THE NONLINEAR EVOLUTION OF RARE EVENTS

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ABSTRACT

In this paper I consider the nonlinear evolution of a rare density fluctuation in a random density field with Gaussian fluctuations, and I rigorously show that it follows the spherical collapse dynamics applied to its mean initial profile. This result is valid for any cosmological model and is independent of the shape of the power spectrum. In the early stages of the dynamics the density contrast of the fluctuation is seen to follow with a good accuracy the form

$$\delta = (1 - \delta_L/1.5)^{-1.5} - 1,$$

where $\delta_L$ is the linearly extrapolated overdensity.

I then investigate the validity domain of the rare event approximation in terms of the parameter $\nu = \delta_L/\sigma$ giving the initial overdensity scaled by the rms fluctuation at the same mass scale, and find that it depends critically on the shape of the power spectrum. When the power law index $n$ is lower than $-1$ the departure from the spherical collapse is expected to be small, at least in the early stages of the dynamics, and even for moderate values of $\nu$ ($|\nu| \geq 2$). However, for $n \geq -1$ and whatever the value of $\nu$, the dynamics seems to be dominated by the small-scale fluctuations and the subsequent evolution of the peak may not be necessarily correlated to its initial overdensity. I discuss the implications of these results for the nonlinear dynamics and the formation of astrophysical objects.

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1. INTRODUCTION

The gravitational instability scenario is the most widely accepted answer to the problem of the formation of the present structures of the universe. In such a view the astrophysical objects such as galaxies or clusters of galaxies are thought to be formed by the collapse of initial density fluctuations existing in the density field after the recombination. If it is well known that such perturbations are gravitationally unstable and are bound to grow under the influence of gravity, the complete dynamics of a density perturbation embedded in a fluctuating density field is not understood. Indeed it requires the resolution of nonlinear equations which has not been done yet. However the behavior of particular objects for which the initial conditions are well defined, such as a perturbation with an initial top–hat density profile, can be calculated up to their final collapse. These results have given birth to some theoretical derivations of the mass distribution function of the objects present in our universe with the so called Press and Schechter formalism (Press & Schechter 1974). The main hypothesis on which this formalism is based is that any part of the universe will eventually collapse and form a virialized object according to its initial overdensity and within the time scale given by the spherical collapse model (e.g., Peebles 1980). It has also been suggested that it is more likely that the objects form at the location of the peaks, maxima of the density field (Kaiser 1984). It leads to the peak formalism developed by Bardeen et al. (1986). In any case, however, the time scale required for the formation of a virialized object is thought to follow the spherical collapse model. These formalisms are the only analytical models available to derive the epoch of formation of astrophysical objects such as quasars (Efstathiou & Rees 1988), or to derive the number density of objects like clusters (Peebles, Daly & Juszkiewicz 1989). Except with the use of numerical simulations, this is the only way to constrain the various cosmological models with such observations. It is thus of crucial interest to know whether the spherical collapse is a good approximation for the formation of isolated structures in a fluctuating density background.

Peebles (1990) recently reconsidered this old problem arguing that the existence of small–scale fluctuations should slow down the dynamics of the collapse. In his view the non–radial motions should initiate a “previrialization” process. He indeed presents some numerical experiments in which such a trend appears with a large magnitude. These results may be at variance with what has been previously obtained by Efstathiou et al. (1988) and by Bond et al. (1990) when they checked the validity of the Press and Schechter formula, and also at variance with the results of Little, Weinberg & Park (1991) and Evrard & Crone (1992) in which they explicitly check the influence of a variable cut–off at small scale in the power spectrum, that fails to detect any “previrialization” effect.

In this paper I consider this problem from an analytical point of view by analysing the early stages of the dynamics of a constrained Gaussian random field with perturbation theories. This paper, however, is not devoted to a new
perturbative theory but aims to consistently calculate some statistical quantities describing the dynamics of a given density fluctuation. I focus my interest on the evolution of the volume of such a rare fluctuation during its nonlinear evolution. The volume is simply defined as the volume occupied at any time by the particles that were in the initial density perturbation. Although it is certainly a naïve picture for the formation of an object that likely involves accretion or ejection of matter, such an approximation is in the spirit of the current theories of the mass distribution function for which it is assumed that the matter content of a fluctuation is conserved during the collapse. In the discussion I will comment on the relevance of this hypothesis.

In part 2, I compute the expectation value of the size of a density perturbation in the limit of rare events. It is shown that in such a limit it can be derived even in the fully nonlinear regime and it then follows the spherical collapse solution. Part 3 is devoted to the discussion of the accuracy of this limit, that is, the first corrections expected from this limit case and their magnitudes. The case of a realistic model for the structure formation, i.e. a CDM model, is considered. A large fraction of the mathematical content of the resolution of this problem is given in appendices. The first one is devoted to usual results on the expectation values of non–Gaussian variables, and the second to the computation of two integrals involving the top–hat filter function. The third one is the most important. It contains the rigorous demonstration of the results presented in part 2. The last one is devoted to tedious calculations of interest for the discussion presented in part 3.

2. THE NONLINEAR EVOLUTION OF RARE EVENTS

2.1 The initial conditions

I assume that the structures of the universe form from gravitational instabilities created by Gaussian fluctuations in a pressureless matter density field. The overdensity field \( \delta(\mathbf{r}) \) is then a random field so that its Fourier transform components \( \delta_k \) defined by,

\[
\delta(\mathbf{r}) = \int \frac{d^3k}{(2\pi)^3/2} \delta_k e^{i\mathbf{k} \cdot \mathbf{r}},
\]

are random complex variables with independent phases. The reality constraint of \( \delta(\mathbf{r}) \) implies

\[
\langle \delta_{k_1} \delta_{k_2} \rangle = \delta_3(k_1 + k_2)P(k_1)
\]

where \( \delta_3(\mathbf{k}) \) is the three dimensional Dirac distribution. The power spectrum, \( P(k) \), determines the full process of the structure formation.
Throughout this paper I assume that within a spherical volume of radius $R_0$ there is a given initial overdensity $\delta_i$,

$$\delta_i = \frac{1}{V_0} \int_{V_0} d^3r \, \delta(r).$$ \hspace{1cm} (3)

If the constraint (3) is fulfilled I will say that there is a “peak” in the given volume with the initial overdensity $\delta_i$. This is a rather poor definition that makes sense at least when $\delta_i$ is large in a sense that will be given afterwards. One could have thought of stronger constraints for a given location to be called a peak, and, following Bardeen et al. (1986), required that this volume is at a maximum of the density field. Such a requirement will not change the qualitative results presented throughout this paper as it will be discussed at the end. Moreover the results presented in the following are valid for positive values of $\delta_i$ as well as negative values. So it will describe also the nonlinear evolution of rare voids.

Once a peak is supposed to lie at a given location the resulting constrained random field is not only a function of $P(k)$ but also a function of $\delta_i$. This quantity is a Gaussian variable related to the $\delta_k$. Its variance is given by

$$\sigma_i^2 = \langle \delta_i^2 \rangle = \frac{1}{V_0^2} \int_{V_0} d^3r \, d^3r' \langle \delta(r) \delta(r') \rangle$$

\hspace{1cm} $= \frac{1}{V_0^2} \int_{V_0} d^3r \, d^3r' \int \frac{d^3k}{(2\pi)^{3/2}} \frac{d^3k'}{(2\pi)^{3/2}} \langle \delta_k \delta_k' \rangle e^{ik \cdot r + ik' \cdot r'}.$ \hspace{1cm} (4)

For convenience I introduce the Fourier transform of the top–hat window function,

$$\frac{1}{V_0} \int_{V_0} d^3r \, e^{ik \cdot r} = \frac{3}{(kR_0)^3} [\sin(kR_0) - kR_0 \cos(kR_0)] \equiv W_{\text{TH}}(kR_0).$$ \hspace{1cm} (5)

The resulting variance for the initial overdensity is

$$\sigma_i^2 = \int \frac{d^3k}{(2\pi)^3} W_{\text{TH}}^2(kR_0) P(k).$$ \hspace{1cm} (6)

The purpose of this paper is to study the expected dynamics of the random field once the constraint (3) has been set. One obvious and direct thing to do is to consider the expectation values of the Fourier components of the random field. In the following I denote $\langle \cdot \rangle_{\delta_i}$ the expectation value of any random quantity when the constraint (3) is set. A priori any such quantity depends both on the power spectrum and on the value of $\delta_i$. The variables $\delta_i$ and $\delta_k$ are both Gaussian and they are correlated together, so that

$$\langle \delta_k \rangle_{\delta_i} = \frac{\langle \delta_k \delta_i \rangle}{\langle \delta_i^2 \rangle} \delta_i$$

\hspace{1cm} $= \frac{P(k) W_{\text{TH}}(kR_0)}{(2\pi)^{3/2} \sigma_i^2} \delta_i.$ \hspace{1cm} (7)
The mean initial field exhibits a symmetric density perturbation, the profile of which is determined by the shape of the power spectrum. The expected local overdensity within a distance $r$ from the center of the perturbation is

$$\langle \delta(< r) \rangle_{\delta_i} = \frac{\int d^3k/(2\pi)^3 P(k) W_{\text{TH}}(k R_0) W_{\text{TH}}(k r)}{\sigma_i^2} \delta_i$$

and the local overdensity at a distance $r$ from the center can be easily derived from this formula. The nonlinear evolution of a spherically symmetric perturbation with this mean density profile is also straightforward to compute: the trajectory of the matter at a distance $r$ of the center depends only on the matter contained inside the radius $r$. The evolution of the profile of the object is shown in Fig. 1 as a function of time for various initial power spectra. There are a priori no reasons for the nonlinear evolution of these mean profiles to reproduce the mean nonlinear profiles. However, for a very dense perturbation one would expect that the dynamics of the collapse somehow follows this behavior. This is the central problem addressed in this paper.

### 2.2 The equations of motion

In the nonlinear evolution of the objects with the previous profiles, the size of the perturbation turns out to be the only quantity that is independent of the shape of the power spectrum. This is the reason why, also in the nonlinear constrained dynamics, I focus my interest on it. The size of the perturbation is assumed to be simply given by the volume occupied by the matter that was initially inside the radius $R_0$ all along the nonlinear evolution. A Lagrangian calculation is the most adequate approach for the purpose of this calculation. In the initial unperturbed field, fluctuations are assumed to create initial random displacements that will be amplified by the gravitational dynamics. The unperturbed comoving positions are denoted $q$, and the exact comoving positions, $x$, are related to $q$ through a displacement field $\Psi(t, q)$,

$$x = q + \Psi(t, q).$$

The volume $V(t)$ subsequently occupied by the particles in the real space is related to the displacement field through space derivatives. Indeed it reads

$$V(t) = \int_V d^3x = \int_{V_0} \left| \frac{\partial x}{\partial q} \right| d^3q.$$

The jacobian of the transformation $J(t, q)$ defined by

$$J(t, q) = \left| \frac{\partial x}{\partial q} \right|$$

(11)
is then the quantity of interest for the calculation of the volume. It is the determinant of the matrix the elements of which are

\[
\frac{\partial x_i}{\partial q_j} = \delta_{ij} + \frac{\partial \Psi_i}{\partial q_j} = \delta_{ij} + \Psi_{i,j}
\] (12)

where \(\delta_{ij}\) is the Kronecker symbol. The motion equation of the point being at the position \(x\) leads straightforwardly with the Poisson equation to

\[
\nabla_x \ddot{x} + 2 \frac{\dot{a}}{a} \nabla_x \dot{x} = -4\pi G (\rho(x) - \bar{\rho}).
\] (13)

The density \(\rho(x)\) at the position \(x\) is given by the matter conservation constraint

\[
|J(x)|\rho(x) = \bar{\rho}.
\] (14)

In the following I assume that \(J(x)\) is positive so that \(|J(x)| = J(x)\). This assumption is valid in the quasilinear regime since \(J(x)\) is then close to 1, but breaks after the first shell crossing where \(J(x)\) is zero. With this assumption we then have

\[
J(t, q) \left( \nabla_x \ddot{x} + 2 \frac{\dot{a}}{a} \nabla_x \dot{x} \right) = -4\pi G \bar{\rho} \left[ 1 - J(t, q) \right].
\] (15)

We assume that the time \(t_i\) at which the initial fluctuations start to grow is arbitrarily close to the origin. The decaying modes of any kind are then subsequently erased so that the only purely growing modes remain at the present time. This is likely to be an unjustified approximation for the study of the virialization processes but we are only interested here in the collapse prior to the virialization. The time dependence of the growing mode in the linear approximation will be denoted \(D(t)\). It is just proportional to the expansion factor in case of an Einstein–de Sitter universe but in general depends on the cosmological parameters (see Lahav et al. 1991 for a review). Another consequence of crucial interest is that the rotational modes are suppressed, since they are simply expected to be diluted with the expansion (e.g., Peebles 1980). I then assume that the displacement field is non–rotational, that is

\[
\nabla_x \times \dot{\Psi}(x) = 0.
\] (16)

The resolution of the dynamics in such an approach is entirely driven by the behavior of the displacement field, so that the motion equations have to be written only in term of \(\Psi(t, q)\) and its derivatives. The equation (11) then reads

\[
J(t, q) = 1 + \nabla_q \cdot \Psi + \frac{1}{2} \left[ (\nabla_q \cdot \Psi)^2 - \sum_{ij} \Psi_{i,j} \Psi_{j,i} \right]
\]

\[
+ \frac{1}{6} \left[ (\nabla_q \cdot \Psi)^3 - 3 \nabla_q \cdot \Psi \sum_{ij} \Psi_{i,j} \Psi_{j,i} + 2 \sum_{ijk} \Psi_{i,j} \Psi_{j,k} \Psi_{k,i} \right].
\] (17)
where $\nabla q$ is the gradient taken relatively to $q$ and the summations are made over the three spatial components. The equation (15) involves a time derivative operator that I denote $\mathcal{T}(\cdot)$ and that is defined by

$$\mathcal{T}(A) = \dddot{A} + \frac{\dot{a}}{a} \ddot{A}.$$  

(18)

Expressed in term of the displacement field only the equation (15) reads

$$\mathcal{T}(\nabla q \cdot \Psi) + \nabla q \cdot \Psi \mathcal{T}(\nabla q \cdot \Psi) - \sum_{ij} \Psi_{i,j} \mathcal{T}(\Psi_{j,i}) + \frac{1}{2} (\nabla q \cdot \Psi)^2 \mathcal{T}(\nabla q \cdot \Psi) - \frac{1}{2} \mathcal{T}(\nabla q \cdot \Psi) \sum_{ij} [\Psi_{i,j} \Psi_{j,i} - \nabla q \cdot \Psi \mathcal{T}(\Psi_{i,j}) \Psi_{j,i}] + \sum_{ij} \Psi_{i,j} \Psi_{j,k} \mathcal{T}(\Psi_{k,i})$$

$$= -4\pi G \rho [1 - J(t, q)].$$  

(19)

The equations (17) and (19) do not allow a complete resolution of the dynamics. However, the displacement field is fully determined as soon as the non–rotational constraint (16) has been set. The resolution of these equations would lead to a complete knowledge of the nonlinear dynamics (before shell crossings). I am interested here in a particular quantity, i.e. the volume occupied by a peak all along its nonlinear evolution.

2.3 The expectation value of the volume

The volume of the object is obviously a non–Gaussian random variable and a complicated function of the random variables $\delta_k$, which are correlated to the initial linear overdensity $\delta_i$. In this part I am interested in the expectation value of the volume, $\langle V(t) \rangle_{\delta_i}$, for a random density field that follows the constraint (3). This quantity is a priori a function of $V_0$, of the power spectrum $P(k)$, of the initial overdensity $\delta_i$ and of time. Once the time and the shape of the power spectrum have been specified this is a function of $\delta_i$ and $\sigma_i$ that contains the normalization of the initial fluctuations. Equivalently it is a function of the linearly extrapolated overdensity $\delta_L$,

$$\delta_L = \frac{D(t)}{D(t_i)} \delta_i,$$  

(20)

where $D(t)$ is the time dependence of the growing mode, and the linearly extrapolated rms fluctuations at the same scale,

$$\sigma_L = \frac{D(t)}{D(t_i)} \sigma_i.$$  

(21)

The regime that is investigated concerns the collapse of rare peaks or the expansion of rare troughs before the first shell crossing. As a result the linear overdensity of
the object, $\delta_L$, is at most of the order of a few units but the rms fluctuation at the same scale, $\sigma_L$, is small. The rare peak limit then implies that the expectation value of $V(t)$ has to be taken for a vanishing value of $\sigma_L$ and a fixed value of $\delta_L$.

Following the appendix A, such expectation value is given by (Eqs. A.13, A.16)

$$\langle V(t) \rangle_{\delta_L} = \int_{-\infty}^{+\infty} dz \mathcal{G}_V(t, \sigma_L, \delta_L + iz) \exp \left[ -\frac{z^2}{2 \sigma_L^2} \right] / \int_{-\infty}^{+\infty} dz \exp \left[ -\frac{z^2}{2 \sigma_L^2} \right]$$

with

$$\mathcal{G}_V(t, \sigma_L, y) = \sum_{p=0}^{\infty} \frac{\langle V(t) \delta_L^p \rangle_c}{\sigma_L^{2p}} \frac{y^p}{p!},$$

since $\delta_L$ is a Gaussian variable. The mean volume is then a complicated function of $\delta_L$ and $\sigma_L$ which I want to calculate in the limit $\sigma_L \to 0$. The leading term in the numerator of the ratio (22) is given by a saddle point approximation (around $z \approx 0$). The relation (22) then reads

$$\langle V(t) \rangle_{\delta_L} \approx \mathcal{G}_V(t, \sigma_L = 0, \delta_L).$$

I would like to stress that this expression is exact in the rare peak limit but involves all orders of the perturbation theory through the function $\mathcal{G}_V(t, \sigma_L = 0, \delta_L)$ (but assuming no shell crossing and irrotationality). If the rare peak approximation were released, the expression (24) would no longer be valid. The part 3 is devoted to an analysis of the accuracy of this limit, its real meaning, and its validity domain.

Each term involved in the expression of $\mathcal{G}_V(t, \sigma_L, \delta_L)$ has then to be calculated when $\sigma_L = 0$ (as we will see each term is actually finite in spite of the denominators appearing in Eq. (23)). The quantities of interest $\langle V(t) \delta_L^p \rangle_c$ have then to be calculated at the lowest order in $\sigma_L$ (here $\delta_L$ is not a fixed parameter but a Gaussian random variable).

The random variable $V(t)$ can be expanded relatively to $\delta_k$,

$$V(t) = \sum_{p=0}^{\infty} V^{(p)}(t)$$

where $V^{(p)}(t)$ contains exactly $p$ factors of $\delta_k$ in its expression. The magnitude of $V^{(p)}(t)$ is

$$V^{(p)}(t) \sim V_0 \delta_L^p.$$
variables and then requires the product of at least $2p$ random Gaussian variables. As a result the leading contribution to this cumulant when the rms fluctuation is small comes from the order $p$ of $V(t)$. These arguments are the same that lead to the scaling behavior of the $p$ order cumulants of the distribution function of the density at large scale (Fry 1986, Bernardeau 1992). The leading contribution at small $\sigma_L$ is the exact result in the rare peak approximation, so that in such a case we get

$$j_p(t) \equiv \langle V(t)\delta_L^p \rangle_c/V_0 = \langle V^{(p)}(t)\delta_L^p \rangle_c/V_0$$

and this quantity is of the order of

$$j_p(t) \sim \sigma_i^{2p}. \quad (27)$$

The function $G_V(t, \sigma_L = 0, \delta_L)$ is then a finite number for finite values of $\delta_L$.

At this stage the most difficult part is the derivation of the function $G_V(t, 0, y)$ (denoted $G_V(t, y)$ in the following). It cannot be done by a simple perturbative calculation since all orders of the dynamics are involved. As it is a rather complicated and long demonstration I present it only as an appendix (appendix C). It contains absolutely no other approximations than the ones presented previously and it gives the values of the whole series of $j_p(t)$ as written in the equation (26).

The result of the appendix C is quite simple and shows that the function $G_V(t, y)$ follows the differential equation,

$$\frac{d^2}{dt^2} \left( R_0 [G_V(t, y)/V_0]^{1/3} \right) = \frac{-(G \rho + \Lambda/4\pi) V_0}{\left( R_0 [G_V(t, y)/V_0]^{1/3} \right)^2}, \quad (28)$$

which is the motion equation describing a spherical collapse, where $\left( R_0 [G_V(t, y)/V_0]^{1/3} \right)$ is the size of the object ($\Lambda$ is a possible cosmological constant). The behavior of $G_V(t, y)$ for small values of $y$, $(G_V(t, y) = V_0[1 - y + \ldots])$ ensures that the corresponding overdensity is simply $y$.

The function $G_V(t, \delta_L)$ that gives the expectation volume of a peak of overdensity $\delta_L$ is then simply the volume occupied at the time $t$ by an object having the same initial overdensity and initial comoving volume, according to the spherical collapse model. This result is independent of the cosmological parameters ($\Omega$ and $\Lambda$). As it can be noticed, the dependence on the power spectrum completely vanished and an individual collapse is expected to follow the same dynamics whatever the shape of the power spectrum. This result, however, is not robust, as it is discussed in part 3. It is quite comforting to see that the spherical collapse has something to do with the collapse of an object embedded in a fluctuating universe.

### 2.4 The nonlinear evolution of the profile, of the density
The previous calculation deals with the expectation value of the volume $V$ occupied by the particles that were in the initial volume $V_0$. One obviously could have considered the particles that were initially within the radius $\lambda R_0$ (whatever $\lambda$) and derive the expectation value of the volume occupied by these particles. The same kind of calculations can be done for this more general problem and the result is that they occupy the volume $\lambda^3 V_0 G_V[t, \langle \delta(<\lambda R_0) \rangle_{\delta_L}]$ which is the exact nonlinear evolution of the mean profile (this result is not a consequence of the previous one but its demonstration is nearly identical). The result is also valid for underdense areas so that the spherical model solution is expected to describe as well the collapse of rare peaks as the expansion of rare troughs.

I propose a simple analytical expression for the evolution of the volume valid as long as the density contrast is not too large:

$$\langle V(t) \rangle_{\delta_L} = V_0 \left[ 1 - \frac{\delta_L}{1.5} \right]^{1.5},$$

(29a)

where $\delta_L$ is the linear extrapolation of the initial overdensity (eq. [20]). The dependence with the cosmological parameters is entirely contained in the time dependence of $D(t)$. The form (29a) is exact for $\Omega \to 0$, $\Lambda = 0$ and is a good fit for any values of $\Omega$ and $\Lambda$, especially for the underdense regions ($\delta_L < 0$). It fails, however, to reproduce with a good accuracy the position of the singularity (Fig. 2).

The mean overdensity of the object, simply given by $\langle V/V(t) \rangle_{\delta_L} - 1$, can also be calculated in the rare event limit. It involves a similar generating function $G_\delta(t, y)$ that can be calculated in the rare event limit (appendix C). The result in such a case is quite simple and reads, $G_\delta(t, y) = V_0/G_V(t, y) - 1$, so that the mean density is simply given by the inverse of the mean volume and one can use the relationship,

$$\langle \delta \rangle_{\delta_L} = \left[ 1 - \frac{\delta_L}{1.5} \right]^{-1.5} - 1.$$

(29b)

As a result it is shown that in the rare event limit the density fluctuations behave as if they had the mean initial profile described in the section 2.1. In particular the density fluctuations are expected to be spherical (otherwise the timescale of the collapse would not be the one of the spherical collapse) and the outskirts of clusters for instance are well described by such an approach (while the inner part is subject to processes that are beyond the scope of these calculations). This result may not be surprising. It is simply that the rare event constraint of the random Gaussian field (3) is strong enough to determine the shape of the initial profile and its subsequent dynamics. What has been obtained here is a rigourous demonstration that in the rare event limit, in a field with Gaussian fluctuations, the dynamics of a constained field follows the spherical collapse up to the first singularity.
(that is at $\delta_L \approx 1.68$ for an Einstein–de Sitter universe). The subsequent evolution cannot be studied with a perturbation theory (the reason is that the determinant $J(x)$ is not necessarily positive and taking the absolute value cannot be obtained by means of a perturbative expansion).

In practice however, the astrophysical objects may approach such a limit but never reach it exactly. This is the problem addressed in the third part of this work.

### 3. THE ACCURACY OF THE RARE EVENT LIMIT

The interest of the previous calculations is not only that we recover a known dynamics but that it is now viewed as a limit process. The accuracy of this limit can now be checked for each step of the calculation. It can be evaluated either by the departure of the exact mean collapse from the spherical collapse, or by the magnitude of the scattering around such a behavior. Firstly I examine the first corrections to the spherical collapse that appear when the rare event limit is released. The parameter that is supposed to be large is defined by

$$\nu = \frac{\delta_L}{\sigma_L}. \quad (30)$$

The probability distribution of $\nu$ is normal and rare events correspond to large values of $\nu$, either positive or negative. In case of clusters for instance the values for $\nu$ derived from the observations are of the order of 2 to 4 (Peebles et al. 1989). For quasars the values required for $\nu$ may be greater. In the following I will then be concerned by the values of $\nu$ of this order of magnitude or greater.

#### 3.1 The corrections to the mean collapse

The mean value of the volume can be expanded with respect to $\sigma_L$,

$$\langle V \rangle_{\delta_L} = \langle V \rangle_{\delta_L}^{(0)} + \sigma_L^2 \langle V \rangle_{\delta_L}^{(2)} + \ldots$$

The term $\langle V \rangle_{\delta_L}^{(0)}$ is the one that has been calculated previously in the rare event limit. The first correction is given by the term $\sigma_L^2 \langle V \rangle_{\delta_L}^{(2)}$ (the order 1 in $\sigma_L$ is identically zero). To discuss the results in term of $\nu$ this correction can be written as $\delta_L^2/\nu^2 \langle V \rangle_{\delta_L}^{(2)}$ which due to this change of variable gives a correction starting at the quadratic order in $\delta_L$. The evaluation of $\langle V \rangle_{\delta_L}^{(2)}$ has to be done by a perturbative calculation around the rare event limit, that is around $\sigma_L = 0$. During the calculations of part 2 the $\sigma_L = 0$ limit has been invoked two times, once to derive the expression of the mean value of $V(t)$ as given by (22), then to calculate $j_p$, in Eq. (26). It would be possible to calculate (22) exactly but the derivation
of the parameters \( j_p \) is more complicated and cannot be done in general. Actually it will not be possible to derive \( \langle V(2) \rangle_{\delta L} \) exactly but only the first two terms of its expansion with respect to \( \delta_L \). These terms are partly given by the first correction in \( \sigma_L^2 \) of \( j_0 \) and \( j_1 \). Indeed in general the value of \( j_p \) can be seen as an expansion with respect to \( \delta_k \),

\[
V_0 j_p(t) = \langle V^{(p)}(t) \delta_L^p \rangle_c + \langle V^{(p+2)}(t) \delta_L^p \rangle_c + ...
\]  

(31)

The first term of this expansion is the one already calculated, and it has been shown to correspond to the spherical collapse. The corrective terms cannot be calculated as easily and the calculation has to be made by hand for each of the terms. For \( p = 0 \) the situation is extremely simple since the corrections are all identically zero. It just means that the ensemble average of the volume simply follows the comoving volume. The first non–trivial term appears for \( p = 1 \). The leading term is simply the one coming from the linear approximation and the first correction is then

\[
j_1(t) = -\sigma_L^2 + \frac{1}{V_0} \langle V^{(3)}(t) \delta_L \rangle_c + ...
\]  

(32)

The calculation of this corrective term is presented in the appendix D in the case of an Einstein–de Sitter universe. According to the relation (D.19) the expression of \( j_1 \) is then

\[
j_1(t) = -\sigma_L^2 \left[ 1 - \frac{4\sigma_L^2}{21\sigma_i^2} \int \frac{d^3k}{(2\pi)^3} W^2(k R_0) P(k) \int \frac{d^3k'}{(2\pi)^3} P(k') f(k'/k) \right]
\]  

(33)

where the corrective coefficient \( C_{\text{exp}} \) is given by

\[
C_{\text{exp}} = \frac{4 \int \frac{d^3k}{(2\pi)^3} W^2(k R_0) P(k) \int \frac{d^3k'}{(2\pi)^3} P(k') f(k'/k)}{21 \left[ \int \frac{d^3k}{(2\pi)^3} W^2(k R_0) P(k) \right]^2}
\]  

(34)

and \( f(\tau) \) is the function defined in (D.20). It is plotted in Fig. 3. The corrective term appears to be the product of one non–dimensional parameter, \( C_{\text{exp}} \), and a time dependent factor which is simply the linear extrapolated value of the variance at the scale \( R_0 \) for the present time. This allows one to derive the exact form of \( \langle V(t) \rangle_{\delta k} \) up to the third order in \( \delta_L \) as a function of \( \nu \). The results are presented for an Einstein–de Sitter universe. In such a case we have

\[
j_2(t) = \frac{8}{21}\sigma_i^4, \quad j_3(t) = \frac{20}{189}\sigma_i^6.
\]  

(35)
The resulting form for the expectation value of the volume when the corrections coming from the exact calculation of the integral (22) are included, is

\[
\langle V(t) \rangle_{\delta L} = V_0 \left[ 1 - \delta_L + \frac{4}{21}(1 - 1/\nu^2) \delta_L^2 + \frac{10}{567} \left[ 1 + \frac{-3 + 567/10 \cdot C_{\text{exp}}}{\nu^2} \right] \delta_L^3 + \ldots \right].
\] (36)

The dynamics of the spherical collapse is recovered in the limit \( \nu \to \infty \). The accuracy of this limit then depends on the values taken by \( C_{\text{exp}} \). As can be seen from the equation (34) the value of this coefficient is a function of the shape of the power spectrum. In case \( P(k) \) has a power–law shape

\[ P(k) \propto k^n \] (37)

the resulting values for \( C_{\text{exp}} \) are given in Fig. 4a. The most remarkable feature is that the coefficient diverges for \( n \geq -1 \). This divergence comes from the large values of \( k \) and is due to the shape of the function \( f(\tau) \) for \( \tau \to \infty \) \((f(\tau) \sim 5/8\tau^2 \) when \( \tau \to \infty \)). The small scale fluctuations are then responsible for the infinite value of \( C_{\text{exp}} \) when \( n \geq -1 \), and for such power spectra the rare event limit can never be reached, \textit{whatever the value of } \( \nu \). In order to check the importance of this divergence, I also considered the case of a power spectrum with a cut–off \( k_c \) for large values of \( k \). I then give in Fig. 4b the behavior of \( C_{\text{exp}} \) when the position of the cut–off varies compared to the inverse of the size of the fluctuation. The results are given for various values of \( n \). It appears that for \( n = -1 \), which is the critical case, the divergence is only logarithmic so that a very large dynamical range is required for such an effect to be noticed. For \( n = 0 \) the divergence is more rapid (as \([R_0 k_c]^2\)) and such an effect could be seen in numerical simulations.

When \( n < -1 \) the corrective coefficient is small and the values of \( \nu \) of a few units are enough to ensure a behavior close to the spherical case. For \( n = -2 \) we have \( C_{\text{exp}} \approx 0.11 \). In Fig. 5, I present the expected density of the object (the mere inverse of the volume) as a function of the initial overdensity for \( \nu = 4 \) and \( \nu = 2 \). The curves correspond to

\[
\langle V_0 / V(t) \rangle_{\delta L} = 1 + \delta_L + \frac{17}{21}(1 + \frac{4}{17\nu^2}) \delta_L^2 + \frac{341}{567} \left[ 1 + \frac{246/341 - 567/341 \cdot C_{\text{exp}}}{\nu^2} \right] \delta_L^3 + \ldots
\] (38)

The relation (38) has not been obtained as a mere expansion of the inverse of the relation (36) but as an exact determination of \( \langle V_0 / V(t) \rangle_{\delta L} \) up to the third order. It happens that it is the same. The departures from the spherical collapse (at the same order) remain small.

For the known astrophysical objects this is then of crucial interest to know the value of \( n \) at their mass scale. For galaxies and quasars a direct observation is not possible since the universe is now fully nonlinear at their mass scale, but
according to most of the theories based on the gravitational instability scenario, \( n \) is likely to be of the order of \(-2\) or smaller. For the clusters the situation is more critical. Direct measurements of the mass fluctuations through galaxy counts give \( n \approx -1.3 \) according to Peacock (1991) for the APM angular survey, \( n \approx -1.4 \) according to Fisher et al. (1993) for the IRAS galaxy survey. The use of the temperature distribution of the clusters allows one (with the Press and Schechter formalism) to do a direct estimation of the shape of \( P(k) \) for the matter field. It leads, according to Henry & Arnaud to \( n \approx -1.71 + 0.65 - 0.35 \) and according to Oukbir & Blanchard (1992) to \( n \approx -2 \) independently of the density of the universe. It then seems that for the clusters it is also fair to use the spherical collapse model.

The case of the CDM model is considered in Fig. 6. The coefficient \( C_{\text{exp}} \) is computed with

\[
P(k) \propto \frac{k}{(1 + \alpha k + \beta k^{3/2} + \gamma k^2)^2},
\]

where

\[
\alpha = 1.71 l, \quad \beta = 9.0 l^{3/2}, \quad \gamma = 1.0 l^2 \quad \text{and} \quad l = 4.0 \quad \text{corresponding to} \quad h = 0.5, \quad \Omega = 1. \quad \text{(Davis et al. 1985)}.
\]

The corrective coefficient remains essentially finite for the masses of interest, so that the spherical collapse model is expected to hold for such a shape of power spectra.

### 3.2 Fluctuations around the mean behavior

The purpose of this part is to derive the value of \( \langle V^2(t) \rangle_{\delta L} - \langle V(t) \rangle_{\delta L}^2 \) to evaluate the magnitude of the scattering around the mean collapse.

First of all we can calculate this expression in the same limit, i.e. the rare event approximation. According to the appendix A, this function is given by

\[
\langle V^2(t) \rangle_{\delta L} = \int_{-\infty}^{+\infty} dz \left[ G_{V^2}^2(t, \delta L + iz) + G_{V^2}^2(t, \delta L + iz) \right] \exp \left[ -\frac{z^2}{2 \sigma^2_L} \right] \frac{1}{\int_{-\infty}^{+\infty} dz \exp \left[ -\frac{z^2}{2 \sigma^2_L} \right]} \]

(40)

with

\[
G_{V^2}(t, y) = \sum_{p=0}^{\infty} \frac{\langle V^2(t) \delta^p_{L} \rangle_c y^p}{\sigma^2_{L} p!}.
\]

In the rare event limit, and for the same reason as before we have,

\[
\langle V^2(t) \delta^p_{L} \rangle_c \approx \sum_{r=1}^{p+1} \langle V^{(r)}(t) V^{(p+2-r)}(t) \delta^p_{L} \rangle_c.
\]

(41)

The latter expression can be calculated exactly. The result given in the appendix C is quite simple and reads,

\[
G_{V^2}(t, y) \approx \sigma^2_{L} \left[ \frac{d}{dy} G_V(t, y) \right]^2.
\]

(42)
This result is only valid in the rare event approximation: the function $G_V$ that appears in this expression corresponds to the spherical collapse and the first corrections to (42) are of the order of $\sigma_L^4$. The calculation of the expression (40) has then to be made in a consistent way. We obtain

$$\langle V^2(t) \rangle_{\delta_L} = G_V^2(t, \delta_L) + 2 \sigma_L^2 G_V(t, \delta_L) \frac{d}{dy} G_V(t, \delta_L) \left( \frac{d}{dy} G_V(t, \delta_L) \right)^2 + \ldots$$

(43)

In the first term $G_V$ does not only correspond to the spherical collapse but also contains corrections of the order of $\sigma_L^2$ when the approximation (24) is released. The last term corresponds to the first correction of (40) around the saddle point position (obtained by an expansion of $G_V^2(t, y)$ around $z = 0$). The first neglected terms are of the order of $\sigma_L^4$. At the same order we have

$$\langle V(t) \rangle_{\delta_L}^2 = G_V^2(t, \delta_L) - 2 \sigma_L^2 G_V(t, \delta_L) \frac{d}{dy} G_V(t, \delta_L) + \ldots$$

(44)

The equations (42-44) imply that all the terms that are independent of $\sigma_L$ or proportional to $\sigma_L^2$ vanish, leading to a scattering of the form of,

$$\left[ \langle V^2(t) \rangle_{\delta_L} - \langle V(t) \rangle_{\delta_L}^2 \right]^{1/2} \sim \sigma_L^2 C_{\text{Scat}}(t, \delta_L) V_0,$$

(45)

at the lowest order in $\sigma_L$. The coefficient $C_{\text{Scat}}(t, \delta_L)$ is a finite function, which in case of an Einstein–de Sitter universe is a function of $\delta_L$ only. This result just means that there is no scattering when the rare event approximation is valid: the magnitude of the scattering is just equal to the magnitude of the error between the spherical collapse model and the exact mean behavior.

To be more convincing I derive the value of $C_{\text{Scat}}$ for $\delta_L = 0$. A careful examination of the various contributions leads to the expression,

$$C_{\text{Scat}}(\delta_L = 0) = \frac{1}{V_0^2 \sigma_L^2} \left[ \langle V^{(2)}(t) \rangle^2 - \frac{1}{2} \frac{\langle V(t)^2 \rangle}{\sigma_L^4} \delta_L^2 \right]^{1/2}.$$

(46)

It is calculated in the appendix D and plotted in Fig. 7 as a function of $n$ in case of a power–law spectrum (eq. [37]). Once again the coefficient becomes infinite when $n \geq -1$. For $n \approx -2$, $C_{\text{Scat}} \approx 0.3$ so that the scattering around $\delta_L = 0$ is of the order of

$$\left[ \langle V^2(t) \rangle_{\delta_L} - \langle V(t) \rangle_{\delta_L}^2 \right]^{1/2} \sim 0.3 V_0 \frac{\delta_L^2}{V^2}.$$

(47)
In such a case the fluctuations around the spherical collapse are expected to be small.

The fluctuations around the mean density can also be calculated. They are given by

\[
\left[ \langle \delta^2(t) \rangle_{\delta_L} - \langle \delta(t) \rangle_{\delta_L}^2 \right]^{1/2} \approx C'_{\text{scat}} \frac{\delta_L^2}{\nu^2} \tag{48}
\]

with

\[
C'_{\text{scat}} = \frac{1}{V_0^2 \sigma_L^2} \left[ \langle \left( V^{(1)}(t) \right)^4 \rangle - 2 \langle V^{(2)}(t) \left( V^{(1)}(t) \right)^2 \rangle + \langle \left( V^{(2)}(t) \right)^2 \rangle \right. \\
\left. - \frac{1}{2} \frac{\langle \left( V^{(1)}(t) \right)^2 \delta_L^2 \rangle^2}{\sigma_L^4} - \langle V^{(2)}(t) \delta_L^2 \rangle^2 \right]^{1/2}. \tag{49}
\]

For \( n \approx -2 \) we get \( C'_{\text{scat}} \approx 1 \). The situation obtained there is sketched in Fig. 8: the overdense objects are expected to follow the spherical collapse model with a possible departure and a possible scattering of the same order of magnitude. Then the accuracy of the spherical collapse model depends on the value of \( \nu \). It is not possible to determine the value of \( \delta \) up to which this approximation is valid since the corrective term is known only up to the third order in \( \delta_L \). At this order the discrepancy between the dynamics of the spherical collapse and the real dynamics is seen to be weak as long as \( \nu \) is of the order of a few units (say 2, 3) (eqs. [36, 38]). It implies that the spherical collapse model is relevant at least up to \( \delta \sim 4 \) and for values of \( \nu \) of a few units. Whether this approximation breaks down when \( \delta \) is larger than 4 is a problem not solved by these analytic calculations.

When \( n \geq -1 \), however, the situation is quite different, and the fluctuations are unbounded. It seems that the initial overdensity \( \delta_L \) does not determine at all the subsequent evolution of the object which rather depends on the small-scale fluctuations. The origin of this behavior is discussed in the next section.

### 3.3 Discussion

I want to stress that the results presented here take fully into account the application of a filter function on the density field at any order of the expansion. It represents a considerable improvement compared to previous results. For instance Kofman (1991) proposes a density distribution function based in the Zel’dovich (1970) approximation, thought to be valid when the fluctuations are still small that, however, does not take into account the filtering of the evolved density field. Bernardeau (1992) also presents the expected shape of the density distribution in a quasi–Gaussian approximation (that takes into account the nonlinear behavior of the density field) with the same kind of approximation. Both of these calculations were made for a density in a vanishing volume and appear as approximations to
the exact solution for a finite and large volume. In the calculation presented here, however, the filtering process has been fully taken into account. It turns out that the expected nonlinear evolution of a rare event follows exactly what would have been expected from the spherical model.

It has been stated that the geometrical dependences contained in the second or third order terms may induce a slowing of the dynamics by tidal or shear effects (Juszkiewicz & Bouchet 1992) that would produce a sort of previrialization. The calculations I presented here take into account these geometrical dependences but contain no indications of previrialization, at least in the rare event limit (and as long as \( n < -1 \)). The idea that the small–scale fluctuations may alter the collapse rate of an object was expressed by Peebles (1990). He presented numerical experiments where indeed the growing rate of the fluctuation was slightly lower compared to the spherical collapse model even for objects of final density of the order of 3. This result is at variance with what has been obtained here in the case of moderately rare events. Previrialization however may be present in the final stage of the collapse when the radius of the object starts to decrease for a not extremely rare event.

The extent of the central problem solved in this paper, the expected behavior of a constrained random field, also deserves some comments. In the equation (3) a “peak” has been defined as a mere overdense part of the random density field. The result of part 2, however, holds even for more constraining initial conditions. One could have imposed that the volume is at an extremum of the density field (in case of a rare overdense volume, it is likely to be a maximum). In such a case the initial random field can be seen as a non–homogeneous random Gaussian field for which the relation (1) has been replaced by

\[
\langle \delta_{k_1} \delta_{k_2} \rangle = \delta_3 (k_1 + k_2) P(k_1) - \frac{k_1 \cdot k_2 P(k_1) P(k_2) W(k_1) W(k_2)}{\int \frac{d^3k}{(2\pi)^{3/2}} k^2 W^2(k) P(k)}
\]

(50)

where \( W(k) \) is the shape of the window function in the Fourier space used to determine whether a point is the peak or not (this is not necessarily the top–hat window function). The random field is still Gaussian and isotropic around \( q = 0 \) so that the whole demonstration still holds. The corrections to the rare event limit and in particular the values of the coefficients \( C_{\text{exp}} \) and \( C_{\text{scat}} \), however, are likely to change (and to be reduced). One could have also required that a greater volume, \( V_l \), centered on \( V_0 \) contains a certain overdensity, \( \delta_l \) in order to study the formation of objects in a rich, or a poor, environment as did Bower (1991) in the frame of the Press and Schechter formalism. The conditional random field is still Gaussian and isotropic so that the demonstration still holds, and in the rare peak limit the dynamics follows the spherical collapse model. The only change is that the mean value of \( \delta_L \) is not zero anymore but

\[
\langle \delta_L \rangle = \frac{\int d^3k W_{\text{TH}}(k R_0) W_{\text{TH}}(k R_l) P(k)}{\int d^3k W_{\text{TH}}^2(k R_0) P(k)} \delta_l
\]

(51)
(with an unchanged variance), so that the distribution of $\delta L$ is shifted towards higher or lower values according to the sign of $\delta_1$. This is also of great interest for the Press and Schechter formalism, since as stressed by Bond et al. (1990) and Blanchard, Valls–Gabaud & Mamon (1992), the mass distribution function is based on an assumption on the dynamics of a somehow isolated density perturbation: the initial constraint is not only that a given overdensity is reached in a given volume but also that the surroundings do not form a higher overdense region. This local constraint may be especially important for the low values of $\nu$. This paper does not address, by far, the problem of the exact nonlinear dynamics with such initial conditions when $\nu$ is small so it does not provide a justification for the Press and Schechter formula in its whole generality. The large mass behavior, however, corresponding to the high $\nu$ limit received a strong dynamical justification. The demonstration is unfortunately not complete, since the virialization process remains beyond the scope of these calculations. The work of Thomas & Couchman (1992) suggests that the spherical collapse model is correct up to the maximum expansion of the object but fails to reproduce the contraction rate. This is not that surprising, since in this regime the decaying and rotational modes and the shell crossings that have been neglected are likely to play a major role in the process of relaxation.

The second major phenomenon highlighted in this paper is the existence of a critical value for the power–law index. The existence of an unbound correction when $n \geq -1$ is the signal that a perturbation approach is not safe. Such a result deserves a deeper examination. It turns out that the divergence of $C_{\text{exp}}$ is entirely due to dynamical properties. For instance if we consider the conditional expectation values of the spatial derivatives of the displacement field, we get,

$$\langle \Psi_{i,j}(q) \rangle_{\delta L} = \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{k_i k_j}{k^2} P(k) W_{\text{TH}}(k R_0) \frac{1}{(2\pi)^{3/2} \sigma_i^2} \delta L e^{i k \cdot q} \times \left[ -1 + \frac{4}{21} \left( \frac{D(t)}{D(t_i)} \right)^3 \int \frac{d^3 k'}{(2\pi)^{3/2}} P(k') f(k'/k) + \ldots \right]$$

which is finite only if $P(k) \ll k^{-1}$ when $k \to \infty$ (since $f(\tau) \sim 1/\tau^2$ for $\tau \to \infty$).

As a result when $n \geq -1$ the displacement field is not a derivable quantity: two points that are initially very close may have completely different trajectories. This is the indication that the matter follows a chaotic behavior rather than a mere coherent flow towards the overdense regions. It is confirmed by the derivation of the expectation value of the functional derivative of the displacement field with the initial fluctuation. The result is

$$\left\langle \frac{\delta \Psi_i(q)}{\delta k_i} \right\rangle_{\delta L} = \int \frac{d^3 k}{(2\pi)^{3/2}} \frac{-i k_i}{k^2} e^{i q \cdot k} \left[ -\frac{D(t)}{D(t_i)} + \frac{4}{21} \left( \frac{D(t)}{D(t_i)} \right)^3 \int \frac{d^3 k'}{(2\pi)^{3/2}} P(k') f(k'/k) + \ldots \right]$$
which is also divergent for \( n \geq -1 \). As can be seen in (53) this is only due to the shape of \( f(\tau) \) and not at all to the shape of the filter function. As a result a small change in the initial fluctuations would induce a dramatic change of the behavior of the displacement field. It seems that the particles that were initially in the volume \( V_0 \) do, at least in the first stage of the dynamics, a sort of chaotic diffusion, and have no reason to form ultimately a unique virialized object. This effect is present whatever the value of \( n \) but when \( n < -1 \) the correlation length of the displacement field becomes infinite (as it is for the velocity field) and the displacement affects the whole object without changing its volume. When \( n > -1 \) this is no longer the case and small–scale displacements can disrupt the collapsing peak. That does not mean that no object forms at the position of the peak, but it is a complete nonsense to try to derive its geometrical properties with the trajectories of the particles that were initially inside \( V_0 \). The edge of the volume occupied by the particles is expected to be extremely fuzzy (like a fractal set), and the approach adopted here is totally inadequate to deal with this problem. The use of more constrained initial conditions (such as in eq. [49]) is not expected to fix this problem, since these constraints do not involve the small–scale fluctuations. A systematic study of the dynamics of constrained fields with numerical simulations would be helpful to clarify the situation, but in any case the simple picture used to derive the mass distribution functions, based on the assumption of mass conservation, is severely altered in the \( n \geq -1 \) case.

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APPENDIX A: Conditional expectation values of non-Gaussian variables

This appendix is devoted to a recall of the general form of the distribution functions of Gaussian and non-Gaussian random variables once the series of their moments is known. We consider two random positive variables $V_1$ and $V_2$ for which their high-order correlations and cross-correlations, $\langle V_1^p V_2^q \rangle$, $p, q$ integers, are known. These moments define a two variable generating function

$$\mathcal{M}(\lambda_1, \lambda_2) = \sum_{p=0, q=0}^{\infty} \langle V_1^p V_2^q \rangle \frac{\lambda_1^p \lambda_2^q}{p!q!}. \quad (A.1)$$

Then the two variable distribution function of $V_1$ and $V_2$ reads

$$\eta(V_1, V_2) dV_1 dV_2 = dV_1 dV_2 \int_{-i\infty+0}^{+i\infty+0} \frac{d\lambda_1}{2\pi i} \int_{-i\infty+0}^{+i\infty+0} \frac{d\lambda_2}{2\pi i} \mathcal{M}(\lambda_1, \lambda_2) \exp(-\lambda_1 V_1 - \lambda_2 V_2). \quad (A.2)$$

Such a form can be checked by deriving the moments of the distribution function defined in (A.2). Indeed the expectation value of $V_1^p V_2^q$ is given by

$$\int_0^\infty \eta(V_1, V_2) V_1^p V_2^q dV_1 dV_2 = \int_{-i\infty+0}^{+i\infty+0} \frac{d\lambda_1}{2\pi i} \int_{-i\infty+0}^{+i\infty+0} \frac{d\lambda_2}{2\pi i} \mathcal{M}(\lambda_1, \lambda_2) \bigg| \frac{\partial^p}{\partial \lambda_1^p} \frac{\partial^q}{\partial \lambda_2^q} \mathcal{M}(\lambda_1, \lambda_2) \bigg|_{\lambda_1=0, \lambda_2=0} \equiv \langle V_1^p V_2^q \rangle. \quad (A.3)$$

However it turns out in fact that it is more efficient to work with the cumulants of the distribution, $\langle V_1^p V_2^q \rangle_c$. They are defined recursively from the moments:

$$\langle 1 \rangle_c = 0;$$
$$\langle V \rangle_c = \langle V \rangle;$$
$$\langle V^2 \rangle_c = \langle V^2 \rangle - \langle V \rangle_c^2;$$
$$\langle V_1 V_2 \rangle_c = \langle V_1 V_2 \rangle - \langle V_1 \rangle_c \langle V_2 \rangle_c;$$
$$\langle V^3 \rangle_c = \langle V^3 \rangle - 3 \langle V \rangle_c^3 - 3 \langle V \rangle_c \langle V^2 \rangle_c;$$

$$\cdots$$

In general the cumulant of order $(p, q)$ is obtained in such a way: consider all the partitions of a set of $p + q$ points (except the one constituted by the set itself), take the product of the cumulants of each of the subsets that come from
moments of lower order, then the cumulant is obtained by subtracting all the terms obtained such a way from the value of the moment of order \((p, q)\). Note that the usual arithmetic operations are not allowed on the cumulants and for instance that 
\[
\langle A B C \rangle_c \neq \langle (A B) C \rangle_c
\]
since the first is the cumulant of three random variables and the second of two. The cumulants also define a generating function \(\chi(\lambda_1, \lambda_2)\)
\[
\chi(\lambda_1, \lambda_2) = \sum_{p=0, q=0}^{\infty} \langle V_1^p V_2^q \rangle_c \frac{\lambda_1^p \lambda_2^q}{p!q!}.
\] (A.5)
The functions \(\chi(\lambda_1, \lambda_2)\) and \(M(\lambda_1, \lambda_2)\) are closely related together by the relation (e.g., Balian & Schaeffer 1989),
\[
M(\lambda_1, \lambda_2) = \exp \left[ \chi(\lambda_1, \lambda_2) \right],
\] (A.6)
so that the distribution reads now
\[
\eta(V_1, V_2) dV_1 dV_2 = dV_1 dV_2 \int_{-\infty}^{+\infty} \frac{d\lambda_1}{2\pi i} \int_{-\infty}^{+\infty} \frac{d\lambda_2}{2\pi i} \exp[\chi(\lambda_1, \lambda_2) - \lambda_1 V_1 - \lambda_2 V_2].
\] (A.7)
The general form (A.7) allows one to calculate any statistical quantity related to the two variables \(V_1\) and \(V_2\). One quantity of interest in this paper is the mean value of one knowing the other, \(\langle V_1 \rangle_{V_2}\). This quantity is given by the ratio,
\[
\langle V_1 \rangle_{V_2} = \int_0^\infty dV_1 \frac{\eta(V_1, V_2)}{\int_0^\infty dV_1 \eta(V_1, V_2)} \int_{-\infty}^{+\infty} \frac{d\lambda_2}{2\pi i} \exp[\chi(0, \lambda_2) - \lambda_2 V_2] \left. \frac{\partial}{\partial \lambda_1} \chi(\lambda_1, \lambda_2) \right|_{\lambda_1 = 0}
\] (A.8)
The partial derivative that appears in the result is given by
\[
\left. \frac{\partial}{\partial \lambda_1} \chi(\lambda_1, \lambda_2) \right|_{\lambda_1 = 0} = \sum_{p=0}^{\infty} \langle V_1 V_2^p \rangle_c \frac{\lambda_2^p}{p!}.
\] (A.9)
We then apply the relation (A.9) when \(V_2\) is a Gaussian variable of expectation value 1, and of rms fluctuation \(\sigma\). This is the case for the early cosmological density field, so that we define the random variable \(\delta_L \equiv V_2 - 1\) for such a case. The function \(\chi(0, \lambda_2)\) then takes a simple form since
\[
\chi_G(0, \lambda_2) = \lambda_2 + \frac{1}{2} \sigma^2 \lambda_2^2
\] (A.10)
(the lowscript $G$ is for Gaussian). We can also notice that

$$
\langle V_1 (1 + \delta_L)^p \rangle_c = \langle V_1^p \delta_L \rangle_c. \quad (A.11)
$$

The expectation value of $V_1$ knowing $\delta_L$ is then given by the relation (A.8) and it reads in this particular case (we make the change of variable $y = \sigma^2 \lambda_2$),

$$
\langle V_1 \rangle_{\delta L} = \int_{-i\infty}^{+i\infty} \frac{dy}{2\pi i} \mathcal{G}_{V_1}(y) \exp \left[ \frac{(y^2 - 2y\delta_L)}{2\sigma^2} \right] / \int_{-i\infty}^{+i\infty} \frac{dy}{2\pi i} \exp \left[ \frac{(y^2 - 2y\delta_L)}{(2\sigma^2)} \right] \quad (A.12)
$$

with

$$
\mathcal{G}_{V_1}(y) = \sum_{p=0}^{\infty} \frac{\langle V_1^p \delta_L \rangle_c y^p}{\sigma^{2p} p!}. \quad (A.13)
$$

We can also compute the expectation value of $V_1^2$ knowing $\delta_L$. It is given by the following expression,

$$
\langle V_1^2 \rangle_{\delta L} = \int_{-i\infty}^{+i\infty} \frac{dy}{2\pi i} \left[ \mathcal{G}_{V_1^2}(y) + \mathcal{G}_{V_1}(y) \right] \exp \left[ \frac{(y^2 - 2y\delta_L)}{2\sigma^2} \right] / \int_{-i\infty}^{+i\infty} \frac{dy}{2\pi i} \exp \left[ \frac{(y^2 - 2y\delta_L)}{(2\sigma^2)} \right] \quad (A.14)
$$

with

$$
\mathcal{G}_{V_1^2}(y) = \sum_{p=0}^{\infty} \frac{\langle V_1^{2p} \delta_L \rangle_c y^p}{\sigma^{2p} p!}. \quad (A.15)
$$

The calculation of the integrals (A.12) and (A.14) can be simplified by the change of variable, $y = \delta_L + iz$. As a result we have

$$
\langle V_1 \rangle_{\delta L} = \int_{-\infty}^{+\infty} dz \mathcal{G}_{V_1}(\delta_L + iz) \exp \left[ -\frac{z^2}{2\sigma^2} \right] / \int_{-\infty}^{+\infty} dz \exp \left[ -\frac{z^2}{2\sigma^2} \right] \quad (A.16)
$$

and

$$
\langle V_1^2 \rangle_{\delta L} = \int_{-\infty}^{+\infty} dz \left[ \mathcal{G}_{V_1}(\delta_L + iz) + \mathcal{G}_{V_1^2}(\delta_L + iz) \right] \exp \left[ -\frac{z^2}{2\sigma^2} \right] / \int_{-\infty}^{+\infty} dz \exp \left[ -\frac{z^2}{2\sigma^2} \right]. \quad (A.17)
$$
APPENDIX B: Geometrical properties of the top–hat window function

The purpose is this appendix is to give general properties of the top–hat window function, \( W_{\text{TH}}(x) \). It is defined by

\[
W_{\text{TH}}(x) = \begin{cases} 
1 & \text{if } |x| \leq R_0; \\
0 & \text{otherwise};
\end{cases} \quad (B.1)
\]

for a scale \( R_0 \). The Fourier transform of this function is then given by

\[
W_{\text{TH}}(k R_0) = \frac{3}{(k R_0)^3} (\sin(k R_0) - k R_0 \cos(k R_0)) \quad (B.2)
\]

where \( k \) is the norm of \( k \). The latter function will be widely used for the calculation of the dynamics.

Consider three wave vectors \( \mathbf{k}_1, \mathbf{k}_2 \) and \( \mathbf{k}_3 \) on which you may have to integrate. The properties of interest concern the integration over their angular parts, \( d\Omega_1, d\Omega_2 \) and \( d\Omega_3 \), and are the following:

\[
\int d\Omega_1 d\Omega_2 \ W_{\text{TH}} (|\mathbf{k}_1 + \mathbf{k}_2| R_0) \left[ 1 - \left( \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \right)^2 \right] = (4\pi)^2 \frac{2}{3} W_{\text{TH}}(k_1 R_0) W_{\text{TH}}(k_2 R_0) \quad (B.3)
\]

and

\[
\int d\Omega_1 d\Omega_2 d\Omega_3 \ W_{\text{TH}} (|\mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3| R_0) \\
\times \left[ 1 - \left( \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2} \right)^2 - \left( \frac{\mathbf{k}_2 \cdot \mathbf{k}_3}{k_2 k_3} \right)^2 - \left( \frac{\mathbf{k}_3 \cdot \mathbf{k}_1}{k_3 k_1} \right)^2 + 2 \frac{\mathbf{k}_1 \cdot \mathbf{k}_2}{k_1 k_2 k_3} \frac{\mathbf{k}_2 \cdot \mathbf{k}_3}{k_2 k_3 k_1} \frac{\mathbf{k}_3 \cdot \mathbf{k}_1}{k_3 k_1 k_2} \right] \\
= (4\pi)^3 \frac{8}{9} W_{\text{TH}}(k_1 R_0) W_{\text{TH}}(k_2 R_0) W_{\text{TH}}(k_3 R_0). \quad (B.4)
\]

These properties are essentially some properties of factorizability of the top–hat window function and are only true for this particular window function. It means that the top–hat window function commutes with certain geometrical quantities and that we can replace the angular dependence of \( \mathbf{k}_1, \mathbf{k}_2 \) and \( \mathbf{k}_3 \) by their mean as if there was no window function at all. For instance the mean value of \( 1 - (\mathbf{k}_1 \cdot \mathbf{k}_2 / k_1 k_2)^2 \) in (B.3) is 2/3 in the absence of window function which exactly corresponds to the coefficient 2/3 appearing in (B.3).
The proofs of these properties rely on the fact that $W_{\text{TH}}(k)$ can be written in term of a Bessel function:

$$W_{\text{TH}}(k) = 3\sqrt{\pi/2} \, k^{-3/2} \, J_{3/2}(k)$$  \hspace{1cm} (B.5)$$

(the value of $R_0$ has been set to 1 for simplicity but the demonstrations obviously hold for any value of $R_0$). The window function can then be written (e.g., Gradshteyn & Ryzhik, Eq. 8.532.1),

$$W_{\text{TH}}(|k_1 + k_2|) = 3\pi \sum_{m=0}^{\infty} \left( \frac{3}{2} + m \right) (k_1 k_2)^{-3/2} \, J_{3/2+m}(k_1) \, J_{3/2+m}(k_2) \frac{d}{du} P_{m+1}(-u)$$  \hspace{1cm} (B.6)$$

where $u = k_1 . k_2 / k_1 k_2$ and $P_m$ is a Legendre polynomial.

Then $I_1(k_1, k_2)$ defined as the left part of the relation (B.3) reads

$$I_1(k_1, k_2) = 3\pi \sum_{m=0}^{\infty} \left( \frac{3}{2} + m \right) (k_1 k_2)^{-3/2} \, J_{3/2+m}(k_1) \, J_{3/2+m}(k_2) \left( 4\pi \right)^{(2\pi)}$$  \hspace{1cm} (B.7)$$

$$I_1(k_1, k_2) = -3\pi \sum_{m=0}^{\infty} \left( \frac{3}{2} + m \right) (k_1 k_2)^{-3/2} \, J_{3/2+m}(k_1) \, J_{3/2+m}(k_2) \left( 4\pi \right)^2$$  \hspace{1cm} (B.7)$$

$$I_1(k_1, k_2) = -3\pi \sum_{m=0}^{\infty} \left( \frac{3}{2} + m \right) (k_1 k_2)^{-3/2} \, J_{3/2+m}(k_1) \, J_{3/2+m}(k_2) \left( 4\pi \right)^2$$  \hspace{1cm} (B.7)$$

where $u = k_1 . k_2 / k_1 k_2$ and $P_m$ is a Legendre polynomial.

Then $I_1(k_1, k_2)$ defined as the left part of the relation (B.3) reads

$$I_1(k_1, k_2) = 3\pi \sum_{m=0}^{\infty} \left( \frac{3}{2} + m \right) (k_1 k_2)^{-3/2} \, J_{3/2+m}(k_1) \, J_{3/2+m}(k_2) \left( 4\pi \right)^{(2\pi)}$$  \hspace{1cm} (B.8)$$

which, using the equation (B.5), gives the relation (B.3).

The proof of the relation (B.4) is based on the same kind of calculation but the demonstration is more complicated due to the fact that three different vectors are involved. Two steps are required. First of all let me define $\mathbf{K} = \mathbf{k}_1 + \mathbf{k}_2$. The first step will be to separate $\mathbf{K}$ from $\mathbf{k}_3$. I can then define the angles $\Theta_{12}$, $\Theta_1$, $\Theta_2$ and $\Theta_3$ which are respectively the angles between $\mathbf{k}_1$ and $\mathbf{k}_2$, between $\mathbf{K}$ and $\mathbf{k}_1$, between $\mathbf{K}$ and $\mathbf{k}_2$ and between $\mathbf{K}$ and $\mathbf{k}_3$ (note that $\Theta_{12} = \Theta_1 + \Theta_2$). The vector $\mathbf{k}_3$ is also defined by its recession angle, $\Phi_3$, around the vector $\mathbf{K}$ (they are set to 0
for the vectors $k_1$ and $k_2$). The angular part appearing in the left part of (B.4) is

$$A(\Theta_1, \Theta_2, \Theta_3, \Phi_3) = 1 - \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 - \left( \frac{k_2 \cdot k_3}{k_2 k_3} \right)^2 - \left( \frac{k_3 \cdot k_1}{k_3 k_1} \right)^2$$

$$+ 2 \frac{k_1 \cdot k_2 k_2 \cdot k_3 k_3 \cdot k_1}{k_1^2 k_2^2 k_3^2}$$

$$= 1 - \cos^2(\Theta_{12})$$

$$- \left[ \cos(\Theta_3) \cos(\Theta_1) + \sin(\Theta_3) \sin(\Theta_1) \sin(\Phi_3) \right]^2$$

$$- \left[ \cos(\Theta_3) \cos(\Theta_2) + \sin(\Theta_3) \sin(\Theta_2) \sin(\Phi_3) \right]^2$$

$$+ 2 \cos(\Theta_{12}) \left[ \cos(\Theta_3) \cos(\Theta_1) + \sin(\Theta_3) \sin(\Theta_1) \sin(\Phi_3) \right]$$

$$\times \left[ \cos(\Theta_3) \cos(\Theta_2) + \sin(\Theta_3) \sin(\Theta_2) \sin(\Phi_3) \right].$$

The integration over $\Phi_3$ can be made at this stage since the factor $W_{TH}(|k_1 + k_2 + k_3|)$ is independent of $\Phi_3$. The result reads

$$B(\Theta_1, \Theta_2, \Theta_3) = \int_0^{2\pi} d\Phi_3 \ A(\Theta_1, \Theta_2, \Theta_3, \Phi_3) =$$

$$2\pi \left( 1 - \cos^2(\Theta_{12}) - \cos^2(\Theta_3) \cos^2(\Theta_1) - \frac{1}{2} \sin^2(\Theta_3) \sin^2(\Theta_1) \right)$$

$$- \cos^2(\Theta_3) \cos^2(\Theta_2) - \frac{1}{2} \sin^2(\Theta_3) \sin^2(\Theta_2)$$

$$+ 2 \cos(\Theta_{12}) \left[ \cos^2(\Theta_3) \cos(\Theta_1) \cos(\Theta_2) + \frac{1}{2} \sin^2(\Theta_3) \sin(\Theta_1) \sin(\Theta_2) \right].$$

The function $I_2(k_1, k_2, k_3)$, defined as the left part of (B.4), then reads

$$I_2(k_1, k_2, k_3) = \int d\Omega_1 \ d\Omega_2 \ \sin(\Theta_3) d\Theta_3$$

$$\times 3\pi \sum_{m=0}^{\infty} \frac{3}{2} (\frac{3}{2} + m) (K k_3)^{-3/2} J_{3/2+m}(K) \ J_{3/2+m}(k_3)$$

$$\times \frac{d}{du} P_{m+1}(- \cos(\Theta_3)) \ B(\Theta_1, \Theta_2, \Theta_3).$$

The integration over $\Theta_3$ takes advantage of the fact that $B(\Theta_1, \Theta_2, \Theta_3)$ contains only terms independent of $\Theta_3$ or proportional to $\cos^2(\Theta_3)$. We can then use general
properties of the Legendre polynomials,

\[
\int_{-1}^{1} du \frac{d}{du} P_{m+1}(-u) = 2 \text{ for even values of } m;
\]

\[
\int_{-1}^{1} du \frac{d}{du} P_{m+1}(-u) \, u^2 = 2 \text{ for even values of } m \text{ and } m \neq 0; \quad (B.12)
\]

\[
\int_{-1}^{1} du \frac{d}{du} P_{1}(-u) \, u^2 = 2/3;
\]

and these integrals are zero otherwise, for \( u = \cos(\Theta_3) \) to compute (B.11). When \( m \neq 0 \), the two contributions coming from the terms independent of \( \Theta_3 \) and the ones proportional to \( \cos^2(\Theta_3) \) have to be added which is obtained by setting \( \Theta_3 = 0 \) in \( B(\Theta_1, \Theta_2, \Theta_3) \) which gives zero.

For \( m = 0 \) the contribution of the integral has to be calculated directly. Taking advantage of the fact that \( \Theta_{12} = \Theta_1 + \Theta_2 \) we obtain

\[
\int \sin(\Theta_3) d\Theta_3 \frac{d}{du} P_1(-\cos(\Theta_3)) \, B(\Theta_1, \Theta_2, \Theta_3) = \frac{4\pi}{3} \left( 1 - \cos^2(\Theta_{12}) \right)
\]

so that

\[
I_2(k_1, k_2, k_3) = \int d\Omega_1 \, d\Omega_2 \, \frac{3}{2} (Kk_3)^{-3/2} J_{3/2}(K) J_{3/2}(k_3) \frac{4\pi}{3} \left[ 1 - \cos^2(\Theta_{12}) \right]
\]

\[
= \frac{4\pi}{3} \, W_{TH}(k_3) \, I_1(k_1, k_2)
\]

where using the equation (B.3) gives straightforwardly the relation (B.4).
APPENDIX C: Calculation of $G_V(t, y)$, $G_V^2(t, y)$ and $G_\delta(t, y)$

The function $G_V(t, y)$ is defined in equation (23) and will be calculated for $\sigma_L = 0$ that is with the approximation (26) for the series of the coefficient $j_p$. Let me first sketch the principle of this calculation. The starting point is the dynamical equations in Lagrangian space describing the evolution of a pressureless fluid (eqs. [C.2,C.3]). Then they will be written in Fourier space displaying the geometrical dependences examined in the previous appendix. Taking the cumulants of each term (once multiplied by $\delta_L$ at the required power) introduces integrations over the angles of the wave vectors that are then known from the result of the appendix B. As the geometrical dependences vanish we end up with a system of equations of which $G(t, y)$ is solution. The field equation system have thus been replaced by a simple first order differential system.

C.1 Notations and general properties

The fulfillment of this derivation requires the definitions of new quantities presented in the following. Let me however start with a general property of the cumulants widely used throughout this appendix. In this paragraph any quantity denoted $Q^{(p)}$ stands for the order $p$ of $Q$ relatively to the random variables $\delta^k_L$. The property of interest concerns the cumulant of a product. If $Q$ can be written as a product, $Q = F_1 F_2$, the order $p$ of $Q$ is given by $Q^{(p)} = \sum_{r=0}^{p} F_1^{(p-r)} F_2^{(r)}$ and the cumulant it gives when multiplied by $\delta^p_L$ is

$$
\langle Q^{(p)} \delta^p_L \rangle_c = \sum_{r=0}^{p} \frac{p!}{r!(p-r)!} \langle F_1^{(p-r)} \delta^{p-r}_L \rangle_c \langle F_2^{(r)} \delta^r_L \rangle_c. \tag{C.1}
$$

This result is due to the properties of the Gaussian variables. A product of $2p$ Gaussian variables has an expectation value that is written as a summation of $p$ moments of pairs, so that the terms involved in (C.1) separate in two sets: the factors involving $F_1$ and the factors involving $F_2$. The symmetry factor $p!/(r!(p-r)!)$ is due to the number of possibilities you have to separate a set of $p$ points into two sets of $r$ and $p-r$ points.

The starting point of the derivation of $G_V(t, y)$ is the motion equations derived in the main section, eqs. (17, 20), that I recall here

$$
J(t, q) = 1 + \nabla q \cdot \Psi + \frac{1}{2} \left[ (\nabla q \cdot \Psi)^2 - \sum_{ij} \Psi_{i,j} \Psi_{j,i} \right] \\
+ \frac{1}{6} \left[ (\nabla q \cdot \Psi)^3 - 3 \nabla q \cdot \Psi \sum_{ij} \Psi_{i,j} \Psi_{j,i} + 2 \sum_{ijk} \Psi_{i,j,k} \Psi_{j,k,i} \right]. \tag{C.2}
$$
where $\nabla_q$ is the gradient taken relatively to $q$. The second equation is

\[
\mathcal{T}(\nabla_q \Psi) + \nabla_q \Psi \mathcal{T}(\nabla_q \Psi) - \sum_{ij} \Psi_{i,j} \mathcal{T}(\Psi_{j,i}) + \frac{1}{2} (\nabla_q \Psi)^2 \mathcal{T}(\nabla_q \Psi) \\
- \frac{1}{2} \mathcal{T}(\nabla_q \Psi) \sum_{ij} [\Psi_{i,j} \Psi_{j,i} - \nabla_q \Psi \mathcal{T}(\Psi_{i,j}) \Psi_{j,i}] + \sum_{ijk} \Psi_{i,j} \Psi_{j,k} \mathcal{T}(\Psi_{k,i}) \\
= -4\pi G_\sigma [1 - J(t,q)]
\]

(C.3)

where $\mathcal{T}(\cdot)$ is a time derivative operator,

\[
\mathcal{T}(A) = \ddot{A} + 2 \dot{\alpha} \dot{A}.
\]

(C.4)

The displacement field has also to be irrotational in the real space. These equations completely define the time evolution of the displacement field. A calculation of the first orders when the displacement field is considered as a small quantity is given in the appendix D. But such direct perturbative calculations are inadequate to compute the values of $j_p(t)$ as defined in (24). The series of the parameters will be obtained as a functional equation for $G_V(t,y)$.

For convenience I define few other quantities. Similarly to $G_V$ I can consider the function

\[
G_D(t,y) = \sum_{p=0}^{\infty} \left\langle \frac{1}{V_0} \int d^3q \nabla_q \cdot \Psi(t,q) \right\rangle \delta_L^{(p)} \}
\]

(C.5)

and the parameters

\[
d_p(t) \equiv \left\langle \frac{1}{V_0} \int d^3q \nabla_q \cdot \Psi(t,q) \right\rangle \delta_L^{(p)} \}
\]

(C.6)

in the rare peak approximation. I also denote $j_{k}^{(p)}(t)$ and $d_{k}^{(p)}(t)$ the Fourier transform of respectively $J(t,q)$ and $\nabla_q \Psi(t,q)$ at order $p$ in an expansion with respect to the random variables $\delta_k$. All these quantities are time dependent and the operator $\mathcal{T}(\cdot)$ defined in (C.3) can apply to any of them.

The term of order $p$ for the volume then reads

\[
V^{(p)}(t)/V_0 = \int \frac{d^3k}{(2\pi)^{3/2}} W_{TH}(kR_0) j_{k}^{(p)}(t)
\]

(C.7)

and we have

\[
j_p = \int \frac{d^3k}{(2\pi)^{3/2}} \left\langle j_{k}^{(p)}(t) \delta_L^{(p)} \right\rangle \} \ \frac{1}{W_{TH}(kR_0)}.
\]

(C.8)
We have similarly for the divergence of the displacement field,
\[
\frac{1}{V_0} \int_{V_0} d^3q \nabla q \cdot \Psi^{(p)}(t, q) = \int \frac{d^3k}{(2\pi)^{3/2}} W_{TH}(k R_0) d^{(p)}_{k}(t). \tag{C.9}
\]
and
\[
d_p = \int \frac{d^3k}{(2\pi)^{3/2}} \langle d^{(p)}_{k}(t) \delta_{L}^{p} \rangle_c W_{TH}(k R_0). \tag{C.10}
\]
The components of the matrix $\Psi_{i,j}$ also define some cumulants,
\[
s_{p; i,j}(t, q) = \langle \Psi^{(p)}_{i,j}(t, q) \delta_{L}^{p} \rangle_c. \tag{C.11}
\]
We can notice that the parameters $j_p(t)$ can simply be expressed with the quantities $s_{p; i,j}$ since
\[
j_p(t) = \frac{1}{V_0} \int_{V_0} d^3q \sum_i s_{p; i,i}(t, q). \tag{C.12}
\]
Written in $k$-space the equations (C.2) and (C.3) introduce the functions
\[
J_1(k_1, k_2) = W_{TH} (|k_1 + k_2| R_0) \left[ 1 - \left( \frac{k_1.k_2}{k_1 k_2} \right)^2 \right], \tag{C.13}
\]
and
\[
J_2(k_1, k_2, k_3) = W_{TH} (|k_1 + k_2 + k_3| R_0)
\]
\[
	imes \left[ 1 - \left( \frac{k_1.k_2}{k_1 k_2} \right)^2 - \left( \frac{k_2.k_3}{k_2 k_3} \right)^2 - \left( \frac{k_3.k_1}{k_3 k_1} \right)^2 + 2 \left( \frac{k_1.k_2 k_2.k_3 k_3.k_1}{k_1^2 k_2^2 k_3^2} \right) \right]. \tag{C.14}
\]
These functions have been studied in the appendix B where it is shown that
\[
\int d\Omega_1 d\Omega_2 J_1(k_1, k_2) = (4\pi)^2 \frac{2}{3} W_{TH}(k_1 R_0) W_{TH}(k_2 R_0) \tag{C.15}
\]
and
\[
\int d\Omega_1 d\Omega_2 d\Omega_3 J_2(k_1, k_2, k_3) = (4\pi)^3 \frac{2}{9} W_{TH}(k_1 R_0) W_{TH}(k_2 R_0) W_{TH}(k_3 R_0). \tag{C.16}
\]

C.2 The function $G_V(t, y)$

The first step of the demonstration is to remark that the quantities $s_{p; i,j}(t, q)$ are symmetric in $i, j$ and are functions of $|q|$ only. The fact that the initial conditions
are randomly isotropic around $q = 0$ implies directly that $s_{p;i,j}$ is a function of the norm of $q$. Moreover when $i \neq j$, $s_{p;i,j} - s_{p;j,i}$ are the components of a vector (given by the rotational of the displacement field) so that these quantities should be zero for the same reason: no particular direction exists in the initial conditions. Note that this property is correct at the level of these expectation values but not at all for the random displacement field itself. The random variable $\Psi_{i,j} - \Psi_{j,i}$ is not zero although some of its expectation values vanish. The implication of such property is that $s_{p;i,j}$ can be written

$$s_{p;i,j}(t, q) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{k_i k_j}{k^2} s_p(k) e^{ik \cdot q} \quad (C.17)$$

and, according to (C.12), $s(k)$ is then simply given by

$$s_p(k) = \langle d^{(p)}_k(t) \delta^p_L \rangle_c \quad (C.18)$$

We can now transform the equations (C.2) and (C.3) involving the displacement field in two equations between the series $j_p$ and $d_p$.

Let me start with the equation (C.2). This equation has to be true at any order so that for $p \neq 0$,

$$J^{(p)}(t, q) = \nabla q \cdot \Psi^{(p)} + \frac{1}{2} \sum_{r=0}^{p} \left[ \nabla q \cdot \Psi^{(p-r)} \nabla q \cdot \Psi^{(r)} - \sum_{ij} \Psi^{(p-r)}_{i,j} \Psi^{(r)}_{j,i} \right]$$

$$+ \frac{1}{6} \sum_{r=0, s=0}^{r=p, s=p-r} \left[ \nabla q \cdot \Psi^{(p-r-s)} \nabla q \cdot \Psi^{(r)} \nabla q \cdot \Psi^{(s)} \right. - 3 \nabla q \cdot \Psi^{(p-r-s)} \sum_{ij} \Psi^{(r)}_{i,j} \Psi^{(s)}_{j,i} + 2 \sum_{ijk} \Psi^{(p-r-s)}_{i,j,k} \Psi^{(r)}_{j,k} \Psi^{(s)}_{k,i} \right] \quad (C.19)$$

The equation (C.19) can now be transformed in an equation between the cumulants $j_p$ and $s_{p;i,j}$ by multiplying it by $\delta^p_L$ and taking the cumulant of each
functions

The equation \( (C.1) \), \( (C.17) \) and \( (C.18) \) we obtain

\[
\int \frac{d^3 k}{(2\pi)^3/2} \langle J^{(p)}_{k} \delta_{L} \rangle_c e^{i k \cdot q} = \int \frac{d^3 p}{(2\pi)^3/2} e^{i p \cdot q} \frac{d^3 k_1}{(2\pi)^3/2} \frac{d^3 k_2}{(2\pi)^3/2} \frac{d^3 k_3}{(2\pi)^3/2} \left[ 1 - \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \right] e^{i(k_1 + k_2) \cdot q} \]

\[
+ \frac{1}{2} \sum_{r=0}^{p} \frac{p!}{r!(p-r)!} \int \frac{d^3 k_1}{(2\pi)^3/2} \frac{d^3 k_2}{(2\pi)^3/2} s_{p-r}(k_1) s_r(k_2) \left[ 1 - \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 \right] e^{i(k_1 + k_2) \cdot q} \]

\[
+ \frac{1}{6} \sum_{r=0, s=0}^{p} \frac{p!}{(p-r-s)!r!s!} \int \frac{d^3 k_1}{(2\pi)^3/2} \frac{d^3 k_2}{(2\pi)^3/2} \frac{d^3 k_3}{(2\pi)^3/2} s_{p-r-s}(k_1) s_r(k_2) s_s(k_3) \times \left[ 1 - \left( \frac{k_1 \cdot k_2}{k_1 k_2} \right)^2 - \left( \frac{k_2 \cdot k_3}{k_2 k_3} \right)^2 - \left( \frac{k_3 \cdot k_1}{k_3 k_1} \right)^2 + 2 \frac{k_1 \cdot k_2 \cdot k_3 \cdot k_1}{k_1 k_2 k_3 k_1} \right] e^{i(k_1 + k_2 + k_3) \cdot q} .
\]

The average of \( (C.20) \) over the volume \( V_0 \) introduces the parameters \( j_p, d_p \) and the functions \( J_1 \) and \( J_2 \) defined in \( (C.13) \) and \( (C.14) \):

\[
j_p = d_p + \frac{1}{2} \sum_{r=0}^{p} \frac{p!}{r!(p-r)!} \int \frac{d^3 k_1}{(2\pi)^3/2} \frac{d^3 k_2}{(2\pi)^3/2} s_{p-r}(k_1) s_r(k_2) J_1(k_1, k_2) \]

\[
+ \frac{1}{6} \sum_{r=0, s=0}^{p} \frac{p!}{(p-r-s)!r!s!} \int \frac{d^3 k_1}{(2\pi)^3/2} \frac{d^3 k_2}{(2\pi)^3/2} \frac{d^3 k_3}{(2\pi)^3/2} s_{p-r-s}(k_1) s_r(k_2) s_s(k_3) \times J_2(k_1, k_2, k_3),
\]

which can be integrated out by the use of the properties \( (C.15) \) and \( (C.16) \). We eventually get

\[
j_p = d_p + \frac{1}{3} \sum_{r=0}^{p} \frac{p!}{(p-r)!r!} d_r d_{p-r} + \frac{1}{27} \sum_{r=0, s=0}^{p} \frac{p!}{(p-r-s)!r!s!(p-r-s)!} d_r d_s d_{p-r-s} .
\]

Then, using the definitions \( (20) \) and \( (C.4) \) we can derive a relation between \( \mathcal{G}_V \) and \( \mathcal{G}_D \):

\[
\mathcal{G}_V(t, y)/V_0 = 1 + \mathcal{G}_D(t, y) + \frac{1}{3} \mathcal{G}_D^2(t, y) + \frac{1}{27} \mathcal{G}_D^3(t, y) = [1 + \mathcal{G}_D(t, y)/3]^3 .
\]

The equation \( (C.3) \) leads to another relationship of the same kind. The only qualitative change is the presence of the operator \( \mathcal{T}(.) \). But as it is a linear operator, it commutes with any operations such as the Fourier transform, ensemble average,
and integration over $V_0$. Similar calculations then lead to

$$
T(d_p) + \frac{2}{3} \sum_{r=0}^{p} \frac{p!}{(p-r)!} d_r T(d_{p-r}) + \frac{1}{9} \sum_{r=0, s=0}^{p, s} \frac{p!}{r! s!(p-r-s)!} T(d_r) d_s d_{p-r-s}
$$

$$
= 4\pi G \bar{\rho} j_p
$$

(C.24)

which gives

$$
T[G_D(t, y)] [1 + G_D(t, y)/3]^2 = -4\pi G \bar{\rho} [V_0 - G_V(t, y)].
$$

(C.25)

The equations (C.23) and (C.25) with the definition of $T(.)$ lead to the differential equation

$$
\frac{d^2}{dt^2} \left( R_0 \left[ G_V(t, y)/V_0 \right]^{1/3} \right) = \frac{- \left( G \bar{\rho} + \Lambda/4\pi \right) V_0}{R_0^2 \left[ G_V(t, y)/V_0 \right]^{2/3}}
$$

(C.26)

since the comoving radius $R_0(t)$ is proportional to the expansion factor and satisfies $\ddot{R}_0 = -4\pi/3 \bar{\rho} G R_0 + \Lambda R_0/3$. The equation (C.26) is the differential equation describing a spherical collapse, where $R_0 \left[ G_V(t, y)/V_0 \right]^{1/3}$ is the size of the object. The behavior of $G_V(y)$ for small values of $y$ ensures that the corresponding overdensity is simply $y$. The function $G_V(t, y)$ is then the volume occupied at the time $t$ by an object of linear initial overdensity $y$ and of initial comoving volume $V_0$.

This result is not so surprising since we saw that the application of the top–hat window function just replaces the geometrical dependences that enter in the real dynamics by their means, i.e., their monopole part, and the dynamics of the spherical collapse just corresponds to the case where there is only one monopole.

C.3 The functions $G_{V^2}(t, y)$ and $G_\delta(t, y)$

The purpose of this part is to compute the values of the functions,

$$
G_{V^2}(t, y) = \sum_{p=0}^{\infty} \frac{\langle V^2(t) \delta_L^p \rangle_c}{\sigma_{2p}^L} \frac{y^p}{p!},
$$

$$
G_\delta(t, y) = \sum_{p=0}^{\infty} \frac{\langle [V_0/V(t) - 1] \delta_L^p \rangle_c}{\sigma_{2p}^L} \frac{y^p}{p!},
$$

(C.27)

in the rare peak approximation. The first function involves a cumulant between $p + 2$ variables that requires at least $p + 1$ links between them. It can only be done with the moment of $2p + 2$ Gaussian variables. As a result

$$
\langle V^2(t) \delta_L^p \rangle_c \approx \sum_{r=1}^{r=p+1} \langle V^{(r)}(t) V^{(p+2-r)}(t) \delta_L^p \rangle_c.
$$

(C.28)
The calculation of such quantities is simple but requires a good understanding of the result obtained in the previous part. The order \( p \) of the volume can be written in such a way,

\[
V^{(p)}(t) = V_0 \int \frac{d^3k_1}{(2\pi)^{3/2}} \cdots \frac{d^3k_p}{(2\pi)^{3/2}} \delta_{k_1} \cdots \delta_{k_p} W_{\text{TH}} \left[ |k_1 + \ldots + k_p| R_0 \right] \\
\times J(k_1, \ldots, k_p) \tag{C.29}
\]

where \( J(k_1, \ldots, k_p) \) is a function of the angular parts of the wave vectors \( k_1, \ldots, k_p \) only. The result obtained in the part (C.1) is then a property of the function \( J(k_1, \ldots, k_p) \) that reads

\[
\int d\Omega_1 \ldots d\Omega_p W_{\text{TH}} \left[ |k_1 + \ldots + k_p| R_0 \right] J(k_1, \ldots, k_p) = \\
(4\pi)^{p} \frac{j_p(t)}{\sigma_L^p} W_{\text{TH}}(k_1 R_0) \ldots W_{\text{TH}}(k_p R_0). \tag{C.30}
\]

Indeed the function \( J(k_1, \ldots, k_p) \) turns out to be a combination of the functions \( J_1 \) and \( J_2 \) for which such a factorization has been explicitly established in the appendix B. The relation (C.30) can also be used in (C.28) since the angular parts of the wave vectors appearing in the expressions of \( V^{(r)} \) and of \( V^{(p+2-r)} \) are independent. As a result we simply obtain

\[
\langle V^{2}(t) \delta^p_L \rangle_c = \sum_{r=1}^{r=p+1} \frac{p!}{(r-1)!(p+1-r)!} j_r(t) j_{p+2-r}(t) \tag{C.31}
\]

so that

\[
\mathcal{G}_{V^2}(t, y) = \sigma_L^2 \left[ \frac{d}{dy} \mathcal{G}_{V}(t, y) \right]^2. \tag{C.32}
\]

The same geometrical arguments can be invoked for the calculation of \( \mathcal{G}_{1/V}(t, y) \) that lead to the expression,

\[
\mathcal{G}_{\delta}(t, y) = V_0 / \mathcal{G}_{V}(t, y) - 1. \tag{C.33}
\]
APPENDIX D: Perturbative calculations in Lagrangian coordinates

This Appendix is devoted to the derivation of the third order of the Jacobian in a Lagrangian calculation (see the main text for details), and some expectation values of interest. The principle of these calculations is not new and has been presented elsewhere (see for instance Moutarde et al. 1991, Buchert 1992)

D.1 The three first orders of the displacement field

To each comoving position \( q \) is associated a displacement vector \( \Psi(q) \) giving the present position \( x \) of the particle being at \( q \) initially,

\[
x = q + \Psi(q),
\]

and the displacement field is thought, in a perturbative approach, to be a small correction to the Hubble expansion (contained in \( q \)).

The Jacobian of the transformation between \( q \) and \( x \), \( |\partial x/\partial q| \), gives a direct information on the volume occupied by the particles at any stage of the dynamics (before shell crossing). Its expression is obtained by the calculation of the determinant of the matrix the elements of which are

\[
\frac{\partial x_i}{\partial q_j} = \delta_{ij} + \frac{\partial \Psi_i}{\partial q_j},
\]

where \( \delta_{ij} \) is the Kronecker symbol. It leads to the relation,

\[
J(t, q) = 1 + \nabla_q \cdot \Psi + \frac{1}{2} \left[ (\nabla_q \cdot \Psi)^2 - \sum_{ij} \Psi_{i,j} \Psi_{j,i} \right] + \frac{1}{6} \left[ (\nabla_q \cdot \Psi)^3 - 3 \nabla_q \cdot \Psi \sum_{ij} \Psi_{i,j} \Psi_{j,i} + 2 \sum_{ijk} \Psi_{i,j} \Psi_{j,k} \Psi_{k,i} \right],
\]

where \( \nabla_q \) is the gradient taken relatively to \( q \) and the summations are made over the three spatial components. The divergence of the acceleration field is proportional to the density field so that

\[
J(t, q) \left( \nabla_x \cdot \ddot{x} + 2 \frac{\dot{a}}{a} \nabla_x \cdot \dot{x} \right) = -4\pi G \rho [1 - J(t, q)]
\]
in which a dot denotes a time derivative. In term of the displacement field it reads

\[
\mathcal{T}(\nabla q \cdot \Psi) + \nabla q \cdot \Psi \mathcal{T}(\nabla q \cdot \Psi) - \sum_{ij} \Psi_{i,j} \mathcal{T}(\Psi_{j,i}) + \frac{1}{2} (\nabla q \cdot \Psi)^2 \mathcal{T}(\nabla q \cdot \Psi)
\]

\[-\frac{1}{2} \mathcal{T}(\nabla q \cdot \Psi) \sum_{ij} [\Psi_{i,j} \Psi_{j,i} - \nabla q \cdot \Psi \mathcal{T}(\Psi_{i,j}) \Psi_{j,i}] + \sum_{ijk} \Psi_{i,j} \Psi_{j,k} \mathcal{T}(\Psi_{k,i})
\]

\[= -4\pi G \rho [1 - J(t, q)]
\]

(D.5)

where \( \mathcal{T}(.) \) is a time derivative operator,

\[\mathcal{T}(A) = \ddot{A} + 2 \frac{\dot{a}}{a} \dot{A} \]  

(D.6)

The displacement field is completely determined by the the non–rotational constraint in the \( x \) coordinates, which leads to a third equation,

\[
\dot{\Psi}_{i,j} - \dot{\Psi}_{j,i} + \nabla q \cdot \Psi (\dot{\Psi}_{i,j} - \dot{\Psi}_{j,i}) - \sum_{k} \dot{\Psi}_{i,k} \Psi_{k,j} + \sum_{k} \dot{\Psi}_{j,k} \Psi_{k,i}
\]

\[+ \frac{1}{2} \left[ (\nabla q \cdot \Psi)^2 - \sum_{ij} \Psi_{i,j} \Psi_{j,i} \right] (\dot{\Psi}_{i,j} - \dot{\Psi}_{j,i}) - \nabla q \cdot \Psi \sum_{k} (\dot{\Psi}_{i,k} \Psi_{k,j} - \dot{\Psi}_{j,k} \Psi_{k,i})
\]

\[+ \sum_{kl} (\dot{\Psi}_{i,k} \Psi_{k,l} \Psi_{l,j} - \dot{\Psi}_{j,k} \Psi_{k,l} \Psi_{l,i}) = 0.
\]

(D.7)

In the following I will assume that the cosmological parameters are the ones of an Einstein-de Sitter universe.

The principle of the calculation is the following. We assume that the displacement field is a small quantity. At a given order the elimination of \( J(t, q) \) in (D.3) and (D.5) provides an equation for the divergence of the displacement field. The equation (D.7) then gives the complete information over its non–symmetric part.

The time dependence of the first order solution, \( D(t) \), is solution of the linearized equations in the displacement field. As a result \( D(t) \) is solution of

\[\mathcal{T}(D(t)) = 4\pi G \rho D(t).
\]

(D.8)

There are two solutions for the previous equations: One is an increasing function of time and is proportional to \( a(t) \) the other is a decreasing function of time, proportional to \( t^{-1} \). Since we assume that the fluctuations begin to grow at an arbitrarily small time the growing mode completely dominates the dynamics.
The first order solution then reads
\[ J^{(1)}(t, q) = \nabla_q \cdot \Psi^{(1)}(t, q) = D(t) \epsilon_+(q), \quad (D.9) \]
where \( \epsilon_+(q) \) is a small perturbation in the displacement field corresponding to the growing mode. This quantity is related to the initial overdensity that gives birth to the cosmic structures. The equation (D.7) insures that \( \Psi_{i,j}^{(1)} = \Psi_{j,i}^{(1)} \) so that the displacement field is completely determined by its divergence.

At the second order the equation for the divergence is
\[
\mathcal{T} \left( \nabla_q \cdot \Psi^{(2)} \right) + \nabla_q \cdot \Psi^{(1)}(q,t) \mathcal{T} \left( \nabla_q \cdot \Psi^{(1)} \right) - \Psi_{i,j}^{(1)} \mathcal{T} \left( \Psi_{i,j}^{(1)} \right) = 4\pi G \mathcal{P} \left[ \left( \nabla_q \cdot \Psi^{(1)} \right)^2 - \sum_{ij} \left( \Psi_{i,j}^{(1)} \right)^2 \right].
\]
\[ \quad (D.10) \]
The time dependence of the solution of this equation is \( D^2(t) \) (once the only pure growing mode has been selected). Note that this result is specific to the Einstein-de Sitter universe and is not true in general. The exact result for any value of \( \Omega \) is given by Bouchet et al. (1992). As a result we obtained,
\[
\nabla_q \cdot \Psi^{(2)}(t, q) = -\frac{3}{14} \left[ \left( \nabla_q \cdot \Psi^{(1)} \right)^2 - \sum_{ij} \left( \Psi_{i,j}^{(1)} \right)^2 \right],
\]
\[ \quad (D.11) \]
Once again the equation (D.7) insures that \( \Psi_{i,j}^{(2)} = \Psi_{j,i}^{(2)} \).

The third order is calculated the same way. Its time dependence is simply \( D(t)^3 \) and the solution reads
\[
\nabla_q \cdot \Psi^{(3)}(t, q) = -\frac{5}{9} \left[ \nabla_q \cdot \Psi^{(1)} \nabla_q \cdot \Psi^{(2)} - \sum_{ij} \Psi_{i,j}^{(1)} \Psi_{i,j}^{(2)} \right]
= -\frac{1}{18} \left[ \left( \nabla_q \cdot \Psi^{(1)} \right)^3 - 3 \nabla_q \cdot \Psi^{(1)} \sum_{ij} \left( \Psi_{i,j}^{(1)} \right)^2 + \sum_{ijk} \Psi_{i,j}^{(1)} \Psi_{j,k}^{(1)} \Psi_{k,i}^{(1)} \right],
\]
\[ J^{(3)}(t, q) = \frac{4}{9} \left[ \nabla_q \cdot \Psi^{(1)} \nabla_q \cdot \Psi^{(2)} - \sum_{ij} \Psi_{i,j}^{(1)} \Psi_{i,j}^{(2)} \right]
+ \frac{1}{9} \left[ \left( \nabla_q \cdot \Psi^{(1)} \right)^3 - 3 \nabla_q \cdot \Psi^{(1)} \sum_{ij} \left( \Psi_{i,j}^{(1)} \right)^2 + \sum_{ijk} \Psi_{i,j}^{(1)} \Psi_{j,k}^{(1)} \Psi_{k,i}^{(1)} \right].
\]
\[ (D.12) \]
Note that $\Psi^{(3)}_{i,j} \neq \Psi^{(3)}_{j,i}$ so that a complete determination of $\Psi^{(3)}_{i,j}$ cannot be derived from (D.12).

\section*{D.2 Calculation of $\langle V^{(3)}(t)\delta L \rangle$}

The second part of this appendix is devoted to the calculation of $\langle V^{(3)}(t)\delta L \rangle$, where $V^{(3)}(t)$ is the integral of $J^{(3)}$ over the volume $V_0$. In the wave vectors space these quantities are given by integrations over $k$ with the filter function $W_{TH}(k R_0)$ as defined in the appendix B. The Fourier transforms, $j^{(1)}_k$, of the first order of the Jacobian are defined by

$$J^{(1)}(q) = \int \frac{d^3 k}{(2\pi)^{3/2}} j^{(1)}_k \exp(iq.k). \quad (D.13)$$

As a result the integral of $J^{(1)}$ over the volume is given by

$$V^{(1)} = V_0 \int \frac{d^3 k}{(2\pi)^{3/2}} j^{(1)}_k W_{TH}(k R_0) \quad (D.14)$$

and

$$V^{(3)} = V_0 \int \frac{d^3 k_1}{(2\pi)^{3/2}} \frac{d^3 k_2}{(2\pi)^{3/2}} \frac{d^3 k_3}{(2\pi)^{3/2}} W_{TH}(|k_1 + k_2 + k_3| R_0) \times j^{(1)}_{k_1} j^{(1)}_{k_2} j^{(1)}_{k_3} \left[ -\frac{2}{21} \left( 1 - \frac{(k_1 \cdot (k_2 + k_3))^2}{k_2^2 |k_2 + k_3|^2} \right) \left( 1 - \left[ \frac{k_2 \cdot k_3}{k_2 k_3} \right]^2 \right) + \frac{1}{9} \left( 1 - 3 \left[ \frac{k_2 \cdot k_3}{k_2 k_3} \right]^2 + 2 \frac{k_1 \cdot k_2 \cdot k_3 \cdot k_4}{k_1^2 k_2^2 k_3^2} \right) \right]. \quad (D.15)$$

If the initial density field has Gaussian fluctuations as described in the part 2 the random variables $j^{(1)}_k$ are Gaussian and

$$j^{(1)}_k = \frac{D(t)}{D(t_i)} \delta k,$$

since $\delta(t, q) = [1 - J(t, q)/J(t, q)] \approx -J^{(1)}(t, q)$ at the first order.

The ensemble average of the product $V^{(3)}\delta L$ exhibits the expectation value of the product of four Gaussian random variables,

$$\langle \delta k j^{(1)}_{k_1} j^{(1)}_{k_2} j^{(1)}_{k_3} \rangle = - \left[ \frac{D(t)}{D(t_i)} \right]^3 \langle \delta k \delta k_1 \delta k_2 \delta k_3 \rangle$$

$$= - \left[ \frac{D(t)}{D(t_i)} \right]^3 \left[ \langle \delta k \delta k_1 \rangle \langle \delta k_2 \delta k_3 \rangle + \text{cyc.} \right] \quad (D.16)$$

$$= - \left[ \frac{D(t)}{D(t_i)} \right]^3 \left[ P(k)\delta_3(k + k_1) P(k_2)\delta_3(k_2 + k_3) + \text{cyc.} \right]$$
where cyc. stands for three other terms obtained by a circular permutation of the indices. The last expression holds because the random Gaussian variables describe the density field of a randomly homogeneous and isotropic universe with the power spectrum $P(k)$.

The use of the equations (D.14-16) gives

$$\langle V^{(3)} \delta_L \rangle = \frac{4V_0}{21} \left[ \frac{D(t)}{D(t_i)} \right]^4 \int \frac{d^3k}{(2\pi)^3} W^2_{\text{TH}}(k \, R_0) P(k) \frac{d^3k'}{(2\pi)^3} P(k') \times \left[ 1 - \frac{(k' \cdot (k + k'))^2}{k'^2|k + k'|^2} \right] \left[ 1 - \frac{(k \cdot k')^2}{kk'} \right].$$  \hspace{1cm} (D.17)

This integral can be simplified since the integration over the angular part between $k$ and $k'$ can be made separately. We can define the function $f(\tau)$ by

$$f(\tau) = \frac{1}{2} \int_{-1}^{1} du \frac{(1-u^2)^2}{1+\tau^2+2\tau u},$$  \hspace{1cm} (D.18)

then

$$\langle V^{(3)} \delta_L \rangle = \frac{4V_0}{21} \left[ \frac{D(t)}{D(t_i)} \right]^4 \int \frac{d^3k}{(2\pi)^3} W^2_{\text{TH}}(k \, R_0) P(k) \int \frac{d^3k'}{(2\pi)^3} P(k') f(k'/k).$$  \hspace{1cm} (D.19)

The calculation of $f(\tau)$ can be done analytically and the result reads

$$f(\tau) = \frac{1}{4\tau} \left[ -2A^3 + \frac{10}{3} A + (1 - A^2)^2 \ln \frac{A + 1}{A - 1} \right]$$  \hspace{1cm} (D.20)

$$A = \frac{\tau}{2} + \frac{1}{2\tau}.$$

The shape of the function $f(\tau)$ is given in Fig. 3. Note that $f(\tau) \sim 5/8$ for $\tau \to 0$ and $f(\tau) \sim 5/8\tau^2$ for $\tau \to \infty$, so that the integral (D.19) converges only if $P(k) \ll k^{-1}$ when $k \to \infty$.

D.3 Calculation of $\langle [V^{(2)}(t)]^2 \rangle$

The expression of $V^{(2)}(t)$ derives from the results given in (D.11),

$$V^{(2)}(t) = \frac{2}{7} \left[ \frac{D(t)}{D(t_i)} \right]^2 \int \frac{d^3k_1}{(2\pi)^{3/2}} \frac{d^3k_2}{(2\pi)^{3/2}} j_{k_1}^{(1)} W_{\text{TH}}(k_1 \, R_0) j_{k_2}^{(1)} W_{\text{TH}}(k_2 \, R_0) (1-u^2)$$  \hspace{1cm} (D.21)

where $u = k_1 \cdot k_2 / k_1 k_2$. As a result we have

$$\langle [V^{(2)}(t)]^2 \rangle = 2 \left( \frac{2}{7} \right)^2 \left[ \frac{D(t)}{D(t_i)} \right]^4 \int \frac{d^3k_1}{(2\pi)^3} \frac{d^3k_2}{(2\pi)^3} W^2_{\text{TH}}(|k_1 + k_2| \, R_0) \times P(k_1) P(k_2) (1-u^2)^2.$$  \hspace{1cm} (D.22)
This expression can be simplified by the change of variable

\[ k_+ = k_1 + k_2; \]
\[ k_- = k_1 - k_2; \]  \hspace{1cm} (D.23)

that leads to

\[ \langle [V^{(2)}(t)]^2 \rangle = \left( \frac{4}{7} \right)^2 \left[ \frac{D(t)}{D(t_i)} \right]^4 \int \frac{d^3k_+}{(2\pi)^3} \frac{d^3k_-}{(2\pi)^3} W_{\text{TH}}^2(k_+ R_0) P(k_1) P(k_2) \]
\[ \times \left[ \frac{(1 - v^2)}{(k_+/k_- + k_-/k_+)^2 - 4v^2} \right]^2 \]  \hspace{1cm} (D.24)

where \( v = k_+ k_- / k_+ k_- \). In case of a power–law spectrum with an index \( n \) (eq. [37]) the result reads,

\[ \langle [V^{(2)}(t)]^2 \rangle = \left( \frac{4}{7} \right)^2 4^{-n} \left[ \frac{D(t)}{D(t_i)} \right]^4 \int \frac{d^3k_+}{(2\pi)^3} k_+^{5+2n} W_{\text{TH}}^2(k_+ R_0) \]
\[ \times \int_{-1}^{1} dv \int_{-\infty}^{+\infty} \frac{dr}{(2\pi)^2} (1 - v^2)^2 \left[ (k_+/k_- + k_-/k_+)^2 - 4v^2 \right]^{n/2-2}, \]  \hspace{1cm} (D.25)

which has to be integrated numerically.
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Fig. 1: Evolution of the density profile for an Einstein–de Sitter universe. The initial profile is the expected shape of the profile for various value of the power–law index: \( n = -2 \) for (a), \( n = -1 \) for (b) and \( n = 0 \) for (c). The solid lines correspond to the initial profiles. The \( n = 0 \) case corresponds to a Poisson distribution and the correlation length is 0 which explains this top–hat shape of the initial profile. The dashed lines correspond to the exact nonlinear evolution of these profiles for various time steps, \( \delta L = 0.25, 0.5, 0.80, 1.10 \).

Fig. 2: The nonlinear density as a function of \( \delta L \) (eq. [20]) for various cosmological models. The solid line is the analytic expression \( \delta = (1 - \delta L/1.5)^{-1.5} - 1 \) corresponding to \( \Omega \rightarrow 0, \Lambda = 0 \); the dotted line is for \( \Omega = 1, \Lambda = 0 \); and the dashed line is for \( \Omega = 1, \lambda = \Lambda/3H_0^2 = 1 \).

Fig. 3: The function \( f(\tau) \) (eq. [D.20]) as a function of \( \tau \).

Fig. 4: The corrective coefficient, \( C_{\text{exp}} \) (eq. [34]) as a function of \( n \) for a power–law spectrum (a) and as a function of the cutoff, \( k_c \), for a truncated power–law spectrum and for various values of \( n \) (b). \( R \) is the size of the perturbation.

Fig. 5: Evolution of the density of a fluctuation as a function of the initial overdensity and of the time. The upper solid line corresponds to the spherical model, the second solid line corresponds to the spherical model solution truncated at the third order. The long dashed and small dashed lines correspond to the expected evolution of a fluctuation up to the third order characterized respectively by \( \nu = 4 \) and \( \nu = 2 \). The corrective coefficients are calculated for \( n = -2 \).

Fig. 6: The corrective coefficient as a function of scale in case of a CDM spectrum. For the mass scale of the clusters the correction is still small, so that the spherical collapse model is expected to be a good description for the collapse of a cluster.

Fig. 7: The coefficient \( C_{\text{scat}} \) (eq. [46]) as a function of \( n \) (solid line) and the coefficient \( C_{\text{scat}}' \) (eq. [48]) (dashed line).

Fig. 8: Schematic representation of the situation obtained in the \( n \approx -2 \).
case. The expectation behavior (dashed line) is close to the spherical collapse model (solid line) with possible fluctuations represented by the shaded area, the departure from the spherical collapse and the fluctuations being of the same order of magnitude.