Relativistic Time Dilation and Length Contraction in Discrete Space-Time using a Modified Distance Formula

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Abstract
In this work, the relativistic phenomena of Lorentz contraction and time dilation are derived using a modified distance formula appropriate for discrete space. This new distance formula is different than Pythagoras’s theorem but converges to it for distances large relative to the Planck length. First, four candidate formulas developed by different people over the last 70 years will be considered. Three of the formulas are shown to be identical for conditions that best describe discrete space; this equation is then used in the rest of the paper. It is shown that this new distance formula is applicable to all size-scales —from the Planck length upwards —and solves two major historical problems associated with a discrete space-time model. One problem it solves is maintaining isotropy in discrete space. The second problem it solves is the commonly perceived incompatibility of the model’s concept of an immutable “atom” of space and the Lorentz contraction of this atom required by special relativity. With the new distance formula, it is shown that the Lorentz contraction of the atom of space does not occur regardless of the relative velocities of two reference frames. It is also shown that time dilation of the atom of time does not occur. Also discussed is the possibility of any object being able to travel at the speed of light for specific temporal durations derived in this work.

Keywords: Discrete Space; Discrete Time, Pythagoras Theorem; Special Relativity;

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1. Introduction

Ever since the Greek and medieval philosophers Leucippus, Democritus, Parmenides, Zeno and Maimonides proposed that space and time are discretized [1, 2, 3], debate on this subject has waxed and waned. In the modern age, Werner Heisenberg had a continued interest in discrete space and corresponded with Neils Bohr and Wolfgang Pauli on the concept [3]. Recently, the idea that space and time are discretized, or atomized, has received increased interest by scientists in the fields of mathematical physics [3], loop quantum gravity [4, 5, 6] and pure mathematics [9, 7, 8, 10, 11]. Recently in [12], we considered a simple consequence of discrete space, specifically, how it imposes crystalline order upon Wheeler’s quantum foam [13]. Then using simple solid-state physics, we were able to show that the resulting universe-wide gravity crystal causes measurably anomalous motion of astronomical bodies (i.e., black holes). We broadened our focus in [14] and discussed a wide range of phenomena and issues associated with discrete space-time (ST), including: the problems associated with this model, the solutions to these problems, and the long forgotten debate on time and duration that received so much attention in the early 20th century by intellectual giants: Albert Einstein, Alfred Whitehead and especially Henri Bergson [15].

In this paper, we focus on one of the most important and widely cited problems associated with discrete ST, namely, the incompatibility of the immutability of the “atoms” of space and time with the laws of special relativity (SR). Simply put, by its name and nature, an atom of space needs to be a constant, an immutable value - one cannot simple travel faster and measure this atom to be a different size. The same applies to the atom of time. This issue of the Lorentz-Fitzgerald contraction of the atoms of space and time, and more generally Lorentz invariance within quantum gravity, is of central importance and should be the priority of the the LQG research community rather than just being infrequently studied (two goods papers on the subject are Rovelli’s [5] and Collin’s [6]) or mentioned in passing as an open issue [4]. In the process solving this problem, we will also see that another problem associated with a discrete ST model is solved, namely, maintaining isotropy. To do so however, we must first review the last 70 years of mathematicians/philosophers/physicists proposing modifications to the distance formula such that it is applicable to discrete space.
This paper is organized as follows. In Section 2, four different versions of the distance formula applicable to discrete space are discussed - those proposed by Hermann Weyl [7], Jean Paul Bendegem [8], Peter Forrest [9] and David Crouse [14]. It is discussed how three of these formulas, with some slight modifications and reinterpretations, are identical for conditions that best describe discrete ST, and how this formula differs from Pythagoras’s theorem for small size-scales but converges to it for any distance that can be measured in any practical way. In Section 3, the standard derivations of time dilation and length contraction found in any textbook on SR [16] are performed but where a modified distance formula is used instead of Pythagoras’s theorem. An interesting consequence of these results are discussed in Section 4, namely, how SR in discrete ST allows objects to travel at the speed of light over a specific temporal duration.

2. Distance Formulas for Discrete Space

In 1949, Hermann Weyl constructed his famous Weyl-tile argument against discrete space [7], which states that if space is discrete, then it must be composed of a fixed grid of identical “tiles”, with a tile-to-tile spacing of $\chi$, (Fig. 1). He then argued that the length of the hypotenuse of an isosceles right triangle (i.e., $c$) is equal to the length of the sides of the triangle (i.e., $a$ or $b$). The important point in his argument, is that $c$ is equal to $a$ regardless of the size of the triangle; and because we measure the hypotenuse of any such triangle to be $\sqrt{2}$ times the length of its side, space must not be discrete. One can then develop a distance formula with Weyl’s construction in much the same way as is done with the similar taxicab geometry [17]. In a certain way then, and to our knowledge, Weyl was the first person to propose any modification of the distance formula given by Pythagoras’s theorem (not considering the curved space of general relativity). It went unrecognized at that time and up to now, that even though Weyl’s correction created one problem with the discrete ST model, it solved a different problem with the model. As explained in Section 3, Weyl’s result solved the problem of Lorentz contraction of the ostensibly immutable atom of space. Unfortunately, it took over 35 years until Jean Paul Bendegem pointed us in the right direction towards a resolution of the Weyl-tile argument [8].

In a 1987 paper on Zeno’s paradoxes and the Weyl-tile argument, Bendegem made the four following assumptions about lines in discrete space [8]:

3
Figure 1: The Weyl construction that includes an *a priori* defined lattice. All distances, from the center of one tile to the center of any neighboring tile have to be separated by integer multiples of $\chi$; *no fractional values of $\chi$ are allowed*. Thus the length of the diagonal is equal to the length of the side of the square ($pq = pr = \chi$), regardless of the size of the square (i.e., $pq' = pr' = 3\chi$).

1. In a discrete geometry, all lines must have a constant nonzero width (of integer $N_D$).

2. A line consists of all the squares that are touched during the act of drawing the line (see Fig. 2).

3. The size of the squares is small compared to the macroscopical width of the lines.

4. The length of a line is the sum of the squares constituting that line, modulo the width.

where $N_D$ is an integer and the actual width in units of length is $N_D\chi$ where later in this paper we assign $\chi$ to be $\chi = 3.24 \times 10^{-35}$ m. In his later works [10] and [11], Bendegem seemed to drop the necessity of Assumption 3 stated above and even considered the case where $m = N_D = 1$ as describing and resolving the Weyl-tile argument. Applying this procedure for straight lines, say of $m$ rectangles long and $N_D$ rectangles wide, one obtains the expected result [8]:

4
\[
\text{Length} = m \cdot N_D \pmod{N_D} 
\]
\[
= m
\]

Thus, the length of the base of the triangle shown in Fig. 2 is \(m\) and the height is \(p\). Now concerning the hypotenuse, Bendegem states in [8] that “the hypotenuse can be considered as a vertical pile of \(p\) layers”. This then leads to the equation, given in [8] and more clearly in [10]:

\[
d(a, c) = p \cdot \left\lfloor \frac{N_D}{\sin \alpha} \right\rfloor \text{div} (N_D) 
\]

Note that Eq. (3) is not quite the same as Assumption 2, but is better because it removes ambiguity concerning which rectangles to include in the sum used to calculate \(d(a, c)\). Importantly, in 1995 Bendegem expressed Eq. (3) in a slightly different way, namely, he placed the factor \(p\) into the floor operation in Eq. (3):

\[
d(a, c) = \left\lfloor p \cdot \frac{N_D}{\sin \alpha} \right\rfloor \text{div} (N_D) 
\]

This change is significant because it is a step towards connecting Bendegem’s approach to the approach taken by Crouse in [14] where he eliminated the \textit{a priori} defined grid. To fully connect the two approaches, we must take the additional step of letting \(N_D = 1\) in Eq. (4) (which seems most appropriate for discrete space). Equation (4) then becomes:

\[
d(a, c) = \left\lfloor \frac{p}{\sin \alpha} \right\rfloor = \left\lfloor \sqrt{m^2 + p^2} \right\rfloor 
\]

It is seen that Eqs. (1) and (5) (but not Eq. (3)) converge to Pythagoras’s theorem for large \(m\) or large \(p\). Importantly, one can interpret Eq. (5) somewhat differently, namely, as the number of complete and partial rectangles included along \textit{one tilted column} along the hypotenuse (Fig. 3). The benefit of this modification is that it suggests a solution to the anisotropy problem: when an entity travels from \(a\) to \(b\), the local grid manifests appropriate to that direction of travel, when traveling from \(a\) to \(c\), the local grid manifests appropriate to this different direction of travel. Namely, the grid is not defined \textit{a priori} but comes into existence only \textit{locally} (i.e., within the immediate neighborhood of the particle). Before describing how Crouse in
Figure 2: Bendegem’s method involves a grid and lines with widths $N_D$. The distance from point $a$ to point $c$ is the sum of all the squares (with red dots) within or touched by the lines is divided $N_D$.

[14] took the process of eliminating the fixed grid to its ultimate conclusion, we discuss below a method developed by Peter Forrest [9] that involves an analysis of what it means for points to be “adjacent”.

In 1995, Peter Forrest sought to develop a distance formula appropriate for discrete space that uses “only a single dyadic relation of adjacency” [9]; he pointedly rejected Bendegem’s approach that he thought did not “define distance in terms of adjacency” [9]. Forrest’s approach is interesting and rests on an intriguing, but we think problematic, definition of adjacency [9]. Forrest states that the distance between two points $p$ and $q$ is the chain with the least number of “links” of pairwise adjacent tiles, the first of which is $p$ and the last of which is $q$ [9]. This first part of his formulation is fine, being also used by Weyl, Bendegem and Crouse in their formulations; it is the second part involving a definition of adjacency that is unique to Forrest’s formulation. Forrest considers two points $\langle u, v \rangle$ and $\langle x, y \rangle$ in $E_{2,m}$, as being adjacent if they satisfy the equation:

$$ (u - x)^2 + (v - y)^2 \leq m^2 $$

where $m$ is a scale factor for which Forrest proposes a value of $10^{30}$ as being appropriate for the real space in which we live. There then exists “balls of adjacency” (BoAs), each one containing a large number of points all adjacent to each other. One then finds the chain of pairwise adjacent points from $p$ to $q$ that has the minimum number of links; this chain is then the distance
Alternative forms of Bendegem’s distance formula (i.e., Eq. (5) that uses $N_D = 1$) suggests that the grid does not have a preferred direction. If this is true, then isotropy is maintained.

from $p$ to $q$ (Fig. 4). Two benefits of Forrest’s approach are: the anisotropic nature of the grid can be minimized by letting $m$ be large, and the distances it predicts for large triangles converge to those predicted by Pythagoras’s theorem. However, one major problem with Forrest’s method is that, as formulated, the distances it calculates are generally at least one integer larger than those predicted by Eq. (5). It will be shown in the next section that because of this, distances calculated using Forrest’s method leads to a non-physical result.

One way to fix this discrepancy is to first let $m \to \infty$ and then to construct Forrest’s BoAs slightly differently, as shown in Fig. 5. Besides letting $m \to \infty$, the only other difference between the constructions exploits the ambiguity in the placement and orientation of $c$’s BoA (in black in Fig. 5). This difference is, when calculating $d(a, c)$, place and orient $c$’s BoA in a manner shown in Fig. 5 and then the last link in the shortest chain need only contain points in $c$’s BoA rather than $c$ itself. These points are in the purple shaded area in Fig. 5. This approach eliminates the anisotropy that existed when $m$ is a finite number and yields distances matching those predicted by Eq. (5). With this fix, agreement between Bendegem and Forrest can be achieved, however one would be justified in feeling uneasy with the foundations of these approaches that rely on grids and/or balls.
Figure 4: In Forrest’s approach, a grid is constructed (black dots) and the balls of adjacency (with $m = 2$) are shown along the base (dashed red circles), height (dashed green circles) and hypotenuse (dashed blue circles). The pair-wise adjacent points are shown in orange.

of adjacency. As described next, in 2016 Crouse [14] – being at the time familiar only with Bendegem’s approach leading up to Eq. (3) but not the correct form of the equation (i.e., Eq. (5)), and being totally unfamiliar with Forrest’s approach – derived Eq. (5) in a way that did not involve any a priori grid or BoAs.

Recently, Crouse studied the five most commonly cited problems associated with discrete ST and developed an approach (shown in Fig. 6) and a distance formula that he believes solves all of these problems [14]. Starting from three fundamental postulates that include: 1) what it means for something to be a part of reality, 2) assuming that space and time are non-absolute entities, and 3) space and time quantization, he derived Eq. (5) but expressed it in a slightly different form. One importance difference in Crouse’s approach relative to Bendegem’s and Forrest’s approaches is that Crouse purposefully did not define or use any a priori grid that would destroy isotropy. Ultimately the result was the same though (i.e., Eq. (5)); this equation will be used in the rest of this work. The three-dimensional version of the new distance formula in units of distance is:

$$n\chi > \sqrt{(m\chi)^2 + (p\chi)^2 + (s\chi)^2} - \chi$$

(7)

where $n$ is the smallest integer that satisfies Eq. (7); $m$, $p$ and $s$ are integers; $m\chi$, $p\chi$ and $s\chi$ are the distances along the arbitrarily chosen $x$, $y$ and $z$
Figure 5: A schematic showing a different way of interpreting Forrest’s approach. First, let the spacing between $\langle u, v \rangle$ grid points go to zero, i.e., the grid is infinitely dense. The grid then disappears, and with it, the anisotropy of Forrest’s construction. Also, we rethink when the “chain” (in blue) has reached the endpoint $c$. We do not require the last link in the chain to contain the $\langle u, v \rangle$ point identified as $c$ but only to be within the $c$’s BoA (in black). The $\langle u, v \rangle$ grid points common to both BoA’s (i.e., of $c$ and of the last link in the chain) are shaded in purple.

It is easy to see that Eq. (8) is identical to Eq. (5) and converges to the standard distance formula given by Pythagoras’s theorem for large values of $m$ or $p$. How this equation conserves the immutability of the atoms of space and time is shown in Section 3.

It is seen that all three approaches, while using different starting points and assumptions, lead to the same equation for the calculation of distances in discrete space (Eqs. (5) or (8)). After assigning $N_D$ to be one, and allowing the grid to change orientation dependent on the direction of the measurement, Bendegem’s approach matches that of Crouse’s. It is also seen that within Forrest’s approach are aspects of Bendegem’s and Crouse’s approaches. Forrest’s identification of points $\langle u, v \rangle$ creates a de facto and a priori grid similar to, but playing a lesser role than the grid in Bendegem’s approach. Also, Forrest’s BoAs are similar to the “atoms of space” used by Crouse. However, within Forrest’s BoAs are a multitude of identifiable and
Figure 6: Crouse’s construction assumes non absolute space and an atom of space ($\chi$). A particle can make transitions in any direction, but only by an amount $\chi$. The distance formula Eq. (7) is easily derived by determining how many translations are required along $\Delta x$ such that the leading edge of translated point along the hypotenuse (denoted by $\alpha$) overtakes the trailing edge (denoted by $\theta$) of the sphere that defines point $c$.

unique grid points - something we believe is entirely inconsistent with the concept of an atom of space. Even if the grid is only a mathematical tool, the use of it clouds important aspects of the true nature of discrete ST (e.g., inherent isotropy and possibly ST’s non absolute nature). It is also important to note that Eq. (8) does not satisfy the Triangle Inequality theorem (TI), but Forrest’s original formulation does satisfy the theorem. In fact, Forrest states that this is one of the benefits of discrete ST because “enables us to explain why distance satisfies the triangle inequality” [9]. Forrest achieves compliance with the theorem because the distances it calculates are generally larger than those predicted by Eq. (8). Consider three collinear points labeled with $(x, y)$ coordinates $e = (0, 0)$, $f = (1, 1)$ and $g = (3, 3)$. The TI theorem states that the distances between these points, $d(e, f)$, $d(f, g)$ and $d(e, g)$, satisfy the inequality $d(e, g) \leq d(e, f) + d(f, g)$. Upon studying Fig. 4, one sees that Forrest’s original approach yields $d(e, f) = 2$, $d(f, g) = 3$ and $d(e, g) = 5$; this satisfies the TI theorem. However, Bendegem’s and Crouse’s approach (i.e., Eq. (8)) yields $d(e, f) = 1$, $d(f, g) = 2$ and $d(e, g) = 4$ which does not satisfy the TI theorem. One can speculate that Forrest was pleased with this result (namely, adherence to the TI theorem for the smallest possible distances)
because in [9] he states “to be sure, we would not call a quantitative relation
distance unless it satisfied the triangle inequality”. However, Forrest’s result
of $d(e, f) = 2$ should have raised red flags; it will be shown in the next section
that such a result does not conserve the immutable nature of the atom of
space, as opposed to Eq. (8), which does. In fact, upon contemplation one
realizes that the only models of discrete ST that conserve the atoms of space
and time are ones that violate the TI theorem. Namely, in flat-space, one
always will obtain a larger value when one sums the distances of component
segments relative to the value of the parent segment. Again, this is imposed
upon nature by the need to conserve the immutability of atoms of space and
time but also to have distances converge to values predicted by Pythagoras’s
theorem at macrogeometric scales.

3. Time Dilation and Length Contraction in Discrete Space-Time

While the exact values of the atom of space ($\chi$) and the atom of time ($\tau$)
are not as important as their existence, Crouse nevertheless derived them in
[14] to be:

$$\chi = 2\sqrt{\frac{\hbar G}{c^3}} = 2l_p = 3.24 \times 10^{-35}m$$

(9)

$$\tau = \frac{2l_p}{c} = 2\tau_p = 1.08 \times 10^{-43}s$$

(10)

where $l_p = 1.62 \times 10^{-35}$ m and $\tau_p = 5.39 \times 10^{-44}$ s are the Planck length
and Planck time respectively (note that $\chi/\tau = c$ where $c$ in this case is the
speed of light, not the length of the hypotenuse). If one is unhappy with
his derivation, one can consider other derivations given by [19, 20, 21, 22]
that yield similar values. Crouse provided the derivations of $\chi$ and $\tau$ in [14]
only as a salve to those married to conventional view of space and time; it is
preferred to view $\chi$ and $\tau$ not as derived quantities, but rather as constants
of nature, or within LQG as the minimal eigenvalues of quantum observables
[5]. We now use these values for the atoms of space and time in Eq. (8), and
use the new formula in the derivation of relativistic time dilation and length
contraction.

Consider the standard derivation for time dilation given in any textbook
on SR [16] that involves two observers and a “light-clock” on a train traveling
in the $\hat{x}$ direction, as shown in Fig. 6. We update this calculation slightly by
replacing the mirror above the photon emitter with a photon receiver. This change is relatively unimportant, but updates this calculation to be inline with Bohm’s and Bondi’s approach to SR [23, 24], and because it allows us to assess shorter time durations. Also, instead of one light-clock with a height \( h \), we have a collection of light-clocks on the train with \( h = p\chi \) with \( p \in \{1, 2, \ldots \} \). All the clocks are at \( x = 0 \) at \( t = t' = 0 \), and are identified from here on according to their value of \( p \). Unprimed coordinates \( \Delta t \) and \( \Delta x \) correspond to temporal durations and spatial extents in the “moving frame” (RF1) for an observer (O1) alongside and stationary relative to the tracks of the train (i.e., the clocks are moving relative to O1). Primed coordinates are used for the “rest reference” (RF2) frame and observer O2 (i.e., the clocks are at rest relative to O2). Now consider the trajectory of a photon from O1’s and O2’s perspectives, and using the fact that a photon travels at a velocity \( c \) in both RFs.

For any clock \( p \), the duration elapsed while a photon travels from the emitter to the receiver is \( \Delta t' = p\chi/c \) in RF2 and \( \Delta t = n\chi/c \) in RF1, during which the clock has moved \( \Delta x = m\chi = v\Delta t \) where \( v \) is the velocity of the train (Fig. 6). Also, and as typical, \( \Delta y = \Delta y' = p\chi \). Thus, we have the standard isosceles right triangle with the lengths of the sides as \( m\chi \) and \( p\chi \) and the length of the hypotenuse as \( n\chi \), with \( v \) and \( \gamma = \Delta t/\Delta t' \) given by:

\[
v = \frac{m}{n} c \quad m \in \{0, 1, 2, \ldots, n\} \quad (11a)
\]
\[
\gamma = \frac{n}{p} \quad (11b)
\]

At this point in the standard calculation one would use Pythagoras’s formula: \( n^2\chi^2 = m^2\chi^2 + p^2\chi^2 \). After using the relations \( m\chi = v\Delta t \), \( n\chi = c\Delta t \), \( p\chi = c\Delta t' \), we can easily solve for the relation between \( \Delta t \) and \( \Delta t' \) as first derived by Einstein, Poincaré and Lorentz:

\[
\Delta t = \frac{1}{\sqrt{1 - v^2/c^2}} \Delta t' = \gamma_E \Delta t' \quad (12)
\]

where the subscript \( E \) denotes Einstein. It is important to note that \( \gamma_E \) in Eq. (12) is independent of \( p \) and only dependent on \( v \). Thus all temporal durations \( \Delta t' \) measured by all the clocks on the train are dilated by this same factor in RF1. The arguments given in any textbook on SR can then be used to describe length contraction:
To be clear, $\Delta t'$ and $\Delta t$ are the time durations as recorded by clocks that are stationary in RF2 and RF1 respectively. $\Delta l$ is the length of a rod, as measured by an observer stationary relative to the rod; this is called the “proper length” of the rod. $\Delta l'$ is the observed length of the rod – the length measured by an observer traveling at a velocity $v$ relative to the rod.

Equations (12) and (13) are the roots of the oft-cited problem concerning the variability of $\chi$ and $\tau$ [3]. Using the standard derivation of time dilation, the atom of time $\tau$ in RF2 is dilated to a larger value in RF1 and the atom of space $\chi$ in RF1 is contracted to smaller value in RF2. These problems – the velocity dependent extent (duration) of the atom of space (time) – are solved using the new distance formula, as described below.

The new derivation of time dilation starts the same way, with light-clocks on the train. However, each light-clock of different heights $p\chi$ will assess different temporal durations of $\Delta t' = p\chi/c$, starting from $\Delta t' = \chi/c = \tau$ (the shortest possible duration) to progressively larger integer multiples of $\tau$. The only other change in the derivation is the use of Eq. (8) instead of Pythagoras’s theorem. To start, you decide the velocity $v$ for which you will calculate $\gamma$; let us consider $v = 0.5c$ as an example (note that $\gamma_E = 1.15$ for this velocity). Note that for particular durations it is not always possible to have $v$ exactly equal to this desired value. In these cases, we choose the
value of \( m \) such that the \( v \) is as close to, but smaller than the desired value. Now one starts the procedure, as described below.

First, all the clocks are set to \( t = t' = 0 \) at which time they all emit a single photon. Then one assess the situation when the receiver for clock \( p = 1 \) detects the photon at \( \Delta t' = \tau \). From the perspective of \( O_1 \), the clock could have either have made a spatial translation of extent \( \Delta x = \chi \) or not moved at all - these are the only two possibilities. If it did move by \( \chi \), then this corresponds to a velocity of \( v = c \) over this duration; if it did not move, the velocity is assessed to be zero. To be consistent with our requirement that \( v \leq 0.5c \), we choose the \( v = 0 \) case. Solving Eq. (8) yields \( n = 1 \), and with \( \gamma = n/p \), we find that \( \gamma = 1 \) for this duration. We next assess the situation at \( \Delta t' = 2\tau \) when the receiver of clock \( p = 2 \) detects its photon. A solution exists for Eq. (8) with \( p = 2 \), \( m = 1 \) and \( n = 2 \), corresponding to a velocity of \( v = 0.5c \) and \( \gamma = 1 \). Thus, a duration of \( \Delta t' = 2\tau \) in RF2 corresponds to the same duration \( \Delta t = 2\tau \) in RF1. For clock \( p = 3 \), no solution to Eq. (8) exists that has \( v = 0.5c \), two solutions with \( v \) closest to \( v = 0.5c \) are: \( \{p, m, n\} = \{3, 1, 3\} \) with \( v = 0.33c \), and \( \{p, m, n\} = \{3, 2, 3\} \) with \( v = 0.67c \). We choose the \( \{3, 1, 3\} \) solution. We continue with this procedure and record the results from the first 15 clocks in Table 1 and the first 50 clocks in Fig. 8. Two important results can be gleaned at this point. First, \( \gamma \) converges to \( \gamma_E \) for temporal durations large relative to \( \tau \), but not monotonically. Second and most important, a temporal duration of \( \tau \) in RF2 is not dilated in RF1, i.e., it has a value of \( \tau \) in RF1. This will be true regardless of the velocity of the train and the clocks, namely, \( \gamma(v, \tau) = 1 \) for any velocity, even for \( v = c \). Thus the immutability of the atom of time is conserved. This is not the case if one uses the distance formula derived by Forrest (his original formulation shown in Fig. 4); doing so yields a \( \gamma \) factor of \( \gamma(c, \tau) = 2 \), and hence the atom of time (space) is dilated (contracted) by a factor of two. Thus Forrest’s original formulation leads to this non physical result; the modified approach (Fig. 5) however is correct, since it predicts the same distances as Bendegem’s and Crouse’s approach (Eq. (8)).

While slightly more complicated, the same basic argument used in conventional SR to connect length contraction and time dilation can be used for SR in discrete ST. Doing so yields Eq. (13) but with \( \gamma_E \) replaced by \( \gamma \). Collecting this formula along with the new time dilation formula, we have:
Table 1: The integer multiples of $\chi$ for the triangles traced out by the photons of the light-clocks, along with the velocity and $\gamma$ factor. The height, base and hypotenuse are relative to $\chi$ and the velocity is relative to $c$. The correspondence between ticks of clocks in RF1 and RF2 are also given.

| Height ($p$) or tick of RF2 clock | Base ($m$) | Hypotenuse ($n$) or tick of RF1 clock | $v$ | $\gamma(v, \Delta t' = p\tau)$ |
|----------------------------------|------------|--------------------------------------|-----|---------------------------------|
| 1                                | 0          | 1                                    | 0   | 1                               |
| 2                                | 1          | 2                                    | 0.5 | 1                               |
| 3                                | 1          | 3                                    | 0.33| 1                               |
| 4                                | 2          | 4                                    | 0.5 | 1                               |
| 5                                | 2          | 5                                    | 0.4 | 1                               |
| 6                                | 3          | 6                                    | 0.5 | 1                               |
| 7                                | 4          | 8                                    | 0.5 | 1.14                            |
| 8                                | 4          | 8                                    | 0.5 | 1                               |
| 9                                | 5          | 10                                   | 0.5 | 1.11                            |
| 10                               | 5          | 11                                   | 0.45| 1.1                             |
| 11                               | 6          | 12                                   | 0.5 | 1.09                            |
| 12                               | 6          | 13                                   | 0.46| 1.08                            |
| 13                               | 7          | 14                                   | 0.5 | 1.08                            |
| 14                               | 8          | 16                                   | 0.5 | 1.14                            |
| 15                               | 8          | 17                                   | 0.47| 1.13                            |

Figure 8: A plot of the correspondence between the ticks of the clock in RF1 and the ticks of the clock in RF2. The two RFs have a relative velocity of $v \approx 0.5c$. The red line has a slope of $\gamma_E = 1.15$. Inset: It is seen that no time dilation occurs for the first 5 ticks (each of duration $\tau$), but then the clocks on the train starts trailing the clock at the station.
\[ \Delta t = \gamma (v, \Delta t') \Delta t' \quad (14) \]
\[ \Delta l = \gamma (v, \Delta t') \Delta l' \quad (15) \]

with \( \Delta t' = p\tau = \Delta l' / c \) in Eq. (15). For the shortest possible spatial extent in RF2, namely, \( \Delta l' = \chi \), one uses \( \gamma(v, \Delta t' = \tau) \) in Eq. (15). But since \( \gamma(v, \tau) = 1 \) regardless of the relative velocities of the two RFs, a \( \Delta l' \) of \( \chi \) in RF2 is measured as \( \chi \) by observers in RF1. Thus, no length contraction occurs and the immutability of the atom of time is conserved.

4. Discussion

Besides the modifications to length contraction and time dilation, a surprising consequence of the new distance formula is that it allows objects to travel at light speed over certain temporal durations. Consider Fig. 7 that shows the isosceles right triangle. The conventional distance formula given by Pythagoras’s theorem does not admit a solution where the lengths of the base and hypotenuse are equal. However, the new distance formula given by Eq. (8) does, as long as the hypotenuse and base are long enough relative to the triangle’s height. To see this, let the velocity of the light-clock be \( v = c \) by setting \( m = n \) in Eq. (8). Next, choose a particular duration in the rest reference frame \( \Delta t' = p\tau \); this sets \( h = p\chi \). Finally, use Eq. (8) to solve for the critical value of \( n \) for which \( n = m \) is possible for this and all greater integer values:

\[ n_{\text{critical}} > \frac{1}{2} (p^2 - 1) \quad (16) \]

For large \( p \), Eq. (16) yields \( p = \sqrt{2n_{\text{critical}}} \), this, along with \( \gamma_{\text{critical}} = n_{\text{critical}} / p \), \( \Delta t' = p\tau \), and \( \Delta t = \gamma_{\text{critical}} \Delta t' \) yields:

\[ \gamma_{\text{critical}} = \sqrt{\frac{\Delta t}{2\tau}} \quad (17) \]

Since the kinetic energy \( (KE) \) of a particle is related to \( \gamma \) according to \( KE = (\gamma - 1) mc^2 \), the energy a particle needs such that it can be measured as traveling at a speed \( c \) over a particular duration \( \Delta t \) is:

\[ KE_{\text{critical}} = \left( \sqrt{\Delta t / 2\tau} - 1 \right) mc^2 \quad (18) \]
It is important to note that we are not predicting faster-than-light travel, or even travel at the speed of light for any situation normally encountered or even possible given modern-day technology. Concerning the former, the rule that a particle transits at most only one $\chi$ per $\tau$ is built into the theory from the very beginning, corresponding to a maximum velocity of $c$. Concerning the later, consider what energy it would take to accelerate an electron to the necessary such that a measurement of its velocity yields $v = c$. Imagine that we can microfabricate two detectors tips that can precisely determine the time that an electron has passed underneath them, but otherwise not perturbing the speed or trajectory of the electron – for example two microfabricated atomic force microscope tips. With currently available microfabrication techniques, a separation between the tips of the two detectors of 10 nm can be achieved. If the electron is traveling at a speed $c$, then $\Delta t = \Delta d/c \approx 0.1$ fs. Equation (18) yields a value of approximately 5,000,000 TeV. This value exceeds what is possible with the most powerful existing particle accelerators by a factor of $10^6$, but may be possible with future accelerators. A more practical and realizable test —if discrete space imposes crystalline order upon Wheeler’s quantum foam—is to observe the motion of black holes [12].

5. Conclusion

It was shown in this work that the discrete space-time model requires a new distance formula. This new formula was first proposed by Bendegem, and independently formulated in a different way by Crouse that explicitly conserves isotropy. This new formula converges to Pythagoras’s theorem for distances large relative to the atom of space $\chi$, but is significantly different for size-scales on the order of the Planck length. When using the new distance formula in the otherwise typical derivations of time dilation and length contraction, one sees that the atoms of space and time are indeed immutable - true constants of nature and independent of the speed of any observer. It was also discussed how this new distance formula allows for temporary travel at the speed of light. The reader is directed to [14] for more details on resolution of the other problems with discrete ST, and to [9] and [11] for discussions on conservation of energy within discrete ST. Also, while it is outside the scope of this work, the results discussed in this paper should be used to generate a new set of Lorentz transformation equations, a clock synchronization scheme applicable to discrete space-time, a new framework
for Minkowski space, discuss how space and time as we perceive them may be emergent properties, and how to include gravity in discrete space-time.

6. Acknowledgements

The authors thank Jean Paul Bendegem on discussions about his approach to calculating distances in discrete space.

7. References

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