On the Dimensional Reduction Procedure

Guido Cognola* and Sergio Zerbini†

Dipartimento di Fisica, Università di Trento
and Istituto Nazionale di Fisica Nucleare,
Gruppo Collegato di Trento, Italia

February 2001

Abstract: The issue related to the so-called dimensional reduction procedure is revisited within the Euclidean formalism. First, it is shown that for symmetric spaces, the local exact heat-kernel density is equal to the reduced one, once the harmonic sum has been successfully performed. In the general case, due to the impossibility to deal with exact results, the short $t$ heat-kernel asymptotics is considered. It is found that the exact heat-kernel and the dimensionally reduced one coincide up to two non trivial leading contributions in the short $t$ expansion. Implications of these results with regard to dimensional-reduction anomaly are discussed.

PACS numbers: 02.30.Tb, 02.70.Hm, 04.62.+v

1 Introduction

Very recently, motivated by applications concerning the black hole physics initiated in [1] and calculation of the effective action after and before the dimensional reduction [2], Frolov, Sutton and Zelnikov have introduced the concept of dimensional-reduction anomaly [3] related to a dimensional reduction procedure. Subsequently, the symmetric space cases have been considered too [4]. Very recently, consequences of the dimensional reduction anomaly in the Schwarzschild spacetime have also been analysed [5].

To begin with, let us recall the general arguments leading to the proposal contained in the above mentioned papers. Within the so-called one-loop approximation in Quantum Field Theory, the Euclidean one-loop effective action $\Gamma$ may be expressed in terms of the functional determinant of an elliptic differential operator $O$, defined on a $D$-dimensional manifold, namely

$$\Gamma \sim \ln \det O. \quad (1.1)$$

The ultraviolet one-loop divergences, which are present, may be regularised by means of analytic regularisations, for example the zeta-function regularisation (for recent reviews, see [6, 7]).

*e-mail: cognola@science.unitn.it
†e-mail: zerbini@science.unitn.it
In the presence of space-times having some symmetries like the D-dimensional space-time is the "warp" product \( M_D = M_P \times \Sigma_Q \), \( \Sigma_Q \) being a Q-dimensional symmetric space with constant curvature, the harmonic analysis on it is normally used in order to dimensionally "reduce" the relevant fluctuation operator \( O \). It turns out that the quantum dimensional-reduced theory, defined by a specific procedure, might be not equivalent to the original one, and this fact is related to the presence of the dimensional-reduction anomaly.

The reason of this possible discrepancy has been explained in [3] as mainly due to the necessity of the regularisation and renormalisation of the effective action in spaces with different dimensions. There, it has also been observed a possible connection with the so called multiplicative anomaly. Regarding this issue, see, for instance, [8, 9, 10].

Let us consider a scalar field \( \Phi \) propagating in the above mentioned space. Thus, one is mainly dealing with a second order self-adjoint non-negative differential operator

\[
L_D = -\Delta_D + m^2 + \xi R_D ,
\]

in which \( \Delta_D \) is the Laplace operator on \( M_D \), \( m^2 \) a possible mass term and \( \xi R_D \) a suitable "potential term", describing the non-minimal coupling with the gravitational field. We will assume that the spectrum is bounded from below. The "exact" theory, namely the non-dimensional reduced one, may be described by the path integral (Euclidean partition function)

\[
Z = \int \mathcal{D}\Phi e^{-\int dV_D \Phi L_D \Phi} = e^{-\Gamma} ,
\]

\( dV_D \) being the infinitesimal volume on \( M_D \). The effective action \( \Gamma \) has to be regularised and may be expressed by means of a zeta-regularised functional determinant \[1, 12, 13\]

\[
\Gamma = -\ln Z = -\frac{1}{2} \left[ \zeta'(0|L_D) + \ln \mu^2 \zeta(0|L_D) \right],
\]

\( \mu^2 \) being the renormalisation parameter. Here, the zeta-function is defined by

\[
\zeta(s|L_D) = \frac{1}{\Gamma(s)} \int_0^\infty dt \, t^{s-1} K_t , \quad K_t = \text{Tr} e^{-tL_D} ,
\]

valid for \( \text{Re} \, s > D/2 \). Here \( \text{Tr} e^{-tL_D} = \sum \lambda_i e^{-t\lambda_i} \), \( \lambda_i \) being the eigenvalues of \( L_D \). As a consequence, \( \zeta(s|L_D) = \sum \lambda_i^{-s} \). If zero modes are present, one has to subtract them, replacing \( \text{Tr} e^{-tL_D} \) with \( \text{Tr} e^{-tL_D} - P_0 \), \( P_0 \) being the projector onto the zero modes.

Of course one may use other regularisation procedures. As an example, the dimensional regularisation is defined by

\[
\Gamma_{\varepsilon} = -\frac{1}{2} \int_0^\infty dt \, t^{\varepsilon-1} \text{Tr} e^{-tL_D/\mu^2} = -\frac{1}{2} \Gamma(\varepsilon) \zeta(\varepsilon|L_D/\mu^2)
\]

\[
= -\frac{1}{2} \left[ \frac{\zeta(0|L_D)}{\varepsilon} + \zeta'(0|L_D) + \ln \mu^2 - \gamma \right] \zeta(0|L_D) + O(\varepsilon) .
\]

Other regularisations may be used with \( t^\varepsilon \) substituted by a suitable regularisation function \( g_{\varepsilon}(t) \) (see, for example [14]). Recall that the zeta-function regularisation is a finite regularisation and corresponds to the choice

\[
g_{\varepsilon}(t) = \frac{d}{d\varepsilon} \left( \frac{t^\varepsilon}{\Gamma(\varepsilon)} \right).
\]
The other ones, as is clear from Eq. (1.6), give the same finite part, modulo a renormalisation, and contain divergent terms as the cutoff parameter $\varepsilon \to 0$ and these divergent terms have to be removed by related counter-terms.

As a consequence, as will be shown in the following, a crucial role is played by the quantity $\text{Tr} e^{-tL_D}$. With regard to this quantity, its short-$t$ asymptotics has been extensively studied. For a second-order operator on a boundary-less $D$-dimensional (smooth) manifold, it reads

$$K_t \simeq \sum_{j=0}^{\infty} A_j(L_D) \, t^{j-D/2},$$

in which $A_j(L_D)$ are the Seeley-DeWitt coefficients, which can be computed with different techniques [15, 16]. The divergent terms appearing in a generic regularisation depend on $A_j(L_D)$.

In the sequel, we shall also deal with local quantities, which can be defined by the local zeta-function. With this regard, it is relevant the local short heat-kernel asymptotics, which reads (see Appendix B)

$$K_t(L_D)(x) = e^{-tL_D}(x) \simeq \frac{1}{(4\pi t)^{D/2}} \sum_{j=0}^{\infty} a_j(x|L_D) \, t^{j-D/2},$$

where $a_j(x|L_D)$ are the local Seeley-DeWitt coefficients. Some of them are well known and read

$$a_0(x|L_D) = 1, \quad a_1(x|L_D) = \frac{R}{6} - X,$n
$$a_2(x|L_D) = \frac{1}{2} [a_1(x|L_D)]^2 + \frac{1}{6} \Delta_D a_1(x|L_D) + \frac{1}{180} \left( \Delta_D R + R^{ijkr}R_{ijkr} - R^{ij}R_{ij} \right),$$

where $X$ is a function which depends on the operator one is dealing with. For example, for the Laplace-like operator in (1.2),

$$X = m^2 + \xi R.$$

It may be convenient to re-sum partially this asymptotic expansion and one has [17]

$$e^{-tL_D}(x) \simeq \sum_{j=0}^{\infty} b_j(x|L_D) \, t^{j-D/2}.$$n

The advantage of the latter expansion with respect to the previous one, is due to the fact that now the $b_j$ coefficients depend on the "potential" $X$ only through its derivatives. In fact one has

$$b_0(x|L_D) = 1, \quad b_1(x|L_D) = 0,$n
$$b_2(x|L_D) = -\frac{1}{6} \Delta_D a_1(x|L_D) + \frac{1}{180} \left( \Delta_D R + R^{ijkr}R_{ijkr} - R^{ij}R_{ij} \right).$$

Further coefficients $b_j(x|L_D)$ are reported in Appendix B.

The local zeta-function is defined by means of the Mellin transform, i.e.

$$\zeta(s|L_D)(x) = \frac{1}{\Gamma(s)} \int_0^{\infty} dt \, t^{s-1} e^{-tL_D}(x).$$

Making use of the local zeta-function, one may evaluate the effective Lagrangian, which is proportional to $\zeta'(0|L_D/\mu^2)(x)$, and the vacuum-expectation value of the quantum fluctuation given by (see for example [18, 19])

$$\langle \Phi(x)^2 \rangle = \lim_{\varepsilon \to 0} \frac{d}{d\varepsilon} \left[ \varepsilon \mu^2 \zeta(1+\varepsilon|L_D) \right],$$

(1.14)
which simplifies when $D$ is odd, due to the fact that in odd dimensions the zeta-function is regular at $s = 1$ and so

$$<\Phi(x)^2> = \zeta(1|L_D). \quad (1.15)$$

Since the exact expression of the local zeta-function is known only in a limited number of cases, in general one has to make use of some approximations. If the coefficient $a_1(x|L_D)$ is very large and negative (this is true if the case of large mass), one may obtain an asymptotic expansion of the local zeta-function by means of the short $t$ expansion \[13\] and the Mellin transform in the form \[14\]

$$\zeta(s|L_D)(x) \simeq \frac{\Gamma(s+\frac{D}{2})}{(4\pi)^{\frac{D}{2}}\Gamma(s)} \left(\frac{a_1(x|L_D)}{D} \right)^{\frac{D}{2}-s} \left(\frac{a_1(x|L_D)}{D} \right)^{s} \sum_{j=2}^{\infty} \frac{\Gamma(s+j-\frac{D}{2})}{(4\pi)^{\frac{D}{2}}\Gamma(s)} \left(-a_1(x|L_D)\right)^{\frac{D}{2}-s-j} b_j(x|L_D). \quad (1.16)$$

The latter expansion directly gives the analytic continuation in the whole complex plane. The global zeta-function can be obtained integrating over the manifold.

The content of the paper is the following. In Sec. 2, the dimensional-reduced theory is introduced and the formalism is developed. In Secs. 3 and 4, the two symmetric spaces $R^D$ and $H^P$ are considered in some detail. In Sec. 5, the general case is considered making use of the heat-kernel asymptotics. The paper ends with the conclusions and two Appendices, where some technical details are reported.

### 2 Dimensional-reduced theory, dimensional reduced heat-kernel and dimensional-reduction anomaly

Here we introduce the dimensional-reduced theory according to \[4\]. We indicate by $\tilde{M}_D$ a $D$-dimensional Riemannian manifold with metric $\tilde{g}_{\mu\nu}$ and coordinates $\tilde{x}^\mu$ ($\mu, \nu = 1, ..., D$) and by $M_P$ and $\tilde{M}_Q$ ($Q = D - P$) two sub-manifolds with coordinates $x^i$ ($i, j = 1, ..., P$) and $\tilde{x}^a$ ($a, b = P + 1, ..., D$) and metrics $g_{ij}$ and $\tilde{g}_{ab}$ respectively, related to $\tilde{g}_{\mu\nu}$ by the warped product

$$d\tilde{s}^2 = \tilde{g}_{\mu\nu}d\tilde{x}^\mu d\tilde{x}^\nu = g_{ij}(x)dx^idx^j + e^{-2\sigma(x)}\tilde{g}_{ab}(\tilde{x})d\tilde{x}^ad\tilde{x}^b. \quad (2.1)$$

Here, $\tilde{M}_Q = \Sigma_Q$ is a constant curvature symmetric space.

We shall use the notation $\tilde{R}^a_{\beta\gamma\delta}$, $R^i_{jmn}$ and $\tilde{R}^a_{bca}$ for Riemann tensors in $\tilde{M}_D$, $M_P$ and $\tilde{M}_Q$ respectively, and similarly for all other quantities. In the Appendix A, one can find the relationship between the geometrical quantities related to the sub-manifolds.

We start with a scalar field $\Phi(\tilde{x})$ in the Riemannian manifold $\tilde{M}_D$. Using Eqs. (A.4) and (A.1) in the appendix A, for the Laplacian-like operator we have

$$L_D \Phi(\tilde{x}) = \tilde{L} \Phi(\tilde{x}) = (-\Delta + \xi \tilde{R} + m^2)\Phi(\tilde{x}) = (L + e^{2\sigma} \tilde{L})\Phi(\tilde{x}), \quad (2.2)$$

where

$$L = -\Delta + Q\sigma^k\nabla_k + \xi \left[R + 2Q\Delta \sigma - Q(Q + 1)\sigma^k\sigma_k\right] + m^2, \quad (2.3)$$

$$\tilde{L} = -\tilde{\Delta}. \quad (2.4)$$
In order to dimensionally reduce the theory, let us introduce the harmonic analysis on $\Sigma_Q$ by means of

$$\hat{L}Y_\alpha(\hat{x}) = \lambda_\alpha Y_\alpha(\hat{x}),$$  \hspace{1cm} (2.5)

$\lambda_\alpha, Y_\alpha$ being the eigenvalues and eigenfunctions of $\hat{L}$ on the symmetric space $\Sigma_Q = \hat{M}_Q$. For any scalar field in $\tilde{M}_D$, we can write

$$\Phi(\tilde{x}) = \sum_{\alpha} \phi_\alpha(x)Y_\alpha(\hat{x})$$  \hspace{1cm} (2.6)

and for the partition function, after integration over $Y_\alpha$ in the classical action,

$$Z^* = \int d[\bar{\phi}] e^{-\int \bar{\phi} \hat{L} \phi \sqrt{\tilde{g}} d^D \tilde{x}} = \prod_{\alpha} Z_\alpha,$$  \hspace{1cm} (2.7)

where

$$Z_\alpha = \int d[\bar{\phi}_\alpha] e^{-\int \bar{\phi}_\alpha L_\alpha \phi_\alpha d^D x}.$$  \hspace{1cm} (2.8)

Here $\bar{\phi} = \sqrt{\tilde{g}} \phi$ and $\bar{\phi}_\alpha = \sqrt{\tilde{g}_\alpha} \phi_\alpha$ are scalar densities of weight $-1/2$ and the dimensional reduced operators read

$$L_\alpha = -\Delta + V + e^{2\sigma} \lambda_\alpha,$$

$$V = m^2 + \xi \left[ R + 2Q\Delta \sigma - Q(Q + 1)\sigma^k \sigma_k \right] - \frac{Q}{2} \Delta \sigma + \frac{Q^2}{4} \sigma^k \sigma_k.$$  \hspace{1cm} (2.9)

In the following, we will denote by an asterix all the quantities associated with the dimensional reduced operators. As a result, we formally have

$$Z^* = \prod_{\alpha} \left( \det \frac{L_\alpha}{\mu^2} \right)^{-1/2}.$$  \hspace{1cm} (2.10)

If we ignore the multiplicative anomaly associated with functional determinants, we have

$$\Gamma^* = -\ln Z^* = \frac{1}{2} \sum_{\alpha} \ln \det \frac{L_\alpha}{\mu^2}.$$  \hspace{1cm} (2.11)

This formal expression may be regularised and renormalized and we have

$$\Gamma_\varepsilon^* = - \frac{\mu^{2\varepsilon}}{2} \sum_{\alpha} \int_0^\infty dt t^{-1} g_\varepsilon(t) \text{Tr} e^{-tL_\alpha}.$$  \hspace{1cm} (2.12)

Removing the cutoff and, for example making use of a finite regularisation, one arrives at

$$\Gamma^* = \frac{1}{2} \sum_{\alpha} \zeta'(0)\left( \frac{L_\alpha}{\mu^2} \right).$$  \hspace{1cm} (2.13)

Within this procedure, a quite natural definition of the dimensional-reduction anomaly is

$$A_{DRA} = \Gamma - \Gamma^*.$$  \hspace{1cm} (2.14)
However, there exists another possible procedure: if we do not remove the ultraviolet cutoff $\varepsilon$, we may interchange the harmonic sum and the integral and arrive at

$$\Gamma_{\varepsilon}^* = -\frac{\mu^{2\varepsilon}}{2} \int_0^\infty dt \, t^{-1} g_\varepsilon(t) K_t^*, \quad (2.15)$$

where we have introduced the dimensionally reduced heat-kernel trace

$$K_t^* = \sum_\alpha \text{Tr} e^{-tL_\alpha}. \quad (2.16)$$

It is clear that within this second procedure, the existence of a non vanishing dimensional reduction anomaly is strictly related to the fact whether the identity

$$K_t^* = K_t \quad (2.17)$$

holds.

In the following Sections, the validity of the identity (2.17) will be investigated.

3 The flat symmetric space $R^D$

In this Section we shall show that, by performing exactly the harmonic sum, the Eq. (2.17) holds.

Let us consider a free massive scalar field in the D-dimensional Euclidean space $R^D$. The relevant operator is

$$L_D = -\Delta_D + m^2 \quad (3.1)$$

and the corresponding heat-kernel reads

$$K_t(L_D) = \frac{e^{-tm^2}}{(4\pi t)^{D/2}} V(R^D), \quad V(R^D) = \int d\tau \int d\vec{x}. \quad (3.2)$$

Here $V(R^D)$ is the (infinite) volume of the manifold.

We can also use spherical co-ordinates. Then

$$ds^2 = d\tau^2 + dr^2 + r^2 d\bar{S}_Q^2, \quad (3.3)$$

where $d\bar{S}_Q^2$ is the measure of the $Q = (D - 2)$-dimensional unitary sphere $S_Q$, which plays the role of the symmetric space $\hat{M}_Q$ in the previous section. In spherical co-ordinates, the volume is given by

$$V(R^D) = \Omega_Q \int \int r^Q dr, \quad \Omega_Q = \frac{2\pi^{Q+1}}{\Gamma(Q/2)},$$

$\Omega_Q$ being the volume of the sphere $S_Q$. The warp factor $e^{-2\sigma} = r^2$ and $V$, in Eq. (2.3), becomes

$$V = m^2 + \frac{Q}{2} \left( \frac{Q}{2} - 1 \right) \frac{1}{r^2}.$$

The spectrum of the Laplacian $\hat{\Delta}$ on the sphere $S_Q$ is well known. In fact, the eigenvalues $\lambda_\alpha \equiv \lambda_l$ and the corresponding degeneration are given by

$$\lambda_l = l(l + Q - 1), \quad l \geq 0, \quad (3.4)$$
\[ D_l^Q = (2l + Q - 1) \frac{(l + Q - 2)!}{(Q - 1)! l!}, \quad D_0^Q = 1. \] (3.5)

From Eq. (2.9) then we directly have
\[ L_l = -\partial_r^2 - \frac{C_l}{r^2} + m^2, \quad C_l = \frac{(2l + Q - 1)^2 - 1}{4}. \] (3.6)

Using the factorisation property of the heat kernel \[7\], for the operator \( L_l \) we get
\[ K_t(L_l) = K_t(-\partial_r^2 + C_l/r^2) = K_t\left(-\partial_r^2 + m^2\right) e^{-tm^2/4\pi t} \int d\tau. \]

The spectrum of \(-\partial_r^2 + C_l/r^2\) is continuous and non-negative. We indicate by \( \lambda^2 \) the eigenvalues and by \( \psi_l \) the eigenfunctions which are given by
\[ \psi_l(r) = \sqrt{r} J_\nu(lr), \quad \nu_l = l + \frac{Q - 1}{2}, \]
\( J_\nu \) being the Bessel functions of the first kind. The heat-kernel density, by definition, is given by
\[ K_t(r|L_l) = \int_0^\infty e^{-t\lambda^2} |\psi_l(r)|^2 \lambda d\lambda = \frac{re^{-r^2/2t}}{2t} I_{\nu_l}\left(\frac{r^2}{2t}\right). \]

For the whole operator we finally get
\[ K_t^*(L_D) = e^{-tm^2/4\pi t} \int d\tau \int \frac{re^{-r^2/2t}}{2t} \sum_{l=0}^\infty D_l^Q I_{\nu_l}\left(\frac{r^2}{2t}\right) dr. \] (3.7)

This must be compared with Eq. (3.3). To this aim we recall some properties of the modified Bessel functions \( I_\nu \) and the Legendre polynomials, that is
\[ \sum_{n=-\infty}^\infty t^n I_n(z) = e^{ze^{t(2+1)/4}} \quad I_n(z) = I_{-n}(z), \quad n = 1, 2, \ldots \] (3.8)
\[ 2\nu I_{\nu}(z) = z [I_{\nu-1}(z) - I_{\nu+1}(z)], \]
\[ \sum_{l=0}^\infty (2l + 1) P_l(t) I_{l+1/2}(z) = \sqrt{\frac{2z}{\pi}} e^{zt}, \quad P_l(1) = 1. \] (3.9)

From the latter equations with \( t = 1 \) we obtain
\[ \sum_{l=0}^\infty D_l^1 I_l(z) = \sum_{n=-\infty}^\infty I_n(z) = e^z, \] (3.10)
\[ \sum_{l=0}^\infty D_l^2 I_{l+1/2}(z) = \sum_{l=0}^\infty (2l + 1) I_{l+1/2}(z) = \sqrt{\frac{2z}{\pi}} e^z. \] (3.11)

Now, by induction, we are able to prove the following equation:
\[ \sum_{l=0}^\infty D_l^Q I_{\nu_l}(z) = \frac{1}{\Gamma(2\nu_0 + 1)} \sum_{\nu = \nu_0}^\infty \frac{\Gamma(\nu + \nu_0)}{\Gamma(\nu + 1 - \nu_0)} 2\nu I_\nu(z) = \frac{z^{\nu_0} e^z}{2^{\nu_0} \Gamma(\nu_0 + 1)}; \] (3.12)
where the sum is over integer or half-integer $\nu$ according to whether $\nu_0 = (Q - 1)/2$ is integer or half-integer.

For $Q = 1$ ($\nu_0 = 0$) and $Q = 2$ ($\nu_0 = 1/2$), the latter equation reduces to (B.10) and (B.11) respectively. Then, let us suppose Eq. (B.12) to be true for a given $\nu_0$ and show that it remains true also for $\nu_0 + 1$. In fact we have

\[ \frac{1}{\Gamma(2\nu_0 + 3)} \sum_{\nu=\nu_0+1}^{\infty} \frac{\Gamma(\nu + \nu_0 + 1)}{\Gamma(\nu - \nu_0)} 2\nu I_{\nu}(z) = \]

\[ \frac{z}{\Gamma(2\nu_0 + 3)} \sum_{\nu=\nu_0}^{\infty} \frac{\Gamma(\nu + \nu_0 + 1)}{\Gamma(\nu - \nu_0)} [I_{\nu-1}(z) - I_{\nu+1}(z)] = \]

\[ \frac{z}{\Gamma(2\nu_0 + 3)} \sum_{\nu=\nu_0}^{\infty} \frac{\Gamma(\nu + \nu_0)}{\Gamma(\nu - \nu_0 + 1)} I_{\nu}(z) [(\nu + \nu_0)(\nu + \nu_0 + 1) - (\nu - \nu_0)(\nu - \nu_0 - 1)] = \]

\[ \frac{z(2\mu_0 + 1)}{2(\nu_0 + 1)} \frac{1}{\Gamma(2\nu_0 + 1)} \sum_{\nu=\nu_0}^{\infty} \frac{\Gamma(\nu + \nu_0)}{\Gamma(\nu - \nu_0 + 1)} 2\nu I_{\nu}(z) = \]

\[ \frac{z^{\nu_0 + 1} e^z}{2^{\nu_0 + 1} \Gamma(\nu_0 + 2)}. \]

Using (B.12) in (B.7) we finally obtain the announced result

\[ K^*_t(L_D) = \frac{e^{tm^2}}{(4\pi t)^{D/2}} \Omega_Q \int d\tau \int r^Q dr = K_t(L_D). \] (3.13)

We conclude this Section observing that the above result also holds for an Euclidean Rindler space, since it can be regarded as the product of an Euclidean space in polar coordinates times an Euclidean transverse space.

4 The non-flat symmetric spaces $H^D$

In this Section, we complete the analysis on the validity of the Eq. (2.17) considering the case $H^D$. For the sake of simplicity, we shall not deal with $S^D$, which is more involved and can be found in [20].

It is convenient to make use of the Poincaré form for the metric of $H^D$, namely

\[ ds^2 = \frac{1}{x^2} \left( d^2 \bar{y} + dx^2 \right). \] (4.1)

Thus, we have the warp product $R \times R^Q$ ($P = 1$, $Q = D - 1$). The manifold is non compact and the spectrum of $L_D = -\Delta_D$ is continuous. The diagonal part of the heat-kernel of $L_D$ on $H^D$ can be obtained directly making use of the Selberg transform (see, for example, [2] and references cited there) and reads

\[ K_t(L_D) = V_D \int_0^\infty d\lambda e^{-t\lambda^2} \Phi_D(\lambda), \] (4.2)

where $V_D$ is the volume of $H^D$ and $\Phi_D(\lambda)$ the Harish-Chandra-Plancherel measure given by

\[ \Phi_D(\lambda) = \frac{2}{(4\pi)^{D/2} \Gamma(D/2)} \frac{\Gamma(i\lambda + (D - 1)/2)^2}{|\Gamma(i\lambda)|^2}. \] (4.3)
Now let us start the computation of $K_t^*(L_D)$. In the Poincaré representation, $\tilde{M}_Q = R^Q$ is again a non compact manifold and the spectrum of the Laplacian $-\tilde{\Delta}_Q$ on it is continuous and formed by $\lambda_\alpha \equiv \lambda_{\tilde{k}} = \tilde{k}^2 > 0$. The reduced harmonic operator is
\[ L_\alpha = L_{\tilde{k}} = -x^2 \partial_x^2 + (Q - 1)x \partial_x + \tilde{k}^2 x^2. \] (4.4)

The related heat-kernel can be computed by means of the Harish-Chandra method [21], namely by solving the generalized eigenfunction equation
\[ \left[-x^2 \partial_x^2 + (Q - 1)x \partial_x + \tilde{k}^2 x^2\right] f_\lambda(x) = \lambda^2 f_\lambda(x). \] (4.5)

The only solutions with the corrected behavior at the infinity are
\[ f_\lambda(x) = x^{Q/2} K_{i\lambda}(x k), \quad k = |\tilde{k}|, \] (4.6)

where $K_z(x)$ is the modified Bessel function. The spectral theorem gives for the local reduced heat-kernel
\[ K_t(x|L_{\tilde{k}}) = \frac{x^Q}{(2\pi)^Q} \int_0^\infty d\lambda \mu(\lambda) e^{-\lambda^2} K_{i\lambda}^2(x k), \] (4.7)

where $\mu(\lambda)$ is the Kontarevich measure, namely
\[ \mu(\lambda) = \frac{2}{\pi |\Gamma(i\lambda)|^2} = \frac{2}{\pi^2} \lambda \sinh \pi \lambda. \] (4.8)

The harmonic sum is now replaced by an integral and we have
\[ K_t^*(x|L_D) = \int_{R^Q} d\tilde{k} K_t(L_{\tilde{k}}) = \frac{x^Q Q^{-1}}{(2\pi)^Q} \int_0^\infty d\lambda \mu(\lambda) e^{-\lambda^2} \int_0^\infty dk k^{Q-1} K_{i\lambda}^2(x k). \] (4.9)

The integration over $k$ involving the modified Bessel function can be done as well as the trivial integration over the coordinates and the result is
\[ K_t^*(L_D) = V_D \frac{2}{(4\pi)^{D/2} \Gamma(D/2)} \int_0^\infty d\lambda \frac{|\Gamma(i\lambda + (D - 1)/2)|^2}{|\Gamma(i\lambda)|^2} e^{-\lambda^2}. \] (4.10)

Recalling Eqs. (4.2) and (4.3), one finally gets
\[ K_t^*(L_D) = K_t(L_D). \] (4.11)

As a consequence, also in the constant curvature space $H^D$, Eq. (2.17) holds.

5 The general case

The result of the previous Sections tell us that it is very crucial to be able to perform the harmonic sum. In general, this is not possible. For this reason, we shall restrict ourselves to the class of non-trivial warped space-time of the kind considered in Sec. 2 and make use of the short $t$ heat-kernel expansion. For the exact theory we have (here $L_D = \tilde{L}$)
\[ K_t(\tilde{L}) = \text{Tr} e^{-t\tilde{L}} \sim \frac{1}{(4\pi t)^{D/2}} \sum_{n=0}^\infty \tilde{a}_n(\tilde{x}|\tilde{L}) t^n, \] (5.1)
with
\[
\begin{align*}
\tilde{a}_1 & = a_1 + e^{2\sigma} \tilde{a}_1 - \frac{Q}{6} \left[ \Delta \sigma - \left( \frac{Q}{2} - 1 \right) \sigma^k \sigma_k \right], \\
\tilde{a}_2 & = a_2 + e^{4\sigma} \tilde{a}_2 + e^{2\sigma} a_1 \tilde{a}_1 - \frac{1}{90} \sigma^k \nabla_k R - \frac{1}{45} \ldots
\end{align*}
\]
(5.2)

(5.3)

where, as in Sec. 2, all quantities with tilde refers to the whole manifold \( \tilde{M}_D \) and all quantities with hat refers to the sub-manifold \( \tilde{M}_Q \).

With regard to the dimensional reduced kernel
\[
K^*_t(\tilde{L}) = \sum_a \text{Tr} e^{-tL_a},
\]
(5.4)

where
\[
L_a = -\Delta + V + e^{2\sigma} \lambda_a,
\]
\[
V = m^2 + \xi \left[ R + 2Q\Delta \sigma - Q(Q + 1)\sigma^k \sigma_k \right] - \frac{Q}{2} \Delta \sigma + \frac{Q^2}{4} \sigma^k \sigma_k,
\]
(5.5)

the short \( t \) expansion can be computed by means of a straightforward computation, which is summarized in the following. Recalling the heat-kernel expansion relations (see Appendix B), we can write
\[
K^*_t(\tilde{L}) \sim \sum_a \frac{1}{(4\pi t)^{D/2}} \int_{\tilde{M}_D} \sqrt{g} d^D \tilde{x} e^{(a_1 - e^{2\sigma} \lambda_a)} \sum_{n=0}^{\infty} b_n(x|L_a) t^n
\]
\[
= \sum_a \frac{1}{(4\pi t)^{D/2}} \int_{\tilde{M}_D} \sqrt{g} d^D \tilde{x} e^{(a_1 - e^{2\sigma} \lambda_a)}
\]
\[
\times \left[ 1 + \sum_{n \geq 2; 0 \leq k \leq 2n/3} \Lambda_{nk} \lambda^k_n t^n \right].
\]
(5.6)

Now we observe that
\[
K^*_t(\tilde{L}) = \frac{1}{(4\pi t)^{Q/2}} \int_{\tilde{M}_Q} \sqrt{g} d^Q \tilde{x} \sum_a e^{-\tau \lambda_a} \sim \frac{1}{(4\pi t)^{Q/2}} \int_{\tilde{M}_Q} \sqrt{g} d^Q \tilde{x} \sum_{l=0}^{\infty} \tilde{a}_l \tau^l,
\]
(5.7)

\[
\frac{d^k K^*_t(\tilde{L})}{d\tau^k} = \frac{(-1)^k}{(4\pi t)^{Q/2}} \int_{\tilde{M}_Q} \sqrt{g} d^Q \tilde{x} \sum_{\alpha} e^{-\tau \lambda_{\alpha}} \lambda_{\alpha}
\]
\[
\sim \frac{1}{(4\pi t)^{Q/2}} \int_{\tilde{M}_Q} \sqrt{g} d^Q \tilde{x} \sum_{l=0}^{\infty} \Gamma(l - Q/2 + 1) \frac{\Gamma(l - Q/2 + 1 - k)}{\Gamma(l - Q/2 + 1 - k)} \tilde{a}_l \tau^{l-k}.
\]
(5.8)

By setting \( \tau = e^{2\sigma} \) and using Eq. (5.8) in Eq. (5.4), we finally obtain
\[
K^*_t(\tilde{L}) \sim \frac{1}{(4\pi t)^{D/2}} \int_{\tilde{M}_D} \sqrt{g} d^D \tilde{x} \sum_{l,j=0}^{\infty} \frac{a^l_j \tilde{a}_l t^{l+j}}{j!} e^{2\sigma}
\]
\[
\times \left[ 1 + \sum_{n \geq 2; 0 \leq k \leq 2n/3} (-1)^k \frac{\Gamma(l - Q/2 + 1)}{\Gamma(l - Q/2 + 1 - k)} \Lambda_{nk} e^{-2\sigma \tau^{l-n-k}} \right]
\]
\[
\sim \frac{1}{(4\pi t)^{D/2}} \sum_{n=0}^{\infty} \tilde{a}_n(x|\tilde{L}) t^n.
\]
(5.9)
By the latter equation we immediately read off the heat kernel coefficients. In particular, we get

\[ a_1^*(\tilde{x}|\tilde{L}) = a_1 + e^{2\sigma}a_1 + \frac{Q}{2}e^{-2\sigma}A_{21} + \frac{Q}{4}(Q + 2)e^{-4\sigma}A_{32}, \]

\[ a_2^*(\tilde{x}|\tilde{L}) = \frac{1}{2}a_1^2 + e^{4\sigma}a_2 + e^{2\sigma}a_1a_1 + A_{20} + \frac{Q}{2}a_1e^{-2\sigma}A_{21} \]

\[ + \frac{Q}{4}(Q - 2)a_1A_{21} + \frac{Q}{2}e^{-2\sigma}A_{31} + \frac{Q}{4}(Q + 2)a_1e^{-4\sigma}A_{32} \]

\[ + \frac{Q}{4}(Q - 2)a_1e^{-2\sigma}A_{32} + \frac{Q}{4}(Q + 2)e^{-4\sigma}A_{42} + \frac{Q}{8}(Q + 2)(Q + 4)e^{-6\sigma}A_{53} \]

\[ + \frac{Q}{16}(Q + 2)(Q + 4)(Q + 6)e^{-8\sigma}A_{64}, \]

Using the relations in Appendices A and B one can show that the latter coefficients \(a_1^*(\tilde{x}|\tilde{L})\) and \(a_2^*(\tilde{x}|\tilde{L})\) exactly coincide with \(\tilde{a}_1\) and \(\tilde{a}_2\), Eqs. (5.2), (5.3).

As a consequence, one has

\[ K_1(\tilde{L}) \simeq \frac{e^{i\tilde{a}_1(\tilde{x}|\tilde{L})}}{(4\pi t)^{D/2}} \left[ 1 + b_2(\tilde{x}|\tilde{L})t^2 + b_3(\tilde{x}|\tilde{L})t^3 + ... \right], \]

\[ K_1^*(\tilde{L}) \simeq \frac{e^{i\tilde{a}_1(\tilde{x}|\tilde{L})}}{(4\pi t)^{D/2}} \left[ 1 + b_2(\tilde{x}|\tilde{L})t^2 + b_3(\tilde{x}|\tilde{L})t^3 + ... \right]. \]

What about \(b_3^*(\tilde{x}|\tilde{L}), b_4^*(\tilde{x}|\tilde{L}), ...?\) Within our short \(t\) approximation, we are not able to say anything about the relationship with \(b_3(\tilde{x}|\tilde{L}), b_4(\tilde{x}|\tilde{L}), ...\) However, it is quite natural to make the conjecture that \(b_n^*(\tilde{x}|\tilde{L}) = b_n(\tilde{x}|\tilde{L})\) for every \(n\) and Eq. (2.17) holds exactly.

**6 Conclusions.**

In this paper, the issue related to the dimensional reduction procedure has been revisited. In the symmetric and constant curvature space-times, as \(R^D\) and \(H^D\), we have shown that the two local diagonal heat-kernels, namely the exact one and the one obtained summing the dimensional reduced harmonic heat-kernels, are equal. This result holds for a generic symmetric space.

In the general case, due to the impossibility to deal with exact quantities, we have used a short \(t\) expansion and a partial re-summation of the heat-kernel expansion. We have conjectured that the exact and total dimensional reduced kernel are equal. Let us discuss about the consequences of this statement.

After the dimensional reduction, as far as the effective action is concerned, the operation of renormalisation (addition of counter-terms and remotion of the cutoff) and the evaluation of the harmonic sum do not commute. If we keep fixed and non vanishing the regularisation parameter, we may perform the harmonic sum, and if (2.17) holds, we may reconstruct the exact partition function, after renormalisation. In such a case, it is evident that no dimensional reduction anomaly occurs.

On the other hand, one may remove the cutoff, adding the necessary counter-terms or using a finite regularisation like the zeta-function and perform the harmonic sum at the end. In this
case, as shown in reference \[3\], one has to correct the result by adding dimensional reduction anomaly terms.

It has to be stressed that the appearance of the reduction anomaly is independent on the regularisation scheme one is dealing with, since the regularised effective action, after the removal of divergences, is the same for all regularisations \[14\].

There exists also a mathematical reason for the necessity of these reduction anomaly terms. In fact, the harmonic sum of the renormalized dimensionally reduced effective action diverges and the dimensional anomaly reduction terms are also necessary to recover the exact and finite result. This is a consequence of the following asymptotic behaviour of the partial renormalized effective action, valid for very large $\lambda_\alpha$ (for the sake of simplicity here $D$ is assumed to be odd)

$$
\zeta'(0|L_\alpha) \simeq \Gamma(-D/2)(V + \lambda_\alpha e^{2\gamma}D/2 + ...)
$$

As a result, the harmonic sum over $\alpha$ is badly divergent! This fact stems also from the necessity of the presence of the multiplicative anomaly, since it also diverges, being associated with a product of an infinite number of dimensional reduced operators \[8, 10\].

As a consequence, any approximation \[22\] based on the truncation in the harmonic sum of the dimensional reduced theory, may lead, with regard to the comparison with the exact theory, to incorrect conclusions (see also the discussions and further references reported in \[23\]).

A Relations between curvatures

Here we write down the relations between curvatures and Laplacian on the manifolds we are dealing with.

For the non vanishing components of connections and curvatures we directly obtain

$$
\begin{align*}
\tilde{\Gamma}^k_{ij} &= \Gamma^k_{ij}, & \tilde{\Gamma}^k_{ab} &= e^{-2\gamma} g^k_{ab}, \\
\tilde{\Gamma}^k_{bc} &= \tilde{\Gamma}^k_{bc}, & \tilde{\Gamma}^k_{bb} &= -\sigma_k g^a_b,
\end{align*}
$$

$$
\begin{align*}
\tilde{R}_{ijmn} &= R_{ijmn}, \\
\tilde{R}_{abcd} &= e^{-2\gamma} \tilde{R}_{abcd} - e^{-4\gamma} (\tilde{g}_a \tilde{g}_b \tilde{g}_c \tilde{g}_d - \tilde{g}_{ad} \tilde{g}_{bc}) , \\
\tilde{R}_{iajb} &= e^{-2\gamma} (\sigma_{ij} - \sigma_i \sigma_j),
\end{align*}
$$

where

$$
\sigma_k = \nabla_k \sigma(x), \quad \sigma^k = g^{kj} \sigma_j, \quad \sigma_{ij} = \nabla_i \nabla_j \sigma(x),$$

are the covariant derivatives, in the metric $g$, of the scalar function $\sigma(x)$. By contraction we get

$$
\begin{align*}
\tilde{R}_{ij} &= \tilde{R}_{ij} + Q (\sigma_{ij} - \sigma_i \sigma_j), \\
\tilde{R}_{ab} &= \tilde{R}_{ab} + e^{-2\gamma} \tilde{g}_{ab} \left( \Delta \sigma - Q \sigma^k \sigma_k \right),
\end{align*}
$$

$$
\tilde{R} = R + e^{2\gamma} \tilde{R} + 2Q \Delta \sigma - Q(Q + 1) \sigma^k \sigma_k,
$$

and also

$$
\tilde{R}^{\alpha\beta\gamma\delta} R_{\alpha\beta\gamma\delta} = R^{ijmn} R_{ijmn} + e^{4\gamma} \tilde{R}_{abcd} \tilde{R}_{abcd} - 4e^{2\gamma} \tilde{R} \left( \sigma^k \sigma_k \right) + 4Q \sigma^{ij} \sigma_{ij} - 8Q \sigma_{ij} \sigma_i \sigma_j + 2 + Q(Q + 1) (\sigma^k \sigma_k)^2,
$$

\[A.2\]
\[
\tilde{R}^{\alpha\beta} \tilde{R}_{\alpha\beta} = R^{ij}R_{ij} + e^{2\sigma} R^{ab}\tilde{R}_{ab} + 2Q R^{ij} (\sigma_{ij} - \sigma_i \sigma_j) \\
+ 2e^{2\sigma} \tilde{R} (\Delta s_i - Q \sigma^k \sigma_k) + Q^2 \sigma^{ij} \sigma_{ij} - 2Q^2 \sigma^i \sigma_j \sigma_i \sigma_j \\
+ Q(\Delta \sigma)^2 - 2Q^2 \sigma^k \sigma_k \Delta \sigma + Q^2 (Q + 1)(\sigma^k \sigma_k)^2.
\]  
(A.3)

Finally, for the Laplacian of any scalar function \( f(\tilde{x}) \) on \( \tilde{M}_D \) we have

\[
\tilde{\Delta} f(\tilde{x}) = (\Delta + e^{2\sigma} \tilde{\Delta} - Q \sigma^k \nabla_k) f(\tilde{x}).
\]  
(A.4)

**B Heat kernel coefficients**

The heat kernel for a Laplacian-like operator \( L = -\Delta + X \) on a \( P \)-dimensional curved manifold without boundary is usually written in the form

\[
K_t(x|L) \sim (4\pi t)^{-P/2} \sum_{n=1}^{\infty} a_n t^{n/2},
\]  
(B.1)

where the spectral coefficients \( a_n \) are computable quantities depending on \( V \), its covariant derivatives and all geometric invariants. There exists also an alternative expansion \([17]\) which has the advantage with respect to the previous one that the expansion coefficients depend on \( V \) only by its covariant derivatives. It reads

\[
K_t(x|L) \sim e^{t a_1} (4\pi t)^{-P/2} \sum_{n=0}^{\infty} b_n t^{n/2}.
\]  
(B.2)

Some coefficients are explicitly known and can be found in Refs. \([17]\). They read \((a_0 = b_0 = 1\) by definition),

\[
a_1 = \frac{R}{6} - X, \quad b_1 = 0,
\]  
(B.3)

\[
b_2 = \frac{1}{6} \Delta a_1 + \frac{1}{180} (\Delta R + R^{ijrs}R_{ijrs} - R^{ij}R_{ij})
\]  
(B.4)

\[
b_3 = \frac{1}{12} \nabla^k X \nabla_k X - \frac{1}{60} \Delta^2 X - \frac{1}{90} R^{ij} \nabla_i \nabla_j X - \frac{1}{30} \nabla^k R \nabla_k X + ...,
\]  
(B.5)

\[
b_4 = \frac{1}{72} (\Delta X)^2 + \frac{1}{90} \nabla^i \nabla^j X \nabla_i \nabla_j X \\
+ \frac{1}{30} \nabla^k X \nabla_k \Delta X + \frac{1}{60} R^{ij} \nabla_i \nabla_j X + ...,
\]  
(B.6)

\[
b_5 = -\frac{1}{72} \nabla^k X \nabla_k X \Delta X - \frac{1}{60} \nabla^i X \nabla^j X \nabla_i \nabla_j X + ...
\]  
(B.7)

\[
b_6 = \frac{1}{288} (\nabla^k X \nabla_k X)^2 + ...,
\]  
(B.8)

where \( ... \) stand for lower terms in \( X \).

The operator we are dealing with in the paper is

\[
L_\alpha = -\Delta + V + e^{2\sigma} \lambda_\alpha
\]
and so we write

\[ b_n = \sum_{0 \leq k \leq 2n/3} \Lambda_n \lambda_k \Lambda_k, \quad n \geq 2. \]

In the latter equation, the restriction on the range of \( k \) can be easily derived by dimensional considerations. Using the above results, for the quantities we need in the paper, after straightforward calculations we get

\[
\Lambda_{20} = \frac{1}{6} \Delta \left( \frac{R}{6} - V \right),
\]

\[
e^{-2\sigma} \Lambda_{21} = -\frac{1}{3} \Delta \sigma - \frac{2}{3} \sigma \lambda_k \sigma_k,
\]

\[
e^{-2\sigma} \Lambda_{31} = -\frac{1}{3} \sigma \nabla_k \left( \frac{R}{6} - V \right) - \frac{1}{90} \sigma \nabla_k R - \frac{1}{45} R_{ij} (7 \sigma_{ij} + 2 \sigma_i \sigma_j) - \frac{1}{30} \Delta^2 \sigma - \frac{1}{15} (\Delta \sigma)^2 - \frac{4}{15} \sigma \lambda_k \Delta \sigma - \frac{1}{15} \left[ 4(\sigma \lambda_k \sigma_k)^2 + 4 \sigma \nabla_k \Delta \sigma + 2 \sigma_{ij} \sigma_{ij} + 8 \sigma_{ij} \sigma_{ij} \right],
\]

\[
e^{-4\sigma} \Lambda_{32} = \frac{1}{3} \sigma \lambda_k \sigma_k
\]

\[
e^{-4\sigma} \Lambda_{42} = \frac{1}{15} R_{ij} \sigma_i \sigma_j + \frac{1}{18} (\Delta \sigma)^2 + \frac{14}{15} (\sigma \lambda_k \sigma_k)^2 + \frac{22}{45} \sigma \lambda_k \Delta \sigma + \frac{2}{15} \sigma \lambda_k \nabla_k \Delta \sigma + \frac{2}{45} \sigma_{ij} \sigma_{ij} + \frac{32}{45} \sigma_{ij} \sigma_{ij},
\]

\[
e^{-6\sigma} \Lambda_{53} = -\frac{1}{9} \sigma \lambda_k \Delta \sigma - \frac{22}{45} (\sigma \lambda_k \sigma_k)^2 - \frac{2}{15} \sigma \lambda_k \sigma_j
\]

\[
e^{-8\sigma} \Lambda_{64} = \frac{1}{18} (\sigma \lambda_k \sigma_k)^2.
\]

References

[1] V. Mukhanov, A. Wipf and A. Zelnikov. Phys. Lett. B 332, 283(1994).

[2] E. Elizalde, S. Naftulin and S. D. Odintsov. Phys. Rev. D49, 2852 (1994); S. Nojiri and S. D. Odintsov. Phys. Lett B463, 57 (1999).

[3] V. Frolov, P. Sutton and A. Zelnikov. Phys. Rev. D 61, 02421, (2000).

[4] P. Sutton. Phys. Rev. D 62, 044033, (2000).

[5] R. Balbinot, A. Fabbri, V. Frolov, P. Nicolini, P. Sutton, A. Zelnikov, it Vacuum polarization in the Schwarzschild spacetime and dimensional reduction. hep-th/0012048.

[6] E. Elizalde, S. D. Odintsov, A. Romeo, A.A. Bytsenko and S. Zerbini. Zeta Regularization Techniques with Applications. World Scientific, Singapore (1994).

[7] A.A. Bytsenko, G. Cognola, L. Vanzo and S. Zerbini. Phys. Rep. 266, 1 (1996).

[8] E. Elizalde, L. Vanzo and S. Zerbini. Commun. Math. Phys. 194, 613 (1998).
[9] E. Elizalde, A. Filippi, L. Vanzo and S. Zerbini. Phys. Rev. \textbf{D57}, 7430 (1998).

[10] E. Elizalde, G. Cognola and S. Zerbini. Nucl. Phys. \textbf{B532}, 407 (1998).

[11] D.B. Ray and I.M. Singer. Ann. Math. \textbf{98}, 154 (1973).

[12] S. W. Hawking. Commun. Math. Phys. \textbf{55}, 133 (1977).

[13] J.S. Dowker and R. Critchley. Phys. Rev. \textbf{D 13}, 3224 (1976).

[14] G. Cognola, K. Kirsten and S. Zerbini. Phys. Rev. \textbf{D 48}, 790 (1993).

[15] B.S. DeWitt. \textit{The Dynamical Theory of Groups and Fields}. Gordon and Breach, New York (1965).

[16] R.T. Seeley. Am. Math. Soc. Prog. Pure Math. \textbf{10}, 172 (1967).

[17] L. Parker and D.J. Toms. Phys. Rev \textbf{D31}, 953 (1985); Phys. Rev \textbf{D31}, 953 (1985).

[18] D. Iellici and V. Moretti. Phys.Lett. \textbf{B435},33 (1998).

[19] D. Binosi and S. Zerbini. J.Math.Phys. \textbf{40}, 5106 (1999).

[20] R. Camporesi. Phys. Rep. \textbf{196}, 1 (1990).

[21] A. A. Bytsenko, G. Cognola and S. Zerbini. Nucl. Phys. \textbf{B458}, 267 (1996).

[22] R. Bousso and S. Hawking. Phys. Rev. \textbf{D56}, 7788 (1997); T. Chiba and M. Siino. Mod. Phys. Lett. \textbf{A 12}, 709 (1997); W. Kummer, H. Lieb and D.V. Vassilevich. Mod. Phys. Lett. \textbf{A 12}, 2683 (1997).

[23] S. Nojiri and S. D. Odintsov. Mod. Phys. Lett. \textbf{A 12}, 2083 (1997); S. Nojiri and S. D. Odintsov. Phys. Rev. \textbf{D57}, 2363 (1998); S. Nojiri and S. D. Odintsov. Phys. Rev. \textbf{D57}, 4847 (1998); S.J. Gates, S. Nojiri, T. Kadoyosi and S. D. Odintsov. Phys. Rev. \textbf{D58}, 084026 (1998); W. Kummer, H. Lieb and D.V. Vassilevich. Phys. Rev. \textbf{D58}, 108501 (1998); S. Nojiri, O. Obregon S. D. Odintsov and K. E. Osetrin. Phys. Rev. \textbf{D60}, 0204008 (1999); R. Balbinot and A. Fabbri Phys. Rev. \textbf{D59}, 044031 (1999); R. Balbinot and A. Fabbri Phys. Lett. \textbf{B459}, 112 (1999); R. Balbinot, A. Fabbri and I. Shapiro. Phys. Rev. Lett. \textbf{83}, 1494 (1999); R. Balbinot, A. Fabbri and I. Shapiro. Nucl. Phys. \textbf{B559}, 301 (1999); W. Kummer, H. Lieb and D.V. Vassilevich. Phys. Rev. \textbf{D60}, 084021 (1999); F. C. Lombardo, F. D. Mazzitelli and J. G. Russo. Phys. Rev. \textbf{D60}, 064007 (1999). R. Balbinot, A. Fabbri, V. Frolov, P. Nicolini, P. Sutton, A. Zelnikov. \texttt{hep-th/0012048}.