On the birational geometry for irreducible symplectic 4-folds related to the Fano schemes of lines

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1 Introduction

A compact Kähler manifold \( X \) is said to be an irreducible symplectic manifold when \( X \) is a simply connected and \( H^0(X, \Omega^2_X) \) is generated by an everywhere non-degenerate holomorphic 2-form \( \sigma_X \). We call such a holomorphic 2-form \( \sigma_X \) a symplectic form. Let \( Y \) be a smooth cubic 4-fold and \( F(Y) \) the set of all lines contained in \( Y \). \( F(Y) \) is said to be the Fano schemes of lines. Beauville and Donagi [B-D] showed that \( F(Y) \) is an irreducible symplectic 4-fold. Examples of irreducible symplectic manifolds are few. Most of them are constructed as the moduli spaces of stable sheaves on K3 surfaces or Abelian surfaces.

Assume that a finite group \( G \) acts on \( F(Y) \). If \( G \) preserves a symplectic form \( \sigma_{F(Y)} \) then there exists a symplectic form \( \tilde{\sigma} \) on the smooth locus of \( X_G := F(Y)/G \). In general \( X_G \) has singular points. Let \( \nu : \tilde{X}_G \to X_G \) be a resolution of \( X_G \). The symplectic 2-form \( \tilde{\sigma} \) lifts to an everywhere non-degenerate 2-form on \( \tilde{X}_G \) if and only if the resolution \( \nu \) is crepant. If \( \nu \) is crepant, then \( \tilde{X}_G \) is also an irreducible symplectic 4-fold. The \( G \)-action on \( F(Y) \) is said to be a good action when \( G \) preserves \( \sigma_{F(Y)} \) and \( X_G \) has a crepant resolution. Can we find good actions on \( F(Y) \)? When it is possible, what is \( \tilde{X}_G \) ?

In this paper, we assume that \( G \) is a subgroup of the automorphism group \( PGL(5) \) of \( \mathbb{P}^5 \) and \( G \) preserves \( Y \). We study two examples of good actions on \( F(Y) \). In the first example, the resolution \( \tilde{X}_G \) is birational to the generalized

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Kummer variety $K^2(A)$ of an Abelian surface $A$. In the second one, $\tilde{X}_G$ is birational to the 2-points Hilbert scheme $\text{Hilb}^2(S)$ of a K3 surface $S$. We have to investigate the $G$-action on symplectic forms on $F(Y)$. The Abel-Jacobi map defines a Hodge isomorphism of type $(-\psi_3,2)$. There is a natural birational map $X$ and $\tilde{X}_G$ to the Hilbert scheme $\text{Hilb}^2(Y)$. Using the identification with $H^0(F(Y),\Omega^2_{F(Y)})$ and $H^1(Y,\Omega^3_Y)$, we can check whether $G$ preserves the symplectic form or not (see Lemma 2.3).

In Section 3, we consider the first example, which was found by Namikawa ([Nam], 1.7.(iv)). Let $Y$ be a smooth cubic 4-fold defined by

$$\{(z_0 : \cdots : z_5) \in \mathbb{P}^2 | f(z_0, z_1, z_2) + g(z_3, z_4, z_5) = 0\},$$

where $f$ and $g$ are homogeneous cubic polynomials. In addition, we consider two elliptic curves defined by $C := \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 | f(z_0, z_1, z_2) = 0\}$ and $D := \{(z_3 : z_4 : z_5) \in \mathbb{P}^2 | g(z_3, z_4, z_5) = 0\}$. Assume that $G$ is the cyclic group $\mathbb{Z}_3$ of order 3. The group $G$ acts on $Y$ in the following way:

$$\mathbb{Z}_3 \times Y \ni (z_0 : \cdots : z_5) \mapsto (z_0 : z_1 : z_2 : \zeta z_3 : \zeta^2 z_4 : \zeta z_5)$$

where $\zeta = \exp(2\pi \sqrt{-1}/3)$. $X_{\mathbb{Z}_3}$ has singularities of type $A_2$ along $C \times D$ and is smooth otherwise. So $X_{\mathbb{Z}_3}$ has a crepant resolution $\tilde{X}_{\mathbb{Z}_3}$ (Lemma 3.1). $\tilde{X}_{\mathbb{Z}_3}$ is birational to $K^2(C \times D)$ for the elliptic curves $C$ and $D$ (Proposition 3.2). There is a natural birational map $\psi : \tilde{X}_{\mathbb{Z}_3} \dashrightarrow K^2(C \times D)$ which is the Mukai flop on disjoint 18 copies of $\mathbb{P}^2$ (Theorem 3.5). Furthermore when $C$ and $D$ are not isogenous, we found 2 more symplectic birational models $X_I$ and $X_{II}$ other than $\tilde{X}_{\mathbb{Z}_3}$ and $K^2(C \times D)$ (Theorem 3.11).

In Section 4, we consider the second example. Let $Y$ be a smooth cubic 4-fold defined by $\{(z_0 : \cdots : z_5) \in \mathbb{P}^5 | z_0^3 + \cdots + z_5^3 = 0\}$. Put

$$M = \left\{ A = \begin{pmatrix} a_1 & \cdots & \cdots & a_6 \\ \end{pmatrix} \in \text{PGL}(5) | \#\{a_i = 1\} = \#\{j | a_j = \zeta\} = 3 \right\}.$$  

Let $G$ be the finite group generated by $M$. Then $G$ is isomorphic to $\mathbb{Z}_3^4$ and the $G$-action on $F(Y)$ turns out to be good. Let $E$ be the elliptic curve defined by $\{(z_0 : z_1 : z_2) \in \mathbb{P}^2 | z_0^3 + z_1^3 + z_2^3 = 0\}$. Since $E$ has a complex multiplication $\zeta \curvearrowright E \ni x \mapsto \zeta x$, the Abelian surface $E \times E$ has the $\mathbb{Z}_3$ action given by $\mathbb{Z}_3 \curvearrowright E \times E \ni (x,y) \mapsto (\zeta x, \zeta^2 y) \in E \times E$. This $\mathbb{Z}_3$ action on $E \times E$ preserves a symplectic form on $E \times E$, and $(E \times E)/\mathbb{Z}_3$ has a crepant resolution $S \rightarrow (E \times E)/\mathbb{Z}_3$. In particular $S$ is a K3 surface. We shall prove that $\tilde{X}_G$ is birational to $\text{Hilb}^2(S)$ (Theorem 4.2).

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2 Preliminaries

In this section we prepare some facts needed later.

2.1 On irreducible symplectic manifolds

Let $A$ be an Abelian surface and $\text{Hilb}^{n+1}(A)$ the Hilbert scheme of $(n+1)$-points on $A$. We define

$\text{Sym}^{n+1}(A) := A^{n+1}/\mathfrak{S}_{n+1}$

where $\mathfrak{S}_{n+1}$ is the $(n+1)$-th symmetric group. Let $\mu$ be the Hilbert-Chow morphism:

$\mu : \text{Hilb}^{n+1}(A) \to \text{Sym}^{n+1}(A)$. 

We define the map $\Sigma : \text{Sym}^{n+1}(A) \to A$ by $\Sigma(\{x_1, \cdots, x_{n+1}\}) := \sum_{i=1}^{n+1} x_i$ and consider the composite $\Sigma \circ \mu$. We define:

$K^n(A) := (\Sigma \circ \mu)^{-1}(0)$. 

We call $K^n(A)$ the generalized Kummer variety of $A$. $K^n(A)$ is an irreducible symplectic manifold \cite{Bea}. We also define $\bar{K}^n(A) := \Sigma^{-1}(0)$. $\bar{K}^n(A)$ is isomorphic to

$\bar{K}^n(A) \cong \{(x_1, \cdots, x_{n+1}) \in A^{n+1} | x_1 + \cdots + x_{n+1} = 0 \}/\mathfrak{S}_{n+1}$. 

For an elliptic curve $C$, we define

$K^2(C) := \{\{p_1, p_2, p_3\} \in \text{Hilb}^3(C) | p_1 + p_2 + p_3 = 0\}$. 

Notice that $K^2(C)$ is isomorphic to $\mathbb{P}^2$.

2.2 Fano schemes

We collect some facts for Fano schemes needed later. Let $Y$ be a smooth cubic 4-fold. Define

$F(Y) := \{l \subset Y | l \cong \mathbb{P}^1, \deg l = 1\}$. 

$F(Y)$ is known to be a smooth projective variety of dimension 4. Furthermore we have the following theorem.

**Proposition 2.1** (Beauville-Donagi\cite{B-D}). *The notation being as above, $F(Y)$ is an irreducible symplectic manifold. $F(Y)$ is deformation equivalent to the Hilbert scheme $\text{Hilb}^2(S)$ of 2-point on a K3 surface $S$.***
The following relation between $H^4(Y, \mathbb{C})$ and $H^2(F(Y), \mathbb{C})$ is important to us.

**Lemma 2.2 ([B-D]).** Let $\mathcal{L}$ be the universal family of lines:

$$\mathcal{L} := \{(l, y) \in F(Y) \times Y | l \ni y\}.$$  

Let $p$ and $q$ be the projections from $\mathcal{L}$ to $F(Y)$ and $Y$ respectively. We define the Abel-Jacobi map $\alpha$ by

$$\alpha : H^4(Y, \mathbb{C}) \to H^2(F(Y), \mathbb{C}), \quad \alpha(\omega) := p_* q^*(\omega).$$

Then $\alpha$ is a Hodge isomorphism of type $(-1, -1)$.

Let $(z_0 : \ldots : z_5)$ be the homogeneous coordinates of $\mathbb{P}^5$, and $f_Y(z_0, \ldots, z_5)$ the defining polynomial of $Y$. Let $Res : H^5(\mathbb{P}^5 - Y, \mathbb{C}) \to H^4(Y, \mathbb{C})$ be the residue map. By Lemma 2.2, we have $\alpha(H^1(Y, \Omega^3_Y)) = H^0(F(Y), \Omega^2_{F(Y)})$. Recall that $H^1(Y, \Omega^3_Y)$ is generated by $Res \frac{\Omega}{f_Y}$ ([Gr1]), where $\Omega = \sum_{i=0}^5 (-1)^i z_i dz_0 \wedge \ldots \wedge dz_i$. From these arguments, we get the following commutative diagram:

$$\begin{array}{ccc}
H^4(Y, \mathbb{C}) & \xrightarrow{\approx} & H^2(F(Y), \mathbb{C}) \\
\uparrow & & \uparrow \\
H^1(Y, \Omega^3_Y) & \xrightarrow{\approx} & H^0(F(Y), \Omega^2_{F(Y)}) \\
\uparrow & & \uparrow \\
\mathbb{C}(Res \frac{\Omega}{f_Y}) & \xrightarrow{\approx} & \mathbb{C}(\sigma_{F(Y)}).
\end{array}$$

### 2.3 Group actions on $F(Y)$

Let $PGL(5)$ be the automorphism group of $\mathbb{P}^5$, and $G$ a finite subgroup of $PGL(5)$. We assume that $G$ preserves $Y$. We need a criterion for the action preserving the symplectic form.

Let us consider the induced action $G^\wedge F(Y)$. Since the Abel-Jacobi map $\alpha$ is $G$-equivariant, the induced action $G^\wedge F(Y)$ preserves the symplectic form $\sigma_{F(Y)}$ if and only if $G$ preserves $Res \frac{\Omega}{f_Y}$. So, we get the following lemma.

**Lemma 2.3.** Assume that $G$ preserves $Y$. Then $G$ preserves the symplectic form $\sigma_{F(Y)}$ on $F(Y)$ if and only if $G$ preserves $Res \frac{\Omega}{f_Y}$. 

4
3 First example

We first recall results of [Nam]. We define a smooth cubic 4-fold \( Y \) by

\[
Y := \{(z_0 : \cdots : z_5) \in \mathbb{P}^5 | f(z_0, z_1, z_2) + g(z_3, z_4, z_5) = 0\}. 
\]

Define two elliptic curves by

\[
C := \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 | f(z_0, z_1, z_2) = 0\}
\]

and

\[
D := \{(z_3 : z_4 : z_5) \in \mathbb{P}^2 | g(z_3, z_4, z_5) = 0\}. 
\]

We put

\[
P_C = \{(z_0 : \cdots : z_5) \in \mathbb{P}^5 | z_3 = z_4 = z_5 = 0\},
\]

and

\[
P_D = \{(z_0 : \cdots : z_5) \in \mathbb{P}^5 | z_0 = z_1 = z_2 = 0\}. 
\]

Notice that \( C \subset P_C \) and \( D \subset P_D \). Assume that \( G \subset PGL(5) \) is the cyclic group \( \mathbb{Z}_3 \) of order 3 generated by

\[
\tau = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \zeta & \zeta \\ 1 & \zeta & \zeta \end{pmatrix},
\]

where \( \zeta = \exp(2\pi \sqrt{-1}/3) \).

Let us consider the natural group action:

\[
\mathbb{Z}_3 \curvearrowright \mathbb{P}^5 \ni (z_0 : z_1 : z_2 : z_3 : z_4 : z_5) \mapsto (z_0 : z_1 : z_2 : \zeta z_3 : \zeta z_4 : \zeta z_5). 
\]

Since \( \mathbb{Z}_3 \) preserves \( Y \), the action induces the \( \mathbb{Z}_3 \)-action on \( F(Y) \).

**Lemma 3.1.** The induced action \( \mathbb{Z}_3 \curvearrowright F(Y) \) preserves the symplectic form \( \sigma_{F(Y)} \), and the singular locus of \( X_{\mathbb{Z}_3} := F(Y)/\mathbb{Z}_3 \) is isomorphic to \( C \times D \). The singularity is \( A_2 \). In particular \( X_{\mathbb{Z}_3} \) has a crepant resolution \( \nu : \tilde{X}_{\mathbb{Z}_3} \to X_{\mathbb{Z}_3} \).

**Proof.** By Lemma 2.3, the induced action \( \mathbb{Z}_3 \curvearrowright F(Y) \) preserves the symplectic form \( \sigma_{F(Y)} \). The singular locus of \( X_{\mathbb{Z}_3} \) is isomorphic to the fixed locus \( \text{Fix}_{\mathbb{Z}_3}(F(Y)) \) of the \( \mathbb{Z}_3 \) action on \( F(Y) \). We remark that \( \text{Fix}_{\mathbb{Z}_3}(Y) \) is the disjoint union \( C \cup D \). If a line \( l \) is in \( \text{Fix}_{\mathbb{Z}_3}(F(Y)) \), then \( \mathbb{Z}_3 \) acts on the line \( l \). \( l \) has two fixed points by the \( \mathbb{Z}_3 \) action. So \( l \in F(Y) \) is in \( \text{Fix}_{\mathbb{Z}_3}(F(Y)) \) if and only if \( l \) passes through a point \( p \in C \) and \( q \in D \). Hence \( \text{Fix}_{\mathbb{Z}_3}(F(Y)) \) is isomorphic to \( C \times D \).
Proposition 3.2 \(\text{(Nam 1.7.(iv))}\). The notation being as above, \(\tilde{X}_{z_3}\) is birational to \(K^2(C \times D)\).

Proof. We first define a rational map \(\psi : \tilde{X}_{z_3} \to K^2(C \times D)\). Let \(l\) be in \(F(Y)\), and \([l]\) the orbit of \(l\) by the \(\mathbb{Z}_3\) action : \([l] = \{l, \tau(l), \tau^2(l)\}\). Let \(W_l\) be the linear space spanned by \(l, \tau(l)\) and \(\tau^2(l)\). If we choose \(l \in F(Y)\) in general, then \(W_l\) is isomorphic to \(\mathbb{P}^3\) and both \(P_C \cap W_l\) and \(P_D \cap W_l\) are lines. Furthermore we may assume that the cubic surface \(S := W_l \cap Y\) is smooth. There are 27 lines in \(S\) \(\text{[Han]}\). Notice that three lines \(l, \tau(l)\) and \(\tau^2(l)\) are contained in \(S\).

We will prove followings.

1. There are three lines \(m_1, m_2, m_3\) such that \(m_i\) does not meet \(m_j\) \((i \neq j)\), and each \(m_i\) meets \(l\) at one point (respectively \(\tau(l), \tau^2(l)\)). See also Figure [1]

2. The set of three lines is unique, and

3. each \(m_i\) \((i = 1, 2, 3)\) is fixed by the \(\mathbb{Z}_3\) action.

Indeed the cubic surface \(S\) is the blow up of \(\mathbb{P}^2\) along 6 points \(a_1, \ldots, a_6\). Let \(E_i\) be the exceptional \((-1)\)-curve over \(a_i\), \(C_i\) the proper transform of the conic passing through 5 points except \(a_i\), and \(L_{ij}\) the proper transform of line passing through \(a_i\) and \(a_j\). We may assume that \(E_1 = l, E_2 = \tau(l)\) and \(E_3 = \tau^2(l)\). By the configuration of lines, each \(m_i\) is given by \(m_1 = C_1, m_2 = C_5\) and \(m_3 = C_6\). There are exactly six lines which does not intersect \(\tau^k(l)\) \((k = 0, 1, 2)\). These six lines are given by \(L_{45}, L_{46}, L_{56}, E_4, E_5\) and \(E_6\). Since \(\mathbb{Z}_3\) acts on the set \(\{E_1, E_2, E_3\}\), \(\mathbb{Z}_3\) acts on the set \(\{L_{45}, L_{46}, L_{56}, E_4, E_5, E_6\}\). Since the order of \(\mathbb{Z}_3\) is 3, \(\mathbb{Z}_3\) acts on the two set \(\{L_{45}, L_{46}, L_{56}\}\) and \(\{E_4, E_5, E_6\}\).
If $E_4$, $E_5$ and $E_6$ are not fixed by $\mathbb{Z}_3$ then all lines in $S$ move by the $\mathbb{Z}_3$ action. Since there are 9 lines in $S$ fixed by the $\mathbb{Z}_3$ action, three lines $E_4$, $E_5$ and $E_6$ are fixed by the $\mathbb{Z}_3$ action. Hence three lines $m_1$, $m_2$ and $m_3$ are fixed by the $\mathbb{Z}_3$ action. In particular $m_i$ ($i = 1, 2, 3$) meet $C$ (resp $D$) at one point.

Put $p_i = m_i \cap C$ and $q_i = m_i \cap D$. Since three points $\{p_1, p_2, p_3\}$ (resp. $\{q_1, q_2, q_3\}$) are in the line $P \cap \ell$, the sum $p_1 + p_2 + p_3$ is $0 \in C$ (resp. $q_1 + q_2 + q_3 = 0 \in D$). So we get an element $\{(p_i, q_i)\}_{i=1}^3$ of $K^2(C \times D)$. We define $\psi$ as:

$$\psi : \tilde{X}_{\mathbb{Z}_3} \rightarrow K^2(C \times D), \psi([\ell]) = \{(p_i, q_i)\}_{i=1}^3.$$ 

We next define a rational map $\varphi : K^2(C \times D) \rightarrow \tilde{X}_{\mathbb{Z}_3}$. Let $\{(p_i, q_i)\}_{i=1}^3$ be in $K^2(C \times D)$. We may assume that $p_i \neq p_j$ and $q_i \neq q_j$ ($i \neq j$). Let $m_i$ ($i = 1, 2, 3$) be the line passing through $p_i$ and $q_i$. Let $M$ be the linear space spanned by $m_1$, $m_2$ and $m_3$. Then $M$ is isomorphic to $\mathbb{P}^3$. Let $S$ be the cubic surface $M \cap Y$. We remark that $S$ is smooth. By the configuration of 27 lines, there are exactly three lines $l_1$, $l_2$ and $l_3$ such that $l_i \cap l_j = \emptyset$ ($i \neq j$), and each $l_i$ meets $m_j$ ($i, j = 1, 2, 3$) at exactly one point. We notice that $m_1$, $m_2$ and $m_3$ are fixed by the $\mathbb{Z}_3$ action on $S$. By the uniqueness of the set of three lines, we have

$$\mathbb{Z}_3 \cap \{l_1, l_2, l_3\} \mapsto \{l_1, l_2, l_3\}.$$ 

Since each $l_i$ is not fixed by the $\mathbb{Z}_3$ action, we may assume that $\tau(l_1) = l_2$.

By the uniqueness of the set of three lines, both $\varphi$ and $\psi$ are birational maps. In particular, $\varphi^{-1} = \psi$.

### 3.1 On the indeterminacy of the birational maps

In Theorem 3.5 we will prove that $\varphi$ and $\psi$ is given as the Mukai flop on 18 disjoint copies of $\mathbb{P}^2$. The key lemma of the proof is Lemma 3.3. In the proof...
of Lemma 3.3, we give the geometric description of the exceptional divisor of \( \nu : \tilde{X}_{Z_3} \to X_{Z_3} \) by using \( G \)-Hilbert scheme. Let \( X \) be a smooth projective variety over \( \mathbb{C} \). We assume that a finite group \( G \) acts on \( X \). Then we define \( G \)-Hilb(\( X \)) as

\[
G \text{-Hilb}(X) := \{ Z \in \text{Hilb}^G(X) | \mathcal{O}_Z \cong \mathbb{C}[G] \text{ (as } G \text{ modules)} \}.
\]

and we call it \( G \)-Hilbert scheme of \( X \). Let \( z \) be in \( \tilde{X}_{Z_3} \) and we assume that \( z \) is not in the exceptional locus \( \text{Exc}(\nu) \) of \( \nu \). Then \( z \) is the orbit of \( l \in F(Y) \) by the \( \mathbb{Z}_3 \) action. Hence \( z \) is identified with \( Z \in \mathbb{Z}_3 \text{-Hilb}(F(Y)) \). Assume that \( z \) is in \( \text{Exc}(\nu) \). On a neighborhood of \( \text{Fix}(\mathbb{Z}_3) \), the \( \mathbb{Z}_3 \) action can be identified with

\[
\mathbb{Z}_3 \langle \text{Spec} \mathbb{C}[x_1, x_2, x_3, x_4], (x_1, x_2, x_3, x_4) \mapsto (x_1, x_2, \zeta x_3, \zeta^2 x_4) \rangle.
\]

By using the \( G \)-graph, we see that \( \mathbb{Z}_3 \text{-Hilb}(\text{Spec} \mathbb{C}[x_1, x_2, x_3, x_4]) \) is isomorphic to \( \mathbb{C}^2 \times \mathbb{Z}_3 \text{-Hilb}(\mathbb{C}^2) \) (See [Nak]). So \( \mathbb{Z}_3 \text{-Hilb}(\text{Spec} \mathbb{C}[x_1, x_2, x_3, x_4]) \) is a crepant resolution of \( \mathbb{C}^4/\mathbb{Z}_3 \). Since this crepant resolution is unique, the element \( z \) of \( \tilde{X}_{Z_3} \) can be identified with the element \( Z \) of \( \mathbb{Z}_3 \text{-Hilb}(F(Y)) \). By pulling back the universal family \( \mathcal{L} \) of \( F(Y) \), we get the family of lines \( \mathcal{L}_Z \) over \( Z \).

\[
\begin{array}{ccc}
\mathcal{L}_Z & \to & \mathcal{L} \subset F(Y) \times \mathbb{P}^5 \to \mathbb{P}^5 \\
\downarrow & & \downarrow \\
Z & \to & F(Y)
\end{array}
\]

The support of \( \mathcal{L}_Z \) is \( l \cup \tau(l) \cup \tau^2(l) \), where \( l \in F(Y) \). Here we put \([l] := l \cup \tau(l) \cup \tau^2(l)\). When \( l \) is not in the fixed locus of the \( \mathbb{Z}_3 \) action, \( \mathcal{L}_Z \) is uniquely determined by \([l]\). If \( l \) is in the fixed locus of the \( \mathbb{Z}_3 \) action, then \( Z \) is not determined uniquely by \([l]\).

Let \( p_2 \) be the second projection \( p_2 : F(Y) \times \mathbb{P}^5 \to \mathbb{P}^5 \). For each \( Z \), take the smallest linear subspace \( L \) contained in \( \mathbb{P}^5 \) such that \( \mathcal{L}_Z \subset p_2^{-1}(L) \). We call this \( L \) the linear space spanned by \( \mathcal{L}_Z \) (or simply \( Z \)). In a generic case, \( Z \) spans \( \mathbb{P}^3 \). In some special cases \( Z \) spans \( \mathbb{P}^2 \).

**Lemma 3.3.** Assume that the line \( l \in F(Y) \) is in the fixed locus of the \( \mathbb{Z}_3 \) action, that is, \( l \) passes through \( p \in C \) and \( q \in D \). We only consider \( Z \in \tilde{X}_{Z_3} \) such that the support of \( \mathcal{L}_Z \) is \( l \). Then \( Z \) is parametrized by the tree \( \mathcal{I} \cup \mathcal{J} \) of two projective lines \( \mathcal{I} \) and \( \mathcal{J} \).

1. Assume that neither \( p \) nor \( q \) are 3-torsion points. Then \( \mathcal{L}_Z \) spans \( \mathbb{P}^3 \), for each \( Z \).

2. Assume that both \( p \) and \( q \) are 3-torsion points. There exists exactly one point on each \( \mathbb{P}^1 \) such that \( \mathcal{L}_Z \) spans \( \mathbb{P}^2 \). These two points are not in the intersection \( \mathcal{I} \cap \mathcal{J} \). For other points, \( \mathcal{L}_Z \) spans \( \mathbb{P}^3 \).
3. Assume that $p$ is a 3-torsion point and $q$ is not a 3-torsion point. There exists exactly one point on the tree such that $L_Z$ spans $\mathbb{P}^2$. Moreover this point is not in the intersection $\mathcal{I} \cap \mathcal{J}$. Otherwise $L_Z$ spans $\mathbb{P}^3$.

When $q$ is a 3-torsion point and $p$ is not a 3-torsion point, we have a similar situation to 3.

Proof. Let $Z \in \tilde{X}_{Z_3}$ be a closed subscheme of length 3 such that the support of $L_Z$ is the line passing through $p \in C$ and $q \in D$. Let $(x : y : z : u : v : w)$ be the homogeneous coordinates of $\mathbb{P}^5$, consider the cubic

$$Y := \{(x : \cdots : w) \in \mathbb{P}^5 | f(x, y, z) + g(u, v, w) = 0\}.$$

Let $C$ and $D$ be elliptic curves defined by

$$C := \{(x : y : z) \in \mathbb{P}^2 | f(x, y, z) = 0\}, \quad D := \{(u : v : w) \in \mathbb{P}^2 | g(u, v, w) = 0\}.$$

We put

$$P_C := \{(x : \cdots : w) \in \mathbb{P}^5 | u = v = w = 0\},$$
and

$$P_D := \{(x : \cdots : w) \in \mathbb{P}^5 | x = y = z = 0\}.$$

1. By changing coordinates, we may assume that

$$p = (0 : 0 : 1)$$
\[ f(x, y, z) = x(x - z)(x - \alpha z) - y^2z, \quad \alpha \in \mathbb{C} - \{0, 1\} \]

$$q = (0 : 0 : 1)$$
\[ g(u, v, w) = u(u - w)(u - \beta w) - v^2w, \quad \beta \in \mathbb{C} - \{0, 1\} \]

So the line $l$ is given by:

$$l = (0 : 0 : \lambda : 0 : 0 : \mu) \text{ where } (\lambda : \mu) \in \mathbb{P}^1.$$

Let $G(2, 6)$ be the Grassmann manifold parameterizing 2-dimensional subspace of $\mathbb{C}^6$. We will give coordinates system around $l$ by using the local coordinates of $G(2, 6)$. We define two hyperplanes

$$\{z = 0\} := \{(x : y : \cdots : w) \in \mathbb{P}^5 | z = 0\},$$
$$\{w = 0\} := \{(x : y : \cdots : w) \in \mathbb{P}^5 | w = 0\}.$$

Then the intersection $\{w = 0\} \cap l$ is $p = (0 : 0 : 1 : 0 : 0 : 0)$ and the intersection $\{z = 0\} \cap l$ is $q = (0 : 0 : 0 : 0 : 0 : 1)$. Let $(u_0, u_1, u_3, u_4)$ be affine coordinates of the hyperplane $\{w = 0\}$ around $p$:

$$(u_0 : u_1 : 1 : u_3 : u_4 : 0) \in \mathbb{P}^4.$$
Let \((v_0, v_1, v_3, v_4)\) be affine coordinates of the hyperplane \(\{z = 0\}\) around \(q:\)
\[
(v_0 : v_1 : 0 : v_3 : v_4 : 1) \in \mathbb{P}^4.
\]

Let \(U_l\) be an open neighborhood of \(l\) in \(G(2, 6)\). The local coordinates of \(U_l\) is
given by \((u_0, u_1, u_3, u_4, v_0, v_1, v_3, v_4)\). Indeed, for each \((u_0, \ldots, v_4) \in U_l\),
the corresponding line \(\tilde{l}\) is given by
\[
(u_0, \ldots, v_4) \leftrightarrow \tilde{l} := \lambda(u_0 : u_1 : 1 : u_3 : u_4 : 0) + \mu(v_0 : v_1 : 0 : v_3 : v_4 : 1)
\]

The line \(\tilde{l} \in G(2, 6)\) is in \(F(Y)\) if and only if \(\tilde{l}\) is contained in \(Y\). Therefore
we have
\[
(f + g)(\tilde{l}) = \lambda^3 F_1 + \lambda^2 \mu F_2 + \lambda \mu^2 F_3 + \mu^3 F_4 = 0, \text{ for } \forall (\lambda : \mu) \in \mathbb{P}^1.
\]
So we get the following equations
\[
\begin{align*}
F_1 & := u_0^3 - (\alpha + 1)u_0^2 + \alpha u_0 - u_1^2 + u_3^3 = 0 \\
F_2 & := 3u_0^2v_0 - (\alpha + 1)2u_0v_0 + \alpha v_0 - 2u_1v_1 + 3u_3v_3 - (\beta + 1)u_3^2 - u_4^2 = 0 \\
F_3 & := 3u_0v_0^2 - (\alpha + 1)v_0^2 - v_1^2 + 3u_3v_3^2 - (\beta + 1)2u_3v_3 + \beta u_3 - 2u_4v_4 = 0 \\
F_4 & := v_0^3 + v_3^3 - (\beta + 1)v_3^2 + \beta v_3 - v_4^2 = 0
\end{align*}
\]

By the implicit function theorem, we can choose the affine coordinates of \(F(Y)\) as \((u_1, u_4, v_1, v_4)\).

Let us consider the exceptional divisor of \(\nu : \tilde{X}_{Z_3} \to X_{Z_3}\). Locally, the \(Z_3\)
action on \(F(Y)\) can be written as
\[
Z_3 \sim (u_1, u_4, v_1, v_4) \leftrightarrow (u_1, \zeta u_4, \zeta^2 v_1, v_4).
\]
The exceptional divisor of \(Z_3\)-\(\text{Hilb}(\text{Spec}\mathbb{C}[u_1, u_4, v_1, v_4]/\mathbb{Z}_3)\) is given by \(E_1 + E_2\), where
\[
E_1 = \{(u_1 - \lambda, v_4 - \mu, u_2^2, u_4v_1, v_3, au_4 + bv_1^2) | (\lambda, \mu) \in \mathbb{C}^2, (a : b) \in \mathbb{P}^1\},
\]
\[
E_2 = \{(u_1 - \lambda, v_4 - \mu, u_4^2, u_4v_1, v_1^2, cu_4^2 + dv_1) | (\lambda, \mu) \in \mathbb{C}^2, (c : d) \in \mathbb{P}^1\}.
\]
Here we identify the sub-schemes with their defining ideals. Since the line \(l\)
corresponds to the origin, when \(Z \in E_1\), the ideal sheaf of \(Z\) in \(G(2, 6)\) is
\[
I_{(a:b)} = (F_1, F_2, F_3, F_4, u_1, v_4, u_4v_1, v_3, au_4 + bv_1^2), \ (a : b) \in \mathbb{P}^1.
\]
When \(Z \in E_2\), the ideal sheaf of \(Z\) in \(G(2, 6)\) is
\[
J_{(c:d)} = (F_1, F_2, F_3, F_4, u_1, v_4, u_4^2, u_4v_1, v_1^3, cu_4^2 + dv_1), \ (c : d) \in \mathbb{P}^1.
\]
By the calculation of generators of ideals, we have

$I_{(a:b)} = (u_0, v_3, v_0, \beta u_3 - v_1^2, u_1, v_4, u_4^2, u_4 v_1, v_1^3, a u_4 + b v_1^2), \ (a : b) \in \mathbb{P}^1$

$J_{(c:d)} = (u_0, u_3, v_0 + u_2, u_1, v_4, u_4^3, u_4 v_1, v_1^2, c u_4 + d v_1), \ (c : d) \in \mathbb{P}^1$

In particular, $Z$ is parametrized by $\mathcal{I} \cup \mathcal{J}$ where $\mathcal{I} = \{I_{(a:b)}|(a : b) \in \mathbb{P}^1\}$ and $\mathcal{J} = \{J_{(c:d)}|(c : d) \in \mathbb{P}^1\}$. Assume that $Z$ is defined by $I_{(a:b)}$. The family of line $\mathcal{L}_Z$ over $Z$ is given by

$\mathcal{L}_Z = \{(0 : \mu v_1 : \lambda : \lambda u_3 : \mu u_4) | u_3^2 = v_1^3 = \beta u_3 - v_1^2 = a u_4 + b v_1^2 = u_4 v_1 = 0\}$.

Since $\beta u_3 - v_1^2 = a u_4 + b v_1^2 = 0$, we get an equation $a u_4 + b \beta u_3 = 0$. So, the linear space $L_{(a:b)}$ spanned by $\mathcal{L}_Z$ is

$L_{(a:b)} = \{(x : y : z : u : v : w) \in \mathbb{P}^5|x = a v + b \beta u = 0\}$.

We remark that $L_{(a:b)}$ is spanned by the tangent line $\{x = 0\} \subset P_C$ of $C$ at $p$ and the line $\{a v + b \beta u = 0\} \subset P_D$ passing through $q \in D$.

Assume that $Z$ is defined by $J_{(c:d)}$. In the same way, the linear space $M_{(c:d)}$ spanned by $\mathcal{L}_Z$ is

$M_{(c:d)} = \{(x : y : z : u : v : w) \in \mathbb{P}^5|u = dy - c \alpha x = 0\}$.

$M_{(c:d)}$ is spanned by the line $\{dy - c \alpha x = 0\} \subset P_C$ passing through $p \in C$ and the tangent line $\{u = 0\} \subset P_D$ of $D$ at $q$. So we can draw the following picture.
2. By changing coordinates, we may assume that

\[ p = (0 : 1 : 0), \quad f(x, y, z) = x(x - z)(x - \alpha z) - y^2z, \quad \alpha \in \mathbb{C} - \{0, 1\} \]

and

\[ q = (0 : 1 : 0), \quad g(u, v, w) = u(u - w)(u - \beta w) - v^2w, \quad \beta \in \mathbb{C} - \{0, 1\}. \]

Let \( U_l \) be a neighborhood of \( l \) in \( G(2, 6) \). Local coordinates of \( U_l \) are given by \((u_0, u_2, u_3, v_0, v_2, v_3, v_5)\). The corresponding line \( \tilde{l} \) to \((u_0, \cdots, v_5)\) is given by

\[ (u_0, \cdots, v_5) \mapsto \tilde{l} := \lambda(u_0 : 1 : u_2 : u_3 : 0 : u_5) + \mu(v_0 : 0 : v_2 : v_3 : 1 : v_5) \]

Since \((f + g)(\tilde{l})\) should be zero for each \((\lambda : \mu) \in \mathbb{P}^1\), we have

\[ (f + g)(\tilde{l}) = \lambda^3F_1 + \lambda^2\mu F_2 + \lambda\mu^2 F_3 + \mu^3F_4 = 0. \]

Let \( dF_i \) be the exterior derivative of \( F_i \) \((i = 1, \cdots, 4)\) at the origin. Then we have

\[ (dF_1 \quad dF_2 \quad dF_3 \quad dF_4) = (-du_2 \quad -dv_2 \quad -du_5 \quad -dv_5). \]

The local coordinates around \( l \in F(Y) \) is given by \((u_0, u_3, v_0, v_3)\). The \( \mathbb{Z}_3 \)

action on \( F(Y) \) is locally given by

\[ \mathbb{Z}_3 \curvearrowright F(Y), (u_0, u_3, v_0, v_3) \mapsto (u_0, \zeta^2u_3, \zeta v_0, v_3). \]

The exceptional divisor of \( \mathbb{Z}_3\text{-Hilb}(\text{Spec}\mathbb{C}[u_0, u_3, v_0, v_3]) \) is given by \( E_1 + E_2 \) where

\[ E_1 = \{(u_0 - \lambda, v_3 - \mu, u_2, u_3v_0, v_0^3, au_3 + bv_0^2) | (\lambda, \mu) \in \mathbb{C}^2, (a : b) \in \mathbb{P}^1\}, \]

\[ E_2 = \{(u_0 - \lambda, v_3 - \mu, u_3, u_3v_0, v_2, cu_3^2 + dv_0) | (\lambda, \mu) \in \mathbb{C}^2, (c : d) \in \mathbb{P}^1\}. \]

When \( Z \in E_1 \), the ideal sheaf of \( Z \) in \( G(2, 6) \) is

\[ I_{(a:b)} = (u_2, u_5, v_2, v_5, u_0, v_3, u_3^2, u_3v_0, v_0^3, au_3 + bv_0^2), (a : b) \in \mathbb{P}^1. \]

When \( Z \in E_2 \), the ideal sheaf of \( Z \) in \( G(2, 6) \) is

\[ J_{(c:d)} = (u_2, u_5, v_2, v_5, u_0, v_3, u_3^2, u_3v_0, v_0^2, cu_3^2 + dv_0), (c : d) \in \mathbb{P}^1. \]

In particular, \( Z \) is parametrized by \( \mathcal{I} \cup \mathcal{J} \) where \( \mathcal{I} = \{ I_{(a:b)} | (a : b) \in \mathbb{P}^1 \} \) and \( \mathcal{J} = \{ J_{(c:d)} | (c : d) \in \mathbb{P}^1 \} \). Assume that \( Z \) is defined by \( I_{(a:b)} \). When \((a : b) \neq (1 : 0), \mathcal{L}_Z \) is

\[ \mathcal{L}_Z = \{(\mu v_0 : \lambda : 0 : \lambda u_3 : \mu : 0) | u_3^2 = u_3v_0 = v_0^3 = au_3 + bv_0^2 = 0\}. \]
The linear space $L_{(a:b)}$ spanned by $L_Z$ is

$$L_{(a:b)} = \{ (x : y : z : u : v : w) \in \mathbb{P}^5 | z = w = 0 \}.$$ 

We remark that $L_{(a:b)}$ is spanned by the tangent line $\{ z = 0 \} \subset P_C$ of $C$ at $p$ and the tangent line $\{ w = 0 \} \subset P_D$ of $D$ at $q$. When $(a, b) = (1, 0)$, $L_Z$ is

$$L_Z = \{ (\mu v_0 : \lambda : 0 : 0 : \mu : 0) | v_0^3 = 0 \}.$$ 

The linear space $L_{(1:0)}$ spanned by $Z$ is

$$L_{(1:0)} = \{ z = u = w = 0 \}.$$ 

The linear space $L_{(1:0)}$ is spanned by the tangent line $\{ z = 0 \} \subset P_C$ of $C$ at $p$ and the point $q \in D$.

Assume that $Z$ is defined by $J_{(c:d)}$. In the same way, when $(c : d) \neq (0 : 1)$, the linear space $M_{(c:d)}$ spanned by $Z$ is

$$M_{(c:d)} = \{ (x : y : z : u : v : w) \in \mathbb{P}^5 | z = w = 0 \}.$$ 

This linear space is spanned by the tangent line $\{ z = 0 \} \subset P_C$ of $C$ at $p$ and the tangent line $\{ w = 0 \} \subset P_D$ of $D$ at $q$. When $(c : d) = (0 : 1)$, $L_Z$ is

$$L_Z = \{ (0 : \lambda : 0 : \lambda u_3 : \mu : 0) | u_3^3 = 0 \}.$$ 

So $M_{(0:1)}$ is

$$\{ z = w = x = 0 \}.$$ 

The linear space $M_{(0:1)}$ is spanned by the point $p \in C$ and the tangent line $\{ w = 0 \} \subset P_D$ of $D$ at $q$.

3. By changing coordinates, we assume that

$$p = (0 : 1 : 0), \quad f(x, y, z) = x(x - z)(x - \alpha z) - y^2 z, \quad \alpha \in \mathbb{C} - \{0, 1\}$$

and

$$q = (0 : 0 : 1), \quad g(u, v, w) = u(u - w)(u - \beta w) - v^2 w, \quad \beta \in \mathbb{C} - \{0, 1\}.$$ 

Let $U_l$ be a neighborhood of $l$ in $G(2, 6)$. Local coordinates of $U_l$ are given by $(u_0, u_2, u_3, u_4, v_0, v_2, v_3, v_4)$. For $(u_0, \cdots, v_4) \in U_l$, the corresponding line $\tilde{l}$ is given by

$$(u_0, \cdots, v_4) \leftrightarrow \tilde{l} := \lambda(u_0 : 1 : u_2 : u_3 : u_4 : 0) + \mu(v_0 : 0 : v_2 : v_3 : v_4 : 1)$$
When \( \bar{l} \in F(Y) \), we have \((f + g)(\bar{l}) = \lambda^3 F_1 + \lambda^2 \mu F_2 + \lambda \mu^2 F_3 + \mu^3 F_4 = 0 \), for each \((\lambda, \mu) \in \mathbb{P}^1\). Let \(dF_1\) be the exterior derivative of \(F_i \) \((i = 1, \cdots, 4)\) at the origin. Then we have
\[
(dF_1 \ dF_2 \ dF_3 \ dF_4) = (-du_2 \ -dv_2 \ \beta du_3 \ \beta dv_3)
\]
So the local coordinates of \(F(Y)\) around \(l\) are given by \((u_0, u_4, v_0, v_4)\). The \(\mathbb{Z}_3\) action is given by:
\[
\mathbb{Z}_3 \cdot F(Y), (u_0, u_4, v_0, v_4) \mapsto (u_0, \zeta u_4, \zeta^2 v_0, v_4).
\]
The exceptional divisor of \(\mathbb{Z}_3\)-\text{Hilb} \((\text{Spec} \mathbb{C}[u_0, u_4, v_0, v_4])\) is given by \(E_1 + E_2\) where
\[
E_1 = \{(u_0 - \lambda, v_4 - \mu, u_4^2, u_4v_0, v_0^3, au_4 + bv_0^2)| (\lambda, \mu) \in \mathbb{C}^2, (a : b) \in \mathbb{P}^1\}
\]
\[
E_2 = \{(u_0 - \lambda, v_4 - \mu, u_4^3, u_4v_0, v_0^2, cu_4^2 + dv_0)| (\lambda, \mu) \in \mathbb{C}^2, (c : d) \in \mathbb{P}^1\}.
\]
If \(Z \in E_1\) then \(Z\) is defined by
\[
I_{(a:b)} = (u_2, u_3, v_2, u_4, v_4, u_4v_0, v_0^3, au_4 + bv_0^2), \ (a : b) \in \mathbb{P}^1
\]
If \(Z \in E_2\) then the ideal sheaf of \(Z\) is
\[
J_{(c:d)} = (u_2, u_3, v_3, u_4^2 + v_2, u_0, v_4, u_4^3, u_4v_0, v_0^2, cu_4^2 + dv_0), \ (c : d) \in \mathbb{P}^1.
\]
In particular, \(Z\) is parametrized by \(\mathcal{I} \cup \mathcal{J}\) where \(\mathcal{I} = \{I_{(a:b)}| (a : b) \in \mathbb{P}^1\}\) and \(\mathcal{J} = \{J_{(c:d)}| (c : d) \in \mathbb{P}^1\}\). Assume that \(Z\) is defined by \(I_{(a:b)}\). When \((a : b) \neq (1 : 0)\), \(\mathcal{L}_Z\) is
\[
\mathcal{L}_Z = \{(\mu v_0 : \lambda : 0 : 0 : \lambda u_4 : \mu)| u_4^2 = u_4v_0 = v_0^3 = au_4 + bv_0^2 = 0\}.
\]
The linear space \(L_{(a:b)}\) spanned by \(\mathcal{L}_Z\) is
\[
L_{(a:b)} = \{z = u = 0\}.
\]
This linear space is spanned by the tangent line \(\{z = 0\} \subset P_C\) of \(C\) at \(p\) and the tangent line \(\{w = 0\} \subset P_D\) of \(D\) at \(q\). When \((a : b) = (1 : 0)\), \(\mathcal{L}_Z\) is
\[
\mathcal{L}_Z = \{(\mu v_0 : \lambda : 0 : 0 : \mu)| v_0^3 = 0\}.
\]
Then the linear space \(L_{(1:0)}\) spanned by \(\mathcal{L}_Z\) is
\[
L_{(1:0)} = \{z = u = v = 0\}.
\]
This linear space is spanned by the tangent line \{z = 0\} ⊂ P_C of C at p and the point q ∈ D.

Assume that Z is defined by \(J_{(c:d)}\). For each \((c : d) ∈ \mathbb{P}^1\), \(L_Z\) is

\[L_Z = (\mu v_0 : \lambda : \mu v_2 : 0 : \lambda u_4 : \mu)|u_4^2 + v_2 = u_4v_0 = v_0^2 = cu_4^2 + dv_0 = 0\].

Similarly, for each \((c : d) ∈ \mathbb{P}^1\), the linear space \(M_{(c:d)}\) spanned by \(L_Z\) is

\[M_{(c:d)} = \{(x : y : z : u : v : w) ∈ \mathbb{P}^5|u = cx - dz = 0\}\].

The linear space is spanned by the line \{cx − dz = 0\} ⊂ P_C passing through the point p and the tangent line \{u = 0\} ⊂ P_D of D at q.

To describe the indeterminacy of \(ψ\) explicitly, we define \(Q_1\) and \(Q_2\) as:

\[Q_1 := \{Z ∈ \bar{X}_3|L_Z \text{ spans } \mathbb{P}^2 \text{ and } L_Z \ni p \text{ where } p ∈ C \text{ and } 3p = 0\}\],

and

\[Q_2 := \{Z ∈ \bar{X}_3|L_Z \text{ spans } \mathbb{P}^2 \text{ and } L_Z \ni q \text{ where } q ∈ D \text{ and } 3q = 0\}\].

Let \(Q := Q_1 ∪ Q_2\). We shall prove that the indeterminacy of \(ψ\) is \(Q\) in Theorem 3.5.

**Lemma 3.4.** If \(L_Z\) spans \(\mathbb{P}^2\), then \(L_Z\) passes through a 3-torsion point of C or D. Furthermore \(Q_1\) is isomorphic to \(\{p ∈ C|3p = 0\} × P_D\), and \(Q_2\) is isomorphic to \(\{q ∈ D|3q = 0\} × P_C\). In particular, \(Q_1\) and \(Q_2\) are respectively isomorphic to 9 disjoint copies of \(\mathbb{P}^2\).

**Proof.** We first prove the first assertion. Let \(l\) be in \(F(Y)\). Since the support of \(L_Z\) is \(l \cup τ(l) \cup τ^2(l)\), we have to consider the following two cases.

1. \(l, τ(l)\) and \(τ^2(l)\) span \(\mathbb{P}^2\).

2. \(l = τ(l) = τ^2(l)\), that is, \(l\) is in the fixed locus of the \(Z_3\) action.

1. Let us consider the first case. Let \(P_l := \langle l, τ(l), τ^2(l) \rangle\) be the plane spanned by \(l, τ(l)\) and \(τ^2(l)\). Then \(Z_3\) acts on \(P_l\). The eigenvalue of the \(Z_3\) action on \(\mathbb{P}^5\) is 1 or \(ζ = \exp(\frac{2\pi i}{3})\). So the eigenvalue of the \(Z_3\) action on the plane \(P_l\) is also 1 or \(ζ\). Therefore we may assume that, for suitable coordinates of \(P_l\), the group action \(Z_3 ∩ P_l\) is given by

\[Z_3 ∩ P_l, (x : y : z) ↦ (x : y : ζz)\].
Therefore the fixed locus $\text{Fix}_{\mathbb{Z}_3}(P_1)$ of the $\mathbb{Z}_3$ action on $P_1$ is the disjoint union of a line $m_{\text{fix}}$ and a point $q_{\text{fix}}$. Since $\text{Fix}_{\mathbb{Z}_3}(\mathbb{P}^5)$ is $P_C \cup P_D$, we may assume that $m_{\text{fix}} = P_1 \cap P_C$, $q_{\text{fix}} = P_1 \cap P_D$. Put $p = l \cap m_{\text{fix}}$. Then $p$ is contained in $\text{Fix}_{\mathbb{Z}_3}(Y)$. Since $\text{Fix}_{\mathbb{Z}_3}(Y) = C \cup D$, $p \in C$. We will prove that the point $p$ is a 3-torsion point of $C$. To prove this, it is enough to show that $m_{\text{fix}}$ meets $C$ only at the point $p$. So we assume that there exists another point $p'$ on $C \cap m_{\text{fix}}$. However $P_1 \cap Y$ must be three lines and these three lines meet at exactly one point $p$. Since $p'$ is in $Y$, this is contradiction. Therefore $p$ is a 3-torsion point of $C$. Hence $l$ passes through a 3-torsion point of $C$.

2. Let us consider the second case. We remark that this case is the limit of the first case. Let $Z$ be in $\tilde{X}_{\mathbb{Z}_3}$ such that the support of $L_Z$ is $l$. By the proof of Lemma 3.3, $L_Z$ which spans $\mathbb{P}^2$ always passes through a 3-torsion point. Hence we get the first assertion.

We shall prove the second assertion for $Q(I)$. Assume that $Z \in Q(I)$. By the above argument, the plane $\langle L_Z \rangle$ spanned by $L_Z$ intersect $P_D$ at one point and $\langle L_Z \rangle \cap P_C$ is the tangent line of $C$ at a 3-torsion point. Conversely we choose a 3-torsion point $p$ of $C$ and a point $q$ from the plane $P_D$. Let $l_C$ be the tangent line of $C$ at $p$, and $\langle l_C, q \rangle$ the plane spanned by $l_C$ and $q$. There uniquely exists $Z \in \tilde{X}_{\mathbb{Z}_3}$ such that $L_Z$ spans $\langle l_C, q \rangle$. This correspondence gives isomorphism between $Q(I)$ and $\{ p \in C|3p = 0 \} \times P_D$. Therefore $Q(I)$ is isomorphic to $\{ p \in C|3p = 0 \} \times P_D$. Similarly we can show that $Q(II)$ is isomorphic to $\{ q \in D|3q = 0 \} \times P_C$. 

To describe the indeterminacy of $\varphi : K^2(C \times D) \dasharrow \tilde{X}_{\mathbb{Z}_3}$, we put

$$\mathbb{P}(I) := \left\{ \{(p, q_1), (p, q_2), (p, q_3)\} \in K^2(C \times D) \middle| 3p = 0 \right\},$$

and

$$\mathbb{P}(II) := \left\{ \{(p_1, q), (p_2, q), (p_3, q)\} \in K^2(C \times D) \middle| 3q = 0 \right\}.$$

We remark that

$$\mathbb{P}(I) \cong \{ p \in C|3p = 0 \} \times P_D^\vee,$$

and

$$\mathbb{P}(II) \cong \{ p \in C|3p = 0 \} \times P_C^\vee,$$

where $P_D^\vee$ (resp. $P_C^\vee$) is the dual of $P_D$ (resp. $P_C$). Moreover, $\mathbb{P}(I)$ and $\mathbb{P}(II)$ are respectively isomorphic to 9 disjoint copies of $\mathbb{P}^2$.

**Theorem 3.5.** The birational map $\psi : \tilde{X}_{\mathbb{Z}_3} \dasharrow K^2(C \times D)$ is decomposed into the Mukai flop on $Q$, and $\varphi$ is decomposed into the Mukai flop on $\mathbb{P}(I) \cup \mathbb{P}(II)$. In particular, the indeterminacy of $\psi$ is $Q$, and the indeterminacy of $\varphi$ is $\mathbb{P}(I) \cup \mathbb{P}(II)$. 

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The proof of Theorem 3.5 is long. Before starting the proof, we would like to explain the strategy of it.

**Step 1.** We will blow up $\tilde{X}_{Z_3}$ along $Q$. We denote it by $\pi : W \to \tilde{X}_{Z_3}$. Now let us consider the situation when an irreducible symplectic 4-fold $X$ contains $P$ which is isomorphic to $\mathbb{P}^2$. Since $\mathbb{P}^2$ is a Lagrangian manifold, the normal bundle $N_{P/X}$ of $P$ in $X$ is isomorphic to the cotangent bundle $\Omega^1_P$ of $P$. Let $B_PX \to X$ be the blow up along $P$. Then the exceptional locus is isomorphic to

$$B_PX \supset \Gamma := \{(x, H) \in P \times P^\vee \mid x \in H \} \to P \subset X,$$

where $P^\vee$ is the dual of $P$. By Fujiki-Nakano contraction theorem, there exists a birational contraction morphism $B_PX \to X^+$ such that the exceptional locus is isomorphic to $\Gamma \to P^\vee$. So, we will blow down $W$ in another direction $\pi^+ : W \to W^+$. Here $W^+$ is a Moishezon 4-fold. We shall prove that $W^+$ is actually a projective manifold in Proposition 3.11.

**Step 2.** We will define a morphism $\mu' : W \to \bar{K}^2(C \times D)$. We will prove that $\mu'$ factors through $W^+$. Namely there is a map $\mu^+ : W^+ \to \bar{K}^2(C \times D)$ such that $\mu^+ \circ \pi^+ = \mu'$.

**Step 3.** We will prove that $W^+$ is isomorphic to $K^2(C \times D)$. The following diagram illustrates three steps above.

Let us consider Step 3. We first construct the blow up $\pi : W \to \tilde{X}_{Z_3}$.

**Proposition 3.6.** Let $G(4, 6)$ be the Grassmann manifold which parametrizes 4 dimensional linear subspaces of $\mathbb{C}^6$. We define $W$ by

$$W := \{(Z, L) \in \tilde{X}_{Z_3} \times G(4, 6) \mid \mathcal{L}_L \subset L, \text{ both } L \cap P_C \text{ and } L \cap P_D \text{ are lines} \}.$$ 

Then the natural projection $\pi : W \to \tilde{X}_{Z_3}$ is the blow up along $Q$. 

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Proof. Let \( \pi' : B_Q\tilde{X}_{Z_3} \to \tilde{X}_{Z_3} \) be the blow up of \( \tilde{X}_{Z_3} \) along \( Q \). Let \( \text{Exc}(\pi') \) be the exceptional divisor of \( \pi' \) and \( \text{Exc}(\pi) \) the exceptional divisor of \( \pi \).

We will define the morphism \( f : B_Q\tilde{X}_{Z_3} \to W \). Outside of the exceptional divisors, we define \( f \) by \( f := \pi^{-1} \circ \pi' \). On \( \text{Exc}(\pi') \), we define \( f \) in the following way. On \( (\pi')^{-1}(Q_{(1)}) \), the morphism \( \pi' \) is defined as follows:

\[
\{ (p,x,H) \in C \times P_D \times P_{D'} | 3p = 0, x \in H \} \xrightarrow{\pi'} \{ (p,x) \in C \times P_D | 3p = 0 \}
\]

\[
\pi'(p,x,H) = (p,x)
\]

By Lemma 3.7 \((p,x)\) determine \( Z_{(p,x)} \in Q_{(1)} \) uniquely. Here let \( l_{C,p} \) be the tangent line of \( C \) at \( p \). Hence we define a morphism \( f : \text{Exc}(\pi') \to \text{Exc}(\pi) \) by \( f(p,x,H) = (Z_{(p,x)}, \langle l_{C,p}, H \rangle) \) where \( \langle l_{C,p}, H \rangle \) is the projective linear space spanned by \( l_{C,p} \) and \( H \). In the same way, we can define \( f \) on \( (\pi')^{-1}(Q_{(1)}) \).

Since \( f \) is bijective, \( W \) is the blow up along \( Q \).

As we remarked, by Fujiki-Nakano contraction theorem, there exists a contraction morphism \( \pi^+ : W \to W^+ \). Note that on \( \text{Exc}(\pi^+) \), \( \pi^+ \) can be identified as

\[
W \supset \text{Exc}(\pi^+) \ni (Z,L) \to L \in G(4,6).
\]

We choose \( w \in \pi^+(\text{Exc}(\pi^+)) \) and fix it. We remark that for each \( (Z,L) \in (\pi^+)^{-1}(w) \), \( L \) is uniquely determined by \( w \).

Next let us consider Step 2. We first define \( \mu' : W \to \tilde{K}^2(C \times D) \) by

\[
\mu'(Z,L) := \{ (p_i, q_i) \}_{i=1}^3 \text{ where } \langle p_i, q_i \rangle \subset L \cap Y, \langle p_i, q_i \rangle \cap L_Z \neq \emptyset.
\]

The map \( \mu' \) is well-defined.

**Lemma 3.7.** The map \( \mu' \) factors through \( W^+ \), that is, there is a map \( \mu^+ : W^+ \to \tilde{K}^2(C \times D) \) such that \( \mu^+ \circ \pi^+ = \mu' \).

**Proof.** Let \( w \) be a point of \( W^+ \). We define \( \mu^+ \) in the following way:

\[
\mu^+(w) := \mu'((\pi^+)^{-1}(w)).
\]

If \( \mu^+ \) is well-defined then the assertion of Lemma 3.7 is clear. So, we will prove that \( \mu'((\pi^+)^{-1}(w)) \) is independent of the fiber \( (\pi^+)^{-1}(w) \). If \( w \) is not in \( \pi^+(\pi^{-1}(Q)) \), then \( (\pi^+)^{-1}(w) \cap \pi^{-1}(Q) = \emptyset \). Since the fiber is unique, \( \mu^+ \) is well-defined.

Assume that \( w \) is in \( \pi^+(\pi^{-1}(Q)) \). Let \( (Z,L) \in (\pi^+)^{-1}(w) \). As we remarked, the linear space \( L \) is uniquely determined by \( w \). Then exactly one of the following two cases will happen.

1. \( L \cap P_C \) is a tangent line of \( C \) at a 3-torsion point.
2. \( L \cap P_D \) is a tangent line of \( D \) at a 3-torsion point.

We may assume that the first case occurs. By Proposition \[3.6\] \((\pi^+)^{-1}(w)\) is parametrized by the line \( L \cap P_D \). Assume that \( L \cap P_D \cap D \) consists of three points \( \{q_1, q_2, q_3\} \). For each \((Z, L) \in (\pi^+)^{-1}(w)\),

\[\mu'(Z, L) = \{(p, q_1), (p, q_2), (p, q_3)\} \in \tilde{K}^2(C \times D).\]

Therefore the map \( \mu^+ \) is well-defined.

We consider Step 3 to complete the proof of Theorem \[3.5\]. We will prove that the map \( \mu^+ \) is isomorphism on the smooth locus of \( \tilde{K}^2(C \times D) \) in the following Lemma.

**Lemma 3.8.** The map \( \mu^+ \) is an isomorphism on the smooth locus of \( \tilde{K}^2(C \times D) \).

**Proof.** Take \( \{(p_i, q_i)\}_{i=1}^3 \) from the smooth locus of \( \tilde{K}^2(C \times D) \). We will prove that the fiber \((\mu^+)^{-1}(\{(p_i, q_i)\})\) (for short \( \{(p_i, q_i)\}_{\mu^+}\)) is one point. Let \( l_C \subset P_C \) (resp. \( l_D \subset P_D \)) be the line passing through three points \( \{p_i\}_{i=1}^3 \) (resp. \( \{q_i\}_{i=1}^3 \)). We have to consider the fibers of \( \mu^+ \) in the following cases.

1. \( \tilde{K}^2(C \times D) \ni \{(p_i, q_i)\}_{i=1}^3 \) where \( p_i \neq p_j \) \((i \neq j)\), and \( q_i \neq q_j \) \((i \neq j)\).
2. \( \tilde{K}^2(C \times D) \ni \{(p_i, q_i)\}_{i=1}^3 \) where \( p_1 = p_2, p_1 \neq p_3 \) and \( q_i \neq q_j \) \((i \neq j)\).
3. \( \tilde{K}^2(C \times D) \ni \{(p_i, q_i)\}_{i=1}^3 \) where \( p_1 = p_2, p_1 \neq p_3 \) and \( q_2 = q_3, q_2 \neq q_1 \).
4. \( \tilde{K}^2(C \times D) \ni \{(p_i, q_i)\}_{i=1}^3 \) where \( p_1 = p_2 = p_3 =: p \), and \( q_i \neq q_j \) \((i \neq j)\).

1. Let us consider the first case. In this case, the fiber \( \{(p_i, q_i)\}_{\mu^+} \) is isomorphic to the fiber \((\mu')^{-1}(\{(p_i, q_i)\})\) (for short \( \{(p_i, q_i)\}_{\mu}\)). So assume that \((Z, L) \in \{(p_i, q_i)\}_{\mu}\). The linear space \( L \) is spanned by \( l_C \) and \( l_D \), and the cubic surface \( L \cap Y \) is smooth. As we wrote in Proposition \[3.2\] \( Z \) consists of three lines \( l, \tau(l) \) and \( \tau^2(l) \). In particular \( Z \) is uniquely determined. Hence the fiber is one point.

2 and 3. Notice that the case 3 is the limit of the case 2. Since \( \{(p_i, q_i)\}_{\mu}\cap \pi^{-1}(Q) = \emptyset \), the fiber \( \{(p_i, q_i)\}_{\mu} \) is isomorphic to the fiber \( \{(p_i, q_i)\}_{\mu^+} \). Let \((Z, L) \in \{(p_i, q_i)\}_{\mu}\). The linear space \( L \) is spanned by \( l_C \) and \( l_D \). Let us consider the cubic surface \( L \cap Y \). There is a line \( l \subset L \cap Y \) which intersects three lines \( \{(p_i, q_i)\}_{i=1}^3 \). In both case 2 and 3, the line \( l \) passes through \( p_1 \) and \( q_3 \). Notice that the line \( l \) is in the fixed locus of the \( \mathbb{Z}_3 \) action on \( F(Y) \). Since there uniquely exists \( Z \) such that \( L_Z \) spans \( L \), the fiber is one point.

4. Let us consider the fourth case. Assume that \((Z, L) \in \{(p_i, q_i)\}_{\mu^+}\). By the similar argument in the proof of Lemma \[3.7\] \( \{(p_i, q_i)\}_{\mu^+} \) is parametrized by the line \( l_D \). Since \( \{(p_i, q_i)\}_{\mu^+} \) is contracted to a point by \( \pi^+ \), the fiber \( \{(p_i, q_i)\}_{\mu^+} \) is one point.
We will use the following result to complete Step 3.

**Proposition 3.9 (F-N).** Let $Y$ be a locally $\mathbb{Q}$ factorial symplectic variety, and suppose that there is a projective crepant resolution of $Y$: $\pi : X \to Y$. If the exceptional divisor of $\pi$ is irreducible, then $\pi$ is a unique crepant resolution, that is, if there is another crepant resolution $\pi' : X' \to Y$ then the natural birational map $\pi' \circ \pi^{-1}$ is an isomorphism.

Since the exceptional divisor of $\mu : K^2(C \times D) \to \bar{K}^2(C \times D)$ is irreducible, it is enough to show that $W^+$ is projective. We will observe the fiber of singular points of $\bar{K}^2(C \times D)$ by $\mu^+$ and prove that $W^+$ is projective.

**Proposition 3.10.** Let $E$ be the exceptional divisor of $\mu^+ : W^+ \to \bar{K}^2(C \times D)$. Then $-E$ is $\mu^+$-ample. In particular $W^+$ is a projective manifold.

**Proof.** We first consider the fiber of singular points by $\mu^+$. By Lemma 3.8, the exceptional divisor of $\pi^+$ is contained in the inverse image of the singular locus of $\bar{K}^2(C \times D)$. We will use the same notation in the proof of Lemma 3.8. By symmetry we have to consider the following three cases.

1. $\{(p_i, q_i)\}_{i=1}^3 \in \bar{K}^2(C \times D)$. $p_1 = p_2 =: p$, $p_1 \neq p_3 = -2p$ and $q_1 = q_2 =: q$, $q_1 \neq q_3 = -2q$.

2. $\{(p_i, q_i)\}_{i=1}^3 \in \bar{K}^2(C \times D)$. $p_1 = p_2 = p_3 =: p$ and $q_1 = q_2 =: q$, $q_1 \neq q_3 = -2q$.

3. $\{(p_i, q_i)\}_{i=1}^3 \in \bar{K}^2(C \times D)$. $p_1 = p_2 = p_3 =: p$ and $q_1 = q_2 = q_3 =: q$

Let us consider the first case. We claim that the fiber $\{(p_i, q_i)\}_{\mu^+}$ is isomorphic to $\mathbb{P}^1$. Since $\{(p_i, q_i)\}_{\mu} \cap \pi^{-1}(Q) = \emptyset$, we have $\{(p_i, q_i)\}_{\mu^+} \cong \{(p_i, q_i)\}_{\mu}$. So, let $(Z, L) \in \{(p_i, q_i)\}_{\mu^+}$. Since the linear space $L$ is spanned by the line $l_C$ and $l_D$, $L$ is unique. There are infinitely many lines in the cubic surface $L \cap Y$ which intersect with two lines $\langle p, q \rangle$ and $\langle -2p, -2q \rangle$. Here we can draw the following picture of the configuration of lines in $L \cap Y$. 

\[\text{Diagram of lines in } L \cap Y\]
The set of lines can be parametrized by the line \( \langle p, q \rangle \), that is, the line \( l \) in \( L \cap Y \) is uniquely determined by \( t \in \langle p, q \rangle \). The cubic surface has singular points along the line \( \langle p, q \rangle \).

Notice that \( \mathbb{Z}_3 \) acts on the line \( \langle p, q \rangle \). The fiber \( \{(p_i, q_i)\}_{\mu'} \) is isomorphic to \( \langle p, q \rangle / \mathbb{Z}_3 \). Indeed let \( l_t \subset L \cap Y \) be the line passing through \( t \in \langle p, q \rangle \). When \( t \in \langle p, q \rangle - \{p, q\} \), \( l_t \) is not in the fixed locus of the \( \mathbb{Z}_3 \) action on \( F(Y) \). Hence there exists uniquely \( Z \in \mathcal{X}_{\mathbb{Z}_3} \) such that the support of \( \mathcal{L}_Z \) is \( l_t \cup \tau(l_t) \cup \tau^2(l_t) \). When \( t = p \), \( l_p \) passes through \( p \) and \( -2q \). Notice that \( l_p \) is in the fixed locus of the \( \mathbb{Z}_3 \) action on \( F(Y) \). By Lemma 3.3, there uniquely exists \( Z \in \mathcal{X}_{\mathbb{Z}_3} \) such that the support of \( \mathcal{L}_Z \) is \( l_p \) and \( \mathcal{L}_Z \) spans \( L \). Similarly, if \( t = q \), then the line \( l_q \) passes through \( -2p \) and \( q \). By \( L \), \( Z \) can be determined uniquely. So \( \{(p_i, q_i)\}_{\mu'} \) is isomorphic to \( \langle p, q \rangle / \mathbb{Z}_3 \).

2. Let us consider the second case. Recall the notation \( \mathcal{I} \) defined in third case of the proof for Lemma 3.3. We will prove that the fiber \( \{(p_i, q_i)\}_{\mu'} \) is parametrized by \( \mathcal{I} \). Let \( (Z, L) \in \{(p_i, q_i)\}_{\mu'} \). Then \( L \) is spanned by \( l_C \) and \( l_D \). Furthermore, the cubic surface \( L \cap Y \) is given by \( \{x : y : z : w \in L | y^3 + z^2w = 0\} \) for suitable homogeneous coordinates of \( L \). The point \( (1 : 0 : 0 : 0) \in L \) correspond to \( p \). Notice that all lines contained in the cubic surface \( L \cap Y \) pass through \( p \).

We claim that the fiber \( \{(p_i, q_i)\}_{\mu'} \) is isomorphic to the tree \( \mathcal{I} \cup l_D \) of two projective line \( \mathcal{I} \) and \( l_D \). Let \( P_t \) be the plane spanned by \( l_C \) and \( t \in l_D \). If \( t \in l, t \in \{p, q, -2q\} \), then \( P_t \cap Y \) is not in the fixed locus. So we can uniquely determine \( Z \) such that the support of \( \mathcal{L}_Z \) is \( P_t \cap Y \). When \( t = -2q \), \( P_t \cap Y \) is the line \( \langle p, -2q \rangle \) passing through \( p \) and \( -2q \). There uniquely exists \( Z \) such that the support of \( \mathcal{L}_Z \) is the line \( \langle p, -2q \rangle \) and \( \mathcal{L}_Z \) spans \( L \). When \( t = q \), \( P_t \cap Y \) is the line \( \langle p, q \rangle \). Here recall that \( I_{(a:b)} \in \mathcal{I} \) is the ideal defined in the third case in the proof of Lemma 3.3. Let \( Z_{(a:b)} \in \mathcal{X}_{\mathbb{Z}_3} \) be the point defined by \( I_{(a:b)} \). For each \( I_{(a:b)} \in \mathcal{I} \), \( (Z_{(a:b)}, L) \) is in the fiber \( \{(p_i, q_i)\}_{\mu'} \). So in the second case, we can not uniquely determine \( Z \). Therefore, \( \{(p_i, q_i)\}_{\mu'} \) is isomorphic to \( \mathcal{I} \cup l_D \).

Since the line \( l_D \) is contracted to a point by \( \pi^+ : W \to W^+ \), the fiber \( \{(p_i, q_i)\}_{\mu'} \) is isomorphic to \( \mathcal{I} \). We draw the picture of the “blow down”.

\[
\begin{array}{ccc}
W & \xrightarrow{\pi^+} & W^+ \\
\mathcal{I} & \downarrow l_D & \mathcal{I} \\
q & \xrightarrow{-2q} & q \\
\{(p_i, q_i)\}_{\mu'} & \downarrow & \{(p_i, q_i)\}_{\mu'}
\end{array}
\]
3. Let us consider the third case. When \((Z, L) \in \{(p_i, q_i)\}_{\mu^+}\), the linear space \(L\) is spanned by \(l_C\) and \(l_D\). Moreover the cubic surface \(L \cap Y\) is three planes \(P_1, P_2\) and \(P_3\). Each plane is sent to another plane by the \(\mathbb{Z}_3\) action. Three planes share the line \(\langle p, q \rangle\). So we choose a plane \(P_1\) and fix it.

For each \(t \in \langle p, q \rangle\), the set of lines passing through \(t\) in \(P_1\) is parametrized by \(P_1\). This parameter means direction of the line passing through \(t\). So we call this parameter \(D_t(s)\) where \(s \in P_1\). For each \(t\), there is a special direction \(D_t(s_0)\) which represent the line \(\langle p, q \rangle\). Notice that \(\langle p, q \rangle\) is in the fixed locus of the \(\mathbb{Z}_3\) action on \(F(Y)\). Here recall that \(I\) and \(J\) defined in the second case of the proof for Lemma 3.3 and let \(I_{(a:b)} \in I\) and \(J_{(c:d)} \in J\). If we consider the blow up \(\tilde{X}_{\mathbb{Z}_3} \rightarrow X_{\mathbb{Z}_3}\), then the tree \(I \cup J\) appears as the exceptional curves.

Now let us consider the limit \(\lim_{s \rightarrow s_0} D_t(s)\) for \(t \in \langle p, q \rangle - \{p, q\}\). This limit is independent of \(t\) and \(\lim_{s \rightarrow s_0} D_t(s)\) coincides with the intersection \(I \cap J\).

If we flop \(\tilde{X}_{\mathbb{Z}_3} \rightarrow W^+\), then \(D_p(s)\) and \(D_q(s)\) are contracted to points. So the fiber \(\{(p_i, q_i)\}_{\mu^+}\) is the contraction of Hirtzebruch surface \(F_n \rightarrow F_n^+\) by \((-n)\)-curve. In particular, the Picard number of the surface \(\{\{(p_i, q_i)\}_{\mu^+}\) is 1.

We can explain these by the following picture.

Let \(E\) be the exceptional divisor of \(\mu^+\). \(E\) has a fibration over the singular points of \(K^2(C \times D)\). A general fiber of the fibration is \(\mathbb{P}^1\) and there are 81 singular fibers \(F_n\). Since the Picard number of \(F_n\) is 1, all curves contracted

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by $\mu^+$ are algebraically equivalent to a general fiber
\[ \xi = (\mu^+)^{-1}\{(p,q), (p,q), (-2p,-2q)\} \].
Since the normal bundle of $\xi$ is trivial, the intersection number $(\xi, E)$ is $-2$. Therefore $-E$ is $\mu^+$ ample. \hfill \square

3.2 Birational symplectic models of $\tilde{X}_{Z_3}$

Finally let us consider other birational symplectic models of $\tilde{X}_{Z_3}$ which are different from $K^2(C \times D)$. We define divisors needed later. Let $E_1$ be the irreducible component of $\text{Exc}(\nu) : \tilde{X}_{Z_3} \to X_{Z_3}$ such that $E_1$ meets $Q(I)$, and $E_1$ does not meet $Q(II)$. Similarly, let $E_2$ be the irreducible component of $\text{Exc}(\nu)$ such that $E_2$ meets $Q(II)$, and $E_2$ does not meet $Q(I)$. We remark that $\text{Exc}(\nu)$ is $E_1 + E_2$. Here let $K^2(C \times D) \dashrightarrow X_I$ be the Mukai flop on $\mathbb{P}(I)$, and $K^2(C \times D) \dashrightarrow X_{II}$ the Mukai flop on $\mathbb{P}(II)$.

**Theorem 3.11.** Assume that $C$ and $D$ are “not” isogenous. Then there are exactly 4 birational models of $\tilde{X}_{Z_3}$, that is, there are 4 projective irreducible symplectic manifolds $K^2(C \times D)$, $\tilde{X}_{Z_3}$, $X_I$ and $X_{II}$ which are birational to $K^2(C \times D)$. The movable cone of $\tilde{X}_{Z_3}$ is decomposed into the ample cones of these 4 models in the following way.

| Face | Morphism                  |
|------|---------------------------|
| $od$ | Contraction of $\mathbb{P}(II)$ |
| $cd$ | $K^2(C \times D) \to K^2(C \times D)$ |
| $ab$ | Contraction of $E_1$ |
| $ae$ | Contraction of $E_2$ |
| $ob$ | Contraction of $Q(II)$ |
| $oe$ | Contraction of $Q(I)$ |
| $bc$ | Contraction of $E_1$ |
| $de$ | Contraction of $E_2$ |

Each codimension 1 face corresponds to the following contraction morphism.
Here $\tilde{E}_1$ is the proper transform of $E_1$ in $X_1$, $\tilde{E}_2$ is the proper transform of $E_2$ in $X_\Pi$.

We will provide the proof into three parts. We always assume that $C$ and $D$ are not isogenous.

**Lemma 3.12.** The ample cone of $K^2(C \times D)$ is given by the following picture, where $H_1$ (resp. $H_2$, $6H_1 + 6H_2 - E$) is the supporting divisor of the vertex $c$ (resp. $d$, $o$).

![Diagram of the ample cone](image)

The divisors $H_1$, $H_2$ and $E$ will be defined in the proof.

**Proof.** Since $C$ and $D$ are not isogenous, the Picard number of $K^2(C \times D)$ is 3. Let $E$ be the exceptional divisor of $\mu : K^2(C \times D) \to \bar{K}^2(C \times D)$. We have the following commutative diagram.

$$
\begin{array}{ccc}
K^2(C) & \xrightarrow{\pi_1} & \bar{K}^2(C \times D) \\
\mu \downarrow & & \downarrow \pi_2 \\
K^2(C \times D) & \xrightarrow{\bar{\pi}_2} & K^2(D)
\end{array}
$$

Let $H_C$ (resp. $H_D$) be the pull back of the tautological line bundle $\mathcal{O}_{\mathbb{P}^2}(1)$ by the isomorphism $K^2(C) \to \mathbb{P}^2$ (resp. $K^2(D) \to \mathbb{P}^2$). We define $H_1$ and $H_2$ by:

$H_1 := \pi_1^*(H_C)$, $H_2 := \pi_2^*(H_D)$.

Let $x$ be a line in $\mathbb{P}(I)$, $y$ a line in $\mathbb{P}(II)$ and $z$ a rational curve which can be contracted by $\mu$. Write $\mu(z) = \{(p, q), (p, q), (2p, 2q)\}$ and assume that neither $p$ nor $q$ are 3-torsion points.
Since \( x \) is contracted by \( \pi_1 \), the intersection number \((x, H_1)\) is zero. Since \( \pi_2 \) is an isomorphism on one connected component of \( \mathbb{P}(I) \), we have \((x, H_2) = 1\). In the same way, \((y, H_2) = 0\), and \((y, H_1) = 1\). Since \( z \) is contracted by \( \pi_1 \) and \( \pi_2 \), we have \((z, H_1) = (z, H_2) = 0\). Since the normal bundle of \( z \) in \( E \) is trivial and \( z \) is isomorphic to \( \mathbb{P}^1 \), \((z, E) = -2\). The intersection \( E \cap \mathbb{P}(I) \) is given by

\[
E \cap \mathbb{P}(I) = \{(p, q), (p, -2q), (p, q) \in K^2(C \times D) | 3p = 0, q \in D\}.
\]

So, \( E \cap \mathbb{P}(I) \) is isomorphic to 9 copies of the curve

\[
\{(q, q, -2q) \in K^2(D) | q \in D\}.
\]

This curve is isomorphic to the dual curve \( D^\vee \) of \( D \). Since \( \deg D^\vee = 6 \), \((x, E) = 6\). Similarly, we have \((y, E) = 6\). From these arguments, we have the following table of intersection numbers.

|   | \( H_1 \) | \( H_2 \) | \( E \) |
|---|-----|-----|-----|
| \( x \) | 0 | 1 | 6 |
| \( y \) | 1 | 0 | 6 |
| \( z \) | 0 | 0 | -2 |

Clearly both \( H_1 \) and \( H_2 \) are nef. We will prove that \( 6H_1 + 6H_2 - E \) is nef. Notice that \( 6H_1 \) (resp. \( 6H_2 \)) is linearly equivalent to \( \pi_1^*C^\vee \) (resp. \( \pi_2^*D^\vee \)). Since \( \pi_1^*C^\vee \) contains the singular locus of \( \bar{K^2(C \times D)} \), \( \pi_1^*C^\vee \) has two irreducible components. Let \( F_1 \) be the proper transform of \( \pi_1^*C^\vee \) and \( F_2 \) the proper transform of \( \pi_2^*D^\vee \). Assume that \( \pi_1^*C^\vee \sim aF_1 + bE \) for some numbers \( a \) and \( b \). Since \((z, F_1) = 1 \) and \((y, F_1) = 0 \), we have \( \pi_1^*C^\vee = 2F_1 + E \). Similarly, \( \pi_2^*D^\vee \sim 2F_2 + E \). Assume that \( 6H_1 + 6H_2 - E \) is not nef. Then there is a curve \( \xi \) such that \((\xi, 6H_1 + 6H_2 - E) < 0\). Since \( 6H_1 + 6H_2 - E \) is linearly equivalent to the following divisors, \( \xi \) should be contained in \( F_1 \cap F_2 \).

\[
6H_1 + 6H_2 - E \sim \pi_1^*C^\vee + 6H_2 - E = 2F_1 + 6H_2
\]

\[
\sim 6H_1 + \pi_2^*D^\vee - E = 6H_1 + 2F_2.
\]

By using the cone theorem, we can contract \( \xi \). Theorem 1 in [Kaw] says that the exceptional locus is covered by rational curves. Since \( F_1 \cap F_2 \) is isomorphic to \( C^\vee \times D^\vee \), \( F_1 \cap F_2 \) does not contain the rational curve. So, \( 6H_1 + 6H_2 - E \) is nef.

By the table of intersection numbers, the face \( \overline{oc} \) (resp. \( \overline{ad} \) and \( \overline{cd} \)) defines the contraction of \( x \) (resp. \( y \) and \( z \)).

**Lemma 3.13.** The ample cone of \( \tilde{X}_{23} \) is given by:

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Each supporting divisor will be defined in the proof.

Proof. Let $K^2(C \times D) \to X_0$ be the contraction of $\mathbb{P} \cup \mathbb{P}$. By Theorem 3.5, there exists a projective morphism $\phi : \tilde{X} \to X_0$, and we have the following diagram:

\[
\begin{array}{ccc}
W & \xrightarrow{\pi} & K^2(C \times D) \\
\downarrow & & \swarrow \phi \\
X_0 & & \tilde{X}
\end{array}
\]

Let $H_{X_0}$ be an ample divisor of $X_0$. Then we may assume that $\phi^*H_{X_0}$ is numerically equivalent to the proper transform of $6H_1 + 6H_2 - E$. Let $x$ be a rational curve in $E_1$ which is contracted by $\nu$, $y$ a rational curve in $E_2$ which is contracted by $\nu$, $z$ a line in $Q(I)$, and $w$ a line in $Q(II)$. Clearly $(x, E_2) = (y, E_1) = 1$. Since the normal bundle of $x$ in $E_1$ (resp. $y$ in $E_2$) is trivial, $(x, E_1) = -2$ (resp. $(y, E_2) = -2$). By the definition of $X_0$, we have $(z, \phi^*H_{X_0}) = (w, \phi^*H_{X_0}) = 0$. Since $Q(I) \cap E_1$ is isomorphic to $D$, we have $(z, E_1) = 3$. Since $Q(I) \cap E_2 = \emptyset$, we have $(z, E_2) = 0$. Similarly, we have $(w, E_2) = 3$ and $(w, E_1) = 0$. Recall divisors $F_1$, $F_2$ and $E$ of $K^2(C \times D)$ defined in the proof for Lemma 3.12. Let $\bar{F}_1$, $\bar{F}_2$ and $\bar{E}$ be a proper transform of $F_1$, $F_2$ and $E$ respectively. By Lemma 3.3, we have $\bar{F}_1 = E_1$ and $\bar{F}_2 = E_2$. Here, we will prove that $(x, \phi^*H_{X_0}) = 2$. Let $Z \in \tilde{X}$ be in the curve $x$. For each $Z$, the support of $\mathcal{L}_Z$ is a line $\langle p_0, q_0 \rangle$ passing through $p_0 \in C$ and $q_0 \in D$. Then $x \cap \bar{E}$ is identified with 4 points $\{q \in D | 2q = q_0\}$. Hence $(x, \bar{E}) = 4$. Since $6H_1 + 6H_2 - E \sim 2F_1 + 2F_2 + E$, we have the following equation:

\[
4 = (x, \bar{E}) = (x, \phi^*H_{X_0} - 2\bar{F}_1 - 2\bar{F}_2) \\
= (x, \phi^*H_{X_0}) - 2(x, E_1) - 2(x, E_2).
\]

Hence we have $(x, \phi^*H_{X_0}) = 2$. In the same way, we have $(y, \phi^*H_{X_0}) = 2$. Therefore we get the following table of intersection numbers.
We will prove that the divisor $\phi^*H_{X_0} + 2E_1 + 2E_2$ defines the morphism $\nu: \tilde{X}_{Z_3} \to X_{Z_3}$. Suppose that the vertex $a$ is generated by $\nu^*H$, where $H$ is an ample divisor of $X_{Z_3}$. Since the Picard number of $\tilde{X}_{Z_3}$ is 3, we put $\nu^*H = a^*H_{X_0} + bE_1 + cE_2$ for some numbers $(a, b, c)$. Since both $(x, \nu^*H) = (y, \nu^*H) = 0$, we have $b = c = 2a$.

We will prove that both $\phi^*H_{X_0} + E_1$ and $\phi^*H_{X_0} + E_2$ are nef. From symmetry it is enough to prove it only for $\phi^*H_{X_0} + E_1$. Since $\phi^*H_{X_0}$ is nef, the curve $\xi$ such that $(\xi, \phi^*H_{X_0} + E_1) < 0$ should be contained in $E_1$. We can contract $\xi$ by the cone theorem, and Theorem 1 in [Kaw] says that such a curve must be rational. Since $E_1$ is $\mathbb{P}^1$ bundle over $C \times D$, it is enough to check that $(x, \phi^*H_{X_0} + E_1) \geq 0$. By the above table of intersection number, we have $(x, \phi^*H_{X_0} + E_1) = 0$.

Therefore $\phi^*H_{X_0} + E_1$ is nef.

By the table of intersection numbers, we have the correspondence between vertices and contraction morphisms in Theorem 3.11.

To complete the proof of Theorem 3.11 we consider the ample cone of $X_{I}$ and $X_{II}$. By symmetry, it is enough to prove it for $X_{I}$.

**Lemma 3.14.** The ample cones of $X_{I}$ and $X_{II}$ are given by:

|       | $\phi^*H_{X_0}$ | $E_1$ | $E_2$ |
|-------|------------------|-------|-------|
| $x$   | 2                | -2    | 1     |
| $y$   | 2                | 1     | -2    |
| $z$   | 0                | 3     | 0     |
| $w$   | 0                | 0     | 3     |

**Proof.** Throughout the proof we use the same notations used in the proof of Lemma 3.13. We only prove that the ample cone of $X_{(I)}$ is a “tetrahedron”. In $\tilde{X}_{Z_3}$, the vertex $b$ corresponds to the contraction of $E_1 \cup Q_{II}$. We denote by $\tilde{E}_1 \subset X_{I}$ the proper transform of $E_1$ and by $\tilde{x} \subset X_{I}$ the proper transform of $x$. Let $K^2(C \times D) \to Z_1$ be a contraction of $\mathbb{P}_{(I)}$. By the construction of
there exists a birational contraction morphism $\psi_I : X_I \to K^2(C)$ such that the following diagram commutes.

In $X_I$, the vertex $c$ corresponds to the contraction morphism $\psi_I$. Hence it is enough to prove that $\tilde{x}$ is contracted by $\psi_I$. Assume that $Z \in \tilde{X}_{Z,I}$ is in the curve $x$. For each $Z \in x$, the support of $L_Z$ is the line $(p, q)$ passing through $p \in C$ ($3p \neq 0$) and $q \in D$ ($3q \neq 0$). Let $\xi \subset K^2(C \times D)$ be the proper transform of $\tilde{x}$. Then the curve $\xi \subset K^2(C \times D)$ is given by:

$$\xi = \{(p, q_1), (p, q_2), (-2p, q) \in K^2(C \times D) | q_1 + q_2 + q = 0\}.$$ 

So $\tilde{x}$ is contracted by $\psi_I$. 

\[4\] Second example

Let $M$ be the finite subset of $PGL(5)$ defined by

$$M := \left\{ A := \begin{pmatrix} a_0 & \cdots & a_4 \\ a_5 \end{pmatrix} \in PGL(5) \right| A \text{ is a diagonal matrix,} \right\}$$

$$\#\{i | a_i = 1\} = 3, \#\{j | a_j = \zeta\} = 3 \cdots (*)$$

Then $\#M = 10$. Let $G$ be the finite group generated by the set $M$. We define $\tau_i (i = 1, \cdots, 4) \in G$ respectively by:

$$\tau_1 := \{a_0 = a_1 = a_2 = 1, a_3 = a_4 = a_5 = \zeta\}$$

$$\tau_2 := \{a_0 = a_1 = a_5 = 1, a_2 = a_3 = a_4 = \zeta\}$$

$$\tau_3 := \{a_0 = a_1 = a_2 = a_3 = 1, a_4 = \zeta, a_5 = \zeta^2\}$$

$$\tau_4 := \{a_0 = a_3 = a_4 = a_5 = 1, a_1 = \zeta, a_2 = \zeta^2\}.$$ 

Then $\tau_1, \tau_2, \tau_3$ and $\tau_4$ are generators of $G$. So $G$ is isomorphic to $\mathbb{Z}_3^\oplus 4$. Let $Y$ be a smooth cubic 4-fold of Fermat type:

$$Y := \{(z_0 : \cdots : z_5) \in \mathbb{P}^5 | z_0^3 + \cdots + z_5^3 = 0\}.$$
Let $C$ be the Fermat type elliptic curve: \{(x : y : z) ∈ \mathbb{P}^2 | x^3 + y^3 + z^3 = 0 \}. We will consider the induced $G$-action on $F(Y)$.

**Lemma 4.1.** The $G$-action preserves the symplectic form $\sigma_{F(Y)}$ and $X_G = F(Y)/G$ has a crepant resolution $\tilde{X}_G$.

**Proof.** By Lemma 2.3, $G$ preserves the symplectic form $\sigma_{F(Y)}$. We put $\alpha_i = \text{diag}(a_0^{(i)}, \ldots, a_5^{(i)}) ∈ M (i = 1, \ldots, 10).$ We may assume that $\alpha_1$ is $τ_1$ and $\alpha_2$ is $τ_2$. Each $\alpha_i$ generates the cyclic group of order 3. Let $\text{Fix}(\alpha_i)(i = 1, \ldots, 10)$ be the fixed locus of the action $⟨\alpha_i⟩ \curvearrowright F(Y)$. Then $\text{Fix}(\alpha_i)$ is isomorphic to $C × C$

Each intersection $\text{Fix}(\alpha_i) \cap \text{Fix}(\alpha_j)(i ≠ j)$ consists of 9 points. Since the $G$-action preserves the symplectic form, singularities of $X_G$ is the product of two $A_2$ singularities at each point of $\text{Fix}(\alpha_i) \cap \text{Fix}(\alpha_j)$. So $X_G$ has a crepant resolution $\tilde{X}_G$.

Since $C$ has a complex multiplication, $\mathbb{Z}_3$ acts on $C$ in the following way:

$$\mathbb{Z}_3 \curvearrowright C, \ x \mapsto \zeta x.$$ We consider the following $\mathbb{Z}_3$ action on $C × C$:

$$\mathbb{Z}_3 \curvearrowright C × C, \ (x, y) \mapsto (\zeta x, \zeta^2 y).$$ Let $S$ be the minimal resolution of $C × C/\mathbb{Z}_3$. Notice that $S$ is a K3 surface.

**Theorem 4.2.** Notations being as above, $\tilde{X}_G$ is birational to $\text{Hilb}^2(S)$.
Proof. To distinguish two elliptic curves \( \{(z_0 : z_1 : z_2) \in \mathbb{P}^2 | z_0^3 + z_1^3 + z_2^3 = 0\} \) and \( \{(z_3 : z_4 : z_5) \in \mathbb{P}^2 | z_3^3 + z_4^3 + z_5^3 = 0\} \), we call them \( C \) and \( D \) respectively. Let \( \mathfrak{S}_3 \) be the symmetric group with degree 3. By Theorem 3.3 we have the following birational map:

\[
\tilde{X}_G \overset{\text{birat.}}{\sim} F(Y)/\langle \tau_1, \tau_2, \tau_3, \tau_4 \rangle \overset{\text{birat.}}{\sim} \tilde{K}^2(C \times D)/\langle \tau_2, \tau_3, \tau_4 \rangle
\]

We will show that \( \left((C \times D)^2/\mathfrak{S}_3\right)/\langle \tau_2, \tau_3, \tau_4 \rangle \) is birational to \( \text{Hilb}^2(S) \). Let \( x_0 \) (resp. \( y_0 \)) be a 3-torsion point of \( C \) (resp. \( D \)) such that \( \zeta x_0 = x_0 \). The induced actions of \( \tau_2, \tau_3 \) and \( \tau_4 \) on \( (C \times D)^2 \) are given by

\[
\begin{align*}
\tau_2^\ast C \times C \times D \times D &\ni (x_1, x_2, y_1, y_2) \mapsto (\zeta x_1, \zeta x_2, \zeta^2 y_1, \zeta^2 y_2) \\
\tau_3^\ast C \times C \times D \times D &\ni (x_1, x_2, y_1, y_2) \mapsto (x_1, x_2, y_1 + y_0, y_2 + y_0) \\
\tau_4^\ast C \times C \times D \times D &\ni (x_1, x_2, y_1, y_2) \mapsto (x_1 + x_0, x_2 + x_0, y_1, y_2).
\end{align*}
\]

Here \( \mathfrak{S}_3 \) is generated by

\( \mathfrak{S}_3 = \langle \sigma, \tau | \sigma^3 = 1, \tau^2 = 1, \sigma \tau = \tau \sigma^2 \rangle \),

and the action \( \mathfrak{S}_3 \) on \( C \times C \times D \times D \) is given by

\[
\begin{align*}
\sigma^\ast C \times C \times D \times D &\ni (x_1, x_2, y_1, y_2) \mapsto (x_2, -x_1 - x_2, y_2, -y_1 - y_2) \\
\tau^\ast C \times C \times D \times D &\ni (x_1, x_2, y_1, y_2) \mapsto (x_2, x_1, y_2, y_1).
\end{align*}
\]

Let us diagonalize the actions \( \sigma^\ast C \times C \). We have the following diagram:

\[
\begin{array}{ccc}
C \times C & \overset{f}{\longrightarrow} & C \times C \\
\sigma \downarrow & & \downarrow \bar{\sigma} \\
C \times C & \overset{f}{\longrightarrow} & C \times C,
\end{array}
\]

where the matrix representations of \( \sigma, f \) and \( \bar{\sigma} \) are respectively

\[
\sigma = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad f = \begin{pmatrix} \zeta^2 & -1 \\ -\zeta & 1 \end{pmatrix}, \quad \bar{\sigma} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}.
\]

For \( \tau \), we have the following diagram:

\[
\begin{array}{ccc}
C \times C & \overset{f}{\longrightarrow} & C \times C \\
\tau \downarrow & & \downarrow \bar{\tau} \\
C \times C & \overset{f}{\longrightarrow} & C \times C,
\end{array}
\]

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where the matrix representations of $\tau$ and $\tilde{\tau}$ are respectively

$$\tau = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\tau} = \begin{pmatrix} \zeta & 0 \\ 0 & \zeta^2 \end{pmatrix}. $$

We remark that $(x_1, x_2) \in C \times C$ is in the kernel of $f$ if and only if $\zeta^2 x_1 = x_2, \zeta x_1 = x_2$. So $\text{Ker} \ f$ is $\{(0, 0), (x_0, x_0), (2x_0, 2x_0)\}$. In same way for $D$, the action $\sigma \circ D \times D$ can be diagonalized. So we have

$$\left( (C \times C \times D \times D)/\langle \sigma, \tau \rangle \right)/\langle \tau_2, \tau_3, \tau_4 \rangle = \left( (C \times C \times D \times D)/\langle \tau_3, \tau_4 \rangle \right)/\langle \tau_2, \tilde{\sigma}, \tilde{\tau} \rangle.$$

Hence

$$\left( (C^2 \times D^2)/\mathfrak{S}_3 \right)/\langle \tau_2, \tau_3, \tau_4 \rangle \cong \left( (C \times C \times D \times D)/\langle \tilde{\sigma}, \tilde{\tau}, \tau_2 \rangle \right)/\langle \tilde{\tau} \rangle.$$

Since $\langle \tilde{\sigma}, \tau_2 \rangle$ is a normal subgroup of $\langle \tilde{\sigma}, \tilde{\tau}, \tau_2 \rangle$, we have

$$(C \times C \times D \times D)/\langle \tilde{\sigma}, \tilde{\tau}, \tau_2 \rangle \cong \left( (C \times C \times D \times D)/\langle \tilde{\sigma}, \tau_2 \rangle \right)/\langle \tilde{\tau} \rangle.$$

We can change generators of $\langle \tilde{\sigma}, \tau_2 \rangle$ from $\{\tilde{\sigma}, \tau_2\}$ to $\{\tau_2 \circ \tilde{\sigma}, \tau_2 \circ \tilde{\sigma}^2\}$. Each action of $\tau_2 \circ \tilde{\sigma}$ and $\tau_2 \circ \tilde{\sigma}^2$ are respectively given by

$$\tau_2 \circ \tilde{\sigma} \circ C \times C \times D \times D \ni (x_1, x_2, y_1, y_2) \mapsto (\zeta^2 x_1, x_2, y_1, \zeta y_2)$$

$$\tau_2 \circ \tilde{\sigma}^2 \circ C \times C \times D \times D \ni (x_1, x_2, y_1, y_2) \mapsto (x_1, \zeta x_2, \zeta^2 y_1, y_2).$$

We identify $C \times C \times D \times D$ with $(C \times D) \times (C \times D)$ in the following way;

$$C \times C \times D \times D \leftrightarrow (C \times D) \times (C \times D)$$

$$(x_1, x_2, y_1, y_2) \leftrightarrow ((x_1, y_2), (x_2, y_1)).$$

The action of $\tilde{\tau}$ on $(C \times D) \times (C \times D)$ is given by

$$\tilde{\tau} \circ (C \times D) \times (C \times D) \ni (x_1, y_2, x_2, y_1) \mapsto (\zeta^2 x_2, \zeta y_1, \zeta x_1, \zeta^2 y_1).$$

So we have

$$\left( (C \times C \times D \times D)/\langle \tilde{\sigma}, \tau_2 \rangle \right)/\langle \tilde{\tau} \rangle \cong \text{Sym}^2\left((C \times D)/\mathbb{Z}_3\right).$$

Here the $\mathbb{Z}_3$ action is the same one as in the first part of this section. So we get the conclusion. \hfill \Box
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