TOTAL NONNEGATIVITY OF INFINITE HURWITZ MATRICES
OF ENTIRE AND MEROMORPHIC FUNCTIONS

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Abstract. In this paper we fully describe functions generating the infinite totally nonnegative Hurwitz matrices. In particular, we generalize the well-known result by Asner and Kemperman on the total nonnegativity of the Hurwitz matrices of real stable polynomials. An alternative criterion for entire functions to generate a Pólya frequency sequence is also obtained. The results are based on a connection between a factorization of totally nonnegative matrices of the Hurwitz type and the expansion of Stieltjes meromorphic functions into Stieltjes continued fractions (regular $C$-fractions with positive coefficients).

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1. Introduction

Functions mapping the upper half-plane of the complex plane into itself ($R$-functions) are well studied and play a significant role in applications. The subclass $S$ of $R$-functions, the functions that are regular and nonnegative over the nonnegative semi-axis (also known as Stieltjes functions) is of particular interest. In this paper we demonstrate a connection of meromorphic $S$-functions with total nonnegativity of corresponding Hurwitz-type matrices (Theorem 1.4). As an application, we study the following problem on the distribution of zeros.

A polynomial with no roots with a positive real part is called quasi-stable. Asner (see [3]) established that the Hurwitz matrix of a real quasi-stable polynomial is totally nonnegative (although there are polynomials with totally nonnegative Hurwitz matrices which are not quasi-stable). A matrix is called totally nonnegative if

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all of its minors are nonnegative. In addition, Kemperman (see [15]) showed that quasi-stable polynomials have totally nonnegative infinite Hurwitz matrices.

It turns out that the replacement of finite Hurwitz matrices with infinite Hurwitz matrices allows us to prove the converse: a polynomial is quasi-stable if its infinite Hurwitz matrix is totally nonnegative. The key to this is given in [9]: a special matrix factorization, which was successfully applied to a closely related problem in [10]. Moreover, when a theorem involves an infinite Hurwitz matrix, it is natural to suggest that it can be generalized to entire functions or power series. The first goal of the present paper is to obtain the following extension of the results from [3], [15] and [9] to power series, including the converse result.

**Theorem 1.1.** Given a power series $f(z) = z^j \sum_{k=0}^{\infty} f_k z^k$ in the complex variable $z$, where $f_0 > 0$ and $j$ is a nonnegative integer, the infinite Hurwitz matrix

$$
\mathcal{H}_f = \begin{pmatrix}
 f_0 & f_2 & f_4 & f_6 & f_8 & \ldots \\
 0 & f_1 & f_3 & f_5 & f_7 & \ldots \\
 0 & f_0 & f_2 & f_4 & f_6 & \ldots \\
 0 & 0 & f_1 & f_3 & f_5 & \ldots \\
 0 & 0 & f_0 & f_2 & f_4 & \ldots \\
 \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
$$

is totally nonnegative if and only if the series $f$ converges to a function of the form

$$
f(z) = C z^j e^{\gamma_1 z + \gamma_2 z^2} \prod_{\mu} \left(1 + \frac{z}{x_{\mu}}\right) \prod_{\nu} \left(1 + \frac{z}{\alpha_{\nu}}\right) \left(1 + \frac{z}{\pi_{\nu}}\right),
$$

where $C, \gamma_1, \gamma_2 \geq 0$, $x_{\mu}, y_{\lambda} > 0$, $\text{Re} \alpha_{\nu} \geq 0$, $\text{Im} \alpha_{\nu} > 0$ and

$$
\sum_{\mu} \frac{1}{x_{\mu}} + \sum_{\nu} \text{Re} \left(\frac{1}{\alpha_{\nu}}\right) + \sum_{\nu} \frac{1}{|\alpha_{\nu}|^2} + \sum_{\lambda} \frac{1}{y_{\lambda}} < \infty.
$$

**Remark 1.** Stating herein that a power series converges, by default we assume it to be convergent in a neighbourhood of the origin. Moreover, where it creates no uncertainties we use the same abbreviation for the series and a function it converges to.

**Remark 2.** It is possible that $\{x_{\mu}\} \cap \{y_{\lambda}\} \neq \emptyset$ in the expression (1.1). If so, the coinciding negative zeros and poles of the function $f(z)$ cancel each other out, while its positive poles remain untouched. For example, although the series $\sum_{k=0}^{\infty} z^k$ satisfies Theorem 1.1, it converges to the function $\frac{1}{1 - z}$ with a unique positive pole. The number of such cancellations may be infinite, however it cannot affect the convergence of involved infinite products.

Our second goal is achieved by Theorem 1.2, which is an extension of [10, Theorem 4.29].

**Theorem 1.2.** A power series $f(z) = \sum_{k=0}^{\infty} f_k z^k$ with $f_0 > 0$ converges to an entire function of the form

$$
f(z) = f_0 e^{\gamma z} \prod_{\nu} \left(1 + \frac{z}{\alpha_{\nu}}\right),
$$

(1.2)
where $\gamma \geq 0$, $\alpha_\nu > 0$ for all $\nu$ and $\sum_\nu \frac{1}{\alpha_\nu} < \infty$, if and only if the infinite matrix

$$D_f = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & f_4 & \cdots \\ 0 & f_1 & 2f_2 & 3f_3 & 4f_4 & \cdots \\ 0 & f_0 & f_1 & 2f_2 & 3f_3 & \cdots \\ 0 & 0 & f_1 & 2f_2 & 3f_3 & \cdots \\ 0 & 0 & 0 & f_0 & f_1 & f_2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \ddots \end{pmatrix}$$

is totally nonnegative.

This theorem complements the following well-known criterion established by Aissen, Edrei, Schoenberg and Whitney.

**Theorem 1.3** ([1, 2, 6], see also [12, Section 8 §5]). Given a formal power series $f(z) = \sum_{k=0}^{\infty} f_k z^k$, $f_0 > 0$, the Toeplitz matrix

$$T(f) = \begin{pmatrix} f_0 & f_1 & f_2 & f_3 & f_4 & \cdots \\ 0 & f_0 & f_1 & 2f_2 & 3f_3 & \cdots \\ 0 & 0 & f_0 & f_1 & 2f_2 & 3f_3 & \cdots \\ 0 & 0 & 0 & f_0 & f_1 & 2f_2 & \cdots \\ 0 & 0 & 0 & 0 & f_0 & \cdots & \ddots & \ddots \end{pmatrix} \quad (1.3)$$

is totally nonnegative if and only if $f$ converges to a meromorphic function of the form:

$$f(z) = f_0 e^{\gamma z} \prod_{\nu} \left( \frac{1 + \frac{z}{\alpha_\nu}}{\prod_\mu \left( 1 - \frac{z}{\beta_\mu} \right)} \right), \quad (1.4)$$

where $\gamma \geq 0$, $\alpha_\nu, \beta_\mu > 0$ for all $\mu, \nu$ and $\sum_\nu \frac{1}{\alpha_\nu} + \sum_\mu \frac{1}{\beta_\mu} < \infty$.

If we require the series $f(z)$ to represent an entire function under the assumptions of Theorem 1.3, we obtain that it has the form (1.2). We prove Theorems 1.1 and 1.2 in Section 4.

A sequence $(f_k)_{k=0}^{\infty}$ is commonly called totally positive (e.g. [2]), or a Pólya frequency sequence (e.g. [12]), whenever the matrix $T(f)$ defined by (1.3) is totally nonnegative. By Theorem 1.3, the general form of its generating function is given by the formula (1.4).

**Definition 1.** We denote by $\mathcal{R}$ ($\mathcal{R}^{-1}$ resp.) the class of all meromorphic$^1$ functions $F(z)$ analytic in the complement of the real axis and such that

$$\frac{\text{Im} F(z)}{\text{Im} z} \geq 0 \quad \left( \text{or} \quad \frac{\text{Im} F(z)}{\text{Im} z} \leq 0 \text{ for } F \in \mathcal{R}^{-1} \text{ resp.} \right)$$

Note that it is a straightforward consequence of the definition that $\mathcal{R}$- and $\mathcal{R}^{-1}$- functions are real (i.e. map the real line into itself). Furthermore, our definition includes real constants (like in [13]) into both classes $\mathcal{R}$ and $\mathcal{R}^{-1}$ although sometimes they are excluded in the literature (e.g. [22]).

$^1$In general, the condition to be meromorphic is replaced by less restrictive $F(\bar{z}) = \overline{F(z)}$. Basic properties of $\mathcal{R}$-functions can be found, for example, in [13] and (for the meromorphic case) in [22]. For brevity’s sake we confine ourselves to meromorphic functions only.
Definition 2. Denote by $\mathcal{S}$ the subclass of $\mathcal{R}$-functions that are regular and non-negative over the nonnegative reals. (Since $\mathcal{S}$-functions are meromorphic, they can have only negative poles and nonpositive zeros.)

Consider the infinite Hurwitz-type matrix (i.e. the matrix of the Hurwitz type)

$$H(p, q) = \begin{pmatrix}
  b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \ldots \\
  0 & a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\
  0 & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\
  0 & 0 & a_0 & a_1 & a_2 & a_3 & \ldots \\
  0 & 0 & b_0 & b_1 & b_2 & b_3 & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad (1.5)$$

where $p(z) = \sum_{k=0}^{\infty} a_k z^k$ and $q(z) = \sum_{k=0}^{\infty} b_k z^k$ are formal power series. Given two arbitrary constants $c$ and $\beta$, we also consider the matrix

$$J(c, \beta) = \begin{pmatrix}
  c & \beta & 0 & 0 & 0 & 0 & \ldots \\
  0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
  0 & 0 & c & \beta & 0 & 0 & \ldots \\
  0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
  0 & 0 & 0 & 0 & c & \beta & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad (1.6)$$

Matrices of this type will appear in our factorizations below.

Finally, for an infinite matrix $A = (a_{ij})_{i,j=1}^{\infty}$ and a fixed number $\rho$, $0 < \rho \leq 1$, we consider the matrix norm

$$\|A\|_\rho := \sup_{i \geq 1} \sum_{j=1}^{\infty} \rho^{i-1}|a_{ij}|.$$

Remark 3. Convergence in this norm implies entry-wise convergence. Moreover, the norm $\|A\|_\rho$ of a matrix $A$ coincides with the norm of the operator

$$A_\rho : x \mapsto A \cdot \text{diag}(1, \rho, \rho^2, \ldots) \cdot x,$$

acting on the space $l_\infty$ of bounded sequences.

Remark 4. Let functions $g(z), p(z), q(z)$ and $g^{(k)}(z), p^{(k)}(z), q^{(k)}(z), k = 1, 2, \ldots$, be holomorphic on $D_\rho := \{z \in \mathbb{C} : |z| \leq \rho\}$. Then the condition

$$\lim_{k \to \infty} \|T(g^{(k)}) - T(g)\|_\rho = 0$$

is equivalent to the uniform convergence of $g^{(k)}(z)$ to $g(z)$ on $D_\rho$, and the condition

$$\lim_{k \to \infty} \|H(p^{(k)}, q^{(k)}) - H(p, q)\|_\rho = 0$$

is equivalent to the uniform convergence of $p^{(k)}(z)$ to $p(z)$ and $q^{(k)}(z)$ to $q(z)$ on $D_\rho$.

Now we can formulate the more important result of this paper concerning properties of $\mathcal{S}$-functions. It has its own value apart from the proofs of Theorems 1.1 and 1.2.

Theorem 1.4. Consider the ratio $F(z) = \frac{n(z)}{p(z)}$ of power series $p(z) = \sum_{k=0}^{\infty} a_k z^k$ and $q(z) = \sum_{k=0}^{\infty} b_k z^k$, normalized by the equality $p(0) = a_0 = 1$. The following conditions are equivalent:
(i) The infinite Hurwitz-type matrix $H(p,q)$ defined by (1.5) is totally nonnegative.

(ii) The matrix $H(p,q)$ possesses the infinite factorization

$$H(p,q) = \lim_{j \to \infty} \left( J(b_0, \beta_0) J(1, \beta_1) \cdots J(1, \beta_j) \right) H(1,1) T(g)$$

converging in $\| \cdot \|_\rho$-norm for some $\rho$, $0 < \rho \leq 1$. Here $b_0 \geq 0$ and the sequence $(\beta_j)_{j \geq 0}$ is nonnegative, has a finite sum and contains no zeros followed by a nonzero entry, that is

$$\beta_0, \beta_1, \ldots, \beta_{\omega-1} > 0, \quad \beta_\omega = \beta_{\omega+1} = \cdots = 0,$$

$$0 \leq \omega \leq \infty, \quad \text{and} \quad \sum_{j=0}^\infty \beta_j < \infty. \quad (1.8)$$

The matrix $T(g)$ denotes a totally nonnegative Toeplitz matrix of the form (1.3) with ones on its main diagonal.

(iii) The ratio $F(z)$ is a meromorphic $S$-function; its numerator $q(z)$ and denominator $p(z)$ are entire functions of genus 0 up to a common meromorphic factor $g(z)$ of the form (1.4), $g(0) = 1$.

Remark 5. Note that

$$J(c,0) H(1,1) = H(1,c). \quad (1.9)$$

If $\omega$ is a finite number in (1.8), then $\beta_{\omega+1} = \beta_{\omega+2} = \cdots = 0$ implying

$$J(b_0, \beta_0) \cdots J(1, \beta_{\omega+1}) H(1,1) = J(b_0, \beta_0) \cdots J(1, \beta_{\omega+1}) J(1, \beta_{\omega+2}) H(1,1) = \cdots.$$ 

As a consequence, the factorization (1.7) can be expressed as follows in this case

$$H(p,q) = \begin{cases} 
J(b_0,0) H(1,1) T(g) = H(1,b_0) T(g) & \text{if } \omega = 0; \\
J(b_0, \beta_0) J(1, \beta_1) \cdots J(1, \beta_{\omega-1}) H(1,1) T(g) & \text{if } 0 < \omega < \infty.
\end{cases} \quad (1.10)$$

Remark 6. The number $\rho$ in Theorem 1.4 can be anywhere in $(0,1] \cap (0, \rho_0)$, here $\rho_0$ denotes the radius of convergence of $g(z)$ (which is positive by Theorem 1.3).

The matrix $T(g)$ from the condition (ii) of Theorem 1.4 is the Toeplitz matrix of the function $g(z)$ from (iii) given by (1.5).

If we require $p(z)$ and $q(z)$ to be entire functions in Theorem 1.4, then the function $g(z)$ has the form (1.2) or $g(z) \equiv 1$ and (1.7) converges in $\| \cdot \|_1$.

Remark 7. In the case $q(0) = b_0 = 0$ it can be convenient to “trim” the matrix $H(p,q)$ by removing its first row and its trivial first column. This corresponds to replacing $J(0, \beta_0)$ in the factorization (3.9) by its diagonal analogue $\text{diag}(1, \beta_0, 1, \beta_0, \ldots)$.

Remark 8. Since entire functions of genus 0 have unique Weierstraß’ representation, it makes sense to consider the greater common divisor of a subset of this class. Accordingly, two entire functions $p$ and $q$ of genus 0 are coprime whenever $\gcd(p,q) \equiv 1$.

Consider the continued fraction

$$b_0 + \frac{\beta_0 z}{1 + \frac{\beta_1 z}{1 + \frac{\beta_2 z}{1 + \cdots + \frac{\beta_{\omega-1} z}{1}}}}, \quad b_0 \geq 0, \quad \beta_0, \beta_1, \ldots, \beta_{\omega-1} > 0, \quad 0 \leq \omega \leq \infty, \quad (1.11)$$

where we combine both finite (terminating) and infinite cases. If the continued fraction is infinite, we assume $\omega = \infty$. The following Corollary (see its proof in
Subsection 2.2) allows us to connect the factorization (1.7) with continued fractions of this type.

**Corollary 1.5.** Let \( F(z) = \frac{q(z)}{p(z)} \) be a meromorphic \( S \)-function, where the entire functions \( p(z) \) and \( q(z) \) are of genus 0. Then it can be expanded into a uniformly convergent continued fraction of the form (1.11) with exactly the same coefficients \( b_0 \) and \( (\beta_j)_{j=0}^{\omega-1}, \sum_{j=0}^{\omega-1} \beta_j < \infty \), as in the factorization (3.9) of the matrix \( H(p,q) \). No other continued fractions of the form

\[
F(z) = c_0 + \frac{c_1 z^{r_1}}{1} + \frac{c_2 z^{r_2}}{1} + \frac{c_3 z^{r_3}}{1} + \ldots + \frac{c_\omega z^{r_\omega}}{1},
\]

where \( c_j \neq 0 \) and \( r_j \in \mathbb{N} \) for \( j = 1, \ldots, \omega, 0 \leq \omega \leq \infty \), can correspond to the Taylor series of \( F(z) \).

**Remark 9.** Corollary 1.5 implies that each pair \((p(z), q(z))\) satisfying Theorem 1.4 determine a unique factorization of the form (1.7).

Let \( p(z) \) and \( q(z) \) be real polynomials. Denote

\[
u(z) := \sum_{k=0}^{n} a_k z^{-k} = z^n p \left( \frac{1}{z} \right) \text{ and } v(z) := \sum_{k=0}^{n} b_k z^{-k} = z^n q \left( \frac{1}{z} \right),
\]

where \( n = \max\{\deg p, \deg q\} \). In this case it is more common to work with the matrix \( \tilde{H}(u,v) := H(p,q) \) instead of \( H(u,v) \).

In fact, Theorem 1.4 extends the following result by Holtz and Tyaglov to meromorphic functions. In [10, Theorems 1.46 and 3.43, Corollaries 3.41–3.42] they established that the matrix \( \tilde{H}(u,v) \) is totally nonnegative if and only if it can be factored as follows

\[
\tilde{H}(u,v) = J(c_0,1) \ldots J(c_j,1) H(1,0) T(g), \quad c_1, \ldots, c_j > 0,
\]

where \( T(g) \) is totally nonnegative and \( g = \gcd(u,v) \). Note that the factorization (1.12) corresponds to (1.10) after the substitutions \( b_0 = c_0, \beta_0 = (c_1)^{-1} \) and \( \beta_{i-1} = (c_{i-1}c_i)^{-1} \) for \( i = 2, \ldots, j \). Moreover, by Theorem 3.44 from [10] the matrix \( \tilde{H}(u,v) \) is totally nonnegative if and only if \( v(z) \) and \( u(z) \) have no positive zeros and \( \frac{v}{u} \in \mathbb{R}^{-1} \). Since

\[
\frac{p(z)}{q(z)} = \frac{v \left( \frac{1}{z} \right)}{u \left( \frac{1}{z} \right)},
\]

we obtain the polynomial analogue of Theorem 1.4.

Earlier, Holtz (see [9]) found that the infinite Hurwitz matrix of a stable polynomial (i.e. a polynomial with no roots with nonnegative real part) has the factorization (1.12) with \( T(g) \) equal to the identity matrix. Additionally, each of the factors \( J(c_j,1) \) corresponds to a step of the Routh scheme. These factorizations coincide because the problems considered in [9] and [10] are closely connected (see, for example, the monographs of Gantmakher [7, Ch. XV] and Wall [21, Chapters IX and X]). In order to deduce Theorem 1.1 from Theorem 1.4, we are using the same underlying connection.

2. Basic facts

Here we consider some facts that are quite significant, although, in fact, they are not new. We put them here to introduce the area and our notation. The
most “non-standard” assertion here is Lemma 2.11, since it reverses the approach of Theorem 2.10.

2.1. S-functions in terms of Hurwitz-type matrices. Consider power series

\[ p(z) = \sum_{k=0}^{\infty} a_k z^k, \quad a_0 = 1, \quad \text{and} \quad q(z) = \sum_{k=0}^{\infty} b_k z^k, \quad b_0 \geq 0. \]  

(2.1)

Let us introduce the following notations

\[ p_0(z) := p(z), \quad p_{-1}(z) := q(z), \quad H := H(p, q), \quad \text{and} \quad H_0 := H(p_0, p_{-1}) = H. \]

Denote the minor of a matrix \( A \) with rows \( i_1, i_2, \ldots, i_k \) and columns \( j_1, j_2, \ldots, j_k \) by

\[ A(i_1 \ i_2 \ \ldots \ i_k)_{j_1 \ j_2 \ \ldots \ j_k}. \]

In addition set

\[ A^{(k)} := A\begin{pmatrix} 2 & 3 & \ldots & k \\ 2 & 3 & \ldots & k \end{pmatrix}. \]  

(2.2)

If the number \( \beta_0 = b_1 - b_0 a_1 = H_0^{(3)} \) is nonzero, we define

\[ p_1(z) := \frac{q(z) - b_0 p(z)}{\beta_0 z} \quad \text{and} \quad H_1 := H(p_1, p_0). \]

Now we can perform the same manipulations with the pair \( p_1(z), p_0(z) \). That is, we can make the next step of the following algorithm.

At the \( j \)th step, \( j = 0, 1, 2, \ldots \), the series \( p_j(z) \) and \( p_{j-1}(z) \) are already defined, as well as the matrix \( H_j = H(p_j, p_{j-1}) \). We set

\[ \beta_j := H_j^{(3)}, \]  

(2.3)

and, if \( \beta_j \) is nonzero, we set

\[ p_{j+1}(z) := \frac{p_{j-1}(z) - p_{j-1}(0)p_j(z)}{\beta_j z} \quad \text{(note that} \ p_{j-1}(0) = 1 \ \text{when} \ j \geq 1) \]  

(2.4)

so that \( H_{j+1} := H(p_{j+1}, p_j) \). These steps can be repeated unless \( \beta_j = 0 \). In Corollary 2.7 we will show that \( \beta_j > 0 \) whenever \( F_j(z) = \frac{p_{j-1}(z)}{p_j(z)} \) represents a non-constant meromorphic S-function. To do this we need some auxiliary facts.

Suppose that \( \beta_i \neq 0, \ i = 0, 1, \ldots, j \) for some nonnegative \( j \), such that the power series \( p_{j-1}(z), p_j(z) \) and \( p_{j+1}(z) \) are defined according to the recurrence formula (2.4).

Lemma 2.1. The identity\(^2\)

\[ H_j \left( \begin{array}{cccc} 2 & 3 & \ldots & k \\ 2 & 3 & \ldots & k \end{array} \right)_{i+1} = \beta_j \left( \begin{array}{c} k+1 \\ i+1 \end{array} \right) H_{j+1} \left( \begin{array}{cccc} 2 & 3 & \ldots & k-1 \\ 2 & 3 & \ldots & k-1 \end{array} \right)_{i}, \]

holds for all \( k = 2, 3, \ldots \) and \( i = k, k+1, \ldots \).

\(^2\)The notation \([a] \) stands for the maximal integer not exceeding \( a \).
Proof. Without loss of generality we consider the case \( j = 0 \), since for higher values of \( j \) the relations (2.3)–(2.4) are analogous. In the case \( k = 2m \)

\[
\beta_m^m H_1 \begin{pmatrix} 2 & 3 & \ldots & 2m-1 & 2m \\ 2 & 3 & \ldots & 2m-1 & i \end{pmatrix} = \beta_m^m H_1 \begin{pmatrix} 1 & 2 & \ldots & 2m-1 & 2m \\ 1 & 2 & \ldots & 2m-1 & i \end{pmatrix} =
\]

\[
\begin{vmatrix}
0 & b_1 - b_0 a_1 & b_2 - b_0 a_2 & \ldots & b_{2m-2} - b_0 a_{2m-2} & b_{1m} - b_0 a_{1m} \\
1 & a_1 & a_2 & \ldots & a_{2m-2} & a_{1m} \\
0 & a_0 & a_1 & \ldots & a_{2m-3} & a_{1m} \\
0 & 0 & a_0 & a_1 & \ldots & a_{2m-3} & a_{1m} \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & \ldots & b_{m-1} - b_0 a_{m-1} & b_{1m} - b_0 a_{1m}
\end{vmatrix} = H_0 \begin{pmatrix} 2 & 3 & \ldots & 2m & 2m+1 \\ 2 & 3 & \ldots & 2m & i+1 \end{pmatrix},
\]

here the equality \( a_0 = 1 \) has been used. For \( k = 2m-1 \) the transformation remains the same. \( \square \)

In particular, if we suppose that \( \beta_0, \beta_1, \ldots, \beta_{k-1} > 0 \) for \( k \geq 3 \), this lemma implies

\[
H_j^{(k)} = \beta_j^k \frac{\beta_j^{k-1}}{\beta_{j+1}^{k-1}} H_j^{(k-1)} = \ldots = \beta_j^{k-3} \frac{\beta_j^{k-2}}{\beta_{j+1}^{k-2}} H_j^{(3)} = \left( \prod_{i=1}^{k-3} \beta_i^{\frac{k-1-i}{i+1}} \right) \frac{\beta_j}{\beta_{j+1}} = \frac{\beta_j}{\beta_{j+1}} \left( \prod_{i=1}^{k-2} \beta_j^{\frac{k-1-i}{i+1}} \right). \tag{2.5}
\]

The next theorem was established by Chebotarev, see [4] and [5, Ch.V §1]; see also the proof of M. Schiffer and V. Bargmann in [22, II.8]. At the same time, it can be derived as a particular case from Nevanlinna’s theory, see [14, Theorem 8].

**Theorem 2.2** ([4, 5, 22, 14]). A real meromorphic function \( F(z) \) regular at the origin is an R-function if and only if it has the form

\[
F(z) = B_0 + B_1 z + \sum_{1 \leq \nu \leq \omega} \left( \frac{A_\nu}{z + \sigma_\nu} - \frac{A_\nu}{\sigma_\nu} \right),
\]

where \( \sum_{1 \leq \nu \leq \omega} \frac{|A_\nu|}{\sigma_\nu^2} < \infty, B_1 > 0 \) and \( A_\nu < 0, \sigma_\nu \in \mathbb{R} \) for \( \nu = 1, 2, \ldots, \omega \). \tag{2.6}

The proof of this theorem relies on the following fact which we will use later.

**Lemma 2.3** (see e.g. [5, Ch.VI §8]). Let entire functions \( q(z) \) and \( p(z) \) have no common zeros and such that \( F = \frac{q}{p} \in \mathcal{R} \) is not a constant. Then \( F'(z) > 0 \) on the real line, the zeros of \( p(z) \) and \( q(z) \) are real, simple and interlacing.

The interlacing property means that between each two consequent zeros of \( p(z) \) there exists a unique root of \( q(z) \) and vice versa. The proof from [5] is based on the behaviour of meromorphic R-functions in neighbourhoods of its zeros and poles. For completeness, we deduce this lemma here from the partial fraction expansion (2.6).
Proof. Let $F(z) = \frac{q(z)}{p(z)}$ have the form (2.6). If $z$ is not real, then
\[
\frac{\text{Im} F(z)}{\text{Im} z} = B_1 + \sum_{1 \leq \nu < \omega} \frac{-A_\nu}{|z + \sigma_\nu|^2} > 0.
\]
Therefore, $F(z)$ (as well as $q(z)$) has no zeros outside the real axis.

Now from (2.6) it follows that $F(z)$ is real and can only have simple poles. Since
\[
F'(z) = B_1 + \sum_{1 \leq \nu < \omega} \frac{-A_\nu}{(z + \sigma_\nu)^2} > 0, \quad z \in \mathbb{R},
\]
the function $F(z)$ grows between any of its two subsequent poles $z_1$ and $z_2$ from $-\infty$ to $+\infty$. So there is one and only one $z_*$ such that $F(z_*) = q(z_*) = 0$. For the same reason, there exists a unique zero of $p(z)$ between any two subsequent zeros of $q(z)$. \qed

The next theorem is a consequence of Grommer's theorem (see [8, §14, Satz III]) and Theorem 2.2. It can be proved by applying the Hurwitz transformation [11] (see also [14, §6.1], [5, Ch.I §7], [10, Theorem 1.5], [7]) to the matrices of the Hankel forms corresponding to $F(z)$.

**Theorem 2.4** (e.g. [5, Ch.V §3]). A meromorphic function $F(z) = \frac{q(z)}{p(z)}$, where $p(z)$ and $q(z)$ are of the form (2.1), is an $R$-function if and only if there exists $l$, $0 \leq l \leq \infty$, such that
\[
H^{2m+1}, m = 1, 2, \ldots, l, \quad H^{2l+3} = H^{2l+5} = \cdots = 0.
\]
Moreover, $l$ is finite if and only if $F(z)$ is a rational function with exactly $l$ poles, counting a pole at infinity (if exists).

Let $\beta_i \neq 0$, $i = 0, 1, \ldots, j$ for some nonnegative $j$, and the power series $p_{j-1}(z)$, $p_j(z)$ and $p_{j+1}(z)$ be defined by the recurrence formula (2.4). Suppose that the ratio $F_j(z) = \frac{p_{j-1}(z)}{p_j(z)}$ of formal power series converges to a meromorphic function. Then there exist entire functions $\tilde{p}_{j-1}(z)$ and $\tilde{p}_j(z)$ with no common zeros such that
\[
F_j(z) = \frac{\tilde{p}_{j-1}(z)}{\tilde{p}_j(z)}, \quad \tilde{p}_{j-1}(0) = p_{j-1}(0) \quad \text{and} \quad \tilde{p}_j(0) = p_j(0) = 1.
\]
Define the power series $g(z) := \frac{p_j(z)}{\tilde{p}_j(z)}$ satisfying $g(0) = 1$. Then
\[
\tilde{p}_j(z) = \frac{p_j(z)}{g(z)} \quad \text{and} \quad \tilde{p}_{j-1}(z) = \frac{p_{j-1}(z)}{g(z)}.
\]
If $F_j \in R$ is a non-constant function, then by Theorem 2.4 the inequality $\beta_j = H_j^{(3)} > 0$ is satisfied. So from the formula (2.4) we find
\[
F_{j+1}(z) := \frac{p_j(z)}{p_{j+1}(z)} = \frac{\beta_j z}{F_j(z) - \tilde{p}_{j-1}(0)}.
\]

**Lemma 2.5.** The ratio $\frac{\tilde{p}_{j+1}(z)}{g(z)}$ converges to the entire function
\[
\tilde{p}_{j+1}(z) := \frac{\tilde{p}_{j-1}(z) - \tilde{p}_{j-1}(0) \tilde{p}_j(z)}{\beta_j z}.
\]
The pairs $(\tilde{p}_j(z), \tilde{p}_{j+1}(z))$ and $(\tilde{p}_{j-1}(z), \tilde{p}_{j+1}(z))$ have no common zeros.
Proof. Dividing (2.4) by \(g(z)\) gives \(\frac{p_{j+1}(z)}{g(z)} = \tilde{p}_{j+1}(z)\), that means the relation (2.7) holds. Consequently,

\[
\hat{p}_{j-1}(z) = \beta_j z \tilde{p}_{j+1}(z) + \hat{p}_{j-1}(0) \tilde{p}_j(z).
\]

Each common zero of any two summands in this equation must be a zero of the third summand. Since the functions \(\tilde{p}_{j-1}(z)\) and \(\tilde{p}_j(z)\) have no common zeros, the pairs \((\tilde{p}_{j-1}(z), \tilde{p}_{j+1}(z))\) and \((\tilde{p}_j(z), \tilde{p}_{j+1}(z))\) also have no common zeros. \(\square\)

This lemma implies that \(F_{j+1}(z)\) represents the meromorphic function

\[
F_{j+1}(z) = \frac{\tilde{p}_j(z)}{\tilde{p}_{j+1}(z)}.
\]

**Lemma 2.6.** If the meromorphic function \(F_j(z)\) is not a constant, then \(F_j \in \mathcal{S}\) if and only if \(F_j, F_{j+1} \in \mathcal{R}\) and \(F_j(0) \geq 0\).

**Proof.** Let \(F_j \in \mathcal{S}\), then Theorem 2.2 gives that it has the form

\[
F_j(z) = B_0 + B_1 z + \sum_{1 \leq \nu \leq \omega} \left( \frac{A_\nu}{z + \sigma_\nu} - \frac{A_\nu}{z - \sigma_\nu} \right) = B_0 + B_1 z + \sum_{1 \leq \nu \leq \omega} \frac{(-A_\nu/\sigma_\nu)}{z + \sigma_\nu}.
\]

where \(\sum_{1 \leq \nu \leq \omega} \frac{|A_\nu|}{\sigma_\nu^2} < \infty\), \(B_0 \geq 0\), \(B_1 > 0\) and \(A_\nu < 0\), \(\sigma_\nu > 0\) for \(\nu = 1, 2, \ldots, \omega\).

It is enough to show that \(F_{j+1}(z)\) is a well-defined \(\mathcal{R}\)-function. Consider the function

\[
G_j(z) := \frac{F_j(-z) - F_j(0)}{-z} = B_1 + \sum_{1 \leq \nu \leq \omega} \frac{(-A_\nu/\sigma_\nu)}{-z - \sigma_\nu} = B_1 + \sum_{1 \leq \nu \leq \omega} \frac{(A_\nu/\sigma_\nu)}{z - \sigma_\nu}.
\]

It has the form (2.6) and, hence, is a meromorphic \(\mathcal{R}\)-function by Theorem 2.2.

The mappings \(z \mapsto \frac{1}{z}\) and \(z \mapsto -z\) are in the class \(\mathcal{R}^{-1}\) (i.e. they map the upper half of the complex plane into the lower half of the complex plane). Since \(\beta_j = H_j^{(3)} > 0\) and \(G_j \in \mathcal{R}\), the function composition

\[
\left( \frac{1}{z} \circ G_j \circ (- \cdot ) \right)(z) = \frac{\beta_j}{G_j(-z)} = \frac{\beta_j z}{F_j(z) - p_{j-1}(0)} = F_{j+1}(z)
\]

is an \(\mathcal{R}\)-function as well.

Conversely, let \(F_j, F_{j+1} \in \mathcal{R}\) and \(F_j(0) \geq 0\). The inequality \(\beta_j > 0\) holds, therefore \(F_{j+1}(z) \not\equiv 0\) and the meromorphic function

\[
G_j(z) := \frac{\beta_j}{F_j(z) - F_j(0)} = \frac{\beta_j z}{F_j(-z)} = \frac{\beta_j}{F_{j+1}(-z)} = \frac{\beta_j z}{F_{j+1}(z)}
\]

is an \(\mathcal{R}\)-function. On one hand, Theorem 2.2 gives

\[
F_j(z) = B_0 + B_1 z + \sum_{1 \leq \nu \leq \omega} \frac{(-A_\nu/\sigma_\nu)}{z + \sigma_\nu}, \text{ where }
\]

\[
\sum_{1 \leq \nu \leq \omega} \frac{|A_\nu|}{\sigma_\nu^2} < \infty, \ B_1 > 0 \text{ and } A_\nu < 0, \ \sigma_\nu \in \mathbb{R} \text{ for } \nu = 1, 2, \ldots, \omega.
\]
such that
\[ G_j(z) = \frac{F_j(-z) - F_j(0)}{-z} = B_1 + \sum_{1 \leq \nu \leq \omega} \frac{(A_{\nu}/\sigma_{\nu})}{z - \sigma_{\nu}}. \]

On the other hand, Theorem 2.2 states that each \( R \)-function has negative residues at its poles. That is, \( \frac{A_{\nu}}{\sigma_{\nu}} < 0 \) for all \( \nu \) since \( G_j \in R \). Therefore, the poles \(-\sigma_{\nu}, \nu = 1, 2, \ldots, \omega\), of the function \( F_j \) are negative. Consequently, \( F_j \in S \). \( \square \)

**Corollary 2.7.** Suppose that for some \( j \geq 0 \) the function \( F_j(z) \) is in the class \( S \). Then \( \beta_j \geq 0 \). The inequality \( \beta_j \geq 0 \) implies \( F_{j+1} \in S \), while the equality \( \beta_j = 0 \) implies that \( F_j(z) \) is constant.

**Proof.** If \( F_j(z) \) is a constant then \( \beta_j = H_j^{(3)} = 0 \) by Theorem 2.4. Let \( F_j(z) \) be a non-constant \( S \)-function. Applying Lemma 2.6 to it gives \( F_{j+1} \in R \). Consequently, \( \beta_{j+1} = 0 \) if \( F_{j+1}(z) \) is a constant and \( \beta_{j+1} > 0 \) if it is not. Moreover, we have \( F_{j+1}(0) = 1 > 0 \). Thus, the corollary holds in the cases of constant \( F_j(z) \) or \( F_{j+1}(z) \).

Suppose that \( \beta_j, \beta_{j+1} > 0 \). Then Lemma 2.1 implies that
\[
H_j^{(2m+3)} = \beta_j \left| \frac{2m+2}{2m+1} \right| \beta_{j+1}^{2m+1} H_j^{(2m+1)} = \beta_j \beta_{j+1}^{m} \beta_{j+2}^{m} H_j^{(2m+1)}, \quad m = 1, 2, \ldots, \ (2.8)
\]
That is, for each natural \( m \) the sign of \( H_j^{(2m+1)} \) coincides with the sign of \( H_j^{(2m+3)} \).

Since \( F_j \in R \), Theorem 2.4 yields \( F_{j+2} \in R \). That is, \( F_{j+1} \in S \) by Lemma 2.6. \( \square \)

**Theorem 2.8.** A meromorphic function \( F(z) = \frac{q(z)}{p(z)} \), where \( p(z) \) and \( q(z) \) are series of the form (2.1), is an \( S \)-function if and only if there exists \( \omega, 2 \leq \omega \leq \infty \), such that
\[
H^{(k)} > 0, \ k = 2, 3, \ldots, \omega \quad \text{and} \quad H^{(\omega+1)} = H^{(\omega+2)} = \cdots = 0. \ (2.9)
\]
Moreover, \( \omega \) is finite if and only if \( F(z) \) is a rational function with exactly \( \left\lfloor \frac{\omega-1}{2} \right\rfloor \) poles, counting a pole at infinity (if exists).

**Proof.** By definition, \( H_j^{(2)} = H^{(2)} = 1 > 0 \). Denote \( p_0(z) := p(z) \) and \( p_{-j}(z) := q(z) \) such that \( F(z) = F_0(z) \). From the recurrence formulas (2.3)–(2.4) we obtain the sequences \( (p_j)_{j=1}^{\omega-2} \) and \( (\beta_j)_{j=0}^{\omega-2} \), where \( \beta_j \neq 0 \) for all \( j = 0, 1, \ldots, \omega-3 \) and \( 2 \leq \omega \leq \infty \). Whenever \( \omega < \infty \) we also have \( \beta_{\omega-2} = 0 \).

Suppose that \( F_0 \in S \). Then \( \beta_j > 0 \) for all \( j = 0, 1, \ldots, \omega-3 \) by Corollary 2.7. Furthermore, the identity (2.5) gives
\[
H^{(j)} = H_0^{(j)} = \prod_{i=1}^{j-2} \beta_i^{\left\lfloor \frac{j-i}{2} \right\rfloor} > 0, \quad j = 3, 4, \ldots, \omega. \ (2.10)
\]

Let \( \omega < \infty \), then \( F_{\omega-2}(z) \) is a constant by Corollary 2.7. Therefore, we have \( H_{\omega-2}^{(3)} = H_{\omega-2}^{(4)} = \cdots = 0 \) since all these minors contain proportional rows. By the identity (2.5), this is equivalent to \( H^{(\omega+1)} = H^{(\omega+2)} = \cdots = 0 \).

So we obtained that \( F \in S \) implies (2.9). The number of poles the function \( F(z) \) has can be determined from Theorem 2.4.

Now suppose that the conditions (2.9) hold. If \( \omega = 2 \) then \( H^{(3)} = H^{(4)} = \cdots = 0 \) and the assertion of this theorem is equivalent to Theorem 2.4. In the case of
3 \leq \omega \leq \infty we have \( \beta_0 > 0 \), so by Lemma 2.1,

\[
H_1^{(2m+1)} = \beta_0^{- \left\lfloor \frac{2m+1}{2} \right\rfloor} H^{(2m+2)} > 0, m = 1, 2, \ldots, \left\lfloor \frac{\omega}{2} \right\rfloor - 1, \quad \text{and}
\]

\[
H_1^{(2m+1)} = \beta_0^{- \left\lfloor \frac{2m+1}{2} \right\rfloor} H^{(2m+2)} = 0, m \geq \left\lfloor \frac{\omega}{2} \right\rfloor.
\]

Hence, the functions \( F(z) \) and \( F_1(z) \) are \( R \)-functions by Theorem 2.4, and Lemma 2.6 yields \( F \in \mathcal{S} \).

\[\square\]

2.2. \( \mathcal{S} \)-functions as continued fractions. A continued fraction of the form

\[
F(z) = c_0 + \frac{c_1 z^{r_1}}{1} + \frac{c_2 z^{r_2}}{1} + \frac{c_3 z^{r_3}}{1} + \cdots + \frac{c_\omega z^{r_\omega}}{1}, \quad (2.11)
\]

where \( c_j \neq 0 \) and \( r_j \in \mathbb{N} \) for \( j = 1, \ldots, \omega, \quad 0 \leq \omega \leq \infty \), is called a (general) \( C \)-fraction. The special case of (2.11) that corresponds to \( r_j = 1 \) for all \( j = 1, \ldots, \omega \) is called a regular \( C \)-fraction. Continued fractions of the form (2.11) are able to represent power series uniquely, that is the following fact is true.

\textbf{Theorem 2.9 (\cite{16}, see also \cite{18, Sätze 3.2–3.5, 3.24]). Each (formal) power series \( F(z) = \sum_{k=0}^{\infty} s_k z^k \) corresponds to a fraction of the form (2.11). This correspondence is set by the following sequence of relations}

\[
F_0(z) = F(z), \quad c_0 = F(0), \quad F_i(z) = \frac{c_i z^{r_i}}{F_{i-1}(z) - F_{i-1}(0)}, \quad i = 1, 2, \ldots, \omega, \quad (2.12)
\]

where \( \omega \leq \infty \) is such that \( F_{i-1}(z) \neq F_{i-1}(0) \) for \( i - 1 < \omega \) and \( F_\omega(z) \equiv F_\omega(0) \). The exponents \( r_i \) are positive integers chosen together with the complex constants \( c_i \) in such a way that \( F_1(0) = 1 \).

If two \( C \)-fractions (finite or infinite) of the form (2.11) correspond to the same power series, then they coincide. A \( C \)-fraction is finite if and only if it corresponds to a rational function (and, hence, represents that function).

Moreover, if an infinite continued fraction of the form (2.11) converges uniformly in a closed region \( T \) containing the origin in its interior, it represents a regular analytic non-rational function of \( z \) throughout the interior of \( T \). Further, the corresponding power series converges to the same function in and on the boundary of the largest circle which can be drawn with its center at the origin, lying wholly within \( T \).

Suppose that \( F(z) = \frac{p(z)}{p(z)} \) is an \( \mathcal{S} \)-function. We again denote \( F_0(z) := F(z), \quad p_0(z) := p(z) \) and \( p_{-1}(z) := q(z) \) and use the recurrence formulæ (2.3)-(2.4) to obtain the sequences \( (p_j)_{j=-1}^{\omega}, (\beta_j)_{j=0}^{\omega} \), where \( \beta_j \neq 0 \) for all \( j = 0, 1, \ldots, \omega - 1 \) and \( -1 \leq \omega \leq \infty \). In the case \( \omega < \infty \) we also have \( \beta_\omega = 0 \).

For each \( j = 0, 1, \ldots, \omega - 1 \), we apply Corollary 2.7, obtaining \( \beta_j > 0 \) and \( F_j \in \mathcal{S} \). If \( \omega \) is a finite number, then \( F_\omega \) is a constant. From the relation (2.4) we have

\[
F_j(z) = \frac{p_j(z)}{p_j(z)} = p_{j-1}(0) + \frac{\beta_j z}{F_{j+1}(z)}, \quad j = 0, 1, \ldots, \omega - 1. \quad (2.13)
\]

These formulæ can be combined into the continued fractions

\[
F(z) = F_0(z) = b_0 + \frac{\beta_0 z}{1 + \frac{\beta_1 z}{1 + \frac{\beta_2 z}{1 + \cdots + \frac{\beta_{\omega-1} z}{1}}}}, \quad (2.14)
\]

\[
F_j(z) = 1 + \frac{\beta_j z}{1 + \frac{\beta_{j+1} z}{1 + \frac{\beta_{j+2} z}{1 + \cdots + \frac{\beta_{\omega-1} z}{1}}}}, \quad (2.15)
\]
Lemma 2.11. Let the functions \( \hat{\beta}_j(z) := \frac{\beta_j(z)}{g(z)} \) be entire functions for \( j = -1, 0, \ldots, \omega \), and for \( j \geq 0 \) \( \hat{\beta}_{j-1}(z) \) and \( \hat{\beta}_j(z) \) have no common zeros (see Lemma 2.5). Observe that the relations (2.13), (2.14) and (2.15) remain the same, if we replace all the series \( p_j(z) \) by the functions \( \hat{\beta}_j(z) \).

It is convenient to study the continued fractions (2.14) and (2.15), using the following.

Theorem 2.10 (Stieltjes, [20, n° 68–69]; see also [18, 21]). Let \( b_0 \geq 0 \). A sequence of positive numbers \( \beta_0, \beta_1, \ldots, \beta_\omega \), \( -1 \leq \omega \leq \infty \), has a finite sum if and only if the continued fraction (2.14) converges to a meromorphic \( S \)-function and its partial numerators and denominators converge to coprime\(^4\) entire functions of genus 0. That is, if the \( j \)th convergent (approximant) to \( F(z) \) is denoted by \( \frac{Q_j(z)}{P_j(z)} \), then for \( j \to \infty \) we have

\[
P_j(z) \to p(z), \quad Q_j(z) \to q(z)
\]

and

\[
b_0 + \frac{\beta_0 z}{1 + \frac{\beta_1 z}{1 + \frac{\beta_2 z}{1 + \cdots}}} = \frac{Q_j(z)}{P_j(z)} \to q(z) = p(z) = F(z),
\]

where \( p(z) \) and \( q(z) \) are coprime entire functions of genus 0. The convergence is uniform on compact subsets of \( \mathbb{C} \) containing no poles of the function \( F(z) \).

To apply this theorem we need to distinguish the case of \( \sum_{j=0}^\infty \beta_j < \infty \) for \( \omega = \infty \).

Lemma 2.11. Let the functions \( \hat{\beta}_1(z) \) and \( \hat{\beta}_0(z) \) be entire of genus 0, coprime and such that their ratio \( S \ni F_1 := \frac{\hat{\beta}_0}{\hat{\beta}_1} \) is not rational. Then \( \hat{\beta}_j(z) \to 1 \) as \( j \to \infty \) uniformly on compact subsets of \( \mathbb{C} \), and \( \sum_{j=1}^\infty \beta_j < \infty \).

Proof. According to (2.7), all the functions \( \hat{\beta}_j(z), \ j = 0, 1, \ldots, \) are entire of genus 0 and \( \hat{\beta}_j(0) = 1 \). Moreover, \( \hat{\beta}_0, \ldots, \hat{\beta}_j \) are coprime (by Lemma 2.5) and hence have only negative zeros (since by Corollary 2.7 \( F_j = \frac{\hat{\beta}_{j-1}}{\hat{\beta}_j} \in S \)). Therefore, the following representation is valid for \( j = 0, 1, \ldots, \)

\[
\hat{\beta}_j(z) = \sum_{k=0}^\infty a_k^{(j)} z^k = \prod_{\nu=1}^\infty \left( 1 + \frac{1}{\sigma_{\nu}^{(j)}} \right),
\]

where \( 0 < \sigma_1^{(j)} \leq \sigma_2^{(j)} \leq \ldots \) and \( \sum_{\nu=1}^\infty \frac{1}{\sigma_{\nu}^{(j)}} < \infty \). The coefficients \( a_k^{(j)}, \ k = 1, 2, \ldots, \) are equal to

\[
a_k^{(j)} = \sum_{\nu=1}^\infty \sum_{i_1=1}^\infty \cdots \sum_{i_{k-1}=1}^\infty \frac{1}{\sigma_1^{(j)} \sigma_2^{(j)} \cdots \sigma_{k-1}^{(j)}}.
\]

\(^3\)The separate convergence of numerators and denominators was shown by Śleszyński in [19].

\(^4\)This fact was obtained by Maillet in [17]; see also [18, p.150].
Note that these sums are convergent, since \(^5\)
\[
a_k^{(j)} < \sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{1}{a_{i_1}^{(j)}} \frac{1}{a_{i_2}^{(j)}} \cdots \frac{1}{a_{i_k}^{(j)}} = \left( a_1^{(j)} \right)^k \quad \text{when } k = 2, 3, 4, \ldots \quad (2.17)
\]

By Lemma 2.3 the zeros of \(\tilde{p}_j(z)\) and \(\tilde{p}_{j-1}(z)\) (which are negative) must be simple and interlacing. In addition, \(F_j'(z) > 0\) for real \(z\) implying that \(0 < \sigma_1^{(j-1)} < \sigma_1^{(j)}\). Hence
\[
0 < \sigma_1^{(j-1)} < \sigma_1^{(j)} < \sigma_2^{(j-1)} < \sigma_2^{(j)} < \sigma_3^{(j-1)} < \sigma_3^{(j)} < \ldots \quad (2.18)
\]
Now we estimate the expression (2.16) using the inequalities (2.18) and obtain that \(0 < a_k^{(j)} < a_k^{(j-1)}\) for \(j = 1, 2, \ldots\), i.e. the sequence of positive numbers \(a_k^{(j)}\) decreases in \(j\) for fixed \(k\). Therefore, there exists a finite \(\lim_{j \to \infty} a_k^{(j)} \neq 0\) dependent on \(k\).

At the same time, the equality (2.7) implies that the first Taylor coefficient \(a_1^{(j)}\) for any \(j\) has the form
\[
a_1^{(j-1)} = \beta_j + a_1^{(j)}.
\]
Consequently,
\[
a_1^{(0)} = \sum_{i=1}^{j} \beta_i + a_1^{(j)} \quad \text{and} \quad \sum_{j=1}^{\infty} \beta_j = a_1^{(0)} - \lim_{j \to \infty} a_1^{(j)}. \quad (2.19)
\]
Therefore, the series \(\sum_{j=0}^{\infty} \beta_j\) converges.

As a consequence, for an arbitrary positive number \(R\) there exists an integer \(j_0(R)\) such that \(\beta_j R < \frac{1}{4}\) for all \(j \geq j_0\). By virtue of Worpitzky’s test (as it stated in [21, p. 45], see also [23]) the continued fraction
\[
F_{j_0}(z) = 1 + \frac{\beta_{j_0} z}{1 + \frac{\beta_{j_0+1} z}{1 + \frac{\beta_{j_0+2} z}{1 + \cdots}}}
\]
converges to an analytic function uniformly in the disk \(|z| < R\). This analytic function coincides with \(F_j(z)\) (since \(F_j(z)\) corresponds to the continued fraction, see (2.15)). Therefore, \(\tilde{p}_{j_0}\) has no zeros in this disk, that is \(R < \sigma_1^{(j_0)} < \sigma_1^{(j)}\), \(j \geq j_0\).

Letting \(R\) tend to infinity, we obtain \(\lim_{j \to \infty} \sigma_1^{(j)} = \infty\). According to (2.16) we have
\[
a_1^{(j)} = \sum_{i=1}^{\infty} \frac{1}{a_i^{(j)}} \quad \text{for} \quad k = 2, 3, 4, \ldots
\]

\(^5\)In fact, even an estimate stronger than (2.17) is valid (cf. [19, p. 105]). For each tuple of distinct numbers \((i_1, i_2, \ldots, i_k)\) there is only one summand \(\left( a_{i_1}^{(j_1)} a_{i_2}^{(j_2)} \cdots a_{i_k}^{(j_k)} \right)^{-1}\) in the right-hand side of (2.16). At the same time, the sum
\[
\sum_{i_1=1}^{\infty} \sum_{i_2=1}^{\infty} \cdots \sum_{i_k=1}^{\infty} \frac{1}{a_{i_1}^{(j_1)}} \frac{1}{a_{i_2}^{(j_2)}} \cdots \frac{1}{a_{i_k}^{(j_k)}}
\]
contains exactly \(k!\) summands of this form. Therefore,
\[
a_k^{(j)} < \frac{1}{k!} \left( a_1^{(j)} \right)^k \quad \text{for} \quad k = 2, 3, 4, \ldots
\]
and each term in this series monotonically tends to zero as \( j \to \infty \). For any \( \varepsilon > 0 \) there exists \( N \) such that

\[
\sum_{i=N+1}^{\infty} \frac{1}{\sigma_i^{(j)}} < \varepsilon,
\]

which for \( j \geq j_0 \) gives

\[
\sum_{i=N+1}^{\infty} \frac{1}{\sigma_i^{(j)}} \leq \sum_{i=N+1}^{\infty} \frac{1}{\sigma_i^{(j_0)}} < \varepsilon.
\]

On the other hand, \( \lim_{j \to \infty} \sum_{i=1}^{N} \frac{1}{\sigma_i^{(j)}} = 0 \), so the coefficient \( a_i^{(j)} \) also vanishes. Now from (2.17) we obtain that \( \tilde{p}_j(z) \to 1 \) as \( j \to \infty \) uniformly on compact subsets of \( \mathbb{C} \).

**Corollary 2.12.** Under the assumptions of Lemma 2.11 there exists a positive number \( M \) independent of \( j \) such that

\[
\| H(\tilde{p}_{j+1}, \tilde{p}_j) \|_1 < M \quad \text{for} \quad j = 0, 1, \ldots,
\]

where the matrix \( H(\tilde{p}_{j+1}, \tilde{p}_j) \) is defined by (1.5), and

\[
\| H(\tilde{p}_{j+1}, \tilde{p}_j) - H(1, 1) \|_1 \overset{j \to \infty}{\longrightarrow} 0.
\]

**Proof.** Since \( a_k^{(j)} \geq a_k^{(j+1)} \geq 0 \) for all \( j, k = 0, 1, \ldots \), we have

\[
\| H(\tilde{p}_{j+1}, \tilde{p}_j) \|_1 = \max \left\{ \sum_{k=0}^{\infty} a_k^{(j+1)}, \sum_{k=0}^{\infty} a_k^{(j)} \right\} = \sum_{k=0}^{\infty} a_k^{(j)} \leq \sum_{k=0}^{\infty} a_k^{(0)} = \tilde{p}_0(1) < \infty.
\]

Observe that \( a_0^{(j)} = 1 \) whenever \( j \geq 0 \). Therefore, by Lemma 2.11 we obtain the required

\[
\| H(\tilde{p}_{j+1}, \tilde{p}_j) - H(1, 1) \|_1 = \max \left\{ \sum_{k=1}^{\infty} a_k^{(j+1)}, \sum_{k=1}^{\infty} a_k^{(j)} \right\} = \sum_{k=1}^{\infty} a_k^{(j)} = \tilde{p}_j(1) - 1 \overset{j \to \infty}{\longrightarrow} 0.
\]

**Corollary 1.5.** Let \( F(z) = \frac{p(z)}{q(z)} \) be a meromorphic \( S \)-function, where the entire functions \( p(z) \) and \( q(z) \) are of genus 0. Then it can be expanded into a uniformly convergent continued fraction of the form (2.14) with the coefficients \( b_0 \) and \( (\beta_j)_{j=0}^{\infty} \),

\[
\sum_{j=0}^{\infty} \beta_j < \infty,
\]

given by (2.3). No other continued fractions of the form (2.11) can correspond to the Taylor series of \( F(z) \).

**Proof.** If \( F = \frac{p}{q} \in S \) then \( F(z) \) can be formally developed into the continued fraction (2.14) with the coefficients \( b_0 = F(0) \) and \( (\beta_j)_{j=0}^{\infty} \) given by (2.3). By Lemma 2.11, the coefficients of this continued fraction satisfy the condition \( \sum_{j=0}^{\infty} \beta_j < \infty \). Consequently, Theorem 2.10 implies that (2.14) converges uniformly on compact sets containing no poles of its limiting function. So the rest of the proof comes from Theorem 2.9: the continued fraction (2.14) corresponds to and converges to \( F(z) \), and there is no other continued fraction of the form (2.11) corresponding to \( F(z) \). 

\[ \square \]
3. Total nonnegativity of Hurwitz-type matrices

This section contains the proof of Theorem 1.4 and Corollary 1.5, preceded by several auxiliary facts.

Suppose that meromorphic functions \( p \) and \( q \) are regular at the origin and have the Taylor expansion (2.1). Consider the Hurwitz-type matrix \( H(p, q) \), defined by (1.5).

Lemma 3.1. If \( \frac{p}{\rho} \in S \) and there exists a meromorphic function \( g(z) \), such that the ratios \( \tilde{p}(z) := \frac{p(z)}{g(z)} \) and \( \tilde{q}(z) := \frac{q(z)}{g(z)} \) are entire of genus 0 and coprime, then the matrix \( H(p, q) \) can be factored as in (1.7), where the numbers \( \beta_j, j = 0, 1, \ldots \), are given by (2.3) (possibly followed by zeros). Moreover,

\[
\|H(p, q) - J(b_0, \beta_0)J(1, \beta_1) \cdots J(1, \beta_j)H(1, 1)T(g)\|_{\rho} \xrightarrow{J \to \infty} 0,
\]

where \( \rho, 0 < \rho \leq 1 \), is such that \( g(z) \) has no poles in the disk \( |z| \leq \rho \).

Proof. For any two matrices \( A = (a_{kl})_{i,j} \) and \( B = (b_{kl})_{i,j} \) such that \( \|A\|_1 < \infty \) and \( \|B\|_\rho < \infty \) the following estimate (implying the existence of the product \( AB \)) is true

\[
\infty > \|A\|_1\|B\|_\rho = \sup_{1 \leq k < \infty} \sum_{i=1}^\infty |a_{kl}| \sup_{1 \leq m < \infty} \sum_{j=1}^\infty |b_{mj}| \rho^{j-1} \geq \sup_{1 \leq k < \infty} \sum_{i=1}^\infty |a_{kl}| \sum_{j=1}^\infty \left| \sum_{l=1}^\infty a_{kl}b_{lj} \right| \rho^{j-1} = \|AB\|_\rho. \tag{3.1}
\]

Now we note that the decomposition

\[
H(p, q) = H(\tilde{p}, \tilde{q})T(g)
\]

is valid. It can be checked by the straightforward multiplication.

Denote \( p_0(z) := \tilde{p}(z) \) and \( p_{-1}(z) := \tilde{q}(z) \). We are now using the algorithm (2.3)–(2.4) to construct the (longest possible) sequence \((p_j, J)_{j=1}^\omega \) of entire functions, \( 0 \leq \omega \leq \infty \). By Corollary 2.7, the corresponding numbers \((\beta_j)_{j=0}^\omega \) satisfy \( \beta_j > 0 \) for all \( j = 0, 1, \ldots \omega - 1 \). In the case of finite \( \omega \) we have \( p_{\omega-1}(z) \equiv p_{\omega-1}(0) = 1 \) and \( \beta_\omega = 0 \); we extend the latter equality by \( \beta_{\omega+1} = \beta_{\omega+2} = \cdots = 0 \).

The identity (3.2) implies the factorization (1.10) in the case of \( q(z) \equiv 0 \) (corresponding to \( \omega = 0 \)). Suppose that \( \omega > 0 \). If we expand \( p_i(z) \) and \( p_{i-1}(z) \), \( i = 0, 1, \ldots \omega \), as follows

\[
p_i(z) = \sum_{k=0}^\infty c_kz^k \quad \text{and} \quad p_{i-1}(z) = \sum_{k=0}^\infty d_kz^k,
\]

then the matrix \( H_{i+1} := H(p_{i+1}, p_i) \) takes the form

\[
H_{i+1} = \begin{pmatrix}
c_0 & c_1 & \frac{1}{\beta_i}(d_1 - d_0c_1) & \frac{1}{\beta_i}(d_2 - d_0c_2) & \frac{1}{\beta_i}(c_0d_3 - d_0c_3) & \cdots \\
0 & c_0 & \frac{1}{\beta_i}(d_1 - d_0c_1) & \frac{1}{\beta_i}(d_2 - d_0c_2) & \frac{1}{\beta_i}(c_0d_3 - d_0c_3) & \cdots \\
0 & 0 & c_1 & \frac{1}{\beta_i}(d_1 - d_0c_1) & \frac{1}{\beta_i}(d_2 - d_0c_2) & \cdots \\
0 & 0 & 0 & c_0 & \frac{1}{\beta_i}(d_1 - d_0c_1) & \frac{1}{\beta_i}(d_2 - d_0c_2) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]
Left-multiplying this matrix by

\[
J(d_0, \beta_i) = \begin{pmatrix}
  d_0 & \beta_i & 0 & 0 & 0 & 0 & \ldots \\
  0 & 0 & 1 & 0 & 0 & 0 & \ldots \\
  0 & 0 & d_0 & \beta_i & 0 & 0 & \ldots \\
  0 & 0 & 0 & 0 & 1 & 0 & \ldots \\
  0 & 0 & 0 & 0 & d_0 & \beta_i & \ldots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix},
\]

we obtain \(H_i\). On putting \(i\) successively equal to 0, 1, \ldots, \(j\) and applying (3.2) we find that

\[
H(p, q) = J(b_0, \beta_0) J(1, \beta_1) \cdots J(1, \beta_j) H(p_{j+1}, p_j) T(g).
\] (3.3)

The finite product of the matrices is well defined and associative, since the matrices \(J(1, \cdot)\) and \(H(1, 1)\) have at most two nonzero entries in each row and column. So if \(\omega\) is a finite number, then for \(j = \omega - 1\) the equality (3.3) coincides with (1.10). Since \(\|T(g)\|_p = g(\rho) < \infty\), from (1.9), (3.1) and Corollary 2.12 we obtain the assertion of the theorem for \(\omega < \infty\).

Suppose that \(\omega = \infty\) and let us prove that the difference between the product in (3.3) and the right-hand side of (1.7) converges to zero as \(j \to \infty\). There exists an index \(j_0 \geq 1\) such that

\[
\sum_{j=j_0}^{\infty} \beta_j < \frac{1}{2}. \quad (3.4)
\]

Let

\[
V := J(b_0, \beta_0) J(1, \beta_1) \cdots J(1, \beta_{j_0-1}) \quad \text{and} \quad U_j := J(1, \beta_{j_0}) \cdots J(1, \beta_j)
\]

for \(j = j_0, j_0 + 1, \ldots\). Then we can express the equality (3.3) as follows

\[
H(p, q) = V U_j H(p_{j+1}, p_j) T(g).
\]

The matrix \(U_j = J(1, \beta_{j_0}) \cdots J(1, \beta_j)\) is upper triangular and has no negative entries since it is a product of upper triangular matrices with nonnegative entries. The diagonal elements of \(U_j\) are the products of corresponding diagonal elements of \(J(1, \beta_{j_0}), \ldots, J(1, \beta_j)\) and, thus, are equal to 1 on the odd rows and 0 on the even ones.

More specifically, denote the entries of \(U_j\) by \(u_{kl}^{(j)}\) so that

\[
U_j = \left( u_{kl}^{(j)} \right)_{k,l=1}^{\infty}.
\]

For \(j \geq j_0, k, m = 1, 2, \ldots\) the equality \(U_{j+1} = U_j J(1, \beta_{j+1})\) implies

\[
u_{k,1}^{(j+1)} = u_{k,1}^{(j)}, \quad u_{k,2m}^{(j+1)} = u_{k,2m-1}^{(j)} \beta_{j+1}, \quad u_{k,2m+1}^{(j+1)} = u_{k,2m}^{(j)} + u_{k,2m+1}^{(j)}.
\]

The following entries for all \(j \geq j_0\) must be zero

\[
u_{k,2m-1}^{(j)} = u_{k,2m}^{(j)} = 0, \quad m = 1, 2, \ldots, \quad k = 2m, 2m + 1, \ldots.
\]

For \(j = j_0\) we have \(U_j = J(1, \beta_j)\), so the nonzero entries of \(U_j\) are only

\[
u_{2m-1,2m}^{(j)} = \beta_j, \quad u_{2m,2m+1}^{(j)} = 1 \quad \text{and} \quad u_{2m+1,2m+1}^{(j)} = 1, \quad \text{where} \quad m = 1, 2, \ldots.
\]
where

Consequently, the following estimate is valid

$$u_{k,2m}^{(j)} + u_{k,2m+1}^{(j)} \leq \left( \sum_{i=j_0}^{j} \beta_i \right)^{m-\left[ \frac{j}{2} \right]}, \quad m = 1, 2, \ldots, \quad k = 1, 2, \ldots, 2m. \quad (3.5)$$

Suppose that (3.5) holds for some $j \geq j_0$, then for $k \leq 2m$ we have

$$u_{k,2m}^{(j+1)} + u_{k,2m+1}^{(j+1)} = u_{k,2m}^{(j)} \beta_{j+1} + \left( u_{k,2m}^{(j)} + u_{k,2m+1}^{(j)} \right) \leq$$

$$u_{k,2m-2}^{(j)} + u_{k,2m-1}^{(j)} \beta_{j+1} + \left( u_{k,2m}^{(j)} + u_{k,2m+1}^{(j)} \right) \leq$$

$$\left( \beta_{j+1} + \sum_{i=j_0}^{j} \beta_i \right) \left( \sum_{i=j_0}^{j} \beta_i \right)^{m-\left[ \frac{j}{2} \right]} \leq \left( \sum_{i=j_0}^{j+1} \beta_i \right)^{m-\left[ \frac{j}{2} \right]}.$$

By induction, the conditions (3.5) hold for all $j \geq j_0$. Therefore, by (3.4),

$$u_{k,2m}^{(j)} + u_{k,2m+1}^{(j)} \leq \left( \sum_{i=j_0}^{\infty} \beta_i \right)^{m-\left[ \frac{j}{2} \right]} \leq 2^{-m+\left[ \frac{j}{2} \right]},$$

where $m = 1, 2, \ldots$ and $k = 1, 2, \ldots, 2m$. As a consequence,

$$\|U\|_1 = \sup_{1 \leq k < \infty} \left( u_{k,1}^{(j)} + \sum_{m=1}^{\infty} \left( u_{k,2m}^{(j)} + u_{k,2m+1}^{(j)} \right) \right) \leq \sum_{m=0}^{\infty} 2^{-m} = 2.$$

Since $u_{k,1}^{(j)} - u_{k,1} = 0$ and $u_{k,2m-1}^{(j)} - u_{k,2m-1} = u_{k,2m-2}^{(j)}$, $m > 1$, we have

$$\|U_{j+1} - U_j\|_1 = \sup_{1 \leq k < \infty} \sum_{m=1}^{\infty} \left( u_{k,2m-1}^{(j)} - u_{k,2m-1}^{(j)} + u_{k,2m}^{(j+1)} - u_{k,2m}^{(j)} \right) \leq$$

$$\sup_{1 \leq k < \infty} \left( \sum_{m=2}^{\infty} u_{k,2m-2}^{(j)} + \sum_{m=1}^{\infty} u_{k,2m}^{(j)} + \sum_{m=1}^{\infty} u_{k,2m}^{(j+1)} \right) \leq$$

$$\sup_{1 \leq k < \infty} \left( 2 \sum_{m=1}^{\infty} u_{k,2m}^{(j)} + \sum_{m=1}^{\infty} u_{k,2m}^{(j+1)} \right) \leq$$

$$2\beta_j \sum_{m=1}^{\infty} u_{k,2m-1}^{(j-1)} + \beta_{j+1} \sum_{m=1}^{\infty} u_{k,2m-1}^{(j)} \leq \beta_j \|U_{j-1}\|_1 + \beta_{j+1}\|U_j\|_1 \leq 4\beta_j + 2\beta_{j+1} \xrightarrow{j \to \infty} 0.$$

That is, $(U_j)_j$ is a Cauchy sequence, hence it converges to its entry-wise limit $U_\ast$.

Using (3.1) we obtain $\|V\|_1 \leq \|J(b_0, \beta_0)\|_1 \|J(1, \beta_1)\|_1 \cdots \|J(1, \beta_{j_0-1})\|_1 < \infty$ and

$$\|V U_j H_j + T(g) - V U_\ast H(1, 1) T(g)\|_\rho \leq$$

$$\|V\|_1 \|U_j H_j + U_\ast H(1, 1) + U_\ast H_j - U_\ast H(1, 1)\|_1 \|T(g)\|_\rho \leq$$

$$\|V\|_1 \|U_j - U_\ast\|_1 \|H_j + H_1 T(g)\|_\rho + \|V\|_1 \|U_\ast\|_1 \|H_j + H_1 - H(1, 1)\_1 T(g)\|_\rho. \quad (3.6)$$
The expansion of $g(z)$ into a power series at the origin converges for $|z| \leq \rho$ absolutely, hence $\|T(g)\|_{\rho} < \infty$. According to Corollary 2.12,

\[ \|H_{j+1}\|_1 = \|H(p_{j+1}, p_j)\|_1 < M \quad \text{and} \quad \|H(p_{j+1}, p_j) - H(1, 1)\|_1 \xrightarrow{j \to \infty} 0, \]

so the right-hand side of (3.6) vanishes and, therefore,

\[ H(p, q) = VU_\ast H(1, 1)T(g), \quad \text{where} \quad VU_\ast = \lim_{j \to \infty} (J(b_0, \beta_0) J(1, \beta_1) \cdots J(1, \beta_j)). \]

\[ \square \]

Remark 10. Applying this lemma to the ratio $\frac{p_{m-1}(z)}{p_0(z)}$, we can explicitly determine the matrix $U_\ast$. Since

\[ u_{k,2m}^{(j+1)} = \beta_{j+1} u_{k,2m-1}^{(j)} \xrightarrow{j \to \infty} 0 \quad \text{for} \quad m = 1, 2, \ldots \quad \text{and} \quad k = 2m, 2m+1, \ldots, \]

from $H_{j_0} = U_\ast H(1, 1)$ we get

\[ U_\ast = H_{j_0} H^T(0, 1), \]

where $A^T$ stands for the transpose of a matrix $A$.

Now consider meromorphic functions $p(z) =: p_0(z)$ and $q(z) =: p_{-1}(z)$, with the power series expansions given by (2.1). Suppose that $\beta_0, \beta_1, \ldots, \beta_{j-1} \neq 0$. Then we can define the functions $p_1(z), p_2(z), \ldots, p_j(z)$ via the formulæ (2.4).

Lemma 3.2. If the Hurwitz-type matrix $H_j := H(p_j, p_{j-1})$ satisfies the conditions

\[ H_j \begin{pmatrix} 2 & 3 & \ldots & k-1 & k \\ 2 & 3 & \ldots & k-1 & i \end{pmatrix} \geq 0, \quad \text{where} \quad k = 2, 3, 4, \ldots \quad \text{and} \quad i = k, k+1, k+2, \ldots, \quad (3.7) \]

then

\[ p_{j-1}(0) p_j(z) \equiv p_{j-1}(z) \iff \beta_j := H_j \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix} = 0. \]

Proof. Without loss of generality we assume $j = 0$. Suppose that $b_0 p_0 \equiv p_{-1}$. Then the minor $\beta_0 = H_0 \begin{pmatrix} 2 & 3 \\ 2 & 3 \end{pmatrix}$ has two proportional rows, and is therefore zero.

The converse we prove by contradiction. Let $\beta_0 = b_1 - a_1 b_0 = 0$ and $p_0(z) \neq b_0 p_{-1}(z)$. Then there exists an integer $i > 1$ such that $b_i \neq b_0 a_i$ (since $p_{-1}(z) \neq 0$). Therefore, according to (3.7) we have

\[ H_0 \begin{pmatrix} 2 & 3 \\ 2 & i+1 \end{pmatrix} = \begin{vmatrix} a_0 & a_i \\ b_0 & b_i \end{vmatrix} > 0. \]

Consequently,

\[ H_0 \begin{pmatrix} 2 & 3 & 4 \\ 2 & 3 & i+2 \end{pmatrix} = \begin{vmatrix} a_0 & a_i & a_i \\ b_0 & b_1 & b_i \\ 0 & a_0 & a_{i-1} \end{vmatrix} = -a_0 \begin{vmatrix} a_0 & a_i \\ b_0 & b_i \end{vmatrix} < 0, \]

which contradicts the conditions (3.7). \[ \square \]

Corollary 3.3. Let meromorphic functions $p(z)$ and $q(z)$ be such that $p(0) = 1$ and $q(0) \geq 0$. If the matrix $H(p, q)$ satisfies the conditions (3.7), then $F := \frac{q}{p} \in S$. 
Proof. For \( q(z) \equiv 0 \) this corollary is obvious. Suppose that \( q(z) \not\equiv 0 \). Set \( p_0(z) := p(z) \) and \( p_{-1}(z) := q(z) \) such that \( F(z) = F_0(z) \).

Suppose that for some \( j \geq 0 \) we have constructed the sequences \( p_{-1}(z), \ldots, p_{j}(z) \) and \( \beta_0, \ldots, \beta_{j-1} > 0 \). By Lemma 2.1 the matrix \( H_j \) satisfies (3.7). Therefore, according to Lemma 3.2, we have two mutually exclusive possibilities: \( \beta_j > 0 \) and \( p_{j-1}(0) \cdot p_{j}(z) = p_{j-1}(z) \). Consider the latter case. We have \( H_j^{(2)} = p_j(0) = 1 \) and \( H_j^{(3)} = H_j^{(4)} = \cdots = 0 \) with the notation (2.2). Additionally, the numbers \( \beta_i \) are positive for all \( i = 0, 1, \ldots, j - 1 \). By virtue of Lemma 2.1 the minors \( H_0^{(k)} \) are positive for \( k = 2, \ldots, j + 2 \), and zero for \( k > j + 2 \). So by Theorem 2.8 the considered function \( F(z) \) is a rational \( S \)-function.

If \( \beta_j > 0 \) we can define the function \( p_{j+1}(z) \) by (2.4). According to Lemma 2.1 the matrix \( H_{j+1} \) satisfies (3.7). So we can make the next step of this algorithm. If this process is infinite, by Lemma 2.1 all the principal minors \( H_0^{(k)} \) are positive. Hence \( F \in S \). □

**Theorem 1.4.** Consider the ratio \( F(z) = \frac{q(z)}{p(z)} \) of power series \( p(z) = \sum_{k=0}^{\infty} a_k z^k \) and \( q(z) = \sum_{k=0}^{\infty} b_k z^k \), normalized by the equality \( p(0) = a_0 = 1 \). The following conditions are equivalent:

(i) The infinite Hurwitz-type matrix

\[
H(p, q) = \begin{pmatrix} b_0 & b_1 & b_2 & b_3 & b_4 & b_5 & \ldots \\ 0 & a_0 & a_1 & a_2 & a_3 & a_4 & \ldots \\ 0 & b_0 & b_1 & b_2 & b_3 & b_4 & \ldots \\ 0 & 0 & a_0 & a_1 & a_2 & a_3 & \ldots \\ 0 & 0 & b_0 & b_1 & b_2 & b_3 & \ldots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}
\]

(3.8)

is totally nonnegative.

(ii) The matrix \( H(p, q) \) possesses the infinite factorization

\[
H(p, q) = \lim_{j \to \infty} \left( J(b_0, \beta_0) J(1, \beta_1) \cdots J(1, \beta_j) \right) H(1, 1) T(g),
\]

(3.9)

converging in \( \| \cdot \|_\rho \)-norm for some \( 0 < \rho \leq 1 \). Here \( b_0 \geq 0 \) and the sequence \( \beta_j \) is nonnegative, has a finite sum and contains no zeros followed by a nonzero entry, that is

\[
\beta_0, \beta_1, \ldots, \beta_{\omega-1} > 0, \quad \beta_\omega = \beta_{\omega+1} = \cdots = 0,
\]

\[
0 \leq \omega \leq \infty, \quad \text{and} \quad \sum_{j=0}^{\infty} \beta_j < \infty.
\]

The matrix \( T(g) \) denotes a totally nonnegative Toeplitz matrix of the form (1.3) with ones on its main diagonal.

(iii) The ratio \( F(z) \) is a meromorphic \( S \)-function; its numerator \( q(z) \) and denominator \( p(z) \) are entire functions of genus 0 up to a common meromorphic factor \( g(z) \) of the form (1.4), \( g(0) = 1 \).

Proof. We are proving as follows: (iii) \( \implies \) (ii) \( \implies \) (i) \( \implies \) (iii).

The implication (iii) \( \implies \) (ii) is a consequence of Lemma 3.1 since total positivity of the matrix \( T(g) \) is provided by Theorem 1.3.
The factors in (3.9) are totally nonnegative and have at most a finite number of nonzero entries in each column. Therefore, the Cauchy-Binet formula is valid and the products of the form
\[ J(b_0, \beta_0) J(1, \beta_1) \cdots J(1, \beta_\omega) H(1, 1) T(g), \]
where \(0 \leq \omega < \infty\),
are totally nonnegative. Moreover, the minors depend continuously on the matrix entries, so the entry-wise limit of totally nonnegative matrices is totally nonnegative itself. As a consequence, the matrix \(H(p, q)\) is totally nonnegative if the condition (ii) holds. That is the implication (ii) \(\implies\) (i) is true.

Let us show that (i) \(\implies\) (iii). The matrices \(T(p)\) and \(T(q)\) are submatrices of \(H(p, q)\). Thus, if (i) is true, by Theorem 1.3 the series \(p(z)\) and \(q(z)\) converge to meromorphic functions of the form (1.4), so \(F(z)\) is a meromorphic function as well.

Corollary 3.3 yields \(F \in \mathcal{S}\). So according to Lemma 2.3, the zeros (we denote their number by \(\omega_1 \leq \infty\)) and poles (we denote their number by \(\omega_2 \leq \infty\)) of \(F(z)\) are real, simple and interlacing. Moreover, \(F(x) > F(0) \geq 0\) for \(x > 0\), hence \(F(z)\) has only the zeros \(-\tau_1, -\tau_2, \ldots, -\tau_{\omega_1}\) and poles \(-\sigma_1, -\sigma_2, \ldots, -\sigma_{\omega_2}\), satisfying the following condition ((cf. (2.18)))
\[ 0 \leq \tau_1 < \sigma_1 < \tau_2 < \sigma_2 < \tau_3 < \ldots \tag{3.10} \]
Note that (3.10) implies the inequality \(\omega_1 - 1 \leq \omega_2 \leq \omega_1\).

However, the functions \(p(z)\) and \(q(z)\) have the form (1.4), in particular they have no nonpositive poles. Therefore, all the numbers \(-\tau_1, -\tau_2, \ldots, -\tau_{\omega_1}\) are among the zeros of \(q(z)\), while \(-\sigma_1, -\sigma_2, \ldots, -\sigma_{\omega_2}\) are among the zeros of \(p(z)\). As a result,
\[
q(z) = e^{\gamma_1 z} \tilde{q}(z) \frac{\prod_{\nu} \left(1 + \frac{z}{\alpha_{\nu}}\right)}{\prod_{\mu} \left(1 - \frac{z}{\beta_{\mu}}\right)} \quad \text{and} \quad p(z) = e^{\gamma_2 z} \tilde{p}(z) \frac{\prod_{\nu} \left(1 + \frac{z}{\alpha_{\nu}}\right)}{\prod_{\mu} \left(1 - \frac{z}{\beta_{\mu}}\right)},
\]
where \(\gamma_1, \gamma_2, (\alpha_{\nu})_{\nu}\) and \((\beta_{\mu})_{\mu}\) are appropriately chosen positive numbers,
\[
\tilde{q}(z) := b_0 z^j \prod_{\nu=\mu+1}^{\omega_1} \left(1 + \frac{z}{\tau_{\nu}}\right) \quad \text{for} \quad j = 1 - \text{sign} b_0 \quad \text{and} \quad \tilde{p}(z) := \prod_{\mu=1}^{\omega_2} \left(1 + \frac{z}{\sigma_{\mu}}\right).
\]
After cancellations in the fraction \(q(z) \over p(z)\) we obtain \(F(z) = e^{\gamma z} \tilde{q}(z) \over \tilde{p}(z)\), where \(\gamma := \gamma_1 - \gamma_2\).

Let us show \(\gamma = 0\). Set
\[
G(z) := \frac{1}{b_0} e^{-\gamma z} F(z).
\]
For \(\omega_1, \omega_2 < \infty\) we can express the rational function \(G(z)\) as a sum of partial fractions and ascertain that it agrees with the expansion (2.6). So in this case \(G \in \mathcal{S}\).

Suppose that \(\omega_1\) and \(\omega_2\) are infinite. With \(\text{Arg}: \mathbb{C} \setminus \{0\} \rightarrow (-\pi, \pi]\) denoting the principal argument, we obtain the following for \(\text{Im} \, z > 0\) from (3.10)
\[
\pi > \text{Arg} (\tau_0 + z) > \text{Arg} (\sigma_0 + z) > \text{Arg} (\tau_1 + z) > \text{Arg} (\sigma_1 + z) > \cdots > 0.
\]
Whenever \(\nu\) or \(\tau_0\) is positive, we obtain
\[
0 < \text{Arg} \left(1 + \frac{z}{\tau_{\nu}}\right) - \text{Arg} \left(1 + \frac{z}{\sigma_{\nu}}\right) < \text{Arg} \left(1 + \frac{z}{\tau_{\nu}}\right) - \text{Arg} \left(1 + \frac{z}{\tau_{\nu+1}}\right) < \pi.
\tag{3.11}
\]
Therefore, if $\tau_0 > 0$

$$0 < \arg \prod_{\nu=0}^{\infty} \frac{1 + \frac{z}{\tau_{\nu}}}{1 + \frac{z}{\sigma_{\nu}}} \leq \sum_{\nu=0}^{\infty} \arg \left( 1 + \frac{z}{\tau_{\nu}} \right) = \sum_{\nu=0}^{\infty} \left( \arg \left( 1 + \frac{z}{\tau_{\nu}} \right) - \arg \left( 1 + \frac{z}{\sigma_{\nu}} \right) \right) < \arg \left( 1 + \frac{z}{\tau_0} \right) - \lim_{\nu \to \infty} \arg \left( 1 + \frac{z}{\tau_{\nu}} \right) < \pi.$$  \hspace{1cm} (3.12)

That is

$$0 < \arg G(z) < \pi \quad \text{when} \quad \text{Im} z > 0,$$  \hspace{1cm} (3.13)

i.e. $G \in S$ since $G(z)$ is real. If $\tau_0 = 0$ we just replace all instances of $\left( 1 + \frac{z}{\tau_{0}} \right)$ in inequalities (3.11) and (3.12) with $z$ and obtain the same. This method to deduce the estimate (3.13) is taken from [5, Ch.IV §10, Lemma 11].

Now if $\gamma \neq 0$, then $\arg F(\pi i/\gamma) = \arg(-G(\pi i/\gamma))$ and $G \in S$, which contradicts the inclusion $F \in S$. Thus $\gamma = 0$. \hspace{1cm} \Box

4. DISTRIBUTION OF ZEROS AND POLES

First, let us prove the following auxiliary fact.

**Lemma 4.1.** Let $p(z), q(z)$ be the formal power series $p(z) = \sum_{k=0}^{\infty} a_k z^k$ and $q(z) = \sum_{k=0}^{\infty} b_k z^k$ such that $a_0 = 0$ and $b_0 > 0$. The matrix $H(p,q)$ defined by (3.8) is totally nonnegative if and only if $p(z) \equiv 0$ and $q(z)$ converges to a function of the form

$$f(z) = f_0 e^{\gamma z} \prod_{\nu} \left( 1 + \frac{z}{\alpha_{\nu}} \right) \prod_{\mu} \left( 1 - \frac{z}{\beta_{\mu}} \right),$$  \hspace{1cm} (4.1)

where $\gamma > 0, \alpha_{\nu}, \beta_{\mu} > 0$ for all $\mu, \nu$ and $\sum_{\nu} \frac{1}{\alpha_{\nu}} + \sum_{\mu} \frac{1}{\beta_{\mu}} < \infty$.

**Proof.** If $a_0 = 0$ then the total nonnegativity of $H = H(p,q)$ implies

$$0 \leq a_i = -\frac{1}{b_0} \begin{vmatrix} a_0 & a_i \\ b_0 & b_i \end{vmatrix} = -\frac{1}{b_0} H \begin{pmatrix} 2 & 3 \\ 2 & i + 1 \end{pmatrix} \leq 0 \quad \forall i = 1, 2, 3, \ldots,$$

so $p(z) \equiv 0$. The Toeplitz matrix $T(q)$ defined by (1.3) is totally nonnegative as a submatrix of $H(p,q)$. Therefore, by Theorem 1.3, $q(z)$ is of the form (4.1).

Conversely, if $p(z) \equiv 0$ then any nonzero minor of $H(p,q)$ is equal to a minor of $T(q)$. According to Theorem 1.3 the matrix $T(q)$ is totally nonnegative, hence $H(p,q)$ is totally nonnegative as well. \hspace{1cm} \Box

**Theorem 1.2.** A power series $f(z) = \sum_{k=0}^{\infty} f_k z^k$ with $f_0 > 0$ converges to an entire function of the form

$$f(z) = f_0 e^{\gamma z} \prod_{1 \leq \nu \leq \omega} \left( 1 + \frac{z}{\alpha_{\nu}} \right),$$  \hspace{1cm} (4.2)
where \( \gamma \geq 0 \), \( \alpha_\nu > 0 \) for all \( \nu \) and \( \sum_{1 \leq \nu \leq \omega} \frac{1}{\alpha_\nu} < \infty \) for some \( \omega \), \( 0 \leq \omega \leq \infty \), if and only if the infinite matrix

\[
D_f = \begin{pmatrix}
  f_0 & f_1 & f_2 & f_3 & f_4 & \cdots \\
  0 & f_1 & 2f_2 & 3f_3 & 4f_4 & \cdots \\
  0 & f_0 & f_1 & f_2 & f_3 & \cdots \\
  0 & 0 & f_1 & 2f_2 & 3f_3 & \cdots \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}
\]

is totally nonnegative.

**Proof.** Let the matrix \( D_f = H(f', f) \) be totally nonnegative, where \( f'(z) \) denotes the formal derivative of \( f(z) \). If \( f_1 = 0 \), by Lemma 4.1 \( f(z) \equiv f_0 > 0 \), i.e., (4.2) is satisfied.

Suppose that \( f_1 \neq 0 \). Theorem 1.4 implies that \( f(z) \) and \( f'(z) \) converge in a neighbourhood of the origin. Moreover, for some meromorphic function \( g(z) \) of the form (4.1), the functions \( \tilde{f}(z) := \frac{f(z)}{g(z)} \) and \( h := \frac{\tilde{f}'(z)}{g(z)} \) are entire of genus 0, coprime and have only negative zeros. In particular, the poles of \( g(z) \) are positive (if any). Let us show \( g(z) \) has no poles. Observe that in the right-hand side of the expression

\[
h(z) = \frac{\tilde{f}(z)g'(z) + g(z)\tilde{f}'(z)}{g(z)} = \tilde{f}'(z) + \frac{g(z)\tilde{f}'(z)}{g(z)}
\]

the logarithmic derivative of \( g(z) \) is multiplied by a function with no positive zeros. Therefore, each pole of \( g(z) \) must be a pole of \( h(z) \). But \( h(z) \) is an entire function, thus \( g(z) \) is entire and \( f(z) = \tilde{f}(z)g(z) \) can be represented as in (4.2).

Conversely, let \( f(z) \) admit the representation (4.2). If \( f(z) \) is a constant then by Lemma 4.1 the matrix \( D_f \) is totally nonnegative. Suppose now that \( f(z) \) is not a constant and consider its logarithmic derivative

\[
F(z) = \frac{f'(z)}{f(z)} = \gamma + \sum_{1 \leq \nu \leq \omega} \frac{1}{z + \alpha_\nu}, \quad 0 \leq \omega \leq \infty.
\]  

(4.3)

Each summand in the right-hand side of (4.3) is in \( \mathcal{R}^{-1} \), so \( F \in \mathcal{R}^{-1} \). The function \( f(z) \) is non-constant, hence \( F(z) \neq 0 \). Therefore, the function \( \frac{1}{F(z)} \) is an \( \mathcal{R} \)-function. Moreover, \( F(z) \) (and hence \( \frac{1}{F(z)} \)) has only negative poles and zeros. Consequently, \( \frac{1}{F(z)} \) is an \( \mathcal{S} \)-function.

Since \( f(z) \) has the form (4.2), each common zero of \( f(z) \) and \( f'(z) \) is negative. In addition the functions \( e^{-\gamma z} f(z) \) and \( e^{-\gamma z} f'(z) \) are of genus 0. So the function \( \frac{f(z)}{f'(z)} = \frac{1}{F(z)} \) satisfies (iii) in Theorem 1.4. Consequently, the matrix

\[
D_f = H(f', f) = f_1 H \left( \frac{f'}{f}, \frac{f}{f} \right)
\]

is totally nonnegative. \( \square \)
Let \( f(z) = \sum_{k=0}^{\infty} f_k z^k \), \( f_0 > 0 \), be a real entire function. Define its infinite Hurwitz matrix
\[
\mathcal{H}_f = \begin{pmatrix} f_0 & f_2 & f_4 & f_6 & f_8 & \ldots \\
0 & f_1 & f_3 & f_5 & f_7 & \ldots \\
0 & f_0 & f_2 & f_4 & f_6 & \ldots \\
0 & 0 & f_1 & f_3 & f_5 & \ldots \\
0 & 0 & f_0 & f_2 & f_4 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{pmatrix}.
\] (4.4)

For the minors of \( \mathcal{H}_f \) we use the same notation as in Section 2, such that
\[
\mathcal{H}_f^{(k)} = \mathcal{H}_f \begin{pmatrix} 2 & 3 & \ldots & k \\
2 & 3 & \ldots & k \\
\end{pmatrix}.
\]

Grommer in [8, §16, Satz IV] extended the Hurwitz criterion [11] to entire functions. However, he overlooked the condition on common zeros of odd and even parts (which was addressed by Kreĭn in [14]). We only need the following particular case of this extension.

**Theorem 4.2** ([14, Theorem 12], [5, Ch.V §4]). Let a real entire function \( f(z) \), \( f(0) > 0 \), be of genus 1 or 0. Suppose that its even part \( (f(z) + f(-z))/2 \) and its odd part \( (f(z) - f(-z))/2 \) have no common zeros.

Then the function \( f \) can be represented as
\[
f(z) = Ce^{\gamma z} \prod_{1 \leq \mu \leq \omega_1} \left( 1 + \frac{z}{x_\mu} \right) \prod_{1 \leq \nu \leq \omega_2} \left( 1 + \frac{z}{\alpha_\nu} \right) \left( 1 + \frac{z}{\bar{\alpha}_\nu} \right),
\] (4.5)

where \( 0 \leq \omega_1, \omega_2 \leq \infty, \gamma \geq 0 \) and \( C > 0 \), and its zeros satisfy the conditions
\[
x_\mu > 0 \quad \text{for} \ 1 \leq \mu \leq \omega_1, \quad \Re \alpha_\nu > 0, \quad \Im \alpha_\nu > 0 \quad \text{for} \ 1 \leq \nu \leq \omega_2,
\]

if and only if
\[
\mathcal{H}_f^{(2)}(f), \mathcal{H}_f^{(3)}(f), \ldots, \mathcal{H}_f^{(\omega_1+2\omega_2+1)}(f) > 0,
\]
\[
\mathcal{H}_f^{(\omega_1+2\omega_2+2)}(f) = \mathcal{H}_f^{(\omega_1+2\omega_2+3)}(f) = \ldots = 0.
\]

Note that the restriction on the genus of \( f(z) \) implies the additional condition
\[
\sum_{1 \leq \nu \leq \omega_2} \frac{1}{|\alpha_\nu|^2} < \infty.
\]

Based on Theorems 1.4 and 4.2 we deduce the following fact.

**Theorem 1.1.** Given a power series \( f(z) = z^j \sum_{k=0}^{\infty} f_k z^k \), where \( f_0 > 0 \) and \( j \) is a nonnegative integer, the infinite matrix \( \mathcal{H}_f \) defined by (4.4) is totally nonnegative if and only if the series \( f(z) \) converges to a function of the form
\[
f(z) = Cz^{\gamma_1 z + \gamma_2 z^2} \prod_{1 \leq \mu \leq \omega_1} \left( 1 + \frac{z}{x_\mu} \right) \prod_{1 \leq \nu \leq \omega_2} \left( 1 + \frac{z}{\alpha_\nu} \right) \left( 1 + \frac{z}{\bar{\alpha}_\nu} \right) \prod_{1 \leq \lambda \leq \omega_3} \left( 1 + \frac{z}{y_\lambda} \right) \left( 1 - \frac{z}{y_\lambda} \right),
\] (4.6)

for some \( \omega_1, \omega_2, \omega_3 \), \( 0 \leq \omega_1, \omega_2, \omega_3 \leq \infty \). Here \( C > 0 \),
\[
\gamma_1, \gamma_2 \geq 0, \quad x_\mu, y_\lambda > 0, \quad \Re \alpha_\nu \geq 0 \quad \text{and} \quad \Im \alpha_\nu > 0 \quad \text{for all} \ \mu, \nu, \lambda,
\]
\[
\sum_{1 \leq \mu \leq \omega_1} \frac{1}{x_\mu} + \sum_{1 \leq \nu \leq \omega_2} \Re \left( \frac{1}{\alpha_\nu} \right) + \sum_{1 \leq \nu \leq \omega_2} \frac{1}{|\alpha_\nu|^2} + \sum_{1 \leq \lambda \leq \omega_3} \frac{1}{y_\lambda} < \infty.
\] (4.8)
Proof. Suppose that \( f(z) \) is represented as (4.6). We can express it as

\[
  f(z) = C z^j g(z^2) h(z),
\]

where

\[
  h(z) := e^{\gamma z} \prod_{1 \leq \mu \leq \omega_1} \left( 1 + \frac{z}{x_\mu} \right) \prod_{\text{Re} \alpha_\nu > 0} \left( 1 + \frac{z}{\alpha_\nu} \right) \left( 1 + \frac{z}{\bar{\alpha}_\nu} \right)
\]

and (4.9)

\[
  g(z^2) := e^{\gamma z^2} \prod_{\text{Re} \alpha_\nu = 0} \left( 1 + \frac{z^2}{i^2 \alpha_\nu^2} \right) \prod_{1 \leq \lambda \leq \omega_2} \left( 1 - \frac{z^2}{\gamma_\lambda^2} \right).
\]

Note that \( g(z) \) has the form (4.1), so by Theorem 1.3 its Toeplitz matrix \( T(g) \) is totally nonnegative.

Split \( h(z) \) into the odd part \( z h_o(z^2) \) and the even part \( h_e(z^2) \) so that

\[
  h(z) = h_e(z^2) + z h_o(z^2).
\]

The function \( h(z) \) is of genus not exceeding 1 as well as \( h_e(z^2) \) and \( h_o(z^2) \). This implies that the genus of \( h_o(z) \) and \( h_e(z) \) is 0. Indeed, for example, the function \( h_e(z^2) \) with zeros \( \pm \delta_n, n = 1, 2, \ldots \), can be represented as the Weierstraß product

\[
  h_e(z^2) = e^{-z} \prod_n \left( 1 - \frac{z^2}{\delta_n^2} \right) e^{\delta_n z} \left( 1 + \frac{z}{\delta_n} \right) e^{-\delta_n z} = e^{-z} \prod_n \left( 1 - \frac{z^2}{\delta_n^2} \right).
\]

And since it depends only on \( z^2 \), we necessarily have \( c = 0 \) in this representation, which implies

\[
  h_e(z) = \prod_n \left( 1 - \frac{z}{\delta_n} \right).
\]

Let us show \( h_o(z) \) and \( h_e(z) \) are coprime. Denote \( r := \gcd(h_o, h_e) \) if \( h_o(z) \neq 0 \). If \( h_o(z) \equiv 0 \) we set \( r(z) := h_e(z) \). Assume that \( r(z) \neq 1 \). So it has zeros, since its genus is zero and \( r(0) = h_e(0) = 1 \). However, if \( r(0) = 0 \) then \( h(\sqrt{\omega_0}) = h(-\sqrt{\omega_0}) = 0 \). Since one of the points \( \pm \sqrt{\omega_0} \neq 0 \) is in the closed right half of the complex plane (independently of the branch of the square root) we get a contradiction to (4.9).

If \( h_o(z) \equiv 0 \) then \( H(h_o, h_e) = H(0, 1) \) is totally nonnegative. This implies the total nonnegativity of the matrix

\[
  \mathcal{H}_f = C H(g h_o, g h_e) = C H(h_o, h_e) T(g).
\]

If \( h_o(z) \neq 0 \), then the function \( h(z) \) has the form (4.9) and its odd and even parts are coprime. That is, \( h(z) \) satisfies the conditions of Theorem 4.2. Therefore, \( \mathcal{H}_h^{(2)}(h_o, h_e), \mathcal{H}_h^{(3)}(h_o, h_e), \ldots \) is a positive sequence possibly followed by zeros. Thus, by Theorem 2.8, we obtain \( \frac{h_o}{h_e} \in S \). Then the applying of Theorem 1.4 to the function \( \frac{h_o}{h_e} \) gives the total nonnegativity of the matrices \( H(h_o, h_e) \) and, consequently, \( \mathcal{H}_f \).

Let us prove the converse. Suppose that the Hurwitz matrix \( \mathcal{H}_f \) is totally nonnegative. We can split the series \( f_0^{-1} z^{-j} f(z) = \sum_{k=0}^\infty H_{f_0} z^k \) into the even part \( g(z^2) \) and the odd part \( z p(z^2) \) so that \( f(z) \) can be expressed as follows

\[
  f(z) = f_0 z^j \left( g(z^2) + z p(z^2) \right).
\]

It gives \( \mathcal{H}_f = f_0 H(p, q) \).
If $f_1 > 0$, then since the matrix $H(p,q)$ is totally nonnegative, by Theorem 1.4 there exists a meromorphic function $g(z)$ of the form (4.1) such that $	ilde{p}(z) = \frac{g(z)}{\tilde{q}(z)}$ and $\tilde{q}(z) = \tilde{g}(z)$ are coprime entire functions of genus 0. Moreover, the ratio $\frac{\tilde{q}(z)}{\tilde{p}(z)}$ is an $S$-function. Let

$$\tilde{f}(z) := \tilde{q}(z^2) + z\tilde{p}(z^2).$$

By Theorem 2.8 the minors $H^{(2)}_f = H^{(2)}(\tilde{p}, \tilde{q})$, $H^{(3)}_f = H^{(3)}(\tilde{p}, \tilde{q})$, . . . form a positive sequence possibly followed by zeros. Since $\tilde{q}(z)$ and $\tilde{p}(z)$ are coprime, the function $\tilde{f}(z)$ has the form (4.5) by Theorem 4.2.

If $f_1 = 0$ then according to Lemma 4.1, $p(z) \equiv 0$ and $q(z)$ has the form (4.1). Here we set $g(z) := q(z)$ so that $\tilde{f}(z) \equiv 1$.

Now consider both cases $f_1 = 0$ and $f_1 > 0$. We showed that $\tilde{f}(z)$ can be represented as in (4.5), while $g(z)$ can be represented as in (4.1). That is, after the appropriate renaming of zeros and poles, the function $\tilde{f}(z)$ has the form (4.9), the function $g(z^2)$ has the form (4.10) and the conditions (4.7)–(4.8) are satisfied. Thereby

$$f(z) = f_0z^2g(z^2)\tilde{f}(z)$$

can be represented as in (4.6).

\[\blacksquare\]

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