COTANGENT MODELS FOR GROUP ACTIONS ON $b$-POISSON MANIFOLDS

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ABSTRACT. In this article we give a normal form of a $b$-symplectic form in the neighborhood of a compact orbit of a Lie group action on a $b$-symplectic manifold. We establish cotangent models for Poisson actions on $b$-Poisson manifolds and a $b$-symplectic slice theorem. We examine interesting particular instances of group actions on $b$-symplectic manifolds preserving the Poisson structure. Also, we revise the notion of cotangent lift and twisted $b$-cotangent lift introduced in [KM] and provide a generalization of the twisted $b$-cotangent lift. We introduce the notion of $b$-Lie group and the associated $b$-symplectic structures in its $b$-cotangent bundle together with their reduction theory.

1. INTRODUCTION

$b$-Symplectic structures have been intensely studied since their introduction in [GMP11]. In particular a study of their geometry in the presence of symmetries was initiated in [GMP14b] (see also [GLPR]) which yielded global results on the structure of $b$-symplectic manifolds equipped with a class of toric actions preserving the $b$-symplectic form known as $b$-Hamiltonian actions.

Symplectic group actions, or, group actions preserving the symplectic form have been well-studied. In this paper we examine, for the first time, the case of actions by non-commutative compact Lie groups on $b$-symplectic manifolds which preserve the $b$-symplectic form. Such actions are Poisson actions, as they preserve the Poisson structure to a $b$-symplectic form. We focus on local results and give a normal form about group orbits. The resulting normal form resembles the normal form provided by the symplectic slice theorem of Guillemin, Sternberg, Marle, being given by the quotient of a vector bundle $T^*G \times_{G_z} V$, where $T^*G$ is equipped with a "canonical" $b$-symplectic form, $V$ is a symplectic vector space, and $G_z$ is a subgroup of $G$ acting by cotangent lifted action of $G_z$ on $T^*G$ and by linear symplectomorphisms on $V$.

We go on to analyze two types of $b$-symplectic Lie group actions in detail, extending some constructions initiated in [KM]. These particular examples are of interest as they can be used to provide examples of integrable systems on $b$-symplectic manifolds. The first type is the $b$-cotangent lifted action to the $b$-cotangent bundle, which is a $b$-symplectic Lie group action for the canonical $b$-symplectic form on the $b$-cotangent bundle. In particular, we give a natural construction of a $b$-symplectic form on the $b$-cotangent bundle of a "$b$-Lie group". We give the reduction of the $b$-symplectic structure under the associated Lie group action.

To achieve this we analyze the consequences of a cosymplectic manifold having a group action transverse to the symplectic foliation. Examples of these include in particular co-Kähler manifolds as discussed in [BO], which inspired some techniques used here.

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The second type of action is an extension of the idea of the twisted $b$-cotangent lifted actions, as first defined in [KM]. We give a family of $b$-symplectic forms on the cotangent bundle of the torus $T^*\mathbb{T}^n$ for which the cotangent lifted action of the torus on $T^*\mathbb{T}^n$ is Poisson. Here we also study the reduction of the Poisson structure, which unlike the previous case is also a $b$-Poisson structure. These two dichotomic situations represent local and global models for Lie group actions on $b$-symplectic manifolds. The global situation is very rigid, with few examples, due to the restrictive nature of the symmetries. See for instance proposition [39] where we prove that there can be no invariant $b$-symplectic form on the cotangent bundle of a semi-simple Lie group.

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2. Preliminaries

In this section we recall the basics in $b$-symplectic geometry. An extended version of these results with proofs can be found in [GMPT11, GMPS14b, KM] and in references therein. The origin of $b$-manifolds can be found in the book by Melrose on the Atiyah-Patodi-Singer index theorem [Mc] where the calculus on manifolds with boundary was developed. In an attempt to extend the deformation quantization scheme to manifolds with boundary Nest and Tsygan [NT] also used $b$-tangent bundles in the nineties.

Definition 1. A $b$-manifold is a pair $(M, Z)$ of an oriented manifold $M$ and an oriented hypersurface $Z \subset M$. A $b$-vector field on a $b$-manifold $(M, Z)$ is a vector field which is tangent to $Z$ at every point $p \in Z$.

If $f$ is a local defining function for $Z$ on some open set $U \subset M$ and $(f, z_2, \ldots, z_n)$ is a chart on $U$, then the set of $b$-vector fields on $U$ is a free $C^\infty(U)$-module with basis

$$\left( f \frac{\partial}{\partial f}, \frac{\partial}{\partial z_2}, \ldots, \frac{\partial}{\partial z_n} \right).$$

We call the vector bundle associated to this locally free $C^\infty_M$-module the $b$-tangent bundle and denote it $bTM$. We define the $b$-cotangent bundle $bT^*M$ of $M$ to be the vector bundle dual to $bTM$. The sheaf of sections of $\Lambda^k(bT^*M)$ is denoted $b\Omega^k$ and its elements are called $b$-$k$-forms.

The classical exterior derivative $d$ on the complex of (smooth) $k$-forms extends to the complex of $b$-forms in a natural way. Indeed, any $b$-$k$-form $\omega$ can locally be written in the form $\omega = \alpha \wedge \frac{df}{f} + \beta$ where $\alpha \in \Omega^{k-1}, \beta \in \Omega^k$. Here $f$ is a local defining function of $Z$ and $\frac{df}{f}$ is the $b$-one-form dual to $f \frac{\partial}{\partial f}$ in a frame of the form $[1]$. We then define the exterior derivative $d\omega := d\alpha \wedge \frac{df}{f} + d\beta$ (see [GMPT14] for details).

To obtain a Poincaré lemma for $b$-forms, we enlarge the set of smooth functions and consider the set of $b$-functions $bC^\infty(M)$, which consists of functions with values in $\mathbb{R} \cup \{\infty\}$ of the form $c \log|f| + g$, where $c \in \mathbb{R}$, $f$ is a defining function for $Z$, and $g$ is a smooth function. We define the differential operator $d$ on this space in the obvious way: $d(c \log|f| + g) := c \frac{df}{f} + dg$, where $dg$ is the standard de Rham derivative.

Definition 2. Let $(M, Z)$ be a $b$-manifold, where $Z$ is the critical hypersurface as in Definition [1]. Let $\omega \in b\Omega^2(M)$ be a closed $b$-two-form. We say that $\omega$ is $b$-symplectic if $\omega_p$ is of maximal rank as an element of $\Lambda^2(bT^*_p M)$ for all $p \in M$. 
Note that a $b$-symplectic form is symplectic away from $Z$, implying in particular that $M$ has even dimension.

Using Moser’s trick and adjusting some classical results from symplectic geometry, we get the corresponding Darboux theorem for the $b$-symplectic case:

**Theorem 3** ($b$-Darboux theorem, [GMP14]). Let $\omega$ be a $b$-symplectic form on $(M^{2n}, Z)$. Let $p \in Z$. Then we can find a local coordinate chart

\[(x_1, y_1, \ldots, x_n, y_n)\]

centered at $p$ such that the hypersurface $Z$ is locally defined by $y_1 = 0$ and

\[\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.\]

2.1. **Geometry of the critical hypersurface.** $b$-Symplectic manifold are a special class of Poisson manifolds and so have an induced symplectic foliation. The connected components of $M \setminus Z$ are open symplectic leaves of dimension $2n$; on $Z$ we obtain a corank 1 Poisson structure, which moreover is a cosymplectic structure: there is a closed one-form $\eta \in \Omega(Z)$ such that $\eta \wedge (\iota^* \omega)^n$ is a volume form for $Z$, where $\iota$ is the embedding of $Z$ in $M$. A cosymplectic manifold has a codimension one foliation by symplectic leaves. We define the following:

**Definition 4.** A form $\alpha \in \Omega^1(Z)$ is a defining one-form of the foliation if it is nowhere vanishing and $\iota_L^* \alpha = 0$ for all leaves $L$, $\iota_L$ the canonical inclusion. A form $\omega \in \Omega^2(Z)$ is a defining two-form of the foliation if $\iota_L^* \omega$ is the symplectic form on each leaf of the foliation.

We have the following two conditions to determine when a cosymplectic manifold is the critical hypersurface of a $b$-symplectic manifold.

**Lemma 5.** (Theorem 21 in [GMP11]) The cosymplectic manifold $Z$ is the critical hypersurface of a $b$-symplectic manifold $M$ if and only if the defining one and two-forms of the foliation can be chosen to be closed.

We can make the choice of the defining one-form of the foliation unique by employing the following

**Lemma 6.** (Proposition 18 in [GMP11]) The defining one form of the cosymplectic manifold $Z$ can be chosen such that $\alpha(v_{\text{mod}}) = 1$. The defining two-form of the foliation can be chosen so that $\iota_{v_{\text{mod}}} \beta = 0$.

When referring to “the” defining one and two forms of the foliation, we understand that we are referring to those forms satisfying the above conditions.

As for any Poisson manifold, we can associate to a function $h \in C^\infty(M)$ the Hamiltonian vector field $X_h = \Pi(dh, \cdot)$. Note that Hamiltonian vector fields in the sense above are always tangent to the symplectic leaves. The following construction yields a vector field that is transverse to the symplectic leaves inside $Z$:

**Definition 7** (Modular vector field). Let $M$ be a Poisson manifold and $\Omega$ a volume form on $M$. The associated modular vector field is defined as the derivation:

\[C^\infty(M) \rightarrow \mathbb{R} : f \mapsto \mathcal{L}_{X_f} \frac{\Omega}{\Omega}.\]

It can be shown (see for instance [We97]) that this is indeed a derivation and, moreover, a Poisson vector field. Furthermore, for different choices of volume form $\Omega$, the resulting vector fields only differ by a Hamiltonian vector field.
Proposition 8. (Theorem 12 in [GMP14]) Assume that the critical hypersurface $Z$ is compact and connected and that one of the symplectic leaves $L$ inside $Z$ is compact. Then $Z$ is the mapping torus of the flow of any choice of modular vector field $u$:

$$Z = ([L \times [0, k]] / \sim_{(\phi(x), k)})$$

where the time-$t$ flow of $u$ is translation by $t$ in the second coordinate. The number $k \in \mathbb{R}^+$ is called the modular period of $Z$, and $\phi$ is the time-$k$ flow of $u$. In particular, all the symplectic leaves inside $Z$ are symplectomorphic.

2.2. Torus actions on $b$-symplectic manifolds, modular weights and moment maps. Let $\mathbb{T}^r$ act on a $b$-symplectic manifold $(M^{2n}, \omega)$ and denote $t = \text{Lie}(\mathbb{T}^r)$. Given $X \in t$ we write $X_M$ for the fundamental vector field of $X$.

Following [GMPS14b] we say that this action is $b$-Hamiltonian if the elements, $X \in t$ satisfy

$$\iota_X \omega = \partial \phi, \quad \phi \in \mathcal{C}_\infty(M),$$

That is to say,

$$\iota_X \omega = c_i(X) \partial (\log |f|) + dg$$

in a tubular neighborhood of the connected components $Z_i$ of $Z$ for $g \in \mathcal{C}_\infty(M)$.

The map $v_i : t \to \mathbb{R} : X \mapsto c_i(X)$ is called the modular weight of $Z_i$ and depends on the connected components $Z_i$. However, one can show ([GMPS14b], §2.3) that either all of them are non vanishing or all of them are vanishing.

As it happens for standard Hamiltonian actions in symplectic manifolds we can introduce the notion of moment map in the context of $b$-symplectic manifolds.

2.3. $b$-Cotangent lifts. We recall the standard cotangent lift on Lie groups:

**Definition 9.** Let $G$ be a Lie group and let $M$ be any smooth manifold. Given a group action $\rho : G \times M \to M$, we define its cotangent lift as the action on $T^* M$ given by $\hat{\rho} g = \rho^*(g^{-1})$. We then have a commuting diagram

$$
\begin{array}{ccc}
T^* M & \xrightarrow{\hat{\rho}} & T^* M \\
\pi \downarrow & & \downarrow \pi \\
M & \xrightarrow{\rho} & M
\end{array}
$$

where $\pi$ is the canonical projection from $T^* M$ to $M$.

It is a well-known fact that any cotangent lifted action is Hamiltonian with respect to the canonical symplectic structure on $T^* M$, see [GS].

In this paper we will discuss two interpretations of cotangent lifts in the $b$-case. In [GMP14] it was noted that, analogous to the symplectic case, the $b$-cotangent bundle comes equipped with a canonical $b$-symplectic form.

**Definition 10.** Let $(M, Z)$ be a $b$-manifold. Then we define a $b$-one-form $\lambda$ on $bT^*G$, considered as a $b$-manifold with critical hypersurface $bT^*G|_Z$, in the following way:

$$\langle \lambda_p, v \rangle := \langle p, (\pi_p)_*(v) \rangle, \quad p \in bT^*G, v \in bT_p(bT^*G)$$

We call $\lambda$ the $b$-Liouville form. The negative differential

$$\omega = -d\lambda$$

is the canonical $b$-symplectic form on $bT^*G$. 
Using this form the canonical b-cotangent lift was defined as follows [KM]:

**Definition 11.** Consider the b-cotangent bundle $^{b}T^*M$ endowed with the canonical b-symplectic structure. Moreover, assume that the action of $G$ on $M$ preserves the hypersurface $Z$, i.e. $\rho_g$ is a b-map for all $g \in G$. Then the lift of $\rho$ to an action on $^{b}T^*M$ is well-defined:

$$\hat{\rho} : G \times^{b}T^*M \to ^{b}T^*M : (g, p) \mapsto \rho_g^*(p).$$

Moreover, it is b-Hamiltonian with respect to the canonical b-symplectic structure on $^{b}T^*M$. We call this action together with the underlying canonical b-symplectic structure the canonical b-cotangent lift.

In [KM] another $b$-symplectic form is discussed, called the **twisted $b$-symplectic form**. For the special case of a torus:

**Definition 12.** Consider the cotangent bundle of the torus $T^*\mathbb{T}^n$ endowed with the standard coordinates $(\theta, a)$, $\theta \in \mathbb{T}^n$ and define the following one-form on the complement of the hypersurface $Z = \{a_1 = 0\}$ of $T^*\mathbb{T}^n$

$$\omega = c \log |a_1| d\theta_1 + \sum_{i=2}^n a_i d\theta_i.$$

This is called the **twisted Liouville one-form** (with modular period $c \in \mathbb{R}^+\). The negative differential of this form extends to a $b$-symplectic form on $T^*\mathbb{T}^n$, which we call the **twisted $b$-symplectic form** on $T^*\mathbb{T}^n$ given in coordinates as

$$\omega_{tw,c} := \frac{c}{a_1} d\theta_1 \wedge da_1 + \sum_{i=2}^n d\theta_i \wedge da_i.$$

We note that the critical hypersurface of such a form, in contrast to the canonical case, does not correspond to a hypersurface of the base manifold. Instead, the hypersurface is defined by a hypersurface in of each fiber of the cotangent bundle. Accordingly, we will refer to forms of this nature as **forms with singularity in the fiber**.

### 2.4. The Symplectic Slice Theorem

If $W$ is a subspace of a symplectic vector space, let $W^\omega$ denote the symplectic orthogonal of $W$. Let $G$ be a compact group acting on a manifold $M$, $z$ a point in $M$ and denote by $O_z$ the orbit of $z$ and by $G_z$ the isotropy group of $z$. The vector space $V_z = T_zO_z^\omega / T_zO_z$ is symplectic, with the symplectic form induced by the form on $T_zM$ i.e. $\omega([v], [w]) = \omega_z(v, w)$ for any $v = \pi(w)$, $\pi$ the projection to the quotient. We have a natural representation $G_z$ on the vector space $V_z$ given by the action of $d\rho_h(z)$. Note that this action is automatically symplectic.

Equip the manifold $T^*G \times_{G_z} V_z$ with a symplectic form which is the sum of the canonical form on $T^*G$ and the linear symplectic form on $V$ formed above. Consider the action of $G_z$ on $T^*G$ given by the cotangent lifted action of the right action of $G_z$ on $G$ and form the quotient bundle $T^*G \times_{G_z} V_z$. The quotient manifold $T^*G \times_{G_z} V_z$ is symplectic. Moreover the action of $G$ on the bundle induced by the cotangent lifted action of $G$ on $T^*G$ is Hamiltonian with respect to the canonical symplectic form. In fact, according to the symplectic slice theorem, this construction gives a model of the action of the group $G$ on $M$ close to the orbit $O_z$.

**Theorem 13.** (Guillemin-Sternberg [GS84], Marle [Ma]) Let $(M, \omega, G)$ be a symplectic manifold together with a Hamiltonian group action. Let $z$ be a point in $M$ such that $O_z$ is contained in the zero level set of the momentum map. Denote $G_z$ the isotropy group and $O_z$ the orbit of $z$. There is a $G$-equivariant symplectomorphism from a neighbourhood of the zero section of the bundle $T^*G \times_{G_z} V_z$ equipped with the above symplectic model to a neighbourhood of the orbit $O_z$. 
3. COTANGENT LIFTS AS MODELS FOR TRANSVERSE POISSON $S^1$ ACTIONS

In this section we will provide a prototype of a ”$b$-symplectic slice theorem” or, in other words, a normal form for Poisson $S^1$ actions in a neighbourhood of an orbit. We will discuss those orbits contained in the critical set of the $b$-symplectic manifold which are transverse to the symplectic foliation, as normal form in other cases can be obtained from the ordinary symplectic slice theorem. We look at a neighbourhood of a connected component of $Z$. If we assume the manifold is orientable, then there is a neighbourhood of $Z$ diffeomorphic to a product $M \cong Z \times (-\epsilon, \epsilon)$. We do not require the transverse Poisson action to be globally $b$-Hamiltonian, though we will prove that every transverse Poisson $S^1$ action is $b$-Hamiltonian in the neighbourhood of an orbit.

By the slice theorem there is an $S^1$-equivariant diffeomorphism from an open neighbourhood $\mathcal{U}$ of $O_z$ to an open neighbourhood of the zero section of $S^1/\mathcal{T}_m \cong S^1 \times G_z T_z M/T_z(S^1 \cdot z)$, $G_z$ the isotropy group of $z$. However, it is not immediately obvious how to equip the manifold $S^1 \times G_z T_z M/T_z(S^1 \cdot z)$ in such a way that the defined equivariant diffeomorphism becomes a Poisson morphism.

The Moser-type theorems available in the $b$-symplectic case can be used to define the desired equivariant diffeomorphism. It will be shown that $b$-symplectic forms on a manifold, $M$, equipped with a canonical $S^1$ action so that the respective defining one and two forms of the $b$-symplectic forms agree on an $S^1$ orbit can be made to be equivariantly $b$-symplectomorphic in a neighbourhood of that orbit. To bring the defining one and two forms into a simple form we lift the Poisson structure to a finite cover and define the Poisson structure as a quotient Poisson structure of a paracompact space.

Proposition 14. Let $M$ come equipped with a transverse Poisson $S^1$ action. Then $Z$ has the structure of a $\mathcal{L}$-symplectic manifold and moreover has a finite cover $\tilde{Z} := S^1 \times \mathcal{L}$, $\mathcal{L}$ a leaf of the foliation, equipped with an $S^1$ action given by translation in the first coordinate for which the projection $p : S^1 \times \mathcal{L} \to Z$ is equivariant.

Proof. A transverse Poisson $S^1$ action preserves the symplectic foliation and so sends symplectic leaves to symplectic leaves by symplectomorphisms. Let $\mathcal{L}_0$ be a leaf of the foliation and choose a fundamental vector field $X$ associated to the $S^1$ action such that the time-$t$ flow of $X$ satisfies $\Phi_X^1(\mathcal{L}_0) = \mathcal{L}_0$ and $\Phi_X^t(\mathcal{L}_0) \neq \mathcal{L}_0$ for any $0 < t < 1$. As in the proof of Proposition 15 [GMP11], the flow of such a vector field $\mathcal{L}$ equips $Z$ with the structure of a symplectic mapping torus with symplectic holonomy map $\Phi_X^1$. Let $k$ be the least time such that $\Phi_X^k = \text{Id}$. Note that $k$ must be an integer as we have $\Phi_X^k(\mathcal{L}_0) = \mathcal{L}_0 = (\Phi_X^1)^k(\mathcal{L}_0)$.

Now consider the product $S^1 \times \mathcal{L}_0$ and let $l$ be a point in $S^1 \times \mathcal{L}_0$. The projection map $p : S^1 \times \mathcal{L}_0 \to Z$ given by

$$p(\theta, l) = \Phi_X^{\theta k}(l)$$

gives the required projection, which is equivariant with respect to the $S^1$ action on $\tilde{Z} := S^1 \times \mathcal{L}_0$ given by

$$S^1 \times \tilde{Z} \to \tilde{Z} \quad \quad \quad (a, \theta, l) \mapsto (\theta + a, l)$$
Remark 15. By Lemma 1.3 of [Sa], for the above result to hold we only need to check that an orbit of the \( S^1 \) action is transverse at a single point \( z \) to a leaf through \( z \).

Proposition 16. Let \( M \) come equipped with a transverse Poisson \( S^1 \) action. Then the Poisson structure on \( Z \) is given by the quotient of a Poisson structure on \( S^1 \times \mathcal{L} \) by the action of a finite cyclic group.

Proof. Consider the action of \( \mathbb{Z}_k \) on \( S^1 \times \mathcal{L} \) given by

\[
\rho_m(t, l) = \left( t - \frac{2\pi}{m}, \Phi^m_{X}(l) \right)
\]

The orbit space of the quotient is the mapping torus

\[
\frac{[0, 1] \times \mathcal{L}}{(0, x) \sim (1, \Phi^1_{X}(x))}
\]

and so is diffeomorphic to \( Z \). We will equip it with a quotient Poisson structure which agrees with the Poisson structure on \( Z \). Denote the defining one and two forms of \( Z \) by \( \alpha \) and \( \beta \) respectively. We take as the defining two form of the Poisson structure on \( S^1 \times \mathcal{L} \) the form \( \beta = p^* \beta \), the pullback of the defining two form of the b-symplectic structure on \( Z \) to \( S^1 \times \mathcal{L} \) under the projection given in Proposition 14. Similarly, we take as the defining one form \( \tilde{\alpha} = p^* \alpha \). This defines a cosymplectic structure on \( \mathcal{L} \times S^1 \). \( \tilde{\alpha} \) and \( \beta \) are closed. Moreover \( \tilde{\alpha} \wedge \beta^n = p^*(\alpha \wedge \beta^n) \) and so \( \tilde{\alpha} \wedge \beta^n \) is a volume form on \( S^1 \times \mathcal{L} \), and so \( \tilde{\alpha} \) and \( \beta \) define a cosymplectic structure on \( S^1 \times \mathcal{L} \). \( \tilde{\alpha} \) and \( \beta \) are invariant under such an action as \( p^*_m(p^* \alpha) = p^* \alpha \) and similarly for the defining two form. The Poisson structure on the quotient clearly agrees with the Poisson structure on \( Z \). □

We can extend this cover to a neighbourhood of \( Z \) using the extension theorem of [GMP11].

Corollary 17. Let \( M \cong Z \times (\epsilon, -\epsilon) \) come equipped with a transverse Poisson \( S^1 \) action preserving the b-symplectic form \( \omega \). Then, the Poisson structure on \( M \) is given by the quotient of a Poisson structure on \( \mathcal{L} \times S^1 \times (\epsilon, -\epsilon) \) by the action of a finite cyclic group.

Proof. The Poisson structure of Proposition 16 on \( S^1 \times \mathcal{L} \) satisfies the conditions for \( \tilde{Z} \cong S^1 \times \mathcal{L} \) to be the critical hypersurface of a b-symplectic manifold. Let \( \pi_{\tilde{Z}} : M \to \tilde{Z} \) be the projection onto the first factor of \( M = \tilde{Z} \times (-\epsilon, \epsilon) \). Define, with \( t \in (-\epsilon, \epsilon) \) the b-symplectic form

\[
\tilde{\omega} = \pi^*_{\tilde{Z}} \tilde{\alpha} \wedge \frac{dt}{t} + \pi^*_{\tilde{Z}} \tilde{\beta}
\]

Then \( \omega \) is invariant under the trivial extension of the action of \( \mathbb{Z}_k \) on \( Z \) to \( N = \tilde{Z} \times (-\epsilon, \epsilon) \) and the resulting b-symplectic form on the quotient manifold agrees with \( \omega \) on all \( z \in Z \). By the b-Moser theorem (Theorem 34) of [GMP14], the forms are b-symplectomorphic in a neighbourhood of \( Z \). □

Remark 18. Similarly, any b-Poisson structure with defining one and two forms \( \tilde{\alpha} \) and \( \tilde{\beta} \) equipped with a discrete Poisson group action gives a b-Poisson structure on the base. For such a group action there are well defined one and two forms, \( \alpha \) and \( \beta \), on the base manifold defined by \( p^*(\alpha) = \tilde{\alpha} \) and \( p^*(\beta) = \tilde{\beta} \), where \( p \) is the projection to the quotient. \( \alpha \) and \( \beta \) automatically fulfill the conditions to define a cosymplectic structure on the image of the critical hypersurface. As the group action is discrete, the quotient of the symplectic structure on leaves is likewise symplectic.
3.2. \textit{b-symplectic forms with Z a product}. We now wish to simplify the Poisson structure on the equivariant cover. One obstruction to the \textit{b}-symplectic form being the simple product Poisson structure

\[ \omega = \frac{dt}{t} \wedge d\theta + \pi_L^*(\beta) \]

for \( \beta \) is a symplectic form on a leaf \( \mathcal{L} \) and \( \pi_L \) the projection \( \mathcal{L} \times S^1 \to \mathcal{L} \), comes from \textit{horizontal components} in the defining two form. These can be thought of as coming from \( H^1(\mathcal{L}) \). We note in particular that for the above form on \( M \sim S^1 \times \mathcal{L} \times (\epsilon, \epsilon) \) equipped with coordinates \( (\theta, l, t) \) and action \( a \cdot (\theta, l, t) \to (\theta + a, l, t) \), the fundamental vector field \( X \) associated to the action is a multiple of the modular vector field. If the modular class of the modular vector field of some \( b\)-symplectic form \( \omega_0 \) defined on \( M \) differs from the modular class of the fundamental vector field of the \( S^1 \) action, then there can be no equivariant \( b\)-symplectomorphism bringing \( \omega_0 \) into the above form. We provide an example below.

\textbf{Example 19.} Take as \textit{b}-symplectic manifold \( M = T^2 \times S^1 \times \mathbb{R} \) with \textit{b}-symplectic form \( \omega = d\phi \wedge d\psi + d\theta \wedge d\phi + d\theta \wedge \frac{dt}{t} \). The modular vector field is given by \( \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi} \). Whence the following action

\[ S^1 \times M \to M \]

\[ a \cdot (\phi, \psi, \theta, t) \to (\phi, \psi, \theta + a, t) \]

is a transverse Poisson action for which the fundamental vector field is not Hamiltonianly equivalent to \( v_{\text{mod}} \).

In the case that the leaves are simply connected, however, it is possible to eliminate horizontal elements by absorbing them in the defining function of the critical hypersurface.

\textbf{Proposition 20.} Suppose \( Z \cong S^1 \times \mathcal{L}, \mathcal{L} \) a leaf of the symplectic foliation, and suppose furthermore that \( \mathcal{L} \) is simply connected. Then \( \omega \) can be taken to be the \textit{b}-symplectic form

\[ \omega = \frac{df}{f} \wedge \alpha + \pi_L^*(\beta) \]

where \( \beta \) is a symplectic form on a leaf \( \mathcal{L} \) and \( \pi_L \) is the projection onto \( \mathcal{L} \times S^1 \to \mathcal{L} \).

\textbf{Proof.} A \textit{b}-symplectic form on \( Z \cong S^1 \times \mathcal{L} \) equipped with coordinates \( \mathcal{L} \times S^1 \times (-\epsilon, \epsilon) \) can be written

\[ \omega = \frac{dt}{t} \wedge \alpha + d\theta \wedge \eta + \gamma \]

where \( \gamma \) is a symplectic form on a leaf \( \mathcal{L} \). As the leaves of the foliation are simply connected, \( \eta = dh \) for some \( h \in C^\infty(M) \). The function \( f = te^h \) is then a defining function for \( Z \) and moreover

\[ \frac{df}{f} = \frac{dt}{t} + dh \]

Whence we have

\[ \omega = \frac{df}{f} \wedge \alpha + \gamma. \]

\( \square \)

3.3. \textbf{Equivariant Normal Form about an \( S^1 \)-Orbit}. We are now finally ready to construct a basic \textit{b}-symplectic model for \( S^1 \) actions in the neighbourhood of an orbit, in the spirit of the symplectic slice theorem. Let \( V \) be an even dimensional vector space and endow \( T^*S^1 \times V \) with the following symplectic form:

\[ \omega = \frac{df}{f} \wedge d\theta + \pi^*(\gamma), \]

where \( \pi \) is the projection onto \( V \), \( \gamma \) a symplectic form on the symplectic vector space \( V \), and \( f \) a defining function for the \textit{b}-symplectic manifold such that the defined critical hypersurface
Z is of the form $S^1 \times V \subset T^*S^1 \times V$, $S^1$ embedded as the zero section of $T^*S^1$. Let $Z_k$ be a finite subgroup of $S^1$ and suppose $Z_k$ defines a group action by linear symplectomorphisms on $V$. Denote the action of an element $n \in Z_k$ by $\rho_n$. Consider the action of $Z_k$ on $T^*S^1 \times V$ given by the diagonal action, where $Z_k$ acts on $T^*S^1$ by the cotangent lifted action of $Z_k$ on $S^1$ and by linear symplectomorphisms $\rho_n$ on $V$. Consider the quotient Poisson structure on $T^*S^1 \times Z_k \times V$ where the action on $T^*S^1 \times V$ is given by the diagonal action, with the action of $Z_k$ the cotangent lifted action of $Z_k$ on $T^*S^1$, i.e.

$$Z_k \times T^*S^1 \times V \to T^*S^1 \times V \quad (n, \theta, t, x) \mapsto (\theta - \frac{2\pi k}{n}, t, \rho_n(x))$$

We call the resulting quotient $b$-symplectic form on the quotient manifold a $b$-symplectic model.

Our main theorem is as follows:

**Theorem 21.** Let $M \cong Z \times (\epsilon, \epsilon)$ be a $b$-symplectic manifold equipped with a $b$-symplectic form $\omega$ and a transverse Poisson $S^1$ action. Let $z \in Z$. Let $O_z$ be an orbit of the $S^1$ action and let $Z_k$ be the isotropy group of $z$. Then there exists an $S^1$ equivariant neighbourhood $U$ of $O_z$ and an $S^1$ equivariant mapping $\phi : U \to T^*S^1 \times Z_k \times V$ equipped with a $b$-symplectic model, $\omega_0$, where the action of $S^1$ is given by the cotangent lifted action on $T^*S^1$ and $\omega$ is $b$-symplectomorphic to $\phi^*\omega_0$ on $U$.

We first need a slight generalization of the relative $b$-Moser theorem.

**Proposition 22.** Suppose that $\omega_1$ and $\omega_0$ are $b$-symplectic forms on $M$, invariant under an action of $S^1$ on $M$ which is transverse Poisson for $\omega_1$ and $\omega_0$. Denote by $O_z$ the orbit of some $z \in Z$ and suppose that $\omega_1$ and $\omega_0$ coincide at $z$. Then $\omega_1$ and $\omega_0$ are $b$-symplectomorphic in some open neighbourhood $U$ of $O_z$.

**Proof.** As the defining one and two forms associated to $\omega_1$ and $\omega_0$ are invariant under the $S^1$ action, it follows that on $O_z$ we have $\alpha_0 = \alpha_1$ and $\beta_0 = \beta_1$. By the relative Poincare lemma, in a contractible neighbourhood of $O_z$ we have that $\alpha_0 - \alpha_1 = dg$, an exact one form on $U$ and similarly $\beta_0 - \beta_1 = d\eta$, an exact two form on $U$. Whence $\omega_0 - \omega_1 = d(-g_{\frac{df}{1}} + \eta)$. Then $\omega_1 = \omega_0 + (1 - t)\omega_1$ is non degenerate on $O_z$ and so on a neighbourhood of $O_z$. We use this to define a $b$-vector field $v_t$ by $v_t \omega_t = g_{\frac{df}{1}} - \eta$. As $v_t$ is zero on $O_z$, the time-one flow exists in a neighbourhood of $O_z$ and gives the required $b$-symplectomorphism. \qed

Now we are ready to prove Theorem 21.

**Proof.** Let $z \in Z$. The slice theorem gives an $S^1$-equivariant diffeomorphism from an $S^1$ invariant open neighbourhood $U$ of $O_z$ to an open neighbourhood of the zero section of $S^1 / Z_m \cong S^1$ in $S^1 \times Z_m T_z Z / T_z (S^1 \cdot z)$. As the Poisson action is transverse, $T_z Z$ is given by $cX \oplus \partial_{\theta}$, $X$ a fundamental vector field of the $S^1$ action. By choosing an appropriate $S^1$ invariant metric we can assume $T_z Z / T_z (S^1 \cdot z) \cong T_z \mathcal{L}$. Consider a collar neighbourhood of $U$ in $M$, $(-\epsilon, \epsilon) \times U$ Consider the restriction of the $b$-symplectic form $\omega$ to $(-\epsilon, \epsilon) \times U$ and the pull-back of $\omega$ to the neighbourhood of $(-\epsilon, \epsilon) \times S^1 \times Z_m T_z \mathcal{L}$ under the diffeomorphism given by the slice theorem. Note that the symplectic leaves pull back to the fibers of the bundle $S^1 \times Z_m T_z \mathcal{L}$ and the critical set to $\epsilon = 0$. There is an induced Poisson structure on the cover, an open neighbourhood $U$ of $S^1$ in $S^1 \times T_z \mathcal{L} \times (-\epsilon, \epsilon)$, given by Proposition 16.

Let $\alpha$ and $\beta$ be the defining one and two forms of the Poisson structure on the cover. Note that the symplectic leaves of the foliation are given by $T_z \mathcal{L} \cap U$ and the critical hypersurface by $\epsilon = 0$. We have $\alpha = d\theta$. As the $S^1$ neighbourhood is simply connected, $\beta$ is of the form $\beta = d\theta \wedge dh + \gamma$ for $\gamma$ a symplectic form on $T_z \mathcal{L}$, $h$ a function on $T_z \mathcal{L}$. By a judicious choice of defining function $f = te^h$ we can adjust the defining two form to be of the form $\beta = \gamma$. Let $h$ be a linear function
on \( T_z \mathcal{L} \) with \( \hat{h} = dh \) at 0 and \( \hat{f} = te^h \). Choose coordinates \((x, y)\) on \( T_z \mathcal{L} \) at \( z \) such that that the induced symplectic form on \( T_z \mathcal{L} \) is given by \( \sum_i dx_i \wedge dy_i \). Then the form

\[
\hat{\omega} = \frac{df}{f} \wedge d\theta + \sum_i dx_i \wedge dy_i
\]

is a \( b \)-symplectic form on \( S^1 \times T_z \mathcal{L} \times (\epsilon, \epsilon) \).

It is clear that that the differential of the action of \( \mathbb{Z}_k \) acts as the identity on \((\epsilon, \epsilon)\), preserves the subspace \( T_z \mathcal{L} \) and induces a linear symplectomorphism \( \phi \) on \( T_z \mathcal{L} \) for the symplectic form induced by \( \beta \) on \( T_z \mathcal{L} \). Therefore the action of \( \mathbb{Z}_k \) on \( S^1 \times T_z \mathcal{L} \times (\epsilon, \epsilon) \) giving the associated bundle of the slice theorem can be expressed \( m \times (\theta, v, \epsilon) \rightarrow (\theta - \frac{2\pi k}{m}, \phi(v), \epsilon) \). The resulting manifold is equivariantly diffeomorphic to the neighbourhood \( U \) and the defining one and two forms of the pullback of the quotient Poisson structure agree with those of \( \omega \) on the closed manifold \( O_\epsilon \). Whence, by Proposition 22 we have that there is a \( b \)-symplectomorphism. As both \( b \)-symplectic forms are invariant under the group action, the \( b \)-symplectomorphism can be made to be equivariant. \( \square \)

**Example 23.** Consider the following symplectic mapping torus: take as a symplectic leaf a torus with non-trivial isotropy group. We can find examples from integrable systems having a naturally associated \( b \)-invariant under the group action, the \( b \)-symplectic form on \( \mathbb{S}^1 \times T_z \mathcal{L} \times (-\epsilon, \epsilon) \). Proposition 22 we have that there is a \( b \)-symplectomorphism can be expressed by \( \beta \) which can be mapped by \( \gamma \) to \( \mathbb{S}^1 \times T_z \mathcal{L} \times (-\epsilon, \epsilon) \). The quotient Poisson structure agree with those of \( \omega \) on the closed manifold \( O_\epsilon \). Whence, by Proposition 22 we have that there is a \( b \)-symplectomorphism. As both \( b \)-symplectic forms are invariant under the group action, the \( b \)-symplectomorphism can be made to be equivariant. \( \square \)

**Example 24.** We can find examples from integrable systems having a naturally associated \( S^1 \)-action model with non-trivial isotropy group.

Take \( M = bT^* (S^1) \times \mathbb{R}^2 \) endowed with coordinates \((p, x, y)\) and \( b \)-symplectic form \( \omega = \frac{1}{p} dp \wedge d\theta + dx \wedge dy \). Consider the \( b \)-integrable system on \( M \) given by \( F = (\log(p), xy) \). This \( b \)-integrable system has hyperbolic singularities. Now let \( \mathbb{Z}/2\mathbb{Z} \) act on \( M \) in the following way: \((1)* (p, x, y) = (p, \theta, -x', -y') \) observe that this action leaves the hyperbola \( xy = cte \) invariant and switches its branches. The action clearly preserves the \( b \)-integrable system and induces a new integrable system on the quotient space \( M/\sim \). Observe that the first component of the integrable system naturally induces an \( S^1 \)-action given by the \( b \)-Hamiltonian vector field associated to \( \log(p) \). This circle action also descends to the quotient and the model for the circle action has non-trivial isotropy group of order two.

This twisted hyperbolic case in \( b \)-symplectic manifolds is a reminiscent of the twisted hyperbolic construction in the symplectic case in [CM] and [MZ], and it is an invitation to study the invariants of a

\footnote{This example shows up in physical examples and corresponds to the 1:2 resonance (see for instance the example in page 32 of the monograph [E]).}
non-degenerate singularity of a $b$-symplectic manifold. This example can be extended to higher dimensions and the action of a $\mathbb{Z}/2\mathbb{Z}$ can be considered for every hyperbolic block added as long as the corank of the singularity is equal or bigger than one. The situation can be visualized using the curled torus, the picture below showing the structure of the set $p = 0, xy = 0$.

4. Actions by Compact Lie Groups on Cosymplectic Manifolds

We treat the case of more general group actions on a $b$-symplectic manifold close to the critical set. First we prove that only groups of a particular form can act on a $b$-symplectic manifold. For now we will treat group actions on symplectic mapping tori and then extend the results to a neighbourhood of the critical set.

**Proposition 25.** Let $G$ be a group acting on a symplectic mapping torus $M_\phi$. Suppose an element $g \in G$ fixes a leaf of the symplectic foliation $\rho_g(L_0) = L_0$. Then $g$ fixes every leaf of the foliation.

**Proof.** The action of the group $G$ on a symplectic mapping torus $M_\phi$ induces an action of $G$ on the base $S^1$ in the obvious fashion

$$\rho : G \times S^1 \to S^1$$

$$(g, \pi(x)) \mapsto \pi(g(x))$$

And so we have the mapping

$$\Phi : G \times \{x\} \to S^1$$

$$(g, x) \mapsto \rho(g, x)$$

As $G$ is compact and connected its image $\rho(G)$ is a compact subgroup of $\text{Homeo}^+(S^1)$, the group of orientation preserving homeomorphisms of the circle. Whence $\rho(G)$ is topologically conjugate to a compact subgroup of $\text{SO}(2)$. Let $h \in G$ and fix a leaf $\mathcal{L}$. This corresponds to a fixed point $p$ of $h$ in the induced action of $G$ on $S^1$, and so a fixed point for $g \rho_h g^{-1} \in \text{SO}(2)$. Whence $g \rho_h g^{-1} = \text{Id}$ and so $\rho_h = \text{Id}$. This corresponds to $h$ fixing all leaves of $M_\phi$. □

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2This is the second invitation to the theory as the first one can be read in [KM].
We shall call $H = \{ g \in G | \rho_g(L) = L \}$ the leaf preserving subgroup of $G$.

**Proposition 26.** Let $G$ be a group acting on a symplectic mapping torus. Denote by $H$ the subgroup of $G$ $H = \{ g \in G | \rho_g(L) = L \}$, $L$ a leaf of the foliation. Then $H$ is a normal subgroup of $G$.

**Proof.** Let $L$ be a leaf of the symplectic foliation. As the action of $G$ maps symplectic leaves to symplectic leaves, the image of $L$ under the diffeomorphism $\rho_g$ associated to a group element $g$ is a symplectic leaf $L$. It is clear that the image of $L$ under the diffeomorphism associated to $g^{-1}$ is $L$. Whence for any $h \in H$, $g \in G$ we have $\rho_g \circ \rho_h \circ \rho_{g^{-1}}$ maps $L$ to $L$ and so $ghg^{-1} \in H$ as required. □

**Proposition 27.** Let $G$ be a group acting on a symplectic mapping torus. Let $H$ be the subgroup of $G$ which is leaf preserving. Then $H$ is a closed Lie subgroup.

**Proof.** The induced action of $G$ on the base of the mapping torus, $S^1$

$$\rho : G \times S^1 \to S^1$$

$$(g, \pi(x)) \mapsto \pi(g(x))$$

gives rise to the mapping

$$\Phi : G \times \{ x \} \to S^1$$

$$(g, x) \mapsto \rho(g, x).$$

It is clear that the level sets $\Phi^{-1}(\theta)$ for $\theta \in S^1$ are exactly the elements of $G$ which are leaf preserving. Whence $\Phi^{-1}(\theta) = H$ is a closed subgroup of $G$. □

**Proposition 28.** The codimension of $H$ in $G$ is at most one.

**Proof.** Suppose the codimension of $H$ in $G$ was greater than one. Let $\xi, \zeta$ be linearly independent elements of $g/\mathfrak{h}$. By definition $\exp(t\xi), \exp(t\zeta)$ do not fix a chosen leaf $L_0$. Choose $s : \mathbb{R} \to \mathbb{R}$ such that $\exp(t\xi)$ and $\exp(s(t)\zeta)$ lie in the same leaf for some $t < \epsilon$ where $\exp$ is a diffeomorphism. Then $\exp(t\xi)\exp(s(t)\zeta)^{-1}$ is in $L_0$ and so $\exp(t\xi)\exp(s(t)\zeta)^{-1} = \exp(t\nu)$ for some $\nu \in \mathfrak{h}$, a contradiction. □

From now on we assume the group action is transverse to the symplectic foliation, i.e. that the codimension of $H$ is exactly one.

**Proposition 29.** Let $G$ be a compact Lie group acting on a compact $b$-symplectic manifold. Then $G$ is of the form $H \times S^1 \mod \Gamma$, where $\Gamma$ is a finite group.

**Proof.** As $G$ is a compact connected Lie group, $G$ is of the form $K_1 \times \cdots \times K_n \times T^m/\mathbb{Z}$ where each $K_i$ is connected, compact, and simply connected, $T^m = (S^1)^m$ is a compact torus, and $Z$ is a finite subgroup of the center of $G$ satisfying $Z \cap T^m = 1$. Let $\phi : K_1 \times \cdots \times K_n \times T^m \to G$ be the projection to the quotient group. As $H$ is a compact subgroup of $G$ by Proposition 27, $\phi^{-1}(H)$ is a compact normal subgroup $\tilde{H}$ of $\tilde{G} := K_1 \times \cdots \times K_n \times T^m$ of codimension 1 and whence of the form $K_1 \times \cdots \times K_n \times T^{m-1}$. Let $\mathfrak{h}$ be the ideal associated to $\tilde{H}$ in $\text{Lie}(G)$ and choose a complementary ideal $\mathfrak{t}$ such that $\mathfrak{t}$ is of rational slope. Then $\exp(\mathfrak{t})$ is a dimension 1 compact subgroup of $\tilde{G}$ and so isomorphic to $S^1$. Whence $G$ is of the required form. □

**Proposition 30.** The action of $G = H \times S^1 \mod \Gamma$ on the critical set $Z$ lifts to an action of $\tilde{G} = \tilde{H} \times S^1$ on the finite cover of $Z$.

**Proof.** The finite cover of $Z$ is given as the finite cover of Proposition 14 with the $S^1$ action given by the action $\exp(t\mathfrak{h}) \cong S^1$. The action of $\tilde{G}$ on $\tilde{Z}$ is given as the product action of $\tilde{H}$ on a leaf, described by the flow of $\exp(t\mathfrak{h})$ and the action of $S^1$. The action descends to an action of $G = H \times S^1 \mod \Gamma$ on $Z$ where $\Gamma = \exp(\mathfrak{h}) \cap \exp(\mathfrak{t}) \subset G$. □
5. A $b$-symplectic slice theorem

First we will construct the $b$-symplectic model which will give us a normal form for the $b$-symplectic form about a group orbit of $G$. As there is a transverse $S^1$ action on $Z$, we have the existence of a trivial finite cover $\mathcal{L} \times S^1$ of $Z$ equipped with a $\tilde{G} \cong \tilde{H} \times S^1$ action for which the projection is equivariant. Let $z \in \mathcal{L}_0$ be a point in a symplectic leaf of $Z$ and consider the orbit $O^\tilde{H}_z$ of $z$ given by the group action of $\tilde{H}$ on $\mathcal{L}_0$ and the symplectic form induced on $\mathcal{L}_0$. By the symplectic slice theorem, there is a $\tilde{H}$-equivariant neighbourhood $U^\tilde{H}_z$ of $O^\tilde{H}_z$ which is equivariantly symplectomorphic to $T^*\tilde{H} \times_{\tilde{H}_z} V_z$ with the symplectic form on $T^*\tilde{H} \times_{\tilde{H}_z} V_z$ given by Theorem 13. We denote the symplectic form on $T^*\tilde{H} \times_{\tilde{H}_z} V_z$ by $\omega^\tilde{H}$. Consider the $b$-symplectic form on $T^*S^1 \times T^*\tilde{H} \times_{\tilde{H}_z} V_z$ given by

$$\omega = \frac{dt}{t} \land d\theta + \pi^*(\omega^\tilde{H})$$

where $t$ is a defining function for $Z$, $\pi$ is the projection $\pi : T^*S^1 \times T^*\tilde{H} \times_{\tilde{H}_z} V_z \to T^*\tilde{H} \times_{\tilde{H}_z} V_z$ and $\omega^\tilde{H}$ is the symplectic form on $T^*\tilde{H} \times_{\tilde{H}_z} V_z$ given by the symplectic slice theorem. Consider the quotient $b$-Poisson structure on $T^*(S^1 \times \tilde{H}) \times (\tilde{H}_z \times \mathbb{Z}_d)$ given by $\mathbb{Z}_d$ acting on $S^1$ and by linear symplectomorphisms on $V_z$ and $\tilde{H}_z$ acts on $T^*\tilde{H}$ by the cotangent lift of $\tilde{H}$ acting on $\tilde{H}$ by translations and by linear symplectomorphisms on $V_z$. We call this the $b$-symplectic model on the associated vector bundle $T^*(S^1 \times \tilde{H}) \times (\tilde{H}_z \times \mathbb{Z}_d)$ $V_z$.

**Theorem 31.** Let $\tilde{H} \times S^1$ be a compact group acting on a $b$-symplectic manifold $(M, \omega)$ transverse to the symplectic foliation. Let $O_z$ be an orbit of the group action contained in the critical set of $M$. Then there is a neighbourhood $U$ of $O_z \cong S^1 \times \tilde{H} / \tilde{H}_z$ which is $b$-symplectomorphic to a neighbourhood of the zero section of an associated bundle $T^*(S^1 \times \tilde{H}) \times (\tilde{H}_z \times \mathbb{Z}_d)$ $V_z$ equipped with the $b$-symplectic model

$$\omega = \frac{dt}{t} \land d\theta + \pi^*(\omega^\tilde{H}).$$

**Proof.** As there is an $S^1$ action on $Z$, $Z$ has a finite cover $S^1 \times \mathcal{L}_0$ on which the cover of the group $G$ acts. For $z \in \mathcal{L}_0$ consider the orbit in $S^1 \times \mathcal{L}_0$ given by $\{\theta\} \times O_z$, $\theta \in S^1$. Consider the $b$-symplectic structure $\omega$ on $S^1 \times \mathcal{L}$ given by Proposition 16. Consider the $b$-symplectic structure

$$\omega_G = \frac{dt}{t} \land d\theta + \phi^*(\omega^\tilde{H})$$

on $S^1 \times \mathcal{L} \times (-\epsilon, \epsilon)$ for $\phi$ the equivariant symplectomorphism $\phi : T^*\tilde{H} \times_{\tilde{H}_z} V_z$ in a neighbourhood of $O^\tilde{H}_z$ given by the symplectic slice theorem. Then, the defining one and two forms of $\omega_G$ agree with $\omega$ on the closed manifold $S^1 \times \tilde{H} / \tilde{H}_z$ and so there is an equivariant $b$-symplectomorphism $\psi$ in a $G$-equivariant neighbourhood of $(\theta, O_z)$ with $\psi^*(\omega_G) = \omega$. Consider the action of $\mathbb{Z}_d$ on $M$

$$\mathbb{Z}_d \times T^*S^1 \times \mathcal{L} \to T^*S^1 \times \mathcal{L}$$

$$(n, \theta, t, x) \mapsto (\theta - \frac{2\pi}{n}, t, \rho_n(x))$$

as defined in Proposition 16. Let $z$ be a fixed point of the action for some $\rho_n$. As the action of $S^1$ commutes with the action of $\tilde{H}$ on $Z$ we have that $\rho_n$ acts as the identity on $O^\tilde{H}_z$. Moreover, as the action is a $b$-symplectomorphism, the form $\omega$ is invariant with respect to the action. As in Proposition 16, the quotient Poisson structure agrees with the Poisson structure on $Z \times (-\epsilon, \epsilon)$ on the image of a $G$ equivariant neighborhood $U$ of the image of $z$. \hfill \Box

**Remark 32.** Using a result of Ortega and Ratiu (see theorem 7.4.1 in [OR]), which generalizes the symplectic slice theorem to the case of symplectic (rather than simply Hamiltonian) actions, the above normal
form can be seen easily to apply likewise in the case where the action on the symplectic leaves is merely symplectic.

6. $b$-Symplectic Forms with Singularity in the Fiber

We wish to extend the idea of an integrable system associated to cotangent lifted actions. Accordingly, we introduce the concept of a left-invariant $b$-form. First we recall some general facts about Lie groups and set some notation.

Let $G$ be an abelian Lie group. Let $\xi_1 \in \mathfrak{g}$ be any element of the Lie algebra, $\xi_2, \ldots, \xi_n$ chosen so that $\xi_1, \ldots, \xi_n$ form a basis of the Lie algebra. Denote by $\eta_i$ the basis of the Lie algebra dual such that $(\eta_i, \xi_j) = \delta_{ij}$. Denote the associated invariant vector fields $L_{g*}\xi_i$ by $v_i$ and $L_g^*\eta_i$ by $m_i$ respectively. At each point $g \in G$ these give a basis for the tangent and cotangent spaces at $g$.

Consider the singular 2-form on $T^*G$

$$\tilde{\omega} = m_1 \wedge \frac{d(\lambda(v_1))}{\lambda(v_1)} + \sum_{i=2}^{n} m_i \wedge d(\lambda(v_i)),$$

where here we have carelessly denoted by $m_i$ the forms $\pi^*(m_i)$, $\pi$ the canonical projection $T^*G \to G$, and $\lambda$ is the canonical Liouville one form. Elements of $T(T^*G)$ at $(g, p) \in T^*G$ can be written $(v, w)$, with $v \in TG, w \in T(T_g^*G)$. The forms above, then, act on elements on $T^*G$ by $\pi^*(m_i)(v, w) = m_i$.

**Proposition 33.** $\tilde{\omega}$ is a $b$-symplectic form on the cotangent bundle of the abelian Lie group $G$ with critical set $Z = \{(g, p) \in T^*G | \lambda(g, p)(v_1) = 0\}$.

**Proof.** It is clear that $\tilde{\omega}$ is a $b$-form. Expanding in the $m_i$ basis, the coordinates of the Liouville one form are given by $\lambda = \sum_{i=1}^{n} \lambda(v_i)m_i$ and so the canonical symplectic form by $\omega = \sum d(\lambda(v_i)) \wedge m_i$. The $n$-th power of $\tilde{\omega}^n = \frac{1}{\lambda(v_1)} \omega^n$ and so is non-degenerate as a $b$-form. As $G$ is abelian $dm_i(u, v) = \eta_i([L_{g^{-1}*}u, L_{g^{-1}*}v]) = 0$ and so the form is closed as a $b$-form. $\square$

**Remark 34.** Note that $dm_i(u, v) = \eta_i([L_{g^{-1}*}u, L_{g^{-1}*}v])$ and so this is the expression for the canonical symplectic form only for $G$ abelian.

As vector fields on the cotangent bundle generated by the cotangent lifted action of a group on itself are horizontal they are automatically elements of the $b$-tangent bundle $bT(T^*G)$ with critical set $\alpha \in T^*G, \alpha(v_1) = 0$ and we can contract these fields with the $b$-symplectic form above. In fact we have the following

**Proposition 35.** The cotangent lifted action on $T^*G$ with the above symplectic form is $b$-Hamiltonian with moment map given by

$$\mu(g, p) = (\log |\lambda(v_1)|, \lambda(v_2), \ldots, \lambda(v_n))$$

**Proof.** The cotangent lifted action associates to each element of the Lie algebra $\xi$ the unique invariant vector field $w$ with value $\xi$ at $g$. On the restriction to the open leaves $\{z \in T^*M | \lambda_p(v_1) \neq 0\}$, the symplectic form is exact, with primitive $\tilde{\lambda} = \log |\lambda(v_1)|m_1 + \cdots + \lambda(v_n)m_n$. Note that $\tilde{\omega}$ is invariant under the cotangent lifted action of $G$ on $T^*G$ and so $L_w(\tilde{\omega}) = 0$. Whence we have $\iota_w\tilde{\omega} = -d(\tilde{\lambda}(w)) = d(\log |\lambda(v_1)|m_1(w) + \sum \lambda(v_i)m_i(w)) = d(\log |\lambda(v_1)|\eta_1(\xi) + \sum \eta_i(\xi)) = -d(\mu, \xi)$ where $\xi$ is the element of $\mathfrak{g}$ defining $v$. The differential of this $b$-Hamiltonian is $\frac{d(\lambda(v_1))}{\lambda(v_1)}m_1(v) + \sum \lambda(v_i)m_i(v)$ which is a well defined $b$-form on the critical set $\lambda(v_1) = 0$ with associated vector field $w$. Whence the action is $b$-Hamiltonian in the sense of [GMPS14b] (Definition 7). $\square$

\footnote{i.e. they do not have components tangent to the fibers of the cotangent bundle}
**Remark 36.** In [KM](Example 3.3) it was noted that the momentum map of the cotangent lifted action of $\mathbb{T}^n$ on itself is given by the contraction of the one-form $\log |p| d\theta + \sum_i p_i d\theta_i$, called the twisted Liouville one form, with the Hamiltonian vector field. The cotangent lifted action of the torus of the base, Hamiltonian with respect to the differential of this form, was called the twisted cotangent lifted action.

The construction of invariant $b$-symplectic forms on the cotangent bundle of an abelian Lie group generalizes easily to the case of a product $G \times H$ where $G$ is an abelian Lie group. As the cotangent bundle splits as the direct sum of bundles $T^*(G \times H) = \pi_G^*(T^*G) \times \pi_H^*(T^*H)$, $\pi_i$ the projection onto the relevant factor, every form can be written as the pull back of forms on $G$ and $T^*H$. We can get a $b$-symplectic form on $T^*(G \times H)$ by adding the pull back of a $b$-form on $G$ of the type above and the pull back of the canonical symplectic form on $T^*H$.

Equip $T^*(S^1 \times G) \equiv T^*(S^1) \times T^*G$ with coordinates $(\theta, a, x, y), (x, y)$ canonical coordinates on $T^*G$ and denote by $p$ the projection 

$$p : T^*(S^1 \times G) \to T^*G$$

**Definition 37.** We define the $b$-lifted symplectic form on $T^*(S^1 \times G)$ with modular period $c$ as the form 

$$\omega = c \frac{da}{\alpha} \wedge d\theta + p^* \bar{\omega}$$

where $\bar{\omega}$ is the canonical $b$-symplectic form on $T^*G$.

**Proposition 38.** The above $b$-lifted symplectic form has as defining one and two forms $cd\theta$ and $p^* \bar{\omega}$ respectively.

**Proof.** Note that the above form has as modular vector field $\frac{1}{c} \frac{\partial}{\partial \theta}$. As the defining one form of the foliation is given uniquely by $\alpha(v_{mod}) = 1$, we have $\alpha = cd\theta$. As $p \circ \iota_L = \text{Id}$, we have that $p^* \bar{\omega}$ is exactly the defining two form. \qed

We now note that such a construction cannot be generalized further:

**Proposition 39.** There is no invariant $b$-symplectic structure on the cotangent bundle of a semi-simple Lie group.

**Proof.** A $b$-symplectic structure on $T^*G$ gives a critical hypersurface with a codimension one foliation by symplectic manifolds. Moreover, we can choose the defining one-form of such a foliation to be closed [GMPST14b]. If the action of the group on the cotangent bundle is Poisson, such a one form is necessarily invariant by the action of the group. However for an invariant one form $\alpha$ we have $d\alpha(u, v) = \alpha([L_{g^{-1}}u, L_{g^{-1}}v])$. If $\alpha$ is closed, then $\alpha([\eta, \nu]) = 0$ for all $\eta, \nu \in \mathfrak{g}$ and the Lie algebra is necessarily a semidirect product of $\ker(\alpha)$ and the linear subspace given by $\mu \in \mathfrak{g}, \alpha(\mu) \neq 0$. \qed

7. $b$-Lie groups and the canonical $b$-symplectic form on $bT^*G$

In the previous section we constructed $b$-symplectic forms on $T^*G$ which have singularities in the fiber. We now assume that $G$ itself is a $b$-manifold and consider a different construction of a $b$-symplectic form – this time on the $b$-cotangent bundle $bT^*G$ instead of $T^*G$ – which is the natural analogue to the construction of the canonical symplectic form on $T^*G$. This so-called canonical $b$-symplectic form on $bT^*G$ is defined in Definition [10].

In the symplectic case, reducing $T^*G$ by the action of $G$ yields the Lie-Poisson structure on $\mathfrak{g}^*$. In order to lift an action to $bT^*G$ (Definition [11]), we have to demand that the action leaves the critical hypersurface invariant. This motivates us to consider the setting where the critical hypersurface is a Lie subgroup $H$ and we consider the action of $H$ on $G$ by translations.

First we give the relevant definitions and preliminary results:
Definition 40. A b-manifold \((G, H)\), where \(G\) is a Lie group and \(H \subset G\) is a closed subgroup\(^4\)
 is called a b-Lie group.

Example 41. The special Euclidean group of orientation-preserving isometries in the plane is the semidirect product
\[
SE(2) \cong SO(2) \rtimes T(2)
\]
where \(T(2)\) are translations in the plane. Recall that we can identify \(SE(2)\) with matrices of the form
\[
\begin{pmatrix}
A & b \\
0 & 1
\end{pmatrix}, \quad A \in SO(2), b \in \mathbb{R}^2
\]
Then \(T(2)\) (identified with \(\{I\} \times T(2) \subset SE(2)\)) is a closed codimension 1 subgroup and the pair \((SE(2), T(2))\) is a b-Lie group.

Example 42. The Galilean group \(G\) is the group of transformations in space-time \(\mathbb{R}^{3+1}\) (the first three dimensions are interpreted as spatial dimensions and the last one is time) whose elements are given by composition of a spatial rotation \(A \in SO(3)\), uniform motion with velocity \(v \in \mathbb{R}^3\) and translations in space and time by a vector \((a, s) \in \mathbb{R}^{3+1}\). As a matrix group, the elements are given by
\[
\begin{pmatrix}
A & v & a \\
0 & 1 & s \\
0 & 0 & 1
\end{pmatrix}, \quad A \in SO(3), v, a \in \mathbb{R}^3, s \in \mathbb{R}
\]
The subgroup \(H\) given by \(s = 0\) (which corresponds to fixing time) is a closed codimension one subgroup and hence the pair \((G, H)\) is a b-Lie group.

Example 43. We consider the \((2n + 1)\)-dimensional Heisenberg group \(H_{2n+1}(\mathbb{R})\) given by matrices of the form
\[
\begin{pmatrix}
1 & a & c \\
0 & I_n & b \\
0 & 0 & 1
\end{pmatrix}, \quad a \in \mathbb{R}^{1 \times n}, b \in \mathbb{R}^{n \times 1}, c \in \mathbb{R}
\]
The subgroup \(\Gamma\) of matrices of the form
\[
\begin{pmatrix}
1 & 0 & k \\
0 & I_n & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad k \in \mathbb{Z}
\]
is central, hence normal, and so we can consider \(G := H_{2n+1}(\mathbb{R})/\Gamma.\) This is a well-known example of a non-matrix Lie group. Now fixing one component \(a_i = 0\) or \(b_i = 0\) yields a closed codimension one subgroup of \(G.\)

Let us consider the action of \(H\) on \(G\) by left translations. This action is obviously free and since \(H\) is closed, it is also proper. Therefore, the left coset space \(G/H\) can be given the structure of a smooth manifold such that the projection \(\pi : G \rightarrow G/H\) is a smooth submersion. Moreover, it is well-known that \(\pi\) turns \(G\) into a principal \(H\)-bundle.

For future reference we summarize these facts in the following lemma:

Lemma 44. Let \((G, H)\) be a b-Lie group. The projection \(\pi : G \rightarrow G/H\) is a principal \(H\)-bundle; in particular \(G\) is semilocally around \(H\) a product
\[
\pi^{-1}(V) \cong V \times H, \quad [e]_\sim \in V \subset G/H,
\]
where \(\pi\) corresponds to the projection onto the first component.

Note that by taking a coordinate \(\varphi\) on \(V\) centered at \([e]_\sim\), we obtain a global defining function \(\varphi \circ \pi\) for the critical hypersurface \(H.\)

\(^4\)This is equivalent to \(H\) being an embedded Lie subgroup.
7.1. The $H$-action on $bTG$ and $b^*TG$. As in the previous section, let $(G, H)$ be a $b$-Lie group and consider the action of $H$ by left translations.

We can lift this action to $TG$ in the obvious way:

$$H \times TG \rightarrow TG : (h, v_g) \mapsto (L_h)_* v_g.$$ 

This action is again proper and free; therefore the quotient space is a manifold, which we want to describe below.

Let us introduce the subbundle $\mathcal{H}$ of $TG$ whose fibre $\mathcal{H}_g$ at $g \in G$ is given by the corresponding left-shift of the Lie algebra $\mathfrak{h}$ of $H$, $\mathcal{H}_g = (L_g)_* \mathfrak{h}$. Let $\pi_\mathcal{H} : TG \rightarrow \mathcal{H}$ be the projection onto $\mathcal{H}$. Recall that $\pi : G \rightarrow G/H$ induces a surjective bundle morphism $\pi_* : TG \rightarrow T(G/H)$ and at each fibre $T_g G$ the kernel is $\mathcal{H}_g$.

**Proposition 45.** There is a diffeomorphism

$$(TG)/H \sim \mathfrak{h} \times T(G/H)$$

$$[v_g]_\sim \mapsto ((L_{g^{-1}})_* (\pi_\mathcal{H}(v_g)), \pi_*(v_g)).$$

**Proof.** The map is well-defined as it does not depend on the representative of $[v_g]_\sim = \{(L_h)_* (v_g) : h \in H\}$. It is obviously smooth and surjective. If $[v_g]_\sim$ and $[v'_g]_\sim$ have the same image, then $\pi_*(v_g) = \pi_*(v'_g)$ implies $\pi(g) = \pi(g')$ so by choosing a different representative in $[v'_g]_\sim$ we can assume $g = g'$. Then $v_g - v'_g \in \ker(\pi_*) = \mathcal{H}_g$ and combining this with $\pi_\mathcal{H}(v_g) = \pi_\mathcal{H}(v'_g)$ we see $v_g = v'_g$. \qed

The analogous result holds for the action of $H$ on the $b$-tangent bundle,

$$H \times bTG \rightarrow bTG : (h, v_g) \mapsto (L_h)_* v_g.$$ 

Note that this action is well-defined since the left translation by $h \in \mathcal{H}$ preserves $H$ i.e. it is a $b$-map. Moreover we define the projection $\pi_\mathcal{H} : bTG \rightarrow \mathcal{H}$ in the obvious way.

**Proposition 46.** There is a diffeomorphism

$$(bTG)/H \sim \mathfrak{h} \times bT(G/H)$$

$$[v_g]_\sim \mapsto ((L_{g^{-1}})_* (\pi_\mathcal{H}(v_g)), \pi_*(v_g))$$

where $bT(G/H)$ is the $b$-tangent bundle of the one-dimensional $b$-manifold $G/H$ with critical hypersurface $[e]_\sim$. Note that $\pi : (G, H) \rightarrow (G/H, [e]_\sim)$ is a $b$-map and therefore $\pi_* : bTG \rightarrow bT(G/H)$ is well-defined.

The right hand sides of the diffeomorphisms in Proposition 45 and 46 are vector bundles over $G/H$. This makes $TG/H$ resp. $bTG/H$ vector bundles over $G/H$ as well with bundle map $[v_g]_\sim \mapsto \pi(g) \in G/H$.

**Corollary 47.** $(TG)/H$ (resp. $(bTG)/H$) is a vector bundle of rank $n$ over $G/H$ isomorphic to the direct sum of the trivial vector bundle $\mathfrak{h} \times G/H$ with $T(G/H)$ (resp. $bT(G/H)$):

$$(TG)/H \cong (\mathfrak{h} \times G/H) \oplus T(G/H), \quad (bTG)/H \cong (\mathfrak{h} \times G/H) \oplus bT(G/H).$$

7.2. The $b$-cotangent lift. In Definition 11 we introduced the $b$-cotangent lift; in the present setting this is given by the following action on the $b$-cotangent bundle $b^*TG$:

$$H \times b^*TG \rightarrow b^*TG : (h, \alpha_g) \mapsto (L_{h^{-1}})^* \alpha_g.$$ 

The quotient space $(b^*TG)/H$ can be viewed as a vector bundle over $G/H$ which is isomorphic to $(b(TG)/H)^*$ via the identification

$$(b^*TG)/H \sim (b(TG)/H)^* : [\alpha_g]_\sim \mapsto ([v_g]_\sim \mapsto (\alpha_g, v_g)), \quad v_g \in bT_g G.$$
Therefore we can dualize the result for $^bTG/H$ of the previous section to obtain an isomorphism of vector bundles

$$(^bT^*G)/H \cong (\mathfrak{h}^* \times G/H) \oplus ^bT^*(G/H).$$

As smooth manifolds,

$$(^bT^*G)/H \cong \mathfrak{h}^* \times ^bT^*(G/H),$$

where the isomorphism is given by identifying an element of the right hand side $(\alpha, [\beta]_\sim) \in \mathfrak{h}^* \times \mathcal{T}^*_g(G/H)$ with the class of $\omega^*_g(\alpha) + \pi^*_\beta\mathcal{g}_\sim \in ^bT^*_gG$ on the left hand side.

7.3. Reduction of the canonical $b$-symplectic structure. The cotangent bundle $T^*G$ has a canonical symplectic structure, which under the action of $G$ on itself by left translations reduces to the minus Lie-Poisson structure on $T^*G/G \cong \mathfrak{g}^*$.

In Definition 10 we have seen how to endow the $b$-cotangent bundle $^bT^*G$ with a canonical $b$-symplectic structure (with critical hypersurface $^bT^*G|_H$). What is the reduced Poisson structure on $(^bT^*G)/H$?

**Theorem 48.** Let $^bT^*G$ be endowed with the canonical $b$-Poisson structure. Then the Poisson reduction under the cotangent lifted action of $H$ by left translations is

$$(^bT^*G)/H, \Pi_{\text{red}} \cong (\mathfrak{h}^* \times ^bT^*(G/H), \Pi_{\text{L-P}} + \Pi_{\text{b-can}})$$

where $\Pi_{\text{L-P}}$ is the minus Lie-Poisson structure on $\mathfrak{h}^*$ and $\Pi_{\text{b-can}}$ is the canonical $b$-symplectic structure on $^bT^*(G/H)$, where $G/H$ is viewed as a $b$-manifold with critical hypersurface the point $[e]_\sim$.

**Proof.** Let $V \subset G/H$ be open and such that $G$ trivializes as a principal $H$-bundle over $V$ (cf. Lemma 44), i.e.

$$G \supset U := \pi^{-1}(V) \xrightarrow{\sim} H \times V$$

where the projection onto the second component corresponds to the quotient projection $\pi$; in particular the critical hypersurface $H$ gets mapped to $H \times [e]_\sim \subset H \times V$ and the $b$-cotangent bundle over $U$ splits in the following way:

$$^bT^*U \cong T^*H \times ^bT^*V.$$ }

Then the canonical $b$-symplectic structure $\omega_0$ on $^bT^*U$ is the product of the canonical symplectic structure $\omega_1$ on $T^*H$ and the canonical $b$-symplectic structure $\omega_2$ on $^bT^*V$. Denoting the Poisson tensor corresponding to $\omega_i$ by $\Pi_i$,

$$\Pi_0 = \Pi_1 + \Pi_2.$$ 

The action of $H$ on $^bT^*U \cong T^*H \times ^bT^*V$ is given by the standard cotangent lift of left translations by $H$ on $T^*H$ times the identity on $^bT^*V$. For the corresponding quotient projections $\pi_0 : ^bT^*U \to (^bT^*U)/H$ and $\pi_1 : T^*H \to (T^*H)/H$ we therefore have $\pi_0 = \pi_0' \times \text{id}_{^bT^*V}$. Hence the reduced Poisson structure on $(^bT^*U)/H$ is

$$\Pi_{\text{red}} = (\pi_0)'_*\Pi_0 = (\pi_0)'_*(\Pi_1 + \Pi_2) = (\pi_0'_*)\Pi_1 + \Pi_2.$$ 

Now note that $(\pi_0'_*)\Pi_1$ is the minus Lie Poisson structure on $\mathfrak{h}^*$ if we identify $(T^*H)/H \cong \mathfrak{h}^*$.

**Example 49.** We return to Example 42 of the special Euclidean group $SE(2)$. Since $T(2)$ is abelian, the Lie-Poisson structure on the dual of its Lie algebra is zero. Hence $^bT^*(SE(2))$ reduces under the action of $T(2)$ to

$$(^bT^*(SE(2))/T(2), \Pi_{\text{red}}) \cong (\mathbb{R}^2 \times ^bT^*(SO(2)), 0 + \Pi_{\text{b-can}}),$$

where $\Pi_{\text{b-can}}$ is the canonical $b$-Poisson structure on $^bT^*(SO(2))$, i.e. identifying $SO(2) \cong \mathbb{S}^1$ in the usual way and letting $\varphi$ be the angle, $(\varphi, p)$ a $b$-canonical chart in a neighborhood of $\{\varphi = 0\}$, then in these coordinates

$$\Pi_{\text{red}} = \varphi \frac{\partial}{\partial \varphi} \wedge \frac{\partial}{\partial p}.$$
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