Exactly solvable magnet of conformal spins in four dimensions

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We provide the eigenfunctions for a quantum chain of $N$ conformal spins with nearest-neighbor interaction and open boundary conditions in the irreducible representation of $SO(1,5)$ of scaling dimension $\Delta = 2 + i\lambda$ and spin numbers $\ell = \ell' = 0$. The spectrum of the model is separated into $N$ equal contributions, each dependent on a quantum number $Y_a = [a_a, n_a]$ which labels a representation of the principal series. The eigenfunctions are orthogonal and we computed the spectral measure by means of a new star-triangle identity. Any portion of a conformal Feynman diagram with square lattice topology can be represented in terms of separated variables, and we reproduce the all-loop “fishnet” integrals computed by B. Basso and L. Dixon via bootstrap techniques. We conjecture that the proposed eigenfunctions form a complete set and provide a tool for the direct computation of conformal data in the fishnet limit of the supersymmetric $\mathcal{N} = 4$ Yang-Mills theory at finite order in the coupling, by means of a cutting-and-gluing procedure on the square lattice.

INTRODUCTION

The exactly solvable spin magnets [1, 2] constitute a class of condensed matter models of wide interest throughout theoretical and mathematical physics. In particular, the integrable chains of nearest-neighbors interacting spins [3, 4] serve as a tool to encode the symmetries of local or non-local operators in quantum field theory, providing a rich amount of non-perturbative results ranging from the scattering spectrum of high-energy gluons in QCD [5–7] to the conformal data of the supersymmetric $\mathcal{N} = 4$ SYM and $\mathcal{N} = 6$ ABJM theories [8]. The archetype model of this class is the $SU(2)$ Heisenberg magnet of spin $\frac{1}{2}$, which for open boundary conditions is described by the Hamiltonian

$$H_{SU(2)} = \sum_{a=1}^{N-1} \bar{\sigma}_a \cdot \sigma_{a+1}, \quad (1)$$

being $\sigma_a$ the vector of Pauli matrices acting on the space $\mathbb{V}_a = \mathbb{C}^2$. Generalizations of (1) to other symmetry groups are known, including the non-compact $SO(1,5)$ spin chain [9]. The latter model is relevant for the study of covariant quantities in a four-dimensional conformal field theory (CFT) [10]. We consider the homogeneous model in the irreducible unitary representation defined by the scaling dimension $\Delta = 2 + i\lambda$, $\lambda \in \mathbb{R}$, and the $SO(4)$ spins $\ell = \ell' = 0$ [11]. The Hamiltonian operator acts on the Hilbert spaces $\mathbb{V}_a = L^2(x_a, d^4x_a)$ as

$$\mathbb{H} = \sum_{a=1}^{N-1} \left[ 2 \ln x_{aa+1}^2 + \frac{1}{(x_{aa+1}^2)^{i\lambda}} \ln((x_{aa+1}^2)^{i\lambda}) + 2 \ln x_{N0}^2 + \ln(\rho_{f1}^2) \right], \quad (2)$$

where $x_{aa+1} = x_a - x_{a+1}$, $\rho_{f1}^2 = -\partial_a \cdot \partial_a$ and $x_{N+1} = x_N$. The point $x_0$ is effectively a parameter for the model, and we will always omit it from the set of coordinates. The spin chain (2) is the four-dimensional version of the open $SL(2,\mathbb{C})$ Heisenberg magnet which describes the scattering amplitudes of high energy gluons in the Regge limit of QCD [7, 12]. The integrability of (2) is realized by the commutative family of normal operators

$$Q_N(u) = Q_{12}(u) \cdot Q_{23}(u) \cdots Q_{N0}(u), \quad (3)$$

labeled by the spectral parameter $u \in \mathbb{C}$ and where

$$Q_{ij}(u) = (x_{ij}^2)^{-i\lambda}(\rho_{ij}^2)^u(x_{ij}^2)^{u+i\lambda}. \quad (4)$$

By the introduction of the operator

$$\hat{Q}_N(u) = Q_N^*(u - i\lambda)Q_N(-i\lambda),$$

the Hamiltonian $\mathbb{H}$ is recovered from the expansion

$$Q_N(u) + \hat{Q}_N(u) = 2 \cdot \mathbb{1} + u \mathbb{H} + o(u). \quad (5)$$

It follows from (4) and from the commutation relation $[Q_N(u), Q_N(v)] = 0$ at generic $u$ and $v$, that the eigenfunctions of $Q_N$ diagonalize the Hamiltonian (2) as well. The spectra of these operators are labeled by the quantum numbers

$$(\mathbb{Y}_a = 1 + \frac{n_a}{2} + iv_a, \quad Y_a = 1 + \frac{n_a}{2} - iv_a), \quad \nu_a \in \mathbb{R}, \quad n_a \in \mathbb{N}, \quad (5)$$

for $a = 1, \ldots, N$, and we use to write $\mathbb{Y} = (Y_1, \ldots, Y_N)$. The spectral equation for the operator (3) reads

$$Q_N(u) \cdot \Psi^{\alpha\beta}(x|\mathbb{Y}) = \tau_N(u, \mathbb{Y}) \Psi^{\alpha\beta}(x|\mathbb{Y}),$$

where we denote $x = (x_1, \ldots, x_N)$ and $\alpha, \beta$ stand for $2N$ auxiliary complex spinors

$$|\alpha_1\rangle, \ldots, |\alpha_N\rangle \text{ and } |\beta_1\rangle, \ldots, |\beta_N\rangle \in \mathbb{C}^2.$$
The eigenfunctions form an orthogonal set with respect to the quantum numbers \((\mathbf{Y}, \alpha, \beta)\), and the eigenvalue is factorized with respect to the labels (5) into equal contributions

\[
\tau_{\mathbf{Y}}(u, \mathbf{Y}) = \prod_{a=1}^{N} \tau_{1}(u, Y_a), \tag{6}
\]

\[
\tau_1(u, Y_a) = \frac{\Gamma(Y_a - i\lambda/2) \Gamma(Y_a^* + u + i\lambda/2)}{\Gamma(Y_a^* + i\lambda/2) \Gamma(Y_a - u - i\lambda/2)}.
\]

As a consequence of (4) and (6) we obtained the spectrum of the Hamiltonian \(\hat{H}\) as a sum of \(N\) independent terms

\[
\eta_{\mathbf{Y}}(Y) = \sum_{a=1}^{N} \left[ \psi \left( Y_a - i\lambda/2 \right) + \psi \left( Y_a^* + u + i\lambda/2 \right) \right] + \text{c.c.} \tag{7}
\]

Formulas (6),(7) show that the \(N\)-body system defined in (2) gets separated into \(N\) one-particle systems over the quantum numbers (5). In other words, the quantities \((Y_a, |\alpha_a\rangle, |\beta_a\rangle)\) are the separated variables of the system in the sense of [13–16], and the spectrum of (2) and (3) is degenerate due to rotation invariance.

The representation over the separated variables \((\mathbf{Y}, \alpha, \beta)\) is defined for a generic function \(\phi(x) = \phi(x_1, \ldots, x_N)\) by the linear transform

\[
\bar{\phi}(\mathbf{Y}, \alpha, \beta) = \int d\mathbf{x} \Psi^{\alpha\beta}(x|Y)^* \phi(x). \tag{8}
\]

The inverse transform of (8) provides the expansion of \(\phi(x)\) over the basis of eigenfunctions

\[
\phi(x) = \sum_{n} \int d\mathbf{v} \mu(\mathbf{Y}) \int D\alpha D\beta \Psi^{\alpha\beta}(x|Y) \bar{\phi}(\mathbf{Y}, \alpha, \beta), \tag{9}
\]

where the sum runs over the non-negative integers \(n = (n_1, \ldots, n_N)\), the integrations \(d\mathbf{v} = dv_1 \cdots dv_N\) are on the real line and the integration in the space of spinors \(D\alpha = D\alpha_1 \cdots D\alpha_N\) is defined as

\[
\int D\alpha = \int_{\mathbb{C}^2} d\alpha e^{-|\alpha|^2}, \quad |\alpha|^2 = |\alpha^{(1)}|^2 + |\alpha^{(2)}|^2.
\]

The spectral measure in (9) can be extracted from the scalar product of eigenfunctions and it is given by

\[
\mu(\mathbf{Y}) = \frac{1}{N!} \prod_{a=1}^{N} (n_a + 1) \prod_{b \neq a}^{N} \left( \nu_{ab}^2 + \frac{n_a n_b}{4} \right) \left( \nu_{ab}^2 + \frac{(n_a + n_b + 2) + 2}{4} \right)
\]

in the notation \(\nu_{ab} = \nu_a - \nu_b\) and \(n_{ab} = n_a - n_b\).

All considerations done so far can be extended by an accurate analytic continuation of the parameter \(\lambda\) to the imaginary strip \((-2i, +2i)\). In particular, at \(\lambda = i\) each site of the chain carries the representation \(\Delta = 1\), \(\ell = \ell = 0\) of a bare scalar field in four dimensions. In this case at the point \(u = -1\) the operator \(\mathcal{Q}_N(u)\) becomes proportional to the graph-building integral operator for a Feynman diagram of square lattice topology

\[
\mathcal{B}_N \phi(x) = \frac{1}{(2\pi)^4} \int d\mathbf{x} \phi(x') \prod_{a=1}^{N} \frac{1}{x_{a+1}^2 + x_{a+2}^2}, \tag{11}
\]

with \(x = (x_1, \ldots, x_N), x' = (x'_1, \ldots, x'_N)\). Throughout the letter we denote \(x_{ab} = x_a - x'_b\). According to (6) the representation of the operator \(\mathcal{B}_N\) over the separated variables factorizes completely a portion of size \(N \times L\) of the planar fishnet diagram [17] in Fig.1, extending to a 4D space-time the analogue result in two-dimensions of [18].

As a direct application of our results, we computed a specific set of four-point functions of Fishnet CFT [19], providing a direct check to formula (14) of [20], obtained via arguments of AdS/CFT correspondence [21–23].

In the next two sections we present the explicit construction of the eigenfunctions of the model (2) by means of newly found integral identities.

**GENERALIZED STAR-TRIANGLE IDENTITY**

Our construction of a basis of eigenfunctions for \(\mathcal{Q}_N(u)\) follows the logic outlined in [24] for the two-dimensional model, and requires the formulation of certain conformal integral identities in 4D.

First we consider a positive integer \(M \leq N\) and set \(x_0' = 0\) without loss of generality. We will denote \(x = (x_1, \ldots, x_M), x' = (x'_1, \ldots, x'_{M-1})\). Let us introduce the tensors

\[
C^{\alpha\beta}_{\mu_1, \mu_2, \ldots, \mu_M} = \langle \sigma^{(1)} | \sigma_{\mu_1}^1 \sigma_{\mu_2}^2 \cdots \sigma_{\mu_M}^M | \beta \rangle \tag{12}
\]

where the symbols \(\sigma\) and \(\sigma\) are defined in terms of Pauli matrices

\[
\sigma_0 = \sigma_0 = 1, \quad \sigma_k = -\sigma_k = i\sigma_k, \quad k = 1, 2, 3.
\]
The tensors (12) satisfy the light-cone condition

$$t^{\mu_1 \cdots \mu_n} t^{\nu_1 \cdots \nu_n} C^{a \beta}_{\mu_1 \cdots \mu_n} C^{a \beta}_{\nu_1 \cdots \nu_n} p^{\cdots n} = 0,$$

where $t^{\mu_1 \cdots \mu_n}$ are auxiliary tensors and $a = 1, 2, \ldots, M$. This property allows to define a family of degree-$n$ homogeneous harmonic polynomials

$$C^{a\beta}_{M} (x|x')^n = \langle \alpha | \bar{x}_{12} \ldots \bar{x}_{12} | \bar{x}_{M0} | \beta \rangle^n,$$

where $x_{ij} = \sigma \mu x_{\mu i} / |x_{ij}|$ and $\bar{x}_{ij} = \sigma \mu x_{\mu i} / |x_{ij}|$. Under a coordinate inversion $x^\mu \rightarrow x^\mu / x^2$ such harmonic polynomials transform covariantly and it follows that using (13) it is possible to generalize the uniqueness - “star-triangle” - relation for a conformal invariant vertex of three scalar propagators [25] (see also [26, 27] and references therein) to any symmetric traceless representation.

The core of the generalized identity is the mixing operator acting on a pair of symmetric spinors $| \alpha, \alpha' \rangle = (| \alpha \rangle \otimes | \alpha' \rangle) \otimes (| \alpha \rangle \otimes | \alpha' \rangle)$ of degrees $n$ and $n'$ as

$$\langle \alpha, \alpha'| R_{n, n'} (z) | \beta, \beta' \rangle = \left( \frac{z - \frac{|n-n'|}{2}}{\Gamma (z + \frac{|n-n'|}{2} + 1)} \right) \times$$

$$\times \partial^\alpha_\alpha \partial^\alpha_{\alpha'} (1 - s \langle \alpha | \beta \rangle + t \langle \alpha | \beta' \rangle - s \langle \alpha' | \beta \rangle + t \langle \alpha' | \beta' \rangle)^{\frac{a + n + n'}{2}},$$

where upon differentiation we set $s = t = 0$. The operator defined by (14) is a unitary solution of the Yang-Baxter equation and can be obtained via the fusion procedure [28] applied to the Yangian R-matrix $R_{1,1}(z)$.

It follows that for any $n, n' \in \mathbb{N}$ and under the constraint $a + b + c = 4$ the following identity holds

$$\int d^4 x_4 \langle \alpha | x_{12} \bar{x}_{24} x_{41} | \beta \rangle^n \langle \alpha' | \bar{x}_{14} x_{24} x_{12} | \beta' \rangle^{n'} =$$

$$= \frac{A_{n, n'} (a, b, c)}{(x_{23}^2)^{a+b+c}} \times$$

$$\times \langle \alpha \bar{x}_{13}, \alpha' | R_{n, n'} | \beta, \beta' \rangle \Gamma (2 - a + \frac{n}{2}) \Gamma (2 - b + \frac{n}{2}) \Gamma (2 - c + \frac{n}{2}),$$

with the coefficient

$$A_{n, n'} (a, b, c) =$$

$$= \frac{\Gamma (2 - a + \frac{n}{2}) \Gamma (2 - b + \frac{n}{2}) \Gamma (2 - c + \frac{n}{2})}{\Gamma (a + \frac{n}{2}) \Gamma (b + \frac{n}{2}) \Gamma (c - \frac{n}{2}).}$$

Setting $n' = 0$, the identity (15) is equivalent to (A.11) of [29], and setting further $n = 0$ it degenerates to the scalar identity [25].

We point out that (15) is the four-dimensional versions of the 2D star-triangle relation which underlies the solution of the $SL(2, \mathbb{C})$ Heisenberg magnet as in [24, 30].

**EIGENFUNCTIONS CONSTRUCTION**

The eigenfunctions of the open conformal chain (2) can be obtained by a recursive procedure in the number of sites of the system. First of all we introduce the integral operators

$$\hat{\Lambda}^{a\beta}_{M, Y_a} = \langle \alpha | \hat{\Lambda}_{M, Y_a} | \beta \rangle$$

through its kernel

$$\langle \alpha | \hat{\Lambda}^{a\beta}_{M, Y_a} (x | x') \phi (x') \rangle =$$

$$= \frac{C^{a\beta}_{M} (x|x')^{n_a}}{(x_{2M}^2)^{1+\nu + i \lambda / 2}} \prod_{a=1}^{M-1} (x_{2a}^2)^{1+i \lambda / 2} (x_{2a+1}^2)^{i \lambda},$$

which at $M = 1$ reduces to a propagator in the irreducible representation of scaling dimension $\Delta = 2 + i \lambda + 2 i \nu$ and tensors rank $n_a$ of the principal series

$$\hat{\Lambda}^{a\beta}_{1, Y_a} (x_1) = \langle \alpha | \bar{x}_{1} | \beta \rangle^{n_a} \frac{(x_1^2)^{1+i \lambda / 2}}{(x_1^4)^{1+i \lambda / 2}}.$$

Making use of (15) at $n = n_a$, $n' = 0$ we verify that

$$Q_M (u) \hat{\Lambda}^{a\beta}_{M, Y_a} = \tau_1 (u, Y_a) \hat{\Lambda}^{a\beta}_{M, Y_a} Q_M (u),$$

for any $M > 1$, moreover

$$Q_1 (u) \hat{\Lambda}^{a\beta}_{1, Y_a} (x_1) = \tau_1 (u, Y_a) \hat{\Lambda}^{a\beta}_{1, Y_a} (x_1).$$

The iterative application of (17) for the length $M$ going from $N$ to 2, together with the initial condition (18), provides a recursive definition of the eigenfunctions of the model with $N$ sites

$$\psi^{a\beta} (Y | x) = \hat{\Lambda}^{a\beta_N}_{N, Y_N} \ldots \hat{\Lambda}^{a\beta_2}_{2, Y_2} \hat{\Lambda}^{a\beta_1}_{1, Y_1} \prod_{a=1}^{N} \frac{r(Y^a)^{n_a-1}}{\sqrt{2 \pi 2^{N-1}}},$$

where the last factor is a suitable normalization and

$$r (Y) = \frac{\Gamma (Y - i \frac{\lambda}{2}) \Gamma (Y^* - i \frac{\lambda}{2})}{\Gamma (Y^* + i \frac{\lambda}{2}) \Gamma (Y - i \frac{\lambda}{2}).}$$

Such a function has a simple behavior in the permutation of two separated variables $(Y, \alpha, \beta), (Y', \alpha', \beta')$, encoded by the exchange property

$$\hat{\Lambda}^{a\beta}_{M, Y'} \cdot \hat{\Lambda}^{a\beta}_{M-1, Y} = \langle \alpha', \alpha' | \hat{\Lambda}_{M, Y'} \cdot \hat{\Lambda}_{M-1, Y} | \beta', \beta \rangle =$$

$$= \frac{r (Y)}{r (Y')} \langle \alpha, \alpha' | R(z) \hat{\Lambda}_{M, Y} \cdot \hat{\Lambda}_{M-1, Y} | \beta, \beta' \rangle,$$

where $z = i (\nu' - \nu)$ and $R = R_{n, n'}$. Any permutation of the separated variables in (19) can be decomposed into elementary steps of type (20), defining a representation of the symmetric group generators

$$s_k Y = (Y_1, \ldots, Y_{k+1}, Y_k, \ldots, Y_N),$$
on the space of symmetric spinors
\[ s_k(\alpha) = R_{n_k,n_{k+1}}(i\nu_{k+1,k})|\alpha_1,\ldots,\alpha_k,\ldots,\alpha_N,\]
and allowing to state the exchange symmetry
\[ \psi^{\alpha\beta}(Y|x) = 
\psi^{s_k(\alpha,\beta)}(s_k Y|x). \]  
(21)
The scalar product of two eigenfunctions can be written according to (19) in operatorial form, so that it can be reduced to \(N\) factorized single-site contributions of the type
\[ \langle \hat\Lambda_1^{y_1} Y_1^{y'_1} \rangle \cdot \Lambda_1^{y_2} Y_1^{y'_2} = \frac{2\pi^3}{n+1} \delta_{n,n'} \delta(\nu - \nu') \langle \alpha | \alpha' \rangle^N \langle \beta | \beta' \rangle^N, \]
by the iterative application of the property
\[ \left( \hat\Lambda_{M,Y} \right)^\dagger \cdot \Lambda_{M,Y} = \langle \beta', \alpha | \hat\Lambda_{M,Y}^\dagger \cdot \hat\Lambda_{M,Y} | \alpha', \beta \rangle = \frac{r(Y)}{r(Y)} \times \]
\[ \times \pi^4 \, \text{Tr}_{n,n'}[(\alpha | R(z) | \alpha') \hat\Lambda_{M,Y-1}^\dagger \cdot \hat\Lambda_{M,Y-1} | \beta' | R(z) | \beta)], \]
valid under the assumption \( Y \neq Y' \) and where the trace means the cyclic contraction of indices in the space of symmetric spinors. As result the scalar product of two functions (19) takes the form of an orthogonality relation
\[ \frac{\mu(Y)}{N!} \sum_{\pi \in S_N} \delta(Y - \pi(Y')) \langle \alpha | \pi\alpha' \rangle \langle \beta' | \pi\beta \rangle, \]
(22)
where \( S_N \) are the permutations of \( N \) objects and we introduced the compact notation
\[ \delta(Y - Y') = \prod_{a=1}^N \delta_{n_a,n'_a} \delta(\nu_a - \nu'_a). \]
The relations (21),(22) allow to conjecture the completeness of the proposed eigenfunctions (19) and to define the representation of separated variables as in (8),(9).

**CONFORMAL FISHERNET INTEGRALS**

In analogy with the 2D results of [18], employing the results of the previous sections we will compute exactly the four-point correlation function
\[ G_{N,L} = \langle \text{Tr}[\phi^N(x_1)\phi^L(x_2)\phi^N(x_3)\phi^L(x_4)] \rangle, \]
(23)
for any \( N \) and \( L \), where \( \phi_1(x), \phi_2(x) \) are the two complex scalar \( N_c \times N_c \) fields which appear in the Lagrangian of the conformal fishnet theory [19] in four dimensions
\[ \mathcal{L}_0 = N_c \text{Tr}[\partial^\mu \phi_1^\dagger \partial_\mu \phi_1 + \partial^\mu \phi_2^\dagger \partial_\mu \phi_2 + (4\pi)\xi^2 \phi_1^\dagger \phi_1 \phi_2^\dagger \phi_2]. \]
In the planar limit \[ |N_c| \rightarrow \infty \] the only Feynmann diagram which contributes to the perturbative expansion in the coupling \( \xi^2 \) of \( G_{N,L} \) is given by the integral
\[ \int \frac{dz}{(4\pi)^N L} \left( \prod_{a=0}^N \left( \frac{1}{(z_{a,b} - z_{a+1,b})^2} \right) \left( \prod_{b=0}^L \frac{1}{(z_{a,b} - z_{a,b+1})^2} \right) \right), \]
(24)
where the integration measure is \( dz = \prod_{a,b=0}^{N,L} d^4 z_{a,b} \) and we set \( z_0 = x_1, z_{N+1b} = x_3, z_{a0} = x_4, z_{L+1a} = x_2 \). Such a square-lattice integral can be expressed via the graph-building operator (11). Indeed, starting from the fishnet diagram
\[ F_{N,L} = \left( \prod_{a=1}^N z_{a,a+1}^2 \right) \left( \mathbb{B}_N \right)^{L+1} \left( \prod_{a=1}^L \frac{1}{(z_{a} - z_{a-1})} \right), \]
(25)
one can transform it to (24) by the reductions of external points \( z_a \rightarrow x_1, z'_a \rightarrow x_3 \) followed by a conformal transformation. Therefore, as a functions \( G_{N,L}(u,v) \) of the cross-ratios \( u = \frac{x_2 x_3 (x_1^2 - x_4^2)}{x_3 x_4 (x_1^2 - x_2^2)} \) and \( v = \frac{x_1 x_2}{x_3 x_4} \), the planar limit of (23) is equal to \( F_{N,L} \) with reduced external points. According to (6) the integral kernel of \( \mathbb{B}_N(z) \) in the space of separated variables is factorized as
\[ \mathbb{B}_N(Y_1,\ldots,Y_N) = \frac{1}{\pi^{2NL}} \prod_{a=1}^N \left[ \frac{1}{4\nu_a^2 + (1 + n_a^2)} \right]^L. \]
(26)
In order to restore the \( (u,v) \)-dependence of (24) one has first to expand the r.h.s. of (25) over the eigenfunctions via the inverse transform (9). Then, by the appropriate reduction of the external points and upon integration of spinors and normalization by the bare correlator, we get
\[ G_{N,L}(u,v) = \sum_{n \in \mathbb{Z}} \int dv \mu(Y) \prod_{k=1}^N \left| \frac{x_{n+1/k}}{x_k + (n_k + 1)^2/4} \right|^{L+N} \]
where \( u/v = x\tilde{x}, v = 1/\sqrt{(1 - \tilde{x})(1 - x)} \). After the redefinition \( n_k \rightarrow n_k - 1, v_k \rightarrow u_k, x \rightarrow z \) it coincides with the result of [20].

We shall conjecture further applications of the sep-
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Y

setting

An interesting reduction of the correlator (27) is obtained

grams drawn on a three-punctured sphere

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labeled by the separated variables

spirit of “hexagonalisation” techniques [22, 23, 32, 33] we

as explained in [32] and exemplified in Fig.3. In the same

form factor

A

x

|A|^2 \prod_{a=1}^{M_1+M_3} \left[ \frac{1}{\nu_a^2 + (\eta_a + 1)^2} \right] \prod_{b=1}^{M_2+M_5} \left[ \frac{1}{\mu_b^2 + (\eta_b + 1)^2} \right]^{L},

and the form factor A is given by the overlapping of three
eigenfunctions of type (19) at different values of x₀

A = \int dz dz' dz'' \Psi^{\alpha} (z, z')\Psi^{\beta} (z, z')\Psi^{\omega} (z', z''),

(28)

for z = (z₁, ..., zₘ₁), z' = (z'₁, ..., z'ₘ₁), and z'' = (z''₁, ..., z''ₘ₂). Finally, the Feynmann integral is recovered
by gluing the two hexagons via completeness sums

\sim \sum_{n,m,l} \int d\nu \, d\mu \, d\tau \, \mu(Y) \, \mu(Z) \, \mu(U) \int D\alpha \, ... \, DW \, |H|^2.

An interesting reduction of the correlator (27) is obtained

setting L = 0 and degenerating it to the two-point function

⟨\text{Tr}(\phi^N) (x_1) \text{Tr}(\phi^N) (x_3)⟩⟩

for which the planar fishnet lies on a cylinder and it is conformally equivalent to

a “wheel” diagram [19, 34–36].

As a general fact the diagrams describing the planar limit of

development UV divergences, which in our representation

should be contained in the form factor (28). The elaboration of a regularization technique at this level is

an intriguing task as it would enable the direct computation of several conformal data in the Fishnet CFT at

finite order in the coupling.

CONCLUSIONS

We formulated and solved the spin chain of SO(1,5)

conformal spins for any number of sites N and for open

boundary conditions, in the principal series representa-
tion of zero spin [11]. Its integrability is realized by a

commuting family of spectral parameter-dependent op-
erators \(Q_N(u)\) which generate the conserved charges of

the model. The spectrum of the model is separated into

N symmetric contributions, each depending on quantum

numbers which for this reason we call separated vari-
bles. We explained how to construct the eigenfunctions

and prove their orthogonality, extending the logic of [24]
to a four dimensional space-time by means of new inte-

gram identities which generalize the star-triangle relation

[25] to symmetric traceless tensors.

Our results can be analytically continued from the repre-
sentation of the principal series to real scaling dimen-
sions, recovering the graph-building operator - intro-
duced in 2D by the authors and V. Kazakov [18] - for

the Feynmann diagrams of Fishnet CFT [19, 37].

The variant of this graph-builder with periodic boundary was

first introduced in [19] and coincides with the \(B\)-operator of

the Fishchain holographic model [38–40]. Following

the same steps as [18], we computed the planar limit of

the fishnet correlator studied by B. Basso and L. Dixon

providing a direct check of the formula (14) of [20].

The separation of variables (SoV) for non-compact spin
magnets is a topic which recently attracted great attention
[41–46], and SoV features appear in remarkable results

of AdS/CFT integrability, for instance [47, 48]. It

has not escaped our notice that the properties of the pro-

posed eigenfunctions immediately suggest their role in

the SoV of the periodic SO(1,5) spin chain [29], in full

analogy with [30]. Moreover it would be interesting to

apply our methods to the computation of other classes of

Feynmann integrals, for example introducing fermions as

in [49, 50], or considering any space-time dimension and

extending our results to the theory proposed in [51]. In

the latter context, the functions (19) for N = 2 sites have

been derived in a somewhat different form and applied to

the formulation of the Thermodynamic Bethe Ansatz equa-
tions [52].

Finally we have conjectured how, means of a cutting-

and-gluing procedure inspired by [32], certain planar two-

and three-point functions of the Fishnet CFT at finite
coupling get factorized into simple contributions over the separated variables. This observation puts as a compelling future task the regularization of such formulas, in order to compare the results based on the AdS/CFT correspondence to a direct computation.

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