MONODROMY OF A FAMILY OF HYPERSURFACES

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Abstract. Let $Y$ be an $(m+1)$-dimensional irreducible smooth complex projective variety embedded in a projective space. Let $Z$ be a closed subscheme of $Y$, and $\delta$ be a positive integer such that $I_{Z,Y}(\delta)$ is generated by global sections. Fix an integer $d \geq \delta + 1$, and assume the general divisor $X \in |H^0(Y, I_{Z,Y}(d))|$ is smooth. Denote by $H^m(X; \mathbb{Q})_{\text{van}}$ the quotient of $H^m(X; \mathbb{Q})$ by the cohomology of $Y$ and also by the cycle classes of the irreducible components of dimension $m$ of $Z$. In the present paper we prove that the monodromy representation on $H^m(X; \mathbb{Q})_{\text{van}}$ for the family of smooth divisors $X \in |H^0(Y, I_{Z,Y}(d))|$ is irreducible.

RÉSUMÉ. Soit $Y$ une variété projective complexe lisse irréductible de dimension $m+1$, plongée dans un espace projectif. Soit $Z$ un sous-schéma fermé de $Y$, et soit $\delta$ un entier positif tel que $I_{Z,Y}(\delta)$ soit engendré par ses sections globales. Fixons un entier $d \geq \delta + 1$, et supposons que le diviseur général $X \in |H^0(Y, I_{Z,Y}(d))|$ soit lisse. Désignons par $H^m(X; \mathbb{Q})_{\text{van}}$ le quotient de $H^m(X; \mathbb{Q})$ par la cohomologie de $Y$ et par les classes des composantes irréductibles de $Z$ de dimension $m$. Dans cet article nous prouvons que la représentation de monodromie sur $H^m(X; \mathbb{Q})_{\text{van}}$ pour la famille des diviseurs lisses $X \in |H^0(Y, I_{Z,Y}(d))|$ est irréductible.

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1. Introduction

In this paper we provide an affirmative answer to a question formulated in [10].

Let $Y \subseteq \mathbb{P}^N$ $(\text{dim } Y = m + 1)$ be an irreducible smooth complex projective variety embedded in a projective space $\mathbb{P}^N$, $Z$ be a closed subscheme of $Y$, and $\delta$ be a positive integer such that $I_{Z,Y}(\delta)$ is generated by global sections. Assume that for $d \gg 0$ the general divisor $X \in |H^0(Y, I_{Z,Y}(d))|$ is smooth. In the paper [10] it is proved that this is equivalent to the fact that the strata $Z_{(j)} = \{x \in Z : \text{dim} T_x Z = j\}$, where $T_x Z$ denotes the Zariski tangent space, satisfy the following inequality:

\begin{equation}
\text{dim } Z_{(j)} + j \leq \text{dim } Y - 1 \quad \text{for any } j \leq \text{dim } Y.
\end{equation}

This property implies that, for any $d \geq \delta$, there exists a smooth hypersurface of degree $d$ which contains $Z$ ([10], 1.2. Theorem).

It is generally expected that, for $d \gg 0$, the Hodge cycles of the general hypersurface $X \in |H^0(Y, I_{Z,Y}(d))|$ depend only on $Z$ and on the ambient variety $Y$. A very precise conjecture in this direction was made in [10]:
Theorem 1.1. Assume \( \deg X \geq \delta + 1 \). Then the monodromy representation on \( H^m(X; \mathbb{Q})_{\text{van}} \) for the family of smooth divisors \( X \in |H^0(Y, \mathcal{O}_Y(d))| \) containing \( Z \) as above is irreducible.

We denote by \( H^m(X; \mathbb{Q})_{\text{van}} \) the subspace of \( H^m(X; \mathbb{Q})_{\text{van}} \) generated by the cycle classes of the maximal dimensional irreducible components of \( Z \) modulo the image of \( H^m(Y; \mathbb{Q}) \) (using the orthogonal decomposition \( H^m(X; \mathbb{Q}) = H^m(Y; \mathbb{Q}) \perp H^m(X; \mathbb{Q})_{\text{van}} \)) if \( m = 2 \dim Z \), and \( H^m(X; \mathbb{Q})_{\text{van}} = 0 \) otherwise, and we denote by \( H^m(X; \mathbb{Q})_{\text{van}} \) the orthogonal complement of \( H^m(X; \mathbb{Q})_{\text{van}} \) in \( H^m(X; \mathbb{Q})_{\text{van}} \). The conjecture above cannot be strengthened because, even in \( Y = \mathbb{P}^3 \), there exist examples for which \( \dim H^m(X; \mathbb{Q})_{\text{van}} \) is arbitrarily large and the monodromy representation associated to the linear system \( |H^0(Y, \mathcal{I}_{Z, Y}(\delta))| \) is diagonalizable.

The Authors of [10] observed that a proof for such a conjecture would confirm the expectation above and would reduce the Hodge conjecture for the general hypersurface \( X_\ell \in |H^0(Y, \mathcal{I}_{Z, Y}(\delta))| \) to the Hodge conjecture for \( Y \). More precisely, by a standard argument, from Conjecture [1] it follows that when \( m = 2 \dim Z \) and the vanishing cohomology of the general \( X_\ell \in |H^0(Y, \mathcal{I}_{Z, Y}(\delta))| \) (\( d \geq \delta + 1 \)) is not of pure Hodge type \( (m/2, m/2) \), then the Hodge cycles in the middle cohomology of \( X_\ell \) are generated by the image of the Hodge cycles on \( Y \) together with the cycle classes of the irreducible components of \( Z \). So, the Hodge conjecture for \( X_\ell \) is reduced to that for \( Y \) (compare with [10], Corollary 0.5). They also proved that the conjecture is satisfied in the range \( d \geq \delta + 2 \), or for \( d = \delta + 1 \) if hyperplane sections of \( Y \) have non trivial top degree holomorphic forms (10, 0.4. Theorem). Their proof relies on Deligne’s semisimplicity Theorem and on Steenbrink’s Theory for semistable degenerations.

Arguing in a different way, we prove in this paper Conjecture [1] in full. More precisely, avoiding degeneration arguments, in Section 2 we will deduce Conjecture [1] from the following:

**Theorem 1.1.** Fix integers \( 1 \leq k < d \), and let \( W = G \cap X \subset Y \) be a complete intersection of smooth divisors \( G \in |H^0(Y, \mathcal{O}_Y(k))| \) and \( X \in |H^0(Y, \mathcal{O}_Y(d))| \). Then the monodromy representation on \( H^m(X; \mathbb{Q})_{\text{van}} \) for the family of smooth divisors \( X_\ell \in |H^0(Y, \mathcal{O}_Y(d))| \) containing \( W \) is irreducible.

Here we define \( H^m(X; \mathbb{Q})_{\text{van}} \) in a similar way as before, i.e. as the orthogonal complement in \( H^m(X; \mathbb{Q})_{\text{van}} \) of the image \( H^m(X; \mathbb{Q})_{\text{van}} \) of the map obtained by composing the natural maps \( H_m(W; \mathbb{Q}) \to H_m(X; \mathbb{Q}) \cong H^m(X; \mathbb{Q}) \to H^m(X; \mathbb{Q})_{\text{van}} \).

The proof of Theorem [1.1] will be given in Section 4 and consists in a Lefschetz type argument applied to the image of the rational map on \( Y \) associated to the linear system \( |H^0(Y, \mathcal{I}_{W, Y}(\delta))| \), which turns out to have at worst isolated singularities. This approach was started in our paper [2] where we proved a particular case of Theorem [1.1] but the proof given here is independent and much simpler.

We begin by proving Conjecture [1] as a consequence of Theorem [1.1] and next we prove Theorem [1.1].
2. Proof of Conjecture [1] as a consequence of Theorem [1.1]

We keep the same notation we introduced before, and need further preliminaries.

Notations 2.1. (i) Let $V_{\delta} \subseteq H^0(Y, I_{Z,Y}(\delta))$ be a subspace generating $I_{Z,Y}(\delta)$, and $V_d \subseteq H^0(Y, I_{Z,Y}(d))$ ($d \geq \delta + 1$) be a subspace containing the image of $V_{\delta} \otimes H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(d - \delta))$ in $H^0(Y, I_{Z,Y}(d))$. Let $G \in |V_{\delta}|$ and $X \in |V_d|$ be divisors. Put $W := G \cap X$. From condition (1), and [10], 1.2. Theorem, we know that if $G$ and $X$ are general then they are smooth. Moreover, by ([4], p. 133, Proposition 4.26. and proof), we know that if $G$ and $X$ are smooth then $W$ has only isolated singularities.

(ii) In the case $m > 2$, fix a smooth $G \in |V_{\delta}|$. Let $H \in |H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(l))|$ be a general hypersurface of degree $l > 0$, and put $Z' := Z \cap H$ and $G' := G \cap H$. Denote by $V_{\delta}' \subseteq H^0(G', I_{Z,Y}(G';d))$ the restriction of $V_{d}$ on $G'$, and by $V_d' \subseteq H^0(G, I_{Z,Y}(G;d))$ the restriction of $V_d$ on $G$. Since $H^0(G, I_{Z,Y}(G;d)) \subseteq H^0(G', I_{Z,Y}(G';d))$, we may identify $V_{\delta}' = V_d'$. Put $W' := W \cap H \in |V_d'|$. Similarly as we did for the triple $(Y, X, Z)$, using the orthogonal decomposition $H^{m-2}(W';\mathbb{Q}) \perp H^{m-2}(W';\mathbb{Q})_{\text{van}}$, we define the subspaces $H^{m-2}(W';\mathbb{Q})_{\text{van}}$ and $H^{m-2}(W';\mathbb{Q})_{\text{van}}$, of $H^{m-2}(W';\mathbb{Q})$ with respect to the triple $(G', W', Z')$. Passing from $(Y, X, Z)$ to $(G', W', Z')$ will allow us to prove Conjecture [1] arguing by induction on $m$ (see the proof of Proposition 2.2 below).

(iii) Let $\varphi : U' \to |V_d'|$ ($U' \subseteq G \times |V_d'|$) be the universal family parametrizing the divisors $W = G \cap X \in |V_d'|$. Denote by $\sigma : \tilde{U} \to W$ a desingularization of $W$, and by $U \subseteq |V_d'|$ a nonempty open set such that the restriction $(\varphi \circ \sigma)_U : (\varphi \circ \sigma)^{-1}(U) \to U$ is smooth. Next, let $\psi : U'' \to |V_d'|$ ($U'' \subseteq G \times |V_d'|$) be the universal family parametrizing the divisors $W'' = W \cap H \in |V_d'|$, and denote by $U'' \subseteq |V_d'|$ a nonempty open set such that the restriction $\psi_{U''} : \psi^{-1}(U'') \to U''$ is smooth. Shrinking $U$ and $U''$ if necessary, we may assume $U := U = U'' \subseteq |V_d'| = |V_d'|$. For any $t \in U$ put $W_t := \varphi^{-1}(t)$, $\tilde{W}_t := \sigma^{-1}(W_t)$, and $W_t' := \psi^{-1}(t)$. Observe that $W_t \cap \text{Sing}(W) \subseteq \text{Sing}(W_t)$, so we may assume $W_t' = W_t \cap H \subseteq W_t \cap \text{Sing}(W_t) \subseteq W_t$. Denote by $\iota_t$ and $\iota_t'$ the inclusion maps $W'_t \to W_t$ and $W_t' \to W_t$. The pull-back maps $\iota_t^* : H^{m-2}((\varphi \circ \sigma)_{U(t)}, \mathbb{Q}) \to H^{m-2}(\psi_{U(t)}, \mathbb{Q})$ give rise to a natural map $\tilde{\iota} : R^{m-2}(\varphi \circ \sigma)_{U(t)}, \mathbb{Q}) \to R^{m-2}(\psi_{U(t)}, \mathbb{Q})$ between local systems on $U$, showing that $Im(\iota_t^*)$ is globally invariant under the monodromy action on the cohomology of the smooth fibers of $\psi$. Finally, we recall that the inclusion map $\iota_t$ defines a Gysin map $\iota_t^* : H_m(W_t; \mathbb{Q}) \to H_{m-2}(W_t'; \mathbb{Q})$ (see [5], p. 382, Example 19.2.1).

Remark 2.2. Fix a smooth $G \in |V_{\delta}|$, and assume $m \geq 2$. The linear system $|V_d|$ induces an embedding of $G \setminus Z$ in some projective space: denote by $\Gamma$ the image of $G \setminus Z$ through this embedding. Since $G \setminus Z$ is irreducible, then also $\Gamma$ is, and so is its general hyperplane section, which is isomorphic to $(G \cap X) \setminus Z$ via $|V_d|$. So we see that, when $m \geq 2$, for any smooth $G \in |V_{\delta}|$ and any general $X \in |V_d|$, one has that $W \setminus Z$ is irreducible. In particular, when $m > 2$, then also $W$ is irreducible.
Lemma 2.3. Fix a smooth $G \in |V_\delta|$, and assume $m > 2$. Then, for a general $t \in U$, one has $\text{Im}(i_t^*) = \text{Im}(PD \circ i_t^*)$, and the map $PD \circ i_t^*$ is injective (PD means “Poincaré duality”: $H_{m-2}(W_t; \mathbb{Q}) \cong H^{m-2}(W_t^*; \mathbb{Q})$).

Proof. By ([14], p. 385, Proposition 16.23) we know that $\text{Im}(i_t^*)$ is equal to the image of the pull-back $H^{m-2}(W_t \setminus \text{Sing}(W_t); \mathbb{Q}) \rightarrow H^{m-2}(W_t'; \mathbb{Q})$. On the other hand, by ([3], p. 157 Proposition 5.4.4., and p. 158 (PD)) we have natural isomorphisms involving intersection cohomology groups:

\begin{equation}
H^{m-2}(W_t \setminus \text{Sing}(W_t); \mathbb{Q}) \cong IH^{m-2}(W_t)
\end{equation}

So we may identify the pull-back $H^{m-2}(W_t \setminus \text{Sing}(W_t); \mathbb{Q}) \rightarrow H^{m-2}(W_t'; \mathbb{Q})$ with $PD \circ i_t^*$. This proves that $\text{Im}(i_t^*) = \text{Im}(PD \circ i_t^*)$. Moreover, since $W_t'$ is smooth, then $IH^{m-2}(W_t') \cong H^{m-2}(W_t'; \mathbb{Q})$ ([3], p. 157). So, from ([3], we may identify $PD \circ i_t^*$ with the natural map $IH^{m-2}(W_t) \rightarrow IH^{m-2}(W_t \cap H)$, which is injective in view of Lefschetz Hyperplane Theorem for intersection cohomology ([3], p. 158 (I), and p. 159, Theorem 5.4.6) (recall that $W_t' = W_t \cap H$).

We are in position to prove Conjecture [I]

Fix a smooth $G \in |V_\delta|$, and a general $X \in |V_d|$. Put $W = G \cap X$. Since the monodromy group of the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing $W$ is a subgroup of the monodromy group of the family of smooth divisors $X \in |H^0(Y, \mathcal{O}_Y(d))|$ containing $Z$, in order to deduce Conjecture [I] from Theorem [I,3] it suffices to prove that $H^m(X; \mathbb{Q})_{\Sigma Z}^{\text{van}} = H^m(X; \mathbb{Q})_{\Sigma W}^{\text{van}}$. Equivalently, it suffices to prove that $H^m(X; \mathbb{Q})_{\Sigma Z}^{\text{van}} = H^m(X; \mathbb{Q})_{\Sigma W}^{\text{van}}$. This is the content of the following:

Proposition 2.4. For any smooth $G \in |V_\delta|$ and any general $X \in |V_d|$, one has $H^m(X; \mathbb{Q})_{\Sigma Z}^{\text{van}} = H^m(X; \mathbb{Q})_{\Sigma W}^{\text{van}}$.

Proof. First we analyze the cases $m = 1$ and $m = 2$, and next we argue by induction on $m > 2$ (recall that $\dim Y = m + 1$).

The case $m = 1$ is trivial because in this case $\dim Z \leq \dim W = 0$.

Next assume $m = 2$. In this case $\dim Y = 3$ and $\dim Z \leq 1$. Denote by $Z_1, \ldots, Z_h$ ($h \geq 0$) the irreducible components of $Z$ of dimension 1 (if there are). Fix a smooth $G \in |V_\delta|$ and a general $X \in |V_d|$, and put $W = G \cap X = Z_1 \cup \cdots \cup Z_h \cup C$, where $C$ is the residual curve, with respect to $Z_1 \cup \cdots \cup Z_h$, in the complete intersection $W$. By Remark [22] we know that $C$ is irreducible. Then, as (co)cycle classes, $Z_1, \ldots, Z_h, C$ generate $H^2(X; \mathbb{Q})_{\Sigma W}^{\text{van}}$, and $Z_1, \ldots, Z_h$ generate $H^2(X; \mathbb{Q})_{\Sigma Z}^{\text{van}}$. Since $Z_1 + \cdots + Z_h + C = \delta H_X$ in $H^2(X; \mathbb{Q})$ ($H_X = \text{general hyperplane section of } X$ in $\mathbb{P}^N$), and this cycle comes from $H^2(Y; \mathbb{Q})$, then $Z_1 + \cdots + Z_h + C = 0$ in $H^2(X; \mathbb{Q})_{\Sigma W}^{\text{van}}$, and so $H^2(X; \mathbb{Q})_{\Sigma Z}^{\text{van}} = H^2(X; \mathbb{Q})_{\Sigma W}^{\text{van}}$. This concludes the proof of Proposition [2.4] in the case $m = 2$.

Now assume $m > 2$ and argue by induction on $m$. First we observe that the intersection pairing on $H^{m-2}(W_t'; \mathbb{Q})_{\Sigma Z}^{\text{van}}$ is non-degenerate: this follows from Hodge Index Theorem, because the cycles in $H^{m-2}(W_t'; \mathbb{Q})_{\Sigma Z}^{\text{van}}$ are primitive and algebraic.
So we have the following orthogonal decomposition:

\[(3) \quad H^{m-2}(W'; \mathbb{Q}) = H^{m-2}(G'; \mathbb{Q}) \perp H^{m-2}(W'; \mathbb{Q})^\vee \perp H^{m-2}(W'; \mathbb{Q})^\vee_{\perp L}.\]

Let \( J \) be the local system on \( U \) with fibre given by \( H^{m-2}(G'; \mathbb{Q}) \perp H^{m-2}(W'; \mathbb{Q})^\vee \). We claim that:

\[(4) \quad \text{Im}(\mathcal{I}^*) = J.\]

We will prove (4) shortly after. From (4) and Lemma 2.3 we get an isomorphism: \( H_m(W; \mathbb{Q}) \cong H^{m-2}(G'; \mathbb{Q}) \perp H^{m-2}(W'; \mathbb{Q})^\vee \). Taking into account that by Lefschetz Hyperplane Theorem we have \( H^{m-2}(Y; \mathbb{Q}) \cong H^{m-2}(G; \mathbb{Q}) \cong H^{m-2}(G'; \mathbb{Q}), \) and that the Gysin map \( H_m(Z; \mathbb{Q}) \to H_{m-2}(Z'; \mathbb{Q}) \) is bijective (because \( H_m(Z; \mathbb{Q}) \) and \( H_{m-2}(Z'; \mathbb{Q}) \) are simply generated by the components which are of dimension \( m \) or \( m-2 \) of \( Z \) and \( Z' \) (if there are)), one sees that the natural map \( H_m(W; \mathbb{Q}) \to H_m(X; \mathbb{Q}) \cong H^m(X; \mathbb{Q}) \) sends \( H^{m-2}(G'; \mathbb{Q}) \) in \( H^m(Y; \mathbb{Q}) \), and \( H^{m-2}(W'; \mathbb{Q})^\vee \) in \( H^m(X; \mathbb{Q})^\vee \). This proves \( H^m(X; \mathbb{Q})^\vee \cong H^m(X; \mathbb{Q})^\vee \). Since the reverse inclusion is obvious, it follows that \( H^m(X; \mathbb{Q})^\vee = H^m(X; \mathbb{Q})^\vee \).

So, to conclude the proof of Proposition 2.4 it remains to prove claim (4). To this purpose first notice that \( \text{Im}(\mathcal{I}^*) \) contains \( H^{m-2}(W'; \mathbb{Q})^\vee \), because, by Lemma 2.3 we have \( \text{Im}(\mathcal{I}^*) = \text{Im}(PD \circ \mathcal{I}^*) \) and \( \text{Im}(PD \circ \mathcal{I}^*) \supset H^{m-2}(W'; \mathbb{Q})^\vee \) in view of the quoted isomorphism \( H_m(Z; \mathbb{Q}) \cong H_m(Z'; \mathbb{Q}) \). Moreover \( \text{Im}(\mathcal{I}^*) \) contains \( H^{m-2}(G'; \mathbb{Q}) \) because \( H^{m-2}(G'; \mathbb{Q}) \cong H^{m-2}(G; \mathbb{Q}), \) and \( H^{m-2}(G; \mathbb{Q}) \) is contained in \( \text{Im}(\mathcal{I}^*) \). Therefore we obtain \( \text{Im}(\mathcal{I}^*) \supset J \), from which we deduce that \( \text{Im}(\mathcal{I}^*) = J \). In fact, otherwise, since by induction \( H^{m-2}(W_t'; \mathbb{Q})^\vee \) is irreducible, from (3) it would follow that \( \text{Im}(\mathcal{I}^*) = R^{m-2}(\psi_{\mathcal{I}^*}), \mathbb{Q} \). This is impossible because for \( t > 0 \) the dimension of \( H^{m-2}(W_t'; \mathbb{Q}) \) is arbitrarily large (by the way, we notice that the same argument proves that \( J \) is nothing but the invariant part of \( R^{m-2}(\psi_{\mathcal{I}^*}), \mathbb{Q} \)).

3. A Monodromy Theorem

In this section we prove a monodromy theorem (see Theorem 3.1 below), which we will use in next section for proving Theorem 1.1 and that we think of independent interest.

Let \( Q \subseteq P \) be an irreducible, reduced, non-degenerate projective variety of dimension \( m+1 \) (\( m \geq 0 \)), with isolated singular points \( q_1, \ldots, q_s \). Let \( L \in \mathbb{G}(1, \mathbb{P}^*) \) be a general pencil of hyperplane sections of \( Q \), and denote by \( Q_L \) the blowing-up of \( Q \) along the base locus of \( L \), and by \( f: Q_L \to L \) the natural map. The ramification locus of \( f \) is a finite set \( \{q_1, \ldots, q_s\} := \text{Sing}(Q) \cup \{q_{r+1}, \ldots, q_s\} \) where \( \{q_{r+1}, \ldots, q_s\} \) denotes the set of tangencies of the pencil. Set \( a_i := f(q_i), 1 \leq i \leq s \) (compare with [13], p. 304). The restriction map \( f: Q_L \setminus f^{-1}(\{a_1, \ldots, a_s\}) \to L \setminus \{a_1, \ldots, a_s\} \) is a smooth proper map. Hence the fundamental group \( \pi_1(L \setminus \{a_1, \ldots, a_s\}, t) (t = \text{general point of } L) \) acts by monodromy on \( Q_t := f^{-1}(t) \), and so on \( H^m(Q_t; \mathbb{Q}) \). By [11], p. 165–167, we know that \( f: Q_L \setminus f^{-1}(\{a_1, \ldots, a_s\}) \to L \setminus \{a_1, \ldots, a_s\} \) induces an orthogonal decomposition: \( H^m(Q_t; \mathbb{Q}) = I \perp V \), where \( I \) is the subspace of the invariant cocycles, and \( V \) is its orthogonal complement.
In the case $Q$ is smooth, a classical basic result in Lefschetz Theory states that $V$ is generated by “standard vanishing cycles” (i.e. by vanishing cycles corresponding to the tangencies of the pencil). This implies the irreducibility of $V$ by standard classical reasonings (§8, §14). Now we are going to prove that it holds true also when $Q$ has isolated singularities. This is the content of the following Theorem 3.1 for which we didn’t succeed in finding an appropriate reference (for a related and somewhat more precise statement, see Proposition 3.4 below).

**Theorem 3.1.** Let $Q \subseteq \mathbb{P}$ be an irreducible, reduced, non-degenerate projective variety of dimension $m + 1 \geq 1$, with isolated singularities, and $Q_{t}$ be a general hyperplane section of $Q$. Let $H^{m}(Q_{t}; \mathbb{Q}) = I \perp V$ be the orthogonal decomposition given by the monodromy action on the cohomology of $Q_{t}$, where $I$ denotes the invariant subspace. Then $V$ is generated, via monodromy, by standard vanishing cycles.

**Remark 3.2.** (i) For a particular case of Theorem 3.1 see [13], Theorem (2.2).

(ii) When $Q$ is a curve, i.e. when $m = 0$, then Theorem 3.1 follows from the well known fact that the monodromy group is the full symmetric group (see [1], pg. 111). So we assume from now on that $m \geq 1$.

(iii) When $Q$ is a cone over a degenerate and necessarily smooth subvariety of $\mathbb{P}$, then $f : Q_{L} \to L$ has only one singular fiber $f^{-1}(a_{1})$ (i.e. $s = 1$). In this case $\pi_{1}(L\{a_{1}\}, t)$ is trivial. Therefore we have that $H^{m}(Q_{t}; \mathbb{Q}) = I$, $V = 0$, and Theorem 3.1 follows.

Before proving Theorem 3.1 we need some preliminaries. We keep the same notation we introduced before.

**Notations 3.3.** (i) Let $R_{L} \to Q_{L}$ be a desingularization of $Q_{L}$. The decomposition $H^{m}(Q_{t}; \mathbb{Q}) = I \perp V$ can be interpreted via $R_{L}$ as $I = j^{*}(H^{m}(R_{L}; \mathbb{Q}))$ and $V = \ker(H^{m}(Q_{t}; \mathbb{Q}) \to H^{m+2}(R_{L}; \mathbb{Q})) \cong \ker(H_{m}(Q_{t}; \mathbb{Q}) \to H_{m}(R_{L}; \mathbb{Q}))$, where $j$ denotes the inclusion $Q_{t} \subset R_{L}$. Using standard arguments (compare with [14], p. 325, Corollaire 14.23) one deduces a natural isomorphism:

\[
V \cong \text{Im}(H_{m+1}(R_{L} - g^{-1}(t_{1}), Q_{t}; \mathbb{Q}) \to H_{m}(Q_{t}; \mathbb{Q})),
\]

where $g : R_{L} \to L$ denotes the composition of $R_{L} \to Q_{L}$ with $f : Q_{L} \to L$, and $t_{1} \neq t$ another regular value of $g$.

(ii) For any critical value $a_{i}$ of $L$ fix a closed disk $\Delta_{i} \subset L\{t_{1}\} \cong \mathbb{C}$ with center $a_{i}$ and radius $0 < \rho \ll 1$. As in [8], (5.3.1) and (5.3.2), one may prove that $H_{m+1}(R_{L} - g^{-1}(t_{1}), Q_{t}; \mathbb{Q}) \cong \bigoplus_{i=1}^{s} H_{m+1}(g^{-1}(\Delta_{i}), g^{-1}(a_{i} + \rho); \mathbb{Q})$. By [9] we have:

\[
V = V_{1} + \cdots + V_{s},
\]

where we denote by $V_{i}$ the image in $H^{m}(Q_{t}; \mathbb{Q}) \cong H_{m}(g^{-1}(a_{i} + \rho); \mathbb{Q})$ of each $H_{m+1}(g^{-1}(\Delta_{i}), g^{-1}(a_{i} + \rho); \mathbb{Q})$. When $r + 1 \leq i \leq s$, we recognize in each $V_{i}$ the subspace generated by the standard vanishing cocycle $\delta_{i}$ corresponding to a tangent hyperplane section of $Q$ (see [8], [13], [14]).
(iii) Consider again the pencil \( f : Q_L \to L \), and let \( \mathbb{P}_L \) be the blowing up of \( \mathbb{P} \) along the base locus \( B_L \). For any \( i \in \{1, \ldots, s\} \), denote by \( D_i \subset \mathbb{P}_L \) a closed ball with center \( q_i \) and small radius \( \epsilon \). Define \( M_i := Im(H_m(f^{-1}(a_i + \rho) \cap D_i; \mathbb{Q}) \to H_m(f^{-1}(a_i + \rho); \mathbb{Q})) \), with \( 0 < \rho < \epsilon \ll 1 \). Since \( H_m(f^{-1}(a_i + \rho); \mathbb{Q}) \cong H_m(Q; \mathbb{Q}) \cong H^m(Q_i; \mathbb{Q}) \), we may regard \( M_i \subseteq H^m(Q_i; \mathbb{Q}) \). When \( 1 \leq i \leq r \), \( M_i \) represents the subspace spanned by the cocycles “coming” from the singularities of \( Q \), and lying in the Milnor fibre \( f^{-1}(a_i + \rho) \cap D_i \). When \( r + 1 \leq i \leq s \), i.e. when \( a_i \) corresponds to a tangent hyperplane section of \( Q \), then \( V_i = M_i \). In general we have:

\[
V_i \subseteq M_i \quad \text{for any } i = 1, \ldots, s.
\]

This is a standard fact, that one may prove as in ([9], (7.13) Proposition). For Reader’s convenience, we give the proof of property \((7)\) in the Appendix, at the end of the paper.

Now we are going to prove Theorem \( \text{3.1} \)

Proof of Theorem \( \text{3.1} \). Let \( \pi : \mathcal{F} \to \mathbb{P}^* \) (\( \mathcal{F} \subseteq \mathbb{P}^* \times \mathbb{P} \)) be the universal family parametrizing the hyperplane sections of \( Q \subseteq \mathbb{P} \), and denote by \( \mathcal{D} \subseteq \mathbb{P}^* \) the discriminant locus of \( \pi \), i.e. the set of hyperplanes \( H \in \mathbb{P}^* \) such that \( Q \cap H \) is singular.

At least set-theoretically, we have \( \mathcal{D} = Q^* \cup \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_r \), where \( Q^* \) denotes the dual variety of \( Q \), and \( \mathcal{H}_j \) denotes the dual hyperplane of \( q_j \) (compare with [10], p. 303).

When the codimension of \( Q^* \) in \( \mathbb{P}^* \) is 1, denote by \( T_i \) the stalk at \( t \in \mathbb{P}^* \setminus \mathcal{D} \) of the local subsystem of \( R^m(\pi_{|_{\pi^{-1}(\mathbb{P}^* \setminus \mathcal{D})}})_Q \) generated by the vanishing cocycle at general point of \( Q^* \) (compare with [10], p. 373, or [13], p. 306). If the codimension of \( Q^* \) in \( \mathbb{P}^* \) is \( \geq 2 \), put \( T_i := \{0\} \). In order to prove Theorem \( \text{3.1} \), it suffices to prove that \( V = T \) (\( T := T_1 \)). By Deligne Complete Reducibility Theorem ([11], p. 167), we may write \( H^m(Q_i; \mathbb{Q}) = W \oplus T \), for a suitable invariant subspace \( W \). Now we claim the following proposition, which we will prove below:

**Proposition 3.4.** The monodromy representation on the quotient local system with stalk \( H^m(Q_i; \mathbb{Q})/T_i \) at \( t \in \mathbb{P}^* \setminus \mathcal{D} \) is trivial.

By previous Proposition \( \text{3.4} \), it follows that for any \( g \in \pi_1(L \setminus \{a_1, \ldots, a_s\}, t) \) and any \( w \in W \) there exists \( \tau \in T \) such that \( w^g = w + \tau \). Then \( \tau = w^g - w \in T \cap W = \{0\} \), and so \( w^g = w \). Therefore \( W \) is invariant, i.e. \( W \subseteq I \), and since \( T \subseteq V \) and \( H^m(Q_i; \mathbb{Q}) = I \oplus V = W \oplus T \), then we have \( T = V \). \( \square \)

It remains to prove Proposition \( \text{3.4} \). To this aim, we need some preliminaries. We keep the same notation we introduced before.

Consider again the universal family \( \pi : \mathcal{F} \to \mathbb{P}^* \) parametrizing the hyperplane sections of \( Q \subseteq \mathbb{P} \). We will denote by \( H_x \) the hyperplane parametrized by \( x \in \mathbb{P}^* \). Fix a point \( q_i \in \text{Sing}(Q) \) (hence \( i \in \{1, \ldots, r\} \)). For general \( L \), \( q_i \) is not a base point of the pencil defined by \( L \), hence \( Q_L \cong Q \) over \( q_i \). Combined with the inclusion \( Q_L \subseteq \mathcal{F} \), we thus have a natural lift of \( q_i \) to a point of \( \mathcal{F} \), still denoted by \( q_i \).
Remark 3.5. If $Q^*$ is contained in $\mathcal{H}_j$ for some $j \in \{1, \ldots, r\}$, then $Q^*$ is degenerate in $\mathbb{P}^*$, and so $Q = Q^{**}$ is a cone in $\mathbb{P}$. Therefore, if $Q$ is not a cone, then $Q^*$ is not contained in $\mathcal{H}_j$ for any $j \in \{1, \ldots, r\}$. In this case, for a general line $\ell \subseteq \mathcal{H}_i$, the set $\ell \cap Q^*$ is finite, and for any $x \in \ell$, $H_x \cap Q$ has an isolated singularity at $q_i$.

Notations 3.6. (i) Let $\ell \subseteq \mathcal{H}_i$ be a general line. For any $u \in \ell \cap Q^*$, denote by $\Delta_u^0$ an open disk of $\ell$ with center $u$ and small radius. Consider the compact $K := \ell((u \in \ell \cap Q^*, \Delta_u^0)$. In the Appendix below (see Lemma 5.1) we prove that there is a closed ball $D_{q_i} \subseteq \mathbb{P}^* \times \mathbb{P}$, with positive radius and centered at $q_i$, such that for any $x \in K$ the distance function $p \in H_x \cap Q \cap D_{q_i} \rightarrow ||p - q_i|| \in \mathbb{R}$ has no critical points $p \neq q_i$ (we already proved a similar result in [2], Lemma 3.4, (v)). By ([II], pp. 21-28) it follows that for any $x \in K$ there is a closed ball $C_x \subseteq \mathbb{P}^*$ centered at $x$, for which the induced map $z \in \pi^{-1}(C_x) \cap D_{q_i} \rightarrow \pi(z) \in C_x$ is a Milnor fibration, with discriminant locus given by $\mathcal{H}_i \cap C_x$. Since $K$ is compact, we may cover it with finitely many of such $C_x$'s. So we deduce the existence of a connected closed tubular neighborhood $\mathcal{K}$ of $K$ in $\mathbb{P}^*$, such that the map:

$$\pi_{\mathcal{K}} : z \in \pi^{-1}(K) \cap D_{q_i} \rightarrow \pi(z) \in \mathcal{K}$$

defines a $C^\infty$-fiber bundle on $\mathcal{K} \setminus \mathcal{H}_i$, and whose fibre $\pi_{\mathcal{K}}^{-1}(t) = H_t \cap Q \cap D_{q_i}$, $t \in \mathcal{K} \setminus \mathcal{H}_i$, may be identified with the Milnor fibre.

(ii) Let $\mathcal{M}_i$ be the local system with fibre $\mathcal{M}_{i,t}$ at $t \in \mathcal{K} \setminus \mathcal{D}$ given by the image of $H_m(H_t \cap Q \cap D_{q_i}; \mathbb{Q})$ in $H_m(H_t \cap Q; \mathbb{Q}) \cong H^m(Q_t; \mathbb{Q})$. Notice that, for any general pencil $L \in \mathbb{G}(1, \mathbb{P}^*)$, the local system $\mathcal{M}_i$ extends, as a local system, $M_i$ on all $L \cap (\mathcal{K} \setminus \mathcal{D})$ (compare with Notations 3.3 (iii)). In particular we may assume $M_i = \mathcal{M}_{i,t}$.

We are in position to prove Proposition 3.4. We keep the same notation we introduced before.

Proof of Proposition 3.4. As in ([III], proof of Theorem (2.2)), we need to consider only the action of $\pi_1(\mathbb{P}^* \setminus (\bigcup_{1 \leq j \leq r} \mathcal{H}_j), t)$.

Consider the finite set $A := \ell \cap (\bigcup_{j \neq i} \mathcal{H}_j)$, and let $a \in A$ be a point. In view of Remark 3.2 (iii), and Remark 3.5 we may assume that $H_a \cap Q$ has an isolated singularity at $q_i$. Notice that, a priori, it may happen that $a \in \ell \cap Q^*$ and so $a \notin K$. But in any case, since $H_a \cap Q$ has an isolated singularity at $q_i$, as before, for any $a \in A$ we may construct a closed ball $D_{q_i}^{(a)} \subseteq \mathbb{P}^* \times \mathbb{P}$, with positive radius and centered at $q_i$, and a closed ball $C_a \subseteq \mathbb{P}^*$ centered at $a$, for which the induced map

$$z \in \pi^{-1}(C_a) \cap D_{q_i}^{(a)} \rightarrow \pi(z) \in C_a$$

is a Milnor fibration with discriminant locus contained in $\mathcal{H}_i \cup Q^*$. We may assume $D_{q_i} \subseteq D_{q_i}^{(a)}$ for any $a \in A$, and, shrinking the disks $\Delta_u^0 (u \in \ell \cap Q^*)$ if necessary, we may also assume that the interior $K^0$ of $K$ meets the interior $C_a^0$ of each $C_a$. Therefore, in $(K^0 \cap C_a^0) \setminus (\mathcal{H}_i \cup Q^*)$, the bundle (8) appears as a subbundle of (9).

Observe that the image in $H^m(Q_t; \mathbb{Q})/T_t$ of the cohomology of (9) coincides with $(\mathcal{M}_i + T_t)/T_t$ on $(K^0 \cap C_a^0) \setminus (\mathcal{H}_i \cup Q^*)$. This implies that, in a suitable small analytic neighborhood $\mathcal{L}$ of $\ell$ in $\mathbb{P}^*$, the quotient local system $(\mathcal{M}_{i,t} + T_t)/T_t$
extends on all \( \mathcal{L} \setminus \mathcal{D} \). Taking into account Picard-Lefschetz formula, and that the discriminant locus of \([\Theta]\) is contained in \( \mathcal{H}_i \cup \mathcal{Q}^* \), we have that \( \pi_1(\mathbb{P} \setminus \mathcal{D}, t) \) acts trivially on \( (\mathcal{M}_{i,t} + T_i)/T_i \). This holds true for any \( i \in \{1, \cdots, r\} \). Hence, in view of \([\Theta]\) and \([\mathcal{S}]\), it follows that the monodromy action is trivial on \( H^m(\mathcal{Q}_i; \mathbb{Q})/T_i \). This concludes the proof of Proposition \[3.3\]. \[\square\]

By standard classical reasonings as in \([8]\) or \([14]\), from Theorem \[3.1\] we deduce the following:

**Corollary 3.7.** \( V \) is irreducible.

**Proof.** Let \( \{0\} \neq V' \subset V \) be an invariant subspace. As before, we may write \( H^m(\mathcal{Q}_i; \mathbb{Q}) = U \oplus V' \), for a suitable invariant subspace \( U \). Hence we have \( V = (V \cap U) \oplus V' \). On the other hand, one knows that \( V \) is nondegenerate with respect to the intersection form \( < \cdot , \cdot > \) on \( \mathcal{Q}_i \) (\[11\], p.167). Therefore, for some \( i \in \{r+1, \ldots, s\} \), there exists \( \tau \in (V \cap U) \cup V' \) such that \( \tau, \delta_i \neq 0 \) (\( \text{Span}(\delta_i) := \mathcal{V}_i \)). From the Picard-Lefschetz formula it follows that the tangential vanishing cycle \( \delta_i \) lies in \( (V \cap U) \cup V' \). If \( \delta_i \in V \cap U \), then by Theorem \[3.1\] we deduce \( V = V \cap U \) (compare with \([8]\), \([9]\), \([13]\), \([14]\)), and this is in contrast with the fact that \( \{0\} \neq V' \). Hence \( \delta_i \in V' \), and by the same reason \( V' = V \). This proves that \( V \) is irreducible. \[\square\]

### 4. Proof of Theorem 1.1

#### 4.1. The set-up

Consider the rational map \( Y \dashrightarrow \mathbb{P} := \mathbb{P}(H^0(Y, \mathcal{I}_{W,Y}(d))^*) \) defined by the linear system \( [H^0(Y, \mathcal{I}_{W,Y}(d))]| \). By \([5]\), 4.4, such a rational map defines a morphism \( Bl_W(Y) \to \mathbb{P} \). We denote by \( Q \) the image of this morphism, i.e.:

\[
Q := \text{Im}(Bl_W(Y) \to \mathbb{P}).
\]

Set \( E := \mathbb{P}(\mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d)) \). The surjections \( \mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d) \to \mathcal{O}_Y(d) \) and \( \mathcal{O}_Y(k) \oplus \mathcal{O}_Y(d) \to \mathcal{O}_Y(k) \) give rise to divisors \( \Theta \cong Y \subseteq E \) and \( \Gamma \cong Y \subseteq E \), with \( \Theta \cap \Gamma = \emptyset \). The line bundle \( \mathcal{O}_E(\Theta) \) is base point free and the corresponding morphism \( E \to \mathbb{P}(H^0(E, \mathcal{O}_E(\Theta))^*) \) sends \( E \) to a cone over the Veronese variety of \( Y \) (i.e. over \( Y \) embedded via \( |H^0(Y, \mathcal{O}_Y(d-k))| \)) in such a way that \( \Gamma \) is contracted to the vertex \( v_\infty \) and \( \Theta \) to a general hyperplane section. In other words, we may view \( E \), via \( E \to \mathbb{P}(H^0(E, \mathcal{O}_E(\Theta))^*) \), as the blowing-up of the cone over the Veronese variety at the vertex, and \( \Gamma \) as the exceptional divisor (\([8]\), p. 374, Example 2.11.4).

From the natural resolution of \( \mathcal{I}_{W,Y} : 0 \to \mathcal{O}_Y(-k-d) \to \mathcal{O}_Y(-k) \oplus \mathcal{O}_Y(-d) \to \mathcal{I}_{W,Y} \to 0 \), we find that \( Bl_W(Y) = \text{Proj}(\oplus_{d \geq 0} \mathcal{I}_{W,Y}) \) is contained in \( E \), and that \( \mathcal{O}_E(\Theta - d\Lambda) |_{Bl_W(Y)} \cong \mathcal{O}_{Bl_W(Y)}(1) \) (\( \Lambda := \text{pull-back of the hyperplane section of} \ Y \subseteq \mathbb{P}^N \text{ through} \ E \to Y \)). Therefore:

(i) we have natural isomorphisms: \( H^0(Y, \mathcal{I}_{W,Y}(d)) \cong H^0(Y, \mathcal{O}_Y \oplus \mathcal{O}_Y(d-k)) \cong H^0(E, \mathcal{O}_E(\Theta)) \);

(ii) the linear series \( |\Theta| \) cut on \( Bl_W(Y) \) the linear series spanned by the strict transforms \( \bar{X} \) of the divisors \( X \in |H^0(Y, \mathcal{I}_{W,Y}(d))| \), and, sending \( E \) to a cone in \( \mathbb{P} \) over a Veronese variety, restricts to \( Bl_W(Y) \) to the map \( Bl_W(Y) \to Q \) defined
above. Hence we have a natural commutative diagram:

\[
\begin{array}{c}
\text{Bl}_W(Y) \hookrightarrow E \\
\downarrow \\
Y \twoheadrightarrow Q \hookrightarrow \mathbb{P}.
\end{array}
\]

By the same reason \( \Gamma \cap \text{Bl}_W(Y) = \tilde{G} \) (\( \tilde{G} := \text{the strict transform of} \ G \) in \( \text{Bl}_W(Y) \)). Notice that \( \tilde{G} \cong G \) since \( W \) is a Cartier divisor in \( G \). Similarly \( \tilde{X} \cong X \) when \( G \) is not contained in \( X \);

(iii) since \( \{\Theta\} \) contracts \( \Gamma \) to the vertex \( v_\infty \), the map \( \text{Bl}_W(Y) \to Q \) contracts \( \tilde{G} \) to \( v_\infty \in Q \). Furthermore we have \( \text{Bl}_W(Y)\setminus \tilde{G} \cong Q\setminus\{v_\infty\} \) and so the hyperplane sections of \( Q \) not containing the vertex are isomorphic, via \( \text{Bl}_W(Y) \to Q \), to the corresponding divisors \( X \in |H^0(Y, I_{W,Y}(d))| \);

(iv) by (ii) above, \( \tilde{G} \) is a smooth Cartier divisor in \( \text{Bl}_W(Y) \), hence \( \tilde{G} \) is disjoint with \( \text{Sing}(\text{Bl}_W(Y)) \). On the other hand, from (4), p. 133, Proposition 4.2.6. and proof) we know that \( \text{Sing}(W) \) is a finite set. The singularities of \( \text{Bl}_W(Y) \) must be contained in the inverse image of \( \text{Sing}(W) \) via \( \text{Bl}_W(Y) \to Y \): this is a finite set of lines none of which lying in \( \text{Sing}(\text{Bl}_W(Y)) \) because \( \tilde{G} \) meets all such lines. Therefore \( \text{Sing}(\text{Bl}_W(Y)) \) must be a finite set, and so also \( \text{Sing}(Q) \) is. Observe also that \( \tilde{G} \) is isomorphic to the tangent cone to \( Q \) at \( v_\infty \), and its degree is \( k(d-k)^m \deg Y \). Hence \( Q \) is nonsingular at \( v_\infty \) only when \( Y = \mathbb{P}^{m+1} \), \( k = 1 \) and \( d = 2 \). In this case \( X \) is a smooth quadric, therefore \( \dim H^m(X; \mathbb{Q})^{\text{van}}_{\text{Bl}_W(Y)} \leq 1 \), and Theorem 1.1 is trivial. So we may assume \( v_\infty \in \text{Sing}(Q) \).

4.2. The proof. We are going to prove Theorem 1.1 that is the irreducibility of the monodromy action on \( H^m(X; \mathbb{Q})^{\text{van}}_{\text{Bl}_W(Y)} \). The proof consists in an application of previous Corollary 3.7 to the variety \( Q \subseteq \mathbb{P} \) defined in (11). We keep the same notation we introduced in (11).

\textit{Proof of Theorem 1.1.} Consider the variety \( Q \subseteq \mathbb{P} \) defined in (11). By the description of it given in (11), we know that \( Q \) is an irreducible, reduced, non-degenerate projective variety of dimension \( m+1 \geq 2 \), with isolated singularities.

Let \( L \in \mathbb{G}(1, \mathbb{P}^n) \) be a general pencil of hyperplane sections of \( Q \), and denote by \( Q_L \) the blowing-up of \( Q \) along the base locus of \( L \), and by \( f : Q_L \rightarrow L \) the natural map (compare with Section 3). Denote by \( \{a_1, \ldots, a_s\} \subseteq L \) the set of the critical values of \( f \). The fundamental group \( \pi_1(L\setminus\{a_1, \ldots, a_s\}, t) \) \( (t = \text{general point of} \ L) \) acts by monodromy on \( f^{-1}(t) \), and so on \( H^m(f^{-1}(t); \mathbb{Q}) \), and this action induces an orthogonal decomposition: \( H^m(f^{-1}(t); \mathbb{Q}) = I \perp V \), where \( I \) is the subspace of the invariant cocycles, and \( V \) is its orthogonal complement. By Corollary 3.7 we know that \( V \) is irreducible.

On the other hand, in view of (11) we may identify \( f^{-1}(t) \) with a general \( X_t \in |H^0(Y, I_{W,Y}(d))| \), and the action of \( \pi_1(L\setminus\{a_1, \ldots, a_s\}, t) \) with the action induced on \( X_t \) by a general pencil of divisors in \( |H^0(Y, I_{W,Y}(d))| \). So, in order to prove Theorem 1.1 it suffices to prove that \( H^m(X_t; \mathbb{Q})^{\text{van}}_{\text{Bl}_W(Y)} = V \). This is equivalent to prove that \( I = H^m(Y; \mathbb{Q}) + H^m(X_t; \mathbb{Q})^{\text{van}}_{\text{Bl}_W(Y)} \). Since the inclusion \( H^m(Y; \mathbb{Q}) + H^m(X_t; \mathbb{Q})^{\text{van}}_{\text{Bl}_W(Y)} \subseteq I \)
is obvious, to prove Theorem (11) it suffices to prove that:

\[(11) \quad I \subseteq H^m(Y; \mathbb{Q}) + H^m(X_t; \mathbb{Q})^{\text{van}}.\]

To this purpose, let \(B_L \subseteq Q\) be the base locus of \(L\). Since \(v_\infty \notin B_L\), then we may regard \(B_L \subseteq Bl_W(Y)\) via \(Bl_W(Y) \to Q\). Notice that \(B_L \cong X_t \cap M_L\), for a suitable general \(M_L \in |H^0(Y, O_Y(d-k))|\). Let \(B_W(Y)\) be the blowing-up of \(Bl_W(Y)\) along \(B_L\), and consider the pencil \(f_1 : Bl_W(Y) \to L\) induced from the natural map \(B_W(Y)_L \to Q_L\). We have \(Q_L \setminus f_1^{-1}(\{a_1, \ldots, a_s\}) \cong Bl_W(Y)_L \setminus f_1^{-1}(\{a_1, \ldots, a_s\})\).

So, if \(R_L \to B_W(Y)_L\) denotes a desingularization of \(B_W(Y)_L\), then the subspace \(I\) of the invariant cocycles can be interpreted via \(R_L\) as \(I = j^*(H^m(R_L; \mathbb{Q}))\), where \(j\) denotes the inclusion \(X_t \subseteq R_L\).

Denote by \(\tilde{W}\) and \(\tilde{B}_L\) the inverse images of \(W \subseteq Y\) and \(B_L \subseteq Bl_W(Y)\) in \(R_L\). The map \(R_L \to Y\) induces an isomorphism \(\alpha_1 : R_L \setminus (\tilde{W} \cup \tilde{B}_L) \to Y \setminus (W \cup (X_t \cap M_L))\). Consider the following natural commutative diagram:

\[
\begin{array}{ccc}
H^m(R_L; \mathbb{Q}) & \xrightarrow{\rho} & H^m(R_L \setminus (\tilde{W} \cup \tilde{B}_L); \mathbb{Q}) \\
\alpha \downarrow & & \| \alpha_1 \\
H^m(Y; \mathbb{Q}) & \xrightarrow{\rho} & H^m(Y \setminus (W \cup (X_t \cap M_L)); \mathbb{Q}) \\
\beta \downarrow & & \| \beta_1 \\
H^m(X_t; \mathbb{Q}) & \xrightarrow{\rho} & H^m(X_t \setminus (W \cup (X_t \cap M_L)); \mathbb{Q})
\end{array}
\]

where \(\alpha\) is the Gysin map, and fix \(c \in I = j^*(H^m(R_L; \mathbb{Q}))\). Let \(c' \in H^m(R_L; \mathbb{Q})\) such that \(j^*(c') = c\). Since \(\beta_1 \circ \alpha_1 \circ \rho_1 = \rho_3 \circ j^*\), then we have: \(\rho_3(c) = (\rho_3 \circ \beta \circ \alpha)(c')\). Hence we have \(c - \beta(\alpha(c')) \in \text{Ker} \rho_3 = \text{Im}(H^m(X_t, X_t \setminus (W \cup (X_t \cap M_L)); \mathbb{Q}) \to H^m(X_t; \mathbb{Q}))\). Since \(H^m(X_t, X_t \setminus (W \cup (X_t \cap M_L)); \mathbb{Q}) \cong H^m(W \cup (X_t \cap M_L); \mathbb{Q})\) (15, (3), p. 371), we deduce \(c - \beta(\alpha(c')) \in \text{Im}(H_m(W \cup (X_t \cap M_L); \mathbb{Q}) \to H_m(X_t; \mathbb{Q}) \cong H^m(X_t; \mathbb{Q}))\). So to prove (11), it suffices to prove that \(\text{Im}(H_m(W \cup (X_t \cap M_L); \mathbb{Q}) \to H_m(X_t; \mathbb{Q}) \cong H^m(X_t; \mathbb{Q}))\) is contained in \(H^m(Y; \mathbb{Q}) + \text{Im}(H_m(W; \mathbb{Q}) \to H_m(X_t; \mathbb{Q}) \cong H^m(X_t; \mathbb{Q}))\).

Since \(W\) has only isolated singularities, and \(M_L\) is general, then \(W \cap M_L\) and \(X_t \cap M_L\) are smooth complete intersections. From Lefschetz Hyperplane Theorem and Hard Lefschetz Theorem it follows that the natural map \(H_{m-1}(W \cap M_L; \mathbb{Q}) \to H_{m-1}(X_t \cap M_L; \mathbb{Q})\) is injective. Hence, from the Mayer-Vietoris sequence of the pair \((W, X_t \cap M_L)\) we deduce that the natural map \(H_m(W; \mathbb{Q}) \oplus H_m(X_t \cap M_L; \mathbb{Q}) \to H_m(W \cup (X_t \cap M_L); \mathbb{Q})\) is surjective. So to prove (11) it suffices to prove that \(\text{Im}(H_m(X_t \cap M_L; \mathbb{Q}) \to H_m(X_t; \mathbb{Q}) \cong H^m(X_t; \mathbb{Q}))\) is contained in \(H^m(Y; \mathbb{Q})\). And this follows from the natural commutative diagram:

\[
\begin{array}{ccc}
H_m(X_t \cap M_L; \mathbb{Q}) & \cong & H^{m-2}(X_t \cap M_L; \mathbb{Q}) \\
\downarrow & & \rho \\
H_m(X_t; \mathbb{Q}) & \cong & H^m(X_t; \mathbb{Q}) \\
\downarrow & & \| \cap M_L \\
H^m(Y; \mathbb{Q}) & \cong & H^m(Y; \mathbb{Q})\end{array}
\]

taking into account that \(\rho\) is an isomorphism by Lefschetz Hyperplane Theorem. This proves (11), and concludes the proof of Theorem (11)."
5. Appendix

Proof of property \([4]\). First notice that since \(f^{-1}(\Delta_i) - D_i^0 \to \Delta_i\) is a trivial fiber bundle \((D_i^0 := \text{interior of } D_i)\), then the inclusion \((f^{-1}(a), f^{-1}(a) \cap D_i) \subseteq (f^{-1}(\Delta_i), f^{-1}(\Delta_i) \cap D_i)\) induces natural isomorphisms \(H_m(f^{-1}(a), f^{-1}(a) \cap D_i; \mathbb{Q}) \cong H_m(f^{-1}(\Delta_i), f^{-1}(\Delta_i) \cap D_i; \mathbb{Q})\) for any \(a \in \Delta_i\) (use \([12]\), p. 200 and 258). So, from the natural commutative diagram:

\[
\begin{array}{ccc}
H_m(f^{-1}(a_i + \rho); \mathbb{Q}) & \xrightarrow{\beta} & H_m(f^{-1}(a_i + \rho), f^{-1}(a_i + \rho) \cap D_i; \mathbb{Q}) \\
\alpha \downarrow & & \downarrow \\
H_m(f^{-1}(\Delta_i); \mathbb{Q}) & \to & H_m(f^{-1}(\Delta_i), f^{-1}(\Delta_i) \cap D_i; \mathbb{Q}),
\end{array}
\]

we deduce that \(\ker \alpha \subseteq \ker \beta = M_i\).

On the other hand, since the inclusion \(f^{-1}(a_i + \rho) \subseteq f^{-1}(\Delta_i)\) is the composition of the isomorphism \(f^{-1}(a_i + \rho) \cong g^{-1}(a_i + \rho)\) with \(g^{-1}(a_i + \rho) \subseteq g^{-1}(\Delta_i)\), followed by the desingularization \(g^{-1}(\Delta_i) \to f^{-1}(\Delta_i)\), we have: \(V_i \subseteq \ker \alpha\). \(\square\)

Lemma 5.1. Let \(\ell \subseteq \mathcal{H}_i\) be a general line. For any \(u \in \ell \cap Q^*\), denote by \(\Delta_u^\circ\) an open disk of \(\ell\) with center \(u\) and small radius. Consider the compact \(K := \ell \setminus (\bigcup_{a \in \mathbb{Q} \setminus Q^*} \Delta_a^\circ)\). Then there is a closed ball \(D_{q_i} \subseteq \mathbb{P}^n \times \mathbb{P}\), with positive radius and centered at \(q_i\), such that for any \(x \in K\) the distance function \(p \in H_x \cap Q \cap D_{q_i} \to ||p - q_i|| \in \mathbb{R}\) has no critical points \(p \neq q_i\).

Proof. We argue by contradiction. Suppose the claim is false. Then there is a sequence of hyperplanes \(y_n \in K\), \(n \in \mathbb{N}\), converging to some \(x \in K\), and a sequence of critical points \(p_n \neq q_i\) for the distance function on \(H_{y_n} \cap Q\), converging to \(q_i\) (we may assume \(p_n\) is smooth for \(H_{y_n} \cap Q\)). Let \(T_{p_n, q_i}^\ast, T_{p_n, H_{y_n} \cap Q}^\ast\) and \(s_{q_i, p_n}\) be the corresponding sequences of tangent spaces and secants, and denote by \(r_{q_i, p_n} \subseteq s_{q_i, p_n}\) the real line meeting \(q_i\) and \(p_n\). We may assume they converge, and we denote by \(T, T', s\) and \(r\) their limits \((r \subseteq s)\). Since \(p_n\) is a critical point, then \(r_{q_i, p_n}\) is orthogonal to \(T_{p_n, H_{y_n} \cap Q}^\ast\), hence \(r \not\subseteq T'\), and so \(T\) is spanned by \(T' \cup s\) by dimension reasons. Since \(T' \cup s \subseteq H_x\) then \(T \subseteq H_x\), so \(H_x\) contains a limit of tangent spaces of \(Q\), with tangencies converging to \(q_i\). This implies that \(x \in Q^*\), contradicting the fact that \(x \in K\). \(\square\)

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