The convergence of the sums of independent random variables under the sub-linear expectations ∗

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Abstract

Let \( \{X_n; n \geq 1\} \) be a sequence of independent random variables on a probability space \((\Omega, \mathcal{F}, P)\) and \( S_n = \sum_{k=1}^{n} X_k \). It is well-known that the almost sure convergence, the convergence in probability and the convergence in distribution of \( S_n \) are equivalent. In this paper, we prove similar results for the independent random variables under the sub-linear expectations, and give a group of sufficient and necessary conditions for these convergence. For proving the results, the Levy and Kolmogorov maximal inequalities for independent random variables under the sub-linear expectation are established. As an application of the maximal inequalities, the sufficient and necessary conditions for the central limit theorem of independent and identically distributed random variables are also obtained.

Keywords: sub-linear expectation; capacity; independence; Levy maximal inequality; central limit theorem.

AMS 2010 subject classifications: 60F15; 60F05

1 Introduction and main results

The convergence of the sums of independent random variables are well-studied. For example, it is well-known that, if \( \{X_n; n \geq 1\} \) is a sequence of independent random variables on a probability space \((\Omega, \mathcal{F}, P)\), then that the infinite series \( \sum_{n=1}^{\infty} X_n \) is convergent almost surely, that it is convergent in probability and that it is convergent in distribution.

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are equivalent. In this paper, we consider this elementary equivalence under the sub-linear expectations. The general framework of the sub-linear expectation is introduced by Peng \cite{7,8,11} in a general function space by relaxing the linear property of the classical linear expectation to the sub-additivity and positive homogeneity (cf. Definition 1.1 below). The sub-linear expectation does not depend on the probability measure, provides a very flexible framework to model distribution uncertainty problems and produces many interesting properties different from those of the linear expectations. Under Peng’s framework, many limit theorems have been being gradually established recently, including the central limit theorem and weak law of large numbers (cf. Peng \cite{8,9}), the small derivation and Chung’s law of the iterated logarithm (cf. Zhang \cite{13}), the strong law of large numbers (cf. Chen \cite{1,3}, Hu \cite{6}, Zhang \cite{15}, Zhang and Lin \cite{17}), and the law of the iterated logarithm (cf. Chen \cite{2}, Zhang \cite{14}). For the convergence of the infinite series $\sum_{n=1}^{\infty} X_n$, Xu and Zhang \cite{12} gave sufficient conditions of the almost sure convergence for independent random variables under the sub-linear expectation via a three-series theorem, recently. In this paper, we will consider the necessity of these conditions and the equivalence of the almost sure convergence, the convergence in capacity and the convergence in distribution.

In the classical probability space, the Levy maximal inequalities are basic to the study of the almost sure behavior of sums of independent random variables and a key to show that the convergence in probability of $\sum_{n=1}^{\infty} X_n$ implies its almost sure convergence. We will establish Levy type inequalities under the sub-linear expectation. For showing that the convergence in distribution of $\sum_{n=1}^{\infty} X_n$ implies its convergence in probability, the characteristic function is a basic tool. But, under the sub-linear expectation, there is no such tools. We will find a new way to show a similar implication under the sub-linear expectations basing on a Komlogorov type maximal inequality.

As for the central limit theorem, it is well-known that the finite variances and mean zeros are sufficient and necessary for $\frac{\sum_{k=1}^{n} X_k}{\sqrt{n}}$ to converge in distribution to a normal random variable if $\{X_n; n \geq 1\}$ is a sequence of independent and identically distributed random variables on a classical probability space $(\Omega, \mathcal{F}, P)$. Under the sub-linear expectation, Peng \cite{8,9} proved the central limit theorem under the finite $(2+\alpha)$-th moment. By applying a moment inequality and the truncation method, Zhang \cite{14} and Lin and Zhang \cite{5} showed that the moment condition can be weakened to the finite second moment. A nature question
is whether the finite second moment is necessary. In this paper, by applying the max- 
imal inequalities, we will obtain the sufficient and necessary conditions for the central limit 
theorem.

In the remainder of the section, we state some notation. In the next section, we will 
establish the maximal inequalities for random variables under the sub-linear expectation. 
The results on the convergence of the infinite series of random variables will given in Section 
3. The sufficient and necessary conditions for the central limit theorem are given in Section 
4.

We use the framework and notations of Peng [8]. Let \((\Omega, \mathcal{F})\) be a given measurable space 
and let \(H\) be a linear space of real functions defined on \((\Omega, \mathcal{F})\) such that if \(X_1, \ldots, X_n \in H\) 
then \(\varphi(X_1, \ldots, X_n) \in H\) for each \(\varphi \in C_{l,\text{Lip}}(\mathbb{R}_n)\), where \(C_{l,\text{Lip}}(\mathbb{R}_n)\) denotes the linear space 
of local Lipschitz functions \(\varphi\) satisfying 
\[
|\varphi(x) - \varphi(y)| \leq C(1 + |x|^m + |y|^m)|x - y|, \quad \forall x, y \in \mathbb{R}_n,
\]
for some \(C > 0, m \in \mathbb{N}\) depending on \(\varphi\).

\(H\) is considered as a space of “random variables”. In this case we denote \(X \in H\). In the 
paper, we also denote \(C_{b,\text{Lip}}(\mathbb{R}_n)\) the space of bounded Lipschitz functions, \(C_b(\mathbb{R}_n)\) the space 
of bounded continuous functions, and \(C^{1}_b(\mathbb{R}_n)\) the space of bounded continuous functions 
with bounded continuous derivations on \(\mathbb{R}_n\).

**Definition 1.1** A sub-linear expectation \(\hat{E}\) on \(\mathcal{H}\) is a function \(\hat{E} : \mathcal{H} \rightarrow \mathbb{R}\) satisfying the 
following properties: for all \(X, Y \in \mathcal{H}\), we have 

(a) **Monotonicity:** If \(X \geq Y\) then \(\hat{E}[X] \geq \hat{E}[Y]\);

(b) **Constant preserving:** \(\hat{E}[c] = c\);

(c) **Sub-additivity:** \(\hat{E}[X+Y] \leq \hat{E}[X]+\hat{E}[Y]\) whenever \(\hat{E}[X]+\hat{E}[Y]\) is not of the form \(+\infty -\infty\) 
or \(-\infty + \infty\);

(d) **Positive homogeneity:** \(\hat{E}[\lambda X] = \lambda \hat{E}[X], \lambda \geq 0\).

Here \(\mathbb{R} = [-\infty, \infty]\). The triple \((\Omega, \mathcal{H}, \hat{E})\) is called a sub-linear expectation space. Give a 
sub-linear expectation \(\hat{E}\), let us denote the conjugate expectation \(\hat{E}\) of \(\hat{E}\) by 
\[
\hat{E}[X] := -\hat{E}[-X], \quad \forall X \in \mathcal{H}.
\]
From the definition, it is easily shown that $\widehat{E}[X] \leq \widehat{E}[X]$, $\widehat{E}[X+c] = \widehat{E}[X] + c$ and $\widehat{E}[X-Y] \geq \widehat{E}[X] - \widehat{E}[Y]$ for all $X, Y \in \mathcal{H}$ with $\widehat{E}[Y]$ being finite. Further, if $\widehat{E}[|X|]$ is finite, then $\widehat{E}[X]$ and $\widehat{E}[X]$ are both finite.

**Definition 1.2** (i) (Identical distribution) Let $X_1$ and $X_2$ be two $n$-dimensional random vectors defined respectively in sub-linear expectation spaces $(\Omega_1, \mathcal{H}_1, \widehat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \widehat{E}_2)$.

They are called identically distributed, denoted by $X_1 \overset{d}{=} X_2$ if

$$\widehat{E}_1[\varphi(X_1)] = \widehat{E}_2[\varphi(X_2)], \quad \forall \varphi \in C_{b,Lip}(\mathbb{R}_n),$$

where $C_{b,Lip}(\mathbb{R}_n)$ is the space of bounded Lipschitz functions.

(ii) (Independence) In a sub-linear expectation space $(\Omega, \mathcal{H}, \widehat{E})$, a random vector $Y = (Y_1, \ldots, Y_n)$, $Y_i \in \mathcal{H}$ is said to be independent to another random vector $X = (X_1, \ldots, X_m)$, $X_i \in \mathcal{H}$ under $\widehat{E}$ if

$$\widehat{E}[\varphi(X, Y)] = \widehat{E}[\widehat{E}[\varphi(x, Y)]|_{x=X}], \quad \forall \varphi \in C_{b,Lip}(\mathbb{R}_m \times \mathbb{R}_n).$$

Random variables $\{X_n; n \geq 1\}$ are said to be independent, if $X_{i+1}$ is independent to $(X_1, \ldots, X_i)$ for each $i \geq 1$.

In Peng [8, 9, 10], the space of the test function $\varphi$ is $C_{l,Lip}(\mathbb{R}_n)$. Here, the test function $\varphi$ in the definition is limit in the space of bounded Lipschitz functions. When the considered random variables have finite moments of each order, i.e., $\widehat{E}[|X|^p] < \infty$ for each $p > 0$, then the space of test functions $C_{b,Lip}(\mathbb{R}_n)$ can be equivalently extended to $C_{l,Lip}(\mathbb{R}_n)$.

A function $V : \mathcal{F} \to [0, 1]$ is called a capacity if $V(\emptyset) = 0$, $V(\Omega) = 1$ and $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{F}$. Let $(\Omega, \mathcal{H}, \widehat{E})$ be a sub-linear space. We denote a pair $(\mathcal{V}, \mathcal{V})$ of capacities by

$$\mathcal{V}(A) := \inf\{\widehat{E}[\xi] : I_A \leq \xi, \xi \in \mathcal{H}\}, \quad \mathcal{V}(A) := 1 - \mathcal{V}(A^c), \quad \forall A \in \mathcal{F},$$

where $A^c$ is the complement set of $A$. Then

$$\widehat{E}[f] \leq \mathcal{V}(A) \leq \widehat{E}[g], \quad \mathcal{V}(f) \leq \mathcal{V}(A) \leq \mathcal{V}(g), \quad \text{if } f \leq I_A \leq g, f, g \in \mathcal{H}.$$ 

It is obvious that $\mathcal{V}$ is sub-additive, i.e., $\mathcal{V}(A \cup B) \leq \mathcal{V}(A) + \mathcal{V}(B)$. But $\mathcal{V}$ and $\widehat{E}$ are not. However, we have

$$\mathcal{V}(A \cup B) \leq \mathcal{V}(A) + \mathcal{V}(B) \quad \text{and} \quad \widehat{E}[X + Y] \leq \widehat{E}[X] + \widehat{E}[Y]$$
due to the fact that $\mathbb{V}(A^c \cap B^c) = \mathbb{V}(A^c \setminus B) \geq \mathbb{V}(A^c) - \mathbb{V}(B)$ and $\hat{\mathbb{E}}[-X - Y] \geq \hat{\mathbb{E}}[-X] - \hat{\mathbb{E}}[Y]$.

Further, if $X$ is not in $\mathcal{H}$, we define $\hat{\mathbb{E}}[X]$ by

$$\hat{\mathbb{E}}[X] = \inf \{ \hat{\mathbb{E}}[Y] : X \leq Y, Y \in \mathcal{H} \}.$$  

Then $\mathbb{V}(A) = \hat{\mathbb{E}}[I_A]$.

**Definition 1.3** (I) A function $V : \mathcal{F} \to [0, 1]$ is called to be countably sub-additive if

$$V\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} V(A_n), \quad \forall A_n \in \mathcal{F}.$$  

(II) A function $V : \mathcal{F} \to [0, 1]$ is called to be continuous if it satisfies:

(i) Continuity from below: $V(A_n) \uparrow V(A)$ if $A_n \uparrow A$, where $A_n, A \in \mathcal{F}$.

(ii) Continuity from above: $V(A_n) \downarrow V(A)$ if $A_n \downarrow A$, where $A_n, A \in \mathcal{F}$.

It is easily seen that a continuous capacity is countably sub-additive.

## 2 Maximal inequalities

In this section, we establish several inequalities on the maximal sums. The first one is the Levy maximal inequality.

**Lemma 2.1** Let $X_1, \cdots, X_n$ be independent random variables in a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, $S_k = \sum_{i=1}^{k} X_i$, and $0 < \alpha < 1$ be a real number. If there exist real constants $\beta_{n,k}$ such that

$$\mathbb{V}(S_k - S_n \geq \beta_{n,k} + \epsilon) \leq \alpha, \quad \text{for all } \epsilon > 0 \text{ and } k = 1, \cdots, n,$$

then

$$(1 - \alpha)\mathbb{V}\left(\max_{k \leq n} (S_k - \beta_{n,k}) > x + \epsilon\right) \leq \mathbb{V}(S_n > x), \quad \text{for all } x > 0, \epsilon > 0. \quad (2.1)$$

If there exist real constants $\beta_{n,k}$ such that

$$\mathbb{V}(|S_k - S_n| \geq \beta_{n,k} + \epsilon) \leq \alpha, \quad \text{for all } \epsilon > 0 \text{ and } k = 1, \cdots, n,$$

then

$$(1 - \alpha)\mathbb{V}\left(\max_{k \leq n} (|S_k| - \beta_{n,k}) > x + \epsilon\right) \leq \mathbb{V}(|S_n| > x), \quad \text{for all } x > 0, \epsilon > 0. \quad (2.2)$$
Proof. We only give the proof of (2.1) since the proof of (2.2) is similar. Let \( g_\epsilon(x) \) be a function with

\[
g_\epsilon \in C_b^1(\mathbb{R}) \quad \text{and} \quad I_{\{x \geq \epsilon\}} \leq g_\epsilon(x) \leq I_{\{x \geq \epsilon/2\}} \quad \text{for all} \quad x,
\]

where \( 0 < \epsilon < 1/2 \), \( C_b^1(\mathbb{R}) \) is the space of bounded continuous function having bounded continuous derivations. Denote \( Z_k = g_\epsilon (S_k - \beta_{n,k} - x) \), \( Z_0 = 0 \) and \( \eta_k = \prod_{i=1}^k (1 - Z_i) \). Then \( S_n - S_m \) is independent to \((Z_1, \ldots, Z_m)\), and

\[
(1 - \alpha) I\{\max_{k \leq n} (S_k - \beta_{n,k}) > x + \epsilon\}
= (1 - \alpha) \left[ 1 - \prod_{k=1}^n I\{S_k - \beta_{n,k} - x \leq \epsilon\} \right]
\leq (1 - \alpha) \left[ 1 - \eta_n \right] = (1 - \alpha) \left[ \sum_{m=1}^n \eta_{m-1} Z_m \right]
= \sum_{m=1}^n \eta_{m-1} Z_m I\{S_m - S_n < \beta_{n,m} + \epsilon/2\}
+ \sum_{m=1}^n \eta_{m-1} Z_m \left[ 1 - \alpha - I\{S_m - S_n < \beta_{n,m} + \epsilon/2\} \right]
\leq \sum_{m=1}^n \eta_{m-1} Z_m I\{S_n > x\} + \sum_{m=1}^n \eta_{m-1} Z_m \left[ -\alpha + I\{S_m - S_n \geq \beta_{n,m} + \epsilon/2\} \right]
= I\{S_n > x\} + \sum_{m=1}^n \eta_{m-1} Z_m \left[ -\alpha + g_\epsilon/2 (S_m - S_n - \beta_{n,m}) \right],
\]

where the second inequality above is due to the fact that on the event \( \{Z_m \neq 0\} \) and \( \{S_m - S_n < \beta_{n,m} + \epsilon/2\} \) we have \( S_n \geq S_m - (S_m - S_n) > x \). Note

\[
\hat{E} \left[ g_\epsilon/2 (S_m - S_n - \beta_{n,m}) \right] \leq V (S_m - S_n \geq \beta_{n,m} + \epsilon/4) \leq \alpha.
\]

By the independence,

\[
\hat{E} \left[ \eta_{m-1} Z_m \left[ -\alpha + g_\epsilon/2 (S_m - S_n - \beta_{n,m}) \right] \right]
= \hat{E} \left[ \eta_{m-1} Z_m \left\{ -\alpha + \hat{E} [g_\epsilon/2 (S_m - S_n - \beta_{n,m})] \right\} \right] \leq 0.
\]

By the sub-additivity of \( \hat{E} \), it follows that

\[
(1 - \alpha) V \left( \max_{k \leq n} (S_k - \beta_{n,k}) > x + \epsilon \right)
\leq V (S_n > x) + \sum_{m=1}^n \hat{E} \left[ \eta_{m-1} Z_m \left[ -\alpha + g_\epsilon/2 (S_m - S_n - \beta_{n,m}) \right] \right]
\leq V (S_n > x).
\]
The proof is completed. □

The second lemma is on the Kolmogorov type inequality.

**Lemma 2.2** Let \(X_1, \cdots, X_n\) be independent random variables in a sub-linear expectation space \((\Omega, \mathcal{H}, \hat{E})\). Let \(S_k = \sum_{i=1}^{k} X_i\).

(i) Suppose \(|X_k| \leq c, k = 1, \cdots, n\). Then

\[
\mathbb{V}(\max_{k \leq n} |S_k| > x) \geq 1 - \frac{(x + c)^2 + 2x \sum_{k=1}^{n} \{ (\hat{E}[X_k])^+ + (\hat{E}[-X_k])^+ \}}{\sum_{k=1}^{n} \hat{E}[X_k^2]}, \tag{2.4}
\]

for all \(x > 0\).

(ii) Suppose \(X_k \leq c, \hat{E}[X_k] \geq 0, k = 1, \cdots, n\). Then

\[
\mathbb{V}(\max_{k \leq n} S_k > x) \geq 1 - \frac{x + c}{\sum_{k=1}^{n} \hat{E}[X_k]} \text{ for all } x > 0. \tag{2.5}
\]

**Proof.** (i) Let \(g_\epsilon\) be defined as in (2.3). Denote \(Z_k = g_\epsilon(|S_k| - x), Z_0 = 0, \eta_k = \prod_{i=1}^{k} (1 - Z_i)\). Then \(I\{|S_k| \geq x + \epsilon\} \leq Z_k \leq I\{|S_k| > x\}\). Also, \(|S_{k-1}| < x + \epsilon\) and \(|S_k| < |S_{k-1}| + |X_k| \leq x + \epsilon + c\) on the event \(\{\eta_{k-1} \neq 0\}\). So

\[
S_{k-1}^2 \eta_{k-1} + 2S_{k-1} X_k \eta_{k-1} + X_k^2 \eta_{k-1} = S_k^2 \eta_k + S_k^2 \eta_{k-1} Z_k
\]

\[
\leq S_k^2 \eta_k + (x + \epsilon + c)^2 [\eta_{k-1} - \eta_k].
\]

Taking the summation over \(k\) yields

\[
\left( \sum_{k=1}^{n} \hat{E}[X_k^2]\right) \eta_n + \sum_{k=1}^{n} \left( X_k^2 - \hat{E}[X_k^2]\right) \eta_{k-1} \leq \sum_{k=1}^{n} X_k^2 \eta_{k-1}
\]

\[
\leq S_n^2 \eta_n + (x + \epsilon + c)^2 [1 - \eta_n] - 2 \sum_{k=1}^{n} S_{k-1} X_k \eta_{k-1}
\]

\[
\leq (x + \epsilon)^2 \eta_n + (x + \epsilon + c)^2 [1 - \eta_n] - 2 \sum_{k=1}^{n} S_{k-1} X_k \eta_{k-1}
\]

\[
\leq (x + \epsilon + c)^2 - 2 \sum_{k=1}^{n} S_{k-1} X_k \eta_{k-1}.
\]

Write \(B_n^2 = \sum_{k=1}^{n} \hat{E}[X_k^2]\). It follows that

\[
1 - \frac{(x + \epsilon + c)^2}{B_n^2} + \frac{\sum_{k=1}^{n} \left( X_k^2 - \hat{E}[X_k^2]\right) \eta_{k-1}}{B_n^2}
\]

\[
\leq 1 - \eta_n + \frac{2}{B_n^2} \sum_{k=1}^{n} [X_k S_{k-1}^- \eta_{k-1} - X_k S_{k-1}^+ \eta_{k-1}].
\]
Note

\[ \hat{E}[X_k S_{k-1}^{-1} \eta_{k-1}] = \hat{E}[X_k] \hat{E}[S_{k-1}^{-1} \eta_{k-1}] \leq (x + \epsilon)(\hat{E}[X_k])^+, \]

\[ \hat{E}[-X_k S_{k-1}^+ \eta_{k-1}] = \hat{E}[-X_k] \hat{E}[S_{k-1}^+ \eta_{k-1}] \leq (x + \epsilon)(\hat{E}[-X_k])^+ \]

and

\[ \hat{E}\left[ \sum_{k=1}^{n} \left( X_k^2 - \hat{E}[X_k^2] \right) \eta_{k-1} \right] = \hat{E}\left[ \sum_{k=1}^{n} \left( X_k^2 - \hat{E}[X_k^2] \right) \eta_{k-1} \right] \eta_{k-1} + \hat{E}[X_n^2 - \hat{E}[X_n^2]] \]

\[ = \hat{E}\left[ \sum_{k=1}^{n-1} \left( X_k^2 - \hat{E}[X_k^2] \right) \eta_{k-1} \right] = \cdots = 0. \quad (2.6) \]

It follows that

\[ 1 - \frac{(x + \epsilon + c)^2}{B_n^2} - \frac{2(x + \epsilon) \sum_{k=1}^{n} \{ (\hat{E}[X_k])^+ + (\hat{E}[X_k])^+ \}}{B_n^2} \]

\[ \leq \hat{E} [1 - \eta_n] \leq \mathbb{V} \left( \max_{k \leq n} |S_k| > x \right). \]

By letting \( \epsilon \to 0 \), we obtain (2.4). The proof of (i) is completed.

(ii) Redefine \( Z_k \) and \( \eta_k \) by \( Z_k = g_k (S_k - x), \ Z_0 = 0, \ \eta_k = \prod_{i=1}^{k} (1 - Z_i) \). Then \( I \{ S_k \geq x + \epsilon \} \leq Z_k \leq I \{ S_k > x \} \). Also, \( S_{k-1} < x + \epsilon \) and \( S_k = S_{k-1} + X_k < x + \epsilon + c \) on the event \( \{ \eta_{k-1} \neq 0 \} \). So

\[ S_{k-1} \eta_{k-1} + X_k \eta_{k-1} = S_k \eta_k + S_k \eta_{k-1} Z_k \leq S_k \eta_k + (x + \epsilon + c) \eta_{k-1} Z_k. \]

Taking the summation over \( k \) yields

\[ \left( \sum_{k=1}^{n} \hat{E}[X_k] \right) \eta_{n} + \sum_{k=1}^{n} \left( X_k - \hat{E}[X_k] \right) \eta_{k-1} \]

\[ \leq \sum_{k=1}^{n} X_k \eta_{k-1} \leq S_n \eta_n + (x + \epsilon + c) \left[ 1 - \eta_n \right] \]

\[ \leq (x + \epsilon) \eta_n + (x + \epsilon + c) \left[ 1 - \eta_n \right] \leq (x + \epsilon + c). \]

Write \( e_n = \sum_{k=1}^{n} \hat{E}[X_k] \). It follows that

\[ 1 - \frac{(x + \epsilon + c)}{e_n} + \frac{\sum_{k=1}^{n} \left( X_k - \hat{E}[X_k] \right) \eta_{k-1}}{e_n} \leq 1 - \eta_n. \]
Note
\[
\hat{\mathbb{E}} \left[ \sum_{k=1}^{n} \left( X_k - \hat{\mathbb{E}}[X_k] \right) \eta_{k-1} \right] = \hat{\mathbb{E}} \left[ \sum_{k=1}^{n-1} \left( X_k - \hat{\mathbb{E}}[X_k] \right) \eta_{k-1} \right] = \cdots = 0,
\]
similar to (2.6). It follows that
\[
1 - \frac{x + \epsilon + c}{c_n} \leq \hat{\mathbb{E}} \left[ 1 - \eta_n \right] \leq \mathbb{V} \left( \max_{k \leq n} S_k > x \right).
\]
By letting \( \epsilon \to 0 \), we obtain (2.4). The proof is completed. \( \square \)

The following lemma on the bounds of the capacities via moments will be used in the paper.

Lemma 2.3 (\cite{14}) Let \( X_1, X_2, \ldots, X_n \) be independent random variables in \( (\Omega, \mathcal{H}, \mathbb{V}) \). If
\( \hat{\mathbb{E}}[X_k] \leq 0, k = 1, \ldots, n, \) then there exists a constant \( C > 0 \) such that
\[
\mathbb{V}(S_n \geq x) \leq C \frac{\sum_{k=1}^{n} \hat{\mathbb{E}}[X_k^2]}{x^2} \text{ for all } \forall x > 0.
\]

3 The convergence of infinite series

Our results on the convergence of the series \( \sum_{n=1}^{\infty} \) are stated as three theorems. The first one gives the equivalency between the almost sure convergence and the convergence in capacity.

Theorem 3.1 Let \( \{X_n; n \geq 1\} \) be a sequence of independent random variables in a sub-linear expectation space \( (\Omega, \mathcal{H}, \hat{\mathbb{E}}) \), \( S_n = \sum_{k=1}^{n} X_k \), and \( S \) be a random variable in the measurable space \( (\Omega, \mathcal{F}) \).

(i) If \( \mathbb{V} \) is countably sub-additive, and
\[
\mathbb{V} \left( |S_n - S| \geq \epsilon \right) \to 0 \text{ as } n \to \infty \text{ for all } \epsilon > 0,
\]
then
\[
\mathbb{V} \left( \left\{ \omega : \lim_{n \to \infty} S_n(\omega) \neq S(\omega) \right\} \right) = 0.
\]
When (3.2) holds, we call that \( \sum_{n=1}^{\infty} X_n \) is almost surely convergent in capacity, and when (3.1) holds, we call that \( \sum_{n=1}^{\infty} X_n \) is convergent in capacity.

(ii) If \( \mathbb{V} \) is continuous, then (3.2) implies (3.1).
The second theorem gives the equivalency between the convergence in capacity and the convergence in distribution.

**Theorem 3.2** Let \( \{X_n; n \geq 1\} \) be a sequence of independent random variables in a sub-linear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\), \( S_n = \sum_{k=1}^{n} X_k \).

(i) If there is a random variable \( S \) in the measurable space \((\Omega, \mathcal{F})\) such that

\[
\mathbb{V}(|S_n - S| \geq \epsilon) \to 0 \quad \text{as } n \to \infty \quad \text{for all } \epsilon > 0, \tag{3.3}
\]

and \( S \) is tight under \( \mathbb{E} \), i.e., \( \mathbb{E}[I_{\{S \leq x\}}] = \mathbb{V}(|S| > x) \to 0 \) as \( x \to \infty \), then

\[
\mathbb{E}[\phi(S_n)] \to \mathbb{E}[\phi(S)], \quad \phi \in C_b(\mathbb{R}), \tag{3.4}
\]

where \( C_b(\mathbb{R}) \) is the space of bounded continuous functions on \( \mathbb{R} \). When (3.4) holds, we call that \( \sum_{n=1}^{\infty} X_n \) is convergent in distribution.

(ii) Suppose that there is a sub-linear space \((\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})\) and a random variable \( \tilde{S} \) on it such that \( \tilde{S} \) is tight under \( \tilde{\mathbb{E}} \), i.e., \( \tilde{\mathbb{V}}(|\tilde{S}| > x) \to 0 \) as \( x \to \infty \), and

\[
\tilde{\mathbb{E}}[\phi(S_n)] \to \tilde{\mathbb{E}}[\phi(\tilde{S})], \quad \phi \in C_b(\mathbb{R}), \tag{3.5}
\]

then \( S_n \) is a Cauchy sequence in capacity \( \mathbb{V} \), namely

\[
\mathbb{V}(|S_n - S_m| \geq \epsilon) \to 0 \quad \text{as } n, m \to \infty \quad \text{for all } \epsilon > 0. \tag{3.6}
\]

Furthermore, if \( \mathbb{V} \) is countably sub-additive, then on the measurable space \((\Omega, \mathcal{F})\) there is a random variable \( S \) which is tight under \( \mathbb{E} \), such that (3.4) and (3.2) hold.

Recently, Xu and Zhang [12] gave sufficient conditions for \( \sum_{n=1}^{\infty} X_n \) to be convergent almost surely in capacity via three series theorem. The third theorem of us gives the sufficient and necessary conditions for \( S_n \) to be a Cauchy sequence in capacity. For any random variable \( X \) and constant \( c \), we denote \( X^c = (-c) \vee (X \wedge c) \).

**Theorem 3.3** Let \( \{X_n; n \geq 1\} \) be a sequence of independent random variables in \((\Omega, \mathcal{H}, \mathbb{E})\), \( S_n = \sum_{k=1}^{n} X_k \). Then \( S_n \) will be a Cauchy sequence in capacity \( \mathbb{V} \) if the following three conditions hold for some \( c > 0 \).
(S1) \( \sum_{n=1}^{\infty} \mathbb{V}(|X_n| > c) < \infty \),

(S2) \( \sum_{n=1}^{\infty} \hat{E}[X_n^c] \) and \( \sum_{n=1}^{\infty} \hat{E}[-X_n^c] \) are both convergent,

(S3) \( \sum_{n=1}^{\infty} \hat{E}\left[\left(X_n^c - \hat{E}[X_n^c]\right)^2\right] < \infty \) or/and \( \sum_{n=1}^{\infty} \hat{E}\left[\left(X_n^c + \hat{E}[-X_n^c]\right)^2\right] < \infty \).

Conversely, if \( S_n \) is a Cauchy sequence in capacity \( \mathbb{V} \), then (S1),(S2) and (S3) will hold for all \( c > 0 \).

From Theorem 3.3 we have the following three series theorem on the sufficient and necessary conditions for the almost sure convergence of \( \sum_{n=1}^{\infty} X_n \).

**Corollary 3.1** Let \( \{X_n; n \geq 1\} \) be a sequence of independent random variables in \( (\Omega, \mathcal{H}, \hat{E}) \). Suppose that \( \mathbb{V} \) is countably sub-additive. Then \( \sum_{n=1}^{\infty} X_n \) will converge almost surely in capacity if the three conditions (S1),(S2) and (S3) in Theorem 3.3 hold for some \( c > 0 \). Conversely, if \( \mathbb{V} \) is continuous and \( \sum_{n=1}^{\infty} X_n \) is convergent almost surely in capacity, then (i),(ii) and (iii) will hold for all \( c > 0 \).

The sufficiency of (S1), (S2) and (S3) is proved by Xu and Zhang [12], and also follows from Theorem 3.3 and the second part of conclusion of Theorem 3.2 (ii). The necessity follows from Theorem 3.3 and Theorem 3.1 (ii).

The prove Theorems 3.1 and 3.2. We need some more lemmas. The first lemma is a version of Theorem 9 of Peng [10].

**Lemma 3.1** Let \( \{Y_n; n \geq 1\} \) be a sequence of \( d \)-dimensional random variables in a sub-linear expectation space \( (\Omega, \mathcal{H}, \hat{E}) \). Suppose that \( Y_n \) is asymptotically tight, i.e.,

\[
\limsup_{n \to \infty} \hat{E}\left[ I\{\|Y_n\| \leq x\} \right] = \limsup_{n \to \infty} \mathbb{V}(\|Y_n\| > x) \to 0 \quad \text{as} \quad x \to \infty.
\]

Then for any subsequence \( \{Y_{n_k}\} \) of \( \{Y_n\} \), there exist further a subsequence \( \{Y_{n_{k'}}\} \) of \( \{Y_{n_k}\} \) and a sub-linear expectation space \( (\hat{\Omega}, \hat{\mathcal{H}}, \hat{E}) \) with a \( d \)-dimensional random variable \( Y \) on it such that

\[
\hat{E}\left[\phi(Y_{n_{k'}})\right] \to \hat{E}[\phi(Y)] \quad \text{for any} \quad \phi \in C_b(\mathbb{R}^d)
\]

and \( Y \) is tight under \( \hat{E} \).
Proof. Let

\[ \mathbb{E} [\phi] = \limsup_{n \to \infty} \hat{\mathbb{E}} [\phi(Y_n)], \quad \phi \in C_b(\mathbb{R}^d). \]

Then \( \mathbb{E} \) is a sub-linear expectation on the function space \( C_b(\mathbb{R}^d) \) and is tight in sense that for any \( \epsilon > 0 \), there is a compact set \( K = \{ x : \|x\| \leq M \} \) for which \( \mathbb{E} [I_{K^c}] < \epsilon \). With the same argument as in the proof of Theorem 9 of Peng [10], there is a countable subset \( \{ \varphi_j \} \) of \( C_b(\mathbb{R}^d) \) such that for each \( \phi \in C_b(\mathbb{R}^d) \) and any \( \epsilon > 0 \) one can find a \( \varphi_j \) satisfying

\[ \mathbb{E} [\|\phi - \varphi_j\|] < \epsilon. \] (3.7)

On the other hand, for each \( \varphi_j \), the sequence \( \hat{\mathbb{E}} [\varphi_j(Y_n)] \) is bounded and so there is a Cauchy subsequence. Note that the set \( \{ \varphi_j \} \) is countable. By the diagonal choice method, one can find a sequence \( \{ n_k \} \subset \{ n \} \) such that \( \hat{\mathbb{E}} [\varphi_j(Y_{n_k})] \) is a Cauchy sequence for each \( \varphi_j \). Now, we show that \( \hat{\mathbb{E}} [\phi(Y_{n_k})] \) is a Cauchy sequence for any \( \phi \in C_b(\mathbb{R}^d) \). For any \( \epsilon > 0 \), choose a \( \varphi_j \) such that (3.7) holds. Then

\[ \hat{\mathbb{E}} [\phi(Y_{n_k})] - \hat{\mathbb{E}} [\phi(Y_{n_l})] \leq \hat{\mathbb{E}} [\|\varphi_j(Y_{n_k}) - \varphi_j(Y_{n_l})\|] + \hat{\mathbb{E}} [\|\phi(Y_{n_k}) - \varphi_j(Y_{n_k})\|]. \]

Taking the limits yields

\[ \limsup_{k,l \to \infty} \hat{\mathbb{E}} [\phi(Y_{n_k})] - \hat{\mathbb{E}} [\phi(Y_{n_l})] \leq 0 + 2\mathbb{E} [\|\phi - \varphi_j\|] < 2\epsilon. \]

Hence \( \hat{\mathbb{E}} [\phi(Y_{n_k})] \) is a Cauchy sequence for any \( \phi \in C_b(\mathbb{R}^d) \), and then

\[ \lim_{k \to \infty} \hat{\mathbb{E}} [\phi(Y_{n_k})] \text{ exists and is finite for any } \phi \in C_b(\mathbb{R}^d). \] (3.8)

Now, let \( \overline{\Omega} = \mathbb{R}^d \), \( \overline{H} = C_{l,\text{lip}}(\mathbb{R}^d) \). Define

\[ \overline{\mathbb{E}} [\varphi] = \limsup_{k \to \infty} \hat{\mathbb{E}} [\varphi(Y_{n_k})], \quad \varphi \in C_{l,\text{lip}}(\mathbb{R}^d). \]

Then \( (\overline{\Omega}, \overline{H}, \overline{\mathbb{E}}) \) is a sub-linear expectation space. Define the random variable \( Y \) by \( Y(x) = x, \ x \in \overline{\Omega} \). From (3.8) it follows that

\[ \lim_{k \to \infty} \hat{\mathbb{E}} [\phi(Y_{n_k})] = \overline{\mathbb{E}} [\phi(Y)] \text{ for any } \phi \in C_b(\mathbb{R}^d). \]

The proof is completed. □
Lemma 3.2 Let $X$ and $Y$ be random variables in a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{E})$.

Suppose that $Y$ and $X$ are independent ($Y$ is independent to $X$, or $X$ is independent to $Y$), and $X$ is tight, i.e. $\mathbb{V}(|X| \geq x) \to 0$ as $x \to \infty$. If $X + Y \overset{d}{=} X$, then $\mathbb{V}(|Y| \geq \epsilon) = 0$ for all $\epsilon > 0$.

Proof. Without loss of generality, we assume that $Y$ is independent to $X$. We can find a sub-linear expectation space $(\Omega', \mathcal{H}', \hat{E}')$ on which there are independent random variables $X_1, Y_1, Y_2, \cdots, Y_n, \cdots$ such that $X_1 \overset{d}{=} X$, $Y_i \overset{d}{=} Y$, $i = 1, 2, \cdots$. Without loss of generality, assume $(\Omega', \mathcal{H}', \hat{E}') = (\Omega, \mathcal{H}, \hat{E})$. Let $S_k = \sum_{j=1}^{k} Y_k$. Then $X_1 + S_k = X$. So,

$$\max_{k \leq n} \mathbb{V}(|S_k| > x_0) \leq \max_{k \leq n} \mathbb{V}(|X_1 + S_k| > x_0/2) + \mathbb{V}(|X_1| > x_0/2)$$

$$\leq \hat{E} \left[ g_{1/2} \left( \frac{|X_1 + S_k|}{x_0} \right) \right] + \hat{E} \left[ g_{1/2} \left( \frac{|X_1|}{x_0} \right) \right]$$

$$\leq 2 \hat{E} \left[ g_{1/2} \left( \frac{|X_1|}{x_0} \right) \right] \leq 2 \mathbb{V}(|X| \geq x_0/4) < 1/4 \quad (3.9)$$

for $x_0$ large enough, where $g_{\epsilon}$ is defined as in (2.3). By Lemma 2.1,

$$\mathbb{V}(\max_{k \leq n} |S_k| > 2x_0 + \epsilon) \leq \frac{4}{3} \max_{k \leq n} \mathbb{V}(|S_k| > x_0) \leq \frac{4}{3} \cdot 2 \mathbb{V}(|X| \geq x_0/4) < \frac{1}{3} \quad (3.10)$$

It follows that for any $\epsilon > 0,$

$$\mathbb{V}(\max_{k \leq n} |Y_k| > 4x_0 + 2\epsilon) < \frac{1}{3}$$

Let $Z_k = g_{\epsilon}(|Y_k| - 4x_0 - 2\epsilon)$, where $g_{\epsilon}$ is defined as in (2.3). Denote $q = \mathbb{V}(|Y_1| > 4x_0 + 3\epsilon)$. Then $Z_1, Z_2, \cdots, Z_n$ are independent and identically distributed with $\{|Y_k| > 4x_0 + 3\epsilon\} \leq Z_k \leq \{|Y_k| > 4x_0 + 2\epsilon\}$ and $\hat{E}[Z_1] \geq \mathbb{V}(|Y_1| > 4x_0 + 3\epsilon) = q$. Then by Lemma 2.2 (ii),

$$\frac{1}{3} > \mathbb{V} \left( \sum_{k=1}^{n} Z_k \geq 1 \right) \geq 1 - \frac{1 + 1}{\sum_{k=1}^{n} \hat{E}[Z_k]} \geq 1 - \frac{2}{nq}. \quad (3.11)$$

The above inequality holds for all $n$, which is impossible unless $q = 0$. So we conclude that

$$\mathbb{V}(|Y_1| > 4x_0 + \epsilon) = 0 \quad \text{for any } \epsilon > 0.$$ 

Now, let $\tilde{Y}_k = (-5x_0) \lor Y_k \land (5x_0)$, $\tilde{S}_k = \sum_{i=1}^{k} \tilde{Y}_i$. Then $\tilde{Y}_1, \cdots, \tilde{Y}_n$ are independent and identically distributed bounded random variables, $\mathbb{V}(\tilde{Y}_k \neq Y_k) = 0$ and $\mathbb{V}(\tilde{S}_k \neq S_k) = 0$. If $\hat{E}[\tilde{Y}_1] > 0$, then by Lemma 2.2 (ii) again,

$$\mathbb{V}(\max_{k \leq n} S_k \geq 3x_0) = \mathbb{V}(\max_{k \leq n} \tilde{S}_k \geq 3x_0) \geq 1 - \frac{3x_0 + 5x_0}{n\hat{E}[Y_1]},$$
which contradicts to (3.10) when \( n > 12x_0/E[Y_1] \). Hence, \( E[Y_1] \leq 0 \). Similarly, \( E[-Y_1] \leq 0 \).

We conclude that \( E[Y_1] = E[-Y_1] = 0 \). Now, if \( E[Y_1^2] \neq 0 \), then by Lemma 2.2 (i) we have

\[
V(\max_{k \leq n} |S_k| \geq 3x_0) \geq 1 - \frac{(3x_0 + 5x_0)^2}{nE[Y_1^2]},
\]

which contradicts to (3.10) when \( n > 96x_0^2/E[Y_1^2] \). We conclude that \( E[Y_1^2] = 0 \).

Finally, for any \( \epsilon > 0 \) (\( \epsilon < 5x_0 \)),

\[
V(|Y| \geq \epsilon) \leq \frac{E[Y^2 \wedge (5x_0)^2]}{\epsilon^2} = \frac{E[Y_1^2]}{\epsilon^2} = 0.
\]

The proof is completed. □

**Proof of Theorem 3.1** (i) Let \( \epsilon_k = 1/2^k \), \( \delta_k = 1/4^k \). By (3.1), there exists a sequence \( n_1 < n_2 < \cdots < n_k \to \infty \), such that

\[
\max_{n \geq n_k} V(|S_n - S| \geq \epsilon_k) < \delta_k.
\] (3.12)

By the countably sub-additivity of \( V \), we have

\[
V\left( \limsup_{k \to \infty} |S_{n_k} - S| > 0 \right) \leq V\left( \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{|S_{n_k} - S| \geq \epsilon_k\} \right)
\]
\[
\leq \sum_{k=m}^{\infty} V(\{|S_{n_k} - S| \geq \epsilon_k\}) \leq \sum_{k=m}^{\infty} \delta_k \to 0 \text{ as } m \to \infty.
\]

By (3.12), \( \max_{n \geq n_k} V(|S_n - S_{n_k+1}| \geq 2\epsilon_k) < 2\delta_k < 1/2 \). Apply the Levy inequality (2.2) yields

\[
V\left( \frac{\max_{n_k \leq n \leq n_{k+1}} |S_n - S_{n_k}|}{5\epsilon_k} > 2\epsilon_k \right) \leq 2V\left( |S_{n_{k+1}} - S_{n_k}| > 2\epsilon_k \right) < 4\delta_k.
\] (3.13)

By the countably sub-additivity of \( V \) again,

\[
V\left( \limsup_{k \to \infty} \max_{n_k \leq n \leq n_{k+1}} |S_n - S_{n_k}| > 0 \right)
\]
\[
\leq V\left( \bigcap_{m=1}^{\infty} \bigcup_{k=m}^{\infty} \{ \max_{n_k \leq n \leq n_{k+1}} |S_n - S_{n_k}| \geq 5\epsilon_k \} \right)
\]
\[
\leq \sum_{k=m}^{\infty} V\left( \max_{n_k \leq n \leq n_{k+1}} |S_n - S_{n_k}| \geq 5\epsilon_k \right) \leq 4 \sum_{k=m}^{\infty} \delta_k \to 0 \text{ as } m \to \infty.
\]

It follows that

\[
V\left( \limsup_{n \to \infty} |S_n - S| > 0 \right)
\]
\[
\leq V\left( \limsup_{k \to \infty} |S_{n_k} - S| > 0 \right) + V\left( \limsup_{k \to \infty} \max_{n_k \leq n \leq n_{k+1}} |S_n - S_{n_k}| > 0 \right) = 0.
\]
From (3.2) and the continuity of $\mathbb{V}$, it follows that for any $\epsilon > 0$, 
\[
0 \geq \mathbb{V}\left(\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \{|S_m - S| \geq \epsilon\}\right) = \lim_{n \to \infty} \mathbb{V}\left(\bigcup_{m=n}^{\infty} \{|S_m - S| \geq \epsilon\}\right) \geq \limsup_{n \to \infty} \mathbb{V}\left(|S_n - S| \geq \epsilon\right).
\]

(3.1) follows. The proof is completed. \[\Box\]

**Proof of Theorem 3.2.** (i) We first show that (3.4) holds for any bounded uniformly continuous function $\phi$. For any $\epsilon > 0$, there is a $\delta > 0$ such that $|\phi(x) - \phi(y)| < \epsilon$ when $|x - y| < \delta$. It follows that 
\[
\left|\hat{\mathbb{E}}[\phi(S_n)] - \hat{\mathbb{E}}[\phi(S)]\right| \leq \epsilon + 2 \sup_x |\phi(x)| \mathbb{V}(|S_n - S| \geq \delta).
\]
By letting $n \to \infty$ and the arbitrariness of $\epsilon > 0$, we obtain (3.4). Now, suppose that $\phi$ is a bounded continuous function. Then for any $N > 1$, $\phi((-N) \vee x \wedge N)$ is a bounded uniformly continuous function. Hence 
\[
\lim_{n \to \infty} \hat{\mathbb{E}}[\phi((-N) \vee S_n \wedge N)] = \hat{\mathbb{E}}[\phi((-N) \vee S \wedge N)].
\]

On the other hand, 
\[
\left|\hat{\mathbb{E}}[\phi((-N) \vee S \wedge N)] - \hat{\mathbb{E}}[\phi(S)]\right| \leq 2 \sup_x |\phi(x)| \mathbb{V}(|S| > N) \to 0 \text{ as } N \to \infty,
\]
and 
\[
\limsup_{n \to \infty} \left|\hat{\mathbb{E}}[\phi((-N) \vee S_n \wedge N)] - \hat{\mathbb{E}}[\phi(S_n)]\right| 
\leq 2 \sup_x |\phi(x)| \limsup_{n \to \infty} \mathbb{V}(|S_n| \geq N) \leq 2 \sup_x |\phi(x)| \limsup_{n \to \infty} \hat{\mathbb{E}}\left[g_1\left(\frac{|S_n|}{N}\right)\right] 
\leq 2 \sup_x |\phi(x)| \limsup_{n \to \infty} \mathbb{V}(|S| \geq N/2) \to 0 \text{ as } N \to \infty,
\]
where $g_\epsilon$ is defined as in (2.3). Hence, (3.4) holds for a bounded continuous function $\phi$.

(ii) Note 
\[
\mathbb{V}(|S_n - S_m| \geq 2x) \leq \mathbb{V}(|S_n| \geq x) + \mathbb{V}(|S_m| \geq x).
\]
It follows that 
\[
\limsup_{m \geq n \to \infty} \mathbb{V}(|S_n - S_m| \geq 2x) \leq 2 \limsup_{n \to \infty} \mathbb{V}(|S_n| \geq x) 
\leq 2 \limsup_{n \to \infty} \hat{\mathbb{E}}\left[g_1\left(\frac{|S_n|}{x}\right)\right] = \hat{\mathbb{E}}\left[g_1\left(\frac{\bar{S}}{x}\right)\right] \leq 2 \mathbb{V}\left(\bar{S} \geq x/2\right) \to 0 \text{ as } x \to \infty.
\]
Write $Y_{n,m} = (S_n, S_m - S_n)$, then the sequence $\{Y_{n,m}; m \geq n\}$ is asymptotically tight, i.e.,

$$
\limsup_{m \geq n \to \infty} \mathbb{V}(\|Y_{n,m}\| \geq x) \to 0 \quad \text{as} \quad x \to \infty.
$$

By Lemma 3.1, for any subsequence $(n_k, m_k)$ of $(n, m)$, there is further a subsequence $(n_{k'}, m_{k'})$ of $(n_k, m_k)$ and a sub-linear expectation space $(\Omega, \mathcal{F}, \mathbb{E})$ with a random vector $Y = (Y_1, Y_2)$ such that

$$
\hat{\mathbb{E}}[\phi(Y_{n_{k'}, m_{k'}})] \to \mathbb{E}[\phi(Y)], \quad \phi \in C_b(\mathbb{R}^2). \tag{3.14}
$$

Note that $S_{m_{k'}} - S_{n_{k'}}$ is independent to $S_{n_{k'}}$. By Lemma 4.4 of Zhang [13], $Y_2$ is independent to $Y_1$ under $\mathbb{E}$. Let $\phi \in C_{b,Lip}(\mathbb{R})$. By (3.14),

$$
\hat{\mathbb{E}}[\phi(S_{m_{k'}})] \to \mathbb{E}[\phi(Y_1 + Y_2)], \quad \hat{\mathbb{E}}[\phi(S_{n_{k'}})] \to \mathbb{E}[\phi(Y_1)] \tag{3.15}
$$

and

$$
\hat{\mathbb{E}}[\phi(S_{m_{k'}} - S_{n_{k'}})] \to \mathbb{E}[\phi(Y_2)]. \tag{3.16}
$$

On the other hand, by (3.5),

$$
\hat{\mathbb{E}}[\phi(S_{m_{k'}})] \to \hat{\mathbb{E}}[\phi(S)] \quad \text{and} \quad \hat{\mathbb{E}}[\phi(S_{n_{k'}})] \to \hat{\mathbb{E}}[\phi(S)]. \tag{3.17}
$$

Combining (3.15) and (3.17) yields

$$
\mathbb{E}[\phi(Y_1 + Y_2)] = \mathbb{E}[\phi(Y_1)] = \hat{\mathbb{E}}[\phi(S)], \quad \phi \in C_{b,Lip}(\mathbb{R}).
$$

Hence, by Lemma 3.2, we obtain $\mathbb{V}(|Y_2| \geq \epsilon) = 0$ for all $\epsilon > 0$. By choosing $\phi \in C_{b,Lip}(\mathbb{R})$ such that $I_{|x| \geq \epsilon} \leq \phi(x) \leq I_{|x| \geq \epsilon/2}$ in (3.16), we have

$$
\limsup_{k' \to \infty} \mathbb{V}(|S_{m_{k'}} - S_{n_{k'}}| \geq \epsilon) \leq \mathbb{V}(|Y_2| \geq \epsilon/2) = 0.
$$

So, we conclude that for any subsequence $(n_k, m_k)$ of $(n, m)$, there is a further a subsequence $(n_{k'}, m_{k'})$ of $(n_k, m_k)$ such that

$$
\mathbb{V}(|S_{m_{k'}} - S_{n_{k'}}| \geq \epsilon) \to 0 \quad \text{for all} \quad \epsilon > 0.
$$

Hence (3.6) is proved.

Next, suppose that $\mathbb{V}$ is countably sub-additive. Let $\epsilon_k = 1/2^k$, $\delta_k = 1/3^k$. By (3.6), there is a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ such that

$$
\mathbb{V}(|S_{n_{k+1}} - S_{n_k}| \geq \epsilon_k) \leq \delta_k.
$$
Let \( A = \{ \omega : \sum_{k=1}^{\infty} |S_{n_k+1} - S_{n_k}| < \infty \} \). Then

\[
V(A^c) \leq V \left( \sum_{k=K}^{\infty} |S_{n_k+1} - S_{n_k}| \geq \sum_{k=K}^{\infty} \epsilon_k \right) \\
\leq \sum_{k=K}^{\infty} V \left( |S_{n_k+1} - S_{n_k}| \geq \epsilon_k \right) \leq \sum_{k=K}^{\infty} \delta_k \to 0 \text{ as } K \to \infty.
\]

Define \( S = \lim_{k \to \infty} S_{n_k} \) on \( A \), and \( S = 0 \) on \( A^c \). Then

\[
V \left( |S - S_{n_k}| \geq 1/2^{k-1} \right) \leq V(A^c) + V \left( A, \sum_{i=k}^{\infty} |S_{n_{i+1}} - S_{n_i}| \geq \sum_{i=k}^{\infty} \epsilon_i \right) \\
\leq \sum_{i=k}^{\infty} V \left( |S_{n_{i+1}} - S_n| \geq \epsilon_i \right) \leq \sum_{i=k}^{\infty} \delta_i \to 0 \text{ as } k \to \infty.
\]

On the other hand, by (3.18),

\[
V \left( |S_n - S_{n_k}| \geq \epsilon \right) \to 0 \text{ as } n, n_k \to \infty.
\]

Hence

\[
V \left( |S_n - S| \geq \epsilon \right) \leq V \left( |S_n - S_{n_k}| \geq \epsilon/2 \right) + V \left( |S - S_{n_k}| \geq \epsilon/2 \right) \to 0.
\]

(3.1) is proved. Further,

\[
V \left( |S| \geq 2M \right) \leq \limsup_n V \left( |S_n| \geq M \right) + \limsup_n V \left( |S_n - S| \geq M \right)
\]

\[
\leq \tilde{V} \left( |\tilde{S}| \geq M/2 \right) \to 0 \text{ as } M \to \infty.
\]

So, \( S \) is tight. Finally, (3.2) follows from Theorem 3.1.

For showing Theorem 3.3, we need a more lemma.

**Lemma 3.3** Let \( \{X_n; n \geq 1\} \) be a sequence of independent random variables in a sub-linear expectation space \((\Omega, \mathcal{H}, \widehat{E})\) with \(|X_k| \leq c, \widehat{E}[X_k] \geq 0\) and \(\widehat{E}[-X_k] \geq 0, k = 1, 2, \ldots\). Let

\[
S_k = \sum_{i=1}^{k} X_i.
\]

Suppose

\[
\lim_{x \to \infty} \lim_{n \to \infty} V \left( \max_{k \leq n} |S_k| > x \right) < 1.
\]

(3.18)

Then \( \sum_{n=1}^{\infty} \widehat{E}[X_n], \sum_{n=1}^{\infty} \widehat{E}[-X_n] \) and \( \sum_{n=1}^{\infty} \widehat{E}[X_n^2] \) are convergent.

**Proof.** By (3.18), there exist \( 0 < \beta < 1, x_0 > 0 \) and \( n_0 \), such that

\[
V \left( \max_{k \leq n} |S_k| > x \right) < \beta, \text{ for all } x \geq x_0, n \geq n_0.
\]

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By (2.5),
\[ \sum_{k=1}^{n} \mathbb{E}[X_k] \leq \frac{x + c}{1 - \beta}, \text{ for all } x \geq x_0, n \geq n_0. \]

So \( \sum_{k=1}^{\infty} \mathbb{E}[X_k] \) is convergent. Similarly, \( \sum_{k=1}^{\infty} \mathbb{E}[-X_k] \) is convergent.

Now, by (2.4),
\[ \sum_{k=1}^{n} \mathbb{E}[X^2_k] \leq \frac{(x + c)^2 + 2x \sum_{k=1}^{n} \{ (\mathbb{E}[X_k])^+ + (\mathbb{E}[-X_k])^+ \}}{1 - \beta} \]
\[ \leq \frac{(x + c)^2 + 2x \sum_{k=1}^{\infty} \{ \mathbb{E}[X_k] + \mathbb{E}[-X_k] \}}{1 - \beta}, \text{ for all } x \geq x_0, n \geq n_0. \]

So \( \sum_{n=1}^{\infty} \mathbb{E}[X^2_n] \) is convergent. The proof is completed. □

**Proof of Theorem 3.3**

(i) By Lemma 2.3 and the condition (S3),
\[ \mathbb{V} \left( S_n - S_m - \sum_{k=m+1}^{n} \mathbb{E}[X_k] \geq \epsilon \right) \]
\[ \leq C \sum_{k=m+1}^{n} \mathbb{E} \left[ (X_k - \mathbb{E}[X_k])^2 \right] / \epsilon^2 \to 0 \text{ as } n \geq m \to \infty. \]

The convergence of \( \sum_{n=1}^{\infty} \mathbb{E}[X_n] \) implies \( \sum_{k=m+1}^{n} \mathbb{E}[X_k] \to 0 \). It follows that
\[ \lim_{n \geq m \to \infty} \mathbb{V} (S_n - S_m \geq \epsilon) = 0 \text{ for all } \epsilon > 0. \]

On the other hand, note \( \mathbb{E}[X_k] + \mathbb{E}[-X_k] \geq 0 \). The condition (S2) implies \( \sum_{n=1}^{\infty} \left( \mathbb{E}[X_k] + \mathbb{E}[-X_k] \right) < \infty \), and then \( \sum_{n=1}^{\infty} \left( \mathbb{E}[X_k] + \mathbb{E}[-X_k] \right)^2 < \infty \). Hence, by the condition (S3) and the fact that \( \mathbb{E} \left[ (X_k - \mathbb{E}[X_k])^2 \right] \leq \mathbb{E} \left[ (X_k - \mathbb{E}[X_k])^2 \right] + \mathbb{E}[X_k] + \mathbb{E}[-X_k])^2, \]
\[ \sum_{n=1}^{\infty} \mathbb{E} \left[ (X_n - \mathbb{E}[-X_n])^2 \right] < \infty. \]

By considering \(-X_n\) instead of \(X_n\), we have
\[ \lim_{n \geq m \to \infty} \mathbb{V} (-S_n + S_m \geq \epsilon) = 0 \text{ for all } \epsilon > 0. \]

It follows that (3.6) holds, i.e., \(S_n\) is a Cauchy sequence in capacity \(\mathbb{V}\).

(ii) Suppose that \(S_n\) is a Cauchy sequence in capacity \(\mathbb{V}\). Similar to (3.13), by applying the Levy inequality (2.2) we have
\[ \lim_{n \geq m \to \infty} \mathbb{V} \left( \max_{m \leq k \leq n} |S_k - S_m| > \epsilon \right) = 0 \text{ for all } \epsilon > 0. \]  (3.19)
Then
\[
\lim_{n \geq m \to \infty} \mathbb{V} \left( \max_{m \leq k \leq n} |X_k| \geq c \right) = 0 \text{ for all } c > 0. \tag{3.20}
\]

Write \( v_k = \mathbb{V} (|X_k| \geq 2c) \). Similar to (3.11), we have for \( m_0 \) large enough and all \( n \geq m \geq m_0, \)
\[
\frac{1}{3} > \mathbb{V} \left( \max_{m \leq k \leq n} |X_k| \geq c \right) \geq 1 - \frac{2}{\sum_{k=m+1}^{n} v_k}.
\]

It follows that \( \sum_{k=1}^{\infty} v_k < \infty \). The condition (S1) is satisfied for all \( c > 0 \).

Next, we consider (S3). Write \( X_n^c = (-c) \vee X_n \wedge c \) and \( S_n^c = \sum_{k=1}^{n} X_k^c \). Note on the event \( \{\max_{m \leq k \leq n} |X_k| < c\}, \)
\[
\lim_{n \geq m \to \infty} \mathbb{V} \left( \max_{m \leq k \leq n} |S_k^c - S_m^c| > \epsilon \right) = 0 \text{ for all } \epsilon > 0. \tag{3.21}
\]
Let \( Y_1, Y_2', Y_2, Y_3', \ldots, Y_n, Y_1', \ldots \) be independent random variables under the sub-linear expectation \( \hat{\mathbb{E}} \) with \( Y_k \overset{d}{=} Y_k' = X_k^c, k = 1, 2, \ldots \). Then
\[
\{Y_{m+1}, \ldots, Y_n\} \overset{d}{=} \{Y_{m+1}', \ldots, Y_n'\} = \{X_{m+1}^c, \ldots, X_n^c\}.
\]
Let \( T_k = \sum_{i=1}^{k} Y_i \) and \( T_k' = \sum_{i=1}^{k} Y_i' \). By (3.21),
\[
\lim_{n \geq m \to \infty} \mathbb{V} \left( \max_{m \leq k \leq n} |T_k - T_m| > \epsilon \right) = \lim_{n \geq m \to \infty} \mathbb{V} \left( \max_{m \leq k \leq n} |T_k' - T_m'| > \epsilon \right) = 0 \text{ for all } \epsilon > 0. \tag{3.22}
\]
Write \( \tilde{Y}_n = Y_n - Y_n' \) and \( \tilde{T}_n = \sum_{k=1}^{n} \tilde{Y}_k \). Then \( \{\tilde{Y}_n; n \geq 1\} \) is a sequence of independent random variables with \( \mathbb{V} (|\tilde{Y}_n| > 3c) = 0 \). Without loss of generality, we can assume \(|\tilde{Y}_n| \leq 3c\) for otherwise we can replace \( \tilde{Y}_n \) by \( (-3c) \vee \tilde{Y}_n \wedge (3c) \). By (3.22),
\[
\lim_{n \geq m \to \infty} \mathbb{V} \left( \max_{m \leq k \leq n} |\tilde{T}_k - \tilde{T}_m| > 2\epsilon \right) = 0 \text{ for all } \epsilon > 0.
\]

Note \( \hat{\mathbb{E}}[\tilde{Y}_k] = \hat{\mathbb{E}}[\tilde{Y}_k'] = (\hat{\mathbb{E}}[X_k^c] + \hat{\mathbb{E}}[-X_k^c])/2 \geq 0 \). By Lemma 3.3,
\[
\sum_{n=1}^{\infty} \left( \hat{\mathbb{E}}[X_n^c] + \hat{\mathbb{E}}[-X_n^c] \right) \text{ and } \sum_{n=1}^{\infty} \hat{\mathbb{E}}[\tilde{Y}_n^2] \text{ are convergent.}
\]

Note
\[
\hat{\mathbb{E}} \left[ \tilde{Y}_n^2 | Y_n \right] \geq (Y_n - \hat{\mathbb{E}}[Y_n])^2 + \hat{\mathbb{E}} \left[ (Y_n' - \hat{\mathbb{E}}[Y_n'])^2 \right] + 2(Y_n - \hat{\mathbb{E}}[Y_n]) (Y_n' - \hat{\mathbb{E}}[Y_n']).
\]
The condition (S3) is proved.

\[
\hat{E}\left[\sum_{i=1}^{\infty} (X_{n}^{c} - \hat{E}[X_{n}^{c}])^2\right] - 2\{\hat{E}[X_{n}^{c}] + \hat{E}[-X_{n}^{c}]\} \hat{E}\left[\sum_{i=1}^{\infty} (X_{n}^{c} - \hat{E}[X_{n}^{c}])^2\right] \\
\geq 2\hat{E}\left[\sum_{i=1}^{\infty} (X_{n}^{c} - \hat{E}[X_{n}^{c}])^2\right] - 2\{\hat{E}[X_{n}^{c}] + \hat{E}[-X_{n}^{c}]\}.
\]

It follows that

\[
\sum_{n=1}^{\infty} \hat{E}\left[(X_{n}^{c} - \hat{E}[X_{n}^{c}])^2\right] < \infty. \tag{3.23}
\]

Since \(\hat{E}\left[\sum_{i=1}^{\infty} (X_{n}^{c} - \hat{E}[X_{n}^{c}])^2\right] \leq \hat{E}\left[\sum_{i=1}^{\infty} (X_{n}^{c} - \hat{E}[X_{n}^{c}])^2\right] + (\hat{E}[X_{n}^{c}] + \hat{E}[-X_{n}^{c}]^2),\) we also have

\[
\sum_{n=1}^{\infty} \hat{E}\left[\sum_{i=1}^{\infty} (X_{n}^{c} - \hat{E}[X_{n}^{c}])^2\right] < \infty.
\]

The condition (S3) is proved.

Finally, we consider (S2). For any \(\epsilon > 0,\) when \(m, n\) are large enough, \(\sum_{k=m+1}^{n} (\hat{E}[X_{k}^{c}] + \hat{E}[-X_{k}^{c}]) < \epsilon.\) By (3.23) and Lemma 2.3,

\[
\forall \left( S_{n}^{c} - S_{m}^{c} - \sum_{k=m+1}^{n} \hat{E}[X_{k}^{c}] - \hat{E}[-X_{k}^{c}] \geq \epsilon \right)
\]

\[
= \forall \left( S_{n}^{c} - S_{m}^{c} - \sum_{k=m+1}^{n} \hat{E}[X_{k}^{c}] \geq \epsilon - \sum_{k=m+1}^{n} \frac{\hat{E}[X_{k}^{c}] + \hat{E}[-X_{k}^{c}]}{2} \right)
\]

\[
\leq C \sum_{k=m+1}^{n} \frac{\hat{E}\left[(X_{k}^{c} - \hat{E}[X_{k}^{c}])^2\right]}{(\epsilon/2)^2} \rightarrow 0 \text{ as } n \geq m \rightarrow \infty.
\]

Similarly, by considering \(-X_{k}^{c}\) instead of \(X_{k}^{c}\) we have

\[
\forall \left( -S_{n}^{c} + S_{m}^{c} - \sum_{k=m+1}^{n} \hat{E}[-X_{k}^{c}] - \hat{E}[X_{k}^{c}] \geq \epsilon \right) \rightarrow 0 \text{ as } n \geq m \rightarrow \infty.
\]

It follows that, for any \(\epsilon > 0,\)

\[
\forall \left( S_{n}^{c} - S_{m}^{c} - \sum_{k=m+1}^{n} \hat{E}[X_{k}^{c}] - \hat{E}[-X_{k}^{c}] \geq \epsilon \right) \rightarrow 0 \text{ as } n \geq m \rightarrow \infty,
\]

which, together with (3.22), implies

\[
\sum_{k=m+1}^{n} \frac{\hat{E}[X_{k}^{c}] - \hat{E}[-X_{k}^{c}]}{2} \rightarrow 0 \text{ as } n \geq m \rightarrow \infty.
\]

Hence, \(\sum_{n=1}^{\infty} (\hat{E}[X_{k}] - \hat{E}[-X_{k}])\) is convergent. Note that \(\sum_{n=1}^{\infty} (\hat{E}[X_{k}] + \hat{E}[-X_{k}]\) is convergent. We conclude that both \(\sum_{n=1}^{\infty} \hat{E}[X_{k}]\) and \(\sum_{n=1}^{\infty} \hat{E}[-X_{k}]\) are convergent. The proof of (ii) is completed. \(\square.\)
4 Central limit theorem

In this section, we consider the sufficient and necessary conditions for the central limit theorem. We first recall the definition of G-normal random variables which is introduced by Peng [8, 9].

**Definition 4.1** (G-normal random variable) For \( 0 \leq \sigma^2 \leq \sigma^2 < \infty \), a random variable \( \xi \) in a sub-linear expectation space \((\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})\) is called a normal \( N(0, [\sigma^2, \sigma^2]) \) distributed random variable (written as \( \xi \sim N(0, [\sigma^2, \sigma^2]) \) under \( \tilde{\mathbb{E}} \)), if for any \( \varphi \in C_{l,Lip}(\mathbb{R}) \), the function \( u(x,t) = \tilde{\mathbb{E}} [\varphi(x + \sqrt{t}\xi)] \) \((x \in \mathbb{R}, t \geq 0)\) is the unique viscosity solution of the following heat equation:

\[
\partial_t u - G(\partial^2_{xx} u) = 0, \ u(0, x) = \varphi(x),
\]

where \( G(\alpha) = \frac{1}{2}(\sigma^2\alpha^+ - \sigma^2\alpha^-) \).

That \( \xi \) is a normal distributed random variable is equivalent to that, if \( \xi' \) is an independent copy of \( \xi \) (i.e., \( \xi' \) is independent to \( \xi \) and \( \xi \overset{d}{=} \xi' \)), then

\[
\tilde{\mathbb{E}} [\varphi(\alpha\xi + \beta\xi')] = \tilde{\mathbb{E}} [\varphi(\sqrt{\alpha^2 + \beta^2})], \ \forall \varphi \in C_{l,Lip}(\mathbb{R}) \text{ and } \forall \alpha, \beta \geq 0,
\]

(cf. Definition II.1.4 and Example II.1.13 of Peng [9]). We also write \( \eta \overset{d}{=} N(0, [\sigma^2, \sigma^2]) \) if \( \eta \overset{d}{=} \xi \) (as defined in Definition 1.2 (i)) and \( \xi \sim N(0, [\sigma^2, \sigma^2]) \) (as defined in Definition 4.1). By definition, \( \eta \overset{d}{=} \xi \) if and only if for any \( \varphi \in C_{b,Lip}(\mathbb{R}) \), the function \( u(x,t) = \tilde{\mathbb{E}} [\varphi(x + \sqrt{t}\eta)] \) \((x \in \mathbb{R}, t \geq 0)\) is the unique viscosity solution of the equation (4.1). In the sequel, without loss of generality, we assume that the sub-linear expectation spaces \((\tilde{\Omega}, \tilde{\mathcal{H}}, \tilde{\mathbb{E}})\) and \((\Omega, \mathcal{H}, \mathbb{E})\) are the same.

Let \( \{X_n; n \geq 1\} \) be a sequence of independent and identically distributed random variables in a sub-linear expectation space \((\Omega, \mathcal{H}, \mathbb{E})\), \( S_n = \sum_{k=1}^{n} X_k \). Peng [8, 9] proved that, if \( \mathbb{E}[X_1] = \tilde{\mathbb{E}}[-X_1] = 0 \) and \( \tilde{\mathbb{E}}[|X_1|^{2+\alpha}] < \infty \) for some \( \alpha > 0 \), then

\[
\lim_{n \to \infty} \tilde{\mathbb{E}} \left[ \varphi \left( \frac{S_n}{\sqrt{n}} \right) \right] = \tilde{\mathbb{E}} [\varphi(\xi)], \ \forall \varphi \in C_{b}(\mathbb{R}),
\]

where \( \xi \sim N(0, [\sigma^2, \sigma^2]) \), \( \sigma^2 = \tilde{\mathbb{E}}[X_1^2] \) and \( \sigma^2 = \tilde{\mathbb{E}}[X_1^2] \). Zhang [14] showed that \( \tilde{\mathbb{E}}[|X_1|^{2+\alpha}] < \infty \) can be weakened to \( \tilde{\mathbb{E}}[(X_1^2 - c)^+] \to 0 \) as \( c \to \infty \) by applying the moment inequalities of sums of independent random variables and the truncation method. A nature question is
whether $\hat{E}[X_1^2] < \infty$ and $\hat{E}[X_1] = \hat{E}[-X_1] = 0$ are sufficient and necessary for (4.3). The following theorem is our main result.

**Theorem 4.1** Let $\{X_n; n \geq 1\}$ be a sequence of independent and identically distributed random variables in a sub-linear expectation space $(\Omega, \mathcal{F}, \hat{E})$, $S_n = \sum_{k=1}^{n} X_k$. Suppose that

(i) $\lim_{c \to \infty} \hat{E}[X_1^2 \land c]$ is finite;

(ii) $x^2 \mathbb{V}(|X_1| \geq x) \to 0$ as $x \to \infty$;

(iii) $\lim_{c \to \infty} \hat{E}[(-c) \lor X_1 \land c] = \lim_{c \to \infty} \hat{E}[(−c) \lor (−X_1) \land c] = 0$.

Write $\sigma^2 = \lim_{c \to \infty} \hat{E}[X_1^2 \land c]$ and $\sigma^2 = \lim_{c \to \infty} \hat{E}[X_1^2 \land c]$. Then for any $\varphi \in C_b(\mathbb{R})$,

$$
\lim_{n \to \infty} \hat{E}\left[ \varphi \left( \frac{S_n}{\sqrt{n}} \right) \right] = \hat{E} \left[ \varphi(\xi) \right], \quad (4.4)
$$

where $\xi \sim N \left(0, [\sigma^2, \sigma^2]\right)$.

Conversely, if (4.4) holds for any $\varphi \in C_b^1(\mathbb{R})$ and a random variable $\xi$ with $x^2 \mathbb{V}(|\xi| \geq x) \to 0$ as $x \to \infty$, then (i),(ii) and (iii) hold and $\xi \overset{d}{=} N \left(0, [\sigma^2, \sigma^2]\right)$.

Before prove the theorem, we give some remarks on the conditions. Note that $\hat{E}[X_1^2 \land c]$ and $\hat{E}[X_1^2 \land c]$ are non-decreasing in $c$. So, $\sigma^2$ and $\sigma^2$ are well-defined and nonnegative, and are finite if the condition (i) is satisfied. It is easily seen that, for $c_1 > c_2 > 0$,

$$
\left| \hat{E}[X_1^2] - \hat{E}[X_1^2] \right| \leq \hat{E}[|X_1| \land c_1 - c_2] \leq \frac{\sigma^2}{c_2}. \quad (4.5)
$$

So, the condition (i) implies that $\lim_{c \to \infty} \hat{E}[X_1^2]$ and $\lim_{c \to \infty} \hat{E}[-X_1^2]$ exist and are finite.

If $\hat{E}$ is a continuous sub-linear expectation, i.e., $\hat{E}[X_n] \nearrow \hat{E}[X]$ whenever $0 \leq X_n \nearrow X$, and $\hat{E}[X_n] \searrow 0$ whenever $X_n \searrow 0$, $\hat{E}[X_n] < \infty$, then (i) is equivalent to $\hat{E}[X_1^2] < \infty$, (iii) is equivalent to $\hat{E}[X_1] = \hat{E}[-X_1] = 0$, and (ii) is automatically implied by $\hat{E}[X_1^2] < \infty$. In general, the condition $\hat{E}[X_1^2] < \infty$ and (i) with (ii) do not imply each other. However, it is easily verified that, if $\hat{E}[(X_1^2 - c)^+] \to 0$ as $c \to \infty$, then (i) and (ii) are satisfied and (iii) is equivalent to $\hat{E}[X_1] = \hat{E}[-X_1] = 0$.

To prove Theorem 4.1 we need a more lemma.
Lemma 4.1 Let $X_{n1}, \cdots, X_{nn}$ be independent random variables in a sub-linear expectation space $(\Omega, \mathcal{H}, \hat{E})$ with
\[
\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \left\{ |\hat{E}[X_{nk}]| + |\hat{E}[-X_{nk}]| \right\} \to 0,
\]
\[
\frac{1}{n} \sum_{k=1}^{n} \left\{ |\hat{E}[X_{nk}^2] - \sigma^2| + |\hat{E}[X_{nk}^2] - \sigma^2| \right\} \to 0
\]
and
\[
\frac{1}{n^{3/2}} \sum_{k=1}^{n} \hat{E}[|X_{nk}|^3] \to 0.
\]
Then
\[
\lim_{n \to \infty} \hat{E} \left[ \varphi \left( \frac{\sum_{k=1}^{n} X_{nk}}{\sqrt{n}} \right) \right] = \hat{E} \left[ \varphi(\xi) \right], \forall \varphi \in C_b(\mathbb{R}),
\]
where $\xi \sim N(0, [\sigma^2, \sigma^2])$.

This lemma can be proved by refining the arguments of Li and Shi [4] and can also follow from the Lindeberg central limit theorem [16]. We omit the proof here.

Proof of Theorem 4.1. We first prove the sufficient part, i.e., (i),(ii) and (iii) $\implies$ (4.4). Let $X_{nk} = (\sqrt{n} \lor X_k \land \sqrt{n})$. Then for any $\epsilon > 0$,
\[
\frac{1}{n^{3/2}} \sum_{k=1}^{n} \hat{E}[|X_{nk}|^3] = \frac{1}{n^{1/2}} \hat{E}[|X_{n1}|^3] \leq \epsilon \sigma^2 + n \mathbb{V} (|X_1| \geq \epsilon \sqrt{n}) \to 0
\]
as $n \to \infty$ and then $\epsilon \to 0$, by the condition (ii). Also,
\[
\frac{1}{n} \sum_{k=1}^{n} \left\{ |\hat{E}[X_{nk}^2] - \sigma^2| + |\hat{E}[X_{nk}^2] - \sigma^2| \right\}
\]
\[
= |\hat{E} [X_1^2 \land n] - \sigma^2| + |\hat{E} [X_1^2 \land n] - \sigma^2| \to 0,
\]
by (i). Note by (ii) and (i),
\[
\frac{1}{\sqrt{n}} \sum_{k=1}^{n} |\hat{E}[X_{nk}]| = \sqrt{n} |\hat{E}[X_{n1}]|
\]
\[
= \sqrt{n} \lim_{c \to \infty} |\hat{E}[X_{n1}] - \hat{E} [(-c \sqrt{n}) \lor X_1 \land (c \sqrt{n})]|
\]
\[
\leq \sqrt{n} \lim_{c \to \infty} \hat{E} [(|X_1| \land (c \sqrt{n}) - x \sqrt{n})^+] + \sqrt{n} \mathbb{E} [(|X_1| \land (x \sqrt{n}) - \sqrt{n})^+]
\]
\[
\leq \frac{\sigma^2}{x} + xn \mathbb{V} (|X_1| \geq \sqrt{n}) \to 0 \text{ as } n \to \infty \text{ and then } x \to \infty,
\]
and similarly,
\[
\frac{1}{\sqrt{n}} \sum_{k=1}^{n} |\hat{E}[-X_{nk}]| \to 0.
\]
The conditions in Lemma 4.1 are satisfied. We obtain
\[
\lim_{n \to \infty} \hat{E} \left[ \varphi \left( \frac{\sum_{k=1}^{n} X_{nk}}{\sqrt{n}} \right) \right] = \hat{E} [\varphi(\xi)].
\]

It is obvious that
\[
\hat{E} \left[ \varphi \left( \frac{\sum_{k=1}^{n} X_{nk}}{\sqrt{n}} \right) - \varphi \left( \frac{S_n}{\sqrt{n}} \right) \right] \leq \sup_x |\varphi(x)| n \mathbb{V} (|X_1| \geq \sqrt{n}) \to 0.
\]

(4.4) is proved.

Now, we consider the necessary part. Letting \( \varphi = g_\epsilon (|x| - t) \) yields
\[
\limsup_{n \to \infty} \mathbb{V} \left( \frac{|S_n|}{\sqrt{n}} \geq t + \epsilon \right) \leq \mathbb{V} (|\xi| \geq t) \quad \text{for all } t > 0, \epsilon > 0.
\]

So
\[
\limsup_{n \geq m \to \infty} \max_{m \leq k \leq n} \mathbb{V} \left( \frac{|S_k - S_l|}{\sqrt{n}} \geq 2t + \epsilon \right) \leq 2 \mathbb{V} (|\xi| \geq t) \quad \text{for all } t > 0, \epsilon > 0.
\]

Choose \( t_0 \) such that \( \mathbb{V} (|\xi| \geq t_0) < 1/(32) \). Applying the Levy maximal inequality (2.2) yields
\[
\limsup_{n \geq m \to \infty} \mathbb{V} \left( \frac{\max_{m \leq k \leq n} |S_k - S_m|}{\sqrt{n}} \geq 4t \right) < \frac{64}{31} \mathbb{V} (|\xi| \geq t) \quad \text{for all } t > t_0.
\]

Hence
\[
\limsup_{n \geq m \to \infty} \mathbb{V} \left( \frac{\max_{m \leq k \leq n} |X_k|}{\sqrt{n}} \geq 8t \right) < \frac{64}{31} \mathbb{V} (|\xi| \geq t) \quad \text{for all } t > t_0.
\]

Let \( t_1 > t_0 \) and \( m_0 \) such that
\[
\mathbb{V} \left( \frac{\max_{m \leq k \leq n} |S_k - S_m|}{\sqrt{n}} > 4t_1 \right) < \frac{2}{31} \quad \text{for all } m \geq m_0
\]
and
\[
\mathbb{V} \left( \frac{\max_{m \leq k \leq n} |X_k|}{\sqrt{n}} > 8t_1 \right) < \frac{4}{31} \quad \text{for all } m \geq m_0.
\]

Write \( Y_{nk} = (-8t_1) \vee \left( \frac{X_k}{\sqrt{n}} \right) \wedge (8t_1) \). Then by (4.8) and (4.9),
\[
\mathbb{V} \left( \max_{m \leq k \leq n} \sum_{j=m+1}^{k} Y_{nj} > 4t_1 \right) < \frac{2}{31} + \frac{4}{31} < \frac{1}{5} \quad \text{for all } m \geq m_0
\]

If \( \hat{E}[Y_{n1}] > 0 \), then by Lemma 2.2 (ii),
\[
\frac{1}{5} > 1 - \frac{4t_1 + 8t_1}{(n-m)\hat{E}[Y_{n1}]}.
\]

Hence \((n-m)(\hat{E}[Y_{n1}])^+ \leq 15t_1\). Similarly, \((n-m)(\hat{E}[-Y_{n1}])^+ \leq 15t_1\). Hence, by Lemma 2.2 (i), it follows that
\[
\frac{1}{5} > 1 - \frac{(4t_1 + 8t_1)^2 + 8t_1 \left( (n-m)(\hat{E}[Y_{n1}])^+ + (n-m)(\hat{E}[-Y_{n1}])^+ \right)}{(n-m)\hat{E}[Y_{n1}^2]}.\]
We conclude that \((n - m)\hat{E}[Y_{n1}^2] \leq \frac{5}{4}(12^2 + 240)t_1^2\). Choose \(m = n/2\) and let \(n \to \infty\). We have
\[
\lim_{c \to \infty} \hat{E}[X_1^2 \wedge c] = \lim_{n \to \infty} n\hat{E}[Y_{n1}^2] \leq \frac{5}{2}(12^2 + 240)t_1^2.
\]
(i) is proved. Note that (i) implies that \(\lim_{c \to \infty} \hat{E}[X^c_i]\) exists and is finite. Then
\[
\lim_{c \to \infty} \hat{E}[X^c_i] = \limsup_{n \to \infty} \sqrt{n}\hat{E}[Y_{n1}] \leq \limsup_{n \to \infty} \frac{30t_1}{\sqrt{n}} = 0.
\]
Similarly, \(\lim_{c \to \infty} \hat{E}[-X^c_i]\) exists, is finite and not positive. Note \(\hat{E}[-X^c_i] + \hat{E}[X^c_i] \geq 0\). Hence (iii) follows.

Finally, we show (ii). For any given \(0 < \epsilon < 1/2\), by the condition \(x^2\hat{V}(|\xi| \geq x) \to 0\), one can choose \(t_1 > t_0\) such that \(\frac{64}{3\pi}\hat{V}(|\xi| \geq t_1) \leq \frac{\epsilon}{9t_1^2} < 1/2\). Then by (4.7), there is \(m_0\) such that
\[
\forall \left(\frac{\max_{m < k < n}|X_k|}{\sqrt{n}} \geq 8t_1\right) < \frac{\epsilon}{9t_1^2}, \quad n \geq m \geq m_0.
\]
Choose \(Z_k = g_c\left(\frac{|X_k|}{8t_1\sqrt{n}} - 1\right)\) such that \(I\{|X_k| \geq 9t_1\sqrt{n}\} \leq Z_k \leq I\{|X_k| \geq 8t_1\sqrt{n}\}\). Let \(q_n = \hat{V}(|X_1| \geq 9t_1\sqrt{n})\). Then
\[
\forall \left(\frac{\max_{m \leq k \leq n}|X_k|}{\sqrt{n}} \geq 8t_1\right) \geq \hat{E} \left[1 - \prod_{k=m+1}^{n} (1 - Z_k)\right] = 1 - \prod_{k=m+1}^{n} (1 - \hat{E}[Z_k]) \geq 1 - e^{-(n-m)q_n}.
\]
It follows that
\[
n\hat{V}(|X_1| \geq 9t_1\sqrt{n}) \leq 2(n - m)q_n < 2 \times 2 \times \frac{\epsilon}{9t_1^2} \text{ for } m = \lfloor n/2 \rfloor \geq m_0.
\]
Hence
\[
(9t_1\sqrt{n})^2\hat{V}(|X_1| \geq 9t_1\sqrt{n}) < \frac{4\epsilon}{9}, \quad n \geq 2m_0.
\]
When \(x \geq 9t_1\sqrt{2m_0}\), there is \(n\) such that \(9t_1\sqrt{n} \leq x \leq 9t_1\sqrt{n+1}\). Then
\[
x^2\hat{V}(|X_1| \geq x) \leq (9t_1\sqrt{n+1})^2\hat{V}(|X_1| \geq 9t_1\sqrt{n}) \leq \frac{8\epsilon}{9}.
\]
It follows that \(\limsup_{x \to \infty} x^2\hat{V}(|X_1| \geq x) < \epsilon\). (ii) is proved. The proof is now completed. \(\Box\)

\textbf{Remark 4.1} From the proof, we can find that
\[
\lim_{x \to \infty} \limsup_{n \to \infty} \hat{V}\left(\frac{|S_n|}{\sqrt{n}} \geq x\right) = 0
\]
implies (i) and (ii). One may conjecture that,
C1 if (4.4) holds for any $\varphi \in C^1_b(\mathbb{R})$ and a tight random variable $\xi$ (i.e., $\forall (\xi \geq x) \to 0$ as $x \to \infty$), then (i), (ii) and (iii) holds and $\xi \overset{d}{=} N(0, [\sigma^2, \sigma'^2])$.

An equivalent conjecture is that,

C2 if $\xi$ and $\xi'$ are independent and identically distributed tight random variables, and

\[ \hat{E}[\varphi(\alpha \xi + \beta \xi')] = \hat{E}[\varphi(\sqrt{\alpha^2 + \beta^2} \xi)], \quad \forall \varphi \in C^1_b(\mathbb{R}) \quad \text{and} \quad \forall \alpha, \beta \geq 0, \quad (4.11) \]

then $\xi \overset{d}{=} N(0, [\sigma^2, \sigma'^2])$, where $\sigma^2 = \lim_{c \to \infty} \hat{E}[\xi^2 \wedge c]$ and $\sigma'^2 = \lim_{c \to \infty} \hat{E}[\xi'^2 \wedge c]$.

It should be noted that the conditions (4.2) and (4.11) are different. The condition (4.2) implies that $\xi$ have finite moments of each order, but non information about the moments of $\xi$ is hidden in (4.11). As Theorem 4.1, the conjecture C2 is true when $x^2 \forall (\xi \geq x) \to 0$ as $x \to \infty$. In fact, let $X_1, X_2, \ldots$, be independent random variables with $X_k \overset{d}{=} \xi$. Then by (4.11), $S_n/\sqrt{n} \overset{d}{=} \xi$. By the necessary part of Theorem 4.1, the conditions (i), (ii) and (iii) are satisfied. Then by the sufficient part of the theorem, $\xi \overset{d}{=} N(0, [\sigma^2, \sigma'^2])$. We don’t known whether conjectures C1 and C2 are true without assuming any moment conditions. It is very possible that they are not true in general. But finding a counterexample is not an easy task.

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