TIGHT CONTACT SMALL SEIFERT SPACES WITH $e_0 \neq 0, -1, -2$

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Abstract. We classify up to isotopy the tight contact structures on small Seifert spaces with $e_0 \neq 0, -1, -2$.

1. Introduction and Statements of Results

A contact structure $\xi$ on an oriented 3-manifold $M$ is a nowhere integrable tangent plane distribution, i.e., near any point of $M$, $\xi$ is defined locally by a 1-form $\alpha$, s.t., $\alpha \wedge d\alpha \neq 0$. Note that the orientation of $M$ given by $\alpha \wedge d\alpha$ depends only on $\xi$, not on the choice of $\alpha$. $\xi$ is said to be positive if this orientation agrees with the native orientation of $M$, and negative if not. A contact structure $\xi$ is said to be co-orientable if $\xi$ is defined globally by a 1-form $\alpha$. Clearly, a co-orientable contact structure is orientable as a plane distribution, and a choice of $\alpha$ determines an orientation of $\xi$.

Unless otherwise specified, all contact structures in this paper will be co-oriented and positive, i.e., with a prescribed defining form $\alpha$ such that $\alpha \wedge d\alpha > 0$. A curve in $M$ is said to be Legendrian if it is tangent to $\xi$ everywhere. $\xi$ is said to be overtwisted if there is an embedded disk $D$ in $M$ such that $\partial D$ is Legendrian, but $D$ is transversal to $\xi$ along $\partial D$. A contact structure that is not overtwisted is called tight. Overtwisted contact structures are classified up to isotopy by Eliashberg in [2]. Classifying tight contact structures up to isotopy is much more difficult. Such classifications are only known for very limited classes of 3-manifolds. (See, e.g., [3], [4], [5], [6], [7], [12], [13], [15].)

For a small Seifert space $M = M(r_1, r_2, r_3)$, define $e_0(M) = \lfloor r_1 \rfloor + \lfloor r_2 \rfloor + \lfloor r_3 \rfloor$, where $\lfloor x \rfloor$ is the greatest integer not greater than $x$. $e_0(M)$ is an invariant of $M$, i.e., it does not depend on the choice of the representatives $(r_1, r_2, r_3)$.

The following are our main results.

Theorem 1.1. Let $M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3})$ be a small Seifert space, where $p_i$ and $q_i$ are integers, s.t., $p_i \geq 2$, $q_i \geq 1$ and $\text{g.c.d.}(p_i, q_i) = 1$. Assume that, for $i = 1, 2, 3$, $-\frac{q_i}{p_i} = <a_0^{(i)}, a_1^{(i)}, \ldots, a_{m_i}^{(i)}>$, where all $a_j^{(i)}$'s are integers, $a_0^{(i)} = -([\frac{q_i}{p_i}] + 1) \leq -1$ and $a_j \leq -2$ for $j \geq 1$. We denote by $<a_0, a_1, \ldots, a_{m_i}>$ the right hand side of equation (1).

The following are our main results.
Then, up to isotopy, there are exactly \(|(e_0(M) + 1) \prod_{i=1}^{3} \prod_{j=1}^{m_i} (a_j^{(i)} + 1)|\) tight contact structures on \(M\). All these tight contact structures are constructed by Legendrian surgeries of \((S^3, \xi_{st})\), and are therefore holomorphically fillable.

**Theorem 1.2.** Let \(M = M(\frac{q_1}{p_1}, \frac{q_2}{p_2}, e_0 + \frac{q_3}{p_3})\) be a small Seifert space, where \(p_i, q_i\) and \(e_0\) are positive integers, s.t., \(p_i > q_i\) and \(\text{g.c.d.}(p_i, q_i) = 1\). Assume that, for \(i = 1, 2, 3, -\frac{p_i}{q_i} = < b_{0}^{(i)}, b_{1}^{(i)}, \cdots, b_{l_i}^{(i)} >\), where all \(b_{j}^{(i)}\)'s are integers less than or equal to \(-2\). Then, up to isotopy, there are exactly \(|\prod_{i=1}^{3} b_{0}^{(i)} \prod_{j=1}^{l_i} (b_{j}^{(i)} + 1)|\) tight contact structures on \(M\). All these tight contact structures are constructed by Legendrian surgeries of \((S^3, \xi_{st})\), and are therefore holomorphically fillable.

Clearly, Theorems 1.1 and 1.2 give complete classifications of tight contact structures on small Seifert spaces with \(e_0 \neq 0, -1, -2\).

### 2. Continued Fractions

In this section, we establish some properties of continued fractions, which will be used in the proofs of Theorems 1.1 and 1.2.

**Lemma 2.1.** Let \(a_0, a_1, \cdots, a_m\) be real numbers such that \(a_0 \leq -1\), and \(a_j \leq -2\) for \(1 \leq j \leq m\). Define \(\{p_j\}\) and \(\{q_j\}\) by

\[
\begin{cases}
  p_j = -a_j p_{j-1} - p_{j-2}, & j = 0, 1, \cdots, m, \\
  p_{-2} = -1, & p_{-1} = 0,
\end{cases}
\]

\[
\begin{cases}
  q_j = -a_j q_{j-1} - q_{j-2}, & j = 0, 1, \cdots, m, \\
  q_{-2} = 0, & q_{-1} = 1.
\end{cases}
\]

Then, for \(1 \leq j \leq m\), we have

1. \(-\frac{q_j}{p_j} = < a_0, a_1, \cdots, a_j >\),
2. \(p_j \geq p_{j-1} > 0, q_j \geq q_{j-1} > 0\),
3. \(p_j q_{j-1} - p_{j-1} q_j = 1\),
4. \(-\frac{q_j + (a_0 + 1)p_j}{q_{j-1} + (a_0 + 1)p_{j-1}} = < a_j, a_{j-1}, \cdots, a_2, a_1 + 1 >\).

**Proof.** By the definitions of \(\{p_j\}\) and \(\{q_j\}\), we have \(p_0 = 1, q_0 = -a_0, p_1 = -a_1,\) and \(q_1 = a_0a_1 - 1\). Then it’s easy to check that the lemma is true for \(j = 1\). Assume that
the lemma is true for $j - 1 \geq 1$. Then,
\[
< a_0, a_1, \cdots, a_j > = < a_0, a_1, \cdots, a_{j-1} - \frac{1}{a_j} >
\]
\[
= -\frac{(a_{j-1} - \frac{1}{a_j})q_{j-2} - q_{j-3}}{(a_{j-1} - \frac{1}{a_j})p_{j-2} - p_{j-3}}
\]
\[
= -\frac{(a_ja_{j-1} - 1)q_{j-2} + a_jq_{j-3}}{(a_ja_{j-1} - 1)p_{j-2} + a_jp_{j-3}}
\]
\[
= a_j(a_{j-1}q_{j-2} + q_{j-3}) - q_{j-2}
\]
\[
= \frac{a_j(a_{j-1}p_{j-2} + p_{j-3}) - p_{j-2}}{-a_jq_{j-1} - q_{j-2}}
\]
\[
= -\frac{q_j}{p_j}.
\]

Also, since $q_{j-1} \geq q_{j-2} > 0$ and $-a_j \geq 2$, we have $q_j = -a_jq_{j-1} - q_{j-2} \geq 2q_{j-1} - q_{j-2} \geq q_{j-1} > 0$, and, similarly, $p_j \geq p_{j-1} > 0$.

Furthermore, by definitions of \{p_j\} and \{q_j\},
\[
p_jq_{j-1} - p_{j-1}q_j = (-a_jp_{j-1} - p_{j-2})q_{j-1} - p_{j-1}(-a_jq_{j-1} - q_{j-2})
\]
\[
= p_{j-1}q_{j-2} - p_{j-2}q_{j-1}
\]
\[
= 1.
\]

Finally,
\[
-\frac{q_j + (a_0 + 1)p_j}{q_{j-1} + (a_0 + 1)p_{j-1}}
\]
\[
= \frac{(-a_jq_{j-1} - q_{j-2}) + (a_0 + 1)(-a_jp_{j-1} - p_{j-2})}{q_{j-1} + (a_0 + 1)p_{j-1}}
\]
\[
= \frac{a_j(q_{j-1} + (a_0 + 1)p_{j-1}) + (q_{j-2} + (a_0 + 1)p_{j-2})}{q_{j-1} + (a_0 + 1)p_{j-1}}
\]
\[
= a_j - \frac{1}{< a_{j-1}, \cdots, a_2, a_1 + 1 >}
\]
\[
= < a_j, a_{j-1}, \cdots, a_2, a_1 + 1 > .
\]

This shows that the lemma is also true for $j$.

\[\square\]

**Remark 2.2.** In the proof of Theorem 1.1 and 1.2 all the $a_j$’s will be integers, and so will the corresponding $p_j$’s and $q_j$’s be. Then, property (3) in Lemma 2.1 implies that $g.c.d.(p_j, q_j) = 1$.

3. The $e_0 \leq -3$ Case

In the rest of this paper, we let $\Sigma$ be a three hole sphere, and $-\partial \Sigma \times S^1 = T_1 + T_2 + T_3$, where the ”-” sign means reversing the orientation. We identify $T_i$ to $\mathbb{R}^2 / \mathbb{Z}^2$ by identifying the corresponding component of $-\partial \Sigma \times \{pt\}$ to $(1,0)^T$, and $\{pt\} \times S^1$ to $(0,1)^T$. Also, for $i = 1, 2, 3$, let $V_i = D^2 \times S^1$, and identify $\partial V_i$ with $\mathbb{R}^2 / \mathbb{Z}^2$ by identifying a meridian $\partial D^2 \times \{pt\}$ with $(1,0)^T$ and a longitude $\{pt\} \times S^1$ with $(0,1)^T$. 
Following Honda, we call a convex torus minimal if it has only two dividing curves.

**Proof of Theorem 1.1.** Define \( \{ p_j^{(i)} \} \) and \( \{ q_j^{(i)} \} \) by

\[
\begin{align*}
p_j^{(i)} &= -a_j p_{j-1}^{(i)} - p_{j-2}^{(i)}, \quad j = 0, 1, \ldots, m_i, \\
p_{-2}^{(i)} &= -1, \quad p_{-1}^{(i)} = 0,
\end{align*}
\]

\[
\begin{align*}
q_j^{(i)} &= -a_j q_{j-1}^{(i)} - q_{j-2}^{(i)}, \quad j = 0, 1, \ldots, m_i, \\
q_{-2}^{(i)} &= 0, \quad q_{-1}^{(i)} = 1.
\end{align*}
\]

By Lemma 2.1 and Remark 2.2, we have \( p_i = p_{m_i}^{(i)} \) and \( q_i = q_{m_i}^{(i)} \). Let \( u_i = p_{m_i-1}^{(i)} \) and \( v_i = q_{m_i-1}^{(i)} \). Then \( p_i \geq u_i > 0, \ q_i \geq v_i > 0, \) and \( p_i v_i - q_i u_i = 1 \).

Define an orientation preserving diffeomorphism \( \varphi_i : \partial V_i \to T_i \) by

\[
\varphi_i = \left( \begin{array}{cc} p_i & u_i \\ q_i & v_i \end{array} \right).
\]

Then

\[
M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3).
\]

Let \( \xi \) be a tight contact structure on \( M \). We first isotope \( \xi \) to make each \( V_i \) a standard neighborhood of a Legendrian circle \( L_i \) isotopic to the \( i \)-th singular fiber with twisting number \( t_i < -2 \), i.e., \( \partial V_i \) is convex with two dividing curves each of which has slope \( \pm \frac{1}{k} \) when measured in the coordinates of \( \partial V_i \) given above. Let \( s_i \) be the slope of the dividing curves of \( T_i = \partial V_i \) measured in the coordinates of \( T_i \). Then we have that

\[
s_i = \frac{t_i q_i + v_i}{t_i p_i + u_i} = \frac{q_i}{p_i} + \frac{1}{p_i(t_i p_i + u_i)}.
\]

The fact \( t_i < -2 \) implies that \( \left\lfloor \frac{1}{p_i} \right\rfloor < s_i < \frac{1}{p_i} \).

After a possible slight isotopy supported in a neighborhood of \( T_i = \partial V_i \), we assume that \( T_i \) has Legendrian ruling of slope \( \infty \) when measured in the coordinates of \( T_i \). For each \( i \), pick a Legendrian ruling \( L_i \) on \( T_i \). Choose a convex vertical annulus \( A \subset \Sigma \times S^1 \), such that \( \partial A = L_1 \cup L_2 \), and the interior of \( A \) is contained in the interior of \( \Sigma \times S^1 \). By Theorem 1.4 of [18], \( \xi \) does not admit Legendrian vertical circles with twisting number 0. So there must be dividing curves of \( A \) that connect the two boundary components of \( A \). We isotope \( T_1 \) and \( T_2 \) by adding to them the bypasses corresponding to the \( \partial \)-parallel dividing curves of \( A \). Since bypass adding is done in a small neighborhood of the bypass and the original surface, we can keep \( V_i \)'s disjoint during this process. Also \( T_1 \) remains minimal after each bypass adding. After we depleted all the \( \partial \)-parallel dividing curves of \( A \), each of the remaining dividing curves connects the two boundary components of \( A \). So the slopes of the dividing curves of \( T_1 \) and \( T_2 \) after the isotopy are \( s_1' = \frac{k}{k} \) and \( s_2' = \frac{k}{k} \), where \( k \geq 1 \) and \( g.c.d.(k, k) = 1 \) for \( i = 1, 2 \). Since \( \left\lfloor \frac{1}{p_i} \right\rfloor < s_i \), we have that, for \( i = 1, 2 \), \( s_i' \geq \left\lfloor \frac{1}{p_i} \right\rfloor \geq 0 \), and, hence \( k_i \geq 0 \). This is because that, by Lemma 3.15 of [12], \( s_i' < \left\lfloor \frac{1}{p_i} \right\rfloor \) implies \( s_i' = \infty \) which contradicts Theorem 1.4 of [18]. Now, cut \( M \) open along \( A \cup T_1 \cup T_2 \) and round the edges. We get a convex torus isotopic to \( T_3 \) with two dividing curves of slope \( -\frac{k_1 + k_2 + 1}{k} \) when measure in the
coordinates of $T_3$. When measured in the coordinates of $\partial V_3$, these dividing curves have slope $-\frac{kq_i+(k_j+k_3)p_3}{k_i+(k_j+k_3)p_3}$. It’s easy to check that $-\frac{kq_i+(k_j+k_3)p_3}{k_i+(k_j+k_3)p_3} < -\frac{q_3}{p_3}$. So, by Theorem 4.16 of [12], we can isotope $\partial V_3$ so that it has two dividing curves of slope $-\frac{q_3}{p_3}$. Measured in the coordinates of $T_3$, the slope is 0. This implies that the maximal twisting number of a Legendrian vertical circle is $-1$.

After an isotopy of $\xi$, we can find a Legendrian vertical circle $L$ in the interior of $\Sigma \times S^1$ with twisting number $-1$, and, again, make each $V_i$ a standard neighborhood of a Legendrian circle $L_i$ isotopic to the $i$-th singular fiber with twisting number $t_i < -2$. As before, we can assume that $T_i$ has Legendrian ruling of slope $\infty$ when measured in the coordinates of $T_i$. Let $L_i$ be a Legendrian ruling of $T_i$. For each $i$, we choose a convex vertical annulus $A_i \subset \Sigma \times S^1$, s.t., $\partial A_i = L_i \cup \partial V_i$, the interior of $A_i$ is contained in the interior of $\Sigma \times S^1$, and $A_i \cap A_j = L$ when $i \neq j$. $A_i$ has $\partial$-parallel dividing curves on the $L$ side since $t(L)$ is maximal. So the dividing set of $A_i$ consists of two curves connecting $L$ to $L_i$ and possibly some $\partial$-parallel curves on the $L_i$ side. We now isotope $T_i$ by adding to it the bypasses corresponding to these $\partial$-parallel dividing curves, and keep $V_i$’s disjoint in this process. After this isotopy, we get a convex decomposition

$$M = M(-\frac{q_1}{p_1}, -\frac{q_2}{p_2}, -\frac{q_3}{p_3}) \cong (\Sigma \times S^1) \cup (V_1 \cup V_2 \cup V_3)$$

of $M$, where each $T_i$ has two dividing curves of slope $\frac{q_3}{p_3}$ when measured in the coordinate of $T_i$. When measured in coordinates of $\partial V_i$, the slope of the dividing curves becomes $-\frac{q_i - \frac{q_3}{p_3}p_i}{v_i - \frac{q_3}{p_3}u_i} = -\frac{q_i + (a_i^{(3)} + 1)p_i}{v_i + (a_i^{(3)} + 1)u_i}$.

By part (4) of Lemma 5.1 of [13], there are exactly $2 + \lfloor \frac{q_3}{p_1} \rfloor + \lfloor \frac{q_3}{p_2} \rfloor + \lfloor \frac{q_3}{p_3} \rfloor = \lfloor e_0(M) + 1 \rfloor$ tight contact structures on $\Sigma \times S^1$ satisfying the boundary condition and admitting no Legendrian vertical circles with twisting number $0$. By Theorem 2.3 of [12] and part (4) of Lemma 2.1 there are exactly $\lfloor \prod_{j=1}^{m_i} (a_j^{(3)} + 1) \rfloor$ tight contact structures on $V_i$ satisfying the boundary condition. Thus, up to isotopy, there are at most $\lfloor (e_0(M) + 1) \prod_{j=1}^{3} (a_j^{(3)} + 1) \rfloor$ tight contact structures on $M$.

It remains to construct $\lfloor (e_0(M) + 1) \prod_{j=1}^{3} (a_j^{(3)} + 1) \rfloor$ tight contact structures on $M$ by Legendrian surgeries of $(S^3, \xi_{st})$. We begin with the standard surgery diagram of $M = M(-\frac{p_1}{q_1}, -\frac{p_2}{q_2}, -\frac{p_3}{q_3})$. Then, for each $i$, perform an $a_i^{(3)}$-Rolfsen twist on the $\frac{p_i}{q_i}$-component. Since $a_0^{(1)} + a_0^{(2)} + a_0^{(3)} = e_0(M)$ and $\frac{p_i}{q_i} \leq \frac{1}{a_i^{(1)}, \cdots, a_i^{(m_i)}}$, the new surgery coefficients of the four components are $e_0(M)$, $< a_1^{(1)}, \cdots, a_{m_1}^{(1)} >$, $< a_1^{(2)}, \cdots, a_{m_2}^{(2)} >$, and $< a_1^{(3)}, \cdots, a_{m_3}^{(3)} >$. Now, we perform (inverses of) the slam-dunks corresponding to the three continued fractions here, which lead us to the diagram at the bottom of Figure 11. Since the maximal Thurston-Bennequin number of an unknot in $(S^3, \xi_{st})$ is $-1$, there are $\lfloor (e_0(M) + 1) \prod_{j=1}^{3} (a_j^{(3)} + 1) \rfloor$ ways to realize this diagram by Legendrian surgeries. According to Proposition 2.3 of [10] and Theorem 1.2 of [17], each of these Legendrian surgeries gives a non-isotopic tight contact structure on $M$. 


Thus, up to isotopy, there are exactly \(|(e_0(M) + 1) \prod_{i=1}^{3} \prod_{j=1}^{m_i} (a_j^{(i)} + 1)| \) tight contact structures on \(M\).

\[\square\]

4. The \(e_0 \geq 1\) Case

Following [18], we call a tight contact structure on \(\Sigma \times S^1\) appropriate is there are no embedded thickened tori in \(\Sigma \times S^1\) with I-twisting \(\geq \pi\) and parallel to one of the boundary components.

The following lemma is a reformulation of parts (1), (2) and (3) of Lemma 5.1 of [13].

**Lemma 4.1.** Let \(\xi\) be an appropriate contact structure on \(\Sigma \times S^1\) with minimal convex boundary that admits a vertical Legendrian circle with twisting number 0. Assume that dividing curves of \(T_1, T_2\) and \(T_3\) are of slopes \(-1, -1, -n\), respectively, where \(n\) is an integer greater than 1. Then there is a factorization \(\Sigma \times S^1 = L_1 \cup L_2 \cup L_3 \cup (\Sigma' \times S^1)\), where \(L_i's\) are embedded thickened tori with minimal twisting and minimal convex boundary \(\partial L_i = T_i' - T_i\), s.t., dividing curves of \(T_i'\) have slope \(\infty\). The appropriate contact structure \(\xi\) is uniquely determined by the signs of the basic slices \(L_1, L_2\) and \(L_3\). The sign convention here is given by associating \((0, 1)^T\) to \(T_i'\).
Proof. We only prove the last sentence. The rest is just part (1) of Lemma 5.1 of [13]. Let \( \Sigma_0 \) be a properly embedded three hole sphere in \( \Sigma \times S^1 \) isotopic to \( \Sigma \times \{ \text{pt} \} \), and \( \Sigma'_0 = \Sigma_0 \cap (\Sigma' \times S^1) \). We isotope \( \Sigma_0 \) so that \( \Sigma_0 \) and \( \Sigma'_0 \) are convex with Legendrian boundaries that intersect the dividing curves of \( \partial \Sigma \times S^1 \) and \( \partial \Sigma' \times S^1 \) efficiently. Then each component of \( \partial \Sigma'_0 \) intersects the dividing curves of \( \Sigma'_0 \) twice. Since \( \xi \) is appropriate, \( \Sigma'_0 \) has no \( \partial \)-parallel dividing curves. This implies that, up to isotopy relative to boundary and Dehn twists parallel to boundary components, there are only two configurations of dividing curves on \( \Sigma'_0 \). (See Figure 2.) Thus, there are only two tight contact structure on \( \Sigma' \times S^1 \), up to isotopy relative to boundary and full horizontal rotations of each boundary component.

Let \( A_i = \Sigma_0 \cap L_i \). Then the dividing set of each of \( A_1 \) and \( A_2 \) consists of two arcs connecting the two boundary components. And the dividing set of \( A_3 \) consists of two arcs connecting the two boundary components and \( n-1 \) \( \partial \)-parallel arcs on the \( T_3 \) side. From the relative Euler class of \( \xi|_{L_3} \), one can see that the half discs bounded by these \( \partial \)-parallel arcs must be pairwise disjoint and of the sign opposite to that of \( L_3 \). By isotoping \( \Sigma_0 \) relative to \( \Sigma'_0 \), we can freely choose the holonomy of the non-\( \partial \)-parallel dividing curves of each \( A_i \). This implies that, up to isotopy relative to boundary, there are only two possible configurations of dividing curves on \( \Sigma_0 \) when the signs of \( L_i \)'s are given. (See Figure 3)

When the signs of \( L_i \)'s are mixed, we can extend \((\partial \Sigma \times S^1, \xi)\) to a universally tight contact manifold \((\partial \Sigma'' \times S^1, \xi'')\) by gluing to \( T_i \) a basic slice \( L_i'' \) of the same sign as \( L_i \) for each \( i \), where \( L_i'' \) has minimal convex boundary \( \partial L_i'' = T_i - T_i'' \), and the dividing curves of \( T_i'' \) are vertical. Extend \( \Sigma_0 \) across \( L_i'' \) to \( \Sigma_0'' \) so that \( \Sigma_0'' \) is convex with Legendrian boundary intersecting the dividing curves of \( T_i'' \) efficiently. For \( i = 1, 2 \), the dividing set of \( \Sigma_0'' \cap L_i'' \) consists of 1 \( \partial \)-parallel arcs on each boundary component. From the relative Euler class of \( \xi''|_{L_i''} \), we can see that the half discs on \( \Sigma_0'' \cap L_i'' \) bounded by these
Figure 3. Possible configurations of dividing curves on $\Sigma_0$. Here, $n = 3$, and the layer $L_3$ is positive.

$\partial$-parallel arcs are of the same sign as the basic slice $L_i$. The dividing set of $\Sigma_0'' \cap L_3''$ consists of $n$ $\partial$-parallel arcs on the $T_3$ side and 1 $\partial$-parallel arc on the $T_2$ side. From the relative Euler class of $\xi''|_{L_3''}$, we can see that the half discs on $\Sigma_0'' \cap L_3''$ bounded by these $\partial$-parallel arcs are pairwise disjoint and of the same sign as the basic slice $L_3$.

Now, one can see that, after the extension, the two possible configurations of dividing curves on $\Sigma_0$ become the same minimal configuration of dividing curves on $\Sigma_0''$. (See Figure 4.) By Lemma 5.1 of [13], the two configurations correspond to the same universally tight contact structure on $\Sigma \times S^1$. This shows that, when the signs of $L_i$'s are mixed, $\xi$ is uniquely determined by the signs of $L_i$'s. When all the $L_i$'s have the same sign, $\xi$ is virtually overtwisted, and the isotopy type relative to boundary of such a contact structure is determined by the action of the relative Euler class on $\Sigma_0$, which is, in turn, determined by the sign of $L_3$. Thus, when all the $L_i$'s have the same sign, this common sign determines $\xi$.

Proof of Theorem 1.2 Define \{${p_j}^{(i)}$\} and \{${q_j}^{(i)}$\} by

\[
\begin{cases}
{p_j}^{(i)} = -b_j^{(i)}{p_j}^{(i)} - {p_{j-2}}^{(i)}, & j = 0, 1, \ldots, l_i, \\
{p_{-2}}^{(i)} = 0, & {p_{-1}}^{(i)} = 1,
\end{cases}
\]

\[
\begin{cases}
{q_j}^{(i)} = -b_j^{(i)}{q_j}^{(i)} - {q_{j-2}}^{(i)}, & j = 0, 1, \ldots, l_i, \\
{q_{-2}}^{(i)} = -1, & {q_{-1}}^{(i)} = 0.
\end{cases}
\]

By Lemma 2.1 and Remark 2.2 we have $p_i = {p_i}^{(i)}$ and $q_i = {q_i}^{(i)}$. Let $u_i = -{p_i}^{(i)}$ and $v_i = -{q_i}^{(i)}$. Then $p_iv_i - q_ii = 1$. 


Figure 4. After extending to $\Sigma_{0}''$, the two possible configurations become the same. Here, $n = 3$, and the signs of the layers $L_1$, $L_2$ and $L_3$ are $(-, - , +)$, respectively.

Define an orientation preserving diffeomorphism $\varphi_i : \partial V_i \rightarrow T_i$ by

$$\varphi_i = \begin{cases} 
\left( \begin{array}{c}
  p_i \\
  q_i \\
  v_i \\
  u_i
\end{array} \right), & i = 1, 2; \\
\left( \begin{array}{c}
  p_3 \\
  q_3 - \epsilon_0 p_3 \\
  v_3 + \epsilon_0 u_3
\end{array} \right), & i = 3.
\end{cases}$$

Then $M = M(\varphi_1, \varphi_2, \varphi_3) \cong (\Sigma \times S^1) \cup_{(\varphi_1 \cup \varphi_2 \cup \varphi_3)} (V_1 \cup V_2 \cup V_3)$.

Let $\xi$ be a tight contact structure on $M$. By Theorem 1.1 of [18], $\xi$ admits a vertical Legendrian circle $L$ with twisting number 0. We first isotope $\xi$ so that there is a vertical Legendrian circle with twisting number 0 in the interior of $\Sigma \times S^1$, and each $V_i$ is a standard neighborhood of a Legendrian circle $L_i$ isotopic to the $i$-th singular fiber with twisting number $t_i < 0$, i.e., $\partial V_i$ is convex with two dividing curves each of which has slope $\frac{1}{t_i}$ when measured in the coordinates of $\partial V_i$ given above. Let $s_i$ be the slope of the dividing curves of $T_i = \partial V_i$ measured in the coordinates of $T_i$. Then we have that

$$s_i = \begin{cases} 
\frac{-q_i q_i + v_i}{p_i (p_i - u_i)} = -\frac{q_i}{p_i} + \frac{1}{p_i (t_i p_i - u_i)}, & i = 1, 2; \\
\frac{-q_3 (q_3 + \epsilon_0 p_3) + (v_3 + \epsilon_0 u_3)}{p_3 (p_3 - u_3)} = -\epsilon_0 - \frac{q_3}{p_3} + \frac{1}{p_3 (t_3 p_3 - u_3)}, & i = 3.
\end{cases}$$
We choose $t_i << -1$ so that $\frac{1}{b_{0(i)}(i)} < s_i < -\frac{q_i}{p_i}$ for $i = 1, 2$, and $-e_0 + \frac{1}{b_{0(3)}+1} < s_3 < -e_0 - \frac{q_3}{p_3}$. Using the vertical Legendrian circle $L$, we can thicken $V_i$ to $V_i'$, s.t., $V_i$'s are pairwise disjoint, and $T_i' = \partial V_i'$ is a minimal convex torus with vertical dividing curves when measured in coordinates of $T_i$. By Proposition 4.16 of [22], there exits a minimal convex torus $T_i''$ in the interior of $V_i' \setminus V_i$ isotopic to $T_i$ that has dividing curves of slope $\frac{1}{b_{0(i)}+1}$ for $i = 1, 2$, and $-e_0 + \frac{1}{b_{0(3)}+1}$ for $i = 3$. Let $V_i''$ be the solid torus bounded by $T_i''$, and $\Sigma'' \times S^1 = M \setminus (V_1'' \cup V_2'' \cup V_3'')$.

Now we count the tight contact structures on $\Sigma'' \times S^1$ and $V_i''$ that satisfy the given boundary condition. First, we look at $V_i''$. In the coordinates in $\partial V_i'$, the dividing curves of $T_i'' = \partial V_i''$ have slope $\frac{(b_{0(i)}+1)q_i+p_i}{(b_{0(i)}+1)v_i+u_i}$. By part (4) of Lemma 2.1 and the definitions of $u_i, v_i$, we have that $\frac{(b_{0(i)}+1)q_i+p_i}{(b_{0(i)}+1)v_i+u_i} = < b_{0(i)}, b_{i-1}, \ldots, b_2, b_1 + 1 >$. Thus, on each $V_i''$, there are $|\prod_{j=1}^{3} b_{0(j)}(j) + 1|$ tight contact structures that satisfy the given boundary condition. Then we look at $\Sigma'' \times S^1$. The thickened torus $L_i$ bounded by $T_i' - T_i''$ is a continued fraction block consisting of $|b_{0(i)}+1|$ basic slices. Let $L_i'$ be the basic slice in $L_i$ closest to $T_i'$, and $\partial L_i' = T_i' - T_i''$. Note that $T_i''$ is a minimal convex torus with dividing curves of slope $-1$ for $i = 1, 2$, and $-e_0 - 1$ for $i = 3$. Let $\Sigma' \times S^1 = M \setminus (V_1' \cup V_2' \cup V_3')$. By Lemma 4.1, the tight contact structure on $(\Sigma' \times S^1) \cup L_1' \cup L_2' \cup L_3'$ is uniquely determined by the signs of the basic slices $L_i'$. But we can shuffle the signs of the basic slices within a continued fraction block. Let’s shuffle all the positive signs in $L_i$ to the basic slices closest to $T_i'$. Then the sign of $L_i'$ is uniquely determined by the number of positive slices in $L_i$, and so is the number of positive slices in $L_i \setminus L_i'$. Thus, the tight contact structures on $(\Sigma' \times S^1) \cup L_1' \cup L_2' \cup L_3'$ and $L_i \setminus L_i'$ are uniquely determined by these three numbers. But there are only $|b_{0(1)}b_{0(2)}b_{0(3)}|$ ways to choose these three numbers. So there are at most $|b_{0(1)}b_{0(2)}b_{0(3)}|$ tight contact structures on $\Sigma'' \times S^1$ that satisfy the given boundary condition. Altogether, there are at most $|\prod_{i=1}^{3} b_{0(i)}(i) \prod_{j=1}^{3} (b_{0(j)}(j) + 1)|$ tight contact structures on $M$.

It remains to construct $|\prod_{i=1}^{3} b_{0(i)}(i) \prod_{j=1}^{3} (b_{0(j)}(j) + 1)|$ tight contact structures on $M$ by Legendrian surgeries of $(S^3, \xi_{st})$. We begin with the standard surgery diagram of $M = M(\frac{p_1}{q_1}, \frac{p_2}{q_2}, e_0 + \frac{q_3}{p_3})$. Then, perform a slum-dunk between the 0-component and the $-\frac{1}{e_0 + \frac{q_3}{p_3}}$-component, after which the $-\frac{1}{e_0 + \frac{q_3}{p_3}}$-component disappears and the original 0-component becomes a $(e_0 + \frac{q_3}{p_3})$-component. Next we perform a $(1)$-Rolfsen twist on the $(e_0 + \frac{q_3}{p_3})$-component, after which the three components remain trivial and have coefficients $-\frac{p_1}{q_1} - 1$, $-\frac{p_2}{q_2} - 1$ and $-\frac{q_3 + e_0 p_3}{q_3 + (e_0 - 1)p_3}$. But we have

$$\frac{p_1}{q_1} - 1 = < b_{0(1)}(1) - 1, b_{1(1)}(1), \ldots, b_{1(i)}(1) >,$$

$$-\frac{p_2}{q_2} - 1 = < b_{0(2)}(1), b_{1(2)}(2), \ldots, b_{1(i)}(2) >$$
and

\[ \frac{-q_3 + e_0 p_3}{q_3 + (e_0 - 1)p_3} = \left< -2, \cdots, -2, b_0^{(3)} - 1, b_1^{(3)}, \cdots, b_l^{(3)} \right>, \]

where, on the right hand side of the last equation, there are \( e_0 - 1 \) many \(-2\)'s in front of \( b_0^{(3)} - 1 \). Now, we perform (inverses of) the slam-dunks corresponding to these three continued fractions here, which lead us to the diagram at the bottom of Figure 5. Note that all components in this diagram are trivial. Since the maximal Thurston-Bennequin number of an unknot in \( (S^3, \xi_{st}) \) is \(-1\), it’s easy to see that there are \( | \prod_{i=1}^{3} b_0^{(i)} \prod_{j=1}^{l} (b_j^{(i)} + 1) | \) ways to realize this diagram by Legendrian surgeries. According to Proposition 2.3 of [10] and Theorem 1.2 of [17], each of these Legendrian surgeries gives a non-isotopic tight contact structure on \( M \). Thus, up to isotopy, there are exactly \( | \prod_{i=1}^{3} b_0^{(i)} \prod_{j=1}^{l} (b_j^{(i)} + 1) | \) tight contact structures on \( M \).

\[ \square \]

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