Research Article

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From simplicial homotopy to crossed module homotopy in modified categories of interest

Abstract: We address the (pointed) homotopy of crossed module morphisms in modified categories of interest that unify the notions of groups and various algebraic structures. We prove that the homotopy relation gives rise to an equivalence relation as well as to a groupoid structure with no restriction on either domain or co-domain of the corresponding crossed module morphisms. Furthermore, we also consider particular cases such as crossed modules in the categories of associative algebras, Leibniz algebras, Lie algebras and dialgebras of the unified homotopy definition. Finally, as one of the major objectives of this paper, we prove that the functor from simplicial objects to crossed modules in modified categories of interest preserves the homotopy as well as the homotopy equivalence.

Keywords: Crossed module, simplicial object, modified categories of interest, homotopy

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1 Introduction

Categories of interest were introduced to unify definitions and properties of different algebraic categories and different algebras. The first steps of this unification were proposed by P.G. Higgins in [25] under the name of groups with multiple operators (for details, see [33]). Later, the generalized notion, i.e., “categories of interest” was introduced by G. Orzech in [31, 32]. Categories of groups, Lie algebras, Leibniz algebras, associative (commutative) algebras, dialgebras and many others are basic examples of categories of interest. Nevertheless, the categories of cat^{-1}-objects of such algebraic structures are not categories of interest. These categories were introduced in [5] and called “modified categories of interest”, which here are shortly denoted by MCI. According to this definition, every category of interest is a modified category of interest as well. Cat^{-1}-Lie (Leibniz, associative, commutative) algebras and many others or crossed modules of these algebras are all MCI [12, 14–16, 20]. Also, the category of commutative von Neumann regular rings is isomorphic to the category of commutative rings with a unary operation (·)∗ satisfying two axioms defined in [3], which is an MCI.

A crossed module (see [6]) \( \mathcal{C} = (\partial : E \rightarrow G, \triangleright) \) in the category of groups is given by a group homomorphism \( \partial : E \rightarrow G \), together with the action \( \triangleright \) of \( G \) on \( E \), satisfying the following Peiffer–Whitehead relations for all \( e, f \in E \) and \( g \in G \):

first Peiffer–Whitehead relation: \( \partial(ge) = g\partial(e)g^{-1} \),
second Peiffer–Whitehead relation: \( \partial(e) \triangleright f = efe^{-1} \).

Crossed modules of groups were introduced by Whitehead in [35] where they were used for the definition of homotopy systems of connected CW complexes. Later, crossed modules were used for the classification of
algebraic 3-types (subsequently, they were called algebraic 2-types) [2, 27, 28]. Another outcome is that the category of crossed modules is also equivalent to the category of cat³-groups [27] as well as to the categories of interest in the sense of [11, 13]. In MCI, the notion of crossed modules is introduced in [5]. In MCI [34], it is proved that the category of crossed modules is equivalent to the category of cat³-objects. As the categories of some cat³-algebras are not categories of interest but are MCI, in this paper we would prefer to work on this modified category case. Therefore, the entire concept of this paper will also be true under the category of interest aspect as well. Crossed modules also appear in the context of simplicial homotopy theory as they are equivalent to simplicial objects with Moore complex of length 1 in (modified) categories of interest [4, 17] that can be diagrammed as

$$\begin{array}{ccc}
\text{Simp}(\mathcal{C})_{\leq 1} & \xrightarrow{X_1} & \text{XMod}(\mathcal{C}) \\
\downarrow \cosk_1 & & \downarrow \tr_1 \\
\text{Tr}_1 \text{Simp}(\mathcal{C}). & & 
\end{array}$$

(1.1)

An equally well established result of this equivalence is that the homotopy category of n-types is equivalent to the homotopy category of simplicial groups with Moore complex of length n − 1, also called algebraic models for n-types.

The homotopy relation between (pre)crossed module morphisms \(G \rightarrow G'\) was introduced for groups by J. Faria Martins in [21] and for commutative algebras in [1]. In both of these studies we see that the homotopy relation between crossed module morphisms \(G \rightarrow G'\) is an equivalence relation in the general case with no restriction on \(G\) or \(G'\). If we examine this result in the sense of [19], this is indeed an unexpected situation as the homotopy relation of morphisms \(G \rightarrow G'\) gives an equivalence relation when \(G = (\partial : E \rightarrow G, \triangleright)\) is cofibrant. On the other hand, \(G = (\partial : E \rightarrow G, \triangleright)\) is cofibrant [7] if and only if \(G\) is a free group in the well known model category structure in the sense of [30].

In this paper, we address the homotopy theory of crossed module morphisms \(X \rightarrow X'\) in MCI that leads us to defining an equivalence relation. Thus we construct a groupoid structure without any restriction on \(X\) or \(X'\). This case represents an undiscovered model category structure for the category of crossed modules of MCI, where all objects are both fibrant and cofibrant.

As indicated in [9], a functorial relation exists between the categories of associative algebras, Leibniz algebras, Lie algebras, dialgebras and crossed modules in these categories that can be pictured with the diagram below, where all faces are commutative:

$$\begin{array}{ccc}
\text{DiAs} & \xrightarrow{J_i} & \text{As} \\
\downarrow & & \downarrow U \\
\text{Li}_{2} & \xrightarrow{U} & \text{Lb} \\
\downarrow c & & \\
\text{XAs} & \xrightarrow{XU} & \text{XLie}_{1} \\
\downarrow & & \downarrow c \\
\text{XLie} & \xrightarrow{XLie_{2}} & \text{XLb} \\
\downarrow & & \\
\text{XAs} & \xrightarrow{XLie_{1}} & \text{XLie} \\
\end{array}$$

(1.2)

In this connection, we intend to handle these crossed module structures and define the homotopy of morphisms by considering the particular cases of homotopy definition of crossed modules of MCI. On the other hand, one can see that the homotopy definitions given in [1, 21] can also be obtained from our unified homotopy definition. Moreover, we also see that the adjoint crossed module functors given in (1.2) preserve the homotopy relation.
One of the main outcomes of this paper is that the functor $X_1$ in (1.1) preserves the homotopy and, what is more, the homotopy equivalence between simplicial objects and crossed modules in MCI. However, this property can not be extended to a groupoid functor since the groupoid structure over simplicial homotopy has not yet been discovered, even for the cases of groups or algebras.

\section{Preliminaries}

Let us recall the main definitions from \cite{5, 23, 26, 29} that will be used in the sequel.

\subsection{Modified categories of interest}

Let $\mathcal{C}$ be a category of groups with a set of operations $\Omega$ and with a set of identities $\mathcal{E}$ such that $\mathcal{E}$ includes the group identities and, if $\Omega_1$ is the set of $i$-ary operations in $\Omega$, the following conditions hold:

(a) $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$.

(b) The group operations (written additively $0, -, +$) are elements of $\Omega_0$, $\Omega_1$ and $\Omega_2$, respectively. Let $\Omega_2^+ = \Omega_2 \setminus \{+\}$ and $\Omega_1^- = \Omega_1 \setminus \{-\}$. Assume that if $* \in \Omega_2$, then $\Omega_2'$ contains $*^*$ defined by $x *^* y = y * x$ and assume $\Omega_0 = \{0\}$.

(c) For each $* \in \Omega_2^+$, the set $\mathcal{E}$ includes the identity $x * (y + z) = x * y + x * z$.

(d) For each $\omega \in \Omega_1^+$ and $* \in \Omega_2^+$, the set $\mathcal{E}$ includes the identity $\omega(x + y) = \omega(x) + \omega(y)$ and either the identities $\omega(x * y) = \omega(x) * \omega(y)$ or $\omega(x * y) = \omega(x) * y$.

Denote by $\Omega_{15}^+$ the subset of those elements in $\Omega_1^+$ which satisfy the identity $\omega(x * y) = \omega(x) * y$, and by $\Omega_{15}'$ all other unary operations, i.e., those which satisfy the first identity from (d).

Let $\mathcal{C}$ be an object of $\mathcal{C}$ and let $x_1, x_2, x_3 \in \mathcal{C}$.

\textbf{Axiom 1.} For all $* \in \Omega_2^+$ we have

\[ x_1 + (x_2 * x_3) = (x_2 * x_3) + x_1. \]

\textbf{Axiom 2.} For each ordered pair $(*, \omega) \in \Omega_1^+ \times \Omega_2^+$ there is a word $W$ such that

\[ (x_1 * x_2)\omega x_3 = W(x_1 x_2 x_3), (x_1 x_2 x_3) x_1, (x_3 x_2) x_1, x_2 (x_1 x_3), x_2 (x_3 x_2), (x_3 x_1) x_2, (x_3 x_1) x_2, \]

where each juxtaposition represents an operation in $\Omega_2^+$.

\textbf{Definition 2.1.} A category of groups with operations $\mathcal{C}$ satisfying conditions (a)–(d), Axiom 1 and Axiom 2 will be called a “modified category of interest” or MCI for short.

As indicated in \cite{5}, the difference of this definition from the original one of the category of interest is the modification of the latter condition. According to this definition, every category of interest is a modified category of interest as well.

\textbf{Example 2.2.} The categories of (pre)cat$^1$-objects in the categories of Leibniz (Lie, associative) algebras and dialgebras are all modified categories of interest, which are not categories of interest.

\textbf{Example 2.3.} The category of noncommutative Leibniz–Poisson algebras introduced in \cite{10} is neither a category of interest nor a modified category of interest.

\textbf{Definition 2.4.} Let $A, B \in \mathcal{C}$. An extension of $B$ by $A$ is the sequence

\[ 0 \longrightarrow A \overset{i}{\longrightarrow} E \overset{p}{\longrightarrow} B \longrightarrow 0 \]

in which $p$ is surjective and $i$ is the kernel of $p$. We say that the extension is split if there exists a morphism $s : B \to E$ such that $p \circ s = 1_B$. 

Definition 2.5. Suppose that $A$, $B$ are objects of $\mathcal{C}$. We say that we have a set of actions of $B$ on $A$ if there exists a map

$$f_* : A \times B \to A$$

for all $* \in \Omega_2$. A split extension of $B$ by $A$ induces an action of $B$ on $A$ corresponding to the operations in $\mathcal{C}$ as follows:

$$b \cdot a = s(b) + a - s(b),$$
$$b * a = s(b) * a,$$

for all $b \in B$, $a \in A$ and $* \in \Omega_2'$. These actions are called derived actions of $B$ on $A$. Note that we use the notation "*" to denote both the star operation and the star action.

Definition 2.6. Given an action of $B$ on $A$, a semi-direct product $A \rtimes B$ is a universal algebra whose underlying set is $A \times B$ and the operations are defined by

$$\omega(a, b) = (\omega(a), \omega(b)),$$
$$(a', b') + (a, b) = (a' + b' \cdot a, b' + b),$$
$$(a', b') \cdot (a, b) = (a' + a' \cdot b + b' \cdot a, b' \cdot b)$$

for all $a, a' \in A$ and $b, b' \in B$.

Theorem 2.7. An action of $B$ on $A$ is a derived action if and only if $A \rtimes B$ is an object of $\mathcal{C}$.

Proposition 2.8. A set of actions of $B$ on $A$ in $\mathcal{C}_G$ is a set of derived actions [5] if and only if it satisfies the following conditions:

1. $0 \cdot a = a$,
2. $b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2$,
3. $(b_1 + b_2) \cdot a = b_1 \cdot (b_2 \cdot a)$,
4. $b \cdot (a_1 + a_2) = b \cdot a_1 + b \cdot a_2$,
5. $(b_1 + b_2) \cdot a = b_1 \cdot a + b_2 \cdot a$,
6. $(b_1 + b_2) \cdot (a_1 + a_2) = a_1 + a_2$,
7. $(b_1 + b_2) \cdot (a \cdot b) = a \cdot b$,
8. $a_1 \cdot (b \cdot a) = a_1 \cdot a_2$,
9. $b \cdot (b_1 \cdot a) = b \cdot a$,
10. $\omega(b \cdot a) = \omega(b) \cdot \omega(a)$,
11. $\omega(a \cdot b) = \omega(a) \cdot b = a \cdot \omega(b)$ for any $\omega \in \Omega_1'$, and $\omega(a \cdot b) = \omega(a) \cdot \omega(b)$ for any $\omega \in \Omega_2'$,
12. $x \cdot y + z \cdot t = z \cdot t + x \cdot y$

for all $\omega \in \Omega_1'$, $* \in \Omega_2'$, $b, b_1, b_2 \in B, a, a_1, a_2 \in A$ and $x, y, z, t \in A \cup B$ whenever each side of 12 has sense.

2.2 Crossed modules in MCI

We work in an arbitrary modified category of interest $\mathcal{C}$ in the rest of the paper.

Definition 2.9. A crossed module $(E, R, \partial)$ in $\mathcal{C}$ is a morphism $\partial : E \to R$ together with a derived action of $R$ on $E$ such that the following Peiffer–Whitehead relations hold:

(XM1) $\partial(r \cdot e) = r + \partial(e) - r$ and $\partial(r \cdot e) = r \cdot \partial(e)$,
(XM2) $\partial(e) \cdot e' = e + e' - e$ and $\partial(e) \cdot e' = e \cdot e'$,

for all $e, e' \in E, r \in R$ and $* \in \Omega_2'$. Without the second relation, it is called a precrossed module.

From now on, we denote an arbitrary crossed module in $\mathcal{C}$ by $\mathcal{X} = (E, R, \partial)$.

Definition 2.10. Let $\mathcal{X}, \mathcal{X}'$ be two crossed modules. A crossed module morphism $f : \mathcal{X} \to \mathcal{X}'$ is a pair

$$f = (f_1 : E \to E', f_0 : R \to R')$$
of morphisms of $\mathcal{C}$ such that the diagram

$$
\begin{array}{ccc}
E & \xrightarrow{\delta} & R \\
\downarrow{f_1} & & \downarrow{f_0} \\
E' & \xrightarrow{\delta'} & R'
\end{array}
$$

(2.2)

is commutative and also preserves the derived action of $R$ on $E$, namely (for all $e \in E$ and $r \in R$)

$$
f_1(r \cdot e) = f_0(r) \cdot f_1(e),
$$

$$
f_1(r \ast e) = f_0(r) \ast f_1(e).
$$

Consequently, we have the category of crossed modules in $\mathcal{C}$ denoted by $\textbf{XMod}(\mathcal{C})$.

### 2.3 Simplicial objects in MCI

**Definition 2.11.** A simplicial object in $\mathcal{C}$ is a functor: $\Delta^{op} \to \mathcal{C}$, where $\Delta$ is a simplicial indexing category.

An alternative definition of a simplicial object can be given as follows.

**Definition 2.12.** A simplicial object $\mathcal{A}$ in $\mathcal{C}$ is a collection of objects $\{A_n : n \in \mathbb{N}\}$ together with the morphisms

$$
d^n_i : A_n \to A_{n-1}, \quad 0 \leq i \leq n,
$$

$$
s^n_j : A_n \to A_{n+1}, \quad 0 \leq j \leq n,
$$

which are called faces and degeneracies, respectively (we will not use superscripts in the sequel).

These morphisms are required to satisfy the following simplicial identities:

$$
\begin{align*}
\text{(i)} & \quad d_i d_j = d_{j-1} d_i \quad \text{if } i < j, \\
\text{(ii)} & \quad s_i s_j = s_{j+1} s_i \quad \text{if } i \leq j, \\
\text{(iii)} & \quad d_i s_j = s_{j+1} d_i \quad \text{if } i < j, \\
& \quad d_i s_j = d_j s_j = \text{id}, \\
& \quad d_i s_j = s_j d_{i-1} \quad \text{if } j > i + 1.
\end{align*}
$$

(2.3)

Any simplicial object can be pictured as

$$
\mathcal{A} \cong \begin{array}{ccc}
A_3 & \xrightarrow{d_2} & A_2 \\
\downarrow{d_1} & & \downarrow{d_0} \\
A_1 & \xrightarrow{d_0} & A_0 \\
\end{array}
$$

(2.4)

**Definition 2.13.** A simplicial map $f : \mathcal{A} \to \mathcal{B}$ is a set of morphisms $f_n : A_n \to B_n$ commuting with all possible face and degeneracy morphisms, namely

$$
f_q d_i = d_if_{q+1},
$$

$$
f_q s_i = s_if_{q-1},
$$

in the diagram

$$
\begin{array}{ccc}
\mathcal{A} & \xrightarrow{f} & \mathcal{B} \\
\downarrow{f_3} & & \downarrow{f_0} \\
A_3 & \xrightarrow{d_2} & A_2 \\
\downarrow{d_1} & & \downarrow{d_0} \\
A_1 & \xrightarrow{d_0} & A_0 \\
\end{array}
$$

Thus we have the category of simplicial objects in $\mathcal{C}$ that are denoted by $\textbf{Simp}(\mathcal{C})$. 
Definition 2.14. An $n$-truncated simplicial object is a simplicial object with objects $A_i$ ($i \leq n$). Therefore, we get the corresponding category $\text{Tr}_n\text{Simp}(C)$ that can be seen as a full subcategory of $\text{Simp}(C)$.

Definition 2.15. Given a simplicial object $\mathcal{A}$, the Moore complex $(NA, \partial)$ of $\mathcal{A}$ is the chain complex defined by

$$NA_n = \bigcap_{i=0}^{n-1} \ker(\partial_i)$$

with the morphisms $\partial_n : NA_n \to NA_{n-1}$ that are induced from $\partial_n$ by restriction.

Definition 2.16. Let $(NA, \partial)$ be the Moore complex of a simplicial object $\mathcal{A}$. We call it the Moore complex of length $n$ if $NA_i$ is equal to $\{0\}$ for each $i > n$. We denote the category of simplicial objects with Moore complex of length $n$ by $\text{Simp}(C)_{\leq n}$.

2.3.1 Simplicial homotopy

Definition 2.17. Given two simplicial maps $f, g : \mathcal{A} \to \mathcal{B}$, a simplicial homotopy [29] which connects $f$ to $g$ is a collection of morphisms

$$h_i^n : A_n \to B_{n+1}, \quad 0 \leq i \leq n,$$

which satisfies the following identities:

$$
\begin{array}{ll}
(i) & d_0 h_0 = f, \quad d_{q+1} h_q = g, \\
(ii) & d_i h_j = h_{j-1} d_i \quad \text{if} \quad i < j, \\
 & d_{j+1} h_{j+1} = d_j h_j, \\
 & d_i h_j = h_{j-1} d_{i-1} \quad \text{if} \quad i > j, \\
(iii) & s_i h_j = h_{j-1} s_i \quad \text{if} \quad i \leq j, \\
 & s_i h_j = h_j s_{i-1} \quad \text{if} \quad i > j.
\end{array}
$$

(2.5)

In this case, we write $f \xrightarrow{(h_i^n)} g$ or $f \sim g$. All of them fit into the diagram

3 Homotopy of crossed modules in MCI

We fix two arbitrary crossed modules $\mathcal{X} = (E, \mathcal{R}, \partial)$ and $\mathcal{X}' = (E', \mathcal{R}', \partial')$ in $C$.

3.1 Derivation and homotopy

Definition 3.1. Let $f_0 : R \to R'$ be a morphism of $C$. An $f_0$-derivation $s : R \to E'$ is a map satisfying

$$
\begin{cases}
    s(a + b) = (f_0(-b) \cdot s(a)) + s(b), \\
    s(a \cdot b) = f_0(a) \cdot s(b) + f_0(b) \cdot s(a) + s(a) \cdot s(b)
\end{cases}
$$

for all $a, b \in R$. 

Lemma 3.2. If $s$ is an $f_0$-derivation for a given morphism $f_0 : R \to R'$, then

- $s(0) = 0$,
- $s(-a) = f_0(a) \cdot (-s(a))$,
- $s(a + b - a) = f_0(a) \cdot (f_0(-b) \cdot s(a) + s(b)) + s(-a)$

for all $a, b \in R$.

Proof. This is proved by easy calculations. $\square$

Lemma 3.3. Let $f : \mathcal{X} \to \mathcal{X}'$ be a crossed module morphism. Any $f_0$-derivation $s : R \to E'$ can be seen as a unique morphism of $s$ in the following form:

$$\phi : r \in R \mapsto (s(r), f_0(r)) \in E' \times R'.$$

Proof. This follows directly by using (2.1) and (3.1). $\square$

This lemma plays a key role in [1, 24] for the case of 2-crossed modules.

Theorem 3.4. Let $f : \mathcal{X} \to \mathcal{X}'$ be a crossed module morphism and let $s$ be an $f_0$-derivation. Define

$$g_0(r) = f_0(r) + (\partial' \circ s)(r), \quad g_1(e) = f_1(e) + (s \circ \partial)(e), \quad (3.2)$$

where $e \in E$ and $r \in R$. The tuple $g = (g_1, g_0)$ defines a crossed module morphism between $\mathcal{X} \to \mathcal{X}'$.

Proof. Since for all $r, r' \in R$,

$$g_0(r * r') = f_0(r * r') + \partial' s(r * r')$$

$$= f_0(r) + f_0(r') + \partial' (f_0(r) \ast s(r') + f_0(r') \ast s(r) + s(r) \ast s(r'))$$

$$= f_0(r) \ast f_0(r') + \partial' (f_0(r) \ast s(r')) + \partial' (f_0(r') \ast s(r)) + \partial' (s(r) \ast s(r'))$$

$$= f_0(r) \ast f_0(r') + f_0(r) \ast s(r') + f_0(r') \ast s(r) + s(r) \ast s(r')$$

and

$$g_0(r + r') = f_0(r + r') + \partial' s(r + r')$$

$$= f_0(r) + f_0(r') + \partial' ((f_0(r) \ast s(r)) + s(r'))$$

$$= f_0(r) + f_0(r') + \partial' f_0(-r') \ast s(r)) + \partial' s(r')$$

$$= f_0(r) + f_0(r') + \partial' (f_0(r) \ast s(r) + f_0(r') + \partial' s(r'))$$

$$= f_0(r) \ast f_0(r') + f_0(r) \ast s(r') + f_0(r') \ast s(r) + s(r')$$

$$= g_0(r) + g_0(r'),$$

$g_0$ is a morphism of $C$; similarly, $g_1$ as well. It is also easy to check that diagram (2.2) commutes. Finally, $g_1$ preserves the derived actions of $R$ on $E$:

$$g_1(r * e) = f_1(r * e) + s\partial(r * e)$$

$$= f_0(r) \ast f_1(e) + s(\partial(e))$$

$$= f_0(r) \ast f_1(e) + f_0(r) \ast s\partial(e) + f_0\partial(e) \ast s(r) + s(r) \ast s(\partial(e))$$

$$= f_0(r) \ast f_1(e) + f_0(r) \ast s\partial(e) + \partial' f_1(e) \ast s(r) + s(r) \ast s(\partial(e))$$

$$= f_0(r) \ast f_1(e) + f_0(r) \ast s\partial(e) + s(r) \ast s(\partial(e))$$

$$= f_0(r) \ast f_1(e) + f_0(r) \ast s\partial(e) + \partial' s(r) \ast f_1(e) + \partial' s(r) \ast s\partial(e)$$

$$= (f_0(r) \ast \partial' s(r)) \ast (f_1(e) + s\partial(e))$$

$$= g_0(r) \ast g_1(e)$$

for all $r \in R$ and $e \in E$. 
On the other hand,
\[ g_o(r) \cdot g_1(e) = (f_0(r) + \partial' s(r)) \cdot (f_1(e) + s\delta(e)) \]
\[ = (f_0(r) + \partial' s(r)) \cdot f_1(e) + (f_0(r) + \partial' s(r)) \cdot s\delta(e) \]
\[ = f_0(r) \cdot (\partial' s(r)) \cdot f_1(e) + f_0(r) \cdot (\partial' s(r)) \cdot s\delta(e) \]
\[ = f_0(r) \cdot (s(r) + f_1(e) - s(r)) + f_0(r) \cdot (s(r) + s\delta(e) - s(r)) \]
\[ = f_0(r) \cdot s(r) + f_0(r) \cdot f_1(e) + f_0(r) \cdot (-s(r)) + f_0(r) \cdot s\delta(e) + f_0(r) \cdot (-s(r)) \]
\[ = f_0(r) \cdot s(r) + f_0(r) \cdot f_1(e) + f_0(r) \cdot s\delta(e) + f_0(r) \cdot (-s(r)) \]
and also
\[ g_1(r \cdot e) = f_1(r \cdot e) + s\delta(r \cdot e) \]
\[ = f_0(r) \cdot f_1(e) + s(r + \partial(e) - r) \]
\[ = f_0(r) \cdot f_1(e) + f_0(r) \cdot (f_0(\partial(-e)) \cdot s(r) + s\delta(e))) + s(-r) \]
\[ = f_0(r) \cdot f_1(e) + f_0(r) \cdot f_0(\partial(-e)) \cdot s(r) + s\delta(e))) + s(-r) \]
\[ = f_0(r) \cdot f_1(e) + f_0(r) \cdot f_0(\partial(-e)) \cdot s(r) + s\delta(e) + s(-r) \]
\[ = f_0(r) \cdot s(r) + f_0(r) \cdot f_1(e) + f_0(r) \cdot s\delta(e) + f_0(r) \cdot (-s(r)) \]
for all \( r \in R \) and \( e \in E \). By using Lemma 3.2, we obtain
\[ g_1(r \cdot e) = g_o(r) \cdot g_1(e). \]

Therefore, \( g = (g_1, g_o) : \mathcal{X} \to \mathcal{X}' \) is a crossed module morphism.

**Notation 3.5.** To make the formulas more compact, we do not use the symbol “\( \circ \)” to denote the compositions in general.

**Definition 3.6.** In the condition of the previous theorem, we write \( f \xrightarrow{(f_o, s)} g \) or shortly \( f = g \), and say that \( (f_o, s) \) is a homotopy (or derivation) connecting \( f \) to \( g \).

As a consequence of this homotopy definition, we can also provide the following definition.

**Definition 3.7.** Let \( \mathcal{X}, \mathcal{X}' \) be two crossed modules. If there exist crossed module morphisms \( f : \mathcal{X} \to \mathcal{X}' \) and \( g : \mathcal{X}' \to \mathcal{X} \) such that \( f \circ g = \text{id}_{\mathcal{X}'} \) and \( g \circ f = \text{id}_{\mathcal{X}} \), we say that the crossed modules \( \mathcal{X} \) and \( \mathcal{X}' \) are homotopy equivalent, denoted by \( \mathcal{X} \simeq \mathcal{X}' \).

**Remark 3.8.** In the calculations above, we use the second Peiffer–Whitehead relation (Definition 2.9), so that this homotopy definition does not hold for precrossed modules (see details in [21] for the group case).

### 3.2 A groupoid

Now let us construct a groupoid structure which is induced from homotopy of crossed module morphisms in \( \mathbb{C} \).

**Lemma 3.9 (Identity).** Let \( f = (f_1, f_0) \) be a crossed module morphism \( \mathcal{X} \to \mathcal{X}' \). The null function
\[ O_0 : r \in R \mapsto 0_{E'} \in E' \]
defines an \( f_0 \)-derivation connecting \( f \xrightarrow{(f_o, O_0)} f \).

**Proof.** This is proved by easy calculations. \( \square \)

**Lemma 3.10 (Inverse).** Let \( f = (f_1, f_0) \) and \( g = (g_1, g_0) \) be crossed module morphisms \( \mathcal{X} \to \mathcal{X}' \) and let \( s \) be an \( f_0 \)-derivation connecting \( f \xrightarrow{(f_o, s)} g \).
Then the map \( \tilde{s} = -s : R \to E' \) with \( \tilde{s}(r) = -s(r) \) defines a \( g_0 \)-derivation connecting

\[
g \xrightarrow{(g_0, \tilde{s})} f.
\]

**Proof.** Since \( s \) is an \( f_0 \)-derivation connecting \( f \xrightarrow{(f_0, s)} g \), we have

\[
g_0(r) = f_0(r) + (\partial' \circ s)(r), \quad g_1(e) = f_1(e) + (s \circ \partial)(e),
\]

which implies (3.2), i.e.,

\[
f_0(r) = g_0(r) + (\partial' \circ s)(r), \quad f_1(e) = g_1(e) + (s \circ \partial)(e)
\]

for all \( r \in R \) and \( e \in E \). Moreover, \( \tilde{s} \) is a \( g_0 \)-derivation since

\[
\tilde{s}(r + r') = -(s(r + r'))
\]

\[
\tilde{s}(r + r') = -(s(r + r')) = -(f_0(-r') \cdot s(r))
\]

and also

\[
\tilde{s}(r * r') = -(s(r * r')) = -(f_0(r) * s(r')) + f_0(r') * s(r) + s(r) * s(r')
\]

for all \( r, r' \in R \).

**Lemma 3.11 (Concatenation).** Let \( f = (f_1, f_0) \), \( g = (g_1, g_0) \) and \( k = (k_1, k_0) \) be crossed module morphisms between \( X \to X' \), let \( s \) be an \( f_0 \)-derivation connecting

\[
f \xrightarrow{(f_0, s)} g,
\]

and let \( s' \) be a \( g_0 \)-derivation connecting

\[
g \xrightarrow{(g_0, s')} k.
\]

Then the linear map \( (s + s') : R \to E' \) given by \( (s + s')(r) = s(r) + s'(r) \) defines an \( f_0 \)-derivation connecting

\[
f \xrightarrow{(f_0, s + s')} k.
\]

**Proof.** We have derivations \( f \xrightarrow{(f_0, s)} g \) and \( g \xrightarrow{(g_0, s')} k \). By using these homotopies, the prerequisite conditions (3.2) are satisfied with

\[
k_0(r) = f_0(r) + (\partial' \circ (s + s'))(r), \quad k_1(e) = f_1(e) + ((s + s') \circ \partial)(e).
\]
Moreover, we get (for all \(r, r' \in R\))

\[
(s + s')(r + r') = s(r + r') + s'(r + r')
\]

\[
= s_0(-r') \cdot s(r) + s'(r') + (g_0(-r') \cdot s'(r)) + s'(r')
\]

\[
= (f_0(-r') \cdot s(r)) + s'(r') + ((f_0(-r') + \partial' s(-r')) \cdot s'(r)) + s'(r')
\]

\[
= (f_0(-r') \cdot s(r)) + s'(r') + (f_0(-r') \cdot (\partial' s(-r') \cdot s'(r)) + s'(r')
\]

\[
= (f_0(-r') \cdot s(r) + s'(r') + f_0(-r') \cdot (f_0(r') \cdot (-s(r'))) + f_0(-r') \cdot s'(r)
\]

\[
- f_0(-r') \cdot (f_0(r') \cdot (-s(r'))) + s'(r')
\]

\[
= f_0(-r') \cdot s(r) + f_0(-r') \cdot s'(r) + s'(r')
\]

\[
= (f_0(-r') \cdot (s + s'(r))(s + s'(r'))
\]

and also

\[
(s + s')(r * r') = s(r * r') + s'(r * r')
\]

\[
\]

\[
= f_0(r) \cdot s(r') + f_0(r') \star s(r) + s(r) \star s'(r') + g_0(r) \star s'(r) + g_0(r') \star s'(r) + s'(r)
\]

\[
+ (f_0(r') + (\partial' \circ s)(r')) \star s'(r) + s'(r) \star s'(r')
\]

\[
= f_0(r) \cdot s(r') + f_0(r') \star s(r) + s(r) \star s'(r') + f_0(r) \star s'(r')
\]

\[
+ (\partial' \circ s)(r) \star s'(r') + f_0(r') \star s'(r) + (\partial' \circ s)(r') \star s'(r) + s'(r) \star s'(r')
\]

\[
= f_0(r) \cdot s(r') + f_0(r') \star s(r) + s(r) \star s'(r') + f_0(r) \star s'(r') + s(r) \star s'(r')
\]

\[
+ f_0(r') \star s'(r) + s'(r') \star s'(r) + s'(r) \star s'(r')
\]

\[
= f_0(r) \cdot (s + s'(r)) + f_0(r') \star (s + s'(r)) + (s + s'(r)) \star (s + s'(r'))
\]

which proves that \((s + s')\) is an \(f_0\)-derivation connecting \(f \rightarrow k\).

\[\]

**Remark 3.12.** Note that we frequently use Axiom 1, the crossed module axioms (Definitions 2.9 and 2.10), and the facts of Lemma 3.2 in the previous proofs.

Now we can give the following corollary.

**Corollary 3.13.** Let \(X, X'\) be two arbitrary but fixed crossed modules in \(C\). We have a groupoid \(\text{HOM}(X, X')\) whose objects are the crossed module morphisms \(X \rightarrow X'\), the morphisms being their homotopies. In particular, the relation below for crossed module morphisms \(X \rightarrow X'\) is an equivalence relation:

\[
f \approx g \iff \text{there exists an } f_0\text{-derivation } s \text{ connecting } f \text{ to } g.
\]

**Proof.** Follows from Lemmas 3.9, 3.10 and 3.11.

\[\]

## 4 From simplicial homotopy to crossed module homotopy

It is a well-known equivalence from [4] that for a (modified) category of interest \(C\), the category of crossed modules is equivalent to the category of simplicial objects with Moore complex of length 1 with the functors

\[\]

\[\]

\[\]

\[\]
In this section, we intend to explore the homotopical properties of the functor \( X_1 : \text{Simp}(C)_{\leq 1} \to \text{XMod}(C) \), but first we need to recall how the functor \( X_1 \) works.

Suppose that \( A \) is a simplicial object with Moore complex of length 1, as seen on (2.4). We can construct the crossed module \( X_1(A) \) as follows:

(i) Put \( R = NA_0 = A_0 \) and \( E = NA_1 = \ker(d_0) \).

(ii) Define the action of \( R \) on \( E \) as follows:

\[
\begin{align*}
    r \cdot e &= s_0(r) + e - s_0(r), \\
    r \cdot e &= s_0(r) \cdot e.
\end{align*}
\]

(iii) Put \( \partial = d_1 \) (restricted to \( E \)).

Then we get the crossed module \((E, R, \partial)\), namely

\[
\ker(d_0) \xrightarrow{d_1} A_0.
\]

**Theorem 4.1.** The functor \( X_1 \) preserves the homotopy. In other words, let \( A \) and \( B \) be two simplicial objects with Moore complex of length 1, and let \( f, g : A \to B \) be two homotopic simplicial maps with \( h : f \simeq g \). Then

\[
X_1(f) = X_1(g).
\]

**Proof.** Let us define the map

\[
\zeta : h = \{h^n\} \mapsto -s_0f_0 + h_0^0.
\]

We claim that \( \zeta \) leads to the following homotopy of crossed module morphisms:

\[
X_1(f) \xrightarrow{(f_0, \zeta(h))} X_1(g).
\]

Graphically, we can summarize all this as the following diagrams:

![Diagram](image)

Note that we have \( X_1(f_1) = f_1 \) (restriction) and \( X_1(f_0) = f_0 \) for any simplicial map \( f : A \to B \).

(i) First of all, the map \( \zeta \) is well defined since

\[
\begin{align*}
    d_0((-s_0f_0 + h_0^0)(a)) &= d_0(-s_0f_0)(a) + h_0^0(a) \\
    &= -d_0(s_0f_0)(a) + d_0(h_0^0(a)) \\
    &= -(d_0s_0)(f_0(a)) + f_0(a) \\
    &= -f_0(a) + f_0(a) \\
    &= 0_{B_0}
\end{align*}
\]

for all \( a \in A_0 \), which means

\[
\text{Im}(\zeta) \subseteq \ker(d_0) = NB_1.
\]
(ii) Now, $\zeta(h)$ needs to be an $f_0$-derivation which satisfies conditions (3.1). Put $\zeta(h) = H$. We get
\[
H(r + r') = (-s_0 f_0 + h_0^f)(r + r')
\]
\[
= -s_0 f_0 (r + r') + h_0^f (r + r')
\]
\[
= (-s_0 f_0(r) + s_0 f_0(r')) + h_0^f (r) + h_0^f (r')
\]
\[
= -s_0 f_0(r') - s_0 f_0(r) + h_0^f (r) + h_0^f (r')
\]
\[
= -s_0 f_0(r') - s_0 f_0(r) + h_0^f (r) + s_0 f_0(r') - h_0^f (r')
\]
\[
= f_0(-r') \cdot (-s_0 f_0(r) + h_0^f (r)) - s_0 f_0(r') + h_0^f (r')
\]
\[
= (f_0(-r') \cdot H(r)) + H(r').
\]
We also need to show that
\[
H(r * r') = f_0(r) * H(r') + f_0(r') * H(r) + H(r) * H(r').
\] (4.1)
On the left-hand side, we have
\[
H(r * r') = (-s_0 f_0 + h_0^f)(r * r')
\]
\[
= -s_0 f_0(r * r') + h_0^f (r * r')
\]
\[
= -s_0 f_0(r) * s_0 f_0(r') + h_0^f (r) * h_0^f (r'),
\]
while the right-hand side is
\[
f_0(r) * H(r') + f_0(r') * H(r) * H(r')
\]
\[
= f_0(r) * (-s_0 f_0 + h_0^f)(r') + f_0(r') * (-s_0 f_0 + h_0^f)(r) \cdot (-s_0 f_0 + h_0^f)(r')
\]
\[
= s_0 f_0(r) * (-s_0 f_0 + h_0^f)(r') + s_0 f_0(r') * (-s_0 f_0 + h_0^f)(r) \cdot (-s_0 f_0 + h_0^f)(r')
\]
\[
= s_0 f_0(r) * h_0^f (r') - s_0 f_0(r) * s_0 f_0(r') + s_0 f_0(r') * h_0^f (r') - s_0 f_0(r') * s_0 f_0(r')
\]
\[
+ h_0^f (r) * h_0^f (r') - h_0^f (r) * s_0 f_0(r') - s_0 f_0(r) * h_0^f (r') + s_0 f_0(r) * s_0 f_0(r')
\]
\[
= -s_0 f_0(r) * s_0 f_0(r') + h_0^f (r) * h_0^f (r'),
\]
which gives us (4.1).

(iii) Finally, we need to prove that the conditions (3.2) are satisfied. It is clear that
\[
(\partial' \circ H)(a) = d_1(-s_0 f_0 + h_0^f)(a)
\]
\[
= -d_1 s_0 f_0(a) + d_1 h_0^f (a)
\]
\[
= -f_0(a) + d_1 h_0^f (a)
\]
\[
= -f_0(a) + g_0(a),
\]
for all $a \in A_0$, which leads to
\[
g_0(a) = f_0(a) + (\partial' \circ H)(a).
\]
For the second condition required, we first obtain
\[
H \circ \partial = (-s_0 f_0 + h_0^f) d_1
\]
\[
= -s_0 f_0 d_1 + h_0^f d_1
\]
\[
= -s_0 d_0 h_0^f d_1 + h_0^f d_1
\]
\[
= -s_0 d_0 d_2 h_0^f + d_2 h_0^f
\]
\[
= -s_0 d_1 d_0 h_0^f + d_2 h_0^f
\]
\[
= -d_2 s_0 d_0 h_0^f + d_2 h_0^f
\]
\[
= -d_2 s_0 d_0 h_0^f + d_2 h_0^f - g_1 + f_1 - f_1 + g_1
\]
\[
= (-d_2 s_0 d_0 h_0^f + d_2 h_0^f - d_2 h_1^f + d_0 h_0^f) - f_1 + g_1
\]
\[
= d_2(-s_0 d_0 h_0^f + h_0^f - h_1^f + s_1 d_0 h_0^f) - f_1 + g_1,
\] (4.2)
which needs to be equal to $-f_1 + g_1$. 

So we just need to show that
\[
d_2((-s_0d_0h_0^1 + h_0^1 - h_1^1 + s_1d_0h_0^1)(e)) = 0
\] (4.3)
for all \(e \in \text{Ker}(d_0)\).

Since the Moore complex of \(B\) is of length 1, we have \(NB_2 = 0\), which implies
\[
d_2(l) = 0
\]
for all \(l \in NB_2\).
So, if we prove
\[
(-s_0d_0h_0^1 + h_0^1 - h_1^1 + s_1d_0h_0^1)(e) \in NB_2
\]
for all \(e \in \text{Ker}(d_0)\), then (4.3) will be satisfied.

In this case (for all \(e \in \text{Ker}(d_0)\)),
\[
d_0((-s_0d_0h_0^1 + h_0^1 - h_1^1 + s_1d_0h_0^1)(e)) = -d_0s_0d_0h_0^1(e) + d_0h_0^1(e) - d_0h_1^1(e) + d_0s_1d_0h_0^1(e)
\]
\[
= -d_0h_0^1(e) + d_0h_0^1(e) - d_0h_0^1(e) + s_0d_0d_0h_0^1(e)
\]
\[
= -h_0^1d_0(e) + s_0d_0d_1h_0^1(e)
\]
\[
= s_0d_0d_1h_0^1(e)
\]
\[
= s_0d_0h_0^2d_0(e)
\]
\[
= 0,
\]
so that
\[
(-s_0d_0h_0^1 + h_0^1 - h_1^1 + s_1d_0h_0^1)(e) \in \text{Ker}(d_0).
\] (4.4)

Similarly,
\[
d_1((-s_0d_0h_0^1 + h_0^1 - h_1^1 + s_1d_0h_0^1)(e)) = -d_1s_0d_0h_0^1(e) + d_1h_0^1(e) - d_1h_1^1(e) + d_1s_1d_0h_0^1(e)
\]
\[
= -d_0h_0^1(e) + d_1h_0^1(e) - d_1h_1^1(e) + d_0h_0^1(e)
\]
\[
= -d_0h_0^1(e) + d_1h_1^1(e) - d_1h_1^1(e) + d_0h_0^1(e)
\]
\[
= -d_0h_0^1(e) + d_0h_0^1(e)
\]
\[
= 0,
\]
so that
\[
(-s_0d_0h_0^1 + h_0^1 - h_1^1 + s_1d_0h_0^1)(e) \in \text{Ker}(d_1).
\] (4.5)

By (4.4) and (4.5) we have
\[
(-s_0d_0h_0^1 + h_0^1 - h_1^1 + s_1d_0h_0^1)(e) \in NB_2 = \text{Ker}(d_0) \cap \text{Ker}(d_1),
\]
which implies
\[
d_2((-s_0d_0h_0^1 + h_0^1 - h_1^1 + s_1d_0h_0^1)(e)) = 0.
\]

Finally, taking into account (4.2), we get
\[
H \circ \partial = -f_1 + g_1,
\]
i.e.,
\[
g_1(e) = f_1(e) + (H \circ \partial)(e)
\]
for all \(e \in \text{Ker}(d_0)\), and we complete the proof.

Note that in the above calculations we explicitly use Axiom 1, simplicial identities (2.3) and simplicial homotopy identities (2.5). □

Consequently, we can provide the following theorem.
Theorem 4.2. The functor $X_1$ preserves the homotopy equivalence. In other words, if $A$ and $B$ are two simplicial objects with Moore complex of length 1 such that $A \cong B$, then

$$X_1(A) \cong X_1(B).$$

Proof. This follows from Theorem 4.1 and the functorial properties of $X_1$. 

5 Applications

If we handle the category $\mathbb{C}$ as the category of groups, which is an MCI, we exactly get the derivation formula of [21], i.e.,

$$s(gh) = (f_0(h^{-1}) \triangleright s(g))s(h).$$

In this section, we examine the homotopies of crossed module morphisms in the categories of associative algebras, Leibniz algebras, Lie algebras and dialgebras (diassociative algebras), which are other examples of MCI. We refer to [8, 9, 18] to recall these structures. We use the symbol “$\triangleright$” to denote the (left) action in such categories and also use “$\lhd$” whenever the second (right) action is required. Additionally, all algebras will be defined over a fixed commutative ring $\kappa$.

5.1 Associative algebras

Definition 5.1. Let $f_0: R \rightarrow R'$ be an associative algebra (or a bare algebra [22]) homomorphism. An $f_0$-derivation $s: R \rightarrow E'$ is a $\kappa$-linear map satisfying (for all $a, b \in R$)

$$s(ab) = f_0(a) \triangleright s(b) + s(a) \lhd f_0(b) + s(a)s(b). \tag{5.1}$$

Note that this formula is the generalization of the derivation formula of the commutative algebra case considered in [1].

5.2 Leibniz algebras

Definition 5.2. Let $f_0: R \rightarrow R'$ be a Leibniz algebra homomorphism. An $f_0$-derivation $s: R \rightarrow E'$ is a $\kappa$-linear map satisfying (for all $a, b \in R$)

$$s([a, b]) = f_0(a) \triangleright s(b) + s(a) \lhd f_0(b) + [s(a), s(b)]. \tag{5.2}$$

5.3 Lie algebras

Definition 5.3. The notion of the homotopy of crossed modules of Lie algebras is obtained by reducing from Leibniz algebras in the sense of [9]. Therefore, the derivation formula will be (for all $a, b \in R$)

$$s([a, b]) = f_0(a) \triangleright s(b) - f_0(b) \triangleright s(a) + [s(a), s(b)]. \tag{5.3}$$

5.4 Dialgebras

Definition 5.4. Let $f_0: R \rightarrow R'$ be a dialgebra homomorphism. An $f_0$-derivation $s: R \rightarrow E'$ is a $\kappa$-linear map satisfying (for all $a, b \in R$)

$$\begin{cases} s(a \triangleright b) = f_0(a) \triangleright \triangleright s(b) + s(a) \lhd \triangleright f_0(b) + s(a) \triangleright \lhd s(b), \\ s(a \triangleright b) = f_0(a) \triangleright \triangleright s(b) + s(a) \lhd \triangleright f_0(b) + s(a) \triangleright \lhd s(b). \end{cases} \tag{5.4}$$

Note that we have two different right (left) actions corresponding to the dialgebra operations $\triangleright$ and $\triangleright \lhd$. 

Theorem 5.5. Let \( f = (f_1, f_0) \) be any crossed module morphism \( \mathcal{X} \to \mathcal{X}' \) in the category of associative algebras, Leibniz algebras, Lie algebras or dialgebras. In the conditions of the previous derivation definitions, if we define \( g = (g_1, g_0) \) as

\[
g_0(r) = f_0(r) + (s' \circ s)(r), \quad g_1(e) = f_1(e) + (s \circ \delta)(e)
\]

for all \( e \in E \) and \( r \in R \), then \( g \) defines a crossed module morphism between \( \mathcal{X} \to \mathcal{X}' \) and we get the homotopy

\[
f \xrightarrow{(f_0, s)} g.
\]

Corollary 5.6. By considering the homotopy (derivation) definitions (5.1)–(5.4), one can see that the adjoint functors

\[
\begin{array}{cccc}
\text{XDiAs} & \text{XAs} & \text{XLie} & \text{XLb} \\
\text{DiAs} & \text{As} & \text{Lie} & \text{Lb} \\
\text{Ud} & \text{XU} & \text{XLie_1} & \text{XLie_2} \\
\text{c} & \bot & \bot & c
\end{array}
\]

preserve not only the crossed module structure, but also the homotopy relations of crossed module morphisms in the sense of [9].

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