Einstein metrics on tangent bundles of spheres.

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Abstract

We give an elementary treatment of the existence of complete Kähler-Einstein metrics with nonpositive Einstein constant and underlying manifold diffeomorphic to the tangent bundle of the \((n + 1)\)-sphere.

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1 Introduction.

Over the last few years there has been considerable interest in a family of Ricci-flat Kähler metrics discovered by Stenzel \([St]\) with underlying manifold diffeomorphic to the tangent bundle of the \((n + 1)\)-sphere. (The \(n = 2\) case was found earlier by Candelas and de la Ossa \([CD]\) : the \(n = 1\) case is the Eguchi-Hanson metric \([EH]\)). The complex structure is that of the quadric in \(\mathbb{C}^{n+2}\).

One source of interest in physics is that there is a “conifold transition” \([CD]\) – both the Stenzel metrics and another family of Ricci-flat Kähler metrics due to Bérard-Bergery \([BB]\) have a cone as a common degenerate limit. Also, Cvetic, Gibbons, Lü and Pope have recently studied the harmonic forms on these metrics and found an explicit formula for the Stenzel metrics in terms of hypergeometric functions \([CGLP]\).

In an unpublished 1993 preprint \([DS2]\) we gave a rather different description of these metrics, and showed that they also have analogues with negative Einstein constant. The underlying manifold of the latter metrics is still diffeomorphic to \(TS^{n+1}\), though the complex structure is now that of a tube in the quadric rather than the whole quadric.

As several people have expressed interest in this work, we hope that it may be worthwhile to publish it, with some amplification, in the current paper.

We remark that the existence of a canonical complex structure on (a tube in) the tangent bundle of a Riemannian manifold \((M, g)\) is a general fact due to Guillemin-Stenzel \([GS]\) and Lempert-Szöke \([LSz]\, [Sz1]\). Now the existence theorem of Mok-Yau \([MY]\) shows the existence of complete Kähler-Einstein metrics on tubes in \(TM\) of radius smaller than the maximal radius on which the complex structure is defined \([Sz2]\).

In our case (i.e. when \(M\) is the round sphere) we are able to give a much more elementary and concrete treatment, exploiting the fact that the metrics are of cohomogeneity one with
respect to an action of $SO(n+2)$. Our techniques are similar to those of our paper [DS1], where we classified Kähler-Einstein metrics in real dimension four which are of Bianchi IX type.

2 Cohomogeneity one Kähler-Einstein metrics.

Let us consider a Riemannian manifold $(M, g)$, which admits an isometric action of $SO(n+2)$ with principal orbit $SO(n+2)/SO(n)$. We further suppose that this action is of cohomogeneity one, that is, each principal orbit has real codimension one in $M$. It follows that the (real) dimension of $M$ is $2n+2$.

The union of the principal orbits will form an open dense set in $M$. On this set we can write the metric as

$$g = dt^2 + g_t$$

where $t$ is the arclength parameter along a geodesic orthogonal to the group orbits, and $g_t$ is a homogeneous metric on the orbits.

Now, for any homogeneous space $G/H$ with $H$ compact we may identify the tangent space at a point with an $Ad_H$-invariant complement $p$ for $h$ in $g$. With this identification, $G$-invariant metrics on $G/H$ correspond precisely to $Ad_H$-invariant inner products on $p$.

In our case we embed $so(n)$ in $so(n+2)$ so that if $X_{i,j}$ denotes the matrix with $ij$th entry = 1, $ji$th entry = -1 and all other entries zero, then $\{X_{i,j} : j = n+1, n+2, 1 \leq i < j\}$ spans a complement $p$ for $so(n)$ in $so(n+2)$. Now the decomposition of $p$ into irreducible components under the adjoint action of $SO(n)$ is

$$p = p_1 \oplus p_2 \oplus p_3$$

where $p_1, p_2$ are standard $n$-dimensional representations spanned by $\{X_{i,n+1} : 1 \leq i \leq n\}$ and $\{X_{i,n+2} : 1 \leq i \leq n\}$ respectively, and $p_3$ is the trivial representation spanned by $X_{n+1,n+2}$.

We shall consider metrics of the form

$$g_t = a(t)^2 \ B \ p_1 \oplus b(t)^2 \ B \ p_2 \oplus c(t)^2 \ B \ p_3$$

where $B$ is the Killing form on $so(n+2)$ defined by $B(X,Y) = -\frac{1}{2} \Tr XY$. Note that this is not the most general possibility for $g_t$. As $p_1$ and $p_2$ are isomorphic as $SO(n)$-representations it is a definite restriction to assume that $p_1$ and $p_2$ are orthogonal for all $t$.

If $a^2 = b^2$ our metric $g_t$ is obtained using a Riemannian submersion with circle fibre over the flag manifold $SO(n+2)/SO(n) \times SO(2)$ equipped with a Kähler-Einstein metric. If $a^2$ and $b^2$ are not equal, however, $g_t$ does not arise in this way.

It is convenient to introduce at this stage a new transverse coordinate $\zeta$ defined by

$$dt = a^n b^n c \ d\zeta.$$

We shall denote differentiation with respect to $\zeta$ by a prime.
The formulae of Bérand-Bergery [BB] enable us to write down the Einstein equations \( R_{\text{g}} = \Lambda g \) for the cohomogeneity one metric \( g \). They are:

\[
- \frac{1}{a^{2n-2}b^{2n}c^{2}} \left( \frac{a'}{a} \right) + \frac{a^4 - (b^2 - c^2)^2}{2b^2c^2} + n - 1 = \Lambda a^2, \tag{2.1}
\]

\[
- \frac{1}{a^{2n}b^{2n-2}c^{2}} \left( \frac{b'}{b} \right) + \frac{b^4 - (c^2 - a^2)^2}{2a^2c^2} + n - 1 = \Lambda b^2, \tag{2.2}
\]

\[
- \frac{1}{a^{2n}b^{2n}c^{2}} \left( \frac{c'}{c} \right) + \frac{n(c^4 - (a^2 - b^2)^2)}{2a^2b^2} = \Lambda c^2, \tag{2.3}
\]

\[
- \frac{1}{a^n b^n c} \left( \frac{n a'}{a} + \frac{n b'}{b} + \frac{c'}{c} \right) - \frac{1}{a^{2n}b^{2n}c^{2}} \left( n \left( \frac{a'}{a} \right)^2 + n \left( \frac{b'}{b} \right)^2 + \left( \frac{c'}{c} \right)^2 \right) = \Lambda, \tag{2.4}
\]

where \( \Lambda \) is the Einstein constant.

**Remark 2.5** If \( n = 1 \), then (2.1)-(2.4) are a presentation of the Einstein equations for a Bianchi IX metric.

It can be verified by direct calculation that (2.1)-(2.4) hold in particular if \( a, b, c \) satisfy the following three first-order equations

\[
a' = \frac{1}{2} a^n b^{n-1} (b^2 + c^2 - a^2), \tag{2.6}
\]

\[
b' = \frac{1}{2} a^{n-1} b^n (c^2 + a^2 - b^2), \tag{2.7}
\]

\[
c' = \frac{1}{2} na^{n-1} b^{n-1} c \left( a^2 + b^2 - c^2 - \frac{2\Lambda}{n} a^2 b^2 \right). \tag{2.8}
\]

It is this reduction of the Einstein equations which we shall study. The metrics arising from solutions of these equations are precisely those Einstein metrics with respect to which the \( SO(n + 2) \)-invariant almost complex structure \( J \) defined by

\[
J : X_{i,n+1} \mapsto -\frac{a}{b} X_{i,n+2}, \quad \frac{\partial}{\partial t} \mapsto -\frac{1}{c} X_{n+1,n+2} \tag{2.9}
\]

is Kähler.

## 3 Complete metrics.

We shall now demonstrate the existence of solutions to (2.6)-(2.8) which give rise to complete metrics. It will be convenient to introduce a new variable \( u \) defined by

\[
\text{du} = (ab)^{n-1} \text{d}\zeta
\]
so $dt = abc\, du$, and our equations become

$$a_u = \frac{1}{2}a(b^2 + c^2 - a^2), \quad (3.1)$$
$$b_u = \frac{1}{2}b(c^2 + a^2 - b^2), \quad (3.2)$$
$$c_u = \frac{1}{2}nc \left(a^2 + b^2 - c^2 - \frac{2\Lambda}{n}a^2b^2\right), \quad (3.3)$$

Remark 3.4 If $n = 1$ these are the Bianchi IX Kähler-Einstein equations studied by the authors in [DS1].

The critical points of the system (3.1)-(3.3) are precisely the points $(0, K, \pm K), (K, 0, \pm K)$ and $(K, \pm K, 0)$, where $K \in \mathbb{R}$.

Assumption

From now on we shall take $\Lambda$ to be less than or equal to zero.

Subject to this assumption, the linearisation of (3.1)-(3.3) about each critical point (except the origin) has one negative, one positive and one zero eigenvalue. So each of these critical points has an unstable curve.

Let us consider an unstable curve of $(0, K, K)$ where $K$ is nonzero. This will be a solution to (3.1)-(3.3) defined on a maximal interval $(-\infty, \eta)$ for some $\eta$ ($\eta$ may be $\infty$).

It follows from (3.1)-(3.3) that if any one of $a, b$ or $c$ is zero at a point in $(-\infty, \eta)$, then it is identically zero. It is clear that on the unstable curve none of $a, b$ or $c$ is identically zero, so none of them vanishes anywhere on $(-\infty, \eta)$. The metric and the equations (3.1)-(3.3) are invariant under changes of sign of $a, b$ or $c$ so from now on we can assume that $a, b, c$ are all positive on $(-\infty, \eta)$; in particular we can take $K$ to be positive.

The metric is therefore defined on $(-\infty, \eta)$ and to show that it is complete we need to study its behaviour as $u$ tends to $-\infty$ and as $u$ tends to $\eta$.

Note that if $a$ equals $b$ at any point on the unstable curve then $a$ is identically equal to $b$, giving a contradiction. It follows that $b$ is greater than $a$ on the unstable curve, and so (from (3.1)) $a$ is strictly increasing. It now follows from (3.1),(3.2) that $b \over a$ is strictly decreasing, and tends to some limit $L \geq 1$ as $u$ tends to $\eta$.

Lemma 3.5 On an unstable curve of $(0, K, K)$ we have the inequalities

$$c^2 \leq a^2 + b^2 - \frac{2\Lambda}{n}a^2b^2 \quad (3.6)$$
$$b^2 \leq a^2 + c^2. \quad (3.7)$$

Proof

Suppose that $c^2 > a^2 + b^2 - \frac{2\Lambda}{n}a^2b^2$ at $u_0$. The equations (3.2),(3.3) imply that $b$ is increasing and $c$ is decreasing at $u_0$. Recall also that $a$ is increasing. We deduce that $c^2$ is greater than $a^2 + b^2 - \frac{2\Lambda}{n}a^2b^2$ on $(-\infty, u_0)$ and hence that $a, b$ are increasing and $c$ is decreasing on this interval. It follows that $c$ is bounded away from $b$ on $(-\infty, u_0)$. This contradicts the fact that $(a, b, c)$ tends to $(0, K, K)$ as $u$ tends to $-\infty$, so we have established inequality (3.6). The proof of inequality (3.7) is very similar. \qed
Remark 3.8 We deduce from (3.2), (3.3) that \( b, c \) are increasing on \( (-\infty, \eta) \). We remarked earlier that \( a \) is strictly increasing on this interval.

Let us now study the behaviour of the metric as \( u \) tends to \(-\infty\) and as \( u \) tends to \( \eta \).

As \( u \) tends to \(-\infty\)
\[
a \simeq Me^{K^2u}, \quad b \simeq K, \quad c \simeq K
\]
for some constant \( M \).

Choosing \( R = Me^{K^2u} \) as a new coordinate, find that the metric is asymptotically given by
\[
dR^2 + R^2B \mid p_1 + K^2(B \mid p_1 \oplus B \mid p_3)
\]
as \( R \) tends to zero.

Now, \( B \mid p_1 \) defines the standard \( SO(n+1) \)-invariant metric on \( S^n \), while \( B \mid p_2 \oplus B \mid p_3 \) defines the standard \( SO(n+2) \)-invariant metric on \( S^{n+1} \). Therefore we obtain a nonsingular metric by adding in an \((n+1)\)-sphere at \( R = 0 \). In terms of the orbit type of the \( SO(n+2) \) action, the isotropy group jumps at \( R = 0 \) from \( SO(n) \) to \( SO(n+1) \), so an \( n \)-dimensional sphere collapses to a point and the orbit at \( R = 0 \) is the \((n+1)\)-sphere \( SO(n+2)/SO(n+1) \) rather than \( SO(n+2)/SO(n) \). The orbit at \( R = 0 \) is called a \textit{Bolt of the second kind} in the terminology of Gibbons, Page and Pope \cite{GPP}. Note that \( K \) determines and is determined by the volume of the Bolt.

To examine the behaviour of the metric as \( u \) tends to \( \eta \) we introduce a new coordinate \( \rho \), defined by
\[
\rho = 2(ab)\frac{1}{2}.
\]
This is an allowable change of variables as \( ab \) is increasing.

The metric is now given by
\[
W^{-1}d\rho^2 + \frac{1}{4}\rho^2 \left( \frac{a}{b} B \mid p_1 + \frac{b}{a} B \mid p_2 + W B \mid p_3 \right)
\]
where \( W = \frac{c^2}{ab} \). It follows from equations (3.1)-(3.3) that
\[
dW \frac{dW}{d\rho} + \frac{2(n+1)W}{\rho} = \frac{2n}{\rho} \left( \frac{a}{b} + \frac{b}{a} \right) - \Lambda \rho \quad \text{(3.10)}
\]
Recall that the inequalities (3.7) and (3.8) show that \( a, b, c \) are monotonic increasing on \((-\infty, \eta)\). Suppose that the limit \( \lambda \) of \( a \) as \( u \) tends to \( \eta \) is finite. Since \( b/a \) is decreasing, and because of the estimate (3.6), we see that the limits \( \mu, \nu \) of \( b, c \) at \( u = \eta \) are also finite. If \( \eta = \infty \) then \( (\lambda, \mu, \nu) \) is a critical point, which leads to a contradiction as \( \lambda, \mu, \nu \) are all positive. If \( \eta \) is finite, then we also obtain a contradiction because \((-\infty, \eta)\) is by definition a maximal interval on which the solution exists. So we deduce that \( a \), and hence \( \rho \), tends to infinity as \( u \) tends to \( \eta \).

Therefore we must study the asymptotics of the metric (3.3) as \( \rho \) tends to infinity. It follows from (3.10) that
\[
\frac{d}{d\rho} (\rho^{2n+2}W) = 2n\rho^{2n+1} \left( \frac{a}{b} + \frac{b}{a} \right) - \Lambda \rho^{2n+3}.
\]
Solving for $W$, and recalling that $\frac{b}{a}$ decreases monotonically on $(-\infty, \eta)$ to some finite positive limit $L$, we see that $W = O(\rho^2)$ if $\Lambda$ is negative and $W$ is bounded for large $\rho$ if $\Lambda$ is zero. It follows that the geodesic distance $\int_{\rho_0}^\infty \sqrt{W^{-1}} d\rho$ from $\rho = \rho_0$ to $\rho = \infty$ is infinite.

We have shown that the metric is complete. The underlying topological manifold is the total space of a rank $n + 1$ vector bundle $E$ over $S^{n+1}$. In fact the sphere bundle of this vector bundle is the Stiefel manifold $SO(n + 2)/SO(n)$, so $E$ is in fact the tangent bundle of $S^{n+1}$. The Bolt is the zero section of $E$.

We summarise our results in the next theorem.

**Theorem 3.11** The unstable curves of points $(0, K, K)$, where $K$ is nonzero, give complete Einstein metrics with nonpositive Einstein constant on $TS^{n+1}$.

**Remark 3.12** (i) As remarked earlier, our metrics and the equations (3.1)-(3.3) are invariant under changing the sign of any of $a, b, c$, and also under interchanging $a, b$. Therefore the metrics arising from unstable curves of $(0, K, -K)$ or $(K, 0, \pm K)$ where $K$ is nonzero are isometric to those of Theorem 3.11.

(ii) One can also obtain complete metrics by considering the unstable curves of $(K, \pm K, 0)$ where $K$ is nonzero and $n - \Lambda K^2$ is a half-integer. The latter condition is needed to ensure that the metric can be completed by adding a Bolt, which in this case is the flag manifold $SO(n + 2)/(SO(n) \times SO(2))$. The underlying manifold of the complete metric is the total space of a complex line bundle over the Bolt.

However, for these trajectories $a^2$ is identically equal to $b^2$ and the metrics on the $SO(n + 2)$ orbits are obtained by Riemannian submersions with circle fibres. The resulting cohomogeneity one metrics are included in the examples of Bérard-Bergery [BB] (see also Gibbons and Pope [GP], and Pedersen and Poon [P, PP]).

(iii) If $\Lambda = 0$ we have a special solution, given up to translation of $u$ by

$$a = b = \sqrt{\frac{n + 1}{2n}} c = \left(1 - \left(\frac{2n}{n + 1}\right) u\right)^{-1/2}$$

This is the Ricci-flat cone over an Einstein metric on $SO(n + 2)/SO(n)$. For this solution, the trajectory is an unstable curve emanating from $(0, 0, 0)$, hence the metric can be viewed as a limiting case of both the Stenzel metrics and of the Bérard-Bergery metrics, in the limit that the volume of the Bolt ($S^{n+1}$ or $SO(n + 2)/SO(n) \times SO(2)$ respectively) tends to zero.

If $\Lambda < 0$, then, as remarked above, the values of $K$ for which the Bérard-Bergery metric extends over the Bolt are discrete so we cannot take a continuous limit of the Bérard-Bergery metrics.

(iv) The Bolt is complex for the Bérard-Bergery metrics, but totally real for the metrics arising from unstable curves of $(K, 0, K)$. We can see this from (2.9), as in the Bérard-Bergery case the tangent space to the Bolt is the $J$-invariant space $p_1 \oplus p_2$, while in the other case the tangent space is $p_1 \oplus p_3$, which satisfies $(p_1 \oplus p_3) \cap J(p_1 \oplus p_3) = \{0\}$.

(v) In the special case when $n = 1$ and $\Lambda = 0$ the equations are symmetric in $a, b, c$ and the metrics in the two families are both isometric to the hyperkähler Eguchi-Hanson metric—but with respect to two different complex structures in its two-sphere of complex structures.
4 Relation with Stenzel’s work

Stenzel [St] has shown the existence of complete Ricci-flat Kähler metrics on the tangent bundles of spheres by different methods. He considers the complex structure on $T S^{n+1}$ obtained by identifying this space with the quadric in $\mathbb{C}^{n+2}$. Regarding the complement of the zero section in $T S^{n+1}$ as $(0, \infty) \times SO(n+2)/SO(n)$, and taking $r$ as a coordinate on $(0, \infty)$, this complex structure is defined by

$$\tilde{J} : \frac{\partial}{\partial r} \mapsto -X_{n+1,n+2}, \quad X_{i,n+1} \mapsto -\tanh r X_{i,n+2}.$$ 

Stenzel now shows that if $f$ satisfies the differential equation

$$\frac{d}{dr} \left( \left( \frac{df}{dr} \right)^{n+1} \right) = (n+1)k(\sinh r)^n \quad \text{(4.1)}$$

for some constant $k$, and if the derivative of $f$ vanishes at $r = 0$, then the function $\phi$ defined by $\phi(r) = f(2r)$ is the Kähler potential with respect to $\tilde{J}$ for a Ricci-flat metric $\tilde{g}$ on $(0, \infty) \times SO(n+2)/SO(n)$ which extends to the whole of $T S^{n+1}$.

Explicitly, the metric is

$$\tilde{g} = \frac{1}{2} \phi'' dr^2 + \tilde{g}_r, \quad \text{(4.2)}$$

where the homogeneous metric $\tilde{g}_r$ is defined in the notation of section 2 by

$$\frac{1}{2} \left( \phi'(r) \tanh r B |_{p_1} \oplus \phi'(r) \coth r B |_{p_2} \oplus \phi''(r) B |_{p_3} \right). \quad \text{(4.3)}$$

(In these expressions, and for the remainder of the paper, we use a prime to indicate differentiation with respect to $r$).

By virtue of (4.1), the function $\phi$ satisfies

$$\phi''' = n\phi'' \left( \coth r + \tanh r - \frac{\phi''}{\phi'} \right). \quad \text{(4.4)}$$

Defining the variable $u$ by $du = \left(2/\phi'\right)dr$, it is now straightforward to verify that this metric is one of those defined by equations (3.1)-(3.3) with $\Lambda = 0$. Moreover $\phi' \tanh r$ tends to zero as $r$ tends to zero. In order to obtain a metric on $T S^{n+1}$ we need $\phi' \coth r$ and $\phi''$ to tend to equal nonzero limits as $r$ tends to zero. This implies that the solution to (3.1)-(3.3) must be an unstable curve of a critical point $(0, K, \pm K)$ with $K$ nonzero, so the metric is one of the Ricci-flat examples of Theorem 3.11.

Conversely, suppose we have a solution to the equations (3.1)-(3.3) with $\Lambda = 0$ arising as an unstable curve of $(0, K, \pm K)$ where $K$ is nonzero. Letting $r$ satisfy $dr = ab du$, we obtain from (3.1),(3.2) the equations

$$(ab)' = c^2, \quad \left(\frac{a}{b}\right)' = 1 - \left(\frac{a}{b}\right)^2.$$ 

It easily follows that the metric can be put in the form given by (4.2),(4.3). Equation (3.3) is then equivalent to (4.4), which may be integrated to give (4.1). Moreover, the fact that
$(a, b, c)$ tends to $(0, K, \pm K)$ as $u$ tends to $-\infty$ gives the initial condition for equation (4.1). Therefore the Ricci-flat metrics of Theorem 3.11 are precisely the metrics of [St].

This argument also shows that any of the metrics of Theorem 3.11 with negative Einstein constant can be put in the form (4.2), (4.3). Now the coordinate $r$ is defined by $r = \int_{-\infty}^{u} ab \ du$ or equivalently, in the notation of (3.10), $r = \int_{0}^{\rho(u)} \frac{2}{\rho} d\rho$. We saw earlier that $W = O(\rho^2)$ if the Einstein constant is negative, so this integral converges as $\rho$ tends to $\infty$, that is, as $u$ tends to $\eta$. The upshot is that the complex structure for our complete metrics with $\Lambda$ negative is obtained by restricting Stenzel’s complex structure on $TS^{n+1}(=\text{quadric in } \mathbb{C}^{n+2})$ to a tube of finite radius. In particular the Kähler structure extends to the Bolt, giving us a globally defined Kähler structure in the metrics of Theorem 3.11.

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