Unitary cocycles and processes on the full Fock space

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February 13, 2013

Abstract

We consider a unitary cocycle or Schürmann triple on the non-commutative unitary group fixed by a complex matrix which induces an additive free white noise or an additive free Lévy process on the tensor algebra over the full Fock space. A Lévy process on a Voiculescu dual semi-group is given by a generator or Schürmann triple. We will show how a free Lévy process on the non-commutative unitary group fixed by a complex matrix can be obtained by infinitesimally convolving the additive free white noise.

1 Introduction

From an algebraic point of view, a (stochastic) process is a family \((f_t)_{t \in \mathbb{R}}\) of unital \(*\)-algebra homomorphisms \(f_t : B \to A\) on a so called quantum probability space (QPS). This is a pair \((A, \Phi)\) consisting of a unital \(*\)-algebra \(A\) and a state \(\Phi\), that is a positive linear normed functional \(\Phi : A \to \mathbb{C}\) and playing the role of an expectation. A well known example is the space of all linear adjointable maps of the symmetric or Boson Fock space \(\text{Par92}\) where \(\Phi\) denotes the vacuum expectation. Another example is the space of all linear adjointable maps of the full Fock space \(\Gamma(H)\) over the Hilbert space \(H\) and the state \(\Phi(g) := \langle \Omega, g(\Omega) \rangle\) for a linear adjointable map \(g : \Gamma(H) \to \Gamma(H)\). A thorough introduction to quantum probability can be found in \(\text{Par92, Hol01}\).

From a measure-theoretical point of view, a stochastic process \(X\) is a family of measurable maps \(X_t : E \to G\), where \(E\) is a probability space. Given a stochastic process \(X = (X_t)_{t \in \mathbb{R}}\) we get an algebraic process \(f_t : B \to A\) by \(f_t(\varphi) := \varphi \circ X_t\), where \(B := L^\infty(G)\) and \(A := L^\infty(E)\). Since \(L^\infty(G)\) is a commutative unital algebra we would like to stress that \(B\) may well be a non-commutative algebra, sometimes it carries additional structure making it a quantum group, a Hopf algebra or in
this paper a dual group in the sense of Voiculescu [Voi85, Voi87]. For example, the authors of [LS12] investigate quantum groups and their analytic aspects in a series of papers. Essentially, the additional structure for dual groups in this paper will be an associative convolution $\star$ of unital $\ast$-algebra homomorphisms and a unit element $\delta$ with respect to this convolution.

The algebraic version of factorisation also known as independence, i.e. the joined distribution equals the product of the marginal distributions, for unital $\ast$-algebra homomorphisms $f$ and $g$ is $\Phi \circ (f \sqcup g) = (\Phi \circ f) \circ (\Phi \circ g)$. In contrast to the independence in (classical) probability theory there is more than one way to choose the product $\circ$, see the ‘Muraki five’: tensor, free, boolean, monotone and anti-monotone product [Mur03].

A Lévy process should have independent and stationary increments. So from our algebraic point of view, see also [Sch93, BGS05, Fra06], a Lévy process is a family $(f_{s,t})_{s \leq t}$ of unital $\ast$-algebra homomorphisms $f_{s,t} : B \to (A, \Phi)$ which are called increments, i.e. factorise with respect to the convolution $f_{r,t} = f_{r,s} \star f_{s,t}$ for all $r \leq s \leq t$. The increments of different time periods factorise with respect to $\circ$. The expectation of an increment $f_{s,t}$ is stationary, i.e. only depends on $(t-s)$. Moreover, it is (weakly) continuous, i.e. $\Phi \circ f_{s,t}(b)$ converges to $\delta(b)$ when $t$ tends to $s$. We investigate free Lévy processes on the non-commutative unitary group $K\langle d \rangle$ over the full Fock space. This investigation also yields a family $(U_{r,s})_{r \leq s}$ of unitaries, which is indeed a process with independent increments. In this paper, $K\langle d \rangle$ is only considered for finite $d$. What happens when $d$ is infinite can be found in [SS10].

The notion of a cocycle appears on various occasions [Par92]. In this paper, we give Schürmann’s algebraic definition of a unitary cocycle, also called Schürman triple $(\rho, \eta, \Psi)$. We state the proper definition in Section 4.2 where (ii) is a reformulation of a cocycle condition.

We will infinitesimally convolve the additive free white noise $T(I_{s,t})$, see Theorem 5.4 given by a Schürmann triple $(\rho, \eta, \Psi)$ coming from a complex $d \times d$ matrix $L$ as in Theorem 4.3 with respect to the convolution of the non-commutative unitary group $K\langle d \rangle$, show that it is a convergent net and that the limit forms a $\circ$-free Lévy process on $K\langle d \rangle$ over the full Fock space with generator $\Psi$. Finally, infinitesimally convolving the limit by the additive convolution ensures that this Lévy process is cyclic, that is applying the Lévy process repeatedly on the vacuum vector $\Omega$ yields a dense subspace of the full Fock space. It follows that our Lévy process with generator $\Psi$ over the full Fock space is stochastically equivalent to every $\circ$-free Lévy process with the same generator $\Psi$ on $K\langle d \rangle$ over any QPS.

A general framework accommodating any dual semi-group, the ‘Muraki five’ products and any Schürmann triple where a representation theorem over the respective Fock space is aquired by applying the same method is in preparation by the author.
2 Preliminaries

We will consider associative algebras over \( \mathbb{C} \), the field of complex numbers. A *-algebra is an algebra \( A \) with an involution *, i.e. an anti-linear map \( a \mapsto a^* \) on \( A \) such that \((ab)^* = b^*a^* \) and \((a^*)^* = a \). A unital algebra is an algebra containing an element \( 1 \) (called unit element) in \( A \) with \( a1 = a = 1a \).

A partition \( \alpha \) is a finite subset of a closed interval \([S,T] \subset \mathbb{R}\). It consists of elements \( S = t_1 < \cdots < t_n = T \) and is denoted by \( \{S = t_1 < \cdots < t_n = T\} \). Let \( \mathcal{P}([S,T]) \) be the set of all partitions of \([S,T]\) partially ordered by inclusion. For a family of maps \((k_t)_t\) indexed by one parameter \( t \) we write

\[
\sum_{\alpha} k_{\alpha} := k_{t_2-t_1} + \cdots + k_{t_n-t_{n-1}}.
\]

For a family of maps \((k_{s,t})_{s,t}\) indexed by two parameters \( s \) and \( t \) we write

\[
\sum_{\alpha} k_{\alpha} := k_{t_1,t_2} + \cdots + k_{t_{n-1},t_n}.
\]

We use the same notation for other binary operators as well. A family of elements \((\theta_{\alpha})_{\alpha \in \mathcal{P}([S,T])}\) is a convergence net to \( \theta = \lim_{\alpha \in \mathcal{P}([S,T])} \theta_{\alpha} \), i.e. for all \( \epsilon > 0 \) there exists a partition \( \gamma \in \mathcal{P}([S,T]) \) such that \( \|\theta_{\alpha} - \theta\| < \epsilon \) for all \( \alpha \geq \gamma \). The family \((\theta_{\alpha})_{\alpha \in \mathcal{P}([S,T])}\) is a Cauchy net if for all \( \epsilon > 0 \) there exists a partition \( \gamma \in \mathcal{P}([S,T]) \) such that \( \|\theta_{\alpha} - \theta_{\beta}\| < \epsilon \) for all \( \alpha \geq \gamma, \beta \geq \gamma \). Similar to the theory of sequences, a metric space is complete if and only if all Cauchy nets are convergence nets.

We write \( A \sqcup B \) for the free product of the algebras \( A \) and \( B \), and \( A \sqcup_1 B \) for the free product of unital algebras. Moreover, let \( \iota_A \) (resp. \( \iota_B \)) be the canonical embedding. We often identify \( A \) (resp. \( B \)) with its embedding in \( A \sqcup B \) or \( A \sqcup_1 B \). Either one is the co-product in its respective category, namely the categories of algebras and unital algebras [BGS05, Section 2]. Let \( k_1 \Pi k_2 : A_1 \sqcup A_2 \to B_1 \sqcup B_2 \) be the algebra homomorphism defined by \( k_1 \Pi k_2 := (\iota_1 \circ k_1) \sqcup (\iota_2 \circ k_2) \) and \( k_1 \Pi_1 k_2 \) analogously. In the category of pairs formed by \((B, \varphi)\), where \( B \) is an algebra and \( \varphi : B \to \mathbb{C} \) a linear functional we will use an abstract notion of independence.

**Definition 2.1** The notion of independence is given by a tensor structure of the form

\[
(B_1, \varphi_1) \square (B_2, \varphi_2) = (B_1 \sqcup B_2, \varphi_1 \circ \varphi_2)
\]
in the tensor category with inclusions, cf. [Fra06]. This means that $\odot$ satisfies the following conditions: The map $\varphi_1 \odot \varphi_2$ is a linear functional and

$$
(\varphi_1 \odot \varphi_2) \circ \iota_i = \varphi_i \quad i = 1, 2 \quad \text{(UP1)}
$$

$$
(\varphi_1 \odot \varphi_2) \circ \varphi_3 = \varphi_1 \odot (\varphi_2 \circ \varphi_3) \quad \text{(UP2)}
$$

$$
(\varphi_1 \circ k_1) \odot (\varphi_2 \circ k_2) = (\varphi_1 \odot \varphi_2) \circ (k_1 \amalg k_2) \quad \text{(UP3)}
$$

for linear functionals $\varphi_i : B_i \to \mathbb{C}$ ($i = 1, 2, 3$) algebra homomorphism $k_i : A_i \to B_i$, ($i = 1, 2$) and for all $b_1 \in B_1$, $b_2 \in B_2$.

If we assert

$$
(\varphi_1 \odot \varphi_2)(b_1 b_2) = \varphi_1 \odot \varphi_2(b_2 b_1) = \varphi_1(b_1) \cdot \varphi_2(b_2) \quad \text{(UP4)}
$$

or for $*$-algebras equivalently

$$
\widetilde{\varphi}_1, \widetilde{\varphi}_2 \text{ states } \Rightarrow \widetilde{\varphi}_1 \odot \widetilde{\varphi}_2 \text{ state.} \quad \text{(UP4')}\]

then there are exactly five examples of $\odot$, namely the Muraki five: tensor, boolean, free, (anti-)monotone [Mur03]. A brief overview is available in [BGS05, Section 2]. We will only discuss the free product also known as freeness [NS06]. There are two ways to describe the free product: Let $b_1, \ldots, b_m$ be in $B_1$ or $B_2$ ($\subset B_1 \sqcup B_2$) where consecutive elements are contained in different algebras

$$
\varphi_1 \odot \varphi_2(b_1 \cdots b_m) := \sum_{I \notin \{1, \ldots, m\}} (-1)^{m-\#I+1} \varphi_1 \odot \varphi_2 \left( \prod_{k \in I} b_k \prod_{j \notin I} \varphi_1 \circ \varphi_2(b_j) \right) \quad \text{(1)}
$$

as recursion formula with $\varphi_1 \odot \varphi_2 \left( \prod_{k \in \emptyset} b_k \right) := 1 \in \mathbb{C}$. For normed linear maps $\varphi_1, \varphi_2$ of unital algebras $B_1, B_2$ we have

$$
\varphi_1 \odot \varphi_2(b_1 \cdots b_m) = 0 \quad \text{if } (\varphi_1 \odot \varphi_2)(b_k) = 0 \quad \text{for all } k = 1, \ldots, m. \quad \text{(2)}
$$

For unital algebras together with normed linear maps both (1) and (2) are equivalent. Elaborating on (1), consider the example $\prod_{k \in \{1,5,6,12,19\}} b_k = b_1 b_5 b_6 b_{12} b_{19}$ where the product is in $B_1 \sqcup B_2$ and assume $b_1 \in B_1$. It is important to be aware that $b_1 b_5$ is one element in the next step of the recursion because $b_1 b_5$ is a product within the algebra $B_1$.

Two algebra homomorphisms $f : B_1 \to (A, \Phi)$ and $g : B_2 \to (A, \Phi)$ are $\odot$-independent if $\Phi \circ (f \sqcup g) = (\Phi \circ f) \circ (\Phi \circ g)$.

We will reformulate the definition of a dual group given in the original paper by Voiculescu [Voi85] in the following way:

**Definition 2.2** A dual semigroup $(B, \Delta, \delta)$ consists of a unital $*$-algebra $B$ and unital $*$-algebra homomorphisms $\Delta : B \to B \sqcup_1 B$, $\delta : B \to \mathbb{C}$ such that

$$
(\Delta \amalg_1 id_B) \circ \Delta = (id_B \amalg_1 \Delta) \circ \Delta \quad \text{(3)}
$$

$$
(\delta \amalg_1 id_B) \circ \Delta = id_B = (id_B \amalg_1 \delta) \circ \Delta \quad \text{(4)}
$$

holds. If, in addition, there exists a unital $*$-algebra homomorphism $S' : B \to B$ such that $(S' \amalg_1 id_B) \circ \Delta = \delta = (id_B \amalg_1 S') \circ \Delta$ then $B$ is called a dual group.
Example 2.3 (Tensor algebra) Let $V$ be a $*$-vector space, i.e. there exists an anti-linear map $v \mapsto v^*$ with $(v^*)^* = v$. The tensor algebra is the direct sum $T(V) := C1 \oplus V \oplus V^2 \oplus \cdots$ with multiplication $v \cdot w := v \otimes w$ possessing the universal property: For a unital $*$-algebra $A$ and a linear map $R : T(V) \to A$ there exists a unique unital $*$-algebra homomorphism $T(R) : T(V) \to A$ with $T(R) \upharpoonright V = R$. Furthermore, $T(R)$ is given by $T(R)(1) = 1$ and $T(R)(v_1 \otimes \cdots \otimes v_n) = (R(v_1)) \cdots (R(v_n))$.

The tensor algebra is a dual semigroup with $\Delta := T(f)$ and $\delta := T(0)$, where $f : V \to T(V) \cup_1 T(V)$ with $f(v) = v^{(1)} + v^{(2)}$. The vector $v^{(1)}$ (resp. $v^{(2)}$) is in the left (resp. right) $V$ component of $T(V) \cup_1 T(V)$. Together with the extension of the map $v \mapsto -v$ to $S'$ by the universal property the tensor algebra is a dual group.

Example 2.4 (Non-commutative unitary group) Let $K\langle d \rangle$ be the unital $*$-algebra of polynomials in non-commuting indeterminates $x_{kl}$ and $x_{kl}^*$ for $k,l = 1,\ldots,d$ with complex coefficients and relations
\[
(X \cdot X^* - E)_{k,l}, \quad (X^* \cdot X - E)_{k,l} \quad \forall k,l = 1,\ldots,d
\]
where $X_{k,l} := x_{k,l}$, $X_{k,l}^* := x_{l,k}^*$ and $E$ the $d \times d$ identity matrix and involution $x_{kl} \mapsto x_{kl}^*$. A unital $*$-algebra homomorphism on $\mathbb{C} \langle d \rangle$ (the above without relations) is determined by the values of $x_{kl}$ for $k,l = 1,\ldots,d$.

Define $\Delta : \mathbb{C} \langle d \rangle \to K\langle d \rangle \cup_1 K\langle d \rangle$ and $\delta : \mathbb{C} \langle d \rangle \to \mathbb{C}$ by
\[
\Delta (x_{kl}) = (X^{(1)}X^{(2)})_{k,l}, \quad \delta (x_{kl}) = E_{kl}
\]
for all $k,l$. Since the maps $\Delta$ and $\delta$ respect the relations above, $(K\langle d \rangle, \Delta, \delta)$ is a dual semigroup. Moreover, it is a dual group with $S'(x_{kl}) := x_{l,k}^*$.

A convolution of unital $*$-algebra homomorphisms $f$ and $g$ and a convolution of functionals $\varphi_1$ and $\varphi_2$ is defined by
\[
f \ast g := (f \cup_1 g) \circ \Delta \quad \varphi_1 \ast \varphi_2 := (\varphi_1 \ast \varphi_2) \circ \Delta.
\]

The proof of the following theorem transfers the convolution $\ast$ to the coalgebra convolution in the symmetric tensor algebra $(S(V), \ast)$, where $\ast$ is determined by the choice of $\ast$ and $\Delta$ [BGS05, Theorem 3.4]. For a linear map $\Psi : V \to \mathbb{C}$ we define $D(\Psi) : S(V) \to \mathbb{C}$ by $D(\Psi) \upharpoonright V = \Psi$ and zero elsewhere. Due to the fundamental theorem of coalgebras [DN01], the exponential
\[
\exp_*(tD(\Psi)) := \sum_{n=0}^{\infty} \frac{t^n D(\Psi)^{\ast n}}{n!} = S(0) + tD(\Psi) + \frac{t^2}{2} D(\Psi)^{\ast 2} + \cdots
\]
exists point-wise [Sch90, Section 4].
Theorem 2.5 Let \((k_t)_{t \geq 0}\) be a family of functionals given by
\[
k_t := \delta + tT + T_t \quad \forall t \geq 0
\]
with \(\Psi(1) = 0\), \(T_t\) linear, \(R_t(1) = 0\) and assume that for every \(b \in B\) there exist \(C_b, \epsilon_b \geq 0\) with \(|R_t(b)| \leq t^2 C_b\) for all \(t \leq \epsilon_b\). Then
\[
\lim_{t \to 0} \| T k_t(b) \| = \exp \left( (T - S) \Delta \right) \| b \|.
\]

Proof: Recall that \(S(f \ast g) = S(f) \ast S(g)\) \cite{BGS05} Theorem 3.4. For \(s \in B(B_0)\) the following holds
\[
S(k_s)(s) = (S(0) + tD(\Omega) + N_t)(s) = k_t(s)
\]
for \(N_t\) like \(R_t\). This is proven by induction over \(s \in B(B_0)\). Then
\[
T k_t(b) = S(T k_t(b)) = \exp \left( (T - S) \Delta \right) k_t(b)
\]
converges by \cite{SSV10} Lemma 4.2. See also \cite{SV12} Lemma 3.2 \(\square\).

The full Fock space \(\Gamma(H)\) over a complex Hilbert space \(H\) is the direct sum of all tensor powers \(H^\otimes n\) of \(H\), \(n \geq 0\), where \(H^\otimes 0 := C\). When taking the symmetric tensor powers we obtain the well known Fock space or Boson Fock space \cite{Par92}. Define the vacuum vector \(\Omega := 1, 0, \ldots \in \Gamma(H)\). Let \(D\) be a pre-Hilbert space and let \(L_a(D)\) be the unital \(*\)-algebra of all linear adjointable maps from \(D\) to \(D\).

We define the creation and annihilation operators \(A^*, A : D \to L_a(\Gamma(H))\) and the preservation operator \(\Lambda : L_a(D) \to L_a(\Gamma(H))\) on a dense subspace \(D\) of \(H\) by

- \(A^*(d)\Omega := d\) and \(A^*(d)(d_1 \otimes \cdots \otimes d_n) := d \otimes d_1 \otimes \cdots \otimes d_n\),
- \(A(d)\Omega := 0 \in \Gamma(H)\) and \(A(d)(d_1 \otimes \cdots \otimes d_n) := \langle d, d_1 \rangle (d_2 \otimes \cdots \otimes d_n)\),
- \(\Lambda(T)\Omega := 0 \in \Gamma(H)\) and \(\Lambda(T)(d_1 \otimes \cdots \otimes d_n) := (T(d_1)) \otimes \cdots \otimes d_n\).

The creation and annihilation operator are the adjoints of each other and have norm \(\|A^*(h)\| = \|A(h)\| = \|h\|\). See also \cite{BKS97}.

3 Statement of main lemma

We state the main lemma for unital normed algebras. In order to keep the proof of the main Lemma \[3.3\] simple we extract Lemma \[3.1\].

Lemma 3.1 Let \(A\) be a unital normed algebra and \(0 \leq S < T < \infty\). If there are a constant \(C > 0\), \(S = s_1 < \cdots < s_{n+1} = T\) and \(a_1, \ldots, a_n \in A\) for some \(n \in \mathbb{N}\) such that \(\|a_i\| \leq (s_{i+1} - s_i)C\) for all \(i = 1, \ldots, n\). Then
\[
\left\| \prod_{i=1}^n (1 + a_i) - 1 - \sum_{i=1}^n a_i \right\| \leq \frac{1}{2} (T - S)^2 C^2 e^{C(T - S)}.
\]
**Proof:** In order to reshape the product define $c^{(j)}_1 := 1$, $c^{(j)}_j := a_j$ for $j = 1 \ldots n$. This yields $\prod_{i=1}^n (1 + a_i) = \sum_{k=(k_1, \ldots, k_n) \in \{1,2\}^n} \prod_{j=1}^n c^{(j)}_{k_j}$. Let $D$ be the set 

$$\{1,2\}^n \setminus \{(1, \ldots, 1) \cup \{(1, \ldots, 1, 2_j, 1, \ldots, 1), j = 1 \ldots n\}\}.$$

Then

$$\sum_{k=(k_1, \ldots, k_n) \in \{1,2\}^n} \prod_{j=1}^n c^{(j)}_{k_j} - 1 - \sum_{i=1}^n a_i \sum_{k=(k_1, \ldots, k_n) \in D} \prod_{j=1}^n c^{(j)}_{k_j}$$

since the summand equals 1 for $k = (1, \ldots, 1)$ and $a_j$ for $k = (\ldots, 1, 2_j, 1, \ldots)$. We will now prove the estimation stated above. Define $Z := \sum_{i=1}^n ||a_i||$. The constraints on $a_i$ imply that $Z \leq (T - S)C$. Therefore,

$$\left\| \sum_{k=(k_1, \ldots, k_n) \in \{1,2\}^n} \prod_{j=1}^n c^{(j)}_{k_j} \right\| \leq \sum_{i=2}^n \sum_{1 \leq k_1 < \cdots < k_i \leq n} \prod_{j=1}^i ||a_{k_j}||$$

$$\leq e^Z - Z - 1 = \frac{Z^2}{2} (1 + \frac{2Z}{3!} + \frac{2Z^2}{4!} + \cdots) \leq \frac{Z^2}{2} e^Z$$

$$\leq \frac{C^2}{2} (T - S)^2 e^{(T - S)C}$$

since $(1 + ||a_1||) \cdots (1 + ||a_n||) \leq e||a_1|| \cdots e||a_n|| = e^Z$ and $\frac{2}{(n+2)!} \leq \frac{1}{n!}$ for all $n \geq 0$.

\[ \square \]

We proceed with the main lemma. Therefore, consider a unital normed algebra $A$ and constants $C > 0$, $R < S \in \mathbb{R}_+$. Let $(g_{r,s})_{r,s} \in A$ be a family with

$$g_{r,s} := 1 + a_{r,s} \quad \forall r \leq s \in [R, S]$$

satisfying

$$||a_{r,s}|| \leq (s - r)C \quad (5)$$

$$r < s < t \quad \Rightarrow \quad a_{r,t} = a_{r,s} + a_{s,t}. \quad (6)$$

We observe that

$$||g_{r,s}|| \leq (1 + (s - r)C) \leq e^{(s-r)C}$$

$$||g_{t_1,t_2} \cdots ||g_{t_n,t_{n+1}}|| \leq e^{(t_{n+1}-t_1)C}.$$
Definition 3.2 For $\alpha = \{R = t_1 < \ldots < t_{n+1} = S\} \in \mathbb{P}([R,S])$ we define 
$$\Theta_\alpha := (g_{t_1,t_2}) \cdots (g_{t_n,t_{n+1}}).$$

The norm of $\Theta_\alpha$ is bounded by $e^{(S-R)C}$.

Lemma 3.3 The net $(\Theta_\alpha)_{\alpha \in \mathbb{P}([R,S])}$ is a Cauchy net.

Proof: The lemma holds if for all $\epsilon > 0$ there exists a partition $\gamma \in \mathbb{P}([R,S])$ such that $\|\Theta_\alpha - \Theta_\beta\| < \epsilon$ for all $\alpha \geq \gamma, \beta \geq \gamma$. Let $\alpha, \beta, \gamma \in \mathbb{P}([R,S])$ with $\alpha \geq \gamma, \beta \geq \gamma$. Then
$$\|\Theta_\alpha - \Theta_\beta\| = \|\Theta_\alpha - \Theta_\gamma + \Theta_\gamma - \Theta_\beta\| \\ \leq \|\Theta_\alpha - \Theta_\gamma\| + \|\Theta_\gamma - \Theta_\beta\|.$$ 

Let us denote the elements in the partitions $\alpha$ and $\gamma$ by
$$\gamma = \{R = t_1 < \ldots < t_{n+1} = S\}$$
$$\alpha = \{t_1 = s_1^{(1)} < \ldots < s_m^{(1)} = t_2\}$$
$$\cup \{t_2 = s_1^{(2)} < \ldots < s_m^{(2)} = t_3\}$$
$$\vdots$$
$$\cup \{t_n = s_1^{(n)} < \ldots < s_m^{(n)} = t_{n+1}\}.$$ 

Since $\alpha \geq \gamma$ we can define
$$\alpha_i := \alpha \cap [t_i, t_{i+1}] = \{t_i = s_1^{(i)} < \ldots < s_m^{(i)} = t_{i+1}\}$$
for $i = 1 \ldots n$. In particular, $\alpha_i \in \mathbb{P}([t_i, t_{i+1}])$. Consider the factors of $\Theta_\alpha$ between $t_i = s_1^{(i)}$ and $s_m^{(i)} = t_{i+1}$, i.e.
$$d_i := g_{s_1^{(i)}, s_2^{(i)}} \cdots g_{s_{m-1}^{(i)}, s_m^{(i)}}$$
and the factor $g_{t_i, t_{i+1}}$ of $\Theta_\gamma$, i.e.
$$c_i := g_{t_i, t_{i+1}} 1 + \sum_{k=1}^{n-1} (a_{s_k^{(i)}, s_{k+1}^{(i)}})$$
for $i = 1, \ldots, n$. Then
$$\|\Theta_\gamma - \Theta_\alpha\| \\ = \|c_1 \cdots c_n - d_1 \cdots d_n\| \\ = \left\| \sum_{j=1}^{n} c_1 \cdots c_{j-1} (c_j - d_j) d_{j+1} \cdots d_n \right\| \\ \leq \sum_{j=1}^{n} \|c_1\| \cdots \|c_{j-1}\| \|d_j - c_j\| \|d_{j+1}\| \cdots \|d_n\| \\ \leq \sum_{j=1}^{n} e^{C(t_j-t_1+t_{n+1}-t_{j+1})} \|d_j - c_j\|$$ 

(7)
with telescoping sum $c_1 \cdots c_n - d_1 \cdots d_n = \sum_{j=1}^{n} c_1 \cdots c_{j-1} (c_j - d_j) d_{j+1} \cdots d_n$. Applying Lemma 3.1 to $\|d_j - c_j\|$ yields

$$\|d_j - c_j\| \leq \left( \frac{1}{2} (t_{j+1} - t_j)^2 C^2 e^{C(t_{j+1} - t_j)} \right),$$

where $(1 + a_1) \cdots (1 + a_n) := d_j$ and $1 + a_1 + \cdots + a_n := c_j$. Let $\|\gamma\| := \max \{ t_{i+1} - t_i, i = 1 \ldots n \}$ and continue at formula (7), so

$$(7) \leq \frac{1}{2} C^2 e^{C(S-R)} \sum_{j=1}^{n} (t_{j+1} - t_j) \|\gamma\|$$

$$= \frac{1}{2} C^2 e^{C(S-R)} \|\gamma\| (S - R).$$

Therefore,

$$\|\Theta_\alpha - \Theta_\gamma\| + \|\Theta_\gamma - \Theta_\beta\| \leq \|\gamma\| C^2 e^{C(S-R)} (S - R).$$

This tends to 0 if $\|\gamma\| \to 0$.

It remains to show that for every $\epsilon > 0$ there exists a partition $\gamma$ such that $\|\gamma\| C^2 e^{C(S-R)} (S - R) < \epsilon$. Choosing a sufficiently fine equidistant partition $\gamma$ finishes the proof. \(\square\)

## 4 Applications

In this section, we apply the previous main lemma to examples I, II and III. Example I is a special case of example II for $d = 1$. Both, example II and III originate from the same construction namely convolving a family of algebra homomorphisms with respect to the convolution $\star$ given by the comultiplication $\Delta$ of a dual semigroup. In order to see that the matrices from example III arise from this kind of construction, we consider a family of algebra homomorphisms $f_{r,s}$ together with the primitive comultiplication $\Delta_p$ of 2.3 and denote the kernel of the counit $\delta$ by $B_0$:

$$\left( \bigoplus_{i=1}^{n} f_{t_i, t_{i+1}} \right) (\delta_{k,l} \mathbf{1} + x_{k,l} - \delta_{k,l} \mathbf{1}) = \delta_{k,l} \mathbf{1} + \sum_{i=1}^{n} f_{t_i, t_{i+1}} (x_{k,l} - \delta_{k,l} \mathbf{1})$$

$$= (1 - n) \delta_{k,l} \mathbf{1} + \sum_{i=1}^{n} f_{t_i, t_{i+1}} (x_{k,l}).$$

We invite the reader to compare the last expression of this equation to the entries of the matrices the net in example III consists of. In example II, we consider the dual semigroup $K\langle d \rangle$ from 2.4.
4 APPLICATIONS

4.1 Example I

Let $A = B(\text{Fock}(L_2(\mathbb{R}_+) \otimes \mathbb{C}))$. Define

$$a_{r,s} := A^*(\chi_{[r,s]} \otimes 1) - A(\chi_{[r,s]} \otimes 1) - \frac{1}{2}(r-s)\text{id},$$

where $A^*$ and $A$ denote the creation and the annihilation operator on the full Fock space and $\chi_{[r,s]}$ the characteristic function of the interval $[r, s]$. Since the family $(a_{r,s})_{0 \leq r < s}$ satisfies conditions (5) and (6) Lemma 3.3 applies to the net

$$(\Theta_n)_{\alpha \in P([R,S])} := ((id + a_{t_1,t_2}) \cdots (id + a_{t_n,t_{n+1}}))_{\alpha = \{t_1 < \cdots < t_{n+1}\} \in P([R,S])}.$$

Therefore, this net is a Cauchy net.

4.2 Example II

Consider a dual semigroup $(B, \Delta, \delta)$.

**Definition 4.1** A generator $\Psi$ on $(B, \delta)$ is a linear map $\Psi : B \to \mathbb{C}$ with $\Psi(b^*) = \overline{\Psi(b)}$ for all $b \in B$, $\Psi(1) = 0$ and $\Psi(b^*b) \geq 0$ for all $b \in \ker(\delta)$.

**Definition 4.2 (Schürmann triple)** A Schürmann triple $(\rho, \eta, \Psi)$ of $(B, \delta)$ on a pre-Hilbert space $D$ consists of

(i) a unital $\ast$-algebra homomorphism $\rho : B \to L_a(D)$ ($\ast$-representation)

(ii) a surjective, linear map $\eta : B \to D$ with

$$\eta(ab) = \rho(a)\eta(b) + \eta(a)\delta(b)$$

(iii) a generator $\Psi : B \to \mathbb{C}$ with

$$\langle \eta(a), \eta(b) \rangle_D = \Psi((a - \delta(a)1)^* (b - \delta(b)1)).$$

There exists a construction of a Schürmann triple of $\Psi$ which resembles the GNS-construction [Sch90, Prop 4.1]. Let $B$ be the dual semigroup $K\langle d \rangle$ for $d \geq 1$ from example 2.4. The existence of a generator $\Psi$ on $K\langle d \rangle$ is ensured by [Sch93, Theorem 5.1.12]:

**Theorem 4.3** Let $L$ be a complex $d \times d$-matrix for $d \geq 1$. Then there exists a unique generator $\Psi : K\langle d \rangle \to \mathbb{C}$ and a Schürmann triple $(\rho, \eta, \Psi)$ such that $\Psi(x_{kl}) = \frac{1}{2}(L^*L)_{kl}$, $D = \mathbb{C}$, $\eta(x_{kl}) = L_{kl}$, $\eta(x_{kl}^*) = -L_{kl}$ and $\rho(x_{kl}) = \delta_{kl}$ for all $k, l = 1, \ldots, d$. 
Let $\Psi : B \to \mathbb{C}$ be a generator with Schürmann triple $(\rho, \eta, \Psi)$. Consider the full Fock space $\Gamma$ over $L_2(\mathbb{R}_+, \mathbb{D}) \cong D \otimes L_2(\mathbb{R}_+)$ and define a family of unital $\ast$-algebra homomorphisms $h_{r,s} : B \to L_\alpha(\Gamma)$ by

$$h_{r,s}(x_{k,l}) := \delta_{k,l}id + A_{r,s}(\eta(x_{k,l})) + A_{s,r}(\eta((x_{k,l})^*)) + (s - r)\Psi(x_{k,l})id,$$

where $A_{r,s}(d) := A(\chi_{[r,s]} \otimes d)$ and $A_{s,r}(d) := A(\chi_{[r,s]} \otimes d)$. Let $a_{r,s}$ be the matrix with entries

$$(a_{r,s})_{k,l} := h_{r,s}(x_{k,l} - \delta_{k,l}1).$$

The family $(a_{r,s})_{0 \leq r < s}$ satisfies conditions (5) and (6). Applying Lemma 3.3 implies that the net $(\Theta_\alpha)_{\alpha \in \mathbb{P}([R,S])}$ is a Cauchy net, where $g_{r,s} := id + a_{r,s}$. Furthermore, the elements of the matrix $g_{r,s}$ are

$$(g_{r,s})_{k,l} = h_{r,s}((E + (X - E))_{k,l}) = h_{r,s}(x_{k,l}).$$

For a partition $\alpha = \{t_1 < \cdots < t_{n+1}\} \in \mathbb{P}([R,S])$ we get that

$$(\Theta_\alpha)_{k,l} = (g_{t_1,t_2} \cdots g_{t_n,t_{n+1}})_{k,l}$$

$$= (h_{t_1,t_2} \sqcup_1 \cdots \sqcup_1 h_{t_n,t_{n+1}}) \left( (X^{(1)} \cdots X^{(n)})_{k,l} \right)$$

$$= (h_{t_1,t_2} \sqcup_1 \cdots \sqcup_1 h_{t_n,t_{n+1}}) \circ \Delta_n(x_{k,l})$$

$$= (\sqcup_\alpha \ h_\alpha) \circ \Delta_n(x_{k,l}),$$

where $\Delta_2 := \Delta$ and $\Delta_{n+1} := (\Delta_n \sqcup_1 id) \circ \Delta$. Therefore, the matrix entry $(\Theta_\alpha)_{k,l}$ is the evaluation of a unital $\ast$-algebra homomorphism at $x_{k,l}$.

**Definition 4.4** Let $U_{R,S}$ be the limit of the net $(\Theta_\alpha)_{\alpha \in \mathbb{P}([R,S])}$.

The net $(\Theta_\alpha)_{\alpha \in \mathbb{P}([R,S])}$ and its limit $U_{R,S}$ are $d \times d$ matrices with entries in the algebra $L_\alpha(\Gamma(D \otimes L_2(\mathbb{R}_+)))$.

**Theorem 4.5** The family $(U_{R,S})_{0 \leq R \leq S}$ has the following properties:

- $U_{r,s}$ is a unitary matrix for $0 \leq r \leq s$
- $U_{r,s}U_{s,t} = U_{r,t}$ for $0 \leq r < s < t$
- $\lim_{r<s,t \to r} U_{r,s} = E$ for $0 \leq r$. 
4 APPLICATIONS

Proof: Since $U_{R,S}$ is the norm limit of $(\Theta_\alpha)_{\alpha \in \mathbb{P}([R,S])}$, we have

$$(U_{R,S}^* U_{R,S})_{k,l} = \lim_{\alpha \in \mathbb{P}([R,S]), \|\alpha\| \to 0} (\Theta_\alpha^* \Theta_\alpha)_{k,l}.$$ 

Consider a partition $\alpha = \{t_1 < \cdots < t_{n+1}\} \in \mathbb{P}([R,S])$. Then

$$(\Theta_\alpha^* \Theta_\alpha)_{k,l} = \sum_{p=1}^d (\Theta_\alpha^*)_{k,p} (\Theta_\alpha)_{p,l}$$

$$= \sum_{p=1}^d \left( \bigcup_{\alpha} h_\alpha \right) \circ \Delta_n(x^*_{k,p}) \cdot \left( \bigcup_{\alpha} h_\alpha \right) \circ \Delta_n(x_{p,l})$$

$$= \left( \bigcup_{\alpha} h_\alpha \right) \circ \Delta_n\left( \sum_{p=1}^d x^*_{k,p} x_{p,l} \right) = \delta_{k,l} id$$

which implies $U_{R,S}^* U_{R,S} = E$. Analogously, $U_{R,S} U_{R,S}^* = E$. For $1 \leq k, l \leq d$ it holds:

$$(U_{r,s} \cdot U_{s,t})_{k,l} = \left( \lim_{\alpha_1 = \{r = s_1 < \cdots < s_m = s\}, \alpha_2 = \{s = t_1 < \cdots < t_n = t\}} \Theta_{r,s,\alpha_1} \Theta_{s,t,\alpha_2} \right)_{k,l} = (U_{r,t})_{k,l}.$$ 

When withdrawing the term $- \sum_{i=1}^n a_i$ in Lemma 3.1 modifying the proof yields the norm limit $\lim_{r<s,s \to r} U_{r,s} = E.$ 

The $h_{r,s}$ from [10] consist of freely adapted creation and annihilation operators. Thus, $g_{r,s}$ are freely adapted entry-wise. Therefore, the limit $U_{R,S}$ is freely adapted at $D \otimes L_2 ([R,S])$ entry-wise. In order to present a unitary cocycle equation we can replace $\mathbb{R}_+$ by $\mathbb{R}$ in our consideration, especially for $L_2 (\mathbb{R})$ instead of $L_2 (\mathbb{R}_+)$ and use time shifts $s_r$ in $L_\alpha (\Gamma (D \otimes L_2 (\mathbb{R})))$, where

$$s_r (x_1 \otimes \cdots \otimes x_p) = (s'_r (x_1)) \otimes \cdots \otimes (s'_r (x_p))$$

and $s'_r (x)(t) = x(t + r)$ denote the time shifts in $L_2 (\mathbb{R}, D)$. When defining $W_t := U_{0,t}$ the following unitary cocycle equation holds:

$$W_{s+t} = W_s s_t W_s s_t^*$$

4.3 Example III

Conversely, the family of unitaries $U_{r,s}$ yields a net converging to $g_{R,S}$.

Definition 4.6 For $\alpha = \{R = t_1 < \cdots < t_{n+1} = S\} \in \mathbb{P}([R,S])$ define

$$\Upsilon_\alpha := (1 - n) id + \sum_{j=1}^n U_{t_j,t_{j+1}}.$$ 

Theorem 4.7 The net \((Y_\alpha)_{\alpha \in P([R,S])}\) converges to \(g_{R,S}\).

Proof: Let \(\alpha = \{R = t_1 < \cdots < t_{n+1} = S\} \in P([R,S])\) and \(\epsilon > 0\). By (6)

\[(1 - n)id - g_{R,S} = -\sum_{j=1}^{n} g_{t_j,t_{j+1}}\]

holds. For \(i = 1, \ldots, n\) a partition \(\beta_i \in P([t_i,t_{i+1}])\) with

\[\left\| \prod_{\beta_i} g_{\beta_i} - U_{t_i,t_{i+1}} \right\| < \frac{\epsilon}{n}\]

exists due to the convergence of \((\Theta_{\alpha_i})_{\alpha_i \in P([t_i,t_{i+1}])}\) to \(U_{t_i,t_{i+1}}\) in example II. Lemma 3.1 yields

\[\prod_{\beta_i} g_{\beta_i} - g_{t_i,t_{i+1}} \leq \frac{1}{2} (t_{i+1} - t_i)^2 C^2 e^{C(t_{i+1} - t_i)}\]

Therefore,

\[\left\| (1 - n)id + \sum_{j=1}^{n} U_{t_j,t_{j+1}} - g_{R,S} \right\| = \sum_{j=1}^{n} \left\| U_{t_j,t_{j+1}} \pm \left( \prod_{\beta_j} g_{\beta_j} \right) - g_{t_j,t_{j+1}} \right\| \leq \epsilon + \sum_{j=1}^{n} \left\| \prod_{\beta_j} g_{\beta_j} - g_{t_j,t_{j+1}} \right\| \leq \epsilon + \sum_{j=1}^{n} \frac{1}{2} (t_{j+1} - t_j)^2 C^2 e^{C(t_{j+1} - t_j)} \leq \epsilon + \frac{1}{2} \|\alpha\| (S - R) C^2 e^{C(S-R)}\]

It remains to show that for \(\delta > 0\) there exists a partition \(\alpha\) with

\[\frac{1}{2} \|\alpha\| (S - R) C^2 e^{C(S-R)} < \frac{\delta}{2}\]

Let \(\epsilon < \frac{\delta}{2}\) and choose a sufficiently fine equidistant partition \(\alpha\). This finishes the proof. \(\square\)

5 Application to free Lévy processes

This section connects the previous sections by using the families \((h_{r,s})_{0 \leq r \leq s}\) and \((U_{R,S})_{0 \leq R \leq S}\) from example II and III to construct a free Lévy process over \(K(d)\) on the full Fock space. We define free (quantum) Lévy processes over a dual semigroup on a quantum probability space.
Definition 5.1 (Lévy process) Let \((B, \Delta, \delta)\) be a dual semigroup. A family \((f_{r,s})_{0 \leq r \leq s}\) of unital \(*\)-algebra homomorphisms \(f_{r,s} : (B, \Delta, \delta) \to (A, \Phi)\) with a state \(\Phi : A \to \mathbb{C}\) such that

(i) \(r < s < t \Rightarrow f_{r,t} = f_{r,s} \circ f_{s,t}\) and \(f_{r,s}(b) = \delta(b)id\ \forall b \in B\)

(ii) \(n \geq 1 \in \mathbb{N}, 0 \leq t_1 < \cdots < t_{n+1} \Rightarrow \)

\[\Phi \circ (f_{t_1,t_2} \sqcup \cdots \sqcup f_{t_n,t_{n+1}})(\cdot) = (\Phi \circ f_{t_1,t_2}) \circ \cdots \circ (\Phi \circ f_{t_n,t_{n+1}})(\cdot)\]

(iii) \(r < s \Rightarrow \Phi \circ f_{r,s} = \Phi \circ f_{0,r-s}\)

(iv) \(b \in B \Rightarrow \lim_{t \to 0} \Phi \circ f_{0,t}(b) = \delta(b)\)

is called a free Lévy process.

Remark 5.2 For an arbitrary \(\circ\) from Definition 2.1 and (UP4) the definition is the same.

Theorem 5.3 The family of marginal distributions \(\varphi_t := \Phi \circ f_{0,t}\) for \(t \geq 0\) is a (weakly) continuous convolution \((w.r.t \ \sqcup)\) semigroup of states. The derivation \(\lim_{t \to 0} \frac{1}{t}(\varphi_t(b) - \delta(b)) = \Psi(b)\) exists and defines a generator \(\Psi\).

Proof: See [BGS05, Prop4.4, Theorem4.6].

Theorem 5.4 (additive free white noise) Consider the dual semigroup \(T(V)\) from example 2.3. For a generator \(\Psi : T(V) \to \mathbb{C}\) and a Schürmann triple \((\rho, \eta, \Psi)\) we define a unital \(*\)-algebra homomorphism \(T(I_{s,t}) : T(V) \to L_a(\Gamma)\) by the linear map

\[I_{s,t}(v) := A_{s,t}^* \circ \eta(v) + \Lambda_{s,t}(\rho(v)) + A_{s,t} \circ \eta(v^*) + (t-s)\Psi(v)id \ \forall v \in V.\]

The family \((T(I_{r,s}))_{0 \leq r \leq s}\) is a cyclic Lévy process with respect to freeness and \(\Psi\) is the generator of the marginal distribution. Moreover, it is cyclic for \(\mathbb{C}1 \oplus V\).

Proof: See [GSS92] and [Fra06].

Example 5.5 We now apply Theorem 5.4 to \((K(d), \Delta, \delta)\):
Let \(C := \ker(\delta)\). Then \(B := K(d) \cong \mathbb{C}1 \oplus C\) and \(B \sqcup_1 B \cong \mathbb{C}1 \oplus C \sqcup C\). Since \(\delta(x_{k,l}) = \delta_{k,l}\) the elements \(x_{k,l}^+ - \delta_{k,l}1\) are in the kernel of \(\delta\). We now evaluate \(T(I_{s,t}) : T(C) \to L_a(\Gamma)\) at \(x_{k,l} - \delta_{k,l}1\):

\[I_{s,t}(x_{k,l} - \delta_{k,l}1) := A_{s,t}^* \circ \eta(x_{k,l} - \delta_{k,l}1) + \Lambda_{s,t}(\rho(x_{k,l} - \delta_{k,l}1)) + A_{s,t} \circ \eta((x_{k,l} - \delta_{k,l}1)^*) + (t-s)\Psi(x_{k,l} - \delta_{k,l}1)id\]

\[= A_{s,t}^* \circ \eta(x_{k,l}) + \Lambda_{s,t}(\rho(x_{k,l} - \delta_{k,l}1)) + A_{s,t} \circ \eta((x_{k,l})^*) + (t-s)\Psi(x_{k,l})id.\]
since $\eta(1) = 0 = \Psi(1)$. If $\rho(x_{k,l}) = \delta_{k,l}1$ then
\[
T(I_{r,s})(\delta_{k,l}1 + (x_{k,l} - \delta_{k,l}1)) = h_{r,s}(x_{k,l}).
\]
We conclude this example by stressing that the $h_{r,s}$ from example II are free.

For a generator and a Schürmann triple induced by a complex $d \times d$ matrix $L$ as in Theorem 4.3 we know that $\rho(x_{k,l}) = \delta_{k,l}1$.

**Construction of a Lévy process over $K\langle d \rangle$**

Let $\Psi : K\langle d \rangle :\to \mathbb{C}$ be generator and let $(\rho, \eta, \Psi)$ be a Schürmann triple with
\[
\rho(x_{k,l}) = \delta_{k,l}1 \quad \text{for } k, l = 1, \ldots, d. \tag{13}
\]

Define a family of unital $*$-algebra homomorphisms by
\[
f_{r,s} : \mathbb{C}\langle d \rangle \to (L_a(\Gamma), \langle \Omega, (\cdot)\Omega \rangle)
\]
\[
f_{r,s}(x_{k,l}) = (U_{r,s})_{k,l}
\]
and $f_{r,s}(b) := \delta(b)i$. Due to the unitaries $U_{r,s}$ the $f_{r,s}$ respect the relations of $K\langle d \rangle$ and we get a family of unital $*$-algebra homomorphisms $(f_{r,s})_{r \leq s}$ on $K\langle d \rangle$.

**Lemma 5.6** For all $b \in K\langle d \rangle$
\[
f_{R,S}(b) = \lim_{\alpha \in \mathbb{P}([R,S]) \atop \alpha = \{R = t_1 < \cdots < t_{n+1} = S\}} \left( h_{t_1,t_2} \sqcup \cdots \sqcup h_{t_n,t_{n+1}} \right) \circ \Delta_n(b) \tag{14}
\]
holds.

**Proof**: Since there are only unital $*$-algebra homomorphisms involved, it suffices to prove the claim for monomials $M := x_1^{z_1} \cdots x_m^{z_m}$, where $z_j \in \mathbb{N}$ and $x_j = x_{k_j,l_j}$ or $x_j = x_{k_j,l_j}^*$ for some $0 < k_j, l_j \leq d$. We then use (12) to obtain the following equation:
\[
f_{R,S}(M) = (f_{R,S}(x_1))^{z_1} \cdots (f_{R,S}(x_k))^{z_k} = \prod_{j=1}^{k} (f_{R,S}(x_j))^{z_j}
\]
\[
= \lim_{\alpha \in \mathbb{P}([R,S]) \atop \alpha = \{R = t_1 < \cdots < t_{n+1} = S\}} \prod_{j=1}^{k} \left( \left( h_{t_1,t_2} \sqcup \cdots \sqcup h_{t_n,t_{n+1}} \right) \circ \Delta_n(x_j) \right)^{z_j}
\]
\[
= \lim_{\alpha \in \mathbb{P}([R,S]) \atop \alpha = \{R = t_1 < \cdots < t_{n+1} = S\}} \left( h_{t_1,t_2} \sqcup \cdots \sqcup h_{t_n,t_{n+1}} \right) \circ \Delta_n\left( \prod_{j=1}^{k} x_j^{z_j} \right).
\]

$\square$
Theorem 5.7 The family \((f_{r,s})_{0 \leq r \leq s}\) is a Lévy process with respect to freeness and \(\Psi\) is the generator of the marginal distribution.

Proof:

(i) Evolution: \(f_{r,s} \ast f_{s,t}(x_{k,l}) = f_{r,s} \sqcup f_{s,t}(X^{(1)}X^{(2)})_{k,l}\)

\[
= f_{r,s} \sqcup f_{s,t} \left( \sum_{n=1}^{d} x_{k,n}x_{n,l}^{(2)} \right) = \sum_{n=1}^{d} f_{r,s}(x_{k,n})f_{s,t}(x_{n,l})
\]

\[
= (U_{r,s} \cdot U_{s,t})_{k,l} = (U_{r,t})_{k,l} = f_{r,t}(x_{k,l}).
\]

(ii) Freeness: The continuity of the scalar product \(\langle \cdot, \cdot \rangle\) implies that the net \(\left( \Phi \circ \mathbf{U}_{h_{\alpha}} \right)_{\alpha=1}^{\infty}(a_1 \ldots a_m)\) converges to \(\left( \Phi \circ \mathbf{U}_{f_{t_i,t_{i+1}}} \right)(a_1 \ldots a_m)\) with partition \(\alpha = \alpha_1 \cup \ldots \cup \alpha_n \in \mathcal{P}(I_1 \cup \ldots \cup I_n)\). The free \(\circ\)-product evaluated at a point is a polynomial, see \([1]\). Due to the continuity of polynomials the net \(\left( \bigodot_{i=1}^{n} \Phi \circ h_{\alpha_i} \right)_{\alpha=1}^{\infty}(a_1 \ldots a_m)\) converges to \(\left( \bigodot_{i=1}^{n} \Phi \circ f_{t_{i+1},t_i} \right)(a_1 \ldots a_m)\). Since \(h_{r,s}\) are free we get \(\Phi \circ \mathbf{U}_{h_{\alpha}}(a_1 \ldots a_m) = \left( \bigodot_{i=1}^{n} \Phi \circ h_{\alpha_i} \right)(a_1 \ldots a_m)\).

(iii) Stationarity: Use \([14]\), the freeness and the stationarity of \(h_{r,s}\).

(iv) Weak continuity: Use \([14]\), the freeness, the stationarity and the weak continuity of \(h_{r,s}\), then \((\text{UP3})\), the counit property and \((\text{UP1})\).

Generator: Observe that

\[
\Phi \circ h_{t_i,t_{i+1}} = \delta + (t_{i+1} - t_i) \Psi + R_{t_{i+1},t_i} = k_{t_{i+1},t_i}.
\]

where \(R_t\) and \(k_t\) as in Theorem 2.5. Then use \([14]\) and the freeness of the \(h_i\)’s to get the marginal distribution

\[
\Phi \circ f_{0,t} = \lim_{\text{net}, \alpha} \Phi \circ h_{\alpha} = \lim_{\text{net}, \alpha} \Psi \circ h_{\alpha} = \lim_{\text{net}, \alpha} \Psi k_{\alpha} = \exp_s(tD(\Psi))
\]

and \(\lim_{t \to 0} \frac{1}{t} \left( -\delta + \exp_s(tD(\Psi)) \right)(b) = \lim_{t \to 0} \frac{1}{t} \left( -\delta + \delta + t \Psi + O(t^2) \right)(b) = \Psi(b)\) yields the generator \(\Psi\).

\(\square\)

A Lévy process on the full Fock space is called cyclic if the evaluation of the unital \(*\)-subalgebra generated by \(f_{r,s}(b)\) for all \(0 \leq r \leq s\) and \(b \in B\) at the vacuum vector \(\Omega\) is dense in the full Fock space. From example III, we know that \((1 - n)\delta_{k,l}1 + \sum_{i=1}^{n} f_{t_i,t_{i+1}}(x_{k,l})\) converges to \(h_{R,S}(x_{k,l})\).

\((1 - n)\delta_{k,l}1 = f_{0,1}((1 - n)\delta_{k,l}1), h_{R,S}(x_{k,l}) = T(I_{R,S})(x_{k,l})\) and the \(T(I_{R,S})\) are cyclic over \(\mathbb{C}1 \oplus B_0\), we get that

\[\text{Corollary 5.8} \quad \text{The free Lévy process } f_{r,s} \text{ is cyclic.}\]
By convolving $h_{r,s}$ in example II, we have transferred the Lévy process properties of the additive free white noise $T(I_{s,t})$ over a tensor algebra to unitaries $U_{R,S}$ which yields the free Lévy process $f_{r,s}$ over $K(d)$ with ‘matrix multiplication’ as comultiplication. Conversely, by convolving $f_{r,s}$ with respect to the additive comultiplication as in example III and by using the cyclic property of $T(I_{r,s})$ the Lévy process $f_{r,s}$ is cyclic.

Two stochastic processes are called stochastically equivalent if their distributions coincide. By the stationarity, the marginal distribution determines the distribution of a Lévy process and by Theorem 5.3 the marginal distribution is determined by a generator. We have shown a representation theorem for generators $\Psi : K(d) \to \mathbb{C}$ with $\rho(x_{k,l}) = \delta_{k,l}1$ (13), that is: A $\odot$-free Lévy process over $K(d)$ on any QPS with that generator is stochastically equivalent to the $\odot$-free Lévy process $(f_{r,s})_{r \leq s}$ on the full Fock space from the previous construction.

Outline

The methods established above expand beyond the setting of this paper. A general framework of computing Lévy processes over dual semigroups $(B, \Delta, \delta)$ using additive Lévy processes $T(I_{r,s})$ which also covers the case of $\rho(x_{k,l}) \neq \delta_{k,l}1$ and yields the representation theorem on Fock spaces not only in the case of freeness, is in preparation by the author.

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