Quantum cosmological solutions:
their dependence on the choice of gauge conditions
and physical interpretation

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Abstract

In “extended phase space” approach to quantum geometrodynamics numerical solutions to Schrödinger equation corresponding to various choice of gauge conditions are obtained for the simplest isotropic model. The “extended phase space” approach belongs to those appeared in the last decade in which, as a result of fixing a reference frame, the Wheeler – DeWitt static picture of the world is replaced by evolutionary quantum geometrodynamics. Some aspects of this approach were discussed at two previous PIRT meetings. We are interested in the part of the wave function depending on physical degrees of freedom. Three gauge conditions having a clear physical meaning are considered. They are the conformal time gauge, the gauge producing the appearance of Λ-term in the Einstein equations, and the one covering the two previous cases as asymptotic limits. The interpretation and discussion of the obtained solutions is given.

1. Introduction

In this paper we present solutions to quantum geometrodynamical Schrödinger equation corresponding to various choice of gauge conditions for the simplest isotropic model. It is widely accepted in quantum geometrodynamics to illustrate general ideas taking simple cosmological models as examples. The reason why physicists working in this field appeal to simple models is that now quantum geometrodynamics is just as far from being a completed theory as it was decades ago. One must confess that hitherto there is no agreement on what “first principles” this theory should be based and what is the form of master equation for a wave function of the Universe. The first version of quantum geometrodynamics, proposed by Wheeler and DeWitt [1, 2], encountered a number of fundamental problems (for discussion, see [3, 4, 5]). The main problem is the so called “frozen formalism”, or the absence of time evolution. It is easy to

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see that the source of the problem of time consists in the application of the Dirac postulates to gravitational field, according to which not the Schrödinger equation but the constraints as conditions on a wave function play the central part in the theory. As a result of impossibility to resolve the problems of the Wheeler–DeWitt quantum geometrodynamics in its own limits, in the last decade there appear a new tendency in the development of the theory which can be called Evolutionary Quantum Gravity. This tendency may be characterized by the two features: firstly, the recognition of the fact that it is impossible to obtain the evolutionary picture of the Universe without fixing a reference frame and, secondly, the rejection of the Wheeler–DeWitt equation and the reestablishment of the role which the Schrödinger equation plays in any quantum theory.

The tendency embraces several approaches (see, for example, [6, 7], where a dust fluid is considered as a good choice to fix a reference frame in quantum gravity), to which the “extended phase space” approach belongs. Some aspects of the latter were discussed at two previous PIRT meetings [8, 9]. The approach is based on a careful analysis of peculiarities of quantization of the Universe as a whole [10, 11]. The analysis showed that quantum geometrodynamics as a mathematically consistent theory failed to be constructed in a gauge invariant way, therefore, the Wheeler–DeWitt equation, being a constraint on a state vector, loses its significance and should be replaced by a gauge dependent Schrödinger equation resulting from the Hamiltonian formulation of the theory in extended phase space. A wave function satisfying the Schrödinger equation is determined on extended configurational space that involves gauge gravitational degrees of freedom equally as physical ones. However, we are actually interested in the part of the wave function depending on physical degrees of freedom only, since this very function defines probability distributions of physical quantities.

In Section 2 we shall describe the model and the Schrödinger equation for the physical part of wave function for the given model. Since the form of the Schrödinger equation is gauge dependent, to obtain descriptions of the Universe corresponding to various gauge conditions (in other words, to various reference frames) one has to solve, in fact, absolutely different differential equations. It naturally leads us to the question, is there any correspondence among solutions of the equations? And how should they be interpreted?

Let us note that while in [6, 7] the authors work with a certain parametrization of gravitational variables (as a rule, it is the Arnowitt–Deser–Misner parametrization [12]) and some “privileged” reference frame, our approach, though was applied to cosmological models with finite degrees of freedom, aimed at including arbitrary parametrizations and a wide enough class of gauge conditions. We shall consider three gauge conditions having a clear physical meaning: the conformal time gauge, the gauge producing the appearance of $\Lambda$-term in the Ein-
stein equations, and the one covering the two previous cases as asymptotic limits. For a closed
universe, the first and third gauges gives rise to a discrete Hamiltonian spectrum, while the
second gauge leads to a continuous spectrum. From a pure methodical viewpoint, the first and
third cases are much easier to be treated, and in Section 3 numerical solution for these cases
will be presented, meantime the second case admits qualitative consideration only. Section 4
contains physical interpretation and conclusions.

2. The model and the Schrödinger equation for the physical part
of the wave function

The action for a closed isotropic universe is

\[ S = -\int dt \left( \frac{1}{2} \dot{a}^2 - \frac{1}{2} Na \right) + S_{(\text{mat})} + S_{(gf)}, \]  

\[ S_{(\text{mat})} = -\int dt Na^3 \varepsilon(a), \quad S_{(gf)} = \int dt \pi_0 \left( \dot{N} - \frac{df}{da} \right). \]  

Matter fields are described in this model phenomenologically, without a clear indication on the
nature of the fields. The dependence of its energy density \( \varepsilon(a) \) on the scale factor \( a \) determines
its equation of state, namely, for the power dependence \( \varepsilon(a) = \frac{\varepsilon_0}{a^n} \), the equation of state is
known to be \( p_{(\text{mat})} = \left( \frac{n}{3} - 1 \right) \varepsilon_{(\text{mat})} \), \( \varepsilon_0 \) is a constant whose dimensionality in the Plank units
is \( \rho_{Pl} n_{Pl} \). Since we are interested in early enough stages of the Universe evolution, we shall
suppose that the Universe was filled with radiation with the equation of state \( p_{(\text{mat})} = \frac{1}{3} \varepsilon_{(\text{mat})} \),
i.e.

\[ \varepsilon(a) = \frac{\varepsilon_0}{a^n}. \]  

\( S_{(gf)} \) is a gauge-fixing part of the action, its variation giving rise to gauge dependent terms in the
Einstein equations. In ordinary quantum theory this terms are to be excluded by asymptotic
boundary conditions. As was argued in [10], in the case of the Universe with a non-trivial
topology, which, in general, does not possess asymptotic states, making use of asymptotic
boundary condition is not justified.

If so, the gauge-fixing action describes a subsystem of the Universe, some medium, whose
state is determined by a chosen gauge. In (2.2) a differential form of the gauge condition

\[ N - f(a) = 0 \]  

is used. The equation of state for this subsystem is

\[ p_{(\text{obs})} = \frac{1}{3} \frac{f'(a)}{f(a)} \alpha \varepsilon_{(\text{obs})}. \]
The index (obs) indicates that this subsystem corresponds to an observer studying the Universe evolution in his reference frame.

The action (2.1) is a particular case of the action for a cosmological model with a finite number degrees of freedom considered in [8, 11]. The Schrödinger equation for the physical part of the wave function looks like

$$\left[ -\frac{1}{2}\sqrt{\frac{N}{a}} \frac{d}{da} \left( \sqrt{\frac{N}{a}} \frac{d\Psi}{da} \right) + \frac{1}{2} Na\Psi - Na^3\varepsilon(a)\Psi \right]_{N=f(a)} = E\Psi. \quad (2.6)$$

From the classical point of view, $E$ is given by

$$E = -\int \sqrt{-g} T^0_{0(\text{obs})} \, d^3x. \quad (2.7)$$

$T^\nu_{\mu(\text{obs})}$ is a quasi energy-momentum tensor obtained by variation of the gauge-fixing action; it is not a real tensor in the sense that it depends on a gauge condition. $T^\nu_{\mu(\text{obs})}$ describes the subsystem of the observer in the gauged Einstein equations [8]. It can be shown that the integral (2.7) of $T^0_{0(\text{obs})}$ taken over space is a conserved quantity for the class of gauge conditions (2.4). Thus, $E$ characterizes the energy of the observer subsystem.

It may be said that on a phenomenological level this approach takes into account interaction between the observer subsystem and the physical Universe. The interaction causes rebuilding of energy balance of two subsystems. It is expected that at the late stage of the Universe evolution, when the Universe is well described by General Relativity, gauge effects are negligible, and the values of $E$ must be very close, if not equal, to zero. However, at the early quantum stage $E$ may have essentially non-zero values, and the exploration of its spectrum is the main task of this work.

Now we consider several gauge conditions.

1. The conformal time gauge $N = a$. The equation of state of the observer subsystem is the same as that of the matter: $p_{(\text{obs})} = \frac{1}{3}\varepsilon_{(\text{obs})}$. Substituting $N = a$ and (2.3) in (2.6), we get

$$-\frac{1}{2} \frac{d^2\Psi}{da^2} + \frac{1}{2} a^2\Psi - \varepsilon_0\Psi = E\Psi. \quad (2.8)$$

After redefinition

$$E + \varepsilon_0 \rightarrow E \quad (2.9)$$

we obtain the equation

$$-\frac{1}{2} \frac{d^2\Psi}{da^2} + \frac{1}{2} a^2\Psi = E\Psi. \quad (2.10)$$

Therefore, Eq. (2.10) describes the Universe filled with a “substance” with the equation of state $p = \frac{1}{3}\varepsilon$. Just some part of the energy of this substance may be due to a usual matter while the other part may be due to gauge, or observer, effects.
It was shown in [13] that Eq. (2.10) can be obtained in the limits of the Wheeler – DeWitt quantum geometrodynamics by rewriting of the Wheeler – DeWitt equation $H \Psi = 0$ as a Schrödinger-like equation $\tilde{H} \Psi = E \Psi$. Under additional requirements, that imply choosing a certain gauge condition and including a certain kind of matter into the model, the classical Hamiltonian constraint $H = 0$ can be presented in a new form, $\tilde{H} = E$, $H = \tilde{H} - E$, where $E$ is a conserved quantity which appears from phenomenological consideration of this kind of matter. So, in this approach, $E = \varepsilon_0$, i.e. $E$ is entirely due to the usual matter (radiation).

On the other side, the need for making a choice of gauge to rewrite the Wheeler – DeWitt equation in the special form $\tilde{H} \Psi = E \Psi$ witnesses to gauge noninvariance of the Wheeler – DeWitt theory. As was already emphasized above, the Wheeler – DeWitt equation loses its meaning, and it seems to be reasonable rejecting it rather trying to hold it by any means.

The effective potential $U(a) = \frac{1}{2} a^2$ is given at Fig. 1(a).

\begin{figure}
\centering
\begin{tabular}{ccc}
\hspace{0.2cm} & a) $N = a$ & b) $N = \frac{1}{a^3}$, $\varepsilon_0 = \frac{1}{50}$ & c) $N = a + \frac{1}{a^3}$, $\varepsilon_0 = \frac{1}{50}$ \\
\end{tabular}
\caption{The effective potentials for Eqs. (2.10), (2.11), (2.13).}
\end{figure}

2. $Na^3 = 1$. The gauge is believed to produce the appearance of $\Lambda$-term in the Einstein equations, since it is the analog of a more general condition $\det |g^{\mu\nu}| = 1$. The equation of state $p_{\text{(obs)}} = -\varepsilon_{\text{(obs)}}$. The Schrödinger equation takes the form

$$- \frac{1}{2a^4} \frac{d^2\Psi}{da^2} + \frac{1}{a^3} \frac{d\Psi}{da} + \frac{1}{2a^2} \Psi - \frac{\varepsilon_0}{a^4} \Psi = E \Psi. \quad (2.11)$$

Here $\varepsilon_0$ characterizes a contribution of the matter fields (radiation). If one includes into the model de Sitter false vacuum with the equation of state $p_{\text{(obs)}} = -\varepsilon_{\text{(obs)}}$ and the dependence $\varepsilon(a) = \varepsilon_0$, it does not affect the form of the equation (2.11) after redefinition (2.9). Then one could say that vacuum energy as well as gauge effects are responsible for eigenvalues of $E$.

The effective potential $U(a) = \frac{1}{2a^2} - \frac{\varepsilon_0}{a^4}$ depends on the parameter $\varepsilon_0$. According to modern cosmological notions, the Universe was created in a metastable state under the barrier depicted
at Fig. 1(b) and then tunneled through the barrier. The smaller the parameter \( \varepsilon_0 \) is, the higher and narrower the barrier becomes. There is a non-zero probability for arbitrary large values of the scale factor \( a \); it means that the Universe may expand to infinity in spite of the sign “+” we have put before the second term in (2.1), which corresponds to the closed model. It demonstrates that a naive correspondence between the kind of a cosmological model and the form of the effective potential has no grounds.

3. \( N = a + \frac{1}{a^3} \). This gauge covers the two previous cases as asymptotic limits. The equation of state is

\[
p_{(\text{obs})} = \frac{1}{3} a^4 - \frac{1}{3} a^4 + \frac{1}{3} \varepsilon_{(\text{obs})}.
\]  

At \( a \to 0 \) the equation gives \( p_{(\text{obs})} = -\varepsilon_{(\text{obs})} \); at \( a \to \infty \) it gives \( p_{(\text{obs})} = \frac{1}{3} \varepsilon_{(\text{obs})} \). Again, after redefinition (2.9) the Schrödinger equation looks like following

\[
-\frac{1}{2} \left( 1 + \frac{1}{a^4} \right) \frac{d^2 \Psi}{da^2} + \frac{1}{a^2} \frac{d \Psi}{da} + \frac{1}{2} a^2 \Psi + \frac{1}{2} a^{-2} \Psi - \frac{\varepsilon_0}{a^4} \Psi = E \Psi.
\]  

(2.13)

It is easy to check that Eqs. (2.11), (2.10) are the asymptotic limits of (2.13) at \( a \to 0 \) and \( a \to \infty \) respectively. In this case the Universe is believed to be filled by some mixture of matter and vacuum. In consequence of the redefinition (2.9), the value of \( E \) is due to matter contribution as well as gauge effects. Like in a previous case, the effective potential \( U(a) = \frac{1}{2} a^2 + \frac{1}{2} a^2 - \frac{\varepsilon_0}{a^4} \) depends on the parameter \( \varepsilon_0 \) and depicted at Fig. 1(c). The barrier at small \( a \) disappears when \( \varepsilon_0 = 0 \) and \( \varepsilon_0 \geq 0.1 \). The potential for some value of \( \varepsilon_0 \) is shown at Fig. 2. One can see that the potentials of Eq. (2.11) (green graph) and of Eq. (2.10) (blue graph) are asymptotic forms of the potential of Eq. (2.13).

\[ \begin{align*}
\text{Fig.2. The effective potentials for some values of } \varepsilon_0. \\
a) \varepsilon_0 = 0 \\
b) \varepsilon_0 = \frac{1}{150} \\
c) \varepsilon_0 = \frac{1}{2}
\end{align*} \]
3. Numerical solutions

The Hamiltonian operators in Eqs. (2.10), (2.13) have a discrete spectrum, and one meets no technical difficulties to obtain numerical solutions to these equations. The operator in (2.6) is Hermitian for an arbitrary gauge (2.4) if the measure in Hilbert space of solution is taken to be

\[ M(a) = \sqrt{\frac{a}{f(a)}}. \]  

(3.1)

One can see that the measure, like the equation itself, is gauge-dependent.

The standard method of finding eigenvalues and eigenfunctions consists in the expansion onto a basis functions which are orthonormal on the interval \([0, \infty]\) with the measure (3.1):

\[ \Psi(a) = \sum_n c_n \psi_n^s(a); \] 

(3.2)

\[ \psi_n^s(a) = \sqrt{\frac{n!}{(n+s)!}} \frac{1}{\sqrt{M(a)}} a^{\frac{s}{2}} L_n^s(a) = \sqrt{\frac{n!}{(n+s)!}} \left( \frac{f(a)}{a} \right)^{\frac{1}{2}} a^{\frac{s}{2}} L_n^s(a); \] 

(3.3)

\[ \int_0^\infty \psi_n^{s*}(a) \psi_m^s(a) M(a) da = \delta_{nm}, \] 

(3.4)

\( L_n^s(a) \) are Laguerre polynomials. The problem is reduced to finding eigenvalues and eigenvectors of the Hamiltonian matrix in the basis (3.3). The more terms are held in the expansion (3.2), the higher the precision is. The results of calculations of first five eigenvalues are presented at Table 1.

| \( \varepsilon_0 \) | Eq. (2.10), \( N = a \) | 1.5 | 3.5 | 5.5 | 7.50001 | 9.50008 |
|-----------------|---------------------|-----|-----|-----|---------|---------|
| 0               | 2.87886             | 5.32668 | 7.66977 | 9.9591 | 12.2175 |
| 1/500           | 2.87846             | 5.32635 | 7.66947 | 9.95882 | 12.2173 |
| 1/150           | 2.87754             | 5.32558 | 7.66877 | 9.95817 | 12.2166 |

| \( \varepsilon_0 \) | Eq. (2.13), \( N = a + \frac{1}{a^3} \) | 1.5 | 3.5 | 5.5 | 7.50001 | 9.50008 |
|-----------------|---------------------|-----|-----|-----|---------|---------|
| 1/2             | 2.77519             | 5.24152 | 7.59315 | 9.88783 | 12.1496 |
| 1              | 2.66102             | 5.15088 | 7.51266 | 9.81349 | 12.0792 |
| 3              | 2.04887             | 4.72486 | 7.14847 | 9.48369 | 11.7714 |
| 4              | 1.59368             | 4.47069 | 6.94071 | 9.29951 | 11.602  |
| 5              | 0.972188            | 4.1924 | 6.71849 | 9.1044  | 11.4236 |
| 7              | -1.07592            | 3.59902 | 6.25063 | 8.69468 | 11.0497 |
One can see that for Eq. (2.10), \( N = a \) the spectrum is equidistant, the difference between eigenvalues is equal to 2 in the Plank units (the deviation from this value is entirely due to calculation inaccuracy).

In the case of Eq. (2.13), \( N = a + \frac{1}{a^3} \), the eigenvalues do not differ significantly for \( \varepsilon_0 \leq \frac{1}{50} \) and converge to limiting values at \( \varepsilon_0 = 0 \). For \( \varepsilon_0 > \frac{1}{50} \) the spectrum levels tend to go down into the potential pit. The schematic picture of the spectrum is shown at Fig.3.

![Graphs showing spectrum levels for different potentials](image)

**Fig.3.** The spectrum levels for some potentials.

Fig. 4–6 pictures the probability distributions for the first (ground state), third and fifth solutions to Eq. (2.10) and Eq. (2.13) when \( \varepsilon_0 = \frac{1}{150} \) and \( \varepsilon_0 = 7 \). One can see that at the qualitative level the probability distributions do not significantly differ. The peak of the probability distribution in the all cases tends to shift to large values of the scale factor \( a \) for larger eigenvalues of \( E \). One could expect this result since the matter and gauge effects contribute to the value of \( E \). So, when the energy of matter increases, there may be enough probability for the scale factor to reach large values.
Fig. 4. The probability distributions for solutions to Eq. (2.10), $N = a$

$a) \ |\Psi_0|^2, \ E_0 = 1.5$

$b) \ |\Psi_2|^2, \ E_2 = 5.5$

$c) \ |\Psi_4|^2, \ E_4 = 9.5008$

Fig. 5. The probability distributions for solutions to Eq. (2.13), $N = a + \frac{1}{a^3}$, $\varepsilon_0 = \frac{1}{150}$

$a) \ |\Psi_0|^2, \ E_0 = 2.87754$

$b) \ |\Psi_2|^2, \ E_2 = 7.66877$

$c) \ |\Psi_4|^2, \ E_4 = 12.2166$

Fig. 6. The probability distributions for solutions to Eq. (2.13), $N = a + \frac{1}{a^3}$, $\varepsilon_0 = 7$
4. Concluding remarks

We should recognize that we have considered a very simple model and the obtained results are not of high degree of generality. The present work is just a small step “to find the way”.

We have seen that the second gauge condition, $N = \frac{1}{a^3}$, leads to a continuous spectrum of eigenvalues of the Schrödinger equation (2.6), while the two other gauges, $N = a$ and $N = a + \frac{1}{a^3}$, leads to a discrete spectrum, in other words, the second case is substantially different. It seems that one should seek for the reason in the structure of spacetime. Indeed, the gauge $N = \frac{1}{a^3}$ corresponds to the Universe in which the interval of proper time between two subsequent spacelike hypersurfaces tends to zero as $a \to \infty$, meantime it is not the case for the two other gauges. Since any gauge condition determines the form of the effective potential, this circumstance require a more careful exploration. It would be interesting to study the gauge $N = 1 + \frac{1}{a^3}$, for which at $a \to \infty$ the reference frame becomes a synchronous one ($N = 1$) and the equation of state of the observer subsystem at $a \to \infty$ is that of dust: $p_{(\text{obs})} = 0$.

The resemblance of probability distributions for solutions to Eqs. (2.10), (2.13) also deserves our attention. It demonstrates that one can reveal some relation among solutions for certain classes of gauge conditions. Let us note that this problem is well-known in the Wheeler – DeWitt quantum geometrodynamics, and the question how solutions to the Wheeler – DeWitt equation are related, was discussed as soon as parametrization noninvariance of this theory had been realized. Then Halliwell [14] proposed to restrict the class of admissible parametrizations. Since parametrization and gauge conditions have a unified interpretation [5], it implies also a restriction of the class of admissible gauge conditions, i.e. such an approach implies that it is permissible to describe the Universe in only one or several “privileged” reference frames. This way seems to be artificial since we do not know for sure what reference frame is privileged. Our point of view is that we face a new problem of finding classes of gauge conditions within which solutions to the Schrödinger equation are stable enough with respect to a choice of gauges, the determination of the classes seems to be inseparable from our understanding of spacetime structure.

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