EXISTENCE AND BV-REGULARITY FOR NEUTRON TRANSPORT EQUATION IN NON-CONVEX DOMAIN

YAN GUO AND XIONGFENG YANG

Abstract. This paper considers the neutron transport equation in bounded domain with a combination of the diffusive boundary condition and the in-flow boundary condition. We firstly study the existence of solution in any fixed time by $L^2 - L^\infty$ method, which was established to study Boltzmann equation in [9]. Based on the uniform estimates of the solution, we also consider the BV-regularity of the solution in non-convex domain. A cut-off function, which aims to exclude all the characteristics emanating from the grazing set $\mathcal{E}_B$, has been constructed precisely.

1. Introduction

The neutron transport equation is a type of radiative transport equation, which is a balance statement that the neutrons conserved. This equation is commonly used to determine the behavior of nuclear reactor cores and experimental or industrial neutron beams. For more details, see [5]. In this paper, we consider the following neutron transport equation

\begin{equation}
\frac{\partial u}{\partial t} + v \cdot \nabla u + \Sigma(x, v)u = Ku + q(t, x, v).
\end{equation}

The function $u(t, x, v)$ represents a density of the number of particles. $\Sigma(x, v) \geq 0$ describes the effective total cross section, which is a given positive function of $x$ and $v$, the given operator $K$ is defined as

\begin{equation}
Ku(x, v) = \int_{V} k(x, v, v')u(t, x, v') dv',
\end{equation}

The nonnegative kernel $k(x, v, v')$ models a transfer of a density of numbers of neutrons from one speed to another. It depends on the state of the material at the point $x$, $v$, $v'$, and it is isotropic if the kernel only depends on the variables $v$ and $v'$. $q(t, x, v)$ is a source of a finite total number of neutrons at each moment $t$.

In bounded domain, the equation describes the evolution of a population of neutrons in a domain $\Omega$ occupied by a medium which interacts with the neutrons. Here $\Omega$ be a bounded open and connected subset of $\mathbb{R}^3$, $\partial \Omega$ is denoted as its boundary. The domain $V$ is the velocity space, which generally is the form $V = \{v \in \mathbb{R}^3 | a \leq |v| \leq b\}$ or a finite union of spheres.

There are extensive developments in the study of the neutron transport equation. The existence of the solution for both steady neutron transport equation and time-dependent neutron transport equation have been studied in [7], [20], [21] by constructing a maximal and a minimal solutions. An abstract theorem is also established in [1]. The existence of the solution for the neutron transport equation was constructed in the Banach spaces $L^p$, $1 \leq p < \infty$.

2000 Mathematics Subject Classification. 35H10, 76P05, 84C40.

Key words and phrases. Neutron transport equation, Existence, BV-regularity, Non-convex domain.

Y. Guo’s research is supported in part by NSFC grant 10828103, NSF grant DMS 1209437, Simon Research Fellowship and BICMR. X.F. Yang is supported by NNSF of China under the Grant 11171212 and the SJTU’s SMC Projection A.
it means that \( L^\infty \) solution has not been treated. For the asymptotic expansions in transport theory, we refer to [2], [13], [14], [15] and [22]. Most of the above works considered the neutron transport while the neutron flux entering \( \Omega \) at each point of \( \partial \Omega \) is zero, that is, the zero in-flow boundary condition. Later, the existence of the neutron transport equation with different types of boundary condition has also appeared in [16], [17], [18], [19] and [24] and referees therein. Moreover, the authors [3] had studied the existence and the asymptotic expansion of the solution for neutron transport equation with in-flow boundary condition together with a weak diffusive boundary condition by the probabilistic theory. In [23], the very recent result show the asymptotic expansion of the solution for the neutron transport equation in 2-D unit disc, which gave a more precise approximation of the solution around the boundary by modifying the Milne problem.

In the following, we list some assumptions on the phase space \( \Omega \times V \). We assume that the boundary \( \partial \Omega \) is locally a graph of a given \( C^2 \) function: for each point \( x_0 \in \partial \Omega \), there exist \( r > 0 \) and a \( C^2 \) function \( \eta : \mathbb{R}^2 \to \mathbb{R} \) such that, up to a rotation and relabeling, we have

\[
\begin{align*}
\partial X &\cap B(x_0; r) = \{ x \in B(x_0; r) : x_3 = \eta(x_1, x_2) \}, \\
\partial X &\cap B(x_0; r) = \{ x \in B(x_0; r) : x_3 > \eta(x_1, x_2) \}.
\end{align*}
\]

In this case, the outward normal direction \( n \) at \( x \in \partial \Omega \) can be expressed as

\[
n(x_1, x_2) = \frac{1}{\sqrt{1 + |\nabla_x \eta(x_1, x_2)|^2}} \left( \partial_{x_1} \eta(x_1, x_2), \partial_{x_2} \eta(x_1, x_2), 1 \right).
\]

The domain \( \Omega \) is called a **strictly non-convex domain** if there exists at least one point \( x_0 \in \partial \Omega \) and nonzero \( u \in \mathbb{R}^2 \) such that (1.3)-(1.4) hold and

\[
\sum_{i,j=1.2} u_i u_j \partial_i \partial_j \eta(x_0) < 0.
\]

The phase boundary in the phase space \( \Omega \times V \) is denoted as \( \gamma = \partial \Omega \times V \), and we split it into the outgoing boundary \( \gamma_+ \), the incoming boundary \( \gamma_- \), and the grazing boundary \( \gamma_0 \)

\[
\begin{align*}
\gamma_+ &= \{ (x, v) \in \partial \Omega \times V : n(x) \cdot v \geq 0 \}, \\
\gamma_0 &= \{ (x, v) \in \partial \Omega \times V : n(x) \cdot v = 0 \}.
\end{align*}
\]

It is known that \( \gamma_+ \) and \( \gamma_- \) (resp \( \gamma_0 \)) are open subsets (resp. closed) of \( \gamma = \partial \Omega \times V \) such that

\[
\gamma = \partial \Omega \times V = \gamma_+ \cup \gamma_0 \cup \gamma_-.
\]

In this paper, we assume that \( \Omega \) is strictly non-convex domain, \( V \) is a bounded domain and it can be locally expressed as (3.12).

Let us explain the difficulty to study the regularity of the kinetic equation in bounded domain with boundary condition. It partly due to the characteristic nature of boundary conditions. To make it clear, we consider the transport equation with the given boundary condition

\[
v \cdot \nabla_x f(x, v) = 0, \quad f|_{\Gamma_-} = g.
\]

Given \((x, v) \in X\), let \([X(s), V(s)] = [X(s; t, x, v), V(s; t, x, v)] = [x - (t - s)v, v]\) be a trajectory for the transport equation:

\[
\frac{dX(s)}{ds} = V(s), \quad \frac{dV(s)}{ds} = 0,
\]

with the initial condition \([X(t; t, x, v), V(t; t, x, v)] = [x, v]\). Then, we solve (1.7) as \( f(x, v) = g(x_b(x, v), v) = g(x - t_b(x, v)v, v) \), where \( t_b(x, v) \geq 0 \) is the backward exit time, or the last
moment at which the back-time straight line \([X(s; 0, x, v), V(s; t, x, v)]\) remains in the interior of \(X\). It is defined as

\[ t_b(x, v) := \sup \left\{ \{0\} \cap \{\tau : x - sv \in X \text{ for all } 0 < s < \tau\} \right\}. \]

The backward exit position on the boundary \(\partial X\) is

\[ x_b(x, v) = x - t_b(x, v)v, \]

and we always have \(v \cdot n(x_b(x, v)) \leq 0\). Similarly the forward exit time \(t_f\) and the forward exit position are defined as

\[ t_f(x, v) := \sup \left\{ \{0\} \cap \{\tau : x + sv \in X \text{ for all } 0 < s < \tau\} \right\}, \]

\[ x_f(x, v) = x + t_f(x, v)v. \]

Generally, it is difficult to determine \(t_b, x_b\) as well as the solution to (1.7) with the diffusive boundary condition. This was solved by introducing the probability measure on the boundary in [9].

There are a few results about the regularity of the solutions to the kinetic equation in bounded domain. The first one has been appeared in [8], Guo constructed the singular solutions of the Vlasov-Maxwell equation on a half line. Recently, Guo [9] developed the \(L^2-L^\infty\) estimate for the solution of Boltzmann equation in convex domain with different boundary conditions, and it was show that the solution are continuous away from the grazing set \(\gamma_0\). It should be pointed out that the domain needs not to be convex for the diffusive reflection condition case. Later, the regularity of the solution for Boltzmann equation was studied in [10]. The authors established the \(C^1\) solution in convex domain and show that the solution should not be \(C^2\). In the above two papers, it could be proved that \(x_b(x, v)\) has singular behavior if \(n(x_b(x, v)) \cdot v = 0\), and the solution might be singular on the set:

\[ \Xi_B := \{(x, v) \in \overline{\Omega} \times V : n(x_b(x, v)) \cdot v = n(x - t_b(x, v)v) \cdot v = 0\}, \]

which is the collection of all the characteristics emanating from the grazing set \(\gamma_0\). In a non-convex domain, Kim [12] discovered that the singularity (discontinuity) of the solution of Boltzmann equation always occurs, and such singularity propagates along the singular set \(\Xi_B\). More precisely, let the concave (singular) grazing boundary in the grazing boundary to be defined as

\[ \gamma_0^S = \{(x, v) \in \gamma_0 : t_b(x, v) \neq 0 \text{ and } t_b(x, -v) \neq 0\} \subset \gamma_0. \]

It was proved that \(\gamma_0^S\) is the only part at which discontinuity can be created or propagates into the interior of the phase space \(\Omega \times V\). So the discontinuity set of the solution in \(\overline{\Omega} \times V\) is

\[ \mathcal{D} = \gamma_0 \cup \{(x, v) \in \overline{\Omega} \times V : (x_b(x, v), v) \in \gamma_0^S\}. \]

It implies that we can not get the classical solution of Boltzmann equation. A nature problem is the regularity of the solutions in non-convex domain. Very recently, the BV-regularity of the solution to Boltzmann equation in non-convex domain has been studied in [11]. Moreover, it was proved that the singular set to the characteristics emanating from the strictly non-convex points

\[ \{(x, v) \in \Xi_B : (x_b(x, v), v) \text{ is a strictly non-convex point}\} \]

is a co-dimension 1 submanifold of \(\Omega \times V\). This means that the BV regularity is the best regularity for Boltzmann equation in the non-convex domain.

Similarly, we expect to establish the existence and BV-regularity of the solution for the neutron transport equation in non-convex domain. The large time behavior and the regularity
of the solution to the Neutron transport equation would be considered in forthcoming papers. In this paper, we assume \( \Omega \) is an non-convex domain and we consider

\[
\frac{\partial u}{\partial t} + v \cdot \nabla u + \Sigma = Ku + q,
\]

with the initial-boundary condition

\[
u(0, x, v) = u_0(x, v), \quad u(t)|_{\gamma} = \mathcal{P}_\gamma u + r.
\]

Here \( r, u_0 \) are given functions and \( \mathcal{P}_\gamma \) is the diffusive reflection: for \( (x, v) \in \gamma_- \),

\[
\mathcal{P}_\gamma u(t, x, v) = c \int_{v' \in V: n(x) \cdot v' > 0} u(t, x, v') \{n(x) \cdot v'\} dv'.
\]

Here the constant \( c \) is normalized as

\[
c \int_{v' \in V: n(x) \cdot v' > 0} \{n(x) \cdot v'\} dv' = 1.
\]

The operator \( \mathcal{P}_\gamma \) could be viewed as function on \( \{v \in V : v \cdot n(x) > 0\} \) for any fixed \( x \in \partial \Omega \), it is a \( L^2_{w-} \) projection with respect to the measure \( |n(x) \cdot v| dv \) for any boundary function \( u \) defined on \( \gamma_+ \).

Before state the main results, we give some notations. We denote \( \| \cdot \|_{\infty} \) the norm of \( L^\infty(\Omega \times V) \), while \( \| \cdot \|_p \) is the norm of the \( L^p(\Omega \times V) \). In particular, \( (\cdot, \cdot) \) is the inner product of the space \( L^2(\Omega \times V) \). We also denote \( | \cdot |_p \) the norm of \( L^p(\partial \Omega \times V) \) and \( | \cdot |_{\gamma, p} \) the norm of \( L^p(\partial \Omega \times V) = L^p(\partial \Omega \times V, d\gamma) \) with \( d\gamma = |n(x) \cdot v| dS_x, dv \) with the surface measure \( dS_x \) on \( \partial X \). We write \( | \cdot |_{\gamma, p} = | \cdot |_{I_{\gamma, p}} \). For a function on \( \Omega \times V \), we denote \( f_\gamma \) to be its trace on \( \gamma \) whenever it exists, \( f \lesssim g \) means \( f = O(g) \).

We show that the \( L^2 \) estimates of the solution for the neutron transport equation with the mixing boundary condition can be obtained by the tracing theorem. These estimates can be applied to achieve the estimates of the solution in \( L^\infty \) norm by using the general characteristics curves of the equation with the same boundary condition. Thus, we get the existence of the solution for the neutron transport equation, which is stated as follows.

**Theorem 1.1.** Let \( \Omega \) be a bounded open subset of \( \mathbb{R}^3 \) with \( \mathcal{C}^2 \) boundary \( \partial \Omega \) as in (1.3)-(1.4). Assume that \( \|u_0\|_{\infty}, \sup_{0 \leq t \leq T} \|r(t)\|_{\infty}, \sup_{0 \leq t \leq T} \|q(t)\|_{\infty} \) are bound for any fixed \( T > 0 \). Suppose further that there exist some constant \( M_a, M_b \), for all \( (x, v) \in \Omega \times V \), it holds

\[
0 \leq \Sigma(x, v) \leq M_a, \quad 0 \leq \int_V k(x, v', v) dv' \leq \int_V k(x, v, v') dv' \leq M_b.
\]

Then, there exists a unique solution \( u \in L^\infty([0, T]; L^\infty(\Omega \times V)) \) of the problem (1.15) with (1.16) such that

\[
\|u(t)\|_{\infty} \leq \|u_0\|_{\infty} + \sup_{0 \leq t \leq T} \|r(t)\|_{\infty} + \sup_{0 \leq t \leq T} \|q(t)\|_{\infty}, \quad \text{for all } 0 \leq t \leq T.
\]

Based on the uniform estimate of solution (1.19), we study the BV-regularity of the solutions in non-convex domain. A function \( f \in L^1(\Omega \times V) \) has bounded variation in \( \Omega \times V \) if

\[
\|f\|_{BV} =: \sup \left\{ \int_{\Omega \times V} f \text{div} \psi dx dv : \psi \in C_c^1(\Omega \times V; \mathbb{R}^3), |\psi| \leq 1 \right\} < \infty
\]

The BV space is defined as follow

\[
f \in L^1(\Omega \times V) \ | \ ||f||_{BV} = ||f||_{L^1} + ||f||_{\tilde{BV}} < \infty \}
\]
For the estimate of BV norm of the solution, we should impose some additional regularity on the data. Let \( \partial = (\partial_x, \partial_v) \). For any fixed \( T \), we assume
\[
(1.22) \quad \| u_0 \|_{BV} + \sup_{0 \leq t \leq T} \left[ |r|_{\infty} + |\partial_r r(t)|_1 + |\partial v(t)|_1 + \| q(t) \|_{BV} \right] < \infty, 
\]
and there exist some constant \( M'_a, M'_b \), it holds, for all \((x, v) \in \Omega \times V\)
\[
(1.23) \quad \partial \Sigma(x, v) \leq M'_a, \quad \int_V \partial k(x, v', v) dv' \leq \int_V \partial k(x, v, v') dv' \leq M'_b. 
\]
The second main result in this paper is the following.

**Theorem 1.2.** Let \( \Omega \) is a non-convex domain with \( C^2 \) boundary \( \partial \Omega \) as in (1.3)-(1.4). Suppose that all the conditions in Theorem 1.1 hold. Moreover, we assume that (1.22)-(1.23) hold. Then, there exists a unique solution \( u \in L^\infty([0, T]; BV(\Omega \times V)) \) of the problem (1.15) with (1.16) satisfies, for all \( 0 \leq t \leq T \),
\[
(1.24) \quad \| u(t) \|_{BV} \leq \| u_0 \|_{BV} + \sup_{0 \leq t \leq T} \left[ |r|_{\infty} + |\partial_r r(t)|_1 + |\partial v(t)|_1 + \| q(t) \|_{BV} \right], 
\]
and \( \nabla_{x,v} u dt \) is a Radon measure \( \sigma_t \) on \( \Omega \times V \) such that \( \int_0^T |\sigma_t(\partial \Omega \times V)| dt \leq \| u_0 \|_{BV} + \sup_{0 \leq t \leq T} \left[ |r|_{\infty} + |\partial_r r(t)|_1 + |\partial v(t)|_1 + \| q(t) \|_{BV} \right] \).

Since the boundary operator associated with the diffusive condition is of norm exactly one, the standard theory of transport equation in bounded domains [1] fails. The ideas of the proof of Theorem 1.1 is similar to that in [9] and [6]. Here, we replace the original unknown function \( u \) with \( e^t u \) for any fixed time \( T \). It is more convenient because there is a diffusion term \( \lambda U \) in the equation. To prove the existence of the solution, we first study the equation about \( U \) with a reduced diffusive reflection boundary condition, which is set up to establish a contracting map argument. Then, we take the limit and get the solution based on the uniformly estimates of the sequence. In this paper, we give a more precise estimate of the sequence \( U_m \) (see Lemma 2.6), it can be easily applied to both the bound of the sequence and its convergence.

We now illustrate the main ideas of the proof of Theorem 1.2, which is similar to that in [11]. For simplicity, we assume that \( u \) satisfies the following simpler problem
\[
\partial_t u + v \cdot \nabla_x u + \Sigma u = H, \quad u|_{t=0} = u_0, \quad u|_{\gamma^c} = P_{\gamma} u + r, 
\]
and where \( \Sigma \geq 0, H \) and \( r \) are smooth enough. In general, solutions \( u \) are discontinuous on \( \Omega_B \) and (distributional) derivatives do not exist. In order to take derivatives, we construct some smooth cut-off function \( \chi_\epsilon(x, v) \) vanishing on an open neighborhood of \( \Omega_B \) and consider the following problem
\[
\partial_t u^\epsilon + v \cdot \nabla_x u^\epsilon + \Sigma u^\epsilon = \chi_\epsilon H, \quad u|_{t=0} = \chi_\epsilon u_0, \quad u'_{|\gamma^c} = \chi_\epsilon P_{\gamma} u^\epsilon + \chi_\epsilon r. 
\]
Due to the cut-off \( \chi_\epsilon \), the solution of \( u^\epsilon \) vanishes on some open subset of \( \Omega \times V \) containing the singular set \( \Omega_B \). Therefore \( u^\epsilon \) is smooth and we can apply (distributional) derivatives \( \partial \) to the approximation equation. Once we can show that \( u^\epsilon \) is uniformly bounded in \( L^\infty \) and \( \partial u^\epsilon \) is uniformly bounded in \( L^1(\Omega \times V) \), then we conclude that \( u^\epsilon \) converges to the solution \( u \) weak-* in \( L^\infty \) and \( BV \).

So, we should firstly construct the smooth cut-off function \( \chi_\epsilon \) such that it vanished on an open neighborhood \( O_{\epsilon, \eta} \) of \( \Omega_B \). we can show that \( O_{\epsilon, \eta} \) contains all points whose distance from \( \Omega_B \) is less than \( \epsilon \). Such \( \epsilon \) thickness is important for constructing cut-off functions.
Secondly, we should control the outgoing term for the estimate of the derivatives. For this purpose, we split the outgoing boundary $\gamma_+$ into the (outgoing) almost grazing set

$$
\gamma^\delta_+ := \{(x, v) \in \gamma_+, \ v \cdot n(x) \leq \delta\},
$$
and the (outgoing) non-grazing set

$$
\gamma_+ \setminus \gamma^\delta_+ := \{(x, v) \in \gamma_+, \ v \cdot n(x) \geq \delta\}.
$$
The set $\gamma_+ \setminus \gamma^\delta_+$ contribution can be controlled by the bulk integration and the initial data by the trace theorem. While the $\gamma^\delta_+$ contribution cannot be bounded by the bulk integration and $\int_0^t |\partial u^\epsilon|_{y_1,1}$ of the energy-type estimate directly. Fortunately, we extract an extra small constant in front of the term $\int_0^t |\partial u^\epsilon|_{y_1,1}$ to bound $\partial u^\epsilon$ on $\gamma_-$ by using the Duhamel formula along the trajectory (Double iteration schedule).

The plan of this paper is the following: In section 2, we obtain the solution $U$ with a reduced diffusive reflection boundary condition. Based on the uniformly estimates of the approximation solution in $L^2$, we take the limit and get the solution of the neutron transport equation. For the uniform bound of the approximation solution, we follows the abstract scheme appeared in [6]. Here we give a new estimate in Lemma 2.6 which can be applied to equation. For the uniform bound of the approximation solution, we follows the abstract and purpose, we split the outgoing boundary $\gamma_+$ into the (outgoing) almost grazing set

$$
\gamma^\delta_+ := \{(x, v) \in \gamma_+, \ v \cdot n(x) \leq \delta\},
$$
and the (outgoing) non-grazing set

$$
\gamma_+ \setminus \gamma^\delta_+ := \{(x, v) \in \gamma_+, \ v \cdot n(x) \geq \delta\}.
$$
The set $\gamma_+ \setminus \gamma^\delta_+$ contribution can be controlled by the bulk integration and the initial data by the trace theorem. While the $\gamma^\delta_+$ contribution cannot be bounded by the bulk integration and $\int_0^t |\partial u^\epsilon|_{y_1,1}$ of the energy-type estimate directly. Fortunately, we extract an extra small constant in front of the term $\int_0^t |\partial u^\epsilon|_{y_1,1}$ to bound $\partial u^\epsilon$ on $\gamma_-$ by using the Duhamel formula along the trajectory (Double iteration schedule).

The plan of this paper is the following: In section 2, we obtain the solution $U$ with a reduced diffusive reflection boundary condition. Based on the uniformly estimates of the approximation solution in $L^2$, we take the limit and get the solution of the neutron transport equation. For the uniform bound of the approximation solution, we follows the abstract scheme appeared in [6]. Here we give a new estimate in Lemma 2.6 which can be applied to obtain both the bound of the sequence and its convergence. In section 3, we firstly construct the desired $\varepsilon$-neighborhood of the singular set and its smooth cut-off functions $\chi_\varepsilon$. Then, we analyse the estimates of $\chi_\varepsilon$ and their derivatives in the bulk and on the boundary. Moreover, the new trace theorem is achieved by using double iteration, and we give the estimates of the approximation sequence $u^\varepsilon_m$ in $L^\infty$ and its derivatives in $L^1(\Omega \times V)$. At last, some useful geometric results will be listed in Section 4.

2. Existence of the solution

In this section, we consider the existence of the solution to (1.15) with the boundary condition (1.16) for all $0 \leq t \leq T$. The result is as follows.

**Proposition 2.1.** Let $T > 0$. Suppose that $\|u_0\|_\infty, \sup_{0 \leq t \leq T} |r(t)|_\infty, \sup_{0 \leq t \leq T} |q|_\infty$ are all bound. Then, there is a solution $u$ of (1.15)-(1.16) such that for any $0 \leq t \leq T$, it satisfies

$$
\|u(t)\|_\infty \leq \|u_0\|_\infty + \sup_{0 \leq t \leq T} (|r(t)|_\infty + |q(t)|_\infty).
$$

Set $U = e^{-\lambda t}u$ for some constant $\lambda \gg 1$ which will be determined later, it satisfies the modified problem

$$
\partial_t U + v \cdot \nabla U + \lambda U = (-\Sigma + K)U + q^1, \ U = u_0,
$$
with $U(t)|_{\gamma_-} = P_{\gamma}U + r^1$, and $q^1 = e^{-\lambda t}q, \ r^1 = e^{-\lambda t}r$. The following result is obvious, we omit the proof here.

**Proposition 2.2.** The problem (1.15)-(1.16) has a unique solution if and only if the problem (2.2) has a unique solution. Moreover, solutions of (1.15) correspond to solutions of (2.2).

The existence of the solution to (2.2) can be obtained from the following proposition.

**Proposition 2.3.** Let $T > 0$. $\|u_0\|_\infty, \sup_{0 \leq t \leq T} |r(t)|_\infty, \sup_{0 \leq t \leq T} |q^1|_\infty$ are all bound. Then, there is a solution $U$ of (2.2) such that for any $0 \leq t \leq T$, it satisfies

$$
\|U(t)\|_\infty \leq \|u_0\|_\infty + \sup_{0 \leq t \leq T} (|r^1(t)|_\infty + |q^1(t)|_\infty).
$$

In the following, we mainly establish the solution to (2.2).
2.1. \(L^2\) estimates of the solution. In order to prove Proposition 2.3, we need to study the \(L^2\) of the solution to (2.2). The estimate are obtained from the following two lemmas. Firstly, we start with the existence of the solution to the simple transport equation. Secondly, we construct the solution to (2.2) by assuming \(\lambda\) is sufficiently large.

**Lemma 2.1.** Let \(\lambda > 0\) and \(T > 0\). Suppose that \(Q \in L^2 \cap L^\infty([0, T] \times \Omega \times V), R \in L^2 \cap L^\infty([0, T] \times \gamma_\lambda), \) \(U_0 \in L^2 \cap L^\infty(\Omega \times V).\) Then, for all \(0 \leq t \leq T,\) there exists a unique solution \(U(t, x, v)\) to

\[
(2.4) \quad [\partial_t + v \cdot \nabla + \lambda]U = Q, \quad U(t)|_{\gamma_-} = R, \quad U|_{t=0} = U_0.
\]

such that

\[
(2.5) \quad \|U(t)\|_2^2 + \int_0^t (\|U(s)\|_2^2 + \|U(s)\|_2^2)ds \leq \|U_0\|_2^2 + \int_0^t (\|R(s)\|_2^2 + \|Q(s)\|_2^2)ds
\]

and

\[
(2.6) \quad \|U(t)\|_\infty + |U(s)|_\infty \leq \|U_0\|_\infty + \sup_{0 \leq s \leq T} |R(s)|_\infty + \int_0^T \|Q(s)\|_\infty ds.
\]

**Proof.** From integration along the characteristic lines of

\[
\frac{dx}{ds} = v \in V \quad \text{and} \quad \frac{dv}{ds} = 0,
\]

the solution of \(U(t, x, v)\) can be rewritten in the integration form as

\[
U(t, x, v) = 1_{[t < t_b]} e^{-\lambda t} U_0(x - tv, v) + 1_{[t \geq t_b]} e^{-\lambda t} R(t - t_b, x_b, v)
\]

\[
(2.7) \quad + \int_{\min[t, t_b(x, v)]}^{t_b(x, v)} e^{-\lambda s} Q(t - s, x - sv, v)ds.
\]

Here \(t_b(x, v)\) and \(x_b\) are defined in (1.9)-(1.10). We then show that \(U(t, x, v)\) is a weak solution of (2.4) in the sense of distributions.

Now, we will establish the \(L^2\) estimate of the solution of (2.4). Multiplying (2.4) with \(U\) and integrating over \([0, T] \times \Omega \times V,\) then Greens formula gives

\[
\|U(t)\|_2^2 + \int_0^t \left(\|U(s)\|_{L^2}^2 + 2\lambda \|U(s)\|_{L^2}^2\right)ds = \|U_0\|_2^2 + \int_0^t \left(\|U(s)\|_{L^2}^2 + 2\langle Q, U \rangle\right)ds.
\]

Since \(\lambda > 0\) and \(U|_{\gamma_-} = R,\) by the Cauchy inequality, we get

\[
(2.8) \quad \|U(t)\|_2^2 + \int_0^t \left(\|U(s)\|_2^2 + \|U(s)\|_2^2\right)ds \leq \|U_0\|_2^2 + \int_0^t \left(\|R(s)\|_{L^2}^2 + \|Q(s)\|_{L^2}^2\right)ds.
\]

This gives the inequality (2.10). The uniqueness of the solution follows from (2.8) when \(U_0 = 0, R = 0\) and \(Q = 0.\) The inequality (2.6) is easily derived from (2.7) since \(\lambda > 0.\) □

In the next lemma, we firstly study the solution of (2.2) with a reduced diffusive reflection boundary condition, which is necessary to establish a contracting map argument. Then, we take the limit and get the solution based on the uniformly estimates of the sequence.

**Lemma 2.2.** Let \(T > 0\) and

\[
(2.9) \quad \lambda > \lambda_0 = 1 + M_d + M_b.
\]
Suppose that \( q^1 \in L^2([0, T] \times \Omega \times V), \ r^1 \in L^2([0, T] \times \gamma_-) \) and \( u_0 \in L^2(\Omega \times V) \). Then, there exists a unique solution \( U \) to (2.2). Moreover, for any \( 0 \leq t \leq T \), the solution satisfies

\[
(2.10) \quad ||U(t)||^2_2 + \int_0^t (||U(s)||^2_2 + ||U(s)||^2_2)ds \leq ||u_0||^2_2 + \int_0^t \left(|r^1(s)|^2_{\gamma_-} + ||q^1(s)||^2_2\right)ds.
\]

**Proof.** Firstly, for any \( j > 0 \), we consider the existence of (2.2) with the reduced diffusive reflection boundary condition

\[
(2.11) \quad U(t, x, v)|_{\gamma_-} = (1 - \frac{1}{j})P_\gamma U + r^1(t, x, v), \quad \text{for} \quad j > 0.
\]

By applying Lemma 2.1 to the following iteration in both \( j \) and \( l \): \( U^0 = u_0 \), and for \( l \geq 0 \),

\[
(2.12) \quad (\partial_t + v \cdot \nabla + \lambda)U^{l+1} = (-\Sigma + K)U^l + q^1, \quad U^{l+1} = u_0,
\]

with \( U^{l+1}|_{\gamma_-} = (1 - \frac{1}{j})P_\gamma U^l + r^l \).

**Step 1.** We fix \( j > 0 \) and take \( l \to \infty \) of the solution of (2.12) with (2.11). Multiply \( U^{l+1} \) on both sides (2.12) and integrate over \([0, T] \times \Omega \times V \), from Greens identity, it holds that

\[
(2.13) \quad ||U^{l+1}(t)||^2_2 + \int_0^t |U^{l+1}(s)|^2_{2, +}ds + 2\lambda \int_0^t ||U^{l+1}(s)||^2_2ds = ||u_0||^2_2 + \int_0^t \left|(-\Sigma + K)U^l + q^1, U^{l+1}\right|ds.
\]

By the choice of \( \lambda \) and \( \lambda_0 \) in (2.9), we derive that

\[
2|(-\Sigma + K)U^l, U^{l+1}| \leq 2||U^{l+1}||_2||(-\Sigma + K)U^l||_2 \leq \lambda_0||U^{l+1}||^2_2 + \lambda_0||U^l||^2_2,
\]

\[
2|q^1, U^{l+1}| \leq (\lambda - \lambda_0)||U^{l+1}||^2_2 + \frac{4}{\lambda - \lambda_0}||q^1||^2_2.
\]

Moreover, there is \( C_j > 0 \) such that

\[
(2.14) \quad |(1 - \frac{1}{j})P_\gamma U^l + r^l|_{2, -}^2 \leq |(1 - \frac{1}{j})P_\gamma U^l|_{2, -}^2 + \frac{1}{2j^2}|P_\gamma U^l|_{2, -}^2 + C_j|r^l|_{2, -}^2.
\]

For simplicity, we denote that

\[
E := ||u_0||^2_2 + \int_0^t \left(|r^1(s)|^2_{\gamma_-} + ||q^1(s)||^2_2\right)ds.
\]

It is easy to know that \( |P_\gamma U^l|_{2, -}^2 \leq |U^l|_{2, -}^2 \). (2.13) derives to

\[
||U^{l+1}(t)||^2_2 + \int_0^t \left[|U^{l+1}(s)|^2_{2, +} + \lambda||U^{l+1}(s)||^2_2\right]ds \leq \int_0^t \left[(1 - \frac{2}{j^2} + \frac{3}{2j^2})|U^l(s)|^2_{2, +} + \lambda_0||U^l(s)||^2_2\right]ds + C_{\lambda, j}E.
\]

Since \( \lambda > \lambda_0 \) and \( 1 - \frac{2}{j} + \frac{3}{2j^2} < 1 \), there is some \( \eta_{\lambda, j} < 1 \) such that

\[
||U^{l+1}(t)||^2_2 + \left(\int_0^t |U^{l+1}(s)|^2_{2, +}ds + \lambda \int_0^t ||U^{l+1}||^2_2ds\right) \leq \eta_{\lambda, j}\left(\int_0^t (|U^l|_{2, +}^2 + \lambda ||U^l||_2^2)ds\right) + C_{\lambda, j}E.
\]
Since $U^0 = u_0$, we iterate again to obtain

$$
\|U^{l+1}(t)\|_2^2 + \left( \int_0^t |U^{l+1}(s)|_{2,+}^2 + \lambda \|U^{l+1}\|_2^2 \right) ds
\leq \eta_{l,j}^2 \left( \int_0^t |U^{l-1}(s)|_{2,+}^2 + \lambda \|U^{l-1}\|_2^2 \right) ds + (1 + \eta_{l,j}) C_{\lambda,j} \mathcal{E}.
$$

\ldots

$$
\leq \eta_{l,j}^l \int (|U^0|_{2,+}^2 + \lambda \|U^0\|_2^2) ds + \frac{1 + \eta_{l,j}^l}{1 - \eta_{l,j}} C_{\lambda,j} \mathcal{E}.
$$

So, we get the following uniform estimates of $U^l$ with respect to $l$

$$
\|U^{l+1}(t)\|_2^2 + \int_0^t |U^{l+1}(s)|_{2,+}^2 ds + \lambda \int_0^t \|U^{l+1}\|_2^2 ds \lesssim_{\lambda,l,T} \mathcal{E}.
$$

Now, taking the difference of $U^{l+1} - U^l$, it satisfies

$$
\left[ \partial_t + v \cdot \nabla + \lambda \right](U^{l+1} - U^l) = (-\Sigma + K)(U^l - U^{l-1}),
$$

with $(U^{l+1} - U^l)|_{\gamma_-} = (1 - \frac{1}{T}) \mathcal{P}_\gamma(U^l - U^{l-1}), (U^{l+1} - U^l)(0) = 0$. Similar to the estimate of (2.15), we yield

$$
\|U^{l+1}(t) - U^l(t)\|_2^2 + \int_0^t \left( \left( (U^{l+1}(s) - U^l(s))_{2,+}^2 + \lambda \|U^{l+1}(s) - U^l(s)\|_2^2 \right) ds
\leq \eta_{l,j}^l \int 0^t \left( ((U^l - U^0)(s))_{2,+}^2 + \lambda \|U^l - U^0\|_2^2 ) ds
$$

From (2.15), we know that $\int_0^t ((U^l - U^0)(s))_{2,+}^2 + \|((U^l - U^0)(s))_{2,+}^2 \| ds < \infty$ for fixed $t$. It concludes that $\{U^l\}_{l=0}^l$ is a Cauchy sequence with respect to $l$ for $\eta_{l,j} < 1$.

Let $l \to \infty$ to obtain $U_j$ as a solution of

$$
(\partial_t + v \cdot \nabla + \lambda) U_j = (\Sigma - K)U_j + q^j, \quad U_j(0) = u_0
$$

with $U_j|_{\gamma_-} = (1 - \frac{1}{T}) \mathcal{P}_\gamma U_j + r^j$ for all $0 \leq t \leq T$. Moreover, the estimate of (2.15) derives to

$$
\|U_j(t)\|_2^2 + \int_0^t \left[ \frac{1}{T} [\mathcal{P}_\gamma U_j(s)]_{2,+}^2 + \|I - \mathcal{P}_\gamma U_j(s)\|_{2,+}^2 \right] ds + \lambda \int_0^t \|U_j(s)\|_2^2 ds \lesssim_{\lambda,l,T} \mathcal{E}.
$$

\textbf{Step 2.} Let $j \to \infty$ for $U_j$. It needs a uniform estimate of $U_j$ w.r.t. $j$. Multiply (2.18) with $U_j$ and integrate over $[0, T] \times \Omega \times V$, then Green’s identity derives to

$$
\|U_j(t)\|_2^2 + \int_0^t |U_j(s)|_{2,+}^2 ds + 2 \int_0^t \lambda \|U_j(s)\|_2^2 ds
$$

$$
= \|u_0\|_2^2 + \int_0^t ((1 - \frac{1}{T}) \mathcal{P}_\gamma U_j + r^j(s))_{2,-}^2 ds + 2 \int_0^t \left[ ((\Sigma - K)U_j, U_j) + (q^j, U_j) \right] ds.
$$
For any \( \eta > 0 \) and \( j > 0 \), \(|(1 - \frac{1}{j})\mathcal{P}_\gamma U_j + r|\|^2_{L^2(\Sigma)}\) can be rewritten as

\[
|(1 - \frac{1}{j})\mathcal{P}_\gamma U_j + r|\|^2_{L^2(\Sigma)} = (1 - \frac{1}{j})^2|\mathcal{P}_\gamma U_j|_{L^2(\Sigma)}^2 + 2(1 - \frac{1}{j}) \int_{\Sigma} \mathcal{P}_\gamma U_j r \gamma d\gamma + |r|_{L^2(\Sigma)}^2
\]

\[
\leq (1 + \eta)|\mathcal{P}_\gamma U_j|_{L^2(\Sigma)}^2 + C|P|_{L^2(\Sigma)}^2.
\]

(2.20)

Because \(|\mathcal{P}_\gamma U_j|_{L^2(\Sigma)}^2 = |\mathcal{P}_\gamma U_j|_{L^2(\Sigma)}^2 + |U_j|_{L^2(\Sigma)}^2 + |(I - \mathcal{P}_\gamma)U_j|_{L^2(\Sigma)}^2\), together with the fact that \(2(\Sigma - K)U_j, U_j\) \(\leq 2\lambda_0||U_j(s)||^2 + 2 \int U_j q^4 \leq (\lambda - \lambda_0)||U_j||^2 + \frac{\lambda}{\lambda_0}||q^4||^2\), we derive that

\[
\|U_j(t)||^2_{L^2(\Sigma)} + \int_0^t \|\mathcal{P}_\gamma U_j(s)||^2_{L^2(\Sigma)} ds + \int_0^t \|\mathcal{P}_\gamma U_j(s)||^2_{L^2(\Sigma)} ds \leq \eta \int_0^t |\mathcal{P}_\gamma U_j(s)||^2_{L^2(\Sigma)} ds + C_{\eta, \lambda, F}E.
\]

(2.21)

Now, we study the estimate of \(\mathcal{P}_\gamma U_j\) by the trace theorem which has appeared in [6] and [10]. For the purpose of it, we consider the boundary contribution

\[
\int_0^t |\mathcal{P}_\gamma U_j(s)||^2_{L^2(\Sigma)} ds = c^2 \int_0^t \int_{\Sigma} \left[ \int_{\{\gamma : n(x) \cdot \gamma > 0\}} U_j(s, x, \gamma) n(x) \cdot \gamma d\gamma \right]^2 \partial_\gamma ds.
\]

We split the domain of inner integration as

\[
\{\gamma' \in V : n(x) \cdot \gamma' > 0\} = \{\gamma' \in V : 0 < n(x) \cdot \gamma' < \epsilon \text{ or } |\gamma'| \leq \epsilon\}
\]

\(\cup\ \{\gamma' \in V : n(x) \cdot \gamma' \geq \epsilon \text{ and } |\gamma'| \geq \epsilon\}.

The first set's contribution (grazing part) of \(\int_0^t |\mathcal{P}_\gamma U_j(s)||^2_{L^2(\Sigma)} ds\) is bounded by the Hölder inequality,

\[
c \int_0^t \int_{\partial \Omega} c \int_V |n \cdot \gamma| d\gamma \int_{\{0 < n \cdot \gamma' < \epsilon \text{ or } |\gamma'| \leq \epsilon\}} |n \cdot \gamma'| d\gamma' \times \int_{\{\gamma' : n \cdot \gamma' > 0\}} |U_j(s, x, \gamma')|^2 |n \cdot \gamma'| d\gamma' dS_x ds
\]

\[
\leq \Omega \epsilon \times \int_0^t \int_{\Sigma} |U_j(s)||^2 d\gamma ds.
\]

(2.22)

Here we have used the fact that

\[
\int_{|n \cdot \gamma'| \leq \epsilon} |n \cdot \gamma'| d\gamma' \leq \epsilon, \quad c \int_V |n \cdot \gamma| d\gamma = 1.
\]

For the bound of the second set's contribution (non-grazing part) of \(\int_0^t |\mathcal{P}_\gamma U_j(s)||^2_{L^2(\Sigma)} ds\), we use Lemma 4.2 and (2.21). From the equation, \((\partial_t + v \cdot \nabla_x)(U_j)^2 = -2\lambda(U_j)^2 - 2U_j(\Sigma - K)U_j + 2U_jq^4\). Taking the absolute value and integrating on \(\Omega \times V\), for \(\lambda \gg 1\), we have

\[
\int_0^t \|((\partial_t + v \cdot \nabla_x)(U_j)^2(s))\|_1 ds \leq 4 \int_0^t \left[\lambda||U_j||^2 + ||q^4||^2\right] ds.
\]
The trace theorem 4.2 gives
\[
\int_0^t |U_j 1_{Y_j \setminus \gamma_j^c}(s)|^2 ds \lesssim_{\epsilon, \Omega, \lambda, T} ||u_0||_2^2 + \int_0^t \left[ \frac{1}{2} ||U_j(s)||_2^2 + ||(\partial_t + v \cdot \nabla_x)(U_j)^2(s)||_1 \right] ds
\]
\[(2.23) \]
From (2.22) and (2.23), we have
\[
\int_0^t |\mathcal{P} \gamma U_j(s)|_{L^2,\Omega}^2 ds \lesssim_{\epsilon, \Omega, \lambda, T} \int_0^t (e|U_j(s)|^2_{L^2,\Omega} + ||U_j(s)||_2^2) ds + \mathcal{E}.
\]
Combining (2.21) and (2.24) with small \( \eta > 0 \) and \( \epsilon > 0 \), we have the following uniform estimate
\[
||U_j(t)||_2^2 + \int_0^t |U_j(s)|_2^2 ds + \lambda \int_0^t ||U_j(s)||_2^2 ds \lesssim_{\epsilon, \Omega, \lambda, T} \mathcal{E}.
\]
By taking a weak limit, we obtain a weak solution \( U \) to (2.2) with the same bound (2.25). Taking the difference, we have
\[
\left( \partial_t + v \cdot \nabla_x + \lambda \right)|U_j - U| = (-\Sigma - K)(U_j - U), \quad [U_j - U](0) = 0
\]
with \([U_j - U]|_{\gamma_+} = \mathcal{P}_\gamma [U_j - U] + \frac{1}{2} \mathcal{P}_\gamma U_j\). Applying (2.25) with \( r^j = \frac{1}{2} \mathcal{P}_\gamma U_j \), we obtain, as \( j \to \infty \)
\[
||U_j(t)||_2^2 + \int_0^t (|U_j(s)|_{L^2,\Omega}^2 + ||U_j(s)||_2^2) ds \lesssim \frac{1}{j^2} \int_0^t |\mathcal{P}_\gamma U_j(s)|^2 ds \to 0
\]
because of (2.19). We yield that the limit \( U \) is \( L^2 \) solution to (2.2) for all \( 0 \leq t \leq T \). Moreover, the estimate of (2.10) is easily obtained from (2.25). The proof of Lemma 2.2 is completed. \( \square \)

2.2. \( L^\infty \) estimate of the solution. We would study the uniform \( L^\infty \) estimates of the solution for the problem (2.2). To bootstrap \( L^2 \) estimate into \( L^\infty \) estimate, we need to define the stochastic cycles for the generalized characteristic lines interacted with the boundary. This method was firstly introduced by Guo in [9], which is a canonical way to treat \( L^\infty \) estimate of the solution to Boltzmann equation with diffusive boundary condition. In the following, we construct the stochastic cycles for neutron transport equation with diffusive boundary condition, which is similar to that for Boltzmann equation in [9] and [6]. Then, we show \( L^\infty \) estimate of the solution to the neutron transport equation (2.2) by this stochastic cycles.

Let \( \mathcal{V}(x) = \{ v' \in D : v' \cdot n(x) > 0 \} \), the probability measure \( d\sigma = d\sigma(x) \) is given by
\[
d\sigma(x) = c(n(x) \cdot v') dv', \quad \text{with} \quad c \int_{\mathcal{V}(x)} d\sigma(x) = 1.
\]

**Definition 2.1.** (Stochastic Cycles). Fix any point \( t > 0 \) and \( (x, v) \notin \gamma_0 \cap \gamma_- \). Let \( (t_0, x_0, v_0) = (t, x, v) \). For \( v_{k+1} \in \mathcal{V}_{k+1} = \{v_{k+1} \cdot n(x_{k+1}) > 0\} \), define the \((k + 1)\) component of the back-time cycle as
\[
(t_{k+1}, x_{k+1}, v_{k+1}) = (t_k - t_b(x_k, v_k), x_b(x_k, v_k), v_{k+1}).
\]
And the stochastic cycle is defined as
\[
X_{st}(s; t, x, v) = \sum_l 1_{[t_{l+1}, t_l]}(s) (x_l + (s - t_l)v_l), \quad V_{st}(s; t, x, v) = \sum_l 1_{[t_{l+1}, t_l]}(s)v_l.
\]
We define the iterated integral for \( k \geq 2 \),

\[
\int_{\prod_{l=1}^{k-1} V_{l}} \prod_{l=1}^{k-1} d\sigma = \prod_{l=1}^{k-1} \left( \int_{V_{l}} d\sigma_{l} \right) \cdots d\sigma_{1}.
\]

Here \( v_{l} (l = 1, 2, \cdots, k) \) are all independent variables and \( t_{k}, x_{k} \) depend on \( t_{l}, x_{l}, v_{l} \) for \( l \leq k - 1 \). The phase space \( \mathcal{V}_{l} \) implicitly depends on \((t, x, v, v_{1}, v_{2}, \cdots, v_{l-1})\). We show that the set in the phase space \( \bigoplus_{l=1}^{k-1} V_{l} \) not reaching \( t = 0 \) after \( k \) bounces is small when \( k \) is large.

**Lemma 2.3.** Fixed \( T > 0 \). For any \( \varepsilon > 0 \), there exists \( k_{0}(\varepsilon, T) \) such that for \( k \geq k_{0} \), for all \((t, x, v) \in [0, T] \times \Omega \times \mathcal{V}_{1} \),

\[
\int_{\prod_{l=1}^{k-1} V_{l}} 1_{\{ t(t, x, v, v_{1}, v_{2}, \cdots, v_{k-1}) > 0 \}} \prod_{l=1}^{k-1} d\sigma_{m} \leq \varepsilon.
\]

**Proof.** Choosing \( 0 < \delta < 1 \) sufficiently small, the non-grazing sets for \( 1 \leq m \leq k - 1 \) is defined as

\[
\mathcal{V}^{\delta}_{m} = \{ v_{m} \in \mathcal{V}_{m} : v_{m} \cdot n(x_{m}) \geq \delta \}.
\]

For any \( m \), by a change of variable \( v_{\parallel} = (n(x_{m}) \cdot v_{m}) n(x) \) and \( v_{\perp} = v_{m} - v_{\parallel} \) for \( |v_{\parallel}| \leq |n(x_{m}) \cdot v_{m}| \leq \delta \), the measure of the grazing set is estimated as

\[
\int_{\mathcal{V}_{m} \setminus \mathcal{V}^{\delta}_{m}} d\sigma_{m} \leq \int_{\mathcal{V}_{m} \setminus \mathcal{V}^{\delta}_{m}} M_{n} \langle v_{m} \rangle d\nu_{m} \leq C \int_{|v_{\parallel}| \leq \delta} d\nu_{\parallel} \int_{\mathbb{R}^{2}} e^{-\frac{|v_{\perp}|^{2}}{2}} d\nu_{\perp} \leq C \delta,
\]

where \( C \) is independent of \( m \). On the other hand, if \( v_{m} \in \mathcal{V}^{\delta}_{m} \), then from diffusive back-time cycle, we have \( x_{m} - x_{m+1} = (t_{m} - t_{m+1}) V_{m} \). From Lemma 4.1, since \( v_{m} \cdot n(x_{m}) \geq \delta \) and \( v_{m} \) is bounded, then \((t_{m} - t_{m+1}) \geq \frac{\delta}{C_{T}} \). Therefore, if \( 0 < t_{k}(t, x, v, v_{1}, v_{2}, \cdots, v_{k-1}) \leq T \), then there can be at most \( \left\lfloor \frac{C_{T}}{\delta} \right\rfloor + 1 \) number of \( v_{m} \in \mathcal{V}^{\delta}_{m} \) for \( 1 \leq m \leq k - 1 \). We therefore have

\[
\int_{\prod_{l=1}^{k-1} V_{l}} 1_{h > 0} d\sigma_{k-1} \cdots d\sigma_{1} \leq \sum_{m=1}^{\left\lfloor \frac{C_{T}}{\delta} \right\rfloor + 1} \int_{\prod_{l=1}^{k-1} V_{l}} \left( \sum_{j=1}^{k-1} \int_{V_{j} \setminus \mathcal{V}^{\delta}_{j}} d\sigma_{j} \right)^{m} \prod_{l=1}^{k-1} d\sigma_{m} \]

Since \( d\sigma_{m} \) is a probability measure \( \int_{\mathcal{V}_{m}} d\sigma_{m} \leq 1 \) and

\[
\left\{ \int_{\mathcal{V}_{j} \setminus \mathcal{V}^{\delta}_{j}} d\sigma_{j} \right\}^{k-m-1} \leq \left\{ \int_{\mathcal{V}_{j} \setminus \mathcal{V}^{\delta}_{j}} d\sigma_{j} \right\}^{k-2-\left\lfloor \frac{C_{T}}{\delta} \right\rfloor} \leq (C \delta)^{k-2-\left\lfloor \frac{C_{T}}{\delta} \right\rfloor}.
\]

But

\[
\sum_{m=1}^{\left\lfloor \frac{C_{T}}{\delta} \right\rfloor + 1} \left( \frac{k - 1}{m} \right) \leq \sum_{m=1}^{\left\lfloor \frac{C_{T}}{\delta} \right\rfloor + 1} \left( \frac{k - 1}{m} \right) m! \leq (k - 1)^{\left\lfloor \frac{C_{T}}{\delta} \right\rfloor + 1} \sum_{m=1}^{\left\lfloor \frac{C_{T}}{\delta} \right\rfloor + 1} \frac{1}{m!} \leq (k - 1)^{\left\lfloor \frac{C_{T}}{\delta} \right\rfloor + 1}.
\]

it deduces that

\[
\int_{\prod_{l=1}^{k-1} V_{l}} 1_{\{ t(t, x, v) > 0 \}} \prod_{l=1}^{k-1} d\sigma_{m} \leq (k - 1)^{\left\lfloor \frac{C_{T}}{\delta} \right\rfloor + 1} (C \delta)^{k-2-\left\lfloor \frac{C_{T}}{\delta} \right\rfloor}.
\]

(2.32)
For $\varepsilon > 0$, (2.31) follows for $C\delta < 1$ and $k \gg \lfloor \frac{C\varepsilon T}{\delta} \rfloor + 1$. For example, we can choose $k = 15\lfloor \frac{C\varepsilon T}{\delta} \rfloor + 1 + 2$ for small $\delta > 0$. \hfill \Box

Let $h$ satisfies the following neutron transport equation with in-flow boundary condition,

$$ \{\partial_t + v \cdot \nabla_x + \lambda + \Sigma\} h = q, \quad h_{|t=0} = h_0, \quad h_{|y_-} = g. $$

For $(x, v) \notin \gamma_0$, we denote its backward exit point as $[t-t_b(x, v), x_b(x, v), v]$. Since $\frac{d}{dt}\{e^{\int_0^t (\lambda + \Sigma) dt}\} h = q$ along the characteristic $\frac{dx}{dt} = v$, $\frac{dv}{dt} = 0$. Thus, if $t - t_b < 0$,

$$ h(t, x, v) = e^{-\int_0^t (\lambda + \Sigma) dt} h_0(x - vt, v) + \int_0^t e^{-\int_s^t (\lambda + \Sigma) d\tau} q(s, x - v(t - \tau), v) d\tau. $$

If $t - t_b > 0$, we have

$$ h(t, x, v) = e^{-\int_0^t (\lambda + \Sigma) dt} g(t - t_b, x_b, v) + \int_{t_b}^t e^{-\int_s^{t_b} (\lambda + \Sigma) d\tau} q(s, x - v(t - s), v) d\tau. $$

Recall the diffusive cycles Definition 2.1, we have the following iteration scheme for the neutron transport equation with the mixing boundary condition. The proof is similar to that of Boltzmann equation in [6], we omit it here.

**Lemma 2.4.** [6] Assume that $h$ satisfies

$$ \{\partial_t + v \cdot \nabla_x + \lambda + \Sigma\} h = q, \quad h_{|t=0} = h_0, \quad h_{|y_-} = \mathcal{P}_Y h + r. $$

Then, for almost every $(x, v) \notin \gamma_0$, if $t_1(t, x, v) \leq 0$,

$$ h(t, x, v) = e^{-\int_0^t (\lambda + \Sigma) dt} h_0(x - vt, v) + \int_0^t e^{-\int_s^t (\lambda + \Sigma) d\tau} q(\tau, x - v(t - \tau), \tau) d\tau. $$

If $t_1(t, x, v) > 0$, then for $k \geq 2$,

$$ h(t, x, v) = \int_{t_1}^t e^{-\int_s^{t_1} (\lambda + \Sigma) ds} q(s, x - v(t - s), v) d\tau + e^{-\int_0^{t_1} (\lambda + \Sigma) dt} r(t_1, x_1, v) $$

$$ + e^{-\int_0^{t_1} (\lambda + \Sigma) dt} \int_{\Pi^{-1}} \mathcal{H} $$

where $H$ is given by

$$ \sum_{m=1}^{k-1} 1_{\{l_{m+1} \leq \tau \leq l_m\}} h_0(x_m - v_m t_m, v_m) d\Sigma_{m}(0) $$

$$ + \sum_{m=1}^{k-1} \int_0^{l_m} 1_{\{l_{m+1} \leq \tau \leq l_m\}} q(\tau, x_m + (\tau - t_m)v_m, v_m) d\Sigma_{m}(\tau) d\tau $$

$$ + \sum_{m=1}^{k-1} \int_{l_{m+1}}^{l_m} 1_{\{l_{m+1} \leq \tau \leq l_m\}} q(\tau, x_m + (\tau - t_m)v_m, v_m) d\Sigma_{m}(\tau) d\tau $$

(2.33)

$$ + 1_{\{l_k > 0\}} h(t_k, x_k, v_{k-1}) d\Sigma_{k-1}(t_k) + \sum_{m=1}^{k-1} 1_{\{l_m > 0\}} d\Sigma_{m}. $$

with $d\Sigma_{m}(0)$, $d\Sigma_{k-1}(t_k)$ are evaluated at $s = 0$ and $s = t_k$ of

(2.34)

$$ d\Sigma_{m}(s) = \{\Pi^{k-1}_{j=1} d\sigma_j\} [e^{\gamma(v)}(s-t_m) d\sigma_{m}] \Pi^{m-1}_{j=1} [e^{-\int_{l_j}^{t_{j+1}} (\lambda + \Sigma) ds} d\sigma_{j}]. $$
and

\[ d\Sigma_m = \prod_{j=m+1}^{k-1} d\sigma_j \left( e^{-\int_{0}^{\infty} \tau r(t_{m+1}, x_{m+1}, v_m) d\sigma_m} \prod_{j=1}^{m-1} \left( e^{-\int_{0}^{\tau} (t_{j+1}, x_{j+1}) d\sigma_j} \right) \right) \]

Now, we consider the solution of (2.12). By Lemma 2.4, we have

\[ U^{l+1}(t, x, v) \leq 1_{\{t_1 \leq 0\}} e^{-\int_{0}^{t_1} (t_{1+1}, x_{1+1}, v_1, v_2)} |u_0(x - t_1, v)| + 1_{\{t_1 \leq 0\}} \int_{0}^{t_1} e^{-\int_{0}^{\tau} (t_{1+1}, x_{1+1}, v_1, v_2)} |KU^{l+1} + q^{l+1}|(\tau, x - (t - \tau)v, v) d\tau 
+ 1_{\{t_1 > 0\}} \int_{t_1}^{\infty} e^{-\int_{0}^{\tau} (t_{1+1}, x_{1+1}, v_1, v_2)} |KU^{l+1} + q^{l+1}|(\tau, x - (t - \tau)v, v) d\tau 
+ 1_{\{t_1 > 0\}} e^{-\int_{0}^{t_1} (t_{1+1}, x_{1+1}, v_1, v_2)} |r^{l+1}(t_1, x_1, v)| + e^{-\int_{0}^{t_1} (t_{1+1}, x_{1+1}, v_1, v_2)} \int_{t_1}^{\infty} |H| d\tau \]

where \( H \) is bounded by

\[ \sum_{m=1}^{k-1} 1_{\{t_{m+1} \leq 0 < t_m\}} |u_0(x_m - v_m t_m, v_m)| d\Sigma_m(0) \]

\[ + \sum_{m=1}^{k-1} \int_{0}^{t_m} 1_{\{t_{m+1} \leq 0 < t_m\}} |KU^{l+1-m} + q^{l+1}|(\tau, x_m + (t - t_m)v_m, v_m)| d\Sigma_m(\tau) d\tau \]

\[ + \sum_{m=1}^{k-1} \int_{t_{m+1}}^{t_m} 1_{\{t_{m+1} > 0\}} |KU^{l+1-m} + q^{l+1}|(\tau, x_m + (t - t_m)v_m, v_m)| d\Sigma_m(\tau) d\tau \]

\[ + 1_{\{t_1 > 0\}} |U^{l+1-k}(t_k, x_k, v_{k-1})| d\Sigma_{k-1}(t_k) \]

\[ + \sum_{m=1}^{k-1} 1_{\{t_{m+1} > 0\}} d\Sigma_m. \]

The estimate of \( U^l \) in \( L^\infty \) is as follows.

**Lemma 2.5.** Suppose that \( ||u_0||_{L^\infty}, \sup_{0 \leq s \leq T} ||r^l(s)||_{L^\infty}, \sup_{0 \leq s \leq T} ||q^l(s)||_{L^\infty} \) are bounded for fixed \( T > 0 \). Then, there exists \( C(k) > 0 \) such that the solution \( U^{l+1} \) of (2.12) satisfies that

\[ \sup_{0 \leq s \leq T} ||U^{l+1}(s)||_{L^\infty} \leq \frac{1}{8} \max_{1 \leq m \leq 2k} \sup_{0 \leq s \leq T} ||U^{l+1-m}(s)||_{L^\infty} + C(k) \max_{1 \leq m \leq 2k} \int_{0}^{T} ||U^{l+1-m}(s)||_{L^2} ds \]

\[ + C(k) \left[ ||u_0||_{L^\infty} + \sup_{0 \leq s \leq T} ||r^l(s)||_{L^\infty} + \sup_{0 \leq s \leq T} ||q^l(s)||_{L^\infty} \right]. \]

**Proof.** We start with \( r \)-contribution in (2.36) and (2.41). Since \( d\sigma_m \) is a probability measure, \( \int_{V_m} d\sigma_m \leq 1 \), from the definition (2.35), the contribution of \( r \) is bounded by

\[ \sum_{m=1}^{k-1} |r^l(t_m, x_m, v)| \leq k \sup_{0 \leq s \leq T} ||r^l(s)||_{L^\infty}. \]

We turn to the \( q^l \)-contribution in (2.36), (2.38) and (2.39). Since \( d\sigma_m \) is a probability measure, all the terms to \( q^l \) is bounded by

\[ \int_{0}^{T} ||q^l(\tau)||_{L^\infty} \left\{ 2 + \sum_{m=1}^{k-1} \int_{t_{m+1} \leq 0 < t_m} 1_{\{t_{m+1} > 0\}} d\Sigma_m \right\} d\tau \]

\[ \leq 2kT \sup_{0 \leq s \leq T} ||q^l(s)||_{L^\infty}. \]
Now, we consider the contribution of the initial data $u_0$ in (2.36) and (2.37). It could be bounded by

$$
(2.45) \quad 1_{\{t_1 \leq 0\}} e^{-\int_{\tau_{(\lambda+\Sigma)}}^{t} \{\int_{\lambda+\Sigma}^{\lambda}} |u_0|_\infty + \int_{\tau_{m+1} \leq 0 < \tau_m} 1_{\{t_\nu \leq 0 < \tau_m\}} |u_0|_\infty d\Sigma_m(0) \leq k |u_0|_\infty.
$$

From Lemma 2.3, (2.40) can be bounded by

$$
(2.46) \quad \leq \sup_{0 \leq t \leq t'} ||U^{l+1-k}(t, x) ||_\infty \int_{\tau_{m+1} \leq 0 < \tau_m} \sum_{m=1}^{k-1} \int_{\tau_{m+1} \leq 0 < \tau_m} |U^{l+1-k}(t, x)| d\Sigma_m(t) \leq \epsilon \sup_{0 \leq t \leq t'} ||U^{l+1-k}(s) ||_\infty.
$$

From (2.43), (2.44), (2.45) and (2.46), we obtain an upper bound that

$$
\begin{align*}
|U^{l+1}(t, x, v)| & \leq 1_{\{t_1 \leq 0\}} \int_{0}^{t} e^{-\int_{\tau_{(\lambda+\Sigma)}}^{\tau} \{\int_{\lambda+\Sigma}^{\lambda}} |KU^{l}(\tau, x - (t - \tau)v, v)| d\tau \\
& \quad + 1_{\{t_1 > 0\}} \int_{t_1}^{t} e^{-\int_{\tau_{(\lambda+\Sigma)}}^{\tau} \{\int_{\lambda+\Sigma}^{\lambda}} |KU^{l}(\tau, x - (t - \tau)v, v)| d\tau \\
& \quad + \sum_{m=1}^{k-1} \int_{\tau_{m+1} \leq 0 < \tau_m} |KU^{l-m}(\tau, X_{cl}(\tau), v_m)| d\Sigma_m(\tau) + A_l(t, x, v)
\end{align*}
$$

(2.47)

with $A_l(t, x, v)$ denotes

$$
A_l(t, x, v) = \epsilon \sup_{0 \leq t \leq t'} ||U^{l+1-k}(s) ||_\infty + C_{k,l} \left( ||u_0||_\infty + \sup_{0 \leq s \leq t'} \{|r^l(s) ||_\infty + ||q^l(s) ||_\infty\} \right).
$$

Recall that the back-time cycle $(s, X_{cl}(s; t, x, v), v)$ denotes $(t_1, x_1, v_1), (t_2, x_2, v_2), \ldots, (t_m, x_m, v_m), \ldots$, we now iterate (2.47) for $l - m$ times to get the representation for $U^{l-m}$ and then plug in $KU^{l-m}(s, X_{cl}(s), v_m)$ to obtain

$$
\begin{align*}
|KU^{l-m}(s, X_{cl}(s), v_m)| & \leq \int_{V} f(X_{cl}(s), v_m, v') |U^{l-m}(s, X_{cl}(s), v')| dv' \\
& \quad \leq \int_{V \times V} \int_{0}^{s} \int_{0}^{s} 1_{\{t_1' \leq 0\}} e^{-\int_{0}^{t_1'} \{\int_{\lambda+\Sigma}^{\lambda}} f(X_{cl}(s) - (s - s_1)v', v_m, v') \\
& \quad \times f(X_{cl}(s) - (s - s_1)v', v', v'') |U^{l-1-m}(s_1, X_{cl}(s) - (s - s_1)v', v'')| ds_1 dv' dv'' \\
& \quad + \int_{V \times V} \int_{\tau_{m+1} < \tau_m} \sum_{p=1}^{k-1} e^{-\int_{\tau_{p+1}}^{\tau_p} \{\int_{\lambda+\Sigma}^{\lambda}} ds_1 1_{\{\tau_{p+1} \leq 0 < \tau_p\}} + ds_1 1_{\{\tau_p > 0\}} \\
& \quad \times f(y, v_m, v') f(y', v', v') |U^{l-1-m-p}(s_1, y, v')| d\Sigma_p(s_1) dv' dv''
\end{align*}
$$

(2.49)

where $y_r = x_r' + (s - t_r')v_r'$.

The total contributions of $A_{l-m-1}$ in (2.47) are obtained via plugging (2.49) with different $l$ into (2.47). Since $\int f(x, v, v') dv' < M_b$, the summation of all contributions of $A_{l-m-1}$ leads
to the bound

\[ 2A_{l-1}(t) \int_0^t e^{-\beta(t-s)(\lambda_1 + \Sigma)} ds + \max_{1 \leq m \leq k} A_{l-m-1}(t) \]

\[ \times \int \prod_{m=1}^{k-1} \sum_{m=1}^{t_m} \left\{ \int_0^{t_m} \mathbf{1}_{l_{m+1} \leq t_m} + \int_{t_m}^{t_m} \mathbf{1}_{l_m > 0} \right\} d\Sigma_m(s) ds + A_l(t) \]

\[ \leq C(k) \max_{0 \leq m \leq k} A_{l-m}(t) \]

(2.50)

\[ \leq \varepsilon \sup_{0 \leq s \leq t} \| U^{l+1-m-k}(s) \|_{\infty} + C_{k,T} \left( \| u_0 \|_{\infty} + \sup_{0 \leq s \leq t} | r^1(s) \|_{\infty} + \sup_{0 \leq s \leq t} \| q^1(s) \|_{\infty} \right) \]

To estimate the \( U^{l-m-1} \) contribution, we separate \( s - s_1 \leq \varepsilon \) and \( s - s_1 \geq \varepsilon \). In the first case, we use the fact \( \int f(x, v, v') dv' < M_b \) and by (2.49) to obtain the small contribution

(2.51)

\[ \varepsilon \max_{1 \leq l \leq k} \sup_{0 \leq s_1 \leq s} \| U^{l-m-l}(s_1) \|_{\infty}. \]

Now let us treat the case of \( s - s_1 \geq \varepsilon \). It can be easily check that \( \int f(x, v, v') dv' \leq M_b < \infty \).

For \( N \gg 1 \), there is some constant \( p(N) \) such that

\[ f_p(x, v', v'') = f_{1|f|<p(N)}. \]

It satisfies that \( \sup_{x,v'_p} \int f(x, v', v'') - f_p(x, v', v'') dv'' \leq \frac{1}{N} \). For any \( l' \). We split \( f(x, v', v'') = \{ f(x, v', v'') - f_p(x, v', v'') \} + f_p(x, v', v'') \). Notice that \( \int f(x, v, v') dv' < M_b \), the first difference of \( f(x, v', v'') \) leads to a small contribution in (2.49)

(2.52)

\[ \frac{1}{N} \max_{1 \leq l \leq k} \sup_{0 \leq s_1 \leq s} \| U^{l-m-l}(s_1) \|_{\infty}. \]

For the remainder main contribution of \( k_p(y_p, v', v'') \) the change of variable \( y_p = x'_p + (s-t'_p)v'_p \) \( (x'_p \) do not depend on \( v'_p \) satisfies \( \frac{dy_p}{dy} \geq \varepsilon^3 \) for \( s - s_1 \geq \varepsilon \). Notice that \( |f_p(y_p, v, v')| \leq p(N) \), the remained part can be estimated by

(2.53)

\[ \frac{p(N)^2}{\varepsilon^3} \int_{\Omega \times V} \| U^{l-m-l}(s_1, y_p, v'') \|_{\infty} \leq \| U^{l-m-l}(s_1) \|_{2}. \]

The estimates (2.51), (2.52), (2.53) give a bound for (2.49) as

(2.54)

\[ |K U^{l+1-m}(s, X_\varepsilon(s), v_m)| \leq \left[ 2 \varepsilon + \frac{C(k)}{N} \max_{1 \leq m \leq k} \sup_{0 \leq s_1 \leq s} \| U^{l-m}(s_1) \|_{\infty} \right. \]

\[ + C_{k,T} \left( \| u_0 \|_{\infty} + \sup_{0 \leq s \leq t} | r^1(s) \|_{\infty} + \sup_{0 \leq s \leq t} \| q^1(s) \|_{\infty} \right) \]

\[ + C_{\varepsilon,N} \max_{1 \leq m \leq 2k} \int_0^s \| U^{l-m}(s_1) \|_{2} ds_1 \]

\[ \equiv B(s). \]
By plugging back (2.50) and (2.54) into (2.47), we have the bounded $|U^{l+1}(t, x, v)|$ by

$$B(t) \left\{ 1_{t_1 \leq 0} \int_0^{t-t_1} e^{-\int_0^{t-t_1} (l+\Sigma)} ds + 1_{t_1 > 0} \int_{t_1}^{t} e^{-\int_0^{t-t_1} (l+\Sigma)} ds \right\}$$

$$+ B(t) e^{-\int_0^{t-t_1} (l+\Sigma)} \sum_{m=1}^{k-1} \left( \int_0^{t_m} 1_{t_{m+1} \leq 0 < t_m} + \int_{t_m}^{t_m+1} 1_{t_{m+1} > 0} \right) d\Sigma_m(s) ds + A_j(t)$$

$$\leq [2\varepsilon + \frac{C(k)}{N}] \max_{1 \leq m \leq 2k} \sup_{0 \leq t \leq s} \|U^{l-m}(s_1)\|_{\infty}$$

$$+ C_{k,T} \left( \|u_0\|_{\infty} + \sup_{0 \leq s \leq t} \|\phi(s)\|_{\infty} + \sup_{0 \leq s \leq t} \|q(s)\|_{\infty} \right)$$

(2.55)$$+ C_{\varepsilon,N} \max_{1 \leq m \leq 2k} \int_0^{t} \|U^{l-m}(s_1)\|_{2} ds_1.$$ 

We then conclude the proof of (2.42) by choosing $\varepsilon$ sufficiently small and $N$ large sufficiently large. The proof of Lemma 2.5 is completed. \qed

Before going to prove the $L^\infty$ estimate of $U^l$, we prove a standard result similar to [10] with more precise estimate. This lemma can be crucial to get the bound of $U^l$ as well as its convergence.

**Lemma 2.6.** Suppose $b_i \geq 0$, $D \geq 0$, $0 \leq \eta \leq 1$ and $B_i = \max\{b_i, \cdots, b_{i-(k-1)}\}$ for fixed $k \in \mathbb{N}$.

(2.56)$$b_{l+1} \leq \frac{1}{8} B_l + D\eta^l \text{ for all } l \geq k.$$ 

Then, for all $p \geq 1$ and $1 \leq m \leq k$, it holds that

(2.57)$$B_{ik+m} \leq \left[ \frac{1}{8} + \frac{7\eta^m}{8} + \frac{7\eta^{k+m}}{8} + \cdots + \frac{7\eta^{(i-2)k+m}}{8} \right] \max\{B_k, \frac{8}{7} D\eta^{k+1}, B_{ik+m}, \frac{8}{7} D\eta^{k+1} \},$$

$$b_{ik+m+1} \leq \left[ \frac{1}{8} + \frac{7\eta^m}{8} + \frac{7\eta^{k+m}}{8} + \cdots + \frac{7\eta^{(i-1)k+m}}{8} \right] \max\{B_k, \frac{8}{7} D\eta^{k+1} \}.$$ 

In particular, when $\eta = 1$, it holds that, for all $l \geq 1$,

(2.58)$$b_l \leq \max\{B_k, \frac{8}{7} D\}.$$ 

**Proof.** We will prove it by induction with responding to $i$. From the definition of $B_k$, we know that $b_1, \cdots, b_k \leq B_k$. Then

$$B_k \leq \max\{B_k, \frac{8}{7} D\eta^{k+1} \}, \quad b_{k+1} \leq \frac{1}{8} B_k + D\eta^{k+1} \leq \max\{B_k, \frac{8}{7} D\eta^{k+1} \},$$

and

$$B_{k+1} = \max\{b_{k+1}, \cdots, b_2\} \leq \max\{B_k, \frac{8}{7} D\eta^{k+1} \},$$

$$b_{k+2} \leq \frac{1}{8} B_{k+1} + D\eta^{k+2} \leq \left( \frac{1}{8} + \frac{7\eta}{8} \right) \max\{B_k, \frac{8}{7} D\eta^{k+1} \} \leq \max\{B_k, \frac{8}{7} D\eta^{k+1} \}.$$ 

Here we have used the fact that $D\eta^{k+1} \leq \frac{7}{8} max\{B_k, \frac{8}{7} D\eta^{k+1} \}$. 


Similarly, for all $1 \leq m \leq k$, it derives to

\[ B_{k+m} = \max \{ b_{k+m}, \cdots, b_{m+1} \} \leq \max \{ B_k, \frac{8}{7} D\eta^{k+1} \} \]

\[ b_{k+m+1} \leq \left( \frac{1}{8} + \frac{7\eta^m}{8} \right) \max \{ B_k, \frac{8}{7} D\eta^{k+1} \} \leq \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}. \]

Since $\eta^m \leq \eta^{m-1}$ for $\eta \leq 1$, we can also derive, for all $1 \leq m \leq k$,

\[ B_{2k+1} = \max \{ b_{2k+1}, \cdots, b_{k+2} \} \leq \left( \frac{1}{8} + \frac{7\eta^m}{8} \right) \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}, \]

\[ b_{2k+2} \leq \frac{1}{8} B_{2k+1} + D\eta^{2k+2} \leq \left( \frac{1}{8^2} + \frac{7\eta^m}{8} + \frac{7\eta^{k+1}}{8} \right) \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}, \]

\[ \cdots \]

\[ B_{2k+m} = \max \{ b_{2k+m}, \cdots, b_{k+m+1} \} \leq \left( \frac{1}{8^m} + \frac{7\eta^m}{8} \right) \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}, \]

\[ b_{2k+m+1} \leq \frac{1}{8} B_{2k+m} + D\eta^{2k+m+1} \leq \left( \frac{1}{8^{m+1}} + \frac{7\eta^m}{8} + \frac{7\eta^{k+m}}{8} \right) \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}. \]

It means that (2.57) is valid for $i = 2$.

Suppose that (2.57) holds for $i = p$, that is, for all $1 \leq m \leq k$

\[ B_{pk+m} \leq \left[ \frac{1}{8^{p-1}} + \left( \frac{7\eta^m}{8^{p-1}} + \frac{7\eta^{k+m}}{8^{p-1}} + \cdots + \frac{7\eta^{(p-2)k+m}}{8} \right) \right] \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}, \]

\[ b_{pk+m+1} \leq \left[ \frac{1}{8^p} + \left( \frac{7\eta^m}{8^p} + \frac{7\eta^{k+m}}{8^p} + \cdots + \frac{7\eta^{(p-1)k+m}}{8} \right) \right] \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}. \]

Then, from (2.56), we have

\[ B_{(p+1)k+1} = \max \{ b_{(p+1)k+1}, \cdots, b_{pk+2} \} \]

\[ \leq \left[ \frac{1}{8^p} \left( \frac{7\eta^m}{8^p} + \frac{7\eta^{k+1}}{8^p} + \cdots + \frac{7\eta^{(p-1)k+1}}{8} \right) \right] \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}, \]

\[ b_{(p+1)k+2} \leq \left[ \frac{1}{8^{p+1}} + \left( \frac{7\eta^m}{8^{p+1}} + \frac{7\eta^{k+1}}{8^{p+1}} + \cdots + \frac{7\eta^{(p-1)k+1}}{8} \right) \right] \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}, \]

\[ \cdots \]

\[ B_{(p+1)k+m} = \max \{ b_{(p+1)k+m}, \cdots, b_{pk+m+1} \} \]

\[ \leq \left[ \frac{1}{8^{p+1}} + \left( \frac{7\eta^m}{8^{p+1}} + \frac{7\eta^{k+m}}{8^{p+1}} + \cdots + \frac{7\eta^{(p-1)k+m}}{8} \right) \right] \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}, \]

\[ b_{(p+1)k+m+1} \leq \left[ \frac{1}{8^{p+2}} + \left( \frac{7\eta^m}{8^{p+2}} + \frac{7\eta^{k+m}}{8^{p+2}} + \cdots + \frac{7\eta^{(p-1)k+m}}{8} \right) \right] \max \{ B_k, \frac{8}{7} D\eta^{k+1} \}. \]

This implies that (2.57) holds for $i = p + 1$ and (2.57) is true for all $p \geq 1$. (2.58) is a consequence of (2.57). The proof of Lemma 2.6 \[ \square \]

Now, we consider the $L^\infty$ estimate of $U^l$. For simplicity, we denote

\[ E = \|u_0\|_\infty \quad + \quad \sup_{0 \leq s \leq T} ||r^l(s)||_\infty \quad + \quad \sup_{0 \leq s \leq T} ||q^l(s)||_\infty. \]

**Lemma 2.7.** Suppose that $\|u_0\|_\infty$, $\sup_{0 \leq s \leq T} ||r^l(s)||_\infty$, $\sup_{0 \leq s \leq T} ||q^l(s)||_\infty$ are bounded for fixed $T > 0$. Then, there exists $C(k) > 0$ such that the solution $U^{l+1}$ of (2.12) satisfies that

\[ \sup_{0 \leq s \leq T} ||U^{l+1}(s)||_\infty \leq \max_{1 \leq m \leq 2k} \int_0^T ||U^{l-m}(s)||_2 ds + E. \]
PROOF. Step 1 Let \( l \to \infty \). From Lemma 2.5 and the estimate (2.15), we can obtain
\[
\sup_{0 \leq s \leq T} \| U^{l+1}(s) \|_\infty \leq \frac{1}{8} \max_{0 \leq m \leq 2k} \sup_{0 \leq s \leq T} \| U^{l-m}(s) \|_\infty + C(k)(E + E).
\]
Set \( b_l = \sup_{0 \leq s \leq T} \| U^l(s) \|_\infty \), \( B_l = \max_{1 \leq m \leq 2k} \sup_{0 \leq s \leq T} \| U^{l-m-1} \|_\infty \), \( \eta = 1 \) and \( D = C(k)(E + E) \), then, (2.57) gives that
\[
(2.60) \quad \sup_{0 \leq s \leq T} \| U^l(s) \|_\infty \leq \max \left\{ B_{2k}, \frac{8}{7} C(k)(E + E) \right\}.
\]
Now, we give the estimate of \( B_{2k} \). Since \( \| Ku \|_{L^\infty} \leq M_0 \| u \|_\infty \) and \( \| \mathcal{P}_\gamma u \|_\infty \leq \| u \|_\infty \), from the iterate scheme (2.12), we have
\[
\| U^{l+1}(t, x, v) \| \leq \begin{cases} 1_{[t_1 \leq t \leq t_2]} e^{-\int_{t_1}^{t_2} (l+\gamma)} \| u_0(x-tv, v) \| \\ + 1_{[t_1 \leq t \leq t_2]} \int_{t_1}^{t_2} e^{-\int_{\tau}^{t_2} (l+\gamma)} \| [KU^l + q^l](\tau, x-(\tau-t)v, v) \| d\tau \\ + 1_{[t_1 \leq t \leq t_2]} \int_{t_1}^{t_2} e^{-\int_{\tau}^{t_2} (l+\gamma)} \| [KU^l + q^l](\tau, x-(\tau-t)v, v) \| d\tau \\ + 1_{[t_1 \leq t \leq t_2]} e^{-\int_{\tau}^{t_2} (l+\gamma)} \left\| [(1 - \frac{1}{\gamma})\mathcal{P}_\gamma U^l + r^l](t_1, x_1, v) \right\| \end{cases}
\]
\[
\leq C_1 \sup_{0 \leq s \leq T} \| U^l(s) \|_\infty + 2E.
\]
So, for fixed \( k \), we get iterate a bound for \( i \leq 2k \) to obtain
\[
\sup_{0 \leq s \leq T} \| U^{m+1}(s) \|_\infty \leq C_1 \sup_{0 \leq s \leq T} \| U^m(s) \|_\infty + 2E \leq C_l \sup_{0 \leq s \leq T} \| U^{m-1}(s) \|_\infty + 2(1 + C_1)E \leq \cdots \leq C_{l+1} \sup_{0 \leq s \leq T} \| U^0(s) \|_\infty + 2[C_l^m + \cdots + C_1 + 1]E
\]
This inequality leads to
\[
(2.61) \quad B_{2k} = \max_{0 \leq m \leq 2k} \sup_{0 \leq s \leq T} \| U^{2k-m} \|_\infty \leq 2[C_l^m + \cdots + C_1 + 1]E < \infty.
\]
Because \( T < \infty \), \( \Omega \) and \( V \) are bounded domains, we know that \( E \leq_{T, \Omega, V} E \). From (2.60), for any \( l \geq 1 \), one get the following uniformly \( L^\infty \) bound of \( U^l \)
\[
(2.62) \quad \sup_{0 \leq s \leq T} \| U^l(s) \|_\infty \leq \max \left\{ B_{2k}, \frac{8}{7} C(k)(E + D) \right\} \leq_{T, \Omega, V} E.
\]
This gives the uniform estimate of the sequences \( U^l \).

On the other hand, the difference \( V^{i+1} = U^{i+1} - U^i \) satisfies (2.16). From Lemma 2.5, we get
\[
\sup_{0 \leq s \leq T} \| V^{i+1}(s) \|_\infty \leq \frac{1}{8} \max_{0 \leq m \leq 2k} \sup_{0 \leq s \leq T} \| V^{i-m}(s) \|_\infty + C(k) \max_{0 \leq m \leq 2k} \int_0^T \| V^{i-m}(s) \|_2 ds.
\]
From the estimate (2.17), we know there is \( \eta_{i,j} \) such that
\[
(2.63) \quad \| V^i(s) \|_2 \leq \eta_{i,j} \int_0^s (\| V^1(s) \|_2^2 + \lambda \| V^1(s) \|_2^2) ds.
\]
Thus there exists constant $C_{k,T}$ such that
\[ C(k) \max_{0 \leq m \leq 2k} \int_0^T \| V^{l-m}(s) \|_2 ds \leq C_{k,T}(\eta_{k,j})^{l/2}. \]

Hence, let $l = 2pk + m$ ($1 \leq m \leq 2k$), (2.57) gives that
\[ \sup_{0 \leq t \leq T} \| V^l(t) \|_\infty \leq \left[ \frac{1}{8^l} + \left( \frac{7\eta_{k,j}^{m/2}}{8^l} + \cdots + \frac{7\eta_{k,j}^{2(l-1)k+m}/2}{8} \right) \right] \times \max \{ \max_{1 \leq i \leq 2k} \sup_{0 \leq t \leq T} \| V^l(t) \|_\infty, \frac{8}{7} C_{k,T}\eta_{k,j}^{(2k+1)/2} \}. \]

Because $\eta_{k,j} < 1$, we know that
\[ \sum_{l=1}^\infty \sup_{0 \leq t \leq T} \| V^l(t) \|_\infty \leq C \max \{ \max_{1 \leq i \leq 2k} \sup_{0 \leq t \leq T} \| V^l(t) \|_\infty, \frac{8}{7} C_{k,T}\eta_{k,j}^{(2k+1)/2} \} < \infty. \]

It means that $\{ U^l \}_{l=1}^\infty$ is a Cauchy series. Hence, there is a limit solution $U^l \to U_j$. $U_j$ is the solution of (2.18). Thus, the difference $U^{l+1} - U_j$ satisfies
\[ (\partial_t + v \cdot \nabla + \lambda + \Sigma)(U^{l+1} - U_j) = K(U^l - U_j), \quad (U^{l+1} - U_j)|_{t=0} = 0, \]
with $(U^{l+1} - U_j)|_{y=\infty} = (1 - \frac{1}{2})\mathcal{P}_\gamma(u^l - U_j)$. By the same argument as above, we can yield that
\[ \sup_{0 \leq t \leq T} \| (U^l - U_j)(s) \|_{L^\infty} \to 0 \text{ as } l \to \infty. \]

**Step 2.** We take $j \to \infty$. Let $U_j$ be the solution to (2.18). Lemma 2.5 implies that
\[ \sup_{0 \leq t \leq T} \| U_j(t) \|_\infty \leq \frac{1}{8} \sup_{0 \leq t \leq T} \| U_j(t) \|_\infty + C(k) \int_0^T \| U_j(s) \|_2 ds + C(k) \left( \| u_0 \|_\infty + \sup_{0 \leq t \leq T} \| r^l(t) \|_\infty + \| \gamma(t) \|_\infty \right). \]

Therefore, by an induction over $j$,
\[ \sup_{0 \leq t \leq T} \| U_j(t) \|_\infty \leq C(k) \int_0^T \| U_j(s) \|_2 ds + C(k)E. \]

Since $\int_0^T \| U_j(s) \|_2 ds$ is bounded from Step 2 of Lemma 2.2, this implies that $\sup_{0 \leq t \leq T} \| U_j(t) \|_\infty$ is uniformly bounded and we obtain the solution $U$. Taking the difference, we have
\[ (\partial_t + v \cdot \nabla + \lambda + \Sigma)(U_j - U) = K(U_j - U) - [U_j](t=0) = 0, \]
with the boundary condition $[U_j](t=\infty) = \mathcal{P}_\gamma(U_j - U) + \frac{1}{\gamma} \mathcal{P}_\gamma U_j$. We regard $\frac{1}{\gamma} \mathcal{P}_\gamma U_j$ as $r^l$ in Lemma 2.5 implies that
\[ \sup_{0 \leq t \leq T} \| (U_j - U)(s) \|_\infty \leq C(k) \int_0^T \| (U_j - U)(s) \|_2 ds + \frac{C(k)}{\gamma} \sup_{0 \leq t \leq T} \| U_j(t) \|_\infty, \]
which goes to zero as $j \to \infty$. We obtain $L^\infty$ solution $U$ to (2.2). \[ \square \]

**The proof of Theorem 1.1:** Let $\lambda$ is large enough such that (2.9) is satisfied. The existence of the solution $U$ of (2.2) and the estimate (2.3) come from Lemma 2.7 immediately. It is exactly the result of Proposition 2.3. Form Proposition 2.2, the solution $u$ of (1.15)-(1.16) is also obtained. Furthermore, (2.1) holds true for all $0 \leq t \leq T$. Proposition 2.1 is proved.
It is exactly the solution to (1.15) with (1.16). Moreover, the estimate (1.19) is valid. This completes the proof of Theorem 1.1. □

3. BV regularity of solution

In this section, we construct an open covering of the singular set \( \mathcal{S}_B \), which is crucial to establish a smooth approximation function that excludes the open covering of \( \mathcal{S}_B \). In particular, the measure of this singular set could be sufficiently small from Lemma 3.1 and Proposition 3.1.

3.1. The neighborhood of singular set. In the following, we construct the neighborhoods of the singular set, which is similar to the Boltzmann equation constructed in [11]. For the completeness of this paper, we present the details here.

Lemma 3.1. For \( 0 \leq \varepsilon \leq \varepsilon_1 \ll 1 \), we construct an open set \( O_{\varepsilon, \varepsilon_1} \subset \Omega \times V \) such that

\[
\mathcal{S}_B \subset O_{\varepsilon, \varepsilon_1}.
\]

There exists \( C_* = C_*(\Omega) \gg 1 \) such that for any \( 0 < \varepsilon \leq \varepsilon_1 \ll 1 \)

\[
\overline{O}_{\varepsilon, \varepsilon_1} \subset O_{\varepsilon, C_*, \varepsilon}.
\]

Moreover, there exist \( C_1 = C(\Omega, C_*) > 0 \), \( C_2 = C_2(\Omega, C_*) > 0 \) such that

\[
\int_{\Omega \times V} 1_{O_{\varepsilon, C_*, \varepsilon}}(x, v) dv dx \leq C_1 \varepsilon,
\]

and

\[
\text{dist}(\overline{\Omega} \times V \setminus O_{\varepsilon, C_*, \varepsilon}, \mathcal{S}_B) > C_2 \varepsilon.
\]

Proof. Construction of \( O_{\varepsilon, \varepsilon_1} \).

Step 1. Decomposition of \( \partial \Omega \). Let us fix \( \theta > 0 \) which will be chosen later. Since the boundary \( \partial \Omega \) is locally a graph of smooth functions, from the finite covering theorem, there exists a finite number \( M_{\Omega, \theta} \) of small open balls \( \mathcal{U}_1, \mathcal{U}_2, \cdots, \mathcal{U}_{M_{\Omega, \theta}} \subset \mathbb{R}^3 \) with \( \text{diam}(\mathcal{U}_m) < 2 \) for all \( m \), such that

\[
\partial \Omega \subset \bigcup_{m=1}^{M_{\Omega, \theta}} [\mathcal{U}_m \cap \partial \Omega] \text{ with } M_{\Omega, \theta} = O(\theta^{-2}),
\]

and for every \( m \), inside \( \mathcal{U}_m \) the boundary \( \mathcal{U}_m \cap \partial \Omega \) is exactly described by a smooth function \( \eta_m \) defined on a (small) open set \( \mathcal{A}_m \subset \mathbb{R}^2 \). Up to rotations and translations and reducing the size of the ball \( \mathcal{U}_m \) we will always assume that

\[
\mathcal{U}_m \cap \partial \Omega = \{(x_1, x_2, \eta_m(x_1, x_2)) \in \mathcal{A}_m \times \mathbb{R}\},
\]

\[
\mathcal{U}_m \cap \Omega = \{(x_1, x_2, x_3) \in \mathcal{A}_m \times \mathbb{R} : x_3 > \eta_m(x_1, x_2)\}
\]

with

\[
(0, 0) \in \mathcal{A}_m \subset [-\theta, \theta] \times [-\theta, \theta], \quad \partial_1 \eta_m(0, 0) = \partial_2 \eta_m(0, 0) = 0.
\]

Therefore, the unit out normal vector at \((0, 0, \eta_m(0, 0))\) is

\[
n(0, 0, \eta_m(0, 0)) = \frac{(\partial_1 \eta_m(0, 0), \partial_2 \eta_m(0, 0), -1)}{\sqrt{1 + |\partial_1 \eta_m(0, 0)|^2 + |\partial_2 \eta_m(0, 0)|^2}} = (0, 0, -1).
\]
Recall that $\partial \Omega$ is locally $C^2$. Then we can choose $\theta > 0$ small enough to satisfy for all $m \in \{1, \cdots, M_\Omega, \theta\}$ such that, for all $(x_1, x_2) \in \mathcal{A}_m$,

$$\sum_{i=1}^{2} |\partial_i \eta_m(x_1, x_2) - \partial_i \eta_m(0, 0)| = \frac{1}{8} \sum_{i,j=1}^{2} |\partial_{ij} \eta_m(x_1, x_2)| \leq C_\eta.$$  

(3.8)

Now we define the lattice point on $\mathcal{A}_m$ as

$$c_{m,i,j,\epsilon} = (\epsilon i, \epsilon j) \text{ for } -N_\epsilon \leq i, j \leq N_\epsilon = O(\epsilon^{-1} \theta).$$  

(3.9)

Then we define the $(i, j)$-rectangular $\mathcal{R}_{m,i,j,\epsilon}$ which is centered at $c_{m,i,j,\epsilon}$ and whose side is $2\epsilon_1$:

$$\mathcal{R}_{m,i,j,\epsilon} = \left\{ (x_1, x_2) : \epsilon i - \epsilon_1 < x_1 < \epsilon i + \epsilon_1, \epsilon j - \epsilon_1 < x_2 < \epsilon j - \epsilon_1 \right\} \cap \mathcal{A}_m. $$  

(3.10)

Note that if $\epsilon_1 \geq \epsilon$ then $\epsilon i - \epsilon_1$ is an open covering of $\mathcal{A}_m$, i.e.

$$\mathcal{A}_m \subset \bigcup_{-N_\epsilon \leq i, j \leq N_\epsilon} \mathcal{R}_{m,i,j,\epsilon} \text{ with } N_\epsilon = O(\epsilon^{-1} \theta).$$  

(3.11)

For each rectangle we define the representative outward normal

$$n_{m,i,j,\epsilon} = \frac{(\partial_1 \eta_m(c_{m,i,j,\epsilon}), \partial_2 \eta_m(c_{m,i,j,\epsilon}), -1)}{\sqrt{1 + |\partial_1 \eta_m(c_{m,i,j,\epsilon})|^2 + |\partial_2 \eta_m(c_{m,i,j,\epsilon})|^2}}.$$  

Let $(\hat{x}_{1,m,i,j,\epsilon}, \hat{x}_{2,m,i,j,\epsilon}) \in \mathbb{S}^2$ be an orthonormal basis of the tangent space of $\partial \Omega$ at $(c_{m,i,j,\epsilon}, \eta_m(c_{m,i,j,\epsilon}))$. Remark that the three vectors $(\hat{x}_{1,m,i,j,\epsilon}, \hat{x}_{2,m,i,j,\epsilon}, n_{m,i,j,\epsilon})$ is an orthonormal basis of $\mathbb{R}^3$ for each $m, i, j, \epsilon$.

**Step 2. Decomposition of $\Omega \times V$** We split the tangent velocity space at $(c_{m,i,j,\epsilon}, \eta_m(c_{m,i,j,\epsilon})) \in \partial \Omega$ as

$$\{ v \in V : v \cdot n_{m,i,j,\epsilon} = 0 \} \subset \bigcup_{l=0}^{L_\epsilon} \Theta_{m,i,j,\epsilon}^{l, 1}, \text{ with } L_\epsilon = O(\frac{1}{\epsilon}),$$

where

$$\Theta_{m,i,j,\epsilon}^{l, 1} := \left\{ r_v \cos \theta_v \cos \phi_v \hat{x}_{1,m,i,j,\epsilon} + r_v \sin \theta_v \cos \phi_v \hat{x}_{2,m,i,j,\epsilon} + r_v \sin \phi_v n_{m,i,j,\epsilon} \in V : |r_v \sin \phi_v| \leq 8 C_\eta \epsilon_1 \max\{|r_v, 1|, |\theta_v - \epsilon_1| \leq \epsilon_1 \text{ for } r_v \geq 0\},$$

(3.12)

with the constant $C_\eta > 0$ form (3.8).

Remark that for $\epsilon_1 \geq \epsilon$,

$$\bigcup_{l=0}^{L_\epsilon} \Theta_{m,i,j,\epsilon}^{l, 1} = \left\{ v \in V : |v \cdot n_{m,i,j,\epsilon}| \leq 8 C_\eta \epsilon_1 \max\{|r_v, 1|\}. \right\}$$  

(3.13)

Now, we are ready to construct the following open sets corresponding to $\mathcal{R}_{m,i,j,\epsilon} \times \Theta_{m,i,j,\epsilon}^{l, 1}$ as

$$O_{m,i,j,\epsilon}^{l, 1} := \bigcup_{x \in \mathcal{X}_{m,i,j,\epsilon}^{l, 1}} B_{\mathbb{R}^3}(x, \epsilon_1) \times \Theta_{m,i,j,\epsilon}^{l, 1},$$

(3.14)

where the index set is defined as

$$\mathcal{X}_{m,i,j,\epsilon}^{l, 1} := \left\{ (x_1, x_2, \eta_m(x_1, x_2)) + \tau [\cos \theta_{1,m,i,j,\epsilon} + \sin \theta_{2,m,i,j,\epsilon}] + sn_{m,i,j,\epsilon} \in \mathbb{R}^3 : (x_1, x_2) \in \mathcal{R}_{m,i,j,\epsilon}, \theta \in (\epsilon l - \epsilon_1, \epsilon l + \epsilon_1), s \in (-\epsilon_1, \epsilon_1) \right\}$$

$$\tau \in [0, t_f((x_1, x_2, \eta_m(x_1, x_2)), \cos \theta_{1,m,i,j,\epsilon} + \sin \theta_{2,m,i,j,\epsilon})].$$  

(3.15)
We denote that \( O_{m,i,j,e_1} \) is an infinite union of open sets and hence is an open set. Finally, we define

\[
O_{e,e_1} := \bigcup_{m,i,j} O_{m,i,j,e,e_1} \bigcup \left\{ \mathbb{R}^3 \times B_{\mathbb{R}^3}(0, \varepsilon_1) \right\},
\]

where \( 1 \leq m \leq M_{\Omega,0} = O(\theta^{-2}) \), \(-N_e \leq i, j \leq N_e = O(\theta e^{-1}) \) and \( 0 \leq L_e = O(\varepsilon^{-1}) \). Since \( O_{e,e_1} \) is a union of open sets, it is also an open set.

With the covering set \( O_{e,e_1} \) on hand, we now prove the properties.

**Proof of (3.1).** Suppose there exists \((x, v) \in \mathcal{S}_B\), by the definition of \( \mathcal{S}_B \) in (1.12), there exists \( y = x_p(x, v) \in \partial \Omega \) such that \( x = y + t_p(x, v)v \) and \( v \cdot n(y) = 0 \) from (1.9) and (1.10). Then \( y \in \mathcal{U}_m \) for some \( m \), that is, \( y = (y_1, y_2, \eta_m(y_1, y_2)) \) and \((y_1, y_2) \in \mathcal{R}_{m,i,j,e,e_1} \) for some \( i, j \).

Firstly, for any \(|v| \geq 1\), we check that

\[
|n_{m,i,j,e} \cdot \frac{v}{|v|}| = \left| n_{m,i,j,e} - n(y_1, y_2, \eta_m(y_1, y_2)) \right| \cdot v + n(y_1, y_2, \eta_m(y_1, y_2)) \cdot \frac{v}{|v|}
\]

\[
\leq \frac{1}{\sqrt{1 + |\nabla \eta_m(c_{m,i,j,e})|^2}} |\nabla \eta_m(c_{m,i,j,e}) - \nabla \eta_m(y_1, y_2), 0)|
\]

\[
+ \frac{\sqrt{1 + |\nabla \eta_m(y_1, y_2)|^2} - \sqrt{1 + |\nabla \eta_m(c_{m,i,j,e})|^2} \sqrt{1 + |\nabla \eta_m(y_1, y_2)|^2}}{\sqrt{1 + |\nabla \eta_m(c_{m,i,j,e})|^2}} |\nabla \eta_m(y_1, y_2), -1|
\]

\[(3.17)
\]

\[
\leq |\nabla \eta_m(c_{m,i,j,e}) - \nabla \eta_m(y_1, y_2)| + \frac{\sqrt{1 + |\nabla \eta_m(y_1, y_2)|^2} - \sqrt{1 + |\nabla \eta_m(c_{m,i,j,e})|^2}}{\sqrt{1 + |\nabla \eta_m(c_{m,i,j,e})|^2}}
\]

\[
\leq 2|\nabla \eta_m(c_{m,i,j,e}) - \nabla \eta_m(y_1, y_2)|,
\]

where we denoted \( \nabla \eta_m(y_1, y_2) = (\partial_{y_1} \eta_m(y_1, y_2), \partial_{y_2} \eta_m(y_1, y_2)) \) and used the fact that

\[
\frac{\sqrt{1 + |\nabla \eta_m(y_1, y_2)|^2}}{\sqrt{1 + |\nabla \eta_m(c_{m,i,j,e})|^2}} \leq \frac{|\nabla \eta_m(y_1, y_2) - \nabla \eta_m(c_{m,i,j,e})|}{\sqrt{1 + |\nabla \eta_m(y_1, y_2)|^2} + \sqrt{1 + |\nabla \eta_m(c_{m,i,j,e})|^2}} \leq \frac{\sqrt{1 + |\nabla \eta_m(y_1, y_2)|^2} - \sqrt{1 + |\nabla \eta_m(c_{m,i,j,e})|^2}}{\sqrt{1 + |\nabla \eta_m(c_{m,i,j,e})|^2} + \sqrt{1 + |\nabla \eta_m(y_1, y_2)|^2}} \leq |\nabla \eta_m(y_1, y_2) - \nabla \eta_m(c_{m,i,j,e})|.
\]

Using (3.8), for \((y_1, y_2) \in \mathcal{R}_{m,i,j,e,e_1}\), we have

\[
|n_{m,i,j,e} \cdot \frac{v}{|v|}| \leq 2 \| \eta_m \|_{C^2(\mathcal{R}_{m,i,j,e,e_1})} |c_{m,i,j,e} - (y_1, y_2)|
\]

\[
\leq 8 \varepsilon_1 \| \eta_m \|_{C^2(\mathcal{R}_{m,i,j,e,e_1})} \leq 8 \varepsilon_1 |\eta_m|_{C^2(\mathcal{A}_m)} \leq 8 C_{\eta} \varepsilon_1.
\]

Secondly, we consider the case \(|v| \leq 1\). From (3.8) and the similar estimates of \(|v| \geq 1\) case, we have

\[
|n_{m,i,j,e} \cdot v| \leq |n(y) \cdot v| + |(n(y_1, y_2) - n_{m,i,j,e}) \cdot v|
\]

\[
\leq 2|\nabla \eta_m(c_{m,i,j,e}) - \nabla \eta_m(y_1, y_2)| \leq 8 C_{\eta} \varepsilon_1.
\]
By (3.13), we conclude that \( v \in \bigcup_{l=1}^{L} \Theta_{m,i,j,e,l} \). Since \((y_1, y_2) \in \mathcal{R}_{m,i,j,e} \in \mathcal{A}_m\), the distance \( s \) in the direction \( n_{m,i,j,e} \) is less than the height \( \sup_{\mathcal{A}_m} |\eta_m| \). From (3.8), we know that \( |\eta_m(x_1, x_2)| \leq |\nabla \eta_m|(x_1, x_2) | \leq \varepsilon_1 \), i.e. \( |s| \leq \varepsilon_1 \), and hence \((x, v) \in O_{e,\varepsilon_1} \).

**Proof of (3.2).** It suffices to show that there exists a constant \( C_\varepsilon \gg 1 \) such that if \((x, v) \in O_{e,\varepsilon_1} \) then \((x, v) \in O_{e,\varepsilon_1} \). Since in the definition (3.16) the union on \( m, i, j, l \) is finite, we have

\[
O_{e,\varepsilon_1} = \bigcup_{m,i,j,l} \Theta_{m,i,j,e,l} \cup \left\{ \mathbb{R}^3 \times B_{\mathbb{R}^3}(0; \varepsilon_1) \right\}
\]

\[= \bigcup_{m,i,j,l} \left[ \bigcup_{x \in \mathcal{X}_{m,i,j,e,l}} B_{\mathbb{R}^3}(x; \varepsilon_1) \times \Theta_{m,i,j,e,l} \right] \cup \left\{ \mathbb{R}^3 \times B_{\mathbb{R}^3}(0; \varepsilon_1) \right\}.
\]

Let \( z \in \bigcup_{x \in \mathcal{X}_{m,i,j,e,l}} B_{\mathbb{R}^3}(x; \varepsilon_1) \). From the definition of closed set, we know that, for given \( \varepsilon_1 \), there exists some \( y \in \bigcup_{x \in \mathcal{X}_{m,i,j,e,l}} B_{\mathbb{R}^3}(x; \varepsilon_1) \) such that \(|z - y| \leq \varepsilon_1 \). Furthermore, we know that there exists some \( x \in \mathcal{X}_{m,i,j,e,l} \) satisfies \(|y - x| \leq \varepsilon_1 \). So, we derive that \(|z - x| \leq |z - y| + |y - x| \leq 2\varepsilon_1 \). That is,

\[
(3.18) \quad \bigcup_{x \in \mathcal{X}_{m,i,j,e,l}} B_{\mathbb{R}^3}(x; \varepsilon_1) \subset \bigcup_{x \in \mathcal{X}_{m,i,j,e,l}} B_{\mathbb{R}^3}(x; C_\varepsilon \varepsilon_1).
\]

On the other hand, from (3.12), \( C_\varepsilon \gg 1 \) and the fact that the vectors \( \hat{x}_{1,m,i,j,e}, \hat{x}_{2,m,i,j,e}, \) and \( n_{m,i,j,e} \) are fixed for given \( m, i, j, \)

\[
\Theta_{m,i,j,e,\varepsilon_1} = \left\{ \nu = r_\nu \cos \theta_\nu \cos \phi_\nu \hat{x}_{1,m,i,j,e} + r_\nu \sin \theta_\nu \cos \phi_\nu \hat{x}_{2,m,i,j,e} + r_\nu \sin \phi_\nu n_{m,i,j,e} \in V : \right.
\]

\[|r_\nu \sin \phi_\nu| \leq 8C_\varepsilon \varepsilon_1 \max \{r_\nu, 1\} \left|\theta_\nu - \varepsilon \ell\right| \leq \varepsilon_1 \text{ for } r_\nu \geq 0 \} \]

\[\subset \left\{ \nu = r_\nu \cos \theta_\nu \cos \phi_\nu \hat{x}_{1,m,i,j,e} + r_\nu \sin \theta_\nu \cos \phi_\nu \hat{x}_{2,m,i,j,e} + r_\nu \sin \phi_\nu n_{m,i,j,e} \in V : \right.\]

\[|r_\nu \sin \phi_\nu| \leq 8C_\varepsilon \varepsilon_1 \max \{r_\nu, 1\}, \left|\theta_\nu - \varepsilon \ell\right| \leq C_\varepsilon \varepsilon_1 \text{ for } r_\nu \geq 0 \}\]

\[
(3.19) \quad = \Theta_{m,i,j,e,C_\varepsilon \varepsilon_1,j}.
\]

Finally, we conclude (3.2) from (3.18)-(3.19),

\[
O_{e,\varepsilon_1} \subset \bigcup_{m,i,j} \left[ \bigcup_{x \in \mathcal{X}_{m,i,j,e,C_\varepsilon \varepsilon_1}} B_{\mathbb{R}^3}(x; C_\varepsilon \varepsilon_1) \times \Theta_{m,i,j,e,C_\varepsilon \varepsilon_1,j} \right] \cup \left\{ \mathbb{R}^3 \times B_{\mathbb{R}^3}(0; C_\varepsilon \varepsilon_1) \right\}
\]

\[= O_{e,C_\varepsilon \varepsilon_1}.
\]

**Proof of (3.3).** From the definition of \( O_{e,\varepsilon_1} \), we deduce that

\[\int \int_{\Omega \times V} 1_{O_{e,C_\varepsilon \varepsilon_1}} dvdx \leq \sum_{m,i,j,l} \int \int_{\Omega \times V} 1_{O_{m,i,j,e,C_\varepsilon \varepsilon_1}} dvdx + m_3(\Omega)O(|\varepsilon|^3) \]

\[\leq M_{\Omega,\varepsilon}(2N_\varepsilon)^2 L_\varepsilon \times \sup_{m,i,j,l} \int \int_{\Omega \times V} 1_{O_{m,i,j,e,C_\varepsilon \varepsilon_1}} dvdx + m_3(\Omega)O(|\varepsilon|^3) \]

\[\leq O(\frac{1}{\varepsilon^3}) \times \sup_{m,i,j,l} \int \int_{\Omega \times V} 1_{O_{m,i,j,e,C_\varepsilon \varepsilon_1}} dvdx + m_3(\Omega)O(|\varepsilon|^3). \]
On the one hand, there is some constant $C_V$ such that $\max\{r_v, 1\} \leq C_V$ because $V$ is a bounded domain. Then, $|r_v \sin \phi_v| \leq 8C_VC_\eta \epsilon_1$ if $v \in \Theta_{m,i,j,e_1,l}$. So, it holds

$$\int_V \mathbf{1}_{\Theta_{m,i,j,e_1,l}} dv \leq \int_r \int_{[\theta_-; \theta_+] \subseteq C_\eta e} \int_{r_v \sin \phi_v \leq 8C_VC_\eta e} r_v^2 \sin \phi_v d\phi_v d\theta_v dr_v \lesssim_{\Omega, V} \epsilon^2.$$ 

On the other hand, we claim that for $\epsilon_1 \geq \epsilon$,

$$m_3 \left( \bigcup_{x \in \chi_{m,i,j,e_1,l}} B_{\mathbb{R}^3}(x; \epsilon_1) \right) \lesssim \epsilon_1^2. \tag{3.21}$$

Without loss of generality, we assume that $i, j, l = 0$. Therefore, $c_{m,i,j,e} = 0$. For simplicity, we denote $e_1 = \hat{x}_{1,m,0,0,e}, e_2 = \hat{x}_{2,m,0,0,e}$, and $e_3 = n_{m,0,0,e}$. Then

$$\chi_{m,i,j,e_1,l} \subset \{(x_1, x_2, \eta m(x_1, x_2)) + \tau [\cos \theta e_1 + \sin \theta e_2] + s e_3 \in \mathbb{R}^3 : \}$

(3.22) 

$$\left( x_1, x_2 \right) \in \mathcal{R}_{m,0,0,e_1}, \theta \in (-\epsilon_1, \epsilon_1),$$

$$\tau \in [0, t_f((x_1, x_2, \eta m(x_1, x_2)), \cos \theta e_1 + \sin \theta e_2)], s \in (-\epsilon_1, \epsilon_1) \right].$$

Let $\text{diam}(\Omega) = \sup_{x,y \in \Omega} |x - y| < +\infty$. Since $\|\cos \theta e_1 + \sin \theta e_2\| = 1$, the exit time $t_f$ satisfies

$$t_f((x_1, x_2, \eta m(x_1, x_2)), \cos \theta e_1 + \sin \theta e_2) \leq \text{diam}(\Omega).$$

From the definition of $\bigcup_{x \in \chi_{m,i,j,e_1,l}} B_{\mathbb{R}^3}(x; \epsilon_1)$, we can check that it is included in the truncated cone with height $\text{diam}(\Omega)$, top radius $[10 + \|\eta\| C^1(\mathcal{A}_m)] \epsilon_1$ and the bottom radius $[10 + \|\eta\| C^1(\mathcal{A}_m) + \text{diam}(\Omega)\|\eta\| C^2(\mathcal{A}_m)] \epsilon_1$. More precisely, it holds that

$$\bigcup_{x \in \chi_{m,i,j,e_1,l}} B_{\mathbb{R}^3}(x; \epsilon_1) \subset \bigcup_{\tau = 0}^{2\text{diam}(\Omega)} B_{\mathbb{R}^3}(\tau e_1; [10 + \|\eta\| C^1(\mathcal{A}_m) + \tau\|\eta\| C^2(\mathcal{A}_m)] \epsilon_1).$$

Therefore, using (3.8), we conclude (3.21)

$$m_3 \left( \bigcup_{x \in \chi_{m,i,j,e_1,l}} B_{\mathbb{R}^3}(x; \epsilon_1) \right) \leq m_3 \left( \bigcup_{\tau = 0}^{2\text{diam}(\Omega)} B_{\mathbb{R}^3}(\tau e_1; [10 + \|\eta\| C^1(\mathcal{A}_m) + \tau\|\eta\| C^2(\mathcal{A}_m)] \epsilon_1) \right)$$

$$\leq 3\text{diam}(\Omega) \left[ 10 + \|\eta\| C^1(\mathcal{A}_m) + \tau\|\eta\| C^2(\mathcal{A}_m) \right]^2 \times (\epsilon_1)^2$$

$$\leq 3\text{diam}(\Omega) \left[ 10 + \frac{1}{8} + C_\eta\text{diam}(\Omega) \right]^2 (\epsilon_1)^2$$

$$\leq \epsilon_1^2.$$ 

Finally, by selecting $\epsilon_1 = C_e \epsilon$ in (3.21), we conclude (3.3) as

$$\int_{\Omega \times V} \mathbf{1}_{O_e,C_e} dv dx \leq O(\frac{1}{\epsilon^3}) m_3 \left( \bigcup_{x \in \chi_{m,i,j,e_1,l}} B_{\mathbb{R}^3}(x; \epsilon_1) \right) \int_V \mathbf{1}_{\Theta_{m,i,j,e_1,l}} dv + m_3(\Omega) O(\epsilon^3) \lesssim \epsilon.$$

Proof of (3.4). Since (3.1) holds for all $\epsilon \leq \epsilon_1$, it suffices to show there exists $C_2 = C_2(C_e) > 0$ such that

$$\text{dist}(\overline{\Omega} \times \mathbb{R}^3 O_{e,C_e}, \overline{O_{e,C_e}}) > C_2 \epsilon. \tag{3.23}$$
By the definition of $O_{ε,ε}$ in (3.16),

\[
\text{dist}(\overline{Ω} \times V \setminus O_{ε,ε}, \overline{O_{ε,ε}}) \\
\geq \inf_{m,i,l} \left\{ |(x, v) - (y, u)| : (x, v) \in (O_{m,i,j,e,l})^c \cap [\mathbb{R}^3 \times (B_{\mathbb{R}^3(0, C, ε)])^c \\
(y, u) \in \overline{O_{m,i,j,e,l}} \cup [\mathbb{R}^3 \times B_{B_{\mathbb{R}^3}(0, ε)]} \right\} \\
\min \left\{ \inf_{m,i,l} \left\{ |(x, v) - (y, u)| : (x, v) \in (O_{m,i,j,e,l})^c \cap [\mathbb{R}^3 \times (B_{\mathbb{R}^3(0, C, ε)])^c \\
(y, u) \in \overline{O_{m,i,j,e,l}} \cap [\mathbb{R}^3 \times B_{B_{\mathbb{R}^3}(0, ε)]} \right\} \right\},
\]

(3.24)

\[
\inf_{m,i,l} \left\{ |(x, v) - (y, u)| : (x, v) \in (O_{m,i,j,e,l})^c \cap [\mathbb{R}^3 \times (B_{\mathbb{R}^3(0, C, ε)])^c \\
(y, u) \in \overline{O_{m,i,j,e,l}} \cap [\mathbb{R}^3 \times B_{B_{\mathbb{R}^3}(0, ε)]} \right\}.
\]

(3.25)

Clearly, we have

\[
(3.24) \geq \inf \{|(x, v) - (y, u)| : (x, v) \in (\mathbb{R}^3 \times (B_{\mathbb{R}^3(0, C, ε)])^c, (y, u) \in \mathbb{R}^3 \times B_{B_{\mathbb{R}^3}(0, ε)}\} \\
\geq \inf \{|v - u| : v \in B_{B_{\mathbb{R}^3}(0, C, ε)}^c, u \in B_{B_{\mathbb{R}^3}(0, ε)}\} \\
= (C_* - 1)ε.
\]

Now, we consider the lower bound of (3.25). Firstly, from the definition of $O_{m,i,j,e,C,e,l}$ in (3.15), we divide $\{(x, v) \in (O_{m,i,j,e,C,e,l})^c\}$ in (3.25) into two parts, we deduce that

\[
(O_{m,i,j,e,C,e,l})^c = \bigcup_{x \in X_{m,i,j,e,C,e,l}} B_{\mathbb{R}^3(x; C, ε)]} \times (\Theta_{m,i,j,e,C,e,l})^c \\
\bigcup_{x \in X_{m,i,j,e,C,e,l}} (B_{\mathbb{R}^3(x; C, ε)]}^c \times V.
\]

Therefore, (3.25) is bounded below by the minimum of the following two numbers

\[
\inf \{|(x, v) - (y, u)| : (x, v) \in \bigcup_{x \in X_{m,i,j,e,C,e,l}} B_{\mathbb{R}^3(x; C, ε)]} \times (\Theta_{m,i,j,e,C,e,l})^c \setminus B_{B_{\mathbb{R}^3}(0; C, ε)]},
\]

(\(y, u) \in \overline{O_{m,i,j,e,C,e,l}} \cap [\mathbb{R}^3 \times B_{B_{\mathbb{R}^3}(0; ε)]}^c)\}

\[
\inf \{|(x, v) - (y, u)| : (x, v) \in \bigcap_{x \in X_{m,i,j,e,C,e,l}} (B_{\mathbb{R}^3(x; C, ε)]}^c \times [\mathbb{R}^3 \setminus B_{B_{\mathbb{R}^3}(0; C, ε)]},
\]

(\(y, u) \in \overline{O_{m,i,j,e,C,e,l}} \cap [\mathbb{R}^3 \times B_{B_{\mathbb{R}^3}(0; ε)]}^c\}.
\]

Secondly, we consider $\{(y, u) \in \overline{O_{m,i,j,e,C,e,l}}\}$. From (3.18) with $ε_1 = ε$ for some $ζ = \zeta(ε, C_*) > 0$ such that

\[
\overline{O_{m,i,j,e,l}} = \bigcup_{x \in X_{m,i,j,e,l}} B_{\mathbb{R}^3(x; ε]}^c \times \Theta_{m,i,j,e,l} \\
\subset \bigcup_{x \in X_{m,i,j,e,l}} B_{\mathbb{R}^3(x; C_ε \frac{C_ε}{2})] \times \Theta_{m,i,j,e,l}}.
\]
So, we conclude that (3.25) is bounded below by the minimum of (A) and (B):

(A) \[ \inf \{ \|(x, v) - (y, u)\| : (x, v) \in \bigcup_{x \in X_{m, i, j, e, l}} B_{\mathbb{R}^3}(x, C, \varepsilon) \times [(\Theta_{m, i, j, e, C, e}) \setminus B_{\mathbb{R}^3}(0; C, \varepsilon)] \}, \]

\[ \quad (y, u) \in \bigcup_{x \in X_{m, i, j, e, l}} B_{\mathbb{R}^3}(x, \frac{C \varepsilon}{2}) \times \left[ \hat{\Theta}_{m, i, j, e, C, e, l} \setminus B_{\mathbb{R}^3}(0; \varepsilon) \right], \]

(B) \[ \inf \{ \|(x, v) - (y, u)\| : (x, v) \in \bigcup_{x \in X_{m, i, j, e, l}} (B_{\mathbb{R}^3}(x, C, \varepsilon))^c \times [\mathbb{R}^3 \setminus B_{\mathbb{R}^3}(0; C, \varepsilon)] \}, \]

\[ \quad (y, u) \in \bigcup_{x \in X_{m, i, j, e, l}} B_{\mathbb{R}^3}(x, \frac{C \varepsilon}{2}) \times \left[ \hat{\Theta}_{m, i, j, e, C, e, l} \setminus B_{\mathbb{R}^3}(0; \varepsilon) \right]. \]

In the following, we firstly prove that (A) \( \geq \varepsilon \). Let \( v \in (\Theta_{m, i, j, e, C, e})^c \setminus B_{\mathbb{R}^3}(0; C, \varepsilon) \). By (3.12), it could be rewritten as

\[ v = r_v \cos \theta_v \cos \phi_v \hat{x}_{l m, i, j, e} + r_v \sin \theta_v \cos \phi_v \hat{x}_{l m, i, j, e} + r_v \sin \phi_v n_{m, i, j, e}, \]

where

(3.26) \[ \min\{r_v, 1\} \sin \phi_v \geq 8C_\eta C, \varepsilon, \text{ or } |\theta_v - l\varepsilon| \geq C, \varepsilon. \]

Let \( u \in \hat{\Theta}_{m, i, j, e, l} \setminus B_{\mathbb{R}^3}(0; \varepsilon) \). Again from (3.12), we have

\[ u = r_u \cos \theta_u \cos \phi_u \hat{x}_{l m, i, j, e} + r_u \sin \theta_u \cos \phi_u \hat{x}_{l m, i, j, e} + r_u \sin \phi_u n_{m, i, j, e}, \]

where

(3.27) \[ \min\{1, r_u\} \sin \phi_u \leq 8C_\eta \varepsilon, \text{ and } |\theta_u - l\varepsilon| \leq \varepsilon. \]

We discuss (A) \( \geq \varepsilon \) in the following cases.

1) If \(|\theta_v - l\varepsilon| \geq C, \varepsilon \) for \( C, \varepsilon \) \( \gg 1 \), then clearly \(|v - u| \geq \varepsilon \) since \(|\theta_u - l\varepsilon| \leq \varepsilon \).

2) For the case \(|\theta_v - l\varepsilon| \leq C, \varepsilon \), it would be divided into several cases.

(a) If \(|r_v|, |r_u| \leq 1 \), then \(|r_v \sin \phi_v| \geq 8C_\eta C, \varepsilon \) from (3.26) and \(|r_u \sin \phi_u| \leq 8C_\eta \varepsilon \) from (3.27). So

\[ |v - u| \geq |(v - u) \cdot n_{m, i, j, e}| \geq |v \cdot n_{m, i, j, e}| - |u \cdot n_{m, i, j, e}| \]

\[ \geq |r_v \sin \phi_v| - |r_u \sin \phi_u| \geq 8C_\eta C, \varepsilon - 8C_\eta \varepsilon \]

\[ \geq \varepsilon. \]

(b) If \(|r_v| \geq 1 \) and \(|r_u| \leq 1 \), then \(|\sin \phi_v| \geq 8C_\eta C, \varepsilon \) from (3.26) and \(|r_u \sin \phi_u| \leq 8C_\eta \varepsilon \) from (3.27). So

\[ |v - u| \geq |(v - u) \cdot n_{m, i, j, e}| \geq |r_v \sin \phi_v| - |r_u \sin \phi_u| \]

\[ \geq |\sin \phi_v| - |r_u \sin \phi_u| \geq 8C_\eta C, \varepsilon - 8C_\eta \varepsilon \]

\[ \geq \varepsilon. \]

(c) If \(|r_v| \leq 1 \) and \(|r_u| \geq 1 \). Then \(|r_v \sin \phi_v| \geq 8C_\eta C, \varepsilon \) from (3.26) and \(|\sin \phi_u| \leq 8C_\eta \varepsilon \) from (3.27). We will discuss it in the following subcases.

Fix \( 0 < C, \varepsilon \ll 1 \ll C_* \). When \(|r_u| \leq C_* - c_* \), then

\[ |v - u| \geq |(v - u) \cdot n_{m, i, j, e}| \geq |v \cdot n_{m, i, j, e}| - |u \cdot n_{m, i, j, e}| \]

\[ \geq |r_v \sin \phi_v| - |r_u \sin \phi_u| \geq 8C_\eta C, \varepsilon - |r_u| \times 8C_\eta \varepsilon \]

\[ = 8C_\eta (C_* - |r_u|) \varepsilon \geq 8C_\eta C, \varepsilon. \]
When \(|r_u| \geq C_* - c_*\), then
\[
|v - u| \geq |[u - (u \cdot n_{m,i,j,e})n_{m,i,j,e}] - [v - (v \cdot n_{m,i,j,e})n_{m,i,j,e}]|
\geq |r_u \cos \phi_u| - |r_v| |\cos \phi_v| \geq |r_u| \sqrt{1 - 64(C_\eta)^2 \varepsilon^2} - |\cos \phi_v|
\geq (C_* - c_*) \sqrt{1 - 64(C_\eta)^2 \varepsilon^2} - 1
\geq 1.
\]

(d) If \(|r_v| \geq 1\) and \(|r_u| \geq 1\). In this case, \(|\sin \phi_v| \geq 8C_\eta C_* \varepsilon\) from (3.26) and \(|\sin \phi_u| \leq 8C_\eta \varepsilon\) from (3.27). Let \(|r_u| = k|r_v|\). We also introduce \(0 < C_\cdot \ll 1 \ll C_*\). When \(k \leq C_* - c_*\), then
\[
|v - u| \geq |r_v| |\sin \phi_u - k \sin \phi_u| \geq |\sin \phi_v - k \sin \phi_u| \geq 8C_\eta C_* \varepsilon.
\]

When \(k \geq C_* - c_*\), one has
\[
|v - u| \geq |[u - (u \cdot n_{m,i,j,e})n_{m,i,j,e}] - [v - (v \cdot n_{m,i,j,e})n_{m,i,j,e}]|
\geq |r_u \cos \phi_u| - |r_v| |\cos \phi_v| \geq k \sqrt{1 - 64(C_\eta)^2 \varepsilon^2} - |\cos \phi_v|
\geq (C_* - c_*) \sqrt{1 - 64(C_\eta)^2 \varepsilon^2} - 1
\geq 1.
\]

Combing all the cases, we deduce \((A) \geq \varepsilon\) for \(\varepsilon\) small enough.

Secondly, we prove \((B) \geq \varepsilon\). It is true due to
\[
(B) \geq \inf \{|x - y| : x \in \bigcap_{z \in X_{m,j,k} \in \mathcal{E}} (B_{R_3}(z, C_\varepsilon \varepsilon)^\circ), y \in \bigcup_{z \in X_{m,i,j} \in \mathcal{E}} B_{R_3}(z, C_\varepsilon \varepsilon/2)\}
\geq \inf \{|x - y| : x \in \bigcap_{z \in X_{m,j,k} \in \mathcal{E}} (B_{R_3}(z, C_\varepsilon \varepsilon)^\circ), y \in \bigcup_{z \in X_{m,i,j} \in \mathcal{E}} B_{R_3}(z, C_\varepsilon \varepsilon/2)\}
\geq \inf \{|x - y| : x \in (B_{R_3}(z, C_\varepsilon \varepsilon)^\circ), y \in B_{R_3}(z, C_\varepsilon \varepsilon/2)\}
\geq \frac{C_\cdot \varepsilon}{2}.
\]
So, the estimate of (3.4) from \((A) \geq \varepsilon\), and \((B) \geq \varepsilon\). immediately. The proof of Lemma 3.1 is completed. \(\square\)

3.2. Construction of cut-off function. Recall the standard mollifier \(\psi : \mathbb{R}^3 \times \mathbb{R}^3 \to [0, \infty)\),
\[
\psi(x, v) = \begin{cases} C \exp\left(\frac{1}{|x^2 + |v|^2 - 1|}\right), & \text{for } |x|^2 + |v|^2 < 1, \\ 0, & \text{for } |x|^2 + |v|^2 \geq 1. \end{cases}
\]
where the constant \(C > 0\) is selected so that \(\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \psi(x, v) dx dv = 1\).
For each \(\varepsilon > 0\), set
\[
(3.28) \quad \psi_\varepsilon(x, v) = \left(\frac{\varepsilon}{\tilde{C}}\right)^{6} \psi\left(\frac{|x|^2 + |v|^2}{\varepsilon^2 / \tilde{C}}\right), \quad \text{with } \tilde{C} \gg C_* \gg 1.
\]
Clearly, \(\psi_\varepsilon\) is smooth and bounded and satisfies
\[
\iint_{\mathbb{R}^3 \times \mathbb{R}^3} \psi_\varepsilon(x, v) dx dv = 1, \quad \text{spt}(\psi_\varepsilon) \subset B_{R_3 \times \mathbb{R}^3}(0; \varepsilon / \tilde{C}).
\]
Definition 3.1. We define a smooth cut-off function \( \chi_\varepsilon : \overline{\Omega} \times V \rightarrow [0, 2] \) as

\[
\chi_\varepsilon(x, v) := \mathbf{1}_{\Omega \times V \cup O_{e,C,e}}(x, v)
= \int_{\mathbb{R}^3 \times \mathbb{R}^3} \mathbf{1}_{\Omega \times V \cup O_{e,C,e}}(y, u) \psi_\varepsilon(x - y, v - u) dydu.
\]

(3.29)

The following properties of the cut-off function are crucial for our analysis.

Proposition 3.1. There exist \( \tilde{C} \gg C_\ast \gg 1 \) in (3.28) and (3.29) and \( \varepsilon_0 = \varepsilon_0(\Omega, V) > 0 \) such that if \( 0 \leq \varepsilon \leq \varepsilon_0 \), then

\[
\Xi_B \subset \{(x, v) \in \overline{\Omega} \times V : \chi_\varepsilon(x, v) = 0\},
\]

and, for either \( \partial = \nabla_x \) or \( \partial = \nabla_v \), it holds that

\[
\int_{\Omega \times V} [1 - \chi_\varepsilon(x, v)] dv dx \leq \varepsilon,
\]

(3.31)

\[
\int_{\Omega \times V} |\partial \chi_\varepsilon(x, v)| dv dx \leq 1.
\]

(3.32)

Proof. This Proposition will be proved by the definition of the cut-off function \( \chi_\varepsilon \) directly. Proof of (3.30). Let \( (x, v) \in \Xi_B \). Due to (3.28) if \( |(x, v) - (y, u)| \geq \varepsilon / \tilde{C} \) then \( \psi_\varepsilon(x, v) = 0 \). Therefore, (3.29) can be rewritten as

\[
\chi_\varepsilon(x, v) = \int_{B_{\tilde{C}}((x,v)/\varepsilon/\tilde{C})} \mathbf{1}_{\Omega \times V \cup O_{e,C,e}}(y, u) \psi_\varepsilon(x - y, v - u) dydu
\]

On the other hand, if \( (y, u) \in B_{\tilde{C}}((x,v)/\varepsilon/\tilde{C}) \), due to (3.1)-(3.2) with \( \varepsilon_1 = \varepsilon \) and \( \tilde{C} \gg C_\ast \gg 1 \), we have \( (y, u) \in \overline{O_{e,C,e}} \subset O_{e,C,e} \) and

\[
\mathbf{1}_{\Omega \times V \cup O_{e,C,e}}(y, u) = 0.
\]

Therefore, we conclude that \( \chi_\varepsilon(x, v) = 0 \) for all \( \Xi_B \) and (3.30) is true.

Proof of (3.31). We use (3.3) with \( \varepsilon_1 = \varepsilon \) to have

\[
\int_{\Omega \times V} \int_{\mathbb{R}^3 \times \mathbb{R}^3} [1 - \mathbf{1}_{\Omega \times V \cup O_{e,C,e}}(y, u)] \psi_\varepsilon(x - y, v - u) dydu dv dx
\]

\[
\leq \int_{\Omega \times V} \mathbf{1}_{\Omega \times V \cup O_{e,C,e}}(y, u) dy \int_{\mathbb{R}^3 \times \mathbb{R}^3} \psi_\varepsilon(x - y, v - u) dv dx
\]

\[
\leq \frac{C_1 \varepsilon}{2} \int_{B_{\tilde{C}}(0,\varepsilon/\tilde{C})} \psi_v(x, v) dv dx
\]

\[
\leq \varepsilon
\]

Proof of (3.32). Note that from a standard scaling argument and (3.28), one has

\[
|\partial \psi_\varepsilon(x, v)| \leq \frac{C_6}{\varepsilon^4} \mathbf{1}_{B_{\tilde{C}}(0,\varepsilon/\tilde{C})}(x, v).
\]
We also note that $\partial \chi_\varepsilon = - \partial [1 - \chi_\varepsilon]$. Therefore, by Lemma 1,

$$\int_{\Omega \times V} |\partial \chi_\varepsilon(x, v)| dv dx = \int \int_{\Omega \times V} \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} [1 - 1_{B(0, \varepsilon)}(y)] \partial \psi_\varepsilon(x - y, v - u) dudy dv dx \leq \int \int_{\Omega \times V} \int_{\mathbb{R}^3} O(C^6) 1_{B(0, \varepsilon)} dudy dv dx \leq O(\varepsilon) \times O(\varepsilon^{-1}) \leq 1.$$  

This completed the proof of Proposition 3.1. \hfill \Box

**Proposition 3.2.** With the same constant $\tilde{C} \gg C_\varepsilon \gg 1$ as in Proposition 3.1 and $0 < \varepsilon \leq \varepsilon_0$, then

$$\varepsilon \cap [\partial \Omega \times V] \subset \{(x, v) \in \partial \Omega \times V : \chi_\varepsilon(x, v) = 0\}. \tag{3.33}$$

Moreover if $|y, u| \leq \varepsilon/\tilde{C}$ for $\tilde{C} \gg C_\varepsilon \gg 1$,

$$\int \int \int_{\partial \Omega \times y \times 0} 1_{x, C_\varepsilon}(x - y, v - u) |n(x - y) \cdot (v - u)| dudS_x \leq \varepsilon, \tag{3.34}$$

and

$$\int_{\varepsilon} |1 - \chi_\varepsilon(x, v)| dv \leq \Omega_\varepsilon \varepsilon, \tag{3.35}$$

$$\int_{\varepsilon} |\partial \chi_\varepsilon(x, v)| dv \leq \Omega_\varepsilon 1. \tag{3.36}$$

The following fact is crucial to prove Proposition 3.2 and especially (3.34). The proof is similar to that in [11].

**Lemma 3.2.** We fix $m_0 = 1, 2, \ldots, M_{\Omega, 0}$ in (3.5). We may assume (up to rotations and translations) there exists a $C^2$ function $\eta_{m_0} : [-\theta, \theta] \times [-\theta, \theta] \rightarrow \mathbb{R}$, whose graph is the boundary $\mathcal{U} \cap \partial \Omega$. Let $(x_1, x_2) \in \mathcal{A}_{m_0} \cap [-\theta, \theta] \times [-\theta, \theta]$ and $(x_1, x_2) \in \mathcal{R}_{m_0, i_0, j_0, \varepsilon, C_\varepsilon}$ for $|i_0|, |j_0| \leq N_\varepsilon$. Furthermore, we suppose that

- $|v| \leq \frac{\varepsilon}{C}$ and

$$\varepsilon_1 \in O_{\varepsilon, C_\varepsilon}, \tag{3.37}$$

- For large but fixed $s_\varepsilon \gg 1$,

$$-1 \leq n_{m_0}(0, 0) \cdot \frac{v}{|v|} \leq -s_\varepsilon C_\varepsilon \sqrt{\varepsilon}, \text{ with } C_\varepsilon := \sqrt{\frac{8C_\varepsilon}{3} [1 + ||\eta_{m_0}||_{C^2}]} \tag{3.38}$$

Then either $|v| \leq \varepsilon^{1/3}$ or there exists $(i, j) \in [-N_1 + i_0, N_1 + j_0] \times [-N_1 + j_0, N_1 + j_0]$ with

$$N_1 := \frac{8C_\varepsilon}{\sqrt{\varepsilon}}, \quad C_3 := \frac{4C_\varepsilon + 8C_\varepsilon [1 + ||\eta_{m_0}||_{C^2}]}{s_\varepsilon C_2}, \tag{3.39}$$

such that

$$((x_1, x_2, \eta_{m_0}(x_1, x_2) - y, v)) \in \bigcup_{0 \leq \varepsilon \leq L_{\eta}} O_{m_0, i_0, j_0, \varepsilon, C_\varepsilon \varepsilon, I} \cap \overline{\Omega} \times \{v \in \mathbb{R}^3 : |v| \geq \varepsilon^{1/3}\},$$
and
\[
|n_{m_0}(0,0) \cdot v| \leq C_4 \sqrt{\varepsilon} \quad \text{with} \quad C_4 = C_3[1 + \|\eta_{m_0}\|_{C^2(X_{m_0})}].
\]

Proof of Lemma 3.2. Without loss of generality (up to rotations and translations), we may assume
\[
(i_0, j_0) = (0,0) \quad \text{and} \quad \eta_{m_0}(0,0) = 0 \quad \text{and} \quad \nabla \eta_{m_0}(0,0) = 0.
\]
Consider the case of \(|v| \geq \varepsilon^{1/3}\). Since \(((x_1, x_2, \eta_{m_0}(x_1, x_2) - y, v)) \in O_{e,C,e}\) we use the definition of \(O_{e,C,e}\) in (3.16) to have
\[
\text{either} \quad |v| \leq C_4\varepsilon \quad \text{or} \quad (x - y, v) \in \bigcup_{m,j,l} O_{m,i,j,e,C,e,l}.
\]
For small \(0 < \varepsilon \ll 1\), we can exclude the first case of \(|v| \leq C_4\varepsilon\) since \(|v| \geq \varepsilon^{1/3} \Rightarrow C_4\varepsilon\).

Now we consider the latter case in (3.42). In this case, we claim that
\[
((x_1, x_2, \eta_{m_0}(x_1, x_2) - y, v)) \in \bigcup_{i,j,l} O_{m_0,i,j,e,C,e,l}.
\]
From (3.42) and the definition of \(O_{m_0,i,j,e,C,e,l}\) in (3.14), there exist \(m, i, j, l\) such that
\[
((x_1, x_2, \eta_{m_0}(x_1, x_2) - y, v)) \in \bigcup_{p \in X_{m_0,i,j,e,C,e,l}} B_{\mathbb{R}^3}(p; C_4\varepsilon) \times \Theta_{m_0,i,j,e,C,e,l}.
\]
In particular, there exists \(p \in X_{m_0,i,j,e,C,e,l}\) satisfying \(|p - (x_1, x_2, \eta_{m_0}(x_1, x_2) - y)| \leq C_4\varepsilon\). By the definition of \(X_{m_0,i,j,e,C,e,l}\) in (3.15), one has
\[
p = (\overline{p}_1, \overline{p}_2, \eta_{m_0}(\overline{p}_1, \overline{p}_2)) = \overline{p}[\cos \overline{\theta}_{1,m,i,j,e} + \sin \overline{\theta}_{2,m,i,j,e}] + \overline{\sigma}_{m_0,i,j,e},
\]
for some
\[
(\overline{p}_1, \overline{p}_2) \in \mathcal{R}_{m_0,i,j,e,C,e},
\]
\[
\overline{\theta} \in (|\varepsilon - C_4\varepsilon, |\varepsilon + C_4\varepsilon|),
\]
\[
\overline{\tau} \in [0, t_{\overline{\tau}}(\overline{p}_1, \overline{p}_2, \eta_{m_0}(\overline{p}_1, \overline{p}_2)), (\cos \overline{\theta}_{1,m,i,j,e} + \sin \overline{\theta}_{2,m,i,j,e})],
\]
\[
\overline{\sigma} \in [-C_4\varepsilon, C_4\varepsilon].
\]
By the definition of \(t_{\overline{\tau}}\) in (1.11),
\[
z := p - \overline{\sigma}_{m_0,i,j,e} = (\overline{p}_1, \overline{p}_2, \eta_{m_0}(\overline{p}_1, \overline{p}_2)) + \overline{\tau}[\cos \overline{\theta}_{1,m,i,j,e} + \sin \overline{\theta}_{2,m,i,j,e}] \in \Omega.
\]
Then, we have
\[
|z - ((x_1, x_2, \eta_{m_0}(x_1, x_2) - y))|
\]
\[
\leq |z - p| + |p - ((x_1, x_2, \eta_{m_0}(x_1, x_2) - y))|
\]
\[
\leq 2C_4\varepsilon.
\]
From (3.41) and (3.44), and \(|y| \leq \varepsilon/\tilde{C}\), we deduce
\[
|z - (0,0, \eta_{m_0}(0,0))|
\]
\[
\leq |z - ((x_1, x_2, \eta_{m_0}(x_1, x_2) - y))| + |((x_1, x_2, \eta_{m_0}(x_1, x_2)) - (0,0,\eta_{m_0}(0,0))| + |y|
\]
\[
\leq 2C_4\varepsilon + 4C_4\varepsilon(1 + \|\eta_{m_0}\|_{C^1(X_{m_0})}) + \varepsilon/\tilde{C}.
\]
Denote \((\tilde{z}_1, \tilde{z}_2) = (\overline{p}_1, \overline{p}_2)\). By the definition of \(t_{\overline{\tau}}\) in (1.9) and (1.11),
\[
x_{b}(\tilde{z}, \cos \overline{\theta}_{1,m,i,j,e} + \sin \overline{\theta}_{2,m,i,j,e} + 0n_{m,i,j,e}) = (\tilde{z}_1, \tilde{z}_2, \eta_{m_0}(\tilde{z}_1, \tilde{z}_2)).
\]
On the other hand, by the definition of $\Theta_{m,i,j,e,C,e,l}$ in (3.12),
\[
(3.46) \quad \frac{v}{|v|} = \cos \theta_v \cos \phi_v \hat{x}_{1,m,i,j,e} + \sin \theta_v \cos \phi_v \hat{x}_{2,m,i,j,e} + \sin \phi_v n_{m,i,j,e},
\]
with
\[
|\theta_v - l\epsilon| \leq C\epsilon,
\]
\[
|v \cdot n_{m,i,j,e}| \leq 8C\eta C\epsilon \text{ for } \epsilon^{1/3} \leq |v| \leq 1,
\]
\[
\left| \frac{v}{|v|} \cdot n_{m,i,j,e} \right| \leq 8C\eta C\epsilon \text{ for } \epsilon^{1/3} \leq |v| \leq 1.
\]
Therefore, for $0 < \epsilon \ll 1$,
\[
(3.47) \quad \left| \frac{v}{|v|} \cdot n_{m,i,j,e} \right| = |\sin \phi_v| \leq \max\{8C\eta C\epsilon^{2/3}, 8C\epsilon\} \leq 16C\eta C\epsilon^{2/3}.
\]
Now we estimate as
\[
\begin{align*}
n_{m_0}(0,0) \cdot (\cos \tilde{\theta} \hat{x}_{1,m,i,j,e} + \sin \tilde{\theta} \hat{x}_{2,m,i,j,e} + 0n_{m,i,j,e}) \\
\leq n_{m_0}(0,0) \cdot \frac{v}{|v|} + n_{m_0}(0,0) \cdot \left( \frac{v}{|v|} - (\cos \tilde{\theta} \hat{x}_{1,m,i,j,e} + \sin \tilde{\theta} \hat{x}_{2,m,i,j,e} + 0n_{m,i,j,e}) \right).
\end{align*}
\]
We use (3.46)-(3.47), and $\tilde{\theta} \in (l\epsilon - C\epsilon, l\epsilon + C\epsilon)$ to conclude that, for $0 < \epsilon \ll 1$,
\[
\begin{align*}
\left| \frac{v}{|v|} - \cos \tilde{\theta} \hat{x}_{1,m,i,j,e} + \sin \tilde{\theta} \hat{x}_{2,m,i,j,e} + 0n_{m,i,j,e} \right| \\
\leq 2\{ |\cos \theta_v - \cos \tilde{\theta}| + |\cos \theta_v| |\cos \phi_v| - 1| + |\sin \theta_v - \sin \tilde{\theta}| + |\sin \theta_v| |\cos \theta_v - 1| + |\sin \theta_v| \}
\leq 2\{4C\epsilon + 16C\eta C\epsilon^{2/3} + 2(16C\eta C\epsilon)^2 \epsilon^{4/3} \}
\leq 200C\eta C\epsilon^{2/3}.
\end{align*}
\]
Finally from (3.38), for $0 < \epsilon \ll 1$,
\[
\begin{align*}
-1 \leq n_{m_0}(0,0) \cdot (\cos \tilde{\theta} \hat{x}_{1,m,i,j,e} + \sin \tilde{\theta} \hat{x}_{2,m,i,j,e} + 0n_{m,i,j,e}) \leq -s_* \times C_2 \sqrt{\epsilon} + 400C\eta C\epsilon^{2/3} \\
\leq - \frac{s_* \times C_2}{2} \sqrt{\epsilon}.
\end{align*}
\]
Now we are ready to prove the claim (3.43). Denote
\[
\hat{u} := \cos \tilde{\theta} \hat{x}_{1,m,i,j,e} + \sin \tilde{\theta} \hat{x}_{2,m,i,j,e}.
\]
Recall that $|z| \leq (2C_* + 4C_\tau(1 + \|\eta_{m_0}\|_{C^1(\mathcal{A}_{m_0})}) + 1/\tilde{C})\epsilon$ and $z \in \Omega$. Therefore for $0 < \epsilon \ll 1$, the function $\eta_{m_0}$ is defined around $(z_1, z_2)$ and $z_3 \geq \eta_{m_0}(z_1, z_2)$.
We define, for $|\tau| \ll 1$,
\[
(3.49) \quad \Phi(\tau) = z_3 - \hat{u}_3 \tau - \eta_{m_0}(z - \hat{u}_1 \tau, z_2 - \hat{u}_2 \tau), \quad \Phi(0) > 0.
\]
Expanding $\Phi(\tau)$ in $\tau$, form $-\hat{u}_3 = n_{m_0}(0,0) \cdot \hat{u}$ and (3.48), we have
\[
\Phi(\tau) \leq -\hat{u}_3 + |z_3| + |\eta_{m_0}(z_1 - \hat{u}_1 \tau, z_2 - \hat{u}_2 \tau)|
\leq - \frac{s_* \times C_2}{2} \sqrt{\epsilon} \tau + (2C_* + 4C_\tau(1 + \|\eta_{m_0}\|_{C^1(\mathcal{A}_{m_0})}) + 1/\tilde{C})\epsilon
\leq |\eta_{m_0}\|_{C^1(\mathcal{A}_{m_0})}(2C_* + 4C_\tau(1 + \|\eta_{m_0}\|_{C^1(\mathcal{A}_{m_0})}) + 1/\tilde{C})^2 \epsilon^2
\leq |\eta_{m_0}\|_{C^2(\mathcal{A}_{m_0})}|\tau|^2,$
where we have used the fact that
\[
\eta_m(z_1 - \hat{u}_1 \tau, z_2 - \hat{u}_2 \tau) = \eta_m(z_1, z_2) + \int_0^\tau \frac{d}{ds} \eta_m(z_1 - \hat{u}_1 s, z_2 - \hat{u}_2 s) ds
\]
\[
= \eta_m(z_1, z_2) - (\hat{u}_1, \hat{u}_2) \cdot \nabla \eta_m \tau + \int_0^\tau \int_0 s^2 \eta_m(z_1 - \hat{u}_1 s_1, z_2 - \hat{u}_2 s_1) ds_1 ds
\]
\[
\leq \|\eta_m\|_{C^2(\mathcal{A}_{m_0})} \frac{|z|^2}{2} + |(\hat{u}_1, \hat{u}_2) \cdot \nabla \eta_m(0, 0)| |\tau| + \|\eta_m\|_{C^2(\mathcal{A}_{m_0})} (|z||\tau| + \frac{|\tau|^2}{2})
\]
\[
\leq \|\eta_m\|_{C^2(\mathcal{A}_{m_0})} (|z|^2 + |\tau|^2).
\]

Now we plug \(\tau = \frac{C_1 \sqrt{\epsilon}}{s_n}\) with the constant \(C_3\) in (3.39) to have, for \(s_n \gg 1\) and \(0 < \epsilon \ll 1\),
\[
\Phi(\tau) \leq \left[ \frac{C_2 C_3}{2} - (2C_+ 4C_1[1 + \|\eta_m\|_{C^1(\mathcal{A}_{m_0})}] + 1/\tilde{C}) \frac{\|\eta_m\|_{C^2(\mathcal{A}_{m_0})} C_3^2}{(s_n)^2} \right] \epsilon + O(\epsilon^2)
\]
\[
< 0.
\]

By the mean value theorem, there exists at least one \(\tau \in (0, C_3 \sqrt{\epsilon}]\) satisfying \(\Phi(\tau) = 0\). We choose the smallest one if them and denote it as \(\tau_0 \in (0, C_3 \sqrt{\epsilon}]\). By this definition and (3.45), for \(0 < \epsilon \ll 1\),
\[
x_b(z, \hat{u}) = x_b(z, \cos \theta \hat{x}_{1,m,i,j,\epsilon} + \sin \theta \hat{x}_{2,m,i,j,\epsilon})
\]
\[
= z - \tau_0 \hat{u} = (z_1 - \tau_0 \hat{u}_1, z_2 - \tau_0 \hat{u}_2, z_3 - \tau_0 \hat{u}_3).
\]

Therefore, \(x_b(z, \hat{u}) \in \partial \Omega \cap \mathcal{U}_{m_0}\) and this proves the claim (3.43). For \(0 < \epsilon \ll 1\),
\[
|(z_1 - \tau_0 \hat{u}_1, z_2 - \tau_0 \hat{u}_2)| \leq |z| + \tau_0 |\hat{u}| \leq (2C_+ 4C_1[1 + \|\eta_m\|_{C^1(\mathcal{A}_{m_0})}] + 1/\tilde{C}) \epsilon + C_3 \sqrt{\epsilon} \leq 2C_3 \sqrt{\epsilon}.
\]

Moreover, for \(|i - i_0|, |j - j_0| \leq (2C_3 \sqrt{\epsilon}) \epsilon \leq 2C_3 \sqrt{\epsilon} \leq N_1\),
\[
(z_1 - \tau_0 \hat{u}_1, z_2 - \tau_0 \hat{u}_2) \in \mathcal{R}_{m_0,i,j,e,c,i,e}.
\]

Finally, we need to prove (3.40). From (3.47) and (3.39)
\[
\left| n_{m_0}(0, 0) \cdot \frac{v}{|v|} \right| \leq \left| n_{m_0, i, j, e, c, i, e} \cdot \frac{v}{|v|} \right| + \left| (n_{m_0}(0, 0) - n_{m_0, i, j, e, c, i, e}) \cdot \frac{v}{|v|} \right|
\]
\[
\leq 16C_4 C_+ \epsilon^{2/3} + \|n_{m_0}\|_{C^1(\mathcal{A}_{m_0})} [N_1 \epsilon + C_+] \epsilon
\]
\[
\leq 16C_4 C_+ \epsilon^{2/3} + \|n_{m_0}\|_{C^1(\mathcal{A}_{m_0})} [2C_3 \sqrt{\epsilon} + C_+] \epsilon
\]
\[
\leq 10C_3 (1 + \|\eta_m\|_{C^2(\mathcal{A}_{m_0})}) \sqrt{\epsilon}
\]
\[
\leq C_4 \sqrt{\epsilon},
\]
and (3.40) follows. \(\Box\)

Now, we turn to the proof of Proposition 3.2.

**Proof of Proposition 3.2.** The first statement (3.33) is clear from (3.30).
Proof of (3.34). Let \((y, u) \leq \varepsilon/\bar{C}\). We use (3.5) to decompose that
\[
\int_{\partial \Omega} \int_{n_m(x)(y) < 0} 1_{E,C,e}(x - y, v - u)n(x - y) \cdot (v - u)dudS_x \\
\leq \sum_{m=1}^{M_{\Omega,0}} \int_{\partial \Omega} \int_{n_m(x)(y) < 0} 1_{E,C,e}(x - y, v - u)n_m(x - y) \cdot (v - u)dS_x dv \\
\leq M_{\Omega,0} \times \sup_m \int_{\partial \Omega} \int_{n_m(x)(y) < 0} 1_{E,C,e}(x - y, v - u)n_m(x - y) \cdot (v - u)dS_x dv.
\]
For fixed \(m = 1, 2, \cdots, M_{\Omega,0}\), we use (3.6) and (3.11) again to decompose
\[
\int_{\partial \Omega} \int_{n_m(x)(y) < 0} 1_{E,C,e}(x - y, v - u)n_m(x - y) \cdot (v - u)dS_x dv \\
= \sum_{n \in [-C_e, C_e]^2} \int_{[-C_e, C_e]^2} \int_{n_{m}(x)(y) < 0} 1_{E,C,e}(x - y, v - u) \\
\times |n_m(x - y) \cdot (v - u)| \sqrt{1 + |\nabla \eta_m(x, x)|} dx_1 dx_2 dv \\
\leq \frac{\theta^2}{\varepsilon^2} \sup_{n \in [-C_e, C_e]^2} \int_{[-C_e, C_e]^2} \int_{n_{m}(x)(y) < 0} 1_{E,C,e}(x - y, v - u) \\
\times |n_m(x - y) \cdot (v - u)| \sqrt{1 + |\nabla \eta_m(x, x)|} dx_1 dx_2 dv \\
\leq \frac{\theta^2}{\varepsilon^2} \sup_{n \in [-C_e, C_e]^2} \int_{[-C_e, C_e]^2} \int_{n_{m}(x)(y) < 0} 1_{E,C,e}(x - y, v - u) \\
\times |n_m(x - y) \cdot v| \sqrt{1 + |\nabla \eta_m(x, x)|} dx_1 dx_2 dv
\]
where \(n_m(x, x_2) = \frac{1}{\sqrt{1 + |\nabla \eta_m(x, x_2)|}}(\partial_1 \eta_m(x_1, x_2), \partial_2 \eta_m(x_1, x_2), -1)\).

We fix \(i, j\). Without loss of generality (up to rotations and translations), we may assume
\[c_{m,i,j,\varepsilon} = (0, 0), \quad \partial_1 \eta_m(0, 0) = \partial_2 \eta_m(0, 0) = 0, \quad n_{m,i,j,\varepsilon} = (0, 0, -1).
\]
For \((x_1, x_2) \in [-C_e, C_e]^2, (y, u) \leq \varepsilon/\bar{C}\) and \(n_m(x_1, x_2) \cdot (v + u) < 0\), we deduce
\[
n_{m,i,j,\varepsilon} \cdot v = n_m(0, 0) \cdot v = n_m(x_1, x_2) \cdot (v + u) + [n_m(0, 0) \cdot v - n_m(x_1, x_2) \cdot (v + u)]
\leq |n_m(x_1, x_2) \cdot u| + |n_m(x_1, x_2) \cdot v| \leq \frac{\varepsilon}{\bar{C}} + 2C_e \|\eta_m\|_{C^1[-C_e, C_e]}|v|
\leq C_5 \varepsilon,
\]
where \(C_5 = \max\{1/\bar{C}, 2C_e \|\eta_m\|_{C^1[-C_e, C_e]}\text{diam}V\}\). Therefore, from Lemma 3.2, we decompose the domain as follows
\[
\int_{[-C_e, C_e]^2} \int_{n_m(0, 0) \cdot v \leq C_3(1 + |v|) \varepsilon} 1_{E,C,e}(x_1 - y_1, x_2 - y_2, \eta_m(x_1, x_2) - y_3, v) \\
\times |n_m(x - y) \cdot v| \sqrt{1 + |\nabla \eta_m(x_1, x_2)|} dx_1 dx_2 dv \\
\leq \int_{[-C_e, C_e]^2} \left[ \int_{-s, C_2 \sqrt{\varepsilon} \leq n_m(0, 0) \cdot v \leq C_3(1 + |v|) \varepsilon} + \int_{n_m(0, 0) \cdot v \leq -s, C_2 \sqrt{\varepsilon}} \right] \cdots
:= (I) + (II).
\]
We consider (I). If \(-s \cdot C_2 \sqrt{\epsilon} \leq n_m(0, 0) \cdot \frac{v}{|v|} \leq 0\), then, \(0 \leq v_3 = -n_m(0, 0) \cdot v \leq s \cdot C_2 |v| \sqrt{\epsilon}\) and

\[
0 \leq v_3 \leq 2s \cdot C_2 \left|\sqrt{|v_1|^2 + |v_2|^2}\right| \sqrt{\epsilon}, \quad \text{for} \quad 0 < \epsilon \ll 1.
\]

Moreover,

\[
\left|n_m(x - y) \cdot v\right| \leq \left|n_m(0, 0) \cdot v\right| + \left|n_m\right|_{C^1([-C, \epsilon, \epsilon, \epsilon]^2)} (C_+ + \frac{1}{\epsilon}) |v| \epsilon
\]
\[
\leq s \cdot C_2 |v| \sqrt{\epsilon} + 4 \left|n_m\right|_{C^1([-C, \epsilon, \epsilon, \epsilon]^2)} (C_+ + \frac{1}{\epsilon}) |v| \epsilon.
\]

Therefore,

\[
|v_3| = |n_m(0, 0) \cdot v| \leq 2C_5 (1 + \sqrt{|v_1|^2 + |v_2|^2}) \epsilon.
\]

Therefore,

\[
(I) \leq \int_{[-C, \epsilon, \epsilon, \epsilon]^2} \int_{0 \leq v_3 \leq 2s \cdot C_2 \left|\sqrt{|v_1|^2 + |v_2|^2}\right| \sqrt{\epsilon}} \left\{s \cdot C_2 |v| \sqrt{\epsilon} + 4 \left|n_m\right|_{C^1([-C, \epsilon, \epsilon, \epsilon]^2)} (C_+ + \frac{1}{\epsilon}) |v| \epsilon\right\}
\]
\[
+ \int_{[-C, \epsilon, \epsilon, \epsilon]^2} \int_{|v_3| \leq 2C_3 (1 + \sqrt{|v_1|^2 + |v_2|^2}) \epsilon} \cdots
\]
\[
\leq m_2([-C, \epsilon, \epsilon, \epsilon]^2) \times \sqrt{\epsilon} \int_{V'} \sqrt{|v_1|^2 + |v_2|^2} \, dv_1 \, dv_2 \int_{0}^{2s \cdot C_2 \sqrt{\epsilon} \sqrt{|v_1|^2 + |v_2|^2}} \, dv_3
\]
\[
+ m_2([-C, \epsilon, \epsilon, \epsilon]^2) \times \int_{V'} \sqrt{|v_1|^2 + |v_2|^2} \, dv_1 \, dv_2 \int_{0}^{2C_3 (1 + \sqrt{|v_1|^2 + |v_2|^2}) \epsilon} \, dv_3
\]
\[
\leq \epsilon^3,
\]

where \(V'\) is the projection of \(V\) onto the space \((v_1, v_2) \in \mathbb{R}^2\), which is also bound.

Now we decompose (II) according to Lemma 3.2:

\[
(II) = \int_{[-C, \epsilon, \epsilon, \epsilon]^2} \int_{|v| \leq \epsilon^{1/3}} + \int_{[-C, \epsilon, \epsilon, \epsilon]^2} \int_{-1 \leq n_m(0, 0) \cdot \frac{v}{|v|} \leq -s \cdot C_2 \sqrt{\epsilon} \text{ and } |v| \geq \epsilon^{1/3}}
\]

The first term is clearly bounded by \(O(1) \epsilon^3\). For the second term, we use (3.40) to have

\[
\{-1 \leq n_m(0, 0) \cdot \frac{v}{|v|} \leq -s \cdot C_2 \sqrt{\epsilon} \text{ and } |v| \geq \epsilon^{1/3}\} \subset \{n_m(0, 0) \cdot \frac{v}{|v|} \leq C_4 \sqrt{\epsilon} \text{ and } |v| \geq \epsilon^{1/3}\}.
\]

So, we follow the same proof for (3.51) to obtain

\[
(II) \leq \epsilon^3 + \int_{[-C, \epsilon, \epsilon, \epsilon]^2} \int_{\left|n_m(0, 0) \cdot \frac{v}{|v|}\right| \leq C_4 \sqrt{\epsilon}} \left\{C_4 |v| \sqrt{\epsilon} + 4 \left|n_m\right|_{C^1([-C, \epsilon, \epsilon, \epsilon]^2)} (C_+ + \frac{1}{\epsilon}) |v| \epsilon\right\}
\]
\[
\leq \epsilon^3.
\]

We conclude the estimate of (3.34) form (3.51) and (3.52).
Proof of (3.35). Due to the properties of the standard mollifier (3.28), we obtain

\[
\int_{x \in \partial \Omega, n(x) \cdot v < 0} [1 - \chi_\epsilon(x, v)] n(x) \cdot v |dS_x dy
\]

\[
= \int_{x \in \partial \Omega, n(x) \cdot v < 0} \int_{\mathbb{R}^3} [1 - 1_{\Omega(x, v)}(x - y, v - u)] \psi_\epsilon(y, u) |n(x) \cdot v |dy |dS_x dy
\]

\[
\leq \int_{\mathbb{R}^3} \psi_\epsilon(y, u) dy \int_{x \in \partial \Omega, n(x) \cdot v < 0} 1_{\Omega(x, v)}(x - y, v - u) |n(x) \cdot v |dS_x dy
\]

\[
\leq \int_{\mathcal{B}_6(0; \epsilon/\tilde{C})} \psi_\epsilon(y, u) dy \int_{x \in \partial \Omega, n(x) \cdot v < 0} 1_{\Omega(x, v)}(x - y, v - u) |n(x) \cdot v |dS_x dy.
\]

Since $\sqrt{|y|^2 + |u|^2} \leq \epsilon/\tilde{C}$ and $n(x) \cdot v \leq 0$, we have

\[
n(x) \cdot v = n(x - y) \cdot (v - u) + (n(x) - n(x - y)) \cdot v + n(x - y) \cdot u
\]

\[
= n(x - y) \cdot (v - u) + O(\epsilon/\tilde{C})(1 + |v|).
\]

Therefore, we use (3.34) to bounded (3.35) further as

\[
\int_{y_0} [1 - \chi_\epsilon(x, v)] d\gamma = \int_{x \in \partial \Omega, n(x) \cdot v < 0} |1 - \chi_\epsilon(x, v)| |n(x) \cdot v |dS_x dy
\]

\[
\leq \int_{\mathcal{B}_6(0; \epsilon/\tilde{C})} \psi_\epsilon(y, u) dy \int_{x \in \partial \Omega, n(x) \cdot v < 0} 1_{\Omega(x, v)}(x - y, v - u) |n(x - y) \cdot (v - u)| dS_x dy
\]

\[
+ O\left(\frac{\epsilon}{\tilde{C}}\right) \times m_3(\partial \Omega) \times \int_V (1 + |v|) dv
\]

\[
\leq \Omega, \nu \epsilon.
\]
Proof of (3.36). Following the same proof of (3.35), we deduce that
\[
\int_{y_2} |\partial \chi(x, v)| dy = \int \int_{x \in \partial \Omega, n(x) \cdot v < 0} |\partial \chi(x, v)||n(x) \cdot v| dS_x d\nu
\]
\[
= \int \int_{y \in \partial \Omega, n(x) \cdot v < 0} |\partial (1 - \chi(x, v))||n(x) \cdot v| dS_x d\nu
\]
\[
= \int \int_{x \in \partial \Omega, n(x) \cdot v < 0} \int_{y \in \mathbb{R}^3} 1_{O_{x, C, \varepsilon}} (y, u) \partial \psi(x - y, v - u) dudy |n(x) \cdot v| dS_x d\nu
\]
\[
\leq \int \int_{B_{x, 0}(0, \varepsilon)} |\partial \psi(y, u)| dudy
\]
\[
\times \left[ \int \int_{x \in \partial \Omega, n(x) \cdot v < 0} 1_{O_{x, C, \varepsilon}} (x - y, v - u)|n(x) \cdot v| dS_x d\nu
\]
\[
+ O(\varepsilon) \times m_3(\partial \Omega) \times \int_V dv \right]
\]
\[
\leq \frac{1}{\varepsilon} \sup_{(y, u) \in B_{x, 0}(0, \varepsilon)} \left[ \int \int_{x \in \partial \Omega, n(x) \cdot v < 0} 1_{O_{x, C, \varepsilon}} (x - y, v - u)|n(x) \cdot v| dS_x d\nu
\]
\[
+ O(\varepsilon) \times m_3(\partial \Omega) \times \int_V (1 + |v|) dv \right]
\]
\[
\leq 1.
\]
The proof of Proposition 3.2 is completed. \qed

3.3. New Trace Theorem via the Double Iteration. In this section we prove the following geometric result. For the later purpose, we state the result for the sequence of solutions.

Proposition 3.3. Let \( h_0 \in L^1(\Omega \times V) \). Let \( (h^m)_{m \geq 0} \subset L^\infty([0, T]; L^1(\Omega \times V)) \cap L^1([0, T]; L^1(\gamma_+, d\gamma)) \) solve
\[
(\partial_t + v \cdot \nabla_x + \Sigma)h^{m+1} = H^m, \quad h^{m+1}|_{t=0} = h_0,
\]
where \( \Sigma = \Sigma(x, v) \geq 0 \), and such that the following inequality holds for all \( 0 \leq t \leq T \) and \((x, v) \in \gamma_- \),
\[
|h^{m+1}(t, x, v)| \leq C_1 \left( 1 + \frac{1}{n(x) \cdot v} \right) \left[ \int_{n(x) \cdot v > 0} |h^m(t, x, v')||n(x) \cdot v'| dv' + R^m \right],
\]
where \( H^m \in L^1([0, T]; L^1(\Omega \times V)) \) and \( R^m \in L^1([0, T]; L^1(\partial \Omega \times V, dS_x, d\nu)) \).

Then for all \( m \geq 1 \), \( h^{m+1} \in L^1([0, T]; L^1(\gamma_-, d\gamma)) \) and satisfies, for \( \tau, t \in [0, T] \) and \( 0 < \delta \ll 1 \),
\[
\int_\tau^t |h^{m+1}(s)|_{\gamma_-} \leq O(\delta) \int_\tau^t |h^{m-1}(s)|_{\gamma_-, 1} + C_\delta ||h(\tau)||_1
\]
\[
+ C_\delta \max_{i=m, m-1} \int_\tau^t \left\{ ||h^i(s)||_1 + ||R^i(s)||_1 + ||H^i(s)||_1 \right\}.
\]

The proof of this proposition requires the following lemma:

Lemma 3.3. Let \( \Omega \subset \mathbb{R}^3 \) be an open bounded set with a smooth boundary \( \partial \Omega \). For \( k \in \mathbb{N} \), consider the map
\[
\Phi_k : \{(x, v) \in \gamma_+ : n(x_b(x, v)) \cdot v < -1/k\} \rightarrow \{(x_b, v) \in \gamma_- : n(x_b) \cdot v < -1/k\},
\]
\[
(x, v) \rightarrow (x, v) := (x, v) := (x_b(x, v), v).
\]
Then $\Phi_k$ is one-to-one and we have a change of variables formula for all $k \in \mathbb{N}$:

$$1_{(n(x) < -1/k)} |n(x) \cdot v| dS_{\delta} dv = 1_{(n(x) < -1/k)} |n(x) \cdot v| dS_{\delta} dv.$$ 

**Proof of Lemma 3.3**: This lemma deals with the change of variable formula: $(x, v) \rightarrow (x_{0}, x, v)$. The proof is the same as in [11]. We omit it here. □

**Proof of Proposition 3.3**. We now prove the estimate (3.55). Using (3.54), we obtain

$$\int_{\gamma} |h^{m+1}(s)|_{\gamma - 1} := \int_{\tau} \int_{|n(x) < 0|} |h^{m+1}(s, x, v)||n(x) \cdot v| dS_{\delta} dv ds \leq (A) + (B),$$

where

$$(A) := \int_{\tau} \int_{|n(x) > 0|} |h^{m}(s, x, v)||n(x) \cdot v| dS_{\delta} dv ds,$$

$$(B) := \int_{\tau} \int_{|n(x) < 0|} |R^{m}(s, x, v)||1 + |n(x) \cdot v||dS_{\delta} dv ds.$$ 

Clearly the last term (B) is bounded by the RHS of (3.55).

We focus on (A) in the following. We split the outgoing part as $\gamma_+ = \gamma_+^{0} \cup (\gamma_+ \setminus \gamma_+^{0})$, where the almost grazing set $\gamma_+^{0}$ is defined in (1.25) and the non-grazing set $\gamma_+ \setminus \gamma_+^{0}$ is defined in (1.26). Due to Lemma 4.2, the non-grazing part $\gamma_+ \setminus \gamma_+^{0}$ of the integral is bounded as

$$\int_{0}^{\tau} \int_{\gamma_+ \setminus \gamma_+^{0}} |h^{m}(s)| d\gamma_{\tau} \leq \varepsilon_{\tau, \delta, \Omega} ||h^{m}(\tau)||_{1} + \int_{\tau} \left[ |||h^{m}(s)||||_{1} + |||\partial_{r} + v \cdot \nabla + \Sigma h^{m}(s)||||_{1} \right] ds$$

(3.56) $$\leq \varepsilon_{\tau, \delta, \Omega} ||h^{m}(\tau)||_{1} + \int_{\tau} ||h^{m}(s)||_{1} + \int_{\tau} ||H^{m-1}(s)||_{1}.$$ 

It is also bounded by the RHS of (3.55).

Now, we deal with the almost grazing set $\gamma_+^{0}$. We claim that the following truncated term with a number $k \in \mathbb{N}$ is uniformly bounded in $k$ as follows:

$$\int_{\tau} \int_{x_{0} \in \gamma_+^{0} \setminus n(x_{0})} 1_{|1/k < |n(x_{0}) \cdot v|} |h^{m}(s, x, v)||n(x) \cdot v| dS_{\delta} dv ds$$

(3.57) $$\leq O(\delta) \int_{\tau} |h^{m-1}(s)|_{\tau - 1} + C_{\delta} \left[ ||h_{0}||_{1} + \int_{\tau} \left( ||h^{m-1}(s)||_{1} + ||H^{m-1}(s)||_{1} + t||R^{m-1}(s)||_{1} \right) \right].$$

In order to show (3.57), we use the Duhamel formula of the equation (3.53) together with (3.54): for $(x, v) \in \gamma_+^{0}$ and $1/k < |n(x_{0}) \cdot v|$

$$|h^{m}(s, x, v)||1_{|1/k < |n(x_{0}) \cdot v|} \leq 1_{|s - t_{0} < |n(x_{0}) \cdot v|} |h^{m}(\tau, x - (s - \tau)v, v)|$$

$$+ 1_{|1/k < |n(x_{0}) \cdot v|} \int_{\tau}^{s} |H^{m-1}(\tau', x - (s - \tau')v, v)| d\tau'$$

$$+ 1_{|s - t_{0} > \tau} 1_{|1/k < |n(x_{0}) \cdot v|} C_{1} \left( 1 + \frac{1}{n(x_{0}) \cdot v} \right)$$

$$\times \int_{|n(x_{0}) \cdot v| > 0} |R^{m-1}(s - t_{0}, x, v, x_{0}, v_{1})||n(x_{0}) \cdot v_{1}| d\nu_{1}$$

$$+ 1_{|s - t_{0} < \tau} 1_{|1/k < |n(x_{0}) \cdot v|} \left( 1 + \frac{1}{n(x_{0}) \cdot v} \right) |R^{m-1}(s - t_{0}, x, v, x_{0}, v)|.$$
We plug this estimate into the left hand side of (3.57) to have
\[
\int_{\tau}^{\tau'} \int_{x \in \partial \Omega, n(x) \cdot v > 0} \mathbf{1}_{|x(x,v) \in \gamma_+|} \mathbf{1}_{|1/k < |n(h_b(x,v)) \cdot v|}[h^m(s, x, v)]\frac{dS_x}{\delta x} \, ds
\]
(3.58) \leq \int_{\tau}^{\tau'} \int_{x \in \partial \Omega, n(x) \cdot v > 0} \mathbf{1}_{|s = t_b(x,v) \cap (s - \tau)v, v > 0} |h^m(s - \tau, x - (s - \tau)v, v)]\frac{dS_x}{\delta x} \, ds
\]
(3.59) \quad + \int_{\tau}^{\tau'} \int_{x \in \partial \Omega, n(x) \cdot v > 0} \mathbf{1}_{|1/k < |n(h_b(x,v)) \cdot v|}[H^{m-1}(s - \tau, x - (s - \tau)v, v)]\frac{dS_x}{\delta x} \, ds
\]

**Estimate of (3.58):** Note that \( x \in \partial \Omega \) in (3.58). Without loss of generality we may assume that there exists \( \eta : \mathbb{R}^2 \to \mathbb{R} \) such that \( x^3 = \eta(x^1, x^2) \). We apply the following change of variables: for fixed \( v \in \mathbb{R}^3 \),
\[
(x^1, x^2, s) \mapsto y = (x^1 - (s - \tau)v^1, x^2 - (s - \tau)v^2, \eta(x^1, x^2) - (s - \tau)v^3).
\]
It maps \( \mathbb{R}^2 \times [0 \leq s - \tau \leq t_b(x,v)] \) into \( \gamma_+ \). We compute the Jacobian:
\[
\det \left( \frac{\partial(y^1, y^2, y^3)}{\partial(x^1, x^2, s)} \right) = \det \begin{pmatrix}
1 & 0 & -v^1 \\
0 & 1 & -v^2 \\
\partial_x \eta(x^1, x^2) & \partial_x \eta(x^1, x^2) & -v^3
\end{pmatrix} = v \cdot \begin{pmatrix}
\partial_x \eta \\
\partial_x \eta \\
-1
\end{pmatrix} = v \cdot n \sqrt{1 + |\partial_x \eta|^2 + |\partial_x \eta|^2}.
\]
Therefore, such mapping \( (x^1, x^2, s - \tau) \mapsto y \) is one-to-one when \( (x^1, x^2) \in \gamma_+ \) and
\[
[n(x) \cdot v]dS_x ds = [n(x) \cdot v] \sqrt{1 + |\partial_x \eta|^2 + |\partial_x \eta|^2} \, dx^1 dx^2 ds = dy^1 dy^2 dy^3,
\]
and
\[
(3.58) \leq \int_{V}^{\tau'} \int_{\partial \Omega} \mathbf{1}_{|x(x,v) \in \gamma_+|} \mathbf{1}_{|1/k < |n(h_b(x,v)) \cdot v|}[h^m(s, x, v)]\frac{dS_x}{\delta x} \, ds dv
\]
(3.62) \quad \leq \int_{V}^{\tau'} \int_{\partial \Omega} |h^m(s, y, v)|dv dy \leq ||h(s)||_1.
\]

**Estimate of (3.59):** Considering the region of \( ((\tau', s) \in [\tau, t] \times [\tau, t] : \max\{\tau, s - t_b(x,v)\} \leq \tau' \leq s \} \), it is bounded by
\[
(3.63) \quad \int_{V}^{\tau'} dv \int_{\tau}^{\tau'} dt' \int_{\partial \Omega} \mathbf{1}_{|1/k < |n(h_b(x,v)) \cdot v|}[H^{m-1}(s, x, v)]\frac{dS_x}{\delta x} \, ds dv
\]
Note that \( x \in \partial \Omega \), without loss of generality, we may assume that \( x^3 = \eta(x^1, x^2) \) for \( : \mathbb{R}^2 \mapsto \mathbb{R} \). We apply the change of variables: for fixed \( v \in V \) and \( \tau' \in [0, t] \),
\[
(x^1, x^2; s) \mapsto y \equiv (x^1 - (s - \tau')v^1, x^2 - (s - \tau')v^2, \eta(x^1, x^2) - (s - \tau')v^3).
\]
Clearly, it maps $\mathbb{R}^2 \times [\tau', \min\{t, \tau' + t_b(x, v)\}]$ into $\overline{\Omega}$ since $0 \leq (s - \tau') \leq t_b(x, v)$. The Jacobian of this change variable is $|v \cdot n(x)| \sqrt{1 + |\partial_x \eta|^2 + |\partial_x \eta|^2}$ and $|v \cdot n(x)|dS \cdot ds \leq dy$. Applying the change of variables to (3.63) to have

$$
(3.64) \quad (3.59) \leq \int_{\tau}^{t} \int_{\Omega} \int_{\Omega} |H^{m-1}(\tau', y, v)|dydvdt' = \int_{\tau}^{t} \|H^{m-1}(\tau')\|_1 d\tau'.
$$

**Estimate of (3.60):** This part is the most delicate. We rewrite (3.60) as

$$
\int_{\tau}^{t} ds \int_{\partial \Omega} dS_x \int_{V} dv \int_{V} dv_1 1_{\{\tau(x, v) < 0\}} 1_{\{v(x_b(x, v)) > 0\}} 1_{\{v_1 < 1/k \}} \frac{|n(x_0(x, v)) \cdot v_1| |H^{m-1}(s - t_b(x, v), x_b(x, v), v_1)|}{|n(x(x, v)) \cdot v|}.
$$

We apply the following change of variables

$$
s \in [0, t] \mapsto \tilde{s} = s - t_b(x, v) \in [\tau, t - t_b(x, v)] \subset [\tau, t],
$$

where we have used the fact that $s \in [t_b(x, v) + \tau, t]$. Clearly the Jacobian is 1 so that $d\tilde{s} = ds$ and hence

$$
(3.65) \leq \int_{\tau}^{t} d\tilde{s} \int_{\partial \Omega} dS_x \int_{V} dv \int_{V} dv_1 1_{\{\tau(x, v) < 0\}} 1_{\{v(x_b(x, v)) > 0\}} 1_{\{v_1 < 1/k \}} \frac{|n(x_0(x, v)) \cdot v_1| |H^{m-1}(\tilde{s}, x_b(x, v), v_1)|}{|n(x(x, v)) \cdot v|}.
$$

Let us denote $\tilde{x} := x_b(x, v)$. In the case $n_3(x_b(x, v)) \neq 0$, there exists some function $\phi : \mathbb{R}^2 \rightarrow \mathbb{R}$ such that

$$
\tilde{x} = x_b(x, v) = (\tilde{x}_1, \tilde{x}_2, \tilde{x}_3) = (\tilde{x}_1, \tilde{x}_2, \phi(\tilde{x}_1, \tilde{x}_2)) \in \partial \Omega.
$$

Note that since $(x, v) \in \gamma_+$ and $|n(x_b(x, v)) \cdot v| > 1/k$, from Lemma 3.3, the mapping $(x, v) \mapsto (\tilde{x}, v)$ is one-to-one. We apply the change of variables of Lemma 3.3: for $(x, v) \in \gamma_+$ and $|n(x_b(x, v)) \cdot v| = |n(\tilde{x}) \cdot v| > 1/k$, we apply the change of variables

$$
(x, v) \mapsto (\tilde{x}, v) := (x_b(x, v), v).
$$

The Jacobian is

$$
\det \left( \frac{\partial (\tilde{x}, v)}{\partial (x, v)} \right) = \det \left( \frac{\partial \tilde{x}}{\partial x} \right) = \frac{|n(x) \cdot v|}{|n(\tilde{x}) \cdot v|} \sqrt{1 + |\nabla \eta|^2} / \sqrt{1 + |\nabla \phi|^2}, \text{ and } dS_{\tilde{x}} := \frac{|n(x) \cdot v|}{|n(\tilde{x}) \cdot v|} dS_x.
$$

At the same time, we have

$$
t_b(x, v) = t_b(x_b, -v),
$$

$$
x = x_b(x, v) + t_b(x, v)v = x_b(x, v) + t_b(x_b(x, v), -v)v
$$

$$
= x_b(x, v) - t_b(x_b(x, v), -v)(-v) = \tilde{x} - x_b(\tilde{x}, -v)(-v).
$$

So we rewrite $(x, v) \in \gamma_+^\delta$ as

$$
1_{\{(x, v) \in \gamma_+^\delta\}} = 1_{\{0 < n(\tilde{x} - x_b(\tilde{x}, -v)(-v)) + \delta\}}.
$$
Then, from (3.66),
\[
(3.66) \leq \int_{\gamma} d\tilde{s} \int_{V} dv_1 \int_{V} dv \int_{\partial \Omega} dS \int_{0<\langle x-t_b(x,v)\rangle \cdot v<\delta} \mathbf{1}_{[0<\langle \tilde{x}-\tilde{t}_b(\tilde{x},v)\rangle \cdot v<\delta]} \times \mathbf{1}_{[\| n(\tilde{x}) \cdot v_1 \| h^m(\tilde{s}, \tilde{x}, v_1) < \max]} \int_{V} \mathbf{1}_{[0<\langle x-t_b(x,v)\rangle \cdot v<\delta]} d\tilde{v} d\tilde{s}
\]
\[
(3.68) \leq \int_{\gamma} \int_{\partial \Omega} |h^{m-1}(\tilde{s}, \tilde{x}, v_1)| |n(\tilde{x}) \cdot v_1| dS \times \int_{0<\langle x-t_b(x,v)\rangle \cdot v<\delta} \mathbf{1}_{[0<\langle x-t_b(x,v)\rangle \cdot v<\delta]} dv \sup_{\tilde{x} \in \partial \Omega} \int_{V} \mathbf{1}_{[0<\langle x-t_b(x,v)\rangle \cdot v<\delta]} dv
\]
Due to Lemma 4.4, for given \( \delta > 0 \), there exists \( \delta_{\partial \Omega} \) and \( l_{\partial \Omega} \) balls \( B(x_1, r_1), B(x_2, r_2), \ldots, B(x_l, r_l) \) covering \( \partial \Omega \), as well as \( l \) open sets \( O_1, O_2, \ldots, O_l \subset V \), with \( m_3(O_i) \leq \delta \) for all \( 1 \leq i \leq l \) such that
\[
\sup_{\tilde{x} \in \partial \Omega} \int_{V} \mathbf{1}_{[0<\langle x-t_b(x,v)\rangle \cdot v<\delta]} dv \leq \max_{i} \sup_{\tilde{x} \in B(x_i, r_i)} \max_{i} m_3 \{ v \in V : |n_b(\tilde{x}, -v) \cdot (\tilde{v})| \leq \delta \}
\]
\[
\leq \max_{i} m_3(O_i) \leq \delta.
\]
Therefore, for \( 0 < \delta \ll 1 \), such that
\[
(3.60) \leq O(\delta) \int_{\gamma} \int_{\partial \Omega} \int_{V} |h^{m-1}(\tilde{s}, \tilde{x}, v_1)| |n(\tilde{x}) \cdot v_1| dS \times \int_{0<\langle x-t_b(x,v)\rangle \cdot v<\delta} \mathbf{1}_{[0<\langle x-t_b(x,v)\rangle \cdot v<\delta]} d\tilde{v} d\tilde{s}
\]
\[
(3.69) = O(\delta) \int_{\gamma} |h^{m-1}(s)|_{\gamma, 1} ds.
\]

**Estimate of (3.61):** We apply the change of variables \( \tilde{s} = s - t_b(x, v) \) and (3.67), then by using Lemma 4.2 to bound as
\[
(3.70) \quad (3.61) \leq \int_{\gamma} \int_{\partial \Omega} \int_{V} |R^{m-1}(\tilde{s}, \tilde{x}, v)| dS \times dv d\tilde{s} = \int_{\gamma} |R^{m-1}(s)|_{\gamma} ds.
\]
Finally from (3.62), (3.64), (3.69) and (3.70), we prove our claim (3.57).

The last step is to pass a limit \( k \to \infty \). Clearly the sequence is non-decreasing in \( k \):
\[
0 \leq \mathbf{1}_{[1/k < |n(\tilde{x}_b(x,v))|]} |h^{m}(s, x, v)| \leq \mathbf{1}_{[1/(k+1) < |n(\tilde{x}_b(x,v))|]} |h^{m}(s, x, v)|.
\]
For \( \varepsilon > 0 \), we choose \( k \gg 1 \) such that \( 1/k < \varepsilon \). Then
\[
\int_{\gamma} \left[ 1 - \mathbf{1}_{[|n(\tilde{x}_b(x,v'))| < 1/k]} \right] (x, v') dy
\]
\[
\leq \int_{\partial \Omega} \int_{\tilde{n}(\tilde{x}_b(x,v'))} \mathbf{1}_{[|n(\tilde{x}_b(x,v'))| < 1/k]} |n(\tilde{x}_b(x,v'))| \cdot v' |dv' dS_x
\]
\[
\leq \frac{1}{k} \int_{\partial \Omega} \int_{\tilde{n}(\tilde{x}_b(x,v'))} dv' dS_x
\]
\[
\leq \varepsilon.
\]
It concludes that
\[
\mathbf{1}_{[1/k < |n(\tilde{x}_b(x,v))|]} h^{m}(s, x, v) \to |h^{m}(s, x, v)|, \text{ a.e. } (x, v) \in \gamma, \text{ with } dy.
\]
Now, we use the monotone convergence theorem to conclude
\[
(3.71) \int_0^\beta \int_{\gamma} \mathbf{1}_{[1/k < |n(\tilde{x}_b(x,v))|]} |h^{m}(s, x, v)| dy ds \to \int_0^\beta \int_{\gamma} |h^{m}(s, x, v)| dy ds,
\]
as \( k \to \infty \) and therefore \( \int_0^1 \int_{\gamma} |h^m(s, x, v)|dyds \) has the same upper bound of (3.57). Together with (3.56) we conclude (3.55).

\[ \square \]

3.4. **Estimates of the total variation.** The purpose of this subsection is to prove Theorem 2. To give the estimate of solution in total variation, we use following approximation scheme. For \( u_0 \in BV(\Omega \times V) \) and \( \|u_0\|_{\infty} < \infty \) we choose \( u^e_0 \in BV(\Omega \times V) \cap C^0(\Omega \times V) \) satisfying \( \|\tilde{u}_0 - u^e_0\|_{\infty} \to 0 \) and \( \|\nabla_{x,v} \tilde{u}_0\|_1 \to \|u_0\|_{\tilde{BV}}. \) At the same time, for fixed \( 0 \leq t \leq T \), we choose \( \tilde{q} \in BV(\Omega \times V) \cap C^0(\Omega \times V) \) satisfying \( \|\tilde{q} - q\|_{\infty} \to 0 \) and \( \|\nabla_{x,v} \tilde{q}\|_1 \to \|q\|_{\tilde{BV}}. \)

Consider the sequence \( u^{e,m} \) defined by \( u^{e,0} = \chi_e u_0 \) and for all \( m \geq 0, \)

\[
\begin{align*}
\partial_t u^{e,m+1} + v \cdot \nabla_x u^{e,m+1} + \Sigma u^{e,m+1} &= \chi_e[Ku^{e,m} + \tilde{q}], & \text{in } \Omega \times V, \\
u^{e,m+1}(0, x, v) &= \chi_e \tilde{u}_0(x,v), & \text{in } \Omega \times V, \\
u^{e,m+1} &= \chi_e \mathcal{P}_y u^{e,m} + \chi_e r, & \text{on } \gamma_-, 
\end{align*}
\]

where \( \chi_e \) is defined in (3.29).

In order to study the derivatives of \( u^{e,m+1}(t, x, v) \) with respect to \( x, v \), we need to consider the derivatives on the boundary. For the purpose of it, we assume that \( u \) satisfies the following neutron transport equation with the diffusive-inflow boundary condition

\[
(\partial_t + v \cdot \nabla_x + (\Sigma - K))u = q, \quad u|_{\gamma_-} = \mathcal{P}_y u + r.
\]

Let \( \tau_1(x), \tau_2(x) \) be a basis of the tangent space at \( x \in \partial \Omega \) (therefore \( \tau_1(x), \tau_2(x), n(x) \) is an orthonormal basis of \( \mathbb{R}^3 \)), i.e. \( \tau_1(x) \cdot n(x) = \tau_2 \cdot n(x) = 0 \) and \( \tau_1 \times \tau_2 = n(x) \). Define the orthonormal transformation from \( n(x), \tau_1, \tau_2 \) to the standard bases \( (e_1, e_2, e_3) \), i.e. \( Tn(x) = e_1, \)\( T\tau_1(x) = e_2, T\tau_2 = e_2 \) and \( T^{-1} = T^t \) \( \). Upon a change of variable: \( \xi = T^t \xi', \) we have

\[
n(x) \cdot \nu' = n(x) \cdot T^t \xi = n(x) \cdot n(x)^t T^t \xi = [T n(x)]^t \xi = e_1 \cdot \xi = \xi_1,
\]

then denote \( \partial_t \) to be the (tangential) \( \tau_i \)-directional derivative and \( \partial_n \) to be the normal derivative. For all \( (x, v) \in \gamma_- \), both \( t \) and \( v \) derivatives behave nicely for the diffusive boundary condition,

\[
\begin{align*}
(\partial_t u)|_{\gamma_-} &= c \int_{n(x)v' > 0} \partial_t u(t, x, v')|n(x) \cdot v'|dv' + \partial_t r, \\
(\nabla_v u)|_{\gamma_-} &= \nabla_v r.
\end{align*}
\]

From the choice of \( T \) in (3.73),

\[
\int_{n(x)v' > 0} u(t, x, v')|n(x) \cdot v'|dv' = \int_{\xi > 0} u(t, x, T^t(x)\xi)\xi_1 d\xi,
\]

So, we can further take the tangential derivatives \( \partial_{\tau_i} \) (\( i = 1, 2 \)) as, for \( (x, v) \in \gamma_- \),

\[
\begin{align*}
\partial_{\tau_i} u(t, x, v) &= c \int_{\xi > 0} \left[ \partial_{\tau_i} u(t, x, T\xi) + \nabla_v u(t, x, T\xi) \frac{\partial T^t(x)}{\partial \tau_i} \right] \xi_1 d\xi + \partial_{\tau_i} r \\
&= c \int_{n(x)v' > 0} \left[ \partial_{\tau_i} u(t, x, v') + \nabla_v u(t, x, v') \frac{\partial T^t(x)}{\partial \tau_i} T^t v' \right] \\
&= c \int_{n(x)v' > 0} |n(x) \cdot v'|dv' + \nabla_v \frac{\partial T^t(x)}{\partial \tau_i}.
\end{align*}
\]

The difficulty is always the control of the normal spatial derivative \( \partial_n u \). Near the boundary \( \partial \Omega \), it is natural to use the original equation to solve \( \partial_n u \) inside the region, in terms of \( \partial_t u, \nabla_v u \)
Lemma 3.4. \[ \partial_t u(t, x, v) = -\frac{1}{n(x) \cdot v} \{ \partial_t u + \sum_{i=1}^{2} (v \cdot \tau_i) \partial_{\tau_i} u - (\Sigma - K) u + q \}. \]

From (3.74), (3.75) and (3.76), we can express \( \partial_t u \) at \((x, v) \in \gamma_- \) as
\[
\partial_t u(t, x, v) = -\frac{c}{n(x) \cdot v} \left\{ \int_{n(x) \cdot v > 0} \{n(x) \cdot v\} dv' \right\} \times \left( \partial_t u(t, x, v') + \sum_{i=1}^{2} (v \cdot \tau_i) \left[ \partial_{\tau_i} u(t, x, v') + \nabla_v u(t, x, v') \partial T'_{\tau_i}(x) T'\right] \right)
+ \partial_r + \sum_{i=1}^{2} (v \cdot \tau_i) \nabla_r \partial T'_{\tau_i}(x) - [\Sigma - K] u - q] \right\}.
\]

Moreover, the equation gives
\[
\partial_t u(t, x, v') = q - \left[ \sum_{i=1}^{2} (v' \cdot \tau_i) \partial_{\tau_i} + (v' \cdot n) \partial_n + (\Sigma - K) \right] u(t, x, v').
\]

Submitting this equality into (3.77), it derives to
\[
\partial_t u(t, x, v) = -\frac{c}{n(x) \cdot v} \left\{ \int_{n(x) \cdot v > 0} \{n(x) \cdot v\} dv' \right\} \times \left( q - \left[ \sum_{i=1}^{2} (v' \cdot \tau_i) \partial_{\tau_i} + (v' \cdot n) \partial_n + (\Sigma - K) \right] u(t, x, v') \right)
+ \sum_{i=1}^{2} (v \cdot \tau_i) \left[ \partial_{\tau_i} u(t, x, v') + \nabla_v u(t, x, v') \partial T'_{\tau_i}(x) T' \right]
+ \partial_r + \sum_{i=1}^{2} (v \cdot \tau_i) \nabla_r \partial T'_{\tau_i}(x) - [\Sigma - K] u - q] \right\}.
\]

We firstly study the estimates of the derivatives of the solution for the following simpler neutron transport equation with in-flow boundary condition
\[
(3.79) \quad u_t + v \cdot \nabla u + \Sigma u = Q, \quad u(t, x, v)|_{\gamma_-} = R(t, x, v), \quad u(0, x, v) = u_0(x, v).
\]

Lemma 3.4. Assume \( \mathcal{U} \) is an open subset of \( \mathbb{R}^3 \times \mathbb{R}^3 \) such that \( \mathcal{B} \subset \mathcal{U} \). For \((t, x, v) \in [0, T] \times \mathcal{U} \cap \Omega \times V\), we assume
\[
(3.80) \quad u_0(x, v) \equiv 0, \quad R(t, x, v) \equiv 0, \quad Q(t, x, v) \equiv 0.
\]

Assume further that
\[
u \in L^\infty(\Omega \times V), \quad R \in L^\infty([0, T] \times \gamma_-), \quad Q \in L^\infty([0, T] \times \Omega \times V),
\]
and
\[
\nabla_{x,v} u_0 \in L^1(\Omega \times V),
\]
\[
\nabla_{x,v} R, \quad \frac{1}{n(x) \cdot v} \{ -[\partial_t + \sum_{i=1}^{2} (v \cdot \tau_i) \partial_{\tau_i} + \Sigma] R + Q \}, \quad \nabla_v R, \quad \nabla_{x,v} \Sigma \in L^1([0, T] \times \Omega \times V),
\]
\[
\nabla_{x,v} \Sigma, \quad \nabla_{x,v} Q \in L^1([0, T] \times \Omega \times V).
\]
Then there exists a unique solution $u$ to the transport equation (3.79) such that $u \in C^0([0, T] \times \Omega \times V)$ and $\nabla_x u \in C^0([0, T], L^1(\Omega \times V))$ and the traces satisfy

$$\nabla_x u = \nabla_x R, \text{ on } \gamma_-, \quad \nabla_x u(0, x, v) = \nabla_x u_0(x, v), \text{ in } \Omega \times V,$$

where $\nabla_x R$ is defined by

$$\nabla_x R = \sum_{i=1}^2 \tau_i \partial_{x_i} R + \frac{n}{n \cdot v} \left\{ - [\partial_t + \sum_{i=1}^2 (v \cdot \tau_i) \partial_{x_i} + \Sigma] R + Q \right\}.$$

Moreover,

$$\|\nabla_x u(t)\|_1 + \int_0^t \|\nabla_x R\|_{\gamma_-} + \int_0^t \|\Sigma \nabla_x u\|_1$$

(3.81)

$$= \|\nabla_x u_0\|_1 + \int_0^t \|\nabla_x R\|_{\gamma_-} + \int_0^t \int_{\Omega \times V} sgn(\nabla_x u)(\nabla_x Q - \nabla_x \Sigma u),$$

$$\|\nabla_x u(t)\|_1 + \int_0^t \|\nabla_x u\|_{\gamma_-} + \int_0^t \|\Sigma \nabla_x u\|_1$$

(3.82)

$$= \|\nabla_x u_0\|_1 + \int_0^t \|\nabla_x R\|_{\gamma_-} + \int_0^t \int_{\Omega \times V} sgn(\nabla_x u)(\nabla_x Q - \nabla_x u - \nabla_x \Sigma u).$$

**Proof.** We use the Duhamel formula of $u$:

$$u(t, x, v) = 1_{\{t < t_b(x, v)\}} e^{-\int_0^t \nabla_x u_0(x - tv, v) dt} u_0(x - tv, v)$$

(3.83)

$$+ 1_{\{t \geq t_b(x, v)\}} e^{-\int_0^t \nabla_x u_0(x - tv, v) dt} \left[ \int_0^t \nabla_x R(t - s, x - sv, v) ds \right]$$

Recall the derivatives of $x_b$ and $t_b$ in Lemma 4.1, following Proposition 1 in [10], the derivative of $u$ with respect to $x, v$ are

$$\nabla_x u(t, x, v) 1_{\{t \neq t_b\}}$$

$$= 1_{\{t < t_b\}} e^{-\int_0^t \nabla_x u_0(x - tv, v) \, dt} \left\{ \nabla_x u_0(x - tv, v) - \int_0^t \nabla_x \Sigma(t - \tau, x - tv, v) d\tau u_0(x - tv, v) \right\}$$

$$+ 1_{\{t \geq t_b\}} e^{-\int_0^t \nabla_x u_0(x - tv, v) \, dt} \left\{ \sum_{i=1}^2 \tau_i \partial_{x_i} R \right\}$$

$$- \frac{n(x_b)}{n(x_b) \cdot v} \left\{ \partial_t + \sum_{i=1}^2 (v \cdot \tau_i) \partial_{x_i} + \Sigma \right\} R - \frac{n(x_b)}{n(x_b) \cdot v} \left\{ \partial_t + \sum_{i=1}^2 (v \cdot \tau_i) \partial_{x_i} + \Sigma \right\} Q$$

$$- 1_{\{t > t_b\}} e^{-\int_0^t \nabla_x u_0(x - tv, v) \, dt} \left( \int_0^t \nabla_x \Sigma(t - \tau, x - tv, v) d\tau \right) R(t - t_b, x_b, v)$$

$$+ \int_0^{\min\{t, t_b\}} e^{-\int_0^\tau \nabla_x \Sigma(s - \tau, x - tv, v) \, ds} \nabla_x Q(t - s, x - sv, v) \, ds$$

$$- \int_0^{\min\{t, t_b(x, v)\}} e^{-\int_0^\tau \nabla_x \Sigma(s - \tau, x - tv, v) \, ds} \nabla_x Q(t - s, x - sv, v) \, ds.$$
and
\[ \nabla_v u(t, x, v) \mathbf{1}_{[t \neq t_b]} \]
\[ = \mathbf{1}_{[t < t_b]} e^{-\int_0^t \Sigma(t, x, x, v, v) dt} [t \nabla_x u_0 + \nabla_v u_0] (x - tv, v) \]
\[ - \mathbf{1}_{[t < t_b]} e^{-\int_0^t \Sigma(t, x, x, v, v) dt} \left( \int_0^t (\nabla_v \Sigma + \nabla_v \Sigma) (t - \tau, x - tv, v) d\tau \right) u_0 (x - tv, v) \]
\[ - \mathbf{1}_{[t > t_b]} t e^{-\int_0^t \Sigma(t, x, x, v, v) dt} \sum_{i=1}^2 \tau_i \partial_{\tau_i} R \]
\[ - \frac{n(x_b)}{n(x_b) \cdot v} \left[ \partial_{\tau} + \sum_{i=1}^2 (v \cdot \tau_i) \partial_{\tau_i} + \Sigma \right] R - Q \right) (t - t_b, x_b, v) \]
\[ + \mathbf{1}_{[t > t_b]} t e^{-\int_0^t \Sigma(t, x, x, v, v) dt} \nabla_v R (t - t_b, x_b, v) \]
\[ - \mathbf{1}_{[t > t_b]} t e^{-\int_0^t \Sigma(t, x, x, v, v) dt} \left( \int_0^t (\nabla_v \Sigma + \nabla_v \Sigma) (t - \tau, x - tv, v) d\tau \right) R (t - t_b, x_b, v) \]
\[ + \int_0^{\min(t, t_b)} e^{-\int_0^t \Sigma(t, x, x, v, v) dt} \left( \int_0^t (\nabla_v Q - s \nabla_v Q) (t - s, x - sv, v) ds \right) \]
\[ \times Q (t - s, x - sv, v) ds. \]

Therefore, we have, for all \( 0 \leq t \leq T \)
\[ \|\nabla_x u(t) \mathbf{1}_{[t \neq t_b]}\|_1 \leq \|\nabla_x u_0\|_1 + (\|u_0\|_\infty + \|R\|_\infty) \]
\[ + \int_0^t \left\| \sum_{i=1}^2 \tau_i \partial_{\tau_i} R - \frac{n(x_b)}{n(x_b) \cdot v} \left[ \partial_{\tau} + \sum_{i=1}^2 (v \cdot \tau_i) \partial_{\tau_i} + \Sigma \right] R - Q \right\|_{y, 1} \]
\[ + \int_0^t \|\nabla_v Q(s)\|_1 + \int_0^t s \|Q(s)\|_\infty, \]

and
\[ \|\nabla_v u(t) \mathbf{1}_{[t \neq t_b]}\|_1 \leq t \|\nabla_x u_0\|_1 + \|\nabla_v u_0\|_1 + t \|u_0\|_\infty \]
\[ + t \int_0^t \left\| \sum_{i=1}^2 \tau_i \partial_{\tau_i} R - \frac{n(x_b)}{n(x_b) \cdot v} \left[ \partial_{\tau} + \sum_{i=1}^2 (v \cdot \tau_i) \partial_{\tau_i} + \Sigma \right] R - Q \right\|_{y, 1} \]
\[ + \int_0^t \|\nabla_v R\|_{y, 1} + t^2 \sup_{0 \leq s \leq t} \|R(s)\|_{y, \infty} \]
\[ + t \int_0^t \|\nabla_v Q(s)\|_1 + \int_0^t \|\nabla_v Q\|_1 + \int_0^t s \|Q(s)\|_\infty. \]

Since \(u_0, R, \) and \(Q\) have compact supports and the RHS of (3.85) and (3.86) are bounded. Therefore
\[ \partial_t u \mathbf{1}_{[t \neq t_b]} = [\nabla_x u \mathbf{1}_{[t \neq t_b]}, \nabla_v u \mathbf{1}_{[t \neq t_b]}] \in L^\infty([0, T]; L^1(\Omega \times \mathbb{R}^3)). \]

Since \(\partial_t u \equiv 0\) around \(t = t_b,\) clearly \(\partial_t u \mathbf{1}_{[t \neq t_b]}\) is the distributional derivative of \(u.\) Therefore \(\nabla_x u\) and \(\nabla_v u\) lie in \(L^\infty([0, T]; L^1(\Omega \times \mathbb{R}^3)),\) this allows us to apply Lemma 4.2 to compute the traces on the incoming boundary in \(L^1([0, T]; L^1(\gamma, dy))\) (by taking limits of the flow along the characteristics: see the proof of Proposition 1 in [10] for details). Then, by Green’s identity 4.3 we know that \(\nabla_x u\) and \(\nabla_v u\) lie in \(C^0([0, T]; L^1(\Omega \times \mathbb{R}^3)).\) Then we get (3.81) and
Proof of Theorem 1.2. We firstly consider the bound in \( \| \cdot \|_{\infty} \) of the solution for the approximation scheme (3.72). For a fixed \( 0 < \varepsilon \ll 1 \), it is clear that \((u^{e,m})_m\) is a Cauchy series for the norm \( \sup_{0 \leq t \leq T} \| \cdot \|_{\infty} \) for fixed \( 0 < T \) from Theorem 1.1. More precisely, the sequence \((u^{e,m})_m\) satisfy

\[
\|u^{e,m}\|_{\infty} \leq_{T, \Omega, V} \|u_0\|_{\infty} + \sup_{0 \leq t \leq T} |r(s)|_{\infty} + \sup_{0 \leq t \leq T} |q(s)|_{\infty}.
\]

Therefore \( u^{e,m} \rightarrow u^e \) up to subsequence for the norm \( \sup_{0 \leq t \leq T} \| \cdot \|_{\infty} \) and \( u^e \) satisfies (3.72) with both \( u^{e,m+1} \) and \( u^{e,m} \) replaced by \( u^e \) by the Green theorem. Since \( |\chi_\varepsilon| \leq 1 \) for \( 0 < \varepsilon \ll 1 \), \( \sup_{0 \leq t \leq T} \|u^e\|_{\infty} \) is uniformly bounded in \( \varepsilon \) for fixed \( T \). Therefore \( u^e \rightarrow u \) weak \(-\ast\) up to a subsequence and the limiting function \( u \) solves the original neutron transport equation in the sense of distributions.

Secondly, we consider the derivatives of the solution \( u^{e,m} \) of (3.72). Recall that \( BV(\Omega \times V) \) has

i) a compactness property: Suppose \( g^k \in BV \) and \( \sup_k \|g^k\|_{BV} < \infty \), then there exists \( g \in BV \) with \( g^k \rightarrow g \) in \( L^1 \) up to subsequence,

ii) a lower semicontinuity property: Suppose \( g^k \in BV \) and \( g^k \rightarrow g \) in \( L^1_{loc} \) then \( \|g\|_{BV} \leq \lim \inf_{k \rightarrow \infty} \|g^k\|_{BV} \).

Due to the smooth approximation \( \hat{u}_0 \) of the initial datum \( u_0 \) and the cut-off \( \chi_\varepsilon \), \( u^{e,m} \) is smooth by Lemma 3.4. On one hand, by Lemma 3.4 with \( \Sigma \geq 0 \),

\[
\|\partial u^{e,m+1}(t)\|_{1} + \int_{0}^{t} |\partial u^{e,m+1}|_{y_{1},1} \leq |\partial u_0|_{1} + \int_{0}^{t} \left( |\partial u^{e,m+1}|_{\infty} + \|u^{e,m}\|_{\infty} + \|q\|_{BV} \right) + \int_{0}^{t} |\partial u^{e,m+1}(s)|_{y_{1},1} + \int_{0}^{t} \left( \|\partial u^{e,m+1}\|_{1} + \|\partial u^{e,m}\|_{1} \right),
\]

(3.87)

where we have used the assumptions that

\[ M'_a = \|\partial \Sigma\|_{\infty} < \infty, \quad M'_b = \sup_{x,v} \int_{V} |\partial f(x, v, v')|dv' < \infty. \]

From the uniform estimate in (3.86), we obtain

\[
\|\partial u^{e,m+1}(t)\|_{1} + \int_{0}^{t} |\partial u^{e,m+1}|_{y_{1},1} \leq |\partial u_0|_{BV} + \sup_{0 \leq s \leq t} (|r(s)|_{\infty} + |\partial r(s)|_{1} + |\partial r(s)|_{1} + \|q(s)\|_{BV}) + \int_{0}^{t} |\partial u^{e,m+1}(s)|_{y_{1},1} + \int_{0}^{t} \left( \sup_{0 \leq s \leq t} \|\partial u^{e,m+1}(s)\|_{1} + \sup_{0 \leq s \leq t} \|\partial u^{e,m}(s)\|_{1} \right).
\]

(3.88)

On the other hand, by taking derivatives \( \partial \in \{\nabla_x, \nabla_v\} \) to (3.72), we have

\[
[\partial_t + v \cdot \nabla_x + \Sigma(\nabla_x u^{e,m+1})] = -\nabla_x \Sigma u^{e,m+1} + \nabla_x (\chi_\varepsilon K u^{e,m} + \chi_\varepsilon \hat{q}),
\]

\[
[\partial_t + v \cdot \nabla_x + \Sigma(\nabla_v u^{e,m+1})] = -2(\nabla_x + \nabla_v \Sigma) u^{e,m+1} + \nabla_v (\chi_\varepsilon K u^{e,m} + \chi_\varepsilon \hat{q})
\]

\[
\hat{u}^{e,m+1}(0, x, v) = \partial \chi_\varepsilon u_0(x, v) + \chi_\varepsilon \partial u_0(x, v),
\]
and, from (3.72)-(3.78) as well as (3.86), we have, for all \((x, v) \in \gamma_-\),
\[
|\partial u^{e,m+1}(t, x, v)| \leq \left(1 + \frac{1}{|n(x) \cdot v|}\right) \int_{n(x) \cdot v' > 0} |\partial u^{e,m}(t, x, v')| |n(x) \cdot v'| dv' + (1 + \frac{1}{|n(x) \cdot v|}) \left(\|u^{e,m}\|_\infty + \|\bar{q}\|_\infty + |\partial r| + |\partial r|\right).
\]

We apply Proposition 3.3 to bound
\[
\int_0^t |\partial u^{e,m+1}|_{\gamma,-} \leq O(\delta) \int_0^t |\partial u^{e,m-1}|_{\gamma,-} + C_\delta t \sup_{0 \leq s \leq t} \|u^{e,m+1}\|_1 + C_\delta t \max_{i=m, m-1} \sup_{0 \leq s \leq t} \|u^{e,i}(s)\|_1
\]
\[
+C_\delta \left(\|u_0\|_{BV} + \sup_{0 \leq s \leq t} (|r(s)| \infty + |\partial r(s)|_1 + |\partial r(s)|_1 + |q(s)|_{BV})\right).
\]

Finally from (3.88) and (3.89), choosing \(\delta \ll 1\) and \(T_0 := T(u_0)\) is small enough, we have for all \(0 \leq t \leq T_0\)
\[
\|\partial u^{e,m+1}(t)\|_1 + \int_0^t |\partial u^{e,m+1}|_{\gamma,-} \leq \frac{1}{8} \max_{i=m, m-1} \left\{\sup_{0 \leq s \leq t} \|\partial u^{e,i}(s)\|_1 + \int_0^s |\partial u^{e,i}|_{\gamma,-}\right\}
\]
\[
+ C \left(\|u_0\|_{BV} + \sup_{0 \leq s \leq t} (|r(s)| \infty + |\partial r(s)|_1 + |\partial r(s)|_1 + |q(s)|_{BV})\right).
\]

Now, using (2.6) with \(k = 2\), we conclude, for all \(m \in \mathbb{N}\)
\[
\|\partial u^{e,m+1}(t)\|_1 + \int_0^t |\partial u^{e,m+1}|_{\gamma,-} \leq \|u_0\|_{BV} + \sup_{0 \leq s \leq t} (|r(s)| \infty + |\partial r(s)|_1 + |\partial r(s)|_1 + |q(s)|_{BV}).
\]

Now, we pass the to limit in \(m \to \infty\) and then in \(\epsilon \to 0\) to conclude the main theorem when \(0 \leq t \leq T_0\). Repeat the same procedure for \([T_0, 2T_0], [2T_0, 3T_0], \ldots\), to conclude the main theorem for all \(0 \leq t \leq T\). From the compactness and a lower semicontinuity we conclude
\[
\sup_{0 \leq s \leq T} \|u(s)\|_{BV} \leq \|u_0\|_{BV} + \sup_{0 \leq s \leq T} (|r(s)| \infty + |\partial r(s)|_1 + |\partial r(s)|_1 + |q(s)|_{BV}), \quad 0 \leq t \leq T.
\]

For the boundary term we use the weak compactness of measures: If \(\sigma^k\) is a signed Radon measures on \(\partial \Omega \times V\) satisfying \(\sup_k \sigma^k(\partial \Omega \times V) < \infty\) then there exists a Radon measure \(\sigma\) such that \(\sigma^k \to \sigma\) in \(\mathcal{M}\). More precisely we define, for almost-every \(s\), and for any Lebesgue-measurable set \(A \subset \partial \Omega \times V\),
\[
\sigma_{s}^{e,m}(A) = \left(\sigma_{s,x_{1,1}}^{e,m}(A), \sigma_{s,x_{1,2}}^{e,m}(A), \sigma_{s,y_{1,1}}^{e,m}(A), \sigma_{s,y_{1,2}}^{e,m}(A), \sigma_{s,y_{1,2}}^{e,m}(A), \sigma_{s,y_{1,2}}^{e,m}(A)\right)^T
\]
\[
:= \int_A \nabla u^{e,m}(s)d\gamma \in V \times V.
\]

Then there exists a Radon measure \(\sigma_s\) such that \(\sigma_{s}^{e,m} \to \sigma_s\) in \(\mathcal{M}\), i.e.
\[
\int_{\partial \Omega \times V} g d\sigma^{e,m} \to \int_{\partial \Omega \times V} g d\sigma_s \quad \text{for all } g \in C_0^0(\partial \Omega \times V).
\]
It is standard (Hahns decomposition theorem) to decompose $\sigma_s = \sigma_{s,+} - \sigma_{s,-}$ with $\sigma_{s,\pm} \geq 0$. Denote $|\sigma_s|_{M(\Omega)} = \sigma_{s,+}(\partial \Omega \times V) + \sigma_{s,-}(\partial \Omega \times V)$. Then by the lower semicontinuity property of measures we have

$$|\sigma_s|_{M(\Omega)} \leq \liminf |\sigma_s^{e.m}|_{M(\Omega)} = \liminf |\partial u_{s}^{e.m}|_{L^1(\Omega)}.$$ 

So that by (3.90)

$$\int_0^1 |\sigma_s|_{M(\Omega)} ds \leq \|u_0\|_{BV} + \sup_{0 \leq s \leq T} |\tau(s)|_1 + \sup_{0 \leq s \leq T} \|g(s)\|_{\infty}).$$

Due to (3.91), the (distributional) derivatives $\nabla_{x,v}u(s)_\rho$ equal the Radon measure $\sigma_s$ on $\partial \Omega \times \mathbb{R}^3$ in the sense of distributions. 

\[ \square \]

4. Appendix

Lemma 4.1. ([9, 10]). If $v \cdot n(x_0(x, v)) < 0$, then $(t_b(x, v), x_b(x, v))$ are smooth functions of $x, v$ such that

$$\nabla_\xi t_b = \frac{n(x_b)}{v \cdot n(x_b)}, \quad \nabla_\nu t_b = - \frac{t_b n(x_b)}{v \cdot n(x_b)},$$

$$\nabla_\xi x_b = I - \frac{n(x_b)}{v \cdot n(x_b)} \otimes v, \quad \nabla_\nu x_b = -t_b I - \frac{t_b n(x_b)}{v \cdot n(x_b)} \otimes v.$$ 

Let $x_i \in \partial \Omega$ for $i = 1, 2$ and $(t_1, x_1, v)$ and $(t_2, x_2, v)$ be connected with the trajectory $\frac{dX(s)}{ds} = V(s), \frac{dV(s)}{ds} = 0$ which lies inside $\Omega$. Then there exists a constant $C_\xi > 0$ such that

$$|t_1 - t_2| \geq \frac{n(x_1) \cdot v}{C_\xi |v|^2}.$$ 

For the estimate of the outing trace on $\gamma_+ \setminus \gamma^\partial_+$, we need the following trace theorem.

Lemma 4.2. (Outgoing trace theorem,[6]). Assume $\psi \geq 0$. For any small parameter $\delta > 0$, there exists a constant $C_{\delta, T, \Omega}$ such that for any $h \in L^1([0, T] \times \Omega \times V)$ with $(\partial_\nu v \cdot \nabla_\nu + \psi)h$ lying in $L^1([0, T] \times \Omega \times V)$, we have for all $0 \leq t \leq T$,

$$\int_0^T \int_{\gamma_t \setminus \gamma_+^\partial} |h| d\nu d\alpha \leq C_{\delta, T, \Omega} \left[\|h_0\|_1 + \int_0^t \left(\|h(s)\|_1 + \|\partial_\nu v \cdot \nabla_\nu + \psi\| h(s)\|_{\infty}\right) ds\right].$$

Furthermore, for any $(s, x, v) \in [0, T] \times \Omega \times V$, the function $h(s + s', x + x', v, v)$ is absolutely continuous in $s'$ in the interval $[\min\{t_b(x, v), s\}, \min\{t_b(x, -v), T - s\}]$.

Lemma 4.3. (Green Identity, [9], [10]). Let $p \in [1, \infty)$. Assume that $f, (\partial_\nu v \cdot \nabla_\nu + \psi) f \in L^p([0, T] \times \Omega \times V)$ with $\psi \geq 0$ and $f|_{\gamma_+} \in L^p([0, T] \times \partial \Omega \times V, dtd\nu)$. Then $f \in C^0([0, T], L^p(\Omega \times V))$ and $f|_{\gamma_+} \in L^p([0, T] \times \partial \Omega \times V, dtd\nu)$ and for almost every $t \in [0, T]$,

$$\|f\|_p^p + \int_0^t \|f\|_{\nu, p}^p = \|f(0)\|_p^p + \int_0^t \|f\|^p_{\nu, p} + \int_0^t \int_{\Omega \times V} |\partial_\nu v \cdot \nabla_\nu f + \psi f| f|^{p-2} f.$$

The covering lemma has proved in [9, 11], here we have the similar result by replacing $B_N = \{v \in \mathbb{R}^3 : |v| \leq N\}$ with the compact set $V \subset \mathbb{R}^3$.

Lemma 4.4. (Covering Lemma, [9, 11]). Let $\Omega \subset \mathbb{R}^3$ be an open bounded set with a smooth boundary $\partial \Omega$ and $V$ is a compact set in $\mathbb{R}^3$. Then, for all $x \in \partial \Omega$, we have

$$m_3\{v \in V : n(x_0(x, v)) \cdot v = 0\} = 0.$$
Moreover, for any $\varepsilon$, there exist $\delta_{\varepsilon,V} > 0$ and $l = l_{\varepsilon,\Omega,V}$ balls $B(x_1, r_1), B(x_2, r_2), \ldots, B(x_l, r_l)$ with $x_i \in \overline{\Omega}$ and covering $\overline{\Omega}$ (i.e. $\overline{\Omega} \subset \bigcup B(x_i, r_i)$), as well as $l$ open sets $O_{x_1}, O_{x_2}, \ldots, O_{x_l} \subset V$, with $\mu_V(O_{x_i}) < \varepsilon$ for all $1 \leq i \leq l_{\varepsilon,\Omega,V}$ such that for any $x \in \overline{\Omega}$, there exists $i = 1, 2, \ldots, l_{\varepsilon,\Omega,V}$ such that $x \in B(x_i, r_l)$ and

$$|v \cdot n(x_0(x,v))| > \delta_{\varepsilon,V}, \text{ for all } v \notin O_x.$$  

In particular,

$$\bigcup_{x \in B(x_0(x,v))} \left\{ v \in V : |v \cdot n(x_0(x,v))| \leq \delta_{\varepsilon,V} \right\} \subset O_x.$$  

Acknowledgments. The second author would like to express deep thank to S.Q. Liu and F.J. Zhou for the fruitful discussion during the visit to Brown University.

References

[1] Beals, R. and Protopopescu, V., Abstract time-dependent transport equations. J. Math. Anal. Appl. 121 (1987), no. 2, 370-405.

[2] Bardos, C., Santos, R. and Sentis, R., Diffusion approximation and computation of the critical size. Trans. Amer. Math. Soc. 284 (1984), no. 2, 617-649.

[3] Bensoussan, A., Lions, J.-L. and Papanicolaou, G.-C., Boundary layers and homogenization of transport processes. Publ. Res. Inst. Math. Sci. 15 (1979), no. 1, 53-157.

[4] Cercignani, C., Illner, R. and Pulvirenti, M., The mathematical theory of dilute gases. Applied Mathematical Sciences, 106. Springer-Verlag, New York, 1994.

[5] Dautray, R. and Lions, J.-L., Mathematical analysis and numerical methods for science and technology Vol.6, Springer-Verlag, New York, 1993.

[6] Esposito, R., Guo, Y., Kim, C. and Marras, R., Non-isothermal boundary in the Boltzmann theory and Fourier law. Comm. Math. Phys. 323 (2013), no. 1, 177-239.

[7] Gibbs, A.G. Analytical solutions of the neutron transport equation in arbitrary convex geometry J. Math. Phys. 10 (1969), no. 5, 875-890.

[8] Guo, Y., Singular solutions of the Vlasov-Maxwell system on a half line. Arch. Rational Mech. Anal. 131 (1995), no. 3, 241-304.

[9] Guo, Y., Decay and continuity of Boltzmann equation in bounded domains. Arch. Rational Mech. Anal. 197 (2010), no. 3, 713-809.

[10] Guo, Y., Kim, C., Tonon, D. and Trescases, A., Regularity of the Boltzmann equation in convex domains. submitted.

[11] Guo, Y., Kim, C., Tonon, D. and Trescases, A., BV-regularity of the Boltzmann equation in non-convex domain. submitted.

[12] Kim, C., Formation and propagation of discontinuity for Boltzmann equation in non-convex domains. Comm. Math. Phys. 308 (2011), no. 3, 641-701.

[13] Larsen, E. W., A functional-analytic approach to the steady, one-speed neutron transport equation with anisotropic scattering. Comm. Pure Appl. Math. 27 (1974), no. 4, 523-545.

[14] Larsen, E. W., Asymptotic theory of the linear transport equation for small mean free paths. II. SIAM J. Appl. Math. 33 (1977), no. 3, 427-445.

[15] Larsen, E.W. and Keller, J.B. Asymptotic solution of neutron transport problems for small free paths J. Math. Phys. 15 (1974), no. 1, 75-81.

[16] Lods, B., On the spectrum of mono-energetic absorption operator with Maxwell boundary conditions. A unified treatment. Transp. Theory Stat. Phys., 37 (2008), no. 1, 1-37.

[17] Lods, B., A generation theorem for kinetic equations with non-contractive boundary operators. C. R. Acad. Sci. Paris, 335 (2002), no. 7, 655-660.

[18] Lods B., Semigroup generation properties of streaming operators with non-contractive boundary conditions. Mathematical and Computer Modelling, 42 (2005), no. 11, 1441-1462.

[19] Lods, B. and Sbini, M., Stability of the essential spectrum for 2D-transport models with Maxwell boundary conditions. Math. Meth. Appl. Sci. 29 (2006), no. 5, 499-523.
[20] Pao, C.V., *A nonlinear Boltzmann equation in transport theory*. Trans. Amer. Math. Soc. **194** (1974), no. 3, 167-175.

[21] Pao, C.V., *Positive solutions and criticality of the linear and some nonlinear neutron transport problems*. SIAM J. Appl. Math. **32**, (1977), no. 1, 164-176.

[22] Protopopescu, V. and Thevenot, L., *Diffusion approximation for linear transport with multiplying boundary conditions*. SIAM J. Appl. Math. **65** (2005), no. 5, 1657-1676.

[23] Wu, L. and Guo, Y., *Geometric correction for diffusive expansion of steady neutron transport equation*. Commun. Math. Phys. **336** (2015), no. 3, 1473-1553.

[24] Zhang, X.W. and Liang, B.Z, *On the spectrum of a one-velocity transport operator with Maxwell boundary condition*. J. Math. Anal. Appl. **202** (1996), no. 3, 920-939.

(Y. Guo) Division of Applied Mathematics, Brown University, Providence, RI 02812, USA
E-mail address: Yan Guo@sjtu.edu.cn

(X.F. Yang)
Department of Mathematics, MOE-LSC and SHL-MAC, Shanghai Jiao Tong University, Shanghai, 200240, P.R. China
E-mail address: xf-yang@sjtu.edu.cn