ON A FINAL VALUE PROBLEM
FOR A CLASS OF NONLINEAR HYPERBOLIC EQUATIONS
WITH DAMPING TERM

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Abstract. This paper deals with the problem of finding the function $u(x, t)$, $(x, t) \in \Omega \times [0, T]$, from the final data $u(x, T) = g(x)$ and $u_t(x, T) = h(x)$,

$$u_{tt} + a \Delta^2 u_t + b \Delta^2 u = R(u).$$

This problem is known as the inverse initial problem for the nonlinear hyperbolic equation with damping term and it is ill-posed in the sense of Hadamard. In order to stabilize the solution, we propose the filter regularization method to regularize the solution. We establish appropriate filtering functions in cases where the nonlinear source $R$ satisfies the global Lipschitz condition and the specific case $R(u) = |u|^p - 1$, $p > 1$ which satisfies the local Lipschitz condition. In addition, we show that regularized solutions converge to the sought solution under a priori assumptions in Gevrey spaces.

1. Introduction. Let $T$ be a positive number and $\Omega \subset \mathbb{R}^N$, $N \geq 1$ be an open bounded domain with a smooth boundary $\Gamma$ and let $Q_T := \Omega \times [0, T)$. In this work, we consider the following damped nonlinear hyperbolic equation (DNHE for short):

$$u_{tt} + a \Delta^2 u_t + b \Delta^2 u = R(u), \quad \text{in } Q_T,$$

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by adding some terminal conditions
\[ u(x,T) = g(x), \quad u_t(x,T) = h(x), \quad \text{in } \Omega, \tag{1.2} \]
and the Dirichlet boundary condition
\[ u(x,t) = 0, \quad \text{on } \Gamma \times [0,T], \tag{1.3} \]
where \( u(x,t) \), \((x,t) \in Q_T\) denotes the unknown function and the subscripts of \( u \) corresponding to \( t \) indicate the partial derivatives with respect to \( t \), \( \Delta^2 u_t \) denotes the strong material damping term, \( a, b \) are two positive constants, and \( \Delta^2 \) denotes the biharmonic operator. The source \( R(u) \) is the given nonlinear function, and \( g(x), h(x), x \in \Omega \) are given final value functions (usually in \( L^m(\Omega), m \geq 1 \)). The model of a class of DNHE’s appear in many applications in natural sciences, such as, modern material sciences [5], longitudinal motion of an elasto-plastic bar [7] and vibrations of a nonlinear damped beam [2,26].

For the initial value problem of (1.1),
\[ t = 0 : \quad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), \quad x \in \Omega, \]
there are many excellent studies on the properties and asymptotic behavior of the solution on DNHE’s, see e.g. [1, 8–10, 14, 20, 24, 26, 27] and the references cited therein. In [8], the authors considered problem (1.1) with the reaction term \( R(u) = -\Delta f(\Delta u) \) and gave sufficient conditions of a blow-up of the solutions and proved the existence and uniqueness of the local generalized solution for the problem. For a nonlinear beam equation with double damping terms [26],
\[ u_{tt} - u_{xxt} + u_{xxxxxt} = g(u_{xx}), \quad x \in \Omega := (0,1), \quad t > 0, \tag{1.4} \]
where the terms \( u_{xxt} \) and \( u_{xxxxxt} \) denote the strong material and internal dynamic damping respectively, the authors determined sufficient conditions for the blow-up of the solution to the problem with the help of an adapted concavity method. Moreover, global existence of weak solutions, the exponential and uniform decay rates of the solution energy are presented using an integral inequality. Similar types can be found in [20,23] and the references therein.

For an initial value problem on the class of nonlinear wave equations (cf. see e.g. [3,12,16,18,19,25,28] and the references therein)
\[ u_{tt} + \Delta^2 u + au_t = R(u), \quad \text{in } Q_T, \quad a \geq 0, \tag{1.5} \]
and these works study the global existence, the asymptotic behavior of weak solutions and the blow-up of solutions. For the system of nonlinear damped wave equations, we refer the reader to [4,5,21], and in these works, the global existence, uniqueness and continuous dependence results, as well as the finite time blow-up of solutions are proven.

Although the initial value problem of DNHE’s has been studied and there have been many works on this type of problem over the past few decades, the final value problem of DNHE’s have not been considered. It is well-known that Problem (1.1)-(1.3) (from now on, we write (1.1)-(1.2)-(1.3) to mean Problem (1.1)-(1.2)-(1.3)) is ill-posed in the sense of Hadamard i.e. if the solution exists then it does not depend continuously on the final values (Cauchy data). So our goal is to find methods to stabilize the solution to Problem (1.1)-(1.3). Based on the idea of the paper [17],
we propose a general filter method to regularize Problem (1.1)-(1.3). Specifically, we consider two cases. For the source function \( R \) satisfying the global Lipschitz condition, we prove that the regularized solution is well-posed and error estimates on \( L^2(\Omega) \) are obtained. Now \( R(u) = u|u|^{p-1}, p > 1 \), is a specific case that satisfies the local Lipschitz condition. For this type of local source function, the error estimates on \( L^2(\Omega) \) are not achieved. It is therefore of interest to establish Sobolev embeddings and obtain existence and regularity estimates of the regularized solution. Moreover, the error estimates on the Hilbert scale \( \mathcal{H}^{\nu^+}(\Omega) \), for \( \nu^+ \geq 0 \) are proved.

In Section 2, some notations are stated and we also formulate the solution to Problem (1.1)-(1.3) and we prove that its solution is ill-posed. Section 3 contains stability estimates under \textit{a priori} conditions on the solution by the filter regularization method. In Subsection 3.1, stable results with the global Lipschitz condition on \( R \) are presented. For \( R \) satisfying the local Lipschitz condition (of the type \( R(u) = u|u|^{p-1}, p > 1 \)) is discussed in Subsection 3.2, and finally, conclusions are summarised in Section 4.

2. Preliminaries.

2.1. Relevant notations. Denote by \( 0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_j \leq \ldots \nearrow \infty \), the family of eigenvalues and \( \xi_j \) the eigenfunction corresponding to \( \lambda_j \) of the elliptic eigenvalue problem:

\[
\begin{cases}
(-\Delta)^k \xi_j(x) = \lambda_j^2 \xi_j(x), & x \in \Omega, \\
\xi_j(x) = 0, & x \in \partial\Omega,
\end{cases}
\]  

(2.1)

Let \( u(t) = \sum_{j \in \mathbb{N}} u_j(t) \xi_j \) be the Fourier series in \( L^2(\Omega) \) with \( u_j(t) = \langle u(t), \xi_j \rangle \) and we let \( \langle \cdot, \cdot \rangle \) denote the inner product in \( L^2(\Omega) \). The notation \( \| \cdot \|_X \) stands for the norm in the Banach space \( X \). We denote by \( L^p(0,T; X), 1 \leq p \leq \infty, \quad T > 0, \) the Banach space of real-valued measurable functions \( f : (0, T) \to X \) with norm

\[
\|f\|_{L^p(0,T;X)} = \left( \int_0^T \| f(t) \|_X^p \, dt \right)^{\frac{1}{p}}, \quad \text{for } 1 \leq p < \infty,
\]  

(2.2)

\[
\|f\|_{L^\infty(0,T;X)} = \sup_{t \in (0,T)} \| f(t) \|_X, \quad \text{for } p = \infty.
\]  

(2.3)

The norm of the function space \( C^k([0,T]; X), 0 \leq k \leq \infty \) is denoted by

\[
\|f\|_{C^k([0,T];X)} = \sum_{i=0}^{k} \sup_{t \in [0,T]} \| f^{(i)}(t) \|_X < \infty.
\]  

(2.4)

For any \( \nu \geq 0 \), we define the Hilbert scale space

\[
\mathcal{H}^{\nu}(\Omega) = \left\{ f(x) = \sum_{j=1}^{\infty} \langle f, \xi_j \rangle \xi_j \in L^2(\Omega) : \sum_{j=1}^{\infty} \langle f, \xi_j \rangle^2 \lambda_j^{2\nu} < \infty \right\},
\]

which is equipped with the norm \( \| f \|_{\mathcal{H}^{\nu}(\Omega)} = \left( \sum_{j=1}^{\infty} \langle f, \xi_j \rangle^2 \lambda_j^{2\nu} \right)^{\frac{1}{2}}. \) Obviously, we have \( \mathcal{H}^0(\Omega) = L^2(\Omega) \) if \( \nu = 0 \). We denote by \( \mathcal{H}^{-\nu}(\Omega) \) the dual space of \( \mathcal{H}^{\nu}(\Omega) \).
provided that the dual space of $L^2(\Omega)$ is identified with itself, e.g. see [15]. The space $\mathcal{H}^{-\nu}(\Omega)$ is a Hilbert space with respect to the norm
\[
\|f\|_{\mathcal{H}^{-\nu}(\Omega)} = \left( \sum_{n=1}^{\infty} \langle f, \xi_j \rangle_{-\nu,\nu}^2 \lambda_j^{-2\nu} \right)^{1/2},
\]
for $f \in \mathcal{H}^{-\nu}(\Omega)$ where $\langle \cdot, \cdot \rangle_{-\nu,\nu}$ is the dual product between $\mathcal{H}^{-\nu}(\Omega)$ and $\mathcal{H}^{\nu}(\Omega)$.

We note that $\langle f_1, f_2 \rangle_{-\nu,\nu} = \langle f_1, f_2 \rangle$, for $f_1 \in L^2(\Omega), f_2 \in \mathcal{H}^{\nu}(\Omega)$.

The abstract Gevrey class of functions of order $\beta > 0$ and index $r_1, r_2 > 0$ see e.g. [6], is defined by
\[
\mathcal{G}^\beta_{r_1,r_2}(\Omega) := \left\{ f \in L^2(\Omega) : \sum_{j \in \mathbb{N}^*} \langle f, \xi_j \rangle^2 \lambda_j^{2r_1} \exp \left( 2r_2 \lambda_j^\beta \right) < \infty \right\},
\]
and the corresponding norm:
\[
\|f\|_{\mathcal{G}^\beta_{r_1,r_2}(\Omega)} := \left( \sum_{j \in \mathbb{N}^*} \langle f, \xi_j \rangle^2 \lambda_j^{2r_1} \exp \left( 2r_2 \lambda_j^\beta \right) \right)^{1/2}.
\]

2.2. Mild solution of the Problem (1.1)-(1.3) and its instability.

2.2.1. Mild solution. Now, suppose that Problem (1.1)-(1.3) has unique solution, then we are interested in finding the formulation of it. From (1.1)-(1.3), taking the inner product with $\xi_j \in L^2(\Omega)$, we obtain
\[
\begin{cases}
u''(t) + a\lambda_j^2\nu'(t) + b\lambda_j^2\nu_j = \mathcal{R}_j(u), \\
u_j(T) = g_j, \quad u_j'(T) = h_j,
\end{cases}
\]
with $u_j(t) = \langle u(\cdot, t), \xi_j \rangle$, $\mathcal{R}_j(u) = \langle \mathcal{R}(u), \xi_j \rangle$, $g_j = \langle g, \xi_j \rangle$ and $h_j = \langle h, \xi_j \rangle$. For problem (2.7), we have the quadratic characteristic polynomial
\[
Z^2 + a\lambda_j^2Z + b\lambda_j^2 = 0, \quad j \in \mathbb{N}^*,
\]
and solving this equation, we find its solutions as follows
\[
Z_i = \frac{-a\lambda_j^2 + (-1)^i \sqrt{\delta_j}}{2}, \quad \text{and} \quad \delta_j = \lambda_j^2 \left( a^2 \lambda_j^2 - 4b \right), \quad i = 1, 2, \quad \text{for any} \ j \in \mathbb{N}^*,
\]
with the condition $\lambda_j^2 > \frac{\sqrt{\delta_j}}{a}$, for all $j \in \mathbb{N}^*$.

For solving a problem (2.7), by direct calculation, we can say that a function $u$ is a mild solution of (1.1)-(1.3) if $u \in C([0, T]; L^2(\Omega))$ and satisfies the following integral equation
\[
u(t) = \mathcal{S}(T-t)g - \mathcal{P}(T-t)h + \int_t^T \mathcal{P}(s-t)\mathcal{R}(u)(s)ds, \quad 0 \leq t \leq T.
\]
Here, the operators $S(t), P(t)$ are defined as (for $(x,t) \in Q_T$)

$$S(t)w := \sum_{j \in \mathbb{N}^*} \left[ \frac{\kappa_j^+ \exp(\tau) - \kappa_j^- \exp(\tau)}{\sqrt{\delta_j}} \right] \langle w, \xi_j \rangle \xi_j(x),$$  

(2.10)

$$P(t)w := \sum_{j \in \mathbb{N}^*} \left[ \frac{\exp(\tau) - \exp(\tau)}{\sqrt{\delta_j}} \right] \langle w, \xi_j \rangle \xi_j(x),$$  

(2.11)

where we set

$$\kappa_j^\pm := \frac{a \lambda_j^2 \pm \sqrt{\delta_j}}{2}, \quad \text{for } j \in \mathbb{N}^*. \quad (2.12)$$

**2.2.2. Ill-posedness of Problem (1.1)-(1.3).** We shall give an example that proves that the solution (2.9) to Problem (1.1)-(1.3) is ill-posedness in the sense of Hadamard. Let us choose

$$g(x) = g_m(x) := 0, \quad h(x) = h_m(x) := \frac{\xi_m(x)}{\sqrt{\lambda_m}}, \quad x \in \Omega, \quad \text{for any } m \in \mathbb{N}^*, \quad (2.13)$$

$$R_m(u)(x,t) := \sum_{j \in \mathbb{N}^*} \left[ \frac{\exp(-a \lambda_j^2 T)}{2m^{1/2}T^{3/2}} \langle u(\cdot,t), \xi_j \rangle \right] \xi_j(x), \quad (x,t) \in Q_T. \quad (2.14)$$

Then we have the following integral equation is a solution to Problem (1.1)-(1.3) for the data (2.13) and (2.14)

$$u_m(t) = -P(T-t)h_m + \int_t^T P(s-t)R_m(u_m)(s)ds. \quad (2.15)$$

Define the following operator for $v \in C([0,T]; L^2(\Omega))$

$$G\vartheta(t) = -P(T-t)h_m + \int_t^T P(s-t)R_m(\vartheta)(s)ds, \quad (2.16)$$

and we shall prove $G$ is a contraction.

Indeed, taking $\vartheta_1, \vartheta_2$ belonging to $C([0,T]; L^2(\Omega))$, we have
\[
\|G\vartheta_1(t) - G\vartheta_2(t)\|_{L^2(\Omega)} \leq \int_t^T \|\mathcal{P}(s-t) [\mathcal{R}_m(\vartheta_1)(s) - \mathcal{R}_m(\vartheta_2)(s)]\|_{L^2(\Omega)} \, ds
\]
\[
\leq \int_t^T \left[ \sum_{j \in \mathbb{N}^*} \frac{\exp\left((s-t)\kappa_j^+\right) - \exp\left((s-t)\kappa_j^-\right)}{\sqrt{\delta_j}} \times |\mathcal{R}_{mj}(\vartheta_1)(s) - \mathcal{R}_{mj}(\vartheta_2)(s)|^2 \right]^{1/2} \, ds
\]
\[
\leq \int_t^T \left[ \sum_{j \in \mathbb{N}^*} \exp\left(2(s-t)\left(\kappa_j^+ + \kappa_j^-\right)\right) \times \frac{|\exp\left(-(s-t)\kappa_j^-\right) - \exp\left(-(s-t)\kappa_j^+\right)|^2}{\delta_j} |\mathcal{R}_{mj}(\vartheta_1)(s) - \mathcal{R}_{mj}(\vartheta_2)(s)|^2 \right]^{1/2} \, ds
\]
\[
\leq \int_t^T \left[ \sum_{j \in \mathbb{N}^*} \exp\left(2a\lambda_j^2(s-t)\right) \frac{(s-t)^2 |\kappa_j^+ - \kappa_j^-|^2}{\delta_j} \times \frac{\exp\left(-2a\lambda_j^2T\right)}{2mT^3} |\vartheta_{ij}(s) - \vartheta_{2j}(s)|^2 \right]^{1/2} \, ds
\]
\[
\leq \int_t^T \frac{1}{2mT} \|\vartheta_1(\cdot,s) - \vartheta_2(\cdot,s)\|_{L^2(\Omega)} \, ds \leq \frac{1}{2m} \|\vartheta_1 - \vartheta_2\|_{L^\infty(0,T;L^2(\Omega))}, \quad \text{(2.17)}
\]

where we denote
\[
\mathcal{R}_{mj}(\vartheta_i)(s) = \langle \mathcal{R}_m(\vartheta_i)(\cdot,s), \xi_j \rangle, \quad \vartheta_{ij}(s) = \langle \vartheta_i(\cdot,s), \xi_j \rangle, \quad i = 1,2,
\]
and in which we have used \(\kappa_j^+ + \kappa_j^- = a\lambda_j^2\) and \(|\kappa_j^+ - \kappa_j^-| = \sqrt{\delta_j}\) along with the following inequalities
\[
|e^{-a} - e^{-b}| \leq |a - b|, \quad \text{for } a,b > 0,
\]
\[
\exp\left(2a\lambda_j^2(s-t-T)\right) \frac{(s-t)^2}{T^2} \leq 1, \quad \text{for all } j \in \mathbb{N}^*, 0 \leq t \leq s \leq T.
\]

From (2.17), one obtains
\[
\|G\vartheta_1 - G\vartheta_2\|_{L^\infty(0,T;L^2(\Omega))} \leq \frac{1}{2m} \|\vartheta_1 - \vartheta_2\|_{L^\infty(0,T;L^2(\Omega))}, \quad \text{for all } t \in [0,T], \quad \text{(2.18)}
\]

and this leads to \(G\) being a contraction in the space \(L^\infty(0,T;L^2(\Omega))\) for any \(m \in \mathbb{N}^*\). Using the Banach fixed-point theorem, we conclude that \(G\vartheta = \vartheta\) has a unique solution \(u_m \in L^\infty(0,T;L^2(\Omega))\). Next, we prove the solution \(u_m\) given as in (2.15) is not continuously dependent on the data. Indeed, from (2.14) we have
We need to estimate

\[ \|u_m(h, t)\|_{L^2(\Omega)} \geq \|\mathcal{P}(T - t)h_m\|_{L^2(\Omega)} - \left\| \int_0^T \mathcal{P}(s - t)\mathcal{R}_m(u_m(s))\,ds \right\|_{L^2(\Omega)} \]

\[ \geq \|\mathcal{P}(T - t)h_m\|_{L^2(\Omega)} - \|G(u_m(t) - G(0)(t))\|_{L^2(\Omega)} \]

\[ \geq \|\mathcal{P}(T - t)h_m\|_{L^2(\Omega)} - \frac{1}{2m}\|u_m\|_{L^\infty(0,T;L^2(\Omega))}. \]

Based on the conclusion above, we infer that Problem (1.1)-(1.3) is ill-posed in the sense of Hadamard. Therefore, regularization methods to regularize this problem is necessary.
3. Filter regularization method and error estimates. As analyzed above, the solution of Problem (1.1)-(1.3) is ill-posed, so in this section we use an appropriate regularization method to regularize the solution to Problem (1.1)-(1.3). Our purpose is to find bounded operators to replace for \( S(t) \) and \( P(t) \). We consider the following two cases:

- **Case 1**: The filter function \( F_j(\alpha) \) is chosen as follows:
  \[
  F_j(\alpha) := \frac{\exp(-aT\lambda_j^2)}{\alpha a \lambda_j^2 + \exp(-aT\lambda_j^2)}, \quad \forall j \in \mathbb{N}^*, \alpha > 0. \tag{3.1}
  \]
  Let us set the operators \( S_\alpha(t), P_\alpha(t) \) defined as (for \( (x,t) \in Q_T \)):
  \[
  S_\alpha(t)w := \sum_{j \in \mathbb{N}^*} F_j(\alpha) \left( \frac{\kappa_j^+ \exp(t \kappa_j^-) - \kappa_j^- \exp(t \kappa_j^+)}{\sqrt{\delta_j}} \right)(w, \xi_j) \xi_j(x), \tag{3.2}
  \]
  \[
  P_\alpha(t)w := \sum_{j \in \mathbb{N}^*} F_j(\alpha) \left( \frac{\exp(t \kappa_j^+) - \exp(t \kappa_j^-)}{\sqrt{\delta_j}} \right)(w, \xi_j) \xi_j(x). \tag{3.3}
  \]

- **Case 2**: The other filter function \( \tilde{F}_j(\pi(\alpha)) \) is chosen as follows:
  \[
  \tilde{F}_j(\pi(\alpha)) = \begin{cases} 1, & \text{for } \lambda_j \leq \pi, \quad j \in \mathbb{N}^*, \quad \pi > 0, \\ 0, & \text{for } \lambda_j > \pi, \quad j \in \mathbb{N}^*, \end{cases} \tag{3.4}
  \]
  and we have the new operators \( S_{\pi}(t), P_{\pi}(t) \) defined as :
  \[
  S_{\pi}(t)w := \sum_{j \leq \pi} \tilde{F}_j(\pi(\alpha)) \left( \frac{\kappa_j^+ \exp(t \kappa_j^-) - \kappa_j^- \exp(t \kappa_j^+)}{\sqrt{\delta_j}} \right)(w, \xi_j) \xi_j(x) \tag{3.5}
  \]
  \[
  = \sum_{j \leq \pi} \left( \frac{\kappa_j^+ \exp(t \kappa_j^-) - \kappa_j^- \exp(t \kappa_j^+)}{\sqrt{\delta_j}} \right)(w, \xi_j) \xi_j(x), \quad \forall(x,t) \in Q_T. \tag{3.6}
  \]

**Remark 1.** The operators \( S_\alpha(t) \) and \( P_\alpha(t) \) (or \( S_{\pi}(t) \) and \( P_{\pi}(t) \)) satisfy the following two properties:

(I) If \( \alpha > 0 \) (or \( \pi > 0 \)) is fixed, then the operators \( S_\alpha(t) \) and \( P_\alpha(t) \) (or \( S_{\pi}(t) \) and \( P_{\pi}(t) \)) are bounded;

(II) If the parameter \( \alpha > 0 \) is small (or \( \pi > 0 \) is large enough), then the kernel \( F_j(\alpha) \) (or \( \tilde{F}_j(\pi(\alpha)) \)) is close to 1. It follows that \( S_\alpha(t) \) and \( P_\alpha(t) \) (or \( S_{\pi}(t) \) and \( P_{\pi}(t) \)) are close to \( S(t) \) and \( P(t) \), respectively.

3.1. The globally Lipschitz case. We start with the following assumptions:

(A1) Assume that \( R \) satisfies the global Lipschitz condition:
  \[
  \| R(u) - R(w) \|_{L^2(\Omega)} \leq K \| u - w \|_{L^2(\Omega)}, \tag{3.7}
  \]
  for \( (x,t) \in Q_T, \ u, w \in C([0,T];L^2(\Omega)) \) and the constant \( K > 0 \) is independent of \( x,t,u,w \).
(A2) Let $\mathcal{R}(0) = 0$, $\forall(x, t) \in Q_T$ and
\[
\|\mathcal{R}(u)\|_{L^2(\Omega)} \leq K \|u\|_{L^2(\Omega)}. \tag{3.8}
\]

(A3) The exact data $g, h \in L^2(\Omega)$ can only be measured and there will be measurement errors, and we thus would have as data some function $g^\varepsilon$ and $h^\varepsilon$ in $L^2(\Omega)$ for which
\[
\|g^\varepsilon - g\|_{L^2(\Omega)} \leq \varepsilon, \quad \|h^\varepsilon - h\|_{L^2(\Omega)} \leq \varepsilon, \tag{3.9}
\]
the constant $\varepsilon > 0$ represents a bound on the measurement error.

We use the filter function $\mathcal{F}_\alpha (\alpha)$ given in (3.1) to regularize this problem. Indeed, the regularized solution has the following form
\[
u_\alpha (t) = S_\alpha (T - t) g^\varepsilon - P_\alpha (T - t) h^\varepsilon + \int_t^T P_\alpha (s - t) R(u_\alpha^\varepsilon) (s) ds, \quad (x, t) \in Q_T. \tag{3.10}
\]
Here $\alpha := \alpha(\varepsilon) > 0$ is a parameter regularization which satisfies $\lim_{\varepsilon \to 0^+} \alpha = 0$.

For $\mathcal{F}_\alpha (\alpha)$ taken as in (3.1), we can see that property (II) in Remark 1 is satisfied as $\alpha$ goes to 0. For property (I), we consider the following lemma.

**Lemma 3.1.** For $0 \leq t \leq T$ and for $\alpha \in (0, 1)$ satisfying $\alpha < eT$, the following estimates hold:
\[
\|S_\alpha (t)\|_{\mathcal{L}^2 \rightarrow \mathcal{L}^2} \leq C_1 \left( \frac{T \alpha^{-1}}{\log (T \alpha^{-1})} \right)^{\frac{2}{T}}, \tag{3.11a}
\]
\[
\|P_\alpha (t)\|_{\mathcal{L}^2 \rightarrow \mathcal{L}^2} \leq T \left( \frac{T \alpha^{-1}}{\log (T \alpha^{-1})} \right)^{\frac{2}{T}}, \tag{3.11b}
\]
where, $C_1 := C_1(a, b, T) = \left( 2 + 8T^2 \left( \frac{b}{a} \right)^2 \right)^{1/2} > 0$.

**Proof.** We begin with the proof of (3.11a). For $\vartheta \in L^2(\Omega)$, we have
\[
\|S_\alpha (t)\vartheta\|_{L^2(\Omega)}^2 = \sum_{j \in \mathbb{N}^*} \left( \mathcal{F}_\alpha (\alpha)^{\kappa_j^+ \exp (tk_j^-) - \kappa_j^- \exp (tk_j^+)} \langle \vartheta, \xi_j \rangle \right)^2
\]
\[
= \sum_{j \in \mathbb{N}^*} \left| \mathcal{F}_\alpha (\alpha) \right|^2 \exp (2t (\kappa_j^+ + \kappa_j^-)) \left( \kappa_j^+ \exp (-tk_j^+ - \kappa_j^- \exp (-tk_j^-)) \right)^2 \langle \vartheta, \xi_j \rangle^2
\]
\[
= \sum_{j \in \mathbb{N}^*} \left| \mathcal{F}_\alpha (\alpha) \right|^2 \exp (2a \lambda_j^2 t) \left( \kappa_j^+ \exp (-tk_j^+ - \kappa_j^- \exp (-tk_j^-)) \right)^2 \langle \vartheta, \xi_j \rangle^2, \tag{3.12}
\]
where we have used $\kappa_j^+ + \kappa_j^- = a \lambda_j^2$. Using the inequality $(c + d)^2 \leq 2c^2 + 2d^2$, $c, d \in \mathbb{R}$ and the inequality $|e^{-c} - e^{-d}| \leq |c - d|$, $c, d \in \mathbb{R}$ and note that $|\kappa_j^+ - \kappa_j^-| = \sqrt{\delta_j}$, we have the following estimate
\[
\left( \kappa_j^+ \exp(-tk_j^+) - \kappa_j^- \exp(-tk_j^-) \right)^2 \over \delta_j \\
= \left( \exp(-tk_j^+) + \kappa_j^- \over \sqrt{\delta_j} \exp(-tk_j^-) \right)^2 \\
\leq 2 \exp(-2tk_j^+) + 2 |\kappa_j^-|^2 \left( \exp(-tk_j^+) - \exp(-tk_j^-) \right)^2 \over \delta_j \\
\leq 2 + 2 |\kappa_j^-|^2 t^2 \left| \kappa_j^+ - \kappa_j^- \right|^2 = 2 + 2t^2 \left| a\lambda_j^2 - \sqrt{a^2\lambda_j^4 - 4b\lambda_j^2} \right|^2 \\
= 2 + t^2 \left| a\lambda_j^2 + \sqrt{a^2\lambda_j^4 - 4b\lambda_j^2} \right|^2 \leq 2 + 8t^2 \left( {b \over a} \right)^2. \tag{3.13}
\]

Now, we continue to estimate the term \( \exp(a\lambda_j^2 t) \left| \mathcal{F}_j(\alpha) \right| \) in (3.12). One has

\[
\exp\left( a\lambda_j^2 t \right) \left| \mathcal{F}_j(\alpha) \right| = \frac{\exp(-a(T-t)\lambda_j^2)}{a\lambda_j^2 + \exp(-aT\lambda_j^2)} \\
= \left( \frac{\exp(-aT\lambda_j^2)}{a\lambda_j^2 + \exp(-aT\lambda_j^2)} \right)^{T-t} \frac{1}{\left( a\lambda_j^2 + \exp(-aT\lambda_j^2) \right)^{T}} \\
\leq \frac{1}{\left( a\lambda_j^2 + \exp(-aT\lambda_j^2) \right)^{T}},
\]

thanks to the property of the function \( Z(y) := \frac{1}{\kappa y + \exp(-yT)} \leq \frac{T/\kappa}{\log (T/\kappa)} \) for \( 0 < \kappa < eT \). Thus, for \( \alpha < eT \), we get that

\[
\exp\left( a\lambda_j^2 t \right) \left| \mathcal{F}_j(\alpha) \right| \leq \left( {T\alpha^{-1} \over \log (T\alpha^{-1})} \right)^{T}. \tag{3.14}
\]

Plugging (3.13) and (3.14) into (3.12), we get

\[
\left\| \mathbf{S}_\alpha(t) \vartheta \right\|^2_{L^2(\Omega)} \leq \left( {T\alpha^{-1} \over \log (T\alpha^{-1})} \right)^{2T} \left( 2 + 8T^2 \left( {b \over a} \right)^2 \right) \sum_{j \in \mathbb{N}^*} \left\langle \vartheta, \xi_j \right\rangle^2 \\
= C_1^2 \left( {T\alpha^{-1} \over \log (T\alpha^{-1})} \right)^{2T} \left\| \vartheta \right\|^2_{L^2(\Omega)},
\]

where \( C_1 = \left( 2 + 8T^2 \left( {b \over a} \right)^2 \right)^{1/2} > 0 \), and this implies (3.11a).

Next we prove (3.11b). Let \( \vartheta \in L^2(\Omega) \), and using the inequality \( |e^{-a} - e^{-b}| \leq |a - b|, a, b \in \mathbb{R} \) and \( |\kappa_j^+ - \kappa_j^-| = \sqrt{\delta_j} \), we deduce that for \( 0 \leq t \leq T \).
\[ \|P_\alpha(t)\vartheta\|^2_{L^2(\Omega)} = \sum_{j \in \mathbb{N}^*} \left[ |F_j(\alpha)| \frac{\exp(tk_j^+)}{\delta_j} - \exp(tk_j^-) \right]^2 \]
\[ = \sum_{j \in \mathbb{N}^*} |F_j(\alpha)|^2 \exp(2t(\kappa_j^+ + \kappa_j^-)) \frac{\left( \exp(-tk_j^+) - \exp(-tk_j^-) \right)^2}{\delta_j} \langle \vartheta, \xi_j \rangle^2 \]
\[ \leq \sum_{j \in \mathbb{N}^*} |F_j(\alpha)|^2 \exp(2a\lambda^2 t) \frac{t^2 |\kappa_j^+ - \kappa_j^-|^2}{\delta_j} \langle \vartheta, \xi_j \rangle^2 \]
\[ \leq T^2 \left( \frac{T_\alpha^{-1}}{\log(T\alpha^{-1})} \right) ^\frac{T}{2} \|\vartheta\|^2_{L^2(\Omega)}. \] (3.15)

This completes the proof of Lemma 3.1. \[\Box\]

We are now ready to prove that the well-posedness properties of the regularized solution given as in (3.10).

3.1.1. The well-posedness of the regularized solution (3.10). For the constant \( \varphi > 0 \), denote by \( \mathcal{C}^\varphi([0,T]; L^2(\Omega)) \) the function space \( C([0,T]; L^2(\Omega)) \) equipped with the following weighted norm (see [13]):

\[ \|f\|_{\mathcal{C}^\varphi([0,T]; L^2(\Omega))} = \max_{0 \leq t \leq T} \|\exp(-\varphi(T-t))f(t)\|_{L^2(\Omega)}, \quad \forall f \in C([0,T]; L^2(\Omega)). \] (3.16)

**Theorem 3.2 (Well-posedness).**

Assume that \( R \) satisfies assumptions \((A_1)\) and \((A_2)\). Then, the integral equation (3.10) has a unique mild solution \( u_\alpha^\varphi \) in \( \mathcal{C}^\varphi([0,T]; L^2(\Omega)) \) for \( \varphi > 0 \). The solution depends continuously on the final value pair \((g,h)\) in \( C([0,T]; L^2(\Omega)) \). Moreover, if \( g^\varphi, h^\varphi \in L^2(\Omega) \), we also have \( (\forall t \in [0,T]) \)

\[ \|u_\alpha^\varphi(\cdot,t)\|_{L^2(\Omega)} \leq C_2 \left( \|g^\varphi\|_{L^2(\Omega)} + \|h^\varphi\|_{L^2(\Omega)} \right) \left( \frac{T_\alpha^{-1}}{\log(T\alpha^{-1})} \right)^{1 - \frac{2}{r}} \exp(T\varphi(T-t)), \] (3.17)

where the positive constant \( C_2 := \max \{C_1; T\} \), and \( C_1 \) is defined as in Lemma 3.1.

**Proof.** The proof is divided into three parts. In Part 1, we prove the existence of a solution to (3.10). In Part 2, we prove the regularized solution of (3.10) depends continuously on the final datum. The regularity of the solution to (3.10) is given in Part 3.

- **Part 1:** Existence to the integral equation (3.10). For \( \vartheta \in C([0,T]; L^2(\Omega)) \), we consider the following function:

\[ D\vartheta(t) = S_\alpha(T-t)g^\varphi - P_\alpha(T-t)h^\varphi + \int_t^T P_\alpha(s-t)R(\vartheta)(s)ds, \] (3.18)

and we aim to show that the map \( D : \mathcal{C}^\varphi([0,T]; L^2(\Omega)) \to \mathcal{C}^\varphi([0,T]; L^2(\Omega)) \) is a contraction. In fact, we see that for every \( \vartheta_1, \vartheta_2 \in \mathcal{C}^\varphi([0,T]; L^2(\Omega)) \), using Lemma 3.1, we have
\[
\exp(-\varphi(T-t)) \| \mathcal{D} \vartheta_1(t) - \mathcal{D} \vartheta_2(t) \|_{L^2(\Omega)} \\
\leq \int_t^T \exp(-\varphi(T-t)) \| \mathcal{P} \alpha(s-t) (R(\vartheta_1)(s) - R(\vartheta_2)(s)) \|_{L^2(\Omega)} \, ds \\
\leq T \int_t^T \exp(-\varphi(T-t)) \left( \frac{T \alpha^{-1}}{\log(T \alpha^{-1})} \right)^{\frac{s-t}{T}} \| R(\vartheta_1)(\cdot, s) - R(\vartheta_2)(\cdot, s) \|_{L^2(\Omega)} \, ds \\
\leq KT \int_t^T \exp(-\varphi(T-t)) \left( \frac{T \alpha^{-1}}{\log(T \alpha^{-1})} \right)^{\frac{s-t}{T}} \| \vartheta_1(\cdot, s) - \vartheta_2(\cdot, s) \|_{L^2(\Omega)} \, ds \\
\leq KT \int_t^T \exp(-\varphi(s-t)) \left( \frac{T \alpha^{-1}}{\log(T \alpha^{-1})} \right)^{\frac{s-t}{T}} \times \max_{0 \leq s \leq T} \| \exp(-\varphi(T-s)) \| R(\vartheta_1(\cdot, s) - \vartheta_2(\cdot, s)) \|_{L^2(\Omega)} \, ds \\
\leq KT \int_t^T \exp(-\varphi(s-t)) \left( \frac{T \alpha^{-1}}{\log(T \alpha^{-1})} \right)^{\frac{s-t}{T}} \, ds \| \vartheta_1 - \vartheta_2 \|_{\mathcal{C}^\varphi([0,T];L^2(\Omega))] \\
\leq \frac{\alpha^{-1}KT^2}{\varphi \log(T \alpha^{-1})} \| \vartheta_1 - \vartheta_2 \|_{\mathcal{C}^\varphi([0,T];L^2(\Omega))}, \quad (3.19)
\]

where we have used the computation:

\[
\int_t^T \exp(-\varphi(s-t)) \, ds = \frac{1}{\varphi}(1 - \exp(-\varphi(T-t))) \leq \frac{1}{\varphi},
\]

and

\[
\left( \frac{T \alpha^{-1}}{\log(T \alpha^{-1})} \right)^{\frac{s-t}{T}} \leq \frac{T \alpha^{-1}}{\log(T \alpha^{-1})}, \quad 0 \leq t \leq s < T.
\]

Since the right hand side of (3.19) is independent of \( t \), we deduce that

\[
\| \mathcal{D} \vartheta_1 - \mathcal{D} \vartheta_2 \|_{\mathcal{C}^\varphi([0,T];L^2(\Omega))] \leq \frac{\alpha^{-1}KT^2}{\varphi \log(T \alpha^{-1})} \| \vartheta_1 - \vartheta_2 \|_{\mathcal{C}^\varphi([0,T];L^2(\Omega))}, \quad (3.20)
\]

By choosing a sufficiently large \( \varphi \), the last inequality implies

\[
\| \mathcal{D} \vartheta_1 - \mathcal{D} \vartheta_2 \|_{\mathcal{C}^\varphi([0,T];L^2(\Omega))] \leq \rho \| \vartheta_1 - \vartheta_2 \|_{\mathcal{C}^\varphi([0,T];L^2(\Omega))}, \quad \rho \in [0,1). \quad (3.21)
\]

Therefore, by the induction principle, we claim that the mapping \( \mathcal{D} \) of the space \( \mathcal{C}^\varphi([0,T];L^2(\Omega)) \) into itself defined by (3.18) is a contraction. The Banach fixed point theorem implies that \( \mathcal{D} \) has a unique fixed point. Hence, the integral equation (3.10) has a unique solution \( u^*_\alpha \in \mathcal{C}^\varphi([0,T];L^2(\Omega))] \).

- Part 2: Continuous dependence on the final values of the solution to (3.10). For \( u^*_\alpha \) and \( v^*_\alpha \) two regularized solutions (3.10) corresponding to final values \( (g_1, h_1) \) and
(g_2, h_2) respectively, using assumption (A_1) and Lemma 3.1, then we get

\[ \| u_\alpha^\varepsilon (\cdot, t) - v_\alpha^\varepsilon (\cdot, t) \|_{L^2(\Omega)} \]
\[ \leq \| S_\alpha (T - t) (g_1 - g_2) \|_{L^2(\Omega)} + \| P_\alpha (T - t) (h_1 - h_2) \|_{L^2(\Omega)} \]
\[ + \int_t^T \| P_\alpha (s - t) | R(u_\alpha^\varepsilon) (s) - R(v_\alpha^\varepsilon) (s) | \|_{L^2(\Omega)} \, ds \]
\[ \leq (C_1 \| g_1 - g_2 \|_{L^2(\Omega)} + T \| h_1 - h_2 \|_{L^2(\Omega)}) \left( \frac{T \alpha^{-1}}{\log (T \alpha^{-1})} \right)^{\frac{\tau - t}{T}} \]
\[ + T \int_t^T \left( \frac{T \alpha^{-1}}{\log (T \alpha^{-1})} \right)^{\frac{\tau - t}{T}} \| R(u_\alpha^\varepsilon) - R(v_\alpha^\varepsilon) \|_{L^2(\Omega)} \, ds \]
\[ \leq (C_1 \| g_1 - g_2 \|_{L^2(\Omega)} + T \| h_1 - h_2 \|_{L^2(\Omega)}) \left( \frac{T \alpha^{-1}}{\log (T \alpha^{-1})} \right)^{\frac{\tau - t}{T}} \]
\[ + KT \int_t^T \left( \frac{T \alpha^{-1}}{\log (T \alpha^{-1})} \right)^{\frac{\tau - t}{T}} \| u_\alpha^\varepsilon (\cdot, s) - v_\alpha^\varepsilon (\cdot, s) \|_{L^2(\Omega)} \, ds. \]

Multiplying the two sides of the inequality above by \( \left( \frac{T \alpha^{-1}}{\log (T \alpha^{-1})} \right)^{\frac{\tau - t}{T}} > 0 \)

\[ \left( \frac{T \alpha^{-1}}{\log (T \alpha^{-1})} \right)^{\frac{\tau - t}{T}} \| u_\alpha^\varepsilon (\cdot, t) - v_\alpha^\varepsilon (\cdot, t) \|_{L^2(\Omega)} \]
\[ \leq (C_1 \| g_1 - g_2 \|_{L^2(\Omega)} + T \| h_1 - h_2 \|_{L^2(\Omega)}) \]
\[ + KT \int_t^T \left( \frac{T \alpha^{-1}}{\log (T \alpha^{-1})} \right)^{\frac{\tau - t}{T}} \| u_\alpha^\varepsilon (\cdot, s) - v_\alpha^\varepsilon (\cdot, s) \|_{L^2(\Omega)} \, ds, \]

and the Grönwall inequality allows us to infer that

\[ \| u_\alpha^\varepsilon (\cdot, t) - v_\alpha^\varepsilon (\cdot, t) \|_{L^2(\Omega)} \]
\[ \leq C_2 \left( \frac{T \alpha^{-1}}{\log (T \alpha^{-1})} \right)^{\frac{\tau - t}{T}} \exp (TK(T - t)) \left( \| g_1 - g_2 \|_{L^2(\Omega)} + \| h_1 - h_2 \|_{L^2(\Omega)} \right), \]

for all \( t \in [0, T) \) and \( C_2 := \max \{ C_1; 1 \} > 0 \), which was to be shown.

- Part 3: Regularity of the regularized solution to (3.10). Using Lemma 3.1, assumption (A_2), we have that
\[\|u^\varepsilon_\alpha(\cdot, t)\|_{L^2(\Omega)} \leq \|S_\alpha(T - t)g^\varepsilon\|_{L^2(\Omega)} + \|P_\alpha(T - t)h^\varepsilon\|_{L^2(\Omega)} + \int_t^T \|P_\alpha(s - t)R(u^\varepsilon_\alpha)(\cdot, s)\|_{L^2(\Omega)} ds\]

\[\leq C_1 \left( \frac{T\alpha^{-1}}{\log (T\alpha^{-1})} \right)^{\frac{\varepsilon T}{T - t}} \|g^\varepsilon\|_{L^2(\Omega)} + T \left( \frac{T\alpha^{-1}}{\log (T\alpha^{-1})} \right)^{\frac{\varepsilon T}{T - t}} \|h^\varepsilon\|_{L^2(\Omega)} + KT \int_t^T \left( \frac{T\alpha^{-1}}{\log (T\alpha^{-1})} \right)^{\frac{\varepsilon T}{T - t}} \|u^\varepsilon_\alpha(\cdot, s)\|_{L^2(\Omega)} ds.\]

Multiplying both sides by \(\left( \frac{T\alpha^{-1}}{\log (T\alpha^{-1})} \right)^{\frac{\varepsilon T}{T - t}} > 0\), then we get that

\[\left( \frac{T\alpha^{-1}}{\log (T\alpha^{-1})} \right)^{\frac{\varepsilon T}{T - t}} \|u^\varepsilon_\alpha(\cdot, t)\|_{L^2(\Omega)} \leq C_1 \|g^\varepsilon\|_{L^2(\Omega)} + T \|h^\varepsilon\|_{L^2(\Omega)} + KT \int_t^T \left( \frac{T\alpha^{-1}}{\log (T\alpha^{-1})} \right)^{\frac{\varepsilon T}{T - t}} \|u^\varepsilon_\alpha(\cdot, s)\|_{L^2(\Omega)} ds.\]

From the Grönwall inequality, (3.17) is straight-forward. This concludes the proof of the Theorem 3.2.

3.1.2. Error estimates. In this subsection, we estimate the error between the regularized solution \(u^\varepsilon_\alpha\) given as in (3.10) and the sought solution \(u\) as in (2.9). The next theorem will be main result on the error estimate in \(L^2(\Omega)\).

**Theorem 3.3 (\(L^2\)-Estimate).** For \(\beta = 2, r_1 \geq 1, r_2 \geq aT\), let \(\alpha := \alpha(\varepsilon) \in (0, 1)\) such that

\[\lim_{\varepsilon \to 0^+} \alpha = 0, \quad \text{and} \quad \lim_{\varepsilon \to 0^+} \alpha^{-1} \varepsilon \leq P \text{ (finite constant)}.\]

Assume that Problem (1.1)-(1.3) has a unique solution \(u\) satisfying

\[u \in L^\infty(0, T; \mathcal{G}^{\beta}_{r_1, r_2}(\Omega)).\]

Then, the following estimate holds

\[\|u^\varepsilon_\alpha(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \leq \left( 2C_2 P + a \|u\|_{L^\infty(0, T; \mathcal{G}^{\beta}_{r_1, r_2}(\Omega))} \right) \alpha^{\frac{1}{2}} \left( \frac{T}{\log (T\alpha^{-1})} \right)^{1 - \frac{\varepsilon}{2}} \exp (TK(T - t)).\]

(3.25)

for all \(t \in [0, T]\) and the positive constant \(C_2\) is as in Theorem 3.2.
Remark 2. In (3.25), the error estimate is of order $\alpha^{\frac{1}{2}} \left( \frac{T}{\log(T\alpha^{-1})} \right)^{1-\frac{1}{\alpha}}$, $\forall t \in [0, T]$.

We also see that
\[
\lim_{\varepsilon \to 0^+} \alpha^{\frac{1}{2}} \left( \frac{T}{\log(T\alpha^{-1})} \right)^{1-\frac{1}{\alpha}} = 0, \quad t \in [0, T].
\]

- If $t \approx T$, the first term $\alpha^{\frac{1}{2}}$ tends to zero as $\varepsilon$ goes to $0^+$. If $t \approx 0$, the second term $\left( \frac{T}{\log(T\alpha^{-1})} \right)^{1-\frac{1}{\alpha}}$ tends to zero as $\varepsilon$ goes to $0^+$.

- Let us choose a parameter regularization $\alpha = \varepsilon^m$, $m \in [0, 1)$, and then the error estimate is of order $\varepsilon^{\frac{1}{2}} \left( \frac{T}{\log(T\varepsilon^{-m})} \right)^{1-\frac{1}{\alpha}}$ for all $t \in [0, T]$.

Proof. For $u_\alpha^\varepsilon$ is the regularized solution (defined in (3.10)) and the sought solution $u$ (defined in (2.9)) to Problem (1.1)-(1.3), thanks to the triangle inequality we have
\[
\|u_\alpha^\varepsilon(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \leq \|u_\alpha^\varepsilon(\cdot, t) - v_\alpha(\cdot, t)\|_{L^2(\Omega)} + \|v_\alpha(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)},
\]
where the function $v_\alpha \in C([0, T]; L^2(\Omega))$ is defined as in (3.10) with the exact final data pair $(g, h)$. First, we begin with the estimation of $\|u_\alpha^\varepsilon(\cdot, t) - v_\alpha(\cdot, t)\|_{L^2(\Omega)}$. In the argument analogous as in the Theorem 3.2 using assumption $(A_3)$, we have that
\[
\|u_\alpha^\varepsilon(\cdot, t) - v_\alpha(\cdot, t)\|_{L^2(\Omega)} \leq C_2 \left( \|g^\varepsilon - g\|_{L^2(\Omega)} + \|h^\varepsilon - h\|_{L^2(\Omega)} \right) \left( \frac{T\alpha^{-1}}{\log(T\alpha^{-1})} \right)^{1-\frac{1}{\alpha}} \exp(TK(T-t))
\]
\leq 2C_2\varepsilon \left( \frac{T\alpha^{-1}}{\log(T\alpha^{-1})} \right)^{1-\frac{1}{\alpha}} \exp(TK(T-t)),
\]
for all $t \in [0, T)$ and the positive constant $C_2$ is defined in Theorem 3.2.

Next we estimate $\|v_\alpha(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)}$, and we observe that
\[
\|v_\alpha(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \leq \|v_\alpha(\cdot, t) - Q_\alpha u(\cdot, t)\|_{L^2(\Omega)} + \|Q_\alpha u(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)},
\]
where the operator $Q_\alpha$ is defined as
\[
Q_\alpha w = \sum_{j \in \mathbb{N}^*} \mathcal{F}_j(\alpha) \langle w, \xi_j \rangle \xi_j(x).
\]
From (2.9), (3.2), (3.3) and (3.29), we have that
\[
Q_\alpha u(t) = S_\alpha(T-t)g - P_\alpha(T-t)h + \int_t^T P_\alpha(s-t)\mathcal{R}(u)(s)ds.
\]
Estimate the term $J_\alpha^\varepsilon(t)$, using (3.11b) and $(A_2)$, and one has
\[
|J_\alpha^\varepsilon(t)| \leq \int_t^T \|P_\alpha(s-t)\|_{L^2(\Omega)} \|\mathcal{R}(v_\alpha)(s) - \mathcal{R}(u)(s)\|_{L^2(\Omega)} ds
\]
\leq T \int_t^T \left( \frac{T\alpha^{-1}}{\log(T\alpha^{-1})} \right)^{\frac{1}{\alpha}} \|\mathcal{R}(v_\alpha)(\cdot, s) - \mathcal{R}(u)(\cdot, s)\|_{L^2(\Omega)} ds
\leq KT \int_t^T \left( \frac{T\alpha^{-1}}{\log(T\alpha^{-1})} \right)^{\frac{1}{\alpha}} \|v_\alpha(\cdot, s) - u(\cdot, s)\|_{L^2(\Omega)} ds.
\]
We continue to estimate the term \( J_2^a(t) \). From (2.9) and (3.30), we infer that

\[
|J_2^a(t)|^2 = \|(Q_a - I)u(t)\|^2_{L^2(\Omega)} = \sum_{j \in \mathbb{N}^*} |\mathcal{F}_j(a) - 1|^2 \langle u(\cdot, t), \xi_j \rangle^2
\]

\[
= \sum_{j \in \mathbb{N}^*} \left( \frac{\exp (-aT\lambda_j^2)}{\alpha a \lambda_j^2 + \exp (-aT\lambda_j^2)} - 1 \right)^2 \langle u(\cdot, t), \xi_j \rangle^2
\]

\[
= \sum_{j \in \mathbb{N}^*} \left( \frac{\alpha a \lambda_j^2}{\alpha a \lambda_j^2 + \exp (-aT\lambda_j^2)} \right)^2 \langle u(\cdot, t), \xi_j \rangle^2
\]

\[
\leq \sum_{j \in \mathbb{N}^*} \alpha^2 a^2 \lambda_j^4 \exp (2Ta\lambda_j^2) \left( \frac{\exp (-ta\lambda_j^2)}{\alpha a \lambda_j^2 + \exp (-aT\lambda_j^2)} \right)^2 \langle u(\cdot, t), \xi_j \rangle^2
\]

\[
\leq \alpha^2 a^2 \left( \frac{T^{-1}}{\log \frac{T}{T^{-1}}} \right)^{\frac{T-1}{T}} \sum_{j \in \mathbb{N}^*} \langle u(\cdot, t), \xi_j \rangle^2 \lambda_j^4 \exp (2Ta\lambda_j^2)
\]

\[
\leq \alpha^2 a^2 \left( \frac{T^{-1}}{\log \frac{T}{T^{-1}}} \right)^{\frac{T-1}{T}} \|u(\cdot, t)\|^2_{\mathcal{G}_{r_1, r_2}(\Omega)}, \quad (3.32)
\]

for \( t \in [0, T] \) and \( \beta = 2, r_1 \geq 1, r_2 \geq aT \), and where we have used the inequality (in the same way as in the proof of (3.11a) of Lemma 3.1)

\[
\frac{\exp (-ta\lambda_j^2)}{\alpha a \lambda_j^2 + \exp (-aT\lambda_j^2)} = \left( \frac{\exp (-T\alpha \lambda_j^2)}{\alpha a \lambda_j^2 + \exp (-aT\lambda_j^2)} \right)^{\frac{T-1}{T}} \frac{1}{\alpha a \lambda_j^2 + \exp (-aT\lambda_j^2)}
\]

Combining (3.28), (3.31) and (3.32) gives

\[
\|v_\alpha(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \leq \alpha a \left( \frac{T^{-1}}{\log \frac{T}{T^{-1}}} \right)^{\frac{T-1}{T}} \|u\|_{L^\infty(0, T; \mathcal{G}_{r_1, r_2}(\Omega))}
\]

\[
+ KT \int_t^T \left( \frac{T^{-1}}{\log \frac{T}{T^{-1}}} \right)^{\frac{T-1}{T}} \|v_\alpha(\cdot, s) - u(\cdot, s)\|_{L^2(\Omega)} \, ds.
\]

\[
(3.33)
\]

Multiplying both sides of (3.33) by \( \left( \frac{T^{-1}}{\log \frac{T}{T^{-1}}} \right)^{\frac{T-1}{T}} > 0 \) and using the Grönwall inequality, we obtain

\[
\left( \frac{T^{-1}}{\log \frac{T}{T^{-1}}} \right)^{\frac{T-1}{T}} \|v_\alpha(\cdot, t) - u(\cdot, t)\|_{L^2(\Omega)} \leq \alpha a \|u\|_{L^\infty(0, T; \mathcal{G}_{r_1, r_2}(\Omega))} \exp (TK(T - t)).
\]

\[
(3.34)
\]
We can now combine the results of (3.27) with (3.34) to infer that

\[ \|u_\varepsilon^T(t) - u(t)\|_{L^2(\Omega)} \leq \left(2C_2\varepsilon\alpha^{-1} + a\|u\|_{L^\infty(\Omega)}\right) \alpha^\frac{1}{\alpha} \left(\frac{T}{\log \left(\frac{T}{\alpha}\right)}\right)^{1/\alpha} \exp(\log_2(T-t)) , \]

(3.36)

for all \( t \in [0,T] \). The proof of Theorem 3.2 is complete. \( \square \)

3.2. The locally Lipschitz case \( \mathcal{R}(u) = u|u|^{p-1}, \ p > 1 \). We will need the following assumptions:

(A4) The locally Lipschitz condition (coercive-type): there exist a constant \( \ell > 0 \) such that

\[ |\mathcal{R}_{\text{loc}}(u) - \mathcal{R}_{\text{loc}}(w)| \leq \ell (1 + |u|^{p-1} + |w|^{p-1}) |u - w|, \quad \text{for } u, w \in \mathbb{R}, \ p > 1. \]  

(3.37)

(A5) The perturbed datum \( g^\varepsilon, h^\varepsilon \in L^m(\Omega), m \geq 1 \) are the approximate values of \( g, h \in L^m(\Omega) \), respectively and satisfy

\[ \|g^\varepsilon - g\|_{L^m(\Omega)} \leq \varepsilon, \quad \text{and} \quad \|h^\varepsilon - h\|_{L^m(\Omega)} \leq \varepsilon. \]  

(3.38)

Throughout we shall consider Problem (1.1)-(1.3) for the source \( \mathcal{R}_{\text{loc}}(u) = u|u|^{p-1} \). In this case, the problem is still ill-posed, so it is necessary to regularize the solution of Problem (1.1)-(1.3). In the subsection, using the kernel \( \mathcal{F}_j(\Omega) \) defined as in (3.4), we propose the new regularized solution

\[ u_\varepsilon^T(t) = S\pi(T-t)g^\varepsilon - P\pi(T-t)h^\varepsilon + \int_t^T P\pi(s-t)\mathcal{R}_{\text{loc}}(u_\varepsilon^T)(s)ds, \]  

(3.39)

for \( (x,t) \in Q_T \). Here \( g^\varepsilon, h^\varepsilon \in L^m(\Omega), m \geq 1 \) are the approximate datum of \( g, h \in L^m(\Omega) \) and we assume that \( \varepsilon = \varepsilon(z) > 0 \) tends to infinity as \( \varepsilon \) goes to \( 0^+ \).

The following lemmas will play a crucial role in the rest of this paper.

Lemma 3.4. (See [11]) We have the following Sobolev embeddings

\[ \left\{ \begin{array}{ll}
L^m(\Omega) \hookrightarrow \mathcal{H}^\nu(\Omega), & \text{if } -\frac{N}{4} < \nu \leq 0, \ m \geq \frac{2N}{N-4\nu}, \\
\mathcal{H}^\nu(\Omega) \hookrightarrow L^m(\Omega), & \text{if } 0 \leq \nu < \frac{N}{4}, \ m \leq \frac{2N}{N-4\nu}.
\end{array} \right. \]  

(3.40)

Lemma 3.5. Let \( \nu^- \leq 0 < \nu^+ \) satisfying \( \nu^+ - \nu^- \in (0,1) \). For \( u, w \in \mathcal{H}^{\nu^+}(\Omega) \), for each \( M > 0 \) there exists a constant \( K_M > 0 \) such that

\[ \|\mathcal{R}_{\text{loc}}(u) - \mathcal{R}_{\text{loc}}(w)\|_{L^p(\Omega)} \leq K_M \|u - w\|_{\mathcal{H}^{\nu^+}(\Omega)}, \text{ if } \max \left\{ \|u\|_{\mathcal{H}^{\nu^+}(\Omega)}; \|w\|_{\mathcal{H}^{\nu^+}(\Omega)} \right\} \leq M. \]

(3.41)

Moreover, we assume that \( \mathcal{R}_{\text{loc}}(0)(x,t) = 0, \forall (x,t) \in Q_T \) and

\[ \|\mathcal{R}_{\text{loc}}(u)\|_{L^{p\nu^-}(\Omega)} \leq K_M \|u\|_{\mathcal{H}^{\nu^+}(\Omega)}. \]

(3.42)

Proof. Indeed, for \( N \geq 1, p > 1, 0 < \nu^+ < \min \left\{ 1; \frac{(p-1)N}{4p} \right\} \), \( \nu^- = \nu^+ - \zeta \), for some constant \( \zeta \) satisfying

\[ \nu^+ < \zeta < \min \left\{ \nu^+ + \frac{N}{4}; (p-1)\frac{N}{4} + 1 + (p-1)\nu^+ \right\}. \]
It is possible to verify $0 < \nu^+ - \nu^- < 1$, and let us put $\nu^* = \frac{2N}{N+4\nu^-}$ and $\nu^{**} = \frac{-2N}{N+4\nu^-}$. First, using the triangle inequality, one deduces that

$$
\left\| R_{\text{loc}}(u) - R_{\text{loc}}(w) \right\|_{L^{p\nu^*}(\Omega)} = \left\| u|u|^{p-1} - w|w|^{p-1} \right\|_{L^{p\nu^*}(\Omega)}
\leq \ell \left\| u - w \right\|_{L^{p\nu^*}(\Omega)} + \ell \left\| u|u|^{p-1} - w|w|^{p-1} \right\|_{L^{p\nu^*}(\Omega)}
\leq \left( \int_\Omega |u|^{\nu^*} \, dx \right)^{p-1} \left( \int_\Omega |u - w|^{\nu^*} \, dx \right)^{1/p}
\leq \| u \|_{L^{p\nu^*}(\Omega)}^{p\nu^*} \| u - w \|_{L^{p\nu^*}(\Omega)}^{\nu^*}.
$$

From the Hölder inequality, we deduce that

$$
\left\| u|u|^{p-1} - w|w|^{p-1} \right\|_{L^{p\nu^*}(\Omega)} \leq \int_\Omega |u|^{(p-1)\nu^*} |u - w|^{\nu^*} \, dx
\leq \left| \int_\Omega |u|^{p\nu^*} \, dx \right|^{p-1} \left| \int_\Omega |u - w|^{p\nu^*} \, dx \right|^{1/p}
\leq \| u \|_{L^{p\nu^*}(\Omega)}^{p\nu^*} \| u - w \|_{L^{p\nu^*}(\Omega)}^{\nu^*}.
$$

Then, we get

$$
\left\| u|u|^{p-1} - w|w|^{p-1} \right\|_{L^{p\nu^*}(\Omega)} \leq \| u \|_{L^{p\nu^*}(\Omega)}^{p\nu^*} \| u - w \|_{L^{p\nu^*}(\Omega)}.
$$

Similarly, we also obtain that

$$
\left\| w|w|^{p-1} - u|u|^{p-1} \right\|_{L^{p\nu^*}(\Omega)} \leq \left\| w \right\|_{L^{p\nu^*}(\Omega)}^{p\nu^*} \| u - w \|_{L^{p\nu^*}(\Omega)}.
$$

From $p\nu^+ < \zeta < p\nu^- + \frac{N}{4}$, we deduce that $\frac{N}{4} < \nu^- \leq 0$, using Lemma 3.4, we get that

$$
L^{p\nu^*}(\Omega) \hookrightarrow \mathcal{H}^{\nu^-}(\Omega).
$$

This leads to

$$
\left\| R_{\text{loc}}(u) - R_{\text{loc}}(w) \right\|_{\mathcal{H}^{\nu^-}(\Omega)} \leq C_4 \left\| R_{\text{loc}}(u) - R_{\text{loc}}(w) \right\|_{L^{p\nu^*}(\Omega)},
$$

for any $C_4 := C_4(N, \nu^-) > 0$.

Moreover, for $\zeta < \frac{N}{4}(p-1)$ and $\nu^- = p\nu^+ - \zeta$, we observe that:

$$
p\nu^* = \frac{2Np}{N + 4\nu^- - 4p\nu^+} > \frac{2Np}{N + (p-1)N - 4p\nu^+} = \nu^{**},
$$

and $0 < \nu^+ < \frac{(p-1)N}{4p} < \frac{N}{4}$. Then, the following Sobolev embedding holds:

$$
\mathcal{H}^{\nu^+}(\Omega) \hookrightarrow L^{p\nu^*}(\Omega).
$$

Then, we imply that $\| w \|_{L^{p\nu^*}(\Omega)} \leq C_5 \| w \|_{\mathcal{H}^{\nu^+}(\Omega)}$ for all $w \in \mathcal{H}^{\nu^+}(\Omega)$, and $C_5 := C_5(N, p, \nu^-, \nu^+) > 0$. Combine this with (3.43), (3.45), (3.46) and (3.47), and we have

$$
\left\| R_{\text{loc}}(u) - R_{\text{loc}}(w) \right\|_{\mathcal{H}^{\nu^-}(\Omega)}
\leq \ell C_4 C_5 \left( 1 + C_5 \| u \|_{\mathcal{H}^{\nu^+}(\Omega)}^{p\nu^+} + C_5 \| w \|_{\mathcal{H}^{\nu^+}(\Omega)}^{p\nu^+} \right) \| u - w \|_{\mathcal{H}^{\nu^+}(\Omega)}.
$$

It is easily seen that (3.41) is fulfilled for $K_M := \ell C_4 \left( 1, C_5 \right)^2 \left( 1 + 2M^{p-1} \right) > 0$. We shall omit the easy proof of (3.42).

The main tool for our proofs will be the following Lemma.
Lemma 3.6. For $0 \leq t \leq T$ and let $\nu^- \leq 0 < \nu^+$, and we have the following hold:

\[
\|S_{\tau}(t)\|_{\mathcal{L}:H^{\nu+}(\Omega) \to H^{\nu-}(\Omega)} \leq C_1 |\tau|^{\nu^+-\nu^-} \exp\left( a |\tau|^2 t \right), \tag{3.49a}
\]
\[
\|P_{\tau}(t)\|_{\mathcal{L}:H^{\nu+}(\Omega) \to H^{\nu-}(\Omega)} \leq T |\tau|^{\nu^+-\nu^-} \exp\left( a |\tau|^2 t \right). \tag{3.49b}
\]

Proof. We shall begin with the proof of (3.49a). Indeed, for $\vartheta \in H^{\nu^-}(\Omega)$ and let $\nu^- \leq 0 < \nu^+$, similarly for (3.13), we have

\[
\|S_{\tau}(t)\vartheta\|_{H^{\nu+}(\Omega)}^2 = \sum_{j \in \mathbb{N}^+} \gamma_j^2 \left( \kappa_j^+ \exp \left( t\kappa_j^- - \kappa_j^- \exp \left( t\kappa_j^+ \right) \right) (\vartheta, \xi_j) \right)^2 \lambda_j^{2\nu^+}.
\]

For (3.49b), the proof is analogous to (3.15), and one has

\[
\|P_{\tau}(t)\vartheta\|_{H^{\nu+}(\Omega)}^2 = \sum_{j \in \mathbb{N}^+} \gamma_j^2 \frac{\exp(\langle t\kappa_j^+, \vartheta \rangle) - \exp(\langle t\kappa_j^-, \vartheta \rangle)}{\delta_j} (\vartheta, \xi_j)^2 \lambda_j^{2\nu^-}.
\]

For (3.49b), the proof is analogous to (3.15), and one has

\[
\|P_{\tau}(t)\vartheta\|_{H^{\nu+}(\Omega)}^2 \leq C_2 |\tau|^{2\nu^+-\nu^-} \exp(2a|\tau|^2t) \sum_{\lambda_j \leq \pi} (\vartheta, \xi_j)^2 \lambda_j^{2\nu^-}.
\]

The proof is complete. \qed

Theorem 3.7 (Existence-regularity). Suppose that $\mathcal{R}_{loc}$ satisfies the conditions (3.41) and (3.42) for $\frac{N}{2} < \nu^- \leq 0 \leq \nu^+ < \frac{N}{4}$. Let $g^\varepsilon, h^\varepsilon \in L^m(\Omega)$ for $m \leq \frac{2N}{\nu^+ - \nu^-} =: \nu^*$, $N \in \mathbb{N}^*$. Then the integral equation (3.39) has a unique solution $u^\varepsilon_\tau$ in $C_0^2(0, T; H^{\nu^+}(\Omega))$, for $\varepsilon > 0$. Moreover, we also have

\[
\|u^\varepsilon_\tau(\cdot, t)\|_{H^{\nu^+}(\Omega)} \leq C_6^\varepsilon \left( \|g^\varepsilon\|_{L^m(\Omega)} + \|h^\varepsilon\|_{L^m(\Omega)} \right), \tag{3.52}
\]

for all $t \in [0, T]$ and where $C_6^\varepsilon > 0$ depends on $\varepsilon$. 


Proof. For \( \vartheta \in C([0, T]; L^2(\Omega)) \), let us consider the nonlinear mapping

\[
\mathcal{M} \vartheta(t) := S_{\varpi}(T - t)g^\varpi - P_{\varpi}(T - t)h^\varpi + \int_t^T P_{\varpi}(s - t)R_{loc}(\vartheta)(s)ds. \tag{3.53}
\]

Let us pick \( \nu^+, \nu^- \) satisfying \(-\frac{N}{4} < \nu^- \leq 0 \leq \nu^+ < \frac{N}{4} \), for \( \vartheta_i \in \mathcal{H}^{\nu^+}(\Omega) \) \( (i = 1, 2) \), using (3.41) and Lemma 3.6, one obtains (for the constant \( \varphi > 0 \))

\[
\exp (-\varphi(T - t)) \| \mathcal{M} \vartheta_1(t) - \mathcal{M} \vartheta_2(t) \|_{\mathcal{H}^{\nu^+}(\Omega)} \leq \int_t^T \exp (-\varphi(T - t)) \| P_{\varpi}(s - t) (R_{loc}(\vartheta_1)(s) - R_{loc}(\vartheta_2)(s)) \|_{\mathcal{H}^{\nu^+}(\Omega)} ds \leq T |\varpi|^{\nu^+ - \nu^-} \int_t^T \exp \left( (\varphi + a|\varpi|^2) (t - s) \right) \exp (-\varphi(T - s)) \times \| R_{loc}(\vartheta_1)(\cdot, s) - R_{loc}(\vartheta_2)(\cdot, s) \|_{\mathcal{H}^{\nu^+}(\Omega)} ds \leq K_M T |\varpi|^{\nu^+ - \nu^-} \int_t^T \exp \left( (\varphi + a|\varpi|^2) (t - s) \right) \exp (-\varphi(T - s)) \times \| \vartheta_1(\cdot, s) - \vartheta_2(\cdot, s) \|_{\mathcal{H}^{\nu^+}(\Omega)} ds \leq K_M T |\varpi|^{\nu^+ - \nu^-} \| \vartheta_1 - \vartheta_2 \|_{\mathcal{H}^{\nu^+}(\Omega)} \int_t^T \exp \left( (\varphi + a|\varpi|^2) (t - s) \right) ds \leq \frac{K_M T |\varpi|^{\nu^+ - \nu^-}}{\varphi + a|\varpi|^2} \left( 1 - \exp \left( (\varphi + a|\varpi|^2) (T - t) \right) \right) \| \vartheta_1 - \vartheta_2 \|_{\mathcal{H}^{\nu^+}(\Omega)} \leq \frac{K_M T |\varpi|^{\nu^+ - \nu^-}}{\varphi + a|\varpi|^2} \| \vartheta_1 - \vartheta_2 \|_{\mathcal{H}^{\nu^+}(\Omega)}, \tag{3.54}
\]

where we have used that \( 1 - \exp \left( (\varphi + a|\varpi|^2) (T - t) \right) \leq 1 \), for all \( 0 \leq t \leq T \). Hence, we have

\[
\| \mathcal{M} \vartheta_1 - \mathcal{M} \vartheta_2 \|_{\mathcal{H}^{\nu^+}(\Omega)} \leq \frac{K_M T |\varpi|^{\nu^+ - \nu^-}}{\varphi + a|\varpi|^2} \| \vartheta_1 - \vartheta_2 \|_{\mathcal{H}^{\nu^+}(\Omega)}. \tag{3.55}
\]

So we pick \( \varphi \) so large that \( \frac{K_M T |\varpi|^{\nu^+ - \nu^-}}{\varphi + a|\varpi|^2} < 1 \) (noting that \( \nu^+ - \nu^- \leq 1 \)). We can conclude that \( \mathcal{M} \) is a contraction, and by the Banach fixed point theorem, it follows that the equation \( \mathcal{M} \vartheta = \vartheta \) has a unique solution \( u^\varpi \in \mathcal{C}^\varpi([0, T]; \mathcal{H}^{\nu^+}(\Omega)) \).

Moreover, using (3.42) and thanks to Lemma 3.6, and note that for \( m \leq \nu^* = \frac{2N}{N - \nu^-} \), with \(-\frac{N}{4} \leq \nu^- \leq 0 \), since the Sobolev embedding

\[
L^m(\Omega) \hookrightarrow \mathcal{H}^{\nu^-}(\Omega),
\]

we have that \( \| w \|_{\mathcal{H}^{\nu^-}(\Omega)} \leq C_4 \| w \|_{L^m(\Omega)} \), and one obtains
\[ u_\eta^\nu(t) \in H^\nu(\Omega) \]

where \( C_2 \) is defined as in Theorem 3.2. Multiplying both sides of (3.56) by \( \exp(a|\alpha|^2(t - T)) \) and from Grönwall’s inequality, we get that

\[ \exp(a|\alpha|^2(t - T)) \frac{\|u\|^2_\nu(t)}{\|u\|^2_\nu + (\Omega)} \leq (3.57) \]

by putting \( C_\nu := C_2C_4|\alpha|^{\nu-\nu} \exp\left(K_M T|\alpha|^{\nu-\nu} - a|\alpha|^2\right) > 0 \), and this (3.52).

**Theorem 3.8 (\( H^\nu \)-Estimate).** For the constants \( \nu^+, \nu^- \) as in Theorem 3.7, the source term \( R_{loc} \) satisfies (3.41) and we can choose \( \alpha > 0 \) such that \( \lim_{\varepsilon \to 0^+} \alpha = \infty \) and

\[ \lim_{\varepsilon \to 0^+} \varepsilon \exp\left(2C_7|\alpha|^2\right) = 0, \quad \text{for } C_7 = \max\left\{K_M T^2; aT\right\} > 0. \]

Suppose that the final datum \( g^\nu, h^\nu, g, h \in L^m(\Omega) \) satisfy (3.38) and Problem (1.1)-(1.3) has a unique solution \( u \in L^\infty(0,T; \mathcal{G}^\nu_{r_1, r_2}(\Omega)) \) for \( \beta = 2, r_1 \geq 1, r_2 \geq 2aT \). Then

\[ u_\varepsilon^\nu(t) = u^\nu(t)\]

for all \( t \in [0,T] \), and \( C_\varepsilon = 2C_2C_4 > 0 \). Moreover, we can choose \( \alpha := \sqrt{\frac{1}{2C_7} \log\left(\frac{1}{\varepsilon}\right)} \), for \( \eta \in [0,1) \). Then the condition in (3.58) is fulfilled and the error estimate in (3.59) is of order \( \varepsilon^{\frac{2\eta}{1+\eta}} \), \( t \in [0,T] \).

\[ \|u_\varepsilon^\nu(t) - u^\nu(t)\|_{H^\nu(\Omega)} \leq \left( C_\varepsilon |\alpha|^{\nu-\nu} \exp\left(2C_7|\alpha|^2\right) \right) \exp\left(-a|\alpha|^2 t\right), \]

(3.59)

for all \( t \in [0,T] \), and \( C_\varepsilon = 2C_2C_4 > 0 \). Moreover, we can choose

\[ C_\varepsilon := \sqrt{\frac{1}{2C_7} \log\left(\frac{1}{\varepsilon}\right)} \], for \( \eta \in [0,1) \). Then the condition in (3.58) is fulfilled and the error estimate in (3.59) is of order \( \varepsilon^{\frac{2\eta}{1+\eta}} \), \( t \in [0,T] \).

Proof. Using the triangle inequality we obtain

\[ \|u_\varepsilon^\nu(t) - u^\nu(t)\|_{H^\nu(\Omega)} \leq \|u_\varepsilon^\nu(t) - V_\varepsilon^\nu(t)\|_{H^\nu(\Omega)} + \|V_\varepsilon^\nu(t) - u(t)\|_{H^\nu(\Omega)}, \]

(3.60)

where \( V_\varepsilon \in C([0,T]; H^\nu(\Omega)) \) is a function of the form (3.39) with exact datum \( g, h \in L^m(\Omega) \). The proof of Theorem 3.8 is divided into two steps.
**Step 1. Estimate** \( \|u_\pi^\varepsilon(\cdot,t) - V_\pi(\cdot,t)\|_{\mathcal{H}^{\nu^+}(\Omega)} \). Using Lemma 3.5, Lemma 3.6 and for \( m \leq \nu^- \), with \(-\frac{N}{2} < \nu^- \leq 0 \), since the Sobolev embedding \( L^m(\Omega) \hookrightarrow \mathcal{H}^{\nu^-}(\Omega) \), we obtain that

\[
\begin{align*}
\|u_\pi^\varepsilon(\cdot,t) - V_\pi(\cdot,t)\|_{\mathcal{H}^{\nu^+}(\Omega)} & \leq \|S_\pi(T-t) (g^\varepsilon - g)\|_{\mathcal{H}^{\nu^+}(\Omega)} + \|P_\pi(T-t) (h^\varepsilon - h)\|_{\mathcal{H}^{\nu^+}(\Omega)} \\
& \quad + \int_t^T \|P_\pi(s-t) (R_{loc}(u_\pi^\varepsilon)(s) - R_{loc}(V_\pi)(s))\|_{\mathcal{H}^{\nu^+}(\Omega)} \, ds \\
& \leq C_2C_4|\pi|^{\nu^+ - \nu^-} \exp(a|\pi|^2(T-t)) \left( \|g^\varepsilon - g\|_{L^m(\Omega)} + \|h^\varepsilon - h\|_{L^m(\Omega)} \right) \\
& \quad + T|\pi|^{\nu^+ - \nu^-} \exp(a|\pi|^2(s-t)) \|R_{loc}(u_\pi^\varepsilon)(\cdot,s) - R_{loc}(V_\pi)(\cdot,s)\|_{\mathcal{H}^{\nu^-}(\Omega)} \, ds \\
& \leq C_8|\pi|^{\nu^+ - \nu^-} \exp(a|\pi|^2(T-t)) \varepsilon \\
& \quad + K_MT|\pi|^{\nu^+ - \nu^-} \int_t^T \exp(a|\pi|^2(s-t)) \|u_\pi^\varepsilon(s,t) - V_\pi(\cdot,s)\|_{\mathcal{H}^{\nu^+}(\Omega)} \, ds. \quad (3.61)
\end{align*}
\]

Next, let us multiply (3.61) by \( \exp(a|\pi|^2(t-T)) \) and thanks to Grönwall’s inequality, we deduce

\[
\exp(a|\pi|^2(t-T)) \|u_\pi^\varepsilon(\cdot,t) - V_\pi(\cdot,t)\|_{\mathcal{H}^{\nu^+}(\Omega)} \leq C_8|\pi|^{\nu^+ - \nu^-} \varepsilon \exp(K_MT|\pi|^{\nu^+ - \nu^-}(T-t)), \quad (3.62)
\]

Noting that \( \exp(K_MT|\pi|^{\nu^+ - \nu^-}(T-t)) \leq \exp(K_MT^2|\pi|^{\nu^+ - \nu^-}) \), \( \forall t \in [0,T] \), we infer that

\[
\begin{align*}
\|u_\pi^\varepsilon(\cdot,t) - V_\pi(\cdot,t)\|_{\mathcal{H}^{\nu^+}(\Omega)} & \leq C_8|\pi|^{\nu^+ - \nu^-} \varepsilon \exp(K_MT^2|\pi|^{\nu^+ - \nu^-} + a|\pi|^2T) \exp(-a|\pi|^2t) \\
& \leq C_8|\pi|^{\nu^+ - \nu^-} \varepsilon \exp(2C_7|\pi|^2) \exp(-a|\pi|^2t), \quad (3.63)
\end{align*}
\]

where for \( \nu^+ - \nu^- \in (0,1) \) and \( \pi \) is large enough and then we have \( |\pi|^{\nu^+ - \nu^-} \leq |\pi|^2 \).

**Step 2. Estimate** \( \|V_\pi(\cdot,t) - u(\cdot,t)\|_{\mathcal{H}^{\nu^+}(\Omega)} \). For \( u \) a solution of the integral equation (2.9) with the source \( R_{loc}(u) = u|u|^{p-1} \), one has

\[
\begin{align*}
\|V_\pi(\cdot,t) - u(\cdot,t)\|_{\mathcal{H}^{\nu^+}(\Omega)} & \leq \|V_\pi(\cdot,t) - \tilde{Q}_\pi u(\cdot,t)\|_{\mathcal{H}^{\nu^+}(\Omega)} + \|u(\cdot,t) - \tilde{Q}_\pi u(\cdot,t)\|_{\mathcal{H}^{\nu^+}(\Omega)} \\
& =: J_{\bar{F}}(t) + J_{\bar{E}}(t), \quad (\text{respectively}), \quad (3.64)
\end{align*}
\]

where \( \tilde{Q}_\pi \) is defined

\[
\tilde{Q}_\pi w := \sum_{j \in \mathbb{N}^n} \tilde{F}_j(\pi) \langle w, \xi_j \rangle \xi_j(x) = \sum_{j \in \pi} \langle w, \xi_j \rangle \xi_j(x). \quad (3.65)
\]
Using Lemma 3.5 and Lemma 3.6, we estimate the term \( J_3^\gamma(t) \) as follows

\[
|J_3^\gamma(t)| \leq \int_t^T \| P_\gamma(s-t) (R_{t,\text{loc}}(V_\gamma)(s) - R_{t,\text{loc}}(u)(s)) \|_{\mathcal{V}^+} \, ds
\]

\[
\leq T \| \alpha \|_{\mathcal{V}^+} \int_t^T \exp \left( a |\alpha|^2 (s-t) \right) \| R_{t,\text{loc}}(V_\gamma)(\cdot, s) - R_{t,\text{loc}}(u)(\cdot, s) \|_{\mathcal{V}^+} \, ds
\]

\[
\leq K_M T \| \alpha \|_{\mathcal{V}^+} \int_t^T \exp \left( a |\alpha|^2 (s-t) \right) \| V_\gamma(\cdot, s) - u(\cdot, s) \|_{\mathcal{V}^+} \, ds
\]

(3.66)

We continue to estimate \( J_4^\gamma(t) \), and we have

\[
|J_4^\gamma(t)| = \left( \sum_{j>\mathfrak{n}} (u(\cdot, t), \xi_j)^2 \lambda_j^{2\nu^+} \right)^{\frac{1}{2}}
\]

\[
\leq \exp \left( -a |\alpha|^2 (T+t) \right) \left( \sum_{j>\mathfrak{n}} (u(\cdot, t), \xi_j)^2 \lambda_j^{2\nu^+ + 2\sigma} \exp \left( 4a \lambda_j^2 T \right) \right)^{\frac{1}{2}}
\]

\[
\leq \exp \left( -a |\alpha|^2 (T+t) \right) \| u(\cdot, t) \|_{\mathcal{V}^+_{1,\text{loc}}} \cdot \frac{1}{|\alpha|^\sigma}, \quad \text{for any } \sigma > 0,
\]

(3.67)

and for \( \beta = 2, r_1 \geq \frac{\nu^+ + \sigma}{2}, r_2 \geq 2aT \). Combining (3.64), (3.66) and (3.67), multiplying both sides by \( \exp \left( a |\alpha|^2 t \right) > 0 \), and we obtain

\[
\exp \left( a |\alpha|^2 t \right) \| V_\gamma(\cdot, t) - u(\cdot, t) \|_{\mathcal{V}^+} \leq \frac{\exp \left( -a |\alpha|^2 T \right) \| u(\cdot, t) \|_{\mathcal{V}^+_{1,\text{loc}}} \cdot \exp \left( K_M T |\alpha|^{\nu^+ - \nu^-} (T-t) \right)}{|\alpha|^\sigma} + K_M T |\alpha|^{\nu^+ - \nu^-} \int_t^T \exp \left( a |\alpha|^2 s \right) \| V_\gamma(\cdot, s) - u(\cdot, s) \|_{\mathcal{V}^+} \, ds.
\]

(3.68)

Grönwall’s inequality allows to obtain

\[
\exp \left( a |\alpha|^2 t \right) \| V_\gamma(\cdot, t) - u(\cdot, t) \|_{\mathcal{V}^+} \leq \exp \left( a |\alpha|^2 T \right) \| u(\cdot, t) \|_{\mathcal{V}^+_{1,\text{loc}}} \cdot \exp \left( K_M T |\alpha|^{\nu^+ - \nu^-} (T-t) \right)
\]

\[
\leq \frac{\| u(\cdot, t) \|_{\mathcal{V}^+_{1,\text{loc}}} \cdot \exp \left( C_7 \left( |\alpha|^{\nu^+ - \nu^-} - |\alpha|^2 \right) \right)}{|\alpha|^\sigma}.
\]

(3.69)

From \( \nu^+ - \nu^- < 1 < 2 \) then \( |\alpha|^{\nu^+ - \nu^-} \leq |\alpha|^2 \), for \( \alpha \) large enough, and we infer that

\[
\exp \left( C_7 \left( |\alpha|^{\nu^+ - \nu^-} - |\alpha|^2 \right) \right) \leq 1,
\]

so, we get that

\[
\| V_\gamma(\cdot, t) - u(\cdot, t) \|_{\mathcal{V}^+} \leq \frac{\| u(\cdot, t) \|_{\mathcal{V}^+_{1,\text{loc}}} \cdot \exp \left( a |\alpha|^2 t \right)}{|\alpha|^\sigma}, \quad 0 \leq t \leq T.
\]

(3.70)

Combining (3.63) and (3.70) we get (3.52) immediately. 

\[\square\]
4. Conclusions. This paper considers the nonlinear hyperbolic equation with damping term. The problem was discussed using the filter regularization method based on two kernels (3.1) and (3.4) for the resulting nonlinear integral equation (2.9). Convergence and stability estimates were formulated and proved ($L^2(\Omega)$ and $H^\nu(\Omega)$ estimates for the source functions $R$ satisfying the global Lipschitz and local Lipschitz condition, respectively).

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