Homogeneous vacua of (generalized) new massive gravity

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Abstract

We obtain all homogeneous solutions of new massive gravity models on $S^3$ and AdS$_3$ by extending previously known results for the cosmological topologically massive theory of gravity in three dimensions. In all cases, apart from the maximally symmetric vacua, there are axially symmetric (i.e. bi-axially squashed) as well as totally anisotropic (i.e. tri-axially squashed) metrics of a special algebraic type. Transitions among the vacua are modeled by instanton solutions of $(3 + 1)$ Hořava–Lifshitz gravity with an anisotropic scaling parameter $z = 4$.

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1. Introduction

Over the years, there has been considerable interest in toy models of gravitational physics at the classical and quantum levels by focusing, in particular, on theories in $(2 + 1)$ spacetime dimensions (for a modern overview of the subject see, for instance, [1] and references therein). Einstein gravity (with or without a cosmological constant) has no propagating degrees of freedom in three dimensions, and, as such, it appears to be of limited interest at first sight. Nevertheless, it provides a soluble model that has been studied extensively as topological (Chern–Simons) field theory using a natural reformulation in terms of the spin connection [2, 3]. Several interesting questions have been addressed in this context, including the possible resolution of classical singularities and topology changing amplitudes in the quantum theory [4]. Localized matter sources were also included and found to affect the geometry globally rather than locally, thus leading to conical singularities in spacetime [5]. Furthermore, the presence of a cosmological constant allowed the construction of AdS$_3$ black-hole solutions

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and also led to important developments in connection with two-dimensional conformal symmetries [7] as a predecessor of AdS/CFT correspondence. The interest in the subject was revived recently, while searching for conformal field theories dual to pure three-dimensional gravity with a negative cosmological constant [8].

Massive generalizations of three-dimensional gravity provide an interesting twist as they allow propagating degrees of freedom in spacetime. Topologically massive gravity is the prime example obtained by adding a gravitational Chern–Simons term to the usual Einstein–Hilbert action in spirit of topologically massive gauge theories [9, 10]. The model was extended by the addition of a cosmological constant term to cosmological topologically massive gravity and it was further generalized to three-dimensional supergravity [11]. The gravitational Chern–Simons term is odd under parity and as a result the theory exhibits a single massive propagating degree of freedom of a given helicity, whereas the other helicity mode remains massless. Topologically massive gravity appears to be a renormalizable quantum field [12] (but see also [13] for a more recent discussion), which makes it a valuable model. Various solutions have been obtained and studied over the years (see, for instance, [14] and references therein) including AdS3 black holes. The cosmological variant of the theory was also investigated in detail at the chiral point [15], and in the context of AdS/CFT correspondence [16], leading to surprisingly rich mathematical structures that are still under investigation and revived interest in the model.

Another massive generalization of three-dimensional gravity was recently proposed by adding a specific quadratic curvature term to the Einstein–Hilbert action [17, 18]. This term (to be discussed later in detail) was designed to yield upon linearization the Pauli–Fierz action for a massive propagating graviton, and the resulting theory became known as new massive gravity. This model also appears to be a unitary renormalizable quantum field theory in three dimensions [19, 20], but unlike topologically massive gravity, the new theory preserves parity, and, as a result, the gravitons acquire the same mass for both helicity states. Models of this type with quadratic curvature terms are known to be renormalizable in four spacetime dimensions [21], which, in turn, imply power counting super-renormalizability of the corresponding three-dimensional theory. Adding a cosmological constant term yields the cosmological new massive gravity. Further generalization is provided by combining the effect of the gravitational Chern–Simons and the quadratic curvature terms to the Einstein–Hilbert action (with or without a cosmological constant), thus leading to the so-called generalized massive gravity model that encompasses all previously known theories of three-dimensional gravity in a unified framework. The resulting theory exhibits by construction propagating degrees of freedom, but with different masses for the two helicity states of the graviton, which reduce consistently to the graviton modes of the topologically massive and the new massive theories of gravity in different corners of the space of coupling constants [17, 18].

Generalization to three-dimensional supergravity was subsequently considered [22], and the chiral point of the generalized massive gravity in AdS3 was investigated [23]. Also, various solutions of new massive gravity, including AdS3 black holes, have already been constructed in the literature [24, 25], and some aspects of the AdS/CFT correspondence turned out to be on par with the holographic studies of topologically massive gravity [26].

The subject of three-dimensional massive gravity provides an active area of research where new developments or applications are mostly welcome. The landscape of vacua is quite rich and it has not been explored in all generality. Various special classes of solutions already exist in the literature—we do not intend to list them all—and some general methods have been proposed for their explicit construction [27]. Certainly, solutions of new massive gravity, and its generalizations, are less studied compared to topologically massive gravity, as they involve certain fourth-order equations in three-dimensional geometry. Most notably, what is
still lacking is the construction of homogeneous solutions which generalize the maximally symmetric vacuum to anisotropic model geometries. The main purpose of this work is to investigate the class of such locally homogeneous vacua of the generalized massive gravity (with or without a cosmological constant) by focusing, in particular, to Bianchi IX metrics on $S^3$ when the theory is defined in the Euclidean regime. In this case, we will be able to solve the field equations in all generality and obtain configurations with different degree of anisotropy. As will be seen later, these solutions reduce to the homogeneous solutions of topologically massive gravity in the appropriate corner of the space of coupling constants, which have been known for a long time [28–30] (but see also [14]). Likewise, by analytic continuation, we will obtain all homogeneous metrics on AdS$_3$ in the Lorentzian version of the theory.

The interest in the solutions of three-dimensional massive gravity also stems from the fact that they provide the static solutions of $(3 + 1)$ Hořava–Lifshitz gravity. Recall that the non-relativistic theory of gravity proposed recently by Hořava [31] involves the Euclidean action of three-dimensional gravity as superpotential functional (assuming detailed balance). Using the generalized three-dimensional massive gravity model, one obtains a super-renormalizable version of Hořava–Lifshitz gravity in $(3 + 1)$ spacetime dimensions with anisotropic scaling $z = 4$ (see also [32]), whereas restriction to topologically massive gravity yields a non-relativistic theory with an anisotropic scaling parameter $z = 3$. Following [33], it is possible to consider gravitational instantons of Hořava–Lifshitz theory, which are defined as eternal solutions of the gradient flow equations following from the action of three-dimensional massive gravity. Then, instantons with $SU(2)$ isometry, which are the simplest to consider, are determined by first reducing the flow equations to three-dimensional geometries of Bianchi type IX and then classifying all eternal solutions that interpolate between the fixed points. The homogeneous vacua of generalized three-dimensional massive gravity are not only the solutions of the Hořava–Lifshitz gravity, but they also offer the end points (as fixed points of the flow lines) to support such instanton solutions. Therefore, the results we report here can also be used as a starting point to extend the methods of our previous work [33] to the construction of gravitational instantons in non-relativistic theories with a higher anisotropic scaling parameter.

The material of this paper is organized as follows. In section 2, we briefly review the theories of three-dimensional massive gravity, with increasing degree of complexity, and their field equations, setting up the notation and the framework of our study. In section 3, we introduce the Bianchi IX ansatz by considering metrics on $S^3$ with an $SU(2)$ isometry group and solve the field equations of the Euclidean generalized massive gravity. The solutions are classified and tabulated according to the geometric characteristics of the metrics, and the limiting cases of new massive gravity and topologically massive gravity are discussed separately. These results constitute the main body of our work. In section 4, the solutions are extended to the Lorentzian regime by analytic continuation of the metrics and obtain all homogeneous solutions of the generalized massive gravity on AdS$_3$. The resulting configurations are also characterized algebraically using the general classification schemes of Petrov and Serge. In section 5, the results are taken in the context of the Hořava–Lifshitz gravity, and we outline the construction and classification of the corresponding $SU(2)$ gravitational instantons when the anisotropic scaling parameter of the theory is $z = 4$. Finally, in section 6, we present the conclusions and some directions for future work.
2. Massive gravity in three dimensions

The models of three-dimensional massive gravity are based on certain higher order extensions of pure Einstein gravity. We first consider the Einstein–Hilbert action

\[ S_{\text{EH}} = \frac{1}{\kappa^2} \int d^3 x \sqrt{|g|} (R - 2\Lambda), \]

(2.1)

also including the effect of a cosmological constant \(\Lambda\), which can assume any value. The three-dimensional gravitational coupling \(\kappa\) will be normalized to 1 for convenience, but it can be easily reinstated by rescaling the other couplings. We have three massive theories of gravity that are presented in increasing order of complexity.

**Topologically massive gravity.** It is defined by adding the gravitational Chern–Simons term to the Einstein–Hilbert action, following [9, 10]:

\[ S_{\text{TMG}} = S_{\text{EH}} + \frac{\omega}{\kappa} S_{\text{CS}}, \]

(2.2)

where

\[ S_{\text{CS}} = \frac{1}{2} \int d^3 x \sqrt{|g|} \varepsilon^{\mu\nu\rho} \nabla_{\mu} \left( \frac{1}{2} R_{\nu\rho} - \frac{1}{4} R g_{\nu\rho} \right) \]

(2.3)

is written in terms of the usual Levi-Civita connection of the spacetime metric \(g\). Here, \(\varepsilon^{i32} = 1\). Clearly, the Chern–Simons term flips sign under orientation-reversing transformations and the theory is not invariant under parity.

The classical equations of motion are obtained by varying the action with respect to the metric and they read as

\[ R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \frac{\Lambda}{\kappa} g_{\mu\nu} + \frac{\omega}{\kappa} C_{\mu\nu} = 0, \]

(2.4)

where \(C_{\mu\nu}\) is the Cotton tensor of the metric \(g\), which is defined as follows:

\[ C_{\mu\nu} = \frac{\varepsilon_{\mu\rho\sigma}}{\sqrt{|g|}} \nabla_{\rho} \left( R_{\sigma\nu} - \frac{1}{4} R g_{\sigma\nu} \right) \]

(2.5)

and it is a traceless and covariantly conserved symmetric tensor. Taking the trace of equation (2.4) yields \(R = 6\Lambda\) for the classical solutions, whereas the remaining equations of motion can be cast in the form

\[ R_{\mu\nu} - \frac{1}{3} R g_{\mu\nu} + \frac{\omega}{\kappa} C_{\mu\nu} = 0 \]

(2.6)

that is most appropriate for the algebraic (Petrov–Segre) characterization of the corresponding solutions, as will be seen later.

Clearly, the maximally symmetric Einstein metrics which are conformally flat are common solutions of topologically massive gravity with the ordinary Einstein theory.

**New massive gravity.** It is defined by adding a very special quadratic curvature term to the Einstein–Hilbert action following [17, 18]:

\[ S_{\text{NMG}} = S_{\text{EH}} - \frac{1}{m^2} S_{\text{BHT}}, \]

(2.7)

where the new term (assuming the signature \(-++\) in the Lorentzian version of the theory rather than \(+---\) used in the original works)

\[ S_{\text{BHT}} = \int d^3 x \sqrt{|g|} \left( R_{\kappa\lambda} R^{\kappa\lambda} - \frac{3}{8} R^2 \right) \]

(2.8)
is denoted by the initials of its inventors. Unlike topologically massive gravity, this theory preserves parity.

The corresponding classical equations of motion are certain fourth-order equations of the form

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} = 0,$$

(2.9)

where $K_{\mu\nu}$ is the symmetric and covariantly conserved tensor:

$$K_{\mu\nu} = 2 \nabla^2 R_{\mu\nu} - \frac{1}{2} \nabla_{\mu} \nabla_{\nu} R + \frac{9}{2} R R_{\mu\nu} - 8 R_{\mu\kappa} R_{\nu\kappa} - g_{\mu\nu} \left( \frac{1}{2} \nabla^2 R - 3 R_{\kappa\lambda} R^{\kappa\lambda} + \frac{13}{8} R^2 \right).$$

(2.10)

The special feature is that its trace coincides with the Lagrangian density of $S_{BHT}$,

$$K \equiv g^{\mu\nu} K_{\mu\nu} = R_{\kappa\lambda} R^{\kappa\lambda} - \frac{3}{8} R^2,$$

(2.11)

which singles out the special value $q = -3/8$ among the more general combination of the quadratic curvature terms $R_{\kappa\lambda} R^{\kappa\lambda} + q R^2$ in the action.

As before, taking the trace of equation (2.9) yields

$$K + m^2 (R - 6\Lambda) = 0$$

for the classical solutions, whereas the remaining equations of motion can be written as a sum of two traceless tensors, namely

$$R_{\mu\nu} - \frac{1}{3} R g_{\mu\nu} - \frac{1}{2m^2} \left( K_{\mu\nu} - \frac{1}{3} K g_{\mu\nu} \right) = 0,$$

(2.12)

which is also a useful form for the algebraic classification of the corresponding metrics.

**Generalized massive gravity.** It is obtained by combining the Einstein–Hilbert action with both higher order terms in the form

$$S_{GMG} = S_{EH} + \frac{1}{\omega} S_{CS} - \frac{1}{m^2} S_{BHT},$$

(2.13)

thus, providing the most general massive theory of gravity up to four derivative terms. Clearly, it is not invariant under parity, in general, and reduces to all simpler massive gravity models in the appropriate corners of the space of coupling constants.

In this general case, the classical equations of motion take the following form:

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} - \frac{1}{2m^2} K_{\mu\nu} + \frac{1}{\omega} C_{\mu\nu} = 0$$

(2.14)

and include a mixture of second-, third- and fourth-order derivative terms, making them more intricate to study. The trace of equation (2.14) yields $K + m^2 (R - 6\Lambda) = 0$, which is the same condition as for new massive gravity, whereas the remaining components can be organized as the sum of three traceless tensors:

$$R_{\mu\nu} - \frac{1}{3} R g_{\mu\nu} - \frac{1}{2m^2} \left( K_{\mu\nu} - \frac{1}{3} K g_{\mu\nu} \right) + \frac{1}{\omega} C_{\mu\nu} = 0.$$ 

(2.15)

We conclude the general presentation of these models by considering three special limiting cases that may arise in the space of coupling constants. Pure second-order Einstein gravity arises in the limit $\omega \to 0$ with Einstein metrics as vacua. Pure third-order Cotton theory is conformal gravity that arises in the limit $\omega \to 0$ and has conformally flat metrics as vacua; it
admits an alternative Chern–Simons gauge field interpretation based on the conformal group in three dimensions [36]. Finally, a special fourth-order theory arises in the limit $m \to 0$, which has already been studied in the literature as a ghost-free model of gravitation [37]. As will be seen later, these three special limiting cases admit some very simple homogeneous solutions that form the base for other vacua. For more general values of the couplings, the vacua arise by balancing three competing terms of different orders and their form can be rather complex.

3. Homogeneous solutions on $S^3$

In this section, we construct and classify all locally homogeneous solutions of the Euclidean generalized massive gravity on $S^3$. The homogeneous vacua of new massive gravity as well as topologically massive gravity (which are already known in the literature) will follow from the general expressions as limiting cases.

3.1. Bianchi IX model geometries

First, we present some background material for homogeneous geometries with an isometry group $SU(2)$ based on the Bianchi classification (see, for instance, [38]). We will also compute the components of the curvature tensors $R_{ij}$, $C_{ij}$ and $K_{ij}$ for this class of models. Such geometries provide consistent reduction of the field equations to an algebraic system of equations of three variables that turns out to be exactly soluble.

The line element of locally homogeneous geometries on $S^3$ takes the following form:

$$ds^2 = \gamma_1 \sigma_1^2 + \gamma_2 \sigma_2^2 + \gamma_3 \sigma_3^2,$$

(3.1)

using the left-invariant Maurer–Cartan 1-forms of $SU(2)$, $\sigma_i$, which satisfy the relations

$$d\sigma_i + \frac{1}{2} \varepsilon_{ijk} \sigma_j \wedge \sigma_k = 0.$$

(3.2)

More explicitly, in terms of Euler angles ranging as $0 \leq \vartheta \leq \pi, 0 \leq \varphi \leq 2\pi$ and $0 \leq \psi \leq 4\pi$, we have the realization

$$\sigma_1 = \sin \vartheta \sin \psi \, d\varphi + \cos \psi \, d\vartheta$$

$$\sigma_2 = \sin \vartheta \cos \psi \, d\varphi - \sin \psi \, d\vartheta$$

$$\sigma_3 = \cos \vartheta \, d\varphi + d\psi.$$

(3.3)

These metrics are not isotropic in general. The isometry group is enhanced to $SU(2) \times U(1)$ when any two $\gamma_i$’s coincide, forming axisymmetric configurations, and it extends to $SU(2) \times SU(2)$ in the fully isotropic case when all $\gamma_i$’s are equal.

The Ricci tensor $R_{ij}$, the Cotton tensor $C_{ij}$ and the fourth-order tensor $K_{ij}$ given by equation (2.10) are all diagonal on this basis. Their non-vanishing components take the form

$$R_{11} = \frac{1}{2\gamma_2 \gamma_3} \left[ \gamma_1^2 - (\gamma_2 - \gamma_3)^2 \right].$$

(3.4)

$$C_{11} = -\frac{\gamma_1}{2(\gamma_1 \gamma_2 \gamma_3)^{3/2}} \left[ \gamma_1^2(2\gamma_1 - \gamma_2 - \gamma_3) - (\gamma_2 + \gamma_3)(\gamma_2 - \gamma_3)^2 \right].$$

(3.5)

$$K_{11} = -\frac{\gamma_1}{32(\gamma_1 \gamma_2 \gamma_3)^2} \left[ 21(5\gamma_1^4 - 3\gamma_2^4 - 3\gamma_3^4) - 2(\gamma_1^2 \gamma_2^2 + \gamma_1^2 \gamma_3^2 - 3\gamma_2^2 \gamma_3^2) - 20(3\gamma_1^3 (\gamma_2 + \gamma_3) - \gamma_2^3 (\gamma_1 + 3\gamma_3) - \gamma_1 \gamma_2 \gamma_3 (\gamma_1 - \gamma_2 - \gamma_3)) \right].$$

(3.6)
and there are similar expressions for the other components that follow by cyclic permutation of the indices in all three tensors. Also, the Ricci scalar curvature is given by
\[ R = \frac{1}{2\gamma_1\gamma_2\gamma_3} \left[ 2\gamma_1\gamma_2 + 2\gamma_2\gamma_3 + 2\gamma_3\gamma_1 - \gamma_1^2 - \gamma_2^2 - \gamma_3^2 \right]. \] (3.7)
whereas the trace of \( K \) is
\[ K = \frac{1}{32(\gamma_1\gamma_2\gamma_3)^2} \left[ 21(\gamma_1^4 + \gamma_2^4 + \gamma_3^4) - 2(\gamma_1^2\gamma_2^2 + \gamma_1^2\gamma_3^2 + \gamma_2^2\gamma_3^2) \right. \\
- \left. + 20(\gamma_1\gamma_2\gamma_3(\gamma_1 + \gamma_2 + \gamma_3) - \gamma_1^3(\gamma_2 + \gamma_3) - \gamma_2^3(\gamma_1 + \gamma_3) - \gamma_3^3(\gamma_1 + \gamma_2)) \right]. \] (3.8)
The expressions for the components of the Cotton tensor are obtained choosing a particular orientation on \( S^3 \). For the opposite orientation, these expressions flip sign as they are odd under parity.

3.2. Vacua of generalized massive gravity

We are now in a position to examine the reduced field equations and provide the general solution of the resulting algebraic equations for generalized massive gravity on \( S^3 \). The free parameters \( \Lambda, \omega \) and \( m^2 \) are taken to assume any real value at first. In general, we will find three different classes of metrics: maximally isotropic with \( SU(2) \times SU(2) \) isometry, axially symmetric with \( SU(2) \times U(1) \) isometry and totally anisotropic metrics with \( SU(2) \) isometry.

Restrictions on the range of free parameters will be placed in each case separately so that the corresponding metrics have a physical (Euclidean) signature.

**Isotropic solutions.** Setting \( \gamma_1 = \gamma_2 = \gamma_3 = \gamma \), the field equations reduce to a single algebraic equation which has at most two physically acceptable solutions:
\[ \gamma = \frac{1}{8\Lambda} \left( 1 \pm \sqrt{1 - \frac{\Lambda}{m^2}} \right). \] (3.9)
Note that these maximally symmetric solutions are conformally flat and they are independent of \( \omega \), since the Cotton tensor vanishes. For \( m^2 = \Lambda \), there is a single solution with \( \gamma = 1/8\Lambda \), which only makes sense for \( \Lambda > 0 \) (and hence \( m^2 > 0 \)). For \( \Lambda = 0 \), the only physically acceptable solution is \( \gamma = 1/16m^2 \), when \( m^2 > 0 \). On the other hand, for \( m^2 > \Lambda > 0 \), both solutions (3.9) are real and positive, thus leading to isotropic metrics with two different radii. Finally, for \( \Lambda > 0 > m^2 \) or \( m^2 > 0 > \Lambda \), only \( \gamma_+ \) and \( \gamma_- \), respectively, are the physical solutions. In all other cases, there are no isotropic vacua in the theory. All these solutions are clearly isotropic solutions of new massive gravity as well, since they are inert to the Chern–Simons coupling \( \omega \) by their symmetry.

**Axially symmetric solutions.** Imposing axial symmetry sets two coefficients of the metric equal, say \( \gamma_1 = \gamma_2 \neq \gamma_3 \), resulting to partially anisotropic metrics on \( S^3 \) also known as Berger spheres or bi-axially squashed spheres. By permuting the coefficients \( \gamma_i \), one can choose the axis of symmetry along any principal direction of space, which can rotate into each other by \( \mathbb{Z}_3 \) symmetry. The corresponding metric takes the following form in terms of Euler angles:
\[ ds^2 = \gamma_1 (d\theta^2 + \sin^2 \theta \, d\varphi^2) + \gamma_3 (d\psi + \cos \theta \, d\varphi)^2 \] (3.10)
and can be viewed as \( S^1 \) fibration over the base space \( S^2 \) with an squashing parameter \( \gamma_3/\gamma_1 \).

The most efficient way to construct such solutions is by expressing the free parameters \( \omega \) and \( \Lambda \) in terms of the metric coefficients and \( m^2 \); of course, for any given solution, the parameters \( \Lambda \) and \( m^2 \) are related by the trace part of the field equations, \( m^2(R - 6\Lambda) + K = 0 \).
Then, for axially symmetric metrics, one finds that the most general solution is determined by the special equations

\[
\omega = \frac{12 m^2 \gamma_1 \sqrt{\gamma_3}}{4 \gamma_1 (1 - 2 m^2 \gamma_1) - 21 \gamma_3},
\]

\[
\Lambda = \frac{64 m^2 \gamma_1^3 - 40 \gamma_1 \gamma_3 + 21 \gamma_3^2 + 16 \gamma_1^2 (1 - m^2 \gamma_3)}{192 m^2 \gamma_1^4},
\]

which are valid for any non-zero value of \(m^2\). It is possible to invert these relations to express \(\gamma = \gamma(m^2, \omega, \Lambda)\), but the resulting expressions are quite lengthy and not very illuminating in general. For the special cases of new massive gravity and topologically massive gravity, these expressions are much simpler and will be presented later.

There are at most three axially symmetric solutions of the generalized massive gravity, depending on the range of parameters, and, in particular, on the sign of \(\omega\) that flips under parity. Thus, for \(m^2 > \Lambda > 0\), there exist no axially symmetric solutions with \(\omega > 0\) and there are up to three distinct solutions with \(\omega < 0\), which, however, depend on the particular value of \(\omega\). Also, for \(0 < m^2 < \Lambda\), the field equations admit one or two axially symmetric solutions for positive or negative values of \(\omega\), respectively.

**Totally anisotropic solutions.** Configurations of this type are often called tri-axially squashed spheres, and, as it turns out, they only exist for couplings satisfying some very special relations. The first one is most conveniently described as

\[
\frac{\omega}{m^2} = \frac{(2(\gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3) - \gamma_1^2 - \gamma_2^2 - \gamma_3^2)}{2(\gamma_1^3 + \gamma_2^3 + \gamma_3^3 - (\gamma_1 + \gamma_2)(\gamma_1 + \gamma_3)(\gamma_2 + \gamma_3))} \sqrt{\gamma_1 \gamma_2 \gamma_3},
\]

with \(\gamma_1 \neq \gamma_2 \neq \gamma_3\); the cosmological constant \(\Lambda\) is not arbitrary, but it can be determined via the trace part of the field equations \(m^2(R - 6\Lambda) + K = 0\) in terms of the corresponding \(\gamma_i\). Alternatively, using the expression of the Ricci scalar curvature \((3.7)\) and introducing the following cubic combination of the metric coefficients:

\[
Y = \gamma_1^3 + \gamma_2^3 + \gamma_3^3 - (\gamma_1 + \gamma_2)(\gamma_1 + \gamma_3)(\gamma_2 + \gamma_3),
\]

we find that the totally anisotropic metrics satisfy the relation

\[
\frac{\omega}{m^2} = (\gamma_1 \gamma_2 \gamma_3)^{3/2} \frac{R}{\sqrt{Y}}.
\]

This form is particularly useful for discussing the anisotropic solutions of new massive gravity and topologically massive gravity as special cases.

It also turns out that such vacua satisfy the special curvature condition, following from the equations,

\[
5R^2 = 16m^2 \Lambda
\]

assuming non-vanishing (but finite) values of \(\Lambda\) and \(m^2\). This relation may be valid for some axially symmetric solutions as well, but not in general. It implies, in particular, that \(m^2 \Lambda > 0\) is a necessary condition for the existence of totally anisotropic solutions; as for the parameter \(\omega\), it can acquire positive or negative values without spoiling the Euclidean signature of space. Using the trace part of the field equations, we may eliminate \(\Lambda\) and arrive at the equivalent special relation \(15R^2 - 8m^2 R - 8K = 0\). Solving for \(m^2\), we obtain

\[
m^2 = \frac{8(\gamma_1 + \gamma_2 + \gamma_3)}{\gamma_1^3 + \gamma_2^3 + \gamma_3^3 - 2(\gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3)} + \frac{3(\gamma_1^3 + \gamma_2^3 + \gamma_3^3) + 26(\gamma_1 \gamma_2 + \gamma_1 \gamma_3 + \gamma_2 \gamma_3)}{8\gamma_1 \gamma_2 \gamma_3},
\]

\[8 \quad (3.17)\]
Table 1. Geometric characteristics of vacua of the generalized massive gravity.

| Ricci curvature | Chern–Simons parameter | BHT mass parameter | Totally anisotropic | Axially symmetric | Isotropic |
|-----------------|-------------------------|--------------------|---------------------|------------------|-----------|
| $R > 0$ $\omega > 0$ | $m^2 > 0$ | No | $\gamma_1 \geq \gamma_3$ | Yes |
| $R > 0$ $\omega > 0$ | $m^2 < 0$ | Yes | Yes | Yes |
| $R > 0$ $\omega < 0$ | $m^2 > 0$ | Yes | Yes | Yes |
| $R > 0$ $\omega < 0$ | $m^2 < 0$ | Yes | Yes | Yes |
| $R < 0$ $\omega > 0$ | $m^2 < 0$ | No | $\gamma_1 < \gamma_3$ | No |
| $R < 0$ $\omega < 0$ | $m^2 > 0$ | Yes | $\gamma_1 < \gamma_3$ | No |
| $R < 0$ $\omega < 0$ | $m^2 < 0$ | No | $\gamma_1 < \gamma_3$ | No |

which together with condition (3.13) (or (3.15)) provides the most efficient way for the general description of such metrics. The common points of these algebraic relations subsequently determine $\gamma_i$ in terms of the free parameters of the theory, but the resulting expressions are incredibly long and they are omitted from the presentation.

The general characteristics of all homogeneous solutions of the generalized massive gravity are summarized in table 1. All other cases do not materialize.

3.3. Vacua of new massive gravity

We now turn our attention to the corresponding solutions of new massive gravity, which are obtained by taking the limit $|\omega| \to \infty$. There are homogeneous solutions with all possible degrees of anisotropy, as in the general case.

**Isotropic solutions.** The isotropic solutions are independent of $\omega$. As such, they are identical to those of the generalized massive gravity with the same range of the parameters $m^2$ and $\Lambda$.

**Axially symmetric solutions.** In this case, equations (3.11) and (3.12) simplify and yield at most two distinct axially symmetric metrics with coefficients explicitly given by

$$\gamma_1 = \gamma_2 = \frac{4m^2 \pm \sqrt{3m^2(5m^2 + 7\Lambda)}}{m^2(21\Lambda - m^2)},$$

$$\gamma_3 = \frac{24m^2(7\Lambda - 11m^2) \pm 4(21\Lambda - 17m^2)\sqrt{3m^2(5m^2 + 7\Lambda)}}{21m^2(21\Lambda - m^2)^2}.$$ (3.18)

For a positive cosmological constant, only the solution with the plus sign is physically acceptable provided that $\Lambda > m^2 > 0$, whereas for $m^2 < -7\Lambda/5$, there are two axially symmetric solutions. Otherwise, there are no axially symmetric vacua with $\Lambda > 0$. For a negative cosmological constant, only the solution with the plus sign is physically acceptable provided that $21\Lambda < m^2 < 0$, whereas for all other values $m^2 < 0$, there are two axially symmetric solutions. Furthermore, there are no axially symmetric vacua when $m^2 > 0$ and $\Lambda < 0$.

**Totally anisotropic solutions.** The theory admits totally anisotropic solutions provided that both $m^2$ and $\Lambda$ are negative. Such solutions are required to satisfy the special condition $Y = 0$:

$$\gamma_1^3 + \gamma_2^3 + \gamma_3^3 - (\gamma_1 + \gamma_2)(\gamma_1 + \gamma_3)(\gamma_2 + \gamma_3) = 0$$

that results from equation (3.15) as $|\omega| \to \infty$. The totally anisotropic solutions have a positive Ricci scalar curvature that is related to the parameters $\Lambda$ and $m^2$ by equation (3.16), which
is left intact by the limiting procedure. Alternatively, solving for $m^2$, as in the generalized massive gravity, the totally anisotropic vacua are determined by the common points of (3.20) and (3.17) with unequal coefficients $\gamma_i$.

Thus, fixing $m^2 < 0$, we can describe all totally anisotropic vacua of new massive gravity in the parametric form. Solving for $\gamma_i(m^2, \Lambda_1)$ results into some very complicated and lengthy expressions that are also omitted from the presentation.

The general characteristics of all homogeneous solutions of new massive gravity are summarized in table 2.

### 3.4. Vacua of topologically massive gravity

The homogeneous vacua of topologically massive gravity follow from the general discussion by specializing the results to the limiting case $|m^2| \to \infty$. As before, we have solutions with all possible degrees of anisotropy that are listed below.

**Isotropic solutions.** The isotropic metrics are conformally flat, since their Cotton tensor vanishes and therefore the only solution is

$$\gamma = \frac{1}{4\Lambda},$$

provided that $\Lambda > 0$, as in pure Einstein gravity. The positive cosmological constant sets the scale for having a constant curvature (round) metric on $S^3$ with radius $\sim 1/\sqrt{\Lambda}$.

**Axially symmetric solutions.** In this case, the system of equations (3.11) and (3.12) simplifies to

$$\omega = \frac{-3\sqrt{\gamma_1}}{2\gamma_1}, \quad \Lambda = \frac{4\gamma_1 - \gamma_3}{12\gamma_1^2},$$

which, in turn, can be easily solved to yield the metric coefficients of the axially symmetric metrics as

$$\gamma_1 = \gamma_2 = \frac{9}{\omega^2 + 27\Lambda}, \quad \gamma_3 = \frac{36\omega^2}{(\omega^2 + 27\Lambda)^2}.$$

Note that these solutions always exist provided that $\omega$ assumes negative values (with the given choice of orientation made in section 3.1) without other restriction. It can be readily seen from the expression for $\Lambda$ given above that axially symmetric metrics with $4\gamma_1 > \gamma_3$ correspond to positive $\Lambda$ and, hence, to a positive Ricci scalar curvature $R$, whereas for $4\gamma_1 < \gamma_3$ the cosmological constant and the scalar Ricci curvature are both negative. This is a well-known property of the Berger spheres, as one can flip the sign of $R$ by elongating the sphere beyond a critical value. At the critical point $4\gamma_1 = \gamma_3$, the curvature vanishes and so does $\Lambda$ in the theory.

**Totally anisotropic solutions.** By the same token, the totally anisotropic solutions of topologically massive gravity can be easily obtained. Taking $|m^2| \to \infty$, we find that
Table 3. Geometric characteristics of vacua of topologically massive gravity.

| Ricci curvature | Chern–Simons parameter | Totally anisotropic | Axially symmetric | Isotropic |
|-----------------|-------------------------|---------------------|-------------------|-----------|
| $R > 0$         | $\omega > 0$            | No                  | No                | Yes       |
| $R > 0$         | $\omega < 0$            | No                  | Yes               | Yes       |
| $R = 0$         | $\omega < 0$            | Yes                 | $\gamma_1 < \gamma_3$ | No       |
| $R < 0$         | $\omega < 0$            | No                  | $\gamma_1 < \gamma_3$ | No       |

these configurations have a vanishing Ricci scalar curvature, following from equation (3.15), in which case $\Lambda = 0$ and the special condition (3.16) is still satisfied in a limiting sense, as $R = 0$. In this limit, it can also be seen that $Y = 4\gamma_1\gamma_2\gamma_3$, following from equation (3.14) by taking into account the vanishing of the Ricci scalar curvature of such metrics. In addition, equation (3.13) yields the following expression for the Chern–Simons coupling:

$$\omega = -\frac{\gamma_1 + \gamma_2 + \gamma_3}{\sqrt{\gamma_1\gamma_2\gamma_3}}, \quad (3.24)$$

which is in fact required to be negative (with the given choice of orientation) without further restriction. Thus, the totally anisotropic vacua are described as common solutions of $R = 0$ and condition (3.24) demanding that $\gamma_1 \neq \gamma_2 \neq \gamma_3$. We note that even in this limiting case it is not easy to express the metric coefficients $\gamma_i$ in terms of $\omega$ in a closed form. Note, however, that this class of metrics has a common element with the class of axially symmetric solutions the Berger sphere with coefficients $\gamma_1 = \gamma_2 = \gamma_3/4$.

The general characteristics of all homogeneous solutions of topologically massive gravity are summarized in table 3.

The classification of all homogeneous vacua of topologically massive gravity was carried out in the literature a long time ago, including the totally anisotropic solutions [28–30] (but see also [14] for an overview, as well as the more recent work [33]). This justifies the parametrization used earlier for the presentation of the classification scheme of all homogeneous solutions of the generalized massive gravity (see, in particular, equation (3.13)) and it is gratifying to see how these special results are reproduced from our general construction.

3.5. Other special limiting cases

Concluding this section, we discuss three special limiting cases that arise in the space of couplings. The homogeneous solutions we obtain in these cases are rather simple and they form the basis for the more general vacua that arise by competition of the individual terms in the general theory.

First, by taking the limit $\kappa \to 0$, we obtain pure Einstein gravity that exhibits a fully isotropic solution for $\Lambda > 0$, so that $R = 6\Lambda$, and there are no other homogeneous vacua. Next, by taking the limit $\omega \to 0$, we obtain the pure Cotton theory of conformal gravity [36], which exhibits a fully isotropic solution and a degenerate axially symmetric vacuum with $\gamma_1 = \gamma_2 = \infty$ and $\gamma_3 = 0$, which is nevertheless regular provided that the volume of space ($\sim \sqrt{\gamma_1\gamma_2\gamma_3}$) is held finite. The latter metric corresponds to a fully squashed configuration along one of the principal directions of $S^3$ and it is unique up to permutations of the axes. Finally, pure fourth-order gravity follows from the general theory in the limit $m \to 0$ [37]. In this case, by first considering the traceless part of the equations of motion, we find that there is a fully isotropic solution as well as two different axially symmetric solutions, which are
unique up to permutations of the axes. One of them is the degenerate (but regular) metric on the fully squashed $S^3$, as in the pure Cotton theory, and the other is a non-degenerate Berger sphere with $\gamma_1 = \gamma_2$ and $\gamma_3 = 4\gamma_1/21$. However, none of these metrics satisfies the trace part of the classical equations of motion, $K = 0$, and, therefore, pure fourth-order gravity has no regular homogeneous vacua; this is also consistent with the absence of a length scale in the model that can stabilize the vacua, if any.

Turning on all parameters in the generalized massive gravity allows for more complex situations that can balance the effect of different terms and produce the web of the homogeneous vacua we have described above. In all cases, it is convenient to first solve the traceless part of the classical equations of motion and then examine the constraints imposed on the vacua from the trace of the equations to obtain physically acceptable solutions with different characteristics. The traceless part of the equations of motion will also be used in the next section to provide an algebraic classification of the homogeneous metrics on $\text{AdS}_3$ following by analytic continuation.

4. Homogeneous solutions on $\text{AdS}_3$

Analytic continuation of the squashed spheres yields homogeneous solutions of massive gravity on $\text{AdS}_3$ with the Lorentzian signature. $S^3$ is an $S^1$ fibration over $S^2$ and, therefore, there are two inequivalent ways to obtain squashed metrics on $\text{AdS}_3$ depending on the choice of time-like direction, $\tau$. One is associated with time-like squashed metrics by viewing $\text{AdS}_3$ as a time-like fibration over the hyperbolic plane $H_2$ and the other to space-like squashed metrics by viewing $\text{AdS}_3$ as a space-like fibration over $\text{AdS}_2$ space. We will consider both possibilities below and indicate how the classification of homogeneous vacua on $S^3$ carry to homogeneous metrics on $\text{AdS}_3$ with the appropriate choice of coupling constants in the theory.

First, we consider the following analytic continuation and define coordinates $\tau$, $\rho$ and $z$ as

$$\psi = \tau, \quad \vartheta = \frac{\pi}{2} - i \rho, \quad \varphi = -iz.$$ (4.1)

Then, the metric on $\text{AdS}_3$ with positive definite coefficients $\gamma_1, \gamma_2, \gamma_3$ takes the general form

$$ds^2 = \gamma_1 (\cos \tau \, d\rho + \sin \tau \, \cosh \rho \, dz)^2 + \gamma_2 (\sin \tau \, d\rho - \cos \tau \, \cosh \rho \, dz)^2 - \gamma_3 (d\tau + \sinh \rho \, dz)^2.$$ (4.2)

after flipping the overall sign of the metric to have the signature $- + +$. For axially symmetric configurations with $\gamma_1 = \gamma_2$, it specializes to

$$ds^2 = \gamma_1 (d\rho^2 + \cosh^2 \rho \, dz^2) - \gamma_3 (d\tau + \sinh \rho \, dz)^2.$$ (4.3)

This corresponds to the case of time-like squashing, where the base space is $H_2$ with metric $d\rho^2 + \cosh^2 \rho \, dz^2$. If we had considered, instead, the analytic continuation $\vartheta = i \rho$, $\psi = \tau$ and $\varphi = z$, the hyperbolic trigonometric functions $\sinh \rho$ and $\cosh \rho$ would have been exchanged, resulting to time-like squashing of $\text{AdS}_3$ over the base space $H_2$ with metric $d\rho^2 + \sinh^2 \rho \, dz^2$. The two choices are clearly related to each other as they correspond to different coordinate patches on $H_2$ (often called hyperbolic and elliptic solutions, respectively). Here, we choose to work with the former as it provides global coordinates in space.

Next, we consider another analytic continuation by defining coordinates $\tau$, $\rho$ and $z$ as follows:

$$\varphi = \tau, \quad \vartheta = \frac{\pi}{2} - i \rho, \quad \psi = iz.$$ (4.4)

which provide a different choice of the time-like direction as the role of $\tau$ and $z$ coordinates is exchanged. Then, the general homogeneous metric on $\text{AdS}_3$ takes the form
\[ ds^2 = \gamma_1 (\cosh z \, d\rho - \sinh z \, \cosh \rho \, dr)^2 - \gamma_2 (\sinh z \, d\rho - \cosh z \, \cosh \rho \, dr)^2 \\
+ \gamma_3 (dz + \sinh \rho \, dr)^2, \quad (4.5) \]

which specializes for \( \gamma_1 = \gamma_2 \) to the bi-axially squashed metrics

\[ ds^2 = \gamma_1 (d\rho^2 - \cosh^2 \rho \, dr^2) + \gamma_3 (dz + \sinh \rho \, dr)^2. \quad (4.6) \]

As before, we also flip the overall sign of the metric to have the signature \(-++\). These correspond to metrics on AdS_3 with space-like squashing, since the base space is AdS_2 with metric \( d\rho^2 - \cosh^2 \rho \, dr^2 \). We also note for completeness that another choice of coordinate patch on base space with metric \( d\rho^2 - \sinh^2 \rho \, dr^2 \) would have resulted from the analytic continuation \( \vartheta = i\rho, \varphi = \tau \) and \( \psi = z \).

In all cases we obtain AdS_3 vacua with coefficients and parameters given as before for all different types of squashing in the generalized theory of three-dimensional massive gravity and its simpler variants (new and topologically massive gravity), thus making the repetition of equations obsolete. The only difference that should be taken into account, as compared to the corresponding expressions for the homogeneous vacua found in section 3, is that the cosmological constant \( \Lambda_1 \) should be replaced by \(-\Lambda_1 \), since we have chosen the signature \(-++\) on AdS_3 using a sign flip of the metric after analytic continuation; if we had chosen to work with the signature \(+--\) this would not be required. Likewise, the other parameters \( \omega \) and \( m^2 \) also flip sign and they should be replaced by \(-\omega \) and \(-m^2 \). Note, however, that the time-like and space-like squashed metrics are mutually related by exchanging the role of \( \varphi \) and \( \psi \) coordinates on \( S^3 \) (prior to analytic continuation), which in turn imply a change of orientation. Thus, if \( \omega \) is replaced by \(-\omega \) in the case of time-like squashed metrics on AdS_3, as explained above, \( \omega \) will not flip sign in the space-like squashed metrics. With these explanations in mind, we obtain a complete classification of all homogeneous metrics on AdS_3 with an \( SU(1, 1) \) isometry group. Their existence and tabulation as time-like and space-like squashed vacua follows easily from the corresponding tables found in section 3 with the appropriate range of parameters. For the simpler case of topologically massive gravity, which has been studied for a long time, the results are in agreement with those reported in earlier works on the subject [29, 30] (but see also [14] for an overview and many more references to the literature).

Concluding this section, we comment on the algebraic characterization of the spacetime metrics on AdS_3 based on the Petrov and Segre classification (see, for instance [38] for the general scheme as it was initially developed in four spacetime dimensions). In three dimensions, the Petrov classification refers to the Cotton tensor \( C^i_{\ j} \) [39–43] and the Segre classification refers to the traceless Ricci tensor \( S^i_{\ j} = R^i_{\ j} - \delta^i_{\ j} R/3 \) [40, 41, 43, 44]. In either case one views these second rank tensors as linear maps between 3-vectors which are classified according to the number of distinct eigenvalues and the spacetime character of their eigenvectors; we refer to the literature for the details and notations used for the different classes of spacetimes.

Topologically massive gravity is rather special in this context, because the Petrov and Segre classifications coincide by the traceless part of the classical equations of motion (2.6), although the notation used in the literature depends on the particular scheme. It is rather instructive to briefly summarize the results of the algebraic characterization of all homogeneous vacua of topologically massive gravity, following [14]. In this case, determining the eigenvalues of \( S^i_{\ j} \) and their multiplicities is equivalent to finding the scalar invariants

\[ I = S^i_{\ j} S^j_{\ i} = \text{tr}(S^2), \quad J = S^i_{\ j} S^j_{\ k} S^k_{\ i} = \text{tr}(S^3). \quad (4.7) \]
Bi-axially squashed AdS$_3$ metrics are of Petrov type $D$ and one often distinguishes between the time-like and space-like squashed metrics using the notation $D_t$ and $D_s$, respectively. These solutions are denoted by $[11,1]$ and $[1,1,1]$, respectively, in the Segre classification scheme and satisfy the relation $I^3 = 6J^2 \neq 0$. Isotropic solutions are of Petrov type $O$ and of Segre type $[11,1]$ satisfying the special relation $I = J = 0$. Finally, totally anisotropic metrics are of Petrov type $I_R$ and of Segre type $[11,1]$ satisfying the relation $I^3 > 6J^2$. Of course, it is also possible to have other solutions of more general algebraic type but they fall outside the class of homogeneous metrics and we are not going to discuss these here.

For the generalized massive gravity, and its limiting theory of new massive gravity, the three-dimensional analogs of the Petrov and Segre classifications are distinct because $C_{ij}$ is no longer proportional to $S_{ij}$. Still one can classify their homogeneous solutions into algebraic types, which turn out to be identical to those appearing in topologically massive gravity. This can be explicitly checked case by case for AdS$_3$ vacua with all possible degrees of anisotropy and verify that they are of Petrov types $O, D (D_t$ or $D_s)$ and $I_R$. Likewise, one can characterize these vacua by their Segre type and find exactly the same classes in the notation used above.

5. Applications to $z = 4$ Hořava–Lifshitz gravity

We will discuss some applications of our results to Hořava–Lifshitz gravity in $(3 + 1)$ dimensions. This is a non-relativistic theory of gravitation that has been proposed as ultraviolet completion of Einstein’s theory [31], but it also serves as a toy model for transitions among vacua of three-dimensional gravity in spirit of Onsager–Machlup theory for non-equilibrium processes [45].

Spacetime is assumed to be $M_4 = \mathbb{R} \times \Sigma_3$ and the theory is defined using the ADM (Arnowitt–Deser–Misner) decomposition of the metric

$$ds^2 = -N^2 dt^2 + g_{ij}(dx^i + N^i dt)(dx^j + N^j dt).$$ (5.1)

The metric on the spatial slices $\Sigma_3$ is $g_{ij}$, whereas $N$ and $N^i$ are the lapse and shift functions, respectively, which depend on all spacetime coordinates, in general. The infinite-dimensional space of all three-dimensional Riemannian metrics $g_{ij}$ is called superspace and it is endowed with a metric

$$G^{ijk\ell} = \frac{1}{2}(g^{ij}g^{\ell k} + g^{i\ell}g^{jk}) - \lambda g^{ij}g^{\ell k},$$ (5.2)

that generalizes the standard DeWitt metric using an arbitrary parameter $\lambda$ (other than 1). The inverse metric in superspace is

$$ G_{ijk\ell} = \frac{1}{2}(g_{ik}g_{\ell j} + g_{ij}g_{\ell k}) - \frac{\lambda}{3\lambda - 1}g_{ij}g_{\ell k}$$ (5.3)

so that

$$ G^{ijk\ell} G_{k\ell mn} = \frac{1}{2}(\delta^i_m\delta^j_n + \delta^j_m\delta^i_n).$$ (5.4)

The action of Hořava–Lifshitz gravity in $(3 + 1)$ dimensions is written as a sum of kinetic and potential terms. Assuming detailed balance, which is important for our discussion, the action takes the form [31]

$$S_{HL} = \frac{2}{\kappa^2} \int dt \int d^3x \sqrt{N} K_{ij} G^{ijk\ell} K_{k\ell} - \frac{\kappa^2}{2} \int dt \int d^3x \sqrt{N} E^{ij} G_{ij\ell\ell} E^{\ell\ell},$$ (5.5)

$^3$ We use Latin indices $i, j, \ldots$ to indicate that $\Sigma_3$ is always Riemannian here. Before we used Greek indices $\mu, \nu, \ldots$ to allow for both Riemannian and pseudo-Riemannian metrics in the discussion of three-dimensional gravitational theories.
where $K_{ij}$ is the second fundamental form measuring the extrinsic curvature of the spatial slices $\Sigma$ at constant $t$ (not to be confused with the fourth-order tensor $K_{ij}$ of new massive gravity):

$$K_{ij} = \frac{1}{2N} \left( \partial_t g_{ij} - \nabla_i N_j - \nabla_j N_i \right) \quad (5.6)$$

and

$$E^{ij} = -\frac{1}{2\sqrt{g}} \frac{\delta W[g]}{\delta g_{ij}}. \quad (5.7)$$

The four-dimensional gravitational coupling is $\kappa$. The kinetic term contains two time derivatives of the metric $g_{ij}$, and, as such, it is identical to general relativity in the canonical form (though $\lambda$ is taken arbitrary here). The potential term is different, however, as it is derived from a superpotential functional $W$ that is chosen appropriately to render the theory power-counting renormalizable.

In the following, we choose $W[g]$ to be the action functional of Euclidean three-dimensional massive gravity, setting, in general, $W = S_{\text{CMG}}$. Then, the theory has an anisotropy scaling parameter that is $z = 4$, since the highest order term in the potential of $S_{\text{HL}}$ is $K_{ij}K^{ij}$ followed by $C_{ij}C^{ij}$ and $R_{ij}R^{ij}$ as well as other subleading cross terms [32]. We will restrict attention to the so-called projectable case of Hořava–Lifshitz gravity, meaning that the lapse function $N$ associated with the freedom of time reparametrization is restricted to be a function of $t$, whereas the shift functions $N_i$ associated with diffeomorphisms of $\Sigma_3$ can depend on all spacetime coordinates. In view of the applications that will be discussed next, we choose

$$N(t) = 1, \quad N^i(t, x) = 0, \quad (5.8)$$

without great loss of generality.

It is clear that the vacua of three-dimensional massive gravity provide static (i.e. $t$-independent) solutions of Hořava–Lifshitz gravity, which is one of the applications. More importantly, these vacua can also be used to support instanton solutions that interpolate smoothly between different critical points of $W[g]$ and, hence, of the potential functional of the four-dimensional action $S_{\text{HL}}$. Although the description of instanton solutions will be quite general here, following earlier work on the subject [33], specialization to homogeneous vacua of the generalized massive gravity on $\Sigma_3 \simeq S^3$ leads to a classification scheme for all $SU(2)$ gravitational instantons of Hořava–Lifshitz theory with an anisotropy scaling parameter $z = 4$. It also puts the results of section 3 in a wider context and makes them the basis for future developments. The rest of this section outlines this construction, but more details will be presented elsewhere [46].

Let us now consider the Euclidean action of Hořava–Lifshitz theory which is obtained by analytic continuation in time. Furthermore, for technical reasons that will become apparent in a moment, we restrict the parameter $\lambda$ of the superspace metric in the range

$$\lambda < 1/3 \quad (5.9)$$

so that $G^{ij\ell}$ is positive definite. Also, $\Sigma_3$ is assumed to be compact with no boundary, as in the typical case $\Sigma_3 \simeq S^3$ we are considering here. Then, the Euclidean action can be manipulated by standard elementary methods as follows [33]:

$$S_{\text{Eucl}} = \frac{2}{\kappa^2} \int dt \, d^3x \sqrt{g} K_{ij} G^{ij\ell} K_{\ell\ell} + \frac{\kappa^2}{2} \int dt \, d^3x \sqrt{g} E^{ij} G_{ij\ell\ell} E^{\ell\ell}$$

$$= \frac{2}{\kappa^2} \int dt \, d^3x \sqrt{g} \left( K_{ij} \pm \frac{\kappa^2}{2} G_{ijmn} E^{mn} \right) G^{ij\ell\ell} \left( K_{\ell\ell} \pm \frac{\kappa^2}{2} G_{\ell\ell\ell} E^{\ell\ell} \right)$$

$$\equiv 2 \int dt \, d^3x \sqrt{g} K_{ij} E^{ij}. \quad (5.10)$$
taking into proper account all boundary terms. Thus, for a positive definite superspace metric, the Euclidean action appears to be bounded from below by

$$S_{\text{Eucl}}^{\text{HL}} \geq \mp2 \int dt \, d^3x \sqrt{g} K_{ij} E^{ij} = \mp \int dt \, d^3x \sqrt{g} \partial_t g_{ij} = \pm \frac{1}{2} \int dt \frac{dW}{dt}.$$  \hspace{1cm} (5.11)

Extrema of the action are provided by configurations satisfying the following special equations that are first order in time:

$$K_{ij} \equiv \frac{1}{2} \partial_t g_{ij} = \mp \frac{\kappa^2}{2} G_{ijmn} E^{mn},$$ \hspace{1cm} (5.12)

which are the defining equations of instantons.

As the spatial slices evolve in Euclidean time following (5.12), the superpotential functional $W$ changes monotonically. This is easily seen by considering

$$\frac{dW}{dt} = -2 \int d^3x \sqrt{g} E^{ij} \partial_t g_{ij} = \pm 2\kappa^2 \int d^3x \sqrt{g} E^{ij} G_{ijkl} E^{kl},$$ \hspace{1cm} (5.13)

which is the integral of a quadratic quantity when $\lambda < 1/3$ and, therefore, it increases or decreases monotonically depending on the overall sign. Using this observation and by taking the time integral of equation (5.13), it turns out that the lower bound of the Euclidean action $S_{\text{HL}}$ is always positive and it is saturated by the special configurations (5.12). Then, the instanton action is

$$S_{\text{Eucl}}^{\text{HL}} = \frac{1}{2|\Delta W|},$$ \hspace{1cm} (5.14)

where $\Delta W$ denotes the difference of the corresponding values of $W$ at the two end points of the time interval that supports such solutions. Instantons and anti-instantons are associated with the two different sign options, and, therefore, they are mutually related by reversing the arrow of time.

As in ordinary instanton physics, it is also appropriate here to consider solutions with finite Euclidean action only. This is possible provided that there are solutions of equation (5.12) that extrapolate smoothly between degenerate minima of the potential for, otherwise, $W$ may become infinite. This restriction is also imposed by the spacetime interpretation of the solutions of Euclidean Hořava–Lifshitz gravity in order to obtain complete spaces with non-singular metrics. Thus, instantons are naturally associated with eternal solutions of certain higher order geometric flow equations, which are gradient flows of $W$ according to equation (5.12). In particular, setting $W = S_{\text{GMG}}$, we obtain the following evolution equations:

$$\partial_t g_{ij} = -\frac{\kappa^2}{2} \left( R_{ij} - \frac{2\lambda - 1}{2(3\lambda - 1)} R g_{ij} - \frac{\Lambda}{3\lambda - 1} g_{ij} \right) - \frac{\kappa^2}{2\omega} C_{ij} + \frac{\kappa^2}{4m^2} \left( K_{ij} - \frac{\lambda}{3\lambda - 1} K \right),$$ \hspace{1cm} (5.15)

choosing for definiteness one of the two sign options. The instanton solutions correspond to trajectories that connect continuously any two fixed points of the flow, without encountering singularities, as $-\infty < t < +\infty$. They are solely selected by their boundary conditions, having spatial slices with zero extrinsic curvature (equal to the normal derivative $\partial_t g_{ij}$) at the two end points of their Euclidean lifetime\(^4\). All other flow lines of the geometric evolution equation (5.12) do not qualify as instantons and, in general, they become extinct (typically in finite time) by encountering singularities, thus leading to the infinite action $S_{\text{Eucl}}^{\text{HL}}$; they are discarded from our general construction.

\(^4\) Despite appearances, the fixed points of the flow equation (5.15) are independent of $\lambda$ and coincide with the vacua of the generalized massive gravity. The parameter $\lambda$ only affects the form of the flow lines that define the instantons.
The explicit construction and classification of all instanton solutions of $z = 4$ Hořava–Lifshitz theory relies heavily on two open problems. The first is the classification of all vacua of the generalized massive gravity, which serve as end points of the interpolating instanton metrics. The second is the general behavior of higher order curvature flows, as (5.15), and the possible occurrence of singularities that may inflict the trajectories. The standard methods that are available for studying second-order equations are no longer applicable and even the short-time existence of solutions is now questionable, in general. Focusing on homogeneous vacua offers a mini-superspace model to study these problems and obtain concrete results. We have obtained a complete classification of all fixed points as classical solutions on $S^3$ with $SU(2)$ symmetry and at the same time the flow equations reduce consistently to a closed system of ordinary differential equations for the metric coefficients $\gamma_i$ as functions of time. Then, eternal solutions of these equations are in one-to-one correspondence with the $SU(2)$ gravitational instanton solutions of $z = 4$ Hořava–Lifshitz theory. Explicit constructions are possible by extending previous results [33] to fourth-order flows, but the details are more complicated and they will be presented in a separate paper.

The instanton solutions of Hořava–Lifshitz gravity can also be used to describe off-shell transitions among the many different vacua that populate the landscape of massive gravity models. This alternative interpretation is in spirit of Onsager–Machlup theory for non-equilibrium processes in thermodynamics [45]. In this general context, $W$ is the entropy function that changes monotonically in time and it is proportional to the logarithm of the probability of a given fluctuation. The gradient of $W$ is the thermodynamic force measuring the tendency of a system to seek equilibrium. Linearization of the flow equations around the fixed points describe small fluctuations away from equilibrium states, whereas the instanton solutions incorporate nonlinear effects for large transitions between different states of the system. It will be interesting to strengthen the analogy between non-equilibrium processes and geometric flow equations by focusing, in particular, to three-dimensional massive gravity models as a working example and explore its higher dimensional origin by embedding the theory in string or M-theory framework. A renormalization group approach to the instanton solutions might also emerge from this study.

6. Conclusions

We have classified all homogeneous vacua of (generalized) new massive gravity in three dimensions using the Bianchi IX ansatz for Riemannian metrics on $S^3$. We have also obtained the corresponding AdS$_3$ metrics by analytic continuation and characterized them algebraically using the Petrov and Segre schemes. Our results provide generalization of the homogeneous vacua of topologically massive gravity in the presence of a new quadratic curvature term in the action based on the recent proposal [17]. In all cases we found that homogeneous metrics with different degrees of anisotropy can be realized as vacua in certain regions of the parameter space of couplings. The most exotic case is provided by the totally anisotropic (i.e. tri-axially squashed) metrics, which have a special Ricci scalar curvature. Although the explicit form of the metric coefficients are rather cumbersome to present, in general, as functions of the couplings, the action takes particularly a simple form, as can be found (but not shown here). These critical values of the action can be used to compute the instanton action of interpolating configurations among the different vacua and associate them with a probability measure by advancing further the connections with higher order geometric flows.

It should be emphasized that these homogeneous solutions coexist in certain regions of the parameter space of couplings, as summarized in tables 1, 2 and 3. Thus, fixing the couplings $\omega$, $m^2$ and $\Lambda$, one may have isotropic, axially symmetric and totally anisotropic
configurations as distinct classical solutions of three-dimensional gravity. In other regions of the parameter space only some of these vacua can coexist. They all provide the landscape of homogeneous vacua in mini-superspace and, as such, they are not continuously connected to each other. Some of these vacua can coalesce by varying the couplings, in which case their defining relations coincide for special values of $\omega$ and $m^2$, as can be readily seen from the equations. These remarks apply to generalized new massive gravity by extending previously known results for topologically massive gravity. Finally, Hořava–Lifshitz gravity was used as a toy model to study off-shell transitions among these vacua. In this context, we were not concerned with the shortcomings and problems of such alternative theories of gravitation (see for instance [47] for a recent overview and references therein), but we certainly have to face them in detail when applying our results to Euclidean gravity. We hope to return to these problems elsewhere.

In future work, it will be interesting to consider other classes of solutions of the generalized massive gravity and explore more regions in the landscape of vacua. Although this is a rather intricate problem, it is the simplest to address in the context of gravitational theories with propagating degrees of freedom. Another important question is the possibility of embedding such three-dimensional theories in string or M-theory and use them to investigate the structure of the corresponding spacetime configurations in higher dimensions. Although some partial results exist in this direction, in particular for topologically massive gravity [48], the general framework is still lacking. We hope to be able to report on this and related issues elsewhere.

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