\textbf{$\mathcal{PT}$-Symmetric Quantum Mechanics: A Precise and Consistent Formulation}

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Abstract

The physical condition that the expectation values of physical observables are real quantities is used to give a precise formulation of $\mathcal{PT}$-symmetric quantum mechanics. A mathematically rigorous proof is given to establish the physical equivalence of $\mathcal{PT}$-symmetric and conventional quantum mechanics. The results reported in this paper apply to arbitrary $\mathcal{PT}$-symmetric Hamiltonians with a real and discrete spectrum. They hold regardless of whether the boundary conditions defining the spectrum of the Hamiltonian are given on the real line or a complex contour.

1 Introduction

Perhaps one of the most important lessons I have learned during my mathematical education, and highly appreciated in my scientific career, is that “The most difficult statements to prove are those that are false, [1].” In my study of theoretical physics I also learned that a prominent feature of all successful fundamental theories is that they have a rigid structure. This actually constitutes the basis of one of the main arguments for justifying research in string theory.\footnote{There are a handful of string theories that are known not to be inconsistent; slightest variation of these leads to the violation of one or more of the basic physical principles such as unitarity or absence of anomalies.} What is not so well-known or at least well-appreciated, especially among the contributors to the development of $\mathcal{PT}$-symmetric QM, is that the conventional QM itself

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is one of the most rigid physical theories that we have come to establish. This is actually at the heart of the failure of three generations of theoretical physicists to generalize QM in any fundamental way. The purpose of the present paper is to show that $\mathcal{PT}$-symmetric or complex extension of QM \cite{2} is no exception.

As this has been a sensitive issue, I have tried to make my analysis as precise and mathematically rigorous as possible. In order to make the paper reasonably self-contained and avoid potential terminological ambiguities, I have included the relevant mathematical definitions and theorems. These should be easily accessible for a typical theoretical physicist familiar with basic features of QM.

The main motivation for the present study has been my attempts to apply the following principle to $\mathcal{PT}$-symmetric QM: In order to achieve a satisfactory understanding of the virtues of a new theoretical scheme, especially if it lacks experimental support, one is obliged to formulate and state the basic postulates of the theory in a precise language, translate them into mathematical statements, and learn and use the standard mathematical notions and theorems, or develop new mathematical tools if necessary, to assess its viability as a consistent physical theory and to determine its relation to the established theories.

I close this section by a quote from Jurg Fröhlich: “It is possible to do good theoretical physics and at the same time be mathematically rigorous.”\textsuperscript{2}

\section{Basic Mathematical Facts}

The paragraphs of this section have been labelled for future reference.

\textbf{P0. Notations and Conventions:} $\mathbb{N}$, $\mathbb{R}$, and $\mathbb{C}$ denote the sets of natural (including zero), real, and complex numbers, respectively; $\mathbb{R}^+$ stands for the set of strictly positive real numbers; The symbols $\forall$, $\exists$, $^*$ respectively mean ‘for all’, ‘there exists’, and ‘complex conjugate’; $\delta_{mn}$ stands for the Kronecker delta function.

\textbf{P1.} Let $V$ be a complex vector space \cite{3}. A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ is called an \textbf{inner product} on $V$ if it is nondegenerate ($\langle \phi, \psi \rangle = 0$ for all $\phi \in V$ implies $\psi = 0$), Hermitian ($\langle \psi, \phi \rangle = \langle \phi, \psi \rangle^*$), and sesquilinear ($\forall a, b \in \mathbb{C}, \forall \psi, \phi, \chi \in V, \langle \psi, a\phi + b\chi \rangle = a\langle \psi, \phi \rangle + b\langle \psi, \chi \rangle$). An inner product $\langle \cdot, \cdot \rangle$ is called \textbf{positive-definite} if it also satisfies: $\langle \psi, \psi \rangle \in \mathbb{R}^+ \cup \{0\}$ for all $\psi \in V$, and $\langle \psi, \psi \rangle = 0$ implies $\psi = 0$.\textsuperscript{3}

\textbf{P2.} A \textbf{complex inner product space} $N$ is a complex vector space endowed with a positive-definite inner product. The function $\| \cdot \| : V \to \mathbb{R}^+ \cup \{0\}$ defined by $\| \psi \| := \sqrt{\langle \psi, \psi \rangle}$

\textsuperscript{2}Stated during his lecture at Les Houches Summer School on ‘Quantum Field Theory: Perspectives and Prospective,’ June 1998.

\textsuperscript{3}As this condition implies nondegenerateness, usually one does not separately postulate the latter in defining a positive-definite inner product.
is called the norm of $N$. The convergence of sequences $\{\psi_n\}$ in $N$ is defined using the norm: $\psi_n \to \psi$ if $\lim_{n \to \infty} \| \psi_n - \psi \| = 0$. $\{\psi_n\}$ is called a Cauchy sequence if $\forall m, n \geq M$, $\lim_{M \to \infty} \| \psi_n - \psi_m \| = 0$ (i.e., $\forall \epsilon \in \mathbb{R}^+$, $\exists M \in \mathbb{N}$ such that $\forall m, n \geq M$, $\| \psi_n - \psi_m \| < \epsilon$.)

A $\subseteq N$ is said to be a dense subset of $N$ if every $\psi \in N$ is the limit of a sequence of elements of $A$.

P3. A complex Hilbert space $\mathcal{H}$ is a complex inner product space that is norm-complete, i.e., the Cauchy sequences in $\mathcal{H}$ converge. All the Hilbert spaces used in this paper are complex Hilbert spaces.

P4. There is a well-defined procedure to extend an inner product space $N$, in a unique way, to a Hilbert space $\mathcal{H}$ such that $N$ is dense in $\mathcal{H}$, $[4]$. $\mathcal{H}$ is then called the Cauchy-completion of $N$, and $\mathcal{H}$ is said to be Cauchy-completed to $\mathcal{H}$.

P5. A complex separable Hilbert space is a complex Hilbert space that has a countable orthonormal basis. The latter is a subset $\mathcal{B} = \{\beta_n \in \mathcal{H} | n \in \mathbb{N}\}$ of $\mathcal{H}$ satisfying: $\forall m, n \in \mathbb{N}$, $\langle \beta_m, \beta_n \rangle = \delta_{mn}$ and $\forall \psi \in \mathcal{H}$, $\exists a_n \in \mathbb{C}$ such that $\psi = \sum_{n=0}^{\infty} a_n \beta_n$. The typical examples are the space of square-integrable functions: $L^2(\mathbb{R}) := \{ \psi : \mathbb{R} \to \mathbb{C} | \int_{\mathbb{R}} |\psi(x)|^2 dx < \infty \}$ with inner product $\langle \psi | \phi \rangle = \int_{\mathbb{R}} \psi(x)^* \phi(x) dx$, and the space of square-summable sequences: $l_2 := \{ \psi : \mathbb{N} \to \mathbb{C} | \sum_{n=0}^{\infty} |\psi(n)|^2 < \infty \}$ with inner product $\langle \psi, \phi \rangle = \sum_{n=0}^{\infty} \psi(n)^* \phi(n)$.

P6. Let $\mathcal{H}_i$, with $i \in \{1, 2\}$, be inner product spaces having inner products $\langle \cdot, \cdot \rangle_i$. Then the adjoint $L^\dagger : \mathcal{H}_2 \to \mathcal{H}_1$ of a linear operator $L : \mathcal{H}_1 \to \mathcal{H}_2$ is the unique linear map satisfying: $\langle \psi_2, L \psi_1 \rangle_2 = \langle L^\dagger \psi_2, \psi_1 \rangle_1, \forall \psi_i \in \mathcal{H}_i$. $L$ is called unitary if (C1:) $\forall \psi, \phi \in \mathcal{H}_1$ one has $\langle L \psi, L \phi \rangle_2 = \langle \psi, \phi \rangle_1$, $[4]$. C1 is equivalent to (C2:) $L^\dagger L = I_1$, where $I_1$ is the identity operator for $\mathcal{H}_1$. Another equivalent condition is (C3:) $L$ maps elements of any orthonormal basis of $\mathcal{H}_1$ to those of an orthonormal basis of $\mathcal{H}_2$ in a one-to-one and onto manner. A linear operator $K : \mathcal{H}_1 \to \mathcal{H}_1$ is called self-adjoint or Hermitian if $K^\dagger = K$, i.e., $K$ satisfies $\langle \psi, K \phi \rangle_1 = \langle K \psi, \phi \rangle_1, \forall \psi, \phi \in \mathcal{H}_1$.$^5$

P7. Two inner product (in particular Hilbert) spaces $\mathcal{H}_1, \mathcal{H}_2$ are said to be unitarily equivalent if there is a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$. In this case a linear operator $K_1 : \mathcal{H}_1 \to \mathcal{H}_1$ is Hermitian if and only if $K_2 := UK_1U^{-1}$ is a Hermitian operator acting in $\mathcal{H}_2$.

P8. Let $\mathcal{B} = \{\beta_n\}$ be a basis of a separable Hilbert space $\mathcal{H}$ and $K : \mathcal{H} \to \mathcal{H}$ be a linear

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$^4$I was surprised to see the author of [5] referred to C1 as a generalization of the standard definition of unitarity. I was amazed to read a referee report on [4] saying “a huge majority of mathematical physicists” will not accept C1 because it did not imply C2, and that I was “WRONG” not to warn the reader of the difference and to use C2 in another publication that I cited in [4]!

$^5$In most mathematical texts, e.g., [4], what we call a Hermitian operator is referred to as a ‘symmetric operator’. In PT-symmetric QM as described in [2] [7] a symmetric operator means an operator that is represented by a symmetric matrix in the usual position representation.
map. The infinite matrix $K^{(B)}$ with entries $K_{mn}^{(B)}$ defined by $K_{\beta n} = \sum_{m=0}^{\infty} K^{(B)}_{mn} \beta_m$ is called the matrix representation of $K$ in the basis $B$. Now, suppose that $B$ is orthonormal, in which case $K_{mn}^{(B)} = \langle \beta_m, K \beta_n \rangle$. Then, $K$ is Hermitian if and only if $K^{(B)}$ is a Hermitian matrix, i.e., $K_{mn}^{(B)} = K_{nm}^{(B)*}$. For a non-orthonormal basis $B$, there is no relationship between the Hermiticity of $K$ and the Hermiticity of $K^{(B)}$.

P9. The condition that a given basis $B$ of a separable Hilbert space $\mathcal{H}$ be orthonormal fixes the inner product on $\mathcal{H}$ uniquely. This together with the fact that every separable Hilbert space has an orthonormal basis lead to the uniqueness theorem for separable Hilbert spaces [4]: Up to unitary equivalence there is a unique separable Hilbert space, i.e., any pair of separable Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ are related by a unitary operator $U : \mathcal{H}_1 \to \mathcal{H}_2$.

3 \ P\T-Symmetric QM as a Fundamental Theory

3.1 Bender’s Formulation

Consider the $\mathcal{P}\mathcal{T}$-symmetric Hamiltonians of the form $H = p^2 + v(ix)$ where $v : \mathbb{R} \to \mathbb{R}$ is a potential. A typical example is $v(x) = -\mu x^2 - \lambda x^{2+\epsilon}$, with $\mu, \lambda, \epsilon \in \mathbb{R}$, which for $\mu = 0, \lambda = 1$ yields

$$H = p^2 + x^2(ix)^\epsilon.$$  

(1)

There is a certain class of potentials $v$ for which the corresponding Sturm-Liouville problem for the Hamiltonian, $-\phi''(x) + v(ix)\phi(x) = E\phi(x)$, may be shown to lead to a strictly real, positive, discrete and nondegenerate spectrum, provided that the vanishing boundary conditions at infinity are imposed along an appropriate contour $C$ in the complex plane, [8]. For the Hamiltonian (1), this happens whenever $\epsilon \geq 0$, [9].

Suppose that $v$ belongs to this class and $C$ is chosen so that the corresponding spectrum for $H$ is strictly real, discrete, and nondegenerate. Let $\mathcal{B} := \{\phi_n\}$ be a set of $\mathcal{P}\mathcal{T}$-invariant [2] eigenfunctions of $H$ with distinct eigenvalues $E_n$.

The claim upon which the $\mathcal{P}\mathcal{T}$-symmetric QM rests is that one can define a Hilbert space for a quantum system whose dynamics, as determined by the Schrödinger equation: $i \frac{d\psi(t)}{dt} = H\psi(t)$, is unitary [2, 7]. $H$ serves as the quantum Hamiltonian operator for this system. In particular, the states of the system are described by superpositions of $\phi_n$, and its energy levels have energies $E_n$. This implies that, as a (complex) vector space, the space $V$ of state vectors is the span of $\mathcal{B}$. In [2], it is shown that one can define a positive-definite ($\mathcal{C}\mathcal{P}\mathcal{T}$-) inner product $\langle \cdot, \cdot \rangle_+$ on $V$, in such a way that $\mathcal{B}$ is promoted to an orthonormal basis. By construction, this defines an infinite-dimensional complex inner product space $N$

\[\text{For physical reasons, one should also require that the spectrum be bounded from below.}\]
having a countable basis. It is this space that Bender and his coworkers \[2\] identify with the ‘physical Hilbert space’ of the system.

The physical observables were initially defined by Bender et al \[2, 7, 10\] as CPT-invariant operators, i.e., those that commute with CPT, (Def. 1). In \[6\], I showed that this definition was inconsistent with the dynamics of the theory and proposed to identify the observables with Hermitian operators acting in the physical Hilbert space (Def. 2). In their recent Erratum \[11\] to \[2\], Bender et al give an alternative definition, namely (Def. 3:) observables are linear operators $A$ satisfying $A^T = CPT A CPT$, where transposition $^T$ is defined in the usual position representation according to $A^T(x,x') = A(x',x)$.$^7$ In \[12\], I discuss the shortcomings of Def. 3 and show that even if they could be resolved for a given system then Def. 3 reduces to a special case of Def. 2.

Note that Def. 1 formed the basis of the rather appealing idea that in PT-symmetric QM one could “replace the mathematical condition of Hermiticity, whose physical content is somewhat remote and obscure, by the physical condition of space-time and charge-conjugation symmetry. \[2, 7\].” Def. 2 clearly defies this claim. The same is true for Def. 3, because it requires the Hamiltonian to be symmetric, a property as mathematical/non-physical as being Hermitian. In fact, Def. 3 makes explicit use of the operation of transposition $^T$ which, according to \[2, 7\], is related to Hermitian conjugation $^\dagger$ as $A^\dagger = T A^T T$.

### 3.2 A Complete and Consistent Formulation

To make the above statement that ‘$N$ is the physical Hilbert space of the system’ meaningful, one is forced to Cauchy complete $N$ to a Hilbert space $\mathcal{H}$. There is essentially no other mathematically viable alternative that would be consistent with the physical principle that the state vectors $\phi$ are superpositions of (possibly infinite number of) energy eigenfunctions $\phi_n$, i.e., $\phi = \sum_{n=0}^{\infty} a_n \phi_n$ for some $a_n \in \mathbb{C}$. As $N$ is spanned by $\mathcal{B}$ and is dense in $\mathcal{H}$, $\mathcal{B}$ is necessarily a complete orthonormal basis of $\mathcal{H}$. It is also a countable set. Therefore, according to P5, $\mathcal{H}$ is a separable Hilbert space. This in turn implies, in view of P9, that $\mathcal{H}$ is unitarily equivalent to both $L^2(\mathbb{R})$ and $l_2$. As I show below it is easy to construct the unitary operators realizing these equivalences.

To establish the unitary equivalence of $\mathcal{H}$ and $l_2$, I use the basis $\mathcal{B}$. Let $U : \mathcal{H} \to l_2$ be defined by $(U\psi)(n) := \langle \phi_n, \psi \rangle_+ , \forall n \in \mathbb{N}$. A simple calculation shows that $\forall \psi, \phi \in \mathcal{H}$,

$$(U\psi, U\phi) = \sum_{n=0}^{\infty} \langle \phi_n, \psi \rangle_+ \langle \phi_n, \phi \rangle_+ = \langle \psi, \phi \rangle_+ ,$$

where I have used the completeness of $\mathcal{B}$. This proves that $U$ satisfies condition C1 (and hence C2) of P6; it is a unitary operator.

Let $e_n := U\phi_n$, then $e_n(m) = \langle \phi_m, \phi_n \rangle_+ = \delta_{mn}$. This shows that $U$ maps the basis $\mathcal{B}$ onto the standard orthonormal basis $\{e_n\}$ of $l_2$. Furthermore, $U$ maps the Hamiltonian $H$

\[^7\]The position representation $O(x, x')$ of a linear operator $O$ is defined by $(O\psi)(x) := \int_{\mathbb{R}} O(x, x') \psi(x') dx'$.
to the linear operator \( UHU^{-1} \) that acts in \( l_2 \). This is just the matrix representation of \( H \) in the basis \( \mathcal{B} \), \( UHU^{-1} = H^{(\mathcal{B})} \), with entries \( H^{(\mathcal{B})}_{mn} = \langle \phi_m, H \phi_n \rangle = E_n \delta_{mn} \). Because \( E_n \in \mathbb{R} \), \( H^{(\mathcal{B})} \) is a Hermitian matrix. This together with the fact that \( \mathcal{B} \) is orthonormal imply that \( H \) is a Hermitian operator acting in \( \mathcal{H} \). This is clearly consistent with Def. 2 of a physical observable; \( H \) is an observable.

In [6], I show that Def. 2 is the only definition that is consistent with the requirements of the quantum measurement theory. Here I give an alternative and shorter proof of this statement. It uses the following well-known theorem of linear algebra. **Theorem:** Let \( H : N \to N \) be a linear operator acting in an inner product space \( N \) with inner product \( \langle \cdot, \cdot \rangle \). Then \( H \) is Hermitian if and only if for all \( \psi \in N \), \( \langle \psi, H \psi \rangle \) is real [3]. This theorem together with the physical postulate that the expectation values of the observables must be real numbers identify Def. 2 as the only physically acceptable definition of an observable. In the rest of this paper I use Def. 2 for a physical observable.

The unitary operator \( U \) may be used to obtain all the observables of the theory in the original \( \mathcal{PT} \)-symmetric picture, [6]. According to P7, the observables \( O \) acting in the Hilbert space \( \mathcal{H} \) are given by \( O\psi := \sum_{n=0}^{\infty} O_{mn} \langle \phi_m, \psi \rangle_{+} \phi_n \), where \( O_{mn} = O^*_{nm} \in \mathbb{C} \), i.e., \( O \) has a Hermitian matrix representation in the basis \( \mathcal{B} \). Clearly, this is consistent with P8.

As explained in [6] the statement that the Hamiltonian (1) is non-Hermitian stems from the definition used by Bender and his collaborators which is equivalent to saying that \( H \) is not a Hermitian operator as an operator acting in \( L^2(\mathbb{R}) \). Because \( L^2(\mathbb{R}) \) is not the physical Hilbert space for the problem, this statement does not have any physical significance. However, it is also not true that one cannot describe the same physical system using conventional quantum mechanics based on the Hilbert space \( L^2(\mathbb{R}) \).

Let \( U : \mathcal{H} \to L^2(\mathbb{R}) \) be the map \( U\psi := \sum_{n=0}^{\infty} \langle \phi_n, \psi \rangle_{+} \Phi_n \) where \( \mathcal{F} := \{ \Phi_n \} \) is a fixed complete orthonormal basis of \( L^2(\mathbb{R}) \), e.g., one can identify \( \Phi_n \) with the normalized eigenfunctions of the simple harmonic oscillator Hamiltonian with unit mass and frequency (in some appropriate units). One can easily show, using the condition C3 of P6, that \( U \) is a unitary operator, for by construction it maps \( \phi_n \) to \( \Phi_n \) and both \( \mathcal{B} \) and \( \mathcal{F} \) are complete orthonormal bases. A simple calculation also shows that under the action of \( U \), the Hamiltonian \( H \) maps to

\[
\hat{h} := UHU^{-1} = \sum_{n=1}^{\infty} E_n |\Phi_n\rangle \langle \Phi_n|,
\]

where I have used the standard bra-ket notation in \( L^2(\mathbb{R}) \). Clearly \( \hat{h} \) is a Hermitian operator acting in \( L^2(\mathbb{R}) \). Similarly, the observables \( O \) are related to Hermitian operators \( o : L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) according to

\[
O = U^{-1} o U.
\]
So that \( O_{mn} := \langle \phi_m, O \phi_n \rangle = \langle \Phi_m | o | \Phi_n \rangle \).

This completes the formulation of the \( \mathcal{PT} \)-symmetric quantum mechanics: The Hilbert space, the Hamiltonian, and the observables are determined, and the physical interpretation is provided by the standard measurement theory.

### 3.3 Equivalence with Conventional QM

Let \( H, \mathcal{H}, O, \) and \( \mathcal{U} \) be as in the preceding subsection. Consider computing the expectation value \( \mathcal{O}(t) \) of an observable \( O \) in the state described by \( \psi(t) \in \mathcal{H} \) for a given initial state vector \( \psi(t_0) \in \mathcal{H} \). The result is

\[
\mathcal{O}(t) = \frac{\langle \psi(t_0) e^{i(t-t_0)H} O e^{-i(t-t_0)H} \psi(t_0) \rangle_+}{\langle \psi(t_0), \psi(t_0) \rangle_+},
\]

where I have used the fact that the time-evolution operator \( e^{-i(t-t_0)H} \) is a unitary operator acting in \( \mathcal{H} \).

Next, let \( \Psi(t_0) := \mathcal{U} \psi(t_0) \) and use the unitary operator \( \mathcal{U} \) to compute the same quantity. In view of (2), (3) and C1 of P6 we have,

\[
\mathcal{O}(t) = \frac{\langle \Psi(t_0) | e^{i(t-t_0)h} O e^{-i(t-t_0)h} | \Psi(t_0) \rangle}{\langle \Psi(t_0), \Psi(t_0) \rangle}.
\]

Eqs. (4) and (5) show that the \( \mathcal{PT} \)-symmetric description of the physical system that uses \((\mathcal{H}, H, O)\) is physically equivalent to the one that uses \((L^2(\mathbb{R}), h, o)\). This establishes the equivalence of \( \mathcal{PT} \)-symmetric and conventional QM as fundamental physical theories. This equivalence is a manifestation of the rigidity of the basic structure of QM that is referred to in Sec. 1.

The choice between the above two equivalent representations in the description of a physical system is subjective in nature. One might argue that the \( \mathcal{PT} \)-symmetric description is more advantageous, for the Hamiltonian \( H \) has a much simpler form. This is true if one is interested in finding the energy levels of the system. However, in general, for example in computing the expectation value \( \mathcal{O}(t) \), the two representations involve the same level of practical difficulties. Though it might be easier to compute the time-evolution operator for \( H \) in (4), the computation of \( O \) (from \( o \)) and the evaluation of the inner products appearing in (4) are as difficult as the computation of \( o \) (from \( O \)) and \( h \) that appear in (5).

### 4 Conclusion

I have provided a complete formulation of \( \mathcal{PT} \)-symmetric QM and gave a rigorous proof that, as a fundamental physical theory, it is equivalent to the conventional QM. I had previously
used the machinery of pseudo-Hermitian Hamiltonians to establish the existence of a unitarily equivalent Hermitian Hamiltonian associated with a given $\mathcal{PT}$-symmetric Hamiltonian. The construction of the Hilbert space $\mathcal{H}$ and the fact that the Hamiltonian is a Hermitian operator acting in $\mathcal{H}$ have also been discussed in [15]. The results of [14, 15] show the equivalence of the dynamical structure of the $\mathcal{PT}$-symmetric and conventional QM. In the present paper I have given an explicit proof establishing the equivalence of the dynamical as well as the kinematical structures of the two theories. Note that this proof does not rely on the notion of pseudo-Hermiticity.

The results reported in this paper clearly show that one should not expect to get more from the $\mathcal{PT}$-symmetric QM than what conventional QM has to offer. This may sound as a negative and discouraging statement, especially for those (like me) who have invested a great deal of time and effort to understand $\mathcal{PT}$-symmetric QM and helped develop it further. Personally, I would prefer to view the matter from a different angle: The above equivalence also means that $\mathcal{PT}$-symmetric QM is as valuable as the conventional QM. Moreover, this equivalence holds only if one insists in viewing $\mathcal{PT}$-symmetric QM as a fundamental theory. I consider the interesting mathematical content of $\mathcal{PT}$-symmetric QM as sufficient evidence that it may find good uses in the study of various effective theories. It is also important to note that the pseudo-Hermitian operators [13], that were primarily developed to deal with $\mathcal{PT}$-symmetric QM, have already found remarkable applications in relativistic QM [16, 17] and quantum cosmology [16, 18]. A more recent related development has been a formulation of a unitary QM based on a time-dependent Hilbert space [19]. This has led to a new interpretation for geometric phases and revealed an interesting similarity between QM and General Relativity. To these I should also add the applications reported in [20]. Finally, I must emphasize that the results of this paper apply to $\mathcal{PT}$-symmetric QM. The status of $\mathcal{PT}$-symmetric QFT [2, 7, 21] is completely open to future investigations.

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