Gaussian sharp-edge diffraction: a paraxial revisitation of Miyamoto-Wolf’s theory

Riccardo Borghi
Dipartimento di Ingegneria, Università degli Studi “Roma Tre”
Corresponding author: riccardo.borghi@uniroma3.it

A “genuinely” paraxial version of Miyamoto-Wolf’s theory aimed at dealing with sharp-edge diffraction under Gaussian beam illumination is presented. The theoretical analysis is carried out in such a way the well known Young-Maggi-Rubinowicz boundary diffraction wave theory can be extended to deal with Gaussian beams in an apparently straightforward way. The key for achieving such an extension is the introduction of suitable “complex angles” within the integral representations of the geometrical and BDW components of the total diffracted wavefield. Surprisingly enough, such a simple (although not rigorously justified) mathematical generalization seems to work well within the complex Gaussian realm. The resulting integrals provide meaningful quantities that, once suitably combined, give rise to predictions which are in perfect agreement with results already obtained in the past. An interesting and still open theoretical question about how to evaluate “Gaussian geometrical shadows” for arbitrarily shaped apertures is also discussed.

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I. INTRODUCTION

Emil Wolf is universally considered the father of classical coherence theory. He loved telling to his guests (I was not an exception during a lunch at Rochester’s Institute of Optics) the following nice story [1]:

In 1956 Born was already in retirement and I was on a visiting appointment at New York University, still working on our book [2]. One day I received a letter from Born in which he asked me why the manuscript was not yet finished. I wrote back saying that the manuscript is almost completed, except for a chapter on partial coherence on which I was still working. Born replied at once saying, “Wolf, who apart from you is interested in coherence? Leave the chapter out and send the manuscript to the printers.” I finished the chapter anyway and our book was published in 1959, only a few months before the invention of the laser, and many of the reviews of our book which were then appearing stressed that “Principles of Optics” contained an account of coherence theory, which had become of crucial importance to the understanding of some features of laser light.

The careers of several young scientists who started studying optics in the early nineties have considerably been influenced by the big deal of work Wolf and his co-workers produced over almost four decades about optical coherence [3]. It would then be natural to celebrate Wolf’s memory and legacy by speaking about classical coherence theory. However, in the present work I wish pursuing a

maybe unusual different route, which touches a different area of optics Wolf gave fundamental contributions between the sixties and seventies: the so-called boundary diffracted wave (BDW henceforth) theory [2, Ch. 8].

The origin of BDW theory can be traced back to 1802, when Thomas Young first suggested the idea that the boundary of an illuminated aperture should act as a secondary light source emitting waves in all directions [4, 5]. According to Young’s picture, diffraction could be thought of as arising from the superposition of the field produced by clipping the incoming wave through the laws of geometrical optics and of the so-called boundary diffraction wave (BDW) which originates from the aperture edge. Unfortunately, at that time Young’s ideas were not adequately supported from a mathematical point of view. Fresnel’s theory, which mathematically implements Huygens’ superposition principle under paraxial approximation, prevailed. It was only between the end of nineteenth and the beginning of the twentieth century, that Maggi [6] and, independently, Rubinowicz [7], gave Young’s idea the mathematical basis needed for it to rise to the status of a quantitative theory.

The Young-Maggi-Rubinowicz theory concerns only with plane- or spherical-wave illuminations, for which the decomposition of the diffracted wavefield into a “geometrical” plus a BDW component can be obtained in a fairly simple way [8, 10]. On further invoking paraxial propagation, the derivation of the BDW decomposition can be achieved starting from the two-dimensional Fresnel in a way that almost resembles an academic exercise, as first shown by Hannay [11]. Quite recently [12, 14], Hannay’s formulation has been used as an effective starting point for revisiting Fresnel’s diffraction theory within the fairly new theoretical framework of the so-called catastrophe optics [15, 16].

The need of extending BDW theory to deal with impinging wavefields more general than plane or spherical waves was pointed out by Wolf in a celebrated 1960 paper, coauthored with Kerno Miyamoto, where it is

*To Emil Wolf (1922 - 2018), in Memoriam
The researches of Maggi and Rubinowicz showed conclusively the basic correctness of Young’s ideas. However, their analyses were restricted to cases when the wave incident upon the aperture is plane or spherical. Attempts to generalize these results to more general fields have so far not been very successful and doubts have in fact been expressed about the possibility of such a generalization. As a physical model for diffraction, the Young-Maggi-Rubinowicz theory is intrinsically simple and physically appealing. It relates diffraction directly to the true cause of its origin, namely, the presence of the boundary of a diffracting body. It seems hard to believe that no proper generalization to more complicated fields exists.

Among the main motivations for extending the Young-Maggi-Rubinowicz theory, still exposed in [17], it is found that

The possibility of such a generalization is not of academic interest alone as the following remarks will indicate. It is well known that the image of a small source formed by an optical system has as a rule a complicated structure. In consequence, calculations of light distribution in such an image are very laborious.

In any case such a generalization would give a new insight into the physical process of image formation.

In [17] Miyamoto and Wolf developed their extension of the BDW theory to arbitrary impinging wavefields. In particular, they showed that, on working within the Kirchhoff diffraction theory, the wavefield diffracted by an aperture in an opaque plane can be decomposed, under very general conditions, as the superposition of (i) a disturbance originating at the aperture boundary and (ii) a disturbance expressed through the sum of several contributions originated from geometrical singularities suitably located within the aperture. When the illumination is plane or spherical, the number of singularities in the disturbance (ii) reduces to one or zero, depending on the position of the observation point within the geometrical shadow. In this way, the results of the Young-Maggi-Rubinowicz theory is then reproduced. Despite its formal beauty, Miyamoto-Wolf’s theory is not easy to be grasped, especially for readers not equipped with adequate mathematical backgrounds. Also its application to apparently simple incoming disturbances, like for instance Gaussian beams, is far from being trivial, as it was pointed out by some works published between the seventies and the eighties [18–21]. A possible route to develop a more manageable theory is to invoke paraxial approximation from the beginning. This has already been done in [11–14] as far as the Young-Maggi-Rubinowicz theory is concerned, for which the mathematical formulation considerably simplifies within paraxial approximation.

In the present paper a “genuinely paraxial” version of the Miyamoto-Wolf theory will be developed for a Gaussian beam impinging on arbitrarily shaped sharp-edge planar apertures. In particular, on using the well known representation of Gaussian beams in terms of complex point sources, a mathematically simple and geometrically sound theoretical treatment of the scalar diffraction problem within paraxial approximation can be derived in an apparently almost straightforward way. However, such apparent simplicity hides some mathematical subtleties which must be investigated before the application to practical cases. This is exactly the aim of the present work, which is structured as follows: in Sec. I a brief résumé of the paraxial version of the Young-Maggi-Rubinowicz BDW theory developed according to the prescriptions given in [11–14] is given. This will help readers to familiarize with the geometry of the problem and the main notations used throughout the paper. At the same time, it will make the paper reasonably self contained. The general paraxial Gaussian diffraction theory is then carried out in Sec. II where the main analytical results of the work are also presented. In the same section the principal mathematical subtleties of the theory are investigated in the light of some conclusions given by Otis in [18]. In particular, his analysis of the so-called geometrical wavefield (which is the field ascribed to the singularities present at the diffraction aperture) left some unanswered questions which could find an explanation, although not definitive, through the approach carried out in the present paper. The practical implementation of the BDW integrals derived in Sec. I is described in Sec. IV for a single but significant case, namely the diffraction of Gaussian beams by misaligned and tilted circular apertures. Some conclusive words are finally given in Sec. V to illustrate the main open questions (of both theoretical and practical nature) as well as the potential applications of the proposed theoretical approach.

II. PRELIMINARIES: HANNAY’S THEORY FOR PLANE AND SPHERICAL WAVES

First of all it is worth recalling Hannay’s formulation of paraxial BDW theory under plane wave illumination following the formulation and notations used in [12–13].

The geometry of the problem is sketched in Fig. 1: a monochromatic plane wave with wavenumber \( k \) orthogonally impinges on an opaque transverse plane with a sharp-edge aperture \( \mathcal{A} \). A unitary (in suitable units) amplitude of the incident plane wave will be assumed. The disturbance at a distance \( z > 0 \) from the aperture is, within the cylindrical reference frame \((r; z)\), given by the
The form of the following identity:

\[ \text{integral in Eqs. (5) and (6).} \]

On introducing the polar reference frame \((r, \varphi)\) shown in Fig. 2, Eq. (2) then becomes

\[ \psi(r; z) = \int_{\rho \in A} d^2 \rho \exp \left( \frac{ik}{2z} \left( r - \rho \right)^2 \right). \tag{2} \]

where the function \(R\) is a polar representation of the boundary \(\Gamma\). Equation (3) will now be recast in the form of the following identity:

\[ -\frac{ik}{2\pi z} \int_{\rho \in A} d^2 \rho \exp \left( \frac{ik}{2z} (r - \rho)^2 \right) = \psi_G(r) + \psi_{BDW}(r; u) \tag{4} \]

where

\[ \psi_G(r) = \frac{1}{2\pi} \oint_{\Gamma} d\varphi, \tag{5} \]

and

\[ \psi_{BDW}(r; u) = -\frac{1}{2\pi} \oint_{\Gamma} d\varphi \exp \left( \frac{iu}{2} R(\varphi)^2 \right). \tag{6} \]

Here the dimensionless parameter \(u = k\ell^2/z\) will be henceforth identified with the Fresnel number, the parameter \(\ell\) being a sort of “natural” unit length characteristic of the aperture \(A\). For instance, if \(A\) were circularly shaped then \(\ell\) would certainly be identified with the aperture radius.

Equations (1) - (6) will play a key role throughout the present paper. In particular, the function \(\psi_G(r)\) defined into Eq. (5) is called “geometrical wavefield” and coincides with the characteristic function of the aperture \(A\), i.e.,

\[ \psi_G(r) = \begin{cases} 1, & r \in A, \\ 0, & r \notin A. \end{cases} \tag{7} \]

In other words, \(\psi_G(r)\) represents the field produced, according to the laws of geometrical optics, by clipping the incident plane wave by the aperture \(A\).

The wavefield \(\psi_{BDW}\) is that generated by the sharp edge \(\Gamma\) and represents Young’s wavelets which have to be superimposed to \(\psi_G\) to retrieve the total diffracted field. The application of the plane wave paraxial BDW theory has already produced some interesting results. An important connection with catastrophe optics has already been pointed out in [12–14], where it is also shown how to build up analytical estimates of the diffraction field at observation points which are located in the neighborhood of the field singularities. The latter are the geometrical boundary shadow and the caustics produced at the geometrical evolute of the projection of the diffracting aperture on the transverse observation plane. In particular, the asymptotics treatment of the integral (6) promoted in [13] allowed unexpected and interesting features of suitably heart-shaped apertures to be grasped [14]. Shortly after, such peculiar properties have also been experimentally confirmed [22–24].

The “genuinely paraxial” Gaussian version of Miyamoto-Wolf’s theory promised at the beginning of the paper will now be carried out as a suitable generalization of the BDW theory described so far.

### III. GAUSSIAN BEAMS DIFFRACTION BY SHARP-EDGE APERTURES

The first step is to extend the plane-wave paraxial theory developed in the previous section to a spherical wave generated by a point source placed at a distance \(D\) from the aperture plane, as sketched in Fig. 3. Without loss of
generality, the $z$-axis will be chosen to contain the point source.

In this way the impinging wavefield across the aperture $\mathcal{A}$ is then

$$\psi_i(r) = \exp\left(\frac{ik}{2D} z^2\right),$$

where an amplitude factor $1/D$ has been omitted. It is straightforward to prove that the diffracted wavefield at the transverse plane $z > 0$ is [11]

$$\psi(r; z) = \frac{D}{z + D} \exp\left(\frac{ik}{2} z^2\right) \times \left(\frac{ik}{2\pi z D}\right) \int_{r \in \mathcal{A}} d^2 \rho \ exp\left[\frac{ik}{2} z + D \rho \ - \ \frac{D}{z + D} D u\right].$$

where the transverse vector $r_C = \frac{D}{z + D} r$ inside the integral defines the position of the point $P'$ corresponding to the intersection between the aperture plane and the line connecting $S$ and $P$, as sketched in Fig. 3.

Equation (9) shows that the diffracted wavefield is basically the product of (i) the field the source $S$ would produce in absence of the aperture $\mathcal{A}$ at a distance $z$ from the aperture plane and (ii) the field a unit-amplitude plane wave, orthogonally impinging on $\mathcal{A}$, would produce at the observation point $P'$ of a transverse plane placed at the distance $z_C = (z^{-1} + D^{-1})^{-1}$ from the aperture plane. On taking Eqs. (4) - (6) into account, the diffracted field in Eq. (9) can then be recast as follows:

$$\psi(r; z) = \frac{D}{z + D} \exp\left(\frac{ik}{2} z^2\right) \left[\psi_C(r_C) + \psi_{\text{BDW}}(r_C; u_C)\right],$$

with

$$u_C = \frac{k \ell^2}{z_C} = \frac{k \ell^2}{z} \left(1 + \frac{z}{D}\right) = u \left(1 + \frac{z}{D}\right),$$

denoting the new Fresnel number.

Equations (10) and (11) represent the key to introduce the paraxial Gaussian BDW theory. To this end, it must be recalled that the wavefield associated to a Gaussian beam formally coincides with that produced, within paraxial approximation, by a point source placed somewhere at a “complex location.”

Consider the geometrical situation depicted in Fig. 4: a Gaussian beam having spot size $w_0$ has its waist plane at a distance $D$ from the aperture plane. The mean propagation direction of the Gaussian beam does coincide with the $z$-axis of the cylindrical reference frame $(r; z)$. For what has been said above, to derive the diffracted wavefield all we have to do is to replace, into Eqs. (10) and (11), the real quantity $D$ by the complex quantity $D - iL$, where $L = kw_0^2/2$, the so-called Rayleigh length, will be assumed as the “natural” unit to measure all longitudinal distances. Accordingly, on first applying the transformation $D \rightarrow D - iL$ into Eq. (10) and then on formally letting $L = 1$, after straightforward algebra the Gaussian diffracted wavefield can formally be expressed as follows:

$$\psi(r; z) = \frac{1 + iD}{1 + i(z + D)} \exp\left(-\frac{r^2/w_0^2}{1 + i(z + D)}\right) \times \left[\psi_C(r_C) + \psi_{\text{BDW}}(r_C; u_C)\right],$$

where both $r_C$ and $u_C$ are now complex quantities, and precisely

$$\begin{align*}
r_C & = \frac{1 + iD}{1 + i(z + D)} r, \\
u_C & = \frac{1 + i(z + D)}{1 + iD} u.
\end{align*}$$

Note that the Fresnel number $u_C$ can also be expressed in terms of dimensionless quantities as follows (remember...
that \( L = 1 \):

\[
\frac{u}{2} = \frac{1}{z} \left( \frac{\ell}{w} \right)^2 ,
\]

(14)

which appears to be dependent only on the propagation distance measured in terms of the Rayleigh length and the aperture “size” measured in terms of the Gaussian beam spot-size.

Equation (12) is one of the main result of the present paper. It allows Gaussian sharp-edge diffraction to be formally derived from the plane-wave sharp-edge diffraction provided that the functions \( \psi_G \) and \( \psi_{BDW} \), which have been defined into Eqs. (5) and (6) only for real values of their arguments, can be analytically continued into the complex realm. To this end, it must be stressed how the geometrical interpretation of Fig. 2 concerning the angular integration variable \( \varphi \) in Eqs. (5) and (6) seems to be no longer valid, since the position of the observation point \( P \) is now defined by the complex 2D transverse vector of Eq. (13). A possibility to solve this problem is to express the angle \( \varphi \) through a parametric representation, say \( Q = Q(t) \), of the boundary \( \Gamma \). On again referring to Fig. 2, consider the position of a typical point \( t \) on the curve \( \Gamma \) to be function of a parameter \( t \) ranging within a real interval \( I \). Let \( Q = Q(t) \) denotes such parametrization. Then the transverse vector \( \mathbf{R} = \mathbf{PQ} \) will also be a function of \( t \).

On considering two subsequent positions \( Q \) and \( Q' \) infinitely close in \( t \), as sketched in Fig. 5 elementary geometry gives at once

\[
d\varphi = dt \frac{\mathbf{R} \times \dot{\mathbf{R}}}{\mathbf{R} \cdot \mathbf{R}} ,
\]

(15)

where the dot denotes the derivative with respect to the parameter \( t \) and the cross product should be intended as the sole \( z \)-component, being both vectors \( \mathbf{R} \) and \( \dot{\mathbf{R}} \) purely transverse (i.e., lying on the aperture plane). Substitution from Eq. (15) into Eqs. (5) and (6) gives at once

\[
\psi_G = \frac{1}{2\pi} \int_{\Gamma} dt \frac{\mathbf{R} \times \dot{\mathbf{R}}}{\mathbf{R} \cdot \mathbf{R}} ,
\]

(16)

and

\[
\psi_{BDW} = -\frac{1}{2\pi} \int_{\Gamma} dt \frac{\mathbf{R} \times \dot{\mathbf{R}}}{\mathbf{R} \cdot \mathbf{R}} \exp \left( \frac{iu}{2} \mathbf{R} \cdot \mathbf{R} \right) ,
\]

(17)

respectively. Equations (16) and (17) could be evaluated, in principle, also for those complex values of \( r_C \) and \( u_C \) given in Eq. (13). They are the main result of the present analysis.

An important check about the validity of Eqs. (16) and (17) can be done in the spherical wave limit, which is reached on letting the Gaussian beam spot size to tend to zero. Accordingly, since \( L \rightarrow 0 \), the complex factor into Eq. (13) tends to the real limit \( D/(z + D) \), in agreement with Eqs. (9)-(11). From a mere mathematical point of view, to formally consider complex values of the observation point \( P \) into the definition of \( \mathbf{R} \) could allow the quantity \( \mathbf{R} \cdot \mathbf{R} \) to vanish for some real values of \( t \). Accordingly, the integrand inside Eqs. (16) and (17) would be singular.

In a 1974 paper [18], Otis claimed that a Gaussian beam impinging on a typical sharp-edge aperture \( A \) according to the geometry depicted in Fig. 4 should produce a well-defined geometrical shadow, which is obtained simply by projecting the aperture boundary \( \Gamma \) via a suitable radially symmetric hyperboloid [18, Eq. (46)]. In particular, the transverse shape of the boundary shadow at the typical propagation plane \( z > 0 \) should be a replica of \( \Gamma \) scaled by the following (within our dimensionless units) real factor [18, Eq. (47)]:

\[
\sqrt{\frac{1 + (z + D)^2}{1 + D^2}} .
\]

(18)

It must be noted how Otis’ conjecture should then imply the integral in Eq. (16), once evaluated at complex transverse vectors \( r_C = r \exp(\mathbf{i}\varphi) \), to satisfy the following relation:

\[
\psi_G(r \exp(\mathbf{i}\varphi)) = \begin{cases} 1 & r \in A, \\ 0 & r \notin A. \end{cases}
\]

(19)

For circular apertures such conjecture can rigorously be proved. To this aim, consider a circular aperture of radius \( a \), centred on the \( z \)-axis. On letting \( \ell = a \), a simple parametrization of the aperture boundary \( \Gamma \) is

\[
Q(t) = (\cos t, \sin t), \quad t \in [0, 2\pi] .
\]

Moreover, due to the axial symmetry of the problem, it is expected the geometrical field \( \psi_G \) to be a radial function of the (complex) normalized quantity \( \xi \) defined by

\[
\xi = \frac{1 + iD}{1 + i(z + D)} a .
\]

(20)

On expressing Eq. (16) through Cartesian coordinates, it is easily found that

\[
\psi_G(\xi) = \frac{1}{\pi} \int_0^\pi dt \frac{1 - \xi \cos t}{1 + \xi^2 - 2\xi \cos t} , \quad \xi \in \mathbb{C} ,
\]

(21)
and in Appendix A it is shown that
\[
\frac{1}{\pi} \int_0^{\pi} dt \frac{1 - \xi \cos t}{1 + \xi^2 - 2\xi \cos t} = \begin{cases} 
1 & |\xi| < 1, \\
0 & |\xi| > 1, 
\end{cases}
\]
which definitely proves Eq. (19) in the case of circular apertures.

As far as the BDW wavefield is concerned, the substitution of the circle parametrization into Eq. (17) gives at once the integral which has already been derived in [19] on the basis of the original Miyamoto-Wolf theory. To give an idea about the fact that Eqs. (16) and (17) provide meaningful quantities when applied to the study of the Gaussian diffraction by a circular aperture, in Fig. 6 it is shown the transverse field distribution of the wavefield produced, at a propagation distance of one Rayleigh length \( z = 1 \), via the diffraction of a Gaussian beam by a circular aperture placed at the beam waist plane \( D = 0 \) and having the radius coincident with the spot size \( \alpha = 1 \). An identical situation was considered long ago by Takenaka et al. in a paper [20] where asymptotic estimates of the diffracted wavefield were obtained starting from the original Miyamoto-Wolf theory. In particular, Fig. 6 should be compared to Fig. 4 of [20]. The agreement is perfect.

![Field distribution](image.png)

**FIG. 6:** Behaviour of the transverse field distribution of the wavefield produced, at a propagation distance of one Rayleigh length \( z = 1 \), via the diffraction of a Gaussian beam by a circular aperture placed at the beam waist plane \( D = 0 \) and having the radius coincident with the spot size \( \alpha = 1 \). The figure should be compared to Fig. 4 of [20].

The results obtained so far would seem to confirm the Otis conjecture about the geometrical interpretation of the wavefield \( \psi_G \) in Eq. (19). However, things are considerably more cumbersome as they could appear at first sight. Consider a typical aperture \( \Gamma \), sketched in Fig. 7 and let \( Q(t) = [X(t), Y(t)] \) be a suitable parametrization of \( \Gamma \). We ask when the scalar product \( \mathbf{R} \cdot \mathbf{R} \) vanishes on letting the observation point \( P = (\xi, \eta) \) to attain complex values of its coordinates \( \xi \) and \( \eta \) according to Eq. (19). To this end, it should be noted that the integrand singularities are the real solution of the equation \( \mathbf{R} \cdot \mathbf{R} = 0 \), i.e.,
\[
[X(t) - \xi]^2 + [Y(t) - \eta]^2 = 0.
\]
These solutions can formally be written through the implicit form
\[
X(t) - \xi = \pm i[Y(t) - \eta],
\]
which immediately leads to the following necessary condition:
\[
X^2(t) + Y^2(t) = |\xi|^2 + |\eta|^2.
\]
Equation (25) has a clear and simple geometrical interpretation, which is depicted in Fig. 7: circles \( \gamma_m \) and \( \gamma_M \) are both centred on the \( z \)-axis (the Gaussian beam mean propagation direction). The former is the entirely made by points inside \( \Gamma \). On the contrary, the latter is the smallest circle centred on \( z \) which is entirely made by points outside \( \Gamma \). For all observation points between \( \gamma_m \) and \( \gamma_M \), Eq. (25) admits at least one real solution (for example points \( Q_1 \) and \( Q_2 \) in Fig. 7). Figure 7 clearly shows what is the main drawback within a general scenario: the mismatch between the axial symmetry of the incident Gaussian beam and the shape of the diffracting aperture which, apart from the unique case of a coaxial circular hole, cannot share the Gaussian axial symmetry at all. This implies that the geometrical wavefield \( \psi_G \) is expected to be identically and rigorously equal to 1, regardless the phase value \( \varphi \), only for the observation points inside the inner circle \( \gamma_m \). At the same time, it is expected \( \psi_G \) to be identically and rigorously equal to 0, regardless the phase value \( \varphi \), only for observation points outside the outer circle \( \gamma_M \). When the point \( P \) is between the two circles, changing the phase \( \varphi \) will cause Eq. (23) to be satisfied for some real values of \( t \), thus making both integrals into Eqs. (16) and (17) singular. In other words, the geometrical wavefield is expected to display, for a given \( P \), a series of discontinuities on letting \( \varphi \) to vary.

![Geometrical wavefield](image.png)

**FIG. 7:** Geometrical wavefield \( \psi_G \) for a nonsymmetric aperture.
A simple example that can be deal with in analytical terms is the replacement of the circular aperture by an elliptic aperture centred at the Gaussian beam mean direction. Let \( \epsilon \) be the ellipse eccentricity, so that \( \chi = \sqrt{1+\epsilon^2} > 1 \) will denote the ellipse major half-axis, being still unitary the minor half-axis. For simplicity we shall consider the evaluation of the geometrical wavefield \( \psi_G \) at observation points of the form \( P = (\xi,0) \), i.e., along the ellipse major axis. Let \( \phi \) the phase of \( \xi \). On using the ellipse parametrization \( [X(t),Y(t)] = (\chi \cos t, \sin t) \), with \( t \in [0,2\pi] \), the geometrical wavefield turns out to be

\[
\psi_G(\xi) = \frac{1}{2\pi} \int_0^{2\pi} dt \, \frac{\chi - \xi \cos t}{1 + \xi^2 - 2\xi \chi \cos t + (\chi^2 - 1)\cos^2 t}
\]

From Equation (25) it follows at once that the integrand in Eq. (26) will be singular if

\[
1 < |\xi| < \chi ,
\]

as expected from the above analysis. Moreover, for a given value of \( |\xi| \) it is not difficult to show that the value of the phase \( \phi \) at which the discontinuity occurs can be expressed in analytical terms as follows:

\[
\tan \phi = \frac{\sqrt{\chi^2 - |\xi|^2}}{\chi \sqrt{|\xi|^2 - 1}} , \quad 1 < |\xi| < \chi .
\]

A visual check of Eq. (28) is shown in Fig. 8 where a two-dimensional map of the geometrical wavefield \( \psi_G \) for an elliptic aperture with \( \chi = 2 \), numerically evaluated via Eq. (16) at complex observation points \((|\xi| \exp(i\phi),0)\), is shown. Within the grey region it turns out that \( \psi_G = 1 \), whereas within the black region \( \psi_G = 0 \). The white curve is just Eq. (28). For the geometrical configuration of Fig. 2 it is trivial to show, from Eq. (20), that the geometrical field jumps must occur at values of \( \phi \) given by

\[
\tan \phi = \frac{z}{1 + \delta(x + z)} ,
\]

which, together with Eq. (28), gives the position of the geometrical wavefield discontinuity along the ellipse major axis, say \( \xi \), as follows:

\[
\xi = \chi \sqrt{\frac{1 + (z + D)^2}{1 + D^2}} \left[ \frac{1 + \left( \frac{z}{1 + D(z + D)} \right)^2}{1 + \chi^2 \left( \frac{z}{1 + D(z + D)} \right)^2} \right].
\]

To numerically check Eq. (30), in Fig. 9 the field amplitudes of the geometrical (dots), the BDW (open circles), and of the total (geometrical plus BDW) diffracted field (solid curve), are shown as functions of the normalized abscissa \( \xi \) along the major axis of an ellipse with \( \chi = 3/2 \), \( \alpha = 1 \), \( z = 1 \), and \( D = 3 \). It can appreciated the discontinuities of both the geometrical and the BDW wavefields at the value \( \xi \simeq 2.24 \) theoretically predicted by Eq. (30), whereas the total field turns out to be continuous. In the same figure the Otis conjecture about the geometrical shadow boundary (occurring at \( \xi \simeq 2.37 \)) is also shown (vertical dashed line).

IV. GAUSSIAN BEAM DIFFRACTION FROM MISALIGNED AND TITLED CIRCULAR APERTURES AND PLATES

The present section is devoted to illustrate a significant practical application of the theoretical treatment carried out in the previous sections. It is worth starting from some of the beautiful results obtained by Coulson and Becknell in [25, 26] where, in order to experimen-
tally investigate plane-wave diffraction by elliptic opaque plates, suitably titled circular plates were then employed. Such a scenario is depicted in Fig. 10, where the symbol $\theta$ denotes the tilting angle and where the center $C$ of the circular aperture has also been placed off-axis with respect to the Gaussian beam mean propagation axis $z$ to account for misalignments. For small values of the tilting angle $\theta$, the circular aperture can then approximately be viewed from the Gaussian beam waist plane as an off-axis elliptic aperture. The study of the effects of misalignments of a limiting aperture on the transmitted wavefield plays a role of pivotal importance, for instance, for the design of laser communication systems, as it was recently pointed out in [27]. It should be stressed again how the theoretical approach carried out in the present paper allows, in principle, to deal with arbitrarily shaped and displaced apertures. To this end, the implementative easiness of Eqs. (16) and (17) will first be tested by reproducing some of the results obtained in a 1969 paper by Pearson et al., who investigated (theoretically and experimentally) the paraxial diffraction of a Gaussian beam by a semi-infinite edge [28].

![FIG. 10: Misalignement of a circular aperture (or plate).](image)

In Fig. 11 the geometry employed for reproducing the results of [28] is shown. The circular aperture is placed in such a way the beam axis is passing through its edge. A transverse Cartesian reference frame $Oxy$ has been introduced with the $x$-axis passing through the aperture centre $C$. The experimental data we have chosen to reproduce are those shown in Figs. 5 and 6 of [28], whose physical parameters are listed in Tab. I. The beam wavelength is $\lambda = 6328$ Å.

| Parameter | Fig. 5 | Fig. 6 |
|-----------|--------|--------|
| $L$       | 776 (cm) | 96 (cm) |
| $D$       | 1480 (cm) | 101 (cm) |
| $z$       | 100 (cm) | 100 (cm) |

TABLE I: Physical parameters used to produce the results in Figs. 5 and 6 of [28]. The wavelength is $\lambda = 6328$ Å

![FIG. 11: Geometry for reproducing the results of [28].](image)

In Fig. 12, the optical intensity distribution $|\psi_G + \psi_{BDW}|^2$ is plotted as a function of the normalized variable $x/a$ for $\alpha = 5$ (open circles), 10 (dots), and 50 (open squares), together with the exact analytical expression for the infinite edge found in [28], for the experimental values of the parameters given in the second column of Tab. I.

![FIG. 12: Behaviour of the optical intensity distribution $|\psi_G + \psi_{BDW}|^2$ as a function of the normalized variable $x/a$ for $\alpha = 5$ (open circles), 10 (dots), and 50 (open squares), together with the exact analytical expression for the infinite edge found in [28], for the experimental values of the parameters given in the second column of Tab. I.](image)

In doing the above figures, the integrals in Eqs. (16) and (17) have been numerically evaluated simply on parametrizing the aperture as $(1 + \cos t, \sin t)$, $t \in [0, 2\pi]$. As a second (and final) application, the paraxial theory here developed will be now implement in order to explore the finest details of the diffractive patterns produced, within the geometrical shadow, by tilted opaque circular plates illuminated by collimated Gaussian beams. To this end consider the situation depicted in Fig. 10 where in place of the aperture we consider a circular plate tilted...
by a small angle \( \theta \) and placed for simplicity on-axis with respect the mean propagation distance of a collimated Gaussian beam having the spot size equal to the plate radius, i.e., such that \( \alpha = 1 \), and whose waist plane coincides with the plate plane. We shall explore the diffraction patterns close to the beam axis \( z \), where the geometrical wavefield \( \psi_G \) is expected to be null. Let us start with a perfectly aligned circular plate. In the case of a plane-wave illumination, it is well known the diffracted pattern at any transverse plane to display a central bright spot, the celebrated Arago (or Poisson) spot. However, when the impinging field is Gaussianly shaped, the axial spot becomes darker and darker on increasing the propagation distance \( z \). This can be viewed on analytically evaluating the BDW field \( \psi_{BDW} \) on-axis through Eq. (17). Since the Gaussian waist is at the plate plane, i.e., \( D = 0 \), Eq. (17) gives at once the following on-axis intensity distribution:

\[
|\psi_{BDW}|^2 = \frac{\exp(-2\alpha^2 z)}{1 + z^2},
\]

which, of course, becomes unitary in the plane-wave limit \( L \to \infty \). According to catastrophe optics \[15\], the bright axial spot represents a highly unstable field configuration which is made possible only by the perfect axial symmetry of the system composed by the diffracting plate and the illuminating wavefield. Accordingly, even a weak perturbation of such symmetry would be sufficient to produce dramatical topological changes on the resulting diffractive patterns. To visually appreciate these changes induced by the axial symmetry breaking, in Fig. 14 two-dimensional maps of the intensity of the diffracted wavefield close to the \( z \)-axis are shown for a tilted circular plate placed at the waist plane \( (D = 0) \) of a Gaussian beam having \( \alpha = 1 \). The diffraction patterns are generated at the normalized propagation distance \( z = 1/20 \), for \( \theta = 0 \) (a), \( \theta = \pi/10 \) (b), \( \theta = \pi/8 \) (c), and \( \theta = \pi/6 \) (d).

Figure 14a is nothing but a blow-up of the Gaussian Poisson spot produced by the non tilted plate. For non-experts of catastrophe optics it would be far from being trivial to appreciate the topological instability of the Poisson spot, which is led to “explode” into a stable, cusp-shaped configuration even by a small perturbation of the diffraction setup. To this end, in Fig. 15 the two-dimensional map of a bigger portion of the transverse diffractive pattern generated for the tilting angle \( \theta = \pi/6 \) is shown at \( z = 500/35333 \) for a Gaussian beam with \( \alpha = 1 \). The chosen value of \( z \) corresponds to a Fresnel number \( u \) given by 12\( \pi \). This has been done to allow a direct comparison with Fig. 10 of [29], where the plane-wave diffraction by an identically elliptic plate was numerically investigated to reproduce the experimental results shown in Figs. 11 of [26]. It is interesting to note how also for the more realistic Gaussian illumination the BDW wavefield tends to focus onto the cusp-shaped geometrical evolute of the elliptic boundary (the white solid curve), in agreement with the theoretical general prescriptions provided by catastrophe optics \[13\].

V. CONCLUSIONS

Diffraction theory is a milestone of classical optics since more than two centuries. However, in the last few years an unexpectedly renewed interest in new, still unexplored
FIG. 15: Two-dimensional map of the transverse diffractive pattern generated for a tilting angle $\theta = \pi/6$ at $z = 500/35333$ for a Gaussian beam with $\alpha = 1$. The chosen value of $z$ corresponds to a Fresnel number $u$ given by $12\pi$. This has been done to allow a direct comparison with Fig. 10 of [29], where the plane-wave diffraction by an identical elliptic plate was numerically investigated to reproduce the experimental results shown in Figs. 11 of [29]. The white solid curve represents the cusp-shaped geometrical evolute of the elliptic boundary.

aspects of sharp-edge diffraction is grown [23, 24, 30–33]. The recently revisitation of paraxial sharp-edge diffraction developed in [12, 14] has here been employed to propose a paraxial version of Miyamoto-Wolf’s theory for exploring the light diffraction produced by arbitrarily shaped planar apertures (or plates) under Gaussian beam illumination. On invoking the paraxial approximation from the beginning, the mathematical formulation considerably simplifies with respect the theory originally developed in [17]. In particular, both the geometrical and the BDW components of the total diffracted wavefield turn out to be expressed through one-dimensional complex integrals whose practical implementation and numerical evaluation issues are reasonably independent of the aperture (or plate) shape. As a consequence, the present approach could constitute an agile and effective general purpose computational platform to deal with a broad spectrum of different scenarios. Gaussian diffraction from highly nonsymmetric apertures as well as the prediction of light behaviour by image formation systems under realistic conditions of illumination, are only a couple of interesting applicative perspectives.

From a purely theoretical point of view, the analysis carried out on the geometrical shadows produced on illuminating sharp-edge apertures by Gaussian beams seems to confute some important and definitive conclusions conjectured in the past [18]. In particular, we have rigorously proved that the shape of the geometrical shadow turns out to be a perfect scaled replica of the diffracting aperture only for circular holes. For differently shaped apertures it is no longer possible to predict the exact shape of the boundary, even though some numerical experiments seem to confirm that the geometrical component of the diffracted wavefield achieves binary values also under Gaussian illumination. Presently we do not possess a rigorous and definitive conclusive word about such a very interesting and still open theoretical question.

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Appendix A: Proof of Eq. (22)

For room reasons, we shall detail only the case $|\xi| < 1$, leaving to the reader to deal with $|\xi| > 1$. Consider first the following integral:

$$J(\xi) = \frac{1}{\pi} \int_0^{\pi} \frac{dt}{1 - \xi \exp(it)},$$

which, after trivial algebra, can be recast as

$$J(\xi) = \frac{1}{\pi} \int_0^{\pi} dt \frac{1 - \xi \cos t}{1 + \xi^2 - 2\xi \cos t}$$

$$+ \frac{i\xi}{\pi} \int_0^{\pi} dt \frac{\sin t}{1 + \xi^2 - 2\xi \cos t},$$

and where the second integral can be evaluated elementarily, so to have

$$J(\xi) = \frac{1}{\pi} \int_0^{\pi} dt \frac{1 - \xi \cos t}{1 + \xi^2 - 2\xi \cos t} + \frac{i}{\pi} \log \frac{1 + \xi}{1 - \xi}. \quad (A3)$$

Consider now Eq. (A1) in which, due to the fact that $|\xi| < 1$, the integrand can be expanded as a geometric series,

$$\frac{1}{1 - \xi \exp(it)} = \sum_{k=0}^{\infty} \xi^k \exp(ikt). \quad (A4)$$

On substituting from Eq. (A4) into Eq. (A1) and after changing the series with the integral we have

$$J(\xi) = \sum_{k=0}^{\infty} \xi^k \frac{1}{\pi} \int_0^{\pi} dt \exp(ikt), \quad (A5)$$

which, on taking into account that

$$\frac{1}{\pi} \int_0^{\pi} dt \exp(ikt) = \begin{cases} 1, & k = 0, \\ \frac{i}{k\pi} [1 - (-1)^k], & k \neq 0, \end{cases} \quad (A6)$$
eventually gives
\[ J(\xi) = 1 + \frac{2i}{\pi} \sum_{k=0}^{\infty} \frac{\xi^{2k+1}}{2k+1} = \]
\[ = 1 + \frac{i}{\pi} \log \frac{1 - \xi}{1 + \xi}. \]  
\hspace{0.5cm} (A7)

On comparing Eqs. (A2) and (A7), it then follows at once
\[ \frac{1}{\pi} \int_0^\pi dt \frac{1 - \xi \cos t}{1 + \xi^2 - 2\xi \cos t} = 1, \quad |\xi| < 1. \]  
\hspace{0.5cm} (A8)

To deal with the case \( |\xi| > 1 \), it is sufficient to recast Eq. (A1) as follows:
\[ J(\xi) = 1 - \frac{1}{\pi} \int_0^\pi \frac{dt}{1 - \xi^{-1} \exp(-it)}, \]  
\hspace{0.5cm} (A9)

and to apply again the above procedure. In this way it is not difficult to prove that
\[ \frac{1}{\pi} \int_0^\pi dt \frac{1 - \xi \cos t}{1 + \xi^2 - 2\xi \cos t} = 0, \quad |\xi| > 1, \]  
\hspace{0.5cm} (A10)

which, together with Eq. (A8), completes the proof of Eq. (22).

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