OPTIMAL DISTRIBUTED CONTROL FOR A COUPLED PHASE-FIELD SYSTEM

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Abstract. Our aim is to consider a distributed optimal control problem for a coupled phase-field system which was introduced by Cahn and Novick-Cohen. First, we establish that the existence of a weak solution, in particular, we also obtain that a strong solution is uniqueness. Then the existence of optimal controls is proved. Finally we derive that the control-to-state operator is Fréchet differentiable and the first-order necessary optimality conditions involving the adjoint system are discussed as well.

1. Introduction. We are concerned in this paper with the following phase-field system

\[ \frac{\partial v}{\partial t} = -f(u + v) + f(u - v) - \alpha v + h^2 \Delta v, \tag{1.1} \]
\[ \frac{\partial u}{\partial t} = h^2 \Delta (f(u + v) + f(u - v) - h^2 \Delta u), \tag{1.2} \]

where parameters \( \alpha, h \) are constants and the nonlinear term meets certain conditions. Next we provide a brief introduction on the background of (1.1)-(1.2).

The coupled Allen-Cahn/Cahn-Hilliard equations (1.1)-(1.2) were derived in [2], to model simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point. Here \( u \) is the average concentration of one of the components and \( v \) denotes an order parameter. In addition, \( u \) is a conserved quantity. Furthermore, \( h > 0 \) represents the lattice spacing and \( \alpha \) reflects the location of the system within the phase diagram, where \( \alpha \) may be either positive or negative. While the nonlinear term \( f \) is derivative of a double-well potential \( F \).

Brochet, Hilhorst and Novick-Cohen [1] have proved that the well-posedness and the existence of maximal attractors and inertial sets for the usual cubic nonlinear term \( f(s) = s^3 - \beta s \) in three space dimensions for (1.1)-(1.2) with Neumann boundary conditions. Miranville, Saoud and Talhouk [15] studied the existence of an exponential attractor and, as a consequence, the existence of the global attractor with finite fractal dimension for the system (1.1)-(1.2) with \( h = 1 \) and \( \alpha = 0 \). In addition, for domain \( \Omega \) being a \( n \)-dimensional box, Li and Yan [10] prove that

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(1.1)-(1.2) with \( f(s) = s^3 - \frac{1}{2}s \) bifurcates from the trivial solution to \( 3^n - 1 \) non-trivial steady state solution as the control parameter crosses a certain critical value. Besides, the expression of bifurcated solution is also obtained. Then by using the dynamic transition theory, combining with the spectral theorem for general linear completely continuous fields, Wang and Yan [27] proved that the system which is the same as [10] undergoes a continuous transition and bifurcates from a trivial solution to an attractor as the control parameter crosses a certain critical value. In addition, for some special cases, they obtain the stability of the singular points of the attractors, and the topological structure of the attractors is illustrated. Furthermore, Miranville, Saoud and Talhouk [18] also showed the existence and uniqueness of the solution for the system (1.1)-(1.2) with singular potentials. Makki, Miranville and Saoud [14] obtained the asymptotic behavior of a Cahn-Hilliard/Allen-Cahn system coupled with a heat equation based on the type III heat conduction law with singular potentials,

\[
\begin{align*}
\frac{\partial v}{\partial t} - \Delta v + f(u + v) - f(u - v) &= \frac{\partial \alpha}{\partial t}, \\
\frac{\partial u}{\partial t} + \Delta^2 u - \Delta(f(u + v) + f(u - v)) &= 0, \\
\frac{\partial^2 \alpha}{\partial t^2} - \Delta \frac{\partial \alpha}{\partial t} - \Delta \alpha &= -\frac{\partial v}{\partial t}.
\end{align*}
\]

They proved a strict separation property (from the pure states) in one space dimension. Moreover Miranville, Quintanilla and Saoud in [17] discussed that the form of heat equation is based on the usual Fourier law and the type III heat conduction law, respectively. They proved the existence of exponential attractors and, therefore, of finite-dimensional global attractors for the corresponding two cases. We also mention that a similar system with a non-constant mobility was studied in, e.g., [19, 20, 6]. In [6], Dal Passo, Giacomelli and Novick-Cohen proved the existence of weak solutions for the Neumann problem for a degenerate parabolic system. In addition, asymptotics for a similar system was studied in [19]. Then the partial wetting case was studied in [20]. Solving coupled Allen-Cahn/Cahn-Hilliard systems using numerical methods were also studied in, e.g., [21, 28]. Besides, for other similar phase filed models, we can see, e.g., [9] and Caginalp phase-field model [16].

It is well known that optimal control problem has recently become a major issue in fields of mathematics. Many authors are interested to study the higher order parabolic equation or the coupled partial differential equations on this aspect (see, for instance, [3, 12, 30, 31]). In particular, Cavaterra, Rocca and Wu [3] considered a two dimensional simplified Ericksen-Leslie system modelling the incompressible nematic liquid crystal flows and [12] consider a sixth order nonlinear parabolic equation which arising in oil-water surfactant mixtures. Besides, for the model of tumor growth, Colli, Gilardi, Rocca and Sprekels [4] considered the distributed optimal control problems for a diffuse interface model of tumor growth, we also read literatures [8, 23, 24] for this aspect. Colli, Gilardi, Marinoschi and Rocca [5] studied the optimal control problems for a conserved phase-field system with a possibly singular potential. Liu and Zhang [13] discussed the optimal distributed control for a new mechanochemical model in biological patterns.

For the system (1.1)-(1.2), an order parameter is a measure of the degree of order across the boundaries in a phase transition system, which normally ranges between
zero in one phase (usually above the critical point) and nonzero in the other. From a theoretical perspective, order parameters arise from symmetry breaking. When this happens, one needs to introduce one or more extra variables to describe the state of the system, and we can adjust the parameter value to achieve the results we want. Therefore, in this work, we consider mainly that the optimal control problem of (1.1)-(1.2). In particular, we focus on the case of $\alpha > 0$, $h = 1$ and the usual cubic nonlinear term $f(s) = s^3 - \beta s (\beta \in \mathbb{R})$ in [1], that is, the nonlinear term $f(u + v)$ and $f(u - v)$ in this paper are

$$f(u + v) = (u + v)^3 - \beta(u + v) = u^3 + 3u^2v + 3uv^2 + v^3 - \beta u - \beta v,$$

$$f(u - v) = (u - v)^3 - \beta(u - v) = u^3 - 3u^2v + 3uv^2 - v^3 - \beta u + \beta v.$$ 

In other words, we will investigate the following state system

$$\frac{\partial v}{\partial t} = -2v^3 - 6u^2v + (2\beta - \alpha)v + \Delta v + \phi, \quad \text{in } Q, \quad (1.3)$$

$$\frac{\partial u}{\partial t} = \Delta(2u^3 + 6uv^2 - 2\beta u - \Delta u), \quad \text{in } Q, \quad (1.4)$$

where $\phi$ is the control constraint term and $Q \triangleq \Omega \times (0, T)$. Here $\Omega$ is an open bounded, connected domain in $\mathbb{R}^3$ and $T > 0$. Furthermore, the state system (1.3)-(1.4) is equipped with the Dirichlet boundary conditions

$$v = u = \Delta u = 0, \quad \text{on } \Sigma, \quad (1.5)$$

and initial conditions

$$v(0) = v(x, 0) = v_0(x), \quad u(0) = u(x, 0) = u_0(x), \quad \text{in } \Omega, \quad (1.6)$$

where $\Sigma \triangleq \partial \Omega \times (0, T)$ and $\partial \Omega$ is smooth boundary of $\Omega$. Obviously, we can rewrite (1.3)-(1.6) in the equivalent form

$$\frac{\partial v}{\partial t} = -2v^3 - 6u^2v + (2\beta - \alpha)v + \Delta v + \phi, \quad \text{in } Q, \quad (1.7)$$

$$\frac{\partial u}{\partial t} = \Delta \mu, \quad \text{in } Q, \quad (1.8)$$

$$\mu = 2u^3 + 6uv^2 - 2\beta u - \Delta u, \quad \text{in } Q, \quad (1.9)$$

$$v = u = \mu = 0, \quad \text{on } \Sigma, \quad (1.10)$$

$$v(0) = v(x, 0) = v_0(x), \quad u(0) = u(x, 0) = u_0(x), \quad \text{in } \Omega. \quad (1.11)$$

The present paper is built up as follows. In section 2, we will provide notations and some general assumptions which are necessary for the understanding of subsequent results. Section 3 devoted to study the well-posedness of (1.3)-(1.6) and provide a continuous dependence result for global strong solution under proper assumptions on the initial data, which turns out to be necessary in order to eventually analyse the optimal control problem. Finally, we establish the existence of optimal controls and obtain the differentiability properties of the control-to-state operator in section 4. In addition, in this section the first-order necessary optimality conditions for the optimal control problem are discussed as well. We are now turning to discuss the issue for more details.
2. Preliminaries. In the section, we introduce firstly that some notations which are often used in the subsequent section. We set \( H \triangleq L^2(\Omega) \), as usual, Lebesgue norms and the scalar product in \( H \) are defined by \( \| \cdot \|_2 = \| \cdot \|_{L^2(\Omega)} \) and \( (\cdot, \cdot) = (\cdot, \cdot)_{L^2(\Omega)} \), respectively. Moreover we also make use of \( \| \cdot \|_p = \| \cdot \|_{L^p(\Omega)} \) to represent the norm of Lebesgue space \( L^p(\Omega) \). For Sobolev space \( H^2(\Omega) \), we define a new space \( V \triangleq \{ v \in H^2(\Omega) : v = 0 \text{ on } \partial\Omega \} \). For the norm of \( H \), it is given by

\[
\| v \|_H = (\| \Delta v \|_2^2 + \| v \|_2^2)^{\frac{1}{2}}, \quad \text{for all } v \in V.
\]

It follows from Lemma 4.2 of Chapter 3 in [25] that the norms on \( V \) equivalent to the \( H^2 \)-norm. In addition, if \( Y \) is a (real) Banach space, the notation \( \langle \cdot, \cdot \rangle \) will be used to denote the duality pairing between \( Y \) and its dual \( Y' \), while \( (\cdot, \cdot)_Y \) and \( \| \cdot \|_Y \) will denote separately the scalar product and the norm in \( Y \). We further set \( (\cdot, \cdot)_Y \triangleq ((-\Delta)^{-\frac{1}{2}}, (-\Delta)^{-\frac{1}{2}})_2 \), with associated norm \( \| \cdot \|_{Y} \), where \((-\Delta)^{-1} \) denotes the inverse minus Laplace operator associated with Dirichlet boundary conditions. Note that\( \| \cdot \|_{Y} \) is equivalent to the usual \( H^{-1} \)-norm on \((H^1(\Omega))' = (H_0^1(\Omega))' \). Furthermore, \( \tilde{f} \) will stand for the average of \( f \) over \( \Omega \) for every \( f \in Y' \), that is, \( \tilde{f} \triangleq \frac{1}{|\Omega|} \langle f, 1 \rangle \). Here \( |\Omega| \) is the Lebesgue measure of \( \Omega \).

Next, we introduce the Riesz isomorphism \( A : H^1(\Omega) \to (H^1(\Omega))' \) associated to the standard scalar product of Sobolev space \( H^1(\Omega) \), i.e.,

\[
(Au, v) \triangleq (u, v)_{H^1(\Omega)} = \int_{\Omega} (\nabla u \cdot \nabla v + uv)\,dx, \quad \forall \, u, v \in H^1(\Omega), \quad (2.1)
\]

where \((\cdot, \cdot)_{H^1(\Omega)}\) is the scalar product of \( H^1(\Omega) \). We notice that \( Au = -\Delta u + u \) if \( u \in V \) and that the restriction of \( A \) to \( V \) is an isomorphism from \( V \) onto \( H \). Clearly, we know that \( A \) is self-adjoint and strictly positive and also write that

\[
(Au, A^{-1} v^*) = \langle v^*, u \rangle, \quad \forall \, u \in H^1(\Omega) \text{ and } v^* \in (H^1(\Omega))', \quad (2.2)
\]

\[
\langle u^*, A^{-1} v^* \rangle = (u^*, v^*)_{(H^1(\Omega))'}, \quad \forall \, u^*, v^* \in (H^1(\Omega))', \quad (2.3)
\]

where \((\cdot, \cdot)_{(H^1(\Omega))'}\) is the dual scalar product in \((H^1(\Omega))'\) associated to the standard one in \( H^1(\Omega) \), and recall that \( \langle v^*, u \rangle = \int_{\Omega} v^* u \,dx \) if \( v^* \in H \) and we have, for every \( v^* \in H^1(0, T; (H^1(\Omega))') \),

\[
\frac{1}{2} \frac{d}{dt} \| v^* \|_{(H^1(\Omega))'}^2 = \left\langle \frac{\partial v^*}{\partial t}, A^{-1} v^* \right\rangle. \quad (2.4)
\]

It should be noted that we have the dense and continuous embeddings \( V \hookrightarrow H^1(\Omega) \hookrightarrow H \hookrightarrow H' = (H^1(\Omega))' \hookrightarrow V' \). Next, for a prescribed positive constant \( R \), we give an open set \( \Phi_R \subset L^\infty(0, T; H^1(\Omega)) \cap H^1(0, T; H) \), which satisfies \( \Phi_{ad} \subset \Phi_R \) and \( \| \phi \|_{L^\infty(0, T; H^1(\Omega))} + \| \phi \|_{H^1(0, T; H)} \leq R \) for any \( \phi \in \Phi_R \), where the admissible set \( \Phi_{ad} \triangleq \{ \phi \in L^\infty(Q) : \phi_{\min} \leq \phi \leq \phi_{\max} \text{ a.e. in } Q \} \). Hence \( \Phi_{ad} \) is a closed, bounded and convex set in \( \Phi_R \) for any fixed \( R \). We also see easily that \( \Phi_{ad} \) is nonempty and \( \phi_{\min} \in \Phi_{ad} \).

At this point, under the following assumptions

\[
\begin{align*}
\rho_1, \rho_2, \rho_3 & \text{ are the nonnegative constants but not all zero,} \quad (2.5) \\
u_Q & \in L^2(Q) \text{ and } u_\Omega \in L^2(\Omega) \text{ are given target functions,} \quad (2.6) \\
\phi_{\min} & \leq \phi_{\max} \text{ a.e. in } Q \text{ and } \phi_{\min}, \phi_{\max} \in L^\infty(Q), \quad (2.7)
\end{align*}
\]

we now can discuss the following distributed optimal control problem.
(CP) Minimize the cost functional
\[
\mathcal{J}(u, \phi) = \frac{\rho_1}{2} \| u - u_Q \|_{L^2(Q)}^2 + \frac{\rho_2}{2} \| u(T) - u_0 \|_{L^2(\Omega)}^2 + \frac{\rho_3}{2} \| \phi \|_{L^2(Q)}^2,
\]
subject to \( \Phi_d \) and to the state system (1.3)-(1.6) on proper the initial data. We shall be devoted to investigate more details in the subsequent section. Throughout this paper, the same letter \( C \) denotes constants which may change from line to line, or even in same line.

3. Existence and continuous dependence of the strong solution. In this section, we will discuss the well-posedness of (1.3)-(1.6). Since the state system (1.3)-(1.6) is equivalent to (1.7)-(1.11), first of all, we give the existence of weak solution to the state system (1.7)-(1.11). Then we discuss the existence and uniqueness of strong solutions and prove that the strong solution has continuous dependence on the control constraint term \( \phi \).

**Theorem 3.1.** Assume that \( v_0(x), u_0(x) \in W^{1,4}(\Omega) \) are satisfied. Let \( T > 0 \). Then, for every \( \phi \in \Phi_R \), the problem (1.7)-(1.11) admits a weak solution on \([0, T]\) such that

\[
v \in L^\infty(0, T; H^1(\Omega)), \quad u \in L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),
\]

(3.1)

\[
\frac{\partial v}{\partial t}, \frac{\partial u}{\partial t} \in L^2(0, T; V'), \quad \mu \in L^2(0, T; H),
\]

(3.2)

\[
v(0) = v(x, 0) = v_0(x), \quad u(0) = u(x, 0) = u_0(x), \quad \text{for} \quad x \in \Omega,
\]

(3.3)

which satisfies the following identities

\[
\int_0^T \left( \frac{\partial v}{\partial t}, \varphi \right) dt + \int_Q \nabla v \cdot \nabla \varphi dx dt + 2 \int_Q v^3 \varphi dx dt + 6 \int_Q u^2 v \varphi dx dt
\]

\[
+ (\alpha - 2\beta) \int_Q v \varphi dx dt = \int_Q \phi \varphi dx dt,
\]

(3.4)

\[
\int_0^T \left( \frac{\partial u}{\partial t}, \varphi \right) dt - \int_Q \mu \Delta \varphi dx dt = 0,
\]

(3.5)

for all \( \varphi \in L^2(0, T; V), \) where \( \mu = 2u^3 + 6u^2 - 2\beta u - \Delta u \) almost everywhere in \( Q \).

**Proof.** We shall use a Galerkin approximation method to prove the (3.1)-(3.5). Since the injection of \( V \) in \( H \) is compact, by a classical spectral theorem there exist a sequence of eigenvalues \( \lambda_j \) with \( 0 < \lambda_1 \leq \lambda_2 \leq \cdots \) and \( \lambda_j \to \infty \), and a family of eigenfunctions \( \varphi_j \in V \) such that \( A \varphi_j = \lambda_j \varphi_j (j = 1, 2, \cdots) \). The family of \( \varphi_j \) is an orthonormal basis in \( H \) and \( V \). For \( N \in \mathbb{N} \) fixed, let \( P_N \) be the orthogonal projectors in \( H \) onto the \( n \)-dimensional subspace \( V_N \hat{=} \{ \varphi_1, \varphi_2, \cdots, \varphi_N \} \) spanned by the \( n \) eigenfunctions. As for \( v_0(x), u_0(x) \in W^{1,4}(\Omega) \), there exist sequences of scalars \( (v^0_{N,j})_{N=1}^\infty \) and \( (u^0_{N,j})_{N=1}^\infty \) such that

\[
\lim_{N \to \infty} \sum_{j=1}^N v^0_{N,j} \varphi_j = v_0(x) \quad \text{and} \quad \lim_{N \to \infty} \sum_{j=1}^N u^0_{N,j} \varphi_j = u_0(x), \quad \text{in} \quad W^{1,4}(\Omega).
\]

(3.6)

Then we look for three functions of the form

\[
v^N(x, t) = \sum_{j=1}^N a^N_j(t) \varphi_j(x), \quad u^N(x, t) = \sum_{j=1}^N b^N_j(t) \varphi_j(x), \quad \mu^N(x, t) = \sum_{j=1}^N c^N_j(t) \varphi_j(x),
\]
that solve the following approximating problem

\[
\int_{\Omega} \frac{\partial v^N}{\partial t} \cdot \varphi_j(x)dx + \int_{\Omega} \nabla v^N \cdot \nabla \varphi_j(x)dx + 2 \int_{\Omega} (v^N)^3 \varphi_j(x)dx = -6 \int_{\Omega} (u^N)^2 v^N \varphi_j(x)dx + (2\beta - \alpha) \int_{\Omega} v^N \varphi_j(x)dx + \int_{\Omega} \phi \varphi_j(x)dx,
\]

(3.7)

\[
\int_{\Omega} \frac{\partial u^N}{\partial t} \cdot \varphi_j(x)dx = \int_{\Omega} \mu^N \Delta \varphi_j(x)dx,
\]

(3.8)

\[
\int_{\Omega} \mu^N \varphi_j(x)dx + 2\beta \int_{\Omega} u^N \varphi_j(x)dx = \int_{\Omega} \nabla u^N \cdot \nabla \varphi_j(x)dx + \int_{\Omega} 2(u^N)^3 \varphi_j(x)dx + \int_{\Omega} 26(v^N)^2 u^N \varphi_j(x)dx,
\]

(3.9)

\[
v^N(x, 0) = \sum_{j=1}^{N} v^N_{i,j} \varphi_j(x), \quad u^N(x, 0) = \sum_{j=1}^{N} u^N_{i,j} \varphi_j(x),
\]

(3.10)

for \( j = 1, 2, \cdots, N \). It is easy to see that solving the approximation problem (3.7)-(3.10) is equivalent to solving an initial value problem for a linear system of 2N ordinary differential equations in the 2N unknowns \( a^N_j(t), b^N_j(t) \). In this linear system, since all of the functions in \( (a^N_j(t), \cdots, b^N_j(t)) \) are continuous, the Cauchy-Peano Theorem [22] ensures that there exists \( T_N > 0 \) such that this linear system has a solution \( (a^N_j(t), \cdots, b^N_j(t)) \in C^1[0, T_N] \). Hence, the approximate problem (3.7)-(3.10) admits a unique local solution. We now deduce the basic estimates on the sequence of approximating solutions. In particular, these estimates will guarantee that global solution to the approximation problem (3.7)-(3.10) exists. In addition, we use the prime to denote the derivative with respect to time. Multiplying then (3.7) by \( (a^N_j(t))' \), (3.8) by \( c^N_j(t) \), (3.9) by \( (b^N_j(t))' \) and summing over \( j = 1, 2, \cdots, N \), we get the following identities satisfied

\[
\int_{\Omega} \left( \frac{\partial v^N}{\partial t} \right)^2 dx + \int_{\Omega} \nabla v^N \cdot \nabla \frac{\partial v^N}{\partial t} dx = \int_{\Omega} \left[ -2(v^N)^3 - 6(u^N)^2 v^N + (2\beta - \alpha) v^N + \phi \right] \cdot \frac{\partial v^N}{\partial t} dx,
\]

\[
\int_{\Omega} \frac{\partial u^N}{\partial t} \cdot \mu^N dx = \int_{\Omega} \mu^N \cdot \Delta \mu^N dx = -\int_{\Omega} \nabla \mu^N \cdot \nabla \mu^N dx,
\]

\[
\int_{\Omega} \mu^N \frac{\partial u^N}{\partial t} dx = \int_{\Omega} \nabla u^N \cdot \nabla \frac{\partial u^N}{\partial t} dx + \int_{\Omega} [2(u^N)^3 + 6(u^N)^2 v^N - 2\beta u^N] \frac{\partial u^N}{\partial t} dx.
\]

By integrating the above resulting identities in time between 0 and \( t \) and adding them, we obtain

\[
\int_{0}^{t} \int_{\Omega} \left( \frac{\partial v^N}{\partial t} \right)^2 dxds + \int_{0}^{t} \int_{\Omega} \nabla v^N \cdot \nabla \frac{\partial v^N}{\partial t} dxds + \int_{0}^{t} \int_{\Omega} |\nabla \mu^N|^2 dxds
\]

\[
+ \int_{0}^{t} \int_{\Omega} \nabla u^N \cdot \nabla \frac{\partial u^N}{\partial t} dxds + \int_{0}^{t} \int_{\Omega} [2(u^N)^3 \frac{\partial u^N}{\partial t} + 2(v^N)^3 \frac{\partial u^N}{\partial t}] dxds
\]
In addition, Young's inequality implies

\[ \int_0^t \int_\Omega [(\alpha - 2\beta)v^N \partial v^N \over \partial t] dx dt + 2\beta u^N \partial u^N \over \partial t] dx ds = \int_0^t \int_\Omega \Phi \partial v^N \over \partial t] dx ds. \]  

(3.11)

Owing to

\[ \int_0^t \int_\Omega [6(u^N)^2 \partial v^N \over \partial t] + 6(v^N)^2 u^N \partial u^N \over \partial t] dx ds = 3 \int_0^t \int_\Omega \partial [u^N]^2 (v^N)^2] dx ds, \]

Hence we obtain from (3.11) that

\[ \frac{1}{2} \int_\Omega \{u^N(x, t)^4 + [v^N(x, t)^4]dx - \beta \int_\Omega [u^N(x, t)]^2 dx \]

+ \frac{\alpha - 2\beta}{2} \int_\Omega [v^N(x, t)^2 dx + \int_\Omega v^N(x, 0)]^2 dx + \frac{\alpha - 2\beta}{2} \int_\Omega [v^N(x, 0)]^2 dx \]

+ \frac{1}{2} \int_\Omega \{u^N(x, 0)^4 + [v^N(x, 0)]^4]dx + \int_\Omega [v^N(x, 0)]^2 dx.

(3.12)

Owing to Cauchy's inequality, we have

\[ \frac{1}{2} \int_\Omega \phi^2 + (\partial v^N \over \partial t])^2] dx ds + \frac{3}{2} \int_\Omega [(v^N(x, 0))^4 + (u^N(x, 0))^4] dx. \]

In addition, Young's inequality implies

\[ \beta \int_\Omega (u^N(x, t))^2 dx - \alpha - 2\beta \int_\Omega (v^N(x, t))^2 dx + \frac{\alpha - 2\beta}{2} \int_\Omega (v^N(x, 0))^2 dx \]

\[ \leq \frac{1}{4} \int_\Omega [(u^N(x, t))^4 + (v^N(x, t))^4] dx + \int_\Omega [(v^N(x, 0))^4 + (u^N(x, 0))^4] dx + C, \]

where \( C \) denotes a non-negative constant depending on \( \alpha, \beta \) and \( |\Omega| \). We thus deduce from \( \Phi \in \Phi_R, v_0(x), u_0(x) \in W^{1,4}(\Omega) \) and (3.12) that

\[ \|v^N\|_{H^1(0, t; H)} + \|v^N\|_{L^\infty(0, t; H)} + \|v^N\|_{L^\infty(0, t; L^1(\Omega))} + \|\nabla u^N\|_{L^2(0, t; H)} \]

\[ \|\nabla u^N\|_{L^\infty(0, t; H)} + \|u^N\|_{L^\infty(0, t; L^1(\Omega))} \leq C, \]

where henceforth \( C \) denotes a non-negative constant depending on the norms of the initial data and \( \Phi, \Omega \). Obviously, it turns out that the system (3.7)-(3.10) admits a global solution. Let us now control the sequence of the averages of \( \mu^N \). From (3.9) and Young’s inequality we get

\[ |\mu^N, 1| = |2(u^N)^3 + 6(v^N)^2 u^N - 2\beta u^N, 1| \]
classical elliptic regularity result implies Therefore, on account of the homogeneous Dirichlet boundary condition for \( u_N \), we have \( \|\mu^N\|_{L^2(0,T;H^1(\Omega))} \leq C \). (3.14)

Next, we will prove that the sequence of \( u_N \) is controlled in \( L^2(0,T;H^2(\Omega)) \). Indeed, notice first that (3.9) can be written as

\[
\mu^N = -\Delta u_N + [2(u_N)^3 + 6(v_N)^2 u_N - 2\beta u_N],
\]

moreover we also know from Young’s inequality with \( \varepsilon \) and the Sobolev embedding theorem that

\[
\int\Omega [2(u_N)^3 + 6(v_N)^2 u_N - 2\beta u_N]^2 dx \\
= \int\Omega [4(u_N)^6 + 36(v_N)^4(u_N)^2 + 4\beta^2(u_N)^2 + 24(u_N)^4(v_N)^2 - 8\beta(u_N)^4] dx \\
- 24 \int\Omega (v_N)^2(u_N)^2 dx \\
\leq 4 \int\Omega |u_N|^6 dx + 36 \int\Omega (\varepsilon \cdot \frac{|u_N|^6}{3} + \varepsilon^{-\frac{1}{2}} \cdot 2\frac{|u_N|^6}{3}) dx + 4\beta^2 \int\Omega (|u_N)|^2 dx + 24 \int\Omega (\varepsilon \cdot \frac{|v_N|^6}{3} + \varepsilon^{-\frac{1}{2}} \cdot 2\frac{|v_N|^6}{3}) dx + 8\beta \int\Omega (|v_N)|^4 dx \\
+ 12 \int\Omega (|v_N|^4 + |u_N|^4) dx \\
\leq C(\varepsilon) \left[ \int\Omega |v_N|^6 + |u_N|^6 \right] dx + C \leq C(\varepsilon) (||u_N||_{H^1(\Omega)}^6 + ||v_N||_{H^1(\Omega)}^6) + C, \quad (3.15)
\]

here we choose \( \varepsilon = 4 \) and use (3.13), then we can write from (3.15) that

\[
\|2(u_N)^3 + 6(v_N)^2 u_N - 2\beta u_N\|_{L^\infty(0,T;H)} \leq C.
\]

Therefore, on account of the homogeneous Dirichlet boundary condition for \( u_N \), a classical elliptic regularity result implies

\[
\|u_N\|_{L^2(0,T;H^2(\Omega))} \leq C. \quad (3.16)
\]

For any \( \varphi \in L^2(0,T;V) \), we have \( P_N\varphi = \sum_{j=1}^N a_j(t)\varphi_j \), where \( a_j(t) = (\varphi,\varphi_j) \). Next we multiply (3.7) by \( a_j(t) \), (3.8) by \( a_j(t) \) and sum over \( j = 1,2,\cdots,N \). By integrating then in time between 0 and \( T \), we get the following identities satisfied

\[
\left| \int_0^T \int\Omega \frac{\partial v_N}{\partial t} \cdot \varphi dx dt \right| = \left| \int_0^T \int\Omega \frac{\partial u_N}{\partial t} \cdot P_N\varphi dx dt \right| \\
\leq \left| \int_0^T \int\Omega [-2(v_N)^3 - 6(v_N)^2 u_N + (2\beta - \alpha)\varphi + \phi] \cdot P_N\varphi dx dt \right| \\
+ \left| \int_0^T \int\Omega \nabla v_N \cdot \nabla P_N\varphi dx dt \right|, \quad (3.17)
\]
On account of (3.13), (3.16), (3.20) and the Aubin-Lions lemma (see [11]), we deduce that, up to a subsequence \( v^N \) and \( u^N \), not relabeled, and \( v, u \in L^\infty(0,T;H^1(\Omega)) \),
where we use the fact that the embedding $H^1(\Omega) \hookrightarrow L^p(\Omega)$ is compact for $1 \leq p < 6$ and $L^p(\Omega) \hookrightarrow (H^2(\Omega))'$ is continuous for $p \geq 1$. Furthermore, for any $\varphi \in L^2(0, T; H^1(\Omega))$, we obtain from (3.23) that
\[
\lim_{N \to \infty} \int_0^T \int_\Omega \nabla v^N \cdot \varphi dxdt = -\lim_{N \to \infty} \int_0^T \int_\Omega \nabla v \cdot \varphi dxdt
\]
\[
= -\lim_{N \to \infty} \int_0^T \int_\Omega v \cdot \varphi dxdt = \lim_{N \to \infty} \int_0^T \int_\Omega \nabla v \cdot \varphi dxdt,
\]
which implies
\[
\nabla v^N \rightharpoonup \nabla v, \quad \text{weakly in } L^2(0, T; (H^1(\Omega))').
\]
Owing to
\[
\int_0^T \int_\Omega |\nabla v^N|^2 dxdt \leq C\|v^N\|_{L^\infty(0, T; H^1(\Omega))}^2 \leq C,
\]
thus we can find a subsequence of $v^N$ which is again indexed by $n$ such that
\[
\nabla v^N \rightharpoonup \nabla v, \quad \text{weakly in } L^2(Q).
\]
Similarly, we can write that $\nabla u^N$ is bounded in $L^2(Q)$, so we have
\[
\nabla u^N \rightharpoonup \nabla u, \quad \text{weakly in } L^2(Q).
\]
In addition, for any $\varphi \in L^2(0, T; V)$, we deduce from (3.21), Young’s inequality and the Sobolev embedding theorem that
\[
\int_0^T \|(v^N)^2 + v^N v + v^2\|_2d\tau \leq C\int_0^T \|\varphi\|_{L^\infty(\Omega)}^2 (\|v^N\|_{4}^2 + \|v\|_{2}^2)d\tau
\]
\[
\leq C\int_0^T \|\varphi\|_{V'}^2d\tau + C\int_0^T \|v^N\|_{4}^2d\tau + C\int_0^T \|v\|_{2}^2d\tau
\]
\[
\leq C\int_0^T \|\varphi\|_{V'}^2d\tau + C\|v^N\|_{L^\infty(0, T; H^1(\Omega))}^2 + C\|v\|_{L^\infty(0, T; H^1(\Omega))}^2 \leq C.
\]
Hence we obtain that $[(v^N)^2 + v^N v + v^2]\varphi \in L^1(0, T; H)$. So we deduce from (3.21) that
\[
(v^N)^3 \rightharpoonup v^3, \quad \text{weakly in } L^2(0, T; V').
\]
Now we use a different idea to deal with \((u^N)^3\). By (3.24), (3.28), the Hölder inequality and the Sobolev embedding theorem, we get
\[
\int_0^T \| (u^N)^3 - v^3 \|^2_2 dt \\
\leq C \int_0^T \| u^N - u \|^6_6 \| v \|^4_{H^1(\Omega)} + \| u \|^4_{H^1(\Omega)} dt \leq C \int_0^T \| u^N - u \|^2_{H^1(\Omega)} dt.
\]
This implies that the following limit is correct when \(N \to \infty\)
\[
(u^N)^3 \to v^3, \quad \text{strongly in } L^2(Q).
\] (3.32)
Due to
\[
\int_0^T \int_{\Omega} |(u^N)^2 v^N - u^2 v|^2 dx dt \\
= \int_0^T \int_{\Omega} |(u^N)^2 (v^N - v) + u^N v (u^N - u) + uv (u^N - u)|^2 dx dt \\
\leq 2 \int_0^T \int_{\Omega} |(u^N)^2 (v^N - v)|^2 dx dt + 4 \int_0^T \int_{\Omega} |u^N v (u^N - u)|^2 dx dt \\
+ 4 \int_0^T \int_{\Omega} |uv (u^N - u)|^2 dx dt \\
\leq C \int_0^T \left( \int_{\Omega} |u^N|^6 dx \right)^{\frac{2}{3}} \left( \int_{\Omega} |v^N - v|^6 dx \right)^{\frac{1}{3}} dt + C \int_0^T \left( \int_{\Omega} |u^N|^3 dx \right)^{\frac{2}{3}} dt \\
\cdot \left( \int_{\Omega} |u^N - u|^6 dx \right)^{\frac{1}{3}} dt + C \int_0^T \left( \int_{\Omega} |uv|^3 dx \right)^{\frac{2}{3}} \left( \int_{\Omega} |u^N - u|^3 dx \right)^{\frac{1}{3}} dt \\
\leq C \| u^N \|^4_{L^4(0,T;L^6(\Omega))} \| v^N - v \|^2_{L^6(0,T;L^6(\Omega))} + C \| u^N \|^2_{L^3(0,T;L^3(\Omega))} \\
\cdot \| u^N - u \|^2_{L^6(0,T;L^6(\Omega))} + C \| uv \|^2_{L^3(0,T;L^3(\Omega))} \| u^N - u \|^2_{L^6(0,T;L^6(\Omega))} \\
\leq C \| u^N \|^4_{L^4(0,T;H^1(\Omega))} \| v^N - v \|^2_{L^6(0,T;H^1(\Omega))} + C \| u^N \|^2_{L^3(0,T;L^3(\Omega))} \| u^N - u \|^2_{L^6(0,T;L^6(\Omega))} \cdot \| u^N - u \|^2_{L^\infty(0,T;H^1(\Omega))} + C \| uv \|^2_{L^3(0,T;L^3(\Omega))} \| u^N - u \|^2_{L^6(0,T;L^6(\Omega))},
\]
and the Young’s inequality implies
\[
\| u^N v \|^2_{L^3(0,T;L^3(\Omega))} = \left( \int_0^T \int_{\Omega} |u^N v|^3 dx dt \right)^{\frac{2}{3}} \\
\leq C \left( \int_0^T \int_{\Omega} |u^N|^6 dx dt + \int_0^T \int_{\Omega} |v|^6 dx dt \right)^{\frac{2}{3}} \\
\leq C (\| u^N \|^\frac{4}{3}_{L^\infty(0,T;H^1(\Omega))} + \| v \|^\frac{4}{3}_{L^\infty(0,T;H^1(\Omega))}) \leq C.
\]
We can obtain that \(\| uv \|^2_{L^3(0,T;L^3(\Omega))} \leq C\) in same way. According to (3.21) and (3.24), we thus have
\[
(u^N)^2 v^N \to u^2 v, \quad \text{weakly in } L^2(Q).
\] (3.33)
Clearly, we also write by similar discussion that
\[
(v^N)^2 u^N \to v^2 u, \quad \text{weakly in } L^2(Q).
\] (3.34)
Besides, the Poincaré inequality and (3.13) imply
\[
\int_0^T \int_\Omega |\mu^N|^2 \, dx \, dt = C \int_0^T \int_\Omega |\nabla \mu^N|^2 \, dx \, dt \leq C,
\]
we thus can find a subsequence of \( v^N \) which is again indexed by \( n \) such that
\[
\mu^N \to \mu, \quad \text{weakly in } L^2(Q).
\] (3.35)

For any \( \varphi \in L^2(0, T; H^1(\Omega) \cap V) \), we have \( \mathcal{P}_m \varphi = \sum_{j=1}^m a_j(t)\varphi_j \), where \( a_j(t) = (\varphi, \varphi_j) \in L^2(0, T) \). Then \( \mathcal{P}_m \varphi \) converges strongly to \( \varphi \) in \( L^2(0, T; H^1(\Omega) \cap V) \). Now we multiply (3.7), (3.8), (3.9) by \( a_j(t) \) respectively, and integrate over \([0, T]\), so we can write that
\[
\int_0^T \left\langle \frac{\partial v^N}{\partial t} \cdot a_j(t) \varphi_j \right\rangle \, dt + \int_0^T \int_\Omega \nabla v^N \cdot \nabla a_j(t) \varphi_j \, dx \, dt
= \int_0^T \int_\Omega [-2v^N + (2\beta - \alpha)v + \phi] \cdot a_j(t) \varphi_j \, dx \, dt,
\] (3.36)
\[
\int_0^T \left\langle \frac{\partial u^N}{\partial t} \cdot a_j(t) \varphi_j \right\rangle \, dt = \int_0^T \int_\Omega \mu^N \cdot \Delta a_j(t) \varphi_j \, dx \, dt,
\] (3.37)
\[
\int_0^T \int_\Omega \mu^N \cdot a_j(t) \varphi_j \, dx \, dt - \int_0^T \int_\Omega \nabla u^N \cdot \nabla a_j(t) \varphi_j \, dx \, dt
= \int_0^T \int_\Omega [2v^N]^3 + 6(v^N)^2u^N - 2\beta u^N \cdot a_j(t) \varphi_j \, dx \, dt.
\] (3.38)

According to (3.21), (3.22), (3.24), (3.25) and (3.29)-(3.35), as \( N \to \infty \), we see from (3.36)-(3.38) that
\[
\int_0^T \left\langle \frac{\partial v}{\partial t} \cdot a_j(t) \varphi_j \right\rangle \, dt + \int_0^T \int_\Omega \nabla v \cdot \nabla a_j(t) \varphi_j \, dx \, dt
= \int_0^T \int_\Omega [-2v^3 - 6u^2v + (2\beta - \alpha)v + \phi] \cdot a_j(t) \varphi_j \, dx \, dt,
\] (3.39)
\[
\int_0^T \left\langle \frac{\partial u}{\partial t} \cdot a_j(t) \varphi_j \right\rangle \, dt = \int_0^T \int_\Omega \mu \cdot \Delta a_j(t) \varphi_j \, dx \, dt,
\] (3.40)
\[
\int_0^T \int_\Omega \mu \cdot a_j(t) \varphi_j \, dx \, dt \leq \int_0^T \int_\Omega \nabla u \cdot \nabla a_j(t) \varphi_j \, dx \, dt
+ \int_0^T \int_\Omega (2v^3 + 6v^2 - 2\beta u) \cdot a_j(t) \varphi_j \, dx \, dt.
\] (3.41)

By summing over \( j = 1, 2, \cdots, m \) on both sides of (3.39)-(3.41), then taking the limit of these resulting identities, we obtain
\[
\int_0^T \left\langle \frac{\partial v}{\partial t} \cdot \varphi \right\rangle \, dt + \int_0^T \int_\Omega \nabla v \cdot \nabla \varphi \, dx \, dt
= \int_0^T \int_\Omega [-2v^3 - 6u^2v + (2\beta - \alpha)v + \phi] \cdot \varphi \, dx \, dt,
\] (3.42)
\[
\int_0^T \left\langle \frac{\partial u}{\partial t} \cdot \varphi \right\rangle \, dt = \int_0^T \int_\Omega \mu \cdot \Delta \varphi \, dx \, dt,
\] (3.43)
\[
\int_0^T \int_\Omega \mu \cdot \varphi \, dxdt = \int_0^T \int_\Omega (\nabla u \cdot \nabla \varphi + (2u^3 + 6v^2 - 2\beta u) \cdot \varphi) \, dxdt. \tag{3.44}
\]
Furthermore, we write from (3.44) and \(u \in L^2(0, T; H^2(\Omega))\) that
\[
\mu = -\Delta u + 2u^3 + 6v^2 - 2\beta u, \quad \text{a.e. in } Q.
\]
In addition, we also obtain from (3.6) and (3.10) that
\[
v^N(x, 0) \to v_0(x), \quad u^N(x, 0) \to u_0(x), \quad \text{in } W^{1,4}(\Omega), \quad \text{as } N \to \infty.
\]
Finally, we deduce from (3.23) and (3.27) that (3.3) is established. The proof is complete. \(\square\)

**Theorem 3.2.** Assume that \(v_0(x), u_0(x) \in W^{1,4}(\Omega) \cap H^2(\Omega)\) are satisfied. Let \(T > 0\). Then, for every \(\phi \in \Phi_R\), the problem (1.7)-(1.11) admits a strong solution on \([0, T]\) such that
\[
\frac{\partial v}{\partial t} \in L^\infty(0, T; H) \cap L^2(0, T; H^1_0(\Omega)), \quad \frac{\partial u}{\partial t} \in L^\infty(0, T; H^{-1}(\Omega)) \cap L^2(0, T; H^1_0(\Omega)),
\]
\[
v \in L^\infty(0, T; H^2(\Omega)), \quad u \in L^\infty(0, T; H^2(\Omega)), \tag{3.45}
\]
\[
\mu \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad \Delta u \in L^2(0, T; V), \tag{3.46}
\]
which satisfies the following estimate
\[
\left\| \frac{\partial v}{\partial t} \right\|_{L^\infty(0,T;H)} + \left\| \frac{\partial u}{\partial t} \right\|_{L^2(0,T;H^1_0(\Omega))} + \left\| \partial u \right\|_{L^\infty(0,T;H)} + \left\| \partial u \right\|_{L^2(0,T;H^1_0(\Omega))} + \left\| v \right\|_{L^\infty(0,T;H^2(\Omega))} + \left\| u \right\|_{L^\infty(0,T;H^2(\Omega))} + \left\| \Delta u \right\|_{L^2(0,T;V)}
\]
\[
+ \left\| \mu \right\|_{L^\infty(0,T;H)} + \left\| \mu \right\|_{L^2(0,T;V)} \leq C. \tag{3.47}
\]
Furthermore, let \((v_1, u_1, \mu_1)\) and \((v_2, u_2, \mu_2)\) are two strong solutions corresponding to the control terms \(\phi_1\) and \(\phi_2\) respectively. Then for \(\phi_1, \phi_2 \in \Phi_R\) and all \(t \in [0, T]\), we get
\[
\|v_1 - v_2\|_{H^1(0,t;H)} + \|v_1 - v_2\|_{L^\infty(0,t;H^1(\Omega))} + \|u_1 - u_2\|_{L^2(0,t;H^2(\Omega))}
\]
\[
+ \|u_1 - u_2\|_{L^\infty(0,t;H^2(\Omega))} + \|u_1 - u_2\|_{L^\infty(0,t;H^2(\Omega))} + \|u_1 - u_2\|_{L^2(0,t;H^2(\Omega))}
\]
\[
+ \|\mu_1 - \mu_2\|_{L^2(0,t;H)} \leq C\|\phi_1 - \phi_2\|_{L^2(0,t;H)}. \tag{3.48}
\]

**Proof.** First of all, we multiply (1.7) by \(\frac{\partial v}{\partial t}\) and then integrate the resulting identities over \(\Omega \times (0, t)\), namely
\[
\int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dxdt + \int_0^t \int_\Omega \nabla v \cdot \nabla \frac{\partial v}{\partial t} \, dxdt + 2 \int_0^t \int_\Omega v^3 \cdot \frac{\partial v}{\partial t} \, dxdt
\]
\[
= \int_0^t \int_\Omega [-6u^2v + (2\beta - \alpha)v + \phi] \cdot \frac{\partial v}{\partial t} \, dxdt. \tag{3.49}
\]
Owing to (3.13), Young’s inequality and the Sobolev embedding theorem, we get
\[
\int_0^t \int_\Omega \left| \frac{\partial v}{\partial t} \right|^2 \, dxdt \leq 6 \int_0^t \int_\Omega |u^2v| \cdot \left| \frac{\partial v}{\partial t} \right| \, dxdt + |2\beta - \alpha| \int_0^t \int_\Omega \left| v \right| \cdot \left| \frac{\partial v}{\partial t} \right| \, dxdt + \int_0^t \int_\Omega \left| \phi \right| \cdot \left| \frac{\partial v}{\partial t} \right| \, dxdt
\]
\[
\leq \frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dxdt + C \int_0^t \int_\Omega \left| u^2v \right|^2 + |v|^2 + |\phi|^2 \, dxdt
\]
\[
\begin{align*}
\leq \frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dx \, ds + C \int_0^t \int_\Omega \left[ |u|^6 + |v|^6 + |v|^2 + |\phi|^2 \right] \, dx \, ds \\
\leq \frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dx \, ds + C \left( \|u\|_{L^\infty(0,T;H^1(\Omega))} + \|v\|_{L^\infty(0,T;H^1(\Omega))} + \|\phi\|_{L^2(0,T;H^1)} \right) \\
\leq \frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dx \, ds + C.
\end{align*}
\]

Hence (3.49) implies
\[
\int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dx \, ds + \frac{1}{2} \int_\Omega |\nabla v|^2 \, dx + \frac{1}{2} \int_0^t v^4 \, dx
\leq \frac{1}{2} \int_\Omega |\nabla v(0)|^2 \, dx + \frac{1}{2} \int_\Omega (v(0))^4 \, dx + \frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dx \, ds + C.
\]

Recalling \( v(0) = v_0(x) \in W^{1,4}(\Omega) \cap H^2(\Omega) \), therefore we infer that
\[
\|v\|_{H^1(0,T;H)} + \|v\|_{L^\infty(0,T;H^1(\Omega))} \leq C. \tag{3.50}
\]

Next, we rewrite (1.8) in the equivalent form
\[
(-\Delta)^{-1} \frac{\partial u}{\partial t} = -\mu. \tag{3.51}
\]

Now we multiply (1.7) by \((-\Delta) \frac{\partial u}{\partial t}\) and (3.51) by \((-\Delta) \frac{\partial u}{\partial t}\), integrate over \(\Omega\) and then sum the the resulting identities, that is
\[
\begin{align*}
\frac{d}{dt} \left( \frac{1}{2} \|\Delta v\|_2^2 + \frac{1}{2} \|\Delta u\|_2^2 \right) + \left\| \nabla \nabla v \right\|_2^2 + \left\| \frac{\partial u}{\partial t} \right\|_2^2 \\
= \int_\Omega [-2v^3 - 6u^2v^2 + (2\beta - \alpha) v] \cdot (-\Delta) \frac{\partial v}{\partial t} \, dx + \int_\Omega [2u^3 + 6uv^2 - 2\beta u] \cdot \frac{\partial \Delta u}{\partial t} \, dx \\
+ \int_\Omega \phi \cdot (-\Delta) \frac{\partial v}{\partial t} \, dx
\end{align*}
\]
\[
\leq \frac{1}{2} \int_\Omega \left| \frac{\partial \nabla v}{\partial t} \right|^2 \, dx + \frac{1}{2} \int_\Omega \left| \frac{\partial u}{\partial t} \right|^2 \, dx + C \int_\Omega |\nabla [-2v^3 - 6u^2v^2 + (2\beta - \alpha)v]|^2 \, dx \\
+ C \int_\Omega |\Delta [2u^3 + 6uv^2 - 2\beta u]|^2 \, dx + C \int_\Omega |\nabla \phi|^2 \, dx. \tag{3.52}
\]

Hence \( \phi \in \Phi_R \) and (3.52) yield
\[
\frac{d}{dt} \left( \|\Delta v\|_2^2 + \|\Delta u\|_2^2 \right) + \left\| \nabla \nabla v \right\|_2^2 + \left\| \frac{\partial u}{\partial t} \right\|_2^2 \leq Q\left(\|u\|_{H^2(\Omega)}, \|v\|_{H^2(\Omega)}\right), \tag{3.53}
\]

where the same letter \( Q \) denotes monotone increasing (with respect to each argument) functions. Now it follows from the paper [15] that there exists \( T_0 \) such that, for \( t \leq T_0 \), we obtain
\[
\|u(t)\|_{H^2(\Omega)}^2 + \|v(t)\|_{H^2(\Omega)}^2 \leq Q\left(\|u(0)\|_{H^2(\Omega)}, \|v(0)\|_{H^2(\Omega)}\right).
\]

We next need to prove that \( u, v \in L^\infty(T_0, T; H^2(\Omega)) \). First of all, we differentiate equations (1.7), (3.51) and (1.9) with respect to time, namely
\[
\frac{\partial}{\partial t} \left( \frac{\partial v}{\partial t} \right) = -6u^2 \frac{\partial v}{\partial t} - 12uv \frac{\partial u}{\partial t} - 6u^2 \frac{\partial v}{\partial t} + (2\beta - \alpha) \frac{\partial v}{\partial t} + \Delta \frac{\partial v}{\partial t} + \frac{\partial \phi}{\partial t}. \tag{3.54}
\]
On account of Cauchy’s inequality, we have
\[
\frac{\partial}{\partial t} \left( (-\Delta)^{-1} \frac{\partial u}{\partial t} \right) = - \frac{\partial u}{\partial t},
\]
(3.55)
\[
\frac{\partial \mu}{\partial t} = 6u^2 \frac{\partial u}{\partial t} + 12uv \frac{\partial v}{\partial t} + 6v^2 \frac{\partial u}{\partial t} - 2\beta \frac{\partial u}{\partial t} + \Delta \frac{\partial u}{\partial t}.
\]
(3.56)

We multiply (3.54) by \(\frac{\partial v}{\partial t}\), and (3.55) by \(\frac{\partial u}{\partial t}\), integrate over \(\Omega\) and then sum the resulting identities, that is
\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( \left\| \frac{\partial v}{\partial t} \right\|_{L^2}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2 \right) + \left\| \frac{\partial \nabla v}{\partial t} \right\|_{L^2}^2 + \left\| \frac{\partial \nabla u}{\partial t} \right\|_{L^2}^2 + 6 \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 dx \\
+ 6 \int_\Omega u^2 \left( \frac{\partial v}{\partial t} \right)^2 dx + 6 \int_\Omega v^2 \left( \frac{\partial u}{\partial t} \right)^2 dx + 6 \int_\Omega v^2 \left( \frac{\partial \nabla u}{\partial t} \right)^2 dx \\
= \int_\Omega \left[ -24uv \frac{\partial u}{\partial t} \cdot \frac{\partial v}{\partial t} + (2\beta - \alpha) \left( \frac{\partial v}{\partial t} \right)^2 + 2\beta \left( \frac{\partial u}{\partial t} \right)^2 \right] dx + \int_\Omega \frac{\partial \phi}{\partial t} \cdot \frac{\partial v}{\partial t} dx.
\end{aligned}
\]

On account of Cauchy’s inequality, we have
\[
\begin{aligned}
\int_\Omega -24uv \frac{\partial u}{\partial t} \cdot \frac{\partial v}{\partial t} dx &= \int_\Omega \left( -12uv \frac{\partial u}{\partial t} \cdot \frac{\partial v}{\partial t} - 12uv \frac{\partial v}{\partial t} \cdot \frac{\partial u}{\partial t} \right) dx \\
\leq 12 \left[ \int_\Omega \left( \frac{1}{2} u^2 \left( \frac{\partial u}{\partial t} \right)^2 + \frac{1}{2} v^2 \left( \frac{\partial v}{\partial t} \right)^2 \right) \right] dx + 12 \left[ \int_\Omega \left( \frac{1}{2} u^2 \left( \frac{\partial u}{\partial t} \right)^2 + \frac{1}{2} v^2 \left( \frac{\partial v}{\partial t} \right)^2 \right) \right] dx \\
\leq 6 \int_\Omega \left( \frac{\partial u}{\partial t} \right)^2 dx + 6 \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 dx + 6 \int_\Omega \left( \frac{\partial u}{\partial t} \right)^2 dx + 6 \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 dx,
\end{aligned}
\]
and
\[
\int_\Omega \frac{\partial \phi}{\partial t} \cdot \frac{\partial v}{\partial t} dx \leq \frac{1}{2} \int_\Omega \left( \frac{\partial \phi}{\partial t} \right)^2 dx + \frac{1}{2} \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 dx,
\]
which yields,
\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( \left\| \frac{\partial v}{\partial t} \right\|_{L^2}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2 \right) + \left\| \frac{\partial \nabla v}{\partial t} \right\|_{L^2}^2 + \left\| \frac{\partial \nabla u}{\partial t} \right\|_{L^2}^2 + 6 \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 dx \\
\leq C \int_\Omega \left( \frac{\partial \phi}{\partial t} \right)^2 dx + C \int_\Omega \left( \frac{\partial u}{\partial t} \right)^2 dx + C \left\| \frac{\partial v}{\partial t} \right\|_{L^2}^2,
\end{aligned}
\]
where we use the interpolation inequality. We thus deduce that
\[
\begin{aligned}
\frac{1}{2} \frac{d}{dt} \left( \left\| \frac{\partial v}{\partial t} \right\|_{L^2}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2 \right) + 2 \left\| \frac{\partial \nabla v}{\partial t} \right\|_{L^2}^2 + \left\| \frac{\partial \nabla u}{\partial t} \right\|_{L^2}^2 + 6 \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 dx \\
\leq C \int_\Omega \left( \frac{\partial \phi}{\partial t} \right)^2 dx + C \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2 + C \left\| \frac{\partial \nabla u}{\partial t} \right\|_{L^2}^2.
\end{aligned}
\]
(3.57)

Applying the Gronwall inequality and \(\phi \in \Phi_R\) to (3.57) we find that \(\left\| \frac{\partial v}{\partial t} \right\|_{L^2}^2 + \left\| \frac{\partial u}{\partial t} \right\|_{L^2}^2 \in L^\infty(0, T)\). Now we multiply (1.7) by \(-\Delta v\), (3.51) by \(-\Delta u\) and (1.9) by \(-\Delta u\), integrate over \(\Omega\) and then sum the resulting identities, then we obtain from Young’s inequality that
\[
\left\| \Delta u \right\|_{L^2}^2 + \left\| \Delta v \right\|_{L^2}^2.
\]
The (3.58) implies thus we can write from the Sobolev embedding theorem and (3.45) that
\[
\frac{\partial u}{\partial t}(1.8) \text{ by }
\]

\[
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\]

\[
\text{have completed the proof that } u, v \text{ in } L^\infty(T_0, T; H^2(\Omega)). \text{ At this point we have completed the proof that } u, v \text{ in } L^\infty(0, T; H^2(\Omega)) \text{ for } T > 0. \text{ Now we multiply (1.8) by } \frac{\partial u}{\partial t} \text{ and then integrate the resulting identities over } \Omega \times (0, t), \text{ namely}
\]

\[
\int_0^t \int_0^1 \left( \frac{\partial u}{\partial t} \right)^2 dx ds = \int_0^t \int_0^1 \Delta [2u^3 + 6uv^2 - 2\beta u - \Delta u] \cdot \frac{\partial u}{\partial t} ds.
\]

Applying Young’s inequality, we get
\[
\int_0^t \int_0^1 \left( \frac{\partial u}{\partial t} \right)^2 ds \leq - \int_0^t \int_0^1 \Delta u \cdot \frac{\partial \Delta u}{\partial t} ds + \frac{1}{2} \int_0^t \int_0^1 \left( \frac{\partial u}{\partial t} \right)^2 ds + C \int_0^t \int_0^1 \Delta [2u^3 + 6uv^2 - 2\beta u]^2 dx ds,
\]

thus we can write from the Sobolev embedding theorem and (3.45) that
\[
\frac{1}{2} \int_0^t \int_0^1 \left( \frac{\partial u}{\partial t} \right)^2 ds + \int_0^t \int_0^1 \Delta u \cdot \frac{\partial \Delta u}{\partial t} ds \leq C \int_0^T \int_\Omega (\Delta [2u^3 + 6uv^2 - 2\beta u])^2 dx dt
\]

\[
\leq C \int_0^T \int_\Omega (u|\nabla u|^2 + u^2 \Delta u + u^2 \Delta u + u|\nabla v|^2 + uv \Delta u + v \nabla u \cdot \nabla u + \Delta u)^2 dx dt
\]

\[
\leq C \int_0^T \int_\Omega Q(||u||_{L^\infty(\Omega)}, ||v||_{L^\infty(\Omega)}) \int_\Omega (||\nabla u|^2 + \Delta u + \Delta u + |\nabla v|^2 + \Delta v + \nabla u \cdot \nabla v + \Delta u)^2 dx dt
\]

\[
\leq Q(||u||_{L^\infty(0, T; H^2(\Omega))}, ||v||_{L^\infty(0, T; H^2(\Omega))}) \int_0^T \int_\Omega (||\nabla u|^2 + \Delta u + \Delta u + |\nabla v|^2 + \Delta v + \nabla u \cdot \nabla v + \Delta u)^2 dx dt
\]

\[
\leq Q(||u||_{L^\infty(0, T; H^2(\Omega))}, ||v||_{L^\infty(0, T; H^2(\Omega))}) \leq C.
\]

(3.58)

The (3.58) implies
\[
\frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial u}{\partial t} \right)^2 ds + \frac{1}{2} \int_\Omega (\Delta u)^2 dx \leq \frac{1}{2} \int_\Omega (\Delta u(0))^2 dx + C \leq C.
\]

We thus see that
\[
\left\| \frac{\partial u}{\partial t} \right\|_{L^2(0, T; H)} \leq C.
\]

(3.59)
In addition, because of (3.59), we apply parabolic regularity theory to (3.43) when 
$v(0) = v_0(x) \in W^{1,4}(\Omega) \cap H^2(\Omega)$, then we have $\mu \in L^2(0,T;V)$. Next we know from (1.9) that
\[
\frac{\partial \mu}{\partial t} = 6u^2 \frac{\partial u}{\partial t} + 12 uv \frac{\partial v}{\partial t} + 6v^2 \frac{\partial u}{\partial t} - 2\beta \frac{\partial u}{\partial t} - \Delta \frac{\partial u}{\partial t}.
\]
We multiply (1.9) by $\frac{\partial u}{\partial t}$ and then integrate the resulting identity over $\Omega \times (0,t)$, which yields
\[
\int_0^t \int_\Omega \mu \cdot \frac{\partial \mu}{\partial t} \, dx \, ds = \int_0^t \int_\Omega [2u^3 + 6uv^2 - 2\beta u - \Delta u] \cdot \frac{\partial \mu}{\partial t} \, dx, \nonumber
\]
it implies from the Sobolev embedding theorem that
\[
\frac{1}{2} \int_\Omega (\mu(t))^2 \, dx = \frac{1}{2} \int_\Omega (\mu(0))^2 \, dx + \int_0^t \int_\Omega [2u^3 + 6uv^2 - 2\beta u - \Delta u] \cdot \frac{\partial \mu}{\partial t} \, dx \, ds
\]
\[
= \frac{1}{2} \int_\Omega (\mu(0))^2 \, dx + 2\beta \int_\Omega u^2 \, dx + \int_\Omega (u(0))^4 \, dx + 4\beta \int_\Omega (u(0))^4 \, dx - \int_\Omega \beta \frac{\partial u}{\partial t} \, dx
\]
\[
+ 18(\int_\Omega u^2 \, dx - \int_\Omega (u(0))^2 \, dx) + 12\beta \int_\Omega (u(0))^2 (v(0))^2 \, dx - \int_\Omega u^2 (v(0))^2 \, dx
\]
\[
+ 2\beta^2 \int_\Omega u^2 \, dx - \int_\Omega (u(0))^2 \, dx + 12\beta \int_\Omega (u(0))^2 (v(0))^2 \, dx - \int_\Omega u^2 (v(0))^2 \, dx
\]
\[
+ 18\int_\Omega u^2 \, dx - \int_\Omega (u(0))^2 \, dx + 12\int_\Omega u^2 (v(0))^2 \, dx - \int_\Omega (u(0))^4 (v(0))^2 \, dx
\]
\[
+ 2\beta \int_\Omega u^2 \, dx - \int_\Omega (u(0))^2 \, dx + 12\beta \int_\Omega u^2 (v(0))^2 \, dx - \int_\Omega (u(0))^4 (v(0))^2 \, dx
\]
\[
\leq C\|u\|_{H^2(\Omega)}^2 + \|v\|_{H^2(\Omega)}^2 + \|u(0)\|_{H^2(\Omega)}^2 + \|v(0)\|_{H^2(\Omega)}^2,
\]
we thus deduce from (3.45) that $\mu \in L^\infty(0,T;H)$. Moreover we also obtain from (1.9) and (1.10) that
\[
\Delta u = 2u^3 + 6uv^2 - 2\beta u - \mu = 0, \quad \text{a.e. } \Sigma,
\]
\[
\Delta \mu = \Delta(2u^3 + 6uv^2 - 2\beta u - \Delta u) = -\Delta^2 u - 2\beta \Delta u + \Delta(2u^3 + 6uv^2),
\]
the second equality implies that
\[
\Delta^2 u = -\Delta \mu - 2\beta \Delta u + 6u|\nabla u|^2 + 3u^2 \Delta u + v^2 \Delta u + 4v \nabla u \cdot \nabla v
\]
\[
+ 2u|\nabla v|^2 + 2uv \Delta v, \quad \text{a.e. } Q.
\]
Owing to
\[
\int_0^T \int_\Omega \#\Delta \mu - 2\beta \Delta u + 6u|\nabla u|^2 + 3u^2 \Delta u + v^2 \Delta u + 4v \nabla u \cdot \nabla v
\]
\[
+ 2u|\nabla v|^2 + 2uv \Delta v \, dx \, dt
\]
\[
\leq C \int_0^T \int_\Omega (\Delta \mu)^2 \, dx \, dt + C \int_0^T \int_\Omega (\Delta u)^2 \, dx \, dt + C \int_0^T \int_\Omega (\nabla u)^4 \, dx \, dt
\]
\[
+ C \int_0^T \int_\Omega (\nabla u)^2 \, dx \, dt + C \int_0^T \int_\Omega \nabla^4 (\Delta u)^2 \, dx \, dt \leq C\|\mu\|_{L^2(0,T;V)}^2 + C\|u\|_{L^\infty(0,T;H^2(\Omega))}^2 + C\|u\|_{L^2(0,T;H^2(\Omega))}^2 \|u\|_{L^\infty(0,T:H^1(\Omega))}
\]
Next, we rewrite (3.61) like the form (3.51), we multiply (3.60) by $\partial v$, yields

Because of

we apply elliptic regularity theory and then obtain that $\Delta u \in L^2(0, T; V)$. So we prove that (3.47). We now deduce the estimates (3.48). Let $(v_1, u_1, \mu_1)$ and $(v_2, u_2, \mu_2)$ be two strong solutions with the control $\phi_1$ and $\phi_2$, respectively. We set $v = v_1 - v_2$, $u = u_1 - u_2$, $\mu = \mu_1 - \mu_2$ and have

\[
\frac{\partial v}{\partial t} = -2(v_1^3 - v_2^3) - 6(u_1^2 v_1 - u_2^2 v_2) + (2\beta - \alpha)v + \Delta v + \phi, \quad \text{a.e. in } Q, \quad (3.60)
\]

\[
\frac{\partial u}{\partial t} = \Delta u, \quad \text{a.e. in } Q, \quad (3.61)
\]

\[
\mu = 2(u_1^3 - u_2^3) + 6(u_1^2 v_1 - u_2^2 v_2) - 2\beta u - \Delta u, \quad \text{a.e. in } Q, \quad (3.62)
\]

\[
v = u = \mu = 0, \quad \text{a.e. on } \Sigma, \quad (3.63)
\]

\[
v(0) = v(x, 0) = 0, \quad u(0) = u(x, 0) = 0, \quad \text{a.e. in } \Omega. \quad (3.64)
\]

Next, we rewrite (3.61) like the form (3.51), we multiply (3.60) by $\frac{\partial v}{\partial t}$, (3.51) by $\frac{\partial u}{\partial t}$ and (3.62) by $\frac{\partial u}{\partial t}$ and then integrate the resulting identity over $\Omega \times (0, t)$, which yields

\[
\int_0^t \int_{\Omega} \left( \frac{\partial v}{\partial t} \right)^2 dx ds + \int_0^t \int_{\Omega} \nabla v \cdot \nabla \frac{\partial v}{\partial t} dx ds + \int_0^t \int_{\Omega} (-\Delta)^{-\frac{1}{2}} \frac{\partial u}{\partial t} \frac{\partial u}{\partial t} dx ds
\]

\[
+ \int_0^t \int_{\Omega} \nabla u \cdot \nabla \frac{\partial u}{\partial t} dx ds
\]

\[
= \int_0^t \int_{\Omega} \left[ -2(v_1^3 - v_2^3) - 6(u_1^2 v_1 - u_2^2 v_2) + (2\beta - \alpha)v + \phi \right] \frac{\partial v}{\partial t} dx ds
\]

\[
- \int_0^t \int_{\Omega} \left[ 2(u_1^3 - u_2^3) + 6(u_1^2 v_1 - u_2^2 v_2) - 2\beta u \right] \frac{\partial u}{\partial t} dx ds.
\]

Because of

\[
\int_0^t \int_{\Omega} \left[ -2(v_1^3 - v_2^3) - 6(u_1^2 v_1 - u_2^2 v_2) \right] \frac{\partial v}{\partial t} dx ds
\]

\[
- \int_0^t \int_{\Omega} \left[ 2(u_1^3 - u_2^3) + 6(u_1^2 v_1 - u_2^2 v_2) \right] \frac{\partial u}{\partial t} dx ds
\]

\[
= \int_0^t \int_{\Omega} (-\Delta)^{-\frac{1}{2}} \left[ -2(v_1^3 - v_2^3) - 6(u_1^2 v_1 - u_2^2 v_2) \right] \cdot (-\Delta)^{-\frac{1}{2}} \frac{\partial v}{\partial t} dx ds
\]

\[
+ \int_0^t \int_{\Omega} (-\Delta)^{-\frac{1}{2}} \left[ 2(u_1^3 - u_2^3) + 6(u_1^2 v_1 - u_2^2 v_2) \right] \cdot (-\Delta)^{-\frac{1}{2}} \frac{\partial u}{\partial t} dx ds
\]

\[
\leq C \int_0^t \|\nabla [-2(v_1^3 - v_2^3) - 6(u_1^2 v_1 - u_2^2 v_2)]\|_2 \cdot \left\| \frac{\partial v}{\partial t} \right\|_{-1} ds
\]

\[
+ C \int_0^t \|\nabla [-2(u_1^3 - u_2^3) - 6(u_1^2 v_1 - u_2^2 v_2)]\|_2 \cdot \left\| \frac{\partial u}{\partial t} \right\|_{-1} ds,
\]
we thus deduce from the Sobolev embedding theorem and Young's inequality that
\[
C \int_0^t \int_{\Omega} \left( \frac{\partial v}{\partial t} \right)^2 dx dt + \int_0^t \int_{\Omega} \nabla v \cdot \nabla \frac{\partial v}{\partial t} dx dt + C \int_0^t \int_{\Omega} (\Delta)^{-\frac{1}{2}} \frac{\partial^2 v}{\partial t^2} \cdot \frac{\partial u}{\partial t} dx dt
\]
\[
+ \int_0^t \int_{\Omega} \nabla u \cdot \nabla \frac{\partial u}{\partial t} dx dt
\]
\[
\leq C \int_0^t \int_{\Omega} \phi^2 dx ds + C \int_0^t \| \nabla [-2(v_1^3 - v_2^3) - 6(u_1^2v_1 - u_2^2v_2)] \|_2^2 ds
\]
\[
+ C \int_0^t \| \nabla [-2(u_1^3 - u_2^3) - 6(u_1v_1^2 - u_2v_2^2)] \|_2^2 ds + C \int_0^t \int_{\Omega} v^2 dx ds
\]
\[
+ C \int_0^t \int_{\Omega} w^2 dx ds.
\]
(3.65)
Owing to
\[
\| \nabla [-2(v_1^3 - v_2^3) - 6(u_1^2v_1 - u_2^2v_2)] \|_2
\]
\[
= \| \nabla \left[ (u_1 - v_1)^3 - (u_1 + v_1)^3 - (u_2 - v_2)^3 + (u_2 + v_2)^3 \right] \|_2
\]
\[
\leq \| \nabla \left[ (u_1 - v_1)^3 - (u_2 - v_2)^3 \right] \|_2 + \| \nabla \left[ (u_2 + v_2)^3 - (u_1 + v_1)^3 \right] \|_2
\]
\[
\leq \left\| 3 \int_0^1 (u_2 - v_2 + \theta (u_1 - v_1 - u_2 + v_2))^2 d\theta \cdot (u - v) \right\|_2
\]
\[
+ \left\| \nabla \left[ 3 \int_0^1 (u_1 + v_1 + \theta (u_2 + v_2 - u_1 - v_1))^2 d\theta \cdot (-u - v) \right] \right\|_2
\]
\[
\leq \left\| 3 \int_0^1 (u_2 - v_2 + \theta (u_1 - v_1 - u_2 + v_2))^2 d\theta \cdot \nabla (u - v) \right\|_2
\]
\[
+ \left\| 3 \int_0^1 (u_1 + v_1 + \theta (u_2 + v_2 - u_1 - v_1))^2 d\theta \cdot \nabla (-u - v) \right\|_2
\]
\[
+ \| 6 \int_0^1 (u_2 - v_2 + \theta (u_1 - v_1 - u_2 + v_2)) \nabla (u_2 - v_2 + \theta (u_1 - v_1 - u_2 + v_2)) d\theta
\]
\[
\cdot (u - v) \|_2
\]
\[
+ \| 6 \int_0^1 (u_1 + v_1 + \theta (u_2 + v_2 - u_1 - v_1)) \nabla (u_1 + v_1 + \theta (u_2 + v_2 - u_1 - v_1)) d\theta
\]
\[
\cdot (-u - v) \|_2
\]
\[
\leq \left\| 3 \int_0^1 (u_2 - v_2 + \theta (u_1 - v_1 - u_2 + v_2))^2 d\theta \cdot \nabla (u - v) \right\|_2
\]
\[
+ \left\| 3 \int_0^1 (u_1 + v_1 + \theta (u_2 + v_2 - u_1 - v_1))^2 d\theta \cdot \nabla (-u - v) \right\|_2
\]
\[
+ Q \left[ \| u - v \|_2 \| \nabla (u_2 - v_2) \|_2 + \| u - v \| \| \nabla (u_1 - v_1) \|_2 \right]
\]
\[
+ Q \left[ \| u + v \| \| \nabla (u_2 + v_2) \|_2 + \| u + v \| \| \nabla (u_1 + v_1) \|_2 \right]
\]
\[
\leq Q \left( \| \nabla v \|_2 + \| \nabla u \|_2 \right),
\]
where \( Q = Q \left( \| v_1 \|_{H^2(\Omega)}, \| u_1 \|_{H^2(\Omega)}, \| v_2 \|_{H^2(\Omega)}, \| u_2 \|_{H^2(\Omega)} \right) \) and the Sobolev embedding theorem is used again. Similarly, we can obtain by using the same idea that
\[
\| \nabla [-2(u_1^3 - u_2^3) - 6(u_1v_1^2 - u_2v_2^2)] \|_2
\]
Therefore (3.65) implies from (3.64) that
\[
\|\nabla v\|^2_2 + \|\nabla u\|^2_2 + C \int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dx \, ds + C \int_0^t \int_\Omega (-\Delta)^{-1} \frac{\partial u}{\partial t} \cdot \frac{\partial u}{\partial t} \, dx \, ds \\
\leq \int_0^t Q(\|v_1\|_{H^2(\Omega)}, \|u_1\|_{H^2(\Omega)}, \|v_2\|_{H^2(\Omega)}, \|u_2\|_{H^2(\Omega)})(\|\nabla v\|^2_2 + \|\nabla u\|^2_2) \, ds \\
+ C \int_0^t (\|v\|^2_2 + \|u\|^2_2) \, ds + C \int_0^t \int_\Omega \phi^2 \, dx \, ds.
\] (3.66)

Now using the Poincaré inequality, (3.45) and the Gronwall inequality, we obtain
\[
\|u\|_{L^\infty(0,t;H^1(\Omega))} + \|v\|_{L^\infty(0,t;H^2(\Omega))} \leq C\|\phi\|_{L^2(0,t;H)}.
\] (3.67)

It is obvious that we have the uniqueness of solution from (3.67). Next we multiply (3.51) by $-\Delta u$ and (3.62) by $-\Delta u$ and obtain, summing the two resulting equalities,
\[
\int_0^t \int_\Omega [2(u_1^3 - u_3^3) + 6(u_1 v_1^2 - u_2 v_2^2) - 2\beta u] \cdot \Delta u \, dx \, ds.
\] (3.68)

Furthermore, we can write from Cauchy’s inequality and the Sobolev embedding theorem that
\[
\int_0^t \int_\Omega [2(u_1^3 - u_3^3) + 6(u_1 v_1^2 - u_2 v_2^2) - 2\beta u] \cdot \Delta u \, dx \, ds \\
\leq \frac{1}{2} \int_0^t \int_\Omega [2(u_1^3 - u_3^3) + 6(u_1 v_1^2 - u_2 v_2^2) - 2\beta u]^2 \, dx \, ds + \frac{1}{2} \int_0^t \int_\Omega (\Delta u)^2 \, dx \, ds \\
\leq Q(\|u_1\|_{L^\infty(0,t;H^1(\Omega))}, \|u_2\|_{L^\infty(0,t;H^2(\Omega))}, \|v_1\|_{L^\infty(0,t;H^2(\Omega))}, \|v_2\|_{L^\infty(0,t;H^2(\Omega))}) \\
\cdot (\|u\|_{L^\infty(0,t;H^1(\Omega))} + \|v\|_{L^\infty(0,t;H^2(\Omega))}) + \frac{1}{2} \int_0^t \int_\Omega (\Delta u)^2 \, dx \, ds \\
\leq C\|\phi\|^2_{L^2(0,t;H)} + \frac{1}{2} \int_0^t \int_\Omega (\Delta u)^2 \, dx \, ds,
\]

it implies from (3.68) and (3.67) that
\[
\|u\|_{L^\infty(0,t;H)} + \|u\|_{L^\infty(0,t;H^2(\Omega))} \leq C\|\phi\|_{L^2(0,t;H}).
\] (3.69)

We now multiply (3.60) by $\frac{\partial v}{\partial t}$ and then integrate the resulting identity over $\Omega \times (0,t)$, we can see from Young’s inequality that
\[
\int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dx \, ds + \int_0^t \int_\Omega \nabla v \cdot \nabla \frac{\partial v}{\partial t} \, dx \, ds \\
= \int_0^t \int_\Omega [-2(v_1^3 - v_3^3) - 6(u_1^2 v_1 - u_2^2 v_2) + (2\beta - \alpha) v + \phi] \cdot \frac{\partial v}{\partial t} \, dx \, ds \\
\leq C \int_0^t \int_\Omega [-2(v_1^3 - v_3^3) - 6(u_1^2 v_1 - u_2^2 v_2)]^2 \, dx \, ds + C \int_0^t \int_\Omega v^2 \, dx \, ds \\
+ C \int_0^t \int_\Omega \phi^2 \, dx \, ds + \frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dx \, ds \\
\leq Q(\|u_1\|_{L^\infty(0,t;H^2(\Omega))}, \|u_2\|_{L^\infty(0,t;H^2(\Omega))}, \|v_1\|_{L^\infty(0,t;H^2(\Omega))}, \|v_2\|_{L^\infty(0,t;H^2(\Omega))}).
\[ \cdot \left( \|u\|^2_{L^\infty(0,t,H^1(\Omega))} + \|v\|^2_{L^\infty(0,t,H^1(\Omega))} \right) + C\|\phi\|^2_{L^2(0,t,H^1)} + \frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dx \, ds \]

\[ \leq C\|\phi\|^2_{L^2(0,t,H^1)} + \frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial v}{\partial t} \right)^2 \, dx \, ds, \]

which yields

\[ \|v\|_{L^1(0,t,H^1(\Omega))} \leq \|v\|_{L^\infty(0,t,H^1(\Omega))} \leq C\|\phi\|_{L^2(0,t,H^1)}. \quad (3.70) \]

Using (3.60) again, we obtain

\[ \|\Delta v\|_{L^2(0,t,H^1)} \leq \|\frac{\partial v}{\partial t}\|_{L^2(0,t,H^1)} + \|2(v_1^3 - v_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1)} + C\|L\|_{L^2(0,t,H^1)}, \]

it implies from (3.67) that

\[ \|v\|_{L^2(0,t,H^2(\Omega))} \leq C\|\phi\|_{L^2(0,t,H^1)}. \quad (3.71) \]

On account of (3.62), we have

\[ \|\mu\|_{L^2(0,t,H^1)} = \|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1)} \leq \|\Delta u\|_{L^2(0,t,H^1)} + \|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1)} + C\|u\|_{L^2(0,t,H^1)} \leq C\|\phi\|_{L^2(0,t,H^1)}. \quad (3.72) \]

Now we multiply (3.61) by \(A^{-1} u\) and then integrate the resulting identity over \(\Omega\), that is

\[ \int_\Omega \frac{\partial u}{\partial t} \cdot A^{-1} u \, dx = - \int_\Omega (-\Delta \mu + \mu - \mu) \cdot A^{-1} u \, dx \]

\[ = - \int_\Omega A \mu \cdot A^{-1} u \, dx + \int_\Omega \mu \cdot A^{-1} u \, dx. \]

Furthermore, we apply (3.62) and the Sobolev embedding theorem and then receive

\[ \frac{1}{2} \frac{d}{dt} \|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} \leq C \int_\Omega u^2 \, dx + \int_\Omega \left[ 2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2) \right] \cdot u \, dx + \int_\Omega \mu \cdot A^{-1} u \, dx \]

\[ \leq C \int_\Omega u^2 \, dx + \int_\Omega \left[ 2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2) \right]^2 \, dx + C\|\mu\|_{(H^1(\Omega))'} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C \int_\Omega u^2 \, dx + C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} \cdot \|\mu\|_{(H^1(\Omega))'} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} \cdot \|\mu\|_{(H^1(\Omega))'} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} \cdot \|\mu\|_{(H^1(\Omega))'} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]

\[ \leq C\|u\|^2_{L^2(0,t,H^1(\Omega))} + \|v\|^2_{L^2(0,t,H^1(\Omega))} + C\|2(u_1^3 - u_2^3) + 6(u_1 v_1 - u_2 v_2)\|_{L^2(0,t,H^1(\Omega))} \cdot \|u\|_{(H^1(\Omega))'} \]
and we thus obtain from (3.72) that
\[C_1 \leq C_2 (u_1^3 - u_2^3) + 6(u_1 v_1^2 - u_2 v_2^2) \|u\|_{H^1(\Omega)} \cdot \|u\|_{H^1(\Omega)}.\] (3.73)

For \( \chi \in H^1(\Omega) \), we can get
\[
|2(u_1^3 - u_2^3) + 6(u_1 v_1^2 - u_2 v_2^2), \chi| = |2(u_1^3 - u_2^3) + 6u v_1^2 + 6u v(v_1 + v_2), \chi| \leq C\|u\|_2 \|v_1\|_6^3 \|v\|_6 + C\|v\|_2 \|u_2\|_6 \|v\|_6 + C\|u\|_2 \|v_2\|_6 \|v\|_6 \leq C(\|u\|_2 + \|v\|_2) \|\chi\|_{H^1(\Omega)},
\]
we thus deduce
\[
\|2(u_1^3 - u_2^3) + 6(u_1 v_1^2 - u_2 v_2^2)\|_{H^1(\Omega)} \leq C(\|u\|_2 + \|v\|_2).
\]
Hence (3.73) yields
\[
\frac{1}{2} \frac{d}{dt} \|u\|^2_{H^1(\Omega)} + \|u\|^2_{H^1(\Omega)} \leq C\|u\|^2_{H^1(\Omega)} + C\|u\|^2_{H^1(\Omega)} + \|u\|_2 \|v\|_2 \cdot \|u\|_{H^1(\Omega)} \leq C\|u\|^2_{H^1(\Omega)} + \|v\|^2_{H^1(\Omega)} + C\|u\|^2_{H^1(\Omega)}.
\]
According to the Gronwall inequality, we can obtain
\[
\|u\|^2_{H^1(\Omega)} \leq C\|u\|^2_{L^\infty(0,T;H^1(\Omega))} + \|v\|^2_{L^\infty(0,T;H^1(\Omega))} \leq C\|\phi\|^2_{L^2(0,t;H)}.
\] (3.74)

For any \( \chi \in L^2(0,T,V) \), we test (3.61) by \( \chi \), by integration by parts, we deduce from the H"older inequality that
\[
\int_0^T \int_\Omega \frac{\partial u}{\partial t} \cdot \chi dx dt = \int_0^T \int_\Omega \Delta \mu \cdot \chi dx dt = \int_0^T \int_\Omega \mu \cdot \Delta \chi dx dt \leq \int_0^T \left( \int_\Omega \mu^2 dx \right)^{\frac{1}{2}} \left( \int_\Omega (\Delta \chi)^2 dx \right)^{\frac{1}{2}} dt \leq C\|\mu\|_{L^2(0,T;H)} \cdot \|\chi\|_{L^2(0,T;V)},
\]
we thus obtain from (3.72) that
\[
\|\frac{\partial u}{\partial t}\|_{L^2(0,t;V')} \leq C\|\phi\|_{L^2(0,t;H)}. \quad (3.75)
\]
According to (3.67), (3.69)-(3.72), (3.74) and (3.75), we have accomplished the proof of (3.48). It should be noted that we use the same argument as in the proof of Theorem 3.1, we thus only prove a priori estimates for (3.47) and (3.48). This completes the proof of the theorem. \( \square \)

4. The main results and proofs. In this section, we will show that the main results in this article. For the triplet \((v, u, \mu)\) which is the unique strong solution of (1.3)-(1.5) corresponding to the control \( \phi \), we define two function spaces by
\[
\mathcal{H} [v \in H^1(0,T;H) \cap C(0,T;H^1(\Omega)) \cap L^\infty(0,T;H^2(\Omega)); u \in H^1(0,T;H)] \times [u \in C(0,T;H) \cap L^2(0,T;H^2(\Omega)), \Delta u \in L^2(0,T;V)] \times [\mu \in L^\infty(0,T;H) \cap L^2(0,T;V)],
\]
and
\[
\mathcal{Y} [v \in C(0,T;H^1(\Omega)) \cap L^2(0,T;H^2(\Omega)) \times [u \in C(0,T;H) \cap L^2(0,T;H^2(\Omega)) \times [\mu \in L^2(0,T;H)].
\]
We use the notation $S$ to stand for the control-to-state mapping 

$$S : \Phi_R \to \mathcal{H}, \quad \phi \mapsto (v, u, \mu) \triangleq S(\phi).$$

According to Theorem 3.2, we know that $S$ is well-defined and $S$ is Lipschitz continuous from $\Phi_R$ into $\mathcal{Y}$. Next we show that the existence of the optimal control problem (CP). Firstly, for the triplet $(v, u, \mu)$ which is the unique strong solution of (1.3)-(1.6) corresponding to the control $\phi$, we introduce the reduced cost functional

$$\tilde{J} : \Phi_R \to [0, \infty), \quad \phi \mapsto \tilde{J}(\phi) \triangleq J(u, \phi).$$

Hence the optimal control problem (CP) is equivalent to the minimum value of $\tilde{J}(\phi)$ for $\phi \in \Phi_{ad}$. Because the infimum for the reduced cost functional $\tilde{J}(\phi)$ exists and $\Phi_{ad}$ is nonempty, then there exists the minimizing sequence $\{\phi^N\} \subset \Phi_{ad}$ such that

$$\lim_{N \to \infty} \tilde{J}(\phi^N) = \inf_{\phi \in \Phi_{ad}} \tilde{J}(\phi).$$

Due to $\{\phi^N\} \subset \Phi_{ad} \subset \Phi_R$, thus there exists a subsequence of $\{\phi^N\}$ which is again indexed by $n$ such that

$$\{\phi^N\} \rightharpoonup \bar{\phi}, \quad \text{weakly in } \Phi_R, \quad \text{as } N \to \infty.$$ 

We know that $\Phi_{ad}$ is weakly closed since $\Phi_{ad}$ is a closed convex set, hence $\bar{\phi} \in \Phi_{ad}$. On account of the property of the control-to-state mapping $S$, we deduce that $S(\bar{\phi})$ is bounded in $\mathcal{H}$. In other words, we obtain that $(v^N, u^N, \mu^N)$ is bounded in $\mathcal{H}$. Thus there exists a subsequence of $(v^N, u^N, \mu^N)$ which is again indexed by $n$ and $(\tilde{v}, \tilde{u}, \bar{\mu}) \in \mathcal{H}$ such that

$$v^N \rightharpoonup \tilde{v}, \quad u^N \rightharpoonup \tilde{u} \quad \text{weakly-* in } L^\infty(0, T; H^2(\Omega)), \quad \text{as } N \to \infty,$$

$$v^N \to \tilde{v}, \quad u^N \to \tilde{u} \quad \text{weakly in } H^1(0, T; H), \quad \text{as } N \to \infty,$$

$$\Delta u^N \to \Delta \tilde{u}, \quad \mu^N \rightharpoonup \bar{\mu} \quad \text{weakly in } L^2(0, T; V), \quad \text{as } N \to \infty,$$

$$\mu^N \to \bar{\mu}, \quad \text{weakly in } L^\infty(0, T; H), \quad \text{as } N \to \infty.$$ 

By the Aubin-Lions lemma and the compact embedding theorems, we obtain the following strong convergence

$$v^N \to \tilde{v}, \quad \text{strongly in } C([0, T]; L^p(\Omega)), \quad \text{for } 2 \leq p < 6, \quad \text{as } N \to \infty,$$

$$v^N \to \tilde{v}, \quad \text{strongly in } L^2(0, T; H^1(\Omega)) \text{ and a.e. in } Q, \quad \text{as } N \to \infty,$$

$$u^N \to \bar{u}, \quad \text{strongly in } C([0, T]; H^1(\Omega)), \quad \text{as } N \to \infty.$$ 

Applying the above estimates, similarly, we can obtain, as $N \to \infty$,

$$2(u^N)^3 - 2\beta u^N \to 2\tilde{v}^3 - 2\beta \tilde{u}, \quad \text{strongly in } L^2(Q),$$

$$(v^N)^3 \to \bar{v}^3, \quad (u^N)^2 v^N \to \bar{u}^2 \tilde{v}, \quad (v^N)^2 u^N \to \bar{v}^2 \bar{u}, \quad \text{strongly in } L^2(Q),$$

$$\Delta v^N \to \Delta \tilde{v}, \quad \Delta \mu^N \to \Delta \bar{\mu}, \quad \text{weakly in } L^2(Q).$$ 

For $\chi \in L^2(0, T; H)$, we replace $(v, u, \mu)$ by $(v^N, u^N, \mu^N)$ in (1.7)-(1.9) and take the test function $\chi$, then we can pass to the limit by the above convergence, that is

$$\int_0^T \left\langle \frac{\partial \tilde{v}}{\partial t}, \chi \right\rangle dt = \int_0^T \int_\Omega [-2\tilde{v}^3 - 6\bar{u}^2 \tilde{v} + (2\beta - \alpha)\tilde{v} + \Delta \tilde{v} + \bar{\phi}] \chi dx dt,$$

$$\int_0^T \left\langle \frac{\partial \bar{u}}{\partial t}, \chi \right\rangle dt = \int_0^T \int_\Omega \Delta \bar{\mu} \cdot \chi dx dt,$$
Due to $(\bar{v}, \bar{u}, \bar{\mu}) \in \mathcal{H}$, we have $(\bar{v}, \bar{u}, \bar{\mu}) = \bar{\phi}$. Thus the limit is admissible for (CP), and we can receive from the weak lower semicontinuity of the cost functional $J$ that
\[
\lim_{N \to \infty} \inf J(u^N, \phi^N) \geq J(\bar{u}, \bar{\phi}).
\]
In other words, we have
\[
J(\bar{u}, \bar{\phi}) = \inf_{\phi \in \Phi_{ad}} J(u, \phi) = \min_{\phi \in \Phi_{ad}} J(u, \phi).
\]
This implies that (CP) admits a solution $((v, u, \mu), \phi)$ such that $\phi \in \Phi_{ad}$ and $(v, u, \mu) = S(\phi)$ when $v_0(x)$, $u_0(x) \in W^{1,2}(\Omega) \cap H^2(\Omega)$ and (2.5)-(2.7) are satisfied.

Next we start with the definition of Fréchet differentiability for $S$ as follows

**Definition 4.1.** For $\phi \in \Phi_{ad}$, if there exists a linear operator $DS(\phi)$ which satisfies $DS(\phi) \in L(L^2(Q), \mathcal{Y})$ such that
\[
\lim_{\|h\|_{L^2(Q)} \to 0} \frac{\|S(\phi \ast + h) - S(\phi) - DS(\phi)(h)\|_{\mathcal{Y}}}{\|h\|_{L^2(Q)}} = 0. \tag{4.1}
\]
Then, $S : \Phi_R \to \mathcal{Y}$ is Fréchet differentiable.

To prove $S$ is Fréchet differentiable, let $h \in L^2(Q)$ be arbitrary but fixed, then for $(v^*, u^*, \mu^*) = S(\phi^*)$, we firstly consider the following linearized system
\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= -6(v^*)^2 \xi - 6(u^*)^2 \xi + (2\beta - \alpha)\xi - 12v^* u^* \eta + \Delta \xi + h, \quad \text{in } Q, \tag{4.2}
\frac{\partial \eta}{\partial t} &= \Delta \rho, \quad \text{in } Q, \tag{4.3}
\rho &= 6(v^*)^2 \eta + 6(v^*)^2 \eta + 12v^* u^* \xi - 2\beta \eta - \Delta \eta, \quad \text{in } Q, \tag{4.4}
\xi(0) &= \xi(x, 0) = 0, \quad \eta(0) = \eta(x, 0) = 0, \quad \text{in } \Sigma, \tag{4.5}
\end{align*}
\]
\[
\xi(t) = \xi(x, t) = \eta(x, t) = 0, \quad \text{on } \Omega. \tag{4.6}
\]

**Theorem 4.2.** Let $T > 0$. Then, for every $h \in L^2(Q)$, the problem (4.2)-(4.6) admits a unique solution on $[0, T]$ such that
\[
\xi \in H^1(0, T; H) \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \tag{4.7}
\eta \in H^1(0, T; (H^2(\Omega))') \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; V), \tag{4.8}
\rho \in L^2(0, T; H), \tag{4.9}
\]
which satisfies the following identities
\[
\begin{align*}
\frac{\partial \xi}{\partial t} &= -6(v^*)^2 \xi - 6(u^*)^2 \xi + (2\beta - \alpha)\xi - 12v^* u^* \eta + \Delta \xi + h, \quad \text{a.e. } Q, \tag{4.10}
\int_0^T \left< \frac{\partial \eta}{\partial t}, \varphi \right> dt &= \int_Q \rho \cdot \Delta \varphi dx dt, \tag{4.11}
\rho &= 6(v^*)^2 \eta + 6(v^*)^2 \eta + 12v^* u^* \xi - 2\beta \eta - \Delta \eta, \quad \text{a.e. } Q. \tag{4.12}
\end{align*}
\]
for all $\varphi \in L^2(0, T; H^2(\Omega))$, where $\xi(x, 0) = \eta(x, 0) = 0$ almost everywhere in $\Omega$ and $\xi = \eta = 0$ almost everywhere in $\Sigma$. Furthermore, we also have, for $t \in [0, T]$,
\[
\begin{align*}
\|\xi\|_{H^1(0,t;H)} + \|\xi\|_{L^\infty(0,t;H^1(\Omega))} + \|\eta\|_{H^2(0,t;H^2(\Omega))} + \|\rho\|_{H^1(0,t;H^2(\Omega)')} \\
+ \|\eta\|_{L^\infty(0,t;H^1(\Omega))} + \|\eta\|_{L^2(0,t;V)} + \|\rho\|_{L^2(0,t;H)} \leq C \|h\|_{L^2(0,t;H)}. \tag{4.13}
\end{align*}
\]
Proof. Similar to the previous discussion of Theorem 3.1, we shall look for three functions of the following form by applying the Galerkin approximation method

\[ \xi^N(x,t) = \sum_{j=1}^{N} a_j^N(t) \varphi_j(x), \quad \eta^N(x,t) = \sum_{j=1}^{N} b_j^N(t) \varphi_j(x), \quad \rho^N(x,t) = \sum_{j=1}^{N} c_j^N(t) \varphi_j(x), \]

that solve the following approximating problem

\[
\begin{align*}
\int_{\Omega} \frac{\partial \xi^N}{\partial t} \cdot \varphi_j dx + \int_{\Omega} \nabla \xi^N \cdot \nabla \varphi_j dx &= \\
-6(u^*)^2 \xi^N - 6(u^*)^2 \xi^N + (2\beta - \alpha) \xi^N - 12v^* u^* \eta^N \varphi_j dx + \int_{\Omega} h \varphi_j dx, \quad (4.14) \\
\int_{\Omega} \frac{\partial \eta^N}{\partial t} \cdot \varphi_j dx &= \int_{\Omega} \rho^N \cdot \Delta \varphi_j dx, \quad (4.15) \\
\int_{\Omega} \rho^N \varphi_j dx &= \int_{\Omega} \nabla \eta^N \cdot \nabla \varphi_j dx \\
&+ \int_{\Omega} \left[ 6(u^*)^2 \eta^N + 6(u^*)^2 \eta^N + 12v^* u^* \xi^N - 2\beta \eta^N \right] \varphi_j dx, \quad (4.16)
\end{align*}
\]

\[ \xi^N(x,0) = \eta^N(x,0) = 0, \quad (4.17) \]

for \( j = 1, 2, \ldots, N \). It is easy to see that solving the approximate problem (3.7)-(3.10) is equivalent to solving an initial value problem for a linear system of 2N ordinary differential equations in the 2N unknowns \( a_j^N, b_j^N \). In this linear system, since all the coefficients belonging to \( L^\infty(0,T) \), it follows from Carathéodory's theorem that this linear system has a unique solution \( (a_j^N, b_j^N, b_1^N, \ldots, b_j^N) \in (W^{1,\infty}(0,T))^{2N} \). Next, we rewrite (4.15) as the form of (3.51) and multiply after by \( (b_j^N(t))' \), then multiplying (4.14) by \((a_j^N(t))'\), (4.16) by \(-(b_j^N(t))'\) and summing over \( j = 1, 2, \ldots, N \). Then we integrate the resulting identity over \((0,t)\), so we get

\[
\int_{0}^{t} \int_{\Omega} \left( \frac{\partial \xi^N}{\partial t} \right)^2 dx ds + \int_{0}^{t} \int_{\Omega} \nabla \xi^N \cdot \nabla \frac{\partial \xi^N}{\partial t} dx ds = \int_{0}^{t} \int_{\Omega} \left[ (2\beta - \alpha) \xi^N - 6(u^*)^2 \xi^N - 6(u^*)^2 \xi^N - 12v^* u^* \eta^N + h \right] \cdot \frac{\partial \xi^N}{\partial t} dx ds, \quad (4.18)
\]

\[
\begin{align*}
\int_{0}^{t} \int_{\Omega} (-\Delta)^{-1} \frac{\partial \eta^N}{\partial t} \cdot \frac{\partial \eta^N}{\partial t} dx ds &= - \int_{0}^{t} \int_{\Omega} \rho^N \cdot \frac{\partial \eta^N}{\partial t} dx ds, \\
- \int_{0}^{t} \int_{\Omega} \rho^N \cdot \frac{\partial \eta^N}{\partial t} dx ds &= - \int_{0}^{t} \int_{\Omega} \nabla \eta^N \cdot \nabla \frac{\partial \eta^N}{\partial t} dx ds \\
- \int_{0}^{t} \int_{\Omega} \left[ 6(u^*)^2 \eta^N + 6(u^*)^2 \eta^N + 12v^* u^* \xi^N - 2\beta \eta^N \right] \cdot \frac{\partial \eta^N}{\partial t} dx ds.
\end{align*}
\]

By adding them and using the Sobolev embedding theorem and Young's inequality, we deduce

\[
\int_{0}^{t} \int_{\Omega} \left( \frac{\partial \xi^N}{\partial t} \right)^2 dx ds + \int_{0}^{t} \int_{\Omega} \nabla \xi^N \cdot \nabla \frac{\partial \xi^N}{\partial t} dx ds + \int_{0}^{t} \int_{\Omega} \nabla \eta^N \cdot \nabla \frac{\partial \eta^N}{\partial t} dx ds + \int_{0}^{t} \int_{\Omega} \nabla \eta^N \cdot \nabla \frac{\partial \eta^N}{\partial t} dx ds
\]

\[ + \int_{0}^{t} \int_{\Omega} (-\Delta)^{-1} \frac{\partial \eta^N}{\partial t} \cdot \frac{\partial \eta^N}{\partial t} dx ds \]
Applying the Gronwall inequality we find from (4.20) that

\[
\int_0^t \int_\Omega [6(u^*)^2 \xi^N - 6(u^*)^2 \xi^N + (2\beta - \alpha) \xi^N - 12u^* u^* \eta^N + h] \cdot \frac{\partial \xi^N}{\partial t} \, dx \, ds
\]

\[
- \int_0^t \int_\Omega [6(u^*)^2 \eta^N + 6(v^*)^2 \eta^N + 12u^* u^* \xi^N - 2\beta \eta^N] \cdot \frac{\partial \eta^N}{\partial t} \, dx \, ds
\]

\[
\leq Q(\|u^*\|_{L^\infty(0,T;H^2(\Omega))}, \|v^*\|_{L^\infty(0,T;H^2(\Omega))}) \{ \int_0^t \int_\Omega (|\xi^N| + |\eta^N| + |h|) \cdot \left| \frac{\partial \xi^N}{\partial t} \right| \, dx \, ds
\]

\[
\int_0^t \int_\Omega (|\nabla \xi^N| + |\nabla \eta^N|) \cdot \left| (-\Delta)^{-\frac{1}{2}} \frac{\partial \eta^N}{\partial t} \right|^2 \, dx \, ds
\]

\[
+ C \int_0^t \int_\Omega (|\xi^N| + |\eta^N| + |h|)^2 \, dx \, ds + C(\varepsilon) \int_0^t \int_\Omega (|\nabla \xi^N| + |\nabla \eta^N|)^2 \, dx \, ds. \quad (4.19)
\]

By selecting appropriate for \( \varepsilon \), we obtain from (4.19) and the Poincaré inequality that

\[
C \int_0^t \int_\Omega \left( \frac{\partial \xi^N}{\partial t} \right)^2 \, dx \, ds + \int_0^t \int_\Omega \nabla \xi^N \cdot \nabla \frac{\partial \xi^N}{\partial t} \, dx \, ds + \int_0^t \int_\Omega \nabla \eta^N \cdot \nabla \frac{\partial \eta^N}{\partial t} \, dx \, ds
\]

\[
+ C \int_0^t \left\| \frac{\partial \eta^N}{\partial t} \right\|^2 \, ds
\]

\[
\leq C \int_0^t \int_\Omega (|\nabla \xi^N|^2 + |\nabla \eta^N|^2) \, dx \, ds + C \int_0^t \int_\Omega h^2 \, dx \, ds. \quad (4.20)
\]

Applying the Gronwall inequality we find from (4.20) that

\[
\|\xi^N\|_{L^\infty(0,T;H^1(\Omega))} + \|\eta^N\|_{L^\infty(0,T;H^1(\Omega))} \leq C\|h\|_{L^2(0,T;H)}, \quad (4.21)
\]

Furthermore, we obtain from (4.18) and (4.21) that

\[
\int_0^t \int_\Omega \left( \frac{\partial \xi^N}{\partial t} \right)^2 \, dx \, ds + \int_0^t \int_\Omega \nabla \xi^N \cdot \nabla \frac{\partial \xi^N}{\partial t} \, dx \, ds
\]

\[
\leq \frac{1}{2} \int_0^t \int_\Omega \left( \frac{\partial \xi^N}{\partial t} \right)^2 \, dx \, ds + C \int_0^t \int_\Omega |\xi^N|^2 \, dx \, ds + C \int_0^t \int_\Omega |\eta^N|^2 \, dx \, ds
\]

\[
+ C \int_0^t \int_\Omega h^2 \, dx \, ds \leq C\|h\|^2_{L^2(0,T;H)},
\]

which yields

\[
\|\xi^N\|_{L^\infty(0,T;H^1(\Omega))} + \|\xi^N\|_{H^1(0,T;H^1(\Omega))} \leq C\|h\|_{L^2(0,T;H)}. \quad (4.22)
\]

Observe that (4.15) can be written as

\[
\frac{\partial \eta^N}{\partial t} = \Delta \rho^N. \quad (4.23)
\]

Testing (4.23) by \( \eta^N \) and integrating the resulting identity over \( \Omega \times (0,t) \), we find

\[
\int_0^t \int_\Omega \frac{\partial \eta^N}{\partial t} \cdot \eta^N \, dx \, ds = \int_0^t \int_\Omega \Delta \rho^N \cdot \eta^N \, dx \, ds. \quad (4.24)
\]

Noticing that \( \Delta \eta^N \in V_N \), then we test (4.16) by \( \Delta \eta^N \) and integrate the resulting identity over \( \Omega \times (0,t) \), plugging this identity and (4.24) yields the following
inequality
\[
\int_0^t \int_\Omega \frac{\partial \eta^N}{\partial t} \cdot \eta^N dxds + \int_0^t \int_\Omega (\Delta \eta^N)^2 dxds \\
= \int_0^t \int_\Omega \left[ 6(u^*)^2 \eta^N + 6(v^*)^2 \eta^N + 12v^* u^* \xi^N - 2\beta \eta^N \right] \cdot (\Delta \eta^N) dxds
\]

\[
\leq Q(\|u^*\|_{L^\infty(0,T;H^2(\Omega))}, \|v^*\|_{L^\infty(0,T;H^2(\Omega))}) \int_0^t \int_\Omega (|\eta^N| + |\xi^N|) \cdot (\Delta \eta^N) dxds
\]

\[
\leq \frac{1}{2} \int_0^t \int_\Omega (\Delta \eta^N)^2 dxds + C \int_0^t \int_\Omega (|\eta^N|^2 + |\xi^N|^2) dxds
\]

\[
\leq \frac{1}{2} \int_0^t \int_\Omega (\Delta \eta^N)^2 dxds + C(\|\xi^N\|^2_{L^\infty(0,T;H^1(\Omega))} + \|\eta^N\|^2_{L^\infty(0,T;H^1(\Omega))})
\]

where we use (3.45). It implies from (4.21) that
\[
\|\eta^N\|^2_{L^2(0,t;H^2(\Omega))} \leq C \|h\|_{L^2(0,t;H)}.
\]

Furthermore, owing to (4.14), we have
\[
\|\Delta \xi^N\|_{L^2(0,T;H)} = \left\| \frac{\partial \xi^N}{\partial t} + 6(v^*)^2 \xi^N + 6(u^*)^2 \xi^N - (2\beta - \alpha) \xi^N + 12v^* u^* \eta^N - h \right\|_{L^2(0,T;H)}
\]

\[
\leq \left\| \frac{\partial \xi^N}{\partial t} \right\|_{L^2(0,T;H)} + \|6(v^*)^2 \xi^N + 6(u^*)^2 \xi^N\|_{L^2(0,T;H)}
\]

\[
+ \|-(2\beta - \alpha) \xi^N\|_{L^2(0,T;H)} + \|12v^* u^* \eta^N\|_{L^2(0,T;H)} + \|h\|_{L^2(0,T;H)}
\]

\[
\leq C \|h\|_{L^2(0,T;H)} + C \|\xi^N\|_{L^2(0,T;H)} + C \|\eta^N\|_{L^2(0,T;H)}
\]

\[
\leq C \|h\|_{L^2(0,T;H)},
\]

which implies from (4.21) that
\[
\|\xi^N\|^2_{L^2(0,T;H^2(\Omega))} \leq C \|h\|^2_{L^2(0,T;H)}.
\]

According to (4.16), we obtain from (4.21) and (4.25) that
\[
\|\rho^N\|_{L^2(0,T;H)} = \|6(u^*)^2 \eta^N + 6(v^*)^2 \eta^N + 12v^* u^* \xi^N - 3\beta \eta^N \|_{L^2(0,T;H)}
\]

\[
\leq \|6(u^*)^2 \eta^N + 6(v^*)^2 \eta^N + 12v^* u^* \xi^N\|_{L^2(0,T;H)} + \|h\|_{L^2(0,T;H)}
\]

\[
+ \|\Delta \eta^N\|_{L^2(0,T;H)} \leq C \|h\|_{L^2(0,T;H)}.
\]

For any \( \chi \in L^2(0,t;H^2(\Omega)) \), we test (4.23) by \( \chi \) and integrate the resulting identity over \( \Omega \times (0,t) \), we thus deduce from the Höffler inequality that
\[
\int_0^t \int_\Omega \frac{\partial \eta^N}{\partial t} \cdot \chi dxds = \int_0^t \int_\Omega (\Delta \rho^N) \cdot \chi dxds
\]

\[
= \int_0^t \int_\Omega \rho^N \cdot \Delta \chi dxds \leq \|\rho^N\|_{L^2(0,T;H)} \cdot \|\chi\|_{L^2(0,T;H^2(\Omega))},
\]

which yields
\[
\left\| \frac{\partial \eta^N}{\partial t} \right\|_{L^2(0,T;H^2(\Omega))} \leq C \|h\|_{L^2(0,T;H)}.
\]
On the basis of (4.21), (4.22), (4.25)-(4.28), we thus receive from the Aubin-Lions lemma that there exists a subsequence of $\xi^N$, $\eta^N$, $\rho^N$ which is again indexed by $n$ such that, as $N \to \infty$,

\[ \xi^N \rightharpoonup \xi, \quad \text{weakly} -^* \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)), \tag{4.29} \]

\[ \xi^N \to \xi, \quad \text{weakly in} \quad H^1(0,T;H) \cap L^2(0,T;H^2(\Omega)), \tag{4.30} \]

\[ \xi^N \rightharpoonup \xi, \quad \text{strongly in} \quad C([0,T];L^p(\Omega)), \quad \text{for} \quad 2 \leq p < 6, \tag{4.31} \]

\[ \xi^N \to \xi, \quad \text{strongly in} \quad L^2(0,T;H^1(\Omega)) \quad \text{and a.e. in} \quad Q, \tag{4.32} \]

\[ \eta^N \rightharpoonup \eta, \quad \text{weakly} -^* \quad \text{in} \quad L^\infty(0,T;H^1(\Omega)), \tag{4.33} \]

\[ \eta^N \to \eta, \quad \text{weakly in} \quad H^1(0,T;(H^2(\Omega))') \cap L^2(0,T;H^2(\Omega)), \tag{4.34} \]

\[ \eta^N \to \eta, \quad \text{strongly in} \quad C([0,T];L^p(\Omega)), \quad \text{for} \quad \frac{6}{5} \leq p < 6, \tag{4.35} \]

\[ \eta^N \to \eta, \quad \text{strongly in} \quad L^2(0,T;H^1(\Omega)) \quad \text{and a.e. in} \quad Q, \tag{4.36} \]

\[ \rho^N \rightharpoonup \rho, \quad \text{weakly in} \quad L^2(0,T;H). \tag{4.37} \]

Furthermore, we can write from (4.30)-(4.32), (4.34), (4.35) and (3.45) that

\[ -6(u^*)^2\xi^N - 6(u^*)^2\xi^N + (2\beta - \alpha)\xi^N \]

\[ \to -6(u^*)^2\xi - 6(u^*)^2\xi + (2\beta - \alpha)\xi, \quad \text{strongly in} \quad L^2(Q), \tag{4.38} \]

\[ 6(u^*)^2\eta^N + 6(u^*)^2\eta^N - 2\beta \eta^N \]

\[ \to 6(u^*)^2\eta + 6(u^*)^2\eta - 2\beta \eta, \quad \text{strongly in} \quad L^2(Q), \tag{4.39} \]

\[ -12u^*v^*\xi^N \to -12u^*v^*\xi, \quad 12u^*v^*\eta^N \to 12u^*v^*\eta^N, \quad \text{strongly in} \quad L^2(Q), \tag{4.40} \]

\[ \Delta \xi^N \to \Delta \xi, \quad \Delta \eta^N \to \Delta \eta, \quad \rho^N \rightharpoonup \rho, \quad \text{weakly in} \quad L^2(Q). \tag{4.41} \]

Due to (4.30), (4.34), (4.35) and (4.37)-(4.41), we can pass to the limit in the following system

\[
\int_0^T \left( \frac{\partial \xi^N}{\partial t}, \chi \right) dx dt = \int_0^T \int_{\Omega} [-6(u^*)^2\xi^N - 6(u^*)^2\xi^N + (2\beta - \alpha)\xi^N - 12u^*v^*\eta] \chi dx dt
+ \int_0^T \int_{\Omega} (h + \Delta \xi^N) \cdot \chi dx dt, \quad \text{for} \quad \chi \in L^2(0,T;H),
\]

\[
\int_0^T \left( \frac{\partial \eta^N}{\partial t}, \chi \right) dt = \int_0^T \int_{\Omega} \rho^N \cdot \Delta \chi dx dt, \quad \text{for} \quad \chi \in L^2(0,T;H^2(\Omega)),
\]

\[
\int_0^T \int_{\Omega} \rho^N \cdot \Delta \chi dx dt = \int_0^T \int_{\Omega} [6(u^*)^2\eta^N + 6(u^*)^2\eta^N + 12u^*v^*\xi^N] \cdot \chi dx dt
- \int_0^T \int_{\Omega} [2\beta \eta^N + \Delta \eta^N] \cdot \chi dx dt, \quad \text{for} \quad \chi \in L^2(0,T;H).
\]

Hence we know that (4.10), (4.11) and (4.12) are established. Moreover we easily obtain that $\xi(x,0) = \eta(x,0) = 0$ almost everywhere in $\Omega$ and $\xi = \eta = 0$ almost everywhere in $\Sigma$. As for the uniqueness of the solution, we can prove it similarly and thus omit it. This completes the proof. \qed

Under the preceding discussion, we will be able to prove the following theorem.
Theorem 4.3. Assume that $v_0, u_0 \in W^{1,4}(\Omega) \cap H^2(\Omega)$. Then $\mathcal{S}$ is Fréchet differentiable. Furthermore, for any $\phi^* \in \Phi_R$, the Fréchet derivative $DS(\phi^*)$ satisfies

$$DS(\phi^*)h = (\xi, \eta, \rho), \quad \text{for } \forall \ h \in L^2(Q).$$

Here $(\xi, \eta, \rho)$ is the unique solution of the system (4.2) – (4.6).

Proof. Because $\Phi_R$ is an open set, hence for any fixed $\phi^* \in \Phi_R$, there exists some $\varepsilon > 0$ such that $\phi^* + h \in \Phi_R$. It should be noted that $h \in L^2(Q)$ and $\|h\|_{L^2(Q)} \leq \varepsilon$. Then we denote that $w^h = v^h - v^* - \xi, \ y^h = w^h - u^* - \eta$ and $z^h = \mu^h - \mu^* - \rho$, where $(v^h, u^h, \mu^h) = \mathcal{S}(\phi^* + h)$. In addition, we also obtain from Theorem 3.2 and Theorem 4.2 that

$$w^h \in H^1(0, T; H) \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),
\quad y^h \in H^1(0, T; (H^2(\Omega))') \cap L^\infty(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)),
\quad z^h \in L^2(0, T; H).$$

Owing to the Taylor’s theorem with integral remainder, we thus deduce that

$$\begin{align*}
-2(\varepsilon^h)^3 + 2(\varepsilon^*)^3 + 6(\varepsilon^*)^2\xi &= -2[3(\varepsilon^*)^2w^h + (v^h - v^*)^2(v^h + 2v^*)], \\
2(\varepsilon^h)^3 - 2(\varepsilon^*)^3 - 6(\varepsilon^*)^2\eta &= 2[3(\varepsilon^*)^2y^h + (w^h - u^*)^2(u^h + 2u^*)], \\
-6(u^h)^2v^h + 6(\varepsilon^*)^2v^* + 6(\varepsilon^*)^2\xi + 12u^*v^*\eta &= -6(u^*)^2w^h - 12u^*v^h\eta - 6(u^h - u^*)^2v^h - 12u^*\eta(v^h - v^*), \\
6(v^h)^2u^h - 6(v^*)^2u^* - 6(v^*)^2\eta - 12u^*v^*\xi &= 6(v^*)^2y^h + 12u^*v^h\eta - 6(v^h - v^*)^2u^h + 12v^*\xi(u^h - u^*).
\end{align*}$$

Hence we know that $(w^h, y^h, z^h)$ satisfies the following problem

$$\begin{align*}
\frac{\partial w^h}{\partial t} &= \Delta w^h + (2\beta - \alpha)w^h - 2[3(\varepsilon^*)^2w^h + (v^h - v^*)^2(v^h + 2v^*)] - 6(\varepsilon^*)^2w^h \\
- 12u^*v^h\eta - 6(u^h - u^*)^2v^h - 12u^*\eta(v^h - v^*) \quad \text{a.e. } Q, \quad (4.42) \\
\int_0^T \left< \frac{\partial y^h}{\partial t}, \varphi \right> dt &= \int_\Omega z^h \cdot \Delta \varphi dx dt, \quad \text{for all } \varphi \in L^2(0, T; H^2(\Omega)), \quad (4.43) \\
z^h &= -\Delta y^h - 2\partial y^h + 2[3(\varepsilon^*)^2y^h + (u^h - u^*)^2(u^h + 2u^*)] + 6(\varepsilon^*)^2y^h \\
+ 12v^h\eta w^h + 6(v^h - v^*)^2u^h + 12v^*\xi(u^h - u^*) \quad \text{a.e. } Q, \quad (4.44) \\
w^h = y^h = 0, \quad \text{on } \Sigma, \quad (4.45) \\
w^h(0) = w^h(x, 0) = 0, \quad y^h(0) = y^h(x, 0) = 0, \quad \text{a.e. } \Omega. \quad (4.46)
\end{align*}$$

Now we multiply (4.42) by $\frac{\partial w^h}{\partial t}$ and (4.44) by $\frac{\partial y^h}{\partial t}$ and choose $\varphi = (-\Delta)^{-1} \frac{\partial y^h}{\partial t}$ in (4.43), and then integrate the resulting identity over $\Omega \times (0, t)$, we thus obtain from (3.45) that

$$\begin{align*}
\int_0^t \int_\Omega \frac{\partial w^h}{\partial t}^2 dx dt + \int_0^t \int_\Omega \nabla w^h \cdot \nabla \frac{\partial w^h}{\partial t} dx ds + \int_0^t \int_\Omega \left( (-\Delta)^{-\frac{1}{2}} \frac{\partial y^h}{\partial t} \right)^2 dx ds \\
+ \int_0^t \int_\Omega \nabla y^h \cdot \nabla \frac{\partial y^h}{\partial t} dx ds
\end{align*}$$
\[
\int_0^t \left( (2\beta - \alpha)w^h - 2 \left[ 3(v)^2 w^h + (v^h - v^*)^2 (v^h + 2v^*) \right] 
- 6(u^*)^2 w^h - 12u^*v^h y^h - 6 (v^h - u^*)^2 v^h - 12u^*\eta (v^h - v^*) \right) \frac{\partial w^h}{\partial t} \, ds \\
+ \int_0^t \left( 2\beta y^h - 2 \left[ 3(u^*)^2 y^h + (u^h - u^*)^2 (u^h + 2u^*) \right] 
- 6(v^*)^2 y^h - 12v^*u^h w^h - 6 (v^h - v^*)^2 u^h - 12v^*\xi (u^h - u^*) \right) \frac{\partial y^h}{\partial t} \, ds \\
\leq C \int_0^t \left| \left( \frac{\partial w^h}{\partial t} \right) \right| \, ds + C \int_0^t \left| \left( \frac{\partial w^h}{\partial t} \right) \right| \, ds \\
+ C \int_0^t \left| \left( \frac{\partial w^h}{\partial t} \right) \right| \, ds + C \int_0^t \left| \left( \frac{\partial w^h}{\partial t} \right) \right| \, ds \\
+ C \int_0^t \left| \left( \frac{\partial w^h}{\partial t} \right) \right| \, ds + C \int_0^t \left| \left( \frac{\partial w^h}{\partial t} \right) \right| \, ds \\
+ C \int_0^t \left| \left( \frac{\partial w^h}{\partial t} \right) \right| \, ds + C \int_0^t \left| \left( \frac{\partial w^h}{\partial t} \right) \right| \, ds.
\]

Similar to the preceding discussion, we obtain from Young’s inequality and the Sobolev embedding theorem that

\[
C \int_0^t \int_0^t \left( \frac{\partial w^h}{\partial t} \right)^2 \, dx \, ds + C \int_0^t \int_0^t \nabla w^h \cdot \nabla \frac{\partial w^h}{\partial t} \, dx \, ds \\
+ C \int_0^t \int_0^t \left( (-\Delta)^{-\frac{1}{2}} \frac{\partial y^h}{\partial t} \right)^2 \, dx \, ds + C \int_0^t \int_0^t \nabla y^h \cdot \nabla \frac{\partial y^h}{\partial t} \, dx \, ds \\
\leq C \int_0^t \int_0^t \left( |\nabla w^h|^2 + |\nabla y^h|^2 \right) \, dx \, ds + C \int_0^t \int_0^t \left[ |(u^h - u^*)^4 + (v^h - v^*)^4 \right] \, dx \, ds \\
+ C \int_0^t \int_0^t \left\{ |\nabla (v^h - v^*)|^2 + |\nabla (u^h - u^*)|^2 \right\} \, dx \, ds \\
+ C \int_0^t \int_0^t \left\{ |\nabla (v^h - v^*)|^2 + |\nabla (u^h - u^*)|^2 \right\} \, dx \, ds \\
\leq C \int_0^t \int_0^t |\nabla w^h|^2 \, dx \, ds + C \int_0^t \int_0^t |\nabla y^h|^2 \, dx \, ds + C \|h\|_{L^2(0,t;H)}^4.
\]

Applying the Gronwall inequality we find that

\[
\|w^h\|_{L^\infty(0,t;H^1(\Omega))} + \|y^h\|_{L^\infty(0,t;H^1(\Omega))} \leq C \|h\|_{L^2(0,t;H)}^2.
\]

Next, we multiply \( \frac{\partial \omega^h}{\partial t} \) again and then integrate the resulting identity over \( \Omega \times (0,t) \), we have

\[
\int_0^t \int_0^t \left( \frac{\partial w^h}{\partial t} \right)^2 \, dx \, ds + \int_0^t \int \nabla w^h \cdot \nabla \frac{\partial w^h}{\partial t} \, dx \, ds \\
= \int_0^t \int_0^t \left[ (2\beta - \alpha)w^h - 2(3v^*)^2 w^h + (v^h - v^*)^2 (v^h + 2v^*) - 6(v^*)^2 w^h \right] \, dx \, ds.
\]
\[
-12u^*v^yh^h - 6(u^h - u^*)^2v^h - 12u^*\eta(v^h - v^*) \cdot \frac{\partial w^h}{\partial t} dxds
\leq \frac{1}{2} \int_0^t \int_{\Omega} \left( \frac{\partial w^h}{\partial t} \right)^2 dxds + C \int_0^t \int_{\Omega} \left\{ w^h - [(v^*)^2]w^h + (v^h - v^*)^2(v^h + 2v^*) - (u^*)^2w^h - u^*v^yh - (u^h - u^*)^2v^h - u^*\eta(v^h - v^*) \right\}^2 dxds
\leq C \int_0^t \int_{\Omega} (w^h)^2 dxds + C \int_0^t \int_{\Omega} (y^h)^2 dxds + C \int_0^t \int_{\Omega} (u^h - u^*)^4 dxds
+ C \int_0^t \int_{\Omega} (v^h - v^*)^4 dxds + C \int_0^t \int_{\Omega} |\eta|^4 dxds
\leq C\|h\|^4_{L^2(0,t;H)}},
\]
which implies that
\[
\|w^h\|_{L^\infty(0,t;H^1(\Omega))} + \|w^h\|_{H^1(0,t;H)} \leq C\|h\|^2_{L^2(0,t;H)}. \tag{4.48}
\]
Furthermore, owing to (4.42), we also have
\[
\|\Delta w^h\|_{L^2(0,t;H)} \leq \left\| \frac{\partial w^h}{\partial t} \right\|_{L^2(0,t;H)} + \|(\alpha - 2\beta)w^h\|_{L^2(0,t;H)} + \|(6(v^*)^2 + 6(v^*)^2)w^h\|_{L^2(0,t;H)} + \|2(v^h - v^*)^2(v^h + 2v^*) + 12u^*v^yh + 6(u^h - u^*)^2v^h\|_{L^2(0,t;H)}
+ \|12u^*\eta(v^h - v^*)\|_{L^2(0,t;H)} \leq C\|h\|^2_{L^2(0,t;H)},
\]
which implies from (4.47) that
\[
\|w^h\|^4_{L^2(0,t;H^2(\Omega))} \leq C\|h\|^2_{L^2(0,t;H)}. \tag{4.49}
\]
Here we also obtain from (4.47)-(4.49) that
\[
\|w^h\|^4_{C(0,t;H^1(\Omega))} \leq C\|h\|^2_{L^2(0,t;H)}, \tag{4.50}
\]
where we refer to the literature [7]. Next, we set \(\varphi = y^h\) in (4.43), and multiply (4.44) by \(-\Delta y^h\) and then integrate the resulting identity over \(\Omega \times (0,t)\), we have
\[
\int_0^t \int_{\Omega} \frac{\partial y^h}{\partial t} \cdot y^h dxds + \int_0^t \int_{\Omega} (\Delta y^h)^2 dxds
= \int_0^t \int_{\Omega} \left\{ -2\beta y^h + 2[3(u^*)^2y^h + (u^h - u^*)^2(u^h + 2u^*)] + 6(v^*)^2y^h \right\} \cdot \Delta y^h dxds
+ \int_0^t \int_{\Omega} \left[ 12v^*u^hw^h + 6(v^h - v^*)^2u^h + 12v^*\xi(u^h - u^*) \right] \cdot \Delta y^h dxds
\leq \frac{1}{2} \int_0^t \int_{\Omega} (\Delta y^h)^2 dxds + \int_0^t \int_{\Omega} \left[ 12v^*u^hw^h + 6(v^h - v^*)^2u^h + 12v^*\xi(u^h - u^*) \right]^2 dxds
\leq \frac{1}{2} \int_0^t \int_{\Omega} (\Delta y^h)^2 dxds + C\|h\|^4_{L^2(0,t;H)},
\]
which yields, owing to (4.47),
\[
\|y^h\|^4_{L^2(0,t;H^2(\Omega))} \leq C\|h\|^2_{L^2(0,t;H)}. \tag{4.51}
\]
Furthermore, owing to (4.44), (4.47)-(4.49) and (4.51), which gives
\[ \|z^h\|_{L^2(0,t;H)} \leq \| - \Delta y^h - 2\beta y^h + 2[3(u^*)^2y^h + (u^h - u^*)^2(u^h + 2u^*)] + 6(v^*)^2y^h \|_{L^2(0,t;H)} \]
\[ + \| 12v^*u^h w^h + 6(v^h - v^*)^2u^h + 12v^*\xi(u^h - u^*) \|_{L^2(0,t;H)} \]
\[ \leq \| - \Delta y^h \|_{L^2(0,t;H)} + \| - 2\beta y^h + 2[3(u^*)^2y^h + (u^h - u^*)^2(u^h + 2u^*)] \|_{L^2(0,t;H)} \]
\[ + \| 6(v^*)^2y^h + 12v^*u^h w^h + 6(v^h - v^*)^2u^h + 12v^*\xi(u^h - u^*) \|_{L^2(0,t;H)} \]
\[ \leq C\|h\|^2_{L^2(0,t;H)}. \] (4.52)

For any \( \chi \in L^2(0,t;H^2(\Omega)) \), we can deduce from (4.43) and the Hölder inequality that
\[ \int_0^t \int_{\Omega} \frac{\partial y^h}{\partial t} \cdot \chi dx ds = \int_0^t \int_{\Omega} z^h \cdot \Delta \chi dx ds \leq \|z^h\|_{L^2(0,t;H)} \cdot \|\chi\|_{L^2(0,t,H^2(\Omega))}, \]
which yields
\[ \left\| \frac{\partial y^h}{\partial t} \right\|_{L^2(0,t;H^2(\Omega)^{\prime})} \leq C\|h\|^2_{L^2(0,t;H)}. \] (4.53)

Moreover, we deduce from (4.48), (4.53) and the paper [29] that
\[ \|y^h\|_{C(0,t,H)} \leq C\|h\|^2_{L^2(0,t;H)}. \] (4.54)

Therefore, we obtain from (4.47)-(4.54) that (4.1) is valid as \( \|h\|_{L^2(Q)} \to 0. \) Moreover, we obtain from (4.13) that the linear mapping \( h \mapsto (\xi, \eta, \rho) \) is a continuous mapping from \( L^2(Q) \) to \( \mathcal{Y}. \) Hence \( \mathcal{S} \) is Fréchet differentiable and for any \( \phi^* \in \Phi_R, \)
\[ DS(\phi^*)h = (\xi, \eta, \rho), \quad \forall \ h \in L^2(Q). \]

This completes the proof. \( \square \)

In the final section, we will give the first-order necessary optimality condition for the optimal control problem (CP). Furthermore, we declare in the next discussions are conducted in assumptions which are \( v_0(x), u_0(x) \in W^{1,4}(\Omega) \cap H^2(\Omega) \) and (2.5)-(2.7). In order to get the first-order necessary optimality condition for the optimal control problem (CP), we firstly consider the the adjoint system for (CP) with the associate state \((\tilde{v}, \tilde{u}, \tilde{\mu}) = S(\tilde{\phi})\)

\[- \frac{\partial p}{\partial t} - \Delta p + [6(\tilde{u})^2 + 6(\tilde{v})^2 + \alpha - 2\beta]p - 12\tilde{u}\tilde{v}r = 0, \quad \text{in} \ Q, \] (4.55)
\[- \frac{\partial q}{\partial t} + \Delta r - [6(\tilde{u})^2 + 6(\tilde{v})^2 - 2\beta]r + 12\tilde{u}\tilde{v}p = \rho_1(\tilde{u} - u_Q), \quad \text{in} \ Q, \] (4.56)
\[ r - \Delta q = 0, \quad \text{in} \ Q, \] (4.57)
\[ p = q = r = 0, \quad \text{on} \ \Sigma, \] (4.58)
\[ p(T) = 0, \quad q(T) = \rho_2(\tilde{u}(T) - u_{\Omega}), \quad \text{in} \ \Omega. \] (4.59)

Obviously, we set \( \tilde{p}(t) \triangleq p(T - t), \tilde{q}(t) \triangleq q(T - t), \tilde{r}(t) \triangleq r(T - t), \) in order to simplify notations, we still write \( p, q, r \) instead of \( \tilde{p}, \tilde{q}, \tilde{r}. \) Then the adjoint system (4.55)-(4.59) be written as follows
\[ \frac{\partial \tilde{p}}{\partial t} - \Delta \tilde{p} + [6(\tilde{u})^2 + 6(\tilde{v})^2 + \alpha - 2\beta]p - 12\tilde{u}\tilde{v}r = 0, \quad \text{in} \ Q, \] (4.60)
\[
\frac{\partial q}{\partial t} + \Delta r - [6(\bar{u})^2 + 6(\bar{v})^2 - 2\beta]r + 12\bar{u}\bar{r}p = \rho_1(\bar{u} - u_Q), \quad \text{in } Q, \quad (4.61)
\]
\[
r - \Delta q = 0, \quad \text{in } Q, \quad (4.62)
\]
\[
p = q = r = 0, \quad \text{on } \Sigma, \quad (4.63)
\]
\[
p(0) = 0, \quad q(0) = \rho_2(\bar{u}(T) - u_Q), \quad \text{in } \Omega. \quad (4.64)
\]

Now we multiply (4.60) by \(p\), (4.61) by \(q\) and (4.62) by \(-\Delta q\), integrate over \(\Omega \times (0, t)\) and then sum the the resulting identities, then we obtain from Young’s inequality that
\[
\int_0^t \int_\Omega \frac{\partial p}{\partial t} \cdot pdxds + \int_0^t \int_\Omega |\nabla p|^2 dxds + 6 \int_0^t \int_\Omega (\bar{u})^2 p^2 dxds + 6 \int_0^t \int_\Omega (\bar{v})^2 p^2 dxds + 6 \int_0^t \int_\Omega \rho_1(\bar{u} - u_Q) q dxds
\]
\[
+ \int_0^t \int_\Omega \frac{\partial q}{\partial t} \cdot q dxds + \int_0^t \int_\Omega (\Delta q)^2 dxds = \int_0^t \int_\Omega (2\beta - \alpha)p^2 dxds + \int_0^t \int_\Omega 12\bar{u}\bar{r}pdxds + \int_0^t \int_\Omega [6(\bar{u})^2 + 6(\bar{v})^2 - 2\beta]rq dxds
\]
\[
- \int_0^t \int_\Omega 12\bar{u}\bar{r}pdxds + \int_0^t \int_\Omega \rho_1(\bar{u} - u_Q) q dxds \leq \frac{1}{2} \int_0^t \int_\Omega (\Delta q)^2 dxds + C \int_0^t \int_\Omega p^2 dxds + \int_0^t \int_\Omega q^2 dxds
\]
\[
+ C \int_0^t \int_\Omega (\bar{u} - u_Q)^2 dxds. \]

Now using the Gronwall inequality and (2.6), we obtain
\[
\|p\|_{L^\infty(0,T;L^2(\Omega))} + \|p\|_{L^2(0,T;H^1(\Omega))} + \|q\|_{L^2(0,T;H^2(\Omega))} + \|q\|_{L^\infty(0,T;H^1)} + \|\tilde{u}p\|_{L^2(0,T;H^1)} + \|\tilde{v}p\|_{L^2(0,T;H^1)} \leq C. \quad (4.65)
\]

In addition, we test (4.60) by \(\frac{\partial p}{\partial t}\), then we have
\[
\int_0^t \int_\Omega \left(\frac{\partial p}{\partial t}\right)^2 dxds + \int_0^t \int_\Omega \nabla p \cdot \nabla \left(\frac{\partial p}{\partial t}\right) dxds
\]
\[
= \int_0^t \int_\Omega 12\bar{u}\bar{r} \frac{\partial p}{\partial t} dxds - \int_0^t \int_\Omega [6(\bar{u})^2 + 6(\bar{v})^2 + \alpha - 2\beta]p \cdot \frac{\partial p}{\partial t} dxds
\]
\[
\leq \frac{1}{2} \int_0^t \int_\Omega \left(\frac{\partial p}{\partial t}\right)^2 dxds + C \int_0^t \int_\Omega r^2 dxds + C \int_0^t \int_\Omega p^2 dxds
\]
\[
\leq \frac{1}{2} \int_0^t \int_\Omega \left(\frac{\partial p}{\partial t}\right)^2 dxds + \int_0^t \int_\Omega (\Delta q)^2 dxds + C \int_0^t \int_\Omega p^2 dxds
\]
\[
\leq \frac{1}{2} \int_0^t \int_\Omega \left(\frac{\partial p}{\partial t}\right)^2 dxds + C,
\]
where we use (4.65) and the fact that \(r \in L^2(0,T;H^1(\Omega))\) on account of (4.62). Hence we receive
\[
\|p\|_{L^\infty(0,T;H^1(\Omega))} + \|p\|_{H^1(0,T;H)} \leq C. \quad (4.66)
\]

Similarly, we multiply (4.60) by \(-\Delta p\) and then obtain from (4.66) that
\[
\int_0^t \int_\Omega (\Delta p)^2 dxds
\]
where \( \tilde{\varphi} \in \Phi_{ad} \) is an optimal control for \((CP)\). Then the problem

\[ p \in H^1(0, T; H) \cap C(0, T; H^1(\Omega)) \cap L^2(0, T; H^2(\Omega)), \]

\[ q \in H^1(0, T; V') \cap C(0, T; H) \cap L^2(0, T; H^2(\Omega)), \]

\[ r \in L^2(0, T; H), \]

which satisfies the following identities

\[ -\frac{\partial p}{\partial t} - \Delta p + [6(\tilde{u})^2 + 6(\tilde{v})^2 + \alpha - 2\beta]p - 12\tilde{u}\tilde{v}r = 0, \quad a.e. \quad Q, \quad (4.69) \]

\[ r - \Delta q = 0, \quad a.e. \quad Q, \quad (4.70) \]

\[ \int_0^T \left( -\frac{\partial q}{\partial t}, \varphi \right) dt + \int_0^T \int_\Omega r \cdot \Delta \varphi dx dt \]

\[ = \int_0^T \int_\Omega \left[ [6(\tilde{u})^2 + 6(\tilde{v})^2 - 2\beta]r + 12\tilde{u}\tilde{v}p + \rho_1(\tilde{u} - u_Q) \right] \cdot \varphi dx dt, \quad (4.71) \]

for all \( \varphi \in L^2(0, T; V) \), where \( p(T) = 0, \ q(T) = \rho_2(\tilde{u}(T) - u_\Omega) \) almost everywhere in \( \Omega \) and \( p = q = 0 \) almost everywhere in \( \Sigma \).

At this point, we can derive the first-order necessary optimality condition which is related to the adjoint system for the optimal control problem \((CP)\).
Theorem 4.5. For the optimal control problem \((CP)\) with the associate state \((\bar{v}, \bar{u}, \bar{m}) = S(\bar{\phi})\) and the adjoint state system (4.55)-(4.59), we can obtain the variational inequality
\[
\int_0^T \int_\Omega (\phi - \bar{\phi})(p + \rho_3 \bar{\phi}) dx dt \geq 0, \quad \text{for } \forall \phi \in \Phi_{ad}.
\] (4.72)

Proof. First of all, we denote \(\eta\) to stand for the second component of the unique solution to the linearized system (4.2)-(4.6) corresponding to \(h = \phi - \bar{\phi}\). Moreover, we recall that the reduced cost functional \(\bar{J}(\phi) = \bar{J}(\Phi, \phi)\) for any \(\phi \in \Phi_{ad}\), where \(\Phi = u\) is the second component of \(S(\phi)\). In addition, on the basis of \(\Phi_{ad}\) is a convex subset of \(L^2(Q)\) and [26], we can obtain
\[
\bar{J}(\phi)(\phi - \bar{\phi}) \geq 0, \quad \text{for } \forall \phi \in \Phi_{ad}.
\] (4.73)

Furthermore, we obtain from the chain rule that
\[
\bar{J}(\phi) = \mathcal{J}'(\Phi, \phi)(\Phi_{ad}, \bar{\phi}) DS_1(\bar{\phi}) + \mathcal{J}'(\Phi, \phi)(\Phi_{ad}, \bar{\phi}) DS_1(\Phi_{ad}, \bar{\phi})
\]
Here \(\mathcal{J}'(\Phi, \phi)\) is the Fréchet derivative of \(\mathcal{J}(\Phi, \phi)\) with respect to \(\Phi = u\) and \(\phi\) respectively. Besides, we obtain from Theorem 4.3 that
\[
DS_1(\Phi_{ad})(\Phi_{ad}, \bar{\phi}) = \eta.
\]
we thus deduce from \(S_1(\Phi_{ad}) = \bar{u}\) that
\[
\bar{J}(\phi)(\phi - \bar{\phi}) = \mathcal{J}'(\Phi, \phi)(\bar{u}, \bar{\phi})(\eta) + \mathcal{J}'(\Phi, \phi)(\bar{u}, \bar{\phi})(\phi - \bar{\phi}),
\]
which along with (2.8) and (4.73), implies
\[
\rho_1 \int_0^T \int_\Omega (\bar{u} - u) \eta dx dt + \rho_2 \int_\Omega (\bar{u}(T) - u(T)) \eta(T) dx
\]
\[
+ \rho_3 \int_0^T \int_\Omega (\phi - \bar{\phi}) \bar{\phi} dx dt \geq 0, \quad \text{for } \forall \phi \in \Phi_{ad}.
\] (4.74)

By integration by parts, we obtain from (4.10)-(4.12) and (4.69)-(4.71) that
\[
\int_0^T \int_\Omega [12\bar{u} v r 12v^* u^* \eta p - (\phi - \bar{\phi}) p] dx dt = \int_0^T \int_\Omega [\xi - \frac{\partial p}{\partial t} - \Delta p + (6\bar{u}^2 + 6\bar{v}^2 + \alpha - 2\beta)p] + [12v^* u^* \eta - (\phi - \bar{\phi}) p] dx dt
\]
\[
= \int_0^T \int_\Omega \left[\frac{\partial \xi}{\partial t} + 6(v^*)^2 \xi + 6(u^*)^2 \xi + (\alpha - 2\beta)\xi + 12v^* u^* \eta - \Delta \xi - (\phi - \bar{\phi}) p\right] dx dt = 0,
\] (4.75)

\[
\rho_2 \int_\Omega (\bar{u}(T) - u(T)) \eta(T) dx - \int_0^T \left[\frac{\partial q}{\partial t}, \eta\right] dt - \int_0^T \int_\Omega \rho \cdot \Delta q dx dt
\]
\[
= \int_0^T \left[\frac{\partial q}{\partial t}, \eta\right] dt - \int_0^T \int_\Omega \rho \cdot \Delta q dx dt = 0,
\] (4.76)

\[
\int_0^T \int_\Omega \rho \cdot \Delta q dx dt - \int_0^T \int_\Omega [6(u^*)^2 + 6(v^*)^2 - 2\beta] \eta r dx dt - \int_0^T \int_\Omega 12v^* u^* \xi r dx dt
\]
\[
+ \left\{\int_0^T \left[\frac{\partial q}{\partial t}, \eta\right] dt + \int_\Omega [6(u^*)^2 + 6(v^*)^2 - 2\beta] \eta r dx dt - \int_0^T \int_\Omega 12v^* u^* p \eta dx dt
\]
\[ + \int_0^T \int_\Omega \rho_1 (\tilde{u} - u_Q) \eta dx dt \}
\]
\[ = \int_0^T \int_\Omega \rho \cdot \Delta q dx dt - \int_0^T \int_\Omega [6(u^*)^2 + 6(v^*)^2 - 2\beta] \eta r dx dt - \int_0^T \int_\Omega 12v^*u^* \xi r dx dt
\]
\[ + \int_0^T \int_\Omega \Delta \eta \cdot r dx dt
\]
\[ = \int_0^T \int_\Omega \rho r dx dt - \int_0^T \int_\Omega [6(u^*)^2 + 6(v^*)^2 - 2\beta] \eta r dx dt - \int_0^T \int_\Omega 12v^*u^* \xi r dx dt
\]
\[ + \int_0^T \int_\Omega \Delta \eta \cdot r dx dt
\]
\[ = \int_0^T \int_\Omega [\rho - 6(u^*)^2 \eta - 6(v^*)^2 \eta - 12v^*u^* \xi + 2\beta \eta + \Delta \eta] r dx dt
\]
\[ = 0. \quad (4.77)
\]
Let us sum the three identities (4.75)-(4.77). Taking also (4.74) into account, we get
\[ \int_0^T \int_\Omega (\phi - \tilde{\phi})(p + \rho_3 \tilde{\phi}) dx dt \geq 0, \text{ for } \forall \phi \in \Phi_{ad}. \quad (4.78)
\]
The proof of Theorem 4.5 is now finished. \(\square\)

5. Conclusions. We provided some theoretical results for a phase-field system was derived in [2], which is to model simultaneous order-disorder and phase separation in binary alloys on a BCC lattice in the neighborhood of the triple point. A formal asymptotic analysis is carried out in [15]. Our first main result is a well-posedness theorem for weak solutions of the system (1.3)-(1.6). We point out that the existence of a solution is obtained through an approximation scheme which might be helpful also for numerical approximations. Furthermore, we also provide a continuous dependence result of global strong solutions for control term \(\phi\) under proper assumptions on the initial data. A further main result is based on a prior estimates of (1.3)-(1.6), namely, the fact that the existence of optimal controls. More precisely, we demonstrate the differentiability properties of the control-to-state operator. In particular, the first-order necessary optimality conditions for the optimal control problem are discussed as well. We think that optimal distributed control problem of this model is a step towards its development. Moreover, this result may be play a role in designing more appropriate target in production processes, etc.

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