Conservation laws of the system of equations of one-dimensional shallow water over uneven bottom in Lagrange’s variables

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Abstract

The system of equations of one-dimensional shallow water over uneven bottom in Euler’s and Lagrange’s variables is considered. Intermediate system of equations is introduced. Hydrodynamic conservation laws of intermediate system of equations is used to find all first order conservation laws of shallow water equations in Lagrange’s variable for all bottom profiles. The obtained conservation laws are compared with the hydrodynamic conservation laws of the system of equations of one-dimensional shallow water over uneven bottom in Euler’s variables. Bottom profiles are given for which there are additional conservation laws.

Keywords: shallow water, conservation laws, Lagrange’s variable, Noether’s theorem.

1 Introduction

There are various approaches to finding conservation laws of equations of mathematical physics [1–6]. The most widely known method of constructing of conservation laws is based on Noether’s theorem [1]. This method uses symmetries.

Many works are devoted to the construction of conservation laws of equations in hydro- and gas-dynamics [7–11].

The hydrodynamic conservation laws of the one-dimensional shallow water equations over uneven bottom in Euler’s variables were obtained in [12].
In the present work, the first-order conservation laws of the shallow-water equations in Lagrangian’s variables for all bottom profiles are obtained without using of symmetries.

2 Basic Equations

In dimensionless variables, the system of one-dimensional shallow-water equations over an uneven bottom has the following form [13]:

\[ \begin{align*}
    u_t + uu_x + \eta_x &= 0, \\
    \eta_t + ((\eta + h(x))u)_x &= 0.
\end{align*} \]  

(1)

Here \( h(x) \) is the thickness of the unperturbed layer of the liquid, \( u = u(x,t) \) is the depth-average horizontal velocity, \( \eta = \eta(x,t) \) is the deviation of the free surface \( \eta(x,t) + h(x) \geq 0 \). The bottom profile is given by the relation \( z = -h(x) \) (\( z \) is the vertical coordinate).

**Remark 1.** The system of equations (1) is similar to the system of equations of one-dimensional gas dynamics [14, 15].

Using the second equation of the system of equations (1), we introduce a new variable \( m = m(x,t) \) and consider the following system of equations

\[ \begin{align*}
    u_t + uu_x + \rho_x &= h'(x), \\
    m_x &= \rho, \\
    m_t &= -u\rho,
\end{align*} \]  

(2)

where \( \rho = \eta + h(x) \).

From the second and third equations it follows that the variable \( m \) is Lagrangian’s variable due to the relation

\[ \frac{dm}{dt} = m_t + um_x = 0. \]

One can get the equation of one-dimensional shallow water in Lagrange’s variables by choosing \( m \) and \( t \) as independent variables [15]

\[ \frac{x_{tt} - x_{mom}}{x_m^3} = h'(x). \]  

(3)

One-to-one correspondence between the system of equations (2) and the equation (3) is given by the relations

\[ u = x_t, \quad \rho = \frac{1}{x_m}. \]  

(4)
Note that the system of equations \( (2) \) is intermediate system between system of equations \( (1) \) and the equation \( (3) \). The system of equations \( (2) \) is a covering system \( (3) \) for the system of equations \( (1) \).

3 Conservation Laws of the equation in Lagrange’s variables

Under the conservation laws of the system of equations \( (2) \) we understand divergent forms for which the solutions of the system equations \( (2) \) satisfy the relation

\[
D_x(P) + D_t(Q) = 0. \tag{5}
\]

Here \( P, Q \) are functions of independent and dependent variables and their derivatives;

\[
D_x = \frac{\partial}{\partial x} + m_x \frac{\partial}{\partial m} + u_x \frac{\partial}{\partial u} + \rho_x \frac{\partial}{\partial \rho} + \ldots , \quad D_t = \frac{\partial}{\partial t} + m_t \frac{\partial}{\partial m} + u_t \frac{\partial}{\partial u} + \rho_t \frac{\partial}{\partial \rho} + \ldots
\]

are total derivatives in variables \( x \) and \( t \). Conservation laws for which the equality \( (5) \) is satisfied everywhere, we will call trivial conservation laws. The maximum order of derivatives included in the functions \( P \) and \( Q \) will be called the order of the conservation law. The conservation laws of the zero order will be called hydrodynamic. The conservation laws of the equation \( (3) \) are defined similarly.

We recall that conservation laws in divergent form \( (5) \) are equivalent to differential 1-forms \( (3) \)

\[
Q \, dx - P \, dt,
\]

which are closed on solutions of the system \( (2) \).

**Proposition 1.** According to the relations \( (4) \), hydrodynamic conservation law of the system of equations \( (2) \) with functions \( P, Q \) defines the first-order conservation law of the equation \( (3) \) with functions \( \tilde{P} = P - x_t Q, \tilde{Q} = x_m Q \). The opposite is true.

**Proof.** Denote the total derivatives in the variables \( m, t \) as \( \tilde{D}_m \) and \( \tilde{D}_t \), and their restrictions on the equation \( (3) \) as \( \overline{D}_x \) and \( \overline{D}_t \). By the relations \( (4) \), these derivatives are related in the following way

\[
\overline{D}_x = \frac{1}{x_m} \tilde{D}_m, \quad \overline{D}_t = \tilde{D}_t - \frac{x_t}{x_m} \tilde{D}_m.
\]
Then
\[ \overline{D}_x(P) + \overline{D}_t(Q) = \frac{1}{x_m} \hat{D}_m(P) + \hat{D}_t(Q) - \frac{x_t}{x_m} \hat{D}_m(Q) = \]
\[ = \frac{1}{x_m} \left( \hat{D}_m(P - x_t Q) + \hat{D}_t(x_m Q) \right). \]

This implies the validity of the proposition being proved.

Also true the proposition

**Proposition 2.** If the functions \( P \) and \( Q \) in a conservation law of the system of equations (2) are independent of \( m \), then they determine the conservation law of the system of equations (1). All conservation laws of the system of equations (1), except the conservation law
\[ D_t(\eta) + D_x((\eta + h(x))u), \]
are obtained from the conservation laws of the system of equations (2).

Note that finding of the hydrodynamic conservation laws of the system of equations (2) is easier than finding of the first-order conservation laws of the equation (3).

Relation (5) on solutions of the system of equations (2) takes the form
\[ P_x + u_x P_u + \rho_x P_{\rho} + \rho P_{\rho} + Q_t + (h'(x) - \rho_x - uu_x) Q_u - (u \rho_x + u_x \rho) Q_{\rho} - u \rho Q_m \equiv 0. \]

Equating to zero the coefficients of the derivatives \( u_x \) and \( \rho_x \), we obtain the following overdetermined system of linear equations
\[ \begin{align*}
P_u &= \rho Q_{\rho} + u Q_u, \\
P_{\rho} &= u Q_{\rho} + Q_u, \\
P_x &= - \rho P_m + u \rho Q_m - h' Q_u - Q_t.
\end{align*} \tag{6} \]

The overdefined system of equations (6) was investigated on compatibility.

According to the proposition 1, solutions of the system of equations (6) can be compared to conservation laws of the equation (3). Below we provide functions \( \tilde{P} \) and \( \tilde{Q} \), which determine the basis of first-order conservation laws \( \tilde{D}_m(\tilde{P}) + \tilde{D}_t(\tilde{Q}) \) of the equation (3) modulo additive trivial conservation laws for all possible bottom profiles \( h(x) \).

1. **\( h = h(x) \) is arbitrary function.** For any bottom profile \( h(x) \), the equation (3) has conservation laws with functions
\[ \begin{align*}
\tilde{P}_1 &= -\frac{x_t^2}{2} + \frac{1}{x_m} - h(x), \\
\tilde{P}_2 &= x_t \left( \frac{1}{x_m^2} - h^2(x) \right), \\
\tilde{Q}_1 &= x_t x_m, \\
\tilde{Q}_2 &= x_t^2 + x_m \left( \frac{1}{x_m} - h(x) \right)^2.
\end{align*} \]

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2. \( h = a_1 x + a_2 \). In this case additional conservation laws of the equation (3) correspond to functions

\[
\begin{align*}
\tilde{P}_3 &= (a_1 a_2 t^2 - 2 a_2 x) r + t s^2 - a_2^2 t, \\
\tilde{Q}_3 &= 2 tr + \frac{2 a_2 x - a_1 a_2 t^2}{s} + a_1 t^2 - 2x, \\
\tilde{P}_4 &= 2mr^3 + 24tr^2 s^2 - (18x - 9a_1 t^2)rs^2 - 12mr s + 16ts^3, \\
\tilde{Q}_4 &= 16tr(r^2 + 3s) + (9a_1 t^2 - 18x)(r^2 + s) - \frac{6mr^2}{s} - 12m \ln s, \\
\tilde{P}_5 &= 10trs^2 + 2mr^2 + (3a_1 t^2 - 6x)s^2 - 4ms, \\
\tilde{Q}_5 &= 10tr^2 + 10ts + (6a_1 t^2 - 12x)r - \frac{4mr}{s}, \\
\tilde{P}_\infty &= p(r, s) - rq(r, s), \quad \tilde{Q}_\infty = \frac{q(r, s)}{s},
\end{align*}
\]

where \( r = x_t - a_1 t, s = 1/x_m \); functions \( p(r, s), q(r, s) \) are arbitrary solutions of the system of equations

\[
p_r = sq_s + rq_r, \quad p_s = rq_s + q_r.
\]

3.1. \( h = a_1 x^2/2 + a_2 x + a_3, \ a_1 > 0 \). In this case additional conservation laws of the equation (3) correspond to functions

\[
\begin{align*}
\tilde{P}_3 &= e^{-t\sqrt{a_1}}\left(x_t h h' + \frac{\sqrt{a_1}}{2}\left(\frac{1}{x_m^2} - h^2\right)\right), \\
\tilde{Q}_3 &= e^{-t\sqrt{a_1}}(\sqrt{a_1} x_t + (1 - x_m h)h'), \\
\tilde{P}_4 &= e^{t\sqrt{a_1}}\left(x_t h h' - \frac{\sqrt{a_1}}{2}\left(\frac{1}{x_m^2} - h^2\right)\right), \\
\tilde{Q}_4 &= e^{t\sqrt{a_1}}(-\sqrt{a_1} x_t + (1 - x_m h)h').
\end{align*}
\]

3.2. \( h = a_1 x^2/2 + a_2 x + a_3, \ a_1 < 0 \). In this case additional conservation laws of the equation (3) correspond to functions

\[
\begin{align*}
\tilde{P}_3 &= \cos(t\sqrt{-a_1})x_t h h' + \frac{\sqrt{-a_1}}{2}\sin(t\sqrt{-a_1})\left(\frac{1}{x_m^2} - h^2\right), \\
\tilde{Q}_3 &= \sin(t\sqrt{-a_1})\sqrt{-a_1} x_t + \cos(t\sqrt{-a_1})(1 - x_m h)h', \\
\tilde{P}_4 &= \sin(t\sqrt{-a_1})x_t h h' - \frac{\sqrt{-a_1}}{2}\cos(t\sqrt{-a_1})\left(\frac{1}{x_m^2} - h^2\right), \\
\tilde{Q}_4 &= -\cos(t\sqrt{-a_1})\sqrt{-a_1} x_t + \sin(t\sqrt{-a_1})(1 - x_m h)h'.
\end{align*}
\]

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4. \( h = a_1(x + a_2)^{-4/3} + a_3, \ a_1 \neq 0, \ x > -a_2. \) In this case additional conservation laws of the equation (3) correspond to functions

\[
\tilde{P}_3 = -\frac{5tx_t}{x_m^2} - mx_t^2 + \frac{3(x + a_2)}{x_m^2} + 2m\left(\frac{1}{x_m} - h\right),
\]

\[
\tilde{Q}_3 = -5t\left(x_t^2 + \frac{1}{x_m}\right) + 6(x + a_2)x_t + (10h - 8a_3)t + 2mx_tx_m.
\]

The conservation laws for cases 2–4 are additional conservation laws to the conservation laws of the general case 1.

4 Comparison of conservation laws in Euler’s and Lagrangian’s variables

Conservation laws of the equation (3) that do not correspond to the conservation laws of the system of equations (1) are obtained from the conservation laws of the system of equations (2), which depend on the Lagrangian variable \( m. \) The results of the section 3 show that such a conservation laws are exist only in two cases. In the case of \( h = a_1x + a_2 \) conservation laws, which are defined by the functions \((\tilde{P}_4, \tilde{Q}_4)\) and \((\tilde{P}_5, \tilde{Q}_5)\), are not correspond to the conservation laws of the system of equations (1); in the case \( h = a_1(x + a_2)^{-4/3} + a_3 \ (a_1 \neq 0, \ x > -a_2) \) conservation laws, which are defined by the functions \((\tilde{P}_3, \tilde{Q}_3)\), also do not comply with the conservation laws of the system of equations (3). The system of shallow water equations in Eulerian variables (1) has no additional conservation laws in this case [12]. All other first-order conservation laws of the equation (3) are correspond to the conservation laws of the system of equations (1).

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