GEOMETRIC INTERPRETATION OF MURPHY BASES AND AN APPLICATION

URI ONN AND POOJA SINGLA†

Abstract. In this article we study the representations of general linear groups which arise from their action on flag spaces. These representations can be decomposed into irreducibles by proving that the associated Hecke algebra is cellular. We give a geometric interpretation of a cellular basis of such Hecke algebras which was introduced by Murphy in the case of finite fields. We apply these results to decompose representations which arise from the space of submodules of a free module over principal ideal local rings of length two with a finite residue field.

1. Introduction

1.1. Flags of vector spaces. Let $k$ be a finite field and let $n$ be a fixed positive integer. Let $G = \text{GL}_n(k)$ be the group of $n$-by-$n$ invertible matrices over $k$ and let $\Lambda_n$ stand for the set of partitions of $n$. For $\lambda = (\lambda_i) \in \Lambda_n$, written in a non-increasing order, let $l(\lambda)$ denote its length, namely the number of non-zero parts. The set $\Lambda_n$ is a lattice under the opposite dominance partial order, defined by: $\lambda \leq \mu$ if $\sum_{j=1}^{i} \lambda_j \geq \sum_{j=1}^{i} \mu_j$ for all $i \in \mathbb{N}$. Let $\lor$ and $\land$ denote the operations of join and meet, respectively, in the lattice $\Lambda_n$. We call a chain of $k$-vector spaces $k^n = x_{l(\lambda)} \supset x_{l(\lambda)} - 1 \supset \cdots \supset x_0 = (0)$ a $\lambda$-flag if $\dim_k(x_{l(\lambda)} - i + 1/x_{l(\lambda)} - i) = \lambda_i$ for all $1 \leq i \leq l(\lambda)$. Let $X_\lambda = \{(x_{l(\lambda)} - 1, \cdots, x_1) \mid k^n = x_{l(\lambda)} \supset \cdots \supset x_0 = (0) \text{ is a } \lambda\text{-flag}\}$, be the set of all $\lambda$-flags in $k^n$. Let $\mathcal{F}_\lambda$ be the permutation representation of $G$ that arises from its action on $X_\lambda$ ($\lambda \in \Lambda_n$). Specifically, $\mathcal{F}_\lambda = \mathbb{Q}(X_\lambda)$ is the vector space of $\mathbb{Q}$-valued functions on $X_\lambda$ endowed with the natural $G$-action:

$$\rho_\lambda : G \to \text{Aut}_\mathbb{Q}(\mathcal{F}_\lambda)$$

$$g \mapsto [\rho_\lambda(g)f](x) = f(g^{-1}x).$$

Let $\mathcal{H}_\lambda = \text{End}_G(\mathcal{F}_\lambda)$ be the Hecke algebra associated to $\mathcal{F}_\lambda$. The algebra $\mathcal{H}_\lambda$ captures the numbers and multiplicities of the irreducible constituents in $\mathcal{F}_\lambda$. The
notion of Cellular Algebra, to be described in Section 2, was defined by Graham and Lehrer in [5]. Proving that the algebra $\mathcal{H}_\lambda$ is cellular gives, in particular, a classification of the irreducible representations of $\mathcal{H}_\lambda$ and hence also gives the decomposition of $\mathcal{F}_\lambda$ into irreducible constituents. Murphy [8, 9] gave a beautiful description of a cellular basis of the Hecke algebras of type $A_n$ denoted $\mathcal{H}_{R,q}(S_n)$; cf. [7]. For $q = |k|$ one has $\mathcal{H}(1,\ldots,1) \cong \mathcal{H}_{Q,q}(S_n)$. Dipper and James [4] (see also [7]) generalized this basis and constructed cellular bases for the Hecke algebras $\mathcal{H}_\lambda$. The first result in this paper is a new construction of this basis which is of geometric nature. More specifically, the characteristic functions of the orbits of the diagonal $G$-action on $X_\lambda \times X_\mu$ gives a basis of the Hecke modules $N_{\mu \lambda} = \text{Hom}_G(\mathcal{F}_\lambda, \mathcal{F}_\mu)$. For $\mu \leq \lambda$ we allocate a subset of these orbits denoted $\mathcal{C}_{\mu \lambda}$ such that going over all the compositions $\mathcal{C}_{\mu \lambda} \circ \mathcal{C}_{\mu \lambda}$ and all $\mu \leq \lambda$ gives the desired basis. The benefit of this description turns out to be an application in the following setting.

1.2. Flags of $\mathfrak{o}$-modules. Let $\mathfrak{o}$ be a complete discrete valuation ring. Let $\mathfrak{p}$ be the unique maximal ideal of $\mathfrak{o}$ and $\pi$ be a fixed uniformizer of $\mathfrak{p}$. Assume that the residue field $k = \mathfrak{o}/\mathfrak{p}$ is finite. The typical examples of such rings are $\mathbb{Z}_p$ (the ring of $p$-adic integers) and $\mathbb{F}_q[[t]]$ (the ring of formal power series with coefficients over a finite field). We denote by $\mathfrak{o}_\ell$ the reduction of $\mathfrak{o}$ modulo $\mathfrak{p}^\ell$, i.e., $\mathfrak{o}_\ell = \mathfrak{o}/\mathfrak{p}^\ell$. Since $\mathfrak{o}$ is a principal ideal domain with a unique maximal ideal $\mathfrak{p}$, every finite $\mathfrak{o}$-module is of the form $\bigoplus_{i=1}^l \mathfrak{o}_{\lambda_i}$, where $\lambda_i$’s can be arranged so that $\lambda = (\lambda_1, \ldots, \lambda_j) \in \Lambda = \cup \Lambda_n$. Let $\text{GL}_n(\mathfrak{o}_\ell)$ denote the group of $n$-by-$n$ invertible matrices over $\mathfrak{o}_\ell$. We are interested in the following generalization of the discussion in [11]. Let

$$\mathcal{L}(r) = \mathcal{L}(r)(\ell^n) = \{(x_r, \cdots, x_1) \mid \mathfrak{o}_\ell^n \supset x_r \supset \cdots \supset x_0 = (0), \quad x_i \text{ are } \mathfrak{o}\text{-modules}\}$$

be the space of flags of length $r$ of submodules in $\mathfrak{o}_\ell^n$. Let $\Xi \subset \mathcal{L}(r)$ denote an orbit of the $\text{GL}_n(\mathfrak{o}_\ell)$-action on $\mathcal{L}(r)$. Let $\mathcal{F}_\Xi = \mathbb{Q}(\Xi)$ be the corresponding permutation representation of $\text{GL}_n(\mathfrak{o}_\ell)$. One is naturally led to the following related problems:

**Problem A.** Decompose $\mathcal{F}_\Xi$ to irreducible representations.

**Problem B.** Find a cellular basis for the algebra $\mathcal{K}_\Xi = \text{End}_{\text{GL}_n(\mathfrak{o}_\ell)}(\mathcal{F}_\Xi)$.

Few other cases, beside the field case ($\ell = 1$) which is our motivating object, were treated in the literature. The Grassmannian of free $\mathfrak{o}_\ell$-modules, i.e., the case $r = 1$ and $x_1 \simeq \mathfrak{o}_\ell^n$ is treated fully in [11, 2]. The methods therein are foundational to the present paper. Another case which at present admits a very partial solution is the case of complete free flags in $\mathfrak{o}_2^n$; cf. [3]. In this paper we treat the first case which is not free but we restrict ourselves to level 2, that is, we look at the Grassmannian of arbitrary $\mathfrak{o}_2$-modules of type $(2^a1^b)$ in $\mathfrak{o}_2^n$. To solve this problem we are naturally led to consider certain spaces of 2-flags of $\mathfrak{o}_2$-modules as well. We give a complete solution to problems A and B in these cases.
2. Preliminaries

2.1. Hecke algebras and Hecke modules. For \( \lambda, \mu \in \Lambda_n \), we let \( N_{\lambda\mu} = \text{Hom}_G(\mathcal{F}_\mu, \mathcal{F}_\lambda) \) denote the \( \mathcal{H}_\lambda \)-\( \mathcal{H}_\mu \)-bimodule of intertwining \( G \)-maps from \( \mathcal{F}_\mu \) to \( \mathcal{F}_\lambda \). The modules \( N_{\lambda\mu} \), and in particular the algebras \( \mathcal{H}_\lambda \), have natural ‘geometric basis’ indexed by \( X_\lambda \times_G X_\mu \), the space of \( G \)-orbits in \( X_\lambda \times X_\mu \) with respect to the diagonal \( G \)-action. Specifically, for \( \Omega \in X_\lambda \times_G X_\mu \), let

\[
g_\Omega f(x) = \sum_{y : (x, y) \in \Omega} f(y), \quad f \in \mathcal{F}_\mu, \ x \in X_\lambda.
\]

Then \( \{ g_\Omega \mid \Omega \in X_\lambda \times_G X_\mu \} \) is a basis of \( N_{\lambda\mu} \). Let \( M_{\lambda\mu} \) be the set of \( l(\lambda) \)-by-\( l(\mu) \) matrices having non-negative integer entries with column sum equal to \( \mu \) and row sum equal to \( \lambda \), namely

\[
M_{\lambda\mu} = \{ (a_{ij}) \mid a_{ij} \in \mathbb{Z}_{\geq 0}, \sum_{i=1}^{l(\lambda)} a_{ij} = \mu_j, \sum_{j=1}^{l(\mu)} a_{ij} = \lambda_i \}.
\]

Geometrically, the orbits in \( X_\lambda \times_G X_\mu \) characterize the relative positions of \( \lambda \)-flags and \( \mu \)-flags in \( k^n \) and hence are in bijective correspondence with the set \( M_{\lambda\mu} \). The bijection

\[
X_\lambda \times_G X_\mu \leftrightarrow M_{\lambda\mu},
\]

is obtained by mapping the pair \( (x, y) \in X_\lambda \times X_\mu \) to its intersection matrix \( (a_{ij}) \in M_{\lambda\mu} \), defined by

\[
a_{ij} = \dim_k \left( \frac{x_i \cap y_j}{x_i \cap y_{j-1} + x_{i-1} \cap y_j} \right).
\]

2.2. The RSK Correspondence. A Young diagram of a partition \( \mu \in \Lambda_n \) is the set \( [\mu] = \{(i, j) \mid 1 \leq j \leq \mu_i \text{ and } 1 \leq i \leq l(\mu)\} \subset \mathbb{N} \times \mathbb{N} \). One usually represent it by an array of boxes in the plane, e.g. if \( \mu = (3, 2) \) then \( [\mu] = \boxed{3} \boxed{2} \). A \( \mu \)-tableau \( \Theta \) is a labeling of the boxes of \([\mu]\) by natural numbers. The partition \( \mu \) is called the shape of \( \Theta \) and denoted \( \text{Shape}(\Theta) \). A Young tableau is called semistandard if its entries are increasing in rows from left to right and are strictly increasing in columns from top to bottom. A semistandard tableau of shape \( \mu \) with \( \sum \mu_i = n \) is called standard if its entries are integers from the set \( \{1, 2, \ldots, n\} \), each appearing exactly once and strictly increasing from left to right as well. Given partitions \( \nu \) and \( \mu \), a tableau \( \Theta \) is called of ‘shape \( \nu \) and type \( \mu \)’ if it is of shape \( \nu \) and each natural number \( i \) occurs exactly \( \mu_i \) times in its labeling. We denote by \( \text{std}(\nu) \) and \( \text{sstd}(\nu \mu) \) the set of all standard \( \nu \)-tableaux and set of semistandard \( \nu \)-tableaux of type \( \mu \), respectively. We remark that the set \( \text{sstd}(\nu \mu) \) is nonempty if and only if \( \nu \leq \mu \).
The RSK correspondence is an algorithm which explicitly defines a bijection
\[ M_{\lambda\mu} \longleftrightarrow \bigsqcup_{\nu \leq \lambda \land \mu} \text{st}(\nu\lambda) \times \text{st}(\nu\mu), \]
where \( \lambda \land \mu \) is the meet of \( \lambda \) and \( \mu \). For more details on this see [6].

**Definition 2.1.** We say that a \( \nu \)-flag \( y \) is *embedded* in a \( \mu \)-flag \( x \), denoted \( y \hookrightarrow x \), if \( l(\nu) \leq l(\mu) \) and \( y_{(\nu)−i} \subset x_{(\mu)−i} \) for all \( 1 \leq i \leq l(\nu) \). The intersection matrix of each embedding of \( \nu \)-flag into \( \mu \)-flag determines a \( \nu \)-tableau of type \( \mu \) as follows: for any intersection matrix \( E = (a_{ij}) \in M_{\nu\lambda} \), construct the Young tableau with \( a_{ij} \) many \( i \)'s in its \( j \)th row. We call an embedding of \( \nu \)-flag into a \( \mu \)-flag permissible if the \( \nu \)-tableau obtained is semistandard. The set \( M^\circ_{\nu\lambda} \) denotes the subset of \( M_{\nu\lambda} \) consisting of intersection matrices that corresponds to permissible embeddings.

The following gives a reformulation of the RSK correspondence purely in terms of intersection matrices:
\[
M_{\lambda\mu} \longleftrightarrow \bigsqcup_{\nu \leq \lambda \land \mu} M^\circ_{\nu\lambda} \times M^\circ_{\nu\mu}.
\]

For partitions \( \nu \leq \lambda \) of \( n \) we let \( (X_\nu \times X_\lambda)^\circ \) denote the subset of \( X_\nu \times X_\lambda \) which consists of pairs \((z, x)\) such that \( z \) is permissibly embedded in \( x \). The orbits \( (X_\nu \times_G X_\lambda)^\circ \) are therefore parameterized by \( M^\circ_{\nu\lambda} \). This gives a purely geometric reformulation of the RSK correspondence:
\[
X_\lambda \times_G X_\mu \xrightarrow{\tau} \bigsqcup_{\nu \leq \lambda \land \mu} (X_\nu \times_G X_\lambda)^\circ \times (X_\nu \times_G X_\mu)^\circ.
\]
The gist of (2.6) is that both sides have geometric interpretations. We remark that the above bijection is an important reason behind the cellularity of MDJ basis (see Section 3.1).

### 2.3. Cellular Algebras.
Cellular algebras were defined by Graham and Lehrer in [2]. We use the following equivalent formulation from Mathas [7].

**Definition 2.2.** Let \( K \) be a field and let \( A \) be an associative unital \( K \)-algebra. Suppose that \((\zeta, \geq)\) is a finite poset and that for each \( \tau \in \zeta \) there exists a finite set \( \mathcal{T}(\tau) \) and elements \( c^\tau_{st} \in A \) for all \( s, t \in \mathcal{T}(\tau) \) such that \( C = \{ c^\tau_{st} \mid \tau \in \zeta \text{ and } s, t \in \mathcal{T}(\tau) \} \) is a basis of \( A \). For each \( \tau \in \zeta \) let \( \tilde{A}^\tau = \text{Span}_K \{ c^\tau_{uv} \mid \omega \in \zeta, \ \omega > \tau \text{ and } u, v \in \mathcal{T}(\omega) \} \). The pair \((C, \zeta)\) is a cellular basis of \( A \) if

1. The \( K \)-linear map \( * : A \to A \) determined by \( c^\tau_{st} \mapsto c^\tau_{ts} \) (\( \tau \in \zeta, s, t \in \mathcal{T}(\tau) \)) is an algebra anti-homomorphism of \( A \); and,
2. for any \( \tau \in \zeta, \ t \in \mathcal{T}(\tau) \) and \( a \in A \) there exists \( \{ \alpha_v \in K \mid v \in \mathcal{T}(\tau) \} \) such that for all \( s \in \mathcal{T}(\tau) \)

\[
a \cdot c^\tau_{st} = \sum_{v \in \mathcal{T}(\tau)} \alpha_v c^\tau_{sv} \text{ mod } \tilde{A}^\tau.
\]
If \( A \) has a cellular basis then \( A \) is called a cellular algebra.

The result about semisimple cellular algebras which we shall need is the following. Let \( A \) be a semisimple cellular algebra with a fixed cellular basis \( (\mathcal{C} = \{ c_{st}^\tau \}, \zeta) \). For \( \tau \in \zeta \) let \( A^\tau \) be the \( K \)-vector space with basis \( \{ c_{uv}^\mu \mid \mu \in \zeta, \mu \geq \tau \text{ and } u, v \in \mathcal{T}(\mu) \} \). Thus \( \tilde{A}^\tau \subset A^\tau \) and \( A^\tau/\tilde{A}^\tau \) has basis \( c_{st}^\tau + \tilde{A}^\tau \) where \( s, t \in \mathcal{T}(\tau) \). It is easy to prove that \( A^\tau \) and \( \tilde{A}^\tau \) are two sided ideals of \( A \). Further if \( s, t \in \mathcal{T}(\tau) \), then there exists an element \( \alpha_{st} \in K \) such that for any \( u, v \in \mathcal{T}(\tau) \),

\[
c_{us}^\tau c_{tv}^\tau = \alpha_{st} c_{uv}^\tau \mod \tilde{A}^\tau.
\]

For each \( \tau \in \zeta \) the cell modules \( C^\tau \) is defined as the left \( A \) module with \( K \)-basis \( \{ b_t^\tau \mid t \in \mathcal{T}(\tau) \} \) and with the left \( A \) action:

\[
a \cdot b_t^\tau = \sum_{v \in \mathcal{T}(\lambda)} \alpha_v b_v^\tau,
\]

for all \( a \in A \) and \( \alpha_v \) are as given in the Definition 2.2. Furthermore, dual to \( C^\tau \) there exists a right \( A \)-modules \( C^{\tau*} \) which has the same dimension over \( K \) as \( C^\tau \), such that the \( A \)-modules \( C^\tau \otimes_K C^{\tau*} \) and \( A^\tau/\tilde{A}^\tau \) are canonically isomorphic.

**Theorem 2.3.** [2 Lemma 2.2 and Theorem 3.8] Suppose \( \zeta \) is finite. Then \( \{ C^\tau \mid \tau \in \zeta \text{ and } C^\tau \neq 0 \} \) is a complete set of pairwise inequivalent irreducible \( A \)-modules. Let \( \zeta^+ \) be the set of elements \( \tau \in \zeta \) such that \( C^\tau \neq 0 \). Then \( A \cong \bigoplus_{\tau \in \zeta^+} C^\tau \otimes_K C^{\tau*} \).

### 3. Another description of Murphy-Dipper-James bases

#### 3.1. MDJ Bases.

For a positive integer \( n \), let \( S_n \) be the symmetric group of \( \{1, 2, \ldots, n\} \). Let \( S \) be the subset of \( S_n \) consisting of the transpositions \((i, i + 1)\). Let \( R \) be a commutative integral domain and let \( q \) be an arbitrary element of \( R \). The Iwahori-Hecke algebra \( \mathcal{H}_{R,q}(S_n) \) is the free \( R \)-module generated by \( \{ T_\omega \mid \omega \in S_n \} \) with multiplication given by

\[
T_w T_s = \begin{cases} T_{ws} & \text{if } \ell(ws) > \ell(w), \\ qT_{ws} + (q-1)T_w & \text{otherwise,} \end{cases}
\]

where \( \ell(w) \) denotes the length of \( w \in S_n \). Also \( \ast : \mathcal{H}_{R,q}(S_n) \to \mathcal{H}_{R,q}(S_n) \) denotes an algebra anti-involution defined by \( T_\omega^* = T_{\omega^{-1}} \). For a partition \( \mu \), let \( S_\mu \) be the subset of \( S_n \) consisting of all permutations leaving the sets \( \{ \sum_{i=1}^{j-1} \mu_i + 1, \ldots, \sum_{i=1}^{j} \mu_i \} \) invariant for all \( 1 \leq j \leq l(\mu) \) and let \( m_\mu = \sum_{\omega \in S_\mu} T_\omega \). Let \( N_{\lambda, \mu}^q \) denote the free \( R \)-module \( m_\lambda \mathcal{H}_{R,q}(S_n) m_\mu \).

For each partition \( \nu \) of \( n \), let \( \phi^\nu \) be the unique \( \nu \)-standard tableau in which the integers \( \{1, 2, \ldots, n\} \) are entered in increasing order from left to right along the rows of \([\nu]\). For each \( \nu \)-standard tableau \( \theta \) define the permutation matrix \( d(\theta) \) by \( \theta = \phi^\nu d(\theta) \). For any standard \( \nu \)-tableau \( \theta \) and partition \( \mu \) of \( n \) such that \( \nu \leq \mu \),
we obtain a semistandard \( \nu \)-tableau of type \( \mu \), denoted \( \mu(\theta) \), by replacing each
entry \( i \) in \( \theta \) by \( r \) if \( i \) appears in row \( r \) of \( \phi^\mu \). For given partitions \( \mu \) and \( \nu \), let
\( \Theta_1 \in \text{sstd}(\nu, \lambda) \) and \( \Theta_2 \in \text{sstd}(\nu, \mu) \), define
\[
m_{\Theta_1\Theta_2} = \sum_{\theta_1, \theta_2} m_{\theta_1\theta_2},
\]
where \( m_{\theta_1\theta_2} = T_{d(\theta_1)}^* m_{\nu} T_{d(\theta_2)} \) and the sum is over all pairs \( (\theta_1, \theta_2) \) of standard
\( \nu \)-tableau such that \( \lambda(\theta_1) = \Theta_1 \) and \( \mu(\theta_2) = \Theta_2 \). Let
\[
\mathcal{M}_{\mu\lambda} = \{ m_{\Theta_1\Theta_2} \mid \Theta_1 \in \text{sstd}(\nu, \lambda), \Theta_2 \in \text{sstd}(\nu, \mu), \nu \leq \lambda \wedge \mu \},
\]
and for any partition \( \mu \in \Lambda_n \), let \( \Lambda_{\mu} = \{ \nu \in \Lambda_n \mid \nu \leq \mu \} \). Then

**Theorem 3.1** (Murphy, Dipper-James). The set \( (\mathcal{M}_{\mu\lambda}, \Lambda_{\lambda\wedge\mu}) \) is an \( R \)-basis of the
Hecke module \( \mathcal{N}_q^\mu \).

**Proof.** See Mathas [7, Theorem 4.10, Corollary 4.12]. \( \square \)

**Remark 3.2.** The following observation from the proof is important for us. For
any semistandard \( \nu \)-tableau \( \Theta \), let first(\( \Theta \)) be the unique row standard \( \nu \)-tableau
such that \( \lambda(\text{first}(\Theta)) = \Theta \). For \( \Theta \in \text{sstd}(\nu, \lambda) \),
\[
(3.1) \quad G_\Theta := \sum_{\theta \in \text{sstd}(\nu), \lambda(\theta) = \Theta} m_{\nu} T_{d(\theta)} = \sum_{\omega \in S_\nu \sigma S_\lambda} T_\omega,
\]
where \( \sigma \in S_n \) is the unique permutation matrix satisfying \( \sigma = d(\text{first}(\Theta)) \) (see
also the Remark 3.4).

Any partition \( \delta = (\delta_i) \) associates \( l(\delta) \) many \( \delta \)-row (\( \delta \)-column) submatrices with
a given \( n \times n \) matrix \( A \) by taking its rows (columns) from \( \sum_{i=0}^j \lambda_i + 1 \) to \( \sum_{i=0}^j \lambda_{i+1} \)
for all \( 0 \leq j \leq l(\delta) - 1 \).

**Definition 3.3.** (\( \lambda\mu \)-Echelon form) A matrix \( A \) is called in \( \lambda\mu \)-Echelon form if
its associated \( \lambda \)-row and \( \mu \)-column sub-matrices are in row reduced and column
reduced Echelon form respectively.

**Remark 3.4.** The matrices \( \sigma_1 \) and \( \sigma_2 \) appearing in the proof of Theorem 3.1 and
Remark 3.2 are in \( \lambda \nu \) and \( \mu \nu \) Echelon form respectively.

**3.2. Geometric interpretation of the MDJ Bases.** Recall RSK correspon-
dence is an algorithm that explicitly defines the correspondence:
\[
\mathcal{M}_{\lambda\mu} \rightarrow \bigcup_{\nu \in \Lambda_{\lambda\wedge\mu}} (X_\nu \times_G X_\lambda)^{\circ} \times (X_\nu \times_G X_\mu)^{\circ}
\]
\[
\bigcup_{\nu \in \Lambda_{\lambda\wedge\mu}} \mathcal{M}_{\nu\lambda} \times \mathcal{M}_{\nu\mu} \rightarrow \bigcup_{\nu \in \Lambda_{\lambda\wedge\mu}} \text{sstd}(\nu, \lambda) \times \text{sstd}(\nu, \mu),
\]
where the upper left corner consists of intersection matrices, the upper right consists of orbits of permissible embeddings, the lower left corner consists of intersection matrices of permissible embedding and the lower right of semistandard tableaux. For \( \nu \in \Lambda_{\lambda \mu} \) and orbits \( \Omega_1 \in (X_\nu \times_G X_\lambda)^o \), \( \Omega_2 \in (X_\nu \times_G X_\mu)^o \) define

\[
e_{\Omega_1, \Omega_2}^\nu := g_{\Omega_1} \circ g_{\Omega_2},
\]

where \([(x, y)]^\text{op} = [(y, x)]\). Clearly, \( e_{\Omega_1, \Omega_2}^\nu \in N_{\lambda \mu} \). For any partition \( \nu \), let \( P_\nu \) be the stabilizer of standard \( \nu \)-flag in \( G \) and \( B \) be the Borel subgroup of \( G \), that is the subgroup consisting of all invertible upper triangular matrices. Let \( e_{\mu \lambda} = \{ e_{\nu}^\nu \}_{\nu \in \Lambda_{\lambda \mu}, \Omega_1 \in (X_\nu \times_G X_\lambda)^o, \Omega_2 \in (X_\nu \times_G X_\mu)^o} \).

**Theorem 3.5.** The set \( (e_{\mu \lambda}, \Lambda_{\lambda \mu}) \) is a \( Q \)-basis of \( N_{\lambda \mu} \).

**Proof.** We prove this by proving that if \( q \) is the cardinality of field \( k \) then the set \( (e_{\mu \lambda}, \Lambda_{\lambda \mu}) \) coincides with MDJ basis of \( N_{\lambda \mu}^q \) up to a scalar. That is, if the orbits \( \Omega_1 \) and \( \Omega_2 \) correspond to semistandard tableau \( \Theta_1 \) and \( \Theta_2 \) by RSK, then

\[
e_{\Omega_1, \Omega_2}^\nu = \frac{|P_\nu|}{|B|} m_{\Omega_1, \Omega_2}.
\]

By definition of \( m_{\Theta_1, \Theta_2} \) and (3.1),

\[
m_{\Theta_1, \Theta_2} = \sum_{\theta_1, \theta_2} m_{\theta_1, \theta_2} = \sum_{\theta_1, \theta_2} T_{d(\theta_1)}^* m_\nu T_{d(\theta_2)}.
\]

By using the observations:

\[1\) \( m_\nu^* = m_\nu, m_\nu^2 = \sum_{w \in S_\nu} q^{(w)} m_\nu. \]

\[2\) \( \sum_{w \in S_\nu} q^{(w)} = \frac{|P_\nu|}{|B|}. \]

We obtain

\[
m_{\Theta_1, \Theta_2} = \frac{|B|}{|P_\nu|} G_{\Theta_1}^* G_{\Theta_2}.
\]

We claim that if the semistandard tableau \( \Theta_i \) corresponds to the orbit \( \Omega_i \) by RSK then \( G_{\Theta_i} = g_{\Omega_i} \). We argue for \( i = 1 \). We have an isomorphism

\[
\mathcal{H}_{Q, B}(S_n) \cong Q[ B \setminus G / B],
\]

such that a basis element \( T_w \) in the Hecke algebra \( \mathcal{H}_{Q, B}(S_n) \) corresponds to the function \( 1_{BwB} \in Q[ B \setminus G / B] \). The commutativity of the following diagram implies that the sum \( \sum_{w \in S_{\nu \sigma_1} \lambda} T_w \) corresponds to \( 1_{P_\nu \sigma_1 P_\lambda} \).

\[
B \setminus G / B \leftrightarrow S_n
\]

\[
\downarrow
\]

\[
P_\nu \setminus G / P_\lambda \leftrightarrow S_{\nu \setminus S_n / \lambda}
\]

Therefore \( G_{\Theta_i} \) belongs to \( \text{Hom}_G(\mathcal{F}_{\lambda}, \mathcal{F}_{\nu}) \). Let orbit \( \Omega_1 \) corresponds to matrix \( m = (m_{ij}) \in M_{\nu, \lambda}^\nu \) by RSK. Then by its definition, matrix \( \sigma_1 \) is the unique matrix in \( \nu \)-Echelon form such that when viewed as a block matrix having \( (i, j) \)th block
of size $\lambda_i \times \nu_j$ for $1 \leq i \leq l(\lambda)$ and $1 \leq j \leq l(\nu)$ then $m_{ij} = \text{sum of entries of } (i, j)^{th} \text{ block of } \sigma_1$. This implies that if $(y, x) \in \Omega_1$, then there exist full flags $\tilde{y}$ and $\tilde{x}$ that extend the flags $y$ and $x$ respectively such that the intersection matrix of $\tilde{y}$ and $\tilde{x}$ is $\sigma_1$. Therefore,

$$g_{\Omega_1} = 1_{p_\sigma_1 p_\lambda} = G_{\Theta_1}.$$  

Further, the anti-automorphism $\star$ on Hecke algebra $H_{Q,q}(S_n)$ coincides with 'op' on $X_\lambda \times_G X_\lambda$, hence the result.

For $\mu \in \Lambda$, let $H_{\lambda}^\mu = \text{Span}_Q \{ c_{\Omega_1 \Omega_2}^\mu | \nu \leq \mu \}$ and $H_{\lambda}^{\mu^\prime} = \text{Span}_Q \{ c_{\Omega_1 \Omega_2}^{\mu^\prime} | \nu < \mu \}$.

**Proposition 3.6.**

(a) Let $f \in \text{Hom}_G(\mathcal{F}_\lambda, \mathcal{F}_\mu)$ and $h \in \text{Hom}_G(\mathcal{F}_\mu, \mathcal{F}_\lambda)$ then $h \circ f \in H_{\lambda}^\mu$.  

(b) The spaces $H_{\lambda}^\mu$ and $H_{\lambda}^{\mu^\prime}$ are two sided ideal of $H_{\lambda}$.

**Proof.** (a) We prove this by induction on the partially ordered set $\Lambda$. If $\delta = (n)$, the partition with only one part, then $\mathcal{F}_\delta$ is the trivial representation, and it is easily seen that $N_{\lambda\delta}$ and $N_{\delta\lambda}$ are one dimensional. It follows that $H_{\lambda}^\delta = N_{\lambda\delta} \circ N_{\delta\lambda}$ is one dimensional and spanned by $c_{\Omega_1 \Omega_2}^\delta$. This established the basis for the induction. Now assume the assertion is true for any $\nu \in \Lambda$ such that $\nu < \mu$. We prove the result for $\mu$. Let $p_i$ for $1 \leq i \leq r$ be all the permissible embeddings of $\mu$-flags into $\lambda$-flags and let $\Omega_{p_i} \in (X_\mu \times_G X_\lambda)^\delta$ be the orbits corresponding to these embeddings. The orbit corresponding to the identity mapping of $\mu$-flag into itself is denoted by $\Omega_{id}$. Let $N_{\lambda\mu}$ be the subspace of $N_{\lambda\mu}$ generated by the set $\{ c_{\Omega_1 \Omega_2}^\mu | \nu < \mu, \Omega_1 \in (X_\mu \times_G X_\lambda)^\delta, \Omega_2 \in (X_\nu \times_G X_\mu)^\delta \}$. It follows that any $h \in \text{Hom}_G(\mathcal{F}_\mu, \mathcal{F}_\lambda)$ can be written as linear combination $\sum_{i=1}^r \alpha_i \cdot c_{\Omega_{p_i} \Omega_\lambda}^\mu$ mod $N_{\lambda\mu}$. Therefore, it is enough to prove that $c_{\Omega_{p_i} \Omega_\lambda}^\mu f \in H_{\lambda}^\mu$ for all $1 \leq i \leq r$. Arguing similarly for $f$ it suffices to prove that $c_{\Omega_{p_i} \Omega_\lambda}^\mu c_{\Omega_{p_j} \Omega_\lambda}^\mu \in H_{\lambda}^\mu$ for all $1 \leq i, j \leq r$. But since $c_{\Omega_{p_i} \Omega_\lambda}^\mu c_{\Omega_{p_j} \Omega_\lambda}^\mu = c_{\Omega_{p_i} \Omega_\lambda}^\mu$, the result follows.

(b) The fact that $H_{\lambda}^\mu$ is a two-sided ideal follows immediately from (a) and fact that $\{ c_{\Omega_1 \Omega_2}^\mu | \mu \leq \lambda, \Omega_1, \Omega_2 \}$ is a basis by observing that compositions of basis elements of the form $c_{\Omega_1 \Omega_2}^\mu c_{\Omega_1 \Omega_2}^\mu$

lies in $H_{\lambda}^{\mu^\prime \mu}$. Finally, as $H_{\lambda}^{\mu^\prime} = \sum_{\nu < \mu} H_{\lambda}^\nu$, the latter is an ideal as well.

**Theorem 3.7.** The Hecke algebra $H_{\lambda}$ is cellular with respect to $(C_{\lambda\lambda}, \Lambda_{\lambda})$.  


Proof. We have a natural anti-automorphism of the Hecke algebras $H_\lambda$ defined as

$$(e^\mu_{\Omega_1\Omega_2})^* = e^\mu_{\overline{\Omega_2}\Omega_1}.$$  

Proposition 3.6 implies that the criterion 2.7 for cellularity is fulfilled as well. □

Corollary 3.8. There exists a collection $\{U_\lambda \mid \lambda \in \Lambda_n\}$ of inequivalent irreducible representations of $\text{GL}_n(k)$ such that

(1) $F_\lambda = \bigoplus_{\nu \leq \lambda} U^{\lambda \rightarrow \nu}_\nu$;

(2) For every $\mu, \nu \leq \lambda$, one has $\text{dim}_Q \text{Hom}_G(U_\nu, F_\mu) = |\mathcal{M}_{\nu, \mu}|$. That is, the multiplicity of $U_\nu$ in $F_\mu$ is the number of non-equivalent permissible embeddings of a $\nu$-flag in a $\mu$-flag. In particular $U_\nu$ appears in $F_\mu$ with multiplicity one and does not appear in $F_\mu$ unless $\nu \leq \mu$.

We remark that part (2) of Theorem 3.8 gives a characterization of the irreducible representations $U_\lambda$, that is, for each $\lambda \in \Lambda_n$, the representation $U_\lambda$ is the unique irreducible representation which occurs in $F_\lambda$ and do not occur in $F_\mu$ for $\mu \leq \lambda$.

3.3. General flags. In this section we extend our results of previous section to the flags not necessarily associated with partitions. A tuple $c = (c_i)$ of positive integers such that $\sum c_i = n$ is called composition of $n$. The length of $c$, denoted $l(c)$, is the the number of its nonzero parts. By reordering parts of a composition in a decreasing order we obtain the unique partition associated with it. We shall use bar to denote the associated partition. For example if $c = (2,1,2)$, then $\overline{c} = (2,2,1)$. A chain of $k$-vector spaces $x = (k^n = x_{l(c)} \supset x_{l(c)-1} \supset \cdots \supset x_1 \supset x_0 = (0))$ is a $c$-flag if $\text{dim}_k (x_{l(c)-i+1}/x_{l(c)-i}) = c_i$ for all $1 \leq i \leq l(c)$. Let $X_c$ be the space of all $c$-flags and $F_c = \mathbb{Q}(X_c)$. By the theory of representation of symmetric groups and Bruhat decomposition, it follows that for any compositions $c_1$ and $c_2$, the Hecke algebras $H_{c_1} = \text{Hom}_G(F_{c_1}, F_{c_2})$ and $H_{\overline{c}} = \text{Hom}_G(F_{\overline{c}_1}, F_{\overline{c}_2})$ are isomorphic as $G$-modules. By composing this isomorphism with the cellular basis of the Hecke algebra $H_{\overline{c}}$, one obtains the cellular basis of the Hecke algebras $H_c$. This implies that irreducible components of $F_c$ are parameterized by the set of partitions $\lambda \in \Lambda$ such that $\lambda \leq \overline{c}$. In particular this gives the following bijection

$$X_{c_1} \times_G X_{c_2} \longleftrightarrow \bigsqcup_{\nu \in \Lambda, \nu \leq \overline{c}} \left( X_\nu \times_G X_{c_1} \right)^{\circ} \times \left( X_\nu \times_G X_{c_2} \right)^{\circ}$$

for certain subsets $(X_\nu \times_G X_{c_1})^{\circ}$ and $(X_\nu \times_G X_{c_2})^{\circ}$ of $X_\nu \times_G X_{c_1}$ and $X_\nu \times_G X_{c_2}$ respectively. For any $(x, y) \in (X_\nu \times_G X_{c_1})^{\circ}$, we say $x$ has permissible embedding in $y$. Whenever we deal with compositions in later section, by cellular basis and permissible embedding we shall mean the general notions defined in this section.
4. The Module Case

In this section $\mathfrak{o}$ denotes a complete discrete valuation ring with maximal ideal $p$ and fixed uniformizer $\pi$. Assume that the residue field $k = \mathfrak{o}/p$ is finite. We denote by $\mathfrak{o}_\ell$ the reduction of $\mathfrak{o}$ modulo $p^\ell$, i.e., $\mathfrak{o}_\ell = \mathfrak{o}/p^\ell$. Since $\mathfrak{o}$ is a principal ideal domain with a unique maximal ideal $p$, every finite $\mathfrak{o}$-module is of the form $\bigoplus_{i=1}^j \mathfrak{o}_{\lambda_i}$, where the $\lambda_i$’s can be arranged so that $\lambda = (\lambda_i) \in \Lambda = \bigcup \Lambda_n$. The rank of an $\mathfrak{o}$-module is defined to be the length of the associated partition. Note that in this section we use arbitrary partitions rather than partitions of a fixed integer and parameterize different objects than in the previous sections: types of $\mathfrak{o}$-modules rather than types of flags of $k$-vector spaces. Let $\tau$ be the type map which maps each $\mathfrak{o}$-module to its associated partition. The group $\text{GL}_n(\mathfrak{o}_\ell)$ denotes the set of invertible matrices of order $n$ over the ring $\mathfrak{o}_\ell$. Let $L(\mathfrak{o}_\ell) = \{ (x_r, \ldots, x_1) \mid \mathfrak{o}_\ell^r \supset x_r \supset \cdots \supset x_0 = (0), \ x_i \text{ are } \mathfrak{o}\text{-modules} \}$ be the space of flags of modules on length $r$ in $\mathfrak{o}_\ell^r$. There is a natural partial ordering on $L(\mathfrak{o}_\ell)$ defined by $\eta = (y_r, \ldots, y_1) \leq (x_r, \ldots, x_1) = \xi$ if there exist embeddings $\phi_1, \ldots, \phi_r$ such that the diagram

\[
\begin{array}{c}
x_r \\ \uparrow_{\phi_r} \\ \vdots \\ \uparrow_{\phi_1} \\ y_r \\ \vdots \\ y_1
\end{array}
\]

is commutative. Two flags $\xi$ and $\eta$ are called equivalent, denoted $\xi \sim \eta$, if the $\phi_i$’s in the diagram are isomorphisms. For any equivalence class $\Xi = [\xi]$ let $\mathcal{F}_\Xi = \mathbb{Q}(\Xi)$ denote the space of rational valued functions on $\Xi$ endowed with the natural $\text{GL}_n(\mathfrak{o}_\ell)$-action. We use the letter $\Xi$ to denote a set of flags as well as the type of the flags in this set.

En route to developing the language and tools for decomposing the representations $\mathcal{F}_\Xi$ into irreducible representations we treat here the special case $\ell = 2$ and give a complete spectral decomposition for the $\text{GL}_n(\mathfrak{o}_2)$-representations $\mathcal{F}_\Xi$ with $\Xi \subset \mathcal{L}^{(1)}(2^n)$. Recall $\Xi \in \mathcal{L}^{(1)}(2^n)$ consists of all submodules $x \subset \mathfrak{o}_2^n$ with a fixed type $\lambda$. We shall also assume that $n \geq 2(\text{Rank}(x))$ and $\text{Rank}(x) \geq 2(\text{Rank}(\pi x))$. We have a map $\iota : \mathcal{L}^{(1)} \to \mathcal{L}^{(2)}$ given by $y \mapsto (y, \pi y)$ which allows us to identify any module with a (canonically defined) pair of modules. We will see that to find and separate the irreducible constituents of $\mathcal{F}_\lambda$ with $\lambda \in \mathcal{L}^{(1)}$ we need to use a specific set of representations $\mathcal{F}_\eta$ with $\eta \in \mathcal{L}^{(2)}$ such that $\eta \leq \iota(\lambda)$. A similar phenomenon has been observed also in [2]. We remark that for the groups $\text{GL}_n(\mathfrak{o}_2)$, it is known that the dimensions of complex irreducible representations and their numbers in each dimension depend only on the cardinality of residue field of $\mathfrak{o}$, see [11]. For the current setting we shall prove that the numbers and multiplicities of the irreducible constituents of $\mathcal{F}_\lambda$ with $\lambda \in \mathcal{L}^{(1)}$ are independent of the residue field as well, though this is not true in general, see [11, 3]. In this section we shall use the
notation $G$ to denote the group $\text{GL}_n(\mathfrak{q}_2)$, and the group of invertible matrices of order $n$ over the field $k$ is denoted by $\text{GL}_n(k)$.

4.1. Parameterizing set. Let $S_{i(\lambda)} \subset \mathcal{L}^{(2)}$ be the set of tuples $(x_2, x_1)$ satisfying

1. The module $x_1$ has a unique embedding in $x_2$ (up to automorphism).
2. $(x_2, x_1) \leq \iota(y)$, $\tau(y) = \lambda$.
3. $\text{Rank}(x_1) \leq \text{Rank}(x_2/x_1)$.

Let $\mathcal{P}_{i(\lambda)} = S_{i(\lambda)}/\sim$ be the set of equivalence classes in $S_{i(\lambda)}$. The uniqueness of embedding implies that $(x_2, x_1), (y_2, y_1) \in S_{i(\lambda)}$ are equivalent if and only if $\tau(x_2) = \tau(y_2)$ and $\tau(x_1) = \tau(y_1)$. Therefore, $\xi = [(x_2, x_1)] \in \mathcal{P}_{i(\lambda)}$ may be identified with the pair $\mu^{(2)} \supset \mu^{(1)}$ where $\mu^{(2)} = \tau(x_2)$ and $\mu^{(1)} = \tau(x_1)$. Further, if $(x_2, x_1) \in S_{i(\lambda)}$ is such that $\tau(x_2) = \lambda$ and $x_1 = \pi x_2$, then the equivalence class of $(x_2, x_1)$ in $\mathcal{P}_{i(\lambda)}$ is also denoted by $\iota(\lambda)$. For $\xi \in \mathcal{P}_{i(\lambda)}$, let

$$Y_\xi = \{ x \in S_{i(\lambda)} \mid [x] = \xi \}.$$ 

Then $S_{i(\lambda)} = \sqcup_{\xi \in \mathcal{P}_{i(\lambda)}} Y_\xi$. Let $F_\xi = \mathbb{Q}(Y_\xi)$ be the space of rational valued functions on $Y_\xi$. As discussed earlier, the space $\mathcal{P}_{i(\lambda)}$ coincides with $\mathcal{F}_\lambda$. We shall prove that $\mathcal{P}_{i(\lambda)}$ parameterizes the irreducible representations of the space $\mathcal{F}_\lambda$ and in particular satisfies a relation similar to (2.6) (See Proposition 4.2).

4.2. An analogue of the RSK correspondence. For $a \in \mathfrak{q}$ and an $\mathfrak{q}$-module $x$, let $x[a]$ and $ax$ denote the kernel and the image, respectively, of the endomorphism of $x$ obtained by multiplication by $a$. For any $x = (x_2, x_1) \in S_{i(\lambda)}$, the flag of $\pi$-torsion points of $x$, denoted $x_\pi$, is the flag $k^n \supseteq x_2[\pi] \supseteq x_1 \supseteq \pi x_2$. In general this flag may not be associated with a partition but rather a composition. If $x_1 \in S_{i(\lambda)}$ are such that $[x] = [y]$ then the compositions associated with the flags $x_\pi$ and $y_\pi$ are equal. Hence if $[x] = \xi$, then the composition associated with $x_\pi$ is denoted by $c(\xi)$.

Lemma 4.1. There exists a canonical bijection between the sets

$$\{(x_2, x_1), (y_2, y_1) \in Y_{i(\lambda)} \times_G Y_\xi \mid x_2 \cap y_2 \cong k^t, t \in \mathbb{N}\} \leftrightarrow X_{c(\xi)}(\lambda) \times_{\text{GL}_n(k)} X_{c(\xi)}$$

obtained by mapping $[(x_2, x_1), (y_2, y_1)]$ to $[(x_2, x_1)_\pi, (y_2, y_1)_\pi]$.

Proof. Since all the pairwise intersections obtained from the modules $x_2, x_1, y_2$ and $y_1$ are $k$-vector spaces, by taking the flags of the $\pi$-torsion points we obtain a well-defined map from $Y_{i(\lambda)} \times_G Y_\xi$ to $X_{c(\xi)}(\lambda) \times_{\text{GL}_n(k)} X_{c(\xi)}$.

We first prove that this map is injective. Let $(x, y), (x', y') \in Y_{i(\lambda)} \times Y_\xi$ be such that $x_\pi = x'_\pi$ and $y_\pi = y'_\pi$. Assume that $[(x_\pi, y_\pi)] = [(x'_\pi, y'_\pi)]$. This means that there exists an isomorphism $h : \pi \mathfrak{q}_2^n \rightarrow \pi \mathfrak{q}_2^n$ such that $h(x_\pi) = x'_\pi$ and $h(y_\pi) = y'_\pi$. We need to extend $h$ to a map $\tilde{h} : \mathfrak{q}_2^n \rightarrow \mathfrak{q}_2^n$ such that $\tilde{h}(x) = x'$ and $\tilde{h}(y) = y'$. The elements $x$ and $y$ are tuples of the form $(x_2, x_1)$ and $(y_2, y_1)$, respectively. We choose maximal free $\mathfrak{q}_2$-submodules $x_3, x'_3, y_3, y'_3$ of $x_2, x'_2, y_2, y'_2$, respectively.
Since \( n \geq 2(l(\lambda)) \), we can extend the map \( h \) to maps \((x_3 + \pi o_2^n) \rightarrow (x_3' + \pi o_2'^n)\) and 
\((y_3 + \pi o_2^n) \rightarrow (y_3' + \pi o_2'^n)\) in a compatible manner such that these two extensions glue to a unique well-defined map \((x_3 + y_3 + \pi o_2^n) \rightarrow (x_3' + y_3' + \pi o_2'^n)\). The latter can now be extended to an isomorphism \(\tilde{h} : o_2^n \rightarrow o_2'^n\) with the desired properties.

To prove surjectivity we need to find a pair \((x, y) \in Y_{\iota(\lambda)} \times Y_{\xi}\) which maps to a given pair \((u, v) \in X_{c(\iota(\lambda)))} \times X_{c(\xi)}\). This follows at once from the assumption \(n \geq 2(l(\lambda))\).

Let \(x = (k^n = x_t \supset \cdots \supset x_1 \supset x_0 = (0))\) be a flag of \(k\)-vector spaces and \(v\) be a \(k\)-vector space such that \(x_i \supset v\) for all \(i\), then \(x/v\) is the flag \(x/v = (k^n/v = x_t/v \supset \cdots \supset x_1/v \supset (0))\). Let \((x, x_1), y = (y_2, y_1) \in S_{\iota(\lambda)}\) be such that \(x \supseteq y\). Flags of our primal interest are \(x_2/\pi y_2\) and \(y_2/\pi y_2\). Observe that although the flag \(y_2/\pi y_2\) is associated with a partition, the flag \(x_2/\pi y_2\) may only be associated to a composition. We say that \(y\) embeds into \(x\) permissibly if \(y \leq x\) and \(y_2/\pi y_2\) embeds permissibly into \(x_2/\pi y_2\) (see Section 3.3). For \(\eta \leq \xi\), let \((Y_{\eta} \times_G Y_{\xi})^\circ\) denote the set of equivalence classes \([[(y, x)] \in Y_{\eta} \times_G Y_{\xi}\) such that \(y\) embeds permissibly in \(x\).

**Proposition 4.2.** There exists a bijection between the following sets

\[Y_{\iota(\lambda)} \times G Y_{\xi} \leftrightarrow \bigsqcup_{\eta \in S_{\iota(\lambda)}, \eta \leq \xi} (Y_{\eta} \times_G Y_{\iota(\lambda)})^\circ \times (Y_{\eta} \times_G Y_{\xi})^\circ.\]

**Proof.** Let \((x_2, x_1) \in Y_{\iota(\lambda)}\) and \((y_2, y_1) \in Y_{\xi}\) be elements such that \(x_2 \cap y_2 = z_2 \oplus z_1\) such that \(z_2 \cong o_2^n\) and \(z_1 \cong o_1^n\), and denote \(\Omega = [(x_2, x_1), (y_2, y_1)]\). Let \(x_2' = x_2/z_2, y_2' = y_2/z_2, x_1' = x_1/\pi(z_2)\) and \(y_1' = y_1/\pi(z_2)\), then \(x_2' \cap y_2' \cong o_1^n\). By Lemma 4.1 and the RSK correspondence, we obtain the double coset \([[(x_2', x_1'), (y_2', y_1')]\) corresponds to a \(\delta\)-flag \((z_2', z_1')\) for some partition \(\delta = n - s\) with its permissible embeddings \(p_1\) and \(p_2\) into the flags \((x_2', x_1')/\pi(z_2')\) and \((y_2', y_1')/\pi(z_2')\) respectively. By adjoining it with \((o_2^n, o_2'^n)\), we obtain \((u_2, u_1) = (o_2^n \oplus z_2, \pi o_2^n \oplus z_1) \in S_{\iota(\lambda)}\) with permissible embeddings \(p_1\) and \(p_2\) in \((x_2, x_1)\) and \((y_2, y_1)\) respectively. The converse implication follows by combining the RSK correspondence with the definition of permissible embedding.

**Remark 4.3.** Observe that if \(\Omega = [(x_2, x_1), (y_2, y_1)] \in Y_{\iota(\lambda))} \times_G Y_{\xi}\) corresponds to permissible embeddings \(p_1, p_2\) of \((z_2, z_1)\) in \((x_2, x_1)\) and \((y_2, y_1)\), respectively, then it may be associated with \((x_2, x_1)\) in \(\iota(\lambda)\) and \((y_2, y_1)\), respectively, then \(\pi(z_2) \cong \pi(x_2 \cap y_2)\) if \([(z_2, z_1)] = (\nu(2), \nu(1))\), we shall use the notation \(\Omega \nu\nu\nu\) instead of \(\Omega\) to specify this information.

### 4.3. Geometric bases of modules

The modules \(\text{Hom}_G(F_{\xi}, F_{\iota(\lambda)})\) for \(\xi \in \mathbb{P}_{\iota(\lambda)}\), and in particular the Hecke algebras \(H_{\iota(\lambda)} = \text{End}_G(F_{\iota(\lambda)})\), have natural geometric bases indexed by \(Y_{\iota(\lambda)} \times G Y_{\xi}\), the space of \(G\) orbits in \(Y_{\iota(\lambda)} \times Y_{\xi}\) with respect to
diagonal action of $G$. Specifically, let

\[(4.1) \quad g_\Omega f(x) = \sum_{y(x,y) \in \Omega} f(y), \quad f \in \mathcal{F}_\xi, \; x \in Y_{i(\lambda)}; \]

Then \(\{g_\Omega \mid \Omega \in Y_{i(\lambda)} \times_G Y_\xi\}\) is a basis of \(\text{Hom}_G(\mathcal{F}_{i(\lambda)}, \mathcal{F}_\xi)\).

### 4.4. Cellular basis of the Hecke algebras

In this section we determine the cellular basis of the Hecke algebras \(\mathcal{H}_{i(\lambda)}\). Let \(\mathcal{R}\) be a refinement of the partial order on \(S_{i(\lambda)}\) given by: For any \((x_2, x_1), (y_2, y_1) \in S_{i(\lambda)}\), \((x_2, x_1) \geq_\mathcal{R} (y_2, y_1)\) if either \((x_2, x_1) \geq (y_2, y_1)\) or \(\pi x_2 > \pi y_2\). The set \(P_{i(\lambda)}\) inherits this partial order as well and is denoted by \(P_{i(\lambda)}^{\mathcal{R}}\) when considered as partially ordered set under \(\mathcal{R}\). For \(\eta \in P_{i(\lambda)}\) and orbits \(\Omega_1 \in (Y_\eta \times_G Y_\xi)^o\), \(\Omega_2 \in (Y_\eta \times_G Y_{i(\lambda)})^o\) define

\[c_{\Omega_1, \Omega_2}^\eta := g_{\Omega_1^{\mathcal{R}}} g_{\Omega_2}.\]

Then \(c_{\Omega_1, \Omega_2}^\eta \in \text{Hom}_G(\mathcal{F}_{i(\lambda)}, \mathcal{F}_\xi)\). Let

\[\mathcal{C}_{i(\lambda)}^\xi = \{c_{\Omega_1, \Omega_2}^\eta \mid \eta \in P_{i(\lambda)}, \eta \leq \xi, \Omega_1 \in (Y_\eta \times_G Y_{i(\lambda)})^o, \Omega_2 \in (Y_\eta \times_G Y_\xi)^o\}.\]

**Proposition 4.4.** The set \(\mathcal{C}_{i(\lambda)}^\xi\) is a \(\mathbb{Q}\)-basis of the Hecke module \(\text{Hom}_G(\mathcal{F}_{i(\lambda)}, \mathcal{F}_\xi)\).

**Proof.** We shall prove this proposition by proving that the transition matrix between the set \(\mathcal{C}_{i(\lambda)}^\xi\) and the geometric basis \(\{g_\Omega\}\) is upper block diagonal matrix with invertible blocks on the diagonal. Wherever required we also use the notation \(\Omega_{\eta \nu}\) in place of \(\Omega\) (see Remark 4.3). We claim that

\[(4.2) \quad c_{\Omega_1, \Omega_2}^\eta = \sum_{(\Delta_\eta \chi \in Y_{i(\lambda)} \times_G Y_\xi \mid \chi \geq \eta)} a_{\Delta_\eta \chi} g_{\Delta_\eta \chi}.\]

Let \([x,y] = [(x_2, x_1), (y_2, y_1)] = \Delta_\eta \chi\). Indeed, from the definition of \(c_{\Omega_1, \Omega_2}^\eta\) and \(g_{\Delta_\eta \chi}\), it is clear that the coefficient \(a_{\Delta_\eta \chi}\) is given by

\[a_{\Delta_\eta \chi} = |\{z' \in S_{i(\lambda)} \mid [z'] = \eta, ([z', x]) = \Omega_1, ([z', y]) = \Omega_2\}|.\]

Note that if \((z_2, z_1) \in S_{i(\lambda)}\) has permissible embedding in \((x_2, x_1)\) and \((y_2, y_1)\) then \(\pi z_2\) embeds into \(\pi(x_2 \cap y_2)\). For the case \(\pi z_2 \cong \pi(x_2 \cap y_2)\), we claim that the coefficients \(a_{\Delta_\eta \chi}\) are nonzero only if \(\chi \geq \eta\). Let \(z_\pi / \pi z_2\) be a \(\delta\)-flag and \(\bar{\Omega}_1, \bar{\Omega}_2\) correspond to permissible embeddings of \(z_\pi / \pi z_2\) in \(x_\pi / \pi z_2\) and \(y_\pi / \pi z_2\), respectively. Assume that \(x_\pi / \pi z_2\) is a \(c_1\)-flag and \(y_\pi / \pi z_2\) is a \(c_2\)-flag for some compositions \(c_1\) and \(c_2\). If \(\Delta_\chi = [(x_\pi / \pi z_2, y_\pi / \pi z_2)]\), then the coefficient of \(g_{\Delta_\eta \chi}\) in the expression of \(c_{\Omega_1, \Omega_2}^{\bar{\eta}} \in \mathcal{C}_{i(\lambda)}\) is given by

\[\bar{a}_{\Delta_\eta \chi} = |\{z' \in F \mid ([z', x_\pi / \pi z_2]) = \bar{\Omega}_1, ([z', y_\pi / \pi z_2]) = \bar{\Omega}_2\}|.\]

Since by definitions \(a_{\Delta_\eta \chi} = \bar{a}_{\Delta_\eta \chi}\), the coefficient \(a_{\Delta_\eta \chi}\) is non-zero only if \(\chi \geq \eta\). This implies \(\chi \geq_\mathcal{R} \eta\) and completes the proof of (4.2).
By the discussion above we also obtain that by arranging the elements \( c_{\Omega_1, \Omega_2}^\eta \) and \( g_{\Omega} \) for \( \eta \in \mathcal{P}_{i(\lambda)} \) in the relation \( R \), the obtained transition matrix between the set

\[
\{ c_{\Omega_1, \Omega_2}^\eta \mid \eta \in \mathcal{P}_{i(\lambda)}, \Omega_1 \in (Y_\eta \times_G Y_\xi)^\circ, \Omega_2 \in (Y_\eta \times_G Y_{i(\lambda)})^\circ \}
\]

and \( \{ g_\Omega \mid \Omega \in Y_{i(\lambda)} \times_G Y_\xi \} \) is an upper block diagonal matrix with invertible diagonal blocks. Observe that the diagonal blocks are obtained as the transition matrix of certain cellular basis of Hecke algebras corresponding to the space of flags of \( k \)-vector spaces to the corresponding geometric basis. This implies that the set \( \mathcal{C}_{i(\lambda), \xi} \) is a \( \mathbb{Q} \)-basis of \( \text{Hom}_G(\mathcal{F}_{i(\lambda)}, \mathcal{F}_\xi) \).

The operation \( (c_{\Omega_1, \Omega_2}^\eta)^* = c_{\Omega_2, \Omega_1}^\eta \) gives an anti-automorphism of \( \mathcal{H}_{i(\lambda)} \). The \( \mathbb{Q} \)-basis of the modules \( \text{Hom}_G(\mathcal{F}_\xi, \mathcal{F}_{i(\lambda)}) \) and \( \text{Hom}_G(\mathcal{F}_{i(\lambda)}, \mathcal{F}_\xi) \) is given by Proposition \ref{prop:cellular}. This, combined with the arguments given in the proof of Theorem \ref{thm:cellular2} proves

**Theorem 4.5.** The set \( (\mathcal{C}_{i(\lambda), i(\lambda)}, \mathcal{P}_{i(\lambda)}^R) \) is a cellular basis of the Hecke algebra \( \mathcal{H}_{i(\lambda)} \).

**Corollary 4.6.** There exists a collection \( \{ \mathcal{V}_\eta \mid \eta \in \mathcal{P}_{i(\lambda)} \} \) of inequivalent irreducible representations of \( \text{GL}_n(\mathbb{O}_2) \) such that

\[
\mathcal{F}_{i(\lambda)} = \bigoplus_{\eta \in \mathcal{P}_{i(\lambda)}} \mathcal{V}_\eta^{m_\eta},
\]

where \( m_\eta = |(Y_\eta \times_G Y_{i(\lambda)})^\circ| \) is the multiplicity of \( \mathcal{V}_\eta \).

**References**

[1] U. Bader and U. Onn. On some geometric representations of \( \text{GL}(n, \mathbb{O}) \). *Comm. Algebra*, to appear.

[2] U. Bader and U. Onn. Geometric representations of \( \text{GL}(n, \mathbb{R}) \), cellular Hecke algebras and the embedding problem. *J. Pure Appl. Algebra*, 208(3):905–922, 2007.

[3] P. S. Campbell and M. Nevins. Branching rules for unramified principal series representations of \( \text{GL}(3) \) over a \( p \)-adic field. *J. Algebra*, 321(9):2422–2444, 2009.

[4] R. Dipper and G. James. Representations of Hecke algebras of general linear groups. *Proc. London Math. Soc. (3)*, 52(1):20–52, 1986.

[5] J. J. Graham and G. I. Lehrer. Cellular algebras. *Invent. Math.*, 123(1):1–34, 1996.

[6] D. E. Knuth. Permutations, matrices, and generalized Young tableaux. *Pacific J. Math.*, 34:709–727, 1970.

[7] A. Mathas. *Iwahori-Hecke algebras and Schur algebras of the symmetric group*, volume 15 of *University Lecture Series*. American Mathematical Society, Providence, RI, 1999.

[8] G. E. Murphy. On the representation theory of the symmetric groups and associated Hecke algebras. *J. Algebra*, 152(2):492–513, 1992.

[9] G. E. Murphy. The representations of Hecke algebras of type \( A_n \). *J. Algebra*, 173(1):97–121, 1995.

[10] U. Onn, A. Prasad, and L. Vaserstein. A note on Bruhat decomposition of \( \text{GL}(n) \) over local principal ideal rings. *Comm. Algebra*, 34(11):4119–4130, 2006.

[11] P. Singla. On representations of general linear groups over principal ideal local rings of length two. *J. Algebra*, 324(9):2543–2563, 2010.
Department of Mathematics, Ben Gurion University of the Negev, Beer-Sheva
84105 Israel
E-mail address: urionn@math.bgu.ac.il

Department of Mathematics, Ben Gurion University of the Negev, Beer-Sheva
84105 Israel
E-mail address: pooja@math.bgu.ac.il