Adaptive Multi-level Hyper-gradient Descent

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This chapter will discuss learning rate adaptation with newly proposed methods. As we know, one of the most important prospective of training deep neural networks is updating the model parameters with optimizers, while adaptive learning rates can lead to faster convergence and better final performance for deep learning models. There are several widely known human-designed adaptive optimizers such as Adam and RMSProp, gradient based adaptive methods such as hyper-descent and L4, and meta learning approaches including learning to learn. However, the issue of balancing adaptiveness and over-parameterization is still a topic to be addressed. In this chapter, we investigate different levels of learning rate adaptation based on the framework of hyper-gradient descent, and further propose a method that adaptively learns the model parameters for combining different levels of adaptations. Meanwhile, we show the relationship between adding regularization on learning rates and building combinations of different levels of adaptive learning rates. The experiments on several network architectures including feed-forward networks, LeNet-5, ResNet-18, and ResNet-34 show that the proposed multi-level adaptive approach can outperform other adaptive methods in a variety circumstances with statistical significance.

1 Introduction

The basic optimization algorithm for training deep neural networks is gradient descent method (GD), including stochastic gradient descent (SGD), mini-batch gradient descent and batch gradient descent. Model parameters are updated according to the first-order gradients of the empirical risks with respect to the parameters being optimized, while backpropagation is implemented for calculating the gradients of parameters \(\text{Ruder} 2016\). Naïve gradient descent methods apply fixed learning rates without any adaptation mechanisms. However, considering the change of available information during the learning process, SGD with fixed learning rates can result in inefficiency and a waste of computing resources in hyper-parameter searching. One solution is to introduce adaptive updating rules, while the learning rates are still fixed in training. This leads to the proposed methods include AdamGrad \(\text{Duchi et al.} 2011\), RMSProp \(\text{Tieleman and Hinton} 2012\), and Adam \(\text{Kingma and Ba} 2015\). Also there are optimizers aiming at addressing the convergence issue in Adam \(\text{Reddi et al.} 2019\) \(\text{Luo et al.} 2018\), or rectify the variance of the adaptive learning rate \(\text{Liu et al.} 2019\). Other techniques such as lookahead could also achieve variance reduction and stability improvement with negligible extra computational cost \(\text{Zhang et al.} 2019\).

Even though the adaptive optimizers with fixed learning rates can converge faster than SGD in a wide range of tasks, the updating rules are designed manually, while more
hyper-parameters are introduced. Another idea is to use the information of objective function and to update the learning rates as trainable parameters. This set of methods was introduced as automatic differentiation, where the hyper-parameters can be optimized with backpropagation [Maclaurin et al., 2015] [Baydin et al., 2018]. As gradient-based hyper-parameter optimization methods, they can be implemented as an online approach [Franceschi et al., 2017]. With the idea of auto-differentiation, learning rates can be updated in real time with the corresponding derivatives of the empirical risk [Almeida et al., 1998], which can be generated to all types of optimizers for deep neural networks [Baydin et al., 2017]. Another step size adaptation approach called “L4” is based on the linearized expansion of the loss function, which focuses on minimizing the need of learning rate tuning with strong reproducible performance across multiple different architectures [Rolinek and Martius, 2018]. Further more, by addressing the issue of poor generalization performance of adaptive methods, dynamic bound for gradient methods was introduced to build a gradual transition between adaptive method and SGD [Luo et al., 2018].

Another set of approaches train an RNN (recurrent neural network) agent to generate the optimal learning rates in the next step given the historical training information, which is known as “learning to learn” [Andrychowicz et al., 2016]. It empirically outperforms hand-designed optimizers in a variety of learning tasks, but another study shows that it may not be effective for long horizon [Lv et al., 2017]. The generalization ability can be improved by using meta training samples and hierarchical LSTMs (Long Short-Term Memory) [Wichrowska et al., 2017]. Still there are studies focusing on incorporating domain knowledge with LSTM-based optimizers to improve the performance in terms of efficacy and efficiency [Fu et al., 2017].

The limitations of existing algorithms are mainly in the following two aspects: (a) The proposed hyper-descent only focuses on the case of global adaptation of learning rates. Even though the original paper mentions that their approach can be generalized to the case where the learning rate is a vector, it is still necessary to investigate whether different levels of parameterization could make a difference in model performance as well as training efficiency. (b) No constraints or prior knowledge for learning rates are introduced in the framework of hyper-descent, which could be essential in resolving the issue of over-parameterization when a large number of independent learning rates need to be optimized.

In this study, we propose an algorithm based on existing works on hyper-descent but extend it to layer-wise, unit-wise and parameter-wise learning rates adaptation. In addition, we introduce a set of regularization techniques for learning rates for the first time to address the balance of global and local adaptation, which is also helpful in solving the issue of over-parameterization as a large number of learning rates are being learned. Although these regularizers indicate that extra hyper-parameters need to be optimized, the model performance after training could be improved with this setting in a large range of tasks. The main contribution of our study can be summarized as by the following three items:

- We propose an algorithm based on existing works on hyper-gradient descent but extend it to layer-wise, unit-wise and parameter-wise learning rates adaptations.
- We introduce a set of regularization techniques for learning rates for the first time to address the balance of global and local adaptation, which is also helpful in solving the issue of over-parameterization as a large number of learning rates are being learned.
We propose an algorithm for implementing the combination of adaptive learning rates in different levels for model parameter updating.

The structure of this chapter is organized as follows: Section 2 summarizes the related works on auto-differentiation, especially the hyper-descent (HD) algorithms. Section 3 explains the method implemented in extending the existing works. Section 4 shows the results of experiments on different learning tasks with a variety of models. Section 5 discusses the validity of the experiment results and Section 6 concludes the study.

2 Related work

This section is dedicated to reviewing the auto-differentiation and hyper-descent with detailed explanation and math formulas. In the original study of hyper-gradient descent (Baydin et al., 2017), the gradient with respect to the learning rate is calculated by using the updating rule of the model parameters in the last iteration. The gradient descent updating rule for model parameter \( \theta \) can be given by Eq. (1):

\[
\theta_t = \theta_{t-1} - \alpha \nabla f(\theta_{t-1})
\]

Note that \( \theta_{t-1} = \theta_{t-2} - \alpha \nabla f(\theta_{t-2}) \), the gradient of objective function with respect to learning rate can then be calculated:

\[
\frac{\partial f(\theta_{t-1})}{\partial \alpha} = \nabla f(\theta_{t-1}) \cdot \frac{\partial (\theta_{t-2} - \alpha \nabla f(\theta_{t-2}))}{\partial \alpha}
\]

\[
= \nabla f(\theta_{t-1}) \cdot (-\nabla f(\theta_{t-2}))
\]

(2)

A whole learning rate updating rule can be written as:

\[
\alpha_t = \alpha_{t-1} - \beta \frac{\partial f(\theta_{t-1})}{\partial \alpha}
\]

\[
= \alpha_{t-1} + \beta \nabla f(\theta_{t-1}) \cdot \nabla f(\theta_{t-2})
\]

(3)

In a more general perspective, assume that we have an updating rule for model parameters \( \theta_t = u(\Theta_{t-1}, \alpha_t) \). We need to update the value of \( \alpha_t \) towards the optimum value \( \alpha_t^* \) that minimizes the expected value of the objective in the next iteration. The corresponding gradient can be written as:

\[
\frac{\partial \mathbb{E}[f(\theta_t)]}{\partial \alpha_t} = \mathbb{E}[\nabla_\theta f(\theta_t)^T \nabla_\alpha u(\Theta_{t-1}, \alpha_t)],
\]

(4)

where \( u(\Theta_{t-1}, \alpha_t) \) denotes the updating rule of a gradient descent method. Then the additive updating rule of learning rate \( \alpha_t \) can be written as:

\[
\alpha_t = \alpha_{t-1} - \beta \mathbb{E}[\nabla_\theta f(\theta_{t-1})^T \nabla_\alpha u(\Theta_{t-2}, \alpha_{t-1})],
\]

(5)

where \( \mathbb{E}[\nabla_\theta f(\theta_{t-1})] \) is the noisy estimator of \( \nabla_\theta f(\theta_t) \). On the other hand, the multiplicative rule is given by:

\[
\alpha_t = \alpha_{t-1} \left(1 - \beta' \frac{\mathbb{E}[\nabla_\theta f(\theta_{t-1})^T \nabla_\alpha u(\Theta_{t-2}, \alpha_{t-1})]}{\mathbb{E}[\| \nabla_\theta f(\theta_{t-1}) \| \| \nabla_\alpha u(\Theta_{t-2}, \alpha_{t-1}) \|]}ight).
\]

(6)

These two types of updating rules can be implemented in any optimizers including SGD and Adam, denoted by corresponding \( \theta_t = u(\Theta_{t-1}, \alpha_t) \).
3 Multi-level adaptation methods

In this study we propose a combination form of adaptive learning rates, where the final learning rate applied for model parameter updating is the weighted combination of different level of adaptive learning rates, while the combination weights can also be trained with back-propagation. This give the similar effect with adding regularization on learning rates with certain kind of baselines. First we introduce the learning rate adaptation in different levels.

3.1 Layer-wise, unit-wise and parameter-wise adaptation

In the paper of hyper-descent (Baydin et al., 2017), the learning rate is set to be a scalar. However, to make the most of learning rate adaptation, in this study we introduce layer-wise or even parameter-wise updating rules, where the learning rate $\alpha_t$ in each time step is considered to be a vector (layer-wise) or even a list of matrices (parameter-wise). For the sake of simplicity, we collect all the learning rates in a vector: $\alpha_t = (\alpha_1, ..., \alpha_N)^T$.

Correspondingly, the objective $f(\theta)$ is a function of $\theta = (\theta_1, \theta_2, ..., \theta_N)^T$, collecting all the model parameters. In this case, the derivative of the objective function $f$ with respect to each learning rate can be written as:

$$\frac{\partial f(\theta)}{\partial \alpha_t} = \sum_{j=1}^{N} \frac{\partial f(\theta_1, ..., \theta_{i,t-1}, ..., \theta_{n,t-1})}{\partial \theta_j} \frac{\partial \theta_j}{\partial \alpha_t}$$

(7)

where $N$ is the total number of all the model parameters. Eq. (7) can be generalized to group-wise updating, where we associate a learning rate with a special group of parameters, and each parameter group is updated according to its only learning rate. Assume $\theta_t = u(\Theta_{t-1}, \alpha)$ is the updating rule, where $\Theta_t = \{\theta_s\}_{s=0}^h$ and $\alpha$ is the learning rate, then the basic gradient descent method for each group $i$ gives $\theta_i = u(\Theta_{t-1}, \alpha_{i,t-1}) = \theta_{i,t-1} - \alpha_{i,t-1} \nabla_{\theta_i} f(\Theta_{t-1})$. Hence for gradient descent,

$$\frac{\partial f(\theta)}{\partial \alpha_{i,t-1}} = \nabla_{\theta_i} f(\theta_{t-1})^T \nabla_{\alpha_{i,t-1}} u(\Theta_{t-1}, \alpha_t) = -\nabla_{\theta_i} f(\theta_{t-1})^T \nabla_{\theta_i} f(\theta_{t-2}).$$

(8)

Here $\alpha_{i,t-1}$ is a scalar with index $i$ at time step $t-1$, corresponding to the learning rate of the $i$th group, while the shape of $\nabla_{\theta_i} f(\theta)$ is the same as the shape of $\theta_i$.

We particularly consider three special cases: (1) In layer-wise adaptation, $\theta_i$ is the weight matrix of $i$th layer, and $\alpha_i$ is the particular learning rate for this layer. (2) In parameter-wise adaptation, $\theta_i$ corresponds to a certain parameter involved in the model, which can be an element of the weight matrix in a certain layer. (3) We can also introduce unit-wise adaptation, where $\theta_i$ is the weight vector connected to a certain neuron, corresponding to a column or a row of the weight matrix depending on whether it is the input or the output weight vector to the neuron concerned. Baydin et al. (2017) mentioned the case where the learning rate can be considered as a vector, which corresponds to layer-wise adaptation in this paper.

3.2 Regularization on learning rate

For the model involving a large number of learning rates for different groups of parameters, the updating for each learning rate only depends on the average of a small number
of examples. Therefore, when the batch size is also not large, over-parameterization is an issue to be concerned.

The idea in this study is to introduce regularization on learning rates, which can be implemented to control the flexibility of learning rate adaptation. First, for layer-wise adaptation, we can add the following regularization term to the cost function:

\[ L_{lr\_reg\_layer} = \lambda_{layer} \sum_l (\alpha_l - \alpha_g)^2 \]  

where \( l \) is the indices for each layer, \( \lambda_{layer} \) is the layer-wise regularization coefficient, \( \alpha_l \) and \( \alpha_g \) are the layer-wise and global-wise adaptive learning rates. A large \( \lambda_{layer} \) can push the learning rate of each layer towards the average learning rate across all the layers. In the extreme case, this will lead to very similar learning rates for all layers, and the algorithm will be reduced to that in [Baydin et al. 2017].

In addition, we can also consider the case where three levels of learning rate adaptations are involved, including global-wise, layer-wise and parameter-wise adaptation. If we introduce two more regularization terms to control the variation of parameter-wise learning rate with respect to layer-wise learning rate and global learning rates, the regularization loss can be written as:

\[ L_{lr\_reg\_para} = \lambda_{layer} \sum_l (\alpha_l - \alpha_g)^2 + \lambda_{para\_layer} \sum_l \sum_p (\alpha_{pl} - \alpha_l)^2 \]

\[ + \lambda_{para} \sum_l \sum_p (\alpha_{pl} - \alpha_g)^2 \]

where \( p \) represents the index of each parameter within each layer. The second and third terms are the regularization terms pushing each parameter-wise learning rate towards the layer-wise learning rate, and the term of pushing the parameter-wise learning rate towards the global learning rates, while \( \lambda_{para\_layer} \) and \( \lambda_{para\_layer} \) are the corresponding regularization coefficients.

With these regularisation terms, the flexibility and variances of learning rates in different levels can be neatly controlled, while it can reduce to the basement case where a single learning rate for the whole model is used. In addition, there could still be one more regularization for improving the stability across different time steps, which can be used in the original hyper-descent algorithm where the learning rate in each time step is a scalar:

\[ L_{lr\_reg\_ts} = \lambda_{ts} (\alpha_{g,t} - \alpha_{g,t-1})^2 \]

where \( \lambda_{ts} \) is the regularization coefficient to control the difference of learning rates between current step and the last step. With this term, the model with learning rate adaptation will be close to the model with fixed learning rate as large regularization coefficients are used. Thus, we can write the loss function of the full model as:

\[ L_{full} = L_{model} + L_{model\_reg} + L_{lr\_reg} + L_{lr\_reg\_ts} \]
3.3 Updating rules for learning rates

Considering these regularisation terms and take layer-wise adaptation for example, the gradient of the cost function with respect to a specific learning rate \( \alpha_l \) in layer \( l \) can be written as:

\[
\frac{\partial L_{\text{full}}(\theta, \alpha)}{\partial \alpha_l} = \frac{\partial L_{\text{model}}(\theta, \alpha)}{\partial \alpha_l} + \frac{\partial L_{\text{lr,reg}}(\theta, \alpha)}{\partial \alpha_l} = \tilde{\nabla}_\theta f(\theta_{t-1}) \nabla_{\alpha_{l,t-1}} u(\Theta_{t-2}, \alpha_{l-1}) + 2\lambda_{\text{layer}} (\alpha_{l,t} - \alpha_{g,t})
\]

with the corresponding updating rule by naive gradient descent:

\[
\alpha_{l,t} = \alpha_{l,t-1} - \beta \frac{\partial L_{\text{full}}}{\partial \alpha_{l,t}}.
\]

The updating rule for other types of adaptation can be derived accordingly. Notice that the time step index of layer-wise regularization term is \( t \) rather than \( t - 1 \), which ensures that we push the layer-wise learning rates towards the corresponding global learning rates of the current step. If we assume

\[
h_{l,t-1} = -\tilde{\nabla}_\theta f(\theta_{t-1}) \nabla_{\alpha_{l,t-1}} u(\Theta_{t-2}, \alpha_{l,t-1})
\]

then Eq. (14) can be written as:

\[
\alpha_{l,t} = \alpha_{l,t-1} - \beta(-h_{l,t-1} + 2\lambda_{\text{layer}} (\alpha_{l,t} - \alpha_{g,t})).
\]

In Eq. (16), both sides include the term of \( \alpha_{l,t} \), while the natural way to handle this is to solve for the close form of \( \alpha_t \), which gives:

\[
\alpha_{l,t} = \frac{1}{1 + 2\beta\lambda_{\text{layer}}} [\alpha_{l,t-1} + \beta(h_{l,t-1} + 2\lambda_{\text{layer}} \alpha_{g,t})].
\]

In this formula, we still need to calculate \( \alpha_{g,t} \), which is the global average learning rate in the current step. It will be even harder to calculate when there are multiple levels of learning rates, while the regularization still depends on their values in the current step. A more clean and probably computational efficient way of handling Eq. (16) is to introduce approximations to get rid of \( \alpha_{l,t} \) in the right hand side. If we do not consider the effect of regularization terms, the updating rule for layer-wise and global-wise learning rates can be written as:

\[
\hat{\alpha}_{l,t} = \alpha_{l,t-1} + \beta h_{l,t-1},
\]

\[
\hat{\alpha}_{g,t} = \alpha_{g,t-1} + \beta h_{g,t-1}
\]

where \( h_{g,t-1} = -\nabla_{\theta} f(\theta_{t-1}) \nabla_{\alpha_{g,t-1}} u(\Theta_{t-2}, \alpha_{g,t-1}) \) is the global \( h \) for all parameters. We define \( \hat{\alpha}_{l,t} \) and \( \hat{\alpha}_{g,t} \) as the “virtual” layer-wise and global-wise learning rates, where “virtual” means they are calculated based on the equation without regularization, and we do not use them directly for model parameter updating. Instead, we only use them as intermediate variables for calculating the real layer-wise learning rate for model training.

\[
\alpha^*_l,t = \alpha_{l,t-1} + 2\beta h_{l,t-1} - 2\beta\lambda_{\text{layer}} (\hat{\alpha}_{l,t} - \hat{\alpha}_{g,t})
\]

\[
= (1 - 2\beta\lambda_{\text{layer}}) \hat{\alpha}_{l,t} + 2\beta\lambda_{\text{layer}} \hat{\alpha}_{g,t}.
\]

Notice that in Eq. (19), the first two terms is actually a weighted average of the layer-wise learning rate \( \hat{\alpha}_{l,t} \) and global learning rate \( \hat{\alpha}_{l,t} \) at the current time step. Since we hope to push the layer-wise learning rates towards the global one, the parameters should
meet the constraint: $0 < 2\beta \lambda_{layer} < 1$, and thus they can be optimized using hyper-parameter searching within a bounded interval. Moreover, gradient-based optimization on these hyper-parameters can also be applied. Hence both the layer-wise learning rates and the combination proportion of the local and global information can be learned with back propagation. This can be done in online or mini-batch settings. The advantage is that the learning process may be in favor of taking more account of global information in some periods, and taking more local information in some other periods to achieve the best learning performance, which is not taken into consideration by existing learning adaptation approaches.

Now consider the difference between Eq. (16) and Eq. (19):

$$\alpha_{l,t}^* - \alpha_{l,t} = -2\beta \lambda_{layer}((\hat{\alpha}_{l,t} - \alpha_{g,t}) - (\alpha_{l,t} - \alpha_{g,t})).$$  \hspace{1cm} (20)

Based on the setting of multi-level adaptation, on the right-hand side of Eq. (20), global learning rate is updated without regularization $\hat{\alpha}_{g,t} = \alpha_{g,t}$. For the layer-wise learning rates, the difference is given by $\hat{\alpha}_{l,t} - \alpha_{l,t} = 2\beta \lambda_{layer}(\alpha_{l,t} - \alpha_{g,t})$, which corresponds to the gradient with respect to the regularization term. Thus, Eq. (20) can be rewritten as:

$$\alpha_{l,t}^* - \alpha_{l,t} = -2\beta \lambda_{layer}(2\beta \lambda_{layer}(\alpha_{l,t} - \alpha_{g,t}))$$

$$= -4\beta^2 \lambda_{layer}^2 (\alpha_{l,t} - \alpha_{g,t}) = -4\beta^2 \lambda_{layer}^2 (1 - \frac{\alpha_{g,t}}{\alpha_{l,t}}) \alpha_{l,t}$$  \hspace{1cm} (21)

which is the error of the virtual approximation introduced in Eq. (18). If $4\beta^2 \lambda_{layer}^2 << 1$ or $\frac{\alpha_{g,t}}{\alpha_{l,t}} \rightarrow 1$, this approximation becomes more accurate.

Another way for handling Eq. (16) is to use the learning rates for the last step in the regularization term.

$$\alpha_{l,t} \approx \alpha_{l,t-1} - \beta(-h_{l,t-1} + 2\lambda_{layer}(\alpha_{l,t-1} - \alpha_{g,t-1})).$$  \hspace{1cm} (22)

Since we have $\alpha_{l,t} = \hat{\alpha}_{l,t} - 2\beta \lambda_{layer}(\alpha_{l,t} - \alpha_{g,t})$ and $\hat{\alpha}_{l,t} = \alpha_{l,t-1} + \beta h_{l,t-1}$, using the learning rates in the last step for regularization will introduce a higher variation from term $\beta h_{l,t-1}$, with respect to the true learning rates in the current step. Thus, we consider the proposed virtual approximation works better than last-step approximation.

Similar to the two-level’s case, for the three-level regularization shown in Eq. (10), we have:

$$\frac{\partial L_{full}(\theta, \alpha)}{\partial \alpha_{p,t}} = \frac{\partial L_{model}(\theta, \alpha)}{\partial \alpha_{p,t}} + \frac{\partial L_{lb,reg}(\alpha)}{\partial \alpha_{p,t}}$$

$$= -\nabla_{\theta_l} f(\theta_{l-1}) \nabla_{\theta_t} u(\Theta_{l-2}, \alpha_{l-1}) + 2\lambda_2(\alpha_{p,t} - \alpha_{g,t}) + 2\lambda_3(\alpha_{p,t} - \alpha_{l,t})$$  \hspace{1cm} (23)

For the sake of simple derivation, we denote $\lambda_2 = \lambda_{layer}$, and $\lambda_3 = \lambda_{para,layer}$ for the regularization parameters in Eq. (10). The updating rule can be written as:

$$\alpha_{p,t} = \alpha_{p,t-1} - \beta(h_p + 2\lambda_2(\alpha_{p,t} - \alpha_{g,t}) + 2\lambda_3(\alpha_{p,t} - \alpha_{l,t})))$$

$$\approx \hat{\alpha}_{p,t}(1 - 2\beta \lambda_2 - 2\beta \lambda_3) + 2\hat{\alpha}_{l,t} \beta \lambda_3 + 2\hat{\alpha}_{g,t} \beta \lambda_2$$  \hspace{1cm} (24)

where we assume that $\hat{\alpha}_{p,t}$, $\hat{\alpha}_{l,t}$, $\hat{\alpha}_{g,t}$ are independent variables. Define

$$\gamma_1 = 1 - 2\beta \lambda_2 - 2\beta \lambda_3, \gamma_2 = 2\beta \lambda_3, \gamma_3 = 2\beta \lambda_2,$$
we still have:

\[
\alpha = \gamma_1\alpha_p + \gamma_2\alpha_l + \gamma_3\alpha_g,
\]

(25)

\[
\gamma_1 + \gamma_2 + \gamma_3 = 1.
\]

Therefore, in the case of three level learning rates adaptation, the regularization effect can still be considered as applying the weighted combination of different levels of learning rates. This conclusion is invariant of the signs in the absolute operators in Eq. (18).

In general, we can organize all the learning rates in a tree structure. For example, in three level case above, \(\alpha_g\) will be the root node, while \(\{\alpha_l\}\) are the children node at level 1 of the tree and \(\{\alpha_p\}\) are the children node of \(\alpha_l\) as leave nodes at level three of the tree. In a general case, we assume there are \(L\) levels in the tree. In general, we can organize all the learning rates in a tree structure. For example, in three level case above, \(\alpha_g\) will be the root node, while \(\{\alpha_l\}\) are the children node at level 1 of the tree and \(\{\alpha_p\}\) are the children node of \(\alpha_l\) as leave nodes at level three of the tree. In a general case, we assume there are \(L\) levels in the tree. Denote the set of all the paths from the root node to each of leave nodes as \(P\) and a path is denoted by \(p = \{\alpha_1, \alpha_2, ..., \alpha_L\}\) where \(\alpha_1\) is the root node and \(\alpha_L\) is the leave node on the path. On this path, denote ancestors\((i)\) all the ancestor nodes of \(\alpha_i\) along the path, i.e., ancestors\((i)\) = \(\{\alpha_1, ..., \alpha_i-1\}\). We will construct a regularizer to push \(\alpha_i\) towards each of its parents. Then the regularization can be written as

\[
L_{lr,\text{reg}} = \sum_{p \in P} \sum_{\alpha_i \in p} \sum_{\alpha_j \in \text{ancestors}(i)} \lambda_{ij} (\alpha_i - \alpha_j)^2.
\]

(26)

Under this pair-wise \(L_2\) regularization, the updating rule for any leave node learning rate \(\alpha_L\) can be given by the following theorem

**Theorem 1.** Under virtual approximation, effect of adding pair-wise \(L_2\) regularization on different levels of adaptive learning rates \(L_{reg} = \sum_i \sum_{j<i} \lambda_{ij} ||\alpha_i - \alpha_j||^2_2\) is equal to performing a weighted linear combination of virtual learning rates in different levels \(\alpha^* = \sum_i \gamma_i \alpha_i\) with \(\sum_i \gamma_i = 1\), where each component \(\alpha_i\) is calculated by assuming there is no regularization.

*Remarks:* Theorem 1 actually suggests that the similar updating rule can be obtained for the learning rate at the any level on the path. All these have been demonstrated in Algorithm 1 for the three level case.

**Proof.** Consider the learning regularizer

\[
L_{lr,\text{reg}}(\alpha) = \sum_{p \in P} \sum_{\alpha_i \in p} \sum_{\alpha_j \in \text{parents}(i)} \lambda_{ij} (\alpha_i - \alpha_j)^2.
\]

(27)

To apply hyper-gradient descent method to update the learning rate \(\alpha_L\) at level \(L\), we need to work the derivative of \(L_{lr,\text{reg}}\) with respect to \(\alpha_L\), the terms in (27) involving \(\alpha_L\) are only \((\alpha_i - \alpha_j)^2\) where \(\alpha_j\) is an ancestor on the path from the root to the leave node \(\alpha_L\). Hence

\[
\frac{\partial L_{\text{full}}(\theta, \alpha)}{\partial \alpha_{L,t}} = \frac{\partial L_{\text{model}}(\theta, \alpha)}{\partial \alpha_{L,t}} + \frac{\partial L_{lr,\text{reg}}(\alpha)}{\partial \alpha_{L,t}}
\]

\[
= -\hat{\nabla}_\theta f(\theta_{t-1})^T \nabla_{\theta_L} u(\Theta_{t-2}, \alpha_{t-1}) + \sum_{\alpha_j \in \text{ancestors}(L)} 2\lambda L_j (\alpha_{L,t} - \alpha_{j,t}).
\]

(28)

As there are exactly \(L - 1\) ancestors on the path, we can simply use the index \(j =
The corresponding updating function for $\alpha_{n,t}$ is:

$$
\alpha_{L,t} = \alpha_{n,t-1} - \beta(h_L + \sum_{j=1}^{L-1} 2\lambda_{Lj}(\alpha_{L,t} - \alpha_{j,t}))
$$

$$
\approx \hat{\alpha}_{L,t}(1 - 2\beta \sum_{j=1}^{L-1} \lambda_{Lj}\alpha_{n,t}) + \sum_{j=1}^{L-1} (2\beta\lambda_{Lj}\hat{\alpha}_{j,t}))
$$

$$
= \sum_{j=1}^{L} \gamma_j \hat{\alpha}_{j,t}.
$$

where

$$
\gamma_L = 1 - 2\beta \sum_{j=1}^{L-1} \lambda_{Lj},
$$

$$
\gamma_j = 2\beta\lambda_{Lj}, \text{ for } j = 1, 2, ..., L - 1.
$$

This form satisfies $\alpha^*_{L} = \sum_{j=1}^{L} \gamma_j \hat{\alpha}_j$ with $\sum_{j=1}^{L} \gamma_j = 1$. This completes the proof.

### 3.4 Prospective of learning rate combination

Motivated by the analytical derivation in Section 3.3, we can consider the combination of adaptive learning rates in different levels as a substitute of regularization on the differences of learning rates. As a simple case, the combination of global-wise and layer-wise adaptive learning rates can be written as:

$$
\alpha_t = \gamma_1 \hat{\alpha}_{l,t} + \gamma_2 \hat{\alpha}_{g,t},
$$

where $\gamma_1 + \gamma_2 = 1$ and $\gamma_1 \geq 0, \gamma_2 \geq 0$. In a general form, assume that we have $n$ levels, which could include global-level, layer-level, unit-level and parameter-level, etc, we have:

$$
\alpha_t = \sum_{i=1}^{n} \gamma_i \hat{\alpha}_{i,t}.
$$

In a more general form, we can implement non-linear models such as neural networks to model the final adaptive learning rates with respect of the learning rates in different levels.

$$
\alpha_t = g(\hat{\alpha}_{1,t}, \hat{\alpha}_{2,t}, ..., \hat{\alpha}_{n,t}; \theta),
$$

where $\theta$ is the vector of parameters of the non-linear model. In this study, we treat the combination weights $\{\gamma_1, ..., \gamma_n\}$ as trainable parameters as demonstrated in Eq. (32). Figure 1 gives an illustration of the linear combination of three-level hierarchical learning rates. In fact, we only need these different levels of learning rate have a hierarchical relationship, which means the selection of component levels is not fixed. For example, in feed-forward neural networks, we can use parameter level, unit-level, layer level and global level. For recurrent neural networks, the corresponding layer level can either be the “layer of gate” within the cell structure such as LSTM and GRU, or the whole cell in a particular RNN layer. Especially, by “layer of gate” we mean the parameters in each gate of a cell structure share a same learning rate. Meanwhile, for convolutional neural network, we can further introduce “filter level” to replace layer-level if their is no clear layer structure, where the parameters in each filter will share a same learning rate.
As the real learning rates implemented in model parameter updating is a weighted combination, the corresponding Hessian matrices cannot be directly used for learning rate updating. If we take the gradients of the loss with respect to the combined learning rates, and use this to update the learning rate for each parameter, the procedure will be reduced to parameter-wise learning rate updating. To address this issue, we first break down the gradient by the combined learning rate to three levels, use each of them to updated the learning rate in each level, and then calculate the combination by the updated learning rates. Especially, $h_{p,t}$, $h_{l,t}$ and $h_{g,t}$ are calculated by the gradients of model losses without regularization, as is shown in Eq. (34).

$$h_{p,t} = \frac{\partial f(\theta, \alpha)}{\partial \alpha_{p,t}} = -\nabla_{\theta} f(\theta_{t-1}, \alpha)|_{p} \cdot \nabla_{\alpha} u(\Theta_{t-2}, \alpha)|_{p}$$

$$h_{l,t} = \frac{\partial f(\theta, \alpha)}{\partial \alpha_{l,t}} = -\text{tr}(\nabla_{\theta} f(\theta_{t-1}, \alpha)|_{l}^T \nabla_{\alpha} u(\Theta_{t-2}, \alpha)|_{l})$$

$$h_{g,t} = \frac{\partial f(\theta, \alpha)}{\partial \alpha_{t}} = -\sum_{l=1}^{n} \text{tr}(\nabla_{\theta} f(\theta_{t-1}, \alpha)|_{l}^T \nabla_{\alpha} u(\Theta_{t-2}, \alpha)|_{l})$$

(34)

where $h_{t} = \sum_{l} h_{l,t} = \sum_{p} h_{p,t}$ and $h_{l,t} = \sum_{p \in \text{th layer}} h_{p}$ and $f(\theta, \alpha)$ corresponds to the model loss $L_{\text{model}}(\theta, \alpha)$ in Section 3.2. Algorithm 1 is the full updating rules for the newly proposed optimizer with three levels, which can be denoted as combined adaptive multi-level hyper-gradient descent (CAM-HD). where we introduce the general form of gradient descent based optimizers (Reddi et al. 2019, Luo et al. 2018). For SGD, $\phi_{t}(g_{1},...g_{t}) = g_{t}$ and $\psi_{t}(g_{1},...g_{t}) = 1$, while for Adam, $\phi_{t}(g_{1},...g_{t}) = (1 - \beta_{1})\Sigma_{t=1}^{t} \beta_{1}^{-1} g_{t}$ and $\psi_{t}(g_{1},...g_{t}) = (1 - \beta_{2})\text{diag}(\Sigma_{t=1}^{t} \beta_{2}^{-1} g_{t}^2)$. Notice that in each updating time step of Algorithm 1 we re-normalize the combination weights $\gamma_{1}$, $\gamma_{2}$ and $\gamma_{3}$ to make sure that their summation is always 1 even after updating with stochastic gradient-based methods. An alternative way of doing this is to implemented softmax, which require an extra set of intermediate variables $c_{p}$, $c_{l}$ and $c_{g}$ following: $\gamma_{p} = \text{softmax}(c_{p}) = \exp^{c_{p}} / (\exp^{c_{p}} + \exp^{c_{l}} + \exp^{c_{g}})$, etc. Then the updating of $\gamma$s will be convert to the updating of $c$s during training. In addition, the training of $\gamma$s can also be extended to multi-level cases, which means we can have different combination weights in different layers. For the updating rates $\beta_{p}$, $\beta_{l}$ and $\beta_{g}$ of
Algorithm 1: Updating rule of three-level CAM-HD

input: \( \alpha_0, \beta, \delta, T \)

initialization: \( \theta_0, \gamma_{t,0}, \gamma_{2,0}, \gamma_{3,0}, \alpha_{p,0}, \alpha_{l,0}, \alpha_0, \alpha_{t,0}^* = \gamma_{1,0}\alpha_{p,0} + \gamma_{2,0}\alpha_{l,0} + \gamma_{3,0}\alpha_0 \)

for \( t \in 1, 2, \ldots, T \) do

\[
\begin{align*}
g_t &= \nabla_{\theta} f(\theta, \alpha) \\
h_{p,t} &= \frac{\partial f(\theta, \alpha)}{\partial \alpha_{p,t}} = -\nabla_{\theta} f(\theta_{t-1}, \alpha)|_p \cdot \nabla_{\alpha} u(\Theta_{t-2}, \alpha)|_p \\
h_{l,t} &= \frac{\partial f(\theta, \alpha)}{\partial \alpha_{l,t}} = -\text{tr}(\nabla_{\theta} f(\theta_{t-1}, \alpha)|_l^T \nabla_{\alpha} u(\Theta_{t-2}, \alpha)|_l) \\
h_{g,t} &= \frac{\partial f(\theta, \alpha)}{\partial \alpha_{g,t}} = -\sum_{p=1}^\gamma \text{tr}(\nabla_{\theta} f(\theta_{t-1}, \alpha)|_l^T \nabla_{\alpha} u(\Theta_{t-2}, \alpha)|_l) \\
\alpha_{p,t} &= \alpha_{p,t-1} - \beta_p \frac{\partial f(\theta_{t-1})}{\partial \alpha_{p,t-1}} = \alpha_{p,t-1} - \beta_p \gamma_{1,t-1} h_{p,t} \\
\alpha_{l,t} &= \alpha_{l,t-1} - \beta_l \sum_p \frac{\partial f(\theta_{t-1})}{\partial \alpha_{p,t-1}} = \alpha_{l,t-1} - \beta_l \gamma_{2,t-1} \sum_p h_{p,t} = \alpha_{l,t-1} - \beta_l \gamma_{2,t-1} h_{l,t} \\
\alpha_{t} &= \alpha_{t-1} - \beta_t \sum p \sum_{p} \frac{\partial f(\theta)}{\partial \alpha_{p,t-1}} \frac{\partial \alpha_{p,t-1}}{\partial \alpha_{t-1}} = \alpha_{t-1} - \beta_t \gamma_{3,t-1} h_{g,t} \\
\alpha_{t}^* &= \gamma_{1,t-1} \alpha_{t-1} + \gamma_{2,t-1} \alpha_{l,t} + \gamma_{3,t-1} \alpha_{t} \\
\gamma_{1,t} &= \gamma_{1,t-1} - \delta \sum_p \frac{\partial L}{\partial \alpha_{p,t-1}} \frac{\partial \alpha_{p,t-1}}{\partial \gamma_{1,t-1}} = \gamma_{1,t-1} - \delta \sum_p \frac{\partial L}{\partial \alpha_{p,t-1}} \frac{\partial \alpha_{p,t-1}}{\partial \gamma_{1,t-1}} = \gamma_{1,t-1} - \delta \alpha_{p,t-1} \sum_p \frac{\partial L}{\partial \alpha_{p,t-1}} \\
\gamma_{2,t} &= \gamma_{2,t-1} - \delta \sum_p \frac{\partial L}{\partial \gamma_{2,t-1}} = \gamma_{2,t-1} - \sum_p \frac{\partial L}{\partial \gamma_{2,t-1}} = \gamma_{2,t-1} - \delta \alpha_{t-1} \sum_p \frac{\partial L}{\partial \gamma_{2,t-1}} \\
\gamma_{3,t} &= \gamma_{3,t-1} - \delta \sum_p \frac{\partial L}{\partial \gamma_{3,t-1}} = \gamma_{3,t-1} - \sum_p \frac{\partial L}{\partial \gamma_{3,t-1}} = \gamma_{3,t-1} - \delta \alpha_{t-1} \sum_p \frac{\partial L}{\partial \gamma_{3,t-1}} \\
g_1 &= \gamma_1/(\gamma_1 + \gamma_2 + \gamma_3), g_2 = \gamma_2/(\gamma_1 + \gamma_2 + \gamma_3), g_3 = \gamma_3/(\gamma_1 + \gamma_2 + \gamma_3) \\
m_t &= \phi_t(g_1, \ldots, g_t) \\
V_t &= \psi_t(g_1, \ldots, g_t) \\
\theta_t &= \theta_{t-1} - \alpha_{t}^* m_t / \sqrt{V_t}
\end{align*}
\]

end

return \( \theta_T, \gamma_1, T, \gamma_2, T, \gamma_3, T, \alpha_p, T, \alpha_l, T, \alpha_T \)
the learning rates in different level, we set:
\[ \beta_p = n_p \beta, \beta_l = n_l \beta, \beta_g = n \beta \] (35)
where \( \beta \) is a shared parameter. This setting will make the updating steps of learning rates in different levels be in the same scale considering the difference in the number of parameters involved in \( h_{p,t}, h_{l,t}, h_{g,t} \). If we take average based on the number of parameters in Eq. (34) at first, this adjustment is not required.

3.5 Convergence analysis:

The proposed CMA-HD is not an independent optimization method, which can be applied in any kinds of gradient-based methods. Its convergence properties highly depends on the base optimizer that is applied. Here we provide an analysis based on the general prospective of learning rate adaptation (Baydin et al. 2017; Karimi et al. 2016). We have learned that for global-wise learning rate adaptation, if we assume that \( f \) is convex and L-Lipschitz smooth with \( \| \nabla f(\theta) \| < M \) for some fixed \( M \) and all \( \theta \), the learning rate \( \alpha_t \) satisfies:
\[
|\alpha_t| \leq |\alpha_0| + \beta \sum_{i=0}^{t-1} |\nabla f(\theta_{t+1})^T \nabla f(\theta_i)| \leq |\alpha_0| + \beta \sum_{i=0}^{t-1} \| \nabla f(\theta_{t+1}) \| \| \nabla f(\theta_i) \| \\
\leq |\alpha_0| + t \beta M^2
\] (36)
where \( \alpha_0 \) is the initial value of \( \alpha \), and \( \beta \) is the updating rate for hyper-gradient descent. By introducing \( \kappa_{p,l} = \tau(t) \alpha^*_{p,l} + (1 - \tau(t)) \alpha_0 \), where the function \( \tau(t) \) is selected to satisfy \( \tau(t) \to 0 \) as \( t \to \infty \), we have the following convergence theorem.

**Theorem 2.** Convergence under certain assumptions about \( f \) Suppose that \( f \) is convex and L-Lipschitz smooth with \( \| \nabla f(\theta) \| < M \) for some fixed \( M \) and all \( \theta \). Then \( \theta_t \to \theta^* \) if \( \alpha^* < 1/L \) and \( t \cdot \tau(t) \to 0 \) as \( t \to \infty \), where the \( \theta_t \) are generated according to (non-stochastic) gradient descent.

The proposed CMA-HD is not an independent optimization method, which can be applied in any kinds of gradient-based updating rules. Its convergence properties highly depends on the base optimizer that is applied. By referring the discussion on convergence in (Baydin et al. 2017), if we introduce \( \kappa_{p,l} = \tau(t) \alpha^*_{p,l} + (1 - \tau(t)) \alpha_\infty \), where the function \( \tau(t) \) is selected to satisfy \( t \tau(t) \to 0 \) as \( t \to \infty \), and \( \alpha_\infty \) is a selected constant value. Then we demonstrate the convergence analysis for the three level case in the following theorem, where \( \nabla_p \) is the the gradient of target function w.r.t. a model parameter with index \( p \), \( \nabla_l \) is the average gradient of target function w.r.t. a parameters in a layer with index \( l \), and \( \nabla_g \) is the global average gradient of target function w.r.t. all model parameters.

**Theorem 3** (Convergence under mild assumptions about \( f \)). Suppose that \( f \) is convex and L-Lipschitz smooth with \( \| \nabla_p f(\theta) \| < M_p, \| \nabla_l f(\theta) \| < M_l, \| \nabla_g f(\theta) \| < M_g \) for some fixed \( M_p, M_l, M_g \) and all \( \theta \). Then \( \theta_t \to \theta^* \) if \( \alpha_\infty < 1/L \) where \( L \) is the Lipschitz constant for all the gradients and \( t \cdot \tau(t) \to 0 \) as \( t \to \infty \), where the \( \theta_t \) are generated according to (non-stochastic) gradient descent.

**Proof.** We take three-level’s case discussed in Section 3 for example, which includes global level, layer-level and parameter-level. Suppose that the target function \( f \) is convex, L-
Lipschitz smooth in all levels, which gives for all $\theta_1$ and $\theta_2$:

$$||\nabla_p f(\theta_1) - \nabla_p f(\theta_2)|| \leq L_p||\theta_1 - \theta_2||$$
$$||\nabla f(\theta_1) - \nabla f(\theta_2)|| \leq L||\theta_1 - \theta_2||$$
$$||\nabla_g f(\theta_1) - \nabla_g f(\theta_2)|| \leq L_g||\theta_1 - \theta_2||$$

$$L = \max\{L_p, L_t, L_g\}$$

and its gradient with respect to parameter-wise, layer-wise, global-wise parameter groups satisfy $||\nabla_p f(\theta)|| < M_p$, $||\nabla f(\theta)|| < M_t$, $||\nabla_g f(\theta)|| < M_g$ for some fixed $M_p$, $M_t$, $M_g$ and all $\theta$. Then the effective combined learning rate for each parameter satisfies:

$$|\alpha_{p,t}^*| = |\gamma_{p,t-1}\alpha_{p,t} + \gamma_{t-1}\alpha_{t,t} + \gamma_{g,t-1}\alpha_t|$$

$$\leq (\gamma_{p,t-1} + \gamma_{t-1} + \gamma_{g,t-1})\alpha_0 + \beta \sum_{i=0}^{t-1} (\gamma_{p,t-1}n_p \max_p{||\nabla f(\theta_{p,i+1})||})$$

$$+ \gamma_{t-1}n_t \max_l{||\nabla f(\theta_{t,i+1})||} ||\nabla f(\theta_{t,i})|| + \gamma_{g,t-1}||\nabla f(\theta_{g,i+1})|| ||\nabla f(\theta_{g,i})||$$

$$\leq \alpha_0 + \beta \sum_{i=0}^{t-1} (\gamma_{p,t-1}n_p \max_p{||\nabla f(\theta_{p,i+1})||})$$

$$+ \gamma_{t-1}n_t \max_l{||\nabla f(\theta_{t,i+1})||} ||\nabla f(\theta_{t,i})|| + \gamma_{g,t-1}||\nabla f(\theta_{g,i+1})|| ||\nabla f(\theta_{g,i})||$$

$$\leq \alpha_0 + t\beta(n_pM_p^2 + n_tM_t^2 + M_g^2)$$

where $\theta_{t,i}$ refers to the value of parameter indexed by $p$ at time step $i$, $\theta_{t,t}$ refers to the set/vector of parameters in layer with index $l$ at time step $i$, and $\theta_{g,i}$ refers to the whole set of model parameters at time step $i$. In addition, $n_p$ and $n_t$ are the total number of parameters and number of the layers, and we have applied $0 < \gamma_p, \gamma_t, \gamma_g < 1$. This gives an upper bound for the learning rate in each particular time step, which is $O(t)$ as $t \to \infty$.

By introducing $\kappa_{p,t} = \tau(t)\alpha_{p,t}^* + (1 - \tau(t))\alpha_\infty$, where the function $\tau(t)$ is selected to satisfy $t\tau(t) \to 0$ as $t \to \infty$, so we have $\kappa_{p,t} \to \alpha_\infty$ as $t \to \infty$. If $\alpha_\infty < \frac{1}{L}$, for larger enough $t$, we have $1/(L + 1) < \kappa_{p,t} < 1/L$, and the algorithm converges when the corresponding gradient-based optimizer converges for such a learning rate under our assumptions about $f$. This follows the discussion in [Karimi et al., 2016; Sun, 2019].

When we introduce $\kappa_{p,t}$ instead of $\alpha_{p,t}^*$ in Algorithm 1, the corresponding gradients $\frac{\partial L(\theta)}{\partial \alpha_{p,t-1}}$ will also be replaced by $\frac{\partial L(\theta)}{\partial \kappa_{p,t-1}} \frac{\partial \kappa_{p,t-1}}{\partial \alpha_{p,t-1}} \tau(t)$.

4 Experiment

To compare with existing basement method, we use the models of feed-forward neural network, recurrent neural networks and convolutions neural network. The comparison will base on the convergence speed and the minimum of validation cost during a long training period. For each type of model, following experiments will be conducted:

- Adaptive combination of different levels of hyper-descent;
- Hyper-gradient descent with global-wise adaptive learning rates;
- Standard baseline optimizers such as Adam and SGD.
Although there are many other optimizers proposed in recent years, considering the fact that CAM-HD is a general adaptation technique that can be applied in most of gradient-based optimizers in deep learning, we do not compare them in parallel and focus on the effectiveness of adaptation. The criterion for model comparison is the average learning curve on validation set for a number trails in each setting. In each trail, we introduce random initialization as well as random sampling of training data from a larger dataset. We use the recommended learning rates and hyper gradient updating rates in the original work of hyper-gradient descent (Baydin et al., 2017).

4.1 Combination ratio and model performances

First we perform a study on the combination of different level learning rates. The simulations are based on image classification tasks on MNIST and CIFAR10 (LeCun et al., 1998; Krizhevsky et al., 2009). One feed-forward neural network with three hidden layers of size [100, 100, 100] and two convolutional network models including LeNet-5 (LeCun et al., 2015) and ResNet-18 (He et al., 2016), are implemented. The LeNet-5 implemented has two convolutional filters with 6 and 16 5*5 filters in each of them, while no dropout is included for reducing the number of hyper-parameters. We use full training sets of MNIST and CIFAR10 for training and full test sets for validation. Two levels of learning rates are considered in each case, which is global and layer-wise adaptation for FFNN, global and filter-wise adaptation for CNNs. Adam optimizer is implemented in all the three simulations. We change the fixed combination weights of two levels in each case to see the change of model performance in terms of classification accuracy. At this stage we do not conduct the learning of combination weights \( \gamma \) by setting the combination weight updating rate \( \delta \) equals to 0. The learning rate for model parameters in each level is initialized to be 0.001. The mini-batch sizes are set to be 256, while the number of training epoch is 10 for each trials. For all the three simulations, we conduct 10 runs at each combination weights with different parameter initializations, and draw the error bars for standard errors. The result is given in Figure 2. We can see that in all of three cases, the optimal performance is neither at full global level nor full layer/filter level, but a combination of two levels of adaptive learning rates. The optimal combination is different in each simulation: For FFNN, it is around \( \gamma_g = 0.5 \), while for LeNet-5, it is roughly \( \gamma_g = 0.8 \), and for ResNet-18, it is about \( \gamma_g = 0.6 \). This indicates that it is necessary to search or train for \( \gamma \)s in different learning tasks. Still, the differences between the end points and the optimal combination in terms of model performance are with some level of statistical significance. This supports our analysis in Section 3. Although the model performance w.r.t. the combination weights could be non-convex, updating \( \gamma \)s with gradient-based method can reduce the

![Figure 2: The diagram of a model performance under different combination ratios in the case of two-level learning rates.](image-url)
time cost for hyper-parameter searching. Also, in real training processes, it is possible that the learning in favor of different combination weights in different stages, and this requires the online updating of the combination weights.

4.2 FFNN for image classification

This experiment is done with feed-forward neural networks for image classification task on MNIST, which includes 60,000 training examples and 10,000 test examples. We use the full training set for training and the full test set for validation. Here we consider three-level learning rate adaptation of Adam optimizer discussed in Algorithm 1 and compare its performance with baseline optimizers including Adam and Adam-HD. Three FFNN with different hidden layer configurations are implemented, including [100, 100], [1000, 100], and [1000, 1000]. For example, the first configuration [100, 100] means we have two hidden layers and each of them have 100 hidden units. For each layer configuration, experiment is repeated for 30 trials with different random weight initialization by Xavier uniform \(\text{(Glorot and Bengio, 2010).}\) The hyper-parameter setting is as follows: The batch size is set to be 128; The initial learning rate or learning rate for baseline optimizer is set to be 0.001; The betas for Adam is \((0.9, 0.999)\) as recommended default value; The hyper-gradient learning rate is \(10^{-7}\); The initial values for \(\gamma\)s are \(\gamma_1 = 0.3, \gamma_2 = 0.3\) and \(\gamma_3 = 0.4\); The updating rate for \(\gamma\) is 0.1; The number of training epochs for each trial is 30. Here we do not introduce any learning rate decay schedule. The comparison of different optimizer by average learning curves of training loss and validation accuracy, as well as the corresponding standard errors of the sample means, is demonstrated in Figure 3. We can learn that under the recommended hyper-parameter setting for original hyper-gradient descent, the proposed three level Adam-CMA-HD consistently achieves a significant improvement in validation accuracy compared with baseline methods in all the three cases. The similar comparison can be also shown in the curves of training loss, where three level Adam-CMA-HD outperforms the other two baselines as well. Still there is no hint that the two baseline methods can surpass it with a longer training period.

4.3 LeNet-5 for image classification

Another experiment is based on LeNet-5 convolutional network model for image classification on MNIST. The data sampling procedure in each trial is exactly the same as the experiment with FFNN, where a random subset of 5,000 examples are used for training. Here we introduce the combined multi-level adaptation for both stochastic gradient descent (SGD) and Adam optimizer. The learning rate for model parameters are set to be 0.001 for both SGD and Adam. As to the hyper-gradient updating rates, we use \(10^{-3}\) for SGD-based optimizers and \(10^{-7}\) for Adam-based optimizers, which is the same with the original HD paper \(\text{(Baydin et al., 2017).}\) In this experiment we use a two-level combination of adaptive learning rates, including global level and filter level for CNNs, where we initialize \(\gamma_1 = 0.2, \gamma_2 = 0.8,\) and \(\delta = 0.01.\) The learning curves for both training and validation set is given by Figure 4. It is shown that for the task of image classification with LeNet-5, CMA can significantly improve the convergence performances for both SGD and Adam on training and validation sets. For further investigate the performance robustness of the proposed optimizer in LeNet-5, we perform other two image classification experiments on the full datasets of CIFAR10 and SVHN. 50,000 examples in the original training set are used for training, while 10,000 test set are divided to 5,000 examples for validation and 5,000 samples for testing. The mini-batch size is set to be 256, while the learning rate and betas for Adam optimizer, as well as the combination weights initialization and corresponding updating rate are the same with previous experiments. We introduce a decay
function $\tau(t) = \exp^{-st}$ where $s = 0.002$ and $t$ is the number of iterations. The learning curves for 10 epochs with error bars are given in Figure 5. It is shown that when we train LeNet-5 models on CIFAR10, the curve of Adam-CAM-HD for cross entropy loss on training set is lower than that of Adam-HD and Adam, while it achieved a significant improvement on validation set in almost all time steps until all the three curves pass their early stopping points. On the other hand, for SVHN, although the training loss of Adam-CAM-HD seems not outperform other two optimizers, its validation performance shows a significant advantage over other two in later stage of training.

4.4 ResNet for image classification

The last experiment is done with ResNet-34 on CIFAR 10 for image classification task. The training-validation-test division of full CIFAR10 is the same with the last experiment. To set a clean baseline and focus on comparison between optimizers, we use a constant learning rate 0.001 for Adam without learning rate schedule. Here we still set the mini-batch size to be 256 and introduce a decay function $\tau(t)$ with the same decay rate 0.002 as in the last experiment. The hyper-grad updating rate $\beta$ is $10^{-8}$ and the combination weights initialization and corresponding updating rate is also the same with the last experiment. Each optimizer is implemented for 10 runs. The result of learning curves on training and validation data are shown in Figure 6. It is shown that the validation loss curve of Adam CAM-HD lies below other two benchmark optimizers in most of the time with statistical significance. For a longer training period, the three curves may overlap with each other, which means both the baseline Adam-HD and CAM-HD-Adam could not significantly surpass Adam without extra hyper-parameter tuning or introducing other mechanisms, but the proposed CAM-HD-Adam converges faster in early stage and reaches a accuracy level of 85% at first. Still, it is possible that CAM-HD can enlarge the performance improvement when building on top of other adaptive optimizers. Now we demonstrate our finding of SGD-CAM-HD and SGDN-CAM-HD optimizer in comparing with corresponding baseline methods on training set. We use a constant learning rate 0.1 without learning rate schedule. Here we still set the mini-batch size to be 256 and introduce a decay function $\tau(t)$ with the decay rate equals 0.001. The hyper-grad updating rate $\beta$ is $10^{-4}$ for our method and $10^{-3}$ for SGDN-HD as recommended, while the combination weights initialization ($\gamma_1 = 0.2$, $\gamma_2 = 0.8$) is the same with the last experiment, and corresponding updating rate $\delta = 0.001$. Each optimizer is implemented for 5 runs. The learning curves on training data are shown in Figure 7 (c). As we can see, the training loss curves of SGD-CAM-HD and SGDN-CAM-HD lies below the corresponding curves of other two benchmark optimizers for most of the time in 100 epochs. The existing SGDN-HD also provides an significant improvement but not as good as ours. Further experiment has shown that the proposed SGDN-CAM-HD can achieves faster convergence in validation set as well.

5 Discussion

5.1 Learning of combination weights

The following figures give the learning curves of combination weights with respect to the number of training iterations in each experiments. Each curve is averaged by 5 trials with error bars. Figure 9 corresponds to the experiment of FFNN on MNIST in Section 4.2 which is a three level case. We can find that in different FFNN architecture the learning behavior of $\gamma$s are also different. On the other hand, Figure 10 and Figure 11 correspond to the experiments of LeNet-5 in Section 4.3. As is shown in these two figures, for
SGD-CAM-HD and SGDN-CAM-HD, the equilibrium value of combination weights are different from that of Adam-CAM-HD. For LeNet-5 image classification tasks, although the initialization $\gamma_1 = 0.2$, $\gamma_2 = 0.8$, and the updating rate $\delta = 0.01$ are the same, the values of $\gamma_1$ and $\gamma_2$ only change a small proportion when training with Adam-CAM-HD, but the change is much more significant towards larger filter/layer wise adaptation when SGD-CAM-HD or SGDN-CAM-HD are implemented. This indicates that Adam may require high proportion of global level adaptation for the same network architecture.

Notice that although we run each setting for 5 trials, the learning curves of $\gamma$s do not have a large standard error, which suggests they are independent of the initialization of model parameters.

### 5.2 Number of parameters and space complexity

The proposed adaptive optimizer is for efficiently updating the model parameters, while the final model parameters will not be increased by introducing CMA-HD optimizer. However, during the training process, several extra intermediate variables are introduced. For example, in the discussed three-level’s case for feed-forward neural network with $n_{\text{layer}}$ layers, we need to restore $h_{p,t}$, $h_{l,t}$ and $h_{g,t}$, which have the sizes of $S(h_{p,t}) = \sum_{l=1}^{n_{\text{layer}}-1} (n_l + 1) n_{l+1}$, $S(h_{l,t}) = n_{\text{layer}}$ and $S(h_{g,t}) = 1$, respectively, where $n_i$ is the number of units in $i$th layer. Also, learning rates $\alpha_{p,t}$, $\alpha_{l,t}$, $\alpha_{g,t}$ and take the sizes of $S(a_{p,t}) = \sum_{l=1}^{n_{\text{layer}}-1} (n_l + 1) n_{l+1}$, $S(a_{l,t}) = n_{\text{layer}}$, $S(a_{g,t}) = 1$, and $S(a_{g,t}^*) = \sum_{l=1}^{n_{\text{layer}}-1} (n_l + 1) n_{l+1}$, respectively. Also we need a small set of scalar parameters to restore $\gamma_1$, $\gamma_2$ and $\gamma_3$ and other coefficients.

Consider the fact that the training the baseline models, we need to restore model parameters, corresponding gradients, as well as the intermediate gradients during the implementation of chain rule, CAM-HD will take twice of the space for storing intermediate variables in the worst case. For two-level learning rate adaptation considering global and layer-wise learning rates, the extra space complexity by CAM-HD will be one to two orders’ smaller than that of baseline model during training.

### 5.3 Time Complexity

In CMA-HD, we need to calculate gradient of loss with respect to the learning rates in each level, which are $h_{p,t}$, $h_{l,t}$ and $h_{g,t}$ in three-level’s case. However, the gradient of each parameter is already known during normal model training, the extra computational cost comes from taking summations and updating the lowest-level learning rates. In general, this cost is in linear relation with the number of differentiable parameters in the original models. Here we discuss the case of feed-forward networks and convolutional networks.

Recall that for feed-forward neural network the whole computational complexity is:

$$T(n) = O(m \cdot n_{\text{iter}} \cdot \sum_{l=2}^{n_{\text{layer}}} n_l \cdot n_{l-1} \cdot n_{l-2})$$

where $m$ is the number of training examples, $n_{\text{iter}}$ is the iterations of training, $n_l$ is the number of units in the $l$-th layer. On the other hand, when using three-level CAM-HD with, where the lowest level is parameter-wise, we need $n_{\text{layer}}$ element products to calculate $h_{p,t}$ for all layers, one $n_{\text{layer}}$ matrix element summations to calculate $h_{l,t}$ for all layers, as well as a list summation to calculate $h_{g,t}$. In addition, two element-wise summations will also be implemented for calculating $\alpha_{p,t}$ and $\alpha_{p}^*$. Therefore, the extra computational cost
of using CAM-HD is $\Delta T(n) = O(m_b \cdot n_{\text{iter}} \sum_{l=2}^{n_{\text{layer}}} (n_l \cdot n_{l-1} + n_l))$, where $m_b$ is the number of mini-batches for training. Notice that $m/m_b$ is the batch size, which is usually larger than 100. This extra cost is more than one-order smaller than the computation complexity of training a model without learning rate adaptation. For the cases when the lowest level is layer-wise, only one element-wise matrix product is needed in each layer to calculate $h_{l,t}$. For convolutional neural networks, we have learned that the total time complexity of all convolutional layers is [He and Sun, 2015]:

$$O(m \cdot n_{\text{iter}} \cdot \sum_{l=1}^{n_{\text{conv}}_{\text{layer}}} (n_{l-1} \cdot s_l^2 \cdot n_l \cdot m_l^2))$$ (40)

where $l$ is the index of a convolutional layer, and $n_{\text{conv}}_{\text{layer}}$ is the depth (number of convolutional layers). $n_l$ is the number of filters in the $l$-th layer, while $n_{l-1}$ is known as the number of input channels of the $l$-th layer. $s_l$ is the spatial size of the filter. $m_l$ is the spatial size of the output feature map. If we consider convolutional filters as layers, the extra computational cost for CAM-HD in this case is $\Delta T(n) = O(m_b \cdot n_{\text{iter}} \sum_{l=1}^{n_{\text{conv}}_{\text{layer}}} ((n_{l-1} \cdot s_l^2 + 1) \cdot n_l))$, which is still more than one order smaller than the cost of model without learning rate adaptation.

Therefore, for large networks, applying CMA-HD will not significantly increase the computational cost from the theoretical prospective.

6 Conclusion

In this study, we propose an gradient-based learning rate adaptation strategy by introducing hierarchical multiple-level learning rates in deep neural networks, including global-wise, layer-wise and parameter-wise adaptation. By considering the combination of adaptive learning rate in different levels, we further propose a joint algorithm for adaptively learning the combination weight of each level. Experiments on FFNN with three different layer configurations, CNN LeNet-5 and ResNet-18, indicate that these hyper-descent based adaptive methods can outperform the standard ADAM/SGD and original hyper-gradient descent algorithm in many cases in terms of convergence speed or validation performances. Although the advantage is not guaranteed, the optimal optimizer could differ on a case by case basis, which can be treated as a discrete hyper-parameter to be tuned. Also, we find that the code provided by the original author of hyper gradient descent is not exactly the global level adaptation, and our revised version can outperform it with statistical significance [Baydin et al., 2017]. The finding of this research could provide a new prospective of adaptive optimizers and lead to more work on trainable combinations of different optimizers.
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Figure 3: The comparison of learning curves on MNIST with different level of adaptation
Figure 4: The comparison of learning curves of training LeNet-5 on MNIST with different adaptive optimizers
Figure 5: The comparison of learning curves on training LeNet-5 on CIFAR10 and SVHN with different adaptive optimizers

Figure 6: The comparison of learning curves of ResNet-34 on CIFAR10
Figure 7: The comparison of learning curves of ResNet-18 on CIFAR10 with Adaptive SGDNs.

Figure 8: The comparison of learning curves of ResNet-18 on CIFAR10 with Adaptive SGDs.

Figure 9: Learning curves of $\gamma$s for FFNN on MNIST with Adam.
Figure 10: Learning curves of $\gamma$s for LeNet-5 on MNIST with SGD, SGDN and Adam.

Figure 11: Learning curves of $\gamma$s for LeNet-5 on CIFAR10 and SVHN.