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Charged Particles and the Electro-Magnetic Field in Non-Inertial Frames of Minkowski Spacetime: I. Admissible 3+1 Splittings of Minkowski Spacetime and the Non-Inertial Rest Frames.

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Abstract

By using the 3+1 point of view and parametrized Minkowski theories we develop the theory of non-inertial frames in Minkowski space-time. The transition from a non-inertial frame to another one is a gauge transformation connecting the respective notions of instantaneous 3-space (clock synchronization convention) and of the 3-coordinates inside them. As a particular case we get the extension of the inertial rest-frame instant form of dynamics to the non-inertial rest-frame one. We show that every isolated system can be described as an external decoupled non-covariant canonical center of mass (described by frozen Jacobi data) carrying a pole-dipole structure: the invariant mass and an effective spin. Moreover we identify the constraints eliminating the internal 3-center of mass inside the instantaneous 3-spaces.

In the case of the isolated system of positive-energy scalar particles with Grassmann-valued electric charges plus the electro-magnetic field we obtain both Maxwell equations and their Hamiltonian description in non-inertial frames. Then by means of a non-covariant decomposition we define the non-inertial radiation gauge and we find the form of the non-covariant Coulomb potential. We identify the coordinate-dependent relativistic inertial potentials and we show that they have the correct Newtonian limit.

In the second paper we will study properties of Maxwell equations in non-inertial frames like the wrap-up effect and the Faraday rotation in astrophysics. Also the 3+1 description without coordinate-singularities of the rotating disk and the Sagnac effect will be given, with added comments on pulsar magnetosphere and on a relativistic extension of the Earth-fixed coordinate system.
I. INTRODUCTION

As a consequence of many years of research devoted to try to establish a consistent formulation of relativistic mechanics, we have now a description of every isolated system (particles, strings, fields, fluids), admitting a Lagrangian formulation, in arbitrary global inertial or non-inertial frames in Minkowski space-time by means of parametrized Minkowski theories [1, 2, 3, 4] (see Ref.[5] for a review). They allow one to get a Hamiltonian description of the relativistic isolated systems, in which the transition from a non-inertial (or inertial) frame to another one is a gauge transformation generated by suitable first-class Dirac constraints. Therefore, all the admissible conventions for clock synchronization, identifying the instantaneous 3-spaces containing the system and allowing a formulation of the Cauchy problem for the equations of the fields present in the system, turn out to be gauge equivalent.

The only known way to have a global description of non-inertial frames is to choose an arbitrary time-like observer and a 3+1 splitting of Minkowski space-time, namely a foliation with space-like hyper-surfaces (namely an arbitrary clock synchronization convention) with a set of 4-coordinates (observer-dependent Lorentz-scalar radar 4-coordinates $\sigma^A = (\tau; \sigma^r)$, $A = \{\tau, r\}$) adapted to the foliation and having the observer as origin of the 3-coordinates $\sigma^r$ on each instantaneous 3-space $\Sigma_\tau$. The time parameter $\tau$, labeling the leaves of the foliation, is an arbitrary monotonically increasing function of the proper time of the observer. Each such foliation defines a global non-inertial frame centered on the given observer if it satisfies the Møller admissibility conditions [6], [3, 5], and if the instantaneous (in general non-Euclidean) 3-spaces, described by the functions $z^\mu(\tau, \sigma^r)$ giving their embedding in a reference inertial frame in Minkowski space-time, tend to space-like hyper-planes at spatial infinity [3]. The 4-metric $g_{AB}(\tau, \sigma^r) = z_A^\mu(\tau, \sigma^r) \eta_{\mu\nu} z_B^\nu(\tau, \sigma^r)$, $z_A^\mu(\tau, \sigma^r) = \frac{\partial z^\mu(\tau, \sigma^r)}{\partial \sigma^A}$, in the non-inertial frame is a function of the embedding obtained from the flat metric $\eta_{\mu\nu}$ in inertial Cartesian 4-coordinates $x^\mu$ by means of a general coordinate transformation $x^\mu \mapsto \sigma^A = (\tau; \sigma^r)$ with inverse transformation $\sigma^A \mapsto x^\mu = z^\mu(\tau, \sigma^r)$.

If we couple the Lagrangian of the isolated system to an external gravitational field, we replace the external gravitational 4-metric with the embedding-dependent 4-metric of a non-inertial frame and we re-express the components of the isolated system in adapted radar 4-coordinates knowing the instantaneous 3-spaces $\Sigma_\tau$. Scalar particles are described with

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1 For a scalar field $\phi(x)$ we get $\phi(\tau, \sigma^r) = \tilde{\phi}(z(\tau, \sigma^r))$. For the electro-magnetic potential $\tilde{A}_\mu(x)$ and field strength $\tilde{F}_{\mu\nu}(x)$ we get the Lorentz-scalar fields $A_A(\tau, \sigma^r) = \tilde{A}_\mu(z(\tau, \sigma^r)) z^\mu_A(\tau, \sigma^r)$, $F_{AB}(\tau, \sigma^r) = \left( \partial_A A_B - \partial_B A_A \right)(\tau, \sigma^r) = \tilde{F}_{\mu\nu}(z(\tau, \sigma^r)) z^\mu_A(\tau, \sigma^r) z^\nu_B(\tau, \sigma^r)$. Differently from $\tilde{\phi}(x)$ and $\tilde{A}_\mu(x)$, the fields $\phi(\tau, \sigma^r)$ and $A_A(\tau, \sigma^r)$ know the whole instantaneous 3-space $\Sigma_\tau$. Scalar particles are described with
parametrized Minkowski theory for the given isolated system. It is a function of the matter and fields of the isolated system (now described as Lorentz-scalar quantities in a non-inertial frame) and of the embedding \( z^\mu(\tau, \sigma) \) of the instantaneous 3-spaces of the non-inertial frame in Minkowski space-time. The main property of the action functional associated with these Lagrangians is the invariance\(^1\)\(^2\) under frame-preserving diffeomorphisms\(^2\) : this implies that the embeddings are \textit{gauge variables}, so that all Møller-admissible clock synchronization conventions (i.e. any definition of instantaneous 3-spaces in space-times with Lorentz signature) are gauge equivalent.

Inertial frames are the special class of frames, connected by the transformations of the Poincare’ group (the relativity principle), selected by the law of inertia. For every configuration of an isolated system there is a special inertial frame intrinsically selected by the system itself, the \textit{rest frame}, whose instantaneous 3-spaces (the Wigner 3-spaces with Wigner covariance) are orthogonal to the conserved 4-momentum of the configuration. This gives rise to the \textit{rest-frame instant form of the dynamics}. In Ref. [8] there is a full account of the rest-frame instant form for arbitrary isolated systems, with special emphasis on the system of ”\(N\) charged positive-energy scalar particles with mutual Coulomb interaction plus the transverse electro-magnetic field of the radiation gauge” [9]. The particles have Grassmann-valued electric charges (each replaced by a two-level system, charge \(+e\) - charge \(-e\), described by a Clifford algebra, after quantization) so that it is possible

a) to make a ultraviolet regularization of the Coulomb self-energies and to eliminate the loops;

b) to make a infrared regularization killing the emission of soft photons;

c) to allow us to have the Lienard-Wiechert transverse potential and electric field expressible as functions only of the 3-positions and 3-momenta of the particles, independently from the chosen Green function (retarded, advanced, symmetric, ..).

This allows us to have a description of the one-photon exchange diagram by means of a potential in the framework of a well defined Cauchy problem for Maxwell equations.

In the rest-frame instant form there are two realizations of the Poincare’ algebra:
1) An *external* one, in which the isolated system is simulated by means of a *decoupled point particle carrying a pole-dipole structure*: the invariant mass $M$ and the rest spin $\vec{S}$ of the isolated system. This decoupled point particle is described by the canonical frozen Jacobi data of the non-covariant external relativistic 3-center of mass: a non-covariant variable $\vec{z} = Mc\vec{x}_{NW}(0)$ is the Cauchy datum of the Newton-Wigner 3-position $\vec{x}_{NW}(\tau)$ and an a-dimensional 3-velocity $\vec{h} = \vec{P}/Mc$, $\{\xi^i, h^j\} = \delta^{ij}$. This universal (i.e. independent from the isolated system) breaking of manifest Lorentz covariance is irrelevant since the 3-center of mass is decoupled from the internal dynamics. Since the Poincare’ generators are global quantities, the relativistic center of mass (a known function of such generators) is a *global quantity* not locally determinable (see Ref.[8] for the non-local aspects of the Newton-Wigner position). The non-covariant canonical external 4-center of mass (or center of spin) $\vec{R}^\mu(\tau) = (\vec{R}^0(\tau); \vec{R}(\tau))$ and the non-covariant non-canonical external Møller 4-center of energy $\vec{R}^\mu(\tau) = (\vec{R}^0(\tau); \vec{R}(\tau))$ are known functions of $\tau$, $\vec{z}$, $\vec{h}$, $M$, $\vec{S}$ given in Ref.[8]. All these collective variables have the same constant 4-velocity: $\dot{\vec{Y}}^\mu(\tau) = \dot{\vec{x}}^\mu(\tau) = \dot{\vec{R}}^\mu(\tau) = P^\mu/Mc = h^\mu$.

The embedding identifying the Wigner 3-spaces is ($\tau = cT$ is the Lorentz-scalar rest time)

$$z^\mu_{W}(\tau, \sigma^\nu) = Y^\mu(\tau) + \epsilon^\mu_\nu(\vec{h}) \sigma^\nu,$$  \hspace{1cm} (1.1)

where $Y^\mu(\tau)$ is the covariant non-canonical Fokker-Pryce external 4-center of inertia and the 3 space-like 4-vectors $\epsilon^\mu_\nu(\vec{h})$ are determined by the standard Wigner boost $L^\mu_\nu(P, \hat{P})$ for time-like orbits sending the rest form $\hat{P}^\mu = Mc(1; \vec{0})$ of the total momentum into $P^\mu = Mc u^\mu(P) = Mc \epsilon^\mu_\nu(\vec{h}) = Mc(\sqrt{1 + h^2}; \vec{h}) = Mc h^\mu$ (we collect here the various notations used in previous papers), i.e. $\epsilon^0_\nu(\vec{h}) = L^\mu_\nu(P, \hat{P})$. We have $\epsilon^0_\nu(\vec{h}) = \sqrt{1 + h^2}$, $\epsilon^i_\nu(\vec{h}) = h^i$, $\epsilon^0_r(\vec{h}) = -e h_r$, $\epsilon^i_r(\vec{h}) = \delta^i_r - e \frac{h^i h_r}{1 + \sqrt{1 + h^2}}$ (see the next Section for the conventions on the 4-metric).

2) A *unfaithful internal* one inside the Wigner 3-spaces, whose generators are determined by the energy-momentum tensor, obtained from the Lagrangian of the parametrized Minkowski theory associated with the given isolated system. The only non-vanishing generators are $M$ and $\vec{S}$. The vanishing of the internal 3-momentum is the *rest-frame condition*, while the vanishing of the internal (interaction-dependent) Lorentz boosts eliminates the internal 3-center of mass (this avoids a double counting of the center of mass). As a consequence, the dynamics inside the instantaneous Wigner 3-spaces is described *only by Wigner-covariant relative variable and momenta* $(\vec{p}_a(\tau), \vec{\pi}_a(\tau), a = 1, \ldots, N-1, \text{for particles})$. The invariant mass $M$ is the Hamiltonian for the internal Hamilton equations. It is possible
to make an orbit reconstruction [4] for the particles in the form \( \vec{\eta}_i(\tau) = \vec{f}_i(\vec{\rho}_a(\tau), \vec{\pi}_a(\tau)) \) and to determine the world-lines \(^3\)

\[
x^\mu_i(\tau) = z^\mu_W(\tau, \vec{\eta}_i(\tau)) = Y^\mu(\tau) + \epsilon^\mu_r(\vec{h}) f^r_i(\vec{\rho}_a(\tau), \vec{\pi}_a(\tau)).
\]

(1.2)

In this paper we study in detail the properties of global admissible non-inertial frames in Minkowski space-time, generalizing the notions defined in the inertial rest-frame instant form of dynamics. We show that also in non-inertial frames every isolated system can be described as an external decoupled non-covariant canonical center of mass (described by frozen Jacobi data) carrying a pole-dipole structure: the invariant mass and an effective spin. Moreover, following the same methods developed for the inertial rest frame, we identify the constraints eliminating the internal 3-center of mass inside the instantaneous 3-spaces.

In the admissible non-inertial frames the instantaneous 3-spaces are orthogonal to a given fixed 4-vector \( l^\mu_{(\infty)} \) at spatial infinity \(^4\).

Then we will restrict the description to the special family of non-inertial frames, in which the instantaneous 3-spaces tend to Wigner 3-spaces, orthogonal to the conserved 4-momentum of the isolated system, at spatial infinity (i.e. \( l^\mu_{(\infty)} = h^\mu = P^\mu/Mc \)): they are the non-inertial rest frames, a non-inertial extension of the inertial ones. This will allow us to define the non-inertial rest-frame instant form of dynamics. The non-inertial rest frame are the only ones allowed by the equivalence principle in the treatment of canonical metric and tetrad gravity in asymptotically flat and globally hyperbolic space-times without super-translations as shown in Refs. [5, 11].

Even if in a non-covariant way, which is however consistent with the coordinate-dependence of the inertial effects, we will give a unified special relativistic description of many properties of isolated systems in accelerated frames, which are scattered in the literature and treated without a global interpretative framework.

Then, as in Ref.[8], we consider the description of the isolated system of positive-energy scalar particles with Grassmann-valued electric charges plus the electro-magnetic field as a parametrized Minkowski theory. As a consequence we obtain both Maxwell equations and their Hamiltonian description in non-inertial frames.

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\(^3\) They turn out to be covariant non-canonical predictive coordinates: \( \{x^\mu_i(\tau), x^\nu_j(\tau)\} \neq 0 \) for all \( i \) and \( j \), \( \mu \) and \( \nu \). Let us remark that this does not imply a breaking of microcausality, which is preserved at the level of the 3-coordinates \( \vec{\eta}_i(\tau) \).

\(^4\) A preliminary description of particles and of their quantization in a class of such frames was given in Ref.[10]. There we introduced an auxiliary decoupled scalar particle whose 4-momentum coincides with \( l^\mu_{(\infty)} \). Here we will avoid to use this method.
By means of a non-covariant decomposition we define the non-inertial non-covariant radiation gauge: this allows to visualize the non-inertial dynamics of transverse electro-magnetic fields, the electro-magnetic Dirac observables. We find the modification of the Coulomb potential in a non-inertial frame: its non-covariance is due to the same type of coordinate-dependence present in the relativistic inertial potentials, which are explicitly identified for the first time (they are the components of the 4-metric $g_{AB}(\tau, \sigma^i)$) and shown to have the correct Newtonian limit. The final Dirac Hamiltonian will contain not only the invariant mass $M c$ but also the modifications induced by the potentials associated with the inertial effects present in the given non-inertial frame.

In a second paper we will study applications of this 3+1 framework to the description of the rotating disk, the Sagnac effect, the Faraday rotation, the wrap-up effect. A preliminary version of the material contained in these two papers is present in arXiv: 0812.3057.

In Section II we review the admissible 3+1 splittings of Minkowski space-time and the properties of the associated global non-inertial frames (Subsection A), we compare them with the accelerated coordinate systems associated with the 1+3 point of view (Subsection B) and we define the non-covariant notations for the electro-magnetic field in non-inertial frames (Subsection C).

In Section III we study the description of the isolated system "charged scalar positive-energy particles plus the electro-magnetic field" in the framework of parametrized Minkowski theories. In particular we show that in non-inertial frames and also in inertial frames with non-Cartesian coordinates there is no true conservation law for the energy-momentum tensor: like in general relativity one could introduce a coordinate-dependent energy-momentum pseudo-tensor describing the contribution of the foliation associated with the admissible 3+1 splitting of Minkowski space-time. However, reverting to inertial frames, it is possible to find the conserved (Poincare' 4-vector) 4-momentum of the isolated system.

In Section IV we give the Hamiltonian description and the Hamilton equations of the isolated system "charged scalar positive-energy particles plus the electro-magnetic field" in admissible non-inertial frames (Subsection A). Then we introduce the non-covariant radiation gauge for the electro-magnetic field and we find both the inertial forces and the non-inertial expression of the Coulomb potential (Subsection B). Finally we evaluate the non-relativistic limit recovering the Newtonian apparent inertial forces (Subsection C).

In Section V we review the determination of the internal Poincare’ generators and of the constraints eliminating the internal 3-center of mass in the inertial rest frames (Subsection A). Then we show how these results are modified in the special family of the non-inertial rest
frames (Subsections B and C) and in arbitrary admissible non-inertial frames (Subsection D).

In the Conclusions we give an overview of the results obtained in this paper and we list the applications to be discussed in the second paper.

In Appendix A there is the expression of the Landau-Lifschitz non-inertial electromagnetic fields in the 3+1 point of view.

In Appendix B there is a comparison of the covariant and non-covariant decompositions of the electro-magnetic field in non-inertial frames and the definition of the non-covariant radiation gauge.
II. ADMISSIBLE 3+1 SPLITTINGS OF MINKOWSKI SPACE-TIME AND NOTATIONS

We use the signature convention \( \eta_{\mu\nu} = \epsilon (+---), \epsilon = \pm 1 \), for the flat Minkowski metric \( (\epsilon = +1 \) is the particle physics convention, while \( \epsilon = -1 \) is the one of general relativity), since it has been used in Refs.[11] for canonical gravity. Since in Ref. [8] the convention \( \epsilon = +1 \) was used, in this Section we also introduce the notations needed for the treatment of the electro-magnetic field in non-inertial frames.

A. Admissible 3+1 Splittings of Minkowski Space-Time

Let us consider an admissible 3+1 splitting of Minkowski space-time, whose instantaneous 3-spaces \( \Sigma_\tau \) are identified by the embedding \( z^\mu(\tau, \sigma^r) \). The radar 4-coordinates \( \sigma^A = (\tau; \sigma^r) \) are adapted to an arbitrary time-like observer with world-line \( x^\mu(\tau) \) in the reference inertial frame, chosen as the origin of the curvilinear 3-coordinates \( \sigma^r \) on each \( \Sigma_\tau \). The Lorentz-scalar time \( \tau \), with dimensions \([\tau] = [ct] = [l]\), is a monotonically increasing function of the proper time of the observer. Therefore, we can put the embeddings in the following form

\[
z^\mu(\tau, \sigma^u) = x^\mu(\tau) + F^\mu(\tau, \sigma^u) = x^\mu_o + \epsilon^\mu_A \left[ f^A(\tau) + F^A(\tau, \sigma^u) \right], \quad F^\mu(\tau, \vec{o}) = 0,
\]

\[
x^\mu(\tau) = x^\mu_o + \epsilon^\mu_A f^A(\tau).
\]

At spatial infinity \( z^\mu(\tau, \sigma^r) \) must tend in a direction-independent way to a space-like hyper-plane with unit time-like normal \( l^\mu(\infty) = \epsilon^\mu_r \); this implies \( F^\mu(\tau, \sigma^s) \to \epsilon^\mu_r \sigma^r \) with the 3 space-like 4-vectors \( \epsilon^\mu_r(\infty) = \epsilon^\mu_r \) orthogonal to \( l^\mu(\infty) \). The asymptotic orthonormal tetrads \( \epsilon_A^\mu \) are associated to asymptotic inertial observers and satisfy \( \epsilon_A^\mu \eta_{\mu\nu} \epsilon_B^\nu = \eta_{AB} \). Let us remark that the natural notation for the asymptotic tetrads would be \( \epsilon^\mu(A) \). However, for the sake of simplicity we shall use the notation \( \epsilon^\mu_A \) for \( \delta^{(B)}_A \epsilon^\mu_B \).

The time-like observer \( x^\mu(\tau) \), origin of the 3-coordinates on the instantaneous 3-spaces \( \Sigma_\tau \), has the following unit 4-velocity and 4-acceleration (we use the notation \( \dot{x}^\mu(\tau) = \frac{d x^\mu(\tau)}{d\tau} \); it must be \( \epsilon \dot{x}^2(\tau) > 0 \))

\[
u^\mu(\tau) = \frac{\dot{x}^\mu(\tau)}{\sqrt{\epsilon \dot{x}^2(\tau)}} = \epsilon^\mu_A u^A(\tau), \quad u^2(\tau) = \epsilon,
\]

\[
u^A(\tau) = \frac{\dot{f}^A(\tau)}{\sqrt{(\dot{f}^\tau(\tau))^2 - \sum_u (\dot{f}^u(\tau))^2}}, \quad (\dot{f}^\tau(\tau))^2 > \sum_u (\dot{f}^u(\tau))^2,
\]
$a^\mu(\tau) = \frac{du^\mu(\tau)}{d\tau} = e^\mu_A a^A(\tau), \quad a_\mu(\tau) u^\mu(\tau) = 0,$

$$a^A(\tau) = \frac{\dot{f}^A(\tau) \left( \left( \dot{f}^\tau(\tau) \right)^2 - \sum_u \left( \dot{f}^u(\tau) \right)^2 \right) - \dot{f}^A(\tau) \left( \dot{f}^\tau(\tau) \dot{f}^\tau(\tau) - \sum_u \dot{f}^u(\tau) \dot{f}^u(\tau) \right)}{\left( \left( \dot{f}^\tau(\tau) \right)^2 - \sum_u \left( \dot{f}^u(\tau) \right)^2 \right)^{3/2}}.$$  

(2.2)

As a consequence we can write $u^\mu(\tau) = L^\mu_\nu(u(\tau), \vec{\sigma}) \vec{\sigma}^\nu$, $\vec{\sigma}^\nu = \epsilon(1; \vec{0})$, by using the standard Wigner boost for time-like 4-vectors.

Eqs.(2.1) imply

$$z^\mu_\tau(\tau, \sigma^u) = \partial_\tau z^\mu(\tau, \sigma^u) = \dot{z}^\mu(\tau) + \partial_\tau F^\mu(\tau, \sigma^u) = e^\mu_A \left( \dot{f}^A(\tau) + \partial_\tau F^A(\tau, \sigma^u) \right) =$$

$$= (1 + n(\tau, \sigma^u)) l^\mu(\tau, \sigma^u) + h^r(\tau, \sigma^u) n_r(\tau, \sigma^u) z_s^\mu(\tau, \sigma^u),$$

$$z^\mu_r(\tau, \sigma^u) = \partial_r z^\mu(\tau, \sigma^u) = \partial_r F^\mu(\tau, \sigma^u) = e^\mu_A \partial_r F^A(\tau, \sigma^u).$$  

(2.3)

While the 3 independent space-like 4-vectors $z^\mu_\tau(\tau, \sigma^u)$ are tangent to $\Sigma_\tau$, the time-like 4-vector $z^\mu_r(\tau, \sigma^u)$ has been decomposed on them and on the unit normal $l^\mu(\tau, \sigma^u), l^2(\tau, \sigma^u) = \epsilon$, to $\Sigma_\tau$ $(l^\mu(\tau, \sigma^u) z^\mu_r(\tau, \sigma^u) = 0)$. This decomposition defines the lapse and shift functions $N(\tau, \sigma^u) = 1 + n(\tau, \sigma^u) > 0$ and $N_\tau(\tau, \sigma^u) = n_\tau(\tau, \sigma^u)$ (we use the notation of Ref.[11]). At spatial infinity we have: $l^\mu(\tau, \sigma^u) \to l^\mu_{(\infty)} = e^\mu_\nu$, $N(\tau, \sigma^u) \to 1 (n(\tau, \sigma^u) \to 0), n_\tau(\tau, \sigma^u) \to 0$.

The 4-metric induced by the 3+1 splitting is $g_{AB}(\tau, \sigma^u) = z^\mu_A(\tau, \sigma^u) \eta_{\mu\nu} z^\nu_B(\tau, \sigma^u)$ and we have

$$g_{\tau\tau}(\tau, \sigma^u) = \left[ z^\mu_\tau \eta_{\mu\nu} z^\nu_\tau \right](\tau, \sigma^u) =$$

$$\epsilon \left[ \left( \dot{f}^\tau(\tau) + \partial_\tau F^\tau(\tau, \sigma^u) \right)^2 - \sum_u \left( \dot{f}^u(\tau) + \partial_\tau F^u(\tau, \sigma^u) \right)^2 \right] =$$

$$\epsilon \left[ \left( 1 + n(\tau, \sigma^u) \right)^2 - h^r(\tau, \sigma^u) n_r(\tau, \sigma^u) n_\tau(\tau, \sigma^u) \right],$$
\[
g_{\tau\tau}(\tau, \sigma^v) = \left[ z^\mu_{\tau} \, \eta_{\mu\nu} \, z^\nu_{\tau} \right](\tau, \sigma^v) = -\epsilon \left[ \sum_u \left( \dot{f}^u(\tau) + \partial_\tau F^u(\tau, \sigma^v) \right) \partial_\tau F^u(\tau, \sigma^v) - \left( \dot{f}^r(\tau) + \partial_\tau F^r(\tau, \sigma^v) \right) \partial_\tau F^r(\tau, \sigma^v) \right] = -\epsilon n_\tau(\tau, \sigma^v) = g_{\tau\tau}(\tau, \sigma^v) \, n_\tau(\tau, \sigma^v) = -\epsilon h_{\tau\tau}(\tau, \sigma^v) \, n_\tau(\tau, \sigma^v),
\]

\[
g_{rs}(\tau, \sigma^v) = \left[ z^\mu_{\tau} \, \eta_{\mu\nu} \, z^\nu_s \right](\tau, \sigma^v) = -\epsilon \left[ \sum_u \partial_\tau F^u(\tau, \sigma^v) \partial_s F^u(\tau, \sigma^v) - \partial_\tau F^r(\tau, \sigma^v) \partial_s F^r(\tau, \sigma^v) \right] = -\epsilon h_{rs}(\tau, \sigma^v).
\]

(2.4)

While the 3-metric \(g_{rs}\) in \(\Sigma_\tau\) and its inverse \(\gamma_{rs}\) (\(\gamma_{ru} g_{us} = \delta^A_s\)) have signature \(\epsilon (- - -)\), the 3-metric \(h_{rs}\) and its inverse \(h_{rs} = -\epsilon \gamma_{rs}\) (\(h_{ru} h_{us} = \delta^A_s\)) have signature \((+++)\).

For the inverse 4-metric \(g^{AB}\) (\(g^{AC} g_{CB} = \delta^A_B\)) we have

\[
g^{\tau\tau} = \frac{\epsilon}{(1 + n)^2}, \quad g^{\tau r} g^{r s} - g^{\tau s} g^{r s} = -\frac{h^{r s}}{(1 + n)^2},
\]

\[
g^{r r} = -\epsilon \frac{n_r}{(1 + n)^2}, \quad g^{r s} = -\epsilon \left( h^{r s} - \frac{n^r n^s}{(1 + n)^2} \right).
\]

(2.5)

For the determinants we have

\[
\gamma = -\epsilon \det g_{rs} = \det h_{rs} > 0, \quad g = \det g_{AB} < 0, \quad \Rightarrow \sqrt{-g} = (1 + n) \sqrt{\gamma}.
\]

(2.6)

Finally the unit normal to the simultaneity surfaces \(\Sigma_\tau\) has the expression

\[
l^\mu(\tau, \sigma^u) = \left[ \eta^\mu_{\alpha\beta\gamma} \, z_1^\alpha \, z_2^\beta \, z_3^\gamma \right](\tau, \sigma^u) = \left[ \frac{1}{\sqrt{\gamma}} \epsilon^\mu_{\alpha\beta\gamma} \, z_1^\alpha \, z_2^\beta \, z_3^\gamma \right](\tau, \sigma^u) = \epsilon_A l^A(\tau, \sigma^v) = \epsilon_A \eta^{AE} \left( \frac{\epsilon_{EBCD}}{\sqrt{\gamma}} \partial_1 F^B \partial_2 F^C \partial_3 F^D \right)(\tau, \sigma^v) = L^\mu(\ell(\tau, \sigma^v), \ell \, \hat{o}^\mu \, o^\nu), \quad \ell = \epsilon (1; \vec{0}),
\]

\[
l^2(\tau, \sigma^u) = \epsilon, \quad \Rightarrow \left( l^\tau(\tau, \sigma^u) \right)^2 > \sum_u \left( l^\mu(\tau, \sigma^u) \right)^2,
\]

\[
\Rightarrow \eta_{\mu\nu} = \epsilon \left( l_\mu l_\nu - z_{\tau\mu} h^{rs} z_{\sigma^v} \right)(\tau, \sigma^v).
\]

(2.7)
The 3+1 splitting for which \( l^\mu \) is constant, i.e. \( \tau \) - and \( \sigma^r \)-independent, have the instantaneous 3-spaces corresponding to parallel space-like hyper-planes: when the frame is non-inertial these hyper-planes are not equally spaced due to linear acceleration and/or have rotating 3-coordinates, so that they are not Euclidean 3-spaces.

The Wigner boost sending \( \tilde{l}^\mu \) into \( l^\mu(\tau, \sigma^u) (\beta^a_i = -\epsilon_i \beta_i^a) \) has the following expression

\[
L^\mu_\nu(l(\tau, \sigma^u), \tilde{l}) = \begin{vmatrix}
\gamma_l & \gamma_l \beta_i^j \\
\gamma_l \beta_i^j & \delta^j_i + (\gamma_l - 1) \sum_k (\delta^j_i)^{k}\beta^k_i \\
\end{vmatrix} (\tau, \sigma^u),
\]

\[
l^\mu(\tau, \sigma^u) = L^\mu_o(l(\tau, \sigma^u), \tilde{l}) = \gamma_l(\tau, \sigma^u) \left(1; \beta_i^j(\tau, \sigma^u)\right) = \epsilon^\mu_A l^A(\tau, \sigma^u) \equiv \epsilon^\mu_o(l(\tau, \sigma^u)),
\]

\[
\epsilon^\mu_j(l(\tau, \sigma^u)) \equiv L^\mu_j(l(\tau, \sigma^u), \tilde{l}),
\]

\[
\gamma_l = \frac{1}{\sqrt{1 - \sum_u (\beta^u_i)^2}} = l^o = \frac{1}{\sqrt{\gamma}} \epsilon_o^A \eta^{AE} \epsilon_{EBCD} \partial_1 F^B \partial_2 F^C \partial_3 F^D,
\]

\[
\beta_i^j = \gamma_l^{-1} l^i = \frac{\epsilon^A \eta^{AE} \epsilon_{EBCD} \partial_1 F^B \partial_2 F^C \partial_3 F^D}{\epsilon^A \eta^{AE} \epsilon_{EBCD} \partial_1 F^B \partial_2 F^C \partial_3 F^D}.
\]  

The orthonormal tetrads \( \epsilon^\mu_A(l(\tau, \sigma^u)) = L^\mu_A(l(\tau, \sigma^u), \tilde{l}), \eta_{\mu\nu} \epsilon^\mu_A(l(\tau, \sigma^u)) \epsilon^\nu_B(l(\tau, \sigma^u)) = \eta_{AB}, \) are the columns of the Wigner boost.

The Wigner boosts \( L^\mu_\nu(u(\tau), \tilde{u}) \) has a similar parametrization in terms of parameters \( \beta^o_i(\tau) \).

The Møller admissibility conditions \[6\], \[3\], implying that the 3+1 splitting gives rise to a nice foliation of Minkowski space-time with space-like leaves identifying the instantaneous 3-spaces \( \Sigma_\tau \), are

\[
\epsilon g_{rr}(\tau, \sigma^u) = [(1 + n)^2 - h^{rs} n_r n_s](\tau, \sigma^u) > 0, \quad \epsilon g_{rr}(\tau, \sigma^u) = -h_{rr}(\tau, \sigma^u) < 0,
\]

\[
\begin{vmatrix}
g_{rr}(\tau, \sigma^u) & g_{rs}(\tau, \sigma^u) \\
g_{sr}(\tau, \sigma^u) & g_{ss}(\tau, \sigma^u)
\end{vmatrix} = \begin{vmatrix} h_{rr}(\tau, \sigma^u) & h_{rs}(\tau, \sigma^u) \\
h_{sr}(\tau, \sigma^u) & h_{ss}(\tau, \sigma^u)
\end{vmatrix} > 0,
\]

\[
\epsilon \det [g_{rs}(\tau, \sigma^u)] = -\gamma(\tau, \sigma^u) < 0, \quad \Rightarrow \det [g_{AB}(\tau, \sigma^u)] < 0.
\]  

(2.9)
They are restrictions on the functions $F^\mu(\tau, \sigma^u)$ of Eqs.(2.1). When they are satisfied, Eqs.(2.1) define a global (in general non-rigid) non-inertial frame. While linear accelerations are not restricted by Eqs.(2.9), rigid rotations are forbidden [3]. The condition $\epsilon g_{\tau\tau}(\tau, \sigma^u) > 0$ implies that in each point $\sigma^u$ the tangential velocity $\omega(\tau, \sigma^u) r(\tau, \sigma^u)$ is less than $c$: instead with $\omega = \omega(\tau)$, like it happens in rigidly rotating coordinate systems, we get $\epsilon g_{\tau\tau}(\tau, R^u) = 0$ at the distance $R^u$ from the rotation axis where $\omega R = c$, so that the time-like vector $z^\mu(\tau, \sigma^u)$ would become a null vector (the so-called horizon problem of the rotating disk).

Since $1 + n(\tau, \sigma^u) > 0$ gives the proper time distance from $\Sigma_\tau$ to $\Sigma_{\tau+dr}$ along the world-line of the Eulerian observer through $(\tau, \sigma^u)$ with tangent vector $l^\mu(\tau, \sigma^u)$, the condition $1 + n(\tau, \sigma^u) > 0$ implies that $\Sigma_\tau$ and $\Sigma_{\tau+dr}$ intersect nowhere. By continuity this implies that the Møller-admissible 3+1 splittings are nice foliations with space-like leaves tending to space-like hyper-planes at spatial infinity in a direction-independent way.

Since the 3-metric $h_{rs}(\tau, \sigma^u)$ is a real symmetric matrix, it can be diagonalized with a rotation matrix $V(\theta^i(\tau, \sigma^u))$, $V^T = V^{-1}$ ($\theta^i(\tau, \sigma^u)$ are Euler angles). Therefore, by using the notations of Ref.[12] for canonical gravity in the York canonical basis, we can parametrize the 3-metric in the following form 5:

---

5 As shown in Ref.[12] the basic variables of tetrad gravity are not the embedding $z^\mu(\tau, \sigma^u)$ but tetrads $E^\mu_{(\alpha)}(\tau, \sigma^u)$, defined after an admissible 3+1 splitting of the space-time identifying the instantaneous 3-spaces $\Sigma_\tau$. The quantities $z^\mu_A(\tau, \sigma^u)$ are now the transition coefficients from world components of tensors to $\Sigma_\tau$-adapted components in radar coordinates $\sigma^A = (\tau, \sigma^u)$: $E^\mu_{(\alpha)} = z^\mu_A E^A_{(\alpha)}$. The 4-metric tensor is defined by the associated cotetrads: $g_{AB} = E^{(\alpha)}_A \eta_{(\alpha)(\beta)} E^{(\beta)}_B$. The gauge variables of tetrad gravity in the York canonical basis are six parameters of the Lorentz group acting on the flat $(\alpha)$ indices of the tetrads $E^\mu_{(\alpha)}$, the lapse $(1 + n)$ and shift $(n_r)$ functions, the Euler angles $\theta^i$ and the momentum variable conjugate to $\phi^6 = \gamma^{1/2}$, i.e. the trace $3K$ of the extrinsic curvature of the instantaneous 3-space $\Sigma_\tau$. The volume variable $\phi = \gamma^{1/12}$ is determined by the super-hamiltonian constraint. The momenta $\pi^A_i(\theta^i)$, conjugate to $\theta^i$, are determined by the super-momentum constraints. The symmetric 3-metric $h_{rs} = -\epsilon g_{rs}$ can be put in the form $h_{rs} = \sum a \lambda_a V_{ra}(\theta^i) V_{sa}(\theta^i)$, where the eigenvalues (assumed non degenerate) have the expression $\lambda_a = \phi^4 e^{2 \sum a \gamma a R a}$. The two functions $R_a$ describe the two physical degrees of freedom of the gravitational field. A gauge fixing for $\theta^i$ and $3K$ implies the determination of the lapse and shift functions.

Instead in non-inertial frames in Minkowski space-time, where gravity is absent, all the functions $(n, n_r, \gamma, \phi^6, \theta^i, R_a)$ parametrizing the components of the 4-metric $g_{AB}$ of Eq.(2.4) are gauge variables globally described by the embedding $z^\mu(\tau, \sigma^u)$ of Eq.(2.1).

In parametrized Minkowski theories (see the next Section), where the embedding is the basic variable, in absence of matter the super-hamiltonian and super-momentum constraints are replaced by the vanishing of the momentum $\rho_{\mu}(\tau, \sigma^u) \approx 0$, see Eq.(3.10), conjugated to $z^\mu(\tau, \sigma^u)$. If we fix $z^\mu(\tau, \sigma^u)$ like in Eq.(4.1), so that the 3-metric is completely fixed $(\theta^i, \gamma$ and $R_a$ are given), then Eqs.(4.2) of Section IV determine the lapse and shift functions. The extrinsic curvature is determined either from the variation of the unit normal $l^\mu$ to $\Sigma_\tau$ or from $3K_{rs} = \frac{1}{2(1+n)} (n_r|_a + n_s|_r - \partial_r h_{rs})$. 

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13
\[ h_{rs}(\tau, \sigma^u) = -\epsilon g_{rs}(\tau, \sigma^u) = \left( \gamma^{1/3} \sum_a Q_a^2 V_{ra}(\theta^i) V_{sa}(\theta^i) \right)(\tau, \sigma^u) = \sum_a e_{(a)r}(\tau, \sigma^u) e_{(a)s}(\tau, \sigma^u), \]

\[ e_{(a)r} = \gamma^{1/6} Q_a V_{ra}(\theta^i), \quad e_{(a)s} = \gamma^{-1/6} Q^{-1}_a V_{sa}(\theta^i), \]

\[ \gamma = \det h_{rs}, \quad Q_a = e \sum_a \gamma_{aa} R_a, \quad (2.10) \]

where \( e_{(a)r}(\tau, \sigma^u) \) and \( e_{(a)s}(\tau, \sigma^u) \), \( \sum_a e_{(a)r} e_{(a)s} = \delta^r_s \), \( \sum_r e'(a)_r e'(b)_r = \delta_{ab} \) are cotriads and triads on \( \Sigma_r \), respectively. At spatial infinity we have \( e'(a)_r(\tau, \sigma^u) \to \delta^r_a, \ e_{(a)r}(\tau, \sigma^u) \to \delta_{ra} \).

To express \( e_{(a)r} \) in terms of \( \partial_r F^A \), we must find the eigenvalues and the eigenvectors of the matrix \( h_{rs} \) in the form given in Eqs.(2.4).

The three eigenvalues of the 3-metric are \( \lambda_a = \gamma^{1/3} Q^2_a > 0 \). The positivity of the eigenvalues is implied by the Møller conditions (2.9): \( \lambda_1 \lambda_2 \lambda_3 = \gamma > 0, \lambda_1 + \lambda_2 + \lambda_3 = h_{11} + h_{22} + h_{33} > 0, \lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = \begin{vmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{vmatrix} + \begin{vmatrix} h_{11} & h_{13} \\ h_{31} & h_{33} \end{vmatrix} + \begin{vmatrix} h_{22} & h_{23} \\ h_{32} & h_{33} \end{vmatrix} > 0. \)

This implies that the three 4-vectors \( z^\mu(\tau, \sigma^u) \) are space-like for every \( \vec{\sigma} \), so that the unit normal \( l^\mu(\tau, \sigma^u) \) is time-like everywhere on the instantaneous 3-spaces.

The Møller condition \( \epsilon g_{r\tau}(\tau, \sigma^u) > 0 \) of Eqs.(2.9) implies that \( z^\mu(\tau, \sigma^u) \) is everywhere time-like on the instantaneous 3-spaces \( \Sigma_r \).

Let us remark that while the generic 3-spaces \( \Sigma_r \) have a 3-metric with 3 distinct eigenvalues, there is a family of 3+1 splittings with two coinciding eigenvalues of \( h_{rs}(\tau, \sigma^u) \) and another family with all the 3 eigenvalues coinciding; they correspond to the existence of symmetries corresponding to the Killing symmetries of Einstein general relativity.

The lapse and shift functions have the following expressions

\[ 1 + n(\tau, \sigma^u) = \epsilon z^\mu(\tau, \sigma^u) l_\mu(\tau, \sigma^u) = \left( \frac{\epsilon}{\sqrt{\gamma}} \epsilon_{\mu \alpha \beta \gamma} z^\alpha z_1^\beta z_2^\gamma \right)(\tau, \sigma^u) = \left( \dot{f}^\tau(\tau) + \partial_\tau F^\tau(\tau, \sigma^u) \right) l^\tau(\tau, \sigma^u) - \sum_u \left( \dot{f}^u(\tau) + \partial_\tau F^u(\tau, \sigma^u) \right) l^u(\tau, \sigma^u) > 0, \]

\[ n_{(r)}(\tau, \sigma^u) = h_{rs}(\tau, \sigma^u) n^s(\tau, \sigma^u) = \sum_u \left( \dot{f}^u(\tau) + \partial_\tau F^u(\tau, \sigma^u) \right) \partial_r F^u(\tau, \sigma^u) - \left( \dot{f}^\tau(\tau) + \partial_\tau F^\tau(\tau, \sigma^u) \right) \partial_r F^\tau(\tau, \sigma^u). \quad (2.11) \]
Let us also remark that all the information carried by $\epsilon^\mu_A f^A(\tau)$, i.e. the velocity and acceleration of the time-like observer $x^\mu(\tau)$, is hidden in the lapse and shift functions.

The extrinsic curvature of the instantaneous 3-space $\Sigma\tau$ can be evaluated by means of the formula $3K_{rs} = \frac{1}{2(1+n)} (n_{rs} + n_{sr} - \partial_r h_{rs}^s)$, by using the Christoffel symbols associated to $h_{rs}$ for the 3-covariant derivatives $n_{rs}$.

In conclusion the relevant conditions on the functions $f^A(\tau)$, $F^A(\tau, \sigma^u)$ of an admissible 3+1 splitting of Minkowski space-time are $\epsilon \dot{x}^2(\tau) > 0$, $1 + n(\tau, \sigma^u) > 0$, $\epsilon g_{\tau\tau}(\tau, \sigma^u) > 0$ and $\lambda_a(\tau, \sigma^u) > 0$.

Finally Eq.(2.10) suggests that it must be $z^\mu_A(\tau, \sigma^u) = \Lambda^\mu_a(\tau, \sigma^u) \epsilon^a_{(a)}(\tau, \sigma^u)$, where $\Lambda(\tau, \sigma^u)$ is some Lorentz matrix, so that $-\epsilon g_{rs} = \epsilon \eta_{\mu\nu} \Lambda^\mu_a \Lambda^\nu_b \epsilon^a_{(a)} \epsilon^b_{(b)} = -\epsilon \eta_{ab} \epsilon_{(a)} \epsilon_{(b)} = h_{rs}$.

To find $\Lambda(\tau, \sigma^u)$ let us remember that in tetrad gravity in the York canonical basis (see Ref.[12]) the expression of the tetrads adapted to $\Sigma\tau$ (Schwinger time gauge) in terms of the unit normal $l^A$ and of the triads $e^a_{(a)}$ are $\hat{E}_{(a)} = l^A$, $\hat{E}^A_{(a)} = (0; e^r_{(a)})$. In terms of them we have $V^A = (1 + n) \hat{E}_{(a)} + e^r_{(a)} n_r \hat{E}_{(a)} = (1; 0)^A$. The world components of this vector are $\hat{V}^\mu = z^\mu_A V^A = z^\mu_A$, while those of $\hat{E}_{(a)}$ are $\hat{E}_{(a)} = z^\mu_A \hat{E}_{(a)} = z^\mu_A e^r_{(a)}$, so that we get $z^\mu_A = e_{(a)} e_{(a)} \hat{E}_{(a)}$. For the unit normal we have $l^\mu = z^\mu_A l^A$.

In Minkowski space-time our parametrization of the embedding uses the asymptotic tetrads $e^\mu_A$ and we have $z^\mu_A = e^\mu_B \partial_A F_B$ and $l^\mu = e^\mu_A l^A = e^\mu_a(l)$. Therefore a set of tetrads adapted to $\Sigma\tau$ in the point $(\tau, \sigma^u)$ is given by the orthonormal tetrads $e^\mu_A(l(\tau, \sigma^u))$ defined in Eqs.(2.8): they replace the adapted tetrads $l^\mu$, $E_{(a)}$ of tetrad gravity. Therefore, consistently with Eq.(2.10), we must have

$$z^\mu_{(a)}(\tau, \sigma^u) = e^\mu_B \partial_A F^A(\tau, \sigma^u) = e^\mu_a(l(l(\tau, \sigma^u))) e_{(a)}(\tau, \sigma^u).$$

(2.12)

This implies $z^\mu_{(a)} = [1 + n] l^A + e^s_{(a)} n_s e^\mu_a(l) (\tau, \sigma^u) = L^\mu_{\nu}(l(l(\tau, \sigma^u) \hat{A}) G^\mu(\tau, \sigma^u)$ with $G^\mu = (1 + n) e^s_{(a)} n_s)$. Eqs.(2.12) are a set of non-linear partial differential equations for $\partial_r F^A(\tau, \sigma^u)$.

It is difficult to construct explicit examples of admissible 3+1 splittings. Let us consider the following two examples in which the instantaneous 3-spaces are space-like hyper-planes.

A) Rigid non-inertial reference frames with translational acceleration. An example are the following embeddings
\[ z^\mu(\tau, \sigma^u) = x_\mu^0 + \epsilon^\mu r f(\tau) + \epsilon^\mu \sigma^r, \]

\[ g_{\tau\tau}(\tau, \sigma^u) = \epsilon \left( \frac{df(\tau)}{d\tau} \right)^2, \quad g_{\tau r}(\tau, \sigma^u) = 0, \quad g_{r s}(\tau, \sigma^u) = -\epsilon \delta_{r s}. \quad (2.13) \]

This is a foliation with parallel hyper-planes with normal \( l^\mu = \epsilon^\mu r = \text{const.} \) and with the time-like observer \( x^\mu(\tau) = x_\mu^0 + \epsilon^\mu r f(\tau) \) as origin of the 3-coordinates. The hyper-planes have translational acceleration \( \ddot{x}^\mu(\tau) = \epsilon^\mu r \ddot{f}(\tau) \), so that they are not uniformly distributed like in the inertial case \( f(\tau) = \tau \).

B) As shown in Refs. [3], the simplest example of 3+1 splitting, whose instantaneous 3-spaces are space-like hyper-planes carrying admissible differentially rotating 3-coordinates, is given by the embedding \( (\sigma = |\vec{\sigma}|; \epsilon^\mu r \) are the asymptotic space-like axes and the unit normal is \( l^\mu = \epsilon^\mu r = \text{const.}; \alpha_i(\tau, \vec{\sigma}) = F(\sigma) \dot{\alpha}_i(\tau), i = 1, 2, 3, \) are Euler angles; \( R^r_s(\alpha_i(\tau, \sigma)) \) is a rotation matrix satisfying the asymptotic conditions \( R^r_s(\tau, \sigma) \to_{\sigma \to \infty} \delta^r_s, \partial_A R^r_s(\tau, \sigma) \to_{\sigma \to \infty} 0) \)

\[ \Omega^r = \Omega^r(\tau), \quad \Omega^r = \Omega^r(\tau), \quad \ddot{x}^\mu(\tau) \epsilon^\mu r R^r_s(\tau)(\dot{\sigma} \times \dot{\Omega}(\tau)) + \sqrt{\dot{x}^2(\tau) + \dot{x}_\mu(\tau) \epsilon^\mu r R^r_s(\tau)(\dot{\sigma} \times \dot{\Omega}(\tau))^2}. \]

We use the notations \( \sigma^u = \sigma \dot{\sigma}^u, \quad \Omega^r = \Omega^r(\tau), \quad \ddot{\sigma}^2 = \dot{\Omega}^2 = 1. \) At this distance from the rotation axis the tangential rotational velocity becomes equal to the velocity of light. This is the horizon problem of the rotating disk. This pathology is common to most of the rotating coordinate systems quoted after Eq.(2.16) and in Appendices A (see also th first Section of the next paper on the rotating disk). Let us remark that an analogous pathology happens on the event horizon of the Schwarzschild black hole, where the time-like Killing vector of the static space-time becomes light-like: in this case we do not have a coordinate singularity but an intrinsic geometric property of the solution of Einstein’s equations. For the rotating Kerr black hole the same phenomenon happens already at the boundary of the ergosphere [13], as a consequence of the Killing vectors own by this solution. Let us remark that in the existing theory of rotating relativistic stars [14], where differential rotations are replacing the rigid ones in model building, it is assumed that in certain rotation regimes an ergosphere may form [15]: however in this case it is not known whether Killing vectors and a dynamical ergosphere exist, so that the horizon problem, arising if one uses 4-coordinates adapted to the Killing vectors, could be associated to a coordinate singularity like for the rotating disk. In the study of the magnetosphere of pulsars the horizon of the rotating disk is named the light cylinder (see Appendix A).
\[ z^\mu(\tau, \sigma^u) = x^\mu(\tau) + \epsilon_\nu^\mu R^\nu_\mu(\tau, \sigma) \sigma^u, \quad x^\mu(\tau) = x^\mu_0 + f^A(\tau) e^\mu_A, \]

\[ R^\nu_\mu(\tau, \sigma) = R^\nu_\mu(\sigma_i(\tau, \sigma)) = R^\nu_\mu(F(\sigma) \tilde{\alpha}_i(\tau)), \]

\[ 0 < F(\sigma) < \frac{1}{A \sigma}, \quad \frac{d F(\sigma)}{d \sigma} \neq 0 \text{ (Moller conditions)}, \]

\[ z^\mu(\tau, \sigma^u) = \dot{x}^\mu(\tau) - \epsilon_\nu^\mu R^\nu_\mu(\tau, \sigma) \delta^{sw} \epsilon_{wuv} \sigma^u \frac{\Omega^v(\tau, \sigma)}{c}, \]

\[ z^\mu_r(\tau, \sigma^u) = \epsilon_k^\mu R^k_v(\tau, \sigma) \left( \delta^v_r + \Omega^v_{(r)u}(\tau, \sigma) \sigma^u \right), \]

\[ \epsilon g_{\tau\tau}(\tau, \sigma^u) = \epsilon \dot{x}^2(\tau) - 2 \epsilon \dot{x}_\mu(\tau) \epsilon_\nu^\mu R^\nu_\mu(\tau, \sigma) \delta^{sw} \epsilon_{wuv} \sigma^u \frac{\Omega^v(\tau, \sigma)}{c} - \]

\[ - \frac{1}{c^2} \sum_k \epsilon_{krs} \sigma^r \Omega^s(\tau, \sigma) \epsilon_{kuv} \sigma^u \Omega^v(\tau, \sigma), \]

\[ n_r(\tau, \sigma) = - \epsilon g_{\tau\tau}(\tau, \sigma) = - \epsilon \dot{x}_\mu(\tau) \epsilon_\nu^\mu R^k_v(\tau, \sigma) \left( \delta^v_r + \Omega^v_{(r)u}(\tau, \sigma) \sigma^u \right) - \]

\[ - \epsilon_{smn} \sigma^m \frac{\Omega^n(\tau, \sigma)}{c} \left( \delta^r_s + \Omega^s_{(r)u}(\tau, \sigma) \sigma^u \right), \]

\[ h_{rs}(\tau, \sigma^u) = - \epsilon g_{rs}(\tau, \sigma^u) = \delta_{rs} + \left( \Omega^m_{(s)u}(\tau, \sigma) + \Omega^m_{(r)u}(\tau, \sigma) \right) \sigma^u + \]

\[ + \sum_w \Omega^w_{(s)u}(\tau, \sigma) \Omega^w_{(r)u}(\tau, \sigma) \sigma^u \sigma^v, \quad (2.14) \]

where \( \left(R^{-1}(\tau, \sigma) \partial_r R(\tau, \sigma) \right)^u_v = \delta^{um} \epsilon_{mvr} \frac{\Omega^r(\tau, \sigma)}{c}, \partial_r R(\tau, \sigma)^u_v = R^u_n(\tau, \sigma) \delta^{nm} \epsilon_{mvr} \frac{\Omega^r(\tau, \sigma)}{c} \)

with \( \Omega^r(\tau, \sigma) = F(\sigma) \tilde{\Omega}(\tau, \sigma) \tilde{n}^r(\tau, \sigma) \) \( \tilde{n}^r(\tau, \sigma) \) being the angular velocity and with \( \Omega^r_{(r)u}(\tau, \sigma) = R^{-1}(\tau, \sigma) \partial_r R(\tau, \sigma) \). The angular velocity vanishes at spatial infinity and has an upper bound proportional to the minimum of the linear velocity \( v_i(\tau) = \dot{x}_\mu l^\mu \) orthogonal to the space-like hyper-planes. When the rotation axis is fixed and \( \tilde{\Omega}(\tau, \sigma) = \omega = \text{const.} \), a simple choice for the function \( F(\sigma) \) is \( F(\sigma) = \frac{1}{1 + \frac{\omega^2 c^2}{v^2}} \) \( \tilde{n}^r(\tau, \sigma) \) defines the instantaneous rotation axis and \( 0 < \tilde{\Omega}(\tau, \sigma) < 2 \max \left( \dot{\alpha}(\tau), \dot{\beta}(\tau), \dot{\gamma}(\tau) \right) \).

Let us remark that the unit normal is \( l^\mu(\tau, \sigma^u) = \epsilon^\mu_r = \text{const.} \) and the lapse function is \( 1 + n(\tau, \sigma^u) = \epsilon \left( z^\mu_r l^\mu_r \right)(\tau, \sigma^u) = \epsilon \epsilon^\mu_r \dot{x}_\mu(\tau) \).

7 \( \tilde{n}^r(\tau, \sigma) \) defines the instantaneous rotation axis and \( 0 < \tilde{\Omega}(\tau, \sigma) < 2 \max \left( \dot{\alpha}(\tau), \dot{\beta}(\tau), \dot{\gamma}(\tau) \right) \).

8 Nearly rigid rotating systems, like a rotating disk of radius \( \sigma_o \), can be described by using a function \( F(\sigma) \) approximating the step function \( \theta(\sigma - \sigma_o) \).
The embedding (2.14) has been used in the first paper of Ref.[10], on quantum mechanics in non-inertial frames, in the form \( z^\mu(\tau, \sigma^u) = x^\mu(\tau) + F^\mu(\tau, \sigma^u) = \theta(\tau) \epsilon^\mu_r + \mathcal{A}^\tau(\tau, \sigma^u) \epsilon^\mu_r \) with \( x^\mu_r = 0 \), \( \theta(\tau) = f^\tau(\tau) \), \( \mathcal{A}^\tau(\tau, \sigma^u) = f^\tau(\tau) + R^s_r(\tau, \sigma) \sigma^s \), describing the freedom in the choice of the mathematical time \( \tau \) and with the world-line of the time-like observer having the expression \( x^\mu(\tau) = \epsilon^\mu_r \theta(\tau) + \epsilon^\mu_r \mathcal{A}^\tau(\tau, 0) \), namely with \( f^\tau(\tau) = \mathcal{A}^\tau(\tau, 0) \) and \( \dot{f}^\tau(\tau) = \frac{w^\tau(\tau)}{c} \) \((\dot{w}(\tau) \) is the ordinary 3-velocity). If we choose \( \theta(\tau) = \tau \), we get from Eq. (2.2) \( u^\mu(\tau) = \epsilon^\mu_A u^A(\tau) = \frac{\epsilon^\mu_A + \epsilon^\mu_r \frac{w^\tau(\tau)}{c^2}}{\sqrt{1 - \frac{\omega^2(v^s)\sigma^2}{c^4}}} \), \( a^\mu(\tau) = \epsilon^\mu_A u^A(\tau) = \frac{1}{c^2} \sum_n \dot{w}^\mu(\tau) \frac{w^\mu(\tau)}{c^2} \left(1 - \frac{w^2(\tau)}{c^2}\right)^{-3/2} \left(\epsilon^\mu_r + \epsilon^\mu_r \frac{w^\tau(\tau)}{c}\right) \). The lapse function is \( 1 + n(\tau) = \dot{f}^\tau(\tau) \).

To evaluate the non-relativistic limit for \( c \to \infty \), where \( \tau = ct \) with \( t \) the absolute Newtonian time and \( \partial_r = \frac{1}{c} \partial_t \), we choose the gauge function \( F(\sigma) = \frac{1}{1 + \frac{\omega^2(v^s)\sigma^2}{c^4}} \to c \to \infty 1 - \frac{\omega^2\sigma^2}{c^4} + O(c^{-4}) \). This implies

\[
R^a_r(\tau, \sigma) \to c^{-\infty} R^a_r(\tau) - \frac{\omega^2\sigma^2}{c^2} \sum_i \frac{\partial R^a_r(\tau, \sigma)}{\partial \dot{\alpha}_i} |_{F(\sigma) = 1} + O(c^{-4}) =
\]

\[
def R^a_r(\tau) - \frac{\omega^2\sigma^2}{c^2} R^{(1)a}_r(\tau) + O(c^{-4}),
\]

(2.15)

and we can introduce a new 3-velocity \( \vec{v}(\tau) \) by means of \( w^\tau(\tau) = c \vec{f}^\tau(\tau) = R^s_r(\tau) v^s(\tau) \). We have \( \Omega^s_r(\tau, \sigma) = \tilde{\Omega}(\tau) \dot{n}^r(\tau) + O(c^{-1}) \) for the angular velocity and \( \Omega(\tau, \sigma) = 0 + O(c^{-2}) \).

Therefore the corrections to rigidly-rotating non-inertial frames coming from Møller conditions are of order \( O(c^{-2}) \) and become important at the distance from the rotation axis where the horizon problem for rigid rotations appears.

Then, from Eqs. (2.14), (2.4), (2.7) and (2.11) we get

\[
z^\mu(\tau, \sigma^u) \to x^\mu(\tau) + \epsilon^\mu_r R^r_s(\tau) \sigma^s - \frac{\omega^2\sigma^2}{c^2} \epsilon^\mu_r R^{(1)r}_s(\tau) \sigma^s + O(c^{-4}),
\]

\[
z^\mu_r(\tau, \sigma^u) \to \dot{x}^\mu(\tau) + \epsilon^\mu_r \partial_r R^r_s(\tau) \sigma^s + O(c^{-3}) =
\]

\[
= \epsilon^\mu_r + \epsilon^\mu_r \dot{f}^\tau(\tau) + \frac{1}{c} \epsilon^\mu_r R^r_s(\tau) \epsilon_{su} \Omega^u(\tau) \sigma^v + O(c^{-3}),
\]

\[
z^\mu_r(\tau, \sigma^u) \to \epsilon^\mu_r \left[R^r_s(\tau) - \frac{\omega^2}{c^2} R^{(1)r}_u(\tau) \left(\delta^u_r \sigma^2 + 2 \sigma^u \sigma^v \delta_{vr}\right)\right] + O(c^{-4}),
\]

\[
h_{rs}(\tau, \sigma^u) \to \delta_{rs} - 2 \frac{\omega^2}{c^2} \sum_u R^u_r(\tau) R^{(1)u}_v(\tau) \left(\delta^v_r + 2 \sigma^v \sigma^n \delta_{rs}\right) + O(c^{-4}),
\]

\[
n(\tau) = 0, \quad n_s(\tau, \sigma^u) \to \frac{1}{c} \left(\delta_{rs} v^s(\tau) + \epsilon_{rus} \Omega^u(\tau) \sigma^v\right) + O(c^{-3}).
\]

(2.16)
There is the enormous amount of bibliography, reviewed in Ref.[16], about the problems of the rotating disk and of the rotating coordinate systems. Independently from the fact whether the disk is a material extended object or a geometrical congruence of time-like world-lines (integral lines of some time-like unit vector field), the idea followed by many researchers [6, 17, 18] (in Refs.[18] are quoted the attempts to develop electro-dynamics in rotating frames) is to start from the Cartesian 4-coordinates of a given inertial system, to pass to cylindrical 3-coordinates and then to make a either Galilean (assuming a non-relativistic behaviour of rotations at the relativistic level) or Lorentz transformation to comoving rotating 4-coordinates (see the locality hypothesis in the next Subsection), with a subsequent evaluation of the 4-metric in the new coordinates. In other cases [19] a suitable global 4-coordinate transformation is postulated, which avoids the horizon problem. Various authors (see for instance Refs.[20]) do not define a coordinate transformation but only a rotating 4-metric. Just starting from Møller rotating 4-metric [6], Nelson (see the second paper in Ref.[13]) was able to deduce a 4-coordinate transformation implying it.

See the first Section of the second paper for the description of the rotating disk and of the Sagnac effect in the 3+1 framework.

B. Congruences of Time-Like Observers Associated with an Admissible 3+1 Splitting, the 1+3 Point of View and the Locality Hypothesis

Each admissible 3+1 splitting of Minkowski space-time, having the time-like observer \( x^\mu(\tau) \) as origin of the 3-coordinates on the instantaneous 3-spaces \( \Sigma_\tau \), automatically determines two time-like vector fields and therefore two congruences of (in general) non-inertial time-like observers:

i) The time-like vector field \( l^\mu(\tau, \sigma^u) \partial_\mu \) of the normals to the simultaneity surfaces \( \Sigma_\tau \) (by construction surface-forming, i.e. irrotational), whose flux lines are the world-lines \( x^\mu_{l,\tau_0,\sigma^u}(\tau) \), \( u^\mu(\tau) = \frac{x^\mu_{l,\tau_0,\sigma^u}(\tau)}{\sqrt{\epsilon_{l,\tau_0,\sigma^u}(\tau)}} \), \( u^\mu_{l,\tau_0,\sigma^u}(\tau_0) = l^\mu(\tau_0, \sigma^u_0) \), of the so-called (in general non-inertial) Eulerian observers. The simultaneity surfaces \( \Sigma_\tau \) are (in general non-flat) Riemannian 3-spaces in which every physical system is visualized and in each point the tangent space to \( \Sigma_\tau \) is the local observer rest frame of the Eulerian observer through that point. The 3+1 viewpoint of these observers is called hyper-surface 3+1 splitting.

ii) The time-like evolution vector field \( z^\mu(\tau, \sigma) \partial_\mu \) of the normals to the simultaneity surfaces \( \Sigma_\tau \) (by construction surface-forming, i.e. it has non-zero vorticity like in the case of the rotating disk). The observers associated to its flux lines \( x^\mu_{z,\tau_0,\sigma^u}(\tau) = z^\mu(\tau, \sigma^u_0) \), \( u^\mu_{z,\sigma^u}(\tau) = \frac{x^\mu_{z,\tau_0,\sigma^u}(\tau)}{\sqrt{\epsilon_{g_{zz}(\tau, \sigma)}}} \), have the local observer rest frames, the tangent 3-spaces orthogonal to the evolution vector field, not tangent to \( \Sigma_\tau \): there is no notion of 3-space for these observers (1+3 point of view or threading splitting) and
no visualization of the physical system in large. However these observers can use the notion of simultaneity associated to the embedding \( x^\mu(\tau, \vec{\sigma}) \), which determines their 4-velocity. Like for the observer \( x^\mu(\tau) \), their 4-velocity is not parallel to \( l^\mu(\tau, \sigma^\nu) \). The 3+1 viewpoint of these observers is called \textit{slicing 3+1 splitting}.

Every 1+3 point of view considers only a time-like observer (either \( x^\mu(\tau) \) or \( x^\mu_{\ell, \tau, \sigma^\nu}(\tau) \) or \( x^\mu_{\vec{z}, \sigma^\nu}(\tau) \)) and tries to give a description of the physics in a region around the observer’s world-line assumed known. Since there is no global notion of simultaneity, namely of instantaneous 3-space, one identifies the space-like hyper-planes orthogonal to the observer unit 4-velocity \( u^\mu_{\text{obs}}(\tau) \) at every instant \( \tau \) (the observer local rest frames) as local instantaneous 3-spaces \( \Sigma_{\text{obs} \tau} \) (strictly speaking it is a tangent space and not a 3-space). Then one makes a choice of a tetrad \( V^\mu_{\text{obs} A}(\tau) = \left( u^\mu_{\text{obs}}(\tau); V^\mu_{\text{obs} A}(\tau) \right) \), \( \eta_{\mu\nu} V^\mu_{\text{obs} A}(\tau) V^\nu_{\text{obs} B}(\tau) = \eta_{AB} \). The space axes \( V^\mu_{\text{obs} A}(\tau) \) can be chosen arbitrarily, even if often they are chosen as the tangents to three space-like geodesics on \( \Sigma_{\text{obs} \tau} \) at the observer position. After parallel transport of the tetrad to the points of \( \Sigma_{\text{obs} \tau} \) not on the observer world-line one tries to build an \textit{accelerated 4-coordinate system} having the observer as origin of the 3-coordinates [21]. In the case of the tangents to space-like geodesics one builds a local system of Fermi coordinates around the observer world-line [22] (see also Ref.[23] for an updated discussion of Fermi-Walker and Fermi normal coordinates).

The drawback of this construction is that the \( \tau \)-dependent family of hyper-planes \( \Sigma_{\text{obs} \tau} \) will have hyper-planes at different \( \tau \)’s \textit{intersecting} at some distance from the observer world-line, usually estimated by using the so-called \textit{acceleration radii} of the observer. This implies that every system of accelerated 4-coordinates of this type will develop \textit{coordinate singularities} when the hyper-planes intersect. As a consequence it is not possible to formulate a well-posed Cauchy problem for Maxwell equations in these accelerated coordinate systems: they can only be used for evaluating local semi-relativistic inertial effects.

At each instant \( \tau \) the tetrads \( V^\mu_{\text{obs} A}(\tau) \) coincide with some Lorentz matrix \( V^\mu_{\text{obs} A}(\tau) = \Lambda^\mu_{\nu=A}(\tau) \), which connects the reference inertial frame to the \textit{instantaneous comoving inertial frame} associated with the accelerated observer at \( \tau \). A possibility is to use the tetrads \( e^\mu_A(u_{\text{obs}}(\tau)) \) associated with the Wigner boost \( L^\mu_\nu(u_{\text{obs}}(\tau), \vec{u}_{\text{obs}}) \). This fact is at the heart of the \textit{locality hypothesis} [24] according to which an accelerated observer is physically equivalent (for measurements) to a continuous family of hypothetical momentarily comoving inertial observers.

If we parametrize the Lorentz transformation \( \Lambda(\tau) \) as the product of a pure boost with a pure rotation \( \Lambda(\tau) = B(\vec{\beta}(\tau)) R(\alpha(\tau), \beta(\tau), \gamma(\tau)) \) and we call \( R^\nu_s(\tau) = R^\nu_{\text{obs}}(\tau) \), we can write (from Eq.(2.8) we have \( B^{jk}(\vec{\beta}(\tau)) = \delta^{jk} + (\gamma(\tau) - 1) \frac{\beta^{jk}(\tau) - \delta^{jk}(\tau)}{\sqrt{(\beta^\mu(\tau) \beta_\mu(\tau))}} \))
Let us define the angular velocity \( \omega_r(\tau) \) by means of

\[
\frac{dR_r(\tau)}{d\tau} \overset{\text{def}}{=} \epsilon_{uvw} \omega_u(\tau) R_v(\tau).
\]

Even if the observer is connected with the embedding \( z^\mu(\tau, \vec{\sigma}) \), this angular velocity is not related to the angular velocity defined after Eq.(2.14).

Finally, if we write

\[
\frac{dV_\mu^{\text{obs}}(A)(\tau)}{d\tau} = A_{\text{obs}}(A)(B)(\tau) V_\mu^{\text{obs}}(B)(\tau),
\]

\[
\Rightarrow A_{\text{obs}}(A)(B)(\tau) = -A_{\text{obs}}(B)(A)(\tau) = \frac{dV_\mu^{\text{obs}}(A)(\tau)}{d\tau} \eta_{\mu\nu} V_\nu^{\text{obs}}(B)(\tau),
\]

and we introduce the definitions \( a_{\text{obs} r}(\tau) = A_{\text{obs}}(r)(\tau) \), \( \Omega_{\text{obs} r}(\tau) = \frac{1}{2} \epsilon_{uvw} A_{\text{obs} (u)(v)}(\tau) \), then the acceleration radii have the following definition [24]: \( I_1(\tau) = \sum_r \left( \Omega_{\text{obs} r}^2(\tau) - a_{\text{obs} r}^2(\tau) \right) \), \( I_2(\tau) = \sum_r a_{\text{obs} r}(\tau) \Omega_{\text{obs} r}(\tau) \). By means of Eq.(2.17) they can be expressed in terms of the parameters of the Lorentz transformation and their \( \tau \)-derivatives.

Let us remark that, since each instantaneous 3-space \( \Sigma_\tau \) centered on an accelerated observer is in general a non-flat Riemannian 3-manifold with 3-metric \( g_{rs}(\tau, \sigma^u) \) and Riemann 3-curvature tensor \( R_{rsu}^m(\tau, \sigma^u) \), we can look for special 3-coordinate systems on \( \Sigma_\tau \) such that the curvature effects are second order near the observer, mimicking what is done in general relativity [21, 22, 23] to define local inertial frames and to visualize inertial effects. On \( \sigma_\tau \) around the observer, origin of the 3-coordinates and whose world-line (in the flat Minkowski space-time) is a 4-geodesics if the observer is inertial, we can introduce:a) Riemann 3-coordinates such that \( \partial_v g_{rs}(\tau, \sigma^u)|_{\sigma^u=0} = 0 \); b) Riemann normal 3-coordinates (the three coordinate lines are 3-geodesics of \( \Sigma_\tau \)) such that \( \partial_m \partial_n g_{rs}(\tau, \sigma^u)|_{\sigma^u=0} = 0 \) and \( \partial_m \Gamma_{uv}^r(\tau, \sigma^u)|_{\sigma^u=0} \) are proportional to suitable combinations of the 3-curvature tensor.

In this way it is possible to simplify the expression of the special relativistic effects in non-inertial frames near the observer as first order corrections depending on the observer acceleration.

Finally let us remark that given an admissible 3+1 splitting of Minkowski space-time, the infinitesimal spatial length \( dl \) in the instantaneous 3-spaces \( \Sigma_\tau \) is defined by putting \( d\tau = 0 \) in the line element \( ds^2 = g_{AB}(\tau, \sigma^u) d\sigma^A d\sigma^B \), namely we have \( dl^2 = g_{rs}(\tau, \sigma^u) d\sigma^r d\sigma^s \).
This global, but coordinate-dependent, definition has to be contrasted with the local, but coordinate-independent, definition used in the $1+3$ point of view as it is done for instance in Landau-Lifschitz [17]. This definition is only locally valid in the local rest frame of an observer: since there is no notion of instantaneous 3-space it cannot be used in a global way. For a detailed comparison of these two notions of spatial length see Section II of the first paper of Ref.[3].

C. Notations for the Electro-Magnetic Field in Non-Inertial Frames

Let us add some notations for the electro-magnetic field in the non-inertial frames, where the instantaneous 3-space is either curved or flat but with rotating coordinates [in both cases it is not Euclidean and has the 3-metric $h_{rs}$ of signature $(+++)$].

The basic field is the electro-magnetic potential $A_A = (A^r; A^s)$. We have $A^A = (A^r; A^s) = g^{AB} A_B = g^{Ar} A_r + g^{As} A_s$. Instead in inertial frames we have $A^r = \epsilon A_r$, $A^s = -\epsilon A_s$.

In non-inertial frames it is convenient to introduce the following "Euclidean" notation: $\tilde{A}^r = h^{rs} A_s \neq A^r$ (in inertial frames: $\tilde{A}^r = A_r = -\epsilon A^r$)

We shall adopt the following conventions for the electric and magnetic fields in terms of $F_{AB} = \partial_A A_B - \partial_B A_A$ $^9$:

a) In inertial frames we have $^10$

$$E_r = -F_{rr} = F^{rr} = \tilde{E}_r,$$

$$B_r = \frac{1}{2} \epsilon_{r uv} F_{uv} = \frac{1}{2} \epsilon_{r uv} F^{uv} = \tilde{B}_r, \quad F_{uv} = F^{uv} = \epsilon_{uvr} B_r = \epsilon_{uvr} \tilde{B}_r. \quad (2.19)$$

b) In non-inertial frames we put the definitions

$^9$ In the inertial case, where $h_{rs} = \delta_{rs}$ implies $V^r \equiv \tilde{V}^r = V_r$ for the components of 3-vector $\tilde{V}$ not being the vector part of a 4-vector (like $\tilde{E}$ and $\tilde{B}$), we can use the vector notation $\tilde{E} = \{E_r\} = \{\tilde{E}_r\}$, $\tilde{B} = \{B_r\} = \{\tilde{B}_r\}$, $\tilde{E}^2 = \sum_r \tilde{E}_r^2 = \sum_r (\tilde{E}_r^2) = \sum_r (\tilde{B}_r^2) = \sum_r (\tilde{B}_r^2)$, $\tilde{E} \times \tilde{B}_r = \sum_u \epsilon_{ruv} \tilde{E}_u \tilde{B}_v$. Since $\tilde{V}^r = h^{rs} V_s \neq V^r$, we are not going to use the vector notation in non-inertial frames.

$^10$ $\epsilon_{uvr}$ is the Euclidean Levi-Civita tensor with $\epsilon_{123} = 1$; $\epsilon_{uvr}$ is never introduced.
\[ E_r \overset{\text{def}}{=} - F_{rr}, \quad B_r \overset{\text{def}}{=} \frac{1}{2} \epsilon_{rur} F_{uv}, \quad F_{rs} = \epsilon_{rsu} B_u. \quad (2.20) \]

Since we have

\[ F^{AB} = g^{AC} g^{BD} F_{CD} = (g^{Ar} g^{Br} - g^{Ar} g^{Br}) F_{rr} + g^{Ar} g^{Bs} F_{rs} = \]

\[ = (g^{Ar} g^{Br} - g^{Ar} g^{Br}) E_r + \epsilon_{rsu} g^{Ar} g^{Bs} B_u, \]

\[ F^{ru} = (g^{\tau r} g^{\tau u} - g^{\tau r} g^{\tau u}) E_r + \epsilon_{rsu} g^{\tau r} g^{us} B_u = \]

\[ = h^{ur} E_r + \frac{1}{(1 + n)^2} \epsilon_{rsu} n^r h^{us} B_u, \]

\[ F^{uv} = (g^{\tau u} g^{\tau v} - g^{\tau u} g^{\tau v}) E_r + \epsilon_{rsu} g^{\tau u} g^{vs} B_u = \]

\[ = \frac{(h^{ur} n^v - h^{vr} n^u)}{(1 + n)^2} E_r + \epsilon_{rsu} \left( h^{ur} h^{vs} - \frac{n^r (n^v h^{us} - n^u h^{us})}{(1 + n)^2} \right) B_u, \quad (2.21) \]

by analogy with inertial frames we can put

\[ F^{rr} \overset{\text{def}}{=} \tilde{E}^r, \quad \tilde{E}^r = E_r + \epsilon_{rnu} n^u h^{rv} h_{nm} \tilde{B}^m \frac{1}{(1 + n)^2} \neq \tilde{E}^r = h^{rs} E_s, \]

\[ F^{uv} \overset{\text{def}}{=} \epsilon_{uvr} \tilde{B}^r, \quad \tilde{B}^r = \frac{2}{(1 + n)^2} \epsilon_{rur} \tilde{E}^u n^v + \]

\[ + \epsilon_{rur} \epsilon_{ksn} \left( h^{uk} h^{vs} - \frac{n^k (n^v h^{us} - n^u h^{us})}{(1 + n)^2} \right) h_{nm} \tilde{B}^m \neq \tilde{B}^r = h^{rs} B_s. \quad (2.22) \]
III. PARAMETRIZED MINKOWSKI THEORIES AND THE INERTIAL REST-FRAME INSTANT FORM FOR CHARGED PARTICLES PLUS THE ELECTROMAGNETIC FIELD.

In this Section we will give a review of the description of the isolated system "N charged positive-energy scalar particles with Grassmann-valued electric charges plus the electromagnetic field" [9] in the framework of parametrized Minkowski theories [1, 5] (see also the Appendix of the first paper in Refs.[11]).

Let be given an admissible 3+1 splitting of Minkowski space-time centered on a time-like observer \(x^\mu(\tau)\). Let \(\sigma^A = (\tau; \sigma^u)\) be the adapted observer-dependent radar 4-coordinates and \(z^\mu(\tau, \sigma^u)\) the embedding of the instantaneous 3-spaces \(\Sigma_\tau\) into Minkowski space-time as seen from an arbitrary reference inertial observer. Let \(g_{AB}(\tau, \sigma^u) = z^\mu_A(\tau, \sigma^u) \eta_{\mu\nu} z_B^\nu(\tau, \sigma^u)\) be the associated 4-metric.

The electro-magnetic field is described by the Lorentz-scalar potential \(A_A(\tau, \sigma^u)\) knowing the equal-time surface. The field strength is \(F_{AB}(\tau, \sigma^u) = (\partial_A A_B - \partial_B A_A)(\tau, \sigma^u)\).

The scalar positive-energy particles are described by the Lorentz-scalar 3-coordinates \(\eta^i(\tau)\) defined by \(x^\mu_i(\tau) = z^\mu_i(\tau, \eta^u_i(\tau))\), where \(x^\mu_i(\tau)\) are their world-lines. \(Q_i\) are the Grassmann-valued electric charges satisfying \(Q_i^2 = 0, Q_i Q_j = Q_j Q_i \neq 0\) for \(i \neq j\). Each \(Q_i\) is an even bilinear function of a complex Grassmann variable \(\theta_i(\tau)\): \(Q_i = e^{\theta_i^*(\tau)} \theta_i(\tau)\).

As shown in Ref.[9] the description of \(N\) scalar positive-energy particles with Grassmann-valued electric charges plus the electro-magnetic field is done in parametrized Minkowski theories with the action

\[
S = \int d\tau d^3\sigma \mathcal{L}(\tau, \sigma^u) = \int d\tau L(\tau),
\]

\[
\mathcal{L}(\tau, \sigma^u) = \frac{i}{2} \sum_{i=1}^{N} \delta^3(\sigma^u - \eta^u_i(\tau)) \left[ \theta_i^*(\tau) \dot{\theta}_i(\tau) - \dot{\theta}_i^*(\tau) \theta_i(\tau) \right] -
\]

\[
- \sum_{i=1}^{N} \delta^3(\sigma^u - \eta^u_i(\tau)) \left[ m_i c \sqrt{\epsilon [g_{rr}(\tau, \sigma^u) + 2 g_{rt}(\tau, \sigma^u) \dot{\eta}_i^r(\tau) + g_{rs}(\tau, \sigma^u) \dot{\eta}_i^s(\tau) \dot{\eta}_i^s(\tau)] -
\]

\[
- \frac{Q_i(\tau)}{c} \left( A_r(\tau, \sigma^u) + A_r(\tau, \sigma^u) \dot{\eta}_i^r(\tau) \right) \right] -
\]

\[
- \frac{1}{4c} \sqrt{-g(\tau, \sigma^u)} g^{AC}(\tau, \sigma^u) g^{BD}(\tau, \sigma^u) F_{AB}(\tau, \sigma^u) F_{CD}(\tau, \sigma^u). \tag{3.1}
\]
The canonical momenta are (for dimensional convenience we introduce a $c$ factor in the definition of the electro-magnetic momenta)

$$
\rho_\mu(\tau, \sigma^u) = -\epsilon \frac{\partial L(\tau, \sigma^u)}{\partial \dot{z}_\mu^\tau(\tau, \sigma^u)} = \\
\sum_{i=1}^{N} \delta^3(\sigma^u - \eta_i^u(\tau)) m_i c \frac{\dot{z}_{\tau\mu}(\tau, \sigma^u) + \dot{z}_{\tau\mu}(\tau, \sigma^u) \dot{\eta}_i^r(\tau)}{\sqrt{\epsilon [g_{rr}(\tau, \sigma^u) + 2 g_{rr}(\tau, \sigma^u) \dot{\eta}_i^r(\tau) + g_{rs}(\tau, \sigma^u) \dot{\eta}_i^r(\tau) \dot{\eta}_s^r(\tau)]}} + \\
\epsilon \frac{\sqrt{-g(\tau, \sigma^u)}}{4c} \left[ (g^{rr} z_{\tau\mu} + g^{rr} z_{\tau\mu}) g^{AC} g^{BD} F_{AB} F_{CD} - \\
- 2 \left( z_{\tau\mu} (g^{Ar} g^{rC} g^{BD} + g^{AC} g^{Br} g^{rD}) + \\
+ z_{\tau\mu} (g^{Ar} g^{rC} + g^{Ar} g^{rC}) g^{BD} \right) F_{AB} F_{CD} \right](\tau, \sigma^u) = \\
= \left[ (\rho_\mu v^\nu) l_\mu + (\rho_\mu z^\nu) \gamma^{rs} z_{sp\mu} \right](\tau, \sigma^u),
$$

$$
\pi^\tau(\tau, \sigma^u) = c \frac{\partial L}{\partial \dot{A}_\tau(\tau, \sigma^u)} = 0,
$$

$$
\pi^\tau(\tau, \sigma^u) = c \frac{\partial L}{\partial \dot{A}_\tau(\tau, \sigma^u)} = \frac{\gamma(\tau, \sigma^u)}{\sqrt{-g(\tau, \sigma^u)}} \left( F_{rs} - n^u F_{us} \right)(\tau, \sigma^u) = \\
= - \frac{\sqrt{-g(\tau, \sigma^u)}}{1 + n(\tau, \sigma^u)} h^{rs}(\tau, \sigma^u) \left( E_s - \epsilon_{us} n^u B_v \right)(\tau, \sigma^u),
$$

$$
\kappa_{ir}(\tau) = + \frac{\partial L(\tau)}{\partial \dot{\eta}_i^r(\tau)} = \frac{Q_i}{c} A_r(\tau, \eta_i^u(\tau)) - \\
- \epsilon m_i c \frac{g_{rr}(\tau, \eta_i^u(\tau)) + g_{rs}(\tau, \eta_i^u(\tau)) \dot{\eta}_i^r(\tau)}{\sqrt{\epsilon [g_{rr}(\tau, \eta_i^u(\tau)) + 2 g_{rr}(\tau, \eta_i^u(\tau)) \dot{\eta}_i^r(\tau) + g_{rs}(\tau, \eta_i^u(\tau)) \dot{\eta}_i^r(\tau) \dot{\eta}_s^r(\tau)]}},
$$

$$
\pi_{\theta_i}(\tau) = \frac{\partial L(\tau)}{\partial \theta_i^*(\tau)} = -\frac{i}{2} \theta_i^*(\tau), \quad \pi_{\theta_i}(\tau) = \frac{\partial L(\tau)}{\partial \theta_i^*(\tau)} = -\frac{i}{2} \theta_i(\tau).
$$

The following Poisson brackets are assumed
\[ \{ z^\mu(\tau, \sigma^u), \rho_\nu(\tau, \sigma'^u) \} = -\epsilon \eta_\mu^\nu \delta^3(\sigma^u - \sigma'^u), \]

\[ \{ A_A(\tau, \sigma^u), \pi_B^\nu(\tau, \sigma'^u) \} = c \eta_A^B \delta^3(\sigma^u - \sigma'^u), \quad \{ \eta_i^\nu(\tau), \kappa_{js}(\tau) \} = +\delta_{ij} \delta^s_r, \]

\[ \{ \theta_i(\tau), \pi_{\theta_j}(\tau) \} = -\delta_{ij}, \quad \{ \theta^*_i(\tau), \pi_{\theta^*_j}(\tau) \} = -\delta_{ij}. \tag{3.3} \]

The Grassmann momenta give rise to the second class constraints

\[ \pi_{\theta_i} + \frac{i}{2} \theta^*_i \approx 0, \quad \pi_{\theta^*_i} + \frac{i}{2} \theta_i \approx 0, \quad \{ \pi_{\theta_i} + \frac{i}{2} \theta^*_i, \pi_{\theta^*_j} + \frac{i}{2} \theta_j \} = -i \delta_{ij}, \tag{3.4} \]

so that \( \pi_{\theta_i} \) and \( \pi_{\theta^*_i} \) can be eliminated with the help of Dirac brackets

\[ \{ A, B \}^* = \{ A, B \} - i \{ A, \pi_{\theta_i} + \frac{i}{2} \theta^*_i \} \{ \pi_{\theta^*_j} + \frac{i}{2} \theta_i, B \} + \{ A, \pi_{\theta^*_i} + \frac{i}{2} \theta_i \} \{ \pi_{\theta_i} + \frac{i}{2} \theta^*_i, B \}. \tag{3.5} \]

As a consequence, the Grassmann variables \( \theta_i(\tau), \theta^*_i(\tau) \), have the fundamental Dirac brackets (we will still denote it as \( \{..,\} \) for the sake of simplicity)

\[ \{ \theta_i(\tau), \theta_j(\tau) \} = \{ \theta^*_i(\tau), \theta^*_j(\tau) \} = 0, \quad \{ \theta_i(\tau), \theta^*_j(\tau) \} = -i \delta_{ij}. \tag{3.6} \]

If we introduce the energy-momentum tensor of the isolated system (in inertial frames we have \( T_{\perp\perp} = T^{\tau\tau} \) and \( T_{\perp r} = \delta_{rs} T^{rs} \))

\[ T^{AB}(\tau, \sigma^u) = -\frac{2}{\sqrt{g(\tau, \sigma^u)}} \frac{\delta S}{\delta g_{AB}(\tau, \sigma^u)}, \]

\[ T^{\mu\nu} = \varepsilon_A^\mu \varepsilon_B^\nu T^{AB} = l^\mu l^\nu T_{\perp\perp} + (l^\mu z_r^\nu + l^\nu z_r^\mu) \gamma^{rs} T_{\perp s} + z_r^\mu z_s^\nu T^{rs}, \]

\[ T_{\perp\perp} = l_\mu l_\nu T^{\mu\nu} = (1 + n)^2 T^{\tau\tau}, \]

\[ T_{\perp r} = l_\mu z_r^\nu T^{\mu\nu} = -(1 + n) h_{rs} (T^{\tau\tau} n^s + T^{rs}), \]

\[ T_{rs} = z_r^\mu z_s^\nu T^{\mu\nu} = n_r n_s T^{\tau\tau} + (n_r h_{su} + n_s h_{ru}) T^{\tau u} + h_{ru} h_{sv} T^{uv}, \]

26
\[ T_{\perp\perp}(\tau, \sigma^u) = \left( \frac{1}{2 c \sqrt{\gamma}} \left[ \frac{1}{\sqrt{\gamma}} h_{rs} \pi^r \pi^s + \frac{\sqrt{\gamma}}{2} h^{rs} h^{uv} F_{ru} F_{sv} \right] \right)(\tau, \sigma^u) + \]
\[ + \frac{N}{\sqrt{\gamma(\tau, \sigma^u)}} \delta^3(\sigma^u - \eta_\mu^\tau(\tau)) \left( \sqrt{m_i^2 c^2 + h^{rs} \left[ \kappa_{ir}(\tau) - \frac{Q_i}{c} A_r \right] \left[ \kappa_{is}(\tau) - \frac{Q_i}{c} A_s \right]} \right)(\tau, \sigma^u), \]
\[ T_{\perp s}(\tau, \sigma^u) = \left( \frac{F_{rs} \pi^s}{c \sqrt{\gamma}} \right)(\tau, \sigma) - \frac{N}{\sqrt{\gamma(\tau, \sigma^u)}} \delta^3(\sigma^u - \eta_\mu^\tau(\tau)) \left[ \kappa_{is} - \frac{Q_i}{c} A_s \right](\tau, \sigma^u) \]
\[ T_{rs}(\tau, \sigma^u) = \left( h_{ru} h_{sv} \left[ - \frac{\pi^u \pi^v}{\gamma} + \frac{n_u n_v}{(1 + n)^2} \left( \frac{n_m \pi^m}{\gamma} \right)^2 \right] + \right. \]
\[ + \frac{1}{2} h_{rs} \left[ \frac{h_{lm} \pi^l \pi^m}{\gamma} + \frac{1}{2} h_{lm} h^{uv} F_{lu} F_{mv} \right] \left[ \frac{h^{lm} - \frac{n l n_m}{(1 + n)^2}}{F_{rl} F_{sm}} \right](\tau, \sigma^u) + \]
\[ + \frac{N}{\sqrt{\gamma(\tau, \sigma^u)}} \delta^3(\sigma^u - \eta_\mu^\tau(\tau)) \left( \frac{\left[ \kappa_{ir} - \frac{Q_i}{c} A_r \right] \left[ \kappa_{is} - \frac{Q_i}{c} A_s \right]}{\sqrt{m_i^2 c^2 + h^{uv} \left[ \kappa_{iu}(\tau) - \frac{Q_i}{c} A_u \right] \left[ \kappa_{iv}(\tau) - \frac{Q_i}{c} A_v \right]}} \right)(\tau, \sigma^u), \] (3.7)

Then from Eq.(3.2) we get

\[ \rho_\mu(\tau, \sigma^u) = \left( \sqrt{-g} z_{A\mu} T^{\tau A} \right)(\tau, \sigma^u) = \]
\[ = \left( (1 + n)^2 \sqrt{\gamma} T^{\tau r} \right) (\tau, \sigma^u) \]
\[ = \left( \sqrt{\gamma} \left[ \frac{1}{2} T_{\perp\perp} - z_{r\mu} h^{rs} T_{\perp s} \right] \right)(\tau, \sigma^u). \] (3.8)

Let us remark that, since all the dependence on the embeddings is in the 4-metric, the Euler-Lagrange equations for the embeddings \( z^\mu(\tau, \sigma^u) \) associated with the Lagrangian (3.1) are (the symbol \( \overset{\text{\scriptsize{E}}}{=} \) means evaluated on the solutions of the equations of motion)

\[ \frac{\delta S}{\delta z^\mu(\tau, \sigma^u)} = \left( \frac{\partial L}{\partial z^\mu} - \partial_A \frac{\partial L}{\partial z^\mu_A} \right)(\tau, \sigma^u) = 2 \eta_{\mu\nu} \partial_A \left[ \sqrt{-g} T^{AB} z_B^\nu \right](\tau, \sigma^u) = \]
\[ = \left( \sqrt{-g} z_{\mu C} g_{CD} T^{DA : A} \right)(\tau, \sigma^u) \overset{\text{\scriptsize{E}}}{=} 0, \] (3.9)

where \( T_{AB : B}(\tau, \sigma^u) \) is the covariant derivative associated to the 4-metric \( g_{AB}(\tau, \sigma^u) \) induced by the admissible 3+1 splitting of Minkowski space-time.

They may be rewritten in a form valid for every isolated system \( \left( \partial_A T^{AB} z_B^\mu \right)(\tau, \sigma^u) \overset{\text{\scriptsize{E}}}{=} - \left( \frac{1}{\sqrt{-g}} \partial_A \left[ \sqrt{-g} z_B^\mu \right] T^{AB} \right)(\tau, \sigma^u). \) When \( \partial_A \left[ \sqrt{-g} z_B^\mu \right](\tau, \sigma^u) = 0, \) as it happens in inertial
frames in inertial Cartesian coordinates, we get the conservation of the energy-momentum tensor $T^{AB}|_{\text{inertial}}$, i.e. $\partial_A T^{AB}|_{\text{inertial}} \approx 0$. Then, after integrating over a 4-volume bounded by a 3-volume $V_1$ at $\tau_1$, a 3-volume $V_2$ at $\tau_2 > \tau_1$ and a time-like 3-surface $S_{12}$ joining them and with section $S_\tau$, boundary of a 3-volume $V_\tau$, at $\tau$, we get $\frac{d}{d\tau} \int_{V_\tau} d^3\sigma T^{Ar}|_{\text{inertial}}(\tau, \sigma_u) = -\int_{S_\tau} d^2\Sigma B T^{AB}|_{\text{inertial}}(\tau, \sigma_u)$, namely the time-variation of the 4-momentum contained in $V_\tau$ is balanced by the flux of energy-momentum through the boundary $S_\tau$. For infinite volume and suitable boundary conditions we get the conservation of the 4-momentum $P^A = \int_{\Sigma_\tau} d^3\sigma T^{Ar}|_{\text{inertial}}(\tau, \sigma_u)$.

Otherwise, in non-inertial frames and also in inertial frames with non-Cartesian coordinates we do not have a real conservation law, but the equation $T^{AB;}_B(\tau, \sigma_u) \approx 0$, which, like in general relativity, could be rewritten as a conservation law $\partial_B (T^{AB} + t^{AB})(\tau, \sigma_u) \approx 0$ involving a coordinate-dependent energy-momentum pseudo-tensor describing the “energy-momentum” of the foliation associated to the 3+1 splitting. Moreover a quantity as $\int_{\Sigma_\tau} d^3\Sigma B T^{AB}|_{\text{non-inertial}}(\tau, \sigma_u)$ is not a tensor under frame-preserving diffeomorphisms (even when $T^{AB}_{\text{non-inertial}}$ transforms correctly as a tensor density), so that it cannot give rise to a well defined coordinate-independent quantity. However, differently from general relativity where the equivalence principle says that global inertial frames do not exist, in Minkowski space-time it is always possible to revert to inertial frames and to find the standard 4-momentum constant of motion, which is a 4-vector under the Poincare’ transformations connecting inertial frames.

At the Hamiltonian level from Eqs.(3.2) we obtain the following five primary constraints

$$\pi^\tau(\tau, \sigma_u) \approx 0,$$

$$\mathcal{H}_\mu(\tau, \sigma_u) = \rho_\mu(\tau, \sigma_u) - l_\mu(\tau, \sigma_u) \sqrt{\gamma(\tau, \sigma_u)} T_{\perp\perp}(\tau, \sigma_u) + z_{\tau\mu}(\tau, \sigma_u) h^{rs}(\tau, \sigma_u) \sqrt{\gamma(\tau, \sigma_u)} T_{\perp\perp}(\tau, \sigma_u) \approx 0,$$

(3.10)

The Lorentz-scalar primary constraint $\pi^\tau(\tau, \sigma_u) \approx 0$ is a consequence of the invariance of the action under electro-magnetic gauge transformations.

The canonical Hamiltonian $H_c$ is
\[ H_c = \sum_{i=1}^{N} \kappa_{ir}(\tau) \dot{\eta}^i(\tau) + \int d^3\sigma \left[ \frac{1}{c} \pi^\Lambda \partial_\tau A_\Lambda - \rho_\mu z_\mu^\tau - \mathcal{L}(\tau, \sigma^u) \right] = \frac{1}{c} \int d^3\sigma \left[ \partial_\tau \left( \pi^r(\tau, \sigma^u) A_\tau(\tau, \sigma^u) \right) - A_\tau(\tau, \sigma^u) \Gamma(\tau, \sigma^u) \right] = -\frac{1}{c} \int d^3\sigma A_\tau(\tau, \sigma^u) \Gamma(\tau, \sigma^u), \]

(3.11)

after the elimination of a surface term and the introduction of the quantity

\[ \Gamma(\tau, \sigma^u) \equiv \partial_\tau \pi^r(\tau, \sigma^u) + \sum_{i=1}^{N} Q_i \delta^3(\sigma^u - \eta^u_i(\tau)). \]

(3.12)

As a consequence, the Dirac Hamiltonian is

\[ H_D = \int d^3\sigma \left[ \lambda^\mu \mathcal{H}_\mu + \mu \pi^r - \frac{1}{c} A_\tau \Gamma \right](\tau, \sigma^u). \]

(3.13)

Here \( \lambda^\mu(\tau, \sigma^u) \) and \( \mu(\tau, \sigma^u) \) are the Dirac multipliers associated with the primary constraints.

The requirement that the five primary constraints be \( \tau \)-independent, i.e. \( \{\pi^r(\tau, \sigma^u), H_D\} \approx 0, \{\mathcal{H}^\mu(\tau, \sigma^u), H_D\} \approx 0 \), implies only the Gauss’ law secondary constraint

\[ \Gamma(\tau, \sigma^u) \approx 0. \]

(3.14)

The 6 constraints are all first class, since they satisfy the following Poisson brackets

\[ \{\Gamma(\tau, \sigma^u), \pi^r(\tau, \sigma^u')\} = \{\Gamma(\tau, \sigma^u), \mathcal{H}_\mu(\tau, \sigma^u')\} = \{\pi^r(\tau, \sigma^u), \mathcal{H}_\mu(\tau, \sigma^u')\} = 0 \]

\[ \mathcal{H}_\mu(\tau, \sigma^u), \mathcal{H}_\nu(\tau, \sigma^u') \} = \frac{1}{c} \left( [l_\mu z_{r\nu} - l_\nu z_{r\mu}] \frac{\pi^r}{\sqrt{\gamma}} - \right. \]

\[ -z_{u\mu} h^{uv} F_{rs} h^{sv} z_{uv} \right)(\tau, \sigma^u) \Gamma(\tau, \sigma^u) \delta^3(\sigma^u - \sigma'^u) \approx 0. \]

(3.15)

The constraints \( \pi^r(\tau, \sigma^u) \approx 0 \) and \( \Gamma(\tau, \sigma^u) \approx 0 \) are the canonical generators of the electro-magnetic gauge transformations.

Instead the constraints \( \mathcal{H}_\mu(\tau, \sigma^u) \approx 0 \) generate the gauge transformations from an admissible 3+1 splitting of Minkowski space-time to another one. These constraints can be replaced with their projections \( \mathcal{H}_r(\tau, \sigma^u) = \mathcal{H}_\mu(\tau, \sigma^u) z_{\mu}^r(\tau, \sigma^u) \approx 0, \mathcal{H}_\perp(\tau, \sigma^u) = \ldots \)
\( \mathcal{H}_\mu(\tau, \sigma^u) l^\mu(\tau, \sigma^u) \approx 0, \) tangent and normal to the instantaneous 3-space \( \Sigma_\tau \) respectively. Modulo the Gauss law constraint \( \Gamma(\tau, \sigma^u) \approx 0, \) the new constraints satisfy the universal Dirac algebra of the super-hamiltonian and super-momentum constraints of canonical metric gravity (see the first paper in Refs.[11]). The gauge transformations generated by the constraint \( \mathcal{H}_\perp(\tau, \sigma^u) \) change the instantaneous 3-spaces \( \Sigma_\tau \) (i.e. the clock synchronization convention), while those generated by the constraints \( \mathcal{H}_r(\tau, \sigma^u) \) change the 3-coordinates on \( \Sigma_\tau \).

The Hamilton-Dirac equations are

\[
\frac{\partial z^\mu(\tau, \sigma^u)}{\partial \tau} = (1 + n) l^\mu + n^r z^\mu_r(\tau, \sigma^u) \equiv -\epsilon \lambda^\mu(\tau, \sigma^u),
\]

\[
\frac{\partial A_r(\tau, \sigma^u)}{\partial \tau} \circ \{ A_r(\tau, \sigma^u), H_D \} = \mu(\tau, \sigma^u),
\]

\[
\frac{\partial A_r(\tau, \sigma^u)}{\partial \tau} \circ \{ A_r(\tau, \sigma^u), H_D \} = -\int d^3\sigma' \left[ \left( \lambda_\mu l^\mu \sqrt{\gamma} \right)(\tau, \sigma') \{ A_r(\tau, \sigma'), T_{\perp\perp}(\tau, \sigma') \} - \right.
\]

\[
- \left( \lambda_\mu z^\mu_a h^{as} \sqrt{\gamma} \right)(\tau, \sigma') \{ A_r(\tau, \sigma'), T_{\perp s}(\tau, \sigma') \} +
\]

\[
+ \frac{1}{c} A_r(\tau, \sigma') \{ A_r(\tau, \sigma'), \Gamma(\tau, \sigma') \} \bigg],
\]

\[
\frac{\partial \pi^r(\tau, \sigma^u)}{\partial \tau} \circ \{ \pi^r(\tau, \sigma^u), H_D \} = -\int d^3\sigma' \left[ \left( \lambda_\mu l^\mu \sqrt{\gamma} \right)(\tau, \sigma') \{ \pi^r(\tau, \sigma'), T_{\perp\perp}(\tau, \sigma') \} - \right.
\]

\[
- \left( \lambda_\mu z^\mu_a h^{as} \sqrt{\gamma} \right)(\tau, \sigma') \{ \pi^r(\tau, \sigma'), T_{\perp s}(\tau, \sigma') \} \bigg],
\]

\[
\frac{d\eta^\nu_r(\tau)}{d\tau} \circ \{ \eta^\nu_r(\tau), H_D \} = -\int d^3\sigma' \left[ \left( \lambda_\mu l^{mu} \sqrt{\gamma} \right)(\tau, \sigma') \{ \eta^\nu_r(\tau), T_{\perp\perp}(\tau, \sigma') \} - \right.
\]

\[
- \left( \lambda_\mu z^\mu_a h^{as} \sqrt{\gamma} \right)(\tau, \sigma') \{ \eta^\nu_r(\tau), T_{\perp s}(\tau, \sigma') \},
\]

\[
\frac{d\kappa_{ir}(\tau)}{d\tau} \circ \{ \kappa_{ir}(\tau), H_D \} = -\int d^3\sigma' \left[ \left( \lambda_\mu l^{mu} \sqrt{\gamma} \right)(\tau, \sigma') \{ \kappa_{ir}(\tau), T_{\perp\perp}(\tau, \sigma') \} - \right.
\]

\[
- \left( \lambda_\mu z^\mu_a h^{as} \sqrt{\gamma} \right)(\tau, \sigma') \{ \kappa_{ir}(\tau), T_{\perp s}(\tau, \sigma') \} +
\]

\[
+ \frac{1}{c} A_r(\tau, \sigma') \{ \kappa_{ir}(\tau), \Gamma(\tau, \sigma') \} \bigg].
\]
The Grassmann-valued electric charges are constants of the motion, \( \frac{dQ_i(\tau)}{d\tau} = 0 \).

Since the embedding variables \( z^\mu(\tau, \sigma^u) \) are the only configuration variables with Lorentz indices, the ten conserved generators of the Poincaré transformations are:

\[
P^\mu = \int d^3\sigma \rho^\mu(\tau, \sigma^u), \quad J^{\mu\nu} = \int d^3\sigma (z^\mu \rho^\nu - z^\nu \rho^\mu)(\tau, \sigma^u).
\]

The determination of the radiation gauge of the electro-magnetic field in non-inertial frames will be done in the next Section.
IV. THE HAMILTONIAN DESCRIPTION OF CHARGED PARTICLES AND THE ELECTRO-MAGNETIC FIELD IN NON-INERTIAL FRAMES

In this Section we study the system of charged positive-energy scalar particles plus the electro-magnetic field in a given admissible non-inertial frame. Then we define the radiation gauge in non-inertial frames.

A. The Hamilton Equations in an Admissible Non-Inertial Frame.

Let us choose an admissible 3+1 splitting of the type (2.1) by adding the gauge fixing constraints

$$\chi(\tau, \sigma^u) = z^\mu(\tau, \sigma^u) - z^\mu_F(\tau, \sigma^u) \approx 0,$$

$$z^\mu_F(\tau, \sigma^u) = x^\mu(\tau) + F^\mu(\tau, \sigma^u), \quad F^\mu(\tau, 0) = 0, \quad (4.1)$$

to the first class constraints $H^\mu(\tau, \sigma^u) \approx 0$ of Eqs.(3.10).

From the Hamilton-Dirac equations (3.16) we have that the Dirac multipliers $\lambda^\mu(\tau, \sigma^u)$ in the Dirac Hamiltonian (3.13) take the form

$$\lambda^\mu(\tau, \sigma^u) \overset{\circ}{=} -\epsilon \left( \dot{x}^\mu(\tau) + \frac{\partial F^\mu(\tau, \sigma^u)}{\partial \tau} \right) = -\epsilon \, z^\mu_F(\tau, \sigma^u) =$$

$$= -\epsilon \left[ (1 + n_F) l^\mu_F + n^r_F \partial_r F^\mu \right](\tau, \sigma^u),$$

$$-\lambda^\mu l^\mu_F = 1 + n_F, \quad \lambda^\mu z^\mu_F \, l^\mu_F = n^r_F. \quad (4.2)$$

$H^\mu(\tau, \sigma^u) \approx 0$ and $\chi(\tau, \sigma^u) \approx 0$ are second class constraints $^{11}$, which eliminate the variables $z^\mu(\tau, \vec{\sigma})$ and $\rho^\mu(\tau, \sigma^u)$. If we go to Dirac brackets, so that these constraints become strongly zero, the Dirac Hamiltonian does not depend any more upon the constraints $H^\mu(\tau, \sigma^u) \approx 0$.

To find the new Dirac Hamiltonian $H_{DF}$ at the level of Dirac brackets (still denoted $\{,\}$) let us put the Dirac multiplier (4.2) in the Hamilton-Dirac equations (3.16) for all the variables $\mathcal{F} = A_\tau, A_r, \pi^r, \eta^r_i, \kappa_{ir}$ independent from the embeddings and their momenta

$^{11}$ We assume $\{H^\mu(\tau, \sigma^u_1), \chi(\tau, \sigma^u_2)\} \neq 0$ as a restriction of $F^\mu(\tau, \sigma^u)$
\[
\begin{aligned}
\frac{\partial \mathcal{F}(..)}{\partial \tau} &\overset{\circ}{=} \{ \mathcal{F}(..), H_D \} = \\
&= \int d^3 \sigma \{ \mathcal{F}(..), \left( \lambda^\mu \mathcal{H}_\mu + \mu \pi^r - \frac{1}{c} A_\tau \Gamma \right)(\tau, \sigma^u) \} = \\
&\overset{\circ}{=} \int d^3 \sigma \{ \mathcal{F}(..), \left( (1 + n_F) \sqrt{\gamma_F} T_{\perp \perp} + n_F^r \sqrt{\gamma_F} T_{\perp r} + \mu \pi^r - \frac{1}{c} A_\tau \Gamma \right)(\tau, \sigma^u) \} = \\
&\overset{\text{def}}{=} \{ \mathcal{F}(..), H_{DF} \}. 
\end{aligned}
\] (4.3)

As a consequence the new Dirac Hamiltonian is

\[
H_{DF} = \int d^3 \sigma \left( (1 + n_F) \sqrt{\gamma_F} T_{\perp \perp} + n_F^r \sqrt{\gamma_F} T_{\perp r} + \mu \pi^r - \frac{1}{c} A_\tau \Gamma \right)(\tau, \sigma^u) = \\
= \int d^3 \sigma \left( (1 + n_F(\tau, \sigma^u)) \left[ \sqrt{\gamma_F(\tau, \sigma^u)} T_{\perp \perp}'(\tau, \sigma^u) + \\
+ \sum_i \delta^3(\sigma^u - \eta_i^u(\tau)) \left( \sqrt{m_i^2 c^2 + \hbar F_i} \left( \kappa_{iu}(\tau) - \frac{Q_i}{c} A_u \right) \left( \kappa_{iu}(\tau) - \frac{Q_i}{c} A_u \right) \right)(\tau, \sigma^u) \right] \\
+ n_F^r(\tau, \sigma^u) \left[ \frac{1}{c} F_{rs}(\tau, \sigma^u) \pi^s(\tau, \sigma^u) - \sum_i \delta^3(\sigma^u - \eta_i^u(\tau)) \left( \kappa_{iu}(\tau) - \frac{Q_i}{c} A_u(\tau, \sigma^u) \right) \right] \\
+ \mu(\tau, \sigma^u) \pi^r(\tau, \sigma^u) - \frac{1}{c} A_\tau(\tau, \sigma^u) \Gamma(\tau, \sigma^u) \right), 
\] (4.4)

where the energy-momentum tensor is evaluated at \( z^\mu(\tau, \sigma^u) = z_F^\mu(\tau, \sigma^u) \)

\[
\left( \sqrt{\gamma_F} T_{\perp \perp}' \right)(\tau, \sigma^u) = \frac{1}{2c} \left( \frac{1}{\sqrt{\gamma_F(\tau, \sigma^u)}} h_{Frs}(\tau, \sigma^u) \pi^r(\tau, \sigma^u) \pi^s(\tau, \sigma^u) + \\
+ \frac{\sqrt{\gamma_F(\tau, \sigma^u)}}{2} h_F^{rs}(\tau, \sigma^u) h_F^{uw}(\tau, \sigma^u) F_{ru}(\tau, \sigma^u) F_{sw}(\tau, \sigma^u) \right). 
\] (4.5)

The Hamilton-Dirac equations for the particle positions take the form

\[
\dot{\eta}_i^u(\tau) \overset{\circ}{=} \left( (1 + n_F) \left[ h_F^{rs} \left( \kappa_{iu}(\tau) - \frac{Q_i}{c} A_u \right) \left( \kappa_{iu}(\tau) - \frac{Q_i}{c} A_u \right) \right](\tau, \eta_i^u(\tau)) - \\
- n_F^r(\tau, \eta_i^u(\tau)) \right), 
\] (4.6)

which can be inverted to get
\[ \kappa_{ir}(\tau) = \left( \frac{h_{Fs}(\tau, \eta^u_i(\tau)) m_i c \left( \dot{\eta}^e_i(\tau) + n^e_i \right)}{\sqrt{\left(1 + n_F\right)^2 - h_{Fuv} \left( \dot{\eta}^u_i(\tau) + n^u_i \right) \left( \dot{\eta}^v_i(\tau) + n^v_i \right)}} \right)(\tau, \eta^u_i(\tau)) + \\
+ \frac{Q_i}{c} A_r(\tau, \eta^u_i(\tau)). \]  

(4.7)

For the particle momenta we get the Hamilton-Dirac equations

\[ \frac{d}{d\tau} \kappa_{ir}(\tau) \equiv \frac{Q_i}{c} \dot{\eta}^u_i(\tau) \frac{\partial A_u(\tau, \eta^u_i(\tau))}{\partial \eta^e_i} + \frac{Q_i}{c} \frac{\partial A_r(\tau, \eta^u_i(\tau))}{\partial \eta^v_i} + \mathcal{F}_{ir}(\tau), \]

\[ \mathcal{F}_{ir}(\tau) = \left( \frac{m_i c \left[1 + n_F\right]^{-1}}{\sqrt{\left(1 + n_F\right)^2 - h_{Fuv} \left( \dot{\eta}^u_i(\tau) + n^u_i \right) \left( \dot{\eta}^v_i(\tau) + n^v_i \right)}} \right)(\tau, \eta^u_i(\tau)) \\
+ \left( \frac{\partial h_{Fs}(\tau, \eta^u_i(\tau))}{\partial \eta^e_i} \left( \dot{\eta}^e_i(\tau) + n^e_i(\tau, \eta^u_i(\tau)) \right) \left( \dot{\eta}^v_i(\tau) + n^v_i(\tau, \eta^u_i(\tau)) \right) - \frac{\partial n_F(\tau, \eta^u_i(\tau))}{\partial \eta^v_i} \right) + \\
+ \left( \frac{\partial n^e_i(\tau, \eta^u_i(\tau))}{\partial \eta^e_i} h_{Fs}(\tau, \eta^u_i(\tau)) \left( \dot{\eta}^e_i(\tau) + n^e_i(\tau, \eta^u_i(\tau)) \right) \right), \]  

(4.8)

where \( \mathcal{F}_{ir}(\tau) \) denotes a set of relativistic inertial forces.

As a consequence, the second order form of the particle equations of motion implied by Eqs. (4.7) and (4.8) is

\[ \frac{d}{d\tau} \left( \frac{h_{Fs} \ m_i c \left( \dot{\eta}^e_i(\tau) + n^e_i \right)}{\sqrt{\left(1 + n_F\right)^2 - h_{Fuv} \left( \dot{\eta}^u_i(\tau) + n^u_i \right) \left( \dot{\eta}^v_i(\tau) + n^v_i \right)}} \right)(\tau, \eta^u_i(\tau)) \equiv \\
\equiv \frac{Q_i}{c} \left[ \dot{\eta}^u_i(\tau) \left( \frac{\partial A_u(\tau, \eta^u_i(\tau))}{\partial \eta^e_i} - \frac{\partial A_r(\tau, \eta^u_i(\tau))}{\partial \eta^v_i} \right) + \left( \frac{\partial A_r(\tau, \eta^u_i(\tau))}{\partial \eta^v_i} - \frac{\partial A_r(\tau, \eta^u_i(\tau))}{\partial \tau} \right) \right] + \\
+ \mathcal{F}_{ir}(\tau), \]
or

\[
\begin{align*}
& \frac{m_i c}{d\tau} \left( \frac{\dot{n}_s^i(\tau) + n_{sF}^i}{\sqrt{\left(1 + n_{sF}^2\right) - h_{Fuv} \left(\dot{n}_s^i(\tau) + n_{sF}^u\right) \left(\dot{n}_s^i(\tau) + n_{sF}^v\right)}} \right)(\tau, n_{sF}^i(\tau)) \overset{\text{def}}{=} \\
& \overset{\circ}{=} Q_i \sqrt{\left(1 + n_{sF}^2\right) - h_{Fuv} \left(\dot{n}_s^i(\tau) + n_{sF}^u\right) \left(\dot{n}_s^i(\tau) + n_{sF}^v\right)} + \\
& + \left[ \left( \frac{\partial A_r(\tau, n_{sF}^i(\tau))}{\partial n_{sF}^i} \right) - \frac{\partial A_r(\tau, n_{sF}^i(\tau))}{\partial \tau} \right] + \mathcal{F}_i^s(\tau),
\end{align*}
\]

where \( \mathcal{F}_i^s(\tau) \) is the form of inertial forces whose non-relativistic limit to rigid non-inertial frames is evaluated in Subsection C.

If, as in Eqs.(2.20), we define the non-inertial electric and magnetic fields in the form \(^{12}\)

\[
E_r \overset{\text{def}}{=} \left( \frac{\partial A_r}{\partial n_{sF}^i} - \frac{\partial A_r}{\partial \tau} \right) = -F_{rr},
\]

\[
B_r \overset{\text{def}}{=} \frac{1}{2} \epsilon_{uvw} F_{uv} = \epsilon_{uvw} \partial_u A_{\perp v} \Rightarrow F_{uv} = \epsilon_{uvr} B_r,
\]

\(^{12}\) In the inertial case Eqs.(2.19) and (3.2) imply \(\pi^s \overset{\circ}{=} -\delta^{sr} E_r = -\mathcal{E}^s \), so that the components of the energy-momentum tensor are \(T_{\tau\tau} \overset{\circ}{=} \frac{1}{c^2} \left(\mathcal{E}^2 + \mathcal{B}^2\right)\), \(T_{\tau r} \overset{\circ}{=} \frac{1}{c} \left(\mathcal{E} \times \mathcal{B}\right)\).
the homogeneous Maxwell equations, allowing the introduction of the electro-magnetic potentials, have the standard inertial form \( \epsilon_{r_{uv}} \partial_u B_v = 0, \epsilon_{r_{uv}} \partial_v E_v + \frac{1}{c} \frac{\partial P_{rr}}{\partial r} = 0. \)

Then also Eqs.(4.9) take the standard inertial form plus inertial forces

\[
\frac{d}{d\tau} \left( \frac{h_{F rs} m_i c \left( \eta_i^s(\tau) + n_{F}^s \right)}{\sqrt{\left( 1 + n_F \right)^2 - h_{F uv} \left( \dot{\eta}_i^u(\tau) + n_{F}^u \right) \left( \dot{\eta}_i^v(\tau) + n_{F}^v \right)}} \right) (\tau, \eta_i^u(\tau)) = 0
\]

\[
= \frac{Q_i}{c} \left[ E_v + \epsilon_{r_{uv}} \dot{\eta}_i^u(\tau) B_v \right] (\tau, \eta_i^u(\tau)) + \mathcal{F}_{ir}(\tau).
\] (4.11)

The Hamilton-Dirac equations for the electro-magnetic field are

\[
\frac{\partial}{\partial \tau} A_r(\tau, \sigma^u) \equiv c \mu(\tau, \sigma^u),
\]

\[
\frac{\partial}{\partial \tau} A_r(\tau, \sigma^u) \equiv \left( \frac{\partial}{\partial \sigma^r} A_r + \frac{1 + n_F}{\sqrt{\gamma_F}} h_{F rs} \pi^s + n_{F}^s F_{sr} \right)(\tau, \sigma^u),
\]

\[
\frac{\partial}{\partial \tau} \pi^r(\tau, \sigma^u) \equiv \sum_i Q_i \dot{\eta}_i^r(\tau) \delta^3(\sigma^u - \eta_i^u(\tau)) +
\]

\[
+ \left( \frac{\partial}{\partial \sigma^s} \left[ (1 + n_F) \sqrt{\gamma_F} h_{F rs} h_{F uv} F_{uv} - (n_{F}^s \pi^r - n_{F}^r \pi^s) \right] \right)(\tau, \sigma^u). \] (4.12)

Eqs.(4.12) imply

\[
\pi^s(\tau, \sigma^u) = - \left[ - \frac{\sqrt{\gamma_F}}{1 + n_F} \frac{h_{F^{uv}}}{h_{F^{rr}}} (F_{rr} - n_{F}^r F_{uv}) \right] (\tau, \sigma^u) =
\]

\[
= - \sqrt{-g_F(\tau, \sigma^u)} g_{F}^{sA}(\tau, \sigma^u) g_{F}^{sB}(\tau, \sigma^u) F_{AB}(\tau, \sigma^u). \] (4.13)

If we introduce the charge density \( \bar{\rho} \), the charge current density \( \bar{\mathcal{J}}^r \) and the total charge \( Q_{tot} = \sum_i Q_i \) on \( \Sigma_\tau \)

\[
\bar{\rho}(\tau, \sigma^u) = \frac{1}{\sqrt{\gamma_F(\tau, \sigma^u)}} \sum_{i=1}^{N} Q_i \delta^3(\sigma^u - \eta_i^u(\tau)),
\]

\[
\bar{\mathcal{J}}^r(\tau, \sigma^u) = \frac{1}{\sqrt{\gamma_F(\tau, \sigma^u)}} \sum_{i=1}^{N} Q_i \dot{\eta}_i^r(\tau) \delta^3(\sigma^u - \eta_i^u(\tau)),
\]

\[
\Rightarrow Q_{tot} = \int d^3\sigma \sqrt{\gamma_F(\tau, \sigma^u)} \bar{\rho}(\tau, \sigma^u), \] (4.14)
then the last of Eqs. (4.12) can be rewritten in form

\[ \frac{\partial}{\partial \sigma} \pi^r(\tau, \sigma^u) \approx -\sqrt{\gamma F(\tau, \sigma^u)} \rho(\tau, \sigma^u), \]

\[ \frac{\partial \pi^r(\tau, \sigma^u)}{\partial \tau} = \frac{\partial}{\partial \sigma^u} \left[ \sqrt{-g_F} h_F^{su} h_F^{ru} F_{vu} - (n_F^s \pi^r - n_F^r \pi^s) \right] (\tau, \sigma^u) + \sqrt{\gamma F(\tau, \sigma^u)} J_r(\tau, \sigma^u). \] (4.15)

If we introduce the 4-current density \( s^A(\tau, \sigma^u) \)

\[ s^\tau(\tau, \sigma^u) = \frac{1}{\sqrt{-g_F(\tau, \sigma^u)}} \sum_{i=1}^N Q_i \delta^3(\sigma^u - \eta_i^u(\tau)), \]

\[ s^s(\tau, \sigma^u) = \frac{1}{\sqrt{-g_F(\tau, \sigma^u)}} \sum_{i=1}^N Q_i \dot{\eta}_i^r(\tau) \delta^3(\sigma^u - \eta_i^u(\tau)), \] (4.16)

and we use (4.13), then Eqs. (4.15) can be rewritten as manifestly covariant equations for the field strengths as in Ref. [25]

\[ \frac{1}{\sqrt{-g_F(\tau, \sigma^u)}} \frac{\partial}{\partial \sigma^A} \left[ \sqrt{-g_F(\tau, \sigma^u)} g_F^{AB}(\tau, \sigma^u) g_F^{CD}(\tau, \sigma^u) F_{BD}(\tau, \sigma^u) \right] \overset{\circ}{=} -s^C(\tau, \sigma^u). \] (4.17)

Eqs. (4.17) imply the following continuity equation due to the skew-symmetry of \( F_{AB} \)

\[ \frac{1}{\sqrt{-g_F(\tau, \sigma^u)}} \frac{\partial}{\partial \sigma^A} \left[ \sqrt{-g_F(\tau, \sigma^u)} s^C(\tau, \sigma^u) \right] \overset{\circ}{=} 0, \]

or

\[ \frac{1}{\sqrt{\gamma F(\tau, \sigma^u)}} \frac{\partial}{\partial \tau} \left[ \sqrt{\gamma F(\tau, \sigma^u)} \rho(\tau, \sigma^u) \right] + \frac{1}{\sqrt{\gamma F(\tau, \sigma^u)}} \frac{\partial}{\partial \sigma^r} \left[ \sqrt{\gamma F(\tau, \sigma^u)} J_r(\tau, \sigma^u) \right] \overset{\circ}{=} 0, \] (4.18)

so that consistently we recover \( \frac{d}{dt} Q_{\text{tot}} \overset{\circ}{=} 0 \).

See Appendix A for the expression of the Landau-Lifschitz non-inertial electro-magnetic fields [17].
B. The Radiation Gauge for the Electro-Magnetic Field in Non-Inertial Frames.

In Appendix B there is a general discussion about the non-covariant decomposition of the vector potential \( \vec{A}(\tau, \sigma^u) \) and its conjugate momentum \( \vec{\pi}(\tau, \sigma^u) \) (the electric field) into longitudinal and transverse parts in absence of matter. Only with this decomposition we can define a Shanmugadhasan canonical transformation adapted to the two first class constraints generating electro-magnetic gauge transformations and identify the physical degrees of freedom (Dirac observables) of the electro-magnetic field without sources. This method identifies the \textit{radiation gauge} as the natural one from the point of view of constraint theory. Here we extend the construction to the case in which there are charged particles: this will allow us to find the expression of the mutual Coulomb interaction among the charges in non-inertial frames.

As in Eq.(B3) let us introduce the non-covariant flat Laplacian \( \Delta = \sum_r \partial_r^2 \) in the instantaneous non-Euclidean 3-space \( \Sigma_\tau \). We use the non-covariant notation \( \hat{\partial}_r = \delta^{rs} \partial_s \) relying on the positive signature of the 3-metric \( h_{Frs}(\tau, \sigma^u) = -\epsilon g_{Frs}(\tau, \sigma^u) \). Since we have:

\[
\Delta \left( -\frac{1}{4\pi} \frac{1}{\sqrt{\sum_u (\sigma^u - \sigma'^u)^2}} \right) = \delta^3(\sigma^u, \sigma'^u), \quad \text{or} \quad \frac{1}{\Delta} \delta^3(\sigma^u, \sigma'^u) = -\frac{1}{4\pi} \frac{1}{\sqrt{\sum_u (\sigma^u - \sigma'^u)^2}},
\]

with \( \delta^3(\sigma^u, \sigma'^u) \) the delta function for \( \Sigma_\tau \), we can introduce the projectors

\[
P^{rs}(\sigma^u, \sigma'^u) = \delta^{rs} \delta^3(\sigma^u, \sigma'^u) - \hat{\partial}_r \hat{\partial}_s \left( -\frac{1}{4\pi} \frac{1}{\sqrt{\sum_u (\sigma^u - \sigma'^u)^2}} \right) = P^{rs}(\sigma^u) \delta^3(\sigma^u, \sigma'^u),
\]

\[
P^r_{\perp}(\sigma^u) = \delta^{rs} - \hat{\partial}_r \hat{\partial}_s \frac{\delta^3(\sigma^u, \sigma'^u)}{\Delta}. \tag{4.20}
\]

As a consequence the transverse part of the electro-magnetic quantities (\( \hat{\partial}_r A_{\perp r} = \partial_r A_{\perp r} = 0, \hat{\partial}_r \pi^r_{\perp} = 0 \)) are

\[
A_{\perp r}(\tau, \sigma^u) = \delta_{ru} \int d^3\sigma' P^{rs}(\sigma^u, \sigma'^u) A_s(\tau, \sigma'^u) = \delta_{ru} P^{us}_{\perp}(\sigma^u) A_s(\tau, \sigma^u),
\]

\[
\pi^r_{\perp}(\tau, \sigma^u) = \sum_s \int d^3\sigma' P^{rs}(\sigma^u, \sigma'^u) \pi^s(\tau, \sigma'^u) = \sum_s P^{rs}_{\perp}(\sigma^u) \pi^s(\tau, \sigma^u). \tag{4.21}
\]

Therefore the Gauss law constraint (3.12) implies the following decomposition of \( \pi^r(\tau, \sigma^u) \)

\(^{13}\) The delta functions are defined in Appendix B after Eq.(B3).
\[
\pi^r(\tau, \sigma^u) = \pi^r_\perp(\tau, \sigma^u) + \hat{\partial}^r \int d^3 \sigma' \left( -\frac{1}{4\pi} \frac{1}{\sqrt{\sum_u (\sigma'^u - \sigma^u)^2}} \right) \left( \Gamma(\tau, \sigma'^u) - \sum_i Q_i \delta^3(\sigma'^u, \eta_i^u(\tau)) \right)
\] (4.22)

If, following Dirac [26], we introduce the variable canonically conjugate to \(\Gamma(\tau, \sigma^u)\) (it describes a Coulomb cloud of longitudinal photons)

\[
\eta_{em}(\tau, \sigma^u) = -\int d^3 \sigma' \left( -\frac{1}{4\pi} \frac{1}{\sqrt{\sum_u (\sigma'^u - \sigma^u)^2}} \right) \left( \sum_r \hat{\partial}^r A_r(\tau, \sigma'^u) \right),
\]

\[
\{\eta_{em}(\tau, \sigma^u), \Gamma(\tau, \sigma'^u)\} = \delta^3(\sigma^u, \sigma'^u),
\] (4.23)

we have the following non-covariant decomposition of the vector potential

\[
A_r(\tau, \sigma^u) = A_{\perp, r}(\tau, \sigma^u) - \partial_r \eta_{em}(\tau, \sigma^u).
\] (4.24)

If we introduce the following new Coulomb-dressed momenta for the particles

\[
\tilde{\kappa}_{ir}(\tau) = \kappa_{ir}(\tau) + \frac{Q_i}{c} \frac{\partial}{\partial \eta^u_i(\tau)} \eta_{em}(\tau, \eta^u_i(\tau)),
\]

\[
\Rightarrow \quad \kappa_{ir}(\tau) - \frac{Q_i}{c} A_r(\tau, \eta^u_i(\tau)) = \tilde{\kappa}_{ir}(\tau) - \frac{Q_i}{c} A_{\perp, r}(\tau, \eta^u_i(\tau))
\] (4.25)

we arrive at the following non-covariant Shanmugadhasan canonical transformation in non-inertial frames

\[
\begin{array}{ccc}
A_r(\tau, \sigma^u) & \eta^r_i(\tau) & A_{\perp, r}(\tau, \sigma^u) & \eta_{em}(\tau, \sigma^u) & \eta^r_i(\tau) \\
\pi^r(\tau, \sigma^u) & \kappa_{ir}(\tau) & \pi^r_\perp(\tau, \sigma^u) & \Gamma(\tau, \sigma^u) \approx 0 & \tilde{\kappa}_{ir}(\tau)
\end{array}
\]

\[
\{A_{\perp, r}(\tau, \sigma^u), \pi^s_\perp(\tau, \sigma'^u)\} = c P^{rs}(\sigma^u, \sigma'^u) = c P^{rs}_\perp(\sigma^u) \delta^3(\sigma^u, \sigma'^u),
\]

\[
\{\eta^r_i(\tau), \tilde{\kappa}_{is}(\tau)\} = \delta^r_s \delta_{ij}.
\] (4.26)

The electromagnetic part of the Hamiltonian (4.4) can be expressed in terms of the new canonical variables, since we have:
\[
\int d^3\sigma \sqrt{\gamma(\tau, \sigma^u)} \left[ (1 + n_F) T_{\perp\perp} + \frac{n_F^r}{c} F_{r s} \pi^s \right](\tau, \sigma^u) = \\
= \frac{1}{c} W(\eta_i^u(\tau), ..., \eta_N^u(\tau)) + \int d^3\sigma \sqrt{\gamma(\tau, \sigma^u)} \left[ (1 + n_F) \tilde{T}_{\perp\perp} + n_F^r \tilde{T}_{\perp r} \right](\tau, \sigma^u) + \\
+ \frac{1}{c} \int d^3\sigma a_+(\tau, \sigma^u) \Gamma(\tau, \sigma^u) + O(\Gamma^2),
\]

(4.27)

where the energy-momentum tensor has the form

\[
\sqrt{\gamma(\tau, \sigma^u)} \tilde{T}_{\perp\perp}(\tau, \sigma^u) = \frac{1}{c} F_{r s}(\tau, \sigma^u) \pi^s_{\perp}(\tau, \sigma^u) + \\
+ \frac{\sqrt{\gamma_F(\tau, \sigma^u)}}{4c} h_{F r}^s(\tau, \sigma^u) h_{F r}^{u v}(\tau, \sigma^u) F_{r u}(\tau, \sigma^u) F_{s v}(\tau, \sigma^u),
\]

(4.28)

In Eq.(4.27) we have introduced the potentials \((F_{r s} = \partial_r A_{s \perp} - \partial_s A_{r \perp})\)

\[
W(\eta_i^u(\tau), ..., \eta_N^u(\tau)) = \\
= + \int d^3\sigma \frac{h_{F r s}(\tau, \sigma^u)}{2c \sqrt{\gamma_F(\tau, \sigma^u)}} \left( 2\pi^r_{\perp}(\tau, \sigma^u) + \frac{1}{4\pi} \sum_i \frac{\partial}{\partial \sigma^r} \frac{Q_i}{\sqrt{\sum_u (\sigma^u - \eta_i^u(\tau))^2}} \right) + \\
+ \frac{1}{4\pi} \sum_j \frac{\partial}{\partial \sigma^s} \left[ \frac{Q_j}{\sqrt{\sum_u (\sigma^u - \eta_j^u(\tau))^2}} \right] + \\
+ n_F^r(\tau, \sigma^u) F_{r s}(\tau, \sigma^u) \left( \frac{1}{4\pi} \sum_j \frac{\partial}{\partial \sigma^s} \frac{Q_j}{\sqrt{\sum_u (\sigma^u - \eta_j^u(\tau))^2}} \right),
\]

(4.29)

and the function

\[
a_+(\tau, \sigma^u) = \int d^3\sigma' \frac{1}{4\pi \sqrt{\sum_u (\sigma^u - \sigma'^u)^2}} \frac{\partial}{\partial \sigma'^r} \left[ n_F^s(\tau, \sigma'^u) F_{s r}(\tau, \sigma'^u) + \\
+ (1 + n_F(\tau, \sigma'^u)) h_{F r s}(\tau, \sigma'^u) \pi^r_{\perp}(\tau, \sigma'^u) + \frac{1}{4\pi} \sum_j \frac{\partial}{\partial \sigma'^s} \frac{Q_j}{\sqrt{\sum_u (\sigma^u - \eta_j^u(\tau))^2}} \right].
\]

(4.30)
Then, the Dirac Hamiltonian (4.4) has the following form in the new variables

\[ H_{DF} = \sum_i \left( 1 + n_F(\tau, \eta_i^u(\tau)) \right) \times \]

\[ \times \sqrt{m_i^2 c^2 + h_F^r s F(\tau, \eta_i^u(\tau)) \left( \hat{\kappa}_{ir}(\tau) - \frac{Q_i}{c} A_{\perp r}(\tau, \eta_i^u(\tau)) \right) \left( \hat{\kappa}_{is}(\tau) - \frac{Q_i}{c} A_{\perp s}(\tau, \eta_i^u(\tau)) \right)} - \]

\[ - \sum_i n_F(\tau, \eta_i^u(\tau)) \left( \hat{\kappa}_{ir}(\tau) - \frac{Q_i}{c} A_{\perp r}(\tau, \eta_i^u(\tau)) \right) + \]

\[ + \frac{1}{c} W(\eta_i^u(\tau), ..., \eta_N^u(\tau)) + \int d^3 \sigma \sqrt{\gamma(\tau, \sigma^u)} \left[ (1 + n_F) \tilde{T}_{\perp} + n_F \tilde{T}_{\perp} \right](\tau, \sigma^u) \]

\[ + \int d^3 \sigma \mu(\tau, \sigma^u) \pi(\tau, \sigma^u) - \frac{1}{c} \left( \partial_{\tau} A_r(\tau, \sigma^u) - a_r(\tau, \sigma^u) \right) \Gamma(\tau, \sigma^u) \right] + O(\Gamma^2), \]  

(4.31)

In Eq.(4.31) we can discard the term quadratic in the constraint \( \Gamma(\tau, \sigma^u) \approx 0 \), because it is strongly zero according to constraint theory: it does never contribute to the dynamics on the constraint sub-manifold (the only relevant region of phase space for constrained systems).

To get the non-covariant radiation gauge we add the gauge fixing

\[ \eta_{\text{em}}(\tau, \sigma^u) \approx 0, \]  

(4.32)

implying \( A_r \approx A_{\perp r} \) due to Eq.(4.26). The \( \tau \)-constancy, \( \partial_{\tau} \eta_{\text{em}}(\tau, \sigma^u) \approx 0 \), of this gauge fixing, together with the Gauss law constraint \( \Gamma(\tau, \sigma^u) \approx 0 \), implies the secondary gauge fixing

\[ A_r(\tau, \sigma^u) - a_r(\tau, \sigma^u) \approx 0, \]  

(4.33)

so that we get

\[ A_r(\tau, \sigma^u) \approx \int d^3 \sigma' \frac{1}{4\pi} \sqrt{\sum_a (\sigma^u - \sigma'^u)^2} \frac{\partial}{\partial \sigma'^r} \left[ n_F^s(\tau, \sigma'^u) F_{sr}(\tau, \sigma'^u) + \right] \]

\[ + \left( 1 + n_F(\tau, \sigma'^u) \right) h_F r s(\tau, \sigma'^u) \left( \pi^s_{\perp}(\tau, \sigma'^u) + \frac{1}{4\pi} \sum_j \frac{\partial}{\partial \sigma'^s} \frac{Q_j}{\sqrt{\sum_a (\sigma'^u - \eta_j^u(\tau))^2}} \right). \]  

(4.34)

Therefore, in the radiation gauge the magnetic field of Eqs.(2.19) is transverse: \( B_r = \epsilon_{ruv} \partial_v A_{\perp u} \). But the electric field \( E_r = -F_{\tau r} = -\partial_{\tau} A_{\perp r} + \partial_r A_r \) is not transverse: it has
$E_{\perp r} = -\partial_r A_{\perp r}$ as a transverse component. Instead the transverse quantity is $\pi_{\perp}^r$, which coincides with $\delta^s E_{\perp s}$ only in inertial frames, and whose expression in terms of the electric and magnetic fields, determined by Eqs.(4.22) and (3.2), is

$$\pi_{\perp}^r(\tau, \sigma^n) = \left[ \frac{\sqrt{\pi}}{1+n} h^s \left( E_s - \epsilon_{uv} n_u B_v \right) \right](\tau, \sigma^n) + \partial\tau \left( \sum_i \frac{Q_i}{4\pi \sum_u (\sigma^n - \eta^n_{i_u}(\tau))^2} \right).$$

The final form of the Dirac Hamiltonian in the radiation gauge (after the elimination of the variables $\eta_{em}$, $\Gamma$, $A_r$, $\pi^r$ by going to Dirac brackets) is

$$H_{DF} = \sum_i \left( 1 + n_F(\tau, \eta^n_{i_u}(\tau)) \right) \times \left[ \frac{\sqrt{m_i^2 c^2 + h_{F}^a(\tau, \eta^n_{i_u}(\tau)) (\kappa_{ir}(\tau) - \frac{Q_i}{c} A_{\perp r}(\tau, \eta^n_{i_u}(\tau))) (\kappa_{is}(\tau) - \frac{Q_i}{c} A_{\perp s}(\tau, \eta^n_{i_u}(\tau))) - \sum_i n_i^r(\tau, \eta^n_{i_u}(\tau)) (\kappa_{ir}(\tau) - \frac{Q_i}{c} A_{\perp r}(\tau, \eta^n_{i_u}(\tau))) + \frac{1}{c} W(\eta^n_{i_u}(\tau), ..., \eta^n_{N_u}(\tau)) + \int d^2\sigma \sqrt{\gamma(\tau, \sigma^n)} (1 + n_F) \tilde{T}_{\perp \perp} + n_F \tilde{T}_{\perp r} \right)(\tau, \sigma^n)$$

(4.35)

where $\tilde{T}_{\perp AB}$ is given in Eq.(4.28). In $H_{DF}$ the components of $g_{AB}(\tau, \sigma^n)$ are the inertial potentials giving rise to the relativistic inertial forces.

The Hamilton-Dirac equations for the particles are ($\mathcal{F}_{ir}(\tau)$ is defined in Eq.(4.8))

$$\dot{\eta}^r_{i_u}(\tau) \triangleq \left[ \frac{\sqrt{m_i^2 c^2 + h_{F}^a(\tau, \eta^n_{i_u}(\tau)) (\kappa_{ir}(\tau) - \frac{Q_i}{c} A_{\perp r}(\tau, \eta^n_{i_u}(\tau))) - \sum_i n_i^r(\tau, \eta^n_{i_u}(\tau)) (\kappa_{ir}(\tau) - \frac{Q_i}{c} A_{\perp r}(\tau, \eta^n_{i_u}(\tau))) + \frac{1}{c} W(\eta^n_{i_u}(\tau), ..., \eta^n_{N_u}(\tau)) + \mathcal{F}_{ir}(\tau).}{(1 + n_F(\tau, \eta^n_{i_u}(\tau))) \frac{\partial A_{\perp u}(\tau, \eta^n_{i_u}(\tau))}{\partial \eta^r_{i_u}} - \frac{1}{c} \frac{\partial}{\partial \eta^r_{i_u}} W(\eta^n_{i_u}(\tau), ..., \eta^n_{N_u}(\tau)) + \mathcal{F}_{ir}(\tau).} \right]$$

(4.36)

In the second half of Eqs.(4.36) the sum of the inertial 2-body Coulomb potentials is replaced by the non-inertial N-body potential $W(\eta^n_{i_u}(\tau), ..., \eta^n_{N_u}(\tau))$ of Eq.(4.29), which can be shown to have the following property due to Eq.(4.30)

$$\frac{\partial W}{\partial \eta^r_{i_u}} = -Q_i \left( \frac{\partial A_r}{\partial \sigma^n} \right)_{\sigma^n = \eta^n_{i_u}} \approx -Q_i \left( \frac{\partial A_r}{\partial \sigma^n} \right)_{\sigma^n = \eta^n_{i_u}}.$$

(4.37)

In the radiation gauge the electric field of Eq.(2.19) is $E_r \approx -\partial_r A_{\perp r} + \partial_t A_r$. Consistently with Eq.(4.11) we have
The first of Eqs. (4.36) can be inverted to get

$$\bar{\kappa}_{ir} (\tau) = \left( \frac{h_{Fr} m_{ic}(\dot{\eta}_{1}^{s}(\tau) + n_{F}^{s})}{\sqrt{(1 + n_{F})^{2} - h_{Fuv}(\dot{\eta}_{1}^{u}(\tau) + n_{F}^{u})(\dot{\eta}_{1}^{v}(\tau) + n_{F}^{v})}} \right)(\tau, \eta_{i}^{u}(\tau)) + \frac{Q_{i}}{c} A_{\tau r}(\tau, \eta_{i}^{u}(\tau)).$$

(4.39)

See the next Subsection for its expression in a nearly non-relativistic frame.

In the general case to evaluate the integral in Eq. (4.39) we must regularize the function

$$t^{rs}(\sigma^{u}) = \frac{1}{\left(\sum_{u}(\sigma^{u})^{2}\right)^{3/2}} \left(\delta^{rs} - 3 \frac{\sigma^{r} \sigma^{s}}{\sum_{u}(\sigma^{u})^{2}}\right),$$

which is singular at $\sigma^{u} = 0$. By considering it as a distribution, we must give a prescription to define the integral

$$\int d^{3}\sigma t^{rs}(\sigma^{u}) f(\sigma^{u}),$$

where $f(\sigma^{u})$ is a test function. Following Ref. [27], we consider the sphere $S_{R}$ centered in the origin and defined by the relation $\sqrt{\sum_{u}(\sigma^{u})^{2}} < R$ and the space $\Omega_{R}$ external to it of the points such that $\sqrt{\sum_{u}(\sigma^{u})^{2}} \geq R$. The integral is written in the form

$$\int d^{3}\sigma t^{rs}(\sigma^{u}) f(\sigma^{u}) = \int_{S_{R}} d^{3}\sigma t^{rs}(\sigma^{u}) f(\sigma^{u}) + \int_{\Omega_{R}} d^{3}\sigma t^{rs}(\sigma^{u}) f(\sigma^{u}).$$

(4.40)

The first term, containing the singularity, can be shown to have the expression

$$\lim_{R \to 0} \int_{S_{R}} d^{3}\sigma t^{rs}(\sigma^{u}) f(\sigma^{u}) = \frac{4\pi}{3} \delta^{rs} f(0).$$

(4.41)

Regarding the second term in Eq. (4.40) we can define a distribution $\bar{t}^{rs}(\sigma^{u})$ such that the following integral

$$\lim_{R \to 0} \int_{\Omega_{R}} d^{3}\sigma t^{rs}(\sigma^{u}) f(\sigma^{u}) = \int d^{3}\sigma \bar{t}^{rs}(\sigma^{u}) f(\sigma^{u})$$

(4.42)

has no singularity in the origin. As a consequence we get
\[ t^{rs}(\sigma^u) = \frac{4\pi}{3} \delta^{rs} \delta^3(\sigma^u) + \tilde{t}^{rs}(\sigma^u). \] (4.43)

Therefore we get

\[ W(\eta_1^u(\tau), ..., \eta_N^u(\tau)) = \]

\[ = \sum_{i \neq j} \int d^3 \sigma \frac{h_{Frs}(\tau, \sigma^u) (1 + n_F(\tau, \sigma^u))}{2 \sqrt{\gamma_F(\tau, \sigma^u)}} \left( \frac{1}{4\pi} \frac{\partial}{\partial \sigma^r} \frac{Q_i}{\sqrt{\sum_u (\sigma^u - \eta_i^u(\tau))^2}} \right) \left( \frac{1}{4\pi} \frac{\partial}{\partial \sigma^s} \frac{Q_j}{\sqrt{\sum_u (\sigma^u - \eta_j^u(\tau))^2}} \right) + \]

\[ + \int d^3 \sigma \left[ \frac{h_{Frs}(1 + n_F)}{\sqrt{\gamma_F}} \pi^r_{\perp} + n_F F_{rs} \right](\tau, \sigma^u) \left( \frac{1}{4\pi} \sum_j \frac{\partial}{\partial \sigma^s} \frac{Q_j}{\sqrt{\sum_u (\sigma^u - \eta_j^u(\tau))^2}} \right). \] (4.44)

After some integrations by parts we get

\[ W(\eta_1^u(\tau), ..., \eta_N^u(\tau)) = \]

\[ = \sum_{i \neq j} \int d^3 \sigma \frac{h_{Frs}(\tau, \sigma^u) (1 + n_F(\tau, \sigma^u))}{2 \sqrt{\gamma_F(\tau, \sigma^u)}} \left( \frac{1}{16\pi^2} \frac{Q_i Q_j}{\sqrt{\sum_u (\sigma^u - \eta_i^u(\tau))^2}} \right) t^{rs}(\sigma^u - \eta_j^u(\tau)) - \]

\[ + \sum_{i \neq j} \int d^3 \sigma \frac{\partial}{\partial \sigma^s} \left( \frac{h_{Frs}(\tau, \sigma^u) (1 + n_F(\tau, \sigma^u))}{2 \sqrt{\gamma_F(\tau, \sigma^u)}} \right) \left( \frac{1}{4\pi} \frac{Q_i Q_j}{\sqrt{\sum_u (\sigma^u - \eta_i^u(\tau))^2}} \right) - \]

\[ - \int d^3 \sigma \left( \frac{1}{4\pi} \sum_j \frac{Q_j}{\sqrt{\sum_u (\sigma^u - \eta_j^u(\tau))^2}} \right) \frac{\partial}{\partial \sigma^s} \left[ \frac{h_{Frs}(1 + n_F)}{\sqrt{\gamma_F}} \pi^r_{\perp} + n_F F_{rs} \right](\tau, \sigma^u), \]

and then we can get the following form
\[
W(\eta^u_1(\tau), ..., \eta^u_N(\tau)) =
\]
\[
= \sum_{i \neq j} \frac{1}{12\pi} \sum_r \left( \frac{h_{Frr}(\tau, \eta^u_j(\tau)) \left(1 + n_F(\tau, \eta^u_j(\tau))\right)}{2 \sqrt{\gamma_F(\tau, \eta^u_j(\tau))}} \right) \frac{Q_i Q_j}{\sqrt{\sum_u (\eta^u_j(\tau) - \eta^u_i(\tau))^2}} +
\]
\[
+ \sum_{i \neq j} \int d^3\sigma \left( \frac{1}{4\pi} \frac{Q_i Q_j}{\sqrt{\sum_u (\sigma^u - \eta^u_j(\tau))^2}} \right) \left[ \frac{h_{Frs}(\tau, \sigma^u) \left(1 + n_F(\tau, \sigma^u)\right)}{2 \sqrt{\gamma_F(\tau, \sigma^u)}} \right] T^s(\sigma^u - \eta^u_j(\tau)) -
\]
\[
+ \frac{1}{4\pi} \frac{\sigma^r - \eta^r_j(\tau)}{\left(\sum_u (\sigma^u - \eta^u_j(\tau))^2\right)^{3/2}} \frac{\partial}{\partial \sigma^s} \left( \frac{h_{Frs}(\tau, \sigma^u) \left(1 + n_F(\tau, \sigma^u)\right)}{2 \sqrt{\gamma_F(\tau, \sigma^u)}} \right) \right] -
\]
\[
- \int d^3\sigma \left( \frac{1}{4\pi} \sum_j \frac{Q_j}{\sqrt{\sum_u (\sigma^u - \eta^u_j(\tau))^2}} \right) \frac{\partial}{\partial \sigma^s} \left[ \frac{h_{Frs}(\tau, \sigma^u) \left(1 + n_F(\tau, \sigma^u)\right)}{\sqrt{\gamma_F}} \right] \pi^r_\perp + n^r_F F_{rs} (\tau, \sigma^u),
\]
(4.46)

which can be checked to be explicitly symmetric in the exchange of \( \vec{\eta}_i \) with \( \vec{\eta}_j \).

Finally the Hamilton equations for the transverse electro-magnetic fields \( A^I_\perp \) and \( \pi^r_\perp \) in the radiation gauge implied by the Dirac Hamiltonian (4.35) are

\[
\partial_\tau A^I_\perp(\tau, \vec{\sigma}) \triangleq \{ A^I_\perp(\tau, \vec{\sigma}), H_{DF} \} =
\]
\[
= \delta_{rn} P^{nu}_\perp(\vec{\sigma}) \left[ \frac{(1 + n)^3 e_{(a)u}}{3e} \left( \pi^v_\perp - \delta^{vm} \sum_i Q_i \eta_i \frac{\partial c(\vec{\sigma}, \vec{\eta}_i(\tau))}{\partial \sigma^m} \right) \right] +
\]
\[
+ \tilde{n}_{(a)}^3 e_{(a)v} F_{vu}(\tau, \vec{\sigma}),
\]

\[
\partial_\tau \pi^r_\perp(\tau, \vec{\sigma}) \triangleq \{ \pi^r_\perp(\tau, \vec{\sigma}), H_{DF} \} =
\]
\[
= P^{rn}_\perp(\vec{\sigma}) \delta_{nm} \left( \sum_i \eta_i Q_i \delta^3(\vec{\sigma}, \vec{\eta}_i(\tau)) \right) 3e_{(a)v} \left( \frac{(1 + n)^3 e_{(a)u} \tilde{\kappa}_{is}(\tau)}{\sqrt{m^2 c^2 + 3 e_{(a)v} \left( \tilde{\kappa}_{ir}(\tau) - \frac{Q_u}{c} A^r_\perp \right) \delta_{(a)u} \left( \tilde{\kappa}_{is}(\tau) - \frac{Q_u}{c} A^s_\perp \right)}} - \tilde{n}_{(a)} \right) (\tau, \vec{\eta}_i(\tau)) +
\]
\[+ \left[(1+n) \left(3e_3 \varepsilon_3 \varepsilon_{(a)}^v 3e_3 \varepsilon_{(b)}^v \left(3e_{(a)}^e - 3e_{(a)}^e \varepsilon_{(b)}^e \right) \partial_r F_{sv} + \right.\]
\[+ \partial_r \left(3e_3 \varepsilon_3 \varepsilon_{(a)}^v \left(3e_{(a)}^e - 3e_{(a)}^e \varepsilon_{(b)}^e \right) \right) F_{sv} \left.\right) + \]
\[+ \partial_r \left(n^3e_3 \varepsilon_3 \varepsilon_{(b)}^v \left(3e_{(a)}^e - 3e_{(a)}^e \varepsilon_{(b)}^e \right) \right) F_{sv} + \]
\[+ \tilde{n}_{(a)} \left(3e_r \varepsilon_{(a)}^v \partial_r \pi_\perp^m + \partial_r \left(3e_{(a)}^e \pi_\perp^m - \partial_r \left(3e_{(a)}^e \pi_\perp^m + \right)\right) + \right.\]
\[+ \left.\left(3e_r \varepsilon_{(a)}^v \delta^m t - \partial_r \left(3e_{(a)}^e \delta^r t \right) \sum_i \eta_i Q_i \frac{\partial c(\tilde{\sigma}, \tilde{\eta}_i(\tau))}{\partial \sigma^i} \right) + \left.\right.\]
\[+ \left.\left(3e_r \varepsilon_{(a)}^v - 3e_{(a)}^e \delta^r t \right) \sum_i \eta_i Q_i \frac{\partial^2 c(\tilde{\sigma}, \tilde{\eta}_i(\tau))}{\partial \sigma^i \partial \sigma^r} \right) + \right.\]
\[+ \partial_r \tilde{n}_{(a)} \left(3e_r \varepsilon_{(a)}^v - 3e_{(a)}^e \delta^r t \right) \sum_i \eta_i Q_i \frac{\partial c(\tilde{\sigma}, \tilde{\eta}_i(\tau))}{\partial \sigma^i} \right) \right)\). (4.47)

Here \(c(\sigma^u, \sigma^r)^u = \frac{1}{4\pi \sqrt{\Sigma_u (\sigma^u - \sigma^r)^u}}\) and, following the general relativity notation of Ref. [12], the metric has been expressed in terms of triads \(3e_r^r\) and cotriads \(3e_{(a)}^e\) on \(\Sigma_r\) as in Eq. (2.10): \(h_{rr} = \sum_a 3e_{(a)}^r 3e_{(a)}^r, h_{r}^s = \sum_a 3e_r^r 3e_{(a)}^s, \gamma_F = 3e\). The shift functions of Eq. (2.4) are replaced by \(\tilde{n}_{(a)} = n^r 3e_{(a)}^r\).

C. On the Non-Relativistic Limit

Let us consider the nearly non-relativistic limit of the embedding (2.10) given in Eqs. (2.16). It can be done either before or after the choice of the radiation gauge.

Since we have \(h_{rr} = \delta_{rr} + O(c^{-2})\), we can use the vector notation of the inertial frames for the 3-vectors: \(\vec{V} = \left\{ V_r = \vec{V}_r \right\}\) (since \(g_{rr} = e \left(1 - \sum_r \left(n_F^r \right)^2 \right) + O(c^{-2}) = e + O(c^{-2})\), we still have \(V^r = g^{rA} V_A \neq \vec{V}^r\) for 4-vectors \(V_A\)). Therefore we have \(\tilde{\kappa}_i = \{ \tilde{\kappa}_i^r \} \equiv \{ \tilde{\kappa}_i \}, \vec{E} = \{ E_r = \vec{E}_r \} + O(c^{-2}), \vec{B} = \{ B_r = \vec{B}_r \} + O(c^{-2}), \) but \(\vec{A} = \{ A_{\perp} = \vec{A}^\perp \neq \vec{A}_\perp \} + O(c^{-2}).\)

In these rigidly-rotating non-inertial frames the equations of motion (4.9) takes the form (the Newtonian functions are \(\tilde{f}(t) = f(\tau = c t)\); \(\tilde{\Omega}(ct)\) has the components \(\tilde{\Omega}(ct)\) defined after Eq. (2.15))

\[m_i \frac{dt}{dt} \left[ \frac{d \tilde{\eta}_i(ct)}{dt} + \vec{v}(ct) + \vec{\Omega}(ct) \times \tilde{\eta}_i(ct) \right] \equiv Q_i \left[ \vec{E} + \frac{1}{c} \frac{d \tilde{\eta}_i(ct)}{dt} \times \vec{B} \right] (ct, \tilde{\eta}_i(ct)) + \]
\[+ \vec{F}_i(ct), \]
\[\vec{F}_i(ct) = -m_i \vec{\Omega}(ct) \times \left[ \frac{d \tilde{\eta}_i(ct)}{dt} + \vec{v}(ct) + \vec{\Omega}(ct) \times \tilde{\eta}_i(ct) \right]. \] (4.48)
As a consequence the final form of the equations of motion of the particles is

\[ m_i \frac{d^2 \vec{\eta}_i(ct)}{dt^2} = + Q_i \left[ \vec{E} + \frac{d \vec{\eta}_i(ct)}{dt} \times \vec{B} \right] (ct, \vec{\eta}_i(ct)) + \vec{F}^{(in)}_i(ct), \]

\[ \vec{F}^{(in)}_i(ct) = \vec{F}_i(ct) + m_i \frac{d}{dt} \left( \vec{v}(ct) + \vec{\Omega}(ct) \times \vec{\eta}_i(ct) \right) = \]

\[ = -m_i \left[ \vec{\Omega}(ct) \times \left( \vec{\Omega}(ct) \times \vec{\eta}_i(ct) \right) + 2 \vec{\Omega}(ct) \times \frac{d \vec{\eta}_i(ct)}{dt} + \frac{d \vec{\Omega}(ct)}{dt} \times \vec{\eta}_i(ct) + \frac{d \vec{v}(ct)}{dt} + \vec{\Omega}(ct) \times \vec{v}(ct) \right], \]  

(4.49)

\[ \vec{F}^{(in)}(\tau) \] is the sum of all the inertial forces (centrifugal, Coriolis, Jacobi, the two pieces of the linear acceleration) present in Newtonian rigid non-inertial frames.

The equations of motion (4.36), (4.29) of the particles in the radiation gauge become

\[ m_i \frac{d^2 \vec{\eta}_i(\tau)}{d\tau^2} = - \frac{\partial}{\partial \vec{\eta}_i} W(\vec{\eta}_1(\tau), ..., \vec{\eta}_N(\tau)) + Q_i \left[ - \frac{1}{c} \frac{\partial \vec{A}_1}{\partial \tau} + \frac{1}{c} \frac{d \vec{\eta}_i(\tau)}{dt} \times \vec{B} \right] (ct, \vec{\eta}_i(\tau)) + \]

\[ + \vec{\mathcal{F}}^{(in)}_i(\tau), \]  

(4.50)

where the non-inertial Coulomb potential takes the form (\( \tau = ct \))\(^{14} \)

\[ W(\vec{\eta}_1(\tau), ..., \vec{\eta}_N(\tau)) = \]

\[ = + \sum_{i>j} \frac{Q_i Q_j}{4\pi |\vec{\eta}_i(\tau) - \vec{\eta}_j(\tau)|} - \sum_i \frac{Q_i}{c} \left[ \vec{v}(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) + \vec{\Omega}(\tau) \times \vec{\eta}_i(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) \right]. \]

(4.51)

Finally the Hamiltonian (4.35) becomes

\[ a_{\tau}(\tau, \vec{\sigma}) = - \left[ \sum_k \frac{Q_k}{4\pi |\vec{\sigma} - \vec{\eta}_k|} - \frac{\vec{v}}{c} \vec{A}_\perp(\tau, \vec{\sigma}) - \frac{\vec{\Omega}}{c} \times \vec{\sigma} \cdot \vec{A}_\perp(\tau, \vec{\sigma}) \right], \]

\(^{14} \text{In this case from Eq.(4.30) we get} \)

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\[
\hat{H}_R = \sum_i \sqrt{m_i^2 c^2 + \left( \tilde{k}_i(\tau) - \frac{Q_i}{c} \vec{A}_\perp(\tau, \tilde{\eta}_i(\tau)) \right)^2} + \sum_{i>j} \frac{Q_i Q_j}{4 \pi c | \tilde{\eta}_i(\tau) - \tilde{\eta}_j(\tau) |} + \\
+ \frac{1}{2c} \int d^3\sigma \left( \vec{\pi}_\perp^2(\tau, \vec{\sigma}) - \vec{A}_\perp(\tau, \vec{\sigma}) \cdot [\Delta \vec{A}_\perp(\tau, \vec{\sigma})] \right) + \\
- \frac{\vec{v}(\tau)}{c} \cdot \left[ \sum_i \vec{\tilde{k}}_i(\tau) - \frac{1}{c} \int d^3\sigma \vec{\pi}_\perp(\tau, \vec{\sigma}) \times (\vec{\tau} \times \vec{A}_\perp(\tau, \vec{\sigma})) \right] + \\
- \frac{\vec{\Omega}(\tau)}{c} \cdot \left[ \sum_i \vec{\tilde{\eta}}_i(\tau) \times \vec{\tilde{k}}_i(\tau) + \vec{\mathcal{J}}(\tau) \right],
\]

\[
\vec{\mathcal{J}}(\tau) = -\frac{1}{c} \int d^3\sigma \sum_r \pi_r^r(\tau, \vec{\sigma}) \left( \vec{\sigma} \times \vec{\tau} \right) \vec{A}_\perp^r(\tau, \vec{\sigma}) - \vec{A}_\perp(\tau, \vec{\sigma}) \times \vec{\pi}_\perp^r(\tau, \vec{\sigma}),
\]

(4.52)

where \[\vec{\mathcal{J}}(\tau)\] is the total angular momentum of the electro-magnetic field.

It can be checked that this Hamiltonian generates the previous limit of the equations of motion of the particles. In particular the first set of Hamilton equations is

\[
\frac{1}{c} \frac{d \vec{\tilde{\eta}}_i(\tau)}{dt} = \frac{\tilde{k}_i(\tau) - \frac{Q_i}{c} \vec{A}_\perp(\tau, \tilde{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \left( \tilde{k}_i(\tau) - \frac{Q_i}{c} \vec{A}_\perp(\tau, \tilde{\eta}_i(\tau)) \right)^2}} - \frac{\vec{v}(\tau)}{c} - \frac{\vec{\Omega}(\tau)}{c} \times \vec{\tilde{\eta}}_i(\tau).
\]

(4.53)
V. THE INSTANT FORM OF DYNAMICS IN NON-INERTIAL FRAMES AND IN THE INERTIAL AND NON-INERTIAL REST FRAMES.

In this Section we study the problem of the separation of the relativistic non-covariant canonical 4-center of mass of an isolated system from the relative variables describing its dynamics. We first recall how this problem is solved in the inertial rest-frame instant form of dynamics [1, 3, 4, 5, 8]. As said in the Introduction the isolated system is described as a decoupled pseudo-particle (described by the non-covariant canonical variables \( \vec{z} \) and \( \vec{h} \)) carrying a pole-dipole structure given by its invariant mass and its rest spin. On each instantaneous Wigner 3-space, centered on the inertial observer corresponding to the Fokker-Pryce 4-center of inertia, these quantities are functions of the relative variables of the isolated system after the elimination of the internal 3-center of mass. The double counting of the center of mass is avoided by the presence of three pairs of second class constraints: the rest-frame conditions, i.e. the vanishing of the internal 3-momentum, and the vanishing of the internal boosts.

In Subsection A we will show how to get these conditions in the inertial rest frames starting from the embeddings (1.1), from the determination (3.8) of their conjugate momenta and from the Poincare’ generators (3.17).

In Subsection B we will extend this construction to determine the three pairs of second class constraints in an arbitrary admissible non-inertial frame described by the embeddings (2.1) and centered on an arbitrary time-like observer. Again the isolated system can be visualized as a pole-dipole carried by the external decoupled center of mass.

In Subsection C we will define the special family of the non-inertial rest-frames, centered on the inertial Fokker-Pryce 4-center of inertia, and the associated non-inertial rest-frame instant form. They are relevant because they are the only global non-inertial frames allowed by the equivalence principle (forbidding the existence of global inertial frames) in canonical metric and tetrad gravity in globally hyperbolic, asymptotically flat (asymptotically Minkowskian) space-times without super-translations, so to have the asymptotic ADM Poincare’ group [11]. Also in this case we identify the three pairs of second class constraints eliminating the internal 3-center of mass, visualizing the isolated system as a pole-dipole and allowing to describe the dynamics on the instantaneous (non-Euclidean) 3-spaces only in terms of relative variables. Then in Subsection D we show how the Hamiltonian description of Section IV has to be modified if we take this point of view in the description of the isolated system. We also delineate the analogue of this procedure for the general case of Subsection B.
A. The Inertial Rest-Frame Instant Form

As said in the Introduction every configuration of an isolated system, with associated finite Poincare’ generators $P^\mu$, $J^{\mu\nu}$, identifies a unique inertial frame in an intrinsic way: the inertial rest frame whose Euclidean instantaneous 3-spaces (the Wigner 3-spaces) are orthogonal to the conserved 4-momentum $P^\mu$ of the configuration. The embedding corresponding to the inertial rest frame, centered on the Fokker-Pryce center of inertia, is given in Eq.(1.1)

The generators of the external realization of the Poincare’ algebra are (following footnote 10 we use only $\epsilon_{ijk}$; $M$ and $\vec{S}$ have vanishing Poisson brackets with $\vec{z}$ and $\vec{h}$ and we have $\{\vec{S}^i, \vec{S}^j\} = \delta^{in} \delta^{jn} \epsilon_{mnk} \vec{S}^k$)

\[
P^\mu = Mc h^\mu, \quad h^\mu = \left(\sqrt{1 + \vec{h}^2}; \vec{h}\right),
\]
\[
J^i = \delta^{in} \epsilon_{mnk} \left(z^n h^k + \vec{S}^k\right), \quad K^i = -\sqrt{1 + \vec{h}^2} z^i + \frac{\delta^{in} \epsilon_{njk} \vec{S}^j h^k}{1 + \sqrt{1 + \vec{h}^2}}.
\]

while those of the unfaithful internal realization of the Poincare’ algebra determined by the energy-momentum tensor (in inertial frames Eqs.(3.8) imply $T_{\tau\tau} = T^{\tau\tau}$ and $T_{\tau r} = \delta_{rs} T^{\tau s}$) are

\[
Mc = \int d^3\sigma T^{\tau\tau}(\tau, \sigma^u), \quad \vec{S}^r = \frac{1}{2} \delta^{rs} \epsilon_{svu} \int d^3\sigma \sigma^u T^{\tau v}(\tau, \sigma^u),
\]
\[
\mathcal{P}^r = \int d^3\sigma T^{\tau r}(\tau, \sigma^u) \approx 0, \quad \vec{K}^r = -\int d^3\sigma \sigma^r T^{\tau\tau}(\tau, \sigma^u) \approx 0.
\]

The constraints $\vec{\mathcal{P}} \approx 0$ are the rest-frame conditions identifying the inertial rest frame. Having chosen the Fokker-Pryce center of inertia as origin of the 3-coordinates, the (interaction-dependent) constraints $\vec{K} \approx 0$ are their gauge fixing: they eliminate the internal 3-center of mass so not to have a double counting (external, internal). Therefore the isolated system is described by the external non-covariant 3-center of mass $\vec{z}$, $\vec{h}$, and by an internal space of Wigner-covariant relative variables ($M$ and $\vec{S}$ depend only upon them).

Eqs. (5.1) and (5.2) are obtained in the following way. If we put the embedding (1.1), namely $z^\mu(\tau, \sigma^u) = Y^\mu(0) + h^\mu \tau + \epsilon^\mu_n(\vec{h}) \sigma^r = Y^\mu(0) + e^\mu_A(\vec{h}) \sigma^A$, in Eq.(3.8), we get $\rho^\mu(\tau, \sigma^u) \approx$
In this equation we use the notation $\epsilon^\mu(\vec{h}) T^{\tau\tau}(\tau, \sigma^u) = \epsilon^\mu(\vec{h}) T^{A\tau}(\tau, \sigma^u)$. Then the first of Eqs.(3.17) implies $P^\mu = M c h^\mu$ if $M c = \int d^3 \sigma T^{\tau\tau}(\tau, \sigma^u)$ and $P^r = \int d^3 \sigma T^{rr}(\tau, \sigma^u) \approx 0$.

The second of Eqs.(3.17) gives $J^{\mu \nu} = \left( Y(0)\epsilon^\nu(\vec{h}) - Y(0) \epsilon^\nu(\vec{h}) \right) \int d^3 \sigma T^{A\tau}(\tau, \sigma^u) + \epsilon_A^\mu(\vec{h}) \epsilon^\nu_B(\vec{h}) S^{AB}$ with $S^{AB} = \int d^3 \sigma \left( \sigma^A T^{B\tau} - \sigma^B T^{A\tau} \right)(\tau, \sigma^u)$. By using $P^r \approx 0$ we get $J^{\mu \nu} \approx M c \left( Y(0) h^\nu - Y(0) h^\mu \right) + \epsilon_A^\mu(\vec{h}) \epsilon^\nu_B(\vec{h}) S^{AB}$ with $S^{rr} = \int d^3 \sigma \sigma^r T^{rr}(\tau, \sigma^u)$ and $S^{rs} = \int d^3 \sigma \left( \sigma^r T^{s\tau} - \sigma^s T^{r\tau} \right)(\tau, \sigma^u)$. Then, by using the expression of the Fokker-Pryce 4-center of inertia given in Eq.(2.20) of Ref.[8], i.e. $Y(0) = Y(0) + h^\mu \tau$ with $Y(0) = \left( \sqrt{1 + \frac{\vec{h}^2}{M c^2}} \frac{\vec{z} \cdot \vec{h}}{M c} + \frac{\vec{z} \cdot \vec{h}}{M c} \frac{\vec{r}}{M c(1+\sqrt{1+\vec{h}^2})} \right)$, as a function of $\tau, \vec{z}, \vec{h}, M c$ and of $\vec{s}$, and the expression of $\epsilon_A^\mu(\vec{h})$ given after Eq.(1.1), we get:

a) $J^{ij} = z^i h^j - z^j h^i + \delta^i \delta^j \epsilon_{rsk} \int d^3 \sigma \sigma^r T^{r\tau}(\tau, \sigma^u)$, which coincides with Eq.(5.1) if $\vec{S}$ has the expression given in Eq.(5.2):

b) $J^{oi} = -\sqrt{1 + \vec{h}^2} z^i + \delta^{in} \epsilon_{njk} \vec{S}^j h^k + \epsilon^i_A(\vec{h}) \epsilon^j_A(\vec{h}) S^{rr}$, which coincides with Eq.(5.1) if $\mathcal{K}^r = -S^{rr} \approx 0$ as in Eqs.(5.2).

Therefore we have $S^{AB} \approx (\delta^A \delta^B - \delta^A \delta^B) \mathcal{K}^r + \delta^A \delta^B \epsilon_{rsk} \vec{S}^k \approx \delta^A \delta^B \epsilon_{rsk} \vec{S}^k$.

As shown in Ref.[8], the restriction of the embedding $z^\mu(\tau, \sigma^u)$ to the Wigner 3-spaces (1.1) implies the replacement of the Dirac Hamiltonian (3.13) with the new one

$$H_{DW} = M c + \int d^3 \sigma \left( \mu \pi^\tau - A_\tau \Gamma \right)(\tau, \sigma^u). \quad (5.3)$$

Therefore, consistently with Eqs.(5.2), the effective Hamiltonian is the invariant mass of the isolated system, whose conserved rest spin is $\vec{S}$. As already said, the three pairs of second class constraints $\vec{P} \approx 0, \vec{K} \approx 0$, eliminate the internal 3-center of mass.

As shown in Refs.[8, 9], in the rest-frame instant form it is possible to restrict the description of N charged positive-energy particles plus the electro-magnetic field to the radiation gauge (see next Section for the non-inertial case), where all the electro-magnetic quantities are transverse. The mutual Coulomb interaction among the particles appears in this gauge, the Hamiltonian (5.3) reduces to $M c$ and we get the following form of the internal Poincare' generators (5.2) \(^{15}\)

\(^{15}\) In this equation we use the notation $\vec{K}_i(\tau)$ for the Coulomb-dressed momenta $\vec{K}_i(\tau) = \vec{K}_i(\tau) - \frac{\partial n_{\tau,\vec{h}}(\tau)}{\partial \vec{h}_i}$ belonging to the Shannugadhasan canonical basis defined in Eqs.(4.26).
\[ \mathcal{E}_{\text{(int)}} = \mathcal{P}^r = M c^2 = c \int d^3 \sigma \, T^{r r}(\tau, \vec{\sigma}) = \]
\[ = c \sum_{i=1}^N \sqrt{m_i^2 c^2 + \left( \vec{\kappa}_i(\tau) - \frac{Q_i}{c} \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) \right)^2} + \]
\[ + \sum_{i \neq j} \frac{Q_i Q_j}{4\pi \left| \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \right|} + \frac{1}{2} \int d^3 \sigma \left[ \vec{\pi}_\perp^2 + \vec{B}^2 \right](\tau, \vec{\sigma}) = \]
\[ = c \sum_{i=1}^N \left( \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} - \frac{Q_i}{c} \vec{\kappa}_i(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_i(\tau)) \right)^2 + \]
\[ + \sum_{i \neq j} \frac{Q_i Q_j}{4\pi \left| \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \right|} + \frac{1}{2} \int d^3 \sigma \left[ \vec{\pi}_\perp^2 + \vec{B}^2 \right](\tau, \vec{\sigma}), \]
\[ \vec{P}_{\text{(int)}} = \int d^3 \sigma \, T^{r r}(\tau, \vec{\sigma}) = \sum_{i=1}^N \vec{\kappa}_i(\tau) + \frac{1}{c} \int d^3 \sigma \left[ \vec{\pi}_\perp \times \vec{B} \right](\tau, \vec{\sigma}) \approx 0, \]
\[ \mathcal{J}^r_{\text{(int)}} = \vec{S}^r = \frac{1}{2} \delta^{rs} \varepsilon_{\text{su}} \int d^3 \sigma \, \sigma^u \, T^{u r}(\tau, \vec{\sigma}) = \]
\[ = \sum_{i=1}^N \left( \vec{\kappa}_i(\tau) \times \vec{\kappa}_i(\tau) \right)^r + \frac{1}{c} \int d^3 \sigma \left( \vec{\pi}_\perp \times \vec{\pi}_\perp \right)^r(\tau, \vec{\sigma}), \]
\[ \mathcal{K}^r_{\text{(int)}} = \vec{S}^{r r} = -\vec{S}^{r r} = - \int d^3 \sigma \, \sigma^r \, T^{r r}(\tau, \vec{\sigma}) = \]
\[ = - \sum_{i=1}^N \eta^r_i(\tau) \left( \sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)} - \frac{Q_i}{c} \frac{\vec{\kappa}_i(\tau) \cdot \vec{A}_\perp(\tau, \vec{\eta}_i(\tau))}{\sqrt{m_i^2 c^2 + \vec{\kappa}_i^2(\tau)}} \right) + \]
\[ + \frac{1}{c} \sum_{i=1}^N \sum_{j \neq i} Q_i Q_j \left[ \int d^3 \sigma \frac{1}{4\pi \left| \vec{\sigma} - \vec{\eta}_i(\tau) \right|} \frac{\partial}{\partial \sigma^r} \frac{1}{4\pi \left| \vec{\sigma} - \vec{\eta}_i(\tau) \right|} + \right] \]
\[ + \frac{\eta^r_i(\tau)}{4\pi \left| \vec{\eta}_i(\tau) - \vec{\eta}_j(\tau) \right|} \right] - \]
\[ - \frac{1}{c} \sum_{i=1}^N Q_i \int d^3 \sigma \frac{\pi^r_i(\tau, \vec{\sigma})}{4\pi \left| \vec{\sigma} - \vec{\eta}_i(\tau) \right|} - \frac{1}{2c} \int d^3 \sigma \, \sigma^r \left( \vec{\pi}_\perp^2 + \vec{B}^2 \right)(\tau, \vec{\sigma}) \approx 0. \quad (5.4) \]

Note that, as required by the Poincare' algebra in an instant form of dynamics, there are interaction terms both in the internal energy and in the internal Lorentz boosts, but not in the 3-momentum and in the angular momentum.
As shown in Ref.[8], we can reconstruct the original gauge potential $\tilde{A}_\mu(x)$ in the radiation gauge. It has the following form

$$\tilde{A}_\mu(Y^\alpha(\tau) + \epsilon^r_\tau(\vec{h}) \sigma^r) = \frac{P^\mu}{Mc} \sum_i \frac{Q_i}{|\vec{\sigma} - \vec{\eta}_i(\tau)|} - \epsilon^r_\tau(\vec{h}) A^r_\tau(\tau, \sigma^u). \quad (5.5)$$

### B. Amissible Non-Inertial Frames

Let us now see whether in an arbitrary admissible non-inertial frame, centered on an arbitrary non-inertial observer and described by the embeddings (2.1), we can arrive at the same picture of an isolated system as a decoupled external canonical non-covariant center of mass $\vec{z}$, $\vec{h}$, carrying a pole-dipole structure, with the external Poincare' generators given by expressions like Eqs.(5.1) and with the dynamics described by suitable relative variables after an appropriate elimination of the internal 3-center of mass inside the instantaneous 3-spaces. If this is possible, there will be a new expression for the internal invariant mass $M$, a new effective spin $\vec{S}$ (supposed to satisfy the Poisson brackets of an angular momentum and such that $J^i = \delta^im \epsilon_{mnk} (z^m h^k + \vec{S}^k)$) and a new form of the three pairs of second class constraints replacing the expressions given in Eqs.(5.2) for the case of the inertial rest frame centered on the Fokker-Pryce center of inertia.

Now the embeddings (2.1) imply the form (3.8) for the conjugate momenta $\rho^\mu(\tau, \sigma^u)$. Therefore we must evaluate the Poincare’ generators (3.17) by using Eqs.(2.1) and (3.8). By equating the resulting expressions with Eqs.(5.1) we will find the new expression of the invariant mass, of the effective spin and of the second class constraints.

Since the embedding (2.1) depend on the asymptotic tetrads $\epsilon^\mu_A$, we must express them in terms of the tetrads $\epsilon^\mu(\vec{h})$ determined by $P^\mu$ (whose expression is given after Eq.(1.1)): $\epsilon^\mu_A = \Lambda_A^B(\vec{h}) \epsilon^\mu_B(\vec{h})$ with $\Lambda(\vec{h})$ a Lorentz matrix.

Then, by using Eqs.(2.1), (3.8) and (5.1) the first of Eqs.(3.17) becomes

$$P^\mu = Mc h^\mu = Mc \epsilon^\mu(\vec{h}) \approx \epsilon^\mu_A \mathcal{P}^A = \mathcal{P}^A \Lambda_A^B(\vec{h}) \epsilon^\mu_B(\vec{h}) =$$

$$= \mathcal{P}^A \left[ \Lambda_A^\tau(\vec{h}) h^\mu + \Lambda_A^r(\vec{h}) \epsilon^\mu_r(\vec{h}) \right],$$

$$\mathcal{P}^A = \int d^3\sigma \sqrt{\gamma(\tau, \sigma^u)} \left[ T_{\perp \perp} l^A - T_{\perp \perp} h^{sr} \partial_r F^A \right](\tau, \sigma^u), \quad (5.6)$$

with $l^A(\tau, \sigma^u)$ given in Eq.(2.7).

Therefore the invariant mass $M$ and the three constraints $\mathcal{P}^\tau \approx 0$ replacing the rest-frame conditions are
\[ M_c \approx \hat{P}^{A} \Lambda_\tau^r(\vec{h}), \quad \hat{P}^r = \hat{P}^{A} \Lambda_\tau^r(\vec{h}) \approx 0, \quad \Rightarrow \hat{P}^{A} \approx M_c \Lambda_\tau^{A}(\vec{h}). \quad (5.7) \]

If we define

\[
\hat{S}^{AB} = \int d^3\sigma \sqrt{\gamma(\tau, \sigma^u)} \left[ (f^A(\tau) + F^A(\tau, \sigma^u)) \left( T_{\perp B} - T_{\perp s} h^{sr} \partial_r F^B \right)(\tau, \sigma^u) \right] = 
- \left( F^B(\tau) + F^D(\tau, \sigma^u) \right) \left( T_{\perp l^D} - T_{\perp s} h^{sr} \partial_r F^C \right)(\tau, \sigma^u) \right] = 
\]

\[
\hat{S}^{AB} = \int d^3\sigma \sqrt{\gamma(\tau, \sigma^u)} \left[ F^C(\tau, \sigma^u) \left( T_{\perp l^D} - T_{\perp s} h^{sr} \partial_r F^D \right)(\tau, \sigma^u) \right] = 
- \left( f^A(\tau) + f^B(\tau) \right) \hat{P}^A + \hat{S}^{CD} \Lambda C^A(\vec{h}) \Lambda D^B(\vec{h}),
\]

then, by using Eq.(5.7), the second of Eqs.(3.17) becomes

\[
J^{\mu\nu} \approx \left( x^\mu_0 e^\nu_A - x^\nu_0 e^\mu_A \right) \hat{P}^A + e^\mu_A e^\nu_B \hat{S}^{AB} = 
\]

\[
= \hat{P}^B \Lambda_B^A(\vec{h}) \left( x^\mu_0 e^\nu_A(\vec{h}) - x^\nu_0 e^\mu_A(\vec{h}) \right) + \hat{S}^{CD} \Lambda C^A(\vec{h}) \Lambda D^B(\vec{h}) e^\mu_A(\vec{h}) e^\nu_B(\vec{h}) = 
\]

\[
= \hat{P}^A \Lambda_B^D(\vec{h}) \left[ \left( x^\mu_0 + f^B(\tau) \Lambda_B^C(\vec{h}) e^\mu_C(\vec{h}) \right) e^\nu_D(\vec{h}) \right] = 
- \left( x^\nu_0 + f^B(\tau) \Lambda_B^C(\vec{h}) e^\nu_C(\vec{h}) \right) e^\mu_D(\vec{h}) \right] + e^\mu_A(\vec{h}) e^\nu_B(\vec{h}) \hat{S}^{AB} \approx 
\]

\[
\approx M_c \left[ \left( x^\mu_0 + f^B(\tau) \Lambda_B^C(\vec{h}) e^\mu_C(\vec{h}) \right) h^\nu \right] + e^\mu_A(\vec{h}) e^\nu_B(\vec{h}) \hat{S}^{AB}. \quad (5.9)
\]

After some algebra Eqs.(5.1) and (5.9) imply
\[
J^{ij} = z^i h^j - z^j h^i + \delta^{iu} \delta^{jv} \epsilon_{u w k} \hat{S}^k \approx \delta^{im} \epsilon_{mnk} h^n \hat{S}^k \approx \frac{M c}{1 + \sqrt{1 + \vec{h}^2}} \left[ \left( x_o^i \right) + f^B(\tau) \Lambda_B \left( \vec{h} \right) \epsilon_C \left( \vec{h} \right) + \frac{1}{M c} \left[ \epsilon_C \left( \vec{h} \right) \hat{K}^r + \frac{\delta^{im} \epsilon_{mnk} h^n \hat{S}^k}{1 + \sqrt{1 + \vec{h}^2}} \right] \right] \hat{h}^j - \left( x_o^j \right) - f^B(\tau) \Lambda_B \left( \vec{h} \right) \epsilon_C \left( \vec{h} \right) + \frac{1}{M c} \left[ \epsilon_C \left( \vec{h} \right) \hat{K}^r + \frac{\delta^{im} \epsilon_{mnk} h^n \hat{S}^k}{1 + \sqrt{1 + \vec{h}^2}} \right] \hat{h}^i + \delta^{im} \delta^{jn} \epsilon_{mnk} h^n \hat{S}^k = \hat{X}^i \hat{h}^j - \hat{X}^j \hat{h}^i + \delta^{im} \delta^{jn} \epsilon_{uwk} \hat{S}^k \], (5.10)
\]

\[
J^{oi} = -\sqrt{1 + \vec{h}^2} z^i - \frac{\delta^{im} \epsilon_{mnk} h^n \hat{S}^k}{1 + \sqrt{1 + \vec{h}^2}} \approx \frac{M c}{1 + \sqrt{1 + \vec{h}^2}} \left[ x_o^i + f^B(\tau) \Lambda_B \left( \vec{h} \right) \epsilon_C \left( \vec{h} \right) + \sum_r h^r \hat{K}^r \right] \hat{h}^i - \sqrt{1 + \vec{h}^2} \left[ x_o^i + f^B(\tau) \Lambda_B \left( \vec{h} \right) \epsilon_C \left( \vec{h} \right) + \frac{1}{M c} \left[ \epsilon_C \left( \vec{h} \right) \hat{K}^r + \frac{\delta^{im} \epsilon_{mnk} h^n \hat{S}^k}{1 + \sqrt{1 + \vec{h}^2}} \right] \right] - \frac{\delta^{im} \epsilon_{mnk} h^n \hat{S}^k}{1 + \sqrt{1 + \vec{h}^2}} = \hat{X}^o \hat{h}^i - \sqrt{1 + \vec{h}^2} X^i - \frac{\delta^{im} \epsilon_{mnk} h^n \hat{S}^k}{1 + \sqrt{1 + \vec{h}^2}}, (5.11)
\]

where in the last lines we introduced the definition of the quantities \(X^o\) and \(X^i\).

This implies the reformulation of the isolated system as an external center of mass \(\vec{z}, \vec{h}\), plus a pole-dipole structure \(M\) and \(\vec{S}\).

If we solve Eq.(5.11) in \(\vec{z}\), we get \(\vec{z} = \vec{X} - X^o \frac{\hat{h}}{\sqrt{1 + \vec{h}^2}} - \frac{(\hat{S} - \vec{S}) \times \hat{h}}{\sqrt{1 + \vec{h}^2} (1 + \sqrt{1 + \vec{h}^2})}\) (we use a vector notation). If we put this expression in Eq.(5.10), we get the following equation: \(\left[ (\hat{S} - \vec{S}) \times \hat{h} \right] \times \hat{h} = \sqrt{1 + \vec{h}^2} (1 + \sqrt{1 + \vec{h}^2}) \left( \vec{S} - \vec{S} \right) \). It implies \((\vec{S} - \vec{S}) \cdot \hat{h} = 0\) and then we get

\[
\vec{S}^r \approx \hat{S}^r, (5.12)
\]

namely the effective spin \(\vec{S}\) is given by \(\hat{S}^r\) of Eqs.(5.8).

By using Eq.(5.12) inside Eq.(5.11) we get three constraints, eliminating the internal 3-center of mass and allowing to re-express the dynamics inside the instantaneous 3-spaces only in terms of relative variables, which are
\[
M c \left[ x_o^o + f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^o(\vec{h}) + \sum_r \frac{h^r \hat{K}^r}{M c} \right] h^i - \sqrt{1 + \vec{h}^2} \left[ x_o^i - z^i + \\
+ f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^i(\vec{h}) + \frac{1}{M c} \left( \epsilon_C^i(\vec{h}) \hat{K}^r + \frac{\delta^{im} \epsilon_{mnk} h^m \hat{S}^k}{1 + \sqrt{1 + \vec{h}^2}} \right) \right] \approx 0,
\]
\[
\Downarrow
\]
\[
\hat{K}^r \approx M c h^r \left( x_o^o + f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^o(\vec{h}) - \sum_u h^u \left( x_o^u - z^u + f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^u(\vec{h}) \right) \right) - \\
\left( x_o^r - z^r + f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^r(\vec{h}) + \frac{\delta^{rm} \epsilon_{mnk} h^m \hat{S}^k}{M c \left( 1 + \sqrt{1 + \vec{h}^2} \right)} \right)
\]
\[\text{(5.13)}\]

They replace the constraints \( K^r \approx 0 \) of Subsection A.

Now we have \( \hat{S}^{AB} \approx \delta^{Ar} \delta^{Bs} \epsilon_{rsk} \hat{S}^k + (\delta^{Ar} \delta^{Bs} - \delta^{Ar} \delta^{Bs}) \hat{K}^r. \)

Let us remark that that if we put \( \Lambda^A_B(\vec{h}) = \delta^A_B \) and \( x_o^u + f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^u(\vec{h}) = Y^\mu(0) + h^\mu \tau \), then we recover the results of Subsection A for the inertial rest frame centered on the Fokker-Pryce inertial observer.

Instead the conditions \( \Lambda^A_B(\vec{h}) = \delta^A_B \) and \( f^B(\tau) \Lambda_B^C(\vec{h}) \epsilon_C^u(\vec{h}) = h^\mu \tau \), identifying the inertial rest frame centered on the inertial observer \( x_o^\mu + h^\mu \tau \), have the constraints \( K^r \approx 0 \) replaced by Eqs.(5.13).

Equations of the type (5.7), (5.12) and (5.13) holds not only for admissible embeddings with pure differential rotations like the ones of Eq.(2.14), but also for the admissible embeddings with pure linear acceleration. If in Eq.(2.1) we put \( F^r(\tau, \sigma^u) = 0, F^r(\tau, \sigma^u) = \sigma^r \), so that the embedding becomes \( z^\mu(\tau, \sigma^u) = x_o^\mu + \epsilon^\mu_r f^r(\tau) + \epsilon^\mu_r \left( f^r(\tau) + \sigma^r \right) \), the instantaneous 3-spaces are space-like hyper-planes orthogonal to \( l^\mu = \epsilon^\mu_r \) and we get \( h_{rs} = \delta_{rs}, 1 + n(\tau) = \hat{f}^r(\tau), n_r(\tau) = \delta_{rs} \hat{f}^s(\tau) \). In the case of Eq.(2.13), i.e. \( f^r(\tau) = 0 \) and \( f^r(\tau) = f(\tau) \), we get \( 1 + n(\tau) = \hat{f}(\tau), n_r = 0 \). If \( f^r(\tau) = \tau \) and \( f^r(\tau) = a^r = \text{const.} \), we have inertial frames centered on inertial observers: changing \( a^r \) we change the inertial observer origin of the 3-coordinates \( \sigma^r \).

Let us remark that the final Dirac Hamiltonian (4.35) does not coincide with \( M c \) due to the presence of the inertial potentials \( g_{AB}(\tau, \sigma^u) \).
C. The Non-Inertial Rest Frames

The family of non-inertial rest frames for an isolated system consists of all the admissible 3+1 splittings of Minkowski space-time whose instantaneous 3-spaces $\Sigma_\tau$ tend to space-like hyper-planes orthogonal to the conserved 4-momentum of the isolated system at spatial infinity. Therefore they tend to the Wigner 3-spaces (1.1) of the inertial rest frame asymptotically.

These non-inertial frames can be centered on the external Fokker-Pryce center of inertia like the inertial ones and are described by the following embeddings

\[
z^\mu(\tau, \sigma^u) \approx z^\mu_F(\tau, \sigma^u) = Y^\mu(\tau) + u^\mu(\vec{h}) g(\tau, \sigma^u) + \epsilon^\mu_\tau(\vec{h}) [\sigma^r + g^r(\tau, \sigma^u)],
\]

\[
\rightarrow |\vec{\sigma}| \rightarrow \infty z^\mu_W(\tau, \sigma^u) = Y^\mu(\tau) + \epsilon^\mu_\tau(\vec{h}) \sigma^r,
\]

\[
x^\mu(\tau) = z^\mu_F(\tau, 0^u),
\]

\[
g(\tau, 0^u) = g^r(\tau, 0^u) = 0, \quad g(\tau, \sigma^u) \rightarrow |\vec{\sigma}| \rightarrow \infty 0, \quad g^r(\tau, \sigma^u) \rightarrow |\vec{\sigma}| \rightarrow \infty 0. \tag{5.14}
\]

These embeddings are a special case of Eqs.(4.1) with $x^\mu(\tau) = Y^\mu(\tau)$ and $F^\mu(\tau, \sigma^u) = \epsilon^\mu_\tau(\vec{h}) g(\tau, \sigma^u) + \epsilon^\mu_\tau(\vec{h}) [\sigma^r + g^r(\tau, \sigma^u)]$, $\epsilon^\mu_\tau(\vec{h}) = \dot{Y}^\mu(\tau)$.

For the induced metric we have

\[
z^\mu_F(\tau, \sigma^u) \approx z^\mu_F(\tau, \sigma^u) = h^\mu [1 + \partial_\tau g(\tau, \sigma^u)] + \epsilon^\mu_\tau(\vec{h}) \partial_\tau g^r(\tau, \sigma^u),
\]

\[
z^\mu_F(\tau, \sigma^u) \approx z^\mu_F(\tau, \sigma^u) = h^\mu \partial_\tau g(\tau, \sigma^u) + \epsilon^\mu_\tau(\vec{h}) [\delta^r_s + \partial_\tau g^s(\tau, \sigma^u)],
\]

\[
\epsilon g_{F\tau \tau}(\tau, \sigma^u) = [1 + \partial_\tau g(\tau, \sigma^u)]^2 - \sum_r [\partial_\tau g^r(\tau, \sigma^u)]^2 =
\]

\[
= \left[ (1 + n_F)^2 - h_{F \tau}^r n_{F \tau} n_{F \tau} \right](\tau, \sigma^u),
\]

\[
\epsilon g_{F\tau u}(\tau, \sigma^u) = [1 + \partial_\tau g(\tau, \sigma^u)] \partial_u g(\tau, \sigma^u) - \sum_r \partial_\tau g^r(\tau, \sigma^u) [\delta^r_u + \partial_u g^r(\tau, \sigma^u)] =
\]

\[
= \left[ (1 + \partial_\tau g] \partial_u g - \partial_\tau g^u - \sum_r \partial_\tau g^r \partial_u g^r \right)(\tau, \sigma^u) = -n_{F u}(\tau, \sigma^u),
\]
\[ \epsilon g_{uv}(\tau, \sigma^u) = -h_{uv}(\tau, \sigma^u) = \\
= \partial_u g(\tau, \sigma^u) \partial_v g(\tau, \sigma^u) - \sum_r [\delta^r_u + \partial_u g^r(\tau, \sigma^u)] [\delta^r_v + \partial_v g^r(\tau, \sigma^u)] = \\
= -\delta_{uv} + \left( \partial_u g \partial_v g - (\partial_u g^v + \partial_v g^u) - \sum_r \partial_u g^r \partial_v g^r \right)(\tau, \sigma^u), \]

(5.15)

The admissibility conditions of Eqs.(2.9), plus the requirement \( 1 + n_F(\tau, \sigma^u) > 0 \), can be written as restrictions on the functions \( g(\tau, \sigma^u) \) and \( g^r(\tau, \sigma^u) \).

The unit normal \( l_F^\mu(\tau, \sigma^u) \) and the tangent 4-vectors \( z_F^\mu(\tau, \sigma^u) \) to the instantaneous 3-spaces \( \Sigma_\tau \) can be projected on the asymptotic tetrad \( h^\mu = e^\mu_\tau(\vec{h}), e^\mu_r(\vec{h}) \)

\[ z_F^\mu(\tau, \sigma^u) = \left[ \partial_r g h^\mu + \partial_r g^s \epsilon^\mu_s(\vec{h}) \right](\tau, \sigma^u) \]

\[ l_F^\mu(\tau, \sigma^u) = \left[ \frac{1}{\sqrt{\gamma}} \epsilon_{\alpha\beta\gamma} z_{F1}^\alpha z_{F2}^\beta z_{F3}^\gamma \right](\tau, \sigma^u) = \\
= \frac{1}{\sqrt{\gamma(\tau, \sigma^u)}} \left[ \text{det} (\delta^s_r + \partial_r g^s) h^\mu - \\
- \delta^r a \epsilon_{asu} \epsilon_{vwt} \partial_v g \partial_w g^s \partial_t g^u \epsilon^\mu_r(\vec{h}) \right](\tau, \sigma^u), \]

\[ 1 + n_F(\tau, \sigma^u) = \epsilon z_F^\mu(\tau, \sigma^u) l_{F\mu}(\tau, \sigma^u) = \\
= \frac{1}{\sqrt{\gamma(\tau, \sigma^u)}} \left[ (1 + \partial_r g \text{det} (\delta^s_r + \partial_r g^s)) - \\
- \partial_r g^r \epsilon_{asu} \epsilon_{vwt} \partial_v g \partial_w g^s \partial_t g^u \right](\tau, \sigma^u), \]

\[ l_F^2(\tau, \sigma^u) = \epsilon, \Rightarrow \gamma_F(\tau, \sigma^u) = \left[ \left( \text{det} (\delta^s_r + \partial_r g^s) \right)^2 - \\
- 2 \epsilon_{vwt} \partial_v g \partial_w g^s \partial_t g^u \epsilon_{hmn} \partial_h g \partial_m g^s \partial_n g^u \right](\tau, \sigma^u). \]

(5.16)

To define the non-inertial rest-frame instant form we must find the form of the internal Poincare’ generators replacing the ones of the inertial rest-frame one, given in Eqs.(5.2).

Eq.(3.8) and the first of Eqs.(3.17) imply
\[ P^\mu = M c h^\mu = \int d^3 \sigma \rho^\mu(\tau, \sigma^u) \approx \]
\[ = h^\mu \int d^3 \sigma \sqrt{\gamma(\tau, \sigma^u)} \left( \frac{\text{det}(\delta^s_r + \partial_r g^s)}{\sqrt{\gamma}} \right) T_{F \perp \perp} - \]
\[ - \partial_r g h^r_{Fs} T_{F \perp s}(\tau, \sigma^u) + \]
\[ + \epsilon^\mu_u(\tilde{h}) \int d^3 \sigma \left( - \frac{\delta^u a \epsilon_{asr} \epsilon_{vwt} \partial_v g \partial_w g^s \partial_t g^r}{\sqrt{\gamma}} T_{F \perp \perp} - \right. \]
\[ - \left. (\delta^u_r + \partial_r g^u) h^r_{Fs} T_{F \perp s}(\tau, \sigma^u) \right) = \]
\[ = \int d^3 \sigma T^\mu_F(\tau, \sigma^u), \quad (5.17) \]

so that the internal mass and the rest-frame conditions become (Eqs.(5.2) are recovered for the inertial rest frame)

\[ M_c = \int d^3 \sigma \left( \frac{\text{det}(\delta^s_r + \partial_r g^s)}{\sqrt{\gamma}} T_{F \perp \perp} - \partial_r g h^r_{Fs} T_{F \perp s}(\tau, \sigma^u) \right), \]

\[ \hat{P}^u = \int d^3 \sigma \left( - \frac{\delta^u a \epsilon_{asr} \epsilon_{vwt} \partial_v g \partial_w g^s \partial_t g^r}{\sqrt{\gamma}} T_{F \perp \perp} - \right. \]
\[ - \left. (\delta^u_r + \partial_r g^u) h^r_{Fs} T_{F \perp s}(\tau, \sigma^u) \right) \approx 0. \quad (5.18) \]

By using Eqs.(3.17) for the angular momentum we get \( J^{\mu \nu} \approx \int d^3 \sigma \left( \frac{z^\mu_F \rho^\nu_F - z^\nu_F \rho^\mu_F}{\sqrt{\gamma}} \right)(\tau, \sigma^u) \)

with \( \rho^\mu_F(\tau, \sigma^u) = \left[ \sqrt{\gamma_F} \left( T_{\perp \perp}^{\mu F} - T_{\perp s} h^s_{F r} z^r_{F r} \right) \right](\tau, \sigma^u) \), where \( z^s_F, z^r_F \) and \( l^l_F \) are given in Eqs.(5.14), (5.15) and (5.16) respectively. The description of the isolated system as a pole-dipole carried by the external center of mass \( \tilde{z} \) requires that we must identify the previous \( J^{ij} \) and \( J^{oi} \) with the expressions like the ones given in Eqs.(5.1), now functions of \( \tilde{z}, \tilde{h}, M_c \) of Eq.(5.18) and of an effective spin \( \tilde{S} \). This identification will allow to find the effective spin \( \tilde{S} \) and three constraints \( \tilde{K}^r \approx 0 \) eliminating the internal 3-center of mass: in the limit of the inertial rest frame they must reproduce the quantities in Eqs.(5.2).

By using Eqs.(5.18) this procedure implies (\( \tilde{K}^r \) and \( \tilde{S}^r \) are the analogue of the quantities defined in Eqs.(5.8) for the embedding (5.14))
\[ J^{\mu
u} \approx \int d^3\sigma \left( z_F^{\mu} p_F^{\nu} - z_F^{\nu} p_F^{\mu} \right) (\tau, \sigma^\nu) = \]

\[ = Mc \left( Y^\mu(0) h^\nu - Y^\nu(0) h^\mu \right) + \hat{P}^u \left( Y^\mu(0) e_u^\nu(\tilde{h}) - Y^\nu(0) e_u^\mu(\tilde{h}) \right) + \]

\[ + \left( \tau \hat{P}^u + \hat{K}^u \right) \left( h^\mu e^\nu_u(\tilde{h}) - h^\nu e^\mu_u(\tilde{h}) \right) + \delta^{\mu\nu} \epsilon_{nv} \hat{S}^{\mu} e^\nu_u(\tilde{h}) \approx \]

\[ \approx Mc \left( Y^\mu(0) h^\nu - Y^\nu(0) h^\mu \right) + \hat{K}^u \left( h^\mu e^\nu_u(\tilde{h}) - h^\nu e^\mu_u(\tilde{h}) \right) + \]

\[ + \delta^{\mu\nu} \epsilon_{nv} \hat{S}^{\mu} e^\nu_u(\tilde{h}) e^\nu_v(\tilde{h}), \]

so that we get

\[ J^{ij} = z^i h^j - z^j h^i + \delta^{iu} \delta^{jv} \epsilon_{uvw} S^k \approx \]

\[ \approx Mc \left( Y^i(0) h^j - Y^j(0) h^i \right) + \hat{K}^u \left( h^i e^j_u(\tilde{h}) - h^j e^i_u(\tilde{h}) \right) + \]

\[ + \delta^{\mu\nu} \epsilon_{nv} \hat{S}^{\mu} e^0_u(\tilde{h}) e^j_v(\tilde{h}), \]

\[ J^{oi} = -\sqrt{1 + \tilde{h}^2} z^i + \frac{\delta^{im} \epsilon_{njk} \tilde{S}^j \tilde{h}^k}{1 + \sqrt{1 + \tilde{h}^2}} \approx \]

\[ \approx Mc \left( Y^o(0) h^i - Y^i(0) h^o \right) + \hat{K}^u \left( h^o e^i_u(\tilde{h}) - h^i e^o_u(\tilde{h}) \right) + \]

\[ + \delta^{\mu\nu} \delta^{om} \epsilon_{nmv} \hat{S}^{\mu} e^o_u(\tilde{h}) e^i_v(\tilde{h}). \] (5.19)

As a consequence, by using the expression of \( Y^\mu(0) \) given after Eq.(5.2), the constraints eliminating the 3-center of mass and the effective spin are

\[ \hat{K}^u = \int d^3\sigma \left( g \left[ \delta^{uv} \partial_r g T_{F\perp} - (\delta^u_r + \partial_r g^u) h_F^{rs} T_{F\perp s} \right] - \right. \]

\[ \left. - (\sigma^u + g^u) \left[ \frac{det(\delta^r + \partial_r g^r)}{\sqrt{T}} T_{F\perp\perp} - \partial_r g^{rs} T_{F\perp s} \right] \right) (\tau, \sigma^u) \approx 0, \]

\[ \hat{S}^r \approx \hat{S}^r = \frac{1}{2} \delta^{rn} \epsilon_{nsv} \int d^3\sigma \left( (\sigma^u + g^u) \left[ \delta^{sm} \partial_m g T_{F\perp\perp} - (\delta^m_r + \partial_r g^m) h_F^{rs} T_{F\perp s} \right] - \right. \]

\[ \left. - (\sigma^v + g^v) \left[ \delta^{um} \partial_m g T_{F\perp\perp} - (\delta^u_r + \partial_r g^u) h_F^{rs} T_{F\perp s} \right] \right) (\tau, \sigma^u). \] (5.20)

and these formulas allow to recover Eqs.(5.2) of the inertial rest frame.

Therefore the non-inertial rest-frame instant form of dynamics is well defined.
D. The Hamiltonian of the Non-Inertial Rest-Frame Instant Form

We have now to find which is the effective Hamiltonian of the non-inertial rest-frame instant form replacing $Mc$ of the inertial rest-frame one. The gauge fixing (5.20) is a special case of Eqs.(4.1), whose final Dirac Hamiltonian is given in Eq.(4.4) [or in Eq.(4.35) in the radiation gauge].

To be able to impose this gauge fixing, let us put $F_\mu(\tau,\sigma^u) = h_\mu g(\tau,\sigma^u) + \epsilon_\mu(\vec{h}) [\sigma^r + g^r(\tau,\sigma^u)]$ in Eq.(4.1), but let us leave $x^\mu(\tau)$ as an arbitrary time-like observer to be restricted to $Y^\mu(\tau)$ at the end. We will only assume that $x^\mu(\tau)$ is canonically conjugate with $P^\mu = \int d^3\sigma \rho^\mu(\tau,\sigma^u), \{x^\mu(\tau), P^\nu\} = -\epsilon \eta^{\mu\nu}$.

Due to the dependence of $F_\mu(\tau,\sigma^u)$ and of $Y^\mu(\tau)$ on $\vec{h} = \vec{P}/\sqrt{\epsilon P^2}$ we must develop a different procedure for the identification of the Dirac Hamiltonian.

In this case the constraints (3.10) can be rewritten in the following form ($T^\mu_F(\tau,\sigma^u)$ is defined in Eq.(5.17))

$$H^\mu(\tau,\sigma^u) = \tilde{H}^\mu(\tau,\sigma^u) + \delta^3(\sigma^u) \int d^3\sigma_1 H^\mu(\tau,\sigma^u_1) \approx 0, \quad \text{with} \quad \int d^3\sigma \tilde{H}^\mu(\tau,\sigma^u) \equiv 0,$$

$$\Downarrow$$

$$\rho^\mu(\tau,\sigma^u) \approx P^\mu \delta^3(\sigma^u) + \left[ T^\mu_F(\tau,\sigma^u) - \delta^3(\sigma^u) R^\mu_F(\tau) \right] = \delta^3(\sigma^u) H^\mu(\tau) + T^\mu_F(\tau,\sigma^u),$$

$$H^\mu(\tau) = P^\mu - R^\mu_F(\tau) \approx 0, \quad R^\mu_F(\tau) \overset{\text{def}}{=} \int d^3\sigma T^\mu_F(\tau,\sigma^u).$$

(5.21)

In this way the original canonical variables $z^\mu(\tau, \vec{\sigma})$, $\rho^\mu(\tau, \vec{\sigma})$ are replaced by the observer $x^\mu(\tau)$, $P^\mu$ and by relative variables with respect to it.

From Eq.(5.14) we get:

a) the gauge fixing to the constraints $\tilde{H}^\mu(\tau,\sigma^u) \approx 0$ is

$$\psi^\mu_r(\tau,\sigma^u) = \frac{\partial \chi^\mu(\tau,\sigma^u)}{\partial \sigma^r} = \left( z^\mu_{\tau r} - \epsilon^\mu_s(\vec{h}) \left[ \delta^{rs}_{\tau} + \frac{\partial g^s}{\partial \sigma^r} - u^\mu(\vec{h}) \frac{\partial g}{\partial \sigma^r} \right] \right)(\tau,\sigma^u) \approx 0; \quad (5.22)$$
b) the gauge fixing to the constraints $H^\mu(\tau) = P^\mu - \mathcal{R}_F^\mu \approx 0$ is $\chi^\mu(\tau, 0) = z^\mu(\tau, 0) - Y^\mu(\tau) = x^\mu(\tau) - Y^\mu(\tau) \approx 0$.

The gauge fixing (5.22) has the following Poisson brackets with the collective variables $x^\mu(\tau), P^\mu$

\[
\{P^\mu, \psi^\nu(\tau, \sigma^u)\} = 0,
\]

\[
\{x^\mu(\tau), \psi^\nu(\tau, \sigma^u)\} = -\frac{\partial \epsilon^\nu_s(\vec{h})}{\partial P^\mu} (\delta^s_s + \frac{\partial g^s(\tau, \sigma^u)}{\partial \sigma^s}) - \frac{\partial \epsilon^\nu_s(\vec{h})}{\partial P^\mu} \frac{\partial g(\tau, \sigma^u)}{\partial \sigma^s} \neq 0. \quad (5.23)
\]

Therefore $x^\mu(\tau)$ is no more a canonical variable after the gauge fixing $\psi^\nu(\tau, \sigma^u) \approx 0$.

By introducing the notation ($\epsilon^A_{\mu} = \eta^{AB} \epsilon_{B\mu} \Rightarrow \epsilon^A_{\mu}(\vec{h}) = \epsilon_{\mu}(\vec{h})$)

\[
T_F^\mu(\tau, \sigma^u) \overset{def}{=} h^\mu T_F^\mu(\tau, \sigma^u) + \epsilon^A_{\mu}(\vec{h}) T_F^A(\tau, \sigma^u), \quad \Rightarrow T_F^A(\tau, \sigma^u) = \epsilon^A_{\mu}(\vec{h}) T_F^\mu(\tau, \sigma^u), \quad (5.24)
\]

the angular momentum generator of Eq.(3.17) takes the form

\[
J^{\mu\nu} = x^\mu(\tau) P^\nu - x^\nu(\tau) P^\mu + S^{\mu\nu},
\]

\[
S^{\mu\nu} \approx \epsilon^A_{\mu}(\vec{h}) \epsilon^B_{\nu}(\vec{h}) \int d^3 \sigma \left[ (\sigma^r + g^r) T^s - (\sigma^s + g^s) T^r \right](\tau, \sigma^u) + \epsilon^A_{\mu}(\vec{h}) \epsilon^B_{\nu}(\vec{h}) \int d^3 \sigma \left[ (\sigma^r + g^r) T^r + g T^r \right](\tau, \sigma^u) = \epsilon^A_{\mu}(\vec{h}) \epsilon^B_{\nu}(\vec{h}) S^{AB},
\]

\[
S^{rs} = \int d^3 \sigma \left[ (\sigma^r + g^r) T^s - (\sigma^s + g^s) T^r \right](\tau, \sigma^u) \overset{def}{=} \delta^{rs} \epsilon_{nu} J^u, \quad S^{rr} = -S^{rt} = -\int d^3 \sigma \left[ (\sigma^r + g^r) T^r + g T^r \right](\tau, \sigma^u) \overset{def}{=} K^r, \quad (5.25)
\]

where only the constraints $\mathcal{H}^\mu(\tau, \sigma^u) \approx 0$ have been used.

Since we have

\[
\{x^\mu(\tau), S^{\alpha\beta}\} = 0,
\]

\[
\left\{ \frac{\partial z^\mu(\tau, \sigma^u)}{\partial \sigma^r}, S^{\alpha\beta}\right\} = \left(\frac{\partial z^\beta}{\partial \sigma^r} \eta^{\mu\alpha} - \frac{\partial z^\alpha}{\partial \sigma^r} \eta^{\mu\beta}\right)(\tau, \sigma^u) \approx \left(\epsilon^\beta_s(\vec{h}) (\delta^s_s + \frac{\partial g^s}{\partial \sigma^s}) + h^\beta \frac{\partial g}{\partial \sigma^s} \right) \eta^{\mu\alpha}(\tau, \sigma^u), \quad (5.26)
\]

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after the gauge fixing the new canonical variable for the observer becomes

\[ \tilde{x}^\mu(\tau) = x^\mu(\tau) - \frac{1}{2} \epsilon_{\sigma A}(\vec{h}) \frac{\partial \epsilon^A(\vec{h})}{\partial P_\mu} S^\sigma, \quad \{ \tilde{x}^\mu(\tau), \psi^\nu(\tau, \vec{\sigma}) \} = 0. \] (5.27)

If we eliminate the relative variables by going to Dirac brackets with respect to the second class constraint \( \tilde{\mathcal{H}}^\mu(\tau, \sigma^u) \approx 0 \), \( \psi^\mu(\tau, \sigma^u) \approx 0 \), the canonical variables \( z^\mu(\tau, \sigma^u) \), \( \rho^\mu(\tau, \sigma^u) \) are reduced to the canonical variables \( \tilde{x}^\mu(\tau) \), \( P^\mu \).

By defining \( \mathcal{R}_F(\tau) = \epsilon h_\mu \mathcal{R}_F^\mu(\tau) \approx Mc = \sqrt{\epsilon P^2} \), the remaining constraints are

\[ H^\mu(\tau) = h^\mu \left( \sqrt{\epsilon P^2 - \mathcal{R}_f(\tau)} \right) + \epsilon^r(\vec{h}) \hat{P}^r, \]

or

\[ \epsilon h^\mu H_\mu(\tau) = \sqrt{\epsilon P^2 - \mathcal{R}_F(\tau)} \approx 0, \quad \epsilon^r(\vec{h}) H^\mu(\tau) = \hat{P}^r \approx 0. \] (5.28)

Like in Eqs.(4.1) and (4.2), after this reduction the Dirac multiplier \( \lambda^\mu(\tau, \sigma^u) \) in the Dirac Hamiltonian (3.16) becomes

\[ \lambda_\mu(\tau, \sigma^u) = \epsilon h_\mu \left( \lambda_\tau(\tau) - \frac{\partial g^\tau(\tau, \sigma^u)}{\partial \tau} \right) + \epsilon \epsilon^r(\vec{h}) \left( \lambda^r(\tau) - \frac{\partial g^r(\tau, \sigma^u)}{\partial \tau} \right) \]

\[ \overset{\circ}{=} - \epsilon \frac{\partial z^\mu(\tau, \sigma^u)}{\partial \tau} \] (5.29)

At this stage the Dirac Hamiltonian depends only on the residual Dirac multipliers \( \lambda_\tau(\tau) \) and \( \lambda^r(\tau) \)

\[ H_D = \lambda_\tau(\tau) \left( \sqrt{\epsilon P^2 - \mathcal{R}_F} \right) - \lambda^r(\tau) \cdot \vec{P} + \int d^3 \sigma \left( \frac{\partial g^r}{\partial \tau} \mathcal{T}_{F r} + \frac{\partial g^r}{\partial \tau} \mathcal{T}_{F r} \right)(\tau, \sigma^u), \] (5.30)

where we introduced the notation \( \mathcal{T}_{F A}(\tau, \sigma^u) \overset{\text{def}}{=} \epsilon \epsilon_{\mu A}(\vec{h}) \mathcal{T}_F^\mu(\tau, \sigma^u) \) so that \( \mathcal{T}_F^\tau = \mathcal{T}_{F \tau}, \quad \mathcal{T}_F^r = - \epsilon \mathcal{T}_{F r} \).

To implement the gauge fixing \( x^\mu(\tau) - Y^\mu(\tau) \approx 0 \) requires two other steps:

1) Firstly we impose the gauge fixing \( \tilde{x}^\mu(\tau) h_\mu = \epsilon \tau \). It implies \( \lambda_\tau(\tau) = -1 \) and \( \sqrt{\epsilon P^2} = Mc \equiv \mathcal{R}_F \). The Dirac Hamiltonian becomes
\[ H_{FD} = \mathcal{M}c - \mathring{\lambda}(\tau) \cdot \mathring{\bar{P}} + \int d^3\sigma \left[ \mu \pi^r - A_{\tau} \Gamma \right](\tau, \sigma^u), \]

\[ \mathcal{M}c = Mc + \int d^3\sigma \left( \frac{\partial g^r}{\partial \tau} T_{Fr} + \frac{\partial g}{\partial \tau} T_{F \tau} \right)(\tau, \sigma^u). \quad (5.31) \]

2) Then we add the gauge fixing \( \mathring{K}^r \approx 0 \) to the rest-frame conditions \( \mathring{\bar{P}}^r \approx 0 \): this implies \( \mathring{\lambda}(\tau) = 0 \). In this way we get \( x^\mu(\tau) \approx Y^\mu(\tau) \) and we also eliminate the internal 3-center of mass. Having chosen the Fokker-Pryce external 4-center of inertia \( Y^\mu(\tau) \) as origin of the 3-coordinates the constraints \( \mathring{K}^r \approx 0 \) correspond to the requirement \( S^{\tau r} \approx 0 \).

In conclusion the effective Hamiltonian \( \mathcal{M}c \) (modulo electro-magnetic gauge transformations) of the non-inertial rest-frame instant form is not the internal mass \( Mc \), since \( Mc \) describes the evolution from the point of view of the asymptotic inertial observers. There is an additional term interpretable as an inertial potential producing relativistic inertial effects (see Eqs. (5.16) for \( 1 + n_F(\tau, \sigma^u) \) and Eqs. (5.15) for \( n_F(\tau, \sigma^u) \))

\[ \mathcal{M}c = Mc + \int d^3\sigma \left( \frac{\partial g^r}{\partial \tau} T_{Fr} + \frac{\partial g}{\partial \tau} T_{F \tau} \right)(\tau, \sigma^u) = \]

\[ = \int d^3\sigma \left( h_{\mu} \left( 1 + \frac{\partial g}{\partial \tau} \right) + \epsilon_{\mu r} \frac{\partial g^r}{\partial \tau} \right) T_{F \mu}^\tau(\tau, \sigma^u) = \]

\[ = \int d^3\sigma \sqrt{\gamma(\tau, \sigma^u)} \left( (1 + n_F) T_{F \perp \perp} + n_F^r T_{F \perp r} \right)(\tau, \sigma^u) \quad (5.32) \]

where

\[ \sqrt{\gamma(\tau, \sigma^u)} T_{F \perp \perp}(\tau, \sigma^u) = \sqrt{\gamma(\tau, \sigma^u)} T_{F \perp \perp}(\tau, \sigma^u) + \]

\[ + \sum_i \delta(\sigma^u - \eta_i^u) \sqrt{m_i^2 c^2 + h_{F i}^2(\tau, \sigma^u)(\kappa_{ir}(\tau) - Q_i A_r(\tau, \sigma^u))(\kappa_{is}(\tau) - Q_i A_s(\tau, \sigma^u))}, \]

\[ \sqrt{\gamma(\tau, \sigma^u)} T_{F \perp r}(\tau, \sigma^u) = F_{rs}(\tau, \sigma^u) \pi^s(\tau, \sigma^u) - \sum_i \delta(\sigma^u - \eta_i^u) (\kappa_{ir}(\tau) - Q_i A_r(\tau, \sigma^u)), \quad (5.33) \]

with \( T_{\perp \perp}^r \) given in Eq. (4.28).

Let us remark that a similar procedure should be applied also to the gauge fixing (4.1) if we want to reproduce the results of Subsection B for arbitrary non-inertial frames. We do not add these calculations, because they agrees substantially with the results of this Subsection and do not alter the conclusions of Section IV.
VI. CONCLUSION

In this paper we have defined the general theory of non-inertial frames in Minkowski space-time. It is based on Møller-admissible 3+1 splittings of Minkowski space-time (they give conventions for clock synchronization, i.e. for the identification of instantaneous 3-spaces) and on parametrized Minkowski theories for isolated systems admitting a Lagrangian description. The transition from a non-inertial frame to every other one is formalized as a gauge transformation, so that physical results do not depend on how the clock are synchronized.

The Møller conditions, implying the absence of rotational velocities higher than the velocity of light \( c \) and requiring that the three eigenvalues of the non-inertial 3-metric inside the instantaneous Riemannian 3-spaces has three non-null positive eigenvalues, have to be implemented with the following two extra conditions:

a) the lapse function must be positive definite in each point of the instantaneous 3-space, so to avoid the intersection of 3-spaces at different times;

b) the space-like hyper-surfaces corresponding to the Riemannian 3-spaces must become space-like hyper-planes (Euclidean 3-spaces) at spatial infinity with a direction-independent unit normal \( l^\mu_{(\infty)} \) (asymptotic inertial observers to be identified with the fixed stars).

Among the admissible non-inertial frames we identified the non-inertial rest frames, generalizing the inertial rest frames and relevant for canonical gravity [5, 11, 12].

All the properties of the inertial rest-frame instant form of dynamics, studied in details in Refs.[8], have been extended to non-inertial frames. Again every isolated system may be described as a decoupled non-covariant external center of mass carrying a pole-dipole structure: the internal mass of the system and an effective spin (becoming the rest spin in the inertial rest frame). In particular we have found the non-inertial generalization of the second class constraints eliminating the internal 3-center of mass inside the instantaneous 3-spaces.

This theory of non-inertial frames is free by construction from the coordinate singularities of all the approaches to accelerated frames based on the 1+3 point of view, in which the instantaneous 3-spaces are identified with the local rest frames of the observer. The pathologies of this approach are either the horizon problem of the rotating disk (rotational velocities higher than \( c \)), which is still present in all the calculations of pulsar magnetosphere in the form of the light cylinder, or the intersection of the local rest 3-spaces. The main difference between the 3+1 and 1+3 points of view is that the Møller conditions forbid rigid rotations in relativistic theories.
We have done a detailed study of the isolated system of positive-energy scalar particles with Grassmann-valued electric charges plus the electro-magnetic field extending to non-inertial frames its Hamiltonian description given in the inertial rest frame in Ref.[8].

By using a non-covariant (i.e. coordinate-dependent) decomposition of the electromagnetic potential we obtained the non-inertial radiation gauge, in which the electromagnetic field is described by means of transverse quantities (the Dirac observables). This allowed us to find the non-inertial expression of the Coulomb potential, which is now dependent also on the field strengths and the inertial potentials. The non-covariance of the description is natural due to the presence in the Hamiltonian of the relativistic inertial potentials, namely the components $g_{AB}(\tau,\sigma^r)$ of the 4-metric induced by the 3+1 splitting, which are intrinsically coordinate dependent. The non-relativistic limit of the inertial potentials reproduces the standard (again coordinate-dependent) Newtonian ones. The Hamiltonian in non-inertial frames turns out to be the sum of the invariant mass (now coordinate-dependent due to its dependence on the 4-metric) of the system plus terms in the inertial potentials disappearing in the inertial rest frame.

In the second paper we will give the simplest example of 3+1 splitting with differential rotations and we will develop the 3+1 point of view for the rotating disk and the Sagnac effect. Then we will study properties of Maxwell equations in admissible nearly rigidly rotating frames like the wrap-up effect, the Faraday rotation in astrophysics and the pulsar magnetosphere.
APPENDIX A: THE LANDAU-LIFSHITZ NON-INERTIAL ELECTRO-MAGNETIC FIELDS

Sometimes, see for instance Ref.[17], the following generalized non-inertial electric and magnetic fields are introduced

\[ \mathcal{E}^s_{(F)}(\tau, \sigma^u) = - \left[ \frac{\sqrt{\gamma_F}}{\sqrt{1 + n_F}} h^s_{fr} (F_{fr} - n^r_F F_{vr}) \right] (\tau, \sigma^u) \equiv \pi^s_{(F)}(\tau, \sigma^u), \]

\[ \mathcal{B}^{w}_{(F)}(\tau, \sigma^u) = \frac{1}{2} \delta^{uw} \epsilon_{lsr} \left[ (1 + n_F) \sqrt{\gamma_F} h^s_{lr} h^s_{sr} F_{vr} - (n^r_F \pi^u - n^r_F \pi^w) \right] (\tau, \sigma^u), \quad (A1) \]

They allow us to rewrite the Hamilton-Dirac Eqs.(4.15) in the following form (we use a vector notation as in the 3-dimensional Euclidean case)

\[ \partial_r \mathcal{E}^r_{(F)}(\tau, \sigma^u) = \sqrt{\gamma_F} \mathcal{E}^r(\tau, \sigma^u) \equiv \bar{p}(\tau, \sigma^u), \]

\[ \epsilon_{rue} \partial_u \mathcal{B}^e_{(F)}(\tau, \sigma^u) - \frac{\partial \mathcal{E}^r_{(F)}(\tau, \sigma^u)}{\partial \tau} = \sqrt{\gamma_F} \mathcal{B}^e(\tau, \sigma^u) \equiv \bar{J}^e(\tau, \sigma^u), \quad (A2) \]

namely in the same form of the usual source- dependent Maxwell equations in an inertial frame.

Since Eqs.(A1) can be rewritten in the form

\[ \mathcal{E}^s_{(F)}(\tau, \sigma^u) = \left[ + \frac{\sqrt{\gamma_F}}{\sqrt{1 + n_F}} h^s_{fr} E_r - \frac{\sqrt{\gamma_F}}{\sqrt{1 + n_F}} h^s_{fr} \epsilon_{rue} n^u_F B_v \right] (\tau, \sigma^u), \]

\[ \mathcal{B}^{w}_{(F)}(\tau, \sigma^u) = \delta^{uw} \epsilon_{lsr} \left[ \frac{1}{2} (1 + n_F) \sqrt{\gamma_F} h^s_{lr} h^s_{sr} n^u_F B_v - n^r_F \epsilon^e_{vul} B_v + n^s_F E_r \right] (\tau, \sigma^u), \quad (A3) \]

we get the following form of the Maxwell equations for the field strengths \( E_r \) and \( B_r \)

\[ \partial_r E_r(\tau, \sigma^u) = \sqrt{\gamma_F(\tau, \sigma^u)} \left[ \bar{p}(\tau, \sigma^u) - \bar{p}_R(\tau, \sigma^u) \right], \]

\[ \epsilon_{suv} \partial_u B_v(\tau, \sigma^u) - \frac{\partial E_s(\tau, \sigma^u)}{\partial \tau} = \delta_{sr} \sqrt{\gamma_F(\tau, \sigma^u)} \left[ \bar{J}^e(\tau, \sigma^u) - \bar{J}^e_R(\tau, \sigma^u) \right], \quad (A4) \]

where the new charge and current densities are the following functions only of the metric tensor and of the fields \( E_r, B_r \).
\[ \overline{p}_R(\tau, \sigma^u) = \frac{1}{\sqrt{\gamma F(\tau, \sigma^u)}} \partial_\tau \left( \mathcal{E}_F(\tau, \sigma^u) - \delta^{u} E_s(\tau, \sigma^u) \right), \]

\[ \overline{J}_R(\tau, \sigma^u) = \frac{1}{\sqrt{\gamma F(\tau, \sigma^u)}} \left[ - \frac{\partial}{\partial \tau} \left( \mathcal{E}_F(\tau, \sigma^u) - \delta^{rs} E_s(\tau, \sigma^u) \right) + \delta^{rs} \epsilon_{suw} \partial_u \left( B^w(F) - \delta^{wk} B_k \right)(\tau, \sigma^u) \right]. \] (A5)

Instead, as a consequence of Eqs.(4.10), the homogeneous equations take the form

\[ \epsilon_{ruv} \partial_u E_v(\tau, \sigma^s) = - \frac{\partial B_r(\tau, \sigma^s)}{\partial \tau}, \quad \epsilon_{ruv} \partial_u B_v(\tau, \sigma^s) = 0. \] (A6)

By using Eq.(3.2) of the second paper we find the results of the Appendix A of Ref.[28]

\[ \mathcal{E}_F(\tau, \sigma^u) = \vec{E}(\tau, \sigma^u) + \left( \frac{\vec{\Omega}(\tau)}{c} \times \vec{\sigma} \right) \times \vec{B}(\tau, \sigma^u), \]

\[ \mathcal{B}_F(\tau, \sigma^u) = \vec{B} + \left( \frac{\vec{\Omega}(\tau)}{c} \times \vec{\sigma} \right) \times \vec{E}(\tau, \sigma^u) + \left( \frac{\vec{\Omega}(\tau)}{c} \times \vec{\sigma} \right) \times [\left( \frac{\vec{\Omega}(\tau)}{c} \times \vec{\sigma} \right) \times \vec{B}(\tau, \sigma^u)]. \] (A7)

In absence of sources Eqs.(4.17) are the generally covariant equations \( \nabla_\nu F^{\mu\nu} = 0 \), suggested by the equivalence principle, in the 3+1 point of view after having taken care of the asymptotic properties at spatial infinity.

Let us remark that in the case of the nearly rigid limit of the foliation (2.14) (see Section VI) and with \( \vec{\Omega}(\tau) = (0, 0, \vec{\Omega} = \text{const.}) \) Eqs.(A4) and (A6) coincide with Eqs.(9) of Schiff [28] if we identify \( \bar{\rho}_R \) with \( \sigma \) and \( \bar{J}_R^r \) with \( j^r \). This is due to the fact that Schiff’s fields \( \vec{E}, \vec{B} \), have the components coinciding with the covariant fields \( E_r, B_r \) of Eqs.(4.10); these fields obviously differ from the fields (A3) defined in Ref.[17].

Eqs. (A4) and (A5), with the metric associated to the admissible notion of simultaneity (2.14), should be the starting point for the calculations in the magnetosphere of pulsars, where one always assumes a rigid rotation \( \omega \) with the consequent appearance of the so-called light cylinder for \( \omega R = c \) (the horizon problem of the rotating disk). See Refs.[29] based on Schiff’s equations [28] (A4) and (A7) or the more recent literature of Refs. [30]. Instead in Refs.[31] the light cylinder is avoided using the rotating coordinates of Refs.[19], but at the price of a bad behavior at spatial infinity.

These equations also show that the non-inertial electric and magnetic fields \( \mathcal{E}_F \) and \( \mathcal{B}_F \) are not, in general, equal to the fields obtained from the inertial ones \( \vec{E} \) and \( \vec{B} \) with a Lorentz transformations to the comoving inertial system like it is usually assumed following Rohrlich [32] and the locality hypothesis.
APPENDIX B: COVARIANT AND NON-COVARIANT DECOMPOSITIONS OF THE ELECTRO-MAGNETIC FIELD AND THE RADIATION GAUGE IN NON-INERTIAL REST FRAMES.

In inertial frames the identification of the physical degrees of freedom (Dirac observables) of the free electro-magnetic field was done in Refs. [26, 33, 34, 35] by means of the Shanmugadhasan canonical transformation adapted to the first class constraints \( \pi^r(\tau, \sigma^u) \approx 0 \) and \( \Gamma(\tau, \sigma^u) = \partial_r \pi^r(\tau, \sigma^u) \approx 0 \). The final canonical basis identifies the radiation gauge with its transverse fields as the natural one from the point of view of constraint theory.

In the parametrized Minkowski theories of Section III Subsection A, due to the last two lines of Eqs.(3.15), we see that two successive gauge transformations, of generators \( G_i(\tau, \sigma^u) = \lambda_i^\mu(\tau, \sigma^u) H_\mu(\tau, \sigma^u), i = 1, 2 \), do not commute but imply an electro-magnetic gauge transformation. Since the effect of the \( i = 1, 2 \) gauge transformations is to modify the notions of simultaneity, also the definition of the Dirac observables of the electro-magnetic field will change with the 3+1 splitting. In general, given two different 3+1 splittings, the two sets of Dirac observables associated with them will be connected by an electro-magnetic gauge transformation.

Since it is not clear whether it is possible to find a quasi-Shanmugadhasan canonical transformation adapted to \( H_\perp(\tau, \sigma^u) = H_\mu(\tau, \sigma^u) z^\mu(\tau, \sigma^u) \approx 0, \pi^r(\tau, \sigma^u) \approx 0, \Gamma(\tau, \sigma^u) \approx 0 \)\(^{16}\), the search of the electro-magnetic Dirac observables must be done with the following strategy:

i) make the choice of an admissible 3+1 splitting by adding four gauge-fixing constraints determining the embedding \( z^\mu(\tau, \sigma^u) \), so that the induced 4-metric \( g_{AB}(\tau, \sigma^u) \) becomes a numerical quantity and is no more a configuration variable;

ii) find the Dirac observables on the resulting completely fixed simultaneity surfaces \( \Sigma_\tau \) with a suitable Shanmugadhasan canonical transformation adapted to the two remaining electro-magnetic constraints.

Let us remark that a similar scheme has to be followed also in the canonical Einstein-Maxwell theory: only after having fixed a 3+1 splitting (a system of 4-coordinates on the solutions of Einstein’s equations) we can find the Dirac observables of the electro-magnetic field.

This strategy is induced by the fact that, while the Gauss law constraint \( \Gamma(\tau, \sigma^u) = \partial_r \pi^r(\tau, \sigma^u) \approx 0 \) is a scalar under change of admissible 3+1 splittings \(^{17}\), the gauge vec-

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\(^{16}\) \( H_\perp(\tau, \sigma^u) = H_\mu(\tau, \sigma^u) l_\mu(\tau, \sigma^u) \approx 0 \), like an ordinary Hamiltonian, can be included in the adapted Darboux-Shanmugadhasan basis only in case of integrability of the equations of motion.

\(^{17}\) \( \pi^r(\tau, \sigma^u) \) is a vector density like in canonical metric gravity.
tor potential $A_r(\tau, \sigma^u)$ is the pull-back to the base of a connection one-form and can be considered as a tensor only with topologically trivial surfaces $\Sigma_\tau$ (like in the case we are considering). Since a Shanmugadhasan canonical transformation adapted to the Gauss law constraint transforms $\Gamma(\tau, \sigma^u)$ in one of the new momenta, it is not clear how to define a conjugate gauge variable $\eta_{em}(\tau, \sigma^u)$ such that \{ $\eta_{em}(\tau, \sigma^u), \Gamma(\tau, \sigma^u)$ \} = $\delta^3(\sigma^u, \sigma^u_1)$ and two conjugate pairs of Dirac observables having vanishing Poisson brackets with both $\eta_{em}(\tau, \sigma^u)$ and $\Gamma(\tau, \sigma^u)$ when the 3-metric on $\Sigma_\tau$ is not Euclidean ($g_{rs}(\tau, \sigma^u) \neq -\epsilon \delta_{rs}$).

With every fixed type of instantaneous 3-space $\Sigma_\tau$ with non-trivial 3-metric, $g_{rs}(\tau, \sigma^u) \neq -\epsilon \delta_{rs}$, we have to find suitable gauge variable $\eta_{em}(\tau, \sigma^u)$ and the Dirac observables replacing $A_r(\tau, \sigma^u)$ and $\pi^r(\tau, \sigma^u)$.

Let us consider an arbitrary admissible non-inertial frame identified by the embedding $z^\mu_F(\tau, \sigma^u) = x^\mu(\tau) + F^\mu(\tau, \sigma^u)$ of Eq.(4.1). In it the fields $A_r(\tau, \sigma^u)$ and $\pi^r(\tau, \sigma^u)$ admit both a covariant and a non-covariant decomposition.

The covariant decomposition [36] is

$$
\pi^r(\tau, \sigma^u) = \hat{\pi}^r_\perp(\tau, \sigma^u) + \hat{\pi}^r_\parallel(\tau, \sigma^u)
$$

$$
\hat{\pi}^r_\perp(\tau, \sigma^u) = \left( \delta^r_s - \nabla^r_F \frac{1}{\Delta_F} \nabla_F s \right) \pi^s(\tau, \sigma^u) = \left( \delta^r_s - \nabla^r_F \frac{1}{\Delta_F} \partial_s \right) \pi^s(\tau, \sigma^u),
\Rightarrow \nabla_F \hat{\pi}^r_\perp(\tau, \sigma^u) = 0,
$$

$$
\hat{\pi}^r_\parallel(\tau, \sigma^u) = \nabla^r_F \frac{1}{\Delta_F} \nabla_F s \pi^s(\tau, \sigma^u) = \nabla^r_F \frac{1}{\Delta_F} \partial_s \pi^s(\tau, \sigma^u),
$$

$$
A_r(\tau, \sigma^u) = \hat{A}_\perp r(\tau, \sigma^u) + \hat{A}_\parallel r(\tau, \sigma^u),
$$

$$
\hat{A}_\perp r(\tau, \sigma^u) = \left( \delta^r_s - \nabla^r_F \frac{1}{\Delta_F} \nabla_F s \right) A_r(\tau, \sigma^u) \Rightarrow \nabla_F \hat{A}_\perp r(\tau, \sigma^u) = 0,
$$

$$
\hat{A}_\parallel r(\tau, \sigma^u) = \nabla^r_F \frac{1}{\Delta_F} \nabla_F s A_s(\tau, \sigma^u). \tag{B1}
$$

Here $\nabla^r_F$ and $\Delta_F = \nabla^r_F \nabla_F r = \frac{1}{\sqrt{\gamma_F(\tau, \sigma^u)}} \partial_r \left( \sqrt{\gamma_F(\tau, \sigma^u)} \gamma_F^r_s(\tau, \sigma^u) \partial_s \right)$ are the covariant derivative and the Laplace-Beltrami operator associated to the positive 3-metric $h_{Frs}(\tau, \sigma^u)$, respectively. The inverse of Laplace-Beltrami operator $(1/\Delta_F)$ is defined by the fun-
The fundamental solution of the Laplace-Beltrami operator $G(\sigma^u, \sigma'^u)$ \footnote{His existence is assured by existence’s theorem (see for example Ref.[37], but a closed analytic form is not known. A general property of these fundamental solutions is a singularity when the geodesic distance $s(\sigma^u, \sigma'^u)$ between $P = \{\sigma^u\}$ and $Q = \{\sigma'^u\}$ goes to zero $\lim_{s \to 0} G(\sigma^u, \sigma'^u) \mapsto \frac{1}{4\pi s(\sigma^u, \sigma'^u)}$.}} $f(\sigma^u) = \frac{1}{\Delta F} g(\sigma^u) \overset{def}{=} \int d^3\sigma' \sqrt{\gamma(\sigma'^u)} G(\sigma^u, \sigma'^u) g(\sigma'^u)$, such that $\Delta F f(\sigma^u) = g(\sigma^u)$.

Since $\pi^r(\tau, \sigma^u)$ is a vector density, we have $\partial_r \pi^r(\tau, \sigma^u) = \nabla_{F r} \pi^r(\tau, \sigma^u)$: this quantity is a 3-scalar density on $\Sigma_\tau$.

Instead the non-covariant decomposition \cite{1, 5, 9, 30} in a transverse and a longitudinal part ($\hat{\partial}^r \overset{def}{=} \delta^r_s \partial_s$, $\Delta = \partial_r \hat{\partial}^r = \hat{\partial}^2$) is

$$\pi^r(\tau, \sigma^u) = \pi^r_\perp(\tau, \sigma^u) + \pi^r_L(\tau, \sigma^u),$$

$$\pi^r_\perp(\tau, \sigma^u) = \left( \delta^r_s - \hat{\partial}^r \frac{1}{\Delta} \partial_s \right) \pi^s(\tau, \sigma^u) \Rightarrow \partial_r \pi^r_\perp(\tau, \sigma^u) = 0,$$

$$\pi^r_L(\tau, \sigma^u) = \hat{\partial}^r \frac{1}{\Delta} \partial_s \pi^s(\tau, \sigma^u),$$

$$A_r(\tau, \sigma^u) = A_{r\perp}(\tau, \sigma^u) + A_{rL}(\tau, \sigma^u),$$

$$A_{r\perp}(\tau, \sigma^u) = \left( \delta^r_s - \partial_r \frac{1}{\Delta} \hat{\partial}^s \right) A_s(\tau, \sigma^u) \Rightarrow \hat{\partial}^r A_{r\perp}(\tau, \sigma^u) = 0,$$

$$A_{rL}(\tau, \sigma^u) = \partial_r \frac{1}{\Delta} \hat{\partial}^s A_s(\tau, \sigma^u).$$ (B2)

In Eq.(B2) $\hat{\partial}^r A_r = \Delta \eta_{em}$ is a non-covariant quantity.

Here the inverse of Laplacian is defined using the standard (Euclidean-like) fundamental solution: $c(\sigma^u - \sigma'^u) = -\frac{1}{4\pi} \frac{1}{\sqrt{\sum_{u=1}^3 (\sigma^u - \sigma'^u)^2}}$, so that $f(\sigma^u) = \frac{1}{\Delta} g(\sigma^u) \overset{def}{=} \int d^3\sigma' c(\sigma^u - \sigma'^u) g(\sigma'^u)$ and $\Delta f(\sigma^u) = \left( \sum_{r=1}^3 \hat{\partial}^r \partial_r \right) f(\sigma^u) = g(\sigma^u)$.

Eq.(B2) allow us to define the following non-covariant Shanmugadhasan canonical transformation
\[
\begin{array}{c|c|c}
A_A & A_r & \eta_{em} & A_{\perp r} \\
\hline
\pi^A & \pi^r & \eta_{em} & A_{\perp r} \\
\end{array}
\]

\[A_r(\tau, \sigma^u) = -\frac{\partial}{\partial \sigma^r} \eta_{em}(\tau, \sigma^u) + A_{\perp r}(\tau, \sigma^u),\]

\[\pi^r(\tau, \sigma^u) = \pi^r_0(\tau, \sigma^u) + \frac{1}{\Delta} \hat{\partial}^r \Gamma(\tau, \sigma^u),\]

\[\eta_{em}(\tau, \sigma^u) = -\hat{\partial}^r A_r(\tau, \sigma^u),\]

\[A_{\perp r}(\tau, \sigma^u) = \left( \delta^s_r - \partial_r \frac{1}{\Delta} \hat{\partial}^s \right) A_s(\tau, \sigma^u),\]

\[\pi_{\perp}^r(\tau, \sigma^u) = \left( \delta^s_r - \hat{\partial}^r \frac{1}{\Delta} \partial_s \right) \pi^s(\tau, \sigma^u),\]

\[\{\eta_{em}(\tau, \sigma^u), \Gamma(\tau, \sigma^u)\} = \delta^3(\sigma^u, \sigma'^u),\]

\[\{A_{\perp r}(\tau, \sigma^u), \pi_{\perp}^s(\tau, \sigma'^u)\} = c \left( \delta^s_r - \frac{\partial_r \hat{\partial}^s}{\Delta} \right) \delta^3(\sigma^u, \sigma'^u). \quad (B3)\]

If we add the gauge fixing \(\eta_{em}(\tau, \sigma^u) \approx 0\), then its \(\tau\)-constancy implies \(A_r(\tau, \sigma^u) \approx 0\) and we get a non-inertial realization of the non-covariant radiation gauge.

While with the non-covariant decomposition we can easily find a Shanmugadhasan canonical transformation adapted to the Gauss law constraint with the standard canonically conjugate (but non-covariant) Dirac observables \(\vec{A}_\perp\) and \(\vec{\pi}_\perp\) of the radiation gauge, it is not clear whether the covariant decomposition can produce such a canonical basis. In any case, as shown in Ref.[36], the radiation gauge formalism is well defined in both cases if we add suitable gauge fixings.

In the inertial rest-frame instant form reviewed in Section III Subsection B the 3-metric inside the Wigner 3-spaces is \(g_{rs}(\tau, \sigma^u) = -\epsilon h_{rs}(\tau, \sigma^u) = -\epsilon \delta_{rs}\) and the two decompositions coincide.

In Subsection B of Section IV there is the non-covariant Shanmugadhasan canonical transformation in non-inertial frames in presence of charged particles.
Let us remark that in the non-Euclidean 3-space we are using a delta function \( \delta_3(\sigma^u, \sigma'^u) \), with the properties \( \delta_3(\sigma^u, \sigma'^u) = \delta_3(\sigma'^u, \sigma^u) \) and \( \frac{\partial}{\partial \sigma} \delta_3(\sigma^u, \sigma'^u) = -\frac{\partial}{\partial \sigma'} \delta_3(\sigma^u, \sigma'^u) \), such that \( d^3 \sigma' \delta_3(a^u, \sigma'^u) f(\sigma'^u) = f(a^u) \), and not a covariant one \( D_3^3(\sigma^u, \sigma'^u) = \frac{1}{\sqrt{\gamma(\tau, \sigma^u)}} \delta_3(\sigma^u, \sigma'^u) = \frac{1}{\sqrt{\gamma(\tau, \sigma'^u)}} \delta_3(\sigma^u, \sigma'^u) \) such that \( \int d^3 \sigma' \sqrt{\gamma(\tau, \sigma'^u)} D_3^3(a^u, \sigma'^u) f(\sigma'^u) = f(a^u) \).
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