The twelve dimensional super (2 + 2)-brane

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Abstract

We discuss supersymmetry in twelve dimensions and present a covariant supersymmetric action for a brane with worldsheet signature (2, 2), called a super (2 + 2)-brane, propagating in the osp(64,12) superspace. This superspace is explicitly constructed, and is trivial in the sense that the spinorial part is a trivial bundle over spacetime, unlike the twisted superspace of usual Poincaré supersymmetry. For consistency, it is necessary to take a projection of the superspace. This is the same as the projection required for worldvolume supersymmetry. Upon compactification of this superspace, a torsion is naturally introduced and we produce the membrane and type IIB string actions in 11 and 10 dimensional Minkowski spacetimes. In addition, the compactification of the twelve dimensional supersymmetry algebra produces the correct algebras for these theories, including central charges. These considerations thus give the type IIB string and M-theory a single twelve dimensional origin.

1 Introduction

The are five apparently distinct known critical superstring theories in ten spacetime dimensions. They are

(i) type IIA,
(ii) type IIB,
(iii) $E_8 \otimes E_8$ heterotic,
(iv) Spin(32)/Z_2 heterotic,

(v) type I.

M-theory is a conjectured direct precursor of (i) and (iii), and reproduces (ii), (iv) and (v) upon compactification to nine dimensions \[1\]. More precisely, in ten dimensions the Spin(32)/Z_2 heterotic and type I theories are S-dual to each other, whereas the type IIB string is self-dual. In nine dimensions, the two heterotic string theories are equivalent under T-duality, as are the two type II strings. In addition, we find that M-theory compactified on the interval \(I\) and the circle \(S^1\) give the \(E_8 \otimes E_8\) heterotic and IIA string theories respectively. Thus it is M-theory which provides the final link between the five string theories. The mysterious M-theory, which is supposed to exist in a spacetime of eleven dimensions, ought somehow to be related to supergravity, and both two-branes and five-branes with \(N = 1\) worldvolume supersymmetry, [2]. Some time ago, it was realized that it was possible to construct type IIA ten-dimensional supergravity by the dimensional reduction of eleven dimensional theory. Subsequently Duff, Howe, Inami and Stelle, [3] showed that the theory coming from the double dimensional reduction of the classical two-brane from eleven to ten dimensions resulted in the classical representation of the type IIA superstring. It is natural therefore to ask if such a correspondence could be made quantum mechanical. In string theory, the string coupling constant \(\kappa\) is given by \(\kappa = e^{\langle \phi \rangle}\) where \(\phi\) is the dilaton field. The eleven dimensional interpretation of \(\kappa\) requires it to be identified with a component of the eleven dimensional metric. Suppose that \(g_{ab}\) is some eleven dimensional metric on spacetime \(\mathcal{M}\), and there is a Killing vector \(k = \partial/\partial x^{11}\) with a circle action on \(\mathcal{M}\). Identification of \(x^{11}\) with unit period then means that the radius of the Kaluza-Klein circle, \(R\), is determined by \(g_{11,11} = 4\pi^2 R^2\). Comparison of the two supergravity theories then yields

\[g_{11,11} = e^{4\langle \phi \rangle/3}.\]  

Hence, the string coupling is related to the radius of the Kaluza-Klein by

\[R \sim \kappa^{2/3}.\]  

Hence weak coupling string theory corresponds to \(R\) small, whereas infinite coupling corresponds to flat eleven dimensional spacetime.

M-theory is hypothetical the eleven dimensional master theory that provides a complete quantum mechanical version of this picture. Compactification of M-theory on a circle gives rise to type IIA string theory. Horava and Witten subsequently showed that compactification of M-theory on \(S^1/Z_2\) gives rise to the heterotic string with gauge group \(E_8 \otimes E_8\), [4]. A route to M-theory therefore is to explore the eleven dimensional theories which are known to exist.
Effectively two string theories are omitted from this direct unification. They are the heterotic string with $Spin(32)/Z_2$ gauge group, which is equivalent to the type $I$ string in ten dimensions, and the type IIB string. As has been suggested by Vafa, a natural viewpoint would be to describe the type IIB string, that has an $SL(2,Z)$ S-duality, as arising from some kind of theory, called F-theory, in twelve dimensions. Given a manifold $A$ which is a $T^2$ fibration over a manifold $B$, F-theory on $A$ is defined to be equivalent to the type IIB string on $B$. It is suggested that by allowing the RR dilaton and axion to vary on the internal manifold we can explain the $SL(2,Z)$ invariance of the IIB string. The implications of F-theory have been studied by several authors.

The idea of there being twelve dimensions is not a new one: Supersymmetric $p$-branes moving in dimensions higher than eleven have been discussed previously, most notably by Duff and Blencowe, but not in any great detail due to the problems with a straightforward generalisation of the Green-Schwarz strings and membranes (An abortive attempt to construct supersymmetric branes in higher dimensions was made by Blencowe, Duff, Hull and Stelle in 1988). A twelve dimensional connection with the supergravity theories for $D \leq 11$ has been suggested by Hull and the possibility of a full twelve dimensional supergravity theory has been studied previously. More recently we have seen the appearance of several other theories which make use of a twelfth dimension. It is now becoming apparent that twelve dimensions could well have a role to play in the formulation of string and membrane theories. In order to explore this idea more fully it seems necessary to try to find which fundamental extended objects can exist in twelve dimensional spaces. It is the issue of twelve dimensional fundamental $p$-branes which we address in this paper. The results will hopefully shed some light on the possible true nature of a twelfth dimension.

2 Supersymmetric Branes

We now consider the formulation of supersymmetric theories in spacetimes of various dimension and signature. The goal is to discover consistent supersymmetric embeddings of $p$-branes into twelve dimensional spacetimes. In this paper we will define a $(p+q)$-brane to be a surface of signature $(s,t) = (p,q)$. A $p$-brane will be a surface of signature $(s,t)$ with $s + t - 1 = p$. We shall use the word ‘spacetime’ to refer to any background manifold in which the $p$-brane exists. Since a supersymmetric theory requires the inclusion of fermions, we begin with a general discussion of spinors.

2.1 Spinors

We start from the Clifford algebra in $D$ dimensions.
\( \{ \Gamma_\mu, \Gamma_\nu \}^\alpha_\beta = 2\eta_{\mu\nu} \Gamma^\alpha_\beta \) \hspace{1cm} (2.1)

where \( \eta = \text{diag}(-,-,\ldots,-,+,+,\ldots,+) \) is the metric on the tangent space of dimension \( D = S + T \) and signature \( S - T \). Thus there are \( T \) timelike directions, each corresponding to a minus sign in the metric, and \( S \) spacelike dimensions. We restrict ourselves to the cases \( S \geq T \). All the results we obtain are representation independent. We may always choose a representation of the \( \Gamma \) matrices such that \( \Gamma^\mu \) is hermitian (anti-hermitian) for \( \mu \) spacelike (timelike) respectively. The indices \( \mu \) and \( \nu \) run from \( 1 \ldots D \) and \( \alpha \) and \( \beta \) are spinorial indices.

The spin space has \( 2^{\text{int}(D/2)} \) dimensions, where ‘int’ denotes the integer part. If \( S - T \) is even, there are two possible inequivalent choices of charge conjugation matrix, whereas if \( S - T \) is odd there is a unique charge conjugation matrix. In either case, the charge conjugation matrices \( C_\pm \) must satisfy

\[ \tilde{\Gamma}_\mu = \pm C_\pm \Gamma_\mu C_\pm^{-1}, \] \hspace{1cm} (2.2)

where the tilde denotes transpose. If we define \( A = \Gamma^1 \ldots \Gamma^T \), where \( \Gamma^\mu \) is timelike for \( 1 \leq \mu \leq T \), then we have the relationship

\[ \Gamma^{\mu\dagger} = (-1)^T A \Gamma^\mu A^{-1}, \] \hspace{1cm} (2.3)

and so we may define the Dirac conjugate as

\[ \bar{\psi}_D = \psi^\dagger A. \] \hspace{1cm} (2.4)

This is chosen so that \( \bar{\psi}_D \psi \) transforms as a scalar under \( SO(S,T) \) Lorentz transformations. The Majorana conjugate is defined as

\[ \bar{\psi}_M = \bar{\psi}C, \] \hspace{1cm} (2.5)

and a Majorana spinor is one for which \( \bar{\psi}_D = \bar{\psi}_M \), corresponding to a real section of the spin bundle. Clearly, both these two spinors must satisfy the same Dirac equation, which leads to consistency conditions on the \( C_\pm \), namely that Majorana spinors exist iff there exists a \( C_+ \) or \( C_- \) such that

\[ \tilde{C}_+ = (-)^{T + \text{int}((T+1)/2)} C_+ \]

\[ \tilde{C}_- = (-)^{\text{int}((T+1)/2)} C_. \] \hspace{1cm} (2.6)

Since \( \pm \Gamma^\mu \) form equivalent representations of the Clifford algebra, we may make a transformation from one representation to the other by

\[ \Gamma^\mu = \eta B^{-1} \Gamma^\mu B, \quad \eta = \pm 1 \]

\[ \Rightarrow B^* B = \epsilon, \quad \epsilon = \pm 1, \] \hspace{1cm} (2.7)
so that we can write the charge conjugation matrix as $C = \tilde{B}A$. The properties of the matrices $A$ and $B$ then imply that

$$\tilde{\Gamma}_\mu = (-1)^T \eta C \Gamma_\mu C^{-1}$$

$$C^\dagger C = 1$$

$$\tilde{C} = \eta^T (-1)^{(T+1)/2} C.$$  \hfill (2.8)

The possible choices of the numbers $\eta, \epsilon = \pm 1$ depend on the signature of the spacetime as follows

| $\epsilon$ | $\eta$ | $(S - T) \mod 8$ |
|-----------|--------|-----------------|
| $+1$      | $+1$   | 0, 1, 2         |
| $+1$      | $-1$   | 0, 6, 7         |
| $-1$      | $+1$   | 4, 5, 6         |
| $-1$      | $-1$   | 2, 3, 4         |  \hfill (2.9)

We see that both $C_+$ and $C_-$ can be defined if $S - T$ is even, otherwise only one of them exists. The charge conjugation matrix and its inverse are then used to lower and raise spinor indices respectively, so that $\psi^\alpha = \psi_\beta (C^{-1})^{\beta\alpha}$ and $\psi_\alpha = \psi^\beta C_{\beta\alpha}$.

The set of matrices $\Gamma^{(i)\alpha\beta} = \Gamma_{[\mu_1...\mu_i]} C^{-1})^{\alpha\beta}$, where $i$ runs over $1...D$, and the square brackets denote antisymmetrisation, form a basis for the space of $2^{D/2} \times 2^{D/2}$ matrices. The $(\Gamma^{(i)})^{\alpha\beta}$ are each either symmetric or antisymmetric in their spinor indices. By using the relations (2.8) we find that the parity, $\pi$, of these matrices is given by

$$\pi = \eta^T (-1)^{(T+1)/2} \left((-1)^T \eta\right)^i (-1)^{i(i-1)/2}. \hfill (2.10)$$

From the table (2.9) we can find a basis of matrices of given symmetry for a given definition of the $\Gamma$ matrices.

2.2 $N = 1$ superalgebras

Superalgebras play an important role in the formation of $p$-brane theories. A natural supposition is that it should be possible to formulate the theory in flat spacetime in such a way that spacetime supersymmetry is manifest. This is the rationale for the Green-Schwarz approach to the theory of extended objects.

Flat spacetime has the Poincaré group $\text{IO}(S,T)$ as its isometry group. Locally this may be described by the exponential of the Lie algebra generated by the Lorentz rotations $M_{\mu\nu}$ and the translations $P_\mu$

$$[M_{\mu\nu}, M_{\rho\sigma}] = M_{\nu\rho} \eta_{\mu\sigma} + M_{\mu\rho} \eta_{\nu\sigma} - M_{\nu\rho} \eta_{\sigma\mu} - M_{\sigma\mu} \eta_{\nu\rho}$$
\[
[M_{\mu\nu}, P_\rho] = P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho}
\]
\[
[P_\mu, P_\nu] = 0.
\] (2.11)

To implement a simple, \( N = 1 \), spacetime supersymmetry we consider graded algebras generated by \( \{M, P, Q\} \), where \( Q \) is a spinorial Grassman odd generator. Usually the term ‘supersymmetry algebra’ refers to the following extension of the bosonic algebra
\[
[M_{\mu\nu}, Q^\alpha] = -\frac{1}{2} (\Gamma_{\mu\nu})^\alpha_\beta Q^\beta
\]
\[
[P_\mu, Q] = 0
\]
\[
\{Q^\alpha, Q^\beta\} = (\Gamma_\mu)^{\alpha\beta} P^\mu.
\] (2.12)

There are, however, many possible spinorial extensions of the Poincaré algebra, and each will generate some supergroup which may be used to define a spacetime supersymmetry. However, only some of these will correspond to on-shell supersymmetric theories. Nahm [17] investigated supersymmetry algebras satisfying the on-shell condition and restrictions on the spins of the states in spacetimes with Minkowski signature, leading to constraints on possible background spacetimes. We shall initially consider general supersymmetric theories without concerning ourselves with degrees of freedom or signature. The restrictions which arise come purely from the self consistency of the superalgebras, by which we mean that the super-Jacobi identities must be satisfied
\[
[A, [B, C]] = [[A, B], C] - (-)^{bc} [[A, C], B],
\] (2.13)

with
\[
(-)^{bc} = -1 \text{ if } B \text{ and } C \text{ are fermionic,}
\]
\[
= +1 \text{ otherwise.}
\] (2.14)

In addition to this constraint, we shall only require that the graded algebra reduce to the bosonic Poincaré algebra, (2.11), when we set \( Q = 0 \). To discover the possible extensions, we first write down the transformation law of a spinor under a Lorentz transformation. This gives the first expression in (2.12).

We now note that the superalgebra must contain a \( \{Q, Q\} \) term. We need to determine the form of this term. Since the anticommutator must be symmetric in its spinor indices, we expand in terms of a basis for symmetric matrices, which may be found from the expression
\[
\{Q^\alpha, Q^\beta\} = \sum_k \frac{1}{k!} (\Gamma_{\mu_1..\mu_k})^{\alpha\beta} (Z^{(k)})^{\mu_1..\mu_k},
\] (2.15)
where $k$ runs over the values for which $\Gamma^{(k)}$ is symmetric in its spinor indices. This is the approach discussed in [18, 12]. We must then check that all possible super-Jacobi identities hold. This places restrictions on the $Z^{(i)}$ terms. Note that the terms $Z^{(1)}, Z^{(2)}$ have the same degrees of freedom as the generators $P, M$ respectively. Thus, if the term $\Gamma^{(1)} C^{-1}$ is antisymmetric in the spinor indices then the anticommutator of the $Q$ with itself cannot generate momentum, and the theory cannot exhibit spacetime supersymmetry in the usual sense of (2.12). The symmetry of the matrix $\Gamma^{(1)} C^{-1}$ depends on the possible choice of the charge conjugation matrix $C$. In twelve dimensions we may generally choose the charge conjugation matrix to be either $C_+$ or $C_-$. For $C_+$ the symmetric matrices are given by those $\Gamma^{(i)} C^{-1}$ for which $i = 2, 3 \mod 4$. For $C_-$, they occur for the cases $i = 1, 2 \mod 4$. This result is independent of the signature of the twelve dimensional space. Thus we find that in twelve dimensions we may formulate a standard theory of supersymmetry for the choice $C_-$ but not for the choice $C_+$: only for the $C_-$ case may we extend the Poincaré algebra by the terms (2.12) if we identify $Z_\mu$ with $P_\mu$. For a Dirac spinor this result is independent of $S - T$. Imposing the Majorana condition on the spinors leads to restrictions on the possible signature, which are shown in the table for the case $S + T = 12$

| $S$ | $T$ | $(\epsilon, \eta)$ for $C_+$ | $(\epsilon, \eta)$ for $C_-$ |
|-----|-----|-----------------|-----------------|
| 12  | 0   | $-$             | $-$             |
| 11  | 1   | $-$             | $(1, 1)$        |
| 10  | 2   | $(1, 1)$        | $(1, -1)$       |
| 9   | 3   | $(1, -1)$       | $-$             |
| 8   | 4   | $-$             | $-$             |
| 7   | 5   | $-$             | $(1, 1)$        |
| 6   | 6   | $(1, 1)$        | $(1, -1)$       |

The general extension of the Poincaré algebra involves the addition of $Z^{(k)}$ terms other than $Z^{(1)} \sim P$. These $p$-form charges [4] are in some respects similar to central charges which occur only for $N \geq 2$ supersymmetric theories, but in contrast arise from the commutator of the spinor generator with itself. The question may be asked as to which terms, if any, arise naturally. To answer this, we investigate the $p$-form charges in the context of gradings of the de Sitter algebra, given by

$$ [M_{\mu\nu}, M_{\rho\sigma}] = M_{\nu\sigma} \eta_{\mu\rho} + M_{\mu\rho} \eta_{\nu\sigma} - M_{\nu\rho} \eta_{\sigma\mu} - M_{\sigma\mu} \eta_{\nu\rho} $$

$$ [M_{\mu\nu}, P_\rho] = P_\mu \eta_{\nu\rho} - P_\nu \eta_{\mu\rho} $$

$$ [P_\mu, P_\nu] = m M_{\mu\nu} $$

where $m^{-1}$ is the radius of the de Sitter space. The de Sitter algebra is of interest because it reduces to the Poincaré algebra in the limit $m \to 0$. If $m \neq 0$ then we find that the
extra structure leads to restrictions on the possible antisymmetric tensors. Since branes are microscopic objects they are insensitive to large scale structure. It is therefore natural to preserve these restrictions even in the \(m = 0\) limit.

The general grading of the de Sitter algebra includes an anticommutator of the form

\[
\{Q^\alpha, Q^\beta\} = \sum_{\text{symmetric}} \frac{1}{k!} (\Gamma_{\mu_1...\mu_k})^{\alpha\beta} Z^{\mu_1...\mu_k}
\]

(2.18)

We may choose any set of the \(Z\) fields provided that they are consistent with the superalgebra. The restrictions which arise from the super-Jacobi identities are given in [18] for the case of \(C_-\), although a similar analysis may also be done for the case \(C_+\). We find that there are only two consistent super-de Sitter algebras, for each \(C\), without setting to zero some of the possible \(Z^{(i)}\) in the \(\{Q, Q\}\) anticommutator. We now impose the additional constraint that in the infinite radius limit we reproduce the Poincaré algebra structure. Since \(P\) and \(M\) have the same degrees of freedom as \(Z^{(1)}\) and \(Z^{(2)}\) respectively we should identify them in the limit. Any other \(Z^{(i)}\) which remain in the \(m = 0\) limit will be taken to be \(p\)-form charges. Consistency of the limit of the de Sitter algebra with the Poincaré case rules out one of the aforementioned solutions, leaving us with a single possible maximal algebra for each \(C\). For the \(C_-\) case, the algebra is generated by the set \(\{Q, P, M, Z^{(5)}, Z^{(6)}, Z^{(9)}, Z^{(10)}\}\). The only subalgebra of this algebra is simply \(\{Q, P, M\}\). Thus, for the algebra which generates momentum, no \(Z^{(i)}\) terms, for \(i > 2\), are singled out from the full set of \(p\)-forms. For the \(C_+\) case we find the full algebra is generated by the set of generators \(\{Q, M, Z^{(3)}, Z^{(6)}, Z^{(7)}, Z^{(10)}, Z^{(11)}\}\). In this case, the algebra contains only the subalgebra generated by \(\{Q, M, Z^{(6)}, Z^{(10)}\}\). We may suppose that we make the identification \(M_{\mu\nu} = \frac{1}{10!} \epsilon^{\rho_1...\rho_{10}\mu\nu} Z_{\rho_1...\rho_{10}}\), where \(\epsilon\) is the alternating tensor. In the infinite radius limit for this truncation, after rescaling the \(Z^{(2)}\), we obtain the additional non-zero commutation relations

\[
[M_{\sigma\rho}, Z_{\mu_1\mu_2\mu_3\mu_4\mu_5\mu_6}] = \eta_{[\mu_1[p} Z_{\sigma]\mu_2\mu_3\mu_4\mu_5\mu_6]} - \eta_{[\mu_1[a} Z_{\rho]\mu_2\mu_3\mu_4\mu_5\mu_6]} \]

\[
\{Q^\alpha, Q^\beta\} = \frac{1}{2} (\Gamma_{\mu\nu})^{\alpha\beta} M^{\mu\nu} + \frac{1}{6!} (\Gamma_{\mu_1...\mu_6})^{\alpha\beta} Z^{\mu_1...\mu_6}.
\]

(2.19)

It is noteworthy that, after taking the limit, it is possible to include any number of the \(Z^{(i)}\) into the algebra in a similar way to the antisymmetric six index object in (2.19), as types of \(p\)-form charges. The above considerations for de Sitter space show that only the above cases of sets of generators which form subalgebras arise naturally if \(P\) is excluded from the algebra. The only other natural truncation is simply the Poincaré algebra which, of course, generates momentum. It should be stressed that we can only expect these algebras to be completely consistent for an on-shell theory. Off shell we inevitably find anomalies in the super-Jacobi

\[1\text{Details of this calculation are given in appendix A}\]
identities. For the algebra including (2.19) we find that all the super-Jacobi identities are indeed satisfied on-shell.

We now address the issue of the degrees of freedom of the spinorial part of the algebras. As a symmetric matrix in $2^{[D/2]}$ dimensions, \( \{Q, Q\} \) has $\frac{1}{6} \times 64.65 = 2080$ components, since the Majorana spinor has 64 real components. The two maximal sets of generators \( \{Z^{(k)}\} \), for $k = \{1, 2, 5, 6, 9, 10\}$ or $\{2, 3, 6, 7, 10, 11\}$, both also have 2080 components, since each antisymmetric tensor field has $\frac{12!}{(12-k)!k!}$ components. Thus a matching will occur for the case of the maximal algebra. For any truncation of one of the algebras this matching will not occur. However, for the natural non-Poincaré subalgebra which arises, (2.19), generated by \( \{Q, M, Z^{(6)}\} \), we may obtain saturation if we project out half of the spinor degrees of freedom. We define a pair of projection operators \( P^\pm = \frac{1}{2}(1 \pm X) \) where \( X^2 = 1 \), with \( X \) chosen such that \( X^2 = 1 \) and so that \( P^\pm \) are rank 32. Taking the projection of (2.19) and the first two terms in (2.12), and setting \( Q^- = 0 \), we obtain

\[
\begin{align*}
[M_{\mu\nu}, Q^\alpha_+] &= -\frac{1}{2}(\Gamma^+_{\mu\nu})^{\alpha\beta} Q^\beta_+ \\
[P_\mu, Q^\alpha_+] &= 0 \\
\{Q^\alpha_+, Q^\beta_+\} &= \frac{1}{2}(\Gamma^+_{\mu\nu})^{\alpha\beta} M^{\mu\nu} + \frac{1}{6!}(\Gamma^+_{\mu_1...\mu_6})^{\alpha\beta} Z^{+\mu_1...\mu_6},
\end{align*}
\]

where \( (\Gamma^+_{\mu_1...\mu_6})^{\alpha\beta} = P^\alpha_+ P^\beta_- (\Gamma_{\mu_1...\mu_6})^{\gamma\delta} \). The remaining part of the \( Z^{(6)} \) term, \( Z^{+6} \), is now self-dual with respect to the projection operator, in that the complement of \( Z^+ \) in \( Z \) vanishes identically so that \( Z^{+\mu_1...\mu_6} \Gamma^-_{\mu_1...\mu_6} = 0 \). The degrees of freedom now match up: the anticommutator has $32 \times 33 = 528$ degrees of freedom; the self-dual six form and the two form have $\frac{12!}{2 \times 2!} = 462$ and $\frac{12!}{2 \times 1!} = 66$ degrees of freedom respectively. Other than the maximal extension of the Poincaré algebra for each choice of \( C \), this is the only algebra in which the matching of degrees of freedom in the anticommutator term occurs. For completeness, we present an on-shell representation of this algebra for which we treat the \( Z^{(6)} \) term as central

\[
\begin{align*}
P_\mu &= \frac{\partial}{\partial X^\mu}, \\
Q_\alpha &= \frac{\partial}{\partial \theta^\alpha}, \\
M_{\mu\nu} &= X_\mu \frac{\partial}{\partial X^\nu} - X_\nu \frac{\partial}{\partial X^\mu} + \frac{1}{2} \theta_\alpha (\Gamma^{\mu\nu})^{\alpha\beta} \frac{\partial}{\partial \theta^\beta}.
\end{align*}
\]

Note that if we define \( X = i^T \Gamma^{13} \) in the expression for the projectors \( P^\pm \), where \( \Gamma^{13} = \Gamma_1 \ldots \Gamma_{12} \), then \( P \) is the Weyl projector, and self-dual means self-dual with respect to the Hodge star operator. Other choices of projector are perfectly possible, although they will not be Lorentz invariant. This does not imply that the underlying theory containing such spinors is not Lorentz invariant, however, which is an important point to note. For example, consider
a twelve dimensional theory in which arbitrary spinors \(X,Y\) are constrained to satisfy the identity

\[ X_\alpha (\Gamma_{\mu\nu})^{\alpha\beta} Y_\beta = 0 \quad \forall \mu, \nu = 1 \ldots 12. \tag{2.22} \]

This is a completely Lorentz invariant constraint, and hence describes a Lorentz invariant theory. If we wish to employ an explicit representation of the spinors obtained by a projection of Majorana spinors in \(D\) dimensions, then the projector used cannot be Lorentz invariant if \(T = 2\), although it can if \(T = 1\). In some signatures this behaviour may create the naive impression that the underlying theory described by (2.22) is not Lorentz invariant.

We have seen in this section that there are many possible graded extensions of the Poincaré algebra, and so the term ‘super-Poincaré algebra’ is, therefore, sometimes ambiguous. We shall extend the idea of a super-Poincaré algebra to \( \text{so}(S,T)_{\{k\}} \), the algebra defined by (2.11), the first two terms in (2.12) and

\[
\begin{align*}
\{Q,Q\} &= \sum_{k=i_1 \ldots i_n} \frac{1}{k!} \Gamma^{(k)} Z^{(k)} \\
\left[Z_{\mu_1 \ldots \mu_i}, M_{\rho\sigma}\right] &= \eta_{\mu_1\rho} Z_{\sigma|\mu_2 \ldots \mu_i} - \eta_{\mu_1\sigma} Z_{\rho|\mu_2 \ldots \mu_i} \forall i \\
\left[Z^{(i)}, Z^{(i)}\right] &= 0, \quad i, j \neq 2 \\
Z^{(1)} &\equiv P \\
Z^{(2)} &\equiv M. \tag{2.23}
\end{align*}
\]

Each of these superalgebras may correspond to a number of supergroups. Since the bodies of these algebras are themselves Lie algebras, the relationship between the superalgebras and the supergroups seems to be well understood \[20\] and the theory progresses in close analogy with that for bosonic Lie algebras and Lie groups. Our results will only depend local properties and it will therefore be consistent to discuss the corresponding \((N = 1)\) supergroup \( \text{SIO}(S,T)_{\{k\}} \) in this work. We can think of the supergroup as acting on some superspace. A superspace is a \(Z_2\)-graded vector space consisting of a Grassmann even (bosonic) subspace and a Grassmann odd (fermionic) subspace. In a local region of superspace, the supergroup elements can be found via exponentiation of the superalgebra. Superspace is a generalisation of the usual notion of spacetime.

### 2.3 Super \( p \)-branes in an \( \text{so}(S,T)_1 \) invariant background

We now describe the formulation of a \( p + 1 \) dimensional extended object, or \( p \)-brane, moving in a flat 12-dimensional spacetime background which is invariant under the action of a super-Poincaré group locally described by the algebra \( \text{so}(S,T)_1 \). We suppose that there are \( s \)-spacelike directions and \( t \)-timelike directions on the brane so that \( s + t - p = 1 \). In order to allow a consistent embedding of the brane into spacetime we make the restrictions \( s \leq S \) and...
t \leq T$. It is natural to formulate such a theory on the superspace associated with $\text{ISO}(S, T)_1$, which is described by the local coordinates

$$Z^M = (X^\mu, \theta^\alpha),$$

(2.24)

where the $X$ are the bosonic spacetime coordinates and the $\theta$ are anticommuting Majorana spinors. The action of a supergroup element infinitesimally close to the identity gives the following change in the coordinates

$$\delta \theta^\alpha = \psi^\alpha \quad \delta X^\mu = a \bar{\psi} \Gamma^\mu \theta$$

(2.25)

where $\psi$ is a constant Majorana spinor. The parameter $a$ is chosen so the variation in the bosonic coordinates is real. This requires that

$$a^* = \epsilon \eta (-1)^T a,$$

(2.26)

where the values $\eta$ and $\epsilon$ are given in (2.9). From these relationships we can define forms invariant under the action of $\text{SIO}(S, T)_1$

$$\Pi^\mu = dX^\mu - a \bar{\theta} \Gamma^\mu d\theta \quad \Pi^\alpha = d\theta^\alpha.$$  

(2.27)

We can then construct the action for the $p$-brane by considering the pullback of these invariant forms to the worldvolume $[21]$; $dX \rightarrow \partial_i X d\xi^i$, where $\xi$ are the coordinates on the $p$-brane. The analogue of the Nambu-Goto action action obtained from the principle of least action is

$$S_0 = \int d^{p+1} \xi \left[ \det(\Pi_i^\mu \Pi^\nu_j \eta_{\mu\nu}) \right]^{\frac{1}{2}}$$

$$\Pi^\mu_i = \partial_i X^\mu - a \bar{\theta} \Gamma^\mu \partial_i \theta,$$

(2.28)

which may be re-written in first order form, using the Howe and Tucker technique $[22]$, as

$$S_0 = \int d^{p+1} \xi \sqrt{|g|} \left[ \frac{1}{2} g^{ij} \Pi^\mu_i \Pi^\nu_j \eta_{\mu\nu} - \frac{1}{2} (p - 1) \right],$$

(2.29)

where $g^{ij}$ is an auxiliary metric on the $p$-brane. These actions are manifestly spacetime supersymmetric. We also require that there be supersymmetry on the brane. This necessitates a matching between the worldvolume bosonic and fermionic degrees of freedom. Can such a matching occur on a brane in twelve dimensions?

In theories of extended objects in non-Euclidean spacetimes, negative norm states exist due to propagation in timelike directions. To avoid this problem we may fix a gauge using diffeomorphism invariance of the brane to allow only propagation in directions transverse to the worldvolume. This gives us positive norm states if the residual symmetry group $\text{SO}(S - s, T - t)$ is compact, which requires that $T = t$. The gauge fixing leaves $D - p - 1$
transverse coordinates, which correspond to the bosonic degrees of freedom. For a three-brane in twelve dimensions we have eight transverse $X^\mu$. This is the important case which parallels the string and the membrane in ten and eleven dimensions respectively.

We should now count the spinorial degrees of freedom. A Dirac spinor in twelve dimensions has $2^{12/2} = 64$ complex components, and a Majorana spinor has 64 real components. The Majorana spinor has 32 real on-shell degrees of freedom, and this is halved to 16 components by taking the Weyl projection, or any other rank 32 projection. In order to obtain a matching for the 8 bosonic coordinates we need to find an extra fermionic symmetry. To find such a symmetry we must introduce an additional term in the action, called the Wess-Zumino term

$$S_{WZ} = - \int d^{p+1}\xi \left( \frac{2}{(p+1)!} \epsilon^{i_1 \ldots i_{p+1}} B_{i_1 \ldots i_{p+1}} \right)$$

(2.30)

where $B_{i_1 \ldots i_{p+1}}$ are the components of a $p+1$ form $B$ which is the potential for the $p+2$ superspace form $H = dB$. This term is of the correct form since it is spacetime supersymmetric and transforms in the same way as the action $S_0$ (2.28) under a scaling of the superspace coordinates. The full action $S_0 + S_{WZ}$ is then invariant [23, 7], for any choice of signature, under the so called $\kappa$-symmetry transformation

$$\delta \theta = \frac{1}{2}(1 + \Gamma) \kappa, \quad \delta X^\mu = \frac{a}{2} \bar{\theta} \Gamma^\mu (1 + \Gamma) \kappa,$$

(2.31)

where $\kappa$ is a scalar on the brane and a fermion in spacetime. The matrix $\Gamma$ is essentially the Weyl projector on the brane, and is given by

$$\Gamma^\alpha_\beta = \frac{(-1)^{(p+1)(p+2)/4}}{(p+1)! \sqrt{|g|}} \epsilon^{i_1 \ldots i_{p+1}} E_{i_1}^{\mu_1} \ldots E_{i_{p+1}}^{\mu_{p+1}} (\Gamma_{\mu_1 \ldots \mu_{p+1}})^{\alpha}_\beta,$$

(2.32)

where $E^i_\mu$ is the vielbein [7]. $\Gamma$ has the property that $\Gamma^2 = 1$ for certain values of the brane signature, hence $\frac{1}{2}(1 \pm \Gamma)$ may be projection operators, which may in fact be used to gauge away half the spin degrees of freedom. By employing the $\kappa$-symmetry and making use of the equations of motion we obtain the general matching formula

$$D - p - 1 = \frac{1}{4} N,$$

(2.33)

where $N$ is the dimension of the spin space, or its projection if the restriction $\theta \rightarrow \mathcal{P} \theta$ is made on the spinors. This equation may only be satisfied in twelve dimensions if we take $p = 3$ and have eight fermionic degrees of freedom. To count the spinor degrees of freedom we first note that a $\kappa$-symmetry exists if the four dimensional worldvolume may be chosen to be Weyl, which is the case for a brane of signature $(2, 2)$. Imposing the effects of the $\kappa$-symmetry and then imposing the equations of motion and the Majorana condition leaves us with 16 degrees of freedom. To reduce this to eight, and hence obtain worldvolume supersymmetry,
we may act with the Weyl projector on the spacetime spinors. We may do this consistently if the spacetime signature is \((10, 2)\), since Majorana-Weyl spinors may only be defined if \((S - T) \mod 8 = 0\).

We shall now check the consistency of this construction. The \(\kappa\)-symmetry is defined via a Wess-Zumino action involving the \((p + 2)\)-form \(H\)

\[
H = \frac{a}{2p!} \Pi^{\mu_p} \ldots \Pi^{\mu_1} d\bar{\theta} \Gamma_{\mu_1 \ldots \mu_p} d\theta .
\]

For this to be well defined we require that \(H\)-be non-zero and closed. Since the \(d\theta\) are commuting variables, for \(H\) to be not identically zero, \(\Gamma^{(p)}\) must be is symmetric in its spinor indices. If \(dH = 0\) then we must have that

\[
(\mathcal{P} d\bar{\theta} \Gamma_\mu \mathcal{P} d\theta)(\mathcal{P} d\bar{\theta} \Gamma^{\mu_1 \ldots \mu_p} d\theta) = 0 ,
\]

which has solutions only if \(\frac{D - p - 1}{4} N\)

\[
D - p - 1 = \frac{1}{4} N
\]

This condition is known to be sufficient if \(1 \leq D \leq 11\), and it can be shown that the identity holds for certain choices of projector for \(D = 12\) and \(p = 3\), which means that the Wess-Zumino term is well defined. This being so it would appear that we could define the \(\kappa\)-symmetry and hence a super three-brane. However, we actually encounter an inconsistency for the three-brane moving in an \(\text{SIO}(S,T)\) background: \(\Gamma^\mu\) and \(\Gamma^{\mu_1 \mu_2 \mu_3}\) cannot be simultaneously symmetric since \(\Gamma^{(i)}\) and \(\Gamma^{(i+2)}\) always have the opposite parity. This implies that \(H \equiv 0\) if the anticommutator of \(Q\) with itself generates momentum, which means that the Wess-Zumino term is identically equal to zero. This is a problem since the brane is expected to couple to the local version of the supersymmetry theory via the Wess-Zumino form. In addition, we may not define a \(\kappa\)-symmetry either, which means that the degrees of freedom on the worldvolume do not match up and the brane is consequently ill defined.

We have learnt that we may not formulate a three-brane theory in twelve dimensions after all, if we assume invariance of the brane under the standard supersymmetry group \(\text{SIO}(S,T)\). This type of consistency problem does not arise in ten or eleven dimensions for which we must choose \(p = 1, 2\) respectively for the \(H\)-form. Since we can make \(\Gamma^{(1)}\) and \(\Gamma^{(2)}\) symmetric simultaneously, we encounter no problems in the formulation of a string or membrane theory for the usual supersymmetry algebra. We must conclude that in twelve dimensions the situation is more complex than it at first appears. The problem may appear to be insoluble, since \(\Gamma^\mu\) and \(\Gamma^{\mu_1 \mu_2}\) are never simultaneously symmetric. There is a resolution, however, which requires us to relax the notion of what we mean by supersymmetry.

\(^2\)see appendix A
In order to progress we need to carry out the previous procedure using the algebra $\text{sio}(S,T)_2$ which differs from $\text{sio}(S,T)_1$ in that the $\{Q, Q\}$ anticommutator in (2.12) is replaced by
\[
\{Q^\alpha, Q^\beta\} = \frac{1}{2} (\Gamma^\mu_{\mu})^{\alpha\beta} M^\mu\nu.
\] (2.37)
This expression requires $\Gamma^\mu\nu$ to be symmetric, but there is no restriction on $\Gamma^\nu$. We may now define a non-trivial superspace form $H$, since it is possible that $\Gamma^\mu\nu$ and $\Gamma^\mu\nu\rho$ are simultaneously symmetric. Trying to formulate a theory in a new type of supersymmetry background may seem strange, but there is no logical reason why this should not be done.

We must now analyse the consequences of having an anticommutator of the form (2.37) in the supersymmetry algebra; Although $\text{sio}(S,T)_2$ reduces to the Poincaré algebra in the limit where the supersymmetry generators are set to zero, the action on superspace will be quite different since the commutator of the two supersymmetries will generate an $SO(S,T)$ rotation instead of a momentum boost. Clearly we shall need to carefully discuss the action of the group on the spacetime manifold.

### 2.4 Action of the new supergroup $\text{SIO}(S,T)_2$ and the $\text{sio}_2$ superspace

Given a group $G$ with an invariant subgroup $H$, we may define a coset manifold
\[
\mathcal{M} = G/H,
\] (2.38)
where $G$ is the isometry group of the tangent space of the manifold and $H$ is the isotropy subgroup for each point, the group elements locally being given by the exponentiation of the Lie algebra of $G$. Such a construction may be used to generate homogeneous spaces. As an example we note that Minkowski space is given by the quotient of the Poincaré group by the subgroup of Lorentz rotations, $SO(S-1,1)$. Since $G$ is the isometry group of the tangent space of $\mathcal{M}$, the action of an infinitesimal group element leaves $\mathcal{M}$ invariant. From the action of the group on the local coordinates of the manifold we can find invariant one-forms from which an action for the space may be constructed. We now consider such a procedure for the algebra $\text{sio}(S,T)_2$. Since this algebra is generated by $\{M, P, Q\}$, a general group element is given by
\[
G(X, \theta, \omega) = \exp(X^\mu P_\mu + \theta^\alpha Q^\alpha + \frac{1}{2} \omega_{\mu\nu} M^\mu\nu),
\] (2.39)
where $(X, \theta, \omega)$ are parameters in the group space. As an ansatz we shall suppose that the background spacetime is given by the coset
\[
G(X, \theta, \omega) = \exp(X^\mu P_\mu + \theta^\alpha Q^\alpha) \exp\left(\frac{1}{2} \omega_{\mu\nu} M^\mu\nu\right),
\] (2.40)
in which case $(X, \theta)$ become the superspace coordinates. Since we have replaced Minkowski space by a more general coset space, we should question the choice of the subgroup we
quotient by. By considering what happens in de Sitter space (appendix A) we see that the subgroup generated by $M_{\mu\nu}$ is singled out from those generated by the other $Z^{(i)}$, and it is thus natural to write the quotient as in (2.40). We now act on the left of the coset with an infinitesimal group element to give

$$G(\delta X, \delta \theta, 0)G(X, \theta, \omega) = \exp(\delta X^\mu P_\mu + \delta \theta_\alpha Q^\alpha)\exp(X^\mu P_\mu + \theta_\alpha Q^\alpha)\exp\left(\frac{1}{2}\omega^{\mu\nu}M_{\mu\nu}\right), \quad (2.41)$$

which may be rearranged using the Baker-Campbell-Hausdorff formula, given by

$$\exp(\epsilon A)\exp(B) = \exp\left(B + \epsilon A + \epsilon \sum_{n=1}^{\infty} \frac{1}{(n + 1)!}\left([\ldots[A,B],B],\ldots,B\right) + O(\epsilon^2)\right), \quad (2.42)$$

where $\epsilon$ is an infinitesimal superspace parameter. This expression is greatly simplified for the $\text{sio}(S,T)_1$ case because only the first commutator in the series is non-zero. The series consequently terminates, giving rise to the supersymmetry transformations (2.25). This termination does not necessarily occur for the $\text{sio}(S,T)_2$ algebra since the commutator of the $Q$ with itself generates a rotation, which does not trivially commute with any of the generators. We find that the right hand side of (2.41) becomes

$$(\theta + \delta \theta)_\alpha Q^\alpha + (X + \delta X)^\mu P_\mu + C_1 + \sum_{n=2}^{\infty} \frac{1}{(n + 1)!}C_n,$$

with

$$C_1 = \frac{1}{2!}\delta \theta_\alpha(\Gamma^{\mu\nu})^{\alpha\beta}\theta_\beta M_{\mu\nu}, \quad (2.44)$$

where $C_n$ is the commutator of $C_{n-1}$ with the exponent of the first factor of $G$ in the expression (2.40). In general this leads to an infinite series of terms involving $M_{\mu\nu}$. These terms may not be factored out into a coset form and as a result the construction is not generally self-consistent. It can be made so, however, if we make the restriction that the variation in $\theta$ obeys the relationship

$$\delta \theta_\alpha(\Gamma^{\mu\nu})^{\alpha\beta}\theta_\beta M_{\mu\nu} = 0,$$

which corresponds to the vanishing of $C_1$ and hence all the other $C_n$. The identity (2.43) is not satisfied for general spinors $\theta$; to make (2.43) hold, we restrict the form of $\theta$ so that we are dealing with a subset of all possible spinors. The projection onto this subspace is defined by a projection operator $P$ so as to give

$$P_\alpha^{\hat{\alpha}}\delta \theta_\alpha(\Gamma^{\mu\nu})^{\alpha\beta}P_\beta^{\hat{\beta}}\theta_\beta = \delta \theta_\alpha\left(\hat{P}\Gamma^{\mu\nu}C^{-1}P\right)^{\alpha\beta}\theta_\beta = 0,$$

where $\Gamma^{\mu\nu} \equiv \{(\Gamma^{\mu\nu})^{\alpha\beta}\}$. For this to be true for all variations requires us to restrict our spinors so that

$$\phi_\alpha(\Gamma^{\mu\nu})^{\alpha\beta}\psi_\beta = 0 \quad \forall \phi, \psi.$$  \quad (2.47)
This identity will prove to be fundamental in the following work and is the defining feature of the \( \text{sio}_2 \) superspace. The equation itself really defines a particular class of spinors\(^3\), which can, in the cases we are interested in, be obtained via a projection of a general Dirac spinor. Some projectors which satisfy this equation will be discussed in the next section, within the context of compactification to lower dimensions. It is worth noting that the equation (2.46) is not satisfied in an even dimensional space if we choose \( P \) to be the Weyl projector, unless there are an odd number of timelike directions. Thus, for a (10,2) signature theory, an explicit projector used to obtain the spinors which satisfy (2.46) from a Dirac or Majorana spinor will not be Lorentz invariant. In order to satisfy the superspace identity we must in fact couple the Weyl projector with another rank(1/2) projector, which is not a Lorentz invariant procedure. The constraint itself is, however, completely Lorentz invariant and the underlying theory may be fully covariant.

As a corollary to this, if the identity (2.46) is satisfied then the supergroup action induces the very simple \( \text{sio}_2 \) supersymmetry transformations

\[
\delta X = x \quad \delta \theta = \epsilon ,
\]

(2.48)

where \( x \) is a constant commuting vector parameter and \( \epsilon \) is a constant anticommuting spinor parameter. This is very pleasing from a geometrical point of view because the spinorial part of the superspace is now just a trivial bundle over spacetime, a superspace generalisation of \( \mathbb{R}^n \). This is, in a sense, a much more natural result then that obtained for the usual \( \text{sio}(S,T)_1 \) algebra, (2.23), in which the superspace is twisted!

### 2.5 \( \text{sio}_2 \) p-branes

Now that the new superspace has been defined we may construct actions for \( p \)-branes propagating in such backgrounds. The 1-forms invariant under the action of \( \text{sio}(S,T)_2 \) are simply given by

\[
\Pi^A = (dX^\mu, d\theta^\alpha) ,
\]

(2.49)

where \( A \) runs over the even and odd coordinates. Using the pullback of the forms \( \Pi^A \) to the brane, \( \Pi^A_i = (\partial_i X^\mu, \partial_i \theta^\alpha) \), we can write down the superspacetime supersymmetric canonical action

\[
S = \int d^{p+1}\xi \left[ \det(\Pi^A_1 \Pi^B_1 \mathcal{G}_{AB}) \right]^{\frac{1}{2}} ,
\]

(2.50)

where the ‘metric’ on the superspace is given by

\[
\mathcal{G}_{AB} = \begin{pmatrix} \eta & 0 \\ 0 & C \end{pmatrix} .
\]

(2.51)

\(^3\)The definition of these spinors is reminiscent of that for pure spinors
A brief discussion of such metrics is given in [23].

We are now in a position to repeat the analysis for a three-brane action in an $\text{so}_2$ background. Recall that the obstacle to defining such an action in a usual $\text{so}_1$ superspace background was that the Wess Zumino integral vanishes identically in twelve dimensions if the supersymmetries generate momentum, since $\Gamma^{(3)}$ is antisymmetric in such a situation. For $\Gamma^{(3)}$ to be symmetric we must necessarily choose $C = C_+$. In such a case, $\Gamma^{(2)}$ is symmetric, and may therefore appear on the right hand side of the $\{Q, Q\}$ anticommutator. This means that we may define a three-brane Wess-Zumino term if we employ the $\text{so}_2$ superspace.

The new Wess-Zumino action is defined in terms of the trivial forms $\Pi^A$ given by (2.49). Since there is no torsion, we have $d\Pi^A = 0$ and the superspace form $H = \Pi^\mu \Pi^\nu \Pi^\rho d\bar{\theta} \Gamma_{\mu \nu \rho \sigma} d\theta$ is automatically closed. The underlying structure is now self consistent, so all that remains to do is to check the brane degrees of freedom.

Constructing the Wess-Zumino action in the same way as for the (2.30) we find that

$$S_{WZ} = - \int *B, \quad B = \pi^\mu dX^\nu dX^\rho dX^\sigma d\bar{\theta} \Gamma_{\mu \nu \rho \sigma} \theta,$$

(2.52)

where $*B$ is the pullback of the form $B$ to the worldvolume, and $H = dB$. Since the $\text{so}_2$ superspace is flat without torsion, there is no spinorial part to the forms $\Pi^\mu$, hence there cannot be any $\kappa$-symmetry in the sense of (2.31). If we insert a general spinor transformation, then we find that under the variation we obtain the non-vanishing term

$$\delta \theta_\alpha (\Gamma \Gamma_i)_{\alpha \beta} \theta = 0,$$

(2.53)

where $\Gamma$ is defined in (2.32) and $\Gamma_i$ is the pullback of the $\Gamma^\mu$ to the brane. This term does not vanish in general, and a $\kappa$-symmetry may not be used to gauge away half the spinorial degrees of freedom. This is not a problem in the case for which the superspace projector is not satisfied by the Weyl projector. In these situations we need further project out another half of the spinor coordinates to satisfy the supersymmetry constraint (2.40). This extra projection has the same effect as the $\kappa$-symmetry on the fermions and leads to the existence of worldvolume supersymmetry and hence the three-brane. This is a very interesting point: the existence of the worldvolume supersymmetry for the twelve dimensional three-brane, and the consistency of the spacetime supersymmetry are inextricably linked, whereas in the lower dimensional cases they could exist independently.

To conclude this section we reiterate the main result. It is possible to define a consistent supersymmetric three-brane action in twelve spacetime dimensions. Such a theory has eight Bose and eight Fermi degrees of freedom on the worldvolume and is thus a new member of
the octonionic sequence \[^{10}\]_16\], in addition to the ten dimensional superstring and the eleven dimensional supermembrane. In order to define the Wess-Zumino term for the three-brane it is necessary to have \((\Gamma^{\mu\nu\rho})_{\alpha\beta}\) symmetric in the spinor indices, which rules out the usual super-Poincaré group, \(\text{SIO}(S,T)_{1}\) as the isometry group of the background spacetime, and also forces us to make the choice \(C = C_+\). Taking an unusual \(N = 1\) supersymmetric algebraic extension of the Poincaré algebra provides us with a new supersymmetry algebra in twelve dimensions, which we denote by \(sio(S,T)_2\). The action of the corresponding supergroup leads to a natural, trivial superspace in which the brane propagates, if we make an appropriate projection of the spinors. It is by such a projection that we obtain matching of degrees of freedom, and hence supersymmetry, on the brane. This result gives a pleasing self consistency to the construction.

3 Compactification from twelve dimensions

As we have shown in the previous section, it is possible to define a consistent twelve dimensional supersymmetric three-brane theory. What relationship does this brane hold to the traditional string and membrane? To answer this question we consider the simultaneous dimensional reduction of the twelve dimensional theory described by the action \[^{2.50}\] with a Wess-Zumino term. Our goal is to relate the twelve dimensional theory to the string theories and M-theory, in ten and eleven dimensions of Minkowskian signature respectively, by a compactification in which we remove one of the additional timelike directions. Thus we can proceed in three ways: via a timelike reduction to eleven dimensions to give the correct degrees of freedom for the \(M\)-theory two-brane; via a timelike reduction followed by a spacelike reduction to obtain the ten dimensional IIA string, and finally via a double null reduction which produces the ten dimensional IIB string.

Since the three-brane action may only be defined for the choice \(C = C_+\), we are restricted to consider theories with signature \((10,2)\), \((6,6)\) or \((9,3)\) (see table \[^{2.16}\]). The case \((6,6)\) is excluded because there are more timelike directions than can fit on the brane, and hence the system is not classically stable. We shall only consider the case \((S,T) = (10,2)\) in this paper, since this is the case which seems to be directly related to the Minkowski theories in lower dimensions. We shall henceforth refer to the three-brane as being, more precisely, a \((2 + 2)\)-brane. For simplicity we shall work with the following real representation of \(\text{Spin}(10,2)\):

\[
\Gamma_p = \begin{pmatrix} \gamma_p & 0 \\ 0 & -\gamma_p \end{pmatrix}, \quad p = 0 \ldots 9, \quad \Gamma_{11} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \Gamma_{12} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},
\]

(3.1)
where
\[
\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu} \quad \eta_{\mu\nu} = \begin{cases} +1, & \text{if } \mu = \nu = 1 \ldots 9 \text{ or } 11 \\ -1, & \text{if } \mu = \nu = 0 \text{ or } 12 \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}
\]

The $\gamma_p$ are a set of Spin(9, 1) gamma matrices, with $\gamma_0^2 = -1$. We may choose a basis such that the twelve dimensional spacelike (timelike) gamma matrices are hermitian (anti-hermitian) respectively. We may also choose that
\[
\gamma_{11} \equiv \gamma_0 \ldots \gamma_9 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \equiv J. \tag{3.3}
\]

It may easily be shown that the twelve dimensional charge conjugation matrix is given by
\[
C = \begin{pmatrix} 0 & C_{10} \\ C_{10} & 0 \end{pmatrix}, \tag{3.4}
\]

where $C_{10}$ is the ten dimensional $C_-$ for the matrices $\gamma_0, \ldots, \gamma_9$.

We are now interested in the compactification of the twelve dimensional theory. We shall first discuss the effects of the compactification on the spin spaces, and compare these with those for string theory and $M$-theory two-branes. If the twelve dimensional theory is to relate to these theories then it must produce the right spinors upon compactification.

### 3.1 Projection of the spin space

Recall that, in order to define the $(2 + 2)$-brane in twelve dimensions, we must work in the $\text{so}_2$ superspace which must be defined in terms of the restricted spinors satisfying
\[
\delta \theta_\alpha (\Gamma^{\mu\nu})^{\alpha\beta} \theta_\beta M_{\mu\nu} = 0. \tag{3.5}
\]

If we wish not to restrict the possible rotation states, this may be re-written as
\[
\bar{\mathcal{P}} \Gamma^{\mu\nu} C^{-1} \mathcal{P} = 0. \tag{3.6}
\]

In order to assist the compactification to ten and eleven dimensions, we shall consider the cases
\[
\mathcal{P}_{10} = \frac{1}{2} (1 + \Gamma_0 \ldots \Gamma_9) = \frac{1}{2} \begin{pmatrix} I + J & 0 \\ 0 & I + J \end{pmatrix} \quad \text{and} \\
\mathcal{P}_{11} = \frac{1}{2} (1 - \Gamma_0 \ldots \Gamma_9 \Gamma_{11}) = \frac{1}{2} \begin{pmatrix} I & J \\ J & I \end{pmatrix}. \tag{3.7}
\]

These two projections restrict the twelve dimensional spinors to be of the form
\[
\psi_{11} = \begin{pmatrix} \alpha \\ \delta \\ \alpha \\ -\delta \end{pmatrix} \quad \text{and} \quad \psi_{10} = \begin{pmatrix} \alpha \\ 0 \\ \gamma \\ 0 \end{pmatrix}, \tag{3.8}
\]
and both $\mathcal{P}_{10}$ and $\mathcal{P}_{11}$ satisfy the equation (3.3), provided we further project out an additional half of the spinor degrees of freedom. To see this is the case for $\mathcal{P}_{10}$, we first note that 

$$\{\Gamma_0 \ldots \Gamma_9, \Gamma^{\mu \nu}C^{-1}\} = 0 \text{ for } \mu, \nu = 0 \ldots 9 \text{ and } \mu, \nu = 11 \text{ or } 12.$$ 

In these situations, the matrix identity (3.6) is satisfied, since $(1 + \Gamma_0 \ldots \Gamma_9)(1 - \Gamma_0 \ldots \Gamma_9) = 0$, and we therefore are done. It remains to check the case for which we have $\mu = 0 \ldots 9$ and $\nu = 11$ or $12$. These terms fail to satisfy the equation in the same way, since $(\Gamma^{\mu \nu})^{\alpha \beta}$ commutes with $P_{10}$ if one and only one of $\mu$ or $\nu$ takes value of $10$ or $11$. In order to overcome this problem we must further project out another half of the spinor degrees of freedom. Substituting the explicit representation of the gamma matrices we find that the extra condition which needs to be satisfied is that

$$C^{\alpha}_{10}(\beta) = 0,$$

which requires an additional rank($\frac{1}{2}$) projection on the spinors, restricting $\alpha$ and $\beta$ in much the same way as a $\kappa$-symmetry would.

We shall now discuss the relation between the theory with these projectors and $p$-brane theories in lower dimensions by compactification. At first glance it may seem very unlikely that the $(2 + 2)$-brane could possibly reduce down to Green-Schwarz strings and membranes, due to the different types of supersymmetry for these theories. As we shall see, however, this is not so.

In order to proceed, recall that to perform a dimensional reduction in a direction defined by a vector $n$ with a background metric $\eta_{\mu \nu}$, we act on the space with projection operators $h_{\mu \nu} = \eta_{\mu \nu} \pm n_{\mu}n_{\nu}$. We need to explicitly define analogous operators $H^{\alpha \beta}$ to act on the spin space so that we can deduce the nature of the lower dimensional spinors. The form of $H^{\alpha \beta}$ will depend on whether we perform a timelike, spacelike or null reduction. In the compactified spin space we define a new set of Gamma matrices, $\{\hat{\Gamma}_{\mu}\}$, as

$$\hat{\Gamma}_{\mu} = \mathcal{H}_{\mu} \bar{\mathcal{H}},$$

in matrix notation. It is with respect to these lower dimensional gamma matrices that we discuss the chirality of the compactified spinors.

### 3.1.1 Double Null Reduction

We now define two vectors which are null with respect to the $(10,2)$ flat space metric (3.2): $u = \frac{1}{\sqrt{2}}(0, \ldots, 0, 1, 1)$ and $v = \frac{1}{\sqrt{2}}(0, \ldots, 0, 1, -1)$, so that $u^2 = v^2 = 0$ and $u \cdot v = +1$. The spin space double null operators are then given by

$$(H_{D \pm})^{\alpha}_{\beta} = \frac{1}{2}(I^{\alpha}_{\beta} + (\Gamma^{\mu \nu}u_{\mu}u_{\nu})^{\alpha}_{\beta}) = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} \text{ or } \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix},$$

(3.11)
after we substitute the explicit representation of the gamma matrices (3.1). Note that $H_D^2 = H_D$ so we have a true projection operator. Since the $H_D$ is a double null projector, the action of this operator produces spinors in a spacetime of signature $(9, 1)$. Using the expression (3.10) we can evaluate the new ten dimensional gamma matrices, $\hat{\Gamma}$, in a (reducible) 64 dimensional representation. As one would expect, $\hat{\Gamma}_p$ is non-zero if $p = 0 \ldots 9$, and the other two gamma matrices are projected to zero under the effect of $H_D$ yielding

$$\hat{\Gamma}_p = \begin{pmatrix} \gamma_p & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & \gamma_p \end{pmatrix} \quad p = 0 \ldots 9.$$ (3.12)

To discover the effect of the compactification on the chirality of the spinors we must define the new $\hat{\Gamma}_{11}$ matrix in the projected space

$$\hat{\Gamma}_{11} = \hat{\Gamma}_0 \ldots \hat{\Gamma}_9 = \begin{pmatrix} J & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & 0 \\ 0 & J \end{pmatrix}.$$ (3.13)

The compactified spinors are now defined by the relationship

$$\hat{\psi} = H_D \psi.$$ (3.14)

We find that the projected spinor $\psi_{10}$, (3.8), becomes a pair of 16 component spinors of the same chirality, with respect to $\hat{\Gamma}_{11}$, under the action of the $H_{D\pm}$; we have thus obtained the spectrum of the type IIB string theory.

### 3.1.2 Timelike reduction from twelve dimensions

We now perform the reduction along the timelike direction $t = (0, 0, \ldots, 0, 1)$, $t^2 = -1$. The relevant projection operators are given by

$$(H_t^{\pm})_{\alpha \beta} = \frac{1}{2} (I^{\alpha \beta} \pm (\Gamma^{13} t_\mu \Gamma_{\mu})^{\alpha \beta}) = \frac{1}{2} \begin{pmatrix} I & \pm J \\ \pm J & I \end{pmatrix},$$ (3.15)

which happens to be of the same form as the projection operator $P_{11}$. In this case the new eleven dimensional chirality projection matrix is given by

$$\hat{\Gamma}_{12} = (H_t \Gamma_0 \hat{H}_t) \ldots (H_t \Gamma_{11} \hat{H}_t) = \frac{1}{2} \begin{pmatrix} 0 & \pm J \\ \pm J & 0 \end{pmatrix}.$$ (3.16)

Compactification of the projected spinor $\psi_{11}$, gives one zero spinor and a single non-zero eleven dimensional spinor of the form

$$\hat{\psi}_{11} = \begin{pmatrix} \alpha \\ \delta \\ \alpha \\ -\delta \end{pmatrix},$$ (3.17)
which has 32 real components, from the eleven dimensional point of view. These are the
correct degrees of freedom for the M-theory two-brane. From this theory we may further
compactify on a circle or a $\mathbb{Z}_2$ orbifold of the circle to produce the type IIA string or heterotic
$E_8 \otimes E_8$ string respectively \cite{4}.

3.2 Compactification of the Superspace

Although the arguments in the previous section show that the spinor degrees of freedom after
compactification correspond to the IIB string theory and the M-theory, we have not shown
that the compactification yields the correct supersymmetric theory given by the action (2.28),
which is invariant under the action of the super Poincaré group $SIO(S,T)_1$. For the twelve
dimensional theory to truly relate to lower dimensions we must clearly be able to produce
the usual type of supersymmetry in a natural way. The superspace in twelve dimensions is
effectively predetermined by the requirements of the existence of a brane action, in that it
must be the superspace for the supergroup $SIO(10,2)$. We now must investigate the effect
that the compactification has on this superspace. We shall specifically consider the effects of
the double-null compactification (the analysis is similar for the timelike case).

Recall that the basic action in twelve dimensions is given by (2.50),

$$S = \int d^4\xi \det[\Pi_A^i \Pi_B^j \mathcal{G}_{AB}]^{\frac{1}{2}},$$

with

$$\mathcal{G} = \begin{pmatrix} \eta & 0 \\ 0 & C \end{pmatrix},$$

and $\Pi^A = (dX^\mu, d\theta^\alpha)$. Since the fundamental geometric object of the theory is the superspace,
it is natural to consider the effects of the compactification on the bosonic and fermionic
indices together in a superspace projection $\mathcal{H}^{A_B}$, for $A,B = (\mu, \alpha)$. Schematically, $\mathcal{H}^{A_B}$ can be written as

$$\mathcal{H} = \begin{pmatrix} B & Y \\ Z & F \end{pmatrix},$$

where $B \equiv \{B^\mu_\nu\}$ is the projector which acts on the purely bosonic part of the superspace
and $F \equiv \{F^\alpha_\beta\}$ is the projector which acts on the spin space. For example, for the double
null reduction case we have the $B^\mu_\nu = h^\mu_\nu = \eta^\mu_\nu + \frac{1}{2}(u^\mu v_\nu - v^\mu u_\nu)$ and $F^\alpha_\beta = H^\alpha_\beta = \frac{1}{2}(I + \Gamma^{\mu\nu} u_\mu v_\nu)^\alpha_\beta$. For a general compactification with these fermionic and bosonic parts,
the matrices $Y \equiv \{Y^\mu_\alpha\}$ and $Z \equiv \{Z^\alpha_\mu\}$ may be arbitrarily chosen. When Cremmer
and Julia constructed the $SO(8)$ supergravity, \cite{26}, they effectively considered the case for which
the superspace projector was block diagonal. This is by no means a necessary choice, and
in pure bosonic Kaluza-Klein theories would correspond to a trivial compactification. We
shall for the moment leave the terms $Y$ and $Z$ arbitrary, and investigate the effect of the compactification on the $(2+2)$-brane action.

Under the action of the superspace projection, the superspace metric schematically transforms as

$$
\mathcal{G} \rightarrow \hat{\mathcal{G}} = \mathcal{H} \hat{\mathcal{G}} \hat{\mathcal{H}} = \begin{pmatrix}
B & Y \\
Z & F
\end{pmatrix}
\begin{pmatrix}
\eta & 0 \\
0 & C
\end{pmatrix}
\begin{pmatrix}
\tilde{B} & -\tilde{Z} \\
\tilde{Y} & \tilde{F}
\end{pmatrix}
= \begin{pmatrix}
B\eta\tilde{B} + YC\tilde{Y} & -B\eta\tilde{Z} + YC\tilde{F} \\
Z\eta\tilde{B} + F\tilde{Y} & -Z\eta\tilde{Z} + FC\tilde{F}
\end{pmatrix},
$$

(3.21)

where the tilde denotes matrix transpose. The minus sign occurs for the $\tilde{Z}$ in the transpose of $\mathcal{H}$ because we are dealing with supermatrices, as opposed to ordinary matrices. We now make the restriction that the purely bosonic part of the transformed superspace metric should equal the projection of $\eta_{\mu\nu}$ by $B^\mu_{\;\nu}$. We similarly require that the fermionic sector should be the same as the projection of the charge conjugation matrix by the spin projector $H^\alpha_{\;\beta}$. This leads to the constraints on $Y$ and $Z$

$$(YC\tilde{Y})^{\mu\nu} \equiv Y^\mu_{\;\alpha} C_{\alpha\beta} \tilde{Y}^{\beta\nu} = 0$$

$$(Z\eta\tilde{Z})^{\alpha\beta} \equiv Z^\alpha_{\;\mu} \eta_{\mu\nu} \tilde{Z}^{\nu\beta} = 0.$$  

(3.22)

These requirements essentially mean that the squares of $Z$ and $Y$ must be antisymmetric in the bosonic indices and symmetric in the fermionic indices respectively. Thus the equations (3.22) are satisfied if we choose

$$Y^\mu_{\;\beta} = Z^\mu_{\;\beta} = \frac{1}{2}\theta_{\alpha} (\Gamma^\mu)^{\alpha\beta}.$$  

(3.23)

Or course, we could choose that $Y = Z = 0$, but this would lead to another trivial superspace in lower dimensions, not the sio$_1$ that we are seeking. Note that we only require that the two sides of the equation (3.23) be proportional to each other, but we choose the factor of $-\frac{1}{2}$ for convenience. Finally we suppose that all the spacetime indices have been acted upon by $B$ and all the spinorial indices by $F$. This is in effect what is achieved in the compactification

$$FY = BY = Y.$$  

(3.24)

We may now calculate the effects of the transformation (3.21) on the Lagrangian for the action (2.50)

$$\mathcal{L} = \Pi^A \mathcal{G}_{AB} \Pi^B \rightarrow \hat{\mathcal{L}} = \Pi^A \hat{\mathcal{G}}^B_{AB} \mathcal{H}_{BC} \mathcal{H}_D \Pi^D$$

$$= \left( dX^\mu \quad d\theta^\alpha \right) \begin{pmatrix} B_{\mu\nu} & 0 \\ 4\tilde{Y}_{\alpha\nu} & F_{\alpha\beta} \end{pmatrix} \left( dX^\nu \quad d\theta^\beta \right)$$

$$= dX^\mu \eta_{\mu\nu} dX^\nu - 2dX^\mu \bar{\theta} \Gamma_{\mu} d\theta,$$

(3.25)
where the coordinates now all lie in the compactified superspace. The beauty of this construction is that the action principle may now be reformulated as

$$S = \int d^p \xi \det[(\partial_\mu X^\nu - \bar{\theta} \Gamma^\mu \partial_\nu \theta) \eta_{\mu\nu}(\partial_\nu X^\mu - \bar{\theta} \Gamma^\nu \partial_\mu \theta)]^{\frac{1}{2}}, \quad (3.26)$$

which is precisely the form of the $\text{sio}_1$ invariant action (2.28). Thus we have shown that by performing a natural superspace projection on a trivial higher dimensional superspace, we reproduce the usual twisted superspace of standard supersymmetry.

### 3.3 Compactification of the super-algebra

We now turn to the question of the dimensional reduction of the supersymmetry algebra, (2.20), which was is so essential in the construction of super $(2 + 2)$-brane. For the reduction of the $(2 + 2)$-brane consistently to yield the usual Minkowski strings and membranes we must retrieve the $\text{sio}(9,1)_1$ or $\text{sio}(10,1)_1$ algebras after compactification. The important term to discuss is the anticommutator of the spinor generator with itself for the $\text{sio}(10,2)_2$ algebra, after we have taken the projection of the spin space,

$$\{ \mathcal{P}_\alpha Q^\alpha, \mathcal{P}_\beta Q^\beta \} = \frac{1}{2} \mathcal{P}_\alpha (\Gamma_{\mu\nu})^{\hat{\alpha}\hat{\beta}} M_{\mu\nu} \mathcal{P}_\beta. \quad (3.27)$$

The only terms which survive on the right hand side of the projected anticommutator are those for which the matrix identity (3.6) is not satisfied for the projectors $\mathcal{P}_{10}$ or $\mathcal{P}_{11}$. This yields

$$\{ (\mathcal{P}_{10})^\alpha Q^\alpha, (\mathcal{P}_{10})^\beta Q^\beta \} = \frac{1}{2} (\mathcal{P}_{10})^\alpha \left( (\Gamma_{\mu\nu})^{\hat{\alpha}\hat{\beta}} M_{\mu\nu} + (\Gamma_{\mu\nu})^{\hat{\alpha}\hat{\beta}} M_{\mu\nu} \right) (\mathcal{P}_{10})^\beta \quad (3.28)$$

We now need to determine the form of the rotation generator, $M_{\mu\nu}$. A realisation of the algebra $\text{sio}(S,T)_2$ is given by

$$M_{\mu\nu} = X_{[\mu} \partial_{\nu]} + \frac{1}{2} \theta_\alpha (\Gamma_{\mu\nu})^\alpha \partial_\beta, \quad (3.29)$$

where we may choose $Q_\alpha = \partial_\alpha = \frac{\partial}{\partial \theta^\alpha}$. If we consider projecting the spinors by the projectors $\mathcal{P}_{11}$ and $\mathcal{P}_{10}$ then the second term in this expression vanishes identically, in which case we find that

$$M_{\mu\nu} = X_{[\mu} \partial_{\nu]} \quad (3.30)$$

We now perform the dimensional reduction on the anticommutators (3.28), by acting with the operators $(H_{D\pm})^{\alpha}_{\beta} (3.11)$. The double null reduction leads to

$$\{ Q^+_\alpha, Q^+_\beta \} = (\gamma^p)^{\alpha\beta} P_p$$
\[
\{Q_{10}^{-\alpha}, Q_{10}^{-\beta}\} = (\gamma^p)^{\alpha\beta} P_p \\
\{Q_{10}^{+\alpha}, Q_{10}^{+\beta}\} = 0 \\
P_p = \frac{\partial}{\partial X_p},
\]

(3.31)

where \(Q_{10}^{\pm\alpha}\) is a positive/negative chirality Spin(9,1) spinor, and \(\gamma^p_{\pm}\) is a positive/negative projection of the gamma matrix \(\gamma_p\) for which \(p = 0 \ldots 9\). For the timelike compactification to eleven dimensions, one of the projectors \((H_{t\pm})^{\alpha\beta}\) sends everything to zero, the action of the other operator gives

\[
\{Q_{11}^{\alpha}, Q_{11}^{\beta}\} = (\tilde{\gamma}^p)^{\alpha\beta} P_p,
\]

(3.32)

where \(\tilde{\gamma}^p\), for \(p = 1 \ldots 11\), are new gamma matrices obtained by reduction of the twelve dimensional \(\Gamma_p\), for \(p = 1 \ldots 11\). We thus see that the compactifications from the twelve dimensional theory gives us the correct basic \(\text{sio}_1\) superalgebra that we require. This is by virtue of the superspace projection (2.43) required to define the \(\text{sio}_2\) \((2 + 2)\)-brane action. It was precisely this constraint which prevented the appearance of ten or eleven dimensional \((\gamma^p)^{\alpha\beta} M_{pq}\) terms in the projected algebras (3.31) and (3.32). Note that, although \((\Gamma^{12})^{\alpha\beta}\) is compactified to zero in eleven dimensions, the term \((\Gamma^{12p})^{\alpha\beta}\) is not, since the projection operator \(P_{11}\) does not commute with the twelve dimensional charge conjugation matrix \(C\). A similar comment applies to the ten dimensional compactification.

There is now one final question to be asked: what does the full twelve dimensional \(\{Q, Q\}\) anticommutator, (A.1), reduce to in ten and eleven dimensions? The full set of the possible \(\{Z^{(i)}\}\) is given by \(\{Z^{(2)}, Z^{(3)}, Z^{(6)}, Z^{(7)}, Z^{(10)}, Z^{(11)}\}\). Since we are compactifying over only one or two directions, the compactification of the \(Z^{(i)}\) with \(i\) even will all be qualitatively the same; a similar statement holds for the odd case. We thus need to determine the compactification of the \(\Gamma^{\mu_1\mu_2\mu_3}\) term. We find that for the double null reduction, \((\Gamma^{\mu_1\mu_2\mu_3})^{\alpha\beta}\) is projected to \((\Gamma^{p11,12})^{\alpha\beta}\) and \((\Gamma^{p12p3})^{\alpha\beta}\). For the eleven dimensional case we find that only the term \(\Gamma^{12p9}\) survives the projection. For the 7 and 11 index cases we need just add 4 or 8 \(p\)-type indices to these expressions. If we then act with the compactification operators, \(H^{\pm}\), then we find compactified versions of (A.1):

\[
\{Q_{10}^{\pm\alpha}, Q_{10}^{\pm\beta}\} = (\gamma^{p \pm})^{\alpha\beta} P_p + (\gamma^{p_1 \cdots p_5 \pm})^{\alpha\beta} Z_{p_1 \cdots p_5}^{\pm} + (\gamma^{p_1 \cdots p_9 \pm})^{\alpha\beta} Z_{p_1 \cdots p_9}^{\pm} \\
\{Q_{10}^{\mp\alpha}, Q_{10}^{\mp\beta}\} = (\gamma^{p \mp})^{\alpha\beta} \tilde{Z}_p + (\gamma^{p_1 \cdots p_5 \mp})^{\alpha\beta} \tilde{Z}_{p_1 \cdots p_5}^{\pm} + (\gamma^{p_1 \cdots p_9 \mp})^{\alpha\beta} \tilde{Z}_{p_1 \cdots p_9}^{\pm}  \\
+ (\gamma^{p_1 \cdots p_7 \mp})^{\alpha\beta} \tilde{Z}_{p_1 \cdots p_7}^{\pm} + (\gamma^{p_1 \cdots p_9 \mp})^{\alpha\beta} \tilde{Z}_{p_1 \cdots p_9}^{\pm} \\
\{Q_{11}^{\alpha}, Q_{11}^{\beta}\} = (\tilde{\gamma}^{p})^{\alpha\beta} P_p + (\tilde{\gamma}^{p_2})^{\alpha\beta} Z_{p_1 p_2} + (\tilde{\gamma}^{p_1 \cdots p_5})^{\alpha\beta} Z_{p_1 \cdots p_5} + (\tilde{\gamma}^{p_1 \cdots p_6})^{\alpha\beta} Z_{p_1 \cdots p_6} \\
+ (\tilde{\gamma}^{p_1 \cdots p_9})^{\alpha\beta} Z_{p_1 \cdots p_9} + (\tilde{\gamma}^{p_1 \cdots p_10})^{\alpha\beta} Z_{p_1 \cdots p_{10}},
\]

(3.33)

where the \(\tilde{Z}^{(i)}\) are new set of objects which are antisymmetric in the spacetime indices. These arise in the commutator of two ten dimensional spinors of opposite chirality, and thus
correspond to string theory central charges. We now make use of the algebraic equivalence between a $p$-form and a $D - p$-form in these supersymmetry algebras to find that

$$
\begin{align*}
\{Q^\alpha_{\pm 10}, Q^{\pm \beta}_{10}\} &= \left(\gamma^p\right)^{\alpha\beta} P^p + \left(\gamma^{p_1...p_5}\right)^{\alpha\beta} Z^{\pm}_{p_1...p_5} \\
\{Q^\alpha_{10}, Q^{\pm \beta}_{10}\} &= \left(\gamma^p\right)^{\alpha\beta} \tilde{Z}^p + \left(\gamma^{p_1...p_3}\right)^{\alpha\beta} \tilde{Z}^{\pm}_{p_1...p_3} + \left(\gamma^{p_1...p_5}\right)^{\alpha\beta} \tilde{Z}^{\pm}_{p_1...p_5} \\
\{Q^\alpha_{11}, Q^{\beta}_{11}\} &= \left(\tilde{\gamma}^{p}\right)^{\alpha\beta} P^p + \left(\gamma^{p_1p_2}\right)^{\alpha\beta} Z^{p_1p_2} + \left(\gamma^{p_1...p_5}\right)^{\alpha\beta} Z^{p_1...p_5}.
\end{align*}
$$

(3.34)

These supersymmetry algebras (3.34) are precisely those for the IIB and IIA superstrings, and for M-theory, [19], completing our analysis of the dimensional reduction of the $\text{so}(10,2)$ invariant superalgebra.

4 Conclusion

We have presented in this paper the construction of a covariant supersymmetric brane with worldvolume signature $(2,2)$. This brane propagates in a twelve dimensional spacetime with two timelike directions, and for consistency the $(2+2)$-brane must be defined in a new type of flat superspace which has no torsion. In order that this superspace be well defined, the spinors must satisfy an unusual spinor identity. This constraint leads to supersymmetry on the brane. We have shown that this $(2+2)$-brane can be related to the M-theory 2-brane and the type IIB string theory in ten dimensions under compactification, without the need to go to nine dimensions, or to use thirteen dimensions, as in [12]. This is a success since these two theories have until now had no direct single origin. Some speculation along the lines of replacing the brane by a $(2,1)$ string to produce the IIB string and M-theory has been made by Kutasov, Martinec and O’Loughlin [13, 14]. In this work it seems necessary to use a $(2+2)$-brane, which could well be related to the super $(2+2)$-brane presented here.

It would be interesting to study the relationship between the theory presented in this paper and F-theory, since both require the signature of spacetime to be $(10,2)$. F-theory has had some successes and provides a mechanism for explaining S-duality in the IIB string theory. There are clearly some conceptual difficulties with these theories, however, due to there being two timelike directions. For example, the notion of a brane propagating through the spacetime becomes unclear. It has been suggested that a twelfth dimension is merely auxiliary and is simply a clever tool used to discuss lower dimensional theories. It should be stressed, however, that the $(2+2)$-brane is an essentially twelve dimensional object, whose definition was independent of consideration of theories in lower dimensions: all the constraints arise from a discussion of twelve dimensional supersymmetry. This could point towards the existence of a real extra dimension.

It seems plausible that the formalism presented in this paper could be related to AdS
supergravity in eleven dimensions and the massive string theories and supergravity in ten dimensions. This is because the quantities

\[ X^A G_{AB} X^B = \text{constant}, \quad (4.1) \]

arise naturally for the supersymmetry discussed in this paper. This is the supersymmetric version of the constraint used to describe AdS in dimension \((d,1)\) as a hypersurface in dimension \((d+1,2)\).

Perhaps the most serious problem with any theory which involves a spacetime dimension greater than eleven is that we are lacking a local field theory. This is not to say that such a theory does not exist, but merely that all the obvious constructions fail. If we suppose that the twelve dimensional field theory should somehow be related to branes as in ten and eleven dimensions, then the failure of the standard methods to provide an answer should not be a surprise, due to the very different nature of the \((2+2)\)-brane to the traditional \(p\)-branes. For the time being, the resolution of this problem remains a mystery.

### A Subalgebras of the de Sitter algebra

We shall investigate the subalgebras of \(N = 1\) gradings of the de Sitter algebra, which is given in (2.17). We include the anticommutator term

\[ \{Q^\alpha, Q^\beta\} = \sum_{\text{symmetric}} \frac{1}{k!} (\Gamma_{\mu_1...\mu_k})^{\alpha\beta} Z^{\mu_1...\mu_k}. \quad (A.1) \]

The full graded algebra is generated by the basic terms \(\{P, M, Q\}\) and a set of \(Z^{(i)}\). Van Holten and Van Proeyen, [18], checked the consistency of these algebras for given \(Z^{(i)}\). They found that, if we identify \(Z^1\) and \(Z^2\) with \(P\) and \(M\) respectively, in the infinite radius limit we have the following terms in the algebra:

\[ \left[ Z^i, Z^j \right] \sim 2y \left( \frac{i + j - k}{2} \right)! \left( \frac{i - j + k}{2} \right)! \left( \frac{-i + j + k}{2} \right)! Z^k. \quad (A.2) \]

In this equation we have suppressed the indices: it is only the form of these commutators which is important in this discussion.

We start from a maximal set of the \(Z^{(i)}\) and search for subalgebras. In twelve dimensions, for \(C_-\) we have \(\{Z^1, Z^2, Z^5, Z^6, Z^9, Z^{10}\}\) as the possible generators, and for \(C_+\) we have \(\{Z^2, Z^3, Z^6, Z^7, Z^{10}, Z^{11}\}\). We now present some of the commutation relations between the \(\{Z^{(i)}\}\). We always have that

\[ \left[ Z^2, Z^{(i)} \right] \sim Z^{(i)}, \quad (A.3) \]
and for the two different choices of the charge conjugation matrix we have an additional set of commutators. For the $C = C_-$ case we have

\[
\begin{align*}
[Z^1, Z^1] & \sim yZ^2 \\
[Z^1, Z^5] & \sim yZ^6 \\
[Z^1, Z^6] & \sim yZ^5 \\
[Z^1, Z^9] & \sim yZ^{10} \\
[Z^1, Z^{10}] & \sim yZ^{10} \\
[Z^5, Z^6] & \sim y(Z^1 + Z^5 + Z^9) \\
[Z^5, Z^9] & \sim y(Z^6 + Z^{10}) \\
[Z^5, Z^{10}] & \sim y(Z^5 + Z^9) \\
[Z^6, Z^9] & \sim y(Z^3 + Z^9) \\
[Z^6, Z^{10}] & \sim y(Z^6 + Z^{10}) \\
[Z^9, Z^{10}] & \sim y(Z^1 + Z^5 + Z^9) \\
[Z^i, Z^i] & = y(Z^2 + Z^6 + Z^{10}) \text{ for } i \geq 5. \quad (A.4)
\end{align*}
\]

From these terms it is clear that the only subalgebras of the algebra generated by the $\{Z^i\}$, without setting some of the generators to zero, is generated by $\{Z^2, Z^1\}$. This subalgebra is the same as the Poincaré algebra, (2.11), in the infinite radius limit, if we make the scaling

\[
Z^1 = \left(\frac{2y}{m}\right)P, \quad Z^2 = 2yM. \quad (A.5)
\]

For the $C = C_+$ case we find that

\[
\begin{align*}
[Z^3, Z^6] & \sim y(Z^3 + Z^7) \\
[Z^3, Z^7] & \sim y(Z^6 + Z^{10}) \\
[Z^3, Z^{10}] & \sim y(Z^7 + Z^{11}) \\
[Z^3, Z^{11}] & \sim yZ^{10} \\
[Z^6, Z^7] & \sim y(Z^3 + Z^7 + Z^{11}) \\
[Z^6, Z^{10}] & \sim y(Z^6 + Z^{10}) \\
[Z^6, Z^{11}] & \sim y(Z^7 + Z^{11}) \\
[Z^7, Z^{10}] & \sim y(Z^3 + Z^7 + Z^{11}) \\
[Z^7, Z^{11}] & \sim y(Z^6 + Z^{10}) \\
[Z^{10}, Z^{11}] & \sim y(Z^3 + Z^7 + Z^{11})
\end{align*}
\]
\[
\begin{align*}
[Z^3, Z^3] & \sim y(Z^2 + Z^6) \\
[Z^i, Z^j] & \sim y(Z^2 + Z^6 + Z^{10}) \text{ for } i \geq 6
\end{align*}
\] (A.6)

In this case we also find that there is only one subalgebra, which is generated by \{Z^2, Z^6, Z^{10}\}. This is algebraically equivalent to the algebra generated by \{Z^2, Z^6\} since in twelve dimensions \(Z^2\) has the same degrees of freedom as \(Z^{10}\).

Note that for any given algebra, when we take the Poincaré limit we must set \(y\) to be zero, in which case all of the \(Z^{(i)}, i > 2\) terms become central charges. For gradings of the Poincaré algebra, any of the \(Z^{(i)}\) can be introduced: those sets which form subalgebras of the full graded de Sitter algebra have a natural origin.

**B**

We wish to find solutions of the identity (2.35). Since the \(d\theta\) terms are real and commuting, all that is required is to show that

\[
(\Gamma_\mu \mathcal{P})_{(\alpha\beta} (\Gamma_{\mu\nu\rho} \mathcal{P})_{\gamma\delta)} = 0,
\] (B.7)

where the round brackets denote symmetrisation of indices. If the projection operator is the identity, then the equation is automatically satisfied, since \(\Gamma^\mu C\) or \(\Gamma_{\mu\nu\rho} C\) is antisymmetric.

We now look at the situation for which \(\mathcal{P}\) is not the identity. We shall use a basis of the Clifford algebra for which \(C\) anticommutes or commutes with \(\Gamma_\mu\) for \(\mu\) timelike or spacelike respectively. We shall consider signature \((10, 2)\).

Firstly, if we choose \(\mathcal{P}\) to be the Weyl projector, \(\mathcal{P} = \frac{1}{2}(1 + \Gamma_1 \ldots \Gamma_{12})\), then the identity (B.7) is clearly satisfied, since one of the two terms in the expansion is antisymmetric in the spinor indices.

We shall now check the identity for another projector which is of use to us: \(\mathcal{P}_{11} = \frac{1}{2}(1 + X)\), where \(X = \Gamma_{\mu_1} \ldots \Gamma_{\mu_{11}}\), for which exactly one of the \(\mu_1, \ldots, \mu_{11}\) is a timelike index. We shall use \(C_+\) as the charge conjugation matrix, so that \(\Gamma^\mu C\) is antisymmetric in the spinor indices.

We shall now check the identity for another projector which is of use to us: \(\mathcal{P}_{11} = \frac{1}{2}(1 + X)\), where \(X = \Gamma_{\mu_1} \ldots \Gamma_{\mu_{11}}\), for which exactly one of the \(\mu_1, \ldots, \mu_{11}\) is a timelike index. We shall use \(C_+\) as the charge conjugation matrix, so that \(\Gamma^\mu C\) is antisymmetric in the spinor indices.

The identity (B.7) becomes

\[
(\Gamma_a C \mathcal{P})_{(\alpha\beta} (\Gamma_{a\nu\rho} \mathcal{P})_{\gamma\delta)} + (\Gamma_{12} C \mathcal{P})_{(\alpha\beta} (\Gamma_{12\nu\rho} \mathcal{P})_{\gamma\delta)} = 0,
\] (B.8)

where \(a\) takes values from 1 to 11. We now note that \((\Gamma_{12} C X)\) and \((\Gamma_{abc} C X)\) are antisymmetric in the spinor indices, whereas \((\Gamma_{12\nu\rho} C X)\) and \((\Gamma_a C X)\) are symmetric in the spinor indices, thus the identity (B.7) reduces to

\[
(\Gamma_a C X)_{(\alpha\beta} (\Gamma_{a12\nu\rho} \mathcal{P})_{\gamma\delta)} = 0,
\] (B.9)
where $a, b$ run over $1, \ldots, 11$.

Substituting in the particular representation (5.1) of the gamma matrices, we see that this is equation is precisely the matching condition equation (2.35), for a 2-brane moving in eleven dimensional Minkowski spacetime, for which we know that the equation is satisfied. Thus we are done.

A similar procedure applies to case where the projector is $P = \frac{1}{2}(1 + \Gamma_0 \ldots \Gamma_9)$.

References

[1] For recent reviews see:

‘Supermembranes’, M. Duff, hep-th/9611203;

‘$M$-theory (the theory formerly known as strings)’, M. Duff, hep-th/9608117;

‘Lectures on superstring and $M$-theory dualities’, John Schwarz, hep-th/9607201.

[2] ‘Five-branes and $M$-theory on an orbifold’, E. Witten, Nucl. Phys. B463 (1996) 383.

[3] ‘Superstrings in $D = 10$ from supermembranes in $D = 11$’, M. Duff, P. Howe, T. Inami and K. Stelle, Phys. Lett. 191B (1987) 70.

[4] ‘Heterotic and type I string dynamics from eleven-dimensions’, P. Horava and E. Witten, Nucl. Phys. B460 (1996) 506.

[5] ‘Evidence for $F$-Theory’, C. Vafa, Nucl. Phys. B469 (1996) 403.

[6] ‘Orbifold and orientifold compactifications of $F$-theory and $M$-theory to six-dimensions and four-dimensions’, R. Gopakumar and S. Mukhi, Nucl. Phys B479 (1996) 260;

‘Compactifications of $F$-Theory on Calabi–Yau Threefolds – $I$', D. Morrison and C. Vafa, Nucl. Phys. B476 (1996) 195;

‘$F$-theory and orientifolds’, A. Sen, Nucl. Phys. B475 (1996) 562;

‘Low-Energy Analysis of $M$ and $F$ Theories on Calabi-Yau Threefolds’, S. Ferrara, R. Minasian and A. Sagnotti, Nucl. Phys. B474 (1996) 323.

‘Compactifications of $F$-Theory on Calabi–Yau Threefolds – $II$', D. Morrison and C. Vafa, Nucl. Phys. B473 (1996) 74;

‘Phase Transitions In $M$-Theory And $F$-Theory’, E. Witten, Nucl. Phys. B471 (1996) 195;

‘Generalized Green-Schwarz mechanism in $F$-theory’, V. Sadov, Phys. Lett. 388B (1996) 45;
‘F-theory from Dirichlet three-branes’, D. Jatkar and R. Kalyana, Phys. Lett. 388B (1996) 283;
‘An Orientifold from F-theory’, J. Blum and A. Zaffaroni, Phys. Lett. 387B (1996) 71;
‘Probing F-theory with branes’, T. Banks, M. Douglas and N. Seiberg, Phys. Lett. 387B (1996) 278;
‘F-theory on Calabi-Yau fourfolds’, I. Brunner and R. Schimmrigk, Phys. Lett. 387B (1996) 750;
‘F-theory at constant coupling’, K. Dasgupta and S. Mukhi, Phys. Lett. 385B (1996) 125;
‘New branches of string compactifications and their F-theory duals’, G. Aldazabal, A. Font, L. Ibanez and A. Uranga, [hep-th/9607121];
‘A Note on orientifolds and F-theory’, A. Dabholkar and J. Park, [hep-th/9607041];
‘From N=2 strings to F and M-theory’, S. Ketov, [hep-th/9606142];
‘F-theory duals of nonperturbative heterotic E(8) x E(8) vacua in six-dimensions’, P. Candelas, E. Perevalov and G. Rajesh, [hep-th/9606133];
‘Gluino condensation in F-theory’, M. Li, [hep-th/9606091];
‘F-theory orientifolds, M-theory orientifolds, and twisted strings’, J. Blum hepth/9608053;
‘Orientifold and F-theory duals of the CHL strings’, J. Park, hepth/9611119;
‘Unification of M-theory and F theory Calabi-Yau fourfold vacua’, I. Brunner, M. Lynker, R. Schimmrigk, hepth/9610195;
‘Colliding singularities in F-theory and phase transitions’, M. Bershadsky, A. Johansen, hepth/9610111;
‘On the duality between the heterotic string and F-theory in eight dimensions’, G. L. Cardoso, G. Curio, D. Lust, T. Mohaupt, hepth/9609111;
‘A comment on BPS states in F-theory in eight dimensions’, A. Johansen, hepth/9608186;
‘String-like description of gravity and possible applications for F theory’, I. Bandos, hepth/9608093;

[7] ‘Supermembranes and the Signature of Spacetime’, M. Blencowe and M. Duff, Nucl. Phys. B310 (1988) 387.

[8] K. Stelle, private communication.
[9] ‘String dynamics at strong coupling’, C. Hull, Nucl. Phys. B468 (1996) 113.

[10] L. Castellani, P. Fré, F. Giani, P. Van Nieuwenhuizen, Phys. Rev. D26 (1982) 1481;
‘Supergravity theory in twelve dimensions’, M. Wang pub. Rome 1985, Proceedings, General Relativity, Pt. B*, 1459.

[11] ‘Twelve dimensional aspects of four dimensional N=1 type vacua’, M. Bianchi, S. Ferrara, G. Pradisi, A. Sagnotti, Ya.S. Stanev, Phys. Lett. 387B (1996) 64;
‘Supersymmetric Yang-Mills equations in (10+2)-dimensions’, H. Nishino, E. Sezgin, [hep-th/9607185];
‘Black hole entropy reveals a twelfth dimension’, I. Bars, hepth/9610074
‘The target space geometry of N=(2,1) string theory’, A. Lawrence, hep/9605223
‘Self-duality of Born-Infeld action and Dirichlet 3-brane of type IIB superstring theory’, A. Tseytlin, Nucl. Phys. B469 (1996) 51.

[12] ‘S theory’, I. Bars, [hep-th/9607112] (To appear in Phys. Rev. D);
‘Supersymmetry, p-brane duality and hidden space-time dimensions’, I. Bars, Phys. Rev. D54 (1996) 5203;

[13] ‘New principles for string / membrane unification’, D. Kutasov and E. Martinec, Nucl. Phys. B477 (1996) 652;

[14] ‘Vacua of M-theory and N=2 strings’, D. Kutasov, E. Martinec, and M. O’Loughlin, Nucl. Phys. B477 (1996) 675.

[15] ‘An introduction to simple supergravity and the Kaluza-Klein program’, P. Van Nieuwenhuizen, Les Houches lectures 1983, pub. North Holland.

[16] ‘Supersymmetry and the Division Algebras’, T. Kugo and P. Townsend, Nucl. Phys. B221 (1983) 357.

[17] ‘Supersymmetries and their representations’, W. Nahm, Nucl. Phys. B135 (1978).

[18] ‘N=1 Supersymmetry Algebras in d = 2, 3, 4 mod 8’, J. Van Holten and A. Van Proeyen, J. Phys A: Math. Gen. (1982) 3763.

[19] ‘p-brane democracy’, P. Townsend, [hep-th/9507048];

[20] ‘Supermanifolds-second edition’, B De Witt, pub. Cambridge University Press, 1992.
[21] ‘Three Lectures on Supermembranes’, P. Townsend, Conference Proceedings of Trieste, 1988.

[22] P. Howe and R. Tucker, J. Phys A10 (1977) L155, 869; J. Math. Phys 19 (1978) 981.

[23] ‘Hidden Local Supersymmetry in the Superparticle Action’, W. Siegel, Phys. Lett. 128B (1983) 397.

[24] ‘Super p-branes’, A. Achucarro, J. Evans, P. Townsend, and D. Wiltshire, Phys. Lett. 198B (1987) 441.

[25] ‘Simple Supersymmetries’, P. Freund and I. Kaplansky, J.M.P. Vol 17.2 (1976) 228.

[26] ‘The SO(8) supergravity’, E. Cremer and B. Julia, Nucl. Phys. B159 (1979) 141.