Categorifications of the colored Jones polynomial

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Abstract

The colored Jones polynomial of links has two natural normalizations: one in which the \( n \)-colored unknot evaluates to \( [n + 1] \), the quantum dimension of the \( (n + 1) \)-dimensional irreducible representation of quantum \( \mathfrak{sl}(2) \), and the other in which it evaluates to 1. For each normalization we construct a bigraded cohomology theory of links with the colored Jones polynomial as the Euler characteristic.

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The colored Jones polynomial

The Jones polynomial. The Jones polynomial $J(L)$ of an oriented link $L$ in $\mathbb{R}^3$ is determined by the skein relation

$$q^2 J(L_1) - q^{-2} J(L_2) = (q - q^{-1}) J(L_3)$$

for any three links $L_1, L_2, L_3$ that differ as shown in figure 1 and its value on the unknot, which we choose to be $q + q^{-1}$.

![Figure 1:](image)

The Jones polynomial does not depend on the framing of link components, but depends slightly on their orientations. Reversing the orientation of a component $L' \subset L$ multiplies the polynomial by $q^{-6 \cdot lk(L', L \setminus L')}$ where $lk(L', L \setminus L')$ is the linking number of $L'$ with its complement in $L$.

The colored Jones polynomial. We briefly recall the basics about the colored Jones polynomial of links (for more details consult [2], [1], [6], Sections 3,4, and references therein). Given an oriented framed link $L$ whose components are colored (marked) by non-negative integers (or, equivalently, by irreducible representations of $U_q(\mathfrak{sl}_2)$, with integer $n$ corresponding to the $(n + 1)$-dimensional representation $V_n$), the colored Jones polynomial $J_n(L)$ takes values in $\mathbb{Z}[q, q^{-1}]$. The label $n$ stands for the coloring, that is, for a function from the set of components of $L$ to non-negative integers. If all components of $L$ are colored by 1, the invariant is the original Jones polynomial of $L$. If a component of $L$ is colored by 0, deleting this component preserves the value of the colored Jones polynomial.

For a framed knot $K$ we denote by $J_n(K)$ the Jones polynomial of $K$ colored by $n$. Thus, $J_1(K)$ is the original Jones polynomial of $K$ (and does not depend on the framing), while $J_0(K) = 1$ for any knot $K$.

More generally, one could label link components by arbitrary finite-dimensional representations of $U_q(\mathfrak{sl}_2))$. This does not give any extra information since the invariant is additive relative to the direct sum of representations,

$$J_{V \oplus W}(K) = J_V(K) + J_W(K),$$
but allows us to express the colored Jones polynomial via the Jones polynomials of cables. First, the Jones polynomial $J_{V \otimes W}(K)$ of a knot $K$ labelled by $V \otimes W$ equals the Jones polynomial of the 2-cable $K^2$ of $K$ with components labelled by $V$ and $W$. The Jones polynomial $J_{V \otimes n}(K)$ equals the original Jones polynomial of the $n$-cable $K^n$ of $K$.

For generic $q$ the quantum group $U_q(\mathfrak{sl}(2))$ has one irreducible representation $V_n$ in each dimension (we consider representations where $q^H$ has only powers of $q$ as eigenvalues). The Grothendieck group of this category of representations is free abelian with a basis given by images $[V_n], n \geq 0$ of all irreducible representations. We put square brackets around $V_n$ to distinguish a representation from its image in the Grothendieck group. Another basis in the Grothendieck group is given by all tensor powers of $V_1$.

Using the formula $V_n \otimes V_1 \cong V_{n+1} \oplus V_{n-1}$ and induction on $n$, one checks that

$$[V_n] = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} [V_1 \otimes (n-2k)].$$

(1)

Thus, the colored Jones polynomial $J_n(K)$ of a knot $K$ can be expressed via Jones polynomials of its cables:

$$J_n(K) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} J(K^{n-2k}).$$

(2)

This formula generalizes from knots to links, by summing over all connected components of a link:

$$J_n(L) = \sum_{k=0}^{[n/2]} (-1)^{|k|} \binom{n-k}{|k|} J(L^{n-2k}),$$

(3)

where $k$ is a function from the set of link components to non-negative integers.

We define the colored Jones polynomial by formula (3). Since the Jones polynomial (which appears on the right) depends slightly on the orientation of each cable component, we need to specify our choices of orientations. Each component of $L$ is oriented. When forming the $m$-cable of a component, if $m$ is even we orient half of the strands one way and the rest the opposite way. If $m$ is odd, we orient $m+1/2$ strands by way of the original orientation of the component, and the remaining strands the opposite way.

With this definition, $J_n(L)$ does not depend on the orientations of components colored by even integers, while reversing the orientation of an odd-colored component $L'$ multiplies the polynomial by $q^{-6|k(L',L_{\text{odd}} \setminus L')}}$, where $L_{\text{odd}}$ is the sublink of $L$ made of all odd-colored components.
If \( K \) is the 0-framed unknot, 
\[
J_n(K) = [n + 1] \overset{\text{def}}{=} \frac{q^{n+1} - q^{-n-1}}{q - q^{-1}},
\]
the quantum dimension of \( V_n \).

Changing the framing of a component multiplies the colored Jones polynomial by a power of \( q \), as explained in the formula below and figure 2.

\[
\begin{align*}
J_n(L_1) &= q^{-2m(m+1)} J_n(L_0) \quad \text{if } n = 2m, \\
J_n(L_1) &= q^{-2m(m+2)} J_n(L_0) \quad \text{if } n = 2m + 1,
\end{align*}
\]
where \( n \) is the color of the component.

**Reduced colored Jones polynomial.** Another common normalization for the colored Jones polynomial of knots is

\[
\tilde{J}_n(K) \overset{\text{def}}{=} \frac{J_n(K)}{[n + 1]}.
\]

We will call \( \tilde{J}_n(K) \) the *reduced* colored Jones polynomial. The reduced polynomial of the 0-framed unknot is 1. We extend this normalization to colored links with a distinguished component, by

\[
\tilde{J}_n(L) \overset{\text{def}}{=} \frac{J_n(L)}{[n + 1]},
\]
where \( n \) is the color of the distinguished component. The reduced colored Jones polynomial has integral coefficients.
2 First non-trivial examples

Cohomology of cables. Take a framed link \( L \), its cable \( L^n \), and consider the complex \( C(L^n) \) and its cohomology groups \( H(L^n) \) (we refer the reader to [4] for definitions of \( C \) and \( H \)). These groups are invariants of \( L \), and depend non-trivially on its framing. The Euler characteristic of \( H(L^n) \) is the Jones polynomial of the cable \( L^n \). We will use the grading conventions of [5].

Categorification of \( J_2(K) \). We would like to have a cohomology theory of links whose Euler characteristic is the colored Jones polynomial. Let’s start with the first non-trivial example beyond the original Jones polynomial: our link is a framed knot \( K \) colored by the three-dimensional representation \( V_2 \). Note that \( V_2 \) is a direct summand of \( V_1 \otimes V_1 \) and the complementary summand is the trivial representation:

\[
V_1 \otimes V_1 \cong V_2 \oplus V_0.
\]

This formula translates into a simple relation between the Jones polynomials:

\[
J_2(K) = J(K^2) - 1. \tag{4}
\]

To categorify \( J(K^2) \) we take the cohomology of the 2-cable of \( K \). Since \( J_0(K) = 1 \), we simply assume that its categorification is the abelian group \( \mathbb{Z} \) placed in bidegree \((0,0)\).

Formula (4) says that to categorify \( J_2(K) \) we need to ”subtract” \( \mathbb{Z} \) from \( \mathcal{H}(K^2) \). This could be achieved by taking the cone of some map \( \mathcal{H}(K^2) \to \mathbb{Z} \) or of a map going in the opposite direction, from \( \mathbb{Z} \) to \( \mathcal{H}(K^2) \). This map should be natural. In the cohomology theory \( \mathcal{H} \) natural maps come from cobordisms between links. Note that \( \mathbb{Z} \) is the cohomology of the empty link \( \emptyset \) and there is a canonical cobordism in \( \mathbb{R}^3 \times [0,1] \) between \( K^2 \) and \( \emptyset \). In the 2-cable \( K^2 \) two copies of \( K \) run parallel next to each other, and there is a standard embedding of an annulus into \( \mathbb{R}^3 \) with \( K^2 \) as its boundary (the two components of \( K^2 \) will be oppositely oriented if we orient our annulus and induce the orientation onto its boundary). Push the interior of the annulus into the interior of \( \mathbb{R}^3 \times [0,1] \) slightly away from the boundary component \( \mathbb{R}^3 \times \{0\} \) that contains \( K^2 \). The resulting annulus \( S_K \) in \( \mathbb{R}^3 \times [0,1] \) is an oriented cobordism between \( K^2 \) and the empty link. Its Euler characteristic is 0 and it induces a bidegree \((0,0)\) map between bigraded cohomology groups \( \mathcal{H}(K^2) \) and \( \mathcal{H}(\emptyset) \cong \mathbb{Z} \). This map is well-defined up to overall minus sign. On the level of complexes, we have a map \( u : C(K^2) \to \mathbb{Z} \), well-defined (up to the minus sign) in the homotopy category of complexes. We define the complex \( C_2(K) \) as the cone of \( u \), shifted by \([-1]\) (so that \( \mathbb{Z} \) is in degree 1),
and cohomology groups $\mathcal{H}_2(K)$ as the cohomology of $\mathcal{C}_2(K)$. There is a short exact sequence of complexes

$$0 \longrightarrow \mathbb{Z}[-1] \longrightarrow \mathcal{C}_2(K) \longrightarrow \mathcal{C}(K^2) \longrightarrow 0$$

giving rise to a long exact sequence of cohomology groups

$$\longrightarrow \mathcal{H}_2(K) \longrightarrow \mathcal{H}(K^2) \overset{u}{\longrightarrow} \mathbb{Z} \longrightarrow$$

In this sequence at most one boundary map could be non-zero (the one from $\mathbb{Z}$ to $\mathcal{H}_2^{1,0}(K)$). This map is non-zero if and only if $u$ is not surjective. It’s easy to see, and we do it below, that $u(\mathcal{H}(K^2))$ is a subgroup of index 1 or 2 in $\mathbb{Z}$.

Strictly speaking, in the above discussion we should fix a plane diagram $D$ of $K$, the associated diagram $D^2$ of the 2-cable $K^2$, and consider the complex $\mathcal{C}(D^2)$. From two diagrams $D_0$ and $D_1$ of $K$ related by a Reidemeister move for framed knots, we obtain diagrams $D^2_0$ and $D^2_1$ of $K^2$ and a diagram of complexes and homomorphisms

$$\begin{array}{c}
\mathcal{C}(D^2_0) \overset{u_0}{\longrightarrow} \mathbb{Z} \\
\downarrow \quad \quad \quad \downarrow \cong \\
\mathcal{C}(D^2_1) \overset{u_1}{\longrightarrow} \mathbb{Z}
\end{array}$$

which commutes up to overall minus sign. Therefore, the cones of $u_0$ and $u_1$ are homotopy equivalent and isomorphism classes of groups $\mathcal{H}_2^{i,j}(K)$ are invariants of $K$.

Since $S_K$ is also a cobordism from the empty link to $K^2$, it induces a map $u' : \mathbb{Z} \rightarrow \mathcal{C}(K^2)$ and we could alternatively define the cohomology groups of $K$ colored by 2 as the cohomology of the cone of $u'$.

Ideally, we would like the two resulting cohomology theories to be naturally isomorphic. Right off, we could see that they are if 2 is invertible in the base ring. Let $k$ be a commutative ring where 2 is invertible (for instance, a field of characteristic other than 2, or $\mathbb{Z}[\frac{1}{2}]$.) Tensoring $u'$ and $u$ with $k$ we get maps $u' : k \rightarrow \mathcal{C}(K^2) \otimes_{\mathbb{Z}} k$ and $u : \mathcal{C}(K^2) \otimes_{\mathbb{Z}} k \rightarrow k$, and the induced maps on cohomology groups (we don’t bother inventing different notations for these maps). The composition $uu' : k \rightarrow k$ is the value of the invariant on the composition of the two cobordisms given by the annulus $S_K$. This composition is a cobordism in $\mathbb{R}^3 \times [0, 1]$ between empty links, and is isotopic to the
torus standardly embedded into $\mathbb{R}^4$. Its invariant is $\pm 2$, therefore, $uu' = \pm 2$. Since $2$ is invertible, we can decompose
\[
\mathcal{H}(K^2)_k \cong \text{im}(u') \oplus \ker(u),
\]
and derive isomorphisms
\[
\mathcal{H}_2(K)_k \cong \ker(u) \cong \text{coker}(u')
\]
proving that the two definitions lead to isomorphic cohomology theories ($\mathcal{H}(K^2)_k$ in (5) stands for cohomology of $\mathcal{C}(K^2) \otimes_{\mathbb{Z}} k$ and $\mathcal{H}_2(K)_k$ in (6) for the cohomology of the total complex of $0 \to \mathcal{C}(K^2) \otimes_{\mathbb{Z}} k \xrightarrow{u} k \to 0$).

The above argument implies that $u(\mathcal{H}(K^2)) \subset \mathbb{Z}$ has index $1$ or $2$.

**Framing:** The colored Jones polynomial $J_2(K)$ depends in a simple way on the framing of $K$. Change in the framing multiplies the polynomial by $q^{\pm 4}$. The cohomology theory $\mathcal{H}_2(K)$ has a more complicated dependence on the framing. Already when $K$ is the unknot with framing $m > 0$, the rank of $\mathcal{H}(K^2)$ is $2 + 2m$, and that of $\mathcal{H}_2(K)$ is $1 + 2m$. In particular, cohomology of two knots that differ by a framing are not just overall shifts of each other.

In general, if $K$ is a knot and $K_1$ is obtained from $K$ by twisting by a large slope, an overall shift superimposes $\mathcal{H}_2(K)$ and $\mathcal{H}_2(K_1)$ except for a long tail in $\mathcal{H}_2(K_1)$ trailing along two adjacent diagonals in the bigrading plane (the tail is essentially the cohomology of the unknot with a large framing).

**Example:** For the $0$-framed unknot $K$,
\[
\mathcal{H}^{i,j}_2(K) \cong \begin{cases} 
\mathbb{Z} & \text{if } i = 0 \text{ and } j \in \{-2, 0, 2\}, \\
0 & \text{otherwise},
\end{cases}
\]
for both definitions of $\mathcal{H}_2$.

**Note:** The complex $\mathcal{C}_2(K)$ and its cohomology $\mathcal{H}_2(K)$ do not depend on the orientation of $K$.

**Categorification of $J_3(K$).** Direct sum decomposition
\[
V_1^{\otimes 3} \cong V_3 \oplus V_1 \oplus V_1
\]
implies that, for a framed knot $K$,
\[
J_3(K) = J(K^3) - 2J(K).
\]
To categorify $J(K^3)$, we form the 3-cable $K^3$ of $K$, take its cochain complex $C(K^3)$ and its cohomology $H(K^3)$. Choose a plane diagram of $K$ and the corresponding diagram of the cable. Enumerate the components of $K^3$ by 1, 2 and 3 so that 2 is in the middle, and orient component 2 oppositely from 1 and 3, see figure 3.

Figure 3: A close-up of $K^3$

Consider two annuli in the neighbourhood of $K$ in $\mathbb{R}^3$, one with components 1 and 2 as the boundary, and the other with components 2 and 3 as the boundary. Push the interiors of the annuli into $\mathbb{R}^3 \times [0, 1]$ away from the boundary. Two cobordisms from $K_3$ to $K$ result, inducing two maps from $C(K^3)$ to $C(K)$ and two maps on the cohomology, well-defined up to overall minus sign. We can guess that the categorification of $J_3(K)$ is the total complex of the bicomplex

$$0 \rightarrow C(K_3) \rightarrow C(K) \oplus C(K) \rightarrow 0.$$ 

Denote the total complex by $C_3(K)$ and its cohomology by $H_3(K)$. We get a bigraded cohomology theory of knots with $J_3(K)$ as the Euler characteristic. Independence of $H_3(K)$ from the choice of planar diagram is straightforward.

To categorify $J_n(K)$ for arbitrary $n$ we should look for a bicomplex built out of complexes $C(K^{n-2k})$ with binomial multiplicities as in (2), then form the total complex and take its cohomology. This is done in Section 4 while the representation-theoretic counterpart of this construction is worked out in the next section.

\section{A resolution of an irreducible $sl(2)$ representation}

In this section we give a homological interpretation of the formula

$$[V_n] = \sum_{k=0}^{[\frac{n}{2}]} (-1)^k \binom{n-k}{k} [V_1^{\otimes(n-2k)}]$$

\(8\)
describing the image of $V_n$ in the Grothendieck group via those of tensor powers of the defining representation $V_1$. We consider the $q = 1$ case, so that $V_n$'s are representations of the Lie algebra $\mathfrak{sl}(2)$. The case of generic $q$ is identical (by substituting below “$q$-antisymmetrization” for “antisymmetrization”).

The right hand side of (7) has alternating coefficients, due to $(-1)^k$, and we could try to realize it as the Euler characteristic of a complex. We put the direct sum of $\binom{n-k}{k}$ copies of $V_1 \otimes (n-2k)$ in the $k$-th cohomological degree and look for a natural differential

$$
\begin{align*}
(V_1 ^{(n-2k)}) \oplus \binom{n-k}{k} & \xrightarrow{d} (V_1 ^{(n-2(k+1)}) \oplus \binom{n-k-1}{k+1} \\
\end{align*}
$$

The differential reduces the number of $V_1$'s in each tensor power by 2. Since $V_1 \otimes V_1 \cong V_0 \oplus V_2$, there is a unique, up to scaling, surjective homomorphism $h : V_1 \otimes V_1 \to V_0$, and it is simply the antisymmetrization. We will use $h$ to construct the differential.

The binomial coefficient $\binom{n-k}{k}$ equals the number of ways to select $k$ pairs of neighbours from $n$ dots placed on a line, such that each dot appears in at most one pair (the $n = 5, k = 2$ example is depicted in figure 4). We will call these $k$-pairings.

![Figure 4: All three 2-pairings of 5 dots](image)

Notice that the exponent in the tensor power of $V_1$ is the number of dots without a partner (“single” dots). The differential goes in the direction of increasing the number of pairs by 1. A new pair will consists of two adjacent dots, and the differential should contract the two corresponding powers of $V_1$ into the trivial representation.

To formalize, let $I_k$ be the set of $k$-pairings of $n$ dots. For $s \in I_k$ let $(s)$ be the set of single dots in $s$, and $V^s \overset{def}{=} V^{(s)}$ be the tensor product of $V_1$'s, one for each single dot. If $s' \in I_{k+1}$ contains $s$ (each pair in $s$ is also a pair in $s'$), there is a map $h_{s',s} : V^s \to V^{s'}$ given by contracting the two copies of $V_1$ representing the only pair in $s' \setminus s$. Consider a graph with with vertices—elements of $I_k$, over all $k = 0, 1, \ldots, \lfloor \frac{n}{2} \rfloor$, and arrows—inclusions $s \subset s'$ as above. For $n = 4$ this graph is depicted in figure 5. Assigning $V^s$ to the vertex $s$ and the map $h_{s',s}$ to the arrow from $s$ to $s'$ we obtain a commutative diagram of $\mathfrak{sl}(2)$ representations.
We make each square in the diagram anticommute by switching from \( h_{s',s} \) to \((-1)^{(s,s')} h_{s',s}\) where \((s,s')\) is the number of pairs in \(s\) to the left of the only pair in \(s' \setminus s\). For each \(k\) take the direct sum of \(V^s\), for all \(s \in I_k\), and the sum of maps as above. The result is a complex, denoted \(C_n\).

**Example:** The complex \(C_4\) has the form

\[
0 \rightarrow V_1^\otimes 4 \rightarrow V_1^\otimes 2 \oplus V_1^\otimes 2 \oplus V_1^\otimes 2 \rightarrow V_0 \rightarrow 0,
\]

with the differential—the sum of arrows in figure 5, the top left arrow appearing with the minus sign.

**Theorem 1** The complex \(C_n\) is acyclic in non-zero degrees and its degree 0 cohomology is the irreducible \(\mathfrak{sl}(2)\) representation \(V_n\).

**Proof:** \(H^0(C_n)\) is the subrepresentation of \(V_1^\otimes n\) which is the kernel of \(d^0\). This map is the sum of antisymmetrizations of two \(V_1\)'s over all pairs of neighbours. Therefore, \(H^0(C_n)\) is isomorphic to the \(n\)-th symmetric power of \(V_1\), and to \(V_n\).

To prove exactness everywhere else, proceed by induction on \(n\). The short exact sequence

\[
0 \rightarrow C_{n-2}[-1] \rightarrow C_n \rightarrow C_{n-1} \otimes V_1 \rightarrow 0
\]

comes from separating all pairings of \(n\) dots into two types: the one where the leftmost dot belongs to a pair, and the one where it does not. The sum of all first type selections is a subcomplex of \(C_n\) isomorphic to \(C_{n-2}\) shifted one degree to the right, the sum of all second type selections is a quotient complex of \(C_n\) isomorphic to \(C_{n-1} \otimes V_1\).
Induction hypothesis and the long exact sequence of (8) imply $H^i(C_n) = 0$ for $i > 1$ and exactness of

$$0 \rightarrow H^0(C_n) \rightarrow V_1 \otimes H^0(C_{n-1}) \rightarrow H^1(C_{n-2}[-1]) \rightarrow H^1(C_n) \rightarrow 0$$

Substituting $V_m$ for $H^0(C_m)$, we get a short exact sequence

$$0 \rightarrow V_n \rightarrow V_1 \otimes V_{n-1} \rightarrow V_{n-2} \rightarrow H^1(C_n) \rightarrow 0$$

telling us that $H^1(C_n) = 0$. □

Remark: $C_n$ is a resolution of a simple module by a complex of semisimple modules. This is different from the homological algebra framework, where we usually resolve a module by a complex of projective, or injective, or flat modules. The category of finite-dimensional $\mathfrak{sl}(2)$ representations is already semisimple, all additive functors from this category are exact, and there is no need for resolutions from the homological algebra viewpoint. However, $C_n$ seems to be interesting on its own.

4 Categorification of the colored Jones polynomial

In characteristic 2. To avoid the sign ambiguity, in this subsection we work in characteristic 2. Let $\mathbb{F}_2$ be the 2-element field. For a diagram $D$ of a link, we denote the complex $C(D) \otimes_{\mathbb{Z}} \mathbb{F}_2$ by $\mathcal{C}(D)_2$ and its cohomology by $\mathcal{H}(D)_2$.

Start with a framed oriented knot $K$ and its plane diagram $D$. It gives rise to diagrams $D^n$ of cables $K^n$. The cables are oriented as in figure 6 and, in a cross-section of $D^n$, orientations alternate. Choose a cross-section of $D^n$ and enumerate the strands from left to right from 1 to $n$ so that component 1 is oriented in the same way as $D$, component 2 is oppositely oriented, etc.

For a pairing $s \in I_k$ denote by $D^s$ the cable diagram containing only components corresponding to single dots (not in any pair) in $s$, and by $K^s$ the corresponding link (it has $n - 2k$ components). Given an arrow $s \rightarrow s'$, there is a canonical cobordism $S^s_{s'}$ from $K^s$ to $K^{s'}$ given by "contracting" the pair $s' \setminus s$ of neighboring components of $D^s$ via an annulus. This cobordism induces a well-defined (up to homotopy) map $h_{s',s} : \mathcal{C}(D^s)_2 \rightarrow \mathcal{C}(D^{s'})_2$ of complexes, and the induced map on cohomology (also denoted $h_{s',s}$).

Let $C_n(D)_2$ be the complex which in the $k$-th degree is the direct sum of $\mathcal{H}(D^s)_2$ over all $s \in I_k$. The differential is the sum of $h_{s',s}$ over all possible arrows $s \rightarrow s'$. 

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Denote by $\mathcal{H}_n(D)_2$ the cohomology of $\mathcal{C}_n(D)_2$. These groups are bigraded,

$$\mathcal{H}_n(D)_2 = \bigoplus_{i,j \in \mathbb{Z}} \mathcal{H}^{i,j}_n(D)_2.$$

If diagrams $D$ and $D'$ are related by a Reidemeister move II or III, the complexes $\mathcal{C}_n(D)_2$ and $\mathcal{C}_n(D')_2$ are isomorphic. This implies

**Proposition 1** Isomorphism classes of bigraded groups $\mathcal{H}_n(D)_2$ do not depend on the diagram $D$ of a framed knot $K$, and are invariants of $K$. Their Euler characteristic is the colored Jones polynomial $J_n(K)$.

**Remarks:** These groups do not depend on the orientation of $K$.

**From knots to links.** Given a colored link $(L, n)$, we choose its diagram $D$, then do the above procedure for each component of $L$. The result is a multi-dimensional commutative diagram of groups $\mathcal{H}(D^s)_2$ with multiplicities–products of binomials, for various pairings $s$ and associated cables $D^s$ of $D$. Note that $n = (n_1, \ldots, n_r)$, and the pairing $s = (s_1, \ldots, s_r)$ where $s_i$ is a pairing of $n_i$ dots (equivalently, of $n_i$ cables of the $i$-th component). We collapse this commutative diagram into a complex, denoted $\mathcal{C}_n(D)_2$ (in characteristic 2 a commutative square is anticommutative). Its cohomology groups do not depend on $D$, and are invariants of $L$. Their Euler characteristic is $J_n(L)$.

**Cobordisms.** A cobordism in $\mathbb{R}^4$ between colored framed links can be represented by a sequence of its cross-sections, each a generic plane diagram of a colored framed link, with each two consequent cross-sections related by either a Reidemeister II or III move, or by a Morse move, see figure 7. Components involved in a saddle point move should be colored by the same number, see figure 7.

A Reidemeister II or III move from $D$ to $D_0$ induces an obvious isomorphism of groups $\mathcal{H}(D^s)_2$ and $\mathcal{H}(D^s_0)_2$ and of complexes $\mathcal{C}_n(D)_2$ and $\mathcal{C}_n(D_0)_2$. 

Figure 6: Orientations

\[
\begin{array}{ccccccc}
1 & \longrightarrow & - & - & - & - & - \\
2 & \longrightarrow & - & - & - & - & - \\
\ldots & \longrightarrow & - & - & - & - & - \\
\ldots & \longrightarrow & - & - & - & - & - \\
n & \longrightarrow & - & - & - & - & - \\
\end{array}
\]
The unit and counit maps of $A^\otimes n$ induce natural maps between the complexes for the empty link and for the crossingless unknot diagram colored by $n$. We assign these maps to the "birth" and "death" Morse moves.

Suppose that diagrams $D$ and $D_0$ are related by a saddle point move. Consider the case when the move merges two components of $D$ (both labelled by $n$ and denoted $K_1$ and $K_2$) into one component $K$ of $D_0$, see figure 8.

We would like to come up with a natural map $\psi$ of complexes

$$\psi : C_n(D)_2 \rightarrow C_{n_0}(D_0)_2,$$

where $n_0$ is the coloring of $D_0$ induced by the coloring $n$ of $D$. As an abelian group, $C_n(D)_2$ is the direct sum of $\mathcal{H}(D^s)_2$ over all pairings $s$ of $n$. Given $s$, let $s_1$ and $s_2$ be the pairings of $n$ dots which are the restrictions of $s$ to
the components $K_1$ and $K_2$. Let $k_1$ be the number of pairs in $s_1$ and $k_2$ the number of pairs in $s_2$.

If pairs in $s_1$ and $s_2$ have at least one common dot (see figure 9), we set $\psi(\mathcal{H}(D^s)_2) = 0$. Otherwise, $s_1$ and $s_2$ are disjoint and their union $s_1s_2$ is a $(k_1 + k_2)$-pairing of $n$ dots (see figure 10).

To such $s$ we assign a pairing $s_0$ of the $n_0$ cable of $D_0$. This pairing is the same as $s$ on components of $D_0$ that are unchanged during the saddle point move, and $s_1s_2$ on the component $K$ colored by $n$. The map $\psi$ will take $\mathcal{H}(D^s)_2$ to $\mathcal{H}(D_0^{n_0})_2$.

A pair in $s_2$ connects two dots numbered $m$ and $m + 1$ for some $m$ (in the figure 10 example $m = 5$). For each such pair we apply the operator of multiplication by $X$ at strand $m$ of $K_1^{s_1}$ on the cohomology $\mathcal{H}(D^s)_2$ (recall that $X$ is the generator of the ring $A$, and $X^2 = 0$). Likewise, for a pair in $s_1$ connecting two dots numbered $t$ and $t + 1$ (in the figure 10 example $t = 2$), we apply the operator of multiplication by $X$ at strand $t$ of $K_2^{s_2}$. Denote by $\psi_1$ the product of these operators. $\psi_1$ is an endomorphism of $\mathcal{H}(D^s)_2$, a multiplication by $k_1 + k_2$ copies of $X$ at certain $k_1 + k_2$ strands of the cable $D^s$. 

Figure 9: Two pairings with a common dot

Figure 10: Disjoint union of pairings (in this example $k_1 = k_2 = 1$)
For each pair in $s_2$ (connecting dots/strands numbered $m$ and $m + 1$) consider a thin annulus whose boundary is the union of two strands of $K_{s_1}^{m}$ labelled $m$ and $m + 1$. Similarly, for each pair in $s_1$ (connecting two dots numbered $t$ and $t + 1$ for some $t$) consider a thin annulus whose boundary is the union of two strands of $K_{s_2}^{t}$ labelled $t$ and $t + 1$. Figure 12 depicts these annuli schematically for our example. The resulting $k_1 + k_2$ annuli give rise to a cobordism from the cabled link with diagram $D^s$ to a cabled link with $2(k_1 + k_2)$ fewer components, whose diagram $D'$ can be produced by removing these $2(k_1 + k_2)$ components (the "mirror" partners of pairs in $s_1$ and $s_2$) from $D^s$. This cobordism induces a map of cohomology groups from $H(D^s)_2$ to $H(D')_2$, denoted $\psi_2$.

In the diagram $D'$ each strand in the cable $K_{s_1}^{s_2}$ has a matching strand in the cable $K_{s_2}^{s_2}$. There is a canonical cobordism from $D'$ to $D_0^{s_0}$, the composition of saddle point cobordisms for each pair of identically numbered strands in $K_{s_1}^{s_2}$ and $K_{s_2}^{s_2}$ (that is, one saddle point cobordism for each dot not in $s_1 s_2$), for an example see figure 14. Denote by $\psi_3$ the map of cohomology groups from $H(D')_0$ to $H(D_0^{s_0})_0$ induced by this composition of saddle point cobordisms.

To summarize, $\psi_1$ is the multiplication by a power of $X$ at certains strands of the cable $D^s$, $\psi_2$ is induced by annuli contraction cobordisms at "mirrors" of strand pairs in $s_1$ and $s_2$, and $\psi_3$ is induced by saddle point cobordisms of the remaining strands.

Let $\psi = \psi_3 \psi_2 \psi_1$:

$$\psi : H(D^s)_2 \xrightarrow{\psi_1} H(D^s)_2 \xrightarrow{\psi_2} H(D')_2 \xrightarrow{\psi_3} H(D_0^{s_0})_2.$$  

Summing over all pairings $s$ of $n$ we get a homomorphism of abelian groups from $\mathcal{C}_n(D)_2$ to $\mathcal{C}_{n_0}(D_0)_2$.

**Proposition 2** $\psi$ is a homomorphisms of complexes.

Proof is straightforward. □

Figures 11-14 illustrate our construction for the case of $s_1$ and $s_2$ in figure 10.

A similar map $\psi$ can be defined for the case when a saddle point cobordism increases (rather than decreases) the number of components by 1. We leave its construction to the reader.

Thus, to each Reidemeister and Morse move of framed colored link diagrams we can assign a map between the corresponding complexes $\mathcal{C}_n(D)_2$. We conjecture that the induced maps on cohomology of these complexes give an invariant of framed colored link cobordisms.
Figure 11: Part of the diagram $D^s$. Dotted lines show strands of $D^n$ that do not belong to cables $K^s_1$ and $K^s_2$.

Figure 12: Multiply by $X$ at the strand 5 of $K^s_1$ and at the strand 2 of $K^s_2$, and then contract pairs of strands along annuli (these pairs are "mirrors" of dotted pairs).

**Categorification over a field.** Let $\mathbb{F}$ be a field. For a diagram $D$ of a knot $K$ denote by $\mathcal{C}(D)_\mathbb{F}$ the complex $\mathcal{C}(D) \otimes_{\mathbb{Z}} \mathbb{F}$ and its cohomology by $\mathcal{H}(D)_\mathbb{F}$.

For each arrow $s \rightarrow s'$ we have a map

$$h_{s',s}: \mathcal{H}(D^s)_\mathbb{F} \rightarrow \mathcal{H}(D^{s'})_\mathbb{F}$$

well-defined up to overall minus sign. For each square of arrows

\[
\begin{array}{ccc}
  s & \rightarrow & s' \\
  \downarrow & & \downarrow \\
  s'' & \rightarrow & s''' \\
\end{array}
\]

the induced square of maps $h$ either commutes or anticommutes.

**Lemma 1** We can always make all squares of maps $h$ anticommute by changing signs in some of the maps $h_{s',s}$.
Let us call a choice of signs in maps $h$ \textit{satisfactory} if all squares anticommute. If $h$ is satisfactory, the direct sum of $\mathcal{H}(D^s)_\mathbb{F}$ over all possible $s$ with differential—the sum of $h_{s',s}$ over all possible arrows $s \rightarrow s'$ forms a complex. Denote this complex by $C_{n,h}(D)_\mathbb{F}$.

\textbf{Lemma 2} For any two satisfactory choices of signs $h', h''$ the complexes $C_{n,h'}(D)_\mathbb{F}$ and $C_{n,h''}(D)_\mathbb{F}$ are isomorphic.

Proofs of these two lemmas are left to the reader. From lemmas 1 and 2 we derive that the cohomology groups of $C_{n,h}(D)_\mathbb{F}$ do not depend on a satisfactory choice of signs. Denote these groups by $\mathcal{H}_n(D)_\mathbb{F}$. Yet another exercise in sign juggling shows that a Reidemeister move induces an isomorphism of these cohomology groups, and implies

\textbf{Proposition 3} Isomorphism classes of bigraded cohomology groups $\mathcal{H}_n(D)_\mathbb{F}$ are invariants of the framed knot $K$. Their Euler characteristic is the colored Jones polynomial:

$$J_n(K) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \text{rk}(\mathcal{H}_n^{i,j}(K)_\mathbb{F}).$$

\textit{Remark:} Similar considerations work over $\mathbb{Z}$, with the further complication that we cannot pass from the complex $\mathcal{C}(D^s)$ directly to its cohomology.
(since $\mathbb{Z}$ is not a field), and will be forced to work with these complexes throughout. Our diagrams will not be commutative or anticommutative, but rather commutative or anticommutative up to chain homotopy.

**Different definitions.** There are several competing definitions of the complex whose cohomology categorify the colored Jones polynomial of a link. Our maps between complexes assigned to cables go in the direction of reducing the number of strands. We could set up these maps to go in the opposite direction.

From section 3 we know that the algebraic counterpart of the complex $C_n(K)_F$ has cohomology only in degree 0. Taking the hint, we could downsize our complex

$$0 \rightarrow \mathcal{H}(K^n)_F \xrightarrow{d_0} \oplus \mathcal{H}(K^{n-2})_F \rightarrow \ldots$$

by restricting to the subgroup ker($d_0$) of $\mathcal{H}(K^n)_F$.

The above two modifications of $C_n(K)_F$ can be done independently, and give rise to the total of four complexes (including $C_n(K)_F$). We conjecture that if $F$ has characteristic 0, these four complexes have isomorphic cohomology groups.

**Examples:** (a) $n = 2$ case. We’ve observed in Section 2 that all four definitions give isomorphic theories over a field of characteristic different from 2.

(b) $n = 3$ case over $F_2$. The differential in

$$0 \rightarrow \mathcal{H}(D^3)_2 \xrightarrow{d_0} \mathcal{H}(D)_2 \oplus \mathcal{H}(D)_2 \rightarrow 0$$

is surjective, and the composition $d_0 \circ d'_{-1}$ is a permutation, where $d'_{-1}$ is the differential in

$$0 \rightarrow \mathcal{H}(D)_2 \oplus \mathcal{H}(D)_2 \rightarrow \mathcal{H}(D^3)_2 \rightarrow 0.$$

It is easy to derive that all four definitions give isomorphic theories in this case.

In yet another approach, we could consider an $S_n$-action on $\mathcal{H}(K^n)$ induced by permutations (braidings) of strands of the cable $K^n$, and take the $S_n$-invariants under this action. We expect that over $\mathbb{Q}$ this definition would give the same result as each the previous four. One problem with this approach is the projectivity of the $S_n$-action, well-defined only up to sign.
5 Categorification of the reduced colored Jones polynomial

In this section we categorify the reduced colored Jones polynomial \( \tilde{J}_n(L) \). Let’s start with the case of a knot. Given a framed colored knot \((K,n)\), the Jones polynomial \( J_n(K) \) can be computed with the help of the Kauffman bracket rules and the Jones-Wenzl projector, see [2]. The Jones-Wenzl projector is uniquely determined by graphical relations in figure 15.

\[ \begin{aligned}
\text{Figure 15: The Jones-Wenzl projector}
\end{aligned} \]

We denote the projector by \( p_n \) and by \( p'_n \) the projector divided by \([n+1]\).

In this section we assume familiarity with [5] and, in particular, with the ring \( H^n \) and its indecomposable left projective modules \( P_a \), for \( a \in B^n \), where \( B^n \) is the set of crossingless matchings of \( 2n \) points. Positioning the Jones-Wenzl projector in the upper half of the plane we can view it as a function from \( B^n \) to \( \mathbb{Z}[q, q^{-1}] \), by coupling the projector to any crossingless matching, as in figure 16.

The Jones-Wenzl projector evaluates to zero on any crossingless matching except for the one denoted \( e \) in figure 16 on which it takes value \([n+1]\). Therefore, \( p'_n \) is a "delta-function" on \( B^n \) supported on \( e \).

In [5] to any crossingless matching \( a \in B^n \) we assigned an indecomposable left projective graded \( H^n \)-module \( P_a \), and to a tangle \( t \) with no bottom and \( 2n \) top endpoints a complex of projective left \( H^n \)-modules \( F(t) \), well-defined in the homotopy category \( K^n \) of complexes of graded \( H^n \)-modules. To a tangle \( s \) with no top and \( 2n \) bottom endpoints we assigned a complex of projective right \( H^n \)-modules \( F(s) \). Coupling \( t \) with \( s \) corresponds to forming the tensor product, \( F(st) \cong F(s) \otimes_{H^n} F(t) \).

For each \( a \in B^n \) there exists a right \( H^n \)-module, denoted \( a\mathbb{Z} \), isomorphic as a graded abelian group to \( \mathbb{Z} \) placed in degree 0, with the idempotent \( 1_a \in H^n \) acting as the identity on \( a\mathbb{Z} \), and other minimal idempotents acting by 0. If we were working over a field rather than over \( \mathbb{Z} \), this module would
Jones–Wenzl projector \hspace{1cm} \text{Crossingless matchings}

\[
\begin{align*}
\text{a} & = 0 = \text{b} \\
\text{e} & = [n+1] \\
\end{align*}
\]

Coupling:

Figure 16: Projector, two crossingless matchings, and the coupling; \( n = 3 \).

have been the simple quotient of the right projective module \( _aP \), while over \( \mathbb{Z} \) the modules \( _a\mathbb{Z} \) are substitutes for simple modules (for instance, they give a basis in the Grothendieck group of \( H^n \)-modules).

We have

\[
_a\mathbb{Z} \otimes H^n P_b \cong \begin{cases} 
\mathbb{Z} & \text{if } a = b, \\
0 & \text{otherwise.}
\end{cases}
\]

In particular,

\[
e\mathbb{Z} \otimes H^n P_a \cong \begin{cases} 
\mathbb{Z} & \text{if } a = e, \\
0 & \text{otherwise,}
\end{cases}
\]

so that \( e\mathbb{Z} \) has the "delta-function" behaviour when coupled to projective modules. Therefore, in our categorification we can interpret the quotient Jones-Wenzl projector \( p'_n \) as the right \( H^n \)-module \( e\mathbb{Z} \). The ring \( H^n \) has infinite homological dimension, and \( e\mathbb{Z} \) does not have a finite length projective resolution. We do not know any explicit construction of a (necessarily infinite) projective resolution of \( e\mathbb{Z} \).

Remark: For a similar categorification of the Jones-Wenzl projector \( p_n \) we should look for a graded \( H^n \)-module which has a filtration with quotient modules isomorphic to \( e\mathbb{Z}\{-n\} \), \( e\mathbb{Z}\{-n+2}\), \ldots, \( e\mathbb{Z}\{n\} \). Such a module does not exist when \( n > 1 \).

We categorify the reduced colored Jones polynomial of a framed knot \( K \) with the help of \( e\mathbb{Z} \). Turn \( K \) into a tangle \( K^\bullet \) with no bottom and two top
endpoints and form the $n$-cable $K^n_*$ of this tangle. The tangle $K^n_*$ has no bottom and $2n$ top endpoints. Orient it the same way we oriented cables in Section 4. The invariant $\mathcal{F}(K^n_*)$ is a complex of graded projective left $H^n$-modules. Let

$$\tilde{C}_n(K) \overset{\text{def}}{=} \mathbb{Z} \otimes_{H^n} \mathcal{F}(K^n_*)$$

and define the reduced cohomology $\tilde{\mathcal{H}}_n(K)$ of $K$ colored by $n$ as the cohomology of the complex $\tilde{C}_n(K)$. These cohomology groups are bigraded,

$$\tilde{\mathcal{H}}_n(K) = \bigoplus_{i,j \in \mathbb{Z}} \tilde{\mathcal{H}}^{i,j}_n(K).$$

**Proposition 4** Cohomology groups $\tilde{\mathcal{H}}_n(K)$ are invariants of a framed knot $K$. Their Euler characteristic is the reduced colored Jones polynomial of $K$:

$$\tilde{J}_n(K) = \sum_{i,j \in \mathbb{Z}} (-1)^i q^j \operatorname{rk}(\tilde{\mathcal{H}}^{i,j}_n(K))$$

*Remark:* If $K_1$ is obtained from $K_0$ by a frame change as in figure 2 we have

$$\tilde{\mathcal{H}}_n(K_1) \cong \tilde{\mathcal{H}}_n(K_0)[-2m(m+1)][-2m^2] \text{ if } n = 2m,$$

$$\tilde{\mathcal{H}}_n(K_1) \cong \tilde{\mathcal{H}}_n(K_0)[-2m(m+2)][-2m(m+1)] \text{ if } n = 2m+1.$$  

We see that $\tilde{\mathcal{H}}_n$ depends on framing in a simpler way than $\mathcal{H}_n$.

*Examples:*

1. When $n = 0$, reduced cohomology does not depend on the knot and is isomorphic to $\mathbb{Z}$ placed in bidegree $(0,0)$.

2. For $n = 1$ reduced cohomology was defined in [3 Section 3]. $\mathcal{C}(K)$ is a complex of $\mathcal{A}$-modules, and tensoring it with the $\mathcal{A}$-module $\mathbb{Z}$ (where $X \in \mathcal{A}$ acts trivially) gives us the reduced complex $\tilde{\mathcal{C}}(K)$.

3. Reduced cohomology of the $n$-colored 0-framed unknot is $\mathbb{Z}$ placed in bidegree $(0,0)$.

For $n = 1$ the relation between $\mathcal{C}_n(K)$ and $\tilde{C}_n(K)$ takes the form of a short exact sequence

$$0 \to \tilde{C}(K)\{1\} \to \mathcal{C}(K) \to \tilde{C}(K)\{-1\} \to 0$$
of complexes, giving rise to a long exact sequence in cohomology

\[ \ldots \to \tilde{H}^{i,j-1}(K) \to H^{i,j}(K) \to \tilde{H}^{i,j+1}(K) \to \tilde{H}^{i+1,j-1}(K) \to \ldots \]

We do not know how to relate categorifications \( \mathcal{C}_n(K) \) and \( \tilde{C}_n(K) \) for \( n > 1 \).

**Categorification of the reduced colored Jones polynomial of links**

For simplicity, in this section we switch the base ring from \( \mathbb{Z} \) to the 2-element field \( \mathbb{F}_2 \).

Start with a colored link \( (L, n) \) with a distinguished component \( L' \) colored by \( n \). Turn \( L \) into a tangle \( L_\bullet \) with no bottom and 2 top endpoints by cutting a segment out of \( L' \). Taking the \( n \)-cable of \( L_\bullet \) at the component \( L' \) gives us a tangle, denoted \( L^n_\bullet \), with no bottom and \( 2n \) top endpoints.

Apply the construction of section 4 to all components of \( L^n_\bullet \) other than the \( n \) components coming from \( L' \). The result is a complex of projective left \( H^n \)-modules which we denote \( \mathcal{F}_n(L_\bullet) \). Let

\[ \tilde{C}_n(L)_2 \overset{\text{def}}{=} e_{\mathbb{F}_2} \otimes_{H^n} \mathcal{F}_n(L_\bullet) \]

(where \( e_{\mathbb{F}_2} \) is the right \( H^n \)-module \( \mathbb{F}_2 \otimes_{\mathbb{Z}} (e\mathbb{Z}) \)), and define the reduced cohomology \( \tilde{H}_n(L)_2 \) as the cohomology of the complex \( \tilde{C}_n(L)_2 \).

**Proposition 5** Isomorphism classes of cohomology groups \( \tilde{H}_n(L)_2 \) are invariants of a framed colored link \( L \) with a distinguished component \( L' \). Their Euler characteristic is the reduced colored Jones polynomial \( \tilde{J}_n(L) \).

### 6 Problems

(a) For a clean definition of cohomology groups \( H_n(L) \) it is necessary to understand the sign ambiguity of our cohomology theory. This ambiguity forced us into a clumsy definition of \( H_n(K)_F \), for a field \( F \) of characteristic other than 2, see section 4. No matter how we define \( H_n(L) \), taking care of signs is a prerequisite for extending these homology groups to an invariant of (framed and colored) link cobordisms.

(b) Cohomology theory described in section 4 depends nontrivially on the framing, and is expected to give rise to invariants of *framed* cobordisms between framed links. To check the invariance, one needs a list of movie moves for framed cobordisms.
Once (a) and (b) have been dealt with, one can move on to the following problem:

(c) Give a clean definition of cohomology groups $H_n(L)$, preferably over $\mathbb{Z}$, and extend this construction to a functor from the category of (colored, framed, decorated) link cobordisms to the category of bigraded abelian groups.

Here are some other problems that we’d like to mention:

(d) Establish equivalences between various definitions of cohomology groups (over $\mathbb{Q}$) discussed at the end of section 4.

(e) Extend the categorification of the reduced colored Jones polynomial to an invariant of (suitably decorated) link cobordisms.

(f) Relate categorifications of the colored Jones polynomial and the reduced colored Jones polynomial.

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