New Bounds on Optimal Sorting Networks

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Abstract. We present new parallel sorting networks for 17 to 20 inputs. For 17, 19, and 20 inputs these new networks are faster (i.e., they require less computation steps) than the previously known best networks. Therefore, we improve upon the known upper bounds for minimal depth sorting networks on 17, 19, and 20 channels. Furthermore, we show that our sorting network for 17 inputs is optimal in the sense that no sorting network using less layers exists. This solves the main open problem of [D. Bundala & J. Závodný. Optimal sorting networks, Proc. LATA 2014].

1 Introduction

Comparator networks are hardwired circuits consisting of simple gates that sort their inputs. If the output of such a network is sorted for all possible inputs, it is called a sorting network. Sorting networks are an old area of interest, and results concerning their size date back at least to the 50’s of the last century.

The size of a comparator network in general can be measured by two different quantities: the total number of comparators involved in the network, or the number of layers the networks consists of. In both cases, finding optimal sorting networks (i.e., of minimal size) is a challenging task even when restricted to few inputs, which was attacked using different methods.

For instance, Valsalam and Miikkulainen [11] employed evolutionary algorithms to generate sorting networks with few comparators. Minimal depth sorting networks for up to 16 inputs were constructed by Shapiro and Van Voorhis in the 60’s and 70’s, and by Schwiebert in 2001, who also made use of evolutionary techniques. For a presentation of these networks see Knuth [8, Fig.51]. However, the optimality of the known networks for 11 to 16 channels was only shown recently by Bundala and Závodný [4], who partitioned the set of first two layers into equivalence classes and reduced the search to extensions of one representative of each class. They then expressed the existence of a sorting network with less layers extending these representatives in propositional logic and used a SAT solver to show that the resulting formulae are unsatisfiable. Codish, Cruz-Filipe, and Schneider-Kamp [5] simplified the generation of the representatives and independently verified Bundala and Závodný’s result.

For more than 16 channels, not much is known about the minimal depths of sorting networks. Al-Haj Baddar and Batcher [2] exhibit a network sorting 18 inputs using 11 layers, which also provides the best known upper bound on the
minimal depth of a sorting network for 17 inputs. The lowest upper bound on
the size of minimal depth sorting networks on 19 to 22 channels also stems from
a network presented by Al-Haj Baddar and Batcher [1]. For 23 and more inputs,
the best upper bounds to date are established by merging the outputs of smaller
sorting networks with Batcher’s odd-even merge [3], which needs \(\lceil \log n \rceil\) layers
for this merging step.

The known lower bounds are due to Parberry [10] and Bundala and Závodný.
A new insight by Codish, Cruz-Filipe, and Schneider-Kamp [6] into the structure
of the last layers of sorting networks lead to a significant further reduction of
the search space. Despite all this recently resparked interest in sorting networks,
the newly gained insights were insufficient to establish a tight lower bound on the
depth of sorting networks for 17 inputs.

We use the SAT approach by Bundala and Závodný to synthesize new sorting
networks of small depths, and thus provide better upper bounds for 17, 19, and
20 inputs. Furthermore, our improvements upon their method allow us to raise
the lower bound for 17 inputs. Therefore, for the first time after the works of
Shapiro, Van Voorhis, and Schwiebert, we present here a new optimal depth
sorting network.

An overview of the old and new upper and lower bounds for the minimal
depth of sorting networks for up to 20 inputs is presented in Table 1.

| Number of inputs | 1  | 2  | 3  | 4  | 5  | 6  | 7  | 8  | 9  | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 |
|------------------|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|----|
| Old upper bound  | 0  | 1  | 3  | 3  | 5  | 5  | 6  | 6  | 7  | 7  | 8  | 8  | 9  | 9  | 9  | 9  | 9  | 9  | 9  | 9  |
| New upper bound  | 0  | 1  | 3  | 3  | 5  | 5  | 6  | 6  | 7  | 7  | 8  | 8  | 9  | 9  | 9  | 9  | 9  | 9  | 9  | 9  |
| Old lower bound  | 0  | 1  | 3  | 3  | 5  | 5  | 6  | 6  | 7  | 7  | 8  | 8  | 9  | 9  | 9  | 9  | 9  | 9  | 9  | 9  |
| New lower bound  | 0  | 1  | 3  | 3  | 5  | 5  | 6  | 6  | 7  | 7  | 8  | 8  | 9  | 9  | 9  | 9  | 9  | 9  | 9  | 9  |

2 Preliminaries

A comparator is a gate with two inputs \(in_1\) and \(in_2\) and two outputs \(out_{\text{min}}\)
and \(out_{\text{max}}\), that compares, and if necessary rearranges its inputs such that
\(out_{\text{min}} = \min\{in_1, in_2\}\) and \(out_{\text{max}} = \max\{in_1, in_2\}\). Combining zero or more
comparators in a network yields a comparator network. Comparator networks
are usually visualized in a graphical manner in a so-called Knuth diagram as
depicted in ??, Here a comparator connecting two channels is drawn as ,
where by convention the upper output is \(out_{\text{min}}\) and the lower output is \(out_{\text{max}}\).
A maximal set of comparators with respect to inclusion that can perform parallel
comparisons in a comparator network is called a layer. The number of layers of a
network is called the depth of the network. A useful tool to verify that a network
is a sorting network is the “0-1-principle” [8], which states that a comparator
network is a sorting network, if and only if it sorts all binary inputs.

For more details about sorting networks we refer to Knuth [8 Sect. 5.3.4].
3 Improved Techniques

In this section we introduce the new techniques and improvements on existing techniques we used to gain our results. We will stick to the formulation by Bundala and Závodný [4], and introduce new variables if necessary. Furthermore, we will extend a technique introduced in their paper, called subnetwork optimization. It is based on the fact that a sorting network must sort all its inputs, but in order to prove non-existence of sorting networks of a certain depth, it is often sufficient to consider only a subset of all possible inputs, which are not all sorted by any network of this restricted depth. Bundala and Závodný chose subsets of the form

$$T^r = \{0^a x^b \mid |x| = r \text{ and } a + b + |x| = n\}$$

for $$r < n$$, which are inputs having a window of size $$r$$. For an input $$0^a x^b$$ from this set the values on the first $$a$$ channels at any point in the network will always be 0, and those on the last $$b$$ channels will always be 1, which significantly reduces the encoding size for these inputs if $$a$$ and $$b$$ are sufficiently large.

3.1 Prefix optimization

It is a well-known fact that permuting the channels of a sorting network, followed by a repair-procedure called untangling yields another feasible sorting network [10]. In fact, Parberry [10] used a first layer with comparators of the form $$(2i - 1, 2i), 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$, which we will call Pb-style whereas Bundala and Závodný used comparators $$(i, n + 1 - i), 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor$$ in the first layer, which we call BZ-style; see ???. Both versions are equivalent in the sense that if there exists a sorting network $$C^d_n$$, then there exist sorting networks of the same depth with both of these prefixes. Nevertheless, for creating networks obeying a certain prefix as well as proving their non-existence, a given prefix may be hard-coded into the SAT formula. Given a prefix $$P$$ of depth $$|P|$$, the remaining SAT formula encodes the proposition “There is a comparator network on $$n$$ channels of depth $$d - |P|$$ which sorts all outputs of $$P$$”. Interestingly, the outputs of different prefix styles are not equally handy for the SAT encoding. A Pb-style first layer performs compare-and-swap operations between adjacent channels, thus the presorting performed here is more local than the one done by BZ-style first layers. Let
out\( (P) \) denote the set of outputs of a prefix \( P \) on \( n \) layers. Then, the number of channels that actually must be considered in the SAT formula is given by

\[
\sum_{x \in \text{out}(P)} \left( n - \max\{ a \mid x = 0^a x' \} - \max\{ b \mid x = x'1^b \} \right),
\]

i.e., the sum of window-sizes of all outputs of \( P \).

Table 2 shows the impact of these previous deliberations when using a 1-layer-prefix for \( 2 \leq n \leq 17 \) channels.

| \( n \) | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 |
|------|---|---|---|---|---|---|---|---|----|----|----|----|----|----|----|----|
| Pb-style | 0 5 12 44 84 233 408 1016 1704 4013 6564 14948 24060 55358 85296 186992 |
| BZ-style | 0 4 10 36 72 196 358 876 1524 3532 5962 13380 22128 48628 79246 171612 |

Table 3. Impact of prefix style when proving that no sorting network for 16 channels with at most 8 layers exists.

| Prefix style | Overall time (s) | Maximum time (s) |
|--------------|-----------------|------------------|
| Pb           | 22.241          | 326              |
| BZ           | 10.927          | 150              |
| Opt          | 5.492           | 36               |

Table 3 shows running times when proving that no sorting network on 16 channels with 8 layers exists. In this case, we used 2-layer-prefixes according to [5], and proved unfeasibility for each of the 211 distinct prefixes. In the first case, they were permuted and untangled such that the first layer is in Pb-style, whereas the second case has BZ-style first layers. In the third case, we used an evolutionary algorithm to find a prefix such that the number of channels to consider when using 800 distinct outputs of the respective prefix was minimized. As we have \( d - |P| \) Boolean variables for each channel which cannot be hard-coded, this procedure minimizes the number of variables in the resulting SAT formula.

This technique reduced the overall running time by factors of 4.05 and 1.99, and the maximum running times by 9.0 and 4.1, respectively.

3.2 Iterative encoding

As mentioned above, it is usually not necessary to use all \( 2^n \) input vectors to prove lower bounds. In order to take advantage of this fact, we implemented an iterative approach. We start with a formula which describes a feasible comparator.
network and a (potentially empty) set of initial inputs, and iteratively add inputs until either a feasible sorting network has been found, or no network can be found which is able to sort the given set of inputs, as depicted in figure (3). During the iterative process, counter-examples (i.e., inputs that are not sorted by the network created so far) of minimal window-size are chosen.

![Network Creation Diagram](image)

**Fig. 3. Iterative generation of new inputs**

Using this technique, we tested the impact of different prefix style on both running time, and the number of inputs required. Table 4 shows results for one 2-layer-prefix for 16 channels, used to prove that this cannot be extended to a sorting network with 8 layers. Here, more inputs are required to prove that a BZ-style prefix cannot be extended when compared to a Pb-style prefix. Nevertheless, BZ-style prefixes are superior in terms of running time. Interestingly, the process becomes faster when more inputs than actually required were chosen, this is, using slightly more inputs is beneficial.

Next, we turn to improve the SAT encoding.

### 3.3 Improved SAT encoding

We modified the SAT encoding of Bundala and Závodný [4], significantly reducing the number of clauses. A variable $g^k_{i,j}$, with $i < j$, encodes the fact that there is a comparator comparing channels $i$ and $j$ in layer $k$ in the network. Furthermore, the variable $v^k_i$ stores the value on channel $i$ after layer $k$. For completeness' sake

| Initial Inputs | 0 100 200 300 400 500 600 700 800 900 |
|----------------|----------------------------------------|
| Pb Time        | 157 139 128 86 61 56 52 45 34 27 |
| Pb Iterations  | 264 174 72 4 1 1 1 1 1 1 |
| BZ Time        | 88 75 64 47 28 14 13 13 13 19 |
| BZ Iterations  | 358 259 165 66 10 1 1 1 1 1 |
we list the original encoding completely:

\[
\begin{align*}
\text{once}^k(C^d_n) &= \bigwedge_{1 \leq i \neq j \leq n} \left( \neg g^k_{\min(i,j),\max(i,j)} \vee \neg g^k_{\min(i,l),\max(i,l)} \right) \\
\text{valid}(C^d_n) &= \bigwedge_{1 \leq k \leq d, 1 \leq i \leq n} \text{once}^k(C^d_n) \\
\text{used}^k(C^d_n) &= \bigvee_{j < i} g^k_{j,i} \lor \bigvee_{i < j} g^k_{i,j} \\
\text{update}^k(C^d_n) &= \left( \neg \text{used}^k(C^d_n) \rightarrow (v^k_i \leftrightarrow v^{k-1}_i) \right) \land \\
&\quad \bigwedge_{1 \leq j < i} (g^k_{j,i} \rightarrow (v^k_i \leftrightarrow (v^{k-1}_j \vee v^{k-1}_i))) \land \\
&\quad \bigwedge_{i < j \leq n} (g^k_{i,j} \rightarrow (v^k_i \leftrightarrow (v^{k-1}_j \land v^{k-1}_i)))
\end{align*}
\]

Here, \text{once} encodes the fact that each channel may be used only once in one layer, and \text{valid} encodes this constraint for each channel and each layer. The \text{update}-formula describes the impact of comparators on the values stored on each channel after every layer.

\[
\text{sorts}(C^d_n, x) = \bigwedge_{1 \leq i \leq n} (v^0_i \leftrightarrow x_i) \land \bigwedge_{1 \leq k \leq d, 1 \leq i \leq n} \text{used}^k(C^d_n) \land \bigwedge_{1 \leq i \leq n} (v^d_i \leftrightarrow y_i)
\]

The constraint \text{sorts} encodes if a certain input is sorted by the network \(C^d_n\). For this purpose, the values after layer \(d\) (i.e., the outputs of the network) are compared to the vector \(y\), which is a sorted copy of the input \(x\). A sorting network for \(n\) channels on \(d\) layers exists if \(\text{valid}(C^d_n) \land \bigwedge_{x \in \{0,1\}^n} \text{sorts}(C^d_n, x)\) is satisfiable.

Consider an input \(x = 0^a x^1 b^b\), and a comparator \(g^k_{ij}\) with \(i \leq a\). This is, we have \(v^k_i \leftrightarrow 0 \land v^{k-1}_i \equiv 0\), and \(v^k_j \leftrightarrow 0 \lor v^{k-1}_j \equiv v^{k-1}_j\). As the same holds for \(j > n - b\), we have that comparators "leaving" a subnetwork need not be considered for sorting the respective inputs. Furthermore, if \(v^{k-1}_i \equiv 1\) for some \(k\) and \(i\), using any comparator \(g^k_{ij}\) will cause \(v^k_i \equiv 1\). Thus, for each channel \(i\) we introduce \text{oneDown}^k_{ij}\) and \text{oneUp}^k_{ij}-variables which indicate whether there is a comparator \(g^k_{ij}\) for some \(i < j \leq a\) or \(g^k_{ij}\) for some \(i \leq l < j\), respectively.

\[
\begin{align*}
\text{oneDown}^k_{i,j} &\leftrightarrow \bigvee_{i < l \leq j} g^k_{i,l} \quad \text{noneDown}^k_{i,j} \leftrightarrow \neg \text{oneDown}^k_{i,j} \\
\text{oneUp}^k_{i,j} &\leftrightarrow \bigvee_{i \leq l < j} g^k_{i,j} \quad \text{noneUp}^k_{i,j} \leftrightarrow \neg \text{oneUp}^k_{i,j}
\end{align*}
\]

To make use of these new predicates, given an input \(x = 0^a x^1 b\), for all \(a < i \leq n - b\) we add

\[
\begin{align*}
v^{k-1}_i \land \text{noneDown}^k_{i,n-b} &\rightarrow v^k_i \\
\neg v^{k-1}_i \land \text{noneUp}^k_{a+1,i} &\rightarrow \neg v^k_i
\end{align*}
\]
to the formula and remove all update-constraints that are covered by these constraints. This offers several advantages: Firstly, it reduces the size of the resulting formula in terms of both the number of clauses, and the overall number of literals. Secondly, this encoding allows for more propagations, thus, conflicts can be found earlier. Thirdly, it offers a more general perspective on the reasons of a conflict. Table 5 shows the impact of both the new encoding and permuting the prefix, which results in an average speed-up of 8.2, and a speed-up of 17.1 for the hardest prefixes.

Table 5. Results for different settings when proving the non-existence of 8-layer sorting-networks for 16 channels

| Encoding | Prefix-Style | Overall time (s) | Max. time (s) | Variables | Clauses | Literals |
|----------|--------------|-----------------|--------------|-----------|---------|---------|
| Old      | Pb           | 22,241          | 326          | 108,802   | 4,467,201 | 13,977,393 |
|          | BZ           | 10,927          | 150          | 99,850    | 3,996,902 | 12,442,522 |
|          | Opt          | 5,492           | 36           | 84,928    | 3,133,636 | 9,879,588  |
| New      | Pb           | 11,766          | 196          | 110,398   | 2,443,186 | 8,108,501  |
|          | BZ           | 4,359           | 54           | 101,404   | 2,049,744 | 6,799,486  |
|          | Opt          | 2,702           | 19           | 85,652    | 1,504,177 | 4,981,882  |

4 Obtaining new lower bounds

In a first test, we tried to prove that there is no sorting network on 17 channels using at most 9 layers by using a SAT encoding which was almost identical to the one introduced in [4], enriched by constraints on the last layers [6]. Before we broke up this experiment after 48 days, we were able to prove that 381 out of 609 prefixes cannot be extended to sorting networks of depth 9. Showing unsatisfiability of these formulae took 353 · 10^6 seconds of CPU time, with a maximum of 3 · 10^6 seconds.

In a new attempt, we used a modified encoding as described in the previous section. For every equivalence class of 2-layer-prefixes, we chose a representative which minimizes the number of variables in the SAT encoding when using 2,000 distinct inputs. This time, we were able to prove that none of the prefixes can be extend to a sorting network using 9 layers. The overall CPU time for all 609 equivalence classes was 27.63 · 10^6 seconds with a maximum running time of 97,112 seconds. This is a speed up of at least 42.7 concerning the maximum running time, and 20.4 for the average running time. Since the result for all 2-layer-prefixes was unsat, we conclude:

Theorem 1 Any sorting network for \( n \geq 17 \) channels has at least 10 layers.
5 Finding faster networks

Even though SAT encodings for sorting networks as well as SAT solvers themselves have become much better within the last years, generating new, large sorting networks from scratch is still out of their scope. Hence, we extended ideas by Al-Haj Baddar and Batcher [2].

5.1 Using hand-crafted prefixes

A well-known technique for the creation of sorting networks is the generation of partially ordered sets for parts of the input in the first layers. ?? shows comparator networks which create partially ordered sets for 2, 4 and 8 input bits. In the case of \( n = 2 \), the output will always be sorted. For \( n = 4 \) bits, the set of possible output vectors is given by

\[
\{(0 \ 0 \ 0 \ 0)^T, (0 \ 0 \ 0 \ 1)^T, (0 \ 0 \ 1 \ 1)^T, (0 \ 1 \ 0 \ 1)^T, (0 \ 1 \ 1 \ 1)^T, (1 \ 1 \ 1 \ 1)^T\},
\]

i.e., there are 6 possible outputs. Furthermore, the first output bit will equal zero unless all input bits are set to one, and the last output bit will always be set to one unless all input bits equal zero. Similarly, the network for \( n = 8 \) inputs allows for 20 different output vectors. Prefixes of sorting networks which consist of such snippets are referred to as Green filters [7].

5.2 Results

We present two sorting networks improving upon the known upper bounds on the minimal depth of sorting networks. The networks presented in ?? are sorting networks for 17 channels using only 10 layers, which outperform the currently best known network due to Al-Haj Baddar and Batcher [2]. The first three layers of the network on the left are a Green filter on the first 16 channels, the remainder of this network was created by a SAT solver. To create the network on the right, we applied the prefix optimization procedure described earlier to the Green filter prefix. The first version of our solver required 29921 seconds to find this network, whereas our current solver can find these networks in 282 seconds, when using the Green filter, and 60 seconds when using the optimized prefix.

Thus, we can summarize our results in the following theorem:
Theorem 2  The optimum depth for a sorting network on 17 channels is 10.

The network displayed in ?? sorts 20 inputs in 11 parallel steps, which beats the previously fastest network using 12 layers [1]. In the first layer, partially ordered sets of size 2 are created. These are merged to 5 partially ordered sets of size 4 in the second layer. The third layer is used to create partially ordered sets of size 8 for the lowest and highest wires, respectively. These are merged in the fourth layer.

The wires in the middle of the network are connected in order to totally sort their intermediate output. Using this prefix and the necessary conditions on sorting networks depicted above, we were able to create the remaining layers using our iterative, SAT-based approach. Interestingly, the result was created in 588 iterations, thus 587 different input vectors were sufficient.

6 Tools

Our software is based on the well-known SAT solver MiniSAT 2.20. Before starting a new loop of our network creation process, we used some probing-based preprocessing techniques [9] as they were quite successful on this kind
of SAT formulae. MiniSAT uses activity values for clauses which are used for managing the learnt clause database. Here, we changed the value “cla-decay” to 0.9999, which leads to better control on learnt clauses that were not used for a long time. Our experiments were performed on Intel Xeon E5-4640 CPUs clocked at 2.40GHz. The software used for our experiments can be downloaded at [http://www.informatik.uni-kiel.de/~the/SortingNetworks.html](http://www.informatik.uni-kiel.de/~the/SortingNetworks.html).

Acknowledgments

We would like to thank Michael Codish, Luís Cruz-Filipe, and Peter Schneider-Kamp for fruitful discussions on the subject and their valuable comments on an earlier draft of this article. Furthermore, we thank Dirk Nowotka for providing us with the computational resources to run our experiments.

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