CONNECTED AND OUTER-CONNECTED DOMINATION NUMBER OF MIDDLE GRAPHS

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Abstract. In this paper, we study the notions of connected domination number and of outer-connected domination number for middle graphs. Indeed, we obtain tight bounds for this number in terms of the order of the graph $M(G)$. We also compute the outer-connected domination number of some families of graphs such as star graphs, cycle graphs, wheel graphs, complete graphs, complete bipartite graphs and some operation on graphs, explicitly. Moreover, some Nordhaus-Gaddum-like relations are presented for the outer-connected domination number of middle graphs.

Keywords: Connected Domination number, Outer-Connected Domination number, Domination number, Middle graph, Nordhaus-Gaddum-like relation.

1. Introduction

Domination problems and its many generalizations have been intensively studied in graph theory since 1950, see for example [6], [7], [8], [12], [14] and [15]. In this paper, we use standard notation for graphs and we assume that every graph is non-empty, finite, undirected and simple. We refer to [2] as a general reference on the subject.

Given a simple graph $G$, a dominating set of $G$ is a set $S \subseteq V(G)$ such that $N_G[v] \cap S \neq \emptyset$, for any vertex $v \in V(G)$, where $N_G[v]$ is the closed neighborhood of $v$. The domination number of $G$ is the minimum cardinality of a dominating set of $G$ and it is denoted by $\gamma(G)$.

An important subclass of the dominating sets, that is central to this paper, is the class of connected dominating sets introduced in [4].

Definition 1.1. A dominating set $S$ of a graph $G$ is called connected dominating set if the induced subgraph $G[S]$ is connected. The minimum cardinality taken over all connected dominating sets in $G$ is called the connected domination number of $G$ and is denoted by $\gamma_c(G)$. Moreover, a connected dominating set of $G$ of cardinality $\gamma_c(G)$ is called a $\gamma_c$-set of $G$.

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In [3], the authors, taking inspiration from the notion of connected dominating set, introduced the concept of outer-connected dominating set.

**Definition 1.2.** A dominating set $S$ of a graph $G$ is called an outer-connected dominating set if the graph $G - S$ is connected. The minimum cardinality of an outer-connected dominating set of $G$ is called the outer-connected domination number of $G$ and it is denoted by $\gamma^c(G)$.

Following our previous works [9], [10] and [11], the aim of this paper is to study connected dominating sets and outer-connected dominating set of middle graphs. The concept of middle graph of a graph was first introduced in [5] as an intersection graph.

**Definition 1.3.** The middle graph $M(G)$ of a graph $G$ is the graph whose vertex set is $V(G) \cup E(G)$, where two vertices $x, y$ in the vertex set of $M(G)$ are adjacent in $M(G)$ if one of the following holds

1. $x, y \in E(G)$ and $x, y$ are adjacent in $G$;
2. $x \in V(G), y \in E(G)$, and $x, y$ are incident in $G$.

Notice that, by definition, if $G$ is a graph of order $n$ and size $m$, then $M(G)$ is a graph of order $n + m$ and size $2m + |E(L(G))|$, where $L(G)$ is the line graph of $G$.

In order to avoid confusion throughout the paper, we will use a “standard” notation for the vertex set and the edge set of the middle graph $M(G)$. In particular, if $V(G) = \{v_1, v_2, \ldots, v_n\}$, then we set $V(M(G)) = V(G) \cup M$, where $M = \{m_{ij} \mid v_iv_j \in E(G)\}$ and $E(M(G)) = \{v_im_{ij}, v_jm_{ij} \mid v_iv_j \in E(G)\} \cup E(L(G))$.

The paper is organized as follows. In Section 2, we recall few known results on outer-connected domination numbers and domination numbers. In Section 3, we compute the connected domination number of the middle graph of a connected graph. In Section 4, we present some upper and lower bounds for $\gamma^c(M(G))$ in terms of the order of the graph $G$, we relate the outer-connected domination number of $M(G)$ to the edge cover number of $G$ and we compute explicitly $\gamma^c(M(G))$ for several known families of graphs. In Section 5, we compute the outer-connected domination number of the middle graphs of graphs obtained by some special operation. In Section 6, we present some Nordhaus-Gaddum like relations for the outer-connected domination number of middle graphs. We then conclude the paper with a section composed of open problems and conjectures.

2. Preliminaries

In this short section, we recall three results which will be useful for our investigation.
Theorem 2.1 (ρ). If $G$ is a connected graph of order $n$, then
$$\gamma_c(G) \leq n - \delta(G),$$
where $\delta(G)$ is the minimum degree of a vertex in $G$.

Theorem 2.2 (ρ). Let $G$ be a graph with $n \geq 2$ vertices. Assume $G$ has no isolated vertices, then
$$\lceil \frac{n}{2} \rceil \leq \gamma(M(G)) \leq n - 1.$$

Theorem 2.3 (ρ). Let $G$ be a graph of order $n \geq 2$ with no isolated vertex. Then
$$\gamma(M(G)) = \rho(G),$$
where $\rho(G)$ is the edge cover number of $G$, i.e., the minimum cardinality of an edge cover of $G$.

3. Connected domination number of middle graphs

In this section, we calculate the exact value of the connected domination number $\gamma_c(M(G))$ for any connected graph $G$ of order $n \geq 3$.

Theorem 3.1. For any connected graph $G$ of order $n \geq 3$
$$\gamma_c(M(G)) = n - 1.$$

Proof. To fix notation, let $G$ be a connected graph with vertex set $V(G) = \{v_1, \ldots, v_n\}$. Then $V((M(G))) = V(G) \cup M$ where $M = \{m_{ij} | v iv_j \in E(G)\}$. First assume that $G = T$ is a tree. Obviously, $M$ forms a unique minimal connected path in $M(T)$ such that $N_{M(T)}[M] = V(M(T))$. This implies that $M$ is the minimal connected dominating set of $M(T)$ with $|M| = n - 1$, and hence $\gamma_c(M(G)) = n - 1$.

Now assume that $G$ is not tree. Then consider a spanning tree $H$ of $G$, and let $M_1 \subseteq M$ be the vertices subdividing the edges set of $H$ in $M(H)$. Obviously, $M_1$ forms a connected path in $M(H)$ such that $N_{M(H)}[M_1] = V(M(H))$. Consider $M_2 = M \setminus M_1$. Clearly, $M_1$ dominates all the vertices in $M_2$, and hence $N_{M(G)}[M_1] = M_2 \cup V(M(H)) = V(M(G))$. This implies that $M_1$ forms the minimal connected path in $M(G)$ with $|M_1| = n - 1$, and hence, $\gamma_c(M(G)) = n - 1$. □

4. Outer-connected domination number of middle graphs

As we will see in this section, the computation of the outer-connected domination number is more intricate than the one for the connected domination number.

We start our study by describing a lower and an upper bound for the outer-connected domination number of the middle graph.

Theorem 4.1. Let $G$ be a connected graph with $n \geq 2$ vertices. Then
$$\left\lceil \frac{n}{2} \right\rceil \leq \gamma_c(M(G)) \leq n.$$
Proof. If we consider \( D = V(G) \), then \( D \) is an outer-connected dominating set of \( M(G) \), and hence, \( \gamma_c(M(G)) \leq n \), proving the second inequality.

By Theorem 2.2 we have \( \gamma_c(M(G)) \geq \left\lceil \frac{n}{2} \right\rceil \), proving the first inequality. \( \square \)

As an immediate consequence of 2.3, we have the following result.

**Corollary 4.2.** Let \( G \) be a graph of order \( n \geq 2 \) with no isolated vertex. Then

\[
\gamma_c(M(G)) \geq \rho(G).
\]

In the next theorem, we calculate the outer-connected domination number of the middle graph of a tree.

**Theorem 4.3.** Let \( T \) be a tree with \( n \geq 2 \) vertices. Then

\[
\gamma_c(M(T)) = n.
\]

**Proof.** Let \( T \) be a tree of order \( n \) with \( V(T) = \{v_1, \ldots, v_n\} \). Then \( V(M(T)) = V(T) \cup M \) where \( M = \{m_{ij} \mid v_i v_j \in E(T)\} \). Let \( L = \{v_i \in V \mid d_T(v_i) = 1\} \) be the set of leaves of \( T \) with \(|L| = l\), and consider

\[
M_1 = \{m_{ij} \mid v_i \in L \text{ or } v_j \in L\},
\]

\[
M_2 = M \setminus M_1.
\]

If there exists a vertex \( v_i \in L \) such that \( v_i \notin D \), since \( N_{M(T)}[v_i] \cap D \neq \emptyset \), then \( N_{M(T)}[v_i] \cap D = \{m_{ij}\} \) for some \( m_{ij} \in M_1 \). As a consequence \( m_{ij} \in D \) and \( v_i \notin D \), and hence \( M(G) - D \) is disconnected, which is a contradiction. This implies that \( L \subseteq D \) and \(|D \cap L| = l\).

Let \( m_{ij} \in M_2 \) be such that \( m_{ij} \notin D \). Then, obviously \( M(G) - D \) is disconnected, which is a contradiction. As a consequence, \( M_2 \cap D = \emptyset \).

Now since for any \( v_i \in V(T) \setminus L \) we have that \( N_{M(T)}[v_i] \cap D \neq \emptyset \) and \( N_{M(T)}[v_i] \cap D \subseteq M_1 \cup (V(T) \setminus L) \), and for every distinct \( v_i, v_j \in V(T) \setminus L \) we have that \( (N_{M(T)}[v_i] \cap D) \cap (N_{M(T)}[v_j] \cap D) = \emptyset \), this implies that \(|D \cap (M_1 \cup (V(T) \setminus L))| \geq n - l\). Hence

\[
|D| = |D \cap L| + |D \cap (M_1 \cup (V(T) \setminus L))| \geq l + (n - l) = n.
\]

By Theorem 4.1, we conclude that \( \gamma_c(M(T)) = n \). \( \square \)

**Remark 4.4.** By Theorem 4.3, the upper bound described in Theorem 4.1 is tight.

**Corollary 4.5.** If \( T \) is a tree of order \( n \), then

\[
\gamma_c(T) < \gamma_c(M(T)).
\]

**Proof.** By Theorem 4.3 \( \gamma_c(M(T)) = n \). On the other hand, \( \gamma_c(T) \leq n - 1 \) by Theorem 2.1, and hence we obtain the described inequality. \( \square \)
Recall that the line graph \( L(G) \) of a graph \( G \) is the graph with vertex set \( E(G) \), where vertices \( x \) and \( y \) are adjacent in \( L(G) \) if and only if the corresponding edges \( x \) and \( y \) share a common vertex in \( G \). Directly from this definition and Theorem 4.3, we obtain the following result.

**Corollary 4.6.** For any tree \( T \) of order \( n \geq 2 \),

\[
\gamma_c(L(T)) < \gamma_c(M(T)).
\]

**Proof.** By definition \( V(L(T)) = E(T) \) and hence, \( |V(L(T))| = n - 1 \). This clearly implies that \( \gamma_c(L(T)) \leq n - 1 \). Hence \( \gamma_c(L(T)) \leq n - 1 < n = \gamma_c(M(T)) \) by Theorem 4.3. \( \square \)

By Theorem 4.3, we can characterize the trees by looking at the outer-connected domination number of their middle graph.

**Theorem 4.7.** Let \( G \) be a connected graph with \( n \geq 4 \) vertices. Then

\[
\gamma_c(M(G)) = n \text{ if and only if } G \text{ is a tree.}
\]

**Proof.** Assume that \( V(G) = \{v_1, \ldots, v_n\} \). Then \( V(M(G)) = V(G) \cup M \) where \( M = \{m_{ij} \mid v_iv_j \in E(G)\} \). If \( G \) is a tree, then \( \gamma_c(M(G)) = n \), by Theorem 4.3. On the other hand, assume that \( \gamma_c(M(G)) = n \) and \( G \) is not tree. Then \( G \) contains at least a cycle of order \( n \geq 3 \) as an induced subgraph. Without loss of generality, assume that \( G[v_1, v_2, \ldots, v_k] \) is a cycle of length \( k \), for some \( k \geq 3 \). Consider \( D = \{v_3, v_4, \ldots, v_n\} \cup \{m_{12}\} \). Then \( D \) is an outer-connected dominating set of \( M(G) \) with \( |D| = n - 1 \), and hence \( \gamma_c(M(G)) \leq n - 1 \), which is a contradiction. This implies that \( G \) is a tree. \( \square \)

In the next theorem we calculate outer-connected domination number for complete graph \( K_n \) where \( n \geq 3 \). Notice that \( K_2 \) is a tree and hence \( \gamma_c(M(K_2)) = 2 \) by Theorem 4.3.

**Theorem 4.8.** For any complete graph \( K_n \) of order \( n \geq 3 \), we have

\[
\gamma_c(M(K_n)) = \lceil n/2 \rceil
\]

**Proof.** Assume that \( V(M(K_n)) = V(K_n) \cup M \) where \( V(K_n) = \{v_1, \ldots, v_n\} \) and \( M = \{m_{ij} \mid v_iv_j \in E(G)\} \). When \( n = 3 \), it is easy to check that \( \gamma_c(M(K_n)) = 2 \), by considering \( D = \{v_1, m_{23}\} \). Now let \( n \geq 4 \). Assume that \( n \) is even and consider

\[
D = \{m_{12}, m_{34}, \ldots, m_{(n-1)n}\}.
\]

Then \( D \) is an outer-connected dominating set of \( M(G) \) with \( |D| = \lceil n/2 \rceil \). Similarly, if \( n \) is odd, consider

\[
D = \{m_{12}, m_{34}, \ldots, m_{(n-2)(n-1)}, m_{(n-1)n}\}.
\]

Then \( D \) is an outer-connected dominating set of \( M(G) \) with \( |D| = \lceil n/2 \rceil \). This show that \( \gamma_c(M(K_n)) \leq \lceil n/2 \rceil \). On the other hand, by Theorem 4.1, \( \gamma_c(M(K_n)) \geq \lceil n/2 \rceil \). \( \square \)
Remark 4.9. By Theorem 4.8, the lower bound described in Theorem 4.1 is tight.

Theorem 4.10. For any cycle $C_n$ of order $n \geq 3$,
$$\gamma_c(M(C_n)) = n - 1.$$  

Proof. To fix the notation, assume that $V(C_n) = \{v_1, \ldots, v_n\}$ and $E(C_n) = \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_n, v_nv_1\}$. Then $V(M(C_n)) = V(C_n) \cup M$, where $M = \{m_{i(i+1)} \mid 1 \leq i \leq n-1\} \cup \{m_{1n}\}$. Consider $D = \{v_3, \ldots, v_n\} \cup \{m_{12}\}$, then $D$ is an outer-connected dominating set with $|D| = n - 1$, and hence $\gamma_c(M(C_n)) \leq n - 1$.

Let $D$ be a minimal outer-connected dominating set of $M(C_n)$.

If $M \cap D = \emptyset$, then $N_{M(C_n)}[v_i] \cap D \neq \emptyset$ for every $1 \leq i \leq n$, implies that $V \subseteq D$ and hence $|D| \geq n$, contradicting the minimality of $D$.

Assume that $|M \cap D| = 1$. Without loss of generality, we can assume that $m_{12} \in D$. Then $N_{M(C_n)}[v_i] \cap D \neq \emptyset$ for $i \neq 1, 2, 3$, implies that $\{v_3, \ldots, v_n\} \subseteq D$ and so $\gamma_c(M(C_n)) \geq n - 1$, proving our statement.

Assume that $|M \cap D| = 2$. Let $m_{ij}, m_{pq} \in D$ for some $i, j, p, q$.

First, assume that $m_{ij}$ is adjacent to $m_{pq}$ in $M(C_n)$. Without loss of generality, we can assume that $m_{12}, m_{23} \in D$. Since $M(C_n) - D$ is connected, then $v_2 \in D$. Moreover, $N_{M(C_n)}[v_i] \cap D \neq \emptyset$ for $i \neq 1, 2, 3$, implies that $\{v_3, \ldots, v_n\} \subseteq D$ and hence $|D| \geq (n - 4 + 1) + 3 = n$, contradicting the minimality of $D$. Assume now that $m_{ij}$ is non-adjacent to $m_{pq}$ in $M(C_n)$ with $i < j < p < q$. Since $m_{ij}, m_{pq} \notin N_{M(C_n)}[v_k]$ for every $k \in \{1, 2, \ldots, n\} \setminus \{i, j, p, q\}$, then $|D \cap V(C_n)| \geq n - 4$. On the other hand, since $M(C_n) - D$ is connected then $v_j, v_p \in D$. This implies that $|D| = |D \cap V(C_n)| + |D \cap M| \geq (n - 2) + 2 = n$, contradicting the minimality of $D$.

Assume that $|M \cap D| = k \geq 3$. Since $M(C_n) - D$ is connected, then $k < n - 1$. Without loss of generality, we can assume that $M \cap D = \{m_{i_1(i_1+1)}, \ldots, m_{i_k(i_k+1)}\}$ where $i_1 < i_2 < \cdots < i_k$. If $i_j + 1 < i_{j+1}$ for some $1 \leq j \leq k - 1$, then $M(C_n) - D$ is disconnected. This implies that $i_j + 1 = i_{j+1}$ for all $1 \leq j \leq k - 1$. Let $I = \{i, i + 1 \mid m_{i(i+1)} \in D\}$, $V_1 = \{v_i \in V \mid i \notin I\}$. $N_{M(C_n)}[v_i] \cap D \neq \emptyset$ for $v_i \in V_1$, implies that $V_1 \subseteq D$. Moreover, since $M(C_n) - D$ is connected, $v_{i_1+1}, \ldots, v_{i_k} \in D$.

As a consequence, $D = \{m_{i_1(i_1+1)}, \ldots, m_{i_k(i_k+1)}\} \cup V_1 \cup \{v_{i_1+1}, \ldots, v_{i_k}\}$. This implies that $|D| = k + (n - k - 1) + k - 1 = n + k - 2 \geq n + 1$, contradicting the minimality of $D$.

Therefore, $\gamma_c(M(C_n)) = n - 1$.

\[\square\]

Theorem 4.11. For any wheel $W_n$ of order $n \geq 4$,
$$\gamma_c(M(W_n)) = \lfloor n/2 \rfloor.$$  

Proof. To fix the notation, assume $V(W_n) = V = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0v_1, v_0v_2, \ldots, v_0v_{n-1}\} \cup \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_1\}$. Then we have $V(M(W_n)) = V(W_n) \cup M$, where $M = \{m_{0i} \mid 1 \leq i \leq n-1\} \cup \{m_{i(i+1)} \mid 1 \leq i \leq n-2\} \cup \{m_{1(n-1)}\}$. Assume that $n$ is even and
consider\( D = \{m_{12}, m_{34}, \ldots, m_{(n-3)(n-2)}\} \cup \{m_{0(n-1)}\}\). Then \( D \) is an outer-connected dominating set of \( M(G) \) with \( |D| = [n/2] \). Similarly, if \( n \) is odd, consider \( D = \{m_{12}, m_{34}, \ldots, m_{(n-2)(n-1)}\} \cup \{m_{0(n-1)}\}\). Then \( D \) is an outer-connected dominating set of \( M(G) \) with \( |D| = [n/2] \).

This show that \( \gamma_c(M(W_n)) \leq [n/2] \). On the other hand, by Theorem 4.1, \( \gamma_c(M(W_n)) \geq [n/2] \). □

**Theorem 4.12.** Let \( K_{n_1, n_2} \) be the complete bipartite graph with \( n_2 \geq n_1 \geq 2 \). Then
\[
\gamma_c(M(K_{n_1, n_2})) = n_2.
\]

**Proof.** To fix the notation, assume \( V(K_{n_1, n_2}) = \{v_1, \ldots, v_{n_1}, u_1, \ldots, u_{n_2}\} \) and \( E(K_{n_1, n_2}) = \{v_iu_j \mid 1 \leq i \leq n_1, 1 \leq j \leq n_2\} \). Then \( M(K_{n_1, n_2}) = V(K_{n_1, n_2}) \cup M \), where \( M = \{m_{ij} \mid 1 \leq i \leq n_1, 1 \leq j \leq n_2\} \). Let \( D \) be an outer-connected dominating set of \( M(K_{n_1, n_2}) \). Since \( D \) is a dominating set for \( M(K_{n_1, n_2}) \), it has to dominate \( u_1, \ldots, u_{n_2} \) that are all disconnected. This implies that \( \gamma(M(K_{n_1, n_2})) \geq n_2 \). Now since \( D = \{m_{11}, m_{22}, \ldots, m_{n_1}\} \cup \{m_{n_1(n_1+1)}, m_{n_1(n_1+2)}, \ldots, m_{n_1n_2}\} \) is an outer-connected dominating set of \( M(K_{n_1, n_2}) \) with \( |D| = n_1 + n_2 - n_1 = n_2 \), this implies that \( \gamma_c(M(K_{n_1, n_2})) = n_2 \). □

**Theorem 4.13.** Let \( F_n \) be the friendship graph with \( n \geq 2 \). Then
\[
\gamma_c(M(F_n)) = n + 1.
\]

**Proof.** To fix the notation, assume \( V(F_n) = \{v_0, v_1, \ldots, v_{2n}\} \) and \( E(F_n) = \{v_0v_1, v_1v_2, \ldots, v_{2n-1}v_{2n}\} \). Then \( M(F_n) = V(F_n) \cup M \), where \( M = \{m_i \mid 1 \leq i \leq 2n\} \cup \{m_{i(i+1)} \mid 1 \leq i \leq 2n - 1 \) and \( i \) is odd\}.

By Theorem 2.2, \( \gamma(M(F_n)) \geq \lceil \frac{2n+1}{2} \rceil = n + 1 \). Now since \( D = \{m_{i(i+1)} \mid 1 \leq i \leq 2n - 1 \) and \( i \) is odd\} \cup \{v_0\} \) is an outer-connected dominating set for \( M(F_n) \) with \( |D| = n + 1 \), we have \( \gamma_c(M(F_n)) = n + 1 \). □

Putting together Theorems 4.10 and 3.1, we have the following result.

**Corollary 4.14.** There exists a connected graph \( G \) of order \( n \geq 3 \) such that
\[
\gamma_c(M(G)) = \gamma_c(M(G)).
\]

**Remark 4.15.** Comparing Theorem 3.1 and Theorem 4.3 we conclude that for any tree \( T \) we have
\[
\gamma_c(M(T)) < \gamma_c(M(T)).
\]

Similarly, comparing Theorems 3.1 and 4.10 we conclude that for any cycle
\[
\gamma_c(M(C_n)) = \gamma_c(M(C_n)).
\]

Finally, comparing Theorem 3.1 and Theorems 4.8 and 4.11 we conclude that
\[
\gamma_c(M(K_n)) = \gamma_c(M(W_n)) < \gamma_c(M(K_n)) = \gamma_c(M(W_n)).
\]
As a consequence, if $G$ be a connected graph of order $n$, then one may not conclude that
\[ \gamma_c(M(G)) \geq \bar{\gamma}_c(M(G)) \text{ or } \gamma_c(M(G)) \leq \bar{\gamma}_c(M(G)) \]

5. Operation on graphs

In this section, we study the outer-connected domination number for the middle graph of the corona, 2-corona and other types of graphs.

**Definition 5.1.** The corona $G \circ K_1$ of a graph $G$ is the graph of order $2|V(G)|$ obtained from $G$ by adding a pendant edge to each vertex of $G$. The 2-corona $G \circ P_2$ of $G$ is the graph of order $3|V(G)|$ obtained from $G$ by attaching a path of length 2 to each vertex of $G$ so that the resulting paths are vertex-disjoint.

**Theorem 5.2.** For any connected graph $G$ of order $n \geq 2$,
\[ n + [n/2] \leq \bar{\gamma}_c(M(G \circ K_1)) \leq 2n. \]

**Proof.** Assume $V(G) = \{v_1, \ldots, v_n\}$, then $V(G \circ K_1) = \{v_1, \ldots, v_{2n}\}$ and $E(G \circ K_1) = \{v_1v_{n+1}, \ldots, v_nv_{2n}\} \cup E(G)$. As a consequence, $V(M(G \circ K_1)) = V(G \circ K_1) \cup \mathcal{M}$, where $\mathcal{M} = \{m_{i(n+i)} | 1 \leq i \leq n\} \cup \{m_{ij} | v_iv_j \in E(G)\}$. Since $\{v_1, \ldots, v_{2n}\}$ is an outer-connected dominating set of $M(G \circ K_1)$, we have $\bar{\gamma}_c(M(G \circ K_1)) \leq 2n$.

Let $D$ be an outer-connected dominating set of $M(G \circ K_1)$. Assume $v_{n+i} \notin D$ for some $1 \leq i \leq n$, then since $D$ is a dominating set of $M(G \circ K_1)$ this implies that $m_{i(n+i)} \in D$ and so $M(G \circ K_1) - D$ is disconnected, which is a contradiction. As a consequence $D_1 = \{v_{n+1}, \ldots, v_{2n}\} \subseteq D$. Now since \( N_{M(G \circ K_1)}[v] \cap D_1 = \emptyset \) for all $v \in V(M(G))$, by Theorem 2.2, we have
\[ \bar{\gamma}_c(M(G \circ K_1)) \geq n + \gamma(M(G)) \geq n + [n/2]. \]

\[ \square \]

**Remark 5.3.** The upper bound in Theorem 5.2 is tight. In fact, when $G$ is a tree, then $G \circ K_1$ is also a tree and so $\bar{\gamma}_c(M(G \circ K_1)) = 2n$ by Theorem 4.3.

Moreover, also the lower bound in Theorem 5.2 is tight. To see it, consider $G = K_n$ and
\[ D = \{v_{n+i} | 1 \leq i \leq n\} \cup \{m_{12}, m_{34}, \ldots, m_{(n-1)n}\} \]
when $n$ is even, and
\[ D = \{v_{n+i} | 1 \leq i \leq n\} \cup \{m_{12}, m_{34}, \ldots, m_{(n-2)(n-1)m_{(n-1)n}}\} \]
when $n$ is odd. In each case, $D$ is an outer-connected dominating set of $M(K_n \circ K_1)$ with $|D| = n + [n/2]$.

Similarly to Theorem 5.2, we can describe lower and upper bounds for the outer-connected domination number of the middle graph of a 2-corona graph.
Theorem 5.4. For any connected graph $G$ of order $n \geq 2$,
\[ 2n + \lceil n/2 \rceil \leq \gamma_c(M(G \circ P_2)) \leq 3n. \]

Proof. Assume $V(G) = \{v_1, \ldots, v_n\}$, then $V(G \circ P_2) = \{v_1, \ldots, v_{3n}\}$ and $E(G \circ P_2) = \{v_iv_{n+i}, v_{n+i}v_{2n+i} \mid 1 \leq i \leq n\} \cup E(G)$. As a consequence, we have that $V(M(G \circ P_2)) = V(G \circ P_2) \cup M$, where $M = \{m_{i(n+i)}, m_{(n+i)(2n+i)} \mid 1 \leq i \leq n\} \cup \{m_{ij} \mid v_iv_j \in E(G)\}$. Since \{v_1, \ldots, v_{3n}\} is an outer-connected dominating set of $M(G \circ P_2)$, we have \[ \gamma_c(M(G \circ P_2)) \leq 3n. \]

Let $D$ be an outer-connected dominating set of $M(G \circ P_2)$. To prove first inequality, we claim that
\[ |D| = |\{v_{2n+i}, m_{(n+i)(2n+i)} \mid 1 \leq i \leq n\} \cap D| \geq 2n \]
Assume $v_{2n+i} \notin D$ for some $1 \leq i \leq n$. Since $D$ is a dominating set of $M(G \circ P_2)$ this implies that $m_{(n+i)(2n+i)} \in D$ and so $M(G \circ P_2) - D$ is disconnected, which is a contradiction. Hence \{v_{2n+i} \mid 1 \leq i \leq n\} \subseteq D. Now assume $v_{n+i} \notin D$ for some $1 \leq i \leq n$. Since $D$ is a dominating set of $M(G \circ P_2)$ this implies that $m_{(n+i)(2n+i)} \in D$ or $m_{i(n+i)} \in D$. If $m_{i(n+i)} \in D$, then $M(G \circ P_2) - D$ is disconnected, which is a contradiction, and hence $m_{i(n+i)} \in D$. This shows that for every $1 \leq i \leq n$, we have that $v_{n+i} \in D$ or $m_{(n+i)(2n+i)} \in D$. Now since by construction of $D_1$, we have that $N_{M(G \circ P_2)}[v] \cap D_1 = \emptyset$ for all $v \in V(M(G))$, this implies
\[ \gamma_c(M(G \circ P_2)) \geq 2n + \gamma(M(G)) \geq 2n + \lceil n/2 \rceil \]
by Theorem 2.2. 

\[ \square \]

Remark 5.5. The upper bound in Theorem 5.4 is tight. This is because when $G$ is a tree, then also $G \circ P_2$ is a tree and hence $\gamma_c(M(G \circ P_2)) = 3n$ by Theorem 4.3.

Moreover, also the lower bound in Theorem 5.4 is tight. Consider $G = K_n$, and
\[ D = \{v_{n+i}, v_{2n+i} \mid 1 \leq i \leq n\} \cup \{m_{12}, m_{34}, \ldots, m_{(n-1)n}\} \]
when $n$ is even, and
\[ D = \{v_{n+i}, v_{2n+i} \mid 1 \leq i \leq n\} \cup \{m_{12}, m_{34}, \ldots, m_{(n-2)(n-1)m_{(n-1)n}}\} \]
when $n$ is odd. In both cases, $D$ is an outer-connected dominating set of $M(K_n \circ P_2)$ with $|D| = 2n + \lceil n/2 \rceil$.

In the next two theorems, we study the outer-connected domination number of the middle graph of the join of a graph with $\overline{K_p}$.

Theorem 5.6. For any connected graph $G$ of order $n \geq 2$ and any integer $p \geq n$,
\[ \gamma_c(M(G + \overline{K_p})) = p. \]
Proof. Assume $V(G) = \{v_1, \ldots, v_n\}$ and $V(\overline{K}_p) = \{v_{n+1}, \ldots, v_{n+p}\}$. Then $V(M(G+\overline{K}_p)) = V(G+\overline{K}_p) \cup M_1 \cup M_2$ where $M_1 = \{m_{ij} \mid v_i v_j \in E(G)\}$ and $M_2 = \{m_{i(n+j)} \mid 1 \leq i \leq n, 1 \leq j \leq p\}$.

By [9, Theorem 2.15], we have that $\gamma(M(G+\overline{K}_p)) = p$, and hence $\gamma_c(M(G+\overline{K}_p)) \geq \gamma(M(G+\overline{K}_p)) = p$.

On the other hand, if we consider $D = \{m_{i(n+i)} \mid 1 \leq i \leq n\} \cup \{m_{1(n+j)} \mid n+1 \leq j \leq n\}$, then $D$ is an outer-connected dominating set of $M(G+\overline{K}_p)$ with $|D| = p$, and hence $\gamma_c(M(G+\overline{K}_p)) \leq p$. \qed

Theorem 5.7. For any connected graph $G$ of order $n \geq 2$ and any integer $p < n$,
\[ \lceil \frac{n+p}{2} \rceil \leq \gamma_c(M(G+\overline{K}_p)) \leq n. \]

Proof. The first inequality follows directly from Theorem 4.1. On the other hand, using the same notation as in the proof of Theorem 5.6, if we consider $D = \{m_{i(n+i)} \mid 1 \leq i \leq p\} \cup \{v_i \mid p+1 \leq j \leq n\}$, then $D$ is an outer-connected dominating set of $M(G+\overline{K}_p)$ with $|D| = n$, and hence we obtain the second inequality. \qed

Remark 5.8. Both inequalities in Theorem 5.7 are sharp. In fact, if we consider $G = C_4$ and $p = 2$, then a direct computation shows that $\gamma_c(M(C_4 + \overline{K}_2)) = 3 = \lceil \frac{n+p}{2} \rceil$. Similarly, if we consider $G = C_4$ and $p = 3$, then $\gamma_c(M(C_4 + \overline{K}_3)) = 4 = n$.

6. NORDHAUS-GADDUM-LIKE RELATIONS

Finding a Nordhaus-Gaddum-like relation for any parameter in graph theory is one of the traditional works which started after the following theorem by Nordhaus and Gaddum from [13].

Theorem 6.1 ([13]). For any graph $G$ of order $n$, $2\sqrt{n} \leq \chi(G) + \chi(\overline{G}) \leq n + 1$.

In this section, we find Nordhaus-Gaddum-like relations for the outer-connected domination number of middle graphs. In particular, by Theorems 4.1 and 3.1, we have the following result.

Corollary 6.2. Let $G$ be a connected graph with $n \geq 4$ vertices, where $G$ is not a tree. Then
\[ n + \frac{n}{2} - 1 \leq \gamma_c(M(G)) + \gamma_c(M(G)) \leq 2n - 2, \]
\[ \lceil \frac{n}{2} \rceil (n - 1) \leq \gamma_c(M(G)) \cdot \gamma_c(M(G)) \leq (n - 1)^2. \]

Remark 6.3. The upper bounds in Corollary 6.2 are both tight, for example when $G$ is a cycle, by Theorem 4.10 and Theorem 3.1.

Similarly, also the lower bounds in Corollary 6.2 are tight, for example when $G$ is a complete graph $K_n$ or a wheel graph $W_n$, by Theorems 4.8, 4.11 and 3.1.
CONNECTED AND OUTER-CONNECTED DOMINATION NUMBER OF MIDDLE GRAPHS

7. Open problems

We conclude the paper with a series of observations and open problems related to the notion of outer-connected domination number.

By Corollary 4.5, if $G$ is a tree of order $n$, then $\bar{\gamma}_c(G) < \bar{\gamma}_c(M(G))$. On the other hand, by Theorems 4.8, 4.10, 4.11, 4.12, 4.13 and [3], it is easy to see that

$$1 = \bar{\gamma}_c(K_n) = \bar{\gamma}_c(W_n) < \bar{\gamma}_c(M(K_n)) = \bar{\gamma}_c(M(W_n)) = \lceil n/2 \rceil,$$

$$1 = \bar{\gamma}_c(F_n) < \bar{\gamma}_c(M(F_n)) = n + 1,$$

$$2 = \bar{\gamma}_c(K_{n1,n_2}) < \bar{\gamma}_c(M(K_{n1,n_2})) = n_2$$

and

$$n - 2 = \bar{\gamma}_c(C_n) < \bar{\gamma}_c(M(C_n)) = n - 1.$$ These facts all support the following conjecture.

**Conjecture 7.1.** Let $G$ be a graph of order $n \geq 2$. Then

$$\bar{\gamma}_c(G) < \bar{\gamma}_c(M(G)).$$

Similarly to the previous conjecture, it is natural to compare the outer-connected domination number of the middle graph and of the line graph.

By Corollary 4.6, if $T$ is a tree, then $\bar{\gamma}_c(L(T)) < \bar{\gamma}_c(M(T))$. On the other hand we can obtain similar results for some known families.

**Proposition 7.2.** For any cycle $C_n$ of order $n \geq 3$,

$$\bar{\gamma}_c(L(C_n)) < \bar{\gamma}_c(M(C_n)).$$

*Proof.* By definition of line graph, $C_n$ is isomorphic to $L(C_n)$ for every $n \geq 3$. This implies that $\bar{\gamma}_c(C_n) = \bar{\gamma}_c(L(C_n)) = n - 2$ by [3]. On the other hand $\bar{\gamma}_c(M(C_n)) = n - 1$ by Theorem 4.10, and hence $\bar{\gamma}_c(L(C_n)) < \bar{\gamma}_c(M(C_n)).$ □

**Proposition 7.3.** For any wheel $W_n$ of order $n \geq 5$,

$$\bar{\gamma}_c(L(W_n)) < \bar{\gamma}_c(M(W_n)).$$

*Proof.* Let $V(W_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and $E(W_n) = \{v_0v_1, \ldots, v_0v_{n-1}\} \cup \{v_1v_2, v_2v_3, \ldots, v_{n-1}v_1\}$. Then $V(M(W_n)) = V(W_n) \cup \mathcal{M}$, where $\mathcal{M} = \{m_{0i} | 1 \leq i \leq n-1\} \cup \{m_{1(i+1)} | 1 \leq i \leq n-2\} \cup \{m_{1(n-1)}\} = V(L(W_n))$ and $E(L(W_n)) = \{m_{ij} \cap \{i, j\} \cap \{p, q\} = \{1\}$. Assume that $n$ is even and consider $D = \{m_{1(i+1)} | 1 \leq i \leq \lceil n/2 \rceil - 2\} \cup \{m_{01i}\}$. Then $D$ is an outer-connected dominating set of $L(G)$ with $|D| = \lceil n/2 \rceil - 1$. Similarly, if $n$ is odd, consider $D = \{m_{0(i+1)} | 1 \leq i \leq \lceil n/2 \rceil - 1\}$. Then $D$ is an outer-connected dominating set of $L(G)$ with $|D| = \lceil n/2 \rceil - 1$. This shows that $\bar{\gamma}_c(M(L_n)) \leq \lceil n/2 \rceil - 1$. By Theorem 4.11, $\bar{\gamma}_c(L(W_n)) \leq \lceil n/2 \rceil - 1 < \lfloor n/2 \rfloor = \bar{\gamma}_c(M(W_n)).$ □
Proposition 7.4. There exists a connected graph $G$ of order $n = 4$ such that 
$$\gamma_c(L(G)) = \gamma_c(M(G)).$$

Proof. Consider $G = W_4$ with $V(G) = \{v_0, v_1, v_2, v_3\}$ and $E(G) = \{v_0v_1, v_0v_2, v_0v_3, v_1v_2, v_2v_3, v_1v_3\}$. Then $V(M(G)) = V \cup M$ where $M = \{m_{ij} \mid v_iv_j \in E(G)\}$ and $V(L(G)) = M$. Assume that $D$ is a dominating set of $L(G)$ with $|D| = 1$. Then there exists an index $i$ for some $1 \leq i \leq 3$ such that $N_{L(G)}[m_{ij}] \cap D = \emptyset$ which is a contradiction. This implies that $\gamma(L(G)) \geq 2$, and hence that $\gamma_c(L(G)) \geq 2$. Now since $D = \{m_{12}, m_{03}\}$ is an outer-connected dominating set of $L(G)$ with $|D| = 2$, we have $\gamma_c(L(G)) = 2$. By Theorem 4.11 $\gamma_c(L(G)) = \gamma_c(M(G)) = 2$. □

As a consequence of Proposition 7.4, it is natural to ask the following

Problem 7.5. Can we classify the graphs $G$ such that 
$$\gamma_c(L(G)) = \gamma_c(M(G))?$$

In addition, the previous results also all support the following conjecture.

Conjecture 7.6. Let $G$ be a graph of order $n \geq 2$. Then 
$$\gamma_c(L(G)) \leq \gamma_c(M(G)).$$

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References

[1] M. Aouchiche and P. Hansen, A survey of Nordhaus-Gaddum type relations, Discrete Applied Mathematics, 161 (2013), 466–546.
[2] J. A. Bondy and U. S. R. Murty, Graph theory, Graduate texts in mathematics, vol. 244, Springer Science and Media, 2008.
[3] Cyman, J. The outer-connected domination number of a graph. Australasian Journal of Combinatorics 38 (2007), 35-46.
[4] E. Sampathkumar and H. B. Walikar, The connected domination number of a graph, J. Math. Phys. Sci., 13:607-613, 1979.
[5] T. Hamada and I. Yoshimura, Traversability and connectivity of the middle graph of a graph, Discrete Mathematics, 14 (1976) 247–255.
[6] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Fundamentals of Domination in Graphs, Marcel Dekker, New York, 1998.
[7] T. W. Haynes, S. T. Hedetniemi and P. J. Slater, Domination in Graphs: Advanced Topics, Marcel Dekker, New York, 1998.
[8] M. A. Henning and A. Yeo, Total domination in graphs, Springer Monographs in Mathematics, 2013.
[9] F. Kazemnejad, B. Pahlavsay, E. Palezzato and M. Torielli, Domination number of middle graphs. To appear in Transactions on Combinatorics. https://doi.org/10.22108/TOC.2022.131151.1927
[10] F. Kazemnejad, B. Pahlavsay, E. Palezzato and M. Torielli, Total dominator coloring number of middle graphs. To appear in Discrete Mathematics, Algorithms and Applications. https://doi.org/10.1142/S1793830922500768.

[11] F. Kazemnejad, B. Pahlavsay, E. Palezzato and M. Torielli, Total domination number of middle graphs. Electronic Journal of Graph Theory and Applications, 10(1), 275–288, 2022. http://dx.doi.org/10.5614/ejgta.2022.10.1.19.

[12] F. Kazemnejad and S. Moradi, Total Domination Number of Central Graphs, Bulletin of the Korean Mathematical Society, 56(2019), No. 4, pp. 1059-1075.

[13] E. A. Nordhaus and J. W. Gaddum, On complementary graphs, Amer. Math. Monthly, 63 (1956), 175-177.

[14] B. Pahlavsay, E. Palezzato and M. Torielli, 3-tuple total domination number of rook’s graphs. Discussiones Mathematicae Graph Theory. 42, 15-37, 2022. https://doi.org/10.7151/dmgt.2242.

[15] B. Pahlavsay, E. Palezzato and M. Torielli, Domination in latin square graphs. Graphs and Combinatorics, 37(3), 971-985, 2021.

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