1. Introduction

The turbulent transport of the fast $\alpha$ particles was considered negligible in tokamak plasmas [1] due to the fast gyration motion with Larmor radius that is much larger than the correlation length. This leads to a very small amplitude of the gyro-averaged potential, which determines the effective electric drift. However, this problem was reconsidered in the last decade [2–14], in preparation of the deuterium–tritium experiments in JET and ITER. The conclusions are rather dispersed, from completely negligible turbulent transport [15] to diffusion coefficients that can be comparable or even larger than those of plasma ions [16, 17].

Most of the theoretical studies of $\alpha$ particle turbulent transport are self-consistent numerical simulations of turbulence and of $\alpha$ particles turbulent fluxes. Other studies use the test particle approach in numerical simulations based on constructed potentials or on the results of turbulence simulations. The characteristics of the turbulence and of the $\alpha$ particle fluxes are determined in the first case as functions of the macroscopic conditions (gradients, heating power). The second approach obtains the diffusion coefficient of the $\alpha$ particles as a function of the characteristics of the turbulence. It allows to find if the turbulent loss of the $\alpha$ particles can be significant and to identify the corresponding conditions.

This paper is included in the last category. We determine the transport regimes for a realistic model that has the characteristics of the ion temperature gradient (ITG) or of the trapped electron mode (TEM) driven turbulence. It includes a spectrum of potential fluctuations that is modeled using the results of the numerical simulations, the drift of the potential with the effective diamagnetic velocity and the parallel motion. Our semi-analytical statistical approach is based on the decorrelation trajectory method (DTM), which is adapted to the gyrokinetic approximation. We obtain the transport coefficients as a function of the parameters of the turbulence and of the energy of the $\alpha$ particles. According to our results, significant turbulent transport of the $\alpha$ particles can appear only at energies of the order of 100 KeV. We determine the corresponding conditions.

Keywords: turbulence, transport, alpha particle loss

(Some figures may appear in colour only in the online journal)
the parameters and obtain the energy dependence of the α-particles diffusion coefficient. The conclusions are summarized in section 4.

2. Model and statistical method

The turbulent transport of the fast ions is studied in the test particle approach starting from the Newton–Lorentz equations of motion in a stochastic potential $\phi(x, y, z, t)$ and a constant magnetic field $B = Be$, oriented along the $z$ axis. The very large value of the cyclotron frequency $\Omega = qB/\rho_0 \gg 1$ enables a strong simplification of the equation of motion by using the gyrokinetic approximation [23, 24]. The Larmor radius is constant and the gyration motion is approximately uniform in these conditions. Moreover, since the displacements of the guiding centers during the gyration period $\theta = 2\pi/\Omega \rho_0$ are much smaller than the correlation length, the equation for the motion of the guiding centre $\xi$ can be averaged over the cyclotron period at constant $\xi$ and $t$. One obtains

$$\frac{d\xi}{dt} = -\epsilon \frac{\partial\phi(\xi, z, t; \rho_0)}{\partial \xi},$$

(1)

$$\psi(\xi, z, t; \rho_0') = \frac{1}{\theta} \int_0^{\theta+\theta} d\tau \phi(\xi(t) + \rho(\tau), z(t), t),$$

where $\psi$ is the gyro-averaged potential, $\rho(\tau) = \rho_0'(\sin(\Omega \tau), \cos(\Omega \tau))$, $\rho_0' = u\Omega \rho_0$ is the Larmor radius of the particles with the perpendicular velocity $u$, $m_0$ is the ion mass, $q$ is its charge, and $\epsilon_{ij}$ is the antisymmetric tensor ($\epsilon_{12} = 1, \epsilon_{21} = -1, \epsilon_{ii} = 0$).

Thus, the motion of the guiding centers of the fast ions obeys the same equation as in the limit of zero Larmor radius, but with the modified potential (2). Using the Fourier representation of the potential, $\tilde{\phi}(k, z(t), t)$, and performing the time integral, one obtains

$$\psi(\xi, z, t; \rho_0') = \int dk dk' \tilde{\phi}(k, z(t), t) J_0(k \rho_0') \exp[i k \cdot \xi(t)],$$

(3)

where the wave number $k = (k_x, k_z)$ is perpendicular on $B_z$, $k = \sqrt{k_x^2 + k_z^2}$ and $J_0$ is the Bessel function of the first kind. This shows that the gyro-average of the potential $\phi$ determines the multiplication of its Fourier transform by $J_0(k \rho_0')$, which corresponds to the gradual attenuation of the large wave number components of the spectrum as the Larmor radius increases.

The Eulerian correlation (EC) of the averaged potential $\psi(\xi, z, t; \rho_0)$ for a Maxwellian distribution $F_M(u) = \exp(-u^2/\lambda^2) / (2\pi \sqrt{\lambda})$ of particle velocities is

$$E(\xi, z, t; \rho_0) = \int du u F_M(u) \times \left( \psi(\xi, z', t') \psi(\xi + \xi, z' + z, t + t') \right) = \int dk S(k, z, t) \exp(-\rho_0^2 k^2) J_0(\rho_0' k^2) \exp[i k \cdot \xi],$$

(4)

Figure 1. Typical turbulence spectrum (5) with $s(k_1, k_2)$ given by equation (6), $k_1^2 = 1, \lambda_1 = 3, \lambda_2 = 1$ and $n_1 = 1$.

where $\rho_0 = v_0/\Omega \rho_0$, $v_0 = \sqrt{T_0/\rho_0}$ is the thermal velocity of the fast particles with the temperature $T_0$, $\chi$ is the average over the realization of the stochastic potential, $S(k, z, t)$ is the spectrum of the turbulence and $I_0(\chi)$ is the modified Bessel function of the first kind. The limit $\rho_0 = 0$ corresponds to the EC of the potential $\phi$, which is the Fourier transform of the spectrum $S$. The gyro-average determines an effective spectrum that is attenuated at large wave numbers.

The potential of the turbulence $\phi(x_1, x_2, z, t)$ is modeled as a stationary and homogeneous Gaussian stochastic function. Its spectrum $S(k_1, k_2, z, t)$ corresponds to the general characteristics of the ITG or TEM turbulence. The shape of $S$ is similar for both types of turbulence. The main difference is given by typical wave numbers ($k_2 \rho_0 \lesssim 1$ for ITG, $k_2 \rho_0 \approx 1$ for TEM). The spectrum has two symmetrical maxima for $k_2 \rho_0$, zero amplitude for $k_2 = 0$, because the growth rate of these instability vanishes at $k_2 = 0$. We use the simple analytical expression of $S(k)$ that was found in [20] to be in agreement with numerical and experimental results of [21, 22]

$$S(k_1, k_2, z, t) = C \exp\left(-\frac{|z|}{\lambda_1} - \frac{|t|}{\tau_c}\right)$$

(5)

$$s(k_1, k_2) = C \left( k_1 \right)^{n_1} \exp\left(-\frac{k_1^2}{2 \lambda_1^2} - \frac{k_2^2}{2 \lambda_2^2}\right),$$

(6)

and a long range spectrum

$$s(k_1, k_2) = C \left( k_1 \right)^{n_1} \left( \frac{1}{2 \lambda_1^2} - \frac{k_2^2}{2 \lambda_2^2}\right)^{n_2}.$$

(7)

The parameters of these functions are the correlation lengths along each direction $\lambda_i$, $i = 1$ (radial), $i = 2$ (poloidal), $i = 3$ (parallel), the correlation time $\tau_c$ and the dominant wave number $k_2^0$. The positive numbers $n_1, n_2$ determine details of the shape of the spectrum. The main characteristic of the
shape of the spectra given by equations (5)–(7) consists of the two symmetrical peaks, as shown in the example presented in figure 1. The amplitude of the potential fluctuations $\Phi$ is

$$\Phi^2 = C \int \int dk_1 dk_2 \left[s(k_1, k_2 - k_0^2) - s(k_1, k_2 + k_0^2)\right].$$

The amplitude of gyro averaged potential $\psi(\xi, z; t; \rho_0)$, $\Phi^0(\rho_0) \equiv \langle E(0, 0, 0; \rho_0) \rangle$, is a monotonically decreasing function of $\rho_0$, as can be seen in figure 2(a). The decay is small at $\rho_0 < \lambda_2$ and $\Phi^0$ decreases as $1/\sqrt{\rho_0}$ at $\rho_0 > \lambda_2$, as usually obtained.

The general shape of the EC is not changed at large $\rho_0$. In particular, the positive and the negative parts compensate, and the integral over $x_2$ is zero for any $\rho_0$. This property is due to the spectrum (5), which cancels for $k_2 = 0$. The dependence of the normalized correlation $E(t|\Phi^0(\rho_0))$ on the radial direction is practically independent on $\rho_0$, while a weak modification appears in the poloidal direction. As seen in figure 2(b), the width of the EC increases when $\rho_0$ increases from zero to one, and it saturates at larger $\rho_0$ (the curves for $\rho_0 = 1$ and $\rho_0 = 20$ are practically superposed).

We underline that the weak dependence on $\rho_0$ of the shape of the gyro-averaged EC is a characteristic of the two peak spectra (5) of the ITG and TEM turbulence. A much stronger dependence yields from a spectrum with one maximum. It consists of the continuous increase of the widths of the EC in both directions. The cause of the different behavior in the case of the spectrum (5) can be deduced from equation (4).

Due to the function $\exp(-\rho_0^2 k^2 + 2\rho_0^2 k^2)$ that is close to 1 for $k_2 \rho_0 < 1$ and decays as $1/(k_2 \rho_0)$ for $k_2 \rho_0 > 1$, the average over the gyro-motion leads to the attenuation of the large $k$ components of the spectrum, while the small $k$ part of $S$ is not affected. The spectrum (5) decays in the small $k$ range because the modes are stable for $k_2 = 0$. Due to this property, all the components of the spectrum are attenuated at large enough values of $\rho_0$ because the condition $k_2 \rho_0 \gtrsim 1$ applies for all components that correspond to significant (not close to zero) values of $S$. This leads, at large values of $\rho_0$, to the change of the effective EC that consists only in the decay of the amplitude but not in the modification of the shape.

We introduce normalized quantities using as units $\lambda_2$ for the perpendicular displacements, $\lambda_c$ for parallel displacements, $V_0 = \Phi/B \lambda_2$ for the $E \times B$ drift velocity and $\tau_0 = \lambda_c/V_0$ for time. The equations of motion for the normalized quantities (designed by the same symbols as the physical ones) are

$$\frac{d\xi_i}{dt} = -\epsilon_{ij} \frac{1}{B} \frac{\partial \psi(\xi, z; t; \rho)}{\partial \xi_j} + \delta_{ij} V_\lambda, \quad \frac{dz}{dt} = \frac{\tau_0}{\tau_\lambda}$$

where $\tau_\lambda = \lambda_c/u_\alpha$ is the decorrelation time induced by the parallel motion, $u_\alpha$ is the thermal velocity of the $\alpha$ particles, $\rho = \rho_0/\lambda_2$, $V_\lambda = V_0/V_\alpha$, and $V_\lambda$ is the effective diamagnetic velocity.

The time dependent diffusion coefficient $D(t)$ is determined using a semi-analytical approach, the decorrelation trajectory method (DTM) [18, 19]. It is based on a set of trajectories obtained from the EC of the potential, the decorrelation trajectories (DTs). The main idea of this method is to group together trajectories that are similar by imposing supplementary initial conditions. Each group corresponds to a subensemble $S$ of realizations of the stochastic potential that is defined by $\psi(0, 0, 0; \rho) = \psi^0$ and $\psi(t, 0, 0; \rho) = \psi^0$, where $\psi(t, 0, 0; \rho)$ are the components of the $E \times B$ velocity and $\psi^0, \psi^0$ are the supplementary initial conditions. The stochastic velocity in a subensemble $S$ is a Gaussian field that has a space and time dependent average $V^0_S(\xi, z, t)$, which is determined by the EC of the potential. The corresponding average Lagrangian velocity in $S$ is approximated in the DTM by this Eulerian velocity calculated along the average trajectory in $S$. This leads to an equation for the DT, $X^0_S(t) = [X(t), X(t), Z(t)]$, which is the average trajectory in $S$. The Lagrangian correlation of the velocity in the subensemble is determined by the average Lagrangian velocity since the initial velocity is the same for all realizations in $S$. 

Figure 2. The EC (4) of the averaged potential $\psi$ : (a) the dependence of the amplitudes of the potential $\Phi^0$ and of the velocity components $V_i$ on $\rho_0$ and (b) the shape of the EC in the poloidal direction, $E(0, 0, 0; \rho_o)(\Phi^0)^2$ for $\rho_0/\lambda_2 = 0$ (continuous blue line), $\rho_0 = 1$ (dashed-dotted red line) and $\rho_0 = 20$ (dashed black line).
\( \psi(t) = \psi_0 + \phi(t) \)

where \( \psi(t) = [\xi_1(t), \xi_2(t), \xi_3(t)] \) is the trajectory of the guiding center obtained from equation (1). The Lagrangian correlation \( L(t) \) and its time integral that is the time dependent diffusion coefficient \( D(t) \) are obtained as a weighted sum over the subensembles

\[
L(t) = \int \int d\psi d\theta \phi(\psi) P(\psi) \phi(\theta) P(\theta) V_0^3(X^S(t), t),
\]

\[
D(t) = \int \int d\psi d\theta \phi(\psi) P(\psi) \phi(\theta) P(\theta) V_0^3 X_0^S(t),
\]

where \( P(\psi), P(\theta) \) are the Gaussian probabilities of the Eulerian fields.

We use here the fast DTM introduced in [20], which imposes only two supplementary initial conditions: the potential in the starting point of the guiding center trajectories \( \psi(0, 0, 0, \rho, \theta) = \psi_0 \) and the orientation \( \theta_0 \) of the normalized initial velocity. Larger subensembles \( S \) are defined, which include all the subensembles \( S \) that correspond to the values \( \psi, \theta \) and to arbitrary values of the amplitude of the normalized velocity \( u \). Using the change of variables \( V_0^0 = V_0 \cos(\theta_0) \) and \( V_0^1 = V_0 \sin(\theta_0) \), where \( V_0 \) are the amplitudes of the velocity fluctuations, and performing the integral over \( u \) in equation (10), one can show that the average velocity in \( S \) is

\[
V_0^S(X^S(t), t) = -V_0^0 \frac{1}{B} \frac{\partial}{\partial X^S} \phi^S(X^S(t), t),
\]

\[
\phi^S = \frac{8}{\pi} \left( \int \frac{8}{\pi} \cos(\theta_0) E_1}{V_0}\right)^2 - \frac{8}{\pi} \left( \int \frac{8}{\pi} \sin(\theta_0) V_1}{V_0}\right)^2,
\]

where \( E = E(x, z, t; \rho) \) is the gyroaveraged EC (4), \( E_1 = \partial E/\partial x_1, E_2 = \partial^2 E/\partial x_1 \partial x_1 \) are its space derivatives, \( \phi^S(\rho) \) is the amplitude of the gyro-averaged potential and \( V_0(\rho) \) are the amplitudes of the \( E \times B \) stochastic drift in the radial and poloidal directions. The latter are defined by \( V_0(\rho) = \sqrt{-E_2(0, 0, 0, \rho)} \), \( V_2(\rho) = \sqrt{-E_1(0, 0, 0, \rho)} \). Thus, the average velocities \( V_0^S(X, t) \) in \( S \) are obtained from the average potential \( \phi^S(x, z, t) \). The DT in the subensemble \( S, X^S(t) \) is the solution of

\[
\frac{dX_0^S}{dt} = V_0^S(X^S(t), t) + \delta_0 V_0(t).
\]

Equation (11) for the time dependent diffusion coefficients \( D(t) \) in the fast DTM is

\[
D(t) = \frac{V_0}{4} \int_{-\infty}^{\infty} d\psi d\theta \exp \left( \frac{(\psi_0)^2}{2} \right) \int_0^{2\pi} \frac{d\theta}{d\psi} \cos(\theta_0) X_0^S(t).
\]

The advantage of this method is that the number of DTs is strongly reduced, which leads to calculation times that are much smaller (with more than an order of magnitude). As shown in [20], this strong simplification of the DTM leads to variation of the results that are of the order of 10%.

The transport coefficient that determines the turbulent particle flux at large space and time scales is the asymptotic value \( D_\infty = \lim_{t \to \infty} D(t) \).

A computer code was developed for determining \( D(t) \) using equations (12)–(15). It calculates the EC of \( \psi(4) \) and its derivatives that appear in the subensemble average velocity (12) using a fast Fourier transform subroutine. The latter links a uniform grid representation in the \( k \) space to a two-dimensional real space mesh on which the velocity field is computed. The spectrum (5) is used for all the calculations presented in this paper, but it can easily be replaced by other models. The DTs are determined using high order interpolation techniques for the velocity field. The time step automatically adapts so that only the space steps along \( x_1 \) and \( x_2 \) have to be optimized. The condition is provided by the case of unperturbed potential, which yields periodic motion on the contour lines of \( \psi \). The computed trajectories have to remain close to these lines during the whole integration time that can be of hundreds of periods.

3. Fast particle diffusion regimes

The EC of the gyro-average potential (4) contains eight physical parameters, six from the spectrum of the turbulence (5) (the amplitude of the potential fluctuations, \( \Phi \), the dominant wave number \( k_0 \), the correlation lengths \( \lambda_1, \lambda_2, \lambda_3 \), and the correlation time \( \tau_2 \) plus the Larmor radius, and the effective diamagnetic velocity \( V_d \). The latter appears as the drift of the potential in the poloidal direction. They define five dimensionless parameters: the effective Kubo number \( K_d \)

\[
K_d \equiv \frac{\tau_2}{\tau_v}, \quad \tau_d = \frac{\tau_2 \tau_v}{\tau_1 + \tau_2},
\]

the diamagnetic parameter \( K_d \), or equivalently the normalized average velocity \( V_d \)

\[
K_d \equiv \frac{\tau_2}{\tau_1} \equiv \frac{V_2}{V_1}, \quad V_d = \frac{V_2}{V_1},
\]

the dominant wave number \( k_0 \equiv k_0^2/\lambda_2 \), the anisotropy \( a \equiv \lambda_1^2/\lambda_2^2 \) and the normalized Larmor radius \( \rho_0 \equiv \lambda_2/\lambda_2 \). The poloidal time of flight \( \tau_2^0 \equiv \lambda_2/V_2 \) appears in the diamagnetic parameter (17).

A short review of the results obtained in the limit \( \rho \to 0 \) are presented in section 3.1 as a background for the analysis of the Larmor radius effects. The transport regimes at \( \rho \neq 0 \) are presented in section 3.2 and the energy dependence of the particle turbulent loss is discussed in section 3.3.

3.1. Transport regimes in the limit \( \rho \to 0 \)

The diffusion regimes of the electrons and of the ions in the realistic model of the spectrum (5) were studied for the limit of zero Larmor radius [20, 25].

We have found that the characteristics of the transport are strongly influenced by the process of trajectory trapping or eddying in the stochastic potential. The conditions for the

4
existence of this process are determined by the values of $K_d$ and $K_c$.

The diamagnetic parameter $K_c$ (17) is a dimensionless measure of the effective velocity $V_d$, which determines the characteristic time $\tau_d = \lambda_d/V_d$. It is equivalent with an average potential $\lambda_0 V_d$, which adds to the stochastic potential and changes the configuration of the total potential. Bunches of opened contour lines appear on a fraction of the surface that increases from zero (for $V_d = 0$, $K_c = \infty$) to one (for $V_d > V_2$, $K_c < 1$). Trajectory trapping is possible when the bunches of open lines fill only a fraction of the surface, and islands of closed contour lines exist between them. This configuration corresponds to the condition $V_d < V_2$ (or $K_c > 1$).

The effective Kubo number $K_d$ (16) is a measure of the decorrelation of the trajectories from the potential, which is determined by the time variation of the potential and/or by the parallel motion of the particles. Trajectory trapping exists when the decorrelation is weak such that the characteristic time $\tau_d$ is larger than the time of flight in the radial and poloidal directions $\tau^R_0 = \lambda_0 V_i$, which are of the order of the time unit $\tau_0 = \lambda_0/\rho_0$.

The dominant wave number $K_0$ and the anisotropy $\alpha$ influence the shape of the contour lines of the potential and the amplitude of the velocity fluctuations. They essentially influence the parameter that limits the transport regimes.

The special shapes of the spectrum (5) and of the EC (figure 2(b)) lead to similar dependences of $D^\infty$ on $K_d$ for the quasilinear regime ($K_c < 1$) and for the nonlinear regime ($K_c > 1$) [20]. In both cases, $D^\infty = V^2_d/\tau_d$ for small $K_d$, it has a maximum at $K_d = K_{max}$ and eventually it decays as $K_d^{-\nu}$ (see figure 5, the curve for $\bar{\rho} = 0$). $K_{max}$ depends on the transport regime: $K_{max} = K_c \sqrt{\alpha}/\sqrt{\lambda_0^2 + 3}$ for $K_c < 1$, and $K_{max} = \sqrt{\alpha}/\sqrt{\lambda_0^2 + 3}$ for $K_c > 1$. The power $\nu$ also depends on the regime and on the EC of the potential. The strongest decay (large value of $\nu$ close to one) corresponds to the short range spectrum (6), while the long range spectrum (7) leads to smaller $\nu$.

The physical processes that determine the decay of the diffusion coefficient at large $K_d$ are completely different in the quasilinear and nonlinear transport.

When $K_c < 1$, the effect is determined by the special shape of the EC of the potential combined with the existence of the diamagnetic velocity. The EC corresponds to a potential that has positive and negative cells that roughly alternate in the poloidal direction. The drift of the potential with the diamagnetic velocity ($V_d > 0$) determines a fast alternation of the positive and negative potential cells on particle trajectories (with the characteristic time $\tau_d = \lambda_0 V_i$). This leads to radial oscillations with displacements of the order $V_1 \tau_d$ and to the diffusion coefficient $D^\infty \approx (V_1 \tau_d)^2/\tau_d$ that decays as $1/K_d$. We note that, for an usual EC (a decreasing function without negative minima that corresponds to a spectrum with one maximum) or for $V_d = 0$, the diffusion coefficient in the quasilinear regime ($K_c < 1$) saturates at the maximum value instead of decaying. This means that $D^\infty$ is significantly smaller at large $K_d$ in the turbulence with a spectrum (5) than in the usual case.

When $K_c > 1$, the cause of the decrease of $D^\infty$ at large $K_d$ is particle trapping or eddying. The transport is determined only by the trajectories that are not trapped (they are not eddying at times of the order of the decorrelation time). The fraction of these trajectories decreases as $\tau_d$ increases, and thus it determines the decay of the diffusion coefficient with the increase of $K_d$. It is interesting to underline that the transport in the presence of trapping has special properties. In the absence of any decorrelation mechanism ($K_d \rightarrow \infty$), the transport is subdiffusive ($D^\infty \rightarrow 0$). However, this process represents a diffusion reservoir in the sense that any weak decorrelation (with very large characteristic time $\tau_d$) leads to diffusive transport with diffusion coefficient that increases when the decorrelation becomes stronger. This tendency is inverse compared to the usual case where trapping is not present. The increase is due to the release of trapped trajectories and it continues until all the trajectories are free. At stronger decorrelation (smaller $\tau_d$), $D^\infty$ decreases (see figure 5, the curve for $\rho_0 = 0$). The maximum value of $D^\infty(K_d)$ is rather large. We note that this large value imposes to the evolution of turbulence to reach a state with $D^\infty$ situated far enough from the maximum such that the diffusive damping is smaller than the growth rate [26].

### 3.2. Transport regimes at large Larmor radius

The transport at large Larmor radius is formally described by the same equation of motion as in the limit $\bar{\rho} = 0$, but with a modified potential that has the EC (4). It is expected that the transport processes are qualitatively similar in the two cases and that only quantitative differences are determined by the fast gyration motion. This can be seen in the time dependent diffusion coefficient. Typical results can be seen in figure 3 where $D(t)$ is represented for $\bar{\rho} = 0$ (the dashed line) and for several values of $\bar{\rho}$ that label the curves (continuous lines). The shapes of the curves are roughly similar for different values of $\bar{\rho}$. The differences between the curves with different $\bar{\rho}$ are practically independent on time, except for the range $\bar{\rho} < 1$ where a stronger dependence on time can be seen.

The asymptotic diffusion coefficient $D^\infty$ is represented in figure 4 as function of the Larmor radius $\bar{\rho}$ for a set of parameters that correspond to the nonlinear regime and for the short range spectrum (6). One can see that the decay at large $\bar{\rho}$ is approximately as $1/\bar{\rho}$. This behavior is the same as in the quasilinear regime.

The similar dependence on $\bar{\rho}$ in the nonlinear and quasilinear regimes is rather surprising because several works have found a weaker decay in the nonlinear regime (as $1/\bar{\rho}^{0.38}$ in [21]). The cause of the faster decay with $\bar{\rho}$ found here is the spectrum with two maxima (5) of the drift type turbulence. It leads, as $\bar{\rho}$ increases, to only a weak change of the shape of the EC of the gyro-averaged potential. As seen in figure 2(b), the change appears in a limited range of $\bar{\rho}$ ($\bar{\rho} \lesssim 1$). The weaker decay of $D^\infty(\bar{\rho})$ observed in other potentials in the nonlinear regime is essentially due to the increase of the correlation lengths with the increase of $\bar{\rho}$, which leads to the increase of the time of flight. This determines a continuous
In terms of the dimensionless parameter $K_d$ found for continuous and $\rho$ at but the product of these parameters and it increases to values of the order one in that partly compensates the is displaced to $DK_d$ of the fast ions on the decorrelation $\rho$ = partition parameter $\rho_a$ and $\rho$ that label the curves and $K_d$ = 10, $V_d$ = 0.1.

The time dependent diffusion coefficient $D(t)$ for the values of $\bar{\rho}$ that label the curves and $K_d$ = 10, $V_d$ = 0.1.

Figure 4. The asymptotic diffusion coefficient $D^\infty$ as function of the Larmor radius for $K_d$ = 10, $V_d$ = 0.1, $k_0$ = 1, $a$ = 9.

displacement of the position of the maximum of $D^\infty$ towards large $K_d$, and the increase of $D^\infty$ that partly compensates the decay of the amplitude as $\phi^\rho$ becomes $\approx 1/\bar{\rho}$. In the present case, the correlation length of the gyro-averaged potential has a weak increase with $\bar{\rho}$ at values $\bar{\rho} \lesssim 1$ and then it saturates. In these conditions, the decay of $D^\infty(\bar{\rho})$ at $\bar{\rho} \gg 1$ in the nonlinear regime is determined only by the amplitude $\Phi^\rho(\bar{\rho})$ and it scales roughly as $1/\bar{\rho}$.

The dependence of $D^\infty$ of the fast ions on the decorrelation parameter $K_d$ is shown in figure 5. One can see that at $\bar{\rho} = 4$ the diffusion coefficient is smaller than in the zero Larmor radius limit. The dependence on $K_d$ of the fast particle diffusion coefficient is the same as at $\bar{\rho} = 0$ for both limits of small and large $K_d$ (the curves are parallel in these limits). The maximum of $D^\infty(K_d)$ is displaced to larger values of $K_d$ at large $\bar{\rho}$. This effect is produced by the increase of the width of the gyro-averaged EC that appears for $\bar{\rho} \lesssim 1$ and saturates at larger values. The results presented in figure 5 are obtained for a short range spectrum (5) and (6) with $n_1$ = 1. The diffusion regimes for a long range spectrum (5) and (7) are similar to those in figure 5. Quantitative differences appear for the amplitude $V_1$ of the velocity (that is a function of $n_1$, $n_2$ in equation (7)) and for the slope of the decay at large $K_d$, which is smaller in this case.

Figure 5. The asymptotic diffusion coefficient $D^\infty$ as function of the decorrelation parameter $K_d$ for $\bar{\rho} = 0$ and $\bar{\rho} = 4$ (continuous lines) and the ratio $f(K_d)$ defined in equation (18) (dashed line). $V_d = 0.1, k_0 = 1, a = 9$.

The diffusion coefficient at large Larmor radius can be approximated as

$$D^\infty(K_d, \bar{\rho}) \approx D^\infty(K_d, 0) \frac{f(K_d)}{\bar{\rho}},$$

where the function $f(K_d)$ is represented in figure 5 (the dotted line). This equation includes the scaling as $1/\bar{\rho}$ found for the diffusion coefficient in both quasilinear and nonlinear regimes and the dependence on the Kubo number. As seen in figure 5, the function $f(K_d)$ is small in the quasilinear regime ($f(K_d) \approx 0.2$), and it increases to values of the order one in the nonlinear regime. This shows that trapping determines a smaller decay of the ratio $D^\infty(K_d, \bar{\rho})/D^\infty(K_d, 0)$, but it does not modify the scaling in $\bar{\rho}$ for a spectrum with two maxima (5).

Thus, at large Larmor radius, the transport coefficient is significantly reduced compared to the limit $\bar{\rho} = 0$, but the transport regimes and the corresponding physical processes are similar to those for $\bar{\rho} = 0$.

3.3. Alpha particle turbulent loss as function of the energy

Numerical simulations have shown that the main decorrelation mechanism in ITG and TEM turbulence is the parallel ion motion. The turbulent transport of the fast particles, which have much smaller parallel time $\tau_z$, is completely dominated by the parallel decorrelation. The decorrelation parameter (16) becomes $K_d \approx \tau_z/\tau_{0_0}$ in these conditions.

Both the decorrelation time and the Larmor radius are functions of the energy of the $\alpha$ particles. The cooling of the fast particles determines the decrease of $\rho_a$ and the increase of $\tau_z$, but the product of these parameters $\rho_a \tau_z = (m_\alpha/q_\alpha)(\lambda_a/B)$ does not depend on $\alpha$ particle energy. It essentially depends on plasma size through $\lambda_a$. In terms of the dimensionless parameters that characterize fast particle transport

$$\bar{\rho} K_d = p, \quad p \equiv \frac{e\phi}{T_\lambda} \left( \frac{\rho_a}{\lambda_a} \right)^3 \frac{R}{a} \frac{\rho_a}{\lambda_a^3},$$

where $R$ is the major radius and $a$ is plasma radius. This show that $\bar{\rho}$ and $K_d$ cannot be modified independently for
ITG or TEM turbulence. Their product is \( p \), a dimensionless parameter that depends on the normalized amplitude of the turbulence \( e^\Phi T_\rho \) on its normalized poloidal and parallel correlation lengths and on the plasma size parameter \( a/\rho_i \). The typical values of the parameter \( p \) are of the order of a few tens, with larger values for ITER size plasmas than for the present devices due to the factor \( a/\rho_i \).

The dependence of the diffusion coefficient on \( p \) is shown in figure 6 for several values of \( p \) and for short and long range spectra. One can see an important difference between these results and those in figure 4, which shows that the decrease of the energy (of \( \tilde{\rho} \)) determines a monotonous increase of the diffusion coefficient. A maximum radial diffusion appears in figure 6, which has the amplitude and the location dependent on the parameter \( p \) and on the characteristics of the two peak spectra. The shape of these curves is determined by the simultaneous variation of \( \tilde{\rho} \) and \( K_d \) with the energy. The decrease of the energy determines the increase of \( K_d \), which moves from the small decorrelation time regime in figure 4 towards the maximum and further to the nonlinear regime with decaying \( D^\infty(K_d) \). The maxima of the curves in figure 6 correspond to the maximum of \( D^\infty(K_d) \) in figure 4. This maximum is a decreasing function of the energy, and this is reflected in figure 6, which shows that the maximum is smaller when it appears at larger energies.

It is important to underline that similar dependencies on the energy appear in the quasilinear and nonlinear regimes. This behavior is a consequence of the two peak spectrum, which leads to the same dependence of \( D^\infty \) on \( K_d \) and \( \tilde{\rho} \) for \( K_s < 1 \) and for \( K_s > 1 \), as discussed in sections 3.1 and 3.2.

Figure 6 shows that the diffusion coefficient is small for the whole range of \( \alpha \) particle energy at \( p = 40 \), but at smaller values of \( p \) it can reach significant values (comparable to the ion diffusion coefficient). The maximum of \( D^\infty \) is at energies that are larger than plasma ion energy (see the curves for \( p = 10 \)). Since the thermal velocity of the cooled \( \alpha \) particles is smaller than the ion velocity, the diffusion coefficient for the ashes is smaller than that of the Deuterium ions. The maximum losses are at higher energies of the order of 100 KeV.

Figure 6 also shows that the general dependence of the fast particle diffusion coefficient on the energy is the same for the short and long range spectra. The dashed curves for the spectrum (5) and (7) are similar to the continuous curves obtained for the spectrum (5) and (6), except for the position of the maximum that is at smaller energies. The effect is produced by the correlation lengths \( \lambda_i \), which are larger in this case and lead to a larger time of flight.

Thus, \( \alpha \) particle turbulent transport strongly depends on the parameter \( p \), which is determined by the amplitude of the turbulence, by the correlation lengths and by the plasma size. Important turbulent losses do not appear at large values of \( p \) that correspond to large amplitudes of the potential fluctuations, but at smaller \( p \) (of the order \( p \gtrsim 10 \)). The increase of plasma size determines the decrease of the \( \alpha \) particle turbulent losses. These results predict smaller losses in ITER than in JET plasmas, if the characteristics of the turbulence (amplitude and correlation lengths) are not dependent on plasma size.

4. Conclusions

A detailed study of the turbulent transport of the fast \( \alpha \) particles was performed based on the development of the DTM. A realistic model of the turbulence was considered. It was necessary to determine numerically the gyro-averaged Eulerian correlation (EC) and to adapt the DTM code to the discretized EC. The code calculates the time dependent diffusion coefficient from the EC and its derivatives represented on a space mesh, using an interpolation procedure.

The two peak shape of the spectrum of the ITG and TEM turbulence leads to the decay of the diffusion coefficient as \( 1/\rho \) for \( \tilde{\rho} \gtrsim 2 \) for both quasilinear and nonlinear regimes. The decay in the nonlinear regime is faster than in the case of monotonically decreasing Eulerian correlations, as the EC considered in [7]. The difference between the quasilinear and the nonlinear regime is given by a factor that depends on the decorrelation parameter \( K_d \) and is larger at large \( K_d \) in the nonlinear regime (see equation (18) and figure 5).

The parallel motion of the \( \alpha \) particles provides the main decorrelation mechanism in ITG turbulence. The characteristic time for this motion \( \tau_i = \lambda_i/\nu_{\alpha} \) depends on \( \alpha \) particle energy. It is very small when \( \alpha \) particles are born in the nuclear fusion reaction, and it increases by a factor of the order 20 during the cooling process. The dependence of the asymptotic diffusion coefficient on the energy of the \( \alpha \) particles was obtained taking into account both the parallel motion and the variation of the Larmor radius. The combined action of these effects leads to the existence of a maximum diffusion coefficient (figure 6). We have identified a parameter \( p \) (19), which includes the characteristics of the turbulence and the plasma size. The maximum turbulent loss rate and the corresponding energy are functions of \( p \). The details of the two peak spectrum represented by the short or long range dependence on the wave numbers (equations (5)–(7)) have weaker influence on the dependence of the diffusion coefficient on the energy of the \( \alpha \) particles (see figure 6). The diffusion coefficient decreases when \( p \) increases. Significant turbulent losses

![Figure 6. Asymptotic values of the diffusion coefficient \( D^\infty \) as function of the energy for the values of \( p \) that label the curves in the case of a short range spectrum (6) with \( n_1 = 1 \) (continuous lines) and of a long range spectrum (7) with \( n_1 = 1, n_2 = 4 \) (dashed lines). The other parameters are \( k^0_2 = 1 \) and \( a = 9 \).](image)
(comparable to those of the ions) appear at $p \simeq 10$ only when most of the energy of the $\alpha$ particles is lost and their energy is in the range of 100 KeV. As $p \sim a/p$, these results predict much smaller turbulent losses in ITER plasmas than in the case of JET.

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