ANYONIC REALIZATION OF THE
QUANTUM AFFINE LIE ALGEBRAS

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Abstract

We give a realization of quantum affine Lie algebras $\mathcal{U}_q(\hat{A}_{N-1})$ and $\mathcal{U}_q(\hat{C}_N)$ in terms of anyons defined on a one-dimensional chain (or on a two-dimensional lattice), the deformation parameter $q$ being related to the statistical parameter $\nu$ of the anyons by $q = e^{i\pi\nu}$. In the limit of the deformation parameter going to one we recover the Feingold-Frenkel fermionic construction of undeformed affine Lie algebras.

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1 Introduction

Anyons are particles with any statistics that interpolate between fermions and bosons.

In the first quantization scheme, the notion of statistics is related to the symmetry properties of the wave function of \(N\) identical particles, the bosons (resp. fermions) corresponding to symmetric (resp. antisymmetric) wave functions under the exchange of particles. In two dimensions, there exist more possibilities: the anyons. In that case, the wave function of \(N\) identical particles picks up a phase factor under the exchange of particles. More precisely, if we consider a system of \(N\) identical hard-core particles in \(d\) dimensions, the configuration space is \(\mathcal{M}_{N,d} = (\mathbb{R}^d)^N - \Delta)/S_N\) where \(\Delta\) is the set of points of \((\mathbb{R}^d)^N\) with at least two equal coordinates and \(S_N\) is the permutation group of \(N\) elements. The fundamental group \(\pi_1(\mathcal{M}_{N,d}) = S_N\) for \(d > 2\) but \(\pi_1(\mathcal{M}_{N,d}) = B_N\) for \(d = 2\) where \(B_N\) is the braid group. Anyons appear as abelian representations of the braid group\(^2\).

Let us remark that anyons can consistently be defined also on a one-dimensional lattice. For simplicity, this will be used here to construct anyonic realizations of the quantum affine Lie algebras \(\mathcal{U}_q(\widehat{A}_{N-1})\) and \(\mathcal{U}_q(\widehat{C}_N)\). In sect. \([\ ]\) the construction by means of two-dimensional anyons \([2]\) will be briefly recalled.

2 The unitary case: \(\mathcal{U}_q(\widehat{A}_{N-1})\)

The strategy to construct an anyonic realization of the quantum affine Lie algebra \(\mathcal{U}_q(\widehat{A}_{N-1})\) with non-vanishing central charge will be the following: (1) start from the description of \(\mathcal{U}_q(\widehat{A}_{N-1})\) in the Serre–Chevalley basis, (2) find a fermionic realization of \(\widehat{A}_{N-1}\) in terms of creation and annihilation operators, (3) construct anyonic oscillators on a one-dimensional lattice and (4) replace the fermionic oscillators by suitable anyons in the expressions of the simple generators of \(\mathcal{U}_q(\widehat{A}_{N-1})\) in the Serre–Chevalley basis.

Let us denote by \(H_\alpha\) and \(E^\pm_\alpha\) where \(\alpha = 0, 1, \ldots, N-1\) the simple generators of \(\mathcal{U}_q(\widehat{A}_{N-1})\) in the Serre–Chevalley basis. The commutation relations are:

\[
\begin{align*}
[H_\alpha, H_\beta] &= 0 \quad (2.1a) \\
[H_\alpha, E^\pm_\beta] &= \pm a_{\alpha\beta}E^\pm_\beta \quad (2.1b) \\
[E^+_\alpha, E^-_\beta] &= \delta_{\alpha\beta} [H_\alpha]_{\alpha} \quad (2.1c)
\end{align*}
\]

and the quantum Serre relations read as:

\[
\sum_{\ell=0}^{1-a_{\alpha\beta}} (-1)^\ell \left[ 1 - a_{\alpha\beta} \right]_{\alpha} \left( E^\pm_\alpha \right)^{1-a_{\alpha\beta}-\ell} E^\pm_\beta \left( E^\pm_\alpha \right)^{\ell} = 0 \quad (2.2)
\]

\(^2\)for a review on anyons see for instance \([5]\); for a review on anyonic realization of deformed Lie algebras see \([6]\).
where the notations are the standard ones, i.e.

\[
[x]_q = \frac{q^x - q^{-x}}{q - q^{-1}}, \quad \left[ \begin{array}{c} m \\ n \end{array} \right]_q = \frac{[m]_q!}{[n]_q! [m - n]_q!}, \quad [m]_q! = [1]_q \cdots [m]_q \quad (2.3)
\]

\(a_{\alpha\beta}\) being the Cartan matrix of \(\hat{A}_{N-1}\) and \(q_{\alpha} = q\) for all \(\alpha = 0, 1, \ldots, N - 1\).

A fermionic realization of \(U_q(\hat{A}_{N-1})\) in terms of creation and annihilation operators is obtained by taking an infinite number of fermionic oscillators \(c_\rho(r), c_\rho^\dagger(r)\) with \(\rho = 1, \ldots, N\) and \(r \in \mathbb{Z}' = \mathbb{Z} + \frac{1}{2}\), which satisfy the anticommutation relations

\[
\{c_\rho(r), c_\sigma(s)\} = \{c_\rho^\dagger(r), c_\sigma^\dagger(s)\} = 0 \quad \text{and} \quad \{c_\rho^\dagger(r), c_\sigma(s)\} = \delta_{\rho\sigma}\delta_{rs} \quad (2.4)
\]

the number operator being defined as usual by \(n_\rho(r) = c_\rho^\dagger(r)c_\rho(r)\).

These oscillators are equipped with a normal ordered product such that

\[
: c_\rho^\dagger(r)c_\sigma(s) : = \begin{cases} c_\rho^\dagger(r)c_\sigma(s) & \text{if } s > 0 \\ -c_\sigma(s)c_\rho^\dagger(r) & \text{if } s < 0 \end{cases} \quad (2.5)
\]

and therefore

\[
: n_\rho(r) : = \begin{cases} n_\rho(r) & \text{if } r > 0 \\ n_\rho(r) - 1 & \text{if } r < 0 \end{cases} \quad (2.6)
\]

Then a fermionic oscillator realization of the simple generators of \(\hat{A}_{N-1}\) in the Serre–Chevalley basis is given by \((\alpha = 0, 1, \ldots, N - 1)\) (we use small letters for the simple generators of the undeformed Lie algebra \(\hat{A}_{N-1}\))

\[
h_\alpha = \sum_{r \in \mathbb{Z}'} h_\alpha(r) \quad \text{and} \quad e_\alpha^\pm = \sum_{r \in \mathbb{Z}'} e_\alpha^\pm(r) \quad (2.7)
\]

where \((i = 1, \ldots, N - 1)\)

\[
h_i(r) = n_i(r) - n_{i+1}(r) = : n_i(r) : - : n_{i+1}(r) : \quad (2.8a)
\]

\[
h_0(r) = n_N(r) - n_1(r + 1) = : n_N(r) : - : n_1(r + 1) : + \delta_{r,-1/2} \quad (2.8b)
\]

\[
e_i^+(r) = c_i^\dagger(r)c_{i+1}(r), \quad e_i^+ = c_N(r)c_1(r + 1) \quad (2.8c)
\]

\[
e_i^-(r) = c_{i+1}(r)c_i(r), \quad e_i^- = c_1^\dagger(r + 1)c_N(r) \quad (2.8d)
\]

Inserting Eq. \((2.8c)\) into Eq. \((2.7)\) and taking into account that the sum over \(r\) can be split into a sum of two convergent series only after normal ordering, one can check that

\[
h_0 = 1 + \sum_{r \in \mathbb{Z}'} : n_N(r) : - \sum_{r \in \mathbb{Z}'} : n_1(r) : = 1 - \sum_{j=1}^{N-1} h_j, \quad (2.9)
\]

that is the central charge is \(\gamma = 1\).

We now introduce anyons defined on a one-dimensional lattice:

\[
a_\rho(r) = K_\rho(r)c_\rho(r) \quad \text{and} \quad \tilde{a}_\rho(r) = \tilde{K}_\rho(r)c_\rho(r) \quad (1 \leq \rho \leq N) \quad (2.10)
\]
and similar expressions for the conjugated operator $a^\dagger_\rho(r)$ and $\tilde{a}^\dagger_\rho(r)$, where the disorder factors $K_\rho(r)$ and $\tilde{K}_\rho(r)$ are expressed as

$$K_\rho(r) = q^{-\frac{1}{2}} \sum_{t \in \mathbb{Z}'} \varepsilon(t-r) \cdot n_\rho(t); \quad \text{and} \quad \tilde{K}_\rho(r) = q^{\frac{1}{2}} \sum_{t \in \mathbb{Z}'} \varepsilon(t-r) \cdot n_\rho(t): \quad (2.11)$$

using the sign function $\varepsilon(t) = |t|/t$ if $t \neq 0$ and $\varepsilon(0) = 0$.

By a direct calculation, one can prove that the $a$-type operators satisfy the following braiding relations for $r > s$:

$$a_\rho(r)a_\rho(s) + q^{-1}a_\rho(s)a_\rho(r) = 0 \quad a^\dagger_\rho(r)a^\dagger_\rho(s) + q a_\rho(s)a^\dagger_\rho(r) = 0$$

and

$$a_\rho(r)a^\dagger_\rho(r) + a^\dagger_\rho(r)a_\rho(r) = 1$$

$$a_\rho(r)^2 = a^\dagger_\rho(r)^2 = 0 \quad \text{(2.13)}$$

which shows that the operators $a_\rho(r), a^\dagger_\rho(s)$ are indeed anyonic oscillators with statistical parameter $\nu$ such that $q = e^{i\pi \nu}$. The $\tilde{a}$-type anyons have the same statistical parameter $\nu$ but opposite braiding (and ordering) prescription.

**Theorem 1** An anyonic realization of the simple generators of the quantum affine Lie algebra $\mathcal{U}_q(\hat{A}_{N-1})$ with central charge $\gamma = 1$ is given by $(\alpha = 0, 1, \ldots, N - 1)$

$$H_\alpha = \sum_{r \in \mathbb{Z}'} H_\alpha(r) \quad \text{and} \quad E^{\pm}_\alpha = \sum_{r \in \mathbb{Z}'} E^{\pm}_\alpha(r) \quad (2.14)$$

where $(1 \leq i \leq N - 1)$

$$H_i(r) = h_i(r) = :n_i(r) : - : n_i+1(r) :$$

$$E^+_i(r) = a^\dagger_i(r)a_{i+1}(r) \quad E^-_i(r) = a^\dagger_{i+1}(r)a_i(r) \quad (2.15a)$$

$$H_0(r) = h_0(r) = :n_N(r) : - : n_1(r) + 1 : + \delta_{r, -1/2}$$

$$E^+_0(r) = q^{-\frac{1}{2}}(r+1/2) \quad a^\dagger_N(r)a_1(r + 1) \quad E^-_0(r) = q^{\frac{1}{2}}(r+1/2) \quad a^\dagger_1(r + 1)a_N(r) \quad (2.15c)$$

**Proof** We must check that the realization Eqs. $(2.15a-2.15c)$ indeed satisfy the quantum affine Lie algebra $\mathcal{U}_q(\hat{A}_{N-1})$ in the Serre–Chevalley basis.

Fist of all, inserting eqs. $(2.10)$, the expressions $(2.15a-d)$ become

$$E^{\pm}_\alpha(r) = e^{\pm}_\alpha(r)q^{\frac{1}{2}} \sum_{t \in \mathbb{Z}'} \varepsilon(t-r) \cdot h_\alpha(t): \quad (2.16)$$

where the generators $h_\alpha(r)$ and $e^{\pm}_\alpha(r)$ coincide with those defined in eqs. $(2.8a-d)$ corresponding to the undeformed affine Lie algebra $\hat{A}_{N-1}$.

Now, consider the non-extended Dynkin diagram of $\hat{A}_{N-1}$, which corresponds to the set of generators $\{h_\alpha(r), e^{\pm}_\alpha(r)\}$ where $\alpha \neq 0$. For a fixed $r \in \mathbb{Z}'$, the set $\{h_\alpha(r), e^{\pm}_\alpha(r)\}$ for $\alpha \neq 0$ is a representation of $A_{N-1}$ of spin 0 and 1/2, and thus of $\mathcal{U}_q(A_{N-1})$ [4]. Thanks
to eqs. (2.14) and (2.16), $H_\alpha, E_\alpha^\pm$ are the correct coproduct in $U_q(A_{N-1})$. Therefore, the relations (2.1a-c) hold for $\alpha, \beta \neq 0$ (step 1).

We consider then the extended Dynkin diagram of $\hat{A}_{N-1}$ and we delete a dot $\mu$ which is not the affine dot. For a fixed $r \in \mathbb{Z}'$, the set \{h_\alpha(r), e_\alpha^\pm(r)\} for $\alpha \neq \mu$ is a representation of $A_{N-1}$ of spin 0 and 1/2, and thus of $U_q(A_{N-1})$. The eqs. (2.14) and (2.16) give once again the correct coproduct in $U_q(A_{N-1})$ and hence $H_\alpha, E_\alpha^\pm$ form a representation of $U_q(A_{N-1})$.

Therefore, the relations (2.1a-c) hold for $\alpha, \beta \neq \mu$ (step 2).

Finally, in the case $U_q(\hat{A}_1)$ the previous arguments fail (in particular to prove the quantum Serre relations). Such equations can however be checked explicitly by using the braiding properties of the anyonic oscillators $a$ and $\tilde{a}$.

## 3 The symplectic case: $U_q(\hat{C}_N)$

Let us consider now the $U_q(\hat{C}_N)$ case. As in the $U_q(\hat{A}_{N-1})$ case, a guideline is given by the fermionic realization of the (undeformed) affine algebra. Such a fermionic realization in terms of creation and annihilation operators is easily obtained by noticing that the folding of the Dynkin diagram of $\hat{A}_{2N}$ leads to the Dynkin diagram of $\hat{C}_N$ (see figure below).

![Dynkin Diagram](https://example.com/dynkin_diagram.png)

Denoting the simple generators of $\hat{A}_{2N}$ by $h_i, e_i^\pm$ with $i = 0, \ldots, 2N - 1$, the lifting of the automorphism associated to the symmetry of the Dynkin diagram of $\hat{A}_{2N}$ leads to the following expression of the simple generators of $\hat{C}_N$:

$$
\begin{align*}
    h'_0 &= h_0, & h'_i &= h_i + h_{2N-i}, & h'_N &= h_N \\
    e'_0^\pm &= e_i^\pm, & e'_i^\pm &= e_i^\pm + e_{2N-i}^\pm, & e'_N^\pm &= e_N^\pm 
\end{align*} 
$$

(3.1)

In the undeformed case, it is immediate to see that the Serre relations of $\hat{C}_N$ are satisfied (actually one can write them entirely in terms of the Serre relations of $\hat{A}_{2N}$).

Now, we go to the deformed case. The main idea of the construction is to use the folding $\hat{A}_{2N} \rightarrow \hat{C}_N$ to obtain a realization of $U_q(\hat{C}_N)$ in terms of anyons, following with some modifications the procedure used in Ref. [4] to build $U_q(C_N)$. Using the fermionic oscillators $c_\mu(r), c_\mu^+(r)$ of the previous Section, with the same normal ordering prescription Eq. (2.6), one defines the following anyons:

$$
a_\mu(r) = K_\mu(r)c_\mu(r) \quad \text{and} \quad \tilde{a}_\mu(r) = \tilde{K}_\mu(r)c_\mu(r) \quad (1 \leq \mu \leq 2N) 
$$

(3.2)
Theorem 2 An anyonic realization of the simple generators of the quantum affine Lie algebra $U_q(\hat{C}_N)$ with central charge equal to 1 is given by ($\alpha = 0, 1, \ldots, N$)

$$H_{\alpha} = \sum_{r \in \mathbb{Z}'} H_{\alpha}(r) \quad \text{and} \quad E_{\alpha}^\pm = \sum_{r \in \mathbb{Z}'} E_{\alpha}^\pm(r)$$

where

$$H_i(r) = h_i(r) = : n_i(r) : - : n_{i+1}(r) : + : n_{2N-i}(r) : - : n_{2N-i+1}(r) :$$

$$E_i^+(r) = a_i^+(r)a_{i+1}(r) + a_{2N-i}^+(r)a_{2N-i+1}(r)$$

$$E_i^-(r) = \tilde{a}_{i+1}(r)\tilde{a}_i(r) + \tilde{a}_{2N-i+1}(r)\tilde{a}_{2N-i}(r)$$

are associated to the simple short roots of $C_N$ ($i = 1, \ldots, N - 1$),

$$H_N(r) = h_N(r) = : n_N(r) : - : n_{N+1}(r) :$$

$$E_N^+(r) = a_N^+(r)a_{N+1}(r)$$

$$E_N^-(r) = \tilde{a}_{N+1}(r)\tilde{a}_N(r)$$

are associated to the simple long root of $C_N$, and

$$H_0(r) = h_0(r) = : n_{2N}(r) : - : n_1(r+1) : + \delta_{r,-1/2}$$

$$E_0^+(r) = a_{2N}^+(r)a_1(r+1)$$

$$E_0^-(r) = \tilde{a}_1^+(r+1)\tilde{a}_{2N}(r)$$

are associated to the affine root of $C_N$.

Proof We have now to check that the realization Eqs. (3.3), (3.6), (3.7) indeed satisfy the quantum affine Lie algebra $U_q(\hat{C}_N)$ in the Serre–Chevalley basis, i.e. the equations (2.1a–2.1d) for $\alpha, \beta = 0, 1, \ldots, N$, together with the quantum Serre relations (2.2) hold, where $a_{\alpha\beta}$ is now the Cartan matrix of $\hat{C}_N$ and $q_i = q$ for $i = 1, \ldots, N - 1$ (short roots) and $q_0 = q_N = q^2$ (long roots).

The proof is based on the strategy exposed in the previous section. First of all, let us define for $\mu = 1, \ldots, 2N - 1$

$$k_\mu(r) = n_\mu(r) - n_{\mu+1}(r)$$

$$f_\mu^+(r) = c_\mu^+(r)c_{\mu+1}(r)$$

$$f_\mu^-(r) = c_{\mu+1}(r)c_\mu(r) = \left(f_\mu^+(r)\right)^\dagger$$

where the disorder operators are now

$$K_\mu(r) = \tilde{K}_\mu^+(r) = q^{-\frac{1}{2}} \sum_{t \in \mathbb{Z}'} \epsilon^{(t-r)(n_\mu(t) - n_{2N-\mu+1}(t))}$$

$$K_\mu(r) = \tilde{K}_\mu^+(r) = q^{-\frac{1}{2}} \sum_{t \in \mathbb{Z}'} \epsilon^{(t-r)(n_\mu(t) - n_{2N-\mu+1}(t))}$$

for $\mu = 1, \ldots, N$

$$K_\mu(r) = \tilde{K}_\mu^+(r) = q^{-\frac{1}{2}} \sum_{t \in \mathbb{Z}'} \epsilon^{(t-r)(n_\mu(t) - n_{2N-\mu+1}(t))} + n_{2N-\mu+1}(r)$$

for $\mu = N + 1, \ldots, 2N$

(3.3)
and

\[ f_0^+(r) = c_{2N}^+(r)c_1(r + 1) \]  \hspace{1cm} (3.8d)
\[ f_0^-(r) = c_1^+(r + 1)c_{2N}(r) = \left( f_0^+(r) \right)^\dagger \]  \hspace{1cm} (3.8e)

As before, by using the definitions (3.4) of \( a, \tilde{a} \) and taking into account Eq. (3.3), the expressions (3.5), (3.6) and (3.7) simplify as (\( \alpha = 0, 1, \ldots, N \))

\[ E_{\alpha}^+(r) = e_{\alpha}^+(r) q_{\alpha}^+ \sum_{t \in \mathbb{Z}} \varepsilon(t-r) h_{\alpha}(r) \]  \hspace{1cm} (3.9)

where

\[ e_{i}^\pm (r) = f_{i}^\pm (r) q_{2N-i}^\pm + f_{2N-i}^\pm (r) q_{-2N+i}^\pm \]  \hspace{1cm} for \( i = 1, \ldots, N - 1 \)  \hspace{1cm} (3.10a)
\[ e_{N}^\pm (r) = f_{N}^\pm (r) \]  \hspace{1cm} (3.10b)
\[ c_{0}^\pm (r) = f_{0}^\pm (r) \]  \hspace{1cm} (3.10c)

The step 1 is achieved by noticing that for any fixed \( r \in \mathbb{Z}' \), the set \( \{ k_i(r), f_i^\pm (r); i = 1, \ldots, N - 1 \} \) and \( \{ k_{2N-i}(r), f_{2N-i}^\pm (r); i = 1, \ldots, N - 1 \} \) are representations of spin 0 and 1/2 of \( U_q(A_{N-1}) \); therefore the set \( \{ h_i(r), e_i^\pm (r); i = 1, \ldots, N - 1 \} \) is a representation of \( U_q(A_{N-1}) \) corresponding to the coproduct of these two representations. Analogously, \( \{ h_{N}(r), e_{N}^\pm (r) \} \) is a representation of \( A_1 \) of spin 0 and 1/2 and thus of \( U_q(A_1) \). Moreover, from the properties of the fermionic oscillators, one can directly check the quantum Serre relations and Eqs. (2.11c) involving \( h_{N}(r) \) and \( e_{N}^\pm (r) \) and thus prove that the set \( \{ h_{\rho}(r), e_{\rho}^\pm (r); \rho = 1, \ldots, N \} \) is a representation of \( U_q(C_N) \). Then Eqs. (3.4) and (3.3) show that also the set \( \{ H_{\rho}, E_{\rho}^\pm; \rho = 1, \ldots, N \} \) is a representation of \( U_q(C_N) \), obtained by iterated coproduct. \[ \Box \]

For the step 2, one has to check that the equations (2.11e, 2.11c) and (2.2) also hold when \( \alpha, \beta \) can take the value 0. Actually the only non trivial cases are for Eqs. (2.11c) and (2.2) when \( \alpha = 0, \beta = 1 \) or \( \alpha = 1, \beta = 0 \). Thus we have just to show that the set \( \{ H_{\alpha}, E_{\alpha}^\pm; \alpha = 0, 1 \} \) is a representation of \( U_q(C_2) \); as \( h_{0}(r) \) involves oscillators both of the site \( r \) and of the site \( r + 1 \), it is convenient to rewrite Eq. (3.4) rearranging the (convergent) series for \( H_{1} \) and \( E_{1}^\pm \):

\[ H_{1} = \sum_{r \in \mathbb{Z}'} H_{1}^r (r) \hspace{1cm} E_{1}^\pm = \sum_{r \in \mathbb{Z}'} E_{1}^r \pm (r) \]  \hspace{1cm} (3.11)

where

\[ H_{1}^r (r) = h_{1}^r (r) = k_{1}(r + 1) + k_{2N-1}(r) \]  \hspace{1cm} (3.12)

and

\[ E_{1}^r \pm = a_{1}^r (r + 1)a_{2}(r + 1) + a_{2N-1}^r (r)a_{2N}(r) \]  \hspace{1cm} (3.13)
\[ E_{1}^r \pm = \tilde{a}_{1}^r (r + 1)a_{1}(r + 1) + \tilde{a}_{2N}^r (r)a_{2N-1}(r) \]  \hspace{1cm} (3.14)

\[ \Box \]

Let us remark that this anyonic representation of \( U_q(C_N) \) does not require any Gutzwiller projection on the fermionic oscillators and thus it is an improvement of the one given in ref. [4].
Then using the definitions (3.2) of $a, \tilde{a}$ and taking into account Eq. (3.3), one gets

$$E^\pm_1(r) = e^\pm_1(r) q^\pm \sum_{t \in \mathbb{Z}} \varepsilon(t-r) h'_1(t)$$

(3.15)

where

$$e^\pm_1(r) = f_{2N-1}^\pm(r) q^{\pm k_1(r+1)} + f_1^\pm(r+1) q^{-\frac{1}{2} k_{2N-1}(r)}$$

(3.16)

Let us show now that the set $\{h_0(r), e_0^\pm(r), h'_1(r), e'_1(r)\}$ is a representation of $U_q(C_2)$. For a fixed $r \in \mathbb{Z}'$, the triple $\{h'_1(r), e'^\pm_1(r)\}$ is a representation of $U_q(A_1)$, corresponding to the coproduct of $\{k_{2N-1}(r), f_{2N-1}^\pm(r)\}$ and $\{k_1(r+1), f_1^\pm(r+1)\}$. Therefore one has $[e'^\pm_1(r), e'^\pm_1(r)] = [h'_1(r)]_q$. Using elementary properties of fermionic oscillators, one checks the equations $[e_0^+, e^-_1] = [e^-_0, e'^+_1] = 0$ and the quantum Serre relations on the generators $e^\pm_0, e^\pm_1$. Then from Eq. (3.3) for $\alpha = 0$ and Eq. (3.13), it follows that the set $\{H_0, E_0^+, H_1, E_1^+\}$ is a representation of $U_q(C_2)$ and thus Eqs. (2.1a-c) and the Serre relations (2.2) on the generators (3.4) also hold for $\alpha, \beta \in \{0, 1\}$, which concludes the proof. As in the $U_q(\tilde{A}_{N-1})$ case, the central charge is 1 because of the normal ordering prescription Eq. (2.9).

4 Conclusion

We have presented here a method to get an anyonic realization of the quantum affine Lie algebras $U_q(\tilde{A}_{N-1})$ and $U_q(C_N)$ with central charge $\gamma = 1$. Let us emphasize the role of the definition of the ordering of anyons which is crucial in this construction. Representations with vanishing central could be built in the same way using alternative ordering prescriptions. These representations with $\gamma = 0$ and $\gamma = 1$ can be combined together to get representations with arbitrary positive integer central charges which are in general reducible. This composition of representations, with the correct deformed coproduct, is naturally achieved by considering anyons defined on a two-dimensional lattice [4]. It is worthwhile to notice that these anyonic constructions have nothing to do with $q$-deformed oscillators. Finally, let us mention that it is also possible to obtain a supersymmetric version of the construction (see ref. [3] for the case $U_q(\tilde{A}(m, n))$).

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