ODD TRANSGRESSION FOR COURANT ALGEBROIDS

PAUL BRESSLER AND CAMILO RENGIFO

Abstract. The “odd transgression” introduced in (Bressler and Rengifo, 2018) is applied to construct and study the inverse image functor in the theory of Courant algebroids.

1. Introduction

The goal of this note is to demonstrate applications of “odd transgression” introduced in (Bressler and Rengifo, 2018) to the theory of Courant algebroids.

The “odd transgression” functor associates to a Courant algebroid $Q$ a differential-graded (DG) Lie algebroid $\tau Q$ over the de Rham complex equipped with a central section of degree $-2$ which we refer to as a marking. Conversely, a marked DG Lie algebroid $\mathcal{A}$ over the de Rham complex satisfying certain natural vanishing conditions gives rise, by way of taking its component in degree $-1$, to a Courant algebroid $Q(\mathcal{A})$. In particular, the Courant algebroid $Q$ is recovered in this manner from its transgression $\tau Q$. Using the functors $\tau$ and $Q$ one can project various standard constructions in Lie algebroids to the, perhaps less familiar, setting of Courant algebroids.

In this note we apply the above idea to the functor of inverse image of a Lie (respectively, Courant) algebroid under a map manifolds. The construction of inverse image for Courant algebroids has appeared in the literature, at least in special cases; see (Li-Bland and Meinrenken, 2009; Severa and Valach, 2018; Vysoky, 2019). We study the naturality properties of the inverse image functor as well as its behavior with respect to some standard constructions. We formulate and prove descent for Lie (respectively, Courant) algebroids along a surjective submersion.

In addition, we take the opportunity to relate the inverse image for Courant algebroids to the notion of Dirac structure with support and of Courant morphisms of (Alekseev and Xu, 2002) (see also (Bursztyn et al., 2009)).

The paper is organized as follows. In Section 2 we briefly review the requisite notions from the theory of DG manifolds. In Section 3 we recall the “odd transgression” for Courant algebroids of (Bressler and Rengifo, 2018) and further develop its properties. In Section 4 we construct the inverse image functor for $\mathcal{O}$-modules equipped with an “anchor” map to the tangent bundle and study its behavior in compositions of maps. Section 5 is devoted to setting up the descent problem. In Section 6 we apply previously obtained results to the setting of DG Lie algebroids. In Section 7 we construct the inverse image functor for Courant algebroids and study its relationship to various notions of Courant algebroid theory.
1.1. **Notation.** In order to simplify notations in numerous signs we will write “$a$” instead of “$\deg(a)$” in expressions appearing in exponents of $-1$. For example, $(-1)^{ab-1}$ stands for $(-1)^{\deg(a) - \deg(b) - 1}$.

Throughout the paper “manifold” means a $C^\infty$, real analytic or complex manifold. For a manifold $X$ we denote by $\mathcal{O}_X$ the corresponding structure sheaf of **real or complex valued** $C^\infty$, respectively analytic or holomorphic functions. We denote by $\mathcal{T}_X$ (respectively, by $\Omega^k_X$) the sheaf of real or complex valued vector fields (respectively, differential forms of degree $k$) on $X$.

For a sheaf of algebras $R$ on $X$, let $R-\text{Mod}$ denotes the (abelian) category of sheaves of $R$-modules on $X$. For a sheaf $\mathcal{E}$ of $\mathcal{O}_X$-modules, we denote \[ f^* \mathcal{E} := \mathcal{O}_Y \otimes_{f^{-1}\mathcal{O}_X} f^{-1}\mathcal{E} \]
the induced $\mathcal{O}_Y$-module, where $f^{-1}$ denotes the sheaf-theoretic inverse image. The assignment $\mathcal{E} \mapsto f^* \mathcal{E}$ extends to a functor $f^* : \mathcal{O}_X-\text{Mod} \to \mathcal{O}_Y-\text{Mod}$.

For a sheaf $\mathcal{F}$ on $X$, by $a \in \mathcal{F}$ we mean that $a$ is a local section of $\mathcal{F}$, i.e. there is an open set $U$ of $X$ such that $a \in \Gamma(U; \mathcal{F})$.

For a $\mathbb{Z}$-graded object $A$ and $i \in \mathbb{Z}$ we denote by $A^i$ the graded component of $A$ of degree $i$.

2. **DG manifolds**

In what follows we use notation introduced in (Bressler and Rengifo, 2018).

2.1. **The category of DG manifolds.** For the purposes of the present note a **differential-graded manifold (DG-manifold)** is a pair $\mathfrak{X} := (X, \mathcal{O}_X)$, where $X$ is a manifold and $\mathcal{O}_X$ is a sheaf of commutative differential-graded algebras (CDGA) on $X$ locally isomorphic to one of the form $\mathcal{O}_X \otimes S(E)$, where $S(E)$ is the symmetric algebra of a finite-dimensional graded vector space $E$.

Let $\mathfrak{X} = (X, \mathcal{O}_X)$ and $\mathfrak{Y} = (Y, \mathcal{O}_Y)$ be DG-manifolds. A morphism $\phi : \mathfrak{X} \to \mathfrak{Y}$ is a morphism of ringed spaces, which is to say a map $\phi : X \to Y$ of manifolds together with a morphism of differential-graded algebras $\phi^* : \phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ compatible with the canonical map $\phi^{-1}\mathcal{O}_Y \to \mathcal{O}_X$.

We denote the category of DG-manifolds by $\text{dgMan}$. Let $\text{dgMan}^+$ denote the full subcategory of DG-manifolds $\mathfrak{X} = (X, \mathcal{O}_X)$ such that $\mathcal{O}_X^i = 0$ if $i < N$ for some $N \in \mathbb{Z}$.

For $\mathfrak{X} \in \text{dgMan}$ we denote by $\mathcal{O}_X-\text{Mod}$ the category of sheaves of differential-graded modules over the structure sheaf $\mathcal{O}_X$.

**Example 1.** An ordinary manifold is an example of a DG-manifold with the structure sheaf concentrated in degree zero. Each ordinary manifold $X$ determines a DG-manifold $X^\bullet \in \text{dgMan}^+$ defined by $X^\bullet = (X, \Omega^\bullet_X, d)$ and frequently denoted by $T[1]X$ in the literature.
There is a canonical morphism $X \to X^\sharp$ of DG-manifolds defined by the canonical map $\Omega^\bullet_X \to \mathcal{O}_X$.

**Example 2.** Let $\mathbf{i}$ denote the DG-manifold with the underlying space consisting of one point and the DG-algebra of functions $\mathcal{O}_\mathbf{i} = \mathbb{C}[\epsilon]$, the free graded commutative algebra with one generator $\epsilon$ of degree $-1$ and the differential $\partial_\epsilon : \epsilon \mapsto \epsilon = 1$. Note that $\mathbf{i} \in \text{dgMan}^+$. The category $\text{dgMan}^+$ has finite products ([Bressler and Rengifo, 2018], Lemma 2.1). For $X = (X, \mathcal{O}_X), Y = (Y, \mathcal{O}_Y) \in \text{dgMan}^+$ the product is given by $(X \times Y, \mathcal{O}_{X \times Y})$ with $\mathcal{O}_{X \times Y} := \mathcal{O}_{X \times Y} \otimes_{\text{pr}_X^{-1}\mathcal{O}_X \otimes \text{pr}_Y^{-1}\mathcal{O}_Y} \text{pr}_X^{-1}\mathcal{O}_X \otimes \text{pr}_Y^{-1}\mathcal{O}_Y$.

2.2. **The odd path space.** For a manifold $X$ the mapping space $X^\sharp$ is represented by the DG manifold $X^\sharp$ of Example 1 ([Bressler and Rengifo, 2018], Theorem 2.1). The evaluation map $\text{ev} : X^\sharp \times \mathbf{i} \to X$ corresponds to the morphism of “pull-back of functions” $\text{ev}^* : \mathcal{O}_X \to \mathcal{O}_{X^\sharp}[\epsilon] := \mathcal{O}_{X^\sharp \times \mathbf{i}}$ given by $f \mapsto f + df \cdot \epsilon$.

There is a short exact sequence of graded $\mathcal{O}_{X^\sharp}$-modules

$$0 \to \mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} T_X[1] \to T_{X^\sharp} \to \mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} T_X \to 0,$$

where $\mathcal{O}_{X^\sharp} \otimes_{\mathcal{O}_X} T_X[1] \cong T_{X^\sharp/X}$.

2.3. **Immersions and submersions.**

**Proposition 3.** If a map of manifolds $f : Y \to X$ is an immersion (respectively, submersion), then the induced map $f^\sharp : Y^\sharp \to X^\sharp$ is an immersion (respectively, submersion).

**Proof.** The derivative $df^\sharp : T_{Y^\sharp} \to f^\sharp T_{X^\sharp}$ gives rise to the map of short exact sequences

$$
\begin{array}{cccccc}
0 & \longrightarrow & \mathcal{O}_{Y^\sharp} \otimes_{\mathcal{O}_Y} T_Y[1] & \longrightarrow & T_{Y^\sharp} & \longrightarrow & \mathcal{O}_{Y^\sharp} \otimes_{\mathcal{O}_Y} T_Y & \longrightarrow & 0 \\
\downarrow \text{id} \otimes df & & \downarrow df & & \downarrow \text{id} \otimes df & & & & \\
0 & \longrightarrow & \mathcal{O}_{Y^\sharp} \otimes_{f^{-1}\mathcal{O}_X} T_X[1] & \longrightarrow & T_{X^\sharp} & \longrightarrow & \mathcal{O}_{Y^\sharp} \otimes_{f^{-1}\mathcal{O}_X} T_X & \longrightarrow & 0
\end{array}
$$

and the claim follows from the five-lemma.

2.4. **Transgression for $\mathcal{O}$-modules.** We denote by $\text{pr} : X^\sharp \times \mathbf{i} \to X^\sharp$ the canonical projection. The diagram

$$
X^\sharp \times \mathbf{i} \xrightarrow{\text{ev}} X
$$

$$
\downarrow \text{pr}
$$

$$
X^\sharp
$$

gives rise to the functor

$$
(2.4.1) \text{pr}_* \text{ev}^* : \mathcal{O}_X \text{-Mod} \to \mathcal{O}_{X^\sharp} \text{-Mod}.
$$
Since the underlying space of both \(X^2\) and \(X^2 \times \mathfrak{T}\) is equal to \(X\), the functor \(\text{ev}^*: \mathcal{O}_X-\text{Mod} \to \mathcal{O}_{X^2 \times \mathfrak{T}}-\text{Mod}\) is given by \(\text{ev}^* \mathcal{E} = \mathcal{O}_{X^2}[\epsilon] \otimes_{\mathcal{O}_X} \mathcal{E}\) and the effect of the functor \(\text{pr}_*\) amounts to restriction of scalars along the unit map \(\mathcal{O}_{X^2} \to \mathcal{O}_{X^2}[\epsilon]\).

2.5. Lie algebroids. An \(\mathcal{O}_X\)-Lie algebroid structure on an \(\mathcal{O}_X\)-module \(\mathcal{A}\) consists of

1. A structure of a sheaf of \(\mathbb{C}\)-Lie algebras \([\cdot, \cdot]: \mathcal{A} \otimes_{\mathbb{C}} \mathcal{A} \to \mathcal{A}\);
2. An \(\mathcal{O}_X\)-linear map \(\sigma: \mathcal{A} \to T_X\) of Lie algebras called the anchor map.

These data are required to satisfy the compatibility condition (Leibniz rule)

\[ [a, f \cdot b] = \sigma(a)(f) \cdot b + f \cdot [a, b]\]

for \(a,b \in \mathcal{A}\) and \(f \in \mathcal{O}_X\).

A morphism of \(\mathcal{O}_X\)-Lie algebroids \(\phi: \mathcal{A}_1 \to \mathcal{A}_2\) is an \(\mathcal{O}_X\)-linear map of Lie algebras which commutes with respective anchor maps.

With the above definition of morphisms \(\mathcal{O}_X\)-Lie algebroids form a category denoted \(\mathcal{O}_X-\text{LieAlgd}\).

The notion of Lie algebroid generalizes readily to the DG context.

2.6. Transgression for Lie algebroids. Suppose that \(\mathcal{A}\) is an \(\mathcal{O}_X\)-Lie algebroid as in 2.5. It is shown in (Bressler and Rengifo, 2018), 3.5, that the \(\mathcal{O}_{X^2}\)-module \(\text{pr}_* \text{ev}^* \mathcal{A}\) admits a canonical structure of a \(\mathcal{O}_{X^2}\)-Lie algebroid, denoted henceforth by \(\mathcal{A}^\sharp\). Moreover, the assignment \(\mathcal{A} \mapsto \mathcal{A}^\sharp\) extends to a functor

\[ (\quad)^\sharp: \mathcal{O}_X-\text{LieAlgd} \to \mathcal{O}_{X^2}-\text{LieAlgd} \]

which preserves terminal objects, i.e. the canonical map \(T_X^\sharp \to T_{X^2}\) is an isomorphism, and products.

3. Transgression for Courant algebroids

3.1. Marked Lie algebroids. Suppose that \(\mathfrak{X} = (X, \mathcal{O}_X)\) is a DG-manifold.

A marked \(\mathcal{O}_X\)-Lie algebroid is a pair \((\mathcal{A}, \mathfrak{c})\), where \(\mathcal{A}\) is a \(\mathcal{O}_X\)-Lie algebroid and \(\mathfrak{c} \in \Gamma(X; \mathcal{A})\) is a homogeneous central section (i.e. \([\mathfrak{c}, \mathcal{A}] = 0\)). The section \(\mathfrak{c}\) is called a marking.

It is easy to see ((Bressler and Rengifo, 2018), Lemma 3.6) that any (homogeneous) central section belongs to the kernel of the anchor map.

A morphism \(\phi: (\mathcal{A}_1, \mathfrak{c}_1) \to (\mathcal{A}_2, \mathfrak{c}_2)\) is a morphism of Lie algebroids \(\phi: \mathcal{A}_1 \to \mathcal{A}_2\) such that \(\phi(\mathfrak{c}_1) = \mathfrak{c}_2\). In particular, \(\mathfrak{c}_1\) and \(\mathfrak{c}_2\) have the same degree.

With the above definitions marked \(\mathcal{O}_X\)-Lie algebroids and morphisms thereof form a category denoted \(\mathcal{O}_X-\text{LieAlgd}^\star\). The full subcategory of marked \(\mathcal{O}_X\)-Lie algebroids \((\mathcal{A}, \mathfrak{c})\) with \(\deg \mathfrak{c} = n\) is denoted \(\mathcal{O}_X-\text{LieAlgd}^\star_n\).

For a marked Lie algebroid \((\mathcal{A}, \mathfrak{c})\) with \(\deg \mathfrak{c} = n\) the Lie algebroid structure on \(\mathcal{A}\) descends to

\[ \overline{\mathcal{A}} := \text{coker}(\mathcal{O}_X[n] \to \mathcal{A})\]

((Bressler and Rengifo, 2018), Lemma 3.7). The assignment \((\mathcal{A}, \mathfrak{c}) \mapsto \overline{\mathcal{A}}\) extends to a functor

\[ (\quad)^\star: \mathcal{O}_X-\text{LieAlgd}^\star \to \mathcal{O}_X-\text{LieAlgd}\]
Example 4. The structure sheaf $\mathcal{O}_X[n]$ has a canonical structure of a marked $\mathcal{O}_X$-Lie algebroid, whose non trivial part is the marking $id: \mathcal{O}_X[n] \rightarrow \mathcal{O}_X[n]$. In particular, the marking is a morphism of marked Lie algebroids and the $\mathcal{O}_X$-Lie algebroid $\mathcal{O}_X[n]$ is zero.

3.2. $\mathcal{O}_X[n]$-extensions. Suppose that $B$ is a $\mathcal{O}_X$-Lie algebroid.

An $\mathcal{O}_X[n]$-extension of $B$ is a marked $\mathcal{O}_X$-Lie algebroid $(\mathcal{A}, c)$ with deg $c = n$ together with a morphism $\mathcal{A} \rightarrow B$ such that the sequence

$$0 \rightarrow \mathcal{O}_X[n] \overset{c}{\rightarrow} \mathcal{A} \rightarrow B \rightarrow 0$$

is exact. A morphism of $\mathcal{O}_X[n]$-extensions of $B$ is a morphism of marked Lie algebroids which induces the identity map on $B$. Such a map is necessarily an isomorphism. We denote the category (groupoid) of $\mathcal{O}_X[n]$-extensions of $B$ by $[\mathcal{O}_X[n]] \text{Ext}(B)$.

The category $[\mathcal{O}_X[n]] \text{Ext}(B)$ has a canonical structure of a ‘$\mathbb{C}$-vector space in categories’ (hence, in particular, that of a Picard groupoid). Namely, given extensions $\mathcal{A}_1, \ldots, \mathcal{A}_m$ and complex numbers $\lambda_1, \ldots, \lambda_m$ the ‘linear combination’ $\lambda_1 \mathcal{A}_1 + \cdots + \lambda_m \mathcal{A}_m$ is defined by the push-out diagram

$$\begin{array}{ccc}
\mathcal{O}_X[n] \times \cdots \times \mathcal{O}_X[n] & \longrightarrow & \mathcal{A}_1 \times_B \cdots \times_B \mathcal{A}_m \\
\downarrow & & \downarrow \\
\mathcal{O}_X[n] & \longrightarrow & \lambda_1 \mathcal{A}_1 + \cdots + \lambda_m \mathcal{A}_m
\end{array}$$

where the left vertical arrow is given by $(\alpha_1, \ldots, \alpha_m) \mapsto \sum_i \lambda_i \alpha_i$. The bracket on $\lambda_1 \mathcal{A}_1 + \cdots + \lambda_m \mathcal{A}_m$ is characterized by the fact that the right vertical map is a morphism of Lie algebras.

3.3. Courant algebroids. Courant algebroids were introduced in (Liu et al., 1997), (Roytenberg, 2002) and (Bressler and Chervov, 2005). For comparison of the following definition with the one encountered in the literature see Remark 5.

A Courant algebroid is an $\mathcal{O}_X$-module $\mathcal{Q}$ equipped with

(1) a structure of a Leibniz $\mathbb{C}$-algebra

$$\{ , \} : \mathcal{Q} \otimes_{\mathbb{C}} \mathcal{Q} \rightarrow \mathcal{Q};$$

(2) an $\mathcal{O}_X$-linear map of Liebniz algebras (the anchor map)

$$\pi : \mathcal{Q} \rightarrow \mathcal{T}_X;$$

(3) a symmetric $\mathcal{O}_X$-bilinear pairing

$$\langle , \rangle : \mathcal{Q} \otimes_{\mathcal{O}_X} \mathcal{Q} \rightarrow \mathcal{O}_X;$$

(4) an $\mathcal{O}_X$-linear map (the co-anchor map)

$$\pi^\dagger : \Omega^1_X \rightarrow \mathcal{Q}.$$
These data are required to satisfy

\begin{align}
\pi \circ \pi^\dagger &= 0 \\
\{q_1, f q_2\} &= f \{q_1, q_2\} + \pi(q_1)(f)q_2 \\
\langle \{q, q_1\} + \{q_1, q\}, q_2 \rangle &= L_{\pi(q)}\langle q_1, q_2 \rangle \\
\{q, \pi^\dagger(\alpha)\} &= \pi^\dagger(\{L_{\pi(q)}(\alpha)\}) \\
\langle q, \pi^\dagger(\alpha) \rangle &= \iota_{\pi(q)} \alpha \\
\{q_1, q_2\} + \{q_2, q_1\} &= \pi^\dagger(d\langle q_1, q_2 \rangle)
\end{align}

for \( f \in \mathcal{O}_X \) and \( q, q_1, q_2 \in \mathcal{Q} \).

A morphism \( \phi: \mathcal{Q}_1 \to \mathcal{Q}_2 \) of Courant algebroids on \( X \) is an \( \mathcal{O}_X \)-linear map of Leibniz \( \mathbb{C} \)-algebras such that the diagram

\[
\begin{array}{ccc}
\Omega^1_X & \xrightarrow{\pi^\dagger_1} & \mathcal{Q}_1 \\
\downarrow & & \downarrow \pi_1 \\
\Omega^1_X & \xrightarrow{\pi^\dagger_2} & \mathcal{Q}_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \xrightarrow{\phi} & & \\
\downarrow & & \downarrow & & \\
& & \pi_2 & & \\
\end{array}
\]

\[
\begin{array}{ccc}
\Omega^1_X & \xrightarrow{\pi^\dagger_1} & \mathcal{Q}_1 \\
& & \downarrow \phi \\
\Omega^1_X & \xrightarrow{\pi^\dagger_2} & \mathcal{Q}_2 \\
\end{array}
\]

\[
\begin{array}{ccc}
& & \xrightarrow{\phi} & & \\
\downarrow & & \downarrow & & \\
& & \pi_2 & & \\
\end{array}
\]

is commutative.

With the above definitions Courant algebroids on \( X \) and morphisms thereof form a category henceforth denoted \( \mathcal{CA}(X) \).

**Remark 5.** The original definition of Courant algebroid is due to \( \text{(Liu et al., 1997)} \), based on \( \text{(Courant, 1990)} \) and \( \text{(Dorfman, 1993)} \), where the additional assumption of non-degeneracy of the pairing is made. Under the latter assumption the co-anchor map is uniquely determined by the anchor map and, hence, does not appear as a separate item. The broader concept of Courant algebroid as described by the definition was advanced in \( \text{(Bressler, 2007)} \) and permits, for example, regarding the kernel of the anchor map as a Courant algebroid with trivial anchor.

Suppose that \( \mathcal{Q} \) is a Courant algebroid on \( X \). Let

\[
\overline{\mathcal{Q}} = \text{coker}(\pi^\dagger).
\]

The Courant algebroid structure on \( \mathcal{Q} \) descends to a structure on a Lie algebroid on \( \overline{\mathcal{Q}} \). In what follows we refer to the Lie algebroid \( \overline{\mathcal{Q}} \) as the *Lie algebroid associated to the Courant algebroid* \( \mathcal{Q} \).

The assignment \( \mathcal{Q} \mapsto \overline{\mathcal{Q}} \) extends to a functor

\[
(\overline{\cdot}): \mathcal{CA}(X) \longrightarrow \mathcal{O}_X-\text{LieAlgd}.
\]

For a Courant algebroid \( \mathcal{Q} \) the *opposite* Courant algebroid, denoted \( \mathcal{Q}^{\text{op}} \), has the same underlying \( \mathcal{O}_X \)-module as \( \mathcal{Q} \), the same Leibniz bracket (i.e. \( \{\ , \ \}^{\text{op}} = \{\ , \ \} \)), same anchor map \( \pi^{\text{op}} = \pi \), the symmetric pairing \( \langle\ , \ \rangle^{\text{op}} = -\langle\ , \ \rangle \) and the co-anchor map \( \pi^{\text{op}\dagger} = -\pi^\dagger \).
Suppose that $\mathcal{Q}_1$ and $\mathcal{Q}_2$ are Courant algebroids. The Courant algebroid $\mathcal{Q}_1 \mathbin{\text{‡}} \mathcal{Q}_2$ is defined by the push-out square
\[
\begin{array}{c}
\Omega^1_X \times \Omega^1_X \quad \longrightarrow \\
\phantom{\Omega^1_X \times} \downarrow \phantom{\Omega^1_X} \\
\Omega^1_X \quad \longrightarrow
\end{array}
\quad \begin{array}{c}
\mathcal{Q}_1 \times_{\mathcal{T}_X} \mathcal{Q}_2 \\
\phantom{\mathcal{Q}_1 \times_{\mathcal{T}_X}} \downarrow \\
\phantom{\mathcal{Q}_2} \\
\end{array}
\]
and equipped with the component-wise operations.

3.4. **Courant extensions.** Suppose that $\mathcal{A}$ is a $\mathcal{O}_X$-Lie algebroid. A Courant extension of $\mathcal{A}$ is a Courant algebroid $\mathcal{Q}$ together with the identification $\mathcal{Q} \cong \mathcal{A}$ such that the sequence
\[
(3.4.1) \quad 0 \to \Omega^1_X \xrightarrow{\pi^*} \mathcal{Q} \to \mathcal{A} \to 0
\]
is exact.

A morphism $\phi: \mathcal{Q}_1 \to \mathcal{Q}_2$ of Courant extensions of $\mathcal{A}$ is a morphism of Courant algebroids which is compatible with the identifications $\mathcal{Q}_i \cong \mathcal{A}$. We denote the category of Courant extension of $\mathcal{A}$ by $\mathcal{C}\text{Ext}(\mathcal{A})$. A morphism in $\mathcal{C}\text{Ext}(\mathcal{A})$ induces a morphism of associated short exact sequences (3.4.1), hence is an isomorphism of underlying $\mathcal{O}_X$-modules. It is easy to see that the inverse map is, in fact, a morphism of Courant algebroids. Consequently, $\mathcal{C}\text{Ext}(\mathcal{A})$ is a groupoid.

3.5. **Linear algebra.** The category $\mathcal{C}\text{Ext}(\mathcal{A})$ has a canonical structure of a ‘$\mathbb{C}$-vector space in categories’ (hence, in particular, that of a Picard groupoid). Namely, given extensions $\mathcal{Q}_1, \ldots, \mathcal{Q}_n$ and complex numbers $\lambda_1, \ldots, \lambda_n$ the ‘linear combination’ $\lambda_1 \mathcal{Q} \mathbin{\text{‡}} \cdots \mathbin{\text{‡}} \lambda_n \mathcal{Q}_n$ is defined by the push-out diagram
\[
\begin{array}{c}
\Omega^1_X \times \cdots \times \Omega^1_X \quad \longrightarrow \\
\phantom{\Omega^1_X \times} \downarrow \\
\Omega^1_X \quad \longrightarrow
\end{array}
\quad \begin{array}{c}
\mathcal{Q}_1 \times_{\mathcal{A}} \cdots \times_{\mathcal{A}} \mathcal{Q}_n \\
\phantom{\mathcal{Q}_1 \times_{\mathcal{A}}} \downarrow \\
\phantom{\mathcal{Q}_n} \\
\end{array}
\]
where the left vertical arrow is given by $(\alpha_1, \ldots, \alpha_n) \mapsto \sum_i \lambda_i \alpha_i$. The Leibniz bracket on $\lambda_1 \mathcal{Q} \mathbin{\text{‡}} \cdots \mathbin{\text{‡}} \lambda_n \mathcal{Q}_n$ is characterized by the fact that the right vertical map is a morphism of Leibniz algebras.

3.6. **Exact Courant algebroids.** A Courant algebroid $\mathcal{Q}$ is called *exact* if the map $\mathcal{Q} \to \mathcal{T}_X$ is an isomorphism. Thus, an exact Courant algebroid is a Courant extension of $\mathcal{T}_X$. We denote the category of exact Courant algebroids by $\mathcal{E}\mathcal{C}\mathcal{A}(X) := \mathcal{C}\text{Ext}(\mathcal{T}_X)$.

3.7. **Transitive Courant algebroids.** A Courant algebroid is called *transitive* if the associated Lie algebroid is, which is to say, the anchor map is an epimorphism. If $\mathcal{Q}$ is a transitive Courant algebroid, the sequence
\[
0 \to \Omega^1_X \to \mathcal{Q} \to \mathcal{Q} \to 0
\]
is exact.
3.8. Transgression for Courant algebroids. We denote by

\[(3.8.1) \int : \text{pr}_* \text{ev}^* \Omega_X^1 = \mathcal{O}_{X^1}[\epsilon] \otimes_{\mathcal{O}_X} \Omega_X^1 \to \mathcal{O}_{X^1}[2].\]

the map of \(\mathcal{O}_{X^1}\)-modules whose component of degree \(-1\) is the identity map.

For a Courant algebroid \(Q\) the marked Lie algebroid \((\tau Q, c) \in \mathcal{O}_X^{-\text{LieAlgd}_2}\) is given by the \(\mathcal{O}_X^\#\)-module

\[
\tau Q := \text{coker}(\text{pr}_* \text{ev}^* \Omega_X^1) \to \mathcal{O}_{X^1}[2] \oplus \text{pr}_* \text{ev}^* Q,
\]

where \(\int\) is the map \((3.8.1)\). In other words, the square

\[
\begin{array}{ccc}
\text{pr}_* \text{ev}^* \Omega_X^1 & \xrightarrow{\int} & \mathcal{O}_{X^1}[2] \\
\downarrow \text{pr}_* \text{ev}^* (\pi) & & \downarrow \\
\mathcal{O}_{X^1}[2] & \to & \tau Q
\end{array}
\]

is cocartesian.

The anchor map is induced by \(\text{pr}_* \text{ev}^* (\pi)\) and the marking \(c \in \Gamma(X; \tau Q^{-2})\) is the image of \(1 \in \Gamma(X; (\mathcal{O}_{X^1}[2])^{-2})\) under the bottom horizontal map in \((3.8.2)\).

The bracket is the extension by Leibniz rule of

\[
\begin{align*}
(1) \ [\epsilon, \tau Q] &= 0 \\
(2) \ [q \cdot \epsilon, \beta] &= (-1)^{|\beta|} [\beta, q \cdot \epsilon] = t_{\pi(q)} \beta \\
(3) \ [q, \beta] &= -[\beta, q] = L_{\pi(q)} \beta \\
(4) \ [q_1 \cdot \epsilon, q_2 \cdot \epsilon]^{-1, -1} &= \langle q_1, q_2 \rangle \in (\mathcal{O}_{X^1}[2])^{-2} = \mathcal{O}_X \\
(5) \ [q_1, q_2]^{-0, -1} &= \{q_1, q_2\} \cdot \epsilon \\
(6) \ [q_1 \cdot \epsilon, q_2]^{-1, 0} &= -d \langle q_1, q_2 \rangle + \{q_1, q_2\} \cdot \epsilon
\end{align*}
\]

The marked \(\mathcal{O}_{X^1}\)-Lie algebroid \(\tau Q\) enjoys the following properties:

(1) The natural map \(\overline{Q} \to \tau Q\) is an isomorphism.
(2) The canonical map \(Q \to \tau Q^{-1}\) is an isomorphism.
(3) \(Q\) is a Courant extension of \(A\) if and only if \(\tau Q\) is a \(\mathcal{O}_{X^1}[2]\)-extension of \(A^\#\).

The assignment \(Q \mapsto \tau Q\) extends to a functor

\[
\tau : \mathcal{C}A(X) \to \mathcal{O}_{X^1}^{-\text{LieAlgd}_2}.
\]

Suppose that \((\mathcal{B}, c) \in \mathcal{O}_{X^1}^{-\text{LieAlgd}_2}\) satisfies \(\mathcal{B}^i = 0\) for \(i \leq -2\). Then, the derived bracket and the \((-1, -1)\)-component of the Lie bracket together with the components of degree \(-1\) of the anchor map and of \(\mathcal{O}_{X^1}[2] \to \mathcal{B}^{-1}\) with a structure of a Courant algebroid (see [Bressler and Rengifo, 2018], Lemma 6.1). We denote this Courant algebroid structure on \(\mathcal{B}^{-1}\) by \(\mathcal{Q}(\mathcal{B}, c)\).

For any Courant algebroid \(Q\) the marked Lie algebroid \(\tau Q\) satisfies the above requirements. Moreover, \(Q(\tau Q) = Q\).
Proposition 6. Suppose that $A$ is a $\mathcal{O}_X$-Lie algebroid $\mathcal{A}$. The functor $\tau$ restricts to an equivalence of categories

$$\tau : \text{CExt}(\mathcal{A}) \longrightarrow \mathcal{O}[2]\text{Ext}(\mathcal{A}^\sharp)$$

whose quasi-inverse is given by the functor

$$Q : \mathcal{O}[2]\text{Ext}(\mathcal{A}^\sharp) \rightarrow \text{CExt}(\mathcal{A}).$$

3.9. Transgression and linear algebra. Suppose that $A$ is a $\mathcal{O}_X$-Lie algebroid locally free of finite rank over $\mathcal{O}_X$. Then, so is any Courant extension of $\mathcal{A}$. Suppose that $Q_1, \ldots, Q_n \in \text{CExt}(\mathcal{A})$, $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. For $\alpha_1, \ldots, \alpha_n \in \Omega^1_X$ let $\ell(\alpha_1, \ldots, \alpha_n) = \sum \lambda_i \alpha_i$.

The diagram

$$\begin{array}{ccc}
\text{pr}_*\text{ev}^*(\Omega^1_X \times \cdots \times \Omega^1_X) & \longrightarrow & \text{pr}_*\text{ev}^*\Omega^1_X \times \cdots \times \text{pr}_*\text{ev}^*\Omega^1_X \\
\downarrow & & \downarrow \\
\text{pr}_*\text{ev}^*(Q_1 \times_A \cdots \times_A Q_n) & \longrightarrow & \text{pr}_*\text{ev}^*Q_1 \times_A \cdots \times_A \text{pr}_*\text{ev}^*Q_n
\end{array}$$

is commutative with horizontal maps isomorphisms. Both squares in the diagram

$$\begin{array}{ccc}
\text{pr}_*\text{ev}^*(\Omega^1_X \times \cdots \times \Omega^1_X) & \longrightarrow & \text{pr}_*\text{ev}^*Q_1 \times_A \cdots \times_A Q_n \\
\downarrow & & \downarrow \\
\mathcal{O}_X[2] & \longrightarrow & \tau Q_1 \times_A \cdots \times_A \tau Q_n
\end{array}$$

are co-Cartesian, hence so is the outer one and the same holds for the diagram

$$\begin{array}{ccc}
\text{pr}_*\text{ev}^*\Omega^1_X \times \cdots \times \text{pr}_*\text{ev}^*\Omega^1_X & \longrightarrow & \text{pr}_*\text{ev}^*Q_1 \times_A \cdots \times_A \text{pr}_*\text{ev}^*Q_n \\
\downarrow & & \downarrow \\
\mathcal{O}_X[2] \times \cdots \times \mathcal{O}_X[2] & \longrightarrow & \tau Q_1 \times_A \cdots \times_A \tau Q_n \\
\downarrow & & \downarrow \\
\mathcal{O}_X[2] & \longrightarrow & \sum_{i=1}^n \lambda_i \tau Q_i
\end{array}$$

Since the diagram

$$\begin{array}{ccc}
\text{pr}_*\text{ev}^*(\Omega^1_X \times \cdots \times \Omega^1_X) & \longrightarrow & \text{pr}_*\text{ev}^*\Omega^1_X \\
\downarrow & & \downarrow \\
\text{pr}_*\text{ev}^*\Omega^1_X \times \cdots \times \text{pr}_*\text{ev}^*\Omega^1_X & \longrightarrow & \mathcal{O}_X[2] \times \cdots \times \mathcal{O}_X[2] \longrightarrow \mathcal{O}_X[2]
\end{array}$$

is commutative with vertical maps isomorphisms it follows that there is a canonical isomorphism

$$\tau (\sum_{i=1}^n \lambda_i Q_i) \cong \sum_{i=1}^n \lambda_i \tau Q_i.$$
We leave the details of the proof of the following lemma to the reader.

**Proposition 7.** The functors
\[ \tau : C\text{Ext}(\mathcal{A}) \rightarrow \mathcal{O}[2]\text{Ext}(\mathcal{A}) : \mathbb{Q} \]
are morphisms of \( \mathbb{C} \)-vector spaces in categories (and, in particular, of Picard groupoids).

### 4. The inverse image functor

#### 4.1. Inverse image and fiber product for \( \mathcal{O} \)-modules.

Suppose that \( \phi : \mathcal{Y} \rightarrow \mathcal{X} \) is a morphism of DG manifolds and \( A \xrightarrow{s} C \xleftarrow{t} B \) are morphisms in \( \mathcal{O}_\mathcal{X}-\text{Mod} \). The sequence
\[ 0 \rightarrow A \times_C B \rightarrow A \times B \xrightarrow{s-t} C \]
is exact. Assume furthermore that \( A, B \) and \( C \) are locally free of finite rank. In what follows we will consider the following condition on morphisms \( A \xrightarrow{s} C \xleftarrow{t} B \):

\[ (C) \quad A \times_C B = \ker(s-t) \text{ and } \coker(s-t) \text{ are locally free} \]
Condition \( (C) \) is trivially fulfilled if at least one of the two maps \( s \) and \( t \) is an epimorphism.

**Lemma 8.** Suppose that the morphisms \( A \xrightarrow{s} C \xleftarrow{t} B \) satisfy the condition \( (C) \). Then, the canonical morphism \( \phi^*(A \times_C B) \rightarrow \phi^*A \times_{\phi^*C} \phi^*B \) is an isomorphism and the maps \( \phi^*A \xrightarrow{\phi^*(s)} \phi^*C \xleftarrow{\phi^*(t)} \phi^*B \) satisfy the condition \( (C) \).

**Proof.** Applying the functor \( \phi^* \) to the exact sequence
\[ 0 \rightarrow A \times_C B \rightarrow A \times B \xrightarrow{s-t} C \rightarrow \coker(s-t) \rightarrow 0 \]
or locally free \( \mathcal{O}_\mathcal{X} \)-modules we obtain the commutative diagram with exact rows
\[
\begin{array}{ccccccccc}
0 & \longrightarrow & \phi^*(A \times_C B) & \longrightarrow & \phi^*(A \times B) & \longrightarrow & \phi^*C & \longrightarrow & \phi^* \coker(s-t) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \phi^*(s-t) & & \downarrow \phi^* \coker(s-t) & & \downarrow \id & & \\
0 & \longrightarrow & \phi^*A \times_{\phi^*C} \phi^*B & \longrightarrow & \phi^*A \times \phi^*B & \longrightarrow & \phi^*C & \longrightarrow & \coker(\phi^*s - \phi^*t) & \longrightarrow & 0
\end{array}
\]
Since the map \( \phi^*(A \times B) \rightarrow \phi^*A \times \phi^*B \) is an isomorphism it follows that so is the map \( \phi^*(A \times_C B) \rightarrow \phi^*A \times_{\phi^*C} \phi^*B \). \( \square \)

#### 4.2. \( \mathcal{O} \)-modules over \( \mathcal{T} \).

Suppose that \( \mathcal{X} = (X, \mathcal{O}_\mathcal{X}) \) is a DG manifold.

We denote by \( \mathcal{O}_\mathcal{X}-\text{Mod}/\mathcal{T}_\mathcal{X} \) the category of \( \mathcal{O}_\mathcal{X} \)-modules over \( \mathcal{T}_\mathcal{X} \), i.e., the category of pairs \((\mathcal{E}, \pi)\), where \( \mathcal{E} \in \mathcal{O}_\mathcal{X}-\text{Mod} \) and \( \pi : \mathcal{E} \rightarrow \mathcal{T}_\mathcal{X} \) is a morphism \( \mathcal{O}_\mathcal{X}-\text{Mod} \) in referred to as the anchor.

A morphism \( t : (\mathcal{E}_1, \pi_1) \rightarrow (\mathcal{E}_2, \pi_2) \) in \( \mathcal{O}_\mathcal{X}-\text{Mod}/\mathcal{T}_\mathcal{X} \) is a morphism of \( \mathcal{O}_\mathcal{X} \)-modules \( t : \mathcal{E}_1 \rightarrow \mathcal{E}_2 \) such that \( \pi_2 \circ t = \pi_1 \).

The category \( \mathcal{O}_\mathcal{X}-\text{Mod}/\mathcal{T}_\mathcal{X} \) has a terminal object, namely \( \mathcal{T}_\mathcal{X} := (\mathcal{T}_\mathcal{X}, \id) \). The product \( (\mathcal{E}_1, \pi_1) \times (\mathcal{E}_2, \pi_2) \) is represented by \( \mathcal{E}_1 \times_{\mathcal{T}_\mathcal{X}} \mathcal{E}_2 \rightarrow \mathcal{T}_\mathcal{X} \).

We shall call \((\mathcal{E}, \pi)\) transitive if the anchor map is surjective.
4.3. **Inverse image.** Suppose that $\phi: \mathcal{Y} \to \mathcal{X}$ is a morphism of DG manifolds.

For $\mathcal{E} := (\mathcal{E}, \pi) \in \mathcal{O}_X \rightarrow \text{Mod}/T_X$ the object $\phi^+ \mathcal{E} \in \mathcal{O}_Y \rightarrow \text{Mod}/T_Y$ is defined as the left vertical arrow in the pull-back square

$$
\begin{array}{ccc}
\phi^+ \mathcal{E} & \xrightarrow{d\phi} & \phi^* \mathcal{E} \\
\phi^+(\pi) \downarrow & & \downarrow \phi^*(\pi) \\
T_Y & \xrightarrow{d\phi} & \phi^* T_X
\end{array}
$$

(4.3.1)

which is to say $\phi^+ \mathcal{E} = T_Y \times_{\phi^* T_X} \phi^* \mathcal{E}$.

A morphism $t: (\mathcal{E}_1, \pi_1) \to (\mathcal{E}_2, \pi_2)$ in $\mathcal{O}_X \rightarrow \text{Mod}/T_X$ induces the morphism $\phi^+(t): \phi^+ \mathcal{E}_1 \to \phi^+ \mathcal{E}_2$ in $\mathcal{O}_Y \rightarrow \text{Mod}/T_Y$ and the above construction extends to a functor, called pull-back or inverse image under $\phi$

$$
\phi^+: \mathcal{O}_X \rightarrow \text{Mod}/T_X \to \mathcal{O}_Y \rightarrow \text{Mod}/T_Y.
$$

The functor of inverse image preserves terminal objects, that is to say $\phi^+ T_X = T_Y$.

The functor of inverse image preserves transitivity: if $\mathcal{E} := (\mathcal{E}, \pi)$ is transitive then so is $\phi^+ \mathcal{E}$.

By the universal property of $\phi^*$ there is a unique map $\phi^*(\mathcal{E}_1 \times_{\phi^* T_X} \mathcal{E}_2) \to \phi^* \mathcal{E}_1 \times T_Y \phi^* \mathcal{E}_2$. Since fiber products commute with products there is a unique isomorphism $T_Y \times \phi^* T_X (\phi^* \mathcal{E}_1 \times \phi^* \mathcal{E}_2) \to (T_Y \times \phi^* T_X \phi^* \mathcal{E}_1 \times (T_Y \times \phi^* T_X \phi^* \mathcal{E}_2)$.

The composition

$$
T_Y \times \phi^* T_X \phi^*(\mathcal{E}_1 \times T_X \mathcal{E}_2) \to T_Y \times \phi^* T_X (\phi^* \mathcal{E}_1 \times \phi^* T_X \phi^* \mathcal{E}_2) \to (T_Y \times \phi^* T_X \phi^* \mathcal{E}_1 \times (T_Y \times \phi^* T_X \phi^* \mathcal{E}_2)
$$

gives rise to the map

$$
\phi^+(\mathcal{E}_1 \times T_X \mathcal{E}_2) \to \phi^+ \mathcal{E}_1 \times T_Y \phi^+ \mathcal{E}_2
$$

(4.3.2)

**Lemma 9.** Suppose that $(\mathcal{E}_1, \pi_i) \in \mathcal{O}_X \rightarrow \text{Mod}/T_X$, $i = 1, 2$, are locally free. Assume that the maps $\mathcal{E}_1 \xrightarrow{\pi_1} T_X \xleftarrow{\pi_2} \mathcal{E}_2$ satisfy condition $[\mathcal{C}]$. Then, the canonical map $(4.3.2)$ is an isomorphism.

**Proof.** Lemma $\mathcal{S}$ implies that the canonical morphism $\phi^*(\mathcal{E}_1 \times T_X \mathcal{E}_2) \to \phi^* \mathcal{E}_1 \times T_Y \phi^* \mathcal{E}_2$ is an isomorphism. Then the map $(4.3.2)$ is an isomorphism. Moreover, the diagrams

1. $\phi^+(\mathcal{E}_1 \times T_X \mathcal{E}_2) \xrightarrow{\phi^*} \phi^*(\mathcal{E}_1 \times T_X \mathcal{E}_2)$.
2. $\phi^+ \mathcal{E}_1 \times T_Y \phi^+ \mathcal{E}_2 \xrightarrow{\phi^* \mathcal{E}_1 \times \phi^* T_X \phi^* \mathcal{E}_2}$.
3. $\phi^+(\mathcal{E}_1 \times T_X \mathcal{E}_2) \xrightarrow{\phi^* \mathcal{E}_1 \times \phi^* T_X \phi^* \mathcal{E}_2}$.
4. $T_Y$.
are commutative and the result follows. □

4.4. **Inverse image and composition.** Suppose given a map \( \psi : Z \to Y \). Applying \( \psi^* \) to (4.3.1) we obtain the commutative diagram

\[
\begin{array}{ccc}
\psi^* \phi^+ \mathcal{E} & \xrightarrow{\psi^*(d\phi)} & (\phi \circ \psi)^* \mathcal{E} \\
\psi^* \psi^+(\pi) & \downarrow & (\phi \circ \psi)^*(\pi) \\
\mathcal{T}_\psi & \xrightarrow{\psi^*(d\phi)} & (\phi \circ \psi)^* \mathcal{T}_X
\end{array}
\]

which is not Cartesian in general. Combining the definition of \( \psi^+ \) with (4.4.1) we obtain the commutative diagram

\[
\begin{array}{ccc}
\psi^+ \phi^+ \mathcal{E} & \xrightarrow{\tilde{d}\psi} & \psi^+ \phi^+ \mathcal{E} & \xrightarrow{\psi^*(\tilde{d}\phi)} & (\phi \circ \psi)^* \mathcal{E} \\
\psi^+ \phi^+(\pi) & \downarrow & \psi^+ \phi^+(\pi) & \downarrow & (\phi \circ \psi)^* \phi^+(\pi) \\
\mathcal{T}_3 & \xrightarrow{d\psi} & \psi^* \mathcal{T}_\psi & \xrightarrow{\psi^*(\tilde{d}\phi)} & (\phi \circ \psi)^* \mathcal{T}_X
\end{array}
\]

On the other hand, the inverse image functor under the map \( \phi \circ \psi \) is defined by the pullback diagram

\[
\begin{array}{ccc}
(\phi \circ \psi)^+ \mathcal{E} & \xrightarrow{d(\phi \circ \psi)} & (\phi \circ \psi)^* \mathcal{E} \\
(\phi \circ \psi)^+(\pi) & \downarrow & (\phi \circ \psi)^*(\pi) \\
\mathcal{T}_3 & \xrightarrow{d(\phi \circ \psi)} & (\phi \circ \psi)^* \mathcal{T}_X
\end{array}
\]

Since \( \psi^*(d\phi) \circ d\psi = d(\phi \circ \psi) \) the universal property of pull-back provides the canonical map

\[
(\phi \circ \psi)^+ \mathcal{E} \xrightarrow{\psi^+(\tilde{d}\phi)} (\phi \circ \psi)^* \mathcal{E}
\]

which satisfies

\[
d(\phi \circ \psi) \circ c^+_{\phi,\psi} = \psi^*(\tilde{d}\phi) \circ \tilde{d}\psi
\]

\[
(\phi \circ \psi)^*(\pi) \circ c^+_{\phi,\psi} = \psi^+ \phi^+(\pi).
\]

In general the canonical map \( c^+_{\phi,\psi} \) is not an isomorphism.

**Lemma 10.** Suppose that \( (\mathcal{E}, \pi) \in \mathcal{O}_X \text{-Mod}/\mathcal{T}_X \) is locally free and the maps \( \mathcal{T}_\psi \xrightarrow{d\psi} \phi^* \mathcal{T}_X \xleftarrow{\phi^*(\pi)} \phi^* \mathcal{E} \) satisfy condition \( (\star) \). Then, the map (4.4.3) is an isomorphism.

**Proof.** The assumptions and Lemma 8 imply that the diagram (4.4.1) is Cartesian. Thus, both small squares in (4.4.2) are Cartesian hence so is their composition. □
Suppose given yet another map \( \xi : \mathcal{M} \to \mathcal{J} \). In this case both compositions

\[
\mathcal{M} \xrightarrow{\psi \circ \xi} \mathcal{Y} \xrightarrow{\phi} \mathcal{X}, \quad \mathcal{M} \xrightarrow{\xi} \mathcal{J} \xrightarrow{\phi \circ \psi} \mathcal{X}
\]

provide respectively the canonical maps

\[
(4.4.4) \quad c^+_{\phi, \psi, \xi} : (\psi \circ \phi)^+ E \to (\phi \circ \psi \circ \xi)^+ E
\]

\[
(4.4.5) \quad c^+_{\phi \circ \psi, \xi} : \xi^+ (\phi \circ \psi)^+ E \to (\phi \circ \psi \circ \xi)^+ E
\]

These two maps satisfy similar compatibility conditions as the map \( (4.4.3) \), namely

\[
d(\bar{\phi} \circ \bar{\psi} \circ \bar{\xi}) \circ c^+_{\phi, \psi, \xi} = (\psi \circ \xi)^* (\bar{d} \phi) \circ d(\bar{\psi} \circ \bar{\xi})
\]

\[
(\phi \circ \psi \circ \xi)^+(\pi) \circ c^+_{\phi, \psi, \xi} = (\psi \circ \xi)^{\phi^+}(\pi)
\]

and

\[
d(\bar{\phi} \circ \bar{\psi} \circ \bar{\xi}) \circ c^+_{\phi \circ \psi, \xi} = \xi^*(d(\bar{\phi} \circ \bar{\psi})) \circ \bar{d}(\bar{\xi})
\]

\[
(\phi \circ \psi \circ \xi)^+(\pi) \circ c^+_{\phi \circ \psi, \xi} = \xi^+(\phi \circ \psi)^+(\pi)
\]

Applying the functor \( \xi^+ \) to the map \( (4.4.3) \) we obtain the map

\[
(4.4.6) \quad \xi^+(c^+_{\phi, \psi}) : \xi^+ \psi^+ \phi^+ E \to \xi^+(\phi \circ \psi)^+ E
\]

which satisfies

\[
\xi^+ \psi^+ \phi^+(\pi) = \xi^+(\phi \circ \psi)^+(\pi) \circ \xi^+(c^+_{\phi, \psi}).
\]

The inverse image functor \( \xi^+ \) on the object \((\psi^+ \phi^+ E, \psi^+ \phi^+(\pi)) \in \mathcal{O}_3 - \text{Mod}/\mathcal{T}_3\) is defined by the pullback square

\[
\begin{array}{ccc}
\xi^+ \psi^+ \phi^+ E & \xrightarrow{\xi^* \psi^* \phi^*} & \xi^* \psi^+ \phi^+ E \\
\downarrow & & \downarrow \\
\mathcal{T}_{\mathcal{M}} & \xrightarrow{\xi^* \psi^* \phi^+ (\pi)} & \xi^* \mathcal{T}_3
\end{array}
\]

The commutative diagram,

\[
\begin{array}{ccc}
\xi^+ \psi^+ \phi^+ E & \xrightarrow{\xi^* (\psi \circ \phi)} & \xi^* \psi^+ \phi^+ E \\
\downarrow & & \downarrow \\
\mathcal{T}_{\mathcal{M}} & \xrightarrow{\xi^* \psi^* \phi^+ (\pi)} & \xi^* \mathcal{T}_3
\end{array}
\]

and the universal property of pullback provide the canonical map

\[
\xi^+ \psi^+ \phi^+ E \to (\phi \circ \psi \circ \xi)^+ E.
\]
Lemma 11. The identity

\[ c^+_{\phi \circ \psi, \xi} \circ \xi^+(c^+_{\phi, \psi}) = c^+_{\phi \circ \psi, \xi} \circ c^+_{\psi, \xi} : \xi^+ \psi^+ \phi^+ \mathcal{E} \to (\phi \circ \psi \circ \xi)^+ \mathcal{E} \]

holds.

Proof. It is enough to check that both compositions make the following diagram commutative. The computations

\[
d(\phi \circ \psi \circ \xi) \circ c^+_{\phi \circ \psi, \xi} \circ c^+_{\psi, \xi} = \xi^*(d\phi) \circ \xi^*(d\psi) \circ \xi^+ \mathcal{E}
\]

and

\[
d(\phi \circ \psi \circ \xi) \circ c^+_{\phi \circ \psi, \xi} \circ c^+_{\psi, \xi} = \xi^*(d\phi) \circ \xi^*(d\psi) \circ \xi^+ \mathcal{E}
\]

show that \(d(\phi \circ \psi \circ \xi) \circ c^+_{\phi \circ \psi, \xi} \circ c^+_{\psi, \xi} = d(\phi \circ \psi \circ \xi) \circ c^+_{\phi \circ \psi, \xi} \circ c^+_{\psi, \xi}.\) Similarly, the calculations

\[
(\phi \circ \psi \circ \xi)^+(\pi) \circ c^+_{\phi \circ \psi, \xi} \circ c^+_{\psi, \xi} = (\psi \circ \xi)^+ \circ \phi^+(\pi) \circ c^+_{\psi, \xi}
\]

and

\[
(\phi \circ \psi \circ \xi)^+(\pi) \circ c^+_{\phi \circ \psi, \xi} \circ c^+_{\psi, \xi} = (\psi \circ \xi)^+ \circ \phi^+(\pi) \circ c^+_{\psi, \xi}
\]

show that \((\phi \circ \psi \circ \xi)^+(\pi) \circ c^+_{\phi \circ \psi, \xi} \circ c^+_{\psi, \xi} = (\phi \circ \psi \circ \xi)^+(\pi) \circ c^+_{\phi \circ \psi, \xi} \circ c^+_{\psi, \xi}.\) Therefore

\[
c^+_{\phi \circ \psi, \xi} \circ \xi^+(c^+_{\phi, \psi}) = c^+_{\phi \circ \psi, \xi} \circ c^+_{\psi, \xi}.
\]

5. Localization and descent

Our exposition is inspired by (Beilinson and Bernstein, 1993).
5.1. **General framework.** We denote by \( \mathcal{M}an \) the category of manifolds. For \( X \in \mathcal{M}an \) we denote by \( X_{\text{sm}} \) the full subcategory of \( \mathcal{M}an/X \) with objects \( P \xrightarrow{\pi_P} X \), where \( \pi_P \) is a submersion. The category \( X_{\text{sm}} \) is equipped with the Grothendieck topology (the “smooth” topology) with covers given by surjective submersions.

We assume given a category \( \mathcal{P} \) equipped with a functor \( p: \mathcal{P} \to \mathcal{M}an \); for a manifold \( X \) we denote by \( \mathcal{P}_X \) the corresponding fiber, i.e. the full subcategory with objects \( F \in \mathcal{P} \) such that \( p(F) = X \). We assume that the functor \( p \) makes \( \mathcal{P} \) a category prefibered over \( \mathcal{M}an \).

This means, by definition, that:

- For a morphism \( Y \xrightarrow{f} X \) in \( \mathcal{M}an \) there is a functor \( f^\triangledown: \mathcal{P}_X \to \mathcal{P}_Y \) of inverse image along \( f \).
- For a pair of composable morphisms \( Z \xrightarrow{g} Y \xrightarrow{f} X \) there is a morphism of functors \( c_{f,g}^\triangledown: g^\triangledown f^\triangledown \to (f \circ g)^\triangledown \)

which satisfy the associativity constraint

\[
c_{f,h,g}^\triangledown \circ (c_{f,g}^\triangledown) = c_{f,g,h}^\triangledown \circ (c_{g,h}^\triangledown) : h^\triangledown f^\triangledown g^\triangledown F \to (f \circ g \circ h)^\triangledown F,
\]

for any triple of composable morphisms \( W \xrightarrow{h} Z \xrightarrow{g} Y \xrightarrow{f} X \) and any \( F \in \mathcal{P}_X \).

In addition, for each manifold \( X \) we assume given a full subcategory \( \mathcal{P}_X^\flat \) of \( \mathcal{P}_X \). We assume that, if \( Y \xrightarrow{f} X \) is a submersion and \( F \in \mathcal{P}_X^\flat \), then \( f^\triangledown F \in \mathcal{P}_Y^\flat \) and for \( g \) arbitrary the morphism \( c_{f,g}^\triangledown: g^\triangledown f^\triangledown F \to (f \circ g)^\triangledown F \) is an isomorphism.

**Example 12.**

1. Let \( (\mathcal{O}-\text{Mod}/T)^\sharp \) denote the category with objects pairs \((X, \mathcal{E})\), where \( X \) is a manifold and \( \mathcal{E} \in \mathcal{O}_{X^3} \text{-}\text{Mod}/T_{X^3} \). A morphism \( u: (Y, \mathcal{F}) \to (X, \mathcal{E}) \) is a pair \( u = (f, t) \), where \( f: Y \to X \) is a map of manifolds and \( t: \mathcal{F} \to f^* \mathcal{E} \) is a morphism in \( \mathcal{O}_{X^3} \text{-}\text{Mod}/T_{X^3} \).

   The functor 

   \[
   (\mathcal{O}-\text{Mod}/T)^\sharp \to \mathcal{M}an: (X, \mathcal{E}) \mapsto X, (f, t) \mapsto f
   \]

   makes \( (\mathcal{O}-\text{Mod}/T)^\sharp \) a prefibered category over the category \( \mathcal{M}an \) of manifolds.

   Let \( (\mathcal{O}-\text{Mod}^\text{lf}/T)^\sharp_X \) denote the full subcategory of \( (\mathcal{O}-\text{Mod}/T)^\sharp \) with objects locally free of finite rank over \( \mathcal{O}_{X^3} \).

   This is an example of the framework of 5.1 with \( \mathcal{P} = (\mathcal{O}-\text{Mod}/T)^\sharp, \mathcal{P}_X^\flat = (\mathcal{O}-\text{Mod}^\text{lf}/T)^\sharp_X \) and the functor of inverse image defined in 4.3.

2. Lie algebroids, see 6.4
3. Marked Lie algebroids, see 6.5
4. Courant algebroids, see 7.7

5.2. **The category of descent data.** For \( P \xrightarrow{\pi_P} X \) in \( X_{\text{sm}} \) consider the diagram

\[
P \times_X P \xrightarrow{\pi_{ij}} P \xrightarrow{\pi_i} P \xrightarrow{\pi_j} P \xrightarrow{\pi_P} X
\]
where the maps $\pi_i$, $i = 1, 2$ (respectively, $\pi_{ij}$, $1 \leq i < j \leq 3$) denote projections onto the $i^{th}$ (respectively, $i^{th}$ and $j^{th}$) factor.

We denote by $\text{Desc}(P; \mathcal{P}^b) = \text{Desc}(P \xrightarrow{\pi_P} X; \mathcal{P}^b)$ the category with objects pairs $(\mathcal{F}, g_\mathcal{F})$, where

1. $\mathcal{F} \in \mathcal{P}^b_P$;
2. $g_\mathcal{F}: \pi_2^\mathcal{F} \rightarrow \pi_1^\mathcal{F}$ is an isomorphism which satisfies the cocycle condition $\pi_1^\mathcal{F}(g_\mathcal{F}) \circ \pi_2^\mathcal{F}(g_\mathcal{F}) = \pi_3^\mathcal{F}(g_\mathcal{F})$.

A morphism $t: (\mathcal{F}, g_\mathcal{F}) \rightarrow (\mathcal{F}', g_{\mathcal{F}'})$ in $\text{Desc}(P; \mathcal{P}^b)$ is a morphism $t: \mathcal{F} \rightarrow \mathcal{F}'$ in $\mathcal{P}^b_P$ which satisfies $\pi_1^\mathcal{F}(t) \circ g_\mathcal{F} = g_{\mathcal{F}'} \circ \pi_2^\mathcal{F}(t)$.

The assignment $(\mathcal{F}, g_\mathcal{F}) \mapsto \mathcal{F}$ defines a functor $\text{Desc}(P; \mathcal{P}^b) \rightarrow \mathcal{P}^b_P$.

5.3. Localization. For $\mathcal{E} \in \mathcal{P}^b_X$ and $P \xrightarrow{\pi_P} X$ in $X_{sm}$ let $\mathcal{E}_P := \pi_P^* \mathcal{E} \in \mathcal{P}^b_P$. If $Q \xrightarrow{f} P$ is a morphism in $X_{sm}$, the map $c_{\pi_P, f}: \mathcal{E}_Q \rightarrow \mathcal{E}_P$ is an isomorphism. Therefore, the assignment $X_{sm} \ni P \mapsto \mathcal{E}_P$ defines a Cartesian section of $\mathcal{P}/X_{sm}$.

The functor $\pi_P^*: \mathcal{P}^b_X \rightarrow \mathcal{P}^b_P$ lifts to the functor

$$\pi_P^*: \mathcal{P}_X \rightarrow \text{Desc}(P; \mathcal{P}^b)$$

defined by $\mathcal{E} \mapsto (\mathcal{E}_P, g_{\mathcal{E}_P})$, where $g_{\mathcal{E}_P}$ is the composition of the canonical isomorphisms $\pi_2^\mathcal{E}_P \simeq \mathcal{E}_{P \times X_P} \simeq \pi_1^\mathcal{E}_P$.

A morphism $f: Q \rightarrow P$ in $X_{sm}$ induces the functor

$$\widetilde{f}^\mathcal{E}: \text{Desc}(P; \mathcal{P}^b) \rightarrow \text{Desc}(Q; \mathcal{P}^b)$$

defined by $(\mathcal{F}, g_\mathcal{F}) \mapsto (f^\mathcal{E} \mathcal{F}, (f \times_X f)^\mathcal{E}_\mathcal{F})$.

5.4. Descent. We shall say that $P \xrightarrow{\pi_P} X \in X_{sm}$ is classical if $\pi_P$ is a disjoint union of open embeddings.

For any smooth cover $P \xrightarrow{\pi_P} X$ there exists a classical cover $T \xrightarrow{\pi_T} X$ such that $\text{Hom}_{X_{sm}}(T, P) \neq \emptyset$, i.e. there exists a morphism $f: T \rightarrow P$ such that $\pi_T = f \circ \pi_P$. In other words, every smooth cover admits a classical refinement.

We shall say that $\mathcal{P}^b$ has the smooth (respectively, classical) descent property if for any smooth (respectively, classical) cover $P \xrightarrow{\pi_P} X$ the functor $\text{Desc}(P; \mathcal{P}^b)$ is an equivalence.

**Theorem 13.** If $\mathcal{P}^b$ has the classical descent property then it has the smooth descent property.

**Proof.** Suppose that $f: T \rightarrow P$ is a morphism in $X_{sm}$ with $T \xrightarrow{\pi_T} X$ a classical cover. For $(\mathcal{F}, g_\mathcal{F}) \in \text{Desc}(P; \mathcal{P}^b)$ there exist $\mathcal{E} \in \mathcal{P}^b_X$ and an isomorphism $\pi_T^\mathcal{E} \simeq \widetilde{f}^\mathcal{E}(\mathcal{F}, g_\mathcal{F})$.

Consider the diagram

$$
\begin{array}{ccc}
P \times_X T & \rightarrow & T \\
\downarrow_{\pi_P} & \searrow_{\pi_T} \\
P & \rightarrow & X
\end{array}
$$

(5.4.1)
Note that $\text{pr}_P: P \times_X T \to P$ a classical cover of $P$. Since $\pi_P \circ \text{pr}_P = \pi_T \circ \text{pr}_T$, $\text{pr}_2 \circ (\text{id} \times f) = f \circ \text{pr}_T$ and $\text{pr}_1 \circ (\text{id} \times f) = \text{pr}_P$ there are isomorphisms

\[(5.4.2) \quad \widetilde{\text{pr}}_P^\ast \pi_P^! \mathcal{E} \cong \widetilde{\text{pr}}_T^\ast \pi_T^! \mathcal{E} \cong \widetilde{\text{pr}}_T^\ast f^!(\mathcal{F}, g_{\mathcal{F}}) \cong \]

\[\cong (\text{id} \times f)^\ast \widetilde{\text{pr}}_T^\ast (\mathcal{F}, g_{\mathcal{F}}) \xrightarrow{(\text{id} \times f)^\ast (g_{\mathcal{F}})} (\text{id} \times f)^\ast \widetilde{\text{pr}}_1^\ast (\mathcal{F}, g_{\mathcal{F}}) \cong \widetilde{\text{pr}}_P^\ast (\mathcal{F}, g_{\mathcal{F}})\]

Since $\text{pr}_P: P \times_X T \to P$ is a classical cover of $P$ the functor $\widetilde{\text{pr}}_P^\ast$ is an equivalence by assumption. Therefore, an isomorphism $(\mathcal{F}, g_{\mathcal{F}}) \cong \widetilde{\text{pr}}_P^\ast (\mathcal{F}, g_{\mathcal{F}})$ is equivalent to an isomorphism $(\mathcal{F}, g_{\mathcal{F}}) \cong \pi_P^! \mathcal{E}$. Thus, the functor $\pi_P^!$ is essentially surjective.

Since $f^! \circ \pi_P^! \cong \pi_T^!$ is an equivalence, it follows that $\pi_P^!$ is faithful.

Suppose that $(\mathcal{F}_1, g_{\mathcal{F}_1}) \xrightarrow{\phi} (\mathcal{F}, g_{\mathcal{F}_2})$ is a morphism in $\text{Desc}(P; \mathcal{P})$. Then, there exits a morphism $\mathcal{E}_1 \xrightarrow{\psi} \mathcal{E}_2$ in $\mathcal{P}_X$ and isomorphisms $\pi_P^! \mathcal{E}_1 \cong f^!(\mathcal{F}_1, g_{\mathcal{F}_1})$ which intertwine $\phi$ and $\psi$. Proceeding as in the proof of essential surjectivity one obtains the commutative diagram

\[
\begin{array}{ccc}
\widetilde{\text{pr}}_P^\ast \pi_P^! \mathcal{E}_1 & \xrightarrow{\cong} & \widetilde{\text{pr}}_P^\ast (\mathcal{F}_1, g_{\mathcal{F}_1}) \\
\downarrow && \downarrow \text{pr}_P^!(\phi) \\
\widetilde{\text{pr}}_P^\ast \pi_P^! \mathcal{E}_2 & \xrightarrow{\cong} & \widetilde{\text{pr}}_P^\ast (\mathcal{F}_1, g_{\mathcal{F}_2})
\end{array}
\]

Since the functor $\widetilde{\text{pr}}_P^\ast$ is an equivalence by assumption it follows that $\phi = \pi_P^!(\psi)$. Therefore the functor $\pi_P^!$ is full.

\[\text{Corollary 14 (of Theorem 13).} \quad (\mathcal{O} - \text{Mod}^{1T}/\mathcal{T})^2 \text{ has the smooth descent property}.\]

6. INVERSE IMAGE FOR LIE ALGEBROIDS

The anchor map of an $\mathcal{O}_X$-Lie algebroid $\mathcal{A}$ renders the latter as an object of $\mathcal{O}_X - \text{Mod}/\mathcal{T}_X$. This correspondence extends to a functor $\mathcal{O}_X - \text{LieAlg} \to \mathcal{O}_X - \text{Mod}/\mathcal{T}_X$.

6.1. INVERSE IMAGE FOR LIE ALGEBROIDS. For an $\mathcal{O}_X$-Lie algebroid $\mathcal{A}$ and a map of DG-manifolds $\phi: \mathcal{Y} \to \mathcal{X}$ the inverse image $\phi^+ \mathcal{A}$ has a canonical structure of a Lie algebroid on $\mathcal{Y}$. Namely, the bracket on $\phi^+ \mathcal{A}$ is given by

\[
[(f \otimes a, \xi), (g \otimes b, \eta)] = ((-1)^g f g \otimes [a, b] + \xi(g) \otimes b - (-1)^{\xi} \eta(f) \otimes a, [\xi, \eta]),
\]

where $f, g \in \mathcal{O}_\mathcal{Y}$, $a, b \in \mathcal{A}$, $\xi, \eta \in \mathcal{T}_\mathcal{Y}$ are homogeneous elements respectively. Since $(f \otimes a, \xi)$, $(g \otimes b, \eta)$ are homogeneous elements, $a + f = \xi$ and $g + b = \eta$.

For any composition of maps of DG-manifolds

\[
\mathcal{Y} \xrightarrow{\psi} \mathcal{Y} \xrightarrow{\phi} \mathcal{X},
\]

a general element in $\psi^+ \phi^+ \mathcal{A}$ is a finite sum of homogeneous elements of the form $(h \otimes (f \otimes a, \xi), \rho)$, where $h \in \mathcal{O}_\mathcal{Y}$, $f \in \mathcal{O}_\mathcal{Y}$, $a \in \mathcal{A}$, $\xi \in \mathcal{T}_\mathcal{Y}$ and $\rho \in \mathcal{T}_\mathcal{Y}$. In these terms, the canonical
map $c_{\phi,\psi}^+: \psi^+\phi^+\mathcal{A} \to (\phi \circ \psi)^+\mathcal{A}$ in the category $\mathcal{O}_3-\text{Mod}/\mathcal{T}_3$ is equal to

$$c_{\phi,\psi}^+(h \otimes (f \otimes a, \xi), \rho) = (h\psi^*(f) \otimes a, \rho),$$

where,

$$\text{(6.1.1)} \quad (d\psi)(\rho) = h \otimes \xi,$$

see Subsection 4.4.

**Lemma 15.** The map $c_{\phi,\psi}^+$ is a morphism of $\mathcal{O}_3$-Lie algebroids.

**Proof.**

$$[c_{\phi,\psi}^+(h \otimes (f \otimes a, \xi), \rho), c_{\phi,\psi}^+(l \otimes (g \otimes b, \eta), \nu)]$$

$$= [(h\psi^*(f) \otimes a, \rho), (l\psi^*(g) \otimes b, \nu)]$$

$$= (\rho(l)\psi^*(g) \otimes b + (-1)^{\rho l}l\rho(\psi^*(g)) \otimes b + (-1)^{\nu \rho l}h\psi^*(f) \otimes a$$

$$- (-1)^{\nu \rho + \nu h}\psi^*(f) \otimes a + (-1)^{l[f+\alpha(f+g)]}h\psi^*(f) \otimes [a, b], \rho, \nu].$$

On the other hand,

$$c_{\phi,\psi}^+[\psi^*(f)(g) \otimes a, \xi, \eta, \rho, \nu]$$

$$= c_{\phi,\psi}^+[\rho(l) \otimes (g \otimes b, \eta) - (-1)^{\nu \rho l}h\psi^*(f)(g) \otimes a$$

$$- (-1)^{\nu \rho + \nu h}\psi^*(f) \otimes a + (-1)^{l[f+\alpha(f+g)]}h\psi^*(f) \otimes [a, b], \rho, \nu].$$

Comparing term by term the calculations of $[c_{\phi,\psi}^+(h \otimes (f \otimes a, \xi), \rho), c_{\phi,\psi}^+(l \otimes (g \otimes b, \eta), \nu)]$ and $c_{\phi,\psi}^+[\psi^*(f)(g) \otimes a, \xi, \eta, \rho, \nu]$, it remains to show that

$$hl(\psi^*(\xi(g))) \otimes b = l\rho(\psi^*(g)) \otimes b,$$

$$hl(\psi^*(\eta(f))) \otimes a = h\nu(\psi^*(f)) \otimes a.$$

Applying the formula (6.1.1) in the first case one gets,

$$(-1)^{\xi}hl(\psi^*(\xi(g))) \otimes b = (-1)^{l[h+\xi]}l(h\psi^*(\xi(g))) \otimes b = (-1)^{l[\rho]}(h \otimes \xi(g) \otimes b) = l\rho(\psi^*(g)) \otimes b.$$

The second case is analogous.
6.2. **Inverse image for marked Lie algebroids.** Suppose that \( \mathcal{Y} \xrightarrow{\phi} \mathcal{X} \) is a map of DG-manifolds. Let \((\mathcal{A}, c) \in \mathcal{O}_\mathcal{X}^{-\text{LieAlgd}} \) be a marked Lie algebroid on \( \mathcal{X} \), i.e. there is a map of marked Lie algebroids

\[
\mathcal{O}_\mathcal{X}[n] \xrightarrow{\phi} \mathcal{A}.
\]

Since \( \phi^+ \mathcal{O}_\mathcal{X}[n] = \ker(d\phi) \oplus \mathcal{O}_\mathcal{Y}[n] \), there is a canonical map

\[
(6.2.1) \quad \mathcal{O}_\mathcal{Y}[n] \to \ker(d\phi) \oplus \mathcal{O}_\mathcal{Y}[n] = \phi^+ \mathcal{O}_\mathcal{X}[n]
\]

Let \( \phi^+(c) \in \Gamma(Y; \phi^+ \mathcal{A})^{-n} \) denote the image of \( 1 \in \Gamma(Y; \mathcal{O}_\mathcal{Y}) \) under the composition

\[
\mathcal{O}_\mathcal{Y}[n] \xrightarrow{\phi} \phi^+ \mathcal{O}_\mathcal{X}[n] \xrightarrow{\phi^+(c)} \phi^+ \mathcal{A}.
\]

The pair \((\phi^+ \mathcal{A}, \phi^+(c))\) is a marked Lie algebroid denoted \( \phi^+(\mathcal{A}, c) \). The map

\[
\mathcal{O}_\mathcal{Y}[n] \xrightarrow{\phi^+(c)} \phi^+ \mathcal{A}.
\]

is given by \( g \mapsto (g \otimes c, 0) \) for \( g \in \mathcal{O}_\mathcal{Y}[n] \).

The assignment \((\mathcal{A}, c) \mapsto \phi^+(\mathcal{A}, c)\) extends to a functor

\[
\phi^+: \mathcal{O}_\mathcal{X}^{-\text{LieAlgd}} \to \mathcal{O}_\mathcal{Y}^{-\text{LieAlgd}}.
\]

For any composition of maps of DG-manifolds, \( \mathcal{Z} \xrightarrow{\psi} \mathcal{Y} \xrightarrow{\phi} \mathcal{X} \), the morphism \( c^+_{\phi,\psi} \) makes the diagram

\[
\begin{array}{ccc}
\mathcal{O}_\mathcal{Z}[n] & \xrightarrow{-(\phi \circ \psi)^+(c)} & (\phi \circ \psi)^+ \mathcal{A} \\
\downarrow_{\psi^+ \phi^+(c)} & \uparrow_{c^+_{\phi,\psi}} & \\
\psi^+ \phi^+ \mathcal{A}
\end{array}
\]

commutative.

**Lemma 16.** The map \( c^+_{\phi,\psi} \) is a morphism of marked Lie algebroids.

**Proof.** Lemma 15 implies that \( c^+_{\phi,\psi} \) is a morphism of \( \mathcal{O}_\mathcal{Y} \)-Lie algebroids. For \( g \in \mathcal{O}_\mathcal{Z} \)

\[
g \cdot (\phi \circ \psi)^+(c) = (g \otimes c, 0),
\]

by construction. On the other hand, the calculation

\[
c^+_{\phi,\psi}(g \cdot \psi^+ \phi^+(c)) = c^+_{\phi,\psi}(g \otimes (1 \otimes c, 0), 0) = (g \otimes c, 0)
\]

shows that \( g \cdot (\phi \circ \psi)^+(c) = c^+_{\phi,\psi}(g \cdot \psi^+ \phi^+(c)) \), i.e. \( c^+_{\phi,\psi} \) preserves the marking. \( \square \)
6.3. **Inverse image for $\mathcal{O}[n]$-extensions.** Suppose that $\mathcal{Y} \xrightarrow{\phi} \mathcal{X}$ is a map of DG-manifolds and $\mathcal{B}$ is a $\mathcal{O}_X$-Lie algebroid.

**Lemma 17.** Suppose that $(\mathcal{A}, \mathbf{c})$ is a $\mathcal{O}[n]$-extension of $\mathcal{B}$. Then, $\phi^+(\mathcal{A}, \mathbf{c})$ is a $\mathcal{O}[n]$-extension of $\phi^+\mathcal{B}$.

**Proof.** Since both small squares in the commutative diagram

\[
\begin{array}{c}
\phi^+\mathcal{B} \times_{\phi^+\mathcal{B}} \phi^*\mathcal{A} \\
\downarrow \\
\phi^*\mathcal{A}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\phi^+\mathcal{B} \\
\downarrow \\
\phi^*\mathcal{T}_\mathcal{X}
\end{array} \quad \rightarrow \quad \begin{array}{c}
\mathcal{T}_\mathcal{Y}
\end{array}
\]

are Cartesian, so is the large one. Therefore there is a canonical isomorphism $\phi^+\mathcal{A} \cong \phi^+\mathcal{B} \times_{\phi^+\mathcal{B}} \phi^*\mathcal{A}$. Hence, $\phi^+(\mathcal{A}, \mathbf{c})$ is a $\mathcal{O}[n]$-extension of $\phi^+\mathcal{B}$. \qed

Thus, the inverse image functor for marked Lie algebroids induces a functor

\[
(6.3.1) \quad \phi^+: \mathcal{O}[n]\text{Ext}(\mathcal{B}) \rightarrow \mathcal{O}[n]\text{Ext}(\phi^+\mathcal{B})
\]

**Proposition 18.** The functor (6.3.1) is a morphism of $\mathbb{C}$-vector spaces in categories (and, in particular, of Picard groupoids).

**Proof.** We prove the statement in the case $\lambda_1\mathcal{A}_1 + \lambda_2\mathcal{A}_2$. The general case is analogous and is left to the reader.

Recall that for any object $\mathcal{A} \in \mathcal{O}[n]\text{Ext}(\mathcal{B})$ there is the canonical isomorphism $\phi^+\mathcal{A} = \phi^+\mathcal{B} \times_{\phi^+\mathcal{B}} \phi^*\mathcal{A}$. Thus, for a pair of objects $\mathcal{A}_1, \mathcal{A}_2 \in \mathcal{O}[n]\text{Ext}(\mathcal{B})$ there is a sequence of isomorphisms

\[
\phi^+\mathcal{A}_1 \times_{\phi^+\mathcal{B}} \phi^+\mathcal{A}_2 \cong (\phi^+\mathcal{B} \times_{\phi^+\mathcal{B}} \phi^*\mathcal{A}_1) \times_{\phi^+\mathcal{B}} (\phi^+\mathcal{B} \times_{\phi^+\mathcal{B}} \phi^*\mathcal{A}_2) \\
\cong \mathcal{T}_{\mathcal{Y}} \times_{\phi^*\mathcal{T}_{\mathcal{X}}} (\phi^*\mathcal{A}_1 \times_{\phi^*\mathcal{B}} \phi^*\mathcal{A}_2) \cong \mathcal{T}_{\mathcal{Y}} \times_{\phi^*\mathcal{T}_{\mathcal{X}}} \phi^*(\mathcal{A}_1 \times_{\mathcal{B}} \mathcal{A}_2) \cong \phi^+(\mathcal{A}_1 \times_{\mathcal{B}} \mathcal{A}_2).
\]

The inverse image functor $\phi^+$ applied to the commutative diagram

\[
\begin{array}{c}
\mathcal{O}_X[n] \times \mathcal{O}_X[n] \\
\downarrow_{(\alpha_1, \alpha_2) \rightarrow \lambda_1\alpha_1 + \lambda_2\alpha_2} \\
\mathcal{O}_X[n]
\end{array} \quad \rightarrow \quad \begin{array}{c}
\mathcal{A}_1 \times_{\mathcal{B}} \mathcal{A}_2 \\
\downarrow \\
\mathcal{A}_1 + \mathcal{A}_2
\end{array}
\]

and canonical isomorphisms $\phi^+(\mathcal{O}_X[n] \times \mathcal{O}_X[n]) \cong \phi^+\mathcal{O}_X[n] \times \phi^+\mathcal{O}_X[n]$ and $\phi^+\mathcal{A}_1 \times_{\phi^+\mathcal{B}} \phi^+\mathcal{A}_2 \cong \phi^+(\mathcal{A}_1 \times_{\mathcal{B}} \mathcal{A}_2)$ give rise to the commutative diagram

\[
\begin{array}{c}
\mathcal{O}_\mathcal{Y}[n] \times \mathcal{O}_\mathcal{Y}[n] \\
\downarrow_{(\alpha_1, \alpha_2) \rightarrow \lambda_1\alpha_1 + \lambda_2\alpha_2} \\
\mathcal{O}_\mathcal{Y}[n]
\end{array} \quad \rightarrow \quad \begin{array}{c}
\phi^+\mathcal{O}_X[n] \times \phi^+\mathcal{O}_X[n] \\
\downarrow_{\phi^+((\alpha_1, \alpha_2) \rightarrow \lambda_1\alpha_1 + \lambda_2\alpha_2)} \\
\phi^+\mathcal{O}_X[n]
\end{array} \quad \rightarrow \quad \begin{array}{c}
\phi^+\mathcal{A}_1 \times_{\phi^+\mathcal{B}} \phi^+\mathcal{A}_2 \\
\downarrow \\
\phi^+(\mathcal{A}_1 + \mathcal{A}_2)
\end{array}
\]

\[
(6.2.1)
\]
Therefore, there is a unique morphism in $\mathcal{O}[n] \text{Ext}(\phi^+B)$
\begin{equation}
\lambda_1 \phi^+ A_1 + \lambda_2 \phi^+ A_2 \rightarrow \phi^+ (\lambda_1 A_1 + \lambda_2 A_2).
\end{equation}

6.4. Smooth localization for Lie algebroids. Let $(\mathcal{O} - \text{LieAlgd})^\sharp$ denote the category with objects pairs $(X, A)$, where $X$ is a manifold and $A \in \mathcal{O}_X - \text{LieAlgd}$. A morphism $u: (Y, B) \rightarrow (X, A)$ is a pair $u = (f, t)$, where $f: Y \rightarrow X$ is a map of manifolds and $t: B \rightarrow f^+ (A)$ is a morphism in $\mathcal{O}_Y - \text{LieAlgd}$. It follows from Lemma 15 that the forgetful functor $(X, A) \mapsto X$ makes $(\mathcal{O} - \text{LieAlgd})^\sharp$ a category prefibered over $\mathfrak{Man}$.

Let $(\mathcal{O} - \text{LieAlgd}_{1f}^\sharp)_X$ denote the full subcategory of $(\mathcal{O} - \text{LieAlgd})^\sharp_X$ with objects locally free of finite rank over $\mathcal{O}_X$.

This is an example of the framework of 5.1 with $P = (\mathcal{O} - \text{LieAlgd})^\sharp$, $P^\sharp_X = (\mathcal{O} - \text{LieAlgd}_{1f}^\sharp)_X$ and the functor of inverse image defined in [6.1].

**Corollary 19** (of Theorem 13). $(\mathcal{O} - \text{LieAlgd}_{1f}^\sharp)^\sharp$ has the smooth descent property.

6.5. Smooth localization for marked Lie algebroids. Let $(\mathcal{O} - \text{LieAlgd}_{n}^\sharp)^\sharp$ denote the category with objects pairs $(X, (A, c))$, where $X$ is a manifold and $(A, c) \in \mathcal{O}_X - \text{LieAlgd}_{n}^*$. A morphism $u: (Y, (B, b)) \rightarrow (X, (A, c))$ is a pair $u = (f, t)$, where $f: Y \rightarrow X$ is a map of manifolds and $t: (B, b) \rightarrow f^+ (A, c)$ is a morphism in $\mathcal{O}_Y - \text{LieAlgd}_{n}^*$. It follows from Lemma 15 and Lemma 16 that the forgetful functor $(X, (A, c)) \mapsto X$ makes $(\mathcal{O} - \text{LieAlgd}_{n}^*)^\sharp$ a category prefibered over $\mathfrak{Man}$.

Let $(\mathcal{O} - \text{LieAlgd}_{n,1f}^\sharp)_X$ denote the full subcategory of $(\mathcal{O} - \text{LieAlgd}_{n}^*)^\sharp_X$ with objects locally free of finite rank over $\mathcal{O}_X$.

This is an example of the framework of 5.1 with $P = (\mathcal{O} - \text{LieAlgd}_{n}^*)^\sharp$, $P^\sharp_X = (\mathcal{O} - \text{LieAlgd}_{n,1f}^*)^\sharp_X$ and the functor of inverse image defined in [6.2].

**Corollary 20** (of Theorem 13). $(\mathcal{O} - \text{LieAlgd}_{n,1f}^*)^\sharp$ has the smooth descent property.

7. Inverse image for Courant algebroids

7.1. The inverse image functor. Suppose that $f: Y \rightarrow X$ is a map of manifolds. For $Q \in \mathcal{C}A(X)$ the functor of inverse image under $f$ for Courant algebroids denoted
\begin{equation}
f^{++}: \mathcal{C}A(X) \rightarrow \mathcal{C}A(Y)
\end{equation}
is defined by
\begin{equation}
f^{++} Q := Q(f^+ \tau Q).
\end{equation}
in the notation of 3.8

7.2. Explicit description. Suppose that $Q$ is a Courant algebroid on $X$ and $f: Y \rightarrow X$ is a map of manifolds. It transpires from the definition, that $f^{++} Q$ is a sub-quotient of $\Omega^1_Y \oplus f^* Q \oplus T_Y$:
\begin{equation}
\Omega^1_Y \oplus f^* Q \oplus T_Y \leftrightarrow \Omega^1_Y \oplus f^* Q \times f^* T_Y \rightarrow f^{++} Q.
\end{equation}
7.2.1. The symmetric pairing. Let $\langle \ , \ \rangle'$ denote the symmetric pairing on $f^* Q \oplus T_Y$ such that

1. it restricts to the pairing induced by $\langle \ , \ \rangle$ on $f^* Q$,
2. $\langle \mathcal{T}_Y, \mathcal{T}_Y \rangle = 0$,
3. for $\alpha \in f^* \Omega^1_X$, $\xi \in \mathcal{T}_Y$, $(i(\alpha), \xi)' = \iota_\xi df^\gamma(\alpha)$.

Let $\langle \ , \ \rangle''$ denote the unique symmetric pairing on $\Omega_Y^1 \oplus f^* Q \times f^* \tau_X \mathcal{T}_Y$ such that

1. it restricts to the pairing $\langle \ , \ \rangle'$ on $f^* Q \times f^* \tau_X \mathcal{T}_Y$,
2. $\langle \Omega_Y^1, \Omega_Y^1 \rangle = 0$,
3. for $\alpha \in \Omega_Y^1$, $q \in f^* Q \times f^* \tau_X \mathcal{T}_Y$, $\langle \alpha, q \rangle'' = \iota_{\sigma(q)} \alpha$.

Since

$$\langle (df^\gamma(\alpha), -i(\alpha)), 0, (\beta, q, \xi) \rangle'' = 0$$

for all $\alpha \in f^* \Omega^1_X$, $\beta \in \Omega_Y^1$, $(q, \xi) \in f^* Q \times f^* \tau_X \mathcal{T}_Y$, it follows that the pairing $\langle \ , \ \rangle''$ descends to a symmetric pairing on $f^{++} Q$ which we will denote by $\langle \ , \ \rangle$.

7.2.2. The bracket. Let $[\ , \ ]'$ denote the unique operation on $\Omega_Y^1 \oplus f^* Q \oplus T_Y$ defined by the formula

$$[\langle \alpha, h \otimes q, \xi \rangle, \langle \beta, j \otimes p, \eta \rangle]' = (\langle \gamma, \xi \rangle, p, \eta) = (-\iota_\eta d\alpha + L_\xi \beta + j dh \langle q, p \rangle, h j \otimes [q, p] - \iota_\eta dh \otimes q + L_\xi j \otimes p, [\xi, \eta]),$$

where $\alpha, \beta \in \Omega_Y^1$, $h, j \in \mathcal{O}_Y$, $q, p \in Q$ and $\xi, \eta \in \mathcal{T}_Y$.

For any $\gamma \in f^* \Omega^1_X$, the calculation

$$\langle \gamma, \xi \rangle, p, \eta) = \langle df^\gamma(\gamma), -i(\gamma), 0, (\beta, j \otimes p, \eta) \rangle' = (-\iota_\eta d(df^\gamma(\gamma)), -j \otimes [\gamma, p], 0)$$

$$= (-\iota_\eta d(df^\gamma(\gamma)), j \otimes L_{\pi(p)} \gamma - j \otimes \pi^1(d\langle p, \gamma \rangle), 0)$$

$$= (-\iota_\eta d(df^\gamma(\gamma)), j \otimes L_{\pi(p)} \gamma - j \otimes d\pi_\pi(p)(\gamma), 0)$$

shows that the bracket $[\ , \ ]'$ descends to an operation on $f^{++}(Q)$ which we will denoted by $[\ , \ ]$.

**Remark 21.** The construction of inverse image for Courant algebroids specializes to that ([Li-Bland and Meinrenken, 2009]) under suitable transversality assumptions.

7.3. Twisting by a 3-form. Suppose that $Q$ is a Courant algebroid on $X$ and $H$ is a closed 3-form on $X$. Recall that the $H$-twist of $Q$, denoted $Q_H$, is the Courant algebroid whose symmetric pairing and the anchor map coincide with the ones given on $Q$ and the bracket is given by the

$$[q, p]_H = [q, p] + \pi^1(\iota_{\pi(p)} t_{\pi(q)} H),$$

for any $q, p \in Q$.

Suppose that $f : Y \to X$ is a map of manifolds.

**Lemma 22.**

$$(f^{++} Q)_{f^* H} = f^{++} Q_H.$$
Proof. The underlying $\mathcal{O}_Y$-modules of both objects coincide. The bracket on $f^{++}Q_H$ is given by the formula,

$$[(\alpha, h \otimes q, \xi), (\beta, j \otimes p, \eta)] = (-\iota_\eta d\alpha + L_\xi \beta + jdh(q, p), h \otimes [q, p]_H - \iota_\eta dh \otimes q + L_\xi j \otimes p, [\xi, \eta]) = (-\iota_\eta d\alpha + L_\xi \beta + jdh(q, p), h \otimes [q, p] - \iota_\eta dh \otimes q + L_\xi j \otimes p, [\xi, \eta]) + (0, h j \otimes \pi^+(\iota_{\pi(p)}\iota_{\pi(q)}H), 0)$$

On the other hand, the bracket on $(f^{++}Q)_{f^*H}$ is given by the formula

$$[(\alpha, h \otimes q, \xi), (\beta, j \otimes p, \eta)]_{f^*H} = [(\alpha, h \otimes q, \xi), (\beta, j \otimes p, \eta)] + (\iota_{\xi} f^*H, 0, 0) = [(\alpha, h \otimes q, \xi), (\beta, j \otimes p, \eta)] + (0, \pi^+(H(df(\xi), df(\eta), )), 0) = [(\alpha, h \otimes q, \xi), (\beta, j \otimes p, \eta)] + (0, h j \otimes \pi^+(\iota_{\pi(p)}\iota_{\pi(q)}H), 0).$$

$$\square$$

7.4. **Connections.** Recall that a connection $\nabla$ on a Courant algebroid $Q$ on $X$ is splitting $\nabla: \mathcal{T}_X \to Q$ of the anchor map $\pi: Q \to \mathcal{T}_X$ isotropic with respect to the symmetric pairing on $Q$. We denote by $\mathcal{C}(Q)$ the sheaf of locally defined connections on $Q$.

Suppose that $Q$ is a Courant algebroid on $X$, and $f: Y \to X$ is a map of manifolds. A connection $\nabla$ on $Q$ induces a splitting $f^*(\nabla): f^*\mathcal{T}_X \to f^*Q$ of $f^*(\pi)$ and the splitting

(7.4.1) $$\mathcal{T}_Y \times_{f^*\mathcal{T}_X} f^*(\nabla): \mathcal{T}_Y \to \mathcal{T}_Y \times_{f^*\mathcal{T}_X} f^*Q$$

of $\mathcal{T}_Y \times_{f^*\mathcal{T}_X} f^*(\pi)$.

Let $f^{++}(\nabla): \mathcal{T}_Y \to f^{++}Q$ denote the composition

$$\mathcal{T}_Y \xrightarrow{f^{++}(\nabla)} \mathcal{T}_Y \times_{f^*\mathcal{T}_X} f^*Q \to f^{++}Q.$$ 

**Lemma 23.** The map $f^{++}(\nabla)$ is a connection on $f^{++}Q$.

**Proof.** For $\xi \in \mathcal{T}_Y$, $f^{++}(\nabla)(\xi) = (0, f^*(\nabla)(df(\xi)), \xi)$. Then

$$\langle f^{++}(\nabla)(\xi), f^{++}(\nabla)(\eta) \rangle = \langle (0, f^*(\nabla)(df(\xi)), \xi), (0, f^*(\nabla)(df(\eta)), \eta) \rangle = 0.$$ 

$$\square$$

Therefore, the map $f$ induces the morphism of sheaves

(7.4.2) $$f^{++}: f^{-1}\mathcal{C}(Q) \to \mathcal{C}(f^{++}Q).$$

7.5. **Inverse image and linear algebra.** Suppose that $f: Y \to X$ is a map of manifolds and $A \in \mathcal{O}_X - \text{LieAlgd}$.

**Lemma 24.** Suppose that $Q$ is a Courant extension of $A$. Then, $f^{++}Q$ is a Courant extension of $f^+A$.

**Proof.** Follows from Proposition 6 and Lemma 17. 

$$\square$$
Thus, the inverse image functor for Courant algebroids induces a functor
\[(7.5.1)\quad f^{++} : \text{CExt}(\mathcal{A}) \to \text{CExt}(f^+ \mathcal{A})\]

**Proposition 25.** The functor \((7.5.1)\) is a morphism of \(\mathbb{C}\)-vector spaces in categories (and, in particular, of Picard groupoids).

**Proof.** Follows from Proposition 7 and Proposition 18. \(\square\)

7.6. **Exact Courant algebroids.** Suppose that \(\mathcal{Q}\) is an exact Courant algebroid on \(X\). Recall ([Bressler, 2007], 3.7) that the sheaf \(\mathcal{C}(\mathcal{Q})\) of locally defined connections on \(\mathcal{Q}\) together with the curvature map \(c : \mathcal{C}(\mathcal{Q}) \to \Omega^3_{X}^\text{cl}\) is a \((\Omega^2_{X} \to \Omega^3_{X}^\text{cl})\)-torsor and the assignment \(\mathcal{Q} \mapsto (\mathcal{C}(\mathcal{Q}), c)\) defines an equivalence \(\mathcal{ECA}(X) \to (\Omega^2_{X} \to \Omega^3_{X}^\text{cl})\)-torsors of \(\mathbb{C}\)-vector spaces in categories.

Suppose that \(f : Y \to X\) is a map of manifolds. The map \((7.4.2)\) is a morphism of torsors relative to the map \(f^* : f^{-1} \Omega^2_{X} \to \Omega^2_{Y}\), hence induces the morphism of \(\Omega^2_{Y}\)-torsors
\[(7.6.1)\quad f^{++} : f^* \mathcal{C}(\mathcal{Q}) := \Omega^2_{Y} \times_{f^{-1} \Omega^2_{X}} f^{-1} \mathcal{C}(\mathcal{Q}) \to \mathcal{C}(f^{++} \mathcal{Q}).\]

**Lemma 26.** The diagram
\[
\begin{array}{ccc}
  f^{-1} \mathcal{C}(\mathcal{Q}) & \xrightarrow{f^{++}} & \mathcal{C}(f^{++} \mathcal{Q}) \\
  f^*(c) \downarrow & & \downarrow c \\
  f^{-1} \Omega^3_{X}^\text{cl} & \xrightarrow{f^*} & \Omega^3_{Y}^\text{cl}
\end{array}
\]
is commutative.

**Proof.** By definition, for \(\nabla \in \mathcal{C}(\mathcal{Q})\), the curvature \(c(\nabla) \in \Omega^3_{X}^\text{cl}\) is defined by
\[
c(\nabla)(\xi, \eta) = [\nabla(\xi), \nabla(\eta)] - \nabla([\xi, \eta]),
\]
where \(\xi, \eta \in \mathcal{T}_X\).

For \(\xi, \eta \in \mathcal{T}_Y\), the computation
\[
c(f^{++}(\nabla))(\xi, \eta) = [f^{++}(\nabla(\xi)), f^{++}(\nabla(\eta))] - f^{++}(\nabla([\xi, \eta]))
\]
\[
= (0, [f^*(\nabla)(df(\xi)), f^*(\nabla)(df(\eta))), [\xi, \eta]) - (0, f^*(\nabla)(df([\xi, \eta])), [\xi, \eta])
\]
\[
= ((df)^\vee([f^*(\nabla)(df(\xi)), f^*(\nabla)(df(\eta))] - f^*(\nabla)(df([\xi, \eta]))), 0, 0)
\]
\[
= f^*(c(\nabla))(\xi, \eta)
\]
implies the claim. \(\square\)

**Proposition 27.** The diagram
\[
\begin{array}{ccc}
  \mathcal{ECA}(X) & \xrightarrow{(c,c)} & (\Omega^2_{X} \to \Omega^3_{X}^\text{cl})\text{-torsors} \\
  f^{++} & \downarrow & \downarrow f^* \\
  \mathcal{ECA}(Y) & \xrightarrow{(c,c)} & (\Omega^2_{Y} \to \Omega^3_{Y}^\text{cl})\text{-torsors}
\end{array}
\]
is commutative.

Proof. The map (7.6.1) provides the isomorphism $f^* \circ C \cong C \circ f^{++}$. Lemma 26 says that it is a morphism of $(\Omega^2_Y \to \Omega^3_{Y,cl})$-torsors. □

7.7. Smooth descent for Courant algebroids. Let $\mathcal{C}A$ denote the category with objects pairs $(X, Q)$, where $X$ is a manifold and $Q \in \mathcal{C}A(X)$. A morphism $u: (Y, Q') \to (X, Q)$ is a pair $u = (f, t)$, where $f: Y \to X$ is a map of manifolds and $t: Q' \to f^{++}Q$ is a morphism in $\mathcal{C}A(Y)$. Lemma 15 and the definition of the inverse image functor imply that the forgetful functor $(X, Q) \mapsto X$ makes $\mathcal{C}A$ a prefibered category over $\mathbf{Man}$.

Let $\mathcal{C}A_{lf}(X)$ denote the full subcategory of $\mathcal{C}A(X)$ with objects locally free of finite rank over $O_X$. This is an example of the framework of 5.1 with $P = \mathcal{C}A$, $P^\flat_X = \mathcal{C}A_{lf}(X)$ and the functor of inverse image defined in 7.1.

Corollary 28 (of Theorem 13). $\mathcal{C}A_{lf}$ has the smooth descent property.

7.8. Dirac structures with support. Suppose that $Z$ is a submanifold of $X$ and let $i: Z \to X$ denote the embedding. Let $Q$ be a Courant algebroid on $X$ locally free of finite rank over $O_X$. For an $O_X$-module $E$ and a submodule $F \subset i^*E$ we denote by $\tilde{F}$ the sub-module of $E$ defined by the Cartesian square

$$
\begin{array}{ccc}
\tilde{F} & \longrightarrow & E \\
\downarrow & & \downarrow \\
i_*F & \longrightarrow & i_*i^*E
\end{array}
$$

Definition 29 ((Alekseev and Xu, 2002; Bursztyn et al., 2009)). A Dirac structure in $Q$ supported on $Z$ is a sub-bundle $K \subset i^*Q$ which satisfies

1. $K$ is maximal isotropic with respect to the restriction of the symmetric pairing;
2. $K$ is mapped to $T_Z$ under (the restriction of) the anchor map;
3. the sheaf $\tilde{K}$ is closed under the bracket on $Q$.

We denote the collection of Dirac structures in $Q$ supported on $Z$ by $\text{Dir}(Q)$ and set $\text{Dir}(Q) := \text{Dir}_X(Q)$.

Remark 30. The second condition in Definition 29 is equivalent to $K \subset T_Z \times_{i^*T_X} i^*Q \subset i^*Q$.

Let $\mathcal{N}^\gamma_{Z/X}$ denote the conormal bundle defined by the exact sequence

$$
0 \to \mathcal{N}^\gamma_{Z/X} \to i^*\Omega^1_X \xrightarrow{d\gamma} \Omega^1_Z \to 0,
$$
i.e. $\mathcal{N}^\gamma_{Z/X} = \text{ann}(T_Z)$.

Suppose that $K$ is a Dirac structure supported on $Z$. In view of Remark 30 $K + \pi^i(\mathcal{N}^\gamma_{Z/X})$ is isotropic. Therefore, by maximality of $K$, $K = K + \pi^i(\mathcal{N}^\gamma_{Z/X})$, and $\pi^i(\mathcal{N}^\gamma_{Z/X}) \subset K$.

Recall that, by definition, $i^{++}Q = T_Z \times_{i^*T_X} i^*Q/\pi^i(\mathcal{N}^\gamma_{Z/X})$. Let $i^{++}K := K/\pi^i(\mathcal{N}^\gamma_{Z/X})$. 

Proposition 31.

(1) $i^+K$ is a Dirac structure in $i^+Q$.

(2) The assignment $K \mapsto i^+K$ defines a bijection between the set of almost Dirac structures in $Q$ supported on $Z$ and the set of almost Dirac structures in $i^+Q$.

Proof. Given such a $\mathcal{P} \subset i^+Q$, its pre-image in $i^+Q$, i.e. $\mathcal{P} + \pi^+(\mathcal{N}_Z^{\mathcal{P}})$, is a Dirac structure supported on $Z$. □

For $\phi \in \text{Hom}_{\mathcal{A}(X)}(Q_1, Q_2)$ the graph $\Gamma_\phi$ is a subsheaf of $Q_1 \times Q_2$. Since, by definition, $\phi$ induces the identity map on $T_X$, it follows that $\Gamma_\phi \subset Q_1 \times T_X Q_2$. Since, by definition, $\phi$ restricts to the identity map on $\Omega_X$, it follows that $\Gamma_\phi \cap (\Omega_X \times \Omega_X)$ is the diagonal. Therefore, the restriction of the map $Q_1 \times T_X Q_2 \to Q_1 + Q_2$ to $\Gamma_\phi$ is a monomorphism and (the image of) $\Gamma_\phi$ is a Dirac structure in $Q_1 + Q_2$. The assignment $\phi \mapsto \Gamma_\phi$ defines a canonical map $\text{Hom}_{\mathcal{A}(X)}(Q_1, Q_2) \to \text{Dir}(Q_1 + Q_2^{\text{op}})$.

7.9. Courant algebroid morphisms (Alekseev and Xu, 2002; Bursztyn et al., 2009).

Suppose that $f: Y \to X$ is a map of manifolds, $Q_X \in \mathcal{A}(X)$, $Q_Y \in \mathcal{A}(Y)$. Let $\text{pr}_X: Y \times X \to X$ and $\text{pr}_Y: Y \times X \to Y$ denote the projections; let $\gamma_f: Y \to Y \times X$ denote the graph embedding $y \mapsto (y, f(y))$.

The sheaf $\text{pr}_Y^*Q_Y \oplus \text{pr}_X^*Q_X^{\text{op}}$ is endowed with the canonical structure of a Courant algebroid on $Y \times X$ canonically isomorphic to $\text{pr}_Y^+Q_Y + \text{pr}_X^+Q_X^{\text{op}}$.

According to Proposition 31 a Courant algebroid morphism (Alekseev and Xu, 2002; Bursztyn et al., 2009) $K \in \text{Dir}_f(Y)(\text{pr}_Y^+Q_Y + \text{pr}_X^+Q_X^{\text{op}})$ corresponds to the Dirac structure $\gamma_f(Y)^{++}K \in \text{Dir}(\gamma_f^{++}(\text{pr}_Y^+Q_Y + \text{pr}_X^+Q_X^{\text{op}})) \cong \text{Dir}(Q_Y + f^+Q_X^{\text{op}})$ and there is a canonical map $\text{Hom}_{\mathcal{A}(X)}(Q_Y, f^+Q_X) \to \text{Dir}_f(Y)(\text{pr}_Y^+Q_Y + \text{pr}_X^+Q_X^{\text{op}})$.

References

Alekseev A., and Xu P. “Derived brackets and Courant algebroids.” Unpublished (2002).

Bressler, P., and Chervov A. “Courant algebroids” Journal of Mathematical Science, Vol 128, no. 4, (2005): 3030 – 3053. https://arxiv.org/abs/hep-th/0212195.

Bressler P. “The first Pontryagin class.” Compositio Math. 143, (2007): 1127–1163.

Bressler P., and Rengifo C. “On higher-dimensional Courant algebroids.” Letters in Math. Phys. 108, (2018): 2099–2137.

Beilinson A., and Bernstein J. “A proof of Jantzen conjectures.” Advances in Soviet mathematics 16, no. 1 (1993): 1–50.

Bursztyn H., Iglesias Ponte D., and Ševera P. “Courant morphisms and moment maps.” Math. Research Letters 16, no. 2 (2009): 215–232.

Courant T., “Dirac manifolds”. Trans. Amer. Math. Soc., 319 (2), 631–661 (1990).
Dorfman I., “Dirac structures and integrability of nonlinear evolution equations”. Nonlinear Science: Theory and Applications, John Wiley and Sons Ltd., Chichester (1993).

Liu Z.-J., Weinstein A., and Xu P., “Manin triples for Lie bialgebroids”. Journal of Differential Geometry, 45, 547–574 (1997).

Li-Bland D., and Meinrenken E. “Courant algebroids and Poisson geometry.” Int. Math. Research Notices 11, (2009): 2106–2145.

Roytenberg D., “On the structure of graded symplectic supermanifolds and Courant algebroids”. Quantization Poisson bracket and Beyond, Manchester (2001), Th Voronov (ed.), Contemp. Math., Vol. 315, Amer. Math. Soc., Providence, RI (2002).

ˇSevera P., “Some title containing the words ”homotopy” and ”symplectic”, e.g. this one”. Travaux mathematiques. Fasc. XVI, Univ. Luxemb. Luxembourg (2005).

ˇSevera P., and Valach F. “Courant algebroids, Poisson-Lie T-duality, and type II supergravities.” arXiv preprint arXiv:1810.07763 (2018).

Vysoky J. “Hitchhiker’s Guide to Courant Algebroid Relations.” arXiv preprint arXiv:1910.05347 (2019).

Universidad de los Andes, Bogotá, Colombia
E-mail address: paul.bressler@gmail.com

Universidad de La Sabana, Chía, Colombia
E-mail address: camiloregu@unisabana.edu.co