec{f}) = (\prod_{o \in \text{ty}(v_i)} f_{|o|})_{i \in n} = W(\vec{v})(\vec{f}_i) = (\prod_{o \in \text{ty}(v_j)} f_{|o|})_{i \in n}
\]
in \(W(n)\).

**Theorem 3.1.** The 3-functor

\[
\text{Mon}_{st} : 2\text{CAT}_\times \to 2\text{CAT}_\times,
\]
defined in section 2 is represented by the 2-category with finite products \(\mathbb{M}\) defined above. The weight 2-functor

\[
W : \mathbb{M} \to \text{Cat}
\]
is the weight for the objects of monoids in the sense defined in section 2.

**Proof.** The fact that \(\mathbb{M}\) has finite products is left for the reader.

To see the first claim, we shall define the 3-functor

\[
\text{Mon}_{st}(\text{Cat}) \xrightarrow{\zeta_{\text{Cat}}} 2\text{CAT}_\times(\mathbb{M}, \text{Cat})
\]

Let \((\mathbb{M}, \otimes, I, \alpha, \lambda, \rho)\) be a monoidal category in \(\text{Mon}_{st}(\text{Cat})\). We define a 2-functor

\[
\overline{\mathbb{M}} = \zeta_{\text{Cat}}(\mathbb{M}, \otimes, I, \alpha, \lambda, \rho) : \mathbb{M} \to \text{Cat}
\]
as follows. For \(n \in \omega\)

\[
\overline{\mathbb{M}}(n) = M^n.
\]

for \(t : n \to 1\) in \(\mathbb{M}\)

\[
\overline{\mathbb{M}} = \begin{cases} 
I & \text{if } s = t, \\
\pi_k & \text{if } s = k \in n, \\
\otimes \circ \langle s_1, s_2 \rangle & \text{if } s = \langle s_1, s_2 \rangle.
\end{cases}
\]

where \(I : M^n \to M\) is the constant functor with value \(I\). As \(\overline{\mathbb{M}}\) is supposed to preserve finite products, it is enough to define it on morphisms with codomain 1 only.

If \(\sigma : s \Rightarrow t : n \to 1\) is a 2-cell in \(\mathbb{M}\), then \(\overline{\mathbb{M}}(\sigma) : \overline{\pi} \Rightarrow \overline{\tau}\) is the unique formal (i.e., built from \(\alpha, \lambda, \rho\)) natural transformation between functors \(\overline{\pi}\) and \(\overline{\tau}\). Such a natural transformation exists by MacLane’s coherence theorem for monoidal categories.

The definition of \(\zeta_{\text{Cat}}\) on 1- and 2-cells is straightforward and is left for the reader.

In order to show that \(W : \mathbb{M} \to \text{Cat}\) is indeed the weight for the objects of monoids, we shall construct the universal \(W\)-weighted cone \(\tau\) over \(\overline{\mathbb{M}}\) with the vertex being the category of monoids \(\text{mon}(\mathbb{M}) = \text{mon}(\mathbb{M}, \otimes, I, \alpha, \lambda, \rho)\). As \(W\) preserves finite products, it is enough to define the projections \(\tau_\vec{a} : \text{mon}(\mathbb{M}) \to M\) indexed by the objects and morphisms of category \(W(1)\).

For object \(a \in W(1)\) and monoid \((m, \mu, i)\), we put

\[
\tau_\vec{a}(m, \mu, i) = \begin{cases} 
i & \text{if } a = t, \\
m & \text{if } s = 0, \\
\otimes \circ \langle a_1, a_2 \rangle & \text{if } a = \langle a_1 \circ a_2 \rangle.
\end{cases}
\]

\(\tau_\vec{a}\) on homomorphisms of monoids is defined in the obvious way.

We still need to define \(\tau\) on morphisms of \(W(1)\). We shall do it for the morphisms of form \(f : a \to 0\). The unique extension to all the morphisms in \(W(1)\) is again due to the MacLane’s coherence. We define

\[
\tau_\vec{f}(m, \mu, i) = \begin{cases} 
i & \text{if } a = t, \\
1_m & \text{if } s = 0, \\
\mu \circ \langle a_1, a_2 \rangle & \text{if } a = \langle a_1 \circ a_2 \rangle.
\end{cases}
\]

The remaining details of the construction and verifications are left for the reader. \(\square\)
4 The weight for actions of monoids

In this section we construct the weight for action of monoids along an action of a monoidal category object.

The set $BW_{*,n,n'}$ of pointed binary words on letters $n \in \omega + 1$ and $n' \in \omega + 1$ is the least set such that

1. $k \in BW_{*,n,n'}$, if $k \in n'$,
2. $(u \diamond v) \in BW_{*,n,n'}$, if $u \in BW_n$ and $v \in BW_{n,n'}$.

We have a type function $ty_\ast : BW_{*,\omega,\omega} \to \text{Lin}_\ast$ associating to any pointed binary word $t$ the linear order with the right end-point of occurrences of the numbers in $t$. Note that in pointed binary words there is always one occurrence of an underlined number and it is the last occurrence in the word; so it is the end-point in the linear order of occurrences. If $o \in ty_\ast(t)$, then $|o|$ denotes the number or underlined number corresponding to the occurrence $o$.

The 2-Lawvere (finite product) theory $\mathbb{A}$, for actions of monoidal category objects, is a 2-category defined as follows. The objects of $\mathbb{M}$ are pairs of natural numbers. The morphism in $\mathbb{A}$

$$(\vec{u}, \vec{s}) = (u_0, \ldots, u_{n-1}; s_0, \ldots, s_{n'-1}) : (m, m') \longrightarrow (n, n')$$

is an $n$-tuple of binary words in $BW_m$, i.e., $u_i \in BW_m$, for $i \in n$ and $n'$-tuple of pointed binary words in $BW_{*m,m'}$, i.e., $s_j \in BW_{*m,m'}$, for $i \in n'$. When convenient we shorten the notation for morphisms to $(u_i; s_j)_{i \in n, j \in n'}$ or even to $(\vec{u}; \vec{s})$.

The identity is given by

$$(u_0, \ldots, u_{n-1}; s_0, \ldots, s_{n'-1}) : (n, n') \longrightarrow (n, n')$$

where $u_i = i$, for $i \in n$, and $s_{i'} = i'_j$, for $i' \in n'$. The composition of morphisms is defined by simultaneous substitution. Given two morphism

$$(u_0, \ldots, u_{m-1}; s_0, \ldots, s_{m'-1}) : (k, k') \longrightarrow (m, m'), \quad (v_0, \ldots, v_{n-1}; t_0, \ldots, t_{n'-1}) : (m, m') \longrightarrow (n, n')$$

their composition is given by

$$(v_i(j \setminus u_j)_{j \in m}; t_{i'}(j \setminus u_{j'})_{j \in m'; j' \in m'})_{i \in n, i' \in n'} : (k, k') \longrightarrow (n, n').$$

A 2-cell

$$\alpha : (u_i; s_j)_{i \in n, j \in n'} \Rightarrow (v_i; t_j)_{i \in n, j \in n'} : (m, m') \longrightarrow (n, m')$$

exists iff $ty(u_i) = ty(v_i)$, for $i \in n$, and $ty(s_j) = ty(t_j)$, for $j \in n'$.

The category of pointed words $CW_\ast$ has $BW_{*,1,1}$ as the set of objects, and a morphism $f : a \to b$ between two pointed binary words $a, b \in BW_{*,1,1}$ is monotone right end-point preserving function $f : ty(a) \to ty(b)$, i.e., between occurrences of 0's and $\mathbf{\underline{0}}$ (occurrence of 0 can be sent to the occurrence $\mathbf{\underline{0}}$ but not vice versa).

The weight 2-functor

$$W_a : \mathbb{A} \longrightarrow \text{Cat}$$

is defined as follows.

$$W_a(1, 0) = W(1), \quad W_a(0, 1) = CW_\ast.$$  

For $m, m' \in \omega$

$$W_a(m, m') = W_a(1, 0)^m \times W_a(0, 1)^{m'}.$$  

For $(\vec{u}; \vec{s}) : (m, m') \longrightarrow (n, n') \in \mathbb{A}$, the functor

$$W_a(\vec{u}; \vec{s}) : W_a(m, m') \longrightarrow W_a(n, n'),$$

is given, for $(\vec{a}; \vec{b}) \in W_a(m, m')$, by

$$W_a(\vec{u}; \vec{s})(\vec{a}; \vec{b}) = (u_i(j \setminus a_j)_{j \in m}; s_{i'}(j \setminus a_{j'})_{j \in m'; j' \in m'})_{i \in n}.$$
and, on morphism \((f', g) : (\tilde{a}; \tilde{b}) \to (\tilde{a}'; \tilde{b}') \in W_a(m, m')\), is given by

\[
W_a(\tilde{a}; \tilde{b})(f' g) = \left( \prod_{o \in ty(u_i)} f_{|o|} \right) \prod_{o \in ty(u_i), |o| \in \omega} f_{|o|}^{+} \prod_{o \in ty(u_i), |o| \in \omega} g_{|o|} \in n,
\]

where \(\omega = \{ k : k \in \omega \}\). Note that \(W_a(\tilde{a}; \tilde{b})(f' g)\) is well defined as the last summand of the right sum contains exactly one element. To a 2-cell

\[
\alpha : (\tilde{a}; \tilde{b}) \Rightarrow (\tilde{v}; \tilde{t}) : (m, m') \rightarrow (n, n')
\]

the 2-functor \(W_a\) associates a natural transformation

\[
W_a(\alpha) : W_a(\tilde{a}; \tilde{b}) \to W_a(\tilde{v}; \tilde{t}),
\]

so that its component at \((\tilde{a}, \tilde{b}) \in W_a(m, m')\) is

\[
W_a(\alpha)_{\tilde{a}, \tilde{b}, i} = \prod_{o \in ty(u_i)} id_{ty(a_i)} : (u_i(j) a_j)_{j \in m} \rightarrow (v_i(j) a_j)_{j \in m},
\]

for \(i \in n\) and

\[
W_a(\alpha)_{\tilde{a}, \tilde{b}, i'} = \prod_{o \in ty(s_{i'}), |o| = 0} id_{ty(a_{i'})} \prod_{o \in ty(s_{i'}), |o| = 0} id_{ty(b_{i'})} : (s_{i'}(j) a_j b_{j'})_{j \in m} \rightarrow (t_{i'}(j) a_j b_{j'})_{j \in m, j' \in m'},
\]

for \(i' \in n'\).

The fact that \(W_a(\alpha)\) is indeed a natural transformation follows from the commutativity of the following diagram \(\textbf{Cat}\):

\[
\begin{array}{ccc}
W_a(\tilde{a}; \tilde{b})(\tilde{a}; \tilde{b}) & \xrightarrow{W_a(\alpha)_{\tilde{a}, \tilde{b}}} & W_a(\tilde{v}; \tilde{t})(\tilde{a}; \tilde{b}) \\
W_a(\tilde{a}; \tilde{b})(\tilde{f}; \tilde{g}) & \xrightarrow{W_a(\alpha)_{\tilde{a}, \tilde{b}}(\tilde{f}; \tilde{g})} & W_a(\tilde{v}; \tilde{t})(\tilde{f}; \tilde{g}) \\
W_a(\tilde{a}; \tilde{b})(\tilde{a}'; \tilde{b}') & \xrightarrow{W_a(\alpha)_{\tilde{a}', \tilde{b}'}(\tilde{a}'; \tilde{b}')} & W_a(\tilde{v}; \tilde{t})(\tilde{a}'; \tilde{b}')
\end{array}
\]

for any morphism \((\tilde{f}; \tilde{g}) : (\tilde{a}; \tilde{b}) \to (\tilde{a}'; \tilde{b}')\) in \(W_a(n, n')\). And this can be checked in a similar way as in the case of the weight for monoids.

**Theorem 4.1.** The 3-functor

\[
\text{Act}_\otimes : 2\text{CAT}_\otimes \to 2\text{CAT}_\otimes,
\]

for action monoidal categories is represented by the 2-category \(\mathcal{A}\) defined above. The weight 2-functor

\[
W_a : \mathcal{A} \to \text{Cat}
\]

is the weight for the objects of actions of monoids along an action of a monoidal category.

**Proof.** The proof is very similar in spirit to the one concerning monoidal category objects and objects of monoids. We leave the details to the reader. \(\square\)

5 Concluding remarks

We end the paper with two remarks.
Lax monoidal 1-cells and monoids

First remark concerns lax monoidal 1-cells in 2-categories.

We know that in \( \textbf{Cat} \) not only strict but also lax monoidal functors between monoidal categories induce functors between categories of monoids. This phenomenon is still true, if we replace \( \textbf{Cat} \) by any 2-category with finite products \( K \). As we have shown, the objects of monoids have universal properties with respect to strict \( W \)-cones, whereas composition of a lax monoidal functor with a strict \( W \)-cone gives rise to a strict/lax \( W \)-cone, in general. By a strict/lax \( W \)-cone we mean a lax \( W \)-cone such that on projections and diagonal 1-cells (i.e., on the coalgebraic part) the commutations are strict. One can verify that the category \( \text{Cone}_{F,W}(X) \) of (strict) \( W \)-cones over \( F \) preserving finite products with the vertex \( X \) is a coreflexive subcategory of the category \( \text{Cone}_{ls,W}(X) \) of lax/strict \( W \)-cones over \( F \) with the vertex \( X \). This coreflection depends on the fact that we define our monoidal category objects using genuine products, not just a monoidal structure on the 2-category \( K \).

Thus if \( (F,\varphi,\bar{\varphi}) : (C,\otimes,\ldots) \rightarrow (C',\otimes',\ldots) \) is a lax monoidal 1-cell between monoidal category objects in \( K \) that admits objects of monoids, then the coreflection of the composed lax \( W \)-cone on \( C' \) with vertex \( \text{mon}(C,\otimes,\ldots) \)

\[(F,\varphi,\bar{\varphi}) \circ (\text{mon},\sigma)\]

is a strict \( W \)-cone over \( (C',\otimes,\ldots) \) and it induces a 1-cell

\[\text{mon}(F,\varphi,\bar{\varphi}) : \text{mon}(C,\otimes,\ldots) \rightarrow \text{mon}(C',\otimes,\ldots)\]

2-algebraic properties in 2-categories

The second remark concerns the properties related to 2-algebraic structures such as adjunctions, monads, comonads, Kleisli and Eilenberg-Moore objects, monoidal category objects and their actions, objects of monoids, objects of actions that hold in all 2-categories. As such structures can be defined in any 2-category \( K \) (possibly with finite products) as structures in \( K \) that are sent to the corresponding (well known) structures in \( \textbf{Cat} \), many properties of these 2-algebraic structures are inherited directly from \( \textbf{Cat} \).

To give an example, consider the following. The notion of being monadic 1-cell can be transferred verbatim to all 2-categories. It is well known that if the forgetful functor from the category of monoids has a left adjoint, it is automatically monadic. Thus in this case the category of monoids is isomorphic to the Eilenberg-Moore object. The same holds true in any 2-category \( K \) in which these constructions make sense. To see this, we can move comparison functor to \( \textbf{Cat} \) via representable functors. There the statement holds and, since 2-Yoneda \( Y : K \rightarrow \mathbf{2CAT}(K^{op},\textbf{Cat}) \) is conservative, we can conclude that the comparison functor in \( K \) is an isomorphism as well.

6 Appendix: monoidal objects and their actions

6.1 Monoidal objects

In this subsection \( K \) is a 2-category with finite products.

A monoidal category object (0-cell) \( (M,\otimes,I,\alpha,\lambda,\rho) \) in \( K \) consists of

1. one 0-cell \( M \),
2. two 1-cells \( \otimes : M \times M \rightarrow M, I : 1 \rightarrow M \)
3. three invertible 2-cells

\[\alpha : \otimes \circ (1_M \times \otimes) \rightarrow \otimes \circ (\otimes \times 1_M) : M \times M \times M \rightarrow M,\]

\[\lambda : \otimes \circ (I,1_M) \rightarrow 1_M : M \rightarrow M,\]

\[\rho : \otimes \circ (1_M,I) \rightarrow 1_M : M \rightarrow M\]

such that the following two diagrams, presented in the ‘infix notation’, \( \text{MC1} \) and \( \text{MC2} \) of 1- and 2-cells commute. The diagram

\[\text{MC1}\]

\[\text{MC2}\]

\[\text{MC3}\]

\[\text{MC4}\]

\[\text{MC5}\]

\[\text{MC6}\]
commutes in the category $\mathcal{K}(M \times M \times M \times M, M)$, where $\Pi_1, \Pi_2, \Pi_3, \Pi_4 : M \times M \times M \times M \to M$ are the first, the second, the third, and the forth projections, respectively. For example, $\Pi_2 \otimes \Pi_3 : M \times M \times M \times M \to M$ is a 1-cell in $\mathcal{K}$ which is a composition of a projection to the 2nd and 3rd component followed by $\otimes$, i.e.,

$$M \times M \times M \times M \xrightarrow{\langle \Pi_2, \Pi_3 \rangle} M \times M \xrightarrow{\otimes} M$$

The diagram

commutes in $\mathcal{K}(M \times M, M)$. Here, according to the convention $\Pi_1, \Pi_2 : M \times M \to M$ are the first and the second projections, respectively, and, for example, $\Pi_1 \otimes (I \otimes \Pi_2) : M \times M \to M$ is a 1-cell that is a composition displayed below

$$M \times M \xrightarrow{\langle \Pi_1, I, \Pi_2 \rangle} M \times M \times M \xrightarrow{\otimes} M$$

and $\alpha_{\Pi_1, I, \Pi_2}$ is the 2-cell $\alpha$ wiskered along the 1-cell $(\Pi_1, I, \Pi_2)$.

A strict monoidal morphism (1-cell)

$$F : (M, \otimes, I, \alpha, \lambda, \rho) \to (M', \otimes', I', \alpha', \lambda', \rho')$$

is a 1-cell $F : M \to M'$ such that

$$F(I) = I', \quad F(\Pi_1 \otimes \Pi_2) = F(\Pi_1 \otimes \Pi_2)$$

and

$$F(\alpha) = \alpha'_{F, F, F}, \quad F(\lambda) = \lambda'_F, \quad F(\rho) = \rho'_F.$$

A monoidal transformation between two strict monoidal 1-cells

$$\tau : F \to F' : (M, \otimes, I, \alpha, \lambda, \rho) \to (M', \otimes', I', \alpha', \lambda', \rho')$$

is a 2-cell $\tau : F \to F'$ in $\mathcal{K}$ such that

$$\tau_I = 1_{I'}, \quad \tau_{\Pi_1} \otimes \tau_{\Pi_2} = \tau_{\Pi_1 \otimes \Pi_2}.$$

In this way we have defined the 2-category $\textbf{Mon}_{st}(\mathcal{K})$ of monoidal category objects with strict monoidal 1- and 2-cells. It is an object map of a 3-functor

$$\textbf{Mon}_{st} : 2\text{CAT}_\times \to 2\text{CAT}_\times,$$

whose remaining parts of definition are left for the reader.

In $\textbf{Cat}$ monoidal category objects are the usual monoidal categories. Given a monoidal category $(M, \otimes, I, \alpha, \lambda, \rho)$ in $\textbf{Cat}$, we define the object of monoids i.e., the category of monoids in the usual way. The objects are monoids, i.e., triples $(m, \mu : m \otimes m \to m, \iota : I \to m)$ so that the following diagram
commutes. A homomorphism of monoids \( h : (m, \mu, \iota) \to (m', \mu', \iota') \) is a morphism \( h : m \to m' \) such that the diagram

\[
\begin{array}{ccc}
m \otimes m & \xrightarrow{\mu} & m \\
h \otimes h & & h \\
m' \otimes m' & \xrightarrow{\mu'} & m'
\end{array}
\]

commutes. In this way we have defined the category of monoids \( \text{mon}(M, \otimes, I, \alpha, \lambda, \rho) \) over monoidal category \( (M, \otimes, I, \alpha, \lambda, \rho) \) in \( \text{Mon}_{\text{st}}(\text{Cat}) \).

### 6.2 Actions of monoidal category objects

Now fix a 0-cell \( X \) in \( \mathcal{K} \). We shall define the 2-category \( \text{ActMon}_{\text{st}}(\mathcal{K}) \) of actions of monoidal category objects in \( \mathcal{K} \) with strict 1- and 2-cells.

A monoidal action \( (M, \otimes, I, \alpha, \lambda, \rho, X, \ast, \psi, \bar{\psi}) \) of a monoidal category object \( (M, \otimes, I, \alpha, \lambda, \rho) \) on a 0-cell \( X \) consists of

1. a monoidal category object \( (M, \otimes, I, \alpha, \lambda, \rho) \),
2. a 0-cell \( X \),
3. a 1-cell \( \ast : M \times X \to X \),
4. two 2-cells

\[
\psi : \Pi_1 \ast (\Pi_2 \ast \Pi_X) \to (\Pi_1 \otimes \Pi_2) \ast \Pi_X : M \times M \times X \to X, \quad \bar{\psi} : \Pi_X \to (I) \ast \Pi_X : X \to X,
\]

(where \( \Pi_X \) is here and below the projection on the coordinate \( X \)) such that the diagrams \( \text{MA1}, \text{MA2}, \text{MA3} \) commute. The diagram

\[
\begin{array}{ccc}
\Pi_1 \ast ((\Pi_2 \otimes \Pi_3) \ast \Pi_X) & \xrightarrow{\psi_{n_1, n_2, n_3, n_X}} & (\Pi_1 \otimes (\Pi_2 \otimes \Pi_3)) \ast \Pi_X \\
1_{n_1} \ast \psi_{n_2, n_3, n_X} & \xrightarrow{\psi_{n_1, n_2, n_3, n_X}} & (\Pi_1 \otimes \Pi_2) \ast (\Pi_3 \ast \Pi_X)
\end{array}
\]

\( \text{MA1} \)

commutes in \( \mathcal{K}(M \times M \times M \times X, X) \). The two squares

\[
\begin{array}{ccc}
\Pi_1 \ast \Pi_X & \xrightarrow{\lambda_{\Pi_1} \ast 1_{\Pi_X}} & \Pi_1 \ast \Pi_X \\
\bar{\psi}_{\Pi_1, \Pi_X} & \xrightarrow{\lambda_{\Pi_1, \Pi_X}} & \lambda_{\Pi_1} \ast 1_{\Pi_X}
\end{array}
\]

\( \text{MA2} \)

and

\[
\begin{array}{ccc}
(I \ast (\Pi_1 \ast \Pi_X)) & \xrightarrow{\psi_{1, \Pi_1, \Pi_X}} & (I \otimes \Pi_1) \ast \Pi_X \\
\psi_{1, \Pi_1, \Pi_X} & \xrightarrow{\psi_{1, \Pi_1, \Pi_X}} & (I \otimes \Pi_1) \ast \Pi_X
\end{array}
\]

are commutative.
and

\[
\begin{array}{c}
\Pi_1 * \Pi_X \\
\downarrow \text{1}_{\Pi_1} * \text{1}_{\Pi_X} \\
\Pi_1 * (I * \Pi_X) \\
\downarrow \psi_{\Pi_1,I,\Pi_X} \\
(\Pi_1 \otimes I) * \Pi_X
\end{array}
\]

commute in \(\mathcal{K}(M \times X, X)\).

A (strict) morphism (1-cell) of monoidal actions

\[(F, G) : (M, \otimes, I, \alpha, \lambda, \rho, X, *, \psi) \to (M', \otimes', I', \alpha', \lambda', \rho', X', *, \psi')\]

consists of

1. a strict monoidal 1-cell \(F : (M, \otimes, I, \alpha, \lambda, \rho) \to (M', \otimes', I', \alpha', \lambda', \rho')\),
2. 1-cell \(G : X \to X'\)
3. such that
   \[
   F(\Pi_1) *' G(\Pi_X) = G(\Pi_1 * \Pi_X), \quad G(\psi) = \psi_F' \circ F \times G, \quad G(\bar{\psi}) = \bar{\psi}'_G.
   \]

A transformation (2-cell) of strict morphisms of actions

\[\tau : F \to F' : (M, \otimes, I, \alpha, \lambda, \rho, X, *, \psi) \to (M', \otimes', I', \alpha', \lambda', \rho', X', *, \psi')\]

is a pair of 2-cells \(\tau : F \to F'\) and \(\sigma : G \to G'\) such that

\[
\tau_{\Pi_1} *' \sigma_{\Pi_X} = \tau_{\Pi_1 * \Pi_X},
\]

i.e.,

\[
\begin{array}{c}
F(\Pi_1) *' G(\Pi_X) \\
\downarrow \tau_{\Pi_1} *' \sigma_{\Pi_X} \\
F'(\Pi_1) *' G'(\Pi_X)
\end{array}
\]

\[
\begin{array}{c}
\tau_{\Pi_1 * \Pi_X} \\
\downarrow \tau_{\Pi_1 * \Pi_X}
\end{array}
\]

In this way we have defined the 2-category \(\text{Act}_{st}(\mathcal{K})\) of actions of monoidal category objects with strict 1- and 2-cells. It is an object map of a 3-functor

\[\text{Act}_{st} : 2\text{CAT}_x \to 2\text{CAT}_x,\]

whose remaining parts of definition are again left for the reader.

In \(\text{Cat}\) actions of monoidal category objects are the usual actions of monoidal categories. Given an action of monoidal category \((M, \otimes, I, \alpha, \lambda, \rho, X, *, \psi)\) in \(\text{Cat}\), we define category \(\text{act}(M, \otimes, I, \alpha, \lambda, \rho, X, *, \psi)\) the object of actions of monoids along the action of monoidal category \((M, \otimes, I, \alpha, \lambda, \rho)\), i.e., the category of actions of monoids in the usual way. The objects are actions of monoids, i.e., 5-tuples \((m, x, \mu : m \otimes m \to m, \iota : I \to m, a : m * x \to x)\) so that \((m, \mu, \iota)\) is a monoid and moreover the following diagram

\[
\begin{array}{c}
m \otimes (m * x) \\
\downarrow \psi \\
(m \otimes m) * x \\
\downarrow \mu \otimes 1_x \\
m * x \\
\downarrow 1 * 1_x \\
I * x
\end{array}
\]

\[
\begin{array}{c}
1 \otimes a \\
\downarrow \\
m * x \\
\downarrow a \\
x \\
\downarrow \psi
\end{array}
\]

commutes. A homomorphism of actions \((h, k) : (m, x, \mu, \iota, a) \to (m', x', \mu', \iota', a')\) is a morphism of monoids \(h : (m, \mu, \iota) \to (m', \mu', \iota')\) and a morphism \(k : x \to x'\) in \(X\) making the diagram
commute. In this way we have defined the category of actions of monoids monoids \( \text{act}(M, \otimes, I, \alpha, \lambda, \rho, X, *, \psi) \) over an action of a monoidal category \((M, \otimes, I, \alpha, \lambda, \rho, X, *, \psi)\) in \( \text{Mon}_{st}(\text{Cat}) \).

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