THIN SEQUENCES AND THEIR ROLE IN MODEL SPACES AND
DOUGLAS ALGEBRAS

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Abstract. We study thin interpolating sequences \{\lambda_n\} and their relationship to interpolation in the Hardy space \(H^2\) and the model spaces \(K_\Theta = H^2 \ominus \Theta H^2\), where \(\Theta\) is an inner function. Our results, phrased in terms of the functions that do the interpolation as well as Carleson measures, show that under the assumption that \(\Theta(\lambda_n) \to 0\) the interpolation properties in \(H^2\) are essentially the same as those in \(K_\Theta\).

1. Introduction and Motivation

A sequence \(\{\lambda_j\}_{j=1}^\infty\) is an interpolating sequence for \(H^\infty\), the space of bounded analytic functions, if for every \(w \in \ell^\infty\) there is a function \(f \in H^\infty\) such that
\[ f(z_j) = w_j, \quad \text{for all } j \in \mathbb{N}. \]

Carleson’s interpolation theorem says that \(\{z_j\}_{j=1}^\infty\) is an interpolating sequence for \(H^\infty\) if and only if
\[ \delta = \inf_j \delta_j := \inf_j |B_j(\lambda_j)| = \inf_j \prod_{k \neq j} \left| \frac{\lambda_j - \lambda_k}{1 - \lambda_j \lambda_k} \right| > 0, \tag{1.1} \]
where \(B_j(z) := \prod_{k \neq j} \frac{-\lambda_k}{|\lambda_k|} \frac{z - \lambda_k}{1 - \lambda_k z} \)
denotes the Blaschke product vanishing on the set of points \(\{\lambda_k : k \neq j\}\).

In this paper, we consider sequences that (eventually) satisfy a stronger condition than (1.1). A sequence \(\{\lambda_j\} \subset \mathbb{D}\) is thin if
\[ \lim_{j \to \infty} \delta_j := \lim_{j \to \infty} \prod_{k \neq j} \left| \frac{\lambda_j - \lambda_k}{1 - \lambda_k \lambda_j} \right| = 1. \]

Thin sequences are of interest not only because functions solving interpolation for thin interpolating sequences have good bounds on the norm, but also because they are interpolating sequences for a very small algebra: the algebra \(QA = VMO \cap H^\infty\), where \(VMO\) is the space of functions on the unit circle with vanishing mean oscillation [22].

Continuing work in [2] and [9], we are interested in understanding these sequences in different settings. This will require two definitions that are motivated by the work of Shapiro and Shields, [18], in which they gave the appropriate conditions for a sequence to be interpolating for the Hardy space \(H^2\).

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Considering more general Hilbert spaces will require the introduction of reproducing kernels. In a reproducing kernel Hilbert space $\mathcal{H}$, we let $K_{\lambda_n}$ denote the kernel corresponding to the point $\lambda_n$; that is, for each function in the Hilbert space we have that $f(\lambda_n) = \langle f, K_{\lambda_n} \rangle_{\mathcal{H}}$. The concepts of interest are the following.

A sequence $\{\lambda_n\} \subset \Omega \subseteq \mathbb{C}^n$ is said to be an eventual 1-interpolating sequence for a reproducing kernel Hilbert space $\mathcal{H}$, denoted $EIS_{\mathcal{H}}$, if for every $\varepsilon > 0$ there exists $N$ such that for each $\{a_n\} \in \ell^2$ there exists $f_{N,a} \in \mathcal{H}$ with

$$f_{N,a}(\lambda_n) \|K_{\lambda_n}\|_{\mathcal{H}}^{-1} = f_{N,a}(\lambda_n)K_{\lambda_n}(\lambda_n)^{-\frac{1}{2}} = a_n \text{ for } n \geq N \text{ and } \|f_{N,a}\|_{\mathcal{H}} \leq (1 + \varepsilon)\|a\|_{N,\ell^2}.$$

A sequence $\{\lambda_n\}$ is said to be a strong asymptotic interpolating sequence for $\mathcal{H}$, denoted $AIS_{\mathcal{H}}$, if for all $\varepsilon > 0$ there exists $N$ such that for all sequences $\{a_n\} \in \ell^2$ there exists a function $G_{N,a} \in \mathcal{H}$ such that $\|G_{N,a}\|_{\mathcal{H}} \leq \|a\|_{N,\ell^2}$ and

$$\|G_{N,a}(\lambda_n)K_{\lambda_n}(\lambda_n)^{-\frac{1}{2}} - a_n\|_{N,\ell^2} < \varepsilon \|a\|_{N,\ell^2}.$$

Given a (nonconstant) inner function $\Theta$, we are interested in these sequences in model spaces; we define the model space for $\Theta$ an inner function by $K_{\Theta} = H^2 \ominus \Theta H^2$. The reproducing kernel in $K_{\Theta}$ for $\lambda_0 \in \mathbb{D}$ is

$$K_{\lambda_0}^\Theta(z) = \frac{1 - \Theta(\lambda_0)\Theta(z)}{1 - \lambda_0 z}$$

and the normalized reproducing kernel is

$$k_{\lambda_0}^\Theta(z) = \sqrt{\frac{1 - |\lambda_0|^2}{1 - |\Theta(\lambda_0)|^2}} K_{\lambda_0}^\Theta(z).$$

Finally, note that

$$K_{\lambda_0} = K_{\lambda_0}^\Theta + \Theta\overline{\Theta(\lambda_0)}K_{\lambda_0}.$$

We let $P_{\Theta}$ denote the orthogonal projection of $H^2$ onto $K_{\Theta}$.

We consider thin sequences in these settings as well as in Douglas algebras: Letting $L^\infty$ denote the algebra of essentially bounded measurable functions on the unit circle, a Douglas algebra is a closed subalgebra of $L^\infty$ containing $H^\infty$. It is a consequence of work of Chang and Marshall that a Douglas algebra $\mathcal{B}$ is equal to the closed algebra generated by $H^\infty$ and the conjugates of the interpolating Blaschke products invertible in $\mathcal{B}$, $[3, 13]$.

In this paper, we continue work started in $[8]$ and $[9]$ investigating the relationship between thin sequences, $EIS_{\mathcal{H}}$ and $AIS_{\mathcal{H}}$ where $\mathcal{H}$ is a model space or the Hardy space $H^2$. In Section 3, we consider the notion of eventually interpolating and asymptotic interpolating sequences in the model space setting. We show that in reproducing kernel Hilbert spaces of analytic functions on domains in $\mathbb{C}^n$, these two are the same. Given results in $[9]$, this is not surprising and the proofs are similar to those in the $H^\infty$ setting. We then turn to our main result of that section. If we have a Blaschke sequence $\{\lambda_n\}$ in $\mathbb{D}$ and assume that our inner function $\Theta$ satisfies $|\Theta(\lambda_n)| \to 0$, then a sequence is an $EIS_{K_{\Theta}}$ sequence if and only if it is an $EIS_{H^2}$ sequence (and therefore $AIS_{K_{\Theta}}$ sequence if and only if it is an $AIS_{H^2}$).

In Section 3.3 we rephrase these properties in terms of the Carleson embedding constants on the model spaces. Finally, in Section 4, we recall the definition of Douglas algebras and show that appropriate definitions and conditions are quite different in that setting.
2. Preliminaries

Recall that a sequence \( \{x_n\} \) in \( \mathcal{H} \) is complete if \( \text{Span}\{x_n : n \geq 1\} = \mathcal{H} \), and asymptotically orthonormal (AOS) if there exists \( N_0 \) such that for all \( N \geq N_0 \) there are positive constants \( c_N \) and \( C_N \) such that

\[
\begin{aligned}
c_N \sum_{n \geq N} |a_n|^2 & \leq \left\| \sum_{n \geq N} a_n x_n \right\|_{\mathcal{H}}^2 \leq C_N \sum_{n \geq N} |a_n|^2,
\end{aligned}
\]

where \( c_N \to 1 \) and \( C_N \to 1 \) as \( N \to \infty \). If we can take \( N_0 = 1 \), the sequence is said to be an AOB; this is equivalent to being AOS and a Riesz sequence. Finally, the Gram matrix corresponding to \( \{x_j\} \) is the matrix \( G = (\langle x_n, x_m \rangle)_{n,m \geq 1} \).

It is well known that if \( \{\lambda_n\} \) is a Blaschke sequence with simple zeros and corresponding Blaschke product \( B \), then \( \{k_{\lambda_n}\} \), where

\[
k_{\lambda_n}(z) = \frac{(1 - |\lambda_n|^2)^{\frac{1}{2}}}{(1 - \lambda_n z)}
\]

is a complete minimal system in \( K_B \) and we also know that \( \{\lambda_n\} \) is interpolating if and only if \( \{k_{\lambda_n}\} \) is a Riesz basis. The following beautiful theorem provides the connection to thin sequences.

**Theorem 2.1** (Volberg, [21, Theorem 2]). The following are equivalent:

1. \( \{\lambda_n\} \) is a thin interpolating sequence;
2. The sequence \( \{k_{\lambda_n}\} \) is a complete AOB in \( K_B \);
3. There exist a separable Hilbert space \( \mathcal{K} \), an orthonormal basis \( \{e_n\} \) for \( \mathcal{K} \) and \( U, K : \mathcal{K} \to K_B, U \text{ unitary}, K \text{ compact}, U + K \text{ invertible} \), such that \( (U + K)(e_n) = k_{\lambda_n} \) for all \( n \in \mathbb{N} \).

In [4, Section 3] and [2, Proposition 3.2], the authors note that [21, Theorem 3] implies the following.

**Proposition 2.2.** Let \( \{x_n\} \) be a sequence in \( \mathcal{H} \). The following are equivalent:

1. \( \{x_n\} \) is an AOB;
2. There exist a separable Hilbert space \( \mathcal{K} \), an orthonormal basis \( \{e_n\} \) for \( \mathcal{K} \) and \( U, K : \mathcal{K} \to \mathcal{H}, U \text{ unitary}, K \text{ compact}, U + K \text{ left invertible} \), such that \( (U + K)(e_n) = x_n \);
3. The Gram matrix \( G \) associated to \( \{x_n\} \) defines a bounded invertible operator of the form \( I + K \) with \( K \text{ compact} \).

We also have the following, which we will use later in this paper.

**Proposition 2.3** (Proposition 5.1, [2]). If \( \{\lambda_n\} \) is a sequence of distinct points in \( \mathbb{D} \) and \( \{k_{\Theta \lambda_n}\} \) is an AOS, then \( \{\lambda_n\} \) is a thin interpolating sequence.

**Theorem 2.4** (Theorem 5.2, [2]). Suppose \( \sup_{n \geq 1} |\Theta(\lambda_n)| < 1 \). If \( \{\lambda_n\} \) is a thin interpolating sequence, then either

(i) \( \{k_{\Theta \lambda_n}\}_{n \geq 1} \) is an AOB or
(ii) there exists \( p \geq 2 \) such that \( \{k_{\Theta \lambda_n}\}_{n \geq p} \) is a complete AOB in \( K_\Theta \).
3. Hilbert Space Versions

3.1. Asymptotic and Eventual Interpolating Sequences. Let $\mathcal{H}$ be a reproducing kernel Hilbert space of analytic functions over a domain $\Omega \subseteq \mathbb{C}^n$ with reproducing kernel $K_\lambda$ at the point $\lambda \in \Omega$. We define two properties that a sequence $\{\lambda_n\} \subseteq \Omega$ can have.

**Definition 3.1.** A sequence $\{\lambda_n\} \subseteq \Omega$ is an eventual 1-interpolating sequence for $\mathcal{H}$, denoted EIS$_\mathcal{H}$, if for every $\varepsilon > 0$ there exists $N$ such that for each $\{a_n\} \in \ell^2$ there exists $f_{N,a} \in \mathcal{H}$ with

$$f_{N,a}(\lambda_n) \|K_{\lambda_n}\|^{-1}_\mathcal{H} = f_{N,a}(\lambda_n)K_{\lambda_n}(\lambda_n)^{-\frac{1}{2}} = a_n$$ for $n \geq N$ and $\|f_{N,a}\|_\mathcal{H} \leq (1 + \varepsilon)\|a\|_{N,\ell^2}$.

**Definition 3.2.** A sequence $\{\lambda_n\} \subseteq \Omega$ is a strong asymptotic interpolating sequence for $\mathcal{H}$, denoted AIS$_\mathcal{H}$, if for all $\varepsilon > 0$ there exists $N$ such that for all sequences $\{a_n\} \in \ell^2$ there exists a function $G_{N,a} \in \mathcal{H}$ such that $\|G_{N,a}\|_\mathcal{H} \leq \|a\|_{N,\ell^2}$ and

$$\|G_{N,a}(\lambda_n)K_{\lambda_n}(\lambda_n)^{-\frac{1}{2}} - a_n\|_{N,\ell^2} < \varepsilon \|a\|_{N,\ell^2}.$$

**Theorem 3.3.** Let $\mathcal{H}$ be a reproducing kernel space of analytic functions over the domain $\Omega \subseteq \mathbb{C}^n$ with reproducing kernel at the point $\lambda$ given by $K_\lambda$. Then $\{\lambda_n\}$ is an EIS$_\mathcal{H}$ sequence if and only if $\{\lambda_n\}$ is an AIS$_\mathcal{H}$.

**Proof.** If a sequence is an EIS$_\mathcal{H}$, then it is trivially AIS$_\mathcal{H}$, for given $\varepsilon > 0$ we may take $G_{N,a} = \frac{f_{N,a}}{1 + \varepsilon}$.

For the other direction, suppose $\{\lambda_n\}$ is an AIS$_\mathcal{H}$ sequence. Let $\varepsilon > 0$, $N := N(\varepsilon)$, and $\{a_j\} := \{a_j^{(0)}\}$ be any sequence. First choose $f_0 \in \mathcal{H}$ so that for $n \geq N$ we have

$$\|K_{\lambda_n}(\lambda_n)^{-\frac{1}{2}}f_0(\lambda_n) - a_n^{(0)}\|_{N,\ell^2} < \frac{\varepsilon}{1 + \varepsilon} \|a\|_{N,\ell^2}$$

and

$$\|f_0\|_\mathcal{H} \leq \|a\|_{N,\ell^2}.$$

Now let $a_n^{(1)} = a_n^{(0)} - K_{\lambda_n}(\lambda_n)^{-\frac{1}{2}}f_0(\lambda_n)$. Note that $\|a_n^{(1)}\|_{N,\ell^2} < \frac{\varepsilon}{1 + \varepsilon} \|a\|_{N,\ell^2}$. Since we have an AIS$_\mathcal{H}$ sequence, we may choose $f_1$ such that for $n \geq N$ we have

$$\|f_1(\lambda_n)K_{\lambda_n}(\lambda_n)^{-\frac{1}{2}} - a_n^{(1)}\|_{N,\ell^2} < \frac{\varepsilon}{1 + \varepsilon} \|a_n^{(1)}\|_{N,\ell^2} < \left(\frac{\varepsilon}{1 + \varepsilon}\right)^2 \|a\|_{N,\ell^2},$$

and

$$\|f_1\|_\mathcal{H} \leq \|a_n^{(1)}\|_{N,\ell^2} < \left(\frac{\varepsilon}{1 + \varepsilon}\right) \|a\|_{N,\ell^2}.$$

In general, we let

$$a_j^{(k)} = -f_{k-1}(\lambda_j)K_{\lambda_j}(\lambda_j)^{-\frac{1}{2}} + a_j^{(k-1)}$$

so that

$$\|a_j^{(k)}\|_{N,\ell^2} \leq \frac{\varepsilon}{1 + \varepsilon} \|a_j^{(k-1)}\|_{N,\ell^2} \leq \left(\frac{\varepsilon}{1 + \varepsilon}\right)^2 \|a_j^{(k-2)}\|_{N,\ell^2} \leq \cdots \leq \left(\frac{\varepsilon}{1 + \varepsilon}\right)^k \|a\|_{N,\ell^2}$$

and

$$\|f_k\|_\mathcal{H} \leq \|a_j^{(k)}\|_{N,\ell^2} < \left(\frac{\varepsilon}{1 + \varepsilon}\right)^k \|a\|_{N,\ell^2}.$$
Then consider $f(z) = \sum_{k=0}^{\infty} f_k(z)$. Since $f_k(\lambda_j) = \left( a_j^{(k)} - a_j^{(k+1)} \right) K_{\lambda_j}(\lambda_j)^{\frac{1}{2}}$ and $a_j^{(k)} \to 0$ as $k \to \infty$, we have for each $j \geq N$,

$$f(\lambda_j) = a_j^{(0)} K_{\lambda_j}(\lambda_j)^{\frac{1}{2}} = a_j K_{\lambda_j}(\lambda_j)^{\frac{1}{2}}.$$  

Further $\|f\|_H \leq \sum_{k=0}^{\infty} \left( \frac{1}{1+\varepsilon} \right)^k \|a\|_{N,\ell^2} = \frac{1}{1+\varepsilon} \|a\|_{N,\ell^2} = (1+\varepsilon)\|a\|_{N,\ell^2}$. This proves that $\{\lambda_n\}$ is an $EIS_H$ sequence. \hfill \Box

3.2. The Hardy and Model Spaces. We let $\Theta$ denote a nonconstant inner function and apply Theorem 3.3 to the reproducing kernel Hilbert space $K_\Theta$. We also include statements and results about Carleson measures. Given a non-negative measure $\mu$ on $\mathbb{D}$, let us denote the (possibly infinite) constant

$$C(\mu) = \sup_{f \in H^2, f \neq 0} \frac{\|f\|^2_{L^2(\mathbb{D}, \mu)}}{\|f\|^2}$$

as the Carleson embedding constant of $\mu$ on $H^2$ and

$$R(\mu) = \sup_{z \in \mathbb{D}} \frac{\|k_z\|_{L^2(\mathbb{D}, \mu)}}{\|k_z\|_2} = \sup_{z} \|k_z\|_{L^2(\mathbb{D}, \mu)}$$

as the embedding constant of $\mu$ on $k_z$, the normalized reproducing kernel of $H^2$. It is well-known that $C(\mu) \approx R(\mu)$, [16, 17].

**Theorem 3.4.** Let $\{\lambda_n\}$ be an interpolating sequence in $\mathbb{D}$ and let $\Theta$ be an inner function. Suppose that $\kappa := \sup_n |\Theta(\lambda_n)| < 1$. The following are equivalent:

1. $\{\lambda_n\}$ is an $EIS_{H^2}$ sequence;
2. $\{\lambda_n\}$ is a thin interpolating sequence;
3. Either
   (a) $\{k_{\lambda_n}^\Theta\}_{n \geq 1}$ is an AOB, or
   (b) there exists $p \geq 2$ such that $\{k_{\lambda_n}^\Theta\}_{n \geq p}$ is a complete AOB in $K_\Theta$;
4. $\{\lambda_n\}$ is an $AIS_{H^2}$ sequence;
5. The measure
   $$\mu_N = \sum_{k \geq N} (1 - |\lambda_k|^2) \delta_{\lambda_k}$$
   is a Carleson measure for $H^2$ with Carleson embedding constant $C(\mu_N)$ satisfying $C(\mu_N) \to 1$ as $N \to \infty$;
6. The measure
   $$\nu_N = \sum_{k \geq N} \frac{(1 - |\lambda_k|^2)}{\delta_k} \delta_{z_k}$$
   is a Carleson measure for $H^2$ with embedding constant $R_{\nu_N}$ on reproducing kernels satisfying $R_{\nu_N} \to 1$.

Further, (7) and (8) are equivalent to each other and imply each of the statements above. If, in addition, $\Theta(\lambda_n) \to 0$, then (1) - (8) are equivalent.

7. $\{\lambda_n\}$ is an $EIS_{K_\Theta}$ sequence;
8. $\{\lambda_n\}$ is an $AIS_{K_\Theta}$ sequence.
Proof. The equivalence between (7) and (8) is contained in Theorem 3.3. Similarly, this applies to (1) and (4). In [9, Theorem 4.5], the authors prove that (2), (5) and (6) are equivalent. The equivalence between (1), (2), and (4) is contained in [9]. That (2) implies (3) is Theorem 2.4. That (3) implies (2) also follows from results in [2], for if a sequence is an $AOB$ for some $p \geq 2$ it is an $AOS$ for $p \geq 2$ and hence thin by Proposition 2.3 for $p \geq 2$.

This is, of course, the same as being thin interpolating. Thus, we have the equivalence of equations (1), (2), (3), (4), (5), and (6), as well as the equivalence of (7) and (8).

Now we show that (7) and (1) are equivalent under the hypothesis that $\Theta(\lambda_n) \to 0$.

(7) $\Rightarrow$ (1). Suppose that $\{\lambda_n\}$ is an $EIS_{K_\Theta}$ sequence. We will prove that this implies it is an $EIS_{H^2}$ sequence, establishing (1).

Let $\varepsilon > 0$ be given and select $N = N(\varepsilon)$ according to the definition of $\{\lambda_n\}$ being an $EIS_{K_\Theta}$ sequence. Given $a \in \ell^2$, since we have that $\kappa := \sup_n |\Theta(\lambda_n)| < 1$, we have that $\{\tilde{a}_n\} = \{a_n (1 - |\Theta(\lambda_n)|^2)^{-\frac{1}{2}}\} \in \ell^2$. Select $f_a \in K_\Theta \subset H^2$ so that $f_a(\lambda_n) \left(\frac{1 - |\Theta(\lambda_n)|^2}{1 - |\lambda_n|^2}\right)^{-\frac{1}{2}} = \tilde{a}_n = a_n (1 - |\Theta(\lambda_n)|^2)^{-\frac{1}{2}}$ for $n \geq N$ and $\|f_a\|_2 \leq (1 + \varepsilon)\|a\|_{N_1,\ell^2}$. Since $f_a \in K_\Theta$, we have that $f_a \in H^2$, and canceling out the common factor yields that $f_a(\lambda_n)(1 - |\lambda_n|^2)^{-\frac{1}{2}} = a_n$ for all $n \geq N$. Thus $\{\lambda_n\}$ is an $EIS_{H^2}$ sequence as claimed.

(1) $\Rightarrow$ (7). Suppose that $\Theta(\lambda_n) \to 0$ and $\{\lambda_n\}$ is an $EIS_{H^2}$ sequence; equivalently, that $\{\lambda_n\}$ is thin. We want to show that the sequence $\{\lambda_n\}$ is an $EIS_{K_\Theta}$ sequence. First we present some observations.

First, looking at the definition, we see that we may assume that $\varepsilon > 0$ is small, for any choice of $N$ that works for small $\varepsilon$ also works for larger values.

Second, if $f \in H^2$ and we let $\tilde{f} = P_{K_\Theta} f$, then we have that $\|\tilde{f}\|_2 \leq \|f\|_2$ since $P_{K_\Theta}$ is an orthogonal projection. Next, we have $P_{K_\Theta} = P_+ - \Theta P_+ \Theta$, where $P_+$ is the orthogonal projection of $L^2$ onto $H^2$, so letting $T_\Theta$ denote the Toeplitz operator with symbol $\Theta$ we have

\begin{equation}
\tilde{f}(z) = f(z) - \Theta(z)T_\Theta(f)(z).
\end{equation}

In what follows, $\kappa_m := \sup_{n \geq m} |\Theta(\lambda_n)|$ and recall that we assume that $\kappa_m \to 0$.

Since $\{\lambda_n\}$ is an $EIS_{H^2}$ sequence, there exists $N_1$ such that for any $a \in \ell^2$ there exists a function $f_0 \in H^2$ such that

\[ f_0(\lambda_n) = a_n \left(\frac{1 - |\Theta(\lambda_n)|^2}{1 - |\lambda_n|^2}\right)^{\frac{1}{2}} \text{ for all } n \geq N_1 \]

and

\[ \|f_0\|_2 \leq (1 + \varepsilon) \left\|a_k(1 - |\Theta(\lambda_k)|^2)^{\frac{1}{2}}\right\|_{N_1,\ell^2} \leq (1 + \varepsilon) \|a\|_{N_1,\ell^2}. \]
Here we have applied the $EIS_{H^2}$ property to the sequence $\{a_k(1 - |\Theta(\lambda_k)|^2)^{\frac{1}{2}}\} \in \ell^2$. By (3.1) we have that
\[ \tilde{f}_0(\lambda_k) = f_0(\lambda_k) - \Theta(\lambda_k)T_{\Sigma}(f_0)(\lambda_k) \]
\[ = a_k(1 - |\Theta(\lambda_k)|^2)^{\frac{1}{2}}(1 - |\lambda_k|^2)^{-\frac{1}{2}} - \Theta(\lambda_k)T_{\Sigma}(f_0)(\lambda_k) \quad \forall k \geq N_1 \]
and $\|\tilde{f}_0\|_2 \leq \|f_0\|_2 \leq (1 + \varepsilon)\|a\|_{N_1,\ell^2}$. Rearranging the above, for $k \geq N_1$ we have
\[ |\tilde{f}_0(\lambda_k)(1 - |\Theta(\lambda_k)|^2)^{-\frac{1}{2}}(1 - |\lambda_k|^2)^{-\frac{1}{2}} - a_k| \leq \|\Theta(\lambda_k)T_{\Sigma}(f_0)(\lambda_k)(1 - |\Theta(\lambda_k)|^2)^{-\frac{1}{2}}(1 - |\lambda_k|^2)^{-\frac{1}{2}}\|_2 \]
\[ \leq \kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}}\|f_0\|_2 \]
\[ \leq (1 + \varepsilon)\kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}}\|a\|_{N_1,\ell^2}. \]
We claim that $\{a_n^{(1)}\} = \{\tilde{f}_0(\lambda_n)(1 - |\Theta(\lambda_n)|^2)^{-\frac{1}{2}}(1 - |\lambda_n|^2)^{-\frac{1}{2}} - a_n\} \in \ell^2$ and that there is a constant $N_2$ depending only on $\varepsilon$ and the Carleson measure given by the thin sequence $\{\lambda_n\}$ such that
\[ (3.2) \quad \|a^{(1)}\|_{N_2,\ell^2} \leq (1 + \varepsilon)^2\kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}}\|a\|_{N_1,\ell^2}. \]

Since the sequence $\{\lambda_n\}$ is thin and distinct, it hence generates a $H^2$ Carleson measure with norm at most $(1 + \varepsilon)$; that is, we have the existence of $N_2 \geq N_1$ such that $\kappa_{N_2}(1 - \kappa_{N_2}^2)^{-\frac{1}{2}} \leq \kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}}$ and
\[ \|a^{(1)}\|_{N_2,\ell^2} \leq (1 + \varepsilon)\kappa_{N_2}(1 - \kappa_{N_2}^2)^{-\frac{1}{2}}\|Tbf\|_{\ell^2} \]
\[ \leq (1 + \varepsilon)\kappa_{N_2}(1 - \kappa_{N_2}^2)^{-\frac{1}{2}}\|f_0\|_2 \]
\[ \leq (1 + \varepsilon)^2\kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}}\|a\|_{N_1,\ell^2} < \infty, \]
completing the proof of the claim.

We will now iterate these estimates and ideas. Let $\tilde{a}_n^{(1)} = \frac{a_n^{(1)}}{(1 + \varepsilon)^2\kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}}} \text{ for } n \geq N_2$
and $\tilde{a}_n^{(1)} = 0$ otherwise. Then from (3.2) we have that $\|\tilde{a}_n^{(1)}\|_{N_1,\ell^2} = \|\tilde{a}_n^{(1)}\|_{N_2,\ell^2} \leq \|a\|_{N_1,\ell^2}$.

Since $\{\lambda_n\}$ is an $EIS_{H^2}$ we may choose $f_1 \in H^2$ with
\[ f_1(\lambda_n) = a_n^{(1)}(1 - |\Theta(\lambda_n)|^2)^{\frac{1}{2}}(1 - |\lambda_n|^2)^{-\frac{1}{2}} \quad \text{for all } n \geq N_1 \]
and, letting $\tilde{f}_1 = P_{K^\phi}(f_1)$, we have
\[ \|\tilde{f}_1\|_2 \leq \|f_1\|_2 \leq (1 + \varepsilon)\|\tilde{a}_n^{(1)}\|_{N_1,\ell^2} \leq (1 + \varepsilon)\|a\|_{N_1,\ell^2}. \]

As above,
\[ \tilde{f}_1(\lambda_k) = f_1(\lambda_k) - \Theta(\lambda_k)T_{\Sigma}(f_1)(\lambda_k) \]
\[ = a_k^{(1)}(1 - |\Theta(\lambda_k)|^2)^{\frac{1}{2}}(1 - |\lambda_k|^2)^{-\frac{1}{2}} - \Theta(\lambda_k)T_{\Sigma}(f_1)(\lambda_k) \quad \forall k \geq N_1. \]
And, for $k \geq N_1$ we have
\[
\left| \tilde{f}_1(\lambda_k)(1 - |\Theta(\lambda_k)|^2)^{-\frac{1}{2}}(1 - |\lambda_k|^2)^{\frac{1}{2}} - \tilde{a}_n^{(1)} \right| = \left| \Theta(\lambda_k)T_{g}(f_1)(\lambda_k)(1 - |\Theta(\lambda_k)|^2)^{-\frac{1}{2}}(1 - |\lambda_k|^2)^{\frac{1}{2}} \right|
\leq \kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}} \|f_1\|_2
\leq (1 + \varepsilon)\kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}} \|a\|_{N_1,\ell^2}.
\]

Using the definition of $\tilde{a}_{(1)}^{(1)}$, for $k \geq N_2$ one arrives at
\[
\left| \left( (1 + \varepsilon)^2 \kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}} \tilde{f}_1(\lambda_k) + \tilde{f}_0(\lambda_k) \right)(1 - |\Theta(\lambda_k)|^2)^{-\frac{1}{2}}(1 - |\lambda_k|^2)^{\frac{1}{2}} - a \right|
\leq (1 + \varepsilon)^3 \kappa_{N_1}^2(1 - \kappa_{N_1}^2)^{-1} \|a\|_{N_1,\ell^2}.
\]
We repeat one more time to establish the pattern. Define
\[
a^{(2)}_n = \left( (1 + \varepsilon)^2 \kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}} \tilde{f}_1(\lambda_n) + \tilde{f}_0(\lambda_n) \right)(1 - |\Theta(\lambda_n)|^2)^{-\frac{1}{2}}(1 - |\lambda_n|^2)^{\frac{1}{2}} - a_n
\]
for $n \geq N_2$ and $a^{(2)}_n = 0$ otherwise. Observe that this sequence is in $\ell^2$ since we have, for $k \geq N_2$,
\[
|a^{(2)}_n| = (1 + \varepsilon)^2 \kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}} \left| \Theta(\lambda_k)T_{g}(f_1)(\lambda_k)(1 - |\Theta(\lambda_k)|^2)^{-\frac{1}{2}}(1 - |\lambda_k|^2)^{\frac{1}{2}} \right|
\]
and, using the Carleson measure property as before,
\[
\|a^{(2)}_n\|_{N_1,\ell^2} = (1 + \varepsilon)^2 \kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}} \left( \sum_{k \geq N_2} |\Theta(\lambda_k)|^2 |T_{g}(f_1)(\lambda_k)|^2 (1 - |\Theta(\lambda_k)|^2)^{-1}(1 - |\lambda_k|^2)^{2} \right)\frac{1}{2}
\leq (1 + \varepsilon)^3 \kappa_{N_1}^2(1 - \kappa_{N_1}^2)^{-1} \|T_{g}f_1\|_2
\leq (1 + \varepsilon)^3 \kappa_{N_1}^2(1 - \kappa_{N_1}^2)^{-1} \|f_1\|_2
\leq (1 + \varepsilon)^4 \kappa_{N_1}^2(1 - \kappa_{N_1}^2)^{-1} \|a\|_{N_1,\ell^2}.
\]
Set $\tilde{a}_n^{(2)} = -\frac{a^{(2)}_n}{(1 + \varepsilon)^3 \kappa_{N_1}^2(1 - \kappa_{N_1}^2)^{-1}}$ for $n \geq N_2$ and 0 otherwise, so that $\left\| \tilde{a}_n^{(2)} \right\|_{N_1,\ell^2} \leq \|a\|_{N_1,\ell^2}$.

Since $\{\lambda_n\}$ is EIS $H^2$, there exists $f_2 \in H^2$ such that
\[
f_2(\lambda_n) = a^{(2)}_n(1 - |\Theta(\lambda_n)|^2)^{-\frac{1}{2}}(1 - |\lambda_n|^2)^{-\frac{1}{2}}
\]
for all $n \geq N_1$
and, letting $\tilde{f}_2 = P_{K_\Theta}(f_2)$, we have
\[
\|\tilde{f}_2\|_2 \leq \|f_2\|_2 \leq (1 + \varepsilon)\left\| \tilde{a}_n^{(2)} \right\|_{N_1,\ell^2} \leq (1 + \varepsilon) \|a\|_{N_1,\ell^2}.
\]
Additionally, for $k \geq N_2$,
\[
\left| \tilde{f}_2(\lambda_k)(1 - |\Theta(\lambda_k)|^2)^{-\frac{1}{2}}(1 - |\lambda_k|^2)^{\frac{1}{2}} - a^{(2)}_k \right| = \left| \Theta(\lambda_k)T_{g}(f_2)(\lambda_k)(1 - |\Theta(\lambda_k)|^2)^{-\frac{1}{2}}(1 - |\lambda_k|^2)^{\frac{1}{2}} \right|
\leq \kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}} \|f_2\|_2
\leq (1 + \varepsilon)\kappa_{N_1}(1 - \kappa_{N_1}^2)^{-\frac{1}{2}} \|a\|_{N_1,\ell^2}.
\]
Substituting in the definition of the sequence \( \widetilde{a}^{(2)} \) we have, for \( k \geq N_2 \),
\[
\left| \left( 1 + \varepsilon \right)^4 \left( \frac{\kappa_{N_1}}{(1 - \kappa_{N_1}^2)^{\frac{3}{2}}} \right)^2 \tilde{f}_2(\lambda_k) + (1 + \varepsilon)^2 \frac{\kappa_{N_1}}{(1 - \kappa_{N_1}^2)^{\frac{3}{2}}} \tilde{f}_1(\lambda_k) + \tilde{f}_0(\lambda_k) \right| (1 - |\lambda_k|^2)^{\frac{1}{2}} - a_k \\
\leq (1 + \varepsilon)^5 \left( \frac{\kappa_{N_1}}{(1 - \kappa_{N_1}^2)^{\frac{3}{2}}} \right)^3 \|a\|_{N_1,\ell^2}.
\]

We continue this procedure, constructing sequences \( a^{(j)} \in \ell^2 \) and functions \( \tilde{f}_j \in K_\Theta \) such that
\[
\|a^{(j)}\|_{N_1,\ell^2} \leq (1 + \varepsilon)^{2j} \left( \frac{\kappa_{N_1}}{(1 - \kappa_{N_1}^2)^{\frac{3}{2}}} \right)^j \|a\|_{N_1,\ell^2},
\]
\[
\left| \frac{(1 - |\lambda_k|^2)^{\frac{1}{2}}}{(1 - |\Theta(\lambda_k)|^2)^{\frac{1}{2}}} \left( \sum_{i=0}^{j} (1 + \varepsilon)^{2i} \left( \frac{\kappa_{N_1}}{(1 - \kappa_{N_1}^2)^{\frac{3}{2}}} \right)^i \tilde{f}_i(\lambda_k) - a_k \right) \right| \leq (1 + \varepsilon)^{2j+1} \left( \frac{\kappa_{N_1}}{(1 - \kappa_{N_1}^2)^{\frac{3}{2}}} \right)^{j+1} \|a\|_{N_1,\ell^2},
\]
and
\[
\|\tilde{f}_j\|_2 \leq (1 + \varepsilon) \|a\|_{N_1,\ell^2} \text{ for all } j \in \mathbb{N}.
\]
Define
\[
F = \sum_{j=0}^{\infty} (1 + \varepsilon)^{2j} \left( \frac{\kappa_{N_1}}{(1 - \kappa_{N_1}^2)^{\frac{3}{2}}} \right)^j \tilde{f}_j.
\]
Then \( F \in K_\Theta \) since each \( \tilde{f}_j \in K_\Theta \) and, since \( \kappa_m \to 0 \), we may assume that
\[
(1 + \varepsilon)^2 \left( \frac{\kappa_{N_1}}{(1 - \kappa_{N_1}^2)^{\frac{3}{2}}} \right) < 1.
\]
So,
\[
\|F\|_2 \leq \frac{(1 + \varepsilon)}{1 - (1 + \varepsilon)^2 \left( \frac{\kappa_{N_1}}{(1 - \kappa_{N_1}^2)^{\frac{3}{2}}} \right)} \|a\|_{N_1,\ell^2}.
\]
For this \( \varepsilon \), consider \( \varepsilon_M < \varepsilon \) with
\[
\frac{(1 + \varepsilon_M)}{1 - (1 + \varepsilon_M)^2 \left( \frac{\kappa_{N_M}}{(1 - \kappa_{N_M}^2)^{\frac{3}{2}}} \right)} < 1 + \varepsilon.
\]
Then, using the process above, we obtain \( F_M \) satisfying \( F_M \in K_\Theta, \|F_M\|_2 \leq (1 + \varepsilon)\|a\|_{M,\ell^2} \) and \( F_M(\lambda_n)\|K_{\lambda_n}\|^{-1} = a_n \) for \( n \geq M \). Taking \( N(\varepsilon) = M, \) we see that \( F_M \) satisfies the exact interpolation conditions, completing the proof of the theorem. \( \square \)

Remark 3.5. The proof above also gives an estimate on the norm of the interpolating function in the event that \( \sup_n |\Theta(\lambda_n)| \leq \kappa < 1 \), but \( (1 + \varepsilon) \) is no longer the best estimate.
3.3. Carleson Measures in Model Spaces. From Theorem 3.4, (5) and (6), we have a Carleson measure statement for thin sequences in the Hardy space $H^2$. In this section, we obtain an equivalence in model spaces.

We now consider the embedding constants in the case of model spaces. As before, given a positive measure $\mu$ on $\mathbb{D}$, we denote the (possibly infinite) constant

$$C_\Theta(\mu) = \sup_{f \in K_\Theta, f \neq 0} \frac{\|f\|^2_{L^2(\mathbb{D}, \mu)}}{\|f\|_2^2}$$

as the Carleson embedding constant of $\mu$ on $K_\Theta$ and

$$R_\Theta(\mu) = \sup_z \|k_z^\Theta\|^2_{L^2(\mathbb{D}, \mu)}$$

as the embedding constant of $\mu$ on the reproducing kernel of $K_\Theta$ (recall that the kernels $k_z^\Theta$ are normalized). It is known that for general measure $\mu$ the constants $R_\Theta(\mu)$ and $C_\Theta(\mu)$ are not equivalent, [14]. The complete geometric characterization of the measures for which $C_\Theta(\mu)$ is finite is contained in [12]. However, we always have that

$$R_\Theta(\mu) \leq C_\Theta(\mu).$$

For $N > 1$, let

$$\sigma_N = \sum_{k \geq N} \|K_{\lambda_k}^\Theta\|^{-2} \delta_{\lambda_k} = \sum_{k \geq N} \frac{1 - |\lambda_k|^2}{1 - |\Theta(\lambda_k)|^2} \delta_{\lambda_k}.$$

Note that for each $f \in K_\Theta$

$$\|f\|^2_{L^2(\mathbb{D}, \sigma_N)} = \sum_{k=N}^{\infty} \frac{(1 - |\lambda_k|^2)}{(1 - |\Theta(\lambda_k)|^2)} |f(\lambda_k)|^2 = \sum_{k=N}^{\infty} |\langle f, k_{\lambda_k}^\Theta \rangle|^2.$$

and therefore we see that for $f \in K_\Theta$

$$1 \leq R_\Theta(\sigma_N) \leq C_\Theta(\sigma_N).$$

By working in a restricted setting and imposing a condition on $\{\Theta(\lambda_n)\}$ we have the following.

**Theorem 3.6.** Suppose $\Lambda = \{\lambda_n\}$ is a sequence in $\mathbb{D}$ and $\Theta$ is a nonconstant inner function such that $\kappa_m := \sup_{n \geq m} |\Theta(\lambda_n)| \to 0$. For $N > 1$, let

$$\sigma_N = \sum_{k \geq N} \|K_{\lambda_k}^\Theta\|^{-2} \delta_{\lambda_k} = \sum_{k \geq N} \frac{1 - |\lambda_k|^2}{1 - |\Theta(\lambda_k)|^2} \delta_{\lambda_k}.$$

Then the following are equivalent:

1. $\Lambda$ is a thin sequence;
2. $C_\Theta(\sigma_N) \to 1$ as $N \to \infty$;
3. $R_\Theta(\sigma_N) \to 1$ as $N \to \infty$.

**Proof.** We have (2) $\Rightarrow$ (3) by testing on the function $f = k_z^\Theta$ for all $z \in \mathbb{D}$.

We next focus on (1) $\Rightarrow$ (2). Let $f \in K_\Theta$ and let the sequence $a$ be defined by $a_j = \|K_{\lambda_j}^\Theta\|^{-1} f(\lambda_j)$. By (3.3), $\|a\|_{N, \ell^2}^2 = \|f\|^2_{L^2(\mathbb{D}, \sigma_N)}$, and since $\{k_{\lambda_j}^\Theta\}$ is an AOB,

$$\|a\|_{N, \ell^2}^2 = \sum_{j \geq N} \|K_{\lambda_j}^\Theta\|^{-2} |f(\lambda_j)|^2 = \langle f, \sum_{j \geq N} a_j k_{\lambda_j}^\Theta \rangle_{K_\Theta} \leq \|f\|_2 \left\| \sum_{j \geq N} a_j k_{\lambda_j}^\Theta \right\|_{K_\Theta} \leq C_N \|f\|_2 \|a\|_{N, \ell^2}.$$
By (1) and [2, Theorem 5.2], we know that $C_N \to 1$ and since we have established that $\|f\|_{L^2(\mathbb{D}, \sigma_N)} \leq C_N \|f\|_2$, (1) $\Rightarrow$ (2) follows.

Now consider (3) $\Rightarrow$ (1) and compute the quantity $\mathcal{R}_\Theta(\sigma_N)$. In what follows, we let $\Lambda_N$ denote the tail of sequence, $\Lambda_N = \{\lambda_k : k \geq N\}$. Note that $|1 - \pi b| \geq \max\{1 - |a|, 1 - |b|\}$. Using this estimate we see that:

$$
\sup_{z \in \mathbb{D}} \left\| k^\Theta_z \right\|_{L^2(\mathbb{D}, \sigma_N)}^2 = \sup_{z \in \mathbb{D}} \sum_{k \geq N} \frac{(1 - |\lambda_k|^2)(1 - |z|^2)}{(1 - |\Theta(\lambda_k)|^2)(1 - |\Theta(z)|^2)} \left| 1 - \frac{\Theta(z)\Theta(\lambda_k)}{1 - z\lambda_k} \right|^2
$$

$$
\geq \sup_{z \in \mathbb{D}} \sum_{k \geq N} \frac{(1 - |\lambda_k|^2)(1 - |z|^2)}{(1 - z\lambda_k)^2} \left( 1 + |\Theta(z)|(1 + |\Theta(\lambda_k)|) \right)
$$

$$
= \sup_{z \in \Lambda_N} \sum_{k \geq N} \frac{(1 - |\lambda_k|^2)(1 - |z|^2)}{(1 - z\lambda_k)^2} \left( 1 + |\Theta(z)|(1 + |\Theta(\lambda_k)|) \right)
$$

$$
\geq \frac{1}{(1 + \kappa_N)^2} \sup_{z \in \Lambda_N} \sum_{k \geq N} \frac{(1 - |\lambda_k|^2)(1 - |z|^2)}{|1 - z\lambda_k|^2},
$$

By the Weierstrass Inequality, we obtain for $M \geq N$ that

$$
\prod_{k \geq N, k \neq M} \left| \frac{\lambda_k - \lambda_M}{1 - \lambda_k \lambda_M} \right|^2 = \prod_{k \geq N, k \neq M} \left( 1 - \frac{(1 - |\lambda_k|^2)(1 - |\lambda_M|^2)}{|1 - \lambda_k \lambda_M|^2} \right)
$$

$$
\geq 1 - \sum_{k \geq N, k \neq M} \frac{(1 - |\lambda_M|^2)(1 - |\lambda_k|^2)}{|1 - \lambda_k \lambda_M|^2}.
$$

Thus, by (3.4) we have

$$
\frac{1}{(1 + \kappa_N)^2} \sup_{z \in \Lambda_N} \sum_{k \geq N} \frac{(1 - |\lambda_k|^2)(1 - |z|^2)}{|1 - z\lambda_k|^2} \geq \frac{1}{(1 + \kappa_N)^2} \left( \sum_{k \geq N, k \neq M} \frac{(1 - |\lambda_k|^2)(1 - |\lambda_M|^2)}{|1 - \lambda_k \lambda_M|^2} + 1 \right)
$$

$$
\geq \frac{1}{(1 + \kappa_N)^2} \left( 1 - \prod_{k \geq N, k \neq M} \left| \frac{\lambda_k - \lambda_M}{1 - \lambda_k \lambda_M} \right|^2 + 1 \right).
$$

Now by assumption, recalling that $\kappa_N := \sup_{n \geq N} |\Theta(\lambda_n)|$, we have

$$
\lim_{N \to \infty} \sup_{z \in \mathbb{D}} \| k^\Theta_z \|_{L^2(\mathbb{D}, \sigma_N)}^2 = 1 \quad \text{and} \quad \lim_{N \to \infty} \kappa_N = 0,
$$

so

$$
1 = \lim_{N \to \infty} \sup_{z \in \mathbb{D}} \| k^\Theta_z \|_{L^2(\mathbb{D}, \sigma_N)}^2 \geq \lim_{N \to \infty} \frac{1}{(1 + \kappa_N)^2} \left( 1 - \prod_{k \geq N, k \neq M} \left| \frac{\lambda_k - \lambda_M}{1 - \lambda_k \lambda_M} \right|^2 + 1 \right) \geq 1.
$$

Therefore, for any $M \geq N$

$$
\prod_{k \geq N, k \neq M} \left| \frac{\lambda_k - \lambda_M}{1 - \lambda_k \lambda_M} \right| > 1 - \varepsilon \quad \text{as} \quad N \to \infty.
$$

(3.5)
Also, for any \( \varepsilon > 0 \) there is an integer \( N_0 \) such that for all \( M > N_0 \) we have:

\[
(3.6) \prod_{k \geq N_0, k \neq M} \left| \frac{\lambda_k - \lambda_M}{1 - \lambda_k \lambda_M} \right| > 1 - \varepsilon.
\]

Fix this value of \( N_0 \), and consider \( k < N_0 \). Further, for \( k \neq M \) and \( k < N_0 \),

\[
1 - \rho(\lambda_M, \lambda_k)^2 = 1 - \left( 1 - \frac{\lambda_k - \lambda_M}{1 - \lambda_k \lambda_M} \right)^2 = \frac{(1 - |\lambda_M|^2)(1 - |\lambda_k|^2)}{|1 - \lambda_k \lambda_M|^2} \]

\[
= \frac{1 - \Theta(\lambda_M)^2}{|1 - \Theta(\lambda_M)\Theta(\lambda_k)|^2} (1 - |\lambda_k|^2)|k^\Theta_N (\lambda_k)|^2
\]

\[
\leq \frac{1 - \Theta(\lambda_M)^2}{|1 - \Theta(\lambda_M)\Theta(\lambda_k)|^2} \left( \|k^\Theta_N\|_{L^2(D, \sigma_M)}^2 - 1 \right)
\]

\[
\leq \frac{1}{(1 - \kappa_M)^2} \left( \|k^\Theta_N\|_{L^2(D, \sigma_M)}^2 - 1 \right) \to 0 \text{ as } M \to \infty,
\]

since \( 1 \leq \|k^\Theta_N\|_{L^2(D, \sigma_M)}^2 \leq \sup_z \|k^\Theta_N\|_{L^2(D, \sigma_M)}^2 \) and, by hypothesis, we have that \( \kappa_N \to 0 \) and \( \mathcal{R}_\Theta (\sigma_N) \to 1 \). Hence, it is possible to choose an integer \( M_0 \) sufficiently large compared to \( N_0 \) so that for all \( M > M_0 \)

\[
\rho(\lambda_k, \lambda_M) > (1 - \varepsilon)^{1/2} \quad k < N_0
\]

which implies that

\[
(3.7) \prod_{k < N_0} \rho(\lambda_k, \lambda_M) > 1 - \varepsilon.
\]

Now given \( \varepsilon > 0 \), first select \( N_0 \) as above in (3.6). Then select \( M_0 \) so that (3.5) holds. Then for any \( M > M_0 \) by writing the product

\[
\prod_{k \neq M} \rho(\lambda_k, \lambda_M) = \prod_{k < N_0} \rho(\lambda_k, \lambda_M) \prod_{k > N_0, k \neq M} \rho(\lambda_k, \lambda_M) > (1 - \varepsilon)^2.
\]

For the first term in the product we have used (3.7) to conclude that it is greater than \( 1 - \varepsilon \). And for \( M \) sufficiently large, by (3.6), we have that the second term in the product is greater than \( 1 - \varepsilon \) as well. Hence, \( B \) is thin as claimed.

\[\square\]

4. Algebra Version

Theorem 3.4 requires that our inner function satisfy \( \Theta(\lambda_n) \to 0 \) for a thin interpolating sequence \( \{\lambda_n\} \) to be an \( AIS_{K_0} \) sequence. Letting \( B \) denote the Blaschke product corresponding to the sequence \( \{\lambda_n\} \), denoting the algebra of continuous functions on the unit circle by \( C \), and letting \( H^\infty + C = \{ f + g : f \in H^\infty, g \in C \} \) (see [19] for more on this algebra), we can express this condition in the following way: \( \Theta(\lambda_n) \to 0 \) if and only if \( B\Theta \in H^\infty + C \). In other words, if and only if \( B \) divides \( \Theta \) in \( H^\infty + C \), [1, 10]. In interpreting our results below,
it is important to recall that each \( x \in M(H^\infty) \) has a unique extension to a linear functional of norm one and, therefore, we may identify \( M(\mathcal{B}) \) with a subset of \( M(H^\infty) \).

We now consider thin sequences in uniform algebras. We let \( \mathcal{B} \) be a Douglas algebra; that is, a uniformly closed subalgebra of \( L^\infty \) containing \( H^\infty \). Throughout \( M(\mathcal{B}) \) denotes the maximal ideal space of the algebra \( \mathcal{B} \); that is, the set of nonzero continuous multiplicative linear functionals on \( \mathcal{B} \). With the weak-* topology, \( M(\mathcal{B}) \) is a compact Hausdorff space. In this context, the condition we will require (see Theorem 4.5) for an \( EIS_B \) sequence to be the same as an \( AIS_B \) sequence is that the sequence be thin near \( M(\mathcal{B}) \). We take the following as the definition (see [20]):

**Definition 4.1.** An interpolating sequence \( \{\lambda_n\} \) with corresponding Blaschke product \( b \) is said to be thin near \( M(\mathcal{B}) \) if for any \( 0 < \eta < 1 \) there is a factorization \( b = b_1b_2 \) with \( b_1 \) invertible in \( \mathcal{B} \) and

\[
|b_2(z_n)|(1 - |z_n|^2) > \eta
\]

for all \( n \) such that \( b_2(z_n) = 0 \).

We will be interested in two related concepts that a sequence can have. We first introduce a norm on a sequence \( \{a_n\} \in \ell^\infty \) that is induced by a second sequence \( \{\lambda_n\} \) and a set \( \mathcal{O} \supset M(\mathcal{B}) \) that is open in \( M(H^\infty) \). Set \( I_\mathcal{O} = \{n \in \mathbb{Z} : \lambda_n \in \mathcal{O}\} \). Then we define

\[
\|a\|_{\mathcal{O},\ell^\infty} = \sup\{|a_n| : n \in I_\mathcal{O}\}.
\]

**Definition 4.2.** A Blaschke sequence \( \{\lambda_n\} \) is an eventual 1-interpolating sequence in \( \mathcal{B} \), denoted \( EIS_B \), if for every \( \varepsilon > 0 \) there exists an open set \( \mathcal{O} \supset M(\mathcal{B}) \) such that for each \( \{a_n\} \in \ell^\infty \) there exists \( f_{\mathcal{O},a} \in H^\infty \) with

\[
f_{\mathcal{O},a}(\lambda_n) = a_n \text{ for } \lambda_n \in \mathcal{O} \text{ and } \|f_{\mathcal{O},a}\|_{\mathcal{O},\ell^\infty} \leq (1 + \varepsilon)\|a\|_{\mathcal{O},\ell^\infty}.
\]

**Definition 4.3.** A Blaschke sequence \( \{\lambda_n\} \) is a strong asymptotic interpolating sequence in \( \mathcal{B} \), denoted \( AIS_B \), if for all \( \varepsilon > 0 \) there exists an open set \( \mathcal{O} \supset M(\mathcal{B}) \) such that for all sequences \( \{a_n\} \in \ell^\infty \) there exists a function \( G_{\mathcal{O},a} \in H^\infty \) such that \( \|G_{\mathcal{O},a}\|_{\mathcal{O},\ell^\infty} \leq \|a\|_{\mathcal{O},\ell^\infty} \) and

\[
\|G_{\mathcal{O},a}(\lambda_n) - a_n\|_{\mathcal{O},\ell^\infty} < \varepsilon\|a\|_{\mathcal{O},\ell^\infty}.
\]

**Theorem 4.4.** Let \( \mathcal{B} \) be a Douglas algebra. Let \( \{\lambda_n\} \) be a Blaschke sequence of points in \( \mathbb{D} \). Then \( \{\lambda_n\} \) is an \( EIS_B \) sequence if and only if \( \{\lambda_n\} \) is an \( AIS_B \).

**Proof.** If a sequence is an \( EIS_B \), then it is trivially \( AIS_B \), for given \( \varepsilon > 0 \) we may take \( G_{\mathcal{N},a} = \frac{f_{\mathcal{N},a}}{1 + \varepsilon} \).

For the other direction, suppose \( \{\lambda_n\} \) is an \( AIS_B \) sequence. Choose \( \varepsilon \) satisfying \( 0 < \varepsilon < 1 \) and \( \mathcal{O} \supset M(\mathcal{B}) \) and let \( \{a_j\} := \{a_j^{(0)}\} \) be any sequence. Now choose \( f_0 \) so that

\[
\|f_0(\lambda_n) - a_n^{(0)}\|_{\mathcal{O},\ell^\infty} < \varepsilon\|a\|_{\mathcal{O},\ell^\infty}
\]

and

\[
\|f_0\|_{\mathcal{O},\ell^\infty} \leq \|a\|_{\mathcal{O},\ell^\infty}.
\]

Let \( a_n^{(1)} = a_n^{(0)} - f_0(\lambda_n) \). Note that \( \|a^{(1)}\|_{\mathcal{O},\ell^\infty} < \varepsilon\|a\|_{\mathcal{O},\ell^\infty} \). Since we have an \( AIS_B \) sequence, we may choose \( f_1 \) such that

\[
\|f_1(\lambda_n) - a_n^{(1)}\|_{\mathcal{O},\ell^\infty} < \frac{\varepsilon}{1 + \varepsilon}\|a^{(1)}\|_{\mathcal{O},\ell^\infty} < \left(\frac{\varepsilon}{1 + \varepsilon}\right)^2\|a\|_{\mathcal{O},\ell^\infty},
\]
and
\[ \|f_1\|_\infty \leq \|a^{(1)}\|_{\ell^\infty} < \frac{\varepsilon}{1+\varepsilon} \|a\|_{\ell^\infty}. \]

In general, we let
\[ a_j^{(k)} = -f_{k-1}(\lambda_j) + a_j^{(k-1)} \]
so that
\[ \|a^{(k)}\|_{\ell^\infty} \leq \frac{\varepsilon}{1+\varepsilon} \|a^{(k-1)}\|_{\ell^\infty} \leq \left( \frac{\varepsilon}{1+\varepsilon} \right)^2 \|a^{(k-2)}\|_{\ell^\infty} \leq \cdots \leq \left( \frac{\varepsilon}{1+\varepsilon} \right)^k \|a\|_{\ell^\infty} \]
and
\[ \|f_k\|_\infty \leq \|a^{(k)}\|_{\ell^\infty} < \left( \frac{\varepsilon}{1+\varepsilon} \right)^k \|a\|_{\ell^\infty}. \]

Then consider \( f(z) = \sum_{k=0}^\infty f_k(z) \). Since \( f_k(\lambda_j) = a_j^{(k)} - a_j^{(k+1)} \) and \( a_j^{(k)} \to 0 \) as \( k \to \infty \), we have
\[ f(\lambda_j) = a_j^{(0)} = a_j. \]
Further \( \|f\|_\infty \leq \sum_{k=0}^\infty \left( \frac{\varepsilon}{1+\varepsilon} \right)^k \|a\|_{\ell^\infty} = \frac{1}{1-\frac{\varepsilon}{1+\varepsilon}} \|a\|_{\ell^\infty} = (1+\varepsilon)\|a\|_{\ell^\infty}. \]

**Theorem 4.5.** Let \( \{\lambda_n\} \) in \( \mathbb{D} \) be an interpolating Blaschke sequence and let \( B \) be a Douglas algebra. The following are equivalent:

1. \( \{\lambda_n\} \) is an EIS\( B \) sequence;
2. \( \{\lambda_n\} \) is a AIS\( B \) sequence;
3. \( \{\lambda_n\} \) is thin near \( M(B) \);

**Proof.** The equivalence between (1) and (2) is contained in Theorem 4.4.

We next prove that if a sequence is thin near \( M(B) \), then it is an EIS\( B \) sequence. We let \( b \) denote the Blaschke product associated to the sequence \( \{\lambda_n\} \).

Given \( \varepsilon > 0 \), choose \( \gamma \) so that
\[ \frac{1+\sqrt{1-\gamma^2}}{\gamma} < 1+\varepsilon. \]

Choose a factorization \( b = b_1^\gamma b_2^\gamma \) so that \( \overline{b}_1^\gamma \in B \) and \( \delta(b_2^\gamma) = \inf(1 - |\lambda|^2)|b_2^\gamma(\lambda)| > \gamma \). Since \( |b_1^\gamma| = 1 \) on \( M(B) \) and \( \gamma < 1 \), there exists an open set \( \mathcal{O} \supset M(B) \) such that \( |b_1^\gamma| > \gamma \) on \( \mathcal{O} \). Note that if \( b(\lambda) = 0 \) and \( \lambda \in \mathcal{O} \), then \( b_2(\lambda) = 0 \).

The condition on \( b_2^\gamma \) coupled with Earl’s Theorem, [5, 6], gives rise to functions \( \{f_\gamma^k\} \) in \( H^\infty (\mathbb{D}) \), and hence in \( B \) so that
\[ f_\gamma^k(\lambda_k) = \delta_{jk} \text{ whenever } b_2^\gamma(\lambda_k) = 0 \text{ and } \sup_{z \in \mathbb{D}} \left| \sum_j f_\gamma^k(z) \right| \leq \frac{1+\sqrt{1-\gamma^2}}{\gamma}. \]

Now given \( a \in \ell^\infty \), choose the corresponding P. Beurling functions (as in (4.1)) and let
\[ f_\gamma^\alpha = \sum_j a_j f_\gamma^j. \]

By construction we have that \( f_{\mathcal{O},a}(\lambda_n) = a_n \) for all \( \lambda_n \in \mathcal{O} \). Also, by Earl’s estimate (4.1), we have that
\[ \|f_{\mathcal{O},a}\|_\infty \leq (1+\varepsilon)\|a\|_\infty. \]

Thus, (3) implies (1).
Finally, suppose \( \{\lambda_n\} \) is a \( EIS_B \) sequence. Let \( 0 < \eta < 1 \) be given and choose \( \eta_1 \) with \( 1/(1 + \eta_1) > \eta \), a function \( f \in H^\infty \) and \( \mathcal{O} \supset M(\mathcal{B}) \) open in \( M(H^\infty) \) with
\[
f_{\mathcal{O},n}(\lambda_m) = \delta_{nm} \quad \text{for } \lambda_m \in \mathcal{O} \text{ and } \|f\|_{\mathcal{O},n} \leq 1 + \eta.
\]
Let \( b_2 \) denote the Blaschke product with zeros in \( \mathcal{O} \), \( b_1 \) the Blaschke product with the remaining zeros and let
\[
f_{\mathcal{O},n}(z) = \left( \prod_{j \neq n, b_2(\lambda_j) = 0} \frac{z - \lambda_j}{1 - \lambda_j z} \right) h(z),
\]
for some \( h \in H^\infty \). Then \( \|h\|_\infty \leq 1 + \eta_1 \) and
\[
1 = |f_{\mathcal{O},n}(\lambda_n)| = \left| \left( \prod_{j \neq n, b_2(\lambda_j) = 0} \frac{\lambda_n - \lambda_j}{1 - \lambda_j \lambda_n} \right) h(\lambda_n) \right| \leq (1 + \eta_1) \prod_{j \neq n} \left| \frac{\lambda_n - \lambda_j}{1 - \lambda_j \lambda_n} \right|.
\]
Therefore
\[
(1 - |\lambda_n|^2)|b_2(\lambda_n)| = \prod_{j \neq n, b_2(\lambda_j) = 0} \left| \frac{\lambda_n - \lambda_j}{1 - \lambda_j \lambda_n} \right| \geq 1/(1 + \eta_1) > \eta.
\]

Now because we assume that \( \{\lambda_n\} \) is interpolating, the Blaschke product \( b = b_1 b_2 \) with zeros at \( \{\lambda_n\} \) will vanish at \( x \in M(H^\infty) \) if and only if \( x \) lies in the closure of the zeros of \( \{\lambda_n\} \), [11, p. 206] or [7, p. 379]. Now, if we choose \( \mathcal{V} \) open in \( M(H^\infty) \) with \( M(\mathcal{B}) \subset \mathcal{V} \subset \overline{\mathcal{V}} \subset \mathcal{O} \), then \( b_1 \) has no zeros in \( \mathcal{V} \cap \overline{\mathcal{D}} \) and, therefore, no point of \( M(\mathcal{B}) \) can lie in the closure of the zeros of \( b_1 \). So \( b_1 \) has no zeros on \( M(\mathcal{B}) \). Thus we see that \( b_1 \) is bounded away from zero on \( M(\mathcal{B}) \) and, consequently, \( b_1 \) is invertible in \( \mathcal{B} \).

We note that we do not need the full assumption that \( b \) is interpolating; it is enough to assume that \( b \) does not vanish identically on a Gleason part contained in \( M(\mathcal{B}) \). Our goal, however, is to illustrate the difference in the Hilbert space and uniform algebra setting and so we have stated the most important setting for our problem.

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