Dimensional reduction, truncations, constraints and the issue of consistency

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Abstract

A brief overview of dimensional reductions for diffeomorphism invariant theories is given. The distinction between the physical idea of compactification and the mathematical problem of a consistent truncation is discussed, and the typical ingredients of the latter – reduction of spacetime dimensions and the introduction of constraints – are examined. The consistency in the case of group manifold reductions, when the structure constants satisfy the unimodularity condition, is shown together with the associated reduction of the gauge group. The problem of consistent truncations on coset spaces is also discussed and we comment on examples of some remarkable consistent truncations that have been found in this context.

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1 Introduction. Some brief highlights

It seems that the first reference to dimensional reduction (see [1] and [2] as general references) appears in the work of Gunnar Nordström [3] who in 1914 formulated a vector-scalar theory in four dimensions—unifying electromagnetism and a scalar theory of gravitation—starting from Maxwell theory in a five-dimensional flat spacetime. Much more known is the work of Theodor Kaluza [4], published in 1921, who showed that gravity in five dimensions could yield a unified gravity plus Maxwell (plus a scalar which was at that time ignored). It was Oskar Klein [5] who came with the idea of compactifying the fifth dimension on a circle. The –Fourier– expansion in modes, today known as the Kaluza-Klein tower, allowed him to compute the radius of compactification by identifying the charge of the first massive mode with the electric charge. With a single stroke, the quantization of electric charge was given an explanation, and the size of the compact dimension, which turned to be of the order of the Plank length, explained why we only see see effectively four dimensions. The other side of the story is the very wrong result for a mass of such a a mode, which was of the order of the Plank mass$^2$.

A compactification on a 2-sphere, from six dimensions, was considered by Wolfgang Pauli in 1953 (see [7]), were the Yang-Mills field strenght made its appearance as a consequence of the non-Abelian reduction. It was Bryce DeWitt [8] in 1963 who showed in full generality—in fact it was left in his Les Houches lectures as an exercise—the unification of gravity and Yang Mills theories when dimensionally reducing from gravity in higher dimensions.

The consistency problems of dimensional reduction where first raised by Stephen Hawking [9] in 1969 in the context of the study of Bianchi cosmologies by dimensionally reducing the Lagrangian of pure 4-dimensional GR under a three dimensional Lie group. He found that when the trace of the structure constants had some non-vanishing component (Bianchi’s type B models), there was a mismatch between the reduction of the original equations of motion and the new equations of motion derived from the reduced Lagrangian. This fact had been already noticed a little earlier by Schücking, but never published [10]. In a more general framework, the tracelessness of the structure constants as a necessary condition for the consistency of dimensional reductions was formally pointed out by Scherk and Schwarz in 1979, [11], although it has since remained almost forgotten and its interpretation somewhat obscure (in words of the authors of [12], this condition is ”a subtlety which is not obvious from the analysis by Scherk and Schwarz”).

A different explanation of the tracelessness condition—also known as the unimodularity condition because it says that the adjoint representation is unimodular—, again as a necessary one, was given in the same year by MacCallum [13] in the context of the Bianchi models, by examining the boundary conditions required for the correct application of the variational principle for solutions exhibiting certain Killing symmetries.

$^2$See [6] for a nice account.
Let us point out the two different philosophies that underlie the issue of dimensional reduction. On one side there is the compactification approach, whose paramount example today is string theory, where one expects that dimensional reduction from ten dimensions—or eleven in M-theory—, will eventually deliver a four dimensional space-time together with an internal compact manifold—or a more general structure like an orbifold—that carries physical information within. In this approach the fundamental theory is the higher dimensional one, and the compactification procedure should eventually be understood as a physical process driven by some physical principles. The physical consequences that have their origin in the compactified structure and the process of compactification itself, are an essential part of the whole physical picture. Perhaps some of these effects, take for instance the presence of massive Kaluza-Klein modes, may become irrelevant when the theory is examined in the low energy regime, due to the small size of the compactified structure, but the effects are there anyway. This was in fact Klein’s elaboration [5] (expanding in Fourier modes) on the original Kaluza [4] idea, which corresponds to the second approach which we adress now.

On the other side one can conceive the dimensional reduction as a mathematical means to formulate a theory in a given space-time dimension having started with a higher dimensional theory as an auxiliary artifact, the extra dimensions never physically existing. In this case we speak of a consistent truncation form the higher dimensional theory to the lower one. One can start with a higher dimensional theory whose formulation is perhaps simpler and eventually end up with a more complicated theory at a lower dimension with the remarkable advantage of keeping full control of its symmetries (including supersymmetry when appropriate) or even with the right internal symmetries one was willing to implement. In Klein’s interpretation of Kaluza’s work, this means that a truncation is made on the tower of Fourier modes (or their generalizations to non-Abelian groups) to keep only the singlet ones. In this sense, this second approach can be understood as a method of model building, having used the higher dimensional theory as an intermediate device (to use the language of [14]), to be disposed of at the end, that helps to formulate a fundamental theory at a lower dimension. This has proven a useful method in supergravity: in this way, different dimensional reductions of the eleven dimensional supergravity (or the ten dimensional Type IIB) theory have yielded a fairly good recollection of supergravity theories with extended supersymmetry at lower dimensions.

In this contribution we will define what is a consistent truncation (Section 2) and classify it in two types. Next we will show (Section 3) the emergence of the tracelessness condition as a necessary and sufficient condition for such a consistency in the case of group manifolds. In Section 4 we consider the introduction of constraints to further reduce the degrees of freedom and a natural contact is made with the Dirac-Bergmann theory of constrained systems. Finally In Section 5 the difficulties associated to coset reductions are considered and some examples of consistent truncations are given.
2 Truncations

Consider a Lagrangian density $\mathcal{L}$ as the starting point, for a certain number of dimensions of the space-time. We can produce a truncation of it, yielding a reduced Lagrangian $\mathcal{L}_R$, by essentially two methods—or a mixture of both:

\textit{i)} First-type: by reducing the dimension of the space-time (Kaluza-Klein dimensional reduction) while keeping unchanged the number of degrees of freedom attached to every space-time point.

\textit{ii)} Second-type: by introducing constraints that reduce the number of independent fields—or field components—defining the theory.

In both cases we are producing a \textit{truncation} in the field content of the theory. At this point an issue of \textit{consistency} of such a truncation arises. Namely, whether the solutions of the equations of motion (e.o.m.) for the truncated theory—with Lagrangian $\mathcal{L}_R$—are still solutions of the e.o.m. for the original $\mathcal{L}$. This property is expressed graphically as the commutativity of the following diagram,

\[ \frac{\delta \mathcal{L}}{\delta \Phi} \xrightarrow{\text{Red.}} \frac{\delta \mathcal{L}_R}{\delta \Phi} = 0 \iff \frac{\delta \mathcal{L}_R}{\delta \Phi} = 0 \]

A proper definition is the following: A \textit{truncation is said to be consistent when its implementation at the level of the variational principle agrees with that at the level of the equations of motion}, i.e., if both operations commute: first truncate the Lagrangian and then obtain the equations of motion (e.o.m.), or first obtain the equations of motion and then truncate them.

This variational principle perspective of a consistent truncation has been studied in the mathematical literature under the name of \textit{Principle of Symmetric Criticality} in [15] and it has been applied to general relativity in [16] (see also [17]).

Another, weaker concept of consistent truncation, [18], just proceeds through the e.o.m.: one starts with a Lagrangian density and introduces some ansatz for the reduction of the fields, which is then plugged into the e.o.m.. If the original e.o.m. are compatible with such an ansatz, the reduced e.o.m. will be considered as a consistent truncation of the former ones. Unless said otherwise, we will use the terminology of consistent truncations in the strong, first sense.
To the process of truncation of theories, $\mathcal{L} \rightarrow \mathcal{L}_R$, going from top to bottom, there corresponds an opposite process of uplifting of solutions, from bottom to top. The basic result in this respect is that a consistent truncation guarantees that any solution of the $\mathcal{L}_R$ dynamics can be uplifted to a solution of the $\mathcal{L}$ dynamics.

3 First-type truncations

3.1 The unimodularity condition

Here we will answer the following question in the framework of diffeomorphism invariant theories: given the set of solutions of a theory –with Lagrangian density $\mathcal{L}$– that share the same algebra of Killing symmetries –which leave each of these solutions invariant–, is there a reduced variational principle describing exactly such a set? We will prove that the answer is in the positive [19, 20] at least in the case when the Killing vectors are all independent and can be written in a certain set of coordinates as $K_a = K_a^b(y) \partial_y^b$, where $y^a$ are the coordinates along the orbits (which disappear under reduction) and we have assumed that the components $K_a^b$ do not depend on the transversal coordinates $x^\mu$ (which survive the reduction). These Killing vectors are independent generators of left action of a group, forming a Lie algebra

$$[K_a, K_b] = C_{ab}^c K_c.$$ 

Associated with the left action of the group, there are left-invariant vectors, $Y_b = Y_b^c(y) \partial_y^c$, which generate a right action of the group:

$$\mathcal{L}_{K_a} Y_b = [K_a, Y_b] = 0, \quad [Y_a, Y_b] = -C_{ab}^c Y_c.$$ 

One can define the dual forms: $\omega^a = \omega_a^b(y) dy^b$, $\omega^a \cdot Y_b = \delta_a^b$, which satisfy

$$d\omega^a = \frac{1}{2} C_{bc}^a \omega^b \wedge \omega^c, \quad (\mathcal{L}_{K_a} \omega^b) = 0.$$ 

Let us use the basis of forms $\{dx^\mu, \omega^a\}$ to express our objects. The metric for instance, will be written as

$$g = g_{\mu\nu} dx^\mu dx^\nu + g_{ab} \left(A^a_\mu dx^\mu + \omega^a\right) \left(A^b_\nu dx^\nu + \omega^b\right),$$

which is just a way to express the degrees of freedom associated to the metric:

$$g_{\mu\nu}(x, y), \quad g_{ab}(x, y), \quad A^a_\mu(x, y).$$

Now apply the Killing conditions. The $y$-dependences will be eliminated,

$$\mathcal{L}_{K_a} g = 0 \Rightarrow g_{\mu\nu}(x), \quad g_{ab}(x), \quad A^a_\mu(x).$$
Notice that \( \det g = (\det g_{\mu\nu})(\det g_{ab})|\omega|^2 \). If we express the Lagrangian in the new basis

\[
\mathcal{L} = \mathcal{L}(\Phi, \Phi_\mu, \Phi_{\mu\nu}, Y_y(\Phi), Y_a Y_b(\Phi)) =: |\omega| \tilde{\mathcal{L}},
\]

where \( \Phi(x, y) \) is a generic component of a generic field. Then we can define the reduced Lagrangian

\[
\mathcal{L}_R(\Phi, \partial_\mu \Phi, \partial_{\mu\nu} \Phi) := \tilde{\mathcal{L}}(\Phi, \partial_\mu \Phi, \partial_{\mu\nu} \Phi, Y_a \Phi = 0, Y_a Y_b \Phi = 0),
\]

and the Euler-Lagrange derivatives become

\[
\frac{\delta \mathcal{L}}{\delta \Phi} = |\omega| \left( \partial_\mu \tilde{\mathcal{L}} - \partial_\mu \frac{\partial \tilde{\mathcal{L}}}{\partial \Phi_\mu} + \frac{1}{2} \partial_{\mu\nu} \frac{\partial \tilde{\mathcal{L}}}{\partial \Phi_{\mu\nu}} \right.
\]

\[
- (Y_a + C_{ac}^c)(\partial_\mu \tilde{\mathcal{L}} \frac{\partial \tilde{\mathcal{L}}}{\partial Y_a \Phi}) + \frac{1}{2} (Y_b + C_{bd}^d)(Y_a + C_{ac}^c)(\frac{\partial \tilde{\mathcal{L}}}{\partial Y_a Y_b \Phi})
\]

where crucial use has been made of the relation \( \partial_\mu (|\omega| Y_a^c) = C_{ab}^c |\omega| \).

If we now apply the Killing conditions on the fields, \( Y_a \Phi \to 0, \ Y_a Y_b \Phi \to 0 \), we end up with

\[
\left( \frac{\delta \mathcal{L}}{\delta \Phi} \right)_R = |\omega| \left\{ \frac{\delta \mathcal{L}_R}{\delta \Phi} - C_{ac}^c \left( \frac{\partial \tilde{\mathcal{L}}}{\partial Y_a \Phi} \right)_R + \frac{1}{2} C_{ac}^c C_{bd}^d \left( \frac{\partial \tilde{\mathcal{L}}}{\partial Y_a Y_b \Phi} \right)_R \right\}.
\]

Since in general the pieces of the type \( \left( \frac{\partial \mathcal{L}}{\partial Y_a \Phi} \right)_R \) will be different from zero, the last relation proves that

\[
\left( \frac{\delta \mathcal{L}}{\delta \Phi} \right)_R = |\omega| \left( \frac{\delta \mathcal{L}_R}{\delta \Phi} \right) \iff C_{ac}^c = 0, \forall a
\]

which is the unimodularity condition mentioned in the introduction. Abelian, semisimple, and compact groups are examples of groups fulfilling this condition.

### 3.2 Reduction of the diffeomorphisms algebra

Not all the elements of the gauge group will survive the reduction. The gauge group will be reduced to the elements that act internally on the subset of solutions that share the Killing symmetries under which the reduction is taking place. To show the general procedure it is enough to use a 1-form \( \Omega \) satisfying the Killing conditions, and write it in the basis \( \{ dx^\mu, \omega^a \} \),

\[
\Omega = \Omega_\mu(x) dx^\mu + \Omega_a(x) (A^a_\mu(x) dx^\mu + \omega^a).
\]

The active diffeomorphisms expressed in this basis produce the change

\[
\Omega \to \Omega' = \Omega'_\mu(x, y) dx^\mu + \Omega'_a(x, y) (A'^a_\mu(x, y) dx^\mu + \omega^a),
\]

and if we require that the new object \( \Omega' \) still satisfies the Killing conditions, that is,

\[
\Omega' = \Omega'_\mu(x) dx^\mu + \Omega'_a(x) (A'^a_\mu(x) dx^\mu + \omega^a),
\]

5
we will reduce the gauge group. In fact, not all diffeomorphisms will produce these exclusive \(x\)-dependences. Those that do are the diffeomorphisms that survive the reduction procedure, and define the reduced gauge group.

Let us pause a moment to notice a somewhat subtle point. We are expressing both the original objects \(-\Omega-\) and the transformed ones \(-\Omega'\) in the same basis \(\{dx^\mu, \omega^a\}\). This fact is crucial for the results that follow and it is just a choice of a basis. The confusion may come from the fact that the \(\omega^a\)'s are themselves forms that change under an active diffeomorphism –active in the sense that moves the objects but not the coordinates– and one may think of expressing \(\Omega'\) in terms of \(\{dx^\mu, \omega'^a\}\). It is a matter of choice to proceed in one or another way, but the convenient choice for the practice of dimensional reductions is to stay always with the unique basis \(\{dx^\mu, \omega^a\}\). The reason is that the definition of the reduced Lagrangian (1) has been given precisely in terms of this unique basis.

The diffeomorphisms belonging to this reduced gauge group turn out [20] to be of the form (for the infinitesimal generators),

\[
\vec{v} \rightarrow e^\mu(x) \partial_\mu + \eta^a(x) Y_a + \xi^a(y) Y_a
\]

where \(e^\mu(x) \partial_\mu\) generate diffeomorphisms in the reduced manifold; \(\eta^a(x) Y_a\) generate Yang-Mills transformations (and correspond to the inner automorphisms of the Lie algebra of Killing vectors); and finally \(\xi^a(y) Y_a\) generate residual rigid symmetries (corresponding to outer automorphisms).

The transformations generated by the reduced diffeomorphisms are

\[
\begin{align*}
\delta_{\text{Diff}} g_{\mu\nu} &= \mathcal{L}_\vec{v} g_{\mu\nu} \quad \text{(tensor)}, \\
\delta_{\text{Diff}} g_{ab} &= \mathcal{L}_\vec{v} g_{ab} \quad \text{(scalar)}, \\
\delta_{\text{Diff}} A^c_\mu &= \mathcal{L}_\vec{v} A^c_\mu \quad \text{(vector)}.
\end{align*}
\]

The Yang Mills gauge transformations are

\[
\begin{align*}
\delta_{\text{YM}} g_{\mu\nu} &= 0, \\
\delta_{\text{YM}} g_{ab} &= \eta^d (C^c_{da} g_{cb} + C^c_{db} g_{ac}), \\
\delta_{\text{YM}} A^a_\mu &= \partial_\mu \eta^a + A^c_\mu C^a_{cd} \eta^d, \\
\delta_{\text{YM}} \Omega_a &= \eta^d C^c_{da} \Omega_c, \\
\delta_{\text{YM}} \Omega_\mu &= 0.
\end{align*}
\]

And the residual rigid symmetries are

\[
\begin{align*}
\delta_{\text{Res}} g_{\mu\nu} &= 0, \\
\delta_{\text{Res}} g_{ab} &= -(B^c_a g_{cb} + B^c_b g_{ac}),
\end{align*}
\]
\[ \delta_{\text{Res}} A_\mu^a = B_b^a A_\mu^b , \]
\[ \delta_{\text{Res}} \Omega_a = -B_a^b \Omega_b , \]
with \[ C_a^b B_c^e - C_a^c B_b^e + C_{bc}^e B_a^e = 0 . \]

It is worth mentioning the appearance of these residual rigid symmetries, associated with the outer automorphisms of the Lie algebra. In the abelian case these rigid symmetries define the general linear group \( GL(n, R) \).

If our original manifold was \( d + n \)-dimensional and the quotient manifold –under the foliation produced by the \( n \)-dimensional algebra of Killing vectors– is \( d \)-dimensional, the structure of the reduced gauge group can be written as

\[ \text{Diff}(\mathcal{M}^{d+n}) \longrightarrow \left( \text{Diff}(\mathcal{M}^d) \otimes \text{Res.} \right) \wedge YM \]

### 4 Second-type truncations: Constraints

Introducing constraints is the second way to reduce the degrees of freedom. Suppose that a dimensional reduction has been performed, producing some scalar fields \( g_{ab} \) out of the higher dimensional metric. They transform under the adjoint action of the Yang-Mills gauge group,

\[ \delta_{YM} g_{ab} = \eta^{cd} (C_{da}^c g_{cb} + C_{db}^c g_{ac}) , \]

which means that they are in general charged objects under YM. We can get rid of the charged scalars by imposing the constraint

\[ C_{da}^c g_{cb} + C_{db}^c g_{ac} = 0 , \quad (3) \]

but this ad hoc imposition entails consequences that we must control, as we shall see now.

To be specific, consider that we have dimensionally reduced the Einstein-Hilbert action

\[ S^{d+n} = \frac{1}{2\kappa^2} \int d^d x \int d^m y \left| \hat{\gamma}_{\mu\nu} \right|^{1/2} \hat{R} , \]

from \( d + n \) to \( d \) dimensions, under a simple Lie algebra of \( n \) independent Killing vector fields. The –consistently truncated– reduced action becomes \([21],[11]\)

\[ S^d = \frac{1}{2\kappa^2} \int d^d x \left| -g_{\mu\nu} \right|^{1/2} |g_{ab}|^{1/2} \left\{ \hat{R} - \frac{1}{4} F_{\mu\nu}^a F_{\mu\nu}^b g_{ab} + \frac{1}{4} g^{\mu\nu} \mathcal{D}_\mu g_{ab} \mathcal{D}_\nu g^{ab} \right. \]
\[ + \left. g^{\mu\nu} \mathcal{D}_\mu \ln \sqrt{|g_{ab}|} \mathcal{D}_\nu \ln \sqrt{|g_{ab}|} - \frac{1}{4} C_{bc}^a \left[ 2C_{ac}^b g_{bc} \right] + C_{bc}^a g_{aa'} g_{bb'} g^{cc'} \right\} . \quad (4) \]

In this case the constraints (3) amount to the survival of a single, neutral scalar \( \varphi \), and the constraints (3) take the explicit form \( g_{ab} = \varphi h_{ab} \) with \( h_{ab} \) being the Cartan-Killing metric. But imposing this substitution \( g_{ab} = \varphi h_{ab} \) on the reduced lagrangian or on its reduced e.o.m. is not a commutative process in general –in the sense of the diagram introduced in Section 2. In fact there is a mismatch in the equation for the neutral scalar \( \varphi \) in the reduced theory and
the original equations for the scalars $g_{ab}$. To overcome the mismatch, some more constraints are needed. In this case, the new constraints are

$$\frac{1}{4} F^{\mu a} F^b_{\mu} - \frac{1}{4n} (F^{\mu c} F^d_{\mu} h_{cd}) h^{ab} = 0.$$  \tag{5}$$

The interpretation of these new constraints is straightforward within the framework of the Dirac-Bergmann theory of constrained systems [22, 23, 24, 25]. They are in fact the secondary constraints dynamically derived from the primary ones (3). At this point it is necessary that we distinguish, when considering second-type truncations, whether the constraints that are introduced preserve or break the remaining gauge symmetry. In the second case we speak of gauge-fixing constraints. Two theorems are worth considering in this respect. Suppose we start with a theory with Lagrangian $L$ that gets reduced, by way of the introduction of some holonomic constraints, which we call “primary”, to a reduced theory $L_R$.

**Theorem 1** *(for non gauge-fixing constraints)* The theory $L$ plus some added ”primary” holonomic constraints is equivalent to the reduced theory $L_R$ plus the ”secondary” constraints dynamically inherited from the ”primary” ones.

This is what happened in our example above. Indeed, a strong way to satisfy the secondary constraints (5) is to set the YM vector potentials to zero. One can then show that no tertiary constraints appear, which means the we have ended up with a consistent truncation. The price, though, is that of having lost the YM structure. The proof of Theorem 1 is given in [26].

**Theorem 2** *(for gauge-fixing constraints)* The theory $L$ plus some gauge-fixing constraints is equivalent to the reduced theory $L_R$ plus the secondary constraints inherited from those primary constraints whose associated gauge symmetries are killed by the gauge-fixing.

A good example of an application of this theorem is the Polyakov string in the conformal gauge, where the Virasoro constraints must be added by hand because the constraints introduced in the conformal gauge make them disappear in the reduced theory. A proof of this theorem is given in [27] for the case of total gauge-fixings and in Appendix C of [19] for partial ones.

## 5 Reduction on coset spaces

The first-type truncations considered so far do not include the very common case of dependent Killing vector fields, as happens in sphere reductions in general. The hard problem in this case is that there are no left invariant forms—under the group $G$—on the coset $G/H$.

Let us first set some notation. We assume that the Lie algebra can be decomposed $G = \mathcal{H} + \mathcal{K}$ with $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$ and $[\mathcal{H}, \mathcal{K}] \subset \mathcal{K}$. In coordinates such that $y^a$ label the $H$-orbits (coordinates for the coset) and $z^i$ are coordinates along these orbits, a generic element of $G$ is written $g(y^a, z^i) = L(y)h(z)$, where $h(z) \in H$ and $L(y)$ is a coset representative (see [28]).
For the group manifold $G$, the left invariant Lie algebra-valued Maurer-Cartan 1-form $g^{-1}dg$ is written as $g^{-1}dg = \omega^a X_a + \omega^i X_i$. The vectors $K_a, K_i$ are defined as the dual vectors to the right invariant 1-forms extracted from $dg g^{-1}$.

Now for the coset. The Lie algebra-valued 1-form $L^{-1}dL = \theta^a X_a + \theta^i X_i$ allows to isolate the 1-forms $\theta^a = \omega^a(y,0)dy^b$, which are the closer we can get to the situation with independent Killing vectors (group manifold) discussed in section 3. They may be used as a basis of 1-forms for the coset space. The problem now is that, instead of being invariant under the left action of $G$, the forms $\theta^a$ satisfy

$$\mathcal{L}_{K_c} \theta^a = K_c^i(y,0) \omega_i^j(y,0) C_{ac}^{eb} \theta^b \neq 0.$$ 

We can now proceed to write the metric in a way similar to what we did before,

$$g = g_{\mu\nu} dx^\mu dx^\nu + g_{ab} \left( A^a_\mu dx^\mu + \theta^a \right) \left( A^b_\nu dx^\nu + \theta^b \right),$$

but it will not be easy to ensure $\mathcal{L}_{K_c} g = 0$ because of the lack of invariance of the 1-forms $\theta^a$. Implementation of the constraints

$$C_{ia}^{cb} g_{db} + C_{ib}^{ac} g_{ac} = 0, \quad A^b_\nu = 0,$$

will indeed guarantee that $\mathcal{L}_{K_c} g = 0$. In this way one can get consistent truncations on coset spaces as in [29], but the challenge remains to get consistent truncations without eliminating the YM gauge bosons. To describe today’s state of the art, let us quote the authors of [30]: If one attempts a generalization of the reduction idea to a case where the internal manifold is a coset space, such as a sphere, then aside from exceptional cases it is not possible to perform a consistent reduction that retains a finite set of lower-dimensional fields including all the gauge bosons of the isometry group. In those exceptional cases where such a consistent reduction is possible, there is currently no clear understanding, for example from group theory, as to why the consistency is achieved.

All these difficulties notwithstanding, there are some impressive cases of consistent truncations in the literature. In fact they are truncations in the weaker sense mentioned in the introduction, but that does not make them less interesting. Let us mention the reduction of $d = 11$ supergravity to $d = 4$ gauged $N = 8$ supergravity on the coset space $S^7$ in the work of [31]; the reduction of $d = 11$ supergravity to $d = 7$ gauged $N = 2$ supergravity on the coset space $S^4$ in [32, 33]; the reduction of $d = 10$ type IIB supergravity to $d = 5$ gauged $N = 8$ supergravity on the coset space $S^5$ in [34]. Although not dealing with cosets, another interesting case worth mentioning is the reduction of $d = 10$ $N = 1$ supergravity to $d = 4$ gauged $N = 4$ supergravity on the group manifold $SU(2) \times U(1)^3$ in [35, 36]. The uplifting to $d = 10$ of the remarkable solution found by these authors was interpreted in [37], in the context of the gauge/gravity correspondence, as a background dual to $d = 4$, $N = 1$ super Yang-Mills in the infrared.
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