PAULI GRAPHS, RIEMANN HYPOTHESIS, AND GOLDBACH PAIRS

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We consider the Pauli group $P_q$ generated by unitary quantum generators $X$ (shift) and $Z$ (clock) acting on vectors of the $q$-dimensional Hilbert space. It has been found that the number of maximal mutually commuting sets within $P_q$ is controlled by the Dedekind psi function $\psi(q)$ and that there exists a specific inequality involving the Euler constant $\gamma \approx 0.577$ that is only satisfied at specific low dimensions $q \in A = \{2, 3, 4, 5, 6, 8, 10, 12, 18, 30\}$. The set $A$ is closely related to the set $A \cup \{1, 24\}$ of integers that are totally Goldbach, i.e., that consist of all primes $p < n - 1$ with $p$ not dividing $n$ and such that $n - p$ is prime.

In the extreme high-dimensional case, at primorial numbers $N_r$, the Hardy–Littlewood function $R(q)$ is introduced for estimating the number of Goldbach pairs, and a new inequality (Theorem 4) is established for the equivalence to the Riemann hypothesis in terms of $R(N_r)$. We discuss these number-theoretical properties in the context of the qudit commutation structure.

Keywords: Riemann hypothesis, Goldbach pair, generalized Pauli group, qudit commutation structure

1. Introduction

We propose new connections between the Pauli graphs [1], [2], which encode the commutation relations of qudit observables, and prime number theory. We already emphasized that the Dedekind psi function $\psi(q) = q \prod_{p|q} \left(1 + \frac{1}{p}\right)$ (where $p$ is prime) is used to count the number of maximal commuting sets of qudits [1] and corresponds to the Riemann hypothesis (RH) at primorial numbers $q \equiv N_r = 2 \cdot \cdots \cdot p_r$ [3]. Similarly, there exist striking connections between $\psi(q)$ and the Hardy–Littlewood function $g(q) = R(q) \cdot \frac{q}{\ln^2 q}$ for the Goldbach distribution of prime pairs (see Sec. 3 for the definition of $R(q)$). In particular, we observe that $\psi(q)$ corresponds to the so-called totally Goldbach numbers at small $q$ and that $R(q) < \zeta(2) \cdot \psi(q)/q$ also corresponds to the RH at primorial numbers. The Euler constant

$$\gamma = \lim_{n \to \infty} \left( \sum_{k=1}^{n} \frac{1}{k} - \ln n \right) \sim 0.57721$$

via the Mertens formula

$$e^\gamma = \lim_{n \to \infty} \frac{1}{\ln n} \prod_{p \leq n} \left(1 - \frac{1}{p}\right)^{-1}$$

is an important ingredient of all the inequalities involved in this correspondence.

In Sec. 2, we report on the number-theoretical coincidence between $\psi(q)$ and the totally Goldbach numbers at small $q$ and also on the already known theorem relating $\psi(q)$ and the RH at primorial numbers $N_r$. In Sec. 3, we explore this coincidence in detail by referring to the qudit Pauli graphs. In Sec. 4, we establish the relation between $R(q)$ and RH at primorial numbers. In the discussion, we propose the concept of a Goldbach defect for encompassing the statements at low and high $q$.

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2. A number-theoretical coincidence

We start the exposition of our ideas with a few definitions and the already established Theorem 1.

Goldbach’s conjecture, formulated in 1742, is that every even integer greater than 2 is the sum of two primes. To date, it has been checked for \( q \) up to \( 2 \cdot 10^{18} [4] \). A pair \((p_1, p_2)\) of primes such that the even integer \( n = p_1 + p_2 \) is called a Goldbach partition.

**Definition 1.** A positive integer \( n \) is totally Goldbach if \( n - p \) is prime for all primes \( p < n - 1 \) such that \( p \) does not divide \( n \) (except when \( p = n - p \)) [5].

The inequality

\[
\frac{\psi(q)}{q} > e^{\gamma} \ln \ln q \tag{1}
\]

is only satisfied at a totally Goldbach number \( q \in \mathcal{A} \), where

\[
\mathcal{A} = \{2, 3, 4, 5, 6, 8, 10, 12, 18, 30\}. 
\]

The only totally Goldbach numbers not satisfying the inequality are \( q = 1 \) and \( q = 24 \). This follows from a combination of results in [3] and [5] (see [6] for a more general setting).

**Definition 2.** A positive integer \( n \) is almost totally Goldbach of index \( r \) if \( n - p \) is prime for all primes \( p < n - 1 \) such that \( p \) does not divide \( n \) (except when \( p = n - p \)) with \( r \) exceptions.

| Table 1          | set of almost totally Goldbach numbers |
|------------------|----------------------------------------|
| \( \text{index } r \) | \( \mathcal{A}_0 = \{1, 2, 3, 4, 5, 6, 8, 10, 12, 18, 24, 30\} \) |
| 1                | \( \mathcal{A}_1 = \mathcal{A}_0 \cup \{7, 9, 14, 16, 20, 36, 42, 60\} \) |
| 2                | \( \mathcal{A}_2 = \mathcal{A}_1 \cup \{15, 22, 48, 90\} \) |
| 3                | \( \mathcal{A}_3 = \mathcal{A}_2 \cup \{13, 26, 28, 34, 54, 66, 84, 120\} \) |
| 4                | \( \mathcal{A}_4 = \mathcal{A}_3 \cup \{11, 21, 40, 78, 210\} \) |
| 5                | \( \mathcal{A}_5 = \mathcal{A}_4 \cup \{19, 32, 44, 50, 72\} \) |
| 6                | \( \mathcal{A}_6 = \mathcal{A}_5 \cup \{17, 25, 46, 70, 102, 114\} \) |
| 7                | \( \mathcal{A}_7 = \mathcal{A}_6 \cup \{33, 38, 52, 64, 126, 150\} \) |
| 8                | \( \mathcal{A}_8 = \mathcal{A}_7 \cup \{23, 27, 31, 39, 56, 58, 96\} \) |
| 9                | \( \mathcal{A}_9 = \mathcal{A}_8 \cup \{29, 35, 76, 108, 168, 180\} \) |
| 10               | \( \mathcal{A}_{10} = \mathcal{A}_9 \cup \{45, 74, 132, 144\} \) |

Almost totally Goldbach numbers of index \( r \leq 10 \).

Let \( g(n) \) be the number of ways of representing the integer \( n \) as the sum of two primes. The maximum value of \( g(n) \) is indeed less than or equal to the number of primes \( n/2 \leq p \leq n - 1 \). Values of \( n \) such that \( g(n) \) reaches its maximum are in the set

\[
\mathcal{B} = \mathcal{A}_0 \cup \{7, 14, 16, 36, 42, 48, 60, 90, 210\},
\]

where \( \mathcal{A}_0 \) is the set of totally Goldbach numbers [7]. It is not surprising that numbers in \( \mathcal{B} \) that are not totally Goldbach are almost totally Goldbach with a small index \( r \) (as can be seen in Table 1). The first five and the integer 60 have the index 1, while 48 and 90 have the index 2 and 210 has the index 4. For a prime number \( p > 3 \), the index \( r(p) \) is Sloane’s sequence A062302. This number-theoretical coincidence is further explored in Sec. 3 in the context of the qudit commutation structure.