METRIC HYPERGRAPHS AND METRIC LINES
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1 Introduction

Given a metric space \((V, \text{dist})\), we follow [1] in writing \([uvw]\) to signify that \(u, v, w\) are pairwise distinct points of \(V\) and \(\text{dist}(u,v) + \text{dist}(v,w) = \text{dist}(u,w)\). With \(M\) standing for the metric space, it will be convenient to write
\[
\mathcal{E}_M = \{\{x, y, z\} : [zxy] \text{ or } [xzy] \text{ or } [xyz]\}.
\]
Following [2], we say that a 3-uniform hypergraph \((V, \mathcal{E})\) is metric if there is a metric space \(M\) such that \(\mathcal{E} = \mathcal{E}_M\). All induced sub-hypergraphs of metric hypergraphs are metric, and so metric hypergraphs can be characterized as hypergraphs without certain induced sub-hypergraphs, namely, the minimal non-metric ones. Section 3 of [2] presents three minimal non-metric hypergraphs along with the following comment:

If there are only finitely many minimal non-metric hypergraphs, then metric hypergraphs can be recognized in polynomial time. However, it is conceivable that there are infinitely many minimal non-metric hypergraphs and it is not clear whether metric hypergraphs can be recognized in polynomial time.

Our first main result shows that there are indeed infinitely many minimal non-metric hypergraphs. To formulate it, we need the following definition: When \(G\) is a graph with vertex set \(V\) and edge set \(E\), the hypergraph based on \(G\) is the 3-uniform hypergraph with vertex set \(V \cup \{x\}\), where \(x \not\in V\), and hyperedge set consisting of all three-point subsets of \(V\) and all three-point sets \(\{x, u, v\}\) such that \(\{u, v\} \in E\).

**Theorem 1.** For every even integer \(n\) greater than four, the hypergraph based on the cycle \(C_n\) is minimal non-metric.

In 1943, Erdős [7] proved that a set of \(n\) points in the Euclidean plane determines at least \(n\) distinct lines unless these \(n\) points are collinear. In
2006, Chen and Chvátal \[5\] asked whether the same statement holds true more generally in all metric spaces \( M \) with the line \( L_M(xy) \) determined by two points \( x \) and \( y \) defined by

\[
L_M(xy) = \{x, y\} \cup \{z : \{x, y, z\} \in \mathcal{E}_M\}.
\]

Early progress toward the conjecture that this generalization does hold true is surveyed in \[6\]; contributions too recent to be included there are \[3, 4, 8, 11, 12\].

When \( W \) is a set and \( s \) is a positive integer, we let \( \binom{W}{s} \) denote the set of all \( s \)-point subsets of \( W \). We say that an equivalence relation \( \equiv \) on \( \binom{W}{2} \) is a metric-line equivalence if there is a metric space \( M \) on ground set \( W \) such that

\[
L_M(ab) = L_M(cd) \iff \{a, b\} \equiv \{c, d\}
\]

for every choice of two-point subsets \( \{a, b\} \) and \( \{c, d\} \) of \( W \).

How difficult is it to tell which equivalence relations on \( \binom{W}{2} \) are metric-line equivalences and which of them are not? Attempts at answering the Chen-Chvátal question could be only helped by an efficient algorithm for their recognition. Until now, all we have had here was a polynomial-time algorithm that, given an equivalence relation \( \equiv \) on \( \binom{W}{2} \), will in some cases certify that \( \equiv \) is not a metric-line equivalence \[5\,\text{Algorithms G and H}\]. We are going to offer another such certificate.

We say that an equivalence relation \( \equiv \) on \( \binom{V}{2} \) is an obstacle if no metric space \( M \) on a superset \( W \) of \( V \) satisfies \( (1) \) for every choice of two-point subsets \( \{a, b\} \) and \( \{c, d\} \) of \( V \). It may not be obvious that there exist any obstacles at all; our second main result shows that there are infinitely many genuinely different ones. To formulate this result, we need additional definitions again.

We say that an obstacle \( \equiv \) on \( \binom{V}{2} \) is minimal if there is no proper subset \( U \) of \( V \) such that the restriction of \( \equiv \) on \( \binom{U}{2} \) is an obstacle. Given a graph \( G \) with vertex set \( V \) and edge set \( E \), we define the equivalence relation \( ^G \equiv \) on \( \binom{V}{2} \) by

\[
e^G \equiv f \iff (e \in E, f \in E) \quad \text{or} \quad (e \notin E, f \notin E)
\]

**Theorem 2.** If \( G = C_n \) with \( n \) an even integer greater than four, then the equivalence relation \( ^G \equiv \) is a minimal obstacle.
2 Metric and non-metric hypergraphs

A corollary of the following lemma is one of the ingredients of our proof of Theorem [1].

**Lemma 1.** For every odd integer $n$ greater than one, the hypergraph based on $C_n$ is metric.

*Proof.* Writing $n = 2s + 1$, consider the metric space on the ground set $\{0, 1, \ldots, 2s\} \cup \{x\}$ with $\text{dist}(i, j) = j - i$ whenever $0 \leq i < j < n$ and

$$
\text{dist}(x, k) = \begin{cases} 
  s & \text{if } k \text{ is even}, \\
  s + 1 & \text{if } k \text{ is odd}.
\end{cases}
$$

$\square$

The next lemma comes from [10] and has been generalized as [2, Lemma 1].

**Lemma 2.** Let $M$ be a metric space and let $V$ be a subset of its ground set such that $|V| \geq 5$. If every three-point subset of $V$ belongs to $\mathcal{E}_M$, then the elements of $V$ can be renamed as $0, 1, \ldots n - 1$ with $n = |V|$ in such a way that

$$
0 \leq u < v < w < n \Rightarrow [uvw]
$$

**Lemma 3.** Given a metric space $M$ on a ground set $V \cup \{x\}$, where $x \notin V$ and $V = \{0, 1, \ldots n - 1\}$, set

$$
\begin{align*}
D_1 &= \{(j, \ell) : j < \ell \text{ and } [xj\ell]\}, \\
D_2 &= \{(j, \ell) : j < \ell \text{ and } [jx\ell]\}, \\
D_3 &= \{(j, \ell) : j < \ell \text{ and } [j\ell x]\}.
\end{align*}
$$
If (2) holds true, then

\[(j, \ell) \in D_1, \ j < k < \ell \implies (j, k) \in D_1, \ (k, \ell) \in D_1, \] (3)
\[(j, \ell) \in D_3, \ j < k < \ell \implies (j, k) \in D_3, \ (k, \ell) \in D_3, \] (4)
\[(j, \ell) \in D_2, \ i < j < \ell \implies (i, j) \in D_3, \ (i, \ell) \in D_2, \] (5)
\[(j, \ell) \in D_2, \ j < \ell < m \implies (j, m) \in D_2, \ (\ell, m) \in D_1, \] (6)
\[(i, j) \in D_1, \ (j, k) \in D_1 \implies (i, k) \in D_1, \] (7)
\[(i, j) \in D_3, \ (j, k) \in D_3 \implies (i, k) \in D_3, \] (8)
\[(i, k) \in D_2, \ (j, k) \in D_1 \implies (i, j) \in D_2 \text{ or } (j, i) \in D_2. \] (9)
\[(i, k) \in D_2, \ (i, j) \in D_3 \implies (j, k) \in D_2 \text{ or } (k, j) \in D_2, \] (10)

Proof. Implications (3) — (9) are special cases of the easily verifiable

\[[abd], [bcd] \implies [abc], [acd], \] (11)

which has been pointed out first by Menger [9].

Actually, the conclusion of (9) can be strengthened to \((i, j) \in D_2\) since \(j < i\) would contradict (3) with \((j, i, k)\) in place of \((j, k, \ell)\). Similarly, the conclusion of (10) can be strengthened to \((j, k) \in D_2\) since \(k < j\) would contradict (4) with \((i, k, j)\) in place of \((j, k, \ell)\). These niceties are irrelevant to our purpose.

Lemma 4. Let \(G\) be a graph with vertex set \(V\) and edge set \(E\). Let \(\mathcal{H}\) be the hypergraph based on \(G\). Let \(M\) be a metric space \(M\) on ground set \(V \cup \{x\}\) such that

\[V = \{0, 1, \ldots n - 1\} \text{ and } [0 \leq u < v < w < n] \implies [uvw]\]

and \(x \notin V\) and such that a two-point subset \(\{u, v\}\) of \(V\) belongs to \(E\) if and only if \(\{x, u, v\}\) is in \(E_M\). If \(G\) contains no triangle, then

\[E \subseteq \{\{0, 1\}, \{1, 2\}, \ldots \{n - 2, n - 1\}, \{n - 1, 0\}\}.\]

Proof. Lemma 3 guarantees that (3) — (9) are satisfied. Since a two-point subset \(\{u, v\}\) of \(V\) belongs to \(E\) if and only if \(\{x, u, v\}\) is in \(E_M\), we have

\[E = \{\{u, v\} : (u, v) \in D_1 \cup D_2 \cup D_3\}.\]
If $G$ contains no triangle, then (5) and (6) show that every $(j, \ell)$ in $D_1 \cup D_3$ with $j < \ell$ has $\ell = j + 1$ and (3), (4) show that every $(j, \ell)$ in $D_2$ with $j < \ell$ has $j = 0$, $\ell = n - 1$. Therefore

$$D_1 \cup D_3 \subseteq \{(i, i + 1) : 0 \leq i < n - 1\} \text{ and } D_2 \subseteq \{(0, n - 1)\}.$$ 

Proof of Theorem 1. First, we shall deduce a contradiction from the assumption that the hypergraph $H$ based on a $C_n$ with $n$ even and greater than four is metric. For this purpose, let $V$ denote the vertex set of the $C_n$ and let $E$ denote its edge set. If $H$ is metric, then Lemma 2 guarantees that the hypothesis of Lemma 4 with $G = C_n$ is satisfied, and so

$$E = \{\{0, 1\}, \{1, 2\}, \ldots, \{n - 2, n - 1\}, \{n - 1, 0\}\}.$$ 

By (10) with $i = 0$, $j = 1$, $k = n - 1$, we have

$$(0, 1) \in D_1;$$

by (9) with $i = 0$, $j = n - 2$, $k = n - 1$, we have

$$(n - 2, n - 1) \in D_3;$$

by (7) and (8) with $j = i + 1$, $k = i + 2$, we have

$$(i, i + 1) \in D_1 \iff (i + 1, i + 2) \in D_3 \quad \text{whenever } 0 \leq i < n - 2.$$ 

Therefore the ordered pairs $(0, 1)$, $(1, 2)$, $(2, 3)$, $\ldots$, $(n - 2, n - 1)$ alternate between $D_1$ and $D_3$, beginning with $(0, 1)$ in $D_1$ and ending with $(n - 2, n - 1)$ in $D_3$. This contradicts the assumption that $n$ is even.

It remains to be proved that every proper induced sub-hypergraph $H_0$ of $H$ is metric. For this purpose, note that $H_0$ is an induced sub-hypergraph of the complete 3-uniform hypergraph on $n$ vertices, which is trivially metric, or an induced sub-hypergraph of the hypergraph based on the path of order $n - 1$, which (being a sub-hypergraph of the hypergraph based on any larger cycle) is metric by Lemma 1. \qed
The lower bound on \( n \) in Theorem 1 is essential: the hypergraph based on \( C_4 \) is metric. To see this, note that the metric space on ground set \( \{a, b, c, d, x\} \) with metric defined by the chart

|   | a | b | c | d | x |
|---|---|---|---|---|---|
| a | 0 | 1 | 2 | 1 | 2 |
| b | 1 | 0 | 1 | 2 | 3 |
| c | 2 | 1 | 0 | 1 | 2 |
| d | 1 | 2 | 1 | 0 | 3 |
| x | 2 | 3 | 2 | 3 | 0 |

has

\[ [abc], [bcd], [cda], [dab], [xab], [xad], [xcb], [xcd], \]

but none of \( [xac], [axc], [acx], [xbd], [bxd], [bdx] \).

The following lemma is used in the next section in our proof of Theorem 2:

**Lemma 5.** The hypergraph based on the complement \( \overline{P_5} \) of the path of order five is non-metric.

**Proof.** We shall deduce a contradiction from the assumption that the hypergraph \( \mathcal{H} \) based on \( \overline{P_5} \) is metric. For this purpose, let \( V \) denote the vertex set of our \( \overline{P_5} \) and let \( E \) denote its edge set. If \( \mathcal{H} \) is metric, then there is a metric space \( M \) on ground set \( V \cup \{x\} \), where \( x \not\in V \), such that all three-point subsets of \( V \) belong to \( E_M \) and such that a two-point subset \( \{u, v\} \) of \( V \) belongs to \( E \) if and only if \( \{x, u, v\} \in E_M \). By Lemma 2, the elements of \( V \) can be renamed as 0, 1, \ldots, 4 in such a way that

\[ 0 \leq u < v < w < 5 \Rightarrow [uvw] \]

and Lemma 3 guarantees that (3) – (9) are satisfied. Since a two-point subset \( \{u, v\} \) of \( V \) belongs to \( E \) if and only if \( \{x, u, v\} \in E_M \), we have

\[ E = \{\{u, v\} : (u, v) \in D_1 \cup D_2 \cup D_3\}. \]

Since \( \overline{P_5} \) contains only one triangle, (3) and (4) show that every \((j, \ell)\) in \( D_1 \cup D_3 \) has \( \ell \leq j + 2 \) and (5), (6) show that every \((j, \ell)\) in \( D_2 \) has \( \ell - j \geq 3 \). Explicitly, we have

\[
D_1 \cup D_3 \subseteq \{(0, 1), (1, 2), (2, 3), (3, 4), (0, 2), (1, 3), (2, 4)\}, \\
D_2 \subseteq \{(0, 3), (0, 4), (1, 4)\}.
\]
Next, let $T$ denote the unique triangle in our $\overline{P_5}$ and let $P$ denote the three-edge path resulting when the three edges of $T$ are deleted.

Since no edge of $P$ extends to a triangle, implications (3), (4) show that none of $\{0, 2\}, \{1, 3\}, \{2, 4\}$ can belong to $P$ and implications (5), (6) show that neither of $\{0, 3\}, \{1, 4\}$ can belong to $P$. Hence each of the three edges of $P$ must be one of $\{0, 1\}, \{1, 2\}, \{2, 3\}, \{3, 4\}, \{0, 4\}$.

Since $E$ contains only one triangle, implications (1), (4) show that none of $\{1, 2, 4\}, \{1, 3, 4\}, \{0, 2, 4\}$ can be the vertex set of $T$ and implications (5), (6) show that neither of $\{0, 1, 4\}, \{0, 3, 4\}$ can be the vertex set of $T$. Flip symmetry $i \leftrightarrow 4-i$ reduces the remaining five options to the following three.

Option 1: the vertex set of $T$ is $\{0, 1, 2\}$,

Option 2: the vertex set of $T$ is $\{1, 2, 3\}$,

Option 3: the vertex set of $T$ is $\{0, 1, 3\}$.

We are going to eliminate these three options one by one.

**Option 1**: $E = \{\{0, 1\}, \{1, 2\}, \{0, 2\}, \{2, 3\}, \{3, 4\}, \{0, 4\}\}$.

- $(0, 4) \in D_2$ and $\{2, 4\} \not\in E$ force $(0, 2) \not\in D_3$ [and so $(0, 2) \in D_1$] by (10),
- $(0, 2) \in D_1$ and $\{0, 3\} \not\in E$ force $(2, 3) \not\in D_1$ [and so $(2, 3) \in D_3$] by (3),
- $(0, 4) \in D_2$ and $\{0, 3\} \not\in E$ force $(3, 4) \not\in D_1$ [and so $(3, 4) \in D_3$] by (9),
- $(2, 3) \in D_3$ and $(3, 4) \in D_3$ force $(2, 4) \in D_3$ by (8).

However, $(2, 4) \in D_3$ is incompatible with $\{2, 4\} \not\in E$.

**Option 2**: $E = \{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{0, 1\}, \{3, 4\}, \{0, 4\}\}$.

- $(0, 4) \in D_2$ and $\{1, 4\} \not\in E$ force $(0, 1) \not\in D_3$ [and so $(0, 1) \in D_1$] by (10),
- $(0, 1) \in D_1$ and $\{0, 3\} \not\in E$ force $(1, 3) \not\in D_1$ [and so $(1, 3) \in D_3$] by (3),
- $(0, 4) \in D_2$ and $\{0, 3\} \not\in E$ force $(3, 4) \not\in D_1$ [and so $(3, 4) \in D_3$] by (9),
- $(1, 3) \in D_3$ and $(3, 4) \in D_3$ force $(1, 4) \in D_3$ by (8).

However, $(1, 4) \in D_3$ is incompatible with $\{1, 4\} \not\in E$.

**Option 3**: $E = \{\{0, 3\}, \{0, 4\}, \{3, 4\}, \{0, 1\}, \{1, 2\}, \{2, 3\}\}$.

- $(0, 3) \in D_2$ and $\{1, 3\} \not\in E$ force $(0, 1) \not\in D_3$ [and so $(0, 1) \in D_1$] by (10),
- $(0, 1) \in D_1$ and $\{0, 2\} \not\in E$ force $(1, 2) \not\in D_1$ [and so $(1, 2) \in D_3$] by (3),
- $(0, 3) \in D_2$ and $\{0, 2\} \not\in E$ force $(2, 3) \not\in D_1$ [and so $(2, 3) \in D_3$] by (9),
- $(1, 2) \in D_3$ and $(2, 3) \in D_3$ force $(1, 3) \in D_3$ by (8).

However, $(1, 3) \in D_3$ is incompatible with $\{1, 3\} \not\in E$.

By the way, the hypergraph based on $\overline{P_5}$ is minimal non-metric. To verify this, enumerate the vertices of $\overline{P_5}$ as $a, b, c, d, e$ in such a way that the edges
of this $P_5$ are
\[
\{a, b\}, \{b, c\}, \{c, d\}, \{d, a\}, \{e, b\}, \{e, c\}.
\]
Now each of $P_5 - b$ and $P_5 - c$ is a $P_4$; by Lemma 1 the hypergraph based on $P_4$ is metric. Next, $P_5 - e$ is a $C_4$; by the comment following our proof of Theorem 1 the hypergraph based on $C_4$ is metric. Finally, $P_5 - a$ and $P_5 - d$ are isomorphic; to see that the hypergraph based on $P_5 - a$ is metric, note that the metric space on the ground set $\{e, b, c, d, x\}$ with metric defined by the chart

|   | e | b | c | d | x |
|---|---|---|---|---|---|
| e | 0 | 1 | 2 | 3 | 2 |
| b | 1 | 0 | 1 | 2 | 3 |
| c | 2 | 1 | 0 | 1 | 4 |
| d | 3 | 2 | 1 | 0 | 3 |
| x | 2 | 3 | 4 | 3 | 0 |

has
\[
[ebc], [ebd], [ecd], [bcd], [xeb], [xec], [xdc], [xbc],
\]
but none of $[xed], [exd], [edx], [xbd], [bxd], [bdx]$.

To close this section, we present a large family of minimal non-metric hypergraphs. Given graphs $G_1, G_2$ on the same vertex set $V$, we let $\mathcal{H}(G_1, G_2)$ denote the 3-uniform hypergraph with vertex set $V \cup \{x_1, x_2\}$ such that $x_1$ and $x_2$ are distinct points outside $V$ and hyperedge set consisting of all three-point subsets of $V$ and all sets $\{x_i, u, v\}$ with $i \in \{1, 2\}$ and $\{u, v\}$ ranging over edges of $G_i$.

**Theorem 3.** Let $n$ be an integer greater than four and let $V$ be a set of size $n$. Let $E_1$ and $E_2$ be subsets of $\binom{S}{2}$ and for both $i$ in $\{1, 2\}$, let $G_i$ denote $(V, E_i)$. Suppose that $E_1$ and $E_2$ satisfy the following conditions:

- the graph $(V, E_1 \cup E_2)$ consists of a $C_{n-1}$ with an additional isolated vertex, and
- there exists some vertex that has degree equal to 1 in both $G_1$ and $G_2$,

then $\mathcal{H}(G_1, G_2)$ is a minimal non-metric hypergraph.
Proof. First, we are going to deduce a contradiction from the assumption that $\mathcal{H}(G_1, G_2)$ is metric. This assumption means that there is a metric space $M$ on ground set $V \cup \{x_1, x_2\}$ such that all three-point subsets of $V$ belong to $\mathcal{E}_M$ and such that a three-point set $\{x_i, u, v\}$ belongs to $\mathcal{E}_M$ if and only if $\{u, v\} \in E_i$. By Lemma 2, the elements of $V$ can be renamed as $0, 1, \ldots n$ to satisfy (2). Two applications of Lemma 4, one with $x = x_1$ and the other with $x = x_2$, show that $E_1 \cup E_2 \subseteq \{\{0, 1\}, \{1, 2\}, \ldots \{n - 2, n - 1\}, \{n - 1, 0\}\}$. This conclusion contradicts the assumption that $(V, E_1 \cup E_2)$ contains a $C_{n-1}$. It follows that $\mathcal{H}(G_1, G_2)$ is non-metric.

Let us now prove that it is minimal with that property. Let $u$ be a vertex and let $\overline{\mathcal{H}}$ be the subhypergraph of $\mathcal{H}(G_1, G_2)$ induced by $(V \setminus \{u\}) \cup \{x_1, x_2\}$. We will describe a metric space $M$ on $(V \setminus \{u\}) \cup \{x_1, x_2\}$ such that $\mathcal{E}_M = \mathcal{E}(\overline{\mathcal{H}})$. The second assumption of the theorem allows us to suppose, without loss of generality, that

- $V \setminus \{u\} = \{1, \ldots, n - 1\}$,
- $E_1 \cup E_2 \subseteq \{\{i - 1, i\}; i = 2, \ldots, n - 1\} \cup \{\{1, n - 1\}\}$, and
- $\{1, n - 1\} \in E_1 - E_2$, and
- $\{n - 2, n - 1\} \in E_2 - E_1$.

For $i = 1, \ldots, n - 1$ we define

$$
\alpha_i = \begin{cases} 
0 & \text{if } i = 1, \text{ or there is an even number of edges of } G_1 \text{ among the pairs } \{1, 2\}, \ldots, \{i - 1, i\}, \\
1 & \text{otherwise}
\end{cases}
$$

and we define $\beta_i$ in the same way with $G_2$ in place of $G_1$. Set

$$
k = \frac{n - 2 - \alpha_{n - 1}}{2}.
$$

We define the metric space $M$ by setting $\text{dist}(i, j) = j - i$ whenever $1 \leq i < j \leq n - 1$ and $\text{dist}(x_1, i) = k + \alpha_i$ and $\text{dist}(x_2, i) = 2k + \beta_i$ for $i = 1, \ldots, n - 1$ and $\text{dist}(x_1, x_2) = k + 2$. 

9
Let us prove that \( \mathcal{E}(\overline{H}) \subseteq \mathcal{E}_M \). By assumption, \( \{1, n - 1\} \in E(G_1) \), so \( \mathcal{E}(\overline{H}) \) contains the triple \( \{1, x_1, n - 1\} \). Since \( \alpha_1 = 0 \), we have

\[
dist(1, x_1) + \dist(x_1, n - 1) = k + k + \alpha_{n-1} = n - 2 = \dist(1, n - 1),
\]

and so we have \( \{1, x_1, n - 1\} \in \mathcal{E}_M \). If \( i \) is such that \( \{i - 1, i, x_1\} \in \mathcal{E}(\overline{H}) \), then \( \dist(x_1, i - 1) - \dist(x_1, i) \) = 1 = \( \dist(i - 1, i) \), and so we have \( \{x_1, i - 1, i\} \) or \( \{i - 1, i, x_1\} \), and so \( \{i - 1, i, x_1\} \in \mathcal{E}_M \). Similarly, if \( i \) is such that \( \{i - 1, i, x_2\} \in \mathcal{E}(\overline{H}) \), then \( \{i - 1, i, x_2\} \in \mathcal{E}_M \).

Now let us prove the other inclusion. Let us note here that \( n \geq 5 \) guarantees \( k \geq 1 \). If \( i \in \{2, \ldots, n - 1\} \) is such that \( \{i - 1, i, x_1\} \notin \mathcal{E}(\overline{H}) \), then

\[
dist(i - 1, x_1) = dist(i, x_1) \geq k \geq 1 = dist(i - 1, i),
\]

and so \( \{i - 1, i, x_1\} \notin \mathcal{E}_M \). Similarly, \( \{i - 1, i, x_2\} \notin \mathcal{E}(\overline{H}) \) implies \( \{i - 1, i, x_2\} \notin \mathcal{E}_M \).

We have \( \dist(1, n - 1) = n - 2, \dist(1, x_2) \in \{n - 2, n - 3\} \), and \( \dist(n - 1, x_2) \in \{n - 3, n - 2, n - 1\} \). Since \( n \geq 5 \), no combination of these values yields equality in the triangle inequality, and so \( \{1, n - 1, x_2\} \notin \mathcal{E}_M \).

Now suppose that \( 1 \leq i < j \leq n - 1 \) with \( 2 \leq j - i \leq n - 3 \). We have \( \dist(i, j) \geq 2 \), but \( \dist(x_1, i) - \dist(x_1, j) \leq 1 \), so if \( \{i, j, x_1\} \in \mathcal{E}_M \), then it must be \([ix_1j]\). In that case,

\[
n - 3 \geq \dist(i, j) = \dist(i, x_1) + \dist(x_1, j) = 2k + \alpha_i + \alpha_j = n - 2 - \alpha_{n-1} + \alpha_i + \alpha_j \geq n - 3. \hspace{1cm} (12)
\]

We see that \( \dist(i, j) = n - 3 \), and there are two possibilities: either \( (i, j) = (1, n - 2) \), or \( (i, j) = (2, n - 1) \). Using the assumption that \( \{n - 2, n - 1\} \notin E(G_1) \), we get \( \alpha_{n-2} = \alpha_{n-1} \). In either of the two cases, we therefore have \( \alpha_{n-1} = \alpha_j \), and these two values in equation (12) cancel out, so in fact \( \dist(i, j) \geq n - 2 \). This is a contradiction, so \( \{i, j, x_1\} \notin \mathcal{E}_M \).

Even easier, if \( 1 \leq i < j \leq n - 1 \) with \( 2 \leq j - i \leq n - 3 \), then again, \( \dist(x_2, i) = \dist(x_2, j) \leq 1 \) and we only have to rule out \([ix_2j]\). Since \( n \geq 5 \),

\[
dist(i, x_2) + \dist(x_2, j) = 4k + \beta_i + \beta_j \geq 2(n - 3) > n - 2 > \dist(i, j),
\]

showing that \( \{i, j, x_2\} \notin \mathcal{E}_M \).
Finally, for any $i \in \{1, \ldots, n-1\}$, the triple $\{x_1, x_2, i\}$ is not in $\mathcal{E}_M$.

Theorem 3 implies that the number of isomorphism classes of minimal non-metric hypergraphs on $n$ vertices increases exponentially with $n$. Specifically, we have at least $2^{2n-5}/(n-1)$ of such non-isomorphic hypergraphs on $\{0, \ldots, n-1\} \cup \{x_1, x_2\}$.

3 Metric-line equivalences

Lemma 6. Let $G$ be a graph on the vertex set $V$, such that both $G$ and its complement $\overline{G}$ contain at least one edge. If neither the hypergraph based on $G$ nor the hypergraph based on $\overline{G}$ is metric, then the equivalence relation $G \equiv \overline{G}$ is an obstacle.

Proof. Let $G$ satisfy the hypothesis of the lemma. Assuming that $G \equiv \overline{G}$ is not an obstacle, we will prove that at least one of the two hypergraphs based on $G$ and on $\overline{G}$ is metric.

Since $G \equiv \overline{G}$ is not an obstacle, there is a metric space $M$ on a superset $W$ of $V$ such that, for every choice of two-point subsets $\{a, b\}$ and $\{c, d\}$ of $V$, we have

$$L_M(ab) = L_M(cd) \iff \{a, b\} \equiv \{c, d\}.$$  

Let $L$ denote the common value of $L_M(uv)$ with $\{u, v\}$ ranging over the edge set of $G$ and let $L'$ denote the common value of $L_M(uv)$ with $\{u, v\}$ ranging over the edge set of $\overline{G}$.

Let $\{u, v, w\}$ be a three-point subset of $V$ and suppose that $L_M(uv) = L_M(vw)$. Then $w \in L_M(uv)$, so $\{u, v, w\} \in \mathcal{E}_M$. For every triple $\{u, v, w\}$, at least two of the three pairs $\{u, v\}$, $\{v, w\}$, $\{u, w\}$ are in the same class of $G \equiv \overline{G}$, so all three-point subsets of $V$ belong to $\mathcal{E}_M$.

The hypothesis of the theorem guarantees that there is a pair $\{u, v\} \in E(G)$. For any $w \in V - \{u, v\}$, $\{u, v, w\} \in \mathcal{E}_M$, so $w \in L$, and $V \subseteq L$. Similarly, using the fact that $E(\overline{G}) \neq \emptyset$, $V \subseteq L'$. 

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Since $L$ and $L'$ are distinct, their symmetric difference $(L - L') \cup (L' - L)$ is nonempty. Switching $G$ and $\overline{G}$ if necessary, we may assume that $L - L'$ is nonempty. Now the hypergraph based on $G$ is metric. To see this, consider any point $x$ in $L - L'$ and any two-point subset $\{u, v\}$ of $V$. We have seen in the previous paragraph that $x \notin V$. If $\{u, v\}$ is an edge of $G$, then $\{x, u, v\} \in \mathcal{E}_M$ as $x \in L$; if $\{u, v\}$ is an edge of $\overline{G}$, then $\{x, u, v\} \not\in \mathcal{E}_M$ as $x \notin L'$.

**Proof of Theorem 2.** Let $n$ be an even integer greater than four and let $G$ be the cycle $C_n$. By Theorem 1, the hypergraph based on $G$ is not metric; by Lemma 5, the hypergraph based on $\overline{G}$ is not metric; hence, by Lemma 6, the equivalence relation $\equiv$ is an obstacle.

To see that $\equiv$ is a minimal obstacle, consider any proper subset $U$ of $V$. The restriction of $\equiv$ on $\binom{U}{2}$ is $\equiv^F$, where $F$ is the subgraph of $G$ induced by $U$. Since $F$ is an induced subgraph of $P_{n-1}$, Lemma 1 guarantees that there is a metric space $M$ on ground set $U \cup \{x\}$, where $x \notin U$, such that all three-point subsets of $U$ belong to $\mathcal{E}_M$ and such that a two-point subset $\{u, v\}$ of $U$ is an edge of $F$ if and only if $\{x, u, v\} \in \mathcal{E}_M$. Whenever $\{u, v\}$ is a two-point subset of $U$, we have

$$L_M(uv) = \begin{cases} U \cup \{x\} & \text{if } \{u, v\} \text{ is an edge of } F, \\ U & \text{otherwise,} \end{cases}$$

and so $\equiv^F$ is not an obstacle. □

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