Control of Hierarchical Networks by Coupling to an External Chaotic System

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Abstract: We explore the behaviour of chaotic oscillators in hierarchical networks coupled to an external chaotic system whose intrinsic dynamics is dissimilar to the other oscillators in the network. Specifically, each oscillator couples to the mean-field of the oscillators below it in the hierarchy, and couples diffusively to the oscillator above it in the hierarchy. We find that coupling to one dissimilar external system manages to suppress the chaotic dynamics of all the oscillators in the network at sufficiently high coupling strength. This holds true irrespective of whether the connection to the external system is direct or indirect through oscillators at another level in the hierarchy. Investigating the synchronization properties show that the oscillators have the same steady state at a particular level of hierarchy, whereas the steady state varies across different hierarchical levels. We quantify the efficacy of control by estimating the fraction of random initial states that go to fixed points, a measure analogous to basin stability. These quantitative results indicate the easy controllability of hierarchical networks of chaotic oscillators by an external chaotic system, thereby suggesting a potent method that may help design control strategies.

Keywords: Complex networks, Control of chaos, Hierarchical networks, Bifurcation, Basin stability, Synchronization, Suppression of oscillation

1. INTRODUCTION

Science of complex systems is an active area of research that has helped in understanding large interactive systems ranging from man-made systems to natural systems. Examples of such systems include Josephson junction Hadley et al. (1988b,a), chemical reactions Schreiber and Marek (1982); Crowley and Epstein (1989), semiconductor laser Varangis et al. (1997); Hohl et al. (1997), power grids Menck and Kurths (2012), neurons Golomb et al. (2001); Li et al. (2006), circadian pacemaker Daan and Berde (1978) and many other biological processes. These complex systems consist of oscillators, which when uncoupled may be periodic, quasi-periodic or chaotic. On coupling they can exhibit a variety of dynamical behaviour such as synchronization Boccaletti et al. (2002); Arenas et al. (2008) and oscillation quenching or suppression of oscillation.

Suppression of oscillation, in particular, is a phenomenon of special interest for stabilization of steady states in complex systems. Such fixed point dynamics serve as a control mechanism, for instance in coupled lasers where the stabilization plays an important role. On the other hand it is relevant in the study of pathological cases of neuronal disorders such as Alzheimers and Parkinsons disease Selkoe (2000); Tanzi (2005); Caughey and Lansbury Jr (2003). Because of such important application, research on suppression of oscillation has been very active over the years.

In this work, we present control of hierarchical network by coupling to an external chaotic system. This study is an extension of our earlier work that demonstrated the suppression of chaos through coupling to an external chaotic system Chaurasia and Sinha (2017). In that work we investigated the behaviour of an ensemble of chaotic oscillators, coupled diffusively only to one external chaotic system, with the external system coupled to the oscillator group via the mean-field of the oscillators. Remarkably, we found that this external system manages to successfully steer a group of chaotic oscillators onto steady states, at sufficiently high coupling strength, when it is dissimilar to the group of oscillators, rather than identical. The results were independent of number of oscillators connected to the external system.

In this work we consider a hierarchical network, with intrinsically chaotic oscillators at different levels of the hierarchy, and study the emergent dynamics of the network. Figs. 1 and 2 show a schematic such networks, where level 0 represent the external system, which is dissimilar to all other oscillators. The oscillators at each level of hierarchy are connected to oscillators in the level above and below it in the hierarchy. For instance, for a network with two levels (cf. Fig. 1), the oscillators at level 1 are connected to oscillators at level 0 (one level above in the hierarchy) and level 2 (one level below in the hierarchy).

Consider networks with k levels of hierarchy. Examples of k = 2 and k = 3 are shown schematically in Figs. 1
will denote it by the special symbols:

Now we describe the dynamics of the oscillators at different levels of the hierarchy is given by

The coupling of the oscillators at level \( l = 1 \) of the hierarchy is given as follows:

The coupling of the oscillators at the intermediate levels \( l = 2, \ldots k - 1 \) of the hierarchy is given as follows:

The coupling strength is denoted by \( \varepsilon \).

In this work, we consider a Lorenz system in the chaotic region as the external system, i.e. at level 0 of the hierarchical network, described by the dynamical equations:

All other oscillators of the network are Rössler oscillators in the chaotic region, with dynamical equations:

Specifically, we consider the parameters of the Lorenz system to be \( \sigma = 10.0, \beta = 8.0/3.0 \), and \( r = 25.0 \) in Eq.5 and the parameters of the Rössler oscillators to be

\[ f(x,y,z) = - (\omega + \delta (x^2 + y^2)) y - z \]
\[ g(x,y,z) = (\omega + \delta (x^2 + y^2)) x + a y \]
\[ h(x,y,z) = b + z(x - c) \]
ω = 0.41, δ = 0.0026, a = 0.15, b = 0.4 and c = 8.4 in Eq. 6. These parameter sets ensure that each oscillator is in the chaotic region when uncoupled. So in this hierarchical network, the external system at level \( l = 0 \) is a chaotic Lorenz system, while the rest of the oscillators at levels \( l \neq 0 \) are chaotic Rössler oscillators.

2. EMERGENT CONTROLLED DYNAMICS

First we present results for a network with two levels of hierarchy (cf. Fig. 1). Fig. 3 shows the bifurcation diagram of the external Lorenz system and a representative Rössler oscillator at level 1 and 2 of the hierarchical network. We find that the oscillators at all levels of hierarchy can be controlled to a fixed point, at sufficiently high coupling strength, by an external chaotic Lorenz system. The results are unchanged on increasing the number of oscillators at level 2 (i.e. increasing \( N_2 \)), indicating that the control is independent of system size.

![Fig. 3. Bifurcation diagram, with respect to coupling strength \( \epsilon \), of one representative oscillator from the two levels of hierarchy (\( l = 1, 2 \) and the external system (\( l = 0 \)). Colours correspond to the hierarchy level \( l \) of oscillators, as given in Fig. 1. In all bifurcation diagrams (including ones below), we show the \( x \)-variable of the Poincaré section of the phase curves of the oscillators at \( y = y_{mid} \), where \( y_{mid} \) is the midpoint along \( y \)-axis of the span of the oscillator.

Now we check the generality of our results, by investigating a network with an additional level of hierarchy, namely three levels of hierarchy (cf. fig2). Fig. 4 shows the bifurcation diagram of one of the representative oscillators from each level of hierarchy. The emergent behaviour is same for any number of oscillators attached to level 2, forming level 3 of hierarchy. At very low coupling strength, all the oscillators yield their intrinsic chaotic dynamics. Increasing the coupling strength distorts the dynamics, with distortion being different at different levels of hierarchy. When coupling strength is sufficiently high (\( \epsilon \sim 0.4 \)), there is sudden transition from oscillations to fixed points. These fixed points are different at each level of hierarchy. However note that all the oscillations are suppressed at exactly the same steady state for all oscillators at a particular level in the hierarchy.

![Fig. 4. Bifurcation diagram, with respect to coupling strength \( \epsilon \), of one representative oscillator from the three levels of hierarchy, and the external system. Colours correspond to the hierarchy level of the oscillators, as given in Fig. 2.](image)

Fig. 5 shows the projection of phase portraits on \( x - y \) plane of one representative oscillator from each level of hierarchy at intermediate coupling strength \( \epsilon = 0.2 \). The colours correspond to the hierarchy level as illustrated in the schematic diagrams. It is evident that the oscillators at the level nearest to the external system are most distorted. As the oscillators move away from the external system (\( l = 0 \) in the level of hierarchy), the oscillations become close to their intrinsic dynamics. The black dots in the figure show the fixed points obtained at high coupling strength (\( \epsilon = 0.6 \)).

![Fig. 5. Phase portraits of the external system and one representative oscillator from each level of hierarchy. The coloured phase portraits are obtained for \( \epsilon = 0.2 \), and the black dots are the fixed points obtained for \( \epsilon = 0.6 \). Different colours of the phase portraits correspond to the hierarchy level of oscillators, as given in Fig. 2.](image)
3. SYNCHRONIZATION

Now we study the synchronization properties of the system in the case of a network with three hierarchical levels. It is evident from Fig. 4 that the oscillators at each level of hierarchy attain different fixed points, i.e. there is no synchronization across different levels, though there is synchronization within a level. Here we calculate the advent of this synchronization within a level of hierarchy in the network, as a function of coupling strength.

![Fig. 6. Synchronization error ⟨Z⟩ of the Rössler oscillators, with respect to coupling strength, averaged over 1000 initial conditions. Different colours correspond to the hierarchy level of oscillators as shown in Fig. 2.](image)

We calculate the synchronization error of the oscillators at particular levels, averaged over time T, given by:

\[
Z = \frac{1}{T} \sum_{t} \sqrt{(\bar{x})^2_t - (\bar{x})^2_t}
\]  

(7)

where \((\bar{x})_t = \sum_{i=1}^{N_l} x_i(t)\) and \((\bar{x}^2)_t = \sum_{i=1}^{N_l} (x_i(t))^2\) are the average value of \(x\) and \(x^2\) of the oscillators, at an instant of time \(t\), at level \(l\) of the hierarchy. Further we average \(Z\) over different initial states to obtain an ensemble averaged synchronization error \langle Z \rangle.

Fig. 6 shows the synchronization error \(\langle Z \rangle\) with respect to coupling strength. It is evident that oscillators within each level of hierarchy are synchronized when the oscillations are suppressed, i.e. oscillators at the same level evolve to the same fixed points, even though there is no direct coupling between them.

4. BASIN STABILITY OF THE SPATIOTEMPORAL FIXED POINTS

Suppression of oscillations is achieved by each oscillator in the network at the same coupling strength, i.e. control to fixed point occurs at the same critical coupling strength \(\varepsilon_c\) for all oscillators at all levels of hierarchy. However, the value \(\varepsilon_c\) may vary with initial conditions.

We now quantify the efficacy of control to steady states by uniformly sampling a large set of random initial conditions and estimating the fraction BS\(_{\text{fixed}}\) of initial states attracted to spatiotemporal fixed points. This measure is analogous to recently used measures of basin stability Menck et al. (2013) and indicates the size of the basin of attraction for a spatiotemporal fixed point state. BS\(_{\text{fixed}}\) ∼ 1 suggests that the fixed point state is globally attracting, while BS\(_{\text{fixed}}\) ∼ 0 indicates that almost no initial states evolve to stable fixed states.

![Fig. 7. Dependence of the fraction of initial conditions attracted to the spatiotemporal fixed point, BS\(_{\text{fixed}}\), on the coupling strength \(\varepsilon\), averaged over 5000 initial conditions.](image)

We show the variation of BS\(_{\text{fixed}}\) as a function of coupling strength \(\langle Z \rangle\) (cf. Fig. 7). For \(\varepsilon_c < 0.38\), system does not evolve to the fixed point state for any initial condition, while for \(\varepsilon_c > 0.5\) the network gets attracted to a spatiotemporal fixed point for all initial conditions. In the intermediate range of coupling strengths, the network has a finite probability to get attracted to the spatiotemporal fixed point state from a generic random initial state, as indicated by BS\(_{\text{fixed}}\) > 0.

5. CONCLUSION

We investigated the behaviour of an ensemble of chaotic oscillators coupled in a hierarchical network, where at the zeroth level of the hierarchy we have one chaotic external system that is dissimilar to the rest of the oscillators in the network. We have shown that at sufficiently high coupling strengths, this one external system can suppress the oscillations at all levels of the hierarchy. Therefore we have demonstrated a potent method to control chaotic oscillators in a hierarchical network to steady states, using one external dissimilar chaotic system.

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