ANDERSON T-MOTIVES ARE ANALOGS OF
ABELIAN VARIETIES WITH MULTIPLICATION
BY IMAGINARY QUADRATIC FIELDS

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July 27, 2009

Abstract. An analogy between abelian Anderson T-motives of rank $r$ and dimension $n$, and abelian varieties over $\mathbb{C}$ with multiplication by an imaginary quadratic field $K$, of dimension $r$ and of signature $(n, r-n)$, permits us to get two elementary results in the theory of abelian varieties. Firstly, we can associate to this abelian variety a (roughly speaking) $K$-vector space of dimension $r$ in $\mathbb{C}^n$. Secondly, if $n = 1$ then we can define the $k$-th exterior power of these abelian varieties. Probably this analogy will be a source of more results. For example, it would be interesting to find analogs of abelian Anderson T-motives whose nilpotent operator $N$ is not 0.

0. Introduction.

The origin of the present paper is an analogy 1.8 between the following two objects corresponding to the functional and number field case respectively:

A. An abelian Anderson T-motive $M$ of rank $r$ and dimension $n$, pure, uniformizable, having the nilpotent operator $N$ equal to 0 (see Section 1 for the exact definitions and the origin of the analogy),

and

B. An abelian variety $A$ over $\mathbb{C}$ with multiplication by an imaginary quadratic field $K$ (abbreviated as an abelian variety with MIQF), of dimension $r$ and of signature $(n, r-n)$.

Using well-known constructions for abelian Anderson T-motives, we get analogous constructions for abelian varieties with MIQF. Firstly, let us consider the lattice $L(M)$ associated to an abelian Anderson T-motive $M$ of type A:

C. $L(M)$ is an $r$-dimensional $\mathbb{F}_p[\theta]$-lattice in $\mathbb{C}_{\infty}^n$, where $\mathbb{C}_{\infty}$ is the functional analog of $\mathbb{C}$.

Let us consider the lattice $L(A)$ associated to an abelian variety $A$ with MIQF:

D. $L(A)$ is an $r$-dimensional $O_K$-lattice in $\mathbb{C}^r$ ($L(A)$ is not an $O_K$-submodule of $\mathbb{C}^r$ treated as $O_K$-module).

1991 Mathematics Subject Classification. Primary 11G09, 11G10, 11G15, 14K99.

Key words and phrases. abelian Anderson T-motives; abelian varieties with multiplication by an imaginary quadratic field; exterior power of abelian varieties.

Thanks: Alain Genestier and Laurent Fargues attracted my attention to the main analogy of the present paper.
Objects of types \((\text{C})\) and \((\text{D})\) are not quite analogous, isn’t it? The first result of the present paper is the following construction

\textbf{E.} (rough statement): \(A\) defines an \(r\)-dimensional \(O_K\)-submodule of \(\mathbb{C}^n\) (not of \(\mathbb{C}^r\) !).

\((\text{E})\) is an analog of \((\text{C})\).

Secondly, it is known that if \(M\) of type \(A\) has \(n = 1\) then its \(k\)-th exterior power \(\lambda^k(M)\) is also an object of type \(A\). By analogy, we can expect that if \(A\) is an abelian variety with MIQF having \(n = 1\) then \(\lambda^k(A)\) is defined, and is also an abelian variety with MIQF. We really give this definition.

Both these results are of elementary nature, they could be known to Riemann. But probably this analogy will be a source of more results. For example, it would be interesting to define analogs of Anderson T-motives having \(N \neq 0\).

The paper is organized as follows. In Section 1 we recall definitions of abelian Anderson T-motives and we formulate the results which are starting points (using the analogy 1.8) of the results of Sections 2, 3. Section 2 contains the exact statement and the proof of \((\text{E})\). In Section 3 we apply this result to construct exterior powers of abelian varieties with MIQF having \(n = 1\). Formally, these sections are independent of Section 1, i.e. they do not require any knowledge of functional case. In Section 4 we formulate research problems.

\textbf{Section 1. Origin of construction: abelian Anderson T-motives.}

A standard reference for abelian Anderson T-motives is [G], we shall use its notations if possible. Let \(r\) be a power of a prime number, \(\mathbb{F}_r(\theta)\) (resp. \(\mathbb{F}_r((\theta^{-1}))\)) the functional analog of \(\mathbb{Q}\) (resp. of \(\mathbb{R}\)), and \(\mathbb{C}_\infty\) — the completion of the algebraic closure of \(\mathbb{F}_r((\theta^{-1}))\) — the functional analog of \(\mathbb{C}\).

The definition of an abelian Anderson T-motive \(M\) is given in [G], Definitions 5.4.2, 5.4.12 (\(L\) of Goss should be considered as \(\mathbb{C}_\infty\)). Particularly, \(M\) is a free \(\mathbb{C}_\infty[T]\)-module of dimension \(r\) (this number \(r\) is called the rank of \(M\) endowed by a \(\mathbb{C}_\infty\)-skew-linear operator \(\tau\) satisfying some properties. A nilpotent operator \(N = N(M)\) associated to an abelian Anderson T-motive is defined in [G], Remark 5.4.3.2. We shall consider only pure ([G], Definition 5.5.2) uniformizable ([G], Theorem 5.9.14, (3)) T-motives. Its dimension \(n\) is defined in [G], Remark 5.4.13.2 (Goss denotes the dimension by \(\rho\)). Condition \(N = 0\) implies \(n \leq r\). An abelian Anderson T-motive of dimension 1 is the same as a Drinfeld module, they are all pure, uniformizable, and their \(N\) is 0.

If \(N(M) = 0\) then attached to such T-motive is a lattice \(L = L(M)\) which is a free \(r\)-dimensional \(\mathbb{F}_r[\theta]\)-module in \(\mathbb{C}_\infty^n\), and if \(N \neq 0\) then \(L(M)\) is a slightly more complicated object, we do not need to consider details for this case. Inclusion of \(L\) in \(\mathbb{C}_\infty^n\) defines a surjective map

\[
\alpha = \alpha(M) : L \otimes_{\mathbb{F}_r[\theta]} \mathbb{C}_\infty \to \mathbb{C}_\infty^n
\]

having the property

Restriction of \(\alpha\) to \(L \otimes_{\mathbb{F}_r[\theta]} \mathbb{F}_r((\theta^{-1}))\) is injective

\[
(1.1)
\]

\[
(1.2)
\]

\[\text{1} \text{We use notation } r \text{ instead of } \mathbb{F}_r \text{ of } [G] \text{ in order do not confuse with the rank of a T-motive.}\]
Tensor product of abelian Anderson T-motives $M_1$, $M_2$ is simply their tensor product over $\mathbb{C}_\infty[T]$, where the action of $\tau$ is defined by the formula $\tau(m_1 \otimes m_2) = \tau(m_1) \otimes \tau(m_2)$. If both $M_1$, $M_2$ are pure uniformizable then $M_1 \otimes M_2$ is pure uniformizable as well. The same definition holds for exterior (resp. symmetric) powers of $M$. The dual T-motive $M^*$ is defined in [L1].

1.3. It is easy to check that even if $N(M_1)$, $N(M_2)$ are 0 then $N(M_1 \otimes M_2)$, $N(\lambda^k(M_1))$, $N(S^k(M_1))$ are not 0. The only exception: if $M$ is a Drinfeld module $(\iff n = 1)$ then $N(\lambda^k(M)) = 0$, this is an elementary calculation.

There is a natural problem to describe $L(M_1 \otimes M_2)$, $L(\lambda^k(M_1))$, $L(S^k(M_1))$ in terms of $L(M_1)$, $L(M_2)$. It was solved by Anderson (non-published), a formula which is equivalent to this description is stated without proof in [P], end of page 3. See [L1], Remark 4.3.7 for more details and for the proof of the formula of Pink in the case when $N(M_1)$, $N(M_2)$ are 0. Let us state the theorem for the case when all $N$ are 0 (and hence all $L$ are lattices), i.e. $M$ is a Drinfeld module of rank $r$.

We need a

**Definition 1.4.** For a short exact sequence of vector spaces over a field

$$0 \rightarrow B_1 \rightarrow B_2 \rightarrow C \rightarrow 0$$

we define its $k$-th exterior power as the following exact sequence:

$$0 \rightarrow \lambda^k(B_1) \overset{\lambda^k(i)}{\rightarrow} \lambda^k(B_2) \rightarrow C_k \rightarrow 0$$

Now let us consider the exact sequences for $M$, $\lambda^k(M)$

$$0 \rightarrow \text{Ker } \alpha(M) \rightarrow L(M) \overset{\varphi_{r,\theta}}{\otimes} \mathbb{C}_\infty \overset{\alpha(M)}{\rightarrow} \mathbb{C}_\infty \rightarrow 0 \quad (1.5)$$

$$0 \rightarrow \text{Ker } \alpha(\lambda^k(M)) \rightarrow L(\lambda^k(M)) \overset{\varphi_{r,\theta}}{\otimes} \mathbb{C}_\infty \overset{\alpha(\lambda^k(M))}{\rightarrow} \mathbb{C}_\infty^{n(r,k)} \rightarrow 0 \quad (1.6)$$

where maps $\alpha(M)$, $\alpha(\lambda^k(M))$ are from 1.1, and $n(r,k) = \binom{r-1}{k-1}$ is the dimension of $\lambda^k(M)$.

**Theorem 1.7.** There exists a canonical isomorphism from the $k$-th exterior power of (1.5) to (1.6) such that the image of $\lambda^k(L(M))$ in (1.5) is $L(\lambda^k(M))$ in (1.6).

**Proof** is completely analogous to the proofs for the case of dual abelian Anderson T-motives ([L1], Theorem 4.3) and for the case of tensor product of abelian Anderson T-motives having $N = 0$ ([L1], Theorem 4.3.7.1), hence it is omitted. □

1.8. **Origin of the analogy.** We give here only a sketch of definitions; see for example [W] for the exact statements. Let $X$ be a Shimura variety and $G$ a reductive group over $\mathbb{Q}$ associated to $X$ according Deligne. Let $p$ be a prime of good reduction of $X$. $\mathbb{H}_p(X)$ — the $p$-part of the Hecke algebra of $X$ — is isomorphic to $\mathcal{H}(G(\mathbb{Q}_p))$ — the algebra of double cosets $G(\mathbb{Z}_p)gG(\mathbb{Z}_p)$, $g \in G(\mathbb{Q}_p)$. There exists a Levi subgroup $M$ of $G$ having the following property:
The $p$-part of the Hecke algebra of $\tilde{X}$ (the reduction of $X$ at $p$) is isomorphic to $\mathcal{H}(M(\mathbb{Q}_p))$.

See, for example, [W], p. 44, (*) and p. 49, (1.10) for the definition and properties of $M$.

**Example.** If $X$ is a Shimura variety parametrizing abelian varieties with multiplication by an imaginary quadratic field $K$, of dimension $r$ and of signature $(n, r-n)$, then $G(\mathbb{Q}) = GU(n, r-n)(K)$. If $p$ splits in $K/\mathbb{Q}$ then $G(\mathbb{Q}_p) = GL_r(\mathbb{Q}_p)$ and $M(\mathbb{Q}_p) = GL_n(\mathbb{Q}_p) \times GL_{r-n}(\mathbb{Q}_p) \subset G(\mathbb{Q}_p)$ is a subgroup of $(n \times r - n)$-block diagonal matrices.

For the functional case an analog of this theory is conjectural, but preliminary results of [L2] show that for abelian Anderson T-motives of rank $r$ and dimension $n$ we have the same groups: $G = GL_r, M = GL_n \times GL_{r-n} \subset GL_r$.

As a corollary we get that the dimensions of moduli spaces of both types of objects (abelian Anderson T-motives; abelian varieties with MIQF) are equal: they are $n(r-n)$.

The below sections 2, 3 contain constructions of the number field case analogs of the map $\alpha$ of (1.1), and of Theorem 1.7 respectively. From one side, finding of these constructions was inspired by the analogy; from another side, their existence is a support to the analogy.

**2. Abelian varieties with MIQF.** We shall fix an imaginary quadratic field $K = \mathbb{Q}(\sqrt{-\Delta})$. For simplicity, an abelian variety $A$ is treated up to isogeny, and we restrict ourselves only by one fixed polarization form.

The main theorem 2.6 establishes an equivalence of objects of types $B$ and $E$.

**Definitions for the type $B$.** Let $A = V/D_{\mathbb{Z}}, V = \mathbb{C}^r$ be an abelian variety with MIQF, $L = D_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$. Since we consider $A$ up to isogeny, we shall deal only with $L$ and not with $D_{\mathbb{Z}}$. We fix an inclusion $i : K \hookrightarrow \text{End} (A) \otimes_{\mathbb{Z}} \mathbb{Q}$ defining multiplication, and we fix an Hermitian polarization form $H = B + i\Omega$ of $A$ on $V$, where $B$ and $\Omega$ are respectively its real and imaginary parts. There are two structures of $K$-module on $V$: the ordinary one which is the restriction of the $\mathbb{C}$-module structure, and the $*$-structure (multiplication is denoted by $k \ast v, k \in K, v \in V$) coming from $i$. $L$ is a $K$-*module. We choose a basis $\xi_1, ..., \xi_r$ of $L/K$ (notations of [Sh]). According [Sh], p. 157, (11) there exists a matrix $T = \{t_{ij}\} \in M_r(K)$ such that

$$\forall k_1, k_2 \in K, i, j = 1, ..., r \quad \Omega(k_1 \ast \xi_i, k_2 \ast \xi_j) = \text{Tr}_{K/\mathbb{Q}}(k_1 t_{ij} \bar{k}_2) \quad (2.1)$$

([Sh], p. 157, (11)). $T$ has properties

(a) $\bar{T}^t = -T$, i.e. $iT$ is hermitian ([Sh], p. 157, (12)) and

(b) Signature of $iT$ is $(n, r-n)$ ([Sh], p. 160, (25)).

We restrict ourselves by those $A$ whose $T$ (it depends on $\xi_1, ..., \xi_r$) satisfy

$$T = \sqrt{-\Delta} E_{n,r-n} \quad (2.2)$$

where $E_{n,r-n} := \begin{pmatrix} E_n & 0 \\ 0 & -E_{r-n} \end{pmatrix}$.
2.3. Definitions for the type E. We consider the set of triples \((L, H_L, \alpha)\) where

1. \(L\) is a \(K\)-vector space of dimension \(r\);
2. \(H_L\) is a \(K\)-valued Hermitian form on \(L\) of signature \((n, r-n)\) such that there exists a basis of \(L\) over \(K\) satisfying the condition:
   
   the matrix of \(H_L\) in this basis is \(E_{n,r-n}\) \(\ldots\) \(\ldots\)

We denote by \(H_{L,C}\) the extension of \(H_L\) to \(L \otimes_K \mathbb{C}\).

3. \(\alpha: L \otimes_K \mathbb{C} \to \mathbb{C}^n\) is a \(\mathbb{C}\)-linear map such that \(\alpha\) is surjective, and
   
   The restriction of \(-H_{L,C}\) to \(\text{Ker } \alpha\) is a positive definite form. \(\ldots\)

**Remark.** This \(\alpha\) is clearly an analog of \(\alpha\) of 1.1. We see that conditions of surjectivity of \(\alpha\) hold in both cases, while the property 1.2 apparently has no analog in the number field case.

**Theorem 2.6.** There is a 1–1 correspondence between the above \(A, \iota, H\) — objects of type \((B)\) (here \(A\) is up to isogeny, and \(A\) has \(T\) satisfying 2.2), and the above triples \((L, H_L, \alpha)\) — objects of type \((E)\).

**Remark.** If the restriction of \(\alpha\) on \(L \subset L \otimes_K \mathbb{C}\) is injective (this holds for almost all triples \((L, H_L, \alpha)\) ) then the \(K\)-vector space in \(\mathbb{C}^n\) mentioned in \((E)\) is \(\alpha(L)\).

**Proof.** \(L\) is the same for both types (when we consider \(L\) for the type \((E)\), we omit * in \(k * l\)). Let the triple \((A, \iota, H)\) of the type \((B)\) be given. There is a canonical decomposition \(V = V^+ \oplus V^-\) where

\[
V^+ := \{v \in V \mid k \ast v = kv\}
\]

\[
V^- := \{v \in V \mid k \ast v = \bar{kv}\}
\]

We denote by \(\pi_+: V \to V^+\) the projection along \(V^-\). Let \(\beta: L \to V\) be the tautological inclusion.

2.9. The triple \((L, H_L, \alpha)\) corresponding to \((A, \iota, H)\) is constructed as follows. We define

\[
\alpha: L \otimes_K \mathbb{C} \to V^+ = \mathbb{C}^n
\]

by the formula \(\alpha(l \otimes z) = z \cdot \pi_+(\beta(l))\). Formula 2.7 shows that it is well-defined. The form \(H_L\) is defined by the equality

\[
\forall l_1, l_2 \in L \quad \text{Tr}_{K/\mathbb{Q}}(\sqrt{-\Delta}H_L(l_1, l_2)) = \Omega(\beta(l_1), \beta(l_2))\]

**Lemma.** Formula 2.10 really defines \(H_L\) uniquely.

**Proof. Unicity:** we fix \(l_1, l_2 \in L\), and we consider a \(\mathbb{Q}\)-linear form \(\gamma: K \to \mathbb{Q}\) defined by the formula

\[
\gamma(k) = \Omega(\beta(k \ast l_1), \beta(l_2))
\]
There exists the only \(k_0 \in K\) such that \(\gamma(k) = \text{Tr}_{K/Q}(k_0k)\). We see that if 2.10 holds for all pairs \(k \cdot l_1, l_2\), then necessarily \(\sqrt{-\Delta H_L(l_1, l_2)} = k_0\).

**Existence:** Let \(r_1, \ldots, r_r\) be a basis of \(L/K\) such that the corresponding \(T\) has the form 2.2. We define \(H_L\) by the condition that the matrix of \(H_L\) in \(r_1, \ldots, r_r\) is \(E_{n,r-n}\). 2.1 implies that \(H_L\) satisfies 2.10. □

To prove 2.5 we recall a well-known

2.11. **Coordinate description.** The Siegel domain for the present case is the following:

\[
\mathcal{H}_{n,r-n}^3 := \{ z \in M_{n,r-n}(\mathbb{C}) \mid E_n - z\bar{z}^t \text{ is positive hermitian} \} \tag{2.12}
\]

([Sh], p. 162, 2.6). For any \(z \in \mathcal{H}_{n,r-n}^3\) we can construct an above abelian variety \(A_z\) (satisfying 2.2) as follows ([Sh]). We fix a basis \(e_1, \ldots, e_r\) of \(V\) over \(\mathbb{C}\) such that \(e_1, \ldots, e_n\) (resp. \(e_{n+1}, \ldots, e_r\)) is a basis of \(V^+\) (resp. \(V^-\)) over \(\mathbb{C}\). It defines the \(K\)-\(*\)-action on \(V\). Let \(r_*\) (resp. \(e_*\)) be the matrix column of \(r_1, \ldots, r_r\) (resp. \(e_1, \ldots, e_r\)). They satisfy

\[
r_* = Ye_* \quad \text{where} \quad Y = \begin{pmatrix} E_n & z \\ z^t & E_{r-n} \end{pmatrix} \tag{2.13}
\]

([Sh], p. 162, (35), Type IV). \(\Omega\) is defined by 2.1, 2.2. These conditions define \(A_z\).

**Proof of 2.5.** 2.13 implies that elements

\[
\lambda_i := r_{n+i} \otimes 1 - \sum_{k=1}^{n} r_k \otimes z_{ki} \tag{2.14}
\]

\(i = 1, \ldots, r-n\), form a basis of \(\text{Ker} \alpha\). We have

\[
H_L(\lambda_i, \lambda_j) = \sum_{k=1}^{n} z_{ki} \bar{z}_{kj} - \delta_i^j = \{ z^t \bar{z} - E_{r-n} \}_{ij}
\]

hence 2.12 implies 2.5.

So, we have constructed a well-defined map from the set of objects of type (B) to the set of objects of type (E).

To construct the inverse map we need a definition. Let \(W\) be a \(\mathbb{C}\)-vector space. We denote by \(i(W)\) the complex conjugate space together with a map \(i : W \rightarrow i(W)\) which is an isomorphism of \(\mathbb{R}\)-vector spaces and satisfies \(i(zw) = \bar{z}i(w), z \in \mathbb{C}, w \in W\).

Let a triple \((L, H_L, \alpha)\) be given. Let \((\text{Ker} \alpha)^\perp \subset L \otimes_K \mathbb{C}\) be the \(H_L, \mathbb{C}\)-orthogonal space of \(\text{Ker} \alpha\), \(\pi_{\alpha} : L \otimes_K \mathbb{C} \rightarrow \text{Ker} \alpha\) the projection along \((\text{Ker} \alpha)^\perp\), and let us consider the composition \(i \circ \pi_{\alpha} : L \otimes_K \mathbb{C} \rightarrow i(\text{Ker} \alpha)\).

We let \(V = \mathbb{C}^n \oplus i(\text{Ker} \alpha)\) (here \(\mathbb{C}^n\) is the target of \(\alpha\)), and we define the inclusion \(L \hookrightarrow V\) by the formula

\[
l \mapsto (\alpha(l), i \circ \pi_{\alpha}(l))
\]

Shimura uses the formula \(Y = \begin{pmatrix} E_n & z \\ z^t & E_{r-n} \end{pmatrix}\); bar in the (2,1)-th block of \(Y\) disappears because of the different system of notations of [Sh] and of the present paper.
This is an easy exercise for the reader to define polarization on $V$, to prove that we get an abelian variety of type (B) and that this construction is inverse to the one of 2.9. Let us indicate some steps of the proof. Let $A = V/D_\mathbb{Z}$ be given, as above. We need to consider the projection $\pi_- : V \to V^-$ along $V^+$ and a map $i \circ \pi_- : V \to i(V^-)$. The restriction of $i \circ \pi_-$ to $L$ is a map of $K$-vector spaces ($K^*$-structure on $L$ and the ordinary structure on $V^-$), hence it can be extended to a $\mathbb{C}$-linear map $L \otimes_K \mathbb{C} \to i(V^-)$; we denote this extension by $i \circ \pi_-$ as well.

Ker $\alpha$, Ker $i \circ \pi_- \subset L \otimes_K \mathbb{C}$ are $H_{L, \mathbb{C}}$-orthogonal: really, 2.13 implies that

$$\pi_- (x_i) = \sum_{k=1}^{r-n} z_{ik} e_{n+k} \quad (i = 1, \ldots, n)$$

$$\pi_- (x_{n+i}) = e_{n+i} \quad (i = 1, \ldots, r-n)$$

hence

$$i \circ \pi_- (x_i) = i \left( \sum_{k=1}^{r-n} z_{ik} e_{n+k} \right) = \sum_{k=1}^{r-n} \bar{z}_{ik} i(e_{n+k}) \quad (i = 1, \ldots, n)$$

$$i \circ \pi_- (x_{n+i}) = i(e_{n+i}) \quad (i = 1, \ldots, r-n)$$

and hence Ker $i \circ \pi_-$ is generated by elements

$$\mu_i := x_i \otimes 1 - \sum_{k=1}^{r-n} x_{n+k} \otimes \bar{z}_{ik}$$

$i = 1, \ldots, n$. Obviously $H_{L, \mathbb{C}}(\lambda_i, \mu_j) = 0$. Clearly we have the canonical isomorphism $V^- = i(\text{Ker} \ \alpha)$. □

**Remark.** If the reader prefers he can use coordinates: a basis $x_1, \ldots, x_r$ satisfying (2.4) defines a $n \times (r-n)$-matrix $z = \{ z_{ij} \}$ as follows (this is the same as 2.14):

$$\alpha(x_{n+i}) = \sum_{k=1}^{n} z_{ki} \alpha(x_k)$$

(It is easy to prove that $\alpha(x_k)$, $k = 1, \ldots, n$ form a basis of $\mathbb{C}^n$). Condition (2.5) implies that $z \in H^3_{n,r-n}$ (calculations coincide with the ones of the above Proof of 2.5). Let $A = A_z$ be the abelian variety corresponding to $z$. $e_1, \ldots, e_r, x_1, \ldots, x_r$ for it are the same as in 2.11, $\Omega$ and hence $H$ are defined by 2.1, 2.2 uniquely.

**Section 3. Exterior powers of abelian varieties with MIQF having $n = 1$.**

Let $(A, \iota, H)$ be a triple of Theorem 2.6. The associated triple $(L, H_L, \alpha)$ defines an exact sequence of $\mathbb{C}$-vector spaces

$$0 \to \text{Ker} \ \alpha \xrightarrow{i} L \otimes_K \mathbb{C} \xrightarrow{\alpha} \mathbb{C}^n \to 0$$

Let us take the $k$-th exterior power of this exact sequence (Definition 1.4):

$$0 \to \lambda^k(\text{Ker} \ \alpha) \xrightarrow{\lambda^k(i)} \lambda^k(L) \otimes_K \mathbb{C} \xrightarrow{\alpha_k} W_k \to 0$$
There exists an Hermitian form $\lambda^k(H_L)$ on $\lambda^k(L)$; recall that

$$\lambda^k(H_L)(l_1 \wedge \ldots \wedge l_k, l'_1 \wedge \ldots \wedge l'_k) = \det\{H_L(l_i \wedge l'_j)\}$$

It is obvious that if $n = 1$ then the restriction of $(-1)^k \lambda^k(H_L)$ to $\lambda^k(\ker \alpha)$ is positive definite. This means that the triple $\lambda^k(L), (-1)^{k-1} \lambda^k(H_L), \alpha_k$ satisfies conditions of Theorem 2.6 and hence defines an abelian variety (up to isogeny) $\lambda^k(A)$ which is called the $k$-th exterior power of $A$. Its signature is $\left(\binom{r - 1}{k - 1}, \binom{r - 1}{k}\right)$.

**Remark 3.1.** It is easy to see that if $n \neq 1, r - 1$ then this construction cannot be applied to abelian varieties of signature $(n, r - n)$; we cannot also define symmetric powers of abelian varieties with MIQF, as well as their tensor products. This is clearly an analog of 1.3.

**Remark 3.2.** For the functional field case there exists a natural definition of the exterior power of abelian Anderson T-motives and 1.7 is a theorem, while for the number field case there is no such definition hence we must consider the above construction — an analog of the Theorem 1.7 — as the definition of the exterior powers of abelian varieties with MIQF.

4. Open problems.

4.1. Remark 3.1 and (1.3) inspire us to define an object (a generalization of an abelian variety) which is an analog of Anderson T-motives having $N \neq 0$. The tensor product of abelian varieties having multiplication by an imaginary quadratic field should be this new object.

4.2. There is no complete analogy between $\alpha$ of 1.3 and 2.3, (3). For example, in the number field case the restriction of $\alpha$ to $L$ is not always injective but only almost always injective. Maybe it is possible to generalize the notion of abelian Anderson T-motive in order to make the analogy more precise?

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