Singular measures of circle homeomorphisms with two breakpoints\textsuperscript{1}

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Abstract

Let $T_f$ be a circle homeomorphism with two break points $a_b, c_b$ and irrational rotation number $\rho_f$. Suppose that the derivative $Df$ of its lift $f$ is absolutely continuous on every connected interval of the set $S^1\setminus\{a_b, c_b\}$, that $D\log Df \in L^1$ and the product of the jump ratios of $Df$ at the break points is nontrivial, i.e. $\frac{DF_{a_b}(a_b)}{DF_{a_b}(c_b)} \neq 1$. We prove that the unique $T_f$-invariant probability measure $\mu_f$ is then singular with respect to Lebesgue measure $l$ on $S^1$.

1 Introduction

Circle homeomorphisms constitute one important class of one-dimensional dynamical systems. The investigation of their properties was initiated by Poincaré [20], who came across them in his studies of differential equations more than a century ago. Since then interest in these maps never diminished. Circle maps are also important because of their applications to natural sciences (see for instance [8]). Let $T_f$ be an orientation preserving circle homeomorphism with lift $f: \mathbb{R} \to \mathbb{R}$, $f$ continuous, strictly increasing and $f(x+1) = f(x) + 1$, $x \in \mathbb{R}$. We identify the unit circle $S^1 = \mathbb{R}/\mathbb{Z}$ with the half open interval $[0,1)$. The circle homeomorphism $T_f$ is then defined by $T_f x = f(x) \mod 1$, $x \in S^1$. An important conjugacy invariant characteristic of orientation preserving homeomorphisms is the rotation number $\rho(f)$. If $T_f$ is a circle homeomorphism with lift $f$, then the rotation number $\varrho = \rho(f)$ is defined by

$$\rho(f) = \left( \lim_{n \to \infty} \frac{f^n(x)}{n} \right) \mod 1,$$

with $f^n$ the $n$-th iterate of $f$. This limit exists and is independent of the choice of the lift and the point $x \in \mathbb{R}$. If $\varrho$ is irrational, then for sufficiently smooth diffeomorphisms the trajectory of an arbitrary point is dense on the circle, and the diffeomorphism itself can be reduced to the pure rotation $T_{\varrho} x = (x + \varrho) \mod 1$ by an angle $\varrho$ through a change of coordinates. This result was proved by Denjoy [2]. More precisely, Denjoy proved that if $f \in C^1(R^1)$ and $\text{var}(\log Df) < \infty$, then there exists a circle homeomorphism $T_{\varrho}$ such that

$$T_f \circ T_{\varrho} = T_{\varrho} \circ T_f.$$

It is a well known fact that a circle homeomorphism $T_f$ with irrational rotation number $\varrho$ is strictly ergodic i.e. admits an unique $T_f$-invariant probability measure $\mu_f$. Note, that the conjugating map $T_{\varrho}$ and the invariant measure $\mu_f$ are related by $T_{\varrho} x = \mu_f([0,x])$ (see [7]). This last relation implies that regularity properties of the conjugating map $T_{\varrho}$ are

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closely related to the existence of an absolutely continuous invariant measure \( \mu_f \) with a regular density.

The problem of smoothness of the conjugacy of smooth diffeomorphisms is now very well understood (see for instance [1, 19, 10, 14, 15, 23]).

An important result is the one by M. Herman [10]:

**Theorem 1.1.** If \( T_f \) is a \( C^2 \)-diffeomorphism with rotation number \( \rho = \rho(f) \) of bounded type (that means the entries in the continued fraction expansion of \( \rho \) are bounded) and \( T_f \) is close to \( T_0 \) then \( \mu_f \) is absolutely continuous with respect to Lebesgue measure.

Katznelson-Ornstein [14] and Khanin-Sinai [15] gave new proofs and an improved global version of this theorem in showing that it is not necessary to assume that \( T_f \) is close to \( T_0 \):

**Theorem 1.2.** (Katznelson-Ornstein). Let \( T_f \) be an orientation preserving \( C^1 \)-circle diffeomorphism. If \( f \) is absolutely continuous, \( D(\log Df) \in L^p \) for some \( p > 1 \) and the rotation number \( \rho = \rho(f) \) is of bounded type, then the invariant measure \( \mu(f) \) is absolutely continuous with respect to Lebesgue measure.

The result proved by Khanin and Sinai in [15] is the following:

**Theorem 1.3.** (Khanin-Sinai). Let \( T_f \) be a \( C^{2+\varepsilon} \) circle diffeomorphism with \( \varepsilon > 0 \), and let the rotation number \( \rho = \rho(f) \) be a Diophantine number with exponent \( \delta \in (0, \varepsilon) \), i.e., there is a constant \( c(\rho) \) such that

\[
|\rho - \frac{p}{q}| \geq \frac{c(\rho)}{q^{2+\delta}} \text{ for any } \frac{p}{q} \in \mathbb{Q}.
\]

Then the conjugating map \( T_{\varphi} \) belongs to \( C^{1+\varepsilon-\delta} \).

Note, that the condition \( T_f \in C^{2+\varepsilon} \) is sharp, because there is a set of full Lebesgue measure in [0, 1] such that for any rotation number in this set there are \( C^2 \)-diffeomorphisms for which the conjugating map \( T_{\varphi} \) is singular [12].

An important and interesting class of circle homeomorphisms are homeomorphisms with singularities. The simplest among them are critical circle homeomorphisms and homeomorphisms with break points. We call this latter class \( P \)-homeomorphisms. In general their ergodic properties like the invariant measures , their renormalization and also their rigidity properties are different from the properties of diffeomorphisms (see [4] chapter I and IV, [10] chapter VI, [16], [3]).

The invariant measures of critical circle homeomorphisms, that means \( C^3 \)-smooth circle homeomorphisms with a finite number of critical points of polynomial type have been studied in [9]:

**Theorem 1.4.** (Graczyk-Swiatek). Critical circle homeomorphisms with irrational rotation number have an invariant measure singular with respect to Lebesgue measure.

The class of \( P \)-homeomorphisms consists of orientation preserving circle homeomorphisms \( T_f \) which are differentiable away from countable many points, the so called break points, at which left and right derivatives, denoted respectively by \( Df_- \) and \( Df_+ \), exist such that
i) there exist constants \(0 < c_1 < c_2 < \infty\) with \(c_1 < Df(x) < c_2\) for all \(x \in S^1 \setminus BP(f)\), 
\(c_1 < Df_-(x_b) < c_2\) and \(c_1 < Df_+(x_b) < c_2\) for all \(x_b \in BP(f)\), the set of break points of \(f\);

ii) \(logDf\) has bounded variation.

In this case \(logDf, logDf_-, logDf_+\) and \(logDf^{-1}, logDf_{-1}, logDf_{+1}\) all have the same total variation denoted by \(v = Var(logDf)\).

The ratio \(\sigma_f(x_b) = \frac{Df_-(x_b)}{Df_+(x_b)}\) is called the jump ratio of \(T_f\) in \(x_b\).

Piecewise linear (PL) orientation preserving circle homeomorphisms with piecewise constant derivatives are the simplest examples of class \(P\)-homeomorphisms. They occur in many other areas of mathematics such as group theory, homotopy theory and logic via the Thompson groups (see [21]. PL-homeomorphisms were considered first by Herman in [10] as examples of homeomorphisms of arbitrary irrational rotation number which admit no invariant \(\sigma\)–finite measure equivalent to Lebesgue measure.

**Theorem 1.5.** (Herman). A PL-circle homeomorphisms with two break points and irrational rotation number has an invariant measure absolutely continuous with respect to Lebesgue measure if and only if its break points belong to the same orbit.

General (non PL) class \(P\)-homeomorphisms with one break point have been studied by Dzhalilov and Khanin in [5]. The character of their results for such circle maps is quite different from the one for \(C^{2+\varepsilon}\) diffeomorphisms. The main result of [5] is the following:

**Theorem 1.6.**. Let \(T_f\) be a class \(P\)-homeomorphism with one break point \(c_0\). If the rotation number \(\rho_f\) is irrational and \(T_f \in C^{2+\varepsilon}(S\setminus\{c_0\})\) for some \(\varepsilon > 0\), then the \(T_f\)-invariant probability measure \(\mu_f\) is singular with respect to Lebesgue measure \(l\) on \(S^1\), i.e. there exists a measurable subset \(A \subset S^1\) such that \(\mu_f(A) = 1\) and \(l(A) = 0\).

I. Liousse proved in [18] the same result for ”generic” PL-homeomorphisms with irrational rotation number of bounded type. In a next step Dzhalilov and I. Liousse studied in [6] circle homeomorphisms with two break points. Their result is the following:

**Theorem 1.7.**. Let \(T_f\) be a class \(P\)-homeomorphism satisfying the following conditions:

i) \(T_f\) has irrational rotation number \(\rho_f\) of bounded type;

ii) there exists constants \(k_i > 0\) such that \(|Df(x) - Df(y)| \leq k_i|x-y|\) on every continuity interval of \(Df\);

iii) \(T_f\) has two break points not on the same orbit of \(T_f\).

Then the \(T_f\)-invariant probability measure \(\mu_f\) is singular with respect to Lebesgue measure.

In the present paper we continue our study of invariant measures for circle homeomorphisms \(T_f\) with two break points and arbitrary irrational rotation number \(\rho_f\). The main result of our paper is the following:

**Theorem 1.8.**. Let \(T_f\) be a class \(P\)-homeomorphism satisfying the following conditions:

(a) the rotation number \(\rho = \rho_f\) of \(T_f\) is irrational;
(b) \( T_f \) has two break points \( a_b, c_b \) and the product of the jump ratios of \( Df \) at the break points is nontrivial i.e. \( \sigma_f(a_b) \cdot \sigma_f(c_b) \neq 1 \).

(c) \( Df(x) \) is absolutely continuous on every connected interval of \( S^1 \setminus \{a_b, c_b\} \) and the second derivative \( D^2 f \in L^1 \);

Then the \( T_f \)-invariant probability measure \( \mu_f \) is singular with respect to Lebesgue measure.

**Remark 1.9.** Obviously condition (c) is weaker than a Lipschitz condition for \( Df \). In the case when \( T_f \) has two break points on the same orbit our Theorem 1.8 gives a new proof of the result in [5], but with a weaker condition than \( C^{2+\varepsilon} \) on \( T_f \).

A direct consequence of our Theorem 1.8 is

**Theorem 1.10.** Let \( T_f \) be a circle homeomorphism satisfying condition (c) of Theorem 1.8 and the conditions

(a') the rotation number \( \rho_f \) of \( T_f \) is irrational of bounded type;

(b') \( T_f \) has two break points \( a_b, c_b \) with disjoint orbits and \( \sigma(a_b)\sigma(a_b) = 1 \).

Then the \( T_f \)-invariant measure \( \mu_f \) is singular with respect to Lebesgue measure.

## 2 Preliminaries and Notations

Let \( T_f \) be an orientation preserving homeomorphism of the circle with lift \( f \) and irrational rotation number \( \rho = \rho_f \). We take an arbitrary point \( x_0 \in S^1 \) and consider the trajectory of this point under the action of \( T_f \), i.e., the set of points \( \{x_i = T_f^i x_0, i \in \mathbb{Z}\} \). According to a classical theorem of Poincaré (see [7]), the order of the points along the trajectory is the same as in the case of the linear rotation \( T_\rho \) of the circle, i.e. for the sequence \( \{\mathcal{I}_i = \{x_0 + i \rho\}, mod 1, i \in \mathbb{Z}\} \). This important property allows one to define a sequence of natural partitions of the circle closely related to the continued fraction expansion of the number \( \rho \).

We denote by \( \{k_n, n \in \mathbb{N}\} \) the sequence of entries in the continued fraction expansion of \( \rho \), so that \( \rho = [k_1, k_2, ..., k_n, ...] = (1/k_1 + (1/(k_2 + ... + 1/k_n + ...))) \). For \( n \in \mathbb{N} \) we denote by \( p_n/q_n = [k_1, k_2, ..., k_n] \) the convergents of \( \rho \). Their denominators \( q_n \) satisfy the recursion relation \( q_{n+1} = k_{n+1} q_n + q_{n-1}, n \geq 1, q_0 = 1, q_1 = k_1 \).

For an arbitrary point \( x_0 \in S^1 \) denote by \( \Delta_i^{(n)}(x_0) \) the closed interval with endpoints \( x_0 \) and \( x_{q_n} \). For \( n \) odd \( x_{q_n} \) is to the left of \( x_0 \), for \( n \) even it is to the right. Denote by \( \Delta_i^{(n)}(x_0) \) the iterates of the interval \( \Delta_0^{(n)}(x_0) \) under \( T_f \): \( \Delta_i^{(n)}(x_0) = T_f^i \Delta_0^{(n)}(x_0), i \geq 1 \).

It is well known (since Denjoy) that the system of intervals

\[
\xi_n(x_0) = \left\{ \Delta_i^{(n-1)}(x_0), 0 \leq i < q_n; \Delta_j^{(n)}(x_0), 0 \leq j < q_{n-1} \right\}
\]

cover the whole circle and that their interiors are mutually disjoint. The partition \( \xi_n(x_0) \) is called the \( n \)-th dynamical partition of the point \( x_0 \) with generators \( \Delta_0^{(n-1)}(x_0) \) and \( \Delta_0^{(n)}(x_0) \). We briefly recall the structure of the dynamical partitions. The passage from \( \xi_n(x_0) \) to \( \xi_{n+1}(x_0) \) is simple: all intervals of order \( n \) are preserved and each of the intervals \( \Delta_i^{(n-1)}(x_0), 0 \leq i < q_n \), is divided into \( k_{n+1} + 1 \) intervals:

\[
\Delta_i^{(n-1)}(x_0) = \Delta_i^{(n+1)}(x_0) \cup \bigcup_{s=0}^{k_{n+1}-1} \Delta_i^{(n)}(x_0),
\]
The following Lemma plays a key role for studying metrical properties of the homeomorphism $T_f$.

**Lemma 2.1.** Let $T_f$ be a $P$-circle homeomorphism with a finite number of break points $z^{(i)}$, $i = 1, 2, ..., m$ and irrational rotation number $\rho_f$. If $x_0 \in S^1$, $n \geq 1$ and $z^{(i)} \notin \{T^i x_0, 0 \leq i < q_n, \}$ then

$$e^{-v} \leq \prod_{i=0}^{q_n-1} DT^i_f(x_0) \leq e^v,$$

where $v = \text{Var}(\log Df)$.

Inequality (3) is called the Denjoy inequality. The proof of Lemma 2.1 is just like in the case of diffeomorphisms (see for instance [15]). Using Lemma 2.1 it can be shown easily that the lengths of the intervals of the dynamical partition $\xi_n$ in (2) are exponentially small:

**Corollary 2.2.** Let $\Delta^{(n)}$ be an arbitrary element of the dynamical partition $\xi_n(x_0)$. Then

$$l(\Delta^{(n)}) \leq \text{const } \lambda^n,$$

where $\lambda = (1 + e^{-v})^{-1/2} < 1$.

**Definition 2.3.** Two homeomorphisms $T_1$ and $T_2$ of the circle are said to be topologically equivalent if there exists a homeomorphism $T_\varphi : S^1 \to S^1$ such that $T_\varphi(T_1(x)) = T_2(T_\varphi(x))$ for any $x \in S^1$. We call the homeomorphism $T_\varphi$ a conjugating map.

From Corollary 2.2 it follows that the trajectory of each point is dense in $S^1$. This together with the monotonicity of the homeomorphism $T_f$ implies the following generalization of the classical Denjoy theorem.

**Theorem 2.4.** Suppose that a homeomorphism $T_f$ with an irrational rotation number $\rho$ satisfies the conditions of Lemma 2.1. Then $T_f$ is topologically equivalent to the linear rotation $T_\rho$.

**Definition 2.5.** (see [14]) An interval $I = (\tau, t) \subset S^1$ is $q_n$-small and its endpoints $\tau, t$ are $q_n$-close if the system of intervals $T^i_f(I)$, $0 \leq i < q_n$ are disjoint.

It is known that the interval $(\tau, t)$ is $q_n$-small if, depending on the parity of $n$, either $t \leq \tau \leq T^{q_n-1}_f(t)$ or $T^{q_n-1}_f(\tau) \leq t \leq \tau$.

**Definition 2.6.** Let $C > 1$. We call two intervals of $S^1$ $C$-comparable if the ratio of their lengths is in $[C^{-1}, C]$.

Lemma 2.1 then implies

**Corollary 2.7.** Suppose that a homeomorphism $T_f$ with an irrational rotation number $\rho$ satisfies the conditions of Lemma 2.1. Then for any interval $I \subset S^1$ the intervals $I$ and $T^{q_n}_f I$ are $e^v$-comparable. If the interval $I$ is $q_n$-small then $l(T^i_f I) < \text{const } \lambda^n$ for all $i = 1, 2, ..., q_n - 1$. 

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Lemma 2.8. Suppose, that a homeomorphism $T_f$ with an irrational rotation number $\rho$ satisfies the conditions of Lemma 2.1 and $x, y \in S^1$ are $q_n$-close. Then for any $0 \leq l \leq q_n$ the following inequality holds:

\begin{equation}
(5) \quad e^{-v} \leq \frac{Df^l(x)}{Df^l(y)} \leq e^v.
\end{equation}

Proof. Take any two $q_n$-close points $x, y \in S^1$ and $0 \leq l \leq q_n - 1$. Denote by $I$ the open interval with endpoints $x$ and $y$. Because the intervals $T_f^i(I)$, $0 \leq i < q_n$ are disjoint, we obtain

$$|\log Df^{q_n}(x) - \log Df^{q_n}(y)| \leq \sum_{s=0}^{q_n-1} |\log f(T_f^s x) - \log f(T_f^s y)| \leq v,$$

from which inequality (5) follows immediately.

Note that P-homeomorphisms $T_f$ satisfying the conditions of Lemma 2.1 are ergodic with respect to Lebesgue measure, i.e. every $T_f$-invariant set has measure zero or one.

3 Cross-ratio tools

Definition 3.1. The cross-ratio $Cr(z_1, z_2, z_3, z_4)$ of four points $z_i \in \mathbb{R}^1$, $i = 1, 2, 3, 4$, $z_1 < z_2 < z_3 < z_4$ is defined as

$$Cr(z_1, z_2, z_3, z_4) = \frac{(z_2 - z_1)(z_4 - z_3)}{(z_3 - z_1)(z_4 - z_2)}.$$

Definition 3.2. The cross-ratio distortion $Dist(z_1, z_2, z_3, z_4; f)$ of four points $z_i \in \mathbb{R}^1$, $i = 1, 2, 3, 4$, $z_1 < z_2 < z_3 < z_4$ with respect to a strictly increasing function $f$ on $\mathbb{R}$ is defined as

$$Dist(z_1, z_2, z_3, z_4; f) = \frac{Cr(f(z_1), f(z_2), f(z_3), f(z_4))}{Cr(z_1, z_2, z_3, z_4)}.$$

Consider then a function $f : [a, b] \to \mathbb{R}^1$, $[a, b] \subset S^1$ satisfying the following conditions:

(i) $f \in C^1([a, b])$, $Df(x) \geq \text{const} > 0$, $\forall x \in [a, b]$;

(ii) $D^2 f \in L^1([a, b])$.

Fix an arbitrary $\varepsilon > 0$. Since $D^2 f \in L^1([a, b])$, it can be written in the form

\begin{equation}
(6) \quad D^2 f(x) = g_\varepsilon(x) + \theta_\varepsilon(x), \quad x \in [a, b],
\end{equation}

where $g_\varepsilon$ is a continuous function on $[a, b]$ and $\|\theta_\varepsilon\|_{L^1} < \varepsilon$.

Theorem 3.3. Suppose, the function $f = f(x)$ satisfies the above conditions i), ii). For $z_i \in [a, b], \quad i = 1, 2, 3, 4$, with $z_1 < z_2 < z_3 < z_4$, the following estimate holds:

\begin{align*}
|Dist(z_1, z_2, z_3, z_4; f) - 1| & \leq C_1 |z_4 - z_1| \max_{x, t \in [a, b]} |g_\varepsilon(x) - g_\varepsilon(t)| + \\
& + \int_{z_1}^{z_2} |\theta_\varepsilon(y)|dy + C_1 \left( \int_{z_1}^{z_2} |D^2 f(y)|dy \right)^2
\end{align*}

where the constant $C_1 > 0$ depends only on the function $f$.
Proof. Take $z_i \in [a, b] \subset S^1$, $i = 1, 2, 3, 4$, with $z_1 < z_2 < z_3 < z_4$. The following equalities are easy to check:

\[
f(z_k) - f(z_1) = Df(z_1)(z_k - z_1) + \int_{z_1}^{z_k} D^2 f(y)(z_k - y)dy, \quad k = 2, 3;
\]

\[
f(z_4) - f(z_l) = Df(z_l)(z_4 - z_l) - \int_{z_l}^{z_4} D^2 f(y)(y - z_l)dy, \quad l = 2, 3.
\]

Using these relations we obtain:

\[
Cr(f(z_1), f(z_2), f(z_3), f(z_4)) = \frac{f(z_2) - f(z_1)}{f(z_3) - f(z_1)} \cdot \frac{f(z_4) - f(z_3)}{f(z_4) - f(z_2)} = \\
1 + \frac{1}{Df(z_1)(z_2 - z_1)} \int_{z_1}^{z_2} D^2 f(y)(z_2 - y)dy \times \\
1 + \frac{1}{Df(z_1)(z_3 - z_1)} \int_{z_1}^{z_3} D^2 f(y)(z_3 - y)dy \\
\times \\
1 - \frac{1}{Df(z_4)(z_1 - z_2)} \int_{z_2}^{z_4} D^2 f(y)(y - z_2)dy.
\]

Setting

\[
A(a, b) = \frac{b}{Df([a, b])} \int_a^b D^2 f(y)(b - y)dy, \quad \text{and} \quad B(a, b) = \frac{b}{Df([a, b])} \int_a^b D^2 f(y)(y - a)dy
\]

we can hence rewrite \(\text{Dist}(z_1, z_2, z_3, z_4; f)\) in the following form:

\[
\text{Dist}(z_1, z_2, z_3, z_4; f) = \frac{1 + A(z_1, z_2)}{1 + A(z_1, z_3)} \times \frac{1 - B(z_3, z_4)}{1 - B(z_2, z_4)} = \\
(1 + A(z_1, z_2)) \cdot (1 - A(z_1, z_3)) + O(A^2(z_1, z_3)) \cdot (1 - B(z_3, z_4)) \times \\
\times (1 + B(z_2, z_4) + O(B^2(z_2, z_4))).
\]

Therefore

\[
\text{Dist}(z_1, z_2, z_3, z_4; f) = 1 + A(z_1, z_2) - A(z_1, z_3) + B(z_2, z_4) - B(z_3, z_4) + \\
+ O \left( \int_{z_2}^{z_3} D^2 f(y)(y) \right)^2.
\]

Set \(M_1 = 0.5 \left( \inf_{x \in [z_1, z_4]} Df(x) \right)^{-1}.\)

To continue the proof of Theorem 3.3 we need the following

Lemma 3.4. Assume, that the function \(f\) satisfies the conditions of Theorem 3.3. Then for any \(a, b \in S^1\), \(a < b\) the following identities hold:

\[
A(a, b) = \int_a^b \frac{D^2 f(y)}{2 Df(y)} dy + G_1(a, b), \quad B(a, b) = \int_a^b \frac{D^2 f(y)}{2 Df(y)} dy + G_2(a, b),
\]
where

\[ |G_1(a,b)| \leq M_1(b - a) \max_{x,t \in [a,b]} |g_\varepsilon(x) - g_\varepsilon(t)| + \]

\[ + M_1 \int_a^b |\theta_\varepsilon(y)| dy + 2M_1^2 \left( \int_a^b |D^2 f(y)| dy \right)^2, \quad i = 1, 2. \]

(9)

**Proof.** We prove only the identity for \( A(a,b) \), the one for \( B(a,b) \) is similar. Set

\[ G_1(a,b) = A(a,b) - \int_a^b \frac{D^2 f(y)}{2 D f(y)} dy. \]

It is clear that

\[ |G_1(a,b)| \leq |A(a,b) - \int_a^b \frac{D^2 f(y)}{2 D f(a)} dy| + \int_a^b \frac{D^2 f(y)}{2 D f(a)} dy - \int_a^b \frac{D^2 f(y)}{2 D f(y)} dy = \]

\[ = |A(a,b) - \int_a^b \frac{D^2 f(y)}{2 D f(a)} dy| + \frac{1}{2} \left| \int_a^b \frac{D^2 f(y)}{D f(y) D f(a)} dy \int y D^2 f(t) dt \right|. \]

using this and the bound \((D f(x))^{-1} \leq 2 M_1\) we get :

(10)

\[ |G_1(a,b)| \leq |A(a,b) - \int_a^b \frac{D^2 f(y)}{2 D f(a)} dy| + 2M_1^2 \left( \int_a^b \frac{|D^2 f(y)|^2 dy}{} \right). \]

To get finally the estimate (9) for \( G_1(a,b) \) it is sufficient to estimate the difference

\[ A(a,b) - \int_a^b \frac{D^2 f(y)}{2 D f(a)} dy. \]

Using the definition of \( A(a,b) \) and the decomposition (6) we obtain:

\[ \left| A(a,b) - \int_a^b \frac{D^2 f(y)}{2 D f(a)} dy \right| = \left| \frac{1}{D f(a)} \int_a^b \frac{D^2 f(y)}{D f(y)} \left( \frac{b - y}{b - a} - \frac{1}{2} \right) dy \right| = \]

\[ = \frac{1}{D f(a)} \int_a^b \left( g_\varepsilon(y) + \theta_\varepsilon(y) \right) \left( \frac{b - y}{b - a} - \frac{1}{2} \right) dy \leq \frac{1}{D f(a)} \int_a^b \left( \frac{b - y}{b - a} - \frac{1}{2} \right) dy + \]

\[ + \frac{1}{D f(a)} \int_a^b \left| g_\varepsilon(y) - g_\varepsilon(a) \right| \left( \frac{b - y}{b - a} - \frac{1}{2} \right) dy \leq \frac{1}{D f(a)} \int_a^b \left| \theta_\varepsilon(y) \right| \left( \frac{b - y}{b - a} - \frac{1}{2} \right) dy \leq \]

\[ \leq M_1(b - a) \max_{x,t \in [a,b]} |g_\varepsilon(x) - g_\varepsilon(t)| + M_1 \int_a^b \left| \theta_\varepsilon(y) \right| dy. \]

Combining this with the estimate (10) we obtain the estimate (9) for \( G_1(a,b) \) in the Lemma.
We can now finish the proof of Theorem 3.3. Combining (8) with the representations of \( A(a, b) \) and \( B(a, b) \) in Lemma 3.4 we obtain:

\[
\begin{aligned}
\text{Dist}(z_1, z_2, z_3, z_4; f) &= 1 + \int_{z_1}^{z_2} \frac{D^2 f(y)}{2Df(y)} dy + G_1(z_1, z_2) - \int_{z_1}^{z_3} \frac{D^2 f(y)}{2Df(y)} dy - G_1(z_1, z_3) \\
&+ \int_{z_2}^{z_4} \frac{D^2 f(y)}{2Df(y)} dy + G_2(z_3, z_4) - \int_{z_3}^{z_4} \frac{D^2 f(y)}{2Df(y)} dy - G_2(z_3, z_4) + O\left( \frac{\int |D^2 f(y)| dy}{z_1} \right)^2 \\
&= 1 + G_1(z_1, z_2) - G_1(z_1, z_3) + G_2(z_2, z_4) - G_2(z_3, z_4) + O\left( \frac{\int |D^2 f(y)| dy}{z_1} \right)^2.
\end{aligned}
\]

Applying next (9) for the intervals \([z_s, z_{s+1}] \in [z_1, z_4], \ s = 1, 2, 3\) we obtain

\[
|G_1(z_s, z_{s+1})| \leq \frac{M_1}{2} |z_4 - z_1| \max_{[z_1, z_4]} |g_\varepsilon(x) - g_\varepsilon(t)| + \frac{M_1}{2} \int_{z_1}^{z_4} |\theta_\varepsilon(y)| dy + \frac{M_2}{2} \int_{z_1}^{z_4} |f''(y)| dy.
\]

from which the assertion of Theorem 3.3 follows immediately. \(\square\)

Next we consider the case when the interval \([z_1, z_4]\) contains just one break point \(x = x_b\). We estimate the distortion of the cross ratio when the break point lies outside the middle interval \([z_2, z_3]\) i.e. \(x_b \in [z_1, z_2] \cup [z_3, z_4]\).

For \(z_i \in S^1, i = 1, 2, 3, 4\) with \(z_1 < z_2 < z_3 < z_4\) and \(x_b \in [z_1, z_2]\) we set

\[\alpha := z_2 - z_1, \beta := z_3 - z_2, \gamma := z_4 - z_3, \tau := z_2 - x_b, \xi := \frac{\beta}{\alpha}, \ z := \frac{\tau}{\alpha}.\]

**Lemma 3.5.** Assume, the function \(f\) is defined on \([z_1, z_4]\), its derivative \(Df\) is continuous on every connected interval of the set \([z_1, z_4]\) \(\{x_b\}\) and \(D^2 f \in L^1([z_1, z_4])\). Choose \(z_i \in S^1, i = 1, 2, 3, 4\), such that \(z_1 < z_2 < z_3 < z_4\) and \(x_b \in [z_1, z_2]\). Then

\[
|\text{Dist}(z_1, z_2, z_3, z_4; f)| \leq K_1 \int_{z_1}^{z_4} |D^2 f(y)| dy,
\]

where the constant \(K_1 > 0\) depends only on the function \(f\).

**Proof.** By assumption \(x_b \in [z_1, z_2]\). Let the jump ratio of \(Df(x)\) at the point \(x_b\) be \(\sigma(x_b) = \frac{Df_- (x_b)}{Df_+ (x_b)}\). Rewrite then \(\text{Dist}(z_1, z_2, z_3, z_4; f)\) in the form:

\[
\begin{aligned}
\text{Dist}(z_1, z_2, z_3, z_4; f) &= \frac{\text{Cr}(f(z_1), f(z_2), f(z_3), f(z_4))}{\text{Cr}(z_1, z_2, z_3, z_4)} \\
&= \left( \frac{f(z_2) - f(z_1)}{z_2 - z_1} \cdot \frac{|f(z_3) - f(z_1)|}{z_3 - z_1} \right) \left( \frac{f(z_4) - f(z_3)}{z_4 - z_3} \cdot \frac{|f(z_4) - f(z_2)|}{z_4 - z_2} \right).
\end{aligned}
\]
It is easy to check that
\[
f(z_2) - f(z_1) = (f(x_b) - f(z_1)) + (f(z_2) - f(x_b)) =
\]
\[
= \left\{ Df_-(x_b)(x_b - z_1) - \int_{x_1}^{x_b} D^2 f(y)(y - z_1) dy \right\} + \left\{ Df_+(x_b)(z_2 - x_b) + \int_{x_b}^{z_2} D^2 f(y)(z_2 - y) dy \right\} =
\]
\[
= Df_+(x_b)(z_2 - z_1) \left[ \sigma(x_b) + (1 - \sigma(x_b)) \frac{\tau}{\alpha} \right] +
\]
\[
+ Df_+(x_b) \alpha \left\{ \frac{1}{Df_+(x_b)} \int_{x_b}^{z_2} D^2 f(y) \frac{z_2 - y}{z_2 - z_1} dy - \frac{1}{Df_+(x_b)} \int_{z_1}^{x_b} D^2 f(y) \frac{y - z_1}{z_2 - z_1} dy \right\}.
\]
Hence
\[
(13) \quad f(z_2) - f(z_1) = Df_+(x_b)(z_2 - z_1) \left[ \sigma(x_b) + (1 - \sigma(x_b)) \frac{\tau}{\alpha} + r_1(x_b, z_1, z_2) \right];
\]
in analogy we find
\[
f(z_3) - f(z_1) = (f(x_b) - f(z_1)) + (f(z_3) - f(x_b)) =
\]
\[
= \left\{ Df_-(x_b)(x_b - z_1) - \int_{x_1}^{x_b} D^2 f(y)(y - z_1) dy \right\} + \left\{ Df_+(x_b)(z_3 - x_b) + \int_{x_b}^{z_3} D^2 f(y)(z_3 - y) dy \right\} =
\]
\[
= Df_+(x_b)(z_3 - z_1) \left[ \frac{z_3 - x_b}{z_3 - z_1} + \sigma(x_b) \frac{z_1 - x_b}{z_3 - z_1} \right] +
\]
\[
+ Df_+(x_b)(z_3 - z_1) \left\{ \frac{1}{Df_+(x_b)} \int_{x_b}^{z_3} D^2 f(y) \frac{z_3 - y}{z_3 - z_1} dy - \frac{1}{Df_+(x_b)} \int_{z_1}^{x_b} D^2 f(y) \frac{y - z_1}{z_3 - z_1} dy \right\}.
\]
respectively
\[
(14) \quad f(z_3) - f(z_1) = Df_+(x_b)(z_3 - z_1) \left[ \frac{\tau + \beta}{\alpha + \beta} + \sigma(x_b) \frac{\alpha - \tau}{\alpha + \beta} + r_2(x_b, z_1, z_3) \right].
\]
For \( |r_1(x_b, z_1, z_2)| \) and \( |r_2(x_b, z_1, z_3)| \) then the following estimates hold:
\[
(15) \quad |r_1(x_b, z_1, z_2)|, |r_2(x_b, z_1, z_3)| \leq \frac{1}{Df_+(x_b)} \int_{z_1}^{z_3} |D^2 f(y)| dy
\]
Using this, (13) and (14) we get:
\[
\left| \frac{f(z_2) - f(z_1)}{z_2 - z_1} - \frac{f(z_3) - f(z_1)}{z_3 - z_1} \right| \leq \frac{1}{Df_+(x_b)} \int_{z_1}^{z_3} |D^2 f(y)| dy \leq K_2 \int_{z_1}^{z_3} |D^2 f(y)| dy,
\]
with \( \xi \) and \( z \) as defined in (11) and where the constant \( K_2 > 0 \) is depending only on the function \( f \).
Since the interval \([z_2, z_4]\) does not contain the break point \(x_b\), it can easily be shown that
\[
\left| \frac{f(z_4) - f(z_3)}{z_4 - z_3} \cdot \frac{f(z_4) - f(z_2)}{z_4 - z_2} - 1 \right| \leq K_3 \int_{z_2}^{z_4} |D^2 f(y)| dy,
\]
where also the constant \(K_3 > 0\) depends only on \(f\). The last inequality together with the bounds (15) and (16) imply the assertion of Lemma 3.5. \(\square\)

**Remark 3.6.** If the break point \(x = x_b\) belongs to the right interval \([z_3, z_4]\), then one can prove the following estimate:
\[
|\text{Dist}(z_1, z_2, z_3, z_4; f) - \frac{\sigma(x_b) + (1 - \sigma(x_b))\vartheta}{\sigma(x_b) + (1 - \sigma(x_b))\vartheta + \eta}| \leq K_4 \int_{z_1}^{z_4} |D^2 f(y)| dy,
\]
where \(\eta = \frac{z_4 - z_2}{z_4 - z_3}\), \(\vartheta = \frac{x_b - z_4}{z_4 - z_3}\) and the constant \(K_4 > 0\) depends only on the function \(f\).

4 The proofs of Theorem 1.8 and Theorem 1.10

For the proofs of Theorem 1.8 and Theorem 1.10 we need several Lemmas which we formulate next and whose proofs will be given later.

**Lemma 4.1.** Assume that the lift \(\varphi\) of the conjugating homeomorphism \(T_\varphi(x)\) has a positive derivative \(D_\varphi(x_0) = \omega\) at the point \(x = x_0 \in S^1\), and the following conditions hold for \(z_i \in S^1\), \(i = 1, \ldots, 4\) with \(z_1 < z_2 < z_3 < z_4\) and some constant \(R_1 > 1\):

(a) \(R_1^{-1} |z_3 - z_2| \leq |z_2 - z_1| \leq R_1 |z_3 - z_2|\), \(R_1^{-1} |z_3 - z_2| \leq |z_4 - z_3| \leq R_1 |z_3 - z_2|\);

(b) \(\max_{1 \leq i \leq 4} |z_i - x_0| \leq R_1 |z_2 - z_1|\).

Then for any \(\varepsilon > 0\) there exists \(\delta = \delta(\varepsilon) > 0\) such that
\[
|\text{Dist}(z_1, z_2, z_3, z_4; T_\varphi) - 1| \leq C_2 \varepsilon,
\]
if \(z_i \in (x_0 - \delta, x_0 + \delta)\) for all \(i = 1, 2, 3, 4\), where the constant \(C_2 > 0\) depends only on \(R_1\), \(\omega\) and not on \(\varepsilon\).

Suppose that \(D_\varphi(x_0) = \omega\) for some point \(x = x_0, x_0 \in S^1\). Consider its \(n\)–th dynamical partition
\[
\xi_n(x_0) = \left\{ \Delta_i^{(n-1)}(x_0), 0 \leq i < q_n; \Delta_j^{(n)}(x_0), 0 \leq j < q_{n-1} \right\}.
\]

For definiteness suppose, that \(n\) is odd. Then \(\Delta_0^{(n)}(x_0) = [T_f^{q_n} x_0, x_0]\) and \(\Delta_0^{(n-1)}(x_0) = [x_0, T_f^{q_n-1} x_0]\) are its two generators. Denote by \(\overline{a}_b\) and \(\overline{c}_b\) the preimages of \(a_b\) and \(c_b\) in the interval \([T_f^{q_n} x_0, T_f^{q_n-1} x_0]\) such that \(\overline{a}_b = T_f^{-l} a_b\) and \(\overline{c}_b = T_f^{-p} c_b\) for some \(l, p \in [0, q_n]\).

Define next for \(m \in [0, q_n]\)
\[
\xi(m) := \frac{T_m^m z_3 - T_m^m z_2}{T_m^m z_2 - T_m^m z_1}, \quad z(m) := \frac{T_m^m z_2 - T_m^m \overline{a}_b}{T_m^m z_2 - T_m^m z_1},
\]
\[
\eta(m) := \frac{T_m^m z_3 - T_m^m z_2}{T_m^m z_4 - T_m^m z_3}, \quad \vartheta(m) := \frac{T_m^m \overline{c}_b - T_m^m z_3}{T_m^m z_4 - T_m^m z_3}.
\]
The numbers $z(m)$ (if $\xi_b \in [z_1, z_2]$) and $\vartheta(m)$ (if $\xi_b \in [z_3, z_4]$) are called normalized coordinates of the point $T_f \xi_b$. It is clear that the normalized coordinates $z(m)$ (respectively $\vartheta(m)$) change from 0 to 1, when the break point $c_b$ is moving from $T_f^p z_2$ to $T_f^p z_1$ (respectively from $T_f^p z_3$ to $T_f^p z_4$).

**Definition 4.2.** The intervals $\{T_f^j[z_1, z_2], T_f^j[z_2, z_3], T_f^j[z_3, z_4] : 0 \leq j \leq q - 1\}$ cover the break points $a_b$, $c_b$ regularly with constants $C \geq 1$, $\zeta \in [0, 1]$, if

1) the intervals $\{T_f^j[z_1, z_2], 0 \leq j \leq q - 1\}$ cover every break point only once;

2) either $z_2 = T_f^{-l} a_b$ and $T_f^{-p} c_b \in [z_1, z_2]$ or $z_3 = T_f^{-l} a_b$ and $T_f^{-p} c_b \in [z_3, z_4]$ for some $l, p \in [0, q_n)$;

3) $\xi(0) \geq C$ and $z(0) \in [0, \zeta]$ if $\xi_b = T_f^{-l} c_b \in [z_1, z_2]$,

$\eta(0) \geq C$ and $\vartheta(0) \in [0, \zeta]$ if $\xi_b = T_f^{-p} c_b \in [z_3, z_4],$

or if conditions 1)-3) hold for $a_b$ and $c_b$ interchanged.

In order to formulate the next Lemma we introduce the following functions for $x > 0$ and $0 \leq t \leq 1$:

$$G(x) = \frac{\sigma(a_b)(1 + x)}{\sigma(a_b) + x}, \quad F(x, t) = \frac{\sigma(c_b) + (1 - \sigma(c_b))t(1 + x)}{\sigma(c_b) + (1 - \sigma(c_b))t + x}.$$  

**Lemma 4.3.** Suppose that the homeomorphism $T_f$ satisfies the conditions of Theorem 1.8

If the intervals $\{T_f^j[z_1, z_2], T_f^j[z_2, z_3], T_f^j[z_3, z_4] : 0 \leq j \leq q - 1\}$ cover the break points $a_b$, $c_b$ such that either $z_2 = T_f^{-l} a_b$ and $T_f^{-p} c_b \in [z_1, z_2]$ or $z_3 = T_f^{-l} a_b$ and $T_f^{-p} c_b \in [z_3, z_4]$ then

$I) \text{Dist}(z_1, z_2, z_3, z_4; T_f^l a_b) = \left[ G(\xi(l)) + \chi_1 \right] \left[ F(\xi(p), z(p)) + \chi_2 \right] \times \prod_{0 \leq i < q_n \neq l, p} \text{Dist}(T_f^j z_1, T_f^j z_2, T_f^j z_3, T_f^j z_4; T_f) \text{ if } z_2 = \xi_b, \xi_b \in [z_1, z_2]$;

$II) \text{Dist}(z_1, z_2, z_3, z_4; T_f^p c_b) = \left[ G(\eta(l)) + \chi_3 \right] \left[ F(\eta(p), \vartheta(p)) + \chi_4 \right] \times \prod_{0 \leq i < q_n \neq l, p} \text{Dist}(T_f^j z_1, T_f^j z_2, T_f^j z_3, T_f^j z_4; T_f) \text{ if } z_3 = \xi_b, \xi_b \in [z_3, z_4]$.

where

$$|\chi_j| = |\chi_j(z_1, z_2, z_3, z_4)| \leq K_1 \int_{z_1}^{z_4} |D^2 f(y)| dy, \quad 1 \leq j \leq 4.$$  

and the constant $R_5$ does not depend on $n$ and $\varepsilon$

Using Lemma 2.8 we get the following inequalities for all $1 \leq m \leq q_n$:

$$e^{-v}\xi(0) \leq \xi(m) \leq e^{v}\xi(0), \quad e^{-v}z(0) \leq z(m) \leq e^{v}z(0),$$  

$$e^{-v}\eta(0) \leq \eta(m) \leq e^{v}\eta(0), \quad e^{-v}\vartheta(0) \leq \vartheta(m) \leq e^{v}\vartheta(0)$$  

From this it follows, that the normalized coordinates $\xi(m), \eta(m), z(m),$ and $\vartheta(m)$ are uniformly (with respect to $x_0$ and $m$) comparable with the initial normalized coordinates $\xi(0), \eta(0), z(0),$ and $\vartheta(0)$ respectively.
Lemma 4.4. If a circle homeomorphism $T_f$ satisfies the conditions of Lemma 2.8 then there exist for any $x_0 \in S^1$ and any $\delta > 0$ constants $C_0 = C_0(f, \sigma(a_b), \sigma(c_b)) > 1$ and $\zeta_0 = \zeta_0(f, \sigma(a_b), \sigma(c_b)) \in (0, 1)$, such that for all triple of intervals $[z_s, z_{s+1}] \subset (x_0 - \delta, x_0 + \delta)$, $s = 1, 2, 3$, covering the break points $a_b$, $c_b$ regularly with constants $C_0$ and $\zeta_0$ the following relations hold:

$$|G(\xi(I))F(\xi(p), z(p)) - 1| = \frac{\sigma(a_b)\sigma(c_b) - 1}{4}$$

if $z_2 = \overline{a_b}$, $0 \leq \frac{z_2 - \overline{c_b}}{z_2 - z_1} \leq \zeta_0$

respectively

$$|G(\eta(I))F(\eta(p), \vartheta(p)) - 1| = \frac{\sigma(a_b)\sigma(c_b) - 1}{4}$$

if $z_3 = \overline{a_b}$, $0 \leq \frac{\overline{c_b} - z_3}{z_4 - z_3} \leq \zeta_0$.

Lemma 4.5. If the circle homeomorphism $T_f$ satisfies the conditions of Lemma 2.8 there exists for any $x_0 \in S^1$ and any $\delta > 0$ a number $N = N(\delta, x_0) > 1$, such that for all $n > N(\delta, x_0)$, there is a triple of intervals $[z_s, z_{s+1}] \subset (x_0 - \delta, x_0 + \delta)$, $s = 1, 2, 3$ with the following properties:

1) the interval $[z_1, z_4]$ is $q_n$ small;

2) the intervals $[z_s, z_{s+1}]$ and $[T_f^{q_n}z_s, T_f^{q_n}z_{s+1}]$ $s = 1, 2, 3$ satisfy conditions a) and b) of Lemma 4.1 with some constant $R_1 > 1$ depending on $C_0$, $\zeta_0$ and $v$;

3) the intervals $\{T_f^iz_1, z_2, T_f^iz_3, z_4\}$, $0 \leq i \leq q_n - 1$ either cover both break points $a_b, c_b$ regularly with constants $C_0$ and $\zeta_0$, or cover only the break point $a_b$ such that its preimage $\overline{a_b}$ coincides with $z_2$ or $z_3$.

Lemma 4.6. Suppose, that the circle homeomorphism $T_f$ satisfies the conditions of Theorem 1.8 and the intervals $[z_s, z_{s+1}]$, $s = 1, 2, 3$ satisfy conditions 1)-3) of Lemma 4.5. Then the following inequality holds for sufficiently large $n$:

$$|\text{Dist}(z_1, z_2, z_3, z_4; T_f^{q_n}) - 1| > \text{const} > 0$$

where the constant depends only on the function $f$.

After these preparations we can now proceed to the proof of Theorem 1.8. Let $T_f$ be a class $P$-homeomorphism satisfying the conditions of Theorem 1.8. Since its rotation number $\rho_f$ is irrational the $T_f$-invariant measure $\mu_f$ is nonatomic i.e. every one point subset of the circle has zero $\mu_f$-measure. The conjugating map $T_\varphi$ related to $\mu_f$ by $T_\varphi x = \mu_f([0, x])$, $x \in S^1$, is a continuous and monotone increasing function on $S^1$. Hence $T_\varphi$ has a finite derivative almost everywhere (w.r.t. Lebesgue measure) on the circle. We show that $D\varphi(x) = 0$ at all points at which the derivative is defined. Choose an $\varepsilon > 0$ and a triple of intervals $[z_s, z_{s+1}] \subset (x_0 - \delta, x_0 + \delta)$, $s = 1, 2, 3$, satisfying the conditions of Lemma 4.1. It follows from this Lemma and Lemma 4.5 that

\begin{equation}
|\text{Dist}(z_1, z_2, z_3, z_4; T_\varphi) - 1| \leq C_2\varepsilon,
\end{equation}

and

\begin{equation}
|\text{Dist}(T_f^{q_n}z_1, T_f^{q_n}z_2, T_f^{q_n}z_3, T_f^{q_n}z_4; T_\varphi) - 1| \leq C_2\varepsilon.
\end{equation}
By definition

$$\text{Dist}(T^q_1z_1, T^q_1z_2, T^q_1z_3, T^q_1z_4; T^q_\varphi) = \frac{\text{Cr}(T^q_\varphi(T^q_1z_1), T^q_\varphi(T^q_1z_2), T^q_\varphi(T^q_1z_3), T^q_\varphi(T^q_1z_4))}{\text{Cr}(T^q_1z_1, T^q_1z_2, T^q_1z_3, T^q_1z_4)}. \tag{24}$$

Since $T^q_\varphi$ conjugates $T^q_1$ with the linear rotation $T^q_\rho$, we can readily see that

$$\text{Cr}(T^q_\varphi(T^q_1z_1), T^q_\varphi(T^q_1z_2), T^q_\varphi(T^q_1z_3), (T^q_\varphi(T^q_1z_4))) = \text{Cr}(T^q_\varphi z_1, T^q_\varphi z_2, T^q_\varphi z_3, T^q_\varphi z_4))$$

and hence

$$\text{Dist}(T^q_1z_1, T^q_1z_2, T^q_1z_3, T^q_1z_4; T^q_\varphi) = \frac{\text{Cr}(T^q_\varphi z_1, T^q_\varphi z_2, T^q_\varphi z_3, T^q_\varphi z_4))}{\text{Cr}(T^q_1z_1, T^q_1z_2, T^q_1z_3, T^q_1z_4)}$$

This together with $(22)$, $(23)$ and $(24)$ implies

$$|\text{Dist}(z_1, z_2, z_3, z_4; T^q_1) - 1| \leq C_3\varepsilon, \tag{25}$$

where the constant $C_3 > 0$ does not depend on $\varepsilon$ and $n$. But this contradicts Lemma 4.6 according to which

$$|\text{Dist}(z_1, z_2, z_3, z_4; T^q_1) - 1| > \text{const} > 0$$

for sufficiently large $n$. This contradiction proves Theorem 1.8.

5 The proofs of Lemmas 4.1-4.6

Proof of Lemma 4.1 Suppose, that the derivative $D\varphi(x_0)$ exists and $D\varphi(x_0) = \omega > 0$. By the definition of the derivative there exists for any $\varepsilon > 0$ a number $\delta = \delta(x_0; \varepsilon) > 0$, such that for all $x \in (x_0 - \delta, x_0 + \delta)$,

$$\omega - \varepsilon < \frac{\varphi(x) - \varphi(x_0)}{x - x_0} < \omega + \varepsilon. \tag{26}$$

Now take four points $z_i \in (x_0 - \delta, x_0 + \delta)$ satisfying conditions (a) and (b) of Lemma 4.1. Assume that $z_i < x_0$, $1 \leq i \leq 4$. For the other cases Lemma 4.1 can be proved similarly. Relation $(26)$ implies for $x = z_i$, $i = 1, 2, 3, 4$

$$(\omega - \varepsilon)(x_0 - z_i) < \varphi(x_0) - \varphi(z_i) < (\omega + \varepsilon)(x_0 - z_i).$$

This yields the following inequalities:

$$\omega - \varepsilon \frac{(x_0 - z_{s+1}) + (x_0 - z_s)}{z_{s+1} - z_s} \leq \frac{\varphi(z_{s+1}) - \varphi(z_s)}{z_{s+1} - z_s} \leq \omega + \varepsilon \frac{(x_0 - z_{s+1}) + (x_0 - z_s)}{z_{s+1} - z_s} \tag{27}$$

for $s = 1, 2, 3,$ and

$$\omega - \varepsilon \frac{(x_0 - z_{s+2}) + (x_0 - z_s)}{z_{s+2} - z_s} \leq \frac{\varphi(z_{s+2}) - \varphi(z_s)}{z_{s+2} - z_s} \leq \omega + \varepsilon \frac{(x_0 - z_{s+2}) + (x_0 - z_s)}{z_{s+2} - z_s} \tag{28}$$
for \( s = 1, 2 \).

From conditions (a) and (b) of Lemma 4.1 on the other hand it follows that

\[
\max_{1 \leq i \leq 4} \left\{ \frac{x - z_i}{z_2 - z_1}, \frac{x_0 - z_i}{z_3 - z_1}, \frac{x_0 - z_i}{z_4 - z_2}, \frac{x_0 - z_i}{z_4 - z_3} \right\} \leq K_1
\]

where the constant \( K_1 > 0 \) depends on \( R_1 \) and does not depend on \( \varepsilon \).

We rewrite \( \text{Dist}(z_1, z_2, z_3, z_4; T_\varphi) \) in the following form:

\[
\text{Dist}(z_1, z_2, z_3, z_4; T_\varphi) = \frac{T_\varphi z_2 - T_\varphi z_1}{z_2 - z_1} \cdot \frac{T_\varphi z_4 - T_\varphi z_3}{z_4 - z_3} \cdot \frac{z_3 - z_1}{T_\varphi z_3 - T_\varphi z_1} \cdot \frac{z_4 - z_2}{T_\varphi z_4 - T_\varphi z_2}.
\]

The inequalities (27) \& (29) then imply the assertion of Lemma 4.1

**Proof of Lemma 4.3** We consider the case \( z_2 = T_f^{-l} a_b, \) \( T_f^{-l} c_b \in [z_1, z_2], \) \( 0 \leq l, p \leq q_n, \) the case \( z_3 = T_f^{-l} a_b, \) \( T_f^{-p} c_b \in [z_3, z_4], \) \( 0 \leq l, p \leq q_n, \) can be treated similarly. Rewrite the distortion \( \text{Dist}(z_1, z_2, z_3, z_4; T_{\varphi}^{q_n}) \) in the following form

\[
\text{Dist}(z_1, z_2, z_3, z_4; T_{\varphi}^{q_n}) = \text{Dist}(T_f^l z_1, T_f^l z_2, T_f^l z_3, T_f^l z_4; T_f) \times \text{Dist}(T_f^p z_1, T_f^p z_2, T_f^p z_3, T_f^p z_4; T_f) \prod_{0 \leq i < q_n, i \neq l, p} \text{Dist}(T_f^i z_1, T_f^i z_2, T_f^i z_3, T_f^i z_4; T_f).
\]

By assumption only the two intervals \([T_f^l z_1, T_f^l z_2]\) and \([T_f^p z_1, T_f^p z_2]\) contain the break points: namely \( a_b = T_f^l z_2 = a_b, \) and \( c_b \in [T_f^p z_1, T_f^p z_2] \) for some \( l, p \in [0, q_n). \)

Using Lemma 3.5 and the definitions of the functions \( G(x), F(x, t) \) we get

\[
\text{Dist}(T_f^l z_1, T_f^l z_2, T_f^l z_3, T_f^l z_4; T_f) = \frac{\sigma(a_b)(1 + \xi(l))}{\sigma(a_b) + \xi(l)} + \chi_1 = G(\xi(l)) + \chi_1,
\]

\[
\text{Dist}(T_f^p z_1, T_f^p z_2, T_f^p z_3, T_f^p z_4; T_f) = \frac{\sigma(c_b)(1 - \sigma(c_b))z(p)[1 + \xi(p)]}{\sigma(c_b) + (1 - \sigma(c_b))z(p) + \xi(p)} + \chi_2 = F(\xi(p), z(p)) + \chi_2,
\]

with \(|\chi_j| = |\chi_j(z_1, z_2, z_3, z_4)| \leq K_1 \int_{z_1}^{z_4} |D^2 f(y)| dy, \) \( j = 1, 2. \)

This together with (30) imply the assertion of Lemma 4.3

**Proof of Lemma 4.4** We prove only the bound for \( G(\xi(l))F(\xi(p), z(p)) \). The one for \( G(\eta(l))F(\eta(p), \vartheta(p)) \) can be proved similarly. We start rewriting \( G(\xi(l))F(\xi(p), z(p)) \) in the following form:

\[
G(\xi(l))F(\xi(p), z(p)) = \frac{\sigma(a_b)(1 + \xi(l))}{\sigma(a_b) + \xi(l)} \cdot \frac{\sigma(c_b)(1 - \sigma(c_b))z(p)[1 + \xi(p)]}{\sigma(c_b) + (1 - \sigma(c_b))z(p) + \xi(p)} = \left[ \frac{(1 + \xi(l))}{\sigma(a_b) + \xi(l)} \right] \cdot \Phi_1(z(p)) \times \Phi_2(\xi(l), \xi(p), z(p))
\]

where \( z(p) \in [0, 1) \) and \( \xi(l), \xi(p) > 0. \)
It is clear, that \( \Phi_1(0) = \sigma(a_b)\sigma(c_b) \) and \( \Phi_2(\xi(l), \xi(p), z(p)) \) tends to 1 as \( \xi(l), \xi(p) \) tend to \( \infty \). Recall that \( \sigma_a \sigma_b \neq 1 \) by assumption.

Next we discuss the conditions under which the expression \( \Phi_1(z(p))\Phi_2(\xi(l), \xi(p), z(p)) \) stays away from 1. Obviously

\[
(32) \quad |\Phi_1 \Phi_2 - 1| = |(\Phi_1 - 1) + \Phi_1(\Phi_2 - 1)| \geq |\Phi_1 - 1| - \Phi_1 \Phi_2 - 1|.
\]

Using the bounds for \( z(m) \) in (21) we get

\[
|\Phi_1 - 1| = |\sigma(a_b)\sigma(c_b) + (1 - \sigma(c_b))\sigma(a_b)z(p) - 1| \\
\geq |\sigma(a_b)\sigma(c_b) - 1| - |1 - \sigma(c_b)|\sigma(a_b)z(p) \\
\geq |\sigma(a_b)\sigma(c_b) - 1| - |1 - \sigma(c_b)|\sigma(a_b)e^v z(0).
\]

If next \( z(0) \) fulfills the inequality

\[
|\sigma(a_b)\sigma(c_b) - 1| - |1 - \sigma(c_b)|\sigma(a_b)e^v z(0) \geq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{2},
\]

and hence

\[
z(0) \leq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{2e^v|\sigma(a_b)\sigma(c_b) - \sigma(a_b)|},
\]

then we conclude that

\[
(33) \quad |\Phi_1 - 1| \geq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{2}, \text{ if } 0 \leq z(0) \leq \zeta_0,
\]

where

\[
(34) \quad \zeta_0 := \min \left\{ \frac{|\sigma(a_b)\sigma(c_b) - 1|}{2e^v|\sigma(a_b)\sigma(c_b) - \sigma(a_b)|}, 1 \right\}.
\]

Next we determine, under which condition on \( \xi(0) \) the inequality:

\[
(35) \quad \Phi_1\Phi_2 - 1 \leq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{4}
\]

holds true. Obviously, for \( z(p) \in [0, 1] \), one has \( \Phi_1(z(p)) \leq \max \{\sigma(a_b)\sigma(c_b), \sigma(c_b)\} := m_\sigma \), for \( z(p) \in [0, 1] \). Inequality (35) then follows, if

\[
(36) \quad |\Phi_2 - 1| \leq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{4m_\sigma}.
\]

Now, if \( \xi(l) \) and \( \xi(p) \) are sufficiently large, then, since

\[
(37) \quad \Phi_2 - 1 = \frac{(1 + \xi(l))}{\sigma(a_b) + \xi(l)} \cdot \frac{(1 + \xi(p))}{\sigma(c_b) + (1 - \sigma(c_b))z(p) + \xi(p)} - 1,
\]

the right hand side of (37) behaves like \( (1 + O(\frac{1}{\xi(l)}))(1 + O(\frac{1}{\xi(p)})) - 1 \), which can be bounded by \( R_6 \left( \frac{1}{\xi(l)} + \frac{1}{\xi(p)} \right) \) for some constant \( R_6 > 1 \) not depending on \( \xi(l) \) and \( \xi(p) \). On the other hand, according to relations (21), \( \xi(m) \) is for \( m \in (0, q_n) \) comparable with \( \xi(0) \), and hence

\[
(38) \quad |\Phi_2 - 1| \leq R_6 \left( \frac{1}{\xi(l)} + \frac{1}{\xi(p)} \right) \leq 2R_6 e^v \frac{1}{\xi(0)}.
\]
 Hence if

\[ 2R_6e^\nu \frac{1}{\xi(0)} \leq \frac{1}{2} \left| \frac{\sigma(a_b)\sigma(c_b) - 1}{4m_\sigma} \right| \]

respectively

\[ \xi(0) \geq \frac{2R_6e^\nu m_\sigma}{|\sigma(a_b)\sigma(c_b) - 1|}, \]

then inequality (38) holds true. Finally, we define the constant \( C_0 \) by

\[ C_0 := \max \left\{ \frac{2R_6e^\nu m_\sigma}{|\sigma(a_b)\sigma(c_b) - 1|}, 1 \right\}. \]

From (33)-40 the assertion of Lemma 4.4 then follows immediately.

**Proof of Lemma 4.5** Let \( D\phi(x_0) = \omega > 0 \). Fix \( n \geq 1 \). W.l.o.g. we consider the case \( n \) odd, the case \( n \) even can be deduced from the odd case by reversing the orientation of the circle. From the structure of the dynamical partition \( \xi_n(x_0) \) it follows, that both preimages \( \overline{a}_b, \overline{b}_a \) are in the interval \( [T_{f}^{q_n-1}x_0, T_{f}^{q_n-1}x_0] \) with \( \overline{a}_b = T_{f}^{-l}a_b, \overline{b}_a = T_{f}^{-l}b_a \). Since \( \overline{a}_b = T_{f}^{-l}a_b \) for some \( 0 \leq l, p \leq q_n \). Take the point \( \overline{a}_b \) and consider its neighbourhood \( [T_{f}^{q_n-1}\overline{a}_b, T_{f}^{q_n-1}\overline{a}_b] \). By Corollary 2.7 for any \( a, b \in S^1 \) all the intervals \( [a, b], T_{f}^{q_n}[a, b], T_{f}^{-q_n}[a, b] \) are \( e^\nu \)-comparable. Since \( \overline{a}_b \in [T_{f}^{q_n}x_0, T_{f}^{q_n-1}x_0] \), it can easily be shown, that the pairs of intervals \( (T_{f}^{q_n-1}x_0, x_0), (T_{f}^{-q_n-1}\overline{a}_b, \overline{a}_b)) \) and \( ([T_{f}^{-q_n-1}x_0, T_{f}^{q_n-1}x_0], [T_{f}^{-q_n-1}\overline{a}_b, T_{f}^{q_n-1}\overline{a}_b]) \) are \( e^\nu \)-comparable.

Let \( \tau_0 \) be the middle point of the interval \( [T_{f}^{-q_n-1}\overline{a}_b, \overline{a}_b] \). Since \( [\overline{a}_b, T_{f}^{q_n-1}\tau_0] = T_{f}^{q_n-1}[T_{f}^{-q_n-1}\overline{a}_b, \tau_0] \) and \( l([T_{f}^{-q_n-1}\overline{a}_b, \tau_0]) = l([\tau_0, \overline{a}_b]) \) we conclude, that the intervals \( [\tau_0, \overline{a}_b] \) and \( [\overline{a}_b, T_{f}^{q_n-1}\tau_0] \) are \( e^\nu \)-comparable (see figure 1).

Set

\[ d_n := \frac{1}{2} \min \left\{ l([\overline{a}_b, T_{f}^{q_n-1}\overline{a}_b]), l([T_{f}^{-q_n-1}\overline{a}_b, \overline{a}_b]) \right\}. \]

Using this and Corollary 2.7 we get

\[ e^{-\frac{1}{2}} l([\overline{a}_b, T_{f}^{q_n-1}\overline{a}_b]) \leq d_n \leq e^{\nu-\frac{1}{2}} l([\overline{a}_b, T_{f}^{q_n-1}\overline{a}_b]). \]

Figure 1: Comparison of the intervals \([x_0, T_{f}^{q_n-1}x_0] \) and \([\overline{a}_b, T_{f}^{q_n-1}\overline{a}_b] \).

Figure 2: The neighbourhoods \( U_n(\overline{a}_b) \) and \( V_n(\overline{a}_b) \) are comparable with \([T_{f}^{-q_n-1}\overline{a}_b, T_{f}^{q_n-1}\overline{a}_b] \).
Notice, that the interval \([\tau_0, T_f^{q_n-1}\tau_0]\) is one of the two generators of the partition \(\xi_n(\tau_0)\). Hence the intervals \(T_f^i[\tau_0, T_f^{q_n-1}\tau_0], i \in [0, q_n]\) cover the break point \(a_0\) only once. Using the constants \(C_0\) and \(\zeta_0\) in Lemma 4.4 we define two neighbourhoods (see figure 2) of the point \(\overline{a}_b\):

\[
V_n(\overline{a}_b) = (\overline{a}_b - \frac{1}{2}e^{-u}C_0^{-1}d_n, \overline{a}_b + \frac{1}{2}e^{-u}C_0^{-1}d_n),
\]

\[
U_n(\overline{a}_b) = [\overline{a}_b - \frac{1}{2}\zeta_0 l(V_n(\overline{a}_b)), \overline{a}_b + \frac{1}{2}\zeta_0 l(V_n(\overline{a}_b))].
\]

It is clear that \(U_n(\overline{a}_b) \subset V_n(\overline{a}_b) \subset [\tau_0, T_f^{q_n-1}\tau_0]\). The construction of the intervals \([z_s, z_{s+1}]\) will depend of the location of \(\overline{a}_b\) in the interval \(V_n(\overline{a}_b)\). There are two possibilities to consider:

(43) \(\overline{a}_b \not\in U_n(\overline{a}_b)\), i.e. \(\overline{a}_b \in [T_f^{q_n-1}\tau_0, T_f^{q_n-1}\tau_0] \setminus U_n(\overline{a}_b)\) or \(\overline{a}_b \in U_n(\overline{a}_b)\).

Consider the first case, when \(\overline{a}_b \in V_n(\overline{a}_b) \setminus U_n(\overline{a}_b)\). In this case we set

\[
z_2 = \overline{a}_b, z_3 = \overline{a}_b + \frac{1}{4}l(U_n(\overline{a}_b)), z_4 = \overline{a}_b + \frac{1}{2}l(U_n(\overline{a}_b)) \text{ and } z_1 = \overline{a}_b - \frac{1}{4}l(U_n(\overline{a}_b)).
\]

It is easy to see, that the interval \([z_1, z_4]\) is a subset of \([\tau_0, T_f^{q_n-1}\tau_0]\) and it does not contain the break point \(\overline{a}_b\). Next we check, that the intervals \([z_s, z_{s+1}], s = 1, 2, 3\) satisfy properties 1)-3) in Lemma 4.5. The interval \([z_1, z_4]\) is \(q_n\)-small, because \([z_1, z_4] \subset [\tau_0, T_f^{q_n-1}\tau_0]\) which is one of the generators of the dynamical partition \(\xi_n(\tau_0)\). By construction, the length of \([z_1, z_4]\) is equal to \(\frac{4}{7}e^{-u}C_0^{-1}\zeta_0 d_n\), but \(d_n\) is half the length of one of the intervals \([T_f^{-q_n-1}\overline{a}_b, \overline{a}_b] \text{ or } [\overline{a}_b, T_f^{q_n-1}\overline{a}_b]\). Consequently, the length of \([z_1, z_4]\) is \(4e^{-u}C_0\zeta_0^{-1}\zeta_0 d_n\). Hence assumption a) of Lemma 4.1 holds true for those intervals with constant \(e^{-u}\).

Next we check assumption b) of Lemma 4.1. It is easy to see that for all \(i = 1, 2, 3, 4\)

\[
|x_0 - z_i| \leq |x_0 - z_2| + |z_4 - z_1| = |x_0 - z_2| + d_n,
\]

\[
|x_0 - T_f^{q_n}z_i| \leq |x_0 - z_2| + |z_2 - T_f^{q_n}z_2| + |T_f^{q_n}z_4 - T_f^{q_n}z_1|.
\]

The point \(z_2\) belongs to \([T_f^{q_n}x_0, T_f^{q_n-1}x_0]\) \(\subset [T_f^{-q_n-1}x_0, T_f^{q_n-1}x_0]\), which is \(e^{-u}\)-comparable with \([T_f^{-q_n-1}\overline{a}_b, T_f^{q_n-1}\overline{a}_b]\). But the length of this last interval is \(4e^{-u}\)-comparable with \(d_n\).

In complete analogy we can estimate the second expression in (44) by \(12e^{-u}C_0\zeta_0^{-1}d_n\). By assumption, the intervals \(T_f^i[z_1, z_1]\) do not cover the break point \(\overline{a}_b\), but they cover the point \(a_0\) exactly once with \(z_2 = T_f^{-1}a_0\).

Next consider the second case, when \(\overline{a}_b \in U_n(\overline{a}_b)\). There are again two possibilities: If \(\overline{a}_b \in [\overline{a}_b - \frac{1}{2}\zeta_0 l(V_n(\overline{a}_b)), \overline{a}_b]\), we set

\[
z_1 = \overline{a}_b - \frac{1}{2}l(V_n(\overline{a}_b)), z_2 = \overline{a}_b, z_3 = \overline{a}_b + \frac{1}{2}C_0l(V_n(\overline{a}_b)), z_4 = \overline{a}_b + C_0l(V_n(\overline{a}_b)).
\]

Then the length of the first interval \([z_1, z_2]\) is \(\frac{1}{2}e^{-u}C_0^{-1}d_n\) and of the other ones it is equal to \(\frac{1}{2}e^{-u}d_n\). Hence the lengths of these intervals are \(2e^{-u}C_0\)-comparable with \(d_n\). The first
two statements of Lemma 4.5 for these intervals can be checked in complete analogy to the first case in (43).

Next we show, that in the present case the intervals \( \{ T_i^j[z_1, z_2], \ T_i^j[z_2, z_3], \ T_i^j[z_3, z_4], \ 0 \leq i \leq q_n \} \) cover both break points \( a_b, c_b \) regularly with constants \( C_0 \) and \( \zeta_0 \). By construction, these intervals cover both break points exactly once. Moreover we have \( z_2 = \overline{a}_b \) and \( \overline{c}_b \in [z_1, z_2] \). It is easy to see, that \( \zeta(0) = \frac{z_2 - z_1}{z_2 - z_1} = C_0 \). Since \( \zeta_l \in [\overline{a}_b, \overline{c}_b] \), we find that \( \zeta(0) = \frac{z_2 - z_1}{z_2 - z_1} \leq \zeta_0 \). So the intervals \( [z_s, z_{s+1}] \), \( s = 1, 2, 3 \) satisfy the statements of Lemma 4.5.

Consider finally the case, when the break point \( \overline{c}_b \) is in the interval \( [\overline{a}_b, \overline{a}_b + \frac{1}{2} \zeta_0 l(V_n(\tau_0)) \] .

In this case we set
\[
z_1 = \overline{a}_b - C_0 l(V_n(\overline{a}_b)), \ z_2 = \overline{a}_b - \frac{1}{2} C_0 l(V_n(\overline{a}_b)), \ z_3 = \overline{a}_b, \ z_4 = \overline{a}_b + \frac{1}{2} l(V_n(\overline{a}_b)).
\]

The proof of Lemma 4.5 for these intervals \( [z_s, z_{s+1}] \), \( s = 1, 2, 3 \) proceeds now exactly as in the previous case. This concludes the proof of Lemma 4.5.

**Proof of Lemma 4.6.** Assume, that the circle homeomorphism \( T_f \) satisfies the assumptions of Theorem 14.5 and the intervals \( [z_s, z_{s+1}] \), \( s = 1, 2, 3 \) satisfy Lemma 4.5.

Consider first the case when the intervals \( \{ T_i^j[z_1, z_2], T_i^j[z_2, z_3], T_i^j[z_3, z_4], 0 \leq i \leq q_n - 1 \} \) cover both break points \( a_b, c_b \) regularly with constants \( C_0 \) and \( \zeta_0 \). Suppose that \( z_2 = \overline{a}_b = T_f^{-l} a_b \) and \( \overline{c}_b = T_f^{-p} c_b \), for some \( 0 \leq l, p \leq q_n \) and
\[
\frac{z_3 - z_2}{z_2 - z_1} \leq C_0, \ 0 \leq \frac{z_3 - \overline{c}_b}{z_2 - z_1} \leq \zeta_0.
\]

Lemma 4.3 shows that
\[
\text{Dist}(z_1, z_2, z_3, z_4; T_f^n) = [G(\xi(l) + \chi_1)] [F(\xi(p), z_2) + \chi_2] \times
\]
\[
\times \prod_{0 \leq i < q_n \atop i \neq l, p} \text{Dist}(T_i^j z_1, T_i^j z_2, T_i^j z_3, T_i^j z_4; T_f)
\]

if \( z_2 = \overline{a}_b \) and \( \overline{c}_b \in [z_1, z_2] \) where
\[
|\chi_j| = |\chi_j(z_1, z_2, z_3, z_4)| \leq K_1 \int_{z_1}^{z_4} |D^2 f(y)| dy, \ j = 1, 2.
\]

with some constant \( K_1 \) not depending on \( n \) and \( \varepsilon \). Next we estimate the right hand side in equation (40). Fix some \( \varepsilon > 0 \). By assumption, the second derivative \( D^2 f \) of the lift \( f \) belongs to \( L^1(S^1, dl) \). Hence it can be written in the form \( D^2 f(x) = g_\varepsilon(x) + \theta_\varepsilon(x) \) with \( g_\varepsilon \) a continuous function on \( S^1 \) and \( \| \theta_\varepsilon(x) \|_{L^1} < \varepsilon \). By assumption, among the intervals \( T_i^j[z_s, z_{s+1}], 0 \leq i \leq q_n \), only the intervals \( T_i^j[z_1, z_4] \) and \( T_i^p[z_1, z_4] \) contain the break points \( a_b \) respectively \( c_b \).

Obviously
\[
|\prod_{0 \leq i < q_n \atop i \neq l, p} \text{Dist}(T_i^j z_1, T_i^j z_2, T_i^j z_3, T_i^j z_4; T_f) - 1| =
\]
\[
= |\exp\{ \sum_{i=0 \atop i \neq l, b}^{q_n-1} \log(1 + (\text{Dist}(T_i^j z_1, T_i^j z_2, T_i^j z_3, T_i^j z_4; T_f) - 1))\} - 1|
\]

(48)
Next applying Theorem 3.3 we obtain

\[
\left| \text{Dist}(T^i_1 z_1, T^i_2 z_2, T^i_3 z_3, T^i_4 z_4; T_f) - 1 \right| \leq C_1 |T^i_4 z_4 - T^i_2 z_1| \times \\
\max_{x,t \in [T^i_j z_1, T^i_j z_4]} |g_\varepsilon(x) - g_\varepsilon(t)| + C_1 \int_{T^i_j z_1} |\theta_\varepsilon(y)|dy + C_1 \left( \int_{T^i_j z_1} |D^2 f(y)|dy \right)^2
\]

(49)

where the constant \( C_1 > 0 \) depends only on the function \( f \).

For \( D^2 f \in L^1([0,1]) \) the function \( \Psi(x) = \int_0^x < D^2 f(y)|dy \) is absolutely continuous on \( S^1 \). Note that the functions \( \Psi(x) \) and \( g_\varepsilon(x) \) are uniformly continuous on \( S^1 \) because they are continuous on \( S^1 \). Hence there exists \( \delta_0 = \delta_0(\varepsilon) > 0 \) such that for any \( x, t \in S^1 \) with \( |x - t| < \delta_0 \), the inequalities

\[
|\Psi(x) - \Psi(t)| < \varepsilon, \quad |g_\varepsilon(x) - g_\varepsilon(t)| < \varepsilon,
\]

hold true.

By assumption, the interval \([z_1, z_4]\) is \( q_n \)-small. Hence by Corollary 2.7 for all \( 0 \leq i \leq q_n \) we have \( l(T^i_j[z_1, z_4]) \leq \text{const} \lambda^n \), \( 0 \leq i \leq q_n - 1 \), with \( \lambda = (1 + e^{-\nu})^{\frac{1}{2}} < 1 \). Consequently there exists a number \( N_0 \geq N_0(\delta) > 0 \) such that for \( n > N_0 \) and all \( 0 \leq i \leq q_n \) one has \( l(T^i_j[z_1, z_4]) \leq \delta_0 \). This together with (50) implies that for \( n > N_0 \) the following inequalities

\[
|\Psi(x) - \Psi(y)| < \varepsilon, \quad |g_\varepsilon(x) - g_\varepsilon(y)| < \varepsilon,
\]

(51)

hold for all \( x, y \in [T^i_j z_1, T^i_j z_4], \) and all \( 0 \leq i \leq q_n \).

On the other hand, since the interval \([z_1, z_4]\) is \( q_n \)-small, the intervals \( T^i_j[z_1, z_4], \)

\( 0 \leq i \leq q_n \), are non intersecting and trivially

\[
\sum_{i=0}^{q_n} l(T^i_j[z_1, z_4]) \leq 1.
\]

Since \( \| \theta_\varepsilon \|_{L^1} < \varepsilon \), we find, using relations (48)-(51),

\[
\left| \prod_{0 \leq i < q_n, i \neq l, p} \text{Dist}(T^i_1 z_1, T^i_2 z_2, T^i_3 z_3, T^i_4 z_4; T_f) - 1 \right| \leq \\
\leq C_2 \sum_{i=0}^{q_n-1} |\text{Dist}(T^i_j z_1, T^i_j z_2, T^i_j z_3, T^i_j z_4; T_f) - 1| \leq C_2 \varepsilon \sum_{i=0}^{q_n-1} |T^i_j z_4 - T^i_j z_1| + \\
+ C_2 \sum_{i=0}^{q_n-1} \int_{T^i_j(z_1)} |\theta_\varepsilon(y)||dy + C_2 \sum_{i=0}^{q_n-1} |\Psi(T^i_j z_4) - \Psi(T^i_j z_1)| \int_{T^i_j(z_1)} |D^2 f(y)||dy \leq \\
\leq C_2 \left\{ 2\varepsilon + \int_0^1 |\theta_\varepsilon(y)||dy + \varepsilon \int_0^1 |D^2 f_1(y)||dy \right\} \leq C_2 (3 + \|D^2 f\|_{L^1}) \varepsilon
\]

(52)

where the constant depends only on \( f \).
Next we estimate the expression $(G(\xi(l)) + \chi_1)(F(\xi(p),z(p)) + \chi_2)$. where $|\chi_i|$ is bounded above by $K_1\int |D^2 f(y)|dy$, for $j = 1, 2$ with some constant $K_1$ not depending on $n$ and $\varepsilon$. For $n > N_0$ we have $\int |D^2 f(y)|dy = \Psi(z_4) - \Psi(z_1) < \varepsilon$. Since $G(x)$ is bounded for $x > 0$ and $F(x,t)$, is bounded for $x > 0$ and $1 \leq t \leq 1$, it is hence sufficient to estimate the term $G(\xi(l))F(\xi(p),z(p))$ in the above product. By assumption, the intervals $\{T^i_j[z_1, z_2], T^i_j[z_2, z_3] T^i_j[z_3, z_4], 0 \leq i \leq q_n - 1\}$ cover both break points $a_b, c_b$ regularly with constants $C_0$ and $\zeta_0$. Applying Lemma 4.4 we conclude

$$|G(\xi(l))F(\xi(p), z(p))| \leq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{4} > 0,$$

since by assumption the product of the jump ratios of $Df$ at the break points is nontrivial i.e. $\sigma(a_b)\sigma(c_b) \neq 1$. It then follows, that for sufficiently large $n$ and small $\varepsilon$ the inequality

$$|[G(\xi(l)) + \chi_1][F(\xi(p), z(p)) + \chi_2]| \geq \frac{|\sigma(a_b)\sigma(c_b) - 1|}{8}$$

holds true. This together with (53) and (52) imply the assertion of Lemma 4.6 in the case of a regular covering of the two break points.

W.l.o.g. we assume next that the intervals $\{T^i_j[z_1, z_2], T^i_j[z_2, z_3] T^i_j[z_3, z_4], 0 \leq i \leq q_n - 1\}$ cover only the break point $a_b$ with $z_2 = \bar{a}_b = T^{-1}_f a_b$ for some $0 \leq l \leq q_n$ and satisfy properties 1), 2) of Lemma 4.5.

We write $\text{Dist}(z_1, z_2, z_3, z_4; T^q_i)$ in the following form

$$\text{Dist}(z_1, z_2, z_3, z_4; T^q_i) = \text{Dist}(T^i_j[z_1, z_2], T^i_j[z_2, z_3] T^i_j[z_3, z_4]; T_f) \times \prod_{0 \leq i < q_n} \text{Dist}(T^i_j[z_1, z_2], T^i_j[z_2, z_3], T^i_j[z_3, z_4]; T_f).$$

For sufficiently large $n$ and any $\varepsilon > 0$ the product over $i \neq l, p$ in (53) takes it value in an $\varepsilon$-neighbourhood of 1. By assumption, only the interval $T^i_j[z_1, z_4]$ contains the break point $a_b$ with $a_b = T^i_j z_2$. Using Lemma 3.3 we find

$$|\text{Dist}(T^i_j z_1, T^i_j z_2, T^i_j z_3, T^i_j z_4; T_f)| = \frac{\sigma(a_b)(1 + \xi(l))}{\sigma(a_b) + \xi(l)} \leq K_1\int_{z_1} |D^2 f(y)|dy,$$

where the constant $K_1 > 0$ depends only on the function $f$ and where $\xi(l) = \frac{\partial}{\partial l} z_3 - z_2$.

Obviously

$$\frac{\sigma(a_b)(1 + \xi(l))}{\sigma(a_b) + \xi(l)} - 1 = \frac{(\sigma(a_b) - 1)\xi(l)}{\sigma(a_b) + \xi(l)}.$$

Using this and the inequalities $R_2^{-1} \leq \xi(l) \leq R_2$, following from (21), and the comparability of the intervals $[z_s, z_{s+1}]$ for $s = 1, 2, 3$, we obtain

$$R_3^{-1} \leq \frac{(\sigma(a_b) - 1)\xi(l)}{\sigma(a_b) + \xi(l)} \leq R_3.$$
where the constants \( R_i > 0, \ i = 1, 2 \) depend only on \( f \). Finally we obtain
\[
|\text{Dist}(T_f^iz_1, T_f^iz_2, T_f^iz_3, T_f^iz_4; T_f) - 1| \geq \text{const} > 0,
\]
where the constant again depends only on \( f \). As in the first case, the inequality
\[
|\prod_{0 \leq i < q_n \atop i \neq l} \text{Dist}(T_i^fz_1, T_i^fz_2, T_i^fz_3, T_i^fz_4; T_f) - 1| \leq \text{const} \varepsilon.
\]
holds true also in the present case. This together with (53) and (55) proves Lemma 4.6.

**Proof of Theorem 1.10** The idea of the proof of Theorem 1.10 is completely similar to the one of Theorem 1.8. Hence we will only give the construction of the intervals \([z_s, z_{s+1}], \ s = 1, 2, 3\), which play the key role in the proof.

Consider the \( n \)-th dynamical partition \( \xi_n(\overline{a}_b) \) of the preimage \( \overline{a}_b \) of the break point \( a_b \) in the interval \([T_f^{n_0}x_0, T_f^{n_0-1}x_0] \) around the point \( x_0 \), at which there exists a positive derivative \( DT_\psi(x_0) \) of the conjugating homeomorphism \( T_\psi(x) \). Since the rotation number \( \rho_f \) is irrational of bounded type there exists a subsequence \( \{n_k, k = 1, 2, \ldots\} \in \mathbb{N} \) such that for every \( n_k \) the interval \([\overline{a}_b, T_f^{q_{nk}-1}\overline{a}_b]\) respectively the interval \([T_f^{q_{nk}}\overline{a}_b, \overline{a}_b]\) contains the point \( \overline{c}_b = T_f^{-p}\overline{c}_b \) for some \( p \in [0, q_{nk}) \) and such that furthermore \( K_3^{-1} \leq \frac{\overline{c}_b - \overline{a}_b}{T_f^{q_{nk}-1}\overline{a}_b - \overline{c}_b} \leq K_3 \) respectively \( K_3^{-1} \leq \frac{\overline{c}_b - T_f^{-p}\overline{c}_b}{T_f^{q_{nk}}\overline{a}_b - \overline{c}_b} \leq K_3 \) holds, where the constant \( K_3 \) depends only on \( f \). Set
\[
d_{nk} = \min \left\{ |\overline{c}_b - \overline{a}_b|, |T_f^{q_{nk}-1}\overline{a}_b - \overline{c}_b| \right\}
\]
and define the points
\[
z_2 = \overline{c}_b, \ z_1 = \overline{c}_b - \frac{1}{2}d_{nk}, \ z_3 = \overline{c}_b + \frac{1}{2}d_{nk}, \ z_4 = \overline{c}_b + d_{nk}.
\]
As in the proof of Lemma 4.5 it can be checked that the intervals \([z_s, z_{s+1}], \ s = 1, 2, 3\) then satisfy the statements of this Lemma.

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