Constructions and representation theory of BiHom-post-Lie algebras

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Abstract
The main goal of this paper is to give some construction results of BiHom-post-Lie algebras which are a generalization of both post-Lie-algebras and Hom-post-Lie algebras. They are the algebraic structures behind the weighted $O$-operator of BiHom-Lie algebras. They can be also regarded as the splitting into three parts of the structure of a BiHom-Lie algebra. Moreover we develop the representation theory of BiHom-post-Lie algebras on a vector space $V$. We show that there is naturally an induced representation of its sub-adjacent Lie algebra. We give also all 2-dimensional BiHom-post-Lie algebras. We exhibit in this work some important examples of post-Lie algebras and Hom-post-Lie algebras.

Keywords BiHom-post-Lie algebras · BiHom-Lie algebras · Representations · $O$-operator

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1 Introduction

The notion of post-algebras goes back to Rosenbloom in [22] (1942). An equivalent formulation of the class of post-algebras was given by Rousseau in [23] (1969, 1970) which became a starting point for deep research.

Post-Lie algebras have been introduced by Vallette in 2007 [25] in connection with the homology of partition posets and the study of Koszul operads. However, J. L. Loday
studied pre-Lie algebras and post-Lie algebras within the context of algebraic operad triples, see for more details [18, 19]. In the last decade, many works [5, 7, 11, 12, 14, 26] interested in post-Lie algebra structures, motivated by the importance of pre-Lie algebras in geometry and in connection with generalized Lie algebra derivations.

Recently, Post-Lie algebras which are non-associative algebras played an important role in different areas of pure and applied mathematics. They consist of a vector space $A$ equipped with a Lie bracket $[\cdot, \cdot]$ and a binary operation $\triangleright$ satisfying the following axioms

\[
\begin{align*}
  x \triangleright [y, z] &= [x \triangleright y, z] + [y, x \triangleright z], \\
  [x, y] \triangleright z &= a_{\triangleright}(x, y, z) - a_{\triangleright}(y, x, z).
\end{align*}
\]

If the bracket $[\cdot, \cdot]$ is zero, we have exactly a pre-Lie structure. The varieties of pre- and post-Lie algebras play a crucial role in the definition of any pre and post-algebra through black Manin operads product, see details in [2, 15]. Whereas pre-Lie algebras are intimately associated with euclidean geometry, post-Lie algebras occur naturally in the differential geometry of homogeneous spaces, and are also closely related to Cartan’s method of moving frames. Ebrahimi-Fard et al. [11] studied universal enveloping algebras of post-Lie algebras and the free post-Lie algebra. In [3], Bakayoko studied modules over Hom-post-Lie algebras and give some constructions and various twistings.

Motivated by a categorical study of Hom-algebra and new type of categories, the authors of [13] introduced a generalized algebraic structure dealing with two commuting multiplicative linear maps, called BiHom-algebras including BiHom-associative algebras and BiHom-Lie algebras. When the two linear maps are the same, then BiHom-algebras become Hom-algebras in some cases. Many varieties of algebras are generalized to the BiHom-version, see details in [4, 8, 9, 10, 20].

In the present work, we will be interested to the BiHom version of post-Lie algebras which generalize the classical structures. We extend many important results already known to the BiHom case.

The paper is organized as follows. In Sect. 2, we recall some basic definitions and properties of post-Lie algebras and BiHom-Lie algebras. This part is illustrated by some examples of post-Lie algebras and Hom-post-Lie algebras which are original. In Sect. 3, we recall the notion of BiHom-post-Lie algebras. We introduce the notion of an module $\mathcal{K}$-algebra of BiHom-Lie algebra and we will provide some construction results. Section 4 is devoted establish connections between BiHom-tri-dendriform algebras and BiHom-post-Lie algebras. In Sect. 5, we investigate the notion of an $\mathcal{O}$-operator of weight $\lambda$ to construct a BiHom-post-Lie algebra on a given module $\mathcal{K}$-algebra of a BiHom-Lie algebra. In Sect. 6, we give all 2-dimensional BiHom-post-Lie algebras. We end this work by introducing the representation theory of a BiHom-post-Lie algebra and provide some elementary results.

Throughout this paper, all vector spaces are over a field $\mathcal{K}$ of characteristic zero.

## 2 Preliminaries

In what follows, we recall some concepts and facts used in this paper

**Definition 2.1** A (left) post-Lie algebra $(A, [\cdot, \cdot], \triangleright)$ consists of a Lie algebra $(A, [\cdot, \cdot])$ and a binary product $\triangleright : A \times A \to A$ such that, for all elements $x, y, z \in A$ the following relations hold

\[
\begin{align*}
  x \triangleright [y, z] &= [x \triangleright y, z] + [y, x \triangleright z], \\
  [x, y] \triangleright z &= a_{\triangleright}(x, y, z) - a_{\triangleright}(y, x, z).
\end{align*}
\]
Proposition 2.3 We denote them by

\[ [x,y] \triangleright z = as_p(x, y, z) - as_p(y, x, z), \]

where \( as_p(x, y, z) = x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z. \)

It is clear that a post-Lie algebra with an abelian Lie algebra structure reduces to a pre-Lie algebra. If we define \( L_p : A \rightarrow \text{End}(A) \) by \( L_p(x)y = x \triangleright y \), then by Eq. (2.1), \( L_p(x) \) is a derivation on \((A, [\cdot, \cdot]), \triangleright\).

A morphism \( f : (A, [\cdot, \cdot], \triangleright) \rightarrow (A', [\cdot', \cdot'], \triangleright') \) of post-Lie algebras is a linear map satisfying

\[ f([x, y]) = [f(x), f(y)], \quad f(x \triangleright y) = f(x) \triangleright f(y), \quad \forall x, y \in A. \]

In the sequel, we construct some examples of post-Lie algebras.

Example 2.2 We denote them by \( PL_l \) for \( l = 1, 2, 3 \), where \( l \) is a running index of the two-dimensional post-Lie algebras in the basis \{\( e_1, e_2 \)\} and \( a, b, c \) and \( d \) are parameters in \( \mathbb{K} \) (the unspecified \( \triangleright \) products are zeros).

\[
\begin{align*}
PL_1 \\
\{e_1 \triangleright e_2 = ae_1 + e_2, \\
e_2 \triangleright e_1 = be_1 + \frac{b}{a} e_2, \\
e_2 \triangleright e_2 = -(a^2 + ab)e_1 - (a + b)e_2,
\end{align*}
\]

\[
\begin{align*}
\{e_1, e_2\} = & be_1 + \frac{b}{a} e_2, \\
\{e_1, e_1\} = & e_1 + e_2.
\end{align*}
\]

The following well-known result is a special case of splitting of operads (see [2]).

Proposition 2.3 Let \((A, [\cdot, \cdot], \triangleright)\) be a post-Lie algebra. Then the bracket

\[ \{x, y\} = x \triangleright y - y \triangleright x + [x, y] \]

defines a Lie algebra structure on \( A \).

We denote this algebra by \( A^C \) and it is called the sub-adjacent Lie-algebra of \((A, [\cdot, \cdot], \triangleright)\).

Definition 2.4 [24] A representation of a post-Lie algebra \((A, [\cdot, \cdot], \triangleright)\) on a vector space \( V \) is a tuple \((V, \rho, \mu, \nu)\), such that \((V, \rho)\) is a representation of \((A, [\cdot, \cdot])\) and \( \mu, \nu : A \rightarrow \text{gl}(V) \) are linear maps satisfying

\[ \nu([x, y]) = \rho(x)\nu(y) - \rho(y)\nu(x), \]

where \(\nu([x, y]) = x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z.\)
\[
\rho(x \triangleright y) = \mu(x)\rho(y) - \rho(y)\mu(x), 
\]
(2.5) 

\[
\mu([x,y]) = \mu(x)\mu(y) - \mu(x \triangleright y) - \mu(y \triangleright x), 
\]
(2.6) 

\[
\nu(y)\rho(x) = \mu(x)\nu(y) - \nu(y)\mu(x) - \nu(x \triangleright y) + \nu(y)v(x), 
\]
(2.7) 

for any \( x, y \in A \).

Let \((A, [\cdot, \cdot], \triangleright)\) be a post-Lie algebra and \((V, \rho, \mu, \nu)\) a representation of \((A, [\cdot, \cdot], \triangleright)\). By identity (2.6), we deduce that \((V, \mu)\) is a representation of the sub-adjacent Lie algebra \((A, [\cdot, \cdot], \triangleright)\). Further, it is obvious that \((A, ad, L_\triangleright)\) is a representation of \((A, [\cdot, \cdot], \triangleright)\) which is called the adjoint representation, where \(ad(x)(y) = [x, y]\) and \(L_\triangleright(x)(y) = R_\triangleright(y)(x) = x \triangleright y\).

**Proposition 2.5** [24] Let \((V, \rho, \mu, \nu)\) be a representation of a post-Lie algebra \((A, [\cdot, \cdot], \triangleright)\). Then \((V, \rho + \mu - \nu)\) is a representation of the sub-adjacent Lie algebra \((A, [\cdot, \cdot])\) of \((A, [\cdot, \cdot], \triangleright)\).

Recall that (see [16]), a Hom-Lie algebra is a tuple \((A, [\cdot, \cdot], \alpha)\) consisting of a linear space \(A\), a skew-symmetric linear map \([\cdot, \cdot] : \otimes^2 A \rightarrow A\) and a linear map \(\alpha : A \rightarrow A\), satisfying

\[
\text{O}_{x,y,z} \in A [\alpha(x), [y,z]] = 0 
\]
for all \(x,y,z \in A\). It is called multiplicative if \(\alpha [\cdot, \cdot] = [\cdot, \cdot] \circ (\alpha \otimes \alpha)\).

**Definition 2.6** [3] A (left) Hom-post-Lie algebra \((A, [\cdot, \cdot], \triangleright, \alpha)\) consists of a Hom-Lie algebra \((A, [\cdot, \cdot], \alpha)\) and a binary product \(\triangleright : A \times A \rightarrow A\) such that, for all elements \(x, y, z \in A\) the following relations hold

\[
\alpha(x) \triangleright [y,z] = [x \triangleright y, \alpha(z)] + [\alpha(y), x \triangleright z], 
\]
(2.8) 

\[
[x,y] \triangleright \alpha(z) = a_{\alpha}^o(x,y,z) - a_{\alpha}^o(x,y,z), 
\]
(2.9) 

where \(a_{\alpha}^o(x,y,z) = \alpha(x) \triangleright (y \triangleright z) - (x \triangleright y) \triangleright \alpha(z)\).

A Hom-post-Lie algebra \((A, [\cdot, \cdot], \triangleright, \alpha)\) is called multiplicative if

\[
\alpha [\cdot, \cdot] = [\cdot, \cdot] \circ (\alpha \otimes \alpha) \quad \text{and} \quad \alpha \triangleright = \triangleright \circ (\alpha \otimes \alpha).
\]

In the following, we construct examples of Hom-post-Lie algebras.

**Example 2.7** We denote by \(HPL^i_\mathbb{K}\) the Hom-post-Lie algebras, where \(i\) indicates, the item of the two-dimensional Hom-post-Lie algebras in the basis \(\{e_1, e_2\}\), and \(a_i\) for \(i = 1, \ldots, 6\) are parameters in \(\mathbb{K}\). The homomorphism \(\alpha\) is defined by the matrix \(\begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}\). Note that the unspecified \(\triangleright\) products are zeros.
\begin{equation}
\begin{aligned}
HPL_1^2 \\
\alpha(e_1) &= e_1, \\
e_1 &\triangleright e_1 = a_1 e_1, \\
e_2 &\triangleright e_1 = e_2, \\
e_2 &\triangleright e_2 = e_2, \\
[e_1, e_2] &= a_2 e_2.
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
HPL_2^2 \\
\alpha(e_2) &= b_{22} e_2, \\
e_1 &\triangleright e_1 = e_1, \\
e_1 &\triangleright e_2 = a_3 e_1, \\
e_2 &\triangleright e_1 = e_1, \\
[e_1, e_2] &= a_4 e_1.
\end{aligned}
\end{equation}

\begin{equation}
\begin{aligned}
HPL_3^2 \\
\alpha(e_1) &= b_{11} e_1, \\
e_1 &\triangleright e_2 = e_2, \\
e_2 &\triangleright e_1 = a_5 e_2, \\
e_2 &\triangleright e_2 = e_2, \\
[e_1, e_2] &= a_6 e_2.
\end{aligned}
\end{equation}

**Definition 2.8** [13] A BiHom-Lie algebra is a tuple \((A, [\cdot, \cdot], \alpha, \beta)\) consisting of a linear space \(A\), a linear map \([\cdot, \cdot] : \otimes^2 A \to \) \(A\) and two linear map \(\alpha, \beta : A \to A\), satisfying
\begin{align}
\alpha \circ \beta &= \beta \circ \alpha, \\
\alpha([x, y]) &= [\alpha(x), \alpha(y)], \quad \beta([x, y]) = [\beta(x), \beta(y)], \\
[\beta(x), \alpha(y)] &= -[\beta(y), \alpha(x)], \\
\bigcup_{x,y,z \in A} [\beta^2(x), [\beta(y), \alpha(z)]] &= 0,
\end{align}
for any \(x, y, z \in A\). A BiHom-Lie algebra \((A, [\cdot, \cdot], \alpha, \beta)\) is called regular if \(\alpha\) and \(\beta\) are bijective.

**Proposition 2.9** [13] Let \((A, [\cdot, \cdot])\) be a Lie algebra and \(\alpha, \beta\) two commuting morphisms on \(A\). Then \((A, [\cdot, \cdot])_{\alpha, \beta} = [\cdot, \cdot] \circ (\alpha \otimes \beta), \alpha, \beta)\) is a BiHom-Lie algebra.

**Definition 2.10** [13] A representation of a BiHom-Lie algebra \((A, [\cdot, \cdot], \alpha, \beta)\) on a vector space \(V\) with respect to commuting linear maps \(\phi, \psi : V \to V\) is a linear map \(\rho : A \to \text{gl}(V)\), such that for all \(x, y \in A\), the following equalities are satisfied
\begin{align}
\rho(\alpha(x)) \circ \phi &= \phi \circ \rho(x), \\
\rho(\beta(x)) \circ \psi &= \psi \circ \rho(x), \\
\rho([\beta(x), y]) \circ \psi &= \rho(\alpha(\beta(x)) \circ \rho(y) - \rho(\beta(y)) \circ \rho(\alpha(x))).
\end{align}

We denote such a representation by \((V, \rho, \phi, \psi)\). For all \(x \in A\), we define \(ad_x : A \to A\) by
\[ad_x(y) = [x, y], \quad \forall y \in A.\]
Then \( ad : A \rightarrow gl(A) \) is a representation of the BiHom-Lie algebra \((A, [\cdot,\cdot], \alpha, \beta)\) on \(A\) with respect to \(\alpha\) and \(\beta\), which is called the adjoint representation.

**Lemma 2.11** [13] Let \((A, [\cdot,\cdot], \alpha, \beta)\) be a regular BiHom-Lie algebra, \((V, \phi, \psi)\) a vector space with two commuting bijective linear transformations and \(\rho : A \rightarrow gl(V)\) a linear map. Then \((V, \rho, \phi, \psi)\) is a representation of \((A, [\cdot,\cdot], \alpha, \beta)\) if and only if \((A \oplus V, [\cdot,\cdot]_\rho, \alpha + \phi, \beta + \psi)\) is a BiHom-Lie algebra, where \([\cdot,\cdot]_\rho\), \(\alpha + \phi\) and \(\beta + \psi\) are defined by

\[
[x + u, y + v]_\rho = [x, y] + \rho(x)v - \rho(\alpha^{-1}(\beta(y))\phi\psi^{-1}u,
(\alpha + \phi)(x + u) = \alpha(x) + \phi(u),
(\beta + \psi)(x + u) = \beta(x) + \psi(u),
\]

for all \(x, y \in A, u, v \in V\).

Let \((A, [\cdot,\cdot])\) be a Lie algebra, \(\alpha, \beta\) two commuting morphisms on \(A\). Consider a representation \((V, \rho)\) of \(A\) and two commuting linear maps \(\phi, \psi : V \rightarrow V\) satisfying

\[
\rho(\alpha(x))\circ \phi = \phi \circ \rho(x), \quad \rho(\beta(x))\circ \psi = \psi \circ \rho(x).
\]

Define a linear map \(\tilde{\rho} : A \rightarrow gl(V)\) by \(\tilde{\rho}(x)(v) = \rho(\alpha(x))(\psi(v))\).

**Proposition 2.12** With the above notation \((V, \tilde{\rho}, \phi, \psi)\) is a representation of \((A, [\cdot,\cdot]_{\alpha,\beta}, \alpha, \beta)\).

**Proof** Using Lemma 2.11 and Proposition 2.9, we can check that \((V, \tilde{\rho}, \phi, \psi)\) is a representation of \((A, [\cdot,\cdot]_{\alpha,\beta}, \alpha, \beta)\).

### 3 BiHom-post-Lie algebras

In this section, we recall the notion of BiHom-post-Lie algebras (see [20]). We introduce the notion of an \(A\)-module \(\mathbb{K}\)-algebra of BiHom-Lie algebra and we will provide some construction results.

**Definition 3.1** Let \((A, [\cdot,\cdot], \alpha, \beta)\) be BiHom-Lie algebra. Let \((V, [\cdot,\cdot]_V, \phi, \psi)\) be a BiHom-Lie algebra and \(\rho : A \rightarrow gl(V)\) be a linear map. We say that \((V, [\cdot,\cdot]_V, \rho, \phi, \psi, .)\) is an \(A\)-module \(\mathbb{K}\)-algebra if \((V, \rho, \phi, \psi)\) is a representation of \((A, [\cdot,\cdot], \alpha, \beta)\) such that the following compatibility condition holds

\[
\rho(\alpha \beta(x))[u, v]_V = [\rho(\beta(x))u, \psi(v)]_V + [\psi(u), \rho(\alpha(x))v]_V, \tag{3.12}
\]

for all \(x \in A, u, v \in V\). If \(\phi\) and \(\psi\) are bijective, then \((V, [\cdot,\cdot]_V, \rho, \phi, \psi)\) is called regular \(A\)-module \(\mathbb{K}\)-algebra.

It is known that \((A, ad, \alpha, \beta)\) is a representation of \(A\) called the adjoint representation. Then \((A, [\cdot,\cdot], ad, \alpha, \beta)\) is an \(A\)-module \(\mathbb{K}\)-algebra.
Proposition 3.2 Let \((A, [\cdot, \cdot], \alpha, \beta)\) and \((V, [\cdot, \cdot]_V, \phi, \psi)\) be two regular BiHom-Lie algebras and \(\rho : A \to \mathfrak{gl}(V)\) be a linear map. Then \((V, [\cdot, \cdot]_V, \rho, \phi, \psi)\) is a regular A-module \(\kappa\)-algebra if and only if the direct sum \(A \oplus V\) of vector spaces is turned into a BiHom-Lie algebra (the semi-direct sum) by defining a bracket on \(A \oplus V\) by
\[
[x + u, y + v]_\rho = [x, y] + \rho(x)v - \rho(\alpha^{-1} \beta(y))\phi\psi^{-1}u + [u, v]_V,
\]
\[
(\alpha + \phi)(x + u) = \alpha(x) + \phi(u),
\]
\[
(\beta + \psi)(x + u) = \beta(x) + \psi(u),
\]
for all \(x, y \in A, u, v \in V\).

We denote this algebra by \(A \lhd^\alpha,\beta_{\rho,\phi,\psi} V\) (or simply \(A \lhd V\)).

Proof Let \(x, y, z \in A\) and \(a, b, c \in V\)
\[
\mathcal{O}_{x+a,y+b,z+c} = [(\beta + \psi)^2(x + a), [(\beta + \psi)(y + b), (\alpha + \phi)(z + c)]_\rho = [\beta^2(x), [\beta(y), \alpha(z)]] + \rho(\beta^2(x))\rho(\beta(y))\phi(c) - \rho(\beta^2(x))\rho(\beta(z))\phi(b) + \rho(\beta^2(x))\rho(\beta(c))_V - \rho([\alpha^{-1} \beta^2(y), \beta(z)])\phi\psi(a) + [\psi^2(a), \rho(\beta(y))\phi(c)]_V - [\psi^2(a), [\psi(b), \phi(c)]_V] = 0,
\]
if and only if \((V, \rho, \phi, \psi)\) is a representation on \(A\) and satisfies Eq. (3.12).

Definition 3.3 [20] A (left) BiHom-post-Lie algebra is a tuple \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) where \((A, [\cdot, \cdot], \alpha, \beta)\) is a BiHom-Lie algebra such that for any \(x, y, z \in A\)
\[
\alpha(x \triangleright y) = \alpha(x) \triangleright \alpha(y), \quad \beta(x \triangleright y) = \beta(x) \triangleright \beta(y), \quad (3.13)
\]
\[
\alpha \beta(x) \triangleright [y, z] = [\beta(x) \triangleright y, \beta(z)] + [\beta(y), \alpha(x) \triangleright z], \quad (3.14)
\]
\[
[\beta(x), \alpha(y)] \triangleright \beta(z) = a_{a,\beta}(\beta(x), \alpha(y), z) - a_{a,\beta}(\beta(y), \alpha(x), z), \quad (3.15)
\]
where \(a_{a,\beta}(x, y, z) = \alpha(x) \triangleright (y \triangleright z) - (x \triangleright y) \triangleright \beta(z)\). We call \([\cdot, \cdot]\) the torsion and \(\triangleright\) the connection of the (left) BiHom-post-Lie algebra.

If \(\alpha = \beta = \text{id}\), then we recover a (left) post-Lie algebra. In addition, if \([\cdot, \cdot] = 0\), then \(A\) reduces to a BiHom-pre-Lie algebra.

From now on, we will write BiHom-post-Lie algebra instead of (left) BiHom-post-Lie algebra. A BiHom-post-Lie algebra \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) is said to be regular if \(\alpha\) and \(\beta\) are bijective.

Example 3.4 Let \((A, [\cdot, \cdot], \triangleright)\) be a post-Lie algebra and \(\alpha, \beta\) be two commuting morphisms on \(A\). Then \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) is a BiHom-post-Lie algebra, Where, for \(x, y \in A\),
\[
[x, y]_{a,\beta} = [\alpha(x), \beta(y)], \quad x \triangleright_{a,\beta} y = a(x) \triangleright \beta(y), \quad (3.16)
\]

Example 3.5 Let \((A, [\cdot, \cdot], \alpha, \beta)\) be a BiHom algebra. Then \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) is a BiHom-post-Lie algebra, where
\[ x \triangleright y = [y, x], \; \forall x, y \in A. \]

In [17], the authors constructed a Lie algebra from an old post-Lie algebra. In the following proposition, we give the BiHom-version.

**Proposition 3.6** Let \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) be a regular BiHom-post-Lie algebra. Then the bracket

\[ [x, y] = x \triangleright y - \alpha^{-1} \beta(y) \triangleright \alpha \beta^{-1}(x) + [x, y] \tag{3.17} \]

defines a BiHom-Lie algebra structure on \(A\). We denote this algebra by \(A^C\) and we call it the sub-adjacent BiHom-Lie algebra of \(A\).

**Proof** The BiHom-skew symmetry is obvious. We will just check the BiHom-Jacobi identity. For all \(x, y, z \in A\), we have

\[
\{ \beta^2(x), \{ \beta(y), \alpha(z) \} \} = \beta^2(x) \triangleright (\beta(y) \triangleright \alpha(z)) - (\alpha^{-1} \beta^2(y)) \triangleright \alpha \beta(z) \triangleright \alpha \beta(x) \\
+ [\beta^2(x), \beta(y) \triangleright \alpha(z)] - \beta^2(x) \triangleright (\beta(z) \triangleright \alpha(y)) \\
+ (\alpha^{-1} \beta^2(z) \triangleright \beta(y)) \triangleright \alpha \beta(x) - [\beta^2(x), \beta(z) \triangleright \alpha(y)] \\
+ \beta^2(x) \triangleright [\beta(y), \alpha(z)] + [\alpha^{-1} \beta^2(y), \beta(z)] \triangleright \alpha \beta(x) \\
+ [\beta^2(x), [\beta(y), \alpha(z)]]).
\]

Similarly,

\[
\{ \beta^2(y), \{ \beta(z), \alpha(x) \} \} = \beta^2(y) \triangleright (\beta(z) \triangleright \alpha(x)) - (\alpha^{-1} \beta^2(z)) \triangleright \beta(x) \triangleright \alpha \beta(y) \\
+ [\beta^2(y), \beta(z) \triangleright \alpha(x)] - \beta^2(y) \triangleright (\beta(x) \triangleright \alpha(z)) \\
+ (\alpha^{-1} \beta^2(x) \triangleright \beta(z)) \triangleright \alpha \beta(y) - [\beta^2(y), \beta(x) \triangleright \alpha(z)] \\
+ \beta^2(y) \triangleright [\beta(z), \alpha(x)] + [\alpha^{-1} \beta^2(z), \beta(x)] \triangleright \alpha \beta(y) \\
+ [\beta^2(y), [\beta(z), \alpha(x)]]).
\]

By the BiHom-Jacobi identity of BiHom-Lie algebras and (??) and (3.13) we have

\[
\{ \beta^2(x), \{ \beta(y), \alpha(z) \} \} + \{ \beta^2(y), \{ \beta(z), \alpha(x) \} \} + \{ \beta^2(z), \{ \beta(x), \alpha(y) \} \} = 0.
\]

**Remark 3.7** Given a BiHom-post-Lie algebra \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\). Suppose that \(\triangleright\) is commutative in the BiHom-sense, that is \(\beta(x) \triangleright \alpha(y) = \beta(y) \triangleright \alpha(x)\). Then the two Lie brackets \([\cdot, \cdot]\) and \{\cdot, \cdot\} coincide.

Note that, a regular BiHom-algebra \((A, \cdot, \alpha, \beta)\) is called BiHom-Lie algebra admissible if the tuple \((A, [\cdot, \cdot], \alpha, \beta)\) is a BiHom-Lie algebra, where \([x, y] = x \cdot y - \alpha^{-1} \beta(y) \cdot \alpha \beta^{-1}(x)\).
Exactly as in the classical case (see [17]), a regular BiHom-post-Lie algebra gives rise to an BiHom-Lie admissible algebra.

**Corollary 3.8** Let \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) be a regular BiHom-post-Lie algebra. Define the product \(\circ\) as

\[
x \circ y = x \triangleright y + \frac{1}{2} [x, y], \quad \forall x, y \in A. \tag{3.18}
\]

Then \((A, \circ, \alpha, \beta)\) is an BiHom-Lie admissible algebra.

**Proposition 3.9** Let \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) be a regular BiHom-post-Lie algebra. Then \((A, L_\triangleright, \alpha, \beta)\) is a representation of \((A, [\cdot, \cdot], \alpha, \beta)\), where the bracket \([\cdot, \cdot]\) is defined by the identity (3.17).

**Proof** It is obvious to check that \(L_\triangleright (\alpha(x)) \circ \alpha = \alpha \circ L_\triangleright (x)\) and \(L_\triangleright (\beta(x)) \circ \beta = \beta \circ L_\triangleright (x)\). To prove (2.11), we compute as follows. Let \(x, y, z \in A\). Then

\[
L_\triangleright \left( \{ \beta(x), y \} \right) \beta(z) = \{ \beta(x), y \} \triangleright \beta(z)
\]

\[
= (\beta(x) \triangleright y) \triangleright \beta - (\alpha^{-1} \beta(y) \triangleright \alpha(x)) \triangleright \beta + [\beta(x, y) \triangleright \beta(z)]
\]

\[
= (\beta(x) \triangleright y) \triangleright \beta(z) + (\alpha^{-1} \beta(y) \triangleright \alpha(x)) \triangleright \beta(z) + [\alpha \beta(x) \triangleright (y \triangleright z)]
\]

\[
= (\beta(x) \triangleright y) \triangleright \beta(z) - \beta(y) \triangleright (\alpha(x) \triangleright z) + (\alpha^{-1} \beta(y) \triangleright \alpha(x)) \triangleright \beta(z)
\]

\[
= \alpha \beta(x) \triangleright (y \triangleright z) - \beta(y) \triangleright (\alpha(x) \triangleright z)
\]

and the proof is finished.

By a modification of the product \(\triangleright\) in \(A\), we obtain another Bihom-post-Lie algebra (see [17] for the classical case).

**Proposition 3.10** Let \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) be a regular BiHom-post-Lie algebra. Then \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) where

\[
x \triangleright y = x \triangleright y + [x, y], \tag{3.19}
\]

is also a BiHom-post-Lie algebra. In addition \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) and \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) have the same sub-adjacent BiHom-Lie algebra \(A^C\).

**Proof** For all \(x, y, z \in A\), we check that \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) is BiHom-post-Lie algebras. In fact we have

\[
-\alpha \beta(x) \triangleright [y, z] = - (\alpha \beta(x) \triangleright [y, z] - [\alpha \beta(x), [y, z]])
\]

\[
= - (\alpha \beta(x) \triangleright [y, z] - [\beta^2 (\alpha \beta^{-1}(x)), [\beta (\beta^{-1}(y)), \alpha (\alpha^{-1}(z))]])
\]

\[
- [\beta(x) \triangleright y, \beta(z)] = - [\beta(x) \triangleright y + [\beta(x, y), \beta(z)]
\]

\[
= - [\beta(x) \triangleright y, \beta(z)] - [\alpha^{-1} \beta^2 (\beta^{-1}(z)), [\alpha (x), \alpha \beta^{-1}(y)]]
\]

and
\[-[\beta(y), \alpha(x)] \triangleright z = -[\beta(y), \alpha(x) \triangleright z + [\alpha(x), z]]
\]
\[-[\beta(y), \alpha(x) \triangleright z] + [\beta^2(\beta^{-1}(y)), [\beta \alpha^{-1}(z), \alpha(\alpha \beta^{-1}(x))]].\]

Then we obtain

\[-\alpha \beta(x) \triangleright [y, z] + [\beta(x) \triangleright y, \beta(z)] + [\beta(y), \alpha(x) \triangleright z]
\]
\[-\alpha \beta(x) \triangleright [y, z] - [\beta(x) \triangleright y, \beta(z)] - [\beta(y), \alpha(x) \triangleright z]
\]
\[-[\beta^2(\alpha \beta^{-1}(x)), [\beta(\beta^{-1}(y)), \alpha(\alpha^{-1}(z))]])
\]
\+[\alpha^{-1} \beta^2(\beta^{-1}(z)), [\alpha(x), \alpha \beta^{-1}(y))]
\]
\+[\beta^2(\beta^{-1}(y)), [\beta \alpha^{-1}(z), \alpha(\alpha \beta^{-1}(x)))] = 0.

Hence, the first condition hold using (??) and BiHom-Jacobi-Identity. To check the second condition, we have

\[-[\beta(x), \alpha(y)] \triangleright \beta(z) - a\triangleright (\beta(x), \alpha(y), z) + a\triangleright (\beta(x), \alpha(y), z)
\]
\[-[\beta(x), \alpha(y)] \triangleright \beta(z) + \beta(x) \triangleright (\alpha(y) \triangleright z)
\]
\[-(\beta(x) \triangleright \alpha(y)) \triangleright \beta(z) + \alpha \beta(x) \triangleright [\alpha(y), z] + [\alpha \beta(x), \alpha(y) \triangleright z]
\]
\+[\alpha(\alpha(x), z) - [\alpha \beta(x), [\alpha(y), \alpha(x) \triangleright z] - [\alpha \beta(x), \alpha(x), z]]
\]
\+[\beta(y), \alpha(x) \triangleright \beta(z) + [\beta(y) \triangleright \alpha(x), \beta(z)] + [[\beta(y), \alpha(x), \beta(z)]
\]
\[= 0,
\]
where \(a\triangleright (x, y, z) = x \triangleright (y \triangleright z) - (x \triangleright y) \triangleright z.\)

**Theorem 3.11** Let \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) be a regular BiHom-post-Lie algebra. Define the double bracket \([\cdot, \cdot]\) on \(A \times A\) by

\[
\left\langle (a, x), (b, y) \right\rangle = (a \triangleright b - \alpha^{-1} \beta(b) \triangleright \alpha \beta^{-1}(a) + [a, b],
\right.
\]
\[
\quad a \triangleright y - \alpha^{-1} \beta(b) \triangleright \alpha \beta^{-1}(x) + [x, y])
\]

(3.20)

for all \(a, b, x, y \in A\). Then \((A \times A, \left\langle \cdot, \cdot \right\rangle, \alpha^{x^2}, \beta^{x^2})\) is a BiHom-Lie algebra.

**Proof** Let \(x, y, z, a, b, c \in A\). It’s obvious that

\[
\left\langle (\beta(a), \beta(x)), (\alpha(b), \alpha(y)) \right\rangle = -\left\langle (\beta(b), \beta(y)), (\alpha(a), \alpha(x)) \right\rangle.
\]

On the other hand, we have

\[
\bigcap_{(a, x), (b, y), (c, z)} \left\langle (\beta^2(a), \beta^2(x)), \left\langle (\beta(b), \beta(y)), (\alpha(c), \alpha(z)) \right\rangle \right\rangle
\]
\[
= \bigcap_{(a, b, c)} \left\langle \beta^2(a), \{\beta(b), \alpha(c)\} \right\rangle
\]
\[
\bigcap_{(a, x), (b, y), (c, z)} \beta^2(a) \triangleright (\beta(b) \triangleright \alpha(z)) - \beta^2(a) \triangleright (\beta(c) \triangleright \alpha(y)) + \beta^2(a) \triangleright [\beta(y), \alpha(z)]
\]
\[
(\beta(b) \triangleright \alpha(c)) \triangleright \alpha \beta(x) - (\alpha(c) \triangleright \alpha(b)) \triangleright \alpha \beta(x) + [\beta(b), \alpha(c)] \triangleright \alpha \beta(x)
\]
\[
+ [\beta^2(x), \beta(b) \triangleright \alpha(z)] - [\beta^2(x), \beta(c) \triangleright \alpha(y)] + [\beta^2(x), [\beta(y), \alpha(z)]]
\]
\[(0, 0),
\]
where the bracket \{\cdot, \cdot\} is defined in (3.17). Therefore \((A \times A, [\cdot, \cdot], \alpha^{x_2}, \beta^{x_2})\) is a BiHom-Lie algebra.

It is well known that BiHom-assciative, BiHom-pre-Lie and BiHom-Novikov algebras are BiHom-Lie admissible algebras. Another class of BiHom-Lie admissible algebras is the variety of BiHom-LR algebras where the ordinary version is given in [6]. A BiHom-LR algebra is a tuple \((A, \cdot, \alpha, \beta)\) satisfying the following conditions

\[(x \cdot y) \cdot a(z) = (x \cdot z) \cdot a(y), \quad (3.21)\]

\[\beta(x) \cdot (y \cdot z) = \beta(y) \cdot (x \cdot z), \quad \forall x, y, z \in A. \quad (3.22)\]

**Proposition 3.12** Let \((A, \cdot, \alpha, \beta)\) be a regular BiHom-LR algebra. Then \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) is a BiHom-post-Lie algebra, where

\[x \triangleright y = -x \cdot y \quad \text{and} \quad [x, y] = x \cdot y - \alpha^{-1} \beta(y) \cdot \alpha \beta^{-1}(x), \quad \forall x, y \in A. \quad (3.23)\]

**Proof** The BiHom-skew symmetry is obvious. We will check the BiHom-Jacobi identity. For all \(x, y, z \in A\), we have

\[\mathcal{O}_{x,y,z} [\beta^2(x), [\beta(y), a(z)]] = \beta^2(x) \cdot (\beta(y) \cdot a(z)) - (\alpha^{-1} \beta^2(y)) \cdot a \beta(x)
- \beta^2(x) \cdot (\beta(z) \cdot a(y)) + (\alpha^{-1} \beta^2(z)) \cdot \alpha \beta(y)
+ \beta^2(y) \cdot (\beta(z) \cdot a(x)) - (\alpha^{-1} \beta^2(z)) \cdot \beta(x) \cdot \alpha \beta(y)
- \beta^2(y) \cdot (\beta(x) \cdot a(z)) + (\alpha^{-1} \beta^2(x)) \cdot \beta(y) \cdot \alpha \beta(z)
+ \beta^2(z) \cdot (\beta(x) \cdot a(y)) - (\alpha^{-1} \beta^2(x)) \cdot \beta(y) \cdot \alpha \beta(z)
- \beta^2(z) \cdot (\beta(y) \cdot a(x)) + (\alpha^{-1} \beta^2(y)) \cdot \beta(x) \cdot \alpha \beta(z).\]

By the identities (3.21) and (3.23) of BiHom-LR algebra, we have

\[\mathcal{O}_{x,y,z} [\beta^2(x), [\beta(y), a(z)]] = 0.\]

Similarly, we have

\[\alpha \beta(x) \triangleright [y, z] - [\beta(y), \alpha(x) \triangleright z] - [\beta(x) \triangleright y, \beta(z)]
= - (\alpha \beta(x) \cdot [y, z] - [\beta(y), \alpha(x) \cdot z] - [\beta(x) \cdot y, \beta(z)])
= - (\alpha \beta(x) \cdot (y \cdot z) - \alpha \beta(x) \cdot (\alpha^{-1} \beta^2(z)) \cdot \alpha \beta^{-1}(y)) - \beta(y) \cdot (\alpha(x) \cdot z)
+ (\beta(x) \cdot \alpha^{-1} \beta^2(z)) \cdot \alpha(y) - (\beta(x) \cdot y) \cdot \beta(z) + \alpha^{-1} \beta^2(z) \cdot (\alpha(x) \cdot \alpha \beta^{-1}(y)))
= 0.\]

**4 From BiHom-tri-dendriform to BiHom-post-Lie algebras**

In this section, we recall the definition of BiHom-tri-dendriform algebras introduced in [21] which is generalisation of tri-dendriform algebras given in [19]. Then we establish that any BiHom-tri-dendriform algebra give rise to a BiHom-post-Lie algebra.
Definition 4.1 [21] A BiHom-tri-dendriform algebra is a 6-tuple \((A, <, >, \cdot, \alpha, \beta)\), where \(A\) is a linear space and \(<, >, \cdot : A \otimes A \to A\) and \(\alpha, \beta : A \to A\) are linear maps satisfying

\[
\alpha(x \cdot y) = \alpha(x) \cdot \alpha(y), \quad \alpha(x > y) = \alpha(x) > \alpha(y), \quad \alpha(x < y) = \alpha(x) < \alpha(y),
\]

\[
\beta(x \cdot y) = \beta(x) \cdot \beta(y), \quad \beta(x > y) = \beta(x) > \beta(y), \quad \beta(x < y) = \beta(x) < \beta(y),
\]

\[
(x < y) < \beta(z) = \alpha(x) < (y < z + y \cdot z),
\]

\[
(x > y) < \beta(z) = \alpha(x) > (y < z),
\]

\[
(\alpha(x), (y > z) = (x < y + x > y + x \cdot y) > \beta(z),
\]

\[
\alpha(x) \cdot (y > z) = (x < y) \cdot \beta(z),
\]

\[
\alpha(x) \cdot (y \cdot z) = (x > y) \cdot \beta(z),
\]

\[
\alpha(x) \cdot (y < z) = (x \cdot y) < \beta(z),
\]

\[
\alpha(x) \cdot (y \cdot z) = (x \cdot y) \cdot \beta(z),
\]

for all \(x, y, z \in A\). We call \(\alpha\) and \(\beta\) (in this order) the structure maps of \(A\).

A morphism \(f : (A, <, >, \cdot, \alpha, \beta) \to (A', <', >', \cdot', \alpha', \beta')\) of BiHom-tri-dendriform algebras is a linear map \(f : A \to A'\) satisfying \(f(x < y) = f(x) <' f(y), f(x > y) = f(x) >' f(y)\) and \(f(x \cdot y) = f(x) \cdot' f(y)\), for all \(x, y \in A\), as well as \(f \circ \alpha = \alpha' \circ f\) and \(f \circ \beta = \beta' \circ f\).

Example 4.2 Let \(A = \langle e_1, e_2 \rangle\) be a two dimensional vector space. Given the multiplications

\[
e_2 < e_2 = e_2 > e_2 = ae_1, \quad e_2 \cdot e_2 = -ae_1,
\]

where the unspecified products are zeros. The linear maps \(\alpha, \beta : A \to A\) are defined by

\[
\alpha(e_1) = e_1, \quad \alpha(e_2) = e_1 + e_2,
\]

\[
\beta(e_1) = e_1, \quad \beta(e_2) = 2e_1 + e_2.
\]

Then \((A, <, >, \cdot, \alpha, \beta)\) is a regular BiHom-tri-dendriform algebra.

The following result generalizes Proposition 5.13 given in [1].

Proposition 4.3 Let \((A, <, >, \cdot, \alpha, \beta)\) be a regular BiHom-tri-dendriform algebra. Then \((A, [\cdot, \cdot], >, \alpha, \beta)\) is a BiHom-post-Lie algebra, where

\[
[x, y] = x \cdot y - \alpha^{-1} \beta(y) \cdot \alpha \beta^{-1}(x),
\]

\[
x > y = x > y - \alpha^{-1} \beta(y) < \alpha \beta^{-1}(x)
\]
for any $x, y \in A$.

**Proof** For $x, y, z \in A$ we prove that

$$\alpha \beta(x) \triangleright [y, z] = [\beta(y), (\alpha(x) \triangleright z)] + [\beta(x) \triangleright y, \beta(z)]$$

We compute:

$$\alpha \beta(x) \triangleright [y, z] = \alpha \beta(x) \triangleright (y \cdot z) - \alpha \beta(x) \triangleright (\alpha^{-1} \beta(z) \cdot \alpha^{-1}(y))$$

$$= \alpha \beta(x) \triangleright (y \cdot z) - (\alpha^{-1} \beta(y) \cdot \alpha^{-1}(z)) - \alpha^2(x)$$

$$- \alpha \beta(x) \triangleright (\alpha^{-1} \beta(z) \cdot \alpha^{-1}(y)) + (\alpha^{-2} \beta(z) \cdot y) < \alpha^2(x).$$

Then

$$[\beta(y), (\alpha(x) \triangleright z)] = [\beta(y), \alpha(x) \triangleright z] - [\beta(y), \alpha^{-1}(\beta(z)) \cdot \alpha^2(x)]$$

$$= \beta(y) \cdot (\alpha(x) > z) - \beta(x) > \alpha^{-1}(\beta(z)) \cdot \alpha(y)$$

$$- \beta(y) \cdot (\alpha^{-1}(\beta(z)) < \alpha^2(x)) + (\alpha^{-2}(\beta(z) < \alpha(x)) \cdot \alpha(y).$$

Similarly, we have

$$[\beta(x) \triangleright y, \beta(z)] = [\beta(x) > y, \beta(z)] - [\alpha^{-1}(\beta(y) < \alpha(x), \beta(z))$$

$$= \beta(x) > y \cdot \beta(z) - \alpha^{-1}(\beta(y) \cdot \alpha^{-2}(\beta(z) < \alpha^{-1}(\beta(z))$$

$$- (\alpha^{-1}(\beta(y) < \alpha(x)) \cdot \beta(z) + \alpha^{-1}(\beta(z) \cdot y) < \alpha^{-2}(\beta(z))$$

and the equality hold applying the relations (4.31) and (4.32) from Definition 4.1. Similar, we can proof the second assertion by using (4.26)–(4.29).

It is easy to see that Eqs. (3.6), (4.34) and (4.35) fit into the commutative diagram

\begin{center}
\begin{tikzcd}
\text{BiHom-tri-dendriform alg.} & \text{BiHom-associative alg.} \\
[x, y] = x \cdot \alpha^{-1}(\beta(y) \cdot \alpha^{-1}(x)) & x \triangleright y = x \cdot \alpha^{-1}(\beta(y) < \alpha^{-1}(x))
\end{tikzcd}
\end{center}

\begin{center}
\begin{tikzcd}
\text{BiHom-post-Lie alg.} & \text{BiHom-Lie alg.}
\end{tikzcd}
\end{center}

When the operation $\cdot$ of the BiHom-tri-dendriform algebra and the bracket $[\cdot, \cdot]$ of the BiHom-post-Lie algebra are both trivial, we obtain the following commutative diagram (See [13, 20, 21] for more details).

\begin{center}
\begin{tikzcd}
\text{BiHom-dendriform alg.} & \text{BiHom-associative alg.} \\
\end{tikzcd}
\end{center}

\begin{center}
\begin{tikzcd}
\text{BiHom-pre-Lie alg.} & \text{BiHom-Lie alg.}
\end{tikzcd}
\end{center}
5 BiHom-post-Lie algebras and $\mathcal{O}$-operators

In this section, we introduce the definition of an $\mathcal{O}$-operator of weight $\lambda$ on a BiHom-Lie algebra associated to an $A$-module $\mathbb{K}$-algebra, which generalizes the Rota-Baxter operator of weight $\lambda$ defined as a linear operator $R$ on a BiHom-Lie algebra $(A, [\cdot, \cdot], \alpha, \beta)$ such that $R\alpha = \alpha R$, $R\beta = \beta R$ and

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)] + \lambda[x, y]), \quad \forall x, y \in A.$$ 

**Definition 5.1** Let $(A, [\cdot, \cdot], \alpha, \beta)$ be a regular BiHom-Lie algebra and $(V, [\cdot, \cdot]_V, \rho, \phi, \psi)$ be a regular $A$-module $\mathbb{K}$-algebra. A linear map $T : V \to A$ is called an $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated to $(V, [\cdot, \cdot]_V, \rho, \phi, \psi)$ if its satisfies

$$T\phi = \alpha T, \quad T\psi = \beta T,$$

and

$$[T(u), T(v)] = T(\rho(T(u))v - \rho(T(\phi^{-1}\psi(v)))\phi\psi^{-1}(u) + \lambda[u, v]_V),$$

(5.36)
for all $u, v \in V$.

**Example 5.2** An $\mathcal{O}$-operator of weight $\lambda$ associated to $(A, [\cdot, \cdot], ad, \alpha, \beta)$ is just a Rota-Baxter operator on $A$ of the same weight.

**Theorem 5.3** Let $(A, [\cdot, \cdot], \alpha, \beta)$ be a BiHom-Lie algebra and $(V, [\cdot, \cdot]_V, \rho, \phi, \psi)$ a regular $A$-module $\mathbb{K}$-algebra. The linear map $T : V \to A$ is an $\mathcal{O}$-operator of weight $\lambda \in \mathbb{K}$ associated to $(V, [\cdot, \cdot]_V, \rho, \phi, \psi)$. Define two new bilinear operations $\{\cdot, \cdot\}, \triangleright : V \times V \to V$ as follows

$$\{u, v\} = \lambda[u, v]_V, \quad u \triangleright v = \rho(T(u))v,$$

for any $u, v \in V$. Then $(V, [\cdot, \cdot], \triangleright, \phi, \psi)$ is a BiHom-post-Lie algebra and $T$ is a morphism of BiHom-Lie algebras from the associated BiHom-Lie algebra of $(V, [\cdot, \cdot], \triangleright, \phi, \psi)$ to $(A, [\cdot, \cdot], \alpha, \beta)$. Furthermore, $T(V)$ is a BiHom-Lie subalgebra of $(A, [\cdot, \cdot], \alpha, \beta)$ and there is an induced BiHom-post-Lie algebra structure on $T(V)$ given by

$$[T(u), T(v)]_{T(V)} = T(\{u, v\}) \quad T(u) \triangleright_{T(V)} T(v) = T(u \triangleright v), \quad \forall u, v \in V.$$

**Proof** We use the last condition of representation of BiHom-Lie algebras on $\mathbb{K}$-algebra.

$$\phi\psi(a) \triangleright \{b, c\} - \psi(a) \triangleright b, \psi(c) - \{\psi(b), \phi(a) \triangleright c\}$$

$$= \phi\psi(a) \triangleright (\lambda[b, c]_V - \lambda[\rho(T(\phi(a)))b, \psi(c)]_V - \lambda[\psi(b), \rho(T(\phi(a)))c]_V)$$

$$= \lambda(\rho(\phi\beta(T(a)))b, \psi(c))_V - \lambda[\rho(\beta(T(a)))b, \psi(c)]_V - \lambda[\psi(b), \rho(\alpha(T(a)))c]_V$$

$$= 0.$$

Using the condition (2.11) of Definition 2.10, we check
\[\{\psi(a), \phi(b)\} \triangleright \psi(c) - \phi\psi(a) \triangleright (\phi(b) \triangleright c)\]
\[= \rho(T(\rho(T(\phi(b)))T(c)) - \rho(T(\rho(T(\phi^{-1} \psi \phi(b))))\phi(a))\psi(a))\]
\[\triangleright + \rho(T(\psi(a)) \phi(b))\psi(c) - \phi\psi(a)(b \triangleright c) + \psi\phi(b) \triangleright (\phi(a) \triangleright c)\]
\[= \rho(\beta(T(a)), T(\phi(b)))\psi(c) - \rho(\alpha\beta(\phi(b)))\rho(T(\phi(b)))c + \rho(\beta(T(\phi(b))))\rho(\alpha(T(a)))c\]
\[= \rho(\beta(T(a)), T(\phi(b)))\psi(c) - \rho(\alpha\beta(\phi(b)))\rho(T(\phi(b)))c + \rho(\beta(T(\phi(b))))\rho(\alpha(T(a)))c\]
\[= 0.\]

An obvious consequence of Theorem 5.3 is the following construction of a BiHom-post-Lie algebra in terms of Rota-Baxter operator of weight \(\lambda\) on a BiHom-Lie algebra.

**Corollary 5.4** Let \((A, [\cdot, \cdot], \alpha, \beta)\) be a regular BiHom-Lie algebra. Then there exists a compatible BiHom-post-Lie algebra structure on \(A\) if and only if there exists an invertible \(O\) operator of weight \(\lambda \in \mathbb{K}\) on \(A\).

**Proof** Let \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\) be a BiHom-post-Lie algebra and \((A, [\cdot, \cdot], \alpha, \beta)\) be the associated BiHom-Lie algebra. Then the identity map \(id : A \rightarrow A\) is an invertible \(O\)-operator of weight 1 on \((A, [\cdot, \cdot], \alpha, \beta)\) associated to \((A, [\cdot, \cdot], \triangleright, \alpha, \beta)\).

Conversely, suppose that there exists an invertible \(O\)-operator \(T\) (of weight \(\lambda\)) of \((A, [\cdot, \cdot], \alpha, \beta)\) associated to an \(A\)-module \(\mathbb{K}\)-algebra \((V, [\cdot, \cdot], \triangleright, \alpha, \beta)\). Then, using Proposition 5.3, there is a BiHom-post-Lie algebra structure on \(T(V) = A\) given by
\[\{\lambda T([u, v]_V), \lambda T(u) \triangleright \lambda T(v) = T(\rho(T(u))v), \forall u, v \in V.\]

If we set \(x = T(u)\) and \(y = T(v)\), then we get
\[\{x, y\} = \lambda T([T^{-1}(x), T^{-1}(y)]_V), x \triangleright y = T(\rho(x)T^{-1}(y)).\]

This is compatible BiHom-post-Lie algebra structure on \((A, [\cdot, \cdot], \alpha, \beta)\). Indeed,
\[x \triangleright y - \alpha^{-1} \beta(y) \triangleright \alpha \beta^{-1}(x) + \{x, y\}\]
\[= T(\rho(x)T^{-1}(y) - \rho(\alpha^{-1} \beta(y))T^{-1} \alpha \beta^{-1}(x) + \lambda[T^{-1}(x), T^{-1}(y)]_V)\]
\[= [TT^{-1}(x), TT^{-1}(y)] = [x, y].\]

The proof is finished.

**Corollary 5.5** Let \((A, [\cdot, \cdot], \alpha, \beta)\) be a regular BiHom-Lie algebra and the linear map \(R : A \rightarrow A\) is a Rota-Baxter operator of weight \(\lambda \in \mathbb{K}\). Then there exists a BiHom-post-Lie structure on \(A\) given by
\[\{x, y\} = \lambda[x, y], \forall x, y \in A.\]

If in addition, \(R\) is invertible, then there is a BiHom-post-Lie algebra structure on \(A\) given by
\[\{x, y\}^{-1}(x, R^{-1}(y)) \triangleleft x \triangleright y = R([x, R^{-1}(y)]), \forall x, y \in A.\]

**Example 5.6** Let \(L = sl(2, \mathbb{K})\) whose standard basis consists of
\[
X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Then \([H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H\). Define two linear maps \(\alpha, \beta : L \to L\) by
\[
\alpha(X) = \chi^2X, \quad \alpha(Y) = \frac{1}{\chi^2}Y, \quad \alpha(H) = H,
\]
\[
\beta(X) = \gamma^2X, \quad \beta(Y) = \frac{1}{\gamma^2}Y, \quad \beta(H) = H,
\]
where \(\chi, \gamma\) are parameters in \(\mathbb{K}\). Obviously we check that \(\alpha\) and \(\beta\) are two morphisms of the Lie algebra \((L, [\cdot, \cdot])\). Consider the linear map \([\cdot, \cdot]_{a, b} : L \otimes L \to L\)
\[
[a, b]_{a, b} = [\alpha(a), \beta(b)], \quad \text{for all } a, b \in L
\]
defined in the basis \(X, Y, H\) by
\[
[H, X]_{a, b} = 2\gamma^2X, \quad [H, Y]_{a, b} = -\frac{2}{\gamma^2}Y, \quad [X, Y]_{a, b} = \frac{\chi^2}{\gamma^2}H.
\]

Then \(L_{(a, b)} := (L, [\cdot, \cdot]_{a, b}, \alpha, \beta)\) is a BiHom-Lie algebra.

Now, define the linear map \(R : L \to L\) by
\[
R(X) = 0, \quad R(Y) = 4Y \quad \text{and} \quad R(H) = 2H.
\]

Then \(R\) is a Rota-Baxter operator of weight \(-4\) on \(L_{(a, b)}\). Using Corollary 5.5, we can construct a BiHom-Post-Lie algebra on \(L_{(a, b)}\) given by
\[
\{H, X\} = -8\gamma^2X, \quad \{H, Y\} = \frac{8}{\gamma^2}Y, \quad \{X, Y\} = -\frac{4}{\gamma^2}H
\]
and
\[
X \triangleright Y = X \triangleright H = 0, \quad Y \triangleright X = -4\frac{\gamma^2}{\chi^2}H, \quad Y \triangleright H = -\frac{8}{\chi^2}Y, \quad H \triangleright X = 4\gamma^2X, \quad H \triangleright Y = -\frac{4}{\gamma^2}Y.
\]

Example 5.7 Let \((A, [\cdot, \cdot], \alpha, \beta)\) be a BiHom-Lie algebra such that \(A = A_1 \oplus A_2\), where \(A_1\) and \(A_2\) are two BiHom-Lie subalgebras, and the linear map \(R : A \to A\) given by
\[
R(x_1 + x_2) = -\lambda x_2, \quad \forall \ x_1 \in A_1, \ x_2 \in A_2.
\]

It is easy to check that \(R\) is a Rota-Baxter operator of weight \(\lambda \in \mathbb{K}\) on \(A\). Then \((A, \triangleright, [\cdot, \cdot], \alpha, \beta)\) is a BiHom-post-Lie algebra, where
\[
\{x, y\} = \lambda[x, y], \quad x \triangleright y = [R(x), y], \quad \forall \ x, y \in A.
\]

6 Two-dimensional BiHom-Post-Lie algebras

In the following, we present a list of all BiHom-Post-Lie algebras associated to the homomorphisms given with respect to the basis \(\{e_1, e_2\}\) of a 2–dimensional vector space \(\mathbb{V}\), by
the following matrices \(\alpha = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}\) and \(\beta = \begin{pmatrix} b_{11} & 0 \\ 0 & b_{22} \end{pmatrix}\) with \(a_{11} \neq b_{11}\) or \(a_{22} \neq b_{22}\). If
we write $e_i \triangleright e_j = \sum_{k=1}^{2} C_{ij}^k e_k$ and $[e_i, e_j] = \sum_{k=1}^{2} D_{ij}^k$, where $C_{ij}^k, D_{ij}^k$ for $i, j, k = 1, 2, 3$ are parameters in $\mathbb{K}$. Then, using mathematica, we obtain the following non-isomorphic family of BiHom-post-Lie algebras denoted by $BPL_i$, $i = 1, \ldots, 27$. 

\begin{align*}
BPL_1 & : 
\begin{cases}
\alpha(e_1) = e_1, \\
\alpha(e_2) = e_2, \\
\beta(e_2) = e_2, \\
\{e_1 \triangleright e_1\} = C_{11}^1 e_1, \\
\{e_1 \triangleright e_2\} = C_{12}^1 e_2, \\
\{e_2 \triangleright e_2\} = C_{22}^1 e_2, \\
\{e_1, e_1\} = D_{11}^1 e_1, \\
\{e_1, e_2\} = D_{12}^1 e_2,
\end{cases} \\
BPL_2 & : 
\begin{cases}
\alpha(e_1) = e_1, \\
\alpha(e_2) = e_2, \\
\beta(e_1) = e_1, \\
\{e_1 \triangleright e_2\} = C_{12}^1 e_2, \\
\{e_2 \triangleright e_1\} = C_{22}^1 e_1, \\
\{e_2 \triangleright e_2\} = C_{22}^2 e_2, \\
\{e_1, e_2\} = D_{12}^2 e_2, \\
\{e_2, e_2\} = D_{22}^2 e_2,
\end{cases} \\
BPL_3 & : 
\begin{cases}
\alpha(e_1) = e_1, \\
\alpha(e_2) = a_{22} e_2, \\
\beta(e_1) = e_1, \\
\{e_1 \triangleright e_2\} = b_{22} e_2, \\
\{e_2 \triangleright e_1\} = C_{22}^1 e_1, \\
\{e_2 \triangleright e_2\} = C_{22}^2 e_2, \\
\{e_1, e_2\} = D_{12}^2 e_2, \\
\{e_2, e_2\} = D_{22}^2 e_2,
\end{cases} \\
BPL_4 & : 
\begin{cases}
\alpha(e_1) = a_{11} e_1, \\
\alpha(e_2) = e_2, \\
\beta(e_1) = b_{11} e_1, \\
\{e_1 \triangleright e_2\} = b_{11} e_2, \\
\{e_2 \triangleright e_1\} = C_{22}^1 e_1, \\
\{e_2 \triangleright e_2\} = C_{22}^2 e_2, \\
\{e_1, e_2\} = D_{12}^1 e_1, \\
\{e_2, e_2\} = \sum_{k=1}^{2} D_{22}^k e_2,
\end{cases} \\
BPL_5 & : 
\begin{cases}
\alpha(e_1) = a_{11} e_1, \\
\alpha(e_2) = a_{22} e_2, \\
\beta(e_1) = b_{11} e_1, \\
\{e_1 \triangleright e_2\} = b_{11} e_2, \\
\{e_2 \triangleright e_1\} = C_{22}^1 e_1, \\
\{e_2 \triangleright e_2\} = C_{22}^2 e_2, \\
\{e_1, e_2\} = D_{12}^1 e_1, \\
\{e_2, e_2\} = D_{22}^2 e_2,
\end{cases} \\
BPL_6 & : 
\begin{cases}
\alpha(e_1) = a_{11} e_1, \\
\alpha(e_2) = e_2, \\
\beta(e_1) = e_1, \\
\{e_1 \triangleright e_2\} = e_2, \\
\{e_2 \triangleright e_1\} = C_{22}^1 e_1, \\
\{e_2 \triangleright e_2\} = C_{22}^2 e_2, \\
\{e_1, e_2\} = D_{12}^1 e_1, \\
\{e_2, e_2\} = \sum_{k=1}^{2} D_{22}^k e_2,
\end{cases} \\
BPL_7 & : 
\begin{cases}
\alpha(e_1) = a_{11} e_1, \\
\alpha(e_2) = e_2, \\
\beta(e_1) = e_1, \\
\{e_1 \triangleright e_2\} = e_2, \\
\{e_2 \triangleright e_1\} = C_{22}^1 e_1, \\
\{e_2 \triangleright e_2\} = C_{22}^2 e_2, \\
\{e_1, e_2\} = D_{12}^1 e_1, \\
\{e_2, e_2\} = \sum_{k=1}^{2} D_{22}^k e_2,
\end{cases} \\
BPL_8 & : 
\begin{cases}
\alpha(e_1) = a_{11} e_1, \\
\alpha(e_2) = e_2, \\
\beta(e_1) = e_1, \\
\{e_1 \triangleright e_2\} = e_2, \\
\{e_2 \triangleright e_1\} = C_{22}^1 e_1, \\
\{e_2 \triangleright e_2\} = C_{22}^2 e_2, \\
\{e_1, e_2\} = D_{12}^1 e_1, \\
\{e_2, e_2\} = \sum_{k=1}^{2} D_{22}^k e_2,
\end{cases} \\
BPL_9 & : 
\begin{cases}
\alpha(e_1) = e_1, \\
\alpha(e_2) = e_2, \\
\beta(e_1) = e_1, \\
\{e_1 \triangleright e_2\} = e_2, \\
\{e_2 \triangleright e_1\} = C_{22}^1 e_1, \\
\{e_2 \triangleright e_2\} = C_{22}^2 e_2, \\
\{e_1, e_2\} = D_{12}^1 e_1, \\
\{e_2, e_2\} = \sum_{k=1}^{2} D_{22}^k e_2,
\end{cases}
\end{align*}
The multiplications and brackets not mentioned are equal to zero, in addition the unspecified of $\alpha$ and $\beta$ are zeros.

7 Representation theory of BiHom-post-Lie algebras

In this section, we introduce the notion of representations of a BiHom-post-Lie algebra $(A, [~,~], \triangleright, \alpha, \beta)$ on a vector space $V$.

**Definition 7.1** A representation of a BiHom-post-Lie algebra $(A, [~,~], \triangleright, \alpha, \beta)$ on a vector space $V$ is a tuple $(V, \rho, \mu, \nu, \phi, \psi)$, such that $(V, \rho, \phi, \psi)$ is a representation of $(A, [~,~], \alpha, \beta)$ and $\mu, \nu : A \to gl(V)$ are linear maps satisfying

\[
\mu(\alpha(x))\phi = \phi\mu(x), \quad \mu(\beta(x))\psi = \psi\mu(x),
\]

\[
\nu(\alpha(x))\phi = \phi\nu(x), \quad \nu(\beta(x))\psi = \psi\nu(x),
\]

\[
\nu([x, y])\phi\psi = \rho(\beta(x))\nu(y)\phi - \rho(\beta(y))\nu(x)\psi,
\]

\[
\nu([x, y])\phi\psi = \rho(\beta(x))\nu(y)\phi - \rho(\beta(y))\nu(x)\psi,
\]

\[
\rho(\beta(x) \triangleright y)\psi = \mu(\alpha(\beta(y)))\rho(y) - \mu(\beta(y))\alpha(\alpha(x)),
\]

\[
\mu([\beta(x), \alpha(y)])\psi = \mu(\alpha(\beta(x)))\mu(\alpha(\alpha(y))) - \mu(\alpha(\beta(y)))\mu(\alpha(\alpha(x))) - \mu(\beta(x) \triangleright \alpha(y))\psi
\]

\[
\nu(\beta(y))\rho(\beta(x))\phi = \mu(\alpha(\beta(x)))\nu(y)\phi - \nu(\beta(y))\mu(\beta(x))\phi,
\]

\[
\nu(\beta(y))\rho(\beta(x))\phi = \mu(\alpha(\beta(x)))\nu(y)\phi - \nu(\beta(y))\mu(\beta(x))\phi,
\]

for any $x, y \in A$. If $\phi$ and $\psi$ are bijective, then $(V, \rho, \phi, \psi)$ is called a regular representation.

Let $(A, [~,~], \triangleright, \alpha, \beta)$ be a BiHom-post-Lie algebra and $(V, \rho, \mu, \nu, \phi, \psi)$ a representation of $(A, [~,~], \triangleright, \alpha, \beta)$. By identity (7.41), we deduce that $(V, \mu, \phi, \psi)$ is a representation of the sub-adjacent BiHom-Lie algebra $(A, [~,~], \alpha, \beta)$ of $(A, [~,~], \triangleright, \alpha, \beta)$. In addition, it is obvious that $(A, ad, L_\phi, R_\psi, \alpha, \beta)$ is a representation of $(A, [~,~], \triangleright, \alpha, \beta)$ which is called the adjoint representation.

**Theorem 7.2** A tuple $(V, \rho, \mu, \nu, \phi, \psi)$ is a regular representation of a regular BiHom-post-Lie algebra

$(A, [~,~], \triangleright, \alpha, \beta)$ if and only if $(A \bigoplus V, [~,~], \triangleright_{\mu, \nu}, \alpha + \phi, \beta + \psi)$ is a BiHom-post-Lie algebra, where for any $x, y \in A$ and $u, v \in V$

\[
(a + \phi)(x + u) = \alpha(x) + \phi(u),
\]

\[
(\beta + \psi)(x + u) = \beta(x) + \psi(u),
\]

\[
[x + u, y + v] \rho = [x, y] + \rho(x)v - \rho(\alpha^{-1}(\beta(y)))\phi\psi^{-1}(u),
\]

\[
(x + u) \triangleright_{\mu, \nu} (y + v) = x \triangleright y + \mu(x)v + \nu(y)u.
\]
Proof By the conditions (7.41) and (7.42) in Definition 7.1 and the identity (3.13), we have

\[
[(\beta + \psi)(x + a), (\alpha + \phi)(y + b)]_\rho \triangleright_{\mu, \nu} (\beta + \psi)(z + c)
\]

\[
- as_{a + \phi, \beta + \psi}((\beta + \psi)(x + a), (\alpha + \phi)(y + b), z + c)
\]

\[
+ as_{a + \phi, \beta + \psi}((\beta + \psi)(y + b), (\alpha + \phi)(x + a), z + c)
\]

\[
= [\beta(x), \alpha(y)] \triangleright \beta(z) + \mu(\beta(x), \alpha(y))\psi(c) + v(\beta(z))(\rho(\beta(x))\phi(b) - \rho(\beta(y))\phi(a))
\]

\[
- \alpha\beta(x) \triangleright (\alpha(y) \triangleright z) - \mu(\alpha\beta(x))\mu(\alpha(y))c - \mu(\alpha\beta(x))v(z)\phi(b) - v(\alpha(y) \triangleright z)\phi\psi(a)
\]

\[
+ (\beta(x) \triangleright \alpha(y)) \triangleright \beta(z) + \mu(\beta(x) \triangleright \alpha(y))\psi(c) + v(\beta(z))\mu(\beta(x))\phi(b) + v(\beta(z))v(\alpha(y))\psi(a)
\]

\[
+ \alpha\beta(y) \triangleright (\alpha(x) \triangleright z) + \mu(\alpha\beta(y))\mu(\alpha(x))c + \mu(\alpha\beta(y))v(z)\phi(a) + v(\alpha(x) \triangleright z)\phi\psi(b)
\]

\[
- (\beta(y) \triangleright \alpha(x)) \triangleright \beta(z) - \mu(\beta(y) \triangleright \alpha(x))\psi(c) - v(\beta(z))\mu(\beta(y))\phi(a) - v(\beta(z))v(\alpha(x))\psi(b)
\]

\[= 0.\]

Similarly we can check the identity (??) for the algebra \( A \bigoplus V \) using the axioms (7.39)–(7.42) and (??).

Let \((A, [\cdot, \cdot], \triangleright)\) be a post-Lie algebra and \(\alpha, \beta\) be two commuting post-Lie algebra morphisms on \(A\). Consider a representation \((V, \rho, \mu, \nu) (A, [\cdot, \cdot], \triangleright)\) and \(\phi, \psi : V \rightarrow V\) be two commuting linear maps satisfying the following conditions

\[
\rho(\alpha(x))\circ \phi = \phi \circ \rho(x), \quad \rho(\beta(x))\circ \psi = \psi \circ \rho(x),
\]

\[
\mu(\alpha(x))\circ \phi = \phi \circ \mu(x), \quad \mu(\beta(x))\circ \psi = \psi \circ \mu(x),
\]

\[
v(\alpha(x))\circ \phi = \phi \circ v(x), \quad v(\beta(x))\circ \psi = \psi \circ v(x).
\]

Define the three linear maps \(\tilde{\rho}, \tilde{\mu}, \tilde{\nu} : A \rightarrow gl(V)\) by

\[
\tilde{\rho}(x)(v) = \rho(\alpha(x))(\psi(v)), \quad \tilde{\mu}(x)(v) = \mu(\alpha(x))(\psi(v)) \text{ and } \tilde{\nu}(x)(v) = v(\alpha(x))(\psi(v)).
\]

Proposition 7.3 The tuple \((V, \tilde{\rho}, \tilde{\mu}, \tilde{\nu}, \phi, \psi)\) is a representation of \((A, [\cdot, \cdot]_{\alpha, \beta}, \triangleright_{\alpha, \beta}, \alpha, \beta)\).

Proof straightforward.

Let \((V, \rho, \mu, \nu, \phi, \psi)\) be a regular representation of a regular BiHom-post-Lie algebra \((A, [\cdot, \cdot], \triangleright_{\alpha, \beta})\). Define the linear map \(\pi : A \rightarrow gl(V)\) by

\[
\pi(x) : = \rho(x) + \mu(u) - \nu(\alpha \beta^{-1}(x))\phi^{-1}\psi, \quad \forall x \in A.
\] (7.43)

Proposition 7.4 With the above notations, \((V, \pi, \phi, \psi)\) is a representation of the sub-adjacent BiHom-Lie algebra \((A, [\cdot, \cdot], \alpha, \beta)\) of \((A, [\cdot, \cdot], \triangleright_{\alpha, \beta})\).

Proof By Theorem 7.2, we have the semi-direct product BiHom-post-Lie algebra \((A \bigoplus V, [\cdot, \cdot]_\rho, \triangleright_{\mu, \nu, \alpha + \phi, \beta + \psi})\). Consider its sub-adjacent Lie algebra structure \([\cdot, \cdot]^C\), we have
\[ [x + u, y + v]^{\psi} = (x + u) \triangleright_{\mu, \psi} (y + v) - (\alpha^{-1}\beta(y) + \phi^{-1}(\psi(v))) \triangleright_{\mu, \psi} (\alpha\beta^{-1}(x) + \phi\psi^{-1}(u)) + [x + u, y + v], \]

\[ = x \triangleright y + \mu(x)v + v(y)u - \alpha^{-1}\beta(y) \triangleright \alpha\beta^{-1}(x) - \mu(\alpha^{-1}\beta(y))\phi\psi^{-1}(u) - v(\alpha\beta^{-1}(x))\phi^{-1}(\psi(v)) \]

\[ + [x, y] + \rho(x)v - \rho(\alpha^{-1}\beta(y))\phi\psi^{-1}(u) \]

\[ = [x, y] + \pi(x)v - \pi(\alpha^{-1}\beta(y))\phi\psi^{-1}(u). \]

Then \((V, \pi, \phi, \psi)\) is a representation of the sub-adjacent BiHom-Lie algebra \((A, \{\cdot, \cdot\}, \alpha, \beta)\).

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