One-dimensional hemodynamics in vessels with elastic walls, multicomponent approach

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Abstract. A review of hemodynamic models (blood movements in the human arterial system) is carried out taking into account the elasticity of the vessel walls. A generalization of the model is formulated regarding blood as a multicomponent medium.

In the general case, physical processes in the human arterial system are described by the three-dimensional unsteady Navier—Stokes equations for a viscous incompressible fluid together with the equations of dynamics of the elastic shells of blood vessels [1]. However, the use of such a complex multidimensional model requires finding solutions to problems with a free boundary for the Navier—Stokes equations in complex domains. Not to mention the obvious boundless complexity of the theoretical analysis of such models, the search for numerical solutions to such problems is associated with huge computational costs. Therefore, multidimensional models in practice, as a rule, are not used for the global description of the arterial system, but simplified hemodynamic models are used, which are usually unsteady systems of (ordinary or partial) differential equations. Such models make it possible to determine not only qualitative, but also quantitative characteristics of blood movement.

There are many simplified (non-stationary) hemodynamic models. They can be divided into three groups. The roughest approximation of the original three-dimensional (3D) model is the so-called lumped model or the “zero-dimensional” (0D) model [2]. The essence of these models is to liken the blood circulatory system to electrical circuits. In this approach, voltage and electric current are analogs of pressure and volumetric blood flow, and resistors of an electric circuit are analogs of vessels. The second group of simplified models consists of one-dimensional (1D) hemodynamic models. The one-dimensional hemodynamic models are, roughly speaking, the hydraulic approximation of the original 3D model and are obtained by averaging it in the “transverse” direction. As for the 0D models, they are obtained by further averaging the 1D models along the length of the vessel [2,3]. Finally, the third group of simplified models are the so-called multi-scale models [2,3] or hybrid models. They represent a reasonable combination of 3D and 1D models, 1D and 0D models, 3D and 0D models, 2D and 1D models, etc.

We will consider a one-dimensional model of hemodynamics. For the first time, the one-dimensional approximation of hemodynamics was proposed by Euler [4]. We will deal with the one-dimensional model of hemodynamics proposed in [5–9]. In addition, we consider a generalization of this model to the case of blood as a multicomponent medium.

The aforementioned generalized one-dimensional model describes the flow of blood (taking into account its multicomponent nature) in arteries and its interaction with moving walls. Let...
t be time, and \((x, y, z)\) be Cartesian coordinates. We assume that the idealization of the artery is an elastic tube in the form of a straight circular cylinder whose axis coincides with the axis \(z\). This cylinder has a variable cross section due to pressure drops and the wall oscillations caused by them, but has a constant radius in the relaxed state. Consider a piece of this tube of length \(L = \text{const}\) from \(z = z_1\) to \(z = z_2\) (see figure 1). By \(V(t)\) we will denote the spatial domain, which is the indicated tube filled with blood, and by \(C(t), S_1(t)\) and \(S_2(t)\) we mean the lateral surface and the bases of the cylinder \(V(t)\). The domain \(V(t)\) changes with time under the influence of a pulsating fluid (blood). The axial section \(\{z = \text{const}\}\) of the cylinder \(V(t)\) will be denoted by \(S = S(z, t)\), and its boundary by \(l(z, t)\). By \(A = A(z, t)\) we denote the area of the axial section \(S(z, t)\). We also denote by \(m\) the unit external normal vector to \(l\).

![Figure 1. Simplified blood vessel geometry.](image)

The considered one-dimensional model of hemodynamics is described using the following system of partial differential equations [5–9]:

\[
\frac{\partial (A \rho \overline{u}_z)}{\partial t} + \frac{\partial (A \rho \overline{u}_z^2)}{\partial z} = 0, \quad (1)
\]

\[
\frac{\partial (A \rho \overline{u}_z)}{\partial t} + \frac{\partial (A \rho \overline{u}_z^2)}{\partial z} + A \frac{\partial p}{\partial z} = A \rho f_z + \nu \oint_l \left( \frac{\partial u_z}{\partial m} \right) dl + A \nu \left( \frac{\partial^2 u_z}{\partial z^2} \right), \quad (2)
\]

in which \(\rho = \text{const} > 0\) is the density of the blood; \(u_z\) is the axial component of the velocity (it is believed that the velocity components orthogonal to the \(z\) axis are negligible compared to \(u_z\)); \(p\) is the pressure; \(f_z\) is the external mass force which acts in the axial direction; \(\nu\) is the viscosity coefficient; the upper bar in the second term in the left hand side of (1), in the first and second terms in the left hand side of (2), as well as in the last term in the right hand side of (2) means the following operation (averaging):

\[
\overline{\xi} = \frac{1}{A} \int_S \xi \, dS. \quad (3)
\]

Let us also introduce the following notation:

\[
Q = A \overline{u}_z. \quad (4)
\]
Using the approximate equalities
\[
\frac{\nu}{\rho} \int_l \left( \frac{\partial u_z}{\partial m} \right) dl \approx \frac{2\pi a}{\rho} \tau_w, \quad \frac{\partial (A u_z^2)}{\partial z} \approx \frac{\partial}{\partial z} \left( \delta_1 \frac{Q^2}{A} \right)
\]
and the equality
\[
A \nu \left( \frac{\partial^2 u_z}{\partial z^2} \right) = \frac{\nu}{\rho} \frac{\partial Q}{\partial z} \quad \left( u_z \big|_{r=a} = 0, \quad \frac{\partial u_z}{\partial z} \big|_{z=a} = 0 \right),
\]
where
\[
\tau_w = \frac{-2\nu}{(1 - \zeta_c) a} Q + \frac{a}{4} (1 - \zeta_c) \frac{\partial p}{\partial z}, \quad \zeta_c = \left( \max \left( 0, 1 - \sqrt{\frac{2}{\alpha}} \right) \right)^2,
\]
\[
\delta_1 = \frac{2 - 2\zeta_c (1 - \ln \zeta_c)}{(1 - \zeta_c)^2},
\]
a is the local radius, \(\tau_w\) is the wall shear stress, \(\alpha\) is the Womersley number, we get from the equations (1) and (2) the following system of equations
\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0,
\]
\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial z} \left( \delta_1 \frac{Q^2}{A} \right) + \frac{A \frac{\partial p}{\partial z}}{\rho} = A f_z + \frac{2\pi a}{\rho} \tau_w + \frac{\nu}{\rho} \frac{\partial^2 Q}{\partial z^2}.
\]
Consider another version of the one-dimensional model of hemodynamics [3]:
\[
\frac{\partial A}{\partial t} + \frac{\partial Q}{\partial z} = 0,
\]
\[
\frac{\partial Q}{\partial t} + \frac{\partial}{\partial z} \left( \alpha_1 \frac{Q^2}{A} \right) + \frac{A \frac{\partial p}{\partial z}}{\rho} + K_R \frac{Q}{A} = A f_z,
\]
where \(\alpha_1\) is the Coriolis coefficient, \(K_R\) is the viscosity-dependent drag coefficient.

In order to close the system (8), (9) (or (10), (11)) it is necessary to use the constitutive equation for pressure. For the system (8), (9), the constitutive equation is taken in the form [6–8]
\[
p(z, t) = \tilde{p}(A(z, t), z, t).
\]
For example, one can set [8]
\[
p(z, t) = E_s(z) \left( \frac{h(z)}{r_0(z)} \left( 1 - \left( \frac{A_0(z)}{A(z, t)} \right)^{1/2} \right) \right),
\]
where \(E_s(z)\) is the static modulus, \(h(z)\) is the wall thickness, \(r_0(z)\) is the radius of zero pressure, and \(A_0(z)\) is the area of zero pressure.

For the system (10), (11) the constitutive equation for the pressure is accepted in the form [3,10]
\[
p = p_{ext} + \rho \gamma h_0 \frac{\partial^2 \eta}{\partial t^2} - \gamma \frac{\partial \eta}{\partial t} - \alpha \frac{\partial^2 \eta}{\partial z^2} - \tilde{c} \frac{\partial^3 \eta}{\partial t \partial z^2} + G(\eta),
\]
where $\eta$ is the deviation of the wall position from the initial state, $p_{\text{ext}}$ is the pressure on the vessel wall from the outside, $\rho_\omega$ is the density of the wall material, $h_0$ is the wall thickness, $\tilde{\gamma}$ is responsible for the interaction of the wall with the fluid, $\tilde{a}$ characterizes the longitudinal stresses of the wall, $\tilde{c}$ characterizes the viscoelastic properties of the wall, $G(\eta) = b_1 \eta + b_2 \eta^2 + b_3 \eta^3$ is a function that defines the nonlinear elastic wall reaction, and $b_1, b_2, b_3$ are numeric coefficients.

The second term in the right hand side of (14) determines the inertia and it is proportional to the wall acceleration, the third is a viscoelastic term, proportional to the wall velocity, the fourth term corresponds to the longitudinal prestressed state of the vessel, the fifth is another viscoelastic term, the sixth is the function of the elastic wall reaction.

Let us agree that $A = \pi (\eta + R_0)^2$, $q = \frac{Q}{\pi}$ and rewrite the equations (10), (11) in the following form:

$$\frac{\partial q}{\partial t} + \alpha_1 \frac{\partial}{\partial z} \left( \frac{q}{\eta + R_0} \right)^2 + \frac{K_R}{\pi} \frac{q}{(\eta + R_0)^2} + \frac{(\eta + R_0)^2}{\rho} \frac{\partial}{\partial z} \left( p_{\text{ext}} + \rho_\omega h_0 \frac{\partial^2 \eta}{\partial t^2} - \tilde{\gamma} \frac{\partial \eta}{\partial t} - \tilde{a} \frac{\partial^2 \eta}{\partial z^2} - \tilde{c} \frac{\partial^3 \eta}{\partial t \partial z^2} + G(\eta) \right) = (\eta + R_0)^2 f_z. \quad (16)$$

In conclusion, we consider the following generalization of the one-dimensional hemodynamic model (generalization of the system (8), (9)) taking into account that blood is multicomponent (more precisely, we assume that blood consists of $N$ components) [11, 12]:

$$\frac{\partial (A \rho_i)}{\partial t} + \frac{\partial (A \rho_i u_{zi})}{\partial z} = 0, \quad i = 1, \ldots, N, \quad (17)$$

$$\frac{\partial (A \rho_i u_{zi})}{\partial t} + \frac{\partial (A \rho_i u_{zi}^2)}{\partial z} + A \frac{\partial p_i}{\partial z} = A \rho_i f_{zi} + \sum_{j=1}^{N} \nu_{ij} \int \left( \frac{\partial u_{zi}}{\partial m} \right) dl + A \sum_{j=1}^{N} \nu_{ij} \left( \frac{\partial^2 u_{zi}}{\partial z^2} \right), \quad i = 1, \ldots, N, \quad (18)$$

in which $\rho_i$ is the density of the $i$-th component; $u_{zi}$ is the axial projection of the velocity of the $i$-th constituent; $p_i$ is the pressure in the $i$-th component; $f_{zi}$ is the external mass force acting on the $i$-th constituent in the axial direction; $\nu_{ij}$ are the viscosity coefficients that form the symmetric matrix $N > 0$.

For the system (17)–(18), initial boundary value problems are formulated in a standard way. The analysis of the existence and uniqueness of solutions to these problems in the class of strong solutions is currently being carried out by the authors.

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