How to (Un-) Quantum Mechanics

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When compared to quantum mechanics, classical mechanics is often depicted in a specific metaphysical flavour: spatio-temporal realism or a Newtonian “background” is presented as an intrinsic fundamental classical presumption. However, the Hamiltonian formulation of classical analytical mechanics is based on abstract generalized coordinates and momenta: It is a mathematical rather than a philosophical framework.

If the metaphysical assumptions ascribed to classical mechanics are dropped, then there exists a presentation in which little of the purported difference between quantum and classical mechanics remains. This presentation allows to derive the mathematics of relativistic quantum mechanics on the basis of a purely classical Hamiltonian phase space picture. It is shown that a spatio-temporal description is not a condition for but a consequence of objectivity. It requires no postulates.

This is achieved by evading spatial notions and assuming nothing but time translation invariance.

I. INTRODUCTION

Uncountable articles and books have been written on the interpretation of quantum theory and most share a number of assertions, briefly summarized as follows: Firstly, yes, quantum mechanics is by far the most precise and successful theory ever formulated and secondly, no, there is no general agreement on what it tells us about the world. Thirdly, it is usually asserted, that classical mechanics (CM) is intuitive and clear while quantum mechanics (QM) is counter-intuitive, weird and somehow raises questions like “is the moon there when nobody looks?” [1], questions that no layman should expect to be raised by physicists.

Even if there is no agreement concerning the interpretation of QM, there seems to be consensus that QM and CM are very, very different. We shall prove this wrong. From a mathematical perspective, there is no fundamental difference between CM and QM. The mathematics of QM can be obtained from classical notions in a straightforward manner, without axioms or postulates. Our view contradicts the standard presentation of QM (SPQM), which stresses the profound mathematical differences between CM and QM. But the truth is that there are none.

This article is dedicated to those physicists who share the author’s intuition, that both, CM as well as QM, have just been presented wrongly and that the real difference between CM and QM is not mathematical.

A. What is Quantum in Quantum Mechanics?

A typical assertion of the SPQM is the following: “In fact, the basic concepts such as observable, ensemble, state, or yes-no measurement, employed in the ”usual” interpretation of quantum mechanics, are themselves not explainable by known pretheories.” [2]

Of course, the “pretheories” are those theories that do not require anything but “classical” concepts. Then it can be precisely defined what should be regarded as “quantum”, namely what can not be explained using classical physics. Hence if, in what follows, we claim that some equation or fact is not “quantum”, then it is not intended to say that it does not belong to the techniques usually employed when “doing” QM, but that it can indeed be derived on the basis of classical concepts. Therefore, according to the SPQM, it can not be “quantum”.

Axioms are only required for things that can not be established otherwise, and the reader may judge for himself, how much “quantumness” eventually remains that requires an “axiomatic” foundation.

B. No Commandments

Victor Stenger wrote that “most laypeople think of the laws of physics as something like the Ten Commandments – rules governing the behavior of matter imposed by some great lawgiver in the sky. However, no stone tablet has ever been found upon which such laws were either naturally or supernaturally inscribed” [3]. But if laypeople have this understanding, then because the usual presentation of physics suggests this view.

Commandment-like-laws (CLLs) can be descriptive, obtained from a fit to experimental data and as such they are, of course, acceptable. But such CLLs are also preliminary, at least concerning their presentation. The problem of how to “interpret” QM arises, because the SPQM is a collection of CLLs (axioms) which do not originate in experimental facts. This is often presented wrongly: Of course there are experimental facts that awaited a physical explanation and QM is the only theory found so far that is able to predict and describe those facts correctly.

But while the law of free fall might be obtained by a fit to the data, the SPQM can not. It cannot be a fit to data since it is declared to be a completely new theory based on completely different and alien principles. Since these

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principles are described as unexplainable from pretheories, they are established as ad hoc commandments. In short, the SPQM tells us what to do but not why.

Descriptive CLLs are usually regarded as pieces of a puzzle, implying that even if it is yet unknown how the bits can be combined, one expects that eventually all CLLs ought to fit seamlessly into a more encompassing deductive approach. The probably best known example for this process of unification are Maxwell’s equations. Hence scientists are obliged to presume that the presentation of CLLs must eventually be adapted once a larger theoretical framework has been established. This is sometimes called “Ochcham’s razor”.

However leading pioneers of quantum physics recommended to accept the theory and its interpretation as commandments. Heisenberg, for instance, wrote: “We must keep in mind this limited range of applicability of the classical concepts while using them, but we cannot and should not try to improve them.” [4], or Dirac: “One must not picture this reality as containing both the waves and particles together and try to construct a mechanism, acting according to classical laws, which shall correctly describe their connexion and account for the motion of the particles. Any such attempt would be quite opposed to the principles by which modern physics advances.” [5]. It is an oddity (to say the least) that quantum scientists dared to call for loyalty to a theory and its interpretation and to demand that one should not try to improve.

One might say, that these quotes are old and outdated, which is true and false, since the authoritative habitus has not disappeared [6]. As Mermin expressed it: “If I were forced to sum up in one sentence what the Copenhagen interpretation says to me, it would be “Shut up and calculate!”” [7].

But in science there is no place for an authoritative attitude. Humans will always ask for reasons and scientists ought to do so for professional reasons: “The problem with the standard textbook formulations of quantum theory is that the postulates therein impose particular mathematical structures without providing any fundamental reason for this choice” [8].

To refuse commandments therefore serves as our main guiding principle. This principle seems to be both, reasonable and scientifically sensible, but also unrealistic. The SPQM claims that about a dozen postulates are required to establish the basics of QM and we claim that none is really needed?

In a preceding paper we argued that the only logical possibility to elude commandments is to derive the “laws of physics” from a definition of what is essentially meant by “objective physical reality”, a world composed of real physical objects (RPOs) [9]. And indeed we found, to our own surprise, that it is possible to derive major parts of QM from nothing but a definition of physicality from a single fundamental constraint which is our “reality condition”. And this constraint is not even exceptionally deep or profound. It is simple, evident, well-known and straightforward. Gerard t’Hooft expressed it as follows [10]: “[...] in particular string theorists expect that the ultimate laws of physics will contain a kind of logic that is even more mysterious and alien than that of quantum mechanics. I, however, will only be content if the logic is found to be completely straightforward.”

The question of a definition of physicality is also raised by multiverse theories that have been suggested in the past decades, for instance by Tegmark, who conjectured the possibility of many different physical worlds, not only different copies or versions of the same kind of world, but worlds that obey very different physical laws [11, 12]. If not only the known physical world, but many different worlds are conjectured, then these worlds must have something in common that allows to call them “physical”. Hence a simple and straight definition of physicality is required, something that must hold in any thinkable physical world, even in hypothetical worlds which might, according to this idea, be ruled by different and alien physical laws, that might have, for instance, a different number of spatial dimensions or, who knows, no spatial dimensions at all.

II. SETUP

Physics creates models of (parts of) reality. These models allow to “simulate reality”, or, in simple cases, to directly calculate results. Hence it is arguable, that, whatever is physical in a world, should allow for a description by algorithms that predict (probabilities for possible) evolutions of physical quantities in time. Hence the basis of a physical model of reality is a (possibly very long) list of quantities $\psi(\tau)$, that depend on time $\tau$. This is the raw material for a general physical model of any thinkable physical reality.

A. Symmetries and Quantities

One of the first facts children learn about real objects is called object permanence [13], namely that the moon is still there, even if nobody looks. Object permanence does not seem to be general enough to serve as the desired constraint that defines physicality, because macroscopic objects can be disassembled and destroyed, one can break tea cups and burn wooden chairs. But matter can be manipulated only within specific constraints. Objects are made of other objects. Insofar as one can disassemble objects, they can be destroyed, but chemistry found that the amount of matter remains unchanged, even if objects are burned. And even though it is theoretically possible to destroy all individual microscopic objects (particles) that a macroscopic thing is made of by annihilation with a perfect copy made of anti-matter – still the energy remains. This, eventually, is an insight that is as simple as it is not trivial.

The impossibility for a perpetuum mobile that produces net energy is the fundamental constraint for any known
closed physical system or process. According to Einstein “The most satisfactory situation is evidently to be found in cases where the new fundamental hypotheses are suggested by the world of experience itself. The hypothesis of the nonexistence of perpetual motion as a basis for thermodynamics affords such an example for a fundamental hypothesis suggested by experience.”[14].

As Einstein rightly remarked, this principle is suggested by experience, but once it’s depth has been recognized, physicists understood that it has the strength and status of a definition of physical realness itself. If a theory fails to provide conservation of energy then it is unphysical by definition. But if it is possible to define un-physicality on the basis of a conservation “law”, why then should it not be possible to also define physicality? But what exactly is a conservation law?

Emmy Noether, in 1928, discovered the math fact[15], as Stenger puts it, “[...] that coordinate independence was more than just a constraint on the mathematical form of physical laws. She proved that some of the most important physics principles are, in fact, nothing more than tautologies that follow from space-time coordinate independence: energy conservation arises from time translation invariance, linear momentum conservation comes from space translation invariance, and angular momentum conservation is a consequence of space rotation invariance. These conserved quantities were simply the mathematical generators of the corresponding symmetry transformation.”[3] Hence it is a math fact that a conservation law is nothing but a continuous symmetry, the generator of which is a conserved quantity.

The concepts of energy as well as of action can only be defined on the basis of an already elaborated physical theory[4]. This would of course be a theory of the known physical world and not necessarily valid in any hypothetical physical world. How should one know a priori whether these notions are relevant and meaningful in any thinkable physical world? Therefore, if arbitrary physical worlds are considered, these notions are too specific to be used from the start.

However, it is not required to specify the type of the conserved quantity at this point. It suffices to formally refer to some positive definite constant of motion (PDCOM) which serves as a measure of object permanence, because it is not the object itself that is permanent, but some abstract quantity that objects are “charged with”. We can anticipate that this quantity will turn out to be a possible measure of the amount of substance. This requires no postulate: If correct then it should be a mathematical consequence of our approach.

The very idea of a real physical object logically requires at least one positive definite constant of motion (PDCOM). Little more than this will be used to a number of basic “laws” of physics. Without commandments.

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1 It is known to be a non-trivial problem to find non-circular definitions of the central notions of physics.

B. Time

There is no a priori reason to introduce more than a single symmetry, namely constancy in time, aka permanence. Or, to be more precise: If more than one symmetry would be presumed, then one would need to specify how many and why not one more or less.

Time is a primary quantity that remains basically “undefined”. Stanley Goldberg wrote that “Either you know or you don’t know what I mean when I use a phrase like “time passes.””[10]. The “dimension” of time is different from spatial dimensions insofar as it is unique. One can discuss the dimensionality of physical space and one can, within the classical framework, imagine physical worlds with more or less than 3, maybe even zero, spatial dimensions. But it is questionable if it is possible to imagine a physical world without the unique dimension of time: “[...] only one true integer may occur in all of physics. The laws of physics refer to one dimension of time”[17].

Hamilton wrote that “the notion or intuition of order in time is not less but more deep-seated in the human mind, than the notion of intuition of order in space; and a mathematical science may be founded on the former, as pure and as demonstrative as the science founded on the latter. There is something mysterious and transcendent involved in the idea of Time; but there is also something definite and clear: and while Metaphysicians meditate on the one, Mathematicians may reason from the other.”[18].

We shall show that, even though Hamilton’s idea to derive algebra as the science of pure time failed[19], there is nonetheless an algebra of pure (aka proper) time. It reveals the possibility to formulate (central parts of) physics as a tautology[10]. This has absolutely no negative connotation, as Goldberg explained: “Different branches of mathematics have different rules but in all branches, since the rules are predetermined, the conclusion is actually a restatement, in a new form, of the premises. Mathematics, like all formal logic, is tautological. That is not to say that it is uninteresting or that it doesn’t contain many surprises.”[10]. If it is possible to derive (essential parts of) physics from a definition of physicality, then the result is a tautology in Goldberg’s sense.

C. Reason

Generality is maintained by presuming nothing specific, neither about the “nature” of the dynamical variables nor about the “nature” of the conserved quantity. This attitude has been summarized by Hamilton under the name principle of sufficient reason (PSR): “Infer nothing without a ground or reason.”[20]. In his form it
might also be called the principle of insufficient reason. Of course, the PSR contains little more than Stenger’s claim that there are no commandments.

If a no-assumption-approach allows for any conclusion about the nature of the conserved quantity, then it must emerge from the form of equations or symmetries. Only if familiar structures emerge, if equations suggest specific interpretations as nearby, we are authorized to map quantities of the theory to known physical observables, aka to interprete.

The PSR has a bias towards symmetry since nothingness (the void) is the most symmetric state: To assume nothing specific about a number of things or quantities has to be understood as assuming no asymmetry and the PSR forbids to introduce asymmetries, distinctions and classifications without a ground or reason.

Hence the raw material for the simplest physical object contains an arbitrary number $\nu$ of dynamical variables $\psi = (\psi_1, \ldots, \psi_\nu)$ (quantities) that depend on a time, the evolution of which is constrained by a PDCOM $\mathcal{H}(\psi) = \mathcal{H}_0 = \text{const}$. With the condition that $\mathcal{H}(\psi)$ is a constant of motion, it is implied that $\psi$ itself does not contain any other constant, i.e. all variables in the list $\psi$ depend on time so that no linear combination of the elements of $\psi$ may presumed to be constant.

### D. Structure of the Paper

In Sec. III we shall firstly show that in any classical dynamical system, which can be derived from the assumed PDCOM, the number of true dynamical variables is even, i.e. the variables come in pairs and secondly that one can always describe the dynamics, after an appropriate change of variables, by Hamilton’s laws of motion. Hamiltonian dynamics has maximal generality and is not negotiable. If the SPQM suggests that it requires modifications then we shall show that this is wrong.

In Sec. IV we introduce the phase space distribution as the fundamental mathematical representation of physical objects. We show that it suffices to consider the simplest possible description of phase space distributions, namely the matrix of second moments (sloppily called auto-correlation matrix), to derive Heisenberg’s equation of motion. We demonstrate the specific notational convention concerning the use of complex numbers in QM in general and specifically in Dirac’s theory. We explain the general conditions that Hamiltonian physics imposes on the dimensionality of phase spaces.

In Sec. V we use simple group-theoretical considerations, that, when applied to the Dirac algebra, suggest an interpretation in terms of relativistic electrodynamics. We show that this interpretation directly yields the Lorentz force law, the Lorentz transformations and the relativistic energy-momentum-relation. We demonstrate that this framework also enables to derive the Zeeman effect, the spin, and the physics of adiabatic high frequency transitions (Breit-Rabi-model).

In Sec. VI we explain what is meant by “unquantization”: Since the so-called “canonical quantization” can be derived and explained on the basis of classical notions, we simultaneously un-quantize QM in the sense explained above and “quantize” CM. Then we discuss Born’s rule and explain why classicality is compatible with background independence but nonetheless leads with necessity to $3+1$-dimensional geometrical notions.

### III. THE “LAW” OF MOTION

Our inventory consists of a number $\nu$ of dynamical variables $\psi$, subject to change in time $\tau$ and a PDCOM $\mathcal{H}(\psi)$. With the prelimentary simplifying assumption that $\mathcal{H}$ does not explicitely depend on time $\frac{\partial \mathcal{H}}{\partial \tau} = 0$, the physicality constraint can be formulated as follows:

$$\dot{\mathcal{H}} = \sum_{k=1}^{\nu} \frac{\partial \mathcal{H}}{\partial \psi_k} \dot{\psi}_k = 0 \quad (1)$$

where the overdot indicates the temporal derivative. Eq. I can be written in vectorial notation as

$$(\nabla_\psi \mathcal{H}) \cdot \dot{\psi} = 0 \quad (2)$$

where the “$\cdot$” indicates a scalar product. The solution is given by

$$\dot{\psi} = J (\nabla_\psi \mathcal{H}) \quad (3)$$

with some arbitrary $\nu \times \nu$ skew-symmetric matrix $J$. Inserted into Eq. 2 the condition for constancy of $\mathcal{H}$ is fulfilled by the skew-symmetry of $J$ alone.

It is a math fact that if $\lambda$ is an eigenvalue of a real square skew-symmetric matrix $J$, then $-\lambda$ is also an eigenvalue. Hence any skew-symmetric matrix of size $\nu \times \nu$ has at least one vanishing eigenvalue, unless $\nu = 2n$ is even. A vanishing eigenvalue corresponds to a hidden constant in $\psi$. Since this was excluded by definition, $J$ has full rank and $\nu = 2n$ is even or can be reduced to even dimension by an appropriate coordinate transformation. In both cases we can restrict ourselves to an even

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2 Ariel Caticha identified the principle of insufficient reason in Quantum mechanics [2].
number of dynamical variables without loss of generality: In any physical world, the number of dynamical variables required to describe an RPO, is even.

We use the preliminary simplifying assumption that \( H(\psi) \) can be written as a Taylor series of \( \psi \) and initially concentrate on the terms of lowest order. For this case of small oscillations one may skip higher than quadratic terms and translate by \( \psi_0 \) such that linear terms vanish, without loss of generality. The constant term can be excluded as trivial. Then \( H(\psi) \) can be written as

\[
H(\psi) = \frac{1}{2} \psi^T A \psi
\]  

(4)

with a positive definite symmetric matrix \( A \) of size \( 2n \times 2n \). The linearized law of motion (LOM) Eq. 3 then is

\[
\dot{\psi} = J A \psi,
\]  

(5)

where \( \psi \) is a vector of \( 2n \) components.

According to a theorem of linear algebra for every non-singular skew-symmetric matrix \( J \) of size \( 2n \times 2n \) there exists a non-singular matrix \( Q \) such that

\[
Q^T J Q = \text{diag}(\lambda_0, \lambda_1, \eta_0, \ldots, \eta_n)
\]  

(6)

where \( \lambda_k \) are real non-zero constants (the modulus of two eigenvalues) and

\[
\eta_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]  

(7)

Since there is no reason to assume anything specific about the eigenvalues, beyond being non-zero, the PSR recommends the most symmetric case, i.e. all \( \lambda_k \) are equal to unity. In this case \( Q \) is an orthogonal transformation so that

\[
Q^T J Q = I_n \otimes \eta_0 \equiv \gamma_0
\]  

(8)

Note that this transformation is only required to obtain the symplectic unit matrix in a simple form that allows to recognize firstly that \( J^2 = \gamma_0^2 = -1 \) and secondly that the dynamical variables can formally be regarded as canonical pairs \( q_i \) and \( p_i \). One may write \( \psi = (q_1, p_1, q_2, p_2, \ldots, q_n, p_n)^T \). A canonical pair represents the smallest thinkable dynamical system with a PDCOM and is called a degree of freedom (DOF).

Eq 3 can then be written, without loss of generality, in the form of Hamilton’s equations of motion

\[
\dot{q} = \frac{\partial H}{\partial p},
\]

(9)

\[
\dot{p} = -\frac{\partial H}{\partial q},
\]

or, using the linear approximation:

\[
\dot{\psi} = \gamma_0 A \psi = H \psi,
\]  

(10)

where \( \gamma_0 A \) has been replaced by a single matrix \( H \). Since nothing but physicality is assumed, Hamilton’s equations of motion must pop up in any thinkable physical reality in some form.

The matrix \( \gamma_0 \) is the so-called symplectic unit matrix (SUM). A matrix that can be written in the form \( H = \gamma_0 A \) is called Hamiltonian. The transpose of a Hamiltonian matrix is

\[
H^T = A^T \gamma_0^T = \gamma_0 H \gamma_0.
\]  

(11)

We define the “adjunct spinor” \( \bar{\psi} = \psi^T \gamma_0^T \) so that the Hamiltonian (Eq. 4) can be written as

\[
\mathcal{H} = \frac{1}{2} \psi^T A \psi = \frac{1}{2} \bar{\psi} H \psi
\]  

(12)

since \( \gamma_0^T \gamma_0 = 1 \).

We stress again that no assumptions were used to arrive at Hamilton’s equations of motion (EQOM) and no assumption about the meaning of \( q_i \) and \( p_i \) are implied by notation. The use of the symbols “\( q \)” and “\( p \)” is just the convention of Hamiltonian theory. They represent arbitrary pairs of conjugate dynamical variables. And we stress again that they are “classical” in the sense that \( q p - p q = 0 \).

Time, an arbitrary number of dynamical variables, a constant of motion and the PSR are the only required ingredients for the concept of a \( 2n \)-dimensional phase space. Hence the concept of phase space has no intrinsic connection to spatial coordinates or mechanical momenta, but is purely abstract. It is the basis of any physical world.

IV. PHASE SPACE

Almost all classical presentations of quantum mechanics as given by Born and Heisenberg as well as by Schrödinger, Dirac or von Neumann, emphasized the Hamiltonian nature of Quantum theory. Even if the SPQM postulates that in QM the classical Poisson brackets have to be replaced by the commutator of conjugate operators, Birkhoff and von Neumann wrote that there “[...] one concept which quantum theory shares alike with classical mechanics and classical electrodynamics. This is the concept of a mathematical ”phase-space.””

But if CM and QM share the concept of phase space, then the purported fundamental differences between CM and QM must be due to the interpretation, due to the assumed relation between phase space variables and measurable (“observable”) quantities. As we shall demonstrate, this is indeed the only fundamental difference between CM and QM.
The size is usually measured by the (square root of) the second moments to characterize their respective size in either direction. The variable list $\psi$ is formally a “coordinate” in some 2$n$-dimensional phase space and according to what has been said before, this does not imply or suggest any specific interpretation. Then real physical objects (RPOs) are, in the first place, inhabitants of phase spaces. As a single classical mass point makes no tangible object in “physical” space, a single coordinate in phase space makes no sensible object as well. Some kind of distribution is needed, a droplet in phase space.

Since a general distribution $\rho(\psi)$ in phase space implies an infinite amount of information, it is convenient and required to reduce the complexity of the description. A common way to describe phase space distributions is to characterize their respective size in either direction. The size is usually measured by the (square root of) the second moments $\Sigma$. The matrix of second moments $\Sigma$ is given by

$$\Sigma_{ij} \equiv \langle \psi_i \psi_j \rangle = \langle \psi \psi^T \rangle,$$

where the embracing angles indicate some (yet unspecified) average. Without loss of generality, one can write this as a matrix product of some 2$n$ $\times$ $m$ matrix $K$ of the form

$$\Sigma_{ij} = KK^T$$

where $m \geq 2n$.

From Eq. 10 one finds the (linearized) equation of the motion of the autocorrelation matrix:

$$\dot{\Sigma} = (\psi \psi^T + \psi \psi^T) = (H \psi \psi^T + \psi \psi^T H^T) = H \Sigma + \Sigma H^T$$

Multiplication from the right of both sides with $\gamma_0^T$ gives:

$$\dot{\Sigma} \gamma_0^T = H \Sigma \gamma_0^T + \Sigma H^T \gamma_0^T$$

Now we define another Hamiltonian matrix $S$ by $S = \Sigma \gamma_0^T$ and with $\gamma_0^T = -\gamma_0$, $\gamma_0^T \gamma_0 = 1$ and Eq. 11 one obtains

$$\dot{S} = H S - S H \equiv [H, S],$$

which is known as Heisenberg’s equation of motion for operators.

It is still missing the quantum look and feel, namely the unit imaginary and $\hbar$, which are both absent from Eq. 17. But it is wrong to think that these factors are valid indicators for the quantunness of equations. We prove this by simply introducing both from void.

Since all variables in $\psi$ are treated equally they all have the same unit. According to Eq. 10 the elements of the Hamiltonian matrix $H$ have the unit of frequency. No one can prevent us from giving $H$ the unit of energy by multiplication with some conversion factor $\hbar$ with the dimension of action. We then obtain with $\hat{H} = \hbar H$:

$$\dot{\hat{S}} = \frac{1}{\hbar} [\hat{H}, S]$$

Any Hamiltonian matrix that represents stable dynamical systems has purely imaginary eigenvalues. Furthermore, if $\lambda$ is eigenvalue of a Hamiltonian matrix, then $-\lambda$, as well the complex conjugates $\pm \lambda$ are also eigenvalues. Since $\hat{H}$ is by definition a stable non-degenerate Hamiltonian matrix, it can be written as

$$\hat{H} = \epsilon E \text{Diag}(\epsilon_1, -\epsilon_1, \epsilon_2, -\epsilon_2, \ldots, -\epsilon_n) E^{-1}$$

where $E$ is the matrix of eigenvectors and $\epsilon_i = \hbar \omega_i$ are real energy eigenvalues. We introduce another matrix $\hat{H}$ by multiplication with $-i$:

$$\tilde{H} = -i \hat{H} = \epsilon E \text{Diag}(\epsilon_1, -\epsilon_1, \epsilon_2, -\epsilon_2, \ldots, \epsilon_n, -\epsilon_n) E^{-1}$$

which has now real energy eigenvalues so that the unit imaginary appears explicitly:

$$\dot{\hat{S}} = \frac{i}{\hbar} [\hat{H}, S]$$

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5. It is well-known that Hamiltonian motion in phase space corresponds to the flow of an incompressible fluid.

6. That the “quantum state” contains infinite information is the content of a Hardy’s theorem and required to reduce the complexity of these concepts to mass points. The methods of Hamilton and Lagrange are applied in all fields of physics and any kind of dynamical variables, also in those that do not refer to a spatio-temporal description in the first place. The theory of canonical transformations, which allows any kind of transformation that preserves the Hamiltonian equations of motion, is the core concept of analytical mechanics and it is incompatible with a limitation of CM to mass point dynamics.

7. Mathematically it is well-known that reasonable distributions with finite moments are completely and uniquely determined by their moments.

8. We leave aside subtleties of possible interpretations of how to obtain and understand this average. At this point it suffices to agree that one can average over an ensemble of phase space points or some phase space volume. If the system is presumed to be ergodic, the average might also be obtained by integration over time.

9. We shall use the term auto-correlation matrix not only for $\Sigma$ but also for $S$.

10. The autocorrelation matrix $S$ can, up to this point, be given an arbitrary unit.
The introduction of factors that otherwise cancel out can not add anything physical to an equation. Therefore Heisenberg’s operator equation is as such not quantum: we just derived it from classical Hamiltonian mechanics. Without further assumptions it is simply an equation that describes the linearized equations of motion of second moments in some classical phase space. Furthermore Eq. \[17\] proves that commutators are just an algebraic result of considering the evolution of second moments in time. Hence commutators are, as such, not quantum either.

A. Unitary vs. Symplectic Motion

However, the matrix \( H \) is Hamiltonian (not Hermitian), therefore it generates symplectic (and not unitary) evolution in time. Symplectic motion is more general than unitary motion since firstly, unitary motion is always symplectic \([12]\) but secondly, symplectic motion allows for complex eigenvalues \([9]\) which are excluded in unitary motion. But no law of the universe and no commandment forbids complex eigenvalues. Such a law would be superfluous anyway as complex eigenvalues and unitary motion. But no law of the universe and no commandment forbids complex eigenvalues. Such a law would be superfluous anyway as complex eigenvalues and stability are incompatible. Complex eigenvalues may appear (for a limited time) in nature, for instance in case of resonance \([31,32]\), but they are incompatible with long-term stability.

Bender and others have shown that unitarity is not universally required, not even in the case of real eigenvalues \([37–40]\). The corresponding postulate of QM is therefore not universally valid and can be dropped. This is a simple math fact and requires no postulates: the description of stable states (of motion) requires the corresponding mathematical form of eigenvalues. If one insists on the unit imaginary as an indispensable QM factor, then the eigenvalues of a stable system must be real. In stable symplectic motion, which is considered to be classical, the unit imaginary is not written explicitely and the eigenvalues of stable motion are purely imaginary. In the former case one uses the unit imaginary explicitely and writes the frequency as \( \omega = \varepsilon / \hbar \), in the latter case the unit imaginary is used implicitely as the eigenvalues have the form \( \pm i \omega \) with a real valued frequency \( \omega \). But neither nature nor mathematical reason cares much about notational conventions: Neither the explicite use of the unit imaginary nor unitary evolution are quantum.

The derivation of Eq. \[17\] from a PDCOM (the Hamiltonian) suffices to generate a structure preserving symplectic framework in which “probability current conservation” pops up automatically, since symplectic motion is known to conserve the occupied volume of phase space as we shall derive in the next section.

V. PHASE-SPACE ALGEBRA

Let us mention some math facts about Hamiltonian matrices and symplectic motion that are known, but maybe not well-known. We begin with the fact that Eq. \[17\] constitutes a so-called Lax pair. As Peter Lax has shown, if a pair \( S \) and \( H \) of operators obeys Eq. \[17\], then the trace of any power of \( S \) is a constant of motion \([11]\):

\[
\text{Tr}(S^k) = \text{const},
\]

for all \( k \in \mathbb{N} \). This also holds for non-linear operators. Within our approach, both matrices are by definition the product of the skew-symmetric SUM \( \gamma_0 \) and a symmetric positive definite matrix \( A \). According to linear algebra they have a vanishing trace and it can be shown that all odd powers of \( H \) and \( S \) share this property:

\[
\text{Tr}(S^{2k+1}) = 0, \quad (23)
\]

for \( k \in \mathbb{N} \), so that only the even powers in Eq. \[22\] are “non-trivial” constants of motion (COMs). It has been shown elsewhere that the eigenvalues of the autocorrelation matrix are a measure of the occupied phase space volume \([42,43]\). Again: This is a math fact and requires no postulate. Furthermore, in statistical mechanics, a phase space density is as close as can be to a probability density.

As already mentioned, both matrices, the driving matrix \( H \) and the matrix \( S \), have the same structure, namely both are Hamiltonian (Eq. \[11\]). It is well known that such matrices are generators of symplectic motion by the fact that the solution of Eq. \[10\] is the matrix exponential

\[
\psi(\tau) = \exp(\mathbf{H}\tau) \psi(0) = \mathbf{M}(\tau) \psi(0). \quad (24)
\]

It is straightforward to show that a symplectic matrix \( \mathbf{M} = \exp(\mathbf{F}\tau) \) holds:

\[
\mathbf{M} \gamma_0 \mathbf{M}^T = \gamma_0. \quad (25)
\]

A matrix \( \mathbf{N} \) is skew-symplectic if

\[
\mathbf{N} \gamma_0 \mathbf{N}^T = -\gamma_0. \quad (26)
\]

One finds after few steps

\[
\mathbf{S}(\tau) = \mathbf{M}(\tau) \mathbf{S}(0) \mathbf{M}^{-1}(\tau) = \mathbf{M}(\tau) \mathbf{S}(0) \mathbf{M}(-\tau). \quad (27)
\]

The result of Hamiltonian evolution in time, the result of motion, is a symplectic similarity transformation (SST). And since similarity transformations do not change eigenvalues, the eigenvalues are COMs.
A. Eigenvectors and Eigenvalues

It is yet another math fact that commuting matrices share a system of eigenvectors. According to Eq. [17] the matrix $S$ (and hence the second moments) is constant, iff $H$ and $S$ commute. Only diagonal matrices always commute, so that commuting matrices must have the same matrix of eigenvectors $E$:

$$D_S D_F = D_F D_S$$
$$\begin{pmatrix} E S E^{-1} \end{pmatrix} (E H E^{-1}) = (E H E^{-1}) \begin{pmatrix} E S E^{-1} \end{pmatrix}$$

(28)

where $D_S = \text{Diag}(i \varepsilon_1, \ldots, -i \varepsilon_n)$ is the diagonal matrix containing the eigenvalues of $S$.

Therefore eigenvectors and -values play an important role in physics. This is a consequence of constructing observables from second moments on the basis of Hamiltonian mechanics. Again this requires no postulates, neither quantum nor otherwise. Oscillatory systems have eigenvalues – the frequencies – and eigenvectors\footnote{Note that $S$ commutes with analytical functions of $H$: if it commutes with $H$, it commutes with $M = \exp(H_F)$.}. It is a math fact that strongly stable systems must have purely imaginary eigenvalues and complex eigenvectors. And since the eigenvalues come in pairs (or quadruples, if complex), the eigenvectors also come in complex conjugate pairs. This is a math fact about (classical) coupled oscillating systems, subject to linear Hamiltonian motion. No commandment is required.

B. Symplectic Motion is Structure Preserving

Since similarity transformations do not change eigenvalues, this also holds for linear symplectic motion, i.e. SSTs. SSTs also preserve the structure of Hamilton's equations of motion, i.e. the form of the matrix $\gamma_0$. With respect to an RPO it is specifically the dynamical structure which determines the properties, or the type of these objects. The fact that evolution in time is a SST guarantees that Hamiltonian matrices remain Hamiltonian. The exponential of a Hamiltonian matrix is symplectic and the logarithm of a symplectic matrix is Hamiltonian\footnote{Note that $S$ commutes with analytical functions of $H$: if it commutes with $H$, it commutes with $M = \exp(H_F)$.}. A skew-Hamiltonian matrix $C$ is a product of the SUM $\gamma_0$ and an arbitrary skew-symmetric matrices $B = -B^T$:

$$C = \gamma_0 B$$

(29)

such that

$$C^T = -\gamma_0 C \gamma_0$$

(30)

Accordingly the number of linear independent elements $\nu_s$ in a Hamiltonian matrix of size $2n \times 2n$ is

$$\nu_s = n (2n + 1)$$

(31)

and in a skew-Hamiltonian matrix it is $\nu_c$:

$$\nu_c = n (2n - 1)$$

(32)

It is a straightforward exercise to show that the (anti-) commutators of Hamiltonian $(S_i)$ and skew-Hamiltonian $(C_j)$ matrices have (anti-) commutators of the following type:

$$\begin{array}{c}
S_1 S_2 - S_2 S_1 \\
C_1 C_2 - C_2 C_1 \\
C S + S C \\
S^{2n+1}
\end{array}$$

\Rightarrow \text{Hamiltonian}

$$\begin{array}{c}
S_1 S_2 + S_2 S_1 \\
C_1 C_2 + C_2 C_1 \\
C S - S C \\
S^{2n}
\end{array}$$

\Rightarrow \text{skew-Hamiltonian}

(33)

Note that the unit matrix $1$ is skew-Hamiltonian. It is remarkable that it is possible to derive the complex structure\footnote{Note that $S$ commutes with analytical functions of $H$: if it commutes with $H$, it commutes with $M = \exp(H_F)$.} from nothing but symmetry arguments, i.e. from pure physical logic.

C. Observables and Generators

Quantum mechanics is not the first physical theory that requires a reflection about the meaning of a measurement. Also special relativity is a theory that struggles with the meaning of time and length measurements.

Our considerations are based on a definition of physical realness and are therefore incompatible with claims that $\psi$ is somehow “unreal”. In the contrary, $\psi$ was the only ontic thing we presumed at all. This view is confirmed by the no-go-theorem of Pusey, Barrett and Rudolph in which the authors claim that “if a quantum state merely represents information about the real physical state of a system, then experimental predictions are obtained which contradict quantum theory”\footnote{Note that $S$ commutes with analytical functions of $H$: if it commutes with $H$, it commutes with $M = \exp(H_F)$.}. There is a minority report of physicists that do not subscribe the dogma of unreality. Roger Penrose, for instance, wrote “if we are to believe that any one thing is in the quantum formalism is ‘actually’ real[…], then I think it has to be the wavefunction [...]”\footnote{Note that $S$ commutes with analytical functions of $H$: if it commutes with $H$, it commutes with $M = \exp(H_F)$.}, or Lev Vaidman: “The only fundamental physical ontology is the quantum wave function”\footnote{Note that $S$ commutes with analytical functions of $H$: if it commutes with $H$, it commutes with $M = \exp(H_F)$.}.

But though we defy to regard $\psi$ as somehow unreal, it can not be denied that the meaning of $\psi$ is not self-evident. So far we did not consider the physical unit of the variables $\psi$. What type of quantity do these variables represent? The answer could be “Since it’s part of nature, we don’t really know.”\footnote{Note that $S$ commutes with analytical functions of $H$: if it commutes with $H$, it commutes with $M = \exp(H_F)$.}

We can only stress again that formally $\psi$ is a coordinate in phase space. Classical phase spaces coordinates have no fixed units, only the product of canonical pairs is fixed to the unit of action, or angular momentum, respectively.
Hence we can say that a $2n$-dim. volume ($n \geq 1$) of phase space has a unit, single coordinate values don’t. From the PSR it follows that both, the canonical coordinates and momenta forming $\psi$, have the same unit. Though one might formally say that $\sqrt{n}$ would be the nearby unit for a phase space coordinate, this has little practical value.

A unit requires not only a name and a symbol. For a direct measurement it is necessary to have a reference artifact that has a constant property of the same type: a certain weight, length, clock frequency or voltage. But since the variables $\psi$ are supposed to be fundamental, how should such an artifact emerge from a more fundamental level, if there is none, by definition?

Furthermore, $\psi$ is by construction a list of dynamical variables in the literal sense. By definition we required that none of these variables (and no linear combination thereof) can be considered a non-zero constant. Hence there is no constant reference and therefore $\psi$ can not be directly measured. Only available constant quantities can provide a reference, i.e. second or higher even moments like the Hamiltonian $H(\psi)$. Linear Hamiltonian theory is based on a quadratic form, the Hamiltonian, which is constant by construction and provides the reference for all second (and possibly higher even) moments.

Then the use of second (or higher even) moments and correlations to describe the phase space distribution is not only a convenient and natural choice, it is the only possible choice. There is no need to postulate that $\psi$ can not be measured: unless someone presents a solution to the reference problem, we doubt that it has a solution. Mermin asserted that "[...]

Humans are inhabitants of a physical world and have the perspective of insiders. One can not prevent anyone from considering the possibility that some supernatural being, some kind Maxwellian demon, might have a different perspective and is in the possession of a reference that enables to "measure" $\psi$. But from within the physical world, a direct measurement is hardly possible. Some schools of philosophy deny the possibility to presume existence of unmeasurable entities. But we do not suggest that the entities are unmeasurable, we just doubt that the values of the variables in $\psi$ can be directly measured at some time $\tau$.

If one regards this as a reason to exclude the wavefunction from classical physics, then classicality would have to be limited to observable physical quantities as well. However, such a limitation of classicality would be historically untenable: Newton based the fundament of his theory on the existence of something unmeasurable, namely absolute space. Furthermore he suggested a corpuscular theory for light, before any such corpuscle was experimentally detected. Boltzmann used atoms in the kinetic theory of gases before there was sufficient evidence that atoms exist at all. A sober view of physics reveals that there are plenty of entities which can not be observed "directly". This is due to the very method of physics: Physics proceeds by presenting simple but not directly observable common causes. If the cause would be directly observable, it would not need a theory for it.

The reference problem explains many of the difficulties to understand and accept un-quantum physics for inhabitants of physical worlds, if these inhabitants try to establish a physical theory on the basis of measurable quantities alone. This also holds for human beings.

The matrix of second moments $S = \Sigma \gamma_0^\dagger$, and the spinor $\psi$ have very different LOMs. In contrast to the dynamical variables in $\psi$, that can by construction not be constant, the variables in $S$ ("observables") can be constant, if $S$ and $H$ commute. Otherwise they (for instance) oscillate with some frequency and amplitude. The Hamiltonian as a PDCOM is available as a reference quantity so that the correlations of $S$ can always be measured. Hence there are (at least) two different levels of reality, the "spinor" $\psi$ and its auto-correlation matrix $S$.

The matrix $S$ is, like $H$, a Hamiltonian matrix. Since skew-Hamiltonian matrix components do not contribute to the Hamiltonian, they cannot be generators of possible evolutions in time. Correspondingly the autocorrelation matrix $\Sigma$ is symmetric and skew-Hamiltonian matrices have zero "expectation" values. This means that there are further "parameters" emergent in the theory that necessarily vanish and are in this sense "unmeasurable" or "hidden".

A phase space density is constant if it is exclusively a function of COMs. Since only even moments can mathematically generate COMs, a stable phase space density is an even function of $\psi$: $\rho(\psi) = \rho(-\psi)$. Classical statistical mechanics is concerned with many DOFs and in this case, only positive definite values do not cancel by averaging over some thermal ensemble, namely the known PDCOM, so that eventually in this case one finds $\rho = \rho(H)$. The Boltzmann distribution $\rho(H) \propto \exp(-\beta H)$ is such a case and corresponds, using Eq. 4, to a multivariate normal distribution in $\psi$, up to a normalization.

The constraint that only Hamiltonian "operators", parameters of the Hamiltonian matrix, represent observables, might be regarded as the true origin of the Hermiticity condition for complex Dirac matrices. But out classical approach is clearer, more straightforward and also more stringent. Our analysis suggests that fundamental dynamical quantities can not be classical observables and that, instead, second (or higher even) moments are required to obtain observables. Then of course, some strange effects concerning the statistical properties

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15 Given by $\gamma_0 \gamma_\mu \gamma_0 = \gamma_\mu^\dagger$. In the standard presentation of the Dirac electron theory, the adjoint spinor is defined as $\bar{\psi} \equiv \psi^\dagger \gamma_0$, but the multiplication with $\gamma_0$ is not explained.

16 This implies that our approach denies the possibility that classical physics could be fundamental at all.
of observables are unavoidable.  

D. The Pauli Matrices from Hamiltonian Symmetry

DOFs are the basic elements of the dynamical description of real objects. If one considers a single DOF, the matrices \( \gamma_0, H, S \) and \( M \) are of size \( 2 \times 2 \). Consider an arbitrary real \( 2 \times 2 \) matrix \( K \):

\[
K = \begin{pmatrix} a & b \\ c & d \end{pmatrix}
\]  

This parametrization is simple and (sort of) nearby but it does not fit the needs of Hamiltonian theory. Any matrix is the sum of a Hamiltonian and a skew-Hamiltonian matrix. The chosen parameters should belong to either of them. Since Hamiltonian matrices have zero trace, we can easily identify the skew-Hamiltonian part as a multiple of the unit matrix \( \eta_3 = 1_2 \):

\[
K = s_0 \eta_0 + s_1 \eta_1 + s_2 \eta_2 + c \eta_3,
\]

where

\[
\eta_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \eta_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \eta_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad \eta_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1
\]

are the real Pauli matrices (RPMS). \( \eta_0 = (\gamma_0)_{2 \times 2} \) is the SUM for a single DOF. Hence the RPMS provide a parameterization that matches the symmetries relevant in Hamiltonian dynamics:

\[
K = \begin{pmatrix} c + s_2 & s_0 + s_1 \\ s_1 - s_0 & c - s_2 \end{pmatrix}.
\]

An analysis of the properties of these matrices reveals that

\[
(\eta_0)^2 = -1, \quad (\eta_1)^2 = 1, \quad (\eta_2)^2 = 1, \quad (\eta_3)^2 = 1
\]

Furthermore one finds that the three non-trivial matrices mutually anti-commute, i.e. for \( i, j \in \{0, 1, 2\} \):

\[
\eta_i \eta_j + \eta_j \eta_i = 2 \text{Diag}(-1, 1, 1).
\]

All individual RPMS are either symmetric or skew-symmetric, they either pairwise commute or anti-commute, they square to \( \pm 1_2 \) and they are either Hamiltonian or skew-Hamiltonian, symplectic (Eq. 25) or skew-symplectic (Eq. 25). Their trace vanishes except for the unit matrix. The relevant symmetries of the Pauli algebra are given by

\[
\eta_i \eta_j = \pm \eta_j \eta_i, \quad \eta_i^2 = \pm 1, \quad \eta_i = \pm \eta_i^T, \quad \text{Tr}(\eta_i) = 0 \text{ unless } \eta_i = 1
\]

Note the math fact that skew-symmetric matrices \( \eta_i \) square to \(-1\) while symmetric matrices \( \eta_i \) square to \(1\) \(\text{[33]}\). The signature (the sign of the trace of the square) of the Pauli matrices corresponds to their symmetry under matrix transposition.

The type of transformation that these matrices generate (Eq. 23) is the matrix version of Euler’s formula \( e^{i\phi} = \cos \phi + i \sin \phi \):

\[
\exp(\eta_i \phi) = \cos \phi + \eta_i \sin \phi \quad \text{for} \quad \eta_i^2 = -1
\]

\[
\exp(\eta_i \phi) = \cosh \phi + \eta_i \sinh \phi \quad \text{for} \quad \eta_i^2 = +1
\]

Formally trigonometric functions belong to rotations while the hyperbolic functions belong to boosts \(\text{[52, 53]}\). Hence Eq. 42 suggests that it is thinkable to obtain an account of the Lorentz transformations directly on the basis of Hamiltonian algebras: Without considering spatio-temporal notions the Hamiltonian algebra of proper time automatically generates the mathematical means to describe Minkowski type space-times.

Note that only the transformation matrix for rotations is symplectic and orthogonal, while for boosts it is only symplectic. The matrix algebra of a single DOF is the real Pauli algebra. Since we derived the significance of the real Pauli matrices (RPMS) from classical Hamiltonian theory, the Pauli algebra can not be quantum.

E. The Kronecker Product and Hamiltonian Clifford Algebras

Two methods to generalize the Pauli algebra are possible: One can either add more DOFs and analyze the properties of Hamiltonian systems with two, three, four DOF and so on, or one may use a multiplicative approach based on the Kronecker product. The next system, constructed from an additive approach has two DOF and requires the use of the real \( 4 \times 4 \) matrices, i.e. the real Dirac algebra. Three DOF would require \( 6 \times 6 \)-matrices and one can anticipate that the natural symmetries inherited from the real Pauli matrices will be broken.

The multiplicative generalization is based on Kronecker (or tensor) products. The Kronecker product of

\footnote{D.N. Klyshko considered that many if not all “quantum paradoxes” have a common origin, namely the “failure to find a solution to a certain moments’ problem” \text{[53]}.}
two Pauli matrices $A = \{a_{ij}\}$ and $B = \{b_{kl}\}$ is given by:

$$C = A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & a_{11}b_{21} & a_{12}b_{11} & a_{12}b_{21} \\ a_{11}b_{12} & a_{11}b_{22} & a_{12}b_{12} & a_{12}b_{22} \\ a_{21}b_{11} & a_{21}b_{21} & a_{22}b_{11} & a_{22}b_{21} \\ a_{21}b_{12} & a_{21}b_{22} & a_{22}b_{12} & a_{22}b_{22} \end{pmatrix},$$

i.e. the Kronecker product is a method to systematically write down all possible products between all elements of $A$ and $B$, respectively. The general rules of Kronecker matrix products are [54]:

$$\begin{align*}
(A \otimes B)T &= A^T \otimes B^T \\
A \otimes (B + C) &= A \otimes B + A \otimes C \\
(A \otimes B)(C \otimes D) &= AC \otimes BD \\
\text{Tr}(A \otimes B) &= \text{Tr}(A) \text{Tr}(B) \\
(A \otimes B)^{-1} &= A^{-1} \otimes B^{-1}
\end{align*}$$

It is straightforward to verify that Kronecker multiplication preserves the symmetries of Eq. [41].

The Kronecker product allows to construct all Clifford algebras (CAs) with real matrix representations from the real Pauli matrices, i.e. all Hamiltonian Clifford algebras (HCA) [18]. Apparently there is exactly one algebra that follows both rules, which is the only additive and multiplicative generalization of the Pauli algebra, namely the real Dirac algebra.

Before we discuss the Dirac algebra, we shall first give a (very) brief introduction to CAs as they are usually presented, i.e. without reference to Hamiltonian theory, and explain our motivation to restrict us to Clifford algebras with (irreducible) real matrix representation.

VI. CLIFFORD ALGEBRAS

Mathematically Clifford algebras can be defined as generated by a list $e_k, k \in [0 \ldots N - 1]$ of $N$ pairwise anticommuting elements that hold $e_k^2 = \pm 1$. These are called the generators of the Clifford algebra. If one considers $p$ generators that square to 1 and $q = N - p$ generators that square to $-1$, then the algebra is denoted by $Cl(p, q)$ and it has a signature (or metric tensor) $g_{\mu \nu} = \text{Diag}(1, \ldots, 1, -1, \ldots, -1)$ with $p$ positive and $q$ negative entries in the diagonal.

The $N$ generators can be used to obtain new elements by multiplication since products of two (or more) different generators $e_i e_j$ are unique elements, different from the unit element and different from each factor, which square to $\pm 1$. It follows from combinatorics that there are $\binom{N}{k}$ products of $k$ generators, called $k$-vectors, so that one has

$$\sum_k \binom{N}{k} = 2^N$$

elements in total. Hence $m \times m$ matrix representations require at least the same number of independent parameters, so that $m^2 \geq 2^N$ or $m \geq 2^{N/2}$. An isomorphism between a given CA and some corresponding matrix algebra is only possible for even $N$ with $m = 2^{N/2}$.

Clifford algebras are used in various branches of physics, but the Dirac algebra is of special interest since it matches the geometry of $3 + 1$-dimensional spacetime as David Hestenes has shown in many publications (see for instance Ref. [55]). The commutator table of the Dirac algebra also determines the form of the electromagnetic field tensor [56]. Furthermore the anti-commutation properties, enable to naturally explain the vector cross product and therefore to describe the handedness of space [53], a product that has otherwise to be introduced by some commandment.

Certainly CAs are an interesting mathematical topic in themselves and Hestenes has shown, that it is possible to give a presentation of Dirac’s theory without matrix representations [55]. Indeed, from a general mathematical point of view, this question can be outsourced into a special branch of mathematics called representation theory, which is then another interesting topic in its own right. It is a fairly common approach to abstract CAs from their respective matrix representations. And of course CAs do not need Hamiltonian theory to be interesting and useful. They generate geometric spaces even if they are not considered in our specific context of Hamiltonian phase spaces.

But in the dynamical context of our presentation, matrix transposition is an indispensable element that allows to distinguish Hamiltonian from skew-Hamiltonian elements and to analyze their algebra (Eq. [53]). It is essential for the motivation to consider Clifford algebras at all. [19]

A. The Complex Numbers

Consider for instance the case $N = 1$ in which we have, besides the unit element $1$ only one single non-trivial element $e_0$, which gives the algebra $Cl(0, 1)$.

18. As we started out from a conservation law, we are specifically interested in Hamiltonian Clifford algebras, and in Clifford algebras mainly as they preserve the dynamical symmetries of symplectic theory, but we are not specifically interested in Clifford algebras as such.

19. For a discussion of real CAs in the context of linear Hamiltonian theory see [33, 53, 54, 57]. We would have preferred to also cite other authors in this context, but haven’t found much. In Refs. [55, 54], for instance, the name “Hamilton” is mentioned exclusively in context of quaternions. That CAs might be useful in the context of classical Hamiltonian dynamics, is not considered.
If this element squares to $-1$, then the corresponding “Clifford number” (CNs) $x$ has the form $x = a1 + be_0$. The multiplication of two CNs is then

$$xy = (a_x 1 + b_x e_0)(a_y 1 + b_y e_0)$$

$$= (a_x a_y - b_x b_y) 1 + 2(a_x b_y + b_x a_y) e_0. \quad (46)$$

This is the product of two complex numbers, which are hence almost identical to the Clifford algebra $Cl(0,1)$. We say “almost”, because the theory of complex numbers knows the operation of complex conjugation, which has no correspondence in $Cl(0,1)$ unless we refer to real matrix representations.

The operation of complex conjugation however can be naturally obtained from representation theory, namely if we use the real Pauli matrices. Then $e_0 = \eta_0$ is the (only) skew-symmetric element and complex conjugation is identical to matrix transposition, so that in the conventional notation $z = x + iy$ is

$$z = \begin{pmatrix} x & y \\ -y & x \end{pmatrix} \quad (47)$$

and

$$z^* = x - iy = z^T = \begin{pmatrix} x & -y \\ y & x \end{pmatrix}. \quad (48)$$

This means that transposition and complex conjugation can not be properly distinguished with full logical rigour. One might also substitute matrix transposition by a multiplication with the signature of the corresponding Clifford $k$-vectors. But again, the signature is only fixed, if exclusively real representations are used.

It also means that the complex numbers are, regarded from this perspective, a special case of (the algebra of) $2 \times 2$-matrices. As we have shown in Ref. [33], the reduction of the real Pauli algebra to the algebra of the complex numbers corresponds to the reduction to the general LOM of a DOF to normal form, to an harmonic oscillator. It is well known in many branches of physics, for instance in accelerator physics, that the unit circle in the complex plane is the normal form trajectory of the motion of a single DOF so that the complexity of the actual state of affairs can be reduced to a single number, namely the “phase advance” (i.e. time) $\Theta$.

The analysis of normal forms is of course a useful mathematical technique, but one should always keep in mind, that it describes the system in a special coordinate system and that coordinate transformations have a two-fold meaning. They can be understood as passive transformations and, regarded this way, they just concern our mathematical methods to describe a given physical process. But symplectic transformations also describe the full space of physical possibilities, of possible evolutions in time. This space of possibilities is substantially narrowed if we restrict our math to the use of normal forms only.

B. The Unit Imaginary And The Dirac Equation

Foreclosing what we are going to argue below, let us remark that it is not as nearby as often suggested to consider representations “over” the complex numbers as something fundamental [23]. In fact we suspect that the incautious use of complex (or quaternionic) “numbers” substantially contributes to the scrambling of the quantum omelette [24]. To understand this point correctly is indispensable for a successful unscrambling. Let us therefore spend a few words for it.

It is part of the logic of quantum theory that the Schrödinger equation is the non-relativistic approximation of the Dirac equation [62]. Hence the Dirac equation must be regarded as more fundamental than the Schrödinger equation and is the logical basis of quantum theory. Nonetheless, discussions about the interpretation of QM rarely refer to Dirac’s theory. As Hestenes noted, “[it] has long puzzled me is why Dirac theory is almost universally ignored in studies on the interpretation of quantum mechanics, despite the fact that the Dirac equation is widely recognized as the most fundamental equation in quantum mechanics” [64].

Dirac’s theory is required to provide Lorentz covariance, to explain the spin, the gyromagnetic ratio and essential parts of the hydrogen spectrum. The Dirac equation is the basis of QED and QFT. In the current approach it is Dirac’s theory that underlies most of modern physics and it is therefore annoying that students must first and sometimes exclusively undergo the brainwashing [25] of textbook QM before they have the chance to understand that this formalism is essentially classical [26]. It is annoying that most textbooks on quantum theory do not treat the Dirac equation at all or just briefly as a kind of addendum, in the last chapter or the second volume.

The Dirac equation, however, allows for but it does not require the explicite appearance of the unit imaginary. This becomes quite obvious by the fact that the complex Clifford algebra $Cl(1,3)$ can be directly replaced with

20 See also Ref. [61]: “[...] complex numbers are not required in order to describe quantum mechanical systems and their evolution [...]”.

21 “Our present QM formalism is a peculiar mixture describing in part laws of Nature and in part incomplete human information about Nature—all scrambled up together by Bohr into an omelette that nobody has seen how to unscramble.” [26]

22 Murray Gell-Mann is quoted with the following words: “The fact that an adequate philosophical presentation [of quantum physics] has been so long delayed is no doubt caused by the fact that Niels Bohr brainwashed a whole generation of theorists.” [67]

23 The author heart about the fact that the dynamics governing the time evolution of Quantum theory is indeed classical for the first time during his PhD, 1998 or 1999, in a talk given by John Ralston at DESY in Hamburg. The title of the talk was “Spin and the well-dressed Quark”. Like F. Strocchi in 1966, Ralston argued that the time-dependent part of Schrödinger’s equation is identical to Hamilton’s equations of motion [24, 67].
the real Clifford algebra $Cl(3,1)$, just by letting the $\gamma$-matrices “absorb” the unit imaginary. Due to Pauli’s fundamental theorem of the Dirac matrices any choice of Dirac matrices that generates the same metric, can be obtained by similarity transformations from each other and is hence physically equivalent. Hence one can always use the (purely imaginary) Majorana matrices. If we denote the generating elements of the real algebra $Cl(3,1)$ by $\gamma_\mu$ and those of a complex CA $Cl(1,3)$, using the Majorana basis, by $\Gamma_\mu$, then $i\Gamma_\mu = \gamma_\mu$ and we are done. The “complex” version:

$$(i\Gamma_\mu \partial_\mu \pm m)\psi = 0$$

becomes

$$(\gamma_\mu \partial_\mu \pm m)\psi = 0$$

with purely real matrices $\gamma_\mu$. Since this is just a notational issue, it can by no means imply different physics\(^{24}\). The algebraic form that is obtained in energy-momentum space is the eigenvalue equation

$$(\gamma_0 \mathcal{E} + p_x \gamma_1 + p_y \gamma_2 + p_z \gamma_3 \pm im)\psi = 0,$$

where $\lambda = \pm im$ is an eigenvalue of the matrix

$$\mathbf{H} = \gamma_0 \mathcal{E} + p_x \gamma_1 + p_y \gamma_2 + p_z \gamma_3,$$

such that a positive mass corresponds to a purely imaginary eigenvalue $\lambda$, as required in stable linear Hamiltonian systems.\(^{23}\) Moreover, the matrix $\mathbf{H}$ is a Hamiltonian matrix and the Dirac equation is therefore just a special case of Eq.\(^{10}\) hence it is but classical linear Hamiltonian theory, applied to fundamental variables.

This is the proof that the Dirac equation is as such not quantum, as it can be obtained classically. It will be discussed in more detail below. The use of the unit imaginary is pure notation and does not make it quantum either. The use of scaling factor like $\hbar$ also can not be quantum.\(^{72}\) Since the spin is an original result of Dirac’s theory, also spin is not quantum. Hence, if the more general equation, namely the relativistic equation of Dirac, is not quantum, then the non-relativistic approximation, Schrödinger’s equation, can’t be quantum either.

\(^{24}\) Of course, the use of $Cl(3,1)$ instead of $Cl(1,3)$ is accompanied with the use of the of the so-called “east coast metric” (ECM) instead of the “west coast metric” (WCM).\(^{67}\) But though the WCM is used more often, both choices are physically equivalent. As mentioned by Woit, Weinberg preferred the ECM in his presentation of QFT.\(^{70}\) A third notational convention that writes time as a kind of imaginary fourth coordinate in the form of $dx^2 = dx^1 + dx^2 + dx^3 + dx^4$ with $dx_4 = i\, c\, dt$ has also been used, for instance by Sommerfeld in Ref.\(^{71}\) and by Einstein in Ref.\(^{72}\).

### C. Math and Physics

In our view it is important to understand that, though physics requires the use of math (and mathematical logic), mathematics does not require physics, i.e. even if many tools developed in pure mathematics turned out later to be useful in physics, they usually have not been designed for this purpose. Physicists have to re-design and to select the mathematical tools that serves their needs best. It is obvious that nature restricts the mathematics that is useful for physics and the approach of this work is based on a restricted use of the math.

The PSR suggests the equivalence of all variables in $\psi$, until it turned out to be inevitable to break this equivalence formally with the introduction of a skew-symmetric matrix $\gamma_0$, the symplectic unit matrix. The skew-symmetry of $\gamma_0$ suggested to formally introduce canonical pairs. This is the reason why Schrödinger’s non-relativistic equation must be complex: It needs to implement a canonical pair to generate a constant of motion.\(^{68,67}\) We insisted on real-valued dynamical variables $\psi$ and PDCCOM $\mathcal{H}$, but this does not imply to abandon the use of the complex numbers as such.

As stressed before, stable Hamiltonian systems have purely imaginary eigenvalues and complex eigenvectors. This is accepted and known from classical mechanics and is nowhere regarded as an argument for suspicious conclusions about the realness of the dynamical quantities. What is unacceptable however in a classical setting is the a priori use of complex numbers for the coordinates $\psi$ or the conserved quantity $\mathcal{H}$ due to insouciance or by a commandment. We have shown it is not required by the math of QM: the unit imaginary does not generate quanumness.

We can not prevent anyone from using a suggestive notation, but we can doubt that a specific notation is physically relevant. The only logical constraint for $\psi$ is an even number of variables. This alone does not require the use of complex numbers, even if it might be convenient to use them in specific problems.

Though the use of complex numbers for $\psi$ and $\mathcal{H}$ is not wrong per se, but in the context of Dirac’s theory it wrongly suggests the physical equivalence of all sixteen matrices, while the restriction to the reals enables to properly distinguish between ten Hamiltonian and six skew-Hamiltonian components. Also Dirac found and discussed only ten generators (and not sixteen)\(^{74}\).

If intended or not, it seems that the complex notation mainly serves the purpose of scrambling the quantum omelette. But as Ralston argued, there is little in quantum theory that proves the non-reality of the wavefunction: “Bohr and Heisenberg had made up their minds about a philosophy of unreality before the actual quantum theory existed.”

In his book “Der Teil und das Ganze”, published 1969, Heisenberg frankly admitted that already in 1926 he had made up his mind, that nature must be discontinuous. Even 43 years later, he was unable (or unwilling) to pro-
vide arguments in support of this conviction. Furthermore his writings suggests that for him, objectivity was identical to a spatio-temporal description\(^2\). A spatio-temporal description however is not a condition for but a consequence of objectivity.

### D. Hamiltonian Clifford Algebras

The usefulness of Clifford algebras in the context of Hamiltonian theory is due to Eqs. 53\(^3\).

Given we have a set of \(N\) anti-commuting matrices \(\gamma_k, k \in [1..N - 1]\) and the SUM \(\gamma_0\) generated by (repeated) Kronecker multiplication of the real Pauli matrices, then the matrix system has a dimension \(2^N = 2^m \times 2^m\).\(^4\) It follows that also all real Dirac matrices are either Hamiltonian or skew-Hamiltonian:

\[
\begin{align*}
\gamma_0 \gamma_k \gamma_0 &= \pm \gamma_0 (\gamma_0 \gamma_k) \\
&= \pm \gamma_k \\
&= \pm \gamma_k^T
\end{align*}
\]  
\(\text{(53)}\)

Then any matrix \(\gamma_\mu\) that anti-commutes with \(\gamma_0\), holds

\[
\begin{align*}
\gamma_0 \gamma_\mu \gamma_0 &= \gamma_0 (\gamma_0 \gamma_\mu) \\
&= \gamma_\mu
\end{align*}
\]  
\(\text{(54)}\)

(since \(\gamma_0^2 = -1\)) and is therefore either Hamiltonian and symmetric or skew-Hamiltonian and skew-symmetric. This connection between the different symmetries is of severe importance for the theory of Hamiltonian Clifford algebras and has consequences for the general description of \(n\) DOF. It is specifically the mixture of the properties of Clifford algebras and Hamiltonian constraints (Eq. 53), that produces new and complex structures.

As the generators of Clifford algebras all anti-commute (by definition), Eq. 54\(^3\) is such a constraint: In systems in which all generators of the Clifford algebra are also generators of symplectic motion (i.e. Hamiltonian), the metric tensor necessarily has the form \(g_{\mu\nu} = \text{Diag}(1, 1, \ldots, 1, -1)\) and the Clifford algebra has dimension \(\text{Cl}(p, q)\) with \(q = 1\) and \(p = N - 1\). This implies that the formalism reproduces the fact that \textit{time is unique.}\n
The use of Hamiltonian \textit{Clifford} algebras (and not only Hamiltonian algebras) for the parametrization of even moments of phase provides maximal symmetry with respect to the individual variables in \(\psi\) as well as with respect to the individual DOFs \(2^N\). This naturally conforms the requirement of the PSR to treat all elements on equal grounds: the matrix representation of all \(k\)-vector elements \(\gamma_A\) of any real Clifford algebra has one (and only one) entry of \(\pm 1\) in each row and each column while all other elements are identically zero.

Hamiltonian Clifford algebras \(\text{Cl}(p, q)\), i.e. CAs with real representations, exist only for \(77\)

\[
p - q = 0, 1, 2 \text{ mod } 8, \quad (55)
\]

which, since \(N = p + q\) must be even, reduces in our case to

\[
p - q = 0, 2 \text{ mod } 8. \quad (56)
\]

From Eq. 54\(^3\) we derived that, if all generators of \(\text{Cl}(p, q)\) are Hamiltonian (i.e. “observable”), then one has \(q = 1\) and \(p = N - 1\) so that

\[
N - 2 = 0, 2 \text{ mod } 8. \quad (57)
\]

This selects the dimensionalities listed in Tab. I as candidates of special interest within Hamiltonian theory. The simplest algebras are the real Pauli algebra \(\text{Cl}(1, 1)\), and the real Dirac algebra \(\text{Cl}(3, 1)\).

According to Eq. 57\(^3\) there are two sequences of HCAs, given by

\[
N = 8m + 2 \quad (58)
\]

which we call a HCA of the \textit{Pauli type} and

\[
N = 8m + 4 \quad (59)
\]

which we call a HCA of the \textit{Dirac type} where \(m \in \mathbb{N}\). If the dynamical significance of CAs and hence the requirement of a real representation is ignored, then CAs can be defined for practically any dimensionality. In a purely mathematical setting, this might be an interesting generalization, but in a physical context it is the easy road to dynamical misconceptions.

In order to understand the logical and dynamical properties of the individual elements of HCAs of the mentioned dimensionalities, it is important to notice that all generators except the SUM \(\gamma_0\) are symmetric real matrices. Since we have no specific argument to prefer any of them, it is nearby to consider the role of matrices that play a special role by their formal position within the CA. Besides the SUM \(\gamma_0\) and the unit matrix \(1\), any CA

---

\(^{25}\) Latin indices \(\gamma_k\) denote a range \(k = [1, \ldots, N - 1]\), greek indices \(\gamma_\mu\) a range \(\mu = [0, 1, \ldots, N - 1]\).

\(^{26}\) The Hamiltonian algebra of \(6 \times 6\)-matrices as it is usually associated with the classical motion of particles in “physical” space, has less symmetry.
has two unique elements, the first being the \( N \)-vector \( \gamma_\pi \), which is the product of all generators

\[
\gamma_\pi = \prod_{\mu=0}^{N-1} \gamma_\mu
\]

(60)
called pseudo-scalar.

Since \( N \) must be an even integer, the pseudo-scalar anti-commutes with all generators (vector elements), so it therefore commutes with all 2-vectors and anti-commutes again with all 3-vectors and so forth: The pseudo-scalar distinguishes even from odd \( k \)-vectors. Therefore the pseudoscalar of the Dirac algebra induces charge conjugation, namely a change of sign of the bi-vectors only (see Sec. 6 below).

As derived in App. D, the pseudoscalar of Pauli type HCAs is Hamiltonian and symmetric while in Dirac type algebras it is skew-Hamiltonian and skew-symmetric.

Furthermore, as shown in App. D in both, the Pauli type and the Dirac type algebras, only \( k \)-vectors with \( k = 1, 2, 5, 6, 9, 10, \ldots \) are Hamiltonian while \( k \)-vectors with \( k = 3, 4, 7, 8, \ldots \) are skew-Hamiltonian (App. D).

Another special element is the product \( \gamma_0 \gamma_\pi \), which is the product of all Clifford generators except \( \gamma_0 \). This operator anti-commutes with the SUM and \( \gamma_\pi \), but commutes with all other generators of \( Cl(N-1,1) \). It can hence distinguish between the two types of Clifford generators and part of the CPT-theorem [18].

E. The real Dirac algebra

Since the smallest system with some kind of internal dynamics, with interaction, is composed of two DOF and described by the Dirac algebra, it is as fundamental as the real Pauli algebra.

The usefulness of the real Dirac algebra in classical Hamiltonian theory has been described in previous works [56, 57]. It was shown, for instance, that a general block-diagonalization of stable Hamiltonian matrices can be achieved with a Jacobi type iterative algorithm: In each step, two DOF are block-diagonalized based on symplectic similarity transformations using the real Hamiltonian Dirac matrices as generators [57, 78]. Hence the real Hamiltonian Dirac algebra suffices to describe all possible linear interactions between two Hamiltonian DOF.

In the previous section we derived the conditions for a possible isomorphism between real matrix reps of CAs \( Cl(p,q) = Cl(N-1,1) \) and Hamiltonian algebras. Note that the size of the spinor that corresponds to \( Cl(N-1,1) \) is \( 2n = 2^{N/2} \) so that \( Cl(9,1) \) corresponds to a spinor of size \( 2n = 2^5 = 32 \) and an algebra with \( 2^N = 1024 \) elements, \( n(2n+1) = 528 \) Hamiltonian and 496 skew-Hamiltonian elements. These numbers alone clearly indicate that \( Cl(9,1) \) can not represent the simplest possible RPO. But there are more reasons why a RPO must be composed of two DOF, which are discussed elsewhere [33, 48]. Here we restrict us to a short summary of the main points: we stress again that the Dirac algebra with \( 4 \times 4 \)-matrices is the minimal size required to represent the general case of complex eigenvalues. But there is no fundamental reason for nature to exclude those types of dynamical processes that require, maybe for a short time, complex eigenvalues; they belong to the full scope of possibilities.

Secondly, if the number of variables in the spinor is supposed to correspond to the number of variables representing the RPO, then

\[
2n = N = 2^{N/2}
\]

(61)
which has only two solutions, namely \( N = 2 \) or \( N = 4 \). And thirdly, as we shall elaborate now, the system of Clifford generators should determine the structure of the algebra and hence provide the basic web of physical notions uniquely, without ambiguity.

Regarding the real Dirac algebra \( Cl(3,1) \), one has the following unique elements: The SUM \( \gamma_0 \), the pseudoscalar \( \gamma_\pi \), \( \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) and the product of both \( \gamma_0 \).

While there is only a single skew-symmetric element in the real Pauli algebra, the Dirac algebra contains six of them. By Pauli’s fundamental theorem of the Dirac matrices [65] it is allowed to select any of the skew-symmetric matrices to represent the SUM \( \gamma_0 \). Above we have chosen the form \( 1_3 \otimes \eta_0 \). If one choses to use a different skew-symmetric matrix as SUM, this is equivalent to a permutation of the order of the elements in \( \psi \). One obtains \( \psi = (q_1, q_2, p_1, p_2)^T \) in case of \( \gamma_0 = \eta_0 \otimes 1_2 \). Hence the real Dirac matrices have their meaning relative to the initial choice of the SUM.

Next one has to select one of nine symmetric matrices [27], however it is a math fact that \( \gamma_0 \) anti-commutes only with six of them. Hence one has to choose again one out of six matrices [28] and fix it as \( \gamma_1 \). This choice is again arbitrary insofar as all choices give the same physics [56]. But the selection of these two matrices suffices to decide about the type of all remaining matrices, i.e. whether they are vectors, bi-vector and so forth.

According to Eq. 33 and Eq. 32 there are 10 Hamiltonian elements in the Dirac algebra, but we identified only 4 of them, namely the generators (called 1-vectors or simply vectors). In the previous section we have shown that in HCAs in which all generators of the Clifford algebra are Hamiltonian, only \( k \)-vectors for \( k \in \{1, 2, 5, 6, 9, 10, \ldots \} \) are Hamiltonian. Since the highest \( k \)-vector of the Dirac algebra is the pseudo-scalar with \( k = 4 \), a Dirac type Hamiltonian may contain only vector and bi-vector elements. We use \( \gamma_{14} = \gamma_0 \gamma_1 \gamma_2 \gamma_3 \) to denote the pseudo-scalar.

27 The unit matrix can not be a generator of a CA since it commutes with all others.

28 We can’t tell if “god throws dice” or not. Here dice are an option.
The first (symmetric) bi-vectors are given by
\[\begin{align*}
g_4 &= \gamma_0 \gamma_1 \\
g_5 &= \gamma_0 \gamma_2 \\
g_6 &= \gamma_0 \gamma_3,
\end{align*}\]
and the second (skew-symmetric) set by
\[\begin{align*}
g_7 &= \gamma_1 \gamma_4 = \gamma_2 \gamma_3 \\
g_8 &= \gamma_1 \gamma_5 = \gamma_3 \gamma_1 \\
g_9 &= \gamma_1 \gamma_6 = \gamma_1 \gamma_2
\end{align*}\]
The skew-Hamiltonian 3-vector elements are
\[\begin{align*}
g_{10} &= \gamma_1 \gamma_0 = \gamma_1 \gamma_2 \gamma_3 \\
g_{11} &= \gamma_1 \gamma_1 = \gamma_0 \gamma_2 \gamma_3 \\
g_{12} &= \gamma_1 \gamma_2 = \gamma_0 \gamma_3 \gamma_1 \\
g_{13} &= \gamma_1 \gamma_3 = \gamma_0 \gamma_1 \gamma_2
\end{align*}\]
The last element is the scalar \(g_{15} = 1\), e.g. the unit matrix.

Any \(4 \times 4\)-matrix \(M\) can be written as a linear combination of the real Dirac matrices (RDMs):
\[M = \sum_{k=0}^{15} m_k \gamma_k\]
Where a sequential index \(k \in [0, \ldots, 15]\) is used instead of the multi-index convention \(\gamma_{\mu \nu}\).

This means that the Dirac algebra enable, as the real Pauli algebra, for a re-parametrization of the elements of \(4 \times 4\)-matrices, suited to symmetries relevant in abstract Hamiltonian dynamics. This is usually presented in wrong order: It is true that Dirac introduced his matrices with heuristic arguments from the relativistic energy-momentum relation. But it is not forbidden to prefer a logical presentation instead of a historical one.

Since all RDMs besides the unit matrix are orthogonal and have zero trace, one obtains the coefficients \(m_k\) by
\[m_k = \frac{1}{4} \text{Tr}(M \gamma_k^T)\]
The general form of the symplex \(H\) that couples two degrees of freedom is a linear combination of ten symplexes, of 4 vectors and 6 bi-vectors:
\[\begin{align*}
H &= \sum_{k=0}^{9} f_k \gamma_k \\
S &= \sum_{k=0}^{9} s_k \gamma_k
\end{align*}\]
In order to simplify the calculation one may use Eq. (67) to analyze the result of a SST (compare Eq. (27)):
\[S(\tau) = \exp (\gamma_a \tau) S(0) \exp (-\gamma_a \tau)\]

VII. FUNCTION FOLLOWS FORM

We promised that the classical notions of mass, energy and momentum would follow from the logic of the imposed dynamical constraint. Of course it is impossible to provide a logical proof for an interpretation. Interpretations can be consistent and plausible, but not logical or illogical.

As well known in classical mechanics, the generators of canonical transformations correspond to physical quantities. We started with a single PDCOM and apparently this suffices to explain the emergence of 10 quantities that may act as generators of SSTs within the Dirac algebra waiting for an interpretation, four vector components and 3 + 3 bi-vector components. The tri-vectors, the scalar and pseudo-scalar are skew-Hamiltonian, they do not correspond to non-zero correlations, and require no interpretation, at least at this point.

One can construct \(k\)-vectors with even \(k\) from products of \(k\)-vectors for \(k\) even or odd, but one can not obtain \(k\)-vectors with odd \(k\) from products of even \(k\)-vectors: bi-vectors can be obtained multiplicatively from vectors but not vice versa. In other words: the even elements, namely the 0-vectors (scalar), the six bi-vectors, and the 4-vector element (pseudoscalar) form the even subgroup. This holds for all even-dimensional CAs, i.e. with \(N = 2M, M \in \mathbb{N}\).

Hence the algebra forces us to distinguish between the set of quantities associated with the vector elements \((\gamma_0, \gamma_2, \gamma_6)\) and two sets of 3 bi-vector each, namely the symmetric elements \(\gamma_4, \gamma_5, \gamma_6\) and the skew-symmetric elements \(\gamma_4, \gamma_5, \gamma_6\).

This structure suggest to interpret vector components as representing the particle (RPO) and bi-vector components as fields: objects are the sources of fields, fields act on objects. Hence, it is nearby to interpret the six bi-vectors, 3 skew-symmetric and 3 symmetric, as generators of the Lorentz transformations. This is indeed the case.

For a detailed account of the Lorentz transformations as they naturally emerge from \(Cl(3, 1)\), see Refs. [53, 54, 57]. Here we just mention the result, namely that the skew-symmetric bi-vectors \(\gamma_7, \gamma_8\) and \(\gamma_9\) are generators of spatial rotations while the symmetric bi-vectors \(\gamma_4, \gamma_5 \) and \(\gamma_6\) generate Lorentz boosts in the corresponding directions, both a mathematical consequence of Eq. (42) or Eq. (68) respectively. This interpretation is completely

29 This holds for exponentials of non-singular matrices. The general case is described for instance in Ref. [74].
determined by the structure of the Hamiltonian Dirac algebra and by the transformation properties of the quantities under canonical transformations (SSTs).

Hence the structure of the real Dirac algebra is isomorphic to relativistic electrodynamics, it is allowed to interpret the parameters accordingly. The vector parameters of the auto-correlation matrix $S$, are then identified with energy and momentum:

$$s_0 \equiv \mathbf{E}$$  
$$(s_1, s_2, s_3)^T \equiv \mathbf{P}$$  \hspace{1cm} (69)

Hence, self-interaction is, within this linear approximation, unobservable.

Then, given the RPO is in interaction with external fields, one obtains:

$$\dot{\mathbf{P}} + \mathbf{F} = \mathbf{F}_x (\mathbf{P} + \mathbf{F}) - (\mathbf{P} + \mathbf{F}) \mathbf{F}_x$$  \hspace{1cm} (75)

It follows from the commutator table of the Dirac algebra, that this can be splitted into:

$$\dot{\mathbf{P}} = \mathbf{F}_x \mathbf{P} - \mathbf{P} \mathbf{F}_x$$  \hspace{1cm} (76)

which is the Lorentz force equation as we shall show in Sec. [VII D] and secondly

$$\dot{\mathbf{F}} = \mathbf{F}_x \mathbf{F} - \mathbf{F} \mathbf{F}_x$$  \hspace{1cm} (77)

which describes spin precession (see Sec. [VII D] below).

A. Units: The Schwinger Limiting Fields

In Eq. (67) different physical quantities like electromagnetic fields, energy and momentum are added. This is allowed if one uses appropriate natural units. Modern physics identified a number of scaling factors, namely the "speed of light" $c$ for the scale between mass, energy and momentum and between electric and magnetic fields, $\hbar$ for the scale between energy and frequency and the unit charge to scale fields relative to mechanical quantities. A detailed account of how physical constants are understood has been given in Ref. [10].

As we shall show below, according to the EMEQ, the eigenvalues of the Hamiltonian matrix correspond to the mass of a particle. Hence, if the RPO has the mass of the electron $m_e$, then this scales the electromagnetic fields automatically relative to the so-called Schwinger limiting fields $E_S$ and $B_S$, which were first derived by Sauter [80–82] $E_S$ and $B_S$ [83]. Hence the scaling factor between a magnetic field $B$ in SI-units and in units of frequency is of the order $\frac{c}{m}$ and for electric fields $E$ of order $\frac{m}{e}$. The fields as they appear here, are scaled relative to the properties of the RPO, $e$ and $m$.

B. The Eigenvalues of Dirac Hamiltonian

Let us first have a look at the eigenvalues of the Hamiltonian matrix "operator" that follows the parametrizations Eqs. (69) and (70) separately and combined. The trace of a matrix equals the sum of its eigenvalues, the

$$\text{trace of a matrix equals the sum of its eigenvalues,}$$

30 Then, since $\mathbf{H}$ and $\mathbf{S}$ are Hamiltonian, only odd terms can contribute, since only odd powers of a Hamiltonian matrix are Hamiltonian.

31 These fields are, given in SI-units: $E_S = \frac{m^2 c^3}{e \hbar} = 1.323 \cdot 10^{18}$ V/m and $B_S = \frac{m^2 c^2}{e \hbar} = 4.414 \cdot 10^{8}$ T. These values are beyond any technical scale. Only the largest modern pulsed lasers might allow to generate fields of this strength [83].
trace of the squared matrix equals the sum of the squared eigenvalues and so on. From Eq. 28 we know that the trace of any odd power of some Hamiltonian matrix vanishes. Hence only even powers are left, i.e. the second and fourth power:

\[
\begin{align*}
\text{Tr}(\mathbf{H}^2) &= \sum_k \lambda_k^2 \\
\text{Tr}(\mathbf{H}^4) &= \sum_k \lambda_k^4
\end{align*}
\] (78)

which allows to compute the eigenfrequencies \([33, 57, 79]\). The result is given by:

\[
\begin{align*}
K_1 &= -\frac{\text{Tr}(\mathbf{H}^2)}{4} \\
K_2 &= \frac{\text{Tr}(\mathbf{H}^4)}{16} - \frac{K_1^2}{4} \\
\omega_1 &= \sqrt{K_1 + 2\sqrt{K_2}} \\
\omega_2 &= \sqrt{K_1 - 2\sqrt{K_2}} \\
\omega_1^2 \omega_2^2 &= K_1^2 - 4K_2 = \text{Det} (\mathbf{H}) \\
K_1 &= \mathbf{E}^2 + \mathbf{B}^2 - \mathbf{E} \cdot \mathbf{B} \\
K_2 &= (\mathbf{E} \cdot \mathbf{B} + \mathbf{E} \times \mathbf{B})^2 - (\mathbf{E} \cdot \mathbf{B})^2 - (\mathbf{E} \cdot \mathbf{B})^2
\end{align*}
\] (79)

Hamiltonian matrices of stable systems have purely imaginary eigenvalues, corresponding to real frequencies \(\omega_i\), so that for stable systems one has \(K_2 > 0\) and \(K_1 > 2\sqrt{K_2}\).

From this we find that \(K_2 = 0\) when \(\mathbf{E} = \mathbf{B} = 0\), i.e. for pure vectors (Eq. 69)

\[
\omega = \pm \sqrt{\mathbf{E}^2 - \mathbf{P}^2},
\] (80)

and for pure bi-vectors, where \(\mathbf{E} = 0 = \mathbf{P}\), the frequencies are (Eq. 70)

\[
\omega = \pm \sqrt{\mathbf{B}^2 - \mathbf{E}^2 \pm 2\sqrt{(\mathbf{E} \cdot \mathbf{B})^2}}.
\] (81)

The frequencies are invariants under SSTs and hence are Lorentz scalars, i.e. invariant quantities. We therefore know that pure bi-vectors have two relativistic invariants, namely \(\mathbf{B}^2 - \mathbf{E}^2\) and \(\mathbf{E} \cdot \mathbf{B}\) and we know this without any reference to Maxwell’s equations. Furthermore we directly know that a stable bi-vector type oscillation is only possible if \(\mathbf{E} \cdot \mathbf{B} = 0\), since only under this condition one obtains real frequencies (aka purely imaginary eigenvalues) \([52]\).

In the theory of electromagnetic wave propagation one finds that \(\mathbf{B}^2 - \mathbf{E}^2 = 0\) so that, if this is inserted into Eq. 81, apparently electromagnetic fields have no eigenfrequency. This is generally known to be true, it nevertheless leads, in our approach, to a degenerate matrix \(\mathbf{H}\). This can be understood if we consider the frequency of the RPO Eq. 80 which apparently provides a constant and invariant frequency, which equals the mass of the RPO. As is well-known, the frequency of the Dirac spinor that describes a particle at rest is (up to constant scaling factors \(\hbar\) and \(c\)) identical to the mass. Hence the time variable \(\tau\) must be identified with proper time, the time of a co-moving observer. Then it is clear why the electromagnetic bi-vector has a vanishing frequency: electromagnetic waves, regarded from the perspective of a (hypothetical) comoving observer, are indeed static. Expressed in language of special relativity we would say that we can not transform into the co-moving frame of an electromagnetic wave, as it moves with the speed of light. However, this requires no commandment concerning space-time, but is a math fact about boost transformations.

C. Special Relativity in a Nutshell

The analysis of the Dirac algebra leads to the insight that an RPO is essentially described by the vector type elements that are associated with the 4-momentum \((\mathbf{E}, \mathbf{P})\):

\[
\mathbf{H} = \mathbf{E} \gamma_0 + \gamma_1 p_x + \gamma_2 p_y + \gamma_3 p_z
\] (82)

The square of this matrix is

\[
\mathbf{H}^2 = -\mathbf{E}^2 + \mathbf{P}^2 = -m^2
\] (83)

so that

\[
\mathbf{H}^2 \psi = -m^2 \psi,
\] (84)

i.e. the mass \(m\) is proportional to the oscillation frequency, an eigenvalue of \(\mathbf{H}\). It is a constant of motion and a scalar, a 0-vector of the Clifford algebra. Nonetheless we have no unique state of affairs of the RPO. We just selected a “mass shell”. The structure of the Dirac algebra given in the previous section suggests that the RPO as described by Eq. 82 is not in interaction, it is a free “particle”. Is it possible not only to formally derive rotations and boosts, i.e. Lorentz transformations (Ref. [53]), but also a space-time interpretation? This requires to switch to the next level of description, to use another emergent constant of motion as Hamiltonian. The original Hamiltonian described the motion of the spinor \(\psi\). But spinors are not directly measurable. In order to (re-) construct classical physics, we need a relation between observables in which the mass is just a constant “parameter”. It is nearby to reinterpret the equations of motion for observables (Eq. 17) in a Hamiltonian context. Then one obtains the “classical” (relativistic) Hamiltonian of a free RPO. The only unique choice for the Hamiltonian of the RPO is the parameter \(\mathbf{E}\), which then depends on the vector components of the momentum. Hence one obtains the classical relativistic energy-momentum-relation (EMR):

\[
\mathbf{E} = \mathcal{H}(\mathbf{P}) = \sqrt{m^2 + \mathbf{P}^2}.
\] (85)
The use of Hamilton’s equation of motion for the velocity \( \beta \) is then given by

\[
\dot{\beta} = \dot{q} = \nabla_\beta \mathcal{H}(p) = \frac{\vec{P}}{\mathcal{E}}
\]  

(86)

so that inserting the result into Eq. \( 35 \) yields

\[
\mathcal{E} = \gamma m
\]

(87)

where \( \gamma \equiv \frac{1}{\sqrt{1 - \beta^2}} \) and

\[
\vec{p} = m \gamma \vec{\beta}
\]

(88)

Hence, within our approach, it is just another math fact that the velocity \(|\vec{\beta}|\) is limited to 1. This is a property of space-time valid by construction. It is not a consequence of the constancy of the speed of light. Both, a maximal speed for massive objects as well as the Lorentz transformations, emerge from the same Hamiltonian formalism. Even the very concept of “speed” itself can be regarded as a result of this formalism.

There is no commandment and no a priori existent space-time required that determines the energy-momentum relation (EMR), which is itself a consequence of classical Hamiltonian theory and the algebra of proper time.

Since \(|\vec{\beta}| \leq 1\), one may write \( \beta = \tanh(\varepsilon) \), where \( \varepsilon \) is the so-called “rapidity”, and then one obtains \( \gamma = \cosh(\varepsilon), \mathcal{E} = m \cosh(\varepsilon) \) and \( p = m \sinh(\varepsilon) \).

Our approach is based on the Lorentz transformations and is therefore mathematically equivalent with the standard presentation of relativity theory (SPRT). Nonetheless it modifies the SPRT insofar as both, Lorentz transformations and “inertial frames” are notions that require no direct reference to space-time at all. We introduced and explained them in a purely Hamiltonian context. This came out almost automatically and we could not possibly have done otherwise. But relativity is not the central issue of this article and hence we can not elaborate in more detail. We refer to the introduction in Ref. \[84\].

D. The Lorentz Force

The three skew-symmetric Hamiltonian elements \( f_7, f_8, f_9 \) act as generators of rotations in a 3-dimensional parameter space and are therefore gyroscopic quantities as for instance magnetic fields or angular momenta. The three symmetric elements \( f_4, f_5, f_6 \) act as generators of boosts in a 3+1-dimensional parameter space and are hence associated with a linear accelerating quantity like the electric field. In other words, the parametrization by the use of Clifford algebras uncovers a unique structure and establishes certain transformation classes for otherwise unintepretable elements of the Hamiltonian matrices \( \mathbf{S} \) and \( \mathbf{H} \). If the real physical object is represented by the second moments of \( \mathbf{S} \), then the matrix \( \mathbf{H} \) contains the driving terms of the symplectic motion, which must then be called (self-) fields.

Eq. \( 76 \) written explicitly using the EMEQ, yields:

\[
\dot{\mathcal{E}} = \vec{P} \cdot \vec{E}
\]

\[
\dot{\vec{P}} = \mathcal{E} \vec{E} + \vec{P} \times \vec{B}
\]

(89)

which are the Lorentz force equations formulated in proper time \( \tau \). \[33, 48, 53, 57\]. With \( \frac{d}{d\tau} = \gamma \frac{d}{dt} \) one finds (assuming \( c = 1 \)):

\[
\frac{d\mathcal{E}}{d\tau} = m \dot{\vec{v}} \cdot \vec{E}
\]

\[
\frac{d\vec{P}}{d\tau} = m \vec{E} + m \vec{v} \times \vec{B}
\]

(90)

and hence

\[
\frac{d\mathcal{E}}{d\tau} = q \vec{v} \cdot \vec{E}
\]

\[
\frac{d\vec{P}}{d\tau} = q \vec{E} + q \vec{v} \times \vec{B}
\]

(91)

As explained in Sec. \[VII.A \] also the bi-vector fields, like all elements of \( \mathbf{H} \), have a unit of frequency and require a re-scaling by \( e/m \) to obtain their values in SI units. \[34\].

The second part, Eq. \( 77 \) describes the precession of the remaining correlations. We denote the internal bi-vectors by \( \vec{a} \) and \( \vec{s} \) to distinguish them from the external fields \( \vec{E} \) and \( \vec{B} \), so that

\[
\mathbf{F} = \gamma_0 \vec{a} \cdot \vec{\gamma} + \gamma_{14} \gamma_0 \vec{s} \cdot \vec{\gamma},
\]

(93)

and one obtains (Eq. \( 77 \)):

\[
\dot{\vec{a}} = \vec{a} \times \vec{B} + \vec{s} \times \vec{E}
\]

\[
\dot{\vec{s}} = -\vec{a} \times \vec{E} + \vec{s} \times \vec{B}
\]

(94)

If one uses a complex notation \( \vec{\sigma} = \vec{s} + i \vec{a} \) and \( \vec{F} = \vec{B} + i \vec{E} \), then

\[
\dot{\vec{\sigma}} = \vec{\sigma} \times \vec{F}
\]

(95)

which is the equation that describes the precession of the spin in an external field.

---

33 As we argued in Ref. \[10\], scaling factors like these can not be derived logically since they depend on a historical, hence arbitrary, choice of units. They have to be introduced in an ad hoc fashion if equations are to be aligned to the MKS system. See also Sec. \[VIII \].
E. Electromagnetic Waves and Spin (-flips)

The Lorentz force that we derived from Eq. 17 using the EMEQ, refers to static, or at least slowly varying, electromagnetic fields. The relevant frequency scale is given by the mass of the RPO, i.e. 0.511 MeV in case of electrons. Visible light belongs to frequencies of order of eV, i.e. several orders of magnitude below the typical de Broglie frequency of electron waves. Still, the wave length of visible light is in the order of nm, while technical fields, for instance in undulators or spectrometer magnets, vary with macroscopic “wave-length”, i.e. order of mm up to m, again several orders of magnitude larger than the wavelength of light. Hence it is legitimate to assume that the variation of the fields is slow.

How do we treat the case in which (the electromagnetic part of) \( \mathbf{H} \) varies, slowly compared to the de Broglie wave? Of course this depends on the type of variation. For electromagnetic waves, we know from Maxwell’s equations \(^{34} \) that polarized e.m. waves, as seen can be described by a rotating “Dreibein”. If \( \vec{E}(\tau = 0) = \vec{E} e_x \) and \( \vec{B}(\tau = 0) = \vec{B} e_y \), then, for an observer in some “inertial reference frame”, these vectors rotate with frequency \( \Omega \) around the \( z \)-axis. The generator for rotations around the \( z \)-axis is \( \gamma \). Hence the time dependency can be written, according to Eq. 17 as \(^{35} \)

\[
\hat{F} = \frac{\Omega}{2} (\gamma \mathbf{F} - \mathbf{F} \gamma) .
\]  

(96)

It is then possible (see App. B) to represent the time dependency of \( \mathbf{F} \) by adding the term \( \Omega / 2 \gamma \) to \( \mathbf{F} \), effectively the same as a magnetic field component \( B_z = \Omega / 2 \).

Hence a circular polarized electromagnetic wave can be described in this approach by effective electromagnetic terms which give \( \vec{E} \cdot \vec{B} = 0 \) and \( \vec{B}^2 - \vec{E}^2 = \Omega^2 / 4 \), i.e. with an additional energy term that is proportional to frequency \( \Omega^2 / 2 \) (times \( \hbar \), in MKS-units), in agreement with Eq. 81.

Assuming that we describe the RPO in its “rest frame”, then \( \vec{P} = 0 \) and the eigenfrequencies, given by Eq. 79 are:

\[
K_1 = \mathcal{E}^2 + B^2 + \Omega^2 / 4
\]

\[
K_2 = \sqrt{\mathcal{E}^2 B^2} = \mathcal{E}^2 (B^2 + \Omega^2 / 4)
\]

\[
\omega = \pm \sqrt{\mathcal{E}^2 + B^2 + \Omega^2 / 4 \pm 2 \mathcal{E} \sqrt{B^2 + \Omega^2 / 4}}
\]

(97)

so that with \( \omega = m = \text{const} \), assumed here to be positive, one finds two possible eigen-frequencies

\[
m = \mathcal{E} \pm \sqrt{B^2 + \Omega^2 / 4},
\]

(98)

\(^{34} \) We have shown how to derive Maxwell’s equations on the basis of this approach in Ref. \(^{48} \).

\(^{35} \) The factor \( 1/2 \) is required to generate a spatial rotation frequency \( \Omega \) and is a peculiarity of spinors \(^{53} \).
eigenfrequencies and two (pairs of complex conjugate) eigenvectors. The general state of motion is a superposition of these eigenvectors. This is purely classical Hamiltonian physics.

VIII. “CANONICAL” (UN-) QUANTIZATION

We did not yet show how to obtain the so-called canonical quantization, i.e. why a spatial derivative represents the momentum operator and a time derivative an energy operator. This requires to use a second method to construct space-time, “complementary” to the one used in Sec. VII C.

Again this requires almost zero steps. We introduced the phase space density $\rho$ and the matrix of second moments $\Sigma$ of this density. It is nearby and well-known in the theory of probability distributions to use the Fourier transform to represent the moments of a distribution. In Sec. VII C we introduced velocities (and hence space-coordinates by the option to integrate the velocity $\beta$ over time) by the eigenvalue equation. The Fourier transform is mathematically rigorous and directly yields the mechanics of waves. However it requires that the phase space density and the spinor are functions of energy and momentum.

The phase space density $\rho(\mathcal{H})$ of some stable state depends on the constant parameters $\mathcal{E}$ and $p$, that describe a particle (RPO) so that $\rho(\mathcal{H}) = \rho(\mathcal{E}, p)$. Equivalently, the spinor $\psi$ is, in case of stable oscillations, also a function of energy and momentum. Hence we define the four-component spinor $\Psi = \Psi(\mathcal{E}, p) = \psi \sqrt{\rho}$ so that the matrix of second moments can be written as

$$\Sigma = \int \psi \psi^T \rho(\psi) d^4 \psi = \int \Psi \Psi^T d^4 \psi,$$

which is the so-called “density matrix” in the SPQM.

Hence the spinor $\Psi$ is square integrable and therefore has a Fourier transform, which can be written as

$$\tilde{\Psi}(t, \vec{x}) \propto \int \Psi(\mathcal{E}, \vec{p}) \exp(-i \mathcal{E} t + i \vec{p} \cdot \vec{x}) d^4 p.$$  (101)

This requires no postulate. It is just the Fourier transform of a phase space density function and as such not quantum, so that also the operator rule, the so-called “canonical quantization”;

$$\langle \tilde{\Psi}^\dagger(t, \vec{x}) \mathcal{E} \Psi(t, \vec{x}) \rangle = \langle \Psi^\dagger(t, \vec{x}) i \partial_t \Psi(t, \vec{x}) \rangle$$
$$\langle \tilde{\Psi}^\dagger(t, \vec{x}) \vec{p} \Psi(t, \vec{x}) \rangle = -\langle \Psi^\dagger(t, \vec{x}) i \vec{\nabla} \Psi(t, \vec{x}) \rangle$$  (102)

often shortly written as

$$\mathcal{E} = i \partial_t$$
$$\vec{p} = -i \vec{\nabla}$$  (103)

This is not quantum. It is but a special way to compute averages, aka statistical mechanics. By construction, the parametric space-time, represented by $t$ and $\vec{x}$, matches to the framework of SSTs that has been developed, if $t$ and $\vec{x}$ are vector components in a Dirac algebra. In this case, the phase of the Fourier transform is an invariant quantity, i.e. a scalar.

Inspection of the (anti-) commutator tables of the Dirac algebra shows, that, if $\mathcal{P} = \mathcal{E} \gamma_0 + \vec{p} \cdot \vec{\gamma}$ is a vector and $\mathcal{X} = t \gamma_0 + \vec{x} \cdot \vec{\gamma}$ is also a vector, then the anticommutator $(\mathcal{P} \mathcal{X} + \mathcal{X} \mathcal{P})/2$ is a scalar. Hence the anticommutator is generalization of the scalar (“inner”) product. The commutator is, no big surprise, a generalized vector (“outer”) product, which suggests the following convention:

$$\mathcal{P} \cdot \mathcal{X} = (\mathcal{P} \mathcal{X} + \mathcal{X} \mathcal{P})/2 = (\mathcal{E} t + \vec{x} \cdot \vec{p}) \mathbf{1}$$
$$\mathcal{P} \wedge \mathcal{X} = (\mathcal{P} \mathcal{X} - \mathcal{X} \mathcal{P})/2 = (\vec{p} \times \vec{x}) \cdot (\gamma_1 \gamma_4 \gamma_0)$$  (104)

This is just a matter of convenient notation.

The “physical” space, defined this way, is no more the reified fundamental container of everything as in Newtonian physics, but is recognized, as it should be, as a non-entity.

However, the Fourier transform requires that energy $\mathcal{E}$ and momentum $p$ are real-valued, which excludes resonances. Even if we can not elaborate here in detail, but the use of the Fourier transform which allows to obtain a spatio-temporal image of the phase space process, seems unproblematic only in specific circumstances, namely in eigenstates of the energy.

The general autocorrelation matrix requires 10 parameters, while the Fourier transform uses only 4 and hence ignores spin. Hence there are variables and correlations that have no spatio-temporal “location”.

Furthermore, Eq. 69 defines energy and momentum as linear combinations of second moments of $\psi$, i.e. energy and momentum depend on $\psi$, while the Fourier transform is formulated as if $\psi$ was a function of energy and momentum: the dependency is reversed. However, the dependency is not (always) bijective. The eigen-spinors of free Dirac particles are in many textbooks expressed as functions of $(\mathcal{E}, p)$, but this is not the case if one uses the corresponding eigenstates. Or, in other words, there are very likely “loopholes” and it is not far-fetched to assume that these might allow to explain the findings that required to introduce the projection postulate. But in any case it is clear that the most “mysterious” features of QM are, if locality is not presumed to be fundamental, merely technical or mathematical issues. In our presentation they neither suggest nor do they suffice to establish a philosophy of unreality.

Space-time geometry and electromagnetism carry the signature of the simplest possible description of Hamiltonian interaction. This can be taken literally: as shown in Ref. 56, the parametrization of a general $4 \times 4$ Hamiltonian matrix that describes the coupling of two
DOF by the use of Dirac matrices and the EMEQ allows a straightforward analysis and transformation to normal forms. Hence the physical notions of the Dirac algebra, provide the mathematical means to solve the general problem of diagonalizing Hamiltonian matrices. This is remarkable insofar as usually one expects that math is used to solve physical problems and not vice versa.

If physical notions are useful to solve a general math problem, then the two are isomorphic. All possible terms that are allowed by the physicality constraint are parameters of the Hamiltonian matrix and have physical significance. Math and physics are isomorphic, the theory is saturated.

The Dirac electron is described by a wave-function, i.e. by a charge distribution. In App. 32 it is shown that the “fields” generated by the Dirac current density obey Maxwell’s equations. Hence this picture is hence self-consistent. The difference to the “classical” picture of a 4-current density is a matter of the order of the presentation.

Classical metaphysics presumes that space-time is fundamental. A charge distribution can hence be split into infinitesimal parts that are distinguishable by their positions in space. Then these fractions should be able to move independently and, according to the Lorentz force, the parts must repel each other. Hence they can not give rise to some stable distribution. Classical (meta-) physics escaped by postulating point charges. This however implies infinite self-energy so that electromagnetism and space-based physics has a renormalization problem, even if this is rarely explicitly mentioned.

But as we have shown, if the “classical” metaphysical presumptions are dropped, the Lorentz force naturally emerges, but it is a force relevant for statistical averages, namely second moments. It does not refer to the motion of single phase space points, but to changes of second moments of phase space distributions under the influence of external fields. The math used is still classical, but the presumption of space as a fundamental notion has been removed.

In our presentation, classicality is based on a logico-mathematical and not to a metaphysical framework: not space but real physical objects (RPOs) are fundamental. Without having established objects first, we can not meaningfully refer to spatial notions. It would be inconsistent to cut an RPO into different “parts” and to locate them independently in space-time. It makes even less sense to presume that they repel each other by a force that can only be established by the second moments of the complete RPOs phase-space ensemble. This demonstrates that un-quantum physics can not be consistently understood, as long as space is regarded as fundamental and this is the core of “how to” un-quantum mechanics: to accept that space, though real, is not fundamental.

Furthermore, the description of an electron, which is mostly defined by its talent for electromagnetic interaction, must somehow provide the mathematical means to explain this talent. The standard presentation simply postulates that electrons “carry” charge. If we aim to avoid commandments, this is not satisfying. In our presentation, the talent for electromagnetic interaction is provided by the bi-vector elements of the Hamiltonian matrix parameterized by the Dirac algebra.

A. Uncertainty Relations

Heisenberg’s “uncertainty relations” are, depending on the presentation, a consequence of Eq. 103 or of the Fourier transform. They describe yet another math fact 37: A certain width of a distribution in momentum space determines a minimal width of the Fourier-transformed distribution in physical space and vice versa. And yet again, since it is a math fact about Fourier transforms, it requires no postulates and is therefore not quantum.

Griffith writes: “This principle is often discussed in terms of measurements of a particle’s position or momentum, and the difficulty of simultaneously measuring both of these quantities. While such discussions are not without merit […] – they tend to put the emphasis in the wrong place, suggesting that the inequality somehow arises out of peculiarities associated with measurements.” 89. The so-called “uncertainty relations” are not due to measurement uncertainties. Nowhere in the derivation of the uncertainty relations is it required to refer to “distortions by measurements”. There is not even an intrinsic need to speak of “uncertainties” at all, as the width of a distributions is not uncertain. It is just a width.

B. The Born Rule

We still have to analyze Born’s Rule, namely that if \( \Psi^T(\vec{x}, t)\Psi(\vec{x}, t) \) can be regarded as a probability density to “find” a particle at position \( \vec{x} \) and time \( t \) 90. Or, in other words, the transfer of a phase space density \( \rho \) into the parametric space-time, might appear somewhat unclear, since \( \rho \) and not \( \Psi^T \Psi = \psi^T \psi \rho \) is - in our presentation - a density in phase space. So why should the latter now be a valid density in space-time? This question is legitimate and nearby. It has a simple answer: In agreement with (the mathematical principles of) spatio-temporal logic, the continuity equation, which follows from Dirac’s equation 23, suffices to validate the consistency of the spatio-temporal “image” and Born’s rule. Not more and not less. It is not in the scope of the continuity equation to guarantee the absence of other, non-spatial, correlations between objects.

The mathematical correctness of an image, however, does not change the fact that the true postal address

\[ \text{See App. 22} \]
of RPOs is phase space. This means that we can take the mathematical form of QM seriously. The squared “amplitude” merely generates the mathematical form of a substance inhabiting space-time. What we perceive and measure in space-time, is a Fourier spectrum. The Fourier transform is a reversible unitary transformation, an isomorphism with respect to the transformed properties: Constraints imposed in either space have consequences in the other. This justifies to regard the image appearing in space-time as real, but it does not suffice to make it fundamental. In our presentation, the spatio-temporal image of an RPO is like an avatar. It is sometimes claimed that QM and specifically the Born rules requires some kind of non-classical probability theory. Leon Cohen has shown that this is just another myth. We found no convincing reason to think otherwise.

C. Space-Time: The Arena of Avatars

As explained in the previous section, Minkowski’s space-time is, from the perspective of a single RPO, a kind of a holographic screen, it is less fundamental than supposed “classically” - nonetheless it is more than a mere image of reality. For many physicists it seems simply not imaginable to weaken local realism, but from a logical perspective a merely commandment-based space-time concept is scientifically unsatisfactory. Even Einstein, in later years, wrote: “Spacetime does not claim existence on its own, but only as a structural quality of the field.”

Also the experimental tests of Bell’s theorem suggest that nature is on the fundamental level non-local. As Maudlin expressed it: “What Bell proved, and what theoretical physics has not yet properly absorbed, is that the physical world itself is non-local.”

De Haro and de Regt argued that physical theories without the primary assumption of space-time are indeed able, contrary to other claims, to “provide scientific understanding,” though they are - of course - difficult to visualize.

Mariani and Truini suggested exceptional Lie algebras might be at the “foundations of space and time”. They suggest the basic principle that “there is no way of defining space-time without a preliminary concept of interaction”. We agree with them concerning the emergence of space-time from interaction, but we doubt it requires a principle. In this work, such a principle does not appear, since the definition of physicality did not require to specify “where” some object is. The question “where” a thing “is”, can only have relevance in the presence of other objects, i.e. by interaction.

Modern physics invented the notion of background independence: “[..] a classical field theory is background-independent if the structure required to make sense of its equations is itself subject to dynamical evolution, rather than being imposed ab initio. [...] a theory is fully background-independent relative to an interpretation if each physical possibility corresponds to a distinct spacetime geometry; and it falls short of full background-independence to the extent that this condition fails.”

Regarding these criteria we think that our physico-logical approach is, though “classical”, fully background independent. It directly and inevitably leads to the simplest possible physical objects and the first order interaction of these generate the 3+1 dimensional parameter space, that human inhabitants of the constructed physical world call space-time.

The Lorentz transformations are not primarily required to describe “coordinate” transformations between “inertial reference frames”, they are (also) active canonical transformations, changing the physical state of the system under consideration. The issues that many - also renowned - physicists had with special relativity might also be due to an unclear attitude (of Einstein, but also others) towards the ontological status of space-time. Though Einstein suggested in his theory of special relativity that the assumption of a material substance, an aether, is dispensible, he was not able or willing to dispense the hegemony of the Newtonian heritage of absolute space.

According to that view it does not suffice to formulate a theory that allows for the derivation of geometrical notions, aka an emergent space-time, but spatio-temporal notions must inevitably be the most fundamental ones. This philosophy might be called space-time fundamentalism (STF) and Einstein frequently, but not always, appeared as a proponent of STF. STF almost requires a reification of space, or space-time, respectively.

Mermin warned us that the reification of mathematical abstractions used in physics is a “bad habit”. He argues mainly from a pragmatic point of view (which is welcome). However, viewed pragmatically: Does the dogma of unreality of wave-functions enable students to understand QM or does it lead to unnecessary confusion? Is it required and justified claim that “the “orbit” is created by the fact that we observe it” as Heisenberg claimed? Do we really need to accept an interpretation that questions object permanence, despite the fact that it is the fundament of physicality. Does it at least correspond to any practice in physics? Do accelerator physicists provide any measure to “observe” particle beams in order to

---

38 Einstein wrote 1947 in a letter to Born: “I cannot seriously believe in it [quantum mechanics] because the theory cannot be reconciled with the idea that physics should represent a reality in time and space, free from spooky actions at a distance.”

39 For Einstein, local causality was a fundamental requirement.

This (and not determinism) is the core of his concerns regarding QM.

40 “Die "Bahn" entsteht erst dadurch, dass wir sie beobachten”
establish the existence of particle orbits? Of course they don’t. Of course there is no need to do so. As far as we can tell, orbits are established by electro-magnetic fields, not by observation.

We believe that Heisenberg’s philosophy of unreality went beyond any reasonable requirement and his arguments suspiciously oscillate between the “uncertainty principle”, commutators of conjugate pairs and considerations about the limitations of measurement precision, enriched with claims of positivistic nature [103]. What he did, is exactly what Mermin (should have) criticized: He overrated mathematical abstractions, not in support of reification, but of un-reification. Heisenberg scrambled the quantum omelette with ingredients of a theory that wasn’t finalized or established yet. And this is certainly a bad habit.

It is rarely emphasized, but the most important “quantum” effects are macroscopic and not microscopic: The stability of matter, the properties of thermal radiation, ferro-magnetism, the properties of chemical bonds, metallic states (Fermi surfaces), superconductivity, superfluidity and so on. If you carefully think it through, it is far more difficult to find examples of physical systems where “quantum” effects can be safely ignored than otherwise. Nonetheless students are taught that quantum effects are specifically microscopic and somehow weird.

But again: it would be completely wrong to conclude that emergence per se implies a questionable reality status. Temperature is an emergent notion and we think there is consensus that temperature is real. Chemistry emerges from solutions of the Schrödinger equation, i.e. from wave-functions and orbitals. This alone does not suggest that chemical bonds are not real. Thus, even if our presentation of classical un-quantum mechanics refuses to regard space-time as apriori given, this does not mean that we regard space-time as being less real. We just regard it as less fundamental.

Many, maybe most, of the alleged mysteries of QM have been debunked before [73, 104], or their non-classicality has been critically reviewed [32]. We went beyond a mere critic of the standard approach: as we have shown there is little in the mathematical formalism of quantum theory that can not be obtained from classical Hamiltonian mechanics. We stress again that classicality is often misrepresented as some kind of STF: If the objective is to describe motion, then, according to STF it must be motion in space and the classical canonical variables have to be understood literally as space-time coordinates and momenta. But this is not part of the classical physical but part of metaphysical presuppositions.

Classical analytical mechanics is but a mathematical framework. The applicability of the notion of generalized coordinates in the sense of dynamical variables in an abstract phase space is, in principle, unlimited. Abstract variables like those forming the wave function have not been invented by QM. We explained what exactly distinguishes wave functions them from observables and why this is a consequence of fundamentality, using classical logic.

Though the Hamiltonian formalism has, with respect to Eq. 9, a perfect (skew-) symmetry between coordinates and momenta, the only classical Hamiltonian that accounts for this symmetry, is the harmonic oscillator. In the Hamiltonian of a “classical” free particle, only the momenta appear, but there is no classical system, in which only coordinates are used. The derivation of Compton’s scattering formula demonstrates the irrelevance of coordinates in the actual treatment. It requires only the EMR, energy and momentum conservation and the de Broglie scaling relations $E = h \omega$ and $\vec{p} = h \vec{k}$ for it’s derivation. It is the same picture in many branches of physics: Positions obtain their physical relevance exclusively from fields, i.e. from interactions, while the constraints that allow to draw physical conclusions and to make real calculations, are derived in energy-momentum-space. It is only the size of the human visual cortex that underlies the human preference for spatio-temporal (i.e. geometrical) notions.

There is, within the standard presentation of classical physics, no explanation why the symmetry of coordinates and momenta should be broken. Max Born wrote: “This lack of symmetry seems to me very strange and rather improbable. There is strong formal evidence for the hypothesis, which I have called the principle of reciprocity, that the laws of nature are symmetrical with regard to space-time and momentum-energy [...]” [105]. Born was absolutely right, but with respect to the fundamental level of Hamiltonian theory, namely the wavefunction. We explained why and how the Dirac algebra breaks this symmetry between spatial coordinates and mechanical momenta. Born spells out what we mentioned above: the principle of sufficient reason. And his intuition was correct, but for spinorial phase space: here the (skew-) symmetry of the canonical pair is fully valid. But yet again, this requires no independent principle: the PSR completely suffices.

We started without any specific assumptions about any “background”. We did not even ask for it in the first place. The structure of space-time is, in our presentation, a consequence of the unique properties of the Hamiltonian Dirac algebra. Our presentation of un-quantum mechanics is entirely based on simple math facts and straightforward logic. By definition it holds in the most general physical world. Hence, without commandments, the celebrated conjecture that many different physical worlds should be possible [12], looses much of it’s plausibility.

The intrinsic non-locality of un-quantum mechanics explains why it makes only limited sense to ask where an electron or a photon “really” is in space: The electron itself is not located at some specific position in space at all. Because physical ontology is not primarily defined by spatial notions, it is meaningless to ask if it can simultaneously “be” at different positions. Surely it can, since projected into space-time, the electron has no definite location, but “is” a wave. However, the location of the
constraining principles that define its dynamical character, is energy-momentum space. This energy-momentum space emerges from (auto-) correlations that originate elsewhere: in an abstract classical phase space.

IX. PROSPECTS: HIGHER DIMENSIONS

The distinction between Hamiltonian and skew-Hamiltonian elements and the fact that skew-Hamiltonian elements have always, by definition, vanishing expectation values, could be useful to discuss higher dimensional “spaces” as well. As mentioned above, our approach suggests to consider not only the Pauli and Dirac algebras alone, but two series of HCAs given in Tab. II.

Two CAs of this list, namely Cl(9, 1) and Cl(25, 1) are also regarded as interesting from the perspective of string theory [102]. Hence, by the way, we already succeeded to un-string these two algebras, unintentionally [43]. As derived above, Hamilton’s algebra of proper time allows, in principle, to consider many, arbitrarily large and complicated, algebras on the basis of classical phase spaces. But furthermore, HCAs have another feature, namely that they allow to “hide” dimensions. How that?

We found Clifford algebras from symmetries originating in Hamiltonian theory. The analysis of the Hamiltonian Dirac algebra Cl(3, 1) generates a system of quantities and relations that precisely fits to relativistic electrodynamics. It forces us to introduce a 3+1 dimensional parametric space-time. Without the use of Hamiltonian notions however, the Dirac matrices do not uniquely select a specific Clifford algebra: Instead of Cl(3, 1), the algebra of real 4 × 4-matrices is also a “representation” of Cl(2, 2). It was shown, that, if all Clifford generators are supposed to be Hamiltonian, then we are restricted to the algebras listed in Tab. II namely to a single time-like vector element and N − 1 spatial elements. Then it seems, that Cl(9, 1) inevitably leads to a 10-dimensional space-time. But as we argued in Ref. [103], geometric spaces with more than 3 dimensions, when derived from Clifford algebras, have some problematic features. Consider some N-dimensional space-time is supposed to emerge from a HCA. Then there are N − 1 generators of boosts γ0 γ1, γ0 γ2, γ0 γ3, . . . , but there are \( \binom{N-1}{2} \) = (N − 1)(N − 2)/2 generator of rotations. While all generators of boosts mutually anti-commute, this does not hold for all rotators. N − 1 = 3 is the largest number of spatial dimensions which is “homogeneous” in this respect [43]. Hence, from the perspective of HCAs, 3 spatial dimensions are the optimal case.

This suggests to consider HCAs with the dimensionality Cl(3, N − 3). From Bott’s periodicity (Eq. 53) one obtains the condition

\[
p - q = 6 - N = 0, \ 2 \mod 8.
\]

which yields two sequences for \( m \in \mathbb{N} \):

\[
N = 6 + 8 m \\
N = 4 + 8 m
\]

The sequence \( N = 6 + 8 m \) has no real representation in which all generators of the Clifford algebra are Hamiltonian, but HCAs of the Dirac type \( (N = 4 + 8 m) \) allow for a reinterpretation in 3-dimensional space. In Ref. [43] we sketched an interpretation of the algebra of real \( 8 \times 8 \) matrices Cl(3, 3), which lead to the introducing an “internal” degrees of freedom.

The real representation of the Cl(11, 1) is based of the same set of matrices as Cl(3, 9) - they are just ordered and interpreted differently. All Dirac type HCAs with \( m \geq 1 \), i.e. \( Cl(3, N - 3) = Cl(3, 1 + 8 m) \), are algebras that can be obtained from multiple Kronecker products of Dirac matrices. Indeed it has been claimed by Sogami that triple tensor products of Dirac spinors are able to reproduce much of the Standard model of particle physics [107–109], an approach that fits seamlessly to our presentation of un-quantum mechanics (see App. F).

As sketched in App. F the Dirac sequence also emerges in a generalization with higher moments of degree 2 or 3, \( M \in \{3, 5, 7, \ldots \} \) or cross-correlation of multiple Dirac particles. Cl(3, 9), for instance, has 3 symmetric generators and 9 skew-symmetric generators. One of the 9 skew-symmetric generators is the symplectic unit matrix \( \gamma_0 \), which is Hamiltonian by construction. Then the remaining 8 skew-symmetric Clifford generators are skew-Hamiltonian and neither represent observables nor are they generators of SSTs. They are “hidden”.

Given \( \gamma_a, a \neq 0 \) is a skew-symmetric generator of \( Cl(3, 9) \), then \( \gamma_a \) and \( \gamma_0 \) anti-commute (by definition of CA generators) \( \gamma_0 \gamma_a = -\gamma_a \gamma_0 \) so that \( \gamma_a \) is skew-Hamiltonian:

\[
\gamma_0 \gamma_a \gamma_0 \gamma_a = -\gamma_a \gamma_0 \gamma_a \gamma_0 = \gamma_0^2 \gamma_a = -\gamma_a
\]

It is therefore neither a generator of a canonical transformation nor an observable. Furthermore, \( \gamma_a \) is skew-symplectic:

\[
\gamma_a \gamma_0 \gamma_a^T = -\gamma_0 \gamma_a \gamma_a^T = -\gamma_0
\]

since \( \gamma_a \gamma_a^T = 1 \). These are features known from the pseudo-scalar of the Dirac-algebra. In context of the Dirac algebra, the pseudo-scalar represents the charge conjugation operator. This suggests that the skew-Hamiltonian \( \gamma_a \) might be interpreted in a similar way, namely as representing discrete (instead of continuous) symmetry-transformations.

41 To un-string string theory means to keep findings of the theory, but without “strings”.
X. CONCLUSIONS AND OUTLOOK

It was shown that it is classical metaphysics rather than classical mathematics that prevents from the insight that QM is classical. The SPQM attacks the problem by constructing a metaphysical rather than a physical paradox: It asks whether electrons are particles or waves, only in order to demonstrate that the electron can’t be either. The typical conclusion however, that, since it is neither, QM cannot possibly be understood, is untenable.

Why should these two metaphors be the only possible options? As has been shown, it is indeed possible to give a logical account of the “wave-particle duality”, without the use of ad hoc metaphysical assumptions and without changing anything else but notation and presentation. The math is literally the same. The only victim of our presentation is the metaphysical presupposition that space is fundamental. This however is in agreement with the experimental tests of Bell’s theorem; it is a price we have to pay anyhow.

A. Approximations

While there are examples of phase space distributions that are exclusively parametrized by their second moments, for instance (multivariate) Gaussians, there is no reason to presume that this is the only possible case. Hence we have no reason to believe that the second moments alone are sufficient to fully characterize the physical situation. The same holds with respect to the form of the Hamiltonian: While it is often possible and legitimate to use a truncated Taylor series approximation as a simplifying assumption, this alone is no reason to believe that a limitation to second order is necessarily an intrinsic feature of nature. However we believe it is remarkable what can be obtained on the basis of logic and some preliminary simplifying assumptions.

The fact that we needed only second moments to derive many essentials of QM might be taken in support of the position that QM cannot be complete, but this does not mean that one has to presume “hidden” variables. It might suffice to consider higher moments and higher order terms in the Hamiltonian. A first, very incomplete, look at possible generalizations for higher moments, represented by Kronecker products, is given in App. F.

B. Classicality

It was shown that Dirac’s theory is mathematically classical,[42] that QM altogether is mathematically classical: No anti-commutation rules need to (and may) be presumed for the fundamental variables (ψ).

Since observables are, in our presentation, nothing but (auto-) correlations of dynamical variables, followers of a purely information-theoretic approach of QM might feel confirmed. Though it is an intriguing idea to think that “correlations have physical reality; that which they correlate does not” [49], it is difficult to see, what exactly this claim explains that cannot be explained otherwise.

Two methods to introduce “physical” space were considered: the first obtains the “velocity” directly by the use of Hamiltonian mechanics applied to the observables and secondly by a statistical description of moments based on the Fourier transform. The former case suggests properly defined trajectories, and given the landscape of electric and magnetic fields is known, one can integrate the trajectory of the particle (ignoring Heisenberg’s claim) by the Lorentz force. There is little in this approach that seems to suggest quantum features. The second moments are correlations and they are in this sense mathematically exact.[43] The matrix of second moments defines energy and momentum precisely.

The latter approach, the Fourier transform, is the basis of wave mechanics and it is required when the spatial extent of structures closed to the particles trajectory is in the same order of magnitude as the de-Broglie wavelength. Provided one has sympathy for the so-called “wave-particle duality” or Bohr’s “complementarity principle”, one might take the duality of methods as a confirmation of his ideas. However, if a principle can be mathematically derived, then there is few reason to call it a “principle” at all. And of course there is a bridge “principle” connecting both accounts, namely the relativistic energy momentum relation which can likewise be regarded as dispersion relation, where the “group velocity” is given by

\[
\vec{v}_g = \nabla_{\vec{k}} \omega(\vec{k}) = \vec{\beta} = \nabla_{\vec{p}} \mathcal{E}(\vec{p}).
\]

Is all this still classical physics? This depends on the point of view. However, we have shown, as promised, that the difference between CM and QM is not mathematical.

According to our definition, real physical objects are characterized by their permanence which is, translated into the language of physics, symmetry in time. This implies not more and not less than a positive definite constant of motion (PDCOM). We have shown that the primary PDCOM of RPOs is mass, i.e. a form of energy. No commandments so far – just a single constraint.

---

42 A hint by Res Jost, pointing in this direction, even found its way into a celebrated paper of Dirac [22].: “It has been pointed out to me by R. Jost that this group is just the 4-dimensional simplectic group, which is equivalent to the 3+2 de Sitter group.”

The mentioned group is the group of the Dirac matrices, and what Res Jost remarked is that this group is a classical group, subject to classical Hamiltonian equations of motion.

43 No one prevents us from interpreting \(\sqrt{\langle x^2 \rangle} - \langle \langle x^2 \rangle \rangle^{\frac{1}{2}}\) as a measure for the “uncertainty” of \(x^2\). However, we doubt that the same quantity can be interpreted as an “uncertainty” of \(\langle x^2 \rangle\).
Maybe this is what Feynman had in mind when saying “...you know how it always is, every new idea, it takes a generation or two until it becomes obvious that there’s no real problem” [112].

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Appendix A: Unitary Motion is Symplectic

Linear symplectic motion is due to Eq. [5] where A is real symmetric, ψ is real and J is a symplectic unit matrix. Linear unitary motion is given by

\[ i \dot{\psi} = H \psi, \]  

(A1)

where H is hermitian and ψ complex.

If we split a Hermitian matrix H and a complex spinor ψ into its respective real and imaginary parts H = A + iB, such that A = AT and −B = BT, and ψ = φ + iχ, then Eqn. (A1) can be written as follows:

\[ i(\dot{\phi} + i\dot{\chi}) = (A + iB)(\phi + i\chi) \]
\[ i\dot{\phi} - \dot{\chi} = A\phi + iB\phi + iA\chi - B\chi \]
\[ \phi = B\phi + A\chi \]
\[ \dot{\chi} = -A\phi + B\chi \]  

(A2)

Thus, if we compose a real spinor \( \Psi = \begin{pmatrix} \phi \\ \chi \end{pmatrix} \) by the real and imaginary parts of the spinor ψ, then unitary motion has the form:

\[ \dot{\Psi} = \begin{pmatrix} B & A \\ -A & B \end{pmatrix} \Psi \]
\[ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & -B \\ B & A \end{pmatrix} \Psi \]
\[ = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} A & B^T \\ B & A \end{pmatrix} \Psi \]
\[ = \gamma_0 A \Psi \]  

(A3)

where \( \gamma_0 \) is a SUM and A is symmetric. In other words, any unitary law of motion can always be expressed by symplectic motion with specific restrictions for the matrix H as given by Eq. A3.

Appendix B: Periodic Time-Dependent Hamiltonian

In the case of a time-dependent Hamiltonian matrix, the condition for a PDCOM Eq. [1] requires a modification:

\[ \frac{dH}{dt} = \frac{\partial H}{\partial \psi} + \nabla_\psi H \dot{\psi} = 0 \]
\[ 0 = \frac{1}{2} \psi^T A \dot{\psi} + \psi^T A \dot{\psi} = 0 \]  

(B1)

We introduce an additional Hamiltonian matrix G and use the Ansatz

\[ \dot{\psi} = (\gamma_0 A + G) \psi, \]  

(B2)

Inserted into Eq. (B1) this gives:

\[ 0 = \frac{1}{2} \psi^T A \dot{\psi} + \psi^T A (\gamma_0 A + G) \psi \]
\[ 0 = \psi^T \left( \frac{1}{2} A + \gamma_0 A + A G \right) \psi \]  

(B3)

The due to the skew-symmetry of \( A \gamma_0 A \), it follows (as before) \( \psi^T A \gamma_0 A \psi = 0 \). The remaining matrix \( \frac{1}{2} A + A G \) must then also be skew-symmetric to fulfill this condition. Since \( \dot{A} = AT \) and \( \dot{F} = \gamma_0 A \dot{A} \), the condition \( \dot{H} = 0 \) requires that \( A \)

\[ 0 = (\frac{1}{2} \dot{A} + A G)^T + \frac{1}{2} \dot{A} + A G \]
\[ 0 = \dot{A} + \gamma_0 G \gamma_0 A + A G \]  

(B4)

Multiplication with \( \gamma_0 \) from the left yields:

\[ 0 = \gamma_0 \dot{A} - G \gamma_0 A + \gamma_0 A G \]  

(B5)

so that

\[ \dot{F} = G F - F G . \]  

(B6)

Thus, if the time dependence of F can be obtained from Eq. [17] then then we have a kind of level transparency for the driving term G: Concerning the original problem, Eq. [12] suggests that G can be directly added to F.

Appendix C: (Multi-) Spinors in Electrodynamics?

There are different possibilities to represent phase space densities. One possibility has been used so far, namely a density function \( \rho(\psi) \). There is another approach, specifically useful in numerical simulations, namely phase space sampling. This implies to uses not a single spinor ψ, but several, i.e. the column vector ψ is replaced by a multi-column vector with m columns, aka matrix a \( 4 \times m \)-matrix. This approach can also be used to impose a symmetry onto the phase space density [32]. The matrix of second moments Σ is then given in the form of Eq. [14].

Our approach so far concentrated on the description of the simplest RPOs, i.e. matter fields. Electromagnetic waves appeared only as terms that act on RPOs, but not as objects in themselves. Even worse, we found that vector components can not be generated from bi-vectors by Eq. [17]. This still holds, but raises the question of how to define electromagnetic energy and momentum within
Then one obtains a density matrix 
\[ F = \gamma_0 (\vec{E} \cdot \vec{\gamma}) + \gamma_{14} \gamma_0 (\vec{B} \cdot \vec{\gamma}). \]  
(C1)

Eq. (14) gives for an arbitrary Hamiltonian matrix:
\[ FF^T \gamma_0^T = F \gamma_0 F. \]  
(C2)

For pure bi-vectors, this expression yields pure vector components for free electromagnetic fields, aka “photons”:
\[ \frac{1}{2} F \gamma_0 F = \frac{1}{2} (\vec{E}^2 + \vec{B}^2) \gamma_0 + (\vec{E} \times \vec{B}) \cdot \vec{\gamma}. \]  
(C3)

This suggests that there is at least some formal similarity between the spinors (phase space coordinates) that we used to model RPOs and electromagnetic fields.

The matrix \( F \) has written explicitly, the form (5):
\[
F = \begin{pmatrix}
-E_x & B_y + E_z & -B_z + E_y & B_x \\
-B_y + E_z & E_x & -B_x & -B_z - E_y \\
B_z + E_y & B_x & E_x & -B_z - E_y \\
-B_x & B_z - E_y & B_y + E_z & E_x
\end{pmatrix}
\]  
(C4)

In case of free electromagnetic waves, we can choose a coordinate system such that the wave propagates along the z-axis so that \( E_z = B_z = 0 \) and hence:
\[
F = \begin{pmatrix}
-E_x & B_y & E_y & B_x \\
-B_y & E_x & -B_x & -B_z - E_y \\
E_y & B_x & E_x & -B_z - E_y \\
-B_x & -E_y & B_y & E_x
\end{pmatrix}
\]  
(C5)

If we define a spinor \( \phi = (-E_x, -B_y, E_y, -B_z)^T \) as the first column (vector) of \( F \), then the other columns are given by \(-\gamma_0 \phi, -\gamma_\phi \) and \( \gamma_{14} \phi \), respectively, so that \( F \) can be written as a “multispinor” (5):
\[
F = (\phi, -\gamma_0 \phi, -\gamma_\phi, \gamma_{14} \phi)
\]  
(C6)

Then one obtains a density matrix \( FF^T \gamma_0 \) of the form
\[
FF^T \gamma_0 = 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} -P_z & \mathcal{E} \\ -\mathcal{E} & P_z \end{pmatrix},
\]  
(C7)

with \( \mathcal{E} = (B_x^2 + B_y^2 + E_x^2 + E_y^2)/2 \) and \( P_z = E_x B_y - E_y B_x \). That is, the matrix \( F \) is block-diagonal or decoupled.

However, it must be kept in mind that \( F \) is not a multispinor, but a Hamiltonian matrix. This becomes clear if one considers the transformation properties: Spinors transform (like conventional vectors) according to the rule
\[
\Psi' = RF \Psi
\]  
(C8)

while Hamiltonian matrices transform according to
\[
F' = RF R^{-1}
\]  
(C9)

so that
\[
(F\gamma_0 F)' = R F (R^{-1} \gamma_0 R) FR^{-1}. \]  
(C10)

Then the matrix \( FF^T \gamma_0 \) is only a proper Hamiltonian, if \( R^{-1} \gamma_0 R \) equals \( \gamma_0 \). According to Eq. (25) this is the case, if the matrix \( R \) is not only symplectic but also orthogonal, i.e. in case of rotations. But it is not correct in case of boosts (see Eq. (22)). Hence the electromagnetic energy density is not a Minkowski 4-vector. It is part of an object called “stress-energy-tensor” in classical electrodynamics (110). This is due to the fact that the volume element in Minkowski space-time is not an invariant quantity, while the volume element of a phase space is an invariant quantity.

The almost obscene complexity of many spatio-temporal descriptions in physics contrasts with the simplicity of the underlying phase space. Since equations should be the simpler the more fundamental, phase space and not space-time must be regarded as fundamental.

**Appendix D: Pseudoscalars**

The pseudo-scalar is defined by Eq. (63). Is the pseudo-scalar Hamiltonian? The transpose of \( \gamma_\pi \) is
\[
\gamma^T = \prod_{\pi=N-1}^{0} \gamma^T \mu = \gamma^T \pi \gamma^T \pi^{-1} \gamma^T (D1)
\]

If all generators are Hamiltonian, this gives:
\[
\gamma^T = (\gamma_0 \gamma_N \gamma_{N-1} \gamma_0) (\gamma_0 \gamma_N \gamma_{N-2} \gamma_0) \cdots (\gamma_0 \gamma_0 \gamma_0) = (-1)^{N-1} \gamma_0 \gamma_{N-1} \gamma_{N-2} \cdots \gamma_0 \gamma_0 \]  
(D2)

A re-sorting of the order of the bracketed product requires a certain number of commutations of the factors and each commutation is accompanied the a reversal of the sign. The number of permutations required to reverse the order of \( N \) matrix factors is \( N (N-1)/2 \), so that
\[
\gamma^T = (-1)^{N-1+N(N-1)/2} \gamma_0 \gamma_{N-1} \gamma_0 \]  
(D3)

The sign is hence positive and \( \gamma_\pi \) is Hamiltonian, if the exponent is an even integer. Since \( N = 2m \) is even,
\[
N - 1 + N(N-1)/2 = m - 1 + 2 m^2 \]  
(D4)

the even term \( 2m^2 \) can be skipped, so that \( \gamma_\pi \) is Hamiltonian if \( m - 1 = N/2 - 1 \) is even, i.e. of the Pauli type (Eq. (59)):
\[
N/2 - 1 = 4 m.
\]  
(D5)

The pseudo-scalar of Dirac type algebras (Eq. (59)) is skew-Hamiltonian, since \( N/2 - 1 \) is odd:
\[
N/2 - 1 = 4 m + 1.
\]  
(D6)
This means that the Pauli algebra provides no criterium to distinguish between the even generator of $Cl(1,1)$ and the pseudo-scalar: The real Pauli algebra is not uniquely defined by the Hamiltonian properties.

According to Eq. (D2) general $k$-vectors are Hamiltonian if the exponent $k + (k - 1)/2 = \frac{k^2 + k - 2}{2}$ is even. It is quickly verified that this condition can be written as

$$\frac{k^2 + k - 2}{2} = 2m \Rightarrow k^2 + k = 4m + 2 \quad (D7)$$

and has a solution for integer $m$ for $k = 4j + 1$ and $k = 4j + 2$, but not for $k = 4j$ and $k = 4j + 3$. Hence in HCA's that are generated from Hamiltonian generators, only $k$-vectors with $k = 1, 2, 5, 6, 9, 10, \ldots$ are Hamiltonian while $k$-vectors with $k = 3, 4, 7, 8, \ldots$ are skew-Hamiltonian.

**Appendix E: Real Dirac Theory**

1. Dirac Current Conservation

The real Dirac equation (Eq. 20) is:

$$0 = \left( \partial_\mu \gamma_\mu \pm m \psi \right)$$

$$0 = \left( \gamma_0 \partial_t \gamma^0 + \gamma^j \cdot \nabla \right) \psi \pm m \psi \quad (E1)$$

Since here we discuss spinors in “physical space” instead of energy-momentum space, spinors are complex (Eq. 10A) and the “adjunct” spinor is $\bar{\psi} = \psi^t \gamma^0$ where the superscript $t$ stands for the transposed complex conjugate. Matrix transposition gives:

$$0 = \left( \partial_\mu \psi^t \gamma^t_\mu \pm m \psi^t \right)$$

$$0 = \left( \partial_\mu \bar{\psi} \right) + \left( \nabla \bar{\psi} \right) \cdot \left( \gamma_0 \gamma^0 \right) \pm m \bar{\psi} \gamma_0 \quad (E2)$$

where we used the fact that $\gamma_k = \gamma^t_k$, $\gamma_0 = -\gamma^t_0$ and the anti-commutation rules. Multiplication with $\gamma_0$ from the right yields

$$0 = \left( \partial_\mu \bar{\psi} \right) \gamma_0 + \left( \nabla \bar{\psi} \right) \cdot \gamma \mp m \bar{\psi} \quad (E3)$$

so that, using $\bar{\psi} \psi = 0$:

$$0 = \bar{\psi} \left( \gamma_0 \partial_t + \gamma^j \cdot \nabla \right) \psi$$

$$0 = \left( \partial_t \bar{\psi} \right) \gamma_0 \psi + \left( \nabla \bar{\psi} \right) \cdot \gamma \psi \quad (E4)$$

The sum of both equations then yields the conserved current:

$$0 = \partial_t \bar{\psi} \gamma_0 \psi + \nabla \cdot \left( \bar{\psi} \gamma \psi \right) \quad (E5)$$

One obtains an electric 4-current ($\rho_e, j_e$) by multiplication with the scaling factor $\pm e$:

$$\rho_e = \pm e \bar{\psi} \gamma_0 \psi$$

$$j_e = \pm e \bar{\psi} \gamma \psi \quad (E6)$$

2. Maxwell’s Equations From Dirac Theory

This section provides evidence that the EMEQ (Eq. 74) is consistent, i.e. that it is sensible to identify the bi-vector elements of the Dirac algebra with electric and magnetic fields, respectively. We will show this by showing that the bi-vector elements $\vec{E}$ and $\vec{B}$ obey Maxwell’s equations.

a. Gauss Law

In Ref. 48 we derived Maxwell’s equations from the Hamiltonian Dirac algebra. But it is also possible to use Dirac’s equation in order to show, that the Dirac current (Eq. 70) is compatible with Maxwell’s equations:

$$0 = -\partial_t \psi + \gamma_0 \gamma^j \cdot \nabla \psi \pm m \gamma_0 \gamma^0 \psi$$

$$0 = \partial_t \bar{\psi} + \left( \nabla \bar{\psi} \right) \cdot \gamma_0 \gamma^0 \pm m \bar{\psi} \gamma_0 \gamma^0 \psi \quad (E7)$$

so that

$$0 = -\bar{\psi} \partial_t \psi + \bar{\psi} \gamma_0 \gamma^j \cdot \nabla \psi \pm m \bar{\psi} \gamma_0 \gamma^0 \psi$$

$$0 = \left( \partial_t \bar{\psi} \right) \psi + \left( \nabla \bar{\psi} \right) \cdot \gamma_0 \gamma^0 \pm m \bar{\psi} \gamma_0 \gamma^0 \psi \quad (E8)$$

The sum yields

$$0 = \left( \partial_t \bar{\psi} \right) \psi - \bar{\psi} \partial_t \psi + \nabla \cdot \left( \bar{\psi} \gamma_0 \gamma^j \psi \right) \pm 2m \bar{\psi} \gamma_0 \psi \quad (E9)$$

From Eq. 103 we take that the first term can be written as $2i \vec{E} \psi \psi$, which vanishes due to the algebraic identity $\bar{\psi} \psi = 0$. Then one has

$$0 = \frac{e}{2m} \nabla \cdot \left( \bar{\psi} \gamma_0 \gamma^j \psi \right) \pm e \bar{\psi} \gamma_0 \psi \quad (E10)$$

which gives Gauss’ law (for electron and positrons):

$$\vec{E} = \pm \rho_e \quad (E11)$$

where $\vec{E} = \frac{e}{2m} \left( \bar{\psi} \gamma_0 \gamma^j \psi \right)$ is the electric field.

b. Gauss Law for Magnetism

In order to show that the magnetic field is free of sources, we multiply with the pseudo-scalar:

$$0 = -\gamma_1 \gamma_4 \partial_t \psi + \gamma_1 \gamma_4 \gamma_0 \gamma^j \cdot \nabla \psi \pm m \gamma_4 \gamma_0 \gamma^0 \psi$$

$$0 = \partial_t \psi \gamma_1 \gamma_4 + \left( \nabla \psi \right) \cdot \gamma_4 \gamma_0 \gamma^0 \pm m \psi \gamma_4 \gamma_0 \gamma^0 \psi \quad (E12)$$

so that

$$0 = -\bar{\psi} \gamma_1 \gamma_4 \left( \partial_t \psi \right) + \bar{\psi} \gamma_1 \gamma_4 \gamma_0 \gamma^j \cdot \nabla \psi \pm m \bar{\psi} \gamma_1 \gamma_4 \gamma_0 \gamma^0 \psi$$

$$0 = \left( \partial_t \bar{\psi} \right) \gamma_1 \gamma_4 \psi + \left( \nabla \bar{\psi} \right) \cdot \gamma_1 \gamma_4 \gamma_0 \gamma^0 \psi \pm m \bar{\psi} \gamma_1 \gamma_4 \gamma_0 \gamma^0 \psi \quad (E13)$$

The terms containing the time derivatives again vanish by Eq. 103 as in case of Gauss Law. However, this time
the “mass term” also vanishes when the equations are added:

\[ \vec{\nabla} \cdot (\bar{\psi} \gamma_4 \gamma_0 \vec{\gamma} \psi) = 0, \]  

(E14)

which gives

\[ \vec{\nabla} \cdot \vec{B} = 0. \]  

(E15)

where \( \vec{B} = \frac{e}{2m} (\bar{\psi} \gamma_4 \gamma_0 \vec{\gamma} \psi) \) is the magnetic field.

c. Ampere’s Law

If we regard the time derivative of the electric field, we obtain:

\[
\frac{2\mu_0}{c} \partial_t \vec{E} = (\partial_t \bar{\psi}) \gamma_0 \vec{\gamma} \psi + \bar{\psi} \gamma_0 \vec{\gamma} (\partial_t \psi) \\
= \left( \mp m \bar{\psi} \gamma_0 - (\vec{\nabla} \bar{\psi}) \cdot \gamma_0 \vec{\gamma} \right) \gamma_0 \vec{\gamma} \psi \\
+ \bar{\psi} \gamma_0 \vec{\gamma} \gamma_0 \vec{\gamma} \psi \pm m \gamma_0 (\vec{\nabla} \psi) \\
= \pm \frac{2\mu_0}{e} j_e - \left( (\vec{\nabla} \bar{\psi}) \cdot \vec{\gamma} \right) \gamma_0 \vec{\gamma} \psi \\
+ \bar{\psi} \vec{\gamma} \vec{\nabla} \psi.
\]

Let us consider the time-component of the remaining terms of the right side:

\[
\bar{\psi} \gamma_1 (\gamma_1 \partial_x + \gamma_2 \partial_y + \gamma_3 \partial_z) \psi \\
- (\partial_x \bar{\psi} \gamma_1 + \partial_y \bar{\psi} \gamma_2 + \partial_z \bar{\psi} \gamma_3) \gamma_1 \psi \\
= \bar{\psi} \partial_x \psi + \bar{\psi} \gamma_1 \gamma_2 \partial_y \psi + \bar{\psi} \gamma_1 \gamma_3 \partial_z \psi \\
- (\partial_x \bar{\psi} \gamma_1) \gamma_2 \gamma_3 \gamma_1 \psi \\
= \bar{\psi} \partial_x \psi - (\partial_y \bar{\psi} \gamma_1 \gamma_2) \gamma_1 \psi + \partial_x \bar{\psi} \gamma_3 \gamma_1 \psi \\
= \bar{\psi} \partial_x \psi - (\partial_y \bar{\psi} \gamma_1 \gamma_2) \gamma_1 \psi + \partial_x \bar{\psi} \gamma_3 \gamma_1 \psi \\
= \bar{\psi} \partial_x \psi - (\partial_y \bar{\psi} \gamma_1 \gamma_2) \gamma_1 \psi + \partial_x \bar{\psi} \gamma_3 \gamma_1 \psi
\]

(E17)

Once again, it follows from Eq. 103 that the first term vanishes and one obtains Ampere’s law:

\[ \vec{\nabla} \times \vec{B} - \partial_t \vec{E} = \mp j_e. \]  

(E18)

We leave Faraday’s Law as an exercise.

Appendix F: Higher Even Moments

Second moments are averages of quadratic forms and can either be represented in the form of the \( \Sigma \)-matrix or alternatively by the use of Kronecker products \( \psi \otimes \psi \). The simplest spinor \( \psi = (q, p)^T \) for instance generates a spinor \( \psi \otimes \psi = (q^2, q p, p q, p^2)^T \), two different spinors give \( \psi_1 \otimes \psi_2 = (q_1, q_2, p_1, p_2, q_1, q_2, p_1, p_2)^T \).

The rules for the Kronecker product “\( \otimes \)” are given in Eq. 14. If we define the second order spinor according to \( \psi_2 = \psi \otimes \psi \), the 4th-order moments can be written in matrix form according to

\[ \Sigma_4 = \langle \psi_2 \psi_2^\dagger \rangle. \]  

(F1)

The spinor \( \psi_2 \) and the matrix \( \Sigma_4 \) are not free of redundancy, since \( q \) and \( p \) commute. However, the use of Kronecker products allows to stay within the algebraic framework as described for the case of simple spinors.

For (skew-) Hamiltonian matrices \( S (C) \) one finds:

\[
(S_1 \otimes S_2)^T = (S_1^\dagger \otimes S_2^\dagger) \psi_1 \otimes (S_2 \otimes \gamma_0) \psi_2 \\
= (\psi_1 \otimes \gamma_0) (S_1 \otimes S_2) (\gamma_0 \otimes \gamma_0) \psi_2 \\
(C_1 \otimes C_2)^T = -(\gamma_0 C_1 \gamma_0) \otimes (\gamma_0 C_2 \gamma_0) \\
= (\gamma_0 \otimes \gamma_0) (C_1 \otimes C_2) (\gamma_0 \otimes \gamma_0)
\]  

(F2)

As mentioned above and explained in Refs. 33, 48, the constitutive properties of the SUM \( \gamma_0 \) are, that it must be skew-symmetric, orthogonal and that it squares to \( -1 \), which is not fulfilled by \( \gamma_0 \otimes \gamma_0 \), but by \( \gamma_0 \otimes \gamma_0 \otimes \gamma_0 \), or more general: The moments of order \( D = 2 d \) for \( d \) odd, lead automatically to symplectic motion, if the basic spinors are subject to symplectic EQOMs.

1. Fourth Order Moments

Given that the fundamental EQOM are linear, e.g. are given by Eq. 3 one finds the simple generalization, starting with \( \psi_1 = F \psi_1 \) and \( \psi_2 = G \psi_2 \):

\[
\phi = \psi_1 \otimes \psi_2 \\
\phi = \psi_1 \otimes \psi_2 + \psi_1 \otimes \psi_2 \\
= (F \otimes 1 + 1 \otimes G) (\psi_1 \otimes \psi_2)
\]

(F3)

such that the EQOM for the second moments are linear as well with the driving matrix \( H \) given by

\[
H = F \otimes 1 + 1 \otimes G \equiv F \oplus G
\]

(F4)

which is called Kronecker sum. As well known from linear algebra, the matrix exponential holds:

\[
\exp(F \oplus G) = \exp(F) \otimes \exp(G).
\]

(F5)

From Eq. 17 one finds:

\[
\frac{d}{dt} (S_1 \otimes S_2) = (\dot{S}_1 \otimes S_2) + (S_1 \otimes \dot{S}_2) \\
= (F \oplus G) (S_1 \otimes S_2) \\
= (S_1 \otimes S_2) (F \oplus G)
\]

(F6)

such that with \( S = S_1 \otimes S_2 \) we may again write:

\[
\dot{S} = HS - SH.
\]

(F7)

The transpose of the driving matrix \( H^T \) is given by:

\[
H^T = F^T \otimes 1 + 1 \otimes G^T \\
= \gamma_0 F \gamma_0 \otimes 1 + 1 \otimes G \gamma_0 \\
= -(\gamma_0 \otimes \gamma_0) (F \otimes 1 + 1 \otimes G) (\gamma_0 \otimes \gamma_0) \\
= -(\gamma_0 \otimes \gamma_0) H (\gamma_0 \otimes \gamma_0)
\]

(F8)

Obviously \( H \) obeys a new criterion and is neither obviously Hamiltonian nor skew-Hamiltonian, since the matrix \( (\gamma_0 \otimes \gamma_0) \) is not skew-symmetric and can hence not
be interpreted as a symplectic unit matrix in the above sense. Therefore $\mathbf{H}$ is not a (higher order) Hamiltonian matrix, though the trace of $\mathbf{H}$ is zero. Nevertheless the EQOM are of a form that constitutes a Lax pair. The corresponding constants of motion are then again
\[
\text{Tr}(S^k) = \text{Tr} \left( (S_1 \otimes S_2)^k \right) = \text{const.} \quad \text{(F9)}
\]
Also Eq. (F3) can be derived within the framework of Hamiltonian motion as we will show in the following. If we write $\tilde{\gamma}_0 = \gamma_0 \otimes \gamma_0$ (where $\gamma_0^2 = 1$ and $\tilde{\gamma}_0^2 = \tilde{\gamma}_0$) and the Hamiltonian $\mathcal{H} (\phi)$ according to
\[
\mathcal{H} = \phi^T \tilde{\gamma}_0 \mathbf{H} \phi,
\]
then we obtain
\[
\mathcal{H} = \phi^T \tilde{\gamma}_0 \mathbf{H} \phi + \phi^T \tilde{\gamma}_0 \mathbf{H} \phi = \phi^T \mathbf{H}^T \tilde{\gamma}_0 \mathbf{H} \phi + \phi^T \tilde{\gamma}_0 \mathbf{H}^2 \phi = \phi^T (-\tilde{\gamma}_0 \mathbf{H} \tilde{\gamma}_0 \mathbf{H} + \tilde{\gamma}_0 \mathbf{H}^2) \phi = \phi^T (-\tilde{\gamma}_0 \mathbf{H}^2 + \tilde{\gamma}_0 \mathbf{H}^2) \phi = 0
\]
and the Hamiltonian function $\mathcal{H}$ of the fourth order moments vanishes, if the first order motion is symplectic.

The transfer matrix is given by Eq. (F5) and is given by
\[
\mathbf{M} = \mathbf{M}_1 \otimes \mathbf{M}_2. \quad \text{(F13)}
\]
Since $\mathbf{M}_1$ and $\mathbf{M}_2$ are symplectic, it follows that
\[
\mathbf{M} \tilde{\gamma}_0 \mathbf{M}^T = (\mathbf{M}_1 \otimes \mathbf{M}_2) (\tilde{\gamma}_0 \otimes \gamma_0) (\mathbf{M}_1^T \otimes \mathbf{M}_2^T) = (\mathbf{M}_1 \tilde{\gamma}_0 \mathbf{M}_1^T) \otimes (\mathbf{M}_2 \gamma_0 \mathbf{M}_2^T) = \tilde{\gamma}_0 \gamma_0 = \tilde{\gamma}_0 \quad \text{(F14)}
\]

Hence, though $\tilde{\gamma}_0$ is not a symplectic unit matrix (since $\tilde{\gamma}_0^2 = +1$) and though $\mathbf{M}$ is not symplectic, nonetheless $\mathbf{M}$ obeys an equation that is equivalent to Eq. (25).

2. Eigenvalues of Kronecker Sums

It is a known result in matrix analysis that the eigenvalues of the Kronecker sum of $\mathbf{F}$ and $\mathbf{G}$ are sums of eigenvalues of $\mathbf{F}$ and $\mathbf{G}$. More precisely, if $\mathbf{f}$ is eigenvector of $\mathbf{F}$ with eigenvalue $f$ and $\mathbf{g}$ is eigenvector of $\mathbf{G}$ with eigenvalue $g$, then $\mathbf{f} \otimes \mathbf{g}$ is eigenvector of $\mathbf{F} \otimes \mathbf{G}$ with eigenvalues $f + g$ [111]. $\mathbf{F}$ and $\mathbf{G}$ are Hamiltonian matrices and for such matrices it is known that if $f$ is an eigenvalue of $\mathbf{F}$, then $-f, \tilde{f}$ and $-\tilde{f}$ are also eigenvalue of $\mathbf{F}$ [23]. Thus the 4-th order moments contain the frequencies $f + g$ and $f - g, -f + g$ and $-f - g$.

In case of two single degrees of freedom, the normal forms are:
\[
\mathbf{F} = \begin{pmatrix} \omega_1 \gamma_0 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{(F15)}
\]
\[
\mathbf{G} = \omega_2 \gamma_0
\]
so that
\[
\mathbf{H} = \mathbf{F} \oplus \mathbf{G} = \begin{pmatrix} 0 & \omega_2 & \omega_1 & 0 \\ -\omega_2 & 0 & 0 & \omega_1 \\ -\omega_1 & 0 & 0 & \omega_2 \\ 0 & -\omega_1 & -\omega_2 & 0 \end{pmatrix} \quad \text{(F16)}
\]
The eigenvalues of $\mathbf{H}$ are $\pm |\omega_1 + \omega_2|$ and $\pm |\omega_1 - \omega_2|$.

Eq. (F8) seemingly suggests the introduction of complex numbers and of a symplectic unit matrix $(\gamma_0)_{i2} = i (\gamma_0 \otimes \gamma_0)$, but there is no way to derive the EQOM from a non-zero real-valued Hamiltonian function.

3. Sixth Order Moments

It is quite obvious that the next even order $\phi \equiv \psi_1 \otimes \psi_2 \otimes \psi_3$, based on the definitions
\[
\dot{\psi}_1 = \mathbf{F} \psi_1 \quad \dot{\psi}_2 = \mathbf{G} \psi_2
\]
\[
\psi_3 = \mathbf{H} \psi_3 \quad (\gamma_0)_{3} = \gamma_0 \otimes \gamma_0 \otimes \gamma_0 \quad \text{(F17)}
\]
again leads to
\[
\mathbf{J} = \mathbf{F} \otimes \mathbf{1} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{G} \otimes \mathbf{1} + \mathbf{1} \otimes \mathbf{1} \otimes \mathbf{H} = \mathbf{F} \oplus \mathbf{G} \oplus \mathbf{H}
\]
\[
\mathbf{S} = \mathbf{S}_1 \otimes \mathbf{S}_2 \otimes \mathbf{S}_3 \quad \text{(F18)}
\]
which again are symplectic laws of motion
\[
\dot{\mathbf{S}} = \mathbf{J} \mathbf{S} - \mathbf{S} \mathbf{J},
\]
with the Lax pair $\mathbf{S}$ and $\mathbf{J}$ and the respective constants of motion. The generalization of these findings is obvious:

All spinors $\psi_k = \prod_{i=0}^{k-1} \otimes \psi_i$ with $k$ odd that are composed of equal sized spinors $\psi_i$, each of which subject to symplectic motion, are again subject to symplectic motion. Spinors with $k$ even produce constants of motion, but the linearized Hamiltonian from which they can be derived, is identically zero.

4. Symplectic High Order Moments

As we argued above, the simplest non-trivial and hence fundamental algebra is the real Dirac algebra and the
size of the corresponding spinor is \(2n = 4\). Hence if spinors for higher moments are composed as a Kronecker product from an uneven number \(k = 2m + 1\) of simple spinors, then they fulfill the constraints for symplectic motion, if all individual spinors do. For the fundamental spinor size of \(2n = 4\) this means that spinors composed from Kronecker-products corresponding to these moments have the size \(4^k = 4^{2m+1}\) and hence the corresponding matrices have the size \((4^k)^2 = 4^{4m+2} = 2^{8m+4}\). This matrix size corresponds to Clifford algebras \(Cl(N - 1, 1)\) with \(N = 8m + 4\), i.e. HCAs or the Dirac type.

Real Dirac spinors, that are constructed from even Kronecker products with an even number \(2m\) of Dirac spinors, have the size \(4^{2m} = 2^{4m}\), which corresponds to real matrix reps with \(2^k\) independent elements. However there exists no real Clifford algebra \(Cl(N - 1, 1)\) of size \(N = 8m\). And vice versa, the case \(N = 2 + 8m\) has no correspondence in higher order moments or higher order correlations. One could say that HCAs of the Pauli type can not be Kronecker-decomposed.

5. Higher Order Hamiltonian

Given a Hamiltonian is of higher than 2nd order, then it can be written as a Taylor series according to

\[
\mathcal{H}(\psi) = \frac{1}{2!} \psi^T \mathbf{A} \psi + \frac{1}{3!} \mathbf{B}_{ijk} \psi_i \psi_j \psi_k + \ldots \tag{F20}
\]

One can argue that in stable (static) systems all terms of odd order vanish, i.e. the Hamiltonian is invariant under \(\psi \rightarrow -\psi\), then:

\[
\mathcal{H}(\psi) = \frac{1}{2!} \psi^T \mathbf{A} \psi + \frac{1}{4!} \mathbf{C}_{ijkl} \psi_i \psi_j \psi_k \psi_l + \ldots \tag{F21}
\]

Since the SUM is orthogonal \(\gamma_0^T \gamma_0 = 1\), the second order term \(\mathcal{H}_2\) can be written as

\[
\mathcal{H}_2(\psi) = \frac{1}{2!} \bar{\psi}^T \mathbf{F} \psi \tag{F22}
\]

where \(\bar{\psi}^T = \psi^T \gamma_0^T\). Using the Kronecker product we define \(\gamma_2 = \psi \otimes \psi\) and \(\bar{\gamma}_2^T = \psi^T \gamma_0^T \otimes \psi^T \gamma_0^T\) such that the fourth order term is

\[
\mathcal{H}_4(\psi_2) = \frac{1}{4!} \bar{\psi}_2^T \mathbf{F}_2 \psi_2 \tag{F23}
\]

where \(\mathbf{F}_2 = (\gamma_0 \otimes \gamma_0) \mathbf{A}_2\) with a symmetric matrix \(\mathbf{A}_2\). The sixth order term is then written as

\[
\mathcal{H}_6(\psi_3) = \frac{1}{6!} \bar{\psi}_3^T \mathbf{F}_3 \psi_3 \ , \tag{F24}
\]

and so on. If the spinor \(\psi\) is of Dirac size \(2n = 4\), or, more generally, if \((2n)^2 = 2^{2N}\), then there exists a complete real matrix system that represents some Clifford algebra with elements \(\Gamma_k, k \in [0, \ldots, 2^{N} - 1]\) such that any matrix \(\mathbf{F}\) can by written as \(\mathbf{F} = \sum_k f_k \Gamma_k\) and the tensor products can be written as

\[
\mathbf{F}_2 = \sum_{jk} f_{jk} \Gamma_j \otimes \Gamma_k \ , \quad \mathbf{F}_3 = \sum_{ijk} f_{ijk} \Gamma_i \otimes \Gamma_j \otimes \Gamma_k \ , \tag{F25}
\]

so that

\[
\mathcal{H}_4 = \frac{1}{8!} \sum_{jk} f_{jk} \psi^T \otimes \bar{\psi}_{j}^T (\Gamma_j \otimes \Gamma_k) \psi \otimes \psi = \frac{1}{8!} \sum_{jk} f_{jk} \psi^T \Gamma_j (\psi^T \Gamma_k \psi) (\psi \otimes \psi) \\
\mathcal{H}_6 = \frac{1}{16!} \sum_{ijk} f_{ijk} \psi^T (\psi^T \Gamma_i (\psi^T \Gamma_j (\psi^T \Gamma_k \psi) (\psi \otimes \psi \otimes \psi))
\]

such that all terms (of the last line) vanish unless \((\gamma_0^T \Gamma_i)\) and \(\gamma_0^T \Gamma_j\) and \(\gamma_0^T \Gamma_k\) are symmetric matrices, respectively.

The number of symmetric matrices of size \(2n \times 2n\) is \(2n(2n+1)/2\), so that the number of non-vanishing terms in the Hamiltonian \(\mathcal{H}_{2k}\) has an upper limit of

\[
[n (2n + 1)]^k \tag{F27}
\]

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