Quantum dynamics corresponding to the classical BKL scenario

Andrzej Góźdź,1,* Aleksandra Pędraκ,2,† and Włodzimierz Piechocki2,‡

1Institute of Physics, Maria Curie-Skłodowska University, pl. Marii Curie-Skłodowskiej 1, 20-031 Lublin, Poland
2Department of Fundamental Research, National Centre for Nuclear Research, Pasteura 7, 02-093 Warszawa, Poland

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Abstract

We quantize the solution to the Belinski-Khalatnikov-Lifshitz (BKL) scenario using the affine coherent states quantization method. Quantization smears the gravitational singularity avoiding its localization in the configuration space. Classical chaotic behaviour of the BKL scenario becomes enhanced at the quantum level. Our results strongly suggest that the generic singularity of general relativity can be avoided at quantum level giving support to the expectation that quantum gravity has good chance to be a regular theory.

* andrzej.gozdz@umcs.lublin.pl
† aleksandra.pedrak@ncbj.gov.pl
‡ wlodzimierz.piechocki@ncbj.gov.pl

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I. Introduction

The Belinski, Khalatnikov and Lifshitz (BKL) conjecture states that general relativity includes the solution with generic gravitational singularity \([1, 2]\). The evolution towards the BKL singularity, the so-called BKL scenario, consists of the deterministic dynamics turning into stochastic process near the generic singularity. A fundamental question arises whether that singularity can be avoided in the corresponding quantum dynamics.

The evolution process presented in \([1, 2]\) is complicated and difficult to map into quantum evolution. There exists well defined and comparatively simple prototype of the BKL scenario that can be used alternatively in the derivation of the BKL conjecture. It has been presented in \([3]\) (see \([4, 5]\) for more details). We have verified
that prototype’s dynamics leads to the gravitational singularity [6, 7]. That dynamics is generically unstable turning into chaotic process near the singularity [8].

Quantization of the dynamics presented in [3] can be used in the examination of the fate of the corresponding quantum dynamics. In fact, we have already quantized that prototype with the conclusion that quantization of the dynamics leads to avoiding gravitational singularity [10, 11]. In these papers, we have quantized Hamilton’s dynamics. However, since quantization is known to be an ambiguous procedure, we have decided to examine the robustness of these results by making use of a completely different quantization method. That method, applied recently to the quantization of the Schwarzschild spacetime [12], includes quantization of the temporal and spatial variables on the same footing. The rationale for such dealing is that the distinction between time and space variables violates the general covariance of arbitrary transformations of temporal and spatial coordinates.

The results of the present paper, in the context of resolving the cosmological singularity, are similar to the results of the paper [12] addressing the issue of the astrophysical singularity of an isolated object. In both cases quantization smears the singularity avoiding its localization in configuration space. The new phenomenon we deal with in the present paper is the fate of the classical stochasticity of the BKL scenario at quantum level.

The paper is organized as follows: In Sec. II we recall the main results of the paper [8] to have our paper self-contained. Sec. III presents the main aspects of the coherent states quantization method adopted to our gravitational system. In Sec. IV we quantize the solution to the BKL scenario. Stochastic aspects of quantum evolution are presented in Sec. V. We conclude in the last section.

II. SOLUTION TO THE BKL SCENARIO

To have the paper self-contained, we recall in this section the main results of Ref. [8] to be used later.

The BKL scenario is defined to be [3, 5]

\[
\begin{align*}
\frac{d^2 \ln a}{dt^2} &= \frac{b}{a} - a^2, \\
\frac{d^2 \ln b}{dt^2} &= a^2 - \frac{b}{a} + \frac{c}{b}, \\
\frac{d^2 \ln c}{dt^2} &= a^2 - \frac{c}{b},
\end{align*}
\]

subject to the constraint

\[
\frac{d \ln a}{dt}+ \frac{d \ln b}{dt}+ \frac{d \ln c}{dt} = a^2 + \frac{b}{a} + \frac{c}{b},
\]

where \(a = a(t), b = b(t)\) and \(c = c(t)\) are the so-called directional scale factors, while \(t\) is the time variable.
Eqs. (1)–(2) have been derived from the general dynamics of the Bianchi VIII and IX models under the condition that near the singularity the following strong inequalities are satisfied \[ a \gg b \gg c > 0. \] (3)

It has been found in \[8\] that the analytical solutions to Eqs. (1)–(2), for \( t > t_0 \), read
\( a(t) = \frac{3}{t - t_0}, \quad b(t) = \frac{30}{(t - t_0)^3}, \quad c(t) = \frac{120}{(t - t_0)^5}, \) (4)
where \( t - t_0 \neq 0 \) and \( t_0 \) is an arbitrary real number. Thus, the solutions are parameterized by the number \( t_0 \in \mathbb{R} \).

The solution (4) corresponds, for instance in the case \( t > t_0 \) and \( t_0 < 0 \), to the following choice of the initial data
\( a(0) = -3 \, t_0^{-1}, \quad \dot{a}(0) = -3 \, t_0^{-2}, \)
\( b(0) = -30 \, t_0^{-3}, \quad \dot{b}(0) = -90 \, t_0^{-4}, \)
\( c(0) = -120 \, t_0^{-5}, \quad \dot{c}(0) = -600 \, t_0^{-6}. \) (5)

The stability analyses carried out in \[8\] have shown that the solution (4) is unstable against small perturbation. More precisely, substituting the following functions
\( a(t) = 3(t - t_0)^{-1} + \epsilon \alpha(t), \)
\( b(t) = 30(t - t_0)^{-3} + \epsilon \beta(t), \)
\( c(t) = 120(t - t_0)^{-5} + \epsilon \gamma(t), \) into (1)–(2) leads, in the first order in the small parameter \( \epsilon \), to the following solution of the resulting equations
\( \alpha(t) = \exp(-\theta/2)[K_1 \cos(\omega_1 t + \varphi_1) + K_2 \cos(\omega_2 t + \varphi_2)] + K_3 \exp(-2\theta), \) (7a)
\( \beta(t) = \exp(-5\theta/2) \left[ \left(4 + 6\sqrt{6}\right) K_1 \cos(\omega_1 t + \varphi_1) \\
\quad + \left(4 - 6\sqrt{6}\right) K_2 \cos(\omega_2 t + \varphi_2) \right] + 30K_3 \exp(-4\theta), \) (7b)
\( \gamma(t) = -4 \exp(-9\theta/2) \left[ \left(26 + 9\sqrt{6}\right) K_1 \cos(\omega_1 t + \varphi_1) \\
\quad + \left(26 - 9\sqrt{6}\right) K_2 \cos(\omega_2 t + \varphi_2) \right] + 200K_3 \exp(-6\theta), \) (7c)
where \( \theta = \ln(t - t_0) \). The two frequencies read
\( \omega_1 = \frac{1}{2} \sqrt{95 - 24\sqrt{6}}, \quad \omega_2 = \frac{1}{2} \sqrt{95 + 24\sqrt{6}}, \) (8)
where \( K_1, K_2, K_3, \varphi_1, \) and \( \varphi_2 \) are constants.

The manifold \( \mathcal{M} \) defined by \( \{ K_1, K_2, K_3, \varphi_1, \varphi_2 \} \) is a submanifold of \( \mathbb{R}^5 \). The solution defined by (6) and (7) corresponds to the choice of the set of the initial data \( \mathcal{N} \) that is a small neighborhood of the initial data (5). \( \mathcal{N} \) is a submanifold of \( \mathbb{R}^5 \) as (5) defines five independent constants due to the constraint (2). Therefore, it is clear that (7) presents a generic solution as the measures of both \( \mathcal{M} \) and \( \mathcal{N} \) are nonzero. The exact solution (4) alone is of zero-measure in the space of all possible solutions to Eqs. (1)–(2).

The relative perturbations \( \alpha/a, \beta/b, \) and \( \gamma/c \) grow proportionally as \( \exp\left(\frac{1}{2} \theta\right) \). The multiplier \( 1/2 \) plays the role of a Lyapunov exponent, describing the rate of their divergence. Since it is positive, the evolution of the system towards the gravitational singularity \( (\theta \to +\infty) \) is chaotic. The transition towards the chaos occurs if the evolution begins with the initial data which belong, for instance, to the neighbourhood of the conditions (5).

III. AFFINE COHERENT STATES QUANTIZATION

We propose to quantize the classical BKL scenario by using the integral quantization called the affine coherent states quantization (ACSQ) (see, e.g., [9] and references therein). We have recently applied this approach in the context of cosmology [10, 11] and astrophysics [13].

In general relativity time and position in space are treated on the same level, however, in quantum mechanics time is considered as a parameter enumerating events, not as a quantum observable. In this paper, in quantum description we treat time and position on the same footing. They are related to operators obtained by the affine coherent states quantization. This idea requires introducing the notion of an extended classical configuration space by including time as an additional coordinate. The correspondence between the classical time and position is done by comparing their classical values with expectation values of their quantum counterparts.

To begin with, we introduce two configuration spaces defined as follows: the classical gravitational configuration space \( T_{BKL} \)

\[
T_{BKL} := \{ (t, a, b, c) : (t, a, b, c) \in \mathbb{R} \times \mathbb{R}_+^3 \},
\]  
where \( \mathbb{R}_+ = (0, +\infty) \), and the affine configuration space \( T \)

\[
T = \{ \xi = (\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) : \xi \in (\mathbb{R} \times \mathbb{R}_+) \times (\mathbb{R} \times \mathbb{R}_+) \times (\mathbb{R} \times \mathbb{R}_+) \},
\]  
where every pair \( (\xi_k, \xi_{k+1}) \), \( (k = 1, 3, 5) \), parameterizes the affine group \( \text{Aff}(\mathbb{R}) \).
The variables with even indices correspond to the scale factors $\xi_2 = a$, $\xi_4 = b$, $\xi_6 = c$. Because $a, b, c > 0$ and $\xi_1, \xi_3, \xi_5 \in \mathbb{R}$, the configuration space $T$ parametrizes the simple product of 3 affine groups $\text{Aff}(\mathbb{R}) \times \text{Aff}(\mathbb{R}) \times \text{Aff}(\mathbb{R}) := G$ to be used in the affine quantization.

As the observational data are parameterized by a single time parameter, the variables $\xi_1$, $\xi_3$, $\xi_5$ should be mapped onto a single variable representing time.

The affine group $\text{Aff}(\mathbb{R})$ is known to have two nontrivial unitary irreducible representations in the Hilbert space $\mathcal{H}_x := L^2(\mathbb{R}_+, d\nu(x))$, where $d\nu(x) := dx/x$. We choose the one defined as follows

$$U(\xi_k, \xi_{k+1})\Psi(x) = e^{i\xi_k x}\Psi(\xi_{k+1} x), \quad (11)$$

where, $k = 1, 3, 5$ and $\langle x|\Psi \rangle =: \Psi(x) \in \mathcal{H}_x$. The action (11) corresponds to the standard parametrization of the affine group $\text{Aff}(\mathbb{R})$ defined by the multiplication law

$$(\xi_k, \xi_{k+1}) \cdot (\xi_k', \xi_{k+1}') := (\xi_k + \xi_{k+1}\xi_k', \xi_{k+1}\xi_{k+1}') \in \text{Aff}(\mathbb{R}). \quad (12)$$

The left invariant measure on the group $\text{Aff}(\mathbb{R})$ reads

$$d\mu(\xi_k, \xi_{k+1}) := d\xi_k \frac{d\xi_{k+1}}{\xi_{k+1}^2}, \quad (13)$$

and the corresponding invariant integration over the affine group is defined to be

$$\int_{\text{Aff}(\mathbb{R})} d\mu(\xi_k, \xi_{k+1}) := \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi_k \int_{0}^{\infty} d\xi_{k+1}/\xi_{k+1}^2. \quad (14)$$

It is clear that the group $G$ has the unitary irreducible representation in the Hilbert space $\mathcal{H} = \mathcal{H}_{x_1} \otimes \mathcal{H}_{x_2} \otimes \mathcal{H}_{x_3} = L^2(\mathbb{R}_+^3, d\nu(x_1, x_2, x_3))$, where $d\nu(x_1, x_2, x_3) = d\nu(x_1)d\nu(x_2)d\nu(x_3)$. It enables defining the continuous family of affine coherent states, $\langle x_1, x_2, x_3|\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6 \rangle := \langle x_1|\xi_1, \xi_2\rangle\langle x_2|\xi_3, \xi_4\rangle\langle x_3|\xi_5, \xi_6 \rangle$, as follows

$$\mathcal{H} \ni \langle x_1, x_2, x_3|\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6 \rangle := U(\xi)\Phi_0(x_1, x_2, x_3), \quad (15)$$

where $U(\xi) := U(\xi_1, \xi_2)U(\xi_3, \xi_4)U(\xi_5, \xi_6)$, and $|\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6 \rangle := |\xi_1, \xi_2\rangle|\xi_3, \xi_4\rangle|\xi_5, \xi_6 \rangle$ and where

$$\mathcal{H} \ni \Phi_0(x_1, x_2, x_3) = \Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3). \quad (16)$$

In (16) the vectors $\Phi_k(x_k) \in L^2(\mathbb{R}_+, d\nu(x_k))$, $k = 1, 2, 3$ are the so-called fiducial vectors. They are required to satisfy the two conditions

$$\int_{0}^{\infty} \frac{dx}{x} |\Phi_k(x)|^2 = 1, \quad (17)$$

6
and

\[ A_{\phi_{l}} := \int_{0}^{\infty} \frac{dx}{x^2} |\Phi_{k}(x)|^2 < \infty. \quad (18) \]

where \( l = 1 \) for \( k = 1 \), \( l = 3 \) for \( k = 2 \) and \( l = 5 \) for \( k = 3 \). The fiducial vectors are the free “parameters” of this quantization scheme.

Finally, we have

\[
U(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6)\Phi_0(x_1, x_2, x_3) = \\
U(\xi_1, \xi_2)U(\xi_3, \xi_4)U(\xi_5, \xi_6)\Phi_1(x_1)\Phi_2(x_2)\Phi_3(x_3) = \\
e^{i(\xi_1 x_1 + \xi_3 x_2 + \xi_5 x_3)}\Phi_1(\xi_2 x_1)\Phi_2(\xi_4 x_2)\Phi_3(\xi_6 x_3). \quad (19)
\]

The irreducibility of the representation leads to the resolution of the unity in \( \mathcal{H} \) as follows

\[
\frac{1}{A_{\phi}} \int_{G} d\mu(\xi) |\xi\rangle \langle \xi| := \bigotimes_{k=1,3,5} \frac{1}{A_{\phi}} \int_{\text{Aff}(\mathbb{R})} d\mu(\xi_k, \xi_{k+1}) |\xi_k, \xi_{k+1}\rangle \langle \xi_k, \xi_{k+1}| \\
= \frac{1}{A_{\phi_1}A_{\phi_2}A_{\phi_5}} \int_{\text{Aff}(\mathbb{R})} d\mu(\xi_1, \xi_2) |\xi_1, \xi_2\rangle \langle \xi_1, \xi_2| \bigotimes \int_{\text{Aff}(\mathbb{R})} d\mu(\xi_3, \xi_4) |\xi_3, \xi_4\rangle \langle \xi_3, \xi_4| \\
\bigotimes \int_{\text{Aff}(\mathbb{R})} d\mu(\xi_5, \xi_6) |\xi_5, \xi_6\rangle \langle \xi_5, \xi_6| = \hat{1}_1 \otimes \hat{1}_2 \otimes \hat{1}_3 = \hat{1}, \quad (20)
\]

where \( d\mu(\xi) = \Pi_{k=1,3,5} d\mu(\xi_k, \xi_{k+1}) \).

The resolution (20) can be used for mapping an observable \( f : T \rightarrow \mathbb{R} \) into an operator \( \hat{f} : \mathcal{H} \rightarrow \mathcal{H} \) as follows [12]

\[
\hat{f} := \frac{1}{A_{\phi}} \int_{G} d\mu(\xi) |\xi\rangle f(\xi) \langle \xi| \\
= \frac{1}{A_{\phi_1}A_{\phi_2}A_{\phi_5}} \int_{\text{Aff}(\mathbb{R})} d\mu(\xi_1, \xi_2) \int_{\text{Aff}(\mathbb{R})} d\mu(\xi_3, \xi_4) \int_{\text{Aff}(\mathbb{R})} d\mu(\xi_5, \xi_6) \\
|\xi_1, \xi_2; \xi_3, \xi_4; \xi_5, \xi_6\rangle f(\xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6) \langle \xi_1, \xi_2, \xi_3, \xi_4, \xi_5, \xi_6|, \quad (21)
\]

where \( A_{\phi} := A_{\phi_1}A_{\phi_2}A_{\phi_5} \).

Expectation values and variances of quantum observables are the most important characteristics which allow to compare quantum and classical worlds. Expectation values of quantum observables correspond to classical values of measured quantities and their variances describe quantum smearing of these observables.

A general form of expectation value of the observable \( \hat{f} \) obtained from the classical function \( f \), while the quantum system is in the state \( |\Psi\rangle \), reads

\[
\langle \hat{f}; \Psi \rangle := \langle \Psi | \hat{f} | \Psi \rangle = \frac{1}{A_{\phi}} \int_{G} d\mu(\xi) f(\xi) |\langle \xi| \langle \xi| |^2. \quad (22)
\]

7
The variance of an observable $\hat{f}$ defined as $\text{var}(\hat{f}; \Psi) := \langle (\hat{f} - \langle \hat{f}; \Psi \rangle)^2 \rangle_{\Psi}$ is more difficult for calculations because it requires $\langle \hat{f}^2; \Psi \rangle$ which involves an overlap between the coherent states, and usually depends on the fiducial vector explicitly:

$$
\langle (\hat{f})^2; \Psi \rangle := \langle \Psi | (\hat{f})^2 | \Psi \rangle = \frac{1}{A_\Phi} \int_G d\mu(\xi) \frac{1}{A_\Phi} \int_G d\mu(\xi') \langle \Psi | \xi \rangle f(\xi) \langle \xi | \xi' \rangle f(\xi') \langle \xi' | \Psi \rangle.
$$

(23)

The variance $\text{var}(\hat{f}; \Psi)$ can be rewritten as

$$
\text{var}(\hat{f}; \Psi) = \langle \hat{f}^2; \Psi \rangle - \langle \hat{f}; \Psi \rangle^2.
$$

(24)

The elementary variables $\xi_k$ $(k = 1, 2, \ldots, 6)$, can be mapped into the quantum operators

$$
\hat{\xi}_k = \frac{1}{A_\Phi} \int_G d\mu(\xi) |\xi\rangle \xi_k \langle \xi|.
$$

(25)

For every $k$ the above equality (25) reduces to integration over a single affine group. The other integrations give the unit operators in two remaining spaces $H_{x_l}$, $l \neq k$. For example,

$$
\hat{\xi}_2 = \frac{1}{A_{\Phi_1}} \int_{\text{Aff}(\mathbb{R})_1} d\mu(\xi_1, \xi_2) |\xi_1, \xi_2\rangle \xi_2 \langle \xi_1, \xi_2| \otimes \hat{1}_{x_1} \otimes \hat{1}_{x_3}.
$$

(26)

To deal with a single time variable at quantum level, one needs to choose a model of time in the configuration space $T$, defined by (10). In general, it can be introduced either as a real function or distribution, $T : T \to \mathbb{R}$. Its quantization leads to the time operator $\hat{T}$. However, we can impose the appropriate constraints to have the common time variable for all three operators $\hat{\xi}_1$, $\hat{\xi}_3$, and $\hat{\xi}_5$. In this paper we realise that option assuming that the only allowed quantum states $\Psi$ of our BKL system are the states which satisfy the condition

$$
\langle \Psi | \hat{\xi}_1 | \Psi \rangle = \langle \Psi | \hat{\xi}_3 | \Psi \rangle = \langle \Psi | \hat{\xi}_5 | \Psi \rangle,
$$

(27)

which is consistent with the choice of the configuration space in the form (9). It means, we require the same expectation values for all 3 operators which represent three “times” related to appropriate quantum observables $\hat{a} = \hat{\xi}_2$, $\hat{b} = \hat{\xi}_4$, and $\hat{c} = \hat{\xi}_6$.

**IV. QUANTIZATION OF THE SOLUTION TO BKL SCENARIO**

The above quantization scheme can be now applied to the solutions (7) of the BKL scenario$^1$

$$
a(t) = \bar{a}(t) + \epsilon \alpha(t), \quad b(t) = \bar{b}(t) + \epsilon \beta(t), \quad c(t) = \bar{c}(t) + \epsilon \gamma(t),
$$

(28)

$^1$ In what follows we denote by $\bar{a}$, $\bar{b}$ and $\bar{c}$ the solution (4).
ascribing to them appropriate quantum states and the corresponding operators. In quantum mechanics, contrary to classical one, one needs two kinds of objects to describe physical world. These are quantum observables represented by either appropriate operators or operator valued measures and quantum states being the vectors in Hilbert space, or the so called density operators. In the classical mechanics the functions on either configuration or phase space are at the same time states and observables.

To quantize solutions of the BKL scenario, we already have the elementary observables $\hat{\xi}_k$, however we have to find the appropriate family of states related to the solutions (28). The classical solutions are represented by tree time dependent functions. In the configuration space $T$ we have 6 variable, where 3 of them $\xi_1, \xi_3, \xi_5$ represent the time in the state space which satisfies the condition (27). As is commonly known, in quantum mechanics the classical observables can be related to their quantum counterparts by the corresponding expectation values. This idea leads directly to the conditions for a family of states $\{\Psi_\tau(x_1, x_2, x_3) = \langle x_1, x_2, x_3|\Psi_\tau\rangle, \tau \in \mathbb{R}\}$ parameterized by the evolution parameter $\tau$, which in general can be different from the classical time $t$.

The states $|\Psi_\tau\rangle$ are required to satisfy the following conditions [12]:

$$
\langle \Psi_\tau|\hat{\xi}_k|\Psi_\tau\rangle = t, \quad k = 1, 3, 5 \\
\langle \Psi_\tau|\hat{\xi}_2|\Psi_\tau\rangle = a(t) \\
\langle \Psi_\tau|\hat{\xi}_4|\Psi_\tau\rangle = b(t) \\
\langle \Psi_\tau|\hat{\xi}_6|\Psi_\tau\rangle = c(t).
$$

The equations (29) represent the single time constraint (27). The parameter $\tau$ labels the allowed family of states. It has to be related to the classic time $t$ as a one-to-one monotonic function, i.e., $\tau_1 \leq \tau_2 \Leftrightarrow t_1 \leq t_2$ in Eq.(29).

In what follows, we determine the states $|\Psi_\tau\rangle$ satisfying the conditions (29)–(32). This will enable examination of the issue of the fate of the gravitational singularity and chaos of the BKL scenario at the quantum level.

### A. Evolving wave packets

In our paper we consider two kinds of wave packets satisfying the conditions (29)–(32). The first kind are the affine coherent states themselves. The second type is a set of modified “exponential” wave packets, which represent a dense set of states in the Hilbert space $\mathcal{H}$. 

9
1. Coherent states (CS) and expectation values

One can verify that considered coherent states generated by a single affine group satisfy, due to the results of the recent paper [12], the following equations

\[ \langle \xi_k, \xi_{k+1} | \hat{\xi}'_l | \xi_k, \xi_{k+1} \rangle = \xi_l, \text{ where } l = k, k + 1; \ k = 1, 3, 5, \]  

(33)

where the operators \( \hat{\xi}'_l \) are defined to be

\[ \hat{\xi}'_l := \frac{1}{2\pi A_\Phi} \int_{\mathbb{R}} d\xi_k \int_{\mathbb{R}^+} \frac{d\xi_{k+1}}{\xi^2_{k+1}} |\xi_k, \xi_{k+1}\rangle \langle \xi_k, \xi_{k+1}|, \]

(34)

and where \( l = k, k + 1, k = 1, 3, 5. \)

The conditions (33) are the consistency conditions between the affine group parametrization and the configuration space of the quantized physical system [12].

This implies that the coherent states generated by the product of 3 affine groups also satisfy the consistency condition

\[ \langle \xi | \hat{\xi}_l | \xi \rangle = \frac{1}{A_\phi} \int_{G} d\mu(\xi) |\xi_l\rangle \langle \xi_l| = \xi_l, \text{ where } l = 1, 2, \ldots, 6. \]  

(35)

Therefore, the coherent states

\[ |CS_\epsilon; t\rangle := |t, \tilde{a}(t) + \epsilon \alpha(t); t, \tilde{b}(t) + \epsilon \beta(t); t, \tilde{c}(t) + \epsilon \gamma(t)\rangle \]  

(36)

satisfy the equation of motions (29)–(32). In such case, we propose to use \( \tau = t \), i.e., the classical time as a label of the evolving family of quantum states. Realization of (36) as a wave packet constructed in the space of square integrable functions \( L^2(\mathbb{R}^3, d\nu(x_1, x_2, x_3)) \) reads

\[ \Psi_{CS}(t, x_1, x_2, x_3) = \langle x_1, x_2, x_3 |CS_\epsilon; t\rangle \]  

\[ = e^{it(x_1+x_2+x_3)} \Phi_1(a(t)x_1) \Phi_2(b(t)x_2) \Phi_3(c(t)x_3). \]  

(37)

2. The modified exponential packet (E) and expectation values

Let us consider the set of Gaussian distribution wave packets (with modified exponential part)

\[ \Psi_n(x; \tau, \gamma) = N x^n \exp \left[ i\tau x - \frac{\gamma^2 x^2}{2} \right], \ N^2 = \frac{2\gamma^n}{(n-1)!}. \]  

(38)
which according to [12] is dense in $L^2(\mathbb{R}_+, d\nu(x))$.

In the space $L^2(\text{Aff}(\mathbb{R}), d\mu(\xi, \xi_{k+1}))$ the expectation values and variances of the operators $\hat{\xi}_k$ and $\hat{\xi}_{k+1}$ have the following values

$$\langle \Psi_n | \hat{\xi}_k | \Psi_n \rangle = \tau, \quad (39)$$

$$\langle \Psi_n | \hat{\xi}_{k+1} | \Psi_n \rangle = \frac{1}{A_\phi} \frac{\Gamma(n - \frac{1}{2})}{(n - 1)!} \gamma, \quad (40)$$

$$\text{var}(\hat{\xi}_k; \Psi_n) = \frac{4n - 3}{4(n - 1)^2} \gamma^2, \quad (41)$$

$$\text{var}(\hat{\xi}_{k+1}; \Psi_n) = \frac{1}{A_\phi^2} \left( \frac{1}{n - 1} - \frac{\Gamma(n - \frac{1}{2})^2}{(n - 1)!^2} \right) \gamma^2. \quad (42)$$

In the space $L^2(\mathbb{R}_+^3, d\nu(x_1, x_2, x_3))$ we take the corresponding wave packets in the form

$$\Psi_{n_1, n_3, n_5}(x_1, x_2, x_3; \tau_1, \tau_3, \gamma_3, \gamma_5) = \Psi_{n_1}(x_1; \tau_1, \gamma_1) \Psi_{n_3}(x_2; \tau_3, \gamma_3) \Psi_{n_5}(x_3; \tau_5, \gamma_5). \quad (43)$$

To meet the properties (29)–(32) for the wave packets $\Psi_{n_1, n_3, n_5}$, we can chose the parameters $\tau_k$ and $\gamma_k$ as follows

$$\tau_1 = \tau_2 = \tau_3 = t, \quad (44)$$

$$\gamma_k = A_\phi \frac{(n_k - 1)!}{\Gamma(n_k - \frac{1}{2})} \cdot f_k(t), \quad k = 1, 3, 5, \quad (45)$$

where

$$f_k(t) = \begin{cases} \tilde{a}(t) + \epsilon \alpha(t), \quad k = 1 \\ \tilde{b}(t) + \epsilon \beta(t), \quad k = 3 \\ \tilde{c}(t) + \epsilon \gamma(t), \quad k = 5 \end{cases}. \quad (46)$$

B. Variances in the Hilbert space $\mathcal{H}$

1. Using coherent states

The variance of the operators $\hat{\xi}_k$, $\hat{\xi}_{k+1}$, $k = 1, 3, 5$ in coherent states (36) read

$$\text{var}(\hat{\xi}_k; |CS; t\rangle) = \langle \hat{\xi}_k^2 \rangle_0 f_k(t)^2, \quad (47)$$

$$\text{var}(\hat{\xi}_{k+1}; |CS; t\rangle) = \left( \langle \hat{\xi}_{k+1}^2 \rangle_0 - 1 \right) f_k(t)^2, \quad (48)$$

where

$$\langle \hat{\xi}_{i}^2 \rangle_0 = \langle 0, 1 | \hat{\xi}_{i}^2 | 0, 1 \rangle, \quad i = 1, 2, \ldots 6. \quad (49)$$

For more details concerning the r.h.s. of (49) see App. D of [12].
2. Using exponential wave packet

The corresponding results for the wave packets (43), under the conditions (44)-(44), read
\begin{align}
\text{var}(\hat{\xi}_k; \Psi_{n_1,n_3,n_5}) &= A_k f_k(t)^2, \quad (50) \\
\text{var}(\hat{\xi}_{k+1}; \Psi_{n_1,n_3,n_5}) &= B_k f_k(t)^2, \quad (51)
\end{align}

where
\begin{align}
A_k &= A_{\Phi_k}^2 \frac{(4n_k - 3)(n_k - 1)!(n_k - 2)!}{4\Gamma \left(n_k - \frac{1}{2}\right)^2}, \quad (52) \\
B_k &= \frac{(n_k - 1)!(n_k - 2)!}{\Gamma \left(n_k - \frac{1}{2}\right)^2} - 1. \quad (53)
\end{align}

V. STOCHASTIC ASPECTS OF QUANTUM EVOLUTION

The results of recent paper [8] give strong support to the expectation that near the generic gravitational singularity the evolution becomes chaotic. In what follows we examine that fundamental property of the BKL scenario at quantum level.

To make direct comparison with the results of [8], we split the variances (51) into the contributions from unperturbed and perturbed states. We have:
\begin{align}
f_2(t)^2 &= (\tilde{a}(t) + \epsilon\alpha(t))^2 = \tilde{a}(t)^2 + 2\epsilon\tilde{a}(t)\alpha(t) + \epsilon^2\alpha(t)^2 \simeq \tilde{a}(t)^2 + 2\epsilon\tilde{a}(t)\alpha(t), \quad (54) \\
f_4(t)^2 &= (\tilde{b}(t) + \epsilon\beta(t))^2 = \tilde{b}(t)^2 + 2\epsilon\tilde{b}(t)\beta(t) + \epsilon^2\beta(t)^2 \simeq \tilde{b}(t)^2 + 2\epsilon\tilde{b}(t)\beta(t), \quad (55) \\
f_6(t)^2 &= (\tilde{c}(t) + \epsilon\gamma(t))^2 = \tilde{c}(t)^2 + 2\epsilon\tilde{c}(t)\gamma(t) + \epsilon^2\gamma(t)^2 \simeq \tilde{c}(t)^2 + 2\epsilon\tilde{c}(t)\gamma(t). \quad (56)
\end{align}

The corresponding dimensionless functions describing relative quantum perturbations are defined to be
\begin{align}
k_k := \frac{\text{var}(\hat{\xi}_{k+1}; \Psi_{\text{pert}}) - \text{var}(\hat{\xi}_{k+1}; \Psi_{\text{unpert}})}{\text{var}(\hat{\xi}_{k+1}; \Psi_{\text{unpert}})}, \quad k = 1, 2, 3, \quad (57)
\end{align}

where \(\Psi_{\text{pert}}\) and \(\Psi_{\text{unpert}}\) denote perturbed and unperturbed wave packets, respectively.
Figure 1: The $t$ dependence of quantum perturbation defined by (58)-(60) for $K_1 = K_2 = 0.01$, $K_3 = 0$, $\phi_1 = \phi_2 = 0$, $t_0 = 0$, $\epsilon = 0.01$. The plot presents the parametric curve $\{\kappa_a(t), \kappa_b(t), \kappa_c(t)\}$, where $t \in (0.01, 35)$.

The explicit form of (57), up to the 1-st order in $\epsilon$, reads:

$$
\kappa_a(t) = \frac{2\epsilon \tilde{a}(t)\alpha(t)}{\tilde{a}(t)^2} = 2\epsilon \frac{\alpha(t)}{\tilde{a}(t)}, \quad (58)
$$

$$
\kappa_b(t) = \frac{2\epsilon \tilde{b}(t)\beta(t)}{\tilde{b}(t)^2} = 2\epsilon \frac{\beta(t)}{\tilde{b}(t)}, \quad (59)
$$

$$
\kappa_c(t) = \frac{2\epsilon \tilde{c}(t)\gamma(t)}{\tilde{c}(t)^2} = 2\epsilon \frac{\gamma(t)}{\tilde{c}(t)}. \quad (60)
$$

It is clear that the relative perturbations (58)–(60) are the same for the coherent states and the exponential wave packets.

Figure 1 presents the parametric curve visualizing the relative quantum perturbations. The time dependence of the expectation values of $\hat{\xi}_2$ operator and corresponding variances of unperturbed and perturbed solutions are presented in Fig. 2. The plots for $\hat{\xi}_4$ and $\hat{\xi}_6$ operators would look similarly so that we do not present them.
Figure 2: The $t$ dependence of the expectation value of the operator $\hat{\xi}_2$ defined by (40), (45) for $K_1 = K_2 = 0.01$, $K_3 = 0$, $\phi_1 = \phi_2 = 0$, $t_0 = 0$, $n_1 = 3$. Axis of $t$ is in logarithmic scale. The left panel correspond to unperturbed solution ($\epsilon = 0$), the right panel correspond to perturbed solution ($\epsilon = 0.01$). The blue area defines the points for which distance from expected value is smaller than $\sqrt{\text{var}(\xi_2; \Psi_n)}$ defined by (51) (the distance is counted along fixed $t$ line).

VI. CONCLUSIONS

It has been shown [8] that the perturbed classical solution near the gravitational singularity exhibits some elements of the chaotic behaviour. Our results present the quantum dynamics corresponding to the BKL scenario. Fig. 1 shows that the relative quantum perturbations grow as the system evolves towards the singularity, which is consistent with the corresponding classical evolution, see Fig. 2 of [8]. The classical stochastic behaviour becomes enhanced while approaching the singularity.

The calculated variances, presented in Fig. 2, are always non-zero. The quantum randomness amplifies the deterministic classical chaos. This supports the hypothesis that in the region corresponding to the neighbourhood of the classical singularity the dynamics, both classical and quantum, enters the stochastic phase. The oscillatory behaviour of the expectation value of the quantum scale factor increases as $t \to \infty$, which is consistent with the classical BKL scenario [2, 5].

The non-zero variance removes the singularity from the quantum evolution. The smeared dynamics leads to the conclusion that the probability of hitting the sin-
gularity is equal to zero. The quantization procedure including quantization of the temporal and spatial variables on the same footing has enabled the construction of consistent non-singular quantum theory.

The results of the present paper support our previous results \cite{10, 11} concerning the fate of the BKL singularity at the quantum level. The BKL conjecture states that general relativity (GR) includes generic gravitational singularity. Our results strongly suggest that the generic singularity of GR can be avoided at quantum level so that one can expect that a theory of quantum gravity (to be constructed) has good chance to be regular.

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