Extending free group action on surfaces

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Abstract

The present work introduces new perspectives in order to extend finite group actions from surfaces to 3-manifolds. We consider the Schur multiplier associated to a finite group $G$ in terms of principal $G$-bordism in dimension two, called $G$-cobordism. We are interested in settle down the conjecture that every free action of a finite group on a closed oriented surface extends to a non-necessarily free action on a 3-manifold. We show this conjecture is true for abelian, dihedral, symmetric and alternating groups. We also show that this holds for non-necessarily free actions of abelian (under certain conditions) and dihedral groups on a closed oriented surface.

Introduction

The purpose of the present work is to establish enough evidence to settle down the conjecture:

Let $G$ be a finite group with a free action on a closed oriented surface, there exists a 3-manifold $M$ with a non-necessarily free action of $G$, with $\partial M = S$, such that the action over $M$ restricts to the action over $S$.

At present, if the group is abelian or dihedral this conjecture is true and the extension is in terms of a 3-dimensional handlebody, see the work of Reni-Zimmermann [RZ96] and Hidalgo [Hid94]. Moreover, we know that the conjecture follows for any group with trivial Schur multiplier, such as the cyclic groups, groups of deficiency zero, etc, see [Kar87].

The approach of this work is by means of the concept of $G$-cobordisms in dimension two, which are diffeomorphism classes of principal $G$-bundles over surfaces [GS16]. We

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say that a $G$-cobordism is extendable if some representative given by a principal $G$-bundle over a surface is the boundary of a 3-dimensional manifold $M$ with an action of $G$.

For instance, for finite abelian groups the conjecture is straightforward, see Theorem 6. This is because we can decompose any $G$-cobordism in small pieces given by $G$-cobordisms over a closed surface of genus one, and in this case, we show that every $G$-cobordism over a closed surface of genus one is extendable, see Proposition 4. For the dihedral group $D_{2n}$, we focus in the case $n = 2k$ since the Schur multiplier $\mathcal{M}(D_{2n})$ vanishes for $n = 2k + 1$. In the even case $n = 2k$, we obtain that every $D_{2n}$-cobordism can be written as a product of the generator with base space of genus one, which is induced by a reflection and rotation 180 degrees, see Corollary 11. For the symmetric group $S_n$, the Schur multiplier is not trivial for $n > 3$, in that case it is equal to $\mathbb{Z}_2$. In fact, there is a generator with base space of genus one, which is induced by any two disjoint transpositions, see Proposition 13. A similar argument works for the alternating group $A_n$, where for $n = 6, 7$, we need to use the Sylow theory of the Schur multiplier, see Proposition 9.

In the case when we have non necessarily free actions on surfaces, our methods are applicable. The fixed points are classified in two types given by the ones induced by hyperelliptic involutions and pairs of ramification points with complementary monodromies (signature $> 2$). For this actions we give an extension process, where we apply that the free actions of finite groups are extendable. For dihedral groups, we reduce any action to a finite product of an specific generator, then we extend the action for this generator and for the product surface that connects the products. The results of this paragraph are already known in the work of Reni-Zimmermann [RZ96] and Hidalgo [Hid94].

This article is organized as follows: in Section 1 we review the concept of $G$-cobordism and the Schur multiplier, as well as the relationship between them. In Section 2 we give explicit generators for the Schur multiplier of the dihedral, the symmetric and the alternating groups. Finally, in Section 3 we construct the extensions of the free actions on closed oriented surfaces for the dihedral, symmetric and alternating groups. Additionally, for non necessarily free actions on closed oriented surfaces for abelian (under certain conditions) and dihedral groups, we construct the extensions given by solid handlebodies.

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## 1 Preliminaries

In this section we review in detail the definition and properties of the theory of $G$-cobordisms introduced in [GS16, Seg12]. In addition, we discuss some important fact of the Schur multiplier.
1.1 \textit{G}-cobordisms

\textbf{Notation 1.} In all the article $G$ denotes a finite group and $1 \in G$ the neutral element. Also, in this work we consider right actions of the group $G$. In addition, all the surfaces are oriented.

A \textit{principal $G$-bundle} over a topological space $X$, consists of a fiber bundle $E \to X$, with a free and transitive action of $G$ over each fiber. A \textit{$G$-cobordism} between two principal $G$-bundles $P \to S$ and $P' \to S'$, is a principal $G$-bundle $\epsilon : Q \to M$ with diffeomorphisms for the boundaries $\partial M \cong S \sqcup -S'$ and $\partial Q \cong P \sqcup -P'$, which match with the projections and the restriction of the action.

Two \textit{$G$-cobordisms} $\epsilon : Q \to M$ and $\epsilon' : Q' \to M'$ define the same class if $M$ and $M'$ are equivalent as cobordisms by a diffeomorphism $\phi : M \to M'$, $Q$ and $Q'$ are equivalent as cobordisms by a $G$-equivariant diffeomorphism $\psi : Q \to Q'$, and in addition, we have the commutative diagram

\[
\begin{array}{ccc}
Q & \xrightarrow{\psi} & Q' \\
\downarrow{\epsilon} & & \downarrow{\epsilon'} \\
M & \xrightarrow{\phi} & M'.
\end{array}
\]

(1)

In dimension one, for every $g \in G$, we construct the principal $G$-bundle $P_g \to S^1$ obtained by attaching the ends of $[0, 2\pi] \times G$ via multiplication by $g$, i.e., $(0, h)$ is identified with $(2\pi, gh)$ for every $h \in G$. This construction $P_g = [0, 1] \times G / \sim_g$, projects to the circle by restriction to the first coordinate, and the action $P_g \times G \to P_g$ is defined by right multiplication on the second coordinate. Any principal $G$-bundle over the circle is isomorphic to one of these, and $P_g$ is isomorphic to $P_h$ if and only if $h$ is conjugate to $g$. Throughout the paper we refer to the element $g \in G$, as the monodromy associated to the corresponding principal $G$-bundle $P_g$. In the case of the neutral element of the group $G$, we say that we have trivial monodromy.

In dimension two, every \textit{$G$-cobordism} is described by the gluing of principal $G$-bundles over the cylinder, the pair of pants and the disc:

\begin{itemize}
  \item (Cylinder) for $g, h \in G$, every principal $G$-bundle over the cylinder, with entry $g$ and exit $h$, is given by an element $k \in G$ such that $h = kgk^{-1}$.
  \item (Pair of pants) for $g, h \in G$, we consider the principal $G$-bundle over the pair of pants, which is a $G$-deformation retract\footnote{By a $G$-deformation retract we mean that the homotopy is by means of principal $G$-bundles.} of a principal $G$-bundle over the wedge $S^1 \vee S^1$.
  \item (Disk) there is only one $G$-cobordism over the disk which is a trivial bundle.
\end{itemize}
In Figure 1 we have pictures for the G-cobordisms over the cylinder, the pair of pants and the disc. In the article we take the direction for our cobordism from left to right. Also, every circle is labelled with the correspondent monodromy and for every cylinder we write inside the element of the group with which we do the conjugation.

For a closed connected surface $S$ of genus $n$, there is a normal form for any G-cobordism which depends on a finite sequence of pairs $(g_1, k_1)(g_2, k_2) \cdots (g_n, k_n)$, such that $\prod_{i=1}^{n} [g_i, k_i] = \prod_{i=1}^{n} [k_i, g_i] = 1$, see Figure 3. The explanation of why we have this normal form can be seen in [Seg12]. Something helpful is considering a set of generators for the fundamental group of $S$, given by a sequence of pair of curves labeled by the monodromies $(g_1, k_1)(g_2, k_2) \cdots (g_n, k_n)$. For example, in Figure 2 we represent this situation for a genus one handlebody.

A $G$-cobordism over a closed connected surface can be cut along a simple closed separating curve\footnote{A simple closed curve in a surface is separating if the cut surface is not connected.}, producing a monodromy inside the commutator group, as in the following proposition.
Proposition 2. For a $G$-cobordism over a closed connected surface $S$, the monodromy of every embedded simple closed separating curve in $S$ belongs to the commutator group $[G, G]$.

Proof. We consider a two dimensional handlebody of genus $n$ with one boundary circle. Every principal $G$-bundle depends on elements $g_i, k_i \in G$, for $1 \leq i \leq n$, with monodromy for the boundary circle given by the product $\prod_{i=1}^{n} [k_i, g_i]$. For example, for genus one, every principal $G$-bundle depends on a pair of elements $g, k \in G$, with monodromy $[k, g]$ for the boundary circle, see Figure 2. \qed

Definition 3. A $G$-cobordism of dimension two, over a closed surface, is extendable if for some representative principal $G$-bundle $P \to S$, with the action $\alpha : P \times G \to P$, there exits a 3-dimensional manifold $M$ with boundary $\partial M = P$, with an action of $G$ of the form $\overline{\alpha} : M \times G \to M$, which extends $\alpha$, i.e., we have the commutative diagram

$$
P \times G \xrightarrow{\alpha} P \quad \text{and} \quad M \times G \xrightarrow{\pi} M. \quad (3)
$$

Examples of extendable $G$-cobordisms are the ones given by the trivial bundles, since any surface can be filled by a solid 3-handlebody. Another example is a $G$-cobordism over a closed surface of genus $n$, defined by the sequence $(g_1, k_1) \cdots (g_n, k_n)$, where in each pair at least one the elements is the neutral element. In the case where we have a closed surface of genus one we have the following result.

Proposition 4. Any $G$-cobordism over a closed surface of genus one is extendable.

Proof. Consider a principal $G$-bundle representing the $G$-cobordism over the torus. It is enough to prove for the case in which the total space of the bundle is a connected space. Moreover, the action of $G$ over the total space can be modified by an isotopy resulting in an action which depends completely on a pair of monodromies $(g, k)$ associated to two
curves in the torus which intersect once. Denote by $P_g$ and $P_k$ the two principal $G$-bundles associated to these two curves. As a consequence, we obtain that the action of $G$ over the total space over the torus is given by the product of the total spaces $P_g$ and $P_k$. Because of the assumptions, at least one of $P_g$ or $P_k$ is a connected space and, let us assume that it is $P_g$. The extension of the action of $G$ over the torus is through the solid handlebody constructed as follows. First, consider the disc $D$ as the union $(S^1 \times (0,1]) \cup \{0\}$, where 0 is the center. For each circle $S^1 \times \{r\}$, for $r \in (0,1]$ we take as monodromy the element $g$ and the principal $G$-bundle is $P_g$. We take the center 0 as a fixed point. Thus over the disc we have the rotation $2\pi/|g|$, with $|g|$ the order of $g \in G$. Taking the product of this disc together with the induced principal $G$-bundle $P_k$, we have the required extension which makes the $G$-cobordism extendable. It remains to show that this construction is smooth, however, this is straightforward since for any slice given by the disc the action is smooth and taking the product does not change the smoothness.

Remark: 5. We want to emphasize why the construction given in the previous proposition does not work for closed surfaces with genus $> 1$. The reason is that the fixed points should be a smooth submanifold and this it is not possible for genus $> 1$, since we have points where three lines meet.

Now, we apply the previous results for abelian groups.

**Theorem 6.** For $G$ a finite abelian group, any $G$-cobordism is extendable.

**Proof.** Consider a $G$-cobordism over a connected closed surface. By Proposition 2 we can write this $G$-cobordism as a connected sum of $G$-cobordisms with base space of genus one. We remark that this connected sum is done for the total space along trivial bundles over a circle. Since the connected sum is in the same bordism class as the disjoint union, then in fact we are decomposing this $G$-cobordism as a disjoint union of $G$-cobordisms with base space a closed surface of genus one. Because of Proposition 4 we have that any $G$-cobordism over a closed surface of genus one is extendable, so the theorem follows.

1.2 The Schur multiplier

The study of this theory began in 1904 by Isaai Schur in order to study the projective representations of groups. Nowadays, the Schur multiplier is called in three different ways, the second free bordism group $\Omega_2(G)$, the second homology group $H_2(G, \mathbb{Z})$ and the second cohomology group $H^2(G, \mathbb{C}^*)$.

In this work, for a finite group $G$, we denote by $\mathcal{M}(G)$ the Schur multiplier and in accordance with the context, we take the corresponding interpretation in terms of bordism, homology and cohomology.

The connection between homology and cohomology for the Schur multiplier can be found in [Bro82, Kar87]. For a presentation of the group by $G = \langle F \mid R \rangle$, the Schur
multiplier is written using the Hopf’s integral homology formula

\[ \mathcal{M}(G) \cong \frac{R \cap [F, F]}{[F, R]} . \]  

The relationship with the second free bordism group \( \mathcal{M}(G) \) is as follows. Let \( \langle G, G \rangle \) be the free group on all pairs \( \langle x, y \rangle \), with \( x, y \in G \). There is a natural homomorphism of \( \langle G, G \rangle \) onto the commutator group \([G, G]\), which sends \( \langle x, y \rangle \) into \([x, y]\). Consider the kernel \( Z(G) \) of this homomorphism up to the normal subgroup \( B(G) \) of \( \langle G, G \rangle \) generated by the relations

\[ \langle x, x \rangle \sim 1 , \]  
\[ \langle x, y \rangle \sim \langle y, x \rangle^{-1} , \]  
\[ \langle xy, z \rangle \sim \langle y, z \rangle^x \langle x, z \rangle , \]  
\[ \langle y, z \rangle^x \sim \langle x, [y, z] \rangle \langle y, z \rangle , \]

where \( x, y, z \in G \) and \( \langle y, z \rangle^x = \langle y^x, z^x \rangle = \langle xyx^{-1}, xzx^{-1} \rangle \). Miller [Mil52] shows that the quotient \( Z(G)/B(G) \) is canonically isomorphic with the Hopf’s integral formula (4). In fact, in [Mil52] there are some consequent relations, which we enumerate in the following theorem.

**Theorem 7** ([Mil52]). There are the following relations:

\[ \langle x, yz \rangle \sim \langle x, y \rangle \langle x, z \rangle^y , \]  
\[ \langle x, y \rangle^{(a,b)} \sim \langle x, y \rangle^{[a,b]} , \]  
\[ ([x, y], [a, b]) \sim ([x, y], [a, b]) , \]  
\[ \langle b, b' \rangle \langle a_0, b_0 \rangle \sim \langle [b, b'], a_0 \rangle \langle a_0, [b, b'] \rangle \langle b, b' \rangle , \]  
\[ \langle b, b' \rangle \langle a_0, b_0 \rangle \sim \langle [b, b'], a_0 \rangle \langle a_0, [b, b'] \rangle \langle b, b' \rangle , \]  
\[ \langle b, b' \rangle \langle a, a' \rangle \sim \langle [b, b'], [a, a'] \rangle \langle a, a' \rangle \langle b, b' \rangle , \]  
\[ \langle x^n, x^s \rangle \sim 1 \quad n = 0, \pm 1, \cdots ; s = 0, \pm 1, \cdots , \]

for \( x, y, z, a, b, a', b', a_0, b_0 \in G \).

The connection with bordism, relates the elements of \( Z(G) \) by means of the assignment

\[ \langle x_1, y_1 \rangle \langle x_2, y_2 \rangle \cdots \langle x_n, y_n \rangle \longrightarrow (y_n, x_n) (y_{n-1}, x_{n-1}) \cdots (y_1, x_1) , \]

where the sequence in the right defines the generating monodromies for a \( G \)-cobordism over a closed surface of genus \( n \) as in the previous section.

Indeed, the previous four relations ([5], [6], [7] and [8]), have the following geometric interpretation in bordism:
Figure 4: Reduction of genus through the cutting along a trivial monodromy.

(i) For (5) we consider the $G$-cobordism over the cylinder, which goes from the principal $G$-bundle $P_x$ to itself, through conjugation by $x \in G$. We apply the Dehn twist diffeomorphism and obtain that the conjugation becomes the neutral element $1 \in G$. Now consider a $G$-cobordism over a handlebody of genus one which is defined by the pair $(x, x)$. By the previous argument, the $G$-cobordism is the same as the one defined by the pair $(x, 1)$. Notice that this last $G$-cobordism vanishes in bordism since we can cut along the trivial monodromy eliminating the hole of the handle.

(ii) For (6) we consider the $G$-cobordism defined by the pair $(x, y)$, over a handlebody of genus one. Now we take the mapping cylinder of the $G$-equivariant diffeomorphism, which interchanges the generators producing the $G$-cobordism defined by the pair $(y, x)$. This gives a 3-dimensional principal bordism with boundary the $G$-cobordism, over a connected surface of genus two, defined by the two pairs $(x, y)(y, x)$. Consequently, the correspondence [16] implies that $\langle x, y \rangle \langle y, x \rangle \sim 1$.

(iii) For (7) and (8) we obtain $G$-cobordism, over a handlebody of genus two, where we can find a curve with trivial monodromy which reduces the genus by one. In Figure 4, we represent these identifications, respectively.

Now we shall focus on Sylow theory of the Schur multiplier. Specifically, in the following result.

**Theorem 8** ([Kar87]). Let $P$ be a Sylow $p$-subgroup of $G$ and let $\mathcal{M}(G)$ the $p$-component of the Schur multiplier $\mathcal{M}(G)$. Then the restriction map $\text{res} : \mathcal{M}(G) \to \mathcal{M}(P)$ induces an injective homomorphism $\mathcal{M}(G)_p \to \mathcal{M}(P)_p$, and the corestriction map $\text{cor} : \mathcal{M}(P) \to \mathcal{M}(G)$ induces a surjective homomorphism $\mathcal{M}(P) \to \mathcal{M}(G)_p$. 

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For a subgroup $H \subset G$, the corestriction map in bordism is defined by Conner-Floyd \cite{CF64} as follows. Begin with a closed oriented manifold $M$ with an action $M \times H \to M$. We form the product $M \times G$ on which $H$ acts by the rule $(g,x)h = (xh,h^{-1}g)$ which is a free action. We form the quotient $(M \times G)/H$ and denote by $[x,g]$ the point in the quotient corresponding to $(x,g) \in M \times G$. The group $G$ acts on $(M \times G)/H$ by $[x,g]g = [x,g^g]$. The quotient $(M \times G)/H$, with the action of $G$, is the image of the corestriction of $M$ with the action of $H$. We obtain the following important application in our context.

**Proposition 9.** For a finite group $G$, and $\text{Syl}(G)$ the set of isomorphism classes of Sylow subgroups of $G$. If any element $P$ in $\text{Syl}(G)$ satisfies that any $P$-cobordism is extendable, then any $G$-cobordism is extendable.

**Proof.** For $n = |G|$ and $n = p^km$ with $p \nmid m$, consider an element in $f \in \mathcal{M}(G)_p$, hence the composition

$$F := \text{cor} \circ \text{res} : \mathcal{M}(G)_p \to \mathcal{M}(G)_p,$$

is given by $f \mapsto f^m$, which is an automorphism of $\mathcal{M}(G)_p$. By the assumptions, the restriction $\text{res}(f)$ is extendable by a 3-manifold $M$ with an action of $P$, therefore, applying the corestriction we obtain that $f^m$ is extendable by the 3-manifold $\text{cor}(M)$. Similarly, we can start with $F^{-1}(f)$ and we get that $f$ is extendable and the proposition follows. \qed

## 2 Generators for the Schur multiplier

In this section we give explicit generators for the Schur multiplier for the dihedral, the symmetric and the alternating groups.

### 2.1 Dihedral group

For $n \geq 3$, the dihedral group is the symmetries of the $n$-regular polygon (with $D_2 = \mathbb{Z}_2$, $D_4 = \mathbb{Z}_2 \times \mathbb{Z}_2$), and presentation

$$D_{2n} = \langle a,b : a^2 = 1, b^2 = 1, (ab)^n = 1 \rangle,$$

where $c := ab$ is the rotation $2\pi/n$. The Schur multiplier has the form

$$\mathcal{M}(D_{2n}) = \begin{cases} 0 & n = 2k + 1, \\ \mathbb{Z}_2 & n = 2k. \end{cases}$$

In order to find a generator we show the following.

**Proposition 10.** We obtain the following identifications:
(i) $\langle c^i, c^j \rangle \sim 1$,
(ii) $\langle c^i, ac^j \rangle \sim \langle c, a \rangle^i$,
(iii) $\langle ac^i, c^j \rangle \sim \langle c, a \rangle^{-j}$, and
(iv) $\langle ac^i, ac^j \rangle \sim \langle c, a \rangle^{j-i}$.

Proof. The relation (i) follows by (15). The use of (7), (15) and (9) implies
\begin{align}
\langle c^i, ac^j \rangle &\sim \langle c^i, a \rangle \langle c^i, c^j \rangle a \\
\langle c^i, ac^j \rangle &= \langle ac^{i-1}, ac^j \rangle \sim \langle c^{i-1}, ac^j \rangle c \langle c, ac^j \rangle \sim \langle c^{i-1}, a \rangle \langle c, a \rangle.
\end{align}

Therefore, we obtain the relation
\begin{equation}
\langle c^i, a \rangle \sim \langle c, a \rangle \langle c, a \rangle \cdots \langle c, a \rangle,
\end{equation}
which implies (ii). By (6) we obtain (iii) as follows
\begin{equation}
\langle ac^i, c^j \rangle \sim \langle c^j, ac^j \rangle^{-1} \sim \langle c, a \rangle^{-j}.
\end{equation}
Finally, we use (9) and (5),
\begin{equation}
\langle ac^i, ac^j \rangle \sim \langle c^j, ac^j \rangle a \langle a, ac^j \rangle \sim \langle c^{-1}, a \rangle^i \langle a, c^j \rangle a,
\end{equation}
and by (6) we obtain (iv).

Corollary 11. For $n = 2k$, the group $\mathcal{M}(D_{2n})$ has as representative for the generator the element $\langle c^k, a \rangle$.

2.2 Symmetric group

The symmetric group $S_n$ is the permutation of the set $[n] = \{1, \cdots, n\}$, which is generated by the permutations $(ij)$ with $i, j \in [n]$. For symmetric groups $S_n$, the Schur multiplier is
\begin{equation}
\mathcal{M}(S_n) = \left\{ \begin{array}{ll}
0 & n \leq 3, \\
\mathbb{Z}_2 & n \geq 4.
\end{array} \right.
\end{equation}

Lemma 12. Let $k \in [n]$, and $\langle \sigma_1, \tau_1 \rangle \cdots \langle \sigma_r, \tau_r \rangle$ be a sequence with $\sigma_i, \tau_i \in S_n$, for $i \in \{1, \cdots, r\}$. There exist a positive number $0 \leq s \leq r$ and the following elements:
(i) $a_i, b_i \in S_n$, with $0 \leq i \leq s$, such that all $a_i, b_i$ fix $k$, and
(ii) \( c_j, d_j \in S_n \), with \( 0 \leq j \leq r - s \), such that for each \( j \), at least one of \( c_j, d_j \) does not fix \( k \),

with the relation

\[
\langle \sigma_1, \tau_1 \rangle \cdots \langle \sigma_r, \tau_r \rangle \sim \langle a_1, b_1 \rangle \cdots \langle a_s, b_s \rangle \langle c_1, d_1 \rangle \cdots \langle c_{r-s}, c_{r-s} \rangle .
\] (26)

Moreover, \( s \) is the amount of pairs \( \langle \sigma_i, \tau_i \rangle \) such that both \( \sigma_i \) and \( \tau_i \) fix \( k \).

Proof. It suffices to note that for pairs \( \langle a, b \rangle \) and \( \langle x, y \rangle \), with \( a, b, x, y \in S_n \), such that \( a \) and \( b \) fix \( k \) there is the relation

\[
\langle x, y \rangle \langle a, b \rangle \sim \langle a, b \rangle \langle b, a \rangle \langle x, y \rangle \langle a, b \rangle \sim \langle a, b \rangle \langle x^\langle b,a \rangle, y^\langle b,a \rangle \rangle ,
\]

where we have used (10). An iterative application of this process, allows us to put all terms fixing \( k \) to the left in the sequence. \( \square \)

Proposition 13. Assume \( n \geq 4 \), and take elements \( \sigma_i, \tau_i, \sigma'_j, \tau'_j \in S_n \), for \( 1 \leq i \leq r \) and \( 1 \leq j \leq s \), with the same commutator, i.e.,

\[
[\sigma_1, \tau_1] \cdots [\sigma_r, \tau_r] = [\sigma'_1, \tau'_1] \cdots [\sigma'_s, \tau'_s].
\] (27)

Therefore, for the element \( u := \langle (1, 2), (3, 4) \rangle \), there is the relation

\[
\langle \sigma_1, \tau_1 \rangle \cdots \langle \sigma_r, \tau_r \rangle \sim u^k \langle \sigma'_1, \tau'_1 \rangle \cdots \langle \sigma'_s, \tau'_s \rangle ,
\] (28)

with \( k \in \{0, 1\} \).

Proof. First, we observe the following. From (7) and (9), we can assume that the elements \( \sigma_i, \tau_i, \sigma'_j, \tau'_j \in S_n \) are transpositions. In addition, by (8) every pair \( \langle \sigma, \tau \rangle \), with \( \sigma \) and \( \tau \) disjoint transpositions, is in the same class that the pair \( \langle \sigma_1, \tau_1 \rangle \). Therefore, we can reduce to pairs of the form \( \langle (i, j), (j, k) \rangle \), with \( i, j \) and \( k \) different numbers.

By exhaustion, the proposition follows for the symmetric group \( S_n \), with \( 4 \leq n \leq 6 \). We proceed by induction for \( n \geq 7 \) and we suppose that for \( k < n \), the generator of the Schur multiplier \( M(S_n) \) is given by the element \( u := \langle (1, 2), (3, 4) \rangle \). Set by \( m \) the maximum of \( r \) and \( s \), for the sequences \( \langle \sigma_1, \tau_1 \rangle \cdots \langle \sigma_r, \tau_r \rangle \) and \( \langle \sigma'_1, \tau'_1 \rangle \cdots \langle \sigma'_s, \tau'_s \rangle \). For \( m = 1 \), the proposition follows from the triviality of \( M(S_3) \). Suppose that our proposition follows for sequences with length \( l < m \). We consider the sequence

\[
\langle \sigma_1, \tau_1 \rangle \cdots \langle \sigma_r, \tau_r \rangle (\langle \sigma'_1, \tau'_1 \rangle \cdots \langle \sigma'_s, \tau'_s \rangle)^{-1} \sim \langle \sigma_1, \tau_1 \rangle \cdots \langle \sigma_r, \tau_r \rangle \langle \tau'_s, \sigma'_s \rangle \cdots \langle \tau'_1, \sigma'_1 \rangle
\] (29)

which has trivial commutator and length given by \( M := r+s \leq 2m \). Let \( x \in \{1, \cdots, n\} \) be the number that is fixed by the most terms of the sequence (29). Given that the sequences
have non trivial terms, each term permutes 3 different numbers in \( \{1, 2, \ldots, n\} \). Therefore, the number \( x \) is not fixed by at most \( \frac{3(r+s)}{n} \) terms. Given that \( \frac{3(r+s)}{n} \leq \frac{3(2m)}{7} < m \), \( x \) is not fixed by at most \( m - 1 \) terms. By Lemma \([12]\) we can find an equivalent sequence for \([29]\), with the following form

\[
\langle \alpha_1, \beta_1 \rangle \cdots \langle \alpha_t, \beta_t \rangle \langle \alpha_{t+1}, \beta_{t+1} \rangle \cdots \langle \alpha_M, \beta_M \rangle .
\]  

(30)

with \( M - t < m \) and such that, \( \alpha_i, \beta_i \in S_n \), with \( 0 \leq i \leq t \), fix \( x \); and for \( \alpha_j, \beta_j \in S_n \), with \( t + 1 \leq j \leq M \), at least one does not fix \( x \). Moreover, by the proof of Lemma \([12]\) the elements \( \alpha_i, \beta_i, \alpha'_j, \beta'_j \in S_n \) are again transpositions. Now we consider the sequences

\[
A := \langle \alpha_1, \beta_1 \rangle \cdots \langle \alpha_t, \beta_t \rangle
\]  

(31)

and

\[
B := (\langle \alpha_{t+1}, \beta_{t+1} \rangle \cdots \langle \alpha_M, \beta_M \rangle)^{-1} = \langle \beta_M, \alpha_M \rangle \cdots \langle \beta_{t+1}, \alpha_{t+1} \rangle ,
\]  

(32)

where both sequences have the same commutator. Furthermore, the sequence \( A \) has pairs composed by elements in \( S_{n-1} \) because they fix \( x \). By our induction hypothesis, for \( n \), we conclude that the Schur multiplier \( \mathcal{M}(S_{n-1}) \) is generated by \( u = \langle (1, 2), (3, 4) \rangle \). Therefore, \( A \sim u'C \) for \( i \in \{0, 1\} \) and \( C \) is a sequence of pairs with elements in \( S_{n-1} \). We can take \( C \) to be of length \( < m \), as it is has the same commutator as the chain \( B \) of length \( < m \). By the other induction hypothesis, for \( m \), since \( B \) and \( C \) have length less than \( m \), then there is \( j \in \{0, 1\} \) such that \( B \sim u^jC \). This shows that the product of our initial sequences in \([29]\) is equivalent to \( u^{-j} \) and the proof of the proposition follows.

\[\square\]

Corollary 14. For \( n \geq 4 \), the group \( \mathcal{M}(S_n) \) has as representative for the generator the element \( u := \langle (1, 2), (3, 4) \rangle \).

2.3 Alternating group

The alternating group \( A_n \) is the normal subgroup of \( S_n \) with index 2. The Schur multiplier has the form

\[
\mathcal{M}(A_n) = \begin{cases} 
0 & n \leq 3, \\
\mathbb{Z}_2 & n = 4, 5, \\
\mathbb{Z}_6 & n = 6, 7, \\
\mathbb{Z}_2 & n \geq 8.
\end{cases}
\]  

(33)

Proposition 15. For \( n \geq 4 \), the element \( \langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle \) is nontrivial in \( \mathcal{M}(A_n) \).

Proof. Because of the relations \([7]\) and \([9]\) in \( \mathcal{M}(S_n) \), we have the following

\[
\langle (1, 2)(3, 4), (1, 3)(2, 4) \rangle \sim \langle (3, 4), (2, 3)(1, 4) \rangle \langle (1, 2), (1, 3)(2, 4) \rangle
\]

\[
\sim \langle (3, 4), (2, 3) \rangle \langle (2, 4), (1, 4) \rangle \langle (1, 2), (1, 3) \rangle \langle (2, 3), (2, 4) \rangle
\]

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We also have from [8], [9] and \((2, 4), (1, 4)\) = \((2, 3), (1, 3)\)\textsuperscript{3,4} that

\[
\langle(2, 4), (1, 4)\rangle \sim \langle(3, 4), [\langle 2, 3, (1, 3)\rangle] \rangle \langle(3, 4), (1, 3)\rangle = \langle(3, 4), (1, 2)(1, 3)\rangle \langle(2, 3), (1, 3)\rangle \\
\sim \langle(3, 4), (1, 2)\rangle \langle(3, 4), (2, 3)\rangle \langle(2, 3), (1, 3)\rangle = u \langle(3, 4), (2, 3)\rangle \langle(2, 3), (1, 3)\rangle
\]

As \([3, 4], (2, 3)\] = \([2, 3], (2, 4)\], \([2, 3], (1, 3)\] = \([1, 3], (1, 2)\] and \(\mathcal{M}(S_3) = 0\), we have that \(\langle(3, 4), (2, 3)\rangle \sim \langle(2, 3), (2, 4)\rangle\) and \(\langle(2, 3), (1, 3)\rangle \sim \langle(1, 3), (1, 2)\rangle\). Therefore,

\[
\langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle \sim \langle(2, 3), (2, 4)\rangle \langle(2, 4), (1, 4)\rangle \langle(1, 2), (1, 3)\rangle \langle(2, 3), (2, 4)\rangle \\
\sim u \langle(2, 3), (2, 4)\rangle^2 \langle(1, 3), (1, 2)\rangle \langle(1, 2), (1, 3)\rangle \langle(2, 3), (2, 4)\rangle \\
\sim u \langle(2, 3), (2, 4)\rangle^3 \sim u = \langle(1, 2), (3, 4)\rangle,
\]

where \(\langle(2, 3), (2, 4)\rangle^3\) vanishes since \([2, 3], (2, 4)\]^3 = 1 and \(\mathcal{M}(S_3) = 0\). As a consequence, the element \(\langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle\) is nontrivial in \(\mathcal{M}(S_n)\), and hence it is also nontrivial in \(\mathcal{M}(A_n)\) for \(n \geq 4\).

\[\Box\]

**Corollary 16.** For \(n \geq 4\) and \(n \notin \{6, 7\}\), the group \(\mathcal{M}(A_n)\) has as representative for the generator the element \(\langle(1, 2)(3, 4), (1, 3)(2, 4)\rangle\).

## 3 Extending group actions on surfaces

This section contains the main applications of the theory developed before. We start with free actions of abelian, dihedral, symmetric and alternating groups and then, we show that these actions extend to actions on a 3-manifold. Finally, we see the case of non-necessarily free actions for abelian and dihedral groups.

### 3.1 Free actions on surfaces

Consider a compact oriented surface \(S\) with a (free) group action \(\alpha : S \times G \to S\). We say that the action is extendable if there exists a 3-manifold \(M\) with boundary \(\partial M = S\), with an action of \(G\) of the form \(\overline{\alpha} : M \times G \to M\), which extends \(\alpha\), i.e., we have the commutative diagram

\[
\begin{array}{ccc}
S \times G & \xrightarrow{\alpha} & S \\
\downarrow & & \downarrow \\
M \times G & \xrightarrow{\overline{\alpha}} & M
\end{array}
\]

We proved in Theorem 6 that any free action of a finite abelian group is extendable. For dihedral groups \(D_{2n}\), we have two cases to consider. One is for \(n = 2k + 1\), but since \(\mathcal{M}(D_{4k+2}) = 0\), then any free action is extendable. The other is for \(n = 2k\), where by Corollary 11 the generator of the Schur multiplier is represented by a \(G\)-cobordism over a
closed surface of genus one, therefore, by Proposition 4 these free actions are extendable. Now consider free actions of the symmetric groups $S_n$, since $M(S_n) = 0$ for $n \leq 3$, it remains to prove the extension for $n \geq 4$. Similar as for dihedral groups, by Corollary 14 these free actions are extendable. For the alternating groups $A_n$ the free actions are extendable for $n \leq 3$. Again, for $n \geq 4$ and $n \neq 6, 7$, by Corollary 16 these actions are extendable. In the case of free actions of $A_n$ for $n = 6, 7$, we notice that the Sylow subgroups of $A_6$ and $A_7$ have the following isomorphic types $\{D_8, \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_7\}$ and because of Proposition 9 we obtain that these free actions are extendable.

3.2 Non-necessarily free action on surfaces

Now we consider non-necessarily free actions of finite abelian groups and dihedral groups. The extension of these actions was already given by Reni-Zimmermann [RZ96] with 3-dimensional methods and by Hidalgo [Hid94] with 2-dimensional methods.

Consider an action of a finite abelian group $G$ on a connected closed surface $S$. The possible fixed points are of two types: (i) fixed points produced by an hyperelliptic involution, see Figure 5 for examples, and (ii) ramification points with complementary monodromies (signature $> 2$). The extension is performed in some steps. First, we consider the quotient of the surface by the hyperelliptic involutions in some order and smooth the corners in order to have a smooth surface. The hyperelliptic involutions act in the set of ramification points produced by rotations, hence in the quotient we still have ramifications points coupled in pairs. Then, we connect the complementary monodromies by cylinders in order to have a free action over a surface. The hyperelliptic involutions act in the set of ramification points produced by rotations, hence in the quotient we still have ramifications points coupled in pairs. Then, we connect the complementary monodromies by cylinders in order to have a free action over a surface. Now we disconnect the complementary monodromies by slices in discs, where each one has only one fixed point, by the proof of Proposition 4. Finally, we extend the action to the original surface by an unfolding process using the hyperelliptic involutions in the reverse order that we made the quotient surface. This gives the extension of the action of a finite abelian group to a handlebody.

Finally, we consider an action of a dihedral group $D_{2n}$ on a closed surface $S$. By Propo-
sition \[\mathbf{10}\] the extension problem reduces to a finite product of the same \( D_{2n}\)-cobordism induced by the pair \( \langle c, a \rangle \sim \langle ab, a \rangle \sim \langle b, a \rangle^a \). Thus, it is enough to solve the extension problem for the pair \( \langle b, a \rangle \) and for the \( D_{2n}\)-cobordism over the pair of pants with entries in \([D_{2n}, D_{2n}] = \langle c^2 \rangle\). The last reduces to the extension of \( G\)-cobordisms over pair of pants with \( G = [D_{2n}, D_{2n}] \), which is straightforward because the group is cyclic. While for the first, we consider a representative for the \( D_{2n}\)-cobordism induced by \( \langle b, a \rangle \). Indeed, we lift the actions of the elements \( a \) and \( b \) on an \( n\)-gon, where, there are two cases to consider:

(i) For \( n = 2k \), we consider the disjoint union of two spheres, where each one, is the gluing of two \( n\)-gons by the boundary. Denote by \( T \) the operation of switching from one sphere to the other and by \( S \), the operation of switching from one \( n\)-gon to the other in the same sphere. Thus the action of \( a \) lifts to the composition \( Sa = aS \) and the action of \( b \) lifts to the composition \( Tb = bT \). We obtain \( n = 2k \) fixed points over each sphere plus the north a south poles, hence for each fixed point, we remove a small disc. Then we connect the holes for opposite fixed points with a cylinder and we extend the action. For \( k = 2 \), in Figure 6, we draw an illustrator of this construction.

(ii) For \( n = 2k + 1 \), we consider one sphere as the gluing of two \( n\)-gons by the boundary. Denote by \( S \) the operation of switching from one \( n\)-gon to the other. Thus the action of \( a \) lifts to the composition \( Sa = aS \) and the action of \( b \) lifts to the composition \( Sb = bS \). We obtain \( 2n \) fixed points plus the north a south poles. Then for each fixed point we remove a small disc and we connect the holes for opposite fixed points with a cylinder and we extend the action.

In both cases we can extend the action to a handlebody by filling the spheres and we end the case of dihedral groups.

Figure 6: Representative for \( \langle b, a \rangle \) with \( n = 2k \ (k = 1) \).
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