Leibniz Algebras Graded by Finite Root Systems

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ABSTRACT

There are several researches on Lie algebras and Lie superalgebras graded by finite root systems. In this paper, we study Leibniz algebras graded by finite root systems and obtain some important results in simply-laced cases.

Key Words: Δ-Graded; dialgebras; Steinberg Leibniz algebras.

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1 Introduction

In [17], J.-L. Loday introduced a non-antisymmetric version of Lie algebras, whose bracket satisfies the Leibniz relation (see (2.5)), therefore called Leibniz algebra. The Leibniz relation, combined with antisymmetry, is a variation of the Jacobi identity, hence Lie algebras are anti-symmetric Leibniz algebras. In [19], Loday also introduced an ‘associative’ version of Leibniz algebras, called associative dialgebras, equipped with two binary operations, ⊢ and⊣, which satisfy the five relations (see the axiom (Ass) in section 2). These identities are all variations of the associative law, so associative algebras are dialgebras for which the two products coincide. The peculiar point is that the bracket [a, b] =: a ⊢ b − b ⊣ a defines a Leibniz algebra which is not antisymmetric, unless the left and right products coincide. Hence dialgebras yield a commutative diagram of categories and functors

\[
\begin{array}{ccc}
\text{Dias} & \searrow & \text{Leib} \\
\downarrow & & \downarrow \\
\text{Assoc} & \searrow & \text{Lie}
\end{array}
\]

Steinberg Lie algebras come from Steinberg groups, which are closely connected with K-theory, and play a key role in the study of Lie algebras graded by finite root systems of type $A$. By definition, the Steinberg Lie algebra $\text{st}(n, A)$ over a $K$-algebra $A$ is a Lie algebra generated by symbols $v_{ij}(a), 1 \leq i \neq j \leq n, a \in A$, subject to the relations

1. $v_{ij}(k_1a + k_2b) = k_1v_{ij}(a) + k_2v_{ij}(b)$, for $a, b \in D, k_1, k_2 \in K$;
2. $[v_{ij}(a), v_{kl}(b)] = 0$ if $i \neq l$ and $j \neq k$;
3. $[v_{ij}(a), v_{kl}(b)] = v_{il}(ab)$ if $i \neq l$ and $j = k$.

It is clear that the relation (3) makes sense only if $n \geq 3$.

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From [6] we see that the map \( \eta : a \to v_1(a) \) is one-to-one if and only if \( A \) is an associative algebra for \( n \geq 4 \) and \( A \) is an alternative algebra for \( n = 3 \).

In 1992, S. Berman and R.V. Moody ([5]) studied Lie algebras graded by finite root systems of \( A_l(l \geq 2), D_l(l \geq 4), E_l(l = 6, 7, 8) \) and obtained the structure of a Lie algebra over \( K \) graded by the root system \( \Delta \) of type \( X_l(l \geq 2) (X_l = A_l, D_l, E_l) \).

The universal central extensions of Lie algebras graded by finite root systems were studied in several papers ([3], [7], [10], [8], [1], etc.).

In this paper we shall consider Leibniz algebras graded by finite root systems of types \( A, D \) and \( E \). We also prove that

**Theorem 1.1. (Recognition Theorem).** Let \( L \) be a Leibniz algebra over \( K \) graded by the root system \( \Delta \) of type \( X_l(l \geq 2) (X_l = A_l, D_l, E_l) \).

1. If \( X_l = A_l(l \geq 3) \), then there exists a unital associative \( K \)-dialgebra \( R \) such that \( L \) is centrally isogenous with \( \mathfrak{sl}(l + 1, R) \);
2. If \( X_l = A_l(l = 2) \), then there exists a unital alternative \( K \)-dialgebra \( R \) such that \( L \) is centrally isogenous with \( \mathfrak{sl}(l + 1, R) \), where \( \mathfrak{sl}(n, R) \) is defined in Section 2.4;
3. If \( X_l = D_l(l \geq 4), E_l(l = 6, 7, 8) \), then there exists a unital associative commutative \( K \)-dialgebra \( R \) such that \( L \) is centrally isogenous with \( \hat{\mathfrak{g}} \otimes R \).

**Remark.** Two perfect Lie algebras \( L_1 \) and \( L_2 \) are called *centrally isogenous* if they have the same universal central extension (up to isomorphism).

The paper is organized as follows. In Section 2, we recall some notions of Leibniz algebras and dialgebras. In Section 3, we give the definition of Leibniz algebras graded by finite root systems. In Sections 4 and 5, we mainly prove the Recognition Theorem (Theorem 1.1). Throughout this paper, \( K \) denotes a field of characteristic 0, \( R \) a unital dialgebra over \( K \).

# 2 Dialgebras and Leibniz algebras

We recall the notions of associative dialgebras, alternative dialgebras, Leibniz algebras and their (co)homology as defined in [16]—[19] and [11].

## 2.1 Dialgebras.

**Definition 2.1.** ([19]) A dialgebra \( D \) over \( K \) is a \( K \)-vector space \( D \) with two operations \( \cdot, \triangleright : D \otimes D \to D \), called left and right products.

A dialgebra is called unital if it is given a specified bar-unit: an element \( 1 \in D \) which is a unit for the left and right products only on the bar-side, that is, \( 1 \triangleright a = a = a \cdot 1 \), for any \( a \in D \). A morphism of dialgebras is a \( K \)-linear map \( f : D \to D' \) which preserves the products, i.e. \( f(a \cdot b) = f(a) \cdot f(b) \), where \( \cdot \) denotes either the product \( \cdot \) or the product \( \triangleright \).

**Definition 2.2.** ([19]) A dialgebra \( D \) over \( K \) is called associative if the two operators \( \cdot \) and \( \triangleright \) satisfy the following five axioms:

\[
\text{(Ass)} \quad \begin{cases} 
    a \cdot (b \triangleright c) = (a \cdot b) \cdot c = a \cdot (b \triangleright c), \\
    (a \cdot b) \cdot c = a \cdot (b \cdot c), \\
    (a \cdot b) \triangleright c = a \triangleright (b \cdot c) = (a \cdot b) \triangleright c.
\end{cases}
\]

Denote by \( \text{Dias} \), \( \text{Assoc} \) the categories of associative dialgebras and associative algebras over \( K \) respectively. Then the category \( \text{Assoc} \) is a full subcategory of \( \text{Dias} \).
Obviously, an associative dialgebra is an associative algebra if and only if \( a \triangleright b = a \rhd b = ab \).

The concept of alternative dialgebras was introduced in [11] for the study of the Steinberg Leibniz algebras.

**Definition 2.3.** [11] A dialgebra \( D \) over \( K \) is called alternative if the two operators \( \triangleright \) and \( \rhd \) satisfying the following five axioms:

\[
\begin{align*}
(Alt) & \quad J_\triangleright (a, b, c) = -J_\triangleright (c, b, a), & J_\rhd (a, b, c) = J_\rhd (b, c, a), \\
& \quad J_X(a, b, c) = -J_X(a, c, b), & (a \rhd b) \rhd c = (a \triangleright b) \rhd c, \\
& \quad a \rhd (b \rhd c) = a \triangleright (b \triangleright c),
\end{align*}
\]

where \( J_\triangleright (a, b, c) = (a \triangleright b) \triangleright c - a \triangleright (b \triangleright c) \), \( J_\rhd (a, b, c) = (a \rhd b) \rhd c - a \rhd (b \rhd c) \) and \( J_X(a, b, c) = (a \rhd b) \rhd c - a \rhd (b \rhd c) \).

Obviously, an alternative dialgebra is an alternative algebra if \( a \triangleright b = a \rhd b = ab \). Moreover, the following formulae are clear for an alternative dialgebra according to the definition.

\[
\begin{align*}
J_\triangleright (a, b, b) &= 0, & J_\rhd (a, a, b) &= 0, & J_X(a, b, a) &= 0.
\end{align*}
\]

**Examples.**

1. Obviously, an associative (alternative) dialgebra is an associative (alternative) algebra if and only if \( a \triangleright b = a \rhd b = ab \).

2. **Differential associative (alternative) dialgebra.** Let \( (A, d) \) be a differential associative (alternative) algebra. So by hypothesis, \( d(ab) = (da)b + adb \) and \( d^2 = 0 \). Define left and right products on \( A \) by the formulas

\[
x \rhd y = xdy, \quad x \triangleright y = (dx)y.
\]

Then \( A \) equipped with these two products is an associative (alternative) dialgebra.

3. **Tensor product.** Let \( D \) and \( D' \) be two associative dialgebras, then \( D \otimes D' \) with multiplication \( (a \otimes a') \star (b \otimes b') = (a \ast b) \otimes (a' \ast b') \), \( \ast = \rhd, \triangleright \), is also an associative dialgebra. Especially, if \( D \) is a unital associative dialgebra, then \( M_n(D) = M_n(K) \otimes D \) is also a unital associative dialgebra.

4. **Let \( D \) be an associative (alternative) algebra.** On the module of \( n \)-space \( D = A^\otimes n \) one puts

\[
(x \triangleright y)_i = x_i \left( \sum_{j=1}^n y_j \right), \quad i = 1, \ldots, n \quad \text{and}
\]

\[
(x \rhd y)_i = \left( \sum_{j=1}^n x_j y_i \right), \quad i = 1, \ldots, n.
\]

Then \( (D, \triangleright, \rhd) \) is an associative (alternative) dialgebra. For \( n = 1 \), this is example 1.

### 2.2 Leibniz algebras.

A **Leibniz algebra** [17] \( L \) is a vector space over a field \( K \) equipped with a \( K \)-bilinear map

\[
[-,-] : L \times L \to L
\]

satisfying the Leibniz identity

\[
[x, [y, z]] = [[x, y], z] - [x, [z, y]], \quad \forall \ x, y, z \in L.
\]
Obviously, a Lie algebra is a Leibniz algebra. A Leibniz algebra is a Lie algebra if and only if \([x,x] = 0\) for all \(x \in L\).

Suppose that \(L\) is a Leibniz algebra over \(K\). For any \(z \in L\), we define \(\text{ad} \ z \in \text{End}_K L\) by
\[
\text{ad} \ z(x) = -[x, z], \quad \forall x \in L.
\]
It follows (2.5) that
\[
\text{ad} \ ([x,y]) = [\text{ad} \ z(x), y] + [x, \text{ad} \ z(y)]
\]
for all \(x, y \in L\). This says that \(\text{ad} \ z\) is a derivation of \(L\). We also call it an inner derivation of \(L\).

Similarly, we also have the definition of general derivation of a Leibniz algebra and we denote by \(\text{Inn}(L)\), \(\text{Der}(L)\) the set of all inner derivations, derivations of \(L\) respectively. They are also Leibniz algebras.

Let \(L\) be a Leibniz algebra over \(K\). Consider the boundary map: \(\delta_n : L^\otimes n \to L^\otimes (n-1)\) defined by
\[
\delta_n(x_1 \otimes \cdots \otimes x_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j+1} x_1 \otimes \cdots \otimes x_{i-1} \otimes [x_i, x_j] \otimes x_{i+1} \otimes \cdots \otimes x_n,
\]
where \(\hat{x}_j\) indicates that the term \(x_j\) is omitted. One can show that \(\delta^2 = 0\) (see [20]) and the complex \((L^\otimes n, \delta)\) \((L^0 = K, \delta_1 = 0)\) gives the Leibniz homology \(HL_n(L)\) of the Leibniz algebra \(L\).

Let \(L\) be a Leibniz algebra over \(K\). It is called perfect if \([L,L] = L\). A central extension of \(L\) is a pair \((\hat{L}, \pi)\) where \(\hat{L}\) is a Leibniz algebra and \(\pi : \hat{L} \to L\) is a surjective homomorphism such that \(\text{Ker} \ \pi\) lies in the center of \(\hat{L}\) and the exact sequence \(0 \to \text{Ker} \ \pi \to \hat{L} \to L \to 0\) splits as \(K\)-module. The pair \((\hat{L}, \pi)\) is a universal central extension of \(L\) if for every central extension \((L, \tau)\) of \(L\) there is a unique homomorphism \(\psi : \hat{L} \to L\) for which \(\tau \circ \psi = \pi\). So the universal central extension is unique, up to isomorphism. A Leibniz algebra \(L\) has a universal central extension if and only if \(L\) is perfect. If \((\hat{L}, \pi)\) is the universal central extension of \(L\), then
\[
HL_2(L) \cong \text{Ker} \ \pi.
\]

**Remark.** In [20], [8], [9] and [13], the universal central extensions of many infinite dimensional Lie algebras in the category of Leibniz algebras are determined.

We also denote by \(\text{Leib}\) and \(\text{Lie}\) the categories of Leibniz algebras and Lie algebras over \(K\) respectively.

For any associative dialgebra \(D\), define
\[
[x,y] = x \rightleftharpoons y - y \rightleftharpoons x,
\]
then \(D\) equipped with this bracket is a Leibniz algebra. We denote it by \(D_L\). The canonical map \(D \mapsto D_L\) induces a functor \((-) : \text{Dias} \rightarrow \text{Leib}\).

For a Leibniz algebra \(L\), let \(L_{Lie}\) be the quotient of \(L\) by the ideal generated by the elements \([x,y] + [y,x]\) for all \(x,y \in L\). It is clear that \(L_{Lie}\) is a Lie algebra. The canonical projection \(L \to L_{Lie}\) is universal among the maps from \(L\) to Lie algebras. In other words, the functor \((-)_{Lie} : \text{Leib} \rightarrow \text{Lie}\) is left adjoint to \(\text{inc} : \text{Lie} \rightarrow \text{Leib}\).

Moreover, we have the following commutative diagram of categories and functors
\[
\begin{array}{ccc}
\text{Dias} & \rightarrow & \text{Leib} \\
\downarrow & & \downarrow \\
\text{Assoc} & \rightarrow & \text{Lie}
\end{array}
\]

As in the Lie algebra case, the universal enveloping associative dialgebra of a Leibniz algebra \(L\) is defined as
\[
Ud(L) := (T(L) \otimes L \otimes T(L))/\{(x,y) \rightleftharpoons x - y + y \rightleftharpoons x | x,y \in L\},
\]
where elements $x, y$ of $L$ are regarded as elements in $K \otimes L \otimes K$.

**Proposition 2.4.** [19]

The functor $Ud : \text{Leib} \to \text{Dias}$ is left adjoint to the functor $- : \text{Dias} \to \text{Leib}$.

Let $L$ be a Leibniz algebra, then $M$ is said to be a right $L$-module if $M$ is a $K$-vector space equipped with the action of $L$:

$$[-, -] : M \times L \to M$$

satisfying

$$[m, [x, y]] = [[m, x], y] - [[m, y], x], \forall x, y \in L, m \in M.$$

So for a Lie algebra $\mathfrak{g}$, any right $\mathfrak{g}$-module in the Leibniz algebra case is just the right $\mathfrak{g}$-module in the Lie algebra case.

### 2.3 Lie algebras graded by finite root system.

First we introduce the Steinberg Lie algebra. Steinberg Lie algebras come from Steinberg groups, which are closely connected with K-theory.

If $R$ is a (nonassociative) ring with 1, and $n \geq 3$, the **Steinberg group** (see [6]) $\text{St}_n(R)$ is the group generated by the symbols $x_{ij}(a), 1 \leq i \neq j \leq n, a \in R$, subject to the relations

$$x_{ij}(a + b) = x_{ij}(a)x_{ij}(b), \text{ for } a, b \in R, k_1, k_2 \in K;$$

$$[x_{ij}(a), x_{kl}(b)] = 1 \text{ if } i \neq l \text{ and } j \neq k;$$

$$[x_{ij}(a), x_{kl}(b)] = x_{il}(ab) \text{ if } i \neq l \text{ and } j = k,$$

where $[x, y] = xyx^{-1}y^{-1}$ is the group commutator.

By definition, the **Steinberg Lie algebra** (see [6]) $\mathfrak{st}(n, A)$ $(n \geq 3)$ over a $K$-algebra $A$ is a Lie algebra generated by symbols $u_{ij}(a), 1 \leq i \neq j \leq n, a \in A$, subject to the relations

$$u_{ij}(k_1 a + k_2 b) = k_1 u_{ij}(a) + k_2 u_{ij}(b), \text{ for } a, b \in A, k_1, k_2 \in K;$$

$$[u_{ij}(a), u_{kl}(b)] = 0 \text{ if } i \neq l \text{ and } j \neq k;$$

$$[u_{ij}(a), u_{kl}(b)] = u_{il}(ab) \text{ if } i \neq l \text{ and } j = k.$$

Define $i_n(A) = \{ a \in A \mid u_{ij}(a) = 0 \}$. Clearly, it is an ideal of $A$ and does not depend on the choice of $i \neq j$.

From [6], we see that $i_n(A) = 0$ if and only if $A$ is an associative algebra for $n \geq 4$ and $A$ is an alternative algebra for $n = 3$.

By definition, a $K$-algebra $A$ is called **alternative** if $(a, b, c) = -(c, b, a) = (b, c, a)$, where $(a, b, c) = (ab)c - a(bc)$.

A Lie algebra $L$ over a field $K$ of characteristic 0 is **graded by the (reduced) root system** (see [5]) $\Delta$ or is $\Delta$-graded if

1. $L$ contains as a subalgebra a finite-dimensional simple Lie algebra $\mathfrak{g} = H \oplus \bigoplus_{\alpha \in \Delta} \mathfrak{g}_\alpha$ whose root system is $\Delta$ relative to a split Cartan subalgebra $H = \mathfrak{g}_0$;

2. $L = \bigoplus_{\alpha \in \Delta \cup \{0\}} L_\alpha$, where $L_\alpha = \{ x \in L \mid [h, x] = \alpha(h)x, \forall h \in H \}$ for $\alpha \in \Delta \cup \{0\}$; and

3. $L_0 = \sum_{\alpha \in \Delta} [L_\alpha, L_{-\alpha}]$.

**Theorem 2.5.** [5] Let $L$ be a Lie algebra over $K$ graded by the root system $\Delta$ of type $X_l (l \geq 2)$ $(X_l = A_l, D_l, E_l)$.
(1) If \( X_l = A_l, l \geq 3 \), then there exists a unital associative \( K \)-algebra \( A \) such that \( L \) is centrally isogenous with \( sl(l + 1, A) \).

(2) If \( X_l = A_l, l = 2 \), then there exists a unital alternative \( K \)-algebra \( A \) such that \( L \) is centrally isogenous with \( sl(l + 1, A) \).

(3) If \( X_l = D_l (l \geq 4) \), \( E_l (l = 6, 7, 8) \), then there exists a unital associative commutative \( K \)-algebra \( A \) such that \( L \) is centrally isogenous with \( \mathfrak{g} \otimes A \).

2.4 Steinberg Leibniz algebras

The matrix Leibniz algebra \( \mathfrak{gl}(n, D) \) is generated by all \( n \times n \) matrices with coefficients from a unital associative dialgebra \( D \), and \( n \geq 3 \) with the bracket

\[
[E_{ij}(a), E_{kl}(b)] = \delta_{jk} E_{il}(a \cdot b) - \delta_{il} E_{kj}(b \cdot a),
\]

for all \( a, b \in D \), where \( E_{ij}(a) \) is the \( n \times n \) matrix with coefficient \( a \) on \((i, j)\)-th position and 0 in all others.

Clearly, \( \mathfrak{gl}(n, D) \) is a Leibniz algebra. If \( D \) is an associative algebra, then \( \mathfrak{gl}(n, D) \) becomes a Lie algebra.

Now we consider the subalgebra \( \mathfrak{sl}(n, D) := [\mathfrak{gl}(n, D), \mathfrak{gl}(n, D)] \), which is called the special linear Leibniz algebra with coefficients in \( D \), of \( \mathfrak{gl}(n, D) \).

By definition, the special linear Leibniz algebra \( \mathfrak{sl}(n, D) \) has generators \( E_{ij}(a), 1 \leq i \neq j \leq n, a \in D \), which satisfy the following relations:

\[
E_{ij}(a), E_{kl}(b) = 0 \quad \text{if} \quad i \neq l \quad \text{and} \quad j \neq k;
\]

\[
E_{ij}(a), E_{ki}(b) = E_{il}(a \cdot b) \quad \text{if} \quad i \neq l \quad \text{and} \quad j = k;
\]

\[
E_{ij}(a), E_{kl}(b) = -E_{kj}(b \cdot a) \quad \text{if} \quad i = l \quad \text{and} \quad j \neq k.
\]

The Steinberg Leibniz algebra was first introduced in [20] for associative algebras and in [11] for associative dialgebras. By definition, the Steinberg Leibniz algebra \( \mathfrak{sl}(n, D) \) is a Leibniz algebra generated by symbols \( v_{ij}(a), 1 \leq i \neq j \leq n, a \in D \), subject to the relations

\[
v_{ij}(k_1a + k_2b) = k_1v_{ij}(a) + k_2v_{ij}(b), \quad \text{for} \quad a, b \in D, \quad k_1, k_2 \in K; \quad (4)
\]

\[
v_{ij}(a) = 0, \quad \text{if} \quad i \neq l \quad \text{and} \quad j \neq k; \quad (5)
\]

\[
v_{ij}(a), v_{kl}(b) = v_{il}(a \cdot b) \quad \text{if} \quad i \neq l \quad \text{and} \quad j = k; \quad (6)
\]

\[
v_{ij}(a), v_{kl}(b) = -v_{kj}(b \cdot a) \quad \text{if} \quad i = l \quad \text{and} \quad j \neq k. \quad (7)
\]

It is clear that the relations (6)–(7) make sense only if \( n \geq 3 \).

Let \( H_{ij}(a, b) := [v_{ij}(a), v_{ji}(b)] \) for \( 1 \leq i \neq j \leq n, a, b \in D \), and \( H \) the submodule of \( \mathfrak{sl}(n, D) \) generated by \( H_{ij}(a, b), i \neq j, a, b \in D \). Define \( i_n(D) = \{ a \in D \mid v_{ij}(a) = 0 \} \). Clearly, it is an ideal of \( D \) and does not depend on the choice of \( i \neq j \). The same consideration as in Steinberg Lie algebras, we have

**Proposition 2.6.** [11] For a unital dialgebra \( D \), \( i_n(D) = 0 \) in \( \mathfrak{sl}(n, D) \) if and only if \( D \) is associative for \( n \geq 4 \) and \( D \) is alternative for \( n = 3 \).

The Steinberg Leibniz algebra \( \mathfrak{sl}(n, D) \) with \( n \geq 3 \) is perfect.

The homomorphism \( \psi \) of Leibniz algebras

\[
\psi : \mathfrak{sl}(n, D) \to \mathfrak{sl}(n, D)
\]

by the rule \( \psi(v_{ij}(a)) = E_{ij}(a) \) is a surjective homomorphism.

**Theorem 2.7.** [11] If \( n \geq 3 \), then \( \mathfrak{sl}(n, D, \psi) \) is the universal central extension of the Leibniz algebra \( \mathfrak{sl}(n, D) \) with kernel \( HHS_1(D) \) for a unital associative dialgebra \( D \), where \( HHS_1(D) \) is the first homology group of chain complex \( (CS_\ast(D), d) \) defined in [6] (or see [11]) .
3 Leibniz algebras graded by finite root systems

Definition 3.1. A Leibniz algebra $L$ over a field $K$ of characteristic 0 is graded by the (reduced) root system $\Delta$ or is $\Delta$-graded if

1. $L$ contains as a subalgebra a finite-dimensional simple Lie algebra $\hat{\mathfrak{g}} = H \oplus \bigoplus_{\alpha \in \Delta} \hat{\mathfrak{g}}_\alpha$ whose root system is $\Delta$ relative to a split Cartan subalgebra $H = \mathfrak{h}_0$;
2. $L = \bigoplus_{\alpha \in \Delta \cup \{0\}} L_\alpha$, where $L_\alpha = \{x \in L \mid \text{ad } h(x) = -[x, h] = \alpha(h)x, \forall h \in H\}$ for $\alpha \in \Delta \cup \{0\}$; and
3. $L_0 = \sum_{\alpha \in \Delta} [L_\alpha, L_{-\alpha}]$.

Remarks.

1. The conditions for being a $\Delta$-graded Leibniz algebra imply that $L$ is a direct sum of finite-dimensional irreducible right $\hat{\mathfrak{g}}$-modules whose highest weights are roots, hence either the highest long root or short root or 0.
2. If $L$ is $\Delta$-graded, then $L$ is perfect. Indeed, the result follows from $L_\alpha = [L_\alpha, H]$ for all $\alpha \in \Delta$ and (3) as above.
3. The Steinberg Leibniz algebra $\mathfrak{sl}(n, D)$ is graded by the root system of type $A_{n-1}$. Let $D$ be a commutative dialgebra, then the Leibniz algebra $\mathfrak{g} \otimes D$ and its central extensions are graded by the root system of type $\hat{\mathfrak{g}}$ (see [12]).

Now we shall prove the Recognition Theorem (Theorem 1.1). So from now on, we always set $\Delta$ to be the root system of type $A_1$ ($l \geq 2$), $D_l$ ($l \geq 4$) or $E_l$ ($l = 6, 7, 8$).

Such as that in [5], we also have the following results.

Definition 3.2. An ordered pair $(\beta, \gamma) \in \Delta \times \Delta$ is an $A_2$-pair if $(\beta, \gamma) = -1$. Thus $(\beta, \gamma)$ is an $A_2$-pair if and only if it is a base for an $A_2$ subroot system of $\Delta$. Two $A_2$-pairs $(\beta, \gamma)$ and $(\beta', \gamma')$ are equivalent, written $(\beta, \gamma) \sim (\beta', \gamma')$, if there is an element $w$ of the Weyl group $W$ of $\Delta$ such that $\beta' = w\beta$ and $\gamma' = w\gamma$. The equivalent class of $(\beta, \gamma)$ is denoted by $[(\beta, \gamma)]$.

Lemma 3.3. [5] (1) If $\Delta$ is of type $D$ or $E$, then there is only one equivalent class of $A_2$-pairs.

2. If $\Delta$ is of type $A$, then there are exactly two equivalent classes of $A_2$-pairs and furthermore, if $(\beta, \gamma)$ is an $A_2$-pair, then

$$(\beta, \gamma) \sim (-\gamma, -\beta), \quad (\beta, \gamma) \not\sim (\gamma, \beta).$$

3. In all cases if $(\beta, \gamma)$ and $(\gamma, \delta)$ are $A_2$-pairs with $(\beta \mid \delta) = 0$, then

$$(\beta, \gamma) \sim (\gamma, \delta) \sim (\beta, \gamma + \delta) \sim (\beta + \gamma, \delta).$$

Remark. In type $A_l$, for definiteness, we distinguish the two classes as follows. We choose a base $\Pi = \{\alpha_1, \ldots, \alpha_l\}$ for $\Delta$ once and for all with Coxeter-Dynkin diagram $\alpha_1 \circ \cdots \circ \alpha_{l-1} \circ \alpha_l$. Then $[(\alpha_1, \alpha_2)]$ is the positive class and an $A_2$-pair $(\alpha, \beta) \in [\alpha_1, \alpha_2]$ is called positive pair, $[(\alpha_2, \alpha_1)]$ is the negative class and an $A_2$-pair $(\alpha, \beta) \in [\alpha_2, \alpha_1]$ is called negative pair. For convenience, any $A_2$-pair in types $D_l$ and $E_l$ is either positive and negative.

Let $L$ be a Leibniz algebra graded by $\Delta$ and $\hat{\mathfrak{g}} \subset L$ the split simple Lie algebra of Definition 3.1. Let $\{e_\alpha, H_i \mid \alpha \in \Delta, i = 1, \ldots, l\}$ be a Chevalley basis of $\hat{\mathfrak{g}}$ so that $e_\alpha \in L_\alpha$ for any $\alpha \in \Delta$, and $\{H_i, i = 1, \ldots, l\}$ a basis of $H$. Let $G = \langle \exp te_\alpha \mid t \in K \rangle$ be the corresponding simply connected Chevalley group. For each $\alpha \in \Delta$, $\{e_\alpha, e^\alpha = [e_\alpha, e_{-\alpha}], e_{-\alpha} \}$ is an $\mathfrak{sl}_2$-triplet. Let

$$n_\alpha(t) = \exp te_\alpha \exp(-t^{-1}e_{-\alpha}) \exp te_\alpha$$

and set

$$N = \langle n_\alpha(t) \mid \alpha \in \Delta, t \in K^\times \rangle,$$
where $K^\times$ is the set of non-zero elements of $K$.

Let $h_\alpha(t) = n_\alpha(t)n_\alpha(1)^{-1}$ and $T = \langle h_\alpha(t) \mid \alpha \in \Delta, t \in K^\times \rangle$. Then

$$T \triangleleft N, N/T \cong W.$$  

Clearly, from the $\Delta$-grading $\text{ad} \, e_\alpha$ and $\text{ad} \, e_{-\alpha}$ are nilpotent on $L$, so we can define a homomorphism

$$\text{Ad} : G \to \text{Aut}(L) \text{ by } \exp te_\alpha \to \exp \text{ad} te_\alpha, \alpha \in \Delta, t \in K.$$  

Next, recall that if $M$ is any integrable (right) $\hat{\mathfrak{g}}$-module (one on which $e_\alpha$ acts locally nilpotent for all $\alpha \in \Delta$) with weight space decomposition $M = \oplus M_\lambda$ relative to $\hat{\mathfrak{g}}$, then, letting $r_\alpha$ denote the reflection $r_\alpha \lambda = \lambda - (\lambda, \alpha^\vee)\alpha$, we have

$$n_\alpha(t)M_\lambda = M_{r_\alpha \lambda} \text{ for all weights } \lambda, \text{ and}$$

the action of $h_\alpha(t)$ restricted to $M_\lambda$ is a scalar-multiplication by $e^{(\lambda, \alpha^\vee)}$.  

In particular, the adjoint representation restricted to $\hat{\mathfrak{g}}$ acts on $L$ in this way. Fix any $\alpha \in \Delta$. Let $W^\alpha$ denote the stabilizer of $\alpha$ in $W$ and let $N^\alpha := W^\alpha T$. Then

$$W^\alpha = \langle r_\beta \mid \beta \in \Delta, (\beta|\alpha) = 0 \rangle, \quad \text{(3.3)}$$

$$N^\alpha = \langle n_\beta(1) \mid \beta \in \Delta, (\beta|\alpha) = 0 \rangle \cdot T. \quad \text{(3.4)}$$

Fix any $\beta \in \Delta$ and choose $w \in W$ with $w\alpha = \beta$. Choose any $n \in N$ with $nT \leftrightarrow w$ in the isomorphism $N/T \cong W$. Then $\text{Ad}(n)L_\alpha = L_\beta$ by (3.1). Let the restriction of $\text{Ad}(n)$ to $L_\beta$ be denoted by $\tilde{\lambda}_{\beta, \alpha}$. If also $w'\alpha = \beta$ and $n'T \leftrightarrow w'$, then $w^{-1}w' \in W^\alpha$ and $n^{-1}n' \in N^\alpha$. Thus $n' \in nN^\alpha = n<n_\alpha(1) \mid \gamma \in \Delta, (\beta|\alpha) = 0>\cdot T$. Elements of $T$ acts as scalar multiplications on $L_\alpha$.

Furthermore,

$$\Rightarrow \text{Ad} \exp(te_{\pm \gamma}) \text{ acts as the identity on } L_{\alpha}, \forall t \in K^\times \Rightarrow n_\gamma(1) \text{ acts as the identity on } L_{\alpha}.$$  

This establishes that $\tilde{\lambda}_{\beta, \alpha}$ is determined, up to a nonzero scalar multiple, by $\alpha$ and $\beta$ and does not otherwise depend on our choice of $w$ or $n$. Since also $n_\beta = \hat{\mathfrak{g}}_\beta$, $ne_\alpha = \varepsilon e_\beta$ for some $\varepsilon \in K^\times$ (actually $\varepsilon = \pm 1$).

**Definition 3.4.** $\lambda_{\beta, \alpha} : L_\alpha \to L_\beta$ is the $K$-linear map $\varepsilon^{-1}\tilde{\lambda}_{\beta, \alpha}$.

Clearly, $\lambda_{\beta, \alpha}$ depends only on $\alpha, \beta$ and the choice of Chevally basis. We see that

$$\lambda_{\alpha, \beta} = \lambda_{\beta, \alpha}^{-1}, \quad \text{(3.5)}$$

$$\lambda_{\alpha, \beta} \lambda_{\beta, \gamma} = \lambda_{\alpha, \gamma}, \quad \text{(3.6)}$$

$$\lambda_{\alpha, \alpha} = 1. \quad \text{(3.7)}$$

With $\alpha$ fixed as above, define $R = L_\alpha$ (as a $K$-space). Eventually, $R$ will have a life of its own, independent of $\alpha$. For this reason, given $r \in R$, we shall write it as $e_\alpha(r)$ when we think of it as being in $L_\alpha$. We identify $K$ as a subspace of $L_\alpha$ by $ae_\alpha = e_\alpha(a)$ for some $a \in K$.

Now if $\beta \in \Delta$ is arbitrary, then the $K$-linear map $\lambda_{\beta, \alpha} : L_\alpha \to L_\beta$ is defined and we define

$$e_\beta(r) = \lambda_{\beta, \alpha} e_\alpha(r). \quad \text{(3.8)}$$

From (3.7), we see that this definition is consistent when $\beta = \alpha$. From (3.5) and (3.6), we see that (3.8) holds for all pairs of roots $\alpha, \beta \in \Delta$.

We note that for all $a, b \in K, r, s \in R$, we have $e_\beta(ar + bs) = ae_\beta(r) + be_\beta(s)$. This allows us to write $x_\beta(r), r \in R$, for any $x_\beta \in \hat{\mathfrak{g}}_\beta$ unambiguously. Clearly,

$$x_\beta(r) = x_\beta(s) \iff r = s.$$
Lemma 3.5. For all \( n \in N, \beta \in \Delta, r \in R \), we have

\[
(\text{Ad } n)(e_\beta(r)) = ((\text{Ad } n)e_\beta)(r).
\]

Proof. Let \( nT \leftrightarrow w \in W \) and \( \gamma = w\beta \), so both sides of the asserted equality lie in \( L_\gamma \). Since \( \lambda_{\gamma, \beta} \) equals, up to a scalar, \( \text{Ad } (n)|_{L_\beta} = b\lambda_{\gamma, \alpha} = \text{Ad } n(\lambda_{\beta, \alpha}) \), for some \( b \in K^\times \). Then

\[
be_\gamma = b\lambda_{\gamma, \alpha}e_\alpha = (\text{Ad } n)\lambda_{\beta, \alpha}e_\alpha = (\text{Ad } n)e_\beta
\]

and hence

\[
(\text{Ad } n)e_\beta(r) = (\text{Ad } n)\lambda_{\beta, \alpha}e_\alpha(r) = b\lambda_{\gamma, \alpha}e_\alpha(r) = be_\gamma(r) = ((\text{Ad } n)e_\beta)(r).
\]

Now we set

\[
[e_\beta(r), e_\gamma(s)] = [e_\beta, e_\gamma](m_{(\beta, \gamma)}(r, s)).
\]  

(3.9)

Lemma 3.6. If \((\beta', \gamma') \in [(\beta, \gamma)]\), then \(m_{(\beta', \gamma')} = m_{(\beta, \gamma)}\).

Proof. Let \( m' = m_{(\beta', \gamma')}, m = m_{(\beta, \gamma)}\). Choose \( w \in W \) with \( \beta' = w\beta \) and \( \gamma' = w\gamma \). Let \( a, b, c, \varepsilon, \varepsilon' \in K^\times \) be chosen such that

\[
a \text{Ad } (n)e_\beta = e_{\beta'}, b \text{Ad } (n)e_\gamma = e_{\gamma'}, c \text{Ad } (n)e_{\beta+\gamma} = e_{\beta'+\gamma'},
\]

\[
[e_\beta, e_\gamma] = \varepsilon e_{\beta'+\gamma'}, [e_{\beta'}, e_{\gamma'}] = \varepsilon' e_{\beta'+\gamma'}.
\]

Since \( \text{Ad } (n) \) is an automorphism, \( \varepsilon' = abc^{-1}\varepsilon \). Now for \( r, s \in R \),

\[
[e_{\beta'}, e_{\gamma'}]m'(r, s) = [e_{\beta'}(r), e_{\gamma'}(s)] = [e_{\beta}(r), e_{\gamma}(s)]
\]

\[
= [\lambda_{\beta', \gamma'}e_{\beta}(r), \lambda_{\beta', \gamma'}e_{\gamma}(s)] = [a \text{Ad } (n)e_{\beta}(r), b \text{Ad } (n)e_{\gamma}(s)] = ab \text{Ad } (n)[e_\beta(r), e_\gamma(s)]
\]

\[
= ab \text{Ad } (n)[e_\beta, e_\gamma]([m(r, s)]) = ab \varepsilon e_{\beta'+\gamma'}[m(r, s)] = \varepsilon' e_{\beta'+\gamma'}([m(r, s)])
\]

Thus \( m_{(\beta', \gamma')} = m_{(\beta, \gamma)} \) and \( m = m' \).

From the above Lemma, we can define two multiplications on \( R \):

1. For a positive \( A_2 \)-pair \((\beta, \gamma)\) (see the Remark after Lemma 3.3), we define \( \dashv : R \times R \to R \) given by

\[
[e_\beta(r), e_\gamma(s)] = [e_\beta, e_\gamma](r \dashv s).
\]

2. For a negative \( A_2 \)-pair \((\beta, \gamma)\), we define \( \vdash : R \times R \to R \) given by

\[
[e_\beta(r), e_\gamma(s)] = [e_\beta, e_\gamma](s \vdash r).
\]

Clearly from Lemmas 3.5 and 3.6, we know that the above definition is well-defined on \( D \). Moreover, if \( \Delta \) is of type \( D_l \) \((l \geq 4)\) or \( E_l \) \((l = 6, 7, 8)\), then \( r \dashv s = s \vdash r \), i.e., \( R \) is a commutative dialgebra since \( m_{\beta, \gamma}(r, s) = m_{\gamma, \beta}(r, s) \).

Let \((\alpha, \beta)\) be a \( A_2 \)-pair. Then \( r_\beta \alpha = \alpha + \beta \). Suppose that \( e_\alpha, e_\beta \) have already been chosen as our Chevalley basis. Let \( n_\beta = \exp \exp \exp \exp (e_{e_\beta})(e_{-e_\beta}) \exp \exp \exp \exp (e_{-e_\beta})(e_{-e_\beta}) \exp \exp \exp \exp (e_{e_\beta})(e_{-e_\beta}) \). Then by definition, we get that

\[
n_\beta e_\alpha(r) = \exp \exp \exp \exp (e_{e_\beta})(e_{-e_\beta})\exp \exp \exp \exp (e_{e_\beta})(e_{-e_\beta}) = -[e_\alpha(r), e_\beta].
\]
Thus
\[(n_{\beta}e_{\alpha})(r) = -[e_{\alpha}(r), e_{\beta}].\]

In particular,
\[n_{\beta}e_{\alpha} = -[e_{\alpha}, e_{\beta}].\]

The element \(e_{\alpha} \in L_{\alpha}\) defines an element of \(R\) that has been identified with \(1 \in K : e_{\alpha}(1) = 1e_{\alpha}\).

For any \(\beta \in \Delta\) we have \(e_{\beta}(1) = \lambda_{\beta,\alpha}(e_{\alpha}(1)) = e_{\beta}\), so every basis element \(e_{\beta}\) defines the same element of \(R\).

Note that 1 is bar-unit of \(R\). For a positive \(A_{2}\)-pair \((\alpha, \beta)\), we have \([e_{\alpha}(r), e_{\beta}(1)] = [e_{\alpha}, e_{\beta}](r \dashv 1)\). But
\[
[e_{\alpha}(r), e_{\beta}(1)] = [e_{\alpha}(r), e_{\beta}] = -n_{\beta}(e_{\alpha}(r)) = -(n_{\beta}e_{\alpha})(r) = [e_{\alpha}, e_{\beta}](r).
\]

Thus \(r \dashv 1 = r\). Similarly, for a negative \(A_{2}\)-pair \((\alpha, \beta)\), we can prove that \(1 \vdash r = r\). So \(R\) is a unital dialgebra.

Next, we investigate the associativity of the multiplication on \(R\).

Now we assume that \(l \geq 3\). Let \((\beta, \gamma)\) and \((\gamma, \delta)\) be two positive \(A_{2}\)-pairs with \((\beta, \delta) = 0\) (such pairs exist if \(l \geq 3\)). By Lemma 3.3 (3), \(m_{(\beta,\gamma)} = m_{(\gamma,\delta)}\). By the Jacobi identity and \([L_{\beta}, L_{\delta}] = 0\), we obtain that
\[
[[e_{\beta}(r), e_{\gamma}(s)], e_{\delta}(t)] = [e_{\beta}(r), [e_{\gamma}(s), e_{\delta}(t)]],
\]
\[
[e_{\beta}(r), [e_{\gamma}(s), e_{\delta}(t)]] = -[[e_{\beta}(r), e_{\gamma}(s)], e_{\delta}(t)],
\]
\[
[[e_{\gamma}(s), e_{\beta}(r)], e_{\delta}(t)] = -[[e_{\gamma}(s), e_{\beta}(r)], e_{\delta}(s)].
\]

It follows that \((r \dashv s) \vdash t = r \dashv (s \vdash t) = r \dashv (s \vdash t)\) and \((r \vdash s) \vdash t = r \vdash (s \vdash t)\). Similarly, we can prove the left identities in the axiom (Ass). The \(l = 2\) case will be handled in \(\S 5.2\).

From the above, we have proved the following theorem.

**Theorem 3.7.** Let \(L\) be a \(\Delta\)-graded Leibniz algebra over a field \(K\) of characteristic 0, where \(\Delta\) is a simply-laced finite indecomposable root system of rank \(l \geq 2\). Let \(\hat{\mathfrak{g}}\) be the associated split simple Lie subalgebra with root system \(\Delta\). For any root \(\alpha \in \Delta\) and let \(R = L_{\alpha}\) as a \(K\)-vector space. Relative to a Chevalley basis \(\{ e_{\beta} \mid \beta \in \Delta \} \cup \{ H_{i} \mid i = 1, \cdots, l \}\), define the map \(\lambda_{\beta, \alpha} : L_{\alpha} \rightarrow L_{\beta}\) of Definition 3.4 and the element \(e_{\beta}(r), r \in R\) of (3.8). Then \(R\) is a unital \(K\)-dialgebra. Moreover, \(R\) is associative if the root \(l \geq 3\). If \(\Delta\) is of type \(D\) or \(E\), then \(R\) is commutative.

### 4 Centrally isogenous of \(\Delta\)-graded Leibniz algebra

Let \(L\) and \(L'\) be \(\Delta\)-graded Leibniz algebras over \(K\) for the same finite root system \(\Delta\). Then their associated split simple subalgebras \(\hat{\mathfrak{g}}\) and \(\hat{\mathfrak{g}}'\) are isomorphic. Simply denoted them by \(\hat{\mathfrak{g}}\), so \(\hat{\mathfrak{g}}\) is a subalgebra both of \(L\) and \(L'\).

**Definition 4.1.** A homomorphism \(\varphi : L \rightarrow L'\) is \(\Delta\)-homomorphic if \(\varphi|_{\hat{\mathfrak{g}}} = \text{id}_{\hat{\mathfrak{g}}}\).

Let \(\varphi : L \rightarrow L'\) be a \(\Delta\)-homomorphism. From the definition, it follows at once that \(\varphi(L_{\alpha}) \subset L'_{\alpha}\) for all \(\alpha \in \Delta \cup \{0\}\). Let \(R\) and \(R'\) be the \(K\)-dialgebras associated to \(L\) and \(L'\) respectively defined by a certain choice of Chevalley basis for \(\hat{\mathfrak{g}}\). For each \(\alpha \in \Delta\), we have \(R = L_{\alpha} \xrightarrow{\varphi_{\alpha}} L'_{\alpha} = R'\), so \(\varphi\) determines a map
\[
\varphi_{\alpha} : R \rightarrow R'.
\]
Proposition 4.2. (1) \( \tilde{\varphi}_\alpha : R \to R' \) is independent of the choice of \( \alpha \in \Delta \) (so we can denote it by \( \tilde{\varphi} \)) and is a homomorphism of dialgebras. Furthermore, \( \tilde{\varphi} \) is injective (resp. surjective, bijective) if \( \varphi \) is injective (resp. surjective, bijective).

(2) If \( \tilde{\varphi} \) is an isomorphism, then the \( \Delta \)-homomorphism \( \varphi \) is central and \( L \) and \( L' \) are centrally isogenous.

Proof. (1) Let \( \{e_\beta\}_{\beta \in \Delta} \cup \{H_i\}_{i=1} \) be the Chevalley basis of \( \tilde{g} \). We have \( L_\alpha = \{e_\alpha(r) \mid r \in R\} \) and \( L'_\alpha = \{e_\alpha(r) \mid r \in R'\} \). Furthermore, \( ke_\alpha = e_\alpha(k) \) for all \( k \in K \) and \( e_\alpha(k_1r + k_2s) = k_1e_\alpha(r) + k_2e_\alpha(s) \) for all \( r, s \in R, R' \), \( k_1, k_2 \in K \). By definition, \( \varphi(e_\alpha(r)) = e_\alpha(\tilde{\varphi}_\alpha(r)) \) for all \( r \in R \), and it follows that \( \tilde{\varphi}_\alpha \) is \( K \)-linear and maps 1 \( \in R \) to 1 \( \in R' \).

Now suppose that \( (\alpha, \beta) \) is a positive \( A_2 \)-pair. Then
\[
[e_\alpha, e_\beta](\tilde{\varphi}_{\alpha + \beta}(r + s)) = \varphi([e_\alpha, e_\beta](r + s)) = \varphi([e_\alpha(r), e_\beta(s)]) = [e_\alpha(\tilde{\varphi}_\alpha(r)), e_\beta(\tilde{\varphi}_\beta(s))] = [e_\alpha, e_\beta](\tilde{\varphi}_\alpha(r) + \tilde{\varphi}_\beta(s)).
\]
Thus
\[
\tilde{\varphi}_{\alpha + \beta}(r + s) = \tilde{\varphi}_\alpha(r) + \tilde{\varphi}_\beta(s).
\]
Similarly, we have
\[
\tilde{\varphi}_{\alpha + \beta}(r - s) = \tilde{\varphi}_\alpha(r) - \tilde{\varphi}_\beta(s).
\]
With \( s = 1 \), we get \( \tilde{\varphi}_{\alpha + \beta}(r) = \tilde{\varphi}_\alpha(r) \). From this, it is easy to see that \( \tilde{\varphi} \) is independent of \( \alpha \). Thus, (4.1) and (4.2) show the homomorphism property of \( \tilde{\varphi} \). The remaining parts of (1) are obvious.

(2) If \( \tilde{\varphi} \) is an isomorphism, then \( \tilde{\varphi}_\alpha \) is an isomorphism for each \( \alpha \in \Delta \), so \( \text{Ker} \varphi \subset L_0 \). Since \( [L_\alpha, \text{Ker} \varphi] \subset L_\alpha \cap \text{Ker} \varphi = \{0\} \), we see that \( \text{Ker} \varphi \) lies in the center of \( L \) from Definition 3.1. Since \( L \) and \( L' \) are perfect and \( L' \cong L/Z \) for some central ideal \( Z \), they have the same universal central extension. □

Proposition 4.3. Let \( L \) be a Leibniz algebra graded by \( \Delta \) and \( (u, \varphi) \) the universal central extension of \( L \). Then \( u \) is graded by \( \Delta \) and has the same associated dialgebra as that of \( L \). Furthermore, \( \varphi \) is a \( \Delta \)-homomorphism and \( \varphi : u_\alpha \to L_\alpha \) is a homomorphism for all \( \alpha \in \Delta \). In particular, \( \text{Ker} \varphi \subset u_0 \).

Proof. It is well known that \( \tilde{g} \) is centrally closed in the category of Leibniz algebras ([9]). Thus the central extension \( \varphi : \varphi^{-1}(\tilde{g}) \to \tilde{g} \) splits and we may view \( \tilde{g} \) as a subalgebra of \( u \). In particular, \( H \) is a subalgebra of \( u \). We define
\[
\tilde{u}_\alpha := \varphi^{-1}(L_\alpha), \quad \alpha \in \Delta \cup \{0\},
\]
\[
u_\alpha := \begin{cases} 
\tilde{u}_\alpha, & \alpha \in \Delta, \\
u_0, & \alpha = 0. 
\end{cases}
\]
For all \( h_1, h_2 \in H, x \in \tilde{u}_\alpha \), we have
\[
\text{ad} h_1([x, h_2]) = -[[x, h_2], h_1] = -[[x, h_1], h_2] = [\alpha(h_1)x + z, h_2] = \alpha(h_1)[x, h_2],
\]
where \( z \in \text{Ker} \varphi \). This proves that
\[
u_\alpha \text{ is an \( \alpha \)-weight space for } \text{ad}_u H, \alpha \in \Delta.
\]

(4.3)
It follows that for \( \alpha \in \Delta, u_\alpha \cap \text{Ker} \varphi = \{0\} \) and hence
\[
\varphi|_{u_\alpha} : u_\alpha \to L_\alpha
\]
is an isomorphism of vector spaces.
Let \( x \in \mathfrak{u} \) and write \( x = \sum_{\alpha \in \Delta \cup \{0\}} \tilde{x}_\alpha \), where \( \tilde{x}_\alpha \in \tilde{\mathfrak{u}}_\alpha \). Fix \( h \in H \) so that for all \( \alpha \in \Delta \), \( \alpha(h) \neq 0 \). Then
\[
\tilde{x}_\alpha - \alpha(h)^{-1}[\tilde{x}_\alpha, x] \in \Ker \varphi \subset \tilde{\mathfrak{u}}_0 = \mathfrak{u}_0.
\]
Thus we may rewrite \( x \) as \( \sum_{\alpha \in \Delta \cup \{0\}} x_\alpha \), where \( x_\alpha \in \mathfrak{u}_\alpha \) and \( x_0 \in \mathfrak{u}_0 \). It follows that
\[
\mathfrak{u} = \mathfrak{u}_0 + \sum_{\alpha \in \Delta} \mathfrak{u}_\alpha.
\]

Now
\[
\mathfrak{u}_0 = \tilde{\mathfrak{u}}_0 = \varphi^{-1}(\sum_{\alpha \in \Delta} [L_\alpha, L_{-\alpha}]) = \sum_{\alpha \in \Delta} [\tilde{\mathfrak{u}}_\alpha, \tilde{\mathfrak{u}}_{-\alpha}] + \Ker \varphi.
\]
Since \( \tilde{\mathfrak{u}}_\alpha = \mathfrak{u}_\alpha + \Ker \varphi \) from (4.3), we have
\[
\mathfrak{u}_0 = \sum_{\alpha \in \Delta} [\mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}] + \Ker \varphi.
\]
This proves that \( \mathfrak{u}_0 \) is a 0-eigenspace for \( H \). Now we see that \( \mathfrak{u}_\alpha \) is exactly the \( \alpha \)-eigenspace for \( H \) for all \( \alpha \in \Delta \cup \{0\} \) and \( [\mathfrak{u}_\alpha, \mathfrak{u}_\beta] \subset \mathfrak{u}_{\alpha + \beta} \), whenever \( \alpha + \beta \in \Delta \cup \{0\} \). Also since \( \mathfrak{u} = [\mathfrak{u}, \mathfrak{u}] \), we see that
\[
\mathfrak{u}_0 = \sum_{\alpha \in \Delta} [\mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}] + [\mathfrak{u}_0, \mathfrak{u}_0].
\]
But
\[
[\mathfrak{u}_0, \mathfrak{u}_0] = \sum_{\alpha, \beta \in \Delta} [[\mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}], [\mathfrak{u}_\beta, \mathfrak{u}_{-\beta}]] \subset \sum_{\gamma \in \Delta} [[\mathfrak{u}_\gamma, \mathfrak{u}_{-\gamma}],
\]
so
\[
\mathfrak{u}_0 = \sum_{\alpha \in \Delta} [\mathfrak{u}_\alpha, \mathfrak{u}_{-\alpha}].
\]
This proves that \( \mathfrak{u} \) is graded by \( \Delta \). From (4.4) and the construction of \( \varphi \), we see that \( \varphi \) is a \( \Delta \)-homomorphism and \( \tilde{\varphi} \) is an isomorphism. Thus the dialgebra associated to \( \mathfrak{u} \) is the same as that associated to \( L \). \( \square \)

**Proposition 4.4.** Let \( L \) be a Leibniz algebra graded by \( \Delta \), \( Z \) the center of \( L \) and \( Z' \) any subspace of \( Z \). Then

1. \( Z \subset L_0 \).
2. \( L/Z' \) has a unique structure as a Leibniz algebra graded by \( \Delta \) that makes the natural map \( \pi : L \rightarrow L/Z' \) a \( \Delta \)-homomorphism.
3. For all \( \alpha \in \Delta \), \( L_\alpha \cong (L/Z')_\alpha \) as vector spaces.
4. The dialgebras associated to \( L \) and \( L/Z' \) are isomorphic by the map \( \tilde{\pi} \) induced by \( \pi \) as in Proposition 4.2.

**Proof.** Since for all \( h \in H \), \( \text{ad} \, h|_{L_\alpha} \) is a scalar by \( \langle \alpha, h \rangle \), we see that \( Z \subset L_0 \). The remaining results are now obvious. \( \square \)

**Proposition 4.5.** Let \( L \) and \( L' \) be centrally isogenous Leibniz algebras and suppose that \( L \) is graded by a finite root system \( \Delta \). Then \( L' \) is also graded by \( \Delta \) and in such a way that the associated dialgebras are isomorphic.

**Proof.** Let \( \mathfrak{u} \) be a universal central extension of \( L \). By Proposition 4.3, \( \mathfrak{u} \) is graded by \( \Delta \). By assumption, \( L' \cong \mathfrak{u}/Z_1 \) for some subspace of the center of \( \mathfrak{u} \). By Proposition 4.4, \( L' \) is graded by \( \Delta \). The associated dialgebras are isomorphic by Propositions 4.3 and 4.4. \( \square \)

In short, all Leibniz algebras in a given isogeny class are \( \Delta \)-graded if one of them is, and all have isomorphic root spaces for all \( \alpha \in \Delta \). They differ only by central elements in the 0-root space.
5 Proof of Recognition Theorem

In this section we shall complete the proof of the Recognition Theorem. We use the results and notation of §3 and §4 freely.

5.1 The Recognition Theorem for types $D$ and $E$

Now we assume that $L$ is a Leibniz algebra graded by $\Delta$, where $\Delta$ is a finite root system of type $D_l$, $l \geq 4$, or $E_6, E_7, E_8$.

Lemma 5.1. Let $\alpha, \beta \in \Delta$, $r, s \in R$ and set $\rho =: \text{ad} ([e_\alpha(r), e_\alpha(s)])$. Then

$$\rho(e_\beta(t)) = \langle \beta, \alpha \rangle e_\beta((t \dashv r) \vdash s).$$

Proof. Case 1. $(\alpha|\beta) = -1$: Then $\beta - \alpha \notin \Delta$ and we have

$$\rho(e_\beta(t)) = -[e_\beta(t), [e_\alpha(r), e_\alpha(s)]] = -[[e_\beta, e_\alpha](t \dashv r), e_\alpha(s)].$$

Case 2. $(\alpha|\beta) = 1$: Then $(-\alpha|\beta) = -1$ and we may use Case 1.

Case 3. $(\alpha|\beta) = 0$: Then neither $\beta + \alpha$ nor $\beta - \alpha$ is a root and both sides of our equation are 0.

Case 4. $\alpha = \beta$: It is easy to find a pair of roots $\gamma, \varepsilon \in \Delta$ with $\gamma + \varepsilon = \alpha$, $\gamma, \varepsilon \notin \{\alpha, -\alpha\}$. Then $e_\beta(t) = e_\alpha(t) = [e_\gamma, e_\varepsilon](t')$ for some $t' \in R$ which is some $K$-multiple of $t$. Applying $\rho$ and using the previous cases,

$$\rho(e_\beta(t)) = \rho([e_\gamma, e_\varepsilon](t')) = \rho([e_\gamma(t'), e_\varepsilon(1)]) = \langle \gamma, \alpha \rangle [e_\gamma(t' \dashv r) \vdash s], e_\varepsilon(1)] + \langle \varepsilon, \alpha \rangle [e_\gamma(t'), e_\varepsilon(r \vdash s)] = \langle \gamma + \varepsilon, \alpha \rangle [e_\gamma, e_\varepsilon](t' \dashv (r \vdash s)) = \langle \alpha, \alpha \rangle e_\beta(t \dashv (r \vdash s)).$$

Case 5. $-\alpha = \beta$: This is similar to Case 4. \qed

We wish to define a homomorphism

$$\phi : L \to t(R, \Delta) = \mathfrak{g} \otimes R \quad \text{(5.1)}$$

by defining

$$\phi(e_\alpha(r)) = e_\alpha \otimes r, \quad \text{for all } \alpha \in \Delta, r \in R. \quad \text{(5.2)}$$

Since $L$ is generated by the subspace $L_\alpha, \alpha \in \Delta$, it is clear that (5.2) uniquely defines $\phi$.

Suppose that $\sum_{\alpha \in \Delta} \sum_{i=1}^{n_\alpha} [e_\alpha(r(\alpha, i)), e_\alpha(s(\alpha, i))] = 0$ for some $r(\alpha, i), s(\alpha, i) \in R$. Then using Lemma 5.1, we have for all $\beta \in \Delta$

$$\sum_{\alpha \in \Delta} \sum_{i=1}^{n_\alpha} \langle \beta, \alpha \rangle e_\beta(r(\alpha, i) \dashv s(\alpha, i)) = 0$$
and hence

\[ \sum_{\alpha \in \Delta} \langle \beta, \alpha^\vee \rangle \sum_{i=1}^{n_\alpha} e_\beta(r(\alpha, i)x - s(\alpha, i)) = 0. \]

Since \( e_\beta(r) = e_\beta(s) \Leftrightarrow r = s \). It follows that in \( H \otimes R \), \( \sum_{\alpha \in \Delta} \sum_{i=1}^{n_\alpha} \alpha^\vee \otimes (r(\alpha, i)x - s(\alpha, i)) = 0 \) (it is killed by all the functionals \( 1 \otimes \beta \)). Thus we may extend \( \phi \) to \( L_0 \) by

\[ \phi: \sum_{\alpha \in \Delta} \sum_{i=1}^{n_\alpha} \{e_\alpha(r(\alpha, i)).e_{-\alpha}(s(\alpha, i)) \} \mapsto \sum_{\alpha \in \Delta} \sum_{i=1}^{n_\alpha} \alpha^\vee \otimes (r(\alpha, i)x - s(\alpha, i)). \]

We check that \( \phi \) is a homomorphism: If \((\alpha, \beta)\) is an \( A_2 \)-pair, then

\[ \phi([e_\alpha(r), e_\beta(s)]) = \phi([e_\alpha, e_\beta](r + s)) = [e_\alpha, e_\beta] \otimes (r + s) = [\phi(e_\alpha(r)), \phi(e_\beta(s))]. \]

If \((\alpha, \beta) \geq 0\), then \( \phi([e_\alpha(r), e_\beta(s)]) = 0 = [\phi(e_\alpha(r)), \phi(e_\beta(s))]. \)

If \( \beta = -\alpha \) then \( \phi([e_\alpha(r), e_{-\alpha}(s)]) = \alpha^\vee \otimes (r \pm s) = [e_\alpha \otimes r, e_\beta \otimes s]. \) Now let \( h = [e_\alpha(r), e_{-\alpha}(s)] \). We have

\[ \phi([e_\beta(t), h]) = -\phi(\beta, \alpha^\vee) e_\beta((t \pm r) \mp s)) = -\langle \beta, \alpha^\vee \rangle e_\beta \otimes ((t \pm r) \mp s) = [e_\beta \otimes t, \alpha^\vee \otimes (r \pm s)] = [\phi(e_\beta(t)), \phi(h)]. \]

and

\[ \phi([e_\beta(t), e_{-\beta}(u)], h]) = -\phi(\beta, \alpha^\vee) e_\beta([u \pm r] \mp s]) + \beta, \alpha^\vee) e_\beta((t \pm r) \mp s), e_{-\beta}(u])
\]

\[ = -\langle \beta, \alpha^\vee \rangle (\beta^\vee \otimes (t \pm ((u \pm r) \mp s)) - \beta^\vee \otimes (((t \pm r) \mp s) \pm u))
\]

\[ = 0 = [\phi([e_\beta(t), e_{-\beta}(u)]), \phi(h)] \]

since \((t \pm r) \mp s \pm u = u \pm (t \pm r) \mp s = (u \pm t) \mp s \pm s = (t \pm u \pm t) \mp s \pm s = t \pm (u \pm r) \mp s\), where we have used the commutativity and associativity of \( R \).

**Proposition 5.2.** The homomorphism \( \phi \) define by (5.1) and (5.2) is a surjective \( \Delta \)-homomorphism and \( \text{Ker} \varphi \) is contained in the center of \( L \).

**Proof.** Since \( L_\alpha = e_\alpha(R) \xrightarrow{\phi} e_\alpha \otimes R \), it is clear that \( \phi \) is bijective on the root spaces of \( L_\alpha, \alpha \in \Delta \) and hence \( \phi \) is surjective and \( \text{Ker} \varphi \subset L_0 \). Thus \([\text{Ker} \varphi, L_\alpha] \subset L_{\alpha} \cap (\text{Ker} \phi) = 0\) and hence \( \text{Ker} \phi \) is central by (2) of Definition 3.1.

We have proved that \( L \) is a central extension of \( \hat{g} \otimes R \), thus proving the third part of the Recognition Theorem 1.1. The universal central extension of \( \hat{g} \otimes R \) is given in [12] for a unital commutative associative dialgebra \( R \).

### 5.2 The Recognition Theorem for type \( A \)

Now we assume that \( L \) is a Leibniz algebra graded by \( \Delta \), where \( \Delta \) is a finite root system of type \( A_l \), \( l \geq 2 \).

Let \( n = l + 1 \) and \( \varepsilon_1, \cdots, \varepsilon_n \) be an orthonormal basis for \( \mathbb{R}^n \). Identify \( \Delta \) with \( \{\varepsilon_i - \varepsilon_j \mid i \neq j\} \) and define a base \( \{\alpha_1, \cdots, \alpha_l\} \) of \( \Delta \) by \( \alpha_i = \varepsilon_i - \varepsilon_{i+1} \). The positive class of \( A_2 \)-pair in \( \Delta \) will be taken as \([\alpha_1, \alpha_2]\).
For $\alpha = \varepsilon_i - \varepsilon_j \in \Delta$ we let $L_{ij} = L_\alpha$. The simple Lie algebra $\mathfrak{g}$ over $K$ of type $A_l$ may be identified with $\mathfrak{sl}(n,K)$. We choose as our Chevalley basis the matrix units $E_{ij}, i \neq j$ and the elements $h_i = [E_{i,i+1}, E_{i+1,i}], i = 1, \ldots, l$. To bring the notation in line with §2.4, we write $e_{ij}$ for $E_{ij}$. Let $R$ be the $K$-dialgebra derived from $L$ with this choice of positive $A_2$-pairs and the given Chevalley basis. Then we have

\[(\alpha_1 - \alpha_2) = \{\varepsilon_i - \varepsilon_j, \varepsilon_j - \varepsilon_k \mid i, j, k \text{ distinct}\}\]

and

\[(\alpha_2 - \alpha_1) = \{\varepsilon_i - \varepsilon_j, \varepsilon_k - \varepsilon_i \mid i, j, k \text{ distinct}\}\]

Thus by results in §4, we have

\[(i) \quad L_{ij} = K\{e_{ij}(r) \mid r \in R, i \neq j\};\]

\[(ii) \quad e_{ij}(k_1 r + k_2 s) = k_1 e_{ij}(r) + k_2 e_{ij}(s);\]

\[(iii) \quad [e_{ij}(r), e_{kl}(s)] = 0 \text{ if } i \neq l \text{ and } j \neq k;\]

\[(iv) \quad [e_{ij}(r), e_{kl}(s)] = e_{il}(r - s) \text{ if } i \neq l \text{ and } j = k;\]

\[(v) \quad [e_{ij}(r), e_{kl}(s)] = -e_{kj}(s + r) \text{ if } i = l \text{ and } j \neq k,\]

for all $r, s \in R, k_1, k_2 \in K$.

Now whether $R$ is associative or not, $L$ is homomorphic to the image of $\mathfrak{sl}(n,R)$, under the map

\[\phi : v_{ij}(r) \to e_{ij}(r), \quad r \in R, i \neq j.\]

In Section 3, we show that $R$ is associative if $l \geq 3$. Similar to the proof of Proposition 3.1 in [11], we can show that $R$ is alternative if $l = 2$.

**Proposition 5.3.** The homomorphism of (5.6) is central.

**Proof.** It is clear that $\mathfrak{sl}(n,R)$ is graded by $\Delta$ and its associated dialgebra is $R$. The homomorphism (5.6) sends $v_{ij}(R) \to e_{ij}(R) = L_{ij}$ isomorphically and hence the $\text{Ker}(\phi) \subset \mathfrak{sl}(n,R)$. The same argument as used in Proposition 5.2 gives that $\phi$ is central. \[\square\]

This completes (1) and (2) of the Recognition Theorem 1.1.

**Remark.** 1. The structures of Leibniz algebras graded by finite root systems of other types are determined in [14] by using the methods in [4] and [2].

2. By Theorem 1.3 in [21] and Theorem 1.1, we can easily obtain the following theorem.

**Theorem 5.4.** A $Q(n)$-graded Leibniz superalgebra over $K$ is centrally isogenous to $\mathfrak{sl}(n+1,D)$, where $D = D_\bar{0} + D_\bar{1}$ is an associative or an alternative (if $n = 2$) unital dialgebra such that there exists $\nu \in D_\bar{1}, \nu^2 = 1$. (The definition of $Q(n)$-graded Leibniz superalgebra immediately follows from Definition 1.3 in [21] and Definition 3.1 in section 3, also see [15]).

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