Partial sum of matrix entries of representations of the symmetric group and its asymptotics

Dario De Stavola *

20th January 2017

Abstract

Many aspects of the asymptotics of Plancherel distributed partitions have been studied in the past fifty years, in particular the limit shape, the distribution of the longest rows, connections with random matrix theory and characters of the representation matrices of the symmetric group. Regarding the latter, we expand a celebrated result of Kerov on the asymptotic of Plancherel distributed characters by studying partial trace and partial sum of a representation matrix. We decompose these objects into a main term and a reminder, proving a central limit theorem for both main terms and a law of large numbers for the partial sum itself. Our main tool is the expansion of symmetric functions evaluated on Jucys-Murphy elements.

1 Introduction

Let \( \lambda \) be a partition of \( n \), in short \( \lambda \vdash n \), represented as a Young diagram in English notation. A filling of the boxes of \( \lambda \) with numbers from 1 to \( n \), increasing towards the right and downwards, is called a standard Young tableau. We call \( \text{dim} \lambda \) the number of standard Young tableaux of shape \( \lambda \). We fix \( n \) and we associate to each \( \lambda \) the probability \( \frac{(\text{dim} \lambda)^2}{n!} \), which defines the Plancherel measure.

Let us recall briefly three results for the study of the asymptotics of Plancherel distributed random partitions. They relate algebraic combinatorics, representation theory of the symmetric group, combinatorics of permutations, and random matrix theory.

1. The partitions of \( n \) index the irreducible representations of the symmetric group \( S_n \). For each \( \lambda \vdash n \) the dimension of the corresponding irreducible representation is \( \text{dim} \lambda \). A natural question concerns the asymptotics of the associated characters when \( \lambda \) is distributed with the Plancherel measure. A central limit theorem was proved with different techniques by Kerov, \cite{Ker93a, IO02}, and Hora, \cite{Hor98}:

\[
\lim_{n \to \infty} n \left| \rho \right| - m_1(\rho) \hat{\chi}_{\lambda}^{\rho} = \prod_{k \geq 2} k^{m_k(\rho)/2} \mathcal{H}_{m_k(\rho)}(\xi_k).
\]

Here \( \rho \) is a partition of \( n \), \( \hat{\chi}_{\lambda}^{\rho} \) is the renormalized character associated to \( \lambda \) calculated on a permutation of cycle type \( \rho \), \( m_k(\rho) \) is the number of parts of \( \rho \) which are equal to \( k \), \( \mathcal{H}_m(x) \) is the \( m \)-th Hermite polynomial, and \( \{\xi_k\}_{k \geq 2} \) are i.i.d. standard gaussian variables.

2. The Robinson-Schensted-Knuth algorithm allows us to interpret the longest increasing subsequence of a uniform random permutation as the first row of a Plancherel distributed partition. This motivates the study of the shape of a random partition. A limit shape result was proved independently by \cite{KV77} and \cite{LS77}, then extended to a central limit theorem by Kerov, \cite{IO02}. For an extensive introduction on the topic, see \cite{Rom15}.

3. More recently it was proved that, after rescaling, the limiting distribution of the longest \( k \) rows of a Plancherel distributed partition \( \lambda \) coincides with the limit distribution of the properly rescaled \( k \) largest eigenvalues of a random Hermitian matrix taken from the Gaussian Unitary Ensemble. See for example \cite{BOO00} and references therein. Such similarities also occur for fluctuations of linear statistics, see \cite{IO02}.

*email: dario.destavola@math.uzh.ch

Institute of Mathematics, University of Zurich, Winterthurerstrasse 190, 8057, Zurich, Switzerland.

Keywords: Jucys-Murphy elements, symmetric functions, symmetric group.
In the aftermath of Kerov’s result \( \text{(1)} \), a natural step in the study of the characters of the symmetric group is to look at the representation matrix rather than just the trace. We consider thus, for a real valued matrix \( A \) of dimension \( N \) and \( u \in [0, 1] \), the partial trace and partial sum defined, respectively, as

\[
PT_u(A) := \sum_{i \leq uN} \frac{A_{i,i}}{N}, \quad PS_u(A) := \sum_{i,j \leq uN} \frac{A_{i,j}}{N}.
\]

We study these values when \( A = \rho^\lambda(\sigma) \) is a representation matrix of \( S_n \), where \( \lambda \) is a partition of \( n \) and \( \sigma \) is a permutation of \( S_r, r \leq n \). We are interested in the asymptotic of \( PT_u(\rho^\lambda(\sigma)) \) and \( PS_u(\rho^\lambda(\sigma)) \) when \( n \) grows. These partial sums are obviously not invariant by isomorphisms of representations, hence we consider an explicit natural construction of irreducible representations (the Young seminormal representation). We define subpartitions \( \mu_j \) of \( \lambda \), denoted \( \mu_j \not\sim \lambda \), as partitions of \( n - 1 \) obtained from \( \lambda \) by removing one box. The seminormal representation, recalled in Section \( \text{2.3} \), allows a decomposition of the partial trace and partial sum of a representation matrix: we will show that there exist \( j \) and \( u \) such that

\[
PT_u^\lambda(\sigma) := PT_u(\pi^\lambda(\sigma)) = \sum_{j < \lambda} \frac{\dim \mu_j}{\dim \lambda} PT_u^{\mu_j}(\sigma) + \frac{\dim \mu_j}{\dim \lambda} PT_u^{\mu_j}(\sigma),
\]

\[
PS_u^\lambda(\sigma) := PS_u(\pi^\lambda(\sigma)) = \sum_{j < \lambda} \frac{\dim \mu_j}{\dim \lambda} PS_u^{\mu_j}(\sigma) + \frac{\dim \mu_j}{\dim \lambda} PS_u^{\mu_j}(\sigma).
\]

Here \( PT_u^{\mu_j}(\sigma) = \bar{\chi}^{\mu_j}(\sigma) \), while

\[
PS_u^\lambda(\sigma) = \sum_{i \leq \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda} =: TS^\lambda(\sigma)
\]

is the total sum of the matrix \( \rho^\lambda(\sigma) \).

In the first (resp. the second) decomposition we call the first term the main term for the partial trace \( MT_u^\lambda(\sigma) \) (resp. main term for the partial sum \( MS_u^\lambda(\sigma) \)) and the second the remainder for the partial trace \( RT_u^\lambda(\sigma) \) (resp. the remainder for the partial sum \( RS_u^\lambda(\sigma) \)). The decompositions show that the behavior of partial trace and partial sum depend on, respectively, total trace and total sum.

As recalled in Equation \( \text{(1)} \), a central limit theorem for total traces, or characters, is well known; we prove a central limit theorem for the total sum (Theorem \( \text{4.2} \)): to each \( \sigma \in S_r \) we associate the two values

\[
m_\sigma := \mathbb{E}_{PL}[TS^\nu(\sigma)] \quad \text{and} \quad v_\sigma := \left( \frac{r}{2} \right) \mathbb{E}_{PL} \left[ \right. x_{\{2,1,\ldots,1\}} \left. TS^\nu(\sigma) \right],
\]

where \( \mathbb{E}_{PL}[X] \) is the average of the random variable \( X \) considered with the Plancherel measure \( (\dim \nu)^2/r! \) for \( \nu \vdash r \).

**Theorem 1.1.** Fix \( \sigma \in S_r \) and let \( \lambda \vdash n \). Then

\[
\sqrt{n} \cdot (TS^\lambda(\sigma) - m_\sigma) \xrightarrow{d} N(0, 2v_\sigma^2),
\]

where \( N(0, 2v_\sigma^2) \) is a normal random variable of variance \( 2v_\sigma^2 \).

The idea is to show that, for \( \sigma \in S_r \), the total sum \( TS^\lambda(\sigma) \) can be written as linear combination of \( \{x^\lambda(\tau)\}_{\tau \in S_r} \).

When investigating the partial trace, we focus on the main term, and we prove the following theorem:

**Theorem 1.2.** let \( \sigma \) be a permutation of cycle type \( \rho \) and \( u \in [0, 1] \). Let \( \{\xi_k\}_{k \geq 2} \) be a sequence of independent standard Gaussian variables. Then

\[
u \sum_{k \geq 2}^{\text{supp}(\sigma)} \frac{\kappa(\rho)}{\kappa(\rho)} \frac{k_{\rho}}{2} H_{\kappa(\rho)}(\xi_k),
\]

where \( |\text{Supp}(\sigma)| \) is the size of the support of the permutation \( \sigma \).
In short, this theorem states that if we condition the partial trace of a representation matrix by the total trace, the partial trace appears to be deterministic. We actually prove multivariate version of the theorems stated in this introduction. In particular, for Theorem 1.2 the joint distributions of \( n^{-\frac{|\text{Supp}(\sigma)|}{2}} MT^\lambda_n(\sigma) \) will converge to Hermite polynomials of the same Gaussian variables for a family \( \{\sigma_i\} \) of permutations and \( \{u_i\} \) of real numbers. Notice that this result generalizes \( T \), although we use Kerov’s result in the proof.

Informally, the main idea to prove Theorem 1.2 is to show that when \( n \) grows,

\[
\sum_{j < j'} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma) \sim \left( \sum_{j < j'} \frac{\dim \mu_j}{\dim \lambda} \right) \hat{\chi}^{\lambda}(\sigma).
\]

To achieve this result we need to estimate the asymptotic of \( \hat{\chi}^{\lambda}(\sigma) - \hat{\chi}^{\mu}(\sigma) \), for \( \lambda \) distributed with the Plancherel measure and \( \mu \not\sim \lambda \). We consider power sums \( p_\nu \), where \( \nu \) is the cycle type of \( \sigma \) in which we remove 1 to each part, calculated on the multiset of contents \( C_\lambda \). We prove in Section 3 that

\[
p_\nu(C_\lambda) - p_\nu(C_\mu) \in o_P(n^{-\frac{|\text{Supp}(\sigma)|}{2}}) \quad \text{implies} \quad \hat{\chi}^{\lambda}(\sigma) - \hat{\chi}^{\mu}(\sigma) \in o_P(n^{-\frac{|\text{Supp}(\sigma)|}{2}}).
\]

This will be accomplished by introducing modified power sums \( \tilde{p}_\nu \) and by analyzing its highest terms with the appropriate filtration. We consider then \( p_\nu(C_\lambda) - p_\nu(C_\mu) \), which can be estimated by an expansion of the power sums.

We cannot unfortunately prove asymptotic results on the remainder \( RT^\lambda_n(\sigma) \), although we conjecture that

\[
n^{-\frac{|\text{Supp}(\sigma)|}{2}} RT^\lambda_n(\sigma) \xrightarrow{d} 0
\]

for \( u \in [0,1] \). In Section 4 we describe a different conjecture, which would imply the one above, involving quotient of dimensions of irreducible representations. We give some numerical evidence. Our conjecture would imply

\[
n^{-\frac{|\text{Supp}(\sigma)|}{2}} PT^\lambda_n(\sigma) \xrightarrow{d} u \cdot \prod_{k \geq 2} k^{m_k(\rho)/2} H_{m_k(\rho)}(\xi_k).
\]

Regarding the partial sum, our results on the total sum and the main term of the partial sum imply a law of large numbers for the partial sum (Theorem 1.3):

**Theorem 1.3.** Let \( u \in \mathbb{R} \), \( \sigma \in S_r \) and \( \lambda \vdash n \). Then

\[
PS^\lambda_n(\sigma) \xrightarrow{d} u \cdot m_\sigma.
\]

It is easy to see that the reminder for the partial sum goes asymptotically to zero, but we do not know how fast. For the same reasons as above, we cannot thus present a central limit theorem for the partial sum. Nevertheless, we prove a central limit theorem for the main term of the partial sum (Corollary 1.12):

**Theorem 1.4.** Let \( u \in \mathbb{R} \), \( \sigma \in S_r \) and \( \lambda \vdash n \). Then

\[
\sqrt{n} \cdot (MS^\lambda_n(\sigma) - u \cdot m_\sigma) \xrightarrow{d} u \cdot N(0,2\nu^2_u).
\]

As for the partial trace case, we show a multivariate generalization of the previous two theorems.

It is worth mentioning that the partial trace \( PT^\lambda_n(A) \) and the partial sum \( PS^\lambda_n(A) \) have been studied by D’Aristotile, Diaconis and Newman in [DDN03] for the case in which \( A \) is a random matrix of the Gaussian Unitary Ensemble (GUE). The authors showed that in this case both partial trace and partial sum, after normalization, converge to Brownian motion, and thus it has a higher degree of randomness than the partial sum and partial trace of a representation matrix.

In section 2 we recall results on the co-transition measure, the Young seminormal representation, partial permutations, shifted symmetric functions and we introduce the partial trace. In section 3 we prove the result concerning the asymptotics of the main term. In section 4 we study the total and partial sum, while in section 5 we describe a conjecture which would imply a convergence result on the partial trace.
2 Preliminaries

2.1 Notation

Set Λ = (λ₁, λ₂, ...) ⊳ n to be a partition of n, i.e. a nonincreasing sequence of positive integers whose sum is n. We associate partitions with Young diagrams represented in English notation, as in the example below. In this setting a box (in symbol □) is an element (a, b) ∈ ℕ × ℕ, and we write □ = (a, b) ∈ Λ if 1 ≤ a ≤ λₖ. We say that a box □ ∈ Λ is a outer corner if there exists another Young diagram with the same shape as Λ without that box. Likewise, a box □ ∈ Λ is a inner corner if there exists a Young diagram with the same shape of Λ together with □. For a box □ = (a, b) the content is defined as c(□) = c(a, b) = a − b. Of particular interest are the contents of outer and inner corners of Λ, and we call them respectively y_j and x_j, ordered in a way such that x₁ < y₁ < x₂ < ... < y_d < x_d+1, where d is the number of outer corners. For a outer corner box □ such that y_j = c(□) for some j, the partition of n − 1 resulting from removing that box is called μ_j, and we write μ_j ⋄ Λ. We call such a μ_j a subpartition of Λ. Similarly, Λ_j indicates the partition of n + 1 obtained from adding the inner corner box of content x_j.

Example 2.1. In Figure 1 we show three Young diagrams. The first is a partition Λ = (6, 4, 3, 3, 2) ⊳ 18 where we stress out a inner corner of content −2 and a outer corner of content 2. The second is a subpartition corresponding to removing the box □ of Λ, while the third has the same shape of Λ with an additional box corresponding to □

We study the irreducible representation π^Λ(σ) associated to Λ, and we call dim Λ its dimension. It is a known fact that the set

\[ \text{SYT}(Λ) := \{ T : T \text{ is a standard Young tableau of shape } Λ \} \]

indexes a basis for π^Λ, so that dim Λ = |SYT(Λ)|.

Example 2.2. We present the set of standard Young tableaux of shape Λ = (3, 2).

\[
T_1 = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 &
\end{array} \quad T_2 = \begin{array}{ccc}
1 & 2 & 4 \\
3 & 5 &
\end{array} \quad T_3 = \begin{array}{ccc}
1 & 3 & 4 \\
2 & 5 &
\end{array} \quad T_4 = \begin{array}{ccc}
1 & 2 & 3 \\
4 & 5 &
\end{array} \quad T_5 = \begin{array}{ccc}
1 & 3 & 5 \\
2 & 4 &
\end{array} .
\]

By setting PL(Λ) = (dim Λ)^2/n! we endow the set of partitions of n with a probability measure, called the Plancherel measure of size n. Throughout the paper we always assume that Λ is distributed with the Plancherel measure.

2.2 Transition and co-transition measures

Fix Λ ⊳ n. Two identities follow from the branching rule (see for example [Sag13, section 2.8]):

\[
\sum_{μ ⊳ Λ} \dim μ = \dim Λ, \quad \sum_{Λ ⊳ Λ} \dim Λ = (n + 1) \dim Λ.
\]

For v ∈ ℝ we can define, respectively, the normalized co-transition distribution and the normalized transition distribution:

\[
F^Λ_{ct}(v) := \sum_{y_j ≤ \sqrt{v}} \frac{\dim μ_j}{\dim Λ}, \quad F^Λ_{tr}(v) := \sum_{x_j ≤ \sqrt{v}} \frac{\dim Λ_j}{(n + 1) \dim Λ}.
\]
In [Ker93b] Kerov proved that the normalized transition distribution converges almost surely to the semicircular distribution, when $\lambda$ is equipped with the Plancherel measure. Building on Kerov’s work, we prove the same result for the normalized co-transition distribution.

**Lemma 2.3.** When the set of partitions of $n$ is equipped with the Plancherel measure, then almost surely

$$\lim_{n \to +\infty} F^{\lambda}_{ct}(v) = \frac{1}{2\pi} \int_{-2}^{v} \sqrt{4 - t^2} \, dt$$

holds for all $|v| \leq 2$.

**Proof.** We prove the lemma by showing that the Stieltjes transform of the normalized co-transition measure associated to the random variable $\lambda(n)$ converges to the Stieltjes transform of the semicircular distribution. It is showed in [GH03, corollary 1] that pointwise convergence of Stieltjes transforms implies convergence in distribution, which again implies convergence of distribution functions at continuity points (recall that the semicircular distribution function is continuous everywhere). We claim thus that

$$ST^{\lambda(n)}_{ct}(u) := \sum_{\nu_j \not\in \lambda} \frac{\dim \mu_j}{\dim \lambda(n)} \frac{1}{u - y_j} \frac{a_{y_j} u - \sqrt{u^2 - 4}}{2} =: ST_{sc}(u).$$

(6)

In [Ker93b, sections 2 and 4], Kerov showed that the Stieltjes transform of the transition measure converges almost surely to the Stieltjes transform of the semicircular distribution. Moreover, he also showed that, if $Q^\lambda(u) := \prod_j (u - y_j)$ and $P^\lambda(u) := \prod_j (u - x_j)$, then

$$ST^{\lambda(n)}_{tr}(u) = \frac{Q^\lambda(u)}{P^\lambda(u)}, \quad ST^{\lambda(n)}_{ct}(u) = u - \frac{P^\lambda(u)}{Q^\lambda(u)}.$$  

Hence

$$ST^{\lambda(n)}_{ct}(u) = u - \frac{1}{ST^{\lambda(n)}_{tr}(u)} \frac{a_{y_j} u - \sqrt{u^2 - 4}}{2} = \frac{u - \sqrt{u^2 - 4}}{2}.$$

This proves (6) and hence the lemma. 

Set

$$(F^{\lambda}_{ct})^* (u) := \sup \{ z \in \mathbb{R} \text{ s.t. } F^{\lambda}_{ct}(z) \leq u \}.$$

We consider $(F^{\lambda}_{ct})^*$ as the inverse of the step function $F^{\lambda}_{ct}$. We want to show that $F^{\lambda}_{ct}(v^\lambda)$ converges to $u$:

**Lemma 2.4.** For $v^\lambda = (F^{\lambda}_{ct})^* (u)$ for a fixed $u$, then

$$F^{\lambda}_{ct}(v^\lambda) = \sum_{y_j \leq v^\lambda \sqrt{\pi}} \frac{\dim \mu_j}{\dim \lambda} \frac{p_j}{u}. $$

Figure 2: Example of graph of $\tilde{u} = F^{\lambda}_{ct}(v)$. 

In [Ker93b] Kerov proved that the normalized transition distribution converges almost surely to the semicircular distribution, when $\lambda$ is equipped with the Plancherel measure. Building on Kerov’s work, we prove the same result for the normalized co-transition distribution.

**Lemma 2.3.** When the set of partitions of $n$ is equipped with the Plancherel measure, then almost surely

$$\lim_{n \to +\infty} F^{\lambda}_{ct}(v) = \frac{1}{2\pi} \int_{-2}^{v} \sqrt{4 - t^2} \, dt$$

holds for all $|v| \leq 2$.

**Proof.** We prove the lemma by showing that the Stieltjes transform of the normalized co-transition measure associated to the random variable $\lambda(n)$ converges to the Stieltjes transform of the semicircular distribution. It is showed in [GH03, corollary 1] that pointwise convergence of Stieltjes transforms implies convergence in distribution, which again implies convergence of distribution functions at continuity points (recall that the semicircular distribution function is continuous everywhere). We claim thus that

$$ST^{\lambda(n)}_{ct}(u) := \sum_{\nu_j \not\in \lambda} \frac{\dim \mu_j}{\dim \lambda(n)} \frac{1}{u - y_j} \frac{a_{y_j} u - \sqrt{u^2 - 4}}{2} =: ST_{sc}(u).$$

(6)

In [Ker93b, sections 2 and 4], Kerov showed that the Stieltjes transform of the transition measure converges almost surely to the Stieltjes transform of the semicircular distribution. Moreover, he also showed that, if $Q^\lambda(u) := \prod_j (u - y_j)$ and $P^\lambda(u) := \prod_j (u - x_j)$, then

$$ST^{\lambda(n)}_{tr}(u) = \frac{Q^\lambda(u)}{P^\lambda(u)}, \quad ST^{\lambda(n)}_{ct}(u) = u - \frac{P^\lambda(u)}{Q^\lambda(u)}.$$  

Hence

$$ST^{\lambda(n)}_{ct}(u) = u - \frac{1}{ST^{\lambda(n)}_{tr}(u)} \frac{a_{y_j} u - \sqrt{u^2 - 4}}{2} = \frac{u - \sqrt{u^2 - 4}}{2}.$$

This proves (6) and hence the lemma. 

Set

$$(F^{\lambda}_{ct})^* (u) := \sup \{ z \in \mathbb{R} \text{ s.t. } F^{\lambda}_{ct}(z) \leq u \}.$$

We consider $(F^{\lambda}_{ct})^*$ as the inverse of the step function $F^{\lambda}_{ct}$. We want to show that $F^{\lambda}_{ct}(v^\lambda)$ converges to $u$:

**Lemma 2.4.** For $v^\lambda = (F^{\lambda}_{ct})^* (u)$ for a fixed $u$, then

$$F^{\lambda}_{ct}(v^\lambda) = \sum_{y_j \leq v^\lambda \sqrt{\pi}} \frac{\dim \mu_j}{\dim \lambda} \frac{p_j}{u}. $$
Proof. We show in Figure 2 an example of a normalized co-transition distribution. Since $F^\lambda_{ct}(v)$ is right continuous then $u \leq F^\lambda_{ct}(v)$. Moreover, since the co-transition distribution is a step function, for each element of the image $\tilde{u} \in F^\lambda_{ct}(R)$ there exists $j$ such that $F^\lambda_{ct}(\frac{\tilde{u}}{\sqrt{n}}) = \tilde{u}$. Call $j$ the index corresponding to $F^\lambda_{ct}(v)$, that is, $F^\lambda_{ct}(\frac{\tilde{u}}{\sqrt{n}}) = F^\lambda_{ct}(v)$. Hence $\sum_{j<k} \dim \mu_j / \dim \lambda \leq u$. Thus

- $\nu^\lambda \leq \frac{\tilde{u}}{\sqrt{n}}$, since $F^\lambda_{ct}(\frac{\tilde{u}}{\sqrt{n}}) \geq u$;
- $\nu^\lambda \geq \frac{\tilde{u}}{\sqrt{n}}$, since for each $\epsilon > 0$, $F^\lambda_{ct}(\frac{\tilde{u}}{\sqrt{n}} - \epsilon) \leq u$.

Thus $\nu^\lambda = \frac{\tilde{u}}{\sqrt{n}}$. Therefore

$$\sum_{j<k} \frac{\dim \mu_j}{\dim \lambda} = F^\lambda_{ct}(\nu^\lambda) - \frac{\dim \mu_j}{\dim \lambda} \leq u \leq F^\lambda_{ct}(\nu^\lambda).$$

Equivalently

$$-\frac{\dim \mu_j}{\dim \lambda} \leq u - F^\lambda_{ct}(\nu^\lambda) \leq 0,$$

and $\max \frac{\dim \mu_j}{\dim \lambda} \to 0$ because of the convergence of the normalized co-transition distribution towards an atom free distribution proved in the previous lemma. This proves the statement. \qed

### 2.3 Young seminormal representation and last letter order

We recall the definition of Young seminormal representation, see [You77] for the original introduction, and [Gre92] for a more modern description. We need three preliminary definitions; set $\lambda \vdash n$, then

1. given a box $\square \in \lambda$ recall that the content $c(\square)$ is the difference between the row index and the column index of the box. For $k \leq n$ we also write $c_k(T)$ for the content of the box containing the number $k$ in the tableau $T$. As instance, in the tableau $T_1$ of Example 2.2 $c_1(T_1) = -1$, $c_1(T_1) = c_2(T_1) = 0$, $c_2(T_1) = 1$ and $c_3(T_1) = 2$.

2. For $k \leq n - 1$ the signed distance between $k$ and $k+1$ in the tableau $T$ is $d_k(T) := c_k(T) - c_{k+1}(T)$. For example $d_1(T_1) = d_2(T_1) = d_1(T_1) = -1$; $d_1(T_1) = 3$.

3. If $k$ and $k+1$ are in different columns and rows we define $(k, k+1)T$ as the standard Young tableau equal to $T$ but with the boxes containing $k$ and $k+1$ inverted. In the previous example $(3, 4)T_1 = T_2$, while $(2, 3)T_1$ is not defined.

**Definition 2.5.** The Young seminormal representation defines the matrix associated to $\pi^\lambda((k, k+1))$ entrywise in the following way: if $T$ and $S$ are standard Young tableaux of shape $\lambda$, then

$$\pi^\lambda((k, k+1))_{T,S} = \begin{cases} 1/d_k(T) & \text{if } T = S; \\ \sqrt{1 - \frac{1}{d_k(T)}}/d_k(T) & \text{if } (k, k+1)T = S; \\ 0 & \text{else.} \end{cases}$$

Notice that the adjacent transpositions $(k, k+1)$, $k = 1, \ldots, n-1$, generate the group $S_n$, hence $\pi^\lambda(\sigma)$ is well defined for all $\sigma \in S_n$.

**Example 2.6.** Consider $\lambda = (3, 2)$, $\sigma = (2, 4, 3)$. We compute $\pi^\lambda((2, 4, 3))$

$$\pi^\lambda((2, 4, 3)) = \pi^\lambda((3, 4)(2, 3)) = \pi^\lambda((3, 4))\pi^\lambda((2, 3))$$

$$= \begin{bmatrix} 1/3 & 0 & 0 & 0 \\ 0 & 1/3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1/2 & \sqrt{3}/4 & 0 \\ 0 & \sqrt{3}/4 & 1/2 & 0 \\ 0 & 0 & 0 & \sqrt{3}/4 \end{bmatrix}$$

$$= \begin{bmatrix} -1/3 & -\sqrt{2}/9 & 0 & 0 \\ \sqrt{2}/9 & -1/6 & 0 & 0 \\ 0 & \sqrt{3}/4 & 1/2 & 0 \\ 0 & 0 & 0 & -1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \sqrt{3}/4 \end{bmatrix}$$
We consider now the second sum of the RHS: we have

**Proof.**

When \( u \) lies in a row in \( T \) which is lower that \( S \); if \( u \) lies in the same row in both tableaux, then we look at the rows containing \( n - 1 \), and so on. Notice that in the list of tableaux of shape \((3, 2)\) of Example 2.2 the tableaux are last letter ordered; also, the entries of the matrices in Example 2.6 are ordered accordingly. We write \( \lambda^\lambda(\pi)_{i,j} \) instead of \( \lambda^\lambda(\sigma)_{T_i, T_j} \), where \( T_i \) is the \( i \)-th tableau of shape \( \lambda \) in the last letter order.

### 2.4 The partial trace and its main term

Let \( \lambda \vdash n \) and \( \sigma \in S_r \) a permutation of the set \( \{1, \ldots, r\} \) with \( r \leq n \). We see \( S_r \) as a subgroup of \( S_n \) by adding fixed points, so that \( \lambda^\lambda(\sigma) \) is well defined. Similarly, if \( \rho \vdash r \leq n \), then we define \( \chi^\rho(\pi) = \chi^\rho_{\rho,j1n-r} \) and \( \hat{\chi}^\lambda(\pi) = \chi^\lambda_{\rho,j1n-r} \).

**Definition 2.7.** Let \( \lambda \vdash n, \sigma \in S_r, u \in [0, 1] \). Define the partial trace as

\[
PT_u^\lambda(\pi) := \sum_{\substack{i < u \dim \lambda \leq \dim \lambda \}} \pi^\lambda(\sigma)_{i,i}.
\]

When \( u = 1 \) then the partial trace correspond to the normalized trace

\[
\hat{\chi}^\lambda(\sigma) = \chi^\lambda(\sigma) \dim \lambda = \sum_{i \leq \dim \lambda} \pi^\lambda(\sigma)_{i,i}.
\]

Our choice of order on the tableaux implies that, as long as \( \sigma \in S_r \) and \( r \leq n - 1 \),

\[
\pi^\lambda(\sigma) = \begin{bmatrix}
\pi^{\mu_1}(\sigma) & 0 & 0 & \cdots \\
0 & \pi^{\mu_2}(\sigma) & 0 & \\
0 & 0 & \pi^{\mu_3}(\sigma) & \\
\vdots & \ddots & \ddots & \\
\end{bmatrix}.
\]

(7)

Hence \( \hat{\chi}^\lambda(\sigma) = \sum_j \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma) \). This decomposition of the total trace can be easily generalized to the partial trace:

**Proposition 2.8 (Decomposition of the partial trace).** Fix \( u \in [0, 1], \lambda \vdash n \) and \( \sigma \in S_r \) with \( r \leq n - 1 \). Set \( (F^\lambda)^*(u) = \frac{\chi^\lambda}{\chi^\lambda} = u^j \) for some \( j \) as in Lemma 2.4. Define

\[
\bar{u} = \frac{\dim \lambda}{\dim \mu_j} \left( u - \sum_{j < j} \dim \mu_j \right) < 1,
\]

then

\[
PT_u^\lambda(\sigma) = \sum_{j < j} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma) + \frac{\dim \mu_j}{\dim \lambda} \chi^\mu_j(\sigma) PT_u^\mu_j(\sigma). 
\]

(8)

**Proof.** We can decompose the partial trace as

\[
PT_u^\lambda(\sigma) = \sum_{i \leq u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda} = \sum_{i \leq \sum_{j < j} \dim \mu_j} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda} + \sum_{\sum_{j < j} \dim \mu_j < u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda}.
\]

The first sum in the RHS is

\[
\sum_{i \leq \sum_{j < j} \dim \mu_j} \frac{\pi^\lambda(\sigma)_{i,i}}{\dim \lambda} = \sum_{j < j} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\sigma).
\]

We consider now the second sum of the RHS: we have \( u \leq \sum_{j < j} \frac{\dim \mu_j}{\dim \lambda} \) by the definition of \( j \), thus

\[
\sum_{j < j} \dim \mu_j < i \leq u \dim \lambda \leq \sum_{j < j} \dim \mu_j.
\]
Hence \( \pi^{\lambda}(\sigma)_{i,i} = \pi^{\mu}(\sigma)_{i,i} \), where \( i = \sum_{j < \tilde{i}} \dim \mu_j \), so that

\[
0 < \tilde{i} \leq u \dim \lambda - \sum_{j < \tilde{i}} \dim \mu_j = \tilde{u} \dim \mu_j.
\]

Therefore

\[
\sum_{j < \tilde{i}} \dim \mu_j < \leq u \dim \lambda \frac{\pi^{\lambda}(\sigma)_{i,i}}{\dim \lambda} = \sum_{\tilde{i} \leq \tilde{u} \dim \mu_j} \frac{\pi^{\lambda}(\sigma)_{i,i}}{\dim \lambda} = \frac{\dim \mu_j}{\dim \lambda} \PT^{\mu_j}_{\tilde{u}}(\sigma),
\]

and the proposition is proved. \( \square \)

We define \( \sum_{y_j \leq u^k} \sqrt{\dim \mu_j} \lambda^{\mu_j}(\sigma) \) to be the main term for the partial trace (denoted \( \MT^{\lambda}(\sigma) \)), while \( \dim \mu J^{\mu_j}(\sigma) \) is called the remainder (denoted \( \RT^{\lambda}(\sigma) \)). The main goal of this paper is to establish the asymptotic behavior of the main term for the partial trace. We settle some notation: if a permutation \( \sigma \) has cycle type \( \sigma \) and the proposition is proved. equivalently, for the number of cycles of \( \sigma \), we define the weights of \( \sigma \) and \( \rho \) as \( w(\sigma) := w(\rho) := |\text{Supp}(\sigma)| = |\rho| - m_1(\rho) \). We recall now a result due to \( \text{[Ker93a]} \), with a complete proof given by [1002] and, independently, by [1052].

**Theorem 2.9.** Fix \( \rho_1 \vdash r_1, \rho_2 \vdash r_2, \ldots \). The asymptotic behavior of the irreducible character \( \chi^{\lambda} \) is given by:

\[
\left\{ u \frac{\pi^{\lambda}(\sigma)}{\dim \lambda} \chi^{\lambda} \right\} \xrightarrow{d} \left\{ \prod_{k \geq 2} k^{m_k(\rho)}/2\mathcal{H}_{m_k(\rho)}(\xi_k) \right\},
\]

where \( \{\xi_k\}_{k \geq 2} \) is a sequence of independent standard Gaussian random variables, and \( \mathcal{H}_m(x) \), \( m \geq 1 \), is the Hermite polynomial of degree \( m \) defined by the recurrence relation \( x\mathcal{H}_m(x) = \mathcal{H}_{m+1}(x) + m\mathcal{H}_{m-1}(x) \) and initial data \( \mathcal{H}_0(x) = 1 \) and \( \mathcal{H}_1(x) = x \).

Our result is the following:

**Theorem 2.10.** Let \( \sigma_1, \sigma_2, \ldots \) be permutations of cycle type respectively \( \rho_1, \rho_2, \ldots \) and \( u_1, u_2, \ldots \in [0,1] \). Let \( \{\xi_k\}_{k \geq 2} \) be a sequence of independent standard Gaussian variables. Then

\[
\left\{ u \frac{\pi^{\lambda}(\sigma)}{\dim \lambda} \chi^{\lambda} \right\} \xrightarrow{d} \left\{ u_1 \prod_{k \geq 2} k^{m_k(\rho)}/2\mathcal{H}_{m_k(\rho)}(\xi_k) \right\}.
\]

Notice that the procedure of Proposition 2.8 can be iterated: consider \( \sigma \in S_r \) and \( s \in \mathbb{N} \) such that \( n - r > s \). Set \( \mu^{(s)} := \lambda \) and \( \mu^{(s-1)} := \mu \), where \( \mu \) is defined according to Proposition 2.8. Similarly set \( u^{(s)} := u \) and \( u^{(s-1)} := u \), then we can rewrite (8) as

\[
\PT^{\mu_{(s)}}_{(s)}(\sigma) = \MT^{\mu_{(s)}}_{u_{(s)}}(\sigma) + \frac{\dim \mu^{(s-1)}}{\dim \lambda} \PT^{\mu_{(s-1)}}_{u_{(s)}(s-1)}(\sigma).
\]

Applying again Proposition 2.8 to the second term of the right hand side, we see that there exists \( \mu^{(s-2)} \not\succ \mu^{(s-1)} \) and \( u^{(s-2)} \in [0,1] \) such that

\[
\PT^{\mu_{(s)}}_{u_{(s)}(s)}(\sigma) + \frac{\dim \mu^{(s-1)}}{\dim \lambda} \PT^{\mu_{(s-1)}}_{u_{(s)}(s-1)}(\sigma) + \frac{\dim \mu^{(s-2)}}{\dim \lambda} \PT^{\mu_{(s-2)}}_{u_{(s)}(s-2)}(\sigma).
\]

By iterating we obtain the following proposition:

**Proposition 2.11.** Let \( \sigma \in S_r \). Set \( s \) such that \( n - s > r \), then there exists a sequence of partitions \( \mu^{(0)} \not\succ \mu^{(1)} \not\succ \ldots \not\succ \mu^{(s)} = \lambda \) and a sequence of real numbers \( 0 < u_0, \ldots, u_s < 1 \) such that

\[
\PT^{\mu_{(s)}}_{u_{(s)}}(\sigma) = \sum_{i=1}^s \frac{\dim \mu^{(i)}}{\dim \lambda} \PT^{\mu_{(i)}}_{u_{(i)}(i)}(\sigma) + \frac{\dim \mu^{(0)}}{\dim \lambda} \PT^{\mu_{(0)}}_{u_{(0)}}(\sigma).
\]
2.5 Partial permutations

In this section we recall some results in the theory of partial permutations, introduced in [IK99]. All the definitions and results in this section can be found in [Fér12].

**Definition 2.12.** A partial permutation is a pair $(\sigma, d)$, where $d \subset \mathbb{N}$ is finite and $\sigma$ is a bijection $d \rightarrow d$. We call $P_n$ the set of partial permutations $(\sigma, d)$ such that $d \subseteq \{1, \ldots, n\}$. The set $P_n$ is endowed with the operation $(\sigma, d) \cdot (\sigma', d') = (\hat{\sigma} \cdot \sigma', d \cup d')$, where $\hat{\sigma}$ is the bijection from $d \cup d'$ to itself defined by $\hat{\sigma}|_d = \sigma$ and $\hat{\sigma}|_{d'} = \text{Id}$; the same holds for $\hat{\sigma}$.

Define the algebra $B_n = \mathbb{C}[P_n]$. There is an action of the symmetric group $S_n$ on $P_n$ defined by $\tau \cdot (\sigma, d) := (\tau \sigma \tau^{-1}, \tau(d))$, and we call $A_n$ the abelian subalgebra of $B_n$ of the invariant elements under this action. Set

$$
\alpha_{\rho, n} := \sum_{d \subseteq \{1, \ldots, n\}, \rho^d \vdash d} (\sigma, d),
$$

where $\rho^d \vdash d$ is a partition of the multiset of cycle type $d$.

**Proposition 2.13.** The family $(\alpha_{\rho, n})_{0 \leq \rho \leq n}$ forms a linear basis for $A_n$.

There is a natural projection $B_{n+1} \rightarrow B_n$ which sends to 0 the partial permutations whose support contains $n + 1$. The projective limit through this projection is called $B_\infty := \varprojlim B_n$, and similarly $A_\infty := \varprojlim A_n$. The family $\alpha_\rho := (\alpha_{\rho, n})_{n \geq 1}$ forms a linear basis of $A_\infty$.

We recall the definition of Jucys-Murphy element, described for example in [Juc66] and [Mur81], and its generalization as a partial permutation.

**Definition 2.14.** The $i$-th Jucys-Murphy element is the element of the group algebra $\mathbb{C}[S_n]$ defined by $J_i := (1, i) + (2, i) + \ldots + (i - 1, i)$, $J_1 = 0$.

The partial Jucys-Murphy element $\xi_i$ is the element of $B_n$ defined by $\xi_i := \sum_{j \leq i} ((j, i), \{j, i\})$, $\xi_1 := 0$.

**Proposition 2.15.** If $f$ is a symmetric function, $f(\xi_1, \ldots, \xi_n) \in A_n$. The sequence $f_n = f(\xi_1, \ldots, \xi_n)$ is an element of the projective limit $A_\infty$, that we denote $\Gamma_f$; moreover, $f \mapsto \Gamma_f$ is an algebra morphism. We define $\Xi$ as the projective limit of the sequence $(\xi_1, \ldots, \xi_n)$ for $n \rightarrow \infty$; equivalently, $\Xi$ is the unique element of $\bigcup_{n=1}^{\infty} \mathbb{C}[S_n]$ such that $f(\Xi) := \Gamma_f$.

For a partition $\nu = (\nu_1, \ldots, \nu_q)$ and $n \geq |\nu|$ we consider $p_\nu(\xi_1, \ldots, \xi_n) = \prod_{i=1}^{q} (\xi_{\nu_1} + \ldots + \xi_{\nu_i})$. Then Féray proved in [Fér12] Proposition 2.5 and proof that

$$
p_\nu(\Xi) = \prod_i m_i(\nu)! \alpha_{\nu+1, i} + \sum_{|\rho|<|\nu|+q} c_\rho \alpha_\rho,
$$

with $n + 1 = (\nu_1 + 1, \ldots, \nu_q + 1)$ and $c_\rho$ non-negative integers.

Let us fix some notation: we write $n^{\downarrow k}$ for the $k$-th falling factorial, that is, $n(n - 1) \cdots (n - k + 1)$, and $z_\rho = \prod_i \rho_i \prod_i m_i(\rho)!$ for the order of the centralizer of a permutation of type $\rho$. We call

$$
K_{\rho \downarrow n-|\rho|} = \frac{1}{\#\{\sigma \text{ of cycle type } \rho \downarrow n-|\rho|\}} \sum \sigma \in \mathbb{C}[S_n],
$$

where the sum runs over the permutations $\sigma \in S_n$ of cycle type $\rho \downarrow n-|\rho|$. We consider the morphism of algebras $\varphi_n : A_n \rightarrow \mathbb{Z}(\mathbb{C}[S_n])$ which sends $(\sigma, d)$ to $\sigma$. On the basis $(\alpha_{\rho, n})$ it acts as follows:

$$
\varphi_n(\alpha_{\rho, n}) = \frac{n^{\downarrow |\rho|}}{z_\rho} K_{\rho \downarrow n-|\rho|} \alpha_{\rho, n}.
$$

In particular

$$
\hat{\lambda}^{\hat{\lambda}}(\varphi_n(\alpha_{\rho, n})) = \frac{1}{z_\rho} \lambda^{\downarrow |\rho|} \hat{\lambda}^{\hat{\lambda}}. \hat{\lambda}^{\hat{\lambda}}(\varphi_n(f(\xi_1, \ldots, \xi_n))) = f(\mathcal{C}_\lambda),
$$

where $f$ is a symmetric function and $\mathcal{C}_\lambda$ is the multisets of contents of the diagram $\lambda$, $\mathcal{C}_\lambda = \{c(\square), \square \in \lambda\}$. 

2.6 Shifted symmetric functions

We recall some results on the algebra of shifted symmetric functions \( \Lambda^* \) and their relations with \( A_\infty \). The results and definitions in this section follow [IK99].

**Definition 2.16.** Let \( \Lambda^*(n) \) be the algebra of polynomials with complex coefficients in \( x_1, \ldots, x_n \) that become symmetric in the new variables \( x'_i = x_i - i \). There is a natural projection \( \Lambda^*(n+1) \to \Lambda^*(n) \) which sends \( x_{n+1} \to 0 \). The algebra of shifted symmetric functions \( \Lambda^* \) is defined as the projective limit of \( \Lambda^*(n) \) according to this projection.

We can apply (shifted) symmetric functions to partitions in the following way: if \( f \) is a (shifted) symmetric function and \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \) a partition, then we set \( f(\lambda) = f(\lambda_1, \lambda_2, \ldots, \lambda_l) \).

As in the symmetric functions algebra, there are several interesting bases for the algebra of shifted symmetric functions. We present one of them:

**Proposition 2.17.** There exists a family of shifted symmetric functions \( \{ p^*_\rho \}_\rho \) such that, for each \( \lambda \vdash n \),

\[
p^*_\rho(\lambda) = \begin{cases} n^{\lambda_1} x^{\lambda_2} \cdots \lambda_{n-r} & \text{if } n \geq r; \\
0 & \text{otherwise.}
\end{cases}
\]

Moreover, the family \( \{ p^*_\rho \}_\rho \), where \( \rho \) runs over all partitions, forms a basis for \( \Lambda^* \).

These functions are called shifted power sums; they were introduced in [OO98, Section 15].

In order to simplify the notation we write \( p^*_k \), where \( k \) is a positive integer, instead of \( p^*_\{(k)\} \). Similarly, \( p^*_k \) replaces \( p^*_\{(k)\} \). As polynomials in the variables \( x_1, \ldots, x_n \), the function \( p^*_\rho \) have degree \( \deg(p^*_\rho(x_1, \ldots, x_n, \ldots)) = |\rho| \). We are interested in two filtrations of the algebra \( \Lambda^* \), which are described in [IK99, section 10];

- \( \deg(p^*_\rho)_1 := |\rho|_1 = |\rho| + m_1(\rho) \). This filtration is usually referred as the Kerov filtration;
- \( \deg(p^*_\rho)_N := |\rho|_N = |\rho| + l(\rho) \), where \( l(\rho) = \sum_{i \in \mathbb{N}} m_i(\rho) \) is the number of parts of \( \rho \).

**Lemma 2.18.** There is an isomorphism \( F : A_\infty \to \Lambda^* \) that sends \( \alpha_\rho \) to \( p^*_\rho / \pi_\rho \).

Our goal in this section is to develop [IO02, Proposition 4.12], which gives information about the top degree term of \( p^*_\rho \) for partitions \( \rho \) and \( \theta \), where top degree refers to the Kerov filtration. Consequently, we obtain results on the product \( \alpha_\rho \cdot \alpha_\theta \) (Corollary 2.23).

We will write \( V^<_k \) for the vector space whose basis is \( \{ p^*_\rho \}_\rho \) such that \( |\rho|_1 < k \). Notice that since \( \deg(\cdot)_1 \) is a filtration,

\[
p^*_\rho \cdot V^<_k \subseteq V^<_1^{\rho_1 + k}.
\]

By abuse of notation we will often write \( p^*_\rho + V^<_1 \) to indicate \( p^*_\rho \) plus a linear combination of elements with \( \deg(\cdot)_1 < k \). We recall a result of Ivanov and Olsianski [IO02, Proposition 4.12]:

**Lemma 2.19.** For any partition \( \rho \) and \( k \geq 2 \),

\[
p^*_\rho p^*_k = p^*_\rho \cdot k \cdot m_k(\rho) \cdot p^*_\{(\rho \setminus k)\cup 1 \}_k + V^<_1 V^<_1^{\rho_1 + k},
\]

where the partition \( \rho \setminus k \) is obtained by removing one part equal to \( k \).

**Lemma 2.20.** For a partition \( \rho \),

\[
p^*_\rho p^*_1 = p^*_\rho + V^<_1 V^<_1^{\rho_1 + 2k}.
\]

**Proof.** We prove the statement by induction on \( k \), the case \( k = 1 \) being proved in [IO02, Proposition 4.11]. This implies that for any \( k \)

\[
p^*_\rho p^*_1 = p^*_\rho + V^<_1 V^<_1^{\rho_1 + 2k}.
\]

Suppose the lemma true for \( k > 1 \), then

\[
p^*_\rho p^*_1 = p^*_\rho p^*_1 p^*_1 + p^*_\rho V^<_1 V^<_1^{\rho_1 + 2k} = p^*_\rho p^*_1 V^<_1 V^<_1^{\rho_1 + 2k}.
\]

where we repetitively used property [11].
We substitute the previous expression in (15) which concludes the proof.

Hence, by applying again Lemma 2.19 where the restriction |\( C \)| holds:

**Proposition 2.22.** Let \( \rho, \theta \) be partitions. Then

\[
p^\rho \cdot \tilde{V}^\theta = p^\rho \cdot \tilde{V}^\theta + \tilde{V} = V^\theta + V^\rho + V^\rho \cdot \tilde{V}^\theta.
\]

**Proposition 2.21.** For a positive integer \( k \) and a partition \( \rho \)

\[
p^\rho \cdot \tilde{V}^\theta = V^\rho_{|\rho| + k} + V^\rho_{|\rho| + k}.
\]

**Proof.** It is enough to show that, for a partition \( \theta \) such that \( |\theta|_1 = k \) and \( m_1(\theta) > 0 \),

\[
p^\rho \cdot p^\rho \subseteq V^\rho_{|\rho| + 1} + V^\rho_{|\rho| + 1}.
\]

Set \( \theta = \tilde{\rho} \cup 1 \). By the previous lemma \( p^\rho = p^\rho \cdot p^\rho + V^\rho_{|\rho| + 2} \) and

\[
p^\rho \cdot p^\rho = p^\rho \cdot p^\rho + V^\rho_{|\rho| + 1} + V^\rho_{|\rho| + 1}.
\]

Set \( C_{1,\theta}^\rho \) to be the structure constants in the basis \( \{p^\rho\} \) of the product \( p^\rho \cdot p^\rho \), that is,

\[
p^\rho \cdot p^\rho = \sum_{|\rho| \leq |\rho| + |\theta|} C_{1,\theta}^\rho p^\rho,
\]

where the restriction \(|\rho| \leq |\rho| + |\theta|\) is a consequence of (11). Thus

\[
p^\rho \cdot p^\rho = \sum_{|\rho| \leq |\rho| + |\theta|} C_{1,\theta}^\rho p^\rho + V^\rho_{|\rho| + \theta} + V^\rho_{|\rho| + \theta}.
\]

This proves (14) and hence the lemma.

**Proposition 2.22.** Let \( \rho, \theta \) be partitions. Then

\[
p^\rho \cdot p^\rho = p^\rho \cdot p^\rho + \tilde{V} = V^\rho_{|\rho| + |\theta|} + V^\rho_{|\rho| + |\theta|} + V^\rho_{|\rho| + |\theta|}.
\]

**Proof.** We prove the statement by induction on the number \( l(\theta) - m_1(\theta) \) of non-one parts of \( \theta \), with the initial case being \( \theta = 1 \).

Consider now the claim true for \( p^\rho \cdot p^\rho \) and set \( \theta = \theta \cup k \) for \( k \geq 2 \). Then by Lemma 2.19

\[
p^\rho \cdot p^\rho = p^\rho \cdot p^\rho + \tilde{V} = V^\rho_{|\rho| + |\theta|} + V^\rho_{|\rho| + |\theta|} + V^\rho_{|\rho| + |\theta|}.
\]

Because of Property (11) and Lemma (2.22),

\[
p^\rho \cdot p^\rho = p^\rho \cdot p^\rho + \tilde{V} = V^\rho_{|\rho| + |\theta|} + V^\rho_{|\rho| + |\theta|} + V^\rho_{|\rho| + |\theta|}.
\]

By the inductive step

\[
p^\rho \cdot p^\rho = p^\rho \cdot p^\rho + \tilde{V} = V^\rho_{|\rho| + |\theta|} + V^\rho_{|\rho| + |\theta|} + V^\rho_{|\rho| + |\theta|}.
\]

Hence, by applying again Lemma 2.19

\[
p^\rho \cdot p^\rho = p^\rho \cdot p^\rho + \tilde{V} = V^\rho_{|\rho| + |\theta|} + V^\rho_{|\rho| + |\theta|} + V^\rho_{|\rho| + |\theta|}.
\]

We substitute the previous expression in (14) which concludes the proof.

We can obtain a similar result in the algebra \( \mathcal{A}_\infty \) by applying the isomorphism \( F^{-1} \) defined in 2.18.

It is easy to see that \( F^{-1}(\tilde{V}^\theta) \) is the space of linear combinations of \( \alpha_\rho \) such that \( |\rho|_1 = k \) and \( m_1(\rho) > 0 \). Similarly \( F^{-1}(V^\rho) \) is the space of linear combinations of \( \alpha_\rho \) such that \( |\rho|_1 < k \).

**Corollary 2.23.** Let \( \rho, \theta \) be partitions. Then

\[
\alpha_\rho \cdot \alpha_\theta = \alpha_{\rho+\theta} + F^{-1}(\tilde{V}^\theta) + F^{-1}(V^\rho).
\]
3 Asymptotic of the main term for the partial trace

In this section we prove Theorem 2.10. The main step is to prove that $\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu \in o_P(n^{-\frac{\omega(\rho)}{\ell(\rho)}})$, where $\rho$ is a fixed partition, $\omega(\rho) = |\rho| - m_1(\rho)$, $\lambda$ is a partition of $n$ and $\mu \not\supset \lambda$.

For a partition $\nu = (\nu_1, \ldots, \nu_q)$ we define slightly modified power sums $\tilde{p}_\nu$; we will show (Lemma 3.3 and Equation (10)) that

$$\hat{\chi}_\rho^\lambda(\varphi_n(\xi_1, \ldots, \xi_n)) - \hat{\chi}_\rho^\mu(\varphi_n(\xi_1, \ldots, \xi_n)) = \tilde{p}_\nu(\mathcal{C}_\lambda) - \tilde{p}_\nu(\mathcal{C}_\mu) \in o_P(n^{-\frac{\omega(\rho)}{\ell(\rho)}}).$$

In order to translate this result on a bound on $\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu$ we need to study the expansion of the modified power sums evaluated on $\Xi$, the infinite set of Jucys-Murphy elements in the theory of partial permutations:

$$\tilde{p}_\nu(\xi_1, \ldots, \xi_n) = \sum_\varsigma c_\varsigma \alpha_{\varsigma;n}.$$

With the right choice of $\nu$ and the right filtration we will prove (Proposition 3.6) that

$$\tilde{p}_\nu(\mathcal{C}_\lambda) - \tilde{p}_\nu(\mathcal{C}_\mu) = \hat{\chi}_\rho^\lambda \circ \varphi_n \left( \sum_\varsigma c_\varsigma \alpha_{\varsigma;n} \right) - \hat{\chi}_\rho^\mu \circ \varphi_n^{-1} \left( \sum_\varsigma c_\varsigma \alpha_{\varsigma;n-1} \right) = \hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu + o_P(n^{-\frac{\omega(\rho)}{\ell(\rho)}}).$$

Comparing (16) and (17), this gives

$$\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu \in o_P(n^{-\frac{\omega(\rho)}{\ell(\rho)}}).$$

Remark 3.1. It is easy to see that $\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu \in o_P(n^{-\frac{\omega(\rho)}{\ell(\rho)}})$, where $\ell(\rho)$ is the number of parts of $\rho$; for example, considering balanced diagrams, in [DFS11] the authors prove an explicit formula for the normalized character, which implies $\hat{\chi}_\rho^\lambda - \hat{\chi}_\rho^\mu \in o_P(n^{-\frac{\omega(\rho)}{\ell(\rho)}})$. Alternatively one can look at the descriptions of normalized characters expressed as polynomials in terms of free cumulants, studied by Biane in [Bia03]. The action of removing a box from a random partition $\lambda$ affects free cumulants in a sense described in [DFS10], which furnish the aforementioned bound. On the other hand we needed a stronger result, namely Proposition 3.6 and for this reason we introduce the modified power sums.

Through the section, $\sigma$ will be a fixed permutation, $\rho$ its cycle type, and we consider $\nu = (\nu_1, \ldots, \nu_q)$, such that $\rho = \nu + \underline{1} = (\nu_1 + 1, \ldots, \nu_q + 1, 1 \ldots, 1)$.

3.1 Modified power sums

Definition 3.2. For a partition $\nu = (\nu_1, \ldots, \nu_q)$ the modified power sum is

$$\tilde{p}_\nu(x_1, x_2, \ldots) = \prod_{i=1}^q \left( p_{\nu_i}(x_1, x_2, \ldots) - \text{Cat} \left( \frac{\nu_i}{2} \right) \cdot \left( \frac{\nu_i}{2} \right)! \cdot \alpha_{(\frac{\nu_i}{2} + 1)} \right),$$

where $\text{Cat}(k) = \binom{2k}{k} \frac{1}{k+1}$ is the $k$-th Catalan number if $k$ is an integer, and 0 otherwise.

Lemma 3.3. Set $\nu = (\nu_1, \ldots, \nu_q)$, then

$$\tilde{p}_\nu(\Xi) = \sum_\varsigma c_\varsigma \alpha_\varsigma,$$

for some non-negative integers $c_\varsigma$. We have

1. the sum runs over the partitions $\varsigma$ such that $|\varsigma|_1 \leq |\nu|_1$;
2. $c_{\nu + \underline{1}} = \prod_i (m_i(\nu)!)$;
3. if $c_\varsigma \neq 0$, $|\varsigma|_1 = |\nu|_1$ and $\varsigma \neq \nu + \underline{1}$ then $m_1(\varsigma) > 0$.

12
Proof.

**Part 1.** Since \( \deg(\alpha_k) = |\varsigma| \) is a filtration, it is enough to prove the statement for \( q = 1 \), that is, \( \nu = (k) \) for some positive integer \( k \). We consider

\[
p_k(\Xi) = \sum_{1 \leq j_1, \ldots, j_k < h} ((j_1, h), (j_1, h)) \cdot \cdots \cdot ((j_k, h), (j_k, h)).
\]

Let \((\sigma, d) = ((j_1, h), \ldots, (j_k, h), \{j_1, \ldots, j_k, h\})\) be a term of the previous sum and let \( \varsigma \vdash |d| \) be the cycle type of \( \sigma \). This is the outline of the proof: firstly, we show an upper bound for \(|\varsigma|\), that is, \(|\varsigma| \leq k + 2\). Then we check some conditions that \( \sigma \) must satisfy in order to have \(|\varsigma| = k + 2 \) (if such a \( \sigma \) exists). We prove that \(|\varsigma| = k + 2 \) if and only if \((\sigma, d) = (1d, d)\) with \( |d| = k/2 + 1 \) and \( k \) even. Finally, we calculate how many times the partial permutation \((1d, d)\), with \( |d| = k/2 + 1 \), appears in the modified power sum \( p_k(\Xi) \). This number is the coefficient of \( \alpha_{(1^2+1)} \) in \( p_k(\Xi) \), and we show that it is equal to \( \text{Cat}\left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)! \).

Since we defined \( \tilde{p}_k(\Xi) = p_k(\Xi) - \text{Cat}\left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)! \cdot \alpha_{(1^2+1)} \) this shows that all the terms in \( \tilde{p}_k(\Xi) \) satisfy \(|\varsigma| \leq k + 1 = |\nu|\), when \( \nu = (k) \).

We first estimate

\[
|\varsigma| = |\{j_1, \ldots, j_k, h\}| + m_1(\varsigma)
\]

\[
= 2\#\{j_i, \text{ s.t. } \sigma j_i = j_i\} + \#\{j_i, \text{ s.t. } \sigma j_i \neq j_i\} + 2\delta(h \text{ is fixed by } \sigma) + \delta(h \text{ is not fixed by } \sigma),
\]

where \( \delta \) is the Kronecker delta. We stress out that, when counting the set cardinalities above, we do not count multiplicities; for example, if

\[
\sigma = (15)(25)(15)(35) = (1)(2)(35),
\]

then

\[
\#\{j_i, \text{ s.t. } \sigma j_i = j_i\} = \#\{1, 2\} = 2,
\]

\[
\#\{j_i, \text{ s.t. } \sigma j_i \neq j_i\} = \#\{3\} = 1,
\]

\( \varsigma = (2, 1, 1) \).

Notice that in the sequence \((j_1, \ldots, j_k)\) all fixed points must appear at least twice, while non fixed points must appear at least once, hence

\[
2\#\{j_i, \text{ s.t. } \sigma j_i = j_i\} + \#\{j_i, \text{ s.t. } \sigma j_i \neq j_i\} \leq k,
\]

while obviously,

\[
2\delta(h \text{ is fixed by } \sigma) + \delta(h \text{ is not fixed by } \sigma) \leq 2.
\]

Thus \(|\varsigma| \leq k + 2 \).

Suppose there exists \( \sigma \) such that \(|\varsigma| = k + 2 \), then from the proof of the inequality \(|\varsigma| \leq k + 2 \) we know that \( \sigma \) satisfies

\( \sigma \) for each \( i, j_i \) appears at most twice;

\[
\text{if it appears exactly twice in the multiset } \{j_1, \ldots, j_k\};
\]

\( \text{if } h \text{ is a fixed point.} \)

We prove by induction on \( k \) that if \(|\varsigma| = k + 2 \) and \( \alpha_k \) appears in the sum \( \text{[19]} \) then \( k \) is even and \( \varsigma = (1^k+1) \). If \( k = 1 \) then \( \varsigma = (2) \) and \(|\varsigma| = 2 < 3 = k + 2 \). If \( k = 2 \) then \( \varsigma = (3) \) or \( \varsigma = (1, 1) \). By our request that \(|\varsigma| = 4 = k + 2 \) we see that we must have \( \varsigma = (1, 1) \). Consider now the statement true up to \( k - 1 \) and \( \sigma = (j_1, h) \cdot \ldots \cdot (j_k, h) \). By property \( \text{[22]} \) \( h \) is fixed, so that \( j_k \) must appear at least twice, and by \( \text{[20]} \) \( j_k \) appears exactly twice. Hence, by \( \text{[21]} \) \( j_k \) is fixed, thus exists a unique \( l < k - 1 \) such that \( j_{l+1} = j_k \) and

\[
\sigma = (j_1, h) \cdot (j_1, h) \cdot (j_2, h) \cdot (j_2, h) \cdot (j_{k-1}, h) (j_k, h) = \tau_1(j_k, h) \tau_2(j_k, h) = \tau_1 \tau_2,
\]

where

\[
\tau_1 = \begin{cases} 
  \text{Id} & \text{if } l = 0 \\
  (j_l, h) \ldots (j_1, h) & \text{if } l > 0
\end{cases}, \quad \tau_2 = \begin{cases} 
  \text{Id} & \text{if } l = k - 2 \\
  (j_{l+2}, h) \ldots (j_{k-1}, j_k) & \text{if } l < k - 2.
\end{cases}
\]

If \( l = 0 \) or \( l = k - 2 \) we can apply induction on either \( \tau_1 \) or \( \tau_2 \) and the result follows, so consider \( 0 < l < k - 2 \). We claim that the sets \( \{j_1, \ldots, j_l\} \) and \( \{j_{l+2}, \ldots, j_k\} \) are disjoint. Notice first that \( j_k \) does not appear in \( \{j_1, \ldots, j_l\} \). Suppose \( j_a = j_b \) for some \( a \leq l < b \), and choose \( b \) minimal with this property. Then \( j_b \) is not fixed by \( \tau_2 \), that is, \( \tau_2(j_b) = j_b \) with either \( b < b \) or \( b = k \). In the first case \((b < b)\), by
minimality of \(b, j\) does not appear in \(\tau_1\); in the second case \((b=k)\) we know that \(j_k\) does not appear in \(\{j_1, \ldots, j_t\}\). Hence in either case \(\sigma(j_b) = j_{b'}\). This is a contradiction, since \(j_b\) appears twice in \(\sigma\) and therefore must be fixed (property (21)). This proves that \(\{j_1, \ldots, j_t\} \cap \{j_{t+1}, \ldots, j_k\} = \emptyset\).

Therefore \(\tau_1\) and \(\tau_2\) respect properties (20), (21) and (22), and we can apply the inductive hypothesis to obtain that \(\tau_1 = \text{Id} = \tau_2\) and both \(l\) and \(k-l-2\) are even. We conclude hence that if \(|\sigma| = k+2\) and \(\alpha_{L}\) appears in the sum (15) then \(\zeta = (1^{k+1})\) and \(k\) even.

We assume now that \(k\) is even and we calculate the coefficient of \(\alpha_{((1^{k+1})+1)}\) in \(p_{\beta}(\Xi)\), which is equal to the coefficient of \((\text{Id},d) = ((j_1,h), \ldots, (j_k,h), \{j_1, \ldots, j_k\})\) in the sum (19), for a fixed \(d\) of cardinality \(k/2 + 1\). In order to do this we count the number of lists \(L = ((j_1,h), \ldots, (j_k,h))\) such that \(\{j_1, \ldots, j_k\} = d\) and \(j_1, \ldots, j_k\) appears exactly twice in \(L\). We call \(L_d\) the set of these lists.

Define a pair set partition of the set \([k] = \{1, \ldots, k\}\) as a set partition \(A = \{\{r_1, s_1\}, \ldots, \{r_{\frac{k}{2}}, s_{\frac{k}{2}}\}\}\) of \([k]\) into pairs. Such a set partition is said to be crossing if \(r_a < r_b < s_a < s_b\) for some \(a, b \leq k/2\), otherwise the set partition is said to be non crossing. Calling \(S_k\) the set of non crossing pair set partitions, it is known that \(|S_k| = \text{Cat}(k/2)\), see [Hor98]. We build a map \(\psi_k : L_d \to S_k\) and we prove that this map is \((\frac{k}{2})\)!-to-one, which implies that \(|L_d| = c_{((1^{k+1})+1)} = \text{Cat}(\frac{k}{2}) \cdot (\frac{k}{2})!\).

Let \(L \in L_d\), \(L = (L_1, \ldots, L_k) = ((j_1,h), \ldots, (j_k,h))\); we have proven that, since \((j_1,h), \ldots, (j_k,h) = \text{Id}\) and \(|d| = |\{j_1, \ldots, j_k\}| = k/2 + 1\), then each element \((j_i,h)\) must appear exactly twice in \(L\). We construct a pair partition \(\psi(L)\) such that a pair \(\{r,s\} \in \psi(L)\) if \(r_s = j_r\). By [Hor98] Lemma 2 this set partition is non crossing. This map is clearly surjective, although not injective: every permutation \(\gamma\) acting on \(d = \{j_1, \ldots, j_k\}\) which fixes \(h\) acts also on \(\psi_d\): \(\gamma(L) = ((\gamma(j_1), h), \ldots, (\gamma(j_k), h))\). Notice that \(h > j_1, \ldots, j_k\) and this is why \(\gamma(h) = h\) in order to have an action on \(L_d\). Moreover \(\psi_k(L) = \psi_k(L')\) if and only if \(L = \gamma(L')\) for some \(\gamma \in S_{d, h}\). Thus \(\psi_k\) is a \((\frac{k}{2})\)!-to-one map, hence \(c_{((1^{k+1})+1)} = \text{Cat}(\frac{k}{2}) \cdot (\frac{k}{2})!\) and the first part of the proof is concluded.

**Part 2.** The second statement is shown in [FGr12 section 2].

**Part 3.** This claim is proven by induction on \(q\), the number of parts of \(\nu\). The initial case is \(q = 1\) and

\[
\tilde{p}_k(\Xi) = \sum_{|\sigma| \leq k+1} c_{\zeta} \alpha_{\zeta}.
\]

We consider the \(\alpha_{\zeta}\)'s appearing in the sum such that \(|\sigma| = k+1\) and \(m_1(\zeta) = 0\), so that \(|\sigma| = k+1\). Expanding the sum in a way similar to Equation (19), we see that \(|\sigma| = k+1 = |\{j_1, \ldots, j_k, h\}|\) and all the elements in this set must be pairwise different. Hence \(\sigma = (j_1, h) \cdot \ldots \cdot (j_k, h) = (h, j_k, \ldots, j_1)\) and \(\zeta = (k+1)\). Thus the statement for \(q = 1\) is proved.

Let \(\nu = (\nu_1, \ldots, \nu_q)\) with \(q \geq 2\). Set \(\tilde{\nu} = (\nu_1, \ldots, \nu_{q-1})\) and suppose, by the induction hypothesis, that the assertion is true for \(\tilde{p}_k(\Xi)\). Then

\[
\tilde{p}_k(\Xi) = \tilde{p}_k(\Xi)\tilde{p}_{\nu_q}(\Xi) = \left( \sum_{|\sigma| \leq |\nu|} c_{\zeta} \alpha_{\zeta} \right) \cdot \left( \sum_{|\sigma| \leq \nu_q+1} c_{\vartheta} \alpha_{\vartheta} \right) = \sum_{|\zeta| \leq |\nu|, |\vartheta| \leq \nu_q+1} c_{\zeta} c_{\vartheta} \alpha_{\zeta \vartheta}.
\]

We apply now Corollary (21) and obtain

\[
\tilde{p}_k(\Xi) = \sum_{|\zeta| \leq |\nu|, |\vartheta| \leq \nu_q+1} c_{\zeta} c_{\vartheta} \alpha_{\zeta \vartheta} + F^{-1}(V_{|\nu|}) + F^{-1}(V_{|\nu|}).
\]

Hence there exists only one term in the previous sum such that \(|\zeta| = |\zeta \cup \vartheta| = |\nu|\) and \(m_1(\zeta) = 0\), that is, \(\zeta = \nu + 1\), and the proof is completed.

Let now \(\lambda \vdash n\) be a random partition distributed with the Plancherel measure. We say that a function \(X\) on the set of partitions is \(X \in o_P(n^0)\) if \(n^{-2}X(\lambda) \xrightarrow{\text{P}} 0\). Similarly, \(X \in O_P(n^0)\) is stochastically bounded by \(n^0\) if for any \(\epsilon > 0\) there exists \(M > 0\) such that \(P_{PL}(|X(\lambda)n^{-2}| > M) \leq \epsilon\). For example, as a consequence of Kerov’s result on the convergence of characters (Theorem (24)), \(\hat{\chi}_\rho \in O_P(n^{-\omega(\rho)/2})\).
If a function $X(\lambda, \mu_j)$ depends also on a subpartition $\mu_j \Rightarrow \lambda$, by $X(\lambda, \mu_j) \in o_P(n^{\beta})$ we mean that $\max_j X(\lambda, \mu_j) \in o_P(n^{\beta})$, and similarly for the notion of stochastic boundedness.

**Lemma 3.4.** For each partition $\rho$ we have that $\hat{\chi}^\lambda(\varphi_n(\alpha,\rho,n)) \in O_P(n^{\frac{-1}{2}\mu})$, where $|\rho|_1 = |\rho| + m_1(\rho)$.

**Proof.** From Equation (9), $\hat{\chi}^\lambda(\varphi_n(\alpha,\rho,n)) = C_{\varphi} n^{\frac{1}{2}|\lambda|} \hat{\chi}^\lambda$. Since $\hat{\chi}^\lambda \in O_P(n^{-w(\rho)/2})$ the lemma follows. □

**Lemma 3.5.** Let $\mu_j \not\Rightarrow \lambda$. In probability

$$\hat{\chi}^\lambda(\varphi_n(\rho,\lambda,\mu_j)) - \hat{\chi}^\mu(\varphi_n-1(\rho,\lambda,\mu_j)) = \hat{\rho}_\mu(\mathcal{C}_\lambda) - \hat{\rho}_\mu(\mathcal{C}_{\mu_j}) \in o_P(n^{\frac{|\mu| - |\lambda|}{2}}).$$

**Proof.** The first equality come from the fact that $\hat{\chi}^\lambda \circ \varphi_n$ applied to a symmetric function of partial Jucys-Murphy elements equals the same symmetric function evaluated on the contents of $\lambda$. We prove now the asymptotic bound. Consider first the case $q = 1$, i.e., $\nu = (k)$ for some positive integer $k$, in which case

$$\hat{\rho}_\mu(\mathcal{C}_\lambda) = \hat{\rho}_\mu(\mathcal{C}_{\mu_j} \cup y_j) = \hat{\rho}_\mu(\mathcal{C}_{\mu_j}) + y_j^k + \text{Cat}(\frac{k}{2} + 1) \left(n^{\frac{1}{2}}(\lambda) + (n-1)^{\frac{1}{2}}(\mu)ight)$$

$$= \hat{\rho}_\mu(\mathcal{C}_{\mu_j}) + y_j^k + \text{Cat}(\frac{k}{2} + 1) \left(n^{1} - 1 \right)^{\frac{1}{2}}.$$ 

Notice that $y_j < \min\{\lambda_1, \lambda_1^j\}$ since $y_j$ is the content of a box of $\lambda$. It is shown in [RomL5] that, with probability that goes to 1, both $\lambda_1$ and $\lambda_1^j$ are smaller than $3\sqrt{n}$, hence for each subpartition $\mu_j$ of $\lambda$, one has $y_j < 3\sqrt{n}$ with probability tending to 1 for $n$ increasing. Therefore we obtain that $\hat{\rho}_\mu(\mathcal{C}_\lambda) - \hat{\rho}_\mu(\mathcal{C}_{\mu_j}) \in O_P(n^{1/2})$ in probability $\in o_P(n^{\frac{|\mu| - |\lambda|}{2}})$.

In the general case $q > 1$ we expand the product, obtaining

$$\hat{\rho}_\mu(\mathcal{C}_\lambda) - \hat{\rho}_\mu(\mathcal{C}_{\mu_j}) = \sum_{A \subseteq \{1,\ldots,q\}} \prod_{i \notin A} \hat{\rho}_\mu(\mathcal{C}_{\mu_i}) \prod_{i \in A} y_i^\nu + \text{Cat}(\frac{|A|}{2}) \left(n^{1} - 1 \right)^{\frac{1}{2}}.$$

We use now Lemma 3.3 and 3.4 which show that the factor $\prod_{i \notin A} \hat{\rho}_\mu(\mathcal{C}_{\mu_i})$ is $O_P(n^{\hat{\chi}(\mathcal{C}_{\mu_i} + |A|)})$ and

$$\prod_{i \in A} y_i^\nu + \text{Cat}(\frac{|A|}{2}) \left(n^{1} - 1 \right)^{\frac{1}{2}} \in O_P(n^{\frac{1}{2} \hat{\chi}(\mathcal{C}_{\mu_i})}).$$

Therefore $\hat{\rho}_\mu(\mathcal{C}_\lambda) - \hat{\rho}_\mu(\mathcal{C}_{\mu_j}) \in O_P(n^{1/2})$, with

$$l = \max_{A \subseteq \{1,\ldots,q\}} \frac{1}{2} \sum_{i \in A} \nu_i + \frac{|A|}{2} < \frac{|\nu| + q}{2}.$$

**Proposition 3.6.** Let $\lambda \cong n$, $\mu \not\Rightarrow \lambda$. Let $\rho \cong r < n$ then

$$\hat{\chi}^\lambda_{\rho} - \hat{\chi}^\mu_{\rho} \in o_P(n^{\frac{-w(\rho)}{2}}).$$

**Proof.** Set $\nu = (\nu_1, \ldots, \nu_q)$ such that $\rho = (\nu_1 + 1, \ldots, \nu_q + 1, 1, 1, \ldots, 1)$. Obviously, $\hat{\chi}^\lambda_{\rho} = \hat{\chi}^\lambda$ and $\hat{\chi}^\mu_{\rho + 1} = \hat{\chi}^\mu_{\rho + 1}$; moreover, $|\nu| + q = wt(\rho)$. We want to prove that $\hat{\chi}^\lambda_{\rho + 1} - \hat{\chi}^\mu_{\rho + 1} \in o_P(n^{\frac{-w(\rho)}{2}})$. We prove the statement by induction on $|\nu| + 1$.

Consider $n^{\frac{-w(\rho)}{2}} (\hat{\rho}_\mu(\mathcal{C}_\lambda) - \hat{\rho}_\mu(\mathcal{C}_{\mu})) \in o_P(n^{1/2})$ because of the previous lemma. It can be rewritten as

$$n^{\frac{-w(\rho)}{2}} \sum_{|\xi| \leq |\nu| + q} \left(\hat{\chi}^\lambda_{\rho} - \hat{\chi}^\mu_{\rho} \circ \varphi_n - \varphi_n - 1 \right)(c_{\xi} \zeta_{\alpha,\xi}) \rightarrow 0,$$

where the coefficients $c_{\xi}$ are described in Lemma 3.3.

Equivalently

$$n^{\frac{-w(\rho)}{2}} (\hat{\rho}_\mu(\mathcal{C}_\lambda) - \hat{\rho}_\mu(\mathcal{C}_{\mu})) = n^{\frac{-w(\rho)}{2}} \sum_{|\xi| \leq |\nu| + q} \frac{C_{\xi}}{z_{\xi}} \left(n^{\frac{1}{2}} \hat{\chi}^\lambda_{\xi} - (n-1)^{\frac{1}{2}} \hat{\chi}^\mu_{\xi}\right)$$

$$= n^{\frac{-w(\rho)}{2}} \sum_{|\xi| \leq |\nu| + q} \frac{C_{\xi}}{z_{\xi}} \left(\hat{\chi}^\lambda_{\xi} + \hat{\chi}^\mu_{\xi} - (n-1)^{\frac{1}{2}} \hat{\chi}^\mu_{\xi}\right).$$

15
We obtain thus that variables considering the partial trace.

respectively we study the sum of the entries whose indices we consider the sum of all the entries in the matrix (before this role was played by the trace), and then that the objects we study really depend on the representation matrix, and change, for example, under representation, up to a certain index (depending on the dimension of the representation). We stress out

In this chapter our goal is to describe the sum of the entries of the matrix associated to a Young seminormal

4 Sum of the entries of an irreducible representation

Proof of Theorem 2.10. We split the previous sum and notice that }

\[ \sum_{i,j} \frac{c_{ij}}{i!j!} (n-1)^{i+j} \lambda \rho \to 0. \]

Indeed

\[ (n-1)^{i+j} \lambda \rho \leq n^{i+j-1} \lambda \rho \quad \text{and} \quad n^{i+j-1} \lambda \rho \to 0 \]

since \( |\lambda| \leq |\rho| + q. \)

We deal with the sum \( n^{i+j} \lambda \rho \sum_{i,j} \frac{c_{ij}}{i!j!} \) \((n-1)^{i+j} (\lambda \rho - \rho \lambda)\). We separate in this sum the terms with \( |\lambda| = |\rho| + q \) and \( \lambda \rho \neq \rho \lambda \), the terms with \( |\lambda| < |\rho| + q \), and the term corresponding to \( \lambda = \rho \lambda \).

- Case \( |\lambda| = |\rho| + q \) and \( \lambda \rho \neq \rho \lambda \): We can apply induction again, to have \( (\lambda \rho - \rho \lambda) \in o_P(n^{-|\lambda|+|\rho|}) \). Therefore

\[ n^{-|\lambda|+|\rho|} \cdot (n-1)^{i+j} (\lambda \rho - \rho \lambda) \in o_P \left( n^{-|\lambda|+|\rho|} \right). \]

We obtain thus that

\[ n^{-|\lambda|+|\rho|} \cdot (n-1)^{|\lambda|+q} \sum_{i,j} \frac{c_{ij}}{i!j!} \frac{m_i(\lambda)}{m_i(\rho)} (\lambda \rho - \rho \lambda) + o_P(1) \to 0, \]

which proves the statement.

**Proof of Theorem 2.10.** Fix \( \sigma \in S_r \) and let \( \rho \) be its cycle type. We have

\[ MT_\sigma^{\lambda}(\rho) = \sum_{y_i \leq v^\delta} \frac{\dim \mu_j}{\dim \lambda} \lambda_{\rho} = \sum_{y_i \leq v^\delta} \frac{\dim \mu_j}{\dim \lambda} \lambda_{\rho} + o_P(1), \]

since \( \sum \dim \mu_j / \dim \lambda \leq 1 \), and the previous proposition. Hence

\[ n^{-|\lambda|+|\rho|} \cdot (n-1)^{|\lambda|+q} \sum_{y_i \leq v^\delta} \frac{\dim \mu_j}{\dim \lambda} \lambda_{\rho} + o_P(1). \]

Finally, by Lemma 2.4 and Theorem 2.9 we obtain that given \( \sigma_1, \sigma_2, \ldots \) permutations of cycle type respectively \( \rho_1, \rho_2, \ldots \) and \( u_1, u_2, \ldots \in [0, 1] \) and calling \( \{\xi_k\}_{k \geq 2} \) a family of independent standard Gaussian variables

\[ \left\{ n^{-|\lambda|+|\rho|} \cdot (n-1)^{|\lambda|+q} \sum_{y_i \leq v^\delta} \frac{\dim \mu_j}{\dim \lambda} \lambda_{\rho_i} + o_P(1) \right\} \overset{d}{\to} \left\{ u_i, \prod_{k \geq 2} k^{m_k(\rho_i)/2} H_{m_k(\rho_i)}(\xi_k) \right\}, \]

for \( i \geq 1. \)

**4 Sum of the entries of an irreducible representation**

In this chapter our goal is to describe the sum of the entries of the matrix associated to a Young seminormal representation, up to a certain index (depending on the dimension of the representation). We stress out that the objects we study really depend on the representation matrix, and change, for example, under isomorphisms of the representation. Some calculations are similar to those in the previous chapter: first we consider the sum of all the entries in the matrix (before this role was played by the trace), and then we study the sum of the entries whose indices \((i, j)\) satisfy \( i \leq u \dim \lambda, j \leq u \dim \lambda \), while before we were considering the partial trace.
4.1 Total sum

**Definition 4.1.** Define the normalized total sum associated to an irreducible representation and a permutation $\sigma \in S_r$ as

$$TS^\lambda(\sigma) := \sum_{i,j \leq \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,j}}{\dim \lambda}$$

The following is the main result of the section:

**Theorem 4.2.** Fix $\sigma_1 \in S_{r_1}, \sigma_2 \in S_{r_2}, \ldots$ and let $\lambda \vdash n$. Define the real numbers

$$m_i := \mathbb{E}^r_{PL}[TS^\nu(\sigma_i)] \quad \text{and} \quad u_i := \left(\frac{r_i}{2}\right) \mathbb{E}^r_{PL}\left[\tilde{\chi}^\nu_{(2,1,\ldots,1)} T S^\nu(\sigma_i)\right].$$

Then

$$\left\{\sqrt{\pi} \cdot (TS^\lambda(\sigma_i) - m_i)\right\} \overset{d}{\to} \{N(0,2v_i^2)\},$$

where $N(0,2v_i^2)$ is a normal random variable of variance $2v_i^2$.

Note that the limit $N(0,2v^2)$ is degenerate in the case $v = 0$. Whether or not $v = 0$ is a nontrivial question. At the end of the section we study the values of $v$ and $m$ when $\sigma$ is an adjacent transposition, and we write the explicit values of $v$ and $m$ for permutations of $S_4$. In order to prove the Theorem, we need some preliminary results.

**Proposition 4.3.** Let $\lambda \vdash n$ and $r \leq n$. There exists a bijection $\phi_r$ between

$$\text{SYT}(\lambda) \overset{\phi_r}{\cong} \bigsqcup_{\nu \vdash r} \text{SYT}(\nu) \times \text{SYT}(\lambda/\nu)$$

**Proof.** Let $T$ be a standard Young tableau of shape $\lambda$, the image $\phi_r(T) = (U, V)$ of $T$ is defined as follows: the boxes of $T$ whose entries are smaller or equal than $r$ identify a tableau $U$ of shape $\nu \subseteq \lambda$. Define now $V$ as a standard Young tableau of skew shape $\lambda/\nu$, and in each box write $a-r$, where $a$ is the value inside the corresponding box in $T$. \qed

**Example 4.4.** Here is an example for $\lambda = (6,4,3,3,3,1)$ and $r = 8$:

\[
\begin{array}{cccccc}
1 & 2 & 5 & 6 & 8 & 13 \\
3 & 7 & 9 & 16 \\
4 & 12 & 17 \\
10 & 15 & 19 \\
11 & 16 & 20 \\
14
\end{array}
\leftrightarrow
\begin{pmatrix}
1 & 2 & 5 & 6 & 8 \\
3 & 7 & 9 \\
4 \\
10 & 12 \\
13 & 15 & 19 \\
14 & 16 & 20
\end{pmatrix}
\begin{pmatrix}
1 & 8 \\
2 & 9 \\
3 & 10 \\
4 & 11 \\
5
\end{pmatrix}
\]

**Lemma 4.5.** Let $\nu \vdash n, \sigma \in S_s$ with $r \leq n$. For two standard Young tableaux $T, T' \in \text{SYT}(\lambda)$, set $\phi_r(T) = (U, V)$ and $\phi_r(T') = (U', V')$, then

$$\pi^\lambda(T,T') = \begin{cases} 
\pi^\nu(U,U') & \text{if } V \neq V' \\
\pi^\nu(U,U') & \text{if } V = V',
\end{cases}$$

where $\nu = \text{sh}(U) = \text{sh}(U')$ is the shape of the tableau $U$ in the second case.

**Proof.** We will prove the lemma by induction on the number of factors in the (minimal) decomposition of $\sigma$ into adjacent transpositions. Recall that if $\sigma = (k, k+1)$, $k < r$, by Definition 2.5

$$\pi^\lambda((k, k+1))_{T, T'} = \begin{cases} 
\frac{1}{d_k(T)} & \text{if } T = T'; \\
\frac{1}{d_k(T)} & \text{if } (k, k+1)T = T'; \\
0 & \text{else.}
\end{cases}$$

Hence if $V \neq V'$ then $\pi^\lambda((k, k+1))_{T, T'} = 0$, otherwise $\pi^\lambda((k, k+1))_{T, T'} = \pi^\nu((k, k+1))_{U, U'}$ since $d_k(T) = d_k(U)$.

Consider a general $\sigma \in S_r$ and write it as $\sigma = (k, k+1)\tilde{\sigma}$. Then

$$\pi^\lambda(T,T') = \sum_{\tilde{\sigma} \in \text{SYT}(\lambda)} \pi^\lambda((k, k+1))_{T,S}\pi^\lambda(\tilde{\sigma})_{S,T'}$$
Proof. Fix the statement can be written as the
Kronecker delta, and same for the RHS by the character orthog onality relation of the first kind.

Proposition 4.8. Let \( \lambda \) and \( \sigma \) be as before, then

\[
TS^\lambda(\sigma) = \sum_{\nu \vdash r} TS^\nu(\sigma) \cdot \frac{\dim \lambda/\nu}{\dim \lambda}
\]

Proof. By the previous lemma:

\[
TS^\lambda(\sigma) = \sum_{T,T' \in \text{SYT}(\lambda)} \frac{\pi^\lambda_T(\sigma)}{\dim \lambda} \cdot \sum_{\nu \vdash r} \sum_{U,U' \in \text{SYT}(\nu)} \frac{\pi^\nu_{U,U'}(\sigma)}{\dim \lambda/\nu} \cdot \frac{\dim \lambda/\nu}{\dim \lambda},
\]

and the conclusion is immediate.

Lemma 4.7. Let \(|\nu| = r \leq n = |\lambda|\), then

\[
\dim \lambda/\nu = \frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \chi^\lambda(\tau).
\]

Proof. Fix \( \nu \), we prove the statement by induction on \( n \). If \( n = r \) then the LHS is equal to \( \delta_{\lambda,\nu} \), the Kronecker delta, and same for the RHS by the character orthog onality relation of the first kind.

Suppose \( r = n - 1 \), then by \((\text{II})\) we have \( \chi^\lambda(\tau) = \sum_{\mu_j \not\supset \lambda} \chi^{\mu_j}(\tau) \) for each \( \tau \in S_{n-1} \). Hence the RHS of the statement can be written as

\[
\frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \chi^\lambda(\tau) = \sum_{\mu_j \not\supset \lambda} \left( \frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \chi^{\mu_j}(\tau) \right) = \sum_{\mu_j \not\supset \lambda} \delta_{\mu_j = \nu} = \delta_{\nu, \lambda},
\]

which is equal to the LHS.

By the inductive hypothesis we consider the statement true for each \( \tilde{\lambda} \vdash n - 1 \). Hence

\[
\dim \lambda/\nu = \sum_{\lambda \vdash n-1} \dim \lambda/\tilde{\lambda} \cdot \dim \tilde{\lambda}/\nu
\]

\[
= \sum_{\lambda \vdash n-1} \left( \frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \chi^{\tilde{\lambda}}(\tau) \right)
\]

\[
= \frac{1}{r!} \sum_{\tau \in S_r} \chi^\nu(\tau) \chi^{\lambda}(\tau).
\]
Theorem 2.9, we have that
\[ \sqrt{\nu} r \] of partitions of \( \tau \) where the first two terms correspond respectively to tableau of shape \( \nu \) and we recognize inside the parenthesis the average \( \mathbb{E}_{PL} [\hat{\chi}^\nu(\tau)TS^\nu(\sigma)] \) taken with the Plancherel measure of partitions of \( r \).

Proof of Theorem 4.2. Consider a permutation \( \sigma \), we write the previous proposition as

\[ TS^\lambda(\sigma) = m + v \cdot \hat{\chi}^\lambda((1, 2)) + \sum_{\substack{\tau \in S_r, \ \text{wt}(\tau) > 1 \ \text{and for all} \ \nu \vdash r \ \text{and for all} \ T, S \ \text{standard Young tableaux of shape} \ \nu, \ d_{r-1}(T') = -d_{r-1}(T), \ \text{where} \ T' \ \text{is a tableau of shape} \ \nu' \ \text{conjugated to} \ T, \ \text{which implies that} \}} \mathbb{E}_{PL} [\hat{\chi}^\nu(\tau)TS^\nu(\sigma)] \] \[ \cdot \hat{\chi}^\lambda(\tau), \]

where the first two terms correspond respectively to \( \tau = \text{Id} \) and the sum of all transpositions. By Kerov’s Theorem 2.9 we have that

\[ \sqrt{n} \sum_{\tau \in S_r, \ \text{wt}(\tau) > 2} \mathbb{E}_{PL} [\hat{\chi}^\nu(\tau)TS^\nu(\sigma)] \] \[ \cdot \hat{\chi}^\lambda(\tau) \overset{p}{\to} 0 \]

and \( \sqrt{n} \cdot v \cdot \hat{\chi}^\lambda((1, 2)) \overset{d}{\to} cN(0, 2) \). Consider now a sequence of permutations \( \sigma_1, \sigma_2, \ldots, \) then

\[ \{ \sqrt{n} \cdot (TS^\lambda(\sigma_i) - m_i) \}_{i \geq 1} = \{ n \cdot v_i \hat{\chi}^\lambda((1, 2)) + a_P(1) \}_{i \geq 1} \overset{d}{\to} \{ N(0, 2 \nu^2) \}_{i \geq 1} \].

Example 4.9. Let \( \sigma = (r - 1, r) \), \( r > 2 \), be an adjacent transposition. In this example we will show that \( m = \mathbb{E}_{PL} [TS^\nu(\sigma)] \) is strictly positive and that \( v := \left( \mathbb{E}_{PL} \left[ \hat{\chi}^\nu(2, 1, \ldots, 1) TS^\nu(\sigma) \right] \right) = 1 \). First, notice that, for all \( \nu \vdash r \) and for all \( T, S \) standard Young tableaux of shape \( \nu, \ d_{r-1}(T') = -d_{r-1}(T), \) where \( T' \) is a tableau of shape \( \nu' \) conjugated to \( T \), which implies that

\[ \pi^\nu(\sigma)_{T', S'} = \begin{cases} -\pi^\nu(\sigma)_{T, S} & \text{if } T = S \\ \pi^\nu(\sigma)_{T, S} & \text{if } T \neq S, \end{cases} \]

and if \( T \neq S \) then \( \pi^\nu(\sigma)_{T, S} \geq 0 \). Hence

\[ m = \sum_{\nu \vdash r} \frac{(\dim \nu)^2}{r!} TS^\nu(\sigma) = \frac{1}{2} \sum_{\nu \vdash r} \frac{(\dim \nu)^2}{r!} (TS^\nu(\sigma) + TS'^\nu(\sigma)). \]

Since \( TS^\lambda(\sigma) = \hat{\chi}^\lambda(\sigma) + \sum_{T,S \in SYT(\nu), \ T \neq S} \frac{\pi^\nu(\sigma)_{T, S}}{\dim \nu} \), then

\[ m = \frac{1}{2} \sum_{\nu \vdash r} \frac{\dim \nu}{r!} \left( \sum_{T,S \in SYT(\nu), \ T \neq S} \pi^\nu(\sigma)_{T, S} + \pi'^\nu(\sigma)_{T', S'} \right) > 0. \]

For \( r > 2 \), at least one summand is nonzero.

Consider now \( v \) and decompose \( TS^\nu(\sigma) \) as above:

\[ v = \left( \frac{r}{2} \right) \left( \mathbb{E}_{PL} \left[ \hat{\chi}^{2, 1, \ldots, 1} \hat{\chi}^\nu(\sigma) \right] + \mathbb{E}_{PL} \left[ \hat{\chi}^{2, 1, \ldots, 1} \sum_{T,S \in SYT(\nu), \ T \neq S} \frac{\pi^\nu(\sigma)_{T, S}}{\dim \nu} \right] \right). \]

By the character relations of the second kind we get

\[ \mathbb{E}_{PL} \left[ \hat{\chi}^{2, 1, \ldots, 1} \hat{\chi}^\nu(\sigma) \right] = \sum_{\nu \vdash r} \frac{(\dim \nu)^2}{r!} \hat{\chi}^{2, 1, \ldots, 1} \hat{\chi}^\nu(\sigma) = \frac{2}{r(r-1)} \].
On the other hand, using $T'$ for the conjugate of $T$ as above,

$$
\mathbb{E}_{\mathcal{PL}} \left[ \chi_{(2,1,\ldots,1)}^{\nu} \sum_{T,S \in \text{SYT}(\nu)} \frac{\pi^{\nu}(\sigma)_{T,S}}{\dim \nu} \right] =
$$

$$
\frac{1}{2} \sum_{\nu \in \nu} \frac{1}{r} \sum_{T,S \in \text{SYT}(\nu)} \left( \chi_{(2,1,\ldots,1)}^{\nu}(\sigma)_{T,S} + \chi_{(2,1,\ldots,1)}^{\nu}(\sigma)_{T,S} \right) = 0
$$

since $\pi^{\nu}(\sigma)_{T,S} = \pi^{\nu}(\sigma)_{T',S'}$ if $T \neq S$ and $\chi_{(2,1,\ldots,1)}^{\nu} = -\chi_{(2,1,\ldots,1)}^{\nu}$. Finally, $\nu = 1$.

Here we write the values of $m = \mathbb{E}_{\mathcal{PL}} [TS^\nu(\sigma)]$ and $v = \left( \sum_{\nu} \pi_{\nu} \right) \mathbb{E}_{\mathcal{PL}} \left[ \chi_{(2,1,\ldots,1)}^{\nu} TS^\nu(\sigma) \right]$ when $\sigma \in S_4$.

| $\sigma$ | Id | (3,4) | (2,3) | (2,4) | (2,3,4) | (1,2,3,4) | (1,2,4,3) | (1,2,4) | (1,3,2) | 1/3 | 1/3 | 1/3 | 0 | -1/3 |
|---------|-----|-------|-------|-------|---------|-----------|-----------|---------|---------|-----|-----|-----|--|-----|
| $\nu$   | 0   | 1/2   | 2/3   | 5/12  | 1/6     | -1/4      | 0         | 1/3     | -3/4   | 1/3 | 1/3 | 1/3 | 0 | 1/2 |

4.2 Partial sum of the entries of an irreducible representation

**Definition 4.10.** Let $\lambda \vdash n$, $\sigma \in S_n$, $u \in \mathbb{R}$. Then

$$PS_u^{\lambda}(\sigma) := \sum_{i,j \leq n} \frac{\pi^\lambda(\sigma)_{i,j}}{\dim \lambda}.$$

We can now argue in a similar way as we did in Section 2.4: summing entries of $\pi^\lambda(\sigma)$ up to a certain index is equivalent to sum all the entries of the submatrices $\pi^{\mu_j}(\sigma)$ for $j < j'$ and the right choice of $j'$, plus a remainder which is again a partial sum. We present the analogous of Proposition 2.8; we omit the proof, since it follows the same argument of the partial trace version:

**Proposition 4.11.** Fix $u \in [0,1]$, $\lambda \vdash n$ and $\sigma \in S_r$ with $r \leq n - 1$. Set $(F_{\lambda}^\sigma)^*(u) = \frac{v_j}{\sqrt{\lambda}} = v^\lambda$. Define

$$\bar{u} = \frac{\dim \lambda}{\dim \mu_j} \left( u - \sum_{j \neq j'} \dim \mu_j \right) < 1,$$

then

$$PS_u^{\lambda}(\sigma) = \sum_{y_j < v^{1/\sqrt{\lambda}}} \frac{\dim \mu_j}{\dim \lambda} TS^{\mu_j}(\sigma) + \frac{\dim \mu_j}{\dim \lambda} PS_u^{\mu_j}(\sigma). \quad (23)$$

As before, we call the main term for the partial sum $MS_u^{\lambda}(\sigma) := \sum_{y_j < v^{1/\sqrt{\lambda}}} \frac{\dim \mu_j}{\dim \lambda} TS^{\mu_j}(\sigma)$ and remainder for the partial sum $RS_u^{\lambda}(\sigma) := \frac{\dim \mu_j}{\dim \lambda} PS_u^{\mu_j}(\sigma)$.

The connection between the main term for the partial trace and the main term for the partial sum is easily described by applying Proposition 4.8:

$$MS_u^{\lambda}(\sigma) = \sum_{y_j < v^{1/\sqrt{\lambda}}} \frac{\dim \mu_j}{\dim \lambda} TS^{\mu_j}(\sigma)$$

$$= \sum_{y_j < v^{1/\sqrt{\lambda}}} \frac{\dim \mu_j}{\dim \lambda} \mathbb{E}_{\mathcal{PL}} \left[ \hat{\chi}^{\nu}(\tau) TS^{\nu}(\sigma) \right] \hat{\chi}^{\mu_j}(\tau)$$

$$= \sum_{\tau \in S_r} \mathbb{E}_{\mathcal{PL}} \left[ \hat{\chi}^{\nu}(\tau) TS^{\nu}(\sigma) \right] \left( \sum_{y_j < v^{1/\sqrt{\lambda}}} \frac{\dim \mu_j}{\dim \lambda} \hat{\chi}^{\mu_j}(\tau) \right)$$

$$= \sum_{\tau \in S_r} \mathbb{E}_{\mathcal{PL}} \left[ \hat{\chi}^{\nu}(\tau) TS^{\nu}(\sigma) \right] MT_u^{\lambda}(\tau), \quad (24)$$

20
We can thus apply Theorem 2.10 on the convergence of the main term for the partial trace to describe the asymptotic of $MS^\lambda_u(\sigma)$; notice that the only term in the previous sum which does not disappear when $\lambda$ increases is the one in which $\tau$ is the identity; moreover, the second highest degree term correspond to $\tau$ being a transposition, and thus of order $O_P(n^{-\frac{1}{2}})$. We get:

**Corollary 4.12.** Set $\sigma_1 \in S_r, \sigma_2 \in S_{r_2}, \ldots, u_1, u_2, \ldots \in [0, 1]$ and let $\lambda \vdash n$. Consider as before

$$m_i := E_{PL}^r [TS^\nu(\sigma_i)]$$

and

$$u_i := \left(\frac{r_i}{2}\right) E_{PL}^r \left[\chi^\nu(\{2, 1, \ldots\}) TS^\nu(\sigma_i)\right].$$

Then

$$\{\sqrt{n} \cdot (MS^\lambda_u(\sigma_1) \cdot m_i) \} \overset{d}{\to} \{u_i \cdot N'(0, 2\nu_i^2)\}.$$ 

**Proof.** For a generic $u$ and $\sigma$, we rewrite (24) as:

$$MS^\lambda_u(\sigma) = m \cdot MT^\lambda_u(Id) + \sum_{\tau \in S_r, \ wt(\tau) = 2} E_{PL}^r [\chi^\nu(\tau) TS^\nu(\sigma)] MT^\lambda_u(\tau) + \sum_{\tau \in S_r, \ wt(\tau) > 2} E_{PL}^r [\chi^\nu(\tau) TS^\nu(\sigma)] MT^\lambda_u(\tau).$$

By Lemma 4.13, $MT^\lambda_u(Id) \overset{P}{\to} u$. By Theorem 2.10

$$n^{-\frac{1}{2}} \sum_{\tau \in S_r, \ wt(\tau) = 2} E_{PL}^r [\chi^\nu(\tau) TS^\nu(\sigma)] MT^\lambda_u(\tau) \overset{d}{\to} v \cdot u \cdot N(0, 2);$$

on the other hand

$$\sum_{\tau \in S_r, \ wt(\tau) > 2} E_{PL}^r [\chi^\nu(\tau) TS^\nu(\sigma)] MT^\lambda_u(\tau) \in o_P(n^{-\frac{1}{2}}).$$

Thus

$$\{\sqrt{n} \cdot (MS^\lambda_u(\sigma_1) \cdot m_i) \} \overset{d}{\to} \{u_i \cdot N'(0, 2\nu_i^2)\}. \quad \Box$$

Although we cannot show a satisfying result on the convergence of the partial trace because we cannot prove that $u \overset{\text{dim } \mu_{(i,j)}}{\sim} \dim \lambda PT^\lambda_{\mu_{(i,j)}}(\sigma) \to 0$, we are more lucky with the partial sum:

**Theorem 4.13.** Set $\sigma \in S_r$ and let $\lambda \vdash n$. Set $m := E_{PL}^r [TS^\nu(\sigma)]$ Then

$$PS^\lambda_u(\sigma) \overset{P}{\to} u \cdot m.$$ 

We will prove the theorem after three lemmas.

**Lemma 4.14.** Let as usual $\sigma \in S_r$ and $\lambda \vdash n$, with $n > r$. Then $\pi^\lambda(\sigma)_{i,j} = 0$ for all $i, j$ such that $|i - j| > r!$

**Proof.** We iteratively use the Formula (7):

$$\pi^\lambda(\sigma) = \bigoplus_{\mu^{(n-1)}, \lambda} \pi^\nu_{\mu^{(n-1)} \lambda}(\sigma) = \bigoplus_{\mu^{(n-2)}, \lambda} \pi^\nu_{\mu^{(n-2)} \mu^{(n-1)}} \cdot \pi^\nu_{\mu^{(r)} \cdots \mu^{(r)} \lambda}(\sigma).$$

Therefore $\pi^\lambda(\sigma)$ is a block matrix such that the only nonzero blocks are those on the diagonal. To conclude, notice that $\dim \mu^{(r)} \leq r!$.

**Lemma 4.15.** With the usual setting of $\lambda$, $\sigma$ and $u$ we have

$$|PS^\lambda_u(\sigma)| \leq 2u \cdot r! \cdot 2^{l(\sigma)},$$

where $l(\sigma)$ is the length of the reduced word of $\sigma$, i.e. the minimal number of adjacent transpositions occurring in the decomposition of $\sigma$. 

21
Proof. We first prove a bound on the absolute value of an entry of the matrix

$$|\pi^\lambda(\sigma)_{T,S}| \leq 2^{l(\sigma)}$$

for each $T,S$ by induction on $l(\sigma)$. The initial case is $\sigma = (k,k+1)$, for which $|\pi^\lambda((k,k+1))_{T,S}| \leq 1$. Suppose $\sigma = (k,k+1)\bar{\sigma}$, then

$$|\pi^\lambda(\sigma)_{T,S}| = \frac{1}{d_k(T)} \pi^\lambda(\bar{\sigma})_{T,S} + \sqrt{1 - \frac{1}{d_k(T)^2} \pi^\lambda(\bar{\sigma})_{(k,k+1),T,S} \delta_{(k,k+1)T,S \in \text{SYT}(\lambda)}} \leq 2^{l(\bar{\sigma})} + 2^{l(\bar{\sigma})} = 2^{l(\sigma)}.$$

We see that

$$|PS^\lambda_u(\sigma)| = \sum_{i,j \leq u \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,j}}{\dim \lambda} \leq 2u \dim \lambda \cdot r! \max_{T,S \in \text{SYT}(\lambda)} \frac{\pi^\lambda(\sigma)_{T,S}}{\dim \lambda} \leq 2^{l(\sigma)+1} u \cdot r!,$$

which allows us to conclude. Notice that we used the previous lemma in the first inequality, which shows that the number of nonzero terms appearing in the sum is bounded by $2u \dim \lambda \cdot r!$.\qed

Lemma 4.16. Let $\sigma \neq \text{Id}$, then $PT^\lambda_u(\sigma) \xrightarrow{p} 0$.

Proof. Recall the decomposition of the partial trace into main term and remainder (Proposition 2.8):

$$PT^\lambda_u(\sigma) = MT^\lambda_u(\sigma) + RT^\lambda_u(\sigma).$$

By Theorem 2.10 if $\sigma \neq \text{Id}$ then $MT^\lambda_u(\sigma) \in o_p(u^{-\frac{\dim \mu^\lambda}{\dim \lambda}}) \subseteq o_p(1)$. On the other hand we can estimate the remainder through the bound on a singular entry calculated in (25):

$$|RT^\lambda_u(\sigma)| \leq 2^{l(\sigma)} \frac{\dim \mu^\lambda}{\dim \lambda} \xrightarrow{p} 0,$$

where $l(\sigma)$ is the length of the reduced word of $\sigma$, $\mu^\lambda$ is the subpartition corresponding to the renormalized content $\frac{\lambda}{\sqrt{\lambda}} = (F^\lambda_{\lambda,\lambda})^*(u)$, and

$$\tilde{u} = \frac{\dim \lambda}{\dim \mu^\lambda} \left( u - \sum_{j<i} \dim \mu_j \right).$$\qed

Proof of Proposition 4.13. We claim that the partial sum $PS^\lambda_u(\sigma)$ and the following quantity are asymptotically close

$$\sum_{\tau \in S_T} E^\tau u \left[ \chi^\nu(\tau)T^{S^\nu}(\sigma) \right] PT^\lambda_u(\tau),$$

that is,

$$\left| PS^\lambda_u(\sigma) - \sum_{\tau \in S_T} E^\tau u \left[ \chi^\nu(\tau)T^{S^\nu}(\sigma) \right] PT^\lambda_u(\tau) \right| \xrightarrow{p} 0.$$

We substitute in the previous expression the decomposition formulas for the partial sum and partial trace, respectively Propositions 4.11 and 2.8 and we simplify according to the equality of Equation (24), so that it remains:

$$\frac{\dim \mu^\lambda}{\dim \lambda} \left| PS^\lambda_{\tilde{u}}(\sigma) - \sum_{\tau \in S_T} E^\tau u \left[ \chi^\nu(\tau)T^{S^\nu}(\sigma) \right] PT^\mu_{\tilde{u}}(\tau) \right|.$$

We recall from (25) that $|\pi^\lambda(\tau)_{T,S}| \leq 2^{l(\tau)}$ for each $T,S$, which implies that

$$|PT^\mu_{\tilde{u}}(\tau)| \leq 2^{l(\tau)} \cdot \tilde{u},$$

hence

$$\frac{\dim \mu^\lambda}{\dim \lambda} \left| PS^\lambda_{\tilde{u}}(\sigma) - \sum_{\tau \in S_T} E^\tau u \left[ \chi^\nu(\tau)T^{S^\nu}(\sigma) \right] PT^\mu_{\tilde{u}}(\tau) \right| \leq \frac{\dim \mu^\lambda}{\dim \lambda} \left( 2\tilde{u} \cdot r! \cdot 2^{l(\sigma)} + \sum_{\tau \in S_T} E^\tau u \left[ \chi^\nu(\tau)T^{S^\nu}(\sigma) \right] 2^{l(\tau)} \cdot \tilde{u} \right) \xrightarrow{p} 0.$$
since the expression inside the parenthesis is bounded and \( \dim \mu_j / \dim \lambda \xrightarrow{\lambda} 0 \).

On the other hand

\[
\sum_{\tau \in S_r} E_{PL} [\chi^r(\tau)TS^\nu(\tau)] PT^\lambda_u(\tau) = \frac{u \dim \lambda}{\dim \lambda} E_{PL} [TS^\nu(\tau)] + o(1) \quad \text{as} \quad u \xrightarrow{\lambda} E_{PL} [TS^\nu(\tau)].
\]

For the same reasons for which we cannot prove convergence of the partial trace, here we cannot show a second order asymptotic, indeed we know that the term \( \frac{u \dim \lambda}{\dim \lambda} PS^\mu_u(\sigma) \) disappear when \( \lambda \) grows, but we do not know how fast. This will be discussed more deeply in the next section.

We can now generalize the concept of partial sum, and Lemma \((1.13)\) allows us to describe its asymptotics.

**Definition 4.17.** Let \( \lambda \vdash n, \sigma \in S_n, u_1, u_2 \in \mathbb{R} \). Define the partial sum of the entries of the irreducible representation matrix associated to \( \pi^\lambda(\sigma) \) stopped at \((u_1, u_2)\) as

\[
PS^\lambda_{u_1, u_2}(\sigma) := \sum_{i \leq u_1 \dim \lambda \quad j \leq u_2 \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,j}}{\dim \lambda}.
\]

**Corollary 4.18.** Let \( u_1, u_2 \in [0,1] \), then

\[
PS^\lambda_{u_1, u_2}(\sigma) \xrightarrow{\lambda} \min\{u_1, u_2\} E_{PL} [TS^\nu(\sigma)]
\]

**Proof.** Suppose \( u_1 < u_2 \), we only need to prove that

\[
|PS^\lambda_{u_1, u_2}(\sigma) - PS^\lambda_{u_1}(\sigma)| = \sum_{i \leq u_1 \dim \lambda \quad j \leq u_2 \dim \lambda} \frac{\pi^\lambda(\sigma)_{i,j}}{\dim \lambda} \xrightarrow{\lambda} 0.
\]

The number of nonzero terms in the above sum is bounded by \( (r!)^2 \) by Lemma \((4.4)\) moreover by Equation \((2.15)\) we have \( |\pi^\lambda(\sigma)_{i,j}| \leq 2^{|\sigma|} \). Therefore

\[
|PS^\lambda_{u_1, u_2}(\sigma) - PS^\lambda_{u_1}(\sigma)| \xrightarrow{\lambda} 0,
\]

and the corollary follows. \(\square\)

## 5 A Conjecture

**Conjecture 5.1.** Set as usual \( \lambda \vdash n \). We say that \( \mu \subseteq \lambda \) if \( \mu \) is a partition obtained from \( \lambda \) by removing boxes.

We conjecture that there exists \( \alpha > 0 \) such that, for all \( s \),

\[
P^n_{PL} \left( \left\{ \lambda : \max_{\mu \subseteq \lambda \vdash n} \frac{\dim \mu}{\dim \lambda} > n^{-\alpha s} \right\} \right) \rightarrow 0.
\]

We run some tests which hint the conjecture to be true for \( s = 1 \); for example, if \( \alpha = 0.2 \) and \( n \leq 70 \), we have

\[
P^n_{PL} \left( \left\{ \lambda : \max_{\mu \subseteq \lambda \vdash n} \frac{\dim \mu}{\dim \lambda} > n^{-0.2} \right\} \right) \leq \begin{cases} 0.2 & \text{if } n \geq 7 \\ 0.1 & \text{if } n \geq 12 \\ 0.05 & \text{if } n \geq 37. \end{cases}
\]

Notice that, for \( s > 1 \),

\[
\max_{\mu \subseteq \lambda \vdash n} \frac{\dim \mu}{\dim \lambda} \leq \prod_{i=1}^{s} \left( \max_{\mu \subseteq \lambda \vdash n, \mu \not\subseteq \mu^{(i)} \vdash n} \frac{\dim \mu}{\dim \mu^{(i)}} \right),
\]

so that it may seem that it is enough to prove the conjecture just for \( s = 1 \). This is not true though, since the sequence \( \mu^{(1)} \prec \cdots \prec \mu^{(s-1)} \) in the RHS above is not Plancherel distributed. Hence we need the Conjecture in the more general form.
We consider which implies that $\dim \lambda$ (because of Theorem 2.10), while with high probability. We choose $\sigma_0$ such that $\dim \lambda = \max_{|\lambda|=\lambda} \dim \mu$. Since $\dim \mu$ decreases as $|\lambda|$ increases, we have $\dim \mu < \sigma_0$ for all $|\lambda| > \lambda$. Therefore, we have

$$
\{PT_u^\lambda(\sigma_i)\} \overset{d}{=} \left\{ u_i \prod_{k \geq 2} k^{m_k(\mu(k))} \mathcal{H}_{m_k(\mu(k))}^2(\xi_k) \right\}.
$$

Proof. Recall from Proposition 2.11 that for any $s < n - r$, there exists a sequence of partitions $\mu^{(0)} \triangleright \mu^{(1)} \triangleright \ldots \triangleright \mu^{(s)} = \lambda$ and a sequence of real numbers $0 \leq c_0, \ldots, c_s < 1$ such that

$$
PT_u^\lambda(\sigma) = \sum_{i=1}^{s} \frac{\dim \mu^{(i)}}{\dim \lambda} MT_u^{\mu^{(i)}}(\sigma) + \frac{\dim \mu^{(0)}}{\dim \lambda} PT_u^{\mu^{(0)}}(\sigma).
$$

We consider

$$
n^{-\frac{\alpha s}{2}} PT_u^\lambda(\sigma) = n^{-\frac{\alpha s}{2}} \sum_{i=1}^{s} \frac{\dim \mu^{(i)}}{\dim \lambda} MT_u^{\mu^{(i)}}(\sigma) + n^{-\frac{\alpha s}{2}} \frac{\dim \mu^{(0)}}{\dim \lambda} PT_u^{\mu^{(0)}}(\sigma).
$$

In the first sum of the right hand side the term corresponding to $i = s$ is

$$
n^{-\frac{\alpha s}{2}} MT_u^{\lambda}(\sigma) \overset{d}{=} \sum_{k \geq 2} k^{m_k(\mu(k))} \mathcal{H}_{m_k(\mu(k))}^2(\xi_k),
$$

due to Theorem 2.10. On the other hand the other terms in the first sum of the RHS are of the form $n^{-\frac{\alpha s}{2}} \frac{\dim \mu^{(i)}}{\dim \lambda} MT_u^{\mu^{(i)}}(\sigma)$ for $i = 1, \ldots, s - 1$, and it is easy to see that $(n - s + 1)^{-\frac{\alpha s}{2}}MT_u^{\mu^{(i)}}(\sigma)$ converge (because of Theorem 2.10), while $\frac{\dim \mu^{(i)}}{\dim \lambda} \in O_P(1)$ because of the convergence of the normalized co-transition distribution towards an atom free distribution (see Lemma 2.3). We study thus the term $n^{-\frac{\alpha s}{2}} \frac{\dim \mu^{(0)}}{\dim \lambda} PT_u^{\mu^{(0)}}(\sigma)$, and Conjecture 5.3 implies that

$$
\frac{\dim \mu^{(0)}}{\dim \lambda} \leq \max_{\mu: |\mu|=\lambda} \frac{\dim \mu}{\dim \lambda} \leq n^{-\alpha s},
$$

with high probability. We choose $s$ such that $\alpha s/2 < \alpha s$. An easy application of Inequality 25 implies that $|PT_u^{\mu^{(0)}}(\sigma)| \leq u^{(0)} \cdot 2^{|\sigma|}$, thus we have

$$
n^{-\frac{\alpha s}{2}} \frac{\dim \mu^{(0)}}{\dim \lambda} PT_u^{\mu^{(0)}}(\sigma) \leq n^{-\frac{\alpha s}{2}} u^{(0)} \cdot 2^{|\sigma|} \to 0,
$$

which implies

$$
n^{-\frac{\alpha s}{2}} \frac{\dim \mu^{(0)}}{\dim \lambda} PT_u^{\mu^{(0)}}(\sigma) \in O_P(1).
$$

Therefore

$$
\{PT_u^\lambda(\sigma_i)\} \overset{d}{=} \{ u_i \prod_{k \geq 2} k^{m_k(\mu(k))} \mathcal{H}_{m_k(\mu(k))}^2(\xi_k) \},
$$

which concludes the proof.

6 Acknowledgments

The author would like to express his gratitude to Valentin Féray for introducing him on the subject, many insightful discussions, and several corrections and suggestions in the development of the paper.

This research was founded by SNSF grant SNF-149461: “Dual combinatorics of Jack polynomials”.

References

[Bia03] Ph. Biane. Characters of symmetric groups and free cumulants. In Asymptotic combinatorics

with applications to mathematical physics (St. Petersburg, 2001), volume 1815 of Lecture Notes

in Math., pages 185–200. Springer, Berlin, 2003.
[BOO00] A. Borodin, A. Okounkov, and G. Olshanski. Asymptotics of Plancherel measures for symmetric groups. *J. Amer. Math. Soc.*, 13(3):481–515, 2000.

[DDN03] A. D’Aristotile, P. Diaconis, and C. Newman. Brownian motion and the classical groups. *Lecture Notes–Monograph Series*, 41:97–116, 2003.

[DFŚ10] M. Dołęga, V. Féray, and P. Śniady. Explicit combinatorial interpretation of Kerov character polynomials as numbers of permutation factorizations. *Adv. in Math.*, 225(1):81–120, 2010.

[Fér12] V. Féray. Partial Jucys–Murphy elements and star factorizations. *European J. Combin.*, 33(2):189–198, 2012.

[FŚ11] V. Féray and P. Śniady. Asymptotics of characters of symmetric groups related to Stanley character formula. *Ann. of Math.*, 173(2):887–906, 2011.

[GH03] J. Geronimo and T. Hill. Necessary and sufficient condition that the limit of Stieltjes transforms is a Stieltjes transform. *J. Approx. Theory*, 121(1):54–60, 2003.

[Gre92] C. Greene. A rational function identity related to the Murnaghan-Nakayama formula for the characters of $S_n$. *J. Algebr. Comb.*, 1(3):235–255, 1992.

[Hor98] A. Hora. Central limit theorem for the adjacency operators on the infinite symmetric group. *Comm. Math. Phys.*, 195(2):405–416, 1998.

[IK99] V. Ivanov and S. Kerov. The algebra of conjugacy classes in symmetric groups, and partial permutations. *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov. (POMI)*, 256(3):95–120, 1999.

[IO02] V. Ivanov and G. Olshanski. Kerov’s central limit theorem for the Plancherel measure on Young diagrams. In *Symmetric functions 2001: surveys of developments and perspectives*, pages 93–151. Springer, 2002.

[Juc66] A. Jucys. On the Young operators of the symmetric groups. *Lithuanian Phys. J.*, VI(2):180–189, 1966.

[Juc74] A. Jucys. Symmetric polynomials and the center of the symmetric group ring. *Reports Math. Phys.*, 5:107–112, 1974.

[Ker93a] S. Kerov. Gaussian limit for the Plancherel measure of the symmetric group. *C. R. Acad. Sci. Paris*, 316:303–308, 1993.

[Ker93b] S. Kerov. Transition probabilities for continual Young diagrams and the Markov moment problem. *Funct. Anal. Appl.*, 27(2):104–117, 1993.

[KV77] S. Kerov and A. Vershik. Asymptotics of the Plancherel measure of the symmetric group and the limiting form of Young tableaux. In *Soviet Math. Dokl*, volume 18, pages 527–531, 1977.

[LS77] B. Logan and L. Shepp. A variational problem for random Young tableaux. *Adv. in Math.*, 26(2):206–222, 1977.

[Mur81] G. Murphy. A new construction of Young’s seminormal representation of the symmetric group. *J. Algebra*, 69:287–291, 1981.

[OO98] A. Okounkov and G. Olshanski. Shifted Schur functions. *St. Petersburg Math. J.*, 9:239–300, 1998.

[Rom15] D. Romik. *The surprising mathematics of longest increasing subsequences*, volume 4 of *Textbooks*. Cambridge University Press, 2015.

[Sag13] B. Sagan. *The symmetric group: representations, combinatorial algorithms, and symmetric functions*, volume 203 of *Graduate Texts in Mathematics*. Springer Science & Business Media, 2013.

[You77] A. Young. *The collected papers of Alfred Young 1873-1940*. Mathematical expositions. University of Toronto Press, 1977.