The Existence of Invariant Tori and Quasiperiodic Solutions of the Nosé–Hoover Oscillator

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In this paper, we consider an equivalent form of the Nosé–Hoover oscillator, \(x' = y, y' = -x - yz, \) and \(z' = y^2 - a, \) where \(a\) is a positive real parameter. Under a series of transformations, it is transformed into a 2-dimensional reversible system about action-angle variables. By applying a version of twist theorem established by Liu and Song in 2004 for reversible mappings, we find infinitely many invariant tori whenever \(a\) is sufficiently small, which eventually turns out that the solutions starting on the invariant tori are quasiperiodic. The discussion about quasiperiodic solutions of such 3-dimensional system is relatively new.

1. Introduction

In 1984, Nosé constructed a system called the Nosé equations to model the interaction of a particle with a heat-bath [1]. Later, in 1986, Posch et al. [2] simplified the Nosé equations by omitting an inessential variable and replaced the residual “momentum” by a “friction coefficient” and then got the Nosé–Hoover oscillator with the following equations of motion:

\[
\begin{align*}
q' &= p, \\
p' &= -q - p\xi, \\
\xi' &= a(p^2 - 1),
\end{align*}
\]

where \(q\) and \(p\) are the coordinate and momentum of the oscillator, respectively; \(\xi\) is a friction coefficient; and \(a\) is a coupling positive real parameter. Specially, for \(a > 0\), by the linear change of variables,

\[
\begin{align*}
q &= \frac{x}{\sqrt{a}}, \\
p &= \frac{y}{\sqrt{a}}, \\
\xi &= z.
\end{align*}
\]

Nosé–Hoover oscillator is transformed into

\[
\begin{align*}
x' &= y, \\
y' &= -x - yz, \\
z' &= y^2 - a,
\end{align*}
\]

which is usually called Sprott A system since it was presented by Sprott [3] with \(a = 1\) as case A in a list of nineteen distinct differential systems with quadratic nonlinearities and having chaotic behavior. In this paper, we are mainly interested in the behaviors of Nosé–Hoover equation for \(a > 0\) small so that we focus on the equivalent form (3) in the sequel.
Nosé–Hoover oscillator possesses rich dynamic behaviors: periodic and quasiperiodic solutions, nested tori, and chaos, all of which have been observed through numerical simulations (see [3–6] and references therein), even this system without equilibrium points. In the last few years, researchers have devoted to the rigorous proofs of these phenomena and achieved some results. In [7, 8], Legoll et al. discussed the "simple" Nosé–Hoover thermostated harmonic oscillator:

\[
\begin{align*}
q' &= \rho, \\
\dot{\rho}' &= -q - \xi \rho, \\
\xi' &= \left(\frac{\rho^2 - T}{M}\right).
\end{align*}
\]

By means of an averaging argument, they reduced the thermostated equations to a nondegenerate twist map to show the existence of KAM tori near the decoupled limit of \( M = \infty \) and \( \xi = 0 \). Subsequently, Butler [9] complemented the result of Legoll et al. and proved the existence of invariant tori near the high-temperature limit \( T = \infty \) with the thermostat mass \( M \) held constant. Although we also pay attention to the existence of invariant tori of Nosé–Hoover equation, our methods are totally different. Compared with the average method mentioned in previous papers, we directly start from the equation itself and give its Poincaré map. Recently, in [10], Messias and Reinol proved the existence of a linearly stable periodic orbit of Sprott A system, which bifurcates from a nonisolated zero-Hopf equilibrium point located at the origin for \( a > 0 \) small, by using the averaging method. Moreover, they showed numerically the existence of nested invariant tori surrounding this periodic orbit, just like in Figure 1.

The present paper explores the existence of invariant tori and quasiperiodic solutions of (3), which is absent of rigorous proof up to now. It is well known that Moser’s twist theorem is a powerful tool to detect the existence of invariant curve (see [11–14] and references therein), but the application of twist theorem on 3-dimensional systems to obtain invariant tori is few. The key point is how to transform a 3-dimensional system into a 2-dimensional one reasonably. Different from the way of reduction dimension by averaging theory in [7], we use the equation itself to eliminate some variables and perform a series of transformations to obtain a 2-dimensional system about action-angle variables. Besides the invariant tori, we further discuss the solutions of the transformed 2-dimensional system starting from the invariant curve. They are quasiperiodic, which corresponds to the quasiperiodic solutions of the original 3-dimensional system with the same frequencies.

Another difficulty brought by system (3) is the absence of area-preserving property or intersection property needed in the twist theorem since it is not a conservative system. Fortunately, we find that the original 3-dimensional system (3) is invariant under the transformation of coordinates \((x, y, z) \rightarrow (-x, -y, z)\), and the transformed 2-dimensional system after every transformation keeps reversible property. Therefore, a twist theorem established in [15] by Liu and Song for reversible systems is valid. In the application of this twist theorem (given in Section 3, Theorem 2), we need to expand the corresponding Poincaré map into the form like (32), which is a tedious work needing much calculation.

Now, we state our main result in the following.

**Theorem 1.** For \( a > 0 \) small enough, system (3) admits an infinite number of invariant tori, which tends to the origin as \( a \rightarrow 0 \) and thus an infinite number of quasiperiodic solutions.

The structure of the paper is as follows. In Section 2, we first briefly introduce some definitions and properties about reversible systems (one can refer to [16] for details). Then, we transform equation (3) into system (14) by a serious of transformations, including cylindrical coordinate change, translation change, scale, and polar coordinate change. Subsequently, we give the expression of Poincaré mapping of (14) and also prove the main result by a twist theorem for reversible system in Section 3. In the last section, numerical simulations of quasiperiodic solutions are given to support our results.

### 2. Transformations on System (3)

#### 2.1. Some Facts on Reversible Systems

Before performing transformations on system (3), we first give some definitions and facts related to reversible systems ([12, 16]).

**Definition 1.** Let \( \Omega \in \mathbb{R}^n \) be open and \( Z = Z(z, t): \Omega \times \mathbb{R} \rightarrow \mathbb{R}^n \) be continuous and \( 2\pi \)-periodic in the time variable \( t \). Moreover, suppose \( G: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is an involution (i.e., \( G \) is a \( C^1 \)-diffeomorphism such that \( G^2 = \text{id}_{\mathbb{R}^n} \)) satisfying \( G(\Omega) = \Omega \). The system

\[
z' = Z(z, t),
\]

is called reversible with respect to \( G \), if

\[
DG(Gz)Z(Gz, t) = -Z(z, t), \quad z \in \Omega, t \in \mathbb{R},
\]

with \( DG \) denoting the Jacobian of \( G \).
Usually, for the two-dimensional system, we are interested in the special involution $G(x, y) = (-x, y)$ and $\mathcal{G}(x, y) = (x, -y)$ for $z = (x, y) \in \mathbb{R}^2$, under which condition (6) is transformed into an obvious form for $G$:

$$
\begin{align*}
Z_1(-x, y, -t) &= Z_1(x, y, t), \\
Z_2(-x, y, -t) &= -Z_2(x, y, t),
\end{align*}
$$

(7)

and for $\mathcal{G}$:

$$
\begin{align*}
Z_1(x, -y, -t) &= -Z_1(x, y, t), \\
Z_2(x, -y, -t) &= Z_2(x, y, t),
\end{align*}
$$

(8)

with $Z = (Z_1, Z_2)$.

**Definition 2.** Let $T(\cdot, t)$ be an invertible transformation of $\Omega$ for every fixed $t$, and suppose that $G$ is an involution with $G(\Omega) = \Omega$. Then, $T$ is called $G$-invariant if

$$
G(T(z, t)) = T(Gz, -t), \quad z \in \Omega, t \in \mathbb{R}.
$$

(9)

Returning to the special example $\mathcal{G}(x, y) = (x, -y)$ from above, condition (9) requires

$$
\begin{align*}
T_1(x, -y, -t) &= T_1(x, y, t), \\
T_2(x, -y, -t) &= -T_2(x, y, t),
\end{align*}
$$

(10)

where $T = (T_1, T_2)$.

Next lemma is useful to show that a transformed version of a reversible system is again reversible.

**Lemma 1.** Assume that (5) is reversible with respect to $G$. If a transformation $T(\cdot, t): \Omega \rightarrow \mathbb{R}^n$ is $G$-invariant, then the transformed system, i.e., the system satisfied by $(z(t) = T(z(t), t)$, is also reversible with respect to $G$.

Finally, reversible systems lead to reversible Poincaré maps, in the following sense.

**Definition 3.** Let $f: \Omega \rightarrow \mathbb{R}^n$ be a homeomorphism onto its image, and let $G$ be an involution. Then, $f$ is called reversible with respect to $G$ on a set $D \subset \Omega$ if

$$
f^{-1} = G \circ f \circ G, \quad \text{in } D.
$$

(11)

**Lemma 2.** Suppose the Cauchy problems associated with (5) are uniquely solvable and let the time $2\pi$-map $P$ and its inverse $P^{-1}$ be defined. Then, $P$ is reversible with respect to $G$, provided that (5) is reversible with respect to $G$.

2.2. Transformed System. Obviously, the equations of motion (3) are invariant under the transformation of coordinates $(x, y, z) \rightarrow (-x, -y, z)$. After a rescaling of time $t$, equation (3) is firstly written into

$$
\begin{align*}
x' &= -y, \\
y' &= x + yz, \\
z' &= a - y^2.
\end{align*}
$$

(12)

Passing to cylindrical coordinates

$$
\begin{align*}
x &= r \cos \theta, \\
y &= r \sin \theta, \\
z &= z,
\end{align*}
$$

(13)

via a short calculation (12) becomes

$$
\begin{align*}
r' &= rz \sin^2 \theta, \\
\theta' &= z \sin \theta \cos \theta + 1, \\
z' &= a - r^2 \sin^2 \theta.
\end{align*}
$$

(14)

Now, we introduce a small quantity by the change in variables

$$
\begin{align*}
r &= \epsilon \rho, \\
z &= \epsilon \tau,
\end{align*}
$$

(15)

with $(\epsilon = \sqrt{a} > 0)$. Then, system (14) is rewritten into

$$
\begin{align*}
\rho' &= \rho \sin^2 \theta, \\
\theta' &= \tau \sin \theta \cos \theta + 1, \\
\tau' &= (1 - \rho^2) \sin^2 \theta.
\end{align*}
$$

(16)

Noticing $(\epsilon \tau \sin \theta \cos \theta + 1 \neq 0)$ for $\epsilon$ small enough and taking $\theta$ as the independent variable, we get

$$
\begin{align*}
\frac{d\rho}{d\theta} &= \frac{\epsilon \rho \sin^2 \theta}{\epsilon \tau \sin \theta \cos \theta + 1}, \\
\frac{d\tau}{d\theta} &= \frac{\epsilon (1 - \rho^2 \sin^2 \theta)}{\epsilon \tau \sin \theta \cos \theta + 1}.
\end{align*}
$$

(17)

It is not difficult to find that equation (17) is reversible under the transformation $(\rho, \tau) \rightarrow (\rho, -\tau)$ since $g_1(\rho, -\tau, -\theta) = -g_1(\rho, \tau, \theta)$ and $g_2(\rho, -\tau, -\theta) = g_2(\rho, \tau, \theta)$ where $g_1$ and $g_2$ denote the functions at the right hands of equation (12), respectively.

Note that the transformations above appeared in [10] either. By the averaging method, Messias and Reinol proved that equation (17) has periodic solutions tending to $(\sqrt{2}, 0)$ as $\epsilon$ tends to 0. Different from the method and result in [10], we focus on the Poincaré mapping to obtain invariant torus. Based on this, the subsequent transformation is carried out after a translation:

$$
\bar{\rho} = \rho - \sqrt{2}, \\
\tau = \tau.
$$

(18)

Correspondingly, equation (17) is transformed into

$$
\begin{align*}
\frac{d\bar{\rho}}{d\theta} &= \frac{\epsilon (\bar{\rho} + \sqrt{2}) \sin^2 \theta}{\epsilon \tau \sin \theta \cos \theta + 1}, \\
\frac{d\tau}{d\theta} &= \frac{\epsilon (1 - (\bar{\rho} + \sqrt{2})^2 \sin^2 \theta)}{\epsilon \tau \sin \theta \cos \theta + 1}.
\end{align*}
$$

(19)
This system is reversible under the transformation $(\mathbf{p}, \tau) \rightarrow (\mathbf{p}, -\tau)$.

Lastly, writing system (19) in polar coordinates

$$
\begin{align*}
\mathbf{p} &= I \cos \varphi, \\
\tau &= I \sin \varphi,
\end{align*}
$$

it becomes

$$
\begin{align*}
\frac{dI}{d\theta} &= -\sqrt{2} \epsilon \sin \varphi \sin^2 \theta (I \cos \varphi + \sqrt{2}) + \epsilon \sin \varphi \\
&\quad + \sqrt{2} \epsilon \cos \varphi - \epsilon (I \cos \varphi + \sqrt{2})(I + \sqrt{2} \cos \varphi) \sin^2 \theta \\
&\quad + \frac{\epsilon}{2} I \sin^2 \varphi \sin 2\theta + \frac{\epsilon}{4} I \cos \varphi \sin \varphi \sin^2 \theta + \frac{\epsilon}{4} I \cos \varphi \sin \varphi \sin^2 \theta \sin^2 \theta + O(\epsilon^4), \\
\frac{d\varphi}{d\theta} &= \frac{\epsilon \cos \varphi - \epsilon (I \cos \varphi + \sqrt{2})(I + \sqrt{2} \cos \varphi) \sin^2 \theta}{I (\epsilon I \sin \varphi \sin \theta \cos \theta + 1)},
\end{align*}
$$

(21)

We also note that system (21) is reversible under the involution $(I, \varphi) \rightarrow (I, -\varphi)$ since $g_3(I, -\varphi, -\theta) = -g_3(I, \varphi, \theta)$ and $g_4(I, -\varphi, -\theta) = g_4(I, \varphi, \theta)$, where

$$
\begin{align*}
g_3 &= -\sqrt{2} \epsilon \sin \varphi \sin^2 \theta (I \cos \varphi + \sqrt{2}) + \epsilon \sin \varphi \\
g_4 &= \frac{\epsilon \cos \varphi - \epsilon (I \cos \varphi + \sqrt{2})(I + \sqrt{2} \cos \varphi) \sin^2 \theta}{I (\epsilon I \sin \varphi \sin \theta \cos \theta + 1)},
\end{align*}
$$

(22)

In this 2-dimensional system (21), $I$ plays the role of action variable, $\varphi$ is the angle variable, and $\theta$ denotes the time variable. We will focus on the Poincaré mapping at $\theta = \pi$ in the sequel.

### 3. Proof of Theorem 1

The key of the whole proof is to figure out the Poincaré mapping induced by (21). Since it involves a lot of tedious calculations by Taylor expansions and numerous notations, we give a sketch of the proof.

**Step 1.** Observing that $g_3$ and $g_4$ defined in (22) are real analytic, $\pi$-periodic with respect to $\theta$ and $2\pi$-periodic with respect to $\varphi$, we do the Taylor expansion of order 3 of (21) at $\epsilon = 0$. Then, there are

$$
\begin{align*}
\frac{dI}{d\theta} &= -\sqrt{2} \epsilon (I \cos \varphi + \sqrt{2}) \sin \varphi \sin^2 \theta + \epsilon \sin \varphi \\
&\quad + \sqrt{2} \epsilon \cos \varphi - \epsilon (I \cos \varphi + \sqrt{2})(I + \sqrt{2} \cos \varphi) \sin^2 \theta + \frac{\epsilon}{2} I \sin^2 \varphi \sin 2\theta + \frac{\epsilon}{4} I \cos \varphi \sin \varphi \sin^2 \theta + \frac{\epsilon}{4} I \cos \varphi \sin \varphi \sin^2 \theta \sin^2 \theta + O(\epsilon^4), \\
\frac{d\varphi}{d\theta} &= \frac{\epsilon \cos \varphi - \epsilon (I \cos \varphi + \sqrt{2})(I + \sqrt{2} \cos \varphi) \sin^2 \theta}{I (\epsilon I \sin \varphi \sin \theta \cos \theta + 1)}.
\end{align*}
$$

(23)

and

$$
\begin{align*}
d\varphi &= \epsilon I^{-1} \cos \varphi - \epsilon I^{-1} (I \cos \varphi + \sqrt{2}) (I + \sqrt{2} \cos \varphi) \sin^2 \theta \\
&\quad - \frac{\epsilon}{4} \sin 2\varphi \sin 2\theta + \frac{\epsilon}{2} (I \cos \varphi + \sqrt{2}) (I + \sqrt{2} \cos \varphi) \sin \varphi \sin^2 \theta \sin^2 \theta + \frac{\epsilon}{4} I (I \cos \varphi + \sqrt{2}) (I + \sqrt{2} \cos \varphi) \sin^2 \theta \sin^2 \theta \sin^2 \theta \\
&\quad + \frac{\epsilon}{4} I \cos \varphi \sin^2 \varphi \sin^2 \theta + O(\epsilon^4).
\end{align*}
$$

(24)

To derive an expression for the corresponding Poincaré map, we set

$$
\begin{align*}
I(\theta; \tau, \varphi) &= I_0 + \epsilon F_1(\theta; \tau, \varphi, \epsilon), \\
\varphi(\theta; \tau, \varphi) &= \phi_0 + \epsilon F_2(\theta; \tau, \varphi, \epsilon),
\end{align*}
$$

(25)

for the solution $(I(\theta), \varphi(\theta))$ of (14) with $(I(0), \varphi(0)) = (I_0, \phi_0)$. Integrating (23) and comparing with (25), we see that

$$
F_1(\theta; \tau, \varphi, \epsilon) = -\sqrt{2} \int_0^\theta \sin(\varphi_0 + \epsilon F_2) \cdot A_1 \sin^2 t dt
$$

$$
+ \int_0^\theta \sin(\varphi_0 + \epsilon F_2) dt
$$

$$
+ \frac{\sqrt{2} \epsilon}{2} \int_0^\theta \sin(\varphi_0 + \epsilon F_2) \cdot \sin 2 t dt
$$

$$
- \frac{\epsilon}{2} \int_0^\theta \sin(\varphi_0 + \epsilon F_2) \cdot \sin 2 t dt
$$

$$
- \frac{\epsilon}{4} \int_0^\theta \sin(\varphi_0 + \epsilon F_2) \cdot \sin 2 t dt
$$

$$
- \frac{\epsilon}{4} \int_0^\theta \sin(\varphi_0 + \epsilon F_2) \cdot \sin 2 t dt
$$

$$
+ \frac{\epsilon}{4} \int_0^\theta (I_0 + \epsilon F_1)^2 A_1 \sin^3 (\varphi_0 + \epsilon F_2)
$$

$$
\cdot \sin^2 t \cdot \sin^2 2 t dt
$$

$$
+ \frac{\epsilon}{4} \int_0^\theta (I_0 + \epsilon F_1)^2 A_1 \sin^3 (\varphi_0 + \epsilon F_2)
$$

$$
\cdot \sin^2 t \cdot \sin^2 2 t dt
$$

$$
+ \frac{\epsilon}{4} \int_0^\theta (I_0 + \epsilon F_1)^2 A_1 \sin^3 (\varphi_0 + \epsilon F_2)
$$

$$
\cdot \sin^2 t \cdot \sin^2 2 t dt + O(\epsilon^4).
$$

(26)
\begin{equation}
F_2(\theta; I_0, \varphi_0, \epsilon) = \int_0^\theta (I_0 + \epsilon F_1)^{-1} \cos(\varphi_0 + \epsilon F_2) dt \\
- \int_0^\theta (I_0 + \epsilon F_1)^{-1} A_1 A_2 \times \sin^2 t dt - \frac{\epsilon}{2} \int_0^\theta \sin(2(\varphi_0 + \epsilon F_2)) \cdot \sin 2t dt \\
+ \frac{\epsilon^2}{4} \int_0^\theta A_1 A_2 \cdot \sin(\varphi_0 + \epsilon F_2) \cdot \sin^2 t \cdot \sin 2t dt \\
- \frac{\epsilon^2}{4} \int_0^\theta (I_0 + \epsilon F_1) \cdot A_1 A_2 \cdot \sin^2(\varphi_0 + \epsilon F_2) \cdot \sin^2 t \cdot \sin^2 2t dt \\
+ \frac{\epsilon^2}{4} \int_0^\theta (I_0 + \epsilon F_1) \cos(\varphi_0 + \epsilon F_2) \cdot \sin^2(\varphi_0 + \epsilon F_2) \cdot \sin^2 2t dt + O(\epsilon^3),
\end{equation}

where

\begin{align*}
A_1 &\equiv (I_0 + \epsilon F_1) \cos(\varphi_0 + \epsilon F_2) + \sqrt{2}, \\
A_2 &\equiv (I_0 + \epsilon F_1) + \sqrt{2} \cos(\varphi_0 + \epsilon F_2).
\end{align*}

We then represent \( F_1 \) and \( F_2 \) according to the order of \( \epsilon \) and simplify the coefficients of each item in the next lemma.

**Lemma 3.** We have

\begin{align*}
F_1(\pi) &= -\frac{\sqrt{2\pi}}{4} I_0 \sin 2\varphi_0 + \epsilon \sqrt{2} \cos \varphi_0 + O(\epsilon^3), \\
F_2(\pi) &= -\frac{\pi}{2} (I_0 \sin \varphi_0 + \sqrt{2} \cos^2 \varphi_0 + \sqrt{2}) \\
&\quad - \epsilon G_1(I_0, \varphi_0) - \epsilon^2 G_2(I_0, \varphi_0) + O(\epsilon^3),
\end{align*}

\begin{equation}
\begin{cases}
I_1 = I_0 - \frac{\sqrt{2\pi}}{4} I_0 \sin 2\varphi_0 \cdot \epsilon + \epsilon^2 L_1(I_0, \varphi_0) + \epsilon^3 L_2(I_0, \varphi_0) + O(\epsilon^4), \\
\varphi_1 = \varphi_0 - \frac{\pi}{2} \left( \sqrt{2} + I_0 \cos \varphi_0 + \sqrt{2} \cos^2 \varphi_0 \right) \epsilon - \epsilon^2 G_1(I_0, \varphi_0) - \epsilon^2 G_2(I_0, \varphi_0) + O(\epsilon^4).
\end{cases}
\end{equation}

In order to complete the proof of Theorem 1, we will show that, for \( \epsilon \) small enough, the Poincaré map \( P \) has an invariant closed curve in the annulus which surrounds the point \((0, 0)\).

According to Moser’s twist theorem [17], the twist map should have twist term, which is independent of initial point and tiny amount \( \epsilon \). However, the twist term in Poincaré map (31) disappears, and the disturbance terms are at least the first-order term of \( \epsilon \). Therefore, we consider the theorem established by Liu and Song for reversible map, which matches the form of (31). We state their result below, and one can refer to [15] for details.

Let \( A = [a, b] \times S^1 \) denote a cylinder with universal cover \([a, b] \times \mathbb{R}\). It will be assumed that a map \( f: A \rightarrow \mathbb{R} \times S^1 \) has a lift, which can be expressed in the following form:

\begin{align*}
\begin{cases}
\dot{\theta}_1 = \theta + \omega + \ell_1(\theta, r; \epsilon) + \epsilon \phi_1(\theta, r; \epsilon), \\
\dot{r}_1 = r + \ell_2(\theta, r; \epsilon) + \epsilon \phi_2(\theta, r; \epsilon).
\end{cases}
\end{align*}

Moreover, \( f \) is reversible with respect to the involution \((\theta, r) \mapsto (-\theta, r)\).

**Theorem 2.** Let \( \omega = 2\pi n \) with an integer \( n \) and the functions \( l_1, l_2, \phi_1, \phi_2 \) satisfy

\begin{align*}
l_1 &\in C^2(A), \\
l_1 &> 0, \\
\frac{\partial l_1}{\partial r} &> 0, \\
\forall (\theta, r) \in A, \\
l_2(\cdot, \cdot), \\
\phi_1(\cdot, \cdot; \epsilon), \\
\phi_2(\cdot, \cdot; \epsilon) &\in C^5(A).
\end{align*}
Figure 2: Quasiperiodic orbits of system (3) near the origin and their $xyz$-coordinates in function of the time $t$ for $a = 10^{-2}$, $a = 10^{-3}$, and $a = 10^{-4}$, respectively.
In addition, we assume that there is a function
$I: A \rightarrow \mathbb{R}$ satisfying

$$I \in C^6(A),$$

$$\frac{\partial I}{\partial r} > 0,$$

$$\forall (\theta, r) \in A,$$

$$l_1(\theta, r) \frac{\partial I}{\partial \theta} (\theta, r) + l_2(\theta, r) \frac{\partial I}{\partial r} (\theta, r) = 0.$$  \hfill (34)

Moreover, suppose that there are two numbers $\bar{a}$ and $\bar{b}$ such that

$$\bar{a} < \bar{a} < \bar{b} < b,$$

$$\hat{T}(\bar{a}) < \hat{T}(\bar{a}) < \hat{T}(\bar{b}) < \hat{T}(b),$$

$$\hat{T}(r) := \min_{\theta \in S^1} I(\theta, r), \quad \hat{T} := \max_{\theta \in S^1} I(\theta, r).$$

Then, there exist $\delta > 0$ and $\Delta > 0$ such that if $\varepsilon < \Delta$ and

$$\|\phi_1\|_{C^1(A)} + \|\phi_2\|_{C^1(A)} < \delta.$$  \hfill (36)

The map $f$ has an invariant curve in $A$. The constants $\delta$ and $\Delta$ are dependent on $a, b, \bar{a}, \bar{b}, l_1, l_2$ and $I$. In particular, $\delta$ is independent of $\varepsilon$.

**Remark 1.** If the signs of $l_1$ and $(\partial l_1/\partial r)$ are inverse, the theorem is still valid.

Then, we turn to verify the conditions in Theorem 2 for the Poincaré map $P$.

**Step 3.** Let choose $A = [0, \varepsilon] \times S^1$, where $\varepsilon$ as before is small. For $(I_0, \phi_0) \in A$, the term $(\pi/2)\varepsilon I_0 \cos \varphi_0$ in (31) can be seen as $O(\varepsilon^2)$. Therefore, we rewrite the Poincaré map (31) in the form of

$$\begin{align*}
I_1 &= I_0 - \frac{\sqrt{2}\pi}{4} I_0 \sin 2\varphi_0 \cdot \varepsilon + O(\varepsilon^2), \\
\varphi_1 &= \varphi_0 - \frac{\sqrt{2}\pi}{2} (1 + \cos^2 \varphi_0) \varepsilon + O(\varepsilon^2).
\end{align*}$$  \hfill (37)

Comparing the Poincaré map (37) with (32), we regard

$$l_1 = -\frac{\sqrt{2}\pi}{2} (1 + \cos^2 \varphi_0),$$

$$l_2 = -\frac{\sqrt{2}\pi}{4} I_0 \sin 2\varphi_0,$$  \hfill (38)

$$\omega = 0.$$  

The high-order terms of $\varepsilon$ are put into $O(\varepsilon^2)$.

It is not difficult to show that, for $(I_0, \varphi_0) \in A$, $l_1 < 0$, $(\partial l_1/\partial I_0) > 0$. Based on the expressions of $l_1$ and $l_2$, we find a function $I = (1/2)\ln(1 + \cos^2 \varphi_0) + \ln I_0$, which is the one satisfying Theorem 2. In fact, there are $(\partial \hat{I}/\partial I_0) = (1/ I_0) > 0$ for $(I_0, \varphi_0) \in A$ and

$$l_1 \frac{\partial \hat{I}}{\partial \varphi_0} + l_2 \frac{\partial \hat{I}}{\partial I_0} = \frac{\sqrt{2}\pi}{2} (1 + \cos^2 \varphi_0) \cdot \frac{\sin \varphi_0 \cos \varphi_0}{1 + \cos^2 \varphi_0}$$

$$-\frac{\sqrt{2}\pi}{4} I_0 \sin 2\varphi_0 \cdot \frac{1}{I_0} = 0.$$  \hfill (39)

Moreover, since $\hat{I} = -(\ln 2/2) + \ln I_0$, $\hat{J} = \ln I_0$ and $f (I_0) \equiv \ln I_0$ monotonically increases for $I_0 > 0$, there must be $\bar{a}, \bar{b} \in (0, \varepsilon)$ such that $\hat{J}(0) < \hat{J}(\bar{a}) < \hat{J}(\bar{b}) < \hat{J}(\bar{b}) < \hat{J}(\bar{b})$. Consequently, by Theorem 2, the Poincaré map $P$ has an invariant closed curve for small enough $\varepsilon$ in another.$\psi$: $T^n(\mathbb{R}/\pi \mathbb{Z}) \rightarrow A = [0, \varepsilon] \times S^1$ of a circle, which is invariant under the map $P$. Furthermore, on this invariant curve, the map $P$ is conjugated to a rotation with number $\omega^*$.  

**Figure 3:** A periodic orbit of system (3) near the origin for $a = 10^{-4}$ and its $xyz$-coordinates in function of time $t$. 

![Graph](image-url)
\[ P \circ \psi(s) = \psi(s + \omega^*), \quad \text{with } s \pmod{\pi}, \]  
(40)

where \( \omega^* \) is not the \( \omega \) in (32), and it satisfies Diophantine condition:

\[ |\omega^* - \frac{p}{q}| \geq c|q|^{-\beta}, \]  
(41)

for all integers \( p \) and \( q \neq 0 \) where \( \beta > 0 \) and \( c > 0 \) are two constants. Then, the solutions of (21) starting at time \( t = 0 \) on this invariant curve determine a \( \pi \)-periodic cylinder in the space \( (t, \phi, \theta) \in A \times \mathbb{R} \). Denote the vector field of (21) by \( X \). Since \( X \) is time \( \pi \)-periodic, the phase space is \( A \times T^n \). Let \( \Phi^t \) with \( \Phi^0 = \Phi \) be the flow of the time-independent vector field \( (X, 1) \) on \( A \times T^n \) and define \( \psi(s, \tau) : T^n \times T^n \rightarrow A \times T^n \) by setting

\[ \psi(s, \tau) = \Phi^t(\psi(s + \tau \omega^*), 0) = (\phi^t \circ \psi(s + \tau \omega^*), \tau), \]  
(42)

where \( \phi^t \) is the flow of the vector field of \( X \) and, therefore, \( P = \phi^1 \). Then,

\[ \psi(s, \tau + \pi) = (\phi^{t+\pi} \circ \psi(s - (\tau + \pi) \omega^*), \tau + \pi) \]
\[ = (\phi^{t} \circ \phi^\pi \circ \psi(s - \tau \omega^* + \pi \omega^*), \tau + \pi) \]
\[ = (\phi^{t} \circ \psi(s - \tau \omega^*), \tau) = \psi(s, \tau) \]
\[ \psi(s + \pi, \tau) = \Phi^t(\psi(s + \pi - \tau \omega^*), 0) = \psi(s, \tau). \]  
(43)

Moreover,

\[ \Phi^{t} \circ \psi(s, \tau) = \psi(s + \omega^* t, \tau + \tau). \]  
(44)

Therefore, the torus \( \psi(T^n \times T^n) \) is quasiperiodic with the frequencies \( (\omega^*, 1) \).

We note that the existence of invariant curves for \( P \) corresponds to the existence of the invariant tori for the three-dimensional flow of (16). By the transformations performed on (3) in Section 2, it is not difficult to find that frequencies of the quasiperiodic solutions of (3) are also \( (\omega^*, 1) \) for \( \varepsilon \) or \( a \) sufficiently small. Then, we finish the proof of Theorem 1.

4. Conclusion

In this paper, we prove that, for \( a > 0 \) small, there exist infinitely many invariant tori. It eventually turns out that the solutions starting at \( t = 0 \) on the invariant tori are quasiperiodic (see Figure 2 with different \( a \)).

The results obtained here confirm what is stated in [18]: around a linearly stable periodic orbit, there are tori on which the orbits move quasiperiodically. The periodic solution of (3) in Figure 3 is verified in [10] in the form of

\[ x(t) = \sqrt{2a} \cos t + \Theta(a), \]
\[ y(t) = -\sqrt{2a} \sin t + \Theta(a), \]
\[ z(t) = \Theta(a). \]  
(45)

Together with the invariant tori found by the twist theorem, we numerically obtain the periodic orbit and the nested invariant tori around it in Figure 1. The orbit is dense and moves quasiperiodically on the invariant tori.

By the twist Theorem 2, there exist invariant curves in the neighbourhood of \( (0, 0) \). On the one hand, for each \( I_0 \in [0, \varepsilon) \), i.e.,

\[ \left( \left( \frac{x_0^2 + y_0^2}{\varepsilon} - \sqrt{2} \right) + \left( \frac{z_0}{\varepsilon} \right)^2 \right)^{1/2} \in [0, \varepsilon], \]  
(46)

where \( \varepsilon = \sqrt{a} \), we obtain an invariant torus. On the other hand, (46) implies that, for sufficiently small \( a \), once there is a initial point \( (x_0, y_0, z_0) \) satisfying (46), it corresponds to an invariant torus. Furthermore, other initial points closer to the origin than \( (x_0, y_0, z_0) \) correspond to invariant tori either. From this point, as \( \varepsilon \rightarrow 0 \), such invariant tori are infinite (see Figure 4 with different initial values for \( a = 10^{-4} \)).

Data Availability

The figure data used to support the findings of this study are included in the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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