The $U(1)_A$ Anomaly and QCD Phenomenology*

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ABSTRACT: The role of the $U(1)_A$ anomaly in QCD phenomenology is reviewed, focusing on the relation between quark dynamics and gluon topology. Topics covered include a generalisation of the Witten-Veneziano formula for the mass of the $\eta'$, the determination of pseudoscalar meson decay constants, radiative pseudoscalar decays and the $U(1)_A$ Goldberger-Treiman relation. Sum rules are derived for the proton and photon structure functions $g_1^p$ and $g_1^\gamma$ measured in polarised deep-inelastic scattering. The first moment sum rule for $g_1^p$ (the ‘proton spin’ problem) is confronted with new data from COMPASS and HERMES on the deuteron structure function and shown to be quantitatively explained in terms of topological charge screening. Proposals for experiments on semi-inclusive DIS and polarised two-photon physics at future $ep$ and high-luminosity $e^+e^-$ colliders are discussed.

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1. Introduction

The $U(1)_A$ anomaly has played an important historical role in establishing QCD as the theory of the strong interactions. The description of radiative decays of the pseudoscalar mesons in the framework of a gauge theory requires the existence of the electromagnetic axial anomaly and determines the number of colours to be $N_c = 3$. The compatibility of the symmetries of QCD with the absence of a ninth light pseudoscalar meson – the so-called ‘$U(1)_A$ problem’ – in turn depends on the contribution of the colour gauge fields to the anomaly. More recently, it has become clear how the anomaly-mediated link between quark dynamics and gluon topology (the non-perturbative dynamics of topologically non-trivial gluon configurations) is the key to understanding a range of phenomena in polarised
QCD phenomenology, most notably the ‘proton spin’ sum rule for the first moment of the structure function \( g_1^p \).

In this paper, based on original research performed in a long-standing collaboration with Gabriele Veneziano, we review the role of the \( U(1)_A \) anomaly in describing a wide variety of phenomena in QCD, ranging from the low-energy dynamics of the pseudoscalar mesons to sum rules in polarised deep-inelastic scattering. The aim is to show how these experiments reveal subtle aspects of quantum field theory, in particular topological gluon dynamics, which go beyond simple current algebra or parton model interpretations.

We begin in section 2 with a brief review of the essential theoretical toolkit: anomalous chiral Ward identities, Zumino transforms, the renormalisation group, and the range of expansion schemes associated with large \( N_c \), notably the OZI approximation. Then, in section 3, we build on Veneziano’s seminal 1979 paper [1] to describe how the pseudoscalar mesons saturate the Ward identities in a way compatible with both the renormalisation group and large-\( N_c \) constraints and derive a generalisation of the famous Witten-Veneziano mass formula for the \( \eta' \) which incorporates, but goes beyond, the original large-\( N_c \) derivation [2, 3].

In section 4, we turn to QCD phenomenology and describe how this intuition on the resolution of the \( U(1)_A \) problem allows a quantitative description of low-energy pseudoscalar meson physics, especially radiative decays, the determination of the pseudoscalar decay constants, and meson-nucleon couplings. We review the \( U(1)_A \) extension of the Goldberger-Treiman formula first proposed by Veneziano [4] as the key to understanding the ‘proton spin’ problem and test an important hypothesis on the origin of OZI violations and their relation to the renormalisation group. Low-energy \( \eta \) and \( \eta' \) physics is currently an active experimental field and we explain the importance of an accurate determination of the couplings \( g_{\eta NN} \) and \( g_{\eta' NN} \) in elucidating the role of gluon topology in QCD.

All of these low-energy phenomena have counterparts in high-energy, polarised deep-inelastic scattering. This enables us to formulate a new sum rule for the first moment of the polarised photon structure function \( g_1^\gamma \) (section 6). The dependence of this sum rule on the invariant momentum of the off-shell target photon measures the form factors of the 3-current AVV Green function and encodes a wealth of information about the realisation of chiral symmetry in QCD, while its asymptotic limit reflects both the electromagnetic and colour \( U(1)_A \) anomalies. We show how this sum rule, which we first proposed in 1992 [5, 6], may soon be tested if the forthcoming generation of high-luminosity \( e^+e^- \) colliders, currently conceived as \( B \) factories, are run with polarised beams [7].

The most striking application of these ideas is, however, to the famous ‘proton spin’ problem, which originated with the observation of the violation of the Ellis-Jaffe sum rule for the first moment of the polarised proton structure function \( g_1^p \) by the EMC collaboration at CERN in 1988. This experiment, and its successors at SLAC, DESY (HERMES) and CERN (SMC, COMPASS) determined the axial charge \( a^0 \) of the proton. In the simple valence-quark parton model, this can be identified with the quark spin and its observed suppression led to an intense experimental and theoretical search over two decades for the origin of the proton spin. In fact, as Veneziano was the first to understand [4], \( a^0 \) does not measure spin in QCD itself and its suppression is related to OZI violations induced by the
The $U(1)_A$ anomaly.

In a series of papers, summarised in section 5, we have shown how $a^0$ decouples from the real angular momentum sum rule for the proton (the form factors for this sum rule are given by generalised parton distributions (GPDs) which can be extracted from less inclusive measurements such as deeply-virtual Compton scattering) and is instead related to the gluon topological susceptibility [8, 9]. The experimentally observed suppression is a manifestation of topological charge screening in the QCD vacuum. In a 1994 paper with Narison [10], using QCD spectral sum rule methods, we were able to compute the slope of the topological susceptibility and give a quantitative prediction for $a^0$. Our prediction, $a^0 = 0.33$, has within the last few months been spectacularly confirmed by the latest data on the deuteron structure function from the COMPASS and HERMES collaborations.

Hopefully, this impressive new evidence for topological charge screening will provide fresh impetus to experimental ‘spin’ physics - first, to verify the real angular momentum sum rule by measuring the relevant GPDs, and second, to pursue the programme of target-fragmentation studies in semi-inclusive DIS at polarised $ep$ colliders which we have proposed as a further test of our understanding of the $g_1^p$ sum rule [11].

This review has been prepared in celebration of the 65th birthday of Gabriele Veneziano. I first met Gabriele when I came to Geneva as a CERN fellow in 1981. In fact, our first interaction was across a tennis court, in a regular Friday doubles match with Daniele Amati and Toine Van Proeyen. I like to think that in those days I could show Gabriele a thing or two about tennis – physics, of course, was a different matter. It has been my privilege through these ensuing 25 years to collaborate with one of the most brilliant and innovative physicists of our generation. But it has also been fun. As all his collaborators will testify, his good humour, generosity to younger colleagues, and enthusiasm in thinking out solutions to the deepest and most fundamental problems in particle physics and cosmology make working with Gabriele not only intellectually rewarding but hugely enjoyable.

In his contribution to the ‘Okubofest’ in 1990 [12], Gabriele concluded an account of the relevance of the OZI rule to $g_1^p$ by hoping that he had ‘made Professor Okubo happy’. In turn, I hope that this review will make Gabriele happy: happy to recall how his original ideas on the $U(1)_A$ problem have grown into a quantitative description of anomalous QCD phenomenology, and happy at the prospect of new discoveries from a rich programme of experimental physics at future polarised colliders. It is my pleasure to join all the contributors to this volume in wishing him a happy birthday.

2. The $U(1)_A$ anomaly and the topological susceptibility

We begin by reviewing some essential features of the $U(1)_A$ anomaly, chiral Ward identities and the renormalisation group, placing particular emphasis on the role of the gluon topological susceptibility. As we shall see, the anomaly provides the vital link between quark dynamics and gluon topology which is essential in understanding a range of phenomena in polarised QCD phenomenology.
2.1 Anomalous chiral Ward identities

An anomaly arises when a symmetry which is present in the classical limit cannot be consistently imposed in a quantum field theory. The original example of an anomaly, and one which continues to have far-reaching implications for the phenomenology of QCD, is the famous Adler-Bell-Jackiw axial anomaly [13, 14, 15], which was first understood in its present form in 1969. In fact, calculations exhibiting what we now recognise as the anomaly had already been performed much earlier by Steinberger in his analysis of meson decays [16] and by Schwinger [17].

Anomalies manifest themselves in a number of ways. The original derivations of the axial anomaly involved the impossibility of simultaneously imposing conservation of both vector and axial currents due to regularisation issues in the AVV triangle diagram in QED. More generally, they arise as anomalous contributions to the commutation relations in current algebra. A modern viewpoint, due to Fujikawa [18], sees anomalies as due to the non-invariance of the fermionic measure in the path integral under transformations corresponding to a symmetry of the classical Lagrangian. In this approach, the result of a chiral transformation $q 	o e^{i\alpha^a T^a q}$ on the quark fields in the QCD generating functional $\mathcal{W}[V_{\mu 5}^a, V_\mu^a, \theta, S_5^a, S^a]$ defined as

$$
e^{i\mathcal{W}} = \int \mathcal{D} A \mathcal{D} \bar{q} q \exp \left[ i \int dx (\mathcal{L}_{\text{QCD}} + V_{\mu 5}^a J_{\mu 5}^a + V^{\mu a} J_\mu^a + \theta Q + S_5^a \phi_5^a + S^a \phi^a) \right]$$

is

$$\int \mathcal{D} A \mathcal{D} \bar{q} q \left[ \partial_\mu W_{\mu 5}^a - \sqrt{24 f} J_{\mu 5}^a - d_{abc} m_b \phi_5^c - \delta \left( \int d^4 x \mathcal{L}_{\text{QCD}} \right) \right] \exp \left[ \ldots \right] = 0 \quad (2.2)$$

The terms in the square bracket are simply those arising from Noether’s theorem, including soft breaking by the quark masses, with the addition of the anomaly involving the gluon topological charge density $Q$. Re-expressing the chiral variation of the elementary fields in terms of a variation with respect to the sources $V_{\mu 5}^a, V_\mu^a, \theta, S_5^a, S^a$ then gives the functional form of the anomalous chiral Ward identities:

$$\partial_\mu W_{\mu 5}^a - \sqrt{24 f} J_{\mu 5}^a - d_{abc} m_b W_{5c}^a$$
$$+ f_{abc} V_{\mu 5}^b W_{\mu 5}^c + f_{abc} V_{\mu 5}^b W_{\mu c}^5 + d_{abc} S_5^b W_{5c}^a - d_{abc} S_5^b W_{5c} = 0 \quad (2.3)$$

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1Our notation follows that of ref.[3]. The currents and pseudoscalar fields $J_{\mu 5}^a, Q, \phi_5^a$ together with the scalar $\phi^a$ are defined by

$$J_{\mu 5}^a = \bar{q} \gamma_\mu \sigma^a q \quad J_\mu^a = \bar{q} \gamma_\mu T^a q \quad Q = \frac{\alpha_s}{8\pi} \text{tr} G_{\mu \nu} \tilde{G}^{\mu \nu}$$
$$\phi_5^a = \bar{q} \gamma_5 T^a q \quad \phi^a = \bar{q} T^a q$$

where $G_{\mu \nu}$ is the field strength for the gluon field. Here, $T^a = \frac{1}{2} \lambda^a$ are flavour $SU(n_f)$ generators, and we include the singlet $U(1)_A$ generator $T^0 = 1/\sqrt{2n_f}$ and let the index $a = 0, i$. With this normalisation, $\text{tr} T^a T^b = \frac{1}{2} \delta^{ab}$ for all the generators $T^a$. This accounts for the rather unconventional factor $\sqrt{2n_f}$ in the anomaly equation but has the advantage of giving a consistent normalisation to the full set of decay constants including the flavour singlets $f^{00f'}$ and $f^{0nf}$. We will only need to consider fields where $i$ corresponds to a generator in the Cartan sub-algebra, so that $a = 3, 8, 0$ for $n_f = 3$ quark flavours. We define $d$-symbols by $\{ T^a, T^b \} = d_{abc} T^c$. For $n_f = 3$, the explicit values are $d_{000} = d_{033} = d_{088} = d_{330} = d_{880} = \sqrt{2/3}, d_{338} = d_{383} = -d_{888} = \sqrt{1/3}$.
where we have abbreviated functional derivatives as suffices. This is the key to all the 
results derived in this section. It makes precise the familiar statement of the anomaly as
\[
\partial^\mu J_{\mu 5}^a - \sqrt{2n_f} Q \delta_{a0} - d_{abc} m_b^b \phi_5^c \sim 0 \tag{2.4}
\]

The chiral Ward identities for two and higher-point Green functions are found by 
taking functional derivatives of eq.(2.3) with respect to the sources. The complete set of 
identities for two-point functions is given in our review [19]. As an example, we find
\[
\partial_\mu W_{V_{\mu 5}^a S_5^b} - \sqrt{2n_f} \delta_{a0} W_{\theta S_5^b} - M_{ac} W_{S_5^a S_5^b} - \Phi_{ab} = 0 \tag{2.5}
\]
which in more familiar notation reads
\[
\partial^\mu \langle 0|T \, J_{\mu 5}^a (0) \phi_5^b |0 \rangle - \sqrt{2n_f} \delta_{a0} \langle 0|T \, Q \, \phi_5^b |0 \rangle - d_{abc} m_b^b \langle 0|T \, \phi_5^c |0 \rangle - d_{abc} \langle \phi^c \rangle = 0 \tag{2.6}
\]

The anomaly breaks the original $U(n_f)_L \times U(n_f)_R$ chiral symmetry to $SU(n_f)_L \times 
SU(n_f)_R \times U(1)_Y / Z_{n_f}^Y$ and the quark condensate spontaneously breaks this further to 
the coset $SU(n_f)_L \times SU(n_f)_R / SU(n_f)_Y$. Goldstone’s theorem follows immediately. In the chiral 
limit, there are $(n_f^2 - 1)$ massless Nambu-Goldstone bosons, which acquire masses of order 
$\sqrt{m}$ for non-zero quark mass. There is no flavour singlet Nambu-Goldstone boson since 
the corresponding current is anomalous.

The zero-momentum Ward identities are especially important here, since they control 
the low-energy dynamics. With the assumption that there are no exactly massless particles 
coupling to the currents, we find
\[
\sqrt{2n_f} \delta_{a0} W_{\theta \theta} + M_{ac} W_{S_5^a \theta} = 0 \\
\sqrt{2n_f} \delta_{a0} W_{\theta S_5^b} + M_{ac} W_{S_5^a S_5^b} + \Phi_{ab} = 0 \tag{2.7}
\]

Another key element of our analysis will be the chiral Ward identities for the effective 
action $\Gamma[V_{\mu 5}^a, V_{\mu}', Q, \phi_5^a, \phi^a]$, defined as the generating functional for vertices which are 1PI 
with respect to the set of fields $Q, \phi_5^a$ and $\phi^a$ but not the currents $J_{\mu 5}^a, J_{\mu}^a$. This is achieved 
using the partial Legendre transform (or Zumino transform):
\[
\Gamma[V_{\mu 5}^a, V_{\mu}', Q, \phi_5^a, \phi^a] = W[V_{\mu 5}^a, V_{\mu}', \theta, S_5^a, S^a] - \int dx \left( \theta Q + S_5^a \phi_5^a + S^a \phi^a \right) \tag{2.8}
\]

\footnote{We use the following $SU(3)$ notation for the quark masses and condensates:
\[
\begin{pmatrix}
m_u & 0 & 0 \\
0 & m_d & 0 \\
0 & 0 & m_s
\end{pmatrix} = \sum_{a=0,3,8} m^a T^a
\]
\[
\begin{pmatrix}
\langle \bar{u} u \rangle & 0 & 0 \\
0 & \langle \bar{d} d \rangle & 0 \\
0 & 0 & \langle \bar{s} s \rangle
\end{pmatrix} = 2 \sum_{a=0,3,8} \langle \phi^a \rangle T^a
\]
where $\langle \phi^a \rangle$ is the VEV of $\phi^a = \bar{q} T^a q$. It is also convenient to use the compact notation
\[
M_{ab} = d_{abc} m^c \quad \Phi_{ab} = d_{abc} \langle \phi^c \rangle$}
The chiral Ward identities for $\Gamma$ are
\[
\partial_\mu \Gamma_{\nu}^{\alpha} - \sqrt{2} n f_\delta a_0 Q - d_{abc} m^b \phi^c_5 + f_{abc} V^b_\mu \Gamma V^c_\mu - d_{abc} \phi^c_5 \Gamma_{\phi^b} + d_{abc} \phi^c \Gamma_\phi^b = 0
\]
(2.9)
Again, the zero-momentum identities for the two-point vertices play an important role:
\[
\Phi_{\alpha\beta} \Gamma_{\phi^c} - \sqrt{2} n f_\delta a_0 = 0
\]
\[
\Phi_{\alpha\beta} \Gamma_{\phi^c} - M_{ab} = 0
\]
(2.10)
These will be used in section 3 to construct an effective action which captures the low-energy dynamics of QCD in the pseudoscalar sector.

2.2 Topological susceptibility

The connection with topology arises through the identification of the gluon operator $Q$ in the anomaly with a topological charge density. $Q$ is a total divergence:
\[
Q = \frac{\alpha_s}{8\pi} \text{tr} G_{\mu\nu} \tilde{G}^{\mu\nu} = \partial_\mu K_\mu
\]
(2.11)
where $K_\mu$ is the Chern-Simons current,
\[
K_\mu = \frac{\alpha_s}{4\pi} \epsilon_{\mu\nu\rho\sigma} \text{tr}(A_\nu G^{\rho\sigma} - \frac{1}{3} g A_\nu [A^\rho, A^\sigma])
\]
(2.12)
Nevertheless, the integral over (Euclidean) spacetime of $Q$ need not vanish. In fact, for gauge field configurations such as instantons which become pure gauge at infinity,
\[
\int d^4x \; Q = n \in \mathbb{Z}
\]
(2.13)
where the integer $n$ is the topological winding number, an element of the homotopy group $\pi_3(SU(N_c))$.

The form of the anomaly is then understood as follows. Under a chiral transformation, the fermion measure in the path integral transforms as (for one flavour)
\[
\mathcal{D}\bar{q} \mathcal{D}q \rightarrow e^{-2i\alpha \int d^4x \bar{\psi}_i \gamma_5 \phi_i} \mathcal{D}\bar{q} \mathcal{D}q = \exp^{-2i\alpha (n_+ - n_-)} \mathcal{D}\bar{q} \mathcal{D}q
\]
(2.14)
where $\phi_i$ is a basis of eigenfunctions of the Dirac operator $\mathcal{D}$ in the background gauge field. The non-zero eigenvalues are chirality paired, so the Jacobian only depends on the difference $(n_+ - n_-)$ of the positive and negative chirality zero modes of $\mathcal{D}$. Finally, the index theorem relates the anomaly to the topological charge density:
\[
\text{ind} \mathcal{D} = n_+ - n_- = \int d^4x \; Q
\]
(2.15)
The topological susceptibility $\chi(p^2)$ is defined as the two-point Green function of $Q$, viz.
\[
\chi(p^2) = i \int d^4x \; e^{ipx} \langle 0 | T Q(x) Q(0) | 0 \rangle
\]
(2.16)
We are primarily concerned with the zero-momentum limit $\chi(0) = W_{\theta\theta}(0)$. Combining eqs.(2.7) gives the crucial Ward identity satisfied by $\chi(0)$:

$$2n_f \chi(0) = M_{0a} W_{S_5 S_5} M_{00} + (M \Phi)_{00}$$

(2.17)

that is,

$$n_f^2 \int dx \langle 0 | T Q(x) Q(0) | 0 \rangle = \int dx \, m^a m^b \langle 0 | T \phi^a_5(x) \phi^b_5(0) | 0 \rangle + m^a \langle \phi^a \rangle$$

(2.18)

Determining exactly how this is satisfied in QCD is at the heart of the Witten-Veneziano approach to the $U(1)_A$ problem [20, 1].

The zero-momentum Ward identities allow us to write a precise form for the topological susceptibility in QCD in terms of just one unknown dynamical constant [21]. To derive this, recall that the matrix of two-point vertices is simply the inverse of the two-point Green function matrix, so in the pseudoscalar sector we have the following inversion formula:

$$\Gamma_{QQ} = -\left( W_{\theta\theta} - W_{\theta S_5} (W_{S_5 S_5})^{-1} W_{S_5 \theta} \right)^{-1}$$

(2.19)

Using the identities (2.7) and (2.17), this implies that at zero momentum

$$\Gamma_{QQ}^{-1} = -\chi \left( 1 - 2n_f \chi (M \Phi)_{00}^{-1} \right)^{-1}$$

(2.20)

and inverting this relation gives

$$\chi = -\Gamma_{QQ}^{-1} \left( 1 - 2n_f \Gamma_{QQ}^{-1} (M \Phi)_{00}^{-1} \right)^{-1}$$

(2.21)

Finally, substituting for $(M \Phi)_{00}^{-1}$ using the definitions above, we find the following important identity which determines the quark mass dependence of the topological susceptibility in QCD:

$$\chi(0) = -A \left( 1 - A \sum_q \frac{1}{m_q(q \bar{q})} \right)^{-1}$$

(2.22)

where we identify the non-perturbative coefficient $A$ as $\Gamma_{QQ}^{-1}$.

Notice immediately how this expression exposes the well-known result that $\chi(0)$ vanishes if any quark mass is set to zero. In section 3, we will see how it also clarifies the role of the $1/N_c$ expansion in the $U(1)_A$ problem.

### 2.3 Renormalisation group

The conserved current corresponding to a non-anomalous symmetry is not renormalised and has vanishing anomalous dimension. However, an anomalous current such as the flavour singlet axial current $J_{\mu 5}^0$ is renormalised. The composite operator renormalisation and mixing in the $J_{\mu 5}^0, Q$ sector is as follows [22]:

$$J_{\mu 5}^{0R} = Z J_{\mu 5B}^0$$

$$Q_R = Q_B - \frac{1}{\sqrt{2n_f}} (1 - Z) \partial_{\mu} J_{\mu 5B}^0$$

(2.23)
Notice the form of the mixing of the operator $Q$ with $\partial^\mu J^0_{\mu 5}$ under renormalisation. This ensures that the combination $(\partial^\mu J^0_{\mu 5} - \sqrt{2nf} Q)$ occurring in the $U(1)_A$ anomaly equation is RG invariant. The chiral Ward identities therefore take precisely the same form expressed in terms of the bare or renormalised operators, making precise the notion of ‘non-renormalisation of the anomaly’. We may therefore interpret the above Ward identities, which were derived in terms of the bare operators, as identities for the renormalised composite operators (and omit the suffix $R$ for notational simplicity).

The renormalisation group equation (RGE) for the generating functional $W[V_{\mu 5}, V^a_{\mu 5}, \theta, S^a_{\mu 5}, S^a]$ follows immediately from the definitions (2.23) of the renormalised composite operators. Including also a standard multiplicative renormalisation $Z_\phi = Z^{-1} m$ for the pseudoscalar and scalar operators $\phi^5_a$ and $\phi^a$ and denoting the anomalous dimensions corresponding to $Z$ and $Z_\phi$ by $\gamma$ and $\gamma_\phi$ respectively, we find

$$D W = \gamma \left( V^0_{\mu 5} - \frac{1}{\sqrt{2nf}} \partial_\mu \theta \right) W^0_{\mu 5} + \gamma_\phi \left( S^a_{\mu 5} W S^a_{\mu 5} + S^a W S^a \right) + \ldots$$

where $D = \left( \mu \frac{\partial}{\partial \mu} + \beta \frac{\partial}{\partial g} - \gamma_m \sum_q m_q \frac{\partial}{\partial m_q} \right) |V_{\theta 5}, S_{\mu 5}, S_{\mu 5}.$

The RGEs for Green functions are found by functional differentiation of eq.(2.24) and can be simplified using the Ward identities. For example, for $W_{\theta \theta}$ we find

$$D W_{\theta \theta} = 2 \gamma W_{\theta \theta} + 2 \gamma_\phi \frac{1}{\sqrt{2nf}} M_{\theta 5} W_{\theta S^a_{\mu 5}} + \ldots$$

At zero momentum, we can then use the first identity in eq.(2.7) to prove that the topological susceptibility $\chi(0)$ is RG invariant,

$$D \chi(0) = 0$$

which is consistent with its explicit expression (2.22).

A similar RGE holds for the effective action $\Gamma[V_{\mu 5}, V^a_{\mu 5}, Q, \phi^5_a, \phi^a]$, which allows the scaling behaviour of the proper vertices involving $Q$ and $\phi^5_a$ to be determined [23, 9, 24]. This reads

$$D \Gamma = \gamma \left( V^0_{\mu 5} - \frac{1}{\sqrt{2nf}} \Gamma Q \partial_\mu \right) V^0_{\mu 5} - \gamma_\phi \left( \phi^5_a \Gamma \phi^5_a + \phi^a \Gamma \phi^a \right) + \ldots$$

An immediate consequence is that $D \Gamma_{QQ} = 0$ at zero momentum, which ensures the compatibility of (2.22) with the RG invariance of $\chi(0)$.

2.4 1/$N_c$, the topological expansion and OZI

The final theoretical input into our analysis of the $U(1)_A$ problem and phenomenological implications of the anomaly concerns the range of dynamical approximation schemes associated with the large-$N_c$ limit. At various points we will refer either to the original large-$N_c$
expansion of ‘t Hooft [25], the topological expansion introduced by Veneziano [26] and the OZI limit [27, 28, 29]. A very clear summary of the distinction between them is given in Veneziano’s ‘Okubofest’ review [12], which we follow here.

In terms of Feynman diagrams, the leading order in the large $N_c$, fixed $n_f$ (‘t Hooft) limit is the most restrictive of these approximations, including only planar diagrams with sources on a single quark line and no further quark loops (Fig. 1).

A better approximation to QCD is the quenched approximation familiar in lattice gauge theory. This is a small $n_f$ expansion at fixed $N_c$, i.e. excluding quark loops but allowing non-planar diagrams (Fig. 2).

An alternative is the topological expansion, which allows any number of internal quark loops, but restricts to planar diagrams at leading order. Provided the sources remain attached to the same quark line, this corresponds to taking large $N_c$ at fixed $n_f/N_c$. This means that quarks and gluons are treated democratically and the order of approximation is determined solely by the topology of the diagrams (Fig. 2).

Finally, the OZI approximation is a still closer match to full QCD with dynamical quarks than either the leading order quenched or topological expansions. Non-planar diagrams and quark loops are retained, but diagrams in which the external sources are connected to different quark loops are still excluded (Fig. 3). This means that amplitudes which involve purely gluonic intermediate states are suppressed. This is the field-theoretic basis for the original empirical OZI rule.

In each of these large-$N_c$ expansions, except the topological expansion where $n_f/N_c$ is fixed, the $U(1)_A$ anomaly does not contribute at leading order. More precisely, the anomalous contribution $\langle 0 | T Q \phi_5^b | 0 \rangle$ in the chiral Ward identity (2.6) is suppressed by
Figure 3: Feynman diagrams allowed (left) and forbidden (right) by the OZI rule.

$O(1/N_c)$ relative to the current term $\langle 0 | T \ J_{\mu\nu}^0 \ \phi^0 | 0 \rangle$. This means that the flavour singlet current is conserved, Goldstone’s theorem applies, and conventional PCAC methods can be used to understand the dynamics of the Green functions with a full set of $(n_f^2 - 1)$ massless bosons in the chiral limit. Taking this as a starting point, we can then learn about the spectral decomposition of the actual QCD Green functions as we relax from the leading-order limits. In particular, this leads us to the famous Witten-Veneziano mass formula for the $\eta'$ meson [20, 1].

The behaviour of the topological susceptibility at large $N_c$ is central to this analysis. It is clear from looking at planar diagrams that at leading order in $1/N_c$, $\chi(0)$ in QCD coincides with the topological susceptibility $\chi(0)_{YM}$ in the corresponding pure Yang-Mills theory. Referring now to the explicit expression (2.22) for $\chi(0)$, large-$N_c$ counting rules give $A = O(1)$ while $\langle \bar{q}q \rangle = O(N_c)$. It follows that for non-zero quark masses,

$$\chi(0) = -A + O(n_f/N_c)$$

(2.28)

where $A = \Gamma_{QQ}$ is identified as $-\chi(0)_{YM} + O(1/N_c)$. On the other hand, if we consider the limit of $\chi(0)$ for $m_q \to 0$ at finite $N_c$, then we have

$$\chi(0)_{m_q \to 0} = 0$$

(2.29)

The ’t Hooft large-$N_c$ limit is therefore not smooth in QCD; the $N_c \to \infty$ and $m_q \to 0$ limits do not commute [20, 1, 21]. This is remedied in the topological expansion, where quark loops are retained and the $O(n_f/N_c)$ contribution in eq.(2.28) allows the smooth chiral limit $\chi(0) \to 0$ even for large $N_c$.

3. ‘$U(1)_{A}$ without instantons’

The $U(1)_{A}$ problem has a long history, pre-dating QCD itself, and has been an important stimulus to new theoretical ideas involving anomalies and gluon topology.

At its simplest, the original ‘$U(1)_{A}$ problem’ in current algebra is relatively straightforwardly resolved by the existence of the anomalous contributions to the chiral Ward identities (anomalous commutators in current algebra) and the consequent absence of a ninth light Nambu-Goldstone boson in $n_f = 3$ QCD. However, a full resolution requires a

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4The existence of a light flavour-singlet Nambu-Goldstone boson would produce a rapid off-shell variation in the $\eta \to 3\pi$ decay amplitude, in contradiction with the experimental data [30].
much more detailed understanding of the dynamics of the pseudoscalar sector and the role of topological fluctuations in the anomalous Green functions.

In this section, we review the analysis of the $U(1)_A$ problem presented by Veneziano in his seminal 1974 paper, 'U(1)$_A$ without instantons' [1]. As well as deriving the eponymous mass formula relating the $\eta'$ mass to the topological susceptibility, the essential problem resolved in ref.[1] is how to describe the dynamics of the Green functions of the pseudoscalar operators in QCD in terms of a spectral decomposition compatible with the $n_f, N_c, \theta$ and quark mass dependence imposed by the anomalous Ward identities.

First, recall that in the absence of the anomaly, there will be light pseudoscalar mesons $\eta^\alpha$ coupling derivatively to the currents with decay constants $f^{a\alpha}$, i.e. $\langle 0 | J^5_{\mu b} | \eta^\alpha \rangle = ip^\mu f^{a\alpha}$. (We use the notation $\eta^\alpha$ to denote the physical mesons $\pi^0, \eta$ and $\eta'$, while the $SU(3)$ index $a = 3, 8, 0$.) The mass matrix $\mu_{\alpha\beta}^2$ satisfies the Dashen, Gell-Mann–Oakes–Renner (DGMOR) relation [35, 36]

$$f^{a\alpha} \mu_{\alpha\beta}^2 f^{T\beta b} = -(M\Phi)_{ab} \quad \text{(no anomaly)} \quad (3.1)$$

The consequences of the anomaly are determined by the interaction of the pseudoscalar fields $\phi^5$ with the topological charge density $Q$ and the subsequent mixing. This gives rise to an additional contribution to the masses. Moreover, we can no longer identify the flavour singlet decay constant by the coupling to $J^0_{\mu 5}$ since this is not RG invariant. Instead, the physical decay constants $f^{a\alpha}$ are defined in terms of the couplings of the $\eta^\alpha$ to the pseudoscalar fields through the relation $f^{a\alpha} \langle 0 | \phi^b_5 | \eta^\alpha \rangle = d_{abc} \langle \phi^c \rangle$. This coincides with the usual definition except in the flavour singlet case.

The most transparent way to describe how all this works is to use an effective action $\Gamma[Q, \phi^5]$ constructed to satisfy the anomalous chiral Ward identities. It is important to emphasise from the outset that this is an effective action in the sense of section 2.1, i.e. the generating functional for vertices which are 1PI with respect to the set of fields $Q, \phi^5$ only. The choice of fields is designed to capture the degrees of freedom essential for the dynamics. A different choice (or linear combination) redefines the physical meaning of the vertices, so it is important that the final choice of fields in $\Gamma$ results in vertices which are most directly related to physical couplings.

The simplest effective action consistent with the anomalous Ward identities and the renormalisation group is

$$\Gamma[Q, \phi^5] = \int dx \left( \frac{1}{2A} Q^2 + Q(\sqrt{2n_f} \delta_{0a} - B_a \partial^2) \Phi^{-1}_{ab} \phi^b + \frac{1}{2} \phi^5_5 \Phi^{-1}_{ac}(M\Phi)_{cd} - C_{cd} \partial^2 \Phi^{-1}_{db} \phi^b \right) \quad (3.2)$$

\[5\] For reviews of the instanton approach to the resolution of the $U(1)_A$ problem, see e.g. refs.[31, 32, 33, 34].

\[6\] Note especially the frequently misunderstood point that the choice of fields in $\Gamma$ is not required to be in any sense a complete set, nor does the restriction to a given set of fields constitute an approximation. Before imposing dynamical simplifications, the identities derived from $\Gamma$ are exact - increasing the set of basis fields simply changes the definitions of the 1PI vertices. The effective action considered here is therefore different from the non-linear chiral Lagrangians incorporating the large-$N_c$ approach to the pseudoscalar mesons constructed by a number of groups. See, for example, refs. [37, 21, 38, 39, 40, 41, 42].
The constants $C_{ab}$ and $B_a$ are related to $\Gamma_{V^a V^b}$ and $\Gamma_{V^a Q}$ respectively. The inclusion of the term with $B_a$ is unusual, but is required for consistency with the RGEs derived from (2.27) beyond zero momentum.

This form of $\Gamma[Q, \phi^a]$ encodes three key dynamical assumptions:

- **Pole dominance.** We assume that the Green functions are dominated by the contribution of single-particle poles associated with the pseudoscalar mesons including the flavour singlet. This extends the usual PCAC assumption to the singlet sector.

- **Smoothness.** We assume that pole-free dynamical quantities such as the decay constants and couplings (1PI vertices) are only weakly momentum-dependent in the range from $p = 0$ to their on-shell values. This allows us to impose relations derived from the zero-momentum Ward identities, provided this is compatible with the renormalisation group.

- **Topology.** There must exist topologically non-trivial fluctuations which can give a non-vanishing value to $\chi(0)|_{YM}$ in pure gluodynamics. This is required to give the non-vanishing coefficient in the all-important $1/2 A_{Q^2}$ term in $\Gamma[Q, \phi^a]$. Notice that we do not require a kinetic term for $Q$, which would be associated with a (presumed heavy) pseudoscalar glueball.

The second derivatives of $\Gamma[Q, \phi^a]$ are

$$
\begin{pmatrix}
\Gamma_{QQ} & \Gamma_{Q\phi^a} \\
\Gamma_{\phi^a Q} & \Gamma_{\phi^a \phi^a}
\end{pmatrix} =
\begin{pmatrix}
A^{-1} & (\sqrt{2n_f}\delta_{0d} + B_d p^2)\Phi_{db}^{-1} \\
\Phi_{ac}^{-1}(\sqrt{2n_f}\delta_{0c} + B_c p^2) & \Phi_{ac}^{-1}((M\Phi)_{cd} + C_{cd} p^2)\Phi_{db}^{-1}
\end{pmatrix}
$$

(3.3)

The corresponding Green functions (composite operator propagators) are given by inversion:

$$
\begin{pmatrix}
W_{\theta\theta} & W_{\theta S^a} \\
W_{S^a \theta} & W_{S^a S^b}
\end{pmatrix} = - \left( \begin{pmatrix}
\Gamma_{QQ} & \Gamma_{Q\phi^a} \\
\Gamma_{\phi^a Q} & \Gamma_{\phi^a \phi^a}
\end{pmatrix} \right)^{-1}
$$

(3.4)

and we find, to leading order in $p^2$,

$$
\begin{align*}
W_{\theta\theta} &= - A \tilde{\Delta}^{-1} \\
W_{\theta S^a} &= W_{S^a \theta} \simeq \sqrt{2n_f} A \Delta^{-1}_{0d} \Phi_{db} \\
W_{S^a S^b} &= - \Phi_{ac} \Delta^{-1}_{cd} \Phi_{db}
\end{align*}
$$

(3.5)

where

$$
\tilde{\Delta} = 1 - (2n_f A\delta_{a0}\delta_{0b} + \sqrt{2n_f} A(\delta_{a0}B_b + B_a\delta_{0b}))p^2 (M\Phi + C p^2)_{ab}^{-1}
$$

(3.6)

and

$$
\Delta_{ab} = \left( C_{ab} - \sqrt{2n_f} A(\delta_{a0}B_b + B_a\delta_{0b}) \right)p^2 + (M\Phi)_{ab} - 2n_f A \delta_{a0}\delta_{0b}
$$

(3.7)

In this form, however, the propagator matrix is not diagonal and the operators are not normalised so as to couple with unit decay constants to the physical states. It is therefore convenient to make a change of variables in $\Gamma$ so that it is written in terms of operators which are more closely identified with the physical states. We do this in two stages, since
the intermediate stage allows us to make direct contact with the discussion in ref.[1] and will play an important role in some of the phenomenological applications considered later.

First, we define rescaled fields \( \hat{\eta}^\alpha \) whose kinetic terms, before mixing with \( Q \), are canonically normalised. That is, we set

\[
\hat{\eta}^\alpha = \hat{f}^{\alpha a} \Phi^{-1} b^a \tag{3.8}
\]

with the ‘decay constants’ \( \hat{f}^{\alpha a} \) defined such that \( \frac{d}{dp^2} \Gamma_{\hat{\eta}^\alpha \hat{\eta}^\beta} |_{p=0} = \delta_{\alpha\beta} \). This implies

\[
(\hat{f} \hat{f}^T)_{ab} = C_{ab} = \left. \frac{d}{dp^2} W_{S_a S_b} \right|_{p=0} \quad \tag{3.9}
\]

where \( D^a = \sqrt{2n_f} \delta_{a0} Q + M_{ab} \phi_b^5 \) is the divergence of the current \( J^a_{\mu5} \). In the chiral limit, this reduces in the flavour singlet sector to

\[
(\hat{f} \hat{f}^T)_{00} = \left. \frac{d}{dp^2} \chi(p^2) \right|_{p=0} = \chi'(0) \quad \tag{3.10}
\]

a result which plays a vital role in understanding the ‘proton spin’ problem. Notice however that the \( \hat{f}^{\alpha a} \) are not RG invariant: in fact, \( D \hat{f}^{\alpha a} = \gamma \delta_{a0} \hat{f}^{\alpha a} \). The effective action \( \Gamma[Q, \hat{\eta}^\alpha] \) is:

\[
\Gamma[Q, \hat{\eta}^\alpha] = \int dx \left( \frac{1}{2A} Q^2 + Q(\sqrt{2n_f} \delta_{a0} - B_a \delta^2)(\hat{f}^{-1})^{\alpha a} \hat{\eta}^\alpha \\
+ \frac{1}{2} \hat{\eta}^\alpha (-\partial^2 + \hat{f}^{-1}T M \Phi \hat{f}^{-1})_{\alpha\beta} \hat{\eta}^\beta \right) \tag{3.11}
\]

In this form, the \( \hat{\eta}^\alpha \) are the canonically normalised fields corresponding to the ‘would-be Nambu-Goldstone bosons’ in the absence of the anomaly, before they acquire an additional anomaly-induced mass. In the framework of the large-\( N_c \) or OZI approximations, they would correspond to true Nambu-Goldstone bosons. The singlet \( \hat{\eta}^0 \) is what we have therefore referred to in our previous papers as the ‘OZI boson’ \( \eta^{OZI} \). As we see later, the naive current algebra relations hold when expressed in terms of the \( \hat{\eta}^\alpha \) and \( f^{\alpha a} \), though these do not correspond to physical states or decay constants.

The physical particle masses are identified with the poles in the two-point Green functions (3.5). We immediately see that due to mixing with the topological charge density \( Q \), the physical pseudoscalar meson mass \( m_{\alpha\beta}^2 \) is shifted from its original value. From the pole in eq.(3.7), we immediately find

\[
f^{\alpha a} m_{\alpha\beta}^2 f^{T \beta b} = -(M \Phi)_{ab} + 2n_f A \delta_{a0} \delta_{b0} \tag{3.12}
\]

where we identify the physical, RG-invariant decay constants as

\[
(f f^T)_{ab} = (\hat{f} \hat{f}^T)_{ab} - \sqrt{2n_f} A (\delta_{a0} B_b + B_a \delta_{b0}) \tag{3.13}
\]

Eq.(3.12) is the key result. It generalises the original DGMOR relations (3.1) to the flavour-singlet sector with the anomaly properly incorporated and the renormalisation
group constraints satisfied. It represents a generalisation of the Witten-Veneziano mass formula which makes no direct reference to large-$N_c$ arguments but depends only on the three dynamical assumptions stated above [2].

With this clarification of the distinction between the physical decay constants $f^{a\alpha}$ and the RG non-invariant $\hat{f}^{a\alpha}$, we can rewrite eq.(3.6) for the topological susceptibility $\chi(p^2) = W_{\theta\theta}(p^2)$ as

$$\chi(p^2) = -A \left[1 - \text{tr}((\hat{f} \hat{f}^T - f f^T)p^2 + 2n_f A \Phi_0)(\hat{f} \hat{f}^T p^2 + M \Phi)^{-1}\right]^{-1}$$

(3.14)

It is clear that in the zero-momentum limit, this expression successfully reproduces eq.(2.22) for $\chi(0)$. For one flavour, the formula simplifies to

$$\chi(p^2) = -A(\hat{f} \hat{f}^T p^2 + M \Phi)\left[ff^T p^2 + M \Phi + 2n_f A\right]^{-1} \quad (n_f = 1)$$

(3.15)

showing clearly the pole at the shifted mass $m^2$ of eq.(3.12). The occurrence of both $\hat{f}^{a\alpha}$ and $f^{a\alpha}$ in these expressions allows them to satisfy the RGE (2.25) for the topological susceptibility, which requires $D\chi(p^2) = O(p^2)$.

The second stage is to make a change of variable which diagonalises the propagator matrix, so as to give the most direct possible relation between the operators and the physical states. Choosing

$$G = Q - W_{\theta S_5} W_{S_5}^{-1} \phi_b^b \simeq Q + \sqrt{2n_f A} \Phi_0^{-1} \phi_5^b$$

$$\eta^a = f^{a\alpha} \phi_b^b$$

(3.16)

defines the effective action $\Gamma[G, \eta^a]$ as

$$\Gamma[G, \eta^a] = \int dx \left(\frac{1}{2A} G^2 + \frac{1}{2} \eta^a (-\partial^2 - m^2)_{\alpha\beta} \eta^\beta\right)$$

(3.17)

with $m_{\alpha\beta}^2$ given by eq.(3.12). The corresponding propagators are

$$\langle 0|T G G|0 \rangle = -A$$

$$\langle 0|T \eta^a \eta^\beta|0 \rangle = \frac{-1}{p^2 - m_{\eta^a}^2} \delta^{a\beta}$$

(3.18)

where with no loss of generality we have taken $m_{\alpha\beta}^2$ diagonal.

Notice also that the states mix in the complementary way to the operators. In particular, the mixing for the states corresponding to eq.(3.16) for the fields $G$ and $\eta^a$ is

$$|G\rangle = |Q\rangle$$

$$|\eta^a\rangle = (f^{-1})^{a\alpha} (\Phi_{ab} |\phi_b^b\rangle - \sqrt{2n_f A} \delta_{a0} |Q\rangle)$$

(3.19)

In this sense, we see that we can regard the physical $\eta'$ (and, with $SU(3)$ breaking, the $\eta$) as an admixture of quark and gluon components, while the unphysical state $|G\rangle$ is purely gluonic.
An immediate corollary is the following relation, which we will use repeatedly in deriving alternative forms of the current algebra identities for the pseudoscalar mesons:

$$\Phi^{\alpha\beta} \frac{\delta}{\delta \phi^{\alpha}} = f^{\alpha\beta} \frac{\delta}{\delta \eta^\alpha} = f^{\alpha\beta} \frac{\delta}{\delta \eta^\alpha} + \sqrt{2n_f A \delta_{\alpha\beta}} \frac{\delta}{\delta G}$$

(3.20)

The formulation in terms of $\Gamma[G, \eta^\alpha]$ is exactly what we need to construct a simple $'U(1)_A$ PCAC' with which to interpret the low-energy phenomenology of the pseudoscalar mesons. We turn to this in the next section.

Here, we focus on the intermediate formulation $\Gamma[Q, \hat{\eta}^\alpha]$ in order to describe Veneziano's analysis of the $'U(1)$ problem in the framework of the large-$N_c$ and topological expansions. The starting point is the anomalous Ward identity (2.18) for the topological susceptibility:

$$n_f^2 \int dx \langle 0| T Q(x) Q(0)|0 \rangle = \int dx \; m^a m^b \langle 0| T \phi^a_\alpha(x) \phi^b_\beta(0)|0 \rangle + m^a \langle \phi^a \rangle$$

(3.21)

The essential problem is how to understand this relation in terms of a spectral decomposition in the context of the $1/N_c$ expansion.

Assuming that $\chi(p^2)_{YM} = -A + O(1/N_c)$ is non-vanishing at $O(1)$, the l.h.s. is $O(n_f^2)$ in leading order in $1/N_c$. On the other hand, the r.h.s. includes the condensate term of $O(n_f^2 m^a)$. To resolve this apparent paradox, we have to go beyond leading order in $1/N_c$ and consider the quark loop contributions which are included in the topological expansion. Although these are formally suppressed by powers of $(n_f/N_c)$, they contain light intermediate states which can enhance the order of the Green function. As we have seen above, these light states are just the ‘OZI bosons’ $|\hat{\eta}^\alpha\rangle$ with masses $\mu^2_{\alpha\beta}$ of $O(n_f m)$. Inserting these intermediate states, we therefore find that:

$$\chi(p^2) = \chi(p^2)_{YM} - \langle 0| Q|\hat{\eta}^\alpha \rangle \frac{1}{(p^2 - \mu^2)_{\alpha\beta}} \langle \hat{\eta}^\beta| Q|0 \rangle + \ldots$$

(3.22)

where the coupling $\langle 0| Q|\hat{\eta}^\alpha \rangle$ is $O(\sqrt{n_f/N_c})$.

Approximating $\chi(p^2)_{YM} \sim -A$ (a low-momentum smoothness assumption) and $\langle 0| Q|\hat{\eta}^\alpha \rangle \sim \sqrt{2n_f A (f^{-1})^{0\alpha}}$, then summing the series of intermediate state contributions, we find

$$\chi(p^2) \simeq - \frac{A}{1 - 2n_f A (f(p^2 - \mu^2) f\tau)^{-1}}$$

(3.23)

This expression reproduces eq.(13) of ref.[1]. Clearly, it is dominated by the physical pseudoscalar pole with anomaly-induced mass given by eq.(3.12). It does not completely recover our more precise expression (3.14) because of the approximation for the coupling of $Q$ to the $|\hat{\eta}^\alpha\rangle$, which misses the subtleties related to the introduction of $B_a$ in the effective action $\Gamma[Q, \eta^\alpha]$ and the distinction of $\hat{f}^{\alpha\alpha}$ and $f^{\alpha\alpha}$. These are effects of higher order in $1/N_c$ but, as we have seen, they are necessary to establish full RG consistency and will prove to be important for phenomenology.

To see how a term with the $O(n_f N_c m)$ dependence of the condensate can arise in $n_f^2 \chi(0)$, notice from eq.(3.12) that the physical pseudoscalar mass squared $m^2_{\eta^\alpha}$ has two
contributions, the first of $O(m)$ from the conventional quark mass term and the new, anomaly-induced contribution of $O(n_f/N_c)$. If we are in a regime where the anomaly contribution dominates ($m < \Lambda_{\text{QCD}}/N_c$), then it follows that the above expression for $\chi(0)$ indeed becomes of $O(n_f^{-1}N_c m)$.

The original Witten-Veneziano mass formula for the $\eta'$ is the large-$N_c$ limit of eq.(3.12). In the chiral limit there is no flavour mixing and the singlet mass is given by

$$m^2_{\eta'} = \frac{1}{(f_{0\eta'})^2} 2n_f A = -\frac{2n_f}{f^2_{\pi}} \chi(0)_{\text{YM}} + O((n_f/N_c)^2)$$

(3.24)

This formula provided the first link between the $\eta'$ mass and gluon topology. For an alternative recent derivation in the context of a $n_f/N_c$ expansion, see also ref.[43].

What we learn from all this is that the Green functions in the anomalous chiral Ward identities admit a consistent spectral decomposition in terms of a full set of $(n_f^2 - 1)$ pseudoscalar mesons, provided they satisfy the generalised DGMOR mass formula (3.12) including the all-important anomaly term. The presence of these light poles can enhance the apparent order of the Green functions, as is familiar with Nambu-Goldstone bosons, and the anomaly-induced $O(n_f/N_c)$ contribution to $m^2_{\eta'}$ is critical in ensuring complete consistency with the Ward identities.

Similar considerations apply to the resolution of apparent paradoxes in the $\theta$-dependence of some Green functions. For example [1], we can show from the anomalous Ward identities that the condensate satisfies

$$\sum_q m_q q^* \langle \bar{q} q \rangle|_{\theta} = m^a \langle \phi^a \rangle = \cos(\theta/n_f) m^a \langle \phi^a \rangle|_{\theta=0}$$

(3.25)

This implies

$$\frac{\partial^2}{\partial \theta^2} m^a \langle \phi^a \rangle|_{\theta=0} = -m^a \int dx \int dy \langle 0 | Q(x) Q(y) \phi^a(0)|0 \rangle
$$

$$= -\frac{1}{n_f} m^a \langle \phi^a \rangle|_{\theta=0}$$

(3.26)

Here, the Green function is superficially of $O(n_f/N_c)$ while the r.h.s. is $O(N_c/n_f)$. The resolution is simply that it contains pseudoscalar intermediate states contributing two light poles with $m^2 \sim O(n_f/N_c)$. So once again we see how the spectral decomposition in terms of the full set of pseudoscalar mesons, including the flavour singlet, ensures consistency with the anomalous Ward identities.

4. Pseudoscalar mesons

This theoretical analysis provides the basis for an extension of the conventional PCAC or chiral Lagrangian description of the phenomenology of the pseudoscalar mesons to the flavour singlet sector. In this section we describe the role of the $U(1)_A$ anomaly in the radiative decays of $\pi^0, \eta$ and $\eta'$ and derive the $U(1)_A$ Goldberger-Treiman relation, first proposed by Veneziano as a resolution of the ‘proton spin’ problem.

7This section is based on the presentation in ref.[3], where we extend and update our original work [2, 44] to include a detailed comparison with experimental data.
4.1 $U(1)_A$ Dashen, Gell-Mann–Oakes–Renner relations

The extension of the DGMOR relations to the $U(1)_A$ sector follows from the application of the three key dynamical assumptions used above (viz. pole dominance by the nonet of pseudoscalar mesons, smoothness of decay constants and couplings over the range from zero to on-shell momentum, and the existence of topologically non-trivial gluon dynamics) to the anomalous chiral Ward identities.

The fundamental $U(1)_A$ DGMOR relation

$$f^{\alpha\beta}m^2_{\alpha\beta}f^{T\beta} = -M_{ac}\Phi_{cb} + 2nfA\delta_{a0}\delta_{b0}$$

has been derived above in the course of the general discussion of the $U(1)_A$ problem. In order to make this section self-contained, we give a brief and direct derivation here.

Recall that the physical meson fields are given as

$$\eta^\alpha = f^{T\alpha}\Phi_{\cdots} - \frac{1}{\sqrt{p^2 - m_{\eta}^2}}[\ldots]$$

It follows immediately that at zero momentum,

$$f^{\alpha\beta}m^2_{\alpha\beta}f = \Phi_{ac}(W_{S^5S^5})^{-1} - \frac{1}{cd}\Phi_{db}$$

Using the chiral Ward identities of section 2 together with the identification (2.21) of the topological susceptibility, we can then show

$$\Phi_{ac}(W_{S^5S^5})^{-1} - \frac{1}{cd}\Phi_{db} = (\Phi M)_{ac}(MW_{S^5S^5}M)_{cd}^{-1}(M\Phi)_{db}$$

proving the result (4.1).

Expanding this out, and assuming the mixed decay constants $f^{0\eta'}, f^{8\pi}, f^{3\eta}, f^{3\eta'}$ are all negligible, we have

$$(f^{0\eta'})^2m_{\eta'}^2 + (f^{0\eta})^2m_{\eta}^2 = -\frac{2}{3}(m_u\langle\bar{u}u\rangle + m_d\langle\bar{d}d\rangle + m_s\langle\bar{s}s\rangle) + 6A$$

$$(f^{8\eta'})^2m_{\eta'}^2 + (f^{8\eta})^2m_{\eta}^2 = -\frac{\sqrt{2}}{3}(m_u\langle\bar{u}u\rangle + m_d\langle\bar{d}d\rangle - 2m_s\langle\bar{s}s\rangle)$$

$$(f^{3\eta'})^2m_{\eta'}^2 + (f^{3\eta})^2m_{\eta}^2 = -\frac{1}{3}(m_u\langle\bar{u}u\rangle + m_d\langle\bar{d}d\rangle + 4m_s\langle\bar{s}s\rangle)$$

$$f_{\eta}^2m_{\eta}^2 = -(m_u\langle\bar{u}u\rangle + m_d\langle\bar{d}d\rangle)$$

and we can add the standard DGMOR relation for the $K^+$,

$$f_K^2m_K^2 = -(m_u\langle\bar{u}u\rangle + m_s\langle\bar{s}s\rangle)$$

We emphasise that these formulae, as well as the radiative decay and $U(1)_A$ Goldberger-Treiman relations derived below, do not depend at all on the $1/N_c$ expansion. In particular,
the constant $A$ appearing in the flavour singlet formula is defined as the non-perturbative parameter determining the topological susceptibility $\chi(0)$ in QCD according to the exact identity (2.22). As explained above, large-$N_c$ ideas do indeed provide a rationale for extending the familiar PCAC assumptions of pole dominance and smoothness to the flavour singlet channel, but these assumptions can be tested independently against experimental data.

The most useful form of these relations for phenomenology is to assume exact $SU(2)$ flavour symmetry and eliminate the quark masses and condensates in favour of $f_\pi, f_K, m_\pi^2$ and $m_K^2$ in the DGMOR relations for the $\eta$ and $\eta'$. This gives

$$ (f^{0\eta'})^2 m_{\eta'}^2 + (f^{0\eta})^2 m_\eta^2 = \frac{1}{3} (f_\pi^2 m_\pi^2 + 2 f_K^2 m_K^2) + 6A \tag{4.9} $$

$$ f^{0\eta'} f^{8\eta'} m_{\eta'}^2 + f^{0\eta} f^{8\eta} m_\eta^2 = \frac{2\sqrt{2}}{3} (f_\pi^2 m_\pi^2 - f_K^2 m_K^2) \tag{4.10} $$

$$ (f^{8\eta'} m_{\eta'}^2 + (f^{8\eta})^2 m_\eta^2 = -\frac{1}{3} (f_\pi^2 m_\pi^2 - 4 f_K^2 m_K^2) \tag{4.11} $$

We can also now clarify the precise relation of these results to the Witten-Veneziano formula for the mass of the $\eta'$ in its non-vanishing quark mass form, viz.

$$ m_{\eta'}^2 + m_\eta^2 - 2m_K^2 = -\frac{6}{f_\pi^2} \chi(0)|_{YM} \tag{4.12} $$

Of course, only the $m_{\pi}^2$ term on the l.h.s. is present in the chiral limit. Substituting in the explicit values for the masses in this formula gives a prediction [1] for the topological susceptibility, $\chi(0) \simeq -(180 \text{ MeV})^4$, which as we see below is remarkably close to the subsequently calculated lattice result.

If we now add the DGMOR formulae (4.9) and (4.11), we find

$$ (f^{0\eta'})^2 m_{\eta'}^2 + (f^{0\eta})^2 m_\eta^2 + (f^{8\eta'})^2 m_{\eta'}^2 + (f^{8\eta})^2 m_\eta^2 - 2f_K^2 m_K^2 = 6A \tag{4.13} $$

which we repeat is valid to all orders in $1/N_c$. To reduce this to its Witten-Veneziano approximation, we impose the large-$N_c$ limit to approximate the QCD topological charge parameter $A$ with $-\chi(0)|_{YM}$ as explained in section 2.4. We then set the ‘mixed’ decay constants $f^{0\eta'}$ and $f^{8\eta'}$ to zero and all the remaining decay constants $f^{0\eta}, f^{8\eta}$ and $f_K$ equal to $f_\pi$. With these approximations, we recover eq.(4.12). Eventually, after we have found explicit experimental values for all these quantities, we will be able to demonstrate quantitatively how good an approximation the large-$N_c$ Witten-Veneziano formula is to the generalised $U(1)_A$ DGMOR relation in full QCD.

4.2 Radiative decay formulae for $\pi^0, \eta, \eta' \to \gamma\gamma$

Radiative decays of the pseudoscalar mesons are of particular interest as they are controlled by the electromagnetic $U(1)_A$ anomaly,

$$ \partial^\mu J_{\mu\nu}^a - M_{ab} \phi_5^b - \sqrt{\frac{2}{n_f}} Q \delta_{a0} - a_m \frac{\alpha}{8\pi} F_{\mu\nu} \tilde{F}_{\mu\nu} = 0 \tag{4.14} $$
where $F_{\mu \nu} = \partial_{\mu} A_{\nu} - \partial_{\nu} A_{\mu}$ is the usual electromagnetic field strength and the anomaly coefficients $a_{\text{em}}$ are determined by the quark charges. The generating functional $\Gamma[V_{\mu \nu}^a, V_{\mu}^a, Q, \phi_0^a, \phi^a, A_{\mu}]$ of 1PI vertices including the photon satisfies the Ward identity

$$\partial_{\mu} \Gamma_{V_{\mu \nu}^a} - \sqrt{2 \pi} \delta_{a 0} \epsilon_{\mu \nu} - a_{\text{em}} \alpha \frac{F^{\mu \nu}}{8 \pi} \tilde{F}_{\mu \nu} - d_{abc} m^b \phi_0^c + f_{abc} V_{\mu \nu}^b \Gamma_{V_{\mu \nu}^c} - d_{abc} \phi_0^b \phi_0^c + d_{abc} \phi_0^c \Gamma_{\phi_0^b} = 0$$

(4.15)

To derive the radiative decay formulae, we first differentiate this identity with respect to the photon field $A_{\mu}$. This gives

$$i p_{\mu} \Gamma_{V_{\mu \nu}^a A^\lambda A^\rho} + \Phi_{ab} \Gamma_{\phi_0^b A^\lambda A^\rho} = - a_{\text{em}} \alpha \frac{\epsilon_{\mu \nu \lambda \rho} k^\mu_1 k^\nu_2}{2}$$

(4.16)

where $k_1, k_2$ are the momenta of the two photons. Notice that the mass term does not contribute directly to this formula. From its definition as 1PI w.r.t. the pseudoscalar fields, the vertex $\Gamma_{V_{\mu \nu}^a A^\lambda A^\rho}$ does not have a pole at $p^2 = 0$, even in the massless limit, so we find simply

$$\Phi_{ab} \Gamma_{\phi_0^b A^\lambda A^\rho} \big|_{p = 0} = - a_{\text{em}} \alpha \frac{\epsilon_{\mu \nu \lambda \rho} k^\mu_1 k^\nu_2}{2}$$

(4.17)

The radiative couplings $g_{\eta^a \gamma \gamma}$ for the physical mesons $\eta^a = \pi_0^0, \eta, \eta'$ are defined as usual from the decay amplitude $\langle \gamma \gamma | \eta^a \rangle$. With the PCAC assumptions already discussed, they can be identified with the 1PI vertices as follows:

$$\langle \gamma \gamma | \eta^a \rangle = - i g_{\eta^a \gamma \gamma} \epsilon_{\mu \nu \lambda \rho} k^\mu_1 k^\nu_2 \epsilon^\lambda (k_1) \epsilon^\rho (k_2) = i \Gamma_{\eta^a A^\lambda A^\rho} \epsilon^\lambda (k_1) \epsilon^\rho (k_2)$$

(4.18)

Re-expressing eq.(4.17) in terms of the canonically normalised ‘OZI bosons’ $\tilde{\eta}^a$, we therefore have the first form of the decay formula,

$$\tilde{f}^a g_{\tilde{\eta}^a \gamma \gamma} = a_{\text{em}} \alpha \frac{\pi}{\pi}$$

(4.19)

Then, rewriting this in terms of the physical pseudoscalar couplings $g_{\eta^a \gamma \gamma}$ and decay constants according to the relation (3.20) gives the final form for the generalised $U(1)_{\text{PCAC}}$ formula describing radiative pseudoscalar decays, incorporating both the electromagnetic and colour anomalies:

$$f_{\alpha} g_{\eta^a \gamma \gamma} + \sqrt{2 \pi} A g_{\gamma \gamma} = a_{\text{em}} \alpha \frac{\pi}{\pi}$$

(4.20)

Expanding this formula, we have

$$f_{0} g_{\eta^a \gamma \gamma} + f_{0} g_{\eta^a \gamma \gamma} + \sqrt{6} A g_{\gamma \gamma} = a_{\text{em}} \alpha \frac{\pi}{\pi}$$

(4.21)

$$f_{8} g_{\eta^a \gamma \gamma} + f_{8} g_{\eta^a \gamma \gamma} = a_{\text{em}} \alpha \frac{\pi}{\pi}$$

(4.22)

$$f_{\pi} g_{\pi \gamma \gamma} = a_{\text{em}} \alpha \frac{\pi}{\pi}$$

(4.23)

where $a_{\text{em}}^0 = \frac{2 \sqrt{2}}{3 \sqrt{3}} N_c$, $a_{\text{em}}^8 = \frac{1}{3 \sqrt{3}} N_c$ and $a_{\text{em}}^3 = \frac{1}{3} N_c$. 
The new element in the flavour singlet decay formula is the gluonic coupling parameter $g_{G\gamma\gamma}$. It takes account of the fact that because of the anomaly-induced mixing with the gluon topological density $Q$, the physical $\eta'$ is not a true Nambu-Goldstone boson so the naive PCAC formulae must be modified. $g_{G\gamma\gamma}$ is not a physical coupling and must be regarded as an extra parameter to be fitted to data, although in view of the identifications in eq.(3.19) it may reasonably be thought of as the coupling of the photons to the gluonic component of the $\eta'$.

The renormalisation group properties of these relations are readily derived from the RGE (2.27) for $\Gamma$. In the ‘OZI boson’ form, the unphysical coupling $g_{\eta'\gamma\gamma}$ satisfies the complementary RGE to the decay constant $\hat{f}^{aa}$ so the combination is RG invariant:

$$D\hat{f}^{aa} = \gamma \delta_{a0} \hat{f}^{aa} \quad D(\hat{f}^{aa} g_{\eta'\gamma\gamma}) = 0 \quad (4.24)$$

In contrast, all the decay constants and couplings in the relation (4.20) can be shown to be separately RG invariant, including the gluonic coupling $g_{G\gamma\gamma}$ [24, 44].

### 4.3 The renormalisation group, OZI and $1/N_c$ : a conjecture

Although these $U(1)_A$ PCAC relations have been derived purely on the basis of the pole dominance and smoothness assumptions, we will nevertheless find it useful in practical applications to exploit their OZI or large-$N_c$ behaviour, in conjunction with the renormalisation group.

The basic idea is that violations of the OZI rule, or equivalently anomalous large-$N_c$ behaviour, are generally related to the existence of the $U(1)_A$ anomaly. Moreover, we can identify the quantities which will be particularly sensitive to the anomaly as those which have RGEs involving the anomalous dimension $\gamma$. We therefore conjecture that the dependence of Green functions and 1PI vertices on $\gamma$ will be an important guide in identifying propagators and couplings which are likely to show violations of the OZI rule and those for which the OZI (or large-$N_c$) limit should be a good approximation [9, 24].

As an example, the large-$N_c$ order of the quantities in the flavour singlet decay relation (4.21) is as follows: $f^{aa} = O(\sqrt{N_c})$ for all the decay constants, $g_{\eta'\gamma\gamma} = O(\sqrt{N_c})$, $g_{G\gamma\gamma} = O(1)$, $a_{em} = O(N_c)$ and the topological susceptibility parameter $A = O(1)$. The renormalisation group behaviour is especially simple, with both the meson and gluonic couplings $g_{\eta'\gamma\gamma}$ and $g_{G\gamma\gamma}$ as well as the decay constants being RG invariant. Putting this together, we find that all the terms in the decay formula are of $O(N_c)$ except the anomalous contribution $Ag_{G\gamma\gamma}$ which is $O(1)$. Since it is RG invariant and independent of the anomalous dimension $\gamma$, we conjecture that it is a quantity for which the OZI (or large-$N_c$) approximation should be reliable so we expect it to be numerically small compared with the other contributions. In the next section, we test this against experiment.

As we shall see later, this conjecture has far-reaching implications for a range of predictions related to the anomaly, particularly in the interpretation of the $U(1)_A$ Goldberger-Treiman relation and associated ideas on the first moment sum rules for $g_1^p$ and $g_1^\gamma$ in deep-inelastic scattering.
4.4 Phenomenology

After all this theoretical development, we finally turn to experiment and use the data on the radiative decays $\eta, \eta' \to \gamma \gamma$ to deduce values for the pseudoscalar meson decay constants $f^{0\eta'}, f^{0\eta}, f^{8\eta'}$ and $f^{8\eta}$ from the set of decay formulae (4.21), (4.22) and $U(1)_A$ DGMOR relations (4.9)-(4.11). We will also find the value of the unphysical coupling parameter $g_{G\gamma\gamma}$ and test the realisation of the $1/N_c$ expansion in real QCD.

The two-photon decay widths are given by

$$\Gamma(\eta'\to \gamma\gamma) = \frac{m_{\eta'}^3}{64\pi}|g_{\eta'\gamma\gamma}|^2$$

(4.25)

The current experimental data, quoted in the Particle Data Group tables [45], are

$$\Gamma(\eta'\to \gamma\gamma) = 4.28 \pm 0.19 \text{ KeV}$$

(4.26)

which is dominated by the 1998 L3 data [46] on the two-photon formation of the $\eta'$ in $e^+e^- \to e^+e^-\pi^+\pi^-\gamma$, and

$$\Gamma(\eta \to \gamma\gamma) = 0.510 \pm 0.026 \text{ KeV}$$

(4.27)

which arises principally from the 1988 Crystal Ball [47] and 1990 ASP [48] results on $e^+e^- \to e^+e^-\eta$. From this data, we deduce the following results for the couplings $g_{\eta'\gamma\gamma}$ and $g_{\eta\gamma\gamma}$:

$$g_{\eta'\gamma\gamma} = 0.031 \pm 0.001 \text{ GeV}^{-1}$$

(4.28)

and

$$g_{\eta\gamma\gamma} = 0.025 \pm 0.001 \text{ GeV}^{-1}$$

(4.29)

which may be compared with $g_{\pi\gamma\gamma} = 0.024 \pm 0.001$ GeV.

We also require the pseudoscalar meson masses:

$$m_{\eta'} = 957.78 \pm 0.14 \text{ MeV} \quad m_{\eta} = 547.30 \pm 0.12 \text{ MeV}$$

$$m_K = 493.68 \pm 0.02 \text{ MeV} \quad m_{\pi} = 139.57 \pm 0.00 \text{ MeV}$$

(4.30)

and the decay constants $f_K$ and $f_{\pi}$. These are defined in the standard way, so we take the following values (in our normalisations) from the PDG [45]:

$$f_K = 113.00 \pm 1.03 \text{ MeV} \quad f_{\pi} = 92.42 \pm 0.26 \text{ MeV}$$

(4.31)

giving $f_K/f_{\pi} = 1.223 \pm 0.012$.

The octet decay constants $f^{8\eta}$ and $f^{8\eta'}$ are obtained from eqs.(4.11) and (4.22). This leaves three remaining equations which determine the singlet decay constants $f^{0\eta'}, f^{0\eta}$ and the gluonic coupling $g_{G\gamma\gamma}$ in terms of the QCD topological susceptibility parameter $A$. This dependence is plotted in Figs. 4 and 5.

To make a definite prediction, we need a theoretical input value for the topological susceptibility. In time, lattice calculations in full QCD with dynamical fermions should
Figure 4: The decay constants $f_{0\eta'}$ and $f_{0\eta}$ as functions of the non-perturbative parameter $A = (x \text{ MeV})^4$ which determines the topological susceptibility in QCD.

Figure 5: This shows the relative sizes of the contributions to the flavour singlet radiative decay formula (4.21) expressed as functions of the topological susceptibility parameter $A = (x \text{ MeV})^4$. The dotted (black) line denotes $\frac{2\sqrt{2}g_{\eta\gamma\gamma}}{\sqrt{2} \alpha_{\text{em}} \pi}$. The dominant contribution comes from the term $f_{0\eta'} g_{\eta'\gamma\gamma}$, denoted by the long-dashed (green) line, while the short-dashed (blue) line denotes $f_{0\eta} g_{\eta\gamma\gamma}$. The contribution from the gluonic coupling, $\sqrt{6}A g_{G\gamma\gamma}$, is shown by the solid (red) line.

be able to determine the parameter $A$. For the moment, however, only the topological susceptibility in pure Yang-Mills theory is known accurately. The most recent value [49] is

$$\chi(0)|_{YM} = -(191 \pm 5 \text{ MeV})^4 = -(1.33 \pm 0.14) \times 10^{-3} \text{ GeV}^4$$  \hspace{1cm} (4.32)$$

This supersedes the original value $\chi(0)|_{YM} \simeq -(180 \text{ MeV})^4$ obtained some time ago [50]. Similar estimates are also obtained using QCD spectral sum rule methods [51]. At this point, therefore, we have to make an approximation and so we assume that the $O(1/N_c)$ corrections in the identification

$$A = \chi(0)|_{YM} + O(1/N_c)$$  \hspace{1cm} (4.33)$$

are numerically small. With this provisional input for $A$, we can then determine the full set of decay constants:

$$f_{0\eta'} = 104.2 \pm 4.0 \text{ MeV} \hspace{1cm} f_{0\eta} = 22.8 \pm 5.7 \text{ MeV}$$

$$f_{8\eta'} = -36.1 \pm 1.2 \text{ MeV} \hspace{1cm} f_{8\eta} = 98.4 \pm 1.4 \text{ MeV}$$  \hspace{1cm} (4.34)$$

and

$$g_{G\gamma\gamma} = -0.001 \pm 0.072 \text{ GeV}^{-4}$$  \hspace{1cm} (4.35)$$
It is striking how close both the diagonal decay constants $f^{0\eta'}$ and $f^{8\eta}$ are to $f_\pi$. Predictably, the off-diagonal ones $f^{0\eta'}$ and $f^{8\eta'}$ are strongly suppressed.

It is also useful to quote these results in the two-angle parametrisation normally used in phenomenology. Defining,

$$
\begin{pmatrix}
  f^{0\eta'} & f^{0\eta} \\
  f^{8\eta'} & f^{8\eta}
\end{pmatrix}
= 
\begin{pmatrix}
  f_0 \cos \theta_0 & -f_0 \sin \theta_0 \\
  f_8 \sin \theta_8 & f_8 \cos \theta_8
\end{pmatrix}
$$

we find

$$
f_0 = 106.6 \pm 4.2 \text{ MeV} \quad f_8 = 104.8 \pm 1.3 \text{ MeV}
$$

that is

$$
\frac{f_0}{f_\pi} = 1.15 \pm 0.05 \quad \frac{f_8}{f_\pi} = 1.13 \pm 0.02
$$

Given these results, we can now investigate how closely our expectations based on OZI or $1/N_c$ reasoning are actually realised by the experimental data. With the input value (4.32) for $A$, the numerical magnitudes and $1/N_c$ orders of the terms in the flavour singlet decay relation are as follows (see Fig. 5):

$$
f^{0\eta'} g_{\eta'\gamma\gamma} [N_c; 3.23] + f^{0\eta} g_{\eta\gamma\gamma} [N_c; 0.57] + \sqrt{6} A g_{G\gamma\gamma} [1; -0.005 \pm 0.23]
$$

The important point is that the gluonic contribution $g_{G\gamma\gamma}$, which is suppressed by a power of $1/N_c$ compared to the others, is also experimentally small. The near-vanishing for the chosen value of $A$ is presumably a coincidence, but we see from Fig. 5 that across a reasonable range of values of the topological susceptibility it is still contributing no more than around 10%, in line with our expectations for a RG-invariant, OZI-suppressed quantity.

It is also interesting to see how the $1/N_c$ approximation is realised in the $U(1)_A$ DG-MOR generalisation (4.13) of the Witten-Veneziano formula (4.12). Here we find

$$
(f^{0\eta'})^2 m_{\eta'}^2 [N_c; 9.96] + (f^{0\eta})^2 m_\eta^2 [N_c; 0.15] + (f^{8\eta'})^2 m_\eta^2 [N_c; 1.19]
$$

$$
+ (f^{8\eta})^2 m_{\eta'}^2 [N_c; 2.90] - 2 f_K^2 m_K^2 [N_c; -6.22] = 6A [1; 7.98]
$$

This confirms the picture that the anomaly-induced contribution of $O(1/N_c)$ to $m_{\eta'}$, which gives a sub-leading $O(1)$ effect in $(f^{0\eta'})^2 m_{\eta'}^2$, is in fact numerically dominant and matched by the $O(1)$ topological susceptibility term $6A$. Away from the chiral limit, the conventional non-anomalous terms are all of $O(N_c)$ and balance as expected. The surprising numerical accuracy of the Witten-Veneziano formula (2.18) is seen to be in part due to a cancellation between the underestimates of $f^{8\eta'}$ (taken to be 0) and $f_K$ (set equal to $f_\pi$). This emphasises, however, that great care must be taken in using the formal order in the $1/N_c$ expansion as a guide to the numerical importance of a physical quantity, especially in the $U(1)_A$ channel.
Nevertheless, the fact that the RG-invariant, OZI-suppressed coupling $g_{G\gamma\gamma}$ is experimentally small is a very encouraging result. It increases our confidence that we are able to identify quantities where the OZI, or leading $1/N_c$, approximation is likely to be numerically good. It also shows that $g_{G\gamma\gamma}$ gives a contribution to the decay formula which is entirely consistent with its picturesque interpretation as the coupling of the photons to the anomaly-induced gluonic component of the $\eta'$. *A posteriori*, the fact that its contribution is at most 10% explains the general success of previous theoretically inconsistent phenomenological parametrisations of $\eta'$ decays in which the naive current algebra formulae omitting the gluonic term are used.

However, while the flavour singlet decay formula is well-defined and theoretically consistent, it is necessarily non-predictive. To be genuinely useful, we would need to find another process in which the same coupling enters. The problem here is that, unlike the decay constants which are universal, the coupling $g_{G\gamma\gamma}$ is process-specific just like $g_{\eta'\gamma\gamma}$ or $g_{\eta\gamma\gamma}$. There are of course many other processes to which our methods may be applied such as $\eta'\to V\gamma$, where $V$ is a flavour singlet vector meson $\rho, \omega, \phi$, or $\eta'\to \pi^+\pi^-\gamma$. The required flavour singlet formulae may readily be written down, generalising the naive PCAC formulae. However, each will introduce its own gluonic coupling, such as $g_{GV\gamma}$. Although strict predictivity is lost, our experience with the two-photon decays suggests that these extra couplings will give relatively small, at most $O(10-20\%)$, contributions if like $g_{G\gamma\gamma}$ they can be identified as RG invariant and $1/N_c$ suppressed. This observation restores at least a reasonable degree of predictivity to the use of PCAC methods in the $U(1)_A$ sector.

### 4.5 $U(1)_A$ Goldberger-Treiman relation

A further classic application of PCAC is to the pseudoscalar couplings of the nucleon. For the pion, the relation between the axial-vector form factor of the nucleon and the pion-nucleon coupling $g_{\pi NN}$ is the famous Goldberger-Treiman relation. Here, we present its generalisation to the flavour singlet sector, which involves the anomaly and gluon topology. This $U(1)_A$ Goldberger-Treiman relation was first proposed by Veneziano [4] in an investigation of the ‘proton-spin’ problem and further developed in refs.[8, 9, 52, 3].

The axial-vector form factors are defined from

$$\langle N|J_{\mu 5}^a|N\rangle = 2m_N \left( G_A^a(p^2)s_\mu + G_P^a(p^2)p.s_p_\mu \right)$$

(4.41)

where $s_\mu = \bar{u}\gamma_\mu\gamma_5u/2m_N$ is the covariant spin vector. In the absence of a massless pseudoscalar, only the form factors $G_A^a(0)$ contribute at zero momentum.

Expressing the matrix element in terms of the 1PI vertices derived from the generating functional $\Gamma[V^a_{\mu 5}, V^a_\mu, Q, \phi_5^a, \phi^a]$, including spectator fields $N, \bar{N}$ for the nucleon, we have

$$\langle N|J_{\mu 5}^a|N\rangle = \bar{u}\left( \Gamma_{V^a_{\mu 5}}\bar{N}N + W_{V^a_{\mu}}\phi_5 \Gamma_{Q\bar{N}N} + W_{V^a_{\mu}}\bar{S}_5 \phi_{5N} \right)u$$

(4.42)

Note that this expansion relies on the specific definition (2.8) of $\Gamma$ as a partial Legendre transform.
We also need the following relation, valid for all momenta, which is derived directly from the fundamental anomalous chiral Ward identity (2.9) for $\Gamma$:

$$\partial_{\mu} \Gamma_{\gamma_5 N N} = - \Phi_{ab} \Gamma_{b_5 N N}$$ (4.43)

Now, taking the divergence of eq.(4.42), using this Ward identity and then taking the zero-momentum limit, noting that the propagators vanish at zero momentum since there is no massless pseudoscalar, gives

$$2m_N G_A^a(0) \bar{u} \gamma_5 u = i\bar{u} \Gamma_{\gamma_5 N N \mid p=0} u$$ (4.44)

The meson-nucleon couplings are related to the 1PI vertices by

$$\langle N | \eta^a N \rangle = g_{\eta^a N N} \bar{u} \gamma_5 u = i\bar{u} \Gamma_{\gamma_5 N N} u$$ (4.45)

Re-expressing eq.(4.44) in terms of the canonically normalised ‘OZI boson’ field $\hat{\eta}^a$, we therefore derive

$$2m_N G_A^a(0) = \hat{f}^{a\alpha} g_{\hat{\eta}^a N N}$$ (4.46)

This relation will be useful to us when we consider the ‘proton spin’ problem.

All that now remains to cast this into its final form is to make the familiar change of variables from $Q, \hat{\eta}^a$ to $G, \eta^a$, where $\eta^a$ are interpreted as the physical mesons. We therefore find the generalised $U(1)_A$ Goldberger-Treiman relation:

$$2m_N G_A^a(0) = f^{a\alpha} g_{\eta^a N N} + \sqrt{2n_f A} g_{G N N} \delta_{a0}$$ (4.47)

For the individual components, this is

- $$2m_N G_A^3 = f_\pi g_{\pi N N}$$ (4.48)
- $$2m_N G_A^8 = f_{8\eta} g_{\eta N N} + f_{8\eta} g_{\eta N N}$$ (4.49)
- $$2m_N G_A^0 = f_{0\eta} g_{\eta N N} + f_{0\eta} g_{\eta N N} + \sqrt{6} A g_{G N N}$$ (4.50)

The renormalisation group properties of these relations are described in great detail in ref.[9]. It is clear that the flavour singlet axial coupling $G_A^0$ satisfies a homogeneous RGE and scales with the anomalous dimension $\gamma$ corresponding to the multiplicative renormalisation of $J_{\mu_5}$. In the form (4.46), RG consistency is simply achieved by

$$\mathcal{D} \hat{f}^{a\alpha} = \gamma \delta_{a0} \hat{f}^{a\alpha} \quad \mathcal{D} g_{\eta^a N N} = 0$$ (4.51)

All the scale dependence is in the decay constant $\hat{f}^{a\alpha}$ while the the coupling $g_{\eta^a N N}$ of the ‘OZI boson’ to the nucleon is RG invariant (in contrast to $g_{\eta^a \gamma_5 \gamma}$). In the final form (4.47) involving the physical decay constants, a careful analysis shows that apart from $G_A^0(0)$ the only other non RG-invariant quantity is the gluonic coupling $g_{G N N}$, which is required to satisfy the following non-homogeneous RGE to ensure the self-consistency of eq.(4.50):

$$\mathcal{D} g_{G N N} = \gamma \left( g_{G N N} + \frac{1}{\sqrt{2nfA}} \frac{1}{A} \hat{f}^{a\alpha} g_{\eta^a N N} \right)$$ (4.52)

The $p \rightarrow 0$ limit is delicate, as is the case for the derivation of the conventional Goldberger-Treiman relation, and should be taken in this order. Literally at $p = 0$, both sides vanish since $\bar{u} \gamma_5 u = 0$.\footnote{The $p \rightarrow 0$ limit is delicate, as is the case for the derivation of the conventional Goldberger-Treiman relation, and should be taken in this order. Literally at $p = 0$, both sides vanish since $\bar{u} \gamma_5 u = 0$.}
The large-$N_c$ behaviour in the flavour singlet relation is as follows: $G_A^0 = O(N_c)$, $f^{\eta}, f^{\eta'} = O(\sqrt{N_c})$, $A = O(1)$, $g_{\eta NN}, g_{\eta' NN} = O(\sqrt{N_c})$, $g_{GNN} = O(1)$. So the final term $A_{GNN}$ is $O(1)$, suppressed by a power of $1/N_c$ compared to all the others, which are $O(N_c)$.

We see that, like $g_{G\gamma\gamma}$, the gluonic coupling $g_{GNN}$ is suppressed at large $N_c$ relative to the corresponding meson couplings. However, unlike $g_{G\gamma\gamma}$ which is RG invariant, $g_{GNN}$ has a complicated RG non-invariance and depends on the anomaly-induced anomalous dimension $\gamma$. The conjecture in section 4.3 then suggests that while the OZI or large-$N_c$ approximation should be a good guide to the value of $g_{G\gamma\gamma}$, we may expect significant OZI violations for $g_{GNN}$. We would therefore not be surprised to find that $g_{GNN}$ makes a sizeable numerical contribution to the $U(1)_A$ Goldberger-Treiman relation.

We now try to test these expectations against the experimental data. We first introduce a notation that has become standard in the literature on deep-inelastic scattering. There, the axial couplings are written as

$$G_A^{3} = \frac{1}{2} a^3, \quad G_A^{8} = \frac{1}{2\sqrt{3}} a^8, \quad G_A^{0} = \frac{1}{\sqrt{6}} a^0$$

(4.53)

where the $a^a$ have a simple interpretation in terms of parton distribution functions.

Experimentally,

$$a^3 = 1.267 \pm 0.004 \quad a^8 = 0.585 \pm 0.025$$

(4.54)

from low-energy data on nucleon and hyperon beta decay. The latest result\(^9\) for $a^0$ quoted by the COMPASS collaboration [53] from deep-inelastic scattering data is

$$a^0 |_{Q^2 \to \infty} = 0.33 \pm 0.06$$

(4.55)

with a similar result from HERMES [54].

The OZI expectation is that $a^0 = a^8$. In the context of DIS, this is a prediction of the simple quark model, where it is known as the Ellis-Jaffe sum rule [57]). We return to this in the context of the ‘proton spin’ problem in section 5 but for now we concentrate on the low-energy phenomenology of the pseudoscalar meson-nucleon couplings.

The original Goldberger-Treiman relation (4.48) gives the following value for the pion-nucleon coupling,

$$g_{\pi NN} = 12.86 \pm 0.06$$

(4.56)

consistent to within about 5% with the experimental value $13.65(13.80) \pm 0.12$ (depending on the dataset used [58]). In an ideal world where $g_{\eta NN}$ and $g_{\eta' NN}$ were both known, we would now verify the octet formula (4.49) then determine the gluonic coupling $g_{GNN}$ from the singlet Goldberger-Treiman relation (4.50). However, the experimental situation with the $\eta$ and $\eta'$-nucleon couplings is far less clear. One would hope to determine these

\(^9\)This supersedes the result $a^0 |_{Q^2 = 4G_{A}V^2} = 0.237^{+0.024}_{-0.029}$ quoted by COMPASS in 2005 [55, 56], which we used as input into our analysis of the phenomenology of the $U(1)_A$ GT relation in ref.[3]. The fits presented here are updated from those of ref.[3] to take account of this. For a further discussion of the experimental situation, see section 5.
Figure 6: These figures show the dimensionless $\eta$-nucleon coupling $g_{\eta N N}$ and the gluonic coupling $g_{G N N}$ in units of GeV$^{-3}$ expressed as functions of the experimentally uncertain $\eta'$-nucleon coupling $g_{\eta' N N}$, as determined from the flavour octet and singlet Goldberger-Treiman relations (4.49) and (4.50).

couplings from the near threshold production of the $\eta$ and $\eta'$ in nucleon-nucleon collisions, i.e. $pp \rightarrow pp\eta$ and $pp \rightarrow pp\eta'$, measured for example at COSY-II [59, 60, 61]. However, the $\eta$ production is dominated by the $N(1535)S_{11}$ nucleon resonance which decays to $N\eta$, and as a result very little is known about $g_{\eta N N}$ itself. The detailed production mechanism of the $\eta'$ is not well understood. However, since there is no known baryonic resonance decaying into $N\eta'$, we may simply assume that the reaction $pp \rightarrow pp\eta'$ is driven by the direct coupling supplemented by heavy-meson exchange. This allows an upper bound to be placed on $g_{\eta' N N}$ and on this basis ref.[62] quotes $g_{\eta' N N} < 2.5$. This is supported by an analysis [63] of very recent data from CLAS [64] on the photoproduction reaction $\gamma p \rightarrow p\eta'$. Describing the cross-section data with a model comprising the direct coupling together with $t$-channel meson exchange and $s$ and $u$-channel resonances, it is found that equally good fits can be obtained for several values of $g_{\eta' N N}$ covering the whole region $0 < g_{\eta' N N} < 2.5$.

In view of this experimental uncertainty, we shall use the octet and singlet Goldberger-Treiman relations to plot the predictions for $g_{\eta N N}$ and $g_{G N N}$ as a function of the ill-determined $\eta'$-nucleon coupling in the experimentally allowed range $0 < g_{\eta' N N} < 2.5$. The results (again taking the value (4.32) for $A$) are given in Fig. 6. In Fig. 7 we have shown the relative magnitudes of the various contributions to the flavour-singlet formula.

What we learn from this is that for values of $g_{\eta' N N}$ approaching the upper end of the experimentally allowed range, the contribution of the OZI-suppressed gluonic coupling $g_{G N N}$ is quite large. The variation of $f^{0\eta'} g_{\eta' N N}$ over the allowed range is compensated almost entirely by the variation of $\sqrt{6} g_{G N N}$, with the $f^{0\eta} g_{\eta N N}$ contribution remaining relatively constant.

For example, if experimentally we found $g_{\eta' N N} \simeq 2.5$, which corresponds to the cross-sections for $pp \rightarrow pp\eta'$ and $\gamma p \rightarrow p\eta'$ being almost entirely determined by the direct coupling, then we would have $g_{\eta N N} \simeq 4.14$ and $g_{G N N} \simeq -31.2$ GeV$^{-3}$. In terms of the contributions to the $U(1)_A$ Goldberger-Treiman relation, this would give (in GeV)

$$2m_N G^0_A[N_c; 0.25] = f^{0\eta'} g_{\eta' N N}[N_c; 0.26] + f^{0\eta} g_{\eta N N}[N_c; 0.09] + \sqrt{6} A g_{G N N}[O(1); -0.10]$$  \hspace{1cm} (4.57)

The anomalously small value of $G^0_A$ compared to its OZI value (the OZI approximation is $2m_N G^0_A|_{OZI} = \sqrt{2} 2m_N G^0_A = 0.45$) is then due to the partial cancellation of the sum of the
Figure 7: This shows the relative sizes of the contributions to the $U(1)_A$ Goldberger-Treiman relation from the individual terms in eq.(4.50), expressed as functions of the coupling $g_{\eta'NN}$. The dotted (black) line denotes $2m_N G_{A}^0$. The long-dashed (green) line is $f_{\eta'}^0 g_{\eta'NN}$ and the short-dashed (blue) line is $f_{\eta}^0 g_{\eta NN}$. The solid (red) line shows the contribution of the novel gluonic coupling, $\sqrt{6}A g_{GNN}$, where $A$ determines the QCD topological susceptibility.

meson-nucleon coupling terms by the gluonic coupling $g_{GNN}$. Although formally $O(1/N_c)$ suppressed, numerically it gives a major contribution to the large OZI violation in $G_{A}^0$. This would give some support to our conjecture and provide further evidence that we are able to predict the location of large OZI violations using the renormalisation group as a guide.

Of course, it may be that experimentally we eventually find a value for $g_{\eta'NN} \simeq 1.5$, in the region where $g_{GNN}$ contributes only around 10% or less. Although surprising, this would open the possibility that all gluonic couplings of type $g_{GXX}$ are close to zero, which could be interpreted as implying that the gluonic component of the $\eta'$ wave function is simply small. Clearly, a reliable determination of $g_{\eta'NN}$, or equivalently $g_{\eta NN}$, would shed considerable light on the $U(1)_A$ dynamics of QCD.

5. Topological charge screening and the ‘proton spin’

So far, we have focused on the implications of the $U(1)_A$ anomaly for low-energy QCD phenomenology. However, the anomaly also plays a vital role in the interpretation of high-energy processes, in particular polarised deep-inelastic scattering.

In this section, we discuss one of the most intensively studied topics in QCD of the last two decades - the famous, but misleadingly named, ‘proton spin’ problem. We review the interpretation initially proposed by Veneziano [4] and developed by us in a series of papers exploring the relation with the $U(1)_A$ GT relation and gluon topology [8, 9, 65]. In subsequent work with Narison, we were able to quantify our prediction by using QCD spectral sum rules to compute the slope $\chi'(0)$ of the topological susceptibility [10, 52]. Remarkably, the most recent experimental data from the COMPASS [53] and HERMES [54] collaborations, released in September 2006, now confirms our original 1994 numerical prediction [10].
5.1 The $g_1^p$ and angular momentum sum rules

The ‘proton spin’ problem concerns the sum rule for the first moment of the polarised proton structure function $g_1^p$. This is measured in polarised DIS experiments through the inclusive processes $\mu p \rightarrow \mu X$ (EMC, SMC, COMPASS at CERN) or $ep \rightarrow eX$ (SLAC, HERMES at DESY) together with similar experiments on a deuteron target. The polarisation asymmetry of the cross-section is expressed as

$$x \frac{d\Delta \sigma}{dx dy} = \frac{Y_p}{2} \frac{16\pi^2 \alpha^2}{s} g_1^p(x, Q^2) + O\left(\frac{M^2 x^2}{Q^2}\right)$$

(5.1)

with conventional notation: $Q^2 = -q^2$ and $x = Q^2/2p_2q$ are the Bjorken variables, where $p_2, q$ are the momenta of the target proton and incident virtual photon respectively, $y = Q^2/xs$ and $Y_p = (2 - y)/y$.

According to standard theory, $g_1^p$ is determined by the proton matrix element of two electromagnetic currents carrying a large spacelike momentum. The sum rule for the first moment of $g_1^p$ is derived from the twist 2, spin 1 terms in the operator product expansion for the currents:

$$J^\lambda(q)J^\rho(-q) \sim 2\epsilon^{\lambda\mu\nu\rho} \frac{q_\mu}{Q^2} \left[ \Delta C_1^{N S}(\alpha_s)\left(J^{3\mu}_5 + \frac{1}{\sqrt{3}} J^{8\mu}_5\right) + \frac{2\sqrt{2} \sqrt{3}}{\sqrt{3}} \Delta C_1^{S}(\alpha_s)J^{0\mu}_5 \right]$$

(5.2)

where $\Delta C_1^{N S}$ and $\Delta C_1^{S}$ are Wilson coefficients and $J^{a\mu}_5 (a = 3, 8, 0)$ are the renormalised axial currents, with the normalisations defined in section 2. It is the occurrence of the axial currents in this OPE that provides the link between the $U(1)_A$ anomaly and polarised DIS. The sum rule is therefore:

$$\Gamma_1^p(Q^2) \equiv \int_0^1 dx \ g_1^p(x, Q^2) = \frac{1}{12} \Delta C_1^{N S}\left(a^3 + \frac{1}{3} a^8\right) + \frac{1}{9} \Delta C_1^{S}a^0(Q^2)$$

(5.3)

where the axial charges $a^3, a^8$ and $a^0(Q^2)$ are defined in terms of the forward proton matrix elements as in eq.(4.53). Here, we have explicitly shown the $Q^2$ scale dependence associated with the RG non-invariance of $a^0(Q^2)$.

Since the flavour non-singlet axial charges are known from low-energy data, a measurement of the first moment of $g_1^p$ amounts to a determination of the flavour singlet $a^0(Q^2)$. At the time of the original EMC experiment in 1988 [66] the theoretical expectation based on the quark model was that $a^0 = a^8$. The resulting sum rule for $g_1^p$ is known as the Ellis-Jaffe sum rule [57]. The great surprise of the EMC measurement was the discovery that in fact $a^0$ is significantly suppressed relative to $a^8$, and indeed the earliest results suggested it could even be zero. However, the reason the result sent shockwaves through both the theoretical and experimental communities (to date, the EMC paper has over 1300 citations) was the interpretation that this implies that the quarks contribute only a fraction of the total spin of the proton.

In fact, this interpretation relies on the simple valence quark model of the proton and is not true in QCD, where the axial charge decouples from the real angular momentum sum rule for the proton. Rather, as we shall show, the suppression of $a^0(Q^2)$ reflects the dynamics of gluon topology and appears to be largely independent of the structure of the
proton itself. Precisely, it is a manifestation of *topological charge screening* in the QCD vacuum.

The angular momentum sum rule is derived by taking the forward matrix element of the conserved angular momentum current $M^\mu\nu\lambda$, defined in terms of the energy-momentum tensor as

$$M^\mu\nu\lambda = x^{[\mu}T^{\lambda]}{}_{\mu} + \partial_\rho X^{\mu\nu\lambda}$$

The inclusion of the arbitrary tensor $X^{\mu\nu\lambda}$ just reflects the usual freedom in QFT of defining conserved currents. This gives us some flexibility in attempting to write $M^\mu\nu\lambda$ as a sum of local operators, suggesting interpretations of the total angular momentum as a sum of ‘components’ of the proton spin. In fact, however, it is not possible to write $M^\mu\nu\lambda$ as a sum of operators corresponding to quark and gluon spin and angular momentum in a gauge-invariant way. The best decomposition is \[67, 68, 69\]

$$M^\mu\nu\lambda = O^\mu\nu\lambda_1 + O^\mu\nu\lambda_2 + O^\mu\nu\lambda_3 + \ldots$$

where the dots denote terms whose forward matrix elements vanish. Here,

$$O^\mu\nu\lambda_1 = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} \bar{q} \gamma_\sigma q = \frac{1}{2} \epsilon^{\mu\nu\lambda\sigma} \sqrt{2n_f} f_{\sigma 5}^0$$

$$O^\mu\nu\lambda_2 = i\bar{q} \gamma_\mu D^\lambda q$$

$$O^\mu\nu\lambda_3 = F^{\mu\rho} F^{\lambda\rho}$$

At first sight, $O^\mu\nu\lambda_1$ looks as if it could be associated with ‘quark spin’, since for free Dirac fermions the spin operator coincides with the axial vector current. $O^\mu\nu\lambda_2$ would correspond to ‘quark orbital angular momentum’, leaving $O^\mu\nu\lambda_3$ as ‘gluon total angular momentum’. Any further decomposition of the gluon angular momentum is necessarily not gauge invariant.

The forward matrix elements of these operators may be expressed in terms of form factors and, as we showed in ref.\[68\], this exhibits an illuminating cancellation. After some analysis, we find:

$$\langle p, s | O^\mu\nu\lambda_1 | p, s \rangle = a_0 m_N \epsilon^{\mu\nu\lambda\sigma} s_\sigma$$

$$\langle p, s | O^\mu\nu\lambda_2 | p, s \rangle = J_q \frac{1}{2m_N} p_\rho p_\sigma (\epsilon^{[\mu} e^{[\lambda]}\nu])_{\rho\sigma} s_\sigma - a_0 m_N \epsilon^{\mu\nu\lambda\sigma} s_\sigma$$

$$\langle p, s | O^\mu\nu\lambda_3 | p, s \rangle = J_g \frac{1}{2m_N} p_\rho p_\sigma (\epsilon^{[\mu} e^{[\lambda]}\nu])_{\rho\sigma} s_\sigma$$

The angular momentum sum rule for the proton is then just

$$\frac{1}{2} = J_q + J_g$$

where the Lorentz and gauge-invariant form factors $J_q$ and $J_g$ may reasonably be thought of as representing quark and gluon total angular momentum. However, even this interpretation is not at all rigorous, not least because $J_q$ and $J_g$ mix under renormalisation and scale as

$$\frac{d}{d \ln Q^2} \begin{pmatrix} J_q \\ J_g \end{pmatrix} = \frac{\alpha_s}{4\pi} \begin{pmatrix} -\frac{8}{3} C_F & \frac{2}{3} n_f \\ \frac{8}{3} C_F & -\frac{2}{3} n_f \end{pmatrix} \begin{pmatrix} J_q \\ J_g \end{pmatrix}$$
Only the total angular momentum is Lorentz, gauge and scale invariant.\textsuperscript{10}  
  
The crucial observation, however, is that the axial charge $a^0$ explicitly cancels from the angular momentum sum rule. $a^0$ is an important form factor, which relates the first moment of $g_1^p$ to gluon topology via the $U(1)_A$ anomaly, but it is not part of the angular momentum sum rule for the proton.  

Just as $a^0$ can be measured in polarised inclusive DIS, the form factors $J_q$ and $J_g$ can be extracted from measurements of unpolarised generalised parton distributions (GPDs) in processes such as deeply-virtual Compton scattering $\gamma^* p \rightarrow \gamma p$. These can also in principle be calculated in lattice QCD. The required identifications with GPDs are given in ref.[68].

5.2 QCD parton model  

Before describing our resolution of the ‘proton spin’ problem, we briefly review the parton model interpretation of the first moment sum rule for $g_1^p$.

In the simplest form of the parton model, the proton structure at large $Q^2$ is described by parton distributions corresponding to free valence quarks only. The polarised structure function is given by

$$ g_1^p(x) = \frac{1}{2} \sum_{i=1}^{n_f} e_i^2 \Delta q_i(x) $$  \hspace{1cm} (5.10)

where $\Delta q_i(x)$ is defined as the difference of the distributions of quarks (and antiquarks) with helicities parallel and antiparallel to the nucleon spin. It is convenient to work with the conventionally-defined flavour non-singlet and singlet combinations $\Delta q^{NS}$ and $\Delta q^S$ (often also written as $\Delta \Sigma$).

In this model, the first moment of the singlet quark distribution $\Delta q^S = \int_0^1 dx \Delta q^S(x)$ can be identified as the sum of the helicities of the quarks. Interpreting the structure function data in this model then leads to the conclusion that the quarks carry only a small fraction of the spin of the proton. There is indeed a real contradiction between the experimental data and the free valence quark parton model.

However, this simple model leaves out many important features of QCD, the most important being gluons, RG scale dependence and the chiral $U_A(1)$ anomaly. When these effects are included, in the QCD parton model, the naive identification of $\Delta q^S$ with spin no longer holds and the experimental results for $g_1^p$ can be accommodated, though not predicted.

In the QCD parton model, the polarised structure function is written in terms of both quark and gluon distributions as follows:

$$ g_1^p(x, Q^2) = \int_x^1 \frac{du}{u} \left[ \Delta C^{NS}(u) \Delta q^{NS}(u, t) + \Delta C^S(x) \Delta q^S(u, t) + \Delta C^g(x) \Delta g(u, t) \right] $$  \hspace{1cm} (5.11)

\textsuperscript{10}For a careful discussion of the parton interpretation of longitudinal and transverse angular momentum sum rules, see ref.[70]. This confirms our assertion that the axial charge $a^0$ is not to be identified with quark helicities in the parton model.
where $\Delta C^S$, $\Delta C^g$ and $\Delta C^{NS}$ are perturbatively calculable functions related to the Wilson coefficients and the quark and gluon distributions have a priori $t = \ln Q^2/\Lambda^2$ dependence determined by the RG evolution, or DGLAP, equations. The first moment sum rule is therefore

$$\Gamma_1^p(Q^2) = \frac{1}{9} \left[ \Delta C_1^{NS} \Delta q^{NS} + \Delta C_1^S \Delta q^S + \Delta C_1^g \Delta g \right]$$ (5.12)

Comparing with eq.(5.3), we see that the axial charge $a^0(Q^2)$ is identified with a linear combination of the first moments of the singlet quark and gluon distributions. It is often, though not always, the case that the moments of parton distributions can be identified in one-to-one correspondence with the matrix elements of local operators. The polarised first moments are special in that two parton distributions correspond to the same local operator.

The RG evolution equations for the first moments of the parton distributions are derived from the matrix of anomalous dimensions for the lowest spin, twist 2 operators. This introduces an inevitable renormalisation scheme ambiguity in the definitions of $\Delta q$ and $\Delta g$, and their physical interpretation is correspondingly nuanced. The choice closest to our own analysis is the ‘AB’ scheme [71] where the parton distributions have the following RG evolution:

$$\frac{d}{d \ln Q^2} \Delta q^{NS} = 0 \quad \frac{d}{d \ln Q^2} \Delta q^S = 0$$

$$\frac{d}{d \ln Q^2} \frac{\alpha_s}{2\pi} \Delta g(Q^2) = \gamma \left( \frac{\alpha_s}{2\pi} \Delta g(Q^2) - \frac{1}{3} \Delta q^S \right)$$ (5.13)

which requires $\Delta C_1^g = \frac{3\alpha_s}{2\pi} \Delta C_1^S$. It is then possible to make the following identifications with the axial charges:

$$a^3 = \Delta u - \Delta d$$

$$a^8 = \Delta u + \Delta d - 2\Delta s$$

$$a^0(Q^2) = \Delta u + \Delta d + \Delta s - \frac{3\alpha_s}{2\pi} \Delta g(Q^2)$$ (5.14)

with $\Delta q^S = \Delta u + \Delta d + \Delta s$. Notice that in the AB scheme, all the scale dependence of the axial charge $a^0(Q^2)$ is assigned to the gluon distribution $\Delta g(Q^2)$.

This was the identification originally introduced for the first moments by Altarelli and Ross [72], and resolves the ‘proton spin’ problem in the context of the QCD parton model. In this scheme, the Ellis-Jaffe sum rule follows from the assumption that in the proton both $\Delta s$ and $\Delta g(Q^2)$ are zero, which is the natural assumption in the free valence quark model. This is equivalent to the OZI approximation $a^0(Q^2) = a^8$. However, in the full QCD parton model, there is no reason why $\Delta g(Q^2)$, or even $\Delta s$, should be zero in the proton. Indeed, given the different scale dependence of $a^0(Q^2)$ and $a^8$, it would be unnatural to expect this to hold in QCD itself.

An interesting conjecture [72] is that the observed suppression in $a^0(Q^2)$ is due overwhelmingly to the gluon distribution $\Delta g(Q^2)$ alone. Although by no means a necessary consequence of QCD, this is a reasonable expectation given that it is the anomaly (which
is due to the gluons and is responsible for OZI violations) which is responsible for the scale dependence in $a^0(Q^2)$ and $\Delta q(Q^2)$ whereas the $\Delta q$ are scale invariant. This would be in the same spirit as our conjecture on OZI violations in low-energy phenomenology in section 4.3. To test this, however, we need to find a way to measure $\Delta q(Q^2)$ itself, rather than the combination $a^0(Q^2)$. The most direct option is to extract $\Delta q(x, Q^2)$ from processes such as open charm production, $\gamma^* g \rightarrow c\bar{c}$, which is currently being intensively studied by the COMPASS [73], STAR [74] and PHENIX [75] collaborations.

5.3 Topological charge screening

We now describe a less conventional approach to deep-inelastic scattering based entirely on field-theoretic concepts. In particular, the role of parton distributions is taken over by the 1PI vertices of composite operators introduced above. (For a review, see ref.[76]).

Once again, the starting point is the use of the OPE to express the moments of a generic structure function $F(x, Q^2)$ as

$$\int_0^1 dx \ x^{-1} F(x, Q^2) = \sum_A C^n_A(Q^2) \langle p|O^n_A(0)|p\rangle$$ (5.15)

where $O^n_A$ denotes the set of lowest twist, spin $n$ operators and $C^n_A(Q^2)$ are the corresponding Wilson coefficients. The next step is to introduce a new set of composite operators $\hat{O}_B$, chosen to encompass the physically relevant degrees of freedom, and write the matrix element as a product of two-point Green functions and 1PI vertices as follows:

$$\int_0^1 dx \ x^{-1} F(x, Q^2) = \sum_A \sum_B C^n_A(Q^2) \langle 0|T O^n_A \hat{O}_B|0\rangle \Gamma_{\hat{O}_Bpp}$$ (5.16)

This decomposition splits the structure function into three parts – first, the Wilson coefficients $C^n_A(Q^2)$ which can be calculated in perturbative QCD; second, non-perturbative but target independent Green functions which encode the dynamics of the QCD vacuum; third, non-perturbative vertex functions which characterise the target by its couplings to the chosen operators $\hat{O}_B$.\footnote{We emphasise again that this decomposition of the matrix elements into products of Green functions and 1PI vertices is exact, independent of the choice of the set of operators $\hat{O}_B$. In particular, it is not necessary for $\hat{O}_B$ to be in any sense a complete set. If a different choice is made, the vertices $\Gamma_{\hat{O}_Bpp}$ themselves change, becoming 1PI with respect to a different set of composite fields. In practice, the set of operators $\hat{O}_B$ should be as small as possible while still capturing the essential degrees of freedom. A good choice can also result in vertices $\Gamma_{\hat{O}_Bpp}$ which are both RG invariant and closely related to low-energy physical couplings.}

Now specialise to the first moment sum rule for $g_1^p$. For simplicity, we first present the analysis for the chiral limit, where there is no flavour mixing. Using the anomaly (2.4), we can express the flavour singlet contribution to the sum rule as

$$\Gamma_1^p(Q^2)_{\text{singlet}} \equiv \int_0^1 dx \ g_1^p(x, Q^2)_{\text{singlet}} = \frac{2}{3} \frac{1}{2m_N} \Delta C^S_1(\alpha_s) \langle p|Q|p\rangle$$ (5.17)

The obvious choice for the operators $\hat{O}_B$ in this case are the flavour singlet pseudoscalars and it is natural to choose the ‘OZI boson’ field $\hat{\eta}^0 = \hat{f}^{00} \frac{1}{\langle qg\rangle} \phi^0_5$, which is normalised so
that $d/dp^2 \Gamma_{\bar{q}q}(p) \big|_{p=0} = 1$. As we have seen in eq.(4.51), the corresponding 1PI vertex is then RG invariant. Writing the 1PI vertices in terms of nucleon couplings as in eq.(4.45), we find
\begin{equation}
\Gamma^p_1(Q^2)_{\text{singlet}} = \frac{2}{32m_N} \Delta C_1^S(\alpha_s) \left( \langle 0| T Q Q |0 \rangle g_{QQN} + \langle 0| T \bar{q}q |0 \rangle g_{\bar{q}q^0NN} \right) \tag{5.18}
\end{equation}

Recalling that the matrix of two-point Green functions is given by the inversion formula
\begin{equation}
\begin{pmatrix}
W_{\theta\theta} & W_{\theta\bar{q}^0} \\
W_{\bar{q}^0}\theta & W_{\bar{q}^0}\bar{q}^0
\end{pmatrix}
= - \begin{pmatrix}
\Gamma_{QQ} & \Gamma_{Q\bar{q}^0} \\
\Gamma_{\bar{q}^0Q} & \Gamma_{\bar{q}^0\bar{q}^0}
\end{pmatrix}^{-1} \tag{5.19}
\end{equation}
and using the normalisation condition for $\bar{q}^0$, we can easily show that at zero momentum,
\begin{equation}
W_{\theta\bar{q}^0}^2 = \frac{d}{dp^2} W_{\theta\bar{q}^0} \big|_{p=0} \tag{5.20}
\end{equation}

Finally, therefore, we can represent the first moment of $g^p_1$ in the following, physically intuitive form:
\begin{equation}
\Gamma^p_1(Q^2)_{\text{singlet}} = \frac{2}{32m_N} \Delta C_1^S(\alpha_s) \left( \chi(0) g_{QQN} + \sqrt{\chi'(0)} g_{\bar{q}q^0NN} \right) \tag{5.21}
\end{equation}
This shows that the first moment is determined by the gluon topological susceptibility in the QCD vacuum as well as the couplings of the proton to the pseudoscalar operators $Q$ and $\bar{q}^0$. In the chiral limit, $\chi(0) = 0$ so the first term vanishes. The entire flavour singlet contribution is therefore simply
\begin{equation}
\Gamma^p_1(Q^2)_{\text{singlet}} = \frac{2}{32m_N} \Delta C_1^S(\alpha_s) \sqrt{\chi'(0)} g_{\bar{q}q^0NN} \tag{5.22}
\end{equation}
The 1PI vertex $g_{\bar{q}q^0NN}$ is RG invariant, and we see from eq.(2.25) that in the chiral limit the slope of the topological susceptibility scales with the anomalous dimension $\gamma$, viz.
\begin{equation}
\frac{d}{d\ln Q^2} \sqrt{\chi'(0)} = \gamma \sqrt{\chi'(0)} \tag{5.23}
\end{equation}
ensuring consistency with the RGE for the flavour singlet axial charge.

The formulae (5.21) and (5.22) are our key result. They show how the first moment of $g^p_1$ can be factorised into couplings $g_{QQN}$ and $g_{\bar{q}q^0NN}$ which carry information on the
proton structure, and Green functions which characterise the QCD vacuum. In the case of $g_1^p$, the Green functions reduce simply to the topological susceptibility $\chi(0)$ and its slope $\chi'(0)$. We now argue that the experimentally observed suppression in the first moment of $g_1^p$ is due not to a suppression in the couplings, but to the vanishing of the topological susceptibility $\chi(0)$ and an anomalously small value for its slope $\chi'(0)$. This is what we refer to as topological charge screening in the QCD vacuum.

The justification follows our now familiar conjecture on the relation between OZI violations and RG scale dependence. We expect the source of OZI violations to be in those quantities which are sensitive to the anomaly, as identified by their scaling dependence on the anomalous dimension $\gamma$, in this case $\chi'(0)$. In contrast, it should be a good approximation to use the OZI value for the RG-invariant vertex $g_{\eta^0NN}$, that is $g_{\eta^0NN} \simeq \sqrt{2} g_{\eta^8NN}$. The corresponding OZI value for $\sqrt{\chi'(0)}$ would be $f_\pi/\sqrt{6}$. This gives our key formula for the flavour singlet axial charge:

$$\frac{a^0(Q^2)}{a^8} \simeq \frac{\sqrt{6}}{f_\pi} \sqrt{\chi'(0)} \quad (5.24)$$

The corresponding prediction for the first moment of $g_1^p$ is

$$\Gamma_1^p(Q^2)_{\text{singlet}} = \frac{1}{9} \Delta C_1^S(\alpha_s) \, a^8 \, \frac{\sqrt{6}}{f_\pi} \sqrt{\chi'(0)} \quad (5.25)$$

The final step is to compute the slope of the topological susceptibility. In time, lattice gauge theory should provide an accurate measurement of $\chi'(0)$. However, this is a particularly difficult correlator for lattice methods since it requires a simulation of QCD with light dynamical fermions and algorithms that implement topologically non-trivial configurations in a sufficiently fast and stable way. Instead, we have estimated the value of $\chi'(0)$ using the QCD spectral sum rule method. Full details and discussion of this computation can be found in refs.[10, 52]. The result is:

$$\sqrt{\chi'(0)} = 26.4 \pm 4.1 \text{ MeV} \quad (5.26)$$

This gives our final prediction for the flavour singlet axial charge and the complete first moment of $g_1^p$:

$$a^0|_{Q^2=10\text{GeV}^2} = 0.33 \pm 0.05 \quad (5.27)$$
$$\Gamma_1^p|_{Q^2=10\text{GeV}^2} = 0.144 \pm 0.009 \quad (5.28)$$

Topological charge screening therefore gives a suppression factor of approximately 0.56 in $a^0$ compared to its OZI value $a^8 = 0.585$.

In the decade since we made this prediction, the experimental measurement has been somewhat lower than this value, in the range $a^0 \simeq 0.20 - 0.25$. This would have suggested there is also a significant OZI violation in the nucleon coupling $g_{\eta^0NN}$ itself, implicating the proton structure in the anomalous suppression of $\Gamma_1^p$. Very recently, however, the COMPASS and HERMES collaborations have published new results on the deuteron structure.
function which spectacularly confirm our picture that topological charge screening in the QCD vacuum is the dominant suppression mechanism.

This new data is shown in Fig. 9. This is based on data collected by COMPASS at CERN in the years 2002-2004 and has only recently been published. The accuracy compared to earlier SMC data at small $x$ is significantly improved and the dip in $xg_d^1$ around $x \sim 10^{-2}$ suggested by the SMC data is no longer present (Fig. 9). This explains the significantly higher value for $a^0$ found by COMPASS compared to SMC. From this data, COMPASS quote the first moment for the proton-neutron average $g_1^N = (g_1^p + g_1^n)/2$ as [53]

$$\Gamma_1^N|_{Q^2=3\text{GeV}^2} = 0.050 \pm 0.003 \text{(stat)} \pm 0.002 \text{(evol)} \pm 0.005 \text{(syst)} \quad (5.29)$$

Extracting the flavour singlet axial charge from the analogue of eq.(5.3) for $\Gamma_1^N$ then gives

$$a^0|_{Q^2=3\text{GeV}^2} = 0.35 \pm 0.03 \text{(stat)} \pm 0.05 \text{(syst)} \quad (5.30)$$

or evolving to the $Q^2 \to \infty$ limit,

$$a^0|_{Q^2\to\infty} = 0.33 \pm 0.03 \text{(stat)} \pm 0.05 \text{(syst)} \quad (5.31)$$

Similar results are found by HERMES, who quote [54]

$$a^0|_{Q^2=5\text{GeV}^2} = 0.330 \pm 0.011 \text{(th)} \pm 0.025 \text{(exp)} \pm 0.028 \text{(evol)} \quad (5.32)$$

The agreement with our prediction (5.27) is striking.

To close this section, we briefly comment on the extension of our analysis beyond the chiral limit. In this case, the operator $\sqrt{2n_f}Q$ in eq.(5.17) is replaced by the full divergence of the flavour singlet axial current, viz. $D^0 = \sqrt{2n_f}Q + d_{abc}m^b \phi^c_1$. Separating the matrix element $\langle p|D^0|p \rangle$ into Green functions and 1PI vertices, we find from the zero-momentum Ward identities that $\langle 0|T D^0 Q|0 \rangle = 0$ so the contribution from $g_{QNN}$ still vanishes. The other Green function is $\langle 0|T D^0 \tilde{g}^\alpha|0 \rangle = -\tilde{f}^{0\alpha}$, so the first moment sum rule becomes

$$\Gamma_1^p(Q^2)_{\text{singlet}} = \frac{1}{9} \frac{1}{2m_N} \Delta C_1^S(\alpha_s) \sqrt{6} \tilde{f}^{0\alpha} g_{QNN} \quad (5.33)$$
It is clear that this is simply an alternative derivation of the $U(1)$ GT relation (4.46) for $a^0$. We could equally use the alternative form (4.47) to write

$$
\Gamma^p_{s} (Q_{2})_{\text{singlet}} = \frac{1}{92m_N} \Delta C^f_{1}(\alpha_s) \sqrt{6} \left( f^{00}_0 g_{\rho NN} + \sqrt{6} A g_{GNN} \right)
$$

(5.34)

Recalling the RGE (4.52) for $g_{GNN}$, we see that this bears a remarkable similarity to the expression for $a^0$ in terms of parton distributions in the AB scheme, eq.(5.14). This was first pointed out in ref.[8, 9].

Manipulating the zero-momentum Ward identities in a similar way to that explained above in the chiral limit now shows that we can express the decay constants $\hat{f}^{a\alpha}$ in terms of vacuum Green functions as follows (see eq.(3.9):

$$
(\hat{f} \hat{f}^T)_{ab} = \frac{d}{dp^2} \langle 0 | T D^a D^b | 0 \rangle |_{p=0}
$$

(5.35)

However, for non-zero quark masses there is flavour mixing amongst the ‘OZI bosons’ $\eta^a$ and we cannot extract the decay constants simply by taking a square root, as was the case in writing $\hat{f}^{00} = \sqrt{\chi^0(0)}$ in the chiral limit. Nevertheless, in ref.[52] we estimated the decay constants and form factors in the approximation where we use eq.(5.35) with the full divergence $D^a$ but neglect flavour mixing. Assuming OZI for the couplings, this gives the estimate

$$
\frac{a^0(Q^2)}{a^8} \simeq \sqrt{6} \frac{\hat{f}^{00}}{\hat{f}^{88}}
$$

(5.36)

where we take

$$
\hat{f}^{00} \simeq \sqrt{\frac{d}{dp^2} \langle 0 | T D^0 D^0 | 0 \rangle |_{p=0}}
\quad \hat{f}^{88} \simeq \sqrt{\frac{d}{dp^2} \langle 0 | T D^8 D^8 | 0 \rangle |_{p=0}}
$$

(5.37)

Evaluating the Green functions using QCD spectral sum rules gives

$$
\begin{align*}
{a^0}_{|Q^2=10\text{GeV}}^2 &= 0.31 \pm 0.02 \\
\Gamma^p_{s} |_{Q^2=10\text{GeV}} &= 0.141 \pm 0.005
\end{align*}
$$

(5.38)

(5.39)

As we have seen in the last section, flavour mixing can be non-negligible in the phenomenology of the pseudoscalar mesons, so we should be a little cautious in over-estimating the accuracy of these estimates. (The quoted errors do not include this systematic effect.) Nevertheless, the fact that they are consistent with those obtained in the chiral limit reinforces our confidence that the flavour singlet axial charge is relatively insensitive to the quark masses and that eqs.(5.27) and (5.28) indeed provide an accurate estimate of the first moment of $g^p_1$.

The observation that the ‘proton spin’ sum rule could be explained in terms of an extension of the Goldberger-Treiman relation to the flavour singlet sector was made in Veneziano’s original paper [4]. This pointed out for the first time that the suppression in $a^0$ was an OZI-breaking effect. Since the Goldberger-Treiman relation connects the pseudovector form factors with the pseudoscalar channel, where it is known that there are large OZI violations for the flavour singlet, it becomes natural to expect similar large OZI violations also in $a^0$. This is the fundamental intuition which we have developed into a quantitative resolution of the ‘proton spin’ problem.
5.4 Semi-inclusive polarised DIS

While the agreement between our prediction for the first moment of $g_1^p$ and experiment is now impressive, it would still be interesting to find other experimental tests of topological charge screening. A key consequence of this mechanism is that the OZI violation observed in $d^0$ is not a property specifically of the proton, but is target independent. This leads us to look for ways to make measurements of the polarised structure functions of other hadronic targets besides the proton and neutron. We now show how this can effectively be done by studying semi-inclusive DIS $eN \rightarrow ehX$ in the target fragmentation region (Fig. 10).

The differential cross-section in the target fragmentation region can be written analogously to eq.(5.1) in terms of fracture functions:

$$x \frac{d\Delta \sigma^{\text{target}}}{dxdydzdt} = \frac{Y_P}{2} \frac{4\pi \alpha^2}{s} \Delta M_1^{hN}(x, z, t, Q^2)$$

(5.40)

where $x = Q^2/2p_2.q$, $x_B = Q^2/2k.q$, $z = p_B^2.q/p_2.q$ so that $1 - z = x/x_B$, and the invariant momentum transfer $t = K^2 = -k^2$, where $k$ is the momentum of the struck parton. For $K^2 \ll Q^2$, $z \simeq E_h/E_N$ (in the photon-nucleon CM frame) is the energy fraction of the target nucleon carried by the detected hadron $h$.

$\Delta M_1^{hN}$ is the fracture function [77] equivalent of the inclusive structure function $g_1^N$, so in the same way as in eq.(5.10) we have

$$\Delta M_1^{hN}(x, z, t, Q^2) = \frac{1}{2} \sum_i e_i^2 \Delta M_i^{hN}(x, z, t, Q^2)$$

(5.41)

Here, $\Delta M_i^{hN}(x, z, t, Q^2)$ is an extended fracture function, introduced by Grazzini, Trentadue and Veneziano [78], which carries an explicit dependence on $t$. One of the advantages of these fracture functions is that they satisfy a simple, homogeneous RG evolution equation analogous to the usual inclusive parton distributions.

Our proposal [11, 79] (see also [80]) is to study semi-inclusive DIS in the kinematical region where the detected hadron $h$ ($\pi$, $K$ or $D$) carries a large target energy fraction, i.e. $z$ approaching 1, with a small invariant momentum transfer $t$. In this region, it is useful...
to think of the target fragmentation process as being simply modelled by a single Reggeon exchange (see Fig. 10), i.e.
\[ \Delta M_1^{hN}(x, z, t, Q^2)_{z \rightarrow 1} \simeq F(t)(1 - z)^{-2\alpha_s(t)} g_1^B(x_B, t, Q^2) \] (5.42)

If we consider ratios of cross-sections, the dynamical Reggeon emission factor \( F(t)(1 - z)^{-2\alpha_s(t)} \) will cancel and we will be able to isolate the ratios of \( g_1^B(x_B, t, Q^2) \) for different effective targets \( B \). Although single Reggeon exchange is of course only an approximation to the more fundamental QCD description in terms of fracture functions (see ref.[81] for a more technical discussion), it shows particularly clearly how observing semi-inclusive processes at large \( z \) with particular choices of \( h \) and \( N \) amounts in effect to performing inclusive DIS on virtual hadronic targets \( B \). Since our predictions will depend only on the \( SU(3) \) properties of \( B \), together with target independence, they will hold equally well when \( B \) is interpreted as a Reggeon rather than a pure hadron state.

The idea is therefore to make predictions for the ratios \( R \) of the first moments of the polarised fracture functions \( \int_0^{1-z} dx \Delta M_1^{hN}(x, z, t, Q^2) \) or equivalently \( \int_0^{1} dx_B g_1^B(x_B, t, Q^2) \) for various reactions. The first moments \( \Gamma_1^B \) are calculated as in eq.(5.3) in terms of the axial charges \( a^3, a^8 \) and \( a^0(Q^2) \) for a state with the \( SU(3) \) quantum numbers of \( B \). We then use topological charge screening to say that
\[ a^0(Q^2) \simeq s(Q^2) a^0|_{OZI}, \]
i.e. the flavour singlet axial charge is suppressed relative to its OZI value by a universal, target-independent, suppression factor \( s(Q^2) \).

From our calculation of \( \sqrt{X'(0)} \) and the experimental results for \( g_1^p \), we have \( s|_{Q^2=10 GeV^2} \simeq 0.33/0.585 = 0.56 \).

Some of the more interesting predictions obtained in ref.[11] are as follows. The ratio
\[ R \left( \frac{en \rightarrow e\pi^+ X}{ep \rightarrow e\pi^- X} \right)_{z \rightarrow 1} \simeq \frac{2s - 1}{2s + 2} \] (5.43)
is calculated by comparing \( \Gamma_1 \) for the \( \Delta^- \) and \( \Delta^{++} \). It is particularly striking because the physical value of \( s(Q^2) \) is close to one half, so the ratio becomes very small. For strange mesons, on the other hand, the ratio depends on whether the exchanged object is in the \( 8 \) (where the reduced matrix elements involve the appropriate \( F/D \) ratio) or \( 10 \) representation, so the prediction is less conclusive, viz.
\[ R \left( \frac{en \rightarrow eK^+ X}{ep \rightarrow eK^0 X} \right)_{z \rightarrow 1} \simeq \frac{2s - 1 - 3(2s - 1)F/D}{2s - 1 - 3(2s + 1)F/D} \] (8) \[ \text{or} \quad \frac{2s - 1}{2s + 1} \] (10) (5.44)

which we find by comparing \( \Gamma_1 \) for either the \( \Sigma^- \) and \( \Sigma^+ \) in the \( 8 \) representation or \( \Sigma^{*-} \) and \( \Sigma^{*+} \) in the \( 10 \). For charmed mesons, we again find
\[ R \left( \frac{en \rightarrow eD^0 X}{ep \rightarrow eD^- X} \right)_{z \rightarrow 1} \simeq \frac{2s - 1}{2s + 2} \] (5.45)
corresponding to the ratio for \( \Sigma_c^0 \) to \( \Sigma_c^{++} \).

At the other extreme, for \( z \) approaching 0, the detected hadron carries only a small fraction of the target nucleon energy. In this limit, the ratio \( R \) of the fracture function moments becomes simply the ratio of the structure function moments for \( n \) and \( p \), i.e. using
the current experimental values, $\mathcal{R}_{z \gtrsim 0} \simeq \Gamma_1^u/\Gamma_1^p = -0.30$. This is to be compared with the corresponding OZI or Ellis-Jaffe value of $-0.12$.

The differences between the OZI, or valence quark model, expectations and our predictions based on topological charge screening can therefore be quite dramatic and should give a very clear experimental signal. In ref.[79], together with De Florian, we analysed the potential for realising these experiments in some detail. Since we require particle identification in the target fragmentation region, fixed-target experiments such as COMPASS or HERMES are not appropriate. The preferred option is a polarised $ep$ collider.

The first requirement is to measure particles at extremely small angles ($\theta \leq 1$ mrad), corresponding to $t$ less than around 1 GeV$^2$. This has already been achieved at HERA in measurements of diffractive and leading proton/neutron scattering using a forward detection system known as the Leading Proton Spectrometer (LPS). The technique for measuring charged particles involves placing detectors commonly known as ‘Roman Pots’ inside the beam pipe itself.

The next point is to notice that the considerations above apply equally to $\rho$ as to $\pi$ production, since the ratios $\mathcal{R}$ are determined by flavour quantum numbers alone. The particle identification requirements will therefore be less stringent, especially as the production of leading strange mesons from protons or neutrons is strongly suppressed. However, we require the forward detectors to have good acceptance for both positive and negatively charged mesons $M = \pi, \rho$ in order to measure the ratio (5.43).

The reactions with a neutron target can be measured if the polarised proton beam is replaced by polarised $^3He$. In this case, if we assume that $^3He = Ap + Bn$, the cross section for the production of positive hadrons $h^+$ measured in the LPS is given by

$$\sigma(^3He \rightarrow h^+) \simeq A\sigma(p \rightarrow h^+) + B\sigma(n \rightarrow p) + B\sigma(n \rightarrow M^+)$$ (5.46)

The first contribution can be obtained from measurements with the proton beam. However, to subtract the second one, the detectors must have sufficient particle identification at least to distinguish protons from positively charged mesons.

Finally, estimates of the total rates [79] suggest that around 1% of the total DIS events will contain a leading meson in the target fragmentation region where a LPS would have non-vanishing acceptance ($z > 0.6$) and in the dominant domain $x < 0.1$. The relevant cross-sections are therefore sufficient to allow the ratios $\mathcal{R}$ to be measured.

The conclusion is that while our proposals undoubtedly pose a challenge to experimentalists, they are nevertheless possible. Given the theoretical importance of the ‘proton spin’ problem and the topological charge screening mechanism, there is therefore strong motivation to perform target fragmentation experiments at a future polarised $ep$ collider [82].

6. Polarised two-photon physics and a sum rule for $g_1^\gamma$

The $U(1)_A$ anomaly plays a vital role in another sum rule arising in polarised deep-inelastic scattering, this time for the polarised photon structure function $g_1^\gamma(x, Q^2, K^2)$. For real photons, the first moment of $g_1^\gamma$ vanishes as a consequence of electromagnetic current
conservation [83]. For off-shell photons, we proposed a sum rule in 1992 [5, 6] whose dependence on the virtual momentum of the target photon encodes a wealth of information about the anomaly, chiral symmetry breaking and gluon dynamics in QCD. This is of special current interest since, given the ultra-high luminosity of proposed e+e− colliders designed as B factories, a detailed measurement of our sum rule is about to become possible for the first time.

6.1 The first moment sum rule for \( g_1^\gamma \)

The polarised structure function \( g_1^\gamma \) is measured in the process \( e^+e^- \to e^+e^-X \), which at sufficiently high energy is dominated by the two-photon interaction shown in Fig. 11. The deep-inelastic limit is characterised by \( Q^2 \to \infty \) with \( x = Q^2/2p_2q \) and \( x_\gamma = Q^2/2k.q \) fixed, where \( Q^2 = -q^2 \), \( K^2 = -k^2 \) and \( s = (p_1 + p_2)^2 \). The target photon is assumed to be relatively soft, \( K^2 \ll Q^2 \).

We are interested in the dependence of the photon structure function \( g_1^\gamma(x_\gamma, Q^2; K^2) \) on the invariant momentum \( K^2 \) of the target photon. Experimentally, this is given by \( K^2 \simeq EE_2\theta_2^2 \) where \( E_2' \) and \( \theta_2 \) are the energy and scattering angle of the target electron. For the values \( K^2 \sim m_\rho^2 \) of interest in the sum rule, the target electron is nearly-forward and \( \theta_2 \) is very small. If it can be tagged, then the virtuality \( K^2 \) is simply determined from \( \theta_2 \); otherwise \( K^2 \) can be inferred indirectly from a measurement of the total hadronic energy.

The total cross-section \( \sigma \) and the spin asymmetry \( \Delta \sigma \) can be expressed formally in terms of ‘electron structure functions’ as follows [5]

\[
\sigma = 2\pi \alpha^2 \frac{1}{s} \int_0^\infty \frac{dQ^2}{Q^2} \int_0^1 \frac{dx}{x^2} \left[ F_2^e \frac{1}{y} \left( 1 - y + \frac{y^2}{2} \right) - F_L^e \frac{y}{2} \right] \quad (6.1)
\]

\[
\Delta \sigma = 2\pi \alpha^2 \frac{1}{s} \int_0^\infty \frac{dQ^2}{Q^2} \int_0^1 \frac{dx}{x} g_1^e \left( 1 - \frac{y}{2} \right) \quad (6.2)
\]

where \( \sigma = \frac{1}{2}(\sigma_++\sigma_-) \) and \( \Delta \sigma = \frac{1}{2}(\sigma_+-\sigma_-) \) with +,− referring to the electron helicities. The parameter \( y = Q^2/xs \ll 1 \) and only the leading order terms are retained below.
These electron structure functions can be expressed as convolutions of the photon structure functions with appropriate splitting functions. In particular, we have

\[ g_1^\gamma(x, Q^2) = \frac{\alpha}{2\pi} \int_0^\infty \frac{dK^2}{K^2} K^2 \int_x^1 \frac{dx_\gamma}{x_\gamma} \Delta P_{\gamma e}(\frac{x}{x_\gamma}) g_1^\gamma(x_\gamma, Q^2; K^2) \]  

(6.3)

where \( \Delta P_{\gamma e}(x) = (2 - x) \). This allows us to relate the \( x_\gamma \)-moments of the photon structure functions to the \( x \)-moments of the cross-sections. For the first moment of \( g_1^\gamma \), we find:

\[ \int_0^1 dx \frac{d^3\Delta\sigma}{dQ^2dxK^2} = 3\frac{\alpha^3}{2sQ^2K^2} \int_0^1 dx_\gamma g_1^\gamma(x_\gamma, Q^2; K^2) \]  

(6.4)

The first moment sum rule follows, as for the proton, by using the OPE (5.2) to express the product of electromagnetic currents for the incident photon in terms of the electromagnetic currents. We define form factors for this fundamental correlator as follows:

\[ -i(0|J_{\mu 5}^\gamma(p)J_\lambda(k_1)J_\rho(k_2)|0) = A_1^a \epsilon_{\mu\lambda\rho\alpha}k_2^\alpha + A_2^a \epsilon_{\mu\lambda\rho\alpha}k_2^\alpha 
+ A_3^a \epsilon_{\mu\lambda\rho\beta}k_1^\beta k_2^\rho + A_4^a \epsilon_{\mu\lambda\rho\beta}k_1^\beta k_2^\rho 
+ A_5^a \epsilon_{\mu\lambda\rho\beta}k_1^\beta k_2^\rho + A_6^a \epsilon_{\mu\lambda\rho\beta}k_1^\beta k_2^\rho \]  

(6.5)

where the six form factors are functions of the invariant momenta, i.e. \( A_1^a = A_1^a(p^2, k_1^2, k_2^2) \). We also abbreviate \( A_1^a(0, k_1^2, k_2^2) = A_1^a(K^2) \).

The first moment sum rule for \( g_1^\gamma \) is then [5]:

\[ \int_0^1 dx_\gamma g_1^\gamma(x_\gamma, Q^2; K^2) = 4\pi\alpha \sum_{a=3,8,0} \Delta C_1^a(Q^2) \left( A_1^a(K^2) - A_2^a(K^2) \right) \]  

(6.6)

where the Wilson coefficients are related to those in eq.(5.3) by \( \Delta C_1^3 = \Delta C_1^{NS} \), \( \Delta C_1^8 = \frac{1}{\sqrt{3}} \Delta C_1^{NS} \) and \( \Delta C_1^0 = \frac{2\sqrt{2}}{3} \Delta C_1^{NS} \).\(^{12}\)

Now, just as the sum rule for the proton structure function \( g_1^p \) could be related to low-energy meson-nucleon couplings via the \( U(1)_A \) Goldberger-Treiman relations, we can relate this sum rule for \( g_1^\gamma \) to the pseudoscalar meson radiative decays using the analysis in section 4.2. Introducing the \textit{off-shell} radiative pseudoscalar couplings for photon virtuality \( K^2 \), we define form factors

\[ F^a(K^2) = 1 - \left( e_{em}^a \frac{\alpha}{\pi} \right)^{-1} \hat{f}_\gamma e^{2\gamma(K^2)} \]  

(6.7)

\(^{12}\)Explicitly,

\[ \Delta C_1^{NS} = \frac{1}{3} \left( 1 - \frac{\alpha_s(Q^2)}{\pi} \right), \quad \Delta C_1^8 = \frac{1}{3} \left( 1 - \frac{\alpha_s(Q^2)}{\pi} \right) \exp \int t(Q) dt' \gamma(\alpha_s(t')) \]

at leading order, where \( t(Q) = \frac{1}{2} \ln \frac{Q^2}{\Lambda^2} \) and \( \gamma = \frac{3}{4} \frac{\alpha^2}{(4\pi)^2} \) is the anomalous dimension corresponding to the \( U(1)_A \) current renormalisation.
or alternatively,

\begin{align*}
F^3(K^2) &= 1 - \left( \frac{a^3_{em}}{\pi} \right)^{-1} f_{\pi g_{\pi\gamma\gamma}}(K^2) \\
F^8(K^2) &= 1 - \left( \frac{a^8_{em}}{\pi} \right)^{-1} \left( f^{s\eta g_{\pi\gamma\gamma}}(K^2) + f^{s\eta' g_{\pi'\gamma\gamma}}(K^2) \right) \\
F^0(K^2) &= 1 - \left( \frac{a^0_{em}}{\pi} \right)^{-1} \left( f^{0\eta g_{\pi\gamma\gamma}}(K^2) + f^{0\eta' g_{\pi'\gamma\gamma}}(K^2) + \sqrt{6} A g_{G\gamma\gamma}(K^2) \right)
\end{align*}

(6.8)

where the \( a^\alpha_{em} \) are the electromagnetic \( U(1)_A \) anomaly coefficients defined earlier. We may then rewrite the sum rule as

\[
\int_0^1 dx_\gamma g^\gamma_1(x_\gamma, Q^2; K^2) = \frac{1}{2} \sum_{a=3,8,0} \Delta C^a_1(Q^2) a^a_{em} F^a(K^2)
\]

(6.9)

The dependence of the \( g^\gamma_1 \) on the invariant momentum \( K^2 \) of the target photon reflects many key aspects of both perturbative and non-perturbative QCD dynamics. For on-shell photons, \( K^2 = 0 \), we have simply [83, 5]

\[
\int_0^1 dx_\gamma g^\gamma_1(x_\gamma, Q^2; K^2 = 0) = 0
\]

(6.10)

This is a consequence of electromagnetic current conservation. This follows simply by taking the divergence of eq.(6.5) and observing that in the limit \( p \to 0 \), both \( A_1 \) and \( A_2 \) are of \( O(K^2) \).\(^{13}\)

In the asymptotic limit where \( K^2 \ll m_p^2 \), a relatively straightforward renormalisation group analysis combined with the anomaly equation shows that, for the flavour non-singlets, the \( A^\alpha_i \) tend to the value \( \frac{4 \pi}{3} a^\alpha_{em} \), while in the flavour singlet sector, \( A^0_i \) has an additional factor depending on the anomalous dimension \( \gamma \). Using the explicit expressions for the Wilson coefficients, we find

\[
\int_0^1 dx_\gamma g^\gamma_1(x_\gamma, Q^2; K^2 \ll m_p^2) = \frac{1}{6 \pi} \left( 1 - \frac{\alpha_s(Q^2)}{\pi} \right) \left( a^3_{em} + \frac{1}{\sqrt{3}} a^8_{em} + 2 \sqrt{2} \frac{a^0_{em}}{\sqrt{3}} \exp \left[ \int_{t(K)}^{t(Q)} dt' \frac{\gamma(a_s(t'))}{a_s(t')} \right] \right)
\]

\[
= \frac{1}{3 \pi} \left[ 1 - \frac{4}{9} \ln \frac{Q^2}{\Lambda^2} + \frac{16}{81} \left( \frac{1}{\ln Q^2/\Lambda^2} - \frac{1}{\ln K^2/\Lambda^2} \right) \right]
\]

(6.11)

The asymptotic limit is therefore determined by the electromagnetic \( U(1)_A \) anomaly, with logarithmic corrections reflecting the anomalous dimension of the flavour singlet current due to the colour \( U(1)_A \) anomaly. (See also ref.[84] for a NNLO analysis.)

\[^{13}\text{Electromagnetic current conservation in eq.(6.5) implies}
\]

\[
A^1_i = A^3_i k^2_1 + A^3_i \frac{1}{2} (p^2 - k^2_1 - k^2_2), \quad A^2_i = A^3_i k^2_1 + A^3_i \frac{1}{2} (p^2 - k^2_1 - k^2_2)
\]

The chiral limit is special since the form factors can have massless poles and is considered in detail in ref.[6]. The sum rule (6.10) still holds.
In between these limits, the first moment of \( g_1^\gamma \) provides a measure of the form factors defining the 3-current AVV Green function, which encodes a great deal of information about the dynamics of QCD, especially the non-perturbative realisation of chiral symmetry \[6\]. Equivalently, in the form (6.9), it measures the momentum dependence of the off-shell radiative couplings of the pseudoscalar mesons as the form factors \( F^a(K^2) \) vary from 0 to 1.

Just as for \( g_1^p \), we can again isolate a dependence on the topological susceptibility through the identification of the flavour singlet decay constant \( f^0 \) in eq.(6.7) with \( \sqrt{\chi'(0)} \) in the chiral limit. This time, however, it is unlikely to be a good approximation to set the corresponding coupling \( g^\gamma_{\pi\gamma\gamma} \) equal to its OZI value since it is not RG invariant. A more promising approximation is to recall from section 4 that the RG invariant gluonic coupling \( g_{G\gamma\gamma}(0) \) is OZI suppressed and likely to be small. This was confirmed by the phenomenological analysis. If we assume this is also true of the off-shell coupling, then we may approximate the sum rule for \( g_1^\gamma \) entirely in terms of the off-shell couplings of the physical mesons \( \pi^0, \eta, \) and \( \eta' \).

In general, the momentum dependence of the form factors \( (A_1^a - A_2^a) \) or \( F^a \) will depend on the fermions contributing to the AVV Green function \[6\]. In the case of leptons, or heavy quarks, the crossover scale as the form factors \( F^a(K^2) \) rise from 0 to 1 with increasing \( K^2 \) will be given by the fermion mass. For the light quarks, however, we expect the crossover scale to be a typical hadronic scale \( \sim m_\rho \) rather than \( m_{u,d,s} \). This can be justified by a rough OPE argument and is consistent with old ideas of vector meson dominance \[6, 85\]. This behaviour would be an interesting manifestation of the spontaneous breaking of chiral symmetry.

Once again, therefore, we see a close relation between the realisation of sum rules in high-energy deep-inelastic scattering and low-energy meson physics. All these issues are discussed at some length in our earlier papers, but here we now turn our attention to the vital question of whether the \( g_1^\gamma \) sum rule can be measured in current or future collider experiments \[7\].

### 6.2 Cross-sections and spin asymmetries at polarised B factories

The spin-dependent cross-sections for the two-photon DIS process \( e^+e^- \rightarrow e^+e^-X \) were analysed in refs.\[5, 7\] taking account of the experimental cuts on the various kinematical parameters. Keeping the lower cut on \( Q^2 \) as a free parameter, we found the following results for the total cross-section and spin asymmetry:

\[
\sigma \simeq 0.5 \times 10^{-8} \frac{1}{Q^2_{\text{min}}} \log \frac{Q^2_{\text{min}}}{\Lambda^2} \left( \log \frac{s}{Q^2_{\text{min}}} \right)^2 \tag{6.12}
\]

and

\[
\frac{\Delta\sigma}{\sigma} = \frac{1}{2} \frac{Q^2_{\text{min}}}{s} \log \frac{s}{4Q^2_{\text{min}}} \left[ 1 + \log \frac{s}{4\Lambda^2} \left( \log \frac{Q^2_{\text{min}}}{\Lambda^2} \right)^{-1} \right] \tag{6.13}
\]

In order to measure the \( g_1^\gamma \) sum rule, we need to find collider parameters such that the spin asymmetry is significant in a kinematic region where the total cross-section is still large.
Figure 12: The left-hand graph shows the total cross-section $\sigma$ (in pb) at SuperKEKB as the experimental cut $Q_{\text{min}}^2$ is varied from 1 to 10 GeV$^2$. The right-hand graph shows the spin asymmetry $\Delta\sigma/\sigma$ over the same range of $Q_{\text{min}}^2$.

A useful statistical measure of the significance of the asymmetry is that $\sqrt{L_\sigma \Delta\sigma/\sigma} \gg 1$, where $L$ is the luminosity.

When we first proposed the first moment sum rule for $g_1^\gamma$, the luminosity available from the then current accelerators was inadequate to allow it to be studied. For example, for a polarised version of LEP operating at $s = 10^4$ GeV$^2$ with an annual integrated luminosity of $L = 100$ pb$^{-1}$, and optimising the cut at $Q_{\text{min}}^2 = 10$ GeV$^2$, we only have $\sigma \simeq 35$ pb and $\Delta\sigma/\sigma \simeq 0.01$. The corresponding annual event rate would be $3.5 \times 10^3$ and the statistical significance $\sqrt{L_\sigma \Delta\sigma/\sigma} \simeq 0.5$, so even a reliable measurement of the spin asymmetry could not be made.

Clearly, a hugely increased luminosity is required and this has now become available with proposals for machines with projected annual integrated luminosities measured in inverse attobarns. However, as noted in ref.[5], if this increased luminosity is associated with increased CM energy, then the $1/s$ factor in the spin asymmetry (6.13) sharply reduces the possibility of extracting $g_1^\gamma$. There is also a competition as $Q_{\text{min}}^2$ is varied between increasing spin asymmetry and decreasing total cross-section. This is particularly evident when we analyse the potential of the ILC [86, 87] for measuring the sum rule [7]. We find that even optimising the $Q_{\text{min}}^2$ cut, the spin asymmetry is still only of order $\Delta\sigma/\sigma \simeq 0.002$ when $\sigma$ itself has fallen to around 15 pb. While, given the high luminosity, this would allow a measurement of the first moment of $g_1^\gamma$ integrated over $K^2$, a detailed study of the $K^2$-dependence of the sum rule requires a much greater spin asymmetry.

This leads us to consider instead the new generation of ultra-high luminosity $e^+e^-$ colliders. Although these are envisaged as $B$ factories, these colliders operating with polarised beams would, as we now show, be extremely valuable for studying polarisation phenomena in QCD. As an example of this class, we take the proposed SuperKEKB collider. (The analysis for PEPII is very similar, the main difference being the additional ten-fold increase in luminosity in the current SuperKEKB proposals.)

SuperKEKB is an asymmetric $e^+e^-$ collider with $s = 132$ GeV$^2$, corresponding to electron and positron beams of 8 and 3.5 GeV respectively. The design luminosity is $5 \times 10^{35}$ cm$^{-2}$s$^{-1}$, which gives an annual integrated luminosity of 5 ab$^{-1}$ [88]. To see the effects of the experimental cut on $Q_{\text{min}}^2$ in this case, we have plotted the total cross-section and the spin asymmetry in Fig. 12, in the range of $Q_{\text{min}}^2$ from 1 to 10 GeV$^2$. In this range $\sigma$ is falling like $1/Q_{\text{min}}^2$ while $\Delta\sigma/\sigma$ rises to what is actually a maximum at $Q_{\text{min}}^2 = 10$ GeV$^2$. 

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Taking $Q_{\text{min}}^2 = 5 \text{ GeV}^2$, we find $\sigma \simeq 12.5 \text{ pb}$ with spin asymmetry $\Delta \sigma/\sigma \simeq 0.1$. The annual event rate is therefore $6.25 \times 10^7$, with $\sqrt{L}\Delta \sigma/\sigma \simeq 750$. This combination of a very high event rate and the large 10% spin asymmetry means that SuperKEKB has the potential not only to measure $\Delta \sigma$ but to access the full first moment sum rule for $g_1$ itself. Recall from eq.(6.4) that to measure $\int dx g_1(x, Q^2; K^2)$ we need not just $\Delta \sigma$ but the fully differential cross-section w.r.t. $K^2$ as well as $x$ and $Q^2$ if the interesting non-perturbative QCD physics is to be accessed. To measure this, we need to divide the data into sufficiently fine $K^2$ bins in order to plot the explicit $K^2$ dependence of $g_1$, while still maintaining the statistical significance of the asymmetry. The ultra-high luminosity of SuperKEKB ensures that the event rate is sufficient, while its moderate CM energy means that the crucial spin asymmetry is not overly suppressed by its $1/s$ dependence.

Our conclusion is that the new generation of ultra-high luminosity, moderate energy $e^+e^-$ colliders, currently conceived as $B$ factories, could also be uniquely sensitive to important QCD physics if run with polarised beams. In particular, they appear to be the only accelerators capable of accessing the full physics content of the sum rule for the first moment of the polarised structure function $g_1(x, Q^2; K^2)$. The richness of this physics, in particular the realisation of chiral symmetry breaking, the manifestations of the axial $U(1)_A$ anomaly and the role of gluon topology, provides a strong motivation for giving serious consideration to an attempt to measure the $g_1$ sum rule at these new colliders.

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References

[1] G. Veneziano, Nucl. Phys. B159, 213 (1979).
[2] G. M. Shore, Nucl. Phys. B569, 107 (2000), [arXiv:hep-ph/9908217].
[3] G. M. Shore, Nucl. Phys. B744, 34 (2006), [arXiv:hep-ph/0601051].
[4] G. Veneziano, Mod. Phys. Lett. A4, 1605 (1989).
[5] S. Narison, G. M. Shore and G. Veneziano, Nucl. Phys. B391, 69 (1993).
[6] G. M. Shore and G. Veneziano, Mod. Phys. Lett. A8, 373 (1993).
[7] G. M. Shore, Nucl. Phys. B712, 411 (2005), [arXiv:hep-ph/0412192].
[8] G. M. Shore and G. Veneziano, Phys. Lett. B244, 75 (1990).
[9] G. M. Shore and G. Veneziano, Nucl. Phys. B381, 23 (1992).
[10] S. Narison, G. M. Shore and G. Veneziano, Nucl. Phys. B433, 209 (1995), [arXiv:hep-ph/9404277].
[11] G. M. Shore and G. Veneziano, Nucl. Phys. B516, 333 (1998), [arXiv:hep-ph/9709213].
[12] G. Veneziano: “The ‘spin’ of the proton and the OZI limit of QCD.” In: From Symmetries to Strings: Forty Years of Rochester Conferences, ed. A. Das (World Scientific, 1990), 13-26.
[13] S. L. Adler, Phys. Rev. 177, 2426 (1969).
[14] J. S. Bell and R. Jackiw, Nuovo Cimento 60A, 47 (1969).
[15] S. L. Adler and W.A. Bardeen, Phys. Rev. 182, 1517 (1969).
[16] J. Steinberger, Phys. Rev. 76, 1180 (1949).
[17] J. Schwinger, Phys. Rev. 82, 664 (1951).
[18] K. Fujikawa, Phys. Rev. Lett. 42, 1195 (1979); Phys. Rev. D21, 2848 (1980), erratum-ibid. D22, 1499 (1980).
[19] G. M. Shore, “U(1)A problems and gluon topology: anomalous symmetry in QCD”, In: Hidden Symmetries and Higgs Phenomena, Zuoz Summer School, Switzerland, 1998, pp 201-223; arXiv:hep-ph/9812354.
[20] E. Witten, Nucl. Phys. B156, 269 (1979).
[21] P. Di Vecchia and G. Veneziano, Nucl. Phys. B171, 253 (1980).
[22] D. Espriu and R. Tarrach, Z. Phys. C16, 77 (1982).
[23] G. M. Shore, Nucl. Phys. B362, 85 (1991).
[24] G. M. Shore and G. Veneziano, Nucl. Phys. B381, 3 (1992).
[25] G. ’t Hooft, Nucl. Phys. B72, 461 (1972).
[26] G. Veneziano, Phys. Lett. 52B, 220 (1974); Nucl. Phys. B117, 519 (1976).
[27] S. Okubo, Phys. Lett. 5, 165 (1963).
[28] G. Zweig, CERN report 8419/TH412 (1964).
[29] J. Iizuka, Prog. Theor. Phys. Suppl. 37-38, 21 (1966).
[30] S. Weinberg, Phys. Rev. D11, 3583 (1975).
[31] G. ’t Hooft, Phys. Rev. D14, 3432 (1976); [Erratum-ibid. D18, 2199 (1978)].
[32] R. J. Crewther, Riv. Nuovo Cim. 2N8, 63 (1979).
[33] G. A. Christos, Phys. Rept. 116, 251 (1984).
[34] G. ’t Hooft, Phys. Rept. 142, 357 (1986).
[35] M. Gell-Mann, R. J. Oakes and B. Renner, Phys. Rev. 175, 2195 (1968).
[36] R. F. Dashen, Phys. Rev. 183, 1245 (1969).
[37] C. Rosenzweig, J. Schechter and C. G. Trahern, Phys. Rev. D21, 3388 (1980).
[38] P. Di Vecchia, F. Nicodemi, R. Pettorino and G. Veneziano, Nucl. Phys. B181, 318 (1981).
[39] K. Kawarabayashi and N. Ohta, Nucl. Phys. B175, 477 (1980).
[40] P. Herrero-Siklody, J. I. Latorre, P. Pascual and J. Taron, Nucl. Phys. B497, 345 (1997); Phys. Lett. B419, 326 (1998).
[41] H. Leutwyler, Nucl. Phys. Proc. Suppl. 64, 223 (1998), [arXiv:hep-ph/9709408].
[42] R. Kaiser and H. Leutwyler, Eur. Phys. J. C17, 623 (2000), [arXiv:hep-ph/0007101].
[43] L. Giusti, G. C. Rossi, M. Testa and G. Veneziano, Nucl. Phys. B628 (2002) 234 [arXiv:hep-lat/0108009].
[44] G. M. Shore, Phys. Scripta T99, 84 (2002), [arXiv:hep-ph/0111165].
[45] Particle Data Group, Review of Particle Properties, Phys. Lett. B592, 1 (2004).
[46] M. Acciarri et al., L3 Collaboration, Phys. Lett. B418, 399 (1998).
[47] D. A. Williams et al., Crystal Ball Collaboration, Phys. Rev. D38, 1365 (1988).
[48] N. A. Roe et al., ASP Collaboration, Phys. Rev. D41, 17 (1990).
[49] L. Del Debbio, L. Giusti and C. Pica, Phys. Rev. Lett. 94, 032003 (2005), [arXiv:hep-th/0407052].
[50] A. Di Giacomo, Nucl. Phys. Proc. Suppl. 23B, 191 (1991).
[51] S. Narison, Phys. Lett. B255, 101 (1991); Z. Phys. C26, 209 (1984).
[52] S. Narison, G. M. Shore and G. Veneziano, Nucl. Phys. B546, 235 (1999), [arXiv:hep-ph/9812333].
[53] V. Y. Alexakhin et al. [COMPASS Collaboration], “The deuteron spin-dependent structure function g1(d) and its first moment,” arXiv:hep-ex/0609038.
[54] A. Airapetian et al. [HERMES Collaboration], “Precise determination of the spin structure function g(1) of the proton, deuteron and neutron,” arXiv:hep-ex/0609039.
[55] G. Mallot, S. Platchkov and A. Magnon, CERN-SPSC-2005-017; SPSC-M-733.
[56] E. S. Ageev et al. [COMPASS Collaboration], Phys. Lett. B612, 154 (2005), [arXiv:hep-ex/0501073].
[57] J. R. Ellis and R. L. Jaffe, Phys. Rev. D9, 1444 (1974), [Erratum-ibid. D10, 1669 (1974)].
[58] D. V. Bugg, Eur. Phys. J. C33, 505 (2004).
[59] P. Moskal, “Hadronic interaction of eta and eta’ mesons with protons,” arXiv:hep-ph/0408162.

[60] S. D. Bass, Phys. Scripta T99, 96 (2002), [arXiv:hep-ph/0111180].

[61] P. Moskal et al., Int. J. Mod. Phys. A20, 1880 (2005), [arXiv:hep-ex/0411052].

[62] P. Moskal et al., Phys. Rev. Lett. 80, 3202 (1998), [arXiv:nucl-ex/9803002].

[63] K. Nakayama and H. Haberzettl, “Analyzing eta’ photoproduction data on the proton at energies of 1.5GeV – 2.3GeV,” arXiv:nucl-th/0507044.

[64] M. Dugger [CLAS Collaboration], “S=0 pseudoscalar meson photoproduction from the proton,” arXiv:nucl-ex/0512005.

[65] G. M. Shore, Nucl. Phys. Proc. Suppl. 39BC, 101 (1995), [arXiv:hep-ph/9410383].

[66] J. Ashman et al., Phys. Lett. B206, 364 (1988); Nucl. Phys. B328, 1 (1990).

[67] R. L. Jaffe and A. Manohar, Nucl. Phys. B337, 509 (1990).

[68] G. M. Shore and B. E. White, Nucl. Phys. B581, 409 (2000), [arXiv:hep-ph/9912341].

[69] G. M. Shore, Nucl. Phys. Proc. Suppl. 96, 171 (2001), [arXiv:hep-ph/0007239].

[70] B. L. G. Bakker, E. Leader and T. L. Trueman, Phys. Rev. D70, 114001 (2004), [arXiv:hep-ph/0406139].

[71] R. D. Ball, S. Forte and G. Ridolfi, Phys. Lett. B378, 255 (1996), [arXiv:hep-ph/9510449].

[72] G. Altarelli and G. G. Ross, Phys. Lett. B212, 391 (1988).

[73] S. Procureur [COMPASS Collaboration], “New measurement of Delta(G)/G at COMPASS,” arXiv:hep-ex/0605043.

[74] R. Fatemi [STAR Collaboration], “Using jet asymmetries to access Delta(G), the gluon helicity distribution of the proton at STAR,” arXiv:nucl-ex/0606007.

[75] Y. Fukao [PHENIX Collaboration], “The overview of the spin physics at RHIC-PHENIX experiment,” AIP Conf. Proc. 842, 321 (2006), [arXiv:hep-ex/0607033].

[76] G. M. Shore, “The proton spin crisis: Another ABJ anomaly?”, In: From the Planck length to the Hubble radius, Erice 1998, ed. A. Zichichi, World Scientific, Singapore, pp 79-105; arXiv:hep-ph/9812355.

[77] L. Trentadue and G. Veneziano, Phys. Lett. B323, 201 (1994).

[78] M. Grazzini, L. Trentadue and G. Veneziano, Nucl. Phys. B519, 394 (1998), [arXiv:hep-ph/9709452].

[79] D. de Florian, G. M. Shore and G. Veneziano, “Target fragmentation at polarized HERA: A test of universal topological charge screening in QCD,” In: Proceedings of the 1997 Workshop with Polarized Protons at Hera, ed. A. de Roeck and T. Gehrmann, Hamburg/Zeuthen 1997, pp 696-703; arXiv:hep-ph/9711353.

[80] G. M. Shore, Nucl. Phys. Proc. Suppl. 64, 167 (1998), [arXiv:hep-ph/9710367].

[81] M. Grazzini, G. M. Shore and B. E. White, Nucl. Phys. B555, 259 (1999), [arXiv:hep-ph/9903530].

[82] A. De Roeck, Nucl. Phys. Proc. Suppl. 105, 40 (2002), [arXiv:hep-ph/0110335].
[83] S.D. Bass, Int. J. Mod. Phys. A7, 6039 (1992).

[84] K. Sasaki, T. Ueda and T. Uematsu, Phys. Rev. D73, 094024 (2006), [arXiv:hep-ph/0604130].

[85] T. Ueda, T. Uematsu and K. Sasaki, Phys. Lett. B640, 188 (2006), [arXiv:hep-ph/0606267].

[86] F. Richard et al., “TESLA: The Superconducting electron positron linear collider with an integrated X-ray laser laboratory. Technical Design Report, Part I”; hep-ph/0106314.

[87] M. Woods et al., “Luminosity, Energy and Polarization Studies for the Linear Collider”, In: Proc. 5th International Workshop on Electron-Electron Interactions at TeV Energies, Santa Cruz, 2003; physics/0403037.

[88] A. G. Akeroyd et al. (SuperKEKB Physics Working Group), “Physics at Super B Factory”; hep-ex/0406071.