Optimization of phononic filters via genetic algorithms

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Abstract. A phononic crystal is commonly characterized by its dispersive frequency spectrum. With appropriate spatial distribution of the constituent material phases, spectral stop bands could be generated. Moreover, it is possible to control the number, the width, and the location of these bands within a frequency range of interest. This study aims at exploring the relationship between unit cell configuration and frequency spectrum characteristics. Focusing on 1D layered phononic crystals, and longitudinal wave propagation in the direction normal to the layering, the unit cell features of interest are the number of layers and the material phase and relative thickness of each layer. An evolutionary search for binary- and ternary-phase cell designs exhibiting a series of stop bands at predetermined frequencies is conducted. A specially formulated representation and set of genetic operators that break the symmetries in the problem are developed for this purpose. An array of optimal designs for a range of ratios in Young’s modulus and density are obtained and the corresponding objective values (the degrees to which the resulting bands match the predetermined targets) are examined as a function of these ratios. It is shown that a rather complex filtering objective could be met with a high degree of success. Structures composed of the designed phononic crystals are excellent candidates for use in a wide range of applications including sound and vibration filtering.

1. Introduction

In phononic crystals, destructive and constructive wave interference mechanisms prevail at characteristic frequency ranges known, respectively, as stop bands (or band gaps) and pass bands. Within stop bands, a medium effectively prohibits wave propagation, and the converse takes place at pass band frequencies. These bands are commonly identified through a frequency spectrum diagram which is a frequency versus wavenumber plot. The topological distribution of constituent material phases within a unit cell directly affects the location and width of stop bands and pass bands in the frequency domain. This provides an opportunity for designing composite structures (formed from these phononic materials) with unique dynamical characteristics that can be utilized for a wide range of applications including sound and vibration filtering.

Utilization of stop-band and pass-band characteristics for material design has been considered in the literature [1-5]. In the study of Hussein et al. [3], a multi-objective genetic algorithm (GA) based on two alternative formulations, zero-one integer and mixed, was used. One objective was to maximize a dynamical performance metric (e.g., wave attenuation) and the other was to minimize the
number of unit cell layers or maximize the dynamical performance of another competing metric. Although the inherent symmetry in the unit cell design problem was addressed, it was not fully capitalized on. In the work of El-Beltagy and Hussein [5], the same problem was revisited with the introduction of a new scale-preserving and repair-free representation that fully utilizes the symmetry of the periodic layered unit cell. Furthermore, multi-objective optimization was replaced by a single objective search for a given number of layers (spanning several values).

In this paper we focus on one of the applications studied in [3], that is, the GA optimization of a unit cell for a target predetermined series of stop bands and pass bands. The resulting phononic crystal can therefore act as multi-band frequency filter. As in [3], attention is narrowed to longitudinal wave propagation in the direction normal to the layering. However, in this work, a range of Young’s modulus ratios and density ratios for up to three material phases are considered, thus adding a material selection component to the design study.

2. Frequency band structure computation
The governing equation of motion for longitudinal wave propagation in a 1D heterogeneous solid is

$$\rho \ddot{u} = \sigma_{xx} + f,$$  

(1)

where \( \rho = \rho(x) \), \( f = f(x,t) \), \( \sigma = \sigma(x,t) \), and \( u = u(x,t) \) denote density, external force, stress, and displacement, respectively. The position coordinate and time are respectively denoted by \( x \) and \( t \). Assuming a linear elastic material, \( \sigma = Eu \), where \( E = E(x) \) is the elastic modulus. We consider a general multi-layered medium where an arbitrary layer \( j \) is positioned between an adjacent layer \( j-1 \) at its left and an adjacent layer \( j+1 \) at its right. The interface between the layers is assumed to be ideal. The \( j \)th layer has thickness \( d^{(j)} \), density \( \rho^{(j)} \), Young’s modulus \( E^{(j)} \), and longitudinal velocity \( c_{p}^{(j)} = (E^{(j)}/\rho^{(j)})^{1/2} \), respectively. The boundary conditions that must be satisfied at the layer interfaces are (i) continuity of the displacement \( u \) and (ii) continuity of the stress \( \sigma \). Furthermore, no external forces are permitted, i.e., \( f = 0 \) in equation (1).

The solution of Eq. (1) in the \( j \)th layer can be written as a superposition of forward and backward travelling waves with harmonic time dependence:

$$u(x,t) = \left[ A_{+}^{(j)} e^{ik_{+}d^{(j)}x} + A_{-}^{(j)} e^{-ik_{+}d^{(j)}x} \right] e^{-i\omega t},$$  

(2)

where \( i = \sqrt{-1} \), \( k_{+}^{(j)} = \omega/c_{p}^{(j)} \), \( \omega \) is the temporal frequency, and \( A_{\pm}^{(j)} \) are complex coefficients. Using Floquet’s theory and the transfer matrix method, the following eigenvalue problem is derived:

$$\left[ T - i e^{id} \right] \begin{bmatrix} u \\ \sigma \end{bmatrix}_{j^{\pm}} = 0,$$  

(3)

where \( T = T_{1}T_{n-1}\cdots T_{1} \), and \( T_{j} \), referred to as a transfer matrix, is defined as

$$T_{j} = \begin{bmatrix} \cos(k^{(j)}d^{(j)}) & (1/Z^{(j)}) \sin(k^{(j)}d^{(j)}) \\ -Z^{(j)} \sin(k^{(j)}d^{(j)}) & \cos(k^{(j)}d^{(j)}) \end{bmatrix}.$$  

(4)

In equation (4), \( Z^{(j)} = \rho^{(j)}c_{p}^{(j)}k_{+}^{(j)} \). A unit cell has \( n \) layers and is of thickness \( d = d^{(1)} + d^{(2)} + \ldots + d^{(n)} \). Equation (3) is solved for the dispersion curves (\( \omega \) versus \( k \)). It is advantageous for these curves to be represented in a non-dimensional form, that is, \( \Omega = \omega d/(E_{n}/\rho_{n})^{1/2} \) for the dimensionless frequency (subscript “n” denotes the low stiffness and low density phase), and \( \xi = \xi_{\text{real}} + i\xi_{\text{imag}} = k \times d \), for the dimensionless wavenumber. More details on this procedure are available in [6] and references within. [7] also treats acoustic phonons in materials with multiple layers in the unit cell.
3. Optimization problem definition and solution method

The objective is to optimize a unit cell for a target predetermined series of stop bands and pass bands. The specific problem studied in [3] Case III/Table 1 is considered. The target is to develop a unit cell with pass bands spanning the frequency ranges $0 < \Omega < 10$, $20 < \Omega < 25$, $35 < \Omega < 40$ and $45 < \Omega < 50$, and stop bands spanning $10 < \Omega < 20$, $25 < \Omega < 35$ and $40 < \Omega < 45$ (see figure 1). The reader is referred to [4] for a formal description of the objective. The optimization method of choice in the present study is genetic algorithms, as in [3,5]. The unit cell design variables are represented by two vectors: $d$, a real vector of the thickness of each unit cell layer, and $t$, an integer vector for the phase contrast for each layer. If we have $m$ material phases, the cardinality of each element in the vector $t$ is $m-1$. More precisely $t^{(i)} \in [0,\ldots,m-1]$ for $i = 1\ldots n-1$. Now we define $\tau^{(i)}$ to be the actual phase type for each layer. Construction of $\tau^{(i)}$ is done uniquely from $t^{(i)}$ using the formula

$$
\tau^{(i)} = \begin{cases} 1 & \text{for } i = 1 \\ (\tau^{(i-1)} + t^{(i-1)}) \mod m + 1 & \text{for } i = 2\ldots n 
\end{cases}
$$

(5)

Without loss of generality, we fix the value of the phase type of the first layer. To prevent having the phase type of the last layer being the same as that of the first layer, we impose the following condition:

- If $\tau^{(n)} = 1$ then $\tau^{(n)} = (\tau^{(n-1)} + (t^{(1)} + 1) \mod (m-1)) \mod m + 1$.  

(6)

A unit cell design can be fully specified by the thickness vector $d$ and the phase contrast vector $t$ in tuple $(d, t)$. As shown in [5], we perform crossover on the parent designs $(d_a, t_a)$ and $(d_b, t_b)$ to generate a new one $(d_c, t_c)$, by randomly selecting one of the thicknesses $d^{(i)}$ and obtaining an intermediate thickness in the new design $d^{(i)}_c = d^{(i)}_a + \alpha(d^{(i)}_b - d^{(i)}_a)$. All layer thicknesses in the new design other than $d^{(i)}_c$ are copied from $d_a$ and scaled in a proportional manner to insure that the new design has a total length of unity. Furthermore, we set $t^{(i)}_c = t^{(i)}_a$ for $i \neq j-1$ and $t^{(i)}_c = t^{(i)}_b$. Mutation proceeds in a similar manner by taking design $(d_a, t_a)$ and mutating it to $(d_c, t_c)$, while preserving both scale and phase order. In a manner similar to the crossover operator, a thickness $d^{(i)}$ is randomly selected and its value modified, whilst also modifying the other thicknesses proportionally in a manner that insures that the mutated design has a total length of unity. The phase of the selected thickness is randomly modified whilst insuring that adjacent layers do not have the same phase. For the GA results presented in the next section, the population size was set to 100 and the number of generations was set to 80. The crossover and the mutation probabilities were taken as 0.9 and 0.02, respectively.

4. Results and conclusion

In the pursuit of obtaining a unit cell design exhibiting the highest possible value of the filtering objective described (and referenced) above, a two-step search process was conducted. The aim is to simultaneously find the best material phase combination and the optimal unit cell layering configuration. The first step involved considering binary material systems and sweeping the GA search through the range of Young's modulus ratios $1 \leq E_2 / E_1 \leq 12$, varying in discrete unit steps, and the range of density ratios, $1 \leq \rho_2 / \rho_1 \leq 12$, also varying in steps of 1. The subscript to $E$ and $\rho$ denote material phase. This sweep totals to 144 combinations, which are effectively 143 GA runs since the case of $E_2 / E_1 = 1$ and $\rho_2 / \rho_1 = 1$ corresponds to a non-dispersive homogenous unit cell. Once an optimal unit cell design was identified, the corresponding $E^{opt\bin}_b = E_2 / E_1$ and $\rho^{opt\bin}_b = \rho_2 / \rho_1$ values for this design were used to define the extreme material property ratios for a ternary system, i.e., $E_3 / E_1 = E^{opt\bin}_b$ and $\rho_3 / \rho_1 = \rho^{opt\bin}_b$. At this point a secondary GA sweep was carried out varying independently the ratio of the properties of the intermediate material phase to that of the low Young's modulus and low density phase, i.e., the ratios $E_2 / E_1$ and $\rho_2 / \rho_1$. This sweep was done in 11 equal discrete steps covering independently the ranges $1 \leq E_2 / E_1 \leq E^{opt\bin}_b$ and $1 \leq \rho_2 / \rho_1 \leq \rho^{opt\bin}_b$, thus providing a total of 119 new ternary unit cell combinations (discounting the case when both $E_2 / E_1$ and $\rho_2 / \rho_1$ are equal to 1 or
the extreme value). The unit cell with the best number and choice of material phases and the best layering configuration (i.e., number and ordering of layers and their thicknesses) emerges as the optimal unit cell design for the stated filtering objective. The frequency band diagram of this design as well as that of the best design obtained in [3] are shown in figure 1. The corresponding unit cell designs are shown in figure 2. It is noteworthy that the optimal unit cell consists of a binary material design with $E_2/E_1 = 6$ and $\rho_2/\rho_1 = 3$, indicating that selecting more phases and/or higher material property contrasts does not necessarily give the best solution for this type of problem.

In conclusion, the results show that the representation and set of GA operators developed provide a powerful tool for optimizing phononic filters for arbitrary ranges of target stop bands and pass bands. Secondly, the optimal material combination is a binary one and does not necessarily consist of the material phases with the highest possible contrast. Finally, it is shown that only a few layers in the unit cell (bottom diagram in figure 2) are sufficient to achieve this filtering function with good precision.

References

[1] Cox S J and Dobson D C 1999 SIAM J. Appl. Math. 59 2108
[2] Burger M, Osher S J and Yablonovitch E 2004 IEICE Trans. Electron. E87C 258
[3] Hussein M I, Hamza K., Hulbert G M, Scott R A and Saitou S 2006 Struct. Multidisc. Optim. 31 60
[4] Lanzillotti-Kimura N D, Fainstein A, Lemaître A and Jusserand B 2006 Appl. Phys. Lett. 88 083113
[5] El-Beltagy M A and Hussein M I Proc. 2006 ASME International Mechanical Eng. Congress and R&D Expo (Chicago, IL, 5-10 Nov. 2006), IMECE2006-15081, CD-ROM, pp. 1
[6] Hussein M I, Hulbert G M and Scott R A 2006 J. Sound Vibration 289 779
[7] Tamura S and Wolfe JP 1988 Phys. Rev. B 38 5610