THE STOCHASTIC SECTOR OF
INTERACTING-FREE QUANTUM FIELD
THEORY

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Abstract

The Quantum Stochastic Limit of a quantum mechanical particle coupled to a quantum field without the neglect of the response details of the interaction (i.e. not making the dipole approximation) is made following the treatment of Accardi and Lu [6] and the corresponding Quantum Stochastic Structure is derived. The stochastic sector for the noise is constructed and is shown to be of a qualitatively new type. We also include a physical discussion on the limit noise which obeys Interacting-Free statistics and include a new shorter proof of the noise convergence and also a new construction of Interacting-Free Fock Space.

1 Introduction.

The theory of stochastic processes has many deep connections with quantum field theory. The path integral approach of Feynman [1], in particular, reveals close analogies between quantum field theory in real time and Brownian motion. An important line of research which has deepened this connection in recent years is that of quantum stochastic approximations: here one considers a test system $S$ (quantum mechanical) coupled to an infinite reservoir $R$ (a Bosonic quantum field), the Hamiltonian for the combined system and reservoir takes the form $H = H_S + H_R + \lambda H_I$ where only the interaction $H_I$ couples $S$ to $R$. A Gaussian state (e.g. vacuum or thermal) is prescribed for the reservoir and one makes a separation of time scales (van Hove limit): time $t$ being rescaled as $t/\lambda^2$ followed by the limit $\lambda \to 0$. In an approach pioneered by Accardi, Frigerio and Lu [2], one constructs suitable collective reservoir fields in which to examine the limiting behaviour of observables and these collective fields have the property of themselves converging to basic quantum stochastic processes (typically quantum Brownian motion). This fact was exploited by Accardi, Lu and Volovich [3] to
establish a (quantum) stochastic sector in quantum field theory.

The original scope of [2] was very limited due to the fact that almost all the standard simplifying assumptions (vacuum state, rotating wave approximation, dipole approximation, etc.) were made in order to make an already complicated problem accessible. However, since then, these assumptions have been removed with relative ease [4]. The connection between the quantum stochastic limit theory (when applied to an atomic system of bound states: i.e. when $H_S$ has discrete spectrum) and the standard application of the *Golden Rule* to the same problem has been explained in Accardi, Gough and Lu [5].

Recently the problem of considering a system with continuous spectrum without recourse to the dipole approximation has been tackled [6]. The surprising feature which emerges is that, by now including all the details of the interaction between $S$ and $R$, the limit quantum noise has a qualitatively new character. Instead of inheriting the Bose statistics of $R$, the noise in fact obeys a non-linear modification of the Free statistics. The originally notion of Free-ness is due to Voiculescu [7] and in the context of quantum stochastic theory was first studied by Kümmerer and Speicher [8]. We shall use the term *Interacting-Free* to describe the noise studied here: the notion of Interacting Fock space over a Fock Module necessary to describe the limit noise was introduced however by Lu [9].

The goal of this paper is to extend the notion of (quantum) stochastic sector so that the interacting-free field limit can be included.

### 1.1 The Physical Model

As system we consider a quantum mechanical particle with spin zero and unperturbed Hamiltonian $H_S$:

$$\dot{H}_S = \frac{p^2}{2m}. \quad (1.1)$$

Here $p$ is canonical momentum with canonical position denoted by $q$: $[q_j, p_l] = i\hbar \delta_{j,l}$.

The reservoir is taken, for transparency, to have spinless Bosonic quanta. We denote by $a^\dagger(k)$ the creation operator for a reservoir quantum of momentum $k$. Along with its adjoint $a(k)$ we have the canonical commutation relations

$$[a(k), a^\dagger(k')] = \delta(k - k'). \quad (1.2)$$

The unperturbed Hamiltonian for the reservoir is taken to be

$$H_R = \int dk \hbar \omega(k)a^\dagger(k)a(k), \quad (1.3)$$

where $\omega(k) \geq 0$ gives the dispersion relation for $R$.

The unperturbed evolution operator for $S + R$ is then

$$V_t^0 = \exp\left\{ \frac{t}{\hbar}(H_S \otimes 1_R + 1_S \otimes H_R) \right\}. \quad (1.4)$$
The interaction between the particle and field takes the form

\[ H_I = D(p)A(q) \]  

where \( D(p) \) is an observable of the system and

\[ A(q) = \int dk \{ g(k)e^{-ik \cdot q} \otimes a^\dagger(k) + \overline{g}(k)e^{ik \cdot q} \otimes a(k) \}. \]  

\( A(q) \) is the potential of the field and naturally depends on the particle’s position \( q \). The form factor \( g \) is taken to be a Schwartz function on \( \mathbb{R} \). We shall assume that \( [D(p), A(q)] = 0 \) so that \( H_I \) is self adjoint.

For the situation of an electron coupled to the QED field, the reservoir quanta (photons) have polarization and we can choose the Coulomb gauge so that \( D(p) \equiv -\frac{e}{\hbar} p \) commutes with \( A(q) \): in this case we would of course have a vector product. In our case, in order to study the field in detail with the only simplifying assumption that the quanta be spinless, we make the assumption that \( D \) is proportional to \( 1_S \) and drop it entirely. This in fact changes very little in the qualitative description of the limiting noise.

We remark that the reverse situation is considered in most other treatments: that is, one assumes that \( D \) is \( p \)-dependent while \( A \) is \( q \)-independent. In such cases, we say that the field is responseless: then there is the replacement

\[ A \mapsto A' = \int dk \{ g(k)a^\dagger(k) + \overline{g}(k)a(k) \}. \]  

In the QED case this is the dipole approximation. Now \( A' \) is trivial and a test system \( S \), under this replacement, cannot obtain any measurable information about the individual modes of the field (because this is precisely the detail which is elided in \( A' \)). The situation of a responseless field has already been studied in the quantum stochastic limit and it is known that a quantum Brownian motion emerges.

Our objective is then to make a study of the responsive interaction

\[ H_I = c^\dagger(g) + c(g), \]  

where we introduce the combined (interacting) fields

\[ c^\dagger(g) = \int dk g(k)e^{-ik \cdot q} \otimes a^\dagger(k), \quad c(g) = \int dk \overline{g}(k)e^{ik \cdot q} \otimes a(k). \]  

The total Hamiltonian is taken to be

\[ H_\lambda = \{ H_S \otimes 1_R + 1_S \otimes H_R \} + \lambda H_I \]  

where \( \lambda \) is a non-zero coupling constant.

The van Hove scaling limit has, in previous applications to quantum stochastic limits, suggested the use of collective operator fields of the following type

\[ c^\sharp_{t,\lambda}(g) := \lambda \int_0^{t/\lambda^2} d\tau V_\tau^0 c^\dagger(g)V_\tau^0. \]  

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The limit of such collective operators in the vacuum field $\Psi_R$ of the reservoir was obtained in [6]. The limiting fields, denoted by $C^\tau(g, t)$, do not satisfy Bose commutation relations but, on account of the response factor $\exp\{\mp ik.q\}$ which couples the system to all modes of the field, satisfy a modified version of the free relations.

The Weyl operators offer a straightforward means to study the unperturbed evolution of the response factors so we review them now. For $a, b \in \mathbb{R}^3$, we define the Weyl operator $W(a, b)$ to be the unitary operator

$$W(a, b) = e^{i(a.p + b.q)}.$$  \hfill (1.12)

They satisfy

1) $W(a, b) = e^{ia.p}e^{ib.q}e^{-iha.b/2} = e^{ib.a}e^{ia.p}e^{iha.b/2}$;

2) $W(a_1, b_1)W(a_2, b_2) = W(a_1 + a_2, b_1 + b_2)\exp\{\frac{i}{2}(a_1.b_2 - a_2.b_1)\}$ or, more generally,

$$W(a_1, b_1)...W(a_n, b_n) = W(\sum_j a_j, \sum_j b_j)\exp\{\frac{i}{2}\sum_j(a_j.b_l - a_l.b_j)\};$$

3) $W(a, b)^\dagger = W(-a, -b)$;

4) Under the unperturbed evolution we have $p_t = p, q_t = q + \frac{t}{m}p$ and so the Weyl operators evolve as shown below

$$W(a, b)_t = e^{i(a.p + b.q)} = e^{i(\frac{t}{m}p + b.q)} = W(a + \frac{t}{m}b, b).$$

Therefore we have

$$V_0^\tau c_\dagger (g) V_0^0 = \int dk g(k)e^{i\omega(k)\tau}W(-\frac{\tau}{m}k, -k) \otimes a_\dagger(k).$$  \hfill (1.13)

## 2 The Quantum Stochastic Sector

The results of Accardi, Lu and Volovich [3] can be summarized as follows:- for $\omega > 0$ let

$$B_{t,\lambda}^\dagger (g) := \lambda \int_{0}^{t/\lambda^2} d\tau \int dk g(k)e^{i[\omega(k) - \omega]\tau}a_\dagger(k).$$  \hfill (2.1)

which is the collective operator describing a responseless field. As $\lambda \to 0$ one shows that $B_{t,\lambda}^\dagger (g)$ converges to Bosonic quantum Brownian motion $B^\dagger (g, t)$ satisfying

$$[B(g, t), B^\dagger (f, s)] = (t \wedge s) (g|f)$$  \hfill (2.2)

where

$$(g|f) := \int_{-\infty}^{+\infty} d\tau \int dk \delta(\omega(k) - \omega)\bar{g}(k)f(k) \equiv \int dk \delta(\omega(k) - \omega)\bar{g}(k)f(k).$$  \hfill (2.3)
Now $\omega$ can be interpreted as a probing frequency: that is $\omega$ is a frequency associated to $D$ under rotating wave approximation and in principle different test systems (having different resonant $\omega$) reveal further information about the reservoir. However in this case the detailed information is restricted by the responseless assumption.

The noise fields $b^\dagger_\lambda(u, k)$ defined by

\[ b^\dagger_\lambda(u, k) := \frac{1}{\lambda} e^{i\omega u/\lambda^2} a^\dagger(k) \]  

so that

\[ B^\dagger_{t, \lambda}(g) \equiv \int_0^t du \int dk g(k) b^\dagger_\lambda(u, k), \]

then converge in the limit $\lambda \to 0$ in the vacuum state to the quantum white noise $b^\dagger(u, k)$ satisfying

\[ [b(u, k), b^\dagger(u', k')] = 2\pi \delta(u - u') \delta(\omega(k) - \omega) \delta(k - k'). \]  

The results of this paper can then be summarized as follows. Introducing the density operators

\[ a^\dagger_\lambda(u, k) := \frac{1}{\lambda} V^{\dagger 0}_{u/\lambda^2} [e^{-ik.q} \otimes a^\dagger(k)] V^{0}_{u/\lambda^2} \]  

we have the limit (in law) $a^\dagger_\lambda(u, k) \to a^\dagger(u, k)$ where

\[ a^\dagger(u, k) = \int_{-\infty}^{+\infty} d\tau e^{i\omega(k)\tau} W(-\frac{\tau}{m}k, -k) \otimes a^\dagger(u, \tau, k) \]  

and $a^\dagger(u, \tau, k)$ satisfy the modified Free relations

\[ a(u, \tau, k)a^\dagger(u', \tau', k') = \delta(u - u')\delta(2\tau - \tau')\delta(k - k'). \]  

The limiting collective operator $C^\dagger(g, t)$ may then be expressed as

\[ C^\dagger(g, t) = \int_0^t du \int_{-\infty}^{+\infty} d\tau \int dk g(k) e^{i\omega(k)\tau} W(-\frac{\tau}{m}k, -k) \otimes a^\dagger(u, \tau, k). \]

Note: we can introduce fields $\alpha^\dagger(\tau, k)$ satisfying the relations

\[ \alpha(\tau, k)\alpha^\dagger(\tau', k') = \delta(2\tau - \tau')\delta(k - k'). \]  

and set

\[ C^\dagger(g, t) = \chi_{[0,t]} \otimes \int_{-\infty}^{+\infty} dk g(k) e^{i\omega(k)\tau} W(-\frac{\tau}{m}k, -k) \otimes a^\dagger(\tau, k); \]

\[ C(g, t) = \chi_{[0,t]} \otimes \int_{-\infty}^{+\infty} dk \overline{g}(k) e^{-i\omega(k)\tau} W(\frac{\tau}{m}k, k) \otimes a(\tau, k) \]  

(2.12)
where $|\alpha>$ is $\text{ket}$ and $<\alpha|$ is $\text{bra}$- for $\alpha \in \mathcal{L}^2(\mathbb{R})$. If we have an ordered product of $\mathcal{C}^\sharp(g_j, t_j)$ then we will have an associated ordered product of bras and kets: the simple algebraic rule applied here is that whenever a bra is immediately to the left of a ket they form a scalar product and can be taken to one side. Thus, for instance,

$$<\alpha| \cdot \beta > := <\alpha, \beta> \equiv \int_{-\infty}^{+\infty} dt \overline{\alpha}(t) \beta(t),$$

$$<\alpha_1| \cdot \beta_1 > : <\alpha_2| \cdot \beta_2 > = <\alpha_1, \beta_1 > <\alpha_2, \beta_2 >$$

while

$$<\alpha_1| \cdot <\alpha_2| \cdot \beta_2 > \cdot \beta_1 > = <\alpha_1, \beta_1 > <\alpha_2, \beta_2 >$$

It is the compatibility of the bra-ket formalism with the free statistics that allows the description of $\mathcal{C}^\dagger(g, t)$ as algebraic tensor product of $\mathcal{L}^2(\mathbb{R})$ with the $W \otimes \alpha^\sharp$-operators (c.f. remark b in section 3).

3 The Limit Processes, $C^\sharp(g, t)$

In the following we shall adopt the convention that $\prod_{j=1}^n X_j = X_n \cdots X_1$ and that, for any operator $X$, $X^0 := X$ while $X^1 := \overline{X}$. A sequence $\varepsilon = \{\varepsilon_{2n}, \ldots, \varepsilon_1\} \in \{0,1\}^{2n}$ will be referred to as non-trivial if $<\Psi_R, \prod_{j=1}^{2n} a^\varepsilon(k_j)\Psi_R >$ is not identically zero. Clearly, there must be an equal number of creators and annihilators if the expectation above is to be non-zero. Suppose $\varepsilon$ is non-trivial and let $M = (m_n, \ldots, m_1)$ denote the set of creator indices (i.e. $\varepsilon_{m_j} = 1$) ordered so that $m_h < m_{h+1}$. Let $M^c$ then denote the (unordered) set of annihilator index positions. To guarantee non-triviality we also require the condition that for all $r = 1, \ldots, 2n$

$$\sharp\{m' \in M^c : m' \leq r\} \leq \max\{h : m_h \leq r\}.$$  \hspace{1cm} (3.1)

The odd correlation functions clearly vanish in the reservoir vacuum. The even correlators are given below. They were first computed by Accardi and Lu [6], however we include here a shorter proof (Theorem 1).

Before this we give a brief account of Free statistics. Let $T$ be a space of test functions with inner product $\langle ., . \rangle$. Let $b(g), b^\dagger(g)$ be operators (for each $g \in T$) and $\Psi$ a vector such that

$$b(g)^\dagger = b^\dagger(g), \ b(g)\Psi = 0$$ \hspace{1cm} (3.2)

and

$$b(g)b^\dagger(f) = \langle g, f \rangle$$ \hspace{1cm} (3.3)

for all $g, f \in T$. The operators $b^\dagger(g)$ are said to satisfy free statistics or free relations. $\Psi$ is referred to as the Fock vacuum vector. An explicit construction can be given on Fock space $\Gamma(T) = \oplus_n^{+\infty}(\otimes^n T)$ over $T$ by taking $b^\dagger(g)$ to be the mapping $\phi_1 \otimes \cdots \otimes \phi_n \mapsto g \otimes \phi_1 \otimes \cdots \otimes \phi_n$. In which case, its adjoint is $b(g) : \phi_1 \otimes \cdots \otimes \phi_n \mapsto (g, \phi_1)\phi_2 \otimes \cdots \otimes \phi_n$. The vacuum vector is then the Fock vacuum $\Psi := 1 \oplus 0 \oplus 0 \ldots$. 

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Now it is easily seen that \( \langle \Psi, \prod_{j=1}^{2n} b^+ (g_j) \Psi \rangle \) is not identically zero provided \( \varepsilon \) is again non-trivial. However the relations (3.3) give that
\[
\langle \Psi, \prod_{j=1}^{2n} b^+ (g_j) \Psi \rangle = \prod_{j=1}^{2n} \langle g_{\overline{m}_j}, g_{m_j} \rangle, \tag{3.4}
\]
where \( \{\overline{m}_n, \ldots, \overline{m}_1\} \) is the unique ordered sequence which agrees with \( M^c \) as a set and satisfies
\begin{enumerate}
  \item \( \overline{m}_j > m_j \)
  \item \( \forall h = 1, \ldots, n \quad \overline{m}_h > m_j \Leftrightarrow \overline{m}_h > \overline{m}_j > m_h \) \tag{3.5}
\end{enumerate}

**Remark:** Condition (i) comes from having to arrange \( \prod_{j=1}^{2n} b^j (g_j) \) in normal order. Note that the logical negation of (ii) also holds, that is if \( m_j \) lies outside of \( \{\overline{m}_n, \ldots, m_n\} \) then so too does \( \overline{m}_j \), and vice versa. The set of \( n \) pairs \( \{\overline{m}_n, m_n\} : h = 1, \ldots, n \} \) as above is called the Wigner or non-crossing pair partition of \( \varepsilon \).

**Remark:** We have already met an example of freeness in our algebraic rule for bras and kets in the last section: the identification
\[
|\alpha\rangle \equiv b^\dagger (\alpha), \quad <\beta| \equiv b (\beta), \tag{3.6}
\]
for \( T \equiv L^2 (\mathbb{R}) \), now makes this rule definite.

**Theorem 1.** Let \( \varepsilon \in \{0,1\}^{2n} \) be non-trivial then
\[
\langle \prod_{j=1}^{2n} C^\varepsilon_j (g_j, T_j) \rangle : = \lim_{\lambda \to 0} <\Psi_R, \prod_{j=1}^{2n} C^\varepsilon_j, \lambda (g_j) \Psi_R > \tag{3.7}
\]
\[
= \prod_{j=1}^{n} (T_{\overline{m}_j} \wedge T_{m_j}) \int_{-\infty}^{\infty} d\tau_{m_1} \ldots \int_{-\infty}^{\infty} d\tau_{m_n}
\]
\[
\times \int d^3 k_1 \ldots \int d^3 k_n \prod_{h=1}^{n} \{\overline{m}_h (k_{m_h}) s_{m_h} \exp \{i[\omega (k_{m_h} + \frac{\hbar}{2m} |k_{m_h}|^2] \tau_{m_h}\}
\]
\[
\times \exp \{-\frac{i}{m} p \sum_{j=1}^{n} k_{m_j} \tau_{m_j} \} \exp \{\frac{ih}{m} \sum_{h, r=1}^{n} \tau_{m_h} k_{m_h} k_{m_r} \chi (m_{\overline{m}_r}) (m_h) \} \tag{3.8}
\]
where \( \{\overline{m}_j, m_j\} : j = 1, \ldots, n \} \) is the Wigner (non-crossing) partition of \( \{1, \ldots, 2n\} \) associated with \( \varepsilon \). If \( \varepsilon \) is trivial then (3.7) vanishes.

The origin of this limit can be explained as follows. In principle the \( 2n \)-point correlations before the limit can be expressed (due to the Bosonic nature
of the reservoir and our choice of a Gaussian state) in terms of all possible pair partitions. Now retaining the response term means that for each emission and absorption of a reservoir quantum we keep the details of the momentum recoil of the system particle, and so enforcing strict momentum conservation. A contracted creation and annihilation pair survives the stochastic limit only if it is energetically balanced: this amounts to the Golden rule. However the only complete set of pair partitions which has all contracted pairs energetically balanced (and here we must have momentum conservation) is the Wigner pair partition, if one exists.

**Proof.** For \( \epsilon \in \{1, 0\} \), we have

\[
e_{T,\lambda}(\epsilon) \equiv \lambda \int_0^{T/\lambda^2} dt \int dk \frac{g^\epsilon(k)}{\omega(k)} W \left( \frac{(-1)^\epsilon \omega(k)}{m} \right) k, (-1)^\epsilon k \rangle \langle a^\epsilon(k).
\]

Here we set

\[
g_0(k) = g(k), \quad g_1(k) = g(k).
\]

For \( \epsilon = \{\epsilon_2, ..., \epsilon_1\} \in \{1, 0\}^n \) non-trivial, we have

\[
\langle \Psi_R, \prod_{j=1}^{2n} e_{T_j,\lambda}(\epsilon_j) g_j \rangle \Psi_R \rangle = \lambda^{2n} \prod_{j=1}^{2n} \left\{ \int_0^{T_j/\lambda^2} dt_j \int d^3 k \ g_j^\epsilon(k) \exp \{i(-1)^\epsilon \omega(k)\tau_j\} \right\}
\]

\[
\times \prod_{l=1}^{2n} W \left( \frac{(-1)^\epsilon \omega(k)}{m} \right) k, (-1)^\epsilon k \rangle < \Psi_R, \prod_{h=1}^{2n} a^{\epsilon_h}(k_h) \Psi_R >.
\]

but

\[
\langle \Psi_R, \prod_{h=1}^{2n} a^{\epsilon_h}(k_h) \rangle \Psi_R > = \sum_{\{M' \equiv M' : m_n' < m_h \forall h\}} \prod_{h=1}^n \delta(k_{m_n'} - k_{m_n})
\]

that is, we sum over all possible pair contractions of creator–annihilator indices \( \{(m_n', m_h) : h = 1, ..., n\} \) where \( M' = (m_1', ..., m_n') \) is equivalent to \( M' \) as a set. As we produce contractions by moving terms to normal order, we clearly need only consider \( m_n' > m_h \) however: that is to say the pair contraction \((m_n', m_h)\) only comes about from having to move the annihilator \( a(k_{m_n'}') \) from the left to the right of \( a^\dagger(k_{m_h}) \).

Therefore we may write

\[
\langle \Psi_R, \prod_{j=1}^{2n} e_{T_j,\lambda}(\epsilon_j) g_j \rangle \Psi_R > =
\]

\[
\sum_{\{M' \equiv M' : m_n' < m_h \forall h\}} \prod_{h=1}^n \left\{ \lambda^2 \int_0^{T_{m_n'}/\lambda^2} d\tau_{m_n'} \int_0^{T_{m_h}/\lambda^2} d\tau_{m_h} \int d^3 k_{m_h} \right\}
\]

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\( \mathcal{G}_{m' h}(k_{m h}) g_{m h}(k_{m h}) \exp \{ i \omega (k_{m h}) \tau_{m h} \} \) \( \prod_{l=1}^{2n} W \left( \frac{(-1)^{\epsilon_l} \tau_l}{m} k_l, (-1)^{\epsilon_l} k_l \right) \), \hspace{1cm} (3.13)

where the product of Weyl operators must be accompanied by the relevant assignment \( k_{m j} = k_{m' j} \), for each \( M' \) considered in the sum.

Now, using the rule for multiplying Weyl operators and mindful of our product convention, we have that

\[
\prod_{l=1}^{2n} W \left( (-1)^{\epsilon_l} \frac{\tau_l}{m} k_l, (-1)^{\epsilon_l} k_l \right) = \exp \left\{ -\frac{i \hbar}{2m} \sum_{1 \leq j < l \leq 2n} (-1)^{\epsilon_j + \epsilon_l} k_j k_l (\tau_j - \tau_l) \right\}
\]

\[
\times W \left( \sum_{1 \leq l \leq 2n} (-1)^{\epsilon_l} \frac{\tau_l}{m} k_l, \sum_{1 \leq l \leq 2n} (-1)^{\epsilon_l} k_l \right) \hspace{1cm} (3.14)
\]

Momentum balance requires that

\[
\sum_{1 \leq l \leq 2n} (-1)^{\epsilon_l} k_l = 0,
\]

so the correlation function is independent of \( q \) and so diagonal in \( p \).

\[
\sum_{1 \leq l \leq 2n} (-1)^{\epsilon_l} \frac{\tau_l}{m} k_l = -\frac{1}{m} \sum_{1 \leq h \leq n} (\tau_{m h} - \tau_{m' h}) k_{m h}.
\]

The phase associated with \( M' \) is then

\[
\frac{-i \hbar}{2m} \sum_{l=1}^{2n} \sum_{j < l} (-1)^{\epsilon_j + \epsilon_l} k_j k_l (\tau_j - \tau_l)
\]

\[
= \frac{-i \hbar}{2m} \sum_{h=1}^{n} \left\{ \sum_{1 \leq j < m' h} (-1)^{\epsilon_j} k_j k_{m' h} (\tau_j - \tau_{m' h}) - \sum_{1 \leq j < m_h} (-1)^{\epsilon_j} k_j k_{m h} (\tau_j - \tau_{m h}) \right\}
\]

\[
= \frac{-i \hbar}{2m} \sum_{h=1}^{n} \left\{ \sum_{\alpha \leq m' h} k_{m' \alpha} k_{m' h} (\tau_{m' \alpha} - \tau_{m' h}) - \sum_{\beta \leq m h} k_{m \beta} k_{m h} (\tau_{m \beta} - \tau_{m h}) \right\}
\]

\[
- \sum_{\gamma \leq m_h} k_{m' \gamma} k_{m h} (\tau_{m' \gamma} - \tau_{m h}) + \sum_{\delta \leq m_h} k_{m' \delta} k_{m h} (\tau_{m' \delta} - \tau_{m h}) \right\} \hspace{1cm} (3.17a)
\]

and putting together the first term with the third and second with fourth

\[
= \frac{-i \hbar}{2m} \sum_{h=1}^{n} \left\{ \sum_{\alpha < m_h} k_{m \alpha} k_{m h} (\tau_{m \alpha} - \tau_{m h}) - \sum_{\beta < m_h} k_{m \beta} k_{m h} (\tau_{m \beta} - \tau_{m h}) \right\}
\]

\[
- \sum_{\gamma < m_h} k_{m' \gamma} k_{m h} (\tau_{m' \gamma} - \tau_{m h}) + \sum_{\delta < m_h} k_{m' \delta} k_{m h} (\tau_{m' \delta} - \tau_{m h}) \right\} \hspace{1cm} (3.17b)
\]
\[ m_h < m'_h < m'_h \]
\[- \sum_{\alpha} k_{m_{\alpha}} k_{m_h} (\tau_{m_{\alpha}} - \tau_{m_h}) - \sum_{\beta} k_{m_{\beta}} k_{m_h} (\tau_{m_{\beta}} - \tau_{m'_h}) \]
\[ \{ \tau_{m_h} - \tau_{m'_h} \} \]  
\[ (3.17b) \]

We now undergo a change of variables
\[ u_{m_h} = \lambda^2 t_{m_h}, \tau_{m_h} = t_{m_h} - t_{m'_h} \]  
\[ (3.18) \]

This gives
\[ < \Psi_R \prod_{j=1}^{2n} c_{T_j, \lambda}^j (g_j) \Psi_R > = \]
\[ \sum_{\{M' \in M: m'_h < m_h \}} \prod_{h=1}^{n} \left\{ \int_{0}^{T_{m_h}} du_{m_h} \int_{-u_{m_h}}^{(T_{m'_h} - u_{m_h}) / \lambda^2} dv_{m_h} \int dk_{m_h} \right. \]
\[ g_{m'_h}(k_{m_h}) g_{m_h}(k_{m_h}) \exp \{ i \omega(k_{m_h}) v_{m_h} \} \} W \left( - \frac{1}{m} \sum_{h=1}^{n} v_{m_h} k_{m_h}, 0 \right) \]
\[ \exp \left\{ \frac{-ih}{2m} \sum_{h=1}^{n} \sum_{\alpha} k_{m_{\alpha}} k_{m_h} v_{m_h} - \sum_{\beta} k_{m_{\beta}} k_{m_h} v_{m_h} \right. \]
\[ + \sum_{\alpha} k_{m_{\alpha}} k_{m_h} (v_{m_{\alpha}} - v_{m_h}) + (u_{m_{\alpha}} - u_{m_h}) / \lambda^2 \]
\[ \left. - \sum_{\beta} k_{m_{\beta}} k_{m_h} (v_{m_{\beta}} - v_{m_h}) + (u_{m_{\beta}} - u_{m_h}) / \lambda^2 \right\} \right] \]  
\[ (3.19) \]

By an application of the Riemann-Lebesgue lemma, we have that the oscillatory factors of the type \( e^{ik^2u/\lambda^2} \) cause the associated term to vanish in the limit \( \lambda \to 0 \). By examining the phase in (3.19) we see that, for each fixed \( h = 1, \ldots, n \) and for any \( \alpha \)
\[ m_h < m_{\alpha} < m'_h \Leftrightarrow m_h < m'_{\alpha} < m'_h \]  
\[ (3.20) \]

but this only possible for the Wigner partition. Hence only \( M' = M' \) survives the limit. Only in this case is the phase term independent of \( u_{m_h}, h = 1, \ldots, n \) and explicitly it equals
\[ \exp \left\{ \frac{-ih}{2m} \sum_{h=1}^{n} \sum_{\alpha} k_{m_{\alpha}} k_{m_h} v_{m_h} - \sum_{\alpha} k_{m_{\alpha}} k_{m_h} v_{m_h} \right. \]
\[ \left. - \sum_{\alpha} k_{m_{\alpha}} k_{m_h} v_{m_{\alpha}} - |k_{m_h}|^2 \right\} \]  
\[ (3.21) \]

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the first three terms can be combined to read as

$$\frac{-i\hbar}{2m} \sum_{h,\alpha=1}^{n} \left\{ \chi(m_{\alpha,n})(m_{h}) - \chi(m_{\alpha,n})(m_{h}) - \chi(m_{\alpha,m_{\alpha}})(m_{h}) \right\} k_{m_{\alpha}} k_{m_{h}} v_{m_{\alpha}}$$

where we reversed the roles of $\alpha$ and $h$ in the third term. The final term is then just

$$\exp\left\{ \frac{i\hbar}{2m} \sum_{h=1}^{n} |k_{m_{h}}|^2 v_{m_{h}} \right\}. \quad (3.23)$$

It is now evident that the $2n$-point function takes the form indicated in the statement of the theorem.

We remark that the form of the correlation functions can be simplified. Let $l_{m_{h}}$ denote the particle’s momentum after the emission vertex $C^{\dagger}(g_{m_{h}}, t_{m_{h}})$, by momentum conservation we have

$$l_{m_{h}} = p - h \sum_{r} k_{m_{r}}, \quad (3.24)$$

that is $l_{m_{h}}$ equals the incoming free momentum $p$ minus the sum of all emitted but not yet reabsorbed reservoir quanta momenta: as the structure is non-crossing this means that we sum over reservoir quanta with momentum $k_{m_{r}}$ which have been emitted before the vertex $m_{r} < m_{h}$ but not yet reabsorbed $m_{r} > m_{h}$. Let $\hbar \Delta(l, k)$ be the energy violation associated with each vertex, that is

$$\hbar \Delta(l, k) := \frac{1}{2m} |l - \hbar k|^2 + \hbar \omega(k) - \frac{1}{2m} |l|^2$$

$$\Rightarrow \Delta(l, k) = -\frac{1}{m} l_{l,k} + \omega(k) + \frac{\hbar}{2m} |k|^2. \quad (3.25)$$

Then we have

$$\langle \prod_{j=1}^{2n} C^{\dagger}(g_{j}, T_{j}) \rangle = \prod_{h=1}^{n} (T_{m_{h}} \wedge T_{m_{h}}) \int_{-\infty}^{+\infty} d\tau_{m_{n}} ... \int_{-\infty}^{+\infty} d\tau_{m_{1}} \int dk_{m_{n}} ... \int dk_{m_{1}}$$

$$\times \prod_{r=1}^{n} \{ \gamma_{m_{r}}(k_{m_{r}}) g_{m_{r}}(k_{m_{r}}) \exp \{ i \Delta(l_{m_{r}}, k_{m_{r}}) \tau_{m_{r}} \} \}. \quad (3.26)$$
\section{Interacting Fock Space}

The theory of Interacting Fock Space was developed in \cite{6,9}. We give a slightly different presentation of it in this section. Let $\mathcal{K} \subset \mathcal{L}^2(\mathbb{R}^3)$ denote the subspace of Schwartz functions such that for all $f, g \in \mathcal{K}$ one has
\begin{equation}
\int_{-\infty}^{+\infty} dt \left| < f, e^{i\Omega t} g > \right| < \infty,
\end{equation}
where $\Omega$ denotes multiplication by $\omega(k)$ on $\mathcal{L}^2(\mathbb{R}^3)$. For $f, g \in \mathcal{K}$ we have shown
\begin{equation}
\langle C(f, t)C^\dagger(g, s) \rangle = t \wedge s \langle f | g \rangle
\end{equation}
where
\begin{equation}
\langle f | g \rangle \equiv \langle f | g \rangle_p = \int_{-\infty}^{+\infty} d\tau \int dk \overline{f}(k)g(k) e^{i\Delta(p, k)\tau}.
\end{equation}

Now $\langle f | g \rangle$ is an element of $\mathcal{P}$, the (commutative) $C^*$-algebra generated by $\{e^{ix.p} : p \in \mathbb{R}^3\}$. The subscript $p$ shall not be displayed in general. We shall denote by $\mathcal{K}_p$ the $\mathcal{P}$-right-linear span of $\mathcal{K}$ and $\mathcal{L}^2_p(\mathbb{R}^3, \mathcal{K})$ the algebraic tensor product of $\mathcal{L}^2(\mathbb{R}^3)$ and $\mathcal{K}_p$. The two point function suggests that we study the bilinear form $\langle . | . \rangle : \mathcal{L}^2(\mathbb{R}^3, \mathcal{K}) \times \mathcal{L}^2(\mathbb{R}^3, \mathcal{K}) \rightarrow \mathcal{P}$ defined by
\begin{equation}
\langle \alpha \otimes f | \beta \otimes g \rangle := \alpha, \beta \in \mathcal{P}, \langle f | g \rangle.
\end{equation}

Next we wish to construct an $n$-particle space over $\mathcal{L}^2(\mathbb{R}^3, \mathcal{K})$ using the $2n$-point function to define the $n$-fold inner product. That is, construct $\mathcal{L}^2(\mathbb{R}^3, \mathcal{K})$ out of $\mathcal{L}^2_p(\mathbb{R}^3, \mathcal{K})$ with
\begin{equation}
\langle \chi_{[0,t_1]} \otimes f_1 \rangle \otimes \ldots \otimes \langle \chi_{[0,t_n]} \otimes f_n \rangle \langle \chi_{[0,s_1]} \otimes g_1 \rangle \otimes \ldots \otimes \langle \chi_{[0,s_n]} \otimes g_n \rangle :=
\end{equation}
\begin{equation}
\langle C(f_n, t_n) \ldots C(f_1, t_1) C^\dagger(g_1, s_1) \ldots C^\dagger(g_n, s_n) \rangle.
\end{equation}

The above $2n$-point function corresponds to the complete rainbow diagram: it equals
\begin{equation}
(s_1 \wedge t_1) \ldots (s_n \wedge t_n) \int_{-\infty}^{+\infty} d\tau_1 \ldots \int_{-\infty}^{+\infty} d\tau_n \int dk_1 \ldots \int dk_k \overline{f_1}(k_1)g_1(k_1) \ldots \overline{f_n}(k_n)g_n(k_n)
\end{equation}
\begin{equation}
e^{i\Delta(p, k_n)\tau_n}e^{i\Delta(p-k_n, k_{n-1})\tau_{n-1}} \ldots e^{i\Delta(p-k_n-k_{n-1}-\ldots-k_2, k_1)\tau_1}.
\end{equation}

However, introducing the transform
\begin{equation}
F = F(k) \mapsto \tilde{F} = \tilde{F}_p := \int_{-\infty}^{+\infty} d\tau \int dk \overline{F}(k) e^{i\Delta(p, k)\tau}
\end{equation}
and the convolution
\begin{equation}
\tilde{G} \ast \tilde{F} = \tilde{G} \ast \tilde{F}_p := \int_{-\infty}^{+\infty} d\tau \int dk \tilde{G}(p-k) F(k) e^{i\Delta(p, k)\tau}
\end{equation}

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\[
= \int_{-\infty}^{\infty} dt_1 \int_{-\infty}^{\infty} dt_2 \int dk_1 \int dk_2 G(k_1)F(k_2)e^{i\Delta(p,k_2)\tau_2}e^{1\Delta(p-k_2,k_1)\tau_1} \quad (4.8)
\]
we can write the correlator more succinctly as
\[
(s_1 \wedge t_1) \ldots (s_n \wedge t_n) (f_1|g_1) \ast \ldots \ast (f_n|g_n)_p. \quad (4.9)
\]
Note that the repeated application of the convolution is not associative and we shall always understand the (inductively defined) convention
\[
\tilde{F}_1 \ast \tilde{F}_2 \ast \ldots \ast \tilde{F}_n := [\tilde{F}_1 \ast \tilde{F}_2 \ast \ldots] \ast \tilde{F}_n. \quad (4.10)
\]
We are therefore lead to the identification
\[
(\alpha_1 \otimes f_1) \circ \ldots \circ (\alpha_n \otimes f_n) \circ (\beta_1 \otimes g_1) \circ \ldots \circ (\beta_n \otimes g_n) :=
\]
\[
< \alpha_1, \beta_1 >_{L^2(\mathbb{R})} \ldots < \alpha_n, \beta_n >_{L^2(\mathbb{R})} (f_1|g_1) \ast \ldots \ast (f_n|g_n), \quad (4.11)
\]
or on absorbing the \(L^2(\mathbb{R})\) term
\[
(\phi_1 \circ \ldots \circ \phi_n|\psi_1 \circ \ldots \circ \psi_n) := (\phi_1|\psi_1) \ast \ldots \ast (\phi_n|\psi_n). \quad (4.12)
\]
The inner product \((.,.)\) on \(\otimes^n L^2_p(\mathbb{R}^3, \mathcal{K})\) does not factor, as in the case with \(\otimes^n L^2_p(\mathbb{R}_p, \mathcal{K})\), and for this reason we refer to \(\otimes^n L^2_p(\mathbb{R}^3, \mathcal{K})\) as the interacting \(n\)-particle space.

So now we have two notions of product on \(\mathcal{P}\): the ordinary \(C^*\)-algebra product and now the non-associative convolution \(\ast\). Likewise, in addition to the usual module product, we can introduce a new product \(\sharp: \mathcal{P} \times L^2_p(\mathbb{R}^3, \mathcal{K}) \rightarrow L^2_p(\mathbb{R}^3, \mathcal{K})\) having the property that, for all \(c, b \in \mathcal{P}\) and \(f, g \in L^2_p(\mathbb{R}^3, \mathcal{K})\),
\[
(f|cg) = c \ast (f|g) \quad (4.13)
\]
and
\[
b \ast cg = (bc)\sharp g. \quad (4.14)
\]
The mapping \(c \mapsto (c\sharp -)\) defines a module homomorphism from \(\mathcal{P}\) to \(\mathcal{B}(L^2_p(\mathbb{R}^3, \mathcal{K}))\).

We note that we can write
\[
(\phi_1 \circ \phi_2|\psi_1 \circ \psi_2) = (\phi_1|\psi) \ast (\phi_2|\psi_2) = (\phi_2| (\phi_1|\psi_1)\sharp \psi_2) \quad (4.15)
\]
and by induction
\[
(\phi_1 \circ \ldots \circ \phi_n|\psi_1 \circ \ldots \circ \psi_n) =
\]
\[
(\phi_n| (\phi_{n-1} \circ \ldots \circ (\phi_1|\psi_1)\sharp \psi_2) \ldots \sharp \psi_{n-1} \ast \psi_n). \quad (4.16)
\]
The interacting Fock space is then defined as
\[
\Gamma_I(\mathcal{L}^2_p(\mathbb{R}^3, \mathcal{K})) := \bigoplus_{n=0}^{\infty} (\otimes^n L^2_p(\mathbb{R}^3, \mathcal{K})), \quad (4.17)
\]
where we take \( \bigotimes^0 \mathcal{L}_P^2(\mathbb{R}^3, \mathcal{K}) = \mathcal{P} \).

The creation operator \( A^\dagger(\phi), \phi \in \mathcal{L}_P^2(\mathbb{R}^3, \mathcal{K}) \), on \( \Gamma_1(\mathcal{L}_P^2(\mathbb{R}^3, \mathcal{K})) \) is then defined by

\[
A^\dagger(\phi) : \bigotimes^n \mathcal{L}_P^2(\mathbb{R}^3, \mathcal{K}) \mapsto \bigotimes^{n+1} \mathcal{L}_P^2(\mathbb{R}^3, \mathcal{K})
\]

\[\psi_1 \circ \ldots \circ \psi_n \mapsto \phi \circ \psi_1 \circ \ldots \circ \psi_n. \quad (4.18)\]

Its formal adjoint is denoted \( A(\phi) \) and we see

\[
(\phi_1 \circ \ldots \circ \phi_{n-1} | A(\phi) \psi_1 \circ \ldots \circ \psi_n) = (\phi \circ \phi_1 \circ \ldots \circ \phi_{n-1} | \psi_1 \circ \ldots \psi_n)
\]

\[= (\phi_{n-1} | \ldots (\phi_1 | (\phi \psi_1) \ldots \psi) \ldots \psi_n). \quad (4.19)\]

As a result we may write the action of the annihilator as

\[
A(\phi) : \bigotimes^n \mathcal{L}_P^2(\mathbb{R}^3, \mathcal{K}) \mapsto \bigotimes^{n-1} \mathcal{L}_P^2(\mathbb{R}^3, \mathcal{K})
\]

\[\psi_1 \circ \ldots \circ \psi_n \mapsto (\phi | \psi_1) \psi_2 \circ \ldots \circ \psi_n. \quad (4.20)\]

To complete our construction of the noise space we need to specify the state; this will just be the expectation in the vacuum state \( \Phi \) given by

\[
\Phi = 1_P \oplus 0 \oplus 0 \oplus 0 \ldots \quad (4.21)
\]

For example, we have the four-point functions

\[
(\Phi | A(\phi_4) A(\phi_3) A^\dagger(\phi_2) A^\dagger(\phi_1) \Phi) = (\phi_3 \circ \phi_4 | \phi_2 \circ \phi_1) = (\phi_3 | \phi_2) * (\phi_4 | \phi_1) \quad (4.22)
\]

and

\[
(\Phi | A(\phi_4) A^\dagger(\phi_3) A(\phi_2) A^\dagger(\phi_1) \Phi) = (\phi_4 | \phi_3) (\phi_2 | \phi_1). \quad (4.23)
\]

The second one is easily computed once one realizes that \( (\Theta | A(\phi) A^\dagger(\psi) \Phi) = (\phi | \psi) (\Theta | \Phi) \) for all \( \Theta \in \Gamma_1(\mathcal{L}_P^2(\mathbb{R}^3, \mathcal{K})) \).

The limit operators \( C^d(g, t) \) are then described mathematically by

\[
C^d(g, t) := A^d(\chi_{[0, t]} \otimes g), \quad (4.24)
\]

with expectation given by \(<.> = (\Phi | \cdot \Phi)\). One easily sees that the correlators, to all orders, are given by this prescription.

5 The Interacting-Free Stochastic Sector of Quantum Field Theory

Define an operator \( A^\dagger(\alpha \otimes f) \), for \( \alpha \in \mathcal{L}_I^2(\mathbb{R}) \) and \( f \in \mathcal{K} \), by

\[
A^\dagger(\alpha \otimes f) := |\alpha > \otimes \int_{-\infty}^{\infty} d\tau \int dk f(k) e^{i\omega(k)\tau} W(-\frac{\tau}{m} k, -k) \otimes a^\dagger(\tau, k) \quad (5.1)
\]
with adjoint

\[ A(\alpha \otimes f) = \langle \alpha | \otimes \int_{-\infty}^{\infty} d\tau \int dk \overline{f}(k)e^{-i\omega(k)\tau}W(\frac{\tau}{m}, k) \otimes \alpha(\tau, k) \]  

(5.2)

where the operators \( \alpha \) satisfy the scaled free relations

\[ \alpha(\tau, k)\alpha(\tau', k') = \delta(\tau' - 2\tau)\delta(k - k'). \]  

(5.3)

Let \( \Phi \) denote the vacuum state

\[ \alpha(\tau, k)\Phi = 0. \]  

(5.4)

The two-point functions are given by

\[ \langle \Phi, A(\alpha_2 \otimes f_2)A^\dagger(\alpha_1 \otimes f_1)\Phi \rangle = \]

\[ \langle \alpha_2 | | \alpha_1 > \int_{-\infty}^{\infty} d\tau_2 \int dk_2 \int_{-\infty}^{\infty} d\tau_1 \int dk_1 \overline{f}_2(k_2)e^{-i\omega(k_2)\tau_2}f_1(k_1)e^{i\omega(k_1)\tau_1} \]

\[ \times W(\frac{\tau_2}{m}, k_2)W(-\frac{\tau_1}{m}, k_1)\delta(k_1 - k_2)\delta(\tau_1 - 2\tau_2) \]

\[ = \langle \alpha_2, \alpha_1 > \int_{-\infty}^{\infty} d\tau \int dk \overline{f}_2(k)e^{-i\omega(k)\tau}f_1(k)e^{i\omega(k)\tau}W(\frac{\tau}{m}, k)W(-\frac{2\tau}{m}, k, -k) \]

\[ = \langle \alpha_2, \alpha_1 > \int_{-\infty}^{\infty} d\tau \int dk \overline{f}_2(k)f_1(k)e^{i\omega(k)\tau}W(\frac{\tau}{m}, k)W(-\frac{2\tau}{m}, k) \]

\[ \equiv \langle \alpha_2, \alpha_1 > (f_2|f_1) \]  

(5.5)

The four point functions are also easily obtained

\[ \langle \Phi, A(\alpha_4 \otimes f_4)A(\alpha_3 \otimes f_3)A^\dagger(\alpha_2 \otimes f_2)A^\dagger(\alpha_1 \otimes f_1)\Phi \rangle = \]

\[ \langle \alpha_4 | | \alpha_3 > \langle \alpha_2 | | \alpha_1 > \int_{-\infty}^{\infty} d\tau_4 \int dk_4 \int_{-\infty}^{\infty} d\tau_1 \int dk_1 \overline{f}_4(k_4)e^{-i\omega(k_4)\tau_4}f_3(k_1)e^{i\omega(k_1)\tau_1} \]

\[ \times \overline{f}_3(k_3)e^{-i\omega(k_3)\tau_3}f_2(k_2)e^{i\omega(k_2)\tau_2} \]

\[ \times W(\frac{\tau_4}{m}, k_4)W(-\frac{\tau_4}{m}, k_4)W(-\frac{\tau_1}{m}, k_1, -k_1)\delta(k_1 - k_4)\delta(\tau_4 - 2\tau_4) \]

\[ = \langle \alpha_4, \alpha_1 > \langle \alpha_3, \alpha_2 > \int_{-\infty}^{\infty} d\tau' \int dk' \int_{-\infty}^{\infty} d\tau \int dk \overline{f}_4(k')f_3(k')f_2(k)e^{i\omega(k)2\tau} \]

\[ \times W(-\frac{\tau}{m}, k)W(-\frac{\tau'}{m}, k,k')e^{i\frac{\omega}{m}|k|}\tau \]

\[ = \langle \alpha_4, \alpha_1 > \langle \alpha_3, \alpha_2 > (f_3|f_2) \times (f_4|f_1). \]  

(5.6)
It is clear that the operators $A^\dagger(\alpha \otimes f)$ defined above reproduce the same correlations as those introduced in the last section.

Acknowledgements: The author is very happy to express gratitude to Prof. Luigi Accardi for many inspiring conversations and for the warm hospitality afforded to him while writing this paper at the Centro Vito Volterra. Thanks also goes to Prof. Yun Gang Lu for giving many insights into theory of Hilbert Modules.

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