Φ(2) Perturbations of WZW Model

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Abstract

We study su(2) \(_k\) WZW model perturbed by a multiplet of primary fields. The theory has a rich variety of particles. Presence of nontrivial decay processes is a peculiarity of the model. We prove integrability by explicit construction of quantum conserved currents. The scattering theory is briefly discussed.

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1. Introduction.

Conformal field theory and integrable field theory in two dimensions are two subjects which attracted a lot of attention in the last years. The reason essentially lies in specific two-dimensional symmetries which lead to exact solutions of the quantum field dynamics. In conformal field theory solvability of a model is provided by a chiral algebra, which includes Virasoro algebra as subalgebra. Examples are W-algebras [1] [2], parafermions [3] [4], Wess-Zumino model [5], etc. When we go away from criticality all the symmetry is usually lost. However, as it was pointed out in [6], for certain classes of perturbations of a conformal field theory the resulting two-dimensional field theory possesses infinite number of mutually commutative conserved charges. The currents corresponding to these charges can be viewed as a deformation of some operators from (enveloping of) chiral algebra. The conserved charges result in a drastically simplified scattering theory - a general S matrix factorizes into product of two-particle S matrices [7] [8]. Factorized scattering preserves the number of particles and the set of on-mass-shell momenta. This great simplification makes it possible to probe a vicinity of a conformal fixed point in detail [9]. Large variety of theories with factorized scattering was explicitly constructed [10] [11] [12] [13] [14] [15]. Many of them are reductions [16] [17] at a special value of coupling constants of quantized classical field theories, e.g. Toda field theories. The factorization is typical for the scattering of solitons of the nonlinear classical field equations integrable by the inverse scattering method. The spectra of conserved charges in such a theories are essentially the same as in the reduced versions. Factorized scattering shows up in semiclassical limit as absence of backward scattering of lumps. Most of the considered models contain one or few stable particles and exhibit no resonances.

In this paper we consider a model containing a bunch of unstable particles, besides a stable ones. From the point of view of conformal field theory the this model is a perturbation of $SU(2)_k$ Wess-Zumino-Witten model by a certain multiplet of primary operators. The perturbation depends in general on four arbitrary parameters. The layout of the paper follows. In section 2 we introduce the necessary notations and recall some basic facts about Wess-Zumino-Witten model. In section 3 we formulate the model, obtain recurrence formula for classical conserved currents and construct $N$-kink solution for the model describing scattering, merge and decay of kinks and their bound states. Section 4 is devoted to the construction of quantum conserved charges. Besides the series of integrals of motion which survive semiclassical limit there is an additional set of conserved charges, not admitting semiclassical limit. A brief consideration of scattering amplitudes is given in section 5.
2. Basics.

In this section we recall basic facts about Wess-Zumino-Witten model and introduce necessary notations.

Consider two-dimensional Wess-Zumino-Witten (WZW) action \[ S_{wzw} = \frac{-k}{16\pi} \int dt dx \, \text{Tr} \left( \partial_\mu gg^{-1} \partial_\mu gg^{-1} \right) \quad (2.1) \]

where field $g(t, x)$ takes value in any semisimple compact group $G$, and $t, x$ - are coordinates of 2-dimensional Euclidean space ($t$ is euclidean time) and $k$ is some dimensionless coupling constant. Here we will deal mostly with $G = SU(2)$. The ambiguous term $\Gamma [g]$ is given by the functional

\[ \Gamma [\tilde{g}] = \frac{1}{24\pi} \int dt dx d\tau \epsilon_{\mu\nu\lambda} \text{Tr} \left( \partial_\mu \tilde{g} \tilde{g}^{-1} \partial_\nu \tilde{g} \tilde{g}^{-1} \partial_\lambda \tilde{g} \tilde{g}^{-1} \right) \quad (2.2) \]

where integration is over 3-dimensional half-space ($t, x, \tau$), $\tau \geq 0$. Boundary conditions $\tilde{g}(t, x, \infty) = 1$, $\tilde{g}(t, x, 0) = g(t, x)$ define the functional $\Gamma$ modulo $2\pi N$ for some integer $N$. With these boundary conditions functional integral is well-defined if $k$ is a positive integer number. It shown in \[3, 20\] the model \[ (2.1) \] is conformally invariant and therefore can be studied by methods of two-dimensional conformal field theory (CFT) \[21\]. In fact this model possesses larger symmetry with respect to current algebra $\sim su(2)_k \times \sim su(2)_k \quad \[20, \]

\[22\]. Namely, the action \[ (2.1) \] is invariant under transformations

\[ g \rightarrow \Omega(z) \, g(z, \bar{z}) \, \bar{\Omega}(\bar{z}) \quad (2.3) \]

where $\Omega(z)$ and $\bar{\Omega}(\bar{z})$ are arbitrary $SU(2)$-valued functions of light-cone variables

\[ z = (t + ix)/2, \quad \bar{z} = (t - ix)/2 \quad (2.4) \]

The field content, anomalous dimensions of the fields and equations for correlation function were studied in \[22\]. Partition functions and applications to string theory were discussed in \[23\]. The infinite dimensional symmetry \[ (2.3) \] of the WZW model is generated by local currents

\[ J = -\frac{k}{4} \partial gg^{-1}, \quad \bar{J} = -\frac{k}{4} g^{-1} \partial g \quad (2.5) \]

which satisfy equations \[4\]

\[ \partial J = \partial \bar{J} = 0 \quad (2.6) \]
The currents $J(z)$ and $\bar{J}(\bar{z})$ obey the following operator product expansion (OPE) \[22\]

$$J^a(z_1) J^b(z_2) = \frac{k}{2} \frac{q^{ab}}{(z_1 - z_2)^2} + \frac{f_c^{ab}}{(z_1 - z_2)} J^c(z_2) + \text{regular terms} \quad (2.7)$$

here tensors $q^{ab}$ and $f_c^{ab}$ are $su(2)$ invariant metric and structure constants. They have components

$$q^{00} = \frac{1}{2} q^{+-} = 1, \quad f^{0+}_0 = 2, \quad f^{0-}_0 = -f^{0-}_0 = 1 \quad (2.8)$$

Let $\mathcal{F}$ be space of mutually local fields of the theory. For any $F(z, \bar{z}) \in \mathcal{F}$ we define an action of operators $J_n^a$, $(a = \pm, 0$ and $n = 0, \pm 1, \pm 2, ..)$ acting in this space

$$J_n^a F(z, \bar{z}) = \oint_{C_z} \frac{d\zeta}{2\pi i} J^a(\zeta)(\zeta - z)^n F(z, \bar{z}) \quad (2.9)$$

As it follows from (2.7) operators $J_n^a$ satisfy commutation relations of $su(2)_k$ Kac-Moody algebra

$$[J_n^a, J_m^b] = f_c^{ab} J_n^c - \frac{km}{2} q^{ab} \delta_{n+m,0} \quad (2.10)$$

Since the theory has conformal invariance energy-momentum tensor is traceless and has only two independent components

$$T(z) = \frac{1}{(k + 2)} q^{ab} : J^a(z) J^b(z) :$$

$$T(\bar{z}) = \frac{1}{(k + 2)} q^{ab} : J^a(\bar{z}) J^b(\bar{z}) : \quad (2.11)$$

Field $T(z)$ satisfy OPE

$$T(z_1) T(z_2) = \frac{c}{2} \frac{1}{(z_1 - z_2)^4} + \frac{2T(z_2)}{(z_1 - z_2)^2} + \frac{\partial T(z_2)}{(z_1 - z_2)} + \text{regular terms} \quad (2.12)$$

Again as in the case with Kac-Moody algebra we can define action of generators $L_n, \ n = 0, \pm 1, \pm 2, ..$ in the space $\mathcal{F}$

$$L_n F(z, \bar{z}) = \oint_{C_z} \frac{d\zeta}{2\pi i} T(\zeta)(\zeta - z)^{n+1} F(z, \bar{z}) \quad (2.13)$$

Operators $L_n$ form Virasoro algebra

$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3 - n)\delta_{n+m,0} \quad (2.14)$$

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with central charge
\[ c = \frac{3k}{k+2} \]  \hspace{1cm} (2.15)
and commute with Kac-Moody generators \( J_m^a \) as
\[ [J_m^a, L_n] = m J_n^a \]  \hspace{1cm} (2.16)

In terms of \( J_n^a \) Virasoro generators \( L_n \) can be expressed as
\[ L_n = \frac{q_{ab}}{k+2} \sum_{m=-\infty}^{+\infty} : J_m^a J_{n-m}^b : \]  \hspace{1cm} (2.17)
where symbol \( : (\cdots) : \) means normal ordering, i.e. operators \( J_m^a \) with \( n < 0 \) to be put to the left. Operators \( \bar{J}_n^a \) and \( \bar{L}_n \) related to the fields \( \bar{J}^a(z), \bar{T}(z) \) can be defined in a similar way.

There are invariant fields \( \Phi^{(j,\bar{j})}_m, \bar{m}, j, \bar{j}; m, \bar{m} = -j, -j+1, \ldots, j-1, j \) in the space \( \mathcal{F} \) which obey equations
\[ J_n^a \Phi^{(j,\bar{j})}_m = 0, \quad n > 0 \]
\[ J_0^+ \Phi^{(j,\bar{j})}_m = [(j-m)(j+m+1)]^{1/2} \Phi^{(j,\bar{j})}_{m+1,\bar{m}} \]
\[ J_0^0 \Phi^{(j,\bar{j})}_m = m \Phi^{(j,\bar{j})}_m \]
\[ J_0^- \Phi^{(j,\bar{j})}_m = [(j+m)(j-m+1)]^{1/2} \Phi^{(j,\bar{j})}_{m-1,\bar{m}} \]  \hspace{1cm} (2.18)
and the “bar”-counterpart of the above equations. Invariant fields \( \Phi^{(j,\bar{j})}_m, \bar{m} \) have dimensions \((\Delta_j, \Delta_{\bar{j}}), \)
\[ \Delta_j = \frac{j(j+1)}{k+2} \]  \hspace{1cm} (2.19)
If \( j = \bar{j} \) we will denote the corresponding field as \( \Phi^{(j)}_m, \bar{m} \). The space \( \mathcal{F} \) can be represented as a sum
\[ \mathcal{F} = \oplus_{j,\bar{j}} [\Phi^{(j,\bar{j})}] \]  \hspace{1cm} (2.20)
where \([\Phi^{(j,\bar{j})}]\) is a space spanned by fields of the form
\[ J_{n_1}^{a_1} J_{n_2}^{a_2} \ldots J_{n_s}^{a_s} \bar{J}_{m_1}^{b_1} \bar{J}_{m_2}^{b_2} \ldots \bar{J}_{m_l}^{b_l} \Phi^{(j,\bar{j})}(z, \bar{z}) \]  \hspace{1cm} (2.21)
In semiclassical limit \( k \to \infty \) fields \( \Phi^{(j,\bar{j})} \) become an \( SU(2) \) matrix \( g^{(j)}(z, \bar{z}) \) in representation of spin \( j \) while fields \( \Phi^{(j)} \) become differential polynomial in matrix elements of \( g^{(j)} \).
3. Perturbation of WZW model. Semiclassical analysis.

Now we turn to perturbation of WZW model. As a perturbation we take operator $O$

$$O = \frac{1}{2\pi} \int dt dx \left( \text{Tr} \left( gAg^{-1}B \right) - \text{Tr} \left( AB \right) \right)$$

(3.1)

In this section we make rotation $t \rightarrow -it$ to Minkowski space. The perturbed action functional become

$$A = S_{wzw} + O$$

(3.2)

here $A$ and $B$ are diagonal matrices

$$A = \text{Diag} \left( a_1, a_2, a_3 \right), \quad B = \text{Diag} \left( b_1, b_2, b_3 \right)$$

(3.3)

and we take $g$ in real spin $j = 1$ representation. Equivalently we can say that $g$ takes values in $SO(3)$ instead of $SU(2)$. If we introduce coordinates $\psi, \vartheta, \varphi$ on $SO(3)$ group manifold

$$g = \begin{pmatrix}
\cos \psi & 0 & \sin \psi \\
-\sin \psi & \cos \psi & 0 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\cos \varphi & 0 & \sin \varphi \\
0 & 1 & 0 \\
-\sin \varphi & 0 & \cos \varphi
\end{pmatrix}
\begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \vartheta & \sin \vartheta \\
0 & -\sin \vartheta & \cos \vartheta
\end{pmatrix}$$

(3.4)

the action functional $A$ takes the form

$$A = \frac{k}{4\pi} \int dx^+ dx^- \left( \partial_+ \psi \partial_- \psi + \partial_+ \varphi \partial_- \varphi + \partial_+ \vartheta \partial_- \vartheta - 2 \sin \varphi \partial_- \vartheta \partial_+ \psi - \kappa^2 U(\psi, \varphi, \vartheta) \right)$$

(3.5)

where the potential $U$ is of extremely complicated form

$$U = \alpha_1 \beta_1 \sin^2 \psi + \alpha_2 \beta_2 \sin^2 \theta + \alpha_3 \beta_3 \sin^2 \varphi + \alpha_2 \beta_1 \sin^2 \psi \sin^2 \vartheta - \alpha_3 \beta_1 \sin^2 \varphi \sin^2 \varphi - \alpha_2 \beta_3 \sin^2 \theta \sin^2 \varphi + \alpha_2 \beta_1 \sin \psi \sin \varphi \sin \vartheta \left( 2 \cos \psi \cos \vartheta + \sin \psi \sin \varphi \sin \vartheta \right),$$

(3.6)

$$x^\pm = (t \pm x)/2, \quad \kappa^2 = 4/k$$

and coupling constants defined as

$$\alpha_1 = a_1 - a_2, \quad \alpha_2 = a_2 - a_3, \quad \alpha_3 = a_1 - a_3, \quad \beta_1 = b_1 - b_2, \quad \beta_2 = b_2 - b_3, \quad \beta_3 = b_1 - b_3$$

(3.7)

The dynamical model described by the above action was first considered in [25].

For general $\alpha_i$ and $\beta_i$ the perturbation completely breaks $\widehat{su(2)}_k \times \widehat{su(2)}_k$ symmetry of the model. Another feature of the model is completely broken $P$-parity, i.e symmetry with respect to reflection of space coordinate $x \rightarrow -x$. WZW can be made invariant under change of sign of $x$ if we simultaneously perform transformation $g \rightarrow g^{-1}$. In the perturbed model we do not have this option because of the specific form of the operator $O$. 


3.1. Lax pair and classical integrals of motion (IM).

In [25] a Lax pair representation for (3.5) was found and it was shown that the model can be solved using inverse scattering method [26]. For our purposes we will use another Lax pair which is more simple and suitable for analysis.

Equations of motion for the perturbed model (3.2)
\[
\partial_+ J = [gAg^{-1}, B] \\
J \equiv -\frac{k}{4}\partial_- gg^{-1}
\]  
are compatibility condition for an auxiliary linear problem
\[
L_- \Psi \equiv \left( \partial_- - \partial_- gg^{-1} - i\lambda^{-1} \kappa B \right) \Psi = 0 \\
L_+ \Psi \equiv \left( \partial_+ - i\kappa gg^{-1} \right) \Psi = 0
\]  
where \( \lambda \) is a spectral parameter. It is remarkable enough that both the equation (3.8) and the operators \( L_\pm \) (but without the spectral parameter \( \lambda \)) appeared also in a context of \( N = 2 \) supersymmetric models [27]. The Lax pair (3.9) is a guaranty for existence of infinite number of conserved charges
\[
Q_a^{(s)} = \int \left( Q_a^{(s)} dx^- + R_a^{(s)} dx^+ \right) \\
\frac{\partial}{\partial t} Q_a^{(s)} = 0
\]  
in the model (3.2) (at least on classical level). In \( Q_a^{(s)} \) \( s \) refers to the spin of the charge and \( a \) distinguish between charges of the same spin. Using the Lax pair we find the recursion formula allowing to construct the densities \( Q_a^{(s)} \)
\[
Q_a^{(s)} = \left( JK^{(s)} \right)_{aa}
\]  
where matrix \( K^{(s)} \) defined recursively
\[
k_{aa}^{(s)} \equiv 0 \\
i\kappa(b_m - b_n)K_{mn}^{(s+1)} = \partial_- K_{mn}^{(s)} + \sum_{s_1+s_2=s} K_{mn}^{(s_1)}Q_{n}^{(s_2)} - \left( JK^{(s)} \right)_{mn} - J_{mn}\delta_{s,0}
\]  
with initial conditions
\[
K^{(0)} = 0, \quad Q^{(0)} = 0
\]
A few first densities are 

\[ Q_1^{(1)} = \text{Tr} (JJ) , \]
\[ Q_2^{(1)} = \text{Tr} (BJJ) , \]
\[ Q_a^{(3)} = - \sum_{p,q,s \neq a} \frac{J_{ap} J_{pq} J_{qs} J_{sa}}{(b_p - b_a)(b_q - b_a)(b_s - b_a)} + \sum_{p \neq a} \frac{J_{ap} J_{pa}}{(b_p - b_a)} \sum_{p \neq a} \frac{J_{ap} J_{pa}}{(b_p - b_a)^2} \] 

(3.14)

Only two of \( Q_a^{(3)} \) are linearly independent. Further analysis shows that allowed set of spins is \( s = \pm 1, \pm 3, \pm 5, \pm 7, \pm 9, \ldots \) and for any value of \( s \) there are two linearly independent conserved charges. Densities with negative \( s \) can be obtained from the positive ones replacing \( J \rightarrow \frac{k}{4} g^{-1} \partial_x g \) and \( b_m \rightarrow a_m \). In the next section we construct quantum conserved charges corresponding to (3.14).

3.2. Kinks.

Now we are going to discuss soliton content of the model. First of all we specify boundary conditions. The natural ones are \( g \rightarrow g_{\pm \infty} \) as \( x \rightarrow \pm \infty \), with \( g_{\pm \infty} \) being some constant matrices. They can be determined from the condition of vanishing of the right hand side of (3.8) as \( x \rightarrow \pm \infty \). There are four allowed vacua

\[ g_0 = \text{Diagonal} (1, 1, 1) \]
\[ g_1 = \text{Diagonal} (-1, -1, 1) \]
\[ g_2 = \text{Diagonal} (1, -1, -1) \]
\[ g_3 = \text{Diagonal} (-1, 1, -1) \]

(3.15)

All these vacua have same energy. Therefore we expect that there are kinks interpolating between them. Let for a moment \( \varphi = 0 \) and \( \vartheta = 0 \) in (1.3). Then we get familiar sine-Gordon (sG) action for the field \( \psi \). It is well known that sG theory has kinks [28], and the corresponding solutions interpolate in our model between \( g_0 \) and \( g_1 \). The same happens if we set another couple of fields to zero. However, there is significant difference between sG kinks and kinks in the model we are considering. Because \( \pi_1 (SO(3)) = \mathbb{Z}_2 \) kinks must have \( \mathbb{Z}_2 \) charge. For example lets take kink \( g(t,x) \) interpolating between \( g_0 \) and \( g_1 \). At given moment \( t, g(x) \) is a path on \( SO(3) \) manifold connecting identity element.
with $g_1$. But $g_1$ is nothing but rotation by angle $\pm \pi$ and kinks with different $Z_2$ charge correspond to different choice of the sign. On the contrary, sG kinks have $Z$ charge.

Probably the easiest way to construct kink solutions is to use Riemann problem \[29\] for the Lax pair \[3.9\]. In this approach solitons are related to $Ψ$’s in \[3.9\] having simple analytical properties in $λ$. Namely,

$$Ψ(x^-, x^+|λ) = Π(x^-, x^+|λ) \cdot \exp \left( iλ^{-1}κBx^- + iλκAx^+ \right)$$  \hspace{1cm} (3.16)

where $Π$ is a meromorphic function of $λ$. If we normalize $Π$ as $Π(x^-, x^+|0) = 1$, then an example of such a solution for $Π$ relevant for scattering of kinks and having $N$ poles in $λ$-plane can be written as

$$[Π(λ)]_{ab} = \frac{\det M^{(ab)}(λ)}{\det M}$$  \hspace{1cm} (3.17)

where matrices $M^{(ab)}(λ)$ and $M$ are $(N + 1) \times (N + 1)$ and $N \times N$ correspondingly

$$M_{ij} = \frac{γ_i (e_i, e_j)}{γ_i + γ_j},$$  \hspace{1cm} (3.18)

and

$$M^{(ab)}(λ) = \begin{pmatrix} δ_{ab} & \lambda e_1^{(b)} & \lambda e_2^{(b)} & \cdots & \lambda e_n^{(b)} \\ e_1^{(a)} & λ + iγ_1 & λ + iγ_2 & \cdots & \lambda + iγ_n \\ e_2^{(a)} & & & & \\ \vdots & & & & \\ e_n^{(a)} & & & & \end{pmatrix}$$  \hspace{1cm} (3.19)

here $γ_i$ are $N$ positive numbers (they are related to rapidities $θ$ of kinks) and $(e_i, e_j)$ is a scalar product of vectors

$$e_n = \exp \left( γ_nκAx^+ - γ_n^{-1}κBx^- \right) \cdot c_n,$$  \hspace{1cm} (3.20)

here $c_n$ are some real 3-dimensional vectors independent of $z$ and $\bar{z}$.

The corresponding solution for $g$ is

$$g(x^-, x^+) = g_a \cdot Π(x^-, x^+|\infty) \cdot g_b$$  \hspace{1cm} (3.21)

and the current $J$ is determined by expansion of $Π$ near $λ = 0$

$$[Π(x^-, x^+|λ)]_{mn} = δ_{mn} + iλ \frac{[J(x^-, x^+)]_{mn}}{(b_m - b_n)} + O(λ^2)$$  \hspace{1cm} (3.22)
The simplest solutions are kinks $K^{(\epsilon)}_{0a}(\theta)$ with topological charge $\epsilon$ interpolating between $g_0$ and $g_a$. They correspond to the case $N = 1$ and one of components of the vector $c$ is zero in \[ (3.21) \]. For example, for $K^{(\epsilon_1)}_{01}(\theta)$

$$g = \begin{pmatrix}
\frac{\sinh \phi_1}{\cosh \phi_1} & \frac{(-1)^{\epsilon_1}}{\cosh \phi_1} & 0 \\
\frac{(-1)^{\epsilon_1}}{\cosh \phi_1} & \frac{-\sinh \phi_1}{\cosh \phi_1} & 0 \\
0 & 0 & 1
\end{pmatrix} \tag{3.23}$$

Here

$$\phi_1 = \frac{k}{2} (\alpha_1 \gamma + \beta_1 \gamma^{-1}) x + \frac{k}{2} (\alpha_1 \gamma - \beta_1 \gamma^{-1}) t + \delta_1 \tag{3.24}$$

$\epsilon_1 = 0, 1$ is $Z_2$ topological charge of the kink and $\delta_1$ is an arbitrary real constant. We see that only field $\psi$ is exited. There are similar solutions for kinks $K^{(\epsilon_2)}_{02}(\theta)$ and $K^{(\epsilon_3)}_{03}(\theta)$ when only fields $\vartheta$ or $\varphi$ are excited. Using the energy-momentum tensor we find that kinks $K^{(\epsilon)}_{0a}(\theta)$ and ones obtained from them by multiplication by $g_n$’s (see \( (3.15) \)) have two-dimensional momenta

$$P^\mu_{(n)} = (M_n \cosh \theta_n, M_n \sinh \theta_n) \tag{3.25}$$

where masses of kinks

$$M_n = \frac{1}{\pi} (k \alpha_n \beta_n)^{1/2} \tag{3.26}$$

and rapidities $\theta_n$ are related to $\gamma$

$$\theta_n = -\ln \gamma + \frac{1}{2} \ln (\beta_n/\alpha_n) \tag{3.27}$$

now we can rewrite phase \( (3.24) \) as

$$\phi_1 = \frac{2\pi M_1}{k} (x \cosh \theta_1 - t \sinh \theta_1) + \delta_1 \tag{3.28}$$

In the following we will assume that

$$a_1 > a_2 > a_3, \quad b_1 > b_2 > b_3 \tag{3.29}$$

In general we have $3N$ free parameters in \( (3.21) \). Restricting components of $c_i$ by condition $c_i^{(1)} \cdot c_i^{(3)} = 0$, $i = 1, 2, .., N$ we obtain $2N$ parametric solution describing scattering of kinks. We give an example of scattering of two different kinks
In full form the corresponding solution is given in Appendix A.

The simple example is decay process $K_{01}^{(e_1)}(\theta_1) K_{13}^{(e_2)}(\theta_2) \to K_{02}^{(e_2)}(\theta_2) K_{23}^{(e_1)}(\theta_1)$ ($N = 1$, $c_1^{(3)} = 0$, $c_2^{(1)} = 0$). From (3.4) and (3.21) we find, for example

$$
\tan \psi = -(-1)^{\epsilon_1} \frac{2e^{\phi_1}(1 + \Delta e^{2\phi_2})}{1 - e^{2\phi_1} + \Delta^2 e^{2\phi_2} - e^{2(\phi_1 + \phi_2)}},
$$

$$
\sin \varphi = (-1)^{\epsilon_1+\epsilon_2} \frac{2(\Delta - 1)e^{\phi_1+\phi_2}}{1 + e^{2\phi_1} + \Delta^2 e^{2\phi_2} + e^{2\phi_1} e^{2\phi_2}},
$$

$$
\tan \theta = -(-1)^{\epsilon_2} \frac{2(e^{2\phi_1} + \Delta)e^{\phi_2}}{1 + e^{2\phi_1} - \Delta^2 e^{2\phi_2} - e^{2(\phi_1 + \phi_2)}},
$$

(3.30)

here

$$
\Delta = \tanh \left( \frac{\theta_1 - \theta_2}{2} - \frac{1}{4} \ln \left( \frac{\alpha_1\beta_2}{\alpha_2\beta_1} \right) \right),
$$

(3.31)

$$
\phi_n = \frac{2\pi M_n}{k} (x \cosh \theta_n - t \sinh \theta_n) + \delta_n
$$

(3.32)

In full form the corresponding solution is given in Appendix A.

What are the rest of parameters responsible for? In general solution (3.21) describes scattering, merge and decay of solitons. The simplest example is decay process $K_{03}^{(e_1+e_2)}(\theta_3) \to K_{02}^{(e_2)}(\theta_2) K_{23}^{(e_1)}(\theta_1)$ ($N = 1$, all components of $c_1$ are nonzero)

$$
g = \begin{pmatrix}
\frac{1 + e^{2\phi_2} - e^{2(\phi_1 + \phi_2)}}{1 + e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}} & \frac{-2(-1)^e^{1+2\phi_2}}{1 + e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}} & \frac{1}{1 + e^{2\phi_2} + D_{m,n} e^{2(\phi_1 + \phi_2)}} \\
\frac{-2(-1)^e^{1+2\phi_2}}{1 + e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}} & \frac{-2(-1)^e^{1+2\phi_2}}{1 + e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}} & \frac{-2(-1)^e^{1+2\phi_2}}{1 + e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}}
\end{pmatrix},
$$

(3.33)

with phases $\phi_n$

$$
\phi_n = \frac{\kappa}{2} \left( \alpha_n \gamma + \beta_n \gamma^{-1} \right) x + \frac{\kappa}{2} \left( \alpha_n \gamma - \beta_n \gamma^{-1} \right) t + \delta_n
$$

(3.34)

Indeed, suppose that $\alpha_3 \gamma - \beta_3 \gamma^{-1} = 0$, $\alpha_2 \gamma - \beta_2 \gamma^{-1} > 0$, and consider worldline $\phi_1 + \phi_2 = \text{const}$. Then if $t \to -\infty$

$$
g \to \begin{pmatrix}
\frac{1 + e^{2\phi_1 + \phi_2}}{1 + e^{2\phi_1 + \phi_2}} & 0 & \frac{-2(-1)^e^{1+2\phi_2}}{1 + e^{2\phi_1 + \phi_2}} \\
0 & 1 & 0 \\
\frac{2(-1)^e^{1+2\phi_2}}{1 + e^{2\phi_1 + \phi_2}} & 0 & \frac{1}{1 + e^{2\phi_1 + \phi_2}}
\end{pmatrix}
$$

(3.35)

i.e. at $t = -\infty$ we have resting kink $K_{03}^{(e_1+e_2)}$. Now consider limit $t \to +\infty$. We first keep $\phi_1 = \text{const}$. In this limit we get

$$
g \to \begin{pmatrix}
\frac{1 - e^{2\phi_1}}{1 + e^{2\phi_1}} & -2(-1)^e^{1+2\phi_2} & 0 \\
-2(-1)^e^{1+2\phi_2} & \frac{1 - e^{2\phi_1}}{1 + e^{2\phi_1}} & 0 \\
0 & 0 & -1
\end{pmatrix}
$$

(3.36)
that is $K_{23}^{(e_1)}$ kink propagating in the positive $x$ direction. Taking limit with $\phi_2 = \text{const}$ we obtain $K_{02}^{(e_2)}$ kink propagating in the negative $x$ direction.

That instability of the heaviest kink $K_{03}$ can be also predicted from the mass spectrum of solitons (3.26)

$$M_3 \geq M_1 + M_2$$

where equality achieved when $A = B$ in (3.2). The solution (3.33) can also be obtained from (3.30) in a limit $\gamma_1 \rightarrow \gamma_2$.

Now we have a very peculiar situation: the model has infinite number of local conserved charges (3.10) and at the same time there are unstable particles. If all particles were stable the standard argument [30] relate the presence of infinite number of local conserved charges to factorization of a scattering matrix into two-particle $S$ matrices. In quantum theory unstable particles are usually excluded from “in” and “out” states, and the only trace of them are resonance poles in $S$-matrix of the theory. From this point of view it is crucial to find quantum counterpart of charges (3.10) to discuss quantum integrability of (3.2), $S$-matrix, etc. We address this problem in the next section.

As in sG model there are bound states of kinks. Solutions describing bound states, scattering processes of bound states and kinks can be obtained from (3.21) by replacement of some poles $\gamma_n$ by pair of poles $\gamma_n \pm i\varepsilon_n$.

3.3. $\tilde{so}(2)$ reduction of the model.

One can note that when one of $\alpha_n$ or $\beta_n$ vanish, the model regains right or left $\tilde{so}(2)$ symmetry. If, for example, $\alpha_3$ and $\beta_3$ vanish simultaneously $\tilde{so}(2) \times \tilde{so}(2)$ symmetry is regained in the model. It means that there is a propagating massless field. It is possible to factor out this mode as in [31], [32] and get another model described by the functional *

$$S = \int d^2z \left( \frac{\partial \mu u}{1 - uu} \frac{\partial \mu \bar{u}}{1 - \bar{u}u} + 2\lambda ((u\bar{u})^2 - u\bar{u}) \right),$$

* This model is closely related to $Z_n$ parafermionic models perturbed by second thermal operator $\varepsilon_2$ [33].
which after change of variables
\[ u = \exp \left( \frac{i \alpha}{2} \right) \sin \left( \frac{\theta}{2} \right), \quad \bar{u} = \exp \left( -\frac{i \alpha}{2} \right) \sin \left( \frac{\theta}{2} \right), \]  
(3.40)
takes the form
\[ S = \frac{1}{4} \int d^2 z \left( \partial^\mu \theta \partial_\mu \theta + \tan^2 \left( \frac{\theta}{2} \right) \partial^\mu \alpha \partial_\mu \alpha + \lambda (\cos 2\theta - 1) \right) \]  
(3.41)
This model is integrable and possesses conserved charges of spin \( s = \pm 1, \pm 3, \pm 5, \ldots \). To describe the densities of these charges we introduce new fields \( \omega \) and \( \gamma \) \[ \bar{32} \] which satisfy equations
\[ \partial \omega = \partial \alpha \frac{\cos \theta}{2 \cos^2 \frac{\theta}{2}}, \quad \partial \omega = \bar{\partial} \alpha \frac{1}{2 \cos^2 \frac{\theta}{2}} \]  
\[ \partial \gamma = \partial \alpha \frac{1}{2 \cos^2 \frac{\theta}{2}}, \quad \bar{\partial} \gamma = \bar{\partial} \alpha \frac{\cos \theta}{2 \cos^2 \frac{\theta}{2}} \]  
(3.42)
The above equations are compatible if equations of motion of (3.41) are being imposed. Using \( \omega \) and \( \gamma \) we construct currents
\[ j^\pm = \left( \partial \theta \pm i \partial \alpha \tan \frac{\theta}{2} \right) e^{\pm i \omega} \]  
\[ \bar{j}^\pm = \left( \bar{\partial} \theta \pm i \bar{\partial} \alpha \tan \frac{\theta}{2} \right) e^{\pm i \gamma} \]  
(3.43)
For \( \lambda = 0 \) equations of motion for (3.41) are equivalent to
\[ \bar{\partial} j^\pm = \partial \bar{j}^\pm = 0 \]  
(3.44)
Local integrals of motion (IM) for (3.41) turn out to be differential polynomials in \( j^\pm \) and \( \bar{j}^\pm \). The first ones, \( s = \pm 1, \pm 3 \) are
\[ Q^{(1)} = j^+ j^- \]  
\[ \bar{Q}^{(1)} = \bar{j}^+ \bar{j}^- \]  
\[ Q^{(3)} = (j^+ j^-)^2 + 2 \partial j^+ \partial j^- \]  
\[ \bar{Q}^{(3)} = (\bar{j}^+ \bar{j}^-)^2 + 2 \bar{\partial} \bar{j}^+ \bar{\partial} \bar{j}^- \]  
(3.45)
4. Quantum Integrals of Motion.

In this section we will construct IMs proving quantum integrability of the model discussed in section 3.

The central object of quantum theory are correlation functions of some operators \( O_n \)
\[
< O_1(z_1, \bar{z}_1) \cdots O_n(z_n, \bar{z}_n) >_A = \int \mathcal{D}\phi \, O_1(z_1, \bar{z}_1) \cdots O_n(z_n, \bar{z}_n) \exp(-A[\phi])
\] (4.1)

where fields \( \phi \) are some “basic” fields, operators \( O_n \) are some local functionals of \( \phi \)’s and their derivatives, and \( A[\phi] \) is an action functional. In this section we are back to euclidean two dimensional space. In our case \( A \) is sum of WZW action (2.1) \( S_{wzw} \), which is conformally invariant and represent fixed point of renormalization group, and perturbation (3.1). We will assume that the structure of space of fields in the perturbed model is the same as in WZW model (2.20). To identify operator \( O \) as an operator in WZW model we use (3.6) to obtain

\[
O = -\frac{1}{2\pi} \int d^2 z \, \eta m \bar{\eta} \Phi^{(2)}_{m, \bar{m}}(z, \bar{z}) - \frac{1}{2\pi} \int d^2 z \, \eta \cdot 1
\]
\[
\equiv O^{(2)} + O^{(0)}
\] (4.2)

where vectors \( \eta^m, \bar{\eta}_{\bar{m}} \) and constant \( \eta \) are given by

\[
\eta_m = \left( \frac{\beta_3}{2}, 0, \frac{\beta_1 - \beta_2}{\sqrt{6}}, 0, \frac{\beta_3}{2} \right), \quad \bar{\eta}_{\bar{m}} = \left( \frac{\alpha_3}{2}, 0, \frac{\alpha_1 - \alpha_2}{\sqrt{6}}, 0, \frac{\alpha_3}{2} \right),
\]
\[
\eta = \frac{1}{3} \text{Tr} A \cdot \text{Tr} B - \text{Tr} (AB)
\] (4.3)

For \( k \geq 6 \) operators \( \Phi^{(2)}_{m, \bar{m}} \) are relevant and we will show soon that the action is well defined for \( k > 10 \) (see discussion of “resonance” condition below). After expansion in \( O^{(2)} \) to the first order we get

\[
< O_1(z_1, \bar{z}_1) \cdots O_n(z_n, \bar{z}_n) >_A = < O_1(z_1, \bar{z}_1) \cdots O_n(z_n, \bar{z}_n) >_{wzw} e^{\frac{V}{2\pi}} -
\]
\[
< O^{(2)} O_1(z_1, \bar{z}_1) \cdots O_n(z_n, \bar{z}_n) >_{wzw} e^{\frac{V}{2\pi}} + \cdots
\] (4.4)

Here \( < \cdots >_{wzw} \) means that average is taken with respect to WZW action and \( V \) is a volume of the base space. Note that the perturbation theory around conformal fixed point is IR divergent and to make it finite we fix \( V \) finite but large. Here we concentrate on equations of motion which are relations between local operators and their derivatives. From this point of view IR divergences are irrelevant.
Suppose we want to calculate $\bar{\partial}$ derivative of some holomorphic (at fixed point) operator, say $Q(z)$, which is local with respect to $\Phi_{m,\bar{m}}^{(2)}(z,\bar{z})$. After taking derivative of (4.4) and neglecting usual contact terms (coming from the first term in (4.4)) we get

$$\bar{\partial}Q(z,\bar{z}) = \eta_m \bar{\eta}_{\bar{m}} \oint_{C_z} d\zeta \Phi_{m,\bar{m}}^{(2)}(\zeta,\bar{z}) Q(z) \quad (4.5)$$

If the residue of OPE

$$\text{Res}(\eta_m \bar{\eta}_{\bar{m}} \Phi_{m,\bar{m}}^{(2)}(z,\bar{z}) Q(w))_{z=w} = \partial R(w,\bar{w}) \quad (4.6)$$

for some operator $R$, then we have constructed quantum conserved current

$$\bar{\partial}Q(z,\bar{z}) = \partial R(w,\bar{w}) \quad (4.7)$$

In general $\bar{\partial}$ derivative of a local operator $Q$ in perturbed model schematically take a form

$$\bar{\partial}Q = (\eta \bar{\eta}) R_1 + (\eta \bar{\eta})^2 R_2 + \cdot \quad (4.8)$$

where $R_n$ are some local fields. Because the structure of the space of fields in the perturbed WZW model was assumed to be the same as in unperturbed model, we conclude that the series in the right hand side of (4.8) terminates at some term. Moreover, all fields $R_n$ must have anomalous dimensions coinciding with the WZW ones. If the anomalous dimension of some field $R_n$ matches the one in WZW it must be added to the right hand side of (4.8). The “resonance” condition can be derived from (4.8) by matching dimensions of operators on both sides of the equation

$$1 - n (1 - \Delta_2) = \Delta_j \quad (4.9)$$

for some $n > 1$ and $j$. The solutions to the above equation corresponds to additional term of order $(\eta \bar{\eta})^n$ in (4.8). The only “resonance” cases are

$$k = 6 \quad j = 0, \quad n = 4 \quad \text{and} \quad j = 1, \quad n = 3$$

$$k = 7, \quad j = 0, \quad n = 3$$

$$k = 8, \quad j = 1, \quad n = 2$$

$$k = 10, \quad j = 0, \quad n = 2 \quad (4.10)$$

For that values of $k$ one need to consider next to the leading order of the conformal perturbation expansion (4.4) to derive equations of motion for local operators. Therefore we will restrict our attention to the case $k > 10$ when the action is well defined and will work within the first order of the conformal perturbation theory.
4.1. Coserved currents in $U \left( \tilde{su}(2)_k \right)$.

Here we will find conserved currents which are composites of Kac-Moody currents $J(z)$, \((2.4)\).

Following \[34\] we define new operators $D_{n,m}$, $n = 0, \pm 1, \pm 2, ..., m = -2, ..., 2$ by its action in current module spanned by fields $J^{a_1}_{n_1} J^{a_2}_{n_2} ... J^{a_s}_{n_s} \cdot 1$

$$D_{m,n} Q(z, \bar{z}) = \oint_{C_z} d\zeta \Phi^{(2)}_{m}(\zeta, \bar{z}) (\zeta - z)^n Q(z) \quad (4.11)$$

where notation $\Phi^{(2)}_{m}(z, \bar{z})$ for the operator $\eta_{\bar{m}} \Phi^{(2)}_{m, \bar{m}}(z, \bar{z})$ was introduced. Using (2.18) one can easily prove the commutations relations

$$[J^+_p, D_{m,n}] = [(j - m)(j + m + 1)]^{1/2} D_{m+1,p+n}$$
$$[J^0_p, D_{m,n}] = mD_{m,p+n} \quad (4.12)$$
$$[J^-_p, D_{m,n}] = [(j + m)(j - m + 1)]^{1/2} D_{m-1,p+n}$$

These operators satisfy simple relations

$$\eta_m D_{m,0} Q(z) = -\bar{\partial}Q(z, \bar{z}),$$
$$D_{m,-n-1} \cdot 1 = \frac{1}{n!} L^n_{-1} \Phi^{(2)}_{m}(z, \bar{z}) \quad (4.13)$$

Now it is straightforward to find $\bar{\partial}$ derivatives of any field from the current module - apply $\eta_m D_{m,0}$ and then move all $D$’s all the way to the right where they transforms into derivatives of $\Phi^{(2)}_{m, \bar{m}}$. Sugavara construction \((2.17)\) allow to express the result entirely in terms of Kac-Moody generators $J^a_n$. Solving after that \((4.7)\) we find quantum integrals of motion for spin $s = 1, 3$. For example

$$Q^{(1)}_1 = \left( J^-_{-1} J^+_{-1} + J^0_{-1} J^0_{-1} \right) \cdot 1$$
$$Q^{(1)}_2 = \left( \eta_{-2} J^-_{-1} J^-_{-1} - \sqrt{6} \eta_0 J^-_{-1} J^-_{-1} + \eta_2 J^+_{-1} J^-_{-1} \right) \cdot 1 \quad (4.14)$$

The expressions for first two nontrivial quantum integrals of motion of spin $s = 3$ are given in Appendix B.

What we found is in perfect agreement with the classical result \((3.14)\). As before there are 2 conserved charges of spin $s = 1$ and 2 charges of spin $s = 3$. Charges with $s = -1$ and $s = -3$ can be obtained from \((4.14)\) and replacing $J^a_n \rightarrow -\bar{J}^a_n$ and $\eta_m \rightarrow \bar{\eta}_m$. It is reasonable to make

Conjecture. For general $k$ the model \((3.2)\) possesses infinite number of mutually commutative local integrals of motion of spins $s = 0, \pm 1, \pm 2, ...$ and for any $s$ there are two independent IM.

Note that the derivation does not feel if $k$ is integer or not.
4.2. Additional set of conserved charges for integer \(k\).

In this section for even \(k\) we derive additional set of integrals of motion.

The spectrum of anomalous dimensions of WZW theory \((2.19)\) contains \(\Delta_k = k/4\). It is possible to show (see Appendix C) that the symmetry algebra of WZW model can be extended by new currents

\[
\Psi_m(z) \equiv \Phi_{m,0}^{(k,0)}(z), \quad \text{and} \quad \bar{\Psi}_\bar{m}(\bar{z}) \equiv \Phi_{0,\bar{m}}^{(0,k)}(\bar{z})
\]

(4.15)

For \(k = 4n\), for example, these fields contribute to the partition function of \(SO(3)_{4n}\) WZW model, \([23]\). We are going to show that certain linear combinations of \(\Psi_m, \bar{\Psi}_{\bar{m}}\) give rise to new set of IM.

To construct new IM’s we need to consider OPE, \((4.16)\),

\[
\eta_m \bar{\eta}_{\bar{m}} \Phi_{m,\bar{m}}^{(2,2)}(z_1, \bar{z}_1) \Psi_n(z_2) = 
C(2, \frac{k}{2}, \frac{k}{2} - 2) \eta_m \bar{\eta}_{\bar{m}} \left( \frac{\beta_{l,m,n}^{l} \Phi_{l,m}^{(4,2)}(z_2, \bar{z}_2)}{(z_1 - z_2)^2} + \beta_{a,l,m,n}^{a,l} J_a^{a,l} \Phi_{l,m}^{(4,2)}(z_2, \bar{z}_2) + \cdots \right)
\]

(4.16)

Here \(C(2, \frac{k}{2}, \frac{k}{2} - 2)\) is a structure constant of OPE (we will not need its exact value), \(\beta_{l,m,n}^{l}\) and \(\beta_{a,l,m,n}^{a,l}\) are some coefficients that can be determined comparing transformation properties of the left and right hand sides of \((4.16)\) under action of \(su(2)_k\). To find \(\beta_{l,m,n}^{l}\) and \(\beta_{a,l,m,n}^{a,l}\) we introduce following \([24]\) new variables \(x, \bar{x}\) and generating functions

\[
J(x, z) = J^{-}(z) + 2x J^{0}(z) - x^2 J^{+}(z)
\]

\[
\Phi^{(j,j)}(x, \bar{x}|z, \bar{z}) = \sum_{m=-j, m=-\bar{j}}^{j, \bar{j}} \left[ C_{2j}^{m+j} C_{2\bar{j}}^{m+j} \right]^{1/2} x^{m+j} \bar{x}^{\bar{m}+\bar{j}} \Phi_{m,\bar{m}}^{(j,\bar{j})}(z, \bar{z})
\]

(4.17)

where \(C_{2j}^{m+j} = (2j)!/(j+m)!(j-m)!\) are binomial coefficients. The singular part of the necessary OPE’s \((2.7)\) take a form

\[
J(x_1, z_1) J(x_2, z_2) = -k \frac{(x_1 - x_2)^2}{(z_1 - z_2)^2} - 2 \frac{x_1 - x_2}{z_1 - z_2} J(x_2, z_2) - \frac{(x_1 - x_2)^2}{z_1 - z_2} \partial J(x_2, z_2)
\]

\[
J(x_1, z_1) \Phi^{(j,j)}(x_2, \bar{x}_2|z_2, \bar{z}_2) = -2j \frac{x_1 - x_2}{z_1 - z_2} + \frac{(x_1 - x_2)^2}{z_1 - z_2} \partial \Phi^{(j,j)}(x_2, \bar{x}_2|z_2, \bar{z}_2)
\]

(4.18)

As in \((2.9)\) we define action of modes \(J_{\alpha}^{a}(x)\)

\[
J_{\alpha}^{a}(x) F(x, \bar{x}|z, \bar{z}) = \frac{1}{\varepsilon_{\alpha}(2 - \alpha^2)} \oint_{C_{x}} \frac{d\zeta}{2\pi i} (\zeta - z)^{\alpha} \oint_{C_{\bar{x}}} \frac{dy}{2\pi i} (y - x)^{-2-a} J^{a}(y, \zeta) F(x, \bar{x}|z, \bar{z})
\]

(4.19)
where $\varepsilon_+ = \varepsilon_0 = -\varepsilon_- = 1$. OPE of primary fields can be written in a form

$$
\Phi^{(j_1,\bar{j}_1)}(x_1, \bar{x}_1 | z_1, \bar{z}_1) \Phi^{(j_2,\bar{j}_2)}(x_2, \bar{x}_2 | z_2, \bar{z}_2) = \sum_{j,\bar{j}} C \left( \begin{array}{ccc} j_1 & j_2 & j \\ \bar{j}_1 & \bar{j}_2 & \bar{j} \end{array} \right) \Phi^{(j,\bar{j})}(x_2, \bar{x}_2 | z_2, \bar{z}_2) \tag{4.20}
$$

where

$$
\left[ \Phi^{(j,\bar{j})} \right] (x_1, \bar{x}_1 | z_1, \bar{z}_1) = R^j_{j_1,j_2} (z_{12}, x_{12}) \bar{R}^{\bar{j}}_{\bar{j}_1,\bar{j}_2} (\bar{z}_{12}, \bar{x}_{12}) \Phi^{(j,\bar{j})}(x_2, \bar{x}_2 | z_2, \bar{z}_2) \tag{4.21}
$$

and $z_{12} = z_1 - z_2$, $x_{12} = x_1 - x_2$, etc. Operators $R^j_{j_1,j_2}$ are defined by series

$$
R^j_{j_1,j_2} (z_{12}, x_{12}) = \sum_{n=0}^{\infty} z^n \sum_{a_i=0, \pm 1} J^{a_1}_{a_1}(x_2) J^{a_2}_{a_2}(x_2) \cdots J^{a_n}_{a_n}(x_n) R_n(a_1, \ldots, a_n) \tag{4.22}
$$

where $R_n(a_1, \ldots, a_n)$ are some differential operators

$$
R_n(a_1, \ldots, a_n) = \sum_{p=0}^{a_n} r_p(a_1, \ldots, a_n | x_{12}) \left( \frac{\partial}{\partial x_2} \right)^p \tag{4.23}
$$

Operators $R_n$ are analogs of coefficients $\beta^l_{m,n}$ and $\beta^{a,l}_{m,n}$ in (4.16). There is, of course, similar expression for $\bar{R}^{\bar{j}}_{\bar{j}_1,\bar{j}_2}$. In the following to simplify notations we will often omit all “bar” arguments and indices as we will work in the holomorphic sector.

Operators $R_n(a_1, \ldots, a_n)$ can be obtained, as we already mentioned, recursively from the requirement that both sides in (4.20) transform in same way under $su(2)_k$. In particular to get the recursion we apply $J^0_n(x)$ with $n = 0$ and $n = 1$ to both sides of (4.20). Action with $J^0_n(x)$ relate different $r_p$’s within same $R_n$ and action with $J^a_n(x)$ allow to relate $r_p$’s for two successive $R_n$. The operator $R_0$ was found in [24]. For $(j_1, j_2, j) = (2, \frac{k}{2}, \frac{k}{2} - 2)$ it is identity operator

$$
R_0 = 1 \tag{4.24}
$$

It is possible to find the next operator $R_1(a)$. The only nonzero coefficients $r_p(a)$ for the case $(j_1, j_2, j) = (2, \frac{k}{2}, \frac{k}{2} - 2)$ are

$$
\begin{align*}
    r_0(0) &= \frac{2(k - 6)}{k - 4}, \\
    r_0(-) &= \frac{4}{k - 4}, \\
    r_1(-) &= \frac{2}{k - 4}
\end{align*} \tag{4.25}
$$
Now, finally, we introduce functions

\[ f(x) = \sum_{p=-2}^{2} f_p x^{2+p} \]

\[ g(x) = \sum_{p=-k/2}^{k/2} g_p x^{k/2+p} \]

\[ h(x) = \sum_{p=-k/2+2}^{k/2} h_p x^{k/2-2+p} \]  

(4.26)

to define operators

\[ \Phi^{(2)}(z) = \int d\mu^{(2)}(x, \bar{x}) f(\bar{x}) \Phi^{(2)}(x, z), \]

\[ \Psi(z) = \int d\mu^{(\bar{\Phi})}(x, \bar{x}) g(\bar{x}) \Phi^{(\bar{\Phi})}(x, z), \]

\[ \Phi^{(\bar{\Phi}-2)}(z) = \int d\mu^{(\bar{\Phi}-2)}(x, \bar{x}) h(\bar{x}) \Phi^{(\bar{\Phi}-2)}(x, z) \]  

(4.27)

where \( d\mu^{(j)} \) is a measure

\[ d\mu^{(j)}(x, \bar{x}) = \frac{1}{\pi} \frac{d^2x}{(1 + x\bar{x})^{2+2j}} \]  

(4.28)

We have arrived at the problem: when the residue \( \text{Res} \left[ \Phi^{(2)}(z_1) \Psi(z_2) \right]_{z_1=z_2} \) is a total derivative

\[ \text{Res} \left[ \Phi^{(2)}(z_1) \Psi(z_2) \right]_{z_1=z_2} = L_{-1} \Phi^{(\bar{\Phi}-2)}(z) ? \]  

(4.29)

The equation (4.29) is equivalent to the system of linear equations

\[ \int d\mu^{(\bar{\Phi})}(x, \bar{x}) g(\bar{x}) \left( (k-6)(a+1)p(x)x^a + p'(x)x^{a+1} - 2p(x)x^{a+1} \frac{\partial}{\partial x} \right) x^{k-2+m} = \]

\[ = \frac{5k-4}{2k+2} \int d\mu^{(\bar{\Phi}-2)}(x, \bar{x}) h(\bar{x}) \left( x^{a+1} \frac{\partial}{\partial x} - \frac{1}{2}(k-4)(a+1)x^a \right) x^{k-2+m} \]

for \( a = \{0, \pm1\}, \ m = \{-\frac{k}{2} + 2, \cdots, \frac{k}{2} - 2\} \)  

(4.30)

where

\[ p(x) = f_2 - f_1 x + f_0 x^2 - f_{-1} x^3 + f_{-2} x^4 \]  

(4.31)

This system has \( 3k - 9 \) equations for \( 2k - 2 \) variables \( g_p, h_p \), i.e. for \( k \geq 8 \) the system is overdetermined. We have found solutions of this system for \( f_{\pm1} = 0 \) and \( k = 6, 7, 8, \cdots, 24 \).
Such a choice of $f_p$'s corresponds to the perturbation (1.2). From analysis of obtained solutions one can draw

**Conjecture.** (i) For any $k = 4n, 4n + 1, 4n + 3$ and $k \geq 11$ the model (3.2) possesses two additional IM which are certain linear combinations of currents $\Psi_m$. (ii) For any $k = 4n + 2 \geq 14$ the model (3.2) possesses three additional IM which are certain linear combinations of currents $\Psi_m$.

Despite that the system (4.30) has also solutions for “resonance” cases $k = 6, 7, 8, 10$, we can not say anything about conservation of the corresponding charges in these cases. For first nonresonance cases $k = 11, 12, 13, 14$ we give solutions for $g_p$'s in Appendix D.

5. Decay of particles.

In section 3 we have constructed the solution describing decay process for the kink with mass $M_3$. The mass of that kink is always larger or equal (if $A = B$) then sum of masses of other, lighter, two kinks (3.38). It is natural to conjecture that while in the corresponding quantum theory the heavy kinks are being excluded from “in” and “out” states they result in resonance poles in $S$-matrix. To justify this point we calculated semiclassical limit of $S$-matrix for the scattering $K_{01}^{(e_1)}(\theta_1) K_{13}^{(e_2)}(\theta_2) \rightarrow K_{02}^{(e_2)}(\theta_2) K_{23}^{(e_1)}(\theta_1)$, (3.30)

$$S_{K_{01}K_{13}}^{(\text{semicl})} \equiv \lim_{n \to \infty} < K_{02}^{(e_2)}(\theta'_2) K_{23}^{(e_1)}(\theta'_1) | K_{01}^{(e_1)}(\theta_1) K_{13}^{(e_2)}(\theta_2) >_{\text{in}} = \left( \frac{16\pi^2}{k} \right)^2 \delta(\theta'_1 - \theta_1) \delta(\theta'_2 - \theta_2) \exp \left( i (A_{K_{01}K_{13}} - A_{K_{01}} - A_{K_{13}}) \right)$$

(5.1)

where $A_g$ is action functional $A$ evaluated on the corresponding kink solution (3.23), (3.30). For any solution $g$ of (3.8) integrals on the left hand side in (5.1) can be done with the use of formula for a variation of $A$

$$\delta A[g] = - \frac{k}{8\pi} \text{Tr} \int dx dt \partial^{\mu} \left( (g^{-1} \delta g)(g^{-1} \partial_{\mu} g) \right)$$

(5.2)

One can obtain

$$S_{K_{01}K_{13}}^{(\text{semicl})}(\theta_{12}) = \left( \frac{16\pi^2}{k} \right)^2 \delta(\theta'_1 - \theta_1) \delta(\theta'_2 - \theta_2) \cdot$$

$$\exp \left( - \frac{i\pi k}{4} - \frac{k}{2\pi} \int_0^\pi d\zeta \ln \left( \frac{1 + e^{\theta_12 - \theta_0 - i\zeta}}{e^{\theta_12 - \theta_0 + e^{-i\zeta}}} \right) \right)$$

(5.3)

here $\theta_0 = \frac{1}{2} \ln ((\alpha_1/\beta_2)/(\alpha_2/\beta_1))$, $\theta_{12} = \theta_1 - \theta_2$. From (5.3) follows that $S_{K_{01}K_{13}}^{(\text{semicl})}(\theta)$ has a line of essential singularities in the physical strip $0 < \text{Im} \theta < \pi$ at $\text{Re} \theta = \theta_0$. We expect
that this singular line corresponds to series of resonance poles which become dense and are “glued” together as $k \to \infty$.

We can also study scattering of “basic” particles $\psi, \vartheta, \varphi$ which enter the action functional $A$. From (3.5) we read their masses

$$
m_\psi = 2 \left( k^{-1} \alpha_1 \beta_1 \right)^{1/2} \\
m_\vartheta = 2 \left( k^{-1} \alpha_2 \beta_2 \right)^{1/2} \\
m_\varphi = 2 \left( k^{-1} \alpha_3 \beta_3 \right)^{1/2}
$$

As in the kink case we obtain

$$m_\varphi \geq m_\psi + m_\vartheta$$

This is an indication of instability of the $\varphi$ particle. Indeed, to the leading order we get

$$\text{out} \langle \vartheta(p_3)\psi(p_2)\varphi(p_1) \rangle_{\text{in}} = i \left( \frac{16\pi^2}{k} \right)^2 k^{-1}(\alpha_1 \beta_2 - \alpha_2 \beta_1) \delta^{(2)}(p_1 - p_2 - p_3)$$

In the rest frame of $\varphi$ particle $\psi$ particle propagate to the right and $\vartheta$ propagate to the left. Due to broken $P$-parity in the theory the amplitude for decay process with interchanged $\psi$ and $\vartheta$ is zero. It is possible to check that to the leading order amplitudes for process $\psi \psi \rightarrow \vartheta \vartheta$ is zero. This illustrates integrability in the tree approximation - nontrivial IMs prohibit such kind of processes. The amplitude for scattering $\psi$ and $\vartheta$ is

$$S_{\vartheta \psi}(\theta_{12}) = \text{out} \langle \psi(\theta'_1)\vartheta(\theta'_2)\vartheta(\theta_2)\psi(\theta_1) \rangle_{\text{in}} =$$

$$\left( \frac{16\pi^2}{k} \right)^2 \left( 1 + \frac{2i}{k \sinh(\theta_{12} \pm \theta_0)} \right) \delta(\theta'_1 - \theta_1) \delta(\theta'_2 - \theta_2)$$

here sign $\pm$ distinguish between $\psi$ particle propagating to the left or to the right in the center-of-mass frame. Due to existence of higher IMs in the theory reflection part of the amplitude is absent in (5.7). The shift of rapidity in (5.7) agrees with the one in semiclassical amplitude for kinks (5.3).
6. Conclusions.

In the paper we studied quantum integrability of \( SU(2)_k \) WZW model perturbed by multiplet of primary fields with isospin \( j = 2 \). The model contains four free parameters. The spectrum of the theory contains kinks and their bound states. Kinks interpolate between different vacua in the theory and has \( Z_2 \) charge. Some of kinks are unstable and we obtained solution describing decay of the heaviest kink into two lighter kinks. Quantum integrability was proven by explicit construction of quantum IMs. It appears that there are two sets of IMs. The first one - IMs belonging to universal enveloping algebra \( U(\mathfrak{su}(2)_k) \). Integrals of motion from this set admit semiclassical limit \( k \to \infty \) and there is recurrent formula for IMs in this limit. The other set are integrals of motion built from chiral components of primary field of isospin \( j = k/2 \) (conformal dimension \( \Delta_{k/2} = \frac{k}{4} \)). Obviously these IMs do not admit semiclassical limit. We also briefly discussed scattering amplitudes for kinks and fundamental particles. We argued that scattering amplitudes for stable kinks and particles possess resonance poles and calculated to the leading order in \( 1/k \) decay amplitude for the heaviest particle. The work on the construction of the exact kink S-matrix of the theory is in progress.

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8. Appendix A.

Full expression for the two kink solution is given by

\[
\begin{align*}
g_{11} &= \frac{1 - e^{2\phi_1} + \Delta^2 e^{2\phi_2} - e^{2(\phi_1 + \phi_2)}}{1 + e^{2\phi_1} + \Delta^2 e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}}, & g_{12} &= \frac{2(-1)^{\epsilon_1} e^{\phi_1} (1 - \Delta e^{2\phi_2})}{1 + e^{2\phi_1} + \Delta^2 e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}} \\
g_{13} &= \frac{2(-1)^{\epsilon_1 + \epsilon_2} (1 + \Delta) e^{\phi_2 + \phi_1}}{1 + e^{2\phi_1} + \Delta^2 e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}}, & g_{21} &= \frac{-2(-1)^{\epsilon_1} e^{\phi_1} (1 + \Delta e^{2\phi_2})}{1 + e^{2\phi_1} + \Delta^2 e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}} \\
g_{22} &= \frac{1 - e^{2\phi_1} - \Delta^2 e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}}{1 + e^{2\phi_1} + \Delta^2 e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}}, & g_{23} &= \frac{-2(-1)^{\epsilon_2} (e^{2\phi_2} - \Delta) e^{\phi_2}}{1 + e^{2\phi_1} + \Delta^2 e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}} \\
g_{31} &= \frac{-2(-1)^{\epsilon_1 + \epsilon_2} (1 - \Delta) e^{\phi_2 + \phi_1}}{1 + e^{2\phi_1} + \Delta^2 e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}}, & g_{32} &= \frac{-2(-1)^{\epsilon_2} (\Delta + e^{2\phi_1}) e^{\phi_2}}{1 + e^{2\phi_1} + \Delta^2 e^{2\phi_2} + e^{2(\phi_1 + \phi_2)}}
\end{align*}
\]
9. Appendix B.

Here we give expressions for first nontrivial IM’s of spin $s = 3$ in universal enveloping algebra $U \left( su(2)_k \right)$

\[
Q^{(3)}_1 = \left( 6 \eta_{-2} J_{-1} J_{-1} J_{-1} J_{-1} + 12 \sqrt{6} \eta_{-2} \eta_0 J_{-1} J_{-1} J_{0} J_{0} + 20 \eta_{-2} \eta_2 J_{-1} J_{+1} J_{+1} + 16 \eta_{-2} \eta_2 J_{-1} J_{0} J_{0} J_{+1} + 4(9 \eta_0^2 + 2 \eta_{-2} \eta_2) J_{-1} J_{0} J_{0} J_{+1} J_{+1} + 12 \sqrt{6} \eta_{-2} \eta_0 J_{-2} J_{-1} J_{0} J_{0} + 6 \eta_{-2} \eta_2 J_{-1} J_{0} J_{+1} J_{+1} + 12 \sqrt{6} (8 - k) \eta_{-2} \eta_0 J_{-2} J_{-1} J_{0} J_{0} + 16(k - 10) \eta_{-2} \eta_2 J_{-2} J_{-1} J_{+1} + 8(k + 2) \eta_{-2} \eta_2 J_{-2} J_{0} J_{+1} J_{+1} + 6 \sqrt{6} (12 - k) \eta_0 \eta_2 J_{-2} J_{+1} J_{+1} + 6 \sqrt{6} (k - 10) \eta_{-2} \eta_0 J_{-2} J_{-2} + 8(k^2 - 12k + 24) \eta_{-2} \eta_2 J_{-2} J_{0} J_{0} + (9(k^2 - 12k + 16) \eta_0^2 + 2(k^2 - 8k - 4) \eta_{-2} \eta_2) J_{-2} J_{0} J_{0} + 8(k^2 - 13k + 24) \eta_{-2} \eta_2 J_{-2} J_{+1} J_{+1} \right) \cdot 1
\]

\[
Q^{(3)}_2 = \left( -6 \eta_{-2} J_{-1} J_{-1} J_{-1} J_{-1} + 12 \sqrt{6} \eta_{-2} \eta_0 J_{-1} J_{-1} J_{-1} J_{+1} - 4(6 \eta_0^2 + 5 \eta_{-2} \eta_2) J_{-1} J_{-1} J_{+1} J_{+1} + 8(3 \eta_0^2 - 2 \eta_{-2} \eta_2) J_{-1} J_{0} J_{0} J_{+1} J_{+1} + 4(3 \eta_0^2 - 2 \eta_{-2} \eta_2) J_{0} J_{0} J_{0} J_{0} J_{+1} J_{+1} + 12 \sqrt{6} \eta_0 \eta_2 J_{-1} J_{0} J_{0} J_{+1} J_{+1} - 6 \eta_{-2} \eta_2 J_{+1} J_{+1} J_{+1} J_{+1} - 72 \sqrt{6} \eta_{-2} \eta_0 J_{-2} J_{-1} J_{0} J_{0} + (12(k - 16) \eta_0^2 - 2(k + 2) \eta_{-2} \eta_2) J_{0} J_{-1} J_{+1} J_{+1} + 8(k - 10)(3 \eta_0^2 - 2 \eta_{-2} \eta_2) J_{-2} J_{0} J_{0} J_{+1} J_{+1} + 36 \sqrt{6} \eta_0 \eta_2 J_{-2} J_{+1} J_{+1} J_{+1} + (6(6 + 10k - k^2) \eta_0^2 - 8(k^2 - 13k + 24) \eta_{-2} \eta_2) J_{-2} J_{+2} J_{+2} + (3(k^2 - 8k - 40) \eta_0^2 - 2(k^2 - 8k - 4) \eta_{-2} \eta_2) J_{0} J_{-2} J_{-2} + 3 \sqrt{6}(k - 10)(k - 2) \eta_{-2} \eta_0 J_{-2} J_{-2} + 3 \sqrt{6}(k - 10)(k - 2) \eta_0 \eta_2 J_{+2} J_{+2} J_{+2} \right) \cdot 1
\]
10. Appendix C.

In this appendix we prove that for integer \( k \) current algebra \( \widehat{su(2)}_k \) can be extended by operators \( \Psi_m \), \((4.17)\).

In section 4.2 we introduced generating functions \( J(x, z) \) and \( \Phi^{(j,j)}(x, \bar{x}, z, \bar{z}) \). Using Knizhnik-Zamolodchikov equation and null-field \( J(x) \Phi^{(k/2)}(x, z) \) one can derive \((24)\) equation for four-point conformal block,

\[
G = \langle \Phi^{(k/2)}(x_1, z_1)\Phi^{(k/2)}(x_2, z_2)\Phi^{(k/2)}(x_3, z_3)\Phi^{(k/2)}(x_4, z_4) \rangle \tag{10.1}
\]

The solution of the equation takes a simple form

\[
G_{34}^{12}(x, z) = (x_{32}x_{14}(x-z))^k (z_{32}z_{14}z(z-1))^{-k/2} \tag{10.2}
\]

where \( x = (x_{12}x_{34})/(x_{14}x_{32}) \), \( z = (z_{12}z_{34})/(z_{14}z_{32}) \). To check associativity of the algebra generated by \( \Phi^{(k/2)}(x, z) \) we fuse \( \Phi \)'s in different ways and compare results. For example, we can fuse pairs \( \Phi^{(k/2)}(x_1, z_1), \Phi^{(k/2)}(x_2, z_2) \) and \( \Phi^{(k/2)}(x_3, z_3), \Phi^{(k/2)}(x_4, z_4) \), or \( \Phi^{(k/2)}(x_1, z_1), \Phi^{(k/2)}(x_3, z_3) \) and \( \Phi^{(k/2)}(x_2, z_2), \Phi^{(k/2)}(x_4, z_4) \). To make functions \( G_{34}^{12}(x, z) \) and \( G_{24}^{13}(1-x, 1-z) \) coincide one should impose “equal-time” commutation relations

\[
\Phi^{(k/2)}(x_1, z_1)\Phi^{(k/2)}(x_2, z_2) = e^{i\pi k/2} \Phi^{(k/2)}(x_2, z_2)\Phi^{(k/2)}(x_1, z_1), \tag{10.3}
\]

With this prescription algebra generated by operators \( \Psi_m \) become associative.
11. Appendix D.

Here solutions for functions $g(x)$, \([4.26]\) are given; $\alpha_{0,\pm1}$, $\alpha$ and $\beta$ are free parameters.

\[ k = 11 \]
\[ g_{\frac{11}{2}} = \frac{1}{3696} f_0 (80 f_0^2 f_{-2} f_2 - 7 f_0^4 - 320 f_{-2}^2 f_2^2) \beta \]
\[ g_{\frac{9}{2}} = \frac{2}{3} f_2^5 \alpha \]
\[ g_{\frac{7}{2}} = \frac{5}{168} f_{-2} (16 f_0^2 f_{-2} f_2 - f_0^4 + 64 f_{-2}^2 f_2^2) \beta \]
\[ g_{\frac{5}{2}} = -5 f_0 f_2^3 \alpha \]
\[ g_{\frac{3}{2}} = -\frac{5}{14} f_0 f_{-2}^2 (f_0^2 + 24 f_{-2} f_2) \beta \]
\[ g_{\frac{1}{2}} = f_2^3 (5 f_0 + 8 f_2 f_2) \alpha \]
\[ g_{-\frac{1}{2}} = f_2^3 (5 f_0 + 8 f_2 f_2) \beta \]
\[ g_{-\frac{3}{2}} = -\frac{5}{14} f_0 f_{-2}^2 (f_0^2 + 24 f_{-2} f_2) \alpha \]
\[ g_{-\frac{5}{2}} = -5 f_0 f_2^4 \beta \]
\[ g_{-\frac{7}{2}} = \frac{5}{168} f_2 (16 f_0^2 f_{-2} f_2 - f_0^4 + 64 f_{-2}^2 f_2^2) \alpha \]
\[ g_{-\frac{9}{2}} = \frac{2}{3} f_{-2}^5 \beta \]
\[ g_{-\frac{11}{2}} = \frac{1}{3696} f_0 (80 f_0^2 f_{-2} f_2 - 7 f_0^4 - 320 f_{-2}^2 f_2^2) \alpha \]

\[ k = 12 \]
\[ g_6 = \frac{1}{1485} f_2^3 (8 f_0^2 \beta + 27 f_{-2} f_2 \beta + 15 f_0^2 \alpha) \]
\[ g_4 = \frac{1}{90} f_{-2} f_2^3 f_0 (8 \beta + 15 \alpha) \]
\[ g_2 = f_{-2}^2 f_2^3 \beta \]
\[ g_0 = f_{-2} f_2^2 f_0 \alpha \]
\[ g_{-2} = f_{-2}^3 f_2^2 \beta \]
\[ g_{-4} = \frac{1}{90} f_{-2}^3 f_2 f_0 (8 \beta + 15 \alpha) \]
\[ g_{-6} = \frac{1}{1485} f_{-2}^3 (8 f_0^2 \beta + 27 f_{-2} f_2 \beta + 15 f_0^2 \alpha) \]
\[ g_{-5} = g_{-3} = g_{-1} = g_1 = g_3 = g_5 = 0 \]
\[ k = 13 \]

\[ g_{13}^{\pm} = \frac{1}{6864} (3f_0^6 - 40f_0^4f_{-2}f_2 + 192f_0^4f_{-2}^2f_2^2 - 512f_{-2}^3f_2^3) \beta \]

\[ g_{11}^\mp = -\frac{4}{11} f_2^6 \alpha \]

\[ g_9^{\pm} = \frac{1}{132} f_0 f_{-2} (f_0^4 - 16f_0^2f_{-2}f_2 + 192f_{-2}^2f_2^2) \beta \]

\[ g_7^\mp = 4f_0f_2^5 \alpha \]

\[ g_4^\pm = \frac{1}{12} f_{-2}^2 (f_0^4 - 48f_0^2f_{-2}f_2 + 64f_{-2}^2f_2^2) \beta \]

\[ g_4^\mp = -f_2^4 (7f_0^2 + 8f_{-2}f_2) \alpha \]

\[ g_4^\pm = 2f_0f_2^3 (f_0^2 + 8f_{-2}f_2) \beta \]

\[ g_{-4}^\pm = 2f_0f_2^3 (f_0^2 + 8f_{-2}f_2) \alpha \]

\[ g_{-2}^\pm = -f_2^4 (7f_0^2 + 8f_{-2}f_2) \beta \]

\[ g_{-2}^\mp = \frac{1}{12} f_2^2 (f_0^4 - 48f_0^2f_{-2}f_2 + 64f_{-2}^2f_2^2) \alpha \]

\[ g_{-2}^\pm = 4f_0f_{-2}^5 \beta \]

\[ g_{-2}^\mp = \frac{1}{132} f_0 f_{-2} (f_0^4 - 16f_0^2f_{-2}f_2 + 192f_{-2}^2f_2^2) \alpha \]

\[ g_{-4}^\pm = -\frac{4}{11} f_2^6 \beta \]

\[ g_{-4}^\mp = \frac{1}{6864} (3f_0^6 - 40f_0^4f_{-2}f_2 + 192f_0^4f_{-2}^2f_2^2 - 512f_{-2}^3f_2^3) \alpha \]
\( k = 14 \)

\[
g_7 = \frac{1}{15015} f_2^3 (5 f_0^3 \alpha_1 + 86 f_{-2} f_0 f_2 \alpha_1 + 35 f_0^2 f_2 \alpha_{-1} + 77 f_{-2} f_2^2 \alpha_{-1})
\]

\[
g_6 = \frac{5}{143} f_2^3 \alpha_0
\]

\[
g_5 = \frac{1}{165} f_{-2} f_2^3 (f_0^2 \alpha_1 + 15 f_{-2} f_2 \alpha_1 + 7 f_0 f_2 \alpha_{-1})
\]

\[
g_4 = -\frac{5}{11} f_0 f_2^2 \alpha_0
\]

\[
g_3 = \frac{1}{15} f_{-2}^2 f_2^3 (f_0 \alpha_1 + 7 f_2 \alpha_{-1})
\]

\[
g_2 = f_2 (f_0^2 + f_{-2} f_2) \alpha_0
\]

\[
g_1 = f_{-2}^2 f_2^3 \alpha_1
\]

\[
g_0 = -\frac{3}{7} f_0 (f_0^2 + 6 f_{-2} f_2) \alpha_0
\]

\[
g_{-1} = f_{-2}^2 f_2^3 \alpha_{-1}
\]

\[
g_{-2} = f_{-2} (f_0^2 + f_{-2} f_2) \alpha_0
\]

\[
g_{-3} = \frac{1}{15} f_{-2}^3 f_2^2 (f_0 \alpha_{-1} + 7 f_{-2} \alpha_1)
\]

\[
g_{-4} = -\frac{5}{11} f_{-2}^2 f_0 \alpha_0
\]

\[
g_{-5} = \frac{1}{165} f_{-2}^3 f_2 (f_0^2 \alpha_{-1} + 15 f_{-2} f_2 \alpha_{-1} + 7 f_{-2} f_0 \alpha_1)
\]

\[
g_{-6} = \frac{5}{143} f_{-2}^3 \alpha_0
\]

\[
g_{-7} = \frac{1}{15015} f_{-2}^3 (5 f_0^3 \alpha_{-1} + 86 f_{-2} f_0 f_2 \alpha_{-1} + 35 f_{-2} f_0^2 \alpha_1 + 77 f_{-2}^2 f_2 \alpha_1)
\]

(11.4)
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