CONJUGACY CLASS CONDITIONS IN LOCALLY COMPACT SECOND COUNTABLE GROUPS

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Abstract. Many non-locally compact second countable groups admit a comeagre conjugacy class. For example, this is the case for $S_\infty$, $\text{Aut}(\mathbb{Q}, <)$, and less trivially $\text{Aut}(\mathbb{R})$ for $\mathbb{R}$ the random graph [12]. In [7], A. Kechris and C. Rosendal ask if a non-trivial locally compact second countable group can admit a comeagre conjugacy class. We answer the question in the negative via an analysis of locally compact second countable groups with topological conditions on a conjugacy class.

1. Introduction

Our goal is to answer a question of Kechris and Rosendal: Can a non-trivial locally compact second countable group admit a comeagre conjugacy class? By an unpublished argument due to K.H. Hofmann, such a group cannot be connected. This, of course, leaves the problematic totally disconnected locally compact case. Here the theory is not nearly as well developed. We therefore begin with an analysis of totally disconnected locally compact second countable (t.d.l.c.s.c.) groups.

Theorem 1.1. Suppose $U$ is a second countable profinite group and $g \in U$. The following are equivalent:

(1) $g^U := \{ ugu^{-1} \mid u \in U \}$ is non-meagre.
(2) $g^U$ is open.
(3) $\mu(g^U) > 0$ where $\mu$ is the normalized Haar measure on $U$.
(4) There is an integer $M > 0$ such that $|C_{U/V}(gV)| \leq M$ for every open normal subgroup $V \triangleleft U$.

$C_{U/V}(gV)$ denotes the centralizer of $gV$ in $U/V$. We extend Theorem 1.1 to the non-compact case.

Corollary 1.2. Suppose $G$ is a t.d.l.c.s.c. group and $g \in G$ is such that $\text{cl}(\langle g \rangle)$ is compact. The following are equivalent:

(1) $g^G$ is open.
(2) $g^G$ is non-meagre.
(3) $g^G$ is non-null.

Applying the above results and two deep theorems in profinite group theory, we eliminate t.d.l.c.s.c. groups as possible candidates for a positive answer to the motivating question.

Theorem 1.3. If $G$ is a non-trivial t.d.l.c.s.c. group and $g^G$ is dense, then $g^G$ is meagre and Haar null.

Date: October 8, 2013.
We conclude by presenting Hofmann’s proof in the connected case to give a complete answer to the question of Kechris and Rosendal.

**Theorem 1.4.** A non-trivial locally compact second countable group does not admit a comeagre conjugacy class.

Along the way, we present an example due to Rosendal showing the existence of an infinite profinite group with a non-meagre conjugacy class, and we sketch the example built by E. Akin, E. Glasner, and B. Weiss of a non-trivial t.d.l.c.s.c. group with a dense conjugacy class.

2. Preliminaries

We fix a few conventions and notations for this paper. All topological groups are assumed to be Hausdorff and all subgroups are taken to be closed unless otherwise noted. To indicate $O$ is an open subgroup of a topological group $G$, we write $O \leq_o G$. For $g \in G$ and $A \subseteq G$, $g^A := \{ag^{-1} \mid a \in A\}$. We use the abbreviations l.c., t.d., and s.c., for “locally compact”, “totally disconnected”, and “second countable”, respectively.

2.1. L.c.s.c. Groups. L.c.s.c. groups are $K_\sigma$ and metrizable by classical results [6]. These groups have a canonical left invariant Borel measure which is unique up to constant multiples called the *Haar measure*. When a subset of a group is said to be non-null, we mean with respect to the Haar measure. A familiarity with the Haar measure and basic properties thereof is assumed; [3] contains a nice, brief introduction.

L.c.s.c. groups are connected by totally disconnected. Indeed, for $G$ a l.c.s.c. group let $G_0$ denote the connected component of the identity. It is easy to see $G_0$ is a closed normal subgroup, and there is a short exact sequence of topological groups

$$1 \to G_0 \to G \to G/G_0 \to 1$$

where $G_0$ is connected and $G/G_0$ is totally disconnected. The study of l.c.s.c. groups thus reduces to the study of connected l.c.s.c. groups and t.d.l.c.s.c. groups.

The study of connected locally compact groups reduces to the study of inverse limits of Lie groups by the celebrated solution to Hilbert’s fifth problem.

**Theorem 2.1** (Gleason, Montgomery, Yamabe, Zippin). A connected locally compact group is pro-Lie.

A group $G$ is a *Lie group* if $G$ has an analytic $K$-manifold structure such that the map $(g, h) \mapsto gh^{-1}$ is analytic where $K$ is either $\mathbb{R}$, $\mathbb{C}$ or some non-discrete ultrametric field. A Lie group $G$ is connected and locally compact if and only if $K$ is either $\mathbb{R}$ or $\mathbb{C}$ and $G$ is finite dimensional. Associated to a Lie group $G$ is the Lie algebra, $\mathfrak{g}$, where $\mathfrak{g}$ is the tangent space at the identity along with a bracket operation. The self action of $G$ by conjugation induces an action of $G$ on $\mathfrak{g}$ by vector space isomorphisms. When $G$ is connected and locally compact, this action is given by the map $Ad : G \to GL(\mathfrak{g})$ where $GL(\mathfrak{g}) = GL_n(K)$ for some finite $n$ and $K$ equal to $\mathbb{R}$ or $\mathbb{C}$.

**Fact 2.2** ([2] III.6.4 Corollary 4). If $G$ is a connected locally compact Lie group, then $Ad : G/Z(G) \to \text{im}(Ad)$ is an isomorphism of Lie groups.

Proofs of the aforementioned properties of Lie groups may be found in [2].

For t.d.l.c. groups, a central theorem is an old result of D. van Dantzig.
**Theorem 2.3** (van Dantzig [4] II.7.7). A t.d.l.c. group admits a basis at the identity of compact open subgroups.

Elements of a t.d.l.c. group which lie in a compact open subgroup are called **periodic**. Note that periodic elements do not necessarily have finite order; we call an element **torsion** if it has finite order. For a t.d.l.c. group $G$, we denote the collection of periodic elements of $G$ by $P_1(G)$. It is easy to see

$$P_1(G) = \{ g \in G \mid \text{cl}(\langle g \rangle) \text{ is compact} \}$$

### 2.2. Profinite Groups.

The compact open subgroups of a t.d.l.c. group given by van Dantzig’s theorem are **profinite**. I.e. they are inverse limits of finite groups. An introduction to the theory of profinite groups may be found in the texts [11] and [14]. We assume a familiarity with profinite groups and merely recall a few relevant definitions and theorems.

Profinite groups have a basis at 1 consisting of open normal subgroups. For a second countable profinite group $U$, we say $(N_i)_{i \in \mathbb{N}}$ is a **normal basis at** 1 for $U$ if each $N_i$ is open and normal, $\bigcap_{i \in \mathbb{N}} N_i = \{1\}$, and $(N_i)_{i \in \mathbb{N}}$ is $\subseteq$-decreasing.

The **Frattini subgroup** of a profinite group $U$, $\Phi(U)$, is the intersection of all maximal proper open subgroups. $\Phi(U)$ is the collection of non-generators of $U$: if $U = H\Phi(U)$ for $H \leq U$, then $U = H$. When $U$ is pro-$p$, an inverse limit of $p$-groups, $\Phi(U)$ has a well understood structure.

**Fact 2.4** ([11] Lemma 2.8.7). If $U$ is pro-$p$, then $\Phi(U) = U^p[U,U]$ where $[U,U]$ is the closure of the commutator subgroup and $U^p$ is the closed subgroup generated by all $p$ powers.

We also note two deep results in profinite group theory.

**Theorem 2.5** (Zel’manov [15]). Every torsion pro-$p$ group is locally finite.

**Theorem 2.6** (Wilson [13]). Let $U$ be a compact Hausdorff torsion group. Then $U$ has a finite series

$$\{1\} = U_0 \leq U_1 \leq \ldots \leq U_n = U$$

of closed characteristic subgroups in which each factor $U_i/U_{i-1}$ is either (1) pro-$p$ for some prime $p$ or (2) isomorphic to a Cartesian product of isomorphic finite simple groups.

### 3. Non-meagre conjugacy classes

We here consider t.d.l.c.s.c. groups with a non-meagre conjugacy class. The key step is to initially consider profinite groups with a non-null conjugacy class.

**Lemma 3.1.** Let $U$ be a profinite group with normalized Haar measure $\mu$. If $h \in U$ is such that $\mu(h^U) > 0$, then

1. For all $N \trianglelefteq_o U$, $|C_{U/N}(hN)| \leq \frac{1}{\mu(h^U)}$.
2. $|C_U(h)| \leq \frac{1}{\mu(h^U)}$, and in particular, $h$ is torsion.

**Proof.** Let $h \in U$ have a non-null conjugacy class and take $N \trianglelefteq_o U$. Take a minimal set of coset representatives $h, k_1hk_1^{-1}, \ldots, k_nhk_n^{-1}$ for $(hN)^{U/N}$ in $U/N$. Certainly,

$$h^U \subseteq hN \cup \ldots \cup k_nhk_n^{-1}N$$

So

$$0 < \mu(h^U) \leq |(hN)^{U/N}|\mu(N)$$
Since $1 = \mu(U) = |U/N|\mu(N)$, we have
\[
|hN|^{U/N}\mu(N) = |U/N : C_{U/N}(hN)|\mu(N)
\]
\[= \frac{|U/N|\mu(N)}{|C_{U/N}(hN)|}
\]
So $|C_{U/N}(hN)| \leq \frac{1}{\mu(U)}$, and it follows $|C_U(h)| \leq \frac{1}{\mu(U)}$.

Via Lemma 3.1, category and measure theoretic notions of size for conjugacy classes in second countable profinite groups agree.

**Theorem 3.2.** Suppose $U$ is a second countable profinite group and $g \in U$. The following are equivalent:

1. $g^U$ is non-meagre.
2. $g^U$ is open.
3. $\mu(g^U) > 0$ where $\mu$ is the normalized Haar measure on $U$.
4. There is $M > 0$ such that $|C_{U/N}(gN)| \leq M$ for all $N \trianglelefteq_U U$.

**Proof.** (1) $\Rightarrow$ (2) Since $g^U$ is non-meagre, it is somewhere dense. Now $g^U$ is also closed since the continuous image of a compact set, and therefore, $g^U$ contains an non-empty open set. Say $O \subseteq g^U$. So $g^U = \bigcup_{u \in U} uOu^{-1}$ and $g^G$ is open.

(2) $\Rightarrow$ (3) is immediate from properties of the Haar measure.

For (3) $\Rightarrow$ (4), let $g \in U$ be as hypothesized and take $N \trianglelefteq_U U$. By Lemma 3.1,

$$|C_{U/N}(gN)| \leq \frac{1}{\mu(g^U)}$$

Fixing $M \geq \frac{1}{\mu(g^U)}$, we have (4).

(4) $\Rightarrow$ (1) Suppose $g \in U$ satisfies (4). Fix $(N_i)_{i \in \mathbb{N}}$ a normal basis at 1 for $U$ and take $M > 0$ which witnesses (4) for $g$. For each $i$, let $A_i \subseteq g^U$ be a minimal set of coset representatives for $(gN_i)^{U/N_i}$ in $U/N_i$.

For each $i \in \mathbb{N}$,

$$g^U \subseteq A_iN_i := \bigcup_{a \in A_i} aN_i$$

and it is easy to check

$$\bigcap_{i \in \mathbb{N}} A_iN_i = g^U$$

Since $(A_iN_i)_{i \in \mathbb{N}}$ is an $\subseteq$-decreasing sequence,

$$\mu(A_iN_i) \to \mu(g^U)$$

by continuity from above for $\mu$.

On the other hand, $|C_{U/N_i}(gN_i)| \leq M$ for any $i \in \mathbb{N}$, and therefore, $|C_{U/N_i}(gN_i)|$ only takes on finitely many values as $i$ varies. By passing to a subsequence, we may assume $|C_{U/N_i}(gN_i)| = k \leq M$ for all $i$. So

$$\mu(A_iN_i) = |(gN_i)^{U/N_i}|\mu(N_i) = \frac{1}{|C_{U/N_i}(gN_i)|} = \frac{1}{k}$$

for each $i$, and it follows $\mu(A_iN_i) = \mu(g^U)$ for all $i$. 
We now fix an $i$ and consider
$$E := (A_i N_i) \setminus g^U$$
Certainly $E$ is open and $\mu(E) = 0$. As the only Haar null open set is $\emptyset$, $E = \emptyset$, and
$$A_i N_i = g^U.$$ Thus, $g^U$ is open and, a fortiori, non-meagre. \qed

We remark that (4) of the above theorem is an algebraic characterization of a profinite group having an open conjugacy class.

We now apply our results for second countable profinite groups to obtain a result for all t.d.l.c.s.c. groups.

**Lemma 3.3.** Let $G$ be a t.d.l.c.s.c. group. If $g^G$ is non-null, then $g^U$ is non-null for any compact open subgroup $U$ of $G$. If $g$ is also periodic, then $g$ is torsion and $g^U$ is open for any compact open subgroup $U$.

**Proof.** Fix $U \leq_o G$ compact. Since $G$ is second countable, there is a countable set $(h_i)_{i \in \mathbb{N}}$ such that $G = \bigcup_{i \in \mathbb{N}} h_i U$. So $g^G = \bigcup_{i \in \mathbb{N}} g^U h_i$, and for some $i \in \mathbb{N}$, $\mu(g^U h_i) > 0$. Fix such an $i$; now
$$0 < \mu(g^U h_i) = \mu(h_i g^U h_i^{-1}) = \mu(g^U) \Delta(h_i^{-1})$$
where $\Delta$ is the modular function. Since the modular function is strictly positive, $\mu(g^U) > 0$ as desired.

If $g$ is also periodic, take $W$ a compact open subgroup containing $g$. By the uniqueness of the Haar measure, $g^W$ is non-null in $W$ with respect to the normalized Haar measure on $W$. Lemma 3.1 then implies $g$ is torsion. Theorem 3.2 further implies $g^W$ is open. Now if $U$ is an arbitrary compact open subgroup, then since $W$ is compact and $U$ open,
$$g^W \subseteq g^{w_1 U} \cup ... \cup g^{w_n U}$$
for some $w_1, ..., w_n \in W$. By the Baire category theorem, $g^{w_i U} = w_i g^U h_i^{-1}$ is non-meagre for some $1 \leq i \leq n$. Certainly, $g^U$ is thereby non-meagre, and since $g^U$ is closed, there is $O \subseteq g^U$ a non-empty open set. So $g^U = \bigcup_{u \in U} uOu^{-1}$, and $g^U$ is open. \qed

**Corollary 3.4.** Suppose $G$ is t.d.l.c.s.c. group and $g$ is periodic. The following are equivalent:

1. $g^G$ is open.
2. $g^G$ is non-meagre.
3. $g^G$ is non-null.

**Proof.** (1) $\Rightarrow$ (2) This follows by the Baire category theorem.

(2) $\Rightarrow$ (3) Suppose $g^G$ is non-meagre. Since $G$ is $K_\sigma$, we may write $G = \bigcup_{i \in \mathbb{N}} K_i$ with the $K_i$ compact. So $g^G = \bigcup_{i \in \mathbb{N}} g^{K_i}$, and the Baire category theorem implies $g^{K_i}$ is non-meagre for some $i$. Fix such an $i$. Now $g^{K_i}$ is also closed and, thereby, has non-empty interior. It follows $\mu(g^{K_i}) > 0$, and we have (3).

(3) $\Rightarrow$ (1) This is immediate from Lemma 3.3. \qed

We remark that it is not clear there are non-discrete examples of the groups discussed in this section. For completeness, we present an example due to C. Rosendal of a non-discrete second countable profinite group with a non-null conjugacy class. The author wishes to express his thanks to Rosendal for allowing this example to be included in the present work.
3.1. Rosendal’s example. Let $D_6$ be the dihedral group of the triangle. Recall

$$D_6 = \langle s, r \mid r^3 = 1, s^2 = 1, \text{ and } sr = r^{-1}s \rangle$$

Every element of $D_6$ is of the form $r^i$ or $sr^i$ for $i = 0, 1, 2$. Form $D^N_6$ and define $G \subseteq D^N_6$ to be the collection of $\alpha$ such that all coordinates of $\alpha$ are of the form $sr^i$ or all coordinates are of the form $r^i$. Certainly, $G$ is a closed subset of $D^N_6$ and is closed under inverses. It is easy to check $G$ is also closed under multiplication.

Let $H \trianglelefteq G$ be the collection of $\alpha \in G$ for which all coordinates are of the form $r^i$. So $H$ is closed and index two in $G$. By an old result of B.J. Pettis [10], $H$ is also open. Now consider the element $\beta \in G$ which is constantly equal to $s$.

Claim. Every $\gamma \in G \setminus H$ is conjugate to $\beta$ by an element of $H$.

Proof. Let $\gamma \in G \setminus H$ and say $\gamma(i) = sr^{j(i)}$ where $j(i) \in \{0, 1, 2\}$. Define $\eta \in H$ as follows:

$$\eta(i) = \begin{cases} 1 & \text{if } j(i) = 0 \\ r^2 & \text{if } j(i) = 1 \\ r & \text{if } j(i) = 2 \end{cases}$$

One checks $\eta \gamma \eta^{-1} = \beta$. □

By the claim, $G \setminus H$ is the conjugacy class of $\beta$, and so $\beta^G$ is open and non-null.

Remark 3.5. We conclude this section with two remarks.

(1) Theorem 3.2 gives a measure and category equivalence which may be independently useful. For example, Theorem 3.2 seems potentially useful to answer the following open question asked by L. Lévi and L. Pyber in [9]: Let $U$ be profinite and put $T_n := \{ x \in U \mid x^n = 1 \}$ for $n \geq 2$. If $\mu(T_n) > 0$, then is $T_n$ non-meagre? For $n = 2$, the question is known to have a positive answer [9].

(2) Rosendal’s example is solvable. It is unknown if all such examples must be virtually solvable. The following partial results are known to the author: (i) Second countable pronilpotent groups with an open conjugacy class are solvable. (ii) Second countable profinite groups $U$ such that $g \in U$ has an open conjugacy class and $|g|$ is prime are virtually solvable.

4. Dense conjugacy classes

In this section, we consider a much different topological condition on a conjugacy class, density.

Lemma 4.1. A torsion pro-$p$ group with an open conjugacy class is finite

Proof. Suppose $U$ is a torsion, pro-$p$ group with an open conjugacy class $h^U$. Since $h^{-1}h^U \subseteq [U, U]$, $[U, U]$ is open, and $\Phi(U)$ is open by Fact 2.4. Let $u_1, \ldots, u_k$ be coset representatives for $\Phi(U)$ in $U$. Plainly,

$$U = \text{cl}((u_1, \ldots, u_n))\Phi(U)$$

So $U = \text{cl}((u_1, \ldots, u_n))$ since $\Phi(U)$ is the collection of non-generators. Zel’manov’s theorem, Theorem 2.5, now implies $U$ is finite. □

Theorem 4.2. If $G$ is a non-trivial t.d.l.c.s.c. group and $g^G$ is dense, then $g^G$ is meagre and null.
Proof. Suppose toward a contradiction $G$ is a non-trivial t.d.l.c.s.c. group and $g^G$ is dense and either non-meagre or non-null. Since $g^G$ is dense, $g \in P_1(G)$. Corollary 3.4 now implies $g^G$ is open, and by Lemma 3.3, $g$ is torsion. Say $|g| = n$ and observe $G$ must have exponent $n$.

Consider $U$ a compact open subgroup of $G$. Since $U$ is torsion, there is a series of characteristic subgroups

\[ \{1\} = U_0 \leq U_1 \leq \ldots \leq U_n = U \]

given by Wilson’s theorem, Theorem 2.6. Let $k < n$ be greatest such that $U_k$ is not open in $U$. Since $U_{k+1}$ is open, $g^{U_{k+1}}$ meets $U_{k+1}$; without loss of generality, $g \in U_{k+1}$. So Lemma 3.3 implies $g^{U_{k+1}}$ is open, and certainly, $U_{k+1}/U_k$ also has an open conjugacy class. By Lemma 4.1, $U_{k+1}/U_k$ cannot be pro-$p$ since else $U_k$ is finite index and, therefore, open by a classical theorem of Pettis [10]. So $U_{k+1}/U_k$ must be isomorphic to a Cartesian product of isomorphic finite simple groups. Say $U_{k+1}/U_k \cong \prod_{i \in I} S_i =: S$ and let $(s_i)_{i \in I} = s \in S$ have an open conjugacy class. Lemma 3.1 implies $s$ has a finite centralizer. However,

\[ C_S(s) = \prod_{i \in I} C_{S_i}(s_i) \]

and each $C_{S_i}(s_i)$ contains at least two elements. So $S$ must be a finite product, and $U_k$ is again open. We have thus contradicted the choice of $k$. \qed

**Corollary 4.3.** If $G$ is a non-trivial, t.d.l.c.s.c. group, then $G$ does not admit a comeagre or co-null conjugacy class.

**Remark 4.4.** Corollary 4.3 shows that a topological analogue of an infinite discrete group with two conjugacy classes, e.g. [5], is impossible.

We note the hypotheses of Theorem 4.2 are not vacuous. Indeed, Akin, Glasner, and Weiss have built an example of such a group in [1]. We include a sketch of their example for completeness.

### 4.1. The example of Akin, Glasner, and Weiss.

Let $J = \{J_i \mid i \in \mathbb{N}\}$ be a sequence of non-empty finite subsets of $\mathbb{N}$ which partition $\mathbb{N}$ and have strictly increasing cardinality. Let $J^k := \bigcup_{i=0}^{k} J_i$ and $K_n$ be the collection of permutations in $Sym(\mathbb{N})$ which setwise stabilize each of $J^n, J_{n+1}, J_{n+2}, \ldots$. So $K_n$ is the collection of permutations of $\mathbb{N}$ which preserve the partition beyond the $n$-th part. It is easy to see $K_n$ is compact as a subset of $Sym(\mathbb{N})$ and $K_n \leq_o K_{n+1}$. Put $G := \bigcup_{i \in \mathbb{N}} K_i$ and give $G$ the inductive topology: $A \subseteq G$ is open if and only if $A \cap K_i$ is open in $K_i$ for all $i$. Akin, Glasner, and Weiss show $G$ is a t.d.l.c.s.c. group. Further, the sets

\[ G(\pi) := \{g \in K_n \mid g \upharpoonright J_n = \pi\} \]

where $n \in \mathbb{N}$ and $\pi \in Sym(J^n)$ vary form a basis for the topology on $G$.

**Claim.** $G$ has a dense conjugacy class.

**Proof.** It is enough to show $G$ is topologically transitive. I.e. for all basic open sets $G(\pi), G(\xi) \subseteq G$ there is $k \in G$ such that $kG(\pi)k^{-1} \cap G(\xi) \neq \emptyset$. Without loss of generality, we may assume $\pi, \xi \in Sym(J^n)$ for some $n$. Choose $k > n$ such that
an injective map $\beta : J^n \to J_k$ and extend $\beta$ to an element of $G$, $b$, by

$$b(i) = \begin{cases} 
\beta(i), & i \in J^n \\
\beta^{-1}(i), & i \in \beta(J^n) \\
id, & \text{else}
\end{cases}$$

It is easy to check that we may find $a \in G(\pi)$ such that $ab \mid f = b\xi \mid f^n$. Now $b^{-1}ab \in b^{-1}G(\pi)b \cap G(\xi)$ as desired. \qed

Remark 4.5. It is worth nothing the above example has ample dense elements. I.e. the diagonal action by conjugation of $G$ on the $n$-th Cartesian power of $G$ has a dense orbit for every $n \geq 1$.

5. THE NON-EXISTENCE OF A COMEAGRE CONJUGACY CLASS

By Corollary 4.3, a non-trivial t.d.l.c.s.c. group does not admit a comeagre conjugacy class. We now eliminate connected groups as candidates for admitting a comeagre class. The connected case is a previously unpublished result of Professor Hofmann. The author wishes to express his thanks to Professor Hofmann for permitting this result to be included in the present work.

To present Hofmann’s result, an old theorem due to W. Burnside is required.

Fact 5.1 (Burnside [8] XVII.3 Corollary 3.3). Let $E$ be a finite dimensional vector space over an algebraically closed field $k$ and $R$ a subalgebra of $\text{End}_k(E)$. If $E$ has no non-trivial proper $R$-invariant subspaces, then $R = \text{End}_k(E)$. We say $E$ is $R$-simple in such a case.

Lemma 5.2. If $G \leq GL_n(\mathbb{C})$ is a closed subgroup with a dense conjugacy class, then $G = \{id\}$.

Proof. We consider general linear groups to be written as matrix groups in the standard basis. Suppose $G \leq GL_n(\mathbb{C})$ and say $A \in G$ has a dense conjugacy class.

Since $A^G$ is dense, we may find $B_i AB_i^{-1} \to id$ with $B_i \in G$. The determinant $\det : M_n(\mathbb{C}) \to \mathbb{C}$ is continuous, and therefore,

$$\det(tI - B_i AB_i) \to \det(tI - id) = (t - 1)^n$$

Since $\det$ is invariant under conjugation by elements of $GL_n(\mathbb{C})$, $\det(tI - B_i AB_i) = \det(tI - A)$ for all $i$, and so $\det(tI - A) = (t - 1)^n$. It follows every $B \in G$ has characteristic polynomial $(t - 1)^n$. By the Cayley-Hamilton theorem, $(B - id)^n = 0$ for all $B \in G$.

Consider $B, C \in G$ and let $Tr$ be the usual trace function. Then,

$$(1) \quad Tr(C(B - id)) = Tr(CB - id) - Tr(C - id) = 0$$

since the trace is linear and $Tr$ vanishes on elements for which some power is zero. Plainly, (1) holds for any linear group over $\mathbb{C}$ with a dense conjugacy class.

We now consider $E$ a least dimension non-trivial $G$-invariant subspace of $\mathbb{C}^n$. Let $r : G \to GL(E)$ be the induced map. Certainly, $\tilde{G} := cld(r(G))$ again has a dense conjugacy class. Further, $E$ is a $\tilde{G}$-simple vector space. Set $R \subseteq \text{End}_\mathbb{C}(E)$ to be the algebra generated by $\tilde{G}$. Since $R$ contains $\tilde{G}$, $E$ is $R$-simple, and $R = \text{End}_\mathbb{C}(E)$ by Burnside’s theorem, Fact 5.1.

Fix $B \in \tilde{G}$ and take $M \in \text{End}_\mathbb{C}(E)$. Since $M = \sum_{i=1}^m \alpha_i A_i$, where $A_i \in \tilde{G}$ and $\alpha_i \in \mathbb{C}$, $Tr(M(B - id)) = 0$ by (1). Consider $M_{ij}$ the matrix which is 1 on the
(i, j)-th entry and 0 else; note $M_{ij} A$ is the matrix of all zeros except for row i which is row j from A. Letting $c_{ik} = \alpha$ be an entry of $(B - Id)$, $M_{kl}(B - Id)$ is such that $\alpha$ is on the diagonal and the rest of the diagonal entries are zero. But

$$0 = Tr(M_{kl}(B - Id)) = \alpha$$

and so $B = Id$. Thus, $\tilde{G}$ is trivial, every element of $G$ fixes $E$, and $E$ is one dimensional. Continuing this process, we obtain $E_1 < ... < E_n = \mathbb{C}^n$ a strictly increasing sequence of $G$-invariant vector subspaces such that $E_{i+1}/E_i$ is dimension one and $G$ acts trivially on $E_{i+1}/E_i$.

We here form $K_i := ker(G \cap E_i)$ for each $1 \leq i \leq n$. Certainly, $K_1 = G$. For $K_2$, $E_2$ has two dimensions; say $E_2 = M \oplus E_1$. Fix $m + e \in E_2$, take $A, B \in G$, and consider their action on $m + e$. Since $G$ acts trivially on $E_1$ and $E_2/E_1$, $A(m + e) = m + \alpha e + e$ and $B(m + e) = m + \beta e + e$ for some $\alpha, \beta \in \mathbb{C}$. Additionally, $A^{-1}(m + e) = m - \alpha e + e$ and $B^{-1}(m + e) = m - \beta e + e$. These observations yield

$$A^{-1}B^{-1}AB(m + e) = A^{-1}B^{-1}(m + \beta e + e)$$

$$= A^{-1}B^{-1}(m + \alpha e + \beta e + e)$$

$$= A^{-1}(m - \beta e + \alpha e + \beta e + e)$$

$$= (m - \alpha e - \beta e + \alpha e + \beta e + e)$$

$$= m + e$$

So $A^{-1}B^{-1}AB \in K_2$, $G/K_2$ is abelian, and $K_2 = G$ since $G/K_2$ has a dense conjugacy class. Continuing in this fashion, $G = K_n$, and $G$ acts trivially on $\mathbb{C}^n$. We have thus demonstrated $G = \{1\}$. \qed

**Theorem 5.3** (Hofmann). A non-trivial connected locally compact group cannot have a dense conjugacy class.

*Proof.* For contradiction, suppose $G$ is a non-trivial connected locally compact group with a dense conjugacy class. By the solution to Hilbert’s fifth problem, $G$ is pro-Lie. We may thereby find a proper closed subgroup $N \triangleleft G$ such that $G/N$ is Lie. Since $N$ is proper, $G/N$ has a dense conjugacy class.

Let $\tilde{G} := Ad(G/N) \leq GL_n(\mathbb{K})$ where $\mathbb{K}$ is either the real or complex field; under the natural inclusion $GL_n(\mathbb{R}) \leq GL_n(\mathbb{C})$, we may assume $\mathbb{K} = \mathbb{C}$. Fact 2.2 implies $\tilde{G}$ has a dense conjugacy class, and therefore, $\tilde{G} = \{1\}$ by Lemma 5.2. So $G/N$ is trivial since abelian with a dense conjugacy class. This contradicts the choice of $N$. \qed

Combining Corollary 4.3 and Theorem 5.3, we have answered Kechris’ and Rosendal’s question:

**Theorem 5.4.** A non-trivial l.c.s.c. group does not admit a comeagre conjugacy class.

*Proof.* Suppose for contradiction $G$ is a l.c.s.c. group and $g^G$ is comeagre. Since $G$ is $K_\sigma$, we may write $G = \bigcup_{i \in \mathbb{N}} K_i$ with the $K_i$ compact sets. So $g^G = \bigcup_{i \in \mathbb{N}} g^{K_i}$, and there is $i \in \mathbb{N}$ such that $g^{K_i}$ is non-meagre by the Baire category theorem. For such an $i$, $g^{K_i}$ is closed and, therefore, must have non-empty interior. It follows $g^G$ is open as well as dense.

Now consider $G_0$, the connected component of the identity. If $G_0$ is non-meagre, then by [10], $G_0$ is open. However, this implies $G_0 = G$ which contradicts Theorem 5.3. If $G_0$ is meagre, form $\tilde{G} := G/G_0$ and let $\tilde{g}$ be the image of $g$ in $\tilde{G}$ under
the usual projection map. By our discussion in the first paragraph, $\tilde{g}^G$ is open and dense in $G$. This contradicts Corollary 4.3. □

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