On Graphs with Equal Domination and Chromatic Transversal Domination Numbers

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Abstract—Let \( \chi(G) \) denote the chromatic number of a graph \( G = (V,E) \). A dominating set \( D \subseteq V(G) \) is a set such that for every vertex \( v \in V(G) \setminus D \) there is at least one neighbor in \( D \). The domination number \( \gamma(G) \) is the least cardinality of a dominating set of \( G \). A chromatic transversal dominating set (CTDS) of a graph \( G \) is a dominating set \( D \) which intersects every color class of each \( \chi \)-partition of \( G \). The chromatic transversal domination number \( \gamma_{ct}(G) \) is the least cardinality of a CTDS of \( G \). In this paper, we characterize cubic graphs, block graphs and cactus graphs with equal domination number and chromatic transversal domination number.

Keywords—domination number, chromatic transversal domination number, cubic graph, block graph, cactus graph.

I. INTRODUCTION

Let \( n \) and \( m \) denote respectively, the order and size of a graph \( G = (V,E) \). \( N_G(v) = \{u \in V(G) : uv \in E(G)\} \) is called the neighborhood of \( v \) in \( G \). The degree \( d_G(v) \) of a vertex \( v \) is the number of vertices in the neighborhood of \( v \) with respect to \( G \). A vertex of degree one is known as a leaf and a vertex adjacent to a leaf is called its support. A strong support vertex is a support adjacent to at least two leaves of \( G \). The complete graph on \( n \) vertices we denote by \( K_n \). Let \( P_n \) and \( C_n \), respectively, denote a path and a cycle of order \( n \).

Michaelraj et al. [5] introduced a hybrid domination parameter, called the chromatic transversal domination number of a graph. Arumugam and Joseph [2] characterized the families of trees, unicyclic graphs and cubic graphs with domination number equal to the connected domination number. Chen et al.[8] further extended this characterization for the families of block graphs and cactus graphs. Ayyaswamy et al. [3] characterized a family of graphs with equal chromatic transversal domination number and connected domination number. In [11], Sahul Hamid introduced the concept of independent transversal domination in graphs. H.A.Ahangar et al. [1] studied the complexity results of this new domination parameter. L.Benedict Michaelraj et al. [7] obtained some characterization results on dominating chromatic partition-covering number of graphs. S. Balamurugan et al.[4] characterized the family of graphs for which neighbourhood chromatic domination number is two.

Motivated by the results in [2,8] we study the equality of domination number and chromatic transversal domination number for the classes of cubic graphs, block graphs and cactus graphs.

A proper coloring of a graph \( G \) is an assignment of colors to the vertices such that adjacent vertices do not share the same color. The chromatic number \( \chi(G) \) is the minimum colors used to color the vertices of \( G \). A partition of \( V(G) \) into \( \chi \)-independent sets \( \{V_1,V_2,\ldots,V_{\chi}\} \) is called a \( \chi \)-partition of \( G \), where each \( V_i \) is the color class that represents the color \( i \) for \( i = 1,2,\ldots,\chi(G) \). A vertex \( v \) of \( G \) is a critical vertex if \( \chi(G-v) < \chi(G) \), where \( \chi(G) \) is the chromatic number of \( G \). If every vertex of \( G \) is critical, then \( SGS \) is called a \( \chi \)-critical graph.

For a subset \( D \subseteq V(G) \), if every vertex of \( V(G) \setminus D \) has a neighbor in \( D \), then \( D \) is called a dominating set of \( G \). The domination number \( \gamma(G) \) is the least cardinality of a dominating set of \( G \). One may refer [8] for a detailed survey about domination in graphs. A chromatic transversal dominating set (CTDS) of a graph \( G \) is a dominating set \( D \) which intersects every color class of each \( \chi \)-partition of \( G \). The chromatic transversal domination number \( \gamma_{ct}(G) \) is the least cardinality of a CTDS of \( G \). For a subset \( S \) of \( V(G) \), \( N_G(S) \) is the collection of vertices adjacent to some vertex in \( S \) and \( N_G(S) = N_G(S) \cup S \). A graph \( G \) is cubic if it is 3-regular.

A vertex \( v \) of a connected graph \( G \) is said to be a cut-vertex if \( G-v \) is disconnected. A connected subgraph \( B \) of \( G \) is a block, if \( B \) has no cut-vertex and every subgraph \( B' \subseteq G \) with \( B \subseteq B' \) and \( B \neq B' \) has at least one cut-vertex. A block \( B \) of \( G \) is called an end-block, if \( B \) contains at most one cut-vertex of \( G \); such cut-vertex is called an end-block cut-vertex. A block graph \( G \) is a connected graph in which every block is complete. A graph \( G \) is called a cactus graph if every edge of \( G \) belongs to at most one cycle. A cycle in a cactus graph \( G \) is called an end cycle if it contains exactly one cut-vertex of \( G \). The clique number \( \omega(G) \) is the order of a maximum clique in \( G \).

**Theorem 1.1** ([6]): Let \( G \) be a connected graph. We have \( \gamma_{ct}(G) = n \) if and only if \( SGS \) is \( \chi \)-critical.

**Theorem 1.2** ([6]): Let \( G \) be a connected bipartite graph of order \( n \geq 3 \) with bipartition \((X,Y)\), where \(|X| \leq |Y|\). Then \( \gamma_{ct}(G) = \chi(G)+1 \) if and only if every vertex of \( X \) is a strong support vertex.
Let $\mathcal{J}$ be a family of trees $T$ such that $T$ is either $K_2$ or $T$ is such that all its support vertices are strong and every vertex at an even distance from a support vertex is also a support vertex.

**Theorem 1.3** ([6])

Let $T$ be a tree. Then $\gamma_s(T) = \gamma(T)$ if and only if $T \in \mathcal{J}$. Let $A$ be the following family of graphs.

![Graphs](image.png)

**Fig.1**

We have the following upper bound on the domination number of a graph.

**Theorem 1.4:**

If $G$ is a connected graph with $\delta(G) \geq 2$ and $G \notin \mathcal{A}$, then $\gamma(G) < \frac{2n}{5}$.

**II. MAIN RESULTS**

**A. Cubic graphs**

Let $G$ be a connected cubic graph with at least one odd cycle $C$. For each cycle $C$ of smallest length in $G$, let $G' = G - N_G[C]$. By $\gamma(G'(C))$ we mean a $\gamma$-set of $G'(C)$ with respect to $G$. That is a $\gamma$-set of $G'(C)$ with respect to $G$ may contain a vertex of $G'(C)$ or a vertex of $N_G[C] \setminus V(C)$ or both.

**Proposition 2.1:** Let $G \neq K_4$ be a connected cubic graph with at least one odd cycle. Let $C$ be the set of all odd cycles of smallest length in $G$. Then $\gamma_{\alpha}(G) = \min\{|C| + \gamma(G'(C)) : C \in \mathcal{C}\}$.

**Proof:** Let $T$ be any subset of $G$. If the subgraph $<T>$ induced by $T$ does not contain an odd cycle, then we can find a $\chi$-partition of $G$ for which all the vertices of $T$ can be colored with at most two colors. This implies $T$ is not a transversal for this $\chi$-partition. Thus any transversal of all $\chi$-partitions of $G$ must contain an odd cycle. Therefore for any $C \in \mathcal{C}$, $\mathcal{T} = V(C) \setminus S$, where $S$ is a $\gamma$-set of $G'(C) = G - N_G[C]$ is a CTDS of $G$. Hence $\gamma_{\alpha}(G) = \min\{|C| + \gamma(G'(C)) : C \in \mathcal{C}\}$.

**Example 2.2:** For this cubic graph $G$ given in fig.2. $C = \{C_1, C_2, C_3, C_4, C_5\}$ where $C_1 = \{v_1, v_2, v_3, v_4, v_5\}$, $C_2 = \{v_1, v_2, v_3, u_2, v_1\}$, $C_3 = \{v_4, v_5, u_5, v_5\}$, $C_4 = \{v_1, v'_1, u_1, v'_2, v_5\}$ and $C_5 = \{v_3, v_4, v'_3, u_7, v'_4\}$.

Let $T$ be a transversal of all $\chi$-partitions of $G$. If $T = C_1$, then a $\gamma$-set of $G'(C_1)$ is $\{u_1, u_5\}$. If $T = C_2$, then a $\gamma$-set of $G'(C_2)$ is $\{v'_4, v'_3, u_6\}$. If $T = C_3$, then a $\gamma$-set of $G'(C_3)$ is $\{v'_5, u_6, v'_4\}$. If $T = C_4$, then a $\gamma$-set of $G'(C_4)$ is $\{v'_5, u_6, v'_4\}$. Similarly, if $T = C_5$, then a $\gamma$-set of $G'(C_5)$ is $\{v'_4, u_6, v'_4\}$. Therefore, $\gamma_{\alpha}(G) = \min\{5 + 2.5 + 3.5 + 4\} = 7$.

**Theorem 2.3:** Let $G$ be a connected cubic graph with at least one odd cycle. Then $\gamma_{\alpha}(G) = \gamma(G)$ if and only if there exists an odd cycle $C$ of smallest length, say $k$ in $G$ such that the following conditions hold:

(i) $G' \neq \emptyset$

(ii) $N_G[C] \setminus V(C)$ is independent with cardinality $k$

(iii) If $S$ is a $\gamma$-set of $G'(C)$ with respect to $G$, then $N[S] \subseteq V(G'(C))$.

**Proof:** Assume that $\gamma_{\alpha}(G) = \gamma(G)$. Then $G \neq K_4$, since $\gamma(K_4) = 1$ whereas $\gamma(K_4) = 4$.

Let $C$ be an odd cycle of smallest length $k$ such that $\gamma_{\alpha}(G) = |V(C)| + \gamma(G'(C))$.

Let $V(C) = \{v_1, v_2, ..., v_k\}$ and

![Graph](image.png)

**Fig.2**
\( N_G(C) \setminus V(C) = \{ v'_1, v'_2, \ldots, v'_r \} \) where \( r \leq k \). Suppose condition (i) fails. That is \( G' = \phi \). Now \( |V(G)| = 2k \) and \( G \not\in A \), in view of Theorem 1.4, \( \gamma(G) \leq \frac{4k}{5} < k \). But \( \gamma_c(G) = k \).

This is a contradiction.

If \( |N_G(C) \setminus V(C)| < k \), then there exist vertices \( v'_i \) and \( v'_j \) such that \( v'_i = v'_j \) for some \( i \) and \( j, i \neq j \). Therefore \( (V(C) \setminus \{v_i, v_j\}) \cup \{v'_i\} \cup S \) is a dominating set of \( G \).

where \( S \) is a set of \( G' \) with respect to \( G \). Hence \( \gamma(G) \leq |C| - 1 + |S| < \gamma_c(G) \), a contradiction. Further, if \( N_G(C) \setminus V(C) \) is not independent, let \( v'_i \) be adjacent to \( v'_j \) in \( G \). Then \( V(C) \setminus \{v'_i\} \cup S \) where \( S \) is a set of \( G' \), is a dominating set with cardinality \( \gamma_c(G) \), a contradiction.

Suppose condition (ii) fails in \( G \). Then there exists a vertex \( v \in S \) such that \( v'_i \in E(G) \) for some \( i = 1,2,3, \ldots, k \). This implies \( (V(C) \setminus \{v_i\}) \cup S \) is a dominating set of \( G \) and so \( \gamma(G) \leq k - 1 + |S| \leq \min\{ |C| + |S| : C \subseteq C \} \), a contradiction.

The converse is obvious.

B. Block graphs

Let \( G \) be a block graph with \( \omega(G) \geq 3 \) and \( K \) be a maximal clique of \( G \). We define the set \( S_K \) to contain vertices \( v \in V(K) \) such that:

(a) \( v \) is either an end-block cut-vertex or

(b) for at least one path \( P \) in \( G - N_G(K) \) from \( v \), the vertex \( w \) in \( P \) with \( d(v, w) = 3 \) is in every \( \gamma \)-set of \( G \).

Question: How to identify such a vertex \( w \) which is in every \( \gamma \)-set of \( G \)?

If \( P \) is a shortest path from \( w \) to a cut-vertex \( w_i \) of an end-block such that every vertex \( w_k \) on \( P \) with \( d(w, w_k) \equiv 0 \pmod{3} \) is in every \( \gamma \)-set of \( G \), in particular, if \( d(w, w_i) \equiv 0 \pmod{3} \), then \( w_i \) is in every \( \gamma \)-set of \( G \).

We give below a few examples satisfying this criteria.

Let \( T_K = \{ v \in K : \text{there is a path } P \text{ from } v \text{ such that the vertex } v_1 \text{ in } N_G(v) \cap P \text{ dominates the vertex } v \text{ and } v_2 \in N_G(v) \cap P \text{ and } U_K = \{ v \in N_G(K) \setminus V(K) : \text{there are at least two paths } P_i \prec v = v_0, v_1, v_2, \ldots, v_r \} \text{ and } P_2 \prec v = u_0, u_1, u_2, \ldots, u_s \} \text{ such that both } v_1 \text{ and } u_1 \text{ are dominated by } v \} \).

Fig. 4: Block graph \( G \) with maximal clique \( K \), end-block \( E \) and a cut-vertex \( w_i \) of two blocks \( \neq K_2 \).

Lemma 3.1: Let \( G \) be a block graph with \( \omega(G) \geq 3 \).

(i) If \( K \) is a maximal clique such that \( |S_K| \) is maximum, then there exists a \( \gamma_c(G) \)-set containing \( K \).

(ii) If no such \( K \) exists, then \( |S_K| = 0 \) or \( |S_K| \) is the same for all maximal cliques \( K \), then for any maximal clique \( K \) for which \( |T_K| \) is minimum, then there exists a \( \gamma_c(G) \)-set containing \( K \).

(iii) If conditions (i) and (ii) fails, then for any maximal clique \( K \) in \( G \), there exists a \( \gamma_c(G) \)-set containing \( K \).

Proof: (i) If \( v \in S_K \) is an end-cut-vertex, then \( v \) is needed to dominate all its neighbors and so \( v \) will be in every \( \gamma \)-set of \( G \). On the other hand, if \( v \) satisfies the condition (b) of \( S_K \), then \( v \) will be in every \( \gamma \)-set of \( G \) and so \( v \) will be dominated by \( w \). Therefore, to dominate \( v_i \), either \( v \) or \( v_1 \) must be included in every \( \gamma \)-set of \( G \). Let \( |S_K| \geq |S_K| \).

Let \( C \) denote the set of all end-cutvertices in \( G - (K_1 \cup K_2) \). Now \( (G - N_G(K_2)) \cup U_{K_1} \) contains \( C \) and \( S_{K_1} \). Similarly, \( (G - N_G(K_1)) \cup U_{K_2} \) contains \( C \) and \( S_{K_2} \). Therefore \( |S_{K_1}| \geq |S_{K_2}| \) implies \( |D_1| \leq |D_2| \) where \( D_1 \) and \( D_2 \) are the \( \gamma \)-sets of \( (G - N_G(K_1)) \cup U_{K_1} \) and \( (G - N_G(K_2)) \cup U_{K_2} \) respectively.
On Graphs With Equal Domination And Chromatic Transversal Domination Numbers

(ii) Let \( K_1 \) and \( K_2 \) be two maximal cliques such that \( |T_{K_1}| < |T_{K_2}| \). Thus among the CTD sets \( V(K_1) \cup E_1 \) and \( V(K_2) \cup E_2 \) of \( G \), \( |V(K_1) \cup E_1| \leq |V(K_2) \cup E_2| \). This implies the condition (ii) is true.

If conditions (i) and (ii) fail, then for any two maximal cliques \( K_1 \) and \( K_2 \), \( |F_1| = |F_2| \) where \( F_1 \) and \( F_2 \) are \( \gamma \)-sets of \( (G - N_0(K_1)) \cup U_{K_1} \) and \( (G - N_0(K_2)) \cup U_{K_2} \) respectively. Therefore, \( |V(K_1) \cup F_1| = |V(K_2) \cup F_2| \) for all \( \gamma \). Hence \( \gamma(G) \) is the domination number of \( G \) with respect to \( \gamma \).

Theorem 3.2: Let \( G \) be a block graph with \( \omega(G) \geq 3 \). Then \( \gamma_{ct}(G) = \gamma(G) \) if and only if there exists a maximal clique \( K \) such that \( S_k \subseteq \gamma(K) = \omega(G) \).

Proof: Let us assume that \( \gamma_{ct}(G) = \gamma(G) \). Let \( K \) be a maximal clique satisfying the condition either (i), (ii) or (iii) of Lemma 3.1. Then \( \gamma_{ct}(G) = |V(K)| + \gamma((G - N_0(K)) \cup U_{K}) \). But \( \gamma(G) = \gamma_{ct}(K) + \gamma((G - N_0(K)) \cup U_{K}) \) where \( \gamma_{ct}(K) \) is the domination number of \( K \) with respect to \( \gamma \). Therefore \( \gamma_{ct}(G) = \gamma(G) \) if and only if \( |V(K)| \equiv \gamma_{ct}(K) \). This implies every vertex of \( K \) dominates at least one of its neighbors. This is possible if and only if \( |S_k| \subseteq |V(K)| = \omega(G) \).

C. Cactus graphs

Let \( G \) be a cactus graph with an odd cycle \( C \) of smallest length. Similar to block graphs we define the sets \( S_C, T_C \) and \( U_C \) as follows:

\[ S_C = \{ v \in C : v \text{ is either a cut-vertex of an end cycle of length } l \equiv 0,2 \pmod{3}, \text{ or at least one path } P \text{ in } G - N_0(C) \text{ from } v, \text{ the vertex } w \text{ in } P \text{ with } d(v,w) = 3 \} \]

is every \( \gamma \)-set of \( G \).

\[ T_C = \{ v \in C : \text{ there is a path } P \text{ from } v \text{ for which the vertex } v_1 \in N_0(v) \cap P \text{ dominates the vertices } v \text{ and } v_2 \in N_0(v_1) \cap P \text{ or } v \text{ is a cut-vertex of an end cycle of length } l \equiv 1 \pmod{3} \} \]

\[ U_C = \{ v \in N_G(C) \setminus V(C) : \text{ there are at least two paths } P_1 = \langle v = v_0, v_1, ..., v_r \rangle \text{ and } P_2 = \langle v = u_0, u_1, ..., u_s \rangle \text{ such that both } v_1 \text{ and } u_1 \text{ are dominated by } v \} \]

Let \( T \) be the set of all odd cycles in a cactus graph \( G \) for which \( S_C \) is maximum.

Lemma 4.1: Let \( G \) be a cactus graph with at least one odd cycle.

(i) If \( C \) is an odd cycle of least length such that \( |S_C| = \min \), then there exists a \( \gamma_{ct}(G) \)-set containing \( C \).

(ii) If no such \( C \) exists, that is, \( |S_C| = 0 \) or there are more than one odd cycle of least length with the same \( |S_C| \), then for any such \( C \) for which \( |T_C| = \min \) there exists a \( \gamma_{ct}(G) \)-set containing \( C \).

(iii) If both conditions (i) and (ii) fail, then for any odd cycle of least length \( C \), there exists a \( \gamma_{ct}(G) \)-set containing \( C \).

Proof: Proof of this lemma is similar to that of Lemma 3.1. For the sake of completeness we prove (ii).

Let \( C_1 \) and \( C_2 \) be two odd cycles in \( T \) of smallest length, say \( k \), such that \( |S_{C_1}| = |S_{C_2}| \) and \( C_1 \) has more number of cut vertices of end cycles of lengths \( l \equiv 1 \pmod{3} \) than those of \( C_2 \). Then \( V(C_1) \cup D_1 \) and \( V(C_2) \cup D_2 \) are CTD-sets of \( G \), where \( D_1 \) and \( D_2 \) are \( \gamma \)-sets of \( (G - N_0(C_1)) \cup U_{C_1} \) and \( (G - N_0(C_2)) \cup U_{C_2} \) respectively. As \( C_2 \) has less number of cut vertices of end cycles of lengths \( l \equiv 1 \pmod{3} \), we have \( |D_2| \leq |D_1| \). This implies \( \gamma_{ct}(G) = |V(C) \cup D| \) where \( C \) is an odd cycle in \( T \) of smallest length \( k \) such that \( C \) has the least number of cut vertices of end cycles of lengths \( l \equiv 1 \pmod{3} \) and \( D \) is a \( \gamma \)-set of \( (G - N_0(C)) \cup U_C \).

Theorem 4.2: Let \( G \) be a cactus graph with at least one odd cycle. Then \( \gamma_{ct}(G) = \gamma(G) \) if and only if there exists an odd cycle \( C \) of least length in \( T \) such that \( |S_C| = |V(C)| \).

Proof: Let us assume that \( \gamma_{ct}(G) = \gamma(G) \). Then there exists an odd cycle CT. By Lemma 4.1,

\[ \gamma_{ct}(G) = |V(C)| + \gamma((G - N_0(C)) \cup U_C) \]. But \( \gamma(G) = \gamma_{ct}(C) + \gamma((G - N_0(C)) \cup U_C) \).

Let \( D \) be a \( \gamma_{ct}(G) \)-set. Suppose \( S_C \neq V(C) \). Then \( T_C \neq \emptyset \). Let \( v \in T_C \). Then we have the following two cases.

Case 1: \( v \) is a cut-vertex of an end cycle of length \( l \equiv 1 \pmod{3} \).

Let \( C' = < v = u_1, u_2, ..., u_{3r+1} > \) be an odd cycle of length \( 3r+1 \) containing \( v \). Then the neighbors of \( v \) in \( C' \) namely \( u_2 \) and \( u_{3r+1} \) are dominated by \( u_3 \) and \( u_{3r} \) respectively. Therefore for any \( \gamma_{ct}(G) \)-set \( D \) containing \( V(C') \), \( D \setminus \{ v \} \) is a \( \gamma \)-set of \( G \). This implies \( \gamma_{ct}(G) > \gamma(G) \), a contradiction.

Case 2: Let \( P = < v = v_0, v_1, ..., v_r > \) be a path in \( G - N_0(C) \) from \( v \) such that \( v \) and \( v_2 \) are dominated by \( v_1 \). Then \( D \setminus \{ v \} \) is a \( \gamma \)-set of \( G \) implying \( \gamma_{ct}(G) > \gamma(G) \), a contradiction. Thus \( |T_C| = 0 \) which implies \( |S_C| = |V(C)| \).
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