On backward uniqueness for parabolic equations when Osgood continuity of the coefficients fails at one point

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Abstract

We prove the uniqueness for backward parabolic equations whose coefficients are Osgood continuous in time for $t > 0$ but not at $t = 0$.

1 Introduction

Let us consider the following backward parabolic operator

$$L = \partial_t + \sum_{i,j=1}^{n} \partial_{x_j} (a_{jk}(t,x) \partial_{x_k}) + \sum_{j=1}^{n} b_j(t,x) \partial_{x_j} + c(t,x).$$

We assume that all coefficients are defined in $[0,T] \times \mathbb{R}_x^n$, measurable and bounded; $(a_{jk}(t,x))_{j,k}$ is a real symmetric matrix for all $(t,x) \in [0,T] \times \mathbb{R}_x^n$ and there exists $\lambda_0 \in (0,1]$ such that

$$\sum_{j,k=1}^{n} a_{jk}(t,x) \xi_j \xi_k \geq \lambda_0 |\xi|^2$$

for all $(t,x) \in [0,T] \times \mathbb{R}_x^n$ and $\xi \in \mathbb{R}_\xi^n$.

Given a functional space $\mathcal{H}$ we say that the operator $L$ has the $\mathcal{H}$–uniqueness property if, whenever $u \in \mathcal{H}$, $Lu = 0$ in $[0,T] \times \mathbb{R}_x^n$ and $u(0,x) = 0$ in $\mathbb{R}_x^n$, then $u = 0$ in $[0,T] \times \mathbb{R}_x^n$. Our choice for $\mathcal{H}$ is the space of functions

$$\mathcal{H} = H^1((0,T), L^2(\mathbb{R}_x^n)) \cap L^2((0,T), H^2(\mathbb{R}_x^n)).$$

This choice is natural, since it follows from elliptic regularity results that the domain of the operator $-\sum_{j,k=1}^{n} \partial_{x_j} (a_{jk}(t,x) \partial_{x_k})$ in $L^2(\mathbb{R}^n)$ is $H^2(\mathbb{R}^n)$ for all $t \in [0,T]$.

In our previous papers [5, 6] we investigated the problem of finding the minimal regularity assumptions on the coefficients $a_{jk}$ ensuring the $\mathcal{H}$–uniqueness property to $L$. Namely, we proved the $\mathcal{H}$–uniqueness property...
for the operator $L$ when the coefficients $a_{jk}$ are Lipschitz continuous in $x$ and the regularity in $t$ is given in terms of a modulus of continuity $\mu$, i.e.
\[
\sup_{s_1, s_2 \in [0,T], x \in \mathbb{R}^n} \frac{|a_{j,k}(s_1, x) - a_{j,k}(s_2, x)|}{\mu(|s_1 - s_2|)} \leq C,
\]
where $\mu$ satisfies the so called Osgood condition
\[
\int_0^1 \frac{1}{\mu(s)} \, ds = +\infty.
\]

Suitable counterexamples show that Osgood condition is sharp for backward uniqueness in parabolic equations: given any non-Osgood modulus of continuity $\mu$, it is possible to construct a backward parabolic equation, whose coefficients are $C^\infty$ in $x$ and $\mu$-continuous in $t$, for which the $\mathcal{H}$-uniqueness property does not hold. In the mentioned counterexamples the coefficients are in fact $C^\infty$ in $t$ for $t \neq 0$, and Osgood continuity fails only at $t = 0$.

In this paper we show that if the loss of Osgood continuity is properly controlled as $t \to 0$, then we can recover the $\mathcal{H}$-uniqueness property for $L$. Our hypothesis reads as follows: given a modulus of continuity $\mu$ satisfying the Osgood condition, we assume that the coefficients $a_{jk}$ are Hölder continuous with respect to $t$ on $[0,T]$, and for all $t \in ]0,T]$ and for all $\epsilon > 0$
\[
\sup_{s_1, s_2 \in [t,T], x \in \mathbb{R}^n} \frac{|a_{j,k}(s_1, x) - a_{j,k}(s_2, x)|}{|s_1 - s_2|} \leq Ct^{-\beta},
\]
where $0 < \beta < 1$. The coefficients $a_{jk}$ are assumed to be globally Lipschitz continuous in $x$. Under such hypothesis we prove that the $\mathcal{H}$-uniqueness property holds for $L$. As in our previous papers [5, 6], the uniqueness result is consequence of a Carleman estimate with a weight function shaped on the modulus of continuity $\mu$. The weight function is obtained as solution of a specific second order ordinary differential equation. In the previous results cited above, the corresponding o.d.e. is autonomous. Here, on the contrary, the time dependent control (1) yields to a non-autonomous o.d.e.. Also, the "Osgood singularity" of $a_{jk}$ at $t = 0$ introduces a number of new technical difficulties which are not present in the fully Osgood-regular situation considered before.

The result is sharp in the following sense: we exhibit a counterexample in which the coefficients $a_{jk}$ are Hölder continuous with respect to $t$ on $[0,T]$, for all $t \in ]0,T]$ and for all $\epsilon > 0$
\[
\sup_{s_1, s_2 \in [t,T], x \in \mathbb{R}^n} \frac{|a_{j,k}(s_1, x) - a_{j,k}(s_2, x)|}{|s_1 - s_2|} \leq Ct^{-(1+\epsilon)},
\]
and the operator $L$ does not have the $\mathcal{H}$-uniqueness property. The borderline case $\epsilon = 0$ in (2) is considered in paper [7]. In such a situation only a very particular uniqueness result holds and the problem remains essentially open.

2
2 Main result

We start with the definition of modulus of continuity.

**Definition 1.** A function \( \mu : [0, 1] \to [0, 1] \) is a modulus of continuity if it is continuous, concave, strictly increasing and \( \mu(0) = 0, \mu(1) = 1 \).

**Remark 1.** Let \( \mu \) be a modulus of continuity. Then
- for all \( s \in [0, 1] \), \( \mu(s) \geq s \);
- on \((0, 1]\), the function \( s \mapsto \frac{\mu(s)}{s} \) is decreasing;
- the limit \( \lim_{s \to 0^+} \frac{\mu(s)}{s} \) exists;
- on \([1, +\infty[\), the function \( \sigma \mapsto \frac{1}{\sigma^2 \mu(\frac{1}{\sigma})} \) is decreasing.

**Definition 2.** Let \( \mu \) be a modulus of continuity and let \( \phi : I \to B \), where \( I \) is an interval in \( \mathbb{R} \) and \( B \) is a Banach space. \( \phi \) is a function in \( C^\mu(I, B) \) if
\[
\| \phi \|_{C^\mu(I, B)} = \| \phi \|_{L^\infty(I, B)} + \sup_{0 < |t-s| < 1} \frac{\| \phi(t) - \phi(s) \|_B}{\mu(|t-s|)} < +\infty.
\]

**Remark 2.** Let \( \alpha \in (0, 1) \) and \( \mu(s) = s^\alpha \). Then \( C^\mu(I, B) = C^{0,\alpha}(I, B) \), the space of Hölder-continuous functions. Let \( \mu(s) = s \). Then \( C^\mu(I, B) = \text{Lip}(I, B) \), the space of bounded Lipschitz-continuous functions.

We introduce the notion of Osgood modulus of continuity.

**Definition 3.** Let \( \mu \) be a modulus of continuity. \( \mu \) satisfies the Osgood condition if
\[
\int_0^1 \frac{1}{\mu(s)} \, ds = +\infty. \tag{3}
\]

**Remark 3.** Examples of moduli of continuity satisfying the Osgood condition are \( \mu(s) = s \) and \( \mu(s) = s \log(e + \frac{1}{s} - 1) \).

We state our main result.

**Theorem 1.** Let \( L \) be the operator
\[
L = \partial_t + \sum_{j,k=1}^n \partial_{x_j} (a_{j,k}(t, x) \partial_{x_k}) + \sum_{j=1}^n b_j(t, x) \partial_{x_j} + c(t, x), \tag{4}
\]
where all the coefficients are supposed to be complex valued, defined in \([0, T] \times \mathbb{R}^n\), measurable and bounded. Let \( (a_{j,k}(t, x))_{j,k} \) be a real symmetric matrix and suppose there exists \( \lambda_0 \in (0, 1) \) such that
\[
\sum_{j,k=1}^n a_{j,k}(t, x) \xi_j \xi_k \geq \lambda_0 |\xi|^2, \tag{5}
\]
for all \((t, x) \in [0, T] \times \mathbb{R}^n\) and for all \(\xi \in \mathbb{R}^n\). Under this condition \(L\) is a backward parabolic operator. Let \(\mathcal{H}\) be the space of functions such that

\[
\mathcal{H} = H^1((0, T), L^2(\mathbb{R}^n_x)) \cap L^2((0, T), H^2(\mathbb{R}^n_x)).
\]

Let \(\mu\) be a modulus of continuity satisfying the Osgood condition. Suppose that there exist \(\alpha \in (0, 1)\) and \(C > 0\) such that,

\begin{enumerate}[i)]  
  \item for all \(j, k = 1, \ldots, n\),
  \[
  a_{j,k} \in C^{0,\alpha}([0, T], L^\infty(\mathbb{R}^n_x)) \cap L^\infty([0, T], \text{Lip}(\mathbb{R}^n_x));
  \]
  \item for all \(j, k = 1, \ldots, n\) and for all \(t \in (0, T]\),
  \[
  \sup_{s_1, s_2 \in \mathbb{R}^n} \frac{|a_{j,k}(s_1, x) - a_{j,k}(s_2, x)|}{\mu(|s_1 - s_2|)} \leq Ct^{\alpha-1}.
  \]
\end{enumerate}

Then \(L\) has the \(\mathcal{H}\)-uniqueness property, i.e. if \(u \in \mathcal{H}\), \(Lu = 0\) in \([0, T] \times \mathbb{R}^n_x\) and \(u(0, x) = 0\) in \(\mathbb{R}^n_x\), then \(u = 0\) in \([0, T] \times \mathbb{R}^n_x\).

### 3 Weight function and Carleman estimate

We define

\[
\phi(t) = \int_1^t \frac{1}{\mu(s)} ds.
\] (9)

The function \(\phi : [1, +\infty[ \to [0, +\infty]\) is a strictly increasing \(C^1\) function and, from Osgood condition, it is bijective. Moreover, for all \(t \in [1, +\infty[\),

\[
\phi'(t) = \frac{1}{t^2 \mu(\frac{1}{t})}.
\]

We remark that \(\phi'(1) = 1\) and \(\phi'\) is decreasing in \([1, +\infty[\), so that \(\phi\) is a concave function. We remark also that \(\phi^{-1} : [0, +\infty[ \to [1, +\infty[\) and, for all \(s \in [0, +\infty[\),

\[
\phi^{-1}(s) \geq 1 + s.
\]

We define

\[
\psi_\gamma(\tau) = \phi^{-1}(\gamma \int_0^\tau (T - s)^{\alpha-1} ds),
\] (10)

where \(\tau \in [0, \gamma T]\).

\[
\phi(\psi_\gamma(\tau)) = \gamma \int_0^{\psi_\gamma(\tau)} (T - s)^{\alpha-1} ds
\]

and

\[
\phi'(\psi_\gamma(\tau))\psi'_\gamma(\tau) = (T - \frac{\tau}{\gamma})^{\alpha-1}.
\]
Then
\[
\psi'_{\gamma}(\tau) = (T - \frac{T}{\gamma})^{\alpha - 1} \cdot (\psi_{\gamma}(\tau))^2 \mu\left(\frac{1}{\psi_{\gamma}(\tau)}\right),
\]
i. e. $\psi_{\gamma}$ is a solution to the differential equation
\[
u'_{\tau} = (T - \frac{T}{\gamma})^{\alpha - 1} \cdot u^2(\tau) \cdot \mu\left(\frac{1}{u(\tau)}\right).
\]
Finally we set, for $\tau \in [0, \gamma T]$,
\[
\Phi_{\gamma}(\tau) = \int_{0}^{\tau} \psi_{\gamma}(\sigma) \, d\sigma.
\]
Remark that, with this definition, $\Phi'_{\gamma}(\tau) = \psi_{\gamma}(\tau)$ and
\[
\Phi''_{\gamma}(\tau) = (T - \frac{T}{\gamma})^{\alpha - 1} \cdot (\psi'_{\gamma}(\tau))^2 \cdot \mu\left(\frac{1}{\Phi'_{\gamma}(\tau)}\right).
\]
In particular, for $t \in (0, \frac{T}{2}]$,
\[
\Phi''_{\gamma}(\gamma(T - t)) = t^{\alpha - 1} \cdot \Phi'_{\gamma}(\gamma(T - t)) \cdot \frac{\mu\left(\frac{1}{\Phi'_{\gamma}(\gamma(T - t))}\right)}{\Phi'_{\gamma}(\gamma(T - t))} \geq t^{\alpha - 1} \geq \left(\frac{T}{2}\right)^{\alpha - 1},
\]
since $\Phi'_{\gamma}(\gamma(T - t)) = \psi_{\gamma}(\gamma(T - t)) \geq 1$ and $\frac{\mu(s)}{s} \geq 1$ for all $s \in (0, 1)$.

We can now state the Carleman estimate.

**Theorem 2.** In the previous hypotheses there exist $\gamma_0 > 0$, $C > 0$ such that
\[
\int_{0}^{\frac{T}{2}} e^{\frac{2}{\gamma} \Phi_{\gamma}(\gamma(T - t))} \|\partial_t u + \sum_{j,k=1}^{n} \partial_{x_j}(a_{j,k}(t,x) \partial_{x_k} u)\|_{L^2(\mathbb{R}^n)}^2 \, dt
\]
\[
\geq C \gamma^{\frac{1}{2}} \int_{0}^{\frac{T}{2}} e^{\frac{2}{\gamma} \Phi_{\gamma}(\gamma(T - t))} (\|\nabla_x u\|_{L^2(\mathbb{R}^n)}^2 + \gamma^2 \|u\|_{L^2(\mathbb{R}^n)}^2) \, dt
\]
for all $\gamma > \gamma_0$ and for all $u \in C_0^\infty(\mathbb{R}^{n+1})$ such that $\text{Supp} \, u \subseteq [0, \frac{T}{2}] \times \mathbb{R}^n$.

The way of obtaining the $\mathcal{H}$-uniqueness from the inequality (14) is a standard procedure, the details of which can be found in [5, Par. 3.4].

**4 Proof of the Carleman estimate**

**4.1 Littlewood-Paley decomposition**

We will use the so called Littlewood-Paley theory. We refer to [2], [3], [9] and [1] for the details. Let $\psi \in C^\infty([0, +\infty[; \mathbb{R})$ such that $\psi$ is non-increasing and
\[
\psi(t) = 1 \quad \text{for} \quad 0 \leq t \leq \frac{11}{10}, \quad \psi(t) = 0 \quad \text{for} \quad t \geq \frac{19}{10}.
\]
We set, for $\xi \in \mathbb{R}^n$,
\[
\chi(\xi) = \psi(|\xi|), \quad \varphi(\xi) = \chi(\xi) - \chi(2\xi).
\]
(15)

Given a tempered distribution $u$, the dyadic blocks are defined by
\[
u_0 = \Delta_0 u = \chi(D)u = \mathcal{F}^{-1}(\chi(\xi)\hat{u}(\xi)),
\]
\[
u_j = \Delta_j u = \varphi(2^{-j}D)u = \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\hat{u}(\xi)) \text{ if } j \geq 1,
\]
where we have denoted by $\mathcal{F}^{-1}$ the inverse of the Fourier transform. We introduce also the operator
\[
S_k u = \sum_{j=0}^{k} \Delta_j u = \mathcal{F}^{-1}(\chi(2^{-k}\xi)\hat{u}(\xi)).
\]

We recall some well known facts on Littlewood-Paley decomposition.

**Proposition 1.** ([4, Prop. 3.1]) Let $s \in \mathbb{R}$. A temperate distribution $u$ is in $H^s$ if and only if, for all $j \in \mathbb{N}$, $\Delta_j u \in L^2$ and
\[
\sum_{j=0}^{+\infty} 2^{j s} \|\Delta_j u\|_{L^2}^2 < +\infty.
\]
Moreover there exists $C > 1$, depending only on $n$ and $s$, such that, for all $u \in H^s$,
\[
\frac{1}{C} \sum_{j=0}^{+\infty} 2^{j s} \|\Delta_j u\|_{L^2}^2 \leq \|u\|_{H^s}^2 \leq C \sum_{j=0}^{+\infty} 2^{j s} \|\Delta_j u\|_{L^2}^2.
\]
(16)

**Proposition 2.** ([8, Lemma 3.2]). A bounded function $a$ is a Lipschitz-continuous function if and only if
\[
\sup_{k \in \mathbb{N}} \|\nabla(S_k a)\|_{L^\infty} < +\infty.
\]
Moreover there exists $C > 0$, depending only on $n$, such that, for all $a \in \text{Lip}$ and for all $k \in \mathbb{N}$,
\[
\|\Delta_k a\|_{L^\infty} \leq C 2^{-k} \|a\|_{\text{Lip}} \quad \text{and} \quad \|\nabla(S_k a)\|_{L^\infty} \leq C \|a\|_{\text{Lip}},
\]
(17)
where $\|a\|_{\text{Lip}} = \|a\|_{L^\infty} + \|\nabla a\|_{L^\infty}$.

4.2 Modified Bony’s paraproduct

**Definition 4.** Let $m \in \mathbb{N} \setminus \{0\}$ and let $a \in L^\infty$. Let $s \in \mathbb{R}$ and let $u \in H^s$. We define
\[
T^m_a u = S_{m-1} a S_{m+1} u + \sum_{k=m-1}^{+\infty} S_k a \Delta_{k+3} u.
\]
We recall some known facts on modified Bony’s paraproduct.

**Proposition 3.** ([4] Prop. 5.2.1 and Th. 5.2.8). Let \( m \in \mathbb{N} \setminus \{0\} \) and let \( a \in L^\infty \). Let \( s \in \mathbb{R} \).

Then \( T_a^m \) maps \( H^s \) into \( H^s \) and there exists \( C > 0 \) depending only on \( n \), \( m \) and \( s \), such that, for all \( u \in H^s \),

\[
\| T_a^m u \|_{H^s} \leq C \| a \|_{L^\infty} \| u \|_{H^s}.
\]  

(18)

Let \( m \in \mathbb{N} \setminus \{0\} \) and let \( a \in \text{Lip} \).

Then \( a - T_a^m \) maps \( L^2 \) into \( H^1 \) and there exists \( C' > 0 \) depending only on \( n \), \( m \), such that, for all \( u \in L^2 \),

\[
\| au - T_a^m u \|_{H^1} \leq C' \| a \|_{\text{Lip}} \| u \|_{L^2}.
\]  

(19)

**Proposition 4.** ([4] Cor. 3.12]) Let \( m \in \mathbb{N} \setminus \{0\} \) and let \( a \in \text{Lip} \). Suppose that, for all \( x \in \mathbb{R}^n \), \( a(x) \geq \lambda_0 > 0 \).

Then there exists \( m \) depending on \( \lambda_0 \) and \( \| a \|_{\text{Lip}} \) such that for all \( u \in L^2 \),

\[
\text{Re} \langle T_a^m u, u \rangle_{L^2, L^2} \geq \frac{\lambda_0}{2} \| u \|_{L^2}^2.
\]  

(20)

A similar result remains valid for valued functions when \( a \) is replaced by a positive definite matrix \((a_{j,k})_{j,k}\).

**Proposition 5.** ([4] Prop. 3.8 and Prop. 3.11 and [5] Prop. 3.8) Let \( m \in \mathbb{N} \setminus \{0\} \) and let \( a \in \text{Lip} \). Let \( (T_a^m)^* \) be the adjoint operator of \( T_a^m \).

Then there exists \( C \) depending only on \( n \) and \( m \) such that for all \( u \in L^2 \),

\[
\| (T_a^m - (T_a^m)^*) \partial_x u \|_{L^2} \leq C \| a \|_{\text{Lip}} \| u \|_{L^2}.
\]  

(21)

We end this subsection with a property which will needed in the proof of the Carleman estimate.

**Proposition 6.** ([5] Prop. 3.8]) Let \( m \in \mathbb{N} \setminus \{0\} \) and let \( a \in \text{Lip} \). Denote by \( [\Delta_k, T_a^m] \) the commutator between \( \Delta_k \) and \( T_a^m \).

Then there exists \( C \) depending only on \( n \) and \( m \) such that for all \( u \in H^1 \),

\[
\sum_{h=0}^{\infty} \| \partial_x ( [\Delta_k, T_a^m] \partial_x u ) \|_{L^2}^2 \leq C \| a \|_{\text{Lip}} \| u \|_{H^1}.
\]  

(22)

### 4.3 Approximated Carleman estimate

We set

\[
v(t, x) = e^{-\frac{\gamma}{2} \Phi_1(\gamma(T-t))} u(t, x).
\]

The Carleman estimate ([14]) becomes: there exist \( \gamma_0 > 0 \), \( C > 0 \) such that

\[
\int_0^T \| \partial_t v + \sum_{j,k=1}^n \partial_x (a_{j,k}(t, x) \partial_x v) + \Phi_1(\gamma(T-t)) v \|^2_{L^2(\mathbb{R}^n_x)} dt \\
\geq C \gamma \frac{1}{2} \int_0^T (\| \nabla_x v \|_{L^2(\mathbb{R}^n_x)}^2 + \gamma \frac{1}{2} \| u \|_{L^2(\mathbb{R}^n_x)}^2) dt,
\]  

(23)
for all $\gamma > \gamma_0$ and for all $v \in C_0^\infty(\mathbb{R}^{n+1})$ such that $\text{Supp } u \subseteq [0, \frac{T}{2}] \times \mathbb{R}_x^n$.

First of all, using Proposition 3 we fix a value for $m$ in such a way that

$$\text{Re } \sum_{j,k} \langle T_{a_{j,k}}^m \partial_x v, \partial_x v \rangle_{L^2, L^2} \geq \frac{\lambda_0}{2} \| \nabla v \|_{L^2},$$

(24)

for all $v \in C_0^\infty(\mathbb{R}^{n+1})$ such that $\text{Supp } u \subseteq [0, \frac{T}{2}] \times \mathbb{R}_x^n$. Next we consider Proposition 3 and in particular from (19) we deduce that (23) will be a consequence of

\[
\int_{0}^{\infty} \| \partial_t v + \sum_{j,k=1}^{n} \partial_j (T_{a_{j,k}}^m \partial_x v) + \Phi'_\gamma(\gamma(T-t)) v \|_{L^2(\mathbb{R}^2)}^2 \, dt \\
\geq C \gamma^{\frac{1}{2}} \int_{0}^{\infty} \left( \| \nabla_x v \|_{L^2(\mathbb{R}^2)}^2 + \gamma \| \nabla v \|_{L^2(\mathbb{R}^2)}^2 \right) \, dt,
\]

(25)

since the difference between (23) and (25) is absorbed by the right side part of (25) with possibly a different value of $C$ and $\gamma_0$. With a similar argument, using (16) and (22), (25) will be deduced from

\[
\int_{0}^{\infty} \| \partial_t v_h + \sum_{j,k=1}^{n} \partial_j (T_{a_{j,k}}^m \partial_x v_h) + \Phi'_\gamma(\gamma(T-t)) v_h \|_{L^2(\mathbb{R}^2)}^2 \, dt \\
\geq C \gamma^{\frac{1}{2}} \int_{0}^{\infty} \left( \| \nabla_x v_h \|_{L^2(\mathbb{R}^2)}^2 + \gamma \| v_h \|_{L^2(\mathbb{R}^2)}^2 \right) \, dt,
\]

(26)

where we have denoted by $v_h$ the dyadic block $\Delta_h v$.

We fix our attention on each of the terms

\[
\int_{0}^{\infty} \| \partial_t v_h + \sum_{j,k=1}^{n} \partial_j (T_{a_{j,k}}^m \partial_x v_h) + \Phi'_\gamma(\gamma(T-t)) v_h \|_{L^2(\mathbb{R}^2)}^2 \, dt.
\]

We have

\[
\int_{0}^{\infty} \| \partial_t v_h + \sum_{j,k=1}^{n} \partial_j (T_{a_{j,k}}^m \partial_x v_h) + \Phi'_\gamma(\gamma(T-t)) v_h \|_{L^2(\mathbb{R}^2)}^2 \, dt \\
= \int_{0}^{\infty} \left( \| \partial_t v_h \|_{L^2}^2 + \| \sum_{j,k=1}^{n} \partial_j (T_{a_{j,k}}^m \partial_x v_h) + \Phi'_\gamma(\gamma(T-t)) v_h \|_{L^2}^2 \\
+ \gamma \Phi''_\gamma(\gamma(T-t)) \| v_h \|_{L^2}^2 + 2 \text{Re } \langle \partial_t v_h, \sum_{j,k=1}^{n} \partial_j (T_{a_{j,k}}^m \partial_x v_h) \rangle_{L^2, L^2} \right) \, dt
\]

(27)

Let consider the last term in (27). We define, for $\varepsilon \in (0, \frac{T}{2}]$,

\[
\tilde{a}_{j,k,\varepsilon}(t, x) = \begin{cases} a_{j,k}(t, x), & \text{if } t \geq \varepsilon, \\
a_{j,k}(\varepsilon, x), & \text{if } t < \varepsilon,
\end{cases}
\]

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and
\[ a_{j,k,\varepsilon}(t, x) = \int_{-\varepsilon}^{\varepsilon} \rho_\varepsilon(s) \partial_{j,k,\varepsilon}(t - s, x) \, ds, \]
where \( \rho \in C_0^\infty(\mathbb{R}) \) with \( \text{Supp} \rho \subseteq [-1, 1] \), \( \int_{\mathbb{R}} \rho(s) \, ds = 1 \), \( \rho(s) \geq 0 \) and \( \rho_\varepsilon(s) = \frac{1}{\varepsilon} \rho(\frac{s}{\varepsilon}) \). With a straightforward computation, form (1) and (8), we obtain
\[ |a_{j,k}(t, x) - a_{j,k,\varepsilon}(t, x)| \leq C \min\{\varepsilon^\alpha, t^{\alpha-1} \mu(\varepsilon)\}, \quad (28) \]
and
\[ |\partial_t a_{j,k,\varepsilon}(t, x)| \leq C \min\{\varepsilon^{\alpha-1}, t^{\alpha-1} \mu(\varepsilon)\}, \quad (29) \]
for all \( j, k = \ldots, n \) and for all \( (t, x) \in [0, \frac{T}{2}] \times \mathbb{R}_x^n \). We deduce
\[
\begin{align*}
\int_0^{\frac{T}{2}} 2 \Re \left( \partial_t v_h, \sum_{j,k=1}^n \partial_{x_j} (T^m_{a_{j,k}} \partial_{x_k} v_h) \right)_{L^2,L^2} dt \\
= -2 \Re \int_0^{\frac{T}{2}} \sum_{j,k=1}^n (\partial_{x_j} \partial_t v_h, T^m_{a_{j,k}} \partial_{x_k} v_h)_{L^2,L^2} dt \\
= -2 \Re \int_0^{\frac{T}{2}} \sum_{j,k=1}^n (\partial_{x_j} \partial_t v_h, (T^m_{a_{j,k}} - T^m_{a_{j,k,\varepsilon}}) \partial_{x_k} v_h)_{L^2,L^2} dt \\
- 2 \Re \int_0^{\frac{T}{2}} \sum_{j,k=1}^n (\partial_{x_j} \partial_t v_h, T^m_{a_{j,k,\varepsilon}} \partial_{x_k} v_h)_{L^2,L^2} dt.
\end{align*}
\]
Now, \( T^m_{a_{j,k}} - T^m_{a_{j,k,\varepsilon}} = T^m_{a_{j,k} - a_{j,k,\varepsilon}} \) and, from (18) and (28),
\[
\|(T^m_{a_{j,k}} - T^m_{a_{j,k,\varepsilon}}) \partial_{x_k} v_h\|_{L^2} = \|T^m_{a_{j,k} - a_{j,k,\varepsilon}} \partial_{x_k} v_h\|_{L^2} \leq C \min\{\varepsilon^\alpha, t^{\alpha-1} \mu(\varepsilon)\}\|\partial_{x_k} v_h\|_{L^2}.
\]
Moreover \( \|\partial_{x_j} v_h\|_{L^2} \leq 2^{h+1} \|v_h\|_{L^2} \) and \( \|\partial_{x_j} \partial_t v_h\|_{L^2} \leq 2^{h+1} \|\partial_t v_h\|_{L^2} \), so that
\[
\begin{align*}
|2 \Re \int_0^{\frac{T}{2}} \sum_{j,k=1}^n (\partial_{x_j} \partial_t v_h, (T^m_{a_{j,k}} - T^m_{a_{j,k,\varepsilon}}) \partial_{x_k} v_h)_{L^2,L^2} dt \\
\leq 2C \int_0^{\frac{T}{2}} \min\{\varepsilon^\alpha, t^{\alpha-1} \mu(\varepsilon)\} \sum_{j,k=1}^n \|\partial_{x_j} \partial_t v_h\|_{L^2} \|\partial_{x_k} v_h\|_{L^2} dt \\
\leq \frac{C}{N} \int_0^{\frac{T}{2}} \|\partial_t v_h\|_{L^2}^2 dt + CN 2^{4(h+1)} \int_0^{\frac{T}{2}} \min\{\varepsilon^\alpha, t^{\alpha-1} \mu(\varepsilon)\} \|v_h\|_{L^2}^2 dt,
\end{align*}
\]
where \( C \) depends only on \( n, m \) and \( \|a_{j,k}\|_{L^\infty} \) and \( N > 0 \) can be chosen arbitrarily.

Similarly
\[
\begin{align*}
-2 \Re \int_0^{\frac{T}{2}} \sum_{j,k=1}^n (\partial_{x_j} \partial_t v_h, T^m_{a_{j,k,\varepsilon}} \partial_{x_k} v_h)_{L^2,L^2} dt \\
= \int_0^{\frac{T}{2}} \sum_{j,k=1}^n (\partial_{x_j} v_h, T^m_{\partial_{a_{j,k,\varepsilon}}} \partial_{x_k} v_h)_{L^2,L^2} dt \\
+ \int_0^{\frac{T}{2}} \sum_{j,k=1}^n (\partial_{x_j} v_h, (T^m_{a_{j,k,\varepsilon}} - (T^m_{a_{j,k,\varepsilon}})^\ast) \partial_{x_k} \partial_t v_h)_{L^2,L^2} dt.
\end{align*}
\]
From (18) and (29) we have
\[
\left| \int_0^T \sum_{j,k=1}^n \langle \partial_{x_j} v_h, T_{a_{j,k}}^m \partial_{x_k} v_h \rangle_{L^2,L^2} dt \right| \\
\leq C 2^{2(h+1)} \int_0^T \min \{ \varepsilon^{\alpha-1}, t^{\alpha-1} \frac{\mu(\varepsilon)}{\varepsilon} \} \| v_h \|_{L^2}^2 dt,
\]
and, from (21),
\[
\left| \int_0^T \sum_{j,k=1}^n \langle \partial_{x_j} v_h, (T_{a_{j,k}}^m - (T_{a_{j,k}}^m)^*) \partial_{x_k} \partial_t v_h \rangle_{L^2,L^2} dt \right| \\
\leq C \int_0^T \| \nabla v_h \|_{L^2} \| \partial_t v_h \|_{L^2} dt \\
\leq \frac{CN}{N} \int_0^T \| \partial_t v_h \|_{L^2}^2 dt + CN 2^{2(h+1)} \int_0^T \| v_h \|_{L^2}^2 dt,
\]
where $C$ depends only on $n$, $m$ and $\| a_{j,k} \|_{\text{Lip}}$ and $N > 0$ can be chosen arbitrarily.

As a conclusion, from (27), we finally obtain
\[
\int_0^T \| \partial_t v_h + \sum_{j,k=1}^n \partial_{x_j}(T_{a_{j,k}}^m \partial_{x_k} v_h) + \Phi'_\gamma(\gamma(T-t)) v_h \|_{L^2(\mathbb{R}^2)}^2 dt \\
\geq \int_0^T \left( \| \sum_{j,k=1}^n \partial_{x_j}(T_{a_{j,k}}^m \partial_{x_k} v_h) + \Phi'_\gamma(\gamma(T-t)) v_h \|_{L^2}^2 + 2^{2(h+1)} \min \{ \varepsilon^{\alpha}, t^{\alpha-1} \frac{\mu(\varepsilon)}{\varepsilon} \} + 1 \right) \| v_h \|_{L^2}^2 dt.
\]

\[(30)\]

### 4.4 End of the proof

We start considering (30) for $h = 0$. We fix $\varepsilon = \frac{1}{2}$. Recalling (13) we have
\[
\int_0^T \| \partial_t v_H + \sum_{j,k=1}^n \partial_{x_j}(T_{a_{j,k}}^m \partial_{x_k} v_0) + \Phi'_\gamma(\gamma(T-t)) v_0 \|_{L^2(\mathbb{R}^2)}^2 dt \\
\geq \int_0^T \left( \varepsilon \Phi''(\varepsilon(T-t)) - C' \right) \| v_0 \|_{L^2}^2 \\
\geq \int_0^T \left( \gamma \left( \frac{T}{2} \right)^{\alpha-1} - C' \right) \| v_0 \|_{L^2}^2 dt.
\]

Choosing a suitable $\gamma_0$, we have that, for all $\gamma > \gamma_0$,
\[
\int_0^T \| \partial_t v_H + \sum_{j,k=1}^n \partial_{x_j}(T_{a_{j,k}}^m \partial_{x_k} v_0) + \Phi'_\gamma(\gamma(T-t)) v_0 \|_{L^2}^2 dt \geq \frac{\gamma}{2} \int_0^T \| v_0 \|_{L^2}^2 dt.
\]

\[(31)\]
We consider (30) for $h \geq 1$. We fix $\varepsilon = 2^{-2h}$. We have

$$\int_0^T \| \partial_t v_h + \sum_{j,k=1}^n \partial_x (T^m_{a,j,k} \partial_{x_k} v_h) + \Phi'_\gamma (\gamma(T - t)) v_h \|_{L^2(\mathbb{R}^2)}^2 dt$$

$$\geq \int_0^T \left( \| \sum_{j,k=1}^n \partial_x (T^m_{a,j,k} \partial_{x_k} v_h) + \Phi'_\gamma (\gamma(T - t)) v_h \|_{L^2}^2 - \Phi'_\gamma (\gamma(T - t)) \| v_h \|_{L^2}^2 \right) dt$$

$$+ \left( \gamma \Phi''_\gamma (\gamma(T - t)) - C(2^{4h} \min\{2^{-2h}, t^{\alpha-1} \mu(2^{-2h})\} + 2^{2h}) \| v_h \|_{L^2}^2 \right) dt$$

$$\geq \int_0^T \left( \| \sum_{j,k=1}^n \partial_x (T^m_{a,j,k} \partial_{x_k} v_h) \|_{L^2}^2 - \Phi'_\gamma (\gamma(T - t)) \| v_h \|_{L^2}^2 \right) dt$$

$$+ \left( \gamma \Phi''_\gamma (\gamma(T - t)) - C(2^{4h} \min\{2^{-2h}, t^{\alpha-1} \mu(2^{-2h})\} + 2^{2h}) \| v_h \|_{L^2}^2 \right) dt.$$

From (21) it is possible to deduce that

$$\| \sum_{j,k=1}^n \partial_x (T^m_{a,j,k} \partial_{x_k} v_h) \|_{L^2}^2 \geq \frac{\lambda_0}{8} 2^{2h} \| v_h \|_{L^2}^2. \quad (32)$$

Suppose first that

$$\Phi'_\gamma (\gamma(T - t)) \leq \frac{\lambda_0}{16} 2^{2h}.$$

From (24) we have

$$\| \sum_{j,k=1}^n \partial_x (T^m_{a,j,k} \partial_{x_k} v_h) \|_{L^2}^2 - \Phi'_\gamma (\gamma(T - t)) \| v_h \|_{L^2}^2 \geq \frac{\lambda_0}{16} 2^{2h} \| v_h \|_{L^2}^2$$

and then, using also (13), we obtain

$$\int_0^T \| \partial_t v_h + \sum_{j,k=1}^n \partial_x (T^m_{a,j,k} \partial_{x_k} v_h) + \Phi'_\gamma (\gamma(T - t)) v_h \|_{L^2(\mathbb{R}^2)}^2 dt$$

$$\geq \int_0^T \left( \| \sum_{j,k=1}^n \partial_x (T^m_{a,j,k} \partial_{x_k} v_h) \|_{L^2}^2 - \Phi'_\gamma (\gamma(T - t)) \| v_h \|_{L^2}^2 \right) dt$$

$$+ \left( \gamma \Phi''_\gamma (\gamma(T - t)) - C(2^{4h} \min\{2^{-2h}, t^{\alpha-1} \mu(2^{-2h})\} + 2^{2h}) \| v_h \|_{L^2}^2 \right) dt$$

$$\geq \int_0^T \left( \frac{\lambda_0}{16} 2^{2h} \| v_h \|_{L^2}^2 + \gamma \left( \frac{T}{2} \right)^{\alpha-1} - C(2^{(4-2\alpha)h}) \| v_h \|_{L^2}^2 \right) dt.$$

Then there exist $\gamma_0 > 0$ and $C > 0$ such that, for all $\gamma \geq \gamma_0$ and for all $h \geq 1$,

$$\int_0^T \| \partial_t v_h + \sum_{j,k=1}^n \partial_x (T^m_{a,j,k} \partial_{x_k} v_h) + \Phi'_\gamma (\gamma(T - t)) v_h \|_{L^2(\mathbb{R}^2)}^2 dt$$

$$\geq C \int_0^T (\gamma + \gamma^2 2^{2h}) \| v_h \|_{L^2}^2 dt \quad (33)$$
Suppose finally that
\[ \Phi'_\gamma(\gamma(T-t)) \geq \frac{\lambda_0}{16} 2^{2h}. \]

From (12), the fact that \( \lambda_0 \leq 1 \) and the properties of the modulus of continuity \( \mu \)
\[ \Phi''(\gamma(T-t)) = t^{a-1}(\Phi'_\gamma(\gamma(T-t)))^2 \mu(\frac{1}{\Phi'_\gamma(\gamma(T-t))}) \]
\[ \geq t^{a-1}(\frac{\lambda_0}{16}) 2^{2h} \mu\left(\frac{16}{\lambda_0} 2^{-2h}\right) \geq t^{a-1}(\frac{\lambda_0}{16}) 2^{2h} \mu(2^{-2h}). \]

and
\[ \Phi''(\gamma(T-t)) = t^{a-1}(\Phi'_\gamma(\gamma(T-t)))^2 \mu(\frac{1}{\Phi'_\gamma(\gamma(T-t))}) \]
\[ = t^{a-1} \Phi'_\gamma(\gamma(T-t)) \frac{\mu(\frac{1}{\Phi'_\gamma(\gamma(T-t))})}{\Phi'_\gamma(\gamma(T-t))} \geq \left(\frac{T}{2}\right)^{a-1}. \]

Consequently
\[ \int_0^T \|Dv_h + \sum_{j,k=1}^n \partial_{x_j}(T_{a,j,k} \partial_{x_k} v_h) + \Phi'_\gamma(\gamma(T-t))v_h\|^2_{L^2(\mathbb{R}^2)} dt \]
\[ \geq \int_0^T (\gamma \Phi''(\gamma(T-t))) - C(2^{4h} \min\{2^{-2h}, t^{a-1} \mu(2^{-2h})\} + 2^{2h}) \|v_h\|^2_{L^2} dt \]
\[ \geq \int_0^T \frac{\gamma}{2} (t^{a-1}(\frac{\lambda_0}{16}) 2^{2h} \mu(2^{-2h}) + \left(\frac{T}{2}\right)^{a-1}) - C (t^{a-1} 2^{4h} \mu(2^{-2h}) + 2^{2h}) \|v_h\|^2_{L^2} dt. \]

Then there exist \( \gamma_0 > 0 \) and \( C > 0 \) such that, for all \( \gamma \geq \gamma_0 \) and for all \( h \geq 1 \),
\[ \int_0^T \|Dv_h + \sum_{j,k=1}^n \partial_{x_j}(T_{a,j,k} \partial_{x_k} v_h) + \Phi'_\gamma(\gamma(T-t))v_h\|^2_{L^2(\mathbb{R}^2)} dt \]
\[ \geq C \gamma \int_0^T (1 + 2^{2h}) \|v_h\|^2_{L^2} dt. \] (34)

As a conclusion, form (31), (33) and (34), there exist \( \gamma_0 > 0 \) and \( C > 0 \) such that, for all \( \gamma \geq \gamma_0 \) and for all \( h \in \mathbb{N} \),
\[ \int_0^T \|Dv_h + \sum_{j,k=1}^n \partial_{x_j}(T_{a,j,k} \partial_{x_k} v_h) + \Phi'_\gamma(\gamma(T-t))v_h\|^2_{L^2(\mathbb{R}^2)} dt \]
\[ \geq C \int_0^T (\gamma + \gamma^2 2^{2h}) \|v_h\|^2_{L^2} dt \] (35)

and (26) follows. The proof is complete.
5 A counterexample

Theorem 3. There exists
\[ l \in \left( \bigcap_{\alpha \in [0,1]} C^{0,\alpha}(\mathbb{R}) \right) \cap C^\infty(\mathbb{R} \setminus \{0\}) \]
with
\[ \frac{1}{2} \leq l(t) \leq \frac{3}{2}, \quad \text{for all } t \in \mathbb{R}, \] (36)
\[ |l'(t)| \leq C_\varepsilon |t|^{-(1+\varepsilon)}, \quad \text{for all } \varepsilon > 0 \text{ and } t \in \{0\}, \] (37)
and there exist \( u, b_1, b_2, c \in C^\infty_b(\mathbb{R} \times \mathbb{R}^2_x) \), with
\[
\text{Supp } u = \{(t, x) \in \mathbb{R}_t \times \mathbb{R}^2_x \mid t \geq 0\},
\]
such that
\[
\partial_t u + \partial_{x_1}^2 u + l \partial_{x_2}^2 u + b_1 \partial_{x_1} u + b_2 \partial_{x_2} u + cu = 0 \quad \text{in } \mathbb{R}_t \times \mathbb{R}^2_x.
\]

Remark 4. Actually the function \( l \) will satisfy
\[
\left| l'(t) \right| < \infty.
\] (38)

From (38) it is easy to obtain (37).

Proof. We will follow the proof of Theorem 1 in [10] (see also Theorem 3 in [5]). Let \( A, B, C, J \) be four \( C^\infty \) functions, defined in \( \mathbb{R} \), with
\[ 0 \leq A(s), B(s), C(s) \leq 1 \quad \text{and} \quad -2 \leq J(s) \leq 2, \quad \text{for all } s \in \mathbb{R}, \]
and
\[ A(s) = 1, \quad \text{for } s \leq \frac{1}{5}, \] \[ A(s) = 0, \quad \text{for } s \geq \frac{1}{4}, \]
\[ B(s) = 0, \quad \text{for } s \leq 0 \text{ or } s \geq 1, \]
\[ B(s) = 1, \quad \text{for } \frac{1}{6} \leq s \leq \frac{1}{2}, \]
\[ C(s) = 0, \quad \text{for } s \leq \frac{1}{4}, \] \[ C(s) = 1, \quad \text{for } s \geq \frac{1}{3}, \]
\[ J(s) = -2, \quad \text{for } s \leq \frac{1}{6} \text{ or } s \geq \frac{1}{2}, \] \[ J(s) = 2, \quad \text{for } \frac{1}{6} \leq s \leq \frac{1}{3}. \]

Let \((a_n)_n\), \((z_n)_n\) be two real sequences such that
\[ -1 < a_n < a_{n+1}, \quad \text{for all } n \geq 1, \quad \text{and} \quad \lim_{n} a_n = 0, \] (39)
\[ 1 < z_n < z_{n+1}, \quad \text{for all } n \geq 1, \quad \text{and} \quad \lim_{n} z_n = +\infty. \] (40)

We define
\[ r_n = a_{n+1} - a_n, \]
\[ q_1 = 0 \quad \text{and} \quad q_n = \sum_{k=2}^{n} z_k r_{k-1}, \quad \text{for } n \geq 2, \]
\[ p_n = (z_{n+1} - z_n) r_n. \]
We require
\[ p_n > 1, \quad \text{for all} \quad n \geq 1. \] (41)

We set
\[ A_n(t) = A\left(\frac{t-a_n}{r_n}\right), \quad B_n(t) = B\left(\frac{t-a_n}{r_n}\right), \]
\[ C_n(t) = C\left(\frac{t-a_n}{r_n}\right), \quad J_n(t) = J\left(\frac{t-a_n}{r_n}\right). \]

We define
\[ v_n(t, x_1) = \exp(-q_n - z_n(t - a_n)) \cos \sqrt{z_n} x_1, \]
\[ w_n(t, x_2) = \exp(-q_n - z_n(t - a_n) + J_n(t)p_n) \cos \sqrt{z_n} x_2, \]
\[ u(t, x_1, x_2) \]
\[ = \begin{cases} 
 v_1(t, x_1), & \text{for } t \leq a_1, \\
 A_n(t)v_n(t, x_1) + B_n(t)w_n(t, x_2) + C_n(t)v_{n+1}(t, x_1), & \text{for } a_n \leq t \leq a_{n+1}, \\
 0, & \text{for } t \geq 0.
\end{cases} \]

The condition
\[ \lim_n \exp(-q_n + 2p_n)z_n^\alpha p_n^\beta r_n^{-\gamma} = 0, \quad \text{for all} \quad \alpha, \beta, \gamma > 0, \] (42)

implies that \( u \in C^\infty_b(\mathbb{R}_t \times \mathbb{R}^2_x) \).

We define
\[ l(t) = \begin{cases} 
 1, & \text{for } t \leq a_1 \text{ or } t \geq 0, \\
 1 + J_n'(t)p_n z_n^{-1}, & \text{for } a_n \leq t \leq a_{n+1}.
\end{cases} \]

\( l \) is a \( C^\infty(\mathbb{R} \setminus \{0\}) \) function. The condition
\[ \sup_n \left\{ p_n r_n^{-1} z_n^{-1} \right\} \leq \frac{1}{2 \| J' \|_{L^\infty}} \] (43)

implies (39), i. e. the operator
\[ L = \partial_t - \partial^2_{x_1} - l(t)\partial^2_{x_2} \]
is a parabolic operator. Moreover \( l \) is in \( \bigcap_{\alpha \in [0,1]} C^{0,\alpha}(\mathbb{R}) \) if
\[ \sup_n \left\{ p_n r_n^{-1-\alpha} z_n^{-1} \right\} < +\infty, \quad \text{for all} \quad \alpha \in [0,1]. \] (44)

Finally, we define
\[ b_1 = -\frac{Lu}{u^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2} \partial_{x_1} u, \]
\[ b_2 = -\frac{Lu}{u^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2} \partial_{x_2} u, \]
\[ c = -\frac{Lu}{u^2 + (\partial_{x_1} u)^2 + (\partial_{x_2} u)^2} u. \]
As in [10] and [5], the functions $b_1, b_2, c$ are in $C^\infty_b(\mathbb{R}_t \times \mathbb{R}^2_\gamma)$ if
\[
\lim_n \exp(-p_n)z_n^\alpha + p_n^\beta r_n^{-\gamma} = 0, \quad \text{for all } \alpha, \beta, \gamma > 0.
\] (45)

We choose, for $j_0 \geq 2$,
\[
a_n = -e^{-\sqrt{\log(n+j_0)}}, \quad z_n = (n+j_0)^3.
\]

With this choice (39) and (40) are satisfied and we have
\[
r_n \sim e^{-\sqrt{\log(n+j_0)}} \frac{1}{(n+j_0)\sqrt{\log(n+j_0)}},
\]
where, for sequences $(f_n)_n, (g_n)_n$, $f_n \sim g_n$ means $\lim_n \frac{f_n}{g_n} = \lambda$, for some $\lambda > 0$. Similarly
\[
p_n \sim e^{-\sqrt{\log(n+j_0)}} \frac{n+j_0}{\sqrt{\log(n+j_0)}}
\]
and condition (41) is verified, for a suitable fixed $j_0$. Remarking that we have, for $j_0$ suitably large,
\[
q_n = \sum_{k=2}^{n} z_k r_{k-1} \geq z_n r_{n-1} \geq \lambda (n+j_0)^{\frac{5}{2}}
\]
and
\[
p_n \leq \lambda (n+j_0)^{\frac{5}{2}}
\]
for some $\lambda > 0$. Finally
\[
p_n r_n^{-1} z_n^{-1} \sim \frac{1}{n+j_0}.
\]

As a consequence (42), (43), (44) and (45) are satisfied for a suitable fixed $j_0$. It remains to check (38). We have
\[
|t'(t)| \leq \|J''\|_{L^\infty} p_n r_n^{-2} z_n^{-1}, \quad \text{for } a_n \leq t \leq a_{n+1}
\]
and consequently
\[
\sup_{t \neq 0} \left( \frac{|t|}{1 + |\log |t||} \right) |t'(t)| = \sup_n \sup_{t \in [a_n, a_{n+1}]} \left( \frac{|t|}{1 + |\log |t||} \right) |t'(t)| \leq \sup_n \left( \frac{a_n}{1 - \log a_n} \right) \|J''\|_{L^\infty} p_n r_n^{-2} z_n^{-1} \leq C.
\]

The conclusion of the theorem is reached simply exchanging $t$ with $-t$. \square
References

[1] Bahouri, Hajer; Chemin, Jean-Yves; Danchin, Raphaël “Fourier analysis and nonlinear partial differential equations”. Grundlehren der Mathematischen Wissenschaften, 343. Springer, Heidelberg, 2011.

[2] Bony, Jean-Michel *Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires*. Ann. Sci. École Norm. Sup. (4) 14 (1981), no. 2, 209–246.

[3] Chemin, Jean-Yves “Fluides parfaits incompressibles”. Astérisque, 230. Société Mathématique de France, Paris, 1995.

[4] Colombini, Ferruccio; Métivier, Guy *The Cauchy problem for wave equations with non Lipschitz coefficients; application to continuation of solutions of some nonlinear wave equations*. Ann. Sci. Éc. Norm. Supér. (4) 41 (2008), no. 2, 177–220.

[5] Del Santo, Daniele; Prizzi, Martino *Backward uniqueness for parabolic operators whose coefficients are non-Lipschitz continuous in time*. J. Math. Pures Appl. (9) 84 (2005), no. 4, 471–491.

[6] Del Santo, Daniele; Prizzi, Martino *A new result on backward uniqueness for parabolic operators*. Ann. Mat. Pura Appl. (4) 194 (2015), no. 2, 387–403.

[7] Del Santo, Daniele; Jäh, Christian *Non-uniqueness and uniqueness in the Cauchy problem of elliptic and backward-parabolic equations*, in "Progress in Partial Differential Equations - Asymptotic Profiles, Regularity and Well-Posedness", M. Ruzhansky, M. Reissig eds., Springer Proceedings in Mathematics and Statistics 44, Springer International Publishing, Basel 2013, pp. 27–52.

[8] Gérard, Patrick; Rauch, Jeffrey *Propagation de la régularité locale de solutions d’équations hyperboliques non linéaires*. (French) [Propagation of the local regularity of solutions of nonlinear hyperbolic equations] Ann. Inst. Fourier (Grenoble) 37 (1987), no. 3, 65–84.

[9] Métivier, Guy “Para-differential calculus and applications to the Cauchy problem for nonlinear systems”. Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series, 5. Edizioni della Normale, Pisa, 2008.

[10] Pliś, Andrzej *On non-uniqueness in Cauchy problem for an elliptic second order differential equation*. Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 11 (1963), 95–100.