The two-dimensional 4-state Potts model in a magnetic field

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Abstract
We present a solution of the nonlinear renormalization group equations leading to the dominant and subdominant singular behaviors of physical quantities (free energy density, correlation length, internal energy, specific heat, magnetization, susceptibility and magnetocaloric coefficient) at the critical temperature in a non-vanishing magnetic field. The solutions (i) lead to exact cancellation of logarithmic corrections in universal amplitude ratios and (ii) prove recently proposed relations among logarithmic exponents.

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1. Introduction, notations and definitions

In a recent paper on the amplitude combinations in the 4-state Potts model [1], we have provided in an appendix a detailed analysis of the nonlinear Salas and Sokal renormalization group (RG) equations [2], leading in particular to the critical behavior of densities of the free energy, the internal energy, the specific heat, the magnetization and the susceptibility in zero external magnetic field.

A similar analysis, that was not written in [1], would allow the calculation of the same quantities (including also the correlation length and magnetocaloric coefficient) at the critical temperature in a non-zero magnetic field. Recently this problem was considered in [3] and this motivated us to revisit our approach and observe that it can (i) complete the RG determination of all critical exponents, (ii) demonstrate the exact cancellation of logarithmic corrections in universal amplitude ratios to all orders and (iii) automatically lead to the scaling laws introduced by Kenna, Johnston and Janke among the logarithmic correction exponents [4–6].
Let us recall first the standard way of deriving universal combinations of critical amplitudes. For this purpose, we will illustrate the case of the non-trivial ratios $R^\pm = A_\pm \Gamma_\pm / B^2_-$ and $R^\pm = \Gamma_\pm D_\pm B^{\pm -1}$ which connect the amplitudes in the high and low temperature phases or at the critical temperature in presence of a magnetic field. We will see that the second ratio, connecting simultaneously temperature scaling and magnetic field scaling will be a source of difficulties when logarithmic corrections are taken into account.

A basic hypothesis in the theory of critical phenomena which relies on the RG analysis is the homogeneity assumption for the singular part of the free energy density [7, 8]

$$f_c(\tau, h) = b^{-D} F_\pm (\kappa \beta b^\nu |\tau|, \kappa \beta^\alpha |h|),$$

(1)

where $D$ is the space dimension, $b$ is the rescaling factor, $\tau$ and $h$ are the relevant thermal and magnetic fields with the corresponding RG eigenvalues $y_\tau$ and $y_h$, $F_\pm (x, y)$ is a universal function of its arguments $x$ and $y$, $\pm$ stands for $T > T_c$ and $T < T_c$, and $\kappa_\tau$ and $\kappa_h$ are non-universal metric factors (which depend e.g. on the lattice symmetry at a given space dimension). The critical behaviors of the magnetization, the susceptibility and the specific heat then follow by taking derivatives with respect to $\tau$ or $h$,

$$M_\tau(0, h) = \kappa_\beta b^{-D+\gamma} \partial_\tau F_\pm (x, y)|_{x=0}, \quad \tau = 0, \quad h \rightarrow 0$$

(2)

$$M_\tau(\tau, 0) = \kappa_\beta b^{-D+\gamma} \partial_\tau F_\pm (x, y)|_{y=0}, \quad \tau \rightarrow 0^-, \quad h = 0$$

(3)

$$\chi_\pm(\tau, h) = \kappa_\beta^2 b^{-D+2\gamma} \partial_\tau^2 F_\pm (x, y), \quad \tau \rightarrow 0, \quad h \rightarrow 0$$

(4)

$$C_\pm(\tau, h) = \kappa_\beta^2 b^{-D+2\gamma} \partial_\tau^2 F_\pm (x, y), \quad \tau \rightarrow 0, \quad h \rightarrow 0.$$  

(5)

The definition of the critical exponents according to the standard terminology follows from the elimination of $x$ and $y$ dependence at the critical temperature, $b = (\kappa_\beta |h|)^{-1/\gamma}$, $\tau = 0$, $h \rightarrow 0$:

$$M_\tau(h) = D_\tau^{-1/\beta} |h|^{1/\beta}, \quad \delta = \frac{y_h}{D}, \quad D_\tau^{-1/\beta} = \kappa_\beta^{1+1/\beta} \partial_\tau F_\pm (0, 1),$$

(6)

or in zero magnetic field, $b = (\kappa_\beta |\tau|)^{-1/\gamma}$, $\tau \rightarrow 0$, $h = 0$:

$$M_\tau(\tau) = B_\tau (\tau)^{\beta}, \quad \beta = \frac{D}{y_\tau}, \quad B_\tau = \kappa_\beta \partial_\tau F_\pm (1, 0), \quad \tau \rightarrow 0^-$$

(7)

$$\chi_\pm(\tau) = \Gamma_\pm |\tau|^{-\gamma}, \quad \gamma = \frac{2y_h - D}{y_\tau}, \quad \Gamma_\pm = \kappa_\beta^2 \kappa_\tau^{-\gamma} \partial_\tau^2 F_\pm (1, 0),$$

(8)

$$C_\pm(\tau) = \frac{A_\pm}{\alpha} |\tau|^{-\alpha}, \quad \alpha = \frac{2y_h - D}{y_\tau}, \quad A_\pm = \kappa_\beta^2 \partial_\tau^2 F_\pm (1, 0).$$

(9)

The subscript $c$, e.g. in $F_c(0, y)$, specifies that the function is evaluated at the critical temperature $\tau = 0$.

The amplitudes are clearly non-universal quantities, but elimination of all non-universal metric factors is possible by forming convenient combinations of these amplitudes which are universal. Notice that $\kappa_h$ disappears from the ratio $\chi_\pm(\tau)/M^2_\pm(\tau)$, then, exploiting the Rushbrooke scaling law $\alpha + 2\beta + \gamma = 2$, we multiply this quantity by $C_\pm(\tau)$ to eliminate also $\kappa_\tau$ and finally we obtain the function

$$R^\pm_c(\tau) = |\tau|^{2} C_\pm(\tau) \chi_\pm(\tau)/M^2_\pm(\tau)$$

(10)

tending to the universal quantity $\partial_\tau^2 F_\pm (1, 0)\partial_\tau^2 F_\pm (1, 0) / (\partial_\tau F_\pm (1, 0))^2 = A_\pm \Gamma_\pm / a b^2$, which establishes the universality of this combination of critical amplitudes. The last equality above follows from the definitions of amplitudes in equations (7), (8) and (9). Clearly, a universal combination is associated to a scaling law, Rushbrooke scaling law in the present case.
But this is not the whole story, since one knows that in some cases (for the 4-state Potts model in two dimensions) logarithmic corrections occur which involve ’hat exponent’ combinations [11, 6], e.g. \( M_\pm (\tau) = B_\pm |\tau|^{\beta (1-\alpha)} (1 - \ln |\tau|)^{\hat{\beta}} \), \( \chi_\pm (\tau) = \Gamma_\pm |\tau|^{-\gamma} (1 - \ln |\tau|)^{\hat{\gamma}} \) and \( C_\pm (\tau) = \frac{A_\pm}{\alpha} |\tau|^{-\nu} (1 - \ln |\tau|)^{\hat{\nu}} \). The combinations \( R^\pm_\pm(\tau) \) in equation (10) now tends toward

\[
R^\pm_\pm(\tau) \rightarrow \frac{A_\pm \Gamma_\pm}{\alpha B^\pm_\pm} (1 - \ln |\tau|)^{\hat{\beta} + \hat{\gamma}},
\]

and, provided that no other log-term appears, we have to impose the scaling relation

\[
\hat{\alpha} - 2\hat{\beta} + \hat{\gamma} = 0
\]

among the exponents describing the logarithmic corrections in order to still guarantee the universality of the combinations \( A_\pm \Gamma_\pm /B^\pm_\pm \).

The same line of reasoning for the other combinations considered, \( R^\pm_\gamma \), is less obvious (and this is the reason why we have chosen this ratio to illustrate our purpose). It is easy to show that in the absence of log-corrections, thanks to the Widom scaling relation \( \gamma = \beta(\delta - 1) \), the function

\[
R^\pm_\gamma (\tau, h) = |\tau| \chi_\pm (\tau) M^{\frac{\beta - 1}{\alpha}} (\tau) M^{-\frac{\beta}{\alpha}} (h)
\]

tends to the universal quantity \( \hat{\alpha}^2 F_\pm (1, 0) (\hat{\delta} F_\pm (1, 0))^{\beta - 1} (\delta F_\pm (0, 1))^{-\delta} \). It follows, using equations (6)–(8), that the combinations \( \Gamma_\pm D_\pm B_\pm^{\beta - 1} \) are universal. On the other hand, when logarithmic corrections occur, one obtains the limiting behavior

\[
R^\pm_\gamma (\tau, h) \rightarrow \Gamma_\pm D_\pm B_\pm^{\beta - 1} (1 - \ln |\tau|)^{\hat{\beta} + \hat{\gamma}} (1 - \ln |h|)^{-\delta \hat{\beta}}
\]

from which one would be tempted to conclude erroneously that \( \hat{\beta} + \hat{\gamma} (\delta - 1) = 0 \) and \( \delta \hat{\beta} = 0 \). This is not correct, as we will see later, because of an interplay between the two types of logarithms in \( |\tau| \) and in \( |h| \) expressed by equation (37) below. We thus have to improve the analysis presented in this introductory section.

2. Renormalization group analysis

In order to solve the problem, we provide below a re-examination of the RG derivation of all scaling quantities, including now the dependence on an external magnetic field along the critical isotherm. Let us first recall the standard definitions of some exponent combinations which will occur below: \( \alpha_\pm = \alpha / \beta \delta, \beta_\pm = \beta / \beta \delta, \chi_\pm = \gamma / \beta \delta, \psi_\pm = \nu / \beta \delta, \beta_\pm = 1 - \alpha_\pm \). For the critical amplitudes we use the notations of [6, 9].

\[
h = 0, \quad \tau \rightarrow 0^+, \quad \tau = 0, \quad h \rightarrow 0^+,
\]

\[
f_\pm (\tau, \psi) = F_\pm |\tau|^{2\alpha - 4 (1 - \ln |\tau|)^{\hat{\alpha}}, \quad f_\pm (h, \psi) = F_\pm |h|^{1 + \alpha - \hat{\alpha}}, (1 - \ln |\tau|)^{\hat{\alpha}},
\]

\[
M(\tau, \psi) = B_\pm |\tau|^{\beta (1 - \ln |\tau|)^{\hat{\beta}}, \quad M(h, \psi) = B_\pm |h|^{1 - \beta (1 - \ln |h|)^{\hat{\beta}}, (1 - \ln |\tau|)^{\hat{\beta}},
\]

\[
E(\tau, \psi) = \frac{A_\pm}{\alpha(1 - \alpha)} |\tau|^{1 - \alpha (1 - \ln |\tau|)^{\hat{\alpha}}, \quad E(h, \psi) = E_\pm |h|^{\alpha (1 - \ln |h|)^{\hat{\alpha}},
\]

\[
\chi(\tau, \psi) = \Gamma_\pm |\tau|^{-\gamma (1 - \ln |\tau|)^{\hat{\gamma}}, \quad \chi(\tau, \psi) = \Gamma_\pm |\tau|^{-\gamma (1 - \ln |\tau|)^{\hat{\gamma}},
\]

\[
C(\tau, \psi) = \frac{A_\pm}{\alpha} |\tau|^{\alpha (1 - \ln |\tau|)^{\hat{\alpha}}, \quad C(h, \psi) = \frac{A_\pm}{\alpha} |h|^{-\alpha (1 - \ln |h|)^{\hat{\alpha}},
\]

\[
m_{\mp}(\tau, \psi) = m_{\mp} |\tau|^{\beta - 1 (1 - \ln |\tau|)^{\hat{\beta}}, \quad m_{\mp}(h, \psi) = m_{\mp} |h|^{\delta - 1 (1 - \ln |h|)^{\hat{\delta}},
\]

\[
\xi(\tau, \psi) = \xi_{\pm} |\tau|^{-\nu (1 - \ln |\tau|)^{\hat{\nu}}, \quad \xi(h, \psi) = \xi_{\pm} |h|^{-\nu (1 - \ln |h|)^{\hat{\nu}},
\]

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In these expressions, $\psi$ denotes an irrelevant field, the role of which is discussed below. The magnetocaloric coefficient $n_\tau$, is often measured in experiments—since it is more singular that the magnetization—but is not an independent quantity.

The nonlinear Salas and Sokal RG equations [2, 11, 10] for the relevant thermal and magnetic fields $\tau$ and $h$ and the marginal dilution field $\psi$, which describe the 4-state Potts model in 2d, are given by

\[
\frac{d\tau}{dl} = (y_\tau + y_{\tau\psi} \psi)\tau, \quad \frac{dh}{dl} = (y_h + y_{h\psi} \psi)h, \quad \frac{d\psi}{dl} = g(\psi),
\]

(23)–(25)

with $l = \ln h$. The fixed point is at $\tau = h = 0$. Starting from initial conditions $\tau(0), h(0)$, the relevant fields $\tau$ and $h$ grow exponentially with $l$, and their behaviors follow from the renormalization flow from $\tau(0) \sim \tau, h(0) \sim h$ in the vicinity of the critical point up to some $\tau(l) = O(1), h(l) = O(1)$ outside the critical region. The function $g(\psi)$ is smooth and may be expanded in powers of $\psi$, $g(\psi) = y_{\psi^2} \psi^2 + y_{\psi^3} \psi^3 + \cdots$. The dilution field $\psi$ being marginal for $q = 4$ in two dimensions, there is no linear term in $g(\psi)$ and along the RG flow $\psi(l)$ remains of order $O(\psi(0))$ and $\psi(0)$ is negative, $|\psi(0)| = O(1)$. The first term in the function $g(\psi)$ was first considered by Nauenberg and Scalapino [10], and later by Cardy, Nauenberg and Scalapino [11], and the second term was introduced by Salas and Sokal [2].

The parameters are known and take the values $y_{\tau\psi} = 3/(4\pi), y_{h\psi} = 1/(16\pi), y_{\psi^2} = 1/\pi$ and $y_{\psi^3} = -1/(2\pi^2)$ [2, 11], while the relevant scaling dimensions are $\gamma_{\tau} = 3/2$ and $\gamma_h = 15/8$.

In zero magnetic field, under a change of the length scale, the singular part of the free energy density and the correlation length transform according to

\[
f_s(\tau(0), h(0), \psi(0)) = e^{-Dl} f_s(\tau(l), 0, \psi(l)), \\
\xi(\tau(0), h(0), \psi(0)) = e^{\xi} \xi(\tau(l), 0, \psi(l)),
\]

(26)–(27)

The thermal behavior in zero magnetic field is obtained by solving equations (23) and (25), while the dependence on the magnetic field obeys

\[
f_s(0, h(0), \psi(0)) = e^{-Dl} f_s(0, h(l), \psi(l)), \\
\xi(0, h(0), \psi(0)) = e^{\xi} \xi(0, h(l), \psi(l)).
\]

(28)–(29)

The thermal behavior in zero magnetic field is obtained by solving equations (23) and (25), while the dependence on the magnetic field along the critical isotherm follows from equations (24) and (25). The two sets of equations have exactly the same structure. We will therefore use a common notation $\varphi$ for the relevant scaling field ($\tau$ or $h$). Starting in the vicinity of the critical point at $\varphi(0) = \varphi$, the field grows under renormalization as $\varphi = \varphi(l) = \varphi e^{\varphi l} + \text{corrections} \sim b^\varphi \varphi + \text{corrections}$. This provides the leading singularities in equations (26)–(29), and the homogeneity assumption approximately takes the usual form $f_s(\varphi, \varphi(0)) = b^{-Dl} f_s(b^\varphi \varphi, \varphi(l))$ and $\xi(\varphi, \varphi(0)) = b^\xi (b^\varphi \varphi, \varphi(l))$. The correction terms will be responsible for the appearance of logarithms in all physical quantities and the purpose of this paper is essentially to analyze in detail the role of the corrections. We shall discuss in particular the relations among the ‘hat-exponents’ introduced by Kenna, Johnston and Janke which are still known only through the scaling laws derived by these authors [6]. Equation (23) (or equation (24)) leads to

\[
\int_0^l \frac{d\varphi}{\varphi} = \ln \frac{\varphi(l)}{\varphi(0)} = \text{const} + \ln \frac{1}{|\varphi|} = y_\varphi l + \psi \int_0^l \psi dl.
\]

(30)
where the last integral is obtained from equation (25) rewritten as
\[
\psi \, dl = \frac{1}{y_\psi^2} \left( \frac{1}{\psi} - \frac{y_\psi^2}{y_\psi^2 + y_\psi^4 \psi} \right) \, d\psi,
\] (31)
thus
\[
\int_0^l \psi \, dl = \frac{1}{y_\psi^2} \ln \left( \frac{\psi(l) \, y_\psi^2 + y_\psi^4 \psi(l)}{\psi(0) \, y_\psi^2 + y_\psi^4 \psi(0)} \right).
\] (32)
Combining equations (30) and (32) we get
\[
l = \text{const} - \frac{1}{y_\psi^2} \ln |\psi(l)| + \frac{y_\psi \psi}{y_\psi^2 + y_\psi^4 \psi(l)} \ln \left( \frac{\psi(0) \, y_\psi^2 + y_\psi^4 \psi(l)}{\psi(l) \, y_\psi^2 + y_\psi^4 \psi(0)} \right).
\] (33)
Apart from the leading \( \psi \)-dependence, all logarithmic corrections which occur in the 4-state Potts model are encoded in the dilution field dependence. The ubiquitous combination where it appears is conveniently denoted as
\[
\zeta = \frac{\psi(l) \, y_\psi^2 + y_\psi^4 \psi(l)}{\psi(0) \, y_\psi^2 + y_\psi^4 \psi(0)},
\] (34)
and remembering that \( \psi \) is either the reduced temperature \( \tau \), or the external magnetic field \( h \),
\[
f_s(\psi, \psi(0)) = \text{const} \times |\psi|^{D \nu \chi_{D\mu}} \psi \, d\psi,
\] (35)
\[
|\xi(\psi, \psi(0))| = \text{const} \times |\psi|^{-1/\nu} \zeta^{\nu \mu},
\] (36)
When we specify \( \psi \), we obtain from equation (33) the functional similarity
\[
|\tau|^{\nu} \zeta^{\mu} \propto |h|^{\nu} \zeta^{\mu},
\] (37)
where it is convenient to denote \( \nu = 1/y_\tau = 2/3 \), \( \nu_c = 1/y_h = 8/15 \), \( \mu = y_\tau \psi / y_\tau y_\psi = 1/2 \), \( \mu_c = y_h \psi / y_h y_\psi = 1/30 \). The free energy density is then written as \( f_s(\tau, \psi) \sim |\tau|^{D \nu} \zeta^{D \mu} \) in zero magnetic field while the field dependence along the critical isotherm is given by \( f_s(h, \psi) \sim |h|^{D \nu} \zeta^{D \mu} \). The other thermodynamic properties follow by derivation with respect to the scaling fields, e.g., \( E(\tau, \psi) = \frac{\partial}{\partial \tau} f_s(\tau, \psi) \) which leads to either \( E(\tau, \psi) \sim |\tau|^{D \nu - 1} \zeta^{D \mu} \) when the magnetic field tends to zero, or to \( E(h, \psi) \sim |h|^{D \nu} \zeta^{D \mu} |\tau|^{-1} \) in the vicinity of the critical isotherm. Using equation (37) when needed, and specifying either \( h = 0 \) or \( \tau = 0 \), we may now collect the following expressions,
\[
h = 0, \quad \tau \to 0^\pm, \quad \tau = 0, \quad h \to 0^\pm,
\] (38)
\[
f_s(\tau, \psi) \sim |\tau|^{D \nu} \zeta^{D \mu}, \quad f_s(h, \psi) \sim |h|^{D \nu} \zeta^{D \mu},
\] (39)
\[
M(\tau, \psi) \sim |\tau|^{D \nu - 1} \zeta^{D \mu} \chi(\psi), \quad M(h, \psi) \sim |h|^{D \nu - 1} \zeta^{D \mu} \chi(\psi),
\] (40)
\[
E(\tau, \psi) \sim |\tau|^{D \nu - 1} \zeta^{D \mu}, \quad E(h, \psi) \sim |h|^{D \nu - 1} \zeta^{D \mu} \chi(\psi),
\] (41)
\[
\chi(\tau, \psi) \sim |\tau|^{D \nu - 1} \zeta^{D \mu} \chi(\psi), \quad \chi(h, \psi) \sim |h|^{D \nu - 1} \zeta^{D \mu} \chi(\psi),
\] (42)
\[
C(\tau, \psi) \sim |\tau|^{D \nu - 1} \zeta^{D \mu}, \quad C(h, \psi) \sim |h|^{D \nu - 1} \zeta^{D \mu},
\] (43)
\[
m_T(\tau, \psi) \sim |\tau|^{D \nu - 1} \zeta^{D \mu}, \quad m_T(h, \psi) \sim |h|^{D \nu - 1} \zeta^{D \mu},
\] (44)
\[
\xi(\tau, \psi) \sim |\tau|^{-\nu} \zeta^{\mu}, \quad \xi(h, \psi) \sim |h|^{-\nu} \zeta^{\mu}.
\] (45)
The values of the leading exponents follow directly,
\[\alpha = 2 - D\nu = \frac{2}{3}, \quad \alpha_c = 2 \frac{\nu_c}{\nu} - D\nu = \frac{8}{15},\]
\[\beta = D\nu - \frac{\nu}{\nu_c} = \frac{1}{12}, \quad \delta = \frac{1}{D\nu_c - 1} = 15,\]
\[\gamma = 2 \frac{\nu}{\nu_c} - D\nu = \frac{7}{6}, \quad \epsilon_c = D\nu_c - \frac{\nu_c}{\nu} = \frac{4}{15},\]
\[\nu = \frac{2}{3}, \quad \nu_c = \frac{8}{15}.\] (46)

### 3. Exponents of logarithmic corrections and scaling relations among them

We now want to explore the values of the ‘hat exponents’ and the link to universal combinations of critical amplitudes. The particular form taken by the function \(\zeta\) follows from the solution of equation (25), combined to equation (33) iterated at the convenient level of approximation (see appendix of [1] for details and [12–18] for different levels of approximation). Keeping only the leading logarithmic behavior for the present context, expression (34) simply yields
\[\zeta \sim (- \ln |\phi|)^{-1} (1 + \text{corrections}) \] (47)
and the exponents of all logarithmic corrections are directly read in equations (39)–(45):
\[\hat{\alpha} = -D\mu = -1, \quad \hat{\alpha}_c = 2 \frac{\mu_c - \mu}{\nu} - D\nu_c = -\frac{22}{15},\]
\[\hat{\beta} = \frac{\mu - \mu_c}{\nu} - D\mu = -\frac{1}{8}, \quad \hat{\delta} = -D\mu_c = -\frac{1}{15},\]
\[\hat{\gamma} = 2 \frac{\mu - \mu_c}{\nu_c} - D\mu = \frac{3}{4}, \quad \hat{\epsilon}_c = \frac{\mu_c - \mu}{\nu} - D\mu_c = -\frac{23}{30},\]
\[\hat{\nu} = \mu = \frac{1}{2}, \quad \hat{\nu}_c = \mu_c = \frac{1}{30}.\] (48)
What appears extremely useful in these expressions is that when defining appropriate effective ratios, the dependence on the quantity \(\zeta\) cancels, due to the scaling relations among the critical exponents. This quantity \(\zeta\) is precisely the only one where the log terms are hidden, and thus we may infer that not only the leading log terms, but all the log terms hidden in the dependence on the marginal dilution field disappear in the conveniently defined effective ratios. For example in effective ratios like those considered in the introduction in equations (10) and (13),
\[R_{\hat{C}}^+(\tau) = \tau^2 \frac{C_{\pm}(\tau, \zeta)\chi_{\pm}(\tau, \zeta)}{M^2(\tau, \zeta)},\] (49)
\[R_{\hat{X}}^+(\tau, h) = |h|\chi_{\pm}(\tau, \zeta)M_{\pm}^{-1}(\tau, \zeta)M_{-\delta}^{-1}(h, \zeta),\] (50)
alI)l corrections to scaling coming from the variable \(\zeta\) disappear, provided that in addition to the scaling relation (12), one more scaling relation is also satisfied
\[\hat{\gamma} + (\delta - 1)\hat{\beta} - \delta\hat{\delta} = 0.\] (51)
The two scaling laws (12) and (51) are verified by the values of the ‘hat exponents’ of the 4-state Potts model given in equations (46).

This solves the problem of the cancellation of the logarithmic corrections identified in equation (14) and the approach is easily extended to the other universal combinations of critical amplitudes. In conclusion this approach provides both the scaling relations among the leading exponents and those among the exponents of the logarithmic corrections.
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