Continuum study of deconfinement at finite temperature

Axel Bender*, David Blaschke†, Yuri Kalinovsky‡ and Craig D. Roberts*

* Physics Division, Bldg. 203, Argonne National Laboratory, Argonne, Illinois 60439-4843
† MPG Arbeitsgruppe “Theoretische Vielteilchenphysik”
Universität Rostock, D-18051 Rostock, Germany
‡ Bogoliubov Laboratory of Theoretical Physics,
Joint Institute for Nuclear Research, 141980 Dubna, Russia

Abstract

Deconfinement and chiral symmetry restoration are explored in a confining, renormalisable, Dyson-Schwinger equation model of two-flavour QCD. An order parameter for deconfinement is introduced and used to establish that, in the chiral limit, deconfinement and chiral symmetry restoration are coincident at $T_c \approx 150\text{ MeV}$. The transitions are second order and each has the same critical exponent: $\beta \approx 0.3$. The deconfinement transition is found to exhibit sensitivity to the current-quark mass. $f_\pi$ and $m_\pi$ change by no more than 10% for $T < 0.7T_c$, however, as $T \to T_c$, thermal fluctuations cause the pion bound state contribution to the four-point quark-antiquark correlation function to disappear.

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The Dyson-Schwinger equations [DSEs] provide a nonperturbative, renormalisable, continuum framework for analysing quantum field theories. An important example is the fermion DSE, which has proven useful in the study of confinement and dynamical chiral symmetry breaking [DCSB] [1]. The DSEs form a tower of coupled equations, which must be truncated to arrive at a tractable problem. Truncations that preserve the global symmetries of a field theory are easy to implement. Preservation of a gauge symmetry is more difficult but there is progress in that direction [2]. The elementary quantities are the Schwinger functions whose analytic properties provide information about confinement and DCSB; e.g., the absence of a Lehmann representation for the two-point dressed-gluon Schwinger function (gluon propagator) entails the absence of asymptotic gluon states; i.e., gluon confinement. The approach is reviewed in Ref. [3].

The finite temperature properties of QCD are important in astrophysics and cosmology, and may be explored in a future-generation heavy-ion accelerator programme. Theoretical tools that can be employed reliably in the nonperturbative study of deconfinement and chiral symmetry restoration are therefore valuable. In Ref. [4] a one-parameter model dressed-gluon propagator was proposed, which provided a good description of π and ρ-meson observables. The single parameter in this model is a mass scale, \( m_t \), that marks the point where the nonperturbative, infrared enhancement found in gluon DSE studies [5] becomes dominant. The model gluon propagator has no Lehmann representation. The calculated quark propagator also has no Lehmann representation; therefore both the gluon and quark are confined. With \( m_t \) fixed at \( T = 0 \), one has a renormalisable DSE-model of QCD, which manifests both confinement and DCSB, whose finite temperature behaviour may provide insight into the finite temperature properties of QCD.

The DSE for the renormalised dressed-quark propagator at finite temperature involves a sum over Matsubara frequencies. In nonperturbative DSE studies, where the analytic structure of the dressed-quark propagator is calculated rather than assumed, it is necessary to perform this sum numerically because the usual analytic methods of evaluating it rely upon the quark propagator having a Lehmann representation, which may not be the case for a confined quark. The absence of a Lehmann representation also complicates, if not precludes, a real-time formulation of the finite temperature theory.

Herein we employ a Euclidean space formulation with \( \{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}, \gamma_\mu = \gamma_\mu^\dagger, \beta \equiv \gamma_4, \gamma \cdot p \equiv \sum_{i=1}^3 \gamma_i p_i, \omega_n \equiv (2n + 1)\pi T \) and \( \Omega_n \equiv 2n\pi T \), where \( T \) is the temperature. The DSE for the renormalised quark propagator, \( S \equiv [-i\gamma \cdot p\sigma_A(p, \omega_n) - i\beta \omega_n\sigma_C(p, \omega_n) + \sigma_B(p, \omega_n)] \), is

\[
S^{-1}(p, \omega_n) = Z_A^2 i\gamma \cdot p + Z_2 (i\beta \omega_n + m_{bm}) + \Sigma'(p, \omega_n),
\]

\( m_{bm} \) is the bare mass and the regularised self energy is

\[
\Sigma'(p, \omega_n) = i\gamma \cdot p \Sigma_A'(p, \omega_n) + i\beta \omega_n \Sigma_C'(p, \omega_n) + \Sigma_B'(p, \omega_n),
\]

with

\[
\Sigma_F'(p, \omega_n) = \int_{m,q} \frac{4}{3} g^2 D_{\mu\nu}(p - q, \omega_n - \omega_m) \times \frac{1}{4} \text{tr} [P_F \gamma_\mu S(q, \omega_m) \Gamma_\nu(q, \omega_m; p, \omega_n)]
\]

(3)
where \( \mathcal{F} = A, B, C; \mathcal{P}_A \equiv -(Z_A^4/p^2)i\gamma \cdot p, \mathcal{P}_B \equiv Z_1, \mathcal{P}_C \equiv -(Z_1/\omega_n)i\beta, \) and \( f^A_{m,q} \equiv T\sum_{m=-\infty}^{\infty} \int d^4q/(2\pi)^4. \) In Eq. (3), \( \Gamma_{\nu}(q, \omega_m; p, \omega_n) \) is the renormalised dressed quark-gluon vertex and \( D_{\mu\nu}(p, \Omega_n) \) is the renormalised dressed-gluon propagator. One can also write

\[
S^{-1}(p, \omega_n) \equiv i\gamma \cdot p A(p, \omega_n) + i\beta \omega_n C(p, \omega_n) + B(p, \omega_n). \tag{4}
\]

In renormalising we require that

\[
S^{-1}(p, \omega_0)|_{p^2 + \omega_0^2 - \mu^2} = i\gamma \cdot p + i\beta \omega_0 + m_R, \tag{5}
\]

which entails that the renormalisation constants are given by: \( Z_2^A(\mu, \Lambda) = 1 - \Sigma'_A(\mu, \omega_0; \Lambda), Z_2(\mu, \Lambda) = 1 - \Sigma_C(\mu, \omega_0; \Lambda), m_R(\mu) = Z_2m_{bm}(\Lambda^2) + \Sigma''_B(\mu, \omega_0; \Lambda), \) and the renormalised self energies are

\[
\mathcal{F}(p, \omega_n; \mu) = \kappa_F + \Sigma'_F(p, \omega_n; \Lambda) - \Sigma_F(p, \omega_n; \Lambda) - \kappa_F - 1 = \kappa_C \text{ and } \kappa_B = m_R(\mu).
\]

It is invalid to neglect \( A(p, \omega_n; \mu) \) and \( C(p, \omega_n; \mu) \) and their dependence on their arguments. In studying confinement the \((p, \omega_n)\)-dependence of \( A \) and \( C \) is qualitatively important since it can conspire with that of \( B \) to eliminate free-particle poles in the quark propagator \([7]\). Further, all calculated observables are quantitatively sensitive to 

\[
\text{In studying confinement the } (p, \omega_n) \text{-dependence of } A \text{ and } C \text{ is qualitatively important since it can conspire with that of } B \text{ to eliminate free-particle poles in the quark propagator} \[7\]. Further, all calculated observables are quantitatively sensitive to } A \text{ and } C; \text{ e.g., in the present study we find } A(0, \omega_n) \sim 1.5 \text{ and } C(0, \omega_n) \sim 1.5 - 1.8 \text{ (increasing with } T \text{) and they return to 1 at the renormalisation point} - \text{neglecting this variation entails } f_\pi \rightarrow 2f_\pi, \text{ which one may not be able to compensate by readjusting a model’s parameters.}
\]

To study the chiral limit one must solve the DSE with \( m_R(\mu) = 0, \text{ not } m_{bm} = 0. \) We denote solution functions obtained in the chiral limit by a subscript “0”; e.g., \( B_0(p, \omega_n; \mu) \) and \( \sigma_{B_0}(p, \omega_n; \mu) \).

The finite-\( T \) gluon propagator is

\[
g^2 D_{\mu\nu}(p, \Omega) = P^L_{\mu\nu}(p, \Omega)\Delta_F(p, \Omega) + P^T_{\mu\nu}(p)\Delta_G(p, \Omega) \tag{7}
\]

in Landau gauge, where

\[
P^T_{\mu\nu}(p) \equiv \begin{cases} 
0; \mu \text{ and/or } \nu = 4, \\
\delta_{ij} - \frac{p_i p_j}{p^2}; \mu, \nu = 1, 2, 3
\end{cases}
\tag{8}
\]

with \( P^T_{\mu\nu}(p) + P^L_{\mu\nu}(p, p_\perp) = \delta_{\mu\nu} - p_\mu p_\nu/\sum_{\alpha=1}^4 p_\alpha p_\alpha; \mu, \nu = 1, \ldots, 4. \) A “Debye-mass” for the gluon appears as a \( T \)-dependent contribution to \( \Delta_F. \)

We employ a parameter-free finite-\( T \) extension of the model dressed-gluon propagator of Ref. \([4]\): \( \Delta_F(p, \Omega) \equiv \mathcal{D}(p, \Omega; m_D) \) and \( \Delta_G(p, \Omega) \equiv \mathcal{D}(p, \Omega; 0); \)

\[
\mathcal{D}(p, \Omega; m) \equiv 4\pi^2 d \times \left[ \frac{2\pi}{T} \frac{m_D^2 \delta_{\Omega_0} \delta^3(p) + 1 - e^{[-(p^2 + \Omega^2 + m^2)/(4m^2)]}}{p^2 + \Omega^2 + m^2} \right], \tag{9}
\]

where \( d = 12/(33 - 2N_f) \) and the “Debye-mass” is \( m_D^2 = \bar{c}T^2, \bar{c} = 4\pi^2 dc, c = (N_c/3 + N_f/6), \) which is included in a manner analogous to that in Ref. \([3]\). There is no \( T \)-dependent mass in \( \Delta_G. \)
The first term in Eq. (9) is an integrable, infrared singularity that generates long-range effects associated with confinement. The second term ensures that, neglecting quantitatively unimportant logarithmic corrections, the propagator has the correct perturbative behaviour at large spacelike arguments. In this model the large-distance, confining effects associated with $\delta_0 \delta^3(p)$ are completely cancelled at small distances and one recovers the perturbative result. $D(p, \Omega; m)$ has no Lehmann representation and hence represents a confined particle, since this ensures the absence of gluon production thresholds in $S$-matrix elements describing colour-singlet to singlet transitions.

Equation (9) is an Ansatz based on $T = 0$ studies augmented by finite-$T$ perturbation theory. Solving the gluon DSE at finite-$T$ would provide additional insight but no such studies are currently available. Without further information; for example, solving the coupled gluon-quark DSE system or lattice estimates of the two-point gluon Schwinger function, we cannot be sure that the $T$-dependence we have introduced provides a qualitatively accurate representation of the temperature evolution of the gluon propagator. The paucity of relevant experimental data means that one cannot presently use experiment to constrain this extrapolation. This is the primary source of uncertainty in our study and we therefore advocate a cautious interpretation of our results near the transition temperature.

The model of Ref. [4] is recovered with two further simplifying specifications. The approximation

$$Z_1 = Z_2 \quad \text{and} \quad Z_1^A = Z_2^A$$

is used. Gauge invariance in finite-$T$ QED entails these Ward identities. In QCD the analogous identities are more complicated and this is only a simplifying truncation, which has proven qualitatively and quantitatively reliable in Landau gauge, $T = 0$ studies. In nonperturbative DSE studies $Z_2$ and $Z_2^A$ are finite and $\approx 1$.

In addition, the rainbow approximation

$$\Gamma_\mu(q, \omega_m; p, \omega_n) = \gamma_\mu$$

is used. In $T = 0$ studies this has proven to be reliable in Landau gauge; i.e., an efficacious phenomenology with a more sophisticated Ansatz only requires a small quantitative modification of the parameters that characterise the small-$k^2$ behaviour of the gluon propagator.

Confinement can be investigated by studying the analytic properties of the two-point Schwinger function of a given excitation. The confinement test proposed in Ref. [6], and used to good effect in Ref. [1], is appropriate to this task. Consider $\Delta_{B_0}(x,0) \equiv (T/2\pi x) \sum_{n=0}^{\infty} \Delta_{B_0}^n(x)$,

$$\Delta_{B_0}^n(x) \equiv \frac{2}{\pi} \int_0^\infty dp \, p \sin(px) \sigma_{B_0}(p, \omega_n).$$

For a free fermion of mass $m$: $\sigma_B(p, \omega_n) = \frac{m}{\sqrt{\omega_n^2 + p^2 + m^2}}$, hence $\Delta_{B_0}^n(x) = m \exp(-x \sqrt{\omega_n^2 + m^2})$. This illustrates that the $n = 0$ term dominates the sum.

For a free fermion, $M(x; T) \equiv -(d/dx) \ln |\Delta_{B_0}^0(x)| = \sqrt{\pi^2 T^2 + m^2}$, a finite constant. Finite-$T$ effects become important for $T \sim m/\pi$. In DSE studies, $m$ is a mass-scale characteristic of dynamical mass generation: $M_{E^{1/4}} \approx 300$ MeV, hence we expect finite-$T$ effects to become noticeable at $T \sim 100$ MeV.
A Schwinger function with complex-conjugate poles, with \( \Re(p^2) < 0 \) (timelike) and \( \Im(p^2) \propto b \), does not have a Lehmann representation; i.e., it represents a confined excitation with no associated asymptotic state. This is the nature of the quark Schwinger function obtained in Ref. [4]. In such cases \( \Delta^0_{B_0}(x) \) has at least one zero. Denote the position of the first zero in \( \Delta^0_{B_0}(x) \) by \( r_1 \), and define \( \nu_0 \equiv 1/r_1 \), then deconfinement is observed if, for some \( T = T_c, \nu_0(T_c) = 0 \); at this point thermal fluctuations have overwhelmed the confinement scale-parameter and the poles have migrated to the real-axis.

We solve the DSE for \( S(p,\omega_n) \) numerically with

\[
m_t = 0.69 \text{ GeV} , \quad m_R(\mu) = 1.1 \text{ MeV} ,
\]

which were fixed in Ref. [4] by requiring a best \( \chi^2 \)-fit to a range of \( \pi \)-meson observables at \( T = 0 \). We use \( d = 4/9 \) in Eq. (9) and renormalise at \( \mu = 9.47 \text{ GeV} \). With these choices our current study has no free parameters. We use \( \bar{\Lambda} \) from Table I of Ref. [4] are reproduced to within 6%.

At finite-\( T \) the pion mass is given by [4]

\[
m_\pi^2 N_\pi^2 = \langle m_R(\mu) \rangle \langle \bar{q}q \rangle_\pi
\]

with \( (\int d\mu_T \equiv (2T)/(\pi^2) \sum_{n=0}^{\infty} \int_0^\Lambda dp \ p^2) \)

\[
\langle m_R(\mu) \langle \bar{q}q \rangle_\pi \rangle \equiv
\]

\[
4 N_c \int d\mu_T \ B_0 \ \left( \sigma_{B_0} - B_0 \left[ \omega_n^2 \sigma_C^2 + p^2 \sigma_A^2 + \sigma_B^2 \right] \right) ,
\]

which vanishes linearly with \( m_R(\mu) \), and

\[
N_\pi^2 = N_c \int d\mu_T \ B_0^2 \ \left\{ \sigma_A^2 - 2 \left[ \omega_n^2 \sigma_C \sigma'_C + p^2 \sigma_A \sigma'_A \\
+ \sigma_B \sigma'_B \right] - \frac{4}{3} p^2 \left[ \omega_n^2 \left( \sigma_C \sigma''_C - (\sigma'_C)^2 \right) \\
+ p^2 \left( \sigma_A \sigma''_A - (\sigma'_A)^2 \right) + \sigma_B \sigma''_B - (\sigma'_B)^2 \right] \right\} ,
\]

which vanishes linearly with \( m_R(\mu) \), and

\[
f_\pi N_\pi = \left. \frac{\partial \sigma_B(p^2, \omega_n)}{\partial p^2} \right|_{p=0} \]

\[
2 N_c \int d\mu_T \ B_0 \ \left\{ \sigma_A \sigma_B + \frac{2}{3} p^2 \left( \sigma'_A \sigma_B - \sigma_A \sigma'_B \right) \right\} .
\]

Equation (14) provides an accurate estimate of the mass obtained in solving the dressed-ladder Bethe-Salpeter equation for the pion [4]. Equations (13)-(17) neglect corrections of order \( m_\pi^2 \), which are unimportant (<3%) for all values of \( T \); i.e., \( m_\pi \) is dominated by a linear response to \( m_R \) at all \( T \).

In deriving these formulae one assumes \( \Gamma_\pi = i\gamma_5 B_0 \), where \( \Gamma_\pi \) is the pion Bethe-Salpeter amplitude, which is a manifestation of Goldstone’s theorem in the DSEs [12]. Since \( B_0 \) is nonzero only if chiral symmetry is dynamically broken, these formulae are not valid above any chiral symmetry restoration temperature, \( T_c \).
We employ the simplest order parameter for DCSB:

\[ \chi \equiv B_0(p = 0, \omega_0). \]  

We plot \( \chi(T) \) and \( \nu_0(T) \) in Fig. 4. The curves, fitted on \( 120 \leq T \text{ (MeV)} \leq 150 \), are of the form \( \alpha (1 - T/T_\nu)^\beta \) with

\[
\begin{align*}
\alpha_\chi &= 1.1 \text{ GeV}, \quad \beta_\chi = 0.33, \quad T_\nu^\chi = 150 \text{ MeV} \\
\alpha_{\nu_0} &= 0.16 \text{ GeV}, \quad \beta_{\nu_0} = 0.30, \quad T_{\nu_0}^\chi = 150 \text{ MeV}.
\end{align*}
\]  

The transitions are coincident and have the same critical exponent, within errors: \( \sim 10\% \).

The massive quark is not deconfined at the same temperature. However, the behaviour of \( \nu_m \) undergoes a qualitative change at \( T_{\nu_m}^\chi = T_c^\chi \), where no contribution to \( B(p, \omega_\pi) \) remains that does not vanish as \( m_\pi \to 0 \). For \( T \leq T_{\nu_m}^\chi \): \( \alpha_{\nu_m} = 0.15 \text{ GeV}, \beta_{\nu_m} = 0.23, T_{\nu_m}^\chi = 160 \text{ MeV}; \) for \( T > T_{\nu_m}^\chi \): \( \alpha_{\nu_m} = 0.26 \text{ GeV}, \beta_{\nu_m} = 0.71, T_{\nu_m}^\chi = 180 \text{ MeV}. \) For \( T > T_c^\chi \), a fit with \( \beta_{\nu_m} = 0.23 \) has a five-times larger standard deviation. The \( u/d \)-quark in this model is therefore deconfined at \( T \sim 180 \text{ MeV} \), suggesting that the deconfinement temperature increases with current-quark mass.

We find that, on \( 120 \leq T \text{ (MeV)} \leq 150, N_\pi^2, N_\pi f_\pi \) and \( \langle m_R(\mu)(\bar{q}q)_\mu \rangle \) are described by \( \alpha (1 - T/T_c)^\beta \), with \( T_c = 150 \text{ MeV} \) in each case and

\[
\begin{align*}
\alpha_{N_\pi^2} &= (0.18 \text{ GeV})^2, \quad \beta_{N_\pi^2} = 1.1, \\
\alpha_{f_\pi N_\pi} &= (0.15 \text{ GeV})^2, \quad \beta_{f_\pi N_\pi} = 0.93, \\
\alpha_{\langle m_R(\mu)(\bar{q}q)_\mu \rangle} &= (0.15 \text{ GeV})^4, \quad \beta_{\langle m_R(\mu)(\bar{q}q) \rangle} = 0.92.
\end{align*}
\]  

In order to avoid round-off errors associated with division near \( T = T_c \) we used these results to determine

\[
\begin{align*}
\alpha_{m_\pi} &= 0.12 \text{ GeV}, \quad \beta_{m_\pi} = -0.11, \\
\alpha_{f_\pi} &= 0.12 \text{ GeV}, \quad \beta_{f_\pi} = 0.36,
\end{align*}
\]  

with \( T_{c m_\pi} = 150 \text{ MeV} = T_{c f_\pi} \). We plot \( m_\pi(T) \) and \( f_\pi(T) \) in Fig. 2.

We explored the finite-\( T \) properties of a renormalisable, confining, DSE-model of QCD [4]. Introducing an order parameter for confinement we found that, in the chiral limit, a deconfinement transition at \( T \approx 150 \text{ MeV} \) is accompanied by the coincident restoration of chiral symmetry. The single model parameter, fixed at \( T = 0 \), is appropriate for two-flavour QCD: the transitions are second order and are not described by mean-field critical exponents. Similar results have been obtained in recent numerical simulations of lattice-QCD at finite-\( T \) [3].

\( f_\pi \) and \( m_\pi \) are weakly sensitive to \( T \) for \( T < 0.7 T_c^\chi \). However, as \( T \) approaches \( T_c^\chi \), the mass eigen value in the pion Bethe-Salpeter equation moves to increasingly larger values, as thermal fluctuations overwhelm attraction in the channel, until at \( T = T_c^\chi \) there is no solution and \( f_\pi \to 0 \). This means that the pion-pole contribution to the four-point, quark-antiquark correlation function disappears; i.e., there is no quark-antiquark pseudoscalar bound state for \( T > T_c^\chi \). This may have important consequences for a wide range of physical observables [4], if borne out by improved studies; e.g., this \( T \)-dependence of \( f_\pi \) and \( m_\pi \) would lead to a \( 20\% \) reduction in the \( \pi \to \mu \nu_\mu \) decay widths at \( T \approx 0.9 T_c^\chi \).
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FIG. 1. The parameters for the curves, \( \chi(T) \) (diamonds) and \( \nu_0(T) \) (circles), are presented in Eq. (19).
FIG. 2. $f_\pi(T)$ (diamonds) and $m_\pi(T)$ (circles) plotted against $T$. See the discussion around Eq. (21).