NEVANLINNA THEORY AND RATIONAL POINTS

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Abstract S. Lang [L] conjectured in 1974 that a hyperbolic algebraic variety defined over a number field has only finitely many rational points, and its analogue over function fields. We discuss the Nevanlinna-Cartan theory over function fields of arbitrary dimension and apply it for Diophantine property of hyperbolic projective hypersurfaces (homogeneous Diophantine equations) constructed by Masuda-Noguchi [MN]. We also deal with the finiteness property of $S$-units points of those Diophantine equations over number fields.

Introduction

S. Lang [L] conjectured in 1974 that a hyperbolic algebraic variety defined over a number field has only finitely many rational points, and its analogue over function fields. For subvarieties of Abelian varieties the function field analogue was dealt with by M. Raynaud [R], and lately G. Faltings [F] proved this conjecture for subvarieties of Abelian varieties over number fields. On the other hand, the author [Nog5] proved the function field analogue in general case (cf. also [Nog1], [Nog2]). See Y. Imayoshi-H. Shiga [IS], M. Zaidenberg [Z], and M. Suzuki [Su1], [Su2] for non-compact versions of this result.

In the case of curves (Fermat, Catalan, Thue equations, etc.) defined over function fields, R.C. Mason [Ma1], J. Silverman [Si] and J. Mueller [Mu] obtained similar or more precise finiteness properties by making use of a different method which relies on the function field analogue of “$abc$-conjecture” of Masser-Oesterlé. The function field analogue of “$abc$-conjecture” was proved in more general form by R.C. Mason [Ma1], [Ma2], J. Voloch [Vo] and W. Brownawell-D. Masser [BrM]. They actually proved a version of “$abc$-conjecture” in several variables (to say, $abc\cdots$ conjecture), which is nothing but a special case of Nevanlinna-Cartan’s second main theorem with truncated counting functions applied to algebraic case (see [C], (3), and §2).

Here we discuss the Nevanlinna-Cartan theory over function fields of arbitrary dimension (cf. J. Wang [W1], [W2] and [W3] for related results), and apply it for Diophantine property of hyperbolic projective hypersurfaces (homogeneous Diophantine equations) constructed by [MN]. We also deal with the finiteness property of $S$-units points of those Diophantine equations over number fields.

Research at MSRI supported in part by NSF grant #DMS 9022140.
Acknowledgement. The present manuscript is based on a talk of the author for Workshop in Nevanlinna Theory and Diophantine Approximation, January 22-26, 1996, Mathematical Sciences Research Institute, University of California, Berkeley. The author is grateful to the institute for the very active and stimulating circumstance, and for the hospitality. His thanks are also due to the co-organizers of the workshop, Professor Pit-Mann Wong and P. Vojta.

1. Function field case

We deal with the Nevanlinna theory over function fields of an arbitrary dimension by making use of the method of Cartan [C] and Nochka [N], which affirmatively proved Cartan’s conjecture (cf. also [Ch] and [Fu2]). Let $k$ be an algebraically closed field of characteristic 0, let $R$ be a smooth projective algebraic variety of dimension $N$ over $k$, and let $K$ denote the rational function field of $R$. To use analytic definitions as well, we set $k = \mathbb{C}$.

We fix a class $\omega$ of an ample line bundle (a Hodge metric form) on $R$. Let $D$ be a divisor on $R$, and define the counting function of $D$ with respect to $\omega$ by

$$N(D; \omega) = \int_D \omega^{N-1}.$$ 

If $\sigma$ is a meromorphic section of some line bundle and $(\sigma)$ denotes the divisor determined by $\sigma$, then we write

$$N(\sigma; \omega) = N((\sigma); \omega).$$

Let $a_i \in K^*$, $1 \leq i \leq m$, and let $((a_j))$ denote the least common multiple of the polar divisors of $a_i$, $1 \leq i \leq m$. We define the height function of $(a_i)_{1 \leq i \leq m}$, by

$$ht((a_j); \omega) = N(((a_i)); \omega).$$

From $a_i \in K^*$, $1 \leq i \leq m$, we obtain a rational mapping $f = [1; a_1, \ldots, a_m] : R \to \mathbb{P}^m(\mathbb{C})$, and define the order function of $f$ by

$$T(f; \omega) = \int_R f^* \Omega \wedge \omega^{N-1},$$

where $\Omega$ denotes the Fubini-Study Kähler form on $\mathbb{P}^m(\mathbb{C})$. The first main theorem is nothing but the Poincaré residue theorem:
(1.1) Theorem. Let the notation be as above. Let $L$ be the line bundle determined by $(a_i)$, and let $\sigma \in \Gamma(R, L)$ be a holomorphic section. Then

$$T(f; \omega) = \text{ht}((a_j); \omega) = N(\sigma; \omega), \quad 0 \leq j \leq m.$$ 

Next we have the “second main theorem with truncated counting functions”:

(1.2) Theorem. Let $f = [\sigma_0, \ldots, \sigma_m] : R \to \mathbb{P}^m(C)$ be a reduced representation of a rational mapping given by holomorphic sections $\sigma_j$ of a line bundle. Let $H_1, \ldots, H_q$ ($q \geq m + 1$) be linear forms on $\mathbb{P}^m(C)$ in general position such that $f^*H_j \neq 0$ for any $j$. Let $r$ denote the rank of $df$ at general point, and let $l$ denote the dimension of the smallest linear subspace of $\mathbb{P}^m(C)$ containing $f(R)$. Then

$$(q - l - 1)T(f; \omega) \leq \sum_{i=1}^{q} N_{l-r+1}(f^*H_i; \omega) + \left\{\frac{(l-r+1)(l-r+2)}{2} + r - 1\right\} N(J; \omega).$$

(1.3)

Here $N_{l-r+1}(H_i(x_j); \omega)$ is the truncated counting function of zeros of $H_i(x_j)$, and $J$ is a divisor used to define the generalized Wronskian bundle $W = [pJ] \otimes L^{l+1}$, and is independent of $f$. Cf. J. T.-Y. Wang [W1], [W2], and [W3] for related results.

Theorem (1.2) plays an important role. Because of the truncated counting functions $N_{l-r+1}(H_i(x_j); \omega)$, we may consider inequality (1.3) as a version of “abc-conjecture” in several variables over function fields of an arbitrary dimension. (cf. [Ma2], [Vo], and [BrM], Corollary I). We derive a “ramification theorem” over function fields (Corollary (2.16)), and “Borel’s theorem” over functions fields:

(1.4) Theorem. Let $x_j \in K^*, 1 \leq j \leq s$, satisfy

$$a_1x_1^d + \cdots + a_sx_s^d = 0 \quad (s \geq 2).$$

Assume

$$(1.5) \quad d > s(s - 2) + (s - 1)^2 \text{ht}((a_1, \ldots, a_s); \omega) + \frac{(s - 1)(s - 2)}{2} N(J; \omega).$$

Then there is a disjoint decomposition $\{1, \ldots, s\} = \bigcup_{\nu=1}^{l} I_\nu$ of indices such that

(i) $|I_\nu| \geq 2$ for all $\nu$;
(ii) for arbitrary two indices $j, k \in I_\nu$, the ratio $x_j/x_k$ is a constant.
(iii) $\sum_{j \in I_\nu} a_jx_j^d = 0$ for all $\nu$. 

We use the above results to study the rational points of a Diophantine equation. Let \( X \subset \mathbb{P}^{n-1}_K \) be a hypersurface defined over \( K \) by equation
\[
a_1 M_1^d(z_1, \ldots, z_n) + \cdots + a_s M_s^d(z_1, \ldots, z_n) = 0,
\]
where \( a_j \in K^* = K \setminus \{0\} \) and \( d \in \mathbb{Z}, > 0 \). We set
\[
Y(P) = \{(u_1, \ldots, u_n) \in \mathbb{P}^{n-1}(k); \sum_j a_j M_j^d(z_1, \ldots, z_n)M_j^d(u_1, \ldots, u_n) = 0,
\text{and } u_j = 0 \text{ if } z_j = 0\}.
\]
Then \( Y(P) \) is a projective variety defined over \( k \). Moreover, we set
\[
\mathcal{R}(P) = \{(z_1u_1, \ldots, z_nu_n) \in \mathbb{P}^{n-1}(K); (u_1, \ldots, u_n) \in Y(P)\} \subset X(K).
\]

(1.7) Main Theorem. Let the notation be as above. Assume that \( \{M_j(z_1, \ldots, z_n)\}_{j=1}^s \) is \( n \)-admissible ([MN]).

(i) Assume that \( d > s(s-2) \). Then the heights \( \text{ht}((z_i); \omega) \) of points of \( X(K) \) are bounded.

(ii) Assume (4) for \( d \). Then all points of \( X(K) \) are defined over \( k \); that is, all points \( x = (x_1, \ldots, x_n) \in X(K) \) are represented by \( x_j \in k \).

(iii) Assume that
\[
d > s!(s! - 2) + \frac{(s! - 1)(s! - 2)}{2}N(J; \omega).
\]
Then there are only finitely many rational points \( P_\mu \in X(K), \mu = 1, \ldots, \mu_0 (< \infty) \) such that
\[
X(K) = \bigcup_{\mu=1}^{\mu_0} \mathcal{R}(P_\mu).
\]
In the proof of (iii) we use the result of E. Bombieri-J. Mueller [BM] in a form generalized by Theorem (1.2).

2. Number field case—\( S \)-unit points

In this section we deal with \( X \) defined by (1.6) over a number field \( F \). We say that a point \( (z_i) \in \mathbb{P}^{n-1}(F) \) is \( S \)-units point if \( z_i \in \mathcal{O}_S^* \) or \( z_i = 0 \), denote by \( X(\mathcal{O}_S^*) \) the set of \( S \)-units point of \( X \subset \mathbb{P}^{n-1}_F \). By making use the same idea as in the proof of the Main Theorem (1.7) we then apply Borel’s Theorem for \( S \)-units to prove
**Theorem.** Let \( \{ M_j(z_1, \ldots, z_n) \}_{j=1}^{s} \) be an \( n \)-admissible set of monomials, and let \( X \) be defined by (1.6) with an arbitrary \( d \geq 1 \). Then \( X(O_S^*) \) is a finite set.

**Remark.** (i) Mahler ([M], p. 724, Folgerung 2) proved Theorem (2.1) in the case of \( n = s = 3 \).

(ii) Let \([F; Q]\) denote the extension degree and let \(|S|\) be the cardinality of \( S \). By making use of the bound obtained by Győry [G1, G2], we have

\[
|X(O_S^*)| \leq (2^n - 1)(2^{35s^2})(s-1)^3|S|.
\]

(iii)(ASMT-conjecture) We give an analogue of Theorem (2.1) over number fields, which is an extension of Schmidt’s linear subspace theorem. This may be called “Arithmetic Second Main Theorem-conjecture” by its nature. Let \( F \) and \( S \) be as above. We define the truncated counting function \( N_\lambda(H(x_i)) \) over the places of \( F \) outside \( S \). Let \( H_j, 1 \leq j \leq q \), be linear forms in general position on \( \mathbb{P}_F^m \). Then for an arbitrary \( \epsilon > 0 \) there are finitely many hyperplanes \( E_\nu \) such that for \((x_i) \in \mathbb{P}_m(F) \setminus \bigcup E_\nu
\]

\[
(q - m - 1 - \epsilon)\text{ht}((x_i)) \leq N_m(H(x_i)).
\]

(iv) The above ASMT-conjecture implies the finiteness of \( X(F) \). In this respect the works of Khoai-Tu [KT] and Sarnak-Wang [SW] are of interest; especially Sarnak-Wang [SW] showed that some hyperbolic smooth hypersurfaces of \( \mathbb{P}^4 \) and \( \mathbb{P}^5 \) constructed by [MN] defined over \( \mathbb{Q} \) has infinitely many rational points over the \( p \)-adic number field \( \mathbb{Q}_p \) for every prime \( p \), and that its Brauer-Manin group known as an obstruction for the Hasse principle is vanishing.

(v) See [MN] for a number of examples.

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