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“Elementary” Number Theory / Théorie “élémentaire” des nombres

On a congruence involving \( q \)-Catalan numbers

Sur une congruence impliquant des \( q \)-nombres de Catalan

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Abstract. Based on a \( q \)-congruence of the author and Petrov, we set up a \( q \)-analogue of Sun–Tauraso’s congruence for sums of Catalan numbers, which extends a \( q \)-congruence due to Tauraso.

Résumé. À partir d’une \( q \)-congruence de l’auteur et Petrov, nous établissons un \( q \)-analogue de la congruence de Sun–Tauraso pour des sommes de nombres de Catalan, qui étend la \( q \)-congruence due à Tauraso.

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1. Introduction

In combinatorics, the Catalan numbers are a sequence of natural numbers, which play an important role in various counting problems. The \( n \)th Catalan number is given by the following binomial coefficient:

\[
C_n = \binom{2n}{n} \frac{1}{n+1} = \binom{2n}{n} - \binom{2n}{2n+1}.
\]

Closely related numbers are the central binomial coefficients \( \binom{2n}{n} \) for \( n \geq 0 \).

Both Catalan numbers and central binomial coefficients satisfy many interesting congruences (see, for instance, [7, 9–11]). In 2011, Sun and Tauraso [11] proved that for primes \( p \geq 5 \),

\[
\sum_{k=0}^{p-1} \binom{2k}{k} \equiv \left( \frac{p}{3} \right) \pmod{p^2},
\]

\[
\sum_{k=0}^{p-1} C_k \equiv \frac{3}{2} \left( \frac{p}{3} \right) - \frac{1}{2} \pmod{p^2},
\]
where \( \left[ \frac{n}{p} \right] \) denotes the Legendre symbol.

In the past few years, \( q \)-analogues of congruences (\( q \)-congruences) for indefinite sums of binomial coefficients as well as hypergeometric series attracted many experts’ attention (see, for example, [2–6, 8, 12, 13]). It is worth mentioning that Guo and Zudilin [6] developed an interesting microscoping method to prove many \( q \)-congruences.

In order to discuss \( q \)-congruences, we first recall some \( q \)-series notation. The \( q \)-binomial coefficients are defined as

\[
\begin{bmatrix} n \\ k \end{bmatrix}_q = \binom{n}{k}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}} \quad \text{if } 0 \leq k \leq n, \\
0 \quad \text{otherwise,}
\]

where the \( q \)-shifted factorial is given by \((a;q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1})\) for \( n \geq 1 \) and \((a;q)_0 = 1\). Moreover, the \( q \)-integers are defined by \([n]_q = (1 - q^n)/(1 - q)\), and the \( n \)th cyclotomic polynomial is given by

\[
\Phi_n(q) = \prod_{1 \leq k \leq n \atop (n,k)=1} (q - e^{2\pi i/n}).
\]

Recently, the author and Petrov [8] established a \( q \)-analogue for (1) as follows:

\[
\sum_{k=0}^{n-1} q^k \cdot \binom{2k}{k}_q \equiv \left( \frac{n}{3} \right) \frac{q^{n^2}}{q^{n^2}} \pmod{\Phi_n(q)^2},
\]

which was originally conjectured by Guo [2] and generalises a \( q \)-congruence of Tauraso [12]. There are several natural \( q \)-analogues of Catalan numbers (see [1]). Here and throughout the paper, we consider the following \( q \)-analogue of Catalan numbers:

\[
C_n(q) = \frac{1}{[n+1]_q} \binom{2n}{n}_q = \binom{2n}{n} - q \binom{2n}{n+1}. \tag{4}
\]

In 2012, Tauraso [12] obtained a weak \( q \)-version of (2) as follows:

\[
\sum_{k=0}^{n-1} q^k C_k(q) \equiv \begin{cases} q^{[n/3]} & \text{if } n \equiv 0, 1 \pmod{3} \\ 1 - q^{[2n-1]/3} & \text{if } n \equiv 2 \pmod{3} \end{cases} \pmod{\Phi_n(q)},
\]

where \([x]\) denotes the integral part of real \( x \). In this note, we aim to set up a \( q \)-analogue of (2) as well as another related \( q \)-congruence for sums of binomial coefficients.

**Theorem 1.** For any positive integer \( n \), the following holds modulo \( \Phi_n(q)^2 \):

\[
\sum_{k=0}^{n-1} q^k C_k(q) \equiv \begin{cases} q^{n^2/3} - q^{2n-1}/3 & \text{if } n \equiv 2 \pmod{3} \\ q^{n^2/3} - q^{n-1}/3(q^n - 1) & \text{if } n \equiv 1 \pmod{3} \end{cases} \pmod{\Phi_n(q)}, \tag{5}
\]

In order to prove (5), we shall establish the following \( q \)-congruence.

**Theorem 2.** For any positive integer \( n \), the following holds modulo \( \Phi_n(q)^2 \):

\[
\sum_{k=0}^{n-1} q^{k+1} \left\lfloor \frac{2k}{k+1} \right\rfloor \equiv \begin{cases} q^{n^2-1}/3 & \text{if } n \equiv 2 \pmod{3}, \\ q^{n-1}/3(q^n - 1) & \text{if } n \equiv 1 \pmod{3} \end{cases} \pmod{\Phi_n(q)}. \tag{6}
\]

It is clear that (5) can be directly deduced from (3), (4) and (6). The remainder of the paper is organized as follows. We first set up a preliminary result in the next section, and prove Theorem 2 in Section 3.
2. An auxiliary result

Lemma 3. For any positive integer \( n \), the following holds modulo \( \Phi_n(q) \):

\[
\sum_{k=1}^{n-1} \frac{(-1)^k q^{\frac{k^2-k}{2}}}{1-q^k} \equiv \begin{cases} 
0 & \text{if } n \equiv 2 \pmod{3}, \\
\frac{n-1}{6} & \text{if } n \equiv 1 \pmod{3}.
\end{cases}
\] (7)

Proof. Note that

\[
\sum_{k=1}^{n-1} (-1)^k \frac{k^2-k}{2} q^k = \sum_{k=0}^{n-3} (-1)^k q^{\frac{k(k+1)}{2}} - \sum_{k=1}^{n-1} (-1)^k q^{\frac{k(k+5)}{2}}.
\]

We shall distinguish two cases to prove (7).

Case 1. \( n \equiv 2 \pmod{3} \). This case is equivalent to

\[
\sum_{k=0}^{n-1} (-1)^k \frac{k^2-k}{2} q^k \equiv \sum_{k=1}^{n} (-1)^k q^{\frac{k^2+k}{2}} \equiv 0 \pmod{\Phi_{3n+2}(q)}.
\] (8)

Let \( \omega \) be a primitive \((3n+2)\)th root of unity. Letting \( k \to n-k \) in the following sum gives

\[
\sum_{k=0}^{n-1} (-1)^k \omega \frac{k^2-k}{2} q^k = n \sum_{k=1}^{n-1} (-1)^{n-k} \omega \frac{(n-k+1)(n-k)}{2} q^{3n-3k+2} = \sum_{k=1}^{n} (-1)^k \omega \frac{3n-3k+2}{2} q^{3n-3k+2} = \sum_{k=1}^{n} (-1)^k \omega \frac{k^2+k}{2},
\]

where we have used the fact that \( \omega \frac{(3n+2)(n+1)}{2} = (-1)^{n+1} \). Thus,

\[
\sum_{k=0}^{n-1} (-1)^k \omega \frac{k^2-k}{2} q^k - \sum_{k=1}^{n} (-1)^k \omega \frac{k^2+k}{2} = 0,
\]

which is equivalent to (8).

Case 2. \( n \equiv 1 \pmod{3} \). Let \( \zeta \) be a primitive \((3n+1)\)th root of unity. It suffices to show that

\[
\sum_{k=0}^{n-1} (-1)^k \zeta \frac{k^2-k}{2} q^k - \sum_{k=1}^{n} (-1)^k \zeta \frac{k^2+k}{2} = \frac{n}{2}.
\] (9)

Note that

\[
\sum_{k=0}^{n-1} (-1)^k \zeta \frac{k^2-k}{2} q^k = \frac{2n}{2} \sum_{k=n+1}^{2n} (-1)^{2n-k} \zeta \frac{(2n-k+1)(2n-k+2)}{2} q^{3n-3k+2} = \sum_{k=n+1}^{2n} (-1)^k \zeta \frac{k^2+k}{2} q^{3n-3k+2} = \sum_{k=n+1}^{2n} (-1)^k \zeta \frac{k^2+k}{2} q^{3n-3k+2},
\]

where we replace \( k \) by \( 2n-k \) in the first step. Thus,

\[
\sum_{k=0}^{n-1} (-1)^k \zeta \frac{k^2-k}{2} q^k - \sum_{k=1}^{n} (-1)^k \zeta \frac{k^2+k}{2} = - \sum_{k=1}^{2n} (-1)^k \zeta \frac{k^2+k}{2} q^{3n-3k+2}.
\] (10)
Furthermore, letting $k \to 2n + 1 - k$ on the right-hand side of (10) gives
\[
\sum_{k=0}^{n-1} \frac{(-1)^k \zeta^{k(3k+5)}/2}{1 - \zeta^{3k+2}} - \sum_{k=1}^{n} \frac{(-1)^k \zeta^{k(3k+5)}/2}{1 - \zeta^{3k}} = -\sum_{k=1}^{2n} \frac{(-1)^{2n+1-k} \zeta^{(2n+1-k)(6n-3k+8)}/2}{1 - \zeta^{3(2n+1-k)}} = -\sum_{k=1}^{2n} \frac{(-1)^{k} \zeta^{k(3k+1)}/2}{1 - \zeta^{3k+1}}.
\]

An identity due to the author and Petrov [8, (2.4)] says
\[
\sum_{k=1}^{2n} \frac{(-1)^k \zeta^{k(3k+1)/2}}{1 - \zeta^{3k+1}} = -\frac{n}{2}.
\]

Then the proof of (9) follows from (11) and (12). \hfill \Box

3. Proof of Theorem 2

Now we are in a position to prove Theorem 2. We recall the following identity:
\[
\sum_{k=0}^{n-1} q^k \left[ \begin{array}{c} 2k \\ k+1 \end{array} \right] = \sum_{k=0}^{n-1} \left( \frac{n-k-1}{3} \right) q^k \left[ 2(n-k)^2 - (n-k)(\frac{n-k-1}{3})-3 \right] \left[ 2n \right] \left[ k \right],
\]

which was proved by Tauraso in a more general form (see [12, Theorem 4.2]). Since $1 - q^n \equiv 0 \pmod{\Phi_n(q)}$, we have
\[
1 - q^{2n} = (1 + q^n)(1 - q^n) \equiv 2(1 - q^n) \pmod{\Phi_n(q^2)}.
\]

It follows that for $1 \leq k \leq n-1$,
\[
\left[ \begin{array}{c} 2n \\ k \end{array} \right] = \frac{(1-q^{2n})(1-q^{2n-1})\cdots(1-q^{2n-k+1})}{(1-q^2)(1-q^3)\cdots(1-q^k)}
\equiv 2(1-q^n) \frac{(1-q^{-1})\cdots(1-q^{-k+1})}{(1-q)(1-q^2)\cdots(1-q^k)} \pmod{\Phi_n(q^2)}
\]
\[
= 2(q^n - 1) \frac{(-1)^k q^{-k(k-1)}}{1-q^k}.
\]

Multiplying both sides of (13) by $q$ and substituting (14) into the right-hand side of (13), we arrive at
\[
\sum_{k=0}^{n-1} q^{k+1} \left[ \begin{array}{c} 2k \\ k+1 \end{array} \right]
= \left( \frac{n-1}{3} \right) q^{1/2} (2n^2-n(n+1)) + \sum_{k=1}^{n-1} \left( \frac{n-k-1}{3} \right) q^{1/2} \left[ 2(n-k)^2 - (n-k)(\frac{n-k-1}{3})-3 \right] \left[ 2n \right] \left[ k \right]
\equiv \left( \frac{n-1}{3} \right) q^{1/2} (2n^2-n(n+1))
\]
\[
+ 2(q^n - 1) \sum_{k=1}^{n-1} \left( \frac{n-k-1}{3} \right) (-1)^k q^{1/2} \left[ 2(n-k)^2 - (n-k)(\frac{n-k-1}{3})-3 \right] \left( \frac{n-k-1}{3} \right) \frac{k(k-1)}{2} \pmod{\Phi_n(q^2)}.
\]
Furthermore,

\[
\sum_{k=1}^{n-1} \left( \frac{n-k-1}{3} \right) (-1)^k q^{\frac{1}{3} \left( 2(n-k)^2 - (n-k) \left( \frac{n-k-1}{3} \right) \right)} \frac{k(k-1)}{1-q^k}
\]

\[
= \sum_{k=1}^{n-1} \left( \frac{k-1}{3} \right) (-1)^n q^{\frac{1}{3} \left( 2k^2 - k \left( \frac{k-1}{3} \right) \right)} \frac{1-q^{n-k}}{1-q^{n-k}}
\]

\[
= \sum_{k=1}^{n-1} \left( \frac{k-1}{3} \right) (-1)^n q^{\frac{1}{3} \left( 2k^2 - k \left( \frac{k-1}{3} \right) \right)} \frac{n(n-1)}{2} \frac{1-q^{(k+1)+nk}}{1-q^{n-k}}
\]

\[
\equiv \sum_{k=1}^{n-1} \left( \frac{k-1}{3} \right) (-1)^k q^{\frac{1}{3} \left( 2k^2 - k \left( \frac{k-1}{3} \right) \right)} \frac{k(k-1)}{1-q^k} \quad (\mod \Phi_n(q)),
\]

where we set \(k \rightarrow n-k\) in the first step. Thus,

\[
\sum_{k=0}^{n-1} q^{k+1} \left[ \frac{2k}{k+1} \right] \equiv \left( \frac{n-1}{3} \right) q^{\frac{1}{3} \left( 2n^2 - n \left( \frac{n-1}{3} \right) \right)} + 2(q^n - 1) \sum_{k=1}^{n-1} \left( \frac{k-1}{3} \right) (-1)^k q^{\frac{1}{3} \left( 2k^2 - k \left( \frac{k-1}{3} \right) \right)} \frac{k(k-1)}{1-q^k} \quad (\mod \Phi_n(q)^2).
\]

We complete the proof of (6) by combining (7) and (16).

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