Asymptotic behavior of solutions of the Dirac system with an integrable potential

August 26, 2020

Łukasz Rzepnicki
Faculty of Mathematics and Computer Science
Nicolaus Copernicus University
ul. Chopina 12/18, 87-100 Toruń
Poland
keleb@mat.umk.pl

Abstract
We consider the Dirac system on the interval $[0,1]$ with a spectral parameter $\mu \in \mathbb{C}$ and a complex-valued potential with entries from $L^p[0,1]$, where $1 \leq p < 2$. We study the asymptotic behavior of its solutions in a stripe $|\text{Im}\mu| \leq d$ for $\mu \to \infty$. These results allows us to obtain sharp asymptotic formulas for eigenvalues and eigenfunctions of Sturm–Liouville operators associated with the aforementioned Dirac system.

keywords: Dirac system, spectral problem, integrable potential, Sturm–Liouville operator

MSC[2010] Primary 34L20, Secondary 34E05

Acknowledgement
The author was supported by NCN grant no. UMO-2017/27/B/ST1/00078.

1 Introduction
Let consider for $x \in [0,1]$, a Cauchy problem

$$D'(x) + J(x)D(x) = A_\mu D(x), \quad D(0) = I,$$  \hspace{1cm} (1.1)

where $A_\mu = i\mu J_0$, and

$$J_0 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad J(x) = \begin{bmatrix} 0 & \sigma_1(x) \\ \sigma_2(x) & 0 \end{bmatrix}, \quad I := \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$  \hspace{1cm} (1.2)

$\mu \in \mathbb{C}$ is a spectral parameter, and for $j = 1,2$ complex-valued functions $\sigma_j$ belong to $L^p[0,1]$, where $1 \leq p < 2$. We study the asymptotic behavior of its solutions $D(x) = D(x,\mu)$ with respect to $\mu$ from a horizontal stripe

$$P_d := \{\mu \in \mathbb{C}: |\text{Im}\mu| \leq d\}.$$
and $\mu \to \infty$.

The solution of (1.1), is a matrix $D$ with entries from the space of absolutely continuous on $[0, 1]$ functions (i.e. from the Sobolev space $W^1_1[0, 1]$) satisfying (1.1) for a.e. $x \in [0, 1]$. In our case, this conditions together with the equation yield that $D$ has entries from $W^1_p[0, 1]$.

This article is an addendum to the paper [7], where the problem (1.1) was analyzed for $\sigma_j \in L_2[0, 1]$, $j = 1, 2$. In that text one can find background for Dirac systems and their connection with Sturm–Liouville problems.

We relay here on the same method as in [7] to use all of its advantages and obtain sharp asymptotic formulas for $D$ and consequently for spectral problems associated with (1.1). In the case when $\sigma_j \in L_p[0, 1]$, $j = 1, 2$, $p > 2$ one can use the results from [7] due to the obvious embedding between $L_p[0, 1]$ spaces. Thus, in this text we restrict ourselves only to $1 \leq p < 2$.

We are interested in the following spectral problem:

$$Y'(x) + J(x)Y(x) = A_\mu Y(x), \quad x \in [0, 1],$$

(1.3)

where $Y = [y_1, y_2]^T$ and

$$y_1(0) = y_2(0), \quad y_1(1) = y_2(1).$$

(1.4)

Conditions (1.4) are an example of strongly regular boundary conditions. The Dirac-type systems or equation (1.3) with a general formulation of regular or strongly regular conditions have been studied recently in many papers and different methods.

In [15] A. M. Savchuk and A. A. Shkalikov derived for $p \geq 1$ basic asymptotic formulas for eigenvalues and for fundamental solutions of the Dirac-type system only with the leading term and the reminders expressed by $\gamma_0$ and $\gamma$ given by (2.22) and (2.23). They obtained their results applying Prüfer’s substitution.

Their result is equivalent to first thesis (2.31) of corollary 2.3. Note that next statement (2.32) is a significant extension of the previous result. Its version for $p = 1$ may be found in remark 2.5. The most general result is the content of lemma 2.2.

Using our method it is also possible to obtain very detailed formulas for eigenvalues and eigenfunctions. In case of the spectral problem associated with (1.1) the eigenvalues admit the representation (3.15)-(3.16) with remainders satisfying (3.17) and (3.18) for $p = 1$ and $1 < p < 2$ respectively. In literature (for instance in [15]) for $1 < p < 2$ one may found results which state that eigenvalues are of the form $\pi n + r_n$, where $(r_n) \in l_q$, and $q$ is conjugated to $p$. Here it is worth to underline that beside the leading term in our asymptotic formulas there occur Fourier coefficients of known functions and the reminder, which belongs to $l_q/2$. Additionally, for $p = 1$ we extend known formulas with $|r_n| < c \Gamma(\pi n)$ (where $\Gamma$ is defined in (2.25)) into more detailed one with the remainder satisfying $|r_n| < c \Gamma^2(\pi n)$.

In the same spirit theorem 3.3 and corollary 3.4 related to eigenfunctions generalize significantly those from literature.

Our method is applicable not only to the spectral problem (1.3)-(1.4) but it works as well for different cases of strongly regular boundary conditions. What is more it may be used to deal with the class of regular boundary conditions (in the sense of Birkhoff).
The articles of A. M. Savchuk and I. V. Sadovnichaya: [11], [12], [13] and [14] may be regarded as a continuation of method from [15] and its application for \( p = 1 \) to problems from the fields of asymptotics formulas and basis properties. Almost all aforementioned works prove or use the same type of results as mentioned before since they deal with the Riesz basis property and very detailed formulas are not needed.

The same aims had M. M. Malamud, A. V. Agibalova and L. L. Oridoroga in [1] for \( p = 2 \) and latter two in [8] for \( p = 1 \). Here the authors used the method of transformation operators.

Whereas in order to study inverse spectral problems S. Albeverio, R. Hryniv and Y. Mykytyuk in [2] investigated a direct spectral problem for the Dirac system in the form

\[
BZ'(x) + Q(x)Z(x) = \mu Z(x), \quad x \in [0, 1],
\]

where

\[
B = \begin{bmatrix}
0 & 1 \\
-1 & 0
\end{bmatrix}, \quad Q(x) = \begin{bmatrix}
q_1(x) & q_2(x) \\
q_2(x) & -q_1(x)
\end{bmatrix}, \quad q_j \in L_p[0, 1], \; j = 1, 2,
\]

with \( p \geq 1 \). They proved also short formulas for fundamental system of solutions, where reminders were expressed in terms of Fourier coefficients for unknown functions from \( L_p \). Furthermore, for the operators associated with the system (1.5) with two kinds of conditions

\[
z_j(1) = z_2(0) = 0, \quad j = 1, 2,
\]

they presented basic formulas for eigenvalues with the same type of reminders. That class of results can be directly derived from our approach with the help of transformation \( Z = UY \), where

\[
U = \begin{bmatrix}
1 & -i \\
-i & 1
\end{bmatrix}.
\]

It leads to the system (1.4) with \( \sigma_1 = q_1 + iq_2 \) and \( \sigma_2 = q_1 - iq_2 \) with appropriate conditions. The relation between different formulations of Dirac systems is explained deeper in [7].

More results concerning different type of problems for the Dirac system with may be found in the series of paper of P. Djakov and B. Mityagin: [4], [5] and [6] or D. V. Puyda [10].

We start with the section concerning asymptotic behavior for solutions of Dirac system. Next, in section 3 we apply these results to the aforementioned spectral problem. For the clarity of exposition some technical results are placed at the end in appendix.

## 2 Dirac system and its solutions

In this section we study the matrix Cauchy problem (1.1) and the behavior of its solution in a special integral form. The idea of this approach was taken from [9] Ch. 1, §24 and developed in [7]. We follow it here directly for similar operators but in different function spaces.
First, we introduce a necessary notation. We use throughout the text a standard symbol \( L^p[0, 1], p \geq 1 \) to denote the space of measurable complex functions integrable with \( p \)-th power with the classical norm
\[
\|f\|_{L^p} = \left( \int_0^1 |f(x)|^p \, dx \right)^{1/p}.
\]
We write \( l^p, p \geq 1 \) for the space of complex sequences summable with \( p \)-th power and endowed with the norm
\[
\|(x_n)\|_p = \left( \sum_{n=1}^\infty |x_n|^p \right)^{1/p}.
\]
\( W^1_p[0, 1] \) is a standard Sobolev space with the derivative in \( L^p[0, 1] \).
If \( X \) is a Banach space, then \( M(X) \) stands for the Banach space of \( 2 \times 2 \) matrices with entries from \( X \) and the norm
\[
\|Q\|_{M(X)} := \sum_{k,j=1}^2 \|Q_{jk}\|_B, \quad Q = [Q_{jk}]_{j,k=1}^2.
\]
We assume throughout the text that \( 1 \leq p < 2 \). Moreover if \( 1 < p < 2 \), then let \( q \) and \( p \) be conjugate exponents and \( r \) be the number from Young’s convolution inequality i.e.
\[
\frac{1}{p} + \frac{1}{q} = 1 \quad \text{and} \quad r = \frac{2}{2 - p}.
\]
If \( p = 1 \), then \( q = \infty \) and \( r = 1 \). Let
\[
\Delta := \{(x, t) \in \mathbb{R}^2 : 0 \leq t \leq x \leq 1\}
\]
and
\[
B := \{f : [0, 1] \times [0, 1] \to \mathbb{C} \text{ a.e.} : \forall x \in [0, 1] f(x, \cdot) \in C([0, 1], L^r), \supp f \subset \Delta \}.
\]
We equip \( B \) with the norm
\[
\|f\|_B := \sup_{x \in [0, 1]} \|f(x, \cdot)\|_{L^r, [0, x]},
\]
so that \( B \) is a Banach space. In particular, directly from the definition if \( f \in B \), then \( f(x, t) = 0 \) for \( 0 \leq x < t \leq 1 \). This comment allows us to underline the property which will be used in the text i.e. for \( f \in B \) there holds
\[
\int_0^x f(x, t) \, dt = \int_0^1 f(x, t) \, dt \in C[0, 1].
\]
We will use the series of constants connected with functions \( \sigma_j, j = 1, 2 \) in our estimations:
\[
a_0 := \max\{\|\sigma_1\|_{L^1}, \|\sigma_2\|_{L^1}\}, \quad a := \|\sigma_1\|_{L^1} \cdot \|\sigma_2\|_{L^1}, \quad a_1 := \|\sigma_1\|_{L^1} + \|\sigma_2\|_{L^1},
\]
(2.4)
and
\[ \tilde{a}_0 := \max\{\|\sigma_1\|_{L_p}, \|\sigma_2\|_{L_p}\}, \quad \tilde{a} := \|\sigma_1\|_{L_p} \cdot \|\sigma_2\|_{L_p}, \quad a_2 := \|\sigma_1\|_{L_p} + \|\sigma_2\|_{L_p}. \] (2.5)

Moreover, let
\[ \sigma_0(x) := |\sigma_1(x)| + |\sigma_2(x)| \in L_p[0, 1]. \] (2.6)

Now we are ready to establish a first crucial property of the solutions to (1.1). The proof of the following lemma relays on technical results related to certain integral operators, which are placed in appendix.

**Lemma 2.1.** Let \( \sigma \in L_p[0,1], 1 \leq p < 2. \)

a) The unique solution \( D = D(x, \mu) \) of Cauchy problem (1.1) can be represented as
\[ D(x, \mu) = e^{xA_\mu} + \int_0^x e^{(x-2t)A_\mu}[J(t) + Q(x, t)]dt, \] (2.7)
where \( Q \in M(B) \) is the unique solution of the integral equation
\[ Q(x, t) = \tilde{J}(x, t) + \int_0^{x-t} J(t + \xi)Q(t + \xi, \xi)d\xi, \] (2.8)
with \( \tilde{J} \in M(B) \) is given by
\[ \tilde{J}(x, t) := \int_0^{x-t} J(t + \xi)J(\xi)d\xi = \int_0^x J(s)J(s - t)ds, \quad (x, t) \in \Delta. \] (2.9)

b) The following estimates hold:
\[ \|Q\|_{M(B)} \leq c, \quad \|Q\|_{M(C[0,1])} \leq c, \quad \mu \in P_d \] (2.10)
with certain constants \( c = c(d, \sigma_1, \sigma_2). \)

**Proof.** Note that the uniqueness of solutions comes from general results on Sturm–Liouville equations (for instance [16, Thm. 1.2.1]). We look for solutions of (1.1) in a special form
\[ D(x, \mu) = e^{xA_\mu}U(x, \mu), \quad U(0, \mu) = I. \] (2.11)

The identity
\[ J(x)e^{xA_\mu} = e^{-2xA_\mu}J(x), \quad a. e. \quad x \in [0,1] \] (2.12)
yield that \( U \) satisfies the Cauchy problem
\[ U'(x, \mu) + e^{-2xA_\mu}J(x)U(x, \mu) = 0, \quad x \in [0,1], \quad U(0, \mu) = I, \]
and this is equivalent to the integral equation
\[ U(x, \mu) = I - \int_0^x e^{-2tA_\mu}J(t)U(t, \mu)dt, \quad x \in [0,1]. \] (2.13)

We will seek for solutions of (2.13) in the form
\[ U(x, \mu) = I + \int_0^x e^{-2tA_\mu}Q_0(x, t)dt, \] (2.14)
where $Q_0 \in M(B)$ does not depend on $\mu$. Inserting (2.14) into (2.13), we obtain
\[
\int_0^x e^{-2tA_\mu} Q_0(x, t) \, dt = - \int_0^x e^{-2tA_\mu} J(t) \, dt
\]
\[
- \int_0^x e^{-2tA_\mu} J(t) \int_0^t e^{-2sA_\mu} Q_0(t, s) \, ds \, dt.
\]
Due to the fact
\[
J_0^2 = I, \quad J_0 J(x) + J(x) J_0 = 0, \quad \text{a.e. } x \in [0, 1],
\]
we get
\[
\int_0^x e^{-2tA_\mu} J(t) \int_0^t e^{-2sA_\mu} Q_0(t, s) \, ds \, dt = \int_0^x e^{-2tA_\mu} \int_0^t e^{-2sA_\mu} J(t) Q_0(t, s) \, ds \, dt
\]
\[
= \int_0^x e^{-2tA_\mu} \int_0^{x-t} J(t + \xi) Q_0(t + \xi, \xi) \, d\xi \, dt,
\]
thus
\[
\int_0^x e^{-2tA_\mu} Q_0(x, t) \, dt = - \int_0^x e^{-2tA_\mu} \left( J(t) + \int_0^{x-t} J(t + \xi) Q_0(t + \xi, \xi) \, d\xi \right) \, dt
\]
for all $x \in [0, 1]$. We conclude that $U$ is a solution of (2.13) if and only if $Q_0 \in M(B)$ is a solution of
\[
Q_0(x, t) = - J(t) - \int_0^{x-t} J(t + \xi) Q_0(t + \xi, \xi) \, d\xi.
\]
Next, setting
\[
Q_0(x, t) = - J(t) + Q(x, t), \quad (x, t) \in \Delta,
\]
and using (2.18), we infer that $Q$ satisfies (2.16). For $Q$ the equation (2.16) can be rewritten in an operator form
\[
Q = J + \tilde{T} Q, \quad \tilde{T} = \begin{bmatrix} 0 & T_{\sigma_1} \\ T_{\sigma_2} & 0 \end{bmatrix},
\]
for the operators $T_{\sigma_1}$ and $T_{\sigma_2}$, defined on $B$ by
\[
(T_{\sigma} f)(x, t) = \int_0^{x-t} \sigma(t + \xi) f(t + \xi, \xi) \, d\xi = \int_t^x \sigma(s) f(s, s - t) \, ds,
\]
where $\sigma \in L_p[0, 1]$. Observe that
\[
\tilde{J}(x, t) = \begin{pmatrix} \tilde{\sigma}_1(x, t) & 0 \\ 0 & \tilde{\sigma}_2(x, t) \end{pmatrix},
\]
where
\[
\tilde{\sigma}_1(x, t) := \int_0^{x-t} \sigma_1(t + \xi) \sigma_2(\xi) \, d\xi, \quad \tilde{\sigma}_2(x, t) := \int_0^{x-t} \sigma_2(t + \xi) \sigma_1(\xi) \, d\xi.
\]
According to lemma 4.1, $\tilde{J} \in M(B)$. What is more, the operators $T_{\sigma}$ are linear and bounded on $B$ due to lemma 4.3. In particular, we have
\[
\|\tilde{T} F\|_{M(B)} \leq a_0 \|F\|_{M(B)}, \quad F \in M(B).
\]
Next observe that
\[ \tilde{T}^{2n} = \begin{bmatrix} T_{n}{12} & 0 \\ 0 & T_{n}{21} \end{bmatrix}, \quad n \in \mathbb{N}, \]
for bounded linear operators \( T_{12} \) and \( T_{21} \) on \( B \) given by
\[ T_{12} := T_\sigma T_{\sigma_2}, \quad T_{21} := T_{\sigma_2} T_\sigma. \]
Therefore by (4.4), we derive
\[ \|\tilde{T}^{2n}F\|_{M(B)} \leq \frac{a^n}{n!} \|F\|_{M(B)}, \quad F \in M(B). \]
We thus see that (2.8) has a unique solution \( Q \in M(B) \) of the form
\[ Q = \sum_{n=0}^{\infty} \tilde{T}^n \tilde{J} = \sum_{n=0}^{\infty} \tilde{T}^{2n}(I + \tilde{T}) \tilde{J}, \quad (2.19) \]
and moreover
\[ \|Q\|_{M(B)} \leq (1 + a_0)e^a\|\tilde{J}\|_{M(B)}. \quad (2.20) \]
Then (2.19) and (4.1) imply (2.10).

Note that from (2.7) and (2.3) we have \( D \in C[0,1] \). Adding together of (2.7) and (2.10), we obtain
\[ \|D\|_{M(C[0,1])} \leq e^d \left( 1 + a_1 + \|Q(x,t)\|_{M(B)} \right), \quad \mu \in P_d. \quad (2.21) \]

We now proceed to derivation of asymptotic formulas for \( D \) with the use of the previous lemma. In what follows we will use different types of estimates for reminders. For fixed \( \sigma_j \in L_p, \ p \geq 1, \ j = 1, 2, \) and \( \mu \in \mathbb{C} \) define
\[ \gamma(\mu) := \sum_{j=1}^{2} \left( \left\| \int_{0}^{x} e^{-2i\mu t} \sigma_j(t)dt \right\|_{L_q} + \left\| \int_{0}^{x} e^{2i\mu t} \sigma_j(t)dt \right\|_{L_q} \right), \quad (2.22) \]
where \( 1/q + 1/p = 1 \). We will need also
\[ \gamma_0(x, \mu) := \sum_{j=1}^{2} \left( \left| \int_{0}^{x} e^{-2i\mu t} \sigma_j(t)dt \right| + \left| \int_{0}^{x} e^{2i\mu t} \sigma_j(t)dt \right| \right), \quad x \in [0,1] \quad (2.23) \]
and
\[ \gamma_1(\mu) := \int_{0}^{1} \sigma_0(s) \gamma_0^2(s, \mu) ds, \quad \gamma_2(\mu) := l_2^2 \gamma^2(\mu) + l_1 \gamma(\mu). \quad (2.24) \]
\[ \Gamma(\mu) := \sum_{j=1}^{2} \left( \sup_{x \in [0,1]} \left| \int_{0}^{x} e^{-2i\mu t} \sigma_j(t)dt \right| + \sup_{x \in [0,1]} \left| \int_{0}^{x} e^{2i\mu t} \sigma_j(t)dt \right| \right). \quad (2.25) \]
Note that \( \Gamma \) is nothing else than \( \gamma \) for \( p = 1 \) and \( q = \infty \).
Observe that the explicit form of $\gamma_0(x, \mu)$ and $\gamma_0(\mu)$ connected with the operator $T$ for all $x \in [0, 1]$, (2.26) and
\[
\gamma_2(\mu) \leq 4e^{4d}a_1^3(a_2 + a_1^2), \quad \gamma_2(\mu) \leq 2a_1e^{2d}(a_2 + 2a_1e^{2d}\|\sigma_0\|_{L_p})\gamma(\mu), \quad (2.27)
\]
In the following lemma we will need
\[
N(x, t) := (\tilde{J} + \tilde{T}\tilde{J})(x, t) \in B. \quad (2.28)
\]
Observe that the explicit form of $N$ is
\[
N(x, t) = \begin{pmatrix}
\tilde{\sigma}_1(x, t) & -(T_{\sigma_1}\tilde{\sigma}_2)(x, t) \\
-(T_{\sigma_2}\tilde{\sigma}_1)(x, t) & \tilde{\sigma}_2(x, t)
\end{pmatrix}.
\]
The very basic but crucial result use mainly the description of some integrals connected with the operator $\tilde{T}$ and its powers stated in lemma 4.6.

**Lemma 2.2.** Let $\sigma_j \in L_p, 1 \leq p < 2$ for $j = 1, 2$. If $D(x, \mu)$ is a solution of (1.1) then
\[
D(x, \mu) = e^{xA_\mu} + D^{(0)}(x, \mu) + D^{(1)}(x, \mu),
\]
where
\[
D^{(0)}(x, \mu) = \int_0^x e^{(x-2t)A_\mu}J(t)dt + \int_0^x e^{(x-2t)A_\mu}N(x, t)dt,
\]
and for all $\mu \in P_d$ and $x \in [0, 1]$,
\[
\|D^{(1)}(x, \mu)\|_{M(C[0,1])} \leq c\gamma_2(\mu),
\]
where $c = c(d, \sigma_1, \sigma_2)$.

**Proof.** Going back to the formulas (2.17) and (2.19) for $D = D(x, \mu)$, $x \in [0, 1]$, $\mu \in P_d$, note that
\[
D(x, \mu) = e^{xA_\mu} + \int_0^x e^{(x-2t)A_\mu}[J(t) + Q(x, t)]dt
\]
\[
= e^{xA_\mu} + \int_0^x e^{(x-2t)A_\mu}J(t)dt + \int_0^x e^{(x-2t)A_\mu}\tilde{J}(x, t)dt
\]
\[
+ \int_0^x e^{(x-2t)A_\mu}(\tilde{T}\tilde{J})(x, t)dt + D^{(1)}(x, \mu)
\]
where
\[
D^{(1)}(x, \mu) = \int_0^x e^{(x-2t)A_\mu}\sum_{n=2}^{\infty}(\tilde{T}^n\tilde{J})(x, t)dt.
\]
Using (2.30) and the inequality (1.12) proved in appendix, we infer that
\[
\|D^{(1)}(x, \mu)\|_{M(C[0,1])} \leq \sum_{n=2}^{\infty}\left\|\int_0^x e^{(x-2t)A_\mu}(\tilde{T}^n\tilde{J})(x, t)dt\right\|_{M(C[0,1])}
\]
\[
\leq 2e^{d}\gamma_2(\mu) \sum_{n=2}^{\infty} \frac{e^{2nd}a_1^{n-2}}{(n-2)!} = 2e^{5d}\exp(e^{2d}a_1)\gamma_2(\mu),
\]
for all $x \in [0, 1]$ and $\mu \in P_d$. \qed
The above lemma leads to sharp asymptotic formulas for \( D \), which are the main result of this section.

**Corollary 2.3.** For every \( d > 0 \) there exist \( c_j = c_j(d, \sigma, \sigma_2) \), \( j = 0, 1 \), such that for all \( x \in [0, 1] \) and \( \mu \in P_d \),

\[
D(x, \mu) = e^{xA_\mu} + R(x, \mu), \tag{2.31}
\]

where

\[
\|R(x, \mu)\|_{M(C([0,1]))} \leq c_0, \quad \|R(x, \mu)\|_{M(C)} \leq c_1(\gamma(\mu) + \gamma_0(x, \mu)).
\]

Moreover,

\[
D(x, \mu) = e^{xA_\mu} + D_0(x, \mu) + R_0(x, \mu), \tag{2.32}
\]

where

\[
D_0(x, \mu) := \int_0^x e^{(x-2t)A_\mu}(J(t) + \tilde{J}(t)) \, dt
\]

and

\[
\|R_0(x, \mu)\|_{M(C)} \leq c_2(\gamma(\mu)\gamma_0(x, \mu) + \gamma_2(\mu)), \quad x \in [0, 1].
\]

**Proof.** Let us start with several simple observations. First of all, remark that

\[
\|(x-2t)A_\mu J(t) \|_{M(C)} + \|\tilde{J}\|_{M(C)} \]

\[
\leq e^d(a_1 + (1 + a_1))(\gamma(\mu)\gamma_0(x, \mu) + \gamma_2(\mu)).
\]

Note also that from

\[
\left\| \int_0^x e^{(x-2t)A_\mu} J(t) \, dt \right\|_{M(C)} \leq e^d\gamma_0(x, \mu), \quad x \in [0, 1],
\]

(1.10) and (1.11) it follows that

\[
\left\| D_0(0, x, \mu) \right\|_{M(C)} \leq e^d\gamma_0(x, \mu) + 2e^d(1 + a_0)\gamma_0(\mu), \quad x \in [0, 1].
\]

Furthermore, by (1.11),

\[
\left\| \int_0^x e^{\mu(x-2t)}TJ_0(x) \, dt \right\|_{M(C)} \leq e^d \left\| \int_0^x e^{-2t}\mu T\tilde{J}(x) \, dt \right\|_{M(C)} \]

\[
\leq e^{3d}((a_2 + 1)(\gamma(\mu)\gamma_0(x, \mu) + \gamma_1(\mu))),
\]

where \( x \in [0, 1] \) and \( \mu \in P_d \). Combining all these inequalities with Lemma 2.2 and the estimates from (2.27), we obtain the required representations for \( D \).

**Remark 2.4.** Note that the explicit formula for \( D_0 \) is the following

\[
D_0(x, \mu) = \begin{pmatrix} r_1(x, \mu) & q_1(x, \mu) \\ q_2(x, \mu) & r_2(x, \mu) \end{pmatrix},
\]

\[
q_1(x, \mu) := \int_0^x e^{i \mu(x-2t)} \sigma_1(t) \, dt, \quad q_2(x, \mu) := \int_0^x e^{-i \mu(x-2t)} \sigma_2(t) \, dt
\]

\[
r_1(x, \mu) := \int_0^x e^{i \mu(x-2t)} \tilde{\sigma}_1(x, t) \, dt, \quad r_2(x, \mu) := \int_0^x e^{-i \mu(x-2t)} \tilde{\sigma}_2(x, t) \, dt
\]

and \( \tilde{\sigma}_j \) are given by (2.18).
Remark 2.5. If $p = 1$, then the remainder $R_0$ from (2.32) satisfies
\[ \|R_0(x, \mu)\|_{M(\mathbb{C})} \leq c_2 \Gamma^2(\mu), \]
where $\Gamma$ is given by (2.25).

3 Spectral problem

We consider spectral problem
\[ Y'(x) + J(x)Y(x) = A_{\mu}Y(x), \quad x \in [0, 1], \]
(3.1)
associated with the matrix problem (1.1) where $Y = [y_1, y_2]^T$ and
\[ y_1(0) = y_2(0), \quad y_1(1) = y_2(1). \]
(3.2)

Let $c = c(x, \mu) = [c_1, c_2]^T$ and $s = s(x, \mu) = [s_1, s_2]^T$ be the solutions of (3.1) satisfying $c_1(0) = 1$, $c_2(0) = 0$ and $s_1(0) = 0$, $s_2(0) = 1$. Then due to conditions (3.2) we find that the eigenvalues are the zeros of
\[ \Phi(\lambda) = c_1(1, \lambda) + s_1(1, \lambda) - c_2(1, \lambda) - s_2(1, \lambda). \]
(3.3)

The eigenfunctions will be of the form:
\[ Y = [y_1, y_2]^T = [c_1(\cdot, \mu_n) + s_1(\cdot, \mu_n), c_2(\cdot, \mu_n) + s_2(\cdot, \mu_n)]^T. \]
(3.4)

The analysis of zeros of (3.3) will now lead us to characterization of eigenvalues.

The standard approach is to derive first basic formula for eigenvalues and then using sharp asymptotic results derive more accurate form. We thus need results related to functions $s$ and $c$ from (2.7). We derive that
\begin{align*}
\Phi(\mu) &= 2i \sin \mu + \int_0^1 e^{(1-2i)t}\mu \left(Q_{11}(1, t) + Q_{12}(1, t) + \sigma_1(t)\right)dt \\
&\quad - \int_0^1 e^{-(1-2i)t}\mu \left(Q_{21}(1, t) + Q_{22}(1, t) + \sigma_2(t)\right)dt.
\end{align*}
(3.5)

Changing variables in integrals we may write
\[ \Phi(\mu) = 2i \sin(\mu) + V(\mu), \]
(3.6)

where
\[ V(\mu) = \int_{-1}^1 e^{i\mu s} f(s)ds - \int_{-1}^1 e^{-i\mu s} g(s)ds \]
(3.7)
and $f, g$ are certain function from $L_2[-1, 1]$.

Note that the identities (3.5) and (3.7) are true not only for $\mu \in P$ but also for all $\mu \in \mathbb{C}$. It is a standard procedure (see for instance [3]) to derive using Rouche Theorem that zeros of $\Phi$ are in the form $\mu_n = \pi n + \bar{\nu}_n$, where $(\bar{\nu}_n)$ is bounded. This conclusion yield that eigenvalues lie in a certain horizontal stripe of the complex plane. We may continue and investigate more precise the behavior of $(\bar{\nu}_n)$.
The formula for $\Phi$ gives us
\[
\sin(\tilde{\mu}_n) = \frac{(-1)^{n+1}}{2i}R(\pi n + \tilde{\mu}_n). \tag{3.8}
\]
This expression converges to zero since the convergence of the integral in follows from Lebesgue–Riemann Lemma and the fact that $\tilde{\mu}_n$ are bounded. Thus $\tilde{\mu}_n \to 0$ when $n \to \infty$. Here ends the reasoning and first claim for $p = 1$.

For $1 < p < 2$ we may continue in order to obtain more information. Using
\[
sin x = x + O(x^3), \quad x \to 0,
\]
and the fact that $\tilde{\mu}_n \to 0$ we obtain
\[
\tilde{\mu}_n = \frac{(-1)^{n+1}}{2i} \int_{-1}^{1} e^{i\tilde{\mu}_n s} e^{i\pi n s} f(s) ds \tag{3.9}
\]
Next, the expansion of the exponential function
\[
e^{\mu t} = 1 + \mu t + O(|\mu|^2), \quad \mu \to 0, \quad |t| \leq 1
\]
leads to a conclusions for one of the integrals
\[
\left| \int_{-1}^{1} e^{i\tilde{\mu}_n s} e^{i\pi n s} f(s) ds \right| = \int_{-1}^{1} e^{i\pi n s} f(s) ds
\]
\[
+ i\tilde{\mu}_n \int_{-1}^{1} e^{i\pi n s} s f(s) ds + O(|\tilde{\mu}_n|^2).
\]
Note that second integral is a product of $\tilde{\mu}_n$ and a Fourier coefficient of the function from $L_p$, hence it would give a sequence from $l_q$, which converges to zero. Consequently, we go back to (3.9) and conclude that $\tilde{\mu}_n$ is a sum of Fourier coefficients for functions from $L_p$, hence $(\tilde{\mu}_n) \in l_q$. Summarizing, we showed that the eigenvalues $\mu_n$ of our spectral problem satisfy
\[
\mu_n = \pi n + \tilde{\mu}_n, \quad (\tilde{\mu}_n) \in l_q. \tag{3.10}
\]
This representation for $1 < p < 2$ and the fact that for $p = 1$ the remainder goes to zero allows us to find in both cases more accurate description of eigenvalues. Recall we showed eigenvalues lie in $P_d$ for a certain $d > 0$, thus we can use asymptotic formulas true in a stripe. The main tool will be the formulas for $c$ and $s$ and consequently for $\Phi$ from corollary 2.3.

We infer that
\[
\Phi(\mu) = 2i \sin \mu + \int_{0}^{1} e^{(1-2t)i\mu} \sigma_1(t) dt + \int_{0}^{1} e^{(1-2t)i\mu} \tilde{\sigma}_1(1, t) dt
\]
\[
- \int_{0}^{1} e^{-(1-2t)i\mu} \sigma_2(t) dt - \int_{0}^{1} e^{-(1-2t)i\mu} \tilde{\sigma}_2(1, t) dt + r(\mu), \tag{3.11}
\]
where
\[
|r(\mu)| \leq c(\gamma(\mu)\gamma_0(1, \mu) + \gamma_2(\mu)) \leq c(\gamma^2(\mu) + \gamma_0^2(1, \mu) + \gamma_1(\mu)).
\]
The representation (3.10), lemma 4.2 and discussion similar to that about eigenvalues yield

\[
2i(-1)^{n+1}\hat{\mu}_n = -\int_0^1 e^{-2\pi i nt}\sigma_1(t)dt \\
+ 2(-1)^{n+1}\int_0^1 \int_0^t \sigma_1(t)\sigma_2(\xi)e^{-2\pi i n\xi}d\xi dt \\
+ \int_0^1 e^{2\pi i n\xi}\sigma_2(t)dt + r(\mu_n),
\]

(3.12)

For \( p = 1 \) we have here \( |r(\mu_n)| \leq c1^2(\pi n) \).

Our last aim is to prove that for \( 1 < p < 2 \) there holds \( (r(\mu_n)) \in l_q/2 \).

In what follows we will use a basic formula for eigenvalues (3.10), a simple inequality

\[
|e^{iz} - 1| \leq |z|, z \in \mathbb{P}
\]

and the Hausdorff–Young inequality. We infer for \( \sigma \in L^p[0,1] \) that

\[
\sum_{n=1}^{\infty} \int_0^x e^{\pm 2\mu_n t}\sigma(t)dt \leq c_q \sum_{n=1}^{\infty} \int_0^x e^{\pm 2\pi i nt}\sigma(t)dt \leq c_q \sum_{n=1}^{\infty} \left( \int_0^x |e^{2i\mu_n t} - 1||\sigma(t)||dt \right)^q \\
\leq c_q ||\sigma||_{L^p[0,1]}^q + c||\sigma||_{L^1[0,1]}^q \sum_{n=1}^{\infty} |\hat{\mu}_n|^q \leq m < \infty,
\]

(3.13)

for any \( x \in [0,1] \). It follows from (3.13) that

\[
\sup_{x \in [0,1]} \sum_{n=1}^{\infty} \gamma_q^p(x, \mu_n) < \infty,
\]

(3.14)

Note that by (3.14)

\[
\sum_{n=1}^{\infty} \gamma_q^p(\mu_n) \leq c \int_0^1 \sum_{n=1}^{\infty} \gamma_q^p(s, \mu_n)ds < \infty.
\]

and

\[
||\gamma_1(\mu_n)||_{l_q/2} \leq \int_0^1 ||\gamma_0^2(s, \mu_n)||_{l_q/2} ds \\
= \int_0^1 ||\sigma_0(s)|| \left( \sum_{n=1}^{\infty} \gamma_q^p(s, \mu_n) \right)^{2/q} ds \leq c||\sigma_0||_{L^1}.
\]

Finally, we obtain

\[
\sum_{n=1}^{\infty} |r_n|^{q/2} < \infty.
\]

Summarizing the discussion above we proved the following fact.
Theorem 3.1. The eigenvalues of the spectral problem (3.1)-(3.2) lie in a certain stripe $P_d$ and admit the representation

$$\mu_n = \pi n + \mu_{0,n} + \rho_n, \quad n = 1, 2, \ldots$$

with

$$\mu_{0,n} = \frac{(-1)^n}{2i} \int_0^1 e^{-2\pi i nt} \sigma_1(t) dt + \frac{(-1)^{n+1}}{2i} \int_0^1 e^{2\pi i nt} \sigma_2(t) dt$$

and for $p = 1$ there holds

$$|\rho_n| < c \Gamma^2(\pi n),$$

where $\Gamma$ is defined in (2.25), whereas for $1 < p < 2$ it is true that

$$\sum_{n=1}^{\infty} |\rho_n|^{q/2} < \infty.$$  

Remark 3.2. Recall that according to lemma 4.1 for every $x \in [0,1]$ functions $\tilde{\sigma}_j(x, \cdot)$ are from $L^r$. If $1 < p \leq \frac{4}{3}$, then $1 < r \leq 2$ and Fourier coefficients of $\tilde{\sigma}_j(x, \cdot)$ are from $l^{q/2}$. Then the representation (3.15) with

$$\sum_{n=1}^{\infty} |\rho_n|^{q/2} < \infty$$

is true but with $\mu_{0,n}$ given by

$$2i(-1)^{n+1} \mu_{0,n} = - \int_0^1 e^{-2\pi i nt} \sigma_1(t) dt + \int_0^1 e^{2\pi i nt} \sigma_2(t) dt.$$  

Now, we can proceed to eigenfunctions. We are going to combine results from the previous theorem with lemma 2.2 and corollary 2.3.

Theorem 3.3. Let $1 < p < 2$ and

$$F_1(x, t) = \sigma_1(t) + \tilde{\sigma}_1(x, t) - (T_{\sigma_1} \tilde{\sigma}_2)(x, t)$$

$$F_2(x, t) = \sigma_2(t) + \tilde{\sigma}_2(x, t) - (T_{\sigma_2} \tilde{\sigma}_1)(x, t).$$

The eigenfunctions of the spectral problem (3.1)-(3.2) admit the representation

$$y_1(x, \mu_n) = e^{i\pi n x} (1 + i\mu_{0,n}x) \left( 1 + \int_0^x e^{-2\pi i nt} F_1(x, t) dt \right)$$

$$- 2i\mu_{0,n} e^{i\pi n x} \int_0^x e^{-2\pi i nt} F_1(x, y) dt + r_1(x, n),$$

$$y_2(x, \mu_n) = e^{-i\pi n x} (1 - i\mu_{0,n}x) \left( 1 + \int_0^x e^{2\pi i nt} F_2(x, t) dt \right)$$

$$+ 2i\mu_{0,n} e^{-i\pi n x} \int_0^x e^{2\pi i nt} F_2(x, t) dt + r_1(x, n),$$
Proof. According to (3.4) eigenfunctions are expressed by solutions $c$ and $s$ in the following way

$$y_1(x, \mu_n) = c_1(x, \mu_n) + s_1(x, \mu_n)$$

and

$$y_2(x, \mu_n) = c_2(x, \mu_n) + s_2(x, \mu_n).$$

The results of lemma 2.29 yield that

$$y_1(x, \mu_n) = e^{i\mu_n x} + \int_0^x e^{(x-2t)i\mu_n} \sigma_1(t) dt$$

$$+ \int_0^x e^{(x-2t)i\mu_n} \sigma_1(x, t) dt - \int_0^x e^{i\mu_n(x-2t)} (T_\sigma_1, \tilde{\sigma}_2)(x, t) dt + \alpha(x, \mu_n),$$

$$y_2(x, \mu_n) = e^{-i\mu_n x} + \int_0^x e^{-(x-2t)i\mu_n} \sigma_2(t) dt$$

$$+ \int_0^x e^{-(x-2t)i\mu_n} \tilde{\sigma}_2(x, t) dt - \int_0^x e^{-i\mu_n(x-2t)} (T_\sigma_2, \tilde{\sigma}_1)(x, t) dt + \beta(x, \mu_n),$$

where

$$|\alpha(x, \mu_n)| + |\beta(x, \mu_n)| \leq c\gamma^2(\mu_n).$$

Repeating once more all arguments used in order to derive formulas for eigenvalues, we obtain the thesis with claimed estimates for remainders. \(\Box\)

It is possible to obtain shorter but less precise formulas for eigenfunctions. This time we use the representation (2.32) and comments from lemma 4.2 to prove the following fact.

**Corollary 3.4.** Let $1 \leq p < 2$, then the eigenfunctions of the spectral problem (3.1)–(3.2) admit the representation

$$y_1(x, \mu_n) = e^{i\pi nx} \left(1 + i\mu_{0,n} x + \int_0^x e^{-2i\pi n t} \sigma_1(t) dt\right)$$

$$+ \int_0^x \int_0^s \sigma_1(s) \sigma_2(\xi) e^{2i\mu_n \xi} d\xi ds + r_1(x, n),$$

$$y_2(x, \mu_n) = e^{-i\pi nx} \left(1 - i\mu_{0,n} x + \int_0^x e^{2i\pi n t} (\sigma_2(t) dt\right)$$

$$+ \int_0^x \int_0^s \sigma_1(\xi) \sigma_2(s) e^{2i\mu_n \xi} d\xi ds + r_2(x, n),$$

(3.20)

where for $1 < p < 2$ we have

$$\sup_{x \in [0,1]} \sum_{n=1}^\infty |r_j(x, n)|^{q/2} < \infty,$$

whereas for $p = 1$ there holds

$$|r_j(x, n)| \leq c\Gamma^2(\pi n).$$
4 Appendix

Lemma 4.1. For every $x \in [0, 1]$ and $j = 1, 2$ the functions $\tilde{\sigma}_j(x, \cdot)$ belong to $L_r[0, 1]$, therefore $\tilde{\sigma}_j \in B$, $j = 1, 2$, and $\tilde{J} \in M(B)$.

Proof. We take $(x, t) \in \Delta$. Let $\tilde{\sigma}_1, \tilde{\sigma}_2$ denote the extension of $\sigma_1$ and $\sigma_2$ by zero outside $[0, 1]$. Note that for every $x \in [0, 1]$ we get

$$
\int_0^{x-t} \sigma_1(t + \xi)\sigma_2(\xi)d\xi = \int_{-\infty}^{\infty} \tilde{\sigma}_1(t + \xi)\tilde{\sigma}_2(\xi)\chi(x - (t + \xi))d\xi
$$

$$
= \int_{-\infty}^{\infty} \tilde{\sigma}_1(t - s)\tilde{\sigma}_2(-s)\chi(x - (t - s))ds
$$

$$
= \left(\tilde{\sigma}_1(\cdot)\chi(x - \cdot) \ast (\tilde{\sigma}_2(\cdot))\right)(t).
$$

We thus have

$$
\| \int_0^{x-t} \sigma_1(t + \xi)\sigma_2(\xi)d\xi \|_{L_r[0, 1]} \leq \| \int_0^{x-t} \tilde{\sigma}_1(t + \xi)\tilde{\sigma}_2(\xi)d\xi \|_{L_r(B)}
$$

$$
\leq \| (\tilde{\sigma}_1(\cdot)\chi(x - \cdot))\|_{L_r(B)}\| \tilde{\sigma}_2(\cdot)\|_{L_r(B)}
$$

$$
\leq \| \sigma_1 \|_{L_r[0, 1]}\| \sigma_2 \|_{L_r[0, 1]},
$$

hence

$$
\| \tilde{\sigma}_j \|_B \leq \bar{a}, \quad j = 1, 2. \quad (4.1)
$$

Clearly, a similar estimate holds for $\tilde{\sigma}_2$ as well.

Therefore, if we consider $\epsilon$ and $x$ such that $0 \leq t \leq x + \epsilon \leq 1$, then repeating the reasoning from the latter inequality, we obtain

$$
\left\| \int_0^{x+\epsilon-t} \sigma_1(t + \xi)\sigma_2(\xi)d\xi - \int_0^{x-t} \sigma_1(t + \xi)\sigma_2(\xi)d\xi \right\|_{L_r[0, 1]}
$$

$$
\leq \| (\tilde{\sigma}_1(\cdot)\chi(x + \epsilon - \cdot) - \chi(x - \cdot))\|_{L_r(B)}\| \tilde{\sigma}_2(\cdot)\|_{L_r(B)}
$$

$$
\leq \int_{\mathbb{R}} |\tilde{\sigma}_1(s)|\chi(x + \epsilon - s) - \chi(x - s)\| ds\| \tilde{\sigma}_2(\cdot)\|_{L_r(B)}.
$$

The integral in the last line converges to zero, if $\epsilon \to 0$, because of Lebesgue Theorem, hence the mapping $x \mapsto \tilde{\sigma}_j(x, \cdot) \in L_r[0, 1]$ is continuous for $j = 1, 2$.

Lemma 4.2. The following identity holds

$$
\int_0^x e^{-2t\mu} \tilde{\sigma}_1(x, t)dt + \int_0^x e^{2t\mu} \tilde{\sigma}_2(x, t)dt = \int_0^x e^{-2\mu s} \sigma_1(\xi)d\xi \int_0^x \sigma_2(s)e^{2\mu s}ds.
$$

Moreover, we have

$$
\left| \int_0^x e^{-2t\mu} \tilde{\sigma}_1(x, t)dt + \int_0^x e^{2t\mu} \tilde{\sigma}_2(x, t)dt \right| \leq c^2_{10}(x, \mu)
$$

and

$$
\int_0^x e^{-2t\mu} \tilde{\sigma}_1(x, t)dt - \int_0^x e^{2t\mu} \tilde{\sigma}_2(x, t)dt
$$

$$
= -2 \int_0^x \int_0^s \sigma_1(\xi)\sigma_2(s)e^{2\mu s}e^{-2\mu \xi}d\xi ds + \alpha_1(\mu),
$$

$$
\int_0^s \sigma_1(\xi)\sigma_2(s)e^{2\mu s}e^{-2\mu \xi}d\xi ds + \alpha_1(\mu),
$$
\[ \int_0^x e^{2i\mu t} \tilde{\sigma}_2(x, t) dt - \int_0^x e^{-2i\mu t} \tilde{\sigma}_1(x, t) dt \\
= -2 \int_0^x \int_0^s \sigma_1(s)\sigma_2(\xi)e^{-2i\mu s} e^{2i\mu \xi} d\xi ds + \alpha_2(\mu), \]

where \( \alpha_j(\mu) = O(\gamma_0^2(x, \mu)) \) for \( j = 1, 2 \).

**Proof.** Note that
\[ \int_0^x e^{-2i\mu t} \tilde{\sigma}_1(x, t) dt = \int_0^x \int_0^s \sigma_1(s)\sigma_2(s) e^{-2i\mu s} e^{2i\mu \xi} d\xi ds \]
\[ \int_0^x e^{2i\mu t} \tilde{\sigma}_2(x, t) dt = \int_0^x \int_0^s \sigma_1(\xi)\sigma_2(s) e^{2i\mu s} e^{-2i\mu \xi} d\xi ds. \]

Observe that the change of variables yield
\[ \int_0^x e^{-2i\mu t} \tilde{\sigma}_1(x, t) dt = \int_0^x \int_0^x \sigma_1(\xi)\sigma_2(s) e^{-2i\mu \xi} e^{2i\mu s} d\xi ds, \]

thus
\[ \int_0^x e^{-2i\mu t} \tilde{\sigma}_1(x, t) dt + \int_0^x e^{2i\mu t} \tilde{\sigma}_2(x, t) dt \\
= \int_0^x \int_0^x \sigma_1(\xi)\sigma_2(s) e^{2i\mu s} e^{-2i\mu \xi} d\xi ds \\
= \int_0^x e^{-2i\mu \xi} \sigma_1(\xi) d\xi \int_0^x \sigma_2(s) e^{2i\mu s} ds. \]

This step shows that
\[ \left| \int_0^x e^{-2i\mu t} \tilde{\sigma}_1(x, t) dt + \int_0^x e^{2i\mu t} \tilde{\sigma}_2(x, t) dt \right| \leq c \gamma_0^2(x, \mu). \]

What is more, then
\[ \int_0^x e^{-2i\mu t} \tilde{\sigma}_1(x, t) dt - \int_0^x e^{2i\mu t} \tilde{\sigma}_2(x, t) dt \\
= \int_0^x e^{-2i\mu \xi} \sigma_1(\xi) d\xi \int_0^x \sigma_2(s) e^{2i\mu s} ds \\
- 2 \int_0^x \int_0^s \sigma_1(\xi)\sigma_2(s) e^{2i\mu s} e^{-2i\mu \xi} d\xi ds, \]

thus
\[ \int_0^x e^{-2i\mu t} \tilde{\sigma}_1(x, t) dt - \int_0^x e^{2i\mu t} \tilde{\sigma}_2(x, t) dt \\
= -2 \int_0^x \int_0^s \sigma_1(\xi)\sigma_2(s) e^{2i\mu s} e^{-2i\mu \xi} d\xi ds \\
+ \alpha_1(\mu), \]

where \( \alpha_1(\mu) = O(\gamma_0^2(x, \mu)) \). Analogously we get the last claim \( \Box \)
Lemma 4.3. The linear operator $T_σ$

\[
(T_σ f)(x, t) = \int_0^{x-t} \sigma(t + ξ) f(t + ξ, ξ) dξ = \int_t^{x} \sigma(s) f(s, s - t) ds, \quad (4.2)
\]

where $σ \in L^p[0, 1]$ is bounded in $B$.

Proof. Note that

\[
\left( \int_0^{x} |(T_σ f)(x, t)|^r dt \right)^{1/r} = \left( \int_0^{x} \left| \int_0^{x} \chi(s - t) \sigma(s) f(s, s - t) ds \right|^r dt \right)^{1/r}
\]

\[
\leq \int_0^{x} |\sigma(s)| \left( \int_0^{s} |f(s, s - t)|^r dt \right)^{1/r} ds
\]

\[
\leq \int_0^{x} |\sigma(s)| ds \sup_{s \in [0, 1]} \left( \int_0^{s} |f(s, τ)|^r dτ \right)^{1/r} ds
\]

\[
\leq \|σ\|_{L^1} \|f\|_B.
\]

(4.3)

For the proof of continuity we take $ε$ and $x$ such that $0 \leq t \leq x + ε \leq 1$.

Then

\[
\| (T_σ f)(x + ε, \cdot) - (T_σ f)(x, \cdot) \|_{L^r[0, 1]} \leq \left( \int_0^{x} \left| \int_0^{x+ε} \sigma(s) f(s, s - t) ds \right|^r dt \right)^{1/r} \]

\[+ \left( \int_0^{x} \left| \int_t^{x+ε} \sigma(s) f(s, s - t) ds \right|^r dt \right)^{1/r}.
\]

First integral may be estimated as follows

\[
\left( \int_0^{x} \left| \int_0^{x+ε} \sigma(s) f(s, s - t) ds \right|^r dt \right)^{1/r} \leq \int_x^{x+ε} |\sigma(s)| \left( \int_0^{s} |f(s, s - t)|^r dt \right)^{1/r} ds
\]

\[\leq \int_x^{x+ε} |\sigma(s)| \left( \int_0^{s} |f(s, τ)|^r dτ \right)^{1/r} ds
\]

\[\leq \|f\|_B \int_x^{x+ε} |\sigma(s)| ds.
\]

and this expression goes to zero whenever $ε$ does.

Second integral can be treated in an analogous way, hence the proof is completed.

Lemma 4.4. The operators $T_{kj}, k, j = 1, 2, \ k \neq j$ satisfy the following estimate

\[
\| T_{kj} f \|_n \leq \frac{a^n}{n!} \|f\|_B, \quad f \in B, \quad n \in \mathbb{N}, \quad k, j = 1, 2, \ k \neq j. \quad (4.4)
\]

Proof. Consider the operator $T_{12}$. Note that directly from third line of (4.3) we
Get
\[
\left( \int_0^x |(T_{12}f)(x,t)|^r \, dt \right)^{1/r} \leq \int_0^x |\sigma_1(s)| \left( \int_0^s |(T_{12}f)(s,\tau)|^r \, d\tau \right)^{1/r} \, ds \\
\leq \int_0^x |\sigma_1(s)| \int_0^s |\sigma_2(\tau)| \left( \int_0^\tau |f(\tau,\xi)|^r \, d\xi \right)^{1/r} \, d\tau \, ds \\
\leq \|f\|_B \int_0^x |\sigma(s)| \int_0^s |\sigma_2(\tau)| \, ds.
\]
(4.5)

Define \( \eta \in C[0,1] \) by
\[
\eta(x) := \int_0^x |\sigma_1(s)| \left( \int_0^s |\sigma_2(\tau)| \, d\tau \right) \, ds, \quad x \in [0,1].
\]
This function is increasing and bounded by \( a = \|\sigma_1\|_L \|\sigma_2\|_L \). It suffices to prove that for all \((x, t) \in \Delta \) and \( n = 1, 2, \ldots, \)
\[
\left( \int_0^x |(T_{12}f)(x,t)|^r \, dt \right)^{1/r} \leq \|f\|_B \eta^n(x), \quad f \in B.
\]
(4.6)

For \( n = 1 \) the estimate (4.5) was shown above. Arguing by induction, suppose that (4.6) holds for some \( n \in \mathbb{N} \). Then for \((x, t) \in \Delta \) and \( f \in B \) from (4.5) we have
\[
\left( \int_0^x |(T_{12}f)(x,t)|^r \, dt \right)^{1/r} \\
\leq \int_0^x |\sigma_1(s)| \int_0^s |\sigma_2(\tau)| \left( \int_0^\tau |(T_{12}f)(\tau,\xi)|^r \, d\xi \right)^{1/r} \, d\tau \, ds \\
\leq \|f\|_B \int_0^x |\sigma_1(s)| \int_0^s |\sigma_2(\tau)| \eta^n(\tau) \, d\tau \, ds \\
\leq \|f\|_B \int_0^x |\sigma_1(s)| \int_0^s |\sigma_2(\tau)| \, d\tau \, \eta^n(s) \, ds \\
= \frac{\|f\|_B}{n!} \int_0^x \eta^n(s) \, ds = \frac{\|f\|_B}{(n+1)!} \eta^{n+1}(x).
\]
Therefore (4.6) hold true and then after taking supremum over \( x \in [0,1] \) we get (4.6).

Next proposition we state below without a proof, since it can be found in [7] Prop. 6.1.

**Proposition 4.5.** If \( \sigma_j \in L_p[0,1] \), \( 1 \leq p < 2 \) and \( F \in M(B) \), then
\[
\int_0^x e^{-2i\mu t}(\tilde{T}F)(x,t) \, dt = -\int_0^x e^{-2i\mu s} J(s) \int_0^s e^{2i\nu \xi} F(s,\xi) \, d\xi \, ds.
\]
(4.7)

Moreover,
\[
\int_0^2 e^{-2i\mu t}(\tilde{T}J)(x,t) \, dt \\
= -\int_0^x e^{2i\mu y} \left( \int_y^x J(z) e^{-2i\mu z} \, dz \int_0^\tau J^T(\tau) e^{-2i\mu \tau} \, d\tau \right) J^T(y) \, dy.
\]
(4.8)
Lemma 4.6. If $\mu \in P_d$, then there hold the following inequalities

$$\left\| \int_0^x e^{-2tA_{\mu}} \tilde{J}(x, t) \, dt \right\|_{M(C[0,1])} \leq 2e^{2d}a_0^2 \gamma(\mu), \quad (4.9)$$

$$\left\| \int_0^x e^{-2tA_{\mu}} (T \tilde{J})(x, t) \, dt \right\|_{M(C[0,1])} \leq 2e^{2d}a_0^2 \gamma(\mu), \quad (4.10)$$

$$\left\| \int_0^x e^{-2tA_{\mu}} (\tilde{T} \tilde{J})(x, t) \, dt \right\|_{M(C[0,1])} \leq 2(a_2 + 1)e^{2d} \left( \gamma(\mu) \gamma_0(x, \mu) + \gamma_1(\mu) \right),$$

$$x \in [0, 1], \quad (4.11)$$

$$\left\| \int_0^x e^{-2tA_{\mu}} (\tilde{T}^n \tilde{J})(x, t) \, dt \right\|_{M(C[0,1])} \leq 2e^{2nd}a_0^{2n-2} \frac{\gamma_2(\mu)}{(n-2)!}, \quad n \geq 2. \quad (4.12)$$

Proof. Note that

$$\left\| \int_0^x e^{-2itA_{\mu}} \tilde{J}(x, t) \, dt \right\|_{M(C)} = \left\| \int_0^x \int_0^s J(s)e^{-2\mu s} J(\xi)e^{2\mu \xi} d\xi ds \right\|_{M(C)}$$

$$= \left\| \int_0^x e^{-2\mu s} \sigma_1(s) \int_0^s e^{2\mu \xi} \sigma_2(\xi) d\xi ds \right\|_{M(C)}$$

$$+ \left\| \int_0^x e^{-2\mu s} \sigma_2(s) \int_0^s e^{2\mu \xi} \sigma_1(\xi) d\xi ds \right\|_{M(C)} \quad (4.13)$$

$$\leq e^{2d} \left\{ \|\sigma_1\|_{L_p} \right\| \int_0^s e^{2\mu \xi} \sigma_2(\xi) d\xi \right\|_{L_q}$$

$$+ \|\sigma_2\|_{L_p} \right\| \int_0^s e^{2\mu \xi} \sigma_1(\xi) d\xi \right\|_{L_q} \leq e^{2d} \max \{ \|\sigma_1\|_{L_p}, \|\sigma_2\|_{L_p} \} \gamma(\mu), \quad x \in [0, 1].$$

We thus proved the estimate (4.10).

Next, from (4.7), if $\mu \in P_d$, $x \in [0, 1]$ and $F \in M(B)$, then

$$\left\| \int_0^x e^{-2itA_{\mu}} (\tilde{T}F)(x, t) \, dt \right\|_{M(C)} \leq e^{2d} \left\| \int_0^x \int_0^s J(s) e^{2\mu \xi} F(s, \xi) d\xi ds \right\|_{M(C)} \right\|_{M(C[0,1])} \quad (4.14)$$

and

$$\left\| \int_0^x e^{-2itA_{\mu}} (\tilde{T}F)(x, t) \, dt \right\|_{M(C[0,1])} \leq e^{2d}a_0 \right\| \int_0^s \int_0^s e^{2\mu \xi} F(s, \xi) d\xi ds \right\|_{M(C[0,1])} \right\|_{M(C[0,1])} \quad (4.15)$$

We use (4.15) and (4.13) to obtain that

$$\left\| \int_0^x e^{-2itA_{\mu}} (\tilde{T} \tilde{J})(x, t) \, dt \right\|_{M(C[0,1])} \leq e^{2d}a_0 \right\| \int_0^x e^{2\mu \xi} \tilde{J}(s, \xi) d\xi \right\|_{M(C[0,1])} \right\|_{M(C[0,1])}$$

$$\leq e^{4d}a_0^2 \gamma(\mu),$$

thus, the estimate (4.10) holds.
Due to the estimate
\[
\left| \int_0^x \sigma_0(s) \gamma_0(y, \mu) \, dy \right| \leq \| \sigma_0 \|_{L_p} \| \gamma_0(y, \mu) \|_{L_q} \leq a_2 \gamma(\mu),
\]
the inequality (4.11) holds if
\[
\left\| \int_0^x e^{-2i\mu t} (\tilde{T} \tilde{J})(x, t) \, dt \right\|_{M(C)} \leq e^{2d} \left( \gamma_1(\mu) + \gamma_0(x, \mu) \int_0^x \sigma_0(s) \gamma_0(y, \mu) \, dy \right).
\]
(4.16)

Whereas using (4.8), (4.9), we have
\[
\text{We proceed by induction. Using (4.14) for } F = \tilde{T} \tilde{J} \text{ and (4.16), we note that}
\]
\[
\left\| \int_0^x e^{-2i\mu t} (\tilde{T} \tilde{J})(x, t) \, dt \right\|_{M(C)} \leq e^{2d} \int_0^x \sigma_0(s) \left| \int_0^s e^{2i\mu \xi} (\tilde{T} \tilde{J})(s, \xi) d\xi \right| \, ds \]
\[
\leq e^{4d} \int_0^x \sigma_0(s) \left( \gamma_0(s, \mu) \int_0^s \sigma_0(y) \gamma_0(y, \mu) \, dy + \gamma_1(\mu) \right) \, ds
\]
\[
\leq e^{4d} \int_0^x \sigma_0(s) \gamma_0(s, \mu) \int_0^s \sigma_0(y) \gamma_0(y, \mu) \, dy \, ds + e^{4d} a_1 \gamma_1(\mu)
\]
\[
\leq e^{4d} \left( \int_0^x \sigma_0(s) \gamma_0(s, \mu) \, ds \right)^2 + e^{4d} a_1 \gamma_1(\mu)
\]
\[
\leq e^{4d} \left( a_2^2 \gamma(\mu) + e^{4d} a_1 \gamma_1(\mu) \right).
\]
Therefore, (4.17) holds for $n = 2$.

Let suppose now that (4.17) holds for some $n \geq 2$. We thus once again use (4.14) to derive
\begin{align*}
\left\| \int_0^x e^{-2i\mu t} (\tilde{T}^{n+1} \tilde{J})(x,t) dt \right\|_{M(C)} &\leq e^{2d} \int_0^x \sigma_0(s) \left\| \int_0^s e^{2i\mu \xi} (\tilde{T}^n \tilde{J})(s,\xi) d\xi \right\|_{M(C)} ds \\
&\leq \frac{e^{2(n+1)d}}{(n-2)!} (\gamma_2(\mu))^n \int_0^x \sigma_0(s) \left( \int_0^s \sigma_0(\tau) d\tau \right)^{n-2} ds \\
&= \frac{e^{2(n+1)d} \gamma_2(\mu)}{(n-1)!} \left( \int_0^x \sigma_0(\tau) d\tau \right)^{n-1}, \quad x \in [0,1],
\end{align*}

thus (4.17) holds also for $n + 1$, and the proof of (4.17) is completed. □

References

[1] M. M. Malamud A. V. Agibalova and L. L. Oridoroga. On the completeness of general boundary value problems for $2 \times 2$ first-order systems of ordinary differential equations. Methods Funct. Anal. Topology, 18:4–18, 2012.

[2] S. Albeverio, R. Hryniv, and Y. Mykytyuk. Inverse spectral problems for Dirac operators with summable potentials. Russian Journal of Math. Physics, 12:406–423, 2005.

[3] R. Bellman and K. L. Cook. Differential-Difference Equations. Academic Press, New York, 1963.

[4] P. Djakov and B. Mityagin. Bari-Markus property for Riesz projections of 1D periodic Dirac operators. Mat. Nachr., 283:443–462, 2010.

[5] P. Djakov and B. Mityagin. Unconditional convergence of spectral decompositions of 1D Dirac operators with regular boundary conditions. Indiana Univ. Math. J., 61:359–398, 2012.

[6] P. Djakov and B. Mityagin. Riesz bases consisting of root functions of 1D Dirac operators. Proc. Amer. Math. Soc., 141:1361–1375, 2013.

[7] A. M. Gomilko and Ł. Rzepnicki. On asymptotic behaviour of solutions of the dirac system and applications to the Sturm-Liouville problem with a singular potential. Journal of Spectral Theory, published online.

[8] A. A. Lunyov and M. M. Malamud. On the Riesz basis property of root vectors system for $2 \times 2$ Dirac type operators. J. Math. Anal. Appl., 441:57–103, 2016.

[9] V. A. Marchenko. Sturm–Liouville Operators and Their Applications. Birkhauser, Basel, 1986.

[10] D. V. Puyda. Inverse Spectral Problems for Dirac Operators with Summable Matrix-Valued Potentials. Integr. Equ. Oper. Theory, 74:417–450, 2012.
[11] I. V. Sadovnichaya. Uniform asymptotics of the eigenvalues and eigenfunctions of the Dirac system with an integrable potential. *Differential Equations*, 52:1000–1010, 2016.

[12] A. M. Savchuk and I. V. Sadovnichaya. Asymptotic Formulas for Fundamental Solutions of the Dirac System with Complex-Valued Integrable Potential. *Differential Equations*, 49:545–556, 2013.

[13] A. M. Savchuk and I. V. Sadovnichaya. The Riesz basis property of generalized eigenspaces for a Dirac system with integrable potential. *Dokl. Math*, 91:309–312, 2015.

[14] A. M. Savchuk and I. V. Sadovnichaya. Estimates of Riesz Constants for the Dirac System with an Integrable Potential. *Differential Equations*, 54:748–757, 2018.

[15] A. M. Savchuk and A. A. Shkalikov. Dirac operator with complex-valued summable potential. *Math. Notes*, 96:777–810, 2014.

[16] A. Zettl. *Sturm–Liouville theory*. American Mathematical Society, Providence, 2005.