Fractional part integral representation for derivatives of a function related to $\ln \Gamma(x + 1)$

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Abstract

For $0 \neq x > -1$ let

$$\Delta(x) = \frac{\ln \Gamma(x + 1)}{x}.$$

Recently Adell and Alzer proved the complete monotonicity of $\Delta'$ on $(-1, \infty)$ by giving an integral representation of $(-1)^n \Delta^{(n+1)}(x)$ in terms of the Hurwitz zeta function $\zeta(s, a)$. We reprove this integral representation in different ways, and then re-express it in terms of fractional part integrals. Special cases then have explicit evaluations. Other relations for $\Delta^{(n+1)}(x)$ are presented, including its leading asymptotic form as $x \to \infty$.

Key words and phrases

Gamma function, digamma function, polygamma function, Hurwitz zeta function, Riemann zeta function, fractional part, integral representation

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Introduction and statement of results

For \(0 \neq x > -1\) let

\[
\Delta(x) = \frac{\ln \Gamma(x + 1)}{x}, \quad \Delta(0) = -\gamma,
\]

where \(\Gamma\) is the Gamma function, \(\gamma = -\psi(1)\) is the Euler constant, and \(\psi(x) = \Gamma'/\Gamma\) is the digamma function. The study of the convexity and monotonicity of the functions \(\Gamma\) and \(\Delta\) and of their derivatives is of interest \([8, 13, 14, 17]\). For instance, the paper \([8]\) gave an analog of the well known Bohr-Mollerup theorem for the function \(\Delta(x)\).

Monotonicity and convexity are very useful properties for developing a variety of inequalities. Completely monotonic functions have applications in several branches, including complex analysis, potential theory, number theory, and probability (e.g., \([5]\)). In \([4]\), \(-\Delta(x)\) was shown to be a Pick function, with integral representation

\[
-\Delta(x) = -\frac{\pi}{4} + \int_{-\infty}^{-1} \left( \frac{1}{t - z} - \frac{t}{t^2 + 1} \right) \frac{dt}{-t}.
\]

I.e., this function is holomorphic in the upper half plane with nonnegative imaginary part.

Recently Adell and Alzer \([2]\) proved the complete monotonicity of \(\Delta'\) on \((-1, \infty)\) by demonstrating the following integral representation.

**Proposition 1.** (Adell and Alzer). For \(x > -1\) and \(n \geq 0\) an integer one has

\[
(-1)^n \Delta^{(n+1)}(x) = (n + 1)! \int_0^1 u^{n+1} \zeta(n + 2, xu + 1) du,
\]

where \(\zeta(s, a)\) is the Hurwitz zeta function (e.g., \([10]\)). The complete monotonicity of \(\Delta'\), the statement \((-1)^n \Delta^{(n+1)}(x) \geq 0\), then follows from \(\zeta(n + 2, xu + 1) \geq 0\) for
We reprove the result (1.2) in two other ways, and in so doing illustrate properties of the ζ function.

**Corollary 1.** We have the following recurrence:

\[
\frac{(-1)^n}{(n+1)!} \Delta^{(n+1)}(x) = \frac{1}{x} \frac{(-1)^n - \Delta^{(n)}(x)}{n!} - \frac{\zeta(n+1, x+1)}{(n+1)x}.
\]  

(1.3)

We then relate cases of (1.2) to fractional part integrals, including the following, wherein we let \( \{x\} = x - [x] \) denote the fractional part of \( x \).

**Proposition 2.** Let \( k \geq 1 \) be an integer. Then we have

\[
\int_0^1 u^{n+1} \zeta(n+2, ku+1) du = \frac{1}{k^{n+2}} \left[ \frac{\{w\}^{n+1}}{w^{n+2}} \int_1^\infty dw + \sum_{j=1}^{k-1} \int_0^\infty \frac{\{x\} \{j\}^{n+1}}{(x+j)^{n+2}} dx \right].
\]  

(1.4)

As a further special case we have

**Corollary 2.** We have

\[
(-1)^n \Delta^{(n+1)}(1) = (n+1)! \int_0^1 y^{n+1} \zeta(n+2, y+1) dy = (n+1)! \int_0^\infty \frac{\{x\}^{n+1}}{(x+1)^{n+2}} dx
\]

\[
= (n+1)! \left[ 1 - \gamma - \sum_{j=2}^{k-1} \frac{1}{j} [\zeta(j) - 1] \right],
\]  

(1.5)

where \( \zeta(s) = \zeta(s, 1) \) is the Riemann zeta function \([7, 10, 15, 16]\).

More generally, we have the following, wherein we put \( P_1(x) = \{x\} - 1/2 \). Let \( {}_2F_1 \) be the Gauss hypergeometric function \([3, 9]\).

**Proposition 3.** We have for integers \( n \geq 0 \)

\[
\int_0^1 u^{n+1} \zeta(n+2, xu+1) du = \frac{1}{2(n+2)} \frac{1}{(x+1)^{n+2}} {}_2F_1 \left( 1, n+2; n+3; \frac{x}{x+1} \right)
\]
\[ + \frac{1}{(n+1)(n+2)(x+1)^{n+1}} {}_2F_1 \left(1, n+1; n+3; \frac{x}{x+1}\right) - \int_0^\infty \frac{1}{(t+1)(t+x+1)^{n+2}} dt. \]

(1.6)

From this Proposition we may then determine the following asymptotic form:

**Corollary 3.** We have

\[ \Delta^{(n+1)}(x) \sim (-1)^n \frac{n!}{(x+1)^{n+1}}, \quad x \to \infty, \quad (1.7) \]

in agreement with Corollary 1.2 of [2]. In fact, the proof shows how higher order terms may be systematically found.

Many expressions may be found for the \( {}_2F_1 \) functions in (1.6) and (2.20) below, and we present a sample of these in an Appendix.

A simple property of \( \Delta \) is given in the following.

**Proposition 4.** We have (a)

\[ \int_0^1 \Delta(x)dx = -\gamma + \sum_{k=2}^\infty \frac{(-1)^k}{k^2} \zeta(k), \quad (1.8a) \]

and (b)

\[ \int_0^1 \Delta^2(x)dx = \gamma^2 - 2\gamma \sum_{k=2}^\infty \frac{(-1)^k}{k^2} \zeta(k) + \sum_{m=4}^\infty \frac{(-1)^m}{(m-1)} \sum_{\ell=2}^{m-2} \frac{\zeta(m-\ell)\zeta(\ell)}{(m-\ell)\ell}. \quad (1.8b) \]

Throughout we let \( \psi^{(j)} \) denote the polygamma functions (e.g., [1]), and we note the relation for integers \( n > 0 \)

\[ \psi^{(n)}(x) = (-1)^{n+1}n!\zeta(n+1, x). \quad (1.9) \]

Therefore, as to be expected, (1.2) could equally well be written as an integral over \( \psi^{(n+1)}(xu + 1) \). The polygamma functions possess the functional equation

\[ \psi^{(j)}(x + 1) = \psi^{(j)}(x) + (-1)^j \frac{j^1}{x^{j+1}}, \quad (1.10) \]
For a very recent development of single- and double-integral and series representations for the Gamma, digamma, and polygamma functions, [6] may be consulted.

Proof of Propositions

Proposition 1. We provide two alternative proofs of this result. The result holds for $n = 0$, and for the first proof we proceed by induction. For the inductive step we have

$$
\Delta^{(n+2)}(x) = \frac{d}{dx}\Delta^{(n+1)}(x)
$$

$$
= (-1)^n(n+1)! \int_0^1 u^{n+1} \frac{d}{dx} \zeta(n + 2, xu + 1) du
$$

$$
= (-1)^{n+1}(n + 2)! \int_0^1 u^{n+2} \zeta(n + 3, xu + 1) du. \quad (2.1)
$$

In the last step, we used $\partial_a \zeta(s, a) = -s \zeta(s + 1, a)$.

We remark that this first method shows that (1.2) may be evaluated by repeated integration by parts, for we have

$$
(n + 1)! \int_0^1 u^{n+1} \zeta(n + 2, xu + 1) du = \frac{(-1)^n}{x^n} \int_0^1 u^{n+1} \left( \frac{\partial}{\partial u} \right)^n \zeta(2, xu + 1) du. \quad (2.2)
$$

Second method. By the product rule we have

$$
\Delta^{(n+1)}(x) = \sum_{j=0}^{n+1} \binom{n+1}{j} \ln \Gamma(x + 1) \zeta(n-j+1, x+1) \frac{(-1)^j j!}{x^{j+1}}
$$

$$
= \sum_{j=0}^{n+1} \binom{n+1}{j} \psi(n-j)(x+1) \frac{(-1)^j j!}{x^{j+1}}. \quad (2.3)
$$

Here, it is understood that $\psi(-1)(x) = \ln \Gamma(x)$. We now apply (1.9) and the integral representation

$$
(n - j)! \zeta(n - j + 1, x + 1) = \int_0^\infty \frac{t^{n-j}e^{-xt}}{e^t - 1} dt, \quad (2.4)
$$

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so that

\[
\Delta^{(n+1)}(x) = (-1)^{n+1} \sum_{j=0}^{n+1} \frac{j!}{x^{j+1}} \int_0^\infty t^{n-j} e^{-xt} \frac{dt}{e^t - 1}
\]

\[
= (-1)^{n+1} \frac{1}{x^{n+2}} \int_0^\infty \frac{e^{-xt}}{(e^t - 1)} \left[e^{xt} \Gamma(n+2, xt) - (n+1)!\right] \frac{dt}{t} \tag{2.5}
\]

where the incomplete Gamma function \(\Gamma(x, y)\) has the property \([9] (p. 941)\)

\[
\Gamma(n+1, x) = n! e^{-x} \sum_{m=0}^n \frac{x^m}{m!} \tag{2.6}
\]

Now we use a Laplace transform,

\[
\int_0^1 u^{n+1} e^{-xu} du = \frac{1}{(xt)^{n+2}}[(n+1)! - \Gamma(n+2, xt)], \tag{2.7}
\]

to write

\[
\Delta^{(n+1)}(x) = (-1)^n \int_0^\infty \frac{e^{-xt}}{(e^t - 1)} t^{n+1} \int_0^1 u^{n+1} e^{-xu} du \frac{dt}{t}
\]

\[
= (-1)^n \int_0^1 u^{n+1} \int_0^\infty \frac{t^{n+1}}{e^t - 1} e^{-xu} \frac{dt}{u} du
\]

\[
= - (-1)^n \int_0^1 u^{n+1} \zeta(n+2, xu + 1) du. \tag{2.8}
\]

By absolute convergence and the Tonelli-Hobson theorem, the interchange of integrations is justified. In the last step, we applied the representation (2.4).

**Corollary 1.** This is proved by integrating by parts in (1.2).

**Remark.** It is possible to find explicit expressions for the values \(\Delta^{(n+1)}(j + 1/2)\) with half-integer argument. This is due to the functional equation (1.10) along with the values \(\psi^{-1}(1/2) = \ln \sqrt{\pi}, \psi(1/2) = -\gamma - 2 \ln 2, \) and \([11] (p. 260)\)

\[
\psi^{(n)}\left(\frac{1}{2}\right) = (-1)^{n+1} n!(2^{n+1} - 1)\zeta(n+1), \quad n \geq 1. \tag{2.9}
\]
We then obtain, for instance, by using (2.3)

\[
\Delta^{(n+1)}\left(-\frac{1}{2}\right) = \sum_{j=0}^{n-1} \frac{(-1)^{n-j}}{(n-j+1)} (2^{n+2} - 2^{j+1}) \zeta(n-j+1) + 2^{n+1}(\gamma + 2 \ln 2) - 2^{n+2} \ln \sqrt{\pi}.
\]  

(2.10)

Similarly, it is possible to find explicit expressions for the values \(\Delta^{(n+1)}(j + 1/4)\) and \(\Delta^{(n+1)}(j + 3/4)\) by using the corresponding values of \(\psi^{(k)}[1]\).

**Proposition 2.** We use two Lemmas.

**Lemma 1.** When the integrals involved are convergent, we have for integrable functions \(f\) and \(g\)

\[
\int_1^\infty f(\{x\})g(x)dx = \int_0^1 f(y) \sum_{\ell=1}^\infty g(y + \ell)dy.
\]  

(2.11)

**Lemma 2.** For \(b > 0, \lambda > 1,\) and \(c \geq 0\) we have for integrable functions \(f\)

\[
\int_0^\infty f\left(\left\{\frac{x}{b}\right\}\right) \frac{dx}{(x+c)^\lambda} = \frac{1}{b^{\lambda-1}} \int_0^1 f(y) \zeta(\lambda, y + c/b)dy.
\]  

(2.12)

This holds when the integrals are convergent.

**Proof.** For Lemma 1 we have

\[
\int_1^\infty f(\{x\})g(x)dx = \sum_{\ell=1}^\infty \int_0^{\ell+1} f(\{x\})g(x)dx = \sum_{\ell=1}^\infty \int_0^1 f(y)g(y + \ell)dy.
\]  

(2.13)

For Lemma 2 we first have

\[
\int_0^\infty f\left(\left\{\frac{x}{b}\right\}\right) g(x)dx = b \int_0^\infty f(\{v\})g(bv)dv
\]
\[ b \sum_{\ell=0}^{\infty} \int_{\ell}^{\ell+1} f(v - \ell) g(bv) dv = b \sum_{\ell=0}^{\infty} \int_{\ell}^{1} f(y) g(b(y + \ell)) dy. \] (2.14)

We now put \( g(x) = 1/(x + c)^\lambda \), so that

\[ \sum_{\ell=0}^{\infty} g[b(y + \ell)] = \frac{1}{b^\lambda} \sum_{\ell=0}^{\infty} \frac{1}{(y + \ell + c/b)^\lambda} = \frac{1}{b^\lambda} \zeta(\lambda, y + c/b). \] (2.15)

**Proof of Proposition 2.** We have for integers \( k \geq 1 \)

\[ \int_{0}^{1} u^{n+1} \zeta(n + 2, ku + 1) du = \frac{1}{k^{n+2}} \int_{0}^{k} v^{n+1} \zeta(n + 2, v + 1) dv \]
\[ = \frac{1}{k^{n+2}} \sum_{\ell=0}^{k-1} \int_{\ell}^{\ell+1} v^{n+1} \zeta(n + 2, v + 1) dv \]
\[ = \frac{1}{k^{n+2}} \sum_{\ell=0}^{k-1} \int_{0}^{1} (w + \ell)^{n+1} \zeta(n + 2, w + \ell + 1) dw. \] (2.16)

We now apply Lemma 2 with \( b = 1, c = \ell + 1, \) and \( f(w) = (w + \ell)^{n+1} \), giving

\[ \int_{0}^{1} u^{n+1} \zeta(n + 2, ku + 1) du = \frac{1}{k^{n+2}} \sum_{\ell=0}^{k-1} \int_{0}^{\infty} \frac{\{x\} + \ell)^{n+1}}{(x + \ell + 1)^{n+2}} dx \]
\[ = \frac{1}{k^{n+2}} \left[ \int_{1}^{\infty} \frac{w^{n+1}}{w^{n+2}} dw + \sum_{\ell=1}^{k-1} \int_{0}^{\infty} \frac{\{x\} + \ell)^{n+1}}{(x + \ell + 1)^{n+2}} dx \right]. \] (2.17)

In the last step we used the periodicity \( \{w - 1\} = \{w\} \).

For Corollary 2, we apply Lemma 2 of [6].

**Proposition 3.** We start from the integral representation

\[ \zeta(s, a) = \frac{a^{-s}}{2} + \frac{a^{1-s}}{s-1} - s \int_{0}^{\infty} \frac{P_1(x)}{(x + a)^{s+1}} dx, \quad \text{Re} s > -1. \] (2.18)

Then

\[ \int_{0}^{1} u^{n+1} \zeta(n + 2, xu + 1) du = \int_{0}^{1} u^{n+1} \left[ \frac{1}{2(xu + 1)^{n+2}} + \frac{1}{(n+1)(xu + 1)^{n+1}} \right] \]
\[-(n+2) \int_0^\infty \frac{P_1(t) dt}{(t+xu+1)^{n+3}} \] du. \quad (2.19) 

By using a standard integral representation of \(2F_1\) (e.g., \([9]\), p. 1040 or \([3]\) p. 65) we have
\[
\int_0^1 u^{n+1} \zeta(n+2, xu+1) du = \frac{1}{2(n+2)} \quad 2F_1(n+2, n+2; n+3; -x) 
+ \frac{1}{(n+1)(n+2)} \quad 2F_1(n+1, n+2; n+3; -x) - \int_0^\infty \frac{1}{(t+1)(t+x+1)^{n+2}} P_1(t) dt. \quad (2.20)
\]

By applying a standard transformation rule \([9]\) (p. 1043) to the \(2F_1\) functions, we obtain the Proposition.

**Corollary 3.** We give the detailed asymptotic forms as \(x \to \infty\) of the hypergeometric functions in (1.6). We easily have that \(2F_1(1, n+1; n+3; 1) = n+2\) and these forms will then show that the corresponding term in (1.6) gives the leading term as \(x \to \infty\). We let \((a)_j = \Gamma(a+j)/\Gamma(a)\) be the Pochhammer symbol. The following expansions are valid for \(|z-1| < 1\) and \(|\arg(1-z)| < \pi\):

\[
2F_1(1, y; 1+y, z) = y \sum_{k=0}^{\infty} \frac{(y)_k}{k!} [\psi(k+1) - \psi(k+y) - \ln(1-z)](1-z)^k, \quad (2.21a)
\]

and

\[
2F_1(1, y; 2+y, z) = y+1 - y(y+1) \sum_{k=0}^{\infty} \frac{(y+1)_k}{k!} [\psi(k+1) - \psi(k+y+1) - \ln(1-z)](1-z)^{k+1}, \quad (2.21b)
\]

where \((y)_0 = 1\). These expansions are the \(n=0\) and \(n=1\) cases of (9.7.5) in \([12]\) (p. 257), respectively. We put \(y = n+1, z = x/(x+1), \ln(1-z) = -\ln(x+1)\) and then
find
\[ _2F_1 \left( 1, n + 1; n + 3; \frac{x}{x + 1} \right) = n + 2 + (n + 1)(n + 2) [\gamma - \ln x + \psi(n + 2)] \frac{1}{x} + O \left( \frac{\ln x}{x^2} \right), \]

(2.22a)

and
\[ _2F_1 \left( 1, n + 2; n + 3; \frac{x}{x + 1} \right) = -(n + 2) [\gamma - \ln x + \psi(n + 2)] + O \left( \frac{\ln x}{x} \right). \]

(2.22b)

The integral term in (1.6) is at most \( O[(x + 1)^{-(n+2)}] \), and is actually much smaller due to cancellation within the integrand, and the Corollary then follows.

Remarks. Of course we have from (1.2) \( \Delta^{(n+1)}(0) = (-1)^n (n+1)! \zeta(n+2)/(n+2) \), in agreement with the expansion (2.26). This special case is recovered from Proposition 3 in the following way. We have \( _2F_1(a,b;c;0) = 1 \) and the representation [16] (p. 14)
\[ \zeta(s) = \frac{1}{s-1} + \frac{1}{2} - s \int_1^\infty \frac{P_1(x)}{x^{s+1}} dx, \]

(2.23)

the \( a = 1 \) case of (2.18), giving the identity \( \int_0^1 u^{n+1} \zeta(n+2,1) du = \zeta(n+2)/(n+2) \).

In connection with Propositions 2 and 3, another representation that might be employed is [7]
\[ \ln \Gamma(x + 1) = (x + \frac{1}{2}) \ln x - x + \frac{1}{2} \ln 2\pi - \int_0^\infty \frac{P_1(t)}{t + x} dt. \]

(2.24)

We may note that representations (2.19) or (2.20), for instance, provide another basis for proving integral representations for \( (-1)^n \Delta^{(n+1)}(x) \) by induction. When using (2.20), we use the derivative property
\[ \frac{d}{dx} _2F_1(a,b;c;-x) = -\frac{ab}{c} _2F_1(a+1,b+1;c+1;-x). \]

(2.25)
The $F_1$ function in (1.6) can be written in other ways, including using a transformation formula [9] (p. 1043), so that

$$F_1\left(1, n + 2; n + 3; \frac{x}{x + 1}\right) = (x + 1) F_1(1, 2; n + 3; -x). \tag{2.26}$$

**Proposition 4.** The result uses the expansion [9] (p. 939)

$$\ln \Gamma(x + 1) = -\gamma x + \sum_{k=2}^{\infty} \frac{(-1)^k}{k} \zeta(k) x^k. \tag{2.27}$$

**Remark.** Let $Ei(x)$ be the exponential integral function (e.g., [9], p. 925). Given the relations [9] (pp. 927, 942)

$$\Gamma(0, x) = -Ei(-x) = -\left( \gamma + \ln x + \sum_{k=1}^{\infty} \frac{(-x)^k}{kk!} \right), \tag{2.28}$$

it is possible to write

$$\int_0^1 \Delta(x) dx = -\gamma - \int_0^\infty \frac{\left[ \gamma - t + \Gamma(0, t) + \ln t \right]}{t(e^t - 1)} dt. \tag{2.29}$$

This follows by inserting a standard integral representation for the values $\zeta(k)$ into the right side of (1.8a).

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Appendix

Here we present illustrative relations for the sort of hypergeometric functions appearing in (1.6) and (2.20).

The contiguous relations [9] (pp. 1044-45) may be readily applied. As well, we have for instance [9] (p. 1043)

\[ 2F_1(n + 2, n + 2; n + 3; -x) = (1 + x)^{-(n+1)} 2F_1(1, 1; n + 3; -x). \]  \( \text{(A.1)} \)

The next result provides a type of recurrence relation in the first parameter of the \(2F_1\) function.

**Proposition A1.** For integers \(n \geq -1\) we have

\[
\int_0^1 \frac{u^{n+1}}{(xu + 1)^{n+2}} du = \frac{1}{(n+2)} 2F_1(n + 2, n + 2; n + 3; -x) \\
= \frac{1}{(n+1)} \left[ \frac{1}{(x+1)^{n+1}} - \frac{1}{(n+2)} 2F_1(n + 1, n + 2; n + 3; -x) \right]. \]  \( \text{(A.2)} \)

**Proof.** With \(v = xu\) in (A.2), we have

\[
\frac{1}{x^{n+2}} \int_0^x \frac{v^{n+1}}{(v + 1)^{n+2}} dv = \frac{1}{x^{n+2}} \int_0^x [(v + 1) - v] \frac{v^{n+1}}{(v + 1)^{n+2}} dv \\
= \frac{1}{x^{n+2}} \left[ \int_0^x \frac{v^{n+1}}{(v + 1)^{n+1}} dv - \int_0^x \frac{v^{n+2}}{(v + 1)^{n+2}} dv \right] \\
= \frac{1}{x^{n+2}} \left[ \int_0^x \frac{v^{n+1}}{(v + 1)^{n+1}} dv - \frac{(n+2)}{(n+1)} \int_0^x \frac{v^{n+1}}{(v + 1)^{n+1}} dv + \frac{1}{(n+1)} \frac{v^{n+2}}{(x+1)^{n+1}} \right], \]  \( \text{(A.3)} \)

where we integrated by parts. Using a standard integral representation for \(2F_1\) [9] (p. 1040) leads to the Proposition.
Proposition A1 may be iterated in the first parameter of the $2F_1$ function. Then the following relation may be applied:

$$\int_0^1 \frac{u^{n+1}}{(xu+1)^2} du = \frac{1}{1+x} - \left(\frac{n+1}{n+2}\right) \ 2F_1(1, n+2; n+3; -x). \quad (A.4)$$

The $2F_1$ functions of concern here may be written with one or more terms containing $\ln(x+1)$. One way to see this is the following. We have for the function of (A.4), first integrating by parts,

$$\int_0^1 \frac{u^{n+1}}{(xu+1)^2} du = -\frac{1}{x} \left[(n+1) \int_0^1 \frac{u^n}{xu+1} du + \frac{1}{x+1}\right]$$

$$= -\frac{1}{x} \left[\frac{(n+1)}{x^{n+1}} \int_0^x \frac{v^n}{v+1} dv + \frac{1}{x+1}\right]$$

$$= -\frac{1}{x} \left[\frac{(n+1)}{x^{n+1}} \int_0^x \left[1 - (1-v^n)\right] dv + \frac{1}{x+1}\right]$$

$$= -\frac{1}{x} \left\{\frac{(n+1)}{x^{n+1}} \left[\ln(x+1) - \int_0^x \frac{(1-v^n)}{v+1} dv\right] + \frac{1}{x+1}\right\}. \quad (A.5)$$

For $0 \leq x \leq 1$ we may note the following simple inequality for the integral of (A.5):

$$\int_0^x \frac{(1-v^n)}{v+1} dv \leq \int_0^x (1-v^n) dv \leq x. \quad (A.6)$$
References

[1] M. Abramowitz and I. A. Stegun, Handbook of Mathematical Functions, Washington, National Bureau of Standards (1964).

[2] J. A. Adell and H. Alzer, A monotonicity property of Euler’s gamma function, Publ. Math. Debrecen (2010).

[3] G. E. Andrews, R. Askey, and R. Roy, Special Functions, Cambridge University Press (1999).

[4] C. Berg and H. L. Pedersen, Pick functions related to the gamma function, Rocky Mtn J. Math. 32, 507-525 (2002).

[5] S. Bochner, Harmonic analysis and the theory of probability, Univ. California Press (1955).

[6] M. W. Coffey, Integral and series representations of the digamma and polygamma functions, arXiv:1008.0040v2 (2010).

[7] H. M. Edwards, Riemann’s Zeta Function, Academic Press, New York (1974).

[8] P. J. Grabner, R. F. Tichy, and U. T. Zimmerman, Inequalities for the gamma function with applications to permanents, Discrete Math. 154, 53-62 (1996).

[9] I. S. Gradshteyn and I. M. Ryzhik, Table of Integrals, Series, and Products, Academic Press, New York (1980).
[10] A. Ivić, The Riemann Zeta-Function, Wiley New York (1985).

[11] K. S. Kölblig, The polygamma function $\psi^{(k)}(x)$ for $x = \frac{1}{4}$ and $x = \frac{3}{4}$, J. Comput. Appl. Math. 75, 43-46 (1996).

[12] N. N. Lebedev, Special functions and their applications, Dover Publications (1972).

[13] F. Qi and C.-P. Chen, A complete monotonicity property of the gamma function, J. Math. Analysis Appl. 296, 603-607 (2004).

[14] F. Qi and B.-N. Guo, Some logarithmically completely monotonic functions related to the gamma function, 47, 1283-1297 (2010).

[15] B. Riemann, Über die Anzahl der Primzahlen unter einer gegebenen Größe, Monats. Preuss. Akad. Wiss., 671 (1859-1860).

[16] E. C. Titchmarsh, The Theory of the Riemann Zeta-Function, 2nd ed., Oxford University Press, Oxford (1986).

[17] H. Vogt and J. Voigt, A monotonicity property of the $\Gamma$-function, J. Inequal. Pure Appl. Math. 3, Art. 73 (2002).