An agent based multi-optional model for the diffusion of innovations

Carlos E. Laciana and Nicolás Oteiza-Aguirre

Grupo de Aplicaciones de Modelos de Agentes (GAMA), Facultad de Ingeniería, Universidad de Buenos Aires, Avenida Las Heras 2214 Ciudad Autónoma de Buenos Aires, C1127AAP, Argentina.
clacian@fi.uba.ar

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Abstract

We propose a model for the diffusion of several products competing in a common market based on the generalization of the Ising model of statistical mechanics (Potts model). Using an agent based implementation we analyze two problems: (i) a three options case, i.e. to adopt a product A, a product B, or non-adoption and (ii) a four option case, i.e. the adoption of product A, product B, both, or none. In the first case we analyze a launching strategy for one of the two products, which delays its launching with the objective of competing with improvements. Market shares reached by each product are then estimated at market saturation. Finally, simulations are carried out with varying degrees of social network topology, uncertainty, and population homogeneity.

1 Introduction

The process of diffusion of innovations has motivated ample academic interest in the decade of the sixties, extending until today. A pioneer work has been the one developed by Bass [1]. Originally, this model was fitted, with enough precision, to the observational data of adoption rates corresponding to many consumer durable goods, obtaining the cumulative S-curves for the number of adopters, in which fast growth is generated by word of mouth between early and late adopters [2]. Later, the Bass model was extended in its use to other products, for example those related to the telecommunications sector [3]. Many of the innovation diffusion models were inspired by models initially developed
for application in different disciplines. The Bass model was not the exception; in that case the analogy was made with epidemic models [4].

The Bass model performs an aggregate description of the behavior of potential decision makers in relation with the adoption or not of an innovation (or technology). In this formulation the set of deciders is assumed to be homogeneous and totally connected, or equivalently, we suppose that each individual is influenced by all remaining decision makers. The basic assumption of the model is that at each point in time, potential buyers are exposed to two kinds of influences: an external influence, in the shape of advertising campaigns carried out by the companies via the mass media; and an internal influence coming from "word-of-mouth" interactions with adopters [5].

The Bass model has the general advantage of its simple application, constituting a description on a macro level that shows the observed global behavior [6]. However, the lack of detail at a microscopic level turns it into a weak prediction instrument, restricting its application to analogies with similar known products [3].

On the other hand, the study of diffusion models at micro level has recently intensified, with special focusing on agent based models (ABMs), as is shown, for example, in the review paper by [7]. In general, ABMs involve two complementary causes for innovation adoption: a) the individual evaluation of the advantages related to the adoption of the new product (or technology) and b) the imitation of behavior of close contacts which are considered as examples to follow by the decision maker. ABMs have the advantage, compared to macro models, that the introduction of the social networks is possible. Those social networks are the routes for interaction between the elements of the system [8].

In the case of the diffusion of only one new product in a potential market, individuals in ABMs must choose between two alternatives: to adopt or not to adopt. It was found that the social system described before, can be obtained by using the analogy with the well-known Ising statistic model [9], which was originally developed in the field of physics to describe the phase transition in ferromagnetic materials [10]. In the physical model the agents represent spins in a regular lattice. Those spins can be in two positions: one in the direction of an external field and the other in the opposite direction. In that model, each agent (ion in the metallic lattice) can be influenced by its neighbors, which generate a local field that induce the orientation of the spin. There are examples in scientific literature in which the Ising model is adapted to modeling the social process of opinion formation [11] [12] and others in which the diffusion of technology is studied [13].

In the Ising model, the interaction is limited to the nearest neighbors in a regular array. However, a social process of communication not necessarily corresponds to an interaction with the geographic proximity, so it is necessary to modify the formalism in order to include irregular networks where the notion of nearness is diffuse. Such is the case of the family of networks known as "small world networks" (SWN) described by Watts and Strogatz [14]. The different networks in the family of SWN are obtained by the variation of only one parameter called rewiring probability. The idea is to reconnect the agents, starting
from a regular network with a probability which is the parameter mentioned before. A rewiring probability of zero corresponds to a regular network, and a probability of one to a totally random network. For values of probability in the interval [0.003, 0.02], the SWN for a two dimensional network, are obtained [15]. SWN have been strongly studied in the scientific literature, observing that they have suitable properties for applications in the social communications and other natural phenomena [16].

At this point we have an Ising-like model equipped with a SWN which lets us study the technology diffusion process corresponding to only one product, as was applied in the previous work [13]. However the generalization of this model to more products, whose physical analogy is the Potts model, has been used in few cases of the social field [17]. The idea of the present work is the exploration of the possibilities of the Potts-like formulation for the analysis of the diffusion process associated to the launching of "M" products in a common market.

The case in which two brands compete for a unique market, is analyzed in ref. [18], by using an agent-based cascade model. Following refs. [19] [20], we can identify two mayor categories of agent-based diffusion models: a) threshold models, in which agents adopt when a specified minimum number of neighbors have adopted, and b) cascade models, where the probability of adoption increases with the number of adopters in the neighborhood, with an exponential mathematical dependency.

One contribution of our proposal, considering the two categories given before, is that this formalism could be thought of as a threshold model, but applicable to "M" options, and particularly applied in this paper to systems of three and four options. In the algorithm we propose, adapted from statistical mechanics applications, agents have an implicit threshold that relates to the perceived effective utility of each of the potential choices. This threshold is a function of the differences of utility of the available options, as is shown in ref. [13]. Moreover, this model can also generate a more stochastic decision process, slightly moving away from the “threshold” category, when uncertainty is considered by varying the temperature parameter.

Yet another contribution of this paper lies in the way agents decide between the available choices, which is embedded in the model. That means that the probability of adoption is dynamically modified in time for each agent, depending of the consumption choices of their particular set of neighbors. This methodology is different from, for example, the ones used by [21] [22], where diffusion is described as a two stage process; first, the units adopt, and then, at each point in time, adopters can decide whether to disadopt, or switch, according to a separate “coin-flip”.

The work is organized as follows: section 2 gives the formalism for "M" options, showing also how the Ising-like (M=2) and the studied cases (M=3 and M=4) can be deduced from it. Section 3 describes the variables of the model, such as: rewiring probability, the utility of each product, the spatial and temporal distribution of early adopters, and the temperature as a measure of uncertainty in the decision. Section 4 gives the details about the implementation of the agent based model. Section 5 shows and performs the analysis of the
results obtained. Section 6 summarizes the main conclusions.

2 Formalism for ”M” options (Potts-like)

Let us consider a population of N agents, which are identified by Greek indexes, and M possible states, which are identified by Latin indexes, for all the agents. The set of states $E$ is written as: $E = \{ \vec{X}_1, ..., \vec{X}_M \}$ with $\vec{X}_i \in R^M$. For simplicity we choose a canonical basis, then $E = \{ (1, 0, ..., 0); (0, 1, 0, ..., 0); ...; (0, 0, ..., 1) \}$.

Introducing the scalar product between a pair of states, we have

$$\vec{X}_i \cdot \vec{X}_j = \delta_{ij},$$

where the term on the r.h.s. is the Kröeneker delta.

Because the agent will be in some of the states of the set $E$, it is convenient to define a time-dependent vectorial function, $\vec{S}_\alpha (t)$, where the discrete variable $\alpha$ identifies each agent, then, $\alpha = 1, 2, ..., N$, and $N$ is the total number of agents. For example, if agent $\alpha$ is in the $k$ state, at time $t$, we have

$$\vec{S}_\alpha (t) = \vec{X}_k.$$ (2)

The interpretation of the state $k$ could be the adoption of a given product (we can directly call it product $k$), then the dimension of the space generated by $E$ coincides with the number of options.

Now we introduce the probability to find the agent $\alpha$, at the instant $t$ in the state $k$, indicated as $P (\vec{S}_\alpha (t) = \vec{X}_k)$. We will assume, in order to re-obtain the Ising model when $N = 2$, that the probability to finding agent $\alpha$ in the $k$ state in a given instant, is given by the Boltzmann-Gibbs distribution, i.e.

$$P \left( \vec{S}_\alpha (t) = \vec{X}_k \right) = \frac{e^{\beta \vec{m}_\alpha \cdot \vec{X}_k}}{\sum_{j=1}^{M} e^{\beta \vec{m}_\alpha \cdot \vec{X}_j}},$$ (3)

where, as in the Ising model, $\beta = 1/\kappa T$, $\kappa$ is the Boltzmann constant and $T$ is the temperature. In our calculation we will consider $\kappa = 1$.

We define vector $\vec{m}_\alpha$ by

$$\vec{m}_\alpha (t) = \frac{1}{V_\alpha (t)} \sum_{\gamma=1}^{N} J_{\alpha \gamma} (t) \vec{S}_\gamma (t) + \vec{m},$$ (4)

$V_\alpha (t)$ is the number of contacts of agent $\alpha$ at time $t$. The coefficients $J_{\alpha \gamma} (t)$ allow the connection between agent $\alpha$ and agent $\gamma$. Those coefficients, in the simpler form, are zeros or ones, where 1 indicates the connection between the respective agents exists. In this case, all the connections are equally weighted. The distribution of the zeros and ones in the matrix $J \equiv [J_{\alpha \gamma}]$ describes the social network topology.
Vector $\mathbf{u}$ is defined as the utility vector. It has a dimension equal to the number of states i.e. $M$, being each one of its elements the utility that corresponds to the election of each one of the $M$ products (or states), so

$$
\mathbf{u} = \left( u \left( \mathbf{X}_1 \right), ..., u \left( \mathbf{X}_k \right), ..., u \left( \mathbf{X}_M \right) \right) .
$$

(5)

The scalar product between $\mathbf{u}$ and the state vector $\mathbf{X}_k$ shows the utility corresponding to that state. Mathematically, this can be expressed as

$$
\mathbf{u} \cdot \mathbf{X}_k = u \left( \mathbf{X}_k \right),
$$

(6)

because is $\mathbf{X}_k = (0, ..., 0, 1, 0, ..., 0)$ with the $k^{th}$ component equal to one.

Dividing in the numerator and denominator of Eq. (3) by $\exp \left( \beta \mathbf{m}_\alpha \cdot \mathbf{X}_k \right)$ the expression can be re-written in the following way to simplify its analysis:

$$
P \left( \mathbf{S}_\alpha (t) = \mathbf{X}_k \right) = \frac{1}{1 + \sum_{j \neq k} e^{-\beta \Delta_{kj}}} ,
$$

(7)

where the exponent has the form

$$
\Delta_{kj} \equiv \mathbf{m}_\alpha \cdot \left( \mathbf{X}_k - \mathbf{X}_j \right)
$$

$$
= \frac{1}{V_\alpha (t)} \sum_{\gamma=1}^N J_{\alpha\gamma} (t) \mathbf{S}_\gamma (t) \cdot \left( \mathbf{X}_k - \mathbf{X}_j \right) + u \left( \mathbf{X}_k \right) - u \left( \mathbf{X}_j \right) ,
$$

(8)

The first term on the r.h.s. of Eq. (8) represents the social contribution due to the imitation effect. If our interest is in the adoption dynamics of product $k$, we can see that when $\mathbf{S}_\gamma (t) \cdot \mathbf{X}_j \neq 0$ we obtain a negative contribution in regards to adoption, which is produced by the existence of some non-adopters within the group of contacts of the considered agent. The other term in Eq. (8) is the difference between the utilities of products $k$ and $j$.

In a more intuitive way $\Delta_{kj}$ can be written as

$$
\Delta_{kj} = \nu_k - \nu_j + u \left( \mathbf{X}_k \right) - u \left( \mathbf{X}_j \right) \equiv \Delta \nu_{kj} + \Delta u_{kj} ,
$$

(9)

where $\nu_k$, $\nu_j$ represent the proportions of adopters of $k$ and $j$ respectively within the set of contacts of the agent considered (nearest neighbors in the particular case of the regular lattice). The first subtraction on the r.h.s. is related with the social imitation process and the second one with the personal differences in the evaluation about the relative advantages of the adoption. As we show in the reference [13], when the commercial target has decision makers with different psychological characteristics, a weight factor multiplying each difference individually of Eq. (9) can be added.

As an example we consider the particular case of $T = 0$ ($\beta = \infty$).
From Eq. (7) the following decision algorithm results

\[ P \left( \vec{S}_\alpha = \vec{X}_k \right) = \begin{cases} 
1 & \text{if } \Delta_{kj} > 0 \ \forall \ j \neq k \\
1/2 & \text{if } \exists \ l \ \text{values of } j \ \text{with } \Delta_{kj} = 0 \ \text{and } \Delta_{kj} > 0 \ \text{for all others} \\
0 & \text{if } \Delta_{kj} < 0 \ \text{for any } j \neq k
\end{cases} \]  

(10)

2.1 Ising-like model as a particular case \((M=2)\)

In this case there are only two states, i.e. \(E = \{(1,0), (0,1)\} \equiv \{\vec{X}_1, \vec{X}_2\}\).

Then Eq. (9) becomes:

\[ \Delta_{12} = \nu_1 - \nu_2 + u(\vec{X}_1) - u(\vec{X}_2). \]

In the simplest case of \(T=0\), using Eq. (10) we obtain

\[ P \left( \vec{S}_\alpha = \vec{X}_1 \right) = \begin{cases} 
1 & \text{if } \Delta_{12} > 0 \\
1/2 & \text{if } \Delta_{12} = 0 \\
0 & \text{if } \Delta_{12} < 0
\end{cases} \]  

(11)

The algorithm given by Eq. (11) has been used in ref. [13] for the calculation of the penetration of a given new product in a potential market. The options in that case were only adoption or non-adoption.

2.2 Decision algorithm for the studied cases \((M=3\) and \(M=4)\)

In what follows, we will adapt Eq. (7) to specific examples, determining the state of each agent according to the probability resulting from it.

2.2.1 Case of three options

We will apply the formalism developed before to the case of three options: the adoption of the product 1, the adoption of the product 2 or the non-adoption. We assume that both products are in competition for the same potential market. Then the set of states will be given by

\[ E \{(1,0,0); (0,1,0); (0,0,1)\} \equiv \{\vec{X}_1; \vec{X}_2; \vec{X}_3\}. \]  

(12)

The adoption probability of agent \(\alpha\), for example, in relation with a generic product 1 in the \(t\) instant, with a noise level in the decision given by \(\beta\), is obtained by the simple application of Eq. (7), obtaining the following expression

\[ P \left( \vec{S}_\alpha (t) = \vec{X}_1 \right) = \frac{1}{1 + e^{-\beta \Delta_{12}} + e^{-\beta \Delta_{13}}}. \]  

(13)
2.2.2 Case of four options

We will also consider an application example with four options, in order to illustrate the adaptability of this formalism to more complex problems. The application is completely analogous to the previous case, except for the additional state in the set:

\[ E \{ (1, 0, 0, 0); (0, 1, 0, 0); (0, 0, 1, 0); (0, 0, 0, 1) \} \equiv \{ \overrightarrow{X}_1; \overrightarrow{X}_2; \overrightarrow{X}_3; \overrightarrow{X}_4 \}. \]  

(14)

Where one of the states corresponds to non-adoption.

The probability of agent \( \alpha \) taking, for example, state \( \overrightarrow{X}_1 \), as deduced from Eq. (7), will be:

\[ P(\overrightarrow{S}_\alpha (t) = \overrightarrow{X}_1) = \frac{1}{1 + e^{-\beta \Delta_{12}} + e^{-\beta \Delta_{13}} + e^{-\beta \Delta_{14}}}. \]  

(15)

3 Variables of the model

The microscopic variables introduced in the diffusion model are related essentially with: the way in which the agents are in communication (mainly by means of the kind of social network), the individual evaluation about the adoption of a new product (personal perception), the initial rate of innovators (mainly influenced by campaigns of advertising) and finally the introduction of an additional parameter (temperature) for the quantification of the uncertainty in the decision process.

3.1 Rewiring probability of the SWN

We will use Watts and Strogatz's [14] method to generate a family of networks, from the regular network to the one totally random, passing by the set of networks known as small world networks (SWN). The SWN has similar properties to the social real networks, such as is stressed in ref. [16].

We begin with a regular lattice in two dimensions, with interaction between the eight nearest neighbors for each agent (neighborhood of Moore), then we consider the possibility of re-connection between an agent and another one outside the neighborhood, introducing a re-wiring probability. This methodology lets us study how the topological changes in the social network influence the adoption process.

3.2 Utility of products in competition

We use the concept of utility in a broad sense as the value that the decision-maker gives to the new product. The functional form of this variable will depend on the type of considered article, i.e. by the different attributes that characterize the product in question. For example in the case of a car the attributes could be: price, efficiency, comfort, safety, speed, etc. Each one of those attributes
could be pondered in a different way depending on the individual. Utility is a convenient concept to describe the subjective personal decision. Another example, totally different to the one given before, are financial assets. It is evident in this case that decision is strongly induced by the risk aversion of each individual [23] [24] [25].

The definition of a utility function is in general a complex task in the field of the economy, for that reason this subject is not analyzed in the present work. In our case we introduce a parameter which takes values between 0 and 1 which plays the role of utility. In our approach the utilities of the two products considered will be respect to the utility of non-adoption.

3.3 Spatial and temporal distribution of early adopters

The technology diffusion model proposed has, for a given agent an implicit adoption threshold, which will be related to the number of adopters within its group of contacts (neighbors for a regular net) and to the personal evaluation about the advantages of the adoption. For example, such it is shown in ref. [13], for the case of the introduction of only one new product (M = 2), when the difference of utility is 0.6, in a regular network with an interaction until 8 neighbors, the adoption threshold is of two adopters in the neighborhood, while for a difference of utility of 0.8 it is 1. Then we can say that the change in the distribution of the initial adopters in space modifies the possibilities to reach, for a given agent, the adoption threshold, just as it is shown in ref. [26], [13] and with it the time in that the product reaches the saturation of the potential market.

The rate at which innovators are generated is also important, as it strongly affects the takeoff time. The innovators will be introduced, in the present work, in a linear way until we reach the 2.5 %, in agreement with ref. [2]. That proceeding was introduced in ref. [27].

3.4 Temperature as a measure of uncertainty in the decision

“Temperature” in the social system considered represents the global uncertainty regarding the decision. With temperature different to 0, via Eq. (7), the decision process about the adoption or not of the new products (or technologies) becomes more stochastic. In fact that effect can be interpreted as noise, due to erratic circumstances which have an influence in the opinion of all agents, in the moment of performing the decision [11], [28]. For example, if the agents are agricultural producers deciding whether to adopt of a new seed, the temperature could represent fluctuations due to epidemics, weather variations, political events, etc. [26]. As consequence of those events the decision is not taken with normality.
4 Agent model implementation

Initially we will consider a regular square grid of 200 X 200, i.e. 40,000 agents. We are not assuming periodic conditions in the boundaries.

Agents will be initially connected in a regular lattice, and through Watts and Strogatz’s [14] method of re-wiring we will replicate small world networks. For the implementation of the model we will use the software known as Anylogic [29].

In all the cases the decision algorithm is coming from Eq. (7).

5 Results

By means of numerical experiments we will study how the penetration, in a common market, of products in competition is influenced by the variation of the following microscopic variables: the probability of rewiring $P_r$, the difference of utilities between adopting and not adopting $\Delta u$, the initial rate of innovators $\gamma$ and the temperature $T$ as a measure of the uncertainty in the decision process.

Also we will analyze the effects, in terms of market penetration, of improving a product, but delaying the launch in relation to another product with no improvement, but that it is launched without delay.

Finally, in order to explicitly show the versatility of the proposed formalism, we will make an experiment with four options. In this case, we will consider two not mutually exclusive products (adopting both of them is a possibility) in a common market, and analyze the influence of variables $T$ and $P_r$ in the diffusion curves.

5.1 Experiments with three options

We will consider the introduction of two products in a common potential market. Therefore in our dynamical system we have three kinds of agents; those that adopt product A, those that adopt product B, and those that do not adopt any of both products. We can call the last option 0. In our proposal the dis-adoption is not included, the following transitions are not allowed: $A \rightarrow B$, $B \rightarrow A$, $A \rightarrow 0$ and $B \rightarrow 0$. The allowed transitions are: $0 \rightarrow A$ and $0 \rightarrow B$.

The probability of occurrence of those transitions is given by Eq. (13), with the states given by $X_1$, $X_2$, $X_3$ corresponding to the options A, B and 0 respectively.

5.1.1 Influence of $P_r$ in the adoption pattern

In this numerical experiment we will consider two very different values for the probability of rewiring, one corresponding to the regular network ($P_r = 0$) and another above the superior limit of the SWN ($P_r = 0.02$). We will also suppose that the evaluation that the agents perform about the advantages of adopting one or another product is the same for both products and for all agents
(homogeneous market) i.e. $\Delta u_A = \Delta u_B = 0.6$ for all the agents. We will also suppose that the innovators are introduced to complete the 2.5% of the market with a temporal rate ($\gamma$) of 125 by tick. The results are shown in Fig. 1 a) and b):

This simulation serves firstly as a way of testing the model, since as the considered products which have the same utility and time of launching, they should present the same adoption curve. That in fact occurs as can be observed in Fig. 1 a,b. Secondly, the adoption distribution pattern in the grid at the end of the adoption process, as can be seen in the figure, is more homogeneous in the case of biggest rewiring probability. Finally, we can observe that the saturation time is smaller in the case of biggest rewiring probability. This behavior corresponds with the fact that the path length and the clustering coefficient of the network are smaller for the biggest rewiring probability [14].

5.1.2 Dependence of the adopter proportion with the innovators generation rate

Following the methodology introduced in ref. [27] the innovators are introduced according to Roger [2] until reaching 2.5% of the market. We want to see how the
way of introducing the innovators affects the proportion of the market reached at the end of the process. In order to study that effect we employ a different incorporation rate of innovators for each product. Product A is introduced at a rate of 125 by tick (i.e. $\gamma = 125$), this way 8 ticks are necessary to complete the total number of innovators, while for the product B we considered values of $\gamma$ from 125 until 1000 (where all innovators are all introduced in the first tick). From an operative point of view, this could mean an aggressive advertising campaign of product B before its launching. The results are shown in the Fig. 2.

We can see from Fig. 2 that the influence of the microscopic variable $\gamma$ is decisive in the possibility of obtaining a bigger portion of the market. Product B almost reaches 75% of the market although the only difference between the products A and B would be, for example, a different advertising campaign in the launching which enables product B to begin the competition with a bigger number of innovators. In the studied example both products have the same difference of utility (0.6) in relation to no adoption. We have repeated the numerical experiment with the variation of other quantities such as the temperature and the rewiring probability without observing considerable differences in the final result. We therefore conclude that the most important effect is the one associated to the variable $\gamma$. 

Figure 2: $n_A$ and $n_B$ at saturation, vs the rate of innovators of product B.
5.1.3 Tradeoff between improvement and time of launching

We will consider two products, A and B, competing for the same potential market. A is launched at time \(t = 0\), while B is launched at a posterior time \(t = t_B\), but, during that time difference, B is improved. In reference [30] two practical possibilities for the use of the extra time previous to the launching are presented, one would be to decrease the unitary cost and the other is our case, where the time is employed to improve the product.

Let us call the difference of utility between adopting product A or not \(\Delta u_A\) and the analogous for product B; \(\Delta u_B\). During time \(t_B\) product B is improved and we assume that the utility \(\Delta u_B\) is an increasing function of \(t_B\), with \(\Delta u_B(t_B = 0) = \Delta u_A\) as an initial condition. Then we propose the following expression for the utility:

\[
\Delta u_B = \Delta u_A + (1 - \Delta u_A) \tanh \left( \frac{t_B}{\tau} \right).
\]

Parameter \(\tau\) gauges the difficulty of improving the product, and has units of time. As we see from Eq. (16), when \(t_B \to \infty\) the utility \(\Delta u_B \to 1\) and when \(t_B \to 0\) the utility \(\Delta u_B \to \Delta u_A\). We assume that the utility is normalized such that \(0 \leq \Delta u \leq 1\).

Numeric experiments will be performed considering \(\Delta u_A = 0.6\), \(P_r = [0, 0.02]\) (the second value of rewiring probability corresponds to the maximum value of the SWN) and \(T = [0, 0.05]\) for the temperature. The trials will be separated in two subsections corresponding to the following cases: a) when the population is homogeneous, in relation to the evaluation of the utility, and b) when the population is heterogeneous. We assign the convenient value of 20/3 to \(\tau\).

**Homogeneous population of decision makers** In this case all the decision makers evaluate both products in the same way. Then, for all the agents, we assume that product A has a utility \(\Delta u_A = 0.6\) and that the advantages of product B are increased respect to A according to Eq. (16), starting from the value 0.6. In this sense, the potential market can be considered as a homogeneous population. Under this supposition the graphics of the Fig. 3 were obtained.

In Fig.3a, that corresponds to \(P_r = 0\) and \(T = 0\), we observe a non-realistic pattern. The maximum difference of market proportion between the products is obtained at \(t_B = 1\). At \(t_B = 2\) the difference diminishes, but it increases at \(t_B = 3\) again. That occurs because the adoption threshold changes in \(t_B = 1\) in the same way for all the agents and it is necessary to wait until \(t_B = 3\) to observe a new threshold change. Therefore, the delay in launching between \(t_B = 1\) and \(t_B = 2\) has no positive effect on the diffusion of product B at all, allowing product A to gain extra market share. That occurs because the decision makers, under the homogeneity assumption, act all in the same way; this behavior would be not realistic.

As will be seen in the next subsection, when a heterogeneous population of deciders is considered, the pattern mentioned before is not observed. Another
Figure 3: $n_A$ and $n_B$ at saturation for different launching times of B, for a homogeneous set of decision makers. In (a) $P_r = 0$ and $T = 0$, in (b) $P_r = 0.02$ and $T = 0$, in (c) $P_r = 0$ and $T = 0.05$, and in (d) $P_r = 0.02$ and $T = 0.05$.

way of introducing heterogeneity and thus avoiding this non-realistic pattern, is to give values different than zero to the rewiring probability $P_r$ or to the temperature, as shown in figures 3 b), 3 c) and 3 d).

When rewiring is performed, the average number of connections per agent remains constant, which in our case is of 8 nearest neighbors. However, at an individual level, this number is not maintained. This fact generates heterogeneity in the population, which is manifested through the individual network of contacts. Therefore, when product B is improved, each agent’s answer is unique, contrary to the ‘in block’ response of a homogeneous population of decision makers, i.e. some of them reach the following threshold later than in the regular lattice compensated by others that reach it earlier. The result of that compensation can be seen on the Fig. 3 b), where the curve flattens out after the first tick compared to Fig. 3 a). Another thing that we can observe in Fig. 3 b) comparatively with Fig. 3 a) is that the critical launch time, time limit below which product B obtains most of the market, is reached one tick before than in the case of 3 a).

We emphasized before that the temperature adds uncertainty to the decision, this means that reaching the threshold doesn’t assure that the adoption is produced and not reaching the threshold doesn’t invalidate the possibility of adopting. As a consequence there is a tendency of reducing the differences between the adopter proportion of A and B. As we can see in Fig. 3 c) the graph is more flattened, the maximum difference of proportions is now for $t_B = 2$ and
Figure 4: $n_A$ and $n_B$ at saturation for different launching times of B, for a heterogeneous set of decision makers. In (a) $P_r = 0$ and $T = 0$, in (b) $P_r = 0.02$ and $T = 0$, in (c) $P_r = 0$ and $T = 0.05$, and in (d) $P_r = 0.02$ and $T = 0.05$.

the critical point, where inversion of the population proportions occurs, is closer to $t_B = 3$. Finally, in Fig. 3 d) the combined effect of $P_r \neq 0$ and $T \neq 0$ is observed. The only difference with Fig. 3 c) is that the maximum difference of proportions appears at $t_B = 1$.

**Heterogeneous population of decision makers** As we mentioned before, when a homogeneous population of potential adopters is considered, the patterns obtained are not very realistic. In this subsection we assume that agents have individual perceptions of the utility of product A, i.e. $\Delta u_A$, has the average value of $<\Delta u_A> = 0.6$, but at an individual level there are some differences given by the following distribution: 40% of the deciders with $\Delta u_A = 0.6$, another 40% with $\Delta u_A = 0.7$ and the other 20% with $\Delta u_A = 0.4$. For product B, the distribution is the same than A, when the delay in the launching is not taken into account, and when it is, the distribution is modified in agreement with Eq. [16].

The same numerical experiments that in the homogeneous case were performed and the results are shown in the Fig. 4.

In Fig. 4 a) we can see that the biggest difference between both populations is produced when the delay in the launching of B is of two ticks, and that the population of B is bigger than that of A until the seven ticks are reached. After that, the relationship between populations is reverted.

In the Fig. 4 a) we can observe the existence, at $t_B = 2$, of an absolute
maximum, contrary to what happens with Fig. 3a), where two local maxima exist. This is not modified in the other experiments: in Fig. 4 b) for example, corresponding to $P_r \neq 0$, the maximum difference of populations between both products happens for a delay of two ticks in the launching of B. The only change is the decrease, in approximately a tick, of the time for the population of the adopters of product A to overcome the population of adopters of product B. This would be a critical point beyond which the launching is no longer convenient.

The variable that produces a drastic change is temperature, as can be observed in figures 4 c) and 4 d). The uncertainty in the decision, quantified by the temperature, distorts the relative advantages of a product compared to the other. This way of introducing uncertainty is the simplest. A more realistic form would be to keep in mind each agent’s differences regarding the risk. However, for simplicity, in this analysis we will suppose that all the agents are affected in the same way. In such sense, temperature can be thought of as a global parameter, like inflation or dollar quotation. The process of decision will become then more stochastic than automatic and causal.

In graph 4 c) we see that the improvement of product B does not assure a great advantage for reaching a larger part of the market. The maximum difference is between $t_B=1$ and $t_B=2$, being approximately of only 10%, while in the example of null temperature it was around 50%, as we can see in graphs 4 a) and b). Moreover, when $T \neq 0$ a delay of 3 or more ticks in the launching is an unfavorable strategy.

In Figures 3 and 4 we can see that improving the product has a positive effect in its market penetration until a “critical time”, where the further delay of the improved product would make it give up the market leadership.

By comparing figures 3 a) and 3 b), we can see that increasing the randomness of the network ($P_r = 0 \rightarrow 0.02$) reduces the critical time. This behavior is observed with the same intensity even in the heterogeneous case, as we can see in figures 4 a) and 4 b), in this case the critical time being one unit larger.

Adding uncertainty to the decision through temperature, strongly affects the value of the critical point. Comparing Fig. 3 a) with 3 c) we can observe how this point decreases by three units. The same effect is seen when considering a more realistic social network ($P_r = 0.02$) in figures 3 b) and 3 c). These behaviors, associated to the change of temperature, show us how a more uncertain scenario adds risk to the launching strategy, making the choice of further development disadvantageous.

It should be noted that in an uncertain scenario ($T > 0$) the type of network becomes less important as can be seen in figures 3 c) and 3 d), where the critical time is not altered.

The observed effects in uncertain scenarios becomes more dramatic for a more realistic model with the possibility of a heterogeneous population, as can be seen by comparing figures 4 a) with 4 c), and 4 b) with 4 d). We can also conclude in this case that the effect of changes on the network topology is negligible when uncertainty is considered.
5.2 Experiments with four options

As an example that shows the versatility of the formalism we propose, we will analyze a case in which the decision-makers choose between four options. These options will be of not mutually exclusive durable goods, such that agents can choose product A, product B, both A and B or non-adoption. We could imagine, as a possible example, that agents are household heads that must decide between buying a mid-range car (A), a high-end car (B) or both. If we are speaking of middle class households, the cost of the purchase and of fixed costs would be determining attributes in the decision process.

Our focus will be in analyzing how the network topology (by varying parameter $P_r$) and the uncertainty of the socio-economic scenario (by varying parameter $T$) affect the final pattern of adoption.

As a simplification we will assume that decision-makers evaluate the options in the same way (we could think of average agents). We will then have all potential consumers that consider a difference of utility $\Delta u_{A,0}$ between adopting product A and non-adoption (0), a $\Delta u_{B,0}$ for product B, and a $\Delta u_{AB,0}$ for the adoption of both products. For a more realistic approach, instead of considering average decision-makers, using distributions of utility that take into account social classes should be used.

The value of the utilities in our case will be assigned arbitrarily, as this is simply an academic example, but in a real problem, a fine analysis of the attributes of each product would be required. The only assumption respecting the chosen values is that, for this specific group of households, the high-end product will be of a lower utility than the mid-range, and that the acquisition of both will have the lower utility. This order should be modified if we considered high class decision-makers.

We will also assume that, considering that A and B are durable goods, dis-adoption is not allowed, but as A and B are not mutually exclusive, $A \rightarrow AB$ and $B \rightarrow AB$ are admitted transitions. We will also consider the $0 \rightarrow AB$ transition, that is, the switch from a state of non-adoption to the acquisition of both products simultaneously.

Therefore, in the decision algorithm given by Eq. (15), intervene, through $\Delta_{jk}$, the differences of utility $\Delta u_{A,0}$, $\Delta u_{B,0}$, and $\Delta u_{AB,0}$, associated to transitions $0 \rightarrow A$, $0 \rightarrow B$, and $0 \rightarrow AB$, and $\Delta u_{AB,A}$, and $\Delta u_{AB,B}$ corresponding to $A \rightarrow AB$ y $B \rightarrow AB$ respectively. These latter differences are calculated using the first, since:

$$\Delta u_{AB,A} = u_{AB} - u_A = \Delta u_{AB,0} - \Delta u_{A,0}. \quad (17)$$

$$\Delta u_{AB,B} = u_{AB} - u_B = \Delta u_{AB,0} - \Delta u_{B,0}. \quad (18)$$

In our example, taking into account the considerations previously mentioned of the order of preference of each of the options, we assign $\Delta u_{A,0} = 0.7$, $\Delta u_{B,0} = 0.65$, and $\Delta u_{AB,0} = 0.6$. 

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We will perform four experiments, with $P_r$ values of 0 and 0.02 (regular lattice and small world network) and $T$ values of 0 and 0.05 (deterministic scenario and moderate uncertainty scenario).

The results are shown in Figures 5 a,b,c and d.

By observing figures 5a and 5b we can see that increasing $P_r$ has a positive effect on the diffusion of product A. This seems reasonable, as it would show that a more efficient network increases the imitation effect, and therefore favors the more massive choice, in our case, the mid-range, more accessible car. This, in turn, affects the diffusion of the high-end car (product B). The number of buyers of both products stays virtually unchanged.

When uncertainty is added to the model (Fig. 5 c)), we observe that some of the consumers of product B decide to adopt A as well. This could infer that the increased uncertainty leads decision-makers to a less-rational behavior, making them choose an operation with higher risk involved.

Finally, in Fig. 5 d) we see how an increase in $P_r$ and $T$ simultaneously, produces a compensation of effects in the proportion of adopters.

An important result of these experiments is related to the differences in times of market saturation for each of the scenarios, which can be evaluated approximately by analyzing the non-adoption curves when varying $P_r$ and $T$. We can see that increasing either of them decreases the time of saturation strongly. In the case of the rewiring probability, the more efficient communication between agents favors the imitation effect, and thus the speed of diffusion. In the case of temperature, some transitions that have null probability of occurrence when
\[ T = 0 \] become slightly possible with \( T > 0 \), and this reduces the diffusion time sharply.

6 Conclusions

A formalism has been developed that allows the study of systems formed by individuals that must decide among several options. This formalism is sufficiently versatile for it to include heterogeneous populations of decision makers deciding between many different options. In our work, that methodology is applied to a problem of three options; the adoption of a product A, a product B, or the non-adoption. It is also applied to a four options problem where the possibilities are: adoption of product A, of product B, of both, or no adoption.

From the numerical experiments performed, for three options case, we can extract the following conclusions:

* As the social network becomes more random, the market becomes saturated with product buyers more quickly.

* A quicker generation of innovators, for example through a bigger investment in publicity, modifies the curve of adoption of the product drastically, reaching as a consequence, in the best of cases more than 70% of the market. The final proportion reached, is slightly affected by the uncertainty in the decision (which is introduced through a parameter analogous to the temperature of the statistical system). The final proportion practically does not change with the modification of the topology of the social network (by means of the rewiring probability). However these factors accelerate the arrival to the equilibrium.

We have also performed numerical experiments related with the diffusion of two products in the same potential market, but where one of the products is launched with a delay, and during this time this product is being improved. We analyze then, when the market is saturated, the advantage obtained by the improved product. In relation to these experiments we conclude the following:

* When a homogeneous set of agents is considered, in relation to their evaluation of the comparative advantages of the products, unrealistic results are obtained, due to the emergence of a collective threshold of decision.

* Temperature, or the inverse of the confidence coefficient \([T]\), is the variable that produces the biggest effect when changed. This variable is associated with the noise or uncertainty in the process of decision, which can be related with certain socio-economic scenarios of high volatility. From the results we can see that temperature causes an appreciable reduction of the advantages of product improvement, favoring the one that was launched first. It also reduces the interval of delay in launching \(t_B\) for which an advantage in the won market proportion is obtained at market saturation.
A topic to investigate in the future, by means of the application of the developed methodology, would be for example, the resulting effect of the investment in advertising (via the innovators generation) in scenarios with certain degree of uncertainty and distributions of decision makers with different risk aversion. This is only one of many possible cases of numerical experiments that are facilitated, without big computing efforts, by agent based modeling.

In regards to the four options problem, we observe that:

* With a larger rewiring probability, the adoption of the product with greater utility is favored. In other words, the reduction in the mean characteristic path length of the network increases the imitation effect, and leads to a massification of the most convenient product.

* An increase in uncertainty (temperature greater than 0) increases the probability of adopting the combination of both products.

The patterns we observe using this model show no unexpected behaviors, however, future investigations should compare these results with real processes, for validation purposes.

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