There are no Cubic Graphs on 26 Vertices with Crossing Number 11

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Abstract We show that no cubic graphs of order 26 have crossing number 11, which proves a conjecture of Ed Pegg Jr and Geoffrey Exoo that the smallest cubic graphs with crossing number 11 have 28 vertices. This result is achieved by first eliminating all girth 3 graphs from consideration, and then using the recently developed QuickCross heuristic to find good embeddings of each remaining graph. In the cases where the embedding found has 11 or more crossings, the heuristic is re-run with a different settings of parameters until an embedding with fewer than 11 crossings is found; this is required in less than 3% of the graphs.

Keywords Crossing number, cubic graphs, conjecture, QuickCross heuristic, Coxeter graph

1 Introduction

In this manuscript we restrict our consideration to simple, undirected, connected graphs. Consider such a graph $G = (V, E)$ where $V$ is the set of vertices, and $E : V \rightarrow V$ is the set of edges. A graph drawing is a mapping of the vertices and edges to a plane. A graph is said to be planar if it is possible to produce a graph drawing for that graph such that edges only intersect at vertices. Henceforth, we will take crossings to describe only those edge intersections that occur away from vertices, and only permit two edges to cross in one place (so if three edges cross simultaneously, it is viewed as three individual crossings). Unquestionably the seminal result in planarity is Kuratowski’s theorem, which states that a graph is planar if and only if it contains no subdivisions of $K_5$ or $K_{3,3}$ [8].
If a graph is non-planar, then one may ask how close to being planar it is. One such measure is the crossing number. For any given drawing of a non-planar graph, there will be a positive number of crossings. Then the crossing number is the minimum number of crossings over all valid drawings of that graph. The crossing number of graph $G$ is denoted $\text{cr}(G)$. For example, the Petersen graph [5] is shown in Figure 1 in two orientations. The first, standard, drawing has five crossing. However, the second drawing has only two, which is known to be the minimum possible.

The problem of finding the crossing number given a graph is called the Crossing Number Problem, and is an NP-hard problem [3], and even determining if the crossing number is less than a given value $k$ is NP-complete. However, the problem is fixed-parameter tractable [7].

A question that may be asked is how small graphs with a requested crossing number may be. If we restrict our consideration to cubic graphs (for which the crossing number problem is still NP-hard [4]), some results are known. In table 1 we indicate the minimum size of cubic graph for each crossing number $k$, and give an example of one such cubic graph of that size.

| $k$ | Min $n$ | Example                  |
|-----|---------|--------------------------|
| 0   | 4       | $K_4$                    |
| 1   | 6       | $K_{3,3}$                |
| 2   | 10      | Petersen graph           |
| 3   | 14      | Heawood graph            |
| 4   | 16      | Möbius-Kantor graph      |
| 5   | 18      | Pappus graph             |
| 6   | 20      | Desargues graph          |
| 7   | 22      | Four (unnamed) graphs    |
| 8   | 24      | McGee graph              |

Table 1 The minimum order of cubic graph for crossing number $k$, and an example of each. For 7 crossings, the four minimal examples are not named graphs, but are displayed in Pegg Jr and Exoo [11].

However, results on minimal cubic graphs with crossing number larger than 8 have, to date, been only conjectured. It is known that the Coxeter graph [2] on 28 vertices has crossing number 11, and the Levi graph [9] on 30 vertices has crossing number 13. However, an open question posed by Pegg Jr and Exoo [11] is whether any cubic graphs of order 26 have crossing number 11. More precisely, they conjectured that (i) the minimal cubic graphs with crossing number 9 and 10 would contain 26 vertices, (ii) the minimal cubic graphs with crossing number 11 would contain 28 vertices, and (iii) the minimal cubic graphs with crossing number 13 would contain 30 vertices. They made no conjecture about whether cubic graphs with crossing number 12 would exist for
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Fig. 2 Possible triangle structures in a cubic graph.

graphs containing 28 vertices. In what follows, we answer question (ii) in the affirmative, and so prove that the Coxeter graph is a minimal example of a cubic graph with crossing number 11.

2 Approach

There are 2,094,480,864 non-isomorphic cubic graphs with 26 vertices [10]. However, with the following simple argument, we can eliminate roughly 80% of these.

**Proposition 1** No cubic graphs on 26 vertices with girth 3 have crossing number 11.

**Proof** Consider any cubic graph \( G \) with girth 3. Then \( G \) contains at least one triangle. Consider the three vertices in one such triangle. There are only three possibilities. Either none of the vertices are involved in a second triangle, two of vertices are, or all three are. In the third case the graph must be \( K_4 \), so we exclude it from consideration. If none of the vertices are involved in a second triangle, it is clear that the triangle could be contracted to a single vertex without altering the crossing number. If two of the vertices are involved in a second triangle, they must form a diamond which can also be contracted to a single vertex without altering the crossing number. In both cases, a smaller cubic graph is obtained with the same crossing number. Noting that cubic graphs with fewer than 26 vertices have crossing number no larger than 8, the result follows immediately. These situations are displayed in Figure 2.

Proposition 1 indicates that we need only to consider cubic graphs on 26 vertices with girth at least 4. There are 432,757,568 such graphs up to isomorphism [10]. For each of these graphs, we used the recently developed QuickCross heuristic [1], a fast algorithm that finds valid embeddings which usually have an optimal or near-optimal number of crossings. For cubic graphs on 26 vertices, QuickCross can process several graphs per second. We partitioned the graphs into sets of 50,000, resulting in roughly 8,000 individual jobs, each of which took up to 3 days to run, and distributed the jobs over 400 cores on a High Performance Computer.

After the initial run with the default settings, embeddings with fewer than 11 crossings were obtained for 422,549,254 of the graphs, leaving approximately ten million graphs to be considered more closely. By altering the parameters and random seeds and re-running the remaining graphs, this number was reduced to under 500,000 graphs. This process was continued until there were less than a hundred graphs remaining, at which point we set up the code to try a hundred different random seeds for each graph and take the best results. This was successful for all graphs, and so we were able to obtain an embedding with fewer than 11 crossings for every graph.
Finally, we performed an additional test to ensure there was no errors arising from potential bugs with the QuickCross heuristic itself. For each graph, we took the embedding found by QuickCross and used it to construct the corresponding planarised graph. We then used the planarity algorithm of Hopcroft and Tarjan to confirm that the resulting graph was indeed planar. In this way, we verified that the embeddings found for each of the graphs were valid. A zip file containing a list of edge crossings, corresponding to a valid embedding with fewer than 11 crossings, for each of the graphs is available upon request.

Note that we could also have eliminated any 1-connected graphs from consideration; in the case of cubic graphs, these are all bridge graphs. It can be easily seen that for these graphs, the bridge (or any bridge, if there is more than one) can be removed, and the crossing number of the two remaining components can be computed and summed to give the crossing number of the original graph. Each of the two remaining components has a degree 2 vertex which can be safely removed (in the sense that its two incident edges are merged into a single edge), and what remains are two cubic graphs containing $a$ and $24 - a$ vertices respectively, for some even $4 \leq a \leq 20$. Hence, from Table 1, it can be easily checked that 1-connected cubic graphs of order 26 have crossing number no greater than 6. However, as we also wanted to stress our heuristic on various special cases to test its robustness, we chose not to exclude 1-connected graphs from the experiment.

References

1. Clancy, K., Haythorpe, M. and Newcombe, A. “An effective crossing minimisation heuristic based on star insertion”, Discrete Applied Mathematics, submitted 2018. Available at: http://arxiv.org/abs/1804.09900
2. Coxeter, H.S.M. “My graph”, Proceeding of the London Mathematical Society, 3(1):117–136, 1983.
3. Garey, M.R., Johnson, D.S. “Crossing number is NP-complete”, SIAM Journal on Algebraic Discrete Methods, 4(3):312–316, 1983.
4. Hlineny, P. “Crossing number is hard for cubic graphs”, Journal of Combinatorial Theory, Series B, 96(4):455–471, 2006.
5. Holton, D.A., Sheehan, J. “The Petersen Graph”, Cambridge University Press, 1993.
6. J. Hopcroft and R. Tarjan. “Efficient planarity testing”, Journal of the ACM, 21(4):549–568, 1974.
7. Kawarabayashi, K., Reed, B. “Computing crossing number in linear time” Proceedings of the 29th Annual ACM Symposium on Theory of Computing, pp.382–390, 2007.
8. Kuratowski, C. “Sur le probleme des courbes gauches en topologie”, Fundamenta Mathematicae, 15(1):271–283, 1930.
9. Levi, F.W. “Finite geometrical systems”, University of Calcutta, 1942.
10. Meringer, M. “Fast Generation of Regular Graphs and Constructions of Cages”, Journal of Graph Theory, 30:137–146, 1999.
11. Pegg Jr, E. and Exoo, G. “Crossing Number Graphs”, The Mathematica Journal, 11(2):161–170, 2009.