Active-Sterile Neutrino Masses and Mixings in $A_4$ Minimal Extended Seesaw Mechanism

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Abstract
Assuming the existence of an eV scale sterile neutrino, we develop a 3 + 1 neutrino mass model using $A_4 \times Z_4 \times Z_2$ symmetry group. Three Higgs fields $H, H', H''$ are considered to give a desired neutrino mass matrix which generates non-zero $\theta_{13}$ and also successfully provides the active-sterile neutrino data in normal hierarchy. We also study the results of kinematic measurements of $\beta$ – decay ($m_\beta$) and neutrinoless double beta decay ($m_{\beta\beta}$) from the model. These results are consistent with latest experimental bounds and found to be in the ranges $0.0784$ eV $< m_\beta < 0.3034$ eV and $0.0034$ eV $< m_{\beta\beta} < 0.0667$ eV respectively.

Keywords Sterile neutrino · Neutrino mixing · $A_4$ models · Active neutrinos · Beyond standard model

1 Introduction
The theory of neutrino masses and mixings has been a very highly exciting field of research in recent years. Observations from SNO, T2K, etc., that neutrinos have masses and oscillate among different flavors, have triggered a new approach in the study of neutrinos beyond the standard model (SM) framework. The three generations of neutrino mass squared differences and leptonic mixing angles have reached the precision measurement status. However, other parameters such as mass hierarchy, absolute neutrino mass, Dirac CP-violating phase are still unknown. The current global fit data for three neutrino oscillations [1] are shown in Table 1.

Apart from these, one interesting anomaly came from the experimental data of LSND [2], which observed excess of electron anti-neutrino ($\bar{\nu}_e$) in a muon anti-neutrino ($\bar{\nu}_\mu$)
beam produced at the Los Alamos laboratory. Another experiment called, MiniBooNE [3] supplemented LSND results and observed an oscillation $\bar{\nu}_\mu$ to $\bar{\nu}_e$ compatible with the LSND data. These results can be interpreted by including more mass eigenstates of neutrino in the three neutrino theory [4]. One of the simplest ways is to add a fourth neutrino state, generally called the sterile neutrino, to the three active neutrinos [5]. The word sterile refers to the fact that such a neutrino cannot have weak interactions in the SM from the requirement of being in agreement with the precision measurement of Z boson decay width at the LEP experiment. However, they can mix with the active neutrinos. It has been shown [6] in the 3 + 1 framework that the new analysis of the MiniBooNE data allows smaller values of active-sterile neutrino mixing for the standard analysis. Super-Kamiokande (SK) has provided upper bounds on sterile neutrino parameter $|U_{\tau 4}| < 0.18$ at 90% CL [7]. IceCube Collaboration [8] has also analyzed light sterile neutrinos from three years of atmospheric neutrino data from the DeepCore detector and provided limits on sterile neutrino mixing at $|U_{\mu 4}|^2 < 0.11$ and $|U_{\tau 4}|^2 < 0.15$ (90% C.L.) for the sterile neutrino mass splitting $\Delta m_{41}^2 = 1.0$ eV$^2$. New data from the reactor and other short and long-baseline neutrino experiments such as MINOS [9], Daya Bay [10] etc., provide new bounds on active-sterile mixings and $\Delta m_{41}^2$. Recently, it has been reported from the MicroBooNE that there is a possibility of absence of an eV-scale sterile neutrino. But, GeV to KeV scale sterile neutrinos are still well-motivated theoretically and do not contradict any existing experiments. Besides, MicroBooNE does not probe the full parameter space of sterile neutrino models hinted at by MiniBooNE and other data, nor do they probe the $\nu_e$ interpretation of the MiniBooNE excess in a model-independent way [11]. Many other ongoing and future long-baseline experiments such as DUNE [12], T2HK [13], T2HKK [14] etc. may provide new insights on neutrino oscillation physics and explore active-sterile mixing. The phenomenology and experimental constraints on 3 + 1 neutrinos have been discussed in [15–24].

Many authors have used discrete symmetries such as $A_n$, $S_n$, $C_n$, $Z_n$, etc. where $n$ is the order of the group, to develop neutrino mass models. Several mechanisms have been used to study possible active-sterile mixing within the seesaw models such as the generic Type-I Seesaw, Neutrino Minimal Standard Model ($\nu$ MSM), Inverse Seesaw, Minimal Extended Seesaw (MES) model, etc. In Refs. [17–19, 25], the authors have

| parameter | best fit ± 1σ | 2σ range | 3σ range |
|------------|--------------|---------|---------|
| $|\Delta m_{21}^2| : [10^{-5} \text{eV}^2]$ | 7.50$^{+0.22}_{-0.20}$ | 7.11–7.93 | 6.94–8.14 |
| $|\Delta m_{31}^2| : [10^{-3} \text{eV}^2]$ (NO) | 2.55$^{+0.02}_{-0.03}$ | 2.49–2.60 | 2.47–2.63 |
| $|\Delta m_{31}^2| : [10^{-3} \text{eV}^2]$ (IO) | 2.45$^{+0.02}_{-0.03}$ | 2.39–2.50 | 2.37–2.53 |
| $\sin^2 \theta_{12}/10^{-1}$ | 3.18 ± 0.16 | 2.86–3.52 | 2.71–3.69 |
| $\sin^2 \theta_{23}/10^{-1}$ (NO) | 5.74 ± 0.14 | 5.41–5.99 | 4.34–6.10 |
| $\sin^2 \theta_{23}/10^{-1}$ (IO) | 5.78$^{+0.10}_{-0.17}$ | 5.41–5.98 | 4.33–6.08 |
| $\sin^2 \theta_{13}/10^{-2}$ (NO) | 2.20$^{+0.069}_{-0.062}$ | 2.069–2.337 | 2.000–2.405 |
| $\sin^2 \theta_{13}/10^{-2}$ (IO) | 2.22$^{+0.064}_{-0.070}$ | 2.086–2.356 | 2.018–2.424 |
| $\delta_{C P}/\pi$ (NO) | 1.08$^{+0.13}_{-0.12}$ | 0.84–1.42 | 0.71–1.99 |
| $\delta_{C P}/\pi$ (IO) | 1.58$^{+0.15}_{-0.16}$ | 1.26–1.85 | 1.11–1.96 |
extensively studied active-sterile mixings in the MES mechanism and their possible
effects on cosmological problems such as baryogenesis, neutrinoless double beta decay,
dark matter, etc.

The main drawback of model building is the presence of a large number of scalar
flavon fields and additional discrete groups which we are trying to reduce in the pre-
sent work. For instance, in Ref. [5], the SM symmetry is extended by $A_4 \times Z_2$ discrete
symmetry with three $A_4$ triplets and three singlets, and it predicts a $\mu - \tau$ symmetric
neutrino mass model and active-sterile mixing. In Ref. [17], SM is extended by add-
ing $A_4 \times Z_4 \times Z_3$ discrete groups with two beyond Standard Model (BSM) Higgs, three
triplet flavons and three singlet flavons while in Ref. [18], the additional discrete group
is reduced to $A_4 \times Z_4$ but increased the BSM particle content of the model to two Higgs,
four triplet flavons and three singlet flavons to study non-zero $\theta_{13}$ of neutrino mixing,
active-sterile mixing and dark matter. In Ref. [25], the author considers a $B - L$ model
with $S_3 \times Z_2 \times Z_2$ symmetry to study active-sterile mixing in normal hierarchy (NH)
mass ordering using five flavon fields. Again in Ref. [26], the authors take $\Delta(96) \times C_2 \times
C_3$ with seven flavons to study TM1 mixing.

In the present work, we use the MES mechanism and extend the SM symmetry by $A_4 \times Z_4 \times Z_2$ with two BSM Higgs doublets, two $A_4$ triplets ($\phi, \psi$) and two $A_4$ singlet flavon
fields ($\zeta, \chi$) to explain active-sterile mixing in the normal hierarchy (NH). A third triplet
flavon ($\eta$) is used to give a perturbation in the Dirac mass matrix, which is essential
to generate non-zero $\theta_{13}$. We assign the group charges to the fields which are different
from other works, and these lead to new interactions and therefore, generate a different
neutrino mass matrix structures. Instead of using special alignments of flavon vacuum
expectation values (v.e.v) to get a desired structure of neutrino mass matrix as in other
papers, we consider here the most general alignments of the triplet flavon along (1,0,0)
and (1,1,1) [27]. The details of the model framework are given in the Section 3. We
develop a (4 × 4) active-sterile neutrino mixing matrix from the model consistent with
current experimental data.

Moreover, there are searches for massive neutrinos, such as kinematic measurements
of $\beta -$decay and searches for neutrinoless double beta decay ($0\nu\beta\beta$) events. The absolute
neutrino mass scale is directly probed from the cut-off of the electron energy spectrum
emitted from $\beta -$decay [28]. The effective neutrino mass $m_\beta$ is given as

$$m_\beta = \left( \sum_{i=1}^{4} |U_{ei}|^2 m_i^2 \right)^{1/2}, \tag{1}$$

where $U$ is the $4 \times 4$ active-sterile neutrino mixing matrix. The effective mass parameter
$m_{\beta\beta}$ from $0\nu\beta\beta$ can be expressed as the sum of mass eigenstates and mixing matrix ele-
ments as

$$m_{\beta\beta} = \left| \sum_{j=1}^{4} |U_{ej}|^2 m_j \right|. \tag{2}$$

We solve these effective mass parameters from the present model. The outline of this paper
is as follows. We have a brief discussion about the MES mechanism in Section 2, followed
by the description of our model in Section 3. In Section 4, we carry out detailed numerical
analysis of the neutrino masses and mixing matrices, leading to the results of our work. We
conclude with a summary and discussion in Section 5.


## 2 Minimal Extended Seesaw Mechanism

In the minimal extended seesaw (MES) mechanism, we take along with the SM particles, three extra right-handed singlet neutrinos ($\nu_R^1, \nu_R^2, \nu_R^3$) and one additional gauge singlet chiral field $S$. It is possible to naturally generate an eV scale sterile neutrino mass with minimal but non-zero mixing with active neutrinos. The mechanism can also be adapted to KeV scale sterile neutrino [5].

The general Lagrangian of neutrino mass terms is given by

$$\mathcal{L} = \bar{\nu}_L M_D \nu_R + \frac{1}{2} \bar{\nu}_R^c M_R \nu_R + \bar{S} M_S \nu_R + h.c.$$  \hspace{1cm} (3)

where $M_D$ and $M_R$ are the Dirac and Majorana mass matrices respectively. $M_S$ is a (1 × 3) sterile neutrino mass matrix arising from the inclusion of only one extra singlet $S$. In the basis $(\nu_L, \nu_R^c, S^c)$, we get a full (7 × 7) matrix given by

$$M^7_{7\times 7} = \begin{pmatrix} 0 & M_D & 0 \\ M_D^T & M_R & M_S^T \\ 0 & M_S & 0 \end{pmatrix}. \hspace{1cm} (4)$$

In analogy to the canonical type-I seesaw, if we take the right-handed neutrino to be much heavier than the electroweak scale of $M_D$, they should be decoupled at low scales. Therefore, we can block-diagonalise the (7 × 7) matrix by using the seesaw formula with the condition $M_R >> M_D$ and get a (4 × 4) neutrino mass matrix in the basis $(\nu_L, S^c)$ as

$$M^4_{4\times 4} = -\begin{pmatrix} M_D M_R^{-1} M_D^T & M_D M_R^{-1} M_S^T \\ M_R^T (M_R^{-1})^T M_D^T & M_R M_R^{-1} M_S^T \end{pmatrix}. \hspace{1cm} (5)$$

We have four light neutrino eigenstates corresponding to three active neutrinos and one sterile neutrino. As we can see in (5), $\det(M^4_{4\times 4}) = 0$. This means that at least one of the four light neutrinos is massless. We further proceed to diagonalize the above 4 × 4 mass matrix with the seesaw condition that $M_D < M_S$, we obtain a leading order of the active neutrino mass matrix $m_\nu$ as well as the sterile mass $m_s$ given as follows.

$$m_\nu \simeq M_D M_R^{-1} M_S^T (M_S M_R^{-1} M_S^T)^{-1} M_S (M_R^{-1})^T M_D^T - M_D M_R^{-1} M_D^T; \hspace{1cm} (6)$$

$$m_s \simeq -M_S M_R^{-1} M_S^T. \hspace{1cm} (7)$$

We can naturally produce a sterile neutrino having mass in the eV scale. For example, if we take $M_D \sim 10^2$ GeV, $M_R \sim 5 \times 10^{14}$ GeV and $M_S \sim 5 \times 10^2$ GeV, we get approximately $m_\nu \sim 0.02$ eV and $m_s \sim 0.5$ eV. Further, as pointed out in Ref. [29], a slightly heavier KeV scale sterile neutrino can also be generated if we increase the $M_S$ mass scale up to TeV scale within the MES mechanism. The present work is focussed on eV scale sterile neutrino physics.

The 3 × 3 active neutrino mass matrix $m_\nu$ can be diagonalized by a unitary 3 × 3 complex matrix $U_{PMNS}$ as [30]

$$m_\nu = U_{PMNS} \text{diag}(m_1, m_2, m_3) U_{PMNS}^T. \hspace{1cm} (8)$$

In MES scheme, for NH: \((m_1 << m_2 < m_3 << m_4)\), the light neutrino masses including the eV mass scale sterile neutrino are given in terms of mass-squared differences as

\[
m_1 = 0; \quad m_2 = \sqrt{\Delta m_{21}^2}; \quad m_3 = \sqrt{\Delta m_{21}^2 + \Delta m_{31}^2}; \quad m_4 = \sqrt{\Delta m_{41}^2},
\]

where \(\Delta m_{ij}^2 = |m_j^2 - m_i^2|\).

\(U_{PMNS}\) can be parameterized using three mixing angles \(\theta_{12}, \theta_{13}, \theta_{23}\) and one CP violating phase \(\delta_{13}\) for Dirac neutrinos and two Majorana phases \(\alpha\) and \(\beta\) for Majorana neutrinos. In PDG convention, the general form of \(U_{PMNS}\) is

\[
U_{PMNS} = \begin{pmatrix}
1 & 0 & 0 & c_{13} \\
0 & c_{23} & s_{23} & 0 \\
0 & -s_{23} & c_{23} & 0 \\
0 & 0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
0 & e^{-i\delta_{13}}s_{13} & 0 & c_{12} \\
0 & 1 & 0 & s_{12} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{pmatrix},
\]

where \(c_{ij} = \cos \theta_{ij}, \quad s_{ij} = \sin \theta_{ij}\) and \(P = \text{diag}(1, e^{ia}, e^{i(\beta+\delta_{13})})\) is the Majorana phase matrix.

In the 3 + 1 mixing framework, the leptonic mixing matrix \(U_{PMNS}\) is not strictly unitary due to contributions from the sterile sector. But, because of minimal active-sterile mixing, we can assume that \(U_{PMNS}\) is unitary at the \(O(10^{-2})\) level \([31]\). The full \(4 \times 4\) neutrino mixing matrix takes the form \([32]\)

\[
V \approx \begin{pmatrix}
(1 - \frac{1}{2}RR^T)U_{PMNS}R & R \\
-R^TU_{PMNS} & 1 - \frac{1}{2}R^TR
\end{pmatrix},
\]

where \(R\) is a \(3 \times 1\) matrix which determines the strength of active-sterile mixing and

\[
R = M_D M_R^{-1} M_S^T (M_S M_R^{-1} M_S^T)^{-1}.
\]

Taking the same mass scales as shown above, we estimate \(R \sim 0.2\) for \(m_1\) in eV scale and \(R \sim 0.1\) for \(m_3\) in KeV scale, both of which are in good agreement with experimental data of active-sterile neutrino mixing.

The \(4 \times 4\) neutrino mixing matrix can also be parameterized by six mixing angles \((\theta_{12}, \theta_{13}, \theta_{23}, \theta_{14}, \theta_{24}, \theta_{34})\), three Dirac phases \((\delta_{13}, \delta_{14}, \delta_{23})\) and three Majorana phases \((\alpha, \beta, \gamma)\) \([20]\).

\[
U^{4\times4} = \begin{pmatrix}
c_{12}c_{13}c_{14} & c_{12}c_{14}s_{12}e^{i\frac{\alpha}{2}} & c_{14}s_{13}e^{i\frac{\gamma}{2}} & s_{14}e^{-i\frac{\beta}{2}} \\
U_{\mu 1} & U_{\mu 2} & U_{\mu 3} & c_{14}s_{24}e^{-i\left(\frac{\pi}{2} - \delta_{14} + \delta_{24}\right)} \\
U_{\tau 1} & U_{\tau 2} & U_{\tau 3} & c_{14}c_{24}s_{34}e^{-i\left(\frac{\pi}{2} - \delta_{14}\right)} \\
U_{s 1} & U_{s 2} & U_{s 3} & c_{14}c_{24}c_{34}e^{-i\left(\frac{\pi}{2} - \delta_{14}\right)}
\end{pmatrix}.
\]

Comparing (10) and (12), we obtain the relations between neutrino mixing angles and the elements of mixing matrix as

\[
\sin^2 \theta_{14} = |V_{e4}|^2
\]

\[
\sin^2 \theta_{24} = \frac{|V_{\mu 4}|^2}{1 - |V_{e4}|^2}
\]
\[
\sin^2 \theta_{34} = \frac{|V_{ed}|^2}{1 - |V_{et}|^2 - |V_{at}|^2} \\
\sin^2 \theta_{12} = \frac{|V_{e2}|^2}{1 - |V_{e3}|^2 - |V_{e1}|^2} \\
\sin^2 \theta_{13} = \frac{|V_{e3}|^2}{1 - |V_{e1}|^2} \\
\sin^2 \theta_{23} = \frac{|V_{e3}|^2(1 - |V_{e1}|^2) - |V_{e1}|^2|V_{e4}|^2}{(1 - |V_{e1}|^2)^2 - |V_{e4}|^2} \tag{15}
\]

where \( V_{ij} \) are the elements of mixing matrix in (10).

One of the important parameters relevant to neutrino sector is the Jarlskog invariant \( J \) which can be calculated from the elements of mixing matrix \( V_{ij} \) as shown in (19). The leptonic Dirac CP violating phase \( \delta_{CP} \) is related to \( J \) through the relation given by

\[
J = \text{Im}[V_{e1}V_{\mu2}V_{e2}^*V_{\mu1}^*] = s_{23}s_{12}s_{13}c_{13} \sin \delta_{CP} \tag{19}
\]

### 3 Description of the Model

This model uses the \( A_4 \) discrete group to develop the neutrino mass matrices along with \( Z_4 \) and \( Z_2 \) symmetry in normal hierarchy (NH) only. \( A_4 \) has four irreducible representations in which three are singlets \((1, 1’, 1’’)\) and one is triplet(3). We take the SM charged lepton doublet \( l \) as triplet under \( A_4 \) and the right-handed charged lepton singlets \((e_R, \mu_R, \tau_R)\) as singlets \((1, 1’, 1’’)\) respectively. It is convenient to extend the SM Higgs \( H \) by adding two more Higgs fields \( H’ \) and \( H’’ \) which are singlets under \( A_4 \) to produce a desired structure of neutrino mass model.

We extend the SM particle content with three right-handed singlet neutrinos \( \nu_{R1}, \nu_{R2}, \nu_{R3} \) and a singlet sterile neutrino field \( S \) with \( A_4 \) charges \( 1’, 1’ \) and \( 1’’ \) respectively. We use two \( A_4 \) triplet flavons \( \psi \) and \( \phi \) for generating \( M_L \) and \( M_D \) and one singlet \( \chi \) which will give the Majorana mass matrix \( M_R \). Another singlet flavon \( \zeta \) is responsible for generating the sterile mass matrix \( M_S \). The full particle contents and their group charges are shown in Table 2. The invariant Lagrangian for the leptonic interactions is given by

\[
\mathcal{L} = \mathcal{L}_{M_L} + \mathcal{L}_{M_D} + \mathcal{L}_{M_R} + \mathcal{L}_{M_S} + h.c. \tag{20}
\]
where,

\[ \mathcal{L}_{M_d} = \frac{y_1}{\Lambda} (\tilde{H}_d^\dagger \psi)_1 e_R + \frac{y_2}{\Lambda} (\tilde{H}_d^\dagger \psi)_1 \mu_R + \frac{y_3}{\Lambda} (\tilde{H}_d^\dagger \psi)_1 \tau_R \]  

(21)

\[ \mathcal{L}_{M_D} = \frac{y_1}{\Lambda} (\tilde{H}_D^\dagger \phi)_1 \nu R_1 + \frac{y_2}{\Lambda} (\tilde{H}_D^\dagger \phi)_1 \nu R_2 + \frac{y_3}{\Lambda} (\tilde{H}_D^\dagger \phi)_1 \nu R_3 \]  

(22)

\[ \mathcal{L}_{M_e} = \frac{1}{2} \lambda_1 \bar{\chi} R_1^c \nu R_1 + \frac{1}{2} \lambda_2 \bar{\chi} R_2^c \nu R_2 + \frac{1}{2} \lambda_3 \bar{\chi} R_3^c \nu R_3 \]  

(23)

\[ \mathcal{L}_{M_S} = \frac{1}{2} k \nu^c \nu R_1 \]  

(24)

The constant \( \Lambda \) denotes the cut-off scale and \( \tilde{H} = \tau_2 H \) (where \( \tau_2 \) is the second Pauli matrix) is used in order to make the Lagrangian gauge invariant whereas \( y_i, y_j, \lambda, \kappa \) where \( i = e, \mu, \tau ; j = 1, 2, 3 \) respectively, are the Yukawa coupling constants.

Terms like \( \frac{1}{\Lambda} (\tilde{H}_d^\dagger \psi)_1 \nu R_1, \frac{1}{\Lambda} (\tilde{H}_D^\dagger \phi)_1 S \) which are allowed in \( A_4 \) can be removed by an additional \( Z_2 \) symmetry. If we choose the \( T \)-diagonal basis of \( A_4 \) along with the most general flavon v.e.v alignments (Appendix A),

\[ \langle \psi \rangle = (v, 0, 0) \quad ; \quad \langle \phi \rangle = (v, v, v) \quad ; \quad \langle \chi \rangle = v \quad ; \quad \langle \zeta \rangle = u, \]  

(25)

then, we get a diagonal charged lepton mass matrix

\[ M_L = \frac{(H)_v}{\Lambda} \text{diag}(y_e, y_\mu, y_\tau). \]  

(26)

The Dirac, Majorana and sterile mass matrices take the following forms

\[ M_D^0 = \frac{(H)_v}{\Lambda} \begin{pmatrix} y_1 & y_2 & 0 \\ y_1 & y_2 & 0 \\ y_1 & y_2 & y_3 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ a & b & 0 \\ a & b & c \end{pmatrix}; \]  

(27)

\[ M_R = v \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{pmatrix}; \]  

(28)

\[ M_S = \begin{pmatrix} ku & 0 & 0 \end{pmatrix}, \]  

(29)

where \( a = \frac{(H)_v}{\Lambda} y_1, \quad b = \frac{(H)_v}{\Lambda} y_2, \quad c = \frac{(H)_v}{\Lambda} y_3, \quad d = \lambda_1 v, \quad e = \lambda_2 v, \quad f = \lambda_3 v. \)

In order to achieve the value of Yukawa coupling in the range below 0.01 and light neutrino masses approximately around 0.02 eV, we consider an approximate input values of the mass scales of the parameters given as, \( \Lambda \approx 10^{14} \text{GeV}, \nu \approx 10^{13} \text{GeV} \) and \( u \approx 10^2 \text{GeV}. \)

Applying MES mechanism with these mass matrices in (6), we obtain the active neutrino mass matrix as

\[ m_\nu^0 = \begin{pmatrix} \frac{b_1^2}{e} & \frac{b_2^2}{e} & \frac{b_3^2}{e} \\ \frac{b_1^2}{e} & \frac{b_2^2}{e} & \frac{b_3^2}{e} \\ \frac{b_1^2}{e} & \frac{b_2^2}{e} & \frac{b_3^2}{e} + c^2 \\ \end{pmatrix}. \]  

(30)
It is easy to see that $m_\nu$ is a $\mu - \tau$ symmetric matrix which give $\theta_{13} = 0$. But, recent experimental data has proven $\theta_{13}$ to be non-zero. In order to generate $\theta_{13} \neq 0$, $M_D$ is modified by adding a perturbation term $M_D'$. We can achieve this if we introduce an $SU(2)_L$ singlet flavon $\eta$ having similar $A_4 \otimes Z_4 \otimes Z_2$ charges as $\phi (3,-i,1)$ with vev alignment of $(v,0,0)$ in our model. The Lagrangian responsible for the perturbation matrix is

$$ L_{M_D'} = \frac{y_4}{\Lambda} (\bar{l}_1 \eta) H^* v_{R1} + \frac{y_4}{\Lambda} (\bar{l}_2 \eta) H^* v_{R2} + \frac{y_4}{\Lambda} (\bar{l}_3 \eta) H^* v_{R3} $$

Then, the perturbation matrix looks like

$$ M_D' = \frac{(H)_v}{\Lambda} \begin{pmatrix} 0 & 0 & y_4 \\ 0 & 0 & 0 \\ y_4 & 0 & 0 \end{pmatrix}. $$

The resultant active neutrino mass matrix with $M_D = M_D' + M_D$ from (6) becomes

$$ m_\nu \simeq - \begin{pmatrix} \frac{b^2}{e} + \frac{c^2}{f} & \frac{b(b+i)}{e} & \frac{b^2}{e} + \frac{c^2}{f} \\ \frac{b}{e} + \frac{ct}{f} & \frac{b(b+i)}{e} & \frac{b^2}{e} + \frac{c^2}{f} \\ \frac{b}{e} + \frac{ct}{f} & \frac{b(b+i)}{e} & \frac{b^2}{e} + \frac{c^2}{f} \end{pmatrix}, $$

where $t = \frac{(H)_v}{\Lambda} y_4$, and $y_4$ is the Yukawa coupling for the perturbation term.

The sterile neutrino mass is obtained from (7) as

$$ m_s \simeq - \left( \frac{g^2}{d} \right), $$

where $g = ku$.

The numerical bounds of the model parameters and mixing parameters obtained from our model will be determined in the next section through numerical analysis.

### 4 Numerical Analysis and Results

To validate the present model, we first try to solve the free parameters by comparing the LHS and RHS of (8) using (33). We take the current $3\sigma$ values of mixing angles, and mass squared differences from Table 1. We vary the unknown Majorana phases in the range $(0,2\pi)$ and we have fixed non-degenerate values for the heavy right-handed neutrino mass parameters $d = e \simeq 10^{13}$ GeV and $f \simeq 5 \times 10^{13}$ GeV. We numerically solve the model parameters $b, c$ and $t$ which satisfy the five independent equations with a tolerance of $O(10^{-2})$. The correlation plots among different models parameters are shown in Fig. 1. We find that the parameter space is very narrow, which can be verified or discarded in future experiments.

Remaining parameter $g$ is solved using the $3\sigma$ bounds on active-sterile mass-squared difference $|\Delta m^2_{41}| \in (0.87, 2.04) \text{eV}^2$ [19, 33, 34] while the parameter $a$ is constrained by the $3\sigma$ bounds of $|U_{1d}|^2 = (0.0098,0.031)$. The values of model parameters are given in Table 3. Now, the model-dependent $4 \times 4$ mixing matrix $V$ which can be developed in (10) can be used to solve other mixing angles using (13)-(18). The variations of different mixing angles with perturbation parameter $t$, are shown in Fig. 2. It is observed that many data points of $t$ are available within the $3\sigma$ ranges of the mixing angles. More data points are concentrated...
at regions $\sin^2 \theta_{23} > 0.50$ which implies that the present model favours higher octant of $\theta_{23}$. In Fig. 3, we show the correlation plots between $\sin^2 \theta_{23}$ with $\sin^2 \theta_{12}$ and $\sin^2 \theta_{13}$. Figure 4 shows the variation of active-sterile mixing elements. Allowed 3$\sigma$ bounds are shown in
the plots and it can be seen that the present model can give values within the experimental bounds. The variations of active-sterile mixing angle $\sin^2 \theta_{14}$ and leptonic mixing parameters $\sin^2 \theta_{13}$, $\sin^2 \theta_{23}$, $\sin^2 \theta_{12}$ and leptonic Dirac CP violating phase $\delta_{CP}$, are shown in Fig. 5. The Dirac CP violating phase is found to be in the range $-1.528 \leq \delta_{CP} \leq 1.451$. We develop the model dependent $4 \times 4$ active-sterile mixing matrix as

$$U = \begin{pmatrix}
0.7598 - 0.8420 & 0.5005 - 0.6013 & 0.1143 - 0.1796 & 0.0990 - 0.1760 \\
0.2407 - 0.5315 & 0.4326 - 0.6882 & 0.6243 - 0.7666 & 0.0990 - 0.1760 \\
0.0241 - 0.7692 & 0.4670 - 0.7060 & 0.5689 - 0.7439 & 0.0028 - 0.3375 \\
0.0990 - 0.1760 & 0.0990 - 0.1760 & 0.0028 - 0.3375 & 0.9143 - 0.9900
\end{pmatrix}$$

We calculate the effective neutrino mass $m_{\beta\beta}$ and $m_\mu$ and plot their variations with other mixing parameters in Fig. 6 and 7 respectively. We find that the mixing angles and effective neutrino mass parameters are within their allowed ranges. The variation of CP-violating Dirac phase $\delta_{CP}$ with the mixing angles is shown in Fig. 8.

### 5 Summary and Discussion

To summarise, we have developed an $A_4$ model supplemented by $Z_4$ and $Z_2$ groups. One singlet sterile neutrino is added to the 3-neutrino theory to explain $3 + 1$ active sterile neutrino masses and mixings. The distinctive feature of this model is that in MES mechanism we have considered only five scalar flavon fields: $\phi, \psi$ responsible for charged lepton mass $M_L$ and $M_D$, a singlet $\chi$ to generate Majorana mass matrix $M_R$ and $\zeta$ for the sterile neutrino mass $M_S$. Addition of another triplet flavon $\eta$ gives the desired breaking of $\mu - \tau$ symmetry.
in the active neutrino mass matrix. The active-sterile mixing matrix \( R \) provides the non-unitary contribution to the active-neutrino mixing matrix \( U_{PMNS} \). By constraining the light neutrino masses \( m_1, m_2, m_3 \) and sterile mass \( m_4 \) from the experimental mass-squared difference and mixing angles at \( 3\sigma \), we have determined the parameters of the model.

We have plotted the bounds on the active sterile mixing \( |V_{\mu 4}|^2 \) and \( |V_{\tau 4}|^2 \) from our model. A large number of data points are concentrated within the allowed ranges. We also calculate the effective mass parameters \( m_\beta \) and \( m_{\beta\beta} \). Their values are obtained in the ranges \( 0.0784 \text{ eV} < m_\beta < 0.3034 \text{ eV} \) and \( 0.0034 \text{ eV} < m_{\beta\beta} < 0.0667 \text{ eV} \) respectively. This is allowed in the latest study for upper bound \( m_\beta < 0.8 \text{ eV} \) at 90\% confidence level recently published by the KATRIN Collaborations [35]. This is also in agreement with the results of the analysis in Ref.[28], where active neutrinos mixing with light sterile sterile neutrino leads to an upper limit of \( m_\beta < 0.09 \text{ eV} \) and \( m_{\beta\beta} < 0.07 \text{ eV} \) at 95\% CL (Fig. 9).

At current status, the bounds on active-sterile mass squared difference \( |\Delta m_{41}|^2 \) is still not known. Many experiments give various constraints, such as, \( |\Delta m_{41}|^2 = 1.7 \text{eV}^2 \) [36], \( |\Delta m_{41}|^2 = 4.5\text{eV}^2 \) [37], \( |\Delta m_{41}|^2 < 10\text{eV}^2 \) [38], \( |\Delta m_{41}|^2 = 7.3 \pm 1.17\text{eV}^2 \) [39], etc. We have chosen a particular bound and performed the numerical analysis.

In this paper, we studied the existence of eV-scale sterile neutrino in NH only. For inverted hierarchy(IH), this model is not able to give solutions of the model parameters that satisfy the tolerance of \( O(10^{-2}) \), which was used in the case of NH. In order to accommodate IH of neutrino mass, there is a need to either change the vacuum alignment of flavon \( \phi \) or introduce new flavon field which will generate a different mass structure in (33) [5, 

![Correlation plots between active-sterile mixing angle $\sin^2 \theta_{14}$ and active neutrino mixing angles $\sin^2 \theta_{13}$, $\sin^2 \theta_{23}$, $\sin^2 \theta_{12}$ and Dirac CP-violating phase $\delta_{CP}$](image-url)
This is beyond the scope of the present paper and we shall address the scenario for IH in our future works. Further studies on the effects of KeV-scale sterile neutrino in the same model and other cosmological problems such as baryogenesis, dark matter, etc. may be carried out in the future communication. In conclusion, we have developed a model framework which is possible to explain the origin of neutrino masses and mixings through discrete symmetry. It can also generate an active-sterile neutrino mixing in the 3 + 1 MES mechanism.

Appendix : A

The v.e.v. alignment of the flavon fields $\phi$ and $\psi$ can be obtained by the minimisation condition of the total flavon potential of the model and by solving the simultaneous equations. Here, we shall consider the triplet flavons and their mutual interactions only. As we can see from Table 2, these flavons have different $Z_4$ charges. Thus, interaction among them are forbidden. The total flavon potential is given by

$$ V = V(\phi) + V(\psi) + V(int.) $$

where

$$ V(\phi) = -m_{\phi}^2 (\phi^\dagger \phi) + \lambda_\phi (\phi^\dagger \phi)^2 $$

Fig. 6 Variations of effective neutrino mass $m_{\beta\beta}$ with mixing angles
and \(V(\nu) = -m^2_\nu (\psi^\dagger \psi) + \lambda_\nu (\psi^\dagger \psi)^2\)

and \(V(\text{int.})\) will be zero in our model.

In component forms, the triplet flavons can be written as

\[
\phi = (\phi_1, \phi_2, \phi_3)
\]

\[
\psi = (\psi_1, \psi_2, \psi_3)
\]

The \(A_4\) product rule is given by [40]

\[
1 \times 1' = 1' ; 1 \times 1'' = 1'' ; 1' \times 1'' = 1 ; 1'' \times 1'' = 1'
\]

and

\[
3 \otimes 3 = 1 \oplus 1' \oplus 1'' \oplus 3_A \oplus 3_S \tag{A2}
\]

If we take two triplets \(\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}\) and \(\begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}\), the multiplication rule in a chosen basis \(^1\) will be
given by the above relation (A2) in which

\(^1\) Here, the \(T\)-diagonal basis is chosen
$$1 = a_1 b_1 + a_2 b_3 + a_3 b_2; \quad \text{(A3)}$$

$$1' = a_3 b_3 + a_1 b_2 + a_2 b_1; \quad \text{(A4)}$$

$$1'' = a_2 b_2 + a_1 b_3 + a_3 b_1; \quad \text{(A5)}$$

Fig. 8 Variations of leptonic Dirac CP violating phase $\delta_{CP}$ with active neutrino mixing angles in (a),(b),(c) and effective neutrino mass $m_{\beta\beta}$ in (d)

Fig. 9 (a). Correlation plots among effective mass parameters $m_{\beta\gamma}$ (b). Correlation plot between $m_{\beta\beta}$ and active sterile mixing angle $\sin^2 \theta_{34}$
\[
3_S = \frac{1}{3} \begin{pmatrix}
2a_1b_1 - a_2b_3 - a_3b_2 \\
2a_2b_3 - a_1b_2 - a_3b_1 \\
2a_3b_2 - a_1b_3 - a_2b_1
\end{pmatrix};
\]

(6)

\[
3_A = \frac{1}{2} \begin{pmatrix}
a_2b_3 - a_3b_2 \\
a_1b_3 - a_2b_1 \\
a_3b_1 - a_1b_3
\end{pmatrix}
\]

(7)

The potential for \( \phi \) becomes

\[
V(\phi) = -m_0^2(\phi^\dagger\phi)_1 + \lambda_0((\phi^\dagger\phi)_1(\phi^\dagger\phi)_1 + (\phi^\dagger\phi)_1(\phi^\dagger\phi)_2 + (\phi^\dagger\phi)_3(\phi^\dagger\phi)_3
+ (\phi^\dagger\phi)_3(\phi^\dagger\phi)_3) + (\phi^\dagger\phi)_3(\phi^\dagger\phi)_3]
\]

\[
= -m_0^2(\phi^\dagger_1\phi_1 + \phi^\dagger_2\phi_2 + \phi^\dagger_3\phi_3 + \lambda_0((\phi^\dagger_1\phi_1 + \phi^\dagger_2\phi_2 + \phi^\dagger_3\phi_3)^2
+ (\phi^\dagger_1\phi_2 + \phi^\dagger_2\phi_1 + \phi^\dagger_3\phi_3 + \phi^\dagger_3\phi_3)^2
+ (\phi^\dagger_2\phi_3 - \phi^\dagger_3\phi_2)^2 + (\phi^\dagger_3\phi_1 - \phi^\dagger_1\phi_3)^2
+ (\phi^\dagger_1\phi_2 - \phi^\dagger_2\phi_2 - \phi^\dagger_3\phi_3)^2 + (\phi^\dagger_2\phi_2 - \phi^\dagger_3\phi_3)^2
+ (\phi^\dagger_2\phi_3 - \phi^\dagger_3\phi_2)^2 + (\phi^\dagger_3\phi_1 - \phi^\dagger_1\phi_3)^2
+ (\phi^\dagger_1\phi_1 - \phi^\dagger_2\phi_2 - \phi^\dagger_3\phi_3)^2
+ (\phi^\dagger_1\phi_1 - \phi^\dagger_2\phi_2 - \phi^\dagger_3\phi_3)^2
+ (\phi^\dagger_2\phi_3 - \phi^\dagger_3\phi_2)^2 + (\phi^\dagger_3\phi_1 - \phi^\dagger_1\phi_3)^2
+ (\phi^\dagger_1\phi_1 - \phi^\dagger_2\phi_2 - \phi^\dagger_3\phi_3)^2
+ (\phi^\dagger_2\phi_3 - \phi^\dagger_3\phi_2)^2 + (\phi^\dagger_3\phi_1 - \phi^\dagger_1\phi_3)^2
+ (\phi^\dagger_1\phi_1 - \phi^\dagger_2\phi_2 - \phi^\dagger_3\phi_3)^2
+ (\phi^\dagger_2\phi_3 - \phi^\dagger_3\phi_2)^2 + (\phi^\dagger_3\phi_1 - \phi^\dagger_1\phi_3)^2
+ (\phi^\dagger_1\phi_1 - \phi^\dagger_2\phi_2 - \phi^\dagger_3\phi_3)^2
+ (\phi^\dagger_2\phi_3 - \phi^\dagger_3\phi_2)^2 + (\phi^\dagger_3\phi_1 - \phi^\dagger_1\phi_3)^2
+ (\phi^\dagger_1\phi_1 - \phi^\dagger_2\phi_2 - \phi^\dagger_3\phi_3)^2
\]

(8)

The minimization condition for the potential is derived by differentiating \( V(\phi) \) with respect to \( \phi_1, \phi_2 \) and \( \phi_3 \) respectively and equating them to zero i.e.

\[
\frac{dV(\phi)}{d\phi_1} = 0
\]

(9)

\[
\frac{dV(\phi)}{d\phi_2} = 0
\]

(10)

\[
\frac{dV(\phi)}{d\phi_3} = 0
\]

(11)

We can simultaneously solve these equations and find out many solutions. For example, two sets of solutions

\[
\phi_1 = \frac{m_\phi}{\sqrt{10\lambda_\phi}}, \quad \phi_2 = 0, \quad \phi_3 = 0
\]

and

\[
\phi_1 = \frac{m_\phi}{2\sqrt{3\lambda_\phi}}, \quad \phi_2 = \frac{m_\phi}{2\sqrt{3\lambda_\phi}}, \quad \phi_3 = \frac{m_\phi}{2\sqrt{3\lambda_\phi}}
\]

satisfy the above three minimization equations. Thus, the v.e.v. alignments

\[
\langle \phi \rangle = \frac{m_\phi}{\sqrt{10\lambda_\phi}}(1, 0, 0); \quad \langle \phi \rangle = \frac{m_\phi}{2\sqrt{3\lambda_\phi}}(1, 1, 1)
\]

are possible for \( \phi \).

The same sets of solutions can be derived for \( \psi \) and \( \eta \) also.

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