Quantum Euler angles and agency-dependent spacetime

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Abstract

Quantum gravity is expected to introduce quantum aspects into the description of reference frames. Here we begin exploring how quantum gravity induced deformations of classical symmetries could modify the transformation laws among reference frames in an effective regime. We invoke the quantum group $SU_q(2)$ as a description of deformed spatial rotations and interpret states of a representation of its algebra as describing the relative orientation between two reference frames. This leads to a quantization of one of the Euler angles and to an aspect of agency dependence: space is reconstructed as a collection of fuzzy points, exclusive to each agent, which depends on their choice of reference frame. Each agent can choose only one direction in which points can be sharp, while points in all other directions become fuzzy in a way that depends on this choice. Two agents making different choices will thus observe the same points with different degrees of fuzziness.

1 Agency-dependent spacetime and spacetime-dependent agency

Observers typically play a somewhat “external” role in both general relativity (GR) and quantum theory. In the former, they are assumed to exert a negligible backreaction on spacetime, which they can thus probe without influencing it. In quantum theory, it is the Heisenberg cut [1, 2] that separates them from the observed system. The observers’ knowledge and choices are described classically, however, there are mutually-incompatible measurements that they can choose to perform (complementarity). Quantum mechanics introduces therefore an important novelty: the operationally-meaningful properties of the observed systems depend on the choices made by the observer. Through the impact of these choices, observers are “agents” for what concerns measurements in quantum mechanics.

When quantum theory and GR are combined in a quantum theory of gravity, one ends up considering spacetime as a quantum object. We may then contemplate the possibility that the agency-dependence of quantum mechanics might challenge the notion of an objective spacetime that all observers agree upon: how spacetime reveals itself to observers may depend on some of their choices. Conversely, the quantum properties of spacetime will likely affect the spectrum of possible operations available to an agent. These observations suggest a picture in which spacetime and the agency of observers affect each other inextricably, so much so that the “externality” idealization, a good working hypothesis in GR and quantum theory, will have to be abandoned in favor of a notion of “internal” observers in quantum gravity [3].

In the present paper, we take a step toward this largely unexplored aspect of quantum gravity. We cannot expect to begin with a complete model of an observer/agent in a quantum spacetime. Therefore, we will be concerned with a particular aspect of this problem: the relative state of the reference frames of two observers, Alice and Bob. In a simplified model, we assume that the two observers are exclusively interested in finding out the relative spatial orientations of their respective laboratories. In standard quantum theory, the rotation matrix relating the two reference frames can be determined by exchanging

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qubits and classical information, as discussed in appendix C. With an arbitrarily large number of exchanges, the two observers can determine the matrix elements with arbitrary precision. However, in the case in which Alice and Bob live in a quantum spacetime, reaching an arbitrary precision may be forbidden.

Quantum group toy model

We will focus on a toy model of quantum gravity in the form of a nonclassical space: this will be the arena in which observers exercise their agency and will attempt to determine the structure of the spacetime they are in. A much-studied candidate for a theory of quantum spacetime is noncommutative geometry [4]. The idea is that, by replacing the (commutative) algebra of functions on a manifold with a noncommutative one, we end up describing a manifold whose points are “fuzzy”, due to the Heisenberg uncertainty principle. In this sense, standard quantum theory can be seen as the noncommutative space one obtains by making the algebra of phase-space functions noncommutative. Recent developments focused on deformed symmetries and quantum groups, in which the algebra of functions on a Lie group is noncommutative [5]. As Lie groups describe the symmetries of classical spaces, quantum groups describe the symmetries of noncommutative geometries; as such they have appeared in several quantum gravity approaches [6–19]. In this regard, since we are only interested in the relative orientation between reference frames, we can model our nonclassical space simply by deforming the rotation group \( SO(3) \), or rather its double cover \( SU(2) \) which describes the rotations of qubits.

To do so, we will be concerned with one of the simplest examples of quantum groups: \( SU_q(2) \) [20], which is the (only) quantum-group deformation of the \( SU(2) \) group. The lowercase \( q \) identifies a dimensionless deformation parameter, such that the case \( q = 1 \) reproduces the undeformed \( SU(2) \) group. In loop quantum gravity models, \( q \) is a function of the dimensionless ratio between the Planck length and the Hubble length scale associated to the cosmological constant [7]. It has been argued that this reflects a minimal possible resolution in angular measurements [12]. One could also imagine a physical scenario in which the dimensionless \( q \) could acquire a dependence on the characteristic energy scale \( E \) of the problem via, for example, the dimensionless ratio \( E/E_P \), with \( E_P \) the Planck energy. This feature could arise in field theories in which \( q \) is a parameter appearing in the lagrangian, and thus would run with the energy as other parameters do. To the best of our knowledge, no concrete example with such features has been constructed, so we do not speculate upon this possibility further.

The mathematical details of \( SU_q(2) \) have been developed extensively [20–23]. We want to further explore the physical features of this group to arrive at a qualitative prediction for the novel “quantum geometrical” effects that observers would see if, in an effective regime of quantum gravity, rotations of reference frames were described by \( SU_q(2) \) transformations.

2 Quantum rotation matrices in \( SU_q(2) \)

2.1 \( SU_q(2) \) algebra and homomorphism with \( SO_q(3) \)

The quantum group \( SU_q(2) \) is defined by considering the algebra of complex functions on \( SU(2) \), denoted by \( C(SU(2)) \) and deforming it in a noncommutative way. The generators \( \{a, c\} \) are regarded as functions on the group, satisfying non-trivial commutation relations

\[
ac = qca \quad \quad ac^* = qa^*c \quad \quad cc^* = c^*c \\
c^*c + a^*a = 1 \quad \quad aa^* - a^*a = (1 - q^2)c^*c.
\]  

(1)

Here, 1 refers to the identity element of the algebra and \( q \) is the deformation parameter assumed to be close to 1 and in particular \( q \lesssim 1 \). Indeed, in the \( q \to 1 \) limit, we obtain the commutative limit and recover the classical description of \( SU(2) \). In passing, we mention that it suffices to consider \( q \lesssim 1 \) since, if \( q > 1 \), the mapping \( a \mapsto a^*, c \mapsto qc^* \) sends the \( SU_q(2) \) algebra to the \( SU_{q^{-1}}(2) \) one.

To establish a first link with the classical picture, we present the generalization of the spin-1/2 representation (21), given by

\[
U_q = \begin{pmatrix} a & -qc^* \\ c & a^* \end{pmatrix} \quad a, c \in C(SU_q(2)) \quad q \in (0, 1).
\]

(2)

We will now construct one of the key ingredients of the analysis which is a \( q \)-analogue of the \( SU(2) \) to \( SO(3) \) homomorphism. In order to do so, we promote the classical homomorphism \( R_{ij} = \frac{1}{2} \text{Tr} \{ \sigma_j U^\dagger \sigma_i U \} \),
where \( \sigma_i \) are the Pauli matrices, to its quantum version by replacing \( U \) with \( U_q \)

\[
(R_q)_{ij} = \frac{1}{2} \text{Tr} \left\{ \sigma_j U_q^\dagger \sigma_i U_q \right\}.
\]  

(3)

By computing these elements explicitly, we obtain

\[
R_q = \begin{pmatrix}
\frac{1}{2}(a^2 - q c^2 + (a^*)^2 - q(c^*)^2) & \frac{i}{2}(-a^2 + q c^2 + (a^*)^2 - q(c^*)^2) & \frac{1}{2}(1 + q^2)(a^* c + c^* a) \\
\frac{i}{2}(a^2 + q c^2 - (a^*)^2 - q(c^*)^2) & \frac{1}{2}(a^2 + q c^2 + (a^*)^2 + q(c^*)^2) & -\frac{i}{2}(1 + q^2)(a^* c - c^* a) \\
-(a c + c^* a) & i(a c - c^* a) & 1 - (1 + q^2)c c^*
\end{pmatrix}
\]

(4)

and one can check that in the commutative limit \( q \to 1 \) this coincides with the matrix obtained via the classical homomorphism. This matrix provides a (co)-representation of \( SO_q(3) \) related to the (co)-representation found in [23] by a similarity transformation. We note that in (3) the cyclic property of the trace operation does not hold because of the noncommutative nature of \( a \) and \( c \). This gives rise to a “quantization ambiguity” which however does not affect our phenomenological results. The relevant technical issues are addressed in appendix A.

The classical analog of (4) describes all possible elements of \( SO(3) \) when varying the complex numbers \( a \) and \( c \) with continuity. Each of these classical matrices describes a possible relative orientation between observers A and B. In the quantum case (4) has no physical meaning when taken alone, but only when paired with a certain state \(|\psi\rangle \in \mathcal{H}\), where \( \mathcal{H} \) is the Hilbert space on which \( a \) and \( c \) act. In the classical case, \( a \) and \( c \) codify information about the alignment of two reference frames, via their angular parametrization (see below and appendix A). For this reason, when being promoted to operators, we interpret the states on which they act as the ones codifying the relative orientation between the reference frames of observers A and B. The physical information about this orientation can be extracted by computing the expectation values \( \langle \psi | (R_q)_{ij} | \psi \rangle \) with their relative uncertainties stemming from the noncommutative nature of these matrix elements, given by

\[
\Delta_{ij} = \sqrt{\langle \psi | (R_q)_{ij}^2 | \psi \rangle - \langle \psi | (R_q)_{ij} | \psi \rangle^2}.
\]

(5)

In this framework, the \( \Delta_{ij} \) do not vanish simultaneously; in general, for a given state \(|\psi\rangle \) introducing a “fuzziness” in the alignment procedure. When this occurs, the latter is affected by an intrinsic uncertainty which cannot be eliminated: A and B are not able to sharply align their reference frames anymore. In light of these arguments, it is crucial to identify \( \mathcal{H} \) and study the representations of operators \( a \) and \( c \) on it.

2.2 \( SU_q(2) \) representations and quantum Euler angles

The representations of algebra (1) have been thoroughly studied in [23]. The Hilbert space containing the two unique irreducible representations of the \( SU_q(2) \) algebra, \( q \in (0, 1) \), is \( \mathcal{H} = \mathcal{H}_\pi \oplus \mathcal{H}_\rho \) where \( \mathcal{H}_\pi = L^2(S^1) \otimes L^2(S^1) \) and \( \mathcal{H}_\rho = L^2(S^1) \). If \( \chi, \phi, \chi \in [0, 2\pi] \) are coordinates on \( S^1 \) and \( |n\rangle \) is the canonical basis of \( \ell^2 \), the algebra of functions on \( SU_q(2) \) is represented as

\[
\rho(a) |\chi\rangle = e^{i\chi} |\chi\rangle \quad \rho(a^*) |\chi\rangle = e^{-i\chi} |\chi\rangle \quad \rho(c) |\chi\rangle = \rho(c^*) |\chi\rangle = 0
\]

(6)

\[
\pi(a) |n, \phi, \chi\rangle = e^{i\sqrt{1 - q^{2n}}} |n - 1, \phi, \chi\rangle \quad \pi(c) |n, \phi, \chi\rangle = e^{i\phi} q^n |n, \phi, \chi\rangle
\]

\[
\pi(a^*) |n, \phi, \chi\rangle = e^{-i\sqrt{1 - q^{2n+2}}} |n + 1, \phi, \chi\rangle \quad \pi(c^*) |n, \phi, \chi\rangle = e^{-i\phi} q^n |n, \phi, \chi\rangle.
\]

(7)

It is not coincidental that quantum number \( \chi \) is common in both representations. We show in appendix D that representation \( \rho \) can be obtained as a limit of representation \( \pi \), in agreement with the fact that in the classical case we only need three real parameters to specify rotations.

Let us give physical meaning to the quantum numbers appearing in the representations, making a comparison between these relations and the classical parametrization of \( a \) and \( c \) for \( U \in SU(2) \)

\[
a = e^{i\eta} \cos \frac{\theta}{2} \quad c = e^{i\delta} \sin \frac{\theta}{2},
\]

(8)

where \( \eta, \delta \) and \( \theta \) are a linear redefinition of Euler angles, as discussed in appendix A. We note that \( \rho(a) \) and \( \rho(c) \) act diagonally, therefore we can make the identifications \( \chi \equiv \eta \) and \( \theta \equiv 0 \). For what concerns
representation $\pi$, $c$ acts diagonally, while $a$ and $a^*$ act as ladder operators. For operator $a$, we added a phase, that is implicitly set to 0 in the literature [20], which does not affect the commutation relations (1) but is needed in order to have a comparison of (7) with (8). From this, we identify $\chi \equiv \eta$, $\phi \equiv \delta$ and, exploiting the fact that $c$ acts diagonally, we are led to the significant result

$$q^n = \sin\left(\frac{\theta(n)}{2}\right) \iff \theta(n) = 2\arcsin(q^n). \quad (9)$$

The Euler angle $\theta$ becomes quantized, a feature captured in fig. 1. The quantization of $\theta$ can also be inferred using $\pi(a)$ (see appendix A).

![Figure 1: Discretized angles (9) computed with $q = 0.9$. $\theta(n)$ is decreasing with $n$ starting from $\pi$ and approaching 0 for large values of $n$. Interestingly, the step between consecutive angles decreases as $n$ gets very larger, and for very large $n$ we can approximate the angular distribution as being continuous. This can easily be verified analytically by computing $\Delta \theta(n) = \theta(n + 1) - \theta(n)$ and taking the limit for $n \to \infty$. Another feature that gives robustness to our proposal is that the angular steps get smaller as $q$ is closer to 1.](image)

### 2.3 Semi-classical rotations

To gain further intuition from the classical picture, where the rotation axis and the angle by which the rotation is performed are specified by three Euler angles, we henceforth focus on “semi-classical” rotations, specified by the three quantum numbers $\theta(n), \phi, \chi$, that describe small deformations of classical rotations defined by these angles. More precisely, the states $|\psi(\theta, \phi, \chi)\rangle$ yielding such deformed rotations should satisfy

$$\begin{cases}
\langle \psi(\theta, \phi, \chi)|(R_q)_{ij}|\psi(\theta, \phi, \chi)\rangle = R_{ij}(\theta, \phi, \chi) + O(1 - q) \\
\Delta_{ij}^q = O(1 - q)
\end{cases} \quad \forall i, j, \quad (10)$$

where $(R_q)_{ij} \in \text{SO}(3)$ and where $\theta$ is one of the allowed values in (9). This prescription constrains the quantum rotations we can construct to the ones in which the $\theta$ Euler angle can only take the allowed values in (9), for a fixed $q$. In order to clarify the meaning of this prescription, we provide some examples.

A first class of states that trivially satisfies requirement (10) is the set of eigenstates $|\chi\rangle \in \mathcal{H}_\rho$. It is easy to see that the expectation value of the quantum rotation matrix (4) in such states describes a rotation of angle $2\chi$ around the $z$-axis, namely:

$$\langle \chi|R_q|\chi\rangle = \begin{pmatrix}
\cos(2\chi) & -\sin(2\chi) & 0 \\
\sin(2\chi) & \cos(2\chi) & 0 \\
0 & 0 & 1
\end{pmatrix} \quad (11)$$

with all the $\Delta_{ij}$ being zero. Thus, rotations around the $z$-axis are classical/continuous: two observers, $A$ and $B$, whose relative orientation is described by $|\chi\rangle \in \mathcal{H}_\rho$, can align themselves sharply.
describe the corresponding semi-classical rotation around an axis in the
selects a rotation around the $x$-axis. These angles specify the axis of rotation also in the quantum case. For the case $\theta = 0$, we build superpositions of states $|n, \phi, \chi\rangle$ centered around a certain $n \in \mathbb{N}$. The expectation values and uncertainties relative to such states will reproduce deformations of classical rotation matrices specified by angles $(\theta(n), \phi, \chi)$, in the sense of (10).

It is worth noticing that there is a special class of such semi-classical rotations for which some simplifications arise. This is the case of rotations with $\theta = \pi$ around axes of the $x - y$ plane. The classical theory predicts that $\chi = 0$ and a generic $\phi$ select a direction in the $x - y$ plane ($\phi = \pi/2$ selects a rotation around the $x$-axis, while $\phi = 0$ selects one around the $y$-axis) around which we rotate an angle $\theta$. As emphasized before, since $\phi, \chi$ behave classically, the quantum numbers associated to these angles specify the axis of rotation also in the quantum case. For the case $\theta(0) = \pi$, building a superposition of basis states is not necessary to satisfy (10) and the basis state $|0, 0, 0\rangle$ is sufficient to describe the corresponding semi-classical rotation around an axis in the $x - y$ plane, specified by $\phi$. The expectation value of $R_\phi$ on such state is

$\langle 0, \phi, 0| R_\phi|0, \phi, 0\rangle = \begin{pmatrix} -q \cos(2\phi) & -q \sin(2\phi) & 0 \\ -q \sin(2\phi) & q \cos(2\phi) & 0 \\ 0 & 0 & -1 \end{pmatrix}$ (14)

while the $\Delta_{ij}$ are non-zero and do not depend on the specific value of $\phi$

$$\Delta_{ij} = \frac{1}{2} \sqrt{(1 - q^2)(1 - q^4)} \begin{pmatrix} \frac{1}{2} \sqrt{(1 - q^2)(1 - q^4)} & \frac{1 + q^2}{2} \sqrt{(1 - q^2)} \\ \frac{1}{2} \sqrt{1 - q^4} & \sqrt{q^2 - q^4} \\ \frac{1}{2} \sqrt{(1 - q^2)(1 - q^4)} & \frac{1 + q^2}{2} \sqrt{(1 - q^2)} \end{pmatrix}$$ (15)

From the above matrices, it can be verified that the state $|0, \phi, 0\rangle$ satisfy (10).

For a generic axis of rotation, the quantum Euler angle $\theta$ will play a non-trivial role both in the determination of the axis itself and in the determination of the rotation angle.

3 Same stars, different skies

Perhaps the most striking implication of our results is that they provide a possible scenario for a novel type of relationship between observers and the spacetime they observe, such that the choices made by the observer in setting up her reference frame affect the spacetime she observes.

We shall argue this by first advocating Einstein’s operational notion of spacetime, whose points have physical meaning only in as much as they label an event there occurring. We use as reference example a network of sources emitting photons (“stars”): each photon emission is a physical point of spacetime. These points of spacetime will be labeled by measured coordinates and uncertainties on those coordinates. The key observation here is that, in our model, these uncertainties are not intrinsic properties of the spacetime points but rather depend on the choices made by the observer. These arguments give rise to a framework in which different skies may originate from the observation of the same stars.

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Figure 2: A semi-quantitative description of the starry skies observed by Alice and Bob. Alice’s z-axis is aligned with the yellow star while Bob’s z-axis is aligned with the green star, so that the yellow star is sharp for Alice while the green star is sharp for Bob. For Alice (Bob) the fuzziness increases as the angular deviation from the yellow (green) star increases and the same stars are observed with different fuzziness because of the relative orientation of the z-axes of the two agents.

3.1 Agency-dependent space

We will apply our formalism to establish the connection between the q-deformation of the SU(2) group and the agency-dependence of space. Before showing this with a concrete numerical example, we will discuss why this property of space is to be expected with a thought experiment.

Consider two observers, Alice and Bob, each equipped with their own set of telescopes, who want to map the starry sky. Each of them chooses their reference frame (in particular their z-axis) to assign coordinates to the stars they observe. We assume that their origins coincide but their z-axes do not. Without loss of generality we can focus on the (y, z)-plane so that, for each observer, any telescope is mapped onto another by a rotation about the x-axis. In the classical case, Alice and Bob can compare their experimental results by rotating their data with a classical rotation matrix. Formally, there exists an element of the SO(3) group that sharply describes the relative orientation between Alice and Bob.

We now analyze how the situation changes in our novel noncommutative framework. We henceforth assume that Alice and Bob can choose the direction along which points can be sharp independently from one another. For definiteness, we focus on observer Alice first. In her task of mapping the starry sky, she focuses on a particular star, which she observes with one of her telescopes. This instrument is generally misaligned with respect to the z-axis she has chosen. In our framework, this means that there exists a state \( |\psi_A \rangle \) connecting the relative orientation between the aforementioned telescope and the z-axis. Following our physical interpretation, \( |\psi_A \rangle \) determines a quantum rotation matrix through the expectation value of \( R_q \) and an uncertainty matrix defined by (5). This means that Alice cannot appreciate the relative orientation between the telescope and her z-axis with arbitrary precision since there will always be some intrinsic uncertainty given by the deformation of the SU(2) group. As a consequence of this, she will not be able to sharply deduce the position of this star in the sky. For that same star, this line of reasoning also applies to Bob but the state \( |\psi_B \rangle \) will generally be different from \( |\psi_A \rangle \), implying a different degree of fuzziness. In turn, this procedure can be applied to any star in the sky so that Alice and Bob each have their own picture of the celestial sphere, as can be seen in fig. 2. The striking result is that the uncertainty associated with each star is not an intrinsic property of spacetime but depends on the choices made by the observers. In particular, different choices of the z-axis give rise to different pictures of the starry sky. The subtle observation is that there is no way for Alice and Bob to define an objective celestial sphere, meaning that the definition of space itself cannot be independent of the observer who reconstructs it. In this sense, we say that Alice and Bob are agents: we are abandoning the idea of objective space, replacing it with a notion of observed space from which we cannot subtract the choices of the observer who infers it.

As mentioned earlier, it is important to note that, even though the z-axis is a preferred direction for observers (only its points can be sharp), our framework produces an isotropic description of space. Indeed, spatial rotations are a (q-deformed) symmetry. There is no direction which is preferred a priori, and the special role played by the z-axis in the reference frame of a given observer is only the result of the choices made by that observer in setting up their frame. The observer chooses freely their preferred z-direction. This should be contrasted with the case of standard spatial anisotropy, in which
Table 1: Dependence of the aperture on the observation angle \( \theta(n) \) using \( q = 0.99 \). The aperture is monotonically increasing as the angles grow in absolute value. The states used to obtain these results are constructed numerically in appendix B. For consistency, the aperture for 180° is not truncated at first order in \((1-q)\) as in (20), but is computed numerically with eq. (17) with \( q = 0.99 \).

From this analysis, it is worth noticing that the fuzziness grows as our quantized Euler angle increases, resulting in a starry sky inferred by Alice, similar to what is depicted in fig. 2. Of course,
the analysis can also be repeated for Bob, who chooses a different z-axis, in principle, and observes the same stars as Alice. The states describing the relative orientation between these stars and his z-axis will be different from the ones characterizing Alice’s frame. The degree of fuzziness he observes for a particular star will be different from the one assigned to it by Alice, resulting in a different inference of the starry sky, as shown in fig. 2.

4 Conclusions

In this work, we have given a physical interpretation of the $SU_q(2)$ quantum group and, from the properties of its representations, derived a framework in which one of the Euler angles becomes quantized and reference frames exhibit fuzziness properties.

The nature of this fuzziness is distinct from the one usually associated with quantum reference frames (QRFs) [24–32], which arises describing the frame itself as an ordinary quantum system. The key difference is the source of the frame fuzziness: for QRFs it is governed by Planck’s constant $\hbar$ and follows from the standard laws of quantum mechanics, while in our scenario it is the deformation of spatial rotations, governed by the parameter $q$, that produces the fuzziness. Accordingly, the states encoding the fuzziness are associated with physical systems (including the frame) in the context of QRFs, while here they describe the relation between two frames. QRFs can be in relative superposition, however, the transformations between them act on the states of the physical systems directly, without being described by a quantum state of their own.

Classically, all observers can objectively reconstruct the sky. In our work, they are agents: using quantum rotations, they reconstruct their own version of the sky based on their choices of the z-axis and there is no operational procedure they can use to reconstruct it objectively. In fact, even if they were to exchange their data, they would need to $q$-rotate it in their coordinates, inevitably introducing the intrinsic uncertainty predicted by the $SU_q(2)$ model.

The next step in this program would be to develop a fully relativistic and relational picture. To do so, we should first consider the case in which more observers are involved, understanding which states can describe the composition of deformed rotations. Then the framework should be extended to the translations and boosts sectors.

The full quantum regime should also be investigated. In our present work, we focused on a semi-classical regime, in which the expectation values and the variances of the $q$-deformed rotation matrix act on classical vectors, to derive some novel qualitative properties. This is a crucial step toward a description of a $q$-deformed version of the alignment procedure described in appendix C. To go further, we need a more developed characterization of physical objects like spinors/qubits and Stern-Gerlach apparatus, as well as a consistent interpretational framework for such “quantum mechanics on a quantum space(time)”. In particular, we would need a description of qubits in a world in which rotations are described by $SU_q(2)$, and this would require facing several open questions: Is it consistent to just assume that the standard 1-qubit Hilbert space describes the spins that Alice sends to Bob? Or does consistency require generalizing the notion of spinors to some noncommutative objects (e.g., see [33])? What does the measurement procedure do to the state of the system? Is the relative orientation of Alice’s and Bob’s laboratories changed by their gaining information about each other while exchanging spins? Or is a quantum spacetime described by $SU_q(2)$ rotations a sufficiently mild deformation of classical spacetime that it still guarantees that two observers can study their relative orientation without changing it?

Besides their conceptual interest, investigating these possibilities might open a path [34,35] towards much-needed experimental tests, which in turn could guide further theoretical development of quantum gravity.

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A Classical SU(2) parametrization and quantization ambiguity

The SU(2) group is defined as the group of $2 \times 2$ unitary matrices with determinant equal to 1. A generic element $U \in SU(2)$ can be parametrized as

$$U = \begin{pmatrix} a & -c^* \\ c & a^* \end{pmatrix},$$

where $a$ and $c$ are two complex numbers satisfying

$$aa^* + cc^* = 1,$$

constraining the degrees of freedom to three real numbers. A commonly used parametrization for $a$ and $c$ in terms of three real numbers is

$$a = e^{i\eta} \cos \frac{\theta}{2} \quad c = e^{i\delta} \sin \frac{\theta}{2},$$

with $\eta, \delta \in [0, 2\pi)$ and $\theta \in [0, \pi)$. These three angles encode all the information needed to uniquely identify an SU(2) element and are useful in identifying the connection of SU(2) to the rotation group SO(3). Indeed, the canonical homomorphism between the two groups is realized via

$$R_{ij} = \frac{1}{2} \text{Tr} \{\sigma_i U \sigma_j U^\dagger\} \equiv \frac{1}{2} \text{Tr} \{U \sigma_j U^\dagger \sigma_i\},$$

where $R_{ij}$ is a rotation matrix, $U$ is parametrized as in (21) and $\sigma_i$ are the Hermitian Pauli matrices. Writing (24) explicitly, we have

$$R = \begin{pmatrix} \frac{1}{2}(a^2 - c^2 + (a^*)^2 - (c^*)^2) & \frac{1}{2}(-a^2 + c^2 + (a^*)^2 - (c^*)^2) & (a^*c + c^*a) \\ \frac{1}{2}(a^2 + c^2 - (a^*)^2 + (c^*)^2) & \frac{1}{2}(a^2 + c^2 + (a^*)^2 + (c^*)^2) & -i(a^*c - c^*a) \\ -(ac + c^*a) & i(ac - c^*a) & 1 - (a^2 + c^2)cc^* \end{pmatrix}.$$

As anticipated in the results section, matrix (4) indeed coincides with this classical one in the $q \to 1$ limit.

Inserting parametrization (23) in (25) we obtain the rotation matrix in terms of trigonometric functions of the three real angles $\eta, \delta$ and $\theta$. These are simply a redefinition of the well known Euler angles $(\alpha, \beta, \gamma)$

$$\theta = \beta \quad \eta = \frac{\alpha + \gamma}{2} \quad \delta = \frac{\pi}{2} - \frac{\alpha - \gamma}{2}.$$

In terms of these, a generic rotation matrix is written as $R(\alpha, \beta, \gamma) = R_z(\alpha)R_x(\beta)R_z(\gamma)$, where $R_z$ and $R_x$ are rotations around the $z$-axis and $x$-axis respectively.

In the quantization procedure, we replace the classical $U$ with the deformed $U_q$ in (25), thus replacing the complex numbers $a$ and $c$ with the operators satisfying (1), and we leave the Pauli matrices untouched. It may be argued that a deformation of the Pauli matrices, such as the one studied in [33], should be considered here. However, doing so leads to non-hermitian operators for the entries of the resulting matrix. With our choice, not only do we get hermitian operators, but we reconstruct a matrix that matches the one obtained in [23] upon doing a change of basis, and this guarantees that we obtain a representation of SO$_q$(3) as explained shortly.

To discuss our results we used matrix (4) which is obtained with the first ordering in (24). Due to the noncommutative nature of operators $a$ and $c$, the cyclic property of the trace, expressed in (24) is not valid anymore. This leads to two possible definitions of the quantum rotation matrix. The alternative to (4), obtained using the second ordering in (24), is

$$P_q = \begin{pmatrix} \frac{1}{2}(a^2 - qc^2 + (a^*)^2 - q(c^*)^2) & \frac{1}{2}(-a^2 + qc^2 + (a^*)^2 - q(c^*)^2) & q(a^*c + c^*a) \\ \frac{1}{2}(a^2 + qc^2 - (a^*)^2 - q(c^*)^2) & \frac{1}{2}(a^2 + q(c^*)^2 + (a^*)^2 + q(c^*)^2) & -iq(a^*c - c^*a) \\ -(ac + c^*a) & i(ac - c^*a) & 1 - (1 + q^2)cc^* \end{pmatrix}.$$

It turns out that $P_q$ and $R_q$ are actually similar matrices: $R_q = MP_qM^{-1}$, with

$$M = \begin{pmatrix} 2^{\frac{1}{2}}(\frac{q}{1+q})^{\frac{1}{2}} & 0 & 0 \\ 0 & 2^{\frac{1}{2}}(\frac{q}{1+q})^{\frac{1}{2}} & 0 \\ 0 & 0 & (\frac{1+q}{2q})^{\frac{1}{2}} \end{pmatrix}.$$
This matrix (with its inverse) reduces to the identity at first order in \((1 - q)\) so that the two matrices are equivalent from a phenomenological point of view. Indeed, since the quantum-gravity expectation is that \(q\) is extremely close to 1, the search for experimental manifestations of our \(q\)-deformation is not likely to ever go beyond the leading \((1 - q)\)-order effects [34]. Notice that the two matrices come from an ambiguity of the “quantization procedure”, namely the ordering of the matrices in (24). This is similar to what happens in standard quantum mechanics where different orderings in the classical observables give rise to different quantum operators which give the same classical limits.

Furthermore, the link with the quantum group \(SO_q(3)\) is established as follows. The vector (co)-representation of \(SU_q(2)\) is linked to the (co)-representation of the quantum group \(SO_q(3)\) via the isomorphism \(C(SO_q(3)) := C(SU_q(2)/\mathbb{Z}_2)\); the representation reads [23]

\[
d_1 = \begin{pmatrix}
(a^*)^2 & -(1 + q^2)a^*c & -qc^2 \\
(-q(c^*)^2 & 1 - (1 + q^2)c^*c & ac \\
-q(c^*)^2 & -(1 + q^2)c^*a & a^2
\end{pmatrix},
\]

In the commutative limit, this matrix approaches a generic rotation matrix written in the so-called complex basis, and can be rewritten in the Cartesian basis by a similarity transformation, with

\[
N = \begin{pmatrix}
-1 & -i & 0 \\
0 & 0 & 1 \\
-1 & i & 0
\end{pmatrix}
\]

acting as the change of basis. By applying this transformation in the \(q\)-deformed case too, one can easily check that \(N^{-1}d_1N\) is identically equal to (4). This clarifies the link to the \(SO_q(3)\) quantum group. Matrices \(P_q\) and \(R_q\) are (co)-representations of \(SO_q(3)\) since they can be obtained by a similarity transformation from (29). Therefore, our ansatz (3) realizes the quantum version of the \(SU(2)\) to \(SO(3)\) homomorphism.

In closing this section, we offer some comments on another quantization ambiguity, concerning our definition of the angle \(\theta(n)\) in (9). There we obtained the definition of \(\sin \left( \frac{\theta(n)}{2} \right)\) by comparing the eigenvalues of \(\pi(c)\) to the classical parametrization of \(c\) in (8). We find that this is consistent with the definition of \(\cos \left( \frac{\theta(n)}{2} \right)\) in terms of \(\pi(a^*a)\), which also acts diagonally. Indeed, on a generic eigenstate \(\ket{n, \phi, \chi}\) (cf. (7)) one has that

\[
\pi(a^*a) \ket{n, \phi, \chi} = (1 - q^{2n}) \ket{n, \phi, \chi} = \cos^2 \left( \frac{\theta(n)}{2} \right) \ket{n, \phi, \chi}.
\]

Furthermore, this is in agreement with the fact that \(a^*a + c^*c = 1\), as can be inferred from (1). Nevertheless, it is noteworthy that since \(a\) and \(a^*\) do not commute (see (1)), one could obtain another result for \(\theta(n)\) upon defining its cosine in terms of \(\pi(aa^*)\), which also acts diagonally (cf. (7)), rather than \(\pi(a^*a)\). As one can easily check, this would lead to allowed values of \(\theta\) which are the same as in our \(\pi(a^*a)\) case, with the only difference that \(\theta = \pi\) would be missing from the spectrum of \(\theta(n)\). With this ordering we would have \(\sin \left( \frac{\theta(n)}{2} \right) = q^{n+1}\) which is merely a shift of \(n\) with respect to the ordering we adopted. Evidently, the main results and the main aspects of physical interpretation of our analysis are unaffected by this quantization ambiguity.

\section*{B Numerical construction of semi-classical states}

In order to construct the semi-classical states, we have resorted to numerical computations, focusing on the case \(q = 0.99\) as illustrative example. However, by increasing the value of \(q\), we have checked that the below states satisfy the condition (10), thus approximating classical rotations with increasing precision.

We will focus on the \((y, z)\)-plane setting \(\chi = 0, \phi = \frac{\pi}{2}\), as can be seen from (26). We want to construct states that semi-classically describe a deformed rotation of a certain angle in this plane. Since \(n\) defines the only angle left, \(\theta\), we choose to build superpositions (12) centered on particular values of \(n\), dubbed \(\bar{n}\), with coefficients \(c_n\) multiplying the states \(\ket{n, \phi, \chi}\) rapidly decreasing as \(\theta(n)\) deviates from \(\theta(\bar{n})\). Our ansatz is that these coefficients have the form of a discretized Gaussian distribution.

The variance is then chosen in the following way. Recalling that

\[
\theta(n) = 2 \arcsin q^n
\]
we can define $\Delta \tilde{n}$ as

$$\Delta \tilde{n} := \frac{dn}{d\theta} \bigg|_{\theta(n)} \Delta \theta.$$  \hfill (33)

We take $\Delta \theta = \frac{\pi}{2} - \arcsin q$, which is just half of the value of the maximum angular deviation and weigh it with the rate of change of $n$ with respect to $\theta$, approximated as the derivative. Therefore, the value of the variance depends on the central value $\bar{n}$. This approximation becomes more and more accurate with increasing values of $n$ for which $\theta(n)$ becomes quasi-continuous.

We then define our superposition coefficients $c_n$ as

$$c_n = \frac{e^{-\frac{(n-\bar{n})^2}{2\Delta^2}}}{\sum_{n=0}^{\infty} e^{-\frac{(n-\bar{n})^2}{2\Delta^2}}}.$$  \hfill (34)

For computational reasons, we do not consider the full superposition going from $n=0$ to $n=\infty$ but we truncate the series by considering the 3-$\sigma$ range of our Gaussian. Namely, the sum goes from $n_{\text{min}}$ to $n_{\text{max}}$, where

$$n_{\text{min}} - \bar{n} = -3\Delta \tilde{n} \quad n_{\text{max}} - \bar{n} = 3\Delta \tilde{n}.$$  \hfill (35)

For relatively small values of $\bar{n}$, the value for $n_{\text{min}}$ might be negative when considering the 3-$\sigma$ range. When this happens, we simply truncate the Gaussian and start the series from $n=0$. We have explicitly verified that this doesn’t affect our results in a significant way.

We have used this algorithm to construct the states used for the numerical analysis of table 1. In what follows, we show the results for the computation of the expectation values of the matrix elements $(R_q)_{ij}$ with their relative uncertainties for states centered around $n = 95, 34, 7, 0$, corresponding to angles $\theta = 45.275^\circ, 90.560^\circ, 137.518^\circ, \theta = 180^\circ$.

- $n = 95, \theta = 45.275^\circ$

$$\langle \psi(45.275^\circ) | R_q | \psi(45.275^\circ) \rangle = \begin{pmatrix} 0.997 & 0.000 & 0.000 \\ 0.000 & 0.695 & 0.708 \\ 0.000 & -0.708 & 0.698 \end{pmatrix}$$  \hfill (36)

$$\Delta R_q = \begin{pmatrix} 0.005 & 0.071 & 0.030 \\ 0.071 & 0.073 & 0.069 \\ 0.030 & 0.069 & 0.073 \end{pmatrix}$$

- $n = 34, \theta = 90.560^\circ$

$$\langle \psi(90.560^\circ) | R_q | \psi(90.560^\circ) \rangle = \begin{pmatrix} 0.990 & 0.000 & 0.000 \\ 0.000 & -0.015 & 0.990 \\ 0.000 & -0.990 & -0.005 \end{pmatrix}$$  \hfill (37)

$$\Delta R_q = \begin{pmatrix} 0.015 & 0.100 & 0.100 \\ 0.100 & 0.099 & 0.004 \\ 0.100 & 0.004 & 0.100 \end{pmatrix}$$

- $n = 7, \theta = 137.518^\circ$

$$\langle \psi(137.518^\circ) | R_q | \psi(137.518^\circ) \rangle = \begin{pmatrix} 0.982 & 0.000 & 0.000 \\ 0.000 & -0.739 & 0.663 \\ 0.000 & -0.663 & -0.721 \end{pmatrix}$$  \hfill (38)

$$\Delta R_q = \begin{pmatrix} 0.025 & 0.067 & 0.173 \\ 0.067 & 0.067 & 0.074 \\ 0.173 & 0.074 & 0.067 \end{pmatrix}$$

- $n=0, \theta = 180^\circ$

$$\langle \psi(180^\circ) | R_q | \psi(180^\circ) \rangle = \begin{pmatrix} 0.99 & 0.0 & 0 \\ 0 & -0.99 & 0 \\ 0 & 0 & -0.99 \end{pmatrix}$$  \hfill (39)

$$\Delta R_q = \begin{pmatrix} 0.014 & 0.014 & 0.140 \\ 0.014 & 0.014 & 0.140 \\ 0.140 & 0.140 & 0 \end{pmatrix}.$$
To obtain the states describing q-deformations of rotations of negative angles \( \zeta < 0 \), we just have to consider the same states but with \( \phi = -\frac{\pi}{2} \). It is straightforward to show that for states of the form (12) with \( \chi = 0 \) having real superposition coefficients like (34), the map \( \phi \mapsto -\phi \) exchanges \((R_q)_{23}\) with \((R_q)_{32}\), leaving all the other elements unchanged, and yields the same uncertainty matrix. Therefore the aperture of the cone in table 1 is the same for opposite angles.

For the \( \theta = 0 \) case, we use the state in the representation \( \rho \) which gives the identity matrix in computing the expectation values which is simply given by \(|\chi = 0\rangle \in H_\rho\). From (11) we thus have

\[
\langle \psi(0^\circ)|R_q|\psi(0^\circ)\rangle = \mathbb{1}_{3\times3} \quad \Delta R_q = 0_{3\times3}.
\]

### C Quantum protocol for the alignment of reference frames in classical space

Let us illustrate a simple (but not necessarily most efficient) protocol for two agents, who reside in a classical Euclidean spatial environment, but do not share a classical reference frame, to synchronize their spatial orientations by communicating qubits, i.e. SU(2) spinors. It is desirable to extend this protocol to the quantum group SU\(_q\)(2), however, we leave this for future research as it requires a better understanding of multi-qubit states and sequential rotations in the presence of the symmetry deformation.

#### C.1 Qubits as SU(2) spinors in standard quantum theory

It is well-known that any single qubit state, represented as a density matrix \( \rho \), can be expanded in the basis of the identity \( \mathbb{1} \) and Hermitian Pauli matrices \( \sigma_i \) as

\[
\rho = \frac{1}{2} (\mathbb{1} + \vec{r} \cdot \vec{\sigma}),
\]

where the expectation values of the spin observables give the components of the Bloch vector:

\[
r_i = \langle \sigma_i \rangle = \text{Tr}\{\rho \sigma_i\}.
\]

The latter relation holds since the Pauli matrices form an orthonormal basis

\[
\text{Tr}\{\sigma_\mu \sigma_\nu\} = 2 \delta_{\mu\nu},
\]

where \( \sigma_0 := \mathbb{1} \). As such, we can also define a 4-dimensional Bloch vector

\[
r_\mu = \text{Tr}\{\rho \sigma_\mu\}
\]

whose zero-component

\[
r_0 = \text{Tr}\rho = 1
\]

is simply the normalization of the state.

Consider now an SU(2) transformation of the quantum state. As it turns out, the density matrix (41) decomposes into an SU(2)-invariant singlet and an SU(2)-covariant triplet term, since for any matrix \( U \in \text{SU}(2) \),

\[
\rho \rightarrow U \rho U^\dagger = \frac{1}{2} (\mathbb{1} + \vec{r} \cdot U \vec{\sigma} U^\dagger) = \frac{1}{2} (\mathbb{1} + \vec{r}' \cdot \vec{\sigma}),
\]

where now the transformed Bloch vector is

\[
\vec{r}' = R \cdot \vec{r}
\]

with

\[
R_{ij} = \frac{1}{2} \text{Tr}\{U \sigma_j U^\dagger \sigma_i\}
\]

an element of \( \text{PSU}(2) \simeq \text{SO}(3) \). In other words, if the quantum state is transformed by an SU(2) matrix, the Bloch vector \( \vec{r}' \) that represents it transforms like a 3-dimensional vector under spatial rotations.

Clearly, the phase of the SU(2) transformations is cancelled and the adjoint action of SU(2) yields a spatial rotation in its fundamental representation. Hence, the normalization \( r_0 \) is an SU(2)-invariant,
Figure 3: Alice (on the left) prepares a set of $N$ qubits (e.g., electron spins) in the spin-up eigenstate of her $x$ axis (e.g., by passing unpolarized electrons through a $x$-oriented Stern–Gerlach apparatus, and selecting only the ones that emerge with their spins up). She then sends these electrons to Bob, whose laboratory is rotated by an unknown amount with respect to hers. Bob divides these $N$ qubits in three groups, and sends each group through a machine that measures the spin along one of his three orthogonal axes (e.g., three perpendicular Stern–Gerlach apparata - in the picture, he is passing the electrons through a $y$-oriented machine). He then counts the number of spin-up and spin-down measurements that each machine reads, and calculates the expectation value of the corresponding observable. Repeating the experiment for a set of $N$ qubits that Alice selected to be polarized along the $y$−, and, respectively, $z$− axes allows Bob to build a statistics of the expectation values of the nine observables associated to each pair of choices of axes made by him and Alice. In the large-$N$ limit, these expectation values tend to the nine components of the rotation matrix $R$ that connects Alice’s reference frame and Bob’s. Notice that, in this illustrative picture the electron beams (in orange) are manipulated with some ”mirrors” (the black cylinders) which are assumed not to have any effect on the qubit states.

while $\vec{r}$ transforms in the adjoint representation of $SU(2)$ and thus under the fundamental representation of $SO(3)$.

In this manner, we can associate to every $SU(2)$ transformation of states or spin observables an $SO(3)$ spatial rotation. In particular, we can write the same qubit state relative to different bases of Pauli matrices $U \vec{\sigma} U^\dagger$ by simply rotating the corresponding Bloch vector accordingly,

$$\vec{r}_\rho' = \text{Tr} \{ \rho U \vec{\sigma} U^\dagger \} = R^{-1} \cdot \vec{r}_\rho = R^{-1} \cdot \text{Tr} \{ \rho \vec{\sigma} \} ,$$

(49) where $\vec{r}_\rho' \text{ and } \vec{r}_\rho$ are both Bloch vectors of the same quantum state $\rho$, however, written relative to different observable bases. Interpreting $U \vec{\sigma} U^\dagger$ and $\vec{\sigma}$ as being the observables corresponding to differently oriented sets of Stern-Gerlach devices in the lab, this rewriting allows us to express how observers with different spatially oriented reference frames will ‘see’ the same quantum state of (possibly an ensemble of) a qubit.

C.2 Alignment protocol between two laboratories by exchanging qubits

Suppose that Alice and Bob have never met before and thus do not know how their reference frames are aligned with respect to each other. Supposing further that they can communicate classically, they can proceed as follows in order to figure out the relation between their descriptions. Alice prepares
three ensembles of qubits, each prepared such that the ensemble state corresponds to a Bloch sphere basis vector. For example, she could prepare three ensembles of $N$ qubits each so that

$$E_{x_A} = \{ \vec{r}_{x_A,n} = (1, 0, 0) \}_{n=1}^N, \quad E_{y_A} = \{ \vec{r}_{y_A,n} = (0, 1, 0) \}_{n=1}^N, \quad E_{z_A} = \{ \vec{r}_{z_A,n} = (0, 0, 1) \}_{n=1}^N,$$

i.e., every qubit of the first ensemble is prepared to be in the ‘up in $x_A$-direction’ pure state, where $x_A$ is Alice’s $x$-direction, and similarly for the other two ensembles. Alice can then send Bob these three ensembles to perform state tomography on them. Bob could proceed as follows: he divides each ensemble in three subensembles and measures all qubits of the first subensemble with a Stern-Gerlach (SG) device oriented in $x_B$-direction, of the second in $y_B$-direction and of the third in $z_B$-direction. This will tell him how to write Alice’s states relative to his frame and, via (49), what the relative SO(3) rotation between their spatial frames is. For example, relative to Alice, the ensemble state of $E_{x_A}$ appears as

$$\rho_{x_A} = \frac{1}{2}(I + \vec{r}_{x_A} \cdot \vec{r}_A) = \frac{1}{2}(I + \sigma_x A).$$

Suppose now that Bob’s measurement apparatus are in the following relation with Alice’s: $\vec{r}_A = U \vec{r}_B U^\dagger$. Then, using (49), we can write the same ensemble state, but now relative to Bob as

$$\rho_{x_B} = \frac{1}{2}(I + \vec{r}_{x_B} \cdot U \sigma_B U^\dagger) = \frac{1}{2}(I + \vec{r}_{x_B} \cdot \vec{r}_B),$$

where

$$\vec{r}_{x_B} = R \cdot \vec{r}_{x_A}$$

and

$$R_{ij} = \frac{1}{2} \text{Tr}\left\{ (\sigma_A)_i U (\sigma_A)_j U^\dagger \right\} \equiv \frac{1}{2} \text{Tr}\left\{ U (\sigma_A)_j U^\dagger (\sigma_A)_i \right\},$$

which is equivalent to (48) if we remember that $U$ are unitary matrices. Bob can measure $\vec{r}_{x_B} = \text{Tr}\{\rho_{x_B} \vec{r}_B\}$ using tomography on $E_{x_A}$, which is why he has to split that ensemble into three subensembles to measure its qubits relative to a basis of SG devices.

Proceeding this ways also with $E_{y_A}, E_{z_A}$, Bob can figure out the Bloch vector basis $\vec{r}_{x_B}, \vec{r}_{y_B}, \vec{r}_{z_B}$ of Alice’s ensemble states relative to his frame and if Alice also tells him how she describes the same states, namely as $\vec{r}_{x_A}, \vec{r}_{y_A}, \vec{r}_{z_A}$, then they can simply compute $R$ from the relation of these two sets of bases. In principle, Bob has to perform an infinite number of measurements to determine the rotation matrix connecting his reference frame with Alice’s with arbitrary precision. From this, the matrix (25) admits a direct, operational interpretation in terms of (54): it is the matrix of the asymptotic $N \to \infty$ values of the averages of the spin measurements that Bob performs on the qubit states sent to him by Alice.

D Representation $\rho$ as limit of representation $\pi$

By adding a phase to the representation $\pi$ discussed in the mathematical literature [20] we have

$$\rho(a) |\chi\rangle = e^{i\chi} |\chi\rangle \quad \rho(a^*) |\chi\rangle = e^{-i\chi} |\chi\rangle \quad \rho(c) |\chi\rangle = \rho(c^*) |\chi\rangle = 0$$

$$\pi(a) |n, \phi, \epsilon\rangle = e^{i\chi} \cos \left( \frac{\theta(n)}{2} \right) |n-1, \phi, \epsilon\rangle \quad \pi(c) |n, \phi, \epsilon\rangle = e^{i\phi} \sin \left( \frac{\theta(n)}{2} \right) |n, \phi, \epsilon\rangle$$

$$\pi(a^*) |n, \phi, \epsilon\rangle = e^{-i\epsilon} \sqrt{1 - q^2 \cos^2 \left( \frac{\theta(n)}{2} \right)} |n+1, \phi, \epsilon\rangle \quad \pi(c^*) |n, \phi, \epsilon\rangle = e^{-i\phi} \sin \left( \frac{\theta(n)}{2} \right) |n, \phi, \epsilon\rangle$$

with our definition of quantized angle, $\theta(n) = 2 \arcsin q^n$. Looking at (55) and (56), it may seem that we have four Euler angles in the representations of $SU_q(2)$, namely the aforementioned $(\chi, \epsilon, \phi, \theta(n))$. This is a subtlety solved by observing that, with the additional $\epsilon$ phase that we introduced in the $\pi$-representation, the representation $\rho$ can be obtained as a somewhat trivial limit of representation $\pi$. Therefore, we have three physical degrees of freedom, with the angles $\chi$ and $\epsilon$ linked by this limit operation, so that we can ultimately set $\epsilon = \chi$. 

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To see this, we need to find a class of states such that
\[ \pi(a) |\psi(\chi)\rangle \to e^{ix} |\psi(\chi)\rangle , \quad \pi(c) |\psi(\chi)\rangle \to 0 \]  
(57)
in this limit. We can show that states of the type (with the identification \( \epsilon = \chi \))
\[ |\psi_n\rangle = \frac{1}{\sqrt{n+1}} \sum_{k=n}^{2n} |k, \phi, \chi\rangle \]
(58)
from the \( \pi \)-representation satisfy requirement (57) for \( n \to \infty \).
In fact, for the operator \( c \) we can immediately see that
\[ \pi(c) |\psi_n\rangle = e^{ix} \frac{1}{\sqrt{n+1}} \sum_{k=n}^{2n} q^k |k, \phi, \chi\rangle \]
and therefore, since \( q \) is slightly less than one
\[ \langle \psi_n | \pi(c^*) \pi(c) |\psi_n\rangle = \frac{1}{n+1} \sum_{k=n}^{2n} (1 - q^{2k+2}) \to 0 , \quad n \to \infty . \]  
(60)
Hence, \( \pi(c) |\psi_n\rangle \overset{n \to \infty}{\to} 0 \). The limit for the operator \( a \) is more subtle; in this case we have that
\[ \pi(a) |\psi_n\rangle = \frac{e^{ix}}{\sqrt{n+1}} \sum_{k=n}^{2n} \sqrt{1 - q^{2k}} |k-1, \phi, \chi\rangle \]
(61)
and we consider the difference
\[ |\psi'_n\rangle = \pi(a) |\psi_n\rangle - e^{ix} |\psi_n\rangle . \]  
(62)
To show that the limit \( \pi(a) |\psi_n\rangle \to e^{ix} |\psi\rangle \) holds we will show that the succession of the norms of the states in (62) has 0 limit. We have that
\[ 0 \leq \langle \psi'_n | \psi'_n \rangle = \langle \psi_n | \pi(a^*) \pi(a) |\psi_n\rangle + \langle \psi'_n | \psi'_n \rangle - 2 \text{Re} \{ e^{ix} \langle \psi_n | \pi(a^*) |\psi_n\rangle \} = \]
\[ = 1 + \frac{1 - q^{2n+2}}{n+1} - \frac{1}{n+1} \sum_{k=n+1}^{2n} \left[ 2 \sqrt{1 - q^{2k}} - (1 - q^{2k}) \right] \leq 1 + \frac{1}{n+1} - \frac{1}{n+1} \sum_{k=n+1}^{2n} (1 - q^{2k}) = \]
\[ = \frac{1}{n+1} \left[ 2 - q^{2n+2} \frac{1 - q^{2n}}{1 - q^2} \right] \overset{n \to \infty}{\to} 0 . \]  
(63)
This observation that representation \( \pi \) converges for certain states to representation \( \rho \) can be extended to generic functions of operators. In general, such functions can be written as a sum of powers of the operators \( \pi(a), \pi(a^*), \pi(c) \) and \( \pi(c^*) \). We now prove that these functions acting on a class of states in representation \( \pi \) give, in a certain limit, the value that the same function yields in the limit of these states with \( \pi(a) \) replaced by \( \rho(a) \) and so on.
In order to do so, we consider a slightly different class of states than the one in (58). In particular we consider
\[ |\Psi_n\rangle = \frac{1}{\sqrt{n+1}} \sum_{k=n!}^{2n!} |k, \phi, \chi\rangle . \]  
(64)
The reason why we make this choice with the factorial will be clear in the following.
Consider first the case in which we have an operator of the form \( \pi(c)^m \pi(a)^l \). If we act with this operator on the state (64) and we compute the norm of the state obtained in this way we get
\[ 0 \leq \langle \Psi_n | \pi(a^*)^l \pi(c)^m \pi(c)^m \pi(a)^l |\Psi_n\rangle \leq \frac{q^{2m(n-1)} \left[ 1 - q^{2m(n+1)} \right]}{(1 - q^{2m})(n!+1)} \overset{n \to \infty}{\to} 0 . \]  
(65)
The same argument holds if we have only powers of the operator \( \pi(c) \) (i.e., \( l = 0 \)) or other combinations involving \( \pi(a^*), \pi(c^*) \) and different orderings. The case with \( m = 0 \), in which the limit above is 1, corresponds to a power of \( \pi(a) \) which will be handled slightly differently in the following.
Now we turn to powers of the operator \( \pi(a) \), i.e. \( \pi(a)^m \). We do something analogous to the argument used for \( \pi(a) \), namely we define the difference state
$$|\Psi'_n\rangle = \pi(a)^m |\Psi_n\rangle - e^{im\chi} |\Psi_n\rangle$$

(66)

and show its norm goes to zero as $n \to \infty$. After some algebraic manipulations, this norm is found to satisfy

$$0 \leq \langle \Psi'_n | \Psi'_n \rangle \leq \frac{2m}{n!+1} + \frac{1}{n!+1} \sum_{k=n!+m}^{2n!} \left[ q^{l(l+1)} q^{2kl} (-1)^l (q^{-2k}; q^2)_l - 1 \right] \xrightarrow{n \to \infty} 0,$$

(67)

where $(q^{-2k}; q^2)_l := \prod_{s=0}^{l-1} (1 - q^{-2k+2s})$ is the $q$-Pochhammer symbol. The result of the limit comes from the fact that the term in the square brackets above is essentially just a sum of powers of $q$.

We have thus shown that a generic function of $\pi(a)$ and $\pi(c)$ operators applied to (a class of) states in the $\pi$ representation converges to the same function applied to a state in $\rho$ representation (a similar argument holds for functions of $\pi(a^*)$ and $\pi(c^*)$ operators). The reason why we considered a state of the form (64) is now clear: it is necessary to have a number of states in the superposition which goes to infinity faster than $m$ does in order to properly do the limit when considering generic functions, in which case we have a series in $m$.

The states (58) can be used also to show directly for the vectorial (co)-representation that it is possible to obtain the rotations about the $z$-axis as a limit in the representation $\pi$. In the same way, it is possible to obtain the identity, i.e. the null rotation, in the same representation.

To obtain a rotation about the $z$-axis, we must have $\phi = 0$ and we have to consider the large $n$ limit. In this limit, setting $\phi = 0$, we can approximate our $\pi$ representation as

$$\pi(a) |n, \phi, \chi\rangle = e^{i\chi} |n-1, \phi, \chi\rangle \quad \pi(c) = \pi(c^*) = 0 \quad \pi(a^*) |n, \phi, \chi\rangle = e^{-i\chi} |n+1, \phi, \chi\rangle .$$

(68)

Therefore, using the states (58), we obtain

$$\langle \psi | R_q | \psi \rangle = \begin{pmatrix} -1+n \cos \chi & -1+n \sin \chi & 0 \\ -\frac{1+n}{1+n} \sin \chi & \frac{1+n}{1+n} \cos \chi & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

(69)

In the large $n$ limit, we see that (69) approximates a rotation matrix about the $z$-axis with greater and greater precision and it can be shown that the variances are equal to 0. Moreover, $\chi = 0$ in (69) gives the identity matrix in the large $n$ limit.