ON THE AVERAGE OF THE NUMBER OF IMAGINARY QUADRATIC FIELDS WITH A GIVEN CLASS NUMBER

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Abstract. Let \( F(h) \) be the number of imaginary quadratic fields with class number \( h \). In this note, we improve the error term in Soundararajan’s asymptotic formula for the average of \( F(h) \). Our argument leads to a similar refinement of the asymptotic for the average of \( F(h) \) over odd \( h \), which was recently obtained by Holmin, Jones, Kurlberg, McLeman and Petersen.

1. Introduction

An important problem in number theory, which goes back to Gauss, is to determine all imaginary quadratic fields with a given class number. Let \( F(h) \) be the number of imaginary quadratic fields with class number \( h \). Then for instance one has \( F(1) = 9 \), which follows from the celebrated solution of Baker-Stark-Heegner to Gauss’ class number 1 problem for imaginary quadratic fields. In [3], Soundararajan studied the quantity \( F(h) \) and determined its average order. More precisely, he proved that for any \( \epsilon > 0 \)

\[
\sum_{h \leq H} F(h) = \frac{3\zeta(2)}{\zeta(3)} H^2 + O_{\epsilon}\left(\frac{H^2}{(\log H)^{1/2-\epsilon}}\right).
\]

The purpose of this note is to improve the error term in this asymptotic formula.

Theorem 1.1. We have

\[
\sum_{h \leq H} F(h) = \frac{3\zeta(2)}{\zeta(3)} H^2 + O\left(\frac{H^2(\log \log H)^3}{\log H}\right).
\]

In a recent work [2], Holmin, Jones, Kurlberg, McLeman and Petersen studied statistics of the class numbers of imaginary quadratic fields. In particular, they used the Cohen-Lenstra heuristics together with the work of Granville and Soundararajan [1] on the distribution of values of \( L(1, \chi_d) \) to formulate a conjecture on the asymptotic nature of \( F(h) \) as \( h \to \infty \) through odd values. They also obtained the analogue of (1.1) for the average of \( F(h) \) over odd values of \( h \), conditionally on the generalized Riemann

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hypothesis GRH. More precisely, they showed that assuming GRH
\begin{equation}
\sum_{h \leq H, \ h \ odd} F(h) = 15 \frac{H^2}{4 \log H} + O_e \left( \frac{H^2}{(\log H)^{3/2 - \epsilon}} \right).
\end{equation}

Unlike (1.1) which is unconditional, the proof of (1.2) uses GRH to bound a certain character sum over primes, which appears in this case due to the fact that when \( d > 8 \), the class number of \( \mathbb{Q}(\sqrt{-d}) \) is odd precisely when \( d \) is prime, by genus theory.

The same argument in our proof of Theorem 1.1 leads to the following refinement of the asymptotic formula (1.2).

**Theorem 1.2.** Assume GRH. Then
\begin{equation}
\sum_{h \leq H, \ h \ odd} F(h) = 15 \frac{H^2}{4 \log H} + O \left( \frac{H^2(\log \log H)^{3}}{(\log H)^{2}} \right).
\end{equation}

The main ingredients in the proofs of (1.1) and (1.2) are asymptotic formulas for the complex moments of the class number \( h(d) \) in a large uniform range (see (2.4) and (2.10) below). Using this approach, the best saving one can hope for in the error terms of Theorems 1.1 and 1.2 will be \( 1/L \), if one can obtain an asymptotic formula for the average of \( h(d)^s \), uniformly in \( s \) such that \(|s| \leq L\). It is also known (see [1]) that the current methods for computing these moments fail when \( L \geq (\log H)(\log \log H)^{2} \). This shows that the saving of \( (\log H)/(\log \log H)^{3} \) in the error terms of Theorems 1.1 and 1.2 constitute (up to the power of \( \log \log H \)) the limit of Soundararajan’s method [3]. In particular, it would be interesting to improve the power of \( \log H \) in the error terms of these results.

2. **Proofs of Theorems 1.1 and 1.2**

Let \( X := H^2 \log \log H \). As in [3], it follows from Theorem 4 of [1] (concerning the distribution of extreme values of \( L(1, \chi_d) \)) together with Tatzuawa’s refinement of the Landau-Siegel Theorem [4] that
\begin{equation}
\sum_{h \leq H} F(h) = \sum_{d \leq X, \ h(-d) \leq H} h + O_A \left( \frac{H^2}{(\log H)^A} \right),
\end{equation}
for any \( A > 0 \), where \( b \) indicates that the sum is over fundamental discriminants \(-d\).

To estimate the main term in (2.1), Soundararajan used the following variant of Perron’s formula
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{x^s}{s} \left( \frac{(1+\delta)^{s+1} - 1}{\delta(s+1)} \right) ds = \begin{cases} 
1 & \text{if } x \geq 1, \\
(1+\delta - 1/x)/\delta & \text{if } (1+\delta)^{-1} \leq x \leq 1, \\
0 & \text{if } 0 < x \leq (1+\delta)^{-1}.
\end{cases}
\]
Our improvement comes from using a different smooth cut-off function, namely
\[
I_{c,\lambda,N}(y) := \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s},
\]
where \(c, \lambda > 0\) are real numbers and \(N\) is a positive integer. We prove

**Lemma 2.1.** Let \(\lambda, c > 0\) be real numbers and \(N\) be a positive integer. Then we have
\[
I_{c,\lambda,N}(y) = \begin{cases} 1 & \text{if } y > 1, \\ \in [0, 1] & \text{if } e^{-\lambda N} \leq y \leq 1, \\ 0 & \text{if } 0 < y < e^{-\lambda N}. \end{cases}
\]

**Proof.** First, we recall Perron’s formula
\[
(2.2) \quad \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \frac{ds}{s} = \begin{cases} 1 & \text{if } y > 1, \\ \frac{1}{y} & \text{if } y = 1, \\ 0 & \text{if } 0 < y < 1. \end{cases}
\]

Then, we observe that
\[
\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} y^s \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s} = \frac{1}{\lambda^N} \int_{0}^{\lambda} \cdots \int_{0}^{\lambda} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (y e^{t_1 + \cdots + t_N})^s \frac{ds}{s} \frac{dt_1 \cdots dt_N}{s}. \]

By (2.2), \(\frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} (y e^{t_1 + \cdots + t_N})^s \frac{ds}{s} \in [0, 1]\) for all values of \(t_i\), and hence \(I_{c,\lambda,N}(y) \in [0, 1]\) for all \(y > 0\). The lemma follows from (2.2) upon noting that \(y e^{t_1 + \cdots + t_N} > 1\) for all \(t_i \in [0, \lambda]\) if \(y > 1\), and \(y e^{t_1 + \cdots + t_N} < 1\) for all \(t_i \in [0, \lambda]\) if \(0 < y < e^{-\lambda N}\). \(\square\)

**Proof of Theorem 1.1.** Let \(c = 1/\log H\), \(N\) be a positive integer, and \(0 < \lambda \leq 1\) be a real number to be chosen later. By (2.1) and Lemma 2.1 we obtain
\[
(2.3) \quad \sum_{h \leq H} \mathcal{F}(h) \leq \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{d \leq X} h^s \frac{H^s}{h(-d)^s} \left( \frac{e^{s \lambda} - 1}{\lambda s} \right)^N \frac{ds}{s} + O_A \left( \frac{H^2}{(\log H)^A} \right) \leq \sum_{h \leq e^{\lambda N}} \mathcal{F}(h).
\]

Let \(T := \log X/(10^4(\log \log X)^2)\). Then, it follows from equation (5) of [3] that
\[
(2.4) \quad \sum_{d \leq X} h(-d)^{-s} = 3\pi^{-s-2} \cdot \mathbb{E} \left( L(1, X)^{-s} \right) \int_{1}^{X} x^{-s/2} dx + O \left( X \exp \left( -\frac{\log X}{5 \log \log X} \right) \right),
\]
for all complex numbers \(s\) with \(\text{Re}(s) = c\) and \(|s| \leq T\), where
\[
(2.5) \quad L(1, X) = \prod_p \left( 1 - \frac{X(p)}{p} \right)^{-1},
\]
and \(\{X(p)\}\) is a sequence of independent random variables taking the value 1 with probability \(p/(2(p+1))\), 0 with probability \(1/(p+1)\), and \(-1\) with probability \(p/(2(p+1))\). Note that \(\mathbb{E}(X(p)) = 0\) and \(\mathbb{E}(X(p)^2) \leq 1\), and hence the random product (2.5) converges almost surely, by Kolmogorov’s three series theorems.
Since \(|e^{\lambda s} - 1| \leq 3\) (if \(H\) is large enough) and \(h(-d) \geq 1\), it follows that the contribution of the region \(|s| > T\) to the integral in (2.3) is

\[
\ll X \left( \frac{3}{\lambda} \right)^N \int_{|s| > T} \frac{|ds|}{|s|^{N+1}} \ll \frac{X}{N} \left( \frac{3}{\lambda T} \right)^N.
\]

Moreover, note that \(|(e^{\lambda s} - 1)/\lambda s| \leq 3\), if \(H\) is large enough. Therefore, if follows from (2.4) that the integral in (2.3) equals

\[
\frac{1}{2\pi i} \int_{|s| \leq T} \frac{3}{\pi^2} \cdot \mathbb{E} (L(1, X)^{-s}) \left( \int_1^X x^{-s/2}dx \right) (\pi H)^s \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s} + \mathcal{E},
\]

where

\[
\mathcal{E} \ll \frac{X}{N} \left( \frac{3}{\lambda T} \right)^N + \frac{3NT}{c} X \exp \left( -\frac{\log X}{5 \log \log X} \right).
\]

Choosing \(\lambda = 10/T\) and \(N = [A \log \log H]\), where \(A > 1\) is a constant, implies that

\[
\mathcal{E} \ll A \frac{H^2}{(\log H)^A}.
\]

Furthermore, extending the main term of (2.6) to \(\int_{\epsilon-i\infty}^{\epsilon+i\infty}\) shows that this integral equals

\[
\frac{1}{2\pi i} \int_{\epsilon-i\infty}^{\epsilon+i\infty} \frac{3}{\pi^2} \cdot \mathbb{E} (L(1, X)^{-s}) \left( \int_1^X x^{-s/2}dx \right) (\pi H)^s \left( \frac{e^{\lambda s} - 1}{\lambda s} \right)^N \frac{ds}{s}
\]

\[+ O_A \left( \mathbb{E} (L(1, X)^{-c}) \frac{X}{N} \left( \frac{3}{\lambda T} \right)^N \right)\]

\[
= \frac{3}{\pi^2} \cdot \mathbb{E} \left( \int_1^X I_{c, \lambda, N} \left( \frac{\pi H}{\sqrt{x}} L(1, X)^{-1} \right) dx \right) + O_A \left( \frac{H^2}{(\log H)^A} \right).
\]

Now, it follows from Lemma 2.1 that for any \(1 \leq x \leq X\) we have

\[
I_{c, \lambda, N} \left( \frac{\pi H}{\sqrt{x}} L(1, X)^{-1} \right) = \begin{cases} 1 & \text{if } \sqrt{x} L(1, X) \leq \pi H, \\ \in [0, 1] & \text{if } \pi H < \sqrt{x} L(1, X) \leq e^{\lambda N} \pi H, \\ 0 & \text{if } \sqrt{x} L(1, X) > \pi H e^{\lambda N}. \end{cases}
\]

Thus we obtain

\[
\mathbb{E} \left( \int_1^X I_{c, \lambda, N} \left( \frac{\pi H}{\sqrt{x}} L(1, X)^{-1} \right) dx \right) = \mathbb{E} \left( \min \left( \frac{\pi^2 H^2}{L(1, X)^2}, X \right) + O \left( \frac{H^2(e^{2\lambda N} - 1)}{L(1, X)^2} \right) \right)
\]

\[= \mathbb{E} \left( \min \left( \frac{\pi^2 H^2}{L(1, X)^2}, X \right) + O \left( \frac{H^2(\log \log H)^3}{\log H} \right) \right).
\]
Finally, using Proposition 1 of \cite{1} (which states that the probability that \( L(1, \chi) \leq \pi^2/(6e^\tau) \) is \( \exp(-e^{-\tau-C_1}/(1+o(1))) \) for some explicit constant \( C_1 \)), we obtain
\[
\mathbb{E} \left( \min \left( \frac{\pi^2 H^2}{L(1, \chi)^2}, X \right) \right) = \pi^2 H^2 \cdot \mathbb{E} \left( L(1, \chi)^{-2} \right) + O_A \left( \frac{H^2}{(\log H)^3} \right) 
= \frac{\pi^2 \zeta(2)}{\zeta(3)} H^2 + O_A \left( \frac{H^2}{(\log H)^3} \right).
\]
Combining this estimate with equations (2.3), (2.6)–(2.9), and noting that \( (He^\lambda N)^2 - H^2 \ll H^2(\log \log H)^3/\log H \) completes the proof. \( \square \)

**Proof of Theorem 1.2.** Let \( X, T \) and \( c \) be as in the proof of Theorem 1.1. Let \( D(x) = \{ p \leq x : p \equiv 3 \pmod{4} \} \). Then similarly to (2.1), one has (see equation 3.6 of \cite{2})
\[
\sum_{h \leq H, \ h \ odd} F(h) = \sum_{p \in D(x)} h(-p) 1 + O_A \left( \frac{H^2}{(\log H)^3} \right).
\]
Let \( \{ Y(p) \} \) be independent random variables taking the values 1 and \(-1\) with equal probabilities 1/2, and define
\[
L(1, Y) = \prod_p \left( 1 - \frac{Y(p)}{p} \right)^{-1}.
\]
To obtain (1.2), the authors of \cite{2} prove that assuming GRH (see Theorem 3.3 of \cite{2}) we have
\[
\sum_{p \in D(x)} L(1, \chi_p)^z = |D(x)| \cdot \mathbb{E} \left( L(1, Y)^z \right) + O_\epsilon \left( x^{1/2+\epsilon} \right),
\]
uniformly for all complex numbers \( z \) such that \( |z| \leq (\log x)/(500(\log \log x)^2) \), where \( L(s, \chi_p) \) is the Dirichlet \( L \)-function attached to the Kronecker symbol \( \chi_p = (\frac{-4}{p}) \). Then, by partial summation together with Dirichlet’s class number formula, they deduced that (see \cite{2}, p. 19)
\[
(2.10) \quad \sum_{p \in D(x)} h(-p)^{-s} = \pi^s \cdot \mathbb{E} \left( L(1, Y)^s \right) \int_1^X x^{-s/2} d|D(x)| + O_\epsilon \left( X^{1/2+\epsilon} \right),
\]
for all complex numbers \( s \) with \( \text{Re}(s) = c \) and \( |s| \leq T \).

The proof of Theorem 1.2 then follows along the same lines of the proof of Theorem 1.1 by using (2.10) instead of (2.4).

\[ \square \]

**References**

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