New Generalization Bounds for Learning Kernels

Corinna Cortes  
Google Research  
New York  
corinna@google.com

Mehryar Mohri  
Courant Institute and  
Google Research  
mohri@cims.nyu.edu

Afshin Rostamizadeh  
Courant Institute  
New York University  
rostami@cs.nyu.edu

Abstract

This paper presents several novel generalization bounds for the problem of learning kernels based on the analysis of the Rademacher complexity of the corresponding hypothesis sets. Our bound for learning kernels with a convex combination of \( p \) base kernels has only a \( \log p \) dependency on the number of kernels, \( p \), which is considerably more favorable than the previous best bound given for the same problem. We also give a novel bound for learning with a linear combination of \( p \) base kernels with an \( L_2 \) regularization whose dependency on \( p \) is only in \( p^{1/4} \).

1 Introduction

Kernel methods are widely used in statistical learning \cite{17, 18}. Positive definite symmetric (PDS) kernels specify an inner product in an implicit Hilbert space where large-margin methods are used for learning and estimation. They can be combined with algorithms such as support vector machines (SVMs) \cite{5, 10, 20} or other kernel-based algorithms to form powerful learning techniques.

But, the choice of the kernel, which is critical to the success of the algorithm, is typically left to the user. Rather than requesting the user to commit to a specific kernel, which may not be optimal for the task, especially if the user’s prior knowledge about the task is poor, learning kernel methods require him only to specify a family of kernels. The learning algorithm then selects both the specific kernel out of that family, and the hypothesis defined with respect to that kernel.

There is a large body of literature dealing with various aspects of the problem of learning kernels, including theoretical questions, optimization problems related to this problem, and experimental results \cite{13, 15, 2, 1, 19, 16, 14, 23, 11, 3, 8, 22, 9}. Some of this previous work considers families of Gaussian kernels \cite{15} or hyperkernels \cite{16}. Non-linear combinations of kernels have been recently considered by \cite{21, 8, 9}. But, the most common family of kernels examined is that of non-negative combinations of some fixed kernels constrained by a trace condition, which can be viewed as an \( L_1 \) regularization \cite{13}, or by an \( L_2 \) regularization \cite{8}.

This paper presents several novel generalization bounds for the problem of learning kernels for the family of convex combinations of base kernels or linear combinations with an \( L_2 \) constraint. One of the first learning bounds given by Lanckriet et al. \cite{13} for the family of convex combinations of \( p \) base kernels is similar to that of Bousquet and Herrmann \cite{6} and has the following form: \( R(h) \leq \hat{R}_p(h) + O(\sqrt{\max_{k=1}^p \text{Tr}(K_k) \max_{i=1}^m (\|K_k\|/\text{Tr}(K_k))/\rho^2}) \) where \( R(h) \) is the generalization error of a hypothesis \( h \), \( R_p(h) \) is the fraction of training points with margin less than or equal to \( \rho \) and \( K_k \) is the kernel matrix associated to the \( k \)th base kernel. This bound was later shown by Srebro and Ben-David \cite{19} to be always larger than one. Another bound by Lanckriet et al. \cite{13} for the family of linear combinations of base kernels was also shown by the same authors to be always larger than one.

But Lanckriet et al. \cite{13} also presented a multiplicative bound for convex combinations of base kernels that is of the form \( R(h) \leq \hat{R}_p(h) + O\left(\sqrt{p/\rho^2}\right) \). This bound converges and can perhaps be viewed as the first informative generalization bound for this family of kernels. However, the dependence of the bound on the number of kernels \( p \) is multiplicative which therefore does not encourage the use of too many base kernels. Srebro and Ben-David \cite{19} presented a generalization bound based on the pseudo-dimension of the family of kernels that significantly improved on this bound. Their bound has the form \( R(h) \leq \hat{R}_p(h) + \tilde{O}\left(\sqrt{p+R^2/\rho^2}\right) \), where the notation \( \tilde{O}(\cdot) \) hides logarithmic terms and where \( R \) is an upper bound on \( K_k(x, x) \).
for all points \( x \) and base kernels \( k_k, k \in [1, p] \). Thus, disregarding logarithmic terms, their bound is only additive in \( p \). Their analysis also applies to other families of kernels. Ying and Campbell \cite{Ying14} also give generalization bounds for learning kernels based on the notion of Rademacher chaos complexity and the pseudo-dimension of the family of kernels used. It is not clear however how their bound compares to that of Srebro and Ben-David. We present new generalization bounds for the family of convex combinations of base kernels that have only a logarithmic dependency on \( p \). Our learning bound is based on a careful analysis of the Rademacher complexity of the hypothesis set considered and has the form: \( R(h) \leq \hat{R}_p(h) + O\left(\sqrt{\frac{(\log p) R^2}{m}}\right) \). Our bound is simpler and contains no other extra logarithmic term. Thus, this represents a substantial improvement over the previous best bounds for this problem. Our bound is also valid for a very large number of kernels, in particular for \( p \gg m \), while the previous bounds were not informative in that case.

We also present new generalization bounds for the family of linear combinations of base kernels with an \( L_2 \) regularization. We had previously given a stability bound for an algorithm extending kernel ridge regression to learning kernels that had an additive dependency with respect to \( p \) \cite{Hastie13} assuming a technical condition of orthogonality on the base kernels. The complexity term of our bound was of the form \( O(1/\sqrt{m + \sqrt{p/m}}) \). Our new learning bound admits only a mild dependency of \( p^{1/4} \) on the number of base kernels.

The next section (Section 2) defines the family of kernels and hypothesis sets we examine. Section 3 presents first a bound on the Rademacher complexity, then a generalization bound for the case of \( L_1 \) constraint and a generalization bond for binary classification directly derived from that result. Similarly, Section 4 presents first a bound on the Rademacher complexity, then a generalization bound for the case of an of \( L_2 \) regularization.

2 Preliminaries

Most learning kernel algorithms are based on a hypothesis set derived from convex combinations of a fixed set of kernels \( K_1, \ldots, K_p \):

\[
H_p = \left\{ \sum_{i=1}^{m} \alpha_i K(x_i, \cdot) : K = \sum_{k=1}^{p} \mu_k K_k, \mu_k \geq 0, \sum_{k=1}^{p} \mu_k = 1, \alpha^\top \Phi \leq 1/\rho^2 \right\}.
\] (1)

Note that linear combinations with possibly negative mixture weights have also been considered in the literature, e.g., \cite{Hastie13}, however these combinations do not ensure that the combined kernel is PDS.

We also consider the hypothesis set \( H_p' \) based on a \( L_2 \) condition on the vector \( \mu \) and defined as follows:

\[
H_p' = \left\{ \sum_{i=1}^{m} \alpha_i K(x_i, \cdot) : K = \sum_{k=1}^{p} \mu_k K_k, \mu_k \geq 0, \sum_{k=1}^{p} \mu_k^2 = 1, \alpha^\top \Phi \leq 1/\rho^2 \right\}.
\] (2)

We bound the empirical Rademacher complexity \( \hat{R}_S(H_p) \) or \( \hat{R}_S(H_p') \) of these families for an arbitrary sample \( S \) of size \( m \), which immediately yields a generalization bound for learning kernels based on this family of hypotheses. For a fixed sample \( S = (x_1, \ldots, x_m) \), the empirical Rademacher complexity of a hypothesis set \( H \) is defined as

\[
\hat{R}_S(H) = \frac{1}{m} \mathbb{E} \left[ \sup_{h \in H} \sum_{i=1}^{m} \sigma_i h(x_i) \right].
\] (3)

The expectation is taken over \( \sigma = (\sigma_1, \ldots, \sigma_m) \) where \( \sigma_i \)'s are independent uniform random variables taking values in \( \{-1, +1\} \).

Let \( h \in H_p \), then

\[
h(x) = \sum_{i=1}^{m} \alpha_i K(x_i, x) = \sum_{k=1}^{p} \sum_{i=1}^{m} \mu_k \alpha_i K_k(x_i, x) = w \cdot \Phi(x),
\] (4)

where \( w = \begin{bmatrix} w_1 \\ \vdots \\ w_p \end{bmatrix} \) with \( w_k = \mu_k \sum_{i=1}^{m} \alpha_i \Phi_k(x_i) \) and \( \Phi(x) = \begin{bmatrix} \Phi_1(x) \\ \vdots \\ \Phi_p(x) \end{bmatrix} \) with \( \Phi_k = K_k(x, \cdot) \), for all \( k \in [1, p] \).

3 Rademacher complexity bound for \( H_p \)

Theorem 1 For any sample \( S \) of size \( m \), the Rademacher complexity of the hypothesis set \( H_p \) can be bounded as follows:

\[
\hat{R}_S(H_p) \leq \frac{\|\tau\|_r}{m \rho} \quad \text{with} \quad \tau = (\sqrt{r \text{Tr}[K_1]}, \ldots, \sqrt{r \text{Tr}[K_p]})^\top,
\] (5)
for any even integer \( r > 0 \). If additionally, \( K_k(x, x) \leq R^2 \) for all \( x \in X \) and \( k \in [1, p] \), then, for \( p > 1 \),

\[
\hat{R}_S(H_p) \leq \sqrt{\frac{2|\log p| R^2 / r^2}{m}}.
\]

**Proof:** Fix a sample \( S \), then \( \hat{R}_S(H_p) \) can be bounded as follows for the hypothesis set of kernel learning algorithms for any \( q, r > 1 \) with \( 1/q + 1/r = 1 \):

\[
\hat{R}_S(H_p) = \frac{1}{m} \mathbb{E} \left[ \sup_{h \in H_p} \sum_{i=1}^{m} \sigma_i h(x_i) \right] = \frac{1}{m} \mathbb{E} \left[ \sup_{w} \sum_{i=1}^{m} \sigma_i \Phi(x_i) \right] \\
\leq \frac{1}{m} \mathbb{E} \left[ \sup_{w} \left( \sum_{k=1}^{p} \| w_k \|^q \right)^{1/q} \left( \sum_{k=1}^{p} \left\| \sum_{i=1}^{m} \sigma_i \Phi_k(x_i) \right\|^{r} \right)^{1/r} \right] \quad \text{(Lemma 5)} \\
= \frac{1}{m} \mathbb{E} \left[ \left( \sum_{k=1}^{p} \left\| \sum_{i=1}^{m} \sigma_i \Phi_k(x_i) \right\|^{r} \right)^{1/r} \right].
\]

We bound each of these two factors separately. The first term can be bounded as follows.

\[
\left( \sum_{k=1}^{p} \| w_k \|^q \right)^{1/q} \leq \sum_{k=1}^{p} \| w_k \|^q \quad \text{(sub-additivity of } x \mapsto x^{1/q}, (1/q) < 1) \]

\[
= \sum_{k=1}^{p} \| \mu_k \| \sum_{i=1}^{m} \sigma_i \Phi_k(x_i) \| \\
\leq \sqrt{\sum_{k=1}^{p} \| \mu_k \| \sum_{i=1}^{m} \sigma_i \Phi_k(x_i) \| ^2} \quad \text{(convexity)} \\
= \sqrt{\sum_{k=1}^{p} \mu_k \alpha^\top K_k \alpha} = \sqrt{\alpha^\top K \alpha} \leq 1/p.
\]

We bound the second term as follows:

\[
\mathbb{E}_\sigma \left[ \left( \sum_{k=1}^{p} \left\| \sum_{i=1}^{m} \sigma_i \Phi_k(x_i) \right\|^{r} \right)^{1/r} \right] \leq \left( \mathbb{E}_\sigma \left[ \sum_{k=1}^{p} \left\| \sum_{i=1}^{m} \sigma_i \Phi_k(x_i) \right\|^{r} \right] \right)^{1/r} \quad \text{(Jensen’s inequality)} \\
= \left( \sum_{k=1}^{p} \mathbb{E}_\sigma \left[ \left\| \sum_{i=1}^{m} \sigma_i \Phi_k(x_i) \right\|^{r} \right] \right)^{1/r}.
\]

Suppose that \( r \) is an even integer, \( r = 2r' \). Then, we can bound the expectation as follows:

\[
\mathbb{E}_\sigma \left[ \left\| \sum_{i=1}^{m} \sigma_i \Phi_k(x_i) \right\|^{r} \right] = \mathbb{E}_\sigma \left[ \left( \sum_{i,j=1}^{m} \sigma_i \sigma_j K_k(x_i, x_j) \right)^{r'} \right] \\
\leq \sum_{1 \leq i_1, \ldots, i_{r'} \leq m} \left| \mathbb{E}_\sigma \left[ \sigma_{i_1} \cdots \sigma_{i_{r'}} \sigma_j \cdots \sigma_{j_{r'}} \right] \right| K_k(x_{i_1}, x_{j_1}) \cdots K_k(x_{i_{r'}}, x_{j_{r'}}) \\
\leq \sum_{1 \leq i_1, \ldots, i_{r'} \leq m} \left| \mathbb{E}_\sigma \left[ \sigma_{i_1} \cdots \sigma_{i_{r'}} \sigma_j \cdots \sigma_{j_{r'}} \right] \right| (K_k(x_{i_1}, x_{i_1}) \cdots K_k(x_{i_{r'}}, x_{i_{r'}}))^{1/2} \\
= \left( \sum_{s_1 + \ldots + s_{r'} = 2r'} \left| \mathbb{E}_\sigma \left[ \sigma_{s_1} \cdots \sigma_{s_m} \right] \right| K_k(x_1, x_1)^{s_1/2} \cdots K_k(x_m, x_m)^{s_m/2} \right)^{1/2} \quad \text{(Cauchy-Schwarz)}
\]
Since $E[\sigma_i] = 0$ for all $i$ and since the Rademacher variables are independent, we can write $E[\sigma_i \ldots \sigma_l] = E[\sigma_i] \ldots E[\sigma_l] = 0$ for any $l$ distinct variables $\sigma_i, \ldots, \sigma_l$. Thus, $E_{\sigma} [\sigma_1^s \ldots \sigma_m^s] = 0$ unless all $s_i$s are even, in which case $E_{\sigma} [\sigma_1^s \ldots \sigma_m^s] = 1$. Therefore, the following inequality holds

$$E_{\sigma} \left[ \left\| \sum_{i=1}^{m} \Phi(x_i) \right\|_{r}^r \right] \leq \sum_{2t_1+\ldots+2t_m=2r'} (2t'_1 \ldots 2t'_m) K_k(x_1, x_1)^{t'_1} \ldots K_k(x_m, x_m)^{t'_m}$$

$$\leq (2r')^{r'} \sum_{t_1+\ldots+t_m=r'} (t'_1 \ldots t'_m) K_k(x_1, x_1)^{t'_1} \ldots K_k(x_m, x_m)^{t'_m} \leq (2r' Tr[K_k])^r' = (r Tr[K_k])^r' / r.$$  

Thus, the Rademacher complexity is bounded by

$$\hat{R}_S(H_p) \leq \frac{\|\tau\|_r}{m\rho} \text{ with } \tau = \left( \sqrt{r Tr[K_1]}, \ldots, \sqrt{r Tr[K_p]} \right)^T, \quad (6)$$

for any even integer $r$.

Assume that $K_k(x, x) \leq R^2$ for all $x \in X$ and $k \in [1, p]$. Then, $Tr[K_k] \leq mR^2$ for any $k \in [1, p]$, thus the Rademacher complexity can be bounded as follows

$$\hat{R}_S(H_p) \leq \frac{1}{m\rho}(\sqrt{r m R^2})^{1/r} = p^{1/r} R^{1/2} \sqrt{\frac{R^2}{m}}.$$  

For $p > 1$, the function $r \mapsto p^{1/r} R^{1/2}$ reaches its minimum at $r_0 = 2 \log p$. This gives

$$\hat{R}_S(H_p) \leq \sqrt{\frac{2e|log p| R^2}{m}}.$$  

It is likely that the constants in the bound of theorem can be further improved. We used a very rough upper bound for the multinomial coefficients. A finer bound using Sterling’s approximation should provide a better result. Remarkably, the bound of the theorem has a very mild dependency with respect to $p$.

The theorem can be used to derive generalization bounds for learning kernels in classification, regression, and other tasks. We briefly illustrate its application to binary classification where the labels $y$ are in $\{-1, +1\}$. Let $R(h)$ denote the generalization error of $h \in H_p$, that is $R(h) = Pr[yh(x) < 0]$. For a training sample $S = ((x_1, y_1), \ldots, (x_m, y_m))$ and any $\rho > 0$, let $\hat{R}_p(h)$ denote the fraction of the training points with margin less than or equal to $\rho$, that is $\hat{R}_p(h) = \frac{1}{m} \sum_{i=1}^{m} 1_{y_i h(x_i) \leq \rho}$. Then, the following result holds.

**Corollary 2** For any $\delta > 0$, with probability at least $1 - \delta$, the following bound holds for any $h \in H_p$:

$$R(h) \leq \hat{R}_p(h) + \frac{2|\tau|}{m\rho} + 2\sqrt{\frac{\log \frac{2}{\delta}}{2m}}.$$  

with $\tau = (\sqrt{r Tr[K_1]}, \ldots, \sqrt{r Tr[K_p]})^T$, for any even integer $r > 0$. If additionally, $K_k(x, x) \leq R^2$ for all $x \in X$ and $k \in [1, p]$, then, for $p > 1$,

$$R(h) \leq \hat{R}_p(h) + 2\sqrt{\frac{2e|log p| R^2}{m}} + 2\sqrt{\frac{\log \frac{2}{\delta}}{2m}}.$$  

**Proof:** With our definition of the Rademacher complexity, for any $\delta > 0$, with probability at least $1 - \delta$, the following bound holds for any $h \in H_p$, [12, 4]:

$$R(h) \leq \hat{R}_p(h) + 2\hat{R}_S(H_p) + 2\sqrt{\frac{\log \frac{2}{\delta}}{2m}}.$$  

We use the following rather rough inequality:

$$(2t'_1 \ldots 2t'_m) = \frac{(2r')!}{(2t'_1)! \ldots (2t'_m)!} \leq \frac{(2r')!}{(t'_1)! \ldots (t'_m)!} \leq \frac{(2r') \ldots (r' + 1) \cdot r'!}{(t'_1)! \ldots (t'_m)!} \leq \frac{(2r')^r' - r'!}{(t'_1)! \ldots (t'_m)!} = (2r')^r' \left( t'_1, \ldots, t'_m \right).$$
Theorem 3

For any sample size \( m \), the Rademacher complexity of the hypothesis set \( H_p' \) can be bounded as follows:

\[
\hat{R}_S(H_p') \leq \frac{\|\tau\|_r}{m^{p/2}} \quad \text{with} \quad \tau = (\sqrt{r \operatorname{Tr}[K_1]}, \ldots, \sqrt{r \operatorname{Tr}[K_p]})^\top, 
\]

for any even integer \( 0 < r \leq 4 \). If additionally, \( K_k(x, x) \leq R^2 \) for all \( x \in X \) and \( k \in [1, p] \), then, for any \( p \geq 1 \),

\[
\hat{R}_S(H_p') \leq 2p^{1/4} \sqrt{R^2 / \rho^2 / m}.
\]

This bound also holds without the condition \( \mu_k \geq 0, k \in [1, p] \), on the hypothesis set \( H_p' \).

**Proof:** We can proceed as in the proof for bounding the Rademacher complexity of \( H_p \), except for bounding the following term:

\[
\left( \sum_{k=1}^{p} \|w_k\|^q \right)^{1/q} = \left[ \sum_{k=1}^{p} \mu_k^q (\alpha^\top K_k \alpha)^{q/2} \right]^{1/q} 
\]

\[
= \left[ \sum_{k=1}^{p} \mu_k^2 \left( \frac{2(q-2)}{q} \alpha^\top K_k \alpha \right)^{q/2} \right]^{1/q} 
\]

\[
\leq \left[ \sum_{k=1}^{p} \mu_k^2 \left( \frac{2(q-2)}{q} \alpha^\top K_k \alpha \right)^{q/2} \right]^{1/q} \quad \text{(convexity)} 
\]

\[
= \sqrt[p]{ \sum_{k=1}^{p} \mu_k^{4(q-1)/q} \alpha^\top K_k \alpha }.
\]

Assume now that \( q > 4/3 \), which implies \( \frac{4(q-1)}{q} < 1 \). Then, since \( \mu_k \in [0, 1] \), this implies \( \mu_k^{4(q-1)/q} \leq \mu_k \).

Thus, for any \( q > 4/3 \), we can write:

\[
\left( \sum_{k=1}^{p} \|w_k\|^q \right)^{1/q} \leq \sqrt[p]{ \sum_{k=1}^{p} \mu_k \alpha^\top K_k \alpha } = \sqrt[p]{\alpha^\top \Sigma \alpha} \leq 1/\rho^2.
\]

Taking the limit \( q \to 4/3 \) shows that the inequality is also verified for \( q = 4/3 \). Thus, as in the proof for \( H_p \), the Rademacher complexity can be bounded as follows

\[
\hat{R}_S(H_p') \leq \frac{\|\tau\|_r}{m^{p/2}} \quad \text{with} \quad \tau = (\sqrt{r \operatorname{Tr}[K_1]}, \ldots, \sqrt{r \operatorname{Tr}[K_p]})^\top, 
\]

but here \( r \) is an even integer such that \( 1/r = 1 - 1/q \geq 1 - 3/4 = 1/4 \), that is \( r \leq 4 \). Assume that \( K_k(x, x) \leq R^2 \) for all \( x \in X \) and \( k \in [1, p] \). Then, \( \operatorname{Tr}[K_k] \leq mR^2 \) for any \( k \in [1, p] \), thus, for \( r = 4 \), the Rademacher complexity can be bounded as follows

\[
\hat{R}_S(H_p') \leq \frac{1}{m^{p/2}} \left( p(\sqrt{4mR^2})^4 \right)^{1/4} = 2p^{1/4} \sqrt{R^2 / \rho^2 / m}.
\]
Thus, in this case, the bound has a mild dependence \((\sqrt[p]{\cdot})\) on the number of kernels \(p\). Proceeding as in the \(L_1\) case leads to the following margin bound in binary classification.

**Corollary 4** For any \(\delta > 0\), with probability at least \(1 - \delta\), the following bound holds for any \(h \in H_p^p\):

\[
R(h) \leq \hat{R}_p(h) + \frac{2\|\tau\|_r}{m\rho} + 2\sqrt{\frac{\log \frac{2}{\delta}}{2m}}.
\]  

(13)

with \(\tau = (\sqrt[r]{\text{Tr}[K_1]}, \ldots, \sqrt[r]{\text{Tr}[K_p]})^\top\), for any even integer \(r \in \{2, 4\}\). If additionally, \(K_k(x, x) \leq R^2\) for all \(x \in X\) and \(k \in [1, p]\), then, for any \(p \geq 1\),

\[
R(h) \leq \hat{R}_p(h) + 4p^{1/4}\sqrt{\frac{R^2/\rho^2}{m^2}} + 2\sqrt{\frac{\log \frac{2}{\delta}}{2m}}.
\]

5 Conclusion

We presented several new generalization bounds for the problem of learning kernels with non-negative combinations of base kernels. Our bounds are simpler and significantly improve over previous bounds. Their very mild dependency on the number of kernels seems to suggest the use of a large number of kernels for this problem. Our experiments with this problem in regression using a large number of kernels seems to corroborate this idea [8]. Much needs to be done however to combine these theoretical findings with the somewhat disappointing performance observed in practice in most learning kernel experiments [7].

References

[1] Andreas Argyriou, Raphael Hauser, Charles Micchelli, and Massimiliano Pontil. A DC-programming algorithm for kernel selection. In *ICML*, 2006.
[2] Andreas Argyriou, Charles Micchelli, and Massimiliano Pontil. Learning convex combinations of continuously parameterized basic kernels. In *COLT*, 2005.
[3] F. Bach. Exploring large feature spaces with hierarchical multiple kernel learning. *NIPS*, 2008.
[4] Peter L. Bartlett and Shahar Mendelson. Rademacher and Gaussian complexities: Risk bounds and structural results. *Journal of Machine Learning Research*, 3:2002, 2002.
[5] Bernhard Boser, Isabelle Guyon, and Vladimir Vapnik. A training algorithm for optimal margin classifiers. In *COLT*, volume 5, 1992.
[6] Olivier Bousquet and Daniel J. L. Herrmann. On the complexity of learning the kernel matrix. In *NIPS*, 2002.
[7] Corinna Cortes. Invited talk: Can learning kernels help performance? In *ICML*, page 161, 2009.
[8] Corinna Cortes, Mehryar Mohri, and Afshin Rostamizadeh. \(L_2\) regularization for learning kernels. In *Proceedings of the 25th Conference on Uncertainty in Artificial Intelligence (UAI 2009)*, Montréal, Canada, June 2009.
[9] Corinna Cortes, Mehryar Mohri, and Afshin Rostamizadeh. Learning non-linear combinations of kernels. In *Advances in Neural Information Processing Systems (NIPS 2009)*, Vancouver, Canada, 2009. MIT Press.
[10] Corinna Cortes and Vladimir Vapnik. Support-Vector Networks. *Machine Learning*, 20(3), 1995.
[11] Tony Jebara. Multi-task feature and kernel selection for SVMs. In *ICML*, 2004.
[12] V. Kolchinskii and D. Panchenko. Rademacher processes and bounding the risk of function learning. In *High Dimensional Probability II*, pages 443–459. preprint, 2000.
[13] Gert Lanckriet, Nello Cristianini, Peter Bartlett, Laurent El Ghaoui, and Michael Jordan. Learning the kernel matrix with semidefinite programming. *JMLR*, 5, 2004.
[14] Darrin P. Lewis, Tony Jebara, and William Stafford Noble. Nonstationary kernel combination. In *ICML*, 2006.
[15] Charles Micchelli and Massimiliano Pontil. Learning the kernel function via regularization. *JMLR*, 6, 2005.
[16] Cheng Soon Ong, Alexander Smola, and Robert Williamson. Learning the kernel with hyperkernels. *JMLR*, 6, 2005.
[17] Bernhard Schölkopf and Alex Smola. *Learning with Kernels*. MIT Press: Cambridge, MA, 2002.
[18] John Shawe-Taylor and Nello Cristianini. *Kernel Methods for Pattern Analysis*. Cambridge Univ. Press, 2004.
[19] Nathan Srebro and Shai Ben-David. Learning bounds for support vector machines with learned kernels. In *COLT*, 2006.
[20] Vladimir N. Vapnik. *Statistical Learning Theory*. John Wiley & Sons, 1998.
[21] Manik Varma and Bodla Rakesh Babu. More generality in efficient multiple kernel learning. In *International Conference on Machine Learning*, 2009.
[22] Yiming Ying and Colin Campbell. Generalization bounds for learning the kernel problem. In *COLT*, 2009.
[23] Alexander Zien and Cheng Soon Ong. Multiclass multiple kernel learning. In *ICML*, 2007.
A Lemma 5

The following lemma is a straightforward version of Hölder’s inequality.

**Lemma 5** Let $q, r > 1$ with $1/q + 1/r = 1$. Then, the following result similar to Hölder’s inequality holds:

$$|w \cdot \Phi(x)| \leq \left( \sum_{k=1}^{p} \|w_k\|^q \right)^{1/q} \left( \sum_{k=1}^{p} \|\Phi_k(x)\|^r \right)^{1/r}.$$  \hspace{1cm} (14)

**Proof:** Let $\Psi_q(w) = (\sum_{k=1}^{p} \|w_k\|^q)^{1/q}$ and $\Psi_r(\Phi(x)) = (\sum_{k=1}^{p} \|\Phi_k(x)\|^r)^{1/r}$, then

$$\frac{|w \cdot \Phi(x)|}{\Psi_q(w) \Psi_r(\Phi(x))} = \frac{\sum_{k=1}^{p} w_k \cdot \Phi_k(x)}{\Psi_q(w) \Psi_r(\Phi(x))} \leq \sum_{k=1}^{p} \frac{w_k}{\Psi_q(w)} \cdot \frac{\Phi_k(x)}{\Psi_r(\Phi(x))} \leq \sum_{k=1}^{p} \frac{\|w_k\|}{\Psi_q(w)} \cdot \frac{\|\Phi_k(x)\|}{\Psi_r(\Phi(x))} \leq \sum_{k=1}^{p} \frac{1}{q} \frac{\|w_k\|^q}{\Psi_q(w)^q} + \frac{1}{r} \frac{\|\Phi_k(x)\|^r}{r \Psi_r(\Phi(x))^r} \leq \frac{1}{q} + \frac{1}{r} = 1.$$