Macroscopic realism and spatiotemporal continuity

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(Dated: February 24, 2010)

Macroscopic realism, as introduced by Leggett and Garg, is the world view in which properties of macroscopic systems exist independent of and are not influenced by measurement. Motivated by classical physical laws such as Newtonian mechanics or Maxwell’s electrodynamics, in this work we add the restrictive postulate that the observables of macroscopic objects are evolved continuously through space and time. Quantum theory violates both macroscopic realism and the continuity assumption. While decoherence or collapse models (e.g. due to a universal noise background or gravitational self energy) can restore macroscopic realism, we show that a continuous spatiotemporal description does not become possible in general. This shines new light on the question how the classical world arises out of the quantum realm.

Classical laws, as they are formulated in mechanics or electrodynamics, (i) give a continuous spatiotemporal evolution of a system’s properties and (ii) are in agreement with the theory-independent concept of macroscopic realism (macrorealism) [1]. For example, the position or angular momentum of a macroscopic body evolve continuously in space and time as governed by classical laws in the form of certain differential equations. These physical properties are macrorealistic, i.e. they can be assumed to exist prior to observation and to be not influenced by them. The predictions of quantum mechanics violate macrorealism. Decoherence is the mechanism by which—through interaction with an environment—the nondiagonal terms of a system’s density matrix are suppressed, turning quantum states into statistical mixtures [2]. We show that, although decoherence can establish macrorealism, it does not necessarily lead to a continuous spatiotemporal evolution of macroscopic variables.

Macrorealism (MR) bases on three postulates [3]:

(1) **Macrorealism per se.** A macroscopic object which has available to it two or more macroscopically distinct states is at any given time in a definite one of those states.

(2) **Non-invasive measurability.** It is possible in principle to determine which of these states the system is in without any effect on the state itself or on the subsequent system dynamics.

(3) **Induction.** The properties of ensembles are determined exclusively by initial conditions (and in particular not by final conditions)."

These assumptions allow to derive the so called Leggett-Garg inequality for temporal correlations, whose violation indicates the non-classicality of a macroscopic object [4]. Notwithstanding recent achievements that could demonstrate quantum interference in large systems [5], the high experimental demands for a demonstration of the violation of macrorealism have not been achieved yet.

One can distinguish various levels of “classicality” for a macroscopic physical system. The most abstract level is the postulation that systems obey the Leggett-Garg inequality and macrorealism (MR). We now introduce the more restrictive notion of classicality as the conjunction of macrorealism and continuity (MR&C). MR&C bases on the three postulates of MR as well as on a fourth one:

(4) **Continuity.** The observables of macroscopic objects evolve continuously through space and time.

Notwithstanding MR&C represents a narrower class of theories than MR, continuity is a natural assumption because even the allegedly discrete and abrupt events in the physical world like the result of a dice toss stem from a continuous evolution of all objects through space and time.

The most restrictive level is classical physics itself where systems obey concrete laws such as Newton’s or Maxwell’s equations. While it is clear that validity of classical physics implies validity of MR&C but not the opposite, we will show that the theory sets obey the strict relation "Classical physics ⊂ MR&C ⊂ MR". This is illustrated in Figure 1 and has interesting consequences for the quantum-to-classical transition, as we will present a situation where due to environmental decoherence (or collapse models [6]) MR is fulfilled but MR&C is not. Since decoherence is extremely hard to avoid, our results should be interpreted as a chance for experiments to demonstrate a certain level of non-classicality despite the action of decoherence.

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*FIG. 1: (Color online.) Illustration of the different levels of classicality: "Macrorealism" (MR) in the Leggett-Garg definition, "Macrorealism and continuity" (MR&C) as put forward in this work, and "Classical physics" with concrete laws of motion.*
In Refs. [7, 8] it was demonstrated that the Leggett-Garg inequality can always be violated under sharp measurements even for arbitrarily large systems, using as an example a quantum spin-\(j\). However, one can speak about a violation of macrorealism only if sufficiently coarse-grained “classical” measurements are employed which distinguish macroscopically distinct states. Under coarse-grained measurements, one can distinguish between two types of time evolutions. For the set of “classical” Hamiltonians, the temporal correlations satisfy macrorealism and even obey classical laws of motion. However, there exist “non-classical” Hamiltonians which allow to violate macrorealism even under coarse-grained measurements. Essentially, they need to build up macroscopic superpositions in time. In this paper, we show that decoherence validates MR. However, whether or not non-classical Hamiltonians lead to a violation of MR&C depends on the concrete model of decoherence. Sufficiently strong thermal (dissipative) environments may establish the validity of MR&C, while even arbitrarily strong dephasing (non-dissipative) environments are not able to accomplish that.

We now introduce the basic mathematical concepts for the further analysis. The (normalized and positive) \(Q\)-distribution of a quantum spin-\(j\) state \(\hat{\rho}\) is given by

\[
Q(\Omega) = \frac{1}{4\pi^{2j+1}} \langle \Omega | \hat{\rho} | \Omega \rangle
\]

where \(\Omega\) the spin coherent states \(|\Omega\rangle\). In a coarse-grained spin measurement, the whole unit sphere is decomposed into a number of mutually disjoint angular regions \(\Omega_k\) where the (polar and azimuthal) angular size of these regions, \(\Delta \Theta\), has to be much larger than the inverse square root of the spin length \(j\): \(\Delta \Theta \gg 1/\sqrt{j}\). A (POVM) coarse-grained measurement with elements \(\hat{P}_k = \frac{2j+1}{4\pi} \sum_{\Omega_k} |\Omega_k\rangle\langle \Omega_k|\) finds out in which of these coarse-grained “slots” the quantum system is \((\sum_k \hat{P}_k = \hat{1})\). The outcome \(k\) is found with probability \(w_k = \text{Tr}[\hat{P}_k \hat{\rho}]\) just via integration over the \(Q\)-distribution, representing a classical ensemble of spins: \(w_k = \int Q(\Omega) d\Omega\). This reflects macrorealism per se.

Upon measurement the state \(\hat{\rho}\) is reduced to \(\hat{\rho}_k = M_k \hat{\rho} M_k^\dagger\) with \(M_k\) the Kraus operators obeying \(M_k^\dagger M_k = \hat{P}_k\). Its corresponding \(Q\)-distribution is \(Q_k = \frac{2j+1}{4\pi} \langle \Omega_k | \hat{\rho}_k | \Omega_k \rangle\). Except near slot borders the \(Q\)-distribution before measurement is the (weighted) mixture of the \(Q\)-distributions of the possible reduced states \(\hat{\rho}_k\):

\[
Q(\Omega) \approx \sum_k w_k Q_k(\Omega)
\]

The time evolution of the system obeys the Leggett-Garg inequality and macrorealism, if the \(Q\)-distribution at any time \(t_f\) without prior measurements is approximately the same as the weighted mixture over all possible outcomes \(k\) of the \(Q\)-distributions that resulted from an intermediate measurement at any time \(t_i\) \((t_i < t_f)\) and then evolved to \(t_f\):

\[
Q(\Omega, t_f) \approx \sum_k w_k Q_k(\Omega, t_f)
\]

This is the necessary and sufficient condition for macrorealism, as it incorporates both macrorealism per se and non-invasive measurability (together with induction). If it is fulfilled, the Leggett-Garg inequality follows [8]. We will use this condition and not the Leggett-Garg inequality to test macrorealism. Under non-classical Hamiltonians we understand those that violate eq. (1), and an example is (in units where \(h = 1\))

\[
\hat{H} = i\omega (|+\rangle\langle -| - |\rangle\langle +|)
\]

Here, \(\omega\) is the precession frequency and \(|+\rangle (-|)\) is the eigenstate of the spin-\(j\) operator’s \(z\)-component with maximal (minimal) eigenvalue, or equivalently, the coherent state pointing to the north (south). (For states of light, nonlinear media may be used to implement similar interactions [11].) Starting from the initial state along north, i.e. \(|\Psi(0)\rangle = |+\rangle\), the Hamiltonian (2) produces an oscillating macroscopic superposition state

\[
|\Psi(t)\rangle = \cos(\omega t) |+\rangle + \sin(\omega t) |-\rangle
\]

The spin effectively behaves as a two-level system, albeit its two states are macroscopically distinct. Given that the spin itself is isolated, coarse-grained measurements or the fact that only measurement apparatuses couple to the environment do not prevent a violation of the Leggett-Garg inequality and macrorealism [8]. But what happens if the system itself is continuously monitored by an environment?

Given the Hamiltonian (2) with the time evolution operator \(\hat{U}_t = \exp(-i\hat{H}t) = \cos(\omega t) (|+\rangle\langle +| + |-\rangle\langle -|) + \sin(\omega t) (|+\rangle\langle -| + |-\rangle\langle +|)\), we approximate the effects of (dephasing) system decoherence by the following simplified model: The initial state along north, \(\hat{\rho}(0) = |+\rangle\langle +|\), freely evolves without decoherence a short time \(\Delta t\) to

\[
\hat{\rho}(\Delta t) = \cos^2(\omega \Delta t) |+\rangle\langle +| + \sin^2(\omega \Delta t) |\rangle\langle \rangle + \text{c.t.}
\]

where the coherence terms “c.t.” are of the form \(|+\rangle\langle -|\) and \(|\rangle\langle +\rangle\). Now we assume that the macroscopic spin system (e.g. say \(j \sim 10^{23}\)) decoheres very rapidly (in the standard pointer basis of \(|+\rangle\) and \(|\rangle\langle +\rangle\)), for instance due to the fact that a single qubit from the environment couples to it in a controlled-not interaction [2], becomes inaccessible immediately afterwards, and does not interact (recohere) with it anymore. If it is impossible to make (joint) measurements on the environmental qubit (and our spin system), the partial trace over the qubit of the total density matrix has to be performed, which kills the coherence terms in eq. (4), leading to the decohered state of the system:

\[
\hat{\rho}(\Delta t) = \cos^2(\omega \Delta t) |+\rangle\langle +| + \sin^2(\omega \Delta t) |\rangle\langle \rangle
\]

Repeating the alternating sequence of free time evolution and rapid decoherence, we obtain the general expression for the (decohered) state at time \(n\Delta t:\)

\[
\hat{\rho}(n\Delta t) = A_n |+\rangle\langle +| + \left(1 - A_{n-1}\right) |\rangle\langle \rangle
\]

The survival probability to find the state along north, \(A_n\), can be retrieved from the recurrence relation \(A_n = c A_{n-1} + (1 - c)(1 - A_{n-1})\) with integer \(n, c \equiv \cos^2(\omega \Delta t)\), and \(A_0 = 1\). If \(\Delta t\) is not too small (to avoid a quantum Zeno-like freezing of the initial state [12] in this model) but smaller than the dynamical timescale of the Hamiltonian, \(\omega^{-1}\), the probability \(A_n\) decays to \(A_{\infty} = \frac{1}{2}\) in a way which can be very well approximated by

\[
A(t) = \frac{1}{2} (1 + e^{-\nu t})
\]
with $\nu \approx 2\sin^2(\omega \Delta t)/\Delta t$ the characteristic decay rate. For times $t \gg \nu^{-1}$ the state asymptotically approaches an equal weight statistical mixture $\hat{\rho}(\infty) = \frac{1}{2}(|+j\rangle\langle +j| + |−j\rangle\langle −j|)$. (In contrast, the probability to find the state along north, i.e. $|+j\rangle$, at time $t$ if no environmental decoherence takes place is given by $\cos^2(\omega t)$, where this characteristic cosine-law allows to violate the Leggett-Garg inequality.)

As we will see below, the results of this simple model are generic for the wide class of exact decoherence models with dephasing environments. The environmental microscopic degrees of freedom drive the system into a mixture, but it does not leave the subspace spanned by $|+j\rangle$ and $|−j\rangle$, and never populates any of the other spin $z$-component eigenstates. The particular decay form of the survival probability is expected to hold very well in all cases where the characteristic decoherence time $\tau_{\text{dec}}$, suppressing off-diagonal elements in a density matrix, is fast compared to the dynamical timescale of the Hamiltonian $\mathcal{H}$, i.e. whenever $\tau_{\text{dec}} \gg \omega^{-1}$.

Let us now investigate what this means for the violation of the Leggett-Garg inequality. We use eq. (5) in its continuous form, i.e. $\hat{\rho}(t) = A(t)|+j\rangle\langle +j| + [1−A(t)]|−j\rangle\langle −j|$, with $A(t)$ the survival probability. If no (coarse-grained) measurement takes place, the spin’s $Q$-distribution at time $t_j$—i.e. the left-hand side of eq. (1)—is given by

$$Q(t_j) = A(t_j)Q_{\text{north}} + [1−A(t_j)]Q_{\text{south}},$$

where $Q_{\text{north}}$ ($Q_{\text{south}}$) is the $Q$-distribution of a spin pointing to the north (south). If a measurement takes place at the intermediate time $t_i$ ($0 < t_i < t_j$), the weighted mixture of the reduced and evolved $Q$-distributions—i.e. the right-hand side of eq. (1)—reads

$$\{A(t_i)A(t_j−t_i) + [1−A(t_i)][1−A(t_j−t_i)]\}Q_{\text{north}} + \{A(t_i)[1−A(t_j−t_i)] + [1−A(t_i)]A(t_j−t_i)\}Q_{\text{south}}.$$  

Without loss of generality, we can set $A(t) = \frac{1}{2}[1+a(t)]$ with some function $a(t)$ such that $A(t)$ is always between 0 and 1. We then find that the non-invasiveness-condition (1), i.e. the equality of (7) and (8), translates into the condition $a(t_j) = a(t_i)a(t_j−t_i)$. This is fulfilled if and only if $a(t)$ has the form $e^{−\varphi t}$, where the solution with negative $\varphi$ is excluded because $a(t)$ must always be between −1 and +1. This means that the only allowed form of $A(t)$ is eq. (6). Therefore, in all decoherence models producing an exponential decay of the survival probability—and only in those—the system’s time evolution under the Hamiltonian $\mathcal{H}$ fulfills the condition (1) for non-invasive measurability, and consequently macrorealism is satisfied.

However, in the case of non-classical Hamiltonians, dephasing decoherence (and therefore collapse models due to a universal noise background or gravitational self energy which also have only a dephasing effect [2]) cannot account for a continuous spatiotemporal description of the macroscopic spin variables. To see this, it is enough to use coarse-grained measurements corresponding to only three different angular regions, one covering the northern part, one the equatorial region, and one the southern part. According to eq. (5), the initial spin along north can be found pointing to the south at some later time, although it did not go through the equatorial region. No classical Hamilton function can achieve such discontinuous “jumps” of a spin vector.

The key point here is that in classical physics as well as in MR&C we have differential equations for observable quantities such as spin directions. Under all circumstances, these equations evolve the observables continuously through real space and time. In quantum mechanics, however, the situation is very different. The Schrödinger equation evolves the state vector continuously through Hilbert space and, under non-classical Hamiltonians, one cannot give a continuous spatiotemporal description of the coarse-grained (macroscopic) observables, even if macrorealism itself is valid.

In the last part, we will investigate two decoherence models in detail, using numerical solutions of the Lindblad master equation [13, 14]:

$$\frac{d}{dt}\hat{\rho}(t) = -i[\hat{H},\hat{\rho}] − \frac{1}{2} \sum_k \{[\hat{L}_k\hat{\rho}\hat{L}_k^\dagger] + [\hat{L}_k^\dagger\hat{\rho}\hat{L}_k]\},$$

with the Lindblad operators $\hat{L}_k$. We assume the spin-$j$ system

![FIG. 2: (Color online.) Snapshots of the probability distribution $P$ for the magnetization along $z$ direction. (a) Dephasing and (b) Thermal environment.](image-url)
of the (dimensionless) magnetization along $z$

\[ \hat{\mathcal{H}} = \frac{\hbar}{2} \omega (\hat{\sigma}_1^x \hat{\sigma}_2^x \ldots \hat{\sigma}_N^x - \hat{\sigma}_1^y \hat{\sigma}_2^y \ldots \hat{\sigma}_N^y), \]

which is, up to a constant factor, the Hamiltonian $\mathcal{H}$. Here, $\hat{\sigma}_k^\pm = \hat{\sigma}_k^x \pm i \hat{\sigma}_k^y$ is the combination of Pauli $x$ and $y$ spin operators.

(i) In the first case, similar to the scenario above, the Lindblad operator corresponds to a local dephasing [14]:

\[ L_{\text{dp}} = \sum_{j=1}^N \gamma_{dp} \hat{\sigma}_j^z \hat{\sigma}_j^z. \]

(ii) In the second case, a thermal environment is modeled by [14]

\[ L_{\text{th}} = \frac{1}{2} \sum_{j=1}^N \gamma_{th} \hat{\sigma}_j^z - \hat{n} \hat{\sigma}_j^+. \]

The coupling parameter for the case of dephasing is set to $\gamma_{dp} = 1$ so that the Lindblad and the Hamiltonian part in the master equation are of the same order. The average number of excitations $\bar{n}$ in the thermal environment is proportional to the temperature. We considered the case when $\bar{n} \gg 1$ and the coupling is such that $\gamma_{th} \bar{n} = 1$. The evolutions for the two cases were computed up to the times that are needed to approximately reach the corresponding stationary distributions.

For the reasons of numerical convenience, the master equation [28] was solved using solutions of an equivalent stochastic nonlinear Schrödinger equation for the quantum trajectories in the system's Hilbert space of pure states. The particular form of the stochastic equation that we have used is the one given by the theory of quantum state diffusion [15], but other forms of equivalent stochastic equations or the direct solutions of [9] would give the same results. The evolution of the expectations for an arbitrary observable is then given by averaging over many stochastic trajectories (10$^3$ in our computation).

The results of the numerical computations are illustrated in the histograms in Figure [2] We took $N = 2j = 10$ spin-$1/2$ particles, initially all with spin along $z$, and computed the values of the (dimensionless) magnetization along $z$, $\hat{m}_z = \frac{1}{2} \sum_{j=1}^N \hat{\sigma}_j^z$.

The possible domain of outcomes $m_z \in [-j, j]$ is divided into intervals and the probability $P$ that, without any intermediate measurement, $m_z$ is in one of the intervals at a particular time is shown. The two columns correspond to snapshots at successive moments in the dimensionless time $\omega t$ with dephasing (left column) and thermal decoherence (right column). Apparently, the evolution of $m_z$ under decoherence by dephasing is discontinuous in the sense that intermediate intervals between north and south are never populated. As in the simple model above, eq. [5], the environment just kills off-diagonal terms in the density matrix. On the other hand, the strong thermal environment actively perturbs the diagonal elements of the matrix and produces a continuous evolution and therefore in principle allows a continuous spatiotemporal description in terms of classical laws of motion.

**Conclusion.**—Figure [8] gives an overview regarding the quantum-to-classical transition. The introduction of the concept macrorealism and continuity (MR&C) in this work is at least of twofold significance: First, it opens up the chance for experiments to demonstrate a certain level of non-classicality despite the action of decoherence. Second, it shows that the collapse models which have been put forward to forbid macroscopic quantum effects [4] are insufficient on their own, as—like dephasing—they only destroy superpositions but would not in general ensure a continuous evolution of macroscopic properties.

This work was supported by the Austrian Science Foundation FWF (SFB, Project No. P19570-N16, and CoQuS), the European Commission through Project QAP (No. 015846), WUS Austria, OAD, and the Serbian Ministry of Science (Contract No. 141003).

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[1] A. J. Leggett and A. Garg, Phys. Rev. Lett. 54, 857 (1985).
[2] W. H. Zurek, Rev. Mod. Phys. 75, 715 (2003); M. Schlosshauer, Rev. Mod. Phys. 76, 1267 (2004).
[3] A. J. Leggett, J. Phys.: Cond. Mat. 14, R415 (2002).
[4] E. Schrödinger, Die Naturwissenschaften 48, 807 (1935).
[5] M. Arndt et al., Nature 401, 680 (1999); J. R. Friedman et al., Nature 406, 43 (2000); B. Julsgaard, A. Kozhekin, and E. S. Polzik, Nature 413, 400 (2001).
[6] G. C. Ghirardi, A. Rimini, and T. Weber, Phys. Rev. D 34, 470 (1986); P. Pearle, Phys. Rev. A 39, 2277 (1989); R. Penrose, Gen. Rel. Grav. 28, 581 (1996).
[7] J. Kofler and Č. Brukner, Phys. Rev. Lett. 99, 180403 (2007).
[8] J. Kofler and Č. Brukner, Phys. Rev. Lett. 101, 090403 (2008).
[9] G. S. Agarwal, Phys. Rev. A 24, 2899 (1981); G. S. Agarwal, Phys. Rev. A 47, 4608 (1993).
[10] J. M. Radcliffe, J. Phys. A: Gen. Phys. 4, 313 (1971); P. W. Atkins and J. C. Dobson, Proc. R. Soc. A 321, 321 (1971).
[11] H. Jeong et al., Phys. Rev. A 70, 061801(R) (2004); H. Jeong, M. Paternostro, and T. C. Ralph, Phys. Rev. Lett. 102, 060403 (2009).
[12] B. Misra and E. C. G. Sudarshan, J. Math. Phys. \textbf{18}, 756 (1977).
[13] G. Lindblad, Commun. Math. Phys. \textbf{48}, 119 (1976).
[14] H.-P. Breuer and F. Petruccione, \textit{The Theory of Open Quantum Systems} (Oxford University Press, Oxford, 2001).
[15] I. C. Percival, \textit{Quantum State Diffusion} (Cambridge University Press, Cambridge, 1999).