COUNTING LATTICE POINTS IN FREE SUMS OF POLYTOPES

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Abstract. We show how to compute the Ehrhart polynomial of the free sum of two lattice polytopes containing the origin \( P \) and \( Q \) in terms of the enumerative combinatorics of \( P \) and \( Q \). This generalizes work of Beck, Jayawant, McAllister, and Braun, and follows from the observation that the weighted \( h^* \)-polynomial is multiplicative with respect to the free sum. We deduce that given a lattice polytope \( P \) containing the origin, the problem of computing the number of lattice points in all rational dilates of \( P \) is equivalent to the problem of computing the number of lattice points in all integer dilates of all free sums of \( P \) with itself.

Let \( P \) and \( Q \) be full-dimensional lattice polytopes containing the origin with respect to lattices \( N_P \cong \mathbb{Z}^{\dim P} \) and \( N_Q \cong \mathbb{Z}^{\dim Q} \) respectively. The free sum (also known as ‘direct sum’) \( P \oplus Q \) is a full-dimensional lattice polytope containing the origin in the lattice \( N_P \oplus N_Q \), defined by:

\[ P \oplus Q = \text{conv}((P \times 0_Q) \cup (0_P \times Q)) \subseteq (N_P \oplus N_Q)_{\mathbb{R}}, \]

where \( \text{conv}(S) \) denotes the convex hull of a set \( S \), \( N_{\mathbb{R}} := N \otimes_{\mathbb{Z}} \mathbb{R} \) for a lattice \( N \), and \( 0_P, 0_Q \) denote the origin in \( N_P, N_Q \) respectively.

The Ehrhart polynomial \( f(P; m) \) of \( P \) is a polynomial of degree \( \dim P \) characterized by the property that \( f(P; m) = \#(mP \cap N_P) \) for all \( m \in \mathbb{Z}_{\geq 0} \) [4]. Our goal is to describe the Ehrhart polynomial of \( P \oplus Q \) in terms of the enumerative combinatorics of \( P \) and \( Q \).

We first observe that \( \{ \#(\lambda P \cap N_P) \mid \lambda \in \mathbb{Q}_{\geq 0} \} \) and \( \{ \#(\lambda Q \cap N_Q) \mid \lambda \in \mathbb{Q}_{\geq 0} \} \) determine \( \{ \#((\lambda (P \oplus Q) \cap (N_P \oplus N_Q)) \mid \lambda \in \mathbb{Q}_{\geq 0} \} \), and hence the set \( \{ \#((m(P \oplus Q) \cap (N_P \oplus N_Q)) \mid m \in \mathbb{Z}_{\geq 0} \} \), which is encoded by the Ehrhart polynomial of \( P \oplus Q \) (see [9] for a partial converse).

Indeed, this follows from the following observation: if \( \partial_{\neq 0} P \) denotes the union of the facets of \( P \) not containing the origin, then, by definition, for any \( \lambda \in \mathbb{Q}_{\geq 0} \):

\[ \#(\partial_{\neq 0}(\lambda P) \cap N_P) = \#(\lambda P \cap N_P) - \max_{0 \leq \lambda' < \lambda} \#(\lambda' P \cap N_P), \]

and

\[ (1) \quad \partial_{\neq 0}(\lambda(P \oplus Q)) = \bigcup_{\substack{\lambda_P, \lambda_Q \geq 0 \\lambda_P + \lambda_Q = \lambda}} \partial_{\neq 0}(\lambda_P P) \times \partial_{\neq 0}(\lambda_Q Q), \]

where the right hand side is a disjoint union.

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It will be useful to express the invariants above in terms of corresponding generating series. Firstly, the Ehrhart polynomial may be encoded as follows:

$$(2) \quad \sum_{m \geq 0} f(P; m)t^m = \frac{h^*(P; t)}{(1-t)^{\dim P+1}},$$

where $h^*(P; t) \in \mathbb{Z}[t]$ is a polynomial of degree at most $\dim P$ with non-negative integer coefficients, called the $h^*$-polynomial of $P$ \cite{10}. Secondly, let $M_P := \text{Hom}(N_P, \mathbb{Z})$ be the dual lattice, and recall that the dual polyhedron $P^\vee$ is defined to be $P^\vee = \{ u \in (M_P)_{\mathbb{R}} \mid \langle u, v \rangle \geq -1 \text{ for all } v \in P \}$. Let

$$(3) \quad r_P := \min \{ r \in \mathbb{Z}_{>0} \mid rP^\vee \text{ is a lattice polyhedron } \}.$$

Note that since $(P \oplus Q)^\vee$ is the Cartesian product $P^\vee \times Q^\vee$, we have $r_{P \oplus Q} = \text{lcm}(r_P, r_Q)$. Then one may associate a generating series encoding $\{ \#(\lambda P \cap N_P) \mid \lambda \in \mathbb{Q}_{\geq 0} \}$:

$$(4) \quad \sum_{\lambda \in \mathbb{Q}_{\geq 0}} \#(\partial_{\neq 0}(\lambda P) \cap N_P)t^\lambda = \frac{\tilde{h}(P; t)}{(1-t)^{\dim P}},$$

where $\tilde{h}(P; t) \in \mathbb{Z}[t^{1/r_P}]$ is a polynomial of degree at most $\dim P$ with fractional exponents and non-negative integer coefficients, called the weighted $h^*$-polynomial of $P$.

**Example 1.** Let $N_P = \mathbb{Z}$ and let $P = [-2, 2]$, $Q = [-1, 3] = P + 1$. Then $r_P = 2, r_Q = 3$, and one may compute:

$$
\begin{align*}
\tilde{h}(P; t) &= 1 + 2t^{1/2} + t, \\
\tilde{h}(Q; t) &= 1 + t^{1/3} + t^{2/3} + t, \\
h^*(P; t) &= h^*(Q; t) = 1 + 3t, \\
\tilde{h}(P \oplus P; t) &= \tilde{h}(P; t)\tilde{h}(P; t) = 1 + 4t^{1/2} + 6t + 4t^{3/2} + t^2, \\
h^*(P \oplus P; t) &= 1 + 10t + 5t^2, \\
\tilde{h}(P \oplus Q; t) &= \tilde{h}(P; t)\tilde{h}(Q; t) \\
&= 1 + t^{1/3} + 2t^{1/2} + t^{2/3} + 2t^{5/6} + 2t \\
&\quad + 2t^{7/6} + t^{1/3} + 2t^{3/2} + t^{5/3} + t^2, \\
h^*(P \oplus Q; t) &= 1 + 8t + 7t^2.
\end{align*}
$$

**Remark 2.** The weighted $h^*$-polynomial was introduced in \cite{11} and generalized in \cite{14} Section 4.3. For the specific definition given in \cite{14}, see the proof of Proposition 2.6 in \cite{11} with $\lambda \equiv 0$ and $s = t$, and note that, roughly speaking, using the denominator $(1-t)^{\dim P}$ in \cite{14} rather than the denominator $(1-t)^{\dim P+1}$ (cf. (2)) corresponds to enumerating lattice points on the boundary of the polytope, rather than those in the polytope itself. For the non-negativity of the coefficients together with a formula to compute $\tilde{h}(P; t)$, see (15) in \cite{11}. For the fact that $\tilde{h}(P; t) \in \mathbb{Z}[t^{1/r_P}]$, see Remark 7 below. Note that it follows from \cite{14} that
if we write \( \tilde{h}(P; t) = \sum_{j \in Q} \tilde{h}_{P,j} t^j \), then the polynomial \( \sum_{i \in \mathbb{Z}} \tilde{h}_{P,i} t^i \) consisting of the terms with integer-valued exponents of \( t \) is precisely the \( h^* \)-polynomial associated to the lattice polyhedral complex determined by the union of the facets of \( P \) not containing the origin.

Moreover, let
\[
\Psi : \bigcup_{r \in \mathbb{Z}_{>0}} \mathbb{R}[t^{1/r}] \to \mathbb{R}[t]
\]
denote the \( \mathbb{R} \)-linear map defined by \( \Psi(t^j) = t^{[j]} \) for all \( j \in \mathbb{Q}_{\geq 0} \). Then we recover the \( h^* \)-polynomial of \( P \) via the formula (see (14) in [11]):
\[
h^*(P; t) = \Psi(\tilde{h}(P; t)).
\]
We also note that when the origin lies in the relative interior of \( P \), we have the symmetry [11, Corollary 2.12]:
\[
\tilde{h}(P; t) = t^{\dim P} \tilde{h}(P; t^{-1}).
\]

**Remark 3.** The following alternative definition of the weighted \( h^* \)-polynomial and surrounding discussion is due to Benjamin Nill. Let \( Q \subset \mathbb{R}^n \) be a full-dimensional rational polytope and let \( k \) be the smallest positive integer such that \( kQ \) is a lattice polytope. Consider the function defined by \( f(Q; m) = \#(mQ \cap \mathbb{R}^n) \) for all \( m \in \mathbb{Z}_{\geq 0} \). In [10], Stanley proved that the associated generating series has the form:
\[
\sum_{m \geq 0} f(Q; m)t^m = \frac{h^*(Q; t)}{(1 - t^k)^{n+1}},
\]
where \( h^*(Q; t) \) is a polynomial of degree at most \( kn + 1 - 1 \) with non-negative integer coefficients, called the associated \( h^* \)-polynomial (see (1) for the case when \( k = 1 \)). If \( P \) is a lattice polytope containing the origin, then, with the notation above, one can verify from the definitions that
\[
\tilde{h}(P; t) = h^*(\frac{1}{r_P} P; t^{1/r_P}).
\]
In particular, it follows that the symmetry (7) is equivalent to the statement that if \( Q \) is a rational polytope containing the origin in its relative interior such that the associated dual polytope is a lattice polytope, then the coefficients of \( h^*(Q; t) \) are symmetric. The latter fact was proved independently by Fiset and Kasprzyk [7].

Then (11) immediately implies the following multiplicative formula.

**Lemma 4.** Let \( P, Q \) be full-dimensional lattice polytopes containing the origin with respect to lattices \( N_P, N_Q \) respectively. Then
\[
\tilde{h}(P \oplus Q; t) = \tilde{h}(P; t) \tilde{h}(Q; t).
\]

Combined with (11), we deduce the following formula for the Ehrhart polynomial of \( P \oplus Q \):
\[
h^*(P \oplus Q; t) = \Psi(\tilde{h}(P; t) \tilde{h}(Q; t)).
\]
Remark 5. Let \( \Theta : \bigcup_{r \in \mathbb{Z}_{>0}} \mathbb{R}[t^{1/r}] \to \mathbb{R}[t] \) denote the \( \mathbb{R} \)-linear map defined by \( \Theta(t^j) = t^{j-\lfloor j \rfloor} \) for all \( j \in \mathbb{Q}_{\geq 0} \). Then [14] Example 4.12] gives an explicit formula for \( \Theta(\tilde{h}(P; t)) \) that we will describe below.

Each facet \( F \) of \( P \) not containing the origin has the form

\[
F = P \cap \{ v \in (N_P)_\mathbb{R} \mid \langle u_F, v \rangle = -m_F \},
\]

where \( u_F \in M_P \) is a primitive integer vector, and \( m_F \in \mathbb{Z}_{>0} \) is the lattice distance of \( F \) from the origin. Then the vertices of \( P' \) are precisely \( \{ \frac{u_F}{m_F} \mid F \text{ facet of } P, 0 \notin F \} \), and hence \( r_P = \text{lcm}(m_F \mid F \text{ facet of } P, 0 \notin F) \). Then

\[
\Theta(\tilde{h}(P; t)) = \sum_{F \text{ facet of } P \atop 0 \notin F} \text{Vol}(F) \sum_{i=0}^{m_F-1} \frac{1}{t^{m_F}},
\]

where \( \text{Vol}(F) \) is defined in Remark 6 below.

Remark 6. For any lattice polytope \( F \), \( h^*(F; 1) \) is equal to the normalized volume \( \text{Vol}(F) \) of \( F \), i.e. after possibly replacing the underlying lattice with a smaller lattice, we may assume that \( F \subseteq N_\mathbb{R} \cong \mathbb{R}^{\dim F} \) for some lattice \( N \), and then \( \text{Vol}(F) \) is \( (\dim F)! \) times the Euclidean volume of \( F \). In the formula in Remark 5, to make the connection with [14] Example 4.12] explicit, observe that \( \text{Vol}(F') = m_F \text{Vol}(F) \), where \( F' \) is the convex hull of \( F \) and the origin.

Remark 7. Remark 5 shows that \( r_P \) is the minimal choice of denominator in the fractional exponents in \( \tilde{h}(P; t) \) in the sense that \( \tilde{h}(P; t) \in \mathbb{Z}[t^{1/r_P}] \) and if \( \tilde{h}(P; t) \in \mathbb{Z}[t^{1/r}] \), then \( r_P \) divides \( r \). For example, \( \tilde{h}(P; t) = h^*(P; t) \) if and only if \( r_P = 1 \).

Remark 8. If \( P \) and \( Q \) contain the origin, but are not full-dimensional, then one may apply the results above after replacing \( N_P \) and \( N_Q \) by their intersections with the linear spans of \( P \) and \( Q \) respectively. If \( P \) contains the origin but not \( Q \), then one may replace \( Q \) with \( Q' = \text{conv}(Q, 0_Q) \) since \( P \oplus Q = P \oplus Q' \).

If neither \( P \) nor \( Q \) contain the origin, but satisfy the property that the affine spans of \( P \) and \( Q \) are strict subsets of the linear spans of \( P \) and \( Q \) respectively, then \( P, Q \) and \( P \oplus Q \) are the unique facets not containing the origin of \( P' = \text{conv}(P, 0_P) \), \( Q' = \text{conv}(Q, 0_Q) \) and \( P' \oplus Q' \) respectively. In this case, by Remark 2 and Lemma 4, \( h^*(P \oplus Q; t) \) is the polynomial consisting of the terms of \( \tilde{h}(P' \oplus Q', t) \) with integer-valued exponents of \( t \).

We deduce a new proof to the following result of Beck, Jayawant and McAllister [3, Theorem 1.3], which itself generalizes a result of Braun [4].

Corollary 9. Let \( P, Q \) be full-dimensional lattice polytopes containing the origin with respect to lattices \( N_P, N_Q \) respectively. Then

\[
h^*(P \oplus Q; t) = h^*(P; t)h^*(Q; t) \iff r_P = 1 \text{ or } r_Q = 1.
\]
Proof. If we write \( \tilde{h}(P; t) = \sum_{j \in \mathbb{Q}} \tilde{h}_{P,j} t^j \), then by (3) and (8),
\[
\begin{align*}
\tilde{h}^*(P \oplus Q; t) &= \sum_{j,j' \in \mathbb{Q}} \tilde{h}_{P,j} \tilde{h}_{Q,j'} t^{[j]+[j']}, \\
\tilde{h}^*(P; t) h^*(Q; t) &= \sum_{j,j' \in \mathbb{Q}} \tilde{h}_{P,j} \tilde{h}_{Q,j'} t^{[j]+[j']},
\end{align*}
\]
If \( r_P = 1 \) or \( r_Q = 1 \), then we have equality. If \( r_P, r_Q > 1 \), then by Remark 5 there exists \( (j, j') \in \mathbb{Q}^2 \) such that \( \tilde{h}_{P,j} \tilde{h}_{Q,j'} > 0 \) and \( 0 < j - \lfloor j \rfloor, j' - \lfloor j' \rfloor \leq 1/2 \). Then \( \lfloor j \rfloor + \lfloor j' \rfloor = \lfloor j + j' \rfloor + 1 \), and the non-negativity of the coefficients of \( \tilde{h}(P; t) \) and \( \tilde{h}(Q; t) \) implies that \( \tilde{h}^*(P \oplus Q; t) \neq \tilde{h}^*(P; t) h^*(Q; t) \).
\( \square \)

Remark 10. A lattice polytope \( P \) satisfying \( r_P = 1 \) and containing the origin in its relative interior is called \textbf{reflexive}. These polytopes have received a lot of attention as, geometrically, they correspond to Fano toric varieties. In particular, they play a central role in Batyrev and Borisov’s construction of mirror pairs of Calabi-Yau varieties [4].

Remark 11. The weighted \( h^* \)-polynomial arises naturally in two distinct geometric situations: Firstly, in the computation of dimensions of the graded pieces of orbifold cohomology groups of toric stacks [11, Theorem 4.3] and, more generally, in the computation of motivic integrals on toric stacks [12, Theorem 6.5]. Secondly, in computations of the action of monodromy on the cohomology of the fiber of a degeneration of complex hypersurfaces (or the associated Milnor fiber) [14, Sections 5, 6, Corollary 5.12]. In particular, taking the free sum of polytopes above corresponds to taking products of the associated toric stacks, and the multiplicativity formula in Lemma 4 may be viewed as a K"unneth formula for the dimensions of the graded pieces of orbifold cohomology groups of toric stacks.

Example 12. In order to provide a wider class of examples of weighted \( h^* \)-polynomials, we consider a class of examples of lattice polytopes used by Payne in [9]. Consider positive integers \( \alpha_0 \geq \alpha_1 \geq \cdots \geq \alpha_d \) with no common factor and let \( N = \mathbb{Z}^{d+1}/(\sum_{i=0}^d \alpha_i e_i = 0) \), where \( e_0, \ldots, e_d \) denotes the standard basis of \( \mathbb{Z}^{d+1} \). Observe that \( N \) is a lattice of rank \( d \) and, if \( P(\alpha_0, \ldots, \alpha_d) \) denotes the convex hull of the images of \( e_0, \ldots, e_d \), then \( P(\alpha_0, \ldots, \alpha_d) \) is a lattice polytope containing the origin in its relative interior. The following formula follows from the proof of [13, Lemma 9.1]:
\[
\tilde{h}(P(\alpha_0, \ldots, \alpha_d); t) = \sum_{i=0}^d \sum_{j=0}^{\alpha_i-1} t^\sum_{0 \leq k \leq d, k \neq i} \left( \frac{j \alpha_k}{\alpha_i} - \frac{j \alpha_k}{\alpha_i} \right) + \sum_{k=i+1}^d \varphi\left( \frac{j \alpha_k}{\alpha_i} \right),
\]
where \( \varphi(x) = 1 \) if \( x \) is an integer and \( \varphi(x) = 0 \), otherwise.

We now consider a partial converse to (8). We will use the following lemma due to Terence Harris [8].

Lemma 13. Let \( f(t) \in \mathbb{R}[t^{1/r}] \) be a polynomial with non-negative coefficients and fractional exponents for some \( r \in \mathbb{Z}_{>0} \). Fix a positive real number \( x \). For any \( n \in \mathbb{Z}_{>0} \), let \( f_n^*(t) := \)
\( \Psi(f(t)^n) \in \mathbb{R}[t] \), where \( \Psi \) is defined in (3), i.e. \( f_n^*(t) \) is obtained from \( f(t)^n \) by rounding up exponents in \( t \). Then

\[
f_n^*(x) \leq f(x)^n \leq \frac{x}{t} - 1f_n^*(x) \text{ if } 0 < x \leq 1,\]

\[
f(x)^n \leq f_n^*(x) \leq \frac{x}{t} - 1f(x)^n \text{ if } x \geq 1.\]

In particular, given any polynomial \( f(t) \in \bigcup_{r \in \mathbb{Z}_{>0}} \mathbb{R}[t^{1/r}] \) with non-negative coefficients and fractional exponents,

\[
f(x) = \lim_{n \to \infty} f_n^*(x)^{1/n},\]

and \( f(t) \) determines and is determined by \( \{f_n^*(t) \mid n \in \mathbb{Z}_{>0}\} \).

**Proof.** First assume that \( x \geq 1 \). When \( n = 1 \), the inequalities \( \Psi(f(t))_{t=x} \leq f(x) \leq x^{1/2} - \Psi(f(t))_{t=x} \) follow from the fact that \( x^{1/2} \leq x^{1/2} \leq x^{1/2}x^{1/2} \) for any \( i \in \mathbb{Z}_{\geq 0} \), and the assumption that the coefficients of \( f(t) \) are non-negative. When \( n \geq 1 \), the inequalities follow by replacing \( f(t) \) with \( f(t)^n \). When \( 0 < x \leq 1, x^{1/2} \leq x^{1/2} \leq x^{1/2}x^{1/2} \) and the result follows similarly. The final statement follows immediately. \( \Box \)

For any positive integer \( n \), let \( P^\oplus \) denote the free sum of \( P \) with itself \( n \) times. By Lemma 1 and (3), one may apply the above lemma with \( f(t) = \tilde{h}(P; t), f(t)^n = \tilde{h}(P^\oplus; t), f_n^*(t) = h^*(P^\oplus; t) \) and \( r = r_P \), to obtain the corollary below.

**Corollary 14.** Let \( P \) be a full-dimensional lattice polytope containing the origin with respect to a lattice \( N_P \). Fix a positive real number \( x \). For any \( n \in \mathbb{Z}_{>0} \), and with \( r_P \) as defined in (3),

\[
h^*(P^\oplus; x) \leq \tilde{h}(P; x)^n \leq \frac{n}{r_P} - \frac{1}{r_P}h^*(P^\oplus; x) \text{ if } 0 < x \leq 1,\]

\[
\tilde{h}(P; x)^n \leq h^*(P^\oplus; x) \leq \frac{x^{1/2} - 1}{r_P} \tilde{h}(P; x)^n \text{ if } x \geq 1.\]

In particular,

\[
\tilde{h}(P; x) = \lim_{n \to \infty} h^*(P^\oplus; x)^{1/n},\]

and \( \tilde{h}(P; t) \) determines and is determined by \( \{h^*(P^\oplus; t) \mid n \in \mathbb{Z}_{>0}\} \).

Note that the final statement above states that the following two sets contain precisely the same information:

\[
\{\#(\lambda P \cap N_P) \mid \lambda \in \mathbb{Q}_{>0}\};
\]

\[
\{\#(mP^\oplus \cap (N_P \oplus \cdots \oplus N_P)) \mid m \in \mathbb{Z}_{>0}, n \in \mathbb{Z}_{>0}\}.
\]

**Remark 15.** From the proof of Corollary 14, \( \tilde{h}(P; t) \) determines and is determined by \( \{h^*(P^\oplus; t) \mid n \in S\} \) for any infinite subset \( S \subseteq \mathbb{Z}_{>0} \).
Finally, the above results together with the central limit theorem describe some of the asymptotic behavior of \( h^*(P^\oplus n; t) \) as \( n \to \infty \). More precisely, let \( X^*_P, \widetilde{X}_P \) be \( \mathbb{R} \)-valued random variables with probability distributions on \( \mathbb{R} \) defined by:

\[
\mathbb{P}(X^*_P = i) = \frac{h^*_{P,i}}{\text{Vol}(P)},
\]

\[
\mathbb{P}(\widetilde{X}_P = j) = \frac{\widetilde{h}_{P,j}}{\text{Vol}(P)},
\]

where \( h^*(P; t) = \sum_{i \in \mathbb{Z}} h^*_{P,i} t^i \) and \( \widetilde{h}(P; t) = \sum_{j \in \mathbb{Q}} \widetilde{h}_{P,j} t^j \), and \( h^*(P; 1) = \widetilde{h}(P; 1) = \text{Vol}(P) \) (see Remark 6). Equivalently, the moment generating functions of \( X^*_P \) and \( \widetilde{X}_P \) are given by:

\[
\mathbb{E}[e^{sX^*_P}] = \frac{1}{\text{Vol}(P)} h^*(P; e^s),
\]

\[
\mathbb{E}[e^{s\widetilde{X}_P}] = \frac{1}{\text{Vol}(P)} \widetilde{h}(P; e^s).
\]

Let \( \mu_P \) and \( \sigma_P \) denote the mean and standard deviation of \( \widetilde{X}_P \) respectively, and let \( \mathcal{N}(\mu, \sigma) \) denote the normal distribution with mean \( \mu \) and variance \( \sigma \).

**Example 16.** When the origin lies in the relative interior of \( P \), (7) implies that \( \mu_P = \frac{\dim P}{2} \).

**Corollary 17.** Let \( P \) be a full-dimensional lattice polytope containing the origin in a lattice \( N_P \). Then

\[
\frac{X^*_P \oplus n - n \mu_P}{\sqrt{n}} \xrightarrow{d} \mathcal{N}(0, \sigma_P),
\]

as \( n \to \infty \), where convergence means convergence in distribution (see Remark 18).

**Proof.** Fix \( s \in \mathbb{R} \). Then by Corollary 14 for any \( n \in \mathbb{Z}_{>0} \),

\[
e^{-s \mu_P \sqrt{n} h^*(P^\oplus n; e^{\sqrt{n}})} \leq e^{-s \mu_P \sqrt{n} h(P; e^{\sqrt{n}})} \leq e^{-s \mu_P \sqrt{n} h^*(P^\oplus n; e^{\sqrt{n}})} \text{ if } s \leq 0,
\]

\[
e^{-s \mu_P \sqrt{n} h(P; e^{\sqrt{n}})} \leq e^{-s \mu_P \sqrt{n} h^*(P^\oplus n; e^{\sqrt{n}})} \leq e^{-s \mu_P \sqrt{n} h(P; e^{\sqrt{n}})} \text{ if } s \geq 0.
\]

If \( \widetilde{X}_1, \ldots, \widetilde{X}_n \) are iid random variables with distribution \( \widetilde{X}_P \), and \( \widetilde{Z}_n := \frac{(\widetilde{X}_1 - \mu_P) + \cdots + (\widetilde{X}_n - \mu_P)}{\sqrt{n}} \), then either a direct computation or invoking the central limit theorem gives:

\[
\lim_{n \to \infty} \mathbb{E}[e^{s\widetilde{Z}_n}] = \lim_{n \to \infty} e^{-s \mu_P \sqrt{n} h(P; e^{\sqrt{n}})} = e^{\frac{(s \mu_P)^2}{2}},
\]

where \( e^{\frac{(s \mu_P)^2}{2}} \) is the moment generating function of \( \mathcal{N}(0, \sigma_P) \). If \( Z^*_n := \frac{X^*_P \oplus n - n \mu_P}{\sqrt{n}} \), then the above inequalities state that

\[
\mathbb{E}[e^{sZ^*_n}] \leq \mathbb{E}[e^{s\widetilde{Z}_n}] \leq e^{\frac{s(1 + r P)}{\sqrt{n}} \mu_P} \mathbb{E}[e^{sZ^*_n}] \text{ if } s \leq 0,
\]

\[
\mathbb{E}[e^{s\widetilde{Z}_n}] \leq \mathbb{E}[e^{sZ^*_n}] \leq e^{\frac{s(1 + r P)}{\sqrt{n}} \mu_P} \mathbb{E}[e^{s\widetilde{Z}_n}] \text{ if } s \geq 0.
\]
Hence $\lim_{n \to \infty} E[e^{sZ_n^*}] = \lim_{n \to \infty} E[e^{s\tilde{Z}_n}] = e^{\frac{(a\tilde{\sigma}P)^2}{2}}$ and the result follows since convergence of the moment generating functions of $Z_n^*$ to the moment generating function of $\mathcal{N}(0,\tilde{\sigma}P)$ implies convergence of the corresponding distributions [5, Theorem 3] (note that all moment generating functions above converge for all $s \in \mathbb{R}$).

**Remark 18.** The convergence in Corollary [17] is defined in terms of the corresponding cumulative distribution functions as follows: for all $x \in \mathbb{R}$, if we write $h^*(P_\oplus n; t) = \sum_{i \in \mathbb{Z}} h_{P_\oplus n_i}^* t$, 

$$F_n(x) = \mathbb{P}\left(\frac{X_{P_\oplus n}^* - n\tilde{\mu}P}{\sqrt{n}} \leq x\right) = \frac{1}{\text{Vol}(P)^n} \sum_{i \leq \sqrt{n}x+n\tilde{\mu}P} h_{P_\oplus n_i}^*,$$

and $\Phi_{\tilde{\sigma}P}(x) = \frac{1}{\sqrt{2\pi\tilde{\sigma}P}} \int_{-\infty}^{x} e^{-\left(\frac{ns^2}{2\tilde{\sigma}P}\right)} ds$, then $\lim_{n \to \infty} F_n(x) = \Phi_{\tilde{\sigma}P}(x)$.

**Example 19.** A lattice polytope $P$ containing the origin is a **standard simplex** if its non-zero vertices form a basis of $N_P$. In this case, $\tilde{h}(P; t) = h^*(P; t) = 1$, $\tilde{\mu}P = \tilde{\sigma}P = 0$ and $P_\oplus n$ is a standard simplex for all $n$.

**Example 20.** Fix a positive integer $n$ and consider the lattice $N = \mathbb{Z}[e^{2\pi i/n}]$. The $n$-th cyclotomic polytope $C_n$ is the convex hull of all $n$-th roots of unity in $N \cong \mathbb{Z}^\varphi(n)$, where $\varphi$ is the Euler totient function. In [2, Theorem 7, Lemma 8, Corollary 9], Beck and Hoosten prove that the lattice points of $C_n$ consist of the $n$-th roots of unity, which are vertices, together with the origin, which is the unique interior lattice point, and they identify $C_n$ with $C_{n\varphi(n)}^{\oplus\text{sqf}(n)}$, where sqf($n$) denotes the square-free part of $n$ i.e. the product of the prime divisors of $n$. Moreover, they prove that $C_n$ is reflexive if $n$ is divisible by at most two odd primes, and they show how to compute $h^*(C_n; t) = \tilde{h}(C_n; t)$ for $n \leq 104$. The smallest value of $n$ for which $h^*(C_n; t)$ is unknown is $n = 105 = 3 \cdot 5 \cdot 7$. We refer the reader to [2] for further results and details.

By [8], and using Beck and Hoosten’s result above, for any positive integer $n$, 

$$h^*(C_n; t) = h^*(C_{n\varphi(n)}^{\oplus\text{sqf}(n)}; t) = \Psi(\tilde{h}(C_{n\varphi(n)}; t)^{\text{sqf}(n)}).$$

It follows from this observation and Remark [15] that for any product of distinct primes $a$, the problem of computing $\{h^*(C_n; t) \mid \text{sqf}(n) = a\}$ is equivalent to the problem of computing $\tilde{h}(C_a; t)$. More precisely, consider a strictly increasing sequence of positive integers $\{n_k\}_{k \in \mathbb{Z}_{>0}}$ satisfying sqf($n_k$) = $a$ for all $k$. Then by Remark [15] $\{h^*(C_{n_k}; t) = h^*(C_{n_k}^{\oplus n_k}; t) \mid k \in \mathbb{Z}_{>0}\}$ determines and is determined by $\tilde{h}(C_a; t)$. Moreover, since $\tilde{\mu}C_a = \frac{\dim C_a}{2} = \frac{\varphi(a)}{2}$ by Example [16] setting $P = C_a$ and $n = \frac{n_k}{a}$ in Corollary [17] implies that 

$$\frac{X^*_{C_{n_k}} - \frac{n_k\varphi(a)}{2a}}{\sqrt{\frac{n_k}{a}}} \overset{d}{\to} \mathcal{N}(0, \tilde{\sigma}C_a),$$

as $k \to \infty$. We note that it is an open problem to compute $\tilde{\sigma}C_a$ for any product of distinct primes $a$. 

8
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