MINIMIZING MOVEMENTS FOR MEAN CURVATURE FLOW OF DROPLETS WITH PRESCRIBED CONTACT ANGLE

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Abstract. We study the mean curvature motion of a droplet flowing by mean curvature on a horizontal hyperplane with a possibly nonconstant prescribed contact angle. Using the minimizing movements method we show the existence of a weak evolution, and its compatibility with a distributional solution. We also prove various comparison results.

1. INTRODUCTION

Historically, capillarity problems attracted attention because of their applications in physics, for instance in the study of wetting phenomena [17, 31], energy minimizing drops and their adhesion properties [49, 1, 20, 18], as well as because of their connections with minimal surfaces, see e.g. [28, 14] and references therein.

In this paper we are interested in the study of the evolution of a droplet flowing on a horizontal hyperplane under curvature driven forces with a prescribed (possibly nonconstant) contact angle. Although there are results in the literature describing the static and dynamic behaviours of droplets [2, 50, 12], not too much seems to be known concerning their mean curvature motion. Various results have been obtained for mean curvature flow of hypersurfaces with Dirichlet boundary conditions [35, 53, 47, 48] and zero-Neumann boundary condition [5, 34, 52, 38]. It is also worthwhile to recall that, when the contact angle is constant, the evolution is related to the so-called mean curvature flow of surface clusters, also called space partitions (networks, in the plane): in two dimensions local well-posedness has been shown in [16], and authors of [39] derived global existence of the motion of grain boundaries close to an equilibrium configuration. See also [43] for related results. In higher space dimensions short time existence for symmetric partitions of space into three phases with graph-type interfaces has been derived in [30, 29]. Very recently, authors of [25] have shown short time existence of the mean curvature flow of three surface clusters.

If we describe the evolving droplet by a set $E(t) \subset \Omega$, $t \geq 0$ the time, where $\Omega = \mathbb{R}^n \times (0, +\infty)$ is the upper half-space in $\mathbb{R}^{n+1}$, the evolution problem we are interested in reads as

$$V = H_{E(t)} \quad \text{on } \Omega \cap \partial E(t)$$

(1.1)

where $V$ is the normal velocity and $H_{E(t)}$ is the mean curvature of $\partial E(t)$, supplied with the contact angle condition on the contact set (the boundary of the wetted area):

$$\nu_{E(t)} \cdot e_{n+1} = \beta \quad \text{on } \Omega \cap \partial E(t) \cap \partial \Omega,$$

(1.2)

where $\nu_{E(t)}$ is the outer unit normal to $\partial \Omega \cap \partial E(t)$ at $\partial \Omega$, and $\beta : \partial \Omega \to [-1, 1]$ is the cosine of the prescribed contact angle. We do not allow $\partial E(t)$ to be tangent to $\partial \Omega$, i.e. we suppose $|\beta| \leq 1 - 2\kappa$ on $\partial \Omega$ for some $\kappa \in (0, \frac{1}{2}]$. Following [38], in Appendix B we show local well-posedness of (1.1)-(1.2).

Short time existence describes the motion only up to the first singularity time. In order to continue the flow through singularities one needs a notion of weak solution. Concerning the case without boundary, there are various notions of generalized solutions, such as Brakke’s varifold-solution [15], the viscosity solution (see [32] and references therein), the Almgren-Taylor-Wang [3] and Luckhaus-Sturzenhecker [41] solution, the minimal barrier solution (see [10] and references therein); see also [37, 26] for other different approaches.

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In the present paper we want to adapt the scheme proposed in [3, 41], and later extended to the notions of minimizing movement and generalized minimizing movement (shortly GMM) by De Giorgi [23] (see also [6, 8]) to solve (1.1)-(1.2). Let us recall the definition.

**Definition 1.1.** Let $S$ be a topological space, $F : S \times S \times [1, +\infty) \times \mathbb{Z} \rightarrow [-\infty, +\infty]$ be a functional and $u : [0, +\infty) \rightarrow S$. We say that $u$ is a generalized minimizing movement associated to $F, S$ (shortly GMM) starting from $a \in S$ if we write $u \in GMM(F, S, \mathbb{Z}, a)$, if there exist $w : [1, +\infty) \times \mathbb{Z} \rightarrow S$ and a diverging sequence $\{\lambda_j\}$ such that

$$
\lim_{j \rightarrow +\infty} w(\lambda_j, [\lambda_j t]) = u(t) \quad \text{for any } t \geq 0,
$$

and the functions $w(\lambda, k), \lambda \geq 1$, $k \in \mathbb{Z}$, are defined inductively as $w(\lambda, k) = a$ for $k \leq 0$ and

$$
F(w(\lambda, k + 1), w(\lambda, k), \lambda, k) = \min_{s \in S} F(s, w(\lambda, k), \lambda, k) \quad \forall k \geq 0.
$$

If $GMM(F, S, \mathbb{Z}, a)$ consists of a unique element it is called a minimizing movement starting from $a$.

In the sequel, we take $S = BV(\Omega, \{0, 1\})$, $F = A_\beta : BV(\Omega, \{0, 1\}) \times BV(\Omega, \{0, 1\}) \times [1, +\infty) \times \mathbb{Z} \rightarrow (-\infty, +\infty)$ defined by

$$
A_\beta(E, E_0, \lambda) = C_\beta(E, \Omega) + \lambda \int_{E \Delta E_0} d_{E_0} dx,
$$

where $E_0 \in BV(\Omega, \{0, 1\})$ is the initial set, $d_{E_0}$ is the distance to $\Omega \cap \partial E_0$ and

$$
C_\beta(E, \Omega) = P(E, \Omega) - \int_{\partial \Omega} \beta \chi_E dH^n
$$

is the capillary functional. If $\Omega = \mathbb{R}^n$ (hence when the term $\int_{\partial \Omega} \beta \chi_E dH^n$ is not present), the weak evolution (GMM) has been studied in [3] and [41], see also [45] for the Dirichlet case. Further when no ambiguity appears we use $GMM(E_0)$ to denote the GMM starting from $E_0 \in BV(\Omega, \{0, 1\})$.

After setting in Section 2 the notation, and some properties of finite perimeter sets, in Section 3 we study the functional $C_\beta(\cdot, \Omega)$ and its level-set counterpart $C_\beta(\cdot, \Omega)$, including lower semicontinuity and coercivity, which will be useful in Section 6. In particular, the map $E \mapsto A_\beta(E, E_0, \lambda)$ is $L^1(\Omega)$-lower semicontinuous if and only if $\|\beta\|_\infty \leq 1$ (Lemma 3.5). Although we can also establish the coercivity of $A_\beta(\cdot, E_0, \lambda)$ (Proposition 3.2), compactness theorems in $BV$ cannot be applied because of the unboundedness of $\Omega$. However, in Theorem 4.1 we prove that if $E_0 \in BV(\Omega, \{0, 1\})$ is bounded and $\|\beta\|_\infty < 1$, then there is a minimizer in $BV(\Omega, \{0, 1\})$ of $A_\beta(\cdot, E_0, \lambda)$, and any minimizer is bounded. In Lemma 4.6 we study the behaviour of minimizers as $\lambda \rightarrow +\infty$. In Proposition 4.4 we show existence of constrained minimizers of $C_\beta(\cdot, \Omega)$, which will be used in the proof of existence of GMMs and in comparison principles. In Appendix A we need to generalize such existence and uniform boundedness results to minimizers of functionals of type $C_\beta(\cdot, \Omega) + \mathcal{V}$ under suitable hypotheses on $\mathcal{V}$.

In Section 5 we study the regularity of minimizers $A_\beta(\cdot, E_0, \lambda)$ (Theorem 5.3). We point out the uniform density estimates for minimizers of $A_\beta(\cdot, E_0, \lambda)$ and constrained minimizers of $C_\beta(\cdot, \Omega)$ (Theorem 5.1 and Proposition 5.8), which are the main ingredients in the existence proof of GMMs (Section 7), and in the proof of coincidence with distributional solutions (Section 8).

In Section 6 we prove the following comparison principle for minimizers of $A_\beta(\cdot, E_0, \lambda)$ (Theorem 6.1): if $E_0, F_0$ are bounded, $E_0 \subseteq F_0$, $\|\beta_1\|_\infty, \|\beta_2\|_\infty < 1$ and $\beta_1 \leq \beta_2$, then

a) there exists a minimizer $F_\lambda^*$ of $A_{\beta_2}(\cdot, F_0, \lambda)$ containing any minimizer of $A_{\beta_1}(\cdot, E_0, \lambda)$;

b) there exists a minimizer $E_\lambda$ of $A_{\beta_2}(\cdot, E_0, \lambda)$ contained in any minimizer of $A_{\beta_1}(\cdot, F_0, \lambda)$; if in addition $\text{dist}(\Omega \cap \partial E_0, \Omega \cap \partial F_0) > 0$, then any minimizer $E_\lambda$ and $F_\lambda$ of $A_{\beta_1}(\cdot, E_0, \lambda)$ and $A_{\beta_2}(\cdot, F_0, \lambda)$ respectively, satisfy $E_\lambda \subseteq F_\lambda$. As a corollary, we show that if $E^+$ is a bounded minimizer of $C_\beta(\cdot, \Omega)$ in the collection $\mathcal{E}(E^+)$ of all finite perimeter sets containing $E^+$, and if $\|\beta\|_\infty < 1$, then for any $E_0 \subseteq E^+$, a minimizer $E_\lambda$ of $A_{\beta_1}(\cdot, E_0, \lambda)$ satisfies $E_\lambda \subseteq E^+$ (Proposition 6.11).

In Section 7 we apply the scheme in Definition 1.1 to the functional $A_\beta(\cdot, E_0, \lambda)$: as in [41, 46] we build a locally $\frac{1}{2}$-Hölder continuous generalized minimizing movement $t \in [0, +\infty) \mapsto E(t) \in BV(\Omega, \{0, 1\})$ starting from a bounded set $E_0 \in BV(\Omega, \{0, 1\})$ (Theorem 7.1). Moreover, using the results of Section 6, we prove that any GMM starting from a bounded set stays bounded. In general,
for two GMMs one cannot expect a comparison principle (for example in the presence of fattening). However, the notions of maximal and minimal GMMs (Definition 7.2) are always comparable if the initial sets are comparable (Theorem 7.3). This requires regularity of minimizers of $A_\beta(\cdot, E_0, \lambda)$ and $C_{\beta}(\cdot, \Omega)$, see Sections 4 and 5. Finally, in Section 8 we prove that, under a suitable conditional convergence assumption and if $1 \leq n \leq 6$, our GMM solution is, in fact, a distributional solution to (1.1)-(1.2).

2. SOME PRELIMINARIES

2.1. Notation. $\chi_F$ stands for the characteristic function of the Lebesgue measurable set $F \subseteq \mathbb{R}^{n+1}$ and $|F|$ denotes its Lebesgue measure. The set of $L^1(\Omega)$-functions having bounded total variation in an open set $\Omega \subseteq \mathbb{R}^{n+1}$ is denoted by $BV(\Omega)$, and

$$BV(\Omega, \{0, 1\}) := \{ E \subseteq \Omega : \chi_E \in BV(\Omega) \}.$$ 

Given $E \subseteq BV(\Omega, \{0, 1\})$ we denote by $P(E, \Omega)$ the perimeter of $E$ in $\Omega$, i.e. $P(E, \Omega) := \int_{\Omega} |D\chi_E|$, by $\partial^* E$ the essential boundary of $E$, and by $\nu_E(x)$ the measure-theoretical exterior normal to $E$ at $x \in \partial^* E$. Since Lebesgue equivalent sets in $\Omega$ have the same perimeter in $\Omega$, we assume that any set $E \subseteq \Omega$ we consider coincides with the set

$$\left\{ x \in \mathbb{R}^{n+1} : \lim_{r \to 0^+} \frac{|B_r(x) \cap E|}{|B_r(x)|} = 1 \right\}$$

of points of density one, where $B_r(x)$ is the ball of radius $r > 0$ centered at $x$. Recall that $\partial^* E = \partial E$. For simplicity, set $P(E, \mathbb{R}^{n+1}) = P(E)$. We say that $E \subseteq \mathbb{R}^{n+1}$ has locally finite perimeter in $\mathbb{R}^{n+1}$, if $P(E, \Omega') < +\infty$ for every bounded open set $\Omega' \subset \mathbb{R}^{n+1}$. The collection of all sets of locally finite perimeter is denoted by $BV_{loc}(\Omega, \{0, 1\})$. We refer to [33, 7] for a complete information about $BV$-functions and sets of finite perimeter.

For a fixed nonempty $E_0 \in BV(\Omega, \{0, 1\})$ set

$$\mathcal{E}(E_0) := \{ E \in BV(\Omega, \{0, 1\}) : E_0 \subseteq E \}, \quad (2.1)$$

which is $L^1(\Omega)$-closed.

Given $\rho > 0$ and $l > 0$ let $C^l_\rho = B_\rho \times (0, l)$ stand for the truncated cylinder in $\mathbb{R}^{n+1}$ of height $l$, whose basis is an open ball $B_\rho \subset \mathbb{R}^n$ centered at the origin of radius $\rho > 0$; also set $\Omega_l := \mathbb{R}^n \times (0, l)$.

2.2. Some properties of sets of finite perimeter. By [21, Theorem II], for every $E \in BV_{loc}(\Omega, \{0, 1\})$ the additive set function $O \mapsto \int_O |D\chi_E|$ defined on the open sets $O \subseteq \Omega$ extends to a measure $B \mapsto \int_B |D\chi_E|$ defined on the Borel $\sigma$-algebra of $\Omega$. Moreover, $P(\cdot, \Omega)$ is strongly subadditive, i.e.

$$P(E \cap F, \Omega) + P(E \cup F, \Omega) \leq P(E, \Omega) + P(F, \Omega) \quad \text{for any } E, F \in BV(\Omega, \{0, 1\}). \quad (2.2)$$

Let $\Omega$ be an open set with Lipschitz boundary and $E \in BV_{loc}(\mathbb{R}^{n+1}, \{0, 1\})$. We denote the interior and exterior traces of the set $E$ on $\partial \Omega$ respectively by $\chi_E^+$ and $\chi_E^-$ and we recall that $\chi_E^+ \in L^1_{loc}(\partial \Omega)$. Moreover, the integration by parts formula holds [21]:

$$\int_{\Omega} \chi_E \text{div } g \, dx = -\int_{\Omega} g \cdot D\chi_E + \int_{\partial \Omega} (\chi_E^+ - \chi_E^-) g \cdot \nu_\Omega \, dH^n \quad \forall g \in C^1_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}), \quad (2.3)$$

where $\nu_\Omega$ is the outer unit normal to $\partial \Omega$.

If $V \subseteq \Omega$ is an open set with Lipschitz boundary, then

$$P(E, \Omega) = P(E, V) + P(E, \Omega \setminus V) + \int_{\Omega \setminus \partial V} |\chi_E^+ - \chi_E^-| \, dH^n.$$ 

The trace set of $E \subseteq \Omega$ on $\partial \Omega$ is denoted by $Tr(E)$. With a slight abuse of notation we set $\chi_{Tr(E)} = \chi_E$. Note that

$$P(E, \overline{\Omega}) := P(E, \Omega) + \int_{\partial \Omega} \chi_E \, dH^n = P(E).$$

In general, even if $E \in BV(\Omega, \{0, 1\})$, the traces $\chi_E^\pm$ are in $L^1_{loc}(\partial \Omega)$, but not in $L^1(\partial \Omega)$. For instance, if $\Omega = (\mathbb{R} \times (0, +\infty)) \cup A \subset \mathbb{R}^2$ and $A = \bigcup_{m=2}^{+\infty} (m - \frac{1}{m^2}, m + \frac{1}{m^2}) \times (-1, 0]$, then
$E = A \in BV(\Omega, \{0, 1\})$, whereas $\mathcal{H}^1(\text{Tr}(E)) = +\infty$. In Lemma 2.1 we show that $\chi_E \in L^1(\partial \Omega)$ for any $E \in BV(\Omega, \{0, 1\})$, provided that $\Omega$ is a half-space.

From now on we fix $\Omega := \mathbb{R}^n \times (0, +\infty)$; we often identify $\partial \Omega = \mathbb{R}^n \times \{0\}$ with $\mathbb{R}^n$, so that $E \subset \partial \Omega$ means $E \subset \mathbb{R}^n$, and $\pi : \Omega \to \partial \Omega$ denotes the projection

$$\pi(\hat{x}, x_{n+1}) := \hat{x}, \quad x = (\hat{x}, x_{n+1}) \in \Omega.$$  

### 2.3 Controlling the trace of a set by its perimeter

The following lemma shows that the $L^1(\partial \Omega)$-norm of the trace of $E \in BV(\Omega, \{0, 1\})$ is controlled by $P(E, \Omega)$.

**Lemma 2.1.** For any $E \in BV(\Omega, \{0, 1\})$ and for any $\beta \in L^\infty(\partial \Omega)$ the relations

$$\left| \int_{\partial \Omega} \beta \chi_E d\mathcal{H}^n \right| \leq \int_{\Omega} |\beta \circ \pi| |D\chi_E| \leq \|\beta\|_{\infty} P(E, \Omega). \tag{2.4}$$

hold. In particular, $P(E) < +\infty$.

**Proof.** The last inequality of (2.4) is immediate. The first inequality is enough to be shown for $\beta \geq 0$.

If $\beta$ is locally Lipschitz, then (2.4) follows from the divergence theorem. Indeed, suppose that $\text{supp}(\beta)$ is compact. Since $\text{div}((\beta \circ \pi)e_{n+1}) = 0$, we have

$$0 = \int_E \text{div}((\beta \circ \pi)e_{n+1}) dx = \int_{\Omega \cap \partial \Omega} (\beta \circ \pi) \nu_E \cdot e_{n+1} d\mathcal{H}^n - \int_{\partial \Omega} \beta \chi_E d\mathcal{H}^n.$$ 

Hence nonnegativity of $\beta$ implies that

$$\int_{\partial \Omega} \beta \chi_E d\mathcal{H}^n \leq \int_{\Omega \cap \partial \Omega} \beta \circ \pi d\mathcal{H}^n = \int_{\Omega} \beta \circ \pi |D\chi_E|.$$ \tag{2.5}

If $\text{supp}(\beta)$ is not compact, we use $\eta_k(|x|)\beta(x)$ in (2.5) instead of $\beta(x)$, where $\eta_k : [0, +\infty) \to [0, +\infty)$ is Lipschitz, linear in $[k, k+1]$, $\eta_k = 1$ in $[0, k]$ and $\eta_k = 0$ in $[k+1, +\infty)$. Now (2.4) follows from the monotone convergence theorem. In particular, when $\beta \equiv 1$ we have

$$P(E) = P(E, \Omega) + \int_{\partial \Omega} \chi_E d\mathcal{H}^n \leq 2P(E, \Omega).$$

Assume that $\beta = \chi_{\hat{O}}$ for some open set $\hat{O} \subseteq \partial \Omega$. Consider a sequence $\{\beta_k\}$ of nonnegative locally Lipschitz functions converging $\mathcal{H}^n$-almost everywhere to $\beta$ on $\partial \Omega$ such that $\beta_k \leq \beta$ and $\text{supp} \beta_k \subseteq \overline{\hat{O}}$. By Fatou’s lemma we get

$$\int_{\partial \Omega} \beta_k \chi_E d\mathcal{H}^n \leq \liminf_{k \to +\infty} \int_{\partial \Omega} \beta_k \chi_E d\mathcal{H}^n \leq \liminf_{k \to +\infty} \int_{\Omega} \beta_k \circ \pi |D\chi_E| \leq \int_{\Omega} \beta \circ \pi |D\chi_E|.$$ 

Finally, if $\beta \in L^\infty(\partial \Omega)$ is any nonnegative function, then the statement of the lemma follows by an approximation argument. \qed

From Lemma 2.1 it follows that $E \in BV(\Omega, \{0, 1\})$ if and only if $E \in BV(\mathbb{R}^{n+1}, \{0, 1\})$.

**Remark 2.2.** If $u \in BV(\Omega)$, then its trace belongs to $L^1(\partial \Omega)$. Indeed, it is well-known that

$$\int_{\Omega} |u| dx = \int_{-\infty}^0 \int_{\partial \Omega} \chi_{\{u < t\}}(x) dx dt + \int_{0}^{+\infty} \int_{\partial \Omega} \chi_{\{u > t\}}(x) dx dt, \tag{2.6}$$

$$\int_{\Omega} |Du| = \int_{-\infty}^0 P(\{u < t\}, \Omega) dt + \int_{0}^{+\infty} P(\{u > t\}, \Omega) dt. \tag{2.7}$$

in particular, $\{u > t\}, \{u < s\} \in BV(\Omega)$ for a.e. $t > 0$ and $s < 0$. Using (2.4) with $\beta \equiv 1$, for a.e. $t > 0$ and $s < 0$ we get

$$\int_{\partial \Omega} \chi_{\{u > t\}} d\mathcal{H}^n \leq P(\{u > t\}, \Omega), \quad \int_{\partial \Omega} \chi_{\{u < s\}} d\mathcal{H}^n \leq P(\{u < s\}, \Omega)$$

and we obtain

$$\int_{\partial \Omega} |u| d\mathcal{H}^n \leq \int_{\Omega} |Du|. $$
Notice that for every \( \beta \in L^\infty(\partial \Omega) \) one has also
\[
\int_{\partial \Omega} \beta u \, dH^n = -\int_0^\infty \int_{\partial \Omega} \beta \chi_{\{u < t\}} \, dH^n \, dt + \int_0^{+\infty} \int_{\partial \Omega} \beta \chi_{\{u > t\}} \, dH^n \, dt. \tag{2.8}
\]

The following lemma is the analog to comparison theorem in [6, page 216]¹.

**Lemma 2.3.** Let \( E_0 \) be a closed convex set such that \( \nu_{E_0} \cdot e_{n+1} \geq 0 \) \( H^n \)-a.e. on \( \Omega \cap \partial E_0 \). Then \( P(E_0, \Omega) \leq P(E, \Omega) \) for every \( E \in \mathcal{E}(E_0) \).

3. CAPILLARY FUNCTIONALS

Let \( \beta \in L^\infty(\partial \Omega) \). The capillary functional \( C_\beta(\cdot, \Omega) : BV(\Omega, \{0, 1\}) \to \mathbb{R} \) and its “level set” version \( C_\beta(\cdot, \Omega) : BV(\Omega) \to \mathbb{R} \) are defined as
\[
C_\beta(E, \Omega) := P(E, \Omega) - \int_{\partial \Omega} \beta \chi_E \, dH^n, \tag{3.1}
\]
and
\[
C_\beta(u, \Omega) := \int_{\Omega} |Du| - \int_{\partial \Omega} \beta u \, dH^n,
\]
respectively. Note that \( C_\beta(\cdot, \Omega) \) is convex, \( C_\beta(u, \Omega) = C_{-\beta}(-u, \Omega) \) for any \( u \in BV(\Omega) \), and \( C_\beta(E, \Omega) = C_\beta(x_E, \Omega) \) for any \( E \in BV(\Omega, \{0, 1\}) \). Moreover, when \( \|\beta\|_\infty \leq 1 \), by (2.4) the functional \( C_\beta(\cdot, \Omega) \) is nonnegative, and the same holds for \( C_\beta(\cdot, \Omega) \) as by (2.6)-(2.8) one has
\[
C_\beta(u, \Omega) = \int_0^0 C_{-\beta}({u < t}, \Omega) \, dt + \int_0^{+\infty} C_\beta({u > t}, \Omega) \, dt. \tag{3.2}
\]

The functional \( C_\beta(\cdot, \Omega) \) will be useful for the comparison principles (Section 6).

3.1. Coercivity and lower semicontinuity. The next lemma is a localized version of [18, Lemma 4], which is needed to prove coercivity of \( C_\beta(\cdot, \Omega) \) and \( C_\beta(\cdot, \Omega) \) and will be frequently used in the proofs (see for example the proofs of Theorem A.3 and Theorem 5.1).

**Lemma 3.1.** Assume that \( \|\beta\|_\infty \leq 1 \) and \( E \in BV(\Omega, \{0, 1\}) \). Then for any open set \( A \subseteq \Omega \) with \( A \in BV_{loc}(\mathbb{R}^{n+1}, \{0, 1\}) \) and
\[
H^n \left( \left[ \pi^{-1}(\pi(A)) \setminus A \right] \cap \Omega \cap \partial^* E \right) = 0 \tag{3.3}
\]
the inequality
\[
P(E, A) - \int_{\partial \Omega} \beta \chi_{E \cap A} \, dH^n \geq \frac{1 - \text{ess sup} \beta}{2} \left[ P(E, A) + \int_{\partial \Omega} \chi_{E \cap A} \, dH^n \right] \tag{3.4}
\]
holds.

**Proof.** Let us first show that if \( F \subset \Omega \) has locally finite perimeter in \( \mathbb{R}^{n+1} \), then
\[
\chi_F \leq \chi_{\pi(F)} \quad H^n \text{-a.e. on } \partial \Omega. \tag{3.5}
\]
Set \( \hat{G} := \{ \hat{x} \in Tr(F) : \chi_{\pi(F)}(\hat{x}) = 0 \} \). For any \( \varepsilon > 0 \) take an open set \( \hat{\Omega} \subseteq \partial \Omega \) such that \( \hat{G} \subseteq \hat{\Omega} \) and \( H^n(\hat{\Omega} \setminus \hat{G}) < \varepsilon \). Since \( H^n(\pi(F) \cap \hat{G}) = 0 \), one has
\[
|F \cap \pi^{-1}(\hat{G})| = \int_{\pi^{-1}(\hat{G})} \chi_F \, dx = \int_0^{+\infty} dx_{n+1} \int_{\hat{G}} \chi_F(\hat{x}, x_{n+1}) \, dH^n(\hat{x}) = \int_0^{+\infty} H^n(\hat{G} \cap \{ x_{n+1} : (\hat{x}, x_{n+1}) \in F \}) \, dx_{n+1} = \int_0^{+\infty} H^n(\hat{G} \cap \pi(F)) \, dx_{n+1} = 0.
\]

¹For any \( E \in BV(\mathbb{R}^{n+1}, \{0, 1\}) \) and any closed convex set \( C \subseteq \mathbb{R}^{n+1} \) the inequality \( P(E \cap C) \leq P(E) \) holds; equality occurs if and only if \( |E \setminus C| = 0 \).
Let $\hat{B}_p \subset \mathbb{R}^n$ denote the ball of radius $\rho > 0$ centered at the origin. Recall that for any $\gamma > 0$ the following estimate [33, page 35] holds:

$$\int_{\hat{O} \cap \hat{B}_p} \chi_F d\mathcal{H}^n \leq P(F, (\hat{O} \cap \hat{B}_p) \times (0, \gamma)) + \frac{1}{\gamma} \int_{(\hat{O} \cap \hat{B}_p) \times (0, \gamma)} \chi_F dx.$$  

Then using $\hat{G} \subseteq \text{Tr}(F)$, we establish

$$\mathcal{H}^n(\hat{G} \cap \hat{B}_p) \leq \int_{\hat{O} \cap \hat{B}_p} \chi_F d\mathcal{H}^n \leq P(F, (\hat{O} \cap \hat{B}_p) \times (0, \gamma))$$

$$+ \frac{1}{\gamma} \int_{(\hat{O} \cap \hat{B}_p) \times (0, \gamma)} \chi_F dx + \frac{1}{\gamma} \int_{((\hat{O} \setminus \hat{G}) \cap \hat{B}_p) \times (0, \gamma)} \chi_F dx$$

$$\leq P(F, \hat{O} \times (0, \gamma)) + \frac{1}{\gamma} |F \cap \pi^{-1}(\hat{G})| + \mathcal{H}^n(\hat{O} \setminus \hat{G}) < P(F, \hat{O} \times (0, \gamma)) + \varepsilon.$$  

Now letting $\varepsilon, \gamma \to 0^+$ we get $\mathcal{H}^n(\hat{G} \cap \hat{B}_p) = 0$ and (3.5) follows from letting $\rho \to +\infty$.

We have

$$\int_{\Omega} \chi_{(A)} \circ \pi \frac{1 + \beta \circ \pi}{2} |D\chi| = \int_{\pi^{-1}(A)} \frac{1 + \beta}{2} |D\chi| = \int_{\pi^{-1}(A)} \frac{1 + \beta}{2} |D\chi|,$$  

where in the second equality we used (3.3). Moreover, from (3.5) with $F = A$ we get

$$\int_{\Omega} \frac{1 + \beta}{2} \chi_{E \cap A} d\mathcal{H}^n = \int_{\Omega} \frac{1 + \beta}{2} \chi_{E} d\mathcal{H}^n \leq \int_{\Omega} \frac{1 + \beta}{2} \chi_{A} d\mathcal{H}^n.$$  

Now, using Lemma 2.1 with $\beta$ replaced with $(1 + \beta)\chi_{(A)}/2$, from (3.6) and (3.7) we obtain

$$\int_{\Omega} \frac{1 + \beta}{2} \chi_{E \cap A} d\mathcal{H}^n \leq \int_{A} \frac{1 + \beta}{2} |D\chi|.$$  

Finally, adding the identities

$$P(E, A) = \int_{A} |D\chi| = \int_{A} \frac{1 - \beta}{2} |D\chi| + \int_{A} \frac{1 + \beta}{2} |D\chi|,$$  

and using (3.8) we deduce

$$P(E, A) - \int_{\Omega} \beta \chi_{E \cap A} d\mathcal{H}^n \geq \int_{A} \frac{1 - \beta}{2} |D\chi| + \int_{\Omega} \frac{1 - \beta}{2} \chi_{E \cap A} d\mathcal{H}^n.$$  

This relation yields (3.4).

**Proposition 3.2 (Coercivity of the capillary functionals).** If $-1 \leq \beta \leq 1 - 2\kappa$ $\mathcal{H}^n$-a.e. on $\partial\Omega$ for some $\kappa \in [0, \frac{1}{2}]$, then

$$\kappa P(E) \leq C_\beta(E, \Omega) \leq P(E) \quad \forall E \in BV(\Omega, \{0, 1\}).$$  

Moreover, if $\|\beta\|_{\infty} \leq 1 - 2\kappa$ for some $\kappa \in [0, \frac{1}{2}]$, then

$$\kappa \int_{\Omega} |Du| \leq C_\beta(u, \Omega) \leq \int_{\Omega} |Du| \quad \forall u \in BV(\Omega).$$  

**Proof.** The inequality $\kappa P(E) \leq C_\beta(E, \Omega)$ follows from Lemma 3.1 with $A = \Omega$. Moreover, by virtue of Lemma 2.1,

$$\|\beta\|_{\infty} \leq 1 \implies C_\beta(E, \Omega) \leq P(E) \quad \forall E \in BV(\Omega, \{0, 1\}).$$  

Now (3.10) follows from the inequalities

$$\kappa P\{\{u < t\}, \Omega\} + \kappa \int_{\partial\Omega} \chi_{\{u < t\}} d\mathcal{H}^n \leq C_{-\beta}\{\{u < t\}, \Omega\} \leq P\{\{u < t\}, \Omega\} + \int_{\partial\Omega} \chi_{\{u < t\}} d\mathcal{H}^n.$$
for a.e. \( t < 0 \) and
\[
\kappa P(\{ u > t \}, \Omega) + \kappa \int_{\partial \Omega} \chi_{\{ u > t \}} \, d\mathcal{H}^n \leq C_\beta(\{ u < t \}, \Omega) \leq P(\{ u > t \}, \Omega) + \int_{\partial \Omega} \chi_{\{ u > t \}} \, d\mathcal{H}^n
\]
for a.e. \( t > 0 \), from (2.6)-(2.8), (3.2) and by [33, Remark 2.14], possibly after extending \( u \) to \( 0 \) outside \( \Omega \).

\section*{Remark 3.3.}
From the proof of Proposition 3.2 it follows that if \( u \geq 0 \), then (3.10) holds for any \( \beta \in L^\infty(\partial \Omega) \) with \(-1 \leq \beta \leq -2\kappa\); if \( u \leq 0 \), (3.10) is valid whenever \(-1 + 2\kappa \leq \beta \leq 1\).

\section*{Remark 3.4.}
If \( \beta > 1 \) on a set of infinite \( \mathcal{H}^n \)-measure, then \( C_\beta(\cdot, \Omega) \) is unbounded from below. Note also that if \( \| \beta \|_\infty \leq 1 \), then \( \emptyset \) is the unique minimizer of \( C_\beta(\cdot, \Omega) \) in \( BV(\Omega, \{0,1\}) \). Indeed, clearly,
\[
0 = C_\beta(\emptyset, \Omega) = \min_{E \in BV(\Omega, \{0,1\})} C_\beta(E, \Omega).
\]
If there were a minimizer \( E \neq \emptyset \) of \( C_\beta(\cdot, \Omega) \), there would exist \( l > 0 \) such that \( |E \setminus \Omega| > 0 \). Now since \( \text{Tr}(E) = \text{Tr}(E \cap \overline{\Omega}) \), by [6, page 216] we have
\[
0 = C_\beta(E, \Omega) > C_\beta(E \cap \overline{\Omega}, \Omega) \geq 0,
\]
a contradiction.

\section*{Lemma 3.5 (Lower semicontinuity).}
Assume that \( \beta \in L^\infty(\partial \Omega) \). Then the functionals \( C_\beta(\cdot, \Omega) \) and \( C_\beta(\cdot, \Omega) \) are \( L^1(\Omega) \)-lower semicontinuous if and only if \( \| \beta \|_\infty \leq 1 \).

\section*{Proof.}
Assume that \( \| \beta \|_\infty \leq 1 \). In this case the lower semicontinuity of \( C_\beta(\cdot, \Omega) \) is proven in [18, Lemma 2]. Let us prove the lower semicontinuity of \( C_\beta(\cdot, \Omega) \). Take \( u_k, u \in BV(\Omega) \) such that \( u_k \to u \) in \( L^1(\Omega) \). By (2.6) we may assume that \( \int_\Omega \{ (u_k < t) \} \Delta \{ u < t \} \, dx \to 0 \) as \( k \to +\infty \) for a.e. \( t \in \mathbb{R} \). Then using the nonnegativity of summands, the lower semicontinuity of \( C_\beta(\cdot, \Omega) \) and Fatou’s Lemma in (3.2) we establish

\[
\liminf_{k \to +\infty} C_\beta(u_k, \Omega) \geq \liminf_{k \to +\infty} \int_{-\infty}^0 C_\beta(\{ u_k < t \}, \Omega) \, dt + \liminf_{k \to +\infty} \int_0^{+\infty} C_\beta(\{ u_k > t \}, \Omega) \, dt
\]

\[
\geq \int_{-\infty}^0 \liminf_{k \to +\infty} C_\beta(\{ u_k < t \}, \Omega) \, dt + \int_0^{+\infty} \liminf_{k \to +\infty} C_\beta(\{ u_k > t \}, \Omega) \, dt
\]

\[
\geq \int_{-\infty}^0 C_\beta(\{ u < t \}, \Omega) \, dt + \int_0^{+\infty} C_\beta(\{ u > t \}, \Omega) \, dt = C_\beta(u, \Omega).
\]

Now assume that \( \| \beta \|_\infty > 1 \), i.e. the set \( \{ \hat{x} \in \partial \Omega : |\beta(\hat{x})| > 1 \} \) has positive \( \mathcal{H}^n \)-measure. Let for some \( \varepsilon, \delta_0 > 0 \) the set \( \hat{A} := \{ \beta > 1 + \varepsilon \} \) satisfy \( |A| \geq \delta_0 \). By Lusin’s theorem, for any \( k \geq \frac{\delta_0}{\varepsilon(1+\varepsilon)} \) there exists \( \beta_k \in C(\partial \Omega) \) such that \( \mathcal{H}^n(\{ \beta \neq \beta_k \}) < \frac{1}{k} \) and \( \| \beta_k \|_\infty \leq \| \beta \|_\infty \). Let \( k \) be so large that \( \mathcal{H}^n(\{ \beta_k > 1 + \varepsilon \}) \geq \delta_0/2 \) and choose an open set \( \hat{O} \subset \{ \beta_k > 1 + \varepsilon \} \) of finite perimeter such that \( \delta_0/4 \leq \mathcal{H}^n(\hat{O}) < +\infty \). Define the sequence of sets \( E_m := \hat{O} \times (0, \frac{1}{m}) \subset \Omega \). Clearly, \( E_m \to 0 \) in \( L^1(\Omega) \) as \( m \to +\infty \). Then, indicating by \( P(\hat{O}) \) the perimeter of \( \hat{O} \) in \( \mathbb{R}^n \), from the relations

\[
C_\beta(E_m, \Omega) = \frac{1}{m} P(\hat{O}) + \mathcal{H}^n(\hat{O}) - \int_{\hat{O}} \beta \, d\mathcal{H}^n
\]

\[
\leq \frac{1}{m} P(\hat{O}) + \mathcal{H}^n(\hat{O}) - \int_{\hat{O}} \beta_k \, d\mathcal{H}^n + \int_{\hat{O}} |\beta - \beta_k| \, d\mathcal{H}^n
\]

\[
\leq \frac{1}{m} P(\hat{O}) - \varepsilon \mathcal{H}^n(\hat{O}) + 2\| \beta \|_\infty \mathcal{H}^n(\hat{O} \cap \{ \beta \neq \beta_k \}) \leq \frac{1}{m} P(\hat{O}) - \frac{\varepsilon \delta_0}{4},
\]

we establish
\[
\liminf_{m \to +\infty} C_\beta(E_m, \Omega) \leq -\frac{\varepsilon \delta_0}{4} < 0 = C_\beta(\emptyset, \Omega).
\]
Since $C_\beta(\chi_E, \Omega) = C_\beta(E, \Omega)$, one has also $\liminf_{m \to +\infty} C_\beta(\chi_{E_m}, \Omega) < 0 = C_\beta(0, \Omega)$. Hence $C_\beta(\cdot, \Omega)$ and $C_\beta(\cdot, \Omega)$ are not $L^1(\Omega)$-lower semicontinuous.

Finally, the case $\hat{B} := \{ \beta < -1 - \epsilon \}$ is of a positive measure can be treated in a similar way.

**Remark 3.6.** If $\Omega$ is an arbitrary bounded open set with Lipschitz boundary and $\|\beta\|_\infty \leq 1$, then the lower semicontinuity of $C_\beta(\cdot, \Omega)$ is a consequence of [4, Theorem 3.4]. In this case $C_\beta(\cdot, \Omega)$ is bounded from below by $-H^n(\partial \Omega)$. Hence again Fatou’s lemma and (3.2) yield lower semicontinuity of $C_\beta(\cdot, \Omega)$.

4. **Capillary Almgren-Taylor-Wang-type functional**

In the sequel, for a given nonempty set $F \subseteq \Omega$, $d_F$ stands for the distance function from the boundary of $\partial F$ in $\Omega$:

$$d_F(x) := \text{dist}(x, \Omega \cap \partial F).$$

The function

$$\tilde{d}_F(x) := \begin{cases} -d_F(x) & \text{if } x \in F, \\
  d_F(x) & \text{if } x \in \Omega \setminus F,
\end{cases}$$

is called the signed distance function from $\partial F$ in $\Omega$ negative inside $F$. The distance from the empty set is assumed to be equal to $+\infty$.

Notice that for $E, F \subseteq \Omega$, $F \neq \emptyset$,

$$\int_{E \Delta F} d_F dx = \int_{E \cap F} \tilde{d}_F dx - \int_{F \setminus E} \tilde{d}_F dx = \int_E \tilde{d}_F dx - \int_F \tilde{d}_F dx,$$

provided $\int_{E \cap F} d_F dx < +\infty$. Moreover, we assume $\int_{E \Delta F} d_F dx := 0$ whenever $|E \Delta F| = 0$.

Given $\beta \in L^\infty(\partial \Omega)$, $E_0 \in BV(\Omega, \{0, 1\})$ and $\lambda \geq 1$, recalling the definition of $C_\beta(\cdot, \Omega)$ in (3.1), we define the capillary Almgren-Taylor-Wang-type functional $A_\beta(\cdot, E_0, \lambda) : BV(\Omega, \{0, 1\}) \to [-\infty, +\infty]$ with contact angle $\beta$, as

$$A_\beta(E, E_0, \lambda) := C_\beta(E, \Omega) + \lambda \int_{E \Delta E_0} d_{E_0} dx,$$

so that

$$A_\beta(E, E_0, \lambda) = P(E, \Omega) + \lambda \int_E \tilde{d}_{E_0} dx - \int_{\partial \Omega} \beta \chi_E dH^n - \lambda \int_{E_0} \tilde{d}_{E_0} dx$$

whenever $\int_{E \cap E_0} d_{E_0} dx < +\infty$.

4.1. **Existence of minimizers of the functional $A_\beta(\cdot, E_0, \lambda)$**. We always suppose that $\lambda \geq 1$ and in this section we assume that

$$\begin{cases} E_0 \in BV(\Omega, \{0, 1\}) \text{ is nonempty and bounded}, \\
  \beta \in L^\infty(\partial \Omega) \text{ and } \exists \kappa \in (0, \frac{1}{2}) : -1 \leq \beta \leq 1 - 2\kappa H^n \text{-a.e on } \partial \Omega.
\end{cases}$$

Hence, there exists a cylinder $C_B^H = \bar{B}_D \times (0, H)$ containing $E_0$ whose basis is an open ball $\bar{B}_D \subset \mathbb{R}^n$ of radius $D > 0$ and height $H = 1 + \max\{x_{n+1} : x = (x', x_{n+1}) \in E_0\}$.

Define

$$R_0 := R_0(n, \kappa, E_0) = D + 1 + \max\left\{ 8^{n^2+n+1} \left( \frac{P(E_0)}{\kappa} \right)^{\frac{n+1}{n}}, 4\mu(\kappa, n) \right\},$$

where $\mu(\kappa, n) = (1/\kappa + 2)^{n+1}$. The proof of the next result is essentially postponed to A, since the main idea does not differ too much from [18].
Theorem 4.1 (Existence of minimizers and uniform bound). Suppose that \( (4.3) \) holds. Then the minimum problem
\[
\inf_{E \in BV(\Omega, \{0,1\})} \mathcal{A}_\beta(E, E_0, \lambda) \tag{4.5}
\]
has a solution \( E_\lambda \). Moreover, any minimizer is contained in \( C^{H}_{R_0} \).

Proof. Let \( f = \lambda \tilde{d}_{E_0} \) and
\[
\mathcal{V} : BV(\Omega, \{0,1\}) \to (-\infty, +\infty], \quad \mathcal{V}(E) := \int_E f \, dx.
\]
Then \( \mathcal{V} \) satisfies Hypothesis A.1 and by Remark A.4 \( R_0 \leq R_0 \). Now the proof directly follows from Theorem A.3. \( \square \)

Remark 4.2. If \( E_0 = \emptyset \), then (4.5) has a unique solution \( E_\lambda = \emptyset \). Moreover, for some choices of \( \lambda \geq 1 \) and \( \emptyset \neq E_0 \in BV(\Omega, \{0,1\}) \), the empty set solves (4.5). For example, let \( B_p \) be the ball centered at \( x \) such that \( x_{n+1} \geq 4p + 4 \). If \( \lambda \rho \leq n \), then as in [19, 11], one can show that \( E_\lambda = \emptyset \) is the unique minimizer of \( \mathcal{A}_\beta(\cdot, B_p, \lambda) \).

Remark 4.3. Let \( F \) minimize \( \mathcal{A}_\beta(\cdot, E_0, \lambda) \) in \( BV(C^{H}_{R_0}, \{0,1\}) \). Then \( F \) is an unconstrained minimizer, i.e.
\[
\mathcal{A}_\beta(F, E_0, \lambda) = \min_{E \in BV(\Omega, \{0,1\})} \mathcal{A}_\beta(E, E_0, \lambda). \tag{4.6}
\]
Indeed, let \( E_\lambda \) be any minimizer of \( \mathcal{A}_\beta(\cdot, E_0, \lambda) \). Clearly, \( \mathcal{A}_\beta(F, E_0, \lambda) \geq \mathcal{A}_\beta(E_\lambda, E_0, \lambda) \). On the other hand, by Theorem 4.1 \( E_\lambda \subseteq C^{H}_{R_0} \) and by minimality of \( F \) in \( C^{H}_{R_0} \) we have \( \mathcal{A}_\beta(F, E_0, \lambda) \leq \mathcal{A}_\beta(E_\lambda, E_0, \lambda) \), which implies (4.6).

Recalling Remark 3.4 and definition (2.1) of \( \mathcal{E}(E_0) \) we have also the following result.

Proposition 4.4 (Existence of constrained minimizers of \( C_\beta \)). Under assumptions \( (4.3) \) the constrained minimum problem
\[
\inf_{E \in BV(\Omega, \{0,1\}), E \in \mathcal{E}(E_0)} C_\beta(E, \Omega) \tag{4.7}
\]
has a solution. In addition, any minimizer \( E^+ \) satisfies \( E^+ \subseteq C^{H}_{R_0} \), where \( R_0 \) is given by (4.4), and \( E^+ \) is also a solution of
\[
\inf_{E \in BV(\Omega, \{0,1\}), E \in \mathcal{E}(E^+)} C_\beta(E, \Omega). \tag{4.8}
\]

Proof. Set
\[
\mathcal{V} : BV(\Omega, \{0,1\}) \to [0, +\infty], \quad \mathcal{V}(E) := \begin{cases} 0 & \text{if } E \in \mathcal{E}(E_0), \\ +\infty & \text{if } E \in BV(\Omega, \{0,1\}) \setminus \mathcal{E}(E_0). \end{cases}
\]
Then \( \mathcal{V} \) satisfies Hypothesis A.1 and \( R_0 \leq R_0 \). Now existence of a minimizer \( E^+ \) of \( C_\beta(\cdot, \Omega) \) in \( \mathcal{E}(E_0) \) and the inclusion \( E^+ \subseteq C^{H}_{R_0} \) follow from Theorem A.3. To show the last statement we observe that the inclusion \( E_0 \subseteq E^+ \) implies \( \mathcal{E}(E^+) \subseteq \mathcal{E}(E_0) \). Hence the minimality of \( E^+ \) yields the inequality \( C_\beta(E^+, \Omega) \leq C_\beta(E_\lambda, \Omega) \) for any \( E \in \mathcal{E}(E^+) \). \( \square \)

Solutions of (4.7) will be called constrained minimizers of \( C_\beta(\cdot, \Omega) \) in \( \mathcal{E}(E_0) \).

Example 4.5. Suppose that \( E_0 \subset \Omega \) is a closed convex set so that \( \nu_{E_0} \cdot e_{n+1} \geq 0 \) \( H^n \)-a.e. on \( \Omega \cap \partial E_0 \). Then for every \( \beta \in L^\infty(\partial \Omega, [-1,0]) \) the set \( E_0 \) is a constrained minimizer of \( C_\beta(\cdot, \Omega) \) in \( \mathcal{E}(E_0) \). Indeed, by Lemma 2.3 \( P(E_0, E_0) \leq P(E, \Omega) \) for all \( E \in \mathcal{E}(E_0) \), therefore
\[
C_\beta(E, \Omega) - C_\beta(E_0, \Omega) = P(E, \Omega) - P(E_0, \Omega) + \int_{\partial \Omega} ( -\beta ) \chi_{E \setminus E_0} \, dH^n \geq 0.
\]

The following lemma shows the behaviour of \( E_\lambda \) as \( \lambda \to +\infty \).

Lemma 4.6 (Asymptotics of \( E_\lambda \) as time goes to \( 0^+ \)). Assume (4.3) and \( |E_0 \setminus E_0| = 0 \). Then any minimizer \( E_\lambda \) satisfies:

a) \( \lim_{\lambda \to +\infty} |E_\lambda \Delta E_0| = 0 \),
b) \( \lim_{\lambda \to +\infty} C_\beta(E_\lambda, \Omega) = C_\beta(E_0, \Omega) \),

c) \( \lim_{\lambda \to +\infty} \lambda \int_{E_\lambda \Delta E_0} dE_0 \, dx = 0 \).

Proof. a) We have

\[
\kappa P(E_\lambda) \leq A_\beta(E_\lambda, E_0, \lambda) \leq A_\beta(E_0, E_0, \lambda) = C_\beta(E_0, \Omega) \leq P(E_0).
\]

Moreover, from \( A_\beta(E_\lambda, E_0, \lambda) \leq P(E_0) \) and (2.4) we get \( \lambda \int_{E_\lambda \Delta E_0} dE_0 \, dx \leq P(E_0) \), hence

\[
\lim_{\lambda \to +\infty} \lambda \int_{E_\lambda \Delta E_0} dE_0 \, dx = 0. \tag{4.9}
\]

Recall from Theorem 4.1 that \( E_\lambda \subseteq C^{H}_{R_0} \) for all \( \lambda \geq 1 \). Hence, by compactness, from every diverging sequence \( \{\lambda_i\} \) we can select a subsequence \( \{\lambda_{i_k}\} \) such that

\[
E_{\lambda_{i_k}} \to E_\infty \quad \text{in} \quad L^1(\Omega)
\]

for some \( E_\infty \in BV(C^{H}_{R_0}, \{0, 1\}) \). From (4.9) we deduce that \( \int_{E_\infty \Delta E_0} dE_0 \, dx = 0 \), and thus, since \( dE_0 \geq 0 \) and by assumption \( [E_0 \setminus E_0] = 0 \), we get \( |E_\infty \Delta E_0| = 0 \). Now arbitrariness of \( \{\lambda_i\} \) implies a).

b) Clearly, \( C_\beta(E_\lambda, \Omega) \leq A_\beta(E_\lambda, E_0, \lambda) \leq C_\beta(E_0, \Omega) \) for all \( \lambda \geq 1 \). Then by a) and by the \( L^1(\Omega) \)-lower semicontinuity of \( C_\beta(\cdot, \Omega) \) (Lemma 3.5) we establish

\[
C_\beta(E_0, \Omega) \leq \liminf_{\lambda \to +\infty} C_\beta(E_\lambda, \Omega) \leq \limsup_{\lambda \to +\infty} C_\beta(E_\lambda, \Omega) \leq C_\beta(E_0, \Omega),
\]

and b) follows.

c) follows from b) and nonnegativity of \( \lambda \int_{E_\lambda \Delta E_0} dE_0 \), since

\[
\limsup_{\lambda \to +\infty} \lambda \int_{E_\lambda \Delta E_0} dE_0 \, dx \leq \lim_{\lambda \to +\infty} |C_\beta(E_0, \Omega) - C_\beta(E_\lambda, \Omega)| = 0.
\]

\[\square\]

5. DENSITY ESTIMATES AND REGULARITY OF MINIMIZERS

In this section we assume that

\[
\begin{cases}
E_0 \in BV(\Omega, \{0, 1\}) \text{ is nonempty and bounded}, \\
\beta \in L^{\infty}(\partial \Omega) \text{ and } \exists \kappa \in (0, \frac{1}{2}] : \|\beta\|_{\infty} \leq 1 - 2\kappa.
\end{cases}
\tag{5.1}
\]

Define

\[
R(n, \kappa) := \left( \frac{2^{n+3} \omega_n + (n+1)\omega_{n+1}}{\omega_{n+1}\kappa^{n+1}} \right)^{\frac{1}{2}}, \quad \gamma(n, \kappa) := \frac{\kappa(n+1)}{R(n, \kappa)^2 + 4\kappa(n+1) + R(n, \kappa)},
\]

and

\[
C(n, \kappa) := (n+1)\omega_{n+1} + 2\omega_n + \frac{\kappa(n+1)}{2} \omega_{n+1}, \quad c(n, \kappa) := c_{n+1}\left(\frac{\kappa}{4}\right)^n,
\]

where \( c_{n+1} \) is the relative isoperimetric constant for the ball, i.e.

\[
c_{n+1} \min\{|B_r \cap F|, |B_r \setminus F|\}^{\frac{n}{n+1}} \leq P(F, B_r), \quad r > 0, \ F \in BV(B_r, \{0, 1\}).
\]

The aim of this section is to prove the following uniform density estimates for minimizers of \( A_\beta(\cdot, E_0, \lambda) \), needed to prove regularity of minimizers (Theorem 5.3) and Proposition 5.7.

**Theorem 5.1.** Assume that \( E_0 \) and \( \beta \) are as in (5.1) and \( E_\lambda \in BV(\Omega, \{0, 1\}) \) is a minimizer of \( A_\beta(\cdot, E_0, \lambda) \). Then either \( E_\lambda = \emptyset \) or

\[
\left(\frac{\kappa}{4}\right)^{n+1} \leq \frac{|E_\lambda \cap B_r(x)|}{\omega_{n+1} r^{n+1}} \leq 1 - \left(\frac{\kappa}{4}\right)^{n+1},
\]

\[
c(n, \kappa) \leq \frac{P(E_\lambda, B_r(x))}{r^n} \leq C(n, \kappa)
\]

(5.5)
for every \( x \in \partial E_\lambda \) and \( r \in (0, \frac{\gamma(n, \kappa)}{\lambda^1/2}) \). In particular,
\[
\mathcal{H}^n(\partial E_\lambda \setminus \partial^s E_\lambda) = 0.
\] (5.6)

We postpone the proof after several auxiliary results. First we show a weaker version of Theorem 5.1: the difference stands in that Proposition 5.2 holds for \( r \leq O(\frac{1}{\lambda^1/2}) \) and \( O(\frac{1}{\lambda}) \) depends on \( E_0 \), whereas Theorem 5.1 is valid for \( r \leq O(\frac{1}{\lambda^1/2}) \) and \( O(\frac{1}{\lambda}) \) is independent of \( E_0 \).

**Proposition 5.2.** Under the assumptions of Theorem 5.1, setting
\[
\Lambda := \Lambda(\lambda, n, \kappa, P(E_0)) = \lambda \operatorname{diam}(\hat{B}_{D+R_0+1} \times (-1, H + 1)),
\]
for any nonempty \( E_\lambda, \ x \in \partial E_\lambda \) and \( r \in (0, \min\{1, \frac{\kappa(n+1)}{2\lambda}\}) \), the density estimates (5.4)-(5.5) hold.

**Proof.** For completeness we give the full proof of the proposition using the methods of [41, 46]. We recall that one could also employ the density estimates for almost minimizers of the capillary functional (see for instance [24, Lemma 2.8]).

Set \( r_0 := \min\{1, \frac{\kappa(n+1)}{2\lambda}\} \), and fix \( x \in \partial^s E_\lambda \). Let \( B_r := B_r(x) \) be the ball of radius \( r \in (0, r_0) \) centered at \( x \), we can choose \( r \) such that
\[
\mathcal{H}^n(\partial B_r \cap \partial E_\lambda) = 0.
\]

First we show that \( E_\lambda \) satisfies
\[
\kappa P(E_\lambda \cap B_r) \leq 2\mathcal{H}^n(E_\lambda \cap \partial B_r) + \Lambda|E_\lambda \cap B_r|.
\] (5.7)

Comparing \( A_\beta(E_\lambda, E_0, \lambda) \) with \( A_\beta(E_\lambda \setminus B_r, E_0, \lambda) \), for a.e. \( s \in (r, r_0) \) we establish
\[
P(E_\lambda, B_s \cap \Omega) - \int_{B_s \cap \partial \Omega} \beta \chi_{E_\lambda \cap B_r} d\mathcal{H}^n + \lambda \int_{E_\lambda \cap B_r} \tilde{d}_{E_0} dy 
\leq P(E_\lambda, (B_s \setminus \overline{B}_r) \cap \Omega) + \mathcal{H}^n(E_\lambda \cap \partial B_r).
\]

Sending \( s \to r^+ \) we get
\[
P(E_\lambda, B_r \cap \Omega) - \int_{B_r \cap \partial \Omega} \beta \chi_{E_\lambda} d\mathcal{H}^n + \lambda \int_{E_\lambda \cap B_r} \tilde{d}_{E_0} dy \leq \mathcal{H}^n(E_\lambda \cap \partial B_r).
\] (5.8)

By Theorem 4.1 \( E_\lambda \subseteq C^H_{R_0} \) and thus, since \( r_0 \leq 1 \), for any \( y \in B_r \)
\[
\lambda|\tilde{d}_{E_0}(y)| \leq \lambda \operatorname{diam}(\hat{B}_{D+R_0+1} \times (-1, H + 1)) = \Lambda.
\] (5.9)

Moreover, using (3.9) for \( E_\lambda \cap B_r \) we get (5.7):
\[
\kappa P(E_\lambda \cap B_r) \leq P(E_\lambda, B_r \cap \Omega) + \lambda \mathcal{H}^n(E_\lambda \cap \partial B_r) - \int_{B_r \cap \partial \Omega} \beta \chi_{E_\lambda} d\mathcal{H}^n
\leq 2\mathcal{H}^n(E_\lambda \cap \partial B_r) + \lambda|E_\lambda \cap B_r|.
\]

Now by the isoperimetric inequality,
\[
P(E_\lambda \cap B_r) \geq (n + 1)\omega_{n+1}^{-\frac{1}{n+1}}|E_\lambda \cap B_r|^{\frac{n}{n+1}}.
\] (5.10)

Set \( m(r) := |E_\lambda \cap B_r| \). Then \( m \) is absolutely continuous, \( m(0) = 0 \), \( m(r) > 0 \) for all \( r > 0 \) and \( m'(r) = \mathcal{H}^n(E_\lambda \cap \partial B_r) \) for a.e. \( r \in (0, r_0) \). Consequently, (5.7) and (5.10) give
\[
\kappa(n+1)\omega_{n+1}^{-\frac{1}{n+1}}m(r)^{\frac{n}{n+1}} \leq 2m'(r) + \Lambda m(r) = 2m'(r) + \Lambda m(r)^{\frac{1}{n+1}}m(r)^{\frac{n-1}{n+1}}.
\] (5.11)

Since \( m(r) \leq \omega_{n+1}r^{n+1} \) and \( r \leq \frac{k(n+1)}{2\Lambda} \), from the last inequality we obtain
\[
\kappa(n+1)\omega_{n+1}^{-\frac{1}{n+1}}m(r)^{\frac{n}{n+1}} \leq m'(r).
\]

Integrating we get the lower volume density estimate
\[
m(r) \geq (\frac{k}{4})^{\frac{n}{n+1}}\omega_{n+1}r^{n+1}, \quad \forall r \in (0, r_0).
\]
Let us prove the upper volume density estimate in (5.4). Since $E_\lambda \subseteq \Omega$ if $x \in \partial \Omega \cap \partial^* E_\lambda$, the inequality
\[
\frac{|B_r \setminus E_\lambda|}{\omega_{n+1} r^{n+1}} \geq \frac{1}{2} > \left( \frac{\kappa}{4} \right)^{n+1} \quad \forall r > 0
\] (5.12)
is trivial. So assume that $x \in \Omega \cap \partial^* E_\lambda$. Since $A_\beta(E_\lambda, E_0, \lambda) \leq A_\beta((E_\lambda \cup B_r) \cap \Omega, E_0, \lambda)$, arguing as in the proof of (5.8) we get
\[
P(E_\lambda, B_r \cap \Omega) + \int_{\partial \Omega} \beta \chi_{(B_r \cap \Omega) \setminus E_\lambda} d\mathcal{H}^n \leq \mathcal{H}^n((\Omega \setminus E_\lambda) \cap \partial B_r) + \lambda \int_{(B_r \cap \Omega) \setminus E_\lambda} \tilde{d}E_0, dy.
\] (5.13)
From the isoperimetric inequality, (3.9), (5.13) and also (5.9), it follows that
\[
\kappa(n+1) \omega_{n+1}^{-1} |(B_r \setminus E_\lambda) \cap \Omega|^{\frac{1}{n+1}} \leq \kappa P((B_r \setminus E_\lambda) \cap \Omega) \leq C_\beta((B_r \setminus E_\lambda) \cap \Omega, \Omega)
\]
\[
\leq P(E_\lambda, B_r \cap \Omega) + \int_{\partial \Omega} \beta \chi_{(B_r \cap \Omega) \setminus E_\lambda} d\mathcal{H}^n + \mathcal{H}^n((\Omega \setminus E_\lambda) \cap \partial B_r)
\]
\[
\leq 2 \mathcal{H}^n((\Omega \setminus E_\lambda) \cap \partial B_r) + \Lambda((B_r \setminus E_\lambda) \cap \Omega).
\] (5.14)

Repeating the same arguments as before we establish
\[
\frac{|B_r \setminus E_\lambda|}{\omega_{n+1} r^{n+1}} \geq \frac{|(B_r \setminus E_\lambda) \cap \Omega|}{\omega_{n+1} r^{n+1}} \geq \left( \frac{\kappa}{4} \right)^{n+1} \quad \forall r \in (0, r_0).
\]

Let us now show (5.5). From (5.8) we get
\[
P(E_\lambda, B_r) = P(E_\lambda, B_r \cap \Omega) + \int_{B_r \cap \partial \Omega} \chi_{E_\lambda} d\mathcal{H}^n
\]
\[
\leq \mathcal{H}^n((\Omega \setminus E_\lambda) \cap \partial B_r) + \int_{B_r \cap \partial \Omega} (1 + \beta) \chi_{E_\lambda} d\mathcal{H}^n + \lambda |E_\lambda \cap B_r|
\]
\[
\leq (n+1) \omega_{n+1} r^n + 2 \omega_n r^n + \omega_{n+1} r^n (\Lambda r)
\]
\[
\leq \left[ (n+1) \omega_{n+1} + 2 \omega_n + \omega_{n+1} \frac{\kappa(n+1)}{2} \right] r^n
\]
for a.e $r \in (0, r_0)$. Since $P(E_\lambda, \cdot)$ is a nonnegative measure, this inequality holds for all $r \in (0, r_0)$. This proves the upper perimeter estimate in (5.5).

The lower perimeter density estimate in (5.5) follows from (5.4) and the relative isoperimetric inequality (see for example [7, page 152]).

**Theorem 5.3 (Regularity of minimizers up to the boundary).** Assume that $E_0$ and $\beta$ satisfy (5.1). Then any nonempty minimizer $E_\lambda$ is open in $\mathbb{R}^{n+1}$ and $\Omega \cap \partial^* E_\lambda$ is an $n$-dimensional manifold of class $C^{2,\alpha}$ for a suitable $\alpha \in (0, 1)$, and $\mathcal{H}^s((\partial E_\lambda \setminus \partial^* E_\lambda) \cap \Omega) = 0$ for all $s > n - 7$. Moreover, if $\beta \in \text{Lip}(\partial \Omega)$, then

a) $\mathcal{H}^n((\partial E_\lambda \cap \partial \Omega) \Delta (\text{Tr}(E_\lambda))) = 0$;

b) $\partial E_\lambda \cap \partial \Omega$ is a set of finite perimeter in $\partial \Omega$ and
\[
\mathcal{H}^{n-1}((\partial (\partial E_\lambda \cap \partial \Omega) \setminus \partial^* (\partial E_\lambda \cap \partial \Omega)) = 0,
\]
where $\partial (\partial E_\lambda \cap \partial \Omega)$ denotes the boundary of $\partial E_\lambda \cap \partial \Omega$ in $\partial \Omega$. Moreover, if $M_\lambda = \Omega \cap \partial E_\lambda$, then
\[
\partial (\partial E_\lambda \cap \partial \Omega) = M_\lambda \cap \partial \Omega.
\]

c) There exists a relatively closed set $\Sigma \subset M_\lambda$ with $\mathcal{H}^{n-1}(\Sigma \cap \partial \Omega) = 0$ such that in a neighborhood of any $x \in (M_\lambda \cap \partial \Omega) \setminus \Sigma$ the set $M_\lambda$ is a $C^{1,1/2}$-manifold with boundary, and
\[
\nu_{E_\lambda} \cdot e_{n+1} = \beta \quad \text{on} \ (M_\lambda \cap \partial \Omega) \setminus \Sigma.
\]
Proof. Since $E_\lambda$ is a minimizer of $A_\beta(\cdot, E_0, \lambda)$ in every ball $B \subset \Omega$, we can apply [44, Theorem 5.2] to prove that $E_\lambda$ is open and $\Omega \cap \partial^* E_\lambda$ is $C^{2,\alpha}$ with $\mathcal{H}^s((\partial^* E_\lambda) \cap \Omega) = 0$ for all $s > n - 7$. Moreover, if $\beta \in \text{Lip}(\partial \Omega)$, by (5.9) the remaining assertions follow from [24, Lemma 2.16, Theorem 1.10].

Remark 5.4. (Compare with [41, Remark 1.4] and [46].)

a) Assume that $x \in \overline{E_\lambda}$ and $r > 0$ are such that $B_r(x) \cap E_0 = \emptyset$. Then $d_{E_0} \geq 0$ in $E_\lambda \cap B_r(x)$ and from (5.8) we get

$$P(E_\lambda, B_r \cap \Omega) = \int_{B_r \cap \partial \Omega} \beta \chi_{E_\lambda} \, d\mathcal{H}^n \leq \mathcal{H}^n(E_\lambda \cap \partial B_r).$$

(5.15)

Then proceeding as in the proof of Proposition 5.2 we get $|E_\lambda \cap B_r| \geq (\kappa/2)^n + 1 + \omega_{n+1} r^{n+1}$. Moreover, from (5.15) it follows that

$$P(E_\lambda, B_r \cap \Omega) \leq \mathcal{H}^n(E_\lambda \cap \partial B_r) + \int_{B_r \cap \partial \Omega} \chi_{E_\lambda} \, d\mathcal{H}^n \leq [(n+1)\omega_{n+1} + \omega_n] r^n.$$

b) Similarly, if $x \in \overline{E_\lambda}$ and $B_r(x) \cap (\Omega \setminus E_0) = \emptyset$, then $|E_\lambda \setminus \partial E_\lambda| \geq (\kappa/2)^n + 1 + \omega_{n+1} r^{n+1}$.

Observe that in both cases $r$ need not be in $(0, \min\{1, \kappa(n+1)/2\})$ and the assumption $x \in \partial E_\lambda$ is not necessary.

The following proposition is the analog of [41, Lemma 2.1] and [46, Proposition 3.2.1].

Proposition 5.5 (L∞-bound for the distance function). Assume that $E_0$ and $\beta$ are as in (5.1) and $E_\lambda \in BV(\Omega, \{0, 1\})$ is a minimizer of $A_\beta(\cdot, E_0, \lambda)$. Then

$$\sqrt{\lambda} \|d_{E_0}\|_{L^\infty(E_\lambda \Delta E_0)} \leq R(n, \kappa).$$

(5.16)

Proof. Let $R := R(n, \kappa)$. Suppose by contradiction that there exist $\varepsilon > 0$, $\lambda > 1$ and $x \in E_\lambda \Delta E_0$ such that $d_{E_0}(x) > (R + \varepsilon)\lambda^{n-1}/2$. Consider first the case $x \in E_\lambda \setminus E_0$. By regularity of $E_\lambda$ (Theorem 5.3) we may assume that $x \in \partial E_\lambda \setminus E_0$. Note that $\partial E_\lambda \setminus E_0 = \emptyset$, where $\partial E_\lambda := B_\rho(x)$, $\rho = (R + \varepsilon)\lambda^{n-1}/2$. Since $A_\beta(E_\lambda, E_0, \lambda) \leq A_\beta(E_\lambda \setminus B_\rho, E_0, \lambda)$, and $d_{E_0}(y) = d_{E_0}(y) \geq \rho$ for any $y \in B_\rho \setminus E_\lambda$, from (5.8) we establish

$$\frac{(R + \varepsilon)\lambda^{n-1}/2}{2} |E_\lambda \cap B_\rho| \leq \lambda \int_{E_\lambda \setminus B_\rho} \tilde{d}_{E_0} \, dy \leq \mathcal{H}^n(E_\lambda \cap \partial B_\rho) + \int_{B_\rho \cap \partial E_\lambda} \beta \chi_{E_\lambda} \, d\mathcal{H}^n \leq [\omega_{n+1} + 1 + \omega_n] \rho^n.$$

This and Remark 5.4 (a) yield

$$\omega_{n+1}(\frac{R + \varepsilon)\lambda^{n+1}}{2^{n+2}} \rho^{n+1} \leq [\omega_{n+1} + 1 + \omega_n] \rho^n,$$

or equivalently, recalling the definition of $\rho$

$$(R + \varepsilon)^2 \leq 2^{n+3} \frac{\omega_n + (n+1)\omega_{n+1}}{\omega_{n+1} \rho^{n+1}} = R^2,$$

which is a contradiction. A similar contradiction is obtained when $x \in E_0 \setminus E_\lambda$. 

Corollary 5.6. Assume (4.3) and $|E_0 \setminus E_0| = 0$. If $\|\beta\|_{\infty} < 1$, then any minimizer $E_\lambda$ satisfies $\Omega \cap \partial E_\lambda \xrightarrow{K} \Omega \cap \partial E_0$ as $\lambda \to +\infty$, where $K$ denotes Kuratowski convergence [40].

Proof. It suffices to show that every diverging sequence $\{\lambda_j\}$ has a subsequence $\{\lambda_j\}$ such that

$$K - \lim_{j \to +\infty} \Omega \cap \partial E_{\lambda_j} = \Omega \cap \partial E_0.$$

Choose any sequence $\lambda_j \to +\infty$. By compactness of closed sets in Kuratowski convergence [40, page 340], there exists a closed set $C \subset \overline{\Omega}$ such that up to a not relabelled subsequence $\Omega \cap \partial E_{\lambda_j} \xrightarrow{K} C$ as $j \to +\infty$. Let us show first that $\Omega \cap \partial E_0 \subseteq C$. Take any $x \in R^{n+1} \setminus C$; we may suppose that $x \in \Omega$. Since $C$ is closed, there exists a ball $B_\rho(x)$ such that $B_\rho(x) \cap C = \emptyset$. Since $\Omega \cap \partial E_{\lambda_j} \xrightarrow{K} C$ as
$j \to +\infty$, we have $B_{\rho}(x) \cap \bar{\Omega} \cap \partial E_{\lambda_j} = \emptyset$ for $j \geq 1$ large enough. Therefore, $P(E_{\lambda_j}, B_{\rho}(x) \cap \Omega) = 0$, and by a) and lower semicontinuity, $P(E_0, B_{\rho}(x) \cap \Omega) = 0$. This yields $B_{\rho/2}(x) \cap \bar{\Omega} \cap \partial E_0 = \emptyset$ and thus $\mathbb{R}^{n+1} \setminus C \subseteq \mathbb{R}^{n+1} \setminus \bar{\Omega} \cap \partial E_0$.

Now suppose that there exists $x \in C \setminus \Omega \cap \partial E_0$. Then there exists $\rho > 0$ such that $B_{\rho}(x) \cap \Omega \cap \partial E_0 = \emptyset$. Since $x \in C$, there exists $x_j \in \Omega \cap \partial E_{\lambda_j}$ such that $x_j \to x$. Choose $j \in \mathbb{N}$ so large that $x_j \in B_{\rho/4}(x)$ and $R(n, \kappa)\lambda_j^{-1/2} < \rho/4$, where $R(n, \kappa)$ is defined in (5.2). By Proposition 5.5, we have

$$d_{E_0}(x_j) \leq R(n, \kappa)\lambda_j^{-1/2} < \frac{\rho}{4}.$$ 

On the other hand, by construction, $d_{E_0}(x) \geq \frac{\rho}{4}$, which leads to a contradiction. This yields $C \subseteq \Omega \cap \partial E_0$.

**Proof of Theorem 5.1.** We repeat the same procedures of the proof of Proposition 5.2 with improved estimates for the volume term of $A_{\beta}(\cdot, E_0, \lambda)$. Let $R := R(n, \kappa)$, $\gamma := \gamma(n, \kappa)$. Fix $x \in \partial^* E_{\lambda}$, and choose $r \in (0, \gamma\lambda^{-1/2})$ such that $\mathcal{H}^n(\partial B_r \cap \partial E_{\lambda}) = 0$. From (5.16) it follows

$$\sup_{(E_{\lambda}(\cap E_0) \cap B_r} d_{E_0} \leq R\lambda^{-1/2}.$$ 

Therefore, using the obvious inequality

$$\sup_{(E_0 \cap E_0) \cap B_r} d_{E_0} \leq 2r + \sup_{(E_0 \cap E_0) \cap B_r} d_{E_0} \leq (2\gamma + R)\lambda^{-1/2},$$ 

from (5.8) we establish that

$$P(E_{\lambda}, B_r \cap \Omega) - \int_{B_r \cap \partial \Omega} \beta x E_{\lambda} d\mathcal{H}^n \leq \mathcal{H}^n(E_{\lambda} \cap \partial B_r) + (R + 2\gamma)\lambda^{1/2}|E_{\lambda} \cap B_r|. \quad (5.17)$$ 

Since $m(r) := |E_{\lambda} \cap B_r| \leq \omega_{n+1}r^{n+1}$ and $r \leq \frac{\gamma}{\sqrt{(n+1)}}$, similarly to (5.11) from (5.17) we deduce

$$\kappa(n + 1)\omega_{n+1}^{-1}m(r)^{n+1} \leq m'(r) + (R + 2\gamma)\lambda^{1/2}r\omega_{n+1}^{-1}m(r)^{n+1},$$ 

for a.e. $r \in (0, \gamma\lambda^{1/2})$.

By the definition of $\gamma$ one has

$$(R + 2\gamma)\lambda^{1/2}r \leq (R + 2\gamma)\gamma = \frac{1}{2}\kappa(n + 1).$$ 

Thus,

$$\frac{\kappa}{4}(n + 1)\omega_{n+1}^{-1}m(r)^{n+1} \leq m'(r)$$ 

for a.e. $r \in (0, \gamma\lambda^{-1/2})$.

Integrating this differential inequality we get the lower volume density estimate in (5.4).

Let us prove the upper volume density estimate in (5.4). Due to (5.12) we may suppose that $x \in \Omega \cap \partial^* E_{\lambda}$. As above one can estimate $d_{E_0}$ in $(B_r \ \setminus E_{\lambda}) \cap \Omega$ as follows:

$$\sup_{\Omega \cap ((B_r \ \setminus E_{\lambda}) \cap E_0)} d_{E_0} \leq 2r + \sup_{E_{\lambda} \\Delta E_0} d_{E_0} \leq (2\gamma + R)\lambda^{-1/2}. \quad (5.18)$$ 

Since $d_{E_0} \leq 0$ in $\Omega \cap ((B_r \ \setminus E_{\lambda}) \cap E_0)$, plugging (5.18) in (5.13) and proceeding as above we establish

$$\frac{\kappa}{4}(n + 1)\omega_{n+1}^{-1}|(B_r \ \setminus E_{\lambda}) \cap \Omega|^n \leq \mathcal{H}^n((\Omega \ \setminus E_{\lambda}) \cap B_r),$$ 

from which the upper volume density estimates in (5.4) follows.

The proof of (5.5) is exactly the same as the proof of perimeter density estimates in Proposition 5.2. Finally, (5.6) is a standard consequence of a covering argument.

Let us prove the following $L^1$-estimate for the minimizers of $A_{\beta}(\cdot, E_0, \lambda)$, the analog of [41, Lemma 1.5] and [46, Proposition 3.2.3]. Notice carefully the exponent $-1/2$ of $\lambda$ in (5.19).

**Proposition 5.7 ($L^1$-estimate).** Assume that $E_0$ and $\beta$ satisfy (5.1) and the uniform volume density estimates (5.4) holds for $E_0$. Then for any minimizer $E_{\lambda}$ of $A_{\beta}(\cdot, E_0, \lambda)$ the estimate

$$|E_{\lambda} \Delta E_0| \leq C_{n,\kappa}P(E_0) \ell - \frac{1}{\ell} \int_{E_{\lambda} \Delta E_0} d_{E_0} dx, \quad \ell \in \left(0, \frac{\gamma(n, \kappa)}{\lambda^{1/2}}\right) \quad (5.19)$$
Proof. Set

\[ A := \{ x \in E \Delta E_0 : d_{E_0}(x) \geq \ell \}, \quad B := \{ x \in E \Delta E_0 : d_{E_0}(x) < \ell \}. \]

By Chebyshev inequality

\[ |A| \leq \frac{1}{\ell} \int_{E \Delta E_0} d_{E_0} \, dx. \]

Let us estimate |B|. Since \( E_0 \) is bounded, by Besicovitch’s covering theorem there exist at most countably many balls \( \{ B_\ell(x_i) \} \), \( x_i \in \partial E_0 \) such that any point of \( \partial E_0 \) belongs to at most \( b(n) \) balls, \( \partial E_0 \subset \bigcup_i B_\ell(x_i) \) and \( B \subset \bigcup_i B_2(x_i) \). Since the balls \( \{ B_\ell(x_i) \} \) cover \( B \), by the density estimates (5.4) and the relative isoperimetric inequality we get

\[ |B_\ell(x)| = 2^{n+1} \omega_{n+1} \ell^{n+1} \leq 2^{n+1} \left( \frac{4}{\kappa} \right)^{n+1} \min \{ |B_\ell(x_i) \cap E_0|, |B_\ell(x_i) \setminus E_0| \} \]

\[ \leq \left( \frac{8}{\kappa} \right)^{n+1} \omega_{n+1}^{n+1} \ell \min \{ |B_\ell(x_i) \cap E_0|, |B_\ell(x_i) \setminus E_0| \} \frac{\lambda}{\kappa} \]

\[ \leq \left( \frac{8}{\kappa} \right)^{n+1} \omega_{n+1}^{n+1} \ell c_{n+1} P(E_0, B_\ell(x_i)). \]

Therefore

\[ |B| \leq \left( \frac{8}{\kappa} \right)^{n+1} \omega_{n+1}^{n+1} c_{n+1} \ell \sum_i P(E_0, B_\ell(x_i)) \leq \left( \frac{8}{\kappa} \right)^{n+1} \omega_{n+1}^{n+1} b(n) c_{n+1} P(E_0) \ell. \]

Now (5.19) follows from the estimates for \(|A|, |B|\) and from \( |E \Delta E_0| \leq |A| + |B| \). \( \square \)

A specific choice of \( \ell \) will be made in the proof of Theorem 7.1. We conclude this section with a proposition about the regularity of minimizers of \( C_\beta(\cdot, \Omega) \).

**Proposition 5.8 (Density estimates for constrained minimizers of \( C_\beta \)).** Assume that \( E_0 \) and \( \beta \) satisfy (5.1) and there exist \( c_1, c_2, \varepsilon \in (0, 1) \) such that for every \( x \in \partial E_0 \) and \( r \in (0, \varepsilon) \) the inequalities

\[ c_1 \leq \frac{|B_r(x) \cap E_0|}{|B_r(x)|} \leq c_2 \]

hold. Let \( E^+ \) be a constrained minimizer of \( C_\beta(\cdot, \Omega) \) in \( E(E_0) \). Then for every \( x \in \partial E^+ \) and \( r \in (0, \varepsilon) \)

\[ c_1 \left( \frac{\kappa}{8} \right)^{n+1} \leq \frac{|B_r(x) \cap E^+|}{|B_r(x)|} \leq 1 - \left( \frac{\kappa}{4} \right)^{n+1}, \]

\[ c_{n+1} \omega_{n+1}^{n} (\kappa/8)^n \leq \frac{P(E^+, B_r(x, r))}{r^n} \leq (n + 1) \omega_{n+1} + \omega_n. \]

In particular, \( \mathcal{H}^n(\partial E^+ \setminus \partial^* E^+) = 0. \)

Proof. Let \( x \in \partial E^+ \), and \( r \in (0, \varepsilon) \) be such that \( \mathcal{H}^n(\partial B_r \cap \partial^* E^+) = 0 \), where \( B_r := B_r(x) \).

We start with the upper volume density estimate in (5.21). We may suppose \( x \in \Omega \cap \partial^* E^+ \), since the case \( x \in \partial \Omega \cap \partial^* E^+ \) is trivial. Using \( C_\beta(E^+, \Omega) \leq C_\beta((E^+ \cup B_r) \cap \Omega, \Omega) \), as in (5.13) we establish

\[ P(E^+, B_r) + \int_{\partial \Omega} \beta_\chi(B_r \setminus E^+ \cap \Omega) \, d\mathcal{H}^n \leq \mathcal{H}^n((\Omega \setminus E^+) \cap \partial B_r). \]

Adding \( \mathcal{H}^n(\partial B_r \cap (\Omega \setminus E^+)) \) to both sides and proceeding as in (5.14) we get

\[ \kappa(n + 1) \omega_{n+1} \omega_{n+1}^{n+1} (B_r \setminus E^+) \cap \Omega \omega_{n+1} \leq 2 \mathcal{H}^n((\Omega \setminus E^+) \cap \partial B_r) \]
and hence as in the proof of Theorem 5.1

$$|B_r \setminus E^+| \geq \left(\frac{\kappa}{4}\right)^{n+1} \omega_{n+1} r^{n+1}.$$  

This implies the lower volume density estimate in (5.21).

The lower volume density estimate is a little delicate, since in general we cannot use the set $E = E^+ \setminus B_r$ as a competitor since it need not belong to $E(0)$. If $d := d_{E_0}(x) = 0$, then $x \in \partial E_0$ and, hence, using $E_0 \cap B_r \subset E^+ \cap B_r$ and the lower volume density estimate for $E_0$ we establish

$$\frac{|E^+ \cap B_r|}{|B_r|} \geq \frac{|E_0 \cap B_r|}{|B_r|} \geq c_1 \geq c_1 \left(\frac{\kappa}{8}\right)^{n+1}.$$  

If $d > 0$ and $r \in (0, \min\{\varepsilon, d\})$, then we may use comparison set $E^+ \setminus B_r$ and as in the proof of (5.4) we obtain

$$\frac{|E^+ \cap B_r|}{|B_r|} \geq \left(\frac{\kappa}{4}\right)^{n+1} \geq c_1 \left(\frac{\kappa}{8}\right)^{n+1}.  \tag{5.23}$$

Suppose $d < \varepsilon$. Since one can extend (5.23) to $(0, d]$ by continuity, if $r \in (d, \min\{2d, \varepsilon\})$, then

$$\frac{|E^+ \cap B_r|}{|B_r|} \geq \frac{|E^\circ \cap B_{r-\varepsilon}|}{|B_{r-\varepsilon}|} \geq \left(\frac{d}{r}\right)^{n+1} \geq \left(\frac{\kappa}{8}\right)^{n+1} \geq c_1 \left(\frac{\kappa}{8}\right)^{n+1}.\]$$

Let $r \in [2d, \varepsilon]$ and $x_0 \in \Omega \cap \partial E_0$ be such that $d = |x - x_0|$. Then using $B(x, r) \supset B(x_0, r - d)$, the lower density estimate for $E_0$ and $r - d \geq r/2$, we obtain

$$\frac{|E^+ \cap B_r|}{|B_r|} \geq \frac{|E_0 \cap B_{r-d}(x_0)|}{|B_{r-d}(x_0)|} \cdot \left(\frac{r-d}{r}\right)^{n+1} \geq c_1 \left(\frac{1}{2}\right)^{n+1} \geq c_1 \left(\frac{\kappa}{8}\right)^{n+1}.$$  

Now the lower perimeter estimate follows from the volume density estimates and the relative isoperimetric inequality. The upper perimeter estimate is obtained from (5.22):

$$P(E^+, B_r) \leq \mathcal{H}^n((\Omega \setminus E^+) \cap \partial B_r) - \int_{\partial \Omega} \beta \chi(B_r \setminus E^+) \cap \partial \Omega \mathcal{H}^n \leq ((n+1)\omega_{n+1} + \omega_n)r^n.$$  

Finally, the relation $\mathcal{H}^n(\partial E^+ \setminus \partial^* E^+) = 0$ is a consequence of the density estimates together with a covering argument.  \[\square\]

\section{Comparison Principles}

The main result of this section is the following comparison between minimizers of $A_\beta(\cdot, E_0, \lambda)$.

\textbf{Theorem 6.1 (Comparison for minimizers of $A_\beta$).} Assume that $E_0, F_0, \beta_1, \beta_2$ satisfy (4.3). Suppose that $E_0 \subseteq F_0$ and $\beta_1 \leq \beta_2$. Then

a) there exists a minimizer $F_\lambda^*$ of $A_{\beta_2}(\cdot, F_0, \lambda)$ containing any minimizer of $A_{\beta_1}(\cdot, E_0, \lambda)$;

b) there exists a minimizer $E_{\lambda_*}$ of $A_{\beta_1}(\cdot, E_0, \lambda)$ contained in any minimizer of $A_{\beta_2}(\cdot, F_0, \lambda)$.

If in addition

$$\operatorname{dist}(\Omega \cap \partial E_0, \Omega \cap \partial F_0) > 0,  \tag{6.1}$$

then all minimizers $E_\lambda$ and $F_\lambda$ of $A_{\beta_1}(\cdot, E_0, \lambda)$ and $A_{\beta_2}(\cdot, F_0, \lambda)$ respectively satisfy

$$E_\lambda \subseteq F_\lambda.$$  

\textbf{Remark 6.2.} We do not exclude the case that either $E_\lambda$ or $F_\lambda$ is empty.

\textbf{Remark 6.3.} For any $E_0, \beta$ satisfying (4.3), using Theorem 6.1 with $\beta_1 = \beta_2 = \beta$ and $F_0 = E_0$, we establish the existence of unique minimizers $E_{\lambda_*}$ and $E_\lambda^*$ of $A_{\beta}(\cdot, E_0, \lambda)$, such that any other minimizer $E_\lambda$ satisfies $E_{\lambda_*} \subseteq E_\lambda \subseteq E_\lambda^*$.

\textbf{Definition 6.4 (Maximal and minimal minimizers).} We call $E_\lambda^*$ and $E_{\lambda_*}$ the maximal and minimal minimizer of $A_{\beta}(\cdot, E_0, \lambda)$ respectively.
Before proving Theorem 6.1 we need the following observations. Given $\beta$ satisfying (4.3), $C = C^h_r, h, r > 0$ and $v \in L^\infty(\Omega)$, $v \geq 0$ a.e. in $\Omega \setminus C$, define the convex functional $B_\beta(\cdot, v, C) : BV(\Omega, [0, 1]) \to (-\infty, +\infty)$, a sort of level-set capillary Almgren-Taylor-Wang-type functional, as

$$B_\beta(u, v, C) = C_\beta(u, \Omega) + \int_\Omega uv \, dx.$$ 

Set

$$R_1(C, v) := r + 1 + \max \left\{ 1 + \frac{C_\beta(C, \Omega)}{\kappa} + \frac{\|v\|_{L^\infty(C)}|C|}{\kappa}, 4\mu(\kappa, n) \right\},$$

(6.2) where $\mu(\kappa, n) = (1/\kappa + 2)^{n+1}$. By Example A.2 the functional

$$\mathcal{V} : BV(\Omega, \{0, 1\}) \to (-\infty, +\infty), \quad \mathcal{V}(E) := \int_E v \, dx$$
satisfies Hypothesis A.1. Thus, by Theorem A.3 the functional $E \in BV(\Omega, \{0, 1\}) \mapsto B_\beta(\chi_E, v, C) \in \mathbb{R}$ has a minimizer, and every minimizer $E_v$ satisfies

$$E_v \subseteq C^h_{R_1(C, v)}.$$ 

(6.3) Notice that by (2.8) and (3.2),

$$B_\beta(u, v, C) = \int_0^1 B_\beta(\chi_{\{u > t\}}, v, C) \, dt \quad \forall u \in BV(\Omega, [0, 1]),$$

(6.4) which yields that $\chi_{E_v}$ is a minimizer of $B_\beta(\cdot, v, C)$ in $BV(\Omega, [0, 1])$.

The following remark is in the spirit of [13, Section 1].

**Remark 6.5 (Minimality of level sets).** From (6.4) it follows that $u \in BV(\Omega, [0, 1])$ is a minimizer of $B_\beta(\cdot, v, C)$ in $BV(\Omega, [0, 1])$ if and only if $\chi_{\{u > t\}}$ is a minimizer of $B_\beta(\cdot, v, C)$ for a.e. $t \in [0, 1]$. Indeed, let for some $u \in BV(\Omega, [0, 1])$ the function $\chi_{\{u > t\}}$ be a minimizer of $B_\beta(\cdot, v, C)$ for a.e. $t \in [0, 1]$. Then for any $w \in BV(\Omega, [0, 1])$ and for a.e. $t \in [0, 1]$ one has $B_\beta(w, v, C) \geq B_\beta(\chi_{\{u > t\}}, v, C)$, therefore,

$$B_\beta(u, v, C) = \int_0^1 B_\beta(\chi_{\{u > t\}}, v, C) \, dt \leq B_\beta(w, v, C).$$

Conversely, if $u \in BV(\Omega, [0, 1])$ is a minimizer of $B_\beta(\cdot, v, C)$, then for a.e. $t \in [0, 1]$ one has $B_\beta(u, v, C) \leq B_\beta(\chi_{\{u > t\}}, v, C)$. Hence, from (6.4) it follows that $B_\beta(\cdot, v, C) = B_\beta(\chi_{\{u > t\}}, v, C)$ for a.e. $t \in [0, 1]$. In particular, if $u \in BV(\Omega, [0, 1])$ is a minimizer of $B_\beta(\cdot, v, C)$, then by (6.3) $\{u > t\} \subseteq C^h_{R_1(C, v)}$ for a.e. $t \in [0, 1]$, i.e. $u = 0$ a.e. in $\Omega \setminus C^h_{R_1(C, v)}$. Hence,

$$\min_{u \in BV(\Omega, [0, 1]), u = 0 \, \text{a.e. in} \, \Omega \setminus C^h_{R_1(C, v)}} B_\beta(u, v, C).$$

(6.5)

**Lemma 6.6.** Let $E_0, \beta$ satisfy (4.3), and $R_0$ be defined as in (4.4). Then $E_\lambda$ is a minimizer of $A_\beta(\cdot, E_0, \lambda)$ if and only if $\chi_{E_\lambda}$ is a minimizer of $B_\beta(\cdot, v^\lambda_{E_0}, C^H_{R_0})$, where $v^\lambda_{E_0} = \lambda \chi_{C^H_{R_0}}$. 

**Proof.** By (4.2) we have

$$A_\beta(E, E_0, \lambda) = B_\beta(\chi_{E}, v^\lambda_{E_0}, C^H_{R_0}) - \lambda \int_{E_0} \vec{d}_{E_0} \, dx \quad \forall E \in BV(C^H_{R_0}, [0, 1]).$$

(6.6)

Now if $E_\lambda$ minimizes $A_\beta(\cdot, E_0, \lambda)$, we have $E_\lambda \subseteq C^H_{R_0}$ (Theorem 4.1) and thus, for any $u \in BV(\Omega, [0, 1])$ with $u = 0$ a.e. in $\Omega \setminus C^H_{R_0}$ from (6.4)-(6.6) we deduce

$$B_\beta(u, v^\lambda_{E_0}, C^H_{R_0}) = \int_0^1 B_\beta(\chi_{\{u > t\}}, v^\lambda_{E_0}, C^H_{R_0}) \, dt = \int_0^1 A_\beta(\{u > t\}, E_0, \lambda) \, dt + \lambda \int_{E_0} \vec{d}_{E_0} \, dx$$

$$\geq \int_0^1 A_\beta(E_\lambda, E_0, \lambda) \, dt + \lambda \int_{E_0} \vec{d}_{E_0} \, dx = B_\beta(\chi_{E_\lambda}, v^\lambda_{E_0}, C^H_{R_0}).$$

By (6.5) $\chi_{E_\lambda}$ is a minimizer of $B_\beta(\cdot, v^\lambda_{E_0}, C^H_{R_0})$. 

17
Conversely, assume that $\chi_{E_\lambda}$ is a minimizer of $B_\beta(\cdot, v_{E_\lambda}^\beta, C_{H_R}^\beta)$, then by (6.6) $E_\lambda \subseteq C_{H_R}^\beta$ is a minimizer of $A_\beta(\cdot, E_0, \lambda)$ in $BV(C_{H_R}^\beta, \{0, 1\})$. Hence, by Remark 4.3 $E_\lambda$ is a minimizer of $A_\beta(\cdot, E_0, \lambda)$.

**Proposition 6.7 (Strong comparison for minimizers of $B_\beta$).** Assume that $v_1, v_2 \in L_{\text{loc}}^\infty(\Omega)$, $v_1 > v_2$ a.e. in $\Omega$ and $v_2 \geq 0$ a.e. in $\Omega \setminus C$. Suppose also that $\beta_1 \leq \beta_2$ satisfy (4.3). Let $u_1, u_2 \in BV(\Omega, [0, 1])$ be minimizers of $B_{\beta_1}(\cdot, v_1, C)$ and $B_{\beta_2}(\cdot, v_2, C)$ respectively. Then $u_1 \leq u_2$ a.e. in $\Omega$.

**Proof.** Adding the inequalities $B_{\beta_1}(u_1, v_1, C) \leq B_{\beta_1}(u_1 \land u_2, v_1, C)$ and $B_{\beta_2}(u_2, v_2, C) \leq B_{\beta_2}(u_1 \lor u_2, v_2, C)$ and using

$$\int_{\Omega} |D(u_1 \land u_2)| + \int_{\Omega} |D(u_1 \lor u_2)| \leq \int_{\Omega} |Du_1| + \int_{\Omega} |Du_2|,$$

we establish

$$\int_{\partial \Omega \cap \{u_1 > u_2\}} (\beta_2 - \beta_1)(u_1 - u_2) dH^n \leq \int_{\{u_1 > u_2\}} (v_2 - v_1)(u_1 - u_2) \, dx.$$

Since $v_1 > v_2$ and $\beta_1 \leq \beta_2$, this inequality holds if and only if $|\{u_1 > u_2\}| = 0$, i.e. $u_1 \leq u_2$ a.e. in $\Omega$.

**Proposition 6.8 (Comparison for minimizers of $B_\beta$).** Assume that $v_1, v_2 \in L_{\text{loc}}^\infty(\Omega)$, $v_1 \geq v_2$ a.e. in $\Omega$ and $v_2 \geq 0$ a.e. in $\Omega \setminus C$. Suppose also that $\beta_1 \leq \beta_2$ satisfy (4.3). Then:

a) there exists a minimizer $u_{1*}$ of $B_{\beta_1}(\cdot, v_1, C)$ such that $u_{1*} \leq u_2$ for any minimizer $u_2$ of $B_{\beta_2}(\cdot, v_2, C)$;

b) there exists a minimizer $u_2^*$ of $B_{\beta_2}(\cdot, v_2, C)$ such that $u_1 \leq u_2^*$ for any minimizer $u_1$ of $B_{\beta_1}(\cdot, v_1, C)$.

**Proof.** a) Take $\varepsilon \in (0, 1)$. Since $v_1 + \varepsilon > v_2$ a.e. in $\Omega$, by Proposition 6.7 any minimizer $u_1^*, u_2 \in BV(\Omega, [0, 1])$ of $B_{\beta_1}(\cdot, v_1 + \varepsilon, C)$ and $B_{\beta_2}(\cdot, v_2, C)$ respectively, satisfies $u_1^* \leq u_2$. Let $R_1 := \max\{R_1(C, v_1), R_1(C, v_2)\}$. By minimality, $B_{\beta_1}(u_1^*, v_1 + \varepsilon, C) \leq B_{\beta_1}(0, v_1 + \varepsilon, C) = 0$, and since by Remark 6.5 $u_1^* = 0$ a.e. in $\Omega \setminus C_{R_1}^\beta$, recalling (3.10) we get

$$\kappa \int_{\Omega} |Du_1^*| \leq (\|v_1\|_{L^\infty(C_{R_1}^\beta)} + 1)|C_{R_1}^\beta| < +\infty.$$

By compactness, there exists $u_{1*} \in BV(\Omega, [0, 1])$ such that, up to a (not relabelled) subsequence, $u_1^* \to u_{1*}$ in $L^1(\Omega)$ and a.e. in $\Omega$ as $\varepsilon \to 0^+$. Then any minimizer $u_2$ of $B_{\beta_2}(\cdot, v_2, C)$ satisfies $u_{1*} \leq u_2$ a.e. in $\Omega$.

It remains to show that $u_{1*}$ is a minimizer of $B_{\beta_1}(\cdot, v_1, C)$. By (6.5) we may consider only those $u \in BV(\Omega, [0, 1])$ with $u = 0$ a.e. in $\Omega \setminus C_{R_1}^\beta$ as a competitor. In this case, the continuity of $u \mapsto \int_{C_{R_1}^\beta} uv \, dx$, the minimality of $u_1^*$ and the lower semicontinuity of $c_{\beta}(\cdot, \Omega)$ imply

$$B_{\beta_1}(u, v_1, C) = \lim_{\varepsilon \to 0^+} B_{\beta_1}(u, v_1 + \varepsilon, C) \geq \liminf_{\varepsilon \to 0^+} B_{\beta_1}(u_1^*, v_1 + \varepsilon, C) \geq \liminf_{\varepsilon \to 0^+} c_{\beta_1}(u_1^*, \Omega) + \int_{C_{R_1}^\beta} u_{1*} v_1 \, dx = B_{\beta_1}(u_{1*}, v_1, C).$$

b) can be proven in a similar manner.

**Proof of Theorem 6.1.** Let $R := \max\{R(E_0), R(F_0)\}$, where $R(E_0)$ and $R(F_0)$ are defined as in (4.4). Then by Theorem 4.1 any minimizer $E_\lambda$ (resp. $F_\lambda$) of $A_{\beta_1}(\cdot, E_0, \lambda)$ (resp. $A_{\beta_2}(\cdot, F_0, \lambda)$) is contained in the cylinder $C := \overline{B_R} \times (0, H)$, where

$$H = 1 + \max\{\max_{(x', x_{n+1}) \in E_0} x_{n+1}, \max_{(x', x_{n+1}) \in F_0} x_{n+1}\}.$$
Set $v_1 := v_1(\lambda, E_0) = \lambda \tilde{d}_{E_0}$ and $v_2 := v_2(\lambda, F_0) = \lambda \tilde{d}_{F_0}$. Since $E_0 \subseteq F_0 \subset \Omega$, we have $\tilde{d}_{E_0} \geq \tilde{d}_{F_0}$. Moreover, by (4.3) there exists a cylinder $C := C^H_D$ such that $v_2 \geq 0$ in $\Omega \setminus C$.

a) Since $v_1 \geq v_2$ and $\beta_1 \leq \beta_2$, by Proposition 6.8 b) there exists a minimizer $u^*_2 := u^*_2(\lambda, F_0)$ of $B_{\beta_2}(\cdot, v_2, C)$ such that any minimizer $u_1$ of $B_{\beta_1}(\cdot, v_1, C)$ satisfies

$$u_1 \leq u^*_2. \tag{6.7}$$

By Remark 6.5 there exists $t \in (0, 1)$ such that $\chi_{\{u^*_2 > t\}}$ is a minimizer of $B_{\beta_2}(\cdot, v_2, C)$. Then, recalling the expression of $v_2$, by Lemma 6.6 $F^*_\lambda := \{u^*_2 > t\}$ is a minimizer of $A_{\beta_2}(\cdot, F_0, \lambda)$. Moreover, if $E_\lambda$ is a minimizer of $A_{\beta_1}(\cdot, E_0, \lambda)$, then by Lemma 6.6 $\chi_{E_\lambda}$ is a minimizer of $B_{\beta_1}(\cdot, v_1, C)$, and by (6.7) $\chi_{E_\lambda} \leq u^*_2$. In particular,

$$E_\lambda = \{\chi_{E_\lambda} > t\} \subseteq \{u^*_2 > t\} =: F^*_\lambda.$$

b) is analogous to a) using Proposition 6.8 a).

The last assertion follows with the same arguments from Lemma 6.6 and Proposition 6.7, since (6.1) implies that $\tilde{d}_{E_0} > \tilde{d}_{F_0}$. \hfill $\square$

One useful case is when $E_0$ is a constrained minimizer of $C_\beta(\cdot, \Omega)$ in $E(E_0)$: in this case $E_0$ acts as a barrier for minimizers of $A_\beta(\cdot, E_0, \lambda)$.

**Proposition 6.9.** Assume that $E_0, \beta_1, \beta_2$ satisfy (4.3). Let $\beta_1 \leq \beta_2$, $E_0$ be a constrained minimizer of $C_{\beta_2}(\cdot, \Omega)$ in $E(E_0)$ and $E_\lambda \in BV(\Omega, \{0, 1\})$ be a minimizer of $A_{\beta_1}(\cdot, E_0, \lambda)$. Then $E_\lambda \subseteq \overline{E_0}$.

**Proof.** Comparing $E_\lambda$ with $E_0 \cap E_\lambda$ we get

$$P(E_\lambda, \Omega) + \lambda \int_{E_\lambda \setminus E_0} \tilde{d}_{E_0} \, dx \leq P(E_\lambda \cap E_0, \Omega) + \int_{\partial \Omega} \beta_1 \chi_{E_\lambda \setminus E_0} \, d\mathcal{H}^n.$$  

From the constrained minimality of $E_0$ we have $C_{\beta_2}(E_0, \Omega) \leq C_{\beta_2}(E_0 \cup E_\lambda, \Omega)$, i.e.

$$P(E_0, \Omega) \leq P(E_0 \cup E_\lambda, \Omega) - \int_{\partial \Omega} \beta_2 \chi_{E_\lambda \setminus E_0} \, d\mathcal{H}^n.$$  

Adding these inequalities we obtain

$$P(E_\lambda, \Omega) + P(E_0, \Omega) + \lambda \int_{E_\lambda \setminus E_0} \tilde{d}_{E_0} \, dx \leq P(E_\lambda \cup E_0, \Omega) + P(E_\lambda \cap E_0, \Omega)$$

$$+ \int_{\partial \Omega} (\beta_1 - \beta_2) \chi_{E_\lambda \setminus E_0} \, d\mathcal{H}^n.$$  

Then the condition $\beta_1 \leq \beta_2$ and (2.2) yield that

$$\lambda \int_{E_\lambda \setminus E_0} \tilde{d}_{E_0} \, dx \leq 0.$$  

Since $\tilde{d}_{E_0} > 0$ outside $\overline{E_0}$, the last inequality is possible only if $|E_\lambda \setminus \overline{E_0}| = 0$, i.e. $E_\lambda \subseteq \overline{E_0}$. \hfill $\square$

Proposition 6.9 gives the following monotonicity principle.

**Proposition 6.10 (Monotonicity).** Assume that $E_0, \beta$ satisfy (4.3), $E_0$ is a constrained minimizer of $C_\beta(\cdot, \Omega)$ in $E(E_0)$ such that $|\overline{E_0} \setminus E_0| = 0$ and $E_\alpha \in BV(\Omega, \{0, 1\})$ is a minimizer of $A_\beta(\cdot, E_0, \alpha)$ for $\alpha \geq 1$. Then $E_\lambda \subseteq E_\mu$ for any $1 \leq \lambda < \mu$. Moreover, every $E_\alpha, \alpha \geq 1$ is also a constrained minimizer of $C_\beta(\cdot, \Omega)$ in $E(E_\alpha)$.

**Proof.** Comparison between $E_\lambda$ and $E_\lambda \cap E_\mu$ gives

$$P(E_\lambda, \Omega) + \lambda \int_{E_\lambda \setminus E_\mu} \tilde{d}_{E_0} \, dx \leq P(E_\lambda \cap E_\mu, \Omega) + \int_{\partial \Omega} \beta \chi_{E_\lambda \setminus E_\mu} \, d\mathcal{H}^n.$$  

Similarly, for $E_\mu$ and $E_\lambda \cup E_\mu$ we have

$$P(E_\mu, \Omega) \leq P(E_\lambda \cup E_\mu, \Omega) + \mu \int_{E_\mu \setminus E_\mu} \tilde{d}_{E_0} \, dx - \int_{\partial \Omega} \beta \chi_{E_\lambda \setminus E_\mu} \, d\mathcal{H}^n.$$  

19
Adding the above inequalities and using \((2.2)\) we obtain
\[
(\lambda - \mu) \int_{E_\lambda \setminus E_\mu} \hat{d}E_\mu \, dx \leq 0.
\]

By hypothesis \(|E_0 \setminus E_0| = 0\), according to Proposition 6.9, \(E_\lambda, E_\mu \subset E_0\). Thus \(\hat{d}E_\mu \leq 0\) in \(E_\lambda \setminus E_\mu\). But since \(\lambda < \mu\), \((6.8)\) is possible only if \(|E_\lambda \setminus E_\mu| = 0\), i.e. \(E_\lambda \subset E_\mu\).

To prove the final assertion take any set \(E \in \mathcal{E}(E_0)\). Then using \(A_\beta(E_\alpha, E_0, \alpha) \leq A_\beta(E_\alpha \cap E_0, E_0, \alpha)\), \(\alpha \int_{(E_0 \cap E) \setminus E_\alpha} dE_\alpha \, dx \geq 0\), and \(E_\alpha \subset E_0 \cap E\), we get
\[
C_\beta(E_\alpha, \Omega) \leq C_\beta(E_\alpha, \Omega) + \alpha \int_{(E_0 \cap E) \setminus E_\alpha} dE_\alpha \, dx \leq C_\beta(E \cap E_0, \Omega).
\]

Moreover, since \(C_\beta(E_0, \Omega) \leq C_\beta(E \cup E_0, \Omega)\), from \((2.2)\) we obtain
\[
C_\beta(E_\alpha, \Omega) + C_\beta(E_0, \Omega) \leq C_\beta(E_0 \cap E, \Omega) + C_\beta(E_0 \cup E, \Omega) \leq C_\beta(E, \Omega) + C_\beta(E_0, \Omega),
\]
i.e. \(C_\beta(E_\alpha, \Omega) \leq C_\beta(E, \Omega)\).

Proposition 6.11 (Comparison between minimizers of \(C_\beta\) and \(A_\beta\)). Suppose that \(E_0\) and \(\beta\) satisfy \((4.3)\).

a) Let \(E^+ \in BV(\Omega, \{0, 1\})\) be a constrained minimizer of \(C_\beta(\cdot, \Omega)\) in \(\mathcal{E}(E_0)\). Then every minimizer \(E_\lambda\) of \(A_\beta(\cdot, E_0, \lambda)\) satisfies \(E_\lambda \subseteq E^+\).

b) Let \(E^+ \in BV(\Omega, \{0, 1\})\) be a bounded constrained minimizer of \(C_\beta(\cdot, \Omega)\) in \(\mathcal{E}(E^+)\). Then for every \(E_0 \subseteq E^+\) and for every minimizer \(E_\lambda\) of \(A_\beta(\cdot, E_0, \lambda)\) one has \(E_\lambda \subseteq E^+\). Moreover, \(E^+\) can be chosen such that \(|E^+ \setminus E^+| = 0\).

Proof. a) By Proposition 4.4 \(E^+\) is a constrained minimizer of \(C_\beta(\cdot, \Omega)\) in \(\mathcal{E}(E^+)\). Let \(E_\lambda^+\) be the maximal minimizer of \(A_\beta(\cdot, E^+, \lambda)\) (Definition 6.4). By Proposition 6.9 we have \(E_\lambda^+ \subseteq E^+\). Take any minimizer \(E_\lambda\) of \(A_\beta(\cdot, E_0, \lambda)\). Since \(E_0 \subseteq E^+\), by Theorem 6.1 a) we have \(E_\lambda \subseteq E_\lambda^+ \subseteq E^+\).

b) The proof of the first part is exactly the same as the proof of a). To prove the second part, we take any \(E_0' \in BV(\Omega, \{0, 1\})\) satisfying the hypotheses of Proposition 5.8 and containing \(E_0\). By Theorem 4.4 there exists a constrained minimizer \(E^+\) of \(C_\beta(\cdot, \Omega)\) in \(\mathcal{E}(E_0')\). In particular, \(E^+\) is bounded, and by Proposition 5.8 \(H^n(\partial E^+) = P(E^+) < +\infty\). Since \(E^+ \setminus E^+ \subseteq E^+\), we have \(|E^+ \setminus E^+| = 0\).

7. Existence of a Generalized Minimizing Movement

Consider the functional \(\widehat{A}_\beta : BV(\Omega, \{0, 1\}) \times BV(\Omega, \{0, 1\}) \times [1, +\infty) \times \mathbb{Z} \to [-\infty, +\infty]\) given by
\[
\widehat{A}_\beta(F, G, \lambda, k) := \begin{cases} 
A_\beta(F, G, \lambda) & \text{if } k > 0, \\
|F \Delta G| & \text{if } k \leq 0.
\end{cases}
\]

For any \(k \in \mathbb{N}\) we build the family of sets \(E_\lambda(k)\) iteratively as follows: \(E_\lambda(0) := E_0\) and \(E_\lambda(k), k \geq 1,\) is a minimizer of \(\widehat{A}_\beta(\cdot, E_\lambda(k-1), \lambda, k)\) in \(BV(\Omega, \{0, 1\})\); notice that existence of minimizers follows from Theorem 4.1.

From now on, we omit the dependence on \(k\) of \(\widehat{A}_\beta\), and we use the notation \(\widehat{A}_\beta(F; G, \lambda)\).

Theorem 7.1 (Existence). Let \(E_0\) and \(\beta\) satisfy \((5.1)\). Then \(GMM(E_0)\) is nonempty, i.e. there exist a map \(t \in [0, +\infty) \mapsto E(t) \in BV(\Omega, \{0, 1\})\) and a diverging sequence \(\{\lambda_j\} \subset [1, +\infty)\) such that
\[
\lim_{j \to +\infty} |E_{\lambda_j}(\lambda_j t) \Delta E(t)| = 0, \quad t \in [0, +\infty),
\]
Moreover, every \(GMM\) \(t \in [0, +\infty) \mapsto E(t)\) starting from \(E_0\) is contained in a bounded set depending only on \(E_0\) and \(\beta\), and belongs to \(C^{1/2}_{\text{loc}}([0, +\infty), L^1(\Omega))\), in the sense that
\[
|E(t) \Delta E(t')| \leq \theta(n, \kappa) P(E_0)|t - t'|^{1/2} \quad \text{for all } t, t' > 0, |t - t'| < 1,
\]
where \( \theta(n, \kappa) = \frac{C_{n, \kappa}^n}{\kappa} + 1 \) and \( C_{n, \kappa} \) is defined in (5.20). If in addition \( |E_0 \setminus E| = 0 \), then (7.2) holds for any \( t, t' \geq 0 \) with \( |t - t'| < 1 \). Finally,

\[
\nu_{\lambda_j}([\lambda_j t]) \mathcal{H}^n \partial^s E_{\lambda_j}([\lambda_j t]) \xrightarrow{w^*} \nu_{\partial E(t)} \mathcal{H}^n \partial^s E(t) \quad \text{for all} \quad t \geq 0 \quad \text{as} \quad \lambda_j \to +\infty. \tag{7.3}
\]

**Proof.** Given \( k \geq 0 \) set \( d_k(\cdot) := \text{dist}(\cdot, \Omega \cap \partial E_\lambda(k)) \). Then for \( k \geq 1 \) the minimality of \( E_\lambda(k) \) entails

\[
A_\beta(E_\lambda(k), E_\lambda(k-1), \lambda) \leq A_\beta(E_\lambda(k-1), E_\lambda(k-1), \lambda),
\]

i.e.

\[
C_\beta(E_\lambda(k), \Omega) + \lambda \int_{E_\lambda(k) \Delta E_\lambda(k-1)} d_{k-1} \, dx \leq C_\beta(E_\lambda(k-1), \Omega). \tag{7.4}
\]

In particular, the sequence \( k \in \mathbb{N} \cup \{0\} \mapsto C_\beta(E_\lambda(k), \Omega) \) is nonincreasing and

\[
C_\beta(E_\lambda(k), \Omega) \leq C_\beta(E_\lambda(0), \Omega) = C_\beta(E_0, \Omega) \leq P(E_0). \tag{7.5}
\]

Let \( t > 0 \) and set \( k = \lfloor \lambda t \rfloor \). Then (3.9) yields

\[
\kappa P(E_\lambda([\lambda t])) \leq C_\beta(E_\lambda([\lambda t]), \Omega) \leq P(E_0). \tag{7.6}
\]

Take \( t_1, t_2 > 0 \), \( t_1 < t_2 \) and let \( \lambda \geq 1 \) be large enough that for some \( k_0, N \in \mathbb{N}, N \geq 3 \)

\[
k_0 = \lfloor \lambda t_1 \rfloor, \quad k_0 + N - 1 = \lfloor \lambda t_2 \rfloor,
\]

i.e.

\[
\frac{k_0}{\lambda} \leq t_1 < \frac{k_0 + 1}{\lambda} < \ldots < \frac{k_0 + N - 1}{\lambda} \leq t_2 < \frac{k_0 + N}{\lambda}.
\]

Then

\[
N - \frac{2}{\lambda} = \frac{k_0 + N - 1 - (k_0 + 1)}{\lambda} \leq t_2 - t_1. \tag{7.7}
\]

Since all \( E_\lambda(s), s \geq 1 \) satisfy uniform density estimates (5.4)-(5.5) (Theorem 5.1), by Proposition 5.7 we have\(^3\)

\[
|E_\lambda([\lambda t_2]) \Delta E_\lambda([\lambda t_1])| = |E_\lambda(k_0 + N - 1) \Delta E_\lambda(k_0)| \leq \sum_{s = k_0}^{k_0 + N - 2} |E_\lambda(s) \Delta E_\lambda(s + 1)| \leq C_{n, \ell} \sum_{s = k_0}^{k_0 + N - 2} P(E_\lambda(s)) \leq \sum_{s = k_0}^{k_0 + N - 2} \frac{P(E_0)}{\kappa} (N - 1). \tag{7.8}
\]

Moreover, for any \( s \in \mathbb{N} \), by (7.4)

\[
\int_{E_\lambda(s+1) \Delta E_\lambda(s)} dE_\lambda(s) \, dx \leq \frac{1}{\lambda} \left( C_\beta(E_\lambda(s), \Omega) - C_\beta(E_\lambda(s + 1), \Omega) \right),
\]

and thus

\[
\sum_{s = k_0}^{k_0 + N - 2} \int_{E_\lambda(s+1) \Delta E_\lambda(s)} dE_\lambda(s) \, dx \leq \frac{1}{\lambda} \sum_{s = k_0}^{k_0 + N - 2} \left( C_\beta(E_\lambda(s), \Omega) - C_\beta(E_\lambda(s + 1), \Omega) \right)
\]

\[
= \frac{1}{\lambda} \left( C_\beta(E_\lambda(k_0), \Omega) - C_\beta(E_\lambda(k_0 + N - 1), \Omega) \right).
\]

\(^3\)Notice that at this point we use \( t_1 > 0 \); since a priori we do not know whether \( E_0 \) satisfies the density estimates, we cannot start summing from \( s = 0 = k_0 \).
Using (7.5) and the nonnegativity of $C_\beta(\cdot, \Omega)$ we get

\[
\sum_{s=k_0}^{k_0+N-2} \int_{E_\lambda(s+1) \Delta E_\lambda(s)} dE_\lambda(s) \, dx \leq \frac{P(E_0)}{\lambda}. \tag{7.10}
\]

Thus, from (7.8), (7.9) and (7.10)

\[
|E_\lambda([\lambda t_1]) \Delta E_\lambda([\lambda t_2])| \leq \frac{C_{n,\kappa} P(E_0)}{\kappa} (N-1) \ell + \frac{P(E_0)}{\lambda \ell}. \tag{7.11}
\]

Now take $\lambda$ so large that

\[
t_2 - t_1 > \frac{1}{\gamma(n, \kappa)^2 \lambda},
\]

so that Proposition 5.7 holds for $\ell = \frac{1}{\lambda(t_2-t_1)^{1/2}}$. From (7.11) and (7.7) we obtain

\[
|E_\lambda([\lambda t_1]) \Delta E_\lambda([\lambda t_2])| \leq \frac{C_{n,\kappa} P(E_0)}{\kappa} \frac{N-2}{\lambda |t_2-t_1|^{1/2}} + \frac{1}{\lambda} \frac{C_{n,\kappa} P(E_0)}{\kappa} |t_2-t_1|^{1/2} + P(E_0) |t_2-t_1|^{1/2}
\]

\[
\leq \theta(n, \kappa) P(E_0) |t_2-t_1|^{1/2} + \frac{1}{\lambda} \frac{C_{n,\kappa} P(E_0)}{\kappa} |t_2-t_1|^{1/2}. \tag{7.12}
\]

By Proposition 6.11 b) there exists a constrained minimizer $E^+ \supseteq E_0$ of $C_\beta(\cdot, \Omega)$ in $E(E^+)$ such that $|E^+ \setminus E^+| = 0$ and $E_\lambda(1) \subseteq E^+$. By induction, we can show that $E_\lambda(k) \subseteq E^+$ for all $k \geq 1$. Consider now an arbitrary diverging sequence $\{\lambda_j\}$. Compactness and a diagonal process yield the existence of a subsequence (still denoted by $\{\lambda_j\}$) such that $E_{\lambda_j}([\lambda_j t])$ converges in $L^1(\Omega)$ to a set $E(t)$ for any rational $t \geq 0$ as $j \to +\infty$.

If $t_1, t_2 \in \mathbb{Q} \cap (0, +\infty)$, with $0 < |t_1 - t_2| < 1$, letting $\lambda_j \to +\infty$ in (7.12) we get

\[
|E(t_1) \Delta E(t_2)| \leq \theta(n, \kappa) P(E_0) |t_2-t_1|^{1/2}. \tag{7.13}
\]

By completeness of $L^1(\Omega)$ we can uniquely extend $\{E(t) : t \in \mathbb{Q} \cap (0, +\infty)\}$ to a family $\{E(t) : t \in (0, +\infty)\}$ preserving the Hölder continuity (7.13) in $(0, +\infty)$. Now we show (7.1). If $t = 0$, $E_0 = E_{\lambda_j}(0) \to E(0)$ in $L^1(\Omega)$ as $j \to +\infty$. If $t > 0$, take any $\varepsilon \in (0, 1)$ and $t_\varepsilon \in \mathbb{Q} \cap (0, +\infty)$ such that $|t - t_\varepsilon| < \varepsilon$. By the choice of $\{\lambda_j\}$, (7.1) holds for $t_\varepsilon$ and thus, using (7.12)-(7.13) we get

\[
\lim_{j \to +\infty} \sup_{|E_{\lambda_j}([\lambda_j t]) \Delta E(t)|} \leq \lim_{j \to +\infty} \sup_{|E_{\lambda_j}([\lambda_j t]) \Delta E_{\lambda_j}([\lambda_j t_\varepsilon])|} + \lim_{j \to +\infty} \sup_{|E_{\lambda_j}([\lambda_j t_\varepsilon]) \Delta E(t)|} + |E(t_\varepsilon) \Delta E(t)|
\]

\[
\leq 2 \theta(n, \kappa) P(E_0) |t - t_\varepsilon|^{1/2} < 2 \theta(n, \kappa) P(E_0) \sqrt{\varepsilon}.
\]

Therefore, letting $\varepsilon \to 0^+$ we get (7.1).

When $|E_0 \setminus E_0| = 0$, for any $t \in (0, 1)$, choosing $\lambda$ sufficiently large, from (7.12) we obtain

\[
|E_\lambda([\lambda t]) \Delta E(0)| \leq |E_\lambda([\lambda t]) \Delta E_\lambda(1)| + |E_\lambda(1) \Delta E_0|
\]

\[
\leq \theta(n, \kappa) P(E_0) |t - \frac{1}{\lambda}|^{1/2} + \frac{1}{\lambda} \frac{C_{n,\kappa} P(E_0)}{\kappa} |t - \frac{1}{\lambda}|^{1/2} + |E_\lambda(1) \Delta E_0|. \tag{7.14}
\]

By Lemma 4.6 a) the last term on the right hand side converges to 0 as $\lambda \to +\infty$. Hence letting $\lambda \to +\infty$ in (7.14) we get the $(1/2)$-Hölder continuity of $t \mapsto E(t)$ in $[0, +\infty)$.

Now let us prove (7.3). We need to show that for any $t \in [0, +\infty)$

\[
\lim_{j \to +\infty} \int_{\partial^* E_{\lambda_j}([\lambda_j t])} \phi \cdot \nu_{E_{\lambda_j}([\lambda_j t])} \, d\mathcal{H}^n = \int_{\partial^* E(t)} \phi \cdot \nu_{E(t)} \, d\mathcal{H}^n \quad \forall \phi \in C_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}).
\]
If \( \phi \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \), by the generalized divergence formula (2.3) and by (7.1) we have

\[
\lim_{j \to +\infty} \int_{\partial^* E_{\lambda_j}(\{\lambda_j t\})} \phi \cdot \nu_{E_{\lambda_j}(\{\lambda_j t\})} d\mathcal{H}^n = \lim_{j \to +\infty} \int_{E_{\lambda_j}(\{\lambda_j t\})} \text{div } \phi d\mathcal{H}^n = \int_{E(t)} \text{div } \phi d\mathcal{H}^n = \int_{\partial^* E(t)} \phi \cdot \nu_{E(t)} d\mathcal{H}^n.
\]  

(7.15)

In general, we approximate \( \phi \in C_c(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \) uniformly with \( \phi_k \in C_c^1(\mathbb{R}^{n+1}, \mathbb{R}^{n+1}) \), \( k \geq 1 \) and use the previous result.

Finally, if \( \{E(t)\}_{t \geq 0} \in GMM(E_0) \), then by construction and Proposition 6.11 b) one has \( E_{\lambda_j}(\{\lambda_j t\}) \subseteq E^+ \), where \( E^+ := E^+(E_0, \beta) \) is a bounded minimizer of \( C_\beta(\cdot, \Omega) \) in \( \mathcal{E}(E^+) \); therefore \( E(t) \subseteq E^+ \) for all \( t \geq 0 \).

\[\square\]

**Definition 7.2 (Maximal and minimal GMM).** Let \( E_0, \beta \) satisfy (5.1), and \( \{\lambda_j\} \) be a diverging sequence such that

\[E^*(t) := \lim_{j \to +\infty} E_{\lambda_j}(\{\lambda_j t\})^* \quad \forall t \geq 0\]

exist in \( L^1(\Omega) \), where \( E_{\lambda_j}(\{\lambda_j t\})^* \) is the maximal minimizer of \( \mathcal{A}_\beta(\cdot, E_{\lambda_j}(\{\lambda_j t\} - 1)^*, \lambda_j) \) with \( (E_0)^* := E_0 \) (Definition 6.4). We call \( E^*(t) \) the maximal GMM associated to the sequence \( \{\lambda_j\} \). Analogously,

\[E_*(t) := \lim_{j \to +\infty} E_{\lambda_j}(\{\lambda_j t\})_* \quad \forall t \geq 0,\]

obtained using the minimal minimizers \( E_{\lambda_j}(\{\lambda_j t\})_* \) of \( \mathcal{A}_\beta(\cdot, E_{\lambda_j}(\{\lambda_j t\} - 1)_*, \lambda_j) \) with \( (E_0)_* := E_0 \), is called the minimal GMM associated to the sequence \( \{\lambda_j\} \).

Observe that if \( t \mapsto E(t) \) is any GMM obtained by the sequence \( \{\lambda_j\} \), then according to the proof of Theorem 7.1 (possibly passing to nonrelabelled subsequences) there exist the maximal GMM \( t \mapsto E^*(t) \) and the minimal GMM \( t \mapsto E_*(t) \) associated to \( \{\lambda_j\} \). Now by Remark 6.3 one has \( E_*(t) \subseteq E(t) \subseteq E^*(t) \) for all \( t \geq 0 \).

**Theorem 7.3 (Comparison principle for maximal and minimal GMM).** Let \( E_0, F_0, \beta_1, \beta_2 \) satisfy (5.1) with \( E_0 \subseteq F_0 \) and \( \beta_1 \leq \beta_2 \). If \( E_*(t) \) and \( F_*(t) \) are minimal GMMs associated to a sequence \( \{\lambda_j\} \), then \( E_*(t) \subseteq F_*(t) \) for all \( t \geq 0 \). Analogously, if \( E^*(t) \) and \( F^*(t) \) are maximal GMMs associated to \( \{\lambda_j\} \), then \( E^*(t) \subseteq F^*(t) \) for all \( t \geq 0 \).

**Proof.** Since \( E_0 \subseteq F_0 \), and \( \beta_1 \leq \beta_2 \), by definition of \( E_{\lambda}(k)^* \) and \( F_{\lambda}(k)^* \) (resp. \( E_{\lambda}(k)_* \) and \( F_{\lambda}(k)_* \)) and by Theorem 6.1, we have \( E_{\lambda_0}(k) \subseteq F_{\lambda_0}(k) \) (resp. \( E_{\lambda}(k)^* \subseteq F_{\lambda}(k)^* \)) which implies \( E_*(t) \subseteq F_*(t) \) (resp. \( E^*(t) \subseteq F^*(t) \)) for all \( t \geq 0 \).

From the proof of Theorem 7.1 and Propositions 6.9 - 6.10 we get the following result (compare with [11]), that could be applied, for instance, to \( E_0 \) as in Example 4.5.

**Theorem 7.4.** Let \( E_0 \) be a constrained minimizer of \( C_\beta(\cdot, \Omega) \) in \( \mathcal{E}(E_0) \) such that \( |E_0 \setminus E_0| = 0 \). Then every maximal (minimal) GMM \( t \mapsto E(t) \) starting from \( E_0 \) satisfies \( E(t) \subseteq E(t') \) provided \( t > t' \geq 0 \).

**Proof.** Applying Propositions 6.9 and 6.10 inductively to maximal minimizers \( E_{\lambda}(k)^* \) of \( \mathcal{A}_\beta(\cdot, E_{\lambda}(k - 1)^*, \lambda) \) we get \( E_{\lambda}(k)^* \subseteq E_{\lambda}(k - 1)^* \) for all \( k \geq 1 \) and \( \lambda \geq 1 \). Hence, if \( t > t' \geq 0 \) then \( E_{\lambda}([\lambda t]^*) \subseteq E_{\lambda}([\lambda t'])^* \). Now the assertion of the theorem follows from (7.1). The arguments for minimal minimizers are the same.

\[\square\]

8. **GMM as a distributional solution**

The aim of this section is to prove that under suitable assumptions GMM is in fact a distributional solution of (1.1)-(1.2). Let us start with the following

**Definition 8.1 (Admissible variation).** A vector field \( X = (X', X_{n+1}) \in C_c^1(\Omega, \mathbb{R}^{n+1}) \) is called admissible if \( X \cdot e_{n+1} = 0 \) on \( \partial \Omega \).
Observe that if \( X \in C^1_c(\Omega, \mathbb{R}^{n+1}) \) is admissible, then for any \( s \in (-\varepsilon, \varepsilon) \) with \( \varepsilon > 0 \) sufficiently small, the vector field \( f_s = \text{Id} + sX \) is a \( C^1 \)-diffeomorphism that satisfies \( f_s(\Omega) = \Omega \), \( f_s(\overline{\Omega}) = \overline{\Omega} \).

**Proposition 8.2 (First variation of \( A_\beta \)).** Suppose that \( E_0, \beta \) satisfy assumptions (5.1) and let \( E \in BV(\Omega, \{0,1\}) \) be bounded with \( \text{Tr}(E) \in BV(\mathbb{R}^n, \{0,1\}) \). Then

\[
\frac{d}{ds}\bigg|_{s=0} A_\beta(f_s(E), E_0, \lambda) = \int_{\Omega \cap \partial^* E} (\text{div} X - \nu_E \cdot (\nabla X)\nu_E) \, d\mathcal{H}^n \quad \text{(8.1)}
\]

where \( \partial^* \text{Tr}(E) \) is the essential boundary of \( \text{Tr}(E) \) on \( \partial \Omega \) and \( \nu'_{\text{Tr}(E)} \) is the outer unit normal to \( \text{Tr}(E) \subset \mathbb{R}^n \).

**Proof.** From [42, Theorem 17.5]

\[
\frac{d}{ds}\bigg|_{s=0} P(f_s(E), \Omega) = \int_{\Omega \cap \partial^* E} (\text{div} X - \nu_E \cdot (\nabla X)\nu_E) \, d\mathcal{H}^n.
\]

Moreover, [42, Theorem 17.8] and the admissibility of \( X \) imply that

\[
\frac{d}{ds}\bigg|_{s=0} \int_{J_{f_s(E)}} \tilde{d}_{E_0} \, dx = \int_{\partial^* E} \tilde{d}_{E_0} \, \nu_E \, d\mathcal{H}^n = \int_{\partial E} \tilde{d}_{E_0} X \cdot \nu_E \, d\mathcal{H}^n.
\]

Finally, since \( \text{Tr}(E) \) is a set of finite perimeter in \( \partial \Omega \equiv \mathbb{R}^n \), again using [42, Theorem 17.8] we get

\[
\frac{d}{ds}\bigg|_{s=0} \beta \chi_{J_{f_s(E)}} d\mathcal{H}^n = \int_{\partial^* \text{Tr}(E)} \beta X' \cdot \nu'_{\text{Tr}(E)} d\mathcal{H}^{n-1}.
\]

\[\square\]

**Remark 8.3.** Under assumptions (5.1) and \( \beta \in \text{Lip}(\partial \Omega) \), if \( E_\lambda \) is a minimizer of \( A_\beta(\cdot, E_0, \lambda) \), and if \( \Omega \cap \partial E_\lambda \) is a \( C^2 \)-manifold with \( \mathcal{H}^n-1 \)-rectifiable boundary, then the mean curvature \( H_{E_\lambda} \) of \( \Omega \cap \partial E_\lambda \) is equal to \( -\lambda \tilde{d}_{E_0} \). Indeed, using the tangential divergence formula for manifolds with boundary we have

\[
\int_{\Omega \cap \partial E_\lambda} (\text{div} X - \nu_{E_\lambda} \cdot (\nabla X)\nu_{E_\lambda}) \, d\mathcal{H}^n = \int_{\Omega \cap \partial E_\lambda} H_{E_\lambda} X \cdot \nu_{E_\lambda} \, d\mathcal{H}^n + \int_{\partial^* \text{Tr}(E_\lambda)} X' \cdot n^\lambda \, d\mathcal{H}^{n-1},
\]

where \( n^\lambda = (n^\lambda, n^\lambda_{n+1}) \) is the outer unit conormal to \( \overline{\Omega \cap \partial E_\lambda} \) at \( \overline{\Omega \cap \partial E_\lambda} \cap \partial \Omega \). By minimality of \( E_\lambda \), we have

\[
\frac{d}{d\lambda} A_\beta(f_s(E_\lambda), E_0, \lambda) = 0, \quad \text{i.e.}
\]

\[
\int_{\Omega \cap \partial E_\lambda} (H_{E_\lambda} + \lambda \tilde{d}_{E_0}) X \cdot \nu_{E_\lambda} \, d\mathcal{H}^n + \int_{\partial^* \text{Tr}(E_\lambda)} X' \cdot (n^\lambda - \beta \nu'_{\text{Tr}(E_\lambda)}) \, d\mathcal{H}^{n-1} = 0.
\]

This implies \( H_{E_\lambda} = -\lambda \tilde{d}_{E_0} \) and \( n^\lambda = \beta \nu'_{\text{Tr}(E_\lambda)} \). Notice that from the latter in particular, we get

\[
\beta = n^\lambda \cdot (\nu'_{\text{Tr}(E_\lambda)}, 0) = \nu_{E_\lambda} \cdot e_{n+1},
\]

accordingly for instance with Theorem 5.3.

Remark 8.3 motivates the following definition [9, 42].

**Definition 8.4 (Distributional mean curvature).** Let \( E \in BV(\Omega, \{0,1\}) \). The function \( H_{E} \in L^1(\Omega \cap \partial^* E; \mathcal{H}^n \mathbf{L}(\Omega \cap \partial^* E)) \) is called distributional mean curvature of \( \Omega \cap \partial^* E \) if for every \( X \in C^1_c(\Omega, \mathbb{R}^{n+1}) \) the generalized tangential divergence formula holds:

\[
\int_{\Omega \cap \partial^* E} (\text{div} X - \nu_E \cdot (\nabla X)\nu_E) \, d\mathcal{H}^n = \int_{\Omega \cap \partial^* E} H_{E} X \cdot \nu_E \, d\mathcal{H}^n.
\]

Given \( x \in \mathbb{R}^{n+1} \) and \( t > 0 \) set

\[
v_\lambda(t, x) := \begin{cases} 
-\lambda \tilde{d}_{E_\lambda}(|\lambda|^{-1})(x) & \text{if } t \geq \frac{1}{\lambda}, \\
0 & \text{if } t \in [0, \frac{1}{\lambda}).
\end{cases}
\]

**Remark 8.5.** By Theorem 5.3, \( \text{Tr}(E_\lambda([\lambda])) \in BV(\mathbb{R}^n, \{0,1\}) \).
The next result relates GMM with distributional solutions of (1.1)-(1.2).

**Theorem 8.6 (GMM is a distributional solution).** Let $E_0, \beta$ satisfy (5.1), $|E_0| = 0$, $\{E(t)\}_{t \geq 0}$ be a GMM starting from $E_0$ obtained along the diverging sequence $\{\lambda_j\}$. Suppose that
\[
\mathcal{H}^n \mathbf{L}(\Omega \cap \partial^* E_{\lambda_j}(\lambda_j t)) \overset{loc}{\underset{\lambda \to \infty}{\to}} \mathcal{H}^n \mathbf{L}(\Omega \cap \partial^* E(t)) \quad \text{as } j \to +\infty \text{ for a.e. } t \geq 0.
\]
Then there exist a function $v : [0, +\infty) \times \Omega \to \mathbb{R}$ with
\[
\int_0^\infty \int_{\Omega \cap \partial^* E(t)} (v)^2 \, d\mathcal{H}^n \, dt \leq \alpha(n, \kappa) P(E_0),
\]
and a (not relabelled) subsequence such that
\[
\lim_{j \to +\infty} \int_0^\infty \int_{\Omega \cap \partial^* E_{\lambda_j}(\lambda_j t)} \phi v_{\lambda_j} \, d\mathcal{H}^n \, dt = \int_0^\infty \int_{\Omega \cap \partial^* E(t)} \phi v \, d\mathcal{H}^n \, dt,
\]
for any $\phi \in C_c(\Omega)$, $\Psi \in C_c((0, +\infty) \times \Omega, \mathbb{R}^{n+1})$, where $\alpha(n, \kappa) := \frac{75(n+1)\omega_{n+1} + \omega_n b(n)}{(n/2)^n n^{n+1}}$. Moreover, $\{E(t)\}_{t \geq 0}$ solves (1.1)-(1.2) with initial datum $E_0$ in the following sense:

(i) for a.e. $t \geq 0$ the set $\Omega \cap \partial^* E(t)$ has distributional mean curvature $H_{E(t)} = v$ and if $1 \leq n \leq 6$, for every $\phi \in C_c^1([0, +\infty) \times \Omega)$:

\[
\int_0^\infty \int_{E(t)} \partial_t \phi \, dx \, dt + \int_{E(0)} \phi(0, x) \, dx = \int_0^\infty \int_{\Omega \cap \partial^* E(t)} \phi H_{E(t)} \, d\mathcal{H}^n \, dt;
\]

(ii) if $\beta \in \text{Lip}(\partial^* \Omega)$ and there exists $h \in L^1_{\text{loc}}([0, +\infty))$ such that
\[
P(\text{Tr}(E_{\lambda_j}(\lambda_j t))) \leq h(t) \quad \text{for all } j \geq 1 \text{ and a.e. } t \geq 0,
\]
then $\text{Tr}(E(t)) \in BV(\mathbb{R}^n, \{0, 1\})$ for a.e. $t > 0$ and
\[
\int_{\Omega \cap \partial^* E(t)} \left( \text{div } X - \nu_{E(t)} \cdot (\nabla X) \nu_{E(t)} \right) \, d\mathcal{H}^n
\]
\[
= \int_{\Omega \cap \partial^* E(t)} H_{E(t)} X \cdot \nu_{E(t)} \, d\mathcal{H}^n + \int_{\partial^* \text{Tr}(E(t))} \beta X' \cdot \nu_{\text{Tr}(E(t))} \, d\mathcal{H}^{n-1}
\]
for every admissible $X \in C_c^1(\Omega, \mathbb{R}^{n+1})$.

The need for assumption (8.3) is not surprising; see [41, 46] for conditional results obtained in other contexts in a similar spirit. We postpone the proof after several auxiliary results.

**Proposition 8.7.** Assume that $E_0$ and $\beta$ satisfy (5.1). Then for any $\lambda \geq 1$ and a.e. $t \geq 1/\lambda$ the function $v_{\lambda}(t, \cdot)$ is the distributional mean curvature of $E_\lambda(\lambda t)$.

**Proof.** Set $E := E_\lambda(\lambda t)$. Remark 8.5 and (8.1) imply that
\[
\int_{\Omega \cap \partial^* E} \left( \text{div } X - \nu_{E} \cdot (\nabla X) \nu_{E} \right) \, d\mathcal{H}^n
\]
Hence, it suffices to prove $v_{\lambda}(t, \cdot) \in L^1(\Omega \cap \partial^* E; \mathcal{H}^n \mathbf{L}(\Omega \cap \partial^* E))$ for a.e. $t \in [1/\lambda, +\infty)$ and since $P(E(t), \Omega) < +\infty$, this follows from Lemma 8.9 below.

**Remark 8.8.** From Definition 8.4, Proposition 8.7 and Lemma 8.9 it follows that
\[
v_{\lambda}(t, x) = H_{E_\lambda(\lambda t)}(t, x) \quad \text{for a.e. } t \geq 1/\lambda \text{ and } \mathcal{H}^n\text{-a.e. } x \in \Omega \cap \partial E_\lambda(\lambda t).
\]
This is a discretized version of equation (1.1).
Lemma 8.9 (Uniform $L^2$-bound of the approximate velocities). Under assumptions (5.1) the inequality
\[ \int_0^{+\infty} \int_{\Omega \cap \partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n \, dt \leq \alpha(n, \kappa) P(E_0) \]
holds.

Proof. The proof is analogous to the proof of [46, Lemma 3.6]. Given $\varepsilon > 0$ and $E \in BV(\Omega, \{0, 1\})$ let
\[ (\partial E)_+ := \{ x \in \mathbb{R}^{n+1} : \text{dist}(x, \Omega \cap \partial E) \leq \varepsilon \}. \]
For $t \in [1, +\infty)$ and $\ell \in \mathbb{Z}$ such that $\ell \leq 1 + \lfloor \log_2(R(n, \kappa)\lambda^{1/2}) \rfloor$, where $R(n, \kappa)$ is given by (5.2), define
\[ K(\ell) = \left\{ x \in \left( \partial E_\lambda([\lambda t] - 1) \right)_+^{R(n, \kappa)\lambda^{-1/2}} : 2^\ell < |v_\lambda(x, t)| \leq 2^{\ell+1} \right\}. \]

By Proposition 5.5 $E_\lambda([\lambda t]) \Delta E_\lambda([\lambda t] - 1) \subseteq \cup_{\ell} K(\ell)$. Take $x \in K(\ell) \cap \Omega \cap \partial E_\lambda([\lambda t])$. Then $B_{2^{\ell-1}}(x) \cap E_\lambda([\lambda t] - 1) = \emptyset$ and hence, by Remark 5.4 the following density estimates hold:
\[ |E_\lambda([\lambda t]) \cap B_{2^{\ell-1}}(x)| \geq \frac{\kappa}{2} \omega_{n+1} \left( \frac{2^{\ell-1}}{\lambda} \right)^{n+1}, \]
\[ \mathcal{H}^n(B_{2^{\ell-1}}(x) \cap \Omega \cap \partial E_\lambda([\lambda t])) \leq \left( (n+1) \omega_{n+1} + \omega_n \right) \left( \frac{2^{\ell-1}}{\lambda} \right)^n. \]

Using $2^{\ell-1} \leq |v_\lambda(y, t)| \leq 5 \cdot 2^{\ell-1}$ for any $y \in B_{2^{\ell-1}}(x)$, from (8.10) we deduce
\[ \int_{B_{2^{\ell-1}}(x) \cap \Omega \cap \partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n \leq 25[(n+1)\omega_{n+1} + \omega_n](2^{\ell-1})^2 \left( \frac{2^{\ell-1}}{\lambda} \right)^n \]
\[ \leq 25[(n+1)\omega_{n+1} + \omega_n] \frac{\lambda}{\kappa/2} \omega_{n+1} \int_{B_{2^{\ell-1}}(x) \cap (E_\lambda([\lambda t]) \Delta E_\lambda([\lambda t] - 1))} |v_\lambda| \, dx. \]

Application of the Besicovitch covering theorem to the collection of balls $\{ B_{2^{\ell-1}}(x) : x \in K(\ell) \cap \partial E_\lambda([\lambda t]) \}$ gives
\[ \int_{K(\ell) \cap \Omega \cap \partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n \leq \frac{25[(n+1)\omega_{n+1} + \omega_n](n) b(n)}{(\kappa/2)^{n+1} \omega_{n+1}} \lambda \int_{\{2^{\ell-1} \leq |v_\lambda| \leq 2^{\ell+2}\} \cap (E_\lambda([\lambda t]) \Delta E_\lambda([\lambda t] - 1))} |v_\lambda| \, dx. \]

Now summing up these inequalities over $\ell \in \mathbb{Z}$ with $\ell \leq 1 + \lfloor \log_2(R(n, \kappa)\lambda^{1/2}) \rfloor$, and using the properties of $K(\ell)$ and the definition of $\alpha(n, \kappa)$ we get
\[ \int_{\Omega \cap \partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n \leq \alpha(n, \kappa) \lambda \int_{E_\lambda([\lambda t]) \Delta E_\lambda([\lambda t] - 1)} |v_\lambda| \, dx. \]

Observe that by (7.4) for any $t \geq 1/\lambda$ one has
\[ \int_{E_\lambda([\lambda t]) \Delta E_\lambda([\lambda t] - 1)} |v_\lambda| \, dx \leq C_\beta(E_\lambda([\lambda t] - 1), \Omega) - C_\beta(E_\lambda([\lambda t]), \Omega). \]

Thus
\[ \int_{\Omega \cap \partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n \leq \alpha(n, \kappa) \lambda \left( C_\beta(E_\lambda([\lambda t] - 1), \Omega) - C_\beta(E_\lambda([\lambda t]), \Omega) \right). \]

Fixing $T > 0$ and integrating this inequality in $t \in [0, T]$ we get
\[ \int_0^T \int_{\Omega \cap \partial E_\lambda([\lambda t])} (v_\lambda)^2 \, d\mathcal{H}^n \, dt \leq \alpha(n, \kappa) \sum_{k=1}^{[T\lambda]+1} \left( C_\beta(E_\lambda(k-1), \Omega) - C_\beta(E_\lambda(k), \Omega) \right) \]
\[ \leq \alpha(n, \kappa) C_\beta(E_0, \Omega) \leq \alpha(n, \kappa) P(E_0), \]

26
where we used (3.9). Now letting \( T \to +\infty \) completes the proof. \( \square \)

**Proposition 8.10.** Let \( E_0, \beta \) satisfy (5.1), \( \lambda \geq 1 \) and \( E^+ \) be as in Proposition 6.11. Then
\[
\lambda \int_{1/\lambda}^{T} |E_{\lambda}([|\lambda|t]) \Delta E_{\lambda}([|\lambda|t] - 1)| dt \leq |E^+| + \frac{P(E_0)}{\gamma(n, \kappa)} + \frac{2^{n+1} \omega_{n+1} \gamma(n, \kappa) b(n)}{\kappa c(n, \kappa)} P(E_0) T
\]  
(8.11)
for any \( T > \frac{1}{\lambda} \). Here \( b(n), \gamma(n, \kappa), c(n, \kappa) \) are defined in Section 5.

**Proof.** Let \( [\lambda T] = N \). Clearly,
\[
\lambda \int_{1/\lambda}^{T} |E_{\lambda}([|\lambda|t]) \Delta E_{\lambda}([|\lambda|t] - 1)| dt = \sum_{k=1}^{N} |E_{\lambda}(k) \Delta E_{\lambda}(k-1)|.
\]
We recall that \( E_{\lambda}(k) \subset E^+ \) for all \( \lambda \geq 1 \) and \( k \geq 0 \), by Proposition 6.11.

If \( k = 1 \), then
\[
|E_{\lambda}(1) \Delta E_{\lambda}(0)| \leq |E^+|.
\]  
(8.12)
Now if \( k \geq 2 \), we write \( E_{\lambda}(k) \Delta E_{\lambda}(k-1) \) as a union of \( A_k \) and \( B_k \), where
\[
A_k = \bigl\{ x \in E_{\lambda}(k) \Delta E_{\lambda}(k-1) : d_{E_{\lambda}(k-1)}(x) > \ell \bigr\},
\]
\[
B_k = \bigl\{ x \in E_{\lambda}(k) \Delta E_{\lambda}(k-1) : d_{E_{\lambda}(k-1)}(x) \leq \ell \bigr\},
\]
where \( \ell := \frac{\gamma(n, \kappa)}{\lambda} \). By Chebyshev inequality \( |A_k| \) can be estimated using (7.4) as
\[
|A_k| \leq \frac{\lambda}{\gamma(n, \kappa)} \int_{E_{\lambda}(k) \Delta E_{\lambda}(k-1)} d_{E_{\lambda}(k-1)} dx \leq \frac{1}{\gamma(n, \kappa)} \left( C_{\beta}(E_{\lambda}(k), \Omega) - C_{\beta}(E_{\lambda}(k), \Omega) \right).
\]
Hence, by (7.6)
\[
\sum_{k=2}^{N} |A_k| \leq \frac{1}{\gamma(n, \kappa)} \sum_{k=2}^{N} \left( C_{\beta}(E_{\lambda}(k), \Omega) - C_{\beta}(E_{\lambda}(k), \Omega) \right) \leq \frac{P(E_0)}{\gamma(n, \kappa)}.
\]
Moreover, by definition \( B_k \) can be covered by the family of balls \( \{ B_{2\ell}(x), x \in \partial E_{\lambda}(k-1) \} \). Thus, by Besicovitch covering theorem we can find at most countably many balls \( \{ B_{\ell}(x), x \in \partial E_{\lambda}(k-1) \} \) covering \( \Omega \cap \partial E_{\lambda}(k-1) \). Hence, the lower density estimate (5.5) for \( E_{\lambda}(k-1) \) used with \( \ell \) implies
\[
|B_{2\ell}(x_j) \cap B_{\ell}(x_j)| \leq (2^{n+1} \omega_{n+1} \ell) \omega_{n+1} \ell \leq \frac{2^{n+1} \omega_{n+1}}{c(n, \kappa)} \ell P(E_{\lambda}(k-1), B_{\ell}(x_j)),
\]
from which it follows that
\[
\sum_{k=2}^{N} |B_k| \leq \sum_{k=2}^{N} \sum_{j \geq 1} |B_{2\ell}(x_j) \cap B_{\ell}(x_j)| \leq \frac{2^{n+1} \omega_{n+1}}{c(n, \kappa)} \ell \sum_{k=2}^{N} \sum_{j \geq 1} P(E_{\lambda}(k-1), B_{\ell}(x_j))
\]
\[
\leq \frac{2^{n+1} \omega_{n+1}}{c(n, \kappa)} \ell \sum_{k=2}^{N} P(E_{\lambda}(k-1), \Omega).
\]
Therefore, using (7.6) and \( N \leq \lambda T \), we get
\[
\sum_{k=2}^{N} |B_k| \leq \frac{2^{n+1} \omega_{n+1} \gamma(n, \kappa)}{\kappa c(n, \kappa)} P(E_0) T.
\]  
(8.13)
Finally, (8.11) follows from (8.12)-(8.13). \( \square \)

The following error estimate is similar to error estimates shown in [41, 46].

**Proposition 8.11 (Error estimate).** Let \( 1 \leq n \leq 6 \). Under assumption (4.3), for every \( \phi \in C^1_c([0, +\infty) \times \Omega) \) the following error-estimate holds:
\[
\lim_{j \to +\infty} \int_{1/\lambda_j}^{+\infty} \lambda_j \left( \int_{\Omega} \left( \chi_{E_{\lambda_j}([|\lambda_j|t])} - \chi_{E_{\lambda_j}([|\lambda_j|t] - 1)} \right) \phi dx - \int_{\Omega \cap \partial E_{\lambda_j}([|\lambda_j|t] - 1)} \tilde{d}E_{\lambda_j}([|\lambda_j|t] - 1) \phi d\mathcal{H}^n \right) dt \to 0.
\]  
(8.14)
Proof. Let us assume that \( \text{supp} \phi \subset [0, T) \times \Omega^\varepsilon \), \( \Omega^\varepsilon := \mathbb{R}^n \times (\varepsilon, +\infty) \subset \Omega \) for some \( \varepsilon, T > 0 \). Let us take \( j \) so large that
\[ 4R\lambda_j^{-1/2} < \varepsilon, \tag{8.15} \]
where \( R := R(n, \kappa) \) is defined in (4.4).

Given an integer \( k \geq 1 \) set
\[ \Delta_k(j) := \int_{k/\lambda_j}^{(k+1)/\lambda_j} \lambda_j \left( \int_\Omega (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}) \phi \, dx \right. - \left. \int_{\Omega \cap \partial E_{\lambda_j}(k)} \tilde{d}_{E_{\lambda_j}(k-1)}(\phi) \, d\mathcal{H}^n \right) \, dt. \]

We need to estimate
\[ \int_{1/\lambda_j}^{T} \lambda_j \left( \int_{\Omega^\varepsilon} (\chi_{E_{\lambda_j}([\lambda_j t])} - \chi_{E_{\lambda_j}([\lambda_j t] - 1)}) \phi \, dx \right. - \left. \int_{\Omega^\varepsilon \cap \partial E_{\lambda_j}([\lambda_j t] - 1)} \tilde{d}_{E_{\lambda_j}(0)}(\phi) \, d\mathcal{H}^n \right) \, dt = \sum_{k=1}^{N} \Delta_k(j), \]
where \( N = [\lambda_j T] \).

First consider \( \Delta_1(j) \). By virtue of Proposition 5.5 and (7.6) we get
\[
|\Delta_1(j)| = \left| \int_{1/\lambda_j}^{2/\lambda_j} \lambda_j \left( \int_{\Omega^\varepsilon} (\chi_{E_{\lambda_j}(1)} - \chi_{E_{\lambda_j}(0)}) \phi \, dx - \int_{\Omega^\varepsilon \cap \partial E_{\lambda_j}(1)} \tilde{d}_{E_{\lambda_j}(0)}(\phi) \, d\mathcal{H}^n \right) \, dt \right| 
\leq \| \phi \|_\infty \left( \| E_{\lambda_j}(1) \Delta E_0 \| + \frac{R(n, \kappa)}{\sqrt{\lambda_j}} P(E_{\lambda_j}(1), \Omega) \right) 
\leq \| \phi \|_\infty \left( \| E_{\lambda_j}(1) \Delta E_0 \| + \frac{R(n, \kappa)}{\kappa \sqrt{\lambda_j}} P(E_0) \right).
\]

By Lemma 4.6, \( \Delta_1(j) \to 0 \) as \( j \to +\infty \).

Now we estimate \( \Delta_k(j) \), \( k \geq 2 \). We use a covering argument with balls centered at the boundaries of \( E_{\lambda_j}(k) \) and \( E_{\lambda_j}(k-1) \).

First we deal with the regions of low curvature. We claim that if \( 1/2 < \sigma_1 < \sigma_2 < 1 \), and \( x \in \Omega^\varepsilon \cap \partial E_{\lambda_j}(k) \) is such that
\[
d_{E_{\lambda_j}(k-1)}(y) \leq \lambda_j^{1-\sigma_2}, \quad \forall y \in B_{R\lambda_j^{-1/2}}(x) \cap (E_{\lambda_j}(k) \Delta E_{\lambda_j}(k-1)), \tag{8.16}\]
then there exists \( \nu := \nu(x) \in S^n \) and a continuous increasing function \( w \in C([0, \infty)) \) with \( w(0) = 0 \) such that
\[
|\nu_{E_{\lambda_j}(k)}(y) - \nu| \leq w(1/\lambda_j) \quad \forall y \in B_{\lambda_j^{-\sigma_1}}(x) \cap \partial E_{\lambda_j}(k), 
|\nu_{E_{\lambda_j}(k-1)}(y) - \nu| \leq w(1/\lambda_j) \quad \forall y \in B_{\lambda_j^{-\sigma_1}}(x) \cap \partial E_{\lambda_j}(k-1). \tag{8.17}\]

Indeed, fix \( r \in (0, \lambda_j^{1-\sigma_1}/2) \). By choice (8.15) of \( j \), the ball \( B_{r\lambda_j^{-\sigma_1}}(x) \) does not intersect \( \partial \Omega \), and hence
\[
P(E_{\lambda_j}(k), B_{r\lambda_j^{-\sigma_1}}(x)) \leq P(F, B_{r\lambda_j^{-\sigma_1}}(x)) + \lambda_j \int_{F \Delta E_{\lambda_j}(k)} dE_{\lambda_j}(k-1) \, dy \]
for every \( F \in BV(\Omega, \{0, 1\}) \) with \( F \Delta E_{\lambda_j}(k) \subset B_{r\lambda_j^{-\sigma_1}}(x) \). Since \( dE_{\lambda_j}(k-1)(\cdot) \) is 1-Lipschitz, by virtue of Proposition 5.5,
\[
d_{E_{\lambda_j}(k-1)}(y) \leq d_{E_{\lambda_j}(k-1)}(x) + |x - y| \leq R\lambda_j^{-1/2} + r\lambda_j^{-\sigma_1} \leq \frac{R + 1}{\lambda_j^{1/2}}, \quad y \in F \Delta E_{\lambda_j}(k), \]
whence
\[
\lambda_j \int_{F \Delta E_{\lambda_j}(k)} dE_{\lambda_j}(k-1) \, dy \leq (R + 1)\lambda_j^{1/2} |F \Delta E_{\lambda_j}(k)|
\]
and
\[
P(E_{\lambda_j}(k), B_{r\lambda_j^{-\sigma_1}}(x)) \leq P(F, B_{r\lambda_j^{-\sigma_1}}(x)) + (R + 1)\lambda_j^{-1/2} |F \Delta E_{\lambda_j}(k)|. \tag{8.18}\]
As $k \geq 2$,
\[P(E_{\lambda_j}(k-1), B_{r\lambda_j^{-\sigma_1}}(x)) \leq P(F, B_{r\lambda_j^{-\sigma_1}}(x)) + (R + 1)\lambda_j^{1/2}|F \Delta E_{\lambda_j}(k-1)|\]
whenever $F \in BV(\Omega, \{0, 1\})$ satisfies $F \Delta E_{\lambda_j}(k-1) \subset \subset B_{r\lambda_j^{-\sigma_1}}(x)$. Set
\[E_{\lambda_j}^1(k) := \frac{E_{\lambda_j}(k) - x}{\lambda_j^{\sigma_1}}, \quad E_{\lambda_j}^\sigma_1(k - 1) := \frac{E_{\lambda_j}(k-1) - x}{\lambda_j^{\sigma_1}}.\]

By virtue of (8.18) these sets satisfy
\[P(E_{\lambda_j}^1(s), B_r) \leq P(F, B_r) + \lambda_j^{\frac{1}{2} - \sigma_1}(R + 1)|E_{\lambda_j}^1(s)\Delta F|\]
for any $r \in (0, \lambda_j^{1/2-\sigma_1})$ and $F \Delta E_{\lambda_j}^1(s) \subset \subset B_r(0)$, $s = k, k - 1$. Hence, $E_{\lambda_j}^1(s)$, $s = k, k - 1$ is an $((R + 1)\lambda_j^{1/2-\sigma_1}, \lambda_j^{1/2-\sigma_1})$-minimizer of the perimeter (see [42, Section 23]). Since $\sigma_1 > 1/2$, $\lambda_j^{\frac{1}{2} - \sigma_1}(R + 1) \to 0$ as $j \to +\infty$ and therefore, by the compactness [42, Proposition 23.13], up to a (not relabelled) subsequence,
\[E_{\lambda_j}^1(s) \to E_s^\sigma_1 \text{ in } L^1_{\text{loc}}(\mathbb{R}^{n+1}) \text{ as } j \to +\infty, \ s = k, k - 1.\]

where $E_s^\sigma_1$, $s = k, k - 1$, is a local minimizer of the perimeter. Since $n \leq 6$, by virtue of [33, Theorem 17.3] $E^\sigma_1$ and $E^{\sigma_1}_{k-1}$ are half-spaces. Moreover, by hypothesis
\[d_{E_{\lambda_j}^1(k-1)}(z) \leq \lambda_j^{\sigma_1 - \sigma_2}, \ \forall z \in B_{\lambda_j^{-1/2}}(0) \cap (E_{\lambda_j}^1(k)\Delta E_{\lambda_j}^1(k-1)),\]
and, therefore, $E^\sigma_1_k = E^{\sigma_1}_{k-1}$, i.e. there exists $\nu \in \mathbb{S}_n$ such that
\[E^\sigma_1_k = E^{\sigma_1}_{k-1} = \{z \in \mathbb{R}^{n+1} : z \cdot \nu < 0\}.\]

By [42, Theorem 26.6] $\nu E_{\lambda_j}^1(s) \to \nu$ uniformly in $B_1(0)$ implying (8.17), which concludes the proof of the claim.

Recall that if $\sigma_1, \sigma_2$ are as above and $x \in \Omega^c \cap \partial E_{\lambda_j}(k)$ satisfies (8.16), then by virtue of [46, Corollary 4.2.2] there exists $C(n) > 0$ such that
\[\int_{C^\rho(x, \nu)}(\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)})dx - \int_{C^\rho(x, \nu) \cap \partial E_{\lambda_j}(k)} dE_{\lambda_j}(k-1) dH^n \leq C(n)w(1/\lambda_j) \int_{C^\rho(x, \nu)} |\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}| dx,\]

where $\rho := \lambda_j^{1/2}/2$, and
\[C^\rho(x, \nu) := \{y \in \mathbb{R}^{n+1} : |(y - x) \cdot \nu| < \rho, \sqrt{|y - x|^2 - |(y - x) \cdot \nu|^2} < \rho\}.\]

Now we estimate $\Delta_k(j)$, $k \geq 2$. Set $\sigma_2 := (1/2, \sigma_2)$ and \[O := \left\{C^\rho(x, \nu(x)) : x \text{ satisfies (8.16)}\right\}.\]

Then by virtue of (8.19),
\[\int_{O} C^\rho(x, \nu) (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}) dy - \int_{O \cap \partial E_{\lambda_j}(k)} dE_{\lambda_j}(k-1) \phi dH^n \leq C(n)w(1/\lambda_j) \left(|E_{\lambda_j}(k)\Delta E_{\lambda_j}(k)| + \lambda_j^{-\sigma_2} P(E_{\lambda_j}(k))\right).\]

It remains to estimate the error in $O^c \cup \bigcup C^\rho_{O}$, By virtue of Proposition 5.5 $O^c \cup \bigcup C^\rho_{O} \cap (E_{\lambda_j}(k)\Delta E_{\lambda_j}(k-1))$ can be covered by the family of balls
\[B := \{B_r(x) : x \in O^c \cap \partial E_{\lambda_j}(k-1)\}, \quad r := R\lambda_j^{-1/2}.\]
By the standard Besicovitch theorem we can extract a finite collection \( \{ B_r(x_i) \} \subset \mathbb{B} \) such that each point of \( \Omega^c \cap (E_{\lambda_j}(k) \Delta E_{\lambda_j}(k - 1)) \) is covered with at most \( b(n) \) elements of \( \{ B_r(x_i) \} \). First we handle the error in each \( B_r(x) \). By the definition, there exists \( y \in O_\rho(x_i) \cap (E_{\lambda_j}(k) \Delta E_{\lambda_j}(k - 1)) \) such that \( d_{E_{\lambda_j}(k-1)}(y) \geq \lambda_j^{-\sigma_2} \). Then clearly,

\[
d_{E_{\lambda_j}(k-1)}(z) \geq \lambda_j^{-\sigma_2}/2, \quad \forall z \in B_{\lambda_j^{-\sigma_2}/2}(y) \subseteq E_{\lambda_j}(k) \Delta E_{\lambda_j}(k - 1).
\]

Applying density estimates in Remark 5.4 to \( B_{\lambda_j^{-\sigma_2}/2}(y) \) we get

\[
\int_{B_{\lambda_j^{-\sigma_2}/2}(y) \cap (E_{\lambda_j}(k) \Delta E_{\lambda_j}(k - 1))} d_{E_{\lambda_j}(k-1)}(z) \geq \left( \frac{\kappa}{2} \right)^{n+1} \omega_{n+1} \left( \frac{1}{2\lambda_j^{\sigma_2}} \right)^{n+2}.
\]

Therefore, by choice of \( \sigma_2 \),

\[
\int_{B_r(x_i)} |\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)}| \, dx \leq \omega_{n+1} \left( R\lambda_j^{-1/2} \right)^{n+1} \leq \left( \frac{4R}{\kappa} \right)^{n+1} 2\lambda_j^{3/4} \int_{B_{\lambda_j^{-\sigma_2}/2}(y) \cap (E_{\lambda_j}(k) \Delta E_{\lambda_j}(k - 1))} d_{E_{\lambda_j}(k-1)}(z) \leq \left( \frac{4R}{\kappa} \right)^{n+1} 2\lambda_j^{3/4} \int_{B_r(x_i) \cap (E_{\lambda_j}(k) \Delta E_{\lambda_j}(k - 1))} d_{E_{\lambda_j}(k-1)}(z).
\]

Moreover, comparing \( E_{\lambda_j}(k) \) to \( E_{\lambda_j}(k) \setminus B_r(x_i) \) (and using \( B_r(x_i) \subset \subset \Omega \)) we find

\[
P(E_{\lambda_j}(k), B_r(x_i)) \leq C(n, \text{diam}(E^+)) (R\lambda_j^{-1/2})^n,
\]

whence, by Proposition 5.5,

\[
\int_{B_r(x_i) \cap \partial E_{\lambda_j}(k)} d_{E_{\lambda_j}(k-1)} \, d\mathcal{H}^n \leq C(n, \text{diam}(E^+)) (R\lambda_j^{-1/2})^{n+1} \omega_{n+1} \left( \frac{4R}{\kappa} \right)^{n+1} 2\lambda_j^{3/4} \int_{B_r(x_i) \cap (E_{\lambda_j}(k) \Delta E_{\lambda_j}(k - 1))} d_{E_{\lambda_j}(k-1)}(z).
\]

Thus

\[
\left| \int_{B_r(x)} (\chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)})\phi \, dy - \int_{B_r(x) \cap \partial E_{\lambda_j}(k)} d_{E_{\lambda_j}(k-1)}(z) \phi \, d\mathcal{H}^n \right| \leq \|\phi\|_\infty C(n, \kappa, \text{diam}(E^+)) \lambda_j^{3/4} \int_{B_r(x) \cap (E_{\lambda_j}(k) \Delta E_{\lambda_j}(k - 1))} d_{E_{\lambda_j}(k-1)}(z),
\]

and therefore,

\[
\left| \int_{\Omega \setminus \bigcup C_i} \chi_{E_{\lambda_j}(k)} - \chi_{E_{\lambda_j}(k-1)} \phi \, dy - \int_{\Omega \setminus \bigcup C_i \cap \partial E_{\lambda_j}(k)} d_{E_{\lambda_j}(k-1)}(z) \eta \phi \, d\mathcal{H}^n \right| \leq b(n) \|\phi\|_\infty C(n, \kappa, \text{diam}(E^+)) \lambda_j^{3/4} \int_{E_{\lambda_j}(k) \Delta E_{\lambda_j}(k - 1)} d_{E_{\lambda_j}(k-1)}(z).
\]

By (7.4)

\[
\lambda_j^{3/4} \int_{E_{\lambda_j}(k) \Delta E_{\lambda_j}(k - 1)} d_{E_{\lambda_j}(k-1)}(z) \leq \lambda^{-1/4} \left( C_\beta(E_{\lambda_j}(1), \Omega) - C_\beta(E_{\lambda_j}(N), \Omega) \right),
\]

30
and thus we have also
\[
\left| \int_{\Omega^j \setminus \bigcup C^\beta_k} (\chi E_{\lambda_j}(k) - \chi E_{\lambda_j}(k-1)) \phi \, dy - \int_{\Omega^j \setminus \bigcup C^\beta_k \cap \partial E_{\lambda_j}(k)} \tilde{d} E_{\lambda_j}(k-1) \eta \phi \, d\mathcal{H}^n \right| \leq b(n) \left| \phi \right|_\infty C(n, \kappa, \text{diam}(E^+))\lambda_j^{-1/4} \left( C_\beta(E_{\lambda_j}(1), \Omega) - C_\beta(E_{\lambda_j}(N), \Omega) \right).
\]
(8.21)

Combining (8.20) and (8.21), we obtain
\[
\left| \Delta_k(j) \right| \leq C(n) w(1/\lambda_j) \left( \left| E_{\lambda_j}(k) \Delta E_{\lambda_j}(k) \right| + \lambda_j^{-\sigma_2} P(E_{\lambda_j}(k)) \right)
+ b(n) \left| \phi \right|_\infty C(n, \kappa, \text{diam}(E^+))\lambda_j^{-1/4} \left( C_\beta(E_{\lambda_j}(1), \Omega) - C_\beta(E_{\lambda_j}(N), \Omega) \right).
\]

Therefore, and we get
\[
\sum_{k=2}^N \left| R_k(j) \right| \leq b(n) \left( C(n) w(1/\lambda_j) \left| \phi \right|_\infty + \left| \nabla \phi \right|_\infty \lambda_j^{-\sigma_1} \right) \sum_{k=2}^N \left| E_{\lambda_j}(k) \Delta E_{\lambda_j}(k-1) \right|
+ b(n) \left| \nabla \phi \right|_\infty \lambda_j^{-\sigma_1-\sigma_2} \frac{(N-1) P(E_0)}{\kappa}
+ b(n) \left| \phi \right|_\infty C(n, \kappa, \text{diam}(E^+))\lambda_j^{-1/4}.
\]

Now applying Proposition 8.10, (7.6), (7.4) and the relation $N \leq T \lambda_j$ we obtain
\[
\sum_{k=2}^N \left| \Delta_k(j) \right| \leq C(n, \kappa, P(E_0), T) \left| \phi \right|_\infty w(1/\lambda_j) + C(n) \left| \phi \right|_\infty \lambda_j^{-\sigma_1} (N-1) P(E_0)
+ b(n) \left| \nabla \phi \right|_\infty \lambda_j^{-\sigma_1-\sigma_2} \frac{P(E_0)}{\kappa} T + b(n) \left| \phi \right|_\infty C(n, \kappa, \text{diam}(E^+))\lambda_j^{-1/4} P(E_0),
\]

This estimate, the assumption $\sigma_1 + \sigma_2 > 1$, and the fact that $R_1(j) \to 0$ as $j \to +\infty$ imply (8.14).

**Proof of Theorem 8.6.** Lemma 8.9, (8.3) and [36, Theorem 4.4.2] imply that there exist a (not relabelled) subsequence and a function $v : [0, +\infty) \times \Omega \to \mathbb{R}$ satisfying (8.4)-(8.6). In particular, from (8.4) it follows that $H_{E(t)} := v(t, \cdot)\chi_{E_{\lambda_j}(t)} E_{\lambda_j}(t) \in L^2(\Omega \cap \partial^* E(t), \mathcal{H}^n \mathbf{L}(\Omega \cap \partial^* E(t)))$ for a.e. $t > 0$. Let us prove that $H_{E(t)}$ is the distributional mean curvature of $E(t)$ for a.e. $t \geq 0$. Fixing $t \geq 0$, by the divergence formula (2.3) for any $\phi \in C_c^1(\mathbb{R}^{n-1} \times \mathbb{R}^{n+1})$ one has
\[
\int_{E_{\lambda_j}(t)} \text{div} \phi \, dx - \int_{\Omega \cap \partial^* E_{\lambda_j}(t)} \phi \cdot \nu_{E_{\lambda_j}(t)} \, d\mathcal{H}^n = \int_{\partial \Omega \cap \partial^* E_{\lambda_j}(t)} \phi_{n+1} \, d\mathcal{H}^n.
\]

Hence, from (7.1) and (7.3) we get
\[
\int_{E(t)} \text{div} \phi \, dx - \int_{\Omega \cap \partial^* E(t)} \phi \cdot \nu_E \, d\mathcal{H}^n = \lim_{j \to +\infty} \int_{\text{Tr}(E_{\lambda_j}(t))} \phi_{n+1} \, d\mathcal{H}^n. \tag{8.22}
\]

The left-hand-side of (8.22) is $\int_{\text{Tr}(E(t))} \phi_{n+1} \, d\mathcal{H}^n$, therefore,
\[
\mathcal{H}^n \mathbf{L} \text{Tr}(E_{\lambda_j}([\lambda_j t])) \xrightarrow{w^*} \mathcal{H}^n \mathbf{L} \text{Tr}(E(t)) \quad \text{as} \quad j \to +\infty. \tag{8.23}
\]

Combining this with (8.3) we get
\[
\mathcal{H}^n \mathbf{L} \partial^* E_{\lambda_j}([\lambda_j t]) \xrightarrow{w^*} \mathcal{H}^n \mathbf{L} \partial^* E(t) \quad \text{as} \quad j \to +\infty \quad \text{for a.e.} \quad t \geq 0.
\]

Take $\eta \in C^1_c([0, +\infty))$ and an admissible $X \in C^1_c(\overline{\Omega}, \mathbb{R}^{n+1})$. By (8.3) and [46, formula (4.2)] for a.e. $t \geq 0$ and for every $F \in C_c(\mathbb{R}^{n+1} \times \mathbb{R}^{n+1})$ one has
\[
\lim_{j \to +\infty} \int_{\Omega \cap \partial^* E_{\lambda_j}(t)} F(x, \nu_{E_{\lambda_j}(t)}(x)) \, d\mathcal{H}^n = \int_{\Omega \cap \partial^* E(t)} F(x, \nu_{E(t)}(x)) \, d\mathcal{H}^n. \tag{8.24}
\]
In particular, taking \( F \in C_c(\overline{\Omega} \times \mathbb{R}^{n+1}) \) such that \( F(x, \xi) = \nabla X(x) - \xi \cdot \nabla X(x) \xi \) in \( \Omega \times \{ |\xi| \leq 2 \} \), by the dominated convergence theorem, (8.2) and (8.6), for \( \Psi(t, x) = \eta(t) X(x) \) we establish
\[
\int_0^{+\infty} \eta(t) \int_{\Omega \cap \partial^* E(t)} F(x, \nu_{E(t)}(x)) dH^n dt = \lim_{j \to +\infty} \int_0^{+\infty} \int_{\Omega \cap \partial^* E_j(\lambda_j t)} \eta(t) F(x, \nu_{E_j(\lambda_j t)}(x)) dH^n dt
\]
\[
= \lim_{j \to +\infty} \int_0^{+\infty} \int_{\Omega \cap \partial^* E_j(\lambda_j t)} v_{\lambda_j} \nu_{E_j(\lambda_j t)} \cdot \Psi(t, x) dH^n dt
\]
\[
= \int_0^{+\infty} \int_{\Omega \cap \partial^* E(t)} v_{\nu_{E(t)}} \cdot \Psi(t, x) dH^n dt = \int_0^{+\infty} \eta(t) \int_{\Omega \cap \partial^* E(t)} H_{E(t)} v_{E(t)} X dH^n dt.
\]
Since \( \eta \in C^1_c([0, +\infty)) \) is arbitrary, for a.e. \( t \geq 0 \) we get
\[
\int_{\Omega \cap \partial^* E(t)} (\nabla X - \nu_{E(t)} \cdot (\nabla X) \nu_{E(t)}) dH^n = \int_{\Omega \cap \partial^* E(t)} H_{E(t)} \nu_{E(t)} X dH^n,
\]
hence \( H_{E(t)} \) is the generalized mean curvature of \( \Omega \cap \partial^* E(t) \).

Let us show (8.7). Take \( \phi \in C^1_c((0, +\infty) \times \Omega) \). By a change of variables we have
\[
\int_1^{+\infty} \left[ \int_{E_{\lambda_j}(\lambda_j t)} \phi dx - \int_{E_{\lambda_j}(\lambda_j t - 1)} \phi dx \right] dt
\]
\[
= \int_1^{+\infty} \int_{E_{\lambda_j}(\lambda_j t)} (\phi(t, x) - \phi(t + 1/\lambda_j, x)) dx dt - \frac{1}{\lambda_j} \int_{E(0)} \phi(x, 0) dx.
\]
Since \( E(0) = E_0 \), from (7.13) we get
\[
\lim_{j \to +\infty} \int_1^{+\infty} \lambda_j \left[ \int_{E_{\lambda_j}(\lambda_j t)} \phi dx - \int_{E_{\lambda_j}(\lambda_j t - 1)} \phi dx \right] dt = - \int_0^{+\infty} \int_{E(t)} \frac{\partial \phi}{\partial t} (t, x) dx dt - \int_{E_0} \phi(x, 0) dx.
\]
Therefore, (8.14), (8.5) and the definition of \( H_{E(t)} \) imply
\[
\int_0^{+\infty} \int_{E(t)} \partial_t \phi dx dt + \int_{E_0} \phi(x, 0) dx = \lim_{j \to +\infty} \int_0^{+\infty} \int_{\Omega \cap \partial^* E_j(\lambda_j t)} v_{\lambda_j} \phi dH^n dt
\]
\[
= \int_0^{+\infty} \int_{\Omega \cap \partial^* E(t)} H_{E(t)} \phi dH^n dt.
\]
(ii) Take an admissible \( X \in C^1_c(\overline{\Omega}, \mathbb{R}^{n+1}) \) and \( \eta \in C^1_c([0, +\infty)) \). From (8.1)
\[
\int_0^{+\infty} \eta(t) \int_{\Omega \cap \partial^* E_j(\lambda_j t)} (\nabla X - \nu_{E_j(\lambda_j t)} \cdot (\nabla X) \nu_{E_j(\lambda_j t)}) dH^n dt
\]
\[
- \int_0^{+\infty} \eta(t) \int_{\Omega \cap \partial^* E_j(\lambda_j t)} v_{\lambda_j} X \cdot \nu_{E_j(\lambda_j t)} dH^n dt
\]
\[
= \int_0^{+\infty} \eta(t) \int_{\partial^* \text{Tr}(E_j(\lambda_j t))} \beta X' \cdot \nu_{\text{Tr}(E_j(\lambda_j t))} dH^{n-1}.
\]
Let \( \{ \lambda_{j_k} \}_{k \geq 1} \) be any subsequence of \( \{ \lambda_j \} \). By the uniform bound (8.8) on the perimeters and by compactness there exists a further subsequence \( \{ \lambda_{j_{k_k}} \}_{k \geq 1} \) of \( \{ \lambda_{j_k} \}_{k \geq 1} \) and a set \( \tilde{F} \in BV(\mathbb{R}^n, \{ 0, 1 \}) \) such that \( \text{Tr}(E_{j_{k_k}}(|j_{k_k} t|)) \rightarrow \tilde{F} \) in \( L^1(\mathbb{R}^n) \) and\(^4\)
\[
\nu_{\text{Tr}(E_{j_{k_k}}(|j_{k_k} t|))} H^{n-1} \sqsubset \partial^* \text{Tr}(E_{j_{k_k}}(|j_{k_k} t|)) \rightharpoonup \nu_{\tilde{F}} H^{n-1} \sqsubset \partial^* \tilde{F}
\]
as \( k \to +\infty \)
for a.e. $t \geq 0$. By (8.23) for every $\phi \in C_c(\mathbb{R}^n)$ we have
\[
\int_{\operatorname{Tr}(E(t)) \setminus \Omega} \phi \, d\mathcal{H}^n = \lim_{k \to +\infty} \int_{\operatorname{Tr}(E_{\lambda_j}(\cdot \setminus \{t\}))} \phi \, d\mathcal{H}^n = \int_{\hat{F}} \phi \, d\mathcal{H}^n.
\]
Whence, $\hat{F} = \operatorname{Tr}(E(t))$. Therefore
\[
\nu_{\operatorname{Tr}(E_{\lambda_j}(\cdot \setminus \{t\}))} \mathcal{H}^{n-1} \{ \partial^* \operatorname{Tr}(E_{\lambda_j}(\cdot \setminus \{t\})) \} \wto \nu_{\operatorname{Tr}(E(t))} \mathcal{H}^{n-1} \{ \partial^* \operatorname{Tr}(E(t)) \} \quad \text{as } j \to +\infty.
\]
Now taking limit in (8.25), using (8.24),(8.6) and applying the dominated convergence theorem on the right-hand-side we get (8.9).

\[\square\]

**APPENDIX A. EXISTENCE OF MINIMIZERS FOR SOME FUNCTIONALS**

In this section we prove an existence result for minimum problems of type
\[
\inf_{E \in BV(\Omega, [0,1])} \mathcal{G}_\beta(E), \quad \mathcal{G}_\beta(E) := C_\beta(E), \quad F(E) := \nu(E), \quad (A.1)
\]
where $F : BV(\Omega, [0,1]) \to (-\infty, +\infty)$. Since $C_\beta(\cdot, \Omega)$ is finite in $BV(\Omega, [0,1])$, the functional $\mathcal{G}_\beta$ is well-defined in $BV(\Omega, [0,1])$. We study (A.1) under the following hypotheses on $F$:

**Hypothesis A.1.**

(a) $F$ is bounded from below in $BV(\Omega, [0,1])$ and there exists a cylinder $C^K_r \subset \Omega$, $K > 1$ such that $F(C^K_r) < +\infty$;

(b) $F(E) \geq F(E \cap C^K_{\rho})$ for any $E \in BV(\Omega, [0,1])$, $\rho \in (r, +\infty)$, and $l \in (K-1, K+1)$;

(c) $F(E) \geq F(E \setminus (C^K_{\rho_1} \setminus C^K_{\rho_2}))$ for any $E \in BV(\Omega, [0,1])$ and $r < \rho_2 < \rho_1 < +\infty$;

(d) $F$ is $L^1(\Omega)$-lower semicontinuous in $BV(\Omega, [0,1])$.

**Example A.2.** Besides (4.8) the following functionals $F : BV(\Omega, [0,1]) \to (-\infty, +\infty)$ satisfy Hypothesis A.1:

1) given $f \in L^1_{\text{loc}}(\Omega)$ with $f \geq 0$ a.e. in $\Omega \setminus C^K_r$ for some $r, l > 0$,
\[
F(E) = \int_E f \, dx.
\]
In particular, we may take $f = \lambda \tilde{d}_{\varnothing}$ with $\varnothing \neq E_0 \in BV(\Omega, [0,1])$ and $E_0 \subset C^K_r$ so that by (4.2) $\mathcal{G}_\beta$ coincides with $A_\beta(\cdot, \varnothing, \lambda) + \int_{E_0} \tilde{d}_{\varnothing} \, dx$.

2) Given a bounded set $E_0 \in BV(\Omega, [0,1])$, $\nu(E) = |E \Delta E_0|^p$, $p > 0$.

Given $F$ satisfying Hypothesis A.1 set
\[
\alpha := \kappa^{-1} \left( \sup_{R>R} \inf_{E \in BV(C^K_R, [0,1])} \mathcal{G}_\beta(E) - \inf \nu \right).
\]
Clearly, $\alpha \leq \mathcal{G}_\beta(C^K_r) - \inf \nu$, hence $\inf \mathcal{G}_\beta < +\infty$.

In view of the previous observation, once we prove the next theorem, the proof of Theorem 4.1 follows.

**Theorem A.3 (Existence of minimizers and uniform bound).** Suppose that Hypothesis A.1 holds. Suppose also $\beta \in L^\infty(\partial \Omega)$ and there exists $\kappa \in (0, \frac{1}{2})$ such that $-1 \leq \beta \leq 1 - 2\kappa$ $\mathcal{H}^{n-1}$ a.e. on $\partial \Omega$.

Then the minimum problem
\[
\inf_{E \in BV(\Omega, [0,1])} \mathcal{G}_\beta(E)
\]
has a solution. Moreover, any minimizer is contained in $C^K_{\mathcal{R}_0}$, where
\[
\mathcal{R}_0 := r + 1 + \max \left\{ 8^{n^2+n+1} \alpha^{n+1} K, 4\mu(\kappa, n) \right\}
\]
and $\mu(\kappa, n)$ is defined in Section 4.1.

\[\textsuperscript{5}\text{One could refine the expression of } \mathcal{R}_0 \text{ using the isoperimetric inequality [22], but we do not need this here.}\]
Remark A.4. In case of Example A.2 1) with \( f = \lambda \tilde{d}_{E_0} \) for some \( C_r^K \supseteq E_0 \),
\[
\kappa \alpha \leq \kappa \sup_{R > r} \inf_{E \in BV(C_r^K \setminus \{0,1\})} A_{\beta}(E, E_0, \lambda) \leq \kappa A_{\beta}(E_0, E_0, \lambda) = \kappa C_{\beta}(E_0, \Omega) \leq \kappa P(E_0).
\]
Hence, \( R_0 \leq R_0 \), where \( R_0 \) is defined in (4.4). The same is true if \( \mathcal{V} \) is as in (4.8).

The assumption on \( \beta \) and the \( L^1(\Omega) \)-lower semicontinuity of \( C_{\beta}(\cdot, \Omega) \) (Lemma 3.5) imply the \( L^1(\Omega) \)-lower semicontinuity of \( G_{\beta} \). Moreover, the coercivity (3.9) of \( C_{\beta}(\cdot, \Omega) \), Hypothesis A.1 (a) and (3.11) imply the coercivity of \( G_{\beta} \):
\[
G_{\beta}(E) \geq \kappa P(E) + \inf \mathcal{V} \quad \forall E \in BV(\Omega, \{0,1\}). \quad (A.3)
\]

The main problem in the proof of existence of minimizers of \( G_{\beta} \) is the lack of compactness due to the unboundedness of \( \Omega \). However, for every \( R > 0 \) inequality (A.3), the compactness theorem in \( BV(C_r^K, \{0,1\}) \) (see for instance [7, Theorems 3.23 and 3.39]) and the lower semicontinuity of \( G_{\beta} \) imply that there exists a solution \( E^R \in BV(C_r^K, \{0,1\}) \) of
\[
\inf_{E \in BV(C_r^K, \{0,1\})} G_{\beta}(E).
\]

To prove Theorem A.3 we mainly follow [18, Section 4], where the existence of volume preserving minimizers of \( C_{\beta}(\cdot, \Omega) \) has been shown. We need two preliminary lemmas. As in [18, Section 3] first we show that one can choose a minimizing sequence consisting of bounded sets.

Lemma A.5 (Truncations with horizontal hyperplanes and vertical cylinders). Suppose that Hypothesis A.1 holds. Then
\[
\inf_{E \in BV(\Omega, \{0,1\})} G_{\beta}(E) = \inf_{E \in BV(\Omega, \{0,1\})} \inf_{R > 0} G_{\beta}(E). \quad (A.4)
\]

Proof. We need two intermediate steps. The first step concerns truncations with horizontal hyperplanes.

Step 1. We have
\[
\inf_{E \in BV(\Omega, \{0,1\})} G_{\beta}(E) = \inf_{E \in BV(\Omega, \{0,1\})} G_{\beta}(E). \quad (A.5)
\]

Indeed, it suffices to show that if \( E \setminus \Omega_{K - \frac{1}{2}} \neq \emptyset \), then
\[
G_{\beta}(E) \geq G_{\beta}(E \cap \Omega_{K - \frac{1}{2}}).
\]

Clearly, \( E \) and \( E \cap \Omega_{K - \frac{1}{2}} \) have the same trace on \( \partial \Omega \) and thus
\[
\int_{\partial \Omega} [1 + \beta] \chi_E dH^n = \int_{\partial \Omega} [1 + \beta] \chi_{E \cap \Omega_{K - \frac{1}{2}}} dH^n.
\]

From the comparison theorem of [6, page 216] we have
\[
P(E) > P(E \cap \Omega_{K - \frac{1}{2}}).
\]

By Hypothesis A.1 (b) we have also
\[
\mathcal{V}(E) \geq \mathcal{V}(E \cap \Omega_{K - \frac{1}{2}}),
\]

therefore from the definition of \( G_{\beta} \) we get even the strict inequality
\[
G_{\beta}(E) > G_{\beta}(E \cap \Omega_{K - \frac{1}{2}}). \quad (A.6)
\]

The second step is more delicate and concerns truncations with the lateral boundary of vertical cylinders.

Step 2. For any \( \varepsilon \in (0,1) \) there exists \( R_\varepsilon > r \) and \( E_\varepsilon \in BV(C_r^K, \{0,1\}) \) such that
\[
G_{\beta}(E_\varepsilon) \leq \inf_{E \in BV(\Omega, \{0,1\})} G_{\beta}(E) + \varepsilon.
\]

Indeed, according to Step 1 and Hypothesis A.1 (a), given \( \varepsilon > 0 \) there exists \( F_\varepsilon \in BV(\Omega_K, \{0,1\}) \) with \( F_\varepsilon \subset \Omega_{K - \frac{1}{4}} \) such that
\[
G_{\beta}(F_\varepsilon) < \inf_{E \in BV(\Omega, \{0,1\})} G_{\beta}(E) + \frac{\varepsilon}{2} < +\infty.
\]
Since $|F_e| < +\infty$, for sufficiently large $R > r$ one has
\[ |F_e \cap (C^K_{R+1} \setminus C^K_R)| = \int_R^{R+1} \mathcal{H}^n(F_e \cap \partial C^K_{\rho}) \, d\rho < \frac{\varepsilon}{2}. \]

Hence there exists $R_e \in (R, R + 1)$ such that
\[ \mathcal{H}^n(F_e \cap \partial C^K_{R_e}) \leq \frac{\varepsilon}{2}, \quad \mathcal{H}^n(\Omega \cap \partial^* F_e \cap \partial C^K_{R_e}) = 0. \]

Now, let $E_e := F_e \cap C^K_{R_e}$. Since $\mathcal{H}^n(\Omega \cap \partial^* F_e \cap \partial C^K_{R_e}) = 0$, we have
\[
P(E_e, \Omega) = P(E_e, \Omega_K) = P(F_e, \Omega_K) + \mathcal{H}^n(F_e \cap C^K_{R_e}) - P(F_e, \Omega_K \setminus C^K_{R_e})
\]
\[ = P(F_e, \Omega) + \mathcal{H}^n(F_e \cap \partial C^K_{R_e}) - P(F_e, \Omega_K \setminus C^K_{R_e}). \tag{A.7} \]

By Hypothesis A.1 (a), $\mathcal{V}(F_e) \geq \mathcal{V}(E_e)$, thus employing (A.7) we get
\[ G_\beta(E_e) \geq G_\beta(E_e) - \mathcal{H}^n(F_e \cap \partial C^K_{R_e}) + P(F_e, \Omega_K \setminus C^K_{R_e}) - \int_{\partial \Omega} \beta \chi_{F_e \setminus C^K_{R_e}} \, d\mathcal{H}^n. \]

By Lemma 3.1 applied with $E = F_e$ and $A = \Omega_K \setminus C^K_{R_e}$, we have
\[ P(F_e, \Omega_K \setminus C^K_{R_e}) - \int_{\partial \Omega} \beta \chi_{F_e \setminus C^K_{R_e}} \, d\mathcal{H}^n \geq 0. \]

Consequently, from the choice of $F_e$ and $R_e$ we get
\[ G_\beta(E_e) \leq G_\beta(F_e) - \mathcal{H}^n(F_e \cap \partial C^K_{R_e}) + \inf_{E \in BV(\Omega, \{0, 1\})} G_\beta(E) + \varepsilon. \]

This concludes the proof of Step 2.

Now, observe that
\[ \inf_{E \in BV(\Omega, \{0, 1\})} G_\beta(E) \leq \inf_{R > 0} \inf_{E \in BV(\Omega, \{0, 1\})} G_\beta(E). \]

On the other hand, since the mapping
\[ R \in (0, +\infty) \mapsto \inf_{E \in BV(\Omega, \{0, 1\})} G_\beta(E) \]

is nonincreasing, Step 2 implies
\[ \inf_{E \in BV(\Omega, \{0, 1\})} G_\beta(E) \geq \inf_{R > 0} \inf_{E \in BV(\Omega, \{0, 1\})} G_\beta(E), \]

therefore (A.4) follows. \hfill \Box

As in [18, Lemma 3] the following lemma holds.

**Lemma A.6 (Good choice of a radius).** Suppose that $\beta$ satisfies (4.3) and Hypothesis A.1 holds. Let $E^R$ be a minimizer of $G_\beta$ in $BV(C^K_R, \{0, 1\})$. Then for any $R > R_0$ there exists $t_R \in [r + 1, R_0]$ such that
\[ \mathcal{H}^n(E^R \cap \partial C^K_{t_R}) = 0. \]

Hence
\[ P(E^R, \Omega) = P(E^R \setminus C^K_{\tau_R}, \Omega) + P(E^R \cap C^K_{t_R}, \Omega). \tag{A.8} \]

**Proof.** The idea of the proof is to cut the $E^R$ with vertical cylinders, similarly to [18, Lemma 5] where cuts with horizontal hyperplanes are performed.

For $R > R_0$ by the isoperimetric-type inequality [21, Theorem VI], (A.3), the minimality of $E^R$ and by the definition of $a$ we have
\[ |E^R|^{\frac{n}{n+1}} \leq P(E^R) \leq \frac{G_\beta(E^R) - \inf \mathcal{V}}{\kappa} = \frac{1}{\kappa} \left( \inf_{E \in BV(C^K_{R_0}, \{0, 1\})} G_\beta(E) + \inf \mathcal{V} \right) \leq a. \]

Thus, for any $0 < a < b$ one has
\[ |E^R \cap (C^K_b \setminus C^K_a)| \leq a^{\frac{n+1}{n}}. \tag{A.9} \]
Take $r + 1 < r_1 < r_2 < r_3 < R_0$ such that
\[ H^n(\Omega \cap \partial^* E^R \cap \partial C_{r_1}^K) = 0, \quad i = 1, 2, 3, \]
and set
\[ v_1 = |E^R \cap (C_{r_2}^K \setminus C_{r_1}^K)|, \quad v_2 = |E^R \cap (C_{r_3}^K \setminus C_{r_2}^K)|, \]
\[ m = \max_{i=1,2,3} H^n(E^R \cap \partial C_{r_i}^K). \]

**Step 1.** We claim that
\[ \min\{v_1, v_2\} \leq \mu m^{\frac{n}{n+1}}, \quad (A.10) \]
where $\mu := \mu(\kappa, n) > 0$.

It suffices to prove that
\[ \frac{n}{n+1} v_1 + \frac{n}{n+1} v_2 \leq 2\mu \frac{n}{n+1} m. \]

We have
\[ \frac{n}{n+1} v_1 \leq P(E^R \cap (C_{r_2}^K \setminus C_{r_1}^K)) \leq P(E^R, C_{r_2}^K \setminus C_{r_1}^K) + H^n(E^R \cap \partial C_{r_1}^K) \]
\[ + H^n(E^R \cap \partial C_{r_2}^K) + \int_{\partial \Omega} \chi_{E^R \cap (C_{r_2}^K \setminus C_{r_1}^K)} dH^n \]
\[ \leq P(E^R, C_{r_2}^K \setminus C_{r_1}^K) + \int_{\partial \Omega} \chi_{E^R \cap (C_{r_2}^K \setminus C_{r_1}^K)} dH^n + 2m. \]

Similarly,
\[ \frac{n}{n+1} v_2 \leq P(E^R, C_{r_3}^K \setminus C_{r_2}^K) + \int_{\partial \Omega} \chi_{E^R \cap (C_{r_3}^K \setminus C_{r_2}^K)} dH^n + 2m. \]

Hence
\[ \frac{n}{n+1} v_1 + \frac{n}{n+1} v_2 \leq P(E^R, C_{r_3}^K \setminus C_{r_1}^K) + \int_{\partial \Omega} \chi_{E^R \cap (C_{r_3}^K \setminus C_{r_1}^K)} dH^n + 4m. \quad (A.11) \]

Comparing $E^R \setminus (C_{r_3}^K \setminus C_{r_1}^K)$ with $E^R$, we get $G_\beta(E^R) \leq G_\beta(E^R \setminus (C_{r_3}^K \setminus C_{r_1}^K))$, therefore from Hypothesis A.1 (c) we obtain
\[ P(E^R) \leq P(E^R \setminus (C_{r_3}^K \setminus C_{r_1}^K)) + \int_{\partial \Omega} [1 + \beta] \chi_{E^R \cap (C_{r_3}^K \setminus C_{r_1}^K)} dH^n. \quad (A.12) \]

Inserting in (A.12) the identity
\[ P(E^R \setminus (C_{r_3}^K \setminus C_{r_1}^K)) = P(E^R) - H^n(E^R \cap \partial C_{r_1}^K) + H^n(E^R \cap \partial C_{r_3}^K) \]
\[ - P(E^R, C_{r_3}^K \setminus C_{r_1}^K) + \int_{\partial \Omega} \chi_{E^R \cap (C_{r_3}^K \setminus C_{r_1}^K)} dH^n, \]
we get
\[ P(E^R, C_{r_3}^K \setminus C_{r_1}^K) = \int_{\partial \Omega} \beta \chi_{E^R \cap (C_{r_3}^K \setminus C_{r_1}^K)} dH^n \leq 2m. \quad (A.13) \]

By Lemma 3.1 applied with $A = C_{r_3}^K \setminus C_{r_1}^K$ and $E = E^R$, the left-hand-side of (A.13) is not less than
\[ \kappa P(E^R, C_{r_3}^K \setminus C_{r_1}^K) + \int_{\partial \Omega} \chi_{E^R \cap (C_{r_3}^K \setminus C_{r_1}^K)} dH^n, \]
hence
\[ P(E^R, C_{r_3}^K \setminus C_{r_1}^K) \leq \frac{2m}{\kappa}. \]

Then from (A.11) it follows that
\[ \frac{n}{n+1} v_1 + \frac{n}{n+1} v_2 \leq \left( \frac{2m}{\kappa} + 4m \right) = 2\mu \frac{n}{n+1} m. \]

This finishes the proof of Step 1.
Before going to Step 2 we need some preliminaries. Choose any \( R \geq \mathcal{R}_0 \). Let \( a_0 = r + 1, \ b_0 = \mathcal{R}_0 \). Given \( r + 1 \leq a_k \leq b_k \leq \mathcal{R}_0, \ k \in \mathbb{N} \), define
\[
v_k = |E^R \cap (C^K_{b_k} \setminus C^K_{a_k})|.
\]
By (A.6) \( E^R \setminus \Omega_{K-\frac{1}{4}} = \emptyset \), hence
\[
|E^R \cap (C^K_b \setminus C^K_a)| = \int_a^b \mathcal{H}^n(E^R \cap \partial C^K_{\rho}) \, d\rho, \quad 0 \leq a < b.
\]
Therefore, for \( h_k = \frac{b_k - a_k}{4} \) it is possible to find \( r_{k,1} \in (a_k, a_k + h_k), \ r_{k,2} \in (\frac{a_k + h_k}{2}, \frac{a_k + h_k}{2} + h_k) \) and \( r_{k,3} \in (b_k - h_k, b_k) \) such that
\[
\mathcal{H}^n(E^R \cap \partial C^K_{r_{k,i}}) \leq \frac{v_k}{h_k}, \quad \mathcal{H}^n(\Omega \cap \partial E^R \cap \partial C^K_{r_{k,i}}) = 0 \quad \text{for } i = 1, 2, 3.
\]
We choose
\[
(a_{k+1}, b_{k+1}) = \begin{cases} (r_{k,1}, r_{k,2}) & \text{if } |E^R \cap (C^K_{r_{k,1}} \setminus C^K_{r_{k,2}})| \leq |E^R \cap (C^K_{r_{k,2}} \setminus C^K_{r_{k,3}})|, \\ (r_{k,2}, r_{k,3}) & \text{if } |E^R \cap (C^K_{r_{k,1}} \setminus C^K_{r_{k,2}})| > |E^R \cap (C^K_{r_{k,2}} \setminus C^K_{r_{k,3}})|. \end{cases}
\]
Let
\[
m_k = \max_{i=1,2,3} \mathcal{H}^n(E^R \cap \partial C^K_{r_{k,i}}),
\]

**Step 2.** Using the definition of \( \mathcal{R}_0 \) we show that
\[
m_k \leq \left( \frac{1}{2} \right)^{\frac{n+1}{n}}.
\]
Indeed, according to (A.10), (A.14) and the definition of \( (a_k, b_k) \) one has
\[
v_{k+1} \leq \mu v_k \frac{n+1}{n}, \quad m_k \leq \frac{v_k}{h_k}.
\]
By construction, \( b_{k+1} - a_{k+1} \geq \frac{b_k - a_k}{8} \), i.e. \( h_{k+1} \geq \frac{h_k}{8} \). By induction one can check that
\[
m_k \leq \left( \sum_{j=1}^{k} \frac{\alpha_j}{\alpha_j - 1} \right)^{1/\alpha_k}
\]
where \( \alpha := \frac{n}{n+1} \). Note that
\[
\sum_{j=1}^{k} j \alpha_j \leq \alpha \sum_{j=1}^{k} j \alpha_j - 1 = \frac{\alpha}{(1-\alpha)^2} = n(n+1).
\]
Since \( h_0 = \frac{\mathcal{R}_0 - r - 1}{4} \) and \( v_0 \leq \frac{n+1}{n} \) by (A.9), the choice of \( \mathcal{R}_0 \) in (A.2) implies \( 8^{n(n+1)} v_0 / h_0 \leq 1/2 \). Moreover \( \left( \frac{\mu}{h_0} \right)^{\sum_{j=1}^{k} \alpha_j} \leq 1 \), since \( \frac{\mu}{h_0} = \frac{4 \mu}{h_0 \mu - 1} \leq 1 \). Now (A.15) follows from these estimates and (A.16).

**Step 3.** Let \( i_k \in \{1, 2, 3\} \) be such that \( m_k = \mathcal{H}^n(E^R \setminus \partial C^K_{r_{k,i_k}}) \). Since \( a_k \leq r_{k,i_k} \leq b_k \), \( \{a_k\} \) is nondecreasing and \( \{b_k\} \) is nonincreasing, there exists \( t_R \in [r + 1, \mathcal{R}_0] \) such that \( r_{k,i_k} \to t_R \) (possibly up to a subsequence). Then, by Step 2,
\[
\mathcal{H}^n(E^R \setminus \partial C^K_{t_R}) = \lim_{k \to +\infty} m_k = 0,
\]
which concludes the proof of the lemma. \( \square \)
Proof of Theorem A.3. Let us prove the existence of a minimizer of \( G_\beta \). For \( R > R_0 \) let \( t_R \in [r + 1, R_0] \) be as in Lemma A.6. Then from (A.8) and \( \mathcal{V}(E^R) \geq \mathcal{V}(E^R \cap C^K_{tr}) \) we get

\[
G_\beta(E^R) \geq G_\beta(E^R \cap C^K_{tr}) + P(E^R \setminus C^K_{tr}, \Omega) - \int_{\partial \Omega} \beta \chi_{E^R \setminus C^K_{tr}} d\mathcal{H}^n. \tag{A.17}
\]

By (3.9) and the isoperimetric-type inequality

\[
P(E^R \setminus C^K_{tr}, \Omega) - \int_{\partial \Omega} \beta \chi_{E^R \setminus C^K_{tr}} d\mathcal{H}^n \geq \kappa P(E^R \setminus C^K_{tr}) \geq \kappa |E^R \setminus C^K_{tr}|^{\frac{n}{n+1}}. \tag{A.18}
\]

Thus from (A.17)

\[
G_\beta(E^R) \geq G_\beta(E^R \cap C^K_{tr}).
\]

Hence, \( F^R := E^R \cap C^K_{tr} \subseteq C^K_{R_0} \) satisfies

\[
\min_{E \in BV(C^K_{R_0}, \{0, 1\})} G_\beta(E) = G_\beta(F^R).
\]

From (3.9) and the minimality of \( F^R \) we get

\[
\kappa P(F^R) \leq C_\beta(F^R, \Omega) \leq G_\beta(F^R) - \inf \mathcal{V} \leq \kappa a,
\]

and thus, by compactness there exists \( E \in BV(C^K_{R_0}, \{0, 1\}) \) such that (up to a subsequence) \( F^R \to E \) in \( L^1(\Omega) \) as \( R \to +\infty \). From the \( L^1(\Omega) \)-lower semicontinuity of \( G_\beta \) and from (A.4) we conclude that \( E \) is a minimizer of \( G_\beta \).

Now we prove that any minimizer \( E \) of \( G_\beta \) satisfies \( E \subseteq C^K_{R_0} \). Arguing as in the proof of (A.6) one can show that \( E \subseteq \Omega_{K - \frac{1}{t}} \).

Claim. There exists \( R > r + 1 \) (possibly depending on \( V \) and \( r \)) such that \( E \subseteq C^K_{R} \).

For any \( \rho > 1 \) such that \( \mathcal{H}^n(\Omega \cap \partial^* E \cap \partial C^K_{\rho}) = 0 \), by the minimality of \( E \) we have \( G_\beta(E) \leq G_\beta(E \cap C^K_{\rho}) \), i.e.

\[
P(E, \Omega_K \setminus \overline{C^K_{\rho}}) - \int_{\partial \Omega} \beta \chi_{E \cap \overline{C^K_{\rho}}} d\mathcal{H}^n \leq \mathcal{H}^n(E \cap \partial C^K_{\rho}). \tag{A.19}
\]

By Lemma 3.1

\[
P(E, \Omega_K \setminus \overline{C^K_{\rho}}) - \int_{\partial \Omega} \beta \chi_{E \cap \overline{C^K_{\rho}}} d\mathcal{H}^n \geq \kappa \left( P(E, \Omega_K \setminus \overline{C^K_{\rho}}) + \int_{\partial \Omega} \chi_{E \cap \overline{C^K_{\rho}}} d\mathcal{H}^n \right). \tag{A.20}
\]

Moreover, by the isoperimetric-type inequality,

\[
|E \setminus \overline{C^K_{\rho}}|^{\frac{n}{n+1}} \leq P(E, \Omega_K \setminus \overline{C^K_{\rho}}) + \mathcal{H}^n(E \cap \partial C^K_{\rho}) + \int_{\partial \Omega} \chi_{E \setminus \overline{C^K_{\rho}}} E d\mathcal{H}^n.
\]

therefore, (A.19) and (A.20) imply

\[
|E \setminus \overline{C^K_{\rho}}|^{\frac{n}{n+1}} \leq \frac{\kappa + 1}{\kappa} \mathcal{H}^n(E \cap \partial C^K_{\rho}). \tag{A.21}
\]

Set \( m(\rho) = |E \setminus \overline{C^K_{\rho}}| \). Clearly, \( m : (1, +\infty) \to [0, |E|] \). Moreover, \( m \) is absolutely continuous, nonincreasing, \( \lim_{\rho \to +\infty} m(\rho) = 0 \) and \( \mathcal{H}^n(E \cap \partial C^K_{\rho}) = -m'(\rho) \) for a.e. \( \rho > r + 1 \). By (A.21)

\[
-m'(\rho) \geq \frac{\kappa + 1}{\kappa} (n + 1) m(\rho)^{\frac{1}{n+1}}. \quad \text{If } E \text{ is unbounded, then } m(\rho) > 0 \text{ for any } \rho > r + 1 \text{, and thus, for any } \rho_1, \rho_2 > r + 1, \rho_1 < \rho_2 \text{ we have}
\]

\[
m(\rho_1)^{\frac{1}{n+1}} - m(\rho_2)^{\frac{1}{n+1}} \geq \frac{\kappa + 1}{\kappa} (\rho_2 - \rho_1).
\]

Now letting \( \rho_2 \to +\infty \) we obtain \( m(\rho_1) = +\infty \), a contradiction. Consequently, there exists \( R > r + 1 \) such that \( m(R) = 0 \), i.e. \( E \subseteq C^K_{R} \).

From the claim it follows that \( E \) is a minimizer of \( G_\beta \) also in \( BV(C^K_{R}, \{0, 1\}) \). By Lemma A.6 we can find \( t_R \in [r + 1, R_0] \) such that \( \mathcal{H}^n(E \cap \partial C^K_{t_R}) = 0 \). Then using \( \mathcal{V}(E) \geq \mathcal{V}(E \cap C^K_{t_R}) \), the relations (A.17) - (A.18) applied with \( E \) in place of \( E^R \) imply

\[
G_\beta(E) \geq G_\beta(E \cap C^K_{t_R}) + \kappa |E \setminus \overline{C^K_{t_R}}|^{\frac{n}{n+1}}.
\]
Therefore, the minimality of $E$ yields $|E \setminus C_R^K| = 0$, i.e. $E \subseteq C_R^K$. Since $t_R \leq R_0$, the conclusion follows. 

**APPENDIX B. LOCAL WELL-POSEDNESS**

In this appendix we sketch the proof of short time existence and uniqueness of smooth hypersurfaces moving with normal velocity equal to their mean curvature in $\Omega$ and meeting the boundary $\partial \Omega$ at a prescribed (not necessarily constant) angle. The following theorem is a generalization of [38, Theorem 1], where short time existence and uniqueness have been proven for constant $\beta$.

**Theorem B.1 (Short time existence and uniqueness).** Let $\beta \in C^{1+\alpha}(\partial \Omega)$, $\|\beta\|_\infty \leq 1 - 2\kappa$, $\kappa \in (0, \frac{1}{2}]$ and $E_0 \subset \Omega$ be a bounded open set such that $\Gamma_0 = \Omega \cap \partial E_0$ is a bounded $C^{3+\alpha}$-hypersurface, $\alpha \in (0,1)$. Assume that $U \subset \mathbb{R}^n$ is a bounded open set with $C^{3+\alpha}$-boundary, $p^0 \in C^{3+\alpha}(\Omega, \mathbb{R}^{n+1})$ is a parametrization of $\Gamma_0$ such that $p^0_{n+1} > 0$ in $U$, $p^0_{n+1} = 0$ on $\partial U$, and

$$-e_{n+1} + \beta(p^0)x_0 = Dp^0[n^0] \quad \text{on } \partial U, \quad (B.1)$$

where $n^0 = (n_1^0, \ldots, n_n^0)$ is the outward unit normal to $\partial U$, $n_0 = \nu(p^0)$ is the outward unit normal of $\Gamma_0$ at $p^0$ and $Dp^0[n^0] = \sum_{j=1}^n n_j^0 p_{\sigma_j}^0$. Then there exists $T_0 = T_0(\|\beta\|_{C^{1+\alpha}}, \|p^0\|_{C^{3+\alpha}}) > 0$, a unique family of bounded open sets $\{ E(t) : t \in [0, T_0] \}$ with a parametrization $p \in C^{1+\alpha/2, 2+\alpha}(\mathbb{R}^3, \mathbb{R}^{n+1})$ of $\Gamma(t) = \Omega \cap \partial E(t)$ solving the parabolic system

$$p_t = \text{trace}(D p \cdot (D p)^T)^{-1} D^2 p) \quad \text{in } (0, T_0) \times U, \quad (B.2)$$

where $(D p \cdot (D p)^T)_{ij} = p_{\sigma_i} \cdot p_{\sigma_j}$ and $(D^2 p)_{ij} = p_{\sigma_i \sigma_j}$, coupled with the initial condition $p(0, \cdot) = p^0$, the boundary conditions

$$\begin{cases} p_{n+1}(t, \cdot) = 0 & \text{on } \partial U \text{ for any } t \in [0, T_0], \\ e_{n+1} \cdot \nu(p(t, \cdot)) = \beta(p(t, \cdot)) & \text{on } \partial U \text{ for any } t \in [0, T_0], \end{cases} \quad (B.3)$$

and the orthogonality conditions

$$Dp^0[n^0] \cdot \tau_i = 0 \quad \text{on } [0, T_0] \times \partial U \quad \text{for every } i = 1, \ldots, n - 1, \quad (B.4)$$

where $\nu(p(t, \cdot))$ is the outward unit normal to $\Gamma(t)$ at $p(t, \cdot)$ and $\tau_{01}, \ldots, \tau_{0n} \in \mathbb{R}^n \times \{ 0 \}$ is a basis for the tangent space of $\Gamma_0 \cap \partial \Omega$ at $p^0$.

**Remark B.2.** Assumption (B.1) on $p^0$ is not restrictive. Indeed, if $q : \partial U \to \Gamma_0 \cap \partial \Omega$ is a $C^{3+\alpha}$-parametrization of the contact set, we may extend it to a sufficiently small tubular neighborhood $S := \{ x \in U : \text{dist}(x, \partial U) < \varepsilon \}$ of $\partial U$ in $U$ with the properties that $q$ is a $C^{3+\alpha}$ diffeomorphism, $q(S) \subset \Gamma_0$ and

$$q(\sigma) = q(\varsigma) + |\sigma - \varsigma| (e_{n+1} - \beta(q(\varsigma)x_0(q(\varsigma)))) + O(1) \quad \text{where } \varsigma \text{ is the projection of } \sigma \text{ on } \partial U. \quad \text{Since } \sigma = \varsigma - |\sigma - \varsigma| n^0(\varsigma), \text{ it follows}$$

$$q(n^0(\varsigma) = -e_{n+1} + \beta(q(\varsigma)x_0(q(\varsigma))),$$

which is (B.1). Now we may arbitrarily extend $q$ to a $C^{3+\alpha}$ diffeomorphism in $\overline{U}$ such that $q(\overline{U}) = \Gamma_0$.

**Remark B.3.** The unit normal to $\Gamma(t)$ at the point $p(t, \sigma_1, \ldots, \sigma_n) \in \Gamma(t)$ can be written with a (standard) abuse of notation $\nu = \nu(p(t, \sigma_1, \ldots, \sigma_n)) = \frac{\partial q}{\partial t}$, where

$$\nu := \nu(p_\sigma) = \det \begin{bmatrix} e_1 & e_2 & \cdots & e_n & e_{n+1} \\
 p_{\sigma_1} & p_{\sigma_2} & \cdots & p_{\sigma_n} \end{bmatrix}. \quad (39)$$
Proof of Theorem B.1. The idea of the proof is standard: first we linearize the equation around the boundary conditions, then prove the existence result for the linearized system and we use a fixed point argument.

Step 1. Let us linearize system (B.2) fixing some \( t_0 > 0 \). Let \( X(t_0) \subseteq C^{1+\alpha/2,2+\alpha}([0,t_0] \times \overline{U}, \mathbb{R}^{n+1}) \) be the nonempty convex set consisting of all functions \( w \in C^{1+\alpha/2,2+\alpha}([0,t_0] \times \overline{U}, \mathbb{R}^{n+1}) \) such that

1. \( w(0, \cdot) = p^0 \),
2. \( w_{n+1}(t, \cdot) = 0 \) on \( \partial U \) for any \( t \in [0,t_0] \),
3. \( \sum_{j=1}^n D_{\sigma_j} w_{\sigma_j} \cdot \tau_0 \equiv 0 \) on \( [0,t_0] \times \partial U \) for every \( i = 1, \ldots, n-1 \).

For \( w \in X(t_0) \) set \( f(t, w) := \text{trace} \left[ \left( (Dw \cdot (Dw)^T) - (Dp^0 \cdot (Dp^0)^T) \right) D^2 w \right] \). Then (B.2) is equivalent to

\[
\dot{w} = \text{trace}[(Dp^0 \cdot (Dp^0)^T)^{-1} D^2 w] + f(t, w).
\]

Notice that

\[
|f(t, w)| \leq c(\|p^0\|_{C^1(\overline{U})}) \|w\|_{C^{0,2}([0,t_0] \times \overline{U})} \|w - p_0\|_{C^{0,1}([0,t_0] \times \overline{U})},
\]

where \( c(\|p^0\|_{C^1(\overline{U})}) > 0 \). Now we linearize the contact angle condition. Since we have \( e_{n+1} \cdot \nu(p^0) = \beta(p^0) \), from Remark B.3 it follows that

\[
e_{n+1} \cdot (\tilde{v}(w_{\sigma}) - \tilde{v}(p^0_{\sigma})) = \beta(w)|\tilde{v}(w_{\sigma})| - \beta(p^0)|\tilde{v}(p^0_{\sigma})|.
\]

Let \( H_1(t, w) := \tilde{v}(w_{\sigma}) - \tilde{v}(p^0_{\sigma}) - D\tilde{v}(p^0_{\sigma})[w_{\sigma} - p^0_{\sigma}], \) where

\[
D\tilde{v} = \begin{bmatrix} D_{p_{\beta_1}} \tilde{v}^1 & D_{p_{\beta_2}} \tilde{v}^2 & \cdots & D_{p_{\beta_n}} \tilde{v}^n \\ D_{p_{\beta_1}} \tilde{v}_2 & D_{p_{\beta_2}} \tilde{v}_2 & \cdots & D_{p_{\beta_n}} \tilde{v}_2 \\ \vdots & \vdots & \ddots & \vdots \\ D_{p_{\beta_1}} \tilde{v}_n^{n+1} & D_{p_{\beta_2}} \tilde{v}_n^{n+1} & \cdots & D_{p_{\beta_n}} \tilde{v}_n^{n+1} \end{bmatrix}, \quad q_{\sigma} = \begin{bmatrix} q_{\sigma_1} \\ q_{\sigma_2} \\ \vdots \\ q_{\sigma_n} \end{bmatrix}, \quad (q_{\sigma_1}, \ldots, q_{\sigma_{n+1}})
\]

and

\[
D\tilde{v}[q_{\sigma}] = \begin{bmatrix} \sum_{i=1}^n D_{p_{\sigma_i}} \tilde{v}_i \cdot q_{\sigma_1} \\ \vdots \\ \sum_{i=1}^n D_{p_{\sigma_i}} \tilde{v}_i^{n+1} \cdot q_{\sigma_1} \\ \vdots \\ \sum_{i=1}^n D_{p_{\sigma_i}} \tilde{v}_i^{n+1} \cdot q_{\sigma_n} \end{bmatrix}.
\]

Clearly, \( |H_1(t, w)| = O\left(\|w - p^0\|^2_{C^{0,1}([0,t_0] \times \overline{U})} \right) \). Moreover,

\[
|\tilde{v}(w_{\sigma})| = |\tilde{v}(p^0_{\sigma})| + \nu(p^0) \cdot D\tilde{v}(p^0_{\sigma})[w_{\sigma} - p^0_{\sigma}] + H_2(t, w)
\]

with \( |H_2(t, w)| = O\left(\|w - p^0\|^2_{C^{0,1}([0,t_0] \times \overline{U})} \right) \). Finally, since \( \beta \in C^{1+\alpha}(\partial \Omega) \) we have

\[
\beta(w)|\tilde{v}(w_{\sigma})| - \beta(p^0)|\tilde{v}(p^0_{\sigma})| = \beta(p^0)\nu(p^0) \cdot D\tilde{v}(p^0_{\sigma})[w_{\sigma} - p^0_{\sigma}] + H_3(t, w),
\]

where \( H_3(t, w) = O\left(\|w - p^0\|^2_{C^{0,1}([0,t_0] \times \overline{U})} \right) \). Thus, (B.5) is equivalent to

\[
(e_{n+1} - \beta(p^0)\nu(p^0)) \cdot D\tilde{v}(p^0_{\sigma})[w_{\sigma}] + (e_{n+1} - \beta(p^0)\nu(p^0)) \cdot D\tilde{v}(p^0_{\sigma})[p^0_{\sigma}] + H_4(t, w),
\]

where \( H_4(t, w) = O\left(\|w - p^0\|^2_{C^{0,1}([0,t_0] \times \overline{U})} \right) \).

Thus we have the following linear parabolic system of equations

\[
L(\sigma, \partial_t, \partial_{\sigma}) w = f \quad \text{in} \quad (0, t_0) \times U
\]

subject to the boundary conditions \( B(\beta)(\sigma, \partial_t, \partial_{\sigma}) w = F(t, \sigma) \) on \( [0, t_0] \times \partial U \), where

\[
F(t, \sigma) = \begin{bmatrix} 0, \ (e_{n+1} - \beta(p^0)\nu(p^0)) \cdot D\tilde{v}(p^0_{\sigma})[p^0_{\sigma}] + H_4(t, w), 0, \ldots, 0 \end{bmatrix}^{T}
\]

subject to the boundary conditions \( B(\beta)(\sigma, \partial_t, \partial_{\sigma}) w = F(t, \sigma) \) on \( [0, t_0] \times \partial U \), where

\[
L(\sigma, \partial_t, \partial_{\sigma}) w = f \quad \text{in} \quad (0, t_0) \times U
\]

subject to the boundary conditions \( B(\beta)(\sigma, \partial_t, \partial_{\sigma}) w = F(t, \sigma) \) on \( [0, t_0] \times \partial U \), where
and, under the notation \( \{g_0\}^{ij} = \{\partial\rho_i/\partial\rho_j\}^{-1} \), \( \tilde{\nu} = \tilde{\nu}(\rho) \), \( \beta(\rho) \) the \((n+1)\times(n+1)\) matrices \( L(\sigma,t,\xi,\zeta) \) and \( B_\beta(\xi,\zeta) \), \( \xi \in \mathbb{R}^n \), \( \zeta \in \mathbb{C} \) are defined as follows:

\[
L(\sigma,\zeta,\xi) := \text{diag} \left( \zeta - \sum_{i,j=1}^{n} g_{ij}^{\xi}\xi_{ij}, \zeta - \sum_{i,j=1}^{n} g_{ij}^{\xi}\xi_{ij}, \ldots, \zeta - \sum_{i,j=1}^{n} g_{ij}^{\xi}\xi_{ij} \right),
\]

\[
B_\beta(\xi,\zeta) := \left[
\begin{array}{ccc}
0 & \cdots & 1 \\
\sum_{k=1}^{n+1} \sum_{i=1}^{n+1} (-\delta_{k,n+1} - \beta_0\nu_0^k) \left[ D(p_{n+1})_0 \right]_{ij} \tilde{\nu}_0^k \xi_i \\
\tau_0 \sum_{i=1}^{n} n_i^0 \xi_i & \cdots & \tau_{n+1} \sum_{i=1}^{n} n_i^0 \xi_i \\
\vdots & \ddots & \vdots \\
\tau_0 \sum_{i=1}^{n} n_i^0 \xi_i & \cdots & \tau_{n+1} \sum_{i=1}^{n} n_i^0 \xi_i 
\end{array}
\right],
\]

where the first row must be intended as \( [0, \ldots, 0, 1] \).

**Step 2.** Now we check the compatibility conditions [51]. Take any \( \zeta \in \partial U \) and let \( \theta \) be in the tangent space of \( \partial U \) at \( \zeta \). Let \( \lambda_0 := \lambda_0(\zeta, \zeta, \theta) \) be a solution of the quadratic equation

\[
h(\lambda; \zeta, \zeta, \theta) := \zeta - \sum_{i,j=1}^{n} g_{ij}^{\xi}\theta_{ij} - 2\lambda \sum_{i,j=1}^{n} g_{ij}^{\xi}\theta_n^{ij} + \lambda^2 \sum_{i,j=1}^{n} g_{ij}^{\xi}\theta_n^{ij} = 0
\]

in \( \lambda \in \mathbb{C} \) with positive imaginary part. Notice that \( \det L = (h(\lambda; \zeta, \zeta, \theta))^{n+1} \) and

\[
\tilde{L} = (\det L) L^{-1} = \text{diag}((h(\lambda; \zeta, \zeta, \theta))^{n}, \ldots, (h(\lambda; \zeta, \zeta, \theta))^n).
\]

In order to prove the compatibility conditions we should prove that the rows of matrix \( B_\beta(\xi, \zeta, \theta - \lambda n_0) \tilde{L}(\xi, \zeta, \theta - \lambda n_0) \) are linearly independent modulo the polynomial \( (\lambda - \lambda_0)^{n+1} \) whenever \( \Re(\zeta) \geq 0 \), \( |\zeta| > 0 \). According to the definitions of \( L \) and \( B_\beta \) one checks [38] that the compatibility conditions equivalent to the conditions

\[
c_1 e_{n+1} + c_2 \tilde{\nu}(\rho) + \sum_{i=1}^{n} c_i + 2\tau_0 \theta_i = 0 \quad \iff \quad c_1 = c_2 = \ldots = c_{n+1} = 0.
\]

Since a basis of the tangent space \( \{\tau_{0i}\}_{i=1}^{n+1} \) of \( \Gamma_0 \cap \partial U \) belongs to the horizontal subspace of \( \mathbb{R}^{n+1} \) and \( \tilde{\nu}(\rho) \) is normal to \( \Gamma_0 \cap \partial U \) at \( \rho \) we have \( c_3 = \ldots = c_{n+1} = 0 \). Moreover, since \( |\beta| \leq 1 - 2\kappa \), and \( \Gamma_0 \) satisfies the contact angle condition, \( e_{n+1} \) and \( \tilde{\nu}(\rho) \) are linearly independent, i.e. \( c_1 = c_2 = 0 \).

**Step 3.** By [51, Theorem 4.9] since \( \partial U \subset C^{3+\alpha}, \beta \in C^{1+\alpha}(\partial U) \) and the compatibility conditions hold, for any \( \tilde{f}, \tilde{F} \in C^{0,\alpha}([0, t_0] \times \overline{U}) \), \( p^0 \in C^{3+\alpha}(\overline{U}) \) there exists a unique solution \( w \in C^{1+\alpha/2, 2+\alpha}([0, t_0] \times \overline{U}) \) such that

\[
w_t = \text{tr}((DP^0 \cdot DP^0)^l - 1 D^2 w) + \tilde{f},
\]

\[
w(0, \cdot) = p^0,
\]

\[
w_{n+1}(t, \cdot) = 0 \quad \text{on} \partial U \quad \text{for any} \ t \in [0, t_0],
\]

\[
(e_{n+1} - \beta(\rho) \nu(\rho)) \cdot D\tilde{\nu}(\rho)[w_\sigma] = (e_{n+1} - \beta(\rho) \nu(\rho)) \cdot D\tilde{\nu}(\rho)[p^0] + \tilde{F}(t, x) \quad \text{on} \ [0, t_0] \times \partial U,
\]

\[
\left( \sum_{j=1}^{n} n_j^0 w_{\sigma_j} \right) \cdot \tau_0 = 0 \quad \text{on} \ [0, t_0] \times \partial U \text{ and } i = 1, \ldots, n - 1.
\]

**Step 4.** Finally, mimicking [27] we can prove the existence of and uniqueness of solution (B.2)-(B.4) in time interval \([0, T_0]\) for some sufficiently small \( T_0 > 0 \) depending on \( \|\beta\|_{C^{1+\alpha}} \) and \( \|p^0\|_{C^{3+\alpha}} \).

We call \( E(t) \) the smooth flow starting from \( E_0 \).
**Proposition B.4 (Comparison for strong solutions).** Let $\beta_1 \in (-1, 1)$, $E_0^{(i)} \subset \Omega$ be bounded sets such that $\Omega \cap \partial E_0^{(i)}$ are $C^{3+\alpha}$ hypersurfaces, and the smooth flows $E_0^{(i)}(t)$ starting from $E_0^{(i)}$ exist in $[0, T_0]$, $i = 1, 2$. If $\beta_1 \leq \beta_2$ and $\text{dist}((\Omega \cap \partial E_0^{(1)}), (\Omega \cap \partial E_0^{(2)})) > 0$, then $\text{dist}((\Omega \cap \partial E_0^{(1)}(t), (\Omega \cap \partial E_0^{(2)}(t))) > 0$ for all $t \in [0, T_0]$.

**Proof.** The proof is an adaptation of the classical one (see for instance [10]).

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