Almost periodic functions in finite-dimensional space with the spectrum in a cone

Favorov S., Udodova O.\textsuperscript{1}

Kharkiv National University
Department of Mechanics and Mathematics
md. Svobody 4, Kharkiv 61077, Ukraine
e-mail: favorov@ilt.kharkov.ua,
Olga.I.Udodova@univer.kharkov.ua

Abstract

We prove that an almost periodic function in finite-dimensional space extends to a holomorphic bounded function in a tube domain with a cone in the base if and only if the spectrum belongs to the conjugate cone. We also prove that an almost periodic function in finite-dimensional space has the bounded spectrum if and only if it extends to an entire function of exponential type.

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A continuous function \( f(z) \) on a strip

\[ S_{a,b} = \{ z = x + iy : x \in \mathbb{R}, \ a \leq y \leq b \} , \ -\infty \leq a \leq b \leq +\infty \]

is called \textit{almost periodic by Bohr} on this strip, if for any \( \varepsilon > 0 \) there exists \( l = l(\varepsilon) \) such that every interval of the real axis of length \( l \) contains a number \( \tau (\varepsilon\text{-almost period for } f(z)) \) with the property

\[ \sup_{z \in S_{a,b}} |f(z + \tau) - f(z)| < \varepsilon. \] (1)

In particular, when \( a = b = 0 \) we obtain the class of almost periodic functions on the real axis.

To each almost periodic function \( f(z) \) assign the Fourier series

\[ \sum_{n=0}^{\infty} a_n(y)e^{i\lambda_n x} , \ \lambda_n \in \mathbb{R} , \]

where \( a_n(y) \) are continuous functions of the variable \( y \in [a, b] \).

In the case \( a = b = 0 \) all exponents \( \lambda_n \) are nonnegative if and only if the function \( f(x) \) extends to the upper half-plane as a holomorphic bounded almost periodic function; the set of all exponents \( \lambda_n \) is bounded if and only if \( f(x) \) extends to the plane \( \mathbb{C} \) as an entire function of the exponential type \( \sigma = \sup_n |\lambda_n| \), which is almost periodic in every horizontal strip of finite width (see [1], [4]).

A number of works connected with almost periodic functions of many variables on a tube set appeared recently (see [2], [6]-[9]). Recall that the set \( T_K \subset \mathbb{C}^m \) is a tube set if

\[ T_K = \{ z = x + iy : x \in \mathbb{R}^m, y \in K \} , \]

where \( K \subset \mathbb{R}^m \) is the base of the tube set.

\textbf{Definition.} (See [6], [9]). A continuous function \( f(z) \), \( z \in T_K \) is called \textit{almost periodic by Bohr} on \( T_K \), if for any \( \varepsilon > 0 \) there exists \( l = l(\varepsilon) \), such that every \( m \)-dimensional cube on \( \mathbb{R}^m \) with the side \( l \) contains at least one point \( \tau \) \((T_K, \varepsilon\text{-almost period from } f(z)) \) with the property

\[ \sup_{z \in T_K} |f(z + \tau) - f(z)| < \varepsilon. \] (2)

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Let \( f, g \) be locally integrable functions on every real plane
\[
\{ z = x + iy_0 : x \in \mathbb{R}^m \}, \ y_0 \in K.
\]

**Definition.** Stepanoff distance of the order \( p \geq 1 \) between functions \( f \) and \( g \) is the value
\[
S_{p,T_K}(f,g) = \sup_{z \in T_K} \left( \int_{[0,1]^m} |f(z + u) - g(z + u)|^p du \right)^{\frac{1}{p}}.
\]

Using this definition, we can extend the concept of almost periodic functions by Stepanoff on a strip (see [4] p.197) to almost periodic functions on a tube set:

**Definition.** A function \( f(z), z \in T_K \), is called almost periodic by Stepanoff on \( T_K \), if for any \( \varepsilon > 0 \) there exists \( l = l(\varepsilon) \) such that every \( m \)-dimensional cube with the side \( l \) contains at least one \( \tau \) \((T_K, \varepsilon, p)\)-almost period by Stepanoff of the function \( f(z) \) with the property
\[
S_{p,T_K}(f(z), f(z + \tau)) < \varepsilon. \tag{3}
\]

The Fourier series for an almost periodic (by Bohr or by Stepanoff) function \( f(z) \) on a set \( T_K \) is the series
\[
\sum_{\lambda \in \mathbb{R}^m} a(\lambda, y)e^{i\langle x, \lambda \rangle}, \tag{4}
\]
where \( \langle x, \lambda \rangle \) is the scalar product on \( \mathbb{R}^m \), and
\[
a(\lambda, y) = \lim_{N \to \infty} \left( \frac{1}{2N} \right)^m \int_{[-N,N]^m} f(x + x' + iy)e^{-i\langle x + x', \lambda \rangle} dx;
\]
this limit exists uniformly in the parameter \( x' \in \mathbb{R}^m \) and does not depend on this parameter (see [6], [8]).

A set of all vectors \( \lambda \in \mathbb{R}^m \) such that \( a(\lambda, y) \neq 0 \) is called the spectrum of \( f(z) \) and is denoted by \( spf \); this set is at most countable, therefore the series (4) can be written in the form
\[
\sum_{n=0}^{\infty} a_n(y)e^{i\langle x, \lambda_n \rangle}.
\]

Note that partial sums of the series (4), generally speaking, do not converge to the function \( f(z) \). However the Bochner-Feyer sums \(^1\)
\[
\sigma_q(z) = \sum_{n=0}^{q-1} k_n^q a_n(y)e^{i\langle x, \lambda_n \rangle}, \ 0 \leq k_n^q < 1, \ k_n^q \to 1 \text{ as } q \to \infty
\]
converge to the function \( f(z) \) uniformly for almost periodic functions by Bohr and with respect to the metric \( S_{p,T_K} \) for almost periodic functions by Stepanoff; in particular, if two functions have the same Fourier series, then these functions coincide identically. For holomorphic almost periodic functions the series (4) can be written in the form
\[
\sum_{n=0}^{\infty} a_n e^{-\langle y, \lambda_n \rangle} e^{i\langle z, \lambda_n \rangle} = \sum_{n=0}^{\infty} a_n e^{i\langle z, \lambda_n \rangle}, \ a_n \in , \tag{6}
\]
(see [8]). Any series of the form (6) is called Dirihlet series.

By \( \Gamma \) we always denote a convex closed cone in \( \mathbb{R}^m \); by \( \hat{\Gamma} \) we denote the conjugate cone to \( \Gamma \):
\[
\hat{\Gamma} = \{ t \in \mathbb{R}^m : \langle t, y \rangle \geq 0 \ \forall y \in \Gamma \},
\]

note that \( \hat{\hat{\Gamma}} = \Gamma \). Also, \( \hat{\Gamma} \) is the interior of a cone \( \Gamma \).

\(^1\)For \( n = 1 \) see [4], for \( n > 1 \) see [8].
Theorem 1. Let \( f(x) \) be an almost periodic function by Bohr on \( \mathbb{R}^m \) with Fourier series
\[
\sum_{n=0}^{\infty} a_n e^{i \langle x, \lambda_n \rangle},
\]
where all the exponents \( \lambda_n \) belong to a cone \( \Gamma \subset \mathbb{R}^m \). Then \( f(x) \) continuously extends to the tube set \( T_\Gamma \) as an almost periodic by Bohr function \( F(z) \) with Fourier series (6). The function \( F(z) \) is holomorphic on the interior \( T_\Gamma \), and for any \( \Gamma' \subset \subset \Gamma \) uniformly w.r.t. \( z \in T_\Gamma \):
\[
\lim_{\|y\| \to \infty} F(z) = a_0,
\]
where \( a_0 \) is the Fourier coefficient corresponding to the exponent \( \lambda_0 = 0 \) (if \( 0 \not\in \text{sp} f \), put \( a_0 = 0 \)). If \( \text{sp} f \subset \Gamma \), then (8) is true uniformly w.r.t. \( z \in T_\Gamma \).

Here the inclusion \( \Gamma' \subset \subset \Gamma \) means that the intersection of \( \Gamma' \) with the unit sphere is contained in the interior of the intersection of \( \Gamma \) with this sphere.

To prove this theorem, we use the following lemmas.

Lemma 1. Suppose that a plurisubharmonic function \( \varphi(z) \) on \( \mathbb{C}^m \) is bounded from above on a set \( T_K \), where \( K \subset \mathbb{R}^m \) is a convex set. Then the function
\[
\psi(y) = \sup_{x \in \mathbb{R}^m} \varphi(x + iy)
\]
is convex on \( K \).

Proof of Lemma 1. Fix \( y_1, y_2 \in K \). The plurisubharmonic on \( \mathbb{C}^m \) function
\[
\varphi_1(z) = \varphi(z) - \frac{\psi(y_2) - \psi(y_1)}{\|y_2 - y_1\|^2} \langle \text{Im} z, y_2 - y_1 \rangle
\]
is bounded from above on the set \( T_{[y_1, y_2]} \). Therefore the subharmonic function \( \varphi_2(w) = \varphi_1((y_2 - y_1)w + iy_1) \) is bounded on the strip \( \{w = u + iv : u \in \mathbb{R}, 0 \leq v \leq 1\} \). Hence, the value of \( \varphi_1 \) at any point of this strip does not exceed
\[
\max \{\sup_{u \in \mathbb{R}} \varphi_2(u), \sup_{u \in \mathbb{R}} \varphi_2(u + i)\} \leq \max \{\sup_{x \in \mathbb{R}^m} \varphi_1(x + iy_1), \sup_{x \in \mathbb{R}^m} \varphi_1(x + iy_2)\}.
\]
Therefore, for any \( z = x + iy \in T_{[y_1, y_2]} \),
\[
\varphi_1(z) \leq \max \{\psi(y_1) - \frac{\psi(y_2) - \psi(y_1)}{\|y_2 - y_1\|^2} \langle y_1, y_2 - y_1 \rangle, \psi(y_2) - \frac{\psi(y_2) - \psi(y_1)}{\|y_2 - y_1\|^2} \langle y_2, y_2 - y_1 \rangle\}
\]
\[
= \frac{\|y_2\|^2 - \langle y_1, y_2 \rangle}{\|y_2 - y_1\|^2} \psi(y_1) + \frac{\|y_1\|^2 - \langle y_1, y_2 \rangle}{\|y_2 - y_1\|^2} \psi(y_2).
\]
Hence for any \( y \in [y_1, y_2] \) we have
\[
\psi(y) \leq \frac{\|y_2\|^2 - \langle y_1, y_2 \rangle - \langle y_2, y_2 - y_1 \rangle}{\|y_2 - y_1\|^2} \psi(y_1) + \frac{\|y_1\|^2 - \langle y_1, y_2 \rangle + \langle y, y_2 - y_1 \rangle}{\|y_2 - y_1\|^2} \psi(y_2).
\]
If \( y = \lambda y_1 + (1 - \lambda) y_2, \lambda \in (0, 1) \), then we obtain the inequality
\[
\psi(\lambda y_1 + (1 - \lambda) y_2) \leq \lambda \psi(y_1) + (1 - \lambda) \psi(y_2).
\]
Therefore, the function \( \psi(y) \) is convex on \( K \).

Lemma 2. Let \( \psi(y) \) be a convex bounded function on a cone \( \Gamma \). Then \( \psi(y) \leq \psi(0) \) for all \( y \in \Gamma \).
Proof of Lemma 2. Since \( \psi(y) \) is convex, we have

\[
\psi(y) \leq \left( 1 - \frac{1}{t} \right) \psi(0) + \frac{1}{t} \psi(ty), \ t > 1.
\]

Taking \( t \to \infty \), we obtain \( \psi(y) \leq \psi(0) \).

\( \blacksquare \)

Proof of Theorem 1. Let \( \sigma_q(x), \ q = 0, 1, 2, \ldots \) be the Bochner-Feyer sums for the series (7). Obviously, these functions are also defined for \( z \in \mathbb{C}^m \). Assume that

\[
\varphi_{q,l}(z) = \log(|\sigma_q(z) - \sigma_l(z)|).
\]

For any fixed \( q \) and \( l, q > l \), the function \( \varphi_{q,l}(z) \) is plurisubharmonic on \( \mathbb{C}^m \). Moreover, for \( z \in T_{\hat{\Gamma}} \) we have \( \langle y, \lambda_n \rangle \geq 0 \) and

\[
|\sigma_q(z) - \sigma_l(z)| \leq |\sigma_q(z)| + |\sigma_l(z)| \leq \sum_{n=0}^{q-1} |a_n| e^{-(y,\lambda_n)} + \sum_{n=0}^{l-1} |a_n| e^{-(y,\lambda_n)} \leq 2 \sum_{n=0}^{q-1} |a_n|.
\]

Consider the function

\[
\psi_{q,l}(y) = \sup_{x \in \mathbb{R}^m} \log(|\sigma_q(z) - \sigma_l(z)|).
\]

Using lemma 1, we obtain that \( \psi_{q,l}(y) \) is convex in \( \hat{\Gamma} \). Therefore, by lemma 2, we have

\[
\sup_{z \in T_{\hat{\Gamma}}} (|\sigma_q(z) - \sigma_l(z)|) \leq \sup_{x \in \mathbb{R}^m} (|\sigma_q(x) - \sigma_l(x)|).
\] (9)

Further, the function \( f(x) \) is almost periodic, therefore the Bochner-Feyer sums converge uniformly on \( \mathbb{R}^m \), and for \( q, l \geq N(\varepsilon) \)

\[
\sup_{x \in \mathbb{R}^m} (|\sigma_q(x) - \sigma_l(x)|) \leq \varepsilon.
\]

Hence, \( \sup_{x \in \mathbb{R}^m} (|\sigma_q(z) - \sigma_l(z)|) \leq \varepsilon \) for all \( z \in T_{\hat{\Gamma}}, \ q, l \geq N(\varepsilon) \).

Thus the Bochner-Feyer sums uniformly converge on \( T_{\hat{\Gamma}} \), their limit is an almost periodic function by Bohr and holomorphic on the interior of \( T_{\hat{\Gamma}} \) with the Dirihlet series (6).

Further, passing to the limit in (9) as \( q \to \infty \), we get

\[
\sup_{z \in T_{\hat{\Gamma}}} (|F(z) - \sigma_l(z)|) \leq \sup_{x \in \mathbb{R}^m} (|f(x) - \sigma_l(x)|).
\]

Choose \( l \) such that the right hand side of this inequality is less than \( \varepsilon \). We have for \( \Gamma' \subset \subset \hat{\Gamma} \)

\[
\sup_{z \in T_{\Gamma'}} |F(z) - a_0| \leq \sup_{z \in T_{\Gamma'}} |F(z) - \sigma_l(z)| + \sup_{z \in T_{\Gamma'}} |\sigma_l(z) - a_0| \leq \varepsilon + \sup_{z \in T_{\Gamma'}} |\sigma_l(z) - a_0|.
\]

Note that for any fixed \( \lambda \in \Gamma \setminus \{0\} \) the value \( \langle y, \lambda_n \rangle \) tends to \( +\infty \) as \( \|y\| \to \infty \), \( y \in \Gamma' \), therefore, we have

\[
|\sigma_l(z) - a_0| = \sum_{j=1}^{l-1} k_j a_j e^{\langle x, \lambda_j \rangle} e^{-(y,\lambda)} \leq \sum_{j=1}^{l-1} |a_j| e^{-(y,\lambda)} \to 0
\]

as \( \|y\| \to \infty \) on \( \Gamma' \). Hence, uniformly w.r.t. \( z \in T_{\Gamma'} \)

\[
\lim_{\|y\| \to \infty} |F(z) - a_0| \leq \varepsilon.
\] (10)

This is true for arbitrary \( \varepsilon > 0 \), then (8) follows.

If \( sp f \subset \hat{\Gamma} \), then for any \( \lambda \in sp f, \ \langle y, \lambda_n \rangle \to +\infty \) as \( \|y\| \to \infty \) uniformly w.r.t. \( y \in \hat{\Gamma} \), therefore (10) is true uniformly w.r.t. \( z \in T_{\hat{\Gamma}} \), and (8) is also true. The theorem has been proved.

\( \blacksquare \)
Theorem 2. Let \( f(x) \) be an almost periodic function by Stepanoff on \( \mathbb{R}^m \) with the Fourier series (7). Let all the exponents \( \lambda_n \) belong to a cone \( \Gamma \subset \mathbb{R}^m \). Then there exists an almost periodic by Stepanoff function \( F(z) \) in the tube set \( T_{\Gamma}^o \) with the Fourier series (6) such that \( F(x) = f(x) \). The function \( F(z) \) is holomorphic almost periodic by Bohr on any domain \( T_{\Gamma + b}^o, b \in \overline{\Gamma} \). Besides, for any cone \( \Gamma' \subset \subset \overline{\Gamma} \) we have uniformly w.r.t. \( z \in T_{\Gamma'}^o \)

\[
\lim_{\|y\| \to \infty} F(z) = a_0,
\]

(11)

where \( a_0 \) is the Fourier coefficient for the exponent \( \lambda = 0 \). If \( sp f \subset \overline{\Gamma} \), then (11) is true uniformly w.r.t. \( z \in T_{\Gamma + b}^o \) for any \( b \in \overline{\Gamma} \).

Proof. To prove the first part of the theorem, we need to replace \( \varphi_{q,l}(z) \) by

\[
\hat{\varphi}_{q,l}(z) = \log \left( \int_{[0,1]^m} |\sigma_q(z+u) - \sigma_l(z+u)|^p du \right)^{\frac{1}{p}}.
\]

Arguing as in the proof of theorem 1, we obtain that the Bochner-Feynman sums converge uniformly in the Stepanoff metric uniformly w.r.t. \( z \in T_{\Gamma}^o \) to an almost periodic function by Stepanoff \( F(z) \) with Fourier series (6).

Let \( b \in \overline{\Gamma} \). The module of the function \( \sigma_q(z) - \sigma_l(z) \) is estimated from above by the mean value on the corresponding ball contained in \( T_{\Gamma}^o \). Using the Hölder inequality, we have

\[
\sup_{x \in \mathbb{R}^m} |\sigma_q(x+bi) - \sigma_l(x+bi)| \leq C \sup_{z \in T_{\Gamma}^o} \left( \int_{[0,1]^m} |\sigma_q(z+u) - \sigma_l(z+u)|^p du \right)^{\frac{1}{p}},
\]

where the constant \( C \) depends only on \( b \) and \( \hat{\Gamma} \).

Applying lemmas 1 and 2 to the functions

\[
\hat{\psi}_{q,l,b}(y) = \sup_{x \in \mathbb{R}^m} \log |\sigma_q(z+bi) - \sigma_l(z+bi)|,
\]

we get that the Bochner-Fourier sums converge uniformly on \( T_{\Gamma + b}^o \) to \( F(z) \), thus \( F(z) \) is holomorphic almost periodic by Bohr in \( T_{\Gamma + b}^o \) for any \( b \in \overline{\Gamma} \).

Then the other statements of the theorem follow from theorem 1.

Now we prove the inverse statements to theorems 1 and 2.

Theorem 3. Suppose that an almost periodic by Bohr function \( f(x) \) continuously extends to the interior of \( T_{\Gamma}^o \) as a holomorphic function \( F(z) \). If \( F(z) \) is bounded on any set \( T_{\Gamma'}^o, \Gamma' \subset \subset \Gamma \), then \( F(z) \) is an almost periodic function by Bohr on \( T_{\Gamma}^o \) and the spectrum of \( F(z) \) is contained in \( \overline{\Gamma} \).

Proof. Take \( \lambda \notin \overline{\Gamma} \). Then there exists \( y_0 \in \overline{\Gamma} \) such that \( \langle y_0, \lambda \rangle < 0 \).

Choose a neighbourhood \( U \subset \overline{\Gamma} \) of \( y_0 \) such that \( \langle y, \lambda \rangle \leq \frac{1}{2}\langle y_0, \lambda \rangle \) for all \( y \in U \). Let \( A \) be any nondegenerate operator in \( \mathbb{R}^m \) such that \( A \) maps all the vectors \( e_1 = (1,0,\ldots,0), \ldots, e_m = (0,0,\ldots,1) \) into \( U \).

The function \( F(Ax) \) is holomorphic and bounded on the set

\[
\{ \zeta = \xi + i\eta \in \mathbb{C}^m : \xi \in \mathbb{R}^m, \eta^j > 0, j = 1,\ldots,m \}
\]

because \( A\{ \eta : \eta^j \geq 0 \} \subset \Gamma \).

If for each coordinates \( \zeta^1,\ldots,\zeta^m \) we change the integration over the segments \(-N \leq \zeta^j \leq N, \eta^j = 0 \) to the integration over the half-circles \( \zeta^j = N e^{i\theta^j}, 0 \leq \theta^j \leq \pi, j = 1,\ldots,m \) we obtain the equality

\[
\left( \frac{1}{2N} \right)^m \int_{[-N,N]^m} F(A\xi) e^{-i(A\xi,\lambda)} d\xi = \left( \frac{i}{2} \right)^m \int_{[0,\pi]^m} F(A Ne^{i\theta}) \prod_{j=1}^m e^{i\theta^j - i\lambda \langle Ac_j, \lambda \rangle} d\theta,
\]

(12)
where $\theta = (\theta^1, \ldots, \theta^m)$, $e^{i\theta} = (e^{i\theta^1}, \ldots, e^{i\theta^m})$.

Since $\langle A e_j, \lambda \rangle < 0$ for $j = 1, \ldots, m$, we see that the integrand in the right-hand side of (12) is uniformly bounded for all $N > 1$. By Lebesgue theorem (12) tends to zero as $N \to \infty$.

Thus
\[
\lim_{N \to \infty} \frac{1}{(2N)^m} \int_{A([-N,N]^m)} f(x) e^{-i(x,\lambda)} dx = 0.
\] (13)

Cover the set $A([-N^2,N^2]^m)$ by cubes $L_j = x_j^' + [-N,N]^m$ such that the interiors of these cubes are not intersected. We may assume that the number of the cubes intersecting the boundary of the set $A([-N^2,N^2]^m)$ is $O(N^{m-1})$ as $N \to \infty$. Taking into account boundedness of the function $f(x)e^{-i(x,\lambda)}$ on $R^m$ and equality (13), we have

\[
\frac{1}{(2N)^{2m}} \int_{\bigcup L_j} f(x) e^{-i(x,\lambda)} dx = \frac{1}{(2N)^{2m}} \left( \int_{A([-N^2,N^2]^m)} f(x) e^{-i(x,\lambda)} dx + O(N^{2m-1}) \right) = o(1) \quad (14)
\]

Since (5) we see that uniformly w.r.t. $x' \in R^m$ as $N \to \infty$

\[
\frac{1}{(2N)^m} \int_{x' + [-N,N]^m} f(x) e^{-i(x,\lambda)} dx = a(\lambda,f) + o(1). \quad (15)
\]

On the other hand, the number of the cubes $L_j$ equals $O(N^m)$ as $N \to \infty$, then the equality $a(\lambda,f) = 0$ follows from (14) and (15). This yields the inclusion $sp f \subset \widehat{\Gamma}$. Using theorem 1 we complete the proof of our theorem.

\[\blacksquare\]

**Theorem 4.** If $F(z)$ is bounded on each set $T_{\Gamma'}$, $\Gamma' \subset \subset \Gamma$, and the nontangential limit value of $F(z)$ as $y \to 0$ is an almost periodic function by Stepanoff on $C^m$, then $F(z)$ extends to $T_{\Gamma'}$ as an almost periodic function by Stepanoff and the spectrum of $F(z)$ is contained in $\widehat{\Gamma}$.

Proof. The proof of this theorem is the same as of theorem 3, but we have to use theorem 2 instead of theorem 1.

\[\blacksquare\]

To formulate further results we need the concept of $P$-indicator. (See, for example, [5], p. 275.)

**Definition.** $P$-indicator of an entire function $F(z)$ on $C^m$ is the function

\[
h_F(y) = \sup_{x \in R^m} \lim_{r \to \infty} \frac{1}{r} \log |F(x + iry)|.
\]

**Theorem 5.** (For $m = 1$ see [3], [4].) Let $f(x)$, $x \in R^m$ be an almost periodic function by Stepanoff with the Fourier series (7), and let $\|\lambda_n\| \leq C < \infty$ for all $n$. Then $F(x)$ extends to $C^m$ as an entire function $F(z)$ of exponential type, which is almost periodic by Bohr on any tube domain in $C^m$ with bounded base; $F(z)$ has the Dirichlet series (6), and $P$-indicator $h_F(y)$ satisfies the equation $h_F(y) = H_{sp f}(-y)$, where $H_{sp f}(\mu) := \sup_{x \in sp} \langle x, \mu \rangle$ is the support function of the set $sp f$.

Proof. Take $\mu \in R^m$ such that $\|\mu\| = 1$. Put

\[
f_\mu(x) = f(x)e^{-i[H_{sp f}(\mu) + \varepsilon]\langle x, \mu \rangle}.
\]

The Fourier series $\sum_{n=0}^{\infty} a_n e^{i(x,\lambda_n - (H_{sp f}(\mu) + \varepsilon)\langle x, \mu \rangle)}$ corresponds to the function $f_\mu(x)$, hence

\[
sp f_\mu \subset \{ x \in R^m : \langle x, \mu \rangle \leq -\varepsilon \}.
\]

Since $sp f_\mu$ is bounded, we obtain for some $\delta > 0$

\[
sp f_\mu(x) \subset \Gamma_{\delta,-\mu} = \{ \lambda \in R^m : \langle \lambda, -\mu \rangle \geq \delta \|\lambda\| \}.
\]
Theorem 2 yields that \( f_\mu(x) \) extends to the interior of the domain \( T_{\hat{\Gamma}_{\delta,-\mu}} \), where

\[
\hat{\Gamma}_{\delta,-\mu} = \{ y : \langle y, -\mu \rangle \geq \sqrt{1 - \delta^2} \| y \| \}
\]

is the conjugate cone to \( \Gamma_{\delta,-\mu} \), as an almost periodic function by Bohr \( F_\mu(z) \). This function is holomorphic on any domain \( T_{\hat{\Gamma}_{\delta,-\mu+b'}} \), with the Dirichlet series

\[
\sum_{n=0}^{\infty} a_n e^{i\langle z, \lambda_n - [H_{sp} f(\mu) + \varepsilon] \rangle},
\]

and \( F_\mu(z) \to 0 \) as \( \| y \| \to \infty \) uniformly w.r.t. \( z \in T_{\Gamma'} \) for any cone \( \Gamma' \subset \hat{\Gamma}_{\delta,-\mu} \). Using (5) we get

\[
|a_n| \leq \sup_{x \in \mathbb{R}^m} |F_\mu(x + iy)| e^{\langle y, \lambda_n \rangle}, \quad y \in \Gamma'.
\]

Put

\[
F(z) := F_\mu(z) e^{i[H_{sp} f(\mu) + \varepsilon] \langle z, \mu \rangle}.
\]

\( F(z) \) is almost periodic on \( T_{\Gamma'} \) with Dirichlet series (6). Therefore it follows from (16) that

\[
|F(z)| \leq C(\Gamma') e^{-[H_{sp} f(\mu) + \varepsilon] \| y \|}, \quad z \in T_{\Gamma'},
\]

(18)

Cover the space \( \mathbb{R}^m \) by the interiors of a finite number of cones \( \Gamma'_1, \ldots, \Gamma'_N \). There exist holomorphic on the interior of \( \Gamma'_k \) almost periodic functions \( f_k(z) \), \( k = 1, \ldots, N \) with identical Dirichlet series (6). Using the uniqueness theorem, we obtain that these functions coincide on the intersections of the cones and thus define a holomorphic function \( F(z) \) on \( \mathbb{C}^m \setminus \mathbb{R}^m \). The Bochner-Feyer sums for \( F(z) \) converge to this function uniformly on any set

\[
\{ z = x + iy : x \in \mathbb{R}^m, \| y \| = r > 0 \}.
\]

Hence, these sums converge on the tube domain \( T_{\| y \| < r} \). Thus \( F(z) \) extends to \( \mathbb{C}^m \) as the holomorphic function, which is almost periodic on any tube set with a bounded base. Owing to the uniqueness of expansion into Fourier series, we have \( F(x) = f(x) \).

Let us prove that \( h_F(y) = H_{sp} f(-y) \). From inequality (18) with \( \mu = -y \) it follows that

\[
h_F(y) \leq \lim_{r \to +\infty} \frac{1}{r} [H_{sp} f(-y) + \varepsilon] \langle ry, y \rangle = H_{sp} f(-y) + \varepsilon.
\]

The functions \( h_F(y) \) and \( H_{sp} f(y) \) are positively homogenous, hence the inequality

\[
h_F(y) \leq H_{sp} f(-y)
\]

is true for all \( y \in \mathbb{R}^m \).

Further, fix \( x, y \in \mathbb{R}^m \). The holomorphic on \( \mathbb{C} \) function \( \varphi(w) = F(x + wy) \) is bounded on the axis \( \operatorname{Im} w = 0 \). Then the estimate

\[
|\varphi(w)| \leq C e^{a \| \text{Im} w \|}
\]

for some \( a > 0 \) and all \( w \in \mathbb{C} \) follows from (18). Using the definition of \( P \)-indicator, we get

\[
\lim_{v \to +\infty} \frac{1}{v} \log |\varphi(iv)| \leq h_F(y).
\]

Therefore the function \( \varphi(w) e^{i(h_F(y)+\varepsilon)w} \) is bounded on the positive part of the imaginary axis. Applying the Fragmen-Lindelof principle to the quadrants \( \operatorname{Re} w \geq 0 \), \( \operatorname{Im} w \geq 0 \) and \( \operatorname{Re} w \leq 0 \), \( \operatorname{Im} w \geq 0 \), we get boundedness of this function on the upper half-plane. Applying the Fragmen-Lindelof principle to the half-plane \( \operatorname{Im} w \geq 0 \), we get the inequality

\[
|\varphi(w)| \leq \left( \sup_{\| \text{Im} w \| = 0} |\varphi(w)| \right) e^{h_F(y) \| \text{Im} w \|} \quad (\text{Im} w > 0).
\]
Hence, for all $z \in \mathbb{C}^m$, we have
\[ |F(z)| \leq \sup_{x \in \mathbb{R}^m} |F(x)|e^{h_F(y)}. \]

Now using formula (17) for coefficients of the Dirichlet series of the function $F(z)$, we get the estimate
\[ |a_n| \leq \sup_{x \in \mathbb{R}^m} |f(x)|e^{h_F(y)+(y,\lambda_n)}. \] (19)

Suppose $\langle y_0, \lambda_n \rangle + h_F(y_0) < 0$ for some $y_0 \in \mathbb{R}^m$. Put $y = ty_0$ in (19) and let $t \to \infty$. We obtain $a_n = 0$. This is impossible because $\lambda_n \in sp f$.

Thus for all $y \in \mathbb{R}^m$ and $\lambda_n \in sp f$ we have $h_F(y) + \langle y, \lambda_n \rangle \geq 0$, hence
\[ H_{spf}(-y) = \sup_{\lambda_n \in sp f} \langle -y, \lambda_n \rangle \leq h_F(y). \]

This completes the proof of the theorem.

The following theorem is inverse to the previous one.

**Theorem 6.** (For $m=1$ see [3], [4].) Let $F(z)$ be an entire function on $\mathbb{C}^m$, $|F(z)| \leq Ce^{b\|z\|}$, let $F(x)$, $x \in \mathbb{R}^m$ be an almost periodic function by Stepanoff with the Fourier series (7). Then $F(z)$ is an almost periodic function by Bohr on any tube domain $T_D \subset \mathbb{C}^m$ with the bounded base, $F(z)$ has the Dirichlet series (6), and $sp F \subset \{ \lambda : \|\lambda\| \leq b \}$.

**Proof.** It follows from theorem 5, that it suffices to prove the inclusion

\[ sp F \subset \{ \lambda : \|\lambda\| \leq b \}. \]

Let the function $F(x)$ be bounded on $\mathbb{R}^m$. Arguing as in theorem 5, we see that for all $z \in \mathbb{C}^m$
\[ |F(z)| \leq \sup_{x \in \mathbb{R}^m} |F(x)|e^{h_F(y)}, \]
where $h_F(y)$ is $P$-indicator for $F(z)$. Further, for all $x \in \mathbb{R}^m$ we have
\[ h_F(y) = \sup_{x \in \mathbb{R}^m} \frac{1}{\lim_{r \to \infty} r} \log |F(x + ir)| \leq \sup_{x \in \mathbb{R}^m} \frac{1}{\lim_{r \to \infty} r} (\log C + b\|x + ir\|) \leq b\|y\|, \]
therefore for all $z \in \mathbb{C}^m$
\[ |F(z)| \leq Ce^{b\|y\|}. \]

Take $\varepsilon > 0$, $\mu \in \mathbb{R}^m$, $\|\mu\| = 1$. Consider the function
\[ F_\mu(z) = F(z)e^{-i(z,\mu)(b+\varepsilon)}. \]

Since $|F_\mu(z)| \leq Ce^{b\|y\|+(b+\varepsilon)(y,\mu)}$ uniformly w.r.t. $x \in \mathbb{R}^m$, then $F_\mu(z)$ is uniformly bounded for $z \in T_{\Gamma_{-\mu}}$, where $\Gamma_{-\mu}$ is the cone $\{ y : \langle y, -\mu \rangle \geq (1 - \frac{\varepsilon}{\|\mu\|})\|y\| \}$. Using theorem 4, we obtain that the spectrum $F_\mu$ is contained in $\widehat{\Gamma}_{-\mu}$ and
\[ sp F = sp F_\mu + (b + \varepsilon)\mu \subset \widehat{\Gamma}_{-\mu} + (b + \varepsilon)\mu. \]

Finally, using the inclusion
\[ \bigcap_{\mu : \|\mu\| = 1} (\widehat{\Gamma}_{-\mu} + (b + \varepsilon)\mu) \subset \{ \lambda : \|\lambda\| \leq b + \varepsilon \} \]
and the arbitrariness of choice of $\varepsilon$ we get the assertion of the theorem in the case of bounded on $\mathbb{R}^m$ function $F(z)$.

Now let the function $F(z)$ be unbounded on $\mathbb{R}^m$. Put for some $N > 0$
\[ g(z) = \frac{1}{N^m} \int_{[0,N]^m} F(z + t)dt. \]

The function $g(z)$ satisfies the estimate on $\mathbb{C}^m$
\[ |g(z)| \leq Ce^{bN\|z\|}. \]
As in the case $m = 1$ (see [4]), we can prove that $g(x)$ is an almost periodic function by Bohr and is bounded on $\mathbb{R}^m$. The function $g(x)$ has the Fourier series

$$
\sum_{n=0}^{\infty} a_n \frac{e^{i\lambda_n^1 N} - 1}{N\lambda_n^1} \cdots \frac{e^{i\lambda_n^m N} - 1}{N\lambda_n^m} e^{i\langle x, \lambda_n \rangle},
$$

where $\lambda_n^j$ are coordinates of the vector $\lambda_n$ (if $\lambda_n^j = 0$, the corresponding multiplier should be replaced by 1).

Using countability of $spF$, we can choose $N$ in such a way that none of the numbers $\lambda_n^j N$ coincides with $2\pi k$, $k \in \mathbb{Z} \setminus \{0\}$. In this case $sp g = sp F$. Applying the proved above statement to the function $g(z)$, we obtain the inclusion

$$
sp F \subset \{ \lambda : \| \lambda \| \leq b \}.
$$

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