Unconventional geometric quantum phase gates with a cavity QED system

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We propose scheme for realizing two-qubit quantum phase gates via unconventional geometric phase shift with atoms in a cavity. In the scheme the atoms interact simultaneously with a highly detuned cavity mode and a classical field. The atoms undergo no transitions during the gate operation, while the cavity mode is displaced along a circle in the phase space, acquiring a geometric phase conditional upon the atomic state. Under certain conditions, the atoms are disentangled with the cavity mode and thus the gate is insensitive to both the atomic spontaneous emission and the cavity decay.

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I. INTRODUCTION

Recently, much attention has been paid to the quantum computers, which are based on the fundamental quantum mechanical principle. The new type of machines can solve some problems exponentially faster than the classical computers [1]. Recent advances in quantum optics have provided powerful tools for quantum information processing. In cavity QED, schemes have been proposed for realizing two-qubit quantum logic gates via both the resonant and dispersive interactions of the atoms with a cavity mode [2]. A quantum phase gate between a cavity mode and an atom has been demonstrated using resonant interaction [3]. On the other hand, a scheme has been proposed for the realization of two-atom entangled states and quantum logic gates within a nonresonant microwave cavity [4]. The scheme does not require the cavity mode to act as the memory and the atoms are coupled via the virtual excitation of the cavity mode. Following the scheme, an experiment has been reported, in which two Rydberg atoms crossing a nonresonant cavity are entangled by coherent energy exchange [5].

The above mentioned schemes are based on dynamic evolution. On the other hand, geometric operation is a promising approach for the implementation of built-in fault-tolerant quantum phase gates. Compared with the dynamic gates, the geometric gates may offer practical advantages since the phase is determined only by the path area, insensitive to the starting state distributions, the path shape, and the passage rate to traverse the close path [6]. Thus, geometric phases may be robust against dephasing [7] and the fidelity of the geometric gates might be significantly higher than that of the dynamical ones, as demonstrated in a recent experiment in the context of trapped ions [8]. There are two approaches to obtain the geometric operations: 1. driving the qubits to undergo appropriate adiabatic cyclic evolutions; 2. displacing a harmonic oscillator along a closed path conditional on the state of the qubits. A gate obtained via the first approach is referred to as a conventional geometric gate, while that obtained via the second one is referred to as a unconventional geometric gate [6]. Schemes have been proposed to construct conventional geometric gates using NMR [9], superconducting nanocircuits [10], and trapped ions [11]. In comparison with the conventional geometric gates, unconventional geometric gates does not require additional operations to cancel dynamical phases and thus simplify experimental operations. The idea of implementing an unconventional geometric gate has been proposed [8,12] and realized [8] in a trapped ion system. However, such gates have not been proposed using other systems.

In this paper we propose a scheme for realizing an unconventional geometric gate in cavity QED. In the scheme the cavity mode is displaced along a closed path depending on the atomic states. By this way the system acquires a phase conditional on the atomic state, producing a phase gate. The atomic spontaneous emission is suppressed since the atoms undergo no transitions during the gate operation. Under certain conditions, the displacement trajectory is a very small circle and thus the cavity mode is disentangled from the atomic system throughout the operation. In this case the gate is insensitive to the cavity decay. As far as we know, it is the first scheme for the implementation of unconventional geometric gates in cavity QED.

This paper is organized as follows. In section 2, we briefly review the unconventional geometric phase caused by the displacement along a closed path in phase space. In section 3, we study the evolution for two identical three-level atoms dispersively interacting with a quantized cavity mode and a classical field. Section 4 is devoted to the unconventional geometric phase gates with the cavity QED system. Conclusions appear in section 5.

II. UNCONVENTIONAL GEOMETRIC PHASE

We first give a brief review of the geometric phase shift due to displacement along an arbitrary path [13,14]. The
displacement operator is given by
\[ D(\alpha) = e^{i\alpha a^+ - \alpha^* a}, \]  
(1)
where \( a^+ \) and \( a \) are the creation and annihilation operators of the harmonic oscillator, respectively. The displacement operators satisfy
\[ D(\beta)D(\alpha) = e^{im(\beta a^+ - \alpha^* a)}D(\alpha + \beta). \]  
(2)
Consider a path consists of \( N \) short straight sections \( \Delta \alpha_j \). Then the total operation is
\[ D_t = D(\Delta \alpha_N) \ldots D(\Delta \alpha_1) = e^{i m \sum_{j=1}^N \Delta \alpha_j} D(\sum_{j=1}^N \Delta \alpha_j). \]  
(3)
An arbitrary path \( \gamma \) can be approached in the limit \( N \to \infty \). Thus, we have
\[ D_t = D(\int_\gamma d\alpha)e^{i\Theta}, \]  
(4)
where
\[ \Theta = Im \{ \int_\gamma \alpha^* d\alpha \}. \]  
(5)
For a closed path we have
\[ D_t = D(0)e^{i\Theta} = e^{i\Theta}. \]  
(6)
Set \( \alpha = x_1 + ix_2 \). Then we have
\[ \Theta = \int (x_1 dx_2 - x_2 dx_1). \]  
(7)
Thus, the absolute value of the phase \( \Theta \) is equal to two times of the area involved by the loop in the phase space. The idea has been used to realize the nonlinear Hamiltonian \( J_y^2 \) in the context of trapped ions, where \( J_y \) is the collective spin operator [15].

III. THE CA VITY QED SYSTEM

We consider two identical three-level atoms, which have one excited state \( |r \rangle \) and two ground states \( |e \rangle \) and \( |g \rangle \). The quantum information is encoded on the states \( |e \rangle \) and \( |g \rangle \) and the state \( |r \rangle \) is an auxiliary state. The transition \( |e \rangle \leftrightarrow |r \rangle \) is driven by a cavity mode with the coupling constant \( g \) and the detuning \( \Delta \) and a classical laser field with a Rabi frequency \( \Omega \) and detuning \( \Delta \) and a classical field. We consider two identical three-level atoms, which are driven by a cavity mode with the coupling constant \( g \) and the detuning \( \Delta \), respectively. The displacement operators satisfy
\[ S_{z,j} = \frac{1}{2} (|r_j \rangle \langle r_j| - |e_j \rangle \langle e_j|), \]  
(8)
\[ S_{z,j}^+ = |r_j \rangle \langle e_j|, \]  
\[ S_{z,j}^- = |e_j \rangle \langle r_j|, \]  
\( a^+ \) and \( a \) are the creation and annihilation operators for the cavity mode. In the case that \( \Delta \gg \Omega, g \), the atoms can not exchange energy with the fields. Then the Hamiltonian of Eq. (1) can be replaced by the effective Hamiltonian, which includes three parts: 1. the Stark shifts; 2. the dipole coupling between the two atoms induced by the cavity mode; 3. the coupling between the atoms and the cavity mode assisted by the classical field. The first two parts have been derived previously [4]. The third part is characterized by the transition \( |r_j, n \rangle \leftrightarrow |r_j, n + 1 \rangle \) and \( |e_j, n \rangle \leftrightarrow |e_j, n + 1 \rangle \). The coefficient for \( |r_j, n \rangle \leftrightarrow |r_j, n + 1 \rangle \), mediated by \( |e_j, n + 1 \rangle \), is given by
\[ \langle r_j, n + 1 | H_I | e_j, n + 1 \rangle \langle e_j, n + 1 | H_I | r_j, n \rangle \]
\[ = \frac{1}{\Delta} \Omega g e^{-i\delta t} \sqrt{n + 1}. \]  
(9)
On the other hand, The effective coupling coefficient for \( |e_j, n \rangle \leftrightarrow |e_j, n + 1 \rangle \) is given by
\[ \langle e_j, n + 1 | H_I | r_j, n \rangle \langle r_j, n | H_I | e_j, n \rangle \]
\[ = \frac{1}{\Delta} \Omega g e^{-i\delta t} \sqrt{n + 1}. \]  
(10)
Therefore, the effective interaction Hamiltonian is
\[ H_i = \sum_{j=1,2} \frac{1}{\Delta} (g^2 a^+ a + \Omega^2 + \Omega g e^{i\delta t} + \Omega g a^+ e^{-i\delta t}) \]
\[ \langle r_j | \langle e_j | - | e_j \rangle \langle e_j | g^2 | r_j \rangle \langle r_j | + \frac{1}{\Delta} g^2 (S_{z,j}^+ S_{z,j}^+ + S_{z,j}^- S_{z,j}^-). \]  
(11)
The first two terms in the first smallest bracket describe the Stark shifts induced by the photons of the cavity field and classical field, respectively, the other two terms in the first smallest bracket describe the coupling between the two atoms and the cavity mode assisted by the classical field. The term \( \frac{1}{\Delta} g^2 | r_j \rangle \langle r_j | \) describes the Stark shift due to the vacuum field. The last two terms describe the dipole coupling between the two atoms induced by the cavity mode.

The time evolution of this system is decided by Schrödinger’s equation:
\[ \frac{d|\psi(t)\rangle}{dt} = H_e |\psi(t)\rangle. \]  
(12)
Perform the unitary transformation
\[ |\psi(t)\rangle = e^{-iH_0 t} |\psi'(t)\rangle, \]  
(13)
with
\[ H_0 = \sum_{j=1,2} \frac{1}{\Delta} (g^2 a^+ a + \Omega^2) (| r_j \rangle \langle r_j | - | e_j \rangle \langle e_j | ) + g^2 | r_j \rangle \langle r_j |. \]  
(14)
Then we obtain
\[ i \frac{d|\psi(t)\rangle}{dt} = H_i|\psi(t)\rangle, \tag{15} \]
where
\[ H_i = \sum_{j=1,2} \frac{\Omega g}{\Delta} \left[ (ae^{i(\delta - \frac{\Delta}{2})t} + a^{\dagger}e^{-i(\delta + \frac{\Delta}{2})t}) |r_j\rangle \langle r_j| \right. \\
\left. -[ae^{i(\delta - \frac{\Delta}{2})t} + a^{\dagger}e^{-i(\delta + \frac{\Delta}{2})t}) |e_j\rangle \langle e_j| \right] + \frac{1}{\Delta} g^2(S_{z}^+ S_{x}^- + S_{z}^- S_{x}^+). \tag{16} \]

The quantum information is encoded onto the two ground electronic state. Under the application of \( H_i \),

\[ |e_j\rangle \rightarrow e^{-iH'_i dt} |e_j\rangle \rightarrow D(\phi) |e_j\rangle \]

where
\[ d\alpha = \frac{\Omega g}{\Delta} e^{-i(\delta + \frac{\Delta}{2}) \alpha} dt, \tag{18} \]

|\phi_{u,v}(t)\rangle (u, v = e, g) denotes the cavity mode state correlated with the qubit state |u_1\rangle |v_2\rangle at the time \( t \). We note that the dipole coupling terms and the terms containing the population operator |r_j\rangle \langle r_j| have no effect on the evolution since the atoms are in the ground states, i.e., \((S_{z}^+ S_{x}^- + S_{z}^- S_{x}^+)|u_1\rangle |v_2\rangle = |r_j\rangle \langle r_j| \otimes |u_1\rangle |v_2\rangle = 0\). In fact, only when one atom is in the excited state |r_j\rangle and the other in the ground state |e_k\rangle the dipole coupling terms affect evolution of the system. On the other hand, the population operator |r_j\rangle \langle r_j| only has effect on the state |r_j\rangle. In the present case, each atom has no probability of being populated in the excited state |r_j\rangle and thus we can ignore the dipole coupling terms and the terms containing the population operator |r_j\rangle \langle r_j|.

**IV. UNCONVENTIONAL GEOMETRIC PHASE GATES WITH THE CA VITY QED SYSTEM**

Assume that the cavity field is initially in the vacuum state |0\rangle. After an interaction time \( \tau \) the evolution operators of the vibrational modes are
\[ |e_j\rangle \rightarrow e^{-i(\phi + \Omega^2 t/\Delta)} |e_j\rangle, \tag{19} \]
where
\[ \alpha = i \int_{0}^{\tau} \frac{\Omega g}{\Delta} e^{-i(\delta + \frac{\Delta}{2}) \alpha} dt \]
\[ = - \frac{\Omega g}{\Delta \delta + g^2}[e^{-i(\delta + \frac{\Delta}{2}) \alpha} - 1], \tag{20} \]
\[ \phi = \text{Im} \int_{\gamma} \alpha^* d\alpha' \]
\[ = - \frac{(\Omega g)^2}{\Delta (\Delta \delta + g^2)}(1 - e^{-i(\delta + \frac{\Delta}{2}) \alpha} \alpha) dt' \]
\[ = - \frac{(\Omega g)^2}{\Delta (\Delta \delta + g^2)}[\pi t - \frac{1}{(\delta + \frac{\Delta}{2})^2}] \sin(\delta + \frac{\Delta}{2}) t], \tag{21} \]
\[ \phi = 4 \Delta \frac{\Omega g}{\Delta} \phi = 4 \phi \tag{22} \]

Under the condition
\[ (\delta + \frac{\Delta}{2}) t = 2\pi, \tag{23} \]
the displacement is along a closed path, returning to the original point in phase space and acquiring a geometric phase conditional upon the electronic states. This leads to
\[ |e_j\rangle \rightarrow e^{i(\phi + \Omega^2 t/\Delta)} |e_j\rangle, \tag{24} \]
\[ |e_j\rangle \rightarrow e^{i(4\phi + 2\Omega^2 t/\Delta)} |e_j\rangle. \tag{25} \]

Using Eq. (13), we obtain the state evolution for the system governed by \( H_i \)
\[ |e_j\rangle \rightarrow e^{i(\phi + \Omega^2 t/\Delta)} |e_j\rangle, \tag{26} \]

After perform the following one-qubit operations
\[ |e_j\rangle \rightarrow e^{-i(\phi + \Omega^2 t/\Delta)} |e_j\rangle, \tag{27} \]
we obtain
\[ |e_j\rangle \rightarrow e^{i(\phi + \Omega^2 t/\Delta)} |e_j\rangle, \tag{28} \]

\[ |e_j\rangle \rightarrow e^{i(\phi + \Omega^2 t/\Delta)} |e_j\rangle, \tag{29} \]
\[ |e_j\rangle \rightarrow e^{i(\phi + \Omega^2 t/\Delta)} |e_j\rangle. \tag{30} \]
Choosing
\[ 2\phi = -\pi, \quad (29) \]
we obtain a \( \pi \)-phase gate. With the choice \( \delta = \frac{g^2}{\Delta} \), \( \Omega = g \), \( g^2 t = \pi \), the conditions (23) and (29) can be satisfied.

We note that the atoms are entangled with the cavity mode during the gate operation and thus it is required that the decoherence time of the cavity mode should be longer than the gate time. However, when \( \frac{\Omega g}{\Delta^2 + g^2} \ll 1 \) the phase-space trajectory is a very small circle and \( D(\alpha) \approx 1 \). In this case the evolution (19) and (21) reduces to (24) and (25) without the requirement of Eq. (23). Thus, the atoms are disentangled from the cavity mode throughout the gate operation and the gate is insensitive to the cavity decay.

We give a brief discussion on the experimental matters. In the scheme the atoms are never populated in the excited states and thus the decoherence mainly arises from the cavity decay. Setting \( \Delta = 10g \), \( \delta = 2g \), and \( \Omega = g \). Then the required operation time is on the order of \( t = \pi \Delta^2 \delta/(2g^4) \approx 10^2/g \). A cavity with a decay rate \( \gamma = g/27 \) is experimentally achievable [16]. The cavity only has a very small probability about \( \left| \frac{\Omega g}{\Delta^2 + g^2} \right|^2 \approx 10^{-3} \) of being excited during the gate operation. Thus the efficient decay time of the cavity is about \( T = 10^3/\gamma \approx 2.7 \times 10^4/g \). The gate error caused by the cavity decay is on the order of \( t/T = 10^{-2} \), much smaller than result reported in the Ref. [3].

V. CONCLUSIONS

In conclusion, we have proposed a scheme for realizing unconventional geometric two-qubit phase gates with atoms in a cavity. In the scheme the atoms interact simultaneously with a highly detuned cavity mode and a classical field. The atoms remain in their ground states during the gate operation, while the cavity mode is displaced along a circle in the phase space, acquiring a geometric phase conditional upon the atomic state. Under certain conditions, the atomic system is disentangled with the cavity mode and thus the gate is insensitive to both the atomic spontaneous emission and the cavity decay.

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