Asymptotic Results for the Two-parameter Poisson-Dirichlet Distribution

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Abstract

The two-parameter Poisson-Dirichlet distribution is the law of a sequence of decreasing non-negative random variables with total sum one. It can be constructed from stable and Gamma subordinators with the two-parameters, \( \alpha \) and \( \theta \), corresponding to the stable component and Gamma component respectively. The moderate deviation principles are established for the two-parameter Poisson-Dirichlet distribution and the corresponding homozygosity when \( \theta \) approaches infinity, and the large deviation principle is established for the two-parameter Poisson-Dirichlet distribution when both \( \alpha \) and \( \theta \) approach zero.

Key words: Poisson-Dirichlet distribution, two-parameter Poisson-Dirichlet distribution, GEM representation, homozygosity, large deviations, moderate deviations.

AMS 1991 subject classifications: Primary: 60F10; Secondary: 92D10.

1 Introduction

For \( \alpha \) in \((0,1)\) and \( \theta > -\alpha \), let \( U_k, k = 1, 2, \cdots \), be a sequence of independent random variables such that \( U_k \) has \( \text{Beta}(1 - \alpha, \theta + k\alpha) \) distribution. Set

\[
X_{1}^{\alpha,\theta} = U_1, \quad X_{n}^{\alpha,\theta} = (1 - U_1) \cdots (1 - U_{n-1})U_n, \quad n \geq 2.
\]

∗Research supported by the Natural Science and Engineering Research Council of Canada
†Research supported by the NSF of China(No.10571139).
Then with probability one
\[ \sum_{k=1}^{\infty} X_k^{\alpha,\theta} = 1, \]
and the law of \((X_1^{\alpha,\theta}, X_2^{\alpha,\theta}, \cdots)\) is called the two-parameter GEM distribution.

Let \(P(\alpha, \theta) = (P_1(\alpha, \theta), P_2(\alpha, \theta), \cdots)\) denote the descending order statistic of \((X_1^{\alpha,\theta}, X_2^{\alpha,\theta}, \cdots)\). The law of \(P(\alpha, \theta)\) is called the two-parameter Poisson-Dirichlet distribution and is denoted by \(\Pi_{\alpha,\theta}\). The well-known one-parameter Poisson-Dirichlet distribution corresponds to \(\alpha = 0\).

For each integer \(m \geq 2\), taking a random sample of size \(m\) from a population with the two-parameter Poisson-Dirichlet distribution. Given the population proportion \(p = (p_1, p_2, \ldots)\), the probability that all samples are of the same type is given by
\[ H_m(p) = \sum_{i=1}^{\infty} p_i^m, \]
which is referred to as the homozygosity of order \(m\).

The main properties of the two-parameter Poisson-Dirichlet distribution are studied in Pitman and Yor [17] including relations to subordinators, Markov chains, Brownian motion and Brownian bridges. The detailed calculations of moments and parameter estimations were carried out in Carlton [2]. In [6] and the references therein one can find connections between two-parameter Poisson-Dirichlet distribution and models in physics including mean-field spin glasses, random map models, fragmentation, and returns of a random walk to origin. The two-parameter Poisson-Dirichlet distribution has also been used in macroeconomics and finance ([1]).

Many properties of the one-parameter Poisson-Dirichlet distribution have generalizations in the two-parameter setting including but not limited to the sampling formula (cf. [8], [16]), the Markov-Krein identity (cf. [2], [18]), and subordinator representation (cf. [13], [17]). Recently, a large deviation principle (henceforth, LDP)is established in [9] for the two-parameter Poisson-Dirichlet distribution when \(\theta\) goes to infinity. This is a generalization to the LDP result for the one-parameter Poisson-Dirichlet distribution in [3]. Our first result here establishes the corresponding moderate deviation principle (henceforth, MDP). This can be viewed as a generalization of the MDP result in [11] to the two-parameter setting. The MDP for the homozygosity is also established generalizing corresponding result in [11]. In order to apply the Campbell’s theorem, we turn to a representation of the two-parameter Poisson-Dirichlet distribution obtained in [16].

When \(\alpha = 0\), the one-parameter Poisson-Dirichlet distribution converges to \(\delta_{(1,0,\ldots)}\) as \(\theta\) goes to zero. The corresponding LDP is established in [10] where a structure called “energy ladder” is revealed. Our second main result generalizes this result to the two-parameter Poisson-Dirichlet distribution when both \(\alpha\) and \(\theta\) go to zero. It turns out that the large deviation speed will depend on \(\alpha\) if it converges to zero at a slower speed than that of \(\theta\).

The current paper is organized as follows. Distributional results are derived in Section 2 using the change of measure formula and the subordinator representation. Section 3 is dedicated to
establishing the MDP for $\Pi_{\alpha, \theta}$ when $\theta$ goes to infinity. The large $\theta$ MDP for the homozygosity is established in Section 4. In Section 5 we prove the LDP for $\Pi_{\alpha, \theta}$ when both $\alpha$ and $\theta$ go to zero.

2 Marginal Distributions

In this section, we derive the marginal distributions of the two-parameter Poisson-Dirichlet distribution. The basic tools are the change of measure formula and the subordinator representation. For general concepts and theorems on MDP and LDP, we will refer to [5].

From now on, the parameter $\theta$ will be assumed to be positive and $\alpha$ is in $(0, 1)$. Let $\{\rho_s, s \geq 0\}$ be a subordinator, an increasing process with stationary independent increment, with no drift component. The Laplace transform of $\rho_s$ is then given by

$$E(\exp(-\lambda \rho_s)) = \exp\left\{s \int_0^{\infty} (e^{-\lambda x} - 1) \Lambda(dx)\right\}, \lambda \geq 0,$$

(2.1)

where $\Lambda$ is the Lévy measure on $(0, +\infty)$ describing the distribution of the jump sizes. Let $V_1(\rho_s) \geq V_2(\rho_s) \geq \cdots$ denote the jump sizes of $\{\rho_s, s \geq 0\}$ over $[0, s)$ in decreasing order.

If

$$\Lambda(dx) = c_\alpha x^{-(1+\alpha)} dx.$$

for some $c_\alpha > 0$, then the subordinator is called a stable subordinator with index $\alpha$ and is denoted by $\{\tau_s, s \geq 0\}$. Without loss of generality, we choose $c_\alpha = \frac{\alpha}{\Gamma(1-\alpha)}$ in this paper.

The next result is from [17].

**Proposition 2.1** (Pitman and Yor). Let $\{\sigma_s : s \geq 0\}$ and $\{\gamma_s : s \geq 0\}$ be two independent subordinators with respective Lévy measures $\alpha C x^{-(\alpha+1)} e^{-x} dx$ and $x^{-1} e^{-x} dx$ for some $C > 0$. Let

$$\zeta(\alpha, \theta) = \frac{\gamma/\alpha}{\theta \Gamma(1-\alpha)}.$$

Then $T = T(\alpha, \theta) = \sigma_{\zeta(\alpha, \theta)}$, and

$$\left(\frac{V_1(T)}{T}, \frac{V_2(T)}{T}, \cdots\right)$$

are independent with respective laws the Gamma($\theta, 1$) distribution and the two-parameter Poisson-Dirichlet distribution $\Pi_{\alpha, \theta}$.

Let $E_{\alpha, \theta}$ denote the expectation with respect to $\Pi_{\alpha, \theta}$. For $n \geq 1$, set

$$C_{\alpha, \theta} = \frac{\Gamma(\theta + 1)}{\Gamma(\frac{\theta}{\alpha} + 1)}, \quad (2.2)$$

$$C_{\alpha, \theta, n} = \frac{\Gamma(\theta + 1) \Gamma(\frac{\theta}{\alpha} + n) \alpha^{n-1}}{\Gamma(\theta + na) \Gamma(\frac{\theta}{\alpha} + 1) \Gamma(1-\alpha)^n}. \quad (2.3)$$

The following change of measure formula is obtained in [15].
Proposition 2.2 (Perman, Pitman and Yor). For any bounded measurable function \( f \) on \( \mathbb{R}^\infty_+ \),

\[
E_{\alpha,\theta}(f(P_1, P_2, \ldots)) = C_{\alpha,\theta}E\left(\tau_1^{-\theta} f \left(\frac{V_1(\tau_1)}{\tau_1}, \frac{V_2(\tau_1)}{\tau_1}, \ldots\right)\right),
\]

where the law of

\[
\left(\frac{V_1(\tau_1)}{\tau_1}, \frac{V_2(\tau_1)}{\tau_1}, \ldots\right)
\]

is \( \Pi_{\alpha,0} \).

Now we are ready to derive the following distributional results.

Theorem 2.3 For each \( \beta > 0 \), define

\[
g_{\alpha,\beta}(x) = P(P_1(\alpha, \beta) \leq x).\tag{2.5}
\]

Then for any \( n \geq 1 \), the joint density function of \( (P_1(\alpha, \theta), \ldots, P_n(\alpha, \theta)) \) is given by

\[
h_{\alpha,\theta,n}(p_1, \ldots, p_n) = C_{\alpha,\theta,n} \frac{(1 - \sum_{i=1}^n p_i)^{\beta + n\alpha - 1}}{(\prod_{i=1}^n p_i)^{1+\alpha}} g_{\alpha,\theta+n\alpha} \left(\frac{p_n}{1 - \sum_{i=1}^n p_i}\right). \tag{2.6}
\]

Proof By Proposition 2.2 and Perman’s formula (cf. [14]), for any non-negative product measurable function \( f \) and any any \( n > 1 \), the joint density function of \( \left(\tau_1, \frac{V_1(\tau_1)}{\tau_1}, \ldots, \frac{V_n(\tau_1)}{\tau_1}\right) \) is given by

\[
\phi_n(t, p_1, \ldots, p_n) = (c_n)^{n-1} \hat{p}_n^{-1} (p_1 \cdot \cdot \cdot p_{n-1})^{-(1+\alpha)} t^{-(\theta+\alpha)\tau_1} \phi_1(t \hat{p}_n, p_n/\hat{p}_n), \tag{2.7}
\]

where

\[
\hat{p}_n = 1 - p_1 - \cdots - p_{n-1}, \tag{2.8}
\]

and \( \phi_1(t, u) \) satisfies

\[
\phi_1(t, u) = c_n t^{-\alpha} u^{-(1+\alpha)} \int_0^{u \wedge 1} \phi_1(t(1-u), v)dv. \tag{2.9}
\]

Integrating out the \( t \) coordinate, it follows from (2.9) that

\[
h_{\alpha,\theta,n}(p_1, \ldots, p_n) = C_{\alpha,\theta}(c_n)^{n-1} \hat{p}_n^{-1} (p_1 \cdot \cdot \cdot p_{n-1})^{-(1+\alpha)} \int_0^\infty t^{-(\theta+(n-1)\alpha)} \phi_1(t \hat{p}_n, p_n/\hat{p}_n) dt
\]

\[
= C_{\alpha,\theta}(c_n)^{n-1} \hat{p}_n^{-\theta+(n-1)\alpha-2} (p_1 \cdot \cdot \cdot p_{n-1})^{-(1+\alpha)} \int_0^\infty s^{-(\theta+(n-1)\alpha)} \phi_1(s, p_n/\hat{p}_n) ds
\]

\[
= C_{\alpha,\theta}(c_n)^n \frac{\hat{p}_n^{\theta+n\alpha-1}}{(p_1 \cdot \cdot \cdot p_{n-1}p_n)^{1+\alpha}} \int_0^{\hat{p}_n} dx \int_0^\infty s^{-\theta+n\alpha} \phi_1(s(1-p_n/\hat{p}_n), x) ds
\]

\[
= C_{\alpha,\theta}(c_n)^n \frac{(\hat{p}_n+1)^{\theta+n\alpha-1}}{(p_1 \cdot \cdot \cdot p_{n-1}p_n)^{1+\alpha}} \int_0^{\hat{p}_n+1} dx \int_0^{\hat{p}_n+1} u^{-\theta+n\alpha} \phi_1(u, x)du
\]

\[
= C_{\alpha,\theta+n\alpha} \frac{(\hat{p}_n+1)^{\theta+n\alpha-1}}{(p_1 \cdot \cdot \cdot p_{n-1}p_n)^{1+\alpha}} g_{\alpha,\theta+n\alpha} \left(\frac{p_n}{1 - \sum_{i=1}^n p_i}\right),
\]
which leads to (2.6).

\[ \square \]

**Remark.** This result appears in Handa [12] where a different proof is used.

**Theorem 2.4** For any \( s > 0 \),

\[
\nu_{\alpha, \theta, 1}(s) = P(V_1(T) \leq s) = \left( 1 + c_\alpha s^{-\alpha} \int_1^\infty z^{-(1+\alpha)} e^{-sz} dz \right)^{-\theta/\alpha}.
\]

**Proof** For each \( s > 0 \), it follows from Proposition 2.1 and the property of the Poisson random measure that

\[
\nu_{\alpha, \theta, 1}(s) = E(P(V_1(T) \leq s|\zeta(\alpha, \theta)))
= E \left( \exp \left\{ -\alpha C \zeta(\alpha, \theta) \int_s^\infty x^{-(\alpha+1)} e^{-x} dx \right\} \right)
= E \left( \exp \left\{ -c_\alpha \gamma/\alpha s^{-\alpha} \int_1^\infty z^{-(\alpha+1)} e^{-sz} dz \right\} \right)
\]

which leads to (2.11).

\[ \square \]

3 Moderate Deviations for the two-parameter Poisson-Dirichlet Distribution

By theorem 6.1 in [12], when \( \theta \) goes to infinity \( P(\alpha, \theta) = (P_1(\alpha, \theta), P_2(\alpha, \theta), \ldots) \) approaches a non-trivial random sequence when scaled by a factor of \( \theta \) and shifted by

\[
\beta(\alpha, \theta) = \log \theta - (\alpha + 1) \log \log \theta - \log \Gamma(1 - \alpha).
\]

In [9], the LDP has been established associated with the limit

\[
\lim_{\theta \to \infty} P(\alpha, \theta) = (0, 0, \ldots).
\]

Replacing the scaling factor by \( a(\theta) \) satisfying

\[
\lim_{\theta \to \infty} \frac{a(\theta)}{\theta} = 0, \quad \lim_{\theta \to \infty} a(\theta) = \infty,
\]

we still have

5
\[
\lim_{\theta \to \infty} a(\theta) \left( P(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta}(1,1,\ldots) \right) \to (0,0,\ldots). \tag{3.2}
\]

The LDP associated with (3.2) is called the MDP for \( P(\alpha, \theta) = (P_1(\alpha, \theta), P_2(\alpha, \theta) \cdots) \). This MDP will be established in this section through a series of lemmas.

The first lemma establishes the MDP for \( V_1(T)/\theta \).

**Lemma 3.1** The MDP holds for \( V_1(T)/\theta \) with speed \( a(\theta) \) and rate function

\[
J_1(x) = \begin{cases} 
x, & x \geq 0 \\
\infty, & \text{otherwise.} 
\end{cases}
\]

**Proof** For any fixed \( x \), we have

\[
P \left\{ a(\theta) \left( \frac{V_1(T) - \beta(\alpha, \theta)}{\theta} \right) \leq x \right\} = P \left( V_1(T) < \frac{\theta}{a(\theta)} x + \beta(\alpha, \theta) \right). \tag{3.3}
\]

Assume that

\[
\lim_{\theta \to \infty} \left[ \frac{\theta}{a(\theta)} x + \beta(\alpha, \theta) \right] = +\infty.
\]

Then it follows from (2.11) that

\[
P \left( V_1(T) < \frac{\theta}{a(\theta)} x + \beta(\alpha, \theta) \right) \sim \left( 1 + \frac{c_\alpha}{\frac{\theta}{a(\theta)} x + \beta(\alpha, \theta)}^{\alpha+1} e^{-\frac{\theta}{a(\theta)} x + \beta(\alpha, \theta)} \right)^{-\theta/\alpha}.
\]

Therefore

\[
\limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( a(\theta) \left( \frac{V_1(T) - \beta(\alpha, \theta)}{\theta} \right) \leq x \right) = \lim_{\theta \to \infty} \frac{a(\theta)}{\theta} \log \left( 1 + \frac{c_\alpha (\log \theta)^{\alpha+1} \Gamma(1-\alpha)}{\theta (\frac{\theta}{a(\theta)} x + \beta(\alpha, \theta))^{\alpha+1}} e^{-\frac{\theta}{a(\theta)} x + \beta(\alpha, \theta)} \right)^{-\theta/\alpha} \tag{3.4}
\]

\[
= \begin{cases} 
0, & x \geq 0 \\
-\infty, & x < 0
\end{cases}.
\]

If there exists a subsequence \( \theta' \) such that the \( \lim_{\theta' \to \infty} \left( \frac{\theta'}{a(\theta')} x + \beta(\alpha, \theta') \right) \) exists in \([-\infty, +\infty)\), then \( x \) must be strictly negative. Since, by Theorem 2.4, \( V_1(T) \) converges to infinity as \( \theta \) converges to infinity, it follows that

\[
\limsup_{\theta' \to \infty} \frac{a(\theta')}{\theta'} \log P \left( a(\theta') \left( \frac{V_1(T) - \beta(\alpha, \theta')}{\theta'} \right) \leq x \right) = -\infty. \tag{3.5}
\]
Putting (3.4) and (3.5) together, one gets
\[
\lim_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( a(\theta) \left( \frac{V_1(T) - \beta(\alpha, \theta)}{\theta} \right) \leq x \right) = 0, \quad x \geq 0,
\]
(3.6)
and
\[
\limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( a(\theta) \left( \frac{V_1(T) - \beta(\alpha, \theta)}{\theta} \right) \leq x \right) = -\infty, \quad x < 0.
\]
(3.7)

For \(x \geq 0\), it follows from (3.3) and (2.11) that
\[
\limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( a(\theta) \left( \frac{V_1(T) - \beta(\alpha, \theta)}{\theta} \right) \in (x - \delta, x + \delta) \right) = -x.
\]
(3.8)

A combination of (3.6) and (3.7) implies that the laws of \(a(\theta) \left( \frac{V_1(T) - \beta(\alpha, \theta)}{\theta} \right)\) is exponentially tight.

Similarly, we can get that for \(x > 0\) and \(\delta > 0\) with \(x - \delta > 0\),
\[
\lim_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( a(\theta) \left( \frac{V_1(T) - \beta(\alpha, \theta)}{\theta} \right) \in (x - \delta, x + \delta) \right) = -x + \delta.
\]
(3.9)

The equality (3.6) combined with (3.7) implies that
\[
\limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( a(\theta) \left( \frac{V_1(T) - \beta(\alpha, \theta)}{\theta} \right) \in (-\delta, \delta) \right) = 0.
\]
(3.10)

The lemma now follows from (3.9), (3.10), and the exponential tightness.
Set
\[ \gamma(\theta) = \frac{a(\theta)\beta(\alpha, \theta)}{\theta}, \]
and, without loss of generality, we can assume that
\[ \lim_{\theta \to \infty} \gamma(\theta) = c \in [0, +\infty]. \]

It is clear that
\[ \frac{a(\theta)}{\gamma^2(\theta)} = \frac{\theta^2}{a(\theta)\beta^2(\alpha, \theta)} \to \infty, \quad \theta \to \infty. \]  

(3.11)

If \( c < \infty \), it follows from Corollary 3.1 in [11] that for any \( L > 0 \)
\[ \limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left\{ \gamma(\theta)\frac{T}{\theta} - 1 \geq L \right\} = -\infty. \]  

(3.12)

For \( c = \infty \), and any \( 1 > \delta > 0 \)
\[ \left\{ \gamma(\theta)\frac{T}{\theta} - 1 \geq L \right\} \subset \left\{ \gamma(\theta)\frac{T}{\theta} - 1 \geq L(1 - \delta) \right\} \bigcup \left\{ \frac{T}{\theta} - 1 \geq \delta \right\}. \]  

(3.13)

Since \( \gamma(\theta) \leq \beta(\alpha, \theta) \) for large \( \theta \) and \( \lim_{\theta \to \infty} \frac{\beta(\alpha, \theta)}{\sqrt{\theta}} = 0 \), it follows from the MDP (Theorem 3.2 in [11]) for \( T/\theta \), and (3.11) that
\[ \limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left\{ \gamma(\theta)\frac{T}{\theta} - 1 \geq (1 - \delta)L \right\} = -\infty, \]  

(3.14)

which combined with Corollary 3.1 in [11] and (3.13) shows that (3.12) still holds in this case. Therefore \( a(\theta) (P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta}) \) and \( \frac{T}{\theta} a(\theta) (\frac{V_1(T)}{\theta} - \frac{\beta(\alpha, \theta)}{\theta}) \) are exponentially equivalent.

Since \( \frac{T}{\theta} a(\theta) (\frac{V_1(T)}{\theta} - \frac{\beta(\alpha, \theta)}{\theta}) \) is exponentially equivalent to \( a(\theta) (\frac{V_1(T)}{\theta} - \frac{\beta(\alpha, \theta)}{\theta}) \) by Lemma 2.1 and Corollary 3.1 in [11], it follows that \( a(\theta) (P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta}) \) and \( a(\theta) (\frac{V_1(T)}{\theta} - \frac{\beta(\alpha, \theta)}{\theta}) \) are exponentially equivalent. Thus we have the following result.

**Lemma 3.2** The MDP holds for \( P_1(\alpha, \theta) \) with speed \( \frac{a(\theta)}{\theta} \) and rate function
\[ J_1(x) = \begin{cases} 
  x, & x \geq 0 \\
  \infty, & \text{otherwise.} 
\end{cases} \]

For each \( n \geq 2 \), we have
Lemma 3.3 The family \( \left\{ P\left( a(\theta)\left( P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) \cdots, P_n(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) \in \cdot \right) : \theta > 0 \right\} \) satisfies a LDP on \( \mathbb{R}^n \) with speed \( \frac{a(\theta)}{\theta} \) and rate function

\[
I_n(x_1, \ldots, x_n) = \begin{cases} 
\sum_{i=1}^{n} x_i, & \text{if } 0 \leq x_n \leq \cdots \leq x_1, \\
+\infty, & \text{otherwise.}
\end{cases}
\]  

(3.15)

Proof It follows from (3.3) that for \( x_1 \geq x_2 \cdots \geq x_n \) and \( \frac{\theta}{a(\theta)}x_n + \beta(\alpha, \theta) > 0 \), the density function \( g_{\alpha, \theta, n}(x_1, \ldots, x_n) \) of \( a(\theta)\left( P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \cdots, P_n(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) \) is

\[
g_{\alpha, \theta, n}(x_1, \ldots, x_n) = \left( \frac{1}{a(\theta)} \right)^n C_{n, \alpha, \theta} \left( \prod_{i=1}^{n} \left( \frac{\theta}{\theta - \left( \frac{\theta}{a(\theta)} \sum_{i=1}^{n} x_i + n(\alpha, \theta) \right)} \right)^{\alpha + 1} \right) \times \left( 1 - \left( \frac{\theta}{a(\theta)} \sum_{i=1}^{n} x_i + n(\alpha, \theta) \right) / \theta \right)^{\theta + n\alpha - 1} g_{\alpha, \theta + n\alpha} \left( \frac{\theta}{\theta - \left( \frac{\theta}{a(\theta)} \sum_{i=1}^{n} x_i + n(\alpha, \theta) \right)} \right). 
\]  

(3.16)

By Theorem 2.4 and direct calculation, for \( x_n > 0 \)

\[
\frac{a(\theta)}{\theta} \log g_{\alpha, \theta + n\alpha} \left( \frac{\theta}{\theta - \left( \frac{\theta}{a(\theta)} \sum_{i=1}^{n} x_i + n(\alpha, \theta) \right)} \right) \to 0.
\]

For \( x_n < 0 \), set

\[
\psi(n, x, \theta, \alpha) = a(\theta) \left( \frac{\theta}{\theta - \left( \frac{\theta}{a(\theta)} \sum_{i=1}^{n} x_i + n(\alpha, \theta) \right)} - \frac{\beta(\alpha, \theta + n\alpha)}{\theta + n\alpha} \right).
\]

Then

\[
g_{\alpha, \theta + n\alpha} \left( \frac{\theta}{\theta - \left( \frac{\theta}{a(\theta)} \sum_{i=1}^{n} x_i + n(\alpha, \theta) \right)} \right) \to P \left( a(\theta) \left( P_1(\alpha, \theta + n\alpha) - \frac{\beta(\alpha, \theta + n\alpha)}{\theta + n\alpha} \right) < \psi(n, x, \theta, \alpha) \right)
\]

and

\[
\lim_{\theta \to -\infty} \psi(n, x, \theta, \alpha) = x_n < 0
\]

which implies that

\[
\lim_{\theta \to -\infty} \frac{a(\theta)}{\theta} \log g_{\alpha, \theta + n\alpha} \left( \frac{\theta}{\theta - \left( \frac{\theta}{a(\theta)} \sum_{i=1}^{n} x_i + n(\alpha, \theta) \right)} \right) = -\infty.
\]
are obtained by approximating the boundary with open subsets away from the boundary.

Noting that $\bigcup a_i \geq 1$ and $\beta_i = 1$ for any $i$, and for any $x$, $x_n > 0$, \n
\[
\lim_{\delta \to 0} \limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( a(\theta) \left( P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) + \cdots + P_n(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) \in B((x_1, \ldots, x_n), \delta) \\
= \lim_{\delta \to 0} \liminf_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( a(\theta) \left( P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) + \cdots + P_n(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) \in B((x_1, \ldots, x_n), \delta) \\
= - \sum_{i=1}^{n} x_i.
\]

and for any $x_n < 0$,

\[
\lim_{\delta \to 0} \limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( a(\theta) \left( P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) + \cdots + P_n(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) \in B((x_1, \ldots, x_n), \delta) \\
= \lim_{\delta \to 0} \liminf_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( a(\theta) \left( P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) + \cdots + P_n(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) \in B((x_1, \ldots, x_n), \delta) \\
= - \infty,
\]

If $x_{r-1} > 0, x_r = 0$ for some $1 \leq r \leq n$, then the upper estimate is obtained from that of $a(\theta) \left( P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) + \cdots + P_{r-1}(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} + P_r(\alpha, \theta) + \frac{\beta(\alpha, \theta)}{\theta}$. The lower estimates when $x_r = 0$ for some $1 \leq r \leq n$ are obtained by approximating the boundary with open subsets away from the boundary.

Noting that $\bigcup^{n}_{i=1} \{ a(\theta) \left( P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) > L \} = \{ a(\theta) \left( P_1(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) > L \}$, it follows that

\[
\lim_{L \to \infty} \limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( \bigcup_{i=1}^{n} \left\{ a(\theta) \left( P_i(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) > L \right\} \right) = -\infty. \quad (3.19)
\]

On the other hand,

\[
\limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left( \bigcup_{i=1}^{n} \left\{ a(\theta) \left( P_i(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) < -L \right\} \right) \leq \limsup_{\theta \to \infty} \frac{a(\theta)}{\theta} \log P \left\{ a(\theta) \left( P_i(\alpha, \theta) - \frac{\beta(\alpha, \theta)}{\theta} \right) \leq -L \right\} = -\infty. \quad (3.20)
\]
These lead to the exponential tightness and the lemma.

Now we are ready to establish the MDP for \((P_1(\alpha, \theta), P_2(\alpha, \theta), \ldots)\).

**Theorem 3.4** For each \(n \geq 1\), the family \(\{P\left(a(\theta)\left(P_1(\theta) - \frac{\beta(\theta)}{\theta}, \ldots, P_n(\theta) - \frac{\beta(\theta)}{\theta}, \ldots\right) \in \cdot : \theta > 0\} \) satisfies a LDP on \(\mathbb{R}^\infty\) with speed \(a(\theta)/\theta\) and rate function

\[
I(x_1, x_2, \ldots) = \begin{cases} 
\sum_{i=1}^{\infty} x_i, & x_1 \geq \cdots \geq 0 \\
\infty, & \text{otherwise.} \end{cases} 
\] 

**Proof** Identify \(\mathbb{R}^\infty\) with the projective limit of \(\mathbb{R}^n, n = 1, \ldots\). Then the theorem follows from Theorem 3.3 in [4] and Lemma 3.3.

\[\square\]

## 4 Moderate Deviations for the Homozygosity

For each \(m \geq 2\), it was shown in [12] that the scaled homozygosity

\[
\sqrt{\theta}\left[\frac{\theta^{m-1}\Gamma(1-\alpha)}{\Gamma(m-\alpha)}H_m(P(\alpha, \theta)) - 1\right] \Rightarrow Z_{\alpha,m}
\]

where \(Z_{\alpha,m}\) is a normal random variable with mean zero and variance

\[
\sigma_{\alpha,m}^2 = \frac{\Gamma(2m-\alpha)\Gamma(1-\alpha)}{\Gamma(m-\alpha)^2} + \alpha - m^2.
\]

It is thus natural to consider the MDP for \(\frac{\theta^{m-1}}{\Gamma(m)}H_m(P(\alpha, \theta))\) or equivalently the LDP for the family \(\{a(\theta)\left[\frac{\theta^{m-1}\Gamma(1-\alpha)}{\Gamma(m-\alpha)}H_m(P(\alpha, \theta)) - 1\right] : \theta > 0\} \) for a scale \(a(\theta)\) satisfying

\[
\lim_{\theta \to \infty} a(\theta) = \infty, \quad \lim_{\theta \to \infty} \frac{a(\theta)}{\sqrt{\theta}} = 0, \quad (4.1)
\]

which is different from (3.1).

The MDP in the case of \(\alpha = 0\) has been established in [11] where that the following additional restriction on \(a(\theta)\) is used: for some \(0 < \epsilon < 1/(2m-1)\),

\[
\liminf_{\theta \to \infty} \frac{a^{1-\epsilon}(\theta)}{\theta^{(m-1)/(2m-1)}} > 0. \quad (4.2)
\]
This condition is also needed for the two-parameter model. As shown in [11], the conditions (4.1) and (4.2) guarantee that there exist \( \tau > 0 \) positive integer \( l \geq 3 \vee (2m-1)e \), and \( r(\theta) \) that grows faster than a positive power of \( \theta \) such that

\[
\lim_{\theta \to \infty} \frac{a(\theta)}{\theta^\tau} = +\infty
\]

and

\[
\lim_{\theta \to \infty} \frac{r(\theta)^{m-1}}{a((l-2)/l)(\theta)} = 0, \quad \lim_{\theta \to \infty} \frac{a^2(\theta)r(\theta)}{\theta} = 0.
\]

For any \( n \geq 1 \), set

\[
G^{(n)}_{\alpha,\theta,r} = \sum_{i=1}^{\infty} V^n_i(T) I\{V_i(T) \leq r(\theta)\},
\]

and

\[
G^{(n)}_{\alpha,\theta} = \sum_{i=1}^{\infty} V^n_i(T),
\]

and

\[
G_{\alpha,\theta,r} = \left( G^{(1)}_{\alpha,\theta,r} - E(G^{(1)}_{\alpha,\theta,r}), G^{(m)}_{\alpha,\theta,r} - E(G^{(m)}_{\alpha,\theta,r}) \right).
\]

For any \( s, t \) in \( \mathbb{R} \), define

\[
\Lambda(s, t) = \frac{1}{2} \left( s^2 + \frac{2\Gamma(m-\alpha)\Gamma(m+1)}{\Gamma(m)\Gamma(1-\alpha)} st + \left( \frac{\Gamma(2m-\alpha)}{\Gamma(1-\alpha)} + \alpha \left( \frac{\Gamma(m-\alpha)}{\Gamma(1-\alpha)} \right)^2 \right) t^2 \right).
\]

It follows by direct calculation that the Fenchel-Legendre transform of \( \Lambda(s, t) \) is given by

\[
\Lambda^*(x, y) = \sup_{s, t} \{sx + ty - \Lambda(s, t)\}
\]

\[
= \frac{\Gamma(1-\alpha)}{2(\Gamma(1-\alpha)\Gamma(2m-\alpha) + (\alpha - m^2)\Gamma^2(m-\alpha))}
\]

\[
\times \left( (\Gamma(2m-\alpha) + \alpha \frac{\Gamma^2(m-\alpha)}{\Gamma(1-\alpha)}) x^2 - 2m\Gamma(m-\alpha)xy + \Gamma(1-\alpha)y^2 \right),
\]

for \( x, y \) in \( \mathbb{R} \).

**Lemma 4.1** The family \( \{\frac{a(\theta)}{\theta} G_{\alpha,\theta,r} : \theta > 0\} \) satisfies a LDP on space \( \mathbb{R}^2 \) with speed \( \frac{a^2(\theta)}{\theta} \) and rate function \( \Lambda^*(\cdot, \cdot) \).

**Proof** For any \( s, t \in \mathbb{R} \), let

\[
g(x) = sx + tx^m
\]

and

\[
\varphi_{\tau}(x) = \frac{g(x)I_{\{x \leq r(\theta)\}}}{a(\theta)}.
\]
It follows by direct calculation that

\[
\int_{0}^{r(\theta)} (e^{\varphi_r(x)} - 1)x^{-(1+\alpha)}e^{-x}dx = \int_{0}^{r(\theta)} \frac{g(x)}{a(\theta)} x^{-(1+\alpha)}e^{-x}dx + \frac{1}{2} \int_{0}^{r(\theta)} \frac{g^2(x)}{a^2(\theta)} x^{-(1+\alpha)}e^{-x}dx + \sum_{k=3}^{l} \frac{1}{k!a^k(\theta)} \int_{0}^{r(\theta)} |sx + tx^m|^k x^{-(1+\alpha)}e^{-x}dx
\]

\[
+ O \left( \sum_{k=l+1}^{\infty} \frac{1}{k!a^k(\theta)} (|s| + |t| \gamma(\theta)^{m-1})^k \Gamma(k-\alpha) \right)
\]

\[
= \int_{0}^{r(\theta)} \frac{g(x)}{a(\theta)} x^{-(1+\alpha)}e^{-x}dx + \frac{1}{2} \int_{0}^{r(\theta)} \frac{g^2(x)}{a^2(\theta)} x^{-(1+\alpha)}e^{-x}dx + o \left( \frac{1}{a^2(\theta)} \right),
\]

which implies that for \( \theta \) large enough,

\[
| \int_{0}^{r(\theta)} (e^{\varphi_r(x)} - 1)x^{-(1+\alpha)}e^{-x}dx | < c^{-1}_\alpha.
\]

By the Campbell’s theorem we get that

\[
E \left( \exp \left\{ \frac{1}{a(\theta)} (sG^{(1)}_{\alpha,\theta,r} + tG^{(m)}_{\alpha,\theta,r}) \right\} \right)
\] = \[ E \left( \exp \left\{ \sum_{i=1}^{\infty} \varphi_r(V_i(T)) \right\} \right)
\] = \[ E \left( E \left( \exp \left\{ \sum_{i=1}^{\infty} \varphi_r(V_i(T)) \right\} | \zeta(\alpha, \theta) \right) \right)
\] = \[ E \left( \exp \left\{ c_\alpha \gamma \left( \frac{\theta}{\alpha} \right) \int_{0}^{r(\theta)} (e^{\varphi_r(x)} - 1)x^{-(1+\alpha)}e^{-x}dx \right\} \right)
\] = \[ \exp \left\{ - \frac{\theta}{\alpha} \log \left( 1 - c_\alpha \int_{0}^{r(\theta)} (e^{\varphi_r(x)} - 1)x^{-(1+\alpha)}e^{-x}dx \right) \right\}.
\]

Putting (4.4) and (4.5) together, we get that

\[
E \left( \exp \left\{ \frac{1}{a(\theta)} (s(G^{(1)}_{\alpha,\theta,r} - E(G^{(1)}_{\alpha,\theta,r})) + t(G^{(m)}_{\alpha,\theta,r} - E(G^{(m)}_{\alpha,\theta,r})) \right\} \right)
\] = \[ \exp \left\{ \frac{\theta c_\alpha}{2\alpha^2(\theta)} \left( c_\alpha \left( \int_{0}^{\infty} g(x)x^{-(1+\alpha)}e^{-x}dx \right)^2 + \int_{0}^{\infty} g^2(x)x^{-(1+\alpha)}e^{-x}dx + o \left( \frac{1}{a^2(\theta)} \right) \right) \right\}
\] = \[ \exp \left( \frac{\theta}{a^2(\theta)} (\Lambda(s, t) + o(\frac{1}{a^2(\theta)})) \right),
\]

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which leads to
\[
\lim_{\theta \to \infty} \frac{a^2(\theta)}{\theta} \log E \left( \exp \left\{ \frac{1}{a(\theta)} \left[ s(G_{\alpha,\theta,r}^{(1)} - E(G_{\alpha,\theta,r}^{(1)})) + t(G_{\alpha,\theta,r}^{(m)} - E(G_{\alpha,\theta,r}^{(m)})) \right] \right\} \right) = \Lambda(s, t). \quad (4.6)
\]

The lemma now follows from (4.3) and the Gärtner-Ellis theorem.

**Lemma 4.2** Set
\[
G_{\alpha,\theta} = \left( T - \theta, G_{\alpha,\theta}^{(m)} - E(G_{\alpha,\theta}^{(m)}) \right).
\]
Then the family \( \{\frac{a(\theta)}{\theta} G_{\alpha,\theta} : \theta > 0\} \) satisfies a LDP with speed \( \frac{a^2(\theta)}{\theta} \) and the rate function \( \Lambda^*(x, y) \).

**Proof** By definition for any \( n \geq 1 \) and any \( \delta > 0 \),
\[
\limsup_{\theta \to \infty} \frac{a^2(\theta)}{\theta} \log P \left( \left| G_{\alpha,\theta,r}^{(m)} - G_{\alpha,\theta}^{(m)} \right| \geq \delta \frac{\theta}{a(\theta)} \right) \\
\leq \limsup_{\theta \to \infty} \frac{a^2(\theta)}{\theta} \log P (V_1(T) \geq r(\theta)) \\
= \limsup_{\theta \to \infty} \frac{a^2(\theta)}{\theta} \log \left( 1 - \left( 1 + \frac{c_\alpha}{r^{\alpha}(\theta)} \int_1^\infty z^{-(1+\alpha)} e^{-r(\theta)z} dz \right)^{-\theta/\alpha} \right) \\
\leq \limsup_{\theta \to \infty} \frac{a^2(\theta)}{\theta} \log \left( 1 - \left( 1 + \frac{c_\alpha}{r^{(1+\alpha)}(\theta)e^{r(\theta)}} \right)^{-\theta/\alpha} \right) \\
= \limsup_{\theta \to \infty} \frac{a^2(\theta)}{\theta} \log \left( \frac{\theta}{\alpha} \log \left( 1 + \frac{c_\alpha}{r^{(1+\alpha)}(\theta)e^{r(\theta)}} \right) \right) \\
\leq - \limsup_{\theta \to \infty} \frac{a^2(\theta) r(\theta)}{\theta} \left( 1 - \frac{\log \theta}{r(\theta)} \right) \\
= - \infty,
\]
which implies that \( \frac{a(\theta)}{\theta} G_{\alpha,\theta,r} \) and \( \frac{a(\theta)}{\theta} G_{\alpha,\theta} \) are exponentially equivalent. Therefore \( \left( \frac{a(\theta)}{\theta} G_{\alpha,\theta}, \frac{a^2(\theta)}{\theta}, \Lambda^* \right) \) satisfies LDP.

Now we are ready to prove the main result of this section.

**Theorem 4.3** The family \( a(\theta) \left( \frac{\rho_{\alpha}^{-1}\Gamma(1-\alpha)}{\Gamma(m-\alpha)} H_m(P(\alpha, \theta)) - 1 \right) \) satisfies a LDP with speed \( \frac{a^2(\theta)}{\theta} \) and rate function \( \frac{\sigma_{\alpha,m}^2}{2a^2_{\alpha,m}} \).
Proof  By direct calculation,
\[
a(\theta) \left( \frac{\theta^{m-1}\Gamma(1-\alpha)}{\Gamma(m-\alpha)} H_m(P(\alpha, \theta)) - 1 \right)
= a(\theta) \left( \frac{\theta^{m-1}G^{(m)}_{\alpha,\theta}}{T^m\Gamma(m-\alpha)/\Gamma(1-\alpha)} - 1 \right)
= a(\theta) \left( \left( \frac{\theta}{T} \right)^m - 1 \right) + \left( \frac{\theta}{T} \right)^m a(\theta) \frac{(G^{(m)}_{\alpha,\theta} - E(G^{(m)}_{\alpha,\theta}))}{\theta\Gamma(m-\alpha)/\Gamma(1-\alpha)}
= \frac{a(\theta)}{\theta} (\theta - T) \sum_{k=1}^{m} \left( \frac{\theta}{T} \right)^k + \left( \frac{\theta}{T} \right)^m a(\theta) \frac{(G^{(m)}_{\alpha,\theta} - E(G^{(m)}_{\alpha,\theta}))}{\theta\Gamma(m-\alpha)/\Gamma(1-\alpha)}.
\]

Noting that for any \( i \geq 1 \) and for any \( \delta > 0 \),
\[
\lim_{\theta \to \infty} \frac{a^2(\theta)}{\theta} \log P \left( \left( \frac{\theta}{T} \right)^i - 1 \geq \delta \right) = -\infty.
\]
It then follows that
\[
a(\theta) \left( \frac{\theta^{m-1}\Gamma(1-\alpha)}{\Gamma(m-\alpha)} H_m(P(\alpha, \theta)) - 1 \right)
\]
and
\[
\frac{a(\theta)(\theta - T)}{\theta} \sum_{k=1}^{m} \left( \frac{\theta}{T} \right)^k + \left( \frac{\theta}{T} \right)^m a(\theta) \frac{(G^{(m)}_{\alpha,\theta} - E(G^{(m)}_{\alpha,\theta}))}{\theta\Gamma(m-\alpha)/\Gamma(1-\alpha)}
\]
are exponentially equivalent, and so they have the same LDP.

The fact that
\[
\inf_{y \in \Gamma(1-\alpha)/m \cdot x = z} \Lambda^*(x, y) = \frac{z}{2\sigma_{\alpha,m}^2},
\]
combined with Lemma 4.2 and the contraction principle implies the theorem.

5  LDP for Small Parameters

Let
\[
\nabla = \left\{ p = (p_1, p_2, \ldots) : p_1 \geq p_2 \geq \cdots \geq 0, \sum_{i=1}^{\infty} p_i \leq 1 \right\}
\]
be equipped with the subspace topology of \([0,1]^{\infty}\), and \( M_1(\nabla) \) be the space of all probability measures on \( \nabla \) equipped with the weak topology. Then \( \Pi_{\alpha,\theta} \) belongs to \( M_1(\nabla) \).

For any \( \delta > 0 \), it follows from the GEM representation (1.1) that
\[
P \left( X_{1,\theta}^\alpha > 1 - \delta \right) \leq P \left( P_1(\alpha, \theta) > 1 - \delta \right).
\]
By direct calculation, we have
\[
\lim_{\alpha + \theta \to 0} P \left( X_1^{\alpha, \theta} > 1 - \delta \right) = 1.
\]
Therefore, \( \Pi_{\alpha, \theta} \) converges in \( M_1(\nabla) \) to \( \delta(1,0,\ldots) \) as \( \alpha + \theta \) converges to zero. In this section, we establish the LDP associated with this limit. This is a two-parameter generalization to the result in [10].

For any \( n \geq 1 \), set
\[
\nabla_n = \left\{ (p_1, \ldots, p_n, 0, 0, \ldots) \in \nabla : \sum_{i=1}^{n} p_i = 1 \right\},
\]
\[
\nabla_\infty = \bigcup_{i=1}^{\infty} \nabla_i,
\]
and
\[
a(\alpha, \theta) = \alpha \lor |\theta|, \quad b(\alpha, \theta) = ( - \log(a(\alpha, \theta)) )^{-1}.
\]

Then we have

**Lemma 5.1** The family of laws of \( \{ P_1(\alpha, \theta) : \alpha + \theta > 0, 0 < \alpha < 1 \} \) satisfies a LDP on \([0, 1]\) as \( a(\alpha, \theta) \) goes to zero with speed \( b(\alpha, \theta) \) and rate function
\[
S_1(p) = \begin{cases} 
0, & p = 1 \\
k, & p \in \left[ \frac{1}{k+1}, \frac{1}{k} \right), k = 1, 2, \ldots \\
\infty, & p = 0.
\end{cases} \tag{5.1}
\]

**Proof** Let \( \{ X_i^{\alpha, \theta} : i = 1, 2, \ldots \} \) be defined in (1.1). For any \( n \geq 1 \), set
\[
\tilde{P}_1^n(\alpha, \theta) = \max\{ X_i^{\alpha, \theta} : 1 \leq i \leq n \}.
\]
Then it follows from direct calculation that for any \( \delta > 0 \)
\[
P\{ P_1(\alpha, \theta) - \tilde{P}_1^n(\alpha, \theta) > \delta \} \leq P\{ (1 - U_1) \cdots (1 - U_n) \geq \delta \} \leq \delta^{-1} \prod_{i=1}^{n} \frac{\theta + i\alpha}{\theta + i\alpha + 1 - \alpha},
\]
which leads to
\[
\limsup_{a(\alpha, \theta) \to 0} b(\alpha, \theta) \log P\{ P_1(\alpha, \theta) - \tilde{P}_1^n(\alpha, \theta) > \delta \} \leq -n.
\]
Thus the families \( \{ \hat{P}_1^n(\alpha, \theta) : 0 < \alpha < 1, \theta + \alpha > 0 \} \) are exponential good approximations to the family \( \{ P_1(\alpha, \theta) : 0 < \alpha < 1, \theta + \alpha > 0 \} \). By the contraction principle, the family \( \{ \hat{P}_1^n(\alpha, \theta) : 0 < \alpha < 1, \theta + \alpha > 0 \} \) satisfies a LDP on \([0,1]\) as \( a(\alpha, \theta) \) goes to zero with speed \( b(\alpha, \theta) \) and rate function

\[
I_n(p) = \begin{cases} 
0, & p = 1 \\
\kappa, & p \in \left[\frac{1}{k+1}, \frac{1}{k}\right), k = 1, 2, \ldots, n - 1 \\
\ell, & \text{else.}
\end{cases}
\]

The lemma now follows from the fact that

\[
S_1(p) = \sup_{\delta > 0} \liminf_{\ell \to \infty} \inf_{|q-p| < \delta} I_n(q).
\]

\[\square\]

**Theorem 5.1** The family \( \{ \Pi_{\alpha, \theta} : \alpha + \theta > 0, 0 < \alpha < 1 \} \) satisfies a LDP on \( \nabla \) as \( a(\alpha, \theta) \) goes to zero with speed \( b(\alpha, \theta) \) and rate function

\[
S(p) = \begin{cases} 
n - 1, & p \in \nabla_n, p_n > 0, n \geq 1 \\
\infty, & p \notin \nabla_\infty.
\end{cases}
\]

\[\text{(5.2)}\]

**Proof** It suffices to establish the LDP for finite dimensional marginal distributions since the infinite dimensional LDP can be derived from the finite dimensional LDP through approximation. For any \( n \geq 2 \), \( (P_1(\alpha, \theta), P_2(\alpha, \theta), \ldots, P_n(\alpha, \theta)) \) and \( (P_1(0, \alpha + \theta), P_2(0, \alpha + \theta), \ldots, P_n(0, \alpha + \theta)) \) have respective joint density functions

\[
h_{\alpha, \theta, n}(p_1, \ldots, p_n) = C_{\alpha, \theta, n} \left( 1 - \sum_{i=1}^{n} p_i \right)^{\theta + n\alpha - 1} P_1(\alpha, n\alpha + \theta) \leq \frac{p_n}{1 - \sum_{i=1}^{n} p_i},
\]

and

\[
g_{\alpha + \theta, n} = (\alpha + \theta)^n \left( 1 - \sum_{i=1}^{n} p_i \right)^{\theta + n\alpha - 1} P_1(0, \alpha + \theta) \leq \frac{p_n}{1 - \sum_{i=1}^{n} p_i}.
\]

Since \( \lim_{a(\alpha, \theta) \to 0} b(\alpha, \theta) \log(\alpha + \theta) = -1 \) and \( \lim_{a(\alpha, \theta) \to 0} b(\alpha, \theta) C_{\alpha, \theta, n} = -n \), it follows from Lemma 2.4 in [10] and Lemma 5.1 that the family of laws of \( (P_1(\alpha, \theta), P_2(\alpha, \theta), \ldots, P_n(\alpha, \theta)) \) satisfies a LDP as \( a(\alpha, \theta) \) goes to zero with speed \( b(\alpha, \theta) \) and rate function

\[
S_n(p_1, \ldots, p_n) = \begin{cases} 
0, & (p_1, p_2, \ldots, p_n) = (1, 0, \ldots, 0) \\
\ell - 1, & 2 \leq \ell \leq n, \sum_{k=1}^{l} p_k = 1, p_l > 0 \\
\ell + S_1 \left( \frac{p_n}{\sum_{i=1}^{n} p_i} \right) \wedge 1, & \sum_{k=1}^{n} p_k < 1, p_n > 0 \\
\infty, & \text{else.}
\end{cases}
\]

\[\square\]
Acknowledgement

Fuqing Gao would like to thank the Department of Mathematics and Statistics at McMaster University for their hospitality during his visit.

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