Aleman-Richter-Sundberg’s Theorem On $P^t(\mu)$-Spaces

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Abstract
Let $\nu$ be a finite complex measure with support in $\mathbb{D}$ and let $C\nu$ denote the Cauchy transform of $\nu$. Suppose that $\nu$ annihilates polynomials in complex variable $z$ and $\nu|_{\partial \mathbb{D}} = h\mu$, where $\mu$ is the normalized Lebesgue measure on $\partial \mathbb{D}$. We show that, for $\epsilon_0 > 0$, $m$-almost all $e^{i\theta} \in \partial \mathbb{D}$, and $a > 0$, when $r$ tends to 1, there exists $E_r \subset B(r e^{i\theta}, \frac{1-r}{r})$ with analytic capacity $\gamma(E_r) < \epsilon_0 \frac{1}{1-i}$ such that $|C\nu(\lambda) - e^{-i\theta}h(e^{i\theta})| \leq a$ area-almost all $\lambda \in B(r e^{i\theta}, \frac{1-r}{r}) \setminus E_r$. Using this result, we provide an alternative proof of Aleman-Richter-Sundberg’s Theorem on nontangential limits in $P^t(\mu)$-Spaces and the index of invariant subspaces.

1 Introduction
Let $\mathcal{P}$ denote the set of polynomials in the complex variable $z$. Let $\mu$ be a compactly supported finite positive measure on the complex plane $\mathbb{C}$, and $1 \leq t < \infty$ with conjugate exponent $t' = \frac{t}{t-1}$. We denote by $P^t(\mu)$ the closure of $\mathcal{P}$ in $L^{t'}(\mu)$. Multiplication by $z$ defines a bounded linear operator on $P^t(\mu)$ which we will denote by $S_\mu$. An invariant subspace of $P^t(\mu)$ is a closed linear subspace $M \subset P^t(\mu)$ such that $S_\mu M \subset M$. For a subset $A \subset \mathbb{C}$, we set $\partial A$ or $\text{clos}(A)$ for its closure, $\text{clos}(A)$ for its complement, and $\chi_A$ for its characteristic function. For $\lambda \in \mathbb{C}$ and $\delta > 0$, we set $B(\lambda, \delta) = \{z : |z - \lambda| < \delta\}$ and $\mathbb{D} = B(0,1)$. Let $m$ be the normalized Lebesgue measure $\frac{dt}{2\pi}$ on $\partial \mathbb{D}$. For $0 < \sigma < 1$ and $z \in \partial \mathbb{D}$, we define the nontangential approach region $\Gamma_\sigma(z)$ to be the interior of the convex hull of $\{z\} \cup B(0,\sigma)$. It is well known that the existence of nontangential limits on a set $E \subset \partial \mathbb{D}$ is independent of $\sigma$ up to sets of $m$-measure zero, so we will write $\Gamma(z) = \Gamma_\frac{1}{2}(z)$ a nontangential approach region. $\lambda \in \mathbb{C}$ is a bounded point evaluation for $P^t(\mu)$ if there exists $C > 0$ such that

$$|p(\lambda)| \leq C||p||_{L^{t'}(\mu)}$$

(1-1)

for all $p \in \mathcal{P}$. We use $bpe(P^t(\mu))$ to denote the set of bounded point evaluations for $P^t(\mu)$. A point $\lambda_0 \in \text{int}(bpe(P^t(\mu)))$ is called an analytic bounded point evaluation for $P^t(\mu)$ if there is a neighborhood $B(\lambda_0, \delta) \subset bpe(P^t(\mu))$ of $\lambda_0$ such that (1-1) is uniformly bounded for $\lambda \in B(\lambda_0, \delta)$. We use $\text{abpe}(P^t(\mu))$ to denote the set of analytic bounded point evaluations for $P^t(\mu)$.

Let $\nu$ be a compactly supported finite measure on $\mathbb{C}$. The Cauchy transform of $\nu$ is defined by

$$C\nu(z) = \int \frac{1}{w-z} d\nu(w)$$

for all $z \in \mathbb{C}$ for which $\int \frac{d|x|}{|w-z|} < \infty$. A standard application of Fubini’s Theorem shows that $C\nu \in L^s_{\text{loc}}(\mathbb{C})$ for $0 < s < 2$, in particular, it is defined for area-almost all $z$, and clearly $C\nu$ is analytic in $\mathbb{C}_\infty \setminus \text{spt}\nu$, where $\text{spt}\nu$ denotes the support of $\nu$ and $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$.

For a compact $K \subset \mathbb{C}$ we define the analytic capacity of $K$ by

$$\gamma(K) = \sup |f'(\infty)|$$

where the sup is taken over those functions $f$ analytic in $\mathbb{C}_\infty \setminus K$ for which $|f(z)| \leq 1$ for all $z \in \mathbb{C}_\infty \setminus K$, and $f'(\infty) = \lim_{z \rightarrow \infty} [f(z) - f(\infty)]$. The analytic capacity of a general $E \subset \mathbb{C}$ is defined to be

$$\gamma(E) = \sup \{\gamma(K) : K \subset E, \ K \text{ compact}\}.$$
Good sources for basic information about analytic capacity are [Garnett (1974), Chapter VIII of Gamelin (1969), Chapter V of Conway (1991), and Tolsa (2014)]. Thomson (1991) proves a remarkable structural theorem for $P^t(\mu)$. 

**Thomson’s Theorem.** There is a Borel partition $\{\Delta_i\}_{i=0}^{\infty}$ of $spt\mu$ such that the space $P^t(\mu|_{\Delta_i})$ contains no nontrivial characteristic functions and 

$$P^t(\mu) = L^t(\mu|_{\Delta_0}) \oplus \bigoplus_{i=1}^{\infty} P^t(\mu|_{\Delta_i}).$$

Furthermore, if $U_i$ is the open set of analytic bounded point evaluations for $P^t(\mu|_{\Delta_i})$ for $i \geq 1$, then $U_i$ is a simply connected region and the closure of $U_i$ contains $\Delta_i$.

Conway and Elias (1993) extends some results of Thomson’s Theorem to the space $R^t(K, \mu)$, the closure of rational functions with poles off $K$ in $L^t(\mu)$, while Brennan (2008) expresses $R^t(K, \mu)$ as a direct sum that includes both Thomson’s theorem and results of (Conway and Elias (1993)). For a compactly supported complex measure $\nu$ of $\mathbb{C}$, by estimating analytic capacity of the set $\{\lambda: |C\nu(\lambda)| \geq c\}$, Brennan (2006, English), Aleman et al. (2009), and Aleman et al. (2011) provide interesting alternative proofs of Thomson’s theorem. Both their proofs rely on X. Tolsa’s deep results on analytic capacity. The author refines the estimations for Cauchy transform, in Lemma 4 of Yang (2018), to study the bounded point evaluations for rationally multicyclic subnormal operators. Our following theorem extends the estimations for Cauchy transform of $\nu$, $spt(\nu) \subset \hat{D}$, near $\partial D$.

**Theorem 1.** Let $\nu$ be a finite complex measure with support in $\hat{D}$. Suppose that $\nu \perp P$ and $\nu|_{\partial D} = hm$. Then for $\epsilon_0 > 0$, $m$-almost all $e^{i\theta} \in \partial D$, and $a > 0$, there exist $0 < \epsilon < 1$ and $E_\theta \subset B(re^{i\theta}, \delta)$ for $r_\theta < r < 1$ and $\delta = \frac{1}{r_\theta}$, such that $\gamma(E_\theta) < \epsilon_0 \delta$ and

$$|C\nu(\lambda) - e^{-i\theta}h(e^{i\theta})| \leq a$$

area-almost all $\lambda \in B(re^{i\theta}, \delta) \setminus E_\theta$.

Because of Thomson’s decomposition, the study of general $P^t(\mu)$ can be reduced to the case where $P^t(\mu)$ is irreducible (contains no nontrivial characteristic functions) and abpe($P^t(\mu)$) is a nonempty simply connected open set whose closure contains $spt\mu$. Olin and Yang (1993) shows that one can use the Riemann Mapping Theorem to further reduce to the case where $abpe(P^t(\mu)) = D$. In this case, Aleman et al. (2009) obtained the following remarkable structural theorem.

**Theorem 2.** (Aleman-Richter-Sundberg’s Theorem) Suppose that $\mu$ is a finite positive measure supported in $\hat{D}$ and is such that $abpe(P^t(\mu)) = D$ and $P^t(\mu)$ is irreducible, and that $\mu(\partial D) > 0$. Then:

(a) If $f \in P^t(\mu)$ then the nontangential limit $f^*(z)$ of $f$ exists for $\mu|_{\partial D}$-almost all $z$, and $f^* = f|_{\partial D}$ as elements of $L^t(\mu|_{\partial D})$.

(b) Every nonzero invariant subspace of $P^t(\mu)$ has index 1.

In this paper, using Theorem 1, we provide an alternative proof of Aleman-Richter-Sundberg’s Theorem (Theorem 2). We present the detail proofs of Theorem 1 and Theorem 2 in section 2. The main difficulty in the proof of Theorem 1 in Aleman et al. (2009), is the proof of the following inequality:

$$\lim_{\Gamma(z) \ni \lambda \to z^+} (1 - |\lambda|^2)^\frac{1}{2} M_\lambda \leq \frac{C}{h(z)^2}$$

(1-2)

for $m$-almost all $z \in \partial D$, where $C$ is some constant and $M_\lambda = \sup_{\rho \in \partial D} \frac{|C(\rho)|}{|\rho|^{1/2}L_1(\rho)}$. Our proof does not depend on the inequality (1-2). However, we will also show that Theorem 4 can be used to prove (1-2).

### 2 The Proofs

A related capacity, $\gamma_+$, is defined for $E \subset \mathbb{C}$ by

$$\gamma_+(E) = \sup \|\mu\|$$

where the sup is taken over positive measures $\mu$ with compact support contained in $E$ for which $\|C\mu\|_{L^\infty(\mathbb{C})} \leq 1$. Since $C\mu$ is analytic in $\mathbb{C} \setminus spt\mu$ and $(C\mu)'(\infty) = \|\mu\|$, we have

$$\gamma_+(E) \leq \gamma(E)$$
Lemma 3. Suppose \( \lambda \) area-almost all for all \( \lambda \) and \( \int \)

(2-1)

for all \( E \subset C \). The following semiaadditivity of analytic capacity is a conclusion of Tolsa’s Theorem.

\[
\gamma \left( \bigcup_{i=1}^{m} E_i \right) \leq A_T \sum_{i=1}^{m} \gamma(E_i)
\]

(2-2)

where \( E_1, E_2, ..., E_m \subset C \).

Let \( \nu \) be a compactly supported finite measure on \( C \). For \( \epsilon > 0 \), \( C_\epsilon \nu \) is defined by

\[
C_\epsilon \nu(z) = \int_{|w-z|>\epsilon} \frac{1}{w-z} d\nu(w),
\]

and the maximal Cauchy transform is defined by

\[
C_\epsilon \nu(z) = \sup_{\epsilon > 0} |C_\epsilon \nu(z)|.
\]

The 1-dimensional radial maximal operator of \( \nu \) (see also (2.7) in Tolsa (2014)) is defined by

\[
M_{R\nu}(z) = \sup_{r>0} \left| \frac{\nu(B(z,r))}{r} \right|
\]

Lemma 1. There is an absolute positive constant \( C_T \), for \( a > 0 \), we have

(1)

\[
\gamma(\{C_\epsilon \nu \geq a\}) \leq \frac{C_T}{a} \|\nu\|
\]

(2-3)

\[
m(\{M_{R\nu} \geq a\}) \leq \frac{C_T}{a} \|\nu\|.
\]

Proof: (1) follows from Proposition 2.1 of Tolsa (2002) and Tolsa’s Theorem (2-1) (also see Tolsa (2014) Proposition 4.16). Theorem 2.6 in Tolsa (2014) implies (2).

The following lemma is due to Lemma 1 in Kriete and Trent (1977).

Lemma 2. Suppose \( \nu \) is a finite positive measure supported on \( D \), then

\[
\lim_{\Gamma(\epsilon i \theta) \ni \lambda \rightarrow e^{i\theta}} \int_D \frac{1 - |\lambda|^2}{|1 - \lambda z|^2} d\nu(z) = 0
\]

for \( m \)-almost every \( e^{i\theta} \).

Lemma 3. Suppose \( \nu \) is a finite measure supported in \( \bar{D} \), \( \nu_1 = \nu|_D \), and \( \nu_2 = \nu|_{\partial D} = h m \). For \( \epsilon_0 > 0 \), let \( e^{i\theta} \in \partial D \), \( a > 0 \), \( \frac{1}{2} < r_0 < 1 \), \( \lambda_0 = r_0 e^{i\theta} \), \( \delta = \frac{\lambda_0 a}{4} \), \( M_1 > 0 \), and \( N = \max(40, \frac{200 M_1}{a}) \) satisfy

\[
M_1 = M_{R\nu}(e^{i\theta}) < \infty,
\]

(2-5)

\[
\left| C_{\nu_2}(\lambda) - C_{\nu_2}(\frac{1}{\lambda_0}) - e^{-i\theta} h(e^{i\theta}) \right| < \frac{a}{4},
\]

(2-6)

and

\[
\int_D \frac{1 - |\lambda|^2}{|1 - \lambda z|^2} d|\nu_1|(z) < \frac{a^2 \epsilon_0^2}{400 C_\nu^2 N M_1},
\]

(2-7)

for all \( \lambda \in B(\lambda_0, \delta) \). Then there exists \( E_\delta \subset B(\lambda_0, \delta) \) such that \( \gamma(E_\delta) < \epsilon_0 \delta \) and

\[
\left| C_{\nu}(\lambda) - C_{\nu}(\frac{1}{\lambda_0}) - e^{-i\theta} h(e^{i\theta}) \right| \leq a
\]

area-almost all \( \lambda \in B(\lambda_0, \delta) \setminus E_\delta \).
Proof: Let $\nu_3 = \frac{\chi\nu_0}{1 - \lambda_0 z}$. For $\epsilon < \delta$ and $\lambda \in B(\lambda_0, \delta)$, we get:

$$B(\lambda, \epsilon) \subset B(\lambda_0, 2\delta) \subset B(e^{i\theta}, N\delta)$$

and

$$|C_{\epsilon} \nu(\lambda) - C_{\epsilon} \nu(1 - \lambda_0)| - e^{-i\theta}h(e^{i\theta})|$$

$$\leq |1 - \lambda_0| \left| \int_{|z - \lambda| > \delta} \frac{d\nu_1}{(z - \lambda)(1 - \lambda_0 z)} \right| + \left| C_{\epsilon} \nu(\lambda) - C_{\epsilon} \nu(1 - \lambda_0) - e^{-i\theta}h(e^{i\theta}) \right|$$

$$\leq 9\delta \int_{B(e^{i\theta}, N\delta)} \frac{d\nu_1}{(z - \lambda)(1 - \lambda_0 z)} + 9\delta \int_{|z - \lambda| > \delta} \frac{d\nu_3}{(z - \lambda)} + \frac{a}{4}$$

$$\leq 9\delta \sum_{k=0}^{\infty} \int_{2^k N\delta < |z - \lambda| < 2^{k+1} N\delta} \frac{1}{|z - \lambda||1 - \lambda_0 z|} d[\nu_1]|(z) + 9\delta |C_{\epsilon} \nu_3(\lambda) + \frac{a}{4}$$

$$\leq 18\delta \sum_{k=0}^{\infty} \frac{2^{k+1} N\delta}{(2^k N\delta - 5\delta)(2^k N\delta - 8\delta)} M_1 = 9\delta |C_{\epsilon} \nu_3(\lambda) + \frac{a}{4}$$

$$\leq \delta' + 9\delta |C_{\epsilon} \nu_3(\lambda)$$

where (2-5), (2-6), and the definition of $N$ are used in above calculation. Let $E_\delta = \{ \lambda : C_{\epsilon} \nu_3(\lambda) \geq \frac{a}{18\delta} \} \cap \bar{B}(\lambda_0, \delta)$, then from (2-8), we get

$$\{ \lambda : |C_{\epsilon} \nu(\lambda) - C_{\epsilon} \nu(1 - \lambda_0) - e^{-i\theta}h(e^{i\theta}) | \geq a \} \cap \bar{B}(\lambda_0, \delta) \subset E_\delta.$$ 

From (2-3), (2-7), and Holder’s inequality, we get

$$\gamma(E_\delta) \leq \frac{18C_7\delta}{a} \int_{B(\epsilon^{i\theta}, N\delta)} \frac{d[\nu_1]}{|1 - \lambda_0 z|^2} \leq \frac{18C_7\delta}{a} \sqrt{N} \left( \int_{B(\epsilon^{i\theta}, N\delta)} \frac{1}{|1 - \lambda_0 z|^2} d[\nu_1]|(z) \right)^{1/2} M_1^+ < \epsilon_0\delta.$$ 

On $B(\lambda_0, \delta) \setminus E_\delta$, for $\epsilon < \delta$, we conclude that

$$|C_{\epsilon} \nu(\lambda) - C_{\epsilon} \nu(1 - \lambda_0) - e^{-i\theta}h(e^{i\theta}) | < a.$$ 

The lemma follows since $\lim_{\epsilon \to 0} C_{\epsilon} \nu(\lambda) = C_{\nu}(\lambda)$ a.e. area.

**Proof of Theorem 1**: Let $E_1 = \{ e^{i\theta} : \int M_R \nu(e^{i\theta}) = \infty \}$, then $m(E_1) = 0$ by Lemma 1 (2). Using Plemelj’s formula (see page 56 of Cima et al. (2006) or Theorem 8.8 in Tolsa (2014)), we can find $E_2 \subset \partial\mathbb{D}$ with $m(E_2) = 0$ such that

$$\lim_{\Gamma(e^{i\theta}) \ni \theta \to e^{i\theta}} C_{\epsilon} \nu_2(z) = \lim_{\Gamma(e^{i\theta}) \ni \theta \to e^{i\theta}} C_{\epsilon} \nu(1 - \frac{z}{2}).$$

for $e^{i\theta} \in \partial\mathbb{D} \setminus E_2$. By Lemma 2 there exists $E_3 \subset \partial\mathbb{D}$ with $m(E_3) = 0$ so that (2-4) holds for $\{|\nu_1|\}$ and $e^{i\theta} \in \partial\mathbb{D} \setminus E_3$. Set $E_0 = E_1 \cup E_2 \cup E_3$. Therefore, for $e^{i\theta} \in \partial\mathbb{D} \setminus E_0$, there exists $0 < \theta < 1$ such that for $\theta < \theta < 1$, $\lambda_0 = re^{i\theta}$, and $\delta = \frac{1}{2}e^{-\theta}$, the conditions (2-5), (2-6), and (2-7) of Lemma 2 are met. The theorem now follows from Lemma 2 since $C_{\nu}(\lambda) = 0$.

The following Lemma is from Lemma B in Aleman et al. (2009) (also see Lemma 3 in Yang (2018)).

**Lemma 4.** There are absolute constants $\epsilon_1 > 0$ and $C_1 < \infty$ with the following property. For $R > 0$, let $E \subset B(\lambda_0, R)$ with $\gamma(E) < R\epsilon_1$. Then

$$|p(\lambda)| \leq \frac{C_1}{R^2} \int_{B(\lambda_0, R) \setminus E} | \frac{p}{\pi} | \frac{dA}{\pi}$$

for all $\lambda \in B(\lambda_0, \frac{R}{2})$ and $p \in A(\lambda_0, R)$, the uniform closure of $P$ in $C(B(\lambda_0, R))$. 

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Now suppose that \( abe(P^k(\mu)) = D \) and \( P^k(\mu) \) is irreducible, and that \( \mu|_{\partial D} = hm \) with \( \mu(\partial D) > 0 \). From Lemma VII.1.7 in [Conway (1991)], we find a function \( G \in P^k(\mu)_c \subset L^f(\mu) \) such that \( G(z) \neq 0 \) for \( \mu \)-almost every \( z \in \partial D \). Every \( f \in P^k(\mu) \) is analytic on \( D \) and

\[
f(\lambda)C(G(\mu))(\lambda) = \int \frac{f(z)}{z-\lambda} G(z)d\mu(z) = C(fG(\mu))(\lambda) \tag{2-9}
\]

for area-almost all \( \lambda \in D \).

**Proof of Theorem 2 (a):** Let \( 1 > \epsilon > 0 \) and \( \epsilon_0 = \frac{\epsilon}{2A_F} \), where \( \epsilon_1 \) is in Lemma 1 and \( A_F \) is from (2-2). For \( f \in P^k(\mu) \), from Theorem 1 we see that for \( m \)-almost all \( e^{i\theta} \) with \( G(e^{i\theta})h(e^{i\theta}) \neq 0, a = \frac{\|e^{i\theta}\|\mu(e^{i\theta})}{2(1+\|e^{i\theta}\|\mu(e^{i\theta}))} > 0 \), there exist \( \frac{1}{2} < \theta_0 < 1, E^1_3 \subset B(re^{i\theta}, \delta), \) and \( E^2_3 \subset B(re^{i\theta}, \delta) \), where \( r_0 < r < 1 \) and \( \delta = \frac{1}{24} \), such that \( \gamma(E^1_3) < \epsilon_0\delta, \gamma(E^2_3) < \epsilon_0\delta, \)

\[
\left| C(G(\mu)) - G(e^{i\theta})e^{-i\theta}h(e^{i\theta}) \right| \leq a
\]

area-almost all \( \lambda \in B(re^{i\theta}, \delta) \setminus E^1_3, \) and

\[
\left| C(fG(\mu))(\lambda) - f(e^{i\theta})G(e^{i\theta})e^{-i\theta}h(e^{i\theta}) \right| \leq a
\]

area-almost all \( \lambda \in B(re^{i\theta}, \delta) \setminus E^2_3 \). Set \( E_3 = E^1_3 \cup E^2_3 \), then from the semiadditivity (2-2), we get

\[
\gamma(E_3) \leq A_F(\gamma(E^1_3) + \gamma(E^2_3)) < \epsilon_1\delta.
\]

Therefore, by (2-9), on area-almost everywhere \( B(re^{i\theta}, \delta) \setminus E_3, \)

\[
\left| \frac{f(\lambda) - f(e^{i\theta})}{f(\lambda)C(G(\mu))(\lambda)} \right| \leq \frac{2}{G(e^{i\theta})h(e^{i\theta})} \left( \left| C(fG(\mu)) - f(e^{i\theta})G(e^{i\theta})e^{-i\theta}h(e^{i\theta}) \right| + \left| C(G(\mu)) - G(e^{i\theta})e^{-i\theta}h(e^{i\theta}) \right| \right)
\]

\[
\leq \epsilon.
\]

Using Lemma 4 for \( p = f - f(e^{i\theta}) \), we get \( |f(\lambda) - f(e^{i\theta})| \leq C_1\epsilon \) for every \( \lambda \in B(re^{i\theta}, \frac{1}{2}) \). Hence,

\[
\lim_{\lambda \to e^{i\theta}} f(\lambda) = f(e^{i\theta}).
\]

This completes the proof.

**Proof of Theorem 2 (b):** Let \( M \) be a nonzero invariant subspace of \( P^k(\mu) \). We must show that \( \text{dim}(M) = 1 \). Let \( n \) be the smallest integer such that \( f(z) = z^n f_0(z) \) for every \( f \in M \) and there exists \( g \in M \) with \( g(z) = z^n g_0(z) \) and \( g_0(0) \neq 0 \). We only need to show \( \frac{f(z)}{f(\lambda)g(z)} \in M \). To do this, it is suffice to show that for \( \phi \in M^\perp \subset L^f(\mu) \), the function

\[
\Phi(\lambda) = \int \frac{g(\lambda)f(z) - f(\lambda)g(z)}{z-\lambda} \phi(z)d\mu(z),
\]

which is analytic on \( D \), is identically zero. In fact, the proof is similar to that of (a). Let \( E \subset \partial D \) so that for \( e^{i\theta} \in E, f \) and \( g \) have nontangential limits at \( e^{i\theta} \), and \( h(e^{i\theta}) > 0 \). By Theorem 2 (a), \( m(E) > 0 \). For \( 1 > \epsilon > 0 \) and \( \epsilon_0 = \frac{\epsilon}{2A_F} \), applying Theorem 1 for \( f\phi\mu, g\phi\mu \) since \( f\phi\mu, g\phi\mu \perp P \) and Theorem 2 (a) for \( f \) and \( g \), we see that for \( e^{i\theta} \in E \) and \( a = \frac{1}{(1+|f(e^{i\theta})|+|g(e^{i\theta})|)(1+|\phi(e^{i\theta})|h(e^{i\theta}))} \), there exist \( \frac{1}{2} < \theta_0 < 1, E^1_3 \subset B(re^{i\theta}, \delta), \) and \( E^2_3 \subset B(re^{i\theta}, \delta) \), where \( r_0 < r < 1 \) and \( \delta = \frac{1}{24} \), such that \( \gamma(E^1_3) < \epsilon_0\delta, \gamma(E^2_3) < \epsilon_0\delta, \)

\[
\left| C(f\phi\mu)(\lambda) - f(e^{i\theta})\phi(e^{i\theta})e^{-i\theta}h(e^{i\theta}) \right| \leq a
\]

area-almost all \( \lambda \in B(re^{i\theta}, \delta) \setminus E^1_3, \) and

\[
\left| C(g\phi\mu)(\lambda) - g(e^{i\theta})\phi(e^{i\theta})e^{-i\theta}h(e^{i\theta}) \right| \leq a,
\]
area-almost all $\lambda \in B(re^{i\theta}, \delta) \setminus E_\delta^2$, $|f(\lambda) - f(e^{i\theta})| < a$ and $|g(\lambda) - g(e^{i\theta})| < a$ on $B(re^{i\theta}, \delta)$. Set $E_\delta = E_\delta^1 \cup E_\delta^2$, then by the semiadditivity (2-2) again, we have $\gamma(E_\delta) < \epsilon_1 \delta$. Therefore, on area-almost everywhere $B(re^{i\theta}, \delta) \setminus E_\delta$,

$$\begin{align*}
|\Phi(\lambda)| &\leq |g(\lambda)| \left| C(f \phi \mu)(\lambda) - f(e^{i\theta})\phi(e^{i\theta})e^{-i\theta}h(e^{i\theta}) \right| + |f(\lambda)| \left| C(g \phi \mu)(\lambda) - g(e^{i\theta})\phi(e^{i\theta})e^{-i\theta}h(e^{i\theta}) \right| \\
&\quad + |f(\lambda)g(e^{i\theta}) - g(\lambda)f(e^{i\theta})|\phi(e^{i\theta})h(e^{i\theta}) \\
&\leq (a + |f(e^{i\theta})| + |g(e^{i\theta})|) (1 + |\phi(e^{i\theta})|) ah(e^{i\theta})a \\
&\leq \epsilon.
\end{align*}$$

Using Lemma 4 for $p = \Phi$, we conclude that $|\Phi(\lambda)| \leq C_1 \epsilon$ for every $\lambda \in B(re^{i\theta}, \delta)$. Hence,

$$\lim_{\Gamma_{\frac{1}{2}}(e^{i\theta}) \ni \lambda \rightarrow e^{i\theta}} \Phi(\lambda) = 0.$$ 

Let $G = \cup_{e^{i\theta} \in \Gamma_{\frac{1}{2}}(e^{i\theta})}$, then $\partial G$ is a rectifiable Jordan curve, $E \subset \partial G$, and $\Phi(\lambda)$ is analytic in $G$. Therefore $\Phi(\lambda) = 0$ since $m(E) > 0$. The theorem is proved.

**Proof of (1-2):** By Theorem 1 for $m$-almost all $e^{i\theta}$ with $G(e^{i\theta})h(e^{i\theta}) \neq 0$, there exist $\frac{1}{2} < r_\theta < 1$ and $E_\delta \subset B(re^{i\theta}, \delta)$, where $r_\theta < r < 1$ and $\delta = \frac{1}{r_\theta^2}$, such that $\gamma(E_\delta) < \epsilon_1 \delta$, and

$$|C(G\mu)(\lambda) - G(e^{i\theta})h(e^{i\theta})| \leq \frac{|G(e^{i\theta})h(e^{i\theta})|}{2}$$

area-almost all $\lambda \in B(re^{i\theta}, \delta) \setminus E_\delta$. We will use $C_1, C_2, \ldots$ for constants in the following calculations. Using Lemma 4 and (2-9), for $\lambda \in B(re^{i\theta}, \frac{1}{2})$ and $p \in \mathcal{P}$, we have

$$|p(\lambda)| \leq \frac{C_1}{\delta} \int_{B(re^{i\theta}, \delta) \setminus E_\delta} |p(z)| \frac{|dA(z)|}{\pi} \leq \frac{C_1}{\delta^2} \int_{B(re^{i\theta}, \delta) \setminus E_\delta} \frac{|C(pG\mu)(z)|}{|C(G\mu)(z)|} \frac{|dA(z)|}{|dA(z)|} \leq \frac{2C_1}{\pi |G(e^{i\theta})h(e^{i\theta})| \delta^2} \int_{B(re^{i\theta}, \delta)} \frac{1}{|z - \lambda|^2} |dA(z)| |G(w)| |d\mu(w)| \leq \frac{C_2}{|G(e^{i\theta})h(e^{i\theta})| \delta} \int |p(w)||G(w)||d\mu(w),$$

and hence,

$$(1 - |\lambda|^2) |p(\lambda)| \leq \frac{C_3}{|G(e^{i\theta})h(e^{i\theta})|} \int |p(w)||G(w)||d\mu(w). \tag{2-10}$$

For $t = 1$, we have

$$\lim_{\Gamma_{\frac{1}{2}}(e^{i\theta}) \ni \lambda \rightarrow e^{i\theta}} (1 - |\lambda|^2) M_\lambda \leq \frac{C_3 \|G\|_{L^\infty(\mu)}}{|G(e^{i\theta})|} \frac{1}{h(e^{i\theta})}. $$

For $t > 1$, replacing $p(z)$ by $(\frac{1}{1 - |\lambda|^2})^{-\frac{t}{2}} p(z)$ in (2-10) and applying Holder’s inequality, we get

$$(1 - |\lambda|^2)^{1 - \frac{t}{2}} |p(\lambda)| \leq \frac{C_3}{|G(e^{i\theta})h(e^{i\theta})|^{\frac{t}{2}}} \left( \int |G(w)|^t \frac{|d\mu(w)|}{|1 - \lambda w|^2} \right)^{\frac{1}{t}},$$

and hence,

$$(1 - |\lambda|^2)^{\frac{t}{2}} M_\lambda \leq \frac{C_3}{|G(e^{i\theta})h(e^{i\theta})|^\frac{t}{2}} \left( \int \frac{1 - |\lambda|^2}{|1 - \lambda w|^2} |G(w)|^t |d\mu(w)| \right)^{\frac{1}{t}}$$

where $\lambda \in B(re^{i\theta}, \frac{1}{2})$. Since $\frac{1 - |\lambda|^2}{|1 - \lambda w|^2}$ is Poisson kernel on $\partial \mathbb{D}$, by Lemma 2 and Fatou’s Theorem, we conclude

$$\lim_{\Gamma_{\frac{1}{2}}(e^{i\theta}) \ni \lambda \rightarrow e^{i\theta}} (1 - |\lambda|^2)^{\frac{t}{2}} M_\lambda \leq \frac{C_3}{|G(e^{i\theta})h(e^{i\theta})|^\frac{t}{2}} \left( \int (G(e^{i\theta})^t h(e^{i\theta}) \right)^{\frac{1}{t}} = \frac{C_3}{h(e^{i\theta})^\frac{t}{2}}$$

for $m$-almost all $e^{i\theta} \in \partial \mathbb{D}$. 

6
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