PERFECT MATCHING INDEX
VS.
CIRCULAR FLOW NUMBER OF A CUBIC GRAPH

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August 12, 2020

Abstract

The perfect matching index of a cubic graph $G$, denoted by $\pi(G)$, is the smallest number of perfect matchings that cover all the edges of $G$. According to the Berge-Fulkerson conjecture, $\pi(G) \leq 5$ for every bridgeless cubic graph $G$. The class of graphs with $\pi \geq 5$ is of particular interest as many conjectures and open problems, including the famous cycle double cover conjecture, can be reduced to it. Although nontrivial examples of such graphs are very difficult to find, a few infinite families are known, all with circular flow number $\Phi_c(G) = 5$. It has been therefore suggested [Electron. J. Combin. 23 (2016), #P3.54] that $\pi(G) \geq 5$ might imply $\Phi_c(G) \geq 5$. In this article we dispel these hopes and present a family of cyclically 4-edge-connected cubic graphs of girth at least 5 (snarks) with $\pi \geq 5$ and $\Phi_c \leq 4 + \frac{2}{3}$.

Keywords: cubic graph, snark, perfect matching, covering, circular flow

AMS subject classifications: 05C21, 05C70, 05C15.

1 Introduction

Cubic graphs that cannot be covered with four perfect matchings have recently attracted considerable attention. The reason for this interest stems from their close relationship to several difficult and long-standing conjectures such as the cycle double cover conjecture, the Berge-Fullkerson conjecture, and others. The current knowledge about these graphs is very limited and examples are extremely rare.

It is well known that every bridgeless cubic graph admits a set of perfect matchings that cover all its edges (see [10]). The smallest number of perfect matchings for such a cover is the perfect matching index and is denoted by $\pi(G)$. Obviously, $\pi(G) \geq 3$ for every bridgeless cubic graph $G$, with equality attained precisely when the graph is 3-edge-colourable. Although no constant upper bound is known, the Berge-Fulkerson conjecture (see [9]) suggests that this number should not exceed 5.
Nontrivial cubic graphs with perfect matching index greater than 4 are very difficult to find. In fact, until 2013, only one cyclically 4-edge-connected cubic graph with $\pi \geq 5$ was known – of course, the Petersen graph. The situation changed after the exhaustive computer search performed by Brinkmann et al. [2] revealed another such graph on 34 vertices (see Figure 1). This graph became a starting point for the construction of three infinite families of graphs with this property, the windmill graphs of Esperet and Mazzuoccolo [4], the treelike snarks of Abreu et al. [1], and a family of Chen [3] similar to the windmill graphs. Esperet and Mazzuoccolo [4] also showed that it is NP-complete to decide for a bridgeless cubic graph $G$ whether $\pi(G) \leq 4$ or $\pi(G) \geq 5$, implying that the family of cubic graphs with $\pi \geq 5$ is sufficiently rich.

Somewhat surprisingly, all graphs with $\pi \geq 5$ known so far have circular flow number at least 5 (see [8, Theorem 9.1]). Recall that the circular flow number of a bridgeless graph $G$, denoted by $\Phi_c(G)$, is the smallest rational number $r$ such that $G$ admits a nowhere-zero $r$-flow. With similar reasons in mind, Abreu at al. [1] and Fiol at al. [6] suggested that cubic graphs critical with respect to perfect matching index (corresponding to Berge’s conjecture) might be also critical with respect to circular flow number (corresponding to Tutte’s 5-flow conjecture). In other words, perfect matching index at least 5 ought to imply circular flow number being at least 5.

In this paper we dispel these expectations and exhibit the first family of cyclically 4-edge-connected cubic graphs of girth at least 5 (nontrivial snarks) with $\pi \geq 5$ and $\Phi_c < 5$. In fact, we provide an infinite family of nontrivial snarks for which $4 + \frac{1}{2} < \Phi_c \leq 4 + \frac{2}{3}$.

Our construction heavily depends on the results of [8]. In that paper we have developed a theory that describes coverings with four perfect matchings as flows whose flow values represent points and outflow patterns represent lines of a tetrahedron in the 3-dimensional projective space $PG(3,2)$ over the 2-element field. The geometric representation of coverings can be used as a powerful tool for the study of graphs that cannot be covered with four perfect matchings and enables a great variety of constructions of such graphs. The main ideas of this theory are reviewed in Section 3, making the present article sufficiently self-contained.

2 Preliminaries

Graphs studied in this paper will be often assembled from smaller building blocks called multipoles. Similarly to graphs, each multipole $M$ has its vertex set $V(M)$, its edge set $E(M)$, and an incidence relation between vertices and edges. Each edge of $M$ has two
Consider a cubic graph which has a covering. A dipole is a multipole whose dangling edges are partitioned into two sets of equal size, called connectors. If the size is \(m\), the dipole is an \((m,m)\)-pole. One of the connectors of a dipole is chosen as its input connector; the other connector is its output connector. In order to avoid ambiguity, connectors of dipoles are endowed with a fixed (but arbitrary) linear order. All multipoles in this paper are cubic, which means that each vertex is incident with exactly three edge-ends.

Free ends of any two dangling edges \(s\) and \(t\) can be coalesced to produce a new edge \(s \ast t\), the junction of \(s\) and \(t\), whose end-vertices are the other end of \(s\) and the other end of \(t\).

Given an \((m,m)\)-pole \(M_1\) and an \((m,m)\)-pole \(M_2\), we can construct a new \((m,m)\)-pole \(M_1 \circ M_2\), the composition of \(M_1\) and \(M_2\), by performing the junction of the \(i\)-th edge of output connector of \(M_1\) with the \(i\)-th edge of the input connector of \(M_2\). The input and the output connectors of \(M_1 \circ M_2\) are inherited from \(M_1\) and \(M_2\), respectively. Composition of dipoles is clearly associative, therefore \((M_1 \circ M_2) \circ M_3 = M_1 \circ (M_2 \circ M_3)\).

An edge-colouring of a graph or a multipole \(X\) is an assignment of colours from a set \(Z\) of colours to the edges of \(X\) in such a way that the edges with adjacent edge ends receive distinct colours. It means that all edge colourings in this paper are proper. A 2-connected cubic graph whose edges cannot be properly coloured with three colours is called a snark. A snark is nontrivial if it is cyclically 4-edge-connected and has girth at least 5.

Given an abelian group \(A\), an \(A\)-flow on a graph \(G\) consists of an orientation of \(G\) and a function \(\phi\): \(E(G) \to A\) such that, at each vertex, the sum of all incoming values equals the sum of all outgoing ones (Kirchhoff’s law). A flow which only uses nonzero elements of the group is said to be nowhere-zero. An integer \(k\)-flow, where \(k \geq 2\) is an integer, is a \(Z\)-flow with value range contained in \(\{0, \pm 1, \ldots, \pm(k-1)\}\).

Finally, we define the total flow through a dipole \(X\) as the sum of flow-values on the dangling edges of the input connector directed towards the dipole; of course, this value coincides with the sum of flow-values on the dangling edges in the output connector of \(X\) directed away from \(X\).

## 3 Tetrahedral flows

Consider a cubic graph which has a covering \(\mathcal{M} = \{P_1, P_2, P_3, P_4\}\) of its edge set with four perfect matchings. One can clearly represent \(\mathcal{M}\) by a mapping

\[
\phi: E(G) \to \mathbb{Z}_2^4
\]

where the \(i\)-th coordinate of the value \(\phi(e)\) equals 1 \(\in \mathbb{Z}_2\) whenever the edge \(e\) does not belong to the perfect matching \(P_i\). It is not difficult to see that \(\phi\) is a nowhere-zero \(\mathbb{Z}_2^4\)-flow on \(G\). It may be a little less obvious that \(\phi\) has an additional geometric structure which can be conveniently described in terms of 3-dimensional projective space over the 2-element field. More importantly, this structure proves useful. In this section we review the main ideas of the theory and refer the reader to our paper [8] for details.

We start with the necessary geometric definitions. The \(n\)-dimensional projective space \(PG(n, 2) = \mathbb{P}_n(\mathbb{F}_2)\) over the 2-element field \(\mathbb{F}_2\) is an incidence geometry whose points can be identified with the nonzero vectors of the \((n + 1)\)-dimensional vector space \(\mathbb{F}_2^{n+1}\).
and lines are formed by the triples \( \{x, y, z\} \) of points such that \( x + y + z = 0 \). Recall that \( PG(2, 2) \) is the Fano plane. Throughout this paper we will mainly encounter the 3-dimensional projective space \( PG(3, 2) \), which has 15 points and 35 lines.

A tetrahedron in \( PG(3, 2) \) is a configuration \( T \) consisting of ten points and six lines spanned by a set \( \{p_1, p_2, p_3, p_4\} \) of four points of \( PG(3, 2) \) in general position; the latter means that the set constitutes a basis of the vector space \( \mathbb{F}_4^2 \). These four points are the corner points of \( T \). Every pair of distinct corner points \( c_1 \) and \( c_2 \) belongs to a unique line \( \{c_1, c_2, c_1 + c_2\} \) in \( T \) whose third point \( c_1 + c_2 \) is its midpoint. Every point \( x \) of \( T \) is assigned its weight, which equals 1 if \( x \) is a corner point and 2 if \( x \) is a midpoint.

Any two distinct points of \( T \) lie on the same line of \( PG(3, 2) \) but not necessarily on a line of \( T \). Those that lie on the same line of \( T \) are collinear in \( T \), otherwise they are non-collinear in \( T \).

For a given a tetrahedron \( T \) in \( PG(3, 2) \) we define a T-flow on a cubic graph \( G \) to be a mapping \( \phi: E(G) \rightarrow P(T) \) from the edge set of \( G \) to the point set of \( T \) such that for each vertex \( v \) of \( G \) the three edges \( e_1, e_2, \) and \( e_3 \) incident with \( v \) receive values that form a line of \( T \); that is, \( \phi(e_1) + \phi(e_2) + \phi(e_3) = 0 \). The last equation actually states that \( \phi \) fulfils the Kirchhoff law, so a T-flow is indeed a flow. A tetrahedral flow on \( G \) is a T-flow for some tetrahedron \( T \) in \( PG(3, 2) \). Note that any T-flow is also a proper edge-colouring.

The following result is a cornerstone of our theory.

**Theorem 3.1.** A cubic graph can have its edges covered with four perfect matchings if and only if it admits a tetrahedral flow. Moreover, there exists a one-to-one correspondence between coverings of \( G \) with four perfect matchings and T-flows, where \( T \) is an arbitrary fixed tetrahedron in \( PG(3, 2) \).

A natural way of applying tetrahedral flows to the study of cubic graphs that cannot be covered with four perfect matchings is by analysing conflicts of tetrahedral flows on the components resulting from the removal of an edge-cut from the graph. If the cut-set has four edges, we can split them into two pairs which can be regarded as the input and the output connectors of a dipole, and inspect how pairs of points of a tetrahedron in \( PG(3, 2) \) are transformed via a tetrahedral flow from the input to the output.

Let us fix a tetrahedron \( T \) with corner points \( p_1, p_2, p_3 \) and \( p_4 \). We distinguish between six types of pairs of points of \( T \), distinct or not, which we treat as geometric shapes.

(i) A line segment is a pair \( \{c_1, c_2\} \) where \( c_1 \) and \( c_2 \) are two distinct corner points of \( T \).

The set of all line segments of \( T \) is denoted by \( ls \).
(ii) A half-line is a pair \( \{c_1, c_1 + c_2\} \) where \( c_1 \) and \( c_2 \) are two distinct corner points of \( T \). The set of all half-lines of \( T \) is denoted by \( \text{hl} \).

(iii) An angle is a pair \( \{c_1 + c_2, c_1 + c_3\} \) where \( c_1, c_2, \) and \( c_3 \) are three distinct corner points of \( T \). The set of all angles of \( T \) is denoted by \( \text{ang} \).

(iv) An altitude is a pair \( \{c_1, c_2 + c_3\} \) where \( c_1, c_2, \) and \( c_3 \) are three distinct corner points of \( T \). The set of all altitudes of \( T \) is denoted by \( \text{alt} \).

(v) An axis is a pair \( \{c_1 + c_2, c_3 + c_4\} \) where \( c_1, c_2, c_3, \) and \( c_4 \) are all four corner points of \( T \) in some order. The set of all axes of \( T \) is denoted by \( \text{ax} \).

(vi) A double point is a degenerate pair \( \{x, x\} \) where \( x \) is any point of \( T \). The set of all degenerate pairs of \( T \) is denoted by \( \text{dpt} \).

The pairs under items (i)-(ii) are collinear, those under (iii)-(v) are non-collinear. The degenerate pairs defined in item (vi) actually occur in two varieties, depending on whether the point \( x \) is a corner point or a midpoint, but both varieties represent the zero flow through a connector, and from this point of view the distinction is irrelevant.

We now define the set of shapes to be the set

\[ \Sigma = \{\text{ls}, \text{hl}, \text{ang}, \text{alt}, \text{ax}, \text{dpt}\} . \]

It can be shown (see [3] Theorem 4.1) that for every pair of points \( \{x, y\} \) of \( T \), distinct or not, there exists a unique element \( s \in \Sigma \) such that \( \{x, y\} \in s \). This element \( s \) is called the shape of \( \{x, y\} \).

The next step is to examine which pairs of shapes can occur on the connectors of a \((2,2)\)-pole equipped with a tetrahedral flow. Consider an arbitrary \((2,2)\)-pole \( X = X(I, O) \) with input connector \( I = \{g_1, g_2\} \) and output connector \( O = \{h_1, h_2\} \), and let \( T \) be a fixed tetrahedron in \( PG(3, 2) \). We say that \( X \) has a transition

\[ \{x, y\} \rightarrow \{x', y'\} \]

or that \( \{x, y\} \rightarrow \{x', y'\} \) is a transition through \( X \), if there exists a \( T \)-flow \( \phi \) on \( X \) such that \( \phi(g_1), \phi(g_2) \in \{x, y\} \) and \( \phi(h_1), \phi(h_2) \in \{x', y'\} \). If \( X \) admits both transitions \( \{x, y\} \rightarrow \{x', y'\} \) and \( \{x', y'\} \rightarrow \{x, y\} \), we write

\[ \{x, y\} \leftrightarrow \{x', y'\} \]

Each transition \( \{x, y\} \rightarrow \{x', y'\} \) through \( X \) between point pairs induces a transition between their shapes. To be more precise, for elements \( s \) and \( t \) of \( \Sigma \) we say that \( X \) has a transition

\[ s \rightarrow t \]

if \( X \) has a transition \( \{x, y\} \rightarrow \{x', y'\} \) such that \( s \) is the shape of \( \{x, y\} \) and \( t \) is the shape of \( \{x', y'\} \). The set of all transitions through \( X \) reduced to their shapes forms a binary relation \( T(X) \) on \( \Sigma \).

For convenience, we refer to the symbols \( \{x, y\} \rightarrow \{x', y'\} \) and \( s \rightarrow t \), with any pair of shapes, as transitions even without any connection to a particular dipole and a tetrahedral flow. There is no danger of confusion with transitions through a dipole defined above, which require the existence of a certain flow through it.

Similarly to dipoles, we can also compose their transition relations. As expected, transitions \( p \rightarrow s \) and \( s \rightarrow t \) of \((2,2)\)-poles \( X_1 \) and \( X_2 \), respectively, give rise to the
transition \( p \to t \) of \( X_1 \circ X_2 \). Conversely, a transition \( p \to q \) through \( X_1 \circ X_2 \) occurs only when there exist transitions \( p \to s \) through \( X_1 \) and \( s \to t \) through \( X_2 \) for a suitable shape \( s \in \Sigma \). These definitions immediately imply that \( T(X_1 \circ X_2) = T(X_1) \circ T(X_2) \).

The following theorem proved in [3, Theorem 5.1] is essentially a consequence of Kirchhoff’s law.

**Theorem 3.2.** All transitions through an arbitrary \((2, 2)\)-pole \( X \) have the form \( s \to s \) for some \( s \in \Sigma \) except possibly the transitions \( ls \to \text{ang} \) or \( \text{ang} \to ls \).

The previous theorem implies that the transition relation \( T(X) \) of every \((2, 2)\)-pole \( X \) is contained in the set

\[
\mathcal{A} = \{ \text{dpt} \to \text{dpt}, \text{hl} \to \text{hl}, \text{alt} \to \text{alt}, \text{ax} \to \text{ax}, \\
\text{ang} \to \text{ang}, \text{ang} \to ls, ls \to \text{ang}, ls \to ls \}. \tag{1}
\]

The elements of \( \mathcal{A} \) will be called **admissible transitions**.

Two types of dipoles are of particular interest. A **decollineator** is a \((2, 2)\)-pole with no transition \( \{x, y\} \to \{x', y'\} \) such that both \( \{x, y\} \) and \( \{x', y'\} \) are collinear. Among the admissible transitions only those of type \( ls \to ls \) and \( hl \to hl \) are collinear, therefore every decollineator \( D \) has its transition relation \( T(D) \) contained in the set

\[
\mathcal{D} = \{ \text{dpt} \to \text{dpt}, \text{alt} \to \text{alt}, \text{ax} \to \text{ax}, \text{ang} \to \text{ang}, \text{ang} \to ls, ls \to \text{ang} \}. \tag{2}
\]

A **deangulator** is a \((2, 2)\)-pole with no transition of the form \( \text{ang} \to \text{ang} \). Decollineators and deangulators are closely related: if \( D_1 \) and \( D_2 \) are decollineators and \( U_1 \) and \( U_2 \) are deangulators, then \( D_1 \circ U_1 \circ D_2 \) is a decollineator and \( U_1 \circ D_1 \circ U_2 \) is a deangulator for each \( i \in \{1, 2\} \), see [3, Proposition 7.3].

The next theorem (see [3, Theorem 5.4]) explains the relationship between decollineators and cubic graphs with perfect matching index at least 5.

**Theorem 3.3.** The following two statements are equivalent for an arbitrary \((2, 2)\)-pole \( X \).

(i) \( X \) is a decollineator, that is, \( X \) admits no collinear transition.

(ii) The cubic graph \( G \) created from \( X \) by adding to \( X \) two adjacent vertices and attaching each of them to a connector of \( X \) has \( \pi(G) \geq 5 \).

### 4 A new family of graphs with \( \pi \geq 5 \)

In this section we present a new family of cubic graphs with perfect matching index at least 5. The reasons for high perfect matching index of its members are quite different from those found in the previously known families, the windmill graphs [4], the treelike snarks [1], the snarks of Chen [3], and in their common generalisation, the Halin snarks, introduced in [8]. While all Halin snarks have circular flow number at least 5 (see [8, Theorem 9.1]), the new family contains an infinite subfamily whose members have circular flow number at most \( 4 + \frac{2}{3} \). The latter property will be established in the next section.

Before proceeding to the construction we need a few definitions.

According to Theorem 3.2, every \((2, 2)\)-pole \( X \) has \( T(X) \subseteq \mathcal{A} \), where \( \mathcal{A} \) is the set of admissible transitions defined by (1). Let \( \mathcal{L} \) be any subset of \( \mathcal{A} \). A \((2, 2)\)-pole \( X \) will be called an \( \mathcal{L} \)-**dipole** if \( T(X) \subseteq \mathcal{L} \). For example, an \( \mathcal{L} \)-dipole with \( \mathcal{L} = \mathcal{A} - \{ \text{hl} \to \text{hl}, \text{ls} \to \text{ls} \} = \mathcal{D} \) is a decollineator and one where \( \mathcal{L} = \mathcal{A} - \{ \text{ang} \to \text{ang} \} \) is a deangulator.
The next two propositions prepare the building blocks for our construction. As we shall see, they are decollineators of the form $D_1 \circ X \circ D_2$, where $D_1$ and $D_2$ are arbitrary decollineators and $X$ is a deangulator with a special transition relation.

**Proposition 4.1.** Let $G$ be a cubic graph with $\pi(G) \geq 5$ containing two 5-cycles $C_1$ and $C_2$ such that $C_1 \cap C_2$ is a path of length 2. Let $e$ and $f$ be the edges of $C_1 \cup C_2$ that are not incident with any vertex of $C_1 \cap C_2$. Let $X$ be a $(2,2)$-pole constructed from $G$ by severing $e$ and $f$ and forming the connectors from the half-edges of the same edge. Then each transition through $X$ has the form

$$hl \to hl, \; ls \to ls, \; alt \to alt, \; \text{and} \; ang \leftrightarrow ls.$$ 

**Proof.** Let $\{x,y\} \to \{x',y'\}$ be an arbitrary transition through $X$, and let $\phi$ be a tetrahedral flow on $X$ that induces it. First observe that $x \neq y$ for otherwise the Kirchhoff law would imply that $x' = y'$ and hence $\phi$ would induce a tetrahedral flow on $G$; this is impossible by Theorem 3.1. Hence $X$ has no transition of the form $dpt \to dpt$. Next we show $X$ has no transition $\{x,y\} \to \{x',y'\}$ where $|x| = |y| = |x'| = |y'| = 2$. Indeed, if it had, then both edges contained in $C_1 \cap C_2$ would be forced to receive values of weight 2 in spite of the fact that they are adjacent (see Figure 3). This excludes from $T(X)$ all admissible transitions involving an axis or an angle except $ang \leftrightarrow ls$. What remains are exactly those transitions that are mentioned in the statement. \hfill $\square$

![Figure 3: Excluding certain transitions in the proof of Proposition 4.1. Edges carrying a value of weight 2 are represented by bold lines.](image)

For the set of transitions mentioned in the statement of Proposition 4.1 we put

$$Q = \{hl \to hl, \; ls \to ls, \; alt \to alt, \; ang \leftrightarrow ls\}.$$ (3)

Note that every $Q$-dipole is a deangulator.

The crucial role in our construction is played by heavy $(2,2)$-poles. We say that an edge $e$ of a multipole is heavy with respect to a tetrahedral flow $\phi$ if the weight of $\phi(e)$ equals 2. A $(2,2)$-pole $X$ is heavy if it has at least two heavy dangling edges for each tetrahedral flow.

The following proposition offers a recipe for constructing heavy $(2,2)$-poles.

**Proposition 4.2.** Let $D_1$ and $D_2$ be decollineators and let $Q$ be a $Q$-dipole, where $Q$ is defined by (3). Then every transition through $D_1 \circ Q \circ D_2$ has the form

$$ls \leftrightarrow ang, \; ang \to ang, \; \text{and} \; alt \to alt.$$ 

In particular, $D_1 \circ Q \circ D_2$ is a heavy dipole.
Proof. Since $D_1$ and $D_2$ are decollineators, we have $T(D_1) \subseteq D$ and $T(D_2) \subseteq D$. It follows that $T(D_1 \circ Q \circ D_2) \subseteq D \circ Q \circ D$, which leaves the transitions listed in the statement. It is straightforward to check that the remaining transitions indeed make $D_1 \circ Q \circ D_2$ a heavy dipole.

For the set of transitions mentioned in the statement of Proposition 4.2 we put

$$R = D \circ Q \circ D = \{ls \leftrightarrow ang, ang \rightarrow ang, alt \rightarrow alt\}. \quad (4)$$

Note that every $R$-pole is a heavy decollineator.

Remark 4.3. Let $D_1 = D_2 = D_{Ps}$ where $D_{Ps}$ is the decollineator obtained from the Petersen graph by removing two adjacent vertices and including two dangling edges in the same connector whenever they were formerly incident with the same vertex. It can be verified that $T(D_{Ps}) = D$. Let $Q_{Ps}$ be the $(2,2)$-pole arising from the Petersen graph by severing two edges at distance 2. It is easy to see that $Q_{Ps}$ satisfies the assumptions of Proposition 4.1 so $T(Q_{Ps}) \subseteq Q$ where $Q$ is the transition set defined by (3). It is not difficult to verify that in fact $T(Q_{Ps}) = Q$. Furthermore, by Proposition 4.2 $T(D_{Ps} \circ Q_{Ps} \circ D_{Ps}) \subseteq R$. Again, it can be checked that $T(D_{Ps} \circ Q \circ D_{Ps}) = R$. Thus $D_{Ps} \circ Q \circ D_{Ps}$ is a heavy dipole.

Now we are ready for the construction of a new family of cubic graphs with $\pi \geq 5$.

Construction. Let $G$ be a cubic graph. A new graph $\tilde{G}$ is constructed as follows.

- Replace every vertex $v$ of $G$ with a pair of independent vertices $v_1$ and $v_2$ in such a way that $\{u_1, u_2\} \cap \{w_1, w_2\} = \emptyset$ whenever $u \neq w$. Any vertex $v_i$ of $\tilde{G}$, where $v$ is a vertex of $G$ and $i \in \{1,2\}$, is called a lift of $v$.
- Replace each edge $e$ of $G$ with a heavy $(2,2)$-pole $X_e = X_e(I,O)$ in such a way that for any two distinct edges $f$ and $h$ the dipoles $X_f$ and $X_h$ are disjoint. The dipoles $X_e$ are called superedges.
- For each edge $e = uv$ attach the dangling edges of the input connector of $X_e$ to distinct vertices in $\{u_1, u_2\}$ and those in the output connector to distinct vertices in $\{v_1, v_2\}$.

The construction of the graph $\tilde{G}$ can be regarded as a special form of superposition. We call $\tilde{G}$ a heavy superposition of $G$. A heavy superposition is said to be basic if each heavy dipole $X_e$ used for the construction is isomorphic to the dipole $D_{Ps} \circ Q_{Ps} \circ D_{Ps}$ from Remark 4.3.

The following theorem is the main result of this section.

Theorem 4.4. Let $G$ be a cubic graph and let $\tilde{G}$ be a heavy superposition of $G$. Then $\pi(\tilde{G}) \geq 5$.

Proof. Assume that $G$ is a cubic graph with $n$ vertices and $m$ edges; clearly, $m = 3n/2$. Suppose to the contrary that $\pi(\tilde{G}) \leq 4$. By Theorem 3.1 $\tilde{G}$ admits a tetrahedral flow, say $\phi$. Let $W$ denote the set of all edges of $\tilde{G}$ that are incident with some lift of a vertex of $G$. Since each lift $v_i$ is incident with exactly one heavy edge with respect to $\phi$, there are exactly $2n$ heavy edges in $W$. On the other hand, the set $W$ can be decomposed into $m$ sets $W_e$ according to which superedge $X_e$ they belong to as their dangling edges. By counting the the average number of heavy dangling edges per superedge we obtain

$$\frac{2n}{m} = \frac{4}{3} < 2.$$
This inequality implies that there exists a superedge $X_e$ with fewer than two heavy dangling edges, contradicting the assumption that all superedges used for the construction of $\tilde{G}$ are heavy. \qed

5 Circular flows vs. perfect matching index

In the preceding section have proved that a heavy superposition $\tilde{G}$ of any cubic graph $G$ has perfect matching index at least 5. We now show that if the superposition is basic and $G$ is 3-edge-colourable, then the circular flow number of $\tilde{G}$ is smaller than 5.

We continue with the pertinent definitions. Given a real number $r \geq 2$, we define a nowhere-zero real-valued $r$-flow as an $\mathbb{R}$-flow $\phi$ such that $1 \leq |\phi(e)| \leq r - 1$ for each edge $e$ of $G$. A nowhere-zero modular $r$-flow is an $\mathbb{R}/r\mathbb{Z}$-flow $\phi$ such that $1 \leq \phi(e)$ (mod $r$) $\leq r - 1$ for each edge $e$. The symbol $x$ (mod $r$) denotes the unique real number $x' \in [0, r) \subseteq \mathbb{R}$ such that $x - x'$ is a multiple of $r$. It is well known that a graph admits a nowhere-zero real-valued $r$-flow if and only if it admits a nowhere-zero modular $r$-flow.

The circular flow number of a graph $G$, denoted by $\Phi_c(G)$, is the infimum of the set of all real numbers $r$ such that $G$ has a nowhere-zero $r$-flow. It is known [7] that the circular flow number of a finite graph is in fact a minimum and a rational number.

Here is the main result of this section.

**Theorem 5.1.** If $\tilde{G}$ is a basic heavy superposition of a 3-edge-colourable cubic graph $G$, then $\tilde{G}$ is a nontrivial snark and

$$4 + \frac{1}{2} < \Phi_c(\tilde{G}) \leq 4 + \frac{2}{3}.$$ 

**Proof.** To prove the lower bound suppose, to the contrary, that $\Phi_c(\tilde{G}) \leq 4 + \frac{1}{2}$. It means that $\tilde{G}$ has a nowhere-zero $(4 + \frac{1}{2})$-flow. In fact, by Theorem 1.1 of [1], $\tilde{G}$ has a nowhere-zero $(4 + \frac{1}{2})$-flow such that every flow value is a rational number of the form $n/2$ for some integer $n$. Let $\phi$ be such a flow. Clearly, $\phi$ can be taken to be a modular $(4 + \frac{1}{2})$-flow. Fix a vertex $v$ in $G$ and let $e_1$, $e_2$, and $e_3$ be the three edges from $G$ incident with $v$. Consider the total flow through the dipole $X_{e_i}$ in the direction from the set $\{v_1, v_2\}$, that is, the sum of flow-values in $\mathbb{R}/(4 + \frac{1}{2})\mathbb{Z}$ on the dangling edges incident with $\{v_1, v_2\}$ directed towards the dipole; let $h_i$ be the value. As the circular flow number of the Petersen graph is 5, the total flow through $D_{P_s}$ lies in the interval $(-1, 1)$ modulo $4 + \frac{1}{2}$, and the total flow through $Q_{P_s}$ is nonzero. Since the dipoles $D_{P_s}$ and $Q_{P_s}$ are sequentially composed in the superedge, the total flow through the superedge belongs to $(-1, 0) \cup (0, 1)$. Taking into account the fact that the flow values are nonzero multiples of $\frac{1}{2}$, we conclude that the total flow through any superedge is either $\frac{1}{2}$ or $-\frac{1}{2}$. In particular, $h_1$, $h_2$, and $h_3$ are all in $\{-\frac{1}{2}, \frac{1}{2}\}$. By the Kirchhoff law, the total outflow from the vertices $v_1$ and $v_2$ is zero, so $h_1 + h_2 + h_3 = 0$, which is clearly impossible. Therefore $\Phi_c(\tilde{G}) > 4 + \frac{1}{2}$, as claimed.

In order to establish the upper bound we construct a nowhere-zero $(4 + \frac{2}{3})$-flow on $\tilde{G}$. To this end, it is sufficient to find an integer 12-flow $\phi$ such that $|\phi(e)| \geq 3$ for each edge $e$ of $\tilde{G}$ and then divide all the values by 3. Let $\{P_1, P_2, P_3\}$ be a 1-factorisation of $G$ induced by a 3-edge-colouring. If an edge $e$ of $G$ belongs to $P_1$, we assign values to the edges of the corresponding superedge $X_e$ according to Figure 1; similarly, if an edge belongs to $P_2$ or $P_3$, the flow values in $X_e$ will be assigned according to Figure 2 and Figure 3 respectively. It is easy to check that the resulting valuation and orientation constitute an integer 12-flow with absolute value not smaller than 3 on each edge of $\tilde{G}$. This gives rise to a nowhere-zero $(4 + \frac{2}{3})$-flow on $\tilde{G}$. 

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Figure 4: A 12-flow on the superedges corresponding to $P_1$

Figure 5: A 12-flow on the superedges corresponding to $P_2$

Figure 6: A 12-flow on the superedges corresponding to $P_3$
At last, we show that $\tilde{G}$ is a nontrivial snark. First note that $\tilde{G}$ is not 3-edge-colourable because $\pi(\tilde{G}) \geq 5$ by Theorem 4.4. Furthermore, the girth of $\tilde{G}$ is obviously 5. Thus it remains to prove that $\tilde{G}$ is cyclically 4-edge-connected. Take an arbitrary cycle-separating edge-cut $S$ in $\tilde{G}$. It is clear that the edges of $S$ cannot all belong to the same superedge. Therefore $S$ intersects at least two superedges, and in each intersected superedge it has at least two edges. Thus $|S| \geq 4$, implying that $\tilde{G}$ is cyclically 4-edge-connected. Summing up, $\tilde{G}$ is a nontrivial snark.

Note that the proof of the lower bound in Theorem 5.1 is valid for all cubic graphs $G$, not necessarily 3-edge-colourable ones. Furthermore, the restriction to a basic superposition is also superfluous.

The following statement is an immediate consequence of Theorems 4.4 and 5.1.

**Corollary 5.2.** There exist infinitely many nontrivial snarks with $\pi \geq 5$ and $\Phi_c < 5$.

**Remark 5.3.** The smallest example with $\pi \geq 5$ and $\Phi_c < 5$ which arises from our construction has 82 vertices. It is constructed by a basic heavy superposition from the cubic graph consisting of two vertices and three parallel edges.

We have shown, contrary to some expectations, that there exist cubic graphs, even nontrivial snarks, with $\pi \geq 5$ and $\Phi_c < 5$. It is natural to ask how small the parameter $\Phi_c$ can be within the family of cubic graphs that cannot be covered with four perfect matchings. We therefore propose the following problem.

**Problem 5.4.** What is the infimum of the set of all real numbers $r$ such that there exists a cubic graph $G$ with $\pi(G) \geq 5$ and $\Phi_c(G) = r$?

An interesting subproblem of Problem 5.4 is to determine whether there exists a constant $c > 4$ such that every cubic graph $G$ with $\pi(G) \geq 5$ has $\Phi_c(G) \geq c$.

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