Strongly coupled $\mathcal{N} = 4$ super Yang-Mills plasma on the Coulomb branch II: Transport coefficients and hard probe parameters

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We study $\mathcal{N} = 4$ super Yang-Mills theory on the Coulomb branch (cSYM) by using its Type IIB supergravity dual. We compute the transport coefficients, and hard probe parameters of $\mathcal{N} = 4$ cSYM at finite temperature $T$. We use the rotating black 3-brane solution of Type IIB supergravity with a single non-zero rotation parameter $r_0$ after analytically continuing $r_0 \rightarrow -i r_0$, and in an ensemble where the Hawking temperature $T$ and a scalar condensate $\langle \mathcal{O} \rangle > r_0^2$ are held fixed. We find that the bulk viscosity to entropy density ratio of the large black hole branch decreases with temperature and has a maxima around the critical temperature $T_c$, while, for the small black hole branch, it increases with temperature. The other transport coefficients and parameters of hard probes, such as the conductivity, jet quenching parameter, drag force, and momentum diffusion coefficients of the large black hole branch increase with temperature and asymptote to their conformal value while, for the small black hole branch, they decrease with temperature.

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I. INTRODUCTION

The AdS/CFT correspondence \cite{1–3} is an important tool to compute the hydrodynamic transport coefficients and hard probe parameters of a strongly coupled plasma \cite{14,15,16}.

In this paper, we use the AdS/ CFT correspondence to study a strongly coupled $\mathcal{N} = 4$ super Yang-Mills plasma on the Coulomb branch. In this branch, a scale $\Lambda$ is generated dynamically through the Higgs mechanism where the scalar particles $\Phi_i$ (i=1...6) of $\mathcal{N} = 4$ SYM acquire a non-zero vacuum expectation value (VEV) that breaks the conformal symmetry, and the gauge symmetry SU($N_c$) to its subgroup $U(1)^{N_c-1}$ but preserves $\mathcal{N} = 4$ supersymmetry and the gauge coupling is not renormalised \cite{14}.

The thermodynamics of $\mathcal{N} = 4$ super Yang-Mills on the Coulomb (cSYM) is investigated in some detail in \cite{15}. In this paper, we will study its hydrodynamic transport coefficients and hard probes by using its dual geometry given by a rotating black 3-brane solution of Type IIB supergravity with a single non-zero rotation parameter $r_0$ \cite{14,15,16,18,23,24}. after analytically continuing $r_0 \rightarrow -i r_0$, and in an ensemble where the Hawking temperature $T$ and a scalar condensate $\langle \mathcal{O} \rangle > r_0^2$ are held fixed \cite{15}.

So far, the computations of the transport coefficients of the rotating black 3-brane have been limited to the grand canonical ensemble (where temperature $T$ and angular velocity $\Omega$ or chemical potential $\mu$ are held fixed), and canonical ensemble (where temperature $T$ and angular momentum density $J$ or charge density $\rho =$ $-J^0$ are held fixed), see \cite{23,26}. In \cite{20,21}, it was shown that for planar rotating black 3-branes the two ensembles have different thermodynamics, for example, there is Hawking-Page transition in the canonical ensemble but not in the grand canonical ensemble.

The outline of this paper is as follows: In section II we write down the 5-dimensional Type IIB supergravity action and its rotating black 3-brane or R-charged black hole solution. In section III, we compute the hydrodynamic transport coefficients, such as shear viscosity, bulk viscosity and conductivity of the rotating black 3-brane solution dual to $\mathcal{N} = 4$ super Yang-Mills on the Coulomb branch (cSYM) at strong coupling.

In section IV, we calculate the the drag force, momentum diffusion coefficient, and jet quenching parameter on the rotating black 3-brane solution.

II. TYPE IIB SUPERGRAVITY ACTION AND BACKGROUND SOLUTION

The action for the $U(1)^3$ consistent truncation of Type IIB supergravity on $S^5$ is given by \cite{27,28,29}, see also \cite{30}

$$S = \frac{1}{16 \pi G_5} \int d^5x \sqrt{-g_5} L_{bulk} \quad (\text{II.1})$$

where

$$L_{bulk} = (R - V) - \frac{1}{2} \sum_{I=1}^{2} (\partial \varphi_I)^2 - \frac{1}{4} R^2 \sum_{a=1}^{3} X^{-2}_{\alpha} (F^a)^2, \quad F^a_{\mu\nu} = \partial_{\mu} A^a_{\nu} - \partial_{\nu} A^a_{\mu}, \quad V = - \frac{4}{R^2} \sum_{a=1}^{3} X^{-1}_{\alpha}, \quad \frac{X_1}{e} = \frac{1}{\sqrt{r^1}}, \quad \frac{X_2}{e} = \frac{1}{\sqrt{r^1} + \frac{1}{\sqrt{r^2}}}, \quad \frac{X_3}{e} = \frac{1}{\sqrt{r^1} + \frac{1}{\sqrt{r^2}}} \quad (\text{II.2})$$

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We have dropped the Chern-Simons term from the action \([\text{II.1]}\) since it does not play any role in our discussion below.

In this paper, we compute the hydrodynamic and hard probe transport coefficients the following rotating black 3-brane solution of the above action \([\text{II.1]}\)\,[22, 23]

\[
ds^2_{(5)} = \frac{r^2}{R^2} H^{1/3} \left( -f \, dt^2 + dx^2 + dy^2 + dz^2 \right) + \frac{H^{-2/3}}{r^2} \, f \, dr^2,
\]

where

\[
f = 1 - \frac{r_h^4 \, H(r_h)}{r^4} \,, \quad H = 1 - \frac{r_0^2}{r^2} \,, \quad \varphi_1 = \frac{1}{\sqrt{6}} \ln H \,, \quad \varphi_2 = \frac{1}{\sqrt{2}} \ln H \,,
\]

\[
A^1 = i \, \frac{r_0 \, r_h^2 \sqrt{H(r_h)}}{R^2} \, H(r) \,, \quad r_h^2 = \frac{1}{2} (r_0^2 + \sqrt{r_0^4 + 4m}) \,, \quad \kappa = \frac{r_h^2}{r_h^4} \quad \text{and} \quad A^2 = A^3 = 0.
\]

Our metric \([\text{II.3]}\) is equivalent to the metric used in \([25]\) after analytically continuing \(r_0 \to -i\sqrt{q} \). Note that having an imaginary gauge potential doesn’t lead to any inconsistencies in the 5-dimensional bulk spacetime, since all physical quantities in the bulk are given in terms of \((\partial_r A^1)_z^2\). In the field theory side, having a finite imaginary chemical potential \(\mu\), means that we are exploring the phase diagram of \(N = 4\) cSYM at finite temperature \(T\) and imaginary chemical potential \(\mu\), see \([32, 41]\) for the study of the QCD phase diagram on the lattice at finite imaginary chemical potential without any inconsistencies.

The Hawking temperature \(T\) of the rotating black 3-brane solution \([\text{II.3]}\) is given by

\[
\frac{T}{\Lambda} = \frac{1 - \frac{1}{2} \kappa}{\sqrt{\kappa - \kappa^2}} \,, \quad \text{(II.6)}
\]

where \(T_0 = \frac{r_h}{\pi R^2}, \Lambda = \frac{r_0}{\pi R^2}, \text{ and } \kappa = \frac{r_h^2}{r_h^4} = \frac{\Lambda^2}{T_r^2}\). We have plotted \(\frac{T}{\Lambda}\) in Fig. 1. We can also invert \((\text{II.6)}\) to find

\[
\kappa = \frac{1 + \frac{T_r^2}{\Lambda^2} \left( 1 + \sqrt{\frac{T_r^2}{\Lambda^2} - 2} \right)}{\frac{1}{2} + \frac{T_r^2}{\Lambda^2}} \,, \quad \text{(II.7)}
\]

and

\[
\frac{T_r^2}{T_r^2} = \frac{2 + \frac{1}{2} \frac{\Lambda^2}{T_r^2} \left( 1 + \sqrt{\frac{T_r^2}{\Lambda^2} - 2} \right)}{1 + \frac{T_r^2}{\Lambda^2} \left( 1 + \sqrt{\frac{T_r^2}{\Lambda^2} - 2} \right)} \,. \quad \text{(II.8)}
\]

Note that in \((\text{II.7)}\) and \((\text{II.8)}\) “−” corresponds to large black hole branch and “+” corresponds to small black hole branch.

The entropy density \(s(T, \Lambda)\), for our ensemble where \(T\) and \(\Lambda\) are held fixed, is given by

\[
s(T, \Lambda) = \frac{A_H}{4 G_5 V_3} = \frac{1}{4 G_5} \sqrt{g_{xx}(r_h) g_{yy}(r_h) g_{zz}(r_h)} \text{ vs. the radius of the horizon } \frac{\rho_h}{\rho_0} \text{, normalized by the energy scale } \Lambda \text{ and rotation parameter } r_0, \text{ respectively.}
\]

\[
\text{FIG. 1: Hawking temperature } \frac{T}{\Lambda} \text{ vs. the radius of the horizon } \frac{\rho_h}{\rho_0} \text{, normalized by the energy scale } \Lambda \text{ and rotation parameter } r_0, \text{ respectively.}
\]

\[
\text{III. HYDRODYNAMIC TRANSPORT COEFFICIENTS OF } N = 4 \text{ CSYM PLASMA}
\]

The transverse metric fluctuation \(h_{xy}(t, z, r)\) decouples from other fluctuations, hence the shear viscosity for a general background metric \(g_{\mu\nu}\) is given by \([32]\)

\[
\eta = \frac{1}{16 \pi G_5} \sqrt{g_{xx}(r_h) g_{yy}(r_h) g_{zz}(r_h)} \frac{g_{xx}(r_h)}{g_{yy}(r_h)} \text{ vs. the radius of the horizon } \frac{\rho_h}{\rho_0} \text{, normalized by the energy scale } \Lambda \text{ and rotation parameter } r_0, \text{ respectively.}
\]

\[
\text{Since, for our background metric } g_{\mu\nu} \text{, } g_{xx} = g_{yy}, \text{ the shear viscosity } \eta \text{ of } N = 4 \text{ cSYM is simply}
\]

\[
\eta = \frac{1}{4 \pi} s \,. \quad \text{(III.11)}
\]

Bulk viscosity \(\zeta\) can be computed by closely following \([12]\). To this end, we first replace \(\varphi_1 \to \frac{1}{2} \varphi_1\) followed by \(\varphi_2 \to \frac{\sqrt{3}}{2} \varphi_1\), to bring the Einstein-Maxwell-scalar part of our action \([\text{II.1]}\) in the same form as the action used in \([12]\), i.e.,

\[
(16 \pi G_5) \sqrt{g_0} \frac{\mathcal{L}}{\sqrt{-g_0}} = (\mathcal{R} - \mathcal{V}(\varphi_1)) - \frac{1}{2} \left( \partial \varphi_1 \right)^2 + ..., \quad \text{(III.12)}
\]
where

\[ \tilde{V}(\tilde{\varphi}_1) = -\frac{4}{R^2} \left( e^{\frac{1}{2}\tilde{\varphi}_1} \left( 1 + \frac{\kappa(1 - \kappa)}{2\kappa^3} (e^{-\frac{3}{2}\tilde{\varphi}_1} - 1)^3 \right) + 2e^{-\frac{1}{2}\tilde{\varphi}_1} \right). \]  

(III.13)

In the \( \tilde{r} = \varphi_1(r) \) gauge, the bulk viscosity \( \zeta \) up to a constant is \[ \frac{\zeta}{\kappa} \propto \frac{1}{4\pi} \overbrace{\tilde{V}'(\tilde{r}_h)}^{(1)} \tilde{r}_h, \]  

(III.14)

where \( ' \) denotes the derivative with respect to \( \tilde{r} = \varphi_1(r) \). Note that, in the gauge \( \tilde{r} = \varphi_1(r) = \frac{2}{\sqrt{6}} \ln H(r) \), the horizon of the black hole is located at \( \tilde{r} = \tilde{r}_h = \frac{2}{\sqrt{6}} \ln H(r_h) = \frac{\kappa}{\sqrt{6}} \ln(1 - \kappa) \) where \( \kappa \) is still given by (II.7). We have plotted \( \zeta \) in Fig. 2.

The conductivity \( \sigma_f \) of a U(1) flavor charge can simply be computed using the general formula [5, 33]

\[ \sigma_f = \frac{1}{g_5^2} \int g_{xx}(r_h)g_{yy}(r_h)g_{zz}(r_h)g^{xx}(r_h) \]  

\[ = \frac{N_c N_f T_0}{4\pi \kappa(1 - \kappa)^{1/6}}, \]  

(III.15)

where \( g_5^2 = \frac{4\pi^2 R}{N_c N_f} \) and a bulk U(1) flavor action of the form \( S_f = -\frac{1}{4g_5^2} \int d^5x \sqrt{-g} F^2 \) which can be derived from the low-energy limit of the Dirac-Born-Infeld action of probe \( N_f \) D7-branes [34]. Note that there is no mixing between the gravitational and flavor gauge field fluctuations.

We have plotted \( \sigma_f \) in Fig. 3.

The conductivity \( \sigma_R \) of a single R-charge can be computed by directly computing the two-point retarded correlation functions \( G^{tt} \) of the spatial component of the R-current \( J^t \), in the presence background \( A^t \) which results in mixing between the gravitational and gauge field fluctuations, and using Kubo formula, i.e.,

\[ \sigma_R = \lim_{\omega \to 0} -\frac{1}{\omega} \text{Im} G^{xx}(\omega, k = 0) = \frac{N_c^2 T_0 (2 - \kappa)^2}{32\pi \sqrt{1 - \kappa}}, \]  

(III.16)

where in the last line we used \( G^{xx} = -\frac{1}{4\pi} (2 - \kappa)^2 N_f T_0 \) which is nothing but Eq.4.34 of [25] after replacing \( \kappa \to -\kappa \), and \( G^{xx} \to \frac{1}{2} G^{xx} \) to compensate for the different normalisation we have for the gauge fields. We have plotted \( \sigma_R \) in Fig. 3.

IV. DRAG FORCE, MOMENTUM DIFFUSION AND JET QUENCHING IN \( \mathcal{N} = 4 \) CSYM PLASMA

The Nambu-Goto (NG) action is

\[ S_{NG} = \int d\tau d\sigma L(h_{ab}) = -\frac{1}{2\pi \alpha'} \int d\tau d\sigma \sqrt{-\text{det} h_{ab}}, \]  

(IV.17)

where the background induced metric on the string \( h_{ab} \) is given by

\[ h_{ab} = g_{\mu\nu} \partial_\mu x^\mu(\tau, \sigma) \partial_\nu x^\nu(\tau, \sigma). \]  

(IV.18)

Using the embedding \( (\tau, \sigma) \Rightarrow (t(\tau, \sigma), 0, 0, x(\tau, \sigma), r = \sigma) \), the background induced metric \( h_{ab}(\tilde{z}, \tilde{z}') \) [IV.18] becomes \( (\equiv d/d\tau, ' \equiv d/d\sigma) \)

\[ h_{ab}(\tilde{x}, \tilde{x}') = g_{\mu\nu} \partial_\mu \tilde{x}(\tau) \partial_\nu \partial_\mu \tilde{x}'(\tau, \sigma). \]  

(IV.19)

Using a particular Ansatz of the form \( t(\tau, \sigma) = \tau + K(\sigma) \) and \( z = v \tau + F(\sigma) \), which represents a “trailing string” configuration moving with velocity \( v \), the background in-

FIG. 2: The bulk viscosity to entropy density ratio \( \zeta/N = 4 \) SYM plasma on the Coulomb branch for both large and small black holes (III.14).

FIG. 3: The conductivity \( \frac{\sigma_f}{T N c N f} \) of a U(1) flavor charge (III.15), and \( \frac{\sigma_R}{T N c} \) of a single R-charge (III.16) of flavored and unfavored \( N = 4 \) SYM plasma, respectively, on the Coulomb branch for both large and small black holes.
duced metric \[ \text{(IV.19)} \] becomes \[ \text{(IV.20)} \]

\[
\begin{align*}
    h_{\tau\tau}(v, x') &= g_{tt} + v^2 g_{xx}, \\
    h_{\sigma\sigma}(v, x') &= g_{tt}(K')^2 + g_{xx}(x')^2 + g_{rr}, \\
    h_{\tau\sigma}(v, x') &= g_{tt}K' + g_{xx}x'v.
\end{align*}
\]

Finding the equation of motion from the action, we have

\[ \partial_\tau \left( \frac{g_{tt}g_{xx}(x' - vK')}{\sqrt{-det h_{\alpha\beta}}} \right) = 0. \] \[ \text{(IV.21)} \]

Requiring \[ h_{\tau\tau}(v, x') = 0 \] to fix this gauge freedom, we have an additional constraint \[ K' = -\frac{g_{xx}}{g_{tt}}x'v, \] which can be used to diagonalize \[ \text{(IV.20)} \] as \[ \text{(IV.22)} \]

\[
\begin{align*}
    h_{\tau\tau}(v, x') &= g_{tt} \left( 1 + v^2 \frac{g_{xx}}{g_{tt}} \right), \\
    h_{\sigma\sigma}(v, x') &= \left( 1 + v^2 \frac{g_{xx}}{g_{tt}} \right) g_{xx}(x')^2 + g_{rr}. \end{align*}
\]

Solving the equation of motion, in this gauge, for \( x' \), we find

\[ (x')^2 = -C^2 g_{tt} \frac{1}{g_{xx}g_{tt}} \frac{1}{1 + v^2 \frac{g_{xx}}{g_{tt}}} (1 + C^2 \frac{g_{tt}}{g_{xx}g_{tt}}) \] \[ \text{(IV.23)} \]

where the integration constant \( C \) is related to the conjugate momenta \( \Pi = \frac{\partial C}{\partial x'} = -C^2 \frac{g_{tt}}{g_{xx}g_{tt}}. \) Since the factor \( 1 + v^2 \frac{g_{xx}}{g_{tt}} \) in \[ \text{(IV.23)} \], for \( v \neq 0 \), vanishes when \( -\frac{g_{tt}(r_s)}{g_{xx}(r_s)} = v^2 \), requiring \( \frac{g_{xx}}{g_{tt}} \) to be positive across \( r = r_s \), the other factor \( 1 + C^2 \frac{g_{tt}}{g_{xx}g_{tt}} \) has to vanish at \( r = r_s \) as well, which will fix the integration constant \( C^2 = -g_{tt}(r_s)g_{xx}(r_s) \) for \( v \neq 0 \).

So, the induced metric \[ \text{(IV.22)} \] for \( v \neq 0 \) becomes

\[
\begin{align*}
    h_{\tau\tau}(v, x') &= g_{tt} \left( 1 - \frac{g_{tt}(r_s)}{g_{xx}(r_s)} \right), \\
    h_{\sigma\sigma}(v, x') &= g_{rr} \left( 1 - \frac{g_{xx}(r_s)g_{tt}(r_s)}{g_{xx}(r_s)g_{tt}(r_s)} \right). \end{align*}
\]

which can be interpreted as a metric of a 2-dimensional black hole with a line element \( ds_{(2)}^2 \) given by

\[ ds_{(2)}^2 = h_{\tau\tau}d\tau^2 + h_{\sigma\sigma}d\sigma^2 = -g_{tt}(-\tilde{f}(r))d\tau^2 + \frac{1}{\tilde{p}(r)}d\sigma^2, \] \[ \text{(IV.25)} \]

where \( \tilde{f}(r) = 1 - \frac{g_{tt}(r_s)}{g_{xx}(r_s)}g_{xx}(r), \tilde{p}(r) = g^{\tau\tau}(1 - \frac{g_{xx}(r)}{g_{xx}(r_s)}g_{tt}(r_s)). \) The radius of the horizon \( r_s \) of the 2-dimensional black hole is found by solving the algebraic equation \( -\frac{g_{tt}(r_s)}{g_{xx}(r_s)} = v^2 \). And, the Hawking temperature of the 2-dimensional black hole denoted as \( T_s \) is

\[ T_s = \frac{1}{4\pi} \sqrt{-g_{tt}(r_s)\tilde{f}(r_s)\tilde{p}'(r_s)}. \] \[ \text{(IV.26)} \]

The drag force is given by \[ \text{(IV.7)} \], see also \[ \text{(III.5)} \],

\[ F_{drag} = -C \frac{g_{xx}(r_s)}{2\pi} \left( 1 + \frac{4\gamma^2}{\pi^2} \right), \]

and the longitudinal momentum diffusion constant per unit time \( \kappa^\parallel(v) \) is \[ \text{(IV.28)} \]

\[ \kappa^\parallel(v) = \frac{T_s}{\pi} \left( \frac{g_{tt}g_{xx}}{g_{xx}} \right) \mid_{r=r_s}, \] \[ \text{(IV.29)} \]

We have plotted \( \kappa^\perp(0) \) and \( \kappa^\parallel(0) \) in Fig. 5. Note from Fig. 5 that \( \kappa^\perp(v) \neq \kappa^\parallel(v) \) even at \( v = 0 \) in \( N = 4 \) cSYM plasma, even though they are equal to each other at \( v = 0 \) in \( N = 4 \) SYM plasma. Also note that, as can be seen in Fig. 5, the difference between \( \kappa^\perp(v) \) and \( \kappa^\parallel(v) \) gets enhanced with increasing \( T \) and \( v \).

The 5-dimensional metric \[ \text{(III.3)} \] can be uplifted to the full 10-dimensional metric as \[ \text{(III.10)} \]

\[
\begin{align*}
    ds_{(10)}^2 &= \frac{r^2}{R^6} \tilde{H}^{1/2} \left( \tilde{f} dt^2 + dx^2 + dy^2 + dz^2 + d\sigma^2 \right) + R^2 \left( \tilde{H}^{-1/2} d\sigma^2 + \tilde{H}^{-1/2} d\omega^2 \right) + \tilde{H}^{-1} \cos^2 \theta d\Omega^2 + 2A^1 H \tilde{H}^{-1/2} R^2 \sin^2 \theta d\phi \right. \\
    &\left. + \frac{r^2}{R^6} \tilde{H}^{-1} \right) \mid_{r=r_s}, \] \[ \text{(IV.30)} \]
and making the analytic continuation $r \rightarrow -r$ of the conformal $N = 4$ SYM plasma \[36\]. We have plotted (IV.32) in Fig. 4.

$f$ and $H$ are the same as in (11.3). Our 10-dimensional metric (IV.30) is equivalent to Eq.2.21 of \[24\] after analytically continuing the rotation parameter $r_0 \rightarrow -ir_0$, and re-writing (IV.30) in terms of $\mu \equiv m^{1/4}$. Note that the $g_{t\phi}$ component of (IV.30) is imaginary and one could make it real by analytically continuing $t \rightarrow -it$ as in \[16\] \[19\]. However, since we are interested in real-time dynamics, such as computation of transport coefficients, we refrain from analytically continuing $t \rightarrow -it$, and we treat our 10-dimensional metric (IV.30) as a complex saddle point. Also note that $g_{t\phi} = A_t^* = 0$ in the extremal limit $r_h = r_0$.

In \[23\] the drag force was studied using the 10-dimensional metric (IV.30), and it was shown that the drag force $F_{\text{drag}(10)}$ is (shown below after re-writing it in terms of $\kappa$, and making the analytic continuation $r_0 \rightarrow -ir_0$ which is equivalent to replacing $\kappa \rightarrow -\kappa$)

$$\dot{H} = \sin^2 \theta + H \cos^2 \theta, \quad \text{and} \quad \hat{f} = 1 - \frac{r_h^4}{r^4} \frac{H(r_h)}{H(r)}, \quad (IV.31)$$

$$F_{\text{drag}(10)} = -\frac{1}{2} \sqrt{\lambda \pi T_0^3 \sqrt{1-\kappa} \gamma v}. \quad (IV.32)$$

Note that (IV.32) is equivalent to the $\gamma \rightarrow \infty$ limit of (IV.27), and has similar $\sqrt{1-\kappa}$ dependence as the entropy density (II.9) indicating that the drag force (IV.32) could be the measure of the color degrees of freedom of the plasma \[36\]. We have plotted (IV.32) in Fig. 4.

And, in \[24\], it was shown that the jet quenching parameter $\hat{q}$, studied using the 10-dimensional metric (IV.30), is (shown below after re-writing it in terms of $\kappa$, and making the analytic continuation $r_0 \rightarrow -ir_0$ which is equivalent to replacing $\kappa \rightarrow -\kappa$)

$$\frac{\hat{q}}{q_0} = \frac{K(1/\sqrt{2})}{K(n)} (2n^2)^2 (2n^2)^{1/2}, \quad (IV.33)$$

where $K(n)$ is the complete elliptic integral of the first kind, $n^2 = 1-\kappa$, $n' = \sqrt{1-n^2}$, and $q_0 = \pi^{3/4}T^{(3/4)} \sqrt{\lambda \pi T^3}$ \[10\] \[19\]. In Mathemtica, the complete elliptic integral of the first kind is implemented using EllipticK[n^2] \[10\]. We have plotted $\hat{q}$ in Fig. 6.

V. CONCLUSION

We have studied the transport coefficients of the non-extremal rotating black 3-brane dual to strongly coupled $N = 4$ cSYM plasma, such as bulk viscosity to entropy density ratio $\frac{\zeta}{s}$ \[III.14\], and conductivity $\sigma$ \[III.15\] \[III.16\], see Fig. 2 and Fig. 3, respectively. We have found that the bulk viscosity of the large black hole has a maxima around $T_c$, and its conductivity $\sigma$ asymptotes to its conformal value starting from below it.

We have also computed the hard probe parameters of $N = 4$ cSYM plasma. We have shown that the drag force $F_{\text{drag}}$, momentum diffusion coefficient $\kappa$, and jet quenching parameter $\hat{q}$ increase with temperature for the large black hole, see Fig. 4, Fig. 5 and Fig. 6.

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