Production of the $X(3872)$ at the Tevatron and the LHC

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Abstract

We predict the differential cross sections for production of the $X(3872)$ at the Tevatron and the Large Hadron Collider from both prompt QCD mechanisms and from decays of $b$ hadrons. The prompt cross section is calculated using the NRQCD factorization formula. Simplifying assumptions are used to reduce the nonperturbative parameters to a single NRQCD matrix element that is determined from an estimate of the prompt cross section at the Tevatron. For $X(3872)$ with transverse momenta greater than about 4 GeV, the predicted cross section is insensitive to the simplifying assumptions. We also discuss critically a recent analysis that concluded that the prompt production rate at the Tevatron is too large by orders of magnitude for the $X(3872)$ to be a weakly-bound charm-meson molecule. We point out that if charm-meson rescattering is properly taken into account, the upper bound is increased by orders of magnitude and is compatible with the observed production rate at the Tevatron.

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I. INTRODUCTION

The $X(3872)$ is a $c\bar{c}$ meson that was discovered by the Belle Collaboration in 2003 through the decay $B^+ \rightarrow K^+ + X$ [1]. Its existence was quickly confirmed by the CDF Collaboration through inclusive production of $X(3872)$ in high energy $p\bar{p}$ collisions [2]. While some of the $X(3872)$'s at the Tevatron are produced by $b$-hadron decays, most are produced promptly by QCD mechanisms [3]. A compelling case can be made that the $X(3872)$ is a loosely-bound charm-meson molecule whose particle content is

$$X = \frac{1}{\sqrt{2}} (D^{*0}D^0 + D^0D^{*0}) \equiv (D^{*0}D^0)_{+}. \quad (1)$$

It has univeral properties that are determined by its binding energy only [4, 5]. They imply that the mean separation of its constituents could be larger than for ordinary hadrons by an order of magnitude or more. Given the identification of the $X(3872)$ as a charm-meson molecule, an obvious question is whether the $X(3872)$ could still be sufficiently robust to be observed at a hadron collider like the Tevatron. This issue has been brought into focus by a recent paper by Bignamini et al. [6] that derives an upper bound on the prompt cross section for a charm-meson molecule in terms of charm-meson-pair cross sections. They apply their upper bound to the $X(3872)$ and conclude that the observed prompt production rate at the Tevatron exceeds their bound by orders of magnitude.

The key to understanding how the $X(3872)$ evades the upper bound of Ref. [6] is rescattering. In this paper, we will show that if rescattering of the charm mesons is properly taken into account, the upper bound of Ref. [6] is increased by orders of magnitude, which brings it into consistency with measurements at the Tevatron. Thus, despite its weak binding, the $X(3872)$ is sufficiently robust to be studied at a hadron collider. Studies by the CDF Collaboration at the Tevatron have provided the strongest constraints thus far on the quantum numbers of the $X(3872)$ [7] and the most precise measurement of its binding energy [8]. At the Large Hadron Collider (LHC) at CERN, the data samples of the $X(3872)$ will be much larger than at the Tevatron and thus will allow some of its properties to be studied with much greater precision.

In this paper, we also predict the differential cross sections for production of $X(3872)$ at the Tevatron and the LHC. We calculate the prompt production rates from QCD mechanisms using the NRQCD factorization formalism [9], which expresses the cross section as the sum of parton cross sections for creating $c\bar{c}$ pairs with vanishing relative momentum.
multiplied by phenomenological constants. We use simplifying assumptions to reduce the phenomenological constants to a single NRQCD matrix element, which is determined using an estimate of the prompt cross section at the Tevatron. We also calculate the rates for production of the $X(3872)$ from $b$-hadron decays using a method that gives the observed production rates of the $J/\psi$ from $b$-hadron decays at the Tevatron.

We begin in Section II by summarizing the case for the $X(3872)$ as a loosely-bound charm-meson molecule and describing some of its universal properties. We also derive a factorization formula that determines the dependence of the production rate on the binding energy of the $X(3872)$. In Section III, we use results from the CDF Collaboration to estimate the cross sections for production of the $X(3872)$ at the Tevatron from both prompt QCD mechanisms and $b$-hadron decays. In Section IV, we consider the relation between the production of the $X(3872)$ and the production of pairs of charm mesons. We point out that if the effects of charm-meson rescattering are taken into account, the upper bound on the cross section for the $X(3872)$ that was derived in Ref. [6] is increased by orders of magnitude. We also present an order-of-magnitude estimate of the $X(3872)$ cross section and show that it is compatible with measurements at the Tevatron. In Section V, we use the NRQCD factorization formula for inclusive production to calculate the differential cross sections for prompt $X(3872)$ at the Tevatron. We also calculate the differential cross sections for $X(3872)$ from $b$-hadron decays at the Tevatron. In Section VI, we calculate the differential cross sections for $X(3872)$ at the LHC from both prompt QCD mechanisms and $b$-hadron decays. Finally, we summarize our results in Section VII.

II. THE $X(3872)$

In this section, we summarize the case for the $X(3872)$ as a loosely-bound charm-meson molecule whose particle content is given in Eq. (1). We also derive a factorization formula for production rates of the $X(3872)$ that reveals how the rate depends on its binding energy.

A. Universal Properties

The only experimental information that is necessary to make the identification of the $X(3872)$ as a loosely-bound charm-meson molecule is the determination of its quantum
numbers and the measurements of its mass. The quantum numbers of the $X(3872)$ are $1^{++}$, which follows from

- the observation of its decay into $J/\psi \gamma$, which implies that it is even under charge conjugation [10, 11],
- analyses of the momentum distributions from its decay into $J/\psi \pi^+\pi^-$, which imply that its spin and parity are $1^+$ or $2^-$ [7, 12],
- either the observation of its decays into $D^0\bar{D}^0\pi^0$ [13], which disfavors $2^-$ because of angular momentum suppression, or the observation of its decay into $\psi(2S)\gamma$ [14], which disfavors $2^-$ because of multipole suppression.

The most recent measurements of the mass of the $X(3872)$ in the $J/\psi \pi^+\pi^-$ decay mode [8, 15, 16] imply that its energy relative to the $D^{*0}\bar{D}^0$ threshold is

$$M_X - (M_{D^{*0}} + M_{D^0}) = -0.30 \pm 0.40 \text{ MeV}. \quad (2)$$

The quantum numbers $1^{++}$ imply that the $X(3872)$ has an S-wave coupling to $D^{*0}\bar{D}^0$. Its tiny energy relative to the $D^{*0}\bar{D}^0$ threshold implies that it is a resonant coupling. Thus the $X(3872)$ is an S-wave threshold resonance.

A system of particles with short-range interactions that are fine-tuned so that pairs of particles have an S-wave threshold resonance is predicted by nonrelativistic quantum mechanics to have universal behavior [17]. The universal properties of the system are determined by the pair scattering length $a$, which is large compared to the length scale $1/\Lambda$ set by the range of the interactions. The universal elastic scattering amplitude is

$$f(E) = \frac{1}{-1/a + \sqrt{-2\mu E - i\epsilon}}, \quad (3)$$

where $E$ is the energy relative to the scattering threshold and $\mu$ is the reduced mass of the pair. This scattering amplitude is applicable in the region $|E| < \Lambda^2/(2\mu)$, with errors that scale as $(2\mu |E|)^{1/2}/\Lambda$. If $a > 0$, one of the universal properties is the existence of a molecule with binding energy $E_X = 1/(2\mu a^2)$. The universal wavefunction for this molecule is

$$\psi_X(r) = \frac{1}{\sqrt{2\pi a}} \exp(-r/a). \quad (4)$$
The universal wavefunction in the momentum representation is

$$\tilde{\psi}_X(k) = \frac{\sqrt{8\pi/a}}{k^2 + 1/a^2},$$  \hspace{1cm} (5)

which is applicable for $k < \Lambda$. The wavefunction in the coordinate representation in Eq. (4) implies that the root-mean-square (rms) separation of the constituents is $r_X = a/\sqrt{2}$.

To apply the universal properties of S-wave threshold resonances to the $X(3872)$, we take the scattering length $a$ to be that for charm mesons in the channel $(D^{*0}\bar{D}^0)_+$ defined in Eq. (1). Assuming $a > 0$, the scattering length $a = (2\mu E_X)^{-1/2}$ can be determined by using the binding energy from Eq. (2). Taking $E_X = 0.30 \pm 0.40$ MeV as the input, we find that the charm mesons in the $X(3872)$ have an astonishingly large rms separation: $r_X = 5.8^{+\infty}_{-2.0}$ fm. A reasonable estimate for the momentum scale $\Lambda$ associated with the range of the interactions between the charm mesons is the pion mass $m_\pi \approx 135$ MeV. The binding energy $E_X$ is very small compared to the corresponding energy scale $m_\pi^2/(2\mu) \approx 9.4$ MeV.

**B. Factorization of the Production Rate**

Production rates of the $X(3872)$ satisfy a factorization formula that is based on separating the momentum scale $\sqrt{2\mu E_X}$ associated with the binding of the $X$ from all the larger momentum scales of QCD [18], which include the pion mass $m_\pi$ and the charm quark mass $m_c$. Our starting point for the derivation of the factorization formula is the differential cross section for a pair of charm mesons with energy near the threshold:

$$d\sigma[D^{*0}\bar{D}^0(k)] = \frac{1}{\text{flux}} \sum_{\text{all}} \int d\phi_{D^{*}\bar{D}+\text{all}} \|T[D^{*0}\bar{D}^0(k) + \text{all}]\|^2 \frac{d^3k}{(2\pi)^32\mu},$$  \hspace{1cm} (6)

where $k$ is the relative momentum of $D^{*0}D^0$ in the center-of-momentum frame of the pair and “all” represents all the other possible particles in the final state. We have made a change of variables from the momenta of $D^{*0}$ and $\bar{D}^0$ to $k$ and the total momentum $P$ of the charm-meson pair. The dependence on $P$ has been suppressed in Eq. (6). In the remaining Lorentz-invariant phase-space measure $d\phi_{D^{*}\bar{D}+\text{all}},$ $D^{*}\bar{D}$ is treated as a single particle with mass $M_{D^{*0}} + M_{D^0} \approx M_X$. According to the Migdal-Watson theorem [19], the dramatic dependence of the T-matrix element in Eq. (6) on $k$ due to final-state interactions resides in a multiplicative factor of the elastic scattering amplitude $f(k^2/2\mu)$ given by Eq. (3). A factorization formula can be derived from Eq. (6) by dividing the T-matrix element by $f$ and
then multiplying the right side by a compensating factor of $|f|^2$. Since $T/f$ is insensitive to the small relative momentum $k$, we can take the limit $k \to 0$ in that factor. The resulting factorization formula is

$$
\frac{d\sigma[D^0\bar{D}^0(k)]}{\text{flux}} \sum_{\text{all}} \int d\phi_{D^*\bar{D}_{+\text{all}}} |T[D^{*0}\bar{D}^0(0) + \text{all}] / f(0)|^2 \times \frac{1}{k^2 + 2\mu E_X} \frac{d^3k}{(2\pi)^3 2\mu}.
$$

(7)

The factorization formula for $D^0\bar{D}^{*0}$ is given by the identical expression. These factorization formulas hold for $k < \Lambda$, where $\Lambda$ is the momentum scale set by the range of the interaction between the charm mesons. They indicate that the cross section integrated over $k$ up to $k_{\text{max}}$ does not have the naive scaling behavior $k_{\text{max}}^3$ expected from phase space. Instead it scales like $k_{\text{max}}$ in the range $\sqrt{2\mu E_X} < k_{\text{max}} < \Lambda$.

In the corresponding factorization formula for the production of $X(3872)$, the short-distance factor is the same as in Eq. (7). The long-distance factor can be deduced by using the fact that the spectral function for the resonance is proportional to the imaginary part of the scattering amplitude in Eq. (3):

$$
\text{Im} f(E) = \frac{\pi}{\mu a} \delta(E + 1/(2\mu a^2)) + \frac{\sqrt{2\mu E}}{1/a^2 + 2\mu E} \theta(E).
$$

(8)

The term with the factor $\theta(E)$ is the combined contribution from $D^{*0}\bar{D}^0$ and $D^0\bar{D}^{*0}$. The term with the delta function is the contribution from $X(3872)$. The long-distance factor in the factorization formula for $D^{*0}\bar{D}^0$ in Eq. (7) is the $\theta(E)$ term in Eq. (8) multiplied by $dE/(4\pi^2)$. In the corresponding factorization formula for $X(3872)$, the long-distance factor is the $\delta$-function term in Eq. (8) multiplied by $dE/(2\pi^2)$ and integrated over $E$. The resulting factorization formula is

$$
\sigma[X(3872)] = \frac{1}{\text{flux}} \sum_{\text{all}} \int d\phi_{D^*\bar{D}_{+\text{all}}} |T[D^{*0}\bar{D}^0(0) + \text{all}] / f(0)|^2 \times \frac{\sqrt{2\mu E_X}}{2\pi \mu}.
$$

(9)

The dependence on the momentum $P$ of the $X(3872)$ has been suppressed, but $P$ can be identified with the total momentum of $D^{*0}\bar{D}^0$ in the short-distance factor.

Since the long-distance factor in Eq. (9) is proportional to $E_X^{3/2}$, the cross section goes to 0 as the binding energy goes to 0. This is in accord with the common intuition that the production of a weakly-bound molecule should be suppressed in high-energy collisions. However the degree of suppression is often overestimated. The common intuition, as exemplified by Ref. [6], is that the suppression factor should scale like $E_X^{3/2}$. In the case of an $S$-wave threshold resonance, there is a much milder suppression factor proportional to $E_X^{1/2}$. 

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A factorization formula equivalent to that in Eq. (9) was previously derived in Ref. [18]. In the factorization formula in Ref. [18], the long-distance factor is also proportional to $E_X^{1/2}$. However the expression for the short-distance factor includes an explicit factor of $\Lambda^2$, where $\Lambda$ is the ultraviolet cutoff of an effective field theory for charm mesons with a large scattering length. That dependence on $\Lambda$ is presumably cancelled by other terms in the short-distance factor that depend implicitly on $\Lambda$. Thus the expression for the short-distance factor in Ref. [18] can not be interpreted literally. The advantage of the factorization formula in Eq. (9) is that the short-distance factor is well-defined.

III. PRODUCTION AT THE TEVATRON

In this section, we use results from the CDF Collaboration to estimate the cross sections for $X(3872)$ in $p\bar{p}$ collisions at the Tevatron from both prompt QCD mechanisms and from decays of $b$ hadrons.

Within a few months of the discovery of the $X(3872)$ by the Belle Collaboration [1], its existence was confirmed by the CDF Collaboration through inclusive production in $p\bar{p}$ collisions at the Tevatron [2]. The $X(3872)$ was also observed at the Tevatron by the $D\phi$ Collaboration [20], who showed that many of its production characteristics are similar to those of the $\psi(2S)$. The CDF Collaboration subsequently showed that for a specific data sample consisting of $X(3872)$ in the decay channel $J/\psi\pi^+\pi^-$ with a modest $p_T$ cut, the production is dominated by prompt QCD mechanisms rather than $b$-hadron decays [3]. The information given about this data sample can be used to estimate the cross section.

In Ref. [3], the CDF collaboration observed both the $X(3872)$ and the $\psi(2S)$ in a $J/\psi\pi^+\pi^-$ data sample with a cut $p_T(J/\psi) > 4$ GeV as well as cuts on the pion momenta. For the $X(3872)$, they applied an additional cut $M(\pi^+\pi^-) > 0.5$ GeV to increase the signal-to-background ratio. For both the $X(3872)$ and $\psi(2S)$, they measured the long-lived fraction $f_{LL}$ that come from the decay of $b$ hadrons: $f_{LL}^X = 16.1 \pm 4.9 \pm 2.0\%$ and $f_{LL}^{\psi} = 28.3 \pm 1.0 \pm 0.7\%$. The complementary fractions $1 - f_{LL}$ are produced promptly by QCD mechanisms. Using the numbers of events reported in Ref. [3], we determine the product of the cross section for $X(3872)$ and its branching fraction into $J/\psi\pi^+\pi^-$ for both
the prompt and $b$-decay mechanisms:

$$
\sigma_{\text{prompt}}[X(3872)] \ Br[X \rightarrow J/\psi \pi^+\pi^-] = (0.0335 \pm 0.0055)(\epsilon_\psi/\epsilon_X)\sigma_{\text{total}}[\psi(2S)], \quad (10a)
$$

$$
\sigma_{b-\text{decay}}[X(3872)] \ Br[X \rightarrow J/\psi \pi^+\pi^-] = (0.00643 \pm 0.0023)(\epsilon_\psi/\epsilon_X)\sigma_{\text{total}}[\psi(2S)], \quad (10b)
$$

where $\epsilon_X$ and $\epsilon_\psi$ are the efficiencies for observing $X(3872)$ and $\psi(2S)$ events in this decay sample. The ratio $\epsilon_\psi/\epsilon_X$ is likely to deviate from 1 by tens of percents rather than factors of 2 [3]. The “total” cross section for $\psi(2S)$ in Eqs. (10) is the sum of the prompt and $b$-decay cross sections. The cross sections in Eqs. (10) should be interpreted as those for production of $X(3872)$ and $\psi(2S)$ within the cuts used to define the two data samples. The cut $p_T(J/\psi) > 4$ GeV implies constraints on the transverse momenta of the $X(3872)$ and $\psi(2S)$. The decay of $X(3872)$ into $J/\psi \pi^+\pi^-$ proceeds through its decay into $J/\psi \rho^*$, where $\rho^*$ is a virtual $\rho$ meson, with $J/\psi$ nearly at rest in the $X$ rest frame. Thus the momentum components of $X$ and $J/\psi$ differ only by the factor $M_X/M_{J/\psi}$. We can therefore interpret the cross section on the left side of Eqs. (10a) as the prompt cross section for $X \rightarrow J/\psi \pi^+\pi^-$ with $p_T(X) > 5.0$ GeV. In the decay $\psi(2S) \rightarrow J/\psi \pi^+\pi^-$, the momentum of the $J/\psi$ in the $\psi(2S)$ rest frame can range up to 0.5 GeV. Using Monte Carlo simulations, we have checked that the cross section for $\psi(2S) \rightarrow J/\psi \pi^+\pi^-$ with $p_T(J/\psi) > 4$ GeV can be approximated by the cross section with $p_T(\psi(2S)) > 5$ GeV within 12% accuracy.

To obtain estimates of the cross sections for $X(3872)$ from Eqs. (10), we need a measurement of the cross section for $\psi(2S)$. The sum of the prompt and $b$-decay cross sections for $\psi(2S)$ in the phase-space region $p_T(\psi(2S)) > 5$ GeV and $|y(\psi(2S))| < 0.6$ has been measured by the CDF Collaboration [21]. The product of this cross section and the branching fraction for $\psi(2S)$ into $\mu^+\mu^-$ is $0.69 \pm 0.01 \pm 0.06$ nb. The branching fraction into muons is $(7.5 \pm 0.8) \times 10^{-3}$ [22]. If we assume that the rapidity distribution for the $X$ and the $\psi(2S)$ are similar in the acceptance region of the CDF detector, we can insert this cross section measurement for $\psi(2S)$ into Eqs. (10). Using the estimate $\epsilon_\psi/\epsilon_X \approx 1$, we find

$$
\sigma_{\text{prompt}}[X(3872)] \ Br[X \rightarrow J/\psi \pi^+\pi^-] \approx 3.1 \pm 0.7 \text{ nb}, \quad (11a)
$$

$$
\sigma_{b-\text{decay}}[X(3872)] \ Br[X \rightarrow J/\psi \pi^+\pi^-] \approx 0.59 \pm 0.23 \text{ nb}. \quad (11b)
$$

We interpret these as the cross sections for $X(3872) \rightarrow J/\psi \pi^+\pi^-$ with $p_T(X) > 5$ GeV and $|y(X)| < 0.6$. The error bars come only from the uncertainties in the $\psi(2S)$ cross sections and the numerical factors in Eqs. (10). They do not include the systematic error from setting
\( \epsilon_{\psi}/\epsilon_X = 1 \), which may be tens of percent. They also do not include the systematic error from approximating the cross section for \( \psi(2S) \rightarrow J/\psi\pi^+\pi^- \) with \( p_T(J/\psi) > 4 \text{ GeV} \) by the cross section with \( p_T(\psi(2S)) > 5 \text{ GeV} \). The estimate for the prompt cross section in Eq. (11a) agrees with the estimate given in Ref. [6].

Existing measurements provide some constraints on the branching fraction for the \( X(3872) \) into \( J/\psi\pi^+\pi^- \). The Babar Collaboration has set a lower bound \( \text{Br} > 0.042 \) at 90\% C.L. [23]. The sum of the measured branching ratios for \( J/\psi\pi^+\pi^-\pi^0 \) [10], \( J/\psi\gamma \) [10, 24], \( D^0\bar{D}^0\pi^0 \) [15, 16] and \( \psi(2S)\gamma \) [24] relative to \( J/\psi\pi^+\pi^- \) is 13.5 \( \pm \) 2.9. If taken at face value, this implies an upper bound on the branching fraction into \( J/\psi\pi^+\pi^- \): \( \text{Br} < 0.093 \) at 90\% C.L. If we allow \( \text{Br}[X \rightarrow J/\psi\pi^+\pi^-] \) to range from 0.042 to 0.093, our estimate of the prompt cross section for \( X(3872) \) from Eq. (11a) ranges from 72 nb to 33 nb, while our estimate of the \( b \)-decay cross section from Eq. (11b) ranges from 14 nb to 6 nb.

IV. CHARM-MESON-PAIR CROSS SECTIONS AND THE X(3872)

In this section, we rederive the upper bound on the cross section for producing a charm-meson molecule presented in Ref. [6]. We point out that their calculation of the upper bound for the \( X(3872) \) is too small by orders of magnitude, because they do not take into account charm-meson rescattering. We also present an alternative method for calculating the upper bound that is much more efficient than the method used in Ref. [6]. Finally, we derive an order-of-magnitude estimate of the cross section for producing \( X(3872) \) and show that this estimate is compatible with the observed prompt production rate at the Tevatron.

A. Upper Bound on Prompt Cross Section

In Ref. [6], Bignamini et al. derived an upper bound on the cross section for producing a charm-meson molecule in terms of the cross section for producing charm-meson pairs integrated over a region of small relative momentum. In the case of prompt production of \( X(3872) \) at the Tevatron, their upper bound was lower than the observed production rate by orders of magnitude, casting doubt on the identification of the \( X(3872) \) as a loosely-bound \( D^*\bar{D}^0 \) molecule.

The starting point in the derivation of the upper bound in Ref. [6] was an expression for
the inclusive cross section for the production of \( X(3872) \) in terms of its momentum-space wavefunction \( \tilde{\psi}_X(k) \):

\[
\sigma[X] = \frac{1}{\text{flux}} \sum_{\text{all}} \int d\phi_{X+\text{all}} \left| \int_{k_{\text{max}}}^{k_{\text{max}}} d^3k \frac{\sqrt{2\mu}}{2\pi^3} T[(D^{*0}D^0)_{+}(k) + \text{ all}] \tilde{\psi}_X(k) \right|^2. \tag{12}
\]

The notation \((D^{*0}D^0)_{+}\) indicates a projection of the charm mesons onto the even-charge-conjugation component defined by the right side of Eq. (1). The integral over the relative momentum \( k \) has been restricted to the region \(|k| < k_{\text{max}}\) in which the integrand has significant support. By applying the Schwartz inequality to the integral over \( k \) in Eq. (12), one obtains an upper bound

\[
\sigma[X] \leq \frac{1}{\text{flux}} \sum_{\text{all}} \int d\phi_{X+\text{all}} \int_{k_{\text{max}}}^{k_{\text{max}}} d^3k \frac{\sqrt{2\mu}}{2\pi^3} T[(D^{*0}D^0)_{+}(k) + \text{ all}]^2 \\
\times \int_{k_{\text{max}}}^{k_{\text{max}}} d^3k \frac{\sqrt{2\mu}}{2\pi^3} \left| \tilde{\psi}_X(k) \right|^2. \tag{13}
\]

The last factor is less than 1, because it is simply the incomplete normalization integral for the wavefunction. The remaining expression on the right side can be expressed as the sum of inclusive cross sections for producing pairs of charm mesons plus interference terms:

\[
\frac{1}{\text{flux}} \sum_{\text{all}} \int d\phi_{X+\text{all}} \int_{k_{\text{max}}}^{k_{\text{max}}} d^3k \frac{\sqrt{2\mu}}{2\pi^3} T[(D^{*0}D^0)_{+}(k) + \text{ all}]^2 \\
\approx \frac{1}{2} \sigma[D^{*0}\bar{D}^0(k < k_{\text{max}})] + \frac{1}{2} \sigma[D^0\bar{D}^{*0}(k < k_{\text{max}})] + \text{(interference)}. \tag{14}
\]

In a high energy collision, an inclusive cross section is summed over many additional particles, which we have represented by “all”. The interference terms between the T-matrix elements for \( D^{*0}\bar{D}^0 \) and \( D^0\bar{D}^{*0} \) should average to 0 upon summing over all these additional particles. Thus the upper bound in Eq. (13) can be written in a very simple form:

\[
\sigma[X] \leq \frac{1}{2} \sigma[D^{*0}\bar{D}^0(k < k_{\text{max}})] + \frac{1}{2} \sigma[D^0\bar{D}^{*0}(k < k_{\text{max}})]. \tag{15}
\]

This inequality holds provided \( k_{\text{max}} \) is large enough to provide most of the region of support for the integral in Eq. (12). In the case of \( p\bar{p} \) collisions, charge conjugation symmetry implies that the two cross sections on the right side of Eq. (15) are equal.

In Ref. [6], the authors assumed that the region of support for the integral over \( k \) in Eq. (12) extends only up to the momentum scale set by the binding momentum \((2\mu E_X)^{1/2}\) of the molecule. The central value of the binding energy from Eq. (2), \( E_X = 0.30 \text{ MeV}, \)
corresponds to a binding momentum of 24 MeV. The authors of Ref. [6] took the upper end-
point of the region of support for the integral in Eq. (12) to be \( k_{\text{max}} = 35 \) MeV. They used
the Monte Carlo event generators Pythia and Herwig to calculate the distribution in the
relative momentum \( k \) for prompt charm-meson pairs \( D^*\bar{D}^0 \) at the Tevatron. The distribu-
tion was normalized using CDF data on the production of \( D^0D^{*-} \) at the Tevatron [28]. For
the charm-meson-pair cross section integrated up to \( k_{\text{max}} = 35 \) MeV, they obtained 0.11 nb
and 0.071 nb using Pythia and Herwig, respectively. These cross sections are more than
two orders of magnitude smaller than our estimate of the prompt cross section for \( X(3872) \)
from Eq. (11a), which is \( 33 - 72 \) nb. They concluded that the \( X(3872) \) is unlikely to be a
charm-meson molecule.

The flaw in the argument of Ref. [6] is that the authors ignored the effects of rescattering
of the charm-meson pair. Rescattering can be important for relative momenta comparable
to or smaller than the scale \( \Lambda \) set by the range of the interaction. For a generic molec-
ular state whose binding momentum is of the order \( \Lambda \), rescattering does not change the
momentum scale. However an S-wave threshold resonance is characterized by a very large
scattering length \( a \) that arises from a fine-tuned balance between the interaction strength of
the constituents and the effects of rescattering. The rescattering effects are so strong that
the cross section for elastic scattering saturates the unitarity bound for relative momentum
in the range between \( (2\mu E_X)^{1/2} \) and \( \Lambda \). These same effects allow charm mesons that are
created with relative momenta as large as \( \Lambda \) to rescatter into small relative momenta of order
\( \sqrt{2\mu E_X} \). For this reason, the region of support for the integral over the relative momentum
in Eq. (12) extends up to the scale \( \Lambda \). This dramatically increases the upper bound on the
cross section given by Eq. (15). Now the charm-meson-pair cross sections calculated using
Monte Carlo methods in Ref. [6] scale like \( k_{\text{max}}^3 \) from the phase space of the charm-meson
pair. If we take \( m_\pi \) as an estimate of the scale \( \Lambda \) and replace the value \( k_{\text{max}} = 35 \) MeV used
in Ref. [6] by \( k_{\text{max}} = c\Lambda \), the charm-meson-pair cross section of 0.07 – 0.11 nb is increased
by a factor of 64\( c^3 \). For reasonable choices of \( c \), this upper bound can be larger than the
estimated prompt cross section of 33 – 72 nb obtained in Section III.

There is an alternative way to see that \( k_{\text{max}} \) in the upper bound in Eq. (15) must be much
larger than the binding momentum of the \( X(3872) \). By the Migdal-Watson theorem [19],
the dramatic dependence of the T-matrix element in Eq. (12) on \( k \) resides in a multiplicative
factor of the elastic scattering amplitude \( f(k^2/2\mu) \). If the region of support of the integral
over $k$ in Eq. (12) was limited to the momentum scale $(2\mu E_X)^{1/2}$, we could replace $f(k^2 / 2\mu)$ by the universal scattering amplitude in Eq. (3) and $\bar{\psi}_X(k)$ by the universal momentum-space wavefunction in Eq. (5). However the resulting integral over $k$ in Eq. (12) is logarithmically ultraviolet divergent. This is a signal that the region of support for this integral is not limited to $k$ of order $\sqrt{2\mu E_X}$. It extends up to the momentum scale set by the range of interactions between the charm mesons.

**B. More efficient calculation of upper bound**

The upper bound on the cross section for the $X(3872)$ in Eq. (15) is the cross section for charm-meson pairs integrated over the relative momentum $k$ up to $k_{\text{max}}$. In Ref. [6], that upper bound was calculated for the case of prompt production of $X(3872)$ at the Tevatron. We now present an alternative calculation of that charm-meson-pair cross section using a method that is much more efficient. In the analysis of Ref. [6], the momentum distribution for charm-meson pairs was calculated by generating more than $5 \times 10^{10}$ 2-to-2 parton-level events and then passing them through Pythia [25] or Herwig [26] for showering and hadronization. Only a tiny fraction of these events include a pair of nearly-collinear charm mesons with substantial $p_T$. These specific events are generated by an outgoing gluon from the 2-to-2 parton collision fragmenting into a charm-quark pair in the showering process, which in turn hadronizes most of the time into a pair of charm mesons. We calculate the charm-meson-pair cross section by generating $gg \rightarrow gc\bar{c}$ parton-level events with MadGaph [27] and then passing them through Pythia for showering and hadronization. This 2-to-3 parton-level process is the dominant mechanism for the production of a pair of charm mesons with small relative momentum and substantial $p_T$. In order to further increase the efficiency, only parton-level events for which the $c\bar{c}$ pair have relative momentum below 2 GeV have been generated. We checked that this reduction of the parton-level phase space does not affect the hadron-level cross section for charm mesons with small relative momentum $k < 600$ MeV. The main advantage of generating only $gg \rightarrow gc\bar{c}$ events is that the momentum distribution in the region of small relative momentum can be calculated with accuracy comparable to that in Ref. [6] by generating fewer events by about a factor of $10^4$.

We follow Ref. [6] in normalizing the Monte Carlo distribution using measurements by the CDF Collaboration of the inclusive cross section for $D^0D^{*-}$ in $p\bar{p}$ collisions at the
FIG. 1: Cross section for the inclusive production of $D^0 D^{*-}$ in $p\bar{p}$ collisions at $\sqrt{s} = 1.96$ TeV. The cross section is integrated over bins of $\Delta \phi$ – the angle between the transverse momenta of $D^0$ and $D^{*-}$. The data points are measurements by the CDF Collaboration at the Tevatron [28]. The histogram is calculated using the Monte Carlo methods described in the text.

The CDF Collaboration measured this cross section in the phase-space region $p_T(D^0), p_T(D^{*-}) > 5.5$ GeV and $|y(D^0)|, |y(D^{*-})| < 1$. The cross section differential in $\Delta \phi$ – the angle between the transverse momenta of the two charm mesons – is shown in Fig. 1. In the analysis of Ref. [6], the Monte Carlo distribution for pairs of charm mesons was normalized to this data. Since the error bars are smallest in the bins of $\Delta \phi$ closest to 90°, this means that in practice the Monte Carlo distribution in Ref. [6] was normalized to the data near 90°. In our analysis, as we are primarily interested in configurations in which two charm mesons are almost collinear, we normalize the Monte Carlo event rate instead to the measured rate in the first bin in $\Delta \phi$, which extends from 0 to 15°. Compared to normalizing the rate near 90°, this increases the normalization by about 60% at the expense of a 20% error from the experimental uncertainty. Our Monte Carlo events for $D^0 D^{*-}$ were generated via the parton-level process $gg \rightarrow gc\bar{c}$, consistent with our procedure for generating $D^{*0} \bar{D}^0$ events. In order to increase the efficiency, we applied the parton-level cuts $p_T(c), p_T(\bar{c}) > 3.5$ GeV, $|y(c)|, |y(\bar{c})| < 2$, and $\Delta \phi(c, \bar{c}) < 90^\circ$, which do not affect the $D^0 D^{*-}$ production rate in the region of interest. The rescaling factor – defined by the ratio
FIG. 2: Cross sections for the inclusive production of $D^{*0}\bar{D}^0$ in $p\bar{p}$ collisions at $\sqrt{s} = 1.96$ TeV. (a) Differential cross section integrated over the region $p_T(D^{*0}\bar{D}^0) > 5$ GeV and $|y(D^{*0}\bar{D}^0)| < 0.6$ as a function of the relative momentum $k$ of $D^{*0}\bar{D}^0$. The histogram is the hadron-level cross section calculated using the Monte Carlo method described in the text. The solid curve is the parton-level differential cross section integrated over the region $p_T(c\bar{c}) > 5$ GeV and $|y(c\bar{c})| < 0.6$ and normalized to the hadron-level cross section in the region $k < 1$ GeV. (b) Cross section obtained by integrating the parton-level differential cross section up to $k_{max}$. The dotted curves are simple phase-space distributions.

of the experimental rate over the Monte Carlo rate in the first bin in $\Delta \phi$ – is 1.62. This same rescaling factor was then applied to the distributions of $D^{*0}\bar{D}^0$ events.

Having fixed the normalization of our Monte Carlo distributions, we proceed to calculate the production rate of $D^{*0}\bar{D}^0$ with small relative momentum at the Tevatron, run II. The differential cross section is displayed as a histogram in the relative momentum $k$ in Fig. 2(a). The distribution, which was obtained by generating about $5 \times 10^6$ events, is as smooth as the distribution in Ref. [6], which was obtained by generating more than $5 \times 10^{10}$ events. For $k$ less than about 300 MeV, the shape of the Monte Carlo distribution is consistent with $k^2$, in accord with simple phase-space suppression.

The results of our calculation using MadGraph and the event generator Pythia can be reproduced approximately using a simpler method that does not require any event generator. We assume that a $D^{*0}\bar{D}^0$ pair with small relative momentum $k$ is formed from a $c\bar{c}$ pair with approximately the same relative momentum $k$. The resulting approximation for the $D^{*0}\bar{D}^0$
cross section is proportional to the parton-level cross section for a $c\bar{c}$ pair:

$$\sigma[D^{*0}\bar{D}^0(k < k_{\text{max}})] \approx \sigma[c\bar{c}(k < k_{\text{max}})] \times F[c\bar{c} \rightarrow D^{*0}\bar{D}^0].$$

(16)

The factor $F[c\bar{c} \rightarrow D^{*0}\bar{D}^0]$ can be interpreted as the probability for a nearly-collinear charm-quark pair to evolve into a nearly-collinear $D^{*0}\bar{D}^0$ pair. The shape of this parton-level distribution agrees well with the hadron-level distribution shown as a histogram in Fig. 2(a). The normalization also agrees if we set $F[c\bar{c} \rightarrow D^{*0}\bar{D}^0] = 0.142$. The resulting distribution is shown as a solid line in Fig. 2(a). Alternatively, the normalizing factor can be determined from the CDF measurement of the cross section for $D^0\bar{D}^{*+}$ in the first bin in $\Delta\phi$ in Fig. 1. The resulting value is $F[c\bar{c} \rightarrow D\bar{D}^*] = 0.118$, which is smaller than the value obtained normalizing to the hadron-level distribution by about 17%.

To obtain the charm-meson-pair cross section $\sigma[D^{0}\bar{D}^{*0}(k < k_{\text{max}})]$ that appears in the upper bound in Eq. (15), the $k$ distribution in Fig. 2(a) must be integrated up to $k_{\text{max}}$. In Fig. 2(b), we show the cross section in Eq. (16) with $F[c\bar{c} \rightarrow D^{*0}\bar{D}^0] = 0.142$ as a function of $k_{\text{max}}$. For the value $k_{\text{max}} = 35$ MeV used in Ref. [6], we obtain a cross section of 0.030 nb, which is smaller than that obtained in Ref. [6] using Pythia by a factor of 3.7.

C. Estimate of the Prompt Cross Section

The factorization formula for the production of $X(3872)$ in Eq. (9) separates the short-distance effects associated with the production of the charm mesons from long-distance effects associated with their binding into the $X(3872)$. The Monte Carlo methods used to calculate the upper bound on the prompt cross section at the Tevatron can be combined with this factorization formula to obtain an order-of-magnitude estimate of the prompt cross section. The Monte Carlo methods use an event generator, such as Pythia or Herwig, to model the hadronization process that produces the charm mesons. The tuning of the parameters of the event generator implicitly takes into account typical hadronic cross sections. It does not take into account any unnaturally large cross section, such as that associated with an S-wave threshold resonance like the $X(3872)$.

The charm-meson pair $D^{*0}\bar{D}^0$ is expected to have a typical hadronic production rate as long as its relative momentum $k$ is comparable to or larger than the momentum scale $\Lambda$ set by the range of the interactions between charm mesons. According to the Migdal-Watson
theorem, the production rate for \( k < \Lambda \) is enhanced because of the factor \( f(k^2/2\mu) = (-1/a-ik)^{-1} \) in the T-matrix element. An event generator like Pythia contains no information about such an enhancement factor. When such an event generator is used, there is an implicit assumption that the T-matrix element \( T[(D^*\bar{D}^0)_+(k) + \text{all}] \) does not vary dramatically for \( |k| < \Lambda \). If this assumption is applied to Eq. (6), we find that the “naive” cross section calculated using an event generator and integrated over \( |k| < \Lambda \) can be approximated by

\[
\sigma_{\text{naive}}[D^*\bar{D}^0(k < \Lambda)] \approx \frac{1}{\text{flux}} \sum_{\text{all}} \int d\phi_{D^*\bar{D}^0} \left| T[D^*\bar{D}^0(|k| = \Lambda) + \text{all}] \right|^2 \times \frac{\Lambda^3}{12\pi^2\mu}, \tag{17}
\]

where the T-matrix element has been evaluated at \( |k| = \Lambda \).

In the short-distance factor of the factorization formula for the \( X(3872) \) in Eq. (9), the term \( T/f \) is evaluated at \( k = 0 \). However this term is insensitive to the relative momentum for \( |k| < \Lambda \), so it can be approximated by its value when \( |k| = \Lambda \). The factorization formula can be expressed in the form

\[
\sigma[X(3872)] \approx \frac{1}{\text{flux}} \sum_{\text{all}} \int d\phi_{D^*\bar{D}^0} \left| T[D^*\bar{D}^0(|k| = \Lambda) + \text{all}] \right|^2 \times \frac{\sqrt{2\mu E_X}}{2\pi\mu}, \tag{18}
\]

where we have used \( \Lambda \gg 1/|a| \) to approximate \( 1/f(\Lambda^2/2\mu) \) by \( \Lambda \). The short-distance factor in Eq. (18) can now be eliminated in favor of the naive charm meson cross section in Eq. (17):

\[
\sigma[X(3872)] \approx \sigma_{\text{naive}}[D^*\bar{D}^0(k < \Lambda)] \times \frac{6\pi\sqrt{2\mu E_X}}{\Lambda}. \tag{19}
\]

If the naive cross section for producing charm mesons has been calculated using an event generator, this expression can be used to estimate the cross section for producing \( X(3872) \).

In Section IV B, we used MadGraph and Pythia to calculate the naive cross section for \( D^*\bar{D}^0 \) at the Tevatron as a function of \( k_{\text{max}} \). Taking the central value \( E_X = 0.30 \text{ MeV} \) for the binding energy from Eq. (2), our estimate of the prompt cross section from Eq. (19) is 1.5, 5.9, and 23 nb for \( \Lambda = m_\pi/2, m_\pi, \) and \( 2m_\pi \), respectively. The upper end of this range of cross sections is compatible with the estimate 33 – 72 nb for the prompt cross section at the Tevatron obtained in Section III.

We can use the estimate of the cross section for \( X(3872) \) in Eq. (19) to obtain a quantitative estimate of the upper endpoint \( k_{\text{max}} \) of the region of support of the integral in Eq. (12). The upper bound in Eq. (15) can be approximated by the naive charm-meson-pair cross section \( \sigma_{\text{naive}}[D^*\bar{D}^0(k < k_{\text{max}})] \) provided \( k_{\text{max}} > \Lambda \):

\[
\sigma[X(3872)] \leq \sigma_{\text{naive}}[D^*\bar{D}^0(k < k_{\text{max}})]. \tag{20}
\]
Demanding that the estimate in Eq. (19) is less than the upper bound in Eq. (20) even if \( \sqrt{2 \mu E_X} \) is as large as \( \Lambda \), we obtain the condition \( k_{\text{max}} > (6 \pi)^{1/3} \Lambda = 2.7 \Lambda \). The naive cross section in Eq. (20) is shown as a function of \( k_{\text{max}} \) in Fig. 2. Setting \( k_{\text{max}} = 2.7 \Lambda \), the upper bound on the cross section for \( X(3872) \) is 4.2, 32, and 230 nb for \( \Lambda = m_\pi/2, m_\pi, \) and \( 2m_\pi \), respectively. This can be much larger than the prompt cross section observed at the Tevatron.

V. NRQCD FACTORIZATION

In this section, we discuss the NRQCD factorization formula for the inclusive production of \( X(3872) \), which expresses the cross section as a sum of parton cross sections multiplied by phenomenological constants. We use simplifying assumptions to reduce the phenomenological constants to a single constant that is determined from our estimate of the prompt cross section at the Tevatron. We then use the NRQCD factorization formula to predict the differential cross section for prompt production of \( X(3872) \) at the Tevatron. We also predict the differential cross section for production of \( X(3872) \) from \( b \)-hadron decays at the Tevatron.

A. NRQCD Factorization Formula

Inclusive production rates for the \( X(3872) \) satisfy an NRQCD factorization formula, which separates the momentum scales of order \( m_c \) and larger that are involved in the creation of the \( c\bar{c} \) pair from all the smaller momentum scales of QCD [9]. The smaller momentum scales include the scale \( m_\pi \) associated with the formation of the charm mesons and the scale \( \sqrt{2 \mu E_X} \) associated with their binding into the \( X(3872) \). The NRQCD factorization formula has the form [29]

\[
\sigma[X(3872)] = \sum_n \hat{\sigma}[c\bar{c}_n] \langle \mathcal{O}^X_n \rangle, \tag{21}
\]

where the sum over \( n \) extends over the color and angular-momentum channels of a \( c\bar{c} \) pair. The short-distance factors \( \hat{\sigma} \) are inclusive cross sections for producing a \( c\bar{c} \) pair with negligible relative momentum in the channel \( n \) together with hard partons. They can be calculated using perturbative QCD. The dependence on the momentum \( \mathbf{P} \) of the \( X(3872) \) has been
suppressed in Eq. (21), but $P$ can be identified with the total momentum of the $c\bar{c}$ pair. The long-distance factor $\langle O_n^H \rangle$ is proportional to the probability for the $c\bar{c}$ pair in the channel $n$ to evolve into the $c\bar{c}$ meson $H$ plus soft partons or hadrons. It is called an NRQCD matrix element, because it can be expressed as a matrix element of a local operator in an effective field theory for the $c\bar{c}$ sector of QCD called nonrelativistic QCD (NRQCD). The NRQCD matrix elements can be treated as phenomenological constants. They are universal, so once they have been determined by fitting experimental results, the NRQCD factorization formula can be used to predict the production rate in other experiments.

The NRQCD factorization formula was originally developed for heavy quarkonium [29]. The relative sizes of the NRQCD matrix elements depend on the angular momentum quantum numbers of the quarkonium state, scaling as specific powers of the typical relative velocity of the heavy quark pair. The pattern of suppression is summarized by a rather intricate set of velocity-scaling rules. The NRQCD factorization formula can also be applied to pairs of charm mesons with small relative momentum [9]. In this case, there should be a hierarchy in the sizes of the NRQCD matrix elements according to their orbital angular momentum. The most important matrix elements should be the S-wave matrix elements. There are four independent S-wave matrix elements: $\langle O_1^X (1S_0) \rangle$, $\langle O_1^X (3S_1) \rangle$, $\langle O_8^X (1S_0) \rangle$, and $\langle O_8^X (3S_1) \rangle$. The subscript 1 or 8 on the operator indicates the color channel: color-singlet or color-octet. The argument is the spectroscopic notation $^{2S+1}L_J$ for the angular-momentum channel. Explicit expressions for these operators in terms of NRQCD fields are given in Ref. [29]. The S-wave matrix elements can be interpreted as probability densities for a $c\bar{c}$ pair created at a point in the specified state to form the hadron $X$ multiplied by spin and color factors. For the four NRQCD matrix elements listed above, the products of the spin and color factors are $2N_c = 6$, $3(2N_c) = 18$, $N_c^2 - 1 = 8$ and $3(N_c^2 - 1) = 24$, respectively.

The large-scattering-length factorization formulas in Eqs. (7) and (9) imply that the relative sizes of the NRQCD matrix elements for $X(3872)$ should be the same as for $D^*0\bar{D}^0$. Truncating the NRQCD factorization formula to the S-wave terms, we get

$$
\sigma[X(3872)] \approx \hat{\sigma}[c\bar{c}_1 (1S_0)] \langle O_1^X (1S_0) \rangle + \hat{\sigma}[c\bar{c}_1 (3S_1)] \langle O_1^X (3S_1) \rangle + \hat{\sigma}[c\bar{c}_8 (1S_0)] \langle O_8^X (1S_0) \rangle + \hat{\sigma}[c\bar{c}_8 (3S_1)] \langle O_8^X (3S_1) \rangle.
$$

A possible binding mechanism for the $X(3872)$ is the accidental fine-tuning of the depth of the interaction potential between the charm mesons $D^*0$ and $\bar{D}^0$ so that there is a bound
state very near the threshold. Since the mesons \(D^{*0}\) and \(D^0\) are color-singlets, the \(c\bar{c}\) pair in the \(D^*\bar{D}\) system has equal probabilities \(1/9\) of being in a color-singlet state or in any of the 8 color-octet states. It is therefore reasonable to assume that the probabilities for formation of the \(X(3872)\) from a color-singlet \(c\bar{c}\) pair and from any of the color-octet \(c\bar{c}\) pairs are equal. Given the normalizations of the NRQCD operators, this assumption translates into a ratio of \(3/4\) between color-singlet and color-octet NRQCD matrix elements:

\[
\langle O_1^{X(3S)} \rangle \approx \frac{3}{4} \langle O_8^{X(3S)} \rangle, \quad \text{(23a)}
\]

\[
\langle O_1^{X(1S)} \rangle \approx \frac{3}{4} \langle O_8^{X(1S)} \rangle. \quad \text{(23b)}
\]

We now consider the dependence of the NRQCD matrix elements on the spin quantum number, which can be spin-triplet or spin-singlet. In the hadronization stage of the formation of the \(X(3872)\), a \(c\bar{c}\) pair evolves into the charm-meson pair \((D^{*0}\bar{D}^0)_+\) with even charge conjugation defined in Eq. (1). Voloshin has pointed out that the \(c\bar{c}\) pair in \((D^{*0}\bar{D}^0)_+\) is necessarily in a spin-triplet state [30]. The approximate heavy-quark spin symmetry of QCD implies that transitions that change the spin state of heavy quarks are suppressed. The state \((D^{*0}\bar{D}^0)_+\) is therefore much more likely to arise from a \(c\bar{c}\) pair that is created at short distances in a spin-triplet state than in a spin-singlet state. As pointed out in Ref. [9], this implies the suppression of the spin-singlet NRQCD matrix elements:

\[
\langle O_1^{X(1S)} \rangle \ll \langle O_1^{X(3S)} \rangle, \quad \text{(24a)}
\]

\[
\langle O_8^{X(1S)} \rangle \ll \langle O_8^{X(3S)} \rangle. \quad \text{(24b)}
\]

One possible mechanism for the binding of \(X(3872)\) that has not been excluded is an accidental fine-tuning of the mass of the charmonium state \(\chi_c(2P)\) to the \(D^{*0}\bar{D}^0\) threshold. The \(c\bar{c}\) state is transformed by its resonant coupling to \(D^{*0}\bar{D}^0\) into a loosely-bound charm-meson molecule with the properties described in Section II. In this case, the relative sizes of the NRQCD matrix elements for the \(X(3872)\) may be governed by the velocity-scaling rules for \(3P_1\) charmonium states. The leading terms in the NRQCD factorization formula are the color-singlet \(3P_1\) and color-octet \(3S_1\) terms [29]:

\[
\sigma[X(3872)] \approx \hat{\sigma}[c\bar{c}\chi_c(3P_1)] \langle O_1^{X(3P_1)} \rangle + \hat{\sigma}[c\bar{c}\chi_c(3S_1)] \langle O_8^{X(3S_1)} \rangle. \quad \text{(25)}
\]
The color-singlet parton cross section $\hat{\sigma}[c\bar{c}_1(3P_1)]$ and the color-octet NRQCD matrix element $\langle O^X_8(3S_1) \rangle$ both depend logarithmically on the NRQCD factorization scale $\Lambda_{\text{NRQCD}}$ in such a way that the dependence cancels between the two terms on the right side of Eq. (25). At any specific value of the momentum $P$ of the $c\bar{c}$ pair, the color-singlet parton cross section $\hat{\sigma}[c\bar{c}_1(3P_1)]$ can be made to vanish by adjusting $\Lambda_{\text{NRQCD}}$. If the production is dominated by regions of $P$ in which the ratio of $\hat{\sigma}[c\bar{c}_1(3P_1)]$ to $\hat{\sigma}[c\bar{c}_8(3S_1)]$ is roughly constant, then $\Lambda_{\text{NRQCD}}$ can be chosen so that the color-octet $3S_1$ term in the factorization formula in Eq. (25) dominates. Thus the S-wave factorization formula in Eq. (22) may be applicable even if the production of $X(3872)$ is dominated by the formation of $\chi_{c1}(2P)$.

B. Determination of the NRQCD matrix elements

The NRQCD matrix elements in the NRQCD factorization formula in Eq. (22) are universal. Once they have been determined by fitting experimental results, the NRQCD factorization formula can be used to predict the inclusive production rates of the $X(3872)$ in other experiments. The estimate in Eq. (11a) for the prompt cross section at the Tevatron provides a single linear constraint on the products of the matrix elements and the branching fraction for $X \to J/\psi \pi^+\pi^-$. In order to derive this constraint, we use MadOnia [31] to evaluate the parton cross sections appearing in Eq. (22) at leading order in $\alpha_s$, which is order $\alpha^3_s$ for $p_T(c\bar{c}) > 0$. We identify the 3-momentum $P$ of the $X(3872)$ with the 3-momentum of the $c\bar{c}$ pair in the center-of-momentum frame of the colliding hadrons. We set the mass of the charm quark to $m_c = 1.5$ GeV, so a $c\bar{c}$ pair with vanishing relative momentum has mass 3 GeV. The phase space region for the $c\bar{c}$ pair is $p_T(c\bar{c}) > 5$ GeV and $|y(c\bar{c})| < 0.6$. We use the parton distribution function Cteq6l1 [32]. The factorization and renormalization scales are set equal to the transverse mass $(p_T^2 + 4m_c^2)^{1/2}$ of the charm-quark pair. The resulting linear constraint on the NRQCD matrix elements from our estimate in Eq. (11a) for the prompt cross section at the Tevatron is

$$\text{Br}[X \to J/\psi \pi^+\pi^-] \left( \langle O^X_8(3S_1) \rangle + 0.159 \langle O^X_8(1S_0) \rangle + 0.085 \langle O^X_1(1S_0) \rangle + 0.00024 \langle O^X_1(3S_1) \rangle \right) = (2.7 \pm 0.6) \times 10^{-4} \text{ GeV}^3. \quad (26)$$

The error bar comes only from the statistical uncertainty in the estimate of the prompt cross section at the Tevatron in Eq. (11a).
Simplifying assumptions can be used to reduce the four independent S-wave NRQCD matrix elements in Eq. (22) to a smaller set. We will consider three different simplifying assumptions that reduce the four matrix elements to a single non-perturbative factor, which we choose to be \( \langle O^X_{3S_1} \rangle \). Our three simplifying assumptions are as follows:

1. **S-wave dominance.** The \( X(3872) \) is equally likely to be formed from any \( c\bar{c} \) pair that is created with small relative momentum in an S-wave state, regardless of the color or spin state of the \( c\bar{c} \) pair. The assumptions on the NRQCD matrix elements are
   \[
   \langle O^X_{1S_0} \rangle = 3 \langle O^X_{3S_1} \rangle,
   \langle O^X_{1S_0} \rangle = 1 \langle O^X_{8S_1} \rangle,
   \langle O^X_{8S_1} \rangle = 1 \langle O^X_{8S_1} \rangle.
   \]
   This is the same pattern of matrix elements as in the color-evaporation model for quarkonium production \([33]\). The linear constraint in Eq. (26) reduces to \( \text{Br} \langle O^X_{8S_1} \rangle \approx 2.5 \times 10^{-4} \text{ GeV}^3 \).

2. **Spin-triplet dominance.** The \( X(3872) \) can be formed only from a \( c\bar{c} \) pair that is created with small relative momentum in a spin-triplet S-wave state, but it is equally likely to be formed if the \( c\bar{c} \) pair is in a color-singlet or color-octet state. The assumptions on the NRQCD matrix elements are
   \[
   \langle O^X_{1S_0} \rangle = 3 \langle O^X_{3S_1} \rangle, \quad \langle O^X_{1S_0} \rangle = \langle O^X_{8S_1} \rangle = 0.
   \]
   The linear constraint in Eq. (26) reduces to \( \text{Br} \langle O^X_{8S_1} \rangle \approx 2.7 \times 10^{-4} \text{ GeV}^3 \).

3. **Color-octet \( 3S_1 \) dominance.** The \( X(3872) \) can be formed only from a \( c\bar{c} \) pair that is created with small relative momentum in a color-octet \( 3S_1 \) state. The assumptions on the NRQCD matrix elements are
   \[
   \langle O^X_{1S_0} \rangle = \langle O^X_{1S_0} \rangle = \langle O^X_{8S_1} \rangle = 0.
   \]
   This simplifying assumption was proposed by Braaten, who suggested that it might give a good approximation to the inclusive production rate of the \( X(3872) \) in high-energy hadron collisions \([9]\). The linear constraint in Eq. (26) reduces to \( \text{Br} \langle O^X_{8S_1} \rangle \approx 2.7 \times 10^{-4} \text{ GeV}^3 \).
FIG. 3: Cross sections for $X(3872) \to J/\psi \pi^+ \pi^-$ in $p\bar{p}$ collisions at $\sqrt{s} = 1.96$ TeV. The graphs are the transverse momentum ($p_T$) distribution for $|y| < 0.6$ (left panel) and the rapidity ($y$) distribution for $p_T > 5$ GeV (right panel). The curves are for prompt production assuming color-octet $^3S_1$ dominance (solid) or S-wave dominance (dotted) and for production from $b$-hadron decay (dashed).

Having obtained estimates of the NRQCD matrix elements for each of our three simplifying assumptions, we can now predict the differential cross sections for prompt production of $X \to J/\psi \pi^+ \pi^-$ at the Tevatron. The distributions of the transverse momentum and the rapidity are shown in Fig. 3. The solid and dotted curves correspond to color-octet $^3S_1$ dominance and S-wave dominance, respectively. The curves for spin-triplet dominance cannot be distinguished from those for color-octet $^3S_1$ dominance. For $p_T$ larger than about 4 GeV, our three simplifying assumptions on the NRQCD matrix elements lead to the same differential cross section. The reason for this is that the order-$\alpha_s^3$ parton cross section $\hat{\sigma}[c\bar{c}S(^3S_1)]$ dominates over that for the other three S-wave channels at large $p_T$. In the color-octet $^3S_1$ channel, the order-$\alpha_s^3$ parton cross section includes a fragmentation contribution that decreases asymptotically as $d\hat{\sigma}/dp_T^2 \sim 1/p_T^4$. For the other three channels, the order-$\alpha_s^3$ parton cross section decreases more rapidly by a factor of $1/p_T^2$ or $1/p_T^4$. They receive fragmentation contributions only at order $\alpha_s^4$ or $\alpha_s^5$. Thus the differences between our three simplifying assumptions might be greater at large $p_T$ if higher order corrections in $\alpha_s$ were included. These corrections are beyond the scope of this paper. As $p_T$ decreases below 4 GeV, the differential cross section for S-wave dominance continues to increase while that from the color-octet $^3S_1$ dominance reaches a maximum and then begins to decrease. The reason the
S-wave dominance cross section increases is that the order-$\alpha_s^3$ parton cross sections in the color-singlet and color-octet $^1S_0$ channels diverge as $p_T \to 0$. At very low transverse momentum, a fixed-order calculation for this channel is not reliable, and one has to resum higher order corrections in $\alpha_s$ associated with soft-gluon radiation from the colliding partons. This resummation is beyond the scope of this paper.

In the differential cross sections in Fig. 3, there is a 23% uncertainty in the normalization from the NRQCD matrix element. There are additional theoretical uncertainties associated with the NRQCD factorization formalism. The normalizations of the parton cross sections are sensitive to the value of $m_c$ and the renormalization and factorization scales. However the shapes of the differential cross section are much less sensitive. Thus variations in these parameters can be largely compensated by changes in the value of $\langle O_8^{X(3S)} \rangle$.

C. Production from $b$-hadron decay

Beside prompt production, the other source of $X(3872)$ in a hadron collider is the feeddown from decays of $b$ hadrons. At the Tevatron, the rate from $b$-hadron decays is smaller than the prompt rate but nevertheless significant. In the sample of $X(3872) \to J/\psi \pi^+ \pi^-$ events studied by the CDF Collaboration in Ref. [3], the fraction of events from decays of $b$ hadrons was $16.1 \pm 4.9 \pm 2.0\%$.

To predict the differential cross section for $X(3872)$ from $b$-hadron decays at the Tevatron, we follow a procedure similar to the one used in Ref. [34] to calculate the momentum distribution of $J/\psi$ from $b$-hadron decays in high energy collisions. The differential cross section for $X(3872)$ is the convolution of the cross section for producing a $b$ quark, a fragmentation function for its hadronization into a $b$ hadron, and the momentum distribution of the $X(3872)$ from the decay of the $b$ hadron. The Monte Carlo program MCFM [35] is used to generate the momentum spectrum of $b$ quarks at next-to-leading order in $\alpha_s$. We set the mass of the $b$ quark to $m_b = 4.75$ GeV. We use the Cteq6.m parton distributions [32], with the factorization and renormalization scales set equal to the transverse mass $(p_T^2 + m_b^2)^{1/2}$ of the $b$ quark. We use the Kartvelishvili fragmentation function [36] with exponent $\alpha = 29.1$ to describe the hadronization of the $b$ quark into a $b$ hadron. There are several ways in which we depart from the procedure used in Ref. [34]. We do not resum the logarithms of $m_b/p_T$ at large transverse momentum. To approximate the momentum distribution of the
$X(3872)$ from the decay of the $b$ hadron, we take the $b$ hadron to be a $B$ meson and we decay it into an $X$ plus a kaon according to an isotropic distribution in the rest frame of the $B$ meson. To normalize the differential cross section for $X(3872)$ from $b$-hadron decays, we use the estimated cross section for $X(3872) \rightarrow J/\psi \pi^+ \pi^-$ at the Tevatron given in Eq. (11b). The normalizing factor can be interpreted as the product of the inclusive branching fraction for a $b$ quark to decay into $X(3872)$ and the branching fraction for $X \rightarrow J/\psi \pi^+ \pi^-$:

$$\text{Br}[b \rightarrow X(3872) + \text{any}] \text{Br}[X \rightarrow J/\psi \pi^+ \pi^-] = (1.9 \pm 0.8) \times 10^{-4}. \quad (30)$$

The error bar comes only from the statistical error in the estimate of the $b$-decay cross section at the Tevatron in Eq. (11b).

Having determined the normalizing factor in Eq. (30), we can predict the differential cross sections for producing $X \rightarrow J/\psi \pi^+ \pi^-$ from $b$ decays at the Tevatron. The predictions are shown as dashed curves in Fig. 3. The shapes of the distributions in $p_T$ and $y$ are predicted to be similar to those for prompt production by the color-octet $3S_1$ mechanism except at low transverse momentum. The fraction of $X(3872)$ from $b$-hadron decay is predicted to increase from about 12% at $p_T = 5$ GeV to about 37% at $p_T = 50$ GeV.

VI. PRODUCTION AT THE LHC

In this section, we predict the differential cross sections for the production of the $X(3872)$ at the Large Hadron Collider (LHC) from both prompt QCD mechanisms and the decays of $b$ hadrons. We consider the production of $X(3872)$ in the phase space regions of the ATLAS and CMS detectors and the LHCb detector.

To predict the differential cross sections for production of $X(3872)$ at the LHC, we use the same methods that were used to calculate the differential cross sections at the Tevatron shown in Fig. 3. We calculate the prompt cross section for $X(3872)$ using the NRQCD factorization formula in Eq. (22). We consider the three simplifying assumptions for the NRQCD matrix elements introduced in Section V B: S-wave dominance defined by Eqs. (27), spin-triplet dominance defined by Eqs. (28), and color-octet $3S_1$ dominance defined by Eq. (29). The normalization of the prompt cross section is determined by the linear constraint in Eq. (26). We calculate the $c\bar{c}$ cross sections using the same parameters as in Section V B. We calculate the cross section for $X(3872)$ from $b$-hadron decays using the method described
FIG. 4: Cross sections for $X(3872) \rightarrow J/\psi \pi^+\pi^-$ in $pp$ collisions at $\sqrt{s} = 7$ TeV. The graphs are the transverse momentum ($p_T$) distribution for $|y| < 2.4$ (left panel) and the rapidity ($y$) distribution for $p_T > 5$ GeV (right panel). The curves are for prompt production assuming color-octet $^3S_1$ dominance (solid) or S-wave dominance (dotted) and for production from $b$-hadron decay (dashed).

in Section V C. The normalization of the $b$ decay cross section is determined by the product of branching fractions given in Eq. (30).

We first consider the production of $X(3872)$ in the phase-space region covered by the CMS and ATLAS detectors. In Fig. 4, we show the $p_T$ distributions integrated over the rapidity range $|y| < 2.4$ and the rapidity distributions integrated over the region $p_T > 5$ GeV. The qualitative behavior of these curves is similar to those for the Tevatron in Fig. 3. The differences in the prompt cross section from different simplifying assumptions for the NRQCD matrix elements are negligible for $p_T$ greater than about 5 GeV. The prompt and $b$-decay cross sections integrated over the region $p_T > 5$ GeV and $|y| < 2.4$ are predicted to be about 49 nb and 8.2 nb, respectively. The fraction of $X(3872)$ events from $b$-hadron decay is predicted to increase from 10% at $p_T = 5$ GeV to 35% at $p_T = 50$ GeV. However there are large uncertainties in the predictions. The statistical uncertainties in the normalizing factors are 23% for the prompt cross section and 39% for the $b$-decay cross section. There are also additional theoretical errors from the masses of the charm and bottom quarks and from the choices of the factorization and renormalization scales. Although these uncertainties are large, they will partly cancel between the extraction of the normalizing factors from the Tevatron data and the calculation of the cross sections at the LHC.
FIG. 5: Cross sections for $X \rightarrow J/\psi \pi^+ \pi^-$ in $pp$ collisions at $\sqrt{s} = 7$ TeV. The graphs are the transverse momentum ($p_T$) distribution for $1.6 < \eta < 5.3$ (left panel) and the pseudorapidity ($\eta$) distribution for $p_T > 0.5$ GeV (right panel). The curves are for prompt production assuming color-octet $^3S_1$ dominance (solid) or S-wave dominance (dotted) and for production from $b$-hadron decay (dashed).

We now move on to the production of $X(3872)$ in the LHCb experiment, which is a forward detector in the pseudorapidity region $1.6 < \eta < 5.3$. In Fig 5, we show the $p_T$ distributions integrated over the pseudorapidity range $1.6 < \eta < 5.3$ and the pseudorapidity distributions integrated over the region $p_T > 0.5$ GeV. The differences in the prompt cross section from different simplifying assumptions for the NRQCD matrix elements are small for $p_T$ greater than about 5 GeV, but they increase at smaller values of $p_T$. The cross section for S-wave dominance is larger than that for color-octet $^3S_1$ dominance by a factor of 2.5 at $p_T = 1.5$ GeV and 6.5 at $p_T = 0.75$ GeV. This leads to the large difference in the normalization of the rapidity distributions for these two cases in Fig. 5. The prompt cross section integrated over the region $p_T > 0.5$ GeV and $1.6 < \eta < 5.3$ is predicted to be about 270 nb and 688 nb in the cases of color-octet $^3S_1$ dominance and S-wave dominance, respectively. The $b$-decay cross section integrated over the same region is predicted to be 14 nb. Assuming color-octet $^3S_1$ dominance, the fraction of $X(3872)$ events from $b$-hadron decay is predicted to increase from 2.2% at $p_T = 0.5$ GeV to 9.0% at $p_T = 5$ GeV and to 20% at $p_T = 20$ GeV. There are large uncertainties in these predictions, especially those that involve production at small $p_T$. 

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VII. SUMMARY

We have analyzed the production of the $X(3872)$ at the Tevatron. We showed that the prompt production rate observed at the Tevatron is compatible with the identification of $X(3872)$ as a loosely-bound charm meson molecule. In Ref. [6], the authors found that the production rate at the Tevatron was too large for a loosely-bound charm-meson molecule by orders of magnitude. The error in their analysis was that they did not take into account effects of rescattering of the charm mesons that are specific to an S-wave threshold resonance.

We have predicted the differential cross sections for the $X(3872)$ at the LHC from both prompt QCD mechanisms and the decays of $b$ hadrons. The prompt cross section for $X(3872)$ was calculated by using the NRQCD factorization formula. We used simplifying assumptions to reduce the nonperturbative factors to a single NRQCD matrix element $\langle O_8^{X(3S_1)} \rangle$, which was determined from an estimate of the prompt cross section at the Tevatron. The cross section for $X(3872)$ from $b$-hadron decay was calculated using a method that gives the correct differential cross section for $J/\psi$ from $b$-hadron decay. The normalizing factor was determined from an estimate of the $b$-decay cross section at the Tevatron.

For the ATLAS and CMS detectors, we predict the production rate to be more than an order of magnitude larger than for the CDF detector at the Tevatron, given the same cut on the $p_T$ of the $X$ and the same integrated luminosity. For the LHCb detector, where a much lower $p_T$ cut is possible, the production rate could be larger by another order of magnitude. Thus the LHC experiments should be able to collect very large data samples of $X(3872)$.

The large data samples of $X(3872)$ at the LHC will allow precise measurements of various properties of this remarkable hadron. It should be possible to measure the branching ratios for decays of $X(3872)$ with a good signature, including $J/\psi \pi^+ \pi^-$, $J/\psi \gamma$, and $\psi(2S) \gamma$. It may also be possible to observe for the first time some rare decay modes of the $X(3872)$, such as $\chi_{cJ}(1P) \pi^+ \pi^-$ [37, 38]. It should be possible to improve on the measurement of the binding energy of $X(3872)$ by the CDF Collaboration [8]. It may even be possible to distinguish the universal line shape of the $X(3872)$ [39, 40] from that of a conventional Breit-Wigner resonance. In conclusion, the experiments at the LHC are sure to add significantly to our understanding of the $X(3872)$. 
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