Stress-strain state of reinforced bimodulus beam on an elastic foundation

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Abstract. The paper provides the calculation theory of an arbitrary supported and arbitrary loaded reinforced beam filled with bimodulus material. The formulas determining normal stresses, bending moments, shear forces, rotation angles and a deflection of a rectangular cross-section beam reinforced with any number of bars aligned parallel to the beam axis have been obtained. The numerical study has been carried out to investigate an influence of a modulus of subgrade reaction on values of maximum normal stresses, maximum bending moments and a maximum deflection of a hinged supported beam loaded with a point force or uniform distributed load. The estimation is based on the method of initial parameters for a beam on elastic foundation and the Bubnov-Galerkin method. Values of maximum deflections, maximum bending moments and maximum stresses obtained by these methods coincide. The numerical studies show that taking into consideration the bimodulus of material leads to the necessity to calculate the strength analysis of both tensile stresses and compressive stresses.

1. Introduction
A wide application in modern technology has been the use of materials having different elastic moduli for tension and compression (bimodulity). Bimodulity is established for alloys of cast iron, bronze, and steel. Strongly expressed property of bimodulity is such widespread building material, as concrete. For some types of fine-grained concrete, the tensile modulus of elasticity is two to three times lower than under compression, for example, AFB-1 concrete: \( E_- = 1.75 \cdot 10^4 \text{ MPa} \), \( E_+ = 0.75 \cdot 10^4 \text{ MPa} \). Differences in the values of elastic characteristics lead to discrepancies in the calculation of stress-strain state without taking into account the bimodule nature of the material. Obtaining an accurate picture of the stress-strain state is extremely important and actual in the calculation of many structures and building structures. Bimodule materials are widely used in engineering, construction and energy industries [1-3]. Vibration behavior of bimodular laminated composite cylindrical panels with various boundary conditions is considered in [4].

2. Materials and Methods
The paper considers an arbitrary supported and arbitrary loaded reinforced beam on the Winkler foundation reinforced with bars aligned parallel to the beam axis. Filler is made of bimodulus material, i.e. an elastic modulus of compression and an elastic modulus of tension are different, however, the material is isotropic. It is proved [1, 2] that the hypothesis and formulas of the strength of materials and the elasticity theory are valid for such materials. The beam is considered as statically indeterminate,
consisting of two layers with different elastic moduli for tension, compression and reinforcement. The compatibility condition for deformations assumes the equality of the radii of curvature of the beam and reinforcement layers [3, 5] presents the formulas of normal stresses, emerged in the filler:

\[ \sigma_{b-} = \frac{M_b E_{b-} z}{E_{b+b} l_{b+b} + E_{b-} l_{b-}} = \frac{M_y E_{b-} z}{E_{b+b} l_{b+b} + E_{b-b} l_{b-b} + n E_a l_a}, \]
\[ \sigma_{b+} = \frac{M_b E_{b+} z}{E_{b+b} l_{b+b} + E_{b-} l_{b-}} = \frac{M_y E_{b+} z}{E_{b+b} l_{b+b} + E_{b-b} l_{b-b} + n E_a l_a}, \] (1)

The rectangular cross-section beam figure 1 is made of fiber foam concrete reinforced with bars aligned parallel to the beam axis. To derive the formulas of normal stresses for the rectangular cross-section beam [6, 7], differences of fiber foam concrete elastic modulus of compression and elastic modulus of tension received as the result of the experiment are taken into account [8].

\[ z_{max+} = h_p = \frac{\sqrt{k} h}{1 + \sqrt{k}}, \quad \sigma_{max b+} = \frac{3(1 + \sqrt{k}) \sqrt{k} h}{k b h^3 + 3(1 + \sqrt{k}) E_a (n_p + l_{a+} + n - l_a) / E_{b+}} [M_{max y}] \] (3)
\[ z_{max-} = h_p = \frac{h}{1 + \sqrt{k}}, \quad \sigma_{max b-} = \frac{3(1 + \sqrt{k}) h}{k b h^3 + 3(1 + \sqrt{k}) E_a (n_p + l_{a+} + n - l_a) / E_{b+}} [M_{max y}] \] (4)

where \( h_p \) - height of the extension zone; \( h_c \) - height of the compression zone; \( k = E_{b-} / E_{b+} \), \( z \) - distance from the neutral line \( \partial y \) to the point in which normal stress is defined; \( M_r \) - bending moment acting relative to the neutral line in any beam cross-section; \( n \) - the number of bars; \( I_{a} \) - cross-section axial moment of inertia of a rebar bar, \( M_n \) - bending moment arising in a rebar bar; \( E_{a} \) - elastic modulus of an extension of rebar bars, \( M_{b-} \) - bending moment arising in a concrete zone of a beam; \( M_{b+} \) - bending moment arising in the concrete tensile zone; \( E_{b+} \) - elastic modulus of the tensile concrete (filler); \( I_{b+} \) - axial moment of inertia in the concrete tensile zone; \( M_{b-} \) - bending moment arising in the concrete compressive zone; \( E_{b-} \) - elastic modulus of the compressive concrete (filler); \( I_{b-} \) - axial moment of inertia in the concrete compressive zone.

3. Results
The calculations were carried out by a known method of initial parameters for a beam on an elastic base (the Krylov method) and the Bubnov-Galerkin method [9] using special orthonormal polynomials constructed on the basis of the classical Legendre polynomials [10, 11].

Using the Bubnov-Galerkin method, the functional takes the form

\[ \int_{-1}^{+1} \left( \frac{\partial v_j(\xi)}{\partial \xi^4} - a^4 [q(\xi) - \alpha v(\xi)] \right) v_j(\xi) d\xi = 0. \] (5)

The deflection function \( v(\xi) = \sum_{j=1}^{N} B_j Q_{j+3}(\xi); \quad v_j(\xi) = Q_{j+3}(\xi). \)
\[ Q_k(\xi) = -\left\{ \frac{(k-3)(k-2)}{2(k-1)} \right\} \frac{1}{2} \left[ P_k(\xi) - P_{k-2}(\xi) \right] + \left\{ \frac{(k-1)k}{2(2k-3)(k-3)(k-2)} \right\} \frac{1}{2} \left[ P_{k-2}(\xi) - P_{k-4}(\xi) \right] \]

\[ k=4,5,6,\ldots., \quad P_{\ell}(\xi) \text{ - Legendre polynomials.} \]

\[ \text{Polynomials } Q(\ ) \text{ satisfy the boundary conditions } Q_{l+3}(\pm 1) = \frac{d^2Q_{l+3}(\pm 1)}{d\xi^2} \text{ and normalization conditions } \int_{-1}^{1} \frac{d^4Q_{l+3}(\xi)}{d\xi^4} Q_{j+3}(\xi)d\xi = \delta_{ij}, \]

\[ \delta_{ij} \text{ - Kronecker symbol.} \]

To determine deflection \( v(x) \), rotation angle \( \psi(x) \), bending moment \( M(x) \), transverse force \( Q(x) \), well-known formulas defining forces and displacements for a beam on elastic foundation expressed of Krylov functions.

4. Discussion

The numerical study using an example of a hinged supported beam on elastic foundation with various moduli of subgrade reaction, elastic moduli of compression and elastic moduli of tension have been conducted. The results of the calculations are presented further.

The beam deflected mode for various elastic moduli of compression, elastic moduli of tension for a modulus of subgrade reaction of an elastic base loaded by concentrated force is considered. The hinged supported beam with both ends fixed (length is \( l=4 \) m) is loaded by concentrated force \( F=10 \) kN. The sizes of cross-section are \( b=20 \) cm; \( h=30 \) cm. The rebar characteristics: \( d=1.2 \) cm; number of bars \( N=4; \) \( E_a = 2.06 \times 10^5 \) MPa. The table 1 shows the results of calculations at \( k_0 = 100 \) MPa/m (sandy-argillaceous ground).

| \( k_0=100 \) MPa/m; \( q=250 \) kN/m | \( E_{b-} \) | \( E_{b+} \) | \( M_{\text{max}} \) | \( v_{\text{max}} \) | \( \sigma_{\text{max}b-} \) | \( \sigma_{\text{max}b+} \) |
|---|---|---|---|---|---|---|
| 2250 | 2250 | 17.8 | 1.37 \times 10^{-2} | 5.93 | 5.93 |
| 5000 | 5000 | 26 | 1.42 \times 10^{-2} | 8.66 | 8.66 |
| 2250 | 5000 | 21 | 1.40 \times 10^{-2} | 8.71 | 5.84 |

Figure 2. Bending moments and the beam bending line diagrams depending on elastic moduli of a filler at \( k_0 = 100 \) MPa/m.

As the Figure 2 and Table 1 show, the maximum value of bending moment is achieved when elastic modulus of tension is assumed as the estimated elastic modulus, the minimum value of bending is achieved for elastic modulus of compression whereas its mean value is accessed when both moduli are
taken into account. Maximum tensile normal tensile normal stress is 1.5 times more than compressive stress if various elastic moduli are considered. Taking into account the fact that the material is bimodulus, the compressive stress of filler takes on the least value, which is 10 % less than the normal stress if the estimated elastic modulus is taken as the elastic modulus of compression.

The results of the calculations are given in the table 2 at $k_0 = 4000$ MPa/m (stiff loams, sand clays and clays).

**Table 2. The results of the calculations at $k_0 = 1000$ MPa/m**

| $E_b$ | $E_b$ | $M_{\text{max}}$ | $v_{\text{max}}$ | $\sigma_{\text{maxb}}$ | $|\sigma_{\text{maxb}}|$ |
|------|-------|-----------------|-----------------|----------------|-----------------|
| 2250 | 2250  | 5.74            | $1.33 \times 10^3$ | 1.91           | 1.91            |
| 5000 | 5000  | 8.51            | $1.33 \times 10^3$ | 2.84           | 2.84            |
| 2250 | 5000  | 6.84            | $1.33 \times 10^3$ | 2.84           | 1.90            |

Figures 3 show diagrams of bending moments and graphs of the curved axis of the beam, depending on the modulus of elasticity of the filler at $k_0 = 1000$ MPa/m.

**Figure 3. Bending moments and the beam bending line diagrams depending on elastic moduli of a filler at $k_0 = 4000$ MPa/m**

All the characteristics of the beam deflected mode decrease with increasing the modulus of subgrade reaction both with taking into account the material bimodulus of the filler and without using differences of the elastic modulus of compression and the elastic modulus of tension. For hard foundations, $k_0 = 1000$ MPa/m. (Table 2, figure 3), taking into account the heteromodularity of concrete practically does not affect deflections, the curves of the bending moments qualitatively correspond to similar graphs for bases with $k_0 = 100$ MPa/m.

The maximum bending moment, taking into account the different modularity of concrete, is 16% greater than the bending moment with the calculation module $E_b = 2250$ MPa and 24% less with the calculation module $E_b = 5000$ MPa.

The beam deflected mode for various elastic moduli of compression, elastic moduli of tension for various moduli of subgrade reaction of an elastic base loaded by concentrated force. Table 3 shows the calculation results for different values of the bedding coefficient of the elastic base $k_0$ at $E_b = 2250$ MPa, $E_{b+} = 5000$ MPa.
Table 3 Results for different values of the bedding coefficient of the elastic base

| $k_0$, MPa/m | 100  | 200  | 500  | 1000 |
|-------------|------|------|------|------|
| $v_{max}$, m | 0.014| 0.00674| 2.66·10^{-3} | 1.33·10^{-3} |
| $M_{max}$, kNm | 21  | 15.2 | 9.72 | 6.84 |
| $\sigma_{max+b}$, MPa | 8.71 | 6.31 | 4.03 | 2.84 |
| $|\sigma_{max+b}|$, MPa | 5.84 | 4.23 | 2.71 | 1.90 |

Figures 6 shows the bending moment diagrams and the curves of the curved axis of the beam for different values of the bed coefficient of the elastic base $k_0$ at $E_b = 2250$ MPa, $E_b +$ = 5000 MPa.

As can be seen from the above numerical calculations (Table 3, figure 4), an increase in the rigidity of the base leads to a reduction in the bending moment, normal stresses and deflection.

5. Conclusion
Analysis of the modulus of elasticity under tension, as well as the analysis of the elasticity modulus in tension and compression, provides a theoretical basis for improving the properties of various types of concrete to increase the load-bearing capacity of building structures [12], and also to carry out refined calculations of strength and rigidity, taking into account the material's heterogeneity. The proposed method makes it possible to investigate the stress-strain state of reinforced beams from a multimodule material under arbitrary loading, for various elastic bases, various geometric and mechanical characteristics of beams. The research results for some bars coincide with the results of the Timoshenko approximate method [13,14]. However, the proposed method provides an opportunity to analyze a deflected mode of any reinforced rectangular cross-section beam on elastic foundation filled with bimodulus filler.

References
[1] Ambartsumyan S A 1982 Heterogeneous theory of elasticity (Moscow: Nauka) 317 p
[2] Beskopylny A N, Veremeenko A and A Yazyev B M 2017 MATEC Web of Conferences 106 04004
[3] Kadomtseva E E and Morgun L V 2013 Engineering journal of Don 2 URL: ivdon.ru/magazine/archive/n2y2013/1655
[4] Khan K, Patel B P and Nath Y 2009 *Journal of sound and vibration* **321** 163-183 pp
[5] Litvinov S V, Beskopyl Ny A N, Trush L and Yaziev S B 2017 MATEC Web of Conferences **106** 04013
[6] Kadomtseva E E and Beskopyl Ny A N 2013 *Engineering journal of Don* **4** URL: ivdon.ru/magazine/archive/n4y2013/1655
[7] Kadomtseva E E, Morgun L V, Beskopylnaya N I, Morgun V N and Berdink Y A 2016 Proceedings of the International conference «Building materials, constructions and designs of XXI century» (St. Petersburg)
[8] Morgun V N, Kurochka P N, Bogatina A U, Kadomtseva E E and Morgun L V 2014 *Stroymaterial Y* **8** 56-59 pp
[9] Philin A P 1981 *Applied mechanics of strained rigid body* V 1 (Moscow Naucka) 832 p
[10] Myshkis A D 2007 *Applied Mathematics for engineers* (Moscow FIZMATLIT) 688 p
[11] Burtevsa S V, Strelnikov G P and Avilkin V I 2012 *Engineering journal of Don* **4** (2) URL: ivdon.ru/magazine/archive/n4p2y2012/1291
[12] 2016 *Proc. of XVII International scientific and technical conference «Contemporary issues of engineering and building industry»* (Tula)
[13] Chepurnenko A S, Andreev V I, Beskopyl Ny A N and Yaziev B M 2016 MATEC Web of Conferences **67** 06059
[14] Beskopyl Ny A N, Liapin A A and Andreev V I 2017 MATEC Web of Conferences **117** 00018