Exact supersymmetry in the relativistic hydrogen atom in general dimensions — supercharge and the generalized Johnson-Lippmann operator

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A Dirac particle in general dimensions moving in a $1/r$ potential is shown to have an exact $\mathcal{N} = 2$ supersymmetry, for which the two supercharge operators are obtained in terms of (a $D$-dimensional generalization of) the Johnson-Lippmann operator, an extension of the Runge-Lenz-Pauli vector that relativistically incorporates spin degrees of freedom. So the extra symmetry ($\text{SU}(2)$) in the quantum Kepler problem, which determines the degeneracy of the levels, is so robust as to accommodate the relativistic case in arbitrary dimensions.

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Introduction — There is an increasing amount of fascination with supersymmetry (SUSY) in various fields of physics. In a broad context SUSY is a kind of (graded) Lie algebra that closes under a combination of commutation and anticommutation. One of the simplest realizations of such a symmetry may be found in one of the oldest problems in physics — motion of an electron in a hydrogen atom, or, classically, Kepler’s problem. It has long been known that there is a hidden symmetry (a dynamical symmetry) in the problem, which is related to an extra conservation (the Runge-Lenz vector). So the symmetry of the problem is higher (SO(4)) than the trivial SO(3), which is the cause of the “accidental” degeneracy (i.e., the energy level independent of $l$) in the spectrum of the hydrogen in non-relativistic quantum mechanics. If we move on to the Dirac electron in hydrogen in the relativistic quantum mechanics, the accidental degeneracy is lifted (since the Runge-Lenz vector, which designates the direction of the perihelion, is no longer conserved). However, the degeneracy is lifted only incompletely, and two-fold degeneracies (for the Dirac-operator quantum number $\kappa \equiv -(2S \cdot L + 1) = \pm(j + 1/2)$ with $j$ being the total angular momentum) remain, which was puzzling.

In 1985 Sukumar made an interesting suggestion that the strange degeneracy may be explained as a supersymmetry in the problem. Subsequently however, Tangerman et al. have pointed out that there is indeed an exact $\mathcal{N} = 2$ supersymmetry for the nonrelativistic hydrogen atom, criticized that Sukumar’s work has not actually constructed supercharges for the relativistic hydrogen atom. On the other hand, analytic studies for the relativistic hydrogen in general spatial dimensions have been developed, and analytic solutions are now obtained.

Given this background, the purpose of the present Letter is to show that an exact $\mathcal{N} = 2$ supersymmetry for the relativistic Dirac particle in a $1/r$ potential in fact exists in general $D$ spatial dimensions. In doing so we have actually constructed the supercharges $\mathcal{Q}_\pm$ (i.e., mutually anticommuting operators that commute with the Hamiltonian) for the Dirac Hamiltonian in $(D + 1)$ dimensions. The symmetry enables us to obtain the lowest eigenenergy and its wave function for each sector of the $D$-dimensional generalization of $\ket{\kappa}$ in a simple and transparent manner, so the problem indeed turns out to be algebraically solvable.

One interesting point is whether the supersymmetry in the hydrogen problem is related to the Runge-Lenz vector that describes the hidden conservation law on the nonrelativistic level. Dahl et al. have shown, for the relativistic hydrogen in $(3+1)$ dimensions, that the Runge-Lenz-Pauli vector (that necessarily involves spin degrees of freedom for the Dirac particle, and is called Johnson-Lippmann operator) may be indeed used to construct supercharges describing the supersymmetry. So the question we address here amounts to whether this extends to general dimensions. We first construct the Johnson-Lippmann operator generalized to the relativistic hydrogen in $(D + 1)$ dimensions, and then show that the generalized operator may actually be used to construct the supercharges. The supercharges are in fact shown to reduce to the $(D + 1)$ dimensional generalization of Runge-Lenz-Pauli vector in the nonrelativistic limit. While the Runge-Lenz-Pauli vector in general dimensions has been discussed in Refs. for the nonrelativistic case, the relativistic one is obtained for the first time. So we shall conclude that the supersymmetry is unexpectedly robust enough to be extended to the most general case, i.e., relativistic case in general dimensions. We can summarize the situation as

\[
\begin{array}{c|cc}
\text{relativistic} & \text{nonrelativistic} & \text{general } D \\
\hline
\text{SUSY} & \text{SUSY} & \text{SUSY} \text{ (present work)} \\
\text{energy levels} & \text{[12]} & \text{[13]} \\
\end{array}
\]

Dirac’s equation — Dirac’s equation for the relativistic hydrogen atom in $(D + 1)$ dimensions is written, in natural units (where $c = \hbar = 1$), as

\[
\{\gamma^0(p_0 - eA_0) + \gamma^i p_i - m\} \Psi(x) = 0
\]

in standard notations, where $\gamma$’s satisfy $\{\gamma^\mu, \gamma^\nu\} = \eta^{\mu\nu} = \text{diag}(1, -1, ..., 1)$, $p_\mu = i\partial_\mu$, $x = (x^0, x^1, ..., x^D)$, and...
sums over repeated indices are implied. We assume a 1/r potential, so we have \( eA_0 = -Z\alpha/r \), where \( \alpha = e^2/(\hbar c) \) is the fine-structure constant and \( Z \) the atomic number. When the lines of electric force are allowed to spread in the \( D \) spatial dimensions the Coulomb potential becomes \( 1/r^{D-2} \), but we focus on the 1/r potential for general dimensions to retain the atomic stability. We come back to this point later. If the time derivative is explicitly written, we have

\[
i \frac{\partial}{\partial t}\Psi(x^0, x^1, \ldots, x^D) = i \gamma^0 \gamma^a \gamma^b \partial a \Psi(x^0, x^1, \ldots, x^D),
\]

where \( H \) is the Hamiltonian and summations are implied for repeated spatial superscripts.

The operators that commute with the Hamiltonian are the total angular momentum (in \( D \) spatial dimensions),

\[
J_{ab} = L_{ab} + \frac{i}{2} \gamma^a \gamma^b, \quad L_{ab} = i x_a \partial_b - i x_b \partial_a,
\]

and the Dirac operator (in \( D \) dimensions),

\[
K \equiv \gamma^0 \left\{ \frac{i}{2} \sum_{a \neq b} \gamma^a \gamma^b L_{ab} + \frac{1}{2} (D - 1) \right\} = \gamma^0 \left\{ J^2 - L^2 - S^2 + \frac{1}{2} (D - 1) \right\},
\]

that is related to the spin-orbit interaction.

**Generalized Johnson-Lippmann operator** — While \( J \) and \( K \) are (usually the only) constants of motion for arbitrary central fields, it is known, for the ordinary \( D = 3 \), that there is an extra operator, a relativistic analogue of the Runge-Lenz-Pauli vector, that commutes with the Hamiltonian, as first constructed by Johnson and Lippmann. We now generalize the Johnson-Lippmann operator to \((D + 1)\) dimensions to define a Johnson-Lippmann-Katsura-Aoki operator,

\[
A \equiv \gamma^{D+1} \gamma^0 \gamma^i \frac{x^i}{r} - \frac{i}{Z\alpha} K \gamma^{D+1}(H - \gamma^0 m).
\]

Here we have defined \( \gamma^{D+1} \), a pseudo-scalar, which is a generalization of \( \gamma^5 \) in \((3+1)\) dimensions, and which satisfies \( \gamma^{D+1} \) satisfies \( \gamma^{D+1} = \gamma^{D+1}, (\gamma^{D+1})^2 = 1, \{\gamma^{D+1}, \gamma^a\} = 0 \). \( \gamma^{D+1} \) is constructed from \( \{\gamma^0, \gamma^1, \ldots, \gamma^D\} \), but its actual form depends on whether the spatial dimension is even or odd.

From the (anti)commutation relations above, we can show, after a rather tedious manipulation, that we have indeed \([H, A] = 0\) with a notable relation between \( H \) and \( A \),

\[
A^2 = 1 + \left( \frac{K}{2\alpha} \right)^2 \left( \frac{H^2}{m^2} - 1 \right).
\]

The expression reduces to the \( D = 3 \) counterpart obtained in Refs. 3, 12.

For \( K \) and \( A \), on the other hand, we can show that

\[
\{A, K\} = 0,
\]

i.e., we end up with mutually anticommuting operators, \( K, A \), that commute with \( H \).

**Construction of the \( N = 2 \) supercharge operators** — We are now in position to construct the generators of the supersymmetry, which was left undone in Ref. 3 and stated in Ref. 12 as an operator interesting to find. We can do so by going back to an \( N = 2 \) supersymmetric quantum mechanical model originally conceived by Witten 12, to which the present Hamiltonian is shown to be formally equivalent. Due to the relation (5), we can take \( \mathcal{H} \equiv A^2 \) as a Hamiltonian of the present problem. We can construct two operators \( Q_1, Q_2 \) that commute with \( \mathcal{H} \),

\[
Q_1 = A, \quad Q_2 = i \frac{AK}{|\kappa|},
\]

where \( \kappa \) is the eigenvalue of \( K \). We have then

\[
\mathcal{H} = Q_1^2 = Q_2^2, \quad \{Q_1, Q_2\} = 0.
\]

If we further define

\[
Q_\pm \equiv \frac{1}{2} (Q_1 \pm iQ_2) = \frac{1}{2} \left( 1 \pm \frac{K}{|\kappa|} \right) A = \frac{1}{2} A(1 \mp \mathcal{P}_\kappa) \quad (9)
\]

with \( \mathcal{P}_\kappa \equiv K/|\kappa| \), we have

\[
Q_+^2 = Q_-^2 = 0, \quad \mathcal{H} = \{Q_+, Q_-\}. \quad (10)
\]

This establishes an equivalence to Witten’s model.

So we can take the eigenvectors \( |n, \pm\rangle \) that satisfy

\[
\mathcal{H}|n, \pm\rangle = E_{n}^{(\pm)}|n, \pm\rangle, \quad \mathcal{P}_\kappa|n, \pm\rangle = \pm |n, \pm\rangle, \quad (11)
\]

since \( \mathcal{H} = A^2 \) commutes with \( K \), and we can talk about the simultaneous eigenvectors.

\( \mathcal{P}_\kappa \) does not commute with \( Q_\pm \), for which we have \( Q_\pm|n, \pm\rangle = 0 \). On the other hand, \([\mathcal{H}, Q_\pm]\) = 0 implies that \( |n, \pm\rangle \) and \( Q_\pm|n, \pm\rangle \) have a degenerate eigenvalue of \( \mathcal{H} \) with different eigenvalues of \( \mathcal{P}_\kappa \). From the relation (6), a zero-eigenstate of \( \mathcal{H} = A^2 \) is the ground state of the original \( \mathcal{H} \) (or, more precisely, the lowest-energy state for each value of \( \kappa \)). Since \( \langle n, \sigma|\mathcal{H}|n, \sigma\rangle = \langle n, \sigma|Q_+, Q_-|n, \sigma\rangle = |Q_+, n, \sigma\rangle^2 + |Q_-, n, \sigma\rangle^2 \geq 0 \) (for \( \sigma = \pm \)), a state is the ground state of \( \mathcal{H} \) if the equality holds in the above inequality. The equality occurs when \( Q_+|0, -\rangle = 0 \) or \( Q_-|0, +\rangle = 0 \). If we go back to the definition of \( Q_\pm \) (eq. 5), the zero-eigenvalue state \( |0\rangle \) should satisfy

\[
A|0\rangle = 0,
\]
since \((1 \mp D_n)|n, \pm\rangle \neq 0\).

**Kernel of \(A\) — One way to establish the existence of such a state is an analytic method. Following Ref. 6, we write the wave function for odd spatial dimensions \(D = 2N + 1\) as

\[
\psi_\kappa(x^1, x^2, \ldots, x^D) = r^{-N} \left( F(r) \phi_\kappa(\Omega) \right)
\]

where \(F(G)\) is the “large (small)” components, \(\phi_\kappa\) is the angular part with angular coordinates \(\Omega\), and \(K\psi_\kappa = \kappa\psi_\kappa\) holds. If we plug this into \(A\psi_\kappa = 0\), we have, after some manipulation,

\[
\left[ \left( \frac{d}{dx} + \text{sign}(\kappa)x \right)^2 - s^2 \right] \left( \frac{f(x)}{g(x)} \right) = 0
\]

(13)

where we have defined \(F(r) \equiv f(x), G(r) \equiv g(x)\) in terms of a dimensionless \(x \equiv (Z\alpha m/|\kappa|)r = (Z/|\kappa|a_0)r\) with \(a_0\) being the Bohr radius.

For the differential equation of second order, there are two independent solutions, \(f(x) \propto x^{-s}e^{-\text{sign}(\kappa)x} \) or \(x^s e^{-\text{sign}(\kappa)x}\) with

\[
\kappa = \pm \lfloor l + \frac{1}{2} \rfloor
\]

where \(l(= 0, 1, \ldots)\) is the orbital angular momentum, but the only normalizable one is the latter with \(\kappa > 0\). With a similar argument for \(g(x)\), we arrive at the kernel of \(A\),

\[
\psi_\kappa \propto x^{s-N} e^{-x} \left( \frac{\phi_\kappa(\Omega)}{\kappa Z} \right)
\]

(14)

as the non-degenerate lowest state (the unpaired levels in Fig.1).

For even \(D = 2N\) we can perform a similar procedure, again following Ref. 6. This time we can put \(\psi_\kappa = r^{-N+1/2}[F(r) \phi_\kappa(\Omega) + iG(r) \phi_{-\kappa}(\Omega)]\), for which the same differential equation results for \(F(r), G(r)\), so we have a non-degenerate ground state, \(\psi_\kappa \propto x^{-s-N+1/2} e^{-x} \{ \phi_\kappa(\Omega) + i[(\kappa - s)/Z\alpha] \phi_{-\kappa}(\Omega) \} \).

**Eigenenergies and the group theory — Eigenenergies may readily be obtained algebraically: As used above, eq. (15) dictates that each of the zero-eigenstates of the supersymmetric Hamiltonian \(\mathcal{H} = A^2\) is the lowest-energy state (for each sector of \(K\)) of the original \(H\). This immediately implies that \(E\), the lowest-lying (within each sector of \(K\)) eigenvalue of \(H\), is

\[
E/m = \sqrt{1 - \left( \frac{Z\alpha}{\kappa} \right)^2} \left[ 1 + \left( \frac{Z\alpha}{s} \right)^2 \right]^{-1/2}, \quad (15)
\]

in agreement with the analytical result in Ref. 6. We can go up the ladder (where the leg corresponds to \(\kappa = \pm |\kappa|\)

FIG. 1: Energy level scheme for the relativistic hydrogen in \(D\) spatial dimensions. The lowest few levels are plotted against the Dirac quantum number \(\kappa\) and \(\tanh D\), where the energy is plotted on a logarithmic scale to make the level splittings clearer. How the \(S(2)\) ladder is obtained with a shift in \(s\) and the supercharge operation \(Q_s\) is indicated by arrows with the sign of \(\kappa\) and the principal quantum number \(n\) indicated.

while the rung spanned by a quantum number we call \(n'\) by making \(s \equiv \sqrt{\kappa^2 - (Z\alpha)^2} \to s + 1\).

So we end up with

\[
E/m = \left[ 1 + \frac{(Z\alpha)^2}{(n - |\kappa| + D - 3/2 + \sqrt{\kappa^2 - (Z\alpha)^2})^2} \right]^{-1/2},
\]

where we assume \(Z\alpha < (D - 1)/2\) for the atomic stability. Here the principal quantum number is given as

\[
n = l + 1 + n'
\]

with \(n' = 0, 1, \ldots\), where each level is doubly degenerated [corresponding to \(\kappa = \pm (l + (D - 1)/2)\) and related by \(Q_s\)], except at the bottom of each ladder at \(n' = 0\) (i.e., \(n = l + 1\)) for which only \(\kappa > 0\) should be taken (Fig.1). We can also see that the inter-dimensional degeneracy, which exists between the levels \((l, D) \to (l \pm 1, D \mp 2)\) noted for the nonrelativistic case 6, persists for the relativistic case, and the supersymmetric ladders live on such a spectrum.

Group-theoretically the present result implies the following. The nonrelativistic hydrogen atom in \(D\) spatial
dimensions has a hidden symmetry (with the Runge-Lenz vector conserved), and the symmetry of the problem is higher (SO($D + 1$)) than the symmetry (SO($D$)) of the space. If we go to the relativistic case the symmetry is degraded, but only partially degraded into SO($D$)$\otimes$S(2) due to the $N = 2$ supersymmetry. So the conjecture, stated in Ref.\[5\], is established here for general dimensions. Namely, the operator in this limit reduces to those defined in $\mathcal{N} = 2$ supersymmetry. Some authors have indeed discussed generalization of the Runge-Lenz-Pauli vector, such as the case of a particle around a magnetic monopole with a vector potential around it. So the exploration of supersymmetry in wider gauge-field models will be an interesting future problem. We wish to thank Dr R. Arita for discussions.

Relation with the Runge-Lenz-Pauli vector — Related to the above, we can note a certain relation between the Johnson-Lippmann-Katsura-Aoki operator ($A$) and the Runge-Lenz-Pauli vector as follows. Since $A$ is Hermitian, we have

$$A = \frac{1}{2}(A + A^\dagger) = -\gamma^{D+1}\sigma^i$$

(16)

$$\times \left[ \frac{1}{2Zm\alpha} \gamma^0(p_jL_{ij} - L_{ij}p^j) - \frac{x^i}{r} \right] + \frac{i}{m}K\gamma^{D+1}\frac{1}{r}.$$ 

This expression reduces, in the nonrelativistic limit, to

$$A \to -\sigma^iM^i,$$

where

$$M^i = \frac{1}{2mZ\alpha}(p_jL_{ij} - L_{ij}p^j) - \frac{x^i}{r}$$

is the nonrelativistic Runge-Lenz-Pauli vector in $D$ spatial dimensions\[10\] and $\sigma^i = \gamma^{D+1}\sigma^i$ the spin operator in $D$ dimensions. Namely, the operator in this limit is the inner product of the Runge-Lenz-Pauli vector ($M$) and the spin ($\sigma$) in $D$ dimensions\[10\].

Summary and discussions — So the higher symmetry for the $1/r$ potential in general dimensions is revealed to be surprisingly robust against the relativistic generalization and is retained as a supersymmetry. The supercharge is indeed related to (a $D$-dimensional generalization of) the Johnson-Lippmann operator.

Given the formula for the general $D$, we are tempted to ask what happens in the $D \to \infty$ limit. In the nonrelativistic case the kinetic energy ($\sim (1/D) \times$ potential energy) is dominated by the potential energy, so the system reduces to a set of harmonic oscillations around the classical potential minima as stressed in Ref.\[11\]. In the relativistic case, however, we see that we cannot eliminate the small component ($G$) in the $D \to \infty$ limit, since the coupling between $F$ and $G$ does not vanish in this limit. This implies that we cannot trivially relate the supersymmetry with a set of oscillators even asymptotically.

Another basic question is whether the supersymmetry is just accidental to the $1/r$ potential. As mentioned above, electromagnetically there is a problem of how we can conceive the $1/r$ potential as the Coulomb potential in general dimensions. We should have $1/r^{D-2}$ potential from Gauss’s law if the lines of electromagnetic force extend over the $D$ spatial dimensions, but there is a well-known Ehrenfest’s 1920 result that the atom becomes unstable for this potential. Some authors argue that we should in fact stick to $1/r$ in general dimensions. Experimentally, a low-dimensional ($D = 2$) case may be interesting as an accessible one.

In a broader context, a supercharge is a kind of generalization of the Dirac operator, so it could be that supersymmetry is shared by a wide class of Dirac-type equations. Some authors\[15\] have indeed discussed generalization of the Runge-Lenz-Pauli vector, such as the case of a particle around a magnetic monopole with a vector potential around it. So the exploration of supersymmetry in wider gauge-field models will be an interesting future problem. We wish to thank Dr R. Arita for discussions.

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