The categories $L$-$\text{Top}_0$ and $L$-$\text{Sob}$ as epireflective hulls

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Abstract

We show that the category $L$-$\text{Top}_0$ of $T_0$-$L$-topological spaces is the epireflective hull of Sierpinski $L$-topological space in the category $L$-$\text{Top}$ of $L$-topological spaces and the category $L$-$\text{Sob}$ of sober $L$-topological spaces is the epireflective hull of Sierpinski $L$-topological space in the category $L$-$\text{Top}_0$.

Key words: Frame, $L$-topology, $T_0$-$L$-topological space, sober $L$-topological space, Epireflective hull

1 Introduction

Rodabaugh [11] extended the notion of sobriety in topology (cf. [6]) to $L$-topology. It is known that the category of sober topological spaces is the epireflective hull of the two-point Sierpinski topological space in the category $\text{Top}_0$ of $T_0$-topological spaces (cf. [10]). Höhle [5] has already pointed out that the category of sober $L$-topological spaces is reflective in the category $L$-$\text{Top}$ of $L$-topological spaces. We show here that the category of sober $L$-topological spaces is in fact the epireflective hull of the ‘Sierpinski $L$-topological space’ in the category $L$-$\text{Top}_0$ of $T_0$-$L$-topological space.

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2 Preliminaries

For the category theoretic notions used here, [1] may be referred. All subcategories are assumed to be full and replete.

Throughout this paper, $L$ denotes a frame (a complete lattice which satisfies the first infinite distributive law: $a \land (b_1 \lor b_2) = (a \land b_1) \lor (a \land b_2)$, with 0 and 1 being its least and largest elements respectively. A frame map is a lattice homomorphism between frames which preserves finite infima and arbitrary suprema. A subset $A \subseteq L$ of a frame $L$ is called a subframe if $A$ is a frame under the order induced by $L$.

We recall some definitions to make the paper self contained.

**Definition 2.1** [5] A family $\tau$ of $L$-sets in a set $X$ is called an $L$-topology on $X$, and the pair $(X, \tau)$ an $L$-topological space, if $\tau$ is a subframe of $L^X$. Furthermore, a map $f : (X, \tau) \to (Y, \delta)$ between two $L$-topological spaces is called continuous if $f^{-1}(\nu) \in \tau$, for every $\nu \in \delta$.

Let $L\text{-Top}$ denote the category of all $L$-topological spaces and their continuous maps.

Given an $L$-topological space $(X, \tau)$ and a subset $Y \subseteq X$, $\tau_Y = \{ \mu \land 1_Y \mid \mu \in \tau \}$ is an $L$-topology on $Y$. The $L$-topological space $(Y, \tau_Y)$ is called a subspace of $(X, \tau)$.

The notions of open map, homeomorphism and embedding in $L\text{-Top}$ are on expected lines.

**Definition 2.2** [1] A concrete category $(\mathcal{C}, U)$ over a category $\mathcal{A}$ is said to be uniquely transportable if for every $X \in \text{ob}\mathcal{C}$ and every $\mathcal{A}$-isomorphism $k : U(X) \to A$ there exists a unique $Y \in \text{ob}\mathcal{C}$ with $U(Y) = A$ such that $k' : X \to Y$ is a $\mathcal{C}$-isomorphism with $U(k') = k$.

**Definition 2.3** [5] A category $\mathcal{C}$ of sets with structures is a uniquely transportable construct $(\mathcal{C}, U)$ in which a class $C_{\mathcal{C}}(X)$ of $\mathcal{C}$-structures' is assigned to every set $X$, which is in one-to-one correspondence with the fibre $\{ A \in \text{ob}\mathcal{C} \mid U(A) = X \}$ of $X$; thus each $\mathcal{C}$-object $A$ can be identified with a pair $(X, s)$ (and called a set with a structure), where $U(A) = X$ and $s \in C_{\mathcal{C}}(X)$.

**Definition 2.4** [1] Given a category $\mathcal{C} = (\mathcal{C}, U)$ of sets with structures,

- a family $\mathcal{F} = \{ f_i : (X, s) \to (Y_i, t_i) \mid i \in I \}$ of $\mathcal{C}$-morphisms is said to be initial (or optimal, as in [5]) if for every $\mathcal{C}$-structured set $(Z, u)$ and a map $g : Z \to X$, $g : (Z, u) \to (X, s)$ if $f_i \circ g : (Z, u) \to (X, s)$, for every $i \in I$.
- if for a family $\mathcal{F} = \{ f_i : X \to U(Y_i, t_i) \mid i \in I \}$ of maps, where $X$ is a set and each $(Y_i, t_i)$ is a $\mathcal{C}$-structured set, there exists $s \in C_{\mathcal{C}}(X)$ such that the family $\{ f_i : (X, s) \to (Y_i, t_i) \mid i \in I \}$ is optimal, then $s$ is called an initial lift (or optimal lift, as in [5]) of the family $\mathcal{F}$.

**Definition 2.5** [5] Given a category $\mathcal{C} = (\mathcal{C}, U)$ of sets with structures, an object $S$ of $\mathcal{C}$ is called a Sierpinski object if for every $X = (X, s) \in \text{ob}\mathcal{C}$, the family of all $\mathcal{C}$-morphisms from $X$ to $S$ is initial.
The category $\mathcal{L}-\text{Top}$ is obviously a category of sets with structure. Moreover it is a topological construct [5]. We shall use $V$ to denote the forgetful functor from $\mathcal{L}-\text{Top}$ to the category of sets and functions, wherever needed.

**Remark 2.1** Let $\zeta$ be a family of $L$-sets in a set $X$. Let $\xi$ be the collection of all finite infima of members of $\zeta$ and $\tau$ be the collection of all suprema of members of $\xi$. It can be verified that $\tau$ is an $L$-topology on $X$. We shall denote it as $< \zeta >$.

**Definition 2.6** Let $\mathcal{F} = \{f_i : X \to \{Y_i, \delta_i\} \mid i \in I\}$ be a family of maps, where $X$ is a set and $\{(Y_i, \delta_i) \mid i \in I\}$ is a family of $L$-topological spaces. Then $\{f_i^{-1}(\mu) \mid \mu \in \delta_i, i \in I\}$ is called the initial $L$-topology on $X$ induced by $\mathcal{F}$.

**Definition 2.7** Given a family $\{(X_i, \tau_i) \mid i \in I\}$ of $L$-topological spaces, the initial $L$-topology on $X (= \prod_{i \in I} X_i)$ induced by all projection maps $\{p_i : X \to (X_i, \tau_i) \mid i \in I\}$ is called the product $L$-topology.

Let $(X, \tau)$ be an $L$-topological space, $Y$ be a set and $f : X \to Y$ be a surjective map. Then $\tau/f = \{v \in L^Y \mid f^*(v) \in \tau\}$ is an $L$-topology on $Y$, called the quotient $L$-topology, and the pair $(Y, \tau/f)$ is called the quotient $L$-topological space with respect to $(X, \tau)$ and $f$. The resulting continuous map $f : (X, \tau) \to (Y, \tau/f)$ is called a quotient map.

Let $\mathcal{C}$ be a category and let $\mathcal{H}$ and $\text{epi}\mathcal{C}$, respectively, denote some class of $\mathcal{C}$-morphisms and the class of all $\mathcal{C}$-epimorphisms.

**Definition 2.8** A $\mathcal{C}$-object $X$ is called $\mathcal{H}$-injective if for every $e : Y \to Z$ in $\mathcal{H}$ and every $\mathcal{C}$-morphism $f : Y \to X$, there exists a $\mathcal{C}$-morphism $g : Z \to X$ such that $g \circ e = f$.

**Definition 2.9** A $\mathcal{C}$-object $X$ is called a cogenerator (called a coseparator in [1]) if for every pair of distinct $f, g \in \mathcal{C}(Y, Z)$, there exists $h \in \mathcal{C}(Z, X)$ such that $h \circ f \neq h \circ g$.

Let $\mathcal{R}$ be a subcategory of $\mathcal{C}$.

**Definition 2.10** [2] $\mathcal{R}$ is said to be epireflective in $\mathcal{C}$ if for each $\mathcal{C}$-object $X$, there exists an epimorphism $r_X : X \to RX$, with $RX \in \text{ob}\mathcal{R}$, such that for each $\mathcal{C}$-morphism $f : X \to Y$, with $Y \in \text{ob}\mathcal{R}$, there exists a unique $\mathcal{R}$-morphism $f^* : RX \to Y$, such that $f^* \circ r_X = f$. If moreover, each $r_X \in \mathcal{R}$ and $f^*$ is a $\mathcal{C}$-isomorphism, whenever $f \in \text{epi}\mathcal{C} \cap \mathcal{R}$, then $\mathcal{R}$ is said to be an $\mathcal{H}$-firm epireflective subcategory of $\mathcal{C}$.

### 3 The Sierpinski $L$-topological Space

Consider the frame $L$. Then $< id_L >$, where $id_L$ is the identity map on $L$, is an $L$-topology on $L$. Call $(L, < id_L >)$ the Sierpinski $L$-topological space and denote it as $L_S$.

The following result is easy to verify.
Proposition 3.1 Let \((X, \tau) \in obL\text{-Top}\). Then \(\mu \in \tau\) iff \(\mu : (X, \tau) \to L_S\) is continuous.

Theorem 3.1 \(L_S\) is a Sierpinski object in \(L\text{-Top}\).

Proof: Let \((X, \tau) \in obL\text{-Top}\) and consider the family \(\mathcal{F} = \{f : (X, \tau) \to L_S \mid f \text{ is continuous}\}\). Let \((Y, \delta) \in obL\text{-Top}\) and let \(g : Y \to X\) be a map such that \(f \circ g : (Y, \delta) \to L_S\) is continuous, for every \(f \in \mathcal{F}\). We have to show that \(g : (Y, \delta) \to (X, \tau)\) is continuous. Let \(\mu \in \tau\). Then by Proposition 3.1, \(\mu : (X, \tau) \to L_S\) is continuous and hence \(\mu \in \mathcal{F}\). So, \(\mu \circ g : (Y, \delta) \to L_S\) is continuous. Again by Proposition 3.1, \(\mu \circ g \in \delta\). But \(\mu^{-1}(\mu) = \mu \circ g\), implying that \(g\) is continuous. Thus \(L_S\) is a Sierpinski object in \(L\text{-Top}\). \(\Box\)

Definition 3.1 [11] An \(L\)-topological space \((X, \tau)\) is called \(T_0\) if for every distinct \(x, y \in X\), there exists some \(\mu \in \tau\) such that \(\mu(x) \neq \mu(y)\).

The Sierpinski \(L\)-topological space \(L_S\) is \(T_0\).

Let \(L\text{-Top}_0\) denote the subcategory of \(L\text{-Top}\), whose objects are \(T_0\)-\(L\)-topological spaces.

From now on, we write ‘injective’ in place of ‘\(\mathcal{H}\)’-injective’, when \(\mathcal{H}\) is the class of all embeddings in \(L\text{-Top}_0\).

Theorem 3.2 \(L_S\) is an injective cogenerator in \(L\text{-Top}_0\).

Proof: Let \(e : (X, \tau) \to (Y, \delta)\) be an embedding in \(L\text{-Top}_0\) and let \(f : (X, \tau) \to L_S\) be a continuous map. Then \(f \in \tau\), implying that \(e^{-1}(f) \in \delta_\delta(X)\). So, there exists \(\nu \in \delta\) such that \(e^{-1}(f) = \nu \lor 1_{\delta}(X)\) and \(\nu \circ e = f\). As \(\nu \in \delta\), \(\nu : (Y, \delta) \to L_S\) is continuous. Thus \(L_S\) is injective.

Consider distinct pair of morphism \(f, g : (X, \tau) \to (Y, \delta)\) in \(L\text{-Top}_0\). Then there exists \(x \in X\) such that \(f(x) \neq g(x)\). As \((Y, \delta)\) is \(T_0\), there exists \(\nu \in \delta\) such that \(\nu(f(x)) \neq \nu(g(x))\) implying that \(\nu \circ f \neq \nu \circ g\). As \(\nu \in \delta\), \(\nu : (Y, \delta) \to L_S\) is continuous. Thus \(L_S\) is a cogenerator. \(\Box\)

4 A characterization of \(L\text{-Top}\)

Manes [8] obtained a characterization of the category \(\text{Top}\) of topological spaces with the help of usual two-point Sierpinski topological space \(2_S\). Srivastava [12] obtained a characterization of the category \(\text{FTop}\) of fuzzy topological spaces with the help of fuzzy Sierpinski space \(L_S\). In this section, we shall give a characterization of \(L\text{-Top}\) with the help of \(L_S\).

Theorem 4.1 A category \((\mathcal{C}, U)\) of sets with structures is concretely isomorphic to \(L\text{-Top}\) iff there exists \((S, u) \in ob\mathcal{C}\) with the underlying set \(S = L\), satisfying the following condition:

1. \((S, u)\) is a Sierpinski object in \(\mathcal{C}\),
2. every family \(\{f_i : X \to U(S, u) \mid i \in I\}\) has an initial lift,
3. the map \(\sup : (S, u)^X \to (S, u)\) is \(\mathcal{C}\)-morphism for every set \(X\) (here \((S, u)^X = \{(S^X, t) \mid t\) is the initial lift of all projection from \(S^X\) to \(U(S, u)\)\).
4. The map $\inf : (S, u)^X \to (S, u)$ is $\mathcal{C}$-morphism for every finite set $X$.

5. for every $(X, s) \in \text{ob}\mathcal{C}$ and for every initial family $\mathcal{F}$ of $\mathcal{C}$-morphisms from $(X, s)$ to $(S, u)$, the initial lift of $\{f : X \to U(S, u) \mid f \in \mathcal{F}\}$ contains every $\mathcal{C}$-morphism $g : (X, s) \to (S, u)$.

**Proof:** Let $(\mathcal{C}, U)$ be concretely isomorphic to $L$-$\text{Top}$. We show that $L$-$\text{Top}$ satisfies (1) – (5). In view of Theorem 3.1, $L_S$ is a Sierpinski object in $L$-$\text{Top}$. So, $L$-$\text{Top}$ satisfies (1). It is easy to see that every family $\mathcal{F} = \{f_i : X \to V(L_S) \mid i \in I\}$ in $L$-$\text{Top}$ has an initial lift, being the initial $L$-topology on $X$ induced by $\mathcal{F}$. So, $L$-$\text{Top}$ satisfies (2). Let $\mu \in L_S^X$ and for $x \in X$, $p_x$ denote the $x$-th projection map from $L^X$ to $V(L_S)$. Then $\sup(\mu) = \bigvee_{x \in X} \mu(x) = \bigvee_{x \in X} p_x(\mu)$ implying that $\sup = \bigvee\{p_x \mid x \in X\}$. As $p_x$ is open in $L_S^X$, $\sup$ is also open in $L_S^X$. Thus $\sup : L_S^X \to L_S$ is continuous, for every set $X$. Similarly, $\inf = \wedge\{p_x \mid x \in X\}$ and therefore $\inf$ is also continuous, for every finite set $X$.

Hence $L$-$\text{Top}$ satisfies (3) and (4). Let $(X, \tau) \in \text{ob} L$-$\text{Top}$ and $\mathcal{F} = \{f \mid f : (X, \tau) \to L_S$ is continuous\} be an initial family. Then $\mathcal{F} \subseteq \tau$. Let $\sigma$ be the initial lift of $\{f : X \to V(L_S) \mid f \in \mathcal{F}\}$. Then $\sigma \subseteq \tau$ and $f : (X, \sigma) \to L_S$ is continuous, for every $f \in \mathcal{F}$. Consider the identity map $id : (X, \sigma) \to (X, \tau)$. As $f \circ id = f$, for every $f \in \mathcal{F}$, $id$ is continuous and hence $\tau \subseteq \sigma$. Thus $L$-$\text{Top}$ satisfies (5).

Conversely, let $(\mathcal{C}, U)$ satisfies the given conditions. We show that $(\mathcal{C}, U)$ and $L$-$\text{Top}$ are concretely isomorphic. For showing this we have to produce two concrete functors $F : \mathcal{C} \to L$-$\text{Top}$ and $G : L$-$\text{Top} \to \mathcal{C}$ which are inverses to each other.

Let $(X, s) \in \text{ob}\mathcal{C}$ and let $\tau_s$ be the initial lift of the family $\mathcal{F} = \{f : X \to V(L_S) \mid f \in \mathcal{F}\}$. Then $\tau_s = \wedge\{f^\mu(\mu) \mid \mu \in L_S^X \}$ and hence $\tau_s = \bigwedge\mathcal{F}$. Now we show that $\mathcal{F}$ is an $L$-topology on $X$. Let $\{f_i \mid i \in I\} \subseteq \mathcal{F}$. Define a map $g : (X, s) \to (S, u)^I$ such that $g(x)(i) = f_i(x)$, for every $x \in X$ and for every $i \in I$. Let $p_i$ denote the $i$-th projection map from $S^I$ to $U(S, u)$. Then for every $i \in I$ and for every $x \in X$, $(p_i \circ g)(x) = p_i(g(x)) = g(x)(i) = f_i(x)$ implying that $p_i \circ g = f_i$, for every $i \in I$. By condition (2), $g$ is $\mathcal{C}$-morphism. For $x \in X$, $(\sup \circ g)(x) = \sup(g(x)) = \bigvee_{i \in I} g(x) = \bigvee_{i \in I} f_i(x)$ implying that $\sup \circ g = \bigvee_{i \in I} f_i$. By using condition (3), $\bigvee_{i \in I} f_i$ is $\mathcal{C}$-morphism and thus $\bigvee_{i \in I} f_i \in \mathcal{F}$. Similarly $\bigwedge_{i \in I} f_i \in \mathcal{F}$, for every finite $I$. Hence $\mathcal{F}$ is an $L$-topology on $X$. Thus $\bigwedge\mathcal{F} = \mathcal{F}$ and therefore $\tau_s = \mathcal{F}$. Let $f : (X, s) \to (Y, u)$ be $\mathcal{C}$-morphism. We show that $f : (X, \tau_s) \to (Y, \tau_u)$ is continuous. Let $\mu \in \tau_s$. Then $\mu : (Y, u) \to (S, u)$ is $\mathcal{C}$-morphism and hence $\mu \circ f : (X, s) \to (S, u)$ is also $\mathcal{C}$-morphism. As $f^\mu(\mu) = \mu \circ f$, $f^\mu(\mu) \in \tau_s$. Thus $f$ is continuous.

Let $(X, \tau) \in \text{ob} L$-$\text{Top}$ and $s_\tau$ be the initial lift of the family $\{\mu : X \to U(S, u) \mid \mu \in \tau\}$. Then $(X, s_\tau) \in \text{ob}\mathcal{C}$ and $\{\mu : (X, s_\tau) \to (S, u) \mid \mu \in \tau\}$ is an initial family. Let $f : (X, \tau) \to (Y, \delta)$ be continuous. We show that $f : (X, s_\tau) \to (Y, s_\delta)$ is $\mathcal{C}$-morphism. For every $\nu \in \delta$, $f^\nu(\nu) = \nu \circ f$.
and hence \( \nu \circ f \in \tau \). As \( \nu \in \delta, \nu : (Y, s_s) \to (S, u) \) is \( \mathscr{C} \)-morphism and \( \nu \circ f \in \tau, f : (X, s_r) \to (S, u) \) is \( \mathscr{C} \)-morphism. Since \( \{ \nu : (Y, s_s) \to (S, u) \mid \nu \in \delta \} \) is an initial family, \( f \) is \( \mathscr{C} \)-morphism.

Now we define two functors \( F : \mathscr{C} \to L\text{-}\text{Top} \) as \( F(X, s) = (X, \tau_s) \) and \( G : L\text{-}\text{Top} \to \mathscr{C} \) as \( G(X, \tau) = (X, s_\tau) \) (both functors have morphisms unchanged at set-theoretic level). It can be verified easily that \( F \) and \( G \) are concrete functors and they are inverses to each other. □

5 Epireflective hull of \( L_S \) in \( L\text{-}\text{Top} \)

The category \( \text{Top}_0 \) of \( T_0 \)-topological spaces is well-known to be the epireflective hull of usual two-point Sierpinski space \( 2_S \) in the category \( \text{Top} \) of topological spaces ([1], page 263). Lowen and Srivastava [7] showed that the category \( \text{FTop}_0 \) of \( T_0 \)-fuzzy topological spaces is the epireflective hull of fuzzy Sierpinski space \( I_S \) in the category \( \text{FTop} \) of fuzzy topological spaces. Analogously, in this section we show that \( L\text{-}\text{Top}_0 \) is the epireflective hull of \( L_S \) in \( L\text{-}\text{Top} \).

For showing that \( L\text{-}\text{Top}_0 \) is the epireflective hull of \( L_S \) in \( L\text{-}\text{Top} \), we first need to identify the epimorphisms and extremal subobjects in \( L\text{-}\text{Top} \). Epimorphisms and extremal subobjects, along with their proofs in \( L\text{-}\text{Top} \) are on familiar lines (as in \( \text{Top} \)).

**Proposition 5.1** Epimorphisms in \( L\text{-}\text{Top} \) are precisely the surjective maps.

**Proposition 5.2** Extremal monomorphisms in \( L\text{-}\text{Top} \) are precisely the embeddings in \( L\text{-}\text{Top} \).

**Corollary 5.1** Extremal subobjects in \( L\text{-}\text{Top} \) are precisely the subspaces.

We shall use the following two results from [9] (Theorem 1 and Theorem 2) to obtaining the epireflective hull of \( L_S \) in the categories \( L\text{-}\text{Top} \) and \( L\text{-}\text{Top}_0 \).

**Theorem 5.1** [9] A subcategory \( \mathcal{B} \) of a category \( \mathcal{A} \) is epireflective in \( \mathcal{A} \) iff it is closed under the formation of products and extremal subobjects in \( \mathcal{A} \).

**Theorem 5.2** [9] Let \( \mathcal{A} \) be a category, \( E \) be a class of \( \mathcal{A} \)-objects and \( \text{RE} \) be the epireflective hull of \( E \) in \( \mathcal{A} \). Then \( A \in \text{obRE} \) iff \( A \) is an extremal subobject of a product of objects of \( E \).

In the above two results, it is assumed that the category \( \mathcal{A} \) has products and \( \mathcal{A} \) is an epi-co-well-powered (epi, extremal mono)-category (cf. [9]).

**Proposition 5.3** \( L\text{-}\text{Top} \) is an epi-co-well-powered (Epi, Extremal mono)-category.

**Proof:** As pointed out earlier, \( L\text{-}\text{Top} \) is a topological construct. Hence by using Theorem 21.16 and Corollary 21.17 of [1], the result follows. □

**Theorem 5.3** \( L\text{-}\text{Top}_0 \) is epireflective in \( L\text{-}\text{Top} \).
Proof: Let \((X, \tau) \in \text{ob} L\text{-Top}\). Define a relation \(\sim\) on \(X\) as follows: for every \(x, y \in X\), \(x \sim y\) iff \(\mu(x) = \mu(y)\), for every \(\mu \in \tau\). It is easy to see that \(\sim\) is an equivalence relation on \(X\). Consider the quotient map \(q_X : X \to \tilde{X}\), where \(\tilde{X} = X/\sim\) and let \(\tilde{\tau}\) be the quotient \(L\)-topology on \(X\) induced by \((X, \tau)\) and \(q_X\). Then \((\tilde{X}, \tilde{\tau}) \in \text{ob} L\text{-Top}_0\). It can be verified easily that the quotient map \(q_X : (X, \tau) \to (\tilde{X}, \tilde{\tau})\) is an epireflection of \((X, \tau)\) in \(L\text{-Top}_0\). □

By using Theorem 5.1, we get the following corollary.

Corollary 5.2 \(L\text{-Top}_0\) is closed under forming products and extremal subobjects in \(L\text{-Top}\).

Proposition 5.4 \((X, \tau) \in \text{ob} L\text{-Top}_0\) iff it is homeomorphic to a subspace of product of copies of \(L_S\).

Proof: Let \((X, \tau) \in \text{ob} L\text{-Top}_0\). Define \(e : (X, \tau) \to L_S^0\) as \(e(x)(\mu) = \mu(x)\), for every \(x \in X\) and for every \(\mu \in \tau\). Let \(p_\mu\) denote the \(\mu\)-th projection map from \(L_S\) to \(L_S\). Then for every \(x \in X\), \((p_\mu \circ e)(x) = p_\mu(e(x)) = e(x)(\mu) = \mu(x)\) implying that \(p_\mu \circ e = \mu\). Thus \(e\) is continuous. For distinct \(x, y \in X\), there exists \(\mu \in \tau\) such that \(\mu(x) \neq \mu(y)\). Hence \(e(x) \neq e(y)\), showing that \(e\) is injective. Let \(\mu \in \tau\). Then for every \(x \in X\), \(e^{-1}(\mu) = \{\mu(y) \mid e(y) = e(x)\} = \mu(x) = e(x)(\mu) = p_\mu(e(x))\) implying that \(e^{-1}(\mu) = p_\mu \wedge 1_e(x)\). As \(p_\mu\) is open in \(L_S\), \(e^{-1}(\mu)\) is open in \(e(X)\). Thus \((X, \tau)\) is homeomorphic to a subspace of product of copies of \(L_S\). By using the Corollary 5.1, Corollary 5.2 and the fact that \(L_S = T_0\), the converse follows. □

In view of Theorem 5.2, Corollary 5.1, and Proposition 5.4 above, we get the following result:

Theorem 5.4 \(L\text{-Top}_0\) is the epireflective hull of \(L_S\) in \(L\text{-Top}\).

6 The sober \(L\)-topological space

Rodabaugh [11] and Höhle [5] introduced the notion of sober \(L\)-topological space in the category \(L\text{-Top}\) as follows.

Let \(A\) be a frame and let \(pt_L A = \{p : A \to L \mid p\) is a frame map\}\}. Define \(\phi_L : A \to L^{pt_L A}\) as \(\phi_L(a)(p) = p(a)\), for every \(a \in A\) and for every \(p \in pt_L A\). Then \(\phi_L(A)\) is an \(L\)-topology on \(pt_L A\) (cf. [11]).

For an \(L\)-topological space \((X, \tau), \tau\) is a frame.

Definition 6.1 [5] An \(L\)-topological space \((X, \tau)\) is said to be sober if for every \(p \in pt_L \tau\), there exists a unique \(x \in X\) such that \(p(\mu) = \mu(x)\), for every \(\mu \in \tau\).

Let \(L\text{-Sob}\) denote the subcategory of \(L\text{-Top}\), whose objects are sober \(L\)-topological spaces.

The following result is easy to verify.

Proposition 6.1 The Sierpinski \(L\)-topological space \(L_S\) is sober.
Proposition 6.2 \[11\] An $L$-topological space $(X, \tau)$ is sober iff $\eta_X : (X, \tau) \to (pt_L, \phi_L(\tau))$ defined as $\eta_X(x)(\mu) = \mu(x)$, for every $x \in X$ and for every $\mu \in \tau$, is a homeomorphism.

Proposition 6.3 \[11\] For an $L$-topological space $(X, \tau)$, $(pt_L, \phi_L(\tau))$ is sober.

Proposition 6.4 \[11\] An $L$-topological space $(X, \tau)$ is $T_0$ iff $\eta_X : (X, \tau) \to (pt_L, \phi_L(\tau))$ defined as $\eta_X(x)(\mu) = \mu(x)$, for every $x \in X$ and for every $\mu \in \tau$, is injective.

In view of Proposition 6.2 and Proposition 6.4, $L\text{-Sob}$ is a subcategory of $L\text{-Top}_0$.

7 Epireflective hull of $L_S$ in $L\text{-Top}_0$

It is known that $L\text{-Sob}$ is reflective in $L\text{-Top}$ (cf. \[11\]5). As $L\text{-Sob}$ is a subcategory of $L\text{-Top}_0$, $L\text{-Sob}$ is also reflective in $L\text{-Top}_0$. Here we show that $L\text{-Sob}$ is epireflective in $L\text{-Top}_0$. Moreover, we show that $L\text{-Sob}$ is, in fact, $\mathcal{M}$-firm epireflective in $L\text{-Top}_0$. For showing this, first we have to identify the epimorphisms in $L\text{-Top}_0$. For $(X, \tau) \in obL\text{-Top}$ and $M \subseteq X$, put $[M] = \cap\{Eq(f, g) \mid f, g : (X, \tau) \to (Y, \delta)$ are morphisms in $L\text{-Top}$ with $(Y, \delta) \in obL\text{-Top}_0$ and $f|M = g|M\}$, where $Eq(f, g) = \{x \in X \mid f(x) = g(x)\}$. $M$ is said to be $[\ ]$-closed if $[M] = M$. It can be verified easily that $[[M]] = [M]$.

Proposition 7.1 A morphism $f : (X, \tau) \to (Y, \delta)$ in $L\text{-Top}_0$ is an epimorphism iff for every $\nu_1, \nu_2 \in \delta$, $f^\rightarrow(\nu_1) = f^\rightarrow(\nu_2)$ implies that $\nu_1 = \nu_2$.

Proof: Let $f : (X, \tau) \to (Y, \delta)$ be an epimorphism in $L\text{-Top}_0$. Let $\nu_1, \nu_2 \in \delta$ such that $f^\rightarrow(\nu_1) = f^\rightarrow(\nu_2)$. Then $\nu_1 \circ f = \nu_2 \circ f$, implying that $\nu_1 = \nu_2$.

Conversely, let for every $\nu_1, \nu_2 \in \delta$, $f^\rightarrow(\nu_1) = f^\rightarrow(\nu_2)$ implies that $\nu_1 = \nu_2$. Let $g, h : (Y, \delta) \to (Z, \Delta)$ be two distinct morphisms in $L\text{-Top}_0$. Then there exists $y \in Y$ such that $g(y) \neq h(y)$. As $(Z, \Delta)$ is $T_0$, there exists $\mu \in \Delta$ such that $\mu(g(y)) \neq \mu(h(y))$. Thus $g^\rightarrow(\mu) \neq h^\rightarrow(\mu)$, implying that $f^\rightarrow(g^\rightarrow(\mu)) \neq f^\rightarrow(h^\rightarrow(\mu))$, i.e., $\mu \circ g \circ f \neq \mu \circ h \circ f$. Hence $g \circ f \neq h \circ f$, showing that $f$ is an epimorphism. □

We shall use the following result, which is a special case of Theorem 1.11 of 9.

Proposition 7.2 A morphism $f : (X, \tau) \to (Y, \delta)$ in $L\text{-Top}_0$ is an epimorphism iff $[f(X)] = Y$.

Call an embedding $e : (X, \tau) \to (Y, \delta)$ in $L\text{-Top}_0$, $[\ ]$-closed if $[e(X)] = e(X)$.

Proposition 7.3 Extremal monomorphisms in $L\text{-Top}_0$ are precisely the $[\ ]$-closed embeddings.
Proof: Let \( f : (X, \tau) \to (Y, \delta) \) be an extremal monomorphism in \( L\Top_0 \). Let \( f' : (X, \tau) \to ([f(X)], \delta_{[f(X)]}) \) be the corestriction of \( f \) onto \([f(X)]\). Then \( f' \) is an epimorphism. Now \( f = i \circ f' \), where \( i : ([f(X)], \delta_{[f(X)]}) \to (y, \delta) \) is the inclusion map, implying that \( f' \) is an isomorphism. Thus \( f \) is \([ \ ]\)-closed embedding.

Conversely, let \( e : (X, \tau) \to (Y, \delta) \) be a \([ \ ]\)-closed embedding in \( L\Top_0 \). Then \( [e(X)] = e(X) \) is an equalizer of some pair of morphism in \( L\Top_0 \), (3). Proposition 1.6). But equalizers are necessarily extremal monomorphisms. So, \( e \) is an extremal monomorphism. \( \Box \)

Corollary 7.1 Extremal subobjects in \( L\Top_0 \) are precisely the \([ \ ]\)-closed subspaces.

Proposition 7.4 \( L\Top_0 \) is an epi-co-well-powered \((\text{Epi}, \text{Extremal mono})\)-category.

Proof: \( L\Top_0 \) is an epireflective subcategory of \( L\Top \) (Theorem 5.3).

Also \( L\Top \) is a topological category. By using Proposition 3.3 of \( 2 \) \( L\Top_0 \) is a well-powered and complete \((\text{Epi}, \text{Extremal mono})\)-category. By using the Example 7.90 (2) of \( 1 \), \( L\Top_0 \) is an epi-co-well-powered and hence the result follows. \( \Box \)

Theorem 7.1 \( L\Sob \) is epireflective in \( L\Top_0 \).

Proof: Let \((X, \tau) \in \ob L\Top_0 \). Then \( \eta_X : (X, \tau) \to (pt\{L, \phi_L(\tau)\}) \) defined as \( \eta_X(x)(\mu) = \mu(x) \), for every \( x \in X \) and for every \( \mu \in \tau \), is the reflection of \((X, \tau) \) in \( L\Sob \) (cf. \( 3 \), Corollary 2.1). Now, we show that \( \eta_X \) is an epimorphism. Let \( \mu_1, \mu_2 \in \tau \) and \( \eta_X(\phi_L(\mu_1)) = \eta_X(\phi_L(\mu_2)) \).

Then for every \( x \in X \), \( \eta_X(\phi_L(\mu_1))(x) = \eta_X(\phi_L(\mu_2))(x) \) implying that \( \phi_L(\mu_1)\eta_X(x) = \phi_L(\mu_2)\eta_X(x) \). Then by definition of \( \phi_L \), \( \eta_X(x)(\mu_1) = \eta_X(x)(\mu_2) \) which implies that \( \mu_1(x) = \mu_2(x) \), for every \( x \in X \). Thus \( \mu_1 = \mu_2 \) implying that \( \phi_L(\mu_1) = \phi_L(\mu_2) \). Thus \( \eta_X \) is an epimorphism. \( \Box \)

By using the Theorem 5.1, we get the following corollary.

Corollary 7.2 \( L\Sob \) is closed under forming products and extremal subobjects in \( L\Top_0 \).

For \((X, \tau) \in \ob L\Top_0 \), \( \eta_X : (X, \tau) \to (pt\{L, \phi_L(\tau)\}) \) defined as \( \eta_X(x)(\mu) = \mu(x) \), for every \( x \in X \) and for every \( \mu \in \tau \), is an open map. As, for \( \mu \in \tau \) and \( p \in pt\{\tau\}, \eta_X(\mu)(p) = \bigvee \{\mu(x) \mid \eta_X(x) = p\} = \bigvee \{\eta_X(x)(\mu) \mid \eta_X(x) = p\} = p(\mu) = \phi_L(\mu)(p) \), showing that \( \eta_X(\mu) \in \phi_L(\tau) \).

Thus in view of Proposition 6.4, \( \eta_X \) is an embedding in \( L\Top_0 \).

Theorem 7.2 \( L\Sob \) is \( \mathcal{H} \)-firm epireflective in \( L\Top_0 \), where \( \mathcal{H} \) denotes the class of all embeddings in \( L\Top_0 \).

Proof: Let \((X, \tau) \in \ob L\Top_0 \). Then \( \eta_X : (X, \tau) \to (pt\{L, \phi_L(\tau)\}) \) is an epimorphic embedding in \( L\Top_0 \). Let \((Y, \delta) \in \ob L\Sob \) and \( f : (X, \tau) \to (Y, \delta) \) be an epimorphic embedding in \( L\Top_0 \). We have to find a unique isomorphism \( f' : (pt\{L, \phi_L(\tau)\}) \to (Y, \delta) \) such that \( f' \circ \eta_X = f \).

Let \( p \in pt\{\tau\}. Then p : \tau \to L \) is a frame map. Define \( p' : \delta \to L \) as \( p'(\nu) = p(\nu \circ f) \), for every \( \nu \in \delta \). It is easy to see that \( p' \) is a frame
map. As \((Y, \delta)\) is sober, there exists a unique \(y \in Y\) such that for every \(\nu \in \delta\), \(p^*(\nu) = \nu(y)\) implying that \(p(\nu \circ f) = \nu(y)\). Put \(f^* (p) = y\). Let \(\nu \in \delta\). Then for every \(p \in pt L \tau\), \(f^*(\nu)(p) = \nu(f^*(p)) = p(\nu \circ f) = p(f^*(\nu)) = \phi L(f^* (\nu))(p)\), implying that \(f^*(\nu) = \phi L(f^* (\nu))\). As \(\nu \in \delta\), \(f^*(\nu) \in \tau\) showing that \(f^*(\nu) \in \phi L(\tau)\). Hence \(f^*\) is continuous. For every \(x \in X\), \((f^* \circ \eta_X)(x) = f^*(\eta_X(x))\). Then for every \(\nu \in \delta\), \(\nu(f^*(\eta_X(x))) = \eta_X(x)(\nu \circ f) = (\nu \circ f)(x) = \nu(f(x))\). By the uniqueness of \(y\) in the definition of \(f^*\), \(f^*(\eta_X(x)) = f(x)\), for every \(x \in X\). Hence \(f^* \circ \eta_X = f\). Now, we show that \(f^*\) is an isomorphism.

For every \(\mu \in \tau\), \(f^* (\mu) \in \delta(x)\) implying that there exists \(\mu_f \in \delta\) such that \(f^* (\mu) = \mu_f \wedge 1_{\delta(x)}\) and \(f^* (\mu_f) = \mu\). As \(f\) is an epimorphism, \(\mu_f\) is unique such that \(f^* (\mu_f) = \mu\). Define \(g : (Y, \delta) \to (ptL \tau, \phi L(\tau))\) as \(g(y)(\mu) = \mu_f(y)\), for every \(y \in Y\) and for every \(\mu \in \tau\). It can be verified easily that \(g(y)\) is a frame map. Now we show that \(g\) is continuous. Let \(\mu \in \tau\). Then for every \(y \in Y\), \(g^* (\phi L(\mu))(y) = \phi L(\mu)(g(y)) = g(y)(\mu) = \mu_f(y)\), showing that \(g^* (\phi L(\mu)) = \mu_f \in \delta\). Thus, \(g\) is continuous. For every \(\nu \in \delta\), \(f^*(\nu) \in \tau\) implying that \(f^* (f^*(\nu)) = f^*(\nu)\) and hence \((f^* (\nu))_f = \nu\). For every \(y \in Y\), \((f^* \circ g)(y) = f^*(g(y))\). Then for every \(\nu \in \delta\), \(\nu(f^*(y)) = g(y)(\nu \circ f) = g(y)(f^*(\nu)) = (f^*(\nu))_f(y) = \nu(y)\).

by the uniqueness of \(y\) in the definition of \(f^*\), \(f^*(g(y)) = y\), for every \(y \in Y\). Hence \(f^* \circ g = id_Y\).

For every \(p \in pt \tau\) and \(\mu \in \tau\), \((g \circ f^*)(p)(\mu) = g(f^*(p))(\mu) = \mu \eta (f^*(p)) = p(\mu \circ f) = p(f^*(\mu)) = p(\mu)\) implying that \(g \circ f^* = id_{pt \tau}\). □

It is well-known that the category \(\text{Sob}\) of sober topological spaces is the epireflective hull of usual two-point Sierpinski space \(2_\emptyset\) in the category \(\text{Top}_0\) of \(T_0\)-topological spaces \([10]\). Srivastava and Khastgir \([13]\) showed that the category \(\text{FSob}\) of fuzzy sober spaces is the epireflective hull of fuzzy Sierpinski space \(I_\emptyset\) in the category \(\text{FTop}_0\) of \(T_0\)-fuzzy topological spaces. Analogously, we show that \(L_\text{Sob}\) is the epireflective hull of \(L_\emptyset\) in \(L_\text{Top}_0\).

**Proposition 7.5** For every \((X, \tau) \in ob L_\text{Top}_0\), \((ptL \tau, \phi L(\tau))\) is a \([\ ]\)-closed subspace of \(L_\emptyset\).

**Proof:** Consider the epimorphic embedding \(\eta_X : (X, \tau) \to (ptL \tau, \phi L(\tau))\) in \(L_\text{Top}_0\). As \((X, \tau) \in ob L_\text{Top}_0\), \((X, \tau)\) can be embedded in \(L_\emptyset\) via the ‘evaluation map’ \(e : (X, \tau) \to L_\emptyset\) defined as \(e(x)(\mu) = \mu(X)\), for every \(x \in X\) and for every \(\mu \in \tau\). Consider the subspace \([e(X)]\) of \(L_\emptyset\). Let \(\psi\) be the corestrict of \(e\) onto \([e(X)]\). Then \(\psi\) is an epimorphic embedding in \(L_\text{Top}_0\). By using Proposition 6.1, Corollary 7.1 and Corollary 7.2, \([e(X)]\) is sober. Then there exists an isomorphism from \((ptL \tau, \phi L(\tau))\) to \([e(X)]\) (by Theorem 7.2). Hence \((ptL \tau, \phi L(\tau))\) is a \([\ ]\)-closed subspace of \(L_\emptyset\). □

**Proposition 7.6** \((X, \tau) \in ob L_\text{Sob}\) iff it is homeomorphic to a \([\ ]\)-closed subspace \(L_\emptyset\).

**Proof:** Let \((X, \tau) \in ob L_\text{Sob}\). Then \(\eta_X : (X, \tau) \to (ptL \tau, \phi L(\tau))\) is a homeomorphism. Hence \((X, \tau)\) is homeomorphic to a \([\ ]\)-closed subspace \(L_\emptyset\). By using Proposition 6.1, Corollary 7.1 and Corollary 7.2, converse follows. □
In view of Theorem 5.2, Corollary 7.1 and Proposition 7.6, we get the following result:

**Theorem 7.3** \(L\)-**Sob** is the epireflective hull of \(L_S\) in \(L\)-**Top**._0.

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**References**

[1] J. Adámek, H. Herrlich, G.E. Strecker, *Abstract and Concrete Categories*, Wiley, 1990.

[2] G.C.L. Brümmer, E. Giuli, H. Herrlich, Epireflections which are completions, *Cahiers Topo. Géom. Diff. Catégories* 33 (1992) 71-93.

[3] G. Castellini, Closure operators, monomorphisms and epimorphisms in categories of groups, *Cahiers Topo. Géom. Diff. Catégories* 27 (1986), 151-167.

[4] J.A. Goguen, \(L\)-fuzzy sets, *J. Math. Anal. Appl.* 18 (1967), 145-174.

[5] U. Höhle, Locales and \(L\)-Topologies, *Mathematik-Arbeitspapiere* 48 (1997), 223-250.

[6] P.T. Johnstone, *Stone spaces*, Cambridge University Press, Cambridge 1982.

[7] R. Lowen, A.K. Srivastava, \(FTS_0\): The epireflective hull of the Sierpinski object in \(FTS\), *Fuzzy Sets and Systems* 29 (1989) 171-176.

[8] E.G. Manes, Compact Hausdorff objects, *General Topology Appl.* 4 (1974) 341-360.

[9] T. Marny, On epireflective subcategories of topological categories, *General Topology Appl.* 10 (1979) 175-181.

[10] L.D. Nel, R.G. Wilson, Epireflections in the category of \(T_0\)-spaces, *Fund. Math.* 75 (1972) 69-74.

[11] S.E. Rodabaugh, Point-Set lattice-theoretic topology, *Fuzzy Sets and Systems* 40 (1991) 297-345.

[12] A.K. Srivastava, Fuzzy Sierpinski Space, *J. Math. Anal. Appl.* 103 (1984) 103-105.

[13] A.K. Srivastava, A.S. Khastrigir, On fuzzy sobriety, *Inform. Sci.* 110 (1998) 195-205.