Spiky strings in $AdS_3 \times S^1$ and their $AdS$ – pp-wave limits

R. Ishizeki, M. Kruczenski and A. Tirziu

Department of Physics, Purdue University,
525 Northwestern Ave., W. Lafayette, IN 47907-2036, USA

A.A. Tseytlin

Blackett Laboratory, Imperial College, London SW7 2AZ, U.K.

(Dated: February 25, 2010)

Abstract

We study a class of classical solutions for closed strings moving in $AdS_3 \times S^1 \subset AdS_5 \times S^5$ with energy $E$ and spin $S$ in $AdS_3$ and angular momentum $J$ and winding $m$ in $S^1$. They have rigid shape with $n$ spikes in $AdS_3$. We find that when $J$ or $m$ are non-zero, the spikes do not end in cusps. We consider in detail a special large $n$ limit in which $S \sim n^2$, $J \sim n$, i.e. $S \gg J \gg 1$, with $\frac{E+S}{n}$, $\frac{E-S}{n}$, $\frac{J}{n}$, $\frac{m}{n}$ staying finite. In that limit the spiky spinning string approaches the boundary of $AdS_5$. We show that the corresponding solution can be interpreted as describing a periodic-spike string moving in $AdS_3$–pp-wave $\times S^1$ background. The resulting expression for the string energy should represent a strong-coupling prediction for anomalous dimension of a class of dual gauge theory states in a particular thermodynamic limit of the $SL(2, R)$ spin chain.

PACS numbers:
I. INTRODUCTION

Remarkable recent progress in understanding the spectrum of $AdS_5 \times S^5$ superstring theory was initiated, in particular, by the study of various classical string solutions. In particular, the energy of the well-known folded spinning string solution in $AdS_3$ [1] describes the dimension of twist two gauge theory operators such as $\text{tr}(\Phi \nabla S \Phi)$ in the limit of large spin $S$. The folded string solution was generalized to $AdS_5 \times S^1$ [2] and quantum corrections to its energy were computed [2–5]. These and similar results on the string theory side aided and tested the construction of all-loop asymptotic Bethe ansatz for anomalous dimensions of the dual gauge theory operators (see, e.g., [6–9]).

The closed folded string solution [1] was generalized to the case of $n$ spikes in $AdS_3$ [10, 11]. The corresponding gauge theory states were argued [10] to represent, in particular, a subclass of higher twist operators in the $SL(2)$ sector of gauge theory [12]. In the large $S$ limit the spikes approach the boundary of $AdS_3$. It was shown in [14] that the motion of spikes in this limit can be described by a string solution in an $AdS_5 – pp$-wave metric:

$$ds^2 = \frac{1}{z^2} \left[ 2 dx_+ dx_- - \mu^2 (z^2 + x_i^2) dx_+^2 + dx_i dx_i + dz^2 \right], \quad i = 1, 2$$  \hspace{1cm} (1)

The metric (1) is obtained by zooming at the near-boundary region of $AdS_5$ while at the same time moving close to the speed of light in an angular direction. This limit thus appears to be relevant for the study of strings and hence dual gauge theory operators with large spin. String theory in an $AdS_5 – pp$-wave $\times S^5$ space is dual to $\mathcal{N} = 4$ SYM in a 4-dimensional pp-wave background with (conformally flat) metric

$$ds_{ft}^2 = 2 dx_+ dx_- - \mu^2 x_i^2 dx_+^2 + dx_i dx_i$$  \hspace{1cm} (2)

Indeed, this is the boundary metric of the space (1). In [15] several solutions for strings moving in the $AdS_3 – pp$-wave space were found. Those strings were ending at the boundary and therefore were dual to various Wilson loops in the field theory in the boundary pp-wave background [2].

In this paper we consider new solutions were the string is entirely in the bulk and therefore should be dual to particular states in the gauge theory. Interestingly, such strings do not move in direction $z$. In usual $AdS_5$ space in Poincaré coordinates this is not possible for extended strings since the curvature of the metric pushes the string towards small $z$. Here this effect is compensated by the extra term in the metric (1) proportional to $dx_+^2$. In fact, the solutions we shall find can be seen as limits of the spiky string solution when the spikes approach the boundary while the number of spikes grows to infinity. As we shall discuss below, the relevant limit that one can take is $n \to \infty$ keeping $E + S$ and $\bar{\gamma} = E - S$ fixed: there is an infinite number of spikes each of which contributes a finite amount $\bar{\gamma}$ to the anomalous dimension $\gamma = n \bar{\gamma} = E - S$.

In [10] it was argued that the spiky string should correspond to an operator of the type

$$\mathcal{O} = \text{tr} \left( \nabla_+^\Phi \nabla_+^\Phi \ldots \nabla_+^\Phi \right)$$  \hspace{1cm} (3)

1 Locally this space is still $AdS_5$. 

2
Such operators can be described by a spin chain with a number of sites \( n \) being the same as the number of spikes. This correspondence was recently emphasized and extended in [26] based also on earlier work of [12, 13]. This suggests that the above large \( n \) limit should be a meaningful thermodynamic limit of such spin chain, describing a strong-coupling asymptotics of the corresponding anomalous dimension.

More precisely, the operators in the \( SL(2) \) sector of planar \( \mathcal{N}=4 \) SYM theory are built out of \( J \) powers of complex scalar \( \Phi \) and \( S \) powers of light-like covariant derivative \( \nabla_+ \), symbolically, \( \text{tr}(\nabla_+^S \Phi^J) \). The eigen-states of the dilatation operator or spin chain Hamiltonian are labeled in addition to \( S \) and \( J \) by other quantum numbers corresponding, e.g., to the number of spikes \( n \) or \( S^1 \subset S^5 \) winding number \( m \) on the dual string theory side. Their scaling dimension may then be written as

\[
E = S + J + \gamma(S, J, m, n; \lambda),
\]

where \( \lambda \) is ’t Hooft coupling. We shall assume that the spin chain length \( J \) is large enough so that one can ignore the wrapping contributions [27], i.e. that \( \gamma \) should have the asymptotic Bethe ansatz [6] description for all values of \( \lambda \). On the perturbative gauge theory side one may consider the limit of large \( S \) (and large \( J \)) at fixed \( \lambda \), e.g., at each order in expansion in \( \lambda < 1 \). To describe the corresponding states in terms of semiclassical strings one is to consider first \( \lambda \gg 1 \) with fixed \( S \equiv \frac{S}{\sqrt{\lambda}} \), \( J \equiv \frac{J}{\sqrt{\lambda}} \) and then take \( S \) large order by order in \( \frac{1}{\sqrt{\lambda}} \) expansion. The two expansions are not a priori the same and may require a certain resummation in order to match.

Here we propose to consider the following special case of the \( S \gg J \gg 1 \) limit on the gauge-theory side (4):

\[
n \gg 1, \quad \frac{E + S}{n^2} = A, \quad \frac{E - S}{n} = B, \quad \frac{J}{n} = K, \quad \frac{m}{n} = k \quad \text{fixed} \quad (5)
\]

This limit is to be taken at fixed \( \lambda \), i.e. the fixed ratios may be functions of \( \lambda \). \(^2\) Equivalently, we assume that for \( n \to \infty \)

\[
E = \frac{1}{2}An^2 + \frac{1}{2}Bn + O(n^0), \quad S = \frac{1}{2}An^2 - \frac{1}{2}Bn + O(n^0),
\]

\[
J = Kn + O(n^0), \quad m = kn + O(n^0). \quad (6)
\]

Then \( E \sim n^2, S \sim n^2, J \sim n \) so that \( S \sim nJ, \) i.e. \( S \gg J \gg 1 \). The operator (3) represents a particular state that may be relevant in such limit having \( J = n \). More generally, one may consider

\[
O = \text{tr} \left( \nabla_+^S \Phi \ nabla_+^S \Phi \ldots \nabla_+^S \Phi \right) \sim \text{tr} \left( \nabla_+^{an} \Phi \ \nabla_+^{an} \Phi \ldots \nabla_+^{an} \Phi \right) \quad (7)
\]

with \( a = \frac{A}{2K} \).

To define a similar limit on the semiclassical string theory side we need again to remember that to have a consistent \( \alpha' \sim \frac{1}{\sqrt{\lambda}} \) expansion we are to take \( \lambda \gg 1 \) first with all the parameters

\(^2\) Since \( S, J, m, n \) should be integers it appears that only \( B \) can be a nontrivial function of \( \lambda \).
characterising a classical string solution like $S = \sqrt{\lambda}$, $J = \sqrt{\lambda}$, $m$ and $n$ being fixed, so that the string energy admits the expansion

$$E = \sqrt{\lambda} E_0(S, J, m, n) + \frac{1}{\sqrt{\lambda}} E_1(S, J, m, n) + \frac{1}{\lambda} E_2(S, J, m, n) + \ldots. \quad (8)$$

Then the analog of the limit (5) in the perturbative string theory expansion is proposed to be

$$n \gg 1, \quad \text{with} \quad \frac{E + S}{n^2} = A, \quad \frac{E - S}{n} = B, \quad \frac{J}{n} = K, \quad \frac{m}{n} = k \quad \text{fixed} \quad (9)$$

where $E = \frac{E_0}{\sqrt{\lambda}}$.

Below we shall consider only the classical string solutions for which the fixed parameters in (9) will not depend on $\sqrt{\lambda}$, i.e. we will be interested in a particular scaling limit in the space of semiclassical parameters.\(^3\) In this limit $E \sim n^2$, $S \sim n^2$, $J \sim n$ so that $S \sim \sqrt{\lambda}n^2 \gg n^2$, $J \sim \sqrt{\lambda}n \gg n$. This limit is obviously different from the one (5) on the gauge theory (spin chain) side but as with other large spin limits the two may happen to be closely connected in certain special cases (like leading terms in large spin expansion).

Having found the exact expression for the dimension $E$ in (4) for all values of $S, J, m, n$ and $\lambda$ one should be able to consider the large $n$ limit either as in (5) or as in (9). For certain terms (like familiar $\ln S$ terms) the predictions of the two limits may differ only by interpolating functions of $\lambda$, but in general to connect the expressions found in the two limits should require a resummation of the corresponding expansions.

Let us mention that in the semiclassical string theory limit that we shall consider below the states with R-charge $J \sim n$ like those in (5) will not be distinguishable from states with $J = 0$ : to have a non-zero semiclassical spin one would need to consider the states with $J \sim \sqrt{\lambda}n \gg n$. In other words, the spiky strings moving only in $\text{AdS}_5$ may still be thought of as corresponding to $\text{SL}(2)$ spin chain operators like (3).

Below we shall also extend the discussion of the spiky strings in [10] to include the angular momentum $J$ (and winding $m$) in a maximal circle $S^1$ in $S^5$. To obtain the solution it turns out to be convenient to use the conformal gauge as in [11, 28]. The resulting solutions are closely related to those discussed in [16]. We shall find that the shape of the string in the $\text{AdS}_3$ space is very similar to the original spiky string with $J = m = 0$. A careful analysis shows, however, that as long as $J$ or $m$ are non-zero the spikes are rounded, namely, they do not end in cusps.

The introduction of $J$ allows for the possibility of taking the large $J$ or fast-string limit as in [2, 17–20]. The leading-order term in the string energy is then described [19, 20] by an effective $\text{SL}(2, R)$ Landau-Lifshitz (LL) model [21, 22] that happens to capture both the fast-moving string limit and the corresponding leading-order semiclassical dynamics of the spin chain on the perturbative field theory side. We shall show how to find the spiky-like solutions directly in the LL model which can then be interpreted either as fast-moving strings or as coherent superpositions of field theory operators.

\(^3\) In principle, one may consider a more general limit in which the fixed parameters may be given by series in inverse string tension like $c_0 + \frac{c_1}{\sqrt{\lambda}} + \frac{c_2}{(\sqrt{\lambda})^2} + \ldots$. That would correspond to a certain resummation of string perturbative expansion.
Another limit of interest is when $\frac{E-S}{n}$ is taken to be much larger than $\frac{J}{n}$. In that case we shall recover the familiar logarithmic scaling of the anomalous dimension.

Finally, we shall also consider the $AdS-pp$-wave limit which for non-zero $J$ corresponds to taking $n \to \infty$ with the ratios in (9) being fixed. Here each spike contributes a finite amount to the anomalous dimension $E - S - J$, the spin $J$ and the winding $m$. In this limit, which should correspond, as discussed above, to a particular thermodynamic limit on the spin chain side, we compute the classical string energy or $\frac{E-S-J}{n}$ as a function of fixed parameters $\frac{E+S}{n^2}, \frac{J}{n}, \frac{m}{n}$. This function should represent a strong-coupling prediction for the thermodynamic limit of the corresponding $SL(2)$ spin chain. It would be very interesting if the methods generalizing those used for the scaling function [6, 8] can be used to reproduce this prediction from the Bethe ansatz.

This rest of this paper is organized as follows. In section 2 we shall review the original spiky string solution with $J = 0$ and show that it admits a consistent large $n$ limit as defined in (9). We shall then demonstrate that the same expression for its energy can be found by considering an infinite rigid string with periodic spikes in the $AdS_3-pp$-wave background (1).

As an aside, in section 3 we shall study a straight string in $AdS-pp$-wave background and show that expanding its energy at large $S$ one is able to reproduce certain terms in the large $S$ expansion of the folded (2-spike) string in $AdS_5$ which is an indication of the utility of the pp-wave picture.

In section 4 we shall describe in detail the construction of the generalization of the $n$-spike solution to the presence of non-zero classical $S^5$ angular momentum $J$ and winding $m$. We shall follow [11] and use a generalized rigid string ansatz in the conformal gauge. We shall show that a non-zero $J$ or $m$ “rounds-up” the spikes and find the (implicit) expression for the energy as a function of the semiclassical parameters $S, J, n, m$. Then in section 5 we shall consider the three special cases: (a) the $J = m = 0$ case when the solution reduces to the original spiky string in $AdS_3$; (b) the fast-string limit with $J \gg 1, \frac{S}{J}$-fixed, which should be reproduced by the Landau-Lifshitz model; (c) the large $n$ limit (9) where $S \gg J \gg 1$ and which should also admit a description in terms of a rigid string in $AdS-pp$-wave background.

In section 6 we shall elaborate on the connection to the Landau-Lifshitz model by presenting the corresponding analogs of the spiky string solution in several different limits. In section 7 we shall demonstrate how to construct the the generalization of the periodic spike solution in $AdS-pp$-wave background from section 2 to the presence of rotation in an extra $S^1 \subset S^5$ and discuss its connection to the large $n$ limit of the solution found in section 4. Appendix contains a list of some useful integrals.

---

4 More precisely, this function is defined implicitly by computing these ratios in terms of the three independent parameters, allowing in principle to obtain any of the four quantities in terms of the other three.

5 As was already mentioned, in the limit we consider $J \to \infty$ so that the wrapping contributions should be absent.
II. LARGE \( n \) LIMIT OF SPIKY STRING AS PERIODIC SPIKE SOLUTION IN 
\( AdS-pp\)-WAVE BACKGROUND

In this section we consider a particular limit of the spiky string [10] which corresponds to taking the number of spikes \( n \) to infinity keeping \((E + S)/n^2\) and \((E - S)/n\) fixed. It turns out that such limit can be also described by a particular spiky solution for a string moving in an \( AdS-pp\)-wave background. This follows from the fact that, at the level of the string solution, this limit is the same as the one shown in [14] to lead to an \( AdS-pp\)-wave metric.

A. Limit of the spiky string solution

Following [10] we consider a rigid string rotating around its center of mass in the \( AdS_3 \) metric

\[
ds^2 = - \cosh^2 \rho \, dt^2 + d\rho^2 + \sinh^2 \rho \, d\theta^2
\]

and described by the ansatz

\[
t = \kappa \tau, \quad \theta = \omega \tau + \sigma, \quad \rho = \rho(\sigma).
\]

Then \( \rho \) satisfies

\[
\frac{dv}{d\sigma} = (1 - v^2) \sqrt{\frac{1 - v_1}{1 - v_0}} \sqrt{\frac{v_0^2 - v^2}{v(v - v_1)}}, \quad v \equiv \frac{1}{\cosh 2\rho}
\]

where \( v_0 \) and \( v_1 \) determine the positions of the spikes and the valleys, namely the maximal and the minimal values of \( \rho \) (or \( v \)). It is then straightforward to compute

\[
\Delta \theta = \frac{2\pi}{n} = 2 \sqrt{\frac{1 - v_0^2}{1 - v_1}} \int_{v_0}^{v_1} \frac{dv}{1 - v^2} \sqrt{\frac{v(v - v_1)}{v_0^2 - v^2}} \quad (13)
\]

\[
\frac{S}{n} = \frac{\sqrt{1 + v_1}}{4\pi v_0} \int_{v_1}^{v_0} \frac{dv}{v(1 + v)} \sqrt{\frac{v_0^2 - v^2}{v(v - v_1)}} \quad (14)
\]

\[
\frac{E}{n} = \frac{1 - v_0^2}{2\pi v_0 \sqrt{1 - v_1}} \int_{v_1}^{v_0} \frac{dv}{1 - v^2} \sqrt{\frac{v(v - v_1)}{v_0^2 - v^2}} + \frac{1 - v_1}{4\pi v_0} \int_{v_1}^{v_0} \frac{dv}{v(1 - v)} \sqrt{\frac{v_0^2 - v^2}{v(v - v_1)}} \quad (15)
\]

Here \( n \) is the number of spikes, \( \Delta \theta \) is the angular distance between spikes, and \( E = 2\pi TS \) and \( S = 2\pi T \mathcal{S} \) are the energy and spin respectively (\( T = \frac{\sqrt{\lambda}}{2\pi} \) is the string tension). The resulting expressions can be written in terms of elliptic functions as in [10] but the above integral representations are more convenient for taking the limit we are interested in here.

Indeed, let us rescale

\[
v_0 \to \epsilon^2 v_0, \quad v_1 \to \epsilon^2 v_1, \quad \epsilon \to 0.
\]

---

6 In [10] a variable \( u = \cosh 2\rho \) was used. Here we find it more convenient to use, instead, \( v = 1/u \).
Then it follows from the above relations that

\[ n \sim \frac{1}{\epsilon^2}, \quad \frac{\mathcal{E} + \mathcal{S}}{n} \sim \frac{1}{\epsilon^2}, \quad \frac{\mathcal{E} - \mathcal{S}}{n} \sim 1 \]  

(17)

Thus we can compute the following finite quantities

\[ \frac{\mathcal{E} + \mathcal{S}}{n^2} \rightarrow \frac{1}{2\pi^2} \int_{v_1}^{v_0} \frac{dv}{v} \sqrt{\frac{v_0^2 - v^2}{v(v - v_1)}} \int_{v_1}^{v_0} \frac{dv'}{v_0} \sqrt{\frac{v'(v' - v_1)}{v_0^2 - v'^2}} \]  

(18)

\[ \frac{\mathcal{E} - \mathcal{S}}{n} \rightarrow \int_{v_1}^{v_0} \frac{dv}{2\pi v_0} \sqrt{\frac{v^2 - v_1^2}{v_0^2 - v^2}} + \int_{v_1}^{v_0} \frac{dv}{2\pi v} \left(1 - \frac{v_1}{2v}\right) \sqrt{\frac{v_0^2 - v^2}{v(v - v_1)}} \]  

(19)

which are clearly invariant under the above rescaling: they depend only on the ratio

\[ b \equiv \frac{v_1}{v_0} \]

(20)

This can be made explicit by writing them in terms of the elliptic functions:

\[ \bar{\mathcal{P}}_- \equiv \frac{E + S}{n^2} = \frac{2T}{\pi b(1 + b)} \left[(1 + b)\mathcal{E}(p) - (2 + b)\mathcal{K}(p) - b^2\Pi(1 - b, p)\right]^2 \]

\[ \bar{\gamma} \equiv \frac{E - S}{n} = \frac{T}{\sqrt{1 + b}} \left[(1 + b)\mathcal{E}(p) - (2 + b)\mathcal{K}(p) + b^2\Pi(1 - b, p)\right] \]  

(21)

where we explicitly included the factor of string tension \( T \) and defined

\[ p = \sqrt{\frac{1 - b}{1 + b}} \]

(22)

\( \mathcal{E}(p), \mathcal{K}(p), \Pi(1 - b, p) \) are the standard elliptic functions (defined as in \([25]\)). Eliminating \( b \) one finds \( \bar{\gamma} \) as a function of \( \bar{\mathcal{P}}_- \).

As was already mentioned in the Introduction, in \([10]\) it was argued that the spiky string should be dual to the operator \([3]\) from the \( SL(2) \) sector of gauge theory (with \( J = n \) which is not distinguishable from zero at the classical string theory level). The large \( n \) limit we just discussed should correspond to a particular thermodynamic limit of the spin chain. It would thus be interesting to reproduce the above strong-coupling expression for \( \gamma \) on the spin chain side.

### B. Periodic spike solution in \( AdS\)–pp-wave background

Let us now show that exactly the same result \([21]\) can be obtained by considering a string moving in an \( AdS\)–pp-wave \([1]\). This space arises as a particular limit of \( AdS_5 \) when one zooms in near the boundary while at the same time moving close to the speed of light along the angular direction. This suggests that the \( AdS\)–pp-wave metric captures all the information necessary to understand such thermodynamic limit. This is considerably more than what was argued for in \([14]\) since there only the leading \( \ln \bar{\mathcal{P}}_- \) dependence of \( E - S \) was considered.
1. Rigid strings in an AdS–pp-wave

As a preparation, let us start by presenting a generic description of the relevant class of solutions in the metric (1), i.e.

\[ ds^2 = \frac{1}{z^2} \left[ 2dx_+dx_- - \mu^2(z^2 + x_i^2)dx_i^2 + dx_i dx_i + d\zeta^2 \right] , \quad x_\pm = \frac{x \pm t}{\sqrt{2}} \]  

(23)

We shall assume that \( x_i = 0 \), i.e. consider strings that move in the subspace spanned by \( x_\pm, z \). To respect the symmetries of the problem we shall make a partial gauge choice by taking

\[ x_+ = \tau. \]  

(24)

In this gauge the string action becomes

\[ I = \int d\sigma d\tau \mathcal{L} = -T \int \frac{d\sigma d\tau}{z^2} \sqrt{x_-'^2 + 2x_-' z' - 2\dot{x}_- z'^2 + \mu^2 z^2 z'^2} \]  

(25)

where dot indicate derivative with respect to \( \tau \) and prime – with respect to \( \sigma \). The equation of motion for \( x_- \) is

\[ \partial_\sigma \left( \frac{x_-' + \dot{z} z'}{z^2 F} \right) - \partial_\tau \left( \frac{z'^2}{z^2 F} \right) = 0 \]  

(26)

and the one for \( z \) is

\[ \partial_\sigma \left( \frac{x_-' z - 2\dot{x}_- z' + \mu^2 z^2 z'}{z^2 F} \right) + \partial_\tau \left( \frac{z' x_-'}{z^2 F} \right) = -\frac{2}{z^3} F + \frac{\mu^2 z'^2}{z F} \]  

(27)

where

\[ F = \sqrt{x_-'^2 + 2x_-' z' - 2\dot{x}_- z'^2 + \mu^2 z^2 z'^2} \]  

(28)

The conserved momenta are\(^7\)

\[ P_+ = \int d\sigma \frac{\partial \mathcal{L}}{\partial \dot{x}_+} = -T \int \frac{d\sigma}{z^2 F} \left( x_-'^2 + \dot{z} z' x_-' - \dot{x}_- z'^2 + \mu^2 z^2 z'^2 \right) \]  

\[ P_- = \int d\sigma \frac{\partial \mathcal{L}}{\partial \dot{x}_-} = T \int \frac{d\sigma}{z^2 F} z'^2 \]  

(29)

2. Periodic spike solution

Besides \( x_+ = \tau \) let us now make the additional gauge choice \( x_- = \sigma \) and look for a solution corresponding to a rigid string moving along \( x = \sqrt{2}(x_+ + x_-) \) with constant velocity \( v \). This implies that

\[ z = z(\xi), \quad \xi \equiv x - vt = x_+ - \frac{v + 1}{v - 1} x_- = \tau - \frac{1}{\eta_0^2} \sigma, \quad \text{with} \quad \eta_0^2 \equiv \frac{v - 1}{v + 1} \]

(30)

\(^7\) To find the expression for \( P_+ \) it is convenient to assume that the gauge is \( x_+ = \kappa \tau \) and compute \( P_+ = \int d\sigma \frac{\partial \mathcal{L}}{\partial \kappa} \) before setting \( \kappa = 1 \).
We defined $\eta_0$ taking into account that, for our solution, $v > 1$. Notice that this does not imply that the string moves faster than light since its actual speed should be measured using the bulk metric. In fact, this implies that the string does not reach the boundary.

With this ansatz, the $x_-$ equation of motion becomes
\[
\partial_\xi \left( \frac{1}{\eta^2_0 z^2 F} \right) = 0
\]
which implies
\[
\partial_\xi z = \frac{\eta^2_0}{\mu z^2} \sqrt{\frac{z^4_0 - z^4}{z^2 - z^2_1}}, \quad z_1 = \sqrt{2} \eta_0 / \mu
\]
where $z_0$ is a constant of integration.

After gluing it with its copies under translations and reflections, the solution takes shape shown in fig.1. Using the equation (29) it is straightforward to compute the conserved quantities:

\[
P_- = \frac{2T}{\mu z^2_0} \int_{z_1}^{z_0} \frac{dz}{z^2} \sqrt{\frac{z^4_0 - z^4}{z^2 - z^2_1}} \]
\[
P_+ = -\frac{2\mu T}{z^2_0} \int_{z_1}^{z_0} dz \left[ z^2 \sqrt{\frac{z^2 - z^2_1}{z^4_0 - z^4}} + \left( 1 - \frac{z^2_1}{2z^2} \right) \sqrt{\frac{z^4_0 - z^4}{z^2 - z^2_1}} \right]
\]
which are given in terms of the position of the valleys $z_0$ and the spikes $z_1$. The integrals can be explicitly done in terms of the elliptic functions:

\[
P_- = \frac{2T}{\mu z^2_0} \frac{1}{b\sqrt{1 + b}} \left[ -bK(p) + (1 + b)E(p) - b^2 \Pi(1 - b, p) \right]
\]
\[
P_+ = \frac{\mu T}{\sqrt{1 + b}} \left[ (1 + b)E(p) - (2 + b)K(p) + b^2 \Pi(1 - b, p) \right]
\]
where
\[
p = \sqrt{\frac{1 - b}{1 + b}}, \quad b = \frac{z^2_1}{z^2_0}
\]
Notice that when $b \to 0$ we get
\[
P_- \approx \frac{2T}{\mu z^2_0} \frac{1}{b}, \quad P_+ \approx \mu T \ln b
\]
so that
\[
P_+ \approx -\mu T \ln P_-
\]
which is the expected result \[14\] at the leading order in the large spin limit $S \to \infty$.

Finally, the separation $\Delta x_-$ (for constant $x_+$) between spikes can be computed as

\[
\Delta x_- = \int d\sigma = \eta^2_0 \int \xi = 2\mu \int_{z_0}^{z_1} z^2 d\xi \sqrt{\frac{z^2 - z^2_1}{z^4_0 - z^4}}
\]
\[
= \frac{\mu z^2_0}{\sqrt{1 + b}} \left[ (1 + b)E(p) - bK(p) - b^2 \Pi(1 - b, p) \right]
\]
This allows us to compute

\[ P_- \Delta x_- = \frac{2T}{b(1+b)} \left[ (1+b)E(p) - bK(p) - b^2 \Pi(1-b,p) \right]^2 \]  \hspace{1cm} (42)

\[ P_+ = \frac{\mu T}{\sqrt{1+b}} \left[ (1+b)E(p) - (2+b)K(p) + b^2 \Pi(1-b,p) \right] \]  \hspace{1cm} (43)

We observe that if we set \( P_- \Delta x_- = \pi \bar{P}_- \), \( P_+ = \mu \bar{\gamma} \) these expressions match the ones in eq. (21), as claimed.

The reason for this \( \Delta x_- \) factor can be understood as follows. In the limit we consider \( n \to \infty \) with \((E+S)/n^2\) being fixed. The number of spikes is related to the angle difference between the spikes that scales as \( \Delta \theta \sim \frac{1}{n} \). Then \((E+S)/n^2 = (E+S)/n \times \Delta \theta\) translates into \( P_- \Delta x_- \) in the pp-wave picture since here \( P_+ = E - S \), \( P_- = -(E + S) \).

### III. STRAIGHT STRING IN ADS–PP-WAVE BACKGROUND

The straight folded ("2-spike") string rotating in \( AdS_5 \) is related, in the large spin limit, to the following simpler solution for a string in \( AdS \)-pp-wave background: the string moves along spatial \( x = \sqrt{2}(x_+ + x_-) \) direction and is extended along \( z \) from a finite distance from a boundary to the horizon, i.e. \( \sigma_1 \leq z < \infty \) (\( \sigma_1 \) is a given constant related to spin of solution in \( AdS_3 \)). If \( \sigma_1 \) is non-zero the string does not touch the boundary. The profile of this solution is presented in figure 2.

The corresponding ansatz for string coordinates is

\[ x_+ = \tau, \quad x_- = V \tau, \quad z = f(\sigma) \]  \hspace{1cm} (44)

The Nambu action is

\[ I = - \int d\tau d\sigma \ L, \quad L = T \frac{f'}{f} \sqrt{\mu^2 f^2 - 2V} \]  \hspace{1cm} (45)

The simplest solution representing a "hanging" string is

\[ z = \sigma, \quad \sigma_1 \leq \sigma < \infty \quad z_1 \equiv \sigma_1 = \frac{\sqrt{2V}}{\mu}. \]  \hspace{1cm} (46)
One can check that all equations of motion are satisfied in this case.

When the “hanging” string touches the boundary, i.e. when \( \sigma_1 = 0 \) (i.e. when \( V = 0 \) and thus the string moves along \( x_- \) only) it was shown in [14] that such a solution reproduces \( \frac{1}{4} \) of the leading large spin asymptotics of the energy of the spinning folded string in \( AdS_3 \).

Here we shall consider a more general case when the string is not touching the boundary but is close to it, i.e. \( \sigma_1 \) is small. The conserved charges are

\[
P_+ = - \int_{\sigma_1}^{\infty} d\sigma \frac{\partial L}{\partial (\partial_x x_+)} = T \int_{\sigma_1}^{\infty} d\sigma \frac{\sigma^2 - \frac{\sigma_1^2}{2}}{\sigma^2 \sqrt{\sigma^2 - \sigma_1^2}} = T \mu \left[ \ln \frac{R + \sqrt{R^2 - \sigma_1^2}}{\sigma_1} - \frac{\sqrt{R^2 - \sigma_1^2}}{2R} \right] \\
= \frac{\mu T}{2} \left( \ln \frac{4R^2}{\sigma_1^4} - 1 \right) + O(\frac{1}{R^4}) \quad \text{ (47)}
\]

\[
P_- = - \int_{\sigma_1}^{\infty} d\sigma \frac{\partial L}{\partial (\partial_x x_-)} = - \frac{T}{\mu} \int_{\sigma_1}^{\infty} d\sigma \frac{1}{\sigma^2 \sqrt{\sigma^2 - \sigma_1^2}} = - \frac{T}{\mu \sigma_1^2} \quad \text{ (48)}
\]

where in \( P_+ \) we have introduced as in [14] a large cutoff \( R \) in \( \sigma \), and expanded in large \( R \). In the limit when \( \sigma_1 \) is small, so that the end of the string is close to the boundary, we find that both \( |P_-| \) and \( P_+ \) are large. Expressing \( \sigma_1 \) in terms of \( P_- \) we obtain

\[
P_+ = \frac{\mu T}{2} \ln |P_-| + \frac{\mu T}{2} \left( \ln \frac{4R^2 \mu}{T} - 1 \right) \quad \text{ (49)}
\]

As was shown in [14] the relation to the folded string in \( AdS_3 \) with energy \( E \) and spin \( S \) can be established by formally identifying

\[
P_+ = E - S, \quad P_- = -(E + S) \quad \text{ (50)}
\]

Then using that \( S = -\frac{1}{2}(P_- + P_+) \) in (49) one obtains

\[
S = \frac{1}{2} |P_-| - \frac{\mu T}{4} \ln |P_-| - \frac{\mu T}{4} \left( \ln \frac{4\mu R^2}{T} - 1 \right) \quad \text{ (51)}
\]
or, inverting this relation,
\begin{equation}
|P_-| = 2S + \frac{\mu T}{2} \ln(2S) + \frac{\mu T}{2} \left( \ln \frac{4\mu R^2}{T} - 1 \right) + \frac{\mu^2 T^2}{8} \ln \frac{S}{S} + \frac{\mu^2 T^2}{8S} \left( \ln \frac{8\mu R^2}{S} - 1 \right) + \ldots
\end{equation}

Inserting $P_-$ back into (49) and expanding in large $S$ gives
\begin{equation}
P_+ = E - S = \frac{\mu T}{2} \ln S + \frac{\mu T}{2} \left( \ln \frac{8\mu R^2}{T} - 1 \right) + \frac{\mu^2 T^2}{8S} \left( \ln S + \ln \frac{8\mu R^2}{T} - 1 \right)
- \frac{\mu^3 T^3}{64S^2} \left[ \ln^2 S + 2 \ln S (\ln \frac{8\mu R^2}{T} - 2) + \ln \frac{\mu R^2}{T} (\ln \frac{64\mu R^2}{T} - 4) + 3 + 3 \ln 2 (3 \ln 2 - 4) \right] + \ldots
\end{equation}

The leading $\ln S$ term here is (with the pp-wave scale parameter set to be $\mu = 1$) the same as $\frac{1}{4}$ of the $\ln S$ term in the folded string energy [14]. To compare higher order terms let us formally replace $T \to 4T$ with $\mu = 1$. We then obtain from (53) ($T = \frac{\sqrt{\lambda}}{2\pi}$)
\begin{equation}
E - S = \frac{\sqrt{\lambda}}{\pi} (\ln S + a) + \frac{\lambda \ln S + a}{2\pi^2} - \frac{\lambda^3/2}{8\pi^3} \frac{(\ln S + a)(\ln S + a - 2)}{S^2} + O \left( \frac{\ln^3 S}{S^3} \right)
\end{equation}
\begin{equation}
a \equiv \ln \frac{4\pi R^2}{\sqrt{\lambda}} - 1
\end{equation}

This may be compared to the corresponding expression for the classical energy of the folded string in $AdS_3$ [23]
\begin{equation}
E - S = \frac{\sqrt{\lambda}}{\pi} (\ln S + b) + \frac{\lambda \ln S + b}{2\pi^2} - \frac{\lambda^3/2}{8\pi^3} \frac{(\ln S + b)(\ln S + b - \frac{5}{2})}{S^2} + O \left( \frac{\ln^3 S}{S^3} \right)
\end{equation}
\begin{equation}
b \equiv \ln \frac{8\pi}{\sqrt{\lambda}} - 1
\end{equation}

The two expressions do have the same structure. We observe that the coefficients of the terms $\ln S$, $\frac{\ln S}{s^k}$ in (54) that do not depend on the cutoff $R$ do match. However, the coefficients of some of the subleading $\frac{1}{s^k}$ terms which depend on $R$ appear to disagree. This is not surprising since we cannot unambiguously fix the cutoff in (49).

This partial matching can be understood as a consequence of the fact that in the large $S$ limit the coefficients of the leading terms at each order in $\frac{1}{s^k}$ receive contributions from the region of the folded string where $\rho$ is large, while the subleading terms are sensitive to the smaller $\rho$ region. The latter is not “seen” in the “pp-wave limit” where one zooms in at the near-boundary part of the $AdS_5$.

IV. SPIKY STRINGS IN $AdS_5$ WITH ANGULAR MOMENTUM IN $S^5$:
GENERAL RELATIONS

Let us now generalize the spiky string solution of [10] to the case of non-zero semiclassical angular momentum $J$ and winding $m$ in $S^5$. This should be important for a detailed
comparison with the states in the \( SL(2) \) sector on the gauge theory side, i.e. the strings should carry spin \( S \) in \( AdS_3 \) and spin \( J \) in \( S^1 \subset S^5 \).

Let us start with the \( AdS_3 \times S^1 \) metric in embedding coordinates and, following [11], use the conformal gauge. Then the string Lagrangian takes the form

\[
L = \frac{1}{2} \left[ - \partial_a Y_0 \partial^a Y_0^* + \partial_a Y_1 \partial^a Y_1^* + \partial_a X \partial^a X^* + \Lambda(|Y_0|^2 - |Y_1|^2 - 1) + \tilde{\Lambda}(|X|^2 - 1) \right] 
\]

(56)

The conformal constraints are

\[
- |\dot{Y}_0|^2 + |\dot{Y}_1|^2 + |\dot{X}|^2 - |Y_0'|^2 + |Y_1'|^2 + |X'|^2 = 0, \quad -\dot{Y}_0 Y_0'^* + \dot{Y}_1 Y_1'^* + \dot{X} X'^* + c.c. = 0
\]

(57)

We shall consider the following rigid string ansatz which is similar to the one used in the \( R \times S^5 \) case in [11]

\[
Y_0 = y_0(u) e^{i w_0 \tau}, \quad Y_1 = y_1(u) e^{i w_1 \tau}, \quad X = x(u) e^{i \nu \tau},
\]

\[
\sigma \equiv \alpha \sigma + \beta \tau
\]

(58)

(59)

With this ansatz the string Lagrangian reduces to the following 1-dimensional integrable mechanical system (here prime is derivative over the argument \( u \))

\[
L = \frac{1}{2} \left[ (\beta^2 - \alpha^2)(|y_0'|^2 - |y_1'|^2 + |x'|^2) + w_0^2 |y_0|^2 - w_1^2 |y_1|^2 - \nu^2 |x|^2 + i w_0 \beta (y_0 y_0'^* - y_0'^* y_0^*)
\]

\[
- i w_1 \beta (y_1 y_1'^* - y_1'^* y_1^*) - i \nu (x x'^* - x' x^*) + \Lambda(|y_0|^2 - |y_1|^2 - 1) + \tilde{\Lambda}(|x|^2 - 1) \right]
\]

(60)

The corresponding conserved Hamiltonian is

\[
H = \frac{1}{2} \left[ (\beta^2 - \alpha^2)(|y_0'|^2 - |y_1'|^2 - |x'|^2) + w_0^2 |y_0|^2 - w_1^2 |y_1|^2 - \nu^2 |x|^2 \right]
\]

(61)

After combining the two conformal constraints they can be written as

\[
(\beta^2 - \alpha^2)(|y_0'|^2 - |y_1'|^2 - |x'|^2) - w_0^2 |y_0|^2 + w_1^2 |y_1|^2 + \nu^2 |x|^2 = 0
\]

(62)

\[
\frac{\beta^2 - \alpha^2}{2 \beta} (-w_0 \xi_0 + w_1 \xi_1 + \nu \xi_2) - w_0^2 |y_0|^2 + w_1^2 |y_1|^2 + \nu^2 |x|^2 = 0
\]

(63)

where

\[
\xi_0 = i(y_0 y_0'' - y_0'^* y_0^*), \quad \xi_1 = i(y_1 y_1'' - y_1'^* y_1^*), \quad \xi_2 = i(x x'' - x' x^*)
\]

(64)

The first constraint (62) is conserved since it is just equivalent to \(-2H = 0\), while the second one (63) is satisfied due to the equations of motion. The equations of motion for \( y_0, y_1, x \) imply

\[
(\beta^2 - \alpha^2)\xi_0' = -2 w_0 \beta (y_0 y_0'^*), \quad (\beta^2 - \alpha^2)\xi_1' = -2 w_1 \beta (y_1 y_1'^*), \quad (\beta^2 - \alpha^2)\xi_2' = -2 \nu \beta (x x'^*)
\]

(65)

Since we consider a closed string, the condition of periodicity in \( \sigma \) implies periodicity in \( u \)

\[
y_0(u) = y_0(u + 2\pi \alpha), \quad y_1(u) = y_1(u + 2\pi \alpha), \quad x(u) = x(u + 2\pi \alpha)
\]
Since \( y_0, y_1 \) are in general complex and \(|y_0|^2 - |y_1|^2 = 1, \ |x|^2 = 1\) we may set
\[
y_0 = r_0(u)e^{i\varphi_0(u)}, \quad y_1 = r_1(u)e^{i\varphi_1(u)}, \quad x = e^{i\psi(u)}, \quad r_0^2 - r_1^2 = 1.
\] (66)

Then the Lagrangian (60) becomes
\[
L = -\frac{1}{2} \left[ (\beta^2 - \alpha^2)(r_0^2 - r_1^2) + r_0^2(\beta^2 - \alpha^2) \left( \varphi_0' + \frac{\beta w_0}{\beta^2 - \alpha^2} \right)^2 - \frac{\alpha^2 r_0^2 w_0^2}{\beta^2 - \alpha^2}
\right.
\]
\[
- r_1^2(\beta^2 - \alpha^2) \left( \varphi_1' + \frac{\beta w_1}{\beta^2 - \alpha^2} \right)^2 + \frac{\alpha^2 r_1^2 w_1^2}{\beta^2 - \alpha^2} - (\beta^2 - \alpha^2) \left( \psi' + \frac{\beta \nu}{\beta^2 - \alpha^2} \right)^2
\]
\[
+ \frac{\alpha^2 \nu^2}{\beta^2 - \alpha^2} \right] + \Lambda(r_0^2 - r_1^2 - 1)
\] (67)

The periodicity conditions (65) imply
\[
r_0(u) = r_0(u + 2\pi \alpha), \quad r_1(u) = r_1(u + 2\pi \alpha)
\] (68)
\[
\varphi_0(u) = \varphi_0(u + 2\pi \alpha) - 2\pi m_0, \quad \varphi_1(u) = \varphi_1(u + 2\pi \alpha) - 2\pi m_1, \quad \psi(u) = \psi(u + 2\pi \alpha) - 2\pi m
\]
where \(m_0, m_1, m\) are integers. Below we shall assume that \(m_0 = 0\) since we consider the global \(AdS_5\) time \(t\) as non-compact.

The equations of motions for \(\varphi_0, \varphi_1, \psi\) can be integrated as
\[
\varphi_0' = -\frac{1}{\beta^2 - \alpha^2} \left( \frac{C_0}{r_0^2} + w_0 \beta \right), \quad \varphi_1' = \frac{1}{\beta^2 - \alpha^2} \left( \frac{C_1}{r_1^2} - w_1 \beta \right), \quad \psi' = \frac{D - \beta \nu}{\beta^2 - \alpha^2}
\] (69)

where \(C_0, C_1, D\) are constants. The equation for \(\psi\) can be integrated again so that in \(S^5\) we just have a rotating string wound on a circle. Denoting the angle in \(S^1 \subset S^5\) by \(\phi\), we have
\[
X = e^{i\phi}, \quad \phi = \nu \tau + \psi = \nu \tau + \frac{(D - \beta \nu)}{\beta^2 - \alpha^2} u
\] (70)

The winding number in \(\phi\) is defined as
\[
2\pi m = \int_0^{2\pi \alpha} du \psi' = \frac{D - \beta \nu}{\beta^2 - \alpha^2} \int du
\] (71)

The condition of having no winding in the \(t\) direction gives the condition
\[
2\pi m_0 = \int_0^{2\pi \alpha} du \varphi_0' = \int_0^{2\pi \alpha} du \left( \frac{C_0}{1 + r_1^2} + w_0 \beta \right) = 0
\] (72)

The effective Lagrangian for \(r_0, r_1\) that reproduces the remaining equations of motion is then
\[
L = -\frac{1}{2} \left[ r_0^2(\beta^2 - \alpha^2) - \frac{C_0^2}{r_0^2(\beta^2 - \alpha^2)} - r_0^2 \alpha^2 \frac{w_0^2}{\beta^2 - \alpha^2} - r_1^2(\beta^2 - \alpha^2) + \frac{C_1^2}{r_1^2(\beta^2 - \alpha^2)}
\right.
\]
\[
+ \frac{r_0^2 \alpha^2 w_0^2}{\beta^2 - \alpha^2} + \frac{D^2}{\beta^2 - \alpha^2} + \frac{\alpha^2 \nu^2}{\beta^2 - \alpha^2} \right] + \Lambda(r_0^2 - r_1^2 - 1)
\] (73)

---

8 We recall that in terms of the coordinates used in (10) \(Y_0 = \cosh \rho e^{i\theta}, \ Y_1 = \sinh \rho e^{i\theta}, \ X = e^{i\phi}\).
The Hamiltonian is

\[ H = -\frac{1}{2}r_0'(\beta^2 - \alpha^2) - \frac{C_0^2}{2r_0'(\beta^2 - \alpha^2)} - \frac{r_0^2\alpha^2w_0^2}{2(\beta^2 - \alpha^2)} + \frac{1}{2}r_1'(\beta^2 - \alpha^2) + \frac{C_1^2}{2r_1'(\beta^2 - \alpha^2)} \]

\[ + \frac{r_1^2\alpha^2w_1^2}{2(\beta^2 - \alpha^2)} + \frac{D^2}{2(\beta^2 - \alpha^2)} + \frac{\alpha^2\nu^2}{2(\beta^2 - \alpha^2)} \]

(74)

The first constraint (62) is just

\[-2H = 0 \]

(75)

The second constraint (63) gives the condition

\[ w_0C_0 + w_1C_1 + D\nu = 0 \]

(76)

In the special case of \( D = 0 \) to which we shall return below the constants \( C_0 \) and \( C_1 \) will have opposite signs as we will assume without loss of generality \( w_0, w_1 \) are positive (one may assume that \( C_1 \) is negative).

The conserved charges are

\[ E = T \int \frac{du}{\alpha} \left( -\frac{\beta C_0}{\beta^2 - \alpha^2} - \frac{\alpha^2\nu_0(1 + r_1^2)}{\beta^2 - \alpha^2} \right) \]

(77)

\[ S = T \int \frac{du}{\alpha} \left( \frac{\beta C_1}{\beta^2 - \alpha^2} - \frac{\alpha^2w_1r_1^2}{\beta^2 - \alpha^2} \right), \quad J = T \int \frac{du}{\alpha} \left( \frac{D\beta}{\beta^2 - \alpha^2} - \frac{\alpha^2\nu}{\beta^2 - \alpha^2} \right) \]

(78)

Using that \( r_0^2 - r_1^2 = 1 \) we get from (75) the equation for \( r_1 \)

\[ (\beta^2 - \alpha^2)r_1' = (1 + r_1^2) \left( \frac{C_0^2}{1 + r_1^2} + \alpha^2w_0^2(1 + r_1^2) - \frac{C_1^2}{r_1^2} - \alpha^2w_1^2r_1^2 - D^2 - \alpha^2\nu^2 \right) \]

(79)

Let us first look for solutions with two turning points \( (r_1' = 0) \) at some finite values of \( r_1 \) (the idea is that this should represent an ark of the spiky string). Considering (79) at large \( r_1 \), we observe that we need to satisfy the condition \( w_0^2 < w_1^2 \) in order for the string not to reach the boundary.

Let us express the equation (79) in terms of the variable \( v \)

\[ v = \frac{1}{1 + 2r_1^2} = \frac{1}{\cosh 2\rho}, \quad 0 \leq v \leq 1 \]

(80)

where we used that in terms of global \( AdS_3 \) coordinate \( \rho \) we have \( r_1 = \sinh \rho \). Then

\[ (\beta^2 - \alpha^2)^2v'^2 = 2v[4C_0^2v^2(1 - v) + \alpha^2w_0^2(1 - v)(1 + v)^2 - 4C_1^2v^2(1 + v) - \alpha^2w_1^2(1 - v)(1 + v) - 2(D^2 + \alpha^2\nu^2)v(1 - v^2)] \]

(81)

Without loss of generality one can set \( \alpha \) in \( u \) to any given value (\( \alpha \) can be absorbed into other parameters). In what follows we shall assume \( \alpha = 1 \). Then the equation (81) becomes

\[ v' = \frac{\sqrt{2vP(v)}}{1 - \beta^2} \]

(82)
where

\[ P(v) = v^3[-4C_0^2 - 4C_1^2 + 2(D^2 + \nu^2) - w_0^2 - w_1^2] + v^2(4C_0^2 - 4C_1^2 - w_0^2 + w_1^2) \\
+ v[w_0^2 + w_1^2 - 2(D^2 + \nu^2)] + w_0^2 - w_1^2 \\equiv [-4C_0^2 - 4C_1^2 + 2(D^2 + \nu^2) - w_0^2 - w_1^2](v - v_1)(v - v_2)(v - v_3) \] (83)

where \( v_n \) are three roots of \( P(v) = 0 \). To have a consistent string solution all roots should be real. We should also take into account the conditions (76), (71), (72) which may be used to eliminate some of the constants in terms of the other constants.

Let us assume that \( P(v) \) has two positive roots \( 0 \leq v_2 \leq v_3 \leq 1 \), and one negative \( v_1 \leq 0 \) (as we shall see below the constants \( C_0, C_1 \) can be always chosen so that this is true). The product of the roots is determined by the parameter

\[ a \equiv -4C_0^2 - 4C_1^2 + 2(D^2 + \nu^2) - w_0^2 - w_1^2 = \frac{w_1^2 - w_0^2}{v_1v_2v_3} \] (85)

We note that then \( a < 0 \) which means that between the two positive roots \( v_2, v_3 \) the polynomial \( P(v) \) is positive. This is our range of interest, meaning that for a physical solution with two turning points we have \( v_2 \leq v \leq v_3 \). The two physical constants that we are to fix are \( v_2, v_3 \). One can then find \( v_1, C_0, C_1 \) in terms of \( v_2, v_3 \). We can write the polynomial as

\[ P(v) = \frac{w_1^2 - w_0^2}{v_1v_2v_3}(v - v_1)(v - v_2)(v - v_3) = -8C_1^2\frac{(v - v_1)(v - v_2)(v - v_3)}{(1 - v_1)(1 - v_2)(1 - v_3)} \] (86)

where now it is understood that \( v_1 \) is not arbitrary but is a function of \( v_2, v_3 \):

\[ v_1 = -\frac{v_2v_3}{v_2 + v_3 + v_2v_3w_0^2 + w_1^2 - 2(\nu^2 + D^2)w_0^2 - w_1^2} \] (87)

The expressions for the constants \( C_0, C_1 \) in term of \( v_1, v_2, v_3 \) are

\[ C_0^2 = \frac{w_0^2 - w_1^2(1 + v_1)(1 + v_2)(1 + v_3)}{v_1v_2v_3}, \quad C_1^2 = \frac{w_0^2 - w_1^2(1 - v_1)(1 - v_2)(1 - v_3)}{v_1v_2v_3} \] (88)

We observe that for a solution satisfying \(-1 \leq v_1 \leq 0 \leq v_2 \leq v_3 \leq 1\), we have \( C_0^2 \geq 0 \) and \( C_1^2 \geq 0 \), i.e. our choice of roots of \( P(v) \) is indeed consistent.

To get solutions with \( n \) spikes we need to glue together a number of \( 2n \) pieces of integrals between a minimum \( (v_2) \) and a maximum \( (v_3) \). In other words, wherever it appears, the integral \( \int du \) is to be replaced by

\[ \int du = 2n \int_{v_2}^{v_3} \frac{dv}{v'} = \frac{2n(1 - \beta^2)}{\sqrt{-2a}}I_1 \] (89)

where \( I_1 \) is defined in Appendix. The winding number \( m \) in (77) becomes

\[ m = \frac{\beta \nu - D}{\pi \sqrt{-2a}}nI_1 \] (90)
Solving for $D$ and using (76) we obtain the equation for $\nu$

$$w_0 C_0 + w_1 C_1 + \nu (\beta \nu - \frac{\pi m \sqrt{-2a}}{n I_1}) = 0 \quad (91)$$

The condition (72) gives an additional relation between the constants, which allows to eliminate one of them, for example, $\beta$

$$2C_0 I_5 + w_0 \beta I_1 = 0 \quad (92)$$

where $I_5$ is defined in Appendix.

For solutions with $n$ spikes the conserved charges are

$$\frac{\pi \mathcal{E}}{n} = \frac{\beta C_0}{\sqrt{-2a}} I_1 + \frac{w_0}{2\sqrt{-2a}} I_3, \quad \frac{\pi \mathcal{S}}{n} = -\frac{\beta C_1}{\sqrt{-2a}} I_1 + \frac{w_1}{2\sqrt{-2a}} I_2, \quad \frac{\pi \mathcal{J}}{n} = \nu - \frac{\beta D}{\sqrt{-2a}} I_1 \quad (93)$$

where the integrals are defined in Appendix and can be written in terms of the elliptic integrals. Here $\mathcal{E} = 2\pi T \mathcal{E}, \quad \mathcal{S} = 2\pi T \mathcal{S}, \quad \mathcal{J} = 2\pi T \mathcal{J}$.

The cartesian coordinate $Y_1$ can be expressed as

$$Y_1 = \sinh \rho \, e^{i \theta}, \quad \theta = w_1 \tau + \int du \, \varphi'_1 \quad (94)$$

The number of spikes can be introduced via $\Delta \theta = \frac{2\pi}{2n}$ at fixed $t = w_0 \tau + \varphi_0(u)$

Here $\Delta \theta$ is the angle between a minimum (valley) and a maximum ($I_6$ is again defined in Appendix)

$$\Delta \theta = \int d\theta = \frac{1}{\beta^2 - \alpha^2} \int du \left( \frac{C_1}{r_1^2} + \frac{w_1}{w_0} \frac{C_0}{1 + r_1^2} \right) = -\frac{2}{\sqrt{-2a}} (C_1 I_6 + \frac{w_1}{w_0} C_0 I_5) \quad (96)$$

To see whether the spikes end in cusps or not we need to evaluate the derivative at the maximum value in $\rho$ or minimum value of $\nu = v_2$ with $t$ fixed

$$\left. \frac{d\rho}{d\theta} \right|_{u = v_2} = \frac{\rho' du}{w_1 d\tau + \varphi'_1 du} \bigg|_{u = v_2} \quad (97)$$

Using that for fixed $t$ we have $dt = w_0 d\tau + \varphi'_0 du = 0$ we get

$$\left. \frac{d\rho}{d\theta} \right|_{u = v_2} = \frac{\rho'}{\varphi'_1 - \frac{w_0}{w_1} \varphi'_0} \bigg|_{u = v_2} \quad (98)$$

Evaluating this expression we obtain

$$\left. \frac{d\rho}{d\theta} \right|_{u = v_2} = \frac{\sqrt{P(v)} \sqrt{1 - v^2}}{\sqrt{2v^3}} \left( \frac{w_0 w_1}{w_0^2 + w_1^2} \frac{1}{\frac{w_0^2 - w_1^2}{w_0^2 + w_1^2} - v} \right) \bigg|_{u = v_2} \quad (99)$$

17
We observe that while $P(v_2) = 0$, the denominator does not vanishes in general. For the particular case when $v_2 = \frac{w_1^2 - w_0^2}{w_0^2 + w_1^2}$ the denominator does vanish but this case corresponds to the situation when the motion is only in $AdS_5$ (see below). Thus in general when the string is moving or extended in $S^5$ the spikes at $v = v_2$ are rounded, i.e. they do not end in cusps.

To illustrate the rounding of spikes we set for simplicity $D = 0$ while keeping $m, \nu$ non-zero, and compute the angle $\theta$ in terms of $v_1, v_2, v_3$ at fixed $t$

$$\theta(v) = \frac{1}{2\sqrt{(1-v_1)(1-v_2)(1-v_3)}} \left[ (1+v_1)(1+v_2)(1+v_3)I_5(v_2,v) 
- (1-v_2)(1-v_2)(1-v_3)\right]$$

For $D = 0$ we have $w_0 C_0 + w_1 C_1 = 0$. The relevant equations simplify as

$$\frac{w_1^2}{w_0^2} = \frac{(1+v_1)(1+v_2)(1+v_3)}{(1-v_1)(1-v_2)(1-v_3)}$$

Using (101) in (87) we get

$$\frac{\nu}{w_0} = \sqrt{\frac{(v_1 + v_2)(v_1 + v_3)(v_2 + v_3)}{v_1 v_2 v_3 (v_1 - 1)(v_2 - 1)(v_3 - 1)}}$$

Solving for $\beta$ in (92) and using this in (90) gives

$$m = -\frac{\nu}{2\pi w_0} n I_5 \sqrt{(1+v_1)(1+v_2)(1+v_3)}$$

The condition $w_1^2 > w_0^2$ along with (101) implies

$$v_1 + v_2 + v_3 + v_1 v_2 v_3 \geq 0, \quad v_1 \geq -\frac{v_2 + v_3}{1 + v_2 v_3}$$

The requirement (104) as well as the condition coming from the positivity of the square root in (102) imply certain ranges for the parameters $v_1, v_2, v_3$.

There are two possible regimes. The first one is represented by $\frac{v_1 + v_3}{1 + v_2 v_3} \geq |v_1| \geq v_3$. This corresponds to spikes at maximal values of $\rho$. In contrast with the case of “true” spikes when the string motion is in $AdS_5$ only \cite{10} here for nonzero $J$ the spikes are rounded. A typical plot in this sector is shown in figure 3. The other regime is with $|v_1| \leq v_2$ and corresponds to spikes at the minimum values of $\rho$, i.e spikes in the interior. For the string moving only in $AdS_5$ this solution was found in \cite{24}. Here again the spikes are rounded due to the presence of $J$. A typical plot is presented in figure 4.

Returning to the general case with $D \neq 0$ we observe that the solution is parametrized by only four independent parameters. For example, all the charges $E, S, J, m$ can be expressed in term of the parameters $\frac{w_1}{w_0}, \frac{\nu}{w_0}, \frac{D}{w_0}, n, v_1, v_2, v_3$. Furthermore, the ratios $\frac{w_1}{w_0}, \frac{\nu}{w_0}, \frac{D}{w_0}$ can be written in terms of $n, v_1, v_2, v_3$ using (85), (76), (96); alternatively, we may use the set of

\footnote{Note that the overall sign in this equation is not relevant since $m$ should be an integer number.}
FIG. 3: Solution in polar coordinates $(\rho, \theta)$ for $n = 3$ spikes for $v_1 = -0.8698$, $v_2 = 0.04$, $v_3 = 0.865$.

The shape of the string near maximal value of the radial coordinate is actually rounded.

$n, \frac{w_1}{w_0}, v_2, v_3$. The explicit expressions are not very illuminating and we will not present them here. Given that there are four independent parameters the energy can be written as

$$E = E(S, J, n, m) .$$  \hspace{1cm} (105)

Without making any further assumptions this expression is rather complicated due to the complicated way the roots $v_1, v_2, v_3$ enter the equations: as in other similar cases \[11, 29\] one is to solve a system of parametric equations containing elliptic integrals.

V. SPIKY STRINGS IN $AdS_5$ WITH ANGULAR MOMENTUM IN $S^5$:

SPECIAL CASES

Below we shall look at few particular cases of the general solution of the previous section. First, we shall demonstrate how to reproduce the original spiky string solution \[10\] of section 2 in the limit when the string is not moving or stretched in $S^5$. Another special case is the fast string limit of large $J$ with $\frac{S}{J}$ fixed in which one recovers the familiar BMN-type \[18\] scaling of the spinning string energy.

Finally, we shall consider the limit in which the spikes move close to the boundary of $AdS_5$. This case corresponds to a generalization of the “$pp$-wave limit” discussed in section
FIG. 4: Solution in polar coordinates ($\rho, \theta$) for $n = 3$ spikes for $v_1 = -0.02, v_2 = 0.04, v_3 = 0.4957$.

The shape of the string near minimal value of the radial coordinate is actually rounded.

2 for the spiky string in $AdS_3$.

A. No motion or stretching in $S^5$

This case corresponds to $m = \nu = 0$. The winding condition (71) then implies $D = 0$. The constraint (76) gives $w_0 C_0 + w_1 C_1 = 0$ which can be solved as

$$C_0 = w_1 f, \quad C_1 = -w_0 f$$  \hspace{1cm} (106)

The polynomial $P(v)$ becomes

$$P(v) = -v^3(1 + 4f^2)(w_0^2 + w_1^2) + v^2(w_1^2 - w_0^2)(1 + 4f^2) + v(w_0^2 + w_1^2) + w_0^2 - w_1^2$$  \hspace{1cm} (107)

with the relevant choice for the roots being ($v_2 \leq v_3$)

$$v_1 = -\frac{1}{\sqrt{1 + 4f^2}}, \quad v_2 = \frac{w_1^2 - w_0^2}{w_0^2 + w_1^2}, \quad v_3 = \frac{1}{\sqrt{1 + 4f^2}}$$  \hspace{1cm} (108)

Interchanging $v_2, v_3$ corresponds to another branch for which the spikes are at the minimal value of $\rho$. In the following we focus on the the branch (108) corresponding to the spikes at the maximal value of $\rho$. 

20
From the condition (99) we observe that in this case we indeed have spikes at $v = v_2$ since $\frac{\partial \mathcal{E}}{\partial n} |_{v=v_2}$ blows up, as expected for the spiky string moving only in $AdS_5$ [10].

Using the condition (92) to eliminate the parameter $\beta$

$$\beta = -\frac{2w_1 f I_5}{w_0 I_1}$$

the conserved charges can be written as

$$\mathcal{E} = \frac{n}{2\pi} \frac{-4w_1^2 f^2 I_5 + w_0^2 I_3}{w_0 \sqrt{2(1 + 4f^2)(w_0^2 + w_1^2)}} , \quad S = \frac{n}{2\pi} \frac{-4w_1 f^2 I_5 + w_1 I_2}{\sqrt{2(1 + 4f^2)(w_0^2 + w_1^2)}}$$

$$\frac{\pi}{n} = \frac{\sqrt{2} f (w_1^2 I_5 - w_0^2 I_6)}{w_0 \sqrt{(1 + 4f^2)(w_0^2 + w_1^2)}}$$

In the arguments above integrals $I_r$ (see Appendix) one should use $v_k$ given by the solution (108).

We end up with two independent parameters $\frac{w_0}{w_1}, f$. This is precisely what one needs to express the energy as $\mathcal{E} = \mathcal{E}(S, n)$. While solving for $\frac{w_0}{w_1}$ and $f$ in terms of $S$ and $n$ analytically is not possible in general, one can solve the parametric equations perturbatively, e.g., for large $S$.

The large $S$ limit corresponds to the small $v_2$ limit. This means $w_1 \rightarrow w_0$. In this limit the integrals $I_5, I_6$ are finite. The number of spikes in this limit can be written as

$$\frac{\pi}{n} = \frac{f}{\sqrt{1 + 4f^2}} (I_5 - I_6)$$

The charges scale as

$$\mathcal{E} - S \approx -\frac{n}{2\pi} \ln v_2 , \quad S \approx \frac{n}{2\pi v_2} , \quad \mathcal{E} - S \approx \frac{n}{2\pi} \ln \frac{S}{n}$$

i.e. in this limit we obtain the expected result

$$E - S = \sqrt{\lambda} \frac{2\pi}{2\pi} n \ln \left( \frac{2\pi}{\sqrt{\lambda} n} \right) + ...$$

with $n = 2$ corresponding to the folded string case.

**B. Fast string limit**

Next, let us consider the fast string limit, i.e. the limit of large $J$ with $\frac{S}{J}$ fixed. In this limit the string energy is expected to match the energy obtained from the corresponding solution in the Landau-Lifshitz model [22] (see next section). This limit corresponds to taking $w_0, w_1, \nu$ being large while the other parameters staying finite, with, e.g., the parameters $v_1, v_2, v_3$ being arbitrary. A particular situation for which we found the direct analog in the
LL model is for $D = 0$. In this case the constraint \((76)\) gives $w_0 C_0 + w_1 C_1 = 0$, i.e. at the leading order $C_0 = -C_1$. Plugging this into \((88)\) we obtain the relation between the $v_1, v_2, v_3$

$$v_1 = -\frac{v_2 + v_3}{1 + v_2 v_3}.$$  \hfill (115)

For the spin $S$ and winding number $m$ we find

$$S = J \frac{I_2}{2 I_1}, \quad m = \frac{n}{2 \pi} \sqrt{1 + v_1 v_2 + v_1 v_3 + v_2 v_3} I_5$$  \hfill (116)

To compute $E - S - J$ we need subleading corrections in $w_0, w_1, \nu$ so we set

$$w_0 = w + \delta w_0, \quad w_1 = w + \delta w_1, \quad \nu = w, \quad w \gg 1 \hfill (117)$$

Using \((85)\) we find for $\delta w_0, \delta w_1$

$$2w(\delta w_0 + \delta w_1) = \frac{w_1^2 - w_0^2}{v_1 v_2 v_3}(v_1 v_2 + v_1 v_3 + v_2 v_3), \quad 2w(\delta w_0 - \delta w_1) = w_0^2 - w_1^2 \hfill (118)$$

Therefore in the fast string limit the parameters $a, C_0$ and all integrals finite.\hfill (121) Then the energy for the $n$-spike solution can be written as

$$\frac{E - S - J}{4\pi n} \approx \frac{\beta C_0 + \beta C_1}{\sqrt{-2a}} I_1 + \frac{w_0 I_3 - w_1 I_2 - 2 \nu I_1}{2 \sqrt{-2a}}$$  \hfill (119)

i.e. as

$$E - S - J \approx \frac{n^2 \lambda}{8\pi^2 J} I_1 \left[ v_1 v_2 v_3 I_+ - (v_1 v_2 + v_1 v_3 + v_2 v_3) I_1 \right]$$  \hfill (120)

where $v_k$ are related to $S/J$ and $m$ via \((115), (116)\). This generalizes a similar expression for the $(S,J)$ folded string \([2]\). Eq. \((120)\) will be reproduced from the Landau-Lifshitz model in the next section.

C. \textit{pp-wave limit}

To obtain the solution in the "pp-wave limit" $n \to \infty$ we are to consider the scaling $(k = 1, 2, 3)$

$$v_k \to \epsilon^2 v_k, \quad \epsilon \to 0 \hfill (121)$$

It turns out that in this scaling limit $\nu$ and $D$ need to be kept fixed, while the constants $C_0, C_1$ scale as

$$C_0 = c_0 \left( \frac{1}{\epsilon^2} + \frac{v_1 + v_2 + v_3}{2} + ... \right), \quad C_1 = -c_0 \left( \frac{1}{\epsilon^2} - \frac{v_1 + v_2 + v_3}{2} + ... \right) \hfill (122)$$

\hfill 10 In the limit under consideration only the leading term in the expansion of $C_1 + C_0$ is relevant. However, $C_0 + C_1 = 0$ for $D = 0$. 

22
where $c_0$ is finite, and $v_k$ here are now finite. We also want $w_0, w_1$ to scale as

$$\frac{w_0 - w_1}{w_0} = r \epsilon^2$$  \hspace{1cm} (123)

where $r$ is finite. Using that $w_0^2 - w_1^2 = 8 \epsilon^2 c_0^2 v_1 v_2 v_3$ we find that $r = \frac{4 \epsilon^2}{w_0^2} v_1 v_2 v_3$. Other useful relations are

$$2(D^2 + \nu^2) - w_0^2 - w_1^2 = 8 \epsilon^2 c_0^2 (v_1 v_2 + v_1 v_3 + v_2 v_3), \quad a = -\frac{8 \epsilon^2 c_0^2}{\epsilon^4}$$  \hspace{1cm} (124)

The constraint (76) becomes, at the leading order in $\epsilon$,

$$c_0 r + c_0(v_1 + v_2 + v_3) + \frac{\nu}{w_0} D = 0$$  \hspace{1cm} (125)

and can be used to eliminate one constant. Using the expressions in Appendix one concludes that under (121) the integrals scale as

$$I_1 \rightarrow \frac{1}{\epsilon^2} I_1, \quad I_2 \rightarrow \frac{1}{\epsilon^4} I_+, \quad I_3 \rightarrow \frac{1}{\epsilon^4} I_+, \quad I_5 \rightarrow I_4, \quad I_6 \rightarrow I_4$$  \hspace{1cm} (126)

where $I_+ = \frac{1}{2}(I_2 + I_3)$. The constraint (92) that determines $\beta$ becomes

$$2c_0 I_4 + w_0 \beta I_1 = 0$$  \hspace{1cm} (127)

The equation for $v$ (82) retains its form in this limit

$$v' = \frac{\sqrt{2v P(v)}}{1 - \beta^2}, \quad P(v) = -8 \epsilon^2 c_0^2 (v - v_1)(v - v_2)(v - v_3)$$  \hspace{1cm} (128)

The cubic polynomial $P(v)$ (83) can be written also as

$$P(v) = v^3(-4C_0^2 - 4C_1^2) + v^2(4C_0^2 - 4C_1^2) + v[w_0^2 + w_1^2 - 2(D^2 + \nu^2)] + w_0^2 - w_1^2$$  \hspace{1cm} (129)

Introducing the parameters

$$w_\pm = \frac{w_1 \pm w_0}{2}, \quad C_\mp = \frac{C_1 \pm C_0}{2}$$  \hspace{1cm} (130)

we find that they scale as

$$C_- \sim O(\epsilon^0), \quad C_+ \sim \frac{1}{\epsilon^2}, \quad w_- \sim \epsilon^2, \quad w_+ \sim O(\epsilon^0)$$  \hspace{1cm} (131)

Then the scaling limit of $P(v)$ takes the form

$$P(v) = -2v^3 C_+^2 - 4C_- C_+ v^2 + 2(w_+^2 - \nu^2 - D^2)v - 4w_+ w_-$$  \hspace{1cm} (132)

The number of spikes $n$ and the winding number $m$ are given by

$$n = \frac{\pi w_0^2}{2c_0^2 v_1 v_2 v_3 I_4} \frac{1}{\epsilon^2}, \quad m = -\frac{2c_0 \nu I_4 + D w_0 I_1}{4 \pi c_0 w_0}$$  \hspace{1cm} (133)
We see that in this scaling limit $n$ grows to infinity while $\frac{m}{n}$ stays finite.

The scaling limit (121) leads also to the following expressions for the conserved charges

$$\frac{\mathcal{E}}{4\pi n} = \left( \frac{w_0 I_+ - c_0}{2 w_0 I_4} \right) \frac{1}{\epsilon^2}, \quad \frac{\mathcal{S}}{4\pi n} = \left( \frac{w_1 I_+ - c_0}{8 c_0} \right) \frac{1}{\epsilon^2}, \quad \frac{\mathcal{J}}{4\pi n} = \frac{w_0 I_1 + 2 D c_0 I_4}{4 c_0 w_0}$$

Thus the ratio $\frac{\mathcal{J}}{n}$ remains finite while $\frac{\mathcal{E}}{n}$ and $\frac{\mathcal{S}}{n}$ diverge in this limit.

We then find also that

$$\frac{\mathcal{E} - \mathcal{S}}{2n} = \frac{c_0}{2 w_0} v_1 v_2 v_3 I_+, \quad \frac{\mathcal{E} + \mathcal{S}}{2n} = \left( \frac{w_0 + w_1}{8 c_0} \right) \frac{1}{\epsilon^2} \sim n$$

We conclude that in this limit

$$\mathcal{S} \gg \mathcal{J} \sim n \gg 1$$

and

$$\frac{\mathcal{E} + \mathcal{S}}{n^2} \sim \frac{\mathcal{E} - \mathcal{S}}{n} \sim \frac{\mathcal{J}}{n} \sim \frac{m}{n} = \text{finite}$$

Let us note that a property of the scaling limit we discussed above is that the number of independent parameters gets reduced by one. Namely, in this $pp$-wave limit there remain only three independent parameters. To see this let us perform the following rescaling

$$v_i \to c v_i, \quad C_+ \to c^{-1} C_+, \quad w_- \to c w_-$$

with all other parameters kept fixed. Under this rescaling the charges and the polynomial (132) remain unchanged, so that all the quantities depend only on the combination $C_+ w_-$ and not separately on $C_+$ and $w_-$. 

VI. FAST SPIKY STRING FROM $SL(2, R)$ LANDAU-LIFSHITZ MODEL

As shown in [21, 22] following [19, 20], taking a large $S^5$ orbital momentum limit of the classical string action one can truncate it to a non-relativistic Landau-Lifshitz (LL) action that should thus be describing the fast-moving string solutions to leading order in expansion in large $J$. The same LL action happens to arise also from the 1-loop dilatation operator in the $SL(2)$ sector of the SYM theory when one restricts consideration to certain “locally-BPS” coherent states present in the thermodynamic limit of large spin chain length $J$. In this limit (which is a generalisation of the BMN limit [17]) the one-loop gauge theory result for the energy happens to match the leading-order term in the fast-string energy [18, 30, 32].

The leading-order fast-string expressions obtained in the previous section should thus follow also from a particular solution of the LL model [22]

$$L = \hat{\eta} \sinh^2 \rho - \frac{1}{2} \left( \rho'^2 + \frac{1}{2} \sinh 2\rho \, \eta'^2 \right)$$

\text{[11] Here we made a rescaling by a factor of $J = \sqrt{\lambda}$ which will be restored later in the expressions for conserved charges.}
We shall use the ansatz
\[ \eta = \omega \tau + f(\sigma), \quad \rho = \rho(\sigma) \] (139)
leading to
\[ \partial_\sigma (f' \sinh^2 \rho \cosh^2 \rho) = 0, \quad \rho'' = \frac{1}{2} \sinh 2 \rho \quad (f'^2 \cosh 2 \rho - 2 \omega) \] (140)
Then
\[ f' = \frac{4C_2}{\sinh^2 2\rho}, \quad \rho' = \sqrt{2C_3 - \omega \cosh 2 \rho - 4C_2^2 \coth^2 2 \rho} \] (141)
where \( C_2, C_3 \) are integration constants. Equivalently, the equation for \( \rho \) can be written as
\[ v' = \sqrt{2vP(v)}, \quad v = \frac{1}{\cosh 2\rho} \] (142)
\[ P(v) = -4C_3v^3 + 2\omega v^2 + 4(C_3 - 2C_2^2)v - 2\omega = -4C_3(v - v_1)(v - v_2)(v - v_3) \] (143)
Now we can follow the same argument as in the previous section. Namely, we assume that \( P(v) \) has two positive roots \( 0 < v_2 < v_3 \), and \( v_1 < 0 \). The product of the roots is related to
\[ b \equiv -2C_3 = \frac{\omega}{v_1v_2v_3} \] (144)
Again, we have \( b < 0 \), so that for a physical solution \( v_2 < v \leq v_3 \). One can find \( v_1, C_2, C_3 \) in terms of \( v_2, v_3 \)
\[ C_2^2 = -\omega (1 - v_2^3)(1 - v_3^2), \quad C_3 = -\frac{\omega}{2v_1v_2v_3}, \quad v_1 = \frac{v_2 + v_3}{1 + v_2v_3} \] (145)
The number of spikes \( n \) is determined by
\[ 2\pi = \Delta \sigma = \int_0^{2\pi} d\sigma = \int \frac{d\rho}{\rho'} = 2n \int_{v_2}^{v_3} dv \frac{d\rho}{d\rho} \frac{1}{\sqrt{vP(v)}} = \frac{n}{\sqrt{-b}} I_1 \] (146)
where \( I_1 \) is given in \([179]\). The spin and energy of this solution are given by
\[ S = J \int_0^{2\pi} d\sigma \frac{\partial L}{\partial \dot{\sigma}} = \frac{2n}{2\pi} J \int_{v_2}^{v_3} dv \frac{\partial L}{\partial \dot{\sigma}} = \frac{nI_2}{4\pi \sqrt{-b}} J \] (147)
\[ E - S - J = \frac{\lambda}{J} \int_0^{2\pi} d\sigma \left( \dot{\eta} \frac{\partial L}{\partial \eta} + \dot{\rho} \frac{\partial L}{\partial \rho} - L \right) = \frac{2n}{2\pi} J \int_{v_2}^{v_3} dv \frac{d\rho}{dv} \left( \dot{\eta} \frac{\partial L}{\partial \eta} + \dot{\rho} \frac{\partial L}{\partial \rho} - L \right) \]
\[ = \frac{n\lambda}{4\pi J} \sqrt{-b} \left[ v_1v_2v_3I_+ - (v_1v_2 + v_2v_3 + v_3v_1)I_1 \right] \] (148)
where \( I_+ = \frac{1}{2}(I_2 + I_3) \), and \( I_1, I_2, I_3 \) are again defined in \([179]\).
The winding number is determined from an additional constraint given in \([22]\)
\[ m = \int_0^{2\pi} d\sigma \quad iV^* \partial_\sigma V + O\left( \frac{1}{J^2} \right) \]
\[ = \frac{n}{\pi} \int_{v_2}^{v_3} dv \frac{d\rho}{dv} \frac{1}{\sqrt{vP(v)}} iV^* \partial_\sigma V + O\left( \frac{1}{J^2} \right) \]
\[ = \frac{n}{2\pi} \sqrt{1 + v_1v_2 + v_2v_3 + v_3v_1} I_5 \] (149)
where $V^r V^r = V_0^2 - |V_1|^2 = 1$ with $V_0 = \cosh \rho$, $V_1 = \sinh \rho \ e^{i\eta}$.

Eliminating $\sqrt{\lambda}$ using (146) we thus end up with

\[
\frac{S}{J} = \frac{I_2}{2I_1},
\]

\[
E - S - J = \frac{n^2 \lambda}{8\pi^2 J} I_1 \left[ v_1 v_2 v_3 I_1 - (v_1 v_2 + v_2 v_3 + v_3 v_1) I_1 \right],
\]

\[
m = \frac{n}{2\pi} \sqrt{1 + v_1 v_2 + v_2 v_3 + v_3 v_1} I_5
\]

Below we shall look at several special cases of these expressions.

**A. Folded string: $n = 2$**

The 2-spike case corresponds to

\[
v_1 = -1, \quad v_3 = 1, \quad v_2 = \text{arbitrary}
\]

Then

\[
\frac{S}{J} = \frac{1}{2} \left( 1 + \frac{1}{v_2} \right) E[q] - 1, \quad q \equiv \sqrt{1 - v_2 \over 1 + v_2}
\]

\[
E - S - J = \frac{4\lambda}{2\pi^2 J} K[q] (K[q] - E[q]),
\]

\[
m = 0
\]

Here

\[
v_2 = {1 \over \cosh 2\rho_2} = {1 \over 1 - 2x_0}, \quad x_0 \equiv -\sinh^2 \rho_2
\]

so that we can transform the elliptic functions as\(^{12}\)

\[
K[\sqrt{x_0}] = \frac{1}{\sqrt{1 - x_0}} K[q], \quad E[\sqrt{x_0}] = \sqrt{1 - x_0} E[q]
\]

Then we finish with

\[
\frac{S}{J} = \frac{E[\sqrt{x_0}]}{K[\sqrt{x_0}]} - 1,
\]

\[
E - S - J = -\frac{2\lambda}{\pi^2 J} K[\sqrt{x_0}] \left( E[\sqrt{x_0}] - (1 - x_0) K[\sqrt{x_0}] \right)
\]

which are the same as equations (B.18) and (B.19) in \[^32\].\(^ {13}\)

\(^{12}\) We use that \[^26\]: $K[i k'] = k' K[k]$, $E[i k'] = E[k]$ where $k' = \sqrt{1 - k'^2}$.

\(^{13}\) Note that ref. \[^32\] used a different definition of elliptic functions: here $K[\sqrt{x_0}]$ is the same as $K(x_0)$ in \[^32\], etc.

12 We use that $K[i k'] = k' K[k]$, $E[i k'] = E[k]$ where $k' = \sqrt{1 - k'^2}$.

13 Note that ref. $[32]$ used a different definition of elliptic functions: here $K[\sqrt{x_0}]$ is the same as $K(x_0)$ in $[32]$, etc.
B. Near-boundary, fixed number of spikes: “long-string” limit

Next, let us look at another particular limit:

\[ v_1 = -v_3, \quad v_2 \to 0 \]

In this case

\[
\frac{S}{J} \simeq \frac{1}{v_2 \ln \frac{8\pi v_3}{v_2}} + O(1),
\]

\[
E - S - J \simeq \frac{n^2 \lambda}{8\pi^2 J} \ln^2 v_2 + O(\ln v_2),
\]

\[
m \simeq \frac{n}{2\pi} \arccos v_3
\]

where we used that for \( v_1 = -(v_2 + v_3)/(1 + v_2 v_3) \) one has

\[
I_5 \simeq \frac{\arccos v_3}{\sqrt{1 - x_3^2}} \quad \text{as} \quad v_2 \to 0
\]

Solving for \( v_2 \) we finish with

\[
E - S - J \simeq \frac{n^2 \lambda}{8\pi^2 J} \ln^2 \frac{S}{J} + O\left(\ln \frac{S}{J}\right).
\]

For \( n = 2 \) this reproduces eq.(1.5) in [3] for the corresponding asymptotics of the energy of the long fast-moving folded string.

C. \( n \to \infty \) limit

To obtain the solution in the analog of the \( pp \)-wave limit let us perform the same rescaling as in the previous section. Again, \( \omega \) is fixed while

\[
b \to \frac{1}{\epsilon_0} b, \quad C_2 \to \frac{1}{\epsilon_3} C_2, \quad C_3 \to \frac{1}{\epsilon_6} C_3, \quad v_1 = -(v_2 + v_3)
\]

The number of the spikes then scales as (cf.(126))

\[
n \to \frac{1}{\epsilon n}
\]

As a result, we find

\[
\bar{m} \equiv \frac{m}{n} = \frac{I_4}{2\pi},
\]

\[
\bar{\gamma} \equiv \frac{E - S - J}{n} = \frac{\lambda}{8\pi^2 J} I_1 \left[ v_1 v_2 v_3 I_+ - (v_1 v_2 + v_2 v_3 + v_3 v_1) I_1 \right]
\]
where $\bar{J} = J/n$, and we used (126) and that $I_+ \to \frac{1}{\lambda} I_+$. We see that $\bar{m}$ and $\bar{\gamma}$ are invariant under the rescaling $v_2 \to c v_2$, $v_3 \to c v_3$, i.e. they depend only on $v_2/v_3$. This means we can express $\bar{\gamma}$ as

$$\bar{\gamma} = \frac{\lambda}{8\pi^2 \bar{J}} f(\bar{m})$$

where $f(\bar{m})$ can, in principle, be computed from (167),(168).

Since the LL model describes also a certain class of coherent gauge-theory states in the thermodynamic limit of the 1-loop gauge-theory $SL(2)$ spin chain, the above expression should represent the 1-loop anomalous dimension of the corresponding “long” “locally-BPS” gauge-theory operator.\(^{14}\)

VII. PERIODIC SPIKES IN ADS–PP-WAVE $\times S^1$ BACKGROUND

Let us now discuss a generalization of the periodic spike solution from section 2 to the case of non-trivial motion in $S^5$.

After taking the limit discussed in [14] the metric becomes an AdS–pp-wave times original $S^5$ (the limit did not affect the 5-sphere). Let us consider the solution in the following 4-dimensional subspace of this limiting space with metric

$$ds^2 = \frac{1}{z^2} (dz^2 + 2dx_+dx_- - \mu^2 z^2 dx_+^2) + d\alpha^2$$

where $\alpha$ is an angle of period $2\phi$ parameterizing a maximal circle $S^1 \subset S^5$. It is straightforward to write down the equations of motion and constraints in conformal gauge for a string moving in such background. To solve them, we propose, by analogy with the discussion in the previous sections, the following ansatz

$$x_\pm = \omega_\pm \tau + \phi_\pm(u), \quad \alpha = \omega \tau + \phi_\alpha(u), \quad z = z(u), \quad u \equiv \sigma + \beta \tau$$

Such solution represents a rigid string that moves along the direction $x = \sqrt{2}(x_+ + x_-)$. In addition, it wraps the $S^1$ as well as moves along it. The equations of motion for $x_\pm$ and $\alpha$ can now be easily integrated to

$$\phi'_\alpha = C_\alpha, \quad \phi'_+ = \frac{1}{1 - \beta^2} \left( \beta \omega_+ + C_+ z^2 \right), \quad \phi'_- = \frac{1}{1 - \beta^2} \left( \beta \omega_- + C_- z^2 + \mu^2 C_+ z^4 \right)$$

where $C_\alpha, C_\pm$ are constants of integration. The conformal gauge constraints are

$$0 = C_- \omega_+ + C_+ \omega_- + \omega D$$

$$\left(1 - \beta^2\right) z'^2 = -\mu^2 C_+ z^6 - 2C_+ C_- z^4 + (\mu^2 \omega^2_+ - \omega^3 - D^2) z^2 - 2\omega_+ \omega_-$$

\(^{14}\) Let us note that the large $n$ limit considered in this subsection is not exactly the same as the pp-wave limit considered above. The assumption made in deriving the LL model is that $J \gg 1$ with $\frac{\hat{J}}{\bar{J}}$ (and $m$) kept fixed, while in the pp-wave limit we had $J \sim n \gg 1$. In general, the fast-string or LL limit and the pp-wave limit do not commute. As a consequence, we cannot set $m = 0$ in the above expression.\(^{169}\)
where, for simplicity, we introduced a new constant $D$ through

$$C_\alpha = \frac{\beta \omega - D}{1 - \beta^2} \quad (175)$$

Introducing the variable $v = z^2$ we can rewrite the second constraint as

$$v' = \frac{\sqrt{2vP(v)}}{1 - \beta^2} \quad (176)$$

$$P(v) = -2\mu^2 C_+ v^3 - 4C_+ C_- v^2 + 2(\mu^2 \omega_+^2 - \omega^2 - D^2)v - 4\omega_+ \omega_- \quad (177)$$

This equation determines the shape of the string $z(u)$ (given which we can then obtain $x_\pm(\sigma, \tau)$). The same equation can be found by taking the limit discussed in section 5C (cf. eq. (132)). Here we rederived this result as a check and also to introduce the ansatz (171) which can be useful to obtain other solutions.

VIII. CONCLUSIONS

In this paper we have analyzed a large class of string solutions in $AdS_3 \times S^1$. We focused in particular on the spiky string solutions in $AdS_3 \times S^1$ and their limits. We found that because of the motion in $S^1 \subset S^5$ the spikes no longer end on cusps.

In the limit of fast spiky string we matched its energy to that of the corresponding solution in the $SL(2, R)$ Landau-Lifshitz model.

Another limit that we considered was the “pp-wave” limit in which the number of spikes $n$ goes to infinity while other conserved quantities scale in certain ways with $n$.

On the gauge theory side this limit corresponds to a particular thermodynamic limit of the $SL(2)$ spin chain. The classical string energy that we found represents a strong-coupling prediction for the corresponding gauge theory anomalous dimension. The implications of this limit deserve further study.

Acknowledgments

We are grateful to P. Argyres, N. Dorey and L. Pando Zayas for useful discussions. M.K. and A.T. were supported in part by NSF under grant PHY-0847322. The work of R.I. was supported in part by the Purdue Research Foundation.

Appendix: Some useful integrals

Here we summarize various integrals that we used above

$$I_1 = \int_{v_2}^{v_3} dv \frac{1}{\sqrt{-v(v-v_1)(v-v_2)(v-v_3)}}, \quad I_2 = \int_{v_2}^{v_3} dv \frac{1 - v}{v \sqrt{-v(v-v_1)(v-v_2)(v-v_3)}}$$
\[ I_3 = \int_{v_2}^{v_3} dv \frac{1 + v}{v \sqrt{-v(v-v_1)(v-v_2)(v-v_3)}} \]  

These integrals can be written in terms of the elliptic integrals

\[ I_1 = \frac{2}{\sqrt{v_3(v_2-v_1)}} K[\sqrt{s}], \quad I_2 = -\frac{2}{v_1 v_2} \sqrt{\frac{v_2-v_1}{v_3}} E[\sqrt{s}] + \frac{2(\frac{1}{m} - 1)}{\sqrt{v_3(v_2-v_1)}} K[\sqrt{s}] \]

\[ I_3 = -\frac{2}{v_1 v_2} \sqrt{\frac{v_2-v_1}{v_3}} E[\sqrt{s}] + \frac{2(\frac{1}{m} + 1)}{\sqrt{v_3(v_2-v_1)}} K[\sqrt{s}] \]

where

\[ s = \frac{v_1(v_2-v_3)}{v_3(v_2-v_1)} \]

Other integrals we used in this paper are

\[ I_4(v_2, v) = \int_{v_2}^{v} dv \frac{v}{\sqrt{-v(v-v_1)(v-v_2)(v-v_3)}} \]

\[ I_5(v_2, v) = \int_{v_2}^{v} dv \frac{v}{(1+v) \sqrt{-v(v-v_1)(v-v_2)(v-v_3)}} \]

\[ I_6(v_2, v) = \int_{v_2}^{v} dv \frac{v}{(1-v) \sqrt{-v(v-v_1)(v-v_2)(v-v_3)}} \]

For particular \( v = v_3 \) these integrals can be written as

\[ I_4 \equiv I_4(v_2, v_3) = \int_{v_2}^{v_3} dv \frac{v}{\sqrt{-v(v-v_1)(v-v_2)(v-v_3)}} \]

\[ = \frac{2v_2}{\sqrt{v_3(v_2-v_1)}} \Pi\left[\frac{v_3-v_2}{v_3}, \sqrt{s}\right] \]

\[ I_5 \equiv I_5(v_2, v_3) = \int_{v_2}^{v_3} dv \frac{v}{(1+v) \sqrt{-v(v-v_1)(v-v_2)(v-v_3)}} \]

\[ = \frac{2v_2}{(1+v_2) \sqrt{v_3(v_2-v_1)} \Pi\left[\frac{v_3-v_2}{v_3(1+v_2)}, \sqrt{s}\right]} \]

\[ I_6 \equiv I_6(v_2, v_3) = \int_{v_2}^{v_3} dv \frac{v}{(1-v) \sqrt{-v(v-v_1)(v-v_2)(v-v_3)}} \]

\[ = \frac{2v_2}{(1-v_2) \sqrt{v_3(v_2-v_1)} \Pi\left[\frac{v_3-v_2}{v_3(1-v_2)}, \sqrt{s}\right]} \]
1. S. S. Gubser, I. R. Klebanov and A. M. Polyakov, “A semi-classical limit of the gauge/string correspondence,” Nucl. Phys. B 636, 99 (2002) [hep-th/0204051].

2. S. Frolov and A. A. Tseytlin, “Semiclassical quantization of rotating superstring in AdS(5) x S(5),” JHEP 0206, 007 (2002) [hep-th/0204226].

3. S. Frolov, A. Tirziu and A. A. Tseytlin, “Logarithmic corrections to higher twist scaling at strong coupling from AdS/CFT,” Nucl. Phys. B 766, 232 (2007) arXiv:hep-th/0611269.

4. R. Roiban, A. Tirziu and A. A. Tseytlin, “Two-loop world-sheet corrections in AdS5 x S5 superstring,” JHEP 0707, 056 (2007) arXiv:0704.3638.

5. R. Roiban and A. A. Tseytlin, “Strong-coupling expansion of cusp anomaly from quantum superstring,” JHEP 0711, 016 (2007) arXiv:0709.0618.

6. N. Beisert, B. Eden and M. Staudacher, “Transcendentality and crossing,” J. Stat. Mech. 0701, P021 (2007) arXiv:hep-th/0610251.

7. M. K. Benna, S. Benvenuti, I. R. Klebanov and A. Scardicchio, “A test of the AdS/CFT correspondence using high-spin operators,” Phys. Rev. Lett. 98, 131603 (2007) arXiv:hep-th/0611135.

8. B. Basso, G. P. Korchemsky and J. Kotanski, “Cusp anomalous dimension in maximally supersymmetric Yang-Mills theory at strong coupling,” Phys. Rev. Lett. 100, 091601 (2008) arXiv:0708.3933.

9. P. Y. Casteill and C. Kristjansen, “The Strong Coupling Limit of the Scaling Function from the Quantum String Bethe Ansatz,” Nucl. Phys. B 785, 1 (2007) arXiv:0705.0890.

10. M. Kruczenski, “Spiky strings and single trace operators in gauge theories,” JHEP 0508, 014 (2005) arXiv:hep-th/0410226.

11. M. Kruczenski, J. Russo and A. A. Tseytlin, “Spiky strings and giant magnons on S5,” JHEP 0610, 002 (2006) arXiv:hep-th/0607044.

12. A. V. Belitsky, A. S. Gorsky and G. P. Korchemsky, “Gauge / string duality for QCD confor-
mal operators,” Nucl. Phys. B 667, 3 (2003) [arXiv:hep-th/0304028].
A. V. Belitsky, A. S. Gorsky and G. P. Korchemsky, “Logarithmic scaling in gauge / string correspondence,” Nucl. Phys. B 748, 24 (2006) [arXiv:hep-th/0601112].
A. V. Belitsky, G. P. Korchemsky and R. S. Pasechnik, “Fine structure of anomalous dimensions in N=4 super Yang-Mills theory,” Nucl. Phys. B 809, 244 (2009) [arXiv:0806.3657].
[13] V. A. Kazakov and K. Zarembo, “Classical / quantum integrability in non-compact sector of AdS/CFT,” JHEP 0410, 060 (2004) [arXiv:hep-th/0410105].
K. Sakai and Y. Satoh, “A large spin limit of strings on AdS(5) x S5 in a non-compact sector,” Phys. Lett. B 645, 293 (2007) [arXiv:hep-th/0607190].
[14] M. Kruczenski and A. A. Tseytlin, “Spiky strings, light-like Wilson loops and pp-wave anomaly,” Phys. Rev. D 77, 126005 (2008) [arXiv:0802.2039].
[15] R. Ishizeki, M. Kruczenski and A. Tirziu, “New open string solutions in AdS5,” Phys. Rev. D 77, 126018 (2008) [arXiv:0804.3438].
[16] H. Hayashi, K. Okamura, R. Suzuki and B. Vicedo, “Large Winding Sector of AdS/CFT,” JHEP 0711, 033 (2007) [arXiv:0709.4033].
[17] D. Berenstein, J. M. Maldacena and H. Nastase, “Strings in flat space and pp waves from N = 4 super Yang Mills,” JHEP 0204, 013 (2002) [arXiv:hep-th/0202021].
[18] S. Frolov and A. A. Tseytlin, “Multi-spin string solutions in AdS(5) x S5,” Nucl. Phys. B 668, 77 (2003) [arXiv:hep-th/0304255].
[19] M. Kruczenski, “Spin chains and string theory,” Phys. Rev. Lett. 93, 161602 (2004) [arXiv:hep-th/0311203].
[20] M. Kruczenski, A. V. Ryzhov and A. A. Tseytlin, “Large spin limit of AdS(5) x S5 string theory and low energy expansion of ferromagnetic spin chains,” Nucl. Phys. B 692, 3 (2004) [arXiv:hep-th/0403120].
M. Kruczenski and A. A. Tseytlin, “Semiclassical relativistic strings in S5 and long coherent operators in N = 4 SYM theory,” JHEP 0409, 038 (2004) [arXiv:hep-th/0406189].
[21] B. J. Stefanski and A. A. Tseytlin, “Large spin limits of AdS/CFT and generalized Landau-Lifshitz equations,” JHEP 0405, 042 (2004) [arXiv:hep-th/0404133].
S. Bellucci, P. Y. Castell, J. F. Morales and C. Sochichiu, “SL(2) spin chain and spinning strings on AdS(5) x S5,” Nucl. Phys. B 707, 303 (2005) [arXiv:hep-th/0409086].
[22] Y. Park, A. Tirziu and A. A. Tseytlin, “Spinning strings in AdS(5) x S5: One-loop correction to energy in SL(2) sector,” JHEP 0503, 013 (2005) [arXiv:hep-th/0501203].
M. Beccaria, V. Forini, A. Tirziu and A. A. Tseytlin, “Structure of large spin expansion of anomalous dimensions at strong coupling,” [arXiv:0809.5234].
[23] E. Mosaffa and B. Safarzadeh, “Dual Spikes: New Spiky String Solutions,” JHEP 0708, 017 (2007) [arXiv:0705.3131].
[24] S. Gradsheteyn, I. Ryzhik, “Table of Integrals Series and Products”, Sixth edition, Academic Press (2000), San Diego, CA, USA, London, UK.
[25] N. Dorey, “A Spin Chain from String Theory,” [arXiv:0805.4387].
N. Dorey and M. Losi, [arXiv:0812.1704] [hep-th].
[26] D. Staudacher, M. Staudacher, “Planar N = 4 gauge theory and the Inozemtsev long range spin chain,” JHEP 0406, 001 (2004) [arXiv:hep-th/0401057].
N. Beisert, V. Dippel and M. Staudacher, “A novel long range spin chain and planar N = 4 super Yang-Mills,” JHEP 0407, 075 (2004) [arXiv:hep-th/0405001].
[27] A. Jevicki and K. Jin, “Solitons and AdS String Solutions,” Int. J. Mod. Phys. A 23, 2289 (2008) [arXiv:0804.0412].
[28] G. Arutyunov, S. Frolov, J. Russo and A. A. Tseytlin, “Spinning strings in AdS(5) x S5 and integrable systems,” Nucl. Phys. B 671, 3 (2003) [arXiv:hep-th/0307191].
G. Arutyunov, J. Russo and A. A. Tseytlin, “Spinning strings in AdS(5) x S5: New integrable system relations,” Phys. Rev. D 69, 086009 (2004) [arXiv:hep-th/0311004].

[30] N. Beisert, J. A. Minahan, M. Staudacher and K. Zarembo, “Stringing spins and spinning strings,” JHEP 0309, 010 (2003) [arXiv:hep-th/0306139].

[31] S. Frolov and A. A. Tseytlin, “Rotating string solutions: AdS/CFT duality in non-supersymmetric sectors,” Phys. Lett. B 570, 96 (2003) [arXiv:hep-th/0306143].

[32] N. Beisert, S. Frolov, M. Staudacher and A. A. Tseytlin, “Precision spectroscopy of AdS/CFT,” JHEP 0310, 037 (2003) [arXiv:hep-th/0308117].