Doubly-Robust Estimation for Unbiased Learning-to-Rank from Position-Biased Click Feedback

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ABSTRACT

Clicks on rankings suffer from position bias: generally items on lower ranks are less likely to be examined – and thus clicked – by users, in spite of their actual preferences between items. The prevalent approach to unbiased click-based Learning-to-Rank (LTR) is based on counterfactual Inverse-Propensity-Scoring (IPS) estimation. Unique about LTR is the fact that standard Doubly-Robust (DR) estimation – which combines IPS with regression predictions – is inapplicable since the treatment variable – indicating whether a user examined an item – cannot be observed in the data.

In this paper, we introduce a novel DR estimator that uses the expectation of treatment per rank instead. Our novel DR estimator has more robust unbiasedness conditions than the existing IPS approach, and in addition, provides enormous decreases in variance: our experimental results indicate it requires several orders of magnitude fewer datapoints to converge at optimal performance. For the unbiased LTR field, our DR estimator contributes both increases in state-of-the-art performance and the most robust theoretical guarantees of all known LTR estimators.

CCS CONCEPTS
• Information systems → Learning to rank.

KEYWORDS
Unbiased Learning to Rank; Counterfactual Learning; Position Bias

1 INTRODUCTION

The basis of recommender systems and search engines are ranking models that aim to provide users with rankings that meet their preferences or help in their search task [21]. The performance of a ranking model is vitally important to the quality of the user experience with a search or recommendation system. Accordingly, the field of Learning-to-Rank (LTR) concerns methods that optimize ranking models [21]; click-based LTR uses logged user interactions to supervise its optimization [14]. However, clicks are biased indicators of user preference [15, 30] because there are many factors beside user preference that influence click behavior. Most importantly, the rank at which an item is displayed is known to have an enormous effect on whether it will be clicked or not [6]. Generally, users do not consider all the items that are presented in a ranking, and instead, are more likely to examine items at the top of the ranking. Consequently, lower-ranked items are less likely to be clicked by users, regardless of whether users actually prefer these items [16]. Therefore, clicks can be more reflective of where an item was displayed during the gathering of data than whether users prefer it. This form of bias is referred to as position-bias [3, 6, 41]; it is extremely prevalent in user clicks on rankings. Correspondingly, this has lead to the introduction of unbiased LTR: methods for click-based optimization that mitigate the effects of position-bias. Wang et al. [40] and Joachims et al. [16] proposed using Inverse-Propensity-Scoring (IPS) estimators [11] to correct for position-bias. By treating the examination probabilities as propensities, IPS can estimate ranking metrics unbiasedly w.r.t. position-bias. This has lead to the inception of the unbiased LTR field, in which IPS estimation has remained the basis for all state-of-the-art methods [1, 2, 25, 26, 38]. However, variance is a large issue with IPS-based LTR and remains an obstacle for its adoption in real-world applications [26].

Outside of LTR, Doubly-Robust (DR) estimators are a widely used alternative for IPS estimation [17, 32], for instance, for optimization in contextual bandit problems [8]. The DR estimator combines an IPS estimate with the predictions of a regression model, such that it is unbiased when per treatment either: the estimated propensity or the regression model is accurate [17]. Additionally, the DR estimator can also bring large decreases in variance if the regression model is adequately accurate [8]. Unfortunately, existing DR estimators are not applicable to unbiased LTR, since the treatment variable – that indicates whether an item was examined or not – cannot be observed in the data. This is the characteristic problem of position-biased clicks: when an item is not clicked, we cannot determine whether the user chose not to interact or the user did not examine it in the first place. Consequently, the unbiased LTR field has not progressed beyond the usage of IPS estimation.

Our main contribution is the first DR estimator that is applicable to position-biased click data. Instead of using the actual treatment variable, which is unobservable in click data, our novel estimator uses the expectation of treatment per rank. Similar to DR estimators for other tasks, it combines the preference predictions of a regression model with IPS estimation. Unlike IPS estimators which are only unbiased with accurate knowledge of the logging policy, our DR estimator requires either the correct logging policy propensity or an accurate regression estimate per item. As a result, our DR estimator has less strict requirements for unbiasedness than IPS estimation. Moreover, it can also provide enormous decreases in variance compared to IPS: our experimental results indicate that the DR estimator requires several orders of magnitude fewer datapoints to converge at optimal performance. In all tested top-5 ranking scenarios, it needs less than $10^6$ logged interactions to reach the performance that IPS reaches at $10^8$ logged interactions. Additionally, when compared to other state-of-the-art methods DR also provides better performance across all tested scenarios. Therefore, the introduction of DR estimation for unbiased LTR contributes the first unbiased LTR estimator that is provenly more robust than IPS, while also improving state-of-the-art performance on benchmark unbiased LTR datasets.
2 RELATED WORK
Optimizing ranking models based on click-data is a well-established concept [14]. Early methods took an online dueling-bandit approach [35, 43] and later an online pairwise approach [23]. The first LTR method with theoretical guarantees of unbiasedness was introduced by Wang et al. [40] and then generalized by Joachims et al. [16]. They assume the probability that a user examines an item only depends on the rank at which it is displayed and that clicks only occur on examined items [6, 41]. Then using counterfactual IPS estimation they correct for the selection bias imposed by the examination probabilities. The introduction of this approach launched the unbiased LTR field: Agarwal et al. [1] expanded the approach for optimizing neural networks. Oosterhuis and de Rijke [25] generalized the approach to also correct for the item-selection-bias in top-k ranking settings by basing the propensities on a stochastic logging policy. Agarwal et al. [2] showed that user behavior shows an additional trust-bias: increased incorrect clicks at higher ranks [15], Vardasbi et al. [38] extended the IPS estimator with affine corrections to correct for this trust-bias. Singh and Joachims [36] use IPS to optimize for a fair distribution of exposure over items. Jagerman et al. [12] consider safe model deployments by bounding user preference w.r.t. item prediction on advertisements placements. However, as Section 4.4 shows an additional possibility that the logging policy is updated during the gathering of data. Wang et al. [39] proposed a Ratio-Propensity-Scoring (RPS) estimator that weights pairs of clicked and non-clicked items by their ratio between the propensities. IPS is an extension of IPS that introduces bias but also reduces variance. In contrast with the rest of the field, Ovaissi et al. [28] propose an adaptation of Heckman’s two-stage method. Besides this one exception and to the best of our knowledge, all methods in the unbiased LTR field are based on IPS.

Interestingly, methods for dealing with position-biased clicks have also been developed outside of the unbiased LTR field. Komiya et al. [18] and Lagrée et al. [19] propose bandit algorithms that use similar IPS estimators for serving ads in multiple on-screen positions at once. Furthermore, Li et al. [20] also propose IPS estimators for the unbiased click-based evaluation of ranking models. This further evidences the widespread usage of IPS estimation for correcting position-biased clicks.

Nevertheless, there is previous work on recommender systems that has applied DR estimators to position-biased clicks: Saito [33] proposed a DR estimator for post-click conversions that estimates how users treat an item after clicking it. Alternatively, Yuan et al. [42] introduced a top-k DR estimator for Click-Through-Rate (CTR) prediction on advertisements placements. However, as Section 4.4 will discuss, these DR estimators reduce to IPS in the standard LTR problem setting. Therefore, we conclude that – to the best of our knowledge – there is no known method that uses DR estimation to correct for position-bias in clicks.

3 PROBLEM DEFINITION
3.1 User Behavior Assumptions
This paper assumes that the probability of a click depends on the user preference w.r.t. item \( d \) and the position (also called rank) \( k \in \{1, 2, \ldots, K\} \) at which \( d \) is displayed. Let \( R_d \equiv P(R = 1 \mid d) \) be the probability that a user prefers item \( d \) and for each \( k \) let \( \alpha_k \in [0, 1] \) and \( \beta_k \in [0, 1] \) such that \( \alpha_k + \beta_k \in [0, 1] \), the probability of \( d \) receiving a click \( C \in \{0, 1\} \) when displayed at position \( k \) is:

\[
P(C = 1 \mid d, k) = \alpha_k P(R = 1 \mid d) + \beta_k = \alpha_k R_d + \beta_k.
\]

This assumption has been derived [38] from a more interpretable user-model proposed by Agarwal et al. [2]. Their model is based on the examination assumption [31]: users first examine an item before they interact with it, i.e. let \( O \in \{0, 1\} \) indicate examination then \( O = 0 \rightarrow C = 0 \). Additionally, they also incorporate the concept of trust bias: users are more likely to click against their preferences on higher ranks because of their trust in the ranking model. This can be modelled by having the probability of a click conditioned on examination vary over \( k \):

\[
\epsilon^+_k = P(C = 1 \mid R = 1, O = 1, k), \quad \epsilon^-_k = P(C = 1 \mid R = 0, O = 1, k).
\]

The proposed user model results in the following click probability:

\[
P(C = 1 \mid d, k) = P(O = 1 \mid k)(\epsilon^+_k R_d + \epsilon^-_k (1 - R_d)),
\]

by comparing Eq. 1 and 3 we see that:

\[
\alpha_k = P(O = 1 \mid k)(\epsilon^+_k - \epsilon^-_k), \quad \beta_k = P(O = 1 \mid k)\epsilon^-_k.
\]

Agarwal et al. [2] provide empirical results that suggest this user-model is more accurate than the previous model that ignores the trust-bias effect: \( \forall k, \beta_k = 0 \) [16, 40, 41]. Since the assumption in Eq. 1 is true in both models, our work is applicable to most settings in earlier unbiased LTR work [2, 12, 16, 25, 26, 28, 36, 38–40].

3.2 Definition of the LTR Goal
The goal of our ranking task is to maximize the probability that a user will click on something they prefer. Let \( \pi \) be the ranking policy to optimize, with \( \pi(k \mid d) \) indicating the probability that \( \pi \) places \( d \) at position \( k \) and let \( O \) indicate the size of the collection of items to be ranked. Most ranking metrics are a weighted sum of item relevances, where the weights \( \omega_k \) depend on the item positions:

\[
\mathcal{R}(\pi) = \mathbb{E}_{y \sim \pi} \left[ \sum_{k=1}^{K} \omega_k R_{y_k} \right] = \sum_{d \in D} \hat{R}_d \sum_{k=1}^{K} \pi(k \mid d) \omega_k.
\]

A traditional metric is Discounted Cumulative Gain (DCG) [13]:

\[
\hat{R}_d^{DCG} = \log_2(k + 1)^{-1},
\]

however, DCG has no clear interpretation in our assumed user model. In contrast, we argue that our metric should actually be motivated by our user behavior assumptions, accordingly, this work will use the weights: \( \omega_k = (\alpha_k + \beta_k) \). This choice results in an easily interpreted metric; for brevity of notation, we first introduce the expected position weight per item:

\[
\omega_d = \mathbb{E}_{y \sim \pi} [\alpha_k d(k) + \beta_k d(k)] = \sum_{k=1}^{K} \pi(k \mid d)(\alpha_k + \beta_k).
\]

using Eq. 1, our ranking metric can then be formulated as:

\[
\mathcal{R}(\pi) = \sum_{d \in D} \omega_d R_d = \mathbb{E}_{y \sim \pi} [P(C = 1, R = 1 \mid d, k)].
\]

This formulation clearly reveals that our chosen metric directly corresponds to the expected number of items that are both clicked and preferred in our assumed user behavior model.

Finally, we note that our main contributions work with any choice of weights \( \omega_k \) and are therefore equally applicable to most
We will investigate both the scenario where the clipping parameter prevents small values, and then inversely weights the result by the unbiased LTR work [16, 40], but the expected correlation between the click probability (position-bias and item-selection-bias [25]):

$$\hat{R}_{\text{IPS}}(\pi) = \frac{1}{N} \sum_{i=1}^{N} \sum_{d \in D} \hat{\gamma}_{d}(d) \cdot (c_i(d) - \hat{\beta}_{k_i(d)}),$$

(12)

where $k_i(d)$ indicates the position of $d$ in the $i$th ranking. In Appendix A, we prove that the IPS estimator is unbiased when both the bias parameters and the logging policy distribution are correctly estimated and clipping has no effect [16, 38]:

$$\hat{\alpha} = \alpha \land \hat{\beta} = \beta \land (\forall d \in D, \hat{\pi}_0(d) = \pi_0(d) \land \hat{\rho}_d \geq \tau)$$

$$\Rightarrow \mathbb{E}_{c, y \sim \pi_0} [\hat{R}_{\text{IPS}}(\pi)] = \mathbb{R}(\pi).$$

Conversely, IPS is biased when clipping does have an effect, even if the bias parameters and logging policy distribution are correctly estimated. Appendix A shows that clipping can provide large variance reductions by preventing small $\hat{\rho}_d$ values [16, 37]. Importantly, the reduction in variance is often much greater than the increase in bias, making this an attractive trade-off that has been widely adopted by the unbiased LTR field [1, 25]. There is currently no known method for variance reduction in IPS-based position-bias correction that does not introduce bias, and thus, in practice unbiased LTR methods are actually often applied in a biased manner.\(^1\)

4.2 The Direct Method

Whereas IPS estimation solely relies on click frequencies and propensities, one can also estimate performance purely based on regression estimates. This approach is known as the Direct-Method (DM) and has been applied successfully to contextual bandits in the past [8]. DM uses a regression model $M$ to predict the rewards of actions directly, instead of estimating these values from the logged click data (as IPS does). For unbiased LTR, $M$ should predict the probability of preference for each item, let $\hat{R}_d$ be the estimate for item $d$: $M(d) = \hat{R}_d \approx R_d$. The DM estimate of $R_\pi$ is:

$$\hat{R}_{\text{DM}}(\pi) = \sum_{d \in D} \hat{\omega}_d \hat{R}_d.$$  

(14)

By considering Eq. 7, 11 and 14, we can clearly see the following condition for unbiasedness:

$$\hat{\alpha} = \alpha \land \hat{\beta} = \beta \land (\forall d \in D, \hat{R}_d = R_d) \Rightarrow \hat{R}_{\text{DM}}(\pi) = \mathbb{R}(\pi).$$

(15)

In other words, both the bias parameters and the regression model have to be accurate for $\hat{R}_{\text{DM}}(\pi)$ to be unbiased. Accordingly, it seems practically infeasible for DM to be unbiased because finding an accurate $M$ appears to be as difficult as the ranking task itself. This reasoning could explain why – to the best of our knowledge – no existing work has applied DM to unbiased LTR.

An advantage of DM over IPS is how non-clicked items are treated: The IPS estimator (Eq. 12) treats items that are not clicked in $D$ as completely irrelevant items. As pointed out in previous work [39], this seems very unfair to items that were never displayed during logging, and this winner-takes-all behavior could potentially explain the high variance of IPS. In contrast, because DM relies on regression estimates it can provide non-zero values to all items, even those never displayed. But DM does not utilize any of the click-data, and thus, DM cannot correct for inaccuracies in the regression estimates, one can also estimate performance purely based on regression estimates.

\(^1\)Oosterhuis and de Rijke [26] show that applying unbiased LTR methods in an online fashion also reduces variance, but this solution does not apply to our problem definition.
estimates. Furthermore, the unbiasedness criteria for DM are much less feasible than those of IPS. Ideally, the advantageous properties of both IPS and DM should be combined in a single estimator, while avoiding the downsides of each approach.

### 4.3 Existing Cross-Entropy Loss Estimation

As discussed, DM requires an accurate regression model to be unbiased or effective. In the ideal situation, one may optimize a regression model using the cross-entropy loss:

$$
\mathcal{L}(\hat{R}) = - \sum_{d \in D} R_d \log(\hat{R}_d) + (1 - R_d) \log(1 - \hat{R}_d).
$$

(16)

However, this loss cannot be computed from the click-data since $\hat{R}_d$ cannot be observed. Luckily, Bekker et al. [4] have introduced an estimator that can be applied to position-biased clicks:

$$
\hat{\mathcal{L}}'(\hat{R}) = -\frac{1}{N} \sum_{i=1}^{N} \sum_{d \in D} \left( \frac{c_i(d)}{\hat{\rho}_d} \log(\hat{R}_d) + \left(1 - \frac{c_i(d)}{\hat{\rho}_d}\right) \log(1 - \hat{R}_d) \right).
$$

(17)

Saito et al. [34] showed that this estimator is effective for recommendation tasks on position-biased click data. $\hat{\mathcal{L}}'$ is unbiased [4, 34] when there is no trust-bias; propensities are accurate and clipping has no effect: $(\forall k, \tilde{x}_k = 0 \land \tilde{a} = a \land (\forall d \in D, \tilde{R}_d = \pi_0(d) \land \tilde{\rho}_d \geq \tau)) \rightarrow \mathbb{E}_{c,y,\pi_0}[\hat{\mathcal{L}}'(\hat{R})] = \mathcal{L}(\hat{R})$.

### 4.4 Existing Doubly-Robust Estimation

Our discussion of IPS and DM concluded with the wish to combine the advantageous properties of both approaches. Doubly-Robust (DR) estimation for statistical inference [17, 32] seems a very promising technique for this desire, especially considering its successful application to contextual bandits [8]. Unfortunately, at first glance, applying DR to unbiased LTR seems impossible because DR estimators use the treatment variable to balance regression estimates and IPS estimation. However, in the LTR setting, the treatment is the examination of the user and we cannot directly observe whether a user has examined a non-clicked item or not. This characteristic problem with position-biased clicks makes standard doubly-robust estimators inapplicable.

Previous work has found two methods to circumvent this issue. Firstly, Saito [33] realized that when estimating post-click conversions, the click signal can be seen as the treatment variable. Therefore, DR estimation can be applied to this task and appears to perform better than IPS [33]. Nevertheless, when this method is applied to predicting click probabilities it reduces to IPS, i.e. if one ignores the post-click conversions and treats every click as having the same value, this estimator reduces to an IPS estimator. Therefore, we conclude that this estimator does not use DR estimation to correct for position-bias. Secondly, Yuan et al. [42] realized that while examination cannot be observed in click-data, it is possible to observe whether an item was displayed inside or outside of a top-$k$. If items outside of a top-$k$ are certainly unobserved, i.e. because they were not displayed to users, then this can be used as a treatment variable. Yuan et al. introduce a DR estimator for clicks in top-$k$ recommendation:

$$
\tilde{\mathcal{R}}_{\text{top-k DR}}(\pi) = \tilde{\mathcal{R}}_{\text{DM}}(\pi) + \frac{1}{N} \sum_{i=1}^{N} \sum_{d \in D} \frac{\hat{\omega}_d}{\hat{\rho}_d} \left( c_i(d) - \hat{\beta}_{k_i(d)} \hat{R}_d - \hat{\alpha}_{k_i(d)} \hat{R}_d \right).
$$

(20)

It uses the regression estimate $\tilde{\mathcal{R}}_{\text{DM}}(\pi)$ as a starting point and adds an IPS estimate of the difference between $\mathcal{R}_\pi$ and $\tilde{\mathcal{R}}_{\text{DM}}(\pi)$ to it [32]. At each interaction, a negative penalty of $\hat{R}_d$ weighted by $\hat{\alpha}_{k_i(d)} / \hat{\rho}_d$ is subtracted. While the sum of penalties remove the value added by $\tilde{\mathcal{R}}_{\text{DM}}(\pi)$ in expectation, the actual value of this sum depends on the correlation between clicks and relevance in the logged data. For instance, an item $d$ that was never displayed to the user will have: $\forall i, \hat{\alpha}_{k_i(d)} = 0$, and therefore, no negative penalties are applied and
the DR estimate for this item will be equal to \( \hat{\mathcal{R}}_d \). Conversely, if clicks on an item \( d' \) are correlated with relevance to a degree of \( \rho_d \), i.e. \( \frac{1}{N} \sum_{i=1}^{N} \delta_{x_i(d')} = \hat{\rho}_d \), then the DR estimate will reduce to an IPS estimate because the sum of penalties is equal to \( \hat{\mathcal{R}}_d \). For items with a degree of correlation in between these examples, the DR estimates will be in between the regression estimates \( \hat{\mathcal{R}}_d \) and IPS. This process allows our DR to unbiasedly utilize both the estimates from a regression model and the preferences that can be inferred from the observed interaction data.

To the best of our knowledge, our proposed estimator is the first DR estimator that uses a soft-treatment variable. The main difference with the estimator for contextual bandits described by Dudík et al. [8] is that the variable indicating which action was taken is replaced with the rank correlation variable \( \hat{\alpha}_{x_i(d)} \). Despite the apparent simplicity of this difference, it makes a very significant contribution to the unbiased LTR field because it enables the first estimator to combine IPS and regression estimates while being unbiased w.r.t. position-bias. Section 7 will empirically show the performance improvements this contribution brings, whereas Appendix B will prove several theoretical advantages DR has over both IPS and DM in terms of bias and variance. We will summarize our main theoretical findings in the remainder of this section.

Our novel DR estimator has broader unbiasedness criteria than both DM and IPS, Theorem B.4 proves the following:

\[
\hat{\alpha} = \alpha \land \hat{\beta} = \beta \land (\forall d \in D, (\hat{\pi}_d(d) = \pi_0(d) \land \hat{\rho}_d \geq \tau) \lor \hat{\mathcal{R}}_d = \mathcal{R}_d) \rightarrow \mathbb{E}_{y,\pi_0}[\hat{\mathcal{R}}_{\text{DR}}(\pi)] = \mathbb{E}_{\mathcal{R}}[\mathcal{R}].
\] (21)

In other words, DR is unbiased when the bias parameters are correctly estimated and per item either the logging policy distribution is correctly estimated and clipping has no effect or the regression estimate is correct. In contrast, remember that IPS needs an accurate \( \hat{\pi}_d(d) \) and clipping to have no effects for all items, and DM needs accurate regression estimates for all items. Therefore, DR is unbiased when either IPS or DM is but can also be unbiased in situations where neither is. Moreover, Theorem B.5 proves our DR estimator has less or equal bias than IPS when \( \hat{\alpha} \) and \( \hat{\beta} \) are accurate and each \( \mathcal{R}_d \) estimate is less than twice the true \( \mathcal{R}_d \):

\[
\left\{ \begin{array}{l}
\hat{\alpha} = \alpha \land \hat{\beta} = \beta \land (\forall d \in D, 0 \leq \hat{\mathcal{R}}_d \leq 2\mathcal{R}_d) \\
\rightarrow |\mathbb{E}_{\mathcal{R}}(\pi) - \hat{\mathcal{R}}_{\text{DR}}(\pi)| \leq |\mathbb{E}_{\mathcal{R}}(\pi) - \mathbb{E}_{\mathcal{R}}[\mathcal{R}(\pi)]|. 
\end{array} \right.
\] (22)

Theorem B.9 proves that under these conditions DR also has less variance than IPS:

\[
\left\{ \begin{array}{l}
\hat{\alpha} = \alpha \land \hat{\beta} = \beta \land (\forall d \in D, 0 \leq \hat{\mathcal{R}}_d \leq 2\mathcal{R}_d) \\
\rightarrow \mathbb{V}[^{\hat{\mathcal{R}}_{\text{DR}}}(\pi)] \leq \mathbb{V}[^{\hat{\mathcal{R}}_{\text{IPS}}}(\pi)]. 
\end{array} \right.
\] (23)

To summarize, in terms of bias and variance, our DR estimator is more robust than both the IPS and the DM estimators: when either of these estimators is unbiased the DR estimator is also unbiased, and in addition, there exist cases where our DR estimator is unbiased and neither IPS nor DM are. Moreover, when the bias parameters are accurate and all regression estimates are between zero and twice the true preferences, we can prove that both the bias and variance of DR are less or equal to those of IPS.

Finally, we note that there is an important exception: our DR estimator is equivalent to IPS when all \( \hat{\alpha}_{x_i(d)} \) are equal to their corresponding \( \rho_d \):

\[
(\forall i, \forall d \in D, \hat{\alpha}_{x_i(d)} = \rho_d) \rightarrow \mathbb{E}_{y,\pi_0}[\hat{\mathcal{R}}_{\text{DR}}(\pi)] = \mathbb{E}_{\mathcal{R}}[\mathcal{R}_0(\pi)].
\] (24)

This situation can only occur when the logging policy \( \pi_0 \) is deterministic and clipping has no effect: \( \forall d \in D, \rho_d \geq \tau \), in all other scenarios, our DR estimator does not reduce to IPS estimation.2

In terms of theory, our novel DR estimator is a breakthrough for the unbiased LTR field: it is the first unbiased estimator that uses DR estimation to correct for position-bias, importantly, this makes it provenly more robust than IPS estimation in terms of both bias and variance. Our DR estimator is applicable in any unbiased LTR setting where IPS can be applied and works with any regression estimates, allowing for widespread adoption across the entire field.

### 5.1 Novel Cross-Entropy Loss Estimation

While our DR estimator can be applied with any regression model, we will propose a novel estimator for the cross-entropy loss to optimize an accurate regression model. There are two issues with the existing \( \hat{L}_t \) estimator (Eq. 17) we wish to avoid: (i) \( \hat{L}_t \) does not correct for trust-bias, and (ii) for any never-displayed item \( i \) the \( \hat{L}_t \) estimate contains the log(1 - \( \hat{\mathcal{R}}_d \)) loss that pushes \( \hat{\mathcal{R}}_d \) towards zero. In other words, \( \hat{L}_t \) penalizes positive \( \hat{\mathcal{R}}_d \) values for items that were never displayed during logging, while it seems more intuitive that a loss estimate should be indifferent to the \( \hat{\mathcal{R}}_d \) values of never-displayed items. We propose the following estimator:

\[
\hat{L}(\hat{\mathcal{R}}) = \frac{1}{N} \sum_{i=1}^{N} \sum_{d \in D} \frac{1}{\hat{\rho}_d} \left( (c_i(d) - \hat{\beta}_{k_i(d)} \log(\hat{\mathcal{R}}_d) + (\hat{\alpha}_{k_i(d)} - c_i(d) + \hat{\beta}_{k_i(d)}) \log(1 - \hat{\mathcal{R}}_d)) \right)
\] (25)

Our novel estimator has \( \hat{\beta} \) corrections to deal with trust-bias and utilizes the \( \hat{\alpha}_{k_i(d)} \) to weight the negative part of the loss: log(1 - \( \hat{\mathcal{R}}_d \)). These weights behave similar to those in the DR estimator (Eq. 20): they can replace the 1 in Eq. 17 because \( \mathbb{E}_{y,\pi_0}[\hat{\alpha}_{k_i(d)}/\hat{\rho}_d] = 1 \) and when an item \( d \) is never displayed the corresponding \( \hat{\mathcal{R}}_d \) does not affect the estimate since in that case: \( \sum_{i=1}^{N} \hat{\alpha}_{k_i(d)} = 0 \). Appendix C proves \( \hat{L} \) is unbiased in the following circumstances:

\[
(\hat{\alpha} = \alpha \land \hat{\beta} = \beta \land (\forall d \in D, \hat{\pi}_d(d) = \pi_0(d) \land \hat{\rho}_d \geq \tau) \lor \hat{\mathcal{R}}_d = \mathcal{R}_d) \rightarrow \mathbb{E}_{y,\pi_0}[\hat{L}(\hat{\mathcal{R}})] = \mathbb{E}_{\mathcal{R}}[\mathcal{L}(\mathcal{R})].
\] (26)

These are the same conditions as those we proved for the IPS estimator (Eq. 13): the bias parameters and the logging policy need to be accurately estimated and clipping should have no effect.

### 6 EXPERIMENTAL SETUP

In order to evaluate our novel DR estimator, we apply the semi-synthetic setup that is common in unbiased LTR [10, 16, 23, 24, 28, 38, 44]. This simulates a web-search scenario by sampling queries and documents from commercial search datasets, while user interactions and rankings are simulated using probabilistic click models. We use the three largest publicly-available LTR industry datasets: Yahoo! WebScope [5], MSLR-WEBS0K [29] and Istella [7]. Each dataset contains queries, preselected documents per query and for the query-document pairs: feature representations and labels indicating expert-judged relevance, with label(\( d \)) ∈ {0, 1, 2, 3, 4}.

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1We acknowledge that there is also a very trivial exception when \( \forall d \in D, \hat{\mathcal{R}}_d = 0 \).

2We acknowledge that there is also a very trivial exception when \( \forall d \in D, \hat{\mathcal{R}}_d = 0 \).
Figure 1: Optimization performance reached on three datasets. Top row: top-5 setting with known $\alpha$ and $\beta$ bias parameters; middle row: top-5 setting with estimated $\hat{\alpha}$ and $\hat{\beta}$; bottom row: full-ranking setting (no cutoff) with known $\alpha$ and $\beta$ bias parameters. Results are means over 20 independent runs, shaded areas indicate the 90% confidence intervals; y-axis: policy performance $R_\pi$ (Eq. 7) on the held-out test-set; x-axis: $N$ the number of displayed rankings in the simulated training set.
we use $P(R = 1 \mid d) = 0.25 \cdot \text{label}(d)$. The queries in the datasets are divided into training, validation and test partitions. Our logging policy is obtained by supervised training on 1% of the training partition [16]. At each interaction $i$, a query is sampled uniformly over the training and validation partitions and a corresponding ranking is sampled from the logging policy. Clicks are simulated using the click model in Eq. 1. We simulate both a top-5 setting, where only five items can be displayed at once, and a full-ranking setting where all items are displayed simultaneously. The parameters for the top-5 setting are based on empirical work by Agarwal et al. [2]: $\alpha^{\text{top-5}} = [0.35, 0.53, 0.55, 0.54, 0.52, 0.0, \ldots]$ and $\beta^{\text{top-5}} = [0.65, 0.26, 0.15, 0.11, 0.08, 0.0, \ldots]$. For the full-ranking setting we use Eq. 4 with: $P(O = 1 \mid k) = (1 + (k - 1)/5)^{-2}$, $\epsilon^+ = 1$, and $\epsilon^- = 0.1 + 0.6/(1 + k/20)$. We simulate both top-5 settings where $\alpha$ and $\beta$ are known and where they are estimated with Expectation-Maximization (EM) [2, 38]. All models are neural networks with two 32-unit hidden layers, applied in Plackett-Luce Ranking models optimized using policy gradients [22]. The only exception is the logging policy in the full-ranking settings which is a determinstic ranker to better match earlier work [1, 16, 38]. Propensities $\rho_d$ use frequentist estimates of the logging policy: $\rho_d(k \mid d) = \sum_{i=1}^{N} I[k_i(d) = k]$, we clip with $z^{\text{top-5}} = 10/\sqrt{N}$ in the top-5 setting and $z^{\text{full}} = 100/\sqrt{N}$ in the full-ranking setting. Early stopping is applied using counterfactual estimates based on clicks on the validation set.

Our results evaluate the DM estimator (Eq. 14) and our DR estimator (Eq. 14) both using the estimates of a regression model optimized by our $\hat{L}$ loss (Eq. 25). Their performance is compared with the following baselines: (i) a naive estimator that ignores bias (Eq. 12 with $\tau = 1$); (ii) IPS (Eq. 12); (iii) Ratio-Propensity-Scoring (RPS) [39]; (iv) DM optimized with $\hat{L}$ (Eq. 14 & 17) from previous work [4, 34]; and (v) top-k DR (Eq. 18) from previous work [42] with regression estimates from the DM of previous work. The estimators receive no information about queries that were not sampled in the training data. To estimate optimal performance, we also optimize a model based on the true labels (full-information).

As an example for clarity, the following procedure is used to evaluate the performance of our DR estimator at $N$ displayed rankings in the top-5 setting with estimated bias parameters: (i) $N$ queries are sampled with replacement from the training and validation partitions, a displayed ranking is generated for each sampled query using the stochastic logging policy. (ii) Clicks on each ranking are simulated using the click model in Eq. 1 and the true $\alpha$ and $\beta$ parameters. (iii) EM is applied to the simulated click data to obtain estimated $\hat{\alpha}$ and $\hat{\beta}$ parameters. (iv) A regression model is optimized using $\hat{L}$, $\hat{\alpha}$, $\hat{\beta}$ and the click data simulated on the training set, $\hat{R}_d$ is calculated for each item. (v) A ranking model is optimized to maximize the DR estimate of its performance, using $\hat{\alpha}$, $\hat{\beta}$, $\hat{R}_d$ and the training click data, early stopping criteria are estimated with the validation click data. (vi) Finally, the true performance $X_F$ of the resulting ranking model is computed on the test-set and added to our results. We repeat each procedure twenty times independently and report the mean results in addition to 90% confidence intervals.

7 RESULTS
Our experimental results are displayed in Figure 1, each plot displays the performance reached using different estimators on varying amounts of simulated interaction data. The figure is split in three rows, each indicating the results of one of the three simulated settings. The displayed results are means over twenty independent runs, 90% confidence intervals are visualized in order to recognize statistically meaningful differences.

We see that in both top-5 settings IPS is unable to reach optimal performance when $N \leq 10^4$ on any of the datasets, an observation also made in previous work [26]. In the top-5 setting with known bias parameters, IPS is theoretically proven to be unbiased and will converge at optimal performance as $N \to \infty$. Consequently, we can conclude that it is high variance which prevents us from observing IPS’s convergence in the top-row of Figure 1.3 This observation illustrates the importance of variance reduction: it is not bias but high variance that prevents IPS from reaching optimal performance with feasible amounts of interaction data. In contrast with the two top-rows, the bottom row shows that IPS can reach optimal performance on all datasets in the full-ranking setting. A likely explanation is that interactions on a complete ranking provide much more information than when only the top-5 can be interacted with. Potentially, the item-selection-bias in the top-5 settings greatly increase the variance of IPS due to the winner-takes-all behavior described in Section 4.2.

Both our DM and DR estimators clearly provide large improvements over IPS: in all settings they outperform IPS when $N \approx 10^4$ and only in the full-ranking setting does IPS catch up around $N \approx 10^6$. Overall DM has lower performance than DR, but in some cases it has comparable or equal performance. On Istella in both top-5 settings the performance difference between our DR and DM estimators is very large; here the performance reached by our DM when $N = 10^8$ is reached by our DR when $N \approx 10^7$. Notably, DM appears to converge on suboptimal performance in the full-ranking setting on Yahoo! and in both top-5 settings on MSLR, indicating that its unbiasedness criteria (Eq. 15) are not met in these situations. In stark contrast, our DR estimator reaches optimal performance in all tested scenarios, which corresponds with its much more robust unbiasedness criteria. Across all settings our DR estimator is the first to converge at optimal performance and provides the highest performance when $N > 10^4$. When compared to IPS in the top-5 settings, we see that the performance of DR when $N \approx 10^6$ is not reached by IPS when $N = 10^7$. This indicates that DR provides an increase in data-efficiency over IPS of a factor greater than 1,000 in the top-5 settings. DM also provides a large increase over IPS albeit smaller than that of DR; our results suggest that the usage of regression estimates provides the greatest reduction in variance. Nevertheless, by utilizing both regression estimates and the available click data our DR estimator clearly provides the most robust and highest performance across all tested settings and datasets.

Despite the large aforementioned advantages, we note that a downside of our DM and DR estimators is that they provide low performance in some settings when $N \leq 10^4$. While this could probably be remedied with safe deployment strategies [12, 27], it seems doubtful to us that in a real-world setting such little data is available that $N \leq 10^4$. Nonetheless, our results show that DM and DR are less resilient to tiny amounts of training data than IPS.

\footnote{Our clipping strategy has no effect in our experimental setting when $N = 10^6$.}
Finally, Figure 1 also displays the performance of the other baseline methods: the naive estimator, RPS and DM & DR from previous work. Unsurprisingly, they are all unable to converge on optimal performance in all settings. This can be explained by the fact that none of these estimators are unbiased in our settings due to trust-bias. The effect of trust-bias appears particularly large in the full-ranking setting, where none of these baselines are able to substantially improve performance over the logging policy (the DR baseline is not even applicable in this setting). Nevertheless, these baselines show that often a decrease in variance can be favourable over unbiasedness, as some of them provide better performance than the unbiased IPS estimator in the top-5 settings on Yahoo! and MSLR. Regardless, due to their bias, they are unable to combine optimal convergence and low variance. It appears that our DR estimator is the only method that is able to combine these properties.

8 CONCLUSION

This paper has introduced the first unbiased estimator that utilizes DR estimation to correct for position-bias in click feedback. Our estimator differs from existing DR estimators by using the expected correlation between clicks and preference per rank, instead of the unobservable examination variable. Additionally, we also proposed a novel cross-entropy loss estimator. In terms of theory, this work has contributed the most robust estimator for LTR yet: our DR estimator is the only method that corrects for position-bias, trust-bias and item-selection bias and has less strict unbiasedness criteria than the prevalent IPS approach. Moreover, our experimental results show that it can provide enormous increases in data-efficiency compared to IPS and better overall performance w.r.t. previous state-of-the-art approaches. Therefore, both our theoretical and empirical results indicate that our DR estimator is the most reliable and effective way to correct for position-bias. Consequently, we think there is large potential in replacing IPS with DR as the new basis for the unbiased LTR field.

Code Resources and Data

To facilitate the reproducibility of the reported results, this work only made use of publicly available data and our experimental implementation is publicly at https://github.com/HarrieO2022-doubly-robust-LTR.

APPENDICES

A BIAS AND VARIANCE OF IPS

**Theorem A.1.** The IPS estimator (Eq. 12) has the following bias:

\[
\begin{align*}
E_{c,y-\tau_0}[\hat{R}_{\text{IPS}}(\pi)] - \mathbb{R}(\pi) &= \sum_{d\in D} \hat{\omega}_d \left( \hat{R}_d - \hat{R}_d \right) + \sum_{d\in D} \hat{\omega}_d \left( \hat{\beta}_d - \hat{\beta}_d \right) \\
&= \sum_{d\in D} \hat{\omega}_d \left( \hat{R}_d - \hat{R}_d \right) + \sum_{d\in D} \hat{\omega}_d \left( \hat{\beta}_d - \hat{\beta}_d \right). 
\end{align*}
\]

**Proof.** Using Eq. 1, 7, 10 and 12 we get the following derivation:

\[
\begin{align*}
E_{c,y-\tau_0}[\hat{R}_{\text{IPS}}(\pi)] &= \sum_{d\in D} \hat{\omega}_d \left( \hat{R}_d - \hat{R}_d \right) + \sum_{d\in D} \hat{\omega}_d \left( \hat{\beta}_d - \hat{\beta}_d \right) \\
&= \mathbb{R}(\pi) + \sum_{d\in D} \hat{\omega}_d \left( \hat{R}_d - \hat{R}_d \right) + \sum_{d\in D} \hat{\omega}_d \left( \hat{\beta}_d - \hat{\beta}_d \right). 
\end{align*}
\]

**Lemma A.2.** By the definitions of \( c \) (Eq. 6) and \( \hat{c} \) (Eq. 11):

\[
(\hat{a} = a \land \hat{b} = b) \rightarrow (\forall d \in D, \hat{\omega}_d = \omega_d).
\]

**Lemma A.3.** By the definitions of \( \hat{a} \) (Eq. 10) and \( \hat{b} \) (Eq. 11):

\[
(\hat{a} = a \land (\forall d \in D, \hat{\pi}_0(d) = \pi_0(d) \land \rho_d \geq \tau)) \rightarrow (\forall d \in D, \hat{\rho}_d = \rho_d).
\]

**Lemma A.4.** Trivially, if the \( \hat{c} \) bias parameters are correct then:

\[
\hat{\beta} = \beta \rightarrow (\forall d \in D, E_{y-\tau_0}[\hat{R}_d] = E_{y-\tau_0}[\beta_d]).
\]

**Corollary A.5.** When \( \hat{a} \) and \( \hat{b} \) are correct IPS has the bias:

\[
(\hat{a} = a \land \hat{b} = b) \rightarrow \mathbb{E}_{c,y-\tau_0}[\hat{R}_{\text{IPS}}(\pi)] - \mathbb{R}(\pi) = \sum_{d\in D} \hat{\omega}_d (\rho_d - \hat{\rho}_d) R_d.
\]

**Proof.** Follows from Theorem A.1 and Lemmas A.2 and A.4.

**Theorem A.6.** The IPS estimator (Eq. 12) is unbiased when \( \hat{a} \) and \( \hat{b} \) are correctly estimated and clipping has no effect:

\[
(\hat{a} = a \land \hat{b} = b \land (\forall d \in D, \hat{\pi}_0(d) = \pi_0(d) \land \rho_d \geq \tau)) \rightarrow E_{c,y-\tau_0}[\hat{R}_{\text{IPS}}(\pi)] = \mathbb{R}(\pi).
\]

**Proof.** Follows from applying Lemma A.3 to Corollary A.5.

**Theorem A.7.** The IPS estimator (Eq. 12) has the variance:

\[
\mathbb{V}[\hat{R}_{\text{IPS}}(\pi)] = \frac{1}{N} \sum_{d\in D} \hat{\omega}_d^2 (\mathbb{V}[c(d)] + \mathbb{V}[\hat{R}_d] - 2 \mathbb{Cov}[c(d), \hat{R}_d])
\]

**Proof.** Follows from Eq. 1 and 12.
Applying Lemma A.3 to Eq. 38 provides proof for Theorem B.4.

**Theorem B.4.** The DR estimator (Eq. 20) is unbiased if the \(\hat{a}\) and \(\hat{b}\) bias parameters are correctly estimated and per item either \(\pi_0(d)\) is correct and clipping has no effect or \(\hat{R}_d\) is correct:

\[
\hat{a} = \alpha \land \hat{b} = \beta \land (\forall d \in D, (\hat{R}_0(d) = \pi_0(d) \land \rho_d \geq \tau) \lor \hat{R}_d = R_d)
\]

\(\implies E_{c,y^n} [\hat{R}_{DR}(\pi)] = R_{\pi}.\) (37)

**Proof.** From Corollary B.3 it clearly follows that the DR estimator is unbiased when the \(\hat{a}\) and \(\hat{b}\) bias parameters are correct and per item \(d\) either \(\hat{R}_d\) or \(R_d\) is correct:

\[
\begin{align*}
\hat{a} &= \alpha \land \hat{b} = \beta \land (\forall d \in D, \hat{R}_d = R_d \lor \hat{R}_d = R_d) \\
&\implies E_{c,y^n} [\hat{R}_{DR}(\pi)] = R_{\pi}.\) (38)
\end{align*}
\]

Applying Lemma A.3 to Eq. 38 provides proof for Theorem B.4. □

**Theorem B.5.** If \(\hat{a}\) and \(\hat{b}\) are correct and the regression model predicts each preference \(\hat{R}_d\) between 0 and twice the true \(\hat{R}_d\) value then the bias of the DR estimator (Eq. 20) is less or equal to that of the IPS estimator (Eq. 12):

\[
\hat{a} = \alpha \land \hat{b} = \beta \land (\forall d \in D, 0 \leq \hat{R}_d \leq 2R_d) \\
\implies |R(\pi) - \hat{R}_{DR}(\pi)| \leq |R(\pi) - R_{IPS}(\pi)|.\) (39)

**Proof.** This follows from comparing Corollary A.5 with B.3. □

**Theorem B.6.** The DR estimator (Eq. 20) has the variance:

\[
\begin{align*}
V[\hat{R}_{DR}(\pi)] &= \frac{1}{N} \sum_{d \in D} \frac{\hat{a}_d^2}{R_d^2} V(c(d)) + V[\hat{R}_k(d)] \\
&+ \hat{R}_d^2 \cdot V[\hat{a}_k(d)] - 2(Cov(c(d), \hat{R}_k(d))) \leq \hat{R}_d \cdot Cov(c(d), \hat{a}_k(d)).
\end{align*}
\]

This follows from Eqs. 1 and 20. □

**Lemma B.7.** The covariance between clicks on an item \(c(d)\) and \(\hat{a}_k(d)\) is (for brevity we use \(E\) to denote \(E_{c,y^n}\))

\[
Cov(c(d), \hat{a}_k(d)) = R_d V[\hat{a}_k(d)] + Cov(\hat{a}_k(d), \hat{R}_k(d)).\) (41)

**Proof.**

\[
Cov(c(d), \hat{a}_k(d)) = E\left[\left(c(d) - E[c(d)]\right)\left(\hat{a}_k(d) - E[\hat{a}_k(d)]\right)\right]
= E\left[\left(c(d) - E[\hat{a}_k(d)]\right)\left(\hat{R}_d - E[\hat{R}_k(d)]\right)\right] \leq E\left[\left(\hat{a}_k(d) - E[\hat{a}_k(d)]\right)\right] \leq \hat{R}_d V[\hat{a}_k(d)] + Cov(\hat{a}_k(d), \hat{R}_k(d)).\]

\[
\begin{align*}
\text{Corollary B.8.} & \text{ If } \hat{a} \text{ and } \hat{b} \text{ are correct then the variance of the DR estimator (Eq. 20) is:}
\hat{a} &= \alpha \land \hat{b} = \beta \land (\forall d \in D, 0 \leq \hat{R}_d \leq 2R_d) \\
&\implies V[\hat{R}_{DR}(\pi)] = \frac{1}{N} \sum_{d \in D} \frac{\hat{a}_d^2}{R_d^2} V[c(d)] \leq \hat{R}_d^2 \cdot 2R_d R_d.
\end{align*}
\]

**Proof.** In Theorem B.9 replace \(\hat{a}\) and \(\hat{b}\) with \(a\) and \(b\) respectively and then use Lemma B.7 to replace \(Cov(c(d), \hat{a}_k(d))\). □

**Theorem B.9.** If \(\hat{a}\) and \(\hat{b}\) are correct and the regression model predicts each preference \(\hat{R}_d\) between 0 and twice the true \(\hat{R}_d\) value then the variance of the DR estimator (Eq. 20) is less or equal to that of the IPS estimator (Eq. 12):

\[
\hat{a} = \alpha \land \hat{b} = \beta \land (\forall d \in D, 0 \leq \hat{R}_d \leq 2R_d) \\
\implies V[\hat{R}_{DR}(\pi)] \leq V[R_{IPS}(\pi)].\) (43)

**Proof.** Comparing Corollary A.5 and B.8 reveals that:

\[
\hat{a} = \alpha \land \hat{b} = \beta \land (\forall d \in D, \hat{R}_d^2 - 2R_d R_d \leq 0) \\
\implies V[\hat{R}_{DR}(\pi)] \leq V[R_{IPS}(\pi)].\) (44)

For a single \(\hat{R}_d\) the following holds:

\[
0 \leq \hat{R}_d \leq 2R_d \implies \hat{R}_d^2 - 2R_d R_d \leq 0.\) (45)

Theorem B.9 follows directly from Eqs. 44 and 45. □

**C. BIASES OF THE CROSS-ENTROPY ESTIMATOR**

**Theorem C.1.** The \(\hat{L}\) estimator (Eq. 25) has the following bias:

\[
E_{c,y^n} \left[\hat{L}(\hat{\pi}) - L(\hat{\pi})\right] = \sum_{d \in D} \frac{1}{\hat{R}_d} (\langle(\beta_d - \rho_d)\hat{R}_d + E_{c,y^n}[\hat{R}_k(d) - \beta_k(d)]\rangle) log(\hat{R}_d) + \langle E_{y^n}[\hat{R}_k(d) - \beta_k(d) - \hat{a}_k(d)] + \hat{b}_d + (\rho_d - \hat{b}_d)R_d\rangle log(1 - \hat{R}_d).
\]

**Proof.** First, we consider the expected value of \(\hat{L}(\hat{\pi})\):

\[
E_{c,y^n} \left[\hat{L}(\hat{\pi})\right] = \sum_{d \in D} \frac{1}{\hat{R}_d} (\langle(\beta_d - \rho_d)R_d + E_{y^n}[\hat{R}_k(d) - \beta_k(d)]\rangle) log(\hat{R}_d) + \langle E_{y^n}[\hat{a}_k(d) - \beta_k(d) + \hat{b}_k(d) - \rho_d R_d]\rangle log(1 - \hat{R}_d).
\]

Subtract Eq. 16 from the result of Eq. 47 to prove Theorem C.1. □

**Lemma C.2.** Following Lemma A.3 and Lemma B.2:

\[
\hat{a} = \alpha \land (\forall d \in D, \hat{R}_n(d) = m(d) \land \rho_d \geq \tau) \\
\implies (\forall d \in D, E_{y^n}[\hat{a}_k(d)] = \hat{R}_d).
\]

**Theorem C.3.** \(\hat{L}(\hat{\pi})\) is unbiased when \(\hat{a}\), \(\hat{b}\) and \(\hat{R}_n\) are correctly estimated and clipping has no effect:

\[
\hat{a} = \alpha \land \hat{b} = \beta \land (\forall d \in D, \hat{R}_n(d) = m(d) \land \rho_d \geq \tau) \\
\implies E_{y^n}[\hat{L}(\hat{\pi})] = L(\hat{\pi}).\) (49)

**Proof.** Theorem C.1 reveals an unbiasedness condition:

\[
\hat{a} = \alpha \land \hat{b} = \beta \land (\forall d \in D, \hat{R}_n(d) = m(d) \land \rho_d \geq \tau) \\
\implies E_{y^n}[\hat{L}(\hat{\pi})] = L(\hat{\pi}).\) (50)

From Lemma A.3, A.4 and C.2 it follows that:

\[
\hat{a} = \alpha \land \hat{b} = \beta \land (\forall d \in D, \hat{R}_n(d) = m(d) \land \rho_d \geq \tau) \\
\implies (\forall d \in D, \hat{R}_d = \rho_d \land E_{y^n}[\hat{R}_k(d)] = E_{y^n}[\hat{R}_k(d)]) \land E_{y^n}[\hat{a}_k(d)] = \hat{R}_d.
\]

Combining Eq. 50 and 51 directly proves Theorem C.3. □
Acknowledgments
This research was partially supported by the Google Research Scholar Program. All content represents the opinion of the authors, which is not necessarily shared or endorsed by their respective employers and/or sponsors.

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