PARAMETRIC SPECTRAL ANALYSIS: SCALE AND SHIFT

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Abstract. We introduce the paradigm of dilation and translation for use in the spectral analysis of complex-valued univariate or multivariate data. The new procedure stems from a search on how to solve ambiguity problems in this analysis, such as aliasing because of too coarsely sampled data (see [8]), or collisions in projected data (see [7]), which may be solved by a translation of the sampling locations.

In Section 2 both dilation and translation are first presented for the classical one-dimensional exponential analysis. In the subsequent Sections 3–7 the paradigm is extended to more functions, among which the trigonometric functions cosine, sine, the hyperbolic cosine and sine functions, the Chebyshev and spread polynomials, the sinc, gamma and Gaussian function, and several multivariate versions of all of the above.

Each of these function classes needs a tailored approach, making optimal use of the properties of the base function used in the considered sparse interpolation problem. With each of the extensions a structured linear matrix pencil is associated, immediately leading to a computational scheme for the spectral analysis, involving a generalized eigenvalue problem and several structured linear systems.

In Section 8 we illustrate the new methods in several examples: fixed width Gaussian distribution fitting, sparse cardinal sine or sinc interpolation, and lacunary or supersparse Chebyshev polynomial interpolation.

Keywords. Prony problems, spectral analysis, parametric methods, sparse interpolation, dilation, translation, structured matrix, generalized eigenvalue problem.

1. Introduction

Exponential analysis and sparse \( n \)-term polynomial interpolation from \( 2n \) or more regularly spaced interpolation points \( t_j \) can be traced back to the exponential fitting method

\[
f(t_j) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i t_j), \quad \alpha_i, \phi_i \in \mathbb{R}, \quad t_j \in \mathbb{R}
\]

of de Prony from the 18-th century [9]. Almost 200 years later this basic problem was reformulated as a generalized eigenvalue problem [12].

More recently, problem statement [1] was partially generalized, on the one hand to the use of non-standard polynomial bases such as the Pochhammer basis and Chebyshev and Legendre polynomials [16,10,13,26,23] and on the other hand to the use of some eigenfunctions of linear operators [22,24,28].
Many of these generalizations are unified in the algebraic framework described in [15]. We generalize (1) to nonlinear sparse interpolation problems of the form

\[ f(t_j) = \sum_{i=1}^{n} \alpha_i g(\phi_i; t_j), \quad \alpha_i, \phi_i \in \mathbb{C}, \quad t_j \in \mathbb{R} \]

for a good collection of parameterized functions \( g(\phi; t) \) and some multivariate formulations thereof. The interpolant is obtained directly from the evaluations \( f(t_j) \) where the \( t_j \) follow a regular interpolation point pattern associated with the specific building blocks \( g(\phi_i; t) \). In addition, for the reconstruction of the underlying model \( f(t) \) from these structured univariate or multivariate samples, we introduce the wavelet inspired paradigm of a selectable dilation and translation of the interpolation points.

The new method follows from a search on how to solve ambiguity problems in exponential analysis, such as aliasing which arises from too coarsely sampled data, or collisions which may occur when handling projected data. Fine or coarse sampling is controlled by the choice of a scale or dilation which allows to stretch and shrink the uniform sampling scheme required for the spectral analysis. The mentioned ambiguity problems can be solved by means of a one- or multidimensional translation of the sampling locations, also called an identification shift [4, 5, 8].

The functions \( g(\phi_i; t) \) and the set of points \( t_j \) we consider, satisfy a discrete generalized eigenfunction relation of the form

\[
L \sum_{k=-L}^{L} a_k g(\phi_i; t_{j+k}) = \lambda_{ij} \sum_{k=-R}^{R} b_k g(\phi_i; t_{j+k}), \quad \lambda_{ij} \in \mathbb{C}.
\]

This property allows to split the nonlinear interpolation problem (2) into the separate computation of the nonlinear parameters \( \phi_i \) on the one hand and the linear \( \alpha_i \) on the other, as in de Prony’s method. Another useful property of the atoms \( g(\phi_i; t) \) that we consider in (3), is that the effect of the scale and the shift on the sampling process is separable, as in \( \exp(\tau+\sigma t) = \exp(\tau) \exp(\sigma t) \).

We present a coherent study undertaken at the occasion of [4, 5, 8] and we incorporate a scale and shift strategy which is new when dealing with more general base functions. We also connect the use of a particular base function \( g(\phi; t) \) to a specific linear matrix pencil of structured matrices in which the function samples are organized. This creates several algorithmic approaches.

Altogether, we solve the sparse nonlinear interpolation problem (2) by reformulating it as a structured generalized eigenvalue problem derived from (3) and one or more structured linear systems of equations. While by the introduction of a scale factor, we loose the uniqueness of the solution in the nonlinear step of the algorithm, we gain the option to stretch, shrink and eventually translate an otherwise uniform scheme of sampling points.
The translation of the sample points in the linear step of the algorithm also allows to restore the uniqueness lost in the nonlinear step.

In each of the subsequent sections on the trigonometric functions, polynomial functions, the Gaussian distribution, some special functions and multivariate versions of all these, a different approach is required to solve the sparse interpolation problem, tailored to the particular properties of the function or function class under consideration.

The list of functions $g(\phi; t)$ that the theory covers, includes the exponential function, the trigonometric functions cosine, sine, the hyperbolic cosine and sine functions, the Chebyshev (1-st, 2-nd, 3-rd, 4-th kind) and spread polynomials, the Gaussian function, the sinc and gamma function, and several multivariate versions of all of the above.

2. Exponential analysis

The mother of all functions to explain the principle with, is the exponential function

$$g(\phi; t) := \exp(\phi t).$$

2.1. Scale and shift scheme. By a combination of [9] and [5] we obtain the following parametric spectral analysis method. Let the signal $f(t)$ be given by

$$f(t) = \sum_{i=1}^{n} \alpha_i \exp(\phi_i t), \quad \alpha_i, \phi_i \in \mathbb{C}. \quad (4)$$

We assume that we can sample $f(t)$ at the equidistant points $t_j = j\Delta$ for $j = 0, 1, 2, \ldots$ with $\Delta \in \mathbb{R}^+$, or more generally at $t_{\tau+j\sigma} = (\tau + j\sigma)\Delta$ with $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Z}$, where the frequency content in (4) is limited by

$$|\mathcal{J}(\phi_i)|\Delta < \pi, \quad i = 1, \ldots, n. \quad (5)$$

More generally $\sigma$ and $\tau$ can belong to $\mathbb{Q}^+$ and $\mathbb{Q}$ respectively, as discussed further below. The values $\sigma$ and $\tau$ are called the scaling factor and shift term respectively. We denote the collected samples by

$$f_{\tau+j\sigma} := f(t_{\tau+j\sigma}), \quad j = 0, 1, 2, \ldots \quad (6)$$

The generalized eigenvalue type equation (3) satisfied by $g(\phi; t) = \exp(\phi t)$ and the $t_j$ is

$$\exp(\phi_j t_{j+1}) = \exp(\phi_j \Delta) \exp(\phi_j t_j).$$

So in (3) $L = 1, R = 0$ and $\lambda_{ij} = \exp(\phi_i(t_{j+1} - t_j))$. From (6) we find that

$$f_{j+1} = \sum_{i=1}^{n} \alpha_i \exp(\phi_i \Delta) \exp(\phi_i j \Delta),$$

or more generally for $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Z}$ that

$$f_{\tau+j\sigma} = \sum_{i=1}^{n} \alpha_i \exp(\phi_i \tau \Delta) \exp(\phi_i j \sigma \Delta). \quad (7)$$
Hence we see that the scaling $\sigma$ and the shift $\tau$ are separated in a natural way when evaluating (4) at $t_{\tau+j\sigma}$, a property that plays an important role in the sequel. The freedom to choose $\sigma$ and $\tau$ when setting up the sampling scheme, allows to stretch, shrink and translate the otherwise uniform progression of sampling points dictated by the sampling step $\Delta$.

2.2. Generalized eigenvalue formulation. The aim is now to estimate the model order $n$, and the parameters $\phi_1, \ldots, \phi_n$ and $\alpha_1, \ldots, \alpha_n$ in (4) from samples $f_j$ at a selection of points $t_j$.

With $n, \sigma \in \mathbb{N}, \tau \in \mathbb{Z}$ we define

$$\tau H_n := \begin{pmatrix} f_{\tau} & f_{\tau+\sigma} & \cdots & f_{\tau+(n-1)\sigma} \\ f_{\tau+\sigma} & \vdots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ f_{\tau+(n-1)\sigma} & \cdots & f_{\tau+(2n-2)\sigma} \end{pmatrix}.$$  

It is well-known that the Hankel matrix $\tau H_n$ can be decomposed as

$$\tau H_n = V_n A_n A_n^T,$$

where

$$V_n = \begin{pmatrix} 1 \\ \exp(\phi_1 \sigma \Delta) & \cdots & \exp(\phi_n \sigma \Delta) \\ \vdots & \ddots & \vdots \\ \exp(\phi_1 (n-1) \sigma \Delta) & \cdots & \exp(\phi_n (n-1) \sigma \Delta) \end{pmatrix},$$

$$A_n = \text{diag}(\alpha_1, \ldots, \alpha_n),$$

$$\Lambda_n = \text{diag}(\exp(\phi_1 \tau \Delta), \ldots, \exp(\phi_n \tau \Delta)).$$

This decomposition on the one hand translates (7) and on the other hand connects it to a generalized eigenvalue problem: the values $\exp(\phi_i \sigma \Delta)$ can be retrieved [12] as the generalized eigenvalues of the problem

$$(\tau H_n) v_i = \exp(\phi_i \sigma \Delta) (\tau H_n) v_i, \quad i = 1, \ldots, n,$$

where $v_i$ are the generalized right eigenvectors. Setting up this generalized eigenvalue problem requires the $2n$ samples $f_{j\sigma}, j = 0, \ldots, 2n-1$. In [9, 12] the choices $\sigma = 1$ and $\tau = 0, 1$ are made for (10) and then, from the generalized eigenvalues $\exp(\phi_i \Delta)$, the complex numbers $\phi_i$ can be retrieved uniquely because of the restriction $|\Im(\phi_i)| \Delta < \pi$. Choosing $\sigma > 1$ offers a number of advantages though [5, 25], among which:

- reconditioning [10, 8] of a possibly ill-conditioned problem statement,
- superresolution [8, 3] in the case of clustered frequencies,
- validation [1] of the exponential analysis output for $n$ and the $\phi_i$.

With $\sigma > 1$ the $\phi_i$ cannot necessarily be retrieved uniquely from the generalized eigenvalues $\exp(\phi_i \sigma \Delta)$ since

$$|\Im(\phi_i)| \sigma \Delta < \sigma \pi.$$

Let us indicate how to solve that problem.
2.3. **Vandermonde structured linear systems.** For chosen $\sigma$ and with $\tau = 0$, the $\alpha_i$ are computed from the interpolation conditions

\[(12) \quad \sum_{i=1}^{n} \alpha_i \exp(\phi_i t_{j\sigma}) = f_{j\sigma}, \quad j = 0, \ldots, 2n - 1, \quad \sigma \in \mathbb{N},\]

either by solving the system in the least squares sense, in the presence of noise, or by solving a subset of $n$ interpolation conditions in case of a noise-free $f(t)$. The samples of $f(t)$ occurring in (12) are the same samples as the ones used to fill the Hankel matrices in (10) with. In a noisy context the Hankel matrices in (10) can also be extended to rectangular $N \times n$ matrices with $N > n$ and the generalized eigenvalue problem can be considered in a least squares sense [2]. In that case the index $j$ in (12) runs from 0 to $N + n - 1$. Note that

\[
\exp(\phi_i t_{j\sigma}) = (\exp(\phi_i \sigma \Delta))^{\frac{j}{\sigma}},
\]

and that for fixed $\sigma$ the coefficient matrix of (12) is therefore a Vandermonde matrix.

Next, for chosen nonzero $\tau$, a shifted set of at least $n$ samples $f_{\tau+j\sigma}$ is interpreted as

\[(13) \quad f_{\tau+j\sigma} = \sum_{i=1}^{n} (\alpha_i \exp(\phi_i \tau \Delta)) \exp(\phi_i j \sigma \Delta), \quad j = k, \ldots, k + n - 1, \quad \tau \in \mathbb{Z},\]

where $k \in \{0, 1, \ldots, n\}$ is fixed. Note that (13) is merely a shifted version of the original problem (4) for $g(\phi_i; t) = \exp(\phi_i t)$, where the effect of the shift is pushed into the coefficients of (4). The latter is possible because of (7). From (13), having the same (but maybe less) coefficient matrix entries as (12), we compute the unknowns $\alpha_i \exp(\phi_i \tau \Delta)$. From $\alpha_i$ and $\alpha_i \exp(\phi_i \tau \Delta)$ we then obtain

\[
\frac{\alpha_i \exp(\phi_i \tau \Delta)}{\alpha_i} = \exp(\phi_i \tau \Delta),
\]

from which again the $\phi_i$ cannot be extracted unambiguously if $\tau > 1$. But the following could be proved [3].

Denote

\[
\begin{align*}
    s_{i,\sigma} &:= \text{sign} \left( \mathcal{J} \left( \text{Ln} \left( \exp(\phi_i \sigma \Delta) \right) \right) \right) \\
    s_{i,\tau} &:= \text{sign} \left( \mathcal{J} \left( \text{Ln} \left( \exp(\phi_i \tau \Delta) \right) \right) \right)
\end{align*}
\]

where $|\mathcal{J} \left( \text{Ln} \left( \exp(\phi_i \sigma \Delta) \right) \right)| \leq \pi$. If $\gcd(\sigma, \tau) = 1$, then the sets

\[
S_i = \left\{ \frac{1}{\sigma \Delta} \text{Ln} \left( \exp(\phi_i \sigma \Delta) \right) + \frac{2\pi i}{\sigma \Delta} \ell, \quad \ell = -s_{i,\sigma} \lfloor \sigma/2 \rfloor, \ldots, 0, \ldots, s_{i,\sigma} \lceil \sigma/2 \rceil - 1 \right\}
\]

and

\[
T_i = \left\{ \frac{1}{\tau \Delta} \text{Ln} \left( \exp(\phi_i \tau \Delta) \right) + \frac{2\pi i}{\tau \Delta} \ell, \quad \ell = -s_{i,\tau} \lfloor \tau/2 \rfloor, \ldots, 0, \ldots, s_{i,\tau} \lceil \tau/2 \rceil - 1 \right\},
\]

which contain all the possible arguments for $\phi_i$ in $\exp(\phi_i \sigma \Delta)$ from (10) and in $\exp(\phi_i \tau \Delta)$ from (13) respectively, have a unique intersection [3]. Here $\text{Ln}(\cdot)$ indicates the principal branch of the complex natural logarithm and,
with $\sigma > 1$, the elements in $S_i$ and $T_i$ have imaginary parts in the open interval $(-\pi/\Delta, \pi/\Delta)$. How to obtain this unique element in the intersection and identify the $\phi_i$ is detailed in [8, 1].

So at this point the nonlinear parameters $\phi_i, i = 1, \ldots, n$ and the linear $\alpha_i, i = 1, \ldots, n$ in [4] are computed through the solution of (10) and (12), and if $\sigma > 1$ also (13).

2.4. Computational variants. Besides having $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Z}$, more general choices are possible [6]. An easy practical generalization is when the scale factor and shift term are rational numbers $\sigma/\rho_1$ and $\tau/\rho_2$ respectively with $\sigma, \rho_1, \rho_2 \in \mathbb{N}$ and $\tau \in \mathbb{Z}$. In that case the condition $\gcd(\sigma, \tau) = 1$ for $S_i$ and $T_i$ to have a unique intersection, is replaced by $\gcd(\sigma, \tau) = 1$ where $\sigma/\rho_1 = \sigma/\rho, \tau/\rho_2 = \tau/\rho$ with $\rho = \text{lcm}(\rho_1, \rho_2)$.

We remark that, although the sparse interpolation problem can be solved from the $2n$ samples $f_{j, \sigma}, j = 0, \ldots, 2n - 1$ when $\sigma = 1$, we need at least an additional $n$ samples at the shifted locations $t_{\tau+j, \sigma}, j = k, \ldots, k+n - 1$ when $\sigma > 1$. The former is Prony’s original problem statement in [9] and the latter is the generalization presented in [5]. The factorisation (9) allows some alternative computational schemes, which may deliver a better numerical accuracy but demand somewhat more samples.

First we remark that the use of a shift $\tau$ can of course be replaced by the choice of a second scale factor $\tilde{\sigma}$ relatively prime with $\sigma$. But this option requires the solution of two generalized eigenvalue problems of which the generalized eigenvalues need to be matched in a combinatorial step. Also, the sampling scheme looks different and requires the $4n - 1$ sampling points

$$\{t_{j, \sigma}, 0 \leq j \leq 2n - 1\} \cup \{t_{j, \tilde{\sigma}}, 0 \leq j \leq 2n - 1\}, \quad \gcd(\sigma, \tilde{\sigma}) = 1.$$ 

A better option is to set up the generalized eigenvalue and eigenvector problem

$$T_i H_n v_i = \exp(\phi_i \tau \Delta) H_n v_i, \quad i = 1, \ldots, n$$

which in a natural way connects each eigenvalue $\exp(\phi_i \tau \Delta)$, bringing forth the set $T_i$, to its associated eigenvector $v_i$ by

$$0_\sigma H_n v_i = \alpha_i (1, \exp(\phi_i \Delta), \ldots, \exp(\phi_i (n-1) \Delta))^T,$$

bringing forth the set $S_i$. The latter is derived from the quotient of any two consecutive entries in the vector $0_\sigma H_n v_i$. Such a scheme requires the $4n - 2$ samples

$$\{t_{j, \tau}, 0 \leq j \leq 2n - 2\} \cup \{t_{\tau+j, \sigma}, 0 \leq j \leq 2n - 2\}, \quad \gcd(\sigma, \tau) = 1.$$ 

Note that the generalized eigenvectors $v_i$ are actually insensitive to the shift $\tau$: the eigenvectors of (10) and (14) are identical. This is a remarkable fact that reappears in each of the subsequent (sub)sections dealing with other choices for $g(\phi_i; \tau)$. 
2.5. **Determining the sparsity.** What can be said about the number of terms \( n \) in (4), which is also called the sparsity? From [11, p. 603] and [14] we know for general \( \sigma \) that

\[
\det_\sigma^0 H_\nu = 0 \text{ accidentally, } \quad \nu < n, \\
\det_\sigma^0 H_n \neq 0, \quad (15) \\
\det_\sigma^0 H_\nu = 0 \quad \nu > n.
\]

A standard approach to make use of this statement is to compute a singular value decomposition of the Hankel matrix \( \sigma H_\nu \) and this for increasing values of \( \nu \). In the presence of noise and/or clustered eigenvalues, this technique is not always reliable and we need to consider rather large values of \( \nu \) for a correct estimate of \( n \) [3] or turn our attention to some validation add-on [1].

With \( \sigma = 1 \) and in the absence of noise, the exponential analysis problem can be solved from 2\( n \) samples for \( \alpha_1, \ldots, \alpha_n \) and \( \phi_1, \ldots, \phi_n \) and at least one additional sample to confirm \( n \). As pointed out already, it may be worthwhile to take \( \sigma > 1 \) and throw in at least an additional \( n \) values \( \tau + j \sigma \) to remedy the aliasing (11). Moreover, if \( \max_i |\phi_i| \) is quite large, then \( \Delta \) may become so small that collecting the samples \( f_j \) becomes expensive and so it may be more feasible to work with a larger sampling interval \( \sigma \Delta \).

We now turn our attention to the identification of other families of parameterized functions and patterns of sampling points that jointly satisfy (3). We distinguish between trigonometric, polynomial and other functions and generalize some to the multivariate case.

### 3. Trigonometric functions

The generalized eigenvalue formulation (10) incorporating the scaling parameter \( \sigma \) was generalized to \( g(\phi_i; t) = \cos(\phi_i t) \) in [10] for integer \( \phi_i \) only. Here we present a more elegant full generalization of (2) for \( \cos(\phi_i t) \) including the use of a shift \( \tau \) as in (13) to restore uniqueness of the solution. In addition we generalize the scale and shift approach to the functions sine, cosine hyperbolic and sine hyperbolic.

#### 3.1. Cosine function.

Let \( g(\phi_i; t) = \cos(\phi_i t) \) with \( \phi_i \in \mathbb{R} \) with

\[
|\phi_i| \Delta < \pi, \quad i = 1, \ldots, n.
\]

Since \( \cos(\phi_i t) = \cos(-\phi_i t) \), we are only interested in the \( |\phi_i|, i = 1, \ldots, n \), disregarding the sign of each \( \phi_i \). With \( t_j = j \Delta \) the relation (3) satisfied by \( g(\phi_i; t) \) and the \( t_j \) is

\[
\frac{1}{2} \cos(\phi_i t_{j+1}) + \frac{1}{2} \cos(\phi_i t_{j-1}) = \cos(\phi_i \Delta) \cos(\phi_i t_j). \\
(17)
\]

So in (3) \( L = 1, R = 0 \) and \( \lambda_{ij} = \cos(\phi_i(t_{j+1} - t_j)) \). We still denote

\[
f_{\tau + j \sigma} := \sum_{i=1}^n \alpha_i \cos(\phi_i(\tau + j \sigma) \Delta), \\
(18)
\]
but because of (17) we now also introduce for fixed chosen $\sigma$ and $\tau$,

$$F_{\tau + j\sigma} := F(\sigma, \tau; t_j) = \frac{1}{2} f_{\tau + j\sigma} + \frac{1}{2} f_{\tau - j\sigma},$$

(19)

$$= \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta) \cos(\phi_i j \sigma \Delta).$$

Relation (17) deals with the case $\sigma = 1$ and $\tau = 1$, while the expression $F_{\tau + j\sigma}$ is a generalization of (17) for general $\sigma$ and $\tau$. Observe the achieved separation in (19) of the scaling $\sigma$ and the shift $\tau$. We emphasize that $\sigma$ and $\tau$ are fixed before defining the function $F(\sigma, \tau; t)$. Otherwise the index $j$ cannot be associated uniquely with the value $1/2(f_{\tau + j\sigma} + f_{\tau - j\sigma})$.

Besides the Hankel structured $\tau \sigma H_n$, we also introduce the Toeplitz structured

$$\tau \sigma T_n := \begin{pmatrix} f_\tau & f_{\tau - \sigma} & \cdots & f_{\tau - (n-1)\sigma} \\ f_{\tau + \sigma} & \ddots & \vdots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ f_{\tau + (n-1)\sigma} & \cdots & f_\tau \end{pmatrix},$$

which for $g(\phi_i; t) = \cos(\phi_i t)$ becomes symmetric when $\tau = 0$. Now consider the structured matrix

(20)

$$\tau \sigma C_n := \frac{1}{4} (\tau \sigma H_n) + \frac{1}{4} (-\tau \sigma H_n) + \frac{1}{4} (\tau \sigma T_n) + \frac{1}{4} (-\tau \sigma T_n),$$

where $-\tau \sigma T_n = \tau \sigma T_n^T$. When $\tau = 0$, the first two matrices in the sum coincide and the latter two do as well. Note that working directly with the cosine function instead of expressing it in terms of the exponential as $\cos x = (\exp(ix) + \exp(-ix))/2$ reduces the size of the involved matrices from $2n$ to $n$.

**Theorem 1.** The matrix $\tau \sigma C_n$ factorizes as

$$\tau \sigma C_n = W_n L_n A_n W_n^T,$$

$$W_n = \begin{pmatrix} \cos(\phi_1 \sigma \Delta) & \cdots & \cos(\phi_n \sigma \Delta) \\ \vdots & \ddots & \vdots \\ \cos(\phi_1 (n-1) \sigma \Delta) & \cdots & \cos(\phi_n (n-1) \sigma \Delta) \end{pmatrix},$$

$$A_n = \text{diag}(\alpha_1, \ldots, \alpha_n),$$

$$L_n = \text{diag}(\cos(\phi_1 \tau \Delta), \ldots, \cos(\phi_n \tau \Delta)).$$

**Proof.** The proof is a verification of the matrix product entry at position $(k + 1, \ell + 1)$ for $k, \ell = 0, \ldots, n - 1$:

$$\frac{1}{4} f_{\tau + (k + \ell)\sigma} + \frac{1}{4} f_{\tau - (k + \ell)\sigma} + \frac{1}{4} f_{\tau + (k - \ell)\sigma} + \frac{1}{4} f_{\tau - (k - \ell)\sigma}$$

$$= \frac{1}{2} \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta) \cos(\phi_i (k + \ell) \sigma \Delta) + \frac{1}{2} \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta) \cos(\phi_i (k - \ell) \sigma \Delta)$$

$$= \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta) \cos(\phi_i k \sigma \Delta) \cos(\phi_i \ell \sigma \Delta).$$

$\square$
This matrix factorization translates (19) and opens the door to the use of a generalized eigenvalue problem: the cosine equivalent of (10) becomes
\begin{equation}
(\sigma C_n) v_i = \cos(\phi_i \sigma \Delta) (\sigma C_n) v_i, \quad i = 1, \ldots, n,
\end{equation}
where \( v_i \) are the generalized right eigenvectors. Setting up (21) takes \( 2n \) evaluations \( f_{j\sigma} \), as in the exponential case. Before turning our attention to the extraction of the \( \phi_i \) from the generalized eigenvalues \( \cos(\phi_i \sigma \Delta) \), we solve two structured systems of interpolation conditions.

The coefficients \( \alpha_i \) in (18) are computed from
\[
\sum_{i=1}^{n} \alpha_i \cos(\phi_i j \sigma \Delta) = f_{j\sigma}, \quad j = 0, \ldots, 2n-1, \quad \sigma \in \mathbb{N}.
\]

Making use of (19), the coefficients \( \alpha_i \cos(\phi_i \tau \Delta) \) are obtained from the shifted interpolation conditions
\begin{equation}
\sum_{i=1}^{n} (\alpha_i \cos(\phi_i \tau \Delta)) \cos(\phi_i j \sigma \Delta) = F_{\tau+j\sigma}, \quad j = k, \ldots, k+n-1, \quad \tau \in \mathbb{Z},
\end{equation}
where \( k \in \{0,1,\ldots,n\} \) is fixed. While for \( \sigma = 1 \) the sparse interpolation problem [2] with \( g(\phi_i; \ell) = \cos(\phi_i \ell) \) can be solved from \( 2n \) samples taken at the points \( t_j = j \Delta, j = 0, \ldots, 2n-1 \), for \( \sigma > 1 \) additional samples are required at the shifted locations \( t_{\tau+j\sigma} = (\tau \pm j \sigma) \Delta \) in order to resolve the ambiguity that arises when extracting the nonlinear parameters \( \phi_i \) from the values \( \cos(\phi_i \sigma \Delta) \). The quotient
\[
\frac{\alpha_i \cos(\phi_i \tau \Delta)}{\alpha_i}, \quad i = 1, \ldots, n
\]
delivers the values \( \cos(\phi_i \tau \Delta), i = 1, \ldots, n \). Neither from \( \cos(\phi_i \sigma \Delta) \) nor from \( \cos(\phi_i \tau \Delta) \) the parameters \( \phi_i \) can necessarily be extracted uniquely when \( \sigma > 1 \) and \( \tau > 1 \). But the following result is proved in the appendix.

If \( \gcd(\sigma, \tau) = 1 \), the sets
\[
S_i = \left\{ \frac{1}{\sigma \Delta} \arccos(\cos(\phi_i \sigma \Delta)) + \frac{2\pi}{\sigma \Delta} \ell, \quad \ell = -\lfloor \sigma/2 \rfloor, \ldots, 0, \ldots, \lfloor \sigma/2 \rfloor - 1 \right\}
\]
and
\[
T_i = \left\{ \frac{1}{\tau \Delta} \arccos(\cos(\phi_i \tau \Delta)) + \frac{2\pi}{\tau \Delta} \ell, \quad \ell = -\lfloor \tau/2 \rfloor, \ldots, 0, \ldots, \lfloor \tau/2 \rfloor - 1 \right\}
\]
containing all the candidate arguments for \( \phi_i \) in \( \cos(\phi_i \sigma \Delta) \) and \( \cos(\phi_i \tau \Delta) \) respectively, have at most two elements in their intersection. Here \( \arccos(\cdot) \in [0, \pi] \) denotes the principal branch of the arccosine function. In case two elements are found, then it suffices to extend (22) to
\[
\sum_{i=1}^{n} (\alpha_i \cos(\phi_i (\sigma + \tau) \Delta)) \cos(\phi_i j \sigma \Delta) = F_{(\sigma+\tau)+j\sigma}, \quad j = k, \ldots, k+n-1,
\]
which only requires the additional sample \( f_{\tau+(k+n)\sigma} \) as \( f_{\tau-(k+n-2)\sigma} \) is already available. From this extension, \( \cos(\phi_i(\sigma + \tau)\Delta) \) can be obtained in the same way as \( \cos(\phi_i\tau\Delta) \). As explained in the appendix, only one of the two elements in the intersection of \( S_i \) and \( T_i \) fits the computed \( \cos(\phi_i(\sigma + \tau)\Delta) \) since \( \gcd(\sigma, \tau) = 1 \) implies that also \( \gcd(\sigma, \tau + \sigma) = 1 = \gcd(\tau, \sigma + \tau) \).

So the unique identification of the \( \phi_i \) can require \( 2n-1 \) additional samples at the shifted locations \( (\tau \pm j\sigma)\Delta, j = 0, \ldots, n-1 \) if the intersections \( S_i \cap T_i \) are all singletons, or \( 2n \) additional samples, namely at \( (\tau \pm j\sigma)\Delta, j = 0, \ldots, n-1 \) and \( (\tau + n\sigma)\Delta \) if at least one of the intersections \( S_i \cap T_i \) is not a singleton.

The factorization in Theorem 1 immediately allows to formulate the following cosine analogue of (15).

**Corollary 1.** For the matrix \( \mathcal{C}_n \) defined in (20) it holds that

\[
\det \mathcal{C}_n = 0, \quad \nu > n.
\]

To round up our discussion, we mention that from the factorization in Theorem 1, it is clear that for the generalized eigenvector \( v_i \) from the different generalized eigenvalue problem

\[
(\mathcal{C}_n) v_i = \cos(\phi_i \tau \Delta) (\mathcal{C}_n) v_i,
\]

holds that

\[
\mathcal{C}_n v_i = \alpha_i (1, \cos(\phi_1 \sigma \Delta), \ldots, \cos(\phi_{n-1} \sigma \Delta))^T.
\]

This immediately leads to a computational variant of the proposed scheme, similar to the one given in Section 2.4 for the exponential function, requiring somewhat more samples though. Let us now turn our attention to other trigonometric functions.

3.2. Sine function. Let \( g(\phi_i; t) = \sin(\phi_i t) \) and let (16) hold. With \( t_j = j\Delta \) the relation (3) satisfied by \( g(\phi_i; t) \) and the \( t_j \) is

\[
\frac{1}{2} \sin(\phi_i t_{j+1}) + \frac{1}{2} \sin(\phi_i t_{j-1}) = \cos(\phi_i \Delta) \sin(\phi_i t_j), \quad \Delta = t_{j+1} - t_j.
\]

So in (3) \( L = 1, R = 0 \) and \( \lambda_{ij} = \cos(\phi_i (t_{j+1} - t_j)) \). We denote

\[
f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i \sin(\phi_i (\tau + j\sigma)\Delta),
\]

and introduce for fixed chosen \( \sigma \) and \( \tau \),

\[
F_{\tau+j\sigma} := F(\sigma, \tau; t_j) = \frac{1}{2} f_{\tau+j\sigma} + \frac{1}{2} f_{-\tau+j\sigma}
\]

\[
= \sum_{i=1}^{n} \left( \frac{1}{2} \alpha_i \cos(\phi_i \tau \Delta) \sin(\phi_i j \sigma \Delta) \right).
\]

We fill the matrices \( \mathcal{H}_n \) and the Toeplitz matrices \( \mathcal{T}_n \) and define

\[
\mathcal{B}_n := \frac{1}{4} (\mathcal{H}_n^+ \mathcal{H}_n) + \frac{1}{4} (\mathcal{H}_n^- \mathcal{H}_n) + \frac{1}{4} (\mathcal{H}_n \mathcal{T}_n) + \frac{1}{4} (\mathcal{T}_n \mathcal{H}_n).
\]
Theorem 2. The structured matrix $\sigma B_n$ factorizes as

$$\sigma B_n = U_n L_n A_n W_n^T,$$

where

$$U_n = \begin{pmatrix} \sin(\phi_1 \sigma \Delta) & \cdots & \sin(\phi_n \sigma \Delta) \\ \vdots & \ddots & \vdots \\ \sin(\phi_1 n \sigma \Delta) & \cdots & \sin(\phi_n n \sigma \Delta) \end{pmatrix},$$

$$W_n = \begin{pmatrix} 1 & \cdots & 1 \\ \cos(\phi_1 \sigma \Delta) & \cdots & \cos(\phi_n \sigma \Delta) \\ \vdots & \ddots & \vdots \\ \cos(\phi_1 (n-1) \sigma \Delta) & \cdots & \cos(\phi_n (n-1) \sigma \Delta) \end{pmatrix},$$

$$A_n = \text{diag}(\alpha_1, \ldots, \alpha_n),$$

$$L_n = \text{diag}(\cos(\phi_1 \tau \Delta), \ldots, \cos(\phi_n \tau \Delta)).$$

Proof. The proof is again a verification of the matrix product entry, at the position $(k, \ell + 1)$ with $k = 1, \ldots, n$ and $\ell = 0, \ldots, n - 1$:

$$\frac{1}{4} f_{\tau + (k+\ell) \sigma} + \frac{1}{4} f_{-\tau + (k+\ell) \sigma} + \frac{1}{4} f_{\tau + (k-\ell) \sigma} + \frac{1}{4} f_{-\tau + (k-\ell) \sigma}
= \frac{1}{2} \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta) \sin(\phi_i (k + \ell) \sigma \Delta) + \frac{1}{2} \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta) \sin(\phi_i (k - \ell) \sigma \Delta)
= \sum_{i=1}^{n} \alpha_i \cos(\phi_i \tau \Delta) \sin(\phi_i k \sigma \Delta) \cos(\phi_i \ell \sigma \Delta). \qed$$

Note that the factorization involves precisely the building blocks in the shifted evaluation \[24\] of the help function $F(\sigma, \tau; t)$. From this decomposition we find that the $\cos(\phi_i \sigma \Delta), i = 1, \ldots, n$ are obtained as the generalized eigenvalues of the problem

$$(\sigma B_n) v_i = \cos(\phi_i \sigma \Delta) (\sigma B_n) v_i, \quad i = 1, \ldots, n.$$ 

We point out that setting up this generalized eigenvalue problem requires to sample $f(t)$ at $t_{(-n+1)\sigma}, \ldots, t_{2n\sigma}$. Since $f(t_{j \sigma}) = -f(t_{-j \sigma})$ and $f(0) = 0$ it costs $2n$ samples. Unfortunately, at this point we cannot compute the $\alpha_i, i = 1, \ldots, n$ from the linear system of interpolation conditions

$$\sum_{i=1}^{n} \alpha_i \sin(\phi_i j \sigma \Delta) = f_{j \sigma}, \quad j = 1, \ldots, 2n,$$

as we usually do, because we do not have the matrix entries $\sin(\phi_i j \sigma \Delta)$ at our disposal. It is however easy to obtain the values $\cos(\phi_i j \sigma \Delta)$ because

$$\cos(\phi_i j \sigma \Delta) = \cos(\pm j \arccos(\cos(\phi_i \sigma \Delta)))$$

where $\arccos(\cos(\phi_i \sigma \Delta))$ returns the principal branch value of the generalized eigenvalues. The proper way to proceed is the following.
From Theorem 2 we get $0^\sigma B_n^T = W_n A_n U_n^T$. Therefore we can obtain the $\alpha_i \sin(\phi_i \sigma \Delta)$ in the first column of $A_n U_n^T$ from the structured linear system

$$W_n \begin{pmatrix} \alpha_1 \sin(\phi_1 \sigma \Delta) \\ \vdots \\ \alpha_n \sin(\phi_n \sigma \Delta) \end{pmatrix} = \begin{pmatrix} b_{11} \\ \vdots \\ b_{1n} \end{pmatrix},$$

where we denote $0^\sigma B_n = (b_{ij})_{i,j=1}^n$. From the generalized eigenvalues $\cos(\phi_i \sigma \Delta)$, $i = 1, \ldots$ and the $\alpha_i \sin(\phi_i \sigma \Delta)$ we can now recursively compute

$$\alpha_i \sin(\phi_i j \sigma \Delta) = \alpha_i \sin(\phi_i (j-1) \sigma \Delta) \cos(\phi_i \sigma \Delta) + \cos(\phi_i (j-1) \sigma \Delta) \alpha_i \sin(\phi_i \sigma \Delta), \quad j = 1, \ldots, n.$$  

The system of shifted linear interpolation conditions

$$\sum_{i=1}^n (\alpha_i \cos(\phi_i \tau \Delta)) \sin(\phi_i j \sigma \Delta) = F_{\tau+j\sigma}, \quad j = k, \ldots, k+n-1, \quad 1 \leq k \leq n+1$$

can then be interpreted as the linear system

$$\sum_{i=1}^n (\alpha_i \sin(\phi_i j \sigma \Delta)) \cos(\phi_i \tau \Delta) = F_{\tau+j\sigma}, \quad j = k, \ldots, k+n-1, \quad 1 \leq k \leq n+1$$

with a coefficient matrix having entries $\alpha_i \sin(\phi_i j \sigma \Delta)$ and with unknowns $\cos(\phi_i \tau \Delta)$. In order to retrieve the $\phi_i$ uniquely from the values $\cos(\phi_i \sigma \Delta)$ and $\cos(\phi_i \tau \Delta)$ with $\gcd(\sigma, \tau) = 1$, one proceeds as in the cosine case. Finally, the $\alpha_i$ are obtained from the expressions $\alpha_i \sin(\phi_i \sigma \Delta)$ after plugging in the correct arguments $\phi_i$ in $\sin(\phi_i \sigma \Delta)$ and dividing by it. So compared to the previous sections, the intermediate computation of the $\alpha_i$ before knowing the $\phi_i$, is replaced by the intermediate computation of the $\alpha_i \sin(\phi_i \sigma \Delta)$. In the end, the $\alpha_i$ are revealed in a division, without the need to solve an additional linear system.

From the factorization in Theorem 2, the following sine analogue of (15) follows immediately.

**Corollary 2.** For the matrix $0^\sigma B_n$ defined in (25) it holds that

$$\det 0^\sigma B_\nu = 0, \quad \nu > n.$$ 

For completeness we mention that one also finds from this factorization that for $v_i$ in the generalized eigenvalue problem

$$(0^\sigma B_n) v_i = \cos(\phi_i \tau \Delta) (0^\sigma B_n) v_i,$$

it holds that

$$0^\sigma B_n v_i = \alpha_i (\sin(\phi_i \sigma \Delta), \ldots, \sin(\phi_i n \sigma \Delta))^T.$$
3.3. Phase shifts in cosine and sine. It is possible to include phase shift parameters in the cosine and sine interpolation schemes. We explain how by working out the sparse interpolation of

\begin{equation}
    f(t) = \sum_{i=1}^{n} \alpha_i \sin(\phi_i t - \psi_i), \quad \psi_i \in \mathbb{R}, \quad \alpha_i, \phi_i \in \mathbb{C}.
\end{equation}

Since

\[ \sin t = \frac{\exp(it) - \exp(-it)}{2i}, \]

we can write each term in (28) as

\[ \alpha_i \sin(\phi_i t - \psi_i) = \alpha_i \exp(-i\psi_i) \exp(i\phi_i t) - \alpha_i \exp(i\phi_i t) \exp(-i\psi_i). \]

So the sparse interpolation of (28) can be solved by considering the exponential sparse interpolation problem

\[ \sum_{i=1}^{2n} \beta_i \exp(i\zeta_i t), \]

where \( \beta_{2i-1} = \alpha_i \exp(-i\psi_i)/(2i), \beta_{2i} = -\alpha_i \exp(i\psi_i)/(2i) \) and \( \zeta_{2i-1} = \phi_i = -\zeta_{2i} \). The computation of the \( \phi_i \) through the \( \zeta_i \) remains separated from that of the \( \alpha_i \) and \( \psi_i \). The latter are obtained as

\[ \tan \psi_i = -i \frac{\beta_{2i} + \beta_{2i-1}}{\beta_{2i} - \beta_{2i-1}}, \]

\[ \alpha_i = -(\beta_{2i} + \beta_{2i-1})/\sin(\psi_i) = -i(\beta_{2i} - \beta_{2i-1})/\cos(\psi_i). \]

3.4. Hyperbolic functions. For \( g(\phi_i; t) = \cosh(\phi_i t) \) the computational scheme parallels that of the cosine and for \( g(\phi_i; t) = \sinh(\phi_i t) \) that of the sine. We merely write down the main issues.

When \( g(\phi_i; t) = \cosh(\phi_i t) \), let

\[ f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i \cosh(\phi_i(\tau + j\sigma)\Delta) \]

and for fixed chosen \( \sigma \) and \( \tau \), let

\[ F_{\tau+j\sigma} := \frac{1}{2} f_{\tau+j\sigma} + \frac{1}{2} f_{\tau-j\sigma} \]

\[ = \sum_{i=1}^{n} \alpha_i \cosh(\phi_i \tau \Delta) \cosh(\phi_i j \sigma \Delta). \]

Subsequently the definition of the structured matrix \( \mathbb{T}_C \) is used and in the factorization of Theorem 1, the cosine function is everywhere replaced by the cosine hyperbolic function.

Similarly, when \( g(\phi_i; t) = \sinh(\phi_i t) \), let

\[ f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i \sinh(\phi_i(\tau + j\sigma)\Delta) \]
and for fixed chosen $\sigma$ and $\tau$, let
\[
F_{\tau+j\sigma} := \frac{1}{2} f_{\tau+j\sigma} + \frac{1}{2} f_{-\tau+j\sigma} = \sum_{i=1}^{n} \alpha_i \cosh(\phi_i \tau \Delta) \sinh(\phi_i j \sigma \Delta).
\]

Now the definition of the structured matrix $B_n$ is used and in the factorization of Theorem 2 the occurrences of $\cos$ are replaced by $\cosh$ and those of $\sin$ by $\sinh$.

4. Polynomial functions

The orthogonal Chebyshev polynomials were among the first polynomial basis functions to be explored for use in combination with a scaling factor $\sigma$, in the context of sparse interpolation in symbolic-numeric computing [10]. We elaborate the topic further for numerical purposes and for lacunary or supersparse interpolation, making use of the scale factor $\sigma$ and the shift term $\tau$. We also extend the approach to other polynomial bases and connect to generalized eigenvalue formulations.

4.1. Chebyshev 1st kind. Let $g(m_i; t) = T_{m_i}(t)$ of degree $m_i$, which is defined by
\[
T_m(t) = \cos(m\theta), \quad t = \cos(\theta), \quad -1 \leq t \leq 1,
\]
and consider the interpolation problem
\[
(29) \quad f(t_j) = \sum_{i=1}^{n} \alpha_i T_{m_i}(t_j), \quad \alpha_i \in \mathbb{C}, \quad m_i \in \mathbb{N}.
\]
The Chebyshev polynomials $T_m(t)$ satisfy the recurrence relation
\[
T_{m+1}(t) = 2t T_m(t) - T_{m-1}(t), \quad T_1(t) = t, \quad T_0(t) = 1.
\]
With
\[
0 \leq m_1 < m_2 < \ldots < m_n < M
\]
we choose $t_j = \cos(j \Delta)$ where $0 < \Delta \leq \pi/M$. Note that the points $t_j$ are allowed to occupy much more general positions than in [10]. The type (3) relation satisfied by the $g(m_i; t) = T_{m_i}(t)$ and the $t_j$ is
\[
\frac{1}{2} T_{m_i}(t_{j+1}) + \frac{1}{2} T_{m_i}(t_{j-1}) = T_{m_i}(\cos \Delta) T_{m_i}(t_j).
\]
If $M$ is extremely large and $n$ is small, in other words if the polynomial is very sparse, then it is a good idea to recover the actual $m_i, i = 1, \ldots, n$ in two tiers as we explain now. Let $\gcd(\sigma, \tau) = 1$. We denote
\[
(30) \quad f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i T_{m_i}(t_{\tau+j\sigma})
\]
and introduce for fixed $\sigma$ and $\tau$,

$$F_{\tau+j\sigma} := F(\sigma, \tau; t_j) = \frac{1}{2} f_{\tau+j\sigma} + \frac{1}{2} f_{\tau-j\sigma}$$

$$= \sum_{i=1}^{n} \alpha_i T_{m_i} (\cos(\tau \Delta)) T_{m_i} (\cos(j \sigma \Delta))$$

in order to separate the effect of $\sigma$ and $\tau$ in the evaluation. With the same matrices $\gamma_i H_{n, \sigma} T_n$ and $\gamma_i C_n$ as in the cosine subsection, now filled with the values $f_{\tau+j\sigma}$ from (30), the values $T_{m_i} (\cos(\sigma \Delta))$ are the generalized eigenvalues of the problem

$$(\sigma C_n) v_i = T_{m_i} (\cos(\sigma \Delta)) (\sigma C_n) v_i, \quad i = 1, \ldots, n.$$

From the values $T_{m_i} (\cos(\sigma \Delta)) = \cos(m_i \sigma \Delta)$ the integer $m_i$ cannot be retrieved unambiguously. We need to find out which of the elements in the set

$$S_i = \left\{ \pm \frac{1}{\sigma \Delta} \arccos(\cos(m_i \sigma \Delta)) + \frac{2\pi}{\sigma \Delta} \ell, \; \ell = 0, \ldots, \sigma - 1 \right\} \cap \mathbb{Z}_M$$

is the one satisfying (41), where $\arccos(\cos(m_i \sigma \Delta))/\sigma \Delta \leq M/\sigma$. Depending on the relationship between $\sigma$ and $M$ (relatively prime, generator, divisor, ... ) the set $S_i$ may contain one or more candidate integers for $m_i$ evaluating to the same value $\cos(m_i \sigma \Delta)$. To resolve the ambiguity we consider the Vandermonde-like system for the $\alpha_i, i = 1, \ldots, n$,

$$\sum_{i=1}^{n} \alpha_i T_{m_i} (\cos(j \sigma \Delta)) = f_{j\sigma}, \quad j = 0, \ldots, 2n - 1,$$

and the shifted problem

$$\sum_{i=1}^{n} (\alpha_i T_{m_i} (\cos(\tau \Delta))) T_{m_i} (\cos(j \sigma \Delta)) = F_{\tau+j\sigma}, \quad j = k, \ldots, k + n - 1, \; \tau \in \mathbb{Z},$$

from which we compute the $\alpha_i T_{m_i} (\cos(\tau \Delta)) = \alpha_i \cos(m_i \tau \Delta)$. Consequently

$$\cos(m_i \tau \Delta) = T_{m_i} (\cos(\tau \Delta))$$

$$= \frac{\alpha_i T_{m_i} \cos(\tau \Delta)}{\alpha_i}, \quad i = 1, \ldots, n$$

If the intersection of the set $S_i$ with the set

$$T_i = \left\{ \pm \frac{1}{\tau \Delta} \arccos(\cos(m_i \tau \Delta)) + \frac{2\pi}{\tau \Delta} \ell, \; \ell = 0, \ldots, \tau - 1 \right\} \cap \mathbb{Z}_M$$

is processed as in Section 3.1, then one can eventually identify the correct $m_i$.

When replacing (31) by

$$(\sigma C_n) v_i = T_{m_i} (\cos(\tau \Delta)) (\sigma C_n) v_i, \quad i = 1, \ldots, n,$$

we find that for $v_i$

$$0 \sigma C_n v_i = \alpha_i (1, T_{m_i} (\cos(\sigma \Delta)), \ldots, T_{m_i} (\cos(n-1 \sigma \Delta)))^T.$$
This offers an alternative algorithm similar to the alternative in Section 3.1 on the cosine function.

4.2. **Chebyshev 2nd, 3rd and 4th kind.** While the Chebyshev polynomials $T_{m_i}(t)$ of the first kind are intrinsically related to the cosine function, the Chebyshev polynomials $U_{m_i}(t)$ of the second kind can be expressed using the sine function:

$$U_m(t) = \frac{\sin((m+1)\theta)}{\sin\theta}, \quad t = \cos\theta, \quad -1 \leq t \leq 1.$$  

Therefore the sparse interpolation problem

$$f(t_j) = \sum_{i=1}^{n} \alpha_i U_{m_i}(t_j), \quad \alpha_i \in \mathbb{C}, \quad m_i \in \mathbb{N}$$

can be solved along the same lines as in Section 4.1 but now using the samples

$$f(t_{\tau+j\sigma}) \sin(\arccos t_{\tau+j\sigma})$$

instead of the $f_{\tau+j\sigma}$, for the sparse interpolation of

$$\sum_{i=1}^{n} \alpha_i \sin((m_i+1)\theta_j) = \sin \theta_j f(t_j), \quad t_j = \cos \theta_j.$$  

In a very similar way the sparse interpolation problems

$$f(t_j) = \sum_{i=1}^{n} \alpha_i V_{m_i}(t_j), \quad \alpha_i \in \mathbb{C}, \quad m_i \in \mathbb{N}$$

and

$$f(t_j) = \sum_{i=1}^{n} \alpha_i W_{m_i}(t_j), \quad \alpha_i \in \mathbb{C}, \quad m_i \in \mathbb{N}$$

can be solved, using the Chebyshev polynomials $V_{m_i}(t)$ and $W_{m_i}(t)$ of the third and fourth kind respectively, given by

$$V_m(t) = \frac{\cos((n+1/2)\theta)}{\cos(\theta/2)}, \quad t = \cos\theta, \quad -1 \leq t \leq 1,$$

$$W_m(t) = \frac{\sin((n+1/2)\theta)}{\sin(\theta/2)}, \quad t = \cos\theta, \quad -1 \leq t \leq 1.$$  

4.3. **Spread polynomials.** Let $g(m_i; t)$ equal the degree $m_i$ spread polynomial $S_{m_i}(t)$ on $[0, 1]$, which is defined by

$$S_m(t) = \sin^2(m\theta), \quad t = \sin^2(\theta), \quad 0 \leq t \leq 1.$$  

The spread polynomials $S_m(t)$ are related to the Chebyshev polynomials of the first kind by $1 - 2tS_m(t) = T_m(1 - 2t)$ and satisfy the recurrence relation

$$S_{m+1}(t) = 2(1-2t)S_m(t) - S_{m-1}(t) + 2t, \quad S_1(t) = t, \quad S_0(t) = 0$$

and the property

$$S_m(t)S_r(t) = \frac{1}{2}S_m(t) + \frac{1}{2}S_r(t) - \frac{1}{4}S_{m+r}(t) - \frac{1}{4}S_{m-r}(t).$$
We consider the interpolation problem

\[ f(t) = \sum_{i=1}^{n} \alpha_i S_{m_i}(t_j), \quad \alpha_i \in \mathbb{C}, \quad m_i \in \mathbb{N}, \]

where \( t_j = \sin^2(j \Delta), j = 0, 1, 2, \ldots \) with \( 0 < \Delta \leq \pi/(2M) \) and

\[ 0 < m_1 < \ldots < m_n < M. \]

The relation of the form (3) satisfied by the current \( g(m_i; t) = S_{m_i}(t) = \sin^2(m_i \arcsin \sqrt{t}) \) is

\[ \frac{1}{2}(S_{m_i} (\sin^2 \Delta) + S_{m_i}(t_j)) - \frac{1}{4}(S_{m_i}(t_{j+1} + S_{m_i}(t_{j-1})) = S_{m_i} (\sin^2 \Delta) S_{m_i}(t_j). \]

As in Section 4.1 we present a two-tier approach, which for \( \sigma \leq 1 \) reduces to one step and avoids the additional evaluations required for the second step. However, as indicated above, the two-tier scheme offers some additional possibilities. We denote

\[ f_{\tau+j\sigma} := \sum_{i=1}^{n} \alpha_i S_{m_i} (\sin^2((\tau + j\sigma) \Delta)) = \sum_{i=1}^{n} \alpha_i S_{m_i} (S_{\tau+j\sigma}(\sin^2 \Delta)). \]

With

\[ F_{\tau+j\sigma} := F(\sigma, \tau; t_j) = \frac{1}{2}(f_{\tau} + f_{j\sigma}) - \frac{1}{4}(f_{\tau+j\sigma} + f_{\tau-j\sigma}) \]

we obtain

\[ F_{\tau+j\sigma} = \sum_{i=1}^{n} \alpha_i S_{m_i} (\sin^2 \tau \Delta) S_{m_i}(\sin^2 j\sigma \Delta). \]

So the effect of the scale factor \( \sigma \) on the one hand and the shift term \( \tau \) on the other can again be separated in the evaluation \( F_{\tau+j\sigma} \).

We introduce the matrices

\[ \sigma J_n := (1/2 f_{k\sigma} + 1/2 f_{k\ell\sigma} - 1/4 f_{(k+\ell)\sigma} - 1/4 f_{(k-\ell)\sigma})_{k,\ell=1}^{n}, \]

\[ \tau K_n := (1/2 F_{k+\sigma} + 1/2 F_{k+\ell\sigma} - 1/4 F_{(k+\ell+\sigma)} - 1/4 F_{(k+\ell-\sigma)})_{k,\ell=1}^{n}. \]

**Theorem 3.** The matrices \( \tau K_n \) and \( \sigma J_n \) factorize as

\[ \tau K_n = R_n L_n A_n R_n^T, \]

\[ \sigma J_n = R_n A_n R_n^T, \]

\[ R_n = \begin{pmatrix} S_{m_1}(\sin^2 \sigma \Delta) & \ldots & S_{m_n}(\sin^2 \sigma \Delta) \\
 & \ldots & \vdots \\
 S_{m_1}(\sin^2 n\sigma \Delta) & \ldots & S_{m_n}(\sin^2 n\sigma \Delta) \end{pmatrix}, \]

\[ A_n = \text{diag}(\alpha_1, \ldots, \alpha_n), \]

\[ L_n = \text{diag}(S_{m_1}(\sin^2 \tau \Delta), \ldots, S_{m_n}(\sin^2 \tau \Delta)) = \text{diag}(\sin^2(m_1 \tau \Delta), \ldots, \sin^2(m_n \tau \Delta)). \]
Proof. The factorization is again verified at the level of the matrix entries, now making use of property (32), which is slightly more particular. □

This factorization paves the way to obtaining the values $S_{m_i}(\sin^2 \sigma \Delta) = \sin^2(m_i \sigma \Delta)$ as the generalized eigenvalues of

$$\left(\sigma K_n\right) v_i = S_{m_i}(\sin^2 \sigma \Delta) \left(\sigma J_n\right) v_i, \quad i = 1, \ldots, n.$$ 

Filling the matrices in this matrix pencil requires $2n + 1$ evaluations $f(j \sigma \Delta)$ for $j = 1, \ldots, 2n + 1$. From these generalized eigenvalues we can not uniquely deduce the values for the indices $m_i$. Instead, we can obtain for each $i = 1, \ldots, n$ the set of elements

$$S_i = \left( \frac{\text{Arcsin}(|\sin(m_i \sigma \Delta)|)}{\sigma \Delta} + \frac{\pi}{\sigma \Delta} \ell, \ell = 0, \ldots, [\sigma/2] - 1 \right) \cup \left( -\frac{\text{Arcsin}(|\sin(m_i \sigma \Delta)|)}{\sigma \Delta} + \frac{\pi}{\sigma \Delta} \ell, \ell = 1, \ldots, [\sigma/2] \right) \cap \mathbb{Z}_M$$

characterising all the possible values for $m_i$ consistent with the sparse spread polynomial interpolation problem. Fortunately, with gcd$(\sigma, \tau) = 1$, we can proceed as follows.

First, the coefficients $\alpha_i$ are obtained from the linear system of interpolation conditions

$$\sum_{i=1}^{n} \alpha_i S_{m_i}(\sin^2 j \sigma \Delta) = f_{j \sigma}, \quad j = 0, \ldots, 2n - 1.$$ 

The additional values $F_{r+j \sigma}$ lead to a second system of interpolation conditions,

$$\sum_{i=1}^{n} \left( \alpha_i S_{m_i}(\sin^2 \tau \Delta) \right) S_{m_i}(\sin^2 j \sigma \Delta) = F_{r+j \sigma}, \quad j = k, \ldots, k + n - 1, \quad 1 \leq k,$n

which delivers the coefficients $\alpha_i S_{m_i}(\sin^2 \tau \Delta)$. Dividing the two solution vectors of these linear systems componentwise delivers the sought after values $S_{m_i}(\sin^2 \tau \Delta)$, $i = 1, \ldots, n$ from which we obtain sets

$$T_i = \left( \frac{\text{Arcsin}(|\sin(m_i \tau \Delta)|)}{\tau \Delta} + \frac{\pi}{\tau \Delta} \ell, \ell = 0, \ldots, [\tau/2] - 1 \right) \cup \left( -\frac{\text{Arcsin}(|\sin(m_i \tau \Delta)|)}{\tau \Delta} + \frac{\pi}{\tau \Delta} \ell, \ell = 1, \ldots, [\tau/2] \right) \cap \mathbb{Z}_M$$

that have the correct $m_i$ in their intersection with the respective $S_i$. The proof of this statement follows a completely similar course as that for the cosine atom $g(\phi_i; t)$ given in the Appendix.

The factorization in Theorem 3 allows to write down a spread polynomial analogue of (15).
Corollary 3. For the matrices $\sigma J_n$ and $0^\sigma K_n$ defined in (33) it holds that
\[
\det_\sigma J_\nu = 0, \quad \nu > n, \\
\det_0 K_\nu = 0, \quad \nu > n.
\]

To round up the discussion we mention that from Theorem 3 and the generalized eigenvalue problem
\[
(\sigma K_n) v_i = S_{m_i} (\sin^2 \tau \Delta) (\sigma J_n) v_i, \quad i = 1, \ldots, n,
\]
we also find that
\[
\sigma J_n v_i = \alpha_i \left( S_{m_i} (\sin^2 \sigma \Delta), \ldots, S_{m_i} (\sin^2 n \sigma \Delta) \right)^T.
\]

At the expense of some additional samples this eigenvalue and eigenvector combination offers again an alternative computational scheme.

5. Distribution functions

In [21, pp. 85–91] Prony’s method is generalized from $g(\phi_i; t) = \exp(\phi_i t)$ with $\phi_i \in \mathbb{C}$ to $g(\phi_i; t) = \exp(-(t - \phi_i)^2)$, to solve the interpolation problem
\[
f(t_j) = \sum_{i=1}^n \alpha_i \exp \left( \frac{(t_j - \phi_i)^2}{2w^2} \right), \quad \alpha_i, \phi_i \in \mathbb{C},
\]
with given fixed Gaussian peak width $w$. Here we further generalize the algorithm to include the new scale and shift paradigm and in the last section we present its multivariate version where the variable $t$ is a radial variable, in other words a (weighted) Euclidean norm of a multi-dimensional vector. Both the univariate and multivariate schemes are useful when modelling phenomena using Gaussian processes. Without loss of generality we put $2w^2 = 1$. The easy adaptation to include a fixed constant width factor in the formulas is left to the reader.

We again assume that (3) holds, but now for $2\Delta$. The discrete generalized eigenfunction relation (3) for $g(\phi_i; t)$ is here
\[
\exp(t_{j+1}^2) \exp(-(t_{j+1} - \phi_i)^2) = \exp(2\phi_i \Delta) \exp(t_j^2) \exp(-(t_j - \phi_i)^2),
\]
with $L = 1, a_1 = \exp(t_{j+1}^2), R = 0, b_0 = \exp(t_j^2)$ and $\lambda_{ij} = \exp(2\phi_i (t_{j+1} - t_j))$.

Let us take a closer look at the evaluation of $f(t)$ at $t_{\tau+j\sigma} = (\tau + j\sigma) \Delta, j = 0, 1, \ldots$ with $\sigma \in \mathbb{N}$ and $\tau \in \mathbb{Z}$:
\[
\exp(-(\tau + j\sigma) \Delta - \phi_i)^2) = \exp(-(\tau \Delta - \phi_i)^2 - j^2 \sigma^2 \Delta^2 - 2(\tau \Delta - \phi_i) j \sigma \Delta).
\]

With the auxiliary function
\[
F(\sigma, \tau; t_j) = \exp(2\tau j \sigma \Delta) \exp(j^2 \sigma^2 \Delta^2) f(t_{\tau+j\sigma})
\]
(34)
\[
= \sum_{i=1}^n \left( \alpha_i \exp(-(\tau \Delta - \phi_i)^2) \right) \exp(2\phi_i j \sigma \Delta)
\]
we obtain a perfect separation of $\sigma$ and $\tau$ and the problem can be solved using Prony’s method. With fixed chosen $\sigma$ and $\tau$, the value $F(\sigma, \tau; t_j)$ is denoted by $F_{\tau+j\sigma}$. 
Theorem 4. The Hankel structured matrix

\[ \tau G_n := \begin{pmatrix} F_\tau & F_{\tau+\sigma} & \ldots & F_{\tau+(n-1)\sigma} \\ F_{\tau+\sigma} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots \\ F_{\tau+(n-1)\sigma} & \ldots & \ddots & F_{\tau+(2n-2)\sigma} \end{pmatrix} \]

factorizes as

\[ \tau G_n = E_n L_n A_n E_n^T, \]

where

\[ E_n = \begin{pmatrix} 1 & \ldots & 1 \\ \exp(2\phi_1\sigma\Delta) & \ldots & \exp(2\phi_n\sigma\Delta) \\ \vdots & \ddots & \vdots \\ \exp(2\phi_1(n-1)\sigma\Delta) & \ldots & \exp(2\phi_n(n-1)\sigma\Delta) \end{pmatrix}, \]

\[ A_n = \text{diag}(\alpha_1 \exp(-\phi_1^2), \ldots, \alpha_n \exp(-\phi_n^2)) \]

\[ L_n = \text{diag}\left(\exp(-\tau^2\Delta^2 + 2\tau\Delta\phi_1), \ldots, \exp(-\tau^2\Delta^2 + 2\tau\Delta\phi_n)\right). \]

Proof. The proof is again by verification of the entry \( F_{\tau+(k+\ell)\sigma} \) in \( \tau G_n \) at position \((k+1, \ell+1)\) for \( k = 0, \ldots, n-1 \) and \( \ell = 0, \ldots, n-1 \).

With \( \tau = 0, \sigma \) the values \( \exp(2\phi_i\sigma\Delta) \) are retrieved as a factor of the generalized eigenvalues of the problem

\[ (\sigma G_n) v_i = \exp(-\sigma^2\Delta^2) \exp(2\phi_i\sigma\Delta) (\sigma G_n) v_i, \quad i = 1, \ldots, n. \]

As we know from the exponential case, the \( \phi_i \) cannot be identified unambiguously from \( \exp(2\phi_i\sigma\Delta) \) when \( \sigma > 1 \). In order to obtain that we turn our attention to two structured linear systems. The first one, where \( \tau = 0, \)

\[ \sum_{i=1}^{n} \alpha_i \exp(-(t_{j\sigma} - \phi_i)^2) = f_{j\sigma}, \quad j = 0, \ldots, 2n-1, \]

delivers the \( \alpha_i \exp(-\phi_i^2) \) after rewriting it as

\[ \sum_{i=1}^{n} \left( \alpha_i \exp(-\phi_i^2) \right) \exp(2\phi_ij\sigma\Delta) = \exp(j^2\sigma^2\Delta^2) f_{j\sigma} = F(\sigma, 0; t_j), \]

\[ j = 0, \ldots, 2n-1. \]

The coefficient matrix of this linear system is Vandermonde structured with entry \( \left( \exp(2\phi_i\sigma\Delta) \right)^j \) at position \((j+1, i)\). The second linear system, where \( \tau > 0, \) delivers the \( \alpha_i \exp(-\tau\Delta - \phi_i)^2 \) through \( \{3.4\}, \)

\[ \sum_{i=1}^{n} \left( \alpha_i \exp(-(\tau\Delta - \phi_i)^2) \right) \exp(2\phi_ij\sigma\Delta) = F_{\tau+j\sigma}, \quad j = k, \ldots, k+n-1. \]
Here the coefficient matrix is structured identically. From both solutions we obtain
\[
\exp(\tau^2 \Delta^2) \frac{\alpha_i \exp(-\tau \Delta - \phi_i^2)}{\alpha_i \exp(-\phi_i^2)}
= \exp(\tau^2 \Delta^2) \frac{\alpha_i \exp(-\tau^2 \Delta^2) \exp(-\phi_i^2) \exp(2\phi_i \tau \Delta)}{\alpha_i \exp(-\phi_i^2)}
= \exp(2\phi_i \tau \Delta).
\]

From the values \(\exp(2\phi_i \sigma \Delta), i = 1, \ldots, n\) and \(\exp(2\phi_i \tau \Delta), i = 1, \ldots, n\) the parameters \(2\phi_i\) can be extracted as explained in Section 2, under the condition that \(\gcd(\sigma, \tau) = 1\).

The values \(\exp(2\phi_i \tau \Delta)\) and \(\exp(2\phi_i \sigma \Delta)\) can also be retrieved respectively from the generalized eigenvalues and the generalized eigenvectors of the alternative problem
\[
(\tau \sigma G_n) v_i = \exp(-\tau^2 \Delta^2) \exp(2\phi_i \tau \Delta) (\sigma G_n^0) v_i, \quad i = 1, \ldots, n,
\]
with
\[
\sigma G_n^0 v_i = \alpha_i (1, \exp(2\phi_i \sigma \Delta), \ldots, \exp(2\phi_i (n - 1) \sigma \Delta))^T,
\]
requiring at least of \(4n - 2\) samples instead of \(3n\) samples.

To conclude, the following analogue of (15) can be given.

**Corollary 4.** For the matrix \(\sigma G_n^0\) given in Theorem 4 it holds that
\[
\det \sigma G_n^0 = 0, \quad \nu > n.
\]

## 6. Some special functions

The sinc function is widely used in digital signal processing, especially in seismic data processing where it is a natural interpolant. There are several similarities between the narrowing sinc function and the Dirac delta function, among which the shape of the pulse. The side lobes of the sinc function mimic the Gibbs phenomenon, which disappears when the sinc function eventually becomes a pulse.

The gamma function first arose in connection with the interpolation problem of finding a function that equals \(n!\) when the argument is a positive integer. Nowadays the function plays an important role in mathematics, physics and engineering, mainly because of the prevalence of integrals of expressions of the form \(g(t) \exp(-h(t))\) which describe processes that decay exponentially in time or space.

### 6.1. The sampling function \(\sin(x)/x\).

Let \(g(\phi_i; t) = \text{sinc}(\phi_i t)\) where \(\text{sinc}(t)\) is historically defined by \(\text{sinc}(t) = \sin(t)/t\). So our sparse interpolation problem is
\[
f(t_j) = \sum_{i=1}^{n} \alpha_i \text{sinc}(\phi_i t_j), \quad t_j = j \Delta,
\]
with the same assumptions for \( \phi \) and \( \Delta \) as in Section 3. In order to solve this inverse problem of identifying the \( \phi \) and \( \alpha \) for \( i = 1, \ldots, n \), we introduce

\[
F(t_j) := j \Delta f(t_j) = \sum_{i=1}^{n} \left( \frac{\alpha_i}{\phi_i} \right) \sin(\phi_i j \Delta)
\]

and apply the technique from Section 3.2 for the separate identification of the nonlinear and linear parameters in the sparse sine interpolation.

6.2. The gamma function \( \Gamma(z) \). With the new exponential analysis tools obtained so far, it is also possible to extend the theory to other functions such as, for instance, the gamma function \( \Gamma(z) \). The function \( g(\phi_i; z) = \Gamma(z + \phi_i) \) with \( z, \phi_i \in \mathbb{C} \), satisfies the discrete generalized eigenfunction relation

\[
(35) \Gamma(\Delta + 1 + \phi_i) = (\Delta + \phi_i) \Gamma(\Delta + \phi_i), \quad \Delta \in \mathbb{C}, \quad \Delta + \phi_i \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}.
\]

Our interest is in the sparse interpolation of

\[
f(z) = \sum_{i=1}^{n} \alpha_i \Gamma(z + \phi_i), \quad z + \phi_i \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}
\]

where the \( \alpha_i, \phi_i, i = 1, \ldots, n \) are unknown. In the sample point \( z = \Delta \) we define

\[
F_0(\Delta) := f(\Delta),
F_j(\Delta) := F_{j-1}(\Delta + 1) - \Delta F_{j-1}(\Delta), \quad j = 1, 2, \ldots
\]

If by the choice of \( \Delta \), one or more of the \( \Delta + \phi_i, i = 1, \ldots, n \) accidentally belong to the set of nonpositive integers, then one cannot sample \( f(z) \) at \( z = \Delta \). In that case a complex shift \( \tau \) can help out. It suffices to shift the arguments \( \Delta + \phi_i \) away from the negative real axis. We then redefine

\[
F_{\tau,j}(\Delta) := F_j(\tau + \Delta), \quad \tau \in \mathbb{C} \setminus (\mathbb{Z} \setminus \{0\})
\]

or

\[
F_{\tau,0}(\Delta) := f(\tau + \Delta),
F_{\tau,j}(\Delta) := F_{\tau,j-1}(\Delta + 1) - (\tau + \Delta) F_{\tau,j-1}(\Delta), \quad j = 1, 2, \ldots
\]

Using (35) we find

\[
F_{\tau,j}(\Delta) = \sum_{i=1}^{n} \alpha_i \phi_i^j \Gamma(\tau + \Delta + \phi_i), \quad j = 0, 1, 2, \ldots
\]

where we identify \( F_j(\Delta) \) with \( F_{0,j} \). As soon as the samples at \( \tau + \Delta + j \) are all well-defined, we can start the algorithm for the computation of the unknown linear parameters \( \alpha_i \) and the nonlinear parameters \( \phi_i \). We introduce

\[
\tau, k_n := \begin{pmatrix}
F_{\tau,k} & \cdots & F_{\tau,k+n-1} \\
\vdots & \ddots & \vdots \\
F_{\tau,k+n-1} & \cdots & F_{\tau,k+2n-2}
\end{pmatrix}.
\]
Theorem 5. The matrix $\tau,1_1 \mathcal{H}_n$ is factored as

$$
\tau,1_1 \mathcal{H}_n = \mathcal{P}_n \mathcal{P}_n Z_n \mathcal{P}_n^T,
$$

where

$$
\mathcal{P}_n = \begin{pmatrix}
1 & \cdots & 1 \\
\phi_1 & \cdots & \phi_n \\
\vdots & \vdots & \vdots \\
\phi_1^{n-1} & \cdots & \phi_n^{n-1}
\end{pmatrix},
$$

$$
Z_n = \text{diag} (\alpha_1 \Gamma(\tau + \Delta + \phi_1), \ldots, \alpha_n \Gamma(\tau + \Delta + \phi_n)),
$$

$$
\mathcal{P}_n = \text{diag}(\phi_1^k, \ldots, \phi_n^k).
$$

Proof. With the matrix factorization given, the proof consists of an easy verification of the matrix product with the matrix $\tau,1_1 \mathcal{H}_n$. □

Filling the matrices $\tau,0_1 \mathcal{H}_n$ and $\tau,1_1 \mathcal{H}_n$ requires the evaluation of $f(z)$ at $z = \tau + \Delta + j, j = 0, \ldots, 2n - 1$ which are points on a straight line parallel with the real axis in the complex plane.

The nonlinear parameters $\phi_i$ are now obtained as the generalized eigenvalues of

$$(\tau,1_1 \mathcal{H}_n)v_i = \phi_i (\tau,0_1 \mathcal{H}_n)v_i, \quad i = 1, \ldots, n,$$

where the $v_i, i = 1, \ldots, n$ are the right generalized eigenvectors. Afterwards the linear parameters $\alpha_i$ are obtained from the linear system of interpolation conditions

$$
\sum_{i=1}^{n} (\alpha_i \Gamma(\tau + \Delta + \phi_i)) \phi_i^j = F_{\tau,j}(\Delta), \quad j = \tau, \ldots, \tau + 2n - 1,
$$

by computing the coefficients $\alpha_i \Gamma(\tau + \Delta + \phi_i)$ and dividing those by the function values $\Gamma(\tau + \Delta + \phi_i)$ which are known because $\Delta, \tau$ and the $\phi_i, i = 1, \ldots, n$ are known.

From Theorem 5 we find that for the generalized eigenvectors of (36) holds that

$$
\tau,1_1 \mathcal{H}_n v_i = \alpha_i (1, \phi_i, \ldots, \phi_i^{n-1})^T.
$$

This allows to validate the computation of the $\phi_i, i = 1, \ldots, n$ obtained as generalized eigenvalues, if desired.

7. Multivariate functions

7.1. Multivariate exponential and trigonometric functions. In [7] the $d$-variate multi-exponential analysis problem of the form

$$(37) \quad f(x_1, \ldots, x_d) = \sum_{i=1}^{n} \alpha_i \exp(\phi_1 x_1 + \ldots + \phi_d x_d)$$

is solved from only $(d + 1)n$ samples, thereby properly generalizing Prony’s univariate method [9], developed in 1795, to the $d$-variate case at last. Essentially, the univariate argument $\phi t$ of the exponential function in (2) is in (37) replaced by the inner product $\langle \phi_i, x \rangle$ where $\phi_i = (\phi_{i1}, \ldots, \phi_{id})$ and
$x = (x_1, \ldots, x_d)$. The multivariate result is obtained by transforming the multivariate problem statement into a generalized eigenvalue problem resulting from a projection onto a one-dimensional subspace, complemented with $d-1$ structured $n \times n$ linear systems accounting for the remaining dimensions.

The projected generalized eigenvalue problem requires $2n$ samples

$$f_{j1} := f(j\Delta), \quad \Delta = (\Delta_1, \ldots, \Delta_d), \quad j = 0, 1, \ldots, 2n - 1$$

with, as in \([5]\), $|\Im((\phi_i, \Delta))| < \pi$. With these one-dimensional samples, Hankel structured matrices as in \([8]\) are filled and the generalized eigenvalue problem

$$(\frac{1}{i} H_n) v_i = \exp((\phi_i, \Delta)) (\frac{1}{i} H_n) v_i, \quad i = 1, \ldots, n$$

is solved for the $\exp((\phi_i, \Delta))$. After extracting $(\phi_i, \Delta)$ from the generalized eigenvalues, the individual $\phi_{i1}, \ldots, \phi_{id}$ for $i = 1, \ldots, n$ cannot yet be identified. For simplicity, we assume that the $\exp((\phi_i, \Delta))$ are mutually distinct. In \([7]\) one finds how to deal with coalescence of some of these values.

The method continues with the computation of the $\alpha_i$ as the solution of the Vandermonde system

$$(38) \quad \sum_{i=1}^{n} \alpha_i \exp(j(\phi_i, \Delta)) = f_{j1}, \quad j = 0, \ldots, 2n - 1.$$  

The $d-1$ linear systems of equations that serve to disentangle the projections $(\phi_i, \Delta)$, result from the use of $d-1$ linearly independent vector shifts $\tau^{(2)}, \ldots, \tau^{(d)}$, much along the lines of what the shift term $\tau$ with $\gcd(\sigma, \tau) = 1$ realizes in the previous sections. Here $\tau^{(k)} = (\tau^{(k)}_1, \ldots, \tau^{(k)}_d)$ with $|\Im((\phi_i, \tau^{(k)}))| < \pi$ and the shifted evaluations are

$$f_{jk} := f(\tau^{(k)} + j\Delta), \quad j = 0, \ldots, n-1, \quad k = 2, \ldots, d.$$  

This explains how the informational usage is only $2n + (d-1)n = (d+1)n$. In some unfortunate cases additional samples are required. For the details of the full algorithm we again refer to \([7]\).

The evaluations $f_{jk}, k = 2, \ldots, d$ with $2 \leq k \leq d$ fixed, at the shifted locations $\tau^{(k)} + j\Delta$, are interpreted as

$$\sum_{i=1}^{n} \left( \alpha_i \exp((\phi_i, \tau^{(k)})) \right) \exp(j(\phi_i, \Delta)) = f_{jk}, \quad j = 0, \ldots, n-1.$$  

In other words, with the samples $f_{jk}$ as right hand side for $k$ fixed and with the first $n$ rows of the same Vandermonde coefficient matrix from \([38]\), we obtain the unknown coefficients $\alpha_i \exp((\phi_i, \tau^{(k)}))$ and subsequently the values $\exp((\phi_i, \tau^{(k)}))$ from the divisions

$$\frac{\alpha_i \exp((\phi_i, \tau^{(k)}))}{\alpha_i}, \quad 2 \leq k \leq d, \quad 1 \leq i \leq n.$$  

We now have extracted all the inner products $(\phi_i, \Delta), (\phi_i, \tau^{(k)})$ for linearly independent vectors $\Delta$ and $\tau^{(k)}, k = 2, \ldots, d$ and so for each $i = 1, \ldots, n$
the individual $\phi_{i1}, \ldots, \phi_{id}$ can be retrieved as the solution of the following regular linear system:

$$
\begin{pmatrix}
\Delta_1 & \cdots & \Delta_d \\
\tau^{(2)}_1 & \cdots & \tau^{(2)}_d \\
\vdots & \ddots & \vdots \\
\tau^{(d)}_1 & \cdots & \tau^{(d)}_d
\end{pmatrix}
\begin{pmatrix}
\phi_{i1} \\
\phi_{i2} \\
\vdots \\
\phi_{id}
\end{pmatrix}
= 
\begin{pmatrix}
\langle \phi_{i1}, \Delta \rangle \\
\langle \phi_{i2}, \Delta \rangle \\
\vdots \\
\langle \phi_{id}, \Delta \rangle
\end{pmatrix},
$$

By combining the scale and shift paradigm of Section 2 with the technique in [7], the multivariate sparse interpolation of (37) can further be generalized to data taken at points $j(\Delta_1, \ldots, \Delta_d), j = 0, 1, 2, \ldots$ of which the location along the line spanned by $(\Delta_1, \ldots, \Delta_d)$ may be stretched or shrunk by some $\sigma$ and translated by some $\tau$. Also, the shift $\tau^{(k)}$ in the sample locations $\tau^{(k)} + j\Delta$ may be stretched or shrunk, where the user has to resort to an analogous approach when the condition $|\Im(\langle \phi_{i1}, \tau^{(k)} \rangle)| < \pi$ is violated.

For the multivariate variant of (14) we introduce for $k = 1, \ldots, d$ the Hankel matrices

$$
\tau^{(k)} H_n := 
\begin{pmatrix}
0, k & f_{\sigma,k} & \cdots & f_{(n-1)\sigma,k} \\
f_{\sigma,k} & \cdots & \cdots & \cdots \\
\vdots & \ddots & \ddots & \cdots \\
f_{(n-1)\sigma,k} & \cdots & \cdots & f_{(2n-2)\sigma,k}
\end{pmatrix},
\quad \tau^{(k)} \in \mathbb{R}^d, \quad \tau^{(1)} = 0.
$$

**Theorem 6.** The matrix $\tau^{(k)} H_n$ for $k = 1, \ldots, d$ factorizes as

$$
\tau^{(k)} H_n = V_n \Lambda_n A_n V_n^T,
$$

where

$$
V_n = 
\begin{pmatrix}
1 & \cdots & 1 \\
\exp(\langle \phi_{i1}, \sigma \Delta \rangle) & \cdots & \exp(\langle \phi_{i1}, \sigma \Delta \rangle) \\
\vdots & \ddots & \vdots \\
\exp(\langle \phi_{i1}, (n-1)\sigma \Delta \rangle) & \cdots & \exp(\langle \phi_{i1}, (n-1)\sigma \Delta \rangle)
\end{pmatrix},
\quad A_n = \text{diag}(\alpha_1, \ldots, \alpha_n),
\quad \Lambda_n = \text{diag}\left(\exp(\langle \phi_{i1}, \tau^{(k)} \rangle), \ldots, \exp(\langle \phi_{in}, \tau^{(k)} \rangle)\right).
$$

**Proof.** With the matrix factors given, the proof is merely an easy verification of the product $V_n \Lambda_n A_n V_n^T$ with the matrix $\tau^{(k)} H_n$. □

This factorisation allows us to obtain the values $\exp(\langle \phi_{i1}, \sigma \Delta \rangle)$ and the values $\exp(\langle \phi_{i1}, \tau^{(k)} \rangle)$ for $k = 2, \ldots, d$ from the generalized eigenvalues and eigenvectors of

$$
\begin{pmatrix}
\tau^{(k)} H_n
\end{pmatrix} v_i = \exp(\langle \phi_{i1}, \tau^{(k)} \rangle) \begin{pmatrix} 0 \\ H_n \end{pmatrix} v_i, \quad k = 2, \ldots, d.
$$

Note that the Hankel matrix $0 H_n$ is the classical Hankel matrix from Section 2 filled here with the one-dimensional samples $f_{j1}$, while the Hankel matrix $\tau^{(k)} H_n$ is the newly introduced one containing the samples at the shifted locations. Also, the matrix $V_n$ is merely the Vandermonde matrix from [9] arising in the factorisation of $\tau^{(k)} H_n$ with the product $\phi_i \sigma \Delta$ replaced by the
inner product $\langle \phi_i, \sigma \Delta \rangle$ as $\phi_i$ and $\Delta$ are now vectors. As a consequence of Theorem 6, a multivariate analogue of (15) is easy to write down as well.

Similarly, making use of the results in Section 3 in combination again with [7], the multivariate sparse interpolation of functions of the form

$$
\sum_{i=1}^{n} \alpha_i \cos (\phi_{i1} x_1 + \ldots + \phi_{id} x_d),
\sum_{i=1}^{n} \alpha_i \sin (\phi_{i1} x_1 + \ldots + \phi_{id} x_d),
\sum_{i=1}^{n} \alpha_i \cosh (\phi_{i1} x_1 + \ldots + \phi_{id} x_d),
\sum_{i=1}^{n} \alpha_i \sinh (\phi_{i1} x_1 + \ldots + \phi_{id} x_d),
$$

becomes available, including the use of a scale factor and a shift term in the sampling directions.

7.2. The multivariate Gaussian function. The technique in [7] can also be combined with the development in Section 5, resulting in the sparse interpolation of

$$
f(x_1, \ldots, x_d) = \sum_{i=1}^{n} \alpha_i \exp \left( -w_1(x_1 - \phi_{i1})^2 - \ldots - w_d(x_d - \phi_{id})^2 \right)
$$

from the minimal number of samples. For ease of notation but without loss of generality, we take $w_i = 1$ and denote $\Delta = (\Delta_1, \ldots, \Delta_d)$, $X_j = j\Delta$ with $j = 0, \ldots, 2n - 1$, $\tau = (\tau_1, \ldots, \tau_d)$, $\sigma = (\sigma_1, \ldots, \sigma_d)$ and $(\tau + j\sigma) \odot \Delta$ for the componentwise product

$$
((\tau_1 + j\sigma_1) \Delta_1, \ldots, (\tau_d + j\sigma_d) \Delta_d).
$$

Similarly to the condition $|\Re(\phi_i \Delta)| < \pi$ we assume that $|\Re((\phi_i, \Delta))| < \pi$ and $|\Re((\phi_i, \tau \odot \Delta))| < \pi$, where in addition $\sigma$ and $\tau$ are linearly independent. Then

$$
f_{\tau+j\sigma} := f((\tau_1 + j\sigma_1) \Delta_1, \ldots, (\tau_d + j\sigma_d) \Delta_d)
= \sum_{i=1}^{n} \alpha_i \exp \left( -\| (\tau + j\sigma) \odot \Delta - \phi_i \|_2^2 \right)
$$

where $\| \cdot \|_2$ is the Euclidean norm. We also introduce

$$
F(\sigma, \tau; X_j) := \exp(2\langle \tau \odot \Delta, j\sigma \odot \Delta \rangle) \exp \left( \| j\sigma \odot \Delta \|_2^2 \right) f_{\tau+j\sigma}
= \sum_{i=1}^{n} \alpha_i \exp \left( -\| \tau \odot \Delta - \phi_i \|_2^2 \right) \exp \left( 2\langle \phi_i, j\sigma \odot \Delta \rangle \right)
$$

(39)

We stress that here and in the sequel the notation is heavily simplified by the introduced componentwise product, as for instance in [39] where we
write
\[
\exp\left(2 \sum_{k=1}^{d} \phi_k j \sigma_k \Delta_k \right) = \exp\left(2(\phi_i, j \sigma \odot \Delta)\right).
\]

For chosen vectors \(\sigma\) and \(\tau\), we denote the evaluations \(F(\sigma, \tau; X_j)\) as \(F_{\tau+j\sigma}\) where \(j = 0, \ldots, 2n - 1\). Most of the one-dimensional interpolation scheme in Section 5 can now be generalized to the \(d\)-dimensional case by replacing the products \(\tau \Delta, j \sigma \Delta\) and \(\phi_i j \sigma \Delta\) by componentwise products and the squares \((\tau \Delta - \phi_i)^2\) and \((\sigma j \Delta)^2\) by the square of the Euclidean norms \(\|\tau \odot \Delta - \phi_i\|_2^2\) and \(\|j \sigma \odot \Delta\|^2\). We thus obtain the following factorization.

**Theorem 7.** The Hankel structured matrix

\[
\tau \sigma G_n := \begin{pmatrix}
F_{\tau} & F_{\tau+j\sigma} & \cdots & F_{\tau+(n-1)\sigma} \\
F_{\tau+j\sigma} & \ddots & \cdots & \vdots \\
\vdots & \ddots & \ddots & \vdots \\
F_{\tau+(n-1)\sigma} & \cdots & F_{\tau+(2n-2)\sigma}
\end{pmatrix}
\]

where \(F_{\tau+j\sigma} = F(\sigma, \tau; X_j)\) as in (39), factorizes as

\[
\tau \sigma G_n = E_n L_n A_n F_n^T,
\]

\[
E_n = \begin{pmatrix}
1 & \cdots & 1 \\
\exp(2(\phi_1, \sigma \odot \Delta)) & \cdots & \exp(2(\phi_n, \sigma \odot \Delta)) \\
\vdots & \ddots & \vdots \\
\exp(2(\phi_1, (n-1) \sigma \odot \Delta)) & \cdots & \exp(2(\phi_n, (n-1) \sigma \odot \Delta))
\end{pmatrix}
\]

\[
A_n = \text{diag}(\alpha_1 \exp(-\|\phi_1\|_2^2), \ldots, \alpha_n \exp(-\|\phi_n\|_2^2))
\]

\[
L_n = \text{diag}(\exp((-\tau \odot \Delta + 2\phi_1, \tau \odot \Delta)), \ldots, \exp((-\tau \odot \Delta + 2\phi_n, \tau \odot \Delta))).
\]

**Proof.** The proof is simply by verification of the entry \(F_{\tau+(k+\ell)\sigma}\) in \(\tau \sigma G_n\) at position \((k+1, \ell+1)\) for \(k = 0, \ldots, n-1\) and \(\ell = 0, \ldots, n-1\), now using the componentwise vector product.

With \(\tau = (0, \ldots, 0)\) the values \(\exp(2(\phi_i, \sigma \odot \Delta))\) are retrieved through the generalized eigenvalues of the problem

\[
(\sigma \sigma G_n) v_i = \exp(-\|\sigma \odot \Delta\|_2^2) \exp(2(\phi_i, \sigma \odot \Delta))(\sigma \sigma G_n) v_i, \quad i = 1, \ldots, n.
\]

The coefficients \(\alpha_i \exp(-\|\phi_i\|_2^2)\) are obtained from the linear system

\[
\sum_{i=1}^{n} (\alpha_i \exp(-\|\phi_i\|_2^2)) \exp(2(\phi_i, j \sigma \odot \Delta)) = F(\sigma, 0; X_j), \quad j = 0, \ldots, 2n - 1.
\]

The coefficient matrix of this linear system is Vandermonde structured with the entry \(\exp(2(\phi_i, j \sigma \odot \Delta))\) at position \((j+1, i)\).

The linear system expressing the interpolation conditions at the shifted sample locations,

\[
\sum_{i=1}^{n} (\alpha_i \exp(-\|\tau \odot \Delta - \phi_i\|_2^2)) \exp(2(\phi_i, j \sigma \odot \Delta)) = F_{\tau+j\sigma},
\]

\(j = 0, \ldots, n-1\)
is Vandermonde structured as well, with \( \exp(2(\phi_i, j\sigma \odot \Delta)) \) at position \((j + 1, i)\). It delivers the unknown coefficients \( \alpha_i \exp\left(-\|\tau \odot \Delta - \phi_i\|^2_2\right) \).

From both solutions and the knowledge of the vectors \( \Delta \) and \( \tau \) we find for each \( i = 1, \ldots, n \),

\[
\alpha_i \exp\left(-\|\tau \odot \Delta - \phi_i\|^2_2\right) = \exp\left(2(\phi_i, \tau \odot \Delta)\right).
\]

The sampling at shifted locations can be executed for a set of \( d - 1 \) shift vectors \( \tau^{(k)} \) that together with the vector \( \sigma \) form a linearly independent set in \( \mathbb{R}^d \). In this way we have at our disposal the values \( \exp(2(\phi_1, \sigma \odot \Delta)) \) and \( \exp(2(\phi_i, \tau^{(k)} \odot \Delta)) \) from which we can extract the \( \phi_i \) since the linear system with left hand side

\[
\begin{pmatrix}
\sigma_1 \Delta_1 & \cdots & \sigma_d \Delta_d \\
\tau_1^{(2)} \Delta_1 & \cdots & \tau_d^{(2)} \Delta_d \\
\vdots & \ddots & \vdots \\
\tau_1^{(d)} \Delta_1 & \cdots & \tau_d^{(d)} \Delta_d \\
\end{pmatrix}
\begin{pmatrix}
\phi_{i1} \\
\phi_{i2} \\
\vdots \\
\phi_{id}
\end{pmatrix}
\]

is regular for each \( i = 1, \ldots, n \).

Because of the high similarity between the one-dimensional and the multidimensional write-up, the formulation of a multivariate analogue of Corollary 4 is left to the reader.

8. Numerical illustrations

We present some examples to illustrate the main novelties of the paper:

- an illustration of sparse interpolation by Gaussian distributions with fixed width but unknown peak locations;
- an illustration of the new generalized eigenvalue formulation for use with several trigonometric functions and the sinc;
- an illustration of the use of the scale and shift strategy for the supersparse interpolation of polynomials.

Our focus is on the mathematical generalizations and not on the numerical issues.

8.1. Fixed width sparse Gaussian fitting. Consider the expression

\[
f(t) = \exp(-(t - 5)^2) + 0.01 \exp(-(t - 4.99)^2),
\]

illustrated in Figure 1, with the parameters \( \alpha_i, \phi_i \in \mathbb{R} \). From the plot it is not obvious that the signal has two peaks.

We first show the output of the widely used \[18\] Matlab state-of-the-art peak fitting program \texttt{peakfit.m}, which calls an unconstrained nonlinear optimization algorithm to decompose an overlapping peak signal into its components \[20\ [19].

In \texttt{peakfit.m} the user needs to supply a guess for the number of peaks and supply this as input. If one does not have any idea on the number of peaks, the usual practice is to try different possibilities and compare the
corresponding results. Of course, a good estimate of the number of peaks may lead to a good fit of the data. In addition to the peak position $\phi_i$, its height $\alpha_i$ and width $w$, the program also returns a goodness-of-fit (GOF).

The peakfit.m algorithm can work without assuming a fixed width, or the width can be passed as an argument. We do the latter as our algorithm also assumes a known fixed peak width $w$.

Let $\Delta = 0.1$ and let us collect 20 samples $f_0, f_1, \ldots, f_{19}$. When passing the width to peakfit.m and guessing the number of peaks, then it returns for 1 peak the estimates

$$\phi_1 = 4.9998944950 \ldots \quad \alpha_1 = 1.0099771180 \ldots \quad \text{GOF} \approx 1.5 \times 10^{-5}.$$  

For 2 peaks it returns

$$\phi_1 = 4.9998944843 \ldots \quad \alpha_1 = 1.0099776250 \ldots$$
$$\phi_2 = -1.5242538810 \ldots \quad \alpha_2 = 1.05 \times 10^{-13} \quad \text{GOF} \approx 5.2 \times 10^{-7}.$$  

Since the result is still not matching our benchmark input parameters, let us push further and supply 100 samples. Then for 1 peak peakfit.m returns

$$\phi_1 = 4.9999009752 \ldots \quad \alpha_1 = 1.009995049 \ldots \quad \text{GOF} \approx 2.5 \times 10^{-5}$$  

and for 2 peaks we get

$$\phi_1 = 4.9999945211 \ldots \quad \alpha_1 = 1.0010660101 \ldots$$
$$\phi_2 = 4.9894206737 \ldots \quad \alpha_2 = 0.0089339897 \ldots \quad \text{GOF} \approx 8.4 \times 10^{-9}.$$  

From the latter very satisfying experiment it is easy to formulate some desired features for a new algorithm:

- built-in guess of the number of peaks in the signal,
- reliable output from a smaller number of samples.

So let us investigate the technique developed in Section 5. Take $\sigma = 1$ and $\tau = 0$ since there is no periodic component in the Gaussian signal, which
has only real parameters. With the 20 samples \( f_0, f_1, \ldots, f_{19} \) we define the samples \( F_j = \exp(j^2\Delta^2) f_j \) and compose the Hankel matrix \( G_{10} \). Its singular value decomposition, illustrated in Figure 2, reveals that the rank of the matrix is 2 and so we deduce that there are \( n = 2 \) peaks.

From the 4 samples \( F_0, F_1, F_2, F_3 \) we obtain through Theorem 7

\[
\begin{align*}
\phi_1 &= 4.999999737 \ldots & \phi_2 &= 4.9899976207 \ldots \\
\alpha_1 &= 1.0000049866 \ldots & \alpha_2 &= 0.0099950129 \ldots 
\end{align*}
\]

The new method clearly provides both an automatic guess of the number of peaks and a reliable estimate of the signal parameters, all from only 20 samples. What remains to be done is to investigate the numerical behaviour of the method on a large collection of different input signals, which falls out of the scope of this paper where we provide the mathematical details. Among other things, it is known that in difficult cases the use of the SVD decomposition is only reliable numerically when a larger number of samples is provided [3]. Other intricacies, which have been studied in the exponential case [8], are the subject of future work.

8.2. Sparse sinc interpolation. Consider the function

\[
f(t) = -10\text{sinc}(145.5t) + 20\text{sinc}(149t) + 4\text{sinc}(147.3t),
\]

plotted in Figure 3, which we sample at \( t_j = j\pi/300 \) for \( j = 0, \ldots, 19 \). The singular value decomposition of \( B_{10} \) filled with the values \( t_j f_j \), of which the log-plot is shown in Figure 4 (left), reveals that \( f(t) \) consists of 3 terms. Remember that the sparse sinc interpolation problem with linear coefficients \( \alpha_i \) and nonlinear parameters \( \phi_i \) transforms into a sparse sine interpolation problem with linear coefficients \( \alpha_i/\phi_i \) and samples \( j\Delta f_j \).

The condition numbers of the matrices \( B_3 \) and \( B_3 \) appearing in the generalized eigenvalue problem

\[
(B_3) v_i = \cos(\phi_i \Delta) (B_3), \quad i = 1, 2, 3
\]
equal respectively $1.6 \times 10^7$ and $7.5 \times 10^6$. To improve the conditioning of the structured matrix we choose $\sigma = 30, \tau = 1$ and resample $f(t)$ at $t_j = 30j\pi/300 = j\pi/10$ for $j = 0, \ldots, 5$. The singular values of $0_{B_{10}}$ are graphed in Figure 4 (right) and the condition numbers of $0_{B_{3}}$ and $1_{B_{3}}$ improve to $1.1 \times 10^3$ and $9.7 \times 10^2$ respectively.

The generalized eigenvalues of the matrix pencil $1_{B_{3}} - \lambda 0_{B_{3}}$ are given by

$$\cos(30\phi_1\Delta) = -0.1564344650400536,$$
$$\cos(30\phi_2\Delta) = -0.9510565162957546,$$
$$\cos(30\phi_3\Delta) = -0.6613118653271576$$

and with these we fill the matrix $W_3$ from Theorem 2. We solve (26) for the values $\alpha_i \sin(\phi_i\sigma\Delta)/\phi_i$, $i = 1, 2, 3$ and further compute

$$\frac{\alpha_i}{\phi_i} \sin(\phi_i j\sigma\Delta) = \frac{\alpha_i}{\phi_i} \sin(\phi_i (j - 1)\sigma\Delta) \cos(\phi_i \sigma\Delta)$$

$$+ \cos(\phi_i (j - 1)\sigma\Delta) \frac{\alpha_i}{\phi_i} \sin(\phi_i \sigma\Delta), \quad j = 1, \ldots, n.$$
At this point the matrix $U_3$ from Theorem 2 can be filled and the $\cos(\phi_i \tau \Delta)$ can be computed from (27), with the right hand side filled with the additional samples $F(31\Delta), F(61\Delta), F(91\Delta)$, where

$$F_{\tau+j\sigma} = \frac{\Delta}{2} (\tau + j\sigma)f_{\tau+j\sigma} + \frac{\Delta}{2} (-\tau + j\sigma)f_{-\tau+\sigma}. $$

Since $\tau = 1$ we obtain the $\phi_i$ directly from the values $\cos(\phi_i \tau \Delta)$:

$$\phi_1 = 145.5000000000, $$
$$\phi_2 = 149.0000000000, $$
$$\phi_3 = 147.3000000000. $$

The linear coefficients $\alpha_i$ are given by

$$\alpha_i = \phi_i \frac{\alpha_i \sin(\phi_i \sigma \Delta)}{\sin(\phi_i \sigma \Delta)}, $$
resulting in

$$\alpha_1 = -9.999999999991, $$
$$\alpha_2 = 19.99999999978, $$
$$\alpha_3 = 4.000000000089. $$

8.3. Supersparse Chebyshev interpolation. We consider the polynomial

$$f(t) = 2T_6(t) + T_7(t) + T_{39999}(t), $$
which is clearly supersparse when expressed in the Chebyshev basis. We sample $f(t)$ at $t_j = \cos(j\Delta)$ where $\Delta = \pi/100000$ with $M = 50000$. The first challenge is maybe to retrieve an indication of the sparsity $n$.

Take $\sigma = 1$ and collect 15 samples $f_j, j = 0, \ldots, 14$ to form the matrix $^{0}C_8$. From its singular value decomposition, computed in double precision arithmetic and illustrated on the log-plot in Figure 5, one may erroneously conclude that $f(t)$ has only 2 terms, a consequence of the fact that, relatively speaking, the degrees $m_1 = 6$ and $m_2 = 7$ are close to one another and appear as one cluster.
Imposing a 3-term model to $f(t)$ instead of the erroneously suggested 2-term one, does not improve the computation as the matrix $0^C_8$ is ill-conditioned with a condition number of the order of $10^{10}$. So 3 generalized eigenvalues cannot be extracted reliably from the samples. For completeness we mention the unreliable double precision results, rounded to integer values: $m_1 = 6, m_2 = 39999, m_3 = 25119$.

Now choose $\sigma = 3125$ and $\tau = 16$. The singular value decomposition of $0^C_{3125}$, shown on the log-plot in Figure 6, reveals that $f(t)$ indeed consists of 3 terms. Also, the conditioning of the involved matrix $0^C_{3125}$ has improved to the order of $10^{3}$.

The distinct generalized eigenvalues extracted from the matrix pencil $1^C_{3125} - \lambda^0_{3125}C_3$ are given by

$$\cos(m_1\sigma\Delta) = 0.999999204093383,$$
$$\cos(m_2\sigma\Delta) = -0.8089800617792506,$$
$$\cos(m_3\sigma\Delta) = -0.007490918959382487.$$

From 3 shifted samples at the arguments $t_{\tau+j\sigma}, j = 0, 1, 2$ we obtain

$$\cos(m_1\tau\Delta) = -0.8084256802389809,$$
$$\cos(m_2\tau\Delta) = 0.9999752362021560,$$
$$\cos(m_3\tau\Delta) = 0.9999818099296417.$$

Building the sets $S_i$ and $T_i$ for $i = 1, 2, 3$ as indicated in Section 4.1, and rounding the result to the nearest integer, does unfortunately not provide singletons for $S_1 \cap T_1, S_2 \cap T_2, S_3 \cap T_3$. We consequently need to consider a second shift, for which we choose $\sigma + \tau = 3141$. With this choice we only need to add the evaluation of $f(\tau+n\sigma)$ to proceed with the computation of

$$\cos(m_1(\sigma+\tau)\Delta) = -0.6780621808989576,$$
$$\cos(m_2(\sigma+\tau)\Delta) = 0.1881836009619241,$$
$$\cos(m_3(\sigma+\tau)\Delta) = 0.3771037932233129.$
Finally, intersecting each \( S_i \cap T_i, i = 1, 2, 3 \) with the solutions provided by the second shift, delivers the correct

\[
m_1 = 6, \\
m_2 = 7, \\
m_3 = 39999.
\]

9. Conclusion

Let us summarize the sparse interpolation formulas obtained in the preceding sections in a table. For each univariate function \( g(\phi_i; t) \) we list in the columns 1 to 4:

1) the minimal number of samples required to solve the sparse interpolation without running into ambiguity problems, meaning for the choice \( \sigma = 1 \),
2) the minimal number of samples required for the choice \( \sigma > 1 \) (if applicable), thereby involving a shift \( \tau \neq 0 \) to restore uniqueness of the solution,
3) the matrix pair \((A, B)\) in the generalized eigenvalue formulation \( Av_i = \lambda_i Bv_i \) of the sparse interpolation problem involving the atom \( g(\phi_i; t) \),
4) the generalized eigenvalues in terms of \( \tau \), as they can be read directly from the structured matrix factorizations presented in the theorems 1–5,
5) the information that can be computed from the associated generalized eigenvectors, as indicated at the end of each (sub)section.

| \( g(\phi_i; t) \) | \# samples | pencil* | \( \lambda_i \) | \( Bv_i \) |
|----------------|------------|---------|----------------|----------|
| \( \exp(\phi_i t) \) | 2\(n\) 3\(n\) | \( \frac{\tau}{\sigma} H_n, 0 H_n \) | \( \exp(\phi_i \tau \Delta) \) | \( \alpha_i \exp(\phi_i \sigma \Delta) \) |
| \( \cos(\phi_i t) \) | 2\(n\) 4\(n\) | \( \frac{\tau}{\sigma} C_n, 0 C_n \) | \( \cos(\phi_i \tau \Delta) \) | \( \alpha_i \cos(\phi_i \sigma \Delta) \) |
| \( \sin(\phi_i t) \) | 2\(n\) 4\(n\) | \( \frac{\tau}{\sigma} B_n, 0 B_n \) | \( \cos(\phi_i \tau \Delta) \) | \( \alpha_i \sin(\phi_i \sigma \Delta) \) |
| \( \cosh(\phi_i t) \) | 2\(n\) 4\(n\) | \( \frac{\tau}{\sigma} C_n, 0 C_n \) | \( \cosh(\phi_i \tau \Delta) \) | \( \alpha_i \cosh(\phi_i \sigma \Delta) \) |
| \( \sinh(\phi_i t) \) | 2\(n\) 4\(n\) | \( \frac{\tau}{\sigma} B_n, 0 B_n \) | \( \cosh(\phi_i \tau \Delta) \) | \( \alpha_i \sinh(\phi_i \sigma \Delta) \) |
| \( T_{m_i}(t) \) | 2\(n\) 4\(n\) | \( \frac{\tau}{\sigma} C_n, 0 C_n \) | \( T_{m_i}(\cos \tau \Delta) \) | \( \alpha_i T_{m_i}(\cos \sigma \Delta) \) |
| \( S_{m_i}(t) \) | 2\(n\) 4\(n\) | \( \frac{\tau}{\sigma} C_n, 0 C_n \) | \( S_{m_i}(\sin \tau \Delta) \) | \( \alpha_i S_{m_i}(\sin \sigma \Delta) \) |
| \( \Gamma(z + \phi_i) \) | 2\(n\) \(X\) | \( \frac{\tau}{\sigma} H_n, 0 H_n \) | \( \phi_i \) | \( \alpha_i \phi_i \) |
| \( \exp(-(t - \phi_i)^2) \) | 2\(n\) 3\(n\) | \( \frac{\tau}{\sigma} C_n, 0 C_n \) | \( \exp(2\phi_i \tau \Delta) \) | \( \alpha_i \exp(2\phi_i \sigma \Delta) \) |

* with \( \cos(\cdot) \) replaced by \( \cosh(\cdot) \) and \( \sin(\cdot) \) replaced by \( \sinh(\cdot) \)
Appendix

To reconstruct a function of the form

\[ f(t) = \sum_{i=1}^{n} \alpha_i \cos(\phi_i t) \]

from equidistantly collected samples \( f_j \) at \( t = j \Delta \), in other words to recover the unknown parameters \( \phi_i \) and coefficients \( \alpha_i \), the sampling step \( \Delta \) needs to satisfy the Shannon-Nyquist constraint

\[ \Delta < \frac{\pi}{\max_{i=1,...,n} |\phi_i|}. \]

Since we do not distinguish \( \phi_i \) from \(-\phi_i\) in this case, we can simply drop the sign information in \( \phi_i \) from here on and write \( \Delta = \frac{\pi}{R} \) with

\[ 0 \leq \max_{i=1,...,n} \phi_i < R. \]

The challenge we consider now is to retrieve the parameters \( \phi_i \) and coefficients \( \alpha_i \) from sub-Nyquist rate collected samples \( f_j \sigma \) at \( t = j \sigma \Delta \) with \( \sigma > 1 \) and the shifted evaluations \( f_{j\sigma + \tau} \) at \( t = (j \sigma + \tau) \Delta \) with \( \gcd(\sigma, \tau) = 1 \). In Section 3.1 we describe how for \( i = 1, \ldots, n \) the values

\[ C_{i,\sigma} := \cos(\phi_i \sigma \Delta), \]
\[ C_{i,\tau} := \cos(\phi_i \tau \Delta) \]

are obtained. The aim is to extract the correct value for \( \phi_i \) from the knowledge of the evaluations \( C_{i,\sigma} \) and \( C_{i,\tau} \), particularly when \( (\sigma \Delta) \max_{i=1,...,n} \phi_i \geq \pi \) and the parameter \( \phi_i \) cannot be obtained uniquely from \( C_{i,\sigma} \) alone. We now discuss the unique identification of this parameter \( \phi_i \) and in doing so we further drop the index \( i \). Let us denote

\[ A_\sigma := \text{Arccos}(C_\sigma), \]
\[ A_\tau := \text{Arccos}(C_\tau) \]

(40)

where \( \text{Arccos}(\cdot) \in [0, \pi] \) indicates the principal value of the inverse cosine function. Knowing that \( 0 \leq A_\sigma, A_\tau \leq \pi \) and that \( 0 \leq \phi \sigma \Delta < \sigma \pi \), we find that all possible positive arguments \( \phi \sigma \Delta \) of \( C_\sigma \) are in \( A_{\sigma,1} \cup A_{\sigma,2} \) with

\[ A_{\sigma,1} := \{ A_\sigma + 2\pi \ell \mid 0 \leq \ell \leq \lfloor \sigma/2 \rfloor - 1 \}, \]
\[ A_{\sigma,2} := \{(2\pi - A_\sigma) \text{sgn}(A_\sigma) + 2\pi \ell \mid 0 \leq \ell \leq \lfloor \sigma/2 \rfloor - 1 \}, \]

where \( \text{sgn}(a) = +1 \) for \( 0 < A_\sigma \leq \pi \) and \( \text{sgn}(0) = 0 \). The set \( A_{\sigma,1} \cup A_{\sigma,2} \) may even contain some candidate arguments of \( C_\sigma \) that do not satisfy the bounds, but this does not create a problem in the identification of the correct \( \phi < R \). Along the same lines, sets \( A_{\tau,1} \) and \( A_{\tau,2} \) can be constructed.

We further denote

\[ \phi_\sigma := \frac{A_\sigma R}{\sigma \pi}, \]
\[ \phi_\tau := \frac{A_\tau R}{\tau \pi}. \]

(41)
Then the possible solutions for $\phi$ to $C_\sigma = \cos(\phi \sigma \Delta)$ are in $\Phi_{\sigma, 1} \cup \Phi_{\sigma, 2}$ where

\[
\Phi_{\sigma, 1} := \{ \phi_\sigma + 2R\ell/\sigma \mid 0 \leq \ell \leq \lfloor \sigma/2 \rfloor - 1 \} \cap [0, R),
\]
\[
\Phi_{\sigma, 2} := \{ (2R/\sigma - \phi_\sigma) \text{sgn}(\phi_\sigma) + 2R\ell/\sigma \mid 0 \leq \ell \leq \lfloor \sigma/2 \rfloor - 1 \} \cap [0, R).
\]

Analogously, the possible solutions to $C_\tau = \cos(\phi \tau \Delta)$ are in $\Phi_{\tau, 1} \cup \Phi_{\tau, 2}$ where

\[
\Phi_{\tau, 1} := \{ \phi_\tau + 2R\ell/\tau \mid 0 \leq \ell \leq \lfloor \tau/2 \rfloor - 1 \} \cap [0, R),
\]
\[
\Phi_{\tau, 2} := \{ (2R/\tau - \phi_\tau) \text{sgn}(\phi_\tau) + 2R\ell/\tau \mid 0 \leq \ell \leq \lfloor \tau/2 \rfloor - 1 \} \cap [0, R).
\]

One statement is obvious: whatever the choice for $\sigma$ and $\tau$, both $\Phi_{\sigma, 1} \cup \Phi_{\sigma, 2}$ and $\Phi_{\tau, 1} \cup \Phi_{\tau, 2}$ contain the unknown value for $\phi$ which produced $C_\sigma$ and $C_\tau$. What remains open is the question whether $(\Phi_{\sigma, 1} \cup \Phi_{\sigma, 2}) \cap (\Phi_{\tau, 1} \cup \Phi_{\tau, 2})$ is a singleton. And in case it is not, we want to find an algorithm that can identify the correct $\phi$.

When either $\phi_\sigma = 0$ or $\phi_\sigma = R/\sigma$ the sets $\Phi_{\sigma, 1}$ and $\Phi_{\sigma, 2}$ coincide. And similarly for $\phi_\tau$. On the other hand, if these sets do not coincide, they are disjoint. So the true value for the unknown parameter $\phi$ can belong to any of the intersections $\Phi_{\sigma, 1} \cap \Phi_{\tau, 1}, \Phi_{\sigma, 1} \cap \Phi_{\tau, 2}, \Phi_{\sigma, 2} \cap \Phi_{\tau, 1}, \Phi_{\sigma, 2} \cap \Phi_{\tau, 2}$. A sequence of lemmas will lead to the conclusion that the four intersections do not deliver more than two distinct elements. Thereafter we indicate how to identify the only true value for the unknown $\phi$.

**Lemma 1.** $i, j \in \{1, 2\} : \Phi_{\sigma, i} \cap \Phi_{\tau, j} \neq \emptyset \implies \#(\Phi_{\sigma, i} \cap \Phi_{\tau, j}) = 1$.

**Proof.** Without loss of generality we prove the statement for $i = 1 = j$, by contraposition. The proof of the other cases is entirely similar. From $\Phi_{\sigma, 1} \cap \Phi_{\tau, 1} \neq \emptyset$ and containing at least two elements, we then find that

$$\exists 0 \leq \ell_1, \ell_2 \leq \lfloor \sigma/2 \rfloor - 1, 0 \leq k_1, k_2 \leq \lfloor \tau/2 \rfloor - 1, \ell_1 \neq \ell_2, k_1 \neq k_2 :$$

\[
\begin{cases}
\phi_\sigma + \ell_1 2R/\sigma = \phi_\tau + k_1 2R/\tau \\
\phi_\sigma + \ell_2 2R/\sigma = \phi_\tau + k_2 2R/\tau.
\end{cases}
\]

This leads to

$$\frac{\ell_1 - \ell_2}{k_1 - k_2} = \frac{\sigma}{\tau}$$

which is a contradiction because $|\ell_1 - \ell_2| < \sigma, |k_1 - k_2| < \tau$ and $\gcd(\sigma, \tau) = 1$. \qed

When the sets $\Phi_{\sigma, 1}$ and $\Phi_{\sigma, 2}$ coincide and the sets $\Phi_{\tau, 1}$ and $\Phi_{\tau, 2}$ do as well, then that unique intersection is $\phi = \phi_\sigma = \phi_\tau = 0$. Because $\gcd(\sigma, \tau) = 1$, other common elements coming from either $\phi_\sigma = 2R/\sigma$ or $\phi_\tau = 2R/\tau$ cannot exist.

We now continue with the situation where either the sets in (42) and (43) or the sets in (44) and (45) do not coincide, so that there are always at least 3 distinct sets in the running. Without loss of generality, we assume that a common element belongs to $\Phi_{\sigma, 1} \cap \Phi_{\tau, 1}$ and we build our reasoning from there.
Lemma 2. $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset \implies \Phi_{\sigma,2} \cap \Phi_{\tau,2} = \emptyset$.

Proof. We know that either $\Phi_{\sigma,1} \cap \Phi_{\sigma,2} \neq \emptyset$ or $\Phi_{\tau,1} \cap \Phi_{\tau,2} \neq \emptyset$ and possibly both, so that $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \Phi_{\sigma,2} \cap \Phi_{\tau,2}$. Again by contraposition, we suppose that $\Phi_{\sigma,2} \cap \Phi_{\tau,2} \neq \emptyset$ and so

$$\exists 0 \leq \ell_1, \ell_2 \leq \lfloor \sigma/2 \rfloor - 1, 0 \leq k_1, k_2 \leq \lfloor \tau/2 \rfloor - 1 :$$

$$\begin{array}{l}
\phi_1 = \phi_\sigma + \ell_1 2R/\sigma = \phi_\tau + k_1 2R/\tau \in \Phi_{\sigma,1} \cap \Phi_{\tau,1}, \\
\phi_2 = 2R/\sigma - \phi_\sigma + \ell_2 2R/\sigma = 2R/\tau - \phi_\tau + k_2 2R/\tau \in \Phi_{\sigma,2} \cap \Phi_{\tau,2}.
\end{array}$$

From this we obtain

$$1 + \ell_1 + \ell_2 = \sigma$$

$$1 + k_1 + k_2 = \tau$$

which can only be true when $1 + \ell_1 + \ell_2 = \sigma$ and $1 + k_1 + k_2 = \tau$. Then

$$\phi_1 + \phi_2 = (1 + \ell_1 + \ell_2) 2R/\sigma = 2R,$$

which contradicts $0 \leq \phi_1, \phi_2 < R$. \hfill \Box

While, assuming $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$, we have seen in Lemma 1 that this intersection is a singleton, and we have seen in Lemma 2 that then $\Phi_{\sigma,2} \cap \Phi_{\tau,2} = \emptyset$, we know nothing so far about the other two intersections $\Phi_{\sigma,1} \cap \Phi_{\tau,2}$ and $\Phi_{\sigma,2} \cap \Phi_{\tau,1}$.

Lemma 3. $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset \implies - (\Phi_{\sigma,2} \cap \Phi_{\tau,1} \neq \emptyset \wedge \Phi_{\sigma,1} \cap \Phi_{\tau,2} \neq \emptyset)$.

Proof. By contraposition we assume that

$$\exists 0 \leq \ell_1, \ell_2 \leq \lfloor \sigma/2 \rfloor - 1, 0 \leq k_1, k_2 \leq \lfloor \tau/2 \rfloor - 1 :$$

$$\begin{array}{l}
\phi_1 = 2R/\sigma - \phi_\sigma + \ell_1 2R/\sigma = \phi_\tau + k_1 2R/\tau \in \Phi_{\sigma,2} \cap \Phi_{\tau,1}, \\
\phi_2 = \phi_\sigma + \ell_2 2R/\sigma = 2R/\tau - \phi_\tau + k_2 2R/\tau \in \Phi_{\sigma,1} \cap \Phi_{\tau,2}.
\end{array}$$

This leads to

$$1 + \ell_1 + \ell_2 = \sigma$$

$$1 + k_1 + k_2 = \tau,$$

which again implies $1 + \ell_1 + \ell_2 = \sigma$ and $1 + k_1 + k_2 = \tau$. Since

$$\phi_1 + \phi_2 = (1 + \ell_1 + \ell_2) 2R/\sigma = 2R,$$

this contradicts $0 \leq \phi_1, \phi_2 < R$. \hfill \Box

We have built our sequence of proofs from Lemma 2 on, without loss of generality, on the fact that $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$ and the fact that $\Phi_{\sigma,1}$ and $\Phi_{\sigma,2}$ on the one hand and $\Phi_{\tau,1}$ and $\Phi_{\tau,2}$ on the other do not collide at the same time. Finally, from Lemma 3 we know that $(\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$ and $\Phi_{\sigma,2} \cap \Phi_{\tau,1} \neq \emptyset)$ or $(\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$ and $\Phi_{\sigma,1} \cap \Phi_{\tau,2} \neq \emptyset)$ cannot occur concurrently, but either one of these cases remains possible.

In general, when at least 3 of the 4 sets $\Phi_{\sigma,1}, \Phi_{\sigma,2}, \Phi_{\tau,1}, \Phi_{\tau,2}$ are distinct, then at most 2 of the 4 intersections

$$\Phi_{\sigma,i} \cap \Phi_{\tau,j}, \quad 1 \leq i, j \leq 2$$
are nonempty, with each of the nonempty intersections being a singleton. Further down we illustrate the actual existence of a case, where two intersections are nonempty and consequently the true value of the unknown $\phi$ cannot be identified from the evaluations $\cos(\phi \sigma \Delta)$ and $\cos(\phi \tau \Delta)$ with $\gcd(\sigma, \tau) = 1$.

In this case we need to collect a third value $C_\rho := \cos(\phi \rho \Delta)$ with $\gcd(\sigma, \rho) = 1$ and $\gcd(\tau, \rho) = 1$. With $A_\rho$ and $\phi_\rho$ defined as in (40) and (41), and $\Phi_{\rho,1}$ and $\Phi_{\rho,2}$ defined as in (42) and (43), we know, as before, that $\Phi_{\rho,1} \cup \Phi_{\rho,2}$ contains the correct value for $\phi$. We also know, because of the remark formulated after the proof of Lemma 1, that at least 5 of the 6 involved sets $\Phi_{\sigma,1}, \Phi_{\sigma,2}, \Phi_{\tau,1}, \Phi_{\tau,2}, \Phi_{\rho,1}, \Phi_{\rho,2}$ are distinct unless $\phi = 0$.

We now inspect

$$\bigcup_{i,j=1}^{2} (\Phi_{\sigma,i} \cap \Phi_{\tau,j}) \cap (\Phi_{\rho,1} \cup \Phi_{\rho,2})$$

$$= \left( \bigcup_{k=1}^{2} \Phi_{\sigma,i_1} \cap \Phi_{\tau,j_1} \cap \Phi_{\rho,k} \right) \cup \left( \bigcup_{k=1}^{2} \Phi_{\sigma,i_2} \cap \Phi_{\tau,j_2} \cap \Phi_{\rho,k} \right)$$

where $i_1, j_1, i_2, j_2$ index the subsets that produce the nonempty intersections of the relatively prime pair $\sigma$ and $\tau$, with either $i_1 \neq i_2$ or $j_1 \neq j_2$ but not both. We have built our sequence of proofs, without loss of generality, on the fact that $i_1 = 1, j_1 = 1$ and have found that it is then possible that $i_2 = 2, j_2 = 1$. We now continue the proofs from that case and inspect the 4 new intersections in (46).

**Lemma 4.** $\Phi_{\rho,1} \cap \Phi_{\rho,2} = \emptyset \land \Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1} \neq \emptyset \implies \Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,2} = \emptyset$.

**Proof.** From Lemma 1, we know that $\Phi_{\sigma,1} \cap \Phi_{\tau,1}$ is a singleton. If that unique element also belongs to $\Phi_{\rho,1}$ then it cannot belong to $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,2}$ when $\Phi_{\rho,1}$ and $\Phi_{\rho,2}$ are disjoint.

**Lemma 5.** $\Phi_{\sigma,1} \cap \Phi_{\sigma,2} = \emptyset \land \Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1} \neq \emptyset \implies \Phi_{\sigma,2} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1} = \emptyset$.

**Proof.** From Lemma 1, we know that $\Phi_{\tau,1} \cap \Phi_{\rho,1}$ is a singleton. If that unique element also belongs to $\Phi_{\sigma,1}$ then it cannot belong to $\Phi_{\sigma,2} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1}$ when $\Phi_{\sigma,1}$ and $\Phi_{\sigma,2}$ are disjoint.

As a consequence of the Lemmas 4 and 5, the unique true $\phi$ is identified in $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1}$ or $\Phi_{\sigma,2} \cap \Phi_{\tau,1} \cap \Phi_{\rho,2}$.

**Lemma 6.** $\# \left( \left( \Phi_{\sigma,1} \cap \Phi_{\tau,1} \cap \Phi_{\rho,1} \right) \cup \left( \Phi_{\sigma,2} \cap \Phi_{\tau,1} \cap \Phi_{\rho,2} \right) \right) = 1$.

**Proof.** We know that either $\phi = 0$ is the unique element in the intersections or at least 2 of the intersections $\Phi_{\sigma,1} \cap \Phi_{\sigma,2}, \Phi_{\tau,1} \cap \Phi_{\tau,2}, \Phi_{\rho,1} \cap \Phi_{\rho,2}$ are empty. So either $\Phi_{\sigma,1} \cap \Phi_{\sigma,2} = \emptyset$ or $\Phi_{\rho,1} \cap \Phi_{\rho,2} = \emptyset$. When applying Lemma 2 to the pair $(\sigma, \rho)$ instead of $(\sigma, \tau)$ the set $\Phi_{\sigma,2} \cap \Phi_{\rho,2}$ if the set $\Phi_{\sigma,1} \cap \Phi_{\rho,1} \neq \emptyset$. Therefore two distinct elements in respectively $\Phi_{\sigma,1} \cap \Phi_{\rho,1}$ and $\Phi_{\sigma,2} \cap \Phi_{\rho,2}$ cannot coexist and solve $C_\tau = \cos(\phi \tau \Delta)$.
So the unknown parameter $\phi$ is identified uniquely from at most 3 values $C_\sigma, C_\tau, C_\rho$ with $\sigma, \tau, \rho$ all mutually prime. An easy choice for $\rho$ is $\rho = \sigma + \tau$ as this minimizes the number of additional samples as explained in Section 3, and also $\text{gcd}(\sigma, \sigma + \tau) = 1 = \text{gcd}(\tau, \sigma + \tau)$ when $\text{gcd}(\sigma, \tau) = 1$.

As promised, we show an example where $\Phi_{\sigma,1} \cap \Phi_{\tau,1} \neq \emptyset$ and $\Phi_{\sigma,2} \cap \Phi_{\tau,1} \neq \emptyset$. Consider $\phi = 70800/1547 < 1000 = R$ with $\Delta = \pi/R$. Choose $\sigma = 299$ and $\tau = 357$ with $\gcd(\sigma, \tau) = 1$.

With $\phi_{\sigma} = 100000/35581$, $\ell = 68 \leq \lceil \sigma/2 \rceil - 1 = 149$, we have $\phi \in \Phi_{\sigma,1}$. With $\phi_{\tau} = 6000/1547$, $\ell = 81 \leq \lceil \tau/2 \rceil - 1 = 178$, we find $\phi \in \Phi_{\tau,1}$. Unfortunately, since $\phi_{\tau} = 2R/\sigma - \phi_{\sigma}$ we also have $\phi_{\tau} \in \Phi_{\sigma,2} \cap \Phi_{\tau,1} \neq \emptyset$.

As a last remark, we add that even replacing (16) by the stricter constraint $|\phi_i|\Delta < \pi/2$, $i = 1, \ldots, n$ does not guarantee that each $\phi$ can be identified from only $C_\sigma$ and $C_\tau$. We illustrate this with a counterexample. Let $\phi = 3300/133 < 50 = R$ with $\Delta = \pi/(2R)$. With $\sigma = 21$ and $\tau = 19$ we find

$$
\phi_{\sigma} = 500/133, \quad (2\pi)/(\sigma \Delta) - \phi_{\sigma} = 2300/399, \\
\phi_{\tau} = 500/133, \quad (2\pi)/(\tau \Delta) - \phi_{\tau} = 900/133.
$$

This leads to $\Phi_{\sigma,1} \cap \Phi_{\tau,1} = \{500/133\}$ and $\Phi_{\sigma,2} \cap \Phi_{\tau,1} = \{3300/133\}$.

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