Structure of Aristotelian Electrodynamics

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Aristotelian electrodynamics (AE) describes the regime of a plasma with a very strong electric field that is not shorted out, with charge current determined completely by pair production and the balance of Lorentz 4-force against curvature radiation reaction. Here it is shown how the principal null directions and associated eigenvalues of the field tensor govern AE, and how force-free electrodynamics arises smoothly from AE when the eigenvalues (and therefore the electric field in some frame) vanish. A criterion for validity of AE and force-free electrodynamics is proposed in terms of a pair of “field curvature scalars” formed from the first derivative of the principal null directions.

I. INTRODUCTION

Gruzinov has proposed that force-free electrodynamics is inadequate for describing some relativistic magnetospheres, because of insufficient pair supply to short out the induced electric fields. In particular he argued that this is the case for “weak pulsars” [1]. In this case, parts of the magnetosphere electric fields. In particular he argued that this is the case for inadequate for describing some relativistic magnetospheres, Gruzinov called the charges in certain quantities. Together they comprise what both degenerate and non-degenerate regions. Both approximations result from neglecting of the mass of the tem that includes both degenerate and non-degenerate regions. The directions of these null eigenvectors are called the principal null directions (PNDs) of the field [9]. The eigenvalue ±E0 is the electric field in a frame in which the electric and magnetic fields are parallel or one of them vanishes. In AE, |E0| ̸= 0, and positive charges travel along one PND while negative charges travel along the other. In FFE, E0 = 0, and the total current 4-vector also lies in the timelike plane spanned by the two PNDs. For null fields there is only one null eigenvector, with vanishing eigenvalue.

In this paper I reformulate AE in a relativistically covariant form, and aim to elucidate the structure of the theory and its interface with force-free electrodynamics (FFE). The key observation is the central role played by the principal null directions of the electromagnetic field. Beyond that there is nothing new here. The covariant form also applies in curved spacetime, so would be particularly useful in systems containing a black hole. The spacetime signature is (+−−−) and the speed of light is sometimes set to c = 1.

II. STRUCTURE OF ELECTROMAGNETIC FIELDS

A non-null electromagnetic field Fαμ at a point has two null eigenvectors k±μ, with opposite eigenvalues,

\[ F^a_{\mu} k^b_{\pm} = \pm E_0 k^a_{\pm}. \tag{1} \]

The directions of these null eigenvectors are the principal null directions (PNDs) of the field [9]. The eigenvalue ±E0 is the electric field in a frame in which the electric and magnetic fields are parallel or one of them vanishes. In AE, |E0| ̸= 0, and positive charges travel along one PND while negative charges travel along the other. In FFE, E0 = 0, and the total current 4-vector also lies in the timelike plane spanned by the two PNDs. For null fields there is only one null eigenvector, with vanishing eigenvalue.

A simple way to understand this structure is to observe that, at a given spacetime point, any electromagnetic field 2-form F can be presented in one of two canonical ways in terms of an adapted orthonormal set of 1-forms \{dt, dx, dy, dz\}:

\[ F^{\text{generic}} = E_0 \, dz \wedge dt + B_0 \, dx \wedge dy, \tag{2} \]

\[ F^{\text{null}} = F_0 \, dz \wedge (dt - dx). \tag{3} \]

The field in (2) has two null eigenvectors, k± = ∂t ± ∂z, with eigenvalues ±E0. The field in (3) has one null eigenvector, ∂t + ∂z, with vanishing eigenvalue. The electric and magnetic fields in (2) are parallel in the Lorentz frame defined by ∂t. The general statement is that they are parallel in any frame lying in the timelike plane spanned by k+. If the field is degenerate, i.e. if F ∧ F = 0 (F · B = 0), then either E0 = 0 or B0 = 0; that is, either E or B will vanish in these “field eigenframes”. If field is null then E and B are perpendicular and have the same magnitude in all frames.

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the spinor factors (or eigenspinors) of the electromagnetic terms of unit vectors $v$ respectively. The PNDs can be specified by a pair of spatial unit vectors $v_\pm$ in a given frame via $k_\pm^a \leftrightarrow (1, v_\pm)$. Gruzinov gives an explicit expression for $v_\pm$ in terms of the electric and magnetic fields:

$$v_\pm = \frac{E \times B \pm (E_0 E + B_0 B)}{E_0^2 + B_0^2}.$$  

[The denominator can also be written more symmetrically as $\frac{1}{2}(E_0^2 + B_0^2 + E^2 + B^2).]

The PNDs can be constructed as quadratic expressions from the spinor factors (or eigenspinors) of the electromagnetic spinor [9]. The Newman-Penrose formalism or other spinor techniques might therefore be useful for analytical and/or numerical studies of AE.

The set of fields modulo Lorentz transformations can be represented as the upper half plane with vertical axis $E_0$ and horizontal axis $B_0$ (see Fig. 1). Degenerate fields have either $E_0 = 0$ or $B_0 = 0$, while null fields have $E_0 = B_0 = 0$. When $E_0 = 0$, the two values $\pm B_0$ label the same set of fields. The moduli space of fields is thus the cone obtained by identifying the positive-$B_0$ half of the dashed line in Fig. 1 with the negative half; the null fields lie at the vertex, and the magnetic and electric degenerate fields correspond to a pair of opposite rays on the cone.

The current 4-vector lies in the plane spanned by the PNDs in both the AE and FFE regimes, though it is determined by different conditions. The distinguishing factor between these regimes is whether the PND eigenvalue is nonzero or zero. In the following we characterize these two regimes.

### III. FORCE-FREE ELECTRODYNAMICS

If $E_0 = 0$, then the field is magnetic or null degenerate. If the field is also strong enough, the inertia of the charges can be neglected, which means that the 4-force on the current can be set to zero,

$$F_{ab}j^b = 0. \quad (7)$$

This condition in turn implies that the field is degenerate. In the magnetic case, the FF current must be a linear combination of the vectors $k^a_{\pm}$, since the latter span the kernel (i.e., the null eigenspace) of $F_{ab}$. In the null case the PNDs coincide, the kernel of $F_{ab}$ is spanned by the unique PND and an orthogonal spacelike direction, and the current must be a linear combination of vectors in those directions. If there is only one sign of charge present, the current cannot be spacelike, so in the case of null fields it must be null.

In either case, somewhat surprisingly, the current is determined by the fields at one time when (7) and Maxwell’s equations are imposed. One therefore has stand-alone evolution equations for magnetically dominated or null force-free fields, without reference to the charge or current densities. In the magnetic case the initial value problem for these equations is well-posed. [10–12].

When is the magnetic field “strong enough” for FFE to apply? One requirement is clearly that the density of charged particles should be low enough to neglect their energy and momentum. Another requirement is that the magnetic field lines not be too curved. Consider a system whose magnetic field lines have radius of curvature $R$ reckoned in an instantaneous field eigenframe. For the current to remain parallel to the magnetic field it must be that during one electron gyroperiod, $\tau_c = \omega_c^{-1} = mc/eB_0$, an electron travels along the field lines a distance small compared to $R$. Allowing the maximum speed $c$, this requires

$$B_0 \gg mc^2/eR. \quad (8)$$

This condition appears simple, but since the curvature radius must be reckoned in the field eigenframe at each point, it hides some complexity.

It would be nice to have an invariant method of computing $R$, given the PND’s. A plausible candidate exists, and is

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1 An electromagnetic field can be represented by the trace-free $2 \times 2$ matrix $\Phi = (E + iB) \cdot \sigma$. The Lorenz group acts by similarity transformation on $\Phi$, so the eigenvalues of $\Phi$ are Lorentz invariant. The properties of the Pauli matrices imply $\Phi^2 = (E + iB) \cdot (E + iB) = (E_0 + iB_0)^2 I$, so the eigenvalues are $\pm (E_0 + iB_0)$. The eigenspinors $\lambda_\pm$ are the square roots of the PNDs of $F_{ab}$, in the sense that $k_\pm^a \leftrightarrow (1, \lambda_\pm \sigma \lambda_\pm)$, with the normalization $\lambda_+^\dagger \lambda_\pm = 1$.

2 I am grateful to Antony Speranza for suggesting that the the invariants (9) might supply the relevant notion of curvature length scale in this context.
rather simple:

\[ R_\pm^{-1} = \frac{\left| (k_\pm \cdot \nabla) k_\pm \right|}{|k_+ \cdot k_-|/2}. \]  

(9)

These “field curvature scalars” have dimensions of inverse length and are invariant under arbitrary rescalings of \( k_\pm \). The two radii \( R_\pm \) coincide if the PNDs are surface forming. For static electric or magnetic fields they coincide and are equal to the curvature of the field lines. More generally, they measure some sort of rate at which one eigendirection bends away in the orthogonal spacelike direction when moving along the other PND. They are equal to twice the magnitudes of the spin coefficients \( \pi \) and \( \tau \) associated with a null tetrad constructed from the PNDs [9].

IV. ARISTOTELIAN ELECTRODYNAMICS

For any field with \( E_0 \neq 0 \), i.e. which is not magnetic or null degenerate, the electric field can accelerate charges without deflection along the spatial directions of the PNDs. In a strong field the charges accelerate until they radiate energy at the same rate as the Lorentz force supplies it to them. The power input is \( \sim eE_0c \), reckoned in a field eigenframe in which the charge in the frame \( c \approx c \), while a charge moving with large Lorentz \( \gamma \) factor on a path following a trajectory with radius of curvature \( R \) emits curvature radiation with power \( \frac{e^2 \gamma^2 c}{2} R^2 \) [13]. Equating the input and output power yields

\[ \gamma = \left( \frac{3E_0 R^2}{2e c} \right)^{1/4}. \]  

The 4-velocity of the charge is nearly parallel to the principal null vector \( k_a^\pm \) if the charge is positive, and to \( k_0^a \) if the charge is negative.

When does AE apply? The electric field should be strong enough that it is a good approximation to treat the current of the charges as running parallel to the principal null directions. If the field is approximately static, it is presumably necessary that the 4-velocity is “nearly lightlike” as reckoned in the approximately static frame, and that the conditions for reaching terminal velocity have been met. Let us use units with \( c = mc^2 = 1 \) for a moment. The conditions just stated are that the terminal gamma factor must be relativistic, say \( (E_0 R^2)^{1/4} > 2 \), and that the particles reach that terminal velocity. This latter condition requires that the voltage drop over a distance \( \sim R \), reckoned in an instantaneous field eigenframe, be greater than the the terminal gamma factor times the rest energy of the electron, \( E_0 R > (E_0 R^2)^{1/4} \), i.e. \( E_0 R^{2/3} > 1 \). The former condition is then automatically met in any relevant situation, since then \( R \gg R_e = 1 \) (the classical electron radius \( e^2/mc^2 \) in these units). This reasoning suggests that the condition of AE applicability is

\[ E_0 > R^{-2/3}. \]  

(10)

Note that the range of applicability grows as \( R \) grows (see Fig. 1). We conjecture that the field curvature scalars (9) provide the correct general interpretation of the length scale \( R \) in (10).

Given a unit timelike vector \( u^a \), we can normalize the principal null vectors by the condition \( k_a^\pm u_a = 1 \). Then, because the charge 4-velocities are parallel to \( k_\pm^a \) in the AE regime, the current in a region where \( E_0 \neq 0 \) takes the form

\[ j^a = \rho_+ k_+^a - \rho_- k_-^a, \]  

(11)

where \( \rho_+ \) and \( -\rho_- \) are the densities of positive and negative charge in the frame \( u^a \). Using the eigenvector property (1), the Lorentz 4-force density acting on the current (11) is

\[ F_{ab} = E_0(\rho_+ k_+^a + \rho_- k_-^a). \]  

(12)

The power deposited into the charges in the frame \( u^a \) is

\[ E_0(\rho_+ + \rho_-). \]

The charge density is determined by \( F_{ab} \) without time derivatives via Gauss’ law, \( \rho = \nabla \cdot E \). Since the current 4-vector \( j^a \) (11) lies in the plane spanned by the two null eigenvalues, the remaining freedom in \( j^a \) is only one function. If all the charges in a given region have the same sign, then the current is null and thus fully determined. If instead both signs of charge are present (nonzero pair multiplicity), the charge densities are determined by pair production and subsequent propagation subject to the continuity equations,

\[ \nabla_a(\rho_+ k_+^a) = \nabla_a(\rho_- k_-^a) = \Gamma. \]  

(13)

The pair creation rate \( \Gamma \) depends on \( E_0 \) and the photon density. Once the form of \( \Gamma \) is specified, Maxwell’s equations together with (11), (13), and initial values for \( \rho_\pm \) determine the time derivatives in terms of the field and charge densities at a given time, so are naively deterministic. (Whether they define a well-posed initial value problem has not been examined, although Gruzinov has evolved them numerically and the solutions seem to behave well [1, 14].)

A. Example: Gruzinov’s Device

To illustrate the workings of AE/FFE, Gruzinov considered an arrangement in two spatial dimensions where FFE and AE regimes coexist. He called this the “Device” [15]. In the Device, opposing FFE Poynting fluxes collide in an AE region and the energy is converted to curvature radiation. He found a stationary solution that transitions from a force-free zone to an AE radiation zone, in analogy with the transition outside a weak pulsar, without any discontinuity. Here I use the Stationary Device to illustrate the formulation given above.

The stationary field can be expressed as

\[ F = E(y)dy \wedge [dt + \beta(x)dx], \]  

(14)
corresponding to an electric field $E$ in the $y$ direction and a magnetic field $-\beta E$ in the $z$ direction. $F$ is a simple 2-form (as in any electromagnetic field in 2+1 dimensions), so it is degenerate. That is, one or both of $E_0$ and $B_0$ vanish.

For $\beta^2 < 1$ the field is electric, so it is $B_0$ that vanishes. There are two PNDs,

$$k_\pm = \partial_t - \beta \partial_x \pm \sqrt{1 - \beta^2} \partial_y,$$

with eigenvalues

$$E_0 = \pm E \sqrt{1 - \beta^2}. \quad (16)$$

The frames in which the magnetic field vanishes are those in the $k_+ k_-$ plane. From (15) we see that these have $x$-coordinate velocity $-\beta$ and any $y$-coordinate velocity with magnitude less than or equal to $\sqrt{1 - \beta^2}$. A positive charge moving in the direction of the null vector $k_+$ feels a Lorentz force proportional to $k_+$. If $\beta = 0$ the spatial direction of such motion is just that of the electric field, $\partial_y$, while if $\beta$ is nonzero there is also a $\partial_x$ component.

For $\beta^2 > 1$ the field is null, and the two PNDs coalesce into a single PND with vanishing eigenvalue. In this case the field takes the form $F = E(y) dy \wedge (dx \pm dt)$, which is a solution of the force-free equations. In fact it has the form of the simple class of null Poynting flux solutions discussed in [5]. For $\beta^2 > 1$ the field is magnetic, but it is not a force-free solution for any nontrivial choice of the functions $E$ and $\beta$, so that case plays no role here.

Let us now consider the AE field equations in the electric case $\beta^2 < 1$. First, the field satisfies Faraday’s law, $dF = 0$, by inspection. The remaining AE equations require that (i) the current computed from $F$ according to Maxwell’s equations is equal to (11) and (ii) the continuity equation (13) holds. Assuming there is no pair creation ($F = 0$), and that only electrons are present, the continuity equation follows from the Maxwell equation, $\partial_t F^{ab} = -\rho_\gamma k^a$. This equation implies $\rho_\gamma = -E^\gamma$, and $E_0/E = -\beta x/\sqrt{1 - \beta^2}$. Since $E$ depends only on $y$, and $\beta$ depends only on $x$, the general solution is given by

$$E = E(0) \exp(-y/\ell), \quad \beta = \sin(x/\ell), \quad (17)$$

where $\ell$ is a constant length. This solution matches smoothly to the force-free Poynting flux solution with $\beta = \pm 1$ at $x/\ell = \pm \pi/2$. The charges and Poynting flux are thus flowing inwards toward $x = 0$ from both sides.

What about the validity of the approximations for the Device? In the FF zone the particle trajectories are straight lines, so the length scale $R$ in (8) goes to infinity, and hence the FF approximation may plausibly be accurate, despite the fact that $B_0$ vanishes. Where the radiation zone begins, $E_0$ starts out zero (16). According to (10), AE therefore applies at the transition only if $R = \infty$, where $R$ is the radius of curvature in an instantaneous field eigenframe. We proposed that the invariant $R_-$ defined in (9) may capture the relevant curvature radius. The vector fields (15) are surface forming, so $R_+ = R_-$, and we find $R^{\pm 1} = \partial_y (1 - \beta^2)^{-1/2}$. This diverges at $x/\ell = \pi/2$, so $R \rightarrow 0$ there, hence for any $E(0)$ there is a region close to $x/\ell = \pi/2$ where the applicability of AE is questionable. Specifically, using (16), the condition (10) becomes $E(0) e^{-y/\ell} > \ell^{-2/3} \sin(x/\ell)/\cos^2(x/\ell)^{2/3}$. 

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