Braiding in Conformal Field Theory and Solvable Lattice Models

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ABSTRACT

Braiding matrices in rational conformal field theory are considered. The braiding matrices for any two block four point function are computed, in general, using the holomorphic properties of the blocks and the holomorphic properties of rational conformal field theory. The braidings of $SU(N)_k$ with the fundamental are evaluated and are used as examples. Solvable interaction round the face lattice models are constructed from these braiding matrices, and their Boltzmann weights are given. This allows, in particular, for the derivation of the Boltzmann weights of such solvable height models.

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In a recent publication [1] one of the authors has described a universal connection between rational conformal field theory and solvable lattice models. The lattice models are built around any rational conformal field theory, $\mathcal{O}$, along with some primary field $x$. The model so obtained, denoted by $\text{IRF}(\mathcal{O}, x)$, is described by the partition function,

$$Z = \sum_{\text{configurations}} \prod_{\text{faces}} w\left(\begin{array}{cc|c} a & b & u \\ c & d & \end{array}\right),$$

(1)

where $a$, $b$, $c$, and $d$ are four primary fields residing on the vertices, the Boltzmann weight $w\left(\begin{array}{cc|c} a & b & u \\ c & d & \end{array}\right)$ is defined for each of the faces, $u$ is a spectral parameter which labels a family of such models, and the product is over all the faces in the theory. The allowed configurations are limited by the so-called admissibility condition, which is

$$N_{ab}^c N_{bx}^c N_{cx}^d N_{dx}^a > 0,$$

(2)

where $N_{ab}^c$ is the fusion coefficient of the rational conformal field theory $\mathcal{O}$ (see ref [1] for more details). Such lattice models have been termed fusion interaction round the face models, or, in short, fusion IRF. The fusion IRF models are solvable if, and only if, the Boltzmann weights obey the star-triangle equation (STE), which implies that the transfer matrices for different spectral parameters, $u$, are commuting, enabling the exact diagonalization of the transfer matrices.

The problem of finding a solution for the STE is very complicated and only a limited number of solutions were known, in general. Further, it is not guaranteed by any means that a given admissibility condition affords such a solution, and indeed most do not. In ref. [1] a general solution for the Boltzmann weights of the fusion IRF models was described, satisfying the STE relation, for any conformal field theory $\mathcal{O}$ and any primary field $x$. The key to the construction of the Boltzmann weights is using the braiding matrix of the RCFT, which interpolates two different ways of writing the four point function. The resulting models automatically satisfy the required STE relation, along with the precise admissibility condition.
It is noteworthy that all the known solvable lattice models with a second order phase transition point can, in fact, be written as fusion IRF models, and thus the above method recovers all the known solvable lattice models, along with providing a riches of new models. This also allows for the direct introduction of RCFT methods in the analysis of lattice models, which is conceptually new.

It is convenient to define the face transfer matrix \( X_i(u) \) for a given lattice model, by the matrix element of the states on the diagonal,

\[
\langle a_1, a_2, \ldots, a_i, \ldots, a_n | X_i(u) | a_1, a_2, \ldots, a_i', \ldots, a_n \rangle = w \left( \frac{a_{i-1}}{a_i} \frac{a'_i}{a_{i+1}} \bigg| u \right),
\]  

(3)

where the rest of the matrix elements vanish. The STE then translates to the fact that the face transfer matrices \( X_i(u) \) obey the Young–Baxter equation,

\[
X_i(u)X_{i+1}(u+v)X_i(v) = X_{i+1}(v)X_i(u+v)X_{i+1}(u),
\]  

\[
X_i(u)X_j(v) = X_j(v)X_i(u) \quad \text{for } |i - j| \geq 2.
\]  

(4)

A special case of the Yang–Baxter equation (4) is obtained when both spectral parameters \( u, v \) tend to \( i \infty \). In this case, eqs. (4) become the well known relations of the braid group. We conclude that \( X_i = \lim_{u \to i \infty} X_i(u) \) is a generator of the braid group, i.e., it implements the braiding of the \( i \) and \( i + 1 \) strands in a braid.

Such a braid group representation arises naturally in rational conformal field theory (for reviews, see for example, [2, 3, 4]). More precisely, consider the chiral blocks of the four point function in the \( s \) channel,

\[
\mathcal{F}_p(z_1, z_2, z_3, z_4) = \langle \psi_i(z_1)\psi_j(z_2)\psi_k(z_3)\psi_l(z_4) \rangle_p,
\]  

(5)

where the \( \psi_i(z) \) are primary fields in the theory. Here \( p \) stands for the primary field exchanged in the \( s \) channel. There is a natural action of the braid group on the conformal blocks by braiding the points \( z_2 \) and \( z_3 \) (or in general any pair of
points). This action is implemented by a finite-dimensional matrix $C$ called the braiding matrix,

$$
F_p(z_1, z_2, z_3, z_4) = \sum_{p'} C_{p,p'} \begin{bmatrix} j & k \\ i & l \end{bmatrix} F_{p'}(z_1, z_3, z_2, z_4),
$$

(6)

where $i$, $j$, $k$ and $l$ label the four primary fields in the correlation function. Define now, for a fixed primary field $x$, the operator,

$$
\langle a, b, c | X_i | a, d, c \rangle = C_{b,d} \begin{bmatrix} x & x \\ a & c \end{bmatrix}.
$$

(7)

Owing to the fact that $C$ implements the braiding of the conformal blocks, it follows that the $X_i$ obey the braid group relations. Further, the non-vanishing of the conformal block, eq. (5), is precisely equivalent to the admissibility condition eq. (2). We conclude that $X_i$ is a valid fusion IRF face transfer matrix. Albeit, we do not have any spectral parameter in it. Thus we need to construct a solution $X_i(u)$ such that $X_i = \lim_{u \to \pm \infty} X_i(u)$, and $X_i(u)$ obey the star-triangle equation. This can be done in a universal, and essentially unique, way, as described in ref. [1]. Here we shall make the simplifying assumption that the field $x$ is a ‘fundamental’ field, i.e., we shall assume that the operator products $x \cdot x$ and $x \cdot \bar{x}$ have precisely two primary fields in each. In this case the face transfer matrices assume a particularly simple form,

$$
X_i(u) = \sin(\lambda - u) + \sin u H_i,
$$

(8)

where $\lambda = \pi(\Delta_1 - \Delta_2)/2$ is the crossing parameter, and $\Delta_i$ are the conformal dimensions of the fields in the product $x \cdot x$. The $H_i$ are defined as

$$
H_i = e^{-i\lambda} - X_i,
$$

(9)

and obey the $A$-type Hecke algebra relations,

$$
H_i H_{i+1} H_i - H_i = H_{i+1} H_i H_{i+1} - H_{i+1},
$$

$$
H_i H_j = H_j H_i \quad \text{for } |i - j| \geq 2,
$$

(10)

$$
H_i^2 = (2 \cos \lambda) H_i,
$$
from which the STE can be readily verified.

Our purpose in this note is to explicitly compute the braiding matrices appearing in rational conformal field theory. This will then provide concrete expressions for the Boltzmann weights of the fusion IRF models. Using the holomorphic properties of the conformal blocks, we will derive a general result for the braiding of any fundamental field, in any rational conformal field theory, expressing the result in terms of conformal dimensions only. The $SU(N)_k$ height models will be constructed for the purpose of illustration.

Let us, at first, demonstrate the technique by specializing to the case of $\text{IRF}(SU(N)_k, N, N) \equiv \text{IRF}(SU(N)_k, [N])$. Here $SU(N)_k$ denotes the corresponding current algebra, or equivalently the corresponding WZNW model. Also, $[N]$ stands for the fundamental representation of $SU(N)$, of highest weight $\Lambda_{[N]} = \Lambda_{(1)} (\Lambda_{(i)}, i = 1, \ldots, N - 1$, denote the fundamental weights of $SU(N)$); this field has indeed only two primaries in its operator product, $[N] \cdot [N] = \psi_1 + \psi_2$, where $\psi_i$ refer to the representations with the highest weight $2\Lambda_{(1)}$ and $\Lambda_{(2)}$, respectively. We want to derive the braiding matrix $B$ by making use of the fact that the analytic properties of the correlation functions of conformal field theory are to a large extent determined by their singularities. Our main task is to analyze the transformation property of the four point function

$$G(z) = \langle \Lambda_a(z)\Lambda_{(1)}(0)\Lambda_{(1)}(1)\Lambda_b(\infty) \rangle$$

under the substitution $z \mapsto 1 - z$. In eq. (11), $\Lambda$ is used as a short-hand notation for the WZNW primary field $\phi^\Lambda$, which carries the irreducible highest weight module with highest weight $\Lambda$.

We are interested in the case where precisely two chiral blocks contribute to $G$. This means that the $SU(N)$ tensor product

$$\Lambda_a \times \Lambda_{(1)} \times \Lambda_{(1)} \times \Lambda_b$$

must contain precisely two singlets. To be able to describe conveniently a necessary
and sufficient condition for this situation, we introduce some notation. First, we write the weight $\Lambda_a$ in the form

$$\Lambda_a = \sum_{j=1}^{n_a} \Lambda(a_j)$$

(13)

where the labels $a_j$ are ordered such that $a_j \leq a_{j+1}$ for all $j = 1,\ldots,n_a - 1$, and analogously for $\Lambda_b$. Next, we define $\Lambda_{(N)} := \Lambda_{(0)} \equiv 0$, and allow for $1 \leq a_j \leq N$ so that without any loss of generality, it can be assumed that the number of terms appearing in eq. (13), and in the corresponding decomposition of $\Lambda_b$ are identical, $n_a = n_b$.

The condition can now be stated as follows: the tensor product (12) contains exactly two singlets iff the weights $\Lambda_a$ and $\Lambda_b$ are related by

$$(\Lambda_b)^* = \Lambda_{a;l,m} := \Lambda_a - \Lambda_{(a_l)} + \Lambda_{(a_l+1)} - \Lambda_{(a_m)} + \Lambda_{(a_m+1)}$$

(14)

for some integers $l$ and $m$, satisfying $1 \leq l, m \leq n_a$ and $l \neq m$ (in particular, the weight defined this way must be dominant, which implies $a_l < a_{l+1}$ and $a_m < a_{m+1}$). If this condition is fulfilled, the families exchanged in the $s$ and $t$ channels (i.e., those which give rise to the chiral blocks for $z \simeq 0$ and $z \simeq 1$, respectively) correspond to the weights

$$\Lambda_1^{(s)} = \Lambda_1^{(t)} = \Lambda_{a;l} \quad \text{and} \quad \Lambda_2^{(s)} = \Lambda_2^{(t)} = \Lambda_{a;m},$$

(15)

respectively, while in the $u$ channel ($z \simeq \infty$) the weights are, of course, $\Lambda_{(2)}$ and $2\Lambda_{(1)}$. In eq. (15), we introduced the notation

$$\Lambda_{a;l} := \Lambda_a - \Lambda_{(a_l)} + \Lambda_{(a_l+1)}$$

(16)

for $0 \leq l < N$. 
The exponents, i.e., the leading singularities of the chiral blocks of $G$, can be expressed in terms of the conformal dimensions $\Delta(\Lambda)$ of the various primary fields encountered above; they read

at $z = 0$:

\[\begin{align*}
\alpha_1^{(0)} &= -\Delta(\Lambda_a) - \Delta(\Lambda_{a; l}), \\
\alpha_2^{(0)} &= -\Delta(\Lambda_a) - \Delta(\Lambda_{a; l}) + \Delta(\Lambda_{a; m}) + 1;
\end{align*}\]

at $z = 1$:

\[\begin{align*}
\alpha_1^{(1)} &= -\Delta(\Lambda_a) - \Delta(\Lambda_{a; l}), \\
\alpha_2^{(1)} &= -\Delta(\Lambda_a) - \Delta(\Lambda_{a; l}) + \Delta(\Lambda_{a; m});
\end{align*}\]

at $z = \infty$:

\[\begin{align*}
\alpha_1^{(\infty)} &= \Delta(\Lambda_a) - \Delta(\Lambda_{a; l, m}) + \Delta(\Lambda_{2}), \\
\alpha_2^{(\infty)} &= \Delta(\Lambda_a) - \Delta(\Lambda_{a; l, m}) + \Delta(2\Lambda_{1}).
\end{align*}\]

The addition of unity to $\alpha_2^{(0)}$ as compared with $\alpha_2^{(1)}$ corresponds [5] to the fact that, with the natural definition of chiral blocks appropriate for $z$ close to a fixed singular point $z_0$, only one of the leading singularities of a WZNW correlation function at $z_0$ can come from the relevant primary field, whereas any other leading singularity must come from a descendant field. In eq. (17) we have chosen $z_0 = 0$ as the distinguished singular point, corresponding to the blocks at $z \approx 0$; below we will also have to consider $z_0 = 1$ which is tantamount to replacing $\alpha_2^{(0)}$ and $\alpha_2^{(1)}$ by $\alpha_2^{(0)} - 1$ and $\alpha_2^{(1)} + 1$, respectively.

The conformal dimensions of WZNW primary fields are given by $\Delta(\Lambda) = \kappa C(\Lambda)$ with $\kappa = \frac{1}{k + \Lambda}$, where $C$ is the eigenvalue of the quadratic Casimir operator. In the notation used in eq. (13), the Casimir eigenvalue is

\[C(\Lambda_a) = \frac{1}{2} \left[ \sum_{j=1}^{n_a} a_j (N - a_j + 2n_a - 2j + 1) - \frac{1}{N} \left( \sum_{j=1}^{n_a} a_j \right)^2 \right].\]

From this formula we deduce, in particular,

\[C(\Lambda_{a;l}) = C(\Lambda_a) - a_l - n_a + \frac{1}{2} (N - \frac{1}{N}) - \frac{1}{N} \sum_{j=1}^{n_a} a_j\]
\[ C(\Lambda_b) \equiv C(\Lambda_{a;l,m}) = C(\Lambda_a) - a_l - a_m - l - m + 2n_a + N - \frac{2}{N}(1 + \sum_{j=1}^{n_a} a_j) \] (20)

if \( \Lambda_{a;l} \) and \( \Lambda_b \) are of the form eq. (16), respectively eq. (14). Inserting this result into eq. (17), one verifies that

\[ \sum_{i=1,2, j=0,1,\infty} \alpha^{(j)}_i = 1, \] (21)

a relation that indeed must be satisfied [6, 5] by the exponents.

Any conformal field theory correlation function with two chiral blocks satisfies a second order linear differential equation. It can be shown [7] that for WZNW correlators, this differential equation does not possess the so-called apparent singularities; as a consequence [6, 5], the chiral blocks are simple combinations of powers and hypergeometric functions, with the parameters determined uniquely by the exponents eq. (17). In the present situation, we find that (up to an overall factor \([z(1-z)]^{\alpha^{(0)}_1}\) which is irrelevant for the considerations below, and which we suppress in the sequel)

\[ G_1(z) \propto \ _2F_1(\gamma, \delta; 1-\alpha; z), \]

\[ G_2(z) \propto z^\alpha \ _2F_1(\alpha + \gamma, \alpha + \delta; 1 + \alpha; z). \] (22)

Here \(_2F_1(a, b; c; z) \equiv \sum_{j=0}^\infty [\Gamma(a + j)\Gamma(b + j)\Gamma(c)/j!\Gamma(a)\Gamma(b)\Gamma(c + j)]z^j\) is the hypergeometric series. The parameters \( \alpha, \gamma \) and \( \delta \) are given by

\[ \alpha = \alpha^{(0)}_2 - \alpha^{(0)}_1 = 1 + \frac{1}{k + N} a_{lm}, \] (23)

with

\[ a_{lm} := a_l - a_m + l - m. \] (24)
and by
\begin{align*}
\gamma &= \alpha_1^{(0)} + \alpha_1^{(1)} + \alpha_1^{(\infty)} = 1 - \alpha - \frac{1}{k+N}, \\
\delta &= \alpha_1^{(0)} + \alpha_1^{(1)} + \alpha_2^{(\infty)} = 1 - \alpha + \frac{1}{k+N}.
\end{align*}  \tag{25}

In particular, the blocks can be expressed in terms of \(\alpha\) (and hence \(a_{lm}\)) alone,
\begin{align*}
G_1(z) &= {_{2}F_{1}}(1 - \alpha - \frac{1}{k+N}, 1 - \alpha + \frac{1}{k+N}; 1 - \alpha; z), \\
G_2(z) &= \frac{1}{k+N} \mathcal{N}_0 z^\alpha \cdot {_{2}F_{1}}(1 - \frac{1}{k+N}, 1 + \frac{1}{k+N}; 1 + \alpha; z). \tag{26}
\end{align*}

Here we defined
\begin{equation}
\mathcal{N}_0 = -e^{\pi i\alpha} \frac{\Gamma(1 - \alpha)\Gamma(\alpha + \frac{1}{k+N})}{\Gamma(\alpha + 1)\Gamma(1 - \alpha + \frac{1}{k+N})} \sqrt{\frac{s(\alpha + \frac{1}{k+N})}{s(\alpha - \frac{1}{k+N})}} \tag{27}
\end{equation}
and
\begin{equation}
s(x) := \sin(\pi x), \tag{28}
\end{equation}

whereby we fixed both the overall normalization, as well as the relative normalization of the two blocks (which affects the normalization of operator product coefficients) in a manner such as to simplify some of the formulae below.

As noted after eq. (17), the functions eq. (26) are the blocks appropriate for \(z \simeq 0\); for \(z \simeq 1\) we have to replace \(\alpha\) by \(\alpha - 1\), leading to the blocks
\begin{align*}
\tilde{G}_1(z) &= \frac{1}{(k+N)(\alpha - 1)} \mathcal{N}_1 \cdot {_{2}F_{1}}(1 - \alpha - \frac{1}{k+N}, 1 - \alpha + \frac{1}{k+N}; 2 - \alpha; z), \\
\tilde{G}_2(z) &= \alpha \mathcal{N}_0 \mathcal{N}_2 z^{\alpha-1} \cdot {_{2}F_{1}}(-\frac{1}{k+N}, \frac{1}{k+N}; \alpha; z). \tag{29}
\end{align*}

Here we again introduced some normalization constants, \(\mathcal{N}_1\) and \(\mathcal{N}_2\), but this time we are not free to fix them at will. Rather, the normalizations are determined uniquely by the condition (33) below. At the present stage, however, \(\mathcal{N}_1\) and \(\mathcal{N}_2\) cannot yet be determined, inasmuch as we have only used properties of linear differential equations.
We have now gathered all the information needed for the construction of the braiding matrix. This is the matrix $B$ which relates the blocks $G_i(1-z)$ to the blocks $\tilde{G}_i(z)$. To obtain $B$, we only need to apply the standard formula expressing the hypergeometric function at $1-z$ in terms of hypergeometric functions at $z$ (see, e.g., [8, p.108]). Using also functional identities of the Gamma function, in particular $\Gamma(x)\Gamma(1-x) = \pi/s(x)$, we can write the result in the form

$$G_i(1-z) = \sum_{j=1}^{2} B_{ij} \tilde{G}_j(z)$$

for $i = 1,2$, with the two by two matrix

$$B = \frac{s(\frac{1}{k+N})}{s(\alpha)} \begin{pmatrix} -N_1^{-1} & -N_2^{-1}e^{-\pi i \rho} \\ -N_1^{-1}e^{\pi i \rho} & N_2^{-1} \end{pmatrix},$$

where

$$\rho = \frac{1}{s(\frac{1}{k+N})} \sqrt{s(\alpha + \frac{1}{k+N}) s(\alpha - \frac{1}{k+N})}.$$ (32)

It remains to fix the normalization constants $N_1$ and $N_2$. This is done using the close relationship between braiding and monodromy. Namely, $B$ must satisfy

$$DB^2 = 1,$$ (33)

where $D$ is the diagonal matrix

$$D = \exp[-\pi i (\Delta(\Lambda_a) + \Delta(\Lambda_b))] \begin{pmatrix} \exp[2\pi i \Delta(\Lambda_{a;l})] & 0 \\ 0 & \exp[2\pi i \Delta(\Lambda_{a;m})] \end{pmatrix},$$ (34)

i.e.,

$$D = -e^{\pi i /N(k+N)} \begin{pmatrix} e^{-\pi i \alpha} & 0 \\ 0 & e^{\pi i \alpha} \end{pmatrix}.$$ (35)

Inserting eq. (34), along with eq. (31), into the relation eq. (33), we conclude
that \( N_1 = \exp\left[\frac{\pi i}{N(k+N)} - \pi i \alpha\right] \) and \( N_2 = \exp\left[\frac{-\pi i}{N(k+N)} + \pi i \alpha\right] \). Hence

\[
B = -s\left(\frac{1}{k+N}\right)s^{-1}(\alpha) e^{-\pi i/N(k+N)} \begin{pmatrix} e^{\pi i \alpha} & \rho \\ \rho & -e^{-\pi i \alpha} \end{pmatrix}.
\]

This result can be expressed in terms of the quantity \( q := e^{2\pi i/(k+N)} \) and the function

\[
[x] := \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} = \frac{s\left(\frac{x}{k+N}\right)}{s\left(\frac{1}{k+N}\right)}.
\]

Eq. (36) then assumes the form

\[
B = \frac{q^{-1/2N}}{[a_{lm}]} \begin{pmatrix} q^{a_{lm}/2} & \sqrt{[a_{lm} + 1][a_{lm} - 1]} \\ \sqrt{[a_{lm} + 1][a_{lm} - 1]} & -q^{-a_{lm}/2} \end{pmatrix}.
\]

Note that for the special case of \( SU(2), N = 2 \), the braiding matrix agrees precisely with the one computed earlier [10]. As in the case of \( SU(2) \), we can now proceed to construct the lattice model IRF(\( SU(N)_k, N, N \)) using the formalism of section 7 of [1]. The crossing parameter is given by (compare eq. (7.27) of [1])

\[
\lambda = \pi (\Delta_1 - \Delta_2)/2 = \frac{\pi}{k+N},
\]

where \( \Delta_1 \) and \( \Delta_2 \) are the dimensions of the two fields exchanged in this channel. The generator of the Hecke algebra is, as usual,

\[
H = q^{-1/2} - e^{-\pi i(\Delta_1 + \Delta_2 - 2\Delta(\Lambda(1)))} B,
\]

and it obeys the usual Hecke relation,

\[
H^2 = \beta H, \quad \text{where } \beta = 2 \cos \lambda.
\]

The Hecke algebra elements are

\[
H_i = \lambda + e_j \begin{pmatrix} \lambda \lambda + e_k \end{pmatrix} (1 - \delta_{jl}) \frac{[s_{jl}(\lambda + e_j) s_{jl}(\lambda + e_k)]^{1/2}}{s_{jl}(\lambda)},
\]

where \( s_{jl}(\lambda) = \sin\left(\frac{\pi}{N} (e_j - e_l) \cdot \lambda\right) \), and where \( e_j = \Lambda_{(a_j)} - \Lambda_{(a_{j+1})} \). This expression corresponds to a well known representation of the Hecke algebra [11]. From this
representation we find, by substituting it into eq. (8), the Boltzmann weights of the trigonometric lattice IRF model, which are,

\[
\begin{align*}
& a_{+e_l} \
& a_{+2e_l} \quad a_{+e_l} = [1 - u], \\
& a_{+e_l} \quad a_{+e_l} = \frac{[a_{lm} + u]}{[a_{lm}]}, \\
& a_{+e_l + e_m} \quad a_{+e_l} = p \left[ u \right] \frac{\sqrt{a_{lm} + 1} [a_{lm} - 1]}{[a_{lm}]},
\end{align*}
\]

(42)

where \( p = \pm 1 \) corresponds to two different solutions (given by \( B \) or its complex conjugate matrix, which is \([1]\) the braiding matrix of \( SU(N)_{-1} \)). Surprisingly, these are precisely the Boltzmann weights of the \( SU(N) \) models described in ref. [12], at the critical limit, \( p = 0 \). This fully illustrates the connection between IRF models and RCFT, giving, in this particular instance, the Boltzmann weights of the solvable lattice models IRF(SU\((N)_{k}, N, N\)).

One can easily extend the results described in this note to other modular invariants of \( SU(N) \), and their extended algebras. It is known that the same blocks appear in all modular invariants, and that the problem of writing the braiding matrices is a simple sesqui-linear re-juxtapositioning of the conformal blocks (see, for example, \([13, 14]\)). Substituting the so obtained braiding matrices into eq. (8) we would find new solvable IRF models, and an explicit solution for their Boltzmann weights. In fact, the Boltzmann weights of such models are yet to be explored. Another approach for the construction of lattice models, based on graph theory, has been advocated by Pasquier (the D-E cases) \([15]\), and generalized to all \( SU(N) \) by di Francesco and Zuber \([16]\). Though the two approaches are bound to be related, the precise connection requires more study. Many of the other modular invariants can be related to quotients (‘orbifolds’) of the corresponding
conformal field theories [17]. The IRF models built on these should correspond to the quotient procedure of the IRF models [18, 19, 20]. More generally, taking first a quotient of an arbitrary rational conformal field theory and then using equation (8) is equivalent to the IRF quotient procedure as applied to the original theory. Note, however, that taking \( x \) to be a twisted field give rise to an entirely new IRF model, which is not a quotient of the original IRF.

Let us turn now to the generalization of the above calculation of the braiding matrix for any two block correlation function. The calculation is a straightforward extension of the one above, where we assume that there are no apparent singularities.

Consider the correlation function

\[
G(z) = \langle \phi(z)\phi_s(0)\phi_t(1)\phi_u(\infty) \rangle, \tag{43}
\]

where \( \phi, \phi_s, \phi_t \) and \( \phi_u \) are four arbitrary primary fields. Further, assume that this correlation function corresponds to two blocks, i.e., it receives contributions from exactly two fields in each channel,

\[
\phi \cdot \phi_a = \phi_1^{(a)} + \phi_2^{(a)}, \tag{44}
\]

where \( a = s, t, \) or \( u \) label the channel. The exponents of the correlator (43) are expressed in terms of the conformal dimensions,

\[
\begin{align*}
\alpha_i^{(0)} &= -\Delta - \Delta_s + \Delta_i^{(s)} + i - 1, \\
\alpha_i^{(1)} &= -\Delta - \Delta_t + \Delta_i^{(t)}, \\
\alpha_i^{(\infty)} &= \Delta - \Delta_u + \Delta_i^{(u)},
\end{align*}
\tag{45}
\]

for \( i = 1, 2 \), where \( \Delta = \Delta(\phi), \Delta_a = \Delta(\phi_a), \) and \( \Delta_i^{(a)} = \Delta(\phi_i^{(a)}) \). The conformal dimensions appearing here are restricted by the exponent sum rule

\[
\sum_{i=1,2} \sum_{j=0,1,\infty} \alpha_i^{(j)} = 1. \tag{46}
\]

The exponents describe the order of the singularities of the correlation function
\( \mathcal{G}(z) \) at the points \( z = 0 \), \( z = 1 \) and \( z = \infty \). Along with the holomorphicity this implies that \( \mathcal{G} \) is given by hypergeometric functions (suppressing again an overall power factor),

\[
\begin{align*}
\mathcal{G}_1(z) &= \,_{2}F_{1}(\gamma, \delta; 1 - \alpha; z), \\
\mathcal{G}_2(z) &= \,_{2}F_{1}(\alpha + \gamma, \alpha + \delta; 1 + \alpha; z),
\end{align*}
\tag{47}
\]

which are the blocks relevant in \( z \). The blocks relevant in \( 1 - z \) are

\[
\begin{align*}
\tilde{\mathcal{G}}_1(z) &= \,_{2}F_{1}(\gamma, \delta; 1 - \beta; z), \\
\tilde{\mathcal{G}}_2(z) &= \,_{2}F_{1}(\beta + \gamma, \beta + \delta; 1 + \beta; z).
\end{align*}
\tag{48}
\]

Here the parameters \( \alpha, \beta, \gamma \) and \( \delta \) are

\[
\begin{align*}
\alpha &= \alpha_2^{(0)} - \alpha_1^{(0)} = \Delta_2^{(s)} - \Delta_1^{(s)} + 1, \\
\beta &= \alpha_2^{(1)} - \alpha_1^{(1)} = \Delta_2^{(t)} - \Delta_1^{(t)}, \\
\gamma &= \alpha_1^{(0)} + \alpha_1^{(1)} + \alpha_1^{(\infty)} = \Delta_1^{(s)} + \Delta_1^{(t)} + \Delta_1^{(u)} - \Delta - \Delta_s - \Delta_t - \Delta_u, \\
\delta &= \alpha_1^{(0)} + \alpha_1^{(1)} + \alpha_2^{(\infty)} = \Delta_1^{(s)} + \Delta_1^{(t)} + \Delta_2^{(u)} - \Delta - \Delta_s - \Delta_t - \Delta_u.
\end{align*}
\tag{49}
\]

From the sum rule (46), the parameters satisfy

\[
\alpha + \beta + \gamma + \delta = 1.
\tag{50}
\]

It is readily verified that the blocks obey the braiding property,

\[
\mathcal{G}_i(1 - z) = \sum_{j=1}^{2} B_{ij} \tilde{\mathcal{G}}_j(z),
\tag{51}
\]

where the braiding matrix is

\[
B = \begin{pmatrix}
\mathcal{N}_1^{-1} \frac{\Gamma(1 - \alpha) \Gamma(\beta)}{\Gamma(\beta + \gamma) \Gamma(\beta + \delta)} & \mathcal{N}_0^{-1} \mathcal{N}_2^{-1} \frac{\Gamma(1 - \alpha) \Gamma(- \beta)}{\Gamma(\gamma) \Gamma(\delta)} \\
\mathcal{N}_0 \mathcal{N}_1^{-1} \frac{\Gamma(1 + \alpha) \Gamma(\beta)}{\Gamma(1 - \gamma) \Gamma(1 - \delta)} & \mathcal{N}_2^{-1} \frac{\Gamma(1 + \alpha) \Gamma(- \beta)}{\Gamma(\alpha + \gamma) \Gamma(\alpha + \delta)}
\end{pmatrix}
\tag{52}
\]

The normalizations \( \mathcal{N}_i \) need to be determined. This is done using the matrix
equation (33), i.e. \((DB)^2 = 1\), where the diagonal matrix \(D\) is given by
\[
D = \begin{pmatrix} d_1 & 0 \\ 0 & d_2 \end{pmatrix} = \exp[-\pi i(\Delta + \Delta_u)] \begin{pmatrix} \exp[2\pi i \Delta_1^{(s)}] & 0 \\ 0 & \exp[2\pi i \Delta_2^{(s)}] \end{pmatrix}.
\] (53)

From the off diagonal elements of this constraint we find
\[
\mathcal{N}_2 = \frac{d_2 \Gamma(\alpha) \Gamma(-\beta) \Gamma(\beta + \gamma) \Gamma(\beta + \delta)}{d_1 \Gamma(-\alpha) \Gamma(\beta) \Gamma(\alpha + \gamma) \Gamma(\alpha + \delta)} \mathcal{N}_1.
\] (54)

For the diagonal elements we then obtain
\[
((DB)^2)_{11} = ((DB)^2)_{22} = (d_1 \mathcal{N}_1^{-1})^2 \frac{\Gamma(1 - \alpha) \Gamma(\beta) \Gamma(\alpha + \gamma) \Gamma(\alpha + \delta)}{\Gamma(\alpha) \Gamma(1 - \beta) \Gamma(\beta + \gamma) \Gamma(\beta + \delta)}.
\] (55)

Thus,
\[
\mathcal{N}_1 = d_1 \sqrt{\frac{\Gamma(1 - \alpha) \Gamma(\beta) \Gamma(\alpha + \gamma) \Gamma(\alpha + \delta)}{\Gamma(\alpha) \Gamma(1 - \beta) \Gamma(\beta + \gamma) \Gamma(\beta + \delta)}},
\] \[
\mathcal{N}_2 = d_2 \sqrt{\frac{\Gamma(1 + \alpha) \Gamma(-\beta) \Gamma(\beta + \gamma) \Gamma(\beta + \delta)}{\Gamma(-\alpha) \Gamma(1 + \beta) \Gamma(\alpha + \gamma) \Gamma(\alpha + \delta)}}.
\] (56)

Inserting the normalizations into \(B\) one finds, in particular,
\[
d_1 B_{11} = d_2 B_{22} = \sqrt{s(\beta + \gamma) s(\beta + \delta)}
\] \[
\frac{s(\alpha) s(\beta)}{s(\alpha) s(\beta)},
\] (57)

i.e. the diagonal elements of \(B\) can be rewritten purely in terms of phases and \(q\)-integers. For the off-diagonal elements there is still the gauge freedom contained in \(\mathcal{N}_0\); choosing
\[
\mathcal{N}_0 = \sqrt{-\frac{\Gamma^2(-\alpha) \Gamma(1 - \gamma) \Gamma(1 - \delta) \Gamma(\alpha + \gamma) \Gamma(\alpha + \delta)}{\Gamma^2(\alpha) \Gamma(\gamma) \Gamma(\delta) \Gamma(\beta + \gamma) \Gamma(\beta + \delta)}} \frac{d_1}{d_2},
\] (58)

we arrive at
\[
B = \sigma \begin{pmatrix} d & \rho \\ \rho & -d^{-1} \end{pmatrix},
\] (59)
with
\[
\begin{align*}
    d & := \sqrt{d_1/d_2}, \\
    \sigma & := \sqrt{s(\beta + \gamma) s(\beta + \delta)} \times \sqrt{\frac{1}{s(\alpha) s(\beta)^2 d_1 d_2}}, \\
    \rho & := \sqrt{-\frac{s(\gamma) s(\delta)}{s(\beta + \gamma) s(\beta + \delta)}}.
\end{align*}
\]

(60)

Using the above formula, eq. (59), yields the Boltzmann weights for any IRF model based on some rational conformal field theory and a two block primary field. In particular, for SU(N) we recover the result derived above, eq. (38). The calculation can be extended along the same lines to more than two blocks. The three block case, which is applicable to many RCFT (e.g., C_n, B_n, D_n with the fundamental fields, which should recover the height models previously described in ref. [12]), has been considered, and will be reported elsewhere [21].

We hope that the derivations described in this paper will be of help in the general understanding of rational conformal field theory, along with elucidating the connection with solvable lattice interaction round the face models and soliton systems as described in ref. [1]. Substituting different RCFT in the general braid formulae described here, gives rise to a variety of new solvable lattice models which are of definite interest, along with describing a host of new integrable field theories in two dimensions.

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REFERENCES

1. D. Gepner, “Foundations of rational field theory, I”, CALT–68–1825, November, 1992

2. J. Fröhlich, in: Como 1987 proceedings, Differential Geometric Methods in Theoretical Physics, p. 219; B. Schroer, same volume, p. 289; J. Fröhlich, in: Cargese Summer Institute 1987, p. 71; L. Alvarez-Gaumé, same volume, p. 1

3. L. Alvarez-Gaumé, C. Gomez, and G. Sierra, in: The Physics and Mathematics of Strings, Knizhnik Memorial Volume, p. 16

4. G. Moore and N. Seiberg, Trieste Spring School 1989, p.1

5. J. Fuchs, Nucl. Phys. B 328 (1989) 585

6. B. Blok and S. Yankielowicz, Nucl. Phys. B 321 (1989) 717

7. J. Fuchs, Nucl. Phys. B 386 (1992) 343

8. A. Erdélyi, ed., *Higher Transcendental Functions* vol. I (McGraw-Hill, New York 1953)

9. K.-H. Rehren and B. Schroer, Nucl. Phys. B 312 (1989) 715

10. A. Tsuchiya and Y. Kanie, Adv. Studies in Pure Math. 16 (1988) 297

11. H. Wenzl, Representations of Hecke algebras and subfactors, Univ. of Pennsylvania Thesis (1985), Invent. Math. 92 (1988) 349

12. M. Jimbo, T. Miwa and M. Okado, Lett. Math. Phys. 14 (1987) 123

13. M.R. Douglas and S.P. Trivedi, Nucl. Phys. B320 (1989) 461

14. J. Fuchs, Phys. Rev. Lett. 62 (1989) 1705; J. Fuchs and A. Klemm, Ann. Phys. 194 (1989) 303

15. V. Pasquier, J. Phys. A20 (1987) L217

16. P. di Francesco and J.B. Zuber, Nucl. Phys. B338 (1990) 602
17. D. Gepner and E. Witten, Nucl. Phys. B278 (1986) 493

18. I. Kostov, Nucl. Phys. B300 [FS22] (1988) 559

19. P. Fendley and P. Ginsparg, Nucl. Phys. B324 (1989) 549

20. P. Fendley, J. Phys. A22 (1989) 4633

21. J. Fuchs, work in progress