A REPRESENTATION FORMULA RELATED TO
SCHRÖDINGER OPERATORS

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Abstract. Let $H = -d^2/dx^2 + V$ be a Schrödinger operator
on the real line, where $V \in L^1 \cap L^2$. We define the perturbed
Fourier transform $\mathcal{F}$ for $H$ and show that $\mathcal{F}$ is an isometry from
the absolute continuous subspace onto $L^2(\mathbb{R})$. This property allows
us to construct a kernel formula for the spectral operator $\varphi(H)$.

Schrödinger operator is a central subject in the mathematical study
of quantum mechanics. Consider the Schrödinger operator
$H = -\Delta + V$ on $\mathbb{R}$, where $\Delta = d^2/dx^2$ and the potential function $V$ is real valued.
In Fourier analysis, it is well-known that a square integrable function
admits an expansion with exponentials as eigenfunctions of $-\Delta$. A
natural conjecture is that an $L^2$ function admits a similar expansion in
terms of “eigenfunctions” of $H$, a perturbation of the Laplacian (see
$[7]$, Ch.XI and the notes), under certain condition on $V$.

The three dimension analogue was proven true by T.Ikebe $[6]$, a
member of Kato’s school, in 1960. Later his result was extended by
Thor to the higher dimension case $[10]$. In one dimension, recent related
results can be found in e.g., Guerin-Holschneider $[5]$, Christ-Kiselev $[4]$ and Benedetto-Zheng $[3]$.

Throughout this paper we assume $V : \mathbb{R} \to \mathbb{R}$ is in $L^1 \cap L^2$. We shall
prove a one-dimensional version of Ikebe’s theorem for $L^2$ functions
(Theorem 1). Theorem 2 presents an integral formula for the kernel of
the spectral operator $\varphi(H)$ for a continuous function $\varphi$ with compact
support. In a sequel to this paper we shall use this explicit formula to
study function spaces associated with $H$ (see $[3]$).

The generalized eigenfunctions $e(x, \xi)$, $\xi \in \mathbb{R}$ of $H$ satisfy

$$(-d^2/dx^2 + V(x))e(x, \xi) = \xi^2 e(x, \xi)$$

in the sense of distributions.

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Definition. The perturbed Fourier transform $\mathcal{F}$ on $L^2$ is given by

$$\mathcal{F}f(\xi) = \lim_{N \to \infty} (2\pi)^{-1/2} \int_{-N}^{N} f(x) e(x, \xi) \, dx,$$

where the convergence is in $L^2$ norm as $N \to \infty$. By Theorem 1, $\mathcal{F}$ is a well-defined isometry from $H_{ac}$ onto $L^2$.

**Theorem 1.** Suppose $V \in L^1 \cap L^2$. Then there exists a family of solutions $e(x, \xi), |\xi| \in [0, \infty) \setminus \mathcal{E}_0, \mathcal{E}_0$ being a bounded closed set of measure zero, to equation (1) with the following properties.

(i) If $f \in L^2$, then there exists an element $\tilde{f} \in L^2$ such that

$$\mathcal{F}f(\xi) = \tilde{f}(\xi) \quad \text{in } L^2.$$

(ii) The adjoint operator $\mathcal{F}^*$ is given by

$$\mathcal{F}^*g = \lim_{N \to \infty} \sum_{i=1}^{N} (2\pi)^{-1/2} \int_{\alpha_i \leq \xi \leq \beta_i} g(\xi) e(x, \xi) \, d\xi$$

in $L^2$, where $[\alpha_i, \beta_i) \subset (0, \infty)$ are a countable collection of disjoint intervals with $[0, \infty) \setminus \mathcal{E}_0$ equal to $\cup_i [\alpha_i, \beta_i)$.

(iii) If $f \in L^2$, then $\|P_{ac}f\|_{L^2} = \|\tilde{f}\|_{L^2}$, where $P_{ac}$ is the projection onto $H_{ac}$, the absolute continuous subspace in $L^2$.

(iv) $\mathcal{F} : L^2 \to L^2$ is a surjection. Moreover, $\mathcal{F}\mathcal{F}^* = Id$ and $\mathcal{F}^*\mathcal{F} = P_{ac}$.

(v) If $f \in D(H)$, then $(Hf)^\sim(\xi) = \xi^2 \tilde{f}(\xi)$ in $L^2$.

Remark 1. The proof is based on the ideas of [6] for 3D. We also use some simplifications as found in Reed and Simon [7] and Simon [8].

Remark 2. If $|e(x, \xi)| \leq C$ a.e. $(x, \xi) \in \mathbb{R}^2$, then we have a “better-looking” form in (ii) of the theorem

$$\mathcal{F}^*g = \lim_{N \to \infty} \sum_{i=1}^{N} (2\pi)^{-1/2} \int_{\alpha_i \leq \xi \leq \beta_i} g(\xi) e(x, \xi) \, d\xi.$$

If $H = \int \lambda dE_{\lambda}$ is the spectral resolution of $H$, define the spectral operator $\varphi(H) := \int \varphi(\lambda) dE_{\lambda}$ by functional calculus. We prove a representation formula for the integral kernel of $\varphi(H)$. 
Let \( \{e_k\}_{k=1}^{\infty} \) be an orthonormal basis in \( \mathcal{H}_p \), the subspace of eigenfunctions in \( L^2 \) for \( H \) and let \( \lambda_k \) be the eigenvalue corresponding to \( e_k \).

**Theorem 2.** Let the operator \( H \) be as in Theorem 1. Suppose \( \varphi : \mathbb{R} \to \mathbb{C} \) is continuous and has a compact support disjoint from \( \mathcal{E}_0^2 := \{ \eta^2 : \eta \in \mathcal{E}_0 \} \). Then for \( f \in L^1 \cap L^2 \)

\[
\varphi(H)f(x) = \int_{-\infty}^{\infty} K(x, y)f(y) \, dy
\]

where \( K = K_{ac} + K_p \),

\[
K_{ac}(x, y) = (2\pi)^{-1} \int_{-\infty}^{\infty} \varphi(\xi^2)e(x, \xi)e(y, \xi) \, d\xi.
\]

and

\[
K_p(x, y) = \sum_k \varphi(\lambda_k)e_k(x)\overline{e_k(y)}.
\]

**Remark 1.** If \( |e(x, \xi)| \leq C \), a.e. \( (x, \xi) \in \mathbb{R}^2 \), then, under the same condition the integral expression (3) is valid for any \( \varphi \in C(\mathbb{R}) \) with compact support.

**Remark 2.** When \( \varphi \) is smooth with rapid decay and \( V \) is compactly supported in \( \mathbb{R}^3 \), a formula of this type appeared in [9] by Tao.

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