Two-photon mediated resonance production in $e^+e^-$ collisions: cross sections and density matrices

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Abstract

Earlier described model amplitudes are used in this paper to evaluate both cross sections and density matrices for two-photon mediated resonance production in $e^+e^-$ collisions. All 25 $q\bar{q}$ low-lying $1S_0$, $3P_J$ and $1D_2$ resonances can thus be treated. Two independent methods are described to obtain the resonance production density matrices and cross sections. These density matrices combined with a resonance decay density matrix give the detailed angular distributions of the resonance decay products. For two particular decays, $\chi_{c2}, \chi_{c1} \to \gamma J/\psi$ the details are given. Several numerical results are presented as well.

1 Introduction

When performing electron-positron collision experiments, there will always be data originating from (virtual) two-photon collisions. In the latter process single resonance production occurs. When present and future experiments collect more and more statistics it becomes worthwhile to have theoretical predictions available for observables, which gradually become experimentally accessible.

In a previous paper [1], hereafter called I, a specific class of such observables namely form factors were studied. In this paper other observables will be focussed upon, such as azimuthal dependences in cross sections and resonance production density matrices. The latter quantity combined with a resonance decay density matrix will give the distribution of the decay products of the resonance.

In order to study the above observables one needs a model for resonance production. In I such a model was described starting from heavy quarks in a non-relativistic bound state. The predicted two-photon decay widths of resonances $R$ turn out to be in reasonable agreement with experiment. Once certain modifications are made, also the widths for light quark mesons become reasonable. Therefore the model of I is chosen as basis to predict observables for the 25 low-lying $1S_0$, $3P_J$ and $1D_2$ resonances. In paper I analytic amplitudes for $\gamma\gamma \to R$ are presented and expressions for form factors are derived. A
number of numerical results for cross sections and $Q_i^2$ dependences can also be found there.

In the present paper numerical studies of azimuthal distributions will be given. Their characteristic features can be understood from various analytic expressions. Whereas this discussion is a simple extension of the results of I, the evaluation of production density matrices requires substantial changes in calculational techniques. Moreover, rotations are required between reference frames favoured by theory and experiment. A discussion of two different evaluation methods will be given in this paper. Finally, in order to make a link to the angular distribution of resonance decay products, also one specific decay mode for two particular resonances will be discussed.

Summarizing, the purpose of the present paper is to discuss theoretical evaluation methods for resonance production density matrices and to present results thereof. Thus the full predictions of the resonance production model of I are in this way completed.

The paper is organized as follows. Section 2 gives the relevant amplitudes for resonance production. The direct method of the evaluation of the observables is discussed in section 3, whereas section 4 gives an extension of the BGMS formalism required for the evaluation of density matrices. After discussing a specific decay model for two resonances in section 5, numerical results for several quantities are presented in section 6.

2 The starting point

In this section the basic ingredients of the evaluation are listed, such as kinematics and matrix elements following from the model of I.

The two-photon mediated production of a resonance $R$ in electron-positron collisions is the reaction

$$e^+(p_1) + e^-(p_2) \rightarrow e^+(p_1') + e^-(p_2') + \gamma(k_1) + \gamma(k_2)$$

$$\rightarrow e^+(p_1') + e^-(p_2') + R(p_R).$$

The four-momenta $p_1 = (E_1, \vec{p}_1)$ and $p_2 = (E_2, \vec{p}_2)$ correspond to the incoming positron and electron respectively, whereas $p_1' = (E_1', \vec{p}_1')$ and $p_2' = (E_2', \vec{p}_2')$ are those of the outgoing positron and electron. The four-momenta of the intermediate photons $k_1 = (\omega_1, \vec{k}_1)$ and $k_2 = (\omega_2, \vec{k}_2)$ are related to the external four-momenta by

$$k_i = p_i - p_i'.$$

These two photons subsequently react to form a resonance with four-momentum $p_R$

$$p_R = k_1 + k_2 = p_1 + p_2 - p_1' - p_2'.$$

The virtuality $Q_i$ of an intermediate photon is defined by

$$Q_i^2 \equiv -k_i^2 = -(p_i - p_i')^2.$$  

As the virtual photons are space-like, the sign in (4) makes the virtualities $Q_i$ real.

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1 We use the standard metric diag$(g_{\mu\nu}) = (+1, -1, -1, -1)$. The totally anti-symmetric Levi-Civita tensor is defined by $\varepsilon_{0123} = +1$. 

2
The invariant mass of the two-photon system is given by
\[ W = p_R^2 = (k_1 + k_2)^2. \] (5)

The total available energy \( \sqrt{s} \) follows from
\[ s = (p_1 + p_2)^2. \] (6)

For a typical two photon event \( W \) is only a small fraction of \( \sqrt{s} \).

Some additional invariants can be defined
\[ s' = (p'_1 + p'_2)^2, \] (7)
\[ u = (p_1 - p'_2)^2, \] (8)
\[ u' = (p_2 - p'_1)^2, \] (9)

and as equivalents of \( Q_i^2 \)
\[ t = (p_1 - p'_1)^2, \] (10)
\[ t' = (p_2 - p'_2)^2. \] (11)

These invariants satisfy
\[ s + s' + t + t' + u + u' = W^2 + 8m_e^2, \] (12)

where \( m_e \) is the electron mass.

In the lab-frame we chose the following parametrization
\[ p_1 = (E_b, 0, 0, -P_b), \]
\[ p_2 = (E_b, 0, 0, P_b), \]
\[ p'_1 = (E'_1, |p'_1| \sin \theta_1 \cos \phi_1, |p'_1| \sin \theta_1 \sin \phi_1, -|p'_1| \cos \theta_1), \] (13)
\[ p'_2 = (E'_2, |p'_2| \sin \theta_2 \cos \phi_2, |p'_2| \sin \theta_2 \sin \phi_2, |p'_2| \cos \theta_2), \]
\[ p_R = (E_R, |p_R| \sin \theta_R \cos \phi_R, |p_R| \sin \theta_R \sin \phi_R, |p_R| \cos \theta_R). \]

This means that the z axis is along the direction of the incoming electron. The x and y axes are chosen such that the xz-plane is the accelerator ring plane and the y axis is perpendicular to that plane and is pointing upwards. In these formulae \( E_b \) is the energy of the incoming leptons and \( P_b \) their momentum. The choice of parameters in (13) is such that we have defined the \( z \)-components of the positron with an explicit minus sign.

This parametrization leads to explicit expressions for the invariants
\[ s = 4E_b^2, \] (14)
\[ Q_i^2 = 2 \left( E_bE'_i - P_b|p'_i| \cos \theta_i - m_e^2 \right) \approx 2E_bE'_i(1 - \cos \theta_i) = 4E_bE'_i \sin^2 \left( \frac{\theta_i}{2} \right), \] (15)
\[ W = \sqrt{2m_e^2 + 4(E_b - E'_1)(E_b - E'_2) - 2E'_1E'_2 - 2|p'_1||p'_2| \cos \theta_{12}} \]
\[ \approx 2\sqrt{\omega_1 \omega_2 - E'_1E'_2 \cos^2 \frac{\theta_{12}}{2}}. \] (16)
The approximations are valid in the limit $E_b > E'_i >> m_e$. In formula (16) the angle $\theta_{12}$ is the angle between the outgoing leptons in the lab-frame

$$\cos \theta_{12} = -\cos \theta_1 \cos \theta_2 - \sin \theta_1 \sin \theta_2 \cos(\phi_1 - \phi_2 + \pi).$$  \hspace{1cm} (17)

The propagators of the intermediate photons will appear as a product of the photon virtualities in the denominator of the matrix elements. This is the most dominant dependence on the photon virtualities. From (15) it follows that the photons are most likely radiated with low energy under a small angle with respect to the incoming lepton.

When the incoming lepton scatters over a large enough angle, so that it is detected in one of the (forward) detectors, this lepton is referred to as a tagged lepton. From the detected lepton the four-momentum, and thus the virtuality, of the associated intermediate photon can be reconstructed.

The two methods of calculation both start from the cross section formula

$$d\sigma = \frac{(2\pi)^4\delta^{(4)}(k_1 + k_2 - P)}{4\sqrt{p_1 \cdot p_2}^2 - m_e^2} \frac{1}{4} \sum |M|^2 \frac{d^3p_1}{(2\pi)^32E'_1} \frac{d^3p_2}{(2\pi)^32E'_2} d\Gamma,$$  \hspace{1cm} (18)

with $d\Gamma$ the invariant phase space element for the resonance. The amplitude $M$ has the structure

$$M = \bar{e}^{\mu} j_1^\mu j_2^\nu M_{\mu\nu},$$  \hspace{1cm} (19)

with

$$j_1^\mu = j_1^\mu(\lambda_1, \lambda'_1) = \bar{\nu}_{\lambda_1}(p_1)\gamma^\mu u_{\lambda'_1}(p_1'),$$

$$j_2^\mu = j_2^\mu(\lambda_2, \lambda'_2) = \bar{\nu}_{\lambda'_2}(p_2')\gamma^\mu u_{\lambda_2}(p_2),$$  \hspace{1cm} (20-21)

or explicitly in the model described (cf. also [3]) in [4]

$$M^{(1)S_0; \lambda_1, \lambda_2, \lambda'_1, \lambda'_2} = \frac{c_1\epsilon^2}{tt'} \varepsilon [k_1, k_2, j_1(\lambda_1, \lambda'_1), j_2(\lambda_2, \lambda'_2)],$$  \hspace{1cm} (22)

$$M^{(3)P_0; \lambda_1, \lambda_2, \lambda'_1, \lambda'_2} = \frac{c_2\epsilon^2}{tt'} ([j_1(\lambda_1, \lambda'_1) \cdot j_2(\lambda_2, \lambda'_2)k_1 \cdot k_2 - j_1(\lambda_1, \lambda'_1) \cdot k_2 j_2(\lambda_2, \lambda'_2) \cdot k_1] (W^2 + k_1 \cdot k_2)$$

$$- j_1(\lambda_1, \lambda'_1) \cdot j_2(\lambda_2, \lambda'_2)tt'),$$  \hspace{1cm} (23)

$$M^{(3)P_1; \lambda_1, \lambda_2, \lambda'_1, \lambda'_2, \lambda_R} = \frac{c_2\epsilon^2}{tt'} (t\varepsilon[\varepsilon^*(\lambda_R), j_1(\lambda_1, \lambda'_1), j_2(\lambda_2, \lambda'_2), k_2]$$

$$+ t'\varepsilon[\varepsilon^*(\lambda_R), j_2(\lambda_2, \lambda'_2), j_1(\lambda_1, \lambda'_1), k_1]),$$  \hspace{1cm} (24)

$$M^{(3)P_2; \lambda_1, \lambda_2, \lambda'_1, \lambda'_2, \lambda_R} = \frac{c_2\epsilon^2}{tt'} (k_1 \cdot k_2 j_{\mu}(\lambda_1, \lambda'_1) j_{2\nu}(\lambda_2, \lambda'_2)$$

$$+ k_{1\mu} k_{2\nu} j_{1}(\lambda_1, \lambda'_1) \cdot j_2(\lambda_2, \lambda'_2)$$

$$- k_{1\mu} j_{2\nu}(\lambda_2, \lambda'_2) j_1(\lambda_1, \lambda'_1) \cdot k_2$$

$$- k_{2\mu} j_{1\nu}(\lambda_1, \lambda'_1) j_2(\lambda_2, \lambda'_2) \cdot k_1) \varepsilon^{*\mu\nu}(\lambda_R),$$  \hspace{1cm} (25)

$^2$In these equations we have introduced the shorthand notation $\varepsilon(p, q, r, s) \equiv \varepsilon^{\alpha\beta\gamma\delta} p_\alpha q_\beta r_\gamma s_\delta$. 


\[ M^{(1)D_2; \lambda_1, \lambda_2, \lambda'_1, \lambda'_2, \lambda_R}) = \frac{e^2}{\mu^2} \varepsilon^{\mu \nu} (\lambda_R) k_{1 \mu} k_{2 \nu} \]
\[ \varepsilon [k_1, k_2, j_1(\lambda_1, \lambda'_1), j_2(\lambda_2, \lambda'_2)] , \]

where

\[ c_1 = g_0, \quad c_2 = 4g_1/W, \quad c_3 = 2\sqrt{6}g_1, \quad c_4 = 4\sqrt{3}Wg_1, \quad c_5 = 8\sqrt{30}g_2. \]

Here we have introduced

\[ g_i = \frac{16e_q^2|R^{(i)}(0)|\alpha}{(s + s' + u + u' - 8m_e^2)^{i+1}} \sqrt{\frac{3\pi}{W}}, \]

with \( R^{(i)}(0) \) the (derivatives of the) radial part of the wave function in the origin and \( e_q \) the fractional quark charge. The actual values are given in I.

Replacing the currents \( j_1, j_2 \) by \( \varepsilon_1 \) and \( \varepsilon_2 \), the polarization vectors of a virtual photon, will give the matrix element of the two-photon reaction, i.e. the second step in reaction (1). That type of matrix element can be simplified by choosing convenient axes to describe the photon polarization. It will be the basis of the method of section 4. In the expressions terms of the form \( \varepsilon_i \cdot k_i \) or \( j_i \cdot k_i \) are omitted, since they vanish in our calculations.

### 3 Direct method of calculation

In this section one way of evaluating the cross section (18) and the related density matrix will be discussed. The motivation to deal with another method in the next section is that we want to obtain the numerical results for observables in two completely independent ways.

The method of this section is a 2 \( \rightarrow \) 3 particle cross section calculation without using the two-photon intermediate step. For the differential cross section alone it would be sufficient to express \( \sum |M|^2 \) in terms of the invariants (18)-(23). In fact, those have been obtained and are given elsewhere [4]. It makes a very fast numerical evaluation of the cross section possible. It is natural to perform this calculation in the laboratory frame as specified in equation (13).

For the density matrices it is more practical to use a rest system of the resonance. Every experimental event can be rotated and boosted back to the resonance rest frame (RRF) by successively rotating the momentum \( \vec{p}_R \) in the direction of the \( z \)-axis and boosting it to rest. Thus the required Lorentz transformation is

\[ L_{LR} = L_B L_\theta L_\phi = \begin{pmatrix}
\gamma & -\gamma \beta \cos \phi \sin \theta & -\gamma \beta \sin \phi \sin \theta & -\gamma \beta \cos \theta \\
0 & \cos \phi \cos \theta & \sin \phi \cos \theta & -\sin \theta \\
0 & -\sin \phi & \cos \phi & 0 \\
-\gamma \beta & \gamma \cos \phi \sin \theta & \gamma \sin \phi \sin \theta & \gamma \cos \theta \\
\end{pmatrix}. \]

In order to calculate the average density matrix in the RRF belonging to a cross section in a certain kinematical region in the laboratory system one proceeds as follows. In the laboratory system one generates weighted events. For this we use the event generator GaGaRes, which is described elsewhere [4]. The event is boosted to the RRF with \( L_{LR} \).
For the momenta in the RRF the amplitude $A_{\lambda_i, \lambda}$ is calculated, where $\lambda_i$ are the lepton helicites and $\lambda$ the resonance helicity. From these amplitudes the normalized density matrix

$$
\rho_{\lambda \lambda'} = \frac{\sum_{\lambda_i} A_{\lambda_i, \lambda} A^*_{\lambda_i', \lambda}}{\sum_{\lambda_i} R_{\lambda_i, \lambda_i} A_{\lambda_i, \lambda} A^*_{\lambda_i, \lambda'}}, \tag{30}
$$

is evaluated for this event. By performing the weighted sum of these individual density matrices and dividing by the total weight one obtains the average density matrix for the production of a resonance. Of course, when events with weight 1 are generated the above procedure becomes simpler. Sometimes unnormalized density matrices are needed, i.e. just the numerator of (30).

It is clear that a method to evaluate amplitudes is needed, preferably a fast technique. For this the Weyl-van der Waerden (WvdW) spinor calculus is a good tool, as has been shown in the literature [5]. Once Dirac spinors, momenta and polarization vectors have been translated into WvdW spinors, the matrix elements (22)-(26) become spinorial quantities.

There now are two ways to continue. The amplitudes can be expressed in terms of spinorial inner products. Evaluating numerically these products and their combinations one obtains the amplitudes as a complex number. Another way is to interpret the spinorial matrix element as the trace of a string of $2 \times 2$ matrices, which can be calculated numerically very fast thanks to efficient matrix multiplication in FORTRAN90. The latter method has been used in our numerical calculations, whereas the former turned out to be useful as independent check. More about the spinorial translation and procedure can be found in [4]. Here we only mention one particular detail, relevant for comparisons and cross checks.

Usually the spin-1 polarization vectors for a particle with momentum

$$
k^\mu = (k^0, |\vec{k}| \sin \theta \cos \phi, |\vec{k}| \sin \theta \sin \phi, |\vec{k}| \cos \theta) = (k^0, |\vec{k}| sc_\phi, |\vec{k}| ss_\phi, |\vec{k}| c), \tag{31}
$$

are chosen to be

$$
\begin{align*}
\varepsilon^\mu_+ &= \frac{1}{\sqrt{2}} (0, \pm cs_\phi + is_\phi, \mp sc_\phi - ic_\phi, \pm s), \\
\varepsilon^\mu_0 &= \frac{k^0}{m} \left( \frac{|\vec{k}|}{k_0}, sc_\phi, ss_\phi, c \right), \tag{32}
\end{align*}
$$

for helicities $\pm 1$ and 0. The spin-2 polarization tensors then usually are taken as

$$
\begin{align*}
\varepsilon^{\mu \nu} (\pm 2) &= \varepsilon^\mu (\pm 1) \varepsilon^\nu (\pm 1), \\
\varepsilon^{\mu \nu} (\pm 1) &= \frac{1}{\sqrt{2}} (\varepsilon^\mu (\pm 1) \varepsilon^\nu (0) + \varepsilon^\mu (0) \varepsilon^\nu (\pm 1)), \\
\varepsilon^{\mu \nu} (0) &= \frac{1}{\sqrt{6}} (\varepsilon^\mu (+1) \varepsilon^\nu (-1) + 2 \varepsilon^\mu (0) \varepsilon^\nu (0) + \varepsilon^\mu (-1) \varepsilon^\nu (1)). \tag{33}
\end{align*}
$$

Of course this follows a certain convention and other choices may be made as can be seen in the literature. In the implementation of the WvdW formalism the conventions of [3] have been used. Unfortunately the polarization vectors and tensors in this reference are different

$$
\begin{align*}
\varepsilon^\mu_\pm (k) &= \frac{e^{\pm i \phi}}{\sqrt{2}} \left( 0, -cc_\phi \pm is_\phi, -cs_\phi \mp ic_\phi, s \right), \\
\varepsilon^\mu_0 &= \frac{k^0}{m} \left( \frac{|\vec{k}|}{k_0}, sc_\phi, ss_\phi, c \right), \tag{34}
\end{align*}
$$

6
\[ \varepsilon^{\mu\nu}(\pm 2) = \varepsilon^\mu(\pm 1)\varepsilon^\nu(\pm 1), \]
\[ \varepsilon^{\mu\nu}(\pm 1) = \frac{1}{\sqrt{2}}(\varepsilon^\mu(\pm 1)\varepsilon^\nu(0) + \varepsilon^\mu(0)\varepsilon^\nu(\pm 1)), \]
\[ \varepsilon^{\mu\nu}(0) = \frac{1}{\sqrt{6}}(-\varepsilon^\mu(1)\varepsilon^\nu(-1) + 2\varepsilon^\mu(0)\varepsilon^\nu(0) - \varepsilon^\mu(-1)\varepsilon^\nu(+1)). \]

When we label the sets of polarization vectors and tensors of equations (32) and (33) with the index \( A \) and the polarization vectors and corresponding tensors in equations (34) and (35) with an index \( B \), the two sets are related by
\[ \varepsilon_{B,\pm}^\mu = \pm e^{\mp i\phi} \varepsilon_{A,\pm}^\mu, \]
\[ \varepsilon_{B,0}^\mu = \varepsilon_{A,0}^\mu, \]
\[ \varepsilon_{B}(\pm 1) = e^{\mp i\phi} \varepsilon_{A}(\pm 1), \]
\[ \varepsilon_{B}(\pm 2) = e^{\pm 2i\phi} \varepsilon_{A}(\pm 2). \]

These formulae give the relations between density matrices in the two conventions. When an event has a density matrix \( \rho_A \), evaluated with the set \( A \), then \( \rho_B \) can be obtained by multiplying each matrix element with a certain phase factor. In a compact notation these factors are summarized as follows
\[ \rho_{\lambda\lambda',B} = e^{i\Delta\phi_{\lambda\lambda'}} \rho_{\lambda\lambda',A}, \]
with for the spin-1 density matrix elements
\[ \Delta\phi_{\lambda\lambda'} = \begin{pmatrix} 0 & \phi & 2\phi + \pi \\ -\phi & 0 & \phi + \pi \\ -(2\phi + \pi) & -(\phi + \pi) & 0 \end{pmatrix}. \]

For the spin-2 density matrix elements we find analogously the set of phase factors
\[ \Delta\phi_{\lambda\lambda'} = \begin{pmatrix} 0 & \phi & 2\phi & 3\phi & 4\phi \\ -\phi & 0 & 2\phi & 3\phi \\ -2\phi & -\phi & 0 & 2\phi \\ -3\phi & -2\phi & -\phi & 0 & \phi \\ -4\phi & -3\phi & -2\phi & -\phi & 0 \end{pmatrix}. \]

### 4 Extended BGMS formalism

A long time ago in a classic paper [2] it was advocated to write the cross section of reaction (\[\square\]) as a two-step process. First virtual photons characterized by a density matrix \( \rho_1 \) are created which in a second step produce the resonance \( R \). In this BGMS formalism the cross section for resonance production (18) takes in first instance the form
\[ d\sigma = \frac{\alpha^2}{Q_1^2 Q_2^2} \rho_1^{\mu\nu} \rho_2^{\mu'\nu'} \delta^{(4)}(k_1 + k_2 - P) \frac{\sum M_{\mu\nu}^* M_{\mu'\nu'}^*}{\sqrt{(p_1 \cdot p_2)^2 - m_R^2}} d^3 p_1^* d^3 p_2^* d\Gamma, \]
where
\[ \rho_i^{\mu\nu} = \frac{1}{2 Q_i^2} \sum j_i^{\mu} j_i^{\mu'} = \frac{2}{Q_i^2} \left( p_i^{\alpha} p_i^{\beta} + p_i^{\alpha} p_i^{\beta} - \frac{1}{2} Q_i^2 g^{\alpha\beta} \right). \]
The index summation in equation (40) is now replaced by a summation over photon helicities $\lambda_i$ and $\lambda'_i$ with values $\pm 1, 0$. Introducing the quantity

$$M_{\lambda'_i, \lambda_i, \lambda'_2, \lambda_2} = \frac{1}{2} (2\pi)^4 \int \delta^4(k_1 + k_2 - p_R) \sum_{\lambda_R} M_{\lambda'_1, \lambda_1, \lambda'_2, \lambda_2, \lambda_R} d\Gamma,$$  

(42)

the cross section can be written as

$$d\sigma = \frac{\alpha^2}{32\pi^4 Q_1^2 Q_2^2} \frac{1}{(p_1 \cdot p_2)^2 - m_e^4} \sum \rho_1^{\lambda_1, \lambda'_1} \rho_2^{\lambda_2, \lambda'_2} M_{\lambda'_1, \lambda_1, \lambda'_2, \lambda_2} \frac{d\vec{p}_1}{E_1'} \frac{d\vec{p}_2}{E_2'}.$$  

(43)

It should be noted that $M_{\lambda'_i, \lambda_i, \lambda'_2, \lambda_2}$ is evaluated in the two-photon rest system, the BGMS frame, where the direction of the photon originating from the positron is taken as $z$-axis. The polarization vectors of both photons now become very simple as can be seen from (32). Also $\lambda_R$ is completely fixed by

$$\lambda_R = \lambda_1 - \lambda_2,$$  

(44)

and the summation in (42) consists of only one term.

The expression for the unnormalized density matrix of an event is obtained from the cross section formula when one introduces another combination $\rho \rho M$.

$$\Sigma_{\lambda \lambda'} = \sum_{\lambda_1, \lambda_2, \lambda'_1, \lambda'_2} \rho_1^{\lambda_1, \lambda'_1} \rho_2^{\lambda_2, \lambda'_2} M_{\lambda'_1, \lambda_1, \lambda'_2, \lambda_2}.$$  

(45)

It should be noted that for a specific $\lambda$ only specific $\lambda_1, \lambda_2$ combinations can contribute. Using the explicit form of the photon density matrices [2] where only a restricted number of elements is independent and using

$$M_{\lambda_1, \lambda_2, \lambda_R} = \eta_R M_{-\lambda_1, -\lambda_2, -\lambda_R}$$  

(46)

where $\eta_R = 1(-1)$ for the “normal” (“abnormal”) $J^P$ series, one obtains expressions for $\Sigma_{\lambda \lambda'}$ in terms of $M_{\lambda'_1, \lambda_2, \lambda_1, \lambda_2}$. For every expression also new quantities are introduced, generalizations of the quantities $\sigma_{AB}$ and $\tau_{AB}$ of [2] with the help of

$$X_{\gamma \gamma} = (k_1 \cdot k_2)^2 - k_1^2 k_2^2.$$  

(47)

The results are

$$\Sigma_{2+2+} = \Sigma_{2-2-} = \rho_1^{++} \rho_2^{++} M_{++--} = 4 \sqrt{X_{\gamma \gamma}} \rho_1^{++} \rho_2^{++} \sigma_{TT}^B,$$  

(48)

$$\Sigma_{++} = \Sigma_{--} = \rho_1^{++} \rho_2^{00} M_{++00} + \rho_1^{00} \rho_2^{++} M_{00++} - 2|\rho_1^{++}||\rho_2^{00}| |M_{00++}| \cos(\phi)$$

$$= 2 \sqrt{X_{\gamma \gamma}} \left( \rho_1^{++} \rho_2^{00} \sigma_{TS} + \rho_1^{00} \rho_2^{++} \sigma_{ST} - 4|\rho_1^{++}||\rho_2^{00}| \cos(\phi) \tau_{TS}^B \right).$$  

(49)
\[ \Sigma_{00} = 2\rho_1^{++}\rho_2^{++}M_{++++} + \rho_1^{00}\rho_2^{00}M_{0000} + 2|\rho_1^+||\rho_2^+|\cos(2\tilde{\phi})M_{++-} - 4|\rho_1^{0+}||\rho_2^{+0}|\cos(\tilde{\phi})M_{00++} \]
\[ = 2\sqrt{X_{\gamma\gamma}} \left(4\rho_1^{++}\rho_2^{++}\sigma_{TT}^T + \rho_1^{00}\rho_2^{00}\sigma_{SS} + 2|\rho_1^+||\rho_2^+|\cos(2\tilde{\phi})\tau_{TT} - 8|\rho_1^{0+}||\rho_2^{+0}|\cos(\tilde{\phi})\tau_{TTS} \right). \]

The various \(\sigma\) and \(\tau\) definitions follow from \([48]-[50]\). In BGMS the following combinations are used
\[ \sigma_{TT} \equiv \sigma_{TT}^A + \sigma_{TT}^B = \frac{1}{4\sqrt{X_{\gamma\gamma}}}(M_{++++} + M_{++-}), \]
\[ \tau_{TS} \equiv \tau_{TS}^A + \tau_{TS}^B = \frac{1}{4\sqrt{X_{\gamma\gamma}}}(M_{++00} + M_{-00+}). \]

In the formulae \(\tilde{\phi}\) is the angle between the two lepton scattering planes in the BGMS frame.

For the off-diagonal elements we do the same. The results are summarized below
\[ \Sigma_{+-} = e^{2i\tilde{\phi}_1} \left(2|\rho_1^{+0}||\rho_2^{0+}|e^{-i\tilde{\phi}}M_{0+00} - |\rho_1^+|\rho_2^{00}M_{-0+0} - \rho_1^{00}|\rho_2^+|e^{-2i\tilde{\phi}}M_{000-} \right), \]
\[ \Sigma_{+0} = ie^{i\tilde{\phi}_1} \left(\rho_1^{++}|\rho_2^{+0}|e^{-i\tilde{\phi}}M_{++00} - |\rho_1^+\rho_2^{00}M_{00++} + |\rho_1^+||\rho_2^+|e^{-2i\tilde{\phi}}M_{--00} + \rho_1^{00}|\rho_2^{+0}|e^{-i\tilde{\phi}}M_{00++} - |\rho_1^{0+}||\rho_2^{0+}|e^{-2i\tilde{\phi}}M_{00++} \right). \]

For the spin-2 resonances one has additionally
\[ \Sigma_{2+1+} = ie^{i\tilde{\phi}_1} \left(\rho_1^{++}|\rho_2^{+0}|e^{-i\tilde{\phi}}M_{++00} - |\rho_1^+\rho_2^{00}M_{00++} \right), \]
\[ \Sigma_{2+0} = -e^{2i\tilde{\phi}_1} \left(\rho_1^{++}|\rho_2^{+0}|e^{-2i\tilde{\phi}}M_{++00} - |\rho_1^+\rho_2^{00}M_{00++} + |\rho_1^+||\rho_2^+|e^{-i\tilde{\phi}}M_{--00} + \rho_1^{00}|\rho_2^{+0}|e^{-2i\tilde{\phi}}M_{00++} \right), \]
\[ \Sigma_{2+1-} = ie^{3i\tilde{\phi}_1} \left(-|\rho_1^+||\rho_2^{+0}|e^{-i\tilde{\phi}}M_{-0+0} + |\rho_1^{00}||\rho_2^+|e^{-2i\tilde{\phi}}M_{0++-} \right), \]
\[ \Sigma_{2+2-} = e^{4i\tilde{\phi}_1}|\rho_1^{++}|\rho_2^{+0}|e^{-2i\tilde{\phi}}M_{--++}. \]

Besides the \(\tilde{\phi}\) dependence, for the off-diagonal elements there is also an overall \(\tilde{\phi}_1\) dependence, \(\tilde{\phi}_1\) being the azimuthal angle of the incoming positron in the BGMS frame.

In analogy with the \(\sigma\) and the \(\tau\) terms we can introduce
\[ \chi_{+0} = \frac{i}{2\sqrt{X_{\gamma\gamma}}}M_{+0++}, \quad \chi_{0+} = \frac{i}{2\sqrt{X_{\gamma\gamma}}}M_{++0+}, \]
\[ \xi_{+0} = \frac{i}{2\sqrt{X_{\gamma\gamma}}}M_{00++}, \quad \xi_{0+} = \frac{i}{2\sqrt{X_{\gamma\gamma}}}M_{000+}, \]
\[ \zeta_{++} = \frac{1}{2\sqrt{X_{\gamma\gamma}}}M_{++--}, \quad \zeta_{00} = \frac{1}{2\sqrt{X_{\gamma\gamma}}}M_{00--}, \]
\[ \zeta_{+0} = \frac{i}{2\sqrt{X_{\gamma\gamma}}}M_{+0--}, \quad \zeta_{0+} = \frac{i}{2\sqrt{X_{\gamma\gamma}}}M_{00+-}. \]
The $\zeta$ functions are only non-vanishing for spin-2 resonances. For all above functions $\sigma_{AB}$, $\ldots$, $\zeta_{AB}$ holds: when $\zeta_{AB} \sim M^{\zeta}_{\lambda_{1}\lambda_{2}\lambda_{3}\lambda_{4}}$ then $\zeta_{BA} \sim M^{\zeta}_{\lambda_{2}\lambda_{1}\lambda_{4}\lambda_{3}}$.

The off-diagonal elements can now be written as

$$\Sigma_{++} = e^{2i\phi_1} \sqrt{X_{\gamma\gamma}} \left( |\rho_1^0|^2 |\rho_2^0|^2 e^{-i\phi} \eta R \tau^B_{TS} - |\rho_1^+|^2 |\rho_2^0|^2 \eta R \sigma_{TS} - |\rho_1^0|^2 |\rho_2^+|^2 e^{-2i\phi} \eta R \sigma_{ST} \right),$$  

$$\Sigma_{+0} = e^{i\phi_1} \sqrt{X_{\gamma\gamma}} \left( |\rho_1^+|^2 |\rho_2^0|^2 e^{-i\phi} \chi_{++} - |\rho_1^0|^2 |\rho_2^0|^2 \eta R e^{i\phi} \chi_{+0} - |\rho_1^0|^2 |\rho_2^+|^2 e^{-2i\phi} \eta R \xi_{+0} \right),$$  

$$\Sigma_{2+1+} = e^{i\phi_1} \sqrt{X_{\gamma\gamma}} \left( |\rho_1^1|^2 |\rho_2^0|^2 e^{-i\phi} \zeta_{++} - |\rho_1^1|^2 |\rho_2^+|^2 \eta R \zeta_{+0} \right),$$  

$$\Sigma_{2+0} = -e^{2i\phi_1} \left( |\rho_1^1|^2 |\rho_2^0|^2 e^{-2i\phi} \zeta_{++} - |\rho_1^0|^2 |\rho_2^0|^2 e^{-i\phi} \zeta_{00} + |\rho_1^0|^2 |\rho_2^+|^2 e^{-i\phi} \zeta_{+0} \right),$$  

$$\Sigma_{2+1-} = e^{3i\phi_1} \sqrt{X_{\gamma\gamma}} \left( -|\rho_1^0|^2 |\rho_2^0|^2 e^{-i\phi} \eta R \zeta_{+0} + |\rho_1^1|^2 |\rho_2^+|^2 e^{-2i\phi} \zeta_{+0} \right),$$  

$$\Sigma_{2+2-} = e^{4i\phi_1} e^{-2i\phi} \sqrt{X_{\gamma\gamma}} |\rho_1^+|^2 |\rho_2^+|^2 \eta R \sigma_{TT}^B.$$  

In these formulae $\eta_R$ is the phase factor arising in equation (66). The $\xi$ functions and $\zeta_{00}$ vanish for the $^3P_1$ resonance ($M_{00,\lambda^\prime}$ and $M_{\lambda^\prime,00}$ vanish).

The other off-diagonal elements can be obtained by using that the density matrix is Hermitian

$$\Sigma_{\lambda^\prime\lambda} = \Sigma_{\lambda\lambda^\prime}^\ast,$$  

and by using that for the polarization vectors in the BGMS formalism the following identity holds

$$\Sigma_{\lambda^\prime\lambda} = (-1)^{\lambda+\lambda^\prime} \Sigma_{-\lambda\lambda^\prime}.$$  

The unnormalized density matrix after integration over the phase space of the outgoing leptons, denoted by $\Sigma_{int}$, is then given by

$$\Sigma_{int}^{\lambda\lambda^\prime} = \int \frac{\alpha^2}{32\pi^3 Q_1 Q_2} \frac{1}{(p_1 \cdot p_2)^2 - m_e^2} \Sigma_{\lambda\lambda^\prime} \frac{d^3 p_1}{E_1} \frac{d^3 p_2}{E_2}.$$  

The trace should equal the total cross section. When we use the expressions for the diagonal elements we indeed obtain the BGMS expression

$$d\sigma = \frac{\alpha^2}{16\pi^3 Q_1 Q_2} \sqrt{\frac{(k_1 \cdot k_2)^2 - k_1^2 k_2^2}{(p_1 \cdot p_2)^2 - m_e^2}} \left[ 4|\rho_1^+|^2 |\rho_2^+|^2 \tau_{TT} \cos(2\phi) + 2|\rho_1^+|^2 |\rho_2^0|^2 \tau_{ST} \cos(2\phi) + 2|\rho_1^0|^2 |\rho_2^+|^2 \tau_{SS} \cos(2\phi) - 8|\rho_1^0|^2 |\rho_2^0|^2 \tau_{TT} \cos(2\phi) \right].$$  

The explicit expressions for quantities $\sigma_{AB}$, $\tau_{AB}$, $\chi_{AB}$, $\xi_{AB}$ and $\zeta_{AB}$ follow from the amplitudes $M_{\lambda_1, \lambda_2, \lambda_3}$ as given in table 1 of I.

In order to have some analytic results we repeat and extend form factor definitions of I and give tables with their forms. Thus we introduce

$$\sigma_{AB} = \delta(P^2 - M^2) \frac{8\pi^2 (2J + 1) \Gamma_{\gamma\gamma}(J^P)}{M} f_{AB}(J^P),$$  

$$\tau_{AB} = \delta(P^2 - M^2) \frac{8\pi^2 (2J + 1) \Gamma_{\gamma\gamma}(J^P)}{M} g_{AB}(J^P).$$
The form factors are given in tables 1 and 2, where the quantities $\kappa$ to the one of the previous section, one should also perform the change (37) for every event.

When one likes to compare the average density matrix, it is calculated in the two-photon rest frame, which is a frame where the resonance is at rest, it is not the RRF of the previous section. For every event one has to relate the axes by a rotation which gives rise to a transformation of the density matrix. The transformation matrices are given in appendix A. When one likes to compare the average density matrix to the one of the previous section, one should also perform the change (37) for every event.

| $J^P$ | $f_{TT}^A$ | $f_{TT}^B$ | $f_{TS}$ | $f_{SS}$ |
|-------|------------|------------|----------|----------|
| 0−    | $\kappa \frac{\rho^2}{2}$ | 0          | 0        | 0        |
| 0+    | $\kappa \left(\frac{X+\nu M^2}{3\nu^2}\right)$ | 0          | 0        | $2\kappa \left(\frac{M^2 \sqrt{Q_1^2 Q_2^2}}{3\nu^2}\right)^2$ |
| 1+    | $\kappa \left(\frac{Q_1^2-Q_2^2}{2\nu}\right)^2$ | 0          | $2\kappa \frac{M^2 Q_2^2}{2\nu} \left(\frac{\nu+Q_1^2}{\nu}\right)^2$ | 0        |
| 2+    | $\kappa \left(\frac{M^2}{2\nu}\right)^2 \kappa \left(\frac{M}{3\nu^2}\right)^2 \left(2Q_1^2 Q_2^2 - \nu (Q_1^2 + Q_2^2)\right)$ | $\kappa \left(\frac{M}{2\nu}\right)^2 Q_1 Q_2 \left(\nu + Q_1^2\right) \left(\nu + Q_2^2\right)$ |
| 2−    | $\kappa \left[\frac{X}{\nu^2}\right]^3$ | 0          | 0        | 0        |

| $J^P$ | $g_{TT}$ | $g_{TS}$ |
|-------|---------|---------|
| 0−    | $-2\kappa \frac{\rho^2}{2}$ | 0    |
| 0+    | $2\kappa \left(\frac{X+\nu M^2}{3\nu^2}\right)$ | $\kappa \left(\frac{M}{3\nu^2}\right)^2 Q_1 Q_2 \left(X + \nu M^2\right)$ |
| 1+    | $-2\kappa \left(\frac{Q_1^2-Q_2^2}{2\nu}\right)^2$ | $\kappa \left(\frac{M}{2\nu}\right)^2 Q_1 Q_2 \left(\nu + Q_1^2\right) \left(\nu + Q_2^2\right)$ |
| 2+    | $\frac{3}{8} \left(\frac{2Q_1^2 Q_2^2 - \nu (Q_1^2 + Q_2^2)}{\nu} - \nu (Q_1^2 + Q_2^2)\right)$ | $\kappa \left(\frac{M}{2\nu}\right)^2 Q_1 Q_2 \left(\nu + Q_1^2\right) \left(\nu + Q_2^2\right)$ |
| 2−    | $-2\kappa \left[\frac{X}{\nu^2}\right]^3$ | 0 |

Table 1: The form factors required for the cross section $f_{AB}$, $g_{AB}$, $A, B = T, S$.

\[ \chi_{AB} = \delta(P^2 - M^2)8\pi^2 \frac{(2J + 1)\Gamma_{\gamma\gamma}(JP)}{M} k_{AB}(JP), \] (75)
\[ \xi_{AB} = \delta(P^2 - M^2)8\pi^2 \frac{(2J + 1)\Gamma_{\gamma\gamma}(JP)}{M} m_{AB}(JP), \] (76)
\[ \zeta_{AB} = \delta(P^2 - M^2)8\pi^2 \frac{(2J + 1)\Gamma_{\gamma\gamma}(JP)}{M} n_{AB}(JP). \] (77)

The form factors are given in tables 1 and 2, where the quantities $\kappa$, $\nu$ and $X$ are defined as

\[ \kappa = \frac{M^2}{2\sqrt{X}}, \] (78)
\[ \nu = k_1 \cdot k_2, \] (79)
\[ X = X_{\gamma\gamma} = \nu^2 - k_1^2 k_2^2. \] (80)

It should be stressed that for every event the matrix $\Sigma$ can now be evaluated. Although it is calculated in the two-photon rest frame, which is a frame where the resonance is at rest, it is not the RRF of the previous section. For every event one has to relate the axes by a rotation which gives rise to a transformation of the density matrix. The transformation matrices are given in appendix A. When one likes to compare the average density matrix to the one of the previous section, one should also perform the change (37) for every event.
\[
\begin{array}{|c|c|c|}
\hline
J^P & k_{+0} & m_{+0} \\
\hline
1^+ & -k \frac{M Q_2}{4 \nu^2} (\nu + Q_1^2)(Q_1^2 - Q_2^2) & 0 \\
2^+ & -k \frac{M Q_2}{4 \nu^2} \left[2Q_1^2 Q_2^2 - \nu(Q_1^2 + Q_2^2)\right] & -k \frac{M^3 Q_1 Q_2^2}{2 \sqrt{3} \nu^3} (\nu - Q_1^2) \\
2^- & 0 & 0 \\
\hline
\end{array}
\]

Table 2: The additional form factors required for the off-diagonal elements. \(k_{AB}, m_{AB}\) and \(n_{AB}\) \((A, B = +, 0)\).

With the above procedure we extend the existing BGMS cross section Monte Carlo program Galuga \([7]\) to a program which can calculate the density matrix \(\Sigma\). We then evaluate for every event generated by this program the density matrix \(\Sigma\) which is then transformed to the experimentally relevant density matrix of section 3. In this way Galuga can evaluate the wanted density matrix. Moreover, we thus have an independent check on the GaGaRes evaluation.

5 Example of a decay model and density matrix

In general the produced resonance \(R\) will decay into some final state \(X\), so that the reaction is

\[
e^+ e^- \rightarrow e^+ e^- R \rightarrow e^+ e^- X.
\]  

(81)

The total amplitude for this process can be written as

\[
\mathcal{M} = \sum_{\lambda R} \xi(P, M) A_{\lambda R} D_{\lambda R}.
\]  

(82)

\(A_{\lambda R}\) describes the two-photon production of a resonance with helicity \(\lambda R\). \(D_{\lambda R}\) describes the decay of the resonance with helicity \(\lambda R\) into the final state \(X\). The factor \(\xi(P, M)\) represents the propagator of the resonance and numerical factors. The total matrix element still depends on the external four-momenta. No implicit integrations over the outgoing momenta have been carried out at this stage. The square of the total matrix element is given by

\[
\sum |\mathcal{M}|^2 = |\xi(P, M)|^2 \sum_{\lambda R, \lambda_R'} A_{\lambda R \lambda_R'} D_{\lambda R \lambda_R'} = |\xi(P, M)|^2 Tr (\mathcal{A} \mathcal{D}^*)
\]  

(83)

The summation on the left hand side represents a summation over the helicities of the initial and final state particles. The quantities \(A_{\lambda R \lambda_R'}\) and \(D_{\lambda R \lambda_R'}\) are \(A_{\lambda R} A_{\lambda_R'}^*\) and \(D_{\lambda R} D_{\lambda_R'}^*\).
summed over the helicities of all particles but the resonance. These are the density matrices for the production and decay of the resonance. For a spin-$J$ resonance the density matrices are formed by $(2J + 1) \times (2J + 1)$-matrices. For actual calculations a choice of polarization vectors and tensors has to be made. The results of this paper are obtained in the conventions \([23]\) and \([25]\).

For $R$ we first take the spin-2 state $\chi_{c2}$ and consider the specific decay mode

$$\chi_{c2}(p_R) \rightarrow \gamma(k_1) J/\psi(k_2). \quad (84)$$

Later on also the possibility of the $J/\psi$ decaying into a lepton pair will be discussed. The choice of this example is prompted by its experimental relevance. Since we expect that a similar $\chi_{c1}$ decay may cause a contamination of the events \((84)\) in an actual experiment we shall also discuss the $\chi_{c1}$ decay at the end of this section.

For the amplitude $\mathcal{M}$ of the decay of the spin-2 resonance we use the amplitude that is given in equation \((23)\)

$$\mathcal{M}(\lambda, \lambda_1, \lambda_2) = \bar{c}_4 \left[ (k_1 \cdot k_2) \varepsilon_{1}^{\mu}(\lambda_1) \varepsilon_{2}^{\nu}(\lambda_2) + (\varepsilon_{1}^{\mu}(\lambda_1) \cdot \varepsilon_{2}^{\nu}(\lambda_2))k_{1}^{\mu}k_{2}^{\nu} \right.$$  
$$- (k_1 \cdot \varepsilon_{2}^{\nu}(\lambda_2)) \varepsilon_{1}^{\mu}(\lambda_1)k_{2}^{\nu} - (k_2 \cdot \varepsilon_{1}^{\mu}(\lambda_1))\varepsilon_{2}^{\nu}(\lambda_2)k_{1}^{\mu}] \varepsilon_{\mu}(\lambda) \quad (85)$$
$$= \bar{c}_4 F_{i}^{\alpha \beta}(\lambda_1) F_{2 \mu \nu}^{*}(\lambda_2) \varepsilon_{\mu}(\lambda),$$

where $F_{i}^{\alpha \beta}$ is the field strength

$$F_{i}^{\alpha \beta}(\lambda_1) = k_{i}^{\alpha} \varepsilon_{1}^{\beta}(\lambda_1) - k_{i}^{\beta} \varepsilon_{1}^{\alpha}(\lambda_1). \quad (86)$$

In these equations the index 1 refers to the photon and 2 refers to the massive $J/\psi$. The $\lambda$’s refer to the helicity of the associated particles. The parameter $\bar{c}_4$ is not specified here, but its complex square will contain the width for the decay of the spin-2 particle into these two spin-1 particles. It will be related to $c_4$.

The density matrix $\mathcal{D}$ for the decay of the resonance is then constructed by

$$\mathcal{D}_{\lambda \lambda'} = \sum_{\lambda_1, \lambda_2} \mathcal{M}(\lambda, \lambda_1, \lambda_2) \mathcal{M}^{*}(\lambda', \lambda_1, \lambda_2). \quad (87)$$

In the evaluation of this density matrix we make use of the invariant spin summations for spin-1 particles. For the massless photon this relation reads

$$\sum_{\lambda_1} \varepsilon_{1, \alpha}(\lambda_1) \varepsilon_{1, \beta}^{*}(\lambda_1) = -g_{\alpha \beta}, \quad (88)$$

whereas for the massive $J/\psi$ it reads

$$\sum_{\lambda_2} \varepsilon_{2, \alpha}(\lambda_2) \varepsilon_{2, \beta}^{*}(\lambda_2) = -g_{\alpha \beta} + \frac{k_{2, \alpha}k_{2, \beta}}{M^2}, \quad (89)$$

where $M$ is its mass. In fact, due to the gauge invariance of the field strength in equation \((84)\) only the $g_{\alpha \beta}$ term in equation \((89)\) contributes. This leads to the following expression for the density matrix for the decay of the resonance

$$\mathcal{D}_{\lambda \lambda'} = |\bar{c}_4|^{2} \left[ (k_1 \cdot k_2) \varepsilon_{1}^{\alpha}(\lambda) \varepsilon_{1, \beta}^{*}(\lambda') + 2(k_1 \cdot \varepsilon(\lambda) \cdot k_1)(k_1 \cdot \varepsilon^{*}(\lambda') \cdot k_1) \right.$$  
$$+ (k_2^{2} + 4(k_1 \cdot k_2))(k_1 \cdot \varepsilon(\lambda) \cdot \varepsilon^{*}(\lambda') \cdot k_1) \right]. \quad (90)$$
In this equation a polarization tensor in between two dots indicates that one Lorentz index has to be contracted with the four-vector on the left side of the tensor whereas the other Lorentz index has to be contracted with the four-vector on the right side of the tensor, i.e.

\[ p \cdot A \cdot q \equiv A^{\mu \nu} p_\mu q_\nu. \] (91)

In the derivation we have used that the final state photon is massless and we have used the properties of the polarization tensor to replace \( k_2 \) in contractions with it by \(-k_1\).

The density matrix for the decay of the \( \chi_{c2} \) is constructed in the rest frame of the resonance with the \( z \) axis as polarization axis and some \( x \) and \( y \) axes. The density matrix will depend on the polar and azimuthal angles of the photon in this reference system. The specific choice of the axes in the \( \chi_{c2} \) rest system is at this point arbitrary. In the applications they should be the same as for the \( \chi_{c2} \) production density matrix. The polarization tensor looks very simple and one can use equations (34) and (35) to verify that the following useful relations hold

\[ \varepsilon^\mu(\lambda)\varepsilon^\nu_\mu(\lambda') = -\delta_{\lambda\lambda'}, \quad \varepsilon^{\mu \nu}(\lambda)\varepsilon^{\ast \mu \nu}(\lambda') \equiv \delta_{\lambda\lambda'}. \] (92)

In this frame the four-momentum of particle 1 can be parametrized by

\[ k_1^\mu = |\vec{k}|(1, \sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) = |\vec{k}|(1, s \cos \phi, s \sin \phi, c). \] (93)

It can be shown that with these definitions the density matrix for the decay of the spin-2 particle can be written in a compact form as the sum of three matrices

\[
D_{\lambda\lambda'} = |\tilde{c}_4|^2 \left[ (k_1 \cdot k_2)^2 \delta_{\lambda\lambda'} + 2|\vec{k}|^4 v_\lambda v_\lambda^* + (k_2^2 + 4(k_1 \cdot k_2))|\vec{k}|^2 R_{\lambda\lambda'} \right]
= |\tilde{c}_4|^2 \left( \frac{M_R^2 - M_\chi^2}{2} \right)^2 \left[ \delta_{\lambda\lambda'} + \frac{1}{2} \alpha^2 v_\lambda v_\lambda^* + (1 + \alpha) R_{\lambda\lambda'} \right].
\] (94)

In this expression \( \delta \) is the Kronecker symbol. The vector \( v \) is given by

\[ v = (v_2, v_1, v_0, v_1, v_2) = \left( \frac{s^2}{2} e^{2i\phi}, -sc e^{i\phi}, \frac{1}{\sqrt{6}}(3e^2 - 1), sc e^{-i\phi}, \frac{s^2}{2} e^{-2i\phi} \right). \] (95)

The tensor \( R \) is given by (the indices run from 2 to \(-2\))

\[
R_{\lambda\lambda'} = \begin{pmatrix}
-\frac{s^2}{2} & \frac{cs}{2} e^{i\phi} & \frac{s^2}{2\sqrt{6}} e^{2i\phi} & 0 & 0 \\
\frac{cs}{2} e^{-i\phi} & -\frac{1+c^2}{4} & \frac{sc}{2\sqrt{6}} e^{i\phi} & \frac{s^2}{4} e^{2i\phi} & 0 \\
\frac{s^2}{2\sqrt{6}} e^{-2i\phi} & \frac{sc}{2\sqrt{6}} e^{-i\phi} & -\frac{3c^2+1}{6} & -\frac{sc}{2\sqrt{6}} e^{i\phi} & \frac{s^2}{2\sqrt{6}} e^{2i\phi} \\
0 & \frac{s^2}{2\sqrt{6}} e^{-2i\phi} & -\frac{sc}{2\sqrt{6}} e^{-i\phi} & -\frac{1+c^2}{4} & -\frac{cs}{2} e^{i\phi} \\
0 & 0 & \frac{s^2}{2\sqrt{6}} e^{-2i\phi} & \frac{cs}{2} e^{-i\phi} & -\frac{s^2}{2} \\
\end{pmatrix}.
\] (96)

In the second line of equation (94) the expansion parameter \( \alpha \) has been introduced.

\[ \alpha = 1 - \frac{M^2}{M_R^2}. \] (97)
whereas for \( \alpha = 6 \) some numerical results for the distributions will be presented. In section 6 and for equation (99)

\[ \chi_{11} \] of the \( J/\psi \) polarization vector of the \( J/\psi \). These distributions can be compared to the experimentally observed decay distribution

\( J/\psi \) produced in \( p\bar{p} \) collisions,

\[ \chi_{12}(k) \rightarrow \gamma(k_1)J/\psi(k_2) \rightarrow \gamma(k_1)l^+(p_1)l^-(p_2). \]  

(104)

The matrix element for this reaction can be obtained from equation (85) by replacing the polarization vector of the \( J/\psi \) by the lepton current

\[ J_\mu^l = \bar{u}(p_2)\gamma^\mu v^l(p_1), \]  

(105)

where \( r \) and \( s \) denote the spins of the final state leptons. In the following the lepton mass \( m_l \) will be neglected, which is a reasonable approximation for this decay.

In the construction of the density matrix for this decay one has to use that for the lepton current one finds

\[ \sum_{r,s} J_{rs}^\mu J_{rs}^{\nu} = 4[p_1^\mu p_2^\nu + p_1^\nu p_2^\mu - \frac{1}{2} k_2^2 g^{\mu\nu}] = 2[k_2^\mu k_2^\nu - l^\mu l^\nu - k_2^2 g^{\mu\nu}]. \]  

(106)
In this equation the vector \( l = p_1 - p_2 \) has been introduced. This has been done as the term proportional to \( k_2^\mu k_2^\nu \) will not contribute to the density matrix. For \( l \) we use the following parametrization

\[
l^\mu = (l_0, l_x, l_y, l_z)
\]

in the above chosen reference frame, where (113) holds. The density matrix now reads

\[
D_{\lambda\lambda'} = |\tilde{c}|^2 \left( \varepsilon^\alpha_{\beta}(\lambda)\varepsilon^*_{\alpha\beta}(\lambda')(k_1 \cdot k_2)^2k_2^2\right.
\]

\[
+ (2k_2^\lambda + l^2)(k_1 \cdot \varepsilon(\lambda) \cdot k_1)(k_1 \cdot \varepsilon^*(\lambda') \cdot k_1)
\]

\[
-(k_1 \cdot k_2)[(k_1 \cdot \varepsilon(\lambda) \cdot k_1)(l \cdot \varepsilon^*(\lambda') \cdot l) + (l \cdot \varepsilon(\lambda) \cdot l)(k_1 \cdot \varepsilon^*(\lambda') \cdot k_1)]
\]

\[
+(2(k_1 \cdot k_2) + k_2^2)(k_1 \cdot \varepsilon(\lambda) \cdot l)(k_1 \cdot \varepsilon^*(\lambda') \cdot l)
\]

\[
+[k_2^2 + 4(k_1 \cdot k_2)k_2^2 + (k_1 \cdot l)^2](k_1 \cdot \varepsilon(\lambda) \cdot \varepsilon^*(\lambda') \cdot k_1)
\]

\[
+(k_1 \cdot k_2)(k_1 \cdot l)[(k_1 \cdot \varepsilon(\lambda) \cdot \varepsilon^*(\lambda') \cdot l) + (l \cdot \varepsilon(\lambda) \cdot \varepsilon^*(\lambda') \cdot k_1)]
\]

\[
+(k_1 \cdot k_2)^2(l \cdot \varepsilon(\lambda) \cdot \varepsilon^*(\lambda') \cdot l)\right).
\]

In the derivation we have used that \((k_2 \cdot l) = 0\) This density matrix can also be written in a more compact form

\[
D_{\lambda\lambda'} = |\tilde{c}|^2 ((k_1 \cdot k_2)^2k_2^2\delta_{\lambda\lambda'}
\]

\[
+ [2k_2^2 + l^2]|\vec{k}|^4 \varepsilon_{\lambda\lambda'} - (k_1 \cdot k_2)|\vec{k}|^2[v_\lambda x_{\lambda'}^* + x_\lambda v_{\lambda'}^*]
\]

\[
+(2(k_1 \cdot k_2) + k_2^2)|\vec{k}|^2w_\lambda w_{\lambda'}^*
\]

\[
+[k_2^4 + 4(k_1 \cdot k_2)k_2^2 + (k_1 \cdot l)^2]|\vec{k}|^2 R_{\lambda\lambda'}
\]

\[
+(k_1 \cdot k_2)(k_1 \cdot l)|\vec{k}|[S_{\lambda\lambda'} + S_{\lambda\lambda'}^*] + (k_1 \cdot k_2)^2 T_{\lambda\lambda'}
\right).
\]

In this density matrix the coefficients can again be written as functions of \( \alpha \)

\[
D_{\lambda\lambda'} = |\tilde{c}|^2 \left( \frac{M^2 - M^2_{R}}{2} \right)^2 M^2 \left( \delta_{\lambda\lambda'} + \frac{\alpha^2}{4} v_{\lambda} v_{\lambda'}^* - \frac{\alpha}{2M^2} (v_{\lambda} x_{\lambda'}^* + x_{\lambda} v_{\lambda'}^*)
\]

\[
+ \frac{1}{M^2} w_\lambda w_{\lambda'}^* + (1 + \alpha + \frac{\alpha^2}{4(1-\alpha)} \cos^2 \theta^*) R_{\lambda\lambda'}
\]

\[
- \frac{\alpha}{2M\sqrt{1-\alpha}} \cos \theta^*(S_{\lambda\lambda'} + S_{\lambda\lambda'}^*) + \frac{1}{M^2} T_{\lambda\lambda'}
\right),
\]

where a convenient expression

\[
k_1 \cdot l = \frac{M^2_{R} - M^2}{2} \cos \theta^*
\]

has been used. The angle \( \theta^* \) is the polar angle of the outgoing \( l^+ \) in the rest system of the \( J/\psi \) where the \( z \) axis is given by the direction of the boost from the RRF to the rest.
In addition, we have also introduced frames of the $J/\psi$, i.e. opposite to the photon direction. In the expressions some additional vectors and tensors have been introduced.

\[ w = \left(\frac{sl_+}{2} e^{i\phi}, -\frac{1}{2}l_z s e^{i\phi} + l_+ c, -\frac{1}{\sqrt{6}} [s(l_x \cos \phi + l_y \sin \phi) - 2cl_z], \frac{1}{2} [l_z s e^{-i\phi} + l_- c], \frac{sl_-}{2} e^{-i\phi}\right), \]  
\[ x = \left(\frac{1}{2}l_+^2, -l_+l_z, -\frac{1}{\sqrt{6}}[l_x^2 + l_y^2 - 2l_z^2], l_- l_z, \frac{1}{2} l_z^2\right), \]  
\[ S_{\lambda\nu} = \begin{pmatrix} -\frac{sl_-}{2} e^{i\phi} & \frac{sl_+}{2} e^{i\phi} & \frac{sl_-}{2} e^{i\phi} & 0 & 0 \\ -\frac{1}{2}l_z s e^{-i\phi} & \frac{l_+}{2\sqrt{6}} & \frac{l_+}{2\sqrt{6}} & \frac{sl_-}{4} e^{i\phi} & 0 \\ \frac{sl_+}{2\sqrt{6}} e^{-i\phi} & \frac{1}{6}[sl_{xy} + 4cl_z] & -\frac{l_+}{2\sqrt{6}} & \frac{sl_+}{2\sqrt{6}} e^{i\phi} & 0 \\ 0 & \frac{sl_-}{4} e^{-i\phi} & -\frac{l_+}{2\sqrt{6}} & -\frac{1}{2}l_z s e^{-i\phi} & -\frac{sl_-}{2} e^{-i\phi} \\ 0 & 0 & -\frac{sl_+}{2\sqrt{6}} e^{-i\phi} & -\frac{sl_-}{2} e^{-i\phi} & -\frac{sl_-}{2} e^{-i\phi}\end{pmatrix}, \]  
\[ T_{\lambda\nu} = \begin{pmatrix} -\frac{1}{2}(l_x^2 + l_y^2) & \frac{l_-l_z}{2} & \frac{l_z^2}{2\sqrt{6}} & 0 & 0 \\ \frac{l_+l_z}{2} & -\frac{1}{4}[l_x^2 + l_y^2] & \frac{l_z^2}{4} & 0 & \frac{l_z^2}{2\sqrt{6}} \\ \frac{l_z^2}{2\sqrt{6}} & \frac{l_+l_z}{2\sqrt{6}} & -\frac{1}{6}[l_x^2 + 3l_z^2] & \frac{l_-l_z}{2\sqrt{6}} & \frac{l_z^2}{2\sqrt{6}} \\ 0 & \frac{l_z^2}{4} & -\frac{l_-l_z}{2\sqrt{6}} & -\frac{1}{4}[l_x^2 + l_y^2] & \frac{l_-l_z}{2\sqrt{6}} \\ 0 & 0 & \frac{l_z^2}{2\sqrt{6}} & -\frac{l_-l_z}{2\sqrt{6}} & -\frac{1}{2}(l_x^2 + l_y^2)\end{pmatrix}. \]  

In the expressions the variables $l_+$ and $l_-$ have been introduced.

\[ l_+ = l_x + il_y, \]  
\[ l_- = l_x - il_y. \]  

In addition, we have also introduced

\[ l_{xy} = l_x \cos \phi + l_y \sin \phi, \]  
\[ l_{cz} = 2cl_- - sl_z e^{-i\phi}, \]  
\[ l_{s+} = 2sl_z e^{i\phi} - cl_+, \]  
\[ l_{sz} = \frac{sl_-}{2} e^{i\phi} + cl_z. \]  

The decay density matrix is now completely specified once the lepton momenta in the $\chi_{c2}$ rest system are inserted in $l_\mu$. It is sometimes convenient to parametrize $l_\mu$ in terms of decay angles $\theta^*$ and $\phi^*$,

\[ s^* = \sin \theta^*, \quad c^* = \cos \theta^*, \quad c_\phi^* = \cos \phi^*, \quad s_\phi^* = \sin \phi^*, \]  

of the $l^+$ in the rest system of the $J/\psi$. 

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The components of the four-momentum $l$ in the RRF then become ($c_\phi = \cos \phi, s_\phi = \sin \phi$, the angles of equation (93))

\[
\begin{align*}
    l_0 &= \frac{M_R^2 - M^2}{2M_R} c^*, \\
    l_x &= M(s^*(c^*c_\phi + s^*s_\phi) - c^*s_\phi), \\
    l_y &= M(s^*(c^*s_\phi + s^*c_\phi) - c^*s_\phi), \\
    l_z &= -M(s^*c_\phi s + c^*c).
\end{align*}
\]  

(122)

Since the $\chi_{c2}$ will be predominantly produced in helicity $\pm 2$ states, we have a closer look at those density matrix elements. In the limit of vanishing $\alpha$ the sum of density matrix elements $\mathcal{D}_{-2-2} + \mathcal{D}_{22}$ reads

\[
\mathcal{D}_{-2-2} + \mathcal{D}_{22} = |\tilde{c}|^2 \left( \frac{M_R^2 - M^2}{2} \right)^2 M^2 \left( 2 + \frac{1}{M^2} (w_{-2}w_{-2}^* + w_2w_2^*) \right)
\]

\[
+ \frac{1}{M^2} (T_{-2-2} + T_{22}) + R_{-2-2} + R_{22}.
\]

(123)

Inserting the expressions for the vectors and the tensors gives

\[
\mathcal{D}_{-2-2} + \mathcal{D}_{22} = |\tilde{c}|^2 \left( \frac{M_R^2 - M^2}{2} \right)^2 M^2 (1 + c^2) \left[ 1 - \frac{l_+ l_-}{2M^2} \right].
\]

(124)

Integrating over the angles associated to the outgoing leptons yields for the sum $\mathcal{D}_{-2-2} + \mathcal{D}_{22}$ a result that is proportional to $1 + c^2$, which is in agreement with the previously found results for the decay of a $\chi_{c2}$ into $J/\psi$ and $\gamma$, equation (102) in the limit $\alpha = 0$. Using the expressions for $l_+$ and $l_-$ results in

\[
\mathcal{D}_{-2-2} + \mathcal{D}_{22} = |\tilde{c}|^2 \left( \frac{M_R^2 - M^2}{2} \right)^2 M^2 (1 + c^2) \left( 1 - \frac{1}{2} \left[ (s^*c_\phi c - c^* s)^2 + (s^*s_\phi^*)^2 \right] \right).
\]

(125)

For helicity-2 predictions these results can be compared to the weight function for the angular distribution, $f_{|\lambda|=2}$, presented in [10] and [11]

\[
f_{|\lambda|=2}(\theta, \theta^*, \phi^*) = \frac{1}{8} A_2^2 (1 + c^2^2)(1 + 6c^2 + c^4) + A_1^4 (1 - c^2^2)(1 - c^4)
\]

\[
+ \frac{3}{4} A_0^4 (1 + c^2^2)(1 - 2c^2 + c^4) + \frac{\sqrt{7}}{4} A_2 A_1 c_\phi^* 2s^*c s(c^3 + 3c)
\]

\[
+ \frac{\sqrt{6}}{4} A_2 A_0 (c_\phi^* - s_\phi^*) s^2 (1 - c^4) - \frac{\sqrt{3}}{2} A_1 A_0 c_\phi^* 2s^*c^* s^3 c.
\]

(126)

A pure electric dipole transition would result in amplitudes $A_0 = 0.316, A_1 = 0.548$ and $A_2 = 0.775$. Experimentally [11] the values $A_0 = 0.21, A_1 = 0.49$ and $A_2 = 0.85$ have been found. It turns out that the compact expression (124) amounts to expression (126) with the pure electric dipole values for $A_1$.

After the above detailed discussion of a $\chi_{c2}$ decay, we now briefly mention a similar $\chi_{c1}$ decay, since it may experimentally contaminate the $\chi_{c2}$ events. Therefore we need a model for the decay

\[
\chi_{c1}(p_R) \rightarrow \gamma(k_1) J/\psi(k_2),
\]

(127)
from which one can derive the density matrix. Again, we use the $\gamma^* \gamma^*$ amplitude where one $\gamma^*$ is replaced by the $J/\psi$. We thus have from (24)

$$\mathcal{M}(\lambda, \lambda_1, \lambda_2) = \tilde{c}_3 M^2 \varepsilon[\varepsilon(\lambda), \varepsilon^*(\lambda_1), \varepsilon^*(\lambda_2), k_1].$$  \hspace{1cm} (128)$$

The density matrix now becomes

$$D_{\lambda \lambda'} = |\tilde{c}_3|^2 M^4 \left\{ \left[ 1 - 2 \frac{(k_1 \cdot k_2)}{M^2} \right] (k_1 \cdot \varepsilon(\lambda))(k_1 \cdot \varepsilon^*(\lambda')) - \frac{(k_1 \cdot k_2)^2}{M^2} (\varepsilon(\lambda) \cdot \varepsilon^*(\lambda')) \right\}.$$

\hspace{1cm} (129)

When $k_1$ is again parametrized as (93), one obtains

$$D_{\lambda \lambda'} = |\tilde{c}_3|^2 M^4 \left\{ \left[ 1 - 2 \frac{(k_1 \cdot k_2)}{M^2} \right] |\vec{k}|^2 u_{\lambda} u_{\lambda'}^* + \frac{(k_1 \cdot k_2)^2}{M^2} \delta_{\lambda \lambda'} \right\},$$

\hspace{1cm} (130)

where

$$u = (u_1, u_0, u_{-1}) = \left( \frac{s}{\sqrt{2}} e^{i \phi}, -c, \frac{s}{\sqrt{2}} e^{-i \phi} \right).$$

\hspace{1cm} (131)

Since in the production the helicity $\pm 1$ states are favoured, the angular decay distribution will now be approximately of the form

$$D_{11} + D_{-1-1} \sim 1 - \frac{2M^2 - M_R^2 c^2}{2M^2 + M_R^2 c^2} \sim 1 - 0.216 \cdot c^2,$$

\hspace{1cm} (132)

which has a behaviour opposite to the decay distribution for the $\chi_{c2}$, e.g. (122). When forward or backward parts of the angular distribution $\gamma J/\psi$ are experimentally not accessible this will affect the $\chi_{c2}$ decay more than the $\chi_{c1}$ decay. Although the GaGaRes predicted two-photon mediated production of $\chi_{c1} \to \gamma J/\psi$ is only 15% of that of $\chi_{c2}$ [4], the $\chi_{c1}$ contribution to the actually seen $\gamma J/\psi$ events may be higher than this 15% because of the opposite decay distribution. In the experimental determination of the $\gamma^* \gamma^*$ production through its $\chi_{c2} \to \gamma J/\psi$ decay it would be worthwhile to estimate the $\chi_{c1}$ contamination of the events.

Such a contamination would somewhat reduce the really present $\gamma^* \gamma^* \to \chi_{c2}$ production and therefore the derived $\chi_{c2} \to \gamma \gamma$ width. This remark is relevant since at present the measured two-photon width is a factor 3 to 4 higher than the determination from $p\bar{p}$ collider experiments [4]. More recently it has been argued that this factor is about 2 to 3 [12].

It is clear from these remarks that a precise measurement of the $\gamma J/\psi$ angular distribution could clarify the above situation. Tools to calculate various density matrices will then be indispensable.

One may wonder whether the above amplitudes predict the ratios of the widths $\Gamma_1$ and $\Gamma_2$ for the decay processes (127) and (84) correctly. This is considered in section 6.4.

6 Numerical Results

In this section numerical results for several quantities are given. They have been obtained by using the event generator GaGaRes [4], while extended GALUGA results served as a check.
6.1 Azimuthal distributions

Before focussing on azimuthal distributions it is useful to show the shape of the cross section \( d\sigma/dQ \) with its marked decrease with \( Q \). This is displayed in figure 1, where every curve has an arbitrary normalization.

Next, results for the \( \Delta \phi = \phi - \phi_c \) distributions for the \( 1S_0 \) and \( 3P_J \) bottomonium states are given for \( \sqrt{s} = 190 \text{ GeV} \). From figures 2a-2d it is seen that there are marked differences between distributions for different resonances. The characteristics can be qualitatively understood when the dominant features of the cross sections are considered. Similar qualitative arguments, but without numerical results have been given in the literature [13] in connection with Pomeron production of resonances. In the first place it is clear from figure 1 that the largest contributions to the cross sections come from the region \( Q_i^2 \ll M^2 \). In those regions \( \Delta \phi \approx \phi \) (cf [2]) such that a qualitative understanding of the \( \phi \) distribution in the BGMS cross section formula (72) is sufficient to explain the features of figures 2a-2d. The behaviour of the BGMS formula for the various resonances now follows from the form factors \( f_{AB} \) and \( g_{AB} \) in table 1 on page 11 and the small \( Q_i^2 \) approximation. One then obtains \( \tilde{\phi} \) distributions, where different photon density matrices play a role. Using the approximate numerical relations valid at \( \sqrt{s} = 190 \text{ GeV} \) and in the small \( Q_i^2 \) region

\[
\rho^{++} \approx \rho^{+-}, \quad \rho^{+0} \approx 1.5\rho^{++}, \quad \rho^{00} \approx 2.3\rho^{++},
\]

one arrives at the following shapes for the \( \tilde{\phi} \) distributions:

\( \eta_b \):

\[
\frac{d\sigma}{d\phi} \sim \kappa \frac{8X}{\nu} \left( \rho_1^{++} + \rho_2^{++} - \rho_1^{+-} - \rho_2^{+-} \cos(2\tilde{\phi}) \right)
\approx \kappa \frac{16X}{\nu^2} \rho_1^{++} \rho_2^{++} \sin^2 \tilde{\phi},
\]

\( \chi_{b0} \):

\[
\frac{d\sigma}{d\phi} \sim 4\kappa \left( \frac{X + \nu M^2}{3\nu^2} \right) \left( \rho_1^{++} + \rho_2^{++} - \rho_1^{+-} - \rho_2^{+-} \cos(2\tilde{\phi}) \right)
\sim 8\kappa \left( \frac{X + \nu M^2}{3\nu^2} \right) \rho_1^{++} \rho_2^{++} \cos^2 \tilde{\phi},
\]

\( \chi_{b1} \):

\[
\frac{d\sigma}{d\phi} \sim 4\kappa \left( \frac{Q_i^2 - Q_{i+1}^2}{2\nu} \right)^2 \left( \rho_1^{++} + \rho_2^{++} - \rho_1^{+-} - \rho_2^{+-} \cos(2\tilde{\phi}) \right)
+ 4\kappa \left( \frac{M^2}{2\nu} \right)^2 \left( \rho_1^{++} \rho_2^{++} Q_1^2 + \rho_1^{00} \rho_2^{00} Q_2^2 - 2\rho_1^{+0} \rho_2^{+0} Q_1 Q_2 \cos \phi \right)
\sim \frac{2\kappa}{\nu^2} \rho_1^{++} \rho_2^{++} \left( M^2(Q_1^2 + Q_2^2) - 2M^2Q_1Q_2 \cos \phi + (Q_1^2 - Q_2^2)^2 \sin^2 \phi \right),
\]

\( \chi_{b2} \):

\[
\frac{d\sigma}{d\phi} \sim \kappa \left( \frac{M^2}{2\nu} \right)^2.
\]

It is clear that the \( \sin^2 \phi \) and \( \cos^2 \phi \) and constant distributions show up in figures 2a,b and d. A more complex structure arises in figure 2c, but also here the \( \cos \phi \) and \( \sin^2 \phi \) distributions can be recognized on a constant background.

The \( 1D_2 \) \( b\bar{b} \) state has not been included in the plot as it would lie exactly on top of the plot for the \( 1S_0 \) \( b\bar{b} \) state.

For completeness we note that for the \( c\bar{c} \) states a similar \( \Delta \phi \) behaviour has been found.
6.2  Density matrices for resonance production

For the $c\bar{c}$ resonances $\chi_{c2}$ and $\chi_{c1}$ we have studied how the different helicities of the resonance contribute to the cross section as a function of the $Q_i^2$ cuts. The results for the diagonal elements in the RRF are given in figure 3. In these plots both $Q_1^2$ and $Q_2^2$ have to be greater than the value given on the horizontal axis. From the right plot in figure 3 one can see that in the absence of cuts the dominant contribution comes from the helicity-2 components. This is in agreement with [14] where a similar dominance was found for the helicity-2 component for the $f_1'(1525)$ resonance. For high $Q_i^2$ cuts the helicity-0 component starts to dominate. In the BGMS-formalism [2] this contribution comes from the $\sigma_{SS}$ component that is proportional to $Q_1^2 Q_2^2$. This result also agrees with theoretical predictions found by Close [15], in which he states that the helicity-0 contribution is of the order $O(tt'M^2)$. From the left plot in figure 3 one can see that in the case of the $\chi_{c1}$ resonance in the absence of cuts the helicity-1 component is dominant. It would be interesting to verify this statement with experimental data. For higher $Q_i^2$ cuts again the helicity-0 component starts to dominate.

6.3  Density matrix for the decay $\chi_{c2} \rightarrow \gamma J/\psi$

In section 5 we have constructed the density matrix for the complete decay of a $\chi_{c2}$ resonance into a $J/\psi$ and a photon, where the $J/\psi$ subsequently decays into electrons or muons. We can now use equation (83) to combine the density matrix for the two-photon production of a $\chi_{c2}$ resonance with the density matrix for the complete decay of the resonance to get the full matrix element squared. The average production density matrix is given in table 3. Again it is clear that the helicity-2 diagonal elements are dominant. The combined production and decay matrix elements [94] have been used to generate the distribution of the angle between the outgoing photon and the boost direction in the RRF. The results are given as data points in figure 4.

In this figure also results are included which are obtained from the assumption of pure helicity-2 production. Under this assumption the $\mathcal{D}_{22} + \mathcal{D}_{-2-2}$ distributions [100], (101) or (103) can be chosen. Comparing the data points of the full calculation and distribution (100) shows that the production is indeed helicity-2 dominated. Comparing the distributions (101) and (103) with the model distribution (100) shows that the model is closer to experiment than the simple dipole distribution.

6.4  The ratio $\Gamma(\chi_{c1} \rightarrow \gamma J/\psi)/\Gamma(\chi_{c2} \rightarrow \gamma J/\psi)$

As the trace of the density matrices for the decay of a $\chi_{cJ}$ resonance into a photon and a $J/\psi$ yields the matrix element squared for this process, we can use equations (94) and (130) to calculate the ratio of the two decay widths. We assume $\tilde{c}_3/\tilde{c}_4 = c_3/c_4$, i.e. we assume that the replacement of a $\gamma^*$ by a $J/\psi$ introduces in both matrix elements (24) and (23) the same scale factor. The mass of the $\chi_{cJ}$ particle is denoted by $M_{RJ}$. Using the expressions for the traces of (94) and (130) yields for the ratio

$$R_{th} = \frac{\Gamma(\chi_{c1} \rightarrow \gamma J/\psi)}{\Gamma(\chi_{c2} \rightarrow \gamma J/\psi)} = 5 \frac{M^2(M_{R1}^2 + M^2)}{M_{R1}^4} \left(\frac{M_{R1}}{M_{R1}}\right)^4 \left(\frac{\alpha}{\beta}\right)^4 \frac{1}{10 - 5\alpha + \alpha^2}.$$  \hspace{1cm} (138)
The normalized density matrix \( \rho_{\lambda'\lambda} = |\rho_{\lambda'\lambda}| e^{i\phi_{\lambda'\lambda}} \) for \( \chi_{c2} \) production at \( \sqrt{s} = 190 \, \text{GeV} \) in the absence of cuts on the external particles. In each entry the absolute value \( |\rho_{\lambda'\lambda}| \) of the density matrix (upper value) and the phase \( \phi_{\lambda'\lambda} \) (lower value) are given. The open boxes are filled by Hermitian conjugation of the other elements.

|          | \( \lambda' = +2 \) | \( \lambda' = +1 \) | \( \lambda' = 0 \) | \( \lambda' = -1 \) | \( \lambda' = -2 \) |
|----------|----------------------|----------------------|----------------------|----------------------|----------------------|
| \( \lambda = +2 \) | 0.47 2.0 \cdot 10^{-4} | 0.012 6.27 5.56 | 0.029 4.72 | \( 0.012 6.27 5.56 \) | \( 0.029 4.72 \) |
| \( \lambda = +1 \) | 0.022 1.53 1.53 | 5.4 \cdot 10^{-4} 4.1 \cdot 10^{-3} 0.023 | 2.41 6.0 \cdot 10^{-4} | \( 0.022 1.53 1.53 \) | \( 2.41 6.0 \cdot 10^{-4} \) |
| \( \lambda = 0 \) | 0.012 4.67 6.27 | 5.4 \cdot 10^{-4} 0.012 | \( 0.012 4.67 6.27 \) |
| \( \lambda = -1 \) | 0.022 1.58 | 2.0 \cdot 10^{-4} 6.0 \cdot 10^{-4} | \( 0.022 1.58 \) |
| \( \lambda = -2 \) | 0.47 | \( 0.47 \) |

where \( \beta \) has been defined like \( \alpha \)

\[
\beta = 1 - \frac{M^2}{M_{R_i}^2} \approx 0.225. \tag{139}
\]

Inserting the numerical values gives

\[
R_{th} \approx 0.8969. \tag{140}
\]

This number should be compared to the experimentally measured ratio

\[
R_{exp} = \frac{\Gamma_{\text{tot}}(\chi_{c1}) Br(\chi_{c1} \to \gamma J/\psi)}{\Gamma_{\text{tot}}(\chi_{c2}) Br(\chi_{c2} \to \gamma J/\psi)} = 0.89 \pm 0.15, \tag{141}
\]

where in the calculation of the error the different contributions to this error are taken to be independent. Using the data selection of [12] this number becomes

\[
R_{exp} = 0.76 \pm 0.26. \tag{142}
\]

These experimental numbers are compatible with the above theoretical prediction.

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A Transforming the density matrices

In section 4 we have constructed the complete density matrix in the BGMS frame. However, from an experimental point of view the RRF is a more convenient reference frame. As the elements of the density matrix depend on the chosen polarization vectors/tensors, the density matrix is frame dependent. When one wants to rotate the quantization axis over the angles $\theta_p$ and $\phi_p$ the density matrix $\rho$ changes into the density matrix $\tilde{\rho}$ according to

$$\tilde{\rho}_{\lambda\lambda'} = A_{\lambda\mu} A^{\lambda'}_{\mu'} \rho_{\mu'\mu}. \quad (143)$$

Thus a density matrix $\rho$ calculated in the BGMS frame can be transformed to the RRF.

In the BGMS frame the boost direction which was needed to get to the $\chi_{c2}$ rest system is characterized by $\theta_p$ and $\phi_p$.

For spin-1 resonances this transformation matrix $A$ can be shown to be

$$A_{\lambda\mu} = \begin{pmatrix}
1 + \frac{c}{2} e^{-i\phi} & \frac{s}{\sqrt{2}} & \frac{1 - c}{2} e^{i\phi} \\
-\frac{s}{\sqrt{2}} e^{-i\phi} & c & \frac{s}{\sqrt{2}} e^{i\phi} \\
\frac{1 - c}{2} e^{-i\phi} & -\frac{s}{\sqrt{2}} & \frac{1 + c}{2} e^{i\phi}
\end{pmatrix} \quad (144)$$

The order of the indices $\lambda, \mu$ is taken to be $(+, 0, -)$. For spin-2 resonances this transformation matrix reads (with order $+2, +1, 0, -1, -2$)

$$A_{\lambda\mu} = \begin{pmatrix}
(1 + \frac{c}{2})^2 e^{-2i\phi} & \frac{(1 + c)s}{2} e^{-i\phi} & \frac{s^2}{2} \sqrt{\frac{3}{2}} \frac{(1 - c)s}{2} e^{i\phi} & \frac{(1 - c)}{2} e^{2i\phi} \\
\frac{(1 + c)s}{2} e^{-2i\phi} & \frac{(1 + c)(2c - 1)}{2} e^{-i\phi} & \frac{c s^2 - 1}{2} \sqrt{\frac{3}{2}} \frac{(1 - c)(2c + 1)}{2} e^{i\phi} & \frac{(1 - c)s}{2} e^{2i\phi} \\
\frac{1}{2} \sqrt{\frac{3}{2}} \frac{s^2}{2} e^{-2i\phi} & -\frac{1}{2} \sqrt{\frac{3}{2}} \frac{s e^{-i\phi}}{2} & \frac{3c^2 - 1}{2} \sqrt{\frac{3}{2}} \frac{s e^{i\phi}}{2} & \frac{1}{2} \sqrt{\frac{3}{2}} \frac{s^2}{2} e^{2i\phi} \\
\frac{1 - c}{2} e^{-2i\phi} & \frac{(1 - c)s}{2} e^{-i\phi} & \frac{s^2}{2} \sqrt{\frac{3}{2}} \frac{(1 + c)(2c - 1)}{2} e^{i\phi} & \frac{(1 + c)s}{2} e^{2i\phi}
\end{pmatrix} \quad (145)$$

In the expressions for the transformation matrices we have introduced the shorthand notation

$$c = \cos(\theta_p), \quad s = \sin(\theta_p), \quad \phi = \phi_p. \quad (146)$$

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Figure 1: The $Q$ spectrum of the photon radiated by the incoming positron for the generated $b \bar{b}$ states, the $\eta_b$ (continuous line), the $\chi_{b0}$ (dashed line), the $\chi_{b1}$ (dotted line) and the $\chi_{b2}$ (dash-dotted line). The events were generated at $\sqrt{s} = 190$ GeV without cuts on the external particles. The vertical scale is logarithmic.
Figure 2: The $\Delta \phi$ spectra for the $b \bar{b}$ states $\eta_b$ (a), $\chi_{b0}$ (b), $\chi_{b1}$ (c) and $\chi_{b2}$ (d). The events have been generated at $\sqrt{s} = 190$ GeV in the absence of additional cuts on the external particles. The vertical scale is linear.
Figure 3: Contributions of the different diagonal density matrix elements of the $\chi_{c1}$ resonance (left plot) and the $\chi_{c2}$ resonance (right plot) to the total cross section at $\sqrt{s} = 91.5$ GeV, for different $Q_i^2$ cuts. (i.e. $Q_i^2 > 0, 1, 5, \ldots, 25$ GeV$^2$.)

Figure 4: The distribution of the polar angle of the outgoing photon in the RRF. The data points show the result from the calculation using the combined matrix element both for the production and decay. The dashed line shows the distribution for the complete density matrix for the decay of the resonance where the production is purely helicity-2. The dotted line represents the distribution according to the pure dipole transition and the dashed-dotted line shows the distribution that has been experimentally observed. The vertical scale is linear.