Raised $k$-Dyck Paths

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June 3, 2022

Abstract

Raised $k$-Dyck paths are a generalization of $k$-Dyck paths that may both begin and end at a nonzero height. In this paper, we develop closed formulas for the number of raised $k$-Dyck paths from $(0, \alpha)$ to $(\ell, \beta)$ for all height pairs $\alpha, \beta \geq 0$, all lengths $\ell \geq 0$, and all $k \geq 2$. We then enumerate raised $k$-Dyck paths with a fixed number of returns to ground, a fixed minimum height, and a fixed maximum height, presenting generating functions (in terms of the generating functions $C_k(t)$ for the $k$-Catalan numbers) when closed formulas aren’t tractable. Specializing our results to $k = 2$ or to $\alpha < k$ reveal connections with preexisting results concerning height-bounded Dyck paths and “Dyck paths with a negative boundary”, respectively.

1 Introduction

For any $k \geq 2$, a $k$-Dyck path of length $\ell$ and height $h$ is an integer lattice path from $(0,0)$ to $(\ell,h)$ that uses steps $\{U = (1,1), D = (1,1-k)\}$ and stays weakly above the line $y = 0$. One may verify that the terminal point of any $k$-Dyck path must satisfy $\ell = h \mod k$. Thus we restrict our attention to $k$-Dyck paths of length $kn + h$ and height $h$, denoting the collection of all $k$-Dyck paths of length $kn + h$ and height $h$ by $D_{kn,h}^k$.

It is well known that $k$-Dyck paths of length $kn$ and height 0 are enumerated by the $k$-Catalan numbers (or Fuss-Catalan numbers), a one-parameter generalization of the Catalan numbers given by $C_k^n = \frac{1}{kn+1} \binom{kn+1}{n}$ for all $k \geq 2$ and $n \geq 0$. In particular, $|D_{kn,0}^k| = C_k^n$ for all $k \geq 2$ and $n \geq 0$. In the case of $k = 2$, this corresponds to the classic combinatorial interpretation of the Catalan numbers by Dyck paths of length $2n$ and height 0. For more information about the $k$-Catalan numbers and their combinatorial interpretations, see Hilton and Pedersen [6] or Heubach, Li and Mansour [5]. For even more details about the classic Catalan numbers, see Stanley [9].

Now let $C_k(t) = \sum_{n=0}^{\infty} C_k^n t^n$ be the ordinary generating function for the $k$-Catalan numbers. As shown by Hilton and Pedersen [6], the $k$-Catalan numbers satisfy $C_k^{n+1} = \sum_{i_1 + \ldots + i_k = n} C_{i_1} \cdots C_{i_k}$ for all $n \geq 0$, implying that these generating functions obey $C_k(t) = tC_k(t)^k + 1$. If we use $[t^n]p(t)$ to denote the coefficient of $t^n$ in the power series $p(t)$, another standard result asserts $|D_{kn,h}^k| = [t^n]C_k(t)^{h+1}$ for all $n, h \geq 0$. See Figure 1 for the decomposition that yields this result.

Also proven by Hilton and Pedersen [6] is that $[t^n]C_k(t)^r = \frac{r}{kn+r} \binom{kn+r}{n} = R_{k,r}(n)$ for all $k \geq 2$, $n \geq 0$, and $r \geq 1$. Here the notation $R_{k,r}(n)$ corresponds to the Raney number (two-parameter Fuss-Catalan number). This gives

\[1\] $k$-Dyck paths of length $kn$ and height $km$ are often referred to as $k$-Dyck paths of “semi-length” $n$ and “semi-height” $m$, with $D_{n,m}^k$ also sometimes being used to refer to such paths.
Figure 1: A $k$-Dyck path $P$ of height $h$ decomposed into a sequence of $h + 1$ paths $P_i$ of height 0, according to the rightmost $U$ steps at each height. Note that some of the $P_i$ may be empty.

$$|D_{n,h}^k| = |P^n|C_k(t)^{h+1} = \frac{h+1}{kn+h+1} \binom{kn+h+1}{n}.$$  

The primary goal of this paper is to generalize the closed formula of (1) to generalized $k$-Dyck paths that may begin (as well as end) at any non-zero height, objects that we informally refer to as “raised $k$-Dyck paths”. These raised $k$-Dyck paths may be interpreted as a natural generalization of the “$k$-Dyck paths with negative boundary” (or $k_1$-Dyck paths) recently investigated by Selkirk [8], Asinowski, Hackl and Selkirk [1], and Prodinger [7], although our results are developed in such a manner that we needn’t restrict our attention to starting heights less than $k$. We then develop closed formulas for the number of raised $k$-Dyck paths with a fixed minimum height and a fixed number of returns. To close the paper, we use our results to derive new generating functions for the number of $k$-Dyck paths of bounded height, a topic where all previous investigations appear to be limited to the $k = 2$ case or are broadly theoretical and don’t account for general starting/ending heights (see [3], [2] for recent discussions concerning $k$-Dyck paths of bounded height).

2 Raised $k$-Dyck paths

Once again fix $k \geq 2$. For any $\alpha, \beta \geq 0$, a raised $k$-Dyck path of length $\ell$ and shape $(\alpha, \beta)$ is an integer lattice path from $(0, \alpha)$ to $(\ell, \beta)$ that uses steps $\{U = (1, 1), D = (1, 1-k)\}$ and stays weakly above the line $y = 0$. The terminal point of any such path must satisfy $\ell = (\beta - \alpha) \mod k$, justifying our restriction to $k$-Dyck paths of length $kn + \beta - \alpha$ and shape $(\alpha, \beta)$. Denote the set of all $k$-Dyck paths of length $kn + \beta - \alpha$ and shape $(\alpha, \beta)$ by $D_{n,(\alpha,\beta)}^k$, and then define $|D_{n,(\alpha,\beta)}^k| = C_{n,(\alpha,\beta)}^k$. Notice that all elements of $D_{n,(\alpha,\beta)}^k$ contain precisely $n + \beta - \alpha$ up steps and $n$ down steps, meaning that the “$n$ index” of a particular path corresponds to its number of $D$ steps.

It is clear that $D_{n,(0,\beta)}^k = D_{n,\beta}^k$. It is also clear that $D_{n,(\beta,\beta)}^k$ is in bijection with integer lattice paths from $(0,0)$ to $(kn,0)$ that use step set $\{U,D\}$ and stay weakly above the line $y = -\beta$. Horizontal reflection then places $D_{n,(\beta,\beta)}^k$ in bijection with the $k_1$-Dyck paths of Selkirk [8] and Asinowski, Hackl and Selkirk [1]. More generally, whenever $\alpha \geq \beta$, the set $D_{n,(\alpha,\beta)}^k$ is in bijection with generalized $k$-Dyck paths from $(0,0)$ to $(kn+\beta,0)$ that stay weakly above $y = -\beta$ and begin with at least $\alpha$ consecutive $U$ steps. This gives additional bijections between our sets and some of the $k_1$-Dyck paths studied by Prodinger [7].

Before proceeding, notice that a trivial path of length $\ell = 0$ only exists when $n = 0$ and $\alpha = \beta$. In this case we have $C_{0,(\beta,\beta)}^k = 1$. Also note that $C_{n,(\alpha,\beta)}^k = 0$ whenever $kn + \beta - \alpha < 0$. This corresponds to the fact that any element of $D_{n,(\alpha,\beta)}^k$ with $\alpha > \beta$ requires some minimal number of $D$ steps in order to end at the correct height.

Fundamental to much of our approach is the following recurrence. In this and subsequent results, we automatically set $C_{n,(\alpha,\beta)}^k = 0$ whenever $\alpha < 0$ or $\beta < 0$. 

Proposition 2.1. For any $k \geq 2$, $n \geq 0$, and $\alpha, \beta \geq 0$ other than $n = 0$ and $\alpha = \beta$,

$$C_{n,(\alpha,\beta)}^k = C_{n,(\alpha+1,\beta)}^k + C_{n-1,(\alpha-k+1,\beta)}^k. $$

Proof. Observe that $n = 0$ and $\alpha = \beta$ corresponds to trivial paths, which cannot be decomposed as outlined below. Excepting that case, let $S_U$ be the subset of $D_{n,(\alpha,\beta)}^k$ including all paths that begin with a $U$ step, and let $S_D$ be the subset of $D_{n,(\alpha,\beta)}^k$ including all paths that begin with a $D$ step. Eliminating the first step of every $P \in S_U$ gives a path of length $kn + \beta - \alpha - 1 = kn + \beta - (\alpha + 1)$ and shape $(\alpha + 1, \beta)$, placing $S_U$ in bijection with $D_{n,(\alpha+1,\beta)}^k$. Eliminating the first step of every $P \in S_D$ gives a path of length $kn + \beta - \alpha - 1 = k(n-1) + \beta - (\alpha - k + 1)$ and shape $(\alpha - k + 1, \beta)$, placing $S_D$ in bijection with $D_{n-1,(\alpha-k+1,\beta)}^k$. \hfill $\Box$

Fully utilizing the recurrence of Proposition 2.1 requires multivariate generating functions. Simultaneously accounting for all shapes $(\alpha, \beta)$, define $C_k(q, r, t) = \sum_{\alpha, \beta, n \geq 0} C_{n,(\alpha,\beta)}^k q^\alpha r^\beta t^n$. For reasons that will become clear in upcoming sections, we separately denote the ordinary generating function for paths of fixed shape $(\alpha, \beta)$ by $C_{k,(\alpha,\beta)}(t) = \sum_{n \geq 0} C_{n,(\alpha,\beta)}^k t^n = [q^\alpha r^\beta]C_k(q, r, t)$.

For fixed shape $(\alpha, \beta)$, observe that the order of $C_{k,(\alpha,\beta)}(t)$ is the smallest non-negative integer $n$ such that $n \geq \frac{\alpha - \beta - 1}{k - 1}$, corresponding to the minimal number of $D$ steps in any path of shape $(\alpha, \beta)$. In particular, if $\alpha \leq \beta$ then $C_{k,(\alpha,\beta)}(t)$ has order 0. When $\alpha \leq \beta$, the minimal coefficient is always $[t^0]C_{k,(\alpha,\beta)}(t) = 1$, corresponding to the unique path of shape $(\alpha, \beta)$ with zero $D$ steps. When $\alpha > \beta$, the minimal coefficient of $C_{k,(\alpha,\beta)}(t)$ may or may not be 1.

Proposition 2.1 may be used to derive the following relationship for $C_k(q, r, t)$:

Theorem 2.2. For any $k \geq 2$,

$$C_k(q, r, t) = \frac{\sum_{i \geq 0} (C_k(t)^{i+1} - q^{i+1}) r^i}{1 - q + q^{k} t}. $$

Proof. The recurrence of Proposition 2.1 is equivalent to $C_{n,(\alpha,\beta)}^k = C_{n,(\alpha-1,\beta)}^k - C_{n-1,(\alpha-k,\beta)}^k$ for all $n \geq 0$ and $\alpha \geq 1$. This suggests a relation that includes $C_k(q, r, t) = qC_k(q, r, t) - q^k t C_k(q, r, t)$. Accounting for the $\alpha = 0$ case, where shape $(0, \beta)$ paths are generated by $C_k(t)^{\beta+1}$, requires an additional $\sum_{i \geq 0} C_k(t)^{i+1} r^i t$ term on the right side. Also accounting for the trivial case of $n = 0$ and $\alpha = \beta$, to which Proposition 2.1 doesn’t apply, we have the full recurrence

$$C_k(q, r, t) = qC_k(q, r, t) - q^k t C_k(q, r, t) + \sum_{i \geq 0} C_k(t)^{i+1} r^i t - \sum_{i \geq 0} q^{i+1} r^i t. $$

(2)

\hfill $\Box$

The generating function of Theorem 2.2 may be used to derive closed formulas for all of the $C_{n,(\alpha,\beta)}^k$, regardless of starting height. In all that follows, we set $\binom{a}{b} = 0$ whenever $a < 0$ or $b < 0$.

Theorem 2.3. For any $k \geq 2$ and $n, \alpha, \beta \geq 0$,

$$C_{n,(\alpha,\beta)}^k = \sum_{i \geq 0} (-1)^i \binom{\beta + 1}{k(n-i) + \beta + 1} \binom{\alpha - (k-1)i}{n-i} \left(\alpha - (k-1)i\right) - (-1)^n \binom{\alpha - \beta - 1 - (k-1)n}{n}. $$

Proof. Specializing the formula of Theorem 2.2 to fixed $\beta$ gives

$$[t^\beta]C_k(q, r, t) = \frac{C_k(t)^{\beta+1} - q^{\beta+1}}{1 - q + q^k t} = \left(C_k(t)^{\beta+1} - q^{\beta+1}\right) \left(1 + (q - q^k t) + (q - q^k t)^2 + \ldots \right). $$

(3)
One may verify that the coefficient of $q^\alpha$ in $(1 + (q - q^k t) + (q - q^k t)^2 + \ldots)$ is $\sum_{i \geq 0} (-1)^i (\alpha - (k-1)i) t^i$. This implies
\[ [q^{\alpha r^\beta}] C_k(q, r, t) = C_k(t)^{\beta+1} \sum_{i \geq 0} (-1)^i \binom{\alpha - (k-1)i}{i} t^i - \sum_{i \geq 0} (-1)^i \binom{\alpha - \beta - 1 - (k-1)i}{i} t^i. \] (4)

As noted in Section 1, $C_k(t)^{\beta+1}$ may be rewritten as $C_k(t)^{\beta+1} = \sum_{i \geq 0} \frac{\beta+1}{k+i+\beta+1} \binom{k+i+\beta+1}{i} t^i$. This transforms the first term from the right side of (1) into a convolution of two power series. Extracting the coefficient of $q^\alpha$ from both terms of (1) yields our formula for $[q^{\alpha r^\beta}] C_k(q, r, t) = C_{n, (\alpha, \beta)}^k$.

Inspecting the formula of Theorem 2.3, observe that the trailing term can only be nonzero when $\alpha > \beta$. Also, at least one of the binomial coefficients from each term of the summation is zero unless $i \leq \min(n, \frac{\alpha}{k})$.

All of this means that the formula of Theorem 2.3 is much simpler when the starting height $\alpha$ is relatively small. In particular, when $0 \leq \alpha \leq k-1$, the leading summation contains only a single nonzero term and we have the following.

**Corollary 2.4.** For any $k \geq 2$, $\beta \geq 0$, and $0 \leq \alpha \leq k-1$,
\[ C_{n, (\alpha, \beta)}^k = \begin{cases} \frac{\beta+1}{kn+\beta+1} \binom{kn+\beta+1}{n} = R_{k, \beta+1}(n) & \text{if } n > 0, \\ 1 & \text{if } n = 0 \text{ and } \alpha \leq \beta, \\ 0 & \text{if } n = 0 \text{ and } \alpha > \beta. \end{cases} \]

**Proof.** When $n > 0$ and $\alpha \leq k-1$, the final term from Theorem 2.3 is always zero and the leading summation simplifies to a single term. When $n = 0$, Theorem 2.3 gives $\frac{\beta+1}{\beta+1} \binom{\beta+1}{0} - \binom{\alpha-\beta-1}{0}$.

Still restricting our attention to $0 \leq \alpha \leq k-1$, we can alternatively begin with (1) to recast Corollary 2.4 in terms of the generating functions $C_{k, (\alpha, \beta)}(t) = [q^{\alpha r^\beta}] C_k(q, r, t)$:

**Corollary 2.5.** For any $k \geq 2$, $\beta \geq 0$, and $0 \leq \alpha \leq k-1$,
\[ C_{k, (\alpha, \beta)}(t) = \begin{cases} C_k(t)^{\beta+1} & \text{if } \alpha \leq \beta, \\ C_k(t)^{\beta+1} - 1 & \text{if } \alpha > \beta. \end{cases} \]

**Proof.** By (1), when $0 \leq \alpha \leq k-1$ we have $[q^{\alpha r^\beta}] C_k(q, r, t) = C_k(t)^{\beta+1} \binom{\alpha}{0} - \binom{\alpha-\beta-1}{0}$.

Corollaries 2.4 and 2.5 place our work in agreement with Selkirk 8 and Asinowski, Hackl and Selkirk [1], assuming we restrict ourselves to their range of $0 \leq \alpha \leq k-1$. In this case, observe that Corollaries 2.4 and 2.5 may also be proven by placing $D_{n, (\alpha, \beta)}^k$ in bijection with $D_{n, (0, \beta)}^k$ via the map that adds $\alpha$ consecutive $U$ steps to the beginning of every $P \in D_{n, (\alpha, \beta)}^k$. This bijection fails when $\alpha > k-1$, since it is no longer the case that every $P \in D_{n, (0, \beta)}^k$ must begin with $\alpha > k-1$ consecutive $U$ steps.

Computation of $C_{k, (\alpha, \beta)}(t)$ becomes increasingly difficult as one extends above $\alpha = k-1$. See Appendix A for a comparison of the sequences generated by $C_{k, (\alpha, 0)}(t)$ to previously-catalogued sequences on OEIS [10], for small $k \geq 2$ and various shapes $(\alpha, \beta)$.
2.1 Raised k-Dyck paths, k=2 case

As with most combinatorial objects related to the $k$-Catalan numbers, investigating raised $k$-Dyck paths becomes much easier in the case of $k = 2$. In this subsection, we present a series of results involving the $C_{n,(\alpha,\beta)}^k$ that hold only when $k = 2$.

The primary reason the $k = 2$ case is simpler is the fact that the left-right reflection of a raised 2-Dyck path still qualifies as a raised 2-Dyck path. In particular, reflecting a 2-Dyck path of length $2n + \beta - \alpha$ and shape $(\alpha,\beta)$ results in a 2-Dyck path of length $2n + \beta - \alpha = 2(n + \beta - \alpha) + \alpha - \beta$ shape $(\beta,\alpha)$. In terms of generating functions, this prompts:

**Proposition 2.6.** For all $\alpha, \beta \geq 0$,

$$C_{2,(\beta,\alpha)}(t) = t^{\beta-\alpha}C_{2,(\alpha,\beta)}(t).$$

Notice that Proposition 2.6 holds even if $\beta - \alpha < 0$. If $\alpha > \beta$, then $C_{2,(\alpha,\beta)}(t)$ has order $\alpha - \beta$ and $t^{\beta-\alpha}C_{2,(\alpha,\beta)}(t)$ is a valid (order 0) power series. When dealing with the $k = 2$ case, Proposition 2.6 allows us to restrict our attention to shapes $(\alpha, \beta)$ where $\beta \geq \alpha$.

Our next result is a replacement of the generating function equation (4) from the proof of Theorem 2.3 that holds only when $k = 2$.

**Theorem 2.7.** For all $\alpha, \beta \geq 0$,

$$C_{2,(\alpha,\beta)}(t) = \sum_{i=0}^{\min(\alpha,\beta)} t^{\alpha-i}C_{2}(t)^{\alpha+\beta+1-2i}.$$

**Proof.** For each $n \geq 0$, we partition $D^2_{n,(\alpha,\beta)}$ into sets $S_{n,0}, \ldots, S_{n,\min(\alpha,\beta)}$, where $S_{n,i}$ includes all paths whose lowest point lies along $y = i$. As shown in Figure 2, every path $P \in S_{i,n}$ may be decomposed into a sequence of $(\alpha - i) + (\beta - i) + 1$ subpaths of shape $(0,0)$. Notice that this decomposition includes $\alpha - i$ “external” down steps that aren’t included within one of the shape-$(0,0)$ subpaths. If we define the generating function $S_i(t) = \sum_{n \geq 0} |S_{n,i}| t^n$, this decomposition implies that $S_i(t) = t^{\alpha-i}C_{2}(t)^{\alpha+\beta+1-2i}$. \hfill \Box

![Figure 2: The decomposition of a path $P \in D^2_{n,(\alpha,\beta)}$ into a sequence of $(\alpha - i) + (\beta - i) + 1$ subpaths of shape $(0,0)$, as referenced in the proof of Theorem 2.7.](image)

If $\beta \geq \alpha$, the formula of Theorem 2.7 may be rewritten as $C_{2,(\alpha,\beta)}(t) = \sum_{i=0}^{\alpha} t^iC_{2}(t)^{\beta-\alpha+2i+1}$. Similarly, if $\alpha \geq \beta$, Theorem 2.7 may be rewritten as $C_{2,(\alpha,\beta)}(t) = \sum_{i=0}^{\beta} t^{\alpha-\beta+i}C_{2}(t)^{\alpha-\beta+2i+1}$. Together these identities ensure $C_{2,(\beta,\alpha)}(t) = t^{\beta-\alpha}C_{2,(\alpha,\beta)}(t)$, placing Theorem 2.7 in agreement with Proposition 2.6.

Temporarily restricting our attention to the case of $\beta \geq \alpha$, also note that we may use the identity $C_{2}(t) = tC_{2}(t)^2 + 1$ to rewrite the formula above as
\[ C_{2,(\alpha,\beta)}(t) = C_2(t)^{\beta-\alpha+1} \sum_{i=0}^\alpha (tC_2(t)^2)^i = C_2(t)^{\beta-\alpha+1} \sum_{i=0}^\alpha (C_2(t) - 1)^i. \] (5)

More significantly, Theorem 2.7 may be used to develop a closed formula for arbitrary \( C^2_{n,(\alpha,\beta)} \), giving a simpler replacement of Theorem 2.3 that holds only when \( k = 2 \).

**Theorem 2.8.** For all \( n, \alpha, \beta \geq 0 \),

\[ C^2_{n,(\alpha,\beta)} = \sum_{i=0}^{\min(\alpha,\beta)} \frac{\alpha + \beta + 1 - 2i}{2n + \beta - \alpha + 1} \left( \frac{2n + \beta - \alpha + 1}{n - \alpha + i} \right). \]

**Proof.** Fixing \( n \geq 0 \) and applying the definition of Raney numbers, we have \([m!t^{\alpha-i}]C_2(t)^{\alpha+\beta+1-2i} = \frac{\alpha+\beta+1-2i}{2(n-\alpha+i) + (\alpha+\beta+1-2i)}(2(n-\alpha+i)+(\alpha+\beta+1-2i)).\) Our closed formula for the \( C^2_{n,(\alpha,\beta)} = [t^n]C_{2,(\alpha,\beta)}(t) \) then follows from the summation of Theorem 2.7.

## 3 Raised k-Dyck paths, filtered by minimum height and returns

For the rest of this paper, we focus upon the enumeration of raised \( k \)-Dyck paths that satisfy additional conditions. We begin by developing formulas for the number of paths \( P \in D^k_{n,(\alpha,\beta)} \) that have a fixed minimum height and paths \( P \in D^k_{n,(\alpha,\beta)} \) that have a certain number of “returns to ground”. Enumerating raised \( k \)-Dyck paths that have a fixed maximum height is delayed until Section 4.

### 3.1 Raised k-Dyck paths, by minimum height

For traditional \( k \)-Dyck paths, all of which necessarily begin at height \( y = 0 \), it is unnecessary to categorize paths according to their minimum \( y \)-coordinate. For raised \( k \)-Dyck paths of shape \((\alpha,\beta)\), this question becomes non-trivial when both \( \alpha > 0 \) and \( \beta > 0 \).

Take any path \( P \in D^k_{n,(\alpha,\beta)} \). If \( P \) stays weakly above \( y = m \), we say that \( P \) is bounded from below by \( m \). Then let \( L^k_{n,(\alpha,\beta)} \) denote the collection of all \( P \in D^k_{n,(\alpha,\beta)} \) that are bounded from below by \( m \). For any such set, there exists a clear bijection between \( L^k_{n,(\alpha,\beta)} \) and \( D^k_{n,(\alpha-m,\beta-m)} \) whereby paths in \( L^k_{n,(\alpha,\beta)} \) are shifted \( m \) units downward. As such, we focus upon enumerating paths that actually obtain a fixed minimum height.

So once again take \( P \in D^k_{n,(\alpha,\beta)} \). If \( P \) is bounded from below by \( m \) yet is not bounded from below by \( m+1 \), meaning \( m \) is the minimum \( y \)-coordinate among all points \((x_i, y_i)\) along \( P \), we say that \( P \) has a minimum height of \( m \). Let \( mD^k_{n,(\alpha,\beta)} \) to denote the set of all raised \( k \)-Dyck paths of length \( kn+\beta-\alpha \) and shape \((\alpha,\beta)\) with minimum height \( m \), and set \( |mD^k_{n,(\alpha,\beta)}| = mC^k_{n,(\alpha,\beta)} \). For fixed shape \((\alpha,\beta)\) and fixed \( m \), define the generating function \( mC_{k,(\alpha,\beta)}(t) = \sum_{n \geq 0} mC^k_{n,(\alpha,\beta)} t^n \).

Obviously, all \( P \in D^k_{n,(\alpha,\beta)} \) have a minimum height that falls in the range \( 0 \leq m \leq \min(\alpha,\beta) \). It follows that \( D^k_{n,(\alpha,\beta)} = \bigcup_{i=0}^{\min(\alpha,\beta)} mD^k_{n,(\alpha,\beta)} \) and hence that \( C_{k,(\alpha,\beta)}(t) = \sum_{i=0}^{\min(\alpha,\beta)} mC_{k,(\alpha,\beta)}(t) \) for all \( k \geq 2 \) and all shapes \((\alpha,\beta)\). By construction, we also have \( mD^k_{n,(\alpha,\beta)} = L^k_{n,(\alpha,\beta)} - L^k_{n,(\alpha+1,\beta)} \). Using the bijection for the \( L^k_{n,(\alpha,\beta)} \) mentioned above, this final fact gives:

**Proposition 3.1.** For any \( k \geq 2 \), \( n, \alpha, \beta \geq 0 \) and \( 0 \leq m \leq \min(\alpha,\beta) \),

\[ mC^k_{n,(\alpha,\beta)} = C^k_{n,(\alpha-m,\beta-m)} - C^k_{n,(\alpha-m-1,\beta-m-1)}. \]
The drawback with Proposition 3.1 is that it relies upon the extremely lengthy formula of Theorem 2.3. This motivates the alternative characterization of $mC_{n,(\alpha, \beta)}^k$ given below, which has the added benefit of relating all our results to enumerations of raised $k$-Dyck paths of shape $(\alpha, 0)$.

**Theorem 3.2.** For any $k \geq 2$, $\alpha, \beta \geq 0$ and $0 \leq m \leq \min(\alpha, \beta)$,

$$mC_{k,(\alpha,\beta)}(t) = C_{k,(\alpha-m,0)}(t)C_k(t)^{\beta-m}.$$  

**Proof.** As shown in Figure 3, every path $P \in mD_{n,(\alpha,\beta)}^k$ may be decomposed according to the rightmost point at its minimum height of $y = m$. When $0 \leq m < \beta$, this decomposition gives the relationship $mC_{k,(\alpha,\beta)}(t) = C_{k,(\alpha-m,0)}(t)C_{k,(0,\beta-m-1)}(t)$. When $m = \beta$, we have the relationship $mC_{k,(\alpha,\beta)}(t) = C_{k,(\alpha-m,0)}(t)$. Both cases simplify to the stated equation. □

![Figure 3: The two possible decompositions of a path $P \in mD_{n,(\alpha,\beta)}^k$ with minimum height $m$, one for $m < \beta$ (left) and one for $m = \beta$ (right). The $(a_i, b_i)$ denote the effective shape of each subpath.](image)

**Corollary 3.4.** For any $k \geq 2$, $n, \alpha, \beta \geq 0$, and $m, \alpha \geq 0$ such that $0 \leq \alpha - m \leq k-1$,

$$mC_{n,(\alpha,\beta)}^k = \begin{cases}
\frac{\beta-m+1}{kn+\beta-m+1}C_{kn+\beta-m+1}(n) & \text{if } m = \alpha, \\
\frac{\beta-m+1}{kn+\beta-m+1}C_{kn+\beta-m+1}(n) - \frac{\beta-m}{kn+\beta-m}(kn+\beta-m) & \text{if } m < \alpha \leq k-1+m.
\end{cases}$$
Proof. By Corollary 3.5 and Theorem 3.2, when \( m = \alpha \) we have \( C_{k,(\alpha-m,0)}(t) = C_k(t) \) and thus that \( mC_{k,(\alpha,\beta)}(t) = C_k(t)^{\beta-m+1} \). Similarly, when \( m < \alpha < k-1+m \) we have \( C_{k,(\alpha-m,0)}(t) = C_k(t) - 1 \) and thus that \( mC_{k,(\alpha,\beta)}(t) = C_k(t)^{\beta-m+1} - C_k(t)^{\beta-m} \). Our closed formulas then follow from the identity \( [t^n]C_k(t)^r = \frac{r^n}{kn+r}(kn) \).

As for the \( k = 2 \) case, in the course of proving Theorem 2.7 we already enumerated paths in \( C_{n,(\alpha,\beta)}^2 \) with minimal height \( m \). It may be verified that the formula below corresponds to \( [t^n]t^{n-m}C_2(t)^{\alpha+\beta+1-2m} = [t^n]C_{2,(\alpha-m)}(t)C_2(t)^{\beta-m} \), placing it in agreement with Theorem 3.2.

**Corollary 3.5.** For any \( n, \alpha, \beta \geq 0 \) and \( 0 \leq m \leq \min(\alpha,\beta) \),

\[
mC_{n,(\alpha,\beta)}^2 = \frac{\alpha + \beta + 1 - 2m}{2n + \beta - \alpha + 1} \left( \frac{2n + \beta - \alpha + 1}{n - \alpha + m} \right).
\]

One unrelated consequence of Theorem 3.2 is the following decomposition of \( C_{k,(\alpha,\beta)}(t) \) into a sum that is indexed by minimal height:

\[
C_{k,(\alpha,\beta)}(t) = \sum_{i=0}^{\min(\alpha,\beta)} C_{k,(\alpha-m,0)}(t)C_k(t)^{\beta-m}.
\] (7)

Comparison of Proposition 3.1 and Theorem 3.2 also gives an unexpected equation whereby shape \((\alpha,\beta)\) paths may enumerated in terms of paths with shapes of the form \((\alpha',0)\) and \((0,\beta')\).

**Corollary 3.6.** For all \( k \geq 2 \) and \( \alpha, \beta \geq 0 \),

\[
C_{k,(\alpha,\beta)}(t) = \sum_{i=0}^{\min(\alpha,\beta)} C_{k,(\alpha-i,0)}(t)C_k(t)^{\beta-i}.
\]

*Proof.* Equating the right sides of Theorem 3.2 and (a generating function-equivalent version of) Proposition 3.1 when \( m = 0 \) gives the relation below, which holds whenever \( \alpha > 0 \) and \( \beta > 0 \):

\[
C_{k,(\alpha,\beta)}(t) = C_{k,(\alpha,0)}(t)C_k(t)^\beta + C_{k,(\alpha-1,\beta-1)}(t).
\] (8)

Repeated application of this relation until \( \alpha = 0 \) or \( \beta = 0 \) yields the desired equation. \( \square \)

### 3.2 Raised k-Dyck paths, by returns

Our next goal is to enumerate paths \( P \in D_{n,(\alpha,\beta)}^k \) with a specific number of “returns to ground”. By a return to ground, we mean a \( D \) step whose right endpoint lies on the line \( y = 0 \). When \( \alpha = 0 \), the initial point \((0,0)\) of a path does not qualify as a return to ground.

Denote the set of all raised \( k \)-Dyck paths of length \( kn + \beta - \alpha \) and shape \((\alpha,\beta)\) with precisely \( \rho \) returns to ground by \( D_{n,(\alpha,\beta),\rho}^k \), and let \( |D_{n,(\alpha,\beta),\rho}^k| = C_{n,(\alpha,\beta),\rho}^k \). As every path in \( D_{n,(\alpha,\beta)}^k \) contains precisely \( n \) down steps, \( C_{n,(\alpha,\beta),\rho}^k \) is \( 0 \) if \( \rho > n \). When \( \alpha > 0 \), we may have \( C_{n,(\alpha,\beta),\rho}^k \) is \( 0 \) even if \( \rho \leq n \).

In this section it is once again beneficial to preemptively fix a shape \((\alpha,\beta)\) and deal with the generating functions \( C_{k,(\alpha,\beta)}(t) = [q^r r^p]C_k(q,r,t) \). Filtering by the number of returns, we then define \( C_{k,(\alpha,\beta)}(t,z) = \sum_{n,\rho \geq 0} C_{n,(\alpha,\beta),\rho}^k q^n z^\rho \).

In the classic case of \( \alpha = 0 \), we quickly recap the standard result. Here, every path in \( D_{n,(0,\beta),\rho}^k \) may be decomposed according to its returns as in Figure 4. This decomposition gives
Proposition 3.7. For any \( k \geq 2 \) and \( \beta \geq 0 \),
\[
C_{k,(0,\beta)}(t, z) = \sum_{i \geq 0} z^i t^i C_k(t)^{\beta+i(k-1)}.
\]

Proof. For paths \( P \in D_{n,(0,\beta)}^k \) with precisely \( \rho \) returns, the decomposition of Figure 4 yields the generating function
\[
C_{k,(0,\beta-\rho)}(t) \left(t C_{k,(0,k-2)}(t)\right)^{\rho} = t^\rho C_k(t)^{\beta} \left(C_k(t)^{k-1}\right)^{\rho}.
\]

Figure 4: The general form of a path \( P \in D_{n,(0,\beta)}^k \) with precisely \( \rho \) returns to ground, along with the effective shape of each subpath.

Theorem 3.8. For any \( k \geq 2 \) and \( \beta, n, \rho \geq 0 \),
\[
C_{n,(0,\beta),\rho}^k = \frac{k \rho + \beta - \rho}{kn + \beta - \rho} \left(\frac{kn + \beta - \rho}{n - \rho}\right).
\]

Proof. By Proposition 3.7, \( C_{n,(0,\beta),\rho}^k = \left[t^n\right] t^\rho C_k(t)^{\beta+\rho(k-1)} = \left[t^n\right] C_k(t)^{\beta+\rho(k-1)}. \)

The case of \( \alpha > 0 \) is similar yet slightly more complex, seeing as elements of \( D_{n,(\alpha,\beta),\rho}^k \) need not have a return to ground. This necessitates two distinct decompositions for elements of \( D_{n,(\alpha,\beta),\rho}^k \), both of which are shown in Figure 5. As with Proposition 3.7, this decomposition prompts

Proposition 3.9. For any \( k \geq 2 \) and \( \beta \geq 0 \) with \( \alpha > 0 \),
\[
C_{k,(\alpha,\beta)}(t, z) = C_{k,(\alpha-1,\beta-1)}(t) + \sum_{i \geq 1} z^i t^i C_{k,(\alpha-1,k-2)}(t) C_k(t)^{\beta+(i-1)(k-1)}.
\]

Proof. The first term corresponds to the first decomposition in Figure 5. The sum corresponds to the second decomposition in Figure 5 where paths \( P \in D_{n,(\alpha,\beta)}^k \) with \( \rho \) returns have generating function
\[
C_{k,(\alpha-1,k-2)}(t) \left(t C_{k,(0,k-2)}(t)\right)^{\rho-1} t C_{k,(0,\beta-1)}(t) = t^\rho C_{k,(\alpha-1,k-2)}(t) \left(C_k(t)^{k-1}\right)^{\rho-1} C_k(t)^{\beta}.
\]

Figure 5: The two possible decompositions for a path \( P \in D_{n,(\alpha,\beta)}^k \) with \( \alpha > 0 \), one for paths with no returns (left) and one for paths with precisely \( \rho > 0 \) returns (right).
Theorem 3.10. For any \( k \geq 2 \) and \( \beta, n, \rho \geq 0 \) with \( \alpha > 0 \),

\[
C_{n, \alpha, \beta, \rho}^k = \begin{cases} 
C_{n, \alpha-1, \beta-1}^k & \text{if } \rho = 0, \\
\sum_{i=0}^{n-\rho} C_{i, \alpha-1, \beta-2}^k R_{k, \beta+i}(\rho-1)(n-i) & \text{if } \rho > 0.
\end{cases}
\]

Proof. Using Proposition 3.9 \( C_{n, \alpha, \beta, \rho}^k = [t^n]C_{k, \alpha, \beta-1}(t) \) when \( \rho = 0 \). For \( \rho > 0 \) we have

\[
C_{n, \alpha, \beta, \rho}^k = [t^n]t^\rho C_{k, \alpha-1, \beta-2}(t) C_k(t)^{\beta+i}(\rho-1)(n-i) = [t^n]C_{k, \alpha-1, \beta-2}(t) C_k(t)^{\beta+i}(\rho-1)(n-i).
\] (9)

Given the complexity of the formula from Theorem 2.3 substituting closed formulas into Theorem 3.10 becomes very lengthy for arbitrary \((\alpha, \beta)\). However, when \( 0 < \alpha \leq k \), we can apply Corollary 2.4 (or Corollary 2.5) to arrive at the much simpler identity shown below.

Corollary 3.11. For any \( k \geq 2 \) and \( \beta, n, \rho \geq 0 \) with \( 0 < \alpha \leq k \),

\[
C_{n, \alpha, \beta, \rho}^k = \begin{cases} 
(\rho+\beta-\rho)(kn+\beta-\rho) & \text{if } 0 < \alpha \leq k-1, \\
(\rho+\beta-\rho)(kn+\beta-\rho)-(\rho+\beta-\rho)(k-1) & \text{if } \alpha = k.
\end{cases}
\]

Proof. Applying Corollary 2.5 to the \( C_{k, \alpha-1, \beta-2}(t) \) terms of Theorem 3.10 note that \( \alpha - 1 \leq k - 2 \) implies \( \alpha \leq k - 1 \), whereas \( \alpha - 1 > k - 2 \) along with \( \alpha - 1 \leq k - 1 \) together imply \( \alpha = k \).

As expected, the \( k = 2 \) case is also comparatively succinct. Not at all expected is that a specialization of Theorem 3.10 to \( k = 2 \) gives a simpler result when \( \rho > 0 \) than when \( \rho = 0 \).

Corollary 3.12. For any \( \beta, n, \rho \geq 0 \) with \( \alpha > 0 \),

\[
C_{n, \alpha, \beta, \rho}^k = \begin{cases} 
\sum_{i=0}^{\min(\alpha-1, \beta-1)} \frac{\alpha + \beta - 1 - 2i}{2n + \beta - \alpha + 1} \left(\frac{2n + \beta - \alpha + 1}{n - \alpha + 1 + i}\right) & \text{if } \rho = 0, \\
\frac{\alpha + \beta + \rho - 1}{2n + \beta - \alpha + 1} \left(\frac{2n + \beta - \alpha + 1}{n - \alpha + 1 + 1}\right) & \text{if } \rho > 0.
\end{cases}
\]

Proof. The \( \rho = 0 \) case follows immediately from an application of Theorem 2.3 to Theorem 3.10. For the \( \rho > 0 \) case, by Proposition 2.6 we have \( C_{n, \alpha, \beta, 0}^2 = [t^n]t^\rho C_{2, \alpha-1}(t) C_2(t)^{\beta+\rho-1} \). Using Proposition 3.9 then gives the following, to which we apply the definition of Raney numbers.

\[
C_{n, \alpha, \beta, \rho}^2 = [t^n]t^\rho \alpha C_{2, \alpha-1}(t) C_2(t)^{\beta+\rho-1} = [t^n]t^{\rho+\alpha-1} C_{2, \alpha}(t) C_2(t)^{\beta+\rho-1}.
\] (10)

4 Raised k-Dyck paths of bounded height

The results of Section 2 may also be used to enumerate (raised) \( k \)-Dyck paths of bounded height. This allows for a derivation of easily-computable generating functions that hold for all \( k \geq 2 \) and shapes \((\alpha, \beta)\), expanding upon the discussions of non-raised, height-bounded lattice paths in Baril and Prodinger \[3\], Bousquet-Mélou \[4\], or Bacher \[2\].
So take any raised $k$-Dyck path $P \in \mathcal{D}_{n,(\alpha,\beta)}^k$. If $P$ stays weakly below $y = M$, we say that $P$ is bounded from above by $M$. We use $U_{n,(\alpha,\beta)}^{k,M}$ to denote the collection of all $P \in \mathcal{D}_{n,(\alpha,\beta)}^k$ that are bounded from above by $M$, and set $|U_{n,(\alpha,\beta)}^{k,M}| = U_{n,(\alpha,\beta)}^{k,M}$. Clearly, $U_{n,(\alpha,\beta)}^{k,M} = 0$ unless $\alpha, \beta \leq M$.

Fixing $0 \leq \alpha, \beta \leq M$, we define the generating function $U_{n,(\alpha,\beta)}^{k,\beta}(t) = \sum_{n=0} U_{n,(\alpha,\beta)}^{k,M} t^n$. The primary goal of this section is to relate the $U_{n,(\alpha,\beta)}^{k,\beta}(t)$ to the generating functions $C_{k,\alpha',\beta'}(t)$ of Section 2 from which one may derive closed formulas for the $U_{n,(\alpha,\beta)}^{k,M}$ using Theorem 2.3.

Before deriving a relationship for general $U_{k,(\alpha,\beta)}(t)$, we consider the special case of $\beta = M$:

**Lemma 4.1.** For any $k \geq 2$, $\alpha \geq 0$, and $M \geq 0$,

$$U_{k,(\alpha,M)}^{M}(t) = \frac{C_{k,(\alpha,M)}(t)}{1 + C_{k,(M+1,M)}(t)}.$$

**Proof.** Every path $P \in \mathcal{D}_{n,(\alpha,M)}^k$ may be decomposed in one of the two ways shown in Figure 6, depending upon whether or not the path rises above $y = M$. This prompts the identity

$$C_{k,(\alpha,M)}(t) = U_{k,(\alpha,M)}^{M}(t) + U_{k,(\alpha,M)}^{M}(t) C_{k,(M+1,M)}(t). \quad (11)$$

Figure 6: The two decompositions for a path $P \in \mathcal{D}_{n,(\alpha,M)}^k$, as used in the proof of Lemma 4.1.

Lemma 4.1 may still be applied to derive our general identity:

**Theorem 4.2.** For any $k \geq 2$, $M \geq 0$, and $0 \leq \alpha, \beta \leq M$,

$$U_{k,(\alpha,\beta)}^{M}(t) = C_{k,(\alpha,\beta)}(t) - \frac{C_{k,(\alpha,M)}(t) C_{k,(M+1,\beta)}(t)}{1 + C_{k,(M+1,M)}(t)}.$$

**Proof.** Via an equivalent decomposition of paths $P \in \mathcal{D}_{n,(\alpha,\beta)}^k$ to that in Figure 6 we have

$$C_{k,(\alpha,\beta)}(t) = U_{k,(\alpha,\beta)}^{M}(t) + U_{k,(\alpha,M)}^{M}(t) C_{k,(M+1,\beta)}(t). \quad (12)$$

Rearranging (12) and applying Lemma 4.1 then gives

$$U_{k,(\alpha,\beta)}^{M}(t) = C_{k,(\alpha,\beta)}(t) - U_{k,(\alpha,M)}^{M}(t) C_{k,(M+1,\beta)}(t) = C_{k,(\alpha,\beta)}(t) - \frac{C_{k,(\alpha,M)}(t) C_{k,(M+1,\beta)}(t)}{1 + C_{k,(M+1,M)}(t)}. \quad (13)$$

Recall that the order of $C_{k,(\alpha,\beta)}(t)$ goes to $\infty$ and $\alpha \to \infty$. This implies that the order of $C_{k,(\alpha,M)}(t) C_{k,(M+1,\beta)}(t)$ goes to $\infty$ as $M \to \infty$, and thus that the order of $\frac{C_{k,(\alpha,M)}(t) C_{k,(M+1,\beta)}(t)}{1 + C_{k,(M+1,M)}(t)}$ goes to $\infty$ as $M \to \infty$. This allows us to conclude that number of initial terms for which $[t^n] U_{k,(\alpha,\beta)}^{M}(t) = \ldots$
$[n^2]C_{k,(\alpha,\beta)}(t)$ goes to $\infty$ and $M \to \infty$, as one would expect for $k$-Dyck paths with an arbitrarily high upper bound.

Also observe that, if $M < k - 1$, then we have both $M + 1 \leq k - 1$ and $\alpha, \beta \leq M$. This means that we can apply Corollary 2.5 to the rightmost term from Theorem 4.2 as below:

$$\frac{C_{k,(\alpha,M)}(t)C_{k,(M+1,\beta)}(t)}{1 + C_{k,(M+1,M)}(t)} = \frac{C_k(t)^{M+1}(C_k(t)^{\beta+1} - 1)}{C_k(t)^{M+1}} = C_k(t)^{\beta+1} - 1. \quad (14)$$

When $M < k - 1$ and $\alpha \leq \beta$, this gives the expected result of $U_{k,(\alpha,\beta)}^M(t) = C_k(t)^{\beta+1} - (C_k(t)^{\beta+1} - 1) = 1$, corresponding to the fact that only the “trivial” path (i.e.- the unique path with zero $D$ steps) stays weakly below $y = M$ when $M < k - 1$. When $M < k - 1$ and $\alpha > \beta$, we similarly get the expected result of $U_{k,(\alpha,\beta)}^M(t) = 0$, reflecting the fact that every path of such a shape $(\alpha, \beta)$ must have at least one $D$ step and thus can’t stay weakly below $y = M$.

Explicit calculations involving the generating function $U_{k,(\alpha,\beta)}^M(t)$ become increasingly difficult when $M \geq k - 1$, but Theorem 4.2 may always be used with with Theorem 2.3 to calculate the cardinalities $U_{n,(\alpha,\beta)}^k(t) = [n^2]U_{k,(\alpha,\beta)}^M(t)$. See Appendix A for explicit calculations of the sequences generated by the $U_{k,(\alpha,\beta)}^M(t)$ for various $k \geq 2$ and small $M$ in the case of $(\alpha, \beta) = (0, 0)$.

For one final application, note that Theorem 4.2 may be used to enumerate the number of raised $k$-Dyck paths that actually obtain a fixed maximum height. This follows immediately from the fact that raised $k$-Dyck paths of maximum height $M$ are precisely those paths that stay weakly below $y = M$ yet fail to stay weakly below $y = M - 1$.

So let $H_{n,(\alpha,\beta)}^k, [n^2]H_{n,(\alpha,\beta)}^k, \beta$ denote the set of all $P \in D_{n,(\alpha,\beta)}^k$, that obtain a maximum height of $M$, and let $|H_{n,(\alpha,\beta)}^k| = H_{n,(\alpha,\beta)}^k$, $H_k(\alpha,\beta)$. In terms of the generating function $H_{k,(\alpha,\beta)}^M(t) = \sum_{n \geq 0} H_{n,(\alpha,\beta)}^M(t)^n$, Theorem 4.2 immediately yields the following result.

**Corollary 4.3.** For any $k \geq 2$, $M \geq 0$ and $0 \leq \alpha, \beta \leq M$,

$$H_{k,(\alpha,\beta)}^M(t) = U_{k,(\alpha,\beta)}^M(t) - U_{k,(\alpha,\beta)}^{M-1}(t) = \frac{C_{k,(\alpha,M-1)}(t)C_{k,(M,\beta)}(t)}{1 + C_{k,(M,M-1)}(t)} - \frac{C_{k,(\alpha,M)}(t)C_{k,(M+1,\beta)}(t)}{1 + C_{k,(M+1,M)}(t)}. \quad (15)$$

As with the $U_{k,(\alpha,\beta)}^M(t)$, the $H_{k,(\alpha,\beta)}^M(t)$ become increasing exhausting to calculate when $M$ becomes large. For $M < k - 1$, it is still easy to verify that we get the expected results of $H_{k,(\alpha,\beta)}^M(t) = 1$ when $\alpha \leq \beta$ and $H_{k,(\alpha,\beta)}^M(t) = 0$ when $\alpha > \beta$. See Appendix A for explicit calculations of the sequences generated by the $H_{k,(\alpha,\beta)}^M(t)$ for various $k \geq 2$ and $M \geq k$ in the case of $(\alpha, \beta) = (0, 0)$.

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\section{Explicit Calculations}

Below are comparisons of the sequences generated by $C_{n,(\alpha,\beta)}(t)$ to preexisting sequences on OEIS, for $k = 2, 3, 4$. All sequences were calculated on Maple 19 via (4) from the proof of Theorem 2.3. All listed sequences are identical up to shifting or the complete absence of (one or more) initial terms.

Notice how Proposition 2.6 ensures that the $k = 2$ table is symmetric along the main diagonal, whereas the $k = 3, 4$ tables are not symmetric along the main diagonal. For all tables, Corollary 2.5 ensures that all sequences with $\alpha \leq k - 1$ correspond to convolutions of the $k-$Catalan numbers.

| $\alpha$ | $\beta = 0$ | $\beta = 1$ | $\beta = 2$ | $\beta = 3$ | $\beta = 4$ | $\beta = 5$ | $\beta = 6$ | $\beta = 7$ |
|----------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0        | A000108     | A000108     | A002057     | A000340     | A003517     | A000588     | A003518     |             |
| 1        | A000108     | A000108     | A000245     | A000245     | A000257     | A000260     | A003517     | A000588     |
| 2        | A000245     | A000245     | A026012     | A026016     | A026013     | A026017     | A026014     | A026018     |
| 3        | A000257     | A026016     | A026020     | A026016     | A026000     | A026014     | A026012     | A026018     |
| 4        | A000340     | A026013     | A026020     |             |             |             |             |             |
| 5        | A003517     | A026017     | A026030     |             |             |             |             |             |
| 6        | A000588     | A026014     | A026027     |             |             |             |             |             |
| 7        | A003518     | A026018     | A026031     |             |             |             |             |             |

Table 1: A comparison of the sequences generated by $C_{2,(\alpha,\beta)}(t)$ to preexisting sequences on OEIS.

| $\alpha$ | $\beta = 0$ | $\beta = 1$ | $\beta = 2$ | $\beta = 3$ | $\beta = 4$ | $\beta = 5$ |
|----------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0        | A001764     | A006013     | A001764     | A006629     | A102893     | A006630     |
| 1        | A001764     | A006013     | A001764     | A006629     | A102893     | A006630     |
| 2        | A001764     | A006013     | A001764     | A006629     | A102893     | A006630     |
| 3        | A334680     |             |             |             |             |             |
| 4        | A336945     | A030983     | A336945     |             |             |             |
| 5        | A334976     | A334977     | A334976     |             |             |             |

Table 2: A comparison of the sequences generated by $C_{3,(\alpha,\beta)}(t)$ to preexisting sequences on OEIS.

| $\alpha$ | $\beta = 0$ | $\beta = 1$ | $\beta = 2$ | $\beta = 3$ | $\beta = 4$ | $\beta = 5$ |
|----------|-------------|-------------|-------------|-------------|-------------|-------------|
| 0        | A002293     | A0069271    | A006632     | A002293     | A196678     | A006633     |
| 1        | A002293     | A0069271    | A006632     | A002293     | A196678     | A006633     |
| 2        | A002293     | A0069271    | A006632     | A002293     | A196678     | A006633     |
| 3        | A002293     | A0069271    | A006632     | A002293     | A196678     | A006633     |
| 4        | A334682     |             |             |             |             |             |
| 5        | A334682     |             |             |             |             |             |

Table 3: A comparison of the sequences generated by $C_{4,(\alpha,\beta)}(t)$ to preexisting sequences on OEIS.
Below are comparisons of the sequences generated by $U_{k,(0,0)}^M(t)$ to preexisting sequences on OEIS, for $k = 2, 3, 4$. All sequences were calculated on Maple 19 via Theorem 4.2 and are identical to the listed sequences up to shifting or the absence of (one or more) initial terms.

|     | $k = 2$ | $k = 3$ | $k = 4$ |
|-----|---------|---------|---------|
| $M = 0$ | 1       | 1       | 1       |
| $M = 1$ | $\{1\}$ | 1       | 1       |
| $M = 2$ | $\{2^n\}$ | $\{1\}$ | 1       |
| $M = 3$ | A001519 | $\{2^n\}$ | $\{1\}$ |
| $M = 4$ | A124302 | $\{3^n\}$ | $(2^n)$ |
| $M = 5$ | A080937 | A001835 | $\{3^n\}$ |
| $M = 6$ | A024175 | A081704 | $(4^n)$ |
| $M = 7$ | A080938 | A083881 | A001253 |
| $M = 8$ | A033191 | –       | –       |
| $M = 9$ | A211216 | –       | A261399 |
| $M = 10$ | –       | –       | A143648 |
| $M = 11$ | –       | –       | –       |
| $M = 12$ | –       | –       | –       |

Table 4: A comparison of the sequences generated by $U_{k,(0,0)}^M(t)$ to preexisting sequences on OEIS. An entry of 1 (without braces) corresponds to the sequence 1, 0, 0, 0, . . .