FUNDAMENTAL GROUP OF NON-SINGULAR LOCUS OF LAURICELLA’S $F_C$

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Abstract. In this paper, we give a set of generators and relations of the fundamental group $\pi_1(Y_n)$ of the non-singular locus $Y_n$ of Lauricella’s hypergeometric function $F_C$.

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1. INTRODUCTION AND MOTIVATION

The Lauricella hypergeometric function $F^{(n)}_C$ of $n$ variable defined by

$$F^{(n)}_C(a, b; c_1, \ldots, c_n; z_1, \ldots, z_n) = \sum_{m_1, \ldots, m_n \in \mathbb{Z}_{\geq 0}} \frac{(a, m_1 + \cdots + m_n)(b, m_1 + \cdots + m_n) z_1^{m_1} \cdots z_n^{m_n}}{(c_1, m_1) \cdots (c_n, m_n) m_1! \cdots m_n!},$$

and has the following integral expression ([G]):

$$(\text{const.}) \int \prod_{k=1}^n t_k^{-c_k} \cdot (1 - \sum_{k=1}^n t_k) \sum_{k=1}^n c_k - a - n \cdot (1 - \sum_{k=1}^n \frac{z_k}{t_k})^{-b} dt_1 \cdots dt_n.$$
Using Cayley technique [GKZ], the function $F_C^{(n)}$ is locally holomorphic on $(z_i) \in (\mathbb{C}^\times)^n$ if the toric hypersurface
\[
\{((t_i), \lambda) \in (\mathbb{C}^\times)^{n+1} \mid \lambda(1 - \sum_{k=1}^{n} t_k) + (1 - \sum_{k=1}^{n} \frac{z_i}{t_k}) = 0\}
\]
is non-degenerate for Newton polyhedra. For non-degeneracy condition, see [T].

Since the non-degeneracy condition for a proper Newton polyhedra is equal to the smoothness of the varieties
\[
\{\lambda(1 + \sum_{i \in I} t_i) + \sum_{j \in J} a_j t_j = 0\},
\]
\[
\{\lambda(\sum_{i \in I} t_i) + 1 + \sum_{j \in J} a_j t_j = 0\}
\]
for $I, J \subset \{1, \ldots, n\}$ and $I \cap J = \emptyset$. Therefore the non-degeneracy condition is equivalent to the smoothness to the open face. Using Jacobian criterion, the singular locus is defined by
\[
1 - \sum_{k=1}^{n} t_k = 0, \\
\lambda - \frac{z_i}{t_i} = 0, \\
\lambda(1 - \sum_{k=1}^{n} t_k) + (1 - \sum_{k=1}^{n} \frac{z_i}{t_k}) = 0.
\]

By setting $\mu^2 = \lambda, x_i^2 = z_i$, and using the first and the second equations, $\mu$ is obtained by
\[
t_i \mu = \epsilon_i x_i, \quad \mu - \sum_{i=1}^{n} \epsilon_i x_i = \mu(1 - \sum_{i=1}^{n} t_i) = 0.
\]
Here $\epsilon_i \in \{-1, 1\}$. Again, using the first and the second equations, the third equation is equal to
\[
0 = 1 - \sum_{i=1}^{n} \frac{x_i^2}{t_i} = 1 - \lambda \sum_{i=1}^{n} t_i = 1 - \lambda = (1 + \mu)(1 - \mu)
\]
\[
= (1 + \sum_{i=1}^{n} \epsilon_i x_i)(1 - \sum_{i=1}^{n} \epsilon_i x_i).
\]

Therefore under the $\mu_2^n$-covering map,
\[
\mathbb{C}^n = \{(x_1, \ldots, x_n) \mid (x_i) \mapsto (x_i^2) \in \mathbb{C}^n = \{(z_1, \ldots, z_n)\}\}
\]
the pull back $Y_n$ of $Y_n$ is given by
\[
Y_n = \{(x_i) \mid \prod_{k=1}^{n} x_k \prod_{\epsilon_i \in \{-1, 1\}}(1 - \sum_{i=1}^{n} \epsilon_i x_i) \neq 0\}.
\]

See also [HT]. Therefore $Y_n \subset \{(z_1, \ldots, z_n)\}$ is isomorphic to $Y_n/\mu_2^n$.

In the study of monodromy of hypergeometric function of type $F_C$, it is a basic problem to give an expression of the fundamental group of $Y_n$. The
The fundamental group of the non-singular locus of Lauricella’s $F_C$ generator and relations of the fundamental group for $n = 2$ and $n = 3$ is determined in [GK]. We prove the following presentation of the fundamental group which is conjectured in [GK].

**Theorem 1.1** (Main Theorem, see Theorem 4.1 and Proposition 4.2). The fundamental group of $Y_n$ is generated by elements $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ with the relations

$$[\Gamma_i, \Gamma_j] = 1, \quad (1 \leq i, j \leq n), \quad (\Gamma_0 \Gamma_i)^2 = (\Gamma_i \Gamma_0)^2, \quad (1 \leq i \leq n),$$

and

$$[M(I)^{-1}\Gamma_0 M(I), M(J)^{-1}\Gamma_0 M(J)] = 1$$

for all subsets $I$ and $J$ of $\{1, \ldots, n\}$ satisfying $I \cap J = \emptyset, I \neq \emptyset, J \neq \emptyset$ and $\#I + \#J \leq n - 1$. Here we set $M(I) = \prod_{i \in I} \Gamma_i$.

For the proof of this theorem, we use a cell complex constructed by Salvetti [S], which is homotopic to the complement of a hyperplane arrangement in $\mathbb{C}^N$ and stable under a group action. The author is grateful for discussions with Y. Goto and K. Matsumoto in “Workshop on Special Varieties in Tambara, 2017”, in Tambara International Seminar House.

2. **Recall of a result of Salvetti**

We recall a construction of 2-skeleton of a cell complex which is homotopic to the complement of real hyperplane arrangement. A finite set $\mathcal{H} = \{H_i\}_{i \in I}$ of complex hyperplanes in $\mathbb{C}^n$ is called a hyperplane arrangement. In this paper, we are interested in the topological space

$$Y = Y(\mathcal{H}) = \mathbb{C}^n - \bigcup_{i \in I} H_i.$$

A hyperplane arrangement is called a real hyperplane arrangement if the defining equations of $H_i$ is defined over $\mathbb{R}$ for all $i \in I$. For a real hyperplane arrangement $\mathcal{H}$, we set $H_i, \mathbb{R} = H_i \cap \mathbb{R}^n$. The set $\{H_i, \mathbb{R}\}_{i \in I}$ is denoted by $\mathcal{H}_R$. A subset of $\mathbb{R}^n$ which can be obtained by the intersection of finite number of $H_i, \mathbb{R}$’s is simply called a linear subset of $\mathcal{H}_R$. As a special case, the total space $\mathbb{R}^n$ is an $n$-dimensional linear subset. Let $L$ be an $i$-dimensional linear subset of $\mathcal{H}_R$. A connected component of the complement of the union of proper linear subsets of $L$ in $\mathbb{R}^n$ is called an $i$-chamber of $\mathcal{H}_R$ and the set of $i$-chambers is denoted by $\text{Ch}_i(\mathcal{H}_R)$. Each $i$-chamber is a convex set.

We define the dual cell complex of $\mathcal{H}_R$ as follows. For each $i$-dimensional chamber $\sigma$, we choose a vertex $v_\sigma$ in the interior of $\sigma$. The set of 0-cell of the dual cell complex is given by $D_{\sigma} = v_\sigma$, where $\sigma$ is an $n$-chamber.

Let $\tau$ be an $(n - 1)$-chamber. Then there exist exactly two $n$-chambers $\tau_1$ and $\tau_2$ such that $\overline{\tau_i} \supset \tau$ for $i = 1, 2$. Here $\overline{\tau}$ is the closure of $\tau$ in $\mathbb{R}^n$. We consider 1-cell $D_\tau$ by considering the union of segments $\Delta(v_{\tau_1}, v_\tau)$ and $\Delta(v_{\tau_2}, v_\tau)$. We continue this procedure to define $2$-cell $D_\sigma$ attached to $(n - 2)$-dimensional chamber as follows. If $\sigma_1, \sigma_2$ and $\sigma$ are $n, (n - 1)$ and $(n - 2)$-chambers, such that

$$\overline{\sigma_1} \supset \overline{\sigma_2} \supset \overline{\sigma},$$
A sequence $F = F(\sigma_1, \sigma_2, \sigma)$ as above is called a (descending) flag of length 3. The triangle $\Delta(v_{\sigma_1}, v_{\sigma_2}, v_{\sigma})$ is called the dual flag $F^*$ of $F$. The union of dual flags containing $v_{\sigma}$ is called the 2-dimensional dual cell $D_{\sigma}$ of $\sigma$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{dual_cell.png}
\caption{Dual cell}
\end{figure}

We recall the construction of the 2-skeleton $X_2$ of the cell complex $X$ after Salvetti [S], which is homotopy equivalent to the space $Y = Y(\mathcal{H})$. The set $C_0(X)$ of 0-cell in $X$ is the set $\{\widetilde{D}_{\sigma}\}_{\sigma \in \text{Ch}_n}$ of the copy $\widetilde{D}_{\sigma}$ of $D_{\sigma}$.

The set $C_1(X)$ of 1-cell consists of $\widetilde{D}_{\sigma, \tau}$ for $\sigma \in \text{Ch}_n, \tau \in \text{Ch}_{n-1}$ such that $\overline{\sigma} \supset \tau$. The $n$-chamber lying on the opposite side of $\sigma$ with respect to the $(n-1)$-chamber $\tau$ is denoted by $\rho_{\tau}(\sigma)$. The attaching map $\partial \widetilde{D}_{\sigma, \tau} \to X_0$ is given by connecting two points $\sigma$ and $\rho_{\tau}(\sigma)$. The 1-cell $\widetilde{D}_{\sigma, \tau}$ is called an arrow from $\sigma$ to $\rho_{\tau}(\sigma)$. The composite of several arrows compatible with the directions is called an oriented path.

The set $C_2(X)$ of 2-cell consists of $\widetilde{D}_{\sigma, \tau}$ for $\sigma \in \text{Ch}_n, \tau \in \text{Ch}_{n-2}$ such that $\overline{\sigma} \supset \tau$. The $n$-chamber lying on the opposite side of $\sigma$ with respect to the $(n-2)$-chamber $\tau$ is denoted by $\rho_{\tau}(\sigma)$ and the vertex in $\rho_{\tau}(\sigma)$ is denoted by $\rho_{\tau}(v_{\sigma})$ (see Figure 2). Then there exist exactly two shortest paths from $v_{\sigma}$ to $\rho_{\tau}(v_{\sigma})$. The attaching map $\partial \widetilde{D}_{\sigma, \tau} \to X_1$ is given by bounding the two shortest paths from $v_{\sigma}$ to $\rho_{\tau}(v_{\sigma})$ (see Figure 2).

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{relations.png}
\caption{Relations}
\end{figure}
Proposition 2.1 (Salvetti). The natural inclusion $X_2 \to Y$ induces an isomorphism of fundamental groups

$$\pi_1(X_2) \to \pi_1(Y).$$

As a consequence, the fundamental groupoid is generated by $\tilde{D}_{\tau_1,\tau_2}$ for $\tau_1 \in \text{Ch}_n, \tau_2 \in \text{Ch}_{n-1}, \overline{\tau_1} \supset \overline{\tau_2}$, and the relation is given by $\tilde{D}_{\sigma_1,\sigma_2}$ for $\sigma_1 \in \text{Ch}_n, \sigma_2 \in \text{Ch}_{n-2}, \overline{\sigma_1} \supset \overline{\sigma_2}$.

3. $F_C$-hyperplane arrangement

3.1. The arrangement $\mathcal{H}_n$. For an element $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \{-1,1\}^n$, we define a hyperplane $H_\epsilon$ by

$$H_\epsilon : \epsilon_1 x_1 + \cdots + \epsilon_n x_n = 1.$$ 

We define $n$-dimensional $F_C$-arrangement $\mathcal{H}_n$ by the union of the set of hyperplanes $\{H_\epsilon\}$ $(\epsilon \in \{-1,1\}^n)$ and that of coordinate hyperplanes

$$L_i : x_i = 0, \quad (i = 1, \ldots, n).$$

The following proposition is used to classify $(n-2)$-chambers in $\mathcal{H}_n$.

Proposition 3.1. (1) Let $\epsilon, \epsilon' \in \{-1,1\}^n$ such that $\# \{i \mid \epsilon_i \neq \epsilon'_i\} \geq 2$ and set

$$H_{\epsilon,\epsilon'} = H_\epsilon \cap H_{\epsilon'}.$$ 

A hyperplane in $\mathcal{H}$ containing $H_{\epsilon,\epsilon'}$ is equal to $H_\epsilon$ or $H_{\epsilon'}$.

(2) For an element $\epsilon \in \{-1,1\}^n$ and an integer $i$ with $1 \leq i \leq n$, we set

$$H_{\epsilon,i} = H_\epsilon \cap L_i.$$ 

A hyperplane in $\mathcal{H}$ containing $H_{\epsilon,i}$ is equal to $L_i, H_\epsilon$ or $H_{g(i)\epsilon}$. Here

$$g(i)(\epsilon_1, \ldots, \epsilon_n) = (\epsilon_1, \ldots, \epsilon_i, \ldots, \epsilon_n).$$

(3) Let $i, j$ be distinct integers such that $1 \leq i, j \leq n$ and set

$$H_{i,j} = L_i \cap L_j.$$ 

A hyperplane in $\mathcal{H}$ containing $H_{i,j}$ is equal to $L_i$ or $L_j$.

3.2. Group action. On the space $Y$, the group $\mu_2^n = \{1, -1\}^n$ acts by

$$g : \mathbb{C}^n \to \mathbb{C}^n : (x_1, \ldots, x_n) \mapsto (g_1 x_1, \ldots, g_n x_n)$$

for $g = (g_1, \ldots, g_n) \in \mu_2^n$. The group $\mu_2^n$ acts on the sets $\text{Ch}_i$. We can choose the set of vertex $\{v_\sigma\}_{\sigma \in \text{Ch}_i}$ so that they are stable under the action of $\mu_2^n$.

Lemma 3.2. On the topological space $X_2$, the action of the group $\mu_2^n$ on $X_2$ is cell-wise and fixed point free.

Proof. The group acts on $\text{Ch}_n$ freely. Therefore it acts freely on the set of 0, 1 and 2-cells. \qed
3.3. Cell complex for the quotient space. We consider topological space \( \overline{X_2} = X_2/\mu_2^n \). Then \( \overline{X_2} \) is a cell complex. We have the following proposition.

**Proposition 3.3.** The natural map \( \pi_1(\overline{X_2}) \to \pi_1(Y/\mu_2^n) \) is an isomorphism.

We describe the cell complex \( \overline{X_2} \) in this subsection. We set \( R_{>0} = \{ x \in \mathbb{R} \mid x > 0 \} \), \( R_{\geq 0} = \{ x \in \mathbb{R} \mid x \geq 0 \} \).

The subset of \( i \)-chambers in \( Ch_i \) contained in \( R_{\geq 0} \) is denoted by \( Ch_i \).

The set \( C_0(\overline{X}) \) of 0-cells in \( \overline{X} \) is identified with \( \{ \tilde{D}_\sigma \mid \sigma \in \overline{Ch}_n \} \). There are the following two kinds of 1-cells in \( \overline{X} \): The image of \( \tilde{D}_{\sigma,\tau} \) such that

1. (type 1, non-closed one cell) \( \tau \subset H_\epsilon, (\epsilon \in \{-1, 1\}^n) \).
2. (type 2, closed one cell) \( \tau \subset L_i, (1 \leq i \leq n) \).

There are three kinds of 2-cells in \( \overline{X_2} \): The image of \( \tilde{D}_{\sigma,\tau} \) such that

1. (type 1, interior disc) \( \tau \subset H_\epsilon \cap H_\epsilon' \).
2. (type 2, boundary disc) \( \tau \subset H_\epsilon \cap L_i \).
3. (type 3, coordinate disc) \( \tau \subset L_i \cap L_j \).

**Definition 3.4.** Let \( \sigma \) be an element in \( \overline{Ch}_n \). We define height \( h(v_\sigma) \) of \( v_\sigma = \tilde{D}_\sigma \) by the number of hyperplanes of the form \( H_\epsilon (\epsilon \in \{-1, 1\}^n) \) separating 0 and \( v_\sigma \). The number \( h(v_\sigma) \) is also denoted by \( h(\sigma) \).

**Proposition 3.5.**

1. A interior disc (type 1) is attached to four 1-cells and contains four 0-cells. The shape of height is as follows.
2. A boundary disc (type 2) is attached to six 1-cells and contains three 0-cells.
3. A coordinate disc (type 3) is attached to two 1-cells and contains one 0-cells.

We define spanning complex which is a slight generalization of spanning tree. A 1-cell \( \tilde{D}_{\sigma,\tau} \) is called a spanning 1-cell if it is type 1 and \( h(\sigma) + 1 = h(\rho(\tau)) \), i.e. \( \rho(\tau) \) is farer from the origin than \( \sigma \). A 2-cell \( \tilde{D}_{\sigma,\tau} \) is called a spanning 2-cell if it is type 1 and \( h(\sigma) \) is the smallest among vertices contained in \( D_\tau \). The union of spanning 1 and 2-cells forms a sub cell complex \( S \) of \( \overline{X_2} \). The complex \( S \) is called the spanning complex of \( \overline{X_2} \).

**Lemma 3.6.** The spanning complex \( S \) is simply connected.

**Proof.** It is identified with a 2-skeleton of the dual cell complex of \( R_{>0}^n \) which is simply connected. \( \square \)

We define \( X_2^{(s)} \) by obtaining contracting a subset \( S \subset \overline{X_2} \) to a point \( s \). By the above proposition, we have

**Proposition 3.7.** The natural map

\[ \pi_1(\overline{X_2}) \to \pi_1(\overline{X_2}^{(s)}, s) \]

is an isomorphism.
Definition 3.8. A 1-cell $\widetilde{D}_{\sigma,\tau}$ in $\overline{X}_2$ is called a generator if it is
(1) type 1 and not spanning, or
(2) type 2.

A generator defines a closed path in $\overline{X}_2^{(s)}$. Then the set of generator
generates the group $\pi_1(\overline{X}_2^{(s)})$.

3.4. Relations for type 1 and type 2.

3.4.1. Type 1 relation. First, we consider a type 1 2-cells $\widetilde{D}_{\sigma,\tau}$ in $\overline{X}$ with
$\tau \subset H_\epsilon \cap H_{\epsilon'}$. The arrows $a, b, c$ and $d$ are spanning 1-cells and $a, b, c$ and $d$
define elements in $\pi_1(\overline{X}_2^{(s)}, s)$. Their relations are given as

$$a = c, \quad b = d, \quad ab = ba.$$  

3.4.2. Type 2 relation. Next we consider a type 2 2-cells $\widetilde{D}_{\sigma,\tau}$ in $\overline{X}$ as in
Figure 4. The arrows $\overline{b}$ and $\overline{c}$ are spanning 1-cells and reduced to one point

in $\overline{X}_2^{(s)}$. We consider a $(n - 2)$-chamber $\tau$ contained in $L_i$. The relations
beginning from $v_1 = v_{\sigma_1}, v_2 = v_{\sigma_2}$ and $v_3 = v_{\sigma_3}$ are the following:

$$\partial \widetilde{D}_{\sigma_1, \tau} : a = d.$$
\[ \partial \widetilde{D}_{\sigma_2, \tau} : ba = dc, \]
\[ \partial \widetilde{D}_{\sigma_3, \tau} : cba = dcb. \]

We can easily check the following proposition.

**Proposition 3.9.** The above relations are equivalent to
\[ (3.2) \quad d = a, \quad c = a^{-1} ba, \quad (ab)^2 = (ba)^2. \]

We consider the above situation and set \( \epsilon = (\epsilon_1, \ldots, \epsilon_n) \) and \( \epsilon' = (\epsilon'_1, \ldots, \epsilon'_n) \). Here \( \epsilon' = g^{(i)}(\epsilon) \) where \( g^{(i)} \) is defined as (3.1). By the definition of height, \( v_1 \) is the closest vertex from the origin. Therefore we have \( \epsilon_j = \epsilon'_j \) if \( j \neq i \) and \( \epsilon_i = 1 \) and \( \epsilon'_i = -1 \).

### 3.5. Definition of \( \gamma_i \) and their relations.

In this subsection, we define \( \gamma_i \) and study their relations.

**Definition 3.10.** We define \( \gamma_i = \widetilde{D}_{\sigma_i, \tau_i} \) for \( i = 0, 1, \ldots, n \) where
\[ \sigma_0 = \{ (x_i) \in \mathbb{R}^n \mid x_i > 0 \quad (0 \leq i \leq n), \quad \sum x_i < 1 \}, \]
\[ \tau_0 = \{ (x_i) \in \mathbb{R}^n \mid x_i > 0 \quad (0 \leq i \leq n), \quad \sum x_i = 1 \}, \]
\[ \tau_i = \{ (x_i) \in \mathbb{R}^n \mid x_i > 0 \quad (0 \leq j \leq n, j \neq i), \quad \sum x_i < 1, \quad x_i = 0 \}. \]

Actually \( \sigma_0 \) is a chamber since if \( x = (x_1, \ldots, x_n) \in \sigma_0 \) and \( \epsilon \neq (1, \ldots, 1) \) then
\[ \sum_i \epsilon x_i < \sum_i x_i < 1, \]
and the point \( x \) is not contained in \( H_{\epsilon, \mathbb{R}} \).

By previous subsection, we have
\[ (3.3) \quad [\gamma_i, \gamma_j] = 1, \quad (1 \leq i, j \leq n), \]
\[ (\gamma_0 \gamma_i)^2 = (\gamma_i \gamma_0)^2, \quad (1 \leq i \leq n). \]

**Proposition 3.11.**

1. Under the notation of figure 1, we have \( a = \gamma_i \) in \( \pi_1(\overline{X_2(s)}) \).

2. Let \( \tau_1 \) and \( \tau_2 \) be two elements in \( \overline{\mathcal{C}ln}_{n-1} \) contained a common hyperplane \( H_{\epsilon, \mathbb{R}} \). Suppose that 1-cells \( \widetilde{D}_{\sigma_1, \tau_1} \) and \( \widetilde{D}_{\sigma_2, \tau_2} \) are generators. Then the paths obtained by them are homotopic to each other. These paths defines a common element in \( \pi_1(\overline{X_2(s)}), s \) which is denoted by \( \gamma_\epsilon \).

3. For \( \epsilon \in \{-1, 1\}^n, \epsilon \neq (-1, \ldots, -1) \), we set
\[ S(\epsilon) = \{ i \mid 1 \leq i \leq n, \epsilon_i = -1 \}, \quad m_\epsilon = \prod_{i \in S(\epsilon)} \gamma_i. \]

Then we have
\[ (3.4) \quad \gamma_\epsilon = m_\epsilon^{-1} \gamma_0 m_\epsilon. \]
**Proof.** (1) We use the first relation (3.2) iteratively and have the statement.
(2) This follows from the relations obtained by a type 1 2-cells. (3) Let \( \epsilon \) be an element in \( \{-1,1\}^n \) and \( \epsilon \neq (-1, \ldots, -1) \). We set \( S(\epsilon) = \{i_1, \ldots, i_k\} \). We consider a chain \( \epsilon^{(0)}, \ldots, \epsilon^{(k)} \in \{-1,1\}^n \) defined by
\[
\epsilon^{(0)} = (1, \ldots, 1), \epsilon^{(1)} = g^{(i_1)}(\epsilon^{(0)}), \epsilon^{(2)} = g^{(i_2)}(\epsilon^{(1)}), \ldots, \epsilon^{(k)} = g^{(i_k)}(\epsilon^{(k-1)}).
\]
Then \( \epsilon^{(k)} = \epsilon \).

**Lemma 3.12.** \( H_{\epsilon(j)} \cap R^+ \neq \emptyset \), \( j = 0, \ldots, k \) and
\[
H_{\epsilon(j)} \cap H_{\epsilon(j-1)} \cap \{(x_l) \in R^n | x_l > 0 \text{ for } l \neq i_j\} \neq \emptyset \quad (i = 1, \ldots, k).
\]

**Proof of Lemma 3.12.** By descending induction, it is enough to prove the lemma for \( j = k \). We set \( \epsilon^{(k)} = (\epsilon_1, \ldots, \epsilon_n) \). First statement holds since \( \epsilon_j = 1 \) for some \( j \). We prove the second statement. Since \( \epsilon_i = -1 \), there exists \( j \neq i_k \) such that \( \epsilon_j = 1 \). Therefore the equation
\[
x_{i_k} = 0, \epsilon_1 x_1 + \cdots + \epsilon_k x_k + \cdots + \epsilon_n x_n = 1.
\]
has a solution satisfying \( x_l > 0 \) for \( l \neq i_j \).

By applying the second relation in (3.2) iteratively, we have statement (3). □

Using Proposition 3.11 (3) and the relation of type 1, we have the following theorem.

**Theorem 3.13.** We have
\[
[m_{\epsilon}^{-1} \gamma_0 m_{\epsilon}, m_{\epsilon'}^{-1} \gamma_0 m_{\epsilon'}] = 1.
\]
for \( \epsilon, \epsilon' \in \{-1,1\}^n \) and \( H_{\epsilon,R} \cap H_{\epsilon',R} \cap R^+ \neq \emptyset \).

4. FUNDAMENTAL RELATION

4.1. Main theorem. In this section, we prove the following theorem.

**Theorem 4.1.** The relations (3.3) and (3.5) are fundamental relations for \( \pi_1(X_2^{(s)}, s) \) with generators \( \gamma_0 \) and \( \gamma_i \) \( (1 \leq i \leq n) \).

We define \( G \) as a group generated by \( \Gamma_0 \) and \( \Gamma_i \) \( (1 \leq i \leq n) \) with the relations
\[
[\Gamma_i, \Gamma_j] = 1, \quad (1 \leq i, j \leq n),
\]
\[
(\Gamma_0 \Gamma_i)^2 = (\Gamma_i \Gamma_0)^2 \quad (1 \leq i \leq n),
\]
and
\[
[M_{\epsilon}^{-1} \Gamma_0 M_{\epsilon}, M_{\epsilon'}^{-1} \Gamma_0 M_{\epsilon'}] = 1.
\]
for \( H_{\epsilon,R} \cap H_{\epsilon',R} \cap R^+ \neq \emptyset \). Here we set \( M_{\epsilon} = \prod_{i \in S(\epsilon)} \Gamma_i \). We define group homomorphisms
\[
\varphi : G \to \pi_1(X_2^{(s)}, s) \quad \text{and} \quad \psi : \pi_1(X_2^{(s)}, s) \to G,
\]
which are inverse to each other.
4.1.1. The definition of $\varphi$. We define $\varphi$ by $\varphi(\Gamma_i) = \gamma_i$ for $i = 0, 1, \ldots, n$. We check that fundamental relations of $G$ are satisfied in $\pi_1(\overline{X}_2(s), s)$. The relation (4.1) is satisfied by the definition of $\varphi$. By the definition of $\varphi$, we have

$$\varphi(M^{-1}_\epsilon \Gamma_0 M_\epsilon) = m^{-1}_\epsilon \gamma_0 m_\epsilon.$$ 

Thus the relation (4.2) is satisfied by Theorem 3.13.

4.1.2. The definition of $\psi$. The group $\pi_1(\overline{X}_2(s), s)$ is generated by type 1 non-spanning arrow $\gamma_{\epsilon, \tau}$ and type 2 generators $\gamma_{i, \tau}$ with the relation of type 1, type 2 and type 3 relations. We set

$$\psi(\gamma_{\epsilon, \tau}) = M^{-1}_\epsilon \Gamma_0 M_\epsilon, \quad \psi(\gamma_{i, \tau}) = \Gamma_i.$$

Type 1 and type 3 relation are satisfied by the fundamental relations of $G$. The first relations of (3.2) is easy to check. The second relation is obtained by the relation between $\epsilon$ and $g(\epsilon)$. We check the third relation of (3.2) by using $\psi(a) = \Gamma_i, \psi(b) = M^{-1}_\epsilon \Gamma_0 M_\epsilon$. Since $\Gamma_i$ and $M_\epsilon$ are commutative in $G$, we have

$$\psi(abab) = \Gamma_i \cdot M^{-1}_\epsilon \Gamma_0 M_\epsilon \cdot \Gamma_i \cdot M^{-1}_\epsilon \Gamma_0 M_\epsilon = M^{-1}_\epsilon \Gamma_0 \Gamma_i \Gamma_0 M_\epsilon$$

and

$$\psi(baba) = M^{-1}_\epsilon \Gamma_0 M_\epsilon \cdot \Gamma_i \cdot M^{-1}_\epsilon \Gamma_0 M_\epsilon \cdot \Gamma_i = M^{-1}_\epsilon \Gamma_0 \Gamma_i \Gamma_0 \Gamma_i M_\epsilon.$$

Thus we have the equality $\psi(abab) = \psi(baba)$. Thus the homomorphism $\psi$ is well defined.

**Proof of Theorem 4.1.** By the definition of $\varphi$ and $\psi$, we see that the homomorphisms $\psi$ and $\varphi$ are inverse to each other. \hfill \square

4.2. Simplification. We modify the relation of (4.2) and get the simpler form cited in [GK]. By Theorem 3.11 and the following proposition, we get Main Theorem 1.1.

**Proposition 4.2.** For a subset $I$ of $\{1, \ldots, n\}$ we set $M(I) = \prod_{i \in I} \Gamma_i$. Under the relation (4.1), the relation (4.2) for $H_\epsilon R \cap H_{\epsilon'} R \cap R_{\geq 0}^n \neq \emptyset$ is equivalent to the following set of relations.

$$[M(I)^{-1} \Gamma_0 M(I), M(J)^{-1} \Gamma_0 M(J)] = 1$$

for all $I, J$ satisfying $I \cap J = \emptyset, I \neq \emptyset, J \neq \emptyset$ and $\#I + \#J \leq n - 1$.

**Proof.** Throughout this proof we assume the commutativity of $\Gamma_1, \ldots, \Gamma_n$. First we assume the condition (4.3) and prove the relation (4.2). We set $K = S(\epsilon) \cap S(\epsilon')$. By the definition of $M_\epsilon$ and the commutativity of $\Gamma_i$, the condition (4.2) can be rewrite as

$$[M_\epsilon^{-1} \Gamma_0 M_\epsilon, M_\epsilon^{-1} \Gamma_0 M_\epsilon] = 1.$$
where $M^*_\epsilon = \prod_{i \in S(\epsilon) - K} \Gamma_i$ and $M^*_{\epsilon'} = \prod_{i \in S(\epsilon') - K} \Gamma_i$. This is one of the conditions in (4.3) by setting $I = S(\epsilon) - K$ and $J = S(\epsilon') - K$. We check that $I$ and $J$ satisfies the required conditions. The condition $I \cap J = \emptyset$ is clear. If $M^*_\epsilon = \emptyset$, then $S(\epsilon) \subset S(\epsilon')$ and this contradicts to the condition $H_{\epsilon, R} \cap H_{\epsilon', R} \cap \mathbb{R}^n_{>0} \neq \emptyset$. If $\# I + \# J = n$, then $\epsilon' = -\epsilon$. This also contradicts to the condition for $\epsilon$ and $\epsilon'$ since $H_{\epsilon, R} \cap H_{-\epsilon, R} = \emptyset$.

Next we assume the condition (4.2) and prove the relation (4.3). Let $I$ and $J$ be subsets in $\{1, \ldots, n\}$ satisfying the condition of (4.3). We define $\epsilon = (\epsilon_1, \ldots, \epsilon_n)$ and $\epsilon' = (\epsilon'_1, \ldots, \epsilon'_n)$ by

$$
\epsilon_i = \begin{cases} 1 & (i \notin I), \\ -1 & (i \in I), \end{cases} \quad \epsilon'_i = \begin{cases} 1 & (i \notin J), \\ -1 & (i \in J). \end{cases}
$$

Then the relation (4.4) becomes the relation (4.3). We check the condition $H_{\epsilon, R} \cap H_{\epsilon', R} \cap \mathbb{R}^n_{>0} \neq \emptyset$. We set $K = \{1, \ldots, n\} - (I \cup J)$. Then we have $K \neq \emptyset$. The system of equations

$$
\begin{align*}
&\sum_{i \notin J} x_i - \sum_{i \in I} x_i = 1, \\
&\sum_{j \notin I} x_j - \sum_{j \in J} x_j = 1
\end{align*}
$$

is equivalent to

$$
\begin{align*}
&\sum_{i \in K} x_i = 1, \\
&\sum_{i \in I} x_i = \sum_{j \in J} x_j.
\end{align*}
$$

Thus it has a solution $x = (x_i) \in \mathbb{R}^n_{>0}$. \qed

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