A Nonlinear Consensus Algorithm Derived from Statistical Physics

Michael Margaliot, Alon Raveh and Yoram Zarai

Abstract—The asymmetric simple exclusion process (ASEP) is an important model from statistical physics describing particles that hop randomly from one site to the next along an ordered lattice of sites, but only if the next site is empty. ASEP has been used to model and analyze numerous multiagent systems with local interactions ranging from ribosome flow along the mRNA to pedestrian traffic.

In ASEP with periodic boundary conditions a particle that hops from the last site returns to the first one. The mean field approximation of this model is referred to as the ribosome flow model on a ring (RFMR). We analyze the RFMR using the theory of monotone dynamical systems. We show that it admits a continuum of equilibrium points and that every trajectory converges to an equilibrium point. Furthermore, we show that it entrains to periodic transition rates between the sites.

When all the transition rates are equal all the state variables converge to the same value. Thus, the RFMR with homogeneous transition rates is a nonlinear consensus algorithm. We describe an application of this to a simple formation control problem.

Index Terms—Nonlinear average consensus, monotone dynamical systems, first integral, asymptotic stability, ribosome flow model, entrainment, asymmetric simple exclusion process, mean field approximation.

I. INTRODUCTION

Distributed multi-agent networks are receiving enormous attention. This seems to be motivated both by the theoretical challenges in analyzing systems with limited and time-varying communication between the agents, and numerous applications including mobile sensor networks and distributed aerospace systems [16]. A fundamental topic in this field is the consensus problem where all the agents need to agree on a certain quantity of interest while restricted by local communication and computation abilities. In the average-consensus problem, the goal is that all the agents end up with a common value that is the average of their initial values.

A consensus algorithm (protocol) is an interaction rule that specifies the information exchange between an agent and its neighbors in the network in order to reach a consensus among all the agents. An important class of algorithms, used for numerous applications, is based on linear interaction rules between the agents [20], [21].

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In this paper, we consider an important model from statistical physics called the asymmetric simple exclusion process (ASEP). ASEP describes particles that hop along an ordered lattice of sites. The dynamics is stochastic: at each time step the particles are scanned, and every particle hops to the next site with some probability if the next site is empty. This simple exclusion principle allows modeling of the interaction between the particles. Note that in particular this prohibits overtaking between particles.

The term “asymmetric” refers to the fact that there is a preferred direction of movement. When the movement is unidirectional, some authors use the term totally asymmetric simple exclusion process (TASEP). ASEP was first proposed in 1968 [11] as a model for the movement of ribosomes along the mRNA strand during gene translation. In this context, the mRNA strand is the lattice and the ribosomes are the particles. Simple exclusion corresponds to the fact that a ribosome cannot move forward if there is another ribosome right in front of it. ASEP has become a paradigmatic model for non-equilibrium statistical mechanics [2], [1]. It is used as the standard model for gene translation [35], and has also been applied to model numerous multiagent systems with local interactions including traffic flow, kinesin traffic, the movement of ants along a trail, pedestrian dynamics and ad-hoc communication networks [24], [27].

The dynamic behavior of ASEP is sensitive to the boundary conditions. In ASEP with periodic boundary conditions the lattice is closed, so that a particle that hops from the last site returns to the first one. In particular, the number of particles on the lattice is conserved. In the open boundary conditions, the lattice boundaries are open and the first and last sites are connected to external particle reservoirs that drive the asymmetric flow of the particles along the lattice.

Recently, the mean field approximation of ASEP with open boundary conditions, called the ribosome flow model (RFM), has been analyzed using tools from systems and control theory [14], [13], [15], [33], [12], [34].

In this paper, we consider the mean field approximation of ASEP with periodic boundary conditions. This is a set of $n$ deterministic nonlinear first-order ordinary differential equations, where $n$ is the number of sites, and each state-variable describes the occupancy level in one of the sites. We refer to this system as the ribosome flow model on a ring (RFMR).

We show that the RFMR admits a continuum of equilibrium points, and that every trajectory converges to an equilibrium point. Furthermore, if the transition rates between the sites are periodic, with a common period $T$, then every trajectory
converges to a periodic solution with period $T$. In other words, the RFMR entrains to the periodic excitation.

In the particular case where all the transition rates are equal all the state variables converge to the same value, namely, the average of all the initial values. In the RFMR, the dynamics of state-variable $x_i$ is local in the sense that it depends only on $x_{i-1}$, $x_i$, and $x_{i+1}$. In other words, information is exchanged between a site and its two nearest neighbors only. Thus, the convergence result implies that the RFMR with homogeneous transition rates is a nonlinear average consensus algorithm. One of the main contributions of this paper is simply in reinterpreting ASEP in the context of consensus algorithms. We describe an application of the theoretical results to a simple formation control problem.

The remainder of this paper is organized as follows. Section II reviews the RFMR. Section III details the main results. Section IV describes the application to formation control. The final section summarizes and describes several variants.

We use standard notation. For an integer $i$, $i_n \in \mathbb{R}^n$ is the $n$-dimensional column vector with all entries equal to $i$. For a matrix $M$, $M'$ denotes the transpose of $M$. Let $|\cdot| : \mathbb{R}^n \to \mathbb{R}_+$ denote the $L_1$ vector norm, that is, $|z|_1 = |z_1| + \cdots + |z_n|$.

II. THE MODEL

The ribosome flow model on a ring (RFMR) is given by

$$
\dot{x}_1 = \lambda_n x_n (1 - x_1) - \lambda_1 x_1 (1 - x_2),
\dot{x}_2 = \lambda_1 x_1 (1 - x_2) - \lambda_2 x_2 (1 - x_3),
\dot{x}_3 = \lambda_2 x_2 (1 - x_3) - \lambda_3 x_3 (1 - x_4),
\vdots
\dot{x}_{n-1} = \lambda_{n-2} x_{n-2} (1 - x_{n-1}) - \lambda_{n-1} x_{n-1} (1 - x_n),
\dot{x}_n = \lambda_{n-1} x_{n-1} (1 - x_n) - \lambda_n x_n (1 - x_1).
$$

(1)

Here $x_i(t)$ is the normalized occupancy level at site $i$ at time $t$, so that $x_i(t) = 0$ [$x_i(t) = 1$] means that site $i$ is completely empty [full] at time $t$. The transition rates $\lambda_1, \ldots, \lambda_n$ are all positive numbers.

To explain this model, consider the equation $\dot{x}_2 = \lambda_1 x_1 (1 - x_2) - \lambda_2 x_2 (1 - x_3)$. The term $\lambda_1 x_1 (1 - x_2)$ represents the flow of particles from site 1 to site 2. This is proportional to the occupancy $x_1$ at site 1 and also to $1 - x_2$, i.e. the flow decreases as site 2 becomes fuller. This is a relaxed version of simple exclusion. The term $\lambda_2 x_2 (1 - x_3)$ represents the flow of particles from site 2 to site 3. The other equations are similar, with the term $\lambda_n x_n (1 - x_1)$ appearing both in the equations for $\dot{x}_1$ and for $\dot{x}_n$ due to the circular structure of the model (see Fig. 1).

The RFMR encapsulates simple exclusion, unidirectional movement along the ring, and the periodic boundary condition of ASEP. This is not surprising, as the RFMR is the mean field approximation of ASEP with periodic boundary conditions (see, e.g., [1, p. R345] and [29, p. 1919]).

Note that we can write (1) succinctly as

$$
\dot{x}_i = \lambda_{i-1} x_{i-1} (1 - x_i) - \lambda_i x_i (1 - x_{i+1}), \quad i = 1, \ldots, n,
$$

where here and below every index is interpreted modulo $n$. Note also that $0_n [1_n]$ is an equilibrium point of (1). Indeed, when all the sites are completely free [completely full] there is no movement of particles between the sites.

For our purposes, it is important to note that the RFMR is a local communication model in the sense that $\dot{x}_k$ depends on $x_{k-1}$, $x_k$, and $x_{k+1}$ only. If we regard $x_k(t)$ as a data value of agent $k$ at time $t$ then updating this data according to (1) requires agent $k$ to communicate with agents $k - 1, k$, and $k + 1$ only.

Denote

$$
C^n := \{ y \in \mathbb{R}^n : y_i \in [0, 1], \ i = 1, \ldots, n \},
$$

i.e., the closed unit cube in $\mathbb{R}^n$. Since the state-variables represent normalized occupancy levels, we always consider initial conditions $x(0) \in C^n$. It is straightforward to verify that $C^n$ is an invariant set of (1), i.e. $x(0) \in C^n$ implies that $x(t) \in C^n$ for all $t \geq 0$.

Note that (1) implies that

$$
\sum_{i=0}^{n} \dot{x}_i(t) = 0, \text{ for all } t \geq 0,
$$

so the total occupancy $H(x) := 1_n^T x$ is conserved:

$$
H(x(t)) = H(x(0)), \quad \text{for all } t \geq 0.
$$

(2)

In other words, the dynamics redistributes the particles between the sites, but without changing the total occupancy level.

Eq. (2) means that we can reduce the $n$-dimensional RFMR into an $(n - 1)$-dimensional model. In particular, the RFMR with $n = 2$ can be explicitly solved. The next example demonstrates this.

**Example 1** Consider (1) with $n = 2$, i.e.

$$
\dot{x}_1 = \lambda_2 x_2 (1 - x_1) - \lambda_1 x_1 (1 - x_2),
\dot{x}_2 = \lambda_1 x_1 (1 - x_2) - \lambda_2 x_2 (1 - x_1).
$$

(3)

We assume that $x(0) \neq 0_2$ and $x(0) \neq 1_2$, as these are equilibrium points of the dynamics. Let $s := x_1(0) + x_2(0)$. Substituting $x_2(t) = s - x_1(t)$ in (3) yields

$$
\dot{x}_1 = \lambda_2 (s - x_1)(1 - x_1) - \lambda_1 x_1 (1 - s + x_1)
= \alpha_2 x_1^2 + \alpha_1 x_1 + \alpha_0,
$$

(4)
where
\[ \alpha_2 := \lambda_2 - \lambda_1, \]
\[ \alpha_1 := (\lambda_1 - \lambda_2)s - \lambda_1 - \lambda_2, \]
\[ \alpha_0 := s\lambda_2. \]

If \( \lambda_1 = \lambda_2 \), then (4) is a linear differential equation and its solution is
\[ x_1(t) = \frac{s}{2}(1 - \exp(-2\lambda_1 t)) + x_1(0)\exp(-2\lambda_1 t), \]
so
\[ x_2(t) = s - x_1(t) \]
\[ = \frac{s}{2}(1 + \exp(-2\lambda_1 t)) - x_1(0)\exp(-2\lambda_1 t) \]
\[ = \frac{s}{2}(1 - \exp(-2\lambda_1 t)) + x_2(0)\exp(-2\lambda_1 t). \]

In particular,
\[ \lim_{t \to \infty} x(t) = (s/2)1_2, \]
i.e., the state-variables converge at an exponential rate to the average of their initial values.

If \( \lambda_1 \neq \lambda_2 \), then (4) is a Riccati equation and solving it yields
\[ x_1(t) = \frac{-\alpha_1 - \sqrt{\Delta} \coth(\sqrt{\Delta}(t - t_0)/2)}{2\alpha_2}, \]
where
\[ \Delta := \alpha_1^2 - 4\alpha_2\alpha_0 = (s - 1)^2(\lambda_1 - \lambda_2)^2 + 4\lambda_1 \lambda_2, \]
\[ t_0 := \frac{2}{\sqrt{\Delta}} \coth^{-1} \left( \frac{2x_1(0)a_2 + \alpha_1}{\sqrt{\Delta}} \right). \]

Note that since the \( \lambda_i \)'s are positive, \( \Delta > 0 \). Also, a straightforward calculation shows that \( t_0 \) is well-defined for all \( x_1(0) \in [0, 1] \). Note that (8) implies that
\[ \lim_{t \to \infty} x(t) = \frac{1}{2\alpha_2} \left[ -\alpha_1 - \sqrt{\Delta} \right] \left[ 2\alpha_2 s + \alpha_1 + \sqrt{\Delta} \right]. \]
The identity
\[ \coth \left( \frac{t}{2\sqrt{\Delta}} \right) - 1 = \frac{2}{\exp(\sqrt{\Delta}t) - 1} \]
implies that for sufficiently large values of \( t \) the convergence is with rate \( \exp(-\sqrt{\Delta}t) \). Thus, the convergence rate depends on \( \lambda_1, \lambda_2, \) and \( s \).

Summarizing, every trajectory follows the straight line from \( x(0) \) to an equilibrium point \( e = e(\lambda_1, \lambda_2, s) \). In particular, if \( a, b \in C^2 \) satisfy \( I_2^t a = I_2^t b \) then the solutions emanating from \( a \) and from \( b \) converge to the same equilibrium point. Fig. 2 depicts the trajectories of the RFMR with \( n = 2 \), \( \lambda_1 = 2 \) and \( \lambda_2 = 1 \) for three initial conditions. □

The next section describes several theoretical results on the RFMR. An application of these results to a consensus problem is described in Section IV.

**III. MAIN RESULTS**

**A. Strong Monotonicity**

A cone \( K \) in \( \mathbb{R}^n \) defines a partial ordering in \( \mathbb{R}^n \) as follows. For two vectors \( a, b \in \mathbb{R}^n \), we write \( a \preceq b \) if \( (b - a) \in K \); \( a \prec b \) if \( a \preceq b \) and \( a \neq b \); and \( a \ll b \) if \( (b - a) \in \text{Int}(K) \). The system \( \dot{y} = f(y) \) is called monotone if \( a \preceq b \) implies that \( y(t, a) \preceq y(t, b) \) for all \( t \geq 0 \). In other words, the flow preserves the partial ordering [26]. It is called strongly monotone if \( a \prec b \) implies that \( y(t, a) \ll y(t, b) \) for all \( t > 0 \).

From here on we consider the particular case where the cone is \( K = \mathbb{R}_+^n \). Then \( a \preceq b \) if \( a_i \leq b_i \) for all \( i \), and \( a \ll b \) if \( a_i < b_i \) for all \( i \). A system that is monotone with respect to this partial order is called cooperative.

The linear average consensus protocol is \( \dot{y} = Ay \), where \( A \) is a Metzler matrix, with zero sum rows. It is well-known that the Metzler property implies that this system is cooperative. The next result shows that the same holds for the RFMR.

**Proposition 1** Let \( x(t, a) \) denote the solution of the RFMR at time \( t \) for the initial condition \( x(0) = a \). For any \( a, b \in C^n \) with \( a \preceq b \) we have
\[ x(t, a) \preceq x(t, b), \quad \text{for all } t \geq 0. \]
Furthermore, if \( a \prec b \) then
\[ x(t, a) \ll x(t, b), \quad \text{for all } t > 0. \]

**Proof.** Write the RFMR (1) as \( \dot{x} = f(x) \). The Jacobian matrix \( J(x) := \frac{\partial f}{\partial x}(x) \) is given in (12). This matrix has nonnegative off-diagonal entries for all \( x \in C^n \). Thus, the RFMR is a cooperative system [26], and this implies (10). Furthermore, it is straightforward to verify that \( J(x) \) is an irreducible matrix for all \( x \in C^n \), and this implies (11) (see, e.g., [26, Ch. 4]). □
B. Stability

The next result shows that every level set $L_s$ of $H$ contains a unique equilibrium point, and that any trajectory of the RFMR emanating from $L_s$ converges to this equilibrium point.

**Theorem 1** Pick $s \in [0, n]$, and let

$$L_s := \{ y \in C^n : 1_s^n y = s \}.$$  

Then $L_s$ contains a unique equilibrium point $e_{L_s}$ of the RFMR and for any $a \in L_s$, 

$$\lim_{t \to \infty} x(t, a) = e_{L_s}.$$ 

Furthermore, for any $0 \leq s < p \leq n$, we have 

$$e_{L_s} \leq e_{L_p}.$$ 

**Proof.** Since the RFMR is a cooperative irreducible system with $H(x) = 1_s^n x$ as a first integral, Thm. 1 follows from the results in [18] (see also [17] and [8] for some related ideas). $\blacksquare$

Note that Thm. 1 implies that the RFMR has a continuum of linearly ordered equilibrium points, namely, $\{e_{L_s} : s \in [0, n]\}$, and also that every solution of the RFMR converges to an equilibrium point.

**Example 2** Consider the RFMR with $n = 3$, $\lambda_1 = 2$, $\lambda_2 = 3$, and $\lambda_3 = 1$. Fig. 3 depicts trajectories of this RFMR for three initial conditions in $L_2$: $[1 1 0]$, $[1 0 1]$, and $[0 1 1]$. It may be observed that all the trajectories converge to the same equilibrium point $e_{L_2} \approx [0.5380 \ 0.6528 \ 0.8091]^T$. Fig. 4 depicts all the equilibrium points of this RFMR. Since $\lambda_2 > \lambda_1$ and $\lambda_2 > \lambda_3$, the transition rate into site 3 is relatively large. As may be observed from the figure this leads to $e_3 \geq e_1$ and $e_3 \geq e_2$ for every equilibrium point $e$. $\Box$

**Example 3** Consider again the RFMR with $n = 2$. Fix $0 < s < p < 2$. Pick $a \in L_s$ and $b \in L_p$. If $\lambda_1 = \lambda_2$ then (7) implies that 

$$e_{L_s} = \lim_{t \to \infty} x(t, a) = (s/2)1_2,$$

$$e_{L_p} = \lim_{t \to \infty} x(t, b) = (p/2)1_2,$$

so clearly (13) holds. Now suppose that $\lambda_1 \neq \lambda_2$. Assume first that $\lambda_2 > \lambda_1$. Denote the coordinates of $e_{L_s}$ [$e_{L_p}$] by $v_1$, $v_2$ [$w_1$, $w_2$]. Recall that $v_1$ is a root of the polynomial 

$$P_s(z) := (\lambda_2 - \lambda_1)z^2 + ((\lambda_1 - \lambda_2)s - \lambda_1 - \lambda_2)z + s\lambda_2$$

(see (4)) satisfying $v_1 \in [0, 1]$. This is a “smiling parabola” satisfying $P_s(0) = s\lambda_2 > 0$ and $P_s(1) = \lambda_1(s - 2) < 0$. Similarly, $w_1 \in [0, 1]$ is a root of a “smiling parabola” $P_p(z)$ satisfying $P_p(0) = p\lambda_2 > 0$ and $P_p(1) = \lambda_1(p - 2) < 0$. Since $p > s$, the graph of $P_p(z)$ lies strictly above the graph
of $P_s(z)$ for all $z \in [0, 1]$. Therefore, $v_1 < w_1$. To show that $v_2 < w_2$ note that $x_2(t,a) = s - x_1(t,a)$ yields
\[
\dot{x}_2(t,a) = -\dot{x}_1(t,a) = -P_s(x_1(t,a)) = -P_s(s - x_2(t,a)).
\]
This implies that $v_2$ is a root of
\[
P_s(z) := -P_s(s - z)
\]
in $[0, 1]$. This is a “frowning parabola” and a calculation yields $P_s(0) = s\lambda_1 > 0$ and $P_s(1) = (s-2)\lambda_2 < 0$. Now $p > s$ implies that the graph of $P_s(z)$ lies strictly below the graph of $P_s(z)$ for all $z \in [0, 1]$, so $v_2 < w_2$. We conclude that $e_{L_s} \ll e_{L_p}$. The analysis in the case $\lambda_2 < \lambda_1$ is similar and again shows that (13) holds. □

C. Contraction

Contraction theory is a powerful tool for analyzing nonlinear dynamical systems (see, e.g., [10], [23]). In a contractive system, the distance between any two trajectories decreases at an exponential rate. It is clear that the RFMR is not a contractive system on $C^n$, with respect to any norm, as it admits more than a single equilibrium point. Nevertheless, the next result shows that the RFMR is non-expanding with respect to the $L_1$ norm.

**Proposition 2** For any $a, b \in C^n$,
\[
|x(t,a) - x(t,b)|_1 \leq |a - b|_1, \quad \text{for all} \ t \geq 0. \quad (14)
\]
In other words, the $L_1$ distance between trajectories can never increase.

*Proof.* Recall that the matrix measure $\mu_1(\cdot) : \mathbb{R}^{n \times n} \to \mathbb{R}$ induced by the $L_1$ norm is
\[
\mu_1(A) = \max\{c_1(A), \ldots, c_n(A)\},
\]
where $c_i(A) := a_{ii} + \sum_{j \neq i} |a_{ij}|$, i.e. the sum of entries in column $i$ of $A$, with the off-diagonal entries taken with absolute value [31]. For the Jacobian of the RFMR, we have $c_i(J(x)) = 0$ for all $i$ and all $x \in C^n$, so $\mu_1(J(x)) = 0$. Now (14) follows from standard results in contraction theory (see, e.g., [23]). ■

**Example 4** Pick $a, b \in C^n$ such that $b \leq a$. By monotonicity, $x(t,b) \leq x(t,a)$ for all $t \geq 0$, so $d(t) := |x(t,a) - x(t,b)|_1 = 1_n(x(t,a) - x(t,b))$. Thus,
\[
\dot{d}(t) = 1_n^T \dot{x}(t,a) - 1_n^T \dot{x}(t,b) = 0 - 0,
\]
so clearly in this case (14) hold with an equality. □

**Example 5** Consider the RFMR with $n = 2$. Pick $a, b \in \{C^2 \setminus \{0_2, 1_2\}\}$ such that $s := 1_2^T a = 1_2^T b$. In other words, $a, b$ both belong to $L_s$. Note that in this case
\[
d(t) := |x_1(t,a) - x_1(t,b)| + |x_2(t,a) - x_2(t,b)|
= |x_1(t,a) - x_1(t,b)| + |s - x_1(t,a) - (s - x_1(t,b))|
= 2|x_1(t,a) - x_1(t,b)|.
\]
In particular, $d(0) = 2|a_1 - b_1|$. If $\lambda_1 = \lambda_2$ then (5) yields $d(t) = 2|a_1 - b_1| \exp(-2\lambda_1 t)$, so clearly (14) holds. If $\lambda_1 \neq \lambda_2$ then (8) yields
\[
d(t) = \frac{\sqrt{\Delta}}{|\alpha_2|} \left| \coth\left(\frac{\sqrt{\Delta}}{2} (t - t_0(b)) \right) - \coth\left(\frac{\sqrt{\Delta}}{2} (t - t_0(a)) \right) \right|
\]
where $t_0(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is defined by
\[
t_0(z) := \frac{2}{\sqrt{\Delta}} \coth^{-1}\left(\frac{2z_1\alpha_2 + \alpha_1}{\sqrt{\Delta}}\right).
\]
Applying (9) and the identity $2 \coth^{-1}(x) = \ln(\frac{3-1}{x})$, for $|x| > 1$, yields
\[
d(t) = d(0)/|\gamma(t)|,
\]
where
\[
\gamma(t) := \frac{1}{4} \exp(\sqrt{\Delta} t) (q(a)q(b) + 1 - q(a) - q(b))
+ \frac{1}{4} \exp(-\sqrt{\Delta} t) (q(a)q(b) + 1 + q(a) + q(b))
+ \frac{1}{2}(1 - q(a)q(b)),
\]
and the function $q(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is defined by
\[
q(z) := \frac{(\lambda_2 - \lambda_1)(z_1 - z_2) - (\lambda_1 + \lambda_2)}{\sqrt{\Delta}}.
\]
Note that $\gamma(0) = 1$. We need to show that $\gamma(t) \geq 1$ for all $t \geq 0$, meaning
\[
(q(a)q(b) + 1)(\cosh(\sqrt{\Delta}t) - 1) \geq (q(a) + q(b)) \sinh(\sqrt{\Delta}t).
\]
(15)
Since $a_i \in (0,1)$ and $b_i \in (0,1)$, $q(a) < 0$ and $q(b) < 0$. Thus (15) holds (with equality only at $t = 0$), so $\gamma(t) > 1$ and, therefore, $d(t) < d(0)$ for all $t > 0$. □

Pick $a \in C^n$, and let $s := 1_n^T a$. Substituting $b = e_{L_s}$ in (14) yields
\[
|x(t,a) - e_{L_s}|_1 \leq |a - e_{L_s}|_1, \quad \text{for all} \ t \geq 0. \quad (16)
\]
This means that the convergence to $e_{L_s}$ is monotone in the sense that the $L_1$ distance to $e_{L_s}$ can never increase. Combining (16) with Theorem 1 implies that every equilibrium point of the RFMR is semistable [5].

D. Entrainment

Consider vehicles moving along a circular road. Traffic flow is controlled by traffic lights located along the road. Assume that all the traffic lights operate at a periodic manner with a common period $T > 0$. A natural question is: will the traffic density and/or traffic flow converge to a periodic pattern with period $T$?

We can model this using the RFMR as follows. We say that a function $f$ is $T$-periodic if $f(t + T) = f(t)$ for all $t$. Assume that the $\lambda_i$s are time-varying functions satisfying:
- there exist $0 < \delta_1 < \delta_2$ such that $\lambda_i(t) \in [\delta_1, \delta_2]$ for all $t \geq 0$ and all $i \in \{1, \ldots, n\}$,
- there exists a (minimal) $T > 0$ such that all the $\lambda_i$s are $T$-periodic.
We refer to the model in this case as the periodic ribosome flow model on a ring (PRFMR).

**Theorem 2** Consider the PRFMR. Fix an arbitrary \( s \in [0, n] \). There exists a unique function \( \phi_s : \mathbb{R}_+ \to \mathbb{C}^n \), that is \( T \)-periodic, and

\[
\lim_{t \to \infty} |x(t, a) - \phi_s(t)| = 0, \quad \text{for all } a \in L_s.
\]

In other words, every level set \( L_s \) of \( H \) contains a unique periodic solution, and every solution of the PRFMR emanating from \( L_s \) converges to this solution. Thus, the PRFMR entrains (or phase locks) to the periodic excitation in the \( \lambda_i \)s.

Note that since a constant function is a periodic function for any \( T \), Thm. 2 implies entrainment to a periodic trajectory in the particular case where one of the \( \lambda_i \)s oscillates and the other are constant. Note also that Thm. 1 follows from Thm. 2.

**Proof of Thm. 2.** Write the PRFMR as \( \dot{x} = f(t, x) \). Then \( f(t, y) = f(t + T, y) \) for all \( t \) and \( y \). Furthermore, \( H(x) = 1_n^t x \) is a first integral of the PRFMR. Now Thm. 2 follows from the results in [28] (see also [7]). \( \Box \)

**Example 6** Consider the RFMR with \( n = 3 \), \( \lambda_1(t) = 3 \), \( \lambda_2(t) = 3 + 2\sin(t + 1/2) \), and \( \lambda_3(t) = 4 - 2\cos(2t) \). Note that all the \( \lambda_i \)s are periodic with a minimal common period \( T = 2\pi \). Fig. 5 shows the solution \( x(t, a) \) for \( a = [0.5 \quad 0.01 \quad 0.9]^\top \). It may be seen that every \( x_i(t) \) converges to a periodic function with period \( 2\pi \). \( \Box \)

**Example 7** Consider the RFMR with \( n = 2 \), \( \lambda_1(t) = 3q(t)/2 \), and \( \lambda_2(t) = q(t)/2 \), where \( q(t) \) is a strictly positive and periodic function. Then (4) becomes

\[
\dot{x}_1 = (-x_1^2 + (s - 2)x_1 + s/2)q.
\]

Assume that

\[
x_1^2(0) < s/2.
\]

It is straightforward to verify that in this case the solution of (17) is

\[
x_1(t) = (s/2) - 1 + z \tanh \left( k + z \int_0^t q(s)ds \right),
\]

where

\[
z := \sqrt{3 + (s - 1)^2}/2,
\]

and

\[
k := \tanh^{-1} ((x_1(0) + 1 - s/2)/z).
\]

Note that (18) implies that \( k \) is well-defined. Suppose, for example, that \( q(t) = 2 + \sin(t) \). Then \( \lambda_1(t), \lambda_2(t) \) are periodic with period \( T = 2\pi \). In this case,

\[
x_1(t) = (s/2) - 1 + z \tanh \left( k + z(2t + 1 - \cos(t)) \right),
\]

and

\[
x_2(t) = s - x_1(t) = (s/2) + 1 - z \tanh \left( k + z(2t + 1 - \cos(t)) \right).
\]

Thus, for every \( a \in L_s \), \( \lim_{t \to \infty} x(t, a) = \phi_s(t) \), where \( \phi_s(t) = \left[(s/2) - 1 + z \quad (s/2) + 1 - z \right] \) (which is of course periodic with period \( T \)). \( \Box \)

**E. The homogeneous case**

Fix an arbitrary \( s \in [0, n] \). To simplify the notation, we just write \( e \) instead of \( e_{L_s} \) from here on. Then

\[
1_n^t e = s, \quad (19)
\]

and since for \( x = e \) the left-hand side of all the equations in (1) is zero,

\[
\lambda_n e_n (1 - e_1) = \lambda_1 e_1 (1 - e_2) = \lambda_2 e_2 (1 - e_3) = \cdots = \lambda_{n-1} e_{n-1} (1 - e_n). \quad (20)
\]

In other words, the steady-state flow \( r := r_{ii+1} = \lambda_i e_i (1 - e_{i+1}) \) for all \( i \).

In general, solving (19) and (20) explicitly seems difficult. In this section, we consider a special case where more explicit results can be derived, namely, the case where

\[
\lambda_1 = \cdots = \lambda_n := \lambda_c,
\]

i.e. all the transition rates are equal, with \( \lambda_c \) denoting their common value. In this case (1) becomes:

\[
\dot{x}_1 = \lambda_c x_n (1 - x_1) - \lambda_c x_1 (1 - x_2),
\]

\[
\dot{x}_2 = \lambda_c x_1 (1 - x_2) - \lambda_c x_2 (1 - x_3),
\]

\[
\vdots
\]

\[
\dot{x}_n = \lambda_c x_{n-1} (1 - x_n) - \lambda_c x_n (1 - x_1). \quad (21)
\]

We refer to this as the homogeneous ribosome flow model on
a ring (HRFMR). Also, (20) becomes
\[ e_n(1 - e_1) = e_1(1 - e_2) \\
\quad = e_2(1 - e_3) \\
\vdots \\
\quad = e_{n-1}(1 - e_n), \tag{22} \]
and it is straightforward to verify that \( e = c1_n, \ c \in \mathbb{R}, \) satisfies (22).

Define the averaging operator \( \text{Ave}(\cdot) : \mathbb{R}^n \to \mathbb{R} \) by \( \text{Ave}(z) := \frac{1}{n} \sum_{i=1}^{n} z_i. \)

**Corollary 1** For any \( a \in C^n \) the solution of the HRFMR satisfies
\[ \lim_{t \to \infty} x(t, a) = \text{Ave}(a)1_n. \]

Note that this implies that the steady-state flow is \( r = \lambda_n \text{Ave}(a)(1 - \text{Ave}(a)). \) Thus, \( r \) is maximized when \( \text{Ave}(a) = 1/2 \) and the maximal value is \( r^* = \lambda_n/4. \)

**Proof of Corollary 1.** Let \( s := 1_n^t a. \) Then \( L_s \) contains \( \text{Ave}(a)1_n \) and this is an equilibrium point. The proof now follows immediately from Thm. 1.

**Remark 1** It is possible also to give a simple and self-contained proof of Corollary 1 using standard tools from the literature on consensus networks. Indeed, pick \( \tau > 0 \) and let \( i \) be an index such that \( x_i(\tau) \geq x_j(\tau) \) for all \( j \neq i. \) Then
\[ \dot{x}_i(\tau) = x_{i-1}(\tau)(1 - x_i(\tau)) - x_i(\tau)(1 - x_{i+1}(\tau)) \leq x_i(\tau)(1 - x_i(\tau)) - x_i(\tau)(1 - x_i(\tau)) = 0. \]

Furthermore, if \( x_i(\tau) > x_j(\tau) \) for all \( j \neq i \) then \( \dot{x}_i(\tau) < 0. \) A similar argument shows that if \( x_i(\tau) \leq x_j(\tau) \) then \( \dot{x}_i(\tau) \geq 0 \) \( \dot{x}_j(\tau) \geq 0. \) Define \( V(\cdot) : \mathbb{R}^n \to \mathbb{R}_+ \) by \( V(y) := \max_i y_i - \min_i y_i. \) Then \( V(x(t)) \) strictly decreases along trajectories of the HRFMR unless \( x(t) = c1_n \) for some \( c \in \mathbb{R}, \) and a standard argument (see, e.g., [9]) implies that the system converges to consensus. Combining this with (2) completes the proof of Corollary 1.

In other words, the HRFMR may be interpreted as a nonlinear average consensus network. Indeed, every state-variable replaces information with its two nearest neighbors on the ring only, yet the dynamics guarantees that every state-variable converges to \( \text{Ave}(a). \)

The physical nature of the underlying model provides a simple explanation for convergence to average consensus. Indeed, the HRFMR may be interpreted as a system of \( n \) water tanks connected in a circular topology through identical pipes. The flow in this system is driven by the imbalance in the water levels, and the state always converges to a homogeneous distribution of water in the tanks. Since the system is closed, this corresponds to average consensus.

1) Convergence rate: The convergence rate of the HRFMR in the vicinity of the equilibrium point \( c1_n \) can be analyzed as follows. Let \( y := x - c1_n. \) Then a calculation shows that the linearized dynamics of \( y \) is given by \( \dot{y} = Qy, \) where
\[ Q := \begin{bmatrix} -1 & c & 0 & 0 & \ldots & 0 & 1 - c \\ 1 - c & -1 & c & 0 & \ldots & 0 & 0 \\ 0 & 1 - c & -1 & c & \ldots & 0 & 0 \\ \vdots \\ c & 0 & 0 & 0 & \ldots & 1 - c & -1 \end{bmatrix}. \]

Using known-results on the eigenvalues of a circulant matrix (see, e.g., [4]) implies that the eigenvalues of \( Q \) are
\[ \lambda_\ell = -1 + cw^{\ell-1} + (1 - c)w^{(\ell-1)(n-1)}, \ \ell = 1, \ldots, n, \]
where \( w := \exp(2\pi \sqrt{-1}/n). \) In particular, \( \lambda_1 = 0. \) The corresponding eigenvector is \( 1_n. \) This is a consequence of the continuum of equilibria in the HRFMR. Also,
\[ \Re(\lambda_\ell) = \cos(2\pi(\ell - 1)(n-1)/n) + c(\cos(2\pi(\ell - 1)/n) - \cos(2\pi(\ell - 1)(n-1)/n)) - 1 \]
\[ = \cos(2\pi(\ell - 1)(n-1)/n) - 1, \]
and this implies that
\[ \Re(\lambda_\ell) \leq \Re(\lambda_2) = \cos(2\pi(n-1)/n) - 1, \]
for \( \ell = 2, \ldots, n. \) Thus, for \( x(0) \) in the vicinity of the equilibrium
\[ |x(t) - c1_n| \leq \exp((\cos(2\pi(n-1)/n) - 1)t)|x(0) - c1_n|. \tag{23} \]
The convergence rate decays with \( n. \) For Example, for \( n = 2, \)
\[ \cos(2\pi(n-1)/n) = -1 \]
and for \( n = 10, \)
\[ \cos(2\pi(n-1)/n) = -1 \approx -0.191. \] In other words, as the length of the chain increases the convergence rate decreases. This is the price paid for the limited communication between the agents.

Our simulations suggest that (23) actually provides a reasonable approximation for the real convergence rate (i.e., not only in the vicinity of the equilibrium point). The next example demonstrates this.

**Example 8** Consider the HRFMR with \( n = 4. \) In this case, \( \Re(\lambda_2) = -1, \) so (23) becomes \( \log(|x(t) - c1_n|) \approx -t + \log(|x(0) - c1_n|). \) Fig. 6 depicts \( \log(|x(t) - (1/4)1_4|) \) for the initial condition \( x(0) = [1 \ 0 \ 0 \ 0]. \) Note that here \( \log(|x(0) - (1/4)1_4|) = \log(\sqrt{3}/4). \) Also shown is the graph of \( -t + \log(\sqrt{3}/4). \) It may be seen that the real convergence rate is slightly faster than the estimate in (23).

**IV. AN APPLICATION: ORBITAL COLLECTIVE MOTION WITH LIMITED COMMUNICATION**

Consider a collection of \( n \) agents moving along a circular ring of radius \( R. \) The location of agent \( k \) at time \( t \) is
\[ [R \cos(\theta_k(t)) \ R \sin(\theta_k(t))]^t, \tag{24} \]
and the dynamics is
\[ \dot{\theta}_k = u_k, \quad k = 1, \ldots, n, \tag{25} \]
i.e. \( u_k \) controls the angular velocity of agent \( k. \)

We say that the agents are in a balanced configuration at time \( t \) if any two neighboring agents along the ring \( p, q, \)
with \( \theta_p(t) - \theta_q(t) \geq 0 \), satisfy \( \theta_p(t) - \theta_q(t) = 2\pi/n \). The goal is to design a control \( u = [u_1, \ldots, u_n]' \) asymptotically driving the system to a balanced configuration. Furthermore, the control must be \textit{local} in the sense that each \( u_k \) should depend only on the state of agent \( k \) and its neighbors. These type of problems arise in the formation control of unmanned autonomous systems (see, e.g., [25]).

In what follows we assume that the agents are numbered such that

\[
0 \leq \theta_1(0) \leq \theta_2(0) \leq \cdots \leq \theta_n(0) < 2\pi.
\]

Proposition 3 Consider (25) with the nonlinear control

\[
u_k = (x_{k+1} - 1)x_k, \quad k = 1, \ldots, n,
\]

where

\[
x_1 := (\theta_1 - \theta_n + 2\pi)/(2\pi), \quad x_k := (\theta_k - \theta_{k-1})/(2\pi), \quad k = 2, \ldots, n.
\]

Then

\[
\lim_{t \to \infty} (\dot{\theta}_i(t) - \dot{\theta}_{i-1}(t)) = 2\pi/n, \quad \text{for all } i.
\]

In other words, the system always converges to a balanced configuration. Note that (27) implies that \( u_k \) only depends on \( \theta_{k-1}, \theta_k \), and \( \theta_{k+1} \). Thus, it can be implemented using local communication requirements.

Proof of Prop. 3. By (26), \( x_k(0) \in [0, 1], k = 1, \ldots, n \), i.e. \( x(0) \in C^n \). Also,

\[
2\pi x_k = \theta_k - \theta_{k-1} = u_k - u_{k-1} = x_{k-1}(1 - x_k) - x_k(1 - x_{k+1}).
\]

This means that the \( x_i \)s follow the dynamics of the HRFMR with \( \Lambda_c = \frac{1}{2\pi} \). By Corollary 1, \( \lim_{t \to \infty} x(t) = \text{Ave}(x(0))1_n = n^{-1}1_n \). Using (28) completes the proof.

Note that since \( x(0) \in C^n \), \( x(t) \in C^n \) for all \( t \geq 0 \). This means in particular that the angular distance between any two neighbors can never change sign, i.e., the dynamics leads to a balanced configuration without changing the relative order of the agents along the ring. Also, note that the term \( (x_{k+1} - 1)x_k \) in (27) is always non-positive.

Combining (25), (27) and Prop. 3 yields

\[
\lim_{t \to \infty} \dot{\theta}_k(t) = \lim_{t \to \infty} u_k(t) = (n^{-1} - 1)n^{-1}, \quad \text{for all } k.
\]

If we change (27) to

\[
u_k = x_k(x_{k+1} - 1) + v, \quad k = 1, \ldots, n,
\]

with \( v \in \mathbb{R} \), then a similar analysis yields that the agents converge to a balanced configuration but now

\[
\lim_{t \to \infty} \dot{\theta}_k(t) = (n^{-1} - 1)n^{-1} + v, \quad \text{for all } k.
\]

Thus, the asymptotic common angular velocity can be shifted to any desired value. The price for that is that all agents must agree beforehand on the common value \( v \). In particular, taking \( v = (1 - n^{-1})n^{-1} \) yields zero asymptotic angular velocity. Note that using this specific value only requires that each agent knows the total number of agents \( n \).

Example 9 Consider the model (24), (25) with \( n = 4 \), \( \theta_1(0) = 0.9\pi \), \( \theta_2(0) = \pi \), \( \theta_3(0) = 1.1\pi \) and \( \theta_4(0) = 1.2\pi \). Fig. 7 depicts \( \theta(t) \) for the control in (30) with \( v = (1 - 4^{-1})4^{-1} = 3/16 \). It may be seen that \( \theta(t) \) converges to \( \bar{\theta} := [0.2768\pi \quad 0.7768\pi \quad 1.2768\pi \quad 1.7768\pi]' \), i.e., to a stationary configuration. Since \( \bar{\theta}_i - \bar{\theta}_{i-1} = 0.5\pi \) for all \( i \), this configuration is also balanced. \( \Box \)

V. Discussion

Various models inspired by physics, such as the Vicsek et al. model [30] and Kuramoto oscillators, have played an
important role in the development of consensus theory (see, e.g., [6, [3]).

The ribosome flow model on a ring (RFMR) is the mean field approximation of ASEP with periodic boundary conditions. In this paper, we reinterpret the RFMR as a nonlinear consensus model. Indeed, the dynamics corresponds to a multi-agent system in which every agent interacts with its two closest neighbors on the ring only. Every solution converges to a stationary state and when all the transition rates are equal this stationary state corresponds to average consensus. A natural question for further research is what are the advantages of this nonlinear average consensus network with respect to the well-known linear average consensus network.

We analyzed the RFMR using tools from monotone dynamical systems theory. Our results show that the RFMR has several nice properties. It is an irreducible cooperative dynamical system admitting a continuum of linearly ordered equilibrium points, and every trajectory converges to an equilibrium point. The RFMR is on the “verge of contraction”, and it entrains to periodic transition rates.

Topics for further research include the following. ASEP with periodic boundary conditions has been studied extensively in the physics literature and many explicit results are known. For example, the time scale until the system relaxes to the (stochastic) steady state is known [1]. A natural research direction is based on extending such results to the RFMR.

For the RFM, that is, the mean-field approximation of ASEP with open boundary conditions, it has been shown that the steady-state translation rate $R$ satisfies the equation

$$0 = f(R),$$

where $f$ is a continued fraction in $R$ [14]. Using the well-known relationship between continued fractions and tridiagonal matrices (see, e.g., [32]) yields that $R^{-1/2}$ is the Perron root of a certain non-negative symmetric tridiagonal matrix with entries that depend on the $\lambda_s$ [22]. This has many applications. For example it implies that $R = R(\lambda_0, \ldots, \lambda_n)$ in the RFM is a concave function on $\mathbb{R}^{n+1}_+ [22].$ An interesting question is whether $R$ in the RFMR can also be described using such equations.

The irreducibility of the Jacobian $J$ plays a crucial role in the proof of global stability for monotone dynamical systems with a first integral [18], [17]. This seems reasonable, as convergence to consensus often requires some kind of connectivity in a corresponding communication graph [16]. An interesting research topic is the generalization of graph-theoretic conditions for convergence to consensus in time-varying linear consensus networks (see, e.g., [19]) to time-varying nonlinear monotone systems.

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