About the Almeida-Thouless transition line in the Sherrington-Kirkpatrick mean field spin glass model

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In this short note, we consider the Sherrington-Kirkpatrick mean field spin glass model \cite{1,2} and we prove that, in the thermodynamic limit $N \to \infty$, the quenched free energy per site is strictly greater than the corresponding replica symmetric approximation \cite{1}, for values of temperature and magnetic field below the Almeida-Thouless line \cite{3}. This is a simple consequence of rigorous bounds, discovered by F. Guerra \cite{4}, which relate the true quenched free energy to the Parisi solution with replica symmetry breaking \cite{5}.

Consider the system at temperature $\beta^{-1}$ and magnetic field $h$, and recall that the Almeida-Thouless critical line is defined by the condition

$$\beta^2 \int d\mu(z) \frac{1}{\cosh^4(z\beta\sqrt{\bar{q}} + \beta h)} = 1,$$

where $d\mu(z)$ is a unit centered Gaussian measure and the Sherrington-Kirkpatrick order parameter $\bar{q}(\beta, h)$ is the unique \cite{6} solution of

$$\bar{q} = \int d\mu(z) \tanh^2(z\beta\sqrt{\bar{q}} + \beta h).$$

The Parisi solution \cite{5} is defined as

$$\bar{\alpha}_P(\beta, h) = \inf_{x \in \mathcal{X}} \bar{\alpha}(\beta, h; x),$$

where $\mathcal{X}$ is the space of functional order parameters, i.e., of non decreasing functions

$$x : q \in [0, 1] \to x(q) \in [0, 1],$$

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and $\bar{\alpha}(\beta, h; x)$ is defined as
\[ \bar{\alpha}(\beta, h; x) = \ln 2 + f(0, h; x, \beta) - \frac{\beta^2}{2} \int_{0}^{1} q x(q) dq. \] (4)

$f(q; x, \beta)$ is the solution of the antiparabolic equation
\[ \partial_q f(q, y; x, \beta) + \frac{1}{2} \left( \partial^2_y f(q, y; x, \beta) + x(q) \partial_y f(q, y; x, \beta) \right)^2 = 0 \] (5)
with final condition
\[ f(1, y; x, \beta) = \ln \cosh(\beta y). \] (6)
The equation for $f$ can be easily solved if $x(q)$ is piecewise constant. For instance, if one takes
\[ \begin{cases} 
  x(q) = 0 & q \in [0, \bar{q}] \\
  x(q) = 1 & q \in (\bar{q}, 1], 
\end{cases} \] (7)
one finds that $\bar{\alpha}(\beta, h, x)$ is the so called replica symmetric solution
\[ \bar{\alpha}(\beta, h) = \ln 2 + \frac{\beta^2}{4} (1 - \bar{q})^2 + \int d\mu(z) \ln \cosh(z\beta\sqrt{\bar{q}} + \beta h). \] (8)

One expects the quenched free energy per site $F_N(\beta, h)$ to be related to the Parisi solution by
\[ -\lim_{N \to \infty} \beta F_N(\beta, h) = \bar{\alpha}_P(\beta, h), \] (9)
where $N$ is the size of the system. While the rigorous proof of this equality has not yet been fully achieved, one can prove (4) that
\[ -\beta F_N(\beta, h) \leq \bar{\alpha}_P(\beta, h), \] (10)
for any value of $N, \beta, h$.

In the following, we employ the result (10) to prove that the thermodynamic limit of the quenched free energy is strictly greater than its replica symmetric approximation, below the Almeida-Thouless line:
\[ -\beta F(\beta, h) \equiv -\beta \lim_{N \to \infty} F_N(\beta, h) < \bar{\alpha}(\beta, h), \] (11)
for
\[ \beta^2 \int d\mu(z) \frac{1}{\cosh^4(z\beta\sqrt{\bar{q}} + \beta h)} > 1. \] (12)
(The limit in (11) exists, thanks to (4). To this purpose, one simply needs to show that, if (12) holds, there exists a functional order parameter $\tilde{x}$ such that $\bar{\alpha}(\beta, h; \tilde{x}) < \bar{\alpha}(\beta, h)$.)
For instance, we choose

\[
\begin{aligned}
\tilde{x}(q) &= 0 \quad q \in [0, \bar{q}]
\tilde{x}(q) &= m \quad q \in (\bar{q}, q]
\tilde{x}(q) &= 1 \quad q \in (q, 1],
\end{aligned}
\]

(13)

where \(0 \leq m \leq 1\) and \(\bar{q} \leq q \leq 1\) and we denote with \(\bar{\alpha}(\beta, h; m, q)\) the corresponding Parisi function \(\bar{\alpha}(\beta, h; \tilde{x})\).

Of course, since \(\bar{\alpha}(\beta, h; 1, q) = \bar{\alpha}(\beta, h)\), it is sufficient to prove that

\[
\partial_m \bar{\alpha}(\beta, h; m, q)|_{m=1} > 0,
\]

for some \(q\). First of all, \(\bar{\alpha}(\beta, h; m, q)\) is easily found to be

\[
\bar{\alpha}(\beta, h; m, q) = \ln 2 + \frac{\beta^2}{2} (1 - q) - \frac{\beta^2}{4} (1 - q^2 + m(q^2 - \bar{q}^2)) + (14)
\]

\[
+ \frac{1}{m} \int d\mu(z') \ln \int d\mu(z) \cosh^m(\beta h + \beta z\sqrt{q - q} + \beta z'\sqrt{q}).
\]

(15)

Next, we compute the derivative with respect to \(m\), keeping \(q\) fixed, and we find

\[
\partial_m \bar{\alpha}(\beta, h; m, q)|_{m=1} \equiv K(\beta, h; q) \equiv
\]

\[
\equiv -\frac{\beta^2}{4}(q^2 - \bar{q}^2) - \int d\mu(z') \ln \int d\mu(z) \cosh(\beta h + \beta z\sqrt{q - q} + \beta z'\sqrt{q})
\]

\[
+ \int d\mu(z') \int d\mu(z) \cosh(\beta h + \beta z\sqrt{q - q} + \beta z'\sqrt{q}) \ln \cosh(\beta h + \beta z\sqrt{q - q} + \beta z'\sqrt{q}).
\]

(16)

(17)

It is clear that, for \(q \downarrow \bar{q}\), the integration over \(z\) disappears, and

\[
K(\beta, h; \bar{q}) = 0.
\]

Therefore, in order to check the sign of \(K(\beta, h; \bar{q})\), we have to expand around \(q = \bar{q}\). By performing the first two derivatives with respect to \(q\), one finds

\[
\partial_q K(\beta, h; q)|_{q=\bar{q}} = 0
\]

and

\[
\partial_q^2 K(\beta, h; q)|_{q=\bar{q}} = -\frac{\beta^2}{2} \left(1 - \frac{1}{\cosh^4(z\beta\sqrt{q} + \beta h)}\right).
\]

(18)

This computation requires a simple integration by parts on a Gaussian variable. It is clear that, when condition (17) holds, \(\partial_q^2 K(\beta, h; q)|_{a=\bar{q}} > 0\), so that

\[
\partial_m \bar{\alpha}(\beta, h; m, q)|_{m=1} > 0,
\]

(19)
at least for $q$ small.

This, together with Guerra’s bound (10), completes the proof of the result (11), i.e., of the instability of the replica symmetric solution.

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