Conditional Symmetry and Reductions for the Two-Dimensional Nonlinear Wave Equation. I. General Case.

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Abstract

We present classification of $Q$-conditional symmetries for the two-dimensional nonlinear wave equations $u_{tt} - u_{xx} = F(t, x, u)$ and the reductions corresponding to these nonlinear symmetries. Classification of inequivalent reductions is discussed.

1 Introduction

Following [1], we discuss conditional symmetries and reductions of the two-dimensional nonlinear wave equation

$$u_{tt} - u_{xx} = F(t, x, u)$$

for the real-valued function $u = u(t, x)$; $t$ is the time variable, $x$ is the space variable. In the equation above and further we will use the following designations for the partial derivatives:

$$u_t = -\frac{\partial u}{\partial t}; \quad u_x = -\frac{\partial u}{\partial x}; \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}; \quad u_{xt} = u_{tx} = \frac{\partial^2 u}{\partial t\partial x}; \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}.$$

Note that the general equation in the class (1) has no invariance operators; however, many well-known particular cases have wide symmetry algebras, see e.g. [2].

The maximal invariance algebra of the equation (1) with general $F = F(u)$ is the Poincaré algebra $AP(1, 1)$ with the basis operators

$$p_t = \frac{\partial}{\partial t}, \quad p_x = \frac{\partial}{\partial x}, \quad J = tp_x + xp_t.$$
The invariance algebras of the equation (1) will also include dilation operators e.g. for $F = \lambda u^k$ or $F = \lambda \exp u$. Equations (1) with e.g. $F = 0$ and $F = \lambda \exp u$ have infinite-dimensional symmetry algebras.

Similarity solutions for the equation (1) can be found by symmetry reduction with respect to non-equivalent subalgebras of its invariance algebras. For studies of symmetry and non-classical solutions of the nonlinear wave equation for various space dimensions see [2]–[9].

Here we present results on classification of $Q$-conditional symmetries for the equation (1) and the relevant reductions in the meaningful cases.

It seems that investigation of conditional symmetry now has fallen out of the mainstream of the symmetry analysis of PDE. We would guess that the reason is that practically all interesting equations for which the problem is manageable (mostly for the evolution equations) have been studied already. However, we believe that this problem remains relevant - first, with respect to investigation of ”difficult, but interesting” equations (e.g. equations with highest derivatives for all variables of the same order, such as the nonlinear wave equation under study), and with respect to investigation of various related aspects (e.g. geometrical aspects and equivalence).

2 What we mean by conditional symmetry

Conditional symmetry in general (additional invariance under arbitrary additional condition) and a narrower concept of the $Q$-conditional invariance (the additional condition has the form $Q u = 0$) were initially discussed in the papers [10]–[14]. Later numerous authors developed these ideas into theory and a number of algorithms for studying symmetry properties of equations of mathematical physics. The importance of investigation of the $Q$-conditional symmetry stems from equivalence of the $Q$-conditional invariance and reducibility of the equations by means of ansatzes determined by such operators $Q$ (see [15]).

Here we will use the following definition of the $Q$-conditional symmetry:

**Definition 1.** The equation $\Phi(x, u, u_1, \ldots, u_l) = 0$, where $u$ is the set of all $k$th-order partial derivatives of the function $u = (u^1, u^2, \ldots, u^m)$, is called
$Q$-conditionally invariant \cite{5} under the operator

$$Q = \xi^i(x, u)\partial_{x^i} + \eta^r(x, u)\partial_{u^r}$$

if there is an additional condition

$$Qu = 0,$$

such that the system of two equations $\Phi = 0$, $Qu = 0$ is invariant under the operator $Q$. All differential consequences of the condition $Qu = 0$ shall be taken into account up to the order $l - 1$.

This definition of the conditional invariance of some equation implies in reality a Lie symmetry (see e.g. the classical texts\cite{16, 17, 18}) of the same equation with a certain additional condition. Conditional symmetries of the multidimensional nonlinear wave equations are specifically discussed in\cite{20, 24, 25}.

3 Previous work on the problem

The particular problem we discuss here was first mentioned by P.Clarkson and E. Mansfield in\cite{21} (the case $f = f(u)$), where the relevant determining equations were written out but not solved, and studies were continued by M. Euler and N. Euler in\cite{22}. In the latter paper the determining conditions for the $Q$-conditional invariance were taken without consideration of the differential consequences of the condition $Qu = 0$, so the resulting operators did not actually present the solution of the problem.

Following\cite{21}, we will consider the equation (in conic variables) equivalent to (1) of the form

$$u_{yz} = f(y, z, u).$$

(3)

We search for the operators of $Q$-conditional invariance in the form

$$Q = a(y, z, u)\partial_y + b(y, z, u)\partial_z + c(y, z, u)\partial_u.$$
4 Background of Classification of Conditional Symmetries

We will base our classification on the procedure of solution of the determining equations for conditional symmetry and then study equivalence within the resulting classes.

Anyway, the equivalence group of the system of the equation (3) and the additional condition

$$a(y, z, u)u_y + b(y, z, u)u_z + c(y, z, u) = 0$$ (5)

determined by the operator (4) is quite narrow, and the standard classification procedure may not be appropriate.

Let us note that the concept of equivalence of $Q$-conditional symmetries was introduced by R. Popovych in [23].

We study the conditional symmetry for the general case $f = f(y, z, u)$. We will not consider the case $f = 0$, as equation (3) in this case has a general solution, so its conditional symmetries may be not quite relevant. There are some interesting special partial cases of the equation (3), first of all when $f = f(u)$ and $f = r(y, z)u$ that will be considered in a future paper.

Considering the system (3), (5), we can see three inequivalent cases to be studied separately:

1. $a = 0$, $b \neq 0$.

Then we can take

$$Q = \partial_z + L(y, z, u)\partial_u$$ (6)

The case $a \neq 0$, $b = 0$ is equivalent to $a = 0$, $b \neq 0$.

In such cases the additional condition reduces equation (3) to a pair of the first-order equations.

2. $a \neq 0$, $b \neq 0$.

In this case we can take

$$Q = \partial_y + K(y, z, u)\partial_z + L(y, z, u)\partial_u.$$ (7)

where $K(y, z, u) \neq 0$. 
This case is obviously the most interesting for consideration. It might seem appropriate to consider separately the case $c = 0$, but it is easy to check that such systems may be equivalent to the general systems within case 2, so it should not be considered separately.

3. $a = 0, b = 0$

This case is trivial and in the case if the original equation and the additional condition are compatible, the additional condition just gives a solution for the equation.

For cases 1-2 the additional condition $Qu = 0$ will be represented respectively by the equations

$$u_z = L(y, z, u),$$  \hspace{1cm} (8)

and

$$u_y + K(y, z, u)u_z = L(y, z, u).$$  \hspace{1cm} (9)

We can start with considering of determining equations for the case 2 with $K \neq 0$, having in mind both cases. The determining equations for the conditional symmetry have the form

$$-K_u^2 + K_uuK = 0,$$  \hspace{1cm} (10)

$$-KL_{uu} + \frac{K_uK_y}{K} + \frac{K_y^2L}{K} + K_u(L_u - K_z) - K_{uy} - LK_{uu} + KK_{zu} = 0,$$  \hspace{1cm} (11)

$$L_{uy} - L_{uz}K + L_{uu}L - \frac{L_uK_y}{K} + \frac{K_yK_z}{K} - K_{yz} = 0,$$

$$3K_uL - \frac{K_uL}{K}(L_u - K_z) + K_uL_z - K_{zu}L = 0,$$  \hspace{1cm} (12)

$$-f_y - Kf_z - Lf_u + Lyz + Lu_zL + Lu_f - \frac{K_y}{K}(L_z - f) - K zf - \frac{K_uL}{K}(L_z - f) = 0$$  \hspace{1cm} (13)

Let us note that these determining equations were first found in [21], though, not studied further.
5 Conditional Symmetry: Main Results

Case 1 - $K = 0$. Here we have equations

$$u_y = L, u_{yz} = f.$$  \(14\)

The determining equations have the form

$$L_{uy} + L_{uu}L = 0,$$

$$-f_y - Lf_u + L_{yz} + L_{uz}L + Luf = 0$$

This case is actually equivalent to a pair of first-order equations

$$u_y = L, u_z = \frac{f - L_z}{L_u}. \quad (15)$$

If we check directly the compatibility conditions they will coincide with the determining equations of conditional invariance.

Case 2.1. $K_u = 0$, $K \neq 0$. The determining equations have the form

$$-KL_{uu} = 0,$$

$$L_{uy} - L_{uz}K + L_{uu}L - \frac{L_uK_y}{K} + \frac{K_yK_z}{K} - Kyz = 0,$$

$$-f_y - Kf_z - Lf_u + L_{yz} + L_{uz}L + Luf - \frac{K_y}{K}(L_z - f) - Kzf = 0$$

We have $K = k(y, z)$, $L = s(y, z)u + d(y, z)$. Using equivalence transformations, we can put $d(y, z) = 0$.

From the determining equations we get

$$k(y, z) = \frac{T_y}{T_z}, \quad (16)$$

$$s(y, z) = \frac{T_{yz}}{T_z}, \quad (17)$$
where \( T = T(y, z) \) is an arbitrary function.

The operator of \( Q \)-conditional symmetry then will be

\[
Q = \partial_y + \frac{T_y}{T_z} \partial_z + \frac{T_y z}{T_z} u \partial_u.
\]

In this case the ansatz reducing equation (3) will have the form

\[
u = \sigma(y, z) \phi(\omega),
\]

where \( \omega = \omega(y, z) \) is a new variable,

\[
T_y \omega_z + T_z \omega_y = 0,
\]

\[
T_y \sigma_z + T_z \sigma_y = \sigma T_{yz}.
\]

The reduced equation will have the form:

\[
\sigma_{yz} \phi + \phi'(\omega_y \sigma_z + \omega_z \sigma_y + \sigma \omega_{yz}) + \phi'' \sigma \omega_y \omega_z = f,
\]

where \( f \) satisfies the relevant conditions (13).

From these conditions we can find the form of the function \( f \) up to equivalence:

\[
f = \frac{T_y T_z}{\sigma^3} \Phi(\omega, \frac{u}{\sigma}) + \frac{\sigma_{yz}}{\sigma} u,
\]

where \( T(y, z) \) is the same arbitrary function entering expressions (16), (17).

At the first glance equation (3) may seem equivalent to some equation of the form \( f = T_y T_z \Phi(\omega, u) \) reducible with the ansatz \( u = \phi(\omega) \). However, generally that is not the case. The criterion for such reduction has the following form:

\[
\sigma_y = k \sigma_z,
\]

where \( k \) is determined by (16).

Let us have a further look at the reduced equation (19). From conditions on \( \omega \) and \( \sigma \) it is easy to check that

\[
\omega_y \sigma_z + \omega_z \sigma_y + \sigma \omega_{yz} = 0,
\]

so the reduced equation will have the form

\[
\phi'' \sigma \omega_y \omega_z = \frac{T_y T_z}{\sigma^3} \Phi(\omega, \phi).
\]
Note that the reduced equations for this case will not include first-order derivatives.

As from conditions on $\omega$ and $\sigma$

$$\sigma^2 = \frac{T_z}{\omega_z} = -\frac{T_y}{\omega_y},$$

we come to the final form of the reduced equation

$$\phi'' = -\Phi(\omega, \phi). \quad (21)$$

Equations of the form (21) include many remarkable ODE, equations for many special functions among them.

**Case 2.2.** $K_u \neq 0$, then $K_{uu}K = K_u^2$, $K = k(y, z)exp(l(y, z)u)$. We can put $k = 1$ and prove from the resulting determining conditions $l_y = l_z = 0$, so we can put $l = 1$.

Then we can found that $L = s(y, z)expu + d(y, z)$. It is possible to reduce this case to $k = 1$, and we get the following determining equation for $f$ with arbitrary $s$ and $d$:

$$f = \frac{1}{3}(s_y + d_z), \quad (22)$$

so $f$ in this case depends only on $y$ and $z$, and the equation $u_{yz} = f(y, z)$ is equivalent to the equation $u_{yz} = 0$.

The conditions for $s$ and $d$ have the form

$$2s_{yz} - sd_z + 2s_y s - d_{zz} = 0, \quad -s_{yy} + 2d_{yz} + s_y d - 2d_z d = 0. \quad (23)$$

**6 Conclusions**

We have considered the equations

$$u_{yz} = f(y, z, u)$$

with $f$ depending on $y, z, u$.

For such general class the only nontrivial case is Case 2.1, $K_u = 0$, $K \neq 0$. 

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We have found that in this case the reduced equation has the form
\[ \phi'' = -\Phi(\omega, \phi), \]
including many remarkable equations.

We have found a general form of the equation (3) that can be reduced to an ODE by means of an ansatz (18) determined by the conditional symmetry operator (7) - \( f \) has to be of the form (20). However, for a general equation it may be not straightforward to determine whether \( f \) can be reduced to such form.

The cases \( f = f(u) \), \( f = r(y, z)u \) require special consideration, and have more inequivalent cases.

Further research may also include study of the general conditional symmetry of the nonlinear wave equation in higher dimensions, as well as description of equivalence classes of conditional symmetries.

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1 Introduction

Following [1], we discuss conditional symmetries and reductions of the two-dimensional nonlinear wave equation

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    u_{tt} - u_{xx} = F(t,x,u)
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for the real-valued function \( u = u(t,x) \); \( t \) is the time variable, \( x \) is the space variable. In the equation above and further we will use the following designations for the partial derivatives:

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    u_t = -\frac{\partial u}{\partial t}; \quad u_x = -\frac{\partial u}{\partial x}; \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}; \quad u_{tx} = u_{xt} = \frac{\partial^2 u}{\partial t \partial x}; \quad u_{xx} = \frac{\partial^2 u}{\partial x^2}.
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Note that the general equation in the class (1) has no invariance operators; however, many well-known particular cases have wide symmetry algebras, see e.g. [2].

The maximal invariance algebra of the equation (1) with general \( F = F(u) \) is the Poincaré algebra \( AP(1,1) \) with the basis operators

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    p_t = \frac{\partial}{\partial t}, \quad p_x = \frac{\partial}{\partial x}, \quad J = tp_x + xp_t.
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The invariance algebras of the equation (1) will also include dilation operators e.g. for $F = \lambda u^k$ or $F = \lambda \exp u$. Equations (1) with e.g. $F = 0$ and $F = \lambda \exp u$ have infinite-dimensional symmetry algebras.

Similarity solutions for the equation (1) can be found by symmetry reduction with respect to non-equivalent subalgebras of its invariance algebras. For studies of symmetry and non-classical solutions of the nonlinear wave equation for various space dimensions see [2]–[9].

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**Definition 1.** The equation $\Phi(x, u, u_1, \ldots, u_l) = 0$, where $u$ is the set of all $k$th-order partial derivatives of the function $u = (u^1, u^2, \ldots, u^m)$, is called
\( Q \)-conditionally invariant \[5\] under the operator

\[ Q = \xi^i(x, u)\partial_{x^i} + \eta^r(x, u)\partial_{u^r} \]

if there is an additional condition

\[ Qu = 0, \] (2)

such that the system of two equations \( \Phi = 0, Qu = O \) is invariant under the operator \( Q \). All differential consequences of the condition \( Qu = 0 \) shall be taken into account up to the order \( l - 1 \).

This definition of the conditional invariance of some equation implies in reality a Lie symmetry (see e.g. the classical texts \[16, 17, 18\]) of the same equation with a certain additional condition. Conditional symmetries of the multidimensional nonlinear wave equations are specifically discussed in \[20, 24, 25\].

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\[ u_{yz} = f(y, z, u). \] (3)

We search for the operators of \( Q \)-conditional invariance in the form

\[ Q = a(y, z, u)\partial_y + b(y, z, u)\partial_z + c(y, z, u)\partial_u. \] (4)
4 Background of Classification of Conditional Symmetries

We will base our classification on the procedure of solution of the determining equations for conditional symmetry and then study equivalence within the resulting classes.

Anyway, the equivalence group of the system of the equation (3) and the additional condition

$$a(y, z, u)u_y + b(y, z, u)u_z + c(y, z, u) = 0$$

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determined by the operator (4) is quite narrow, and the standard classification procedure may not be appropriate.

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Considering the system (3), (5), we can see three inequivalent cases to be studied separately:

1. $a = 0, b \neq 0$.
   Then we can take
   $$Q = \partial_z + L(y, z, u)\partial_u$$
   (6)
   The case $a \neq 0, b = 0$ is equivalent to $a = 0, b \neq 0$.
   In such cases the additional condition reduces equation (3) to a pair of the first-order equations.

2. $a \neq 0, b \neq 0$.
   In this case we can take
   $$Q = \partial_y + K(y, z, u)\partial_z + L(y, z, u)\partial_u.$$ (7)
   where $K(y, z, u) \neq 0$.  


This case is obviously the most interesting for consideration. It might seem appropriate to consider separately the case \( c = 0 \), but it is easy to check that such systems may be equivalent to the general systems within case 2, so it should not be considered separately.

3. \( a = 0, \ b = 0 \)

This case is trivial and in the case if the original equation and the additional condition are compatible, the additional condition just gives a solution for the equation.

For cases 1-2 the additional condition \( Qu = 0 \) will be represented respectively by the equations

\[
 u_z = L(y, z, u), \tag{8}
\]

and

\[
 u_y + K(y, z, u)u_z = L(y, z, u). \tag{9}
\]

We can start with considering of determining equations for the case 2 with \( K \neq 0 \), having in mind both cases. The determining equations for the conditional symmetry have the form

\[
 -K_u^2 + K_{uu}K = 0, \tag{10}
\]

\[
 -KL_{uu} + \frac{K_uK_y}{K} + \frac{K_u^2L}{K} + K_u(L_u - K_z) - K_{uy} - LK_{uu} + KK_{zu} = 0, \tag{11}
\]

\[
 L_{uy} - L_{uz}K + L_{uu}L - \frac{L_uK_y}{K} + \frac{K_yK_z}{K} - K_{yz} -
3K_uL - \frac{K_uL}{K}(L_u - K_z) + K_uL_z - K_{zu}L = 0, \tag{12}
\]

\[
 -f_y - Kf_z - Lf_u + L_{yz} + L_{uz}L + L_uL - \frac{K_y}{K}(L_z - f) - K_zf - \frac{K_uL}{K}(L_z - f) = 0 \tag{13}
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Let us note that these determining equations were first found in [21], though, not studied further.
5 Conditional Symmetry: Main Results

Case 1 - $K = 0$. Here we have equations

$$u_y = L, \quad u_{yz} = f.$$  \hfill (14)

The determining equation has the form

$$-f_y - Lf_u + L_{yz} + L_{uz}L + L_u f + (L_{uy} + L_{uu}L) \frac{f - L_z}{L_u} = 0$$

This case is actually equivalent to a pair of first-order equations

$$u_y = L, \quad u_z = \frac{f - L_z}{L_u}. \hfill (15)$$

If we check directly the compatibility conditions they will coincide with the determining equation of conditional invariance.

Case 2.1. $K_u = 0$, $K \neq 0$. The determining equations have the form

$$-KL_{uu} = 0,$$

$$L_{uy} - L_{uz}K + L_{uu}L - \frac{L_yK_y}{K} + \frac{K_yK_z}{K} - K_{yz} = 0,$$

$$-f_y - Kf_z - Lf_u + L_{yz} + L_{uz}L + L_u f - \frac{K_y}{K}(L_z - f) - K_z f = 0$$

We have $K = k(y, z)$, $L = s(y, z)u + d(y, z)$. Using equivalence transformations, we can put $d(y, z) = 0$.

From the determining equations we get

$$k(y, z) = \frac{T_y}{T_z}, \hfill (16)$$

$$s(y, z) = \frac{T_{yz}}{T_z}, \hfill (17)$$

where $T = T(y, z)$ is an arbitrary function.
The operator of $Q$-conditional symmetry then will be

$$Q = \partial_y + \frac{T_y}{T_z} \partial_z + \frac{T_y}{T_z} u \partial_u.$$  

In this case the ansatz reducing equation (3) will have the form

$$u = \sigma(y, z) \phi(\omega),$$

(18)

where $\omega = \omega(y, z)$ is a new variable,

$$T_y \omega_z + T_z \omega_y = 0,$$

$$T_y \sigma_z + T_z \sigma_y = \sigma T_y z.$$ 

The reduced equation will have the form:

$$\sigma y z \phi + \phi'(\omega y \sigma_z + \omega_z \sigma_y + \sigma \omega y z) + \phi'' \sigma \omega y \omega z = f,$$

(19)

where $f$ satisfies the relevant conditions (13).

From these conditions we can find the form of the function $f$ up to equivalence:

$$f = \frac{T_y T_z}{\sigma^3} \Phi(\omega, u) + \frac{\sigma y z}{\sigma} u,$$

(20)

where $T(y, z)$ is the same arbitrary function entering expressions (16), (17).

At the first glance equation (3) may seem equivalent to some equation of the form $f = T_y T_z \Phi(\omega, u)$ reducible with the ansatz $u = \phi(\omega)$. However, generally that is not the case. The criterion for such reduction has the following form:

$$\sigma y = k \sigma z,$$

where $k$ is determined by (16).

Let us have a further look at the reduced equation (19). From conditions on $\omega$ and $\sigma$ it is easy to check that

$$\omega y \sigma_z + \omega z \sigma_y + \sigma \omega y z = 0,$$

so the reduced equation will have the form

$$\phi'' \sigma \omega y \omega z = \frac{T_y T_z}{\sigma^3} \Phi(\omega, \phi).$$
Note that the reduced equations for this case will not include first-order derivatives.

As from conditions on $\omega$ and $\sigma$

$$\sigma^2 = \frac{T_z}{\omega_z} = -\frac{T_y}{\omega_y},$$

we come to the final form of the reduced equation

$$\phi''' = -\Phi(\omega, \phi).$$

Equations of the form (21) include many remarkable ODE, equations for many special functions among them.

**Case 2.2.** $K_u \neq 0$, then $K_{uu}K = K_u^2$, $K = k(y, z)exp(l(y, z)u)$. We can put $k = 1$ and prove from the resulting determining conditions $l_y = l_z = 0$, so we can put $l = 1$.

Then we can found that $L = s(y, z)exp + d(y, z)$. It is possible to reduce this case to $k = 1$, and we get the following determining equation for $f$ with arbitrary $s$ and $d$:

$$f = \frac{1}{3}(s_y + d_z),$$

so $f$ in this case depends only on $y$ and $z$, and the equation $u_{yz} = f(y, z)$ is equivalent to the equation $u_{yz} = 0$.

The conditions for $s$ and $d$ have the form

$$2s_{yz} - sd_z + 2s_y s - d_{zz} = 0, \quad -s_{yy} + 2d_{yz} + s_y d - 2d_z d = 0.$$  

(23)

6 Conclusions

We have considered the equations

$$u_{yz} = f(y, z, u)$$

with $f$ depending on $y, z, u$.

For such general class the only nontrivial case is Case 2.1, $K_u = 0$, $K \neq 0$. 

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We have found that in this case the reduced equation has the form
\[ \phi'' = -\Phi(\omega, \phi) , \]
including many remarkable equations.

We have found a general form of the equation (3) that can be reduced to an ODE by means of an ansatz (18) determined by the conditional symmetry operator (7) - \( f \) has to be of the form (20). However, for a general equation it may be not straightforward to determine whether \( f \) can be reduced to such form.

The cases \( f = f(u) \), \( f = r(y, z)u \) require special consideration, and have more inequivalent cases.

Further research may also include study of the general conditional symmetry of the nonlinear wave equation in higher dimensions, as well as description of equivalence classes of conditional symmetries.

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