Quantum plasma modification of the Lane-Emden equation for stellar structure

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The proper quantum plasma treatment of the electron gas in degenerate stars such as white dwarfs provides an additional quantum contribution to the electron pressure. The additional pressure term modifies the equation for hydrostatic equilibrium, resulting in the quantum modified Lane-Emden equation for polytropic equation of states. The additional pressure term also modifies the expression for the limiting Chandrasekhar mass of white dwarfs. An approximate solution is derived of the quantum modified Lane-Emden equation for general polytropic indices, and it is demonstrated that the quantum corrections reduce the standard Chandrasekhar mass and enhance the white dwarf radius by negligibly small values only.

I. INTRODUCTION

From a plasma point of view the electron gas of density \( n \) in degenerate stars such as white dwarfs is a nearly collisionless quantum plasma because its central temperature \( T \) is lower than the Fermi temperature \( T_F \), and the quantum coupling parameter \( g_Q = E_c/E_F \) of the ratio of the Coulomb interaction energy \( E_c \) to the Fermi energy \( E_F \) is much smaller than unity. The white dwarf Sirius B has an average mass density of \( 3 \cdot 10^6 \, \text{g cm}^{-3} \) and a central temperature of \( T_c = 7.6 \cdot 10^7 \, \text{K} \) yielding for the temperature ratio

\[
\chi = \frac{T_F}{T} = (3\pi^2)^{2/3} \frac{h^2 n^{2/3}}{2mk_BT} = 3.65 \left( n\lambda_B^3 \right)^{2/3} = 55.7
\]

The ratio \( \chi \) indicates that the thermal de Broglie wavelength of individual plasma particles

\[
\lambda_B = \frac{h}{mv_T} \simeq 3.9 n^{-1/3}, \tag{2}
\]

is of the same order of magnitude as the mean distance between electrons \( n^{-1/3} \). The thermal de Broglie wavelength roughly represents the spatial extension of a particle’s wave function due to the quantum uncertainty. For \( \lambda_B \) comparable to the interparticle distance, because of overlapping electron wave function extensions, individual electrons cannot be treated as pointlike particles as in the classical plasma description, and quantum effects become important. The smallness of the quantum coupling parameter \( g_Q \) assures that collective mean field-effects dominate over binary collisions because the typical electron Fermi energy is much larger than the Coulomb interaction energy with neighbouring electrons.

Here we demonstrate that the proper quantum treatment of the electron gas based on the Wigner distribution function changes the hydrostatic equilibrium equation in white dwarfs leading, however, to a negligible modification of the maximum Chandrasekhar mass of such systems. In the hydrodynamical equations the quantum effects yield an additional pressure term \( P_Q \) to the classical pressure \( P_c \),

\[
P = P_c + P_Q = P_c + \frac{\hbar^2}{2m} \left[ (\nabla n^{1/2})^2 - n^{1/2} \nabla^2 n^{1/2} \right] \quad \tag{3}
\]

in terms of the electron density \( n \). The additional pressure term \( P_Q \) is not listed in Salpeter’s account of the pressure contributions to a zero-temperature degenerate Fermi gas of non-interacting electrons. Here the modification to the standard Lane-Emden equation for polytropic gases and to the limiting Chandrasekhar mass are calculated.

II. QUANTUM MODIFIED EQUATIONS FOR HYDROSTATIC STELLAR EQUILIBRIA

For a plasma at rest (\( \ddot{u} = 0 \)) the hydrodynamical equation \((\text{A10})\) becomes

\[
0 = \ddot{g}(\vec{x}) + \frac{1}{\rho} \nabla P_c - \frac{h^2 Z}{2mm_i} \nabla \left( \frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}} \right) \tag{4}
\]

with the mass density \( \rho = nn_i/Z \) where \( m_i \) denotes the ion mass. Choosing spherical coordinates we obtain the quantum modified equation for hydrostatic equilibria in stars as

\[
0 = g(r) \left[ \frac{1}{\rho(r)} \frac{dP_c}{dr} - \frac{h^2 Z}{2mm_i} \frac{d}{dr} \left( \frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}} \right) \right] \tag{5}
\]

with the operator

\[
\nabla^2 = r^{-2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right), \tag{6}
\]
and the usual gravitational acceleration

\[ g(r) = G\mu(r)/r^2 \]  \hspace{1cm} (7)

due to the enclosed mass

\[ \frac{d\mu}{dr} = 4\pi r^2 \rho(r) \]  \hspace{1cm} (8)

The mass inside the radius \( r \) is given by

\[ M(r) = \int_0^r dr' \frac{d\mu}{dr} = 4\pi \int_0^r (r')^2 \rho(r') \]  \hspace{1cm} (9)

so that the total mass is

\[ M = 4\pi \int_0^R dr \, r^2 \rho(r) \]  \hspace{1cm} (10)

Inserting Eq. (7) into Eq. (5), and differentiating with respect to \( r \) immediately yields the quantum modified stellar structure equation

\[
\frac{d}{dr} \left[ \frac{1}{r^2} \frac{dP_c}{dr} - \frac{h^2 Z}{2mm_i} \frac{d}{dr}\left( \frac{\Sigma_i^2 \rho^{1/2}(r)}{\rho^{1/2}(r)} \right) \right] = -4\pi G\rho^2 \rho(r)
\]

\hspace{1cm} (11)

A. Quantum modified Lane-Emden equation

Following standard procedures\textsuperscript{[24,25]} we adopt the polytropic equation of state

\[ P_c = K \rho^{(1+\alpha)/\alpha} \]  \hspace{1cm} (12)

and substitute \( \rho = \rho_c y_0^\alpha \) (with central density \( \rho_c \)) and \( r = Ax \) with the constant

\[ A = \left[ \frac{(\alpha + 1)K\rho_c^{(1-\alpha)/\alpha}}{4\pi G} \right]^{1/2} \]  \hspace{1cm} (13)

Eq. (11) then becomes the quantum modified Lane-Emden (QMLE) equation for a polytrope of index \( \alpha \)

\[
\frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{dy_0}{dx} \right) + y_0^\alpha = \eta_\alpha \frac{d}{dx} \left( x^2 \frac{dy_0}{dx} \frac{x^2 dy_0^\alpha}{dx} \right)
\]

\hspace{1cm} (14)

where the dimensionless parameter

\[ \eta_\alpha = \frac{\hbar^2 Z}{8\pi Gmm_i\rho_c A^4} = \frac{2\pi G\rho_c^{(\alpha-2)/\alpha}}{m_i K^2(\alpha + 1)^2} \]  \hspace{1cm} (15)

characterizes the strength of the quantum plasma modifications. The term on the right hand side of the QMLE equation (14) reflects the new quantum contribution from the Bohm potential. For \( \eta_\alpha = 0 \) the QMLE-equation reduces to the standard Lane-Emden equation of stellar structure.

The QMLE equation (14) has been derived for flat Newtonian space-time ignoring general relativistic effects. For completeness in Appendix B we derive the relativistic generalization of the QMLE-equation.

B. Quantum modified Chandrasekhar mass

In terms of the substituted variables \( x \) and \( y_0(x) \) the mass \( M \) inside the normalised radius \( x \) reads

\[ M(x) = 4\pi \int_0^{Ax} dr' (r')^2 \rho(r') = 4\pi \rho_c A^3 \int_0^x d\xi y_0^\alpha(\xi) \]  \hspace{1cm} (16)

Inserting the QMLE equation (14) yields

\[ M(x) = -M_0 \left[ x^2 \frac{dy_0}{dx} - \eta_\alpha \left( x^2 \frac{dy_0}{dx} \frac{x^2 dy_0^\alpha}{dx} \right) \right] \]  \hspace{1cm} (17)

with

\[ M_0 = 4\pi \rho_c A^3 \]  \hspace{1cm} (18)

The first zero \( x_1 \) of the solution \( y_0(x_1) = 0 \) of the QMLE equation defines the size of the star. In terms of \( M_0 \), with \( Z = 2 \) and \( m_i = Z \cdot 1836 \, m \), the parameter \( \eta_\alpha \) can be expressed in cgs-units as

\[ \eta_\alpha = \frac{\hbar^2 Z}{2Gmm_iA M_0} = \frac{5.4 \times 10^{13}}{AM_0} = 1.3 \times 10^4 \rho_c^{1/3} \]  \hspace{1cm} (19)

Scaling \( M_0 = 3 \chi M_{\text{sun}} = 6 \chi \cdot 10^{33} \, g \) in solar masses and \( A = a_0 \cdot 10^9 \) cm with the typical white dwarf mass and radius, we obtain very small values of the parameter
The small value of the parameter ηα reflects the small ratio of the Fermi pressure and Bohm pressure terms (second and third term in Eq. (5)). By replacing \((d/dr)\) by \(1/L\), where \(L\) denotes a typical length such as the radius of the star, and using the relativistic expression for the Fermi pressure \(P_c = (3\pi^2)^{1/3}\hbar c\alpha^{1/3}/4\), we find for the ratio of the two terms in terms of the Compton electron wavelength \(\lambda_c = h/(mc) = 3.86 \cdot 10^{-11}\) cm

\[
\frac{T_3}{T_2} = \frac{\hbar^2 n}{2mP_cL^2} = \frac{2}{(3\pi^2)^{1/3}L^2n^{1/3}} \lambda_c
\] (21)

For a white dwarf, typically we have \(n \sim 10^{30}\) cm\(^{-3}\), so that with \(L \sim 10^9\) cm we estimate \(T_3/T_2 \sim 2.5 \cdot 10^{-39}\), in agreement with the estimate (20).

While these rough order of magnitude estimates indicate that the Bohm pressure term is likely a small contributor to modifications of the Chandrasekhar white dwarf mass limit, nevertheless it is appropriate to figure out the effect quantitatively for two main reasons. First, it can happen that the rough estimates given above do not uncover the total effect when it is worked out quantitatively and so it could happen that a small correction has a profound effect on the solution to the non-linear equation. Such effects are not unusual in physics and one truly needs to work out the details in order to be sure that one has not overlooked some subtle component. Second, even when detailed calculations show that the Bohm pressure effect is small, in accord with the rough estimates made, it is then satisfying to note the confirmation of the rough estimates by detailed evaluation. For these two reasons alone it is more than appropriate to evaluate the Bohm pressure effect although, as we will show, the contribution is indeed small. The procedure for evaluating the contribution to the white dwarf mass may also be of relevance to other astrophysical problems and so, as a basic technique, it is also appropriate to spell out the details of how one addresses such non-linear corrections.

Since the pioneering work of Chandrasekhar, it is well known that the interior of white dwarfs is supported by the pressure of completely degenerate electrons. With increasing density the pressure becomes so high that the degeneracy becomes relativistic so that \(P_c = 1.244 \cdot 10^{15}(\rho/\mu_c)^{4/3}\) which corresponds to a polytrope index of \(\alpha = 3\). In this case Eq. (17) yields the quantum-modified Chandrasekhar mass limit

\[
\mathcal{M} = M(x_1, \alpha = 3) = -\frac{4}{\pi^{1/2}} \left(\frac{K}{G}\right)^{3/2}
\]

The first term in Eq. (22) is the standard expression of the Chandrasekhar mass limit whereas the second term represents the modification from the quantum plasma Bohm potential. In order to investigate its effect in enhancing or reducing the standard mass limit one has to compute the first zero \(x_1\) and the shape of the solution \(y_3\) of the QMLE equation.

\[
\left[ x^2 \frac{dy_3}{dx} - \eta_3 x^2 \left( \frac{d}{dx} \left( x^2 \frac{dy_3^{3/2}}{dx} \right) \right) \right]_{x_1}
\] (22)

with

\[
\eta_3 = 4.28 \cdot 10^3 \frac{G^2 \rho_c^{1/3}}{K^2}
\] (23)

The first term in Eq. (22) is the standard expression of the Chandrasekhar mass limit whereas the second term represents the modification from the quantum plasma Bohm potential. In order to investigate its effect in enhancing or reducing the standard mass limit one has to compute the first zero \(x_1\) and the shape of the solution \(y_3\) of the QMLE equation.

### III. APPROXIMATE SOLUTION OF THE QMLE EQUATION FOR GENERAL POLYTROPIC INDICES \(\alpha\) FOR SMALL VALUES OF \(\eta_\alpha\)

Substituting \(t = 1/x\) the QMLE equation (14) reads

\[
t^4 \frac{d^2}{dt^2} \left[ y_0 - \eta_\alpha t^4 \frac{d^2 y_0^{3/2}}{dt^2} \right] + y_0^\alpha = 0
\] (24)

Dropping the index \(\alpha\), i.e. \(y(t) = y_\alpha(t)\) and \(\eta_\alpha = \eta\), and using \(u(t) = [y(t)]^{3/2}\) the QMLE equation (25) reads

\[
\frac{d^2}{dt^2} \left[ y - \eta t^4 \frac{d^2 u}{dt^2} \right] + \frac{u^2}{t^4} = 0
\] (25)

#### A. First-order expansion in \(\eta\)

For very small values of the quantum parameter \(\eta \ll 1\) set

\[
u(t) = u_e(t) + \eta \delta u
\] (26)

where \(u_e\) fulfills the standard Lane-Emden equation

\[
\frac{d^2 u_e^{2/\alpha}}{dt^2} + \frac{u_e^2}{t^4} = 0
\] (27)

The ansatz (26) implies

\[
y(t) = y_e(t) + \eta \delta y = (u_e + \eta \delta u)^{2/\alpha}
\]

\[
\simeq u_e^{2/\alpha} + \frac{2\eta}{\alpha} u_e^{(2-\alpha)/\alpha} \delta u
\] (28)
so that to first order in $\eta$

$$\delta u = \frac{\alpha}{2} t^2 \left(\begin{array}{c} \delta y \\ t_c \end{array}\right)$$

(29)

Using Eqs. (26) and (28) we find for the QMLE equation (25) to first order in $\eta$

$$\frac{d^2}{dt^2} \left[ \begin{array}{c} y_c + \eta \delta y \\ \frac{t^4}{u_c} \frac{d^2 u_c}{dt^2} \end{array} \right] + \left[ \begin{array}{c} u_c^2 + 2 \eta u_c \delta u \\ \frac{t^4}{u_c^2} \frac{d^2 u_c}{dt^2} \end{array} \right] = 0$$

(30)

Using Eqs. (27) and (29) we derive

$$\frac{d^2}{dt^2} \left[ \begin{array}{c} \delta y \\ \frac{t^4}{u_c} \frac{d^2 u_c}{dt^2} \end{array} \right] + \frac{2}{t^4} u_c \delta u =$$

$$\frac{d^2}{dt^2} \left[ \begin{array}{c} \delta y \\ \frac{t^4}{u_c} \frac{d^2 u_c}{dt^2} \end{array} \right] + \frac{\alpha}{t^4} u_c^{2(\alpha-1)/\alpha} \delta y = 0$$

(31)

which, for a known standard solution $u_c$, is a linear second-order differential equation for the deviation $\delta y$.

Both $u(t)$ and the solution $u_c(t)$ of the standard Lane-Emden equation fulfill the boundary conditions

$$u_c(t = \infty) = 1, \quad \frac{du_c}{dt} \bigg|_{t=\infty} = 0,$$

$$u(t = \infty) = 1, \quad \frac{du}{dt} \bigg|_{t=\infty} = 0$$

(32)

implying

$$\delta y(t = \infty) = 0, \quad \frac{d\delta y}{dt} \bigg|_{t=\infty} = 0$$

(33)

With these boundary conditions, integrating equation (31) once over $t$, we obtain

$$\frac{d}{dt} \left[ \begin{array}{c} \delta y \\ \frac{t^4}{u_c} \frac{d^2 u_c}{dt^2} \end{array} \right] = 2 \int_t^\infty dq \ q^{-4} \delta u(q) u_c(q)$$

(34)

B. First-order quantum corrections to the Chandrasekhar mass and radius

In terms of the variable $t = 1/x$ the Chandrasekhar mass (17) reads

$$M(x_1) = 4 \pi \rho_c A^3 \left( \frac{dy_c}{dt} + \eta \frac{d}{dt} \left[ \frac{t^4}{u_c} \frac{d^2 u_c}{dt^2} \right] \right)_{t_1}$$

(35)

where $t_1 = 1/x_1$ denotes the largest zero of the function $u(t_1) = 0$. With the ansätze (26) and (28) we find to first order in $\eta$

$$M(x_1) = 4 \pi \rho_c A^3 \left( \frac{dy_c}{dt} + \eta \frac{d}{dt} \left[ \frac{t^4}{u_c} \frac{d^2 u_c}{dt^2} \right] \right)_{t_1}$$

$$= 4 \pi \rho_c A^3 \left( \frac{dy_c}{dt}_{t_1} + 2 \eta \int_t^\infty dq \ \frac{u_c(q) \delta u(q)}{q^4} \right)$$

(36)

where we inserted equation (31). Eq. (36) represents the first-order correction in $\eta$ of the Chandrasekhar mass.

If $t_c$ denotes the largest zero of $\eta t_c(t_c) = 0$ we set $t_1 = t_c + \delta t$. The boundary condition

$$0 = u(t_1) = u_c(t_c) + \eta \delta u(t_1)$$

together with the Taylor expansion

$$u_c(t_c) \simeq u_c(t_c) + \left( \frac{du_c}{dt} \right)_{t_c} \delta t = \left( \frac{du}{dt} \right)_{t_c} \delta t$$

then yields

$$\delta t = -\eta \left( \frac{du_c}{dt} \right)_{t_c} \simeq -\eta \left( \frac{du}{dt} \right)_{t_c}$$

(37)

so that

$$t_1 = t_c - \eta \tau, \quad \tau = \frac{\delta u(t_c)}{\left( \frac{du}{dt} \right)_{t_c}}$$

(38)

Consequently, the dimensionless radius of the gas sphere

$$x_1 = \frac{1}{x_1} \simeq 1 + \frac{\eta \tau}{t_c^2}$$

(39)

is enhanced (diminished) in comparison to the Chandrasekhar radius $t_c^{-1}$ when $\tau > 0$ ($\tau < 0$).

With the Taylor expansion

$$y_c(t) \simeq y_c(t_c) + \left( \frac{dy_c}{dt} \right)_{t_c} (t - t_c) + \frac{1}{2} \left( \frac{d^2 y_c}{dt^2} \right)_{t_c} (t - t_c)^2$$

we obtain

$$\left( \frac{dy_c}{dt} \right)_{t_c} = \left( \frac{dy_c}{dt} \right)_{t_c} - \eta \frac{d^2 y_c}{dt^2} (t - t_c)$$

(40)

so that the Chandrasekhar mass (30) becomes

$$M(x) = 4 \pi \rho_c A^3 \left( \frac{dy_c}{dt} \right)_{t_c} + \eta B_c$$

(41)

where

$$B_c = 2 \int_t^\infty dq \ \frac{u_c(q) \delta u(q)}{q^4} - \frac{\delta u(t_c)}{\left( \frac{du}{dt} \right)_{t_c} \frac{d^2 y_c}{dt^2} (t - t_c)}$$

(42)

The first term in Eq. (41) represents the standard Chandrasekhar mass. The quantum correction scales linearly proportional to $\eta$. 
C. Approximate solution of the differential equation \((31)\)

It is well-known that the standard Lane-Emden equation, i.e. Eq. \((24)\) with \(\eta = 0\), is analytically solvable only for the values \(\alpha = 0, 1, 5\) but not for the physically interesting case \(\alpha = 3\). However, for general values of \(\alpha\) the standard Lane-Emden equation has the asymptotic solution

\[
y_c(t) \simeq 1 - \frac{1}{6t^2} + \frac{\alpha}{120t^4}
\]

implying

\[
u_c(t) = y_c^{\alpha/2}(t) \simeq 1 - \frac{\alpha}{12t^2} + \frac{\alpha^2}{240t^4},
\]

\[
u_c^{2-(2/\alpha)}(t) \simeq 1 - \frac{\alpha - 1}{6t^2} + \frac{\alpha(\alpha - 1)}{120t^4},
\]

\[
u_c^{1-(2/\alpha)}(t) \simeq 1 - \frac{\alpha - 2}{12t^2} + \frac{\alpha(\alpha - 2)}{240t^4},
\]

and

\[
\frac{d^2u_c}{dt^2} \simeq -\frac{\alpha}{2t^4} \left[ 1 - \frac{\alpha}{6t^2} \right],
\]

respectively. Substituting

\[
\delta y(t) = g(t) + \frac{t^4}{u_c} \frac{d^2u_c}{dt^2} \simeq g(t) - \frac{\alpha}{2} \left[ 1 - \frac{\alpha}{12t^2} \right]
\]

yields for the differential equation \((31)\)

\[
\frac{d^2g}{dt^2} + \frac{\alpha}{t^4} y_c^{2(\alpha-1)/\alpha} g = -\alpha u_c^{1-(2/\alpha)} \frac{d^2u_c}{dt^2}
\]

The boundary conditions \((33)\) imply \(g(t = \infty) = \alpha/2\). Inserting the asymptotics \((41)-(44)\) yield for the differential equation \((49)\)

\[
\frac{d^2g}{dt^2} + \frac{\alpha}{t^4} \left[ 1 - \frac{\alpha - 1}{6t^2} + \frac{\alpha(\alpha - 1)}{120t^4} \right] g \simeq \\
\frac{\alpha^2}{2t^4} \left[ 1 - \frac{3\alpha - 2}{12t^2} \right]
\]

which, at large \(t\), has the asymptotic solution

\[
g(t) \simeq \frac{\alpha}{2} \frac{\alpha^3}{480t^4}
\]

implying from equation \((48)\)

\[
\delta y(t) \simeq \frac{\alpha^2}{24t^2} \left[ 1 - \frac{\alpha}{20t^2} \right]
\]

From equation \((49)\) we obtain accordingly

\[
\delta u = \frac{\alpha}{2} y_c^{(\alpha-2)/\alpha} \delta y \simeq \frac{\alpha^3}{48t^2} \left[ 1 - \frac{4\alpha - 5}{30t^2} \right]
\]

D. Chandrasekhar mass and radius reductions

Collecting terms we obtain for the quantum plasma correction \((42)\) to the standard Chandrasekhar mass the quantity

\[
B_c \simeq \frac{\alpha^2}{8t_c^3} \left[ 1 + \frac{5 - 4\alpha}{30t_c^2} \right]
\]

Likewise, the corrected normalised radius \((39)\) is

\[
x_1 \simeq \frac{1}{t_c} \left[ 1 + \frac{\alpha^2\eta}{8} \left( 1 + \frac{5 - \alpha}{30t_c^2} \right) \right]
\]

For \(\alpha = 3\) the corrected Chandrasekhar mass \((41)\) thus becomes

\[
M(x) = 4\pi \rho_c A^3 \left( \frac{dy_c}{dt} t_c + \frac{9\eta}{8t_c^3} \left[ 1 - \frac{7}{30t_c^2} \right] \right)
\]

whereas the corrected normalised radius \((55)\) is

\[
x_1 \simeq \frac{1}{t_c} \left[ 1 + \frac{9\eta}{8} \left( 1 + \frac{1}{15t_c^2} \right) \right]
\]

Strictly taken, expressions \((54) - (57)\) are only valid for large values of \(t_c \ll 1\) because the corrections have been derived from the corresponding asymptotic solutions. Nevertheless, we use them here for \(t_c = 1/6.89685 = 0.145\), which is not large compared to unity, in order to obtain a first crude estimate of the strength of the quantum corrections. We find that the quantum corrections reduce the standard Chandrasekhar mass by the negligibly small factor

\[
B_c \eta = -3.73 \cdot 10^3 \eta = -3.4 \cdot 10^{-35}(a_0 \chi)^{-1}
\]

and enhance the normalised stellar radius by the negligibly small factor

\[
\frac{9\eta}{8} \left( 1 + \frac{1}{15t_c^2} \right) = 4.69\eta = 4.4 \cdot 10^{-38}(a_0 \chi)^{-1}
\]
IV. SUMMARY AND CONCLUSIONS

The proper quantum plasma treatment of the electron gas in degenerate stars such as white dwarfs provides an additional quantum contribution to the electron pressure. We have calculated how this additional pressure term modifies the equation for hydrostatic equilibrium, resulting in the quantum modified Lane-Emden equation for polytropic equation of states. The additional pressure term also modifies the expression for the limiting Chandrasekhar mass of white dwarfs. We develop an approximate solution of the quantum modified Lane-Emden equation for general polytropic indices. We demonstrate that the quantum corrections reduce the standard Chandrasekhar mass by a negligibly small value of order $O(10^{-35})$ and enhance the stellar white dwarf radius by a negligibly small value of order $O(10^{-38})$. Deviations from charge neutrality in the white dwarf electron-ion plasma are of order $O(10^{-37})$.

Our study is preliminary in two important aspects. First we have used asymptotic expansions of our approximate solution for the QMLE equation for large values of the dimensionless variable $t$ to estimate the quantum corrections to the white dwarf mass and radius. Such can be improved in future work by a numerical solution of the QMLE equation for all values of $t$. Secondly, we have used a nonrelativistic calculation for the additional quantum pressure term to combine it with the relativistic degeneracy electron pressure term when estimating the corrections to the standard Chandrasekhar mass. This shortcoming is currently difficult to fix, as a fully relativistic kinetic theory of quantum plasmas is not completely fleshed out as of yet.

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\[
\frac{\partial n}{\partial t} + \frac{(nu)}{\partial z} = 0, \quad (A2)
\]
and the dynamical electron equation

\[
0 = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial z} + \frac{1}{mn} \frac{d(P_e + P_Q)}{dz} + \frac{e}{m} \frac{d\Phi}{dz} \quad (A3)
\]

with the Fermi pressure $P_e$ and the Bohm pressure

\[
P_Q = \frac{\hbar^2}{2m} \left[ \left( \frac{\partial n^{1/2}}{\partial z} \right)^2 - n^{1/2} \frac{\partial^2 n^{1/2}}{\partial z^2} \right] \quad (A4)
\]

$\Phi$ denotes the electrostatic potential and $n$ and $n_i$ refer to the electron and ion number densities, respectively. With the identity

\[
\frac{1}{nm} \frac{dP_Q}{dz} = \frac{\hbar^2}{2m^2 n} \frac{d}{dz} \left[ \frac{\partial n^{1/2}}{\partial z} \right] \frac{\partial n^{1/2}}{\partial z} \quad (A5)
\]

the dynamical electron equation in full 3-dimensional form reads

\[
0 = \frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla) \vec{u} + \frac{e}{m} \nabla \Phi + \frac{1}{mn} \nabla P_e - \frac{\hbar^2}{2m^2} \nabla \left( \frac{\nabla n^{1/2}}{n^{1/2}} \right) \quad (A6)
\]

Appendix A: Quantum hydrodynamical equations

According to Manfredi and Haas, the quantum hydrodynamic equations of an electron-ion plasma in a 1-dimensional cartesian geometry are given by the Poisson equation

\[
\frac{d^2 \Phi}{dz^2} = 4\pi e [n - n_i], \quad (A1)
\]

the equation of continuity

First we have used asymptotic expansions of our approximate solution for the QMLE equation for large values of the dimensionless variable $t$ to estimate the quantum corrections to the white dwarf mass and radius. Such can be improved in future work by a numerical solution of the QMLE equation for all values of $t$. Secondly, we have used a nonrelativistic calculation for the additional quantum pressure term to combine it with the relativistic degeneracy electron pressure term when estimating the corrections to the standard Chandrasekhar mass. This shortcoming is currently difficult to fix, as a fully relativistic kinetic theory of quantum plasmas is not completely fleshed out as of yet.

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1. Quantum modified hydrostatic equilibria

For static ($\vec{u} = 0$) systems, after adding gravity, the electron dynamical equation reads

$$0 = mn\vec{g}(\vec{x}) + en\nabla\Phi + \nabla P_c - \frac{\hbar^2n}{2m} \nabla \left( \frac{\nabla^2 n^{1/2}}{n^{1/2}} \right) \tag{A7}$$

Because the ion contribution to the pressure is negligibly small, the corresponding ion equation reads

$$0 \simeq m_i n_i \vec{g}(\vec{x}) - Z e n_i \nabla \Phi \tag{A8}$$

Adding equations (A7) and (A8) and using the charge neutrality condition $Z n_i = n$ eliminates the electrostatic potential leaving

$$0 = (mn + m_i n_i) \vec{g}(\vec{x}) + \nabla P_c - \frac{\hbar^2 n}{2m} \nabla \left( \frac{\nabla^2 n^{1/2}}{n^{1/2}} \right) \tag{A9}$$

With the mass density $\rho = m_i n_i + mn \simeq m_i n / Z$ we derive

$$0 = \vec{g}(\vec{x}) + \frac{1}{\rho} \nabla P_c - \frac{\hbar^2 Z}{2m m_i} \nabla \left( \frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}} \right) \tag{A10}$$

which is identical to Eq. (A1). Under charge neutrality the electric field $\vec{E} = -\nabla \Phi$ does not enter the equation for stellar hydrostatic equilibrium.

2. Electric field equation and charge neutrality state

Multiplying equation (A8) with $Z$ and subtracting equation (A7) yields for the electrostatic potential

$$(Z^2 e n_i + e n) \nabla \Phi = (Z m_i n_i - mn) \vec{g}(\vec{x})$$

$$-\nabla P_c + \frac{\hbar^2 n}{2m} \nabla \left( \frac{\nabla^2 n^{1/2}}{n^{1/2}} \right) \tag{A11}$$

Because of the stronger action of the gravitational field on the plasma ions as compared to the plasma electrons, a charge separation results that induces a nonzero electrostatic potential and electric field. Assuming only small perturbations of the charge neutrality condition $Z n_i \simeq n$ then yields

$$\vec{E} = -\nabla \Phi \simeq -\frac{m_i}{e(Z+1)} \vec{g}(\vec{x}) + \frac{\nabla P_c}{e(Z+1)n}$$

$$-\frac{\hbar^2}{2me(Z+1)} \nabla \left( \frac{\nabla^2 \rho^{1/2}}{\rho^{1/2}} \right) = -\frac{m_i}{eZ} \vec{g}(\vec{x}) \tag{A12}$$

where we have inserted Eq. (A9). This electric field provides a net space charge density $\sigma$ that can be estimated by calculating the divergence of Eq. (A12) as

$$\nabla \cdot \vec{E} = 4\pi \sigma = 4\pi e(Z n_i - n) = -\frac{m_i}{eZ} \nabla \cdot \vec{g} \tag{A13}$$

With $\nabla \cdot \vec{g} = -4\pi G \rho$ and $\rho = m_i n_i$ we obtain the positive space charge density

$$\sigma = e(Z n_i - n) = \frac{G m_i^2 n_i}{c^2 e Z^2} = \frac{G m_p^2}{2 e^2 Z^2} = 1.01 \cdot 10^{-37} \tag{A14}$$

that has to be screened by an appropriately negative space charge density on the surface of the star. As relative deviation from the overall charge neutrality state we obtain

$$\frac{Z n_i - n}{Z n_i + n} = \frac{Z n_i - n}{2 Z n_i} = \frac{G m_p^2}{2 e^2 Z^2} = 1.01 \cdot 10^{-37} \tag{A15}$$

if we use $Z = 2$ and $m_i = m_p$. The deviation from the overall charge neutrality state is negligibly small.

Appendix B: Relativistic generalisation of the quantum modified Lane-Emden equation

Assuming only isotropic stresses, the mixed components of the energy-momentum tensor of a spherically symmetric distribution of fluid obeying a polytropic equation of state in a co-moving system read

$$T^1_1 = T^2_2 = T^3_3 = -P \quad T^0_0 = \epsilon = \rho c^2 \tag{B1}$$

where $P$ is the pressure and $\epsilon = \rho c^2$ is the energy density. Using a spherical form of the metric at rest with respect to the fluid distribution

$$ds^2 = e^{\nu(r)} c^2 dt^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2) - e^{\lambda(r)} dr^2 \tag{B2}$$

we obtain the time-independent gravitational equations

$$e^{-\lambda} \left( \frac{1}{r} \frac{d\nu}{dr} + \frac{1}{r^2} \right) - \frac{1}{r^2} = \frac{8\pi G}{c^4} P \tag{B3}$$

$$e^{-\lambda} \left( \frac{1}{r} \frac{d\lambda}{dr} - \frac{1}{r^2} \right) + \frac{1}{r^2} = \frac{8\pi G}{c^4} \rho c^2 \tag{B4}$$

$$e^{-\lambda} \left( \frac{d^2\nu}{dr^2} + \frac{1}{2} \left( \frac{d\nu}{dr} \right)^2 + \frac{1}{r} \left( \frac{d\nu}{dr} \right) \frac{d\lambda}{dr} - \frac{1}{2} \frac{d\lambda}{dr} \frac{d\lambda}{dr} \right) = \frac{16\pi G}{c^4} P \tag{B5}$$
Combining these equations we derive the expression

$$ \frac{dP}{dr} + \frac{1}{2}(P + \rho c^2) \frac{d\nu}{dr} = 0 \quad (B6) $$

which is the general relativistic analogue of the classical form of equation (5). Introducing the quantum correction $P_Q$ in the expression for the pressure $P = P_c + P_Q$ (see equation (A18) and (A20)) equation (B6) yields

$$ \frac{1}{\rho(r)} \frac{dP_c}{dr} - \frac{\hbar^2}{2m_i} \left( \nabla_\rho \rho^{1/2}(r) \right) + \frac{1}{2\rho(r)} \left( P_c + \right. $$

$$ \left. \frac{\hbar^2}{2m_i} \left[ \left( \frac{d\rho^{1/2}(r)}{dr} \right)^2 - \rho^{1/2}(r) \nabla_\rho \rho^{1/2}(r) + \rho c^2 \right] \frac{d\nu}{dr} = 0 \quad (B7) \right) $$

Using the polytropic equation of state (12) and the substitutions $w(\zeta) = \frac{1}{2(\alpha + 1)\sigma} \zeta$, where $A$ is defined by expression (13), $\zeta = r/A$ and the constants $\chi^\alpha = \frac{\hbar^2}{2m_i A^2 c^2 (1 + \sigma y_\alpha)}$, $\sigma = K \rho^{1/\alpha} / c^2$ and inserting equation (B7) into (B3) we derive neglecting terms of order $O(h^4/A^4)$

$$ w + \sigma y_\alpha \frac{dw}{d\zeta} + \frac{1 - 2(\alpha + 1)\sigma w/\zeta}{1 + \sigma y_\alpha} \zeta^2 \frac{dy_\alpha}{d\zeta} = $$

$$ x_\alpha \left( y_\alpha^{-\alpha} \left( 2\sigma y_\alpha + 1 \right) \frac{dw}{d\zeta} + w \right) \left[ y_\alpha^{\alpha/2} \nabla_\zeta y_\alpha^{\alpha/2} - \left( \frac{dy_\alpha^{\alpha/2}}{d\zeta} \right)^2 \right] $$

$$ + \frac{1 - 2(\alpha + 1)\sigma w/\zeta}{\sigma(\alpha + 1)} \zeta^2 \frac{d}{d\zeta} \left( \frac{\nabla_\zeta y_\alpha^{\alpha/2}}{y_\alpha^{\alpha/2}} \right) $$

Equation (B4) yields with the definitions and substitutions given previously

$$ \frac{dw}{d\zeta} = \zeta^2 y_\alpha \quad (B9) $$

In the non-relativistic limit the general relativistic quantum system (B8) and (B9) reduces to equation (14), while for $h \rightarrow 0$ it reduces to the general relativistic Lane-Emden equation of stellar structure (12).