Monochromatic $k$-edge-connection colorings of graphs$^1$

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Abstract

A path in an edge-colored graph $G$ is called monochromatic if any two edges on the path have the same color. For $k \geq 2$, an edge-colored graph $G$ is said to be monochromatic $k$-edge-connected if every two distinct vertices of $G$ are connected by at least $k$ edge-disjoint monochromatic paths, and $G$ is said to be uniformly monochromatic $k$-edge-connected if every two distinct vertices are connected by at least $k$ edge-disjoint monochromatic paths such that all edges of these $k$ paths colored with the same color. We use $mc_k(G)$ and $umc_k(G)$ to denote the maximum number of colors that ensures $G$ to be monochromatic $k$-edge-connected and, respectively, $G$ to be uniformly monochromatic $k$-edge-connected. In this paper, we first conjecture that for any $k$-edge-connected graph $G$, $mc_k(G) = e(G) - e(H) + \lfloor \frac{k}{2} \rfloor$, where $H$ is a minimum $k$-edge-connected spanning subgraph of $G$. We verify the conjecture for $k = 2$. We also prove the conjecture for $G = K_{k+1}$ when $k \geq 4$ is even, and for $G = K_{k,n}$ when $k \geq 4$ is even, or when $k = 3$ and $n \geq k$. When $G$ is a minimal $k$-edge-connected graph, we give an upper bound of $mc_k(G)$, i.e., $mc_k(G) \leq k - 1$, and $mc_k(G) \leq \lfloor \frac{k}{2} \rfloor$ when $G = K_{k,n}$. For the uniformly monochromatic $k$-edge-connectivity, we prove that for all $k$, $umc_k(G) = e(G) - e(H) + 1$, where $H$ is a minimum $k$-edge-connected spanning subgraph of $G$.

Keywords: edge-coloring, monochromatic path, edge-connectivity, monochromatic $k$-edge connection number.

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1 Introduction

All graphs in this paper are simple and undirected. For a graph $G$, we use $V(G), E(G)$ to denote the vertex set and edge set of $G$, respectively, and $e(G)$ the number of edges of $G$. For all other terminology and notation not defined here we follow Bondy and Murty [1].

For a natural number $r$, we use $[r]$ to denote the set $\{1, 2, \cdots, r\}$ of integers. Let $\Gamma : E(G) \rightarrow [r]$ be an edge-coloring of $G$ that allows a same color to be assigned to adjacent edges. For two vertices $u$ and $v$ of $G$, a monochromatic $uv$-path is a $uv$-path of $G$ whose edges are colored with a same color, and $G$ is monochromatic connected if any two distinct vertices of $G$ are connected by a monochromatic path. An edge-coloring $\Gamma$ of $G$ is a monochromatic connection coloring (MC-coloring) if it makes $G$ monochromatic connected. The monochromatic connection number of a connected graph $G$, denoted by $mc(G)$, is the maximum number of colors that are needed in order to make $G$ monochromatic connected. An extremal MC-coloring of $G$ is an MC-coloring that uses $mc(G)$ colors.

The notion monochromatic connection coloring was introduced by Caro and Yuster in [4]. Many results have been obtained; see [3, 6, 10, 14]. For more knowledge on the monochromatic connections of graphs we refer to a survey paper [12]. Gonzalez-Moreno, Guevara, and Montellano-Ballesteros in [5] generalized the above concept to digraphs. Now we introduce the concept of monochromatic $k$-edge-connectivity of graphs. An edge-colored graph $G$ is monochromatic $k$-edge-connected if every two distinct vertices are connected by at least $k$ edge-disjoint monochromatic paths (allow some of the paths to have different colors). An edge-coloring $\Gamma$ of $G$ is a monochromatic $k$-edge-connection coloring (MC$_k$-coloring) if it makes $G$ monochromatic $k$-edge-connected. The monochromatic $k$-edge-connection number, denoted by $mc_k(G)$, of a connected graph $G$ is the maximum number of colors that are needed in order to make $G$ monochromatic $k$-edge-connected. Since we can color all the edges of a $k$-edge-connected graph by distinct colors, $mc_k(G)$ is well-defined. An extremal MC$_k$-coloring of $G$ is an MC$_k$-coloring that uses $mc_k(G)$ colors.

In an edge-colored graph $G$, we say that a subgraph $H$ of $G$ is induced by color $i$ if $H$ is induced by all the edges with a same color $i$ of $G$. If a color $i$ only color one edge of $E(G)$, then we call the color $i$ is a trivial color, and the edge is a trivial edge; otherwise, we call the colors (edges) non-trivial. We call an extremal MC$_k$-coloring a good MC$_k$-coloring of $G$ if the coloring has the maximum number of trivial edges.

Suppose that $X$ is a proper vertex subset of $G$. We use $E(X)$ to denote the set of edges with both ends in $X$. For a graph $G$ and $X \subset V(G)$, to shrink $X$ is to delete all edges in
E(X) and then merge the vertices of X into a single vertex. A partition of a vertex set V is to divide V into some mutual disjoint nonempty sets. Suppose \( \mathcal{P} = \{ V_1, \cdots, V_s \} \) is a partition of \( V(G) \). Then \( G/\mathcal{P} \) is a graph obtained from \( G \) by shrinking every \( V_i \) into a single vertex.

An edge \( e \) of a \( k \)-edge-connected graph \( G \) is \emph{deletable} if \( G \setminus e \) is also a \( k \)-edge-connected graph. A \( k \)-edge-connected graph \( G \) is \emph{minimally \( k \)-edge-connected} if none of its edges is deletable. A \emph{minimal \( k \)-edge-connected spanning subgraph} of \( G \) is a \( k \)-edge-connected spanning graph of \( G \) that does not have any deletable edges. A \emph{minimum \( k \)-edge-connected spanning subgraph} of \( G \) is a minimal \( k \)-edge-connected spanning subgraph of \( G \) that has minimum number of edges. The next result was obtained by Mader.

**Theorem 1.1** (Mader [13]). Let \( G \) be a minimally \( k \)-edge-connected graph of order \( n \). Then

1. \( e(G) \leq k(n - 1) \).
2. every edge \( e \) of \( G \) is contained in a \( k \)-edge cut of \( G \).
3. \( G \) has a vertex of degree \( k \).

The following theorem was proved by Nash-Williams and Tutte independently.

**Theorem 1.2** ([15] [16]). A graph \( G \) has at least \( k \) edge-disjoint spanning trees if and only if \( e(G/\mathcal{P}) \geq k(|G/\mathcal{P}| - 1) \) for any vertex partition \( \mathcal{P} \) of \( V(G) \).

We denote \( \psi(G) = \min_{|\mathcal{P}| \geq 2} \frac{e(G/\mathcal{P})}{|\mathcal{P}| - 1} \), and \( \Psi(G) = \lfloor \psi(G) \rfloor \). Then the Nash-Williams-Tutte theorem can be restated as follows.

**Theorem 1.3.** A graph \( G \) has exactly \( k \) edge-disjoint spanning trees if and only if \( \Psi(G) = k \).

If \( \Gamma \) is an extremal \( MC_k \)-coloring of \( G \), then each color-induced subgraph is connected; otherwise we can recolor the edges of one of its components by a fresh color, and then the new coloring is also an \( MC_k \)-coloring of \( G \), but then the number of colors is increased by one, which contradicts that \( \Gamma \) is extremal.

For the monochromatic \( k \)-edge-connection number of graphs, we conjecture that the following statement is true.

**Conjecture 1.4.** For a \( k \)-edge-connected graph \( G \) with \( k \geq 2 \), \( mc_k(G) = e(G) - e(H) + \lfloor \frac{k}{2} \rfloor \), where \( H \) is a minimum \( k \)-edge-connected spanning subgraph of \( G \).
In Section 2, we will prove that the conjecture is true for $k = 2$, and that it is also true for some special graph classes. We also give a lower bound of $mc_k(G)$ for $2 \leq k \leq \Psi(G)$, and an upper bound of $mc_k(G)$ for minimally $k$-edge-connected graphs with $k \geq 2$.

The following lemma seems easy, but it is useful for some proofs in Section 2.

**Lemma 1.5.** Suppose that $G$ is a 2-edge-connected graph and $H$ is a 2-edge-connected subgraph of $G$. Let $S$ be subset of $E(G)$ whose ends are contained in $V(H)$ such that $S \cap E(H) = \emptyset$. Then $G\setminus S$ is also a 2-edge-connected graph.

**Proof.** We need to show that for any $u,v$ in $G\setminus S$ there are at least two edge-disjoint paths connecting them. From the condition, there are two edge-disjoint $uv$-path $P_1, P_2$ in $G$. Suppose $a_1$ is the first vertex of $V(P_1)$ from $u$ to $v$ contained in $V(H)$, and $a_2$ is the first vertex of $V(P_2)$ from $u$ to $v$ contained in $V(H)$ (if $u \in V(H)$, then $u = a_1 = a_2$); suppose $b_1$ is the last vertex from $u$ to $v$ contained in $V(H)$, and $b_2$ is the last vertex of $V(P_2)$ from $u$ to $v$ contained in $V(H)$ (if $v \in V(H)$, then $v = b_1 = b_2$). Let $L_i = uPa_i$ and $L_{i+2} = b_iPuv$, $i = 1, 2$. Because each of $L_i$ does not contain any edge of $S$ and $H$ is a 2-edge-connected graph, we have that $H \cup \bigcup_{i \in [4]} L_i$ is also a 2-edge-connected graph of $G\setminus S$. Therefore, there are two edge-disjoint $uv$-paths in $G\setminus S$. 

In Section 3, we introduce other version of monochromatic $k$-edge-connection of graphs, i.e., uniformly monochromatic $k$-edge-connection of graphs, and get some results. For details we will state them there.

## 2 Results on the monochromatic $k$-edge-connection number

**Theorem 2.1.** Conjecture 1.4 is true when $G$ and $k$ satisfy one of the following conditions:

1. $k = 2$, i.e., $G$ is a 2-edge-connected graph.
2. $G = K_{k+1}$ where $k \geq 4$ is even.
3. $G = K_{k,n}$ where $k \geq 4$ is even, and $k = 3$ and $n \geq k$.

We restate the first result of Theorem 2.1 as follows.

**Theorem 2.2.** Let $G$ be a 2-edge-connected graph. Then $mc_2(G) = e(G) - e(H) + 1$, where $H$ is a minimum 2-edge-connected spanning subgraph of $G$. 

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The following is the proof of Theorem 2.2. For convenience, we abbreviate the term “monochromatic path” as “path” in the proof.

Let $\Gamma$ be a good $MC_2$-coloring of $G$. Then we denote the set of non-trivial colors of $\Gamma$ by $[r]$, and denote $G_i$ as a subgraph induced by the color $i$; subject to above, let $p(\Gamma) = \sum_{i \in [r]} p(G_i)$ be maximum, where $p(G_i)$ is the number of non-cut edges of $G_i$. It is obvious that each of these edges is contained in some cycles of $G_i$.

Claim 2.3. Each $G_i$ is either a 2-edge-connected graph or a tree.

Proof. Suppose that $G_i$ is neither a 2-edge-connected graph nor a tree, i.e., $G_i$ contains both non-trivial blocks and cut edges. Therefore we can choose a cut edge $e = uv \in E(G_i)$ such that $v$ belongs to a maximal 2-edge-connected subgraph $B$ of $G_i$ (actually, $B$ is the union of some non-trivial blocks). Because $B$ is a 2-edge-connected subgraph of $G_i$, each of its vertices belongs to a cycle. Let $v$ be contained in a cycle $C$ of $B$ and $e' = vw$ be an edge of $C$. Because $e$ is a cut edge of $G_i$, there is just one $uw$-path in $G_i$ (the $uw$-path is $P$). Therefore, there exists another $uw$-path $P'$, which is colored differently from $i$.

If $P'$ is a path colored by $j$, then we can obtain a new coloring $\Gamma'$ of $G$ from $\Gamma$ by recoloring all edges of $G_i - e'$ with $j$. We first prove that $\Gamma'$ is an $MC_2$-coloring of $G$, i.e., we need to prove that for any two vertices $a, b$ of $V(G)$, there are at least two $ab$-paths under $\Gamma'$. If at least one vertex of $a, b$ does not belong to $V(G_i)$, then the two $ab$-paths are colored differently from $i$. Because we just change the color $i$, the two $ab$-paths are not affected; if both of $a, b$ belong to $V(G_i)$ and at least one of them does not belong to $V(B)$, then we can choose a right $ab$-path such that it does not contain $e'$ (under $\Gamma$), and so there are at least two $ab$-paths under $\Gamma'$; if both $a, b \in V(B)$, then the two $ab$-paths under $\Gamma$ (call them $L_1, L_2$) belong to $B$. If $e'$ is not an edge of any $L_1, L_2$, then the two $ab$-paths are not affected. Otherwise, let $e' \in E(L_1)$, and then $L = L_1 - e' + e + P'$ is a trial connecting $a, b$. Because $E(L) \cap E(L_2) = \emptyset$, there are two $ab$-paths under $\Gamma'$.

According to the above, $\Gamma'$ is an $MC_2$-coloring of $G$. If $j \in [r]$ is a non-trivial color, then the number of colors has not changed, but the number of trivial edges is increased by one, which contradicts that $\Gamma$ is good; otherwise, if $j$ is a trivial color, i.e., $uw$ is a trivial edge, then the new coloring $\Gamma'$ is a good $MC_2$-coloring (the number of colors and non-trivial edges have not changed), but compared to $p(\Gamma)$, $p(\Gamma')$ is increased by one, which contradicts that $p(\Gamma)$ is maximum. Therefore, we have proved that $G_i$ is either a 2-edge-connected graph or a tree. 

By Claim 2.3, each $G_i$ is either a 2-edge-connected graph or a tree. Suppose there are $h$ trees and $s = k - h$ 2-edge-connected graphs. W.l.o.g., suppose that $G_1, \cdots, G_s$ are $s$
2-edge-connected graphs and $G_{s+1} = T_1, \cdots, G_k = T_h$ are $h$ trees. $G_i$ colored by $i$ and $F_j$ colored by $s+j$. For convenience, we also call the color of $F_j$ $j$ when there is no confusion.

Claim 2.4. For each $G_i$ and $T_j$, let $e = uv \in E(G_i)$ and $e' = xy \in E(T_j)$. Then at most one of $u, v$ belongs to $V(T_j)$, and at most one of $x, y$ belongs to $V(G_i)$.

Proof. We prove it by contradiction, i.e., suppose that there exist $G_i$ and $T_j$, and there exist $e = uv \in E(G_i)$ and $e' = xy \in E(T_j)$, such that either $u, v \in V(T_j)$ or $x, y \in V(G_i)$.

Case 1: Suppose $u, v \in V(T_j)$. Then we recolor $E(G_i) - e$ by $j$ and keep the color of $e$. We now prove that the new coloring (call it $\Gamma'$) is an extremal $MC_2$-coloring of $G$.

We denote the segment of $uT_jv$ by $L$. For any pair of vertices $a, b$ of $V(G)$, if at least one vertex does not belong to $V(G_i)$, then the two $ab$-paths colored differently from $i$ under $\Gamma$. Because we just change the color $i$, the two $ab$-paths are not affected; if $a, b \in V(G_i)$, because $G_i + L - e$ is also 2-edge-connected, then there are two $ab$-paths (with the same color $j$) under $\Gamma'$. Therefore, $\Gamma'$ is an $MC_2$-coloring, and because the number of colors are not changed, $\Gamma'$ is still an extremal $MC_2$-coloring. However, the number of non-trivial edges is increased ($e$ becomes a trivial edge), which contradicts that $\Gamma$ is good.

Case 2: Suppose $x, y \in V(G_i)$. Then we recolor $E(T_j) - e'$ with $i$ and keep the color of $e'$. We now prove that the new coloring (call it $\Gamma'$) is an extremal $MC_2$-coloring of $G$.

For any vertices pair $a, b$ of $V(G)$, if at least one of $a, b$ does not belong to $V(T_j)$, then the two $ab$-paths colored differently from $j$. Because we just change the color $j$, the two $ab$-paths are not affected; if $a, b \in V(T_j)$ and at least one of $a, b$ does not belong $V(G_i)$, then there is just one $ab$-path of $T_j$ and the other $ab$-paths colored differently from $i$ under $\Gamma$. Because $G_i \cup (T_j \setminus e')$ is connected and all of them colored by $i$ under $\Gamma'$, there are two $ab$-paths under $\Gamma'$; if both $a, b \in V(G_i)$, then there are two $ab$-paths (with the same color $i$) under $\Gamma'$. Above all, $\Gamma'$ is an $MC_2$-coloring of $G$. Because the number of colors are not changed, $\Gamma'$ is an extremal $MC_2$-coloring of $G$. However, the number of non-trivial edges is increased ($e'$ becomes a trivial edge), which contradicts that $\Gamma$ is good. $\blacksquare$

By Claim 2.4, for each edge $e' = xy$ of a $T_j$, the other $xy$-paths belong to some $T_q$; for each edge $e = uv$ of a $G_i$, the other $uv$-paths belong to some $G_l$.

Claim 2.5. $h = 0$, i.e., $G_i$ is a 2-edge-connected graph for any $i \in [r]$.

Proof. If $h \neq 0$, for an edge $e_1 = v_1u_1 \in E(T_1)$, because $P_1 = e_1 = v_1u_1$ is the only $v_1u_1$-path of $T_1$, there exists another $v_1u_1$-path $P_2$, then $|P_2| \geq 2$ (because $G$ is simple),
and therefore the color of $P_2$ is non-trivial. By Claim 2.4, $P_2$ belongs to some $T_j$, w.l.o.g., suppose $j = 2$. Then $e_1 + T_2$ contains a unique cycle $C_1$. Let $f_1 = u_1u_2$ is a pendent edge of $P_2$, and $e_2 = v_2u_2$ is the edge adjacent to $f_1$ in $P_2$. Then there exists a $v_2u_2$-path $P_3$ in $T_3$ and $e_2 + T_3$ contains a unique cycle $C_2$. Let $f_2 = v_2u_3$ is a pendent edge of $P_3$, and $e_3 = v_3u_3$ is the edge adjacent to $f_2$ in $P_3$. By repeating the process, we get a series of trees $T_1, T_2, \ldots,$ paths $P_1, P_2, \ldots$ and edges $f_1 = u_1u_2, f_2 = v_2u_3, \ldots,$ etc. Because there are at most $h < \infty$ trees, there is a $T_d$ which is the first tree appearing before (w.l.o.g., suppose $T_d = T_1$), and the $v_{d-1}u_{d-1}$-path $P_d$ is contained in $T_d = T_1$. Because there are at least two trees in this sequence, we have $d - 1 \geq 2$. Then $f_1 \in T_2, f_2 \in T_3, \ldots, f_{d-2} \in T_{d-1}; P_2 \in T_2, P_3 \in T_3, \ldots, P_d \in T_d = T_1$, etc. $T_1, \ldots, T_{d-1}$ are different trees. Let $H = \bigcup_{i \in \{d-1\}} T_i$.

In order to complete the proof, we need to construct a 2-edge-connected subgraph $T$ of $H$, a connected graph $H'$, and an edge set $B$ of $H$ with $|B| = d - 2$ below.

**Case 1:** $e_1 \notin E(P_d)$.

We have already discussed above that $C_1 = P_2 + e_1, C_2 = P_3 + e_2, \ldots, C_{d-1} = P_d + e_{d-1}$. So, $T = C_1 + C_2 - e_2 + C_3 - e_3 + \cdots + C_{d-1} - e_{d-1} = \bigcup_{i=1}^{d-1} C_i - B$ is a closed trail, where $B = \bigcup_{i=2}^{d-1} e_i$, see Fig. 1(1). Therefore, $T$ is a 2-edge-connected graph. Because the ends of every edge in $B$ belong to $V(T)$, we have that $H' = \bigcup_{i \in \{d-1\}} T_i \setminus B$ is a connected graph.

**Case 2:** $e_1 \in E(P_d)$.

Suppose $F_1, F_2$ are two small trees of $T_3 \setminus e_1$ and let $v_1 \in V(F_1), u_1 \in V(F_2)$. Then there is a $u_{d-1}v_1$-path $L_1$ and a $v_{d-1}u_1$-path $L_2$ (if $u_{d-1}$ connects $u_1$ and $v_{d-1}$ connects $v_1$, the situation is similar). Let

$$T' = v_1e_1u_1P_2u_2P_3u_3 \cdots P_{d-2}u_{d-2}P_{d-1}u_{d-1}L_1v_1$$

and

$$T'' = u_1P_2u_2P_3u_3 \cdots P_{d-2}u_{d-2}P_{d-1}v_{d-1}L_2u_1.$$ 

It is obvious that both of $T'$ and $T''$ are closed trails and

$$T' \cap T'' = u_1P_2u_2 \cdots P_{d-2}u_{d-2}P_{d-1}v_{d-1}$$

is a trail. Therefore, $T = T' \cup T'' = \bigcup_{i=1}^{d-1} C_i - B$ is a 2-edge-connected graph, where $B = \bigcup_{i=2}^{d-2} f_i$, see Fig. 1(2). Because the ends of each edge in $B$ belong to $V(T)$, $H' = \bigcup_{i \in \{d-1\}} T_i \setminus B$ is a connected graph.

In above two cases, $T$ is a 2-edge-connected subgraph of $H$, and $B$ is an edge set of $H$ with $|B| = d - 2$. We recolor each edges of $H - B$ by 1 and recolor each edge of $B$ by different new colors, denote the new coloring of $G$ by $\Gamma'$. Then the total number of colors
is not changed, but the number of trivial colors is increased by $|B| = d - 2 \geq 1$. In order to complete the proof by contradiction, we need to prove that $\Gamma'$ is an $MC_2$-coloring, i.e., we need to prove that for two distinct vertices $x, y$ of $G$, there are 2 edge-disjoint $xy$-paths under $\Gamma'$. There are three cases to discuss.

(I) At least one of $x, y$ does not belong to $V(H)$. Then the two $xy$-paths do not belong to any $T_1, \cdots, T_{d-1}$. Because we just change the colors of $T_1, \cdots, T_{d-1}$, the two $xy$-paths are not affected from $\Gamma$ to $\Gamma'$.

(II) Both of $x, y$ belong to $V(H)$, but at least one of them does not belong to $V(T)$. If there is just one $xy$-path in $H$ under $\Gamma$, then another $xy$-path will not be affected. Because $H'$ is connected, there are also two edge-disjoint $xy$-paths under $\Gamma'$.

(III) Both of $x, y$ belong to $V(T)$. Then because $T$ is a 2-edge-connected graph, there are two edge-disjoint $xy$-paths under $\Gamma'$.

Claim 2.6. $s = 1$, i.e., all the non-trivial edges belong to $G_1$.

Proof. The proof is done by contradiction. If $s \geq 2$, by Claim 2.3 each $G_i$ is a 2-
edge-connected graph. Thus, \( V(G_1) \setminus V(G_2) \neq \emptyset \) and \( V(G_2) \setminus V(G_1) \neq \emptyset \); for otherwise, w.l.o.g. suppose \( V(G_1) \subseteq V(G_2) \). Recoloring all the edges of \( G_1 \) by different new colors, then the new coloring is an \( MC_2 \)-coloring of \( G \) but it has more colors than \( \Gamma \), which contradicts that \( \Gamma \) is extremal.

Let \( a \in V(G_1) \setminus V(G_2) \) and \( b \in V(G_2) \setminus V(G_1) \). Suppose \( G_a = \bigcup_{i \in c_a} G_i \) where \( c_a = \{ i : a \in V(G_i) \} \). Let \( t \) be the minimum integer such that \( V(G_2) \subseteq V(\bigcup_{j \in [t]} G_{i_j}) \) where \( i_j \in c_a \). Then \( t \leq |G_2| \). Recoloring the edges of each \( G_{i_j} \) by \( i_1 \), and recoloring the edges of \( G_2 \) by different new colors. Then the new coloring is an \( MC_2 \)-coloring of \( G \). Because \( e(G_2) \geq |G_2| \geq t \), the number of colors is not decreased. However, the number of trivial colors is increased, which contradicts that \( \Gamma \) is good.

**Claim 2.7.** \( G_1 \) is a minimum 2-edge-connected spanning subgraph of \( G \).

**Proof.** Because \( s = 1 \) and \( h = 0 \), there is just one non-trivial color (call it 1). Then \( G_1 \) is a 2-edge-connected spanning subgraph of \( G \); for otherwise, there is a vertex \( w \notin V(G_1) \), and then there is just one \( uw \)-path (which is a trivial path) for any \( u \in V(G_1) \), a contradiction.

If \( G_1 \) is not minimum, we can choose a minimum 2-edge-connected spanning subgraph \( H \) of \( G \) with \( e(G_1) > e(H) \). Coloring each edge of \( H \) by a same color and coloring the other edges by trivial colors. Then the new coloring is an \( MC_2 \)-coloring of \( G \), but there are more colors than \( \Gamma \), which contradicts that \( \Gamma \) is extremal.

**Proof of Theorem 2.2.** Actually, the theorem can be proved directly by Claims 2.5, 2.6 and 2.7. Because \( \Gamma \) is an extremal \( MC_2 \)-coloring of \( G \), and the non-trivial color-induced subgraph is just \( G_1 \), which is a minimum 2-edge-connected spanning subgraph of \( G \). So, \( mc_2(G) = e(G) - e(H) + 1 \) where \( H \) is a minimum 2-edge-connected spanning subgraph of \( G \).

We have proved that if \( \Gamma \) is a coloring of \( G \) in Theorem 2.2 then there is just one non-trivial color 1 and \( H = G_1 \) is a minimum 2-edge-connected spanning subgraph of \( G \). If \( G \) has \( t \) blocks, then \( H \) also has \( t \) blocks, and each block is a minimum 2-edge-connected spanning subgraph of the corresponding block of \( G \). Furthermore, the number of edges of \( H \) is greater than or equal to \( n + t - 1 \) (equality holds if each block of \( H \) is a cycle). So, the following result is obvious.

**Corollary 2.8.** If \( G \) is a 2-edge-connected graph with \( t \) blocks \( B_1, \ldots, B_t \), then \( mc_2(G) = \sum_{i \in [t]} mc_2(B_i) - t + 1 \), and \( mc_2(G) \leq e(G) - n - t + 2 \).

A cactus is a connected graph where every edge lies in at most one cycle. If \( G \) is a cactus without cut edges, then every edge lies in exactly one cycle. It is obvious that
There are $k$ edge cut of $G$. A minimal $k$-edge-connected graph is also the minimum $k$-edge-connected spanning subgraph of itself, and this fact will not be declared again later.

**Corollary 2.9.** If $G$ is a cactus without cut edge, then $mc_2(G) = 1$.

We have proved the first result of Theorem 2.1. Next we will prove the remaining two results. Before this, we give an upper bound of $mc_k(G)$ for $G$ being a minimal $k$-edge-connected graph. The following lemma is necessary for our later proof.

**Lemma 2.10.** Let $G$ be a minimal $k$-edge-connected graph and $\Gamma$ be an extremal $MC_k$-coloring of $G$ (suppose $mc_k(G) = t$), and let $G_i$ be the subgraph induced by the edges of color $i$, $1 \leq i \leq t$. Then each $G_i$ is a spanning subgraph of $G$.

**Proof.** We prove it by contradiction. Suppose $G_i$ is not a spanning subgraph of $G$. Let $v \notin V(G_i)$. Then for any $u \neq v$, none of the $k$ edge-disjoint monochromatic $uv$-paths is colored by $i$. Let $e$ be an edge colored by $i$. By Theorem 1.1, there exists an edge cut $C(G)$ such that $e \in C(G)$ and $|C(G)| = k$. Then $G \setminus C(G)$ has two components $M_1, M_2$ (in fact, $C(G)$ is a bond of $G$). Let $v \in V(M_1)$ and some $w \in V(M_2)$. Then the $k$ edge-disjoint monochromatic $vw$-paths are retained in $G \setminus e$. However, $C(G) \setminus e$ is an edge cut of $G \setminus e$ that separates $v$ and $w$, and $|C(G) \setminus e| = k - 1$, which contradicts that there are $k$ edge-disjoint monochromatic $vw$-paths in $G \setminus e$.

**Theorem 2.11.** If $G$ is a minimal $k$-edge-connected graph with $k \geq 2$, then $mc_k(G) \leq k - 1$.

**Proof.** We prove it by contradiction. Suppose $mc_k(H) \geq k$. Let $\Gamma$ be an extremal $MC_k$-coloring of $G$. Then by Lemma 2.10, there are at least $k$ edge-disjoint spanning subgraphs of $G$. Because there exists a vertex of $G$ with degree $k$, there are exactly $k$ edge-disjoint spanning subgraphs of $G$, denoted by $G_1, \ldots, G_k$. Because $G$ is a minimal $k$-edge-connected graph, by Theorem 1.1, $e(G) \leq k(n - 1)$, which allows all of $G_1, \ldots, G_k$ to be spanning trees of $G$.

Because $k \geq 2$, there are at least two spanning trees $G_1, G_2$, and so $G_1 \cup G_2$ is a 2-edge-connected spanning subgraph of $G$. Let $e = uv$ be an edge of $G_1$ and let $P_1$ be the $uv$-path of $G_2$. Suppose $e_1 = uu_1$ and $e_2 = vv_1$ are two terminal edges of $P_1$. Let $P_2$ be the $uu_1$-path of $G_1$ and let $P_3$ be the $vv_1$-path of $G_1$.

**Case 1:** If one of $P_2$ and $P_3$ does not contain $e$, w.l.o.g., suppose $P_2$ does not contain $e$. Then $T = uP_3u_1P_1vve$ is a 2-edge-connected graph (in fact, $T$ is a closed trail, see Fig2(1)). Because $u, u_1 \in V(T)$, by Lemma 1.5, $(G_1 \cup G_2) \setminus e_1$ is a 2-edge-connected subgraph of $G$. 

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Case 2: If both $P_2$ and $P_3$ contain $e$, then $T = uevP_2u_1P_1v_1P_3u$ is a 2-edge-connected graph (in fact, $T$ is a closed trail, see Fig.2(2)). Because $u, u_1 \in V(T)$, by Lemma 1.5, $(G_1 \cup G_2) \setminus e_1$ is a 2-edge-connected subgraph of $G$.

\[ \text{Figure 2} \]

The coloring $\Gamma'$ obtained from $\Gamma$ by assigning 1 to the edges of $G_2 \setminus e_1$ and assigning a new color to $e_1$. From above two cases, $(G_1 \cup G_2) \setminus e_1$ is a 2-edge-connected spanning subgraph of $G$ and $G_3, \ldots, G_k$ are spanning subgraph of $G$. So, every two vertices are also connected by $k$ monochromatic paths and the number of colors is not changed, i.e., $\Gamma'$ is also an extremal $MC_k$-coloring of $G$. While $e$ is a single edge, that would contradict that each induced subgraph is spanning by Lemma 2.10.

Before proving the second result of Theorem 2.1, we introduce a well-known result.

**Fact 2.12.** $K_{2n+1}$ can be decomposed into $n$ edge-disjoint Hamiltonian cycles; $K_{2n+2}$ can be decomposed into $n$ edge-disjoint Hamiltonian cycles and a perfect matching.

**Theorem 2.13.** $mc_{2n}(K_{2n+1}) = n$ for $n \geq 2$.

**Proof.** By Fact 2.12, $K_{2n+1}$ can be decomposed into $n$ edge-disjoint Hamiltonian cycles $C_1, \ldots, C_n$. Color each $C_i$ by $i \in [n]$, and then the coloring is an $MC_{2n}$-coloring of $K_{2n+1}$. So, $mc_{2n}(K_{2n+1}) \geq n$.

We need to prove that $mc_{2n}(K_{2n+1}) \leq n$ to complete our proof. The proof is done by contradiction. Suppose $mc_{2n}(K_{2n+1}) = t \geq n + 1$. Let $\Gamma$ be an extremal $MC_{2n}$-coloring of $K_{2n+1}$ and let $G_i$ be the subgraph induced by all the edges with color $i$, $1 \leq i \leq t$. 11
Because $K_{2n+1}$ is a minimal $2n$-edge-connected graph, by Lemma 2.10 we have that each $G_i$ is a spanning subgraph of $G$. If $t \geq 2n$, then
\[
n(2n + 1) = e(K_{2n+1}) = e(\bigcup_{i \in [t]} G_i) \geq 2tn \geq 4n^2,\]
which is a contradiction. Otherwise, if $t < 2n$, then not every $G_i$ is a spanning tree (for otherwise, every two vertices are just connected by $t < 2n$ monochromatic paths). To ensure that every two vertices are connected by at least $2n$ monochromatic paths, there are at least $2n - t$ $G_i$ that are 2-edge-connected. Therefore, the number of edges of $\bigcup_{i \in [t]} G_i$ satisfies
\[
e(\bigcup_{i \in [t]} G_i) \geq (2n + 1)(2n - t) + 2(t - n) \cdot 2n = t(2n - 1) + 2n \geq 2n^2 + 3n - 1.
\]
This contradicts that $\bigcup_{i \in [t]} G_i = K_{2n+1}$ and $e(K_{2n+1}) = n(2n + 1)$.
\[\square\]

Before prove the third result of Theorem 2.1, we introduce another well-known result.

**Fact 2.14.** $K_{2n,2n}$ can be decomposed into $n$ Hamiltonian cycles and $K_{2n+1,2n+1}$ can be decomposed into $n$ Hamiltonian cycles and a perfect matching.

**Theorem 2.15.** If $n \geq k \geq 3$, then $mc_k(K_{k,n}) \leq \left\lfloor \frac{k}{2} \right\rfloor$.

**Proof.** Let $\Gamma$ be an extremal $MC_k$-coloring with $t$ colors and let $G_i$ be the subgraph of $G$ induced by the edges with color $i$. Because $K_{k,n}$ is a minimal $k$-edge-connected graph, by Lemma 2.10 each $G_i$ is a spanning subgraph of $G$. Let $A, B$ be the bipartition (independent sets) of $G$ with $|A| = n$ and $|B| = k$. Then each vertex in $A$ has degree $k$.

We prove that $mc_k(K_{k,n}) \leq \left\lfloor \frac{k}{2} \right\rfloor$ by contradiction. Suppose $mc_k(K_{k,n}) = t \geq \left\lfloor \frac{k}{2} \right\rfloor + 1$. For a vertex $u$ of $A$, let $d_{G_i}(u) = r_i$. Then $\sum_{i \in [t]} r_i = k$ and each $r_i \geq 1$. Because every two vertices of $A$ are connected by $k$ edge-disjoint monochromatic paths, and the degree of every vertex in $A$ is $k$, we have that for each $u \in A$, $d_{G_i}(u) = r_i$. Because $t \geq \left\lfloor \frac{k}{2} \right\rfloor + 1$, there is a color $i$ such that $d_{G_i}(u) = 1$, i.e., all vertices of $A$ are leaves of $G_i$. Because $K_{k,n}$ is a bipartite graph with bipartition $A$ and $B$, $G_i$ is a perfect matching if $n = k$, and $G_i$ is the union of $k$ stars if $n > k$, both of which contradict that $G_i$ is a connected spanning subgraph of $G$. Therefore, $mc_k(K_{k,n}) \leq \left\lfloor \frac{k}{2} \right\rfloor$.
\[\square\]

**Corollary 2.16.** Conjecture 1.4 is true for $G = K_{k,n}$, where $k$ is even and $n \geq k \geq 4$; it is also true for $G = K_{3,n}$, where $k = 3 \leq n$.

**Proof.** If $k = 2l$ is even, then we prove that $mc_k(K_{k,n}) = \left\lfloor \frac{k}{2} \right\rfloor = l$. Actually, we only need to construct an $MC_k$-coloring of $K_{k,n}$ with $l$ colors. Let $A_1$ be a subset of $A$ with
\( k \) vertices and \( A_2 = A - A_1 \), and let \( H \) be the subgraph of \( K_{k,n} \) whose vertex set is \( A_1 \cup B \). Then \( H = K_{k,k} \), and by Fact 2.14 \( H \) can be decomposed into \( l \) Hamiltonian cycles \( \{C_1, \cdots, C_l\} \). Because the degree of each vertex in \( A_2 \) is \( k = 2l \), we mark each two edges incident with \( v \in A_2 \) with \( i, 1 \leq i \leq l \). Let \( E_i \) be the edge set with mark \( i \), and let \( G_i = C_i \cup E_i \). It is obvious that \( G_i \) is a 2-edge-connected spanning graph of \( K_{k,n} \). We color every edge of \( G_i \) by \( i \), and then we find an \( MC_k \)-coloring of \( K_{k,n} \) with \( l \) colors.

Because \( K_{3,n} \) is a minimal 3-edge-connected graph for \( n \geq 3 \), and an \( MC_3 \)-coloring of \( K_{3,n} \) assigns color 1 to all its edges, we have \( mc_3(K_{3,n}) \geq 1 \). By Theorem 2.15 \( mc_3(K_{3,n}) \leq 1 \), and thus \( mc_3(K_{3,n}) = 1 \).

If \( k \leq \Psi(G) \), then \( G \) is \( k \)-edge-connected. By Theorem 1.3 there are \( k \) edge-disjoint spanning trees \( T_1, \cdots, T_k \) of \( G \) and we color \( E(G) \) such that each \( T_i \) is colored by \( i \). Then any two vertices \( u, v \) are connected by at least \( k \) monochromatic \( uv \)-paths with different colors. So, we have the following result.

**Corollary 2.17.** For a graph \( G \) with \( \Psi(G) \geq k \geq 2 \), \( mc_k(G) \geq e(G) - k(n - 2) \).

## 3 Results for uniformly monochromatic \( k \)-edge-connection number

The monochromatic \( k \)-edge-connected graph allows \( k \) edge-disjoint monochromatic paths between any two vertices of the graph. In this section, we generalize the concept of monochromatic \( k \)-edge-connection to uniformly monochromatic \( k \)-edge-connection, and get some results.

An edge-colored \( k \)-edge-connected graph \( G \) is **uniformly monochromatic \( k \)-edge-connected** if every two distinct vertices are connected by at least \( k \) edge-disjoint monochromatic paths of \( G \) such that all these \( k \) paths have the same color. Note that for different pairs of vertices the paths may have different colors. An edge-coloring \( \Gamma \) of \( G \) is a **uniformly monochromatic \( k \)-edge-connection coloring (UMC\(_k\)-coloring)** if it makes \( G \) uniformly monochromatically \( k \)-edge-connected. The **uniformly monochromatic \( k \)-edge-connection number**, denoted by \( umc_k(G) \), of a \( k \)-edge-connected graph \( G \) is the maximum number of colors that are needed in order to make \( G \) uniformly monochromatic \( k \)-edge-connected.

An extremal **UMC\(_k\)-coloring** of \( G \) is an **UMC\(_k\)-coloring** that uses \( umc_k(G) \) colors. We call an extremal **UMC\(_k\)-coloring** a **good UMC\(_k\)-coloring** of \( G \) if the coloring has the maximum number of trivial edges. A uniformly monochromatic \( k \)-edge-connected graph is also a monochromatic connected graph when \( k = 1 \).
Theorem 3.1. Let $G$ be a $k$-edge-connected graph with $k \geq 2$. Then $\text{umc}_k(G) = e(G) - e(H) + 1$, where $H$ is a minimum $k$-edge-connected spanning subgraph of $G$.

We prove the theorem below. For convenience, we abbreviate "monochromatic $uv$-path" as "$uv$-path". Let $\Gamma$ be a good UMC$_k$-coloring of $G$. Then, suppose that the number of non-trivial colors of $\Gamma$ is $t$ and denote the set of them by $[t]$. Let $G_i$ be the subgraph of $G$ induced by the edges with a non-trivial color $i$, $1 \leq i \leq t$. Let $G' = \bigcup_{i \in [t]} G_i$.

Claim 3.2. Each $G_i$ is $k$-edge-connected.

Proof. Let $\pi_i$ denote the set of pairs $(u, v)$ such that there are at least $k$ edge-disjoint $uv$-paths colored by $i \in [t]$. Therefore, any vertex pair $(u, v)$ belongs to some $\pi_i$.

We first prove it by contradiction that each $G_i$ is $k$-edge-connected.

Suppose that $G_i$ is not a $k$-edge-connected graph. Then there exists a bond $C(G_i)$ with $|C(G_i)| \leq k - 1$, and $G \setminus C(G_i)$ has two components $M_1$ and $M_2$. Let $e = uv$ be an edge of $C(G_i)$, $u \in V(M_1)$, $v \in V(M_2)$. Then there are at most $|C(G)| \leq k - 1$ edge-disjoint paths in $G_i$ between $u, v$. Therefore there exists a $j \neq i$ of $[t]$ such that there are at least $k$ edge-disjoint $uv$-paths of $G_j$.

Recolor edges of $G_i - e$ with $j$ and keep the color of $e$, and denote the new coloring of $G$ by $\Gamma'$.

Because any non-trivial color $r \neq i$ is not changed. So, under $\Gamma'$, any pair $(x, y) \in \pi_r$ also have at least $k$ edge-disjoint $xy$-paths colored $r$. For any pair $(x, y) = \pi_i$, if any $k$ edge-disjoint $xy$-paths (Note that $P_1, \ldots, P_k$) of $G_i$ under $\Gamma$ do not contain $e$. Then these $k$ edge-disjoint $xy$-paths are retained. Otherwise, there is a path (Note that $P_1$) contains $e$. We choose a path $P$ of $G_j$ whose terminals are $u, v$. Then $T = (P_1\{e\}) \cup P$ is a trail between $x, y$ and $E(T) \cap \bigcup_{i \neq 1} E(P_i) = \emptyset$. Let $P'$ be a $xy$-path of $T$. Then $P', P_2, \ldots, P_k$ are $k$ edge-disjoint $xy$-paths colored by $j$ (under $\Gamma'$). Therefore, $\Gamma'$ is still an extremal UMC$_k$-coloring of $G$, but then $e$ becomes to a trivial edge, which contradicts that $\Gamma$ is good. So, each $G_i$ is $k$-edge-connected.

By Claim 3.2, because $k \geq 2$, we have $e(G_i) \geq |G_i| \geq 3$. Denote $G_x = \bigcup_{x \in V(G_x)} G_i$, $F_x = G' - G_x$.

Claim 3.3. Each $G_x$ is a $k$-edge-connected spanning subgraph of $G$. Furthermore, $F_x = \emptyset$.

Proof. If there is an $x \in V(G)$ such that $G_x$ is not a spanning subgraph of $G$, then there is a vertex $y \in V(G) \setminus V(G_x)$. Because $G$ is a simple graph and $k \geq 2$, any two vertices
are connected by at least one non-trivial path. It is obvious that there are no non-trivial
\(xy\)-path, a contradiction. Therefore, \(G_x\) is a spanning subgraph of \(G\).

Because each \(G_i\) is \(k\)-edge-connected, \(G_x\) is also \(k\)-edge-connected. Therefore, each
\(G_x\) is a \(k\)-edge-connected spanning subgraph of \(G\).

Now we prove that \(F_x = \emptyset\). Otherwise, if \(F_x \neq \emptyset\), then there is a \(G_j \subseteq F_x\) and \(|G_j| \geq 3\).
Suppose that \(s\) is the minimum number such that \(V(G_j) \subseteq \bigcup_{r \in [s]} G_t\), where \(G_{t1}, \ldots, G_{ts}\)
are contained in \(G_x\). Then, \(s \leq |G_j|\). Because \(k \geq 2\), we have \(e(G_j) \geq |G_j| \geq s\). We have
obtained a new coloring \(\Gamma'\) from \(\Gamma\) by recoloring each \(G_{t1}, \ldots, G_{ts}\) by \(i_1\) and recoloring
each edge of \(G_j\) by different new colors. Because \(G^* = \bigcup_{r \in [s]} G_t\) is \(k\)-edge-connected
graph, each pair \((a, b)\) with \((a, b) \in \{(\pi_{i1}, \ldots, \pi_{is}, \pi_j)\}\) has \(k\)-edge-disjoint \(ab\)-paths colored
\(i_1\) under \(\Gamma'\). It is easy to check that \(\Gamma'\) is a \(UMC_k\)-coloring. Then, the number of colors
is not decreased, but the number of trivial colors is increased by at least \(e(G_j) \geq 3\),
which contradicts that \(\Gamma\) is good. So, \(F_x = \emptyset\).

**Claim 3.4.** \(t = 1\) and \(G_1\) is a minimum \(k\)-edge-connected spanning subgraph of \(G\).

**Proof.** Suppose \(t \geq 2\). Then \(V(G_1) \setminus V(G_2) \neq \emptyset\). Otherwise, if \(V(G_1) \subseteq V(G_2)\), then
\((u, v) \in \pi_2\) when \((u, v) \in \pi_1\). We can recolor all edges of \(G_1\) by fresh colors, and then
the new coloring is also a \(UMC_k\)-coloring of \(G\) but the number of colors is increased,
which contradicts that \(\Gamma\) is extremal. So, \(V(G_1) \setminus V(G_2) \neq \emptyset\), and there is a vertex \(a \in V(G_1) \setminus V(G_2)\), i.e., \(G_2 \not= G_a\), \(G_2 \subseteq F_a\). By Claim 3.3, we have \(F_a = \emptyset\), a contradiction.
Therefore, \(t = 1\), and thus \(G_1 = G_a\) is a spanning subgraph of \(G\).

In fact, \(G_1\) is a minimum \(k\)-edge-connected spanning subgraph of \(G\); otherwise, there
exists a minimum \(k\)-edge-connected spanning subgraph \(H\) of \(G\) such that \(e(H) < e(G_1)\).
Coloring each edge of \(H\) by 1 and coloring the other edges by some different new colors.
Then the coloring is a \(UMC_k\)-coloring of \(G\) with more colors, which contradicts that \(\Gamma\)
is extremal.

**Proof of Theorem 3.1.** We can prove Theorem 3.1 directly by Claim 3.4.

Because any \(k\)-edge-connected graph \(G\) has the minimum degree \(\delta(G) \geq k\), by Theorem 1.1, we have that \(\frac{1}{2}kn \leq e(H) \leq k(n - 1)\), where \(H\) is a minimum \(k\)-edge-connected spanning subgraph of \(G\).

**Corollary 3.5.** For a \(k\)-edge-connected graph \(G\) with \(k \geq 2\), \(e(G) - k(n - 1) + 1 \leq umc_k(G) \leq e(G) - \frac{1}{2}kn + 1\).

By definition, a \(k\)-edge-connected graph \(G\) satisfies that \(umc_k(G) \leq mck(G)\). Therefore,
\(mck(G) \geq e(G) - e(H) + 1\), where \(H\) is a \(k\)-edge-connected spanning subgraph of \(G\).
By this theorem, we also get a result: A graph contains a Hamiltonian cycle if and only if \(umc_2(G) = e(G) - n + 1\).
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