Localization in Lattice Gauge Theory and a New Multigrid Method

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Abstract

We show numerically that the lowest eigenmodes of the 2-dimensional Laplace-operator with SU(2) gauge couplings are strongly localized. A connection is drawn to the Anderson-Localization problem. A new Multigrid algorithm, capable to deal with these modes, shows no critical slowing down for this problem.
It is well-known that the convergence of local relaxation algorithms for solving inhomogeneous linear equations of the form $L\xi = f$ is determined by the lowest eigenmodes of the problem matrix. We studied these modes in an example coming from Lattice Gauge Theory, using

$$L = -\triangle + \delta m^2 - \varepsilon_0.$$  \hspace{1cm} (0.1)

Here $\triangle$ is the 2-dimensional covariant Laplace operator, i.e. the discretized Laplace operator with SU(2)-matrices as nearest-neighbour-couplings (see sect. [3] for a more thorough explanation of this), $\varepsilon_0$ is its lowest eigenvalue and $\delta m^2$ is the critical parameter. The lowest eigenvalue has to be subtracted to make the problem critical—otherwise there would be no critical slowing down and no need to apply a multigrid. This subtraction of a constant does obviously not affect the shape of the eigenvectors. Physically, our model corresponds to a Higgs-doublet with a SU(2)-charge.

It is found that the lowest eigenmodes of this model are strongly localized, i.e. they are appreciably large only in a small region of the grid. This result will be described in the first part of this paper. In the second part a recently proposed multigrid algorithm [1] and its performance for the model problem is investigated. This algorithm works extremely well (it eliminates critical slowing down completely), because it is able to handle the localized modes.

1 Localization

1.1 Localized modes in a model problem

The fundamental form of an inhomogeneous equation is

$$L\xi = f.$$  \hspace{1cm} (1.1)

If we choose $L$ as in eq. (0.1), our equation becomes the propagator equation for a bosonic particle in a SU(2)-gauge field background. The covariant Laplace-operator in stencil-notation is

$$\triangle (z) = \begin{bmatrix} 0 & U_{z,2} & 0 \\ U_{z,-1} & 4 & U_{z,1} \\ 0 & U_{z,-2} & 0 \end{bmatrix},$$  \hspace{1cm} (1.2)

with $U_{z,\mu} \in$ SU(2). The second index denotes the direction of the coupling to the neighbour. The link matrices $U_{z,\mu}$ fulfill $U_{z,-\mu} = U_{z,\mu}^*$. They are distributed according to the Wilson action [2]

$$S_W = \frac{\beta}{4} \sum_P \text{Re tr} \ (1 - U_P).$$  \hspace{1cm} (1.3)

Here $\beta = 4/g^2$ is the inverse coupling and the sum is over all Plaquettes in the lattice. $U_P$ denotes the parallel transport around the Plaquette

$$U_P = U_{z,\nu}^* U_{z+\nu,\mu}^* U_{z+\mu,\nu} U_{z,\mu}$$

with $\mu \neq \nu$.

This distribution leads to a correlation between the gauge field matrices with finite correlation length $\chi$ for finite $\beta$. The case $\beta = 0$ corresponds to a completely random choice of the matrices ($\chi = 0$), for $\beta = \infty$ all matrices are 1 ($\chi = \infty$). In this sense, $\beta$ is a disorder parameter, the smaller $\beta$ the shorter the correlation length and the larger the disorder.

Now we want to study the lowest—and the highest—eigenmodes of this operator. If we look only at the norm of the eigenmodes we can see immediately that the lowest and the highest eigenmodes will look identical because of the following
Theorem: Let $L$ be a $(n \times n)$-matrix with the following properties:

- $L$ has a constant diagonal $\alpha$,
- For all $i, j$ with $i \neq j$ and $i + j$ even: $L_{ij} = 0$.

Then the following statement holds:

If $\xi^k$ is an eigenvector of $L$ to the eigenvalue $\lambda^k$ ($k < n/2$), then the vector $(-)^j \xi_j^k$ is also an eigenvector to $\lambda^{n-k} = 2\alpha - \lambda^k$.

This theorem is also true if the matrix elements are matrices themselves. It applies to the Laplace operator if its matrix elements are ordered in the right way (checkerboard fashion).

Fig. 2 shows the norm of the lowest eigenmode of the covariant Laplace operator on a $64^2$-lattice at $\beta = 1.0$ and on a $32^2$-lattice at $\beta = 5.0$. For the smaller $\beta$-value the localization can be seen clearly, for larger $\beta$ the localization is in a more extended domain.

This of course poses the question whether all modes are localized. To answer this, we calculated all modes on a smaller grid, using a standard library routine. It was found that only a few of the low modes show localization. It might be possible that on very large grids (with very low values of $\beta$) many of the modes are localized, as localization increases with the disorder. But to study this a large computational effort would be needed. (The simple storage of all eigenvectors on a $128^2$-lattice would need about 1 Gigabyte.)

It is expected that the sharpness of localization will increase with the disorder, i.e. with decreasing $\beta$. To study this effect we may look at the participation ratio defined as

$$\alpha^{-1} = \frac{1}{N} \left( \frac{\sum_{i=1}^{N} |\xi_i|^2}{\sum_{i=1}^{N} |\xi_i|^4} \right)^2,$$

where $N$ denotes the number of grid points. This quantity measures the fraction of the lattice over which the peak of the localized state $\xi$ is spread. Fig. 3 shows the participation ratio of the lowest eigenmode as function of $\beta$ calculated on $64^2$-lattices. The dependence of the disorder can be seen clearly. The absolute size of the localized state, measured by $N\alpha^{-1}$, does not depend on the grid size if the grid is larger than the localization peak.

We want to remark here that a similar phenomenon of localization also has been found for the two-dimensional Dirac-equation in a SU(2)-gauge-field and for the same two models in four dimensions.

1.2 Explaining the localization

How can we understand this phenomenon of localization in non-abelian gauge fields?

We will try to relate it to Anderson-localization. Anderson examined a tight-binding-model of an electron in a random potential $V(z)$, leading to a Schrödinger equation

$$(-\Delta_{\text{scalar}} + V(z))\psi = E\psi.$$

Localization of all modes may occur, dependend on the disorder and the dimension. In one dimension, arbitrary small disorder leads to localization, in two dimensions there is a phase transition between weak and strong localization as one increases the disorder, in higher dimensions a transition between non-localized and localized states occurs. A theoretical understanding of this was achieved in the papers. As the Schrödinger operator for this model is again the Laplacian (without a gauge field), we may expect a similarity between our localization problem and Anderson localization, but there are crucial differences: First, our operator is not fully random, because the equilibrated gauge field possesses correlations. Second, our operator shows off-diagonal disorder: It is not the potential that
varies from site to site, but the couplings between the sites. And third, the couplings do not vary in strength, but in orientation in colour space, resulting in a frustrated system.

Nevertheless, in the following we will try to stress similarities between the two models, giving us an—at least intuitive—picture of what happens.

We have to look for a quantity in our model that may play the role of the random potential $V(z)$ in the Anderson problem. There we can extract the value of the potential by calculating the difference between the diagonal element and the norm of the couplings, because this is zero for the Laplace operator without potential, see eq. (1.2). Doing this for our model problem of course only gives $\delta m^2 - \varepsilon_0$, independend of the lattice site. To arrive at a varying quantity, we may look at a blocked version of our operator. The simplest blocking procedure one can think of is the “taxi-driver” blocking. Here four points are blocked to one and all block-quantities are calculated by parallel-transporting the fine-grid-quantities to one of the four points (e.g. the lower left point) via the shortest possible route. (For the upper right point there are two routes, each weighted with a factor 1/2.)

If we use a blocking-operator of this type and calculate $L_{\text{block}} = C^{[j\ 0]} \cdot (-\triangle) \cdot A^{[0\ j]}$, the blocked operator, we still have an operator with fluctuating bonds and a constant diagonal. But now the bonds are fluctuating in strength, so we have to separate the kinetic and the potential part of the diagonal elements by calculating $V(x) = L_{\text{block}}(x, x) - \sum_y \|L_{\text{block}}(x, y)\|$ as explained above. This quantity now is fluctuating on the block lattice, and so we arrived at a situation much more similar to Anderson-localization.

The blocking procedure has a certain arbitrariness (e.g. in the choice of the blocks), but a simple calculation shows that it is mainly the quantity $\|F_{\mu\nu}(z)\|$ that enters into $V(x)$. (This has to be expected taking into account that the quantity involved has to be gauge invariant.) Let us now look at the field strength norm, or, which is more conclusive, at the quantity $W(z) = \sum_{\mu\neq\nu} F_{\mu\nu}(z) F^{\mu\nu}(z)$. Fig. 4 shows this quantity for the same configuration as in figure 2, bottom, and one can see clearly that the localization is in a region where $W(z)$ is small, as should be expected from our argument. This is true for all configurations we looked at: The localization center is always in a region with low field strength sum.

We can support this idea further by looking at the eigenmodes of the following operator, which we called Anderson-Laplace-operator: $D_{AL} = -\triangle_{\text{scalar}} + W(z)$, where $\triangle_{\text{scalar}}$ is the Laplace operator without gauge field. So now we are looking at a true Anderson-localization problem, except that the random potential is not independend at different sites. There is a finite correlation length, as explained in the previous section. Fig. 5 shows the lowest mode of this operator. If compared to fig. 1, bottom, one sees that the center of localization sits at the same place.

In conclusion we can say that our analysis shows that firstly the lowest eigenmodes of the covariant Laplace operator in a SU(2)-gauge field are localized and that secondly this localization occurs where the field strength norm is small. This quantity can be interpreted as a random potential on a block lattice. In this way we were able to draw a connection to Anderson localization.

2 The Iteratively Smoothing Unigrid

2.1 The problem

Disordered systems are among the most interesting and most difficult models in physics. Here we are interested in the numerical solution of—discretized—differential equations for such models. (An
example for this is the inversion of the fermion matrix.) As the critical point of the system is approached, simple local algorithms face the problem of critical slowing down, that is, the nearer one gets to the critical point the slower the convergence.

The use of nonlocal methods may overcome this problem. For the solution of “ordered” problems multigrid methods have been very successful. Even in the disordered case the “Algebraic Multigrid” (AMG) \[11\] was applied with great success to scalar problems like the “random resistor network” \[12\]. But up to now, no generalization of this method has been found that could be used for Lattice Gauge Theory.

In a previous paper \[1\] this problem was tackled and a new algorithm was proposed. Here an improved version will be explained. It will be applied to the model problem described above. This algorithm is a unigrid algorithm. To prepare for the following consideration, in the next section the multigrid method will be reformulated in the unigrid language.

2.2 The Unigrid

Suppose the equation to solve lives on a “fundamental” lattice $\Lambda^0$ with lattice constant $a_0$. We write the equation as

$$L_0^0 \xi^0 = f^0 ,$$

where the index 0 tells us that it is formulated on the fundamental lattice. Inhomogeneous equations of this type also arise when an eigenvalue equation is solved by inverse iteration \[13\]. We will use inverse iteration heavily in our algorithm later on.

One now introduces auxiliary lattices $\Lambda^1, \Lambda^2, \ldots, \Lambda^N$, called layers, with lattice constants $a_j = L^j a_0$, where $L_b$ is the blocking factor (typically, one chooses $L_b = 2$). The last lattice $\Lambda^N$ consists of only one point. The different ways of transferring information between the lattices makes the crucial distinction between a unigrid and a true multigrid method. Let $H^j$ be the space of functions on lattice $\Lambda^j$. Then we introduce grid transfer operators:

- the interpolation operator : $A^{[0,j]} : H^0 \mapsto H^0$ and
- the restriction operator : $C^{[j,0]} : H^0 \mapsto H^j$.

In a true multigrid grid transfer operators are composed from transfer operators that act between adjacent layers. From this it is quite clear that every multigrid can be formulated as a unigrid (instead of transferring information directly from $\Lambda^j$ to $\Lambda^k$ go first to the fundamental lattice and afterwards to the target space), but not vice versa. From the point of view of computational complexity, a unigrid method is inferior because an equal amount of work needs to be done on all layers.

2.3 The meaning of smoothness

The basic principle of multigrid algorithms (for ordered elliptical problems) originates from the observation that after doing local relaxation sweeps, the error gets smooth. Hence it should be possible to represent the error on a coarser lattice, because the intermediate values can be obtained by smooth interpolation. “Good” interpolation operators are therefore known beforehand; for example, one may use linear interpolation. Normally, the smooth modes are the low-lying eigenmodes of the operator.

How can we extend this observation to disordered problems, where no a priori notion of smoothness is given? As explained in \[1\], we propose the following definition of smoothness in a disordered system. It assumes that a fundamental differential operator $L_0$ specifies the problem. In this case

\textbf{Definition:} A function $\xi$ on $\Lambda^0$ is smooth on length scale $a$ when

$$\|L_0 \xi\|^2 \ll \|\xi\|^2 ,$$

in units $a = 1$. 
2.3 The meaning of smoothness

This definition implies that the smoothest function is the lowest eigenmode of $L_0$. So we arrive at the basic principle stated above even for the disordered case: The slow-converging modes, which have to be represented on coarser grids, are smooth. Of course we still have to show that the definition is sensible and can be used to construct a good algorithm.

As a first step we can see that the notion of algebraic smoothness as introduced in the context of the AMG implies smoothness in our sense. The error $e$ of an approximative solution of the differential equation is called “algebraically smooth” if the residual $r$ is small compared to it, so

$$r_i = \sum_j L_{0,j} e_j \ll e_i \Longrightarrow \|L_0 e\| \ll \|e\|.$$  \hspace{1cm} (2.5)

The crucial step in the setup of the algorithm is the choice of the grid transfer operators $A^{0,j}$ and $C^{j,0}$. These operators should be smooth in our sense, because we want to use them to represent a smooth error on a coarser grid. But because our definition of smoothness depends on the problem matrix $L$, they are not given a priori. Instead, we have to compute these operators.

In the following, we will adopt the Galerkin choice $C^{j,0} = A^{0,j} \ast$, where $\ast$ denotes the adjoint, and the coarse-grid-operator $L_j$ will be defined by $L_j = C^{j,0} L_0 A^{0,j}$. So we only need to construct good interpolation operators.

We have to restrict the operators to a part of the lattice, because otherwise interpolation would be too costly. For the time being we will choose the supports of the interpolation operators to be fixed blocks $[x]$ as shown in figure 1. Therefore the operators will possess representations as rectangular matrices with elements $A^{[0,j]}_{zx}$ with $x \in \Lambda^j$ and $z \in \Lambda^0$. The matrix $A^{[0,j]}$ contains all interpolation operators on layer $j$. The single interpolation operator on block $[x]$ is denoted by $A^{0,j}_{:,x}$. It corresponds to one column vector of the matrix operator $A^{0,j}$. $A^{[0,j]}_{:,x}$ vanishes outside the block $[x]$. It has been found that there is little hope in eliminating critical slowing down for the inversion of the Dirac-operator without overlapping blocks [15]. As we intend to apply our algorithm also to this case, we choose overlapping blocks as shown in the figure.

\footnote{It has recently been remarked by Sokal [14] that the operator $L$ does not possess eigenvectors in a strict sense, because it maps a space on its dual and there does not exist a natural scalar product on the two spaces. He proposes to look instead at $B_0^{-1} L$, where $B_0$ is defined by the relaxation algorithm. In case of a matrix with constant diagonal this will not make any difference (as long as we use e.g. Jacobi-iteration), so this does not give problems with our model problem. Remark that this is not true for a true multigrid, because there on a block lattice a non-constant effective mass might be generated. In such an algorithm, the definition of smoothness should take the relaxation algorithm into account, as it is done in the AMG-context.}

Figure 1: Supports of the interpolation operator for the layers 1 and 2. On layer 1, more than one support is drawn to show the overlap.
We now want to look for smooth interpolation operators in the specified sense which fulfill (approximately) the eigenvalue equation restricted to the block \([x]\)

\[
L_0|_x \mathcal{A}_{x,x}^{[0,j]} = \varepsilon_0(x) \mathcal{A}_{x,x}^{[0,j]}. \tag{2.6}
\]

Here \(L_0|_x\) denotes the restriction of \(L_0\) to the block \([x]\). As the interpolation operator must vanish outside the block, we impose Dirichlet boundary conditions. The crucial assumption of our algorithm is that the solution \(\mathcal{A}_{x,x}^{[0,j]}\) of this equation is smooth on length scale \(a_j\). This is true for the scalar Laplace-operator, where the solution is half a sine wave on the block.

If we know such interpolation operators we can start a unigrid algorithm in the usual way: After relaxing on the fundamental layer the error \(e^0\) of the approximative solution \(\tilde{\xi}^0\), defined as \(e^0 = \xi^0 - \tilde{\xi}^0\), should be smooth. It fulfills the error equation \(L_0 e^0 = r^0\), where \(r^0 = f^0 - L_0 \tilde{\xi}^0\) is the residual.

As the error is smooth, it can be obtained by smooth interpolation of a function \(e^1\) living on \(\Lambda^1\):

\[
e^0 = A^{[0,1]} e^1. \tag{2.7}
\]

Inserting this into the error equation yields

\[
L_0 A^{[0,1]} e^1 = r^0 \tag{2.8}
\]

\[
C^{[1,0]} L_0 A^{[0,1]} e^1 = C^{[1,0]} r^0 \tag{2.9}
\]

\[
L_1 e^1 = r^1, \tag{2.10}
\]

which involves only functions and operators on the block lattice. After relaxing on eq. (2.10) we interpolate our estimate \(e^1\) back to the fundamental lattice and replace our approximation by a better one: \(\tilde{\xi} \leftarrow \tilde{\xi} + A^{[0,1]} e^1\). Now the error is expected to be smooth on the length scale \(a_1\) because we have relaxed it on \(\Lambda^1\), and so we can proceed to the next layer. When we come to layer \(\Lambda^j\), the error should be smooth on scale \(a_{j-1}\). After relaxing on this layer, we correct our approximation again: \(\tilde{\xi} \leftarrow \tilde{\xi} + A^{[0,j]} e^j\). By this we have smoothened the error on the larger scale \(a_j\). Eventually we will reach the lattice \(\Lambda^N\) where we can solve the equation exactly and can thereby remove the smoothest mode from the error. The difference to a true multigrid can be seen clearly: The multigrid is defined in a recursive way. (2.11) is solved by going to the next-coarser lattice without correcting the approximate solution \(\tilde{\xi}^0\) on the fundamental lattice.

### 2.4 The ISU-algorithm

But of course the question is: “Where do we get the smooth operators?” Consider for instance \(A^{[0,N]}\). It should satisfy the equation

\[
L_0 A^{[0,N]} = \varepsilon_0 A^{[0,N]}, \tag{2.11}
\]

involving the full unrestricted operator \(L_0\), because the last lattice consists of only one point. If we want to solve this equation with inverse iteration (i.e. by computing \(L_0^{-n} A^{[0,N]}\), start for large \(n\) and an arbitrary starting vector), we will have to solve an equation which seems to be exactly as difficult as our starting point, eq. (2.4).

But this is not so. The worst-converging mode of our starting equation is the mode to the lowest eigenvalue of \(L_0\), but now we want to compute this mode, so it does not contribute to the error of the eigenvalue equation. (We have to do a simple normalization step after each iteration.) Consequently the mode to the second-lowest eigenvalue of \(L_0\) is the one that converges worst and if we could handle this (and all higher modes as well), we could also handle the lowest mode in our inhomogeneous equation (2.3) by solving first eq. (2.11).

The basic idea of our algorithm is that the higher modes are smooth on shorter length-scales. This means that it should be possible to construct them out of pieces which are smooth on these length
scales and have supports on parts of the lattice. So the next-lowest modes $\xi^\text{low}_z$ are representable by linear combination of the interpolation operators $A^0_{x;N-1}$:

$$\xi^\text{low}_z = \sum_{x \in \Lambda^{N-1}} c_y A^0_{x} \xi^0_{x} .$$  

(2.12)

If this is true—and it is for our model problem—we see that the calculation of $A^0_{x;N-1}$ is similar. Again the worst-converging mode is the mode we aim at, the next-higher modes can be represented on smaller blocks. Their calculation is therefore simpler. Finally we arrive at the calculation of $A^0_{x;1}$, having to solve an equation on a $3 \times 3$-lattice. This is easily done. Because of the Dirichlet boundary conditions there is no low-lying mode here.

**Remark:** It might happen that the lowest eigenvalue is much larger than the difference between it and the next eigenvalue. In this case, many inverse iterations have to be done to resolve the two corresponding modes. A possible remedy for this problem is to calculate estimates $\lambda$ of the lowest eigenvalue as we proceed and to invert not $L_0$ but $L_0 - \lambda$. This problem will not arise on the last layer, because otherwise the operator would not be critical and simple relaxation algorithms will suffice to solve the equation.

We identify the site $x \in \Lambda^j$ with the block $[x]$ in $\Lambda^0$ having $x$ at the center. Eq. (2.1) is an eigenvalue equation on $[x]$ for the vector $A^0_{x}$. It can be solved via inverse iteration by our unigrid method, using the already calculated interpolation operators $A^0_{y}$ with supports inside the block. With this we arrive at the following

**Algorithm for calculating smooth interpolation operators:**

1. Choose initial value $A^0_{x;\text{start}}$.
2. Relax on the fundamental lattice on block $[x]$.
3. For all $1 \leq k < j$ do:
   - Calculate the residual $R^0_{x} = L_0[\delta x] A^0_{x} - A^0_{\text{start}}$.
   - Block the residual to layer $\Lambda^k$: $R^0_{x}' = A^0_{k} R^0_{x}$.
   - Calculate $L_0[\delta x] = C^{k0}[\delta x] A^0_{k}$.
   - Determine approximate solution of $L_0[\delta x] = R^0_{x}'$ by relaxation on $[x] \cap \Lambda^k$.
   - Correct $A^0_{x}$:
     $$\begin{align*}
     A^0_{x} &\leftarrow A^0_{x} + \sum_{y \in \Lambda^k} A^0_{0} \delta A^0_{y} .
     \end{align*}$$
4. Normalize the interpolation operator.
5. If approximate solution $A^0_{x}$ good solution of the eigenvalue equation then do next $x$.
   - If approximate solution $A^0_{x}$ good solution of the inhomogeneous eq.
     then set $A^0_{x;\text{start}} \leftarrow A^0_{x}$.
6. go to 2.

We call this method *Iteratively Smoothing Unigrid* or ISU, because it is a unigrid method which computes smooth operators by means of an iterative method (and not directly from the given operator as in the AMG-algorithm).

Now we could try to use the same method to define a true multigrid algorithm: Just replace the eigenvalue equation for $A^0_{x}$ by an equation for $A^j_{x}$ which interpolates between adjacent layers and use $L_{j-1}$ as the operator for this equation. In this way, we would get operators which are smooth with
respect to the blocked differential operator. But this will often not work. To see this, formulate the new true multigrid algorithm in unigrid language. It involves operators \( A^{[0,j]} = A^1 \cdot A^2 \cdots A^j \). But the product of operators which are smooth on different length scales is smooth only on the shorter of these length scales. So we will never get a transfer operator that is smooth on large scales. To put the same fact in another way: In a true multigrid as used for ordered problems, interpolation operators \( A^j \) will smoothly interpolate functions that are smooth on all length scales \( a_k > a_j \) (the usual linear operators for example interpolate constants to constants, regardless of the scale), but in our case \( A^{[0,j]} \) will not be able to do this.

This tells us that our algorithm really is a unigrid algorithm, not just a multigrid in disguise. It is therefore impossible to apply the usual two-grid-analysis to prove convergence. Furthermore, we can not stop the algorithm on a layer \( j < N \), because in this case the modes on the larger scales would not be handled appropriately.

From the above description it is clear that the work involved in calculating good interpolation operators is larger than the time needed for the solution of the equation (2.1) itself. The following table shows the computational costs of the algorithm, compared to a true multigrid and to a local relaxation. Here, \( L \) denotes the grid length and \( d \) is the dimension.

| Algorithm                  | CPU-time  | Storage space |
|----------------------------|-----------|---------------|
| Local relaxation           | \( L^{d+s}(z \approx 2) \) | \( L^d \) |
| True multigrid and AMG     | \( L^d \)  | \( L^d \) |
| ISU-algorithm              | \( L^d \ln^2 L \) | \( L^d \ln L \) |

### 2.5 Performance of ISU

We studied this algorithm for the 2-dimensional bosonic model problem, as described in eqs. (0.1) and (2.1). The subtraction of the lowest eigenvalue makes the problem critical, and we can directly control criticality by tuning \( \delta m^2 \), the lowest eigenvalue of the full problem.

We measured the inverse asymptotic convergence rate \( \tau \) defined as

\[
\tau = \lim_{n \to \infty} \frac{-1}{\ln \varrho_n},
\]

where \( \varrho_n \) is the quotient of the error norms before and after iteration number \( n \). For large \( n \), this quantity approaches a constant.

Fig. 6 shows the inverse asymptotic convergence rate as a function of \( \delta m^2 \) for grid sizes \( 32^2 \cdots 128^2 \) at \( \beta = 1.0 \) in a SU(2)-gauge field background. Absence of critical slowing down can be seen clearly, and the absolute value of \( \tau \) is quite small. (\( \tau = 1 \) corresponds to a reduction of the residual by a factor of \( e \) in one multigrid sweep.) The sweeps are V-cycles with one pre- and one post-relaxation step. The results do not vary appreciably by changing \( \beta \).

**Remark:** To conclude that critical slowing down is absent, it is not sufficient to study only the dependence of \( \delta m^2 \) for fixed grid sizes. Our method ensures correct treatment of the lowest mode with eigenvalue \( \delta m^2 \). But the eigenvalue of the second-lowest mode depends on the grid-size, so the grid-size has to be large enough to make also this eigenvalue fairly low.

Because of the large work involved in calculating the interpolation operators, it has also to be verified that the number of inverse iterations needed for the calculation of the interpolation operators does not grow with the grid size. We found that six multigrid sweeps with one pre- and no postrelaxation step were sufficient for this on every layer, regardless of the lattice size; doing more sweeps and thereby calculating the operators more exactly did not improve the convergence of eq. (2.1).

So it is clear that there is no critical slowing down for the solution of the model problem. Nevertheless the question has to be answered at which grid sizes this will pay off, since the overhead is large. A careful comparison with the conjugate gradient algorithm on CPU-time grounds will have to be done. We think this will be worthwhile only if the ISU works as well for fermions.
To understand this success we can investigate the algorithm more closely. To this end we have calculated all eigenmodes of the covariant Laplace operator (0.1)—using a standard library function—and a solution of eq. (2.1). Now we were able to start the algorithm with an initial value of which we knew the error in advance, and to monitor the behaviour of the error as the algorithm proceeds, always expanding the error into the eigenmodes. So we could check that the fundamental relaxation indeed smoothens the error by eliminating the contribution of the higher modes. The coarser the lattice becomes the lower are the modes which are eliminated, until on the last lattice the contribution of the lowest mode is set exactly to zero.

We also checked that it is indeed possible to represent the low-lying modes by the interpolation operators. We calculated that part of the modes which was orthogonal on all interpolation operators on a given layer. This quantity was small, so the overlap between low-lying modes and interpolation operators is large.

This latter result is not surprising, as we already know that the lowest modes are localized. If a mode consists of a few localized parts it seems clear that we can patch it together from operators which are restricted to a part of the grid. This suggests that localization of low-lying modes is a good prerequisite for convergence of our algorithm.

Fig. 7 shows the interpolation operators $A^{0\,N-1}$ and $A^{0\,N-2}$ belonging to the same gauge field configuration as Fig. 2, bottom. It can be clearly seen that the different support boundaries are able to single out the different localization centers. On smaller blocks, the operators are centered in the middle of the block, because here the restriction through the boundary is too strong.

It is crucial for this possibility of representing the localized modes by the interpolation operators that there exists a block into which the modes fit well. This is not true for other models of disordered systems. Our algorithm was also tried on the “random resistor problem” but here the simple method of fixing the blocks geometrically led to critical slowing down ($\sim 0.7$). This is explained by the fact that there existed parts of eigenmodes which did not fit into any of the blocks and could therefore not be combined from the interpolation operators. (We studied this by again calculating all modes and looking at the shape of the bad-converging ones.) Nevertheless it is possible to generalize our algorithm by choosing the blockcenters in a more sophisticated way, using an algorithm of AMG-type for this step, and then applying the ISU-algorithm. This will be done in the future, using a C++-class library which is developed at the moment.

3 Conclusions

We have seen the localization of the lowest (and highest) modes in a 2-dimensional Lattice Gauge Theory. This phenomenon could be understood by connecting it to the problem of Anderson-Localization.

The performance of a new Multigrid algorithm was studied. For the model problem it showed no critical slowing down. The algorithm is well suited to deal with low-lying localized modes, because good approximations to these modes are used for the interpolation.

The next step will be to investigate localization and the behaviour of the ISU-algorithm for the Dirac-equation. This work is in progress.

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