Functoriality of groupoid quantales. I*

PEDRO RESENDE

Abstract

We provide three functorial extensions of the equivalence between localic étale groupoids and their quantales. The main result is a biequivalence between the bicategory of localic étale groupoids, with bi-actions as 1-cells, and a bicategory of inverse quantal frames whose 1-cells are bimodules. As a consequence, the category \( \text{InvQuF} \) of inverse quantale frames, whose morphisms are the (necessarily involutive) homomorphisms of unital quantales, is equivalent to a category of localic étale groupoids whose arrows are the algebraic morphisms in the sense of Buneci and Stachura. We also show that the subcategory of \( \text{InvQuF} \) with the same objects and whose morphisms preserve finite meets is dually equivalent to a subcategory of the category of localic étale groupoids and continuous functors whose morphisms, in the context of topological groupoids, have been studied by Lawson and Lenz.

Keywords: Groupoids, quantales, covering functors, bi-actions, algebraic morphisms

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Contents

1 Introduction
   1.1 Groupoids and quantales .............................................. 4
   1.2 Groupoid actions ....................................................... 6

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1 Introduction

Locales [11] are a point-free version of topological spaces. An example is the locale $I(A)$ of closed ideals of an abelian C*-algebra $A$, which is an algebraic (lattice-theoretic) object that contains all the information about the spectrum of the algebra. In many contexts locales are more convenient to work with than spaces, especially when points, separation axioms, etc., can be ignored. In such situations locales often lead to more general theorems, in particular theorems that are constructive in the sense of being valid in arbitrary toposes [12]. One can also think of a locale as being a kind of commutative ring (with the underlying abelian group replaced by a sup-lattice). The similarity to commutative algebra goes a long way and it is at the basis of the groupoid representation of Grothendieck toposes [13], in which localic groupoids (i.e., groupoids in the category of locales $\text{Loc}$) arise from toposes via descent.

A generalization of locales is given by quantales [34], which are semigroups in the category of sup-lattices and thus are like noncommutative rings. The idea that some quantales can be regarded as generalized, and C*-algebra related, point-free spaces has been around since the term “quantale” was coined [2,14,16,24,26,35], and there is a particularly good interplay between quantales and groupoids [27,29,31]. Concretely, the quantale of a topological groupoid $G$ (with open domain map) is the topology of the arrow space $G_1$ equipped with pointwise operations of multiplication and involution. This can be regarded as a convolution “algebra”, for if we identify each open subset $U \subset G_1$ with a continuous mapping to Sierpiński space $\tilde{U}: G_1 \to \{0,1\}$. 
we obtain

\[ \tilde{U} \tilde{V} = \tilde{U} * \tilde{V}, \]

where the convolution of two continuous maps \( \phi, \psi : G_1 \to \$ \) is defined by

\[ \phi * \psi(g) = \bigvee_{g = hk} \phi(h) \psi(k). \]

This construction can be carried over to localic groupoids, and the resulting correspondence between groupoids and quantales restricts to a bijection between localic étale groupoids (up to isomorphisms) and inverse quantal frames \([31]\). This is a topological analogue of the dualities of algebraic geometry, with étale groupoids playing the role of “noncommutative varieties”. In particular, any Grothendieck topos coincides, at least in the case of the topos of an étale groupoid, with a category of modules over the quantale of the groupoid \([32]\) (see also \([9]\) for other quantale representations of Grothendieck toposes). However, this analogy is objects-only because the bijection is not functorial with respect to groupoid functors and quantale homomorphisms, and the main aim of this paper is to address this issue.

This functoriality problem is similar to another, well known, one: locally compact groupoids \([28, 30]\) generalize both locally compact groups and locally compact spaces but, if we take groupoid morphisms to be general functors, this generalization is not functorial with respect to convolution algebras and their homomorphisms. In order to see this it suffices to notice that Gelfand duality yields a contravariant functor from compact Hausdorff spaces to C*-algebras, whereas the universal C*-algebra of a discrete group defines a covariant functor. An interpretation of this discrepancy is that a groupoid C*-algebra can be regarded as a description of the space of orbits (in a generalized sense) of the groupoid and that groupoid functors fail to account for this \([8]\). In addition, for two such spaces to be considered “the same” one usually requires the algebras to be only Morita equivalent rather than isomorphic. Accordingly, appropriate definitions of morphism for groupoids, which subsume groupoid functors and map functorially to bimodules, have been defined in terms of bi-actions \([10, 17, 22, 23]\). The idea of a groupoid as a generalized space of orbits is even more explicit in topos theory, since any Grothendieck topos is, in a suitable sense, a quotient of the object space of a groupoid in the 2-category of toposes and geometric morphisms \([19]\). Again, morphisms can be taken to be bi-actions \([5, 20, 21]\).

In the present paper we show that the correspondence between groupoids and quantales is functorial in the bicategorical sense suggested by the above remarks. In order to achieve this we show, in section 4, following preliminary results about groupoid actions in section 3, that the bi-actions of localic
étale groupoids map functorially to quantale bimodules, and that, improving on what would be expected for convolution algebras, this assignment restricts to a biequivalence, namely between the bicategory $\text{Gpd}$ of localic étale groupoids and a bicategory $\text{IQLoc}$ of inverse quantal frames.

As an example, at the end of section 4 we discuss the notion of algebraic morphism of groupoids [3, 4]. Algebraic morphisms are examples of groupoid bi-actions that map functorially and covariantly to homomorphisms of $\text{C}^*$-algebras [4] and to homomorphisms of inverse semigroups [7], and furthermore, as noted in [4], specialize both to group homomorphisms (covariantly) and to continuous maps between topological spaces (contravariantly), hence in a narrower extent suggesting a solution to the functoriality problem addressed in this paper. A corollary of our bicategorical equivalence is that the algebraic morphisms of étale groupoids are “the same” as the homomorphisms of unital (involutive) quantales between the quantales of the groupoids and yield a category $\text{Gpd}_A$ which is equivalent to the category $\text{InvQuF}$ of [31]. For étale groupoids all the above remarks (except those pertaining to $\text{C}^*$-algebras) follow readily from this identification. Another consequence is that there is a covariant functor from a non-trivial category of quantales to $\text{C}^*$-algebras. The existence of such a functor is interesting in its own right, in view of the difficulties that arise with respect to functoriality when studying correspondences between quantales and $\text{C}^*$-algebras [15, 16].

In addition to the above results, and independently from bi-actions and bimodules, we show, in section 2, that the subcategory $\text{IQFrm}$ of $\text{InvQuF}$ whose morphisms are also locale homomorphisms is dually equivalent to a category $\text{Gpd}_C$ of étale groupoids whose morphisms, in the topological context, coincide with the covering functors of [18].

Other functorial aspects of groupoid quantales, for instance regarding Hilsum–Skandalis maps and Morita equivalence, will be addressed in a subsequent paper.

1.1 Groupoids and quantales

We shall use common terminology and notation for sup-lattices, locales, quantales, and groupoids, mostly following [31]. In particular, we shall use the following notation for the structure maps of an étale groupoid $G$

\[
\begin{array}{ccc}
G_2 & \xrightarrow{m} & G_1 \\
& \searrow & \downarrow \cong \\
& & G_0
\end{array}
\]

(writing $d_G, m_G, m_G$, etc., when convenient), where $G_2$ is the pullback of $d$ and $r$ in $\text{Loc}$, and we shall again denote the quantale of $G$ by $\mathcal{O}(G)$. But we shall
not make any distinction between a locale \(X\) regarded as an object of \(Frm\) or as an object of \(Loc = Frm^{op}\), hence limiting the use of the \(O\) notation to groupoid quantales alone. We remark that, keeping up with [31,32], our usage of \(d\) and \(r\) is reversed with respect to the typical conventions for groupoid C*-algebras.

Given an inverse quantal frame \(Q\), we denote the subset 
\[
\downarrow(e) = \{a \in Q \mid a \leq e\}
\]
by \(Q_0\). This is both a locale and a unital involutive subquantale of \(Q\), and we refer to it as the base locale of \(Q\). In order to lighten the notation, we shall denote the inverse semigroup of partial units of \(Q\) by \(Q_I\) instead of \(I(Q)\):
\[
Q_I = \{s \in Q \mid ss^* \lor s^*s \leq e\}.
\]
An important property of any inverse quantal frame \(Q\) is that for all \(b \in Q_0\) and \(a \in Q\) we have
\[
(1.1) \quad ba = b1 \land a \quad \text{and} \quad ab = 1b \land a.
\]
We also recall that the category of inverse quantal frames \(InvQuF\) of [31] has the homomorphisms of unital involutive quantales as morphisms. Indeed these are all the homomorphisms of unital quantales:

**Proposition 1.2** Any homomorphism of unital quantales \(h : Q \to R\) between inverse quantal frames is necessarily involutive.

**Proof.** Since \(h\) is unital it restricts to a homomorphism of inverse semigroups \(Q_I \to R_I\). This necessarily preserves inverses and, since every element of an inverse quantal frame is a join of partial units, the conclusion follows. □

In order to disambiguate some expressions, we shall always write \(X \otimes Y\) (rather than \(X \times Y\)) for the product of the locales \(X\) and \(Y\) in \(Loc\), since this coincides with their tensor product in the category of sup-lattices \(SL\). Accordingly, we extend this notation to morphisms: given two locale maps \(f : X \to X'\) and \(g : Y \to Y'\), their product in \(Loc\) is
\[
f \otimes g : X \otimes Y \to X' \otimes Y'.
\]
The coproduct of \(X\) and \(Y\) in \(Loc\) is the direct sum in \(SL\) and we denote it by \(X \oplus Y\).

The category whose objects are the localic étale groupoids and whose morphisms are the groupoid functors will be denoted by \(Gpd\). We note that an equivalent definition of groupoid functor is provided by the following proposition:
Proposition 1.3 Let $G$ and $H$ be étale groupoids and let

$$f_0 : G_0 \to H_0$$
$$f_1 : G_1 \to H_1$$

be two maps of locales that satisfy the following properties:

$$(1.4) \quad f_1 \circ i = i \circ f_1 ;$$

$$(1.5) \quad f_1 \circ u = u \circ f_0 ;$$

$$(1.6) \quad f_0 \circ d = d \circ f_1 ;$$

$$(1.7) \quad m \circ (f_1 \otimes f_1) \leq f_1 \circ m .$$

Then the pair $(f_0, f_1)$ is a functor of groupoids.

Proof. All we have to do is prove that the above inequality is in fact an equality. In point-set notation this follows from a simple series of inequalities:

$$f_1(x)f_1(y) \leq f_1(xy)f_1(y)^{-1}f_1(y) = f_1(xy)f_1(y)^{-1}f_1(y) \leq f_1(xyy^{-1})f_1(y) = f_1(x)f_1(y) .$$

Converting this to an explicit argument about locale maps is tedious but straightforward.  

1.2 Groupoid actions

We shall mostly follow the conventions of [32] for groupoid actions and their quantale modules. A left action

$$a : G_1 \otimes_{G_0} X \to X$$

of an étale groupoid $G$ on $p : X \to G_0$, where $G_1 \otimes_{G_0} X$ is the pullback of $r$ and $p$, will be denoted by $(X, p)$ or $(X, p, a)$ according to convenience, or simply by $X$ when no confusion will arise. We shall refer to the map $p$ as the anchor map of the action. A right $G$-action is defined similarly, with $X \otimes_{G_0} G_1$ being the pullback of $p$ and $d$ in $\text{Loc}$. The category of $G$-locales and equivariant maps between them is denoted by $G-\text{Loc}$.

In order to simplify notation we shall write $Q$ instead of $\mathcal{O}(G)$. The left $Q$-module associated to an action $(X, p, a)$ is denoted simply by $X$ (rather than $\mathcal{O}(X)$ as in [32]), and we shall continue to refer to it as a (left) $Q$-locale, by which is meant a locale $X$ which is also a unital left $Q$-module satisfying the following anchor condition for all $b \in Q_0$ and $x \in X$:

$$(1.8) \quad bx = b1 \land x .$$
Occasionally, for the sake of clarity, we shall use a dot in order to denote the action, e.g. writing the above condition as

\[ b \cdot x = b \cdot 1 \wedge x. \]

The category of left \( Q \)-locales has the left \( Q \)-locales as objects, and the morphisms are the maps of locales whose inverse images are homomorphisms of left \( Q \)-modules. This category is denoted by \( Q\text{-Loc} \) and it is isomorphic to \( G\text{-Loc} \).

Recall that if \( X \) is a left \( Q \)-locale, its action is a sup-lattice homomorphism \( Q \otimes X \to X \) that factors through another sup-lattice homomorphism \( \alpha \) as

\[ Q \otimes X \to Q \otimes_{Q_0} X \xrightarrow{\alpha} X, \]

and the right adjoint \( \alpha_* \) is given by the following equivalent formulas:

\[
\begin{align*}
\alpha_* (x) &= \bigvee \{ a \otimes y \in Q \otimes_{Q_0} X \mid ay \leq x \} ; \\
\alpha_* (x) &= \bigvee_{s \in Q_x} s \otimes s^* x .
\end{align*}
\]

The latter shows that \( \alpha_* \) preserves joins, and the corresponding groupoid action (now a map in \( \text{Loc} \))

\[ a : G \otimes_{G_0} X \to X \]

is defined by \( a^* = \alpha_* \).

Similar facts hold for right actions, with the above formulas being rewritten as

\[
\begin{align*}
\alpha_* (x) &= \bigvee \{ y \otimes a \in X \otimes_{Q_0} Q \mid ya \leq x \} ; \\
\alpha_* (x) &= \bigvee_{s \in Q_x} xs^* \otimes s .
\end{align*}
\]

We conclude this overview of groupoid actions by looking at a few simple properties of \( Q \)-locales. Eq. (1.8) immediately implies both distributivity and “middle-linearity” of the action of the locale \( Q_0 \) over binary meets, for all \( b \in Q_0 \) and \( x, y \in X \):

\[
\begin{align*}
\text{(1.13)} & \quad b(x \wedge y) = b1 \wedge x \wedge y = (b1 \wedge x) \wedge (b1 \wedge y) = bx \wedge by ; \\
\text{(1.14)} & \quad bx \wedge y = bx \wedge by = x \wedge by .
\end{align*}
\]

The following generalization of this to partial units is not in [32] and will be needed later on:
Proposition 1.15 Let $X$ be a $Q$-locale. For all $s \in Q_I$ and $x, y \in X$, we have

1. $s(x \land y) = sx \land sy$,
2. $s(x \land s^*y) = sx \land y$.

Proof. The inequality $s(x \land y) \leq sx \land sy$ follows immediately from the monotonicity of the action. For the converse inequality, we use the distributivity 1.13 with $b = ss^*$, in order to prove (1)

$$
\begin{align*}
    sx \land sy &= ss^*sx \land ss^*sy = ss^*(sx \land sy) \\
    &\leq s(s^*sx \land s^*sy) = ss^*s(x \land y) \\
    &= s(x \land y).
\end{align*}
$$

Condition (2) follows easily: $sx \land y = ss^*sx \land y = sx \land s^*y = s(x \land s^*y)$. \[\Box\]

2 Functoriality I

We begin by briefly addressing the extent to which the correspondence between étale groupoids and quantales is functorial with respect to groupoid functors, going a bit beyond [31] by showing that, although the assignment from étale groupoids to quantales is not functorial unless quantale homomorphisms are “lax”, the assignment from inverse quantal frames to groupoids is. A similar fact has been noticed in [18], in the context of topological groupoids and inverse semigroups.

2.1 Group homomorphisms

A similar discrepancy to the one we alluded to in the introduction occurs when relating localic groupoids and quantales. On one hand, the (tautological) functor from $\text{Loc}$ to $\text{Frm}$ is contravariant, whereas, on the other hand, it is the covariant powerset functor (rather than the contravariant one) which gives us a functor from the category of groups to the category of unital involutive quantales. More than that, the covariant powerset functor is left adjoint to the functor that to each unital quantale $Q$ assigns its groups of units

$$
Q^\times = \{a \in Q \mid ab = e \text{ for some } b \in Q\},
$$

and thus the group homomorphisms can be identified with homomorphisms of unital quantales:

$$
\text{hom}(G, H) \cong \text{hom}(\varphi(G), \varphi(H)).
$$
Moreover, for each discrete group \( G \) we have \( G \cong \varphi(G)^\times \) (the adjunction is a co-reflection).

On the contrary, the contravariant powerset functor behaves poorly with respect to group homomorphisms:

**Lemma 2.1** The homomorphisms \( f : G \to H \) of discrete groups whose inverse image mappings \( f^{-1} : \varphi(H) \to \varphi(G) \) are homomorphisms of unital quantales are precisely the isomorphisms.

**Proof.** Let \( f : G \to H \) be a homomorphism of discrete groups. If \( f^{-1} \) is a homomorphism of quantales and \( h \in H \) we have

\[
f^{-1}(\{h\})f^{-1}(\{h^{-1}\}) = f^{-1}(\{h\}\{h^{-1}\}) = \ker f .
\]

Therefore \( 1 \in f^{-1}(\{h\})f^{-1}(\{h^{-1}\}) \), which shows that \( h \in f(G) \) and thus \( f \) is surjective.

Conversely, if \( f \) is surjective and \( g \in f^{-1}(\{h\}\{k\}) \) there is \( k_0 \in G \) such that \( f(k_0) = k \) and, setting \( g_1 = gk_0^{-1} \) and \( g_2 = k_0 \), we have

\[
g = g_1g_2 \text{ and } f(g_1) = h \text{ and } f(g_2) = k ,
\]

whence \( g \in f^{-1}(\{h\})f^{-1}(\{k\}) \). This shows that

\[
f^{-1}(\{h\}\{k\}) \subset f^{-1}(\{h\})f^{-1}(\{k\}) ,
\]

and thus \( f^{-1} \) is a homomorphism of quantales.

Finally, the quantale unit is preserved by \( f^{-1} \) if and only if \( \{1\} = \ker f \), i.e., \( f \) is injective. \( \blacksquare \)

This shows that in order to obtain a contravariant functor to the category of unital quantales from a category of étale groupoids whose morphisms are functors, we should either enlarge the class of quantale homomorphisms or severely restrict the class of groupoid functors.

### 2.2 Covering functors

The idea of restricting the class of groupoid functors has been adopted by Lawson and Lenz [18], whose notion of ‘covering functor’, in the context of topological étale groupoids, is equivalent to that of a functor \((f_0, f_1) : G \to H\) such that

\[
f_1^{-1} : \Omega(H_1) \to \Omega(G_1)
\]

is a homomorphism of unital quantales [18, Lemma 2.20]. In this section we see that any homomorphism \( h \) of unital quantales between inverse quantal...
frames equals an inverse image $f_1^*$ for a localic groupoid functor $(f_0, f_1)$ if and only if $h$ preserves finite meets, which gives us a means of extending the definition of covering functor to localic groupoids, as we now explain.

**Definition 2.2** 1. The category $IQFrm$ is the subcategory of $InvQuF$ with the same objects and whose homomorphisms also preserve finite meets.

2. We denote the dual category $IQFrm^{op}$ by $IQLoc$.

**Theorem 2.3** The assignment $Q \mapsto G(Q)$ from inverse quantal frames to étale groupoids extends to a functor $G: IQLoc \to Gpd$.

**Proof.** Let $Q$ and $R$ be inverse quantal frames and let $f: Q \to R$ be a morphism in $IQLoc$. Writing $G$ and $H$ for $G(Q)$ and $G(R)$, respectively, we have, as locales, $G_1 = Q$, $H_1 = R$, $G_0 = Q_0$ and $H_0 = R_0$, with the structure maps of $G$ given in terms of the quantale structure by, for all $a \in G_1$ and $b \in G_0$,

- $u^*(a) = a \land e$
- $d^*(b) = b1$
- $i^*(a) = a^*$.

For $H$ it is similar, and we shall use the same notation for the structure maps of $H$. As a candidate for a groupoid functor we set $f_1 = f$, and $f_0$ is given by defining $f_0^*$ to be the restriction of $f^*$ to $H_0$. (This is well defined because $f^*$ is unital.) We note that since $f^*$ preserves the quantale involution we immediately obtain

\[(2.4) \quad f_1 \circ i = i \circ f_1.\]

Now let us prove the following equalities:

\[(2.5) \quad f_1 \circ u = u \circ f_0;\]
\[(2.6) \quad f_0 \circ d = d \circ f_1.\]

We have: for all $a \in H_1$

\[f_0^*(u^*(a)) = f^*(a \land e) = f^*(a) \land e = u^*(f_1^*(a)),\]

which proves Eq. (2.5); for all $b \in H_0$

\[d^*(f_0^*(b)) = d^*(f^*(b)) = f^*(b)1 = f^*(b1) = f_1^*(d^*(b)).\]

10
which proves Eq. 2.6. By [31, Lemma 5.13] we have

\[(f^* \otimes f^*) \circ m^* \leq m^* \circ f^*,\]

and thus by [13] the pair \((f_0, f_1)\) is a groupoid functor. Finally, the assignment

\[f \mapsto (f_0, f_1)\]

is clearly functorial.

**Definition 2.7** The category \(\text{Gpd}_C\) is the subcategory of \(\text{Gpd}\) whose morphisms are the covering functors, by which we mean the continuous functors \(f : G \to H\) such that

\[f_1^* : \mathcal{O}(H) \to \mathcal{O}(G)\]

is a homomorphism of unital involutive quantales (i.e., \(f_1\) is a morphism in \(\text{IQLoc}\)).

**Corollary 2.8** The categories \(\text{Gpd}_C\) and \(\text{IQLoc}\) are equivalent.

**Proof.** For each inverse quantal frame \(Q\) we have \(\mathcal{O}(\mathcal{G}(Q)) = Q\). And for each étale groupoid \(G\) we have \(\mathcal{G}(\mathcal{O}(G)) \cong G\), where a canonical isomorphism \(\iota_G = (\iota_0, \iota_1) : G \to \mathcal{G}(\mathcal{O}(G))\) in \(\text{Gpd}\) is such that \(\iota_1\) is the identity on \(G_1\) and

\[\iota_0 : G_0 \to Q_0\]

is the codomain restriction of \(u_1 : G_0 \to G_1\). The two assignments \(Q \mapsto \mathcal{G}(Q)\) and \(G \mapsto \mathcal{O}(G)\), which extend to functors as we have seen, together with the two natural transformations

\[\text{id} : I \Rightarrow \mathcal{O} \circ \mathcal{G} \quad \iota : \mathcal{G} \circ \mathcal{O} \Rightarrow I,\]

yield an adjoint equivalence of categories.

**2.3 Lax homomorphisms**

For the sake of completeness let us take a very brief look at an alternative way of obtaining functoriality “on the nose”, namely by enlarging the class of quantale homomorphisms.

We write \(\text{IQLoc}_\ell\) for the extension of \(\text{IQLoc}\) whose objects are the inverse quantal frames and whose morphisms

\[f : R \to Q\]
are the maps of locales such that
\[ f^* (a) f^* (b) \leq f^* (ab) \quad \text{for all } a, b \in Q , \]
\[ f^* (a^*) = f^* (a^*) , \]
\[ e_R \leq h (e_Q) . \]

**Theorem 2.9** The assignment \( G \mapsto \mathcal{O}(G) \) extends to a faithful functor
\[ \mathcal{O} : \text{Gpd} \rightarrow \text{ILoc}_\ell . \]

**Proof.** Let \( f : G \rightarrow H \) be a morphism of \( \text{Gpd} \), and let \( Q = \mathcal{O}(G) \) and \( R = \mathcal{O}(H) \). The assignment \( f \mapsto f_*^1 \) is of course functorial and faithful, so we only have to verify that \( f_*^1 : R \rightarrow Q \) satisfies the three above conditions. The first is a consequence of [31, Lemma 5.13], and the second is an immediate consequence of the fact that functors preserve inverses. The third also holds, as we now explain. The axiom
\[ u_H \circ f_0 = f_1 \circ u_G \]
of groupoid functors implies
\[ f_0^* \circ u_H^* \leq u_G^* \circ f_1^* \]
which, by adjointness, gives us
\[ (u_G) ! \circ f_0^* \circ u_H^* \leq f_1^* . \]
Composing with \( (u_H) ! \) we obtain
\[ (u_G) ! \circ f_0^* \circ u_H^* \circ (u_H) ! \leq f_1^* \circ (u_H) ! , \]
and this, using the unit of the adjunction id \( \leq u_H^* \circ (u_H) ! , \) implies
\[ (u_G) ! \circ f_0^* \leq f_1^* \circ (u_H) ! . \]
Hence,
\[ e_Q = (u_G) ! (1_{G_0}) = (u_G) ! (f_0^* (1_{H_0}) \leq f_1^* ((u_H) ! (1_{H_0})) = f_1^* (e_R) . \]

### 3 Groupoid actions

Let us study some constructions related to orbits of groupoid actions, in the language of quantale modules.
3.1 Orbits

If $G$ is an étale groupoid and $X$ is a left $G$-locale, we can construct the orbit locale of the action as the coequalizer in $\text{Loc}$

$$G_1 \otimes_{G_0} X \xrightarrow{\pi_2} X \xrightarrow{\pi_2} X/G.$$ 

The locale points of $X/G$ can be regarded as being the orbits of the action of $G$ on $X$.

**Definition 3.1** We refer to $X/G$ as the quotient of $X$ by $G$. For a right $G$-locale the corresponding quotient is denoted by $G\backslash X$.

There is a simple description of these quotients in terms of $\mathcal{O}(G)$-modules. We explain this for left actions only, as for right actions everything is similar.

**Definition 3.2** Let $G$ be an étale groupoid with quantale $Q = \mathcal{O}(G)$, and $X$ a left $G$-locale. An element $x \in X$ is invariant if the following equivalent conditions hold (for $X$ regarded as a $Q$-module):

1. For all $a \in Q$ we have $ax \leq x$;
2. For all $s \in Q_I$ we have $sx \leq x$;
3. $1x \leq x$;
4. $1x = x$.

**Theorem 3.3** Let $G$ be an étale groupoid and $X$ a left $G$-locale. The quotient $X/G$ coincides with the set of invariant elements of the action.

**Proof.** First we remark that the invariant elements form an obvious sub-frame $F \subset X$, hence defining a quotient locale as required. It remains to be shown that the following diagram is an equalizer in the category of sets, where $i$ is the frame inclusion:

$$F \xrightarrow{i} X \xrightarrow{\pi_2} G_1 \otimes_{G_0} X.$$ 

In other words, we need to show that $x$ is invariant if and only if

$$\pi_2^*(x) = a^*(x).$$
Let us assume that Eq. (3.4) holds. Using the co-unit of the adjunction 
\( a \dashv a^* \) we conclude that \( x \) is invariant:

\[
1 x = a_i (1 \otimes x) = a_i (\pi_2^* (x)) = a_i (a^* (x)) \leq x .
\]

Conversely, let us assume that \( x \) is invariant. By Eq. (1.9), the condition 
\( 1 x \leq x \) immediately implies that \( 1 \otimes x \leq a^* (x) \). And, by Eq. (1.10), we have, writing \( Q = \mathcal{O}(G) \),

\[
a^* (x) = \bigvee_{s \in Q_x} s \otimes s^* x \leq \bigvee_{s \in Q_x} s \otimes x = 1 \otimes x .
\]

Hence, Eq. (3.4) holds.

We remark that, although this is not needed in what follows, the idea 
that the orbits must be certain “subspaces” can be explicitly conveyed by 
first observing that, as a subframe, \( X/G \) is in fact closed under arbitrary 
meets in \( X \), which means that it also defines a quotient of \( X \) in \( SL \) [13].
This does not correspond to a sublocale of \( X \) because the quotient is not 
taken in \( Frm \). However, by freely adjoining finite meets to \( X \) we obtain 
the lower powerlocale \( P_L X \) (one of several localic notions of “powerspace of 
\( X \)”), whose points can be identified (in an arbitrary topos) with the “weakly 
closed sublocales of \( X \) with open domain” [6] (and coincide, in classical set 
theory, with the closed sublocales of \( X \) — see [33]). Hence, the sup-lattice 
quotient \( X \rightarrow X/G \) extends uniquely to a frame quotient \( P_L X \rightarrow X/G \), 
hence depicting \( X/G \) as a sublocale of \( P_L X \), and allowing us to view the 
orbits of the action as being sublocales of \( X \).

### 3.2 Diagonal actions

Let \( G \) be an étale groupoid with quantale \( Q = \mathcal{O}(G) \). Given right and left \( G \)-
locales \( (X, p, a) \) and \( (Y, q, b) \), we can define on the pullback \( X \otimes_{G_0} Y \) of \( p \) and 
\( q \) (which equals \( X \otimes_{Q_0} Y \)) the diagonal action which, in point-set notation, 
would be given by the formula

\[
g \cdot (x, y) = (x \cdot g^{-1} , g \cdot y) .
\]

Module-theoretically this goes as follows:

**Theorem 3.5** Let \( G \) be an étale groupoid with quantale \( Q = \mathcal{O}(G) \), and let 
\( X \) and \( Y \) be a right \( G \)-locale and a left \( G \)-locale with anchor maps \( p \) and \( q \), 
respectively. The following conditions hold:
1. (Diagonal action.) A left quantale action of $Q$ on $X \otimes_{Q_0} Y$ is defined, for all $x \in X$, $y \in Y$ and $s \in Q_I$, by the condition

\[(3.6) \quad s \cdot (x \otimes y) = xs^* \otimes sy.\]

2. This action makes $X \otimes_{Q_0} Y$ a left $Q$-locale.

Proof. For each $s \in Q_I$, let the mapping

\[f_s : X \oplus Y \to X \otimes_{Q_0} Y\]

be defined by

\[f_s(x, y) = xs^* \otimes sy.\]

This clearly preserves joins in each variable separately. And, for each $b \in Q_0$, the following middle-linearity condition is satisfied:

\[f_s(xb, y) = xbs^* \otimes sy = xs^* sbs^* \otimes sy = xs^* \otimes sy = x^* \otimes ss^* by = x^* \otimes sby = f_s(x, by).\]

Hence, $f_s$ factors uniquely through the sup-lattice homomorphism given by

\[x \otimes y \mapsto xs^* \otimes sy,\]

and thus the semigroup $Q_I$ acts by endomorphisms on $X \otimes_{Q_0} Y$ (the associativity of the action is immediate). Now recall the isomorphism $Q \cong \mathcal{L}'(Q_I)$ [31], where the righthand side is the join-completion of $Q_I$ that preserves the joins of compatible sets (this can be concretely represented as the frame of \textit{compatibly prime ideals} of $Q_I$, which are the downwards-closed subsets of $Q_I$ that are closed under the formation of joins of compatible subsets). In order to show that the action of $Q_I$ extends to the required action of $Q$ it suffices to show that the semigroup action respects such joins. Let then $Z \subset Q_I$ be compatible, i.e., a subset such that for all $s, t \in Z$ we have $st^* \leq e$ and $s^* t \leq e$. Then $\bigvee Z \in Q_I$ and, for all $s, t \in Z$, $x \in X$ and $y \in Y$, we have

\[xs^* \otimes ty = xs^* ss^* \otimes ty = xs^* \otimes ss^* ty \leq xs^* \otimes sy,\]

and thus we obtain

\[
\left(\bigvee Z\right) \cdot (x \otimes y) = x \left(\bigvee Z\right)^* \otimes \left(\bigvee Z\right) y = \bigvee_{s, t \in Z} xs^* \otimes ty
\]

\[= \bigvee_{s \in Z} xs^* \otimes sx = \bigvee (Z \cdot (x \otimes y)).\]
This proves that $X \otimes_{Q_0} Y$ is a left $Q$-module with the action defined by Eq. (3.6). And it is a $Q$-locale because the anchor condition holds: for all $b \in Q_0$ and $\xi = \bigvee_i x_i \otimes y_i \in X \otimes_{Q_0} Y$ we have

$$b \cdot \xi = \bigvee_i x_i \otimes b y_i = \bigvee_i (1b \land x_i) \otimes (b1 \land y_i) = \bigvee_i (1b \land (x_i \land y_i)) = b \cdot (1 \otimes 1) \land \xi.$$ 

### 3.3 Tensor products

Let $G$ be an étale groupoid. Given right and left $G$-locales $(X, p, a)$ and $(Y, q, b)$, a tensor product over $G$ can be defined as a coequalizer in $\text{Loc}$ (cf. [20][21]):

$$X \otimes_{G_0} G_1 \otimes_{G_0} \xrightarrow{(\bot \pi_{12}, \pi_3)} X \otimes_{G_0} Y \xrightarrow{(\pi_1, \bot \pi_{23})} X \otimes_{G} Y.$$ 

Our aim now is to show that this tensor product coincides with the “ring-theoretic” tensor product of $\mathcal{O}(G)$-modules, and our first step will be to show module-theoretically that $X \otimes_{G} Y$ can be given an equivalent definition as the quotient $(X \otimes_{G_0} Y)/G$ by the diagonal action (cf. [22]).

**Lemma 3.8** Let $G$ be an étale groupoid, and $(X, p, a)$ and $(Y, q, b)$ a right and a left $G$-locale, respectively. Then $X \otimes_{G} Y = (X \otimes_{G_0} Y)/G$.

**Proof.** The coequalizer $X \otimes_{G} Y$ can be concretely identified with the sub-frame of $X \otimes_{G_0} Y$ consisting of the elements $\xi$ such that

$$[\pi^*_{12} \circ a^*, \pi_3^*](\xi) = [\pi^*_{1}, \pi^*_{23} \circ b^*](\xi).$$

Using Eqs. (1.10) and (1.12), respectively for $b^*$ and $a^*$, this equality is equivalent, letting $\xi = \bigvee_i x_i \otimes y_i$ and writing $Q = \mathcal{O}(G)$, to

$$\bigvee_i \bigvee_{s \in Q_{I}} x_i s^* \otimes s \otimes y_i = \bigvee_i \bigvee_{s \in Q_{I}} x_i \otimes s \otimes s^* y_i.$$ 

In order to conclude the proof we show that $\xi$ satisfies this equality if and only if it is invariant with respect to the diagonal action. Let us assume that
Eq. (3.9) holds. Then $\xi$ is invariant:

$$1 \xi = \bigvee_{i,s} x_i s^* \otimes y_i = a_i \otimes \text{id} \left( \bigvee_{i,s} x_i \otimes s^* \otimes y_i \right) = a_i \otimes \text{id} (\bigvee_{i,s} x_i s \otimes s^* y_i) = \bigvee_{i,s} x_i s s^* \otimes y_i = \xi .$$

Conversely, assuming that $\xi$ is invariant, Eq. (3.9) holds:

$$\bigvee_{i,s} x_i s \otimes s^* y_i \leq a^* \otimes \text{id} \left( \bigvee_{i,s} x_i s^* \otimes s y_i \right) \quad \text{[by Eq. (1.11)]}$$

$$\leq a^* \otimes \text{id} \left( \bigvee_{i,s} x_i \otimes y_i \right) \quad (s \cdot \xi \leq \xi)$$

$$= \bigvee_{i,s} x_i s^* \otimes s \otimes y_i \quad \text{[by Eq. (1.12)]}$$

$$\leq \text{id} \otimes b^* \left( \bigvee_{i,s} x_i s^* \otimes s y_i \right) \quad \text{[by Eq. (1.9)]}$$

$$\leq \text{id} \otimes b^* \left( \bigvee_{i,s} x_i \otimes y_i \right) \quad (s \cdot \xi \leq \xi)$$

$$= \bigvee_{i,s} x_i \otimes s \otimes s^* y_i \quad \text{[by Eq. (1.10)]} .$$

**Theorem 3.10** Let $G$ be an étale groupoid, and let $X$ and $Y$ be a right $G$-locale and a left $G$-locale as in the previous lemma. Then,

$$X \otimes_G Y = X \otimes_{\mathcal{O}(G)} Y .$$

**Proof.** Let us write $Q$ for $\mathcal{O}(G)$. As a sup-lattice, $X \otimes_Q Y$ is the quotient of $X \otimes_{G_0} Y$ (which equals $X \otimes_{Q_0} Y$) generated by the middle-linearity relations

$$xa \otimes y = x \otimes ay$$

for all $a \in Q$, and it is sufficient to take $a \in Q_I$. By general sup-lattice algebra [13], the sup-lattice quotient can be concretely identified with the subset of $X \otimes_{G_0} Y$ whose elements $\xi$ are closed under the relations; that is, such that for all $x \in X$, $y \in Y$ and $s \in Q_I$ we have

$$x s \otimes y \leq \xi \iff x \otimes sy \leq \xi . \quad (3.11)$$

By [3.8] $X \otimes_G Y$ can be identified with the set of invariant elements for the action Eq. (3.6), so let us show that the invariant elements are the same as those which satisfy the condition (3.11). Let $\xi$ be an invariant element of $X \otimes_{G_0} Y$, i.e., such that $s \cdot \xi \leq \xi$ for all $s \in Q_I$, and let $x \in X$, $y \in Y$, and $s \in Q_I$. If $xs \otimes y \leq \xi$ we obtain

$$x \otimes sy = x \otimes ss^* y = x s s^* \otimes sy = s \cdot (xs \otimes y) \leq s \cdot \xi \leq \xi ,$$

17
and, similarly, if \( x \otimes sy \leq \xi \) we conclude \( xs \otimes y \leq \xi \). Hence, \( \xi \) satisfies (3.11). For the converse, assume that \( \xi = \bigvee_i x_i \otimes y_i \) satisfies (3.11). For all \( i \) and \( s \in Q_x \), we have
\[
x_i s^* s \otimes y_i \leq x_i \otimes y_i \leq \xi
\]
and, using (3.11),
\[
s \cdot (x_i \otimes y_i) = x_i s^* \otimes sy_i \leq \xi .
\]
Hence, \( \xi \) is invariant, and we conclude that \( X \otimes G Y \) coincides, concretely as a subset of \( X \otimes_{G_0} Y \), with \( X \otimes_Q Y \).  

4 Functoriality II

Now we address the main aim of this paper, which is to show that groupoid bi-actions can be identified with a natural notion of bilocale for inverse quantal frames, and to establish an ensuing (bicategorical) equivalence between etale groupoids and inverse quantal frames. Following that, we discuss connections to algebraic morphisms of groupoids in the sense of [3, 4].

4.1 Bimodules

Let \( Q \) and \( R \) be unital quantales. By a \( Q-R \)-bimodule is meant a sup-lattice \( Q X_R \), which can simply be denoted by \( X \), equipped with structures of unital left \( Q \)-module and unital right \( R \)-module that satisfy the associativity condition
\[
(rx)q = r(xq) \quad \text{for all} \ r \in R, \ x \in X, \ q \in Q .
\]
Similarly to rings, we obtain a bicategory [1, sec. 2.5, 5.7]: the 0-cells are the unital quantales, the 1-cells are the bimodules \( Q X_R \); the composition of 1-cells \( Q X_R \) and \( R Y_S \) is given by \( Y \circ X = X \otimes_R Y \); and the 2-cells are the homomorphisms of bimodules, with composition defined as usual. A homomorphism of unital quantales \( h : Q \to R \) can be identified with a \( Q-R \)-bimodule \( X_h \), which is \( R \) with the left \( Q \)-action induced by \( h \) and the right \( R \)-action given by multiplication; there are canonical isomorphisms
\[
X_{h \circ k} \cong X_h \circ X_k ,
\]
and the assignments \( Q \mapsto Q \) and \( h \mapsto X_h \) embed the category of unital quantales in the bicategory.
**Definition 4.1** Let $Q$ and $R$ be inverse quantal frames. A $Q$-$R$-bilocale is a bimodule $\mathbb{Q}X \mathbb{R}$ that is also a locale such that for all $b \in Q_0$, $c \in R_0$ and $x \in X$ the following left and right anchor conditions hold:

\begin{align}
(4.2) & \quad bx = b1 \wedge x \\
(4.3) & \quad xc = 1c \wedge x.
\end{align}

A map of bilocales $f : \mathbb{Q}X \mathbb{R} \to \mathbb{Q}Y \mathbb{R}$ is a map of locales whose inverse image $f^*$ is a homomorphism of bimodules, and the resulting category is denoted by $Q$-$R$-$\text{Loc}$.

It is immediate that any inverse quantal frame $Q$ is a $Q$-$Q$-bilocale, due to Eqs. (1.1). In addition, bilocales behave well with respect to tensor products:

**Lemma 4.4** Let $Q$, $R$, $S$ be inverse quantal frames. The tensor product $X \otimes_R Y$ of bilocales $\mathbb{Q}X \mathbb{R}$ and $\mathbb{R}Y \mathbb{S}$ is a $Q$-$S$-bilocale.

**Proof.** $X \otimes_R Y$ is a $Q$-$S$-bimodule, it is a locale due to 3.10, and it is a bilocale because the left (and the right) anchor condition holds, since for all $b \in Q_0$, $x \in X$ and $y \in Y$ we have

$$b(x \otimes y) = (bx) \otimes y = (b1 \wedge x) \otimes y = (b1) \otimes 1 \wedge x \otimes y = b(1 \otimes 1) \wedge x \otimes y.$$ 

Hence, the following bicategory is well defined:

**Definition 4.5** The bicategory $\text{IQLoc}$ has the inverse quantal frames as 0-cells, the bilocales as 1-cells, and the maps of bilocales as 2-cells. The composition of 1-cells $\mathbb{Q}X \mathbb{R}$ and $\mathbb{R}Y \mathbb{S}$ is defined by

$$Y \circ X = X \otimes_R Y,$$

and the coherence isomorphisms are the maps of bilocales whose inverse images are coherence isomorphisms in the usual “ring” sense.

**Lemma 4.6** The assignments $Q \mapsto Q$ and $h \mapsto X_h$ embed $\text{InvQuF}$ into $\text{IQLoc}$.

**Proof.** All we have to do is prove that if $h : Q \to R$ is a morphism of $\text{InvQuF}$ the bimodule $X_h$ is a bilocale rather than just a bimodule. It is a locale because $R$ is, the right anchor condition follows from Eqs. (1.1), and the left anchor condition holds because $h$ is unital and thus $h(b) \leq e$ for all $b \in Q_0$:

$$b \cdot x = h(b)x = h(b)1 \wedge x = b \cdot 1 \wedge x.$$
4.2 Bi-actions

Let \( G \) and \( H \) be localic étale groupoids. A \( G\)-\( H \)-bilocale is a locale \( G X_H \), which can be simply denoted by \( X \), equipped with a left \( G \)-locale structure \((p,a)\) and a right \( H \)-locale structure \((q,b)\) such that the following diagrams in \( \text{Loc} \) are commutative:

\[
\begin{array}{ccc}
G_1 \otimes G_0 X & \xrightarrow{a} & X \otimes H_0 H_1 & \xrightarrow{b} & X \\
\pi_2 \downarrow & & \downarrow \pi_1 & & \downarrow \pi_1 \\
X & \xrightarrow{q} & H_0 & \xrightarrow{p} & G_0 \\
& & 1 \otimes b & & a \otimes 1 \\
G_1 \otimes G_0 X & \xrightarrow{a} & X \otimes H_0 H_1 & \xrightarrow{b} & X \\
\end{array}
\]

The first two diagrams assert that the anchor map of the \( G \)-locale is invariant under the action of \( H \), and that the anchor map of the \( H \)-locale is invariant under the action of \( G \). Both are in line with the idea that a bilocale may be regarded as being the graph of a binary relation between the “orbit spaces” of \( G \) and \( H \), and they ensure that the third diagram (associativity) makes sense.

A map of bilocales \( f : G X_H \to G Y_H \) is a map of locales that is both a map of left \( G \)-locales and a map of right \( H \)-locales. The resulting category of bilocales is denoted by \( G\)-\( H \)-\( \text{Loc} \). The maps of bilocales are the 2-cells of a bicategory, denoted by \( \text{Gpd} \), whose 0-cells are the étale groupoids and whose 1-cells are the \( G\)-\( H \)-bilocales. The composition of 1-cells is defined by the tensor product: given 1-cells \( G X_H \) and \( H Y_K \) we define

\[ Y \circ X = X \otimes_H Y. \]

The coherence isomorphisms are standard (cf. [20,21]).

**Theorem 4.8** Let \( G \) and \( H \) be étale groupoids. The categories \( G\)-\( H \)-\( \text{Loc} \) and \( \mathcal{O}(G)\)-\( \mathcal{O}(H) \)-\( \text{Loc} \) are isomorphic.

**Proof.** Let us denote \( \mathcal{O}(G) \) and \( \mathcal{O}(H) \) by \( Q \) and \( R \), respectively. Any bilocale \( G X_H \) has both a left \( Q \)-locale structure and a right \( R \)-locale structure, and it is a routine matter to verify that it is a \( Q\)-\( R \)-bilocale because the associativity condition,

\[
(ax)b = a(xb)
\]
for all $a \in Q$, $x \in X$, and $b \in R$, is essentially the direct image version of the associativity diagram of (4.7):

\[
\begin{array}{c}
Q \otimes_{Q_0} X \otimes_{R_0} R \\
\downarrow^{id \otimes b} \\
Q \otimes_{Q_0} X \\
\downarrow^{a} \\
X
\end{array}
\begin{array}{c}
\xrightarrow{a \otimes id} \\
\xrightarrow{b} \\
X \otimes_{R_0} R
\end{array}
\]

Moreover, from the general results on groupoid actions (cf. section 1.2) it follows that a map of locales $f : X \rightarrow Y$ between bilocales $GX_H$ and $GY_H$ is a morphism in $G$-$H$-$\text{Loc}$ if and only if it is a morphism in $Q$-$R$-$\text{Loc}$. Therefore, all that we have left to prove is that every $Q$-$R$-bilocale arises from a (necessarily unique) $G$-$H$-bilocale; that is, that the unique $G$-locale and $H$-locale structures obtained from the $Q$-locale and $R$-locale structures of a $Q$-$R$-bilocale further satisfy the commutativity of the three bilocale diagrams of (4.7). Let $QXR$ be a bilocale, and let $(p, a)$ and $(q, b)$ be, respectively, the unique $G$-locale and $H$-locale structures that it determines. The commutativity of the third diagram of (4.7) follows from reversing the previous argument for associativity: it follows from the commutativity of (4.10), which is equivalent to the bimodule associativity. This kind of argument does not work for the first two diagrams of (4.7) because we are not assuming that $p$ and $q$ are open maps, but we can nevertheless establish their commutativity in terms of inverse images of the locale maps. Let us do this only for the first one,

\[
\begin{array}{c}
G_1 \otimes_{G_0} X \\
\downarrow^{\pi_2} \\
X \\
\downarrow^{q} \\
H_0
\end{array}
\begin{array}{c}
\xrightarrow{a} \\
\xrightarrow{q} \\
X
\end{array}
\]

since the second is proved similarly. Recall [32] that the direct image of the open map $u : H_0 \rightarrow H_1$ restricts to an order isomorphism $u_1 : H_0 \rightarrow R_0$ such that the following triangle commutes in SL:

\[
\begin{array}{c}
X \\
\downarrow^{d^*} \\
H_0 \\
\downarrow^{u_1} \\
R_0
\end{array}
\begin{array}{c}
\xrightarrow{1_x \cdot (-)} \\
\xleftarrow{u_1} \\
R_0
\end{array}
\]

Hence, the commutativity of (4.11) is equivalent to the commutativity of the
diagram in $Frm$

\[
\begin{array}{c}
G_1 \otimes_{G_0} X \xleftarrow{a^*} X \\
\pi_2 \downarrow \quad \downarrow 1_X \cdot (-) \\
X \quad 1_X \cdot (-) R_0 ,
\end{array}
\]

which commutes if and only if for all $c \in R_0$ we have

\[a^*(1_X \cdot c) = 1_Q \otimes (1_X \cdot c) .\]

And the latter condition holds because, on one hand, from Eq. (1.9) (with $\alpha_* = a^*$) and the equality

\[1_Q 1_X c = 1_X c \]

we obtain

\[1_Q \otimes 1_X c \leq a^*(1_X c) ;\]

and, on the other, from Eq. (1.10) we obtain

\[a^*(1_X c) = \bigvee_{s \in Q_x} s \otimes s^* 1_X c \leq 1_Q \otimes 1_X c . \]

Corollary 4.12 The bicategories $Gpd$ and $IQLoc$ are biequivalent.

4.3 Algebraic morphisms

Due to the biequivalence, the embedding $InvQuF \rightarrow IQLoc$ yields a further embedding

\[InvQuF \rightarrow Gpd\]

such that each homomorphism of inverse quantal frames $h : Q \rightarrow R$ maps to a $G(Q)$-$G(R)$-bilocale. Such a bilocale is precisely the same as an algebraic morphism of groupoids in the sense of Buneci and Stachura [3,4]. Moreover, their composition of algebraic morphisms is, up to coherence, the same as that which results from the embedding. But it is strictly associative and therefore defines a category. The definitions can be carried over to very general groupoids:

**Definition 4.13** (Based on [4].) Let $G$ and $H$ be groupoids. By an algebraic morphism from $G$ to $H$ is meant a left action of $G$ on $H_1$ that commutes with right multiplication in $H$. More precisely, an algebraic morphism

\[(p, a) : G \rightarrow H\]
consists of maps $a : G_1 \otimes_{G_0} H_1 \to H_1$ and $p : H_1 \to G_0$ that define a left $G$-locale and make the following diagrams commute:

\[
\begin{array}{cccccc}
G_1 \otimes_{G_0} H_1 & \xrightarrow{a} & H_1 & \xrightarrow{m} & X \\
\downarrow{\pi_2} & & \downarrow{r} & & \\
H_1 & \xrightarrow{r} & H_0 & \xrightarrow{p} & H_1 \\
\end{array}
\]

\[
\begin{array}{cccc}
G_1 \otimes_{G_0} H_1 \otimes_{H_0} H_1 & \xrightarrow{a \otimes 1} & H_1 \otimes_{H_0} H_1 \\
\downarrow{1 \otimes m} & & \downarrow{m} \\
G_1 \otimes_{G_0} H_1 & \xrightarrow{a} & H_1 \\
\end{array}
\]

(4.14)

Given two algebraic morphisms

\[ G \xrightarrow{(p,a)} H \xrightarrow{(q,b)} K , \]

their composition is

\[ (q,b) \circ (p,a) = (p \circ u_H \circ q, c) , \]

where $c : G_1 \otimes_{G_0} K_1 \to K_1$ is defined, in point-set notation, by

\[ g \cdot \gamma k = (g \cdot a u_H(qk)) \cdot b k . \]

This definition applies to internal groupoids in any category with enough pullbacks, for instance topological or localic groupoids, Lie groupoids or, as in [3,4], locally compact groupoids equipped with Haar systems of measures. For localic étale groupoids, denoting the resulting category by $Gpd_A$, we therefore conclude:

**Theorem 4.15** $Gpd_A$ and $InvQuF$ are equivalent categories.

An immediate consequence of this equivalence is, of course, that algebraic morphisms specialize covariantly to homomorphisms of discrete groups and, contravariantly, to locale homomorphisms, as was stated by Buneci and Stachura, whose main goal was to define a (covariant) functor from groupoids to the category of C*-algebras [4]: their functor assigns to each locally compact groupoid $G$ the multiplier algebra of the C*-algebra that arises as the completion of $C_c(G)$ with respect to a norm which is different from either the usual — maximum or reduced — ones. For locally compact étale groupoids (with counting measures) this yields, due to [4,15] a non-trivial example of a functor from quantales to C*-algebras.
Corollary 4.16 There is a covariant functor from the full subcategory of InvQuF whose objects are locally compact locales to the category of $C^*$-algebras and $*$-homomorphisms.

Algebraic morphisms have also been used by Buss, Exel and Meyer [7] in order to define a covariant functor from topological étale groupoids to inverse semigroups. Their functor can be identified, due to [1.15] with the covariant partial units functor $\mathcal{I}$ from spatial inverse quantal frames to inverse semigroups, and therefore it readily extends to localic groupoids.

The results in this section show that for étale groupoids the algebraic morphisms are subsumed by quantale homomorphisms. It is interesting to note that, albeit under completely different terminology, and restricting to discrete groupoids, the idea of defining a morphism of groupoids $G \to H$ to be a homomorphism of quantales $\mathcal{O}(G) \to \mathcal{O}(H)$ can be found in the work of Zakrzewski [37], whose notion of pseudospace (cf. [36]) is based on the idea of replacing the underlying linear space of an associative $*$-algebra by the sup-lattice structure of a powerset, hence leading to algebras that are unital involutive quantales and furthermore, as the author states, are equivalent to discrete groupoids. There is more than one way in which such ideas can be carried over to more general groupoids. For arbitrary open groupoids [29] a definition of morphism $G \to H$ can of course be based on a homomorphism of involutive quantales $\mathcal{O}(G) \to \mathcal{O}(H)$, but additional requirements are needed, in particular due to the absence of multiplicative units. Besides, for groupoids equipped with non-trivial additional structure, such as non-étale Lie groupoids, a homomorphism of quantales only takes the structure of topological groupoid into account. By contrast, algebraic morphisms were proposed by Buneci and Stachura precisely as a way of generalizing Zakrewnski’s ideas to (not necessarily étale) locally compact groupoids, and Zakrzewski’s own extension to the differential setting [38] defines morphisms $G \to H$ of Lie groupoids and symplectic groupoids to be “differential relations”, i.e., submanifolds of $G_1 \times H_1$ satisfying suitable conditions. Our results show that, nevertheless, the identification of groupoid morphisms with quantale homomorphisms is meaningful at least for étale groupoids.

References

[1] J. Bénabou, Introduction to bicategories, Reports of the Midwest Category Seminar, Springer, Berlin, 1967, pp. 1–77. MR0220789 (36 #3841)

[2] F. Borceux, J. Rosický, and G. Van den Bossche, Quantales and $C^*$-algebras, J. London Math. Soc. (2) 40 (1989), no. 3, 398–404. MR1053610 (91d:46075)
[3] M. R. Buneci, *Groupoid categories*, Perspectives in operator algebras and mathematical physics, Theta Ser. Adv. Math., vol. 8, Theta, Bucharest, 2008, pp. 27–40. MR2433025 (2010b:22007)

[4] M. R. Buneci and P. Stachura, *Morphisms of locally compact groupoids endowed with Haar systems* (2005), available at arXiv:math.OA/0511613v1.

[5] M. Bunge, *An application of descent to a classification theorem for toposes*, Math. Proc. Cambridge Philos. Soc. 107 (1990), no. 1, 59–79, DOI 10.1017/S0305004100068365. MR1021873 (90k:18002)

[6] M. Bunge and J. Funk, *Constructive theory of the lower power locale*, Math. Structures Comput. Sci. 6 (1996), no. 1, 69–83, DOI 10.1017/S0960129500000876. MR1386115 (98f:18004)

[7] A. Buss, R. Exel, and R. Meyer, *Inverse semigroup actions as groupoid actions*, Semigroup Forum 85 (2012), no. 2, 227–243, DOI 10.1007/s00233-012-9418-y. MR2969047

[8] A. Connes, *Noncommutative Geometry*, Academic Press Inc., San Diego, CA, 1994. MR1303779 (95j:46063)

[9] H. Heymans and I. Stubbe, *Grothendieck quantaloids for allegories of enriched categories*, Bull. Belg. Math. Soc. Simon Stevin 19 (2012), no. 5, 861–890. MR3009020

[10] M. Hilsam and G. Skandalis, *Morphismes K-orientés d’espaces de feuilles et fonctorialité en théorie de Kasparov (d’après une conjecture d’A. Connes)*, Ann. Sci. École Norm. Sup. (4) 20 (1987), no. 3, 325–390 (French, with English summary). MR925720 (90a:58169)

[11] P. T. Johnstone, *Stone Spaces*, Cambridge Studies in Advanced Mathematics, vol. 3, Cambridge University Press, Cambridge, 1986. Reprint of the 1982 edition. MR861951 (87m:54001)

[12] _____, *The point of pointless topology*, Bull. Amer. Math. Soc. (N.S.) 8 (1983), no. 1, 41–53. MR682820 (84f:01043)

[13] A. Joyal and M. Tierney, *An extension of the Galois theory of Grothendieck*, Mem. Amer. Math. Soc. 51 (1984), no. 309, vii+71. MR756176 (86d:18002)

[14] D. Kruml, *Spatial quantales*, Appl. Categ. Structures 10 (2002), no. 1, 49–62. MR1838084 (2002m:06010)

[15] D. Kruml, J. W. Pelletier, P. Resende, and J. Rosický, *On quantales and spectra of C*-algebras*, Appl. Categ. Structures 11 (2003), no. 6, 543–560. MR2017650 (2004i:46107)

[16] D. Kruml and P. Resende, *On quantales that classify C*-algebras*, Cah. Topol. Géom. Différ. Catég. 45 (2004), no. 4, 287–296 (English, with French summary). MR2108195 (2006b:46096)

[17] N. P. Landsman, *Operator algebras and Poisson manifolds associated to groupoids*, Comm. Math. Phys. 222 (2001), no. 1, 97–116, DOI 10.1007/s002200100496. MR1853865 (2002f:46142)

[18] M. V. Lawson and D. H. Lenz, *Pseudogroups and their étale groupoids*, Adv. Math. 244 (2013), 117–170, DOI 10.1016/j.aim.2013.04.022. MR3077869
I. Moerdijk, The classifying topos of a continuous groupoid. I, Trans. Amer. Math. Soc. 310 (1988), no. 2, 629–668. MR973173 (90a:18005)

______, The classifying topos of a continuous groupoid. II, Cahiers Topologie Géom. Différentielle Catég. 31 (1990), no. 2, 137–168 (English, with French summary). MR1080241 (92c:18003)

______, Toposes and groupoids, Categorical algebra and its applications (Louvain-La-Neuve, 1987), Lecture Notes in Math., vol. 1348, Springer, Berlin, 1988, pp. 280–298, DOI 10.1007/BFb0081366, (to appear in print). MR975977 (89m:18003)

J. Mrčun, Functoriality of the bimodule associated to a Hilsum–Skandalis map, K-Theory 18 (1999), no. 3, 235–253, DOI 10.1023/A:1007773511327. MR1722796 (2001k:22004)

P. S. Muhly, J. N. Renault, and D. P. Williams, Equivalence and isomorphism for groupoid C*-algebras, J. Operator Theory 17 (1987), no. 1, 3–22. MR873460 (88h:46123)

C. J. Mulvey, Êtale groupoids and their quantales, Adv. Math. 208 (2007), no. 1, 199–209. MR2304314 (2008c:22002)

______, Groupoid sheaves as quantale sheaves, J. Pure Appl. Algebra 216 (2012), no. 1, 41–70, DOI 10.1016/j.jpaa.2011.05.002. MR2826418

P. Resende and S. Vickers, Localic sup-lattices and tropological systems, Theoret. Comput. Sci. 305 (2003), no. 1-3, 311–346, DOI 10.1016/S0304-3975(02)00702-8. Topology in computer science (Schloß Dagstuhl, 2000). MR2013577 (2004i:68130)
[35] J. Rosický, *Multiplicative lattices and $C^*$-algebras*, Cahiers Topologie Géom. Différentielle Catég. **30** (1989), no. 2, 95–110 (English, with French summary). MR1004734 (91e:46079)

[36] S. L. Woronowicz, *Pseudospaces, pseudogroups and Pontriagin duality*, Mathematical problems in theoretical physics (Proc. Internat. Conf. Math. Phys., Lausanne, 1979), Lecture Notes in Phys., vol. 116, Springer, Berlin, 1980, pp. 407–412. MR582650 (82e:46079)

[37] S. Zakrzewski, *Quantum and classical pseudogroups. I. Union pseudogroups and their quantization*, Comm. Math. Phys. **134** (1990), no. 2, 347–370. MR1081010 (91m:58012)

[38] ———, *Quantum and classical pseudogroups. II. Differential and symplectic pseudogroups*, Comm. Math. Phys. **134** (1990), no. 2, 371–395. MR1081011 (91m:58013)

CENTRO DE ANÁLISE MATEMÁTICA, GEOMETRIA E SISTEMAS DINÂMICOS
DEPARTAMENTO DE MATEMÁTICA, INSTITUTO SUPERIOR TÉCNICO
UNIVERSIDADE DE LISBOA
AV. ROVISCO PAIS 1, 1049-001 LISBOA, PORTUGAL
E-mail: pmr@math.tecnico.ulisboa.pt