THE HARISH-CHANDRA ISOMORPHISM
FOR REDUCTIVE SYMMETRIC SUPERPAIRS

ALEXANDER ALDRIDGE

Abstract. We consider symmetric pairs of Lie superalgebras which are strongly reductive and of even type, and introduce a graded Harish-Chandra homomorphism. We prove that its image is a certain explicit filtered subalgebra of the Weyl invariants on a Cartan subspace whose associated graded is the image of Chevalley's restriction map on symmetric invariants. This generalises results of Harish-Chandra and V. Kac, M. Gorelik.

1. Introduction

Supermanifolds were developed in the 1970s by Berezin, Kostant and Leites as a rigorous mathematical framework for the quantum field theory of bosonic and fermionic particles. A particular class which appears naturally in connection with the representation theory of Lie supergroups is formed by the symmetric supermanifolds. In physics, they arise as the target spaces of non-linear SUSY $\sigma$-models, for instance, in the spectral theory of disordered systems [Zir96], and more recently, in the study of topological insulators [SRFL09].

Given a symmetric superspace $X = G/K$, a fundamental object is the algebra $D(X)^G$ of $G$-invariant differential operators on $X$. For instance, the study of its $K$-invariant joint eigenfunctions (the spherical superfunctions) should shed light on the regular $G$-representation on superfunctions on $X$. Indeed, in ongoing joint work with J. Hilgert and M. R. Zirnbauer, we are developing the harmonic analysis on $X$ along these lines.

To state our main result, let us fix some notation. Let $(\mathfrak{g}, \mathfrak{k}, \theta)$ be a symmetric pair of complex Lie superalgebras, where the decomposition into $\theta$-eigenspaces is $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. We assume that it is strongly reductive and that there is an even Cartan subspace $\mathfrak{a} \subset \mathfrak{p}_0$ (see the main text for precise definitions).

Let $I(\mathfrak{a}) \subset S(\mathfrak{a})$ be the image of the restriction map $S(\mathfrak{p})^\mathfrak{k} \to S(\mathfrak{a})$. We prove the following generalisation of Harish-Chandra’s celebrated Isomorphism Theorem:

Theorem. There is an explicitly defined filtered subalgebra $J(\mathfrak{a}) \subset S(\mathfrak{a})$ with $\text{gr} J(\mathfrak{a}) = I(\mathfrak{a})$, and a short exact sequence of algebras,

$$0 \longrightarrow (\mathfrak{U}(\mathfrak{g})^\mathfrak{k})^\mathfrak{a} \longrightarrow \mathfrak{U}(\mathfrak{g})^\mathfrak{a} \overset{\Gamma}{\longrightarrow} J(\mathfrak{a}) \longrightarrow 0.$$
In particular, if \((G, K, \theta)\) is any symmetric pair of Lie supergroups whose complexified infinitesimal pair is \((g, k, \theta)\), then the algebra \(D(X)^G\) of complex \(G\)-invariant differential operators on \(X = G/K\) is isomorphic to \(J(a)\).

Here, \(\Gamma\) is the super version of Harish-Chandra’s well-known homomorphism. We stress that below, \(J(a)\) is defined in entirely explicit terms—otherwise the statement of the theorem would be somewhat vacuous. The image \(J(a)\) of Chevalley’s restriction map was determined in joint work with J. Hilgert and M.R. Zirnbauer [AHZ10], and this determination also forms the basis for the description of \(J(a)\).

Our theorem covers the case of \((g, \mathfrak{t}, \theta) = (\mathfrak{t} \oplus \mathfrak{t}, \mathfrak{t}, \text{flip})\), the so-called ‘group type’; for this special case, the theorem is due to V. Kac [Kac84], M. Gorelik [Gor04]. Here, \(I(a) = J(a)\), but this is not true in general. Our method of proof is entirely different from that of Kac, Gorelik.

Rather, we follow Harish-Chandra’s original analytic proof in the even case as closely as possible [HC58, HC84]. His idea is to consider a non-compact real form \((G, K, \theta)\) of \((g, k, \theta)\), and to use the elementary spherical functions defined on \(X = G/K\) to derive the Weyl group invariance of the image of \(\Gamma\).

The graded counterpart of the first step in Harish-Chandra’s proof appears to be impossible at first sight, due to the non-existence of compact (and hence, purely non-compact) real forms in the realm of real Lie superalgebras [Ser83]. However, this can be addressed by using the category of \(cs\) manifolds introduced by J. Bernstein [Ber93, DM99]; it is a full subcategory of complex graded ringed spaces. We discuss \(cs\) manifolds at length in Appendix A, and \(cs\) Lie supergroups in Appendix B. This framework allows us, in Section 2, to prove the existence of non-compact \(cs\) forms (Proposition 2.10), and of a global Iwasawa decomposition (Proposition 2.11).

In Appendix C, we recall the basics of Berezin integration in the framework of \(cs\) manifolds. In particular, we include the definition of the absolute Berezin integral (which is insensitive to changes of orientation in the even variables), and of invariant (absolute) Berezinians on homogeneous \(cs\) manifolds, as developed by J. Hilgert and the author for real supermanifolds [AH10]. In Section 3, we employ these techniques to generalise the integral formulæ for the Iwasawa decomposition. In particular, we introduce certain non-zero joint eigen-superfunctions on \(U(g)^J\), and certain weighted orbital integrals. Compared to the even case, a complication is that \(\int_K 1 \cdot |Dk| = 0\) (reflecting the maximal atypicality of the trivial \(\mathfrak{t}\)-module), so that the technique of ‘invariant integration over \(K\)’ cannot be as liberally employed as is customary for compact Lie groups. We overcome this difficulty by introducing auxiliary superfunctions in Harish-Chandra’s Eisenstein integral. (This amounts to considering joint eigenfunctions in a non-trivial \(K \times K\)-type of \(\Gamma(O_G)\).)

These analytic tools permit the proof of the fact that the image of \(\Gamma\) is invariant under the even Weyl group (Proposition 4.2). The remainder of the proof, in Section 4, of the containment of \(\text{im} \Gamma\) in \(J(a)\) is entirely algebraic. It requires a rank reduction technique pioneered in the even case by Lepowsky [Lep75], and an explicit understanding of the relevant special cases of low rank. This is considerably more difficult than in the even case, where rank one is sufficient and the invariants in rank one are generated by the Casimir. (Both is false in general.)

Once the inclusion of the image of \(\Gamma\) in \(J(a)\) has been established, a simple spectral sequence argument, together with the results from [AHZ10], readily implies the Main Theorem (Theorem 4.19).
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2. The Iwasawa decomposition

In the following, we will liberally employ standard notions of the theory of Lie supergroups, cf. [Lei80, DM99, Kos77, Sch84, Sch79, Kac77]. We will also make use of the notions introduced in Appendices A and B, in particular, the theory of cs manifolds and groups.

2.1. Non-compact cs forms. First, we recall some definitions from [AHZ10].

Definition 2.1. Let \( g \) be a complex Lie superalgebra and \( \theta \) an even involutive automorphism of \( g \). We write \( g = \mathfrak{t} \oplus \mathfrak{p} \) for the \( \theta \)-eigenspace decomposition and say that \((g, \mathfrak{t}, \theta)\) is a symmetric superpair. Occasionally, we may drop \( \theta \) from the notation, although it is fixed as part of the data. We say that it is reductive if \( g_0 \) is reductive in \( g \), the centre \( z(g) \subset g_0 \), and there exists a non-degenerate even \( g \)- and \( \theta \)-invariant supersymmetric bilinear form \( b \) on \( g \) (which is not part of the data).

In the following, we will only consider reductive symmetric superpairs. Mostly, will in fact need to impose a slightly more restrictive condition on \( g \).

Indeed, let \( g \) be a Lie superalgebra such that \( z(g) \subset g_0 \), \( g_0 \) is reductive in \( g \), and \( g \) possesses a \( g \)-invariant non-degenerate even supersymmetric bilinear form \( b \). Then by [Ben00, Proposition 2.1, Lemma 2.3, Theorem 2.1, Corollary 2.1], \( g = \mathfrak{z}(g) \oplus g' \) where \( g' = [g, g] \) if and only if \( g' \) is the direct sum of \( b \)-non-degenerate simple graded ideals. In this case, \( g \) will be called strongly reductive. Any reductive symmetric superpair \((g, \mathfrak{t}, \theta)\) such that \( g \) is a strongly reductive Lie superalgebra will call be called a strongly reductive symmetric superpair.

We say that \((g, \mathfrak{t}, \theta)\) is of even type if there exists \( a \subset g \) such that \( a \) is an even Cartan subspace for \((g, \mathfrak{t}, \theta)\), i.e. \( a \subset \mathfrak{p}_0 \), \( a \) equals its centraliser \( z_p(a) \) in \( \mathfrak{p} \), and \( \text{ad} \ a \) consists of semi-simple endomorphisms of \( g_0 \).

We shall have to choose particular cs supergroup pairs whose underlying Lie superalgebra is part of a reductive symmetric superpair of even type. On the infinitesimal level, the conditions we will need are captured by the following definitions.

Definition 2.2. Let \((g, \mathfrak{t}, \theta)\) be a reductive symmetric superpair. A cs form of \((g, \mathfrak{t})\) is a \( \theta \)-invariant real form \( g_{0,R} \) of \( g_0 \) which is \( b \)-non-degenerate for some choice of the invariant form \( b \). We write \( \mathfrak{t}_{0,R} = g_{0,R} \cap \mathfrak{t} \) and \( \mathfrak{p}_{0,R} = g_{0,R} \cap \mathfrak{p} \).

Let \( \mathfrak{l} \) be a real Lie algebra. Recall from [Bor98, Lemma 4.1, Definition 4.2] that \( \mathfrak{l} \) is called compact if the following equivalent conditions are fulfilled: the set \( \text{ad} \mathfrak{l} \subset \text{End}(\mathfrak{l}) \) consists of semi-simple elements with imaginary spectra; and \( \mathfrak{l} \) is the Lie algebra of a compact real Lie group. More generally, if \( g \) is linear representation on a finite-dimensional real vector space \( V \), then \( \mathfrak{l} \) is called \( g \)-compact if \( g(\mathfrak{l}) \) generates a compact analytic subgroup of \( GL(V) \).

Denoting by \( \text{ad}_g \) the adjoint action of \( g_0 \subset g \) on \( g \), a cs form \( g_{0,R} \) will be called non-compact if \( u_0 = \mathfrak{t}_{0,R} \oplus \mathfrak{p}_{0,R} \) is an \( \text{ad}_g \)-compact real form of \( g_0 \); here, \( \text{ad}_g \) denotes the adjoint action of \( g_0 \) on \( g \).

Given a cs form, a real even Cartan subspace is a subspace \( \mathfrak{a}_R \subset \mathfrak{p}_{0,R} \) whose complexification \( \mathfrak{a} \) is an even Cartan subspace of \((g, \mathfrak{t}, \theta)\).

We will prove the existence of non-compact cs forms. The subtle point is the action of \( z(g_0) \) on \( g \). It can be tamed by the following lemma.
Lemma 2.3. Let $g$ be a strongly reductive Lie superalgebra. There exists an $ad_g$-compact real form of $\mathfrak{j}(g_0)$.

Proof. Let $b$ be a non-degenerate invariant form on $g$. As we have noted, $g'$ is the direct sum of $b$-non-degenerate simple ideals $s$, each of which has $\dim \mathfrak{j}(s_0) \leq 1$. If $\mathfrak{j}(s_0) \neq 0$, then $s_1$ is the direct sum $s_1 = V_1 \oplus V_2$ of simple $s_0$-modules, and by [Sac79, Chapter II, §2.2, Corollary to Theorem 1] there is a unique element $C = C_0 \in \mathfrak{j}(s_0)$ such that $ad C = (-1)^j$ on $V_j$, $j = 1, 2$. The analytic subgroup of $\text{End}(s)$ generated by $iR \cdot ad C$ is isomorphic to $T$.

Take any real form $a$ of $\mathfrak{j}(g)$ and let $b$ be sum of the real linear spans of the elements $iCs \in \mathfrak{j}(s_0)$, for all $b$-non-degenerate simple ideals $s$ of $g'$ with $\mathfrak{j}(s_0) \neq 0$. Define $\mathfrak{j}_R = a \oplus b$. The analytic subgroup of $\text{GL}(g)$ associated with $\mathfrak{j}_R$ is isomorphic to a finite power of $T$. 

Lemma 2.4. Let $(g, t, \theta)$ be a strongly reductive symmetric superpair. Then there exists a non-compact cs form of $(g, t, \theta)$, and for any two such forms, their semi-simple derived algebras are conjugate by an inner automorphism of $g_0$. If $(g, t, \theta)$ is, moreover, of even type, then there exists a real even Cartan subspace for any non-compact cs form of $(g, t, \theta)$.

Proof. Following Lemma 2.3 we may choose an $ad_g$-compact real form $\mathfrak{j}_R$ of $\mathfrak{j}(g_0)$. If $s$ is a $b$-non-degenerate simple ideal of $g'$ with $\mathfrak{j}(s_0) \neq 0$, then $\dim \mathfrak{j}(s_0) = 1$. Since $\mathfrak{j}(s_0)$ is $b$-non-degenerate, it is generated by a $b$-anisotropic vector. Either $s$ is $\theta$-invariant, in which case so is $\mathfrak{j}(s_0)$, or $\theta(s)$ is a distinct but isomorphic $b$-non-degenerate simple ideal of $g'$. In the latter case, we have $\theta(C_s) = \pm C_{\theta(s)}$ for the elements $C = C_s$ considered in the proof of Lemma 2.3 by their mere definition. Hence, by construction, we may assume that $\mathfrak{j}_R$ is $\theta$-invariant and $b$-non-degenerate.

Let $g_0' = [g_0, g_0]$. By [Loo69, pp. 154–155], there exists a $\theta$-invariant compact real form $u'_0$ of $g_0'$ (any real form has a Cartan decomposition compatible with $\theta$, and the compact real form associated with it is $\theta$-stable), and it is unique up to inner automorphisms of $g_0$. We set $g_{0,R} = \mathfrak{j}_R \cap t_0 \oplus u'_0 \cap t_0 \oplus i\mathfrak{j}_R \cap p_0 \oplus iu'_0 \cap p_0$.

Then $g_{0,R}$ is a $\theta$-stable and $b$-non-degenerate real form of $g_0$. Defining $t_{0,R} = g_{0,R} \cap t_0$ and $p_{0,R} = g_{0,R} \cap p_0$, we see that $u_0 = t_{0,R} \oplus ip_{0,R}$ is indeed $ad_g$-compact, so that $g_{0,R}$ is a non-compact cs form of $(g, t, \theta)$.

Assume that $(g, t, \theta)$ is of even type, and that $g_{0,R}$ is a non-compact cs form. There exists a maximal Abelian subspace $\mathfrak{a}_R \subset p_{0,R}$. By assumption, there exists an even Cartan subspace $\mathfrak{a} \subset p_0$. Now $\mathfrak{a}_R \otimes \mathbb{C}$ is the centraliser in $p_0$ of any regular element of $\mathfrak{a}_R$ (such elements exist), and any semi-simple element of $p_0$ is conjugate under the adjoint group of $t_0$ to an element of $\mathfrak{a}$ [Hel84, Chapter III, Proposition 4.16]. Thus, we may assume $\mathfrak{a}_R \otimes \mathbb{C} \subset \mathfrak{a}$. Then $\mathfrak{a} = \mathfrak{j}_{p_0}(\mathfrak{a}_R) \subset \mathfrak{j}_{p_0}(\mathfrak{a}_R) = \mathfrak{a}_R \otimes \mathbb{C}$, and hence the claim. 

2.2. Restricted roots and the Iwasawa decomposition.

2.5. Let $(g, t, \theta)$ be a reductive symmetric superpair of even type, assume given a non-compact cs form $g_{0,R}$ (which always exists if $(g, t, \theta)$ is strongly reductive), let $\mathfrak{a}_R \subset p_{0,R}$ be a real even Cartan subspace, and set $\mathfrak{a} = \mathfrak{a}_R \otimes \mathbb{C}$. We shall fix these data from now on.
By assumption, \( g_0 \) is reductive in \( g \), and \( a \) is a commutative subalgebra consisting of semi-simple elements. In particular, \( g \) is a semi-simple \( a \)-module, and we may decompose it as
\[
\mathfrak{g} = \mathfrak{m} \oplus \mathfrak{a} \oplus \bigoplus_{\lambda \in \Sigma} \mathfrak{g}_\lambda^a \quad \text{where} \quad \mathfrak{m} = \mathfrak{z}_1(a)
\]
is the centraliser of \( a \) in \( \mathfrak{k} \), and for \( \lambda \in a^* \),
\[
\mathfrak{g}_\lambda^a = \{ x \in \mathfrak{g} \mid \forall h \in a : [h, x] = \lambda(h)x \} \quad \text{and} \quad \Sigma = \{ \lambda \in a^* \mid 0 \mid \mathfrak{g}_\lambda^a \neq 0 \}.
\]
We also define \( \mathfrak{g}^+_{j,a} = \mathfrak{g}_j \cap \mathfrak{g}_a^+ \) and \( \Sigma_j = \{ \lambda \in a^* \mid 0 \mid \mathfrak{g}_\lambda^a \neq 0 \} \). Then we have \( \Sigma = \Sigma_0 \cup \Sigma_1 \), but the union may not be disjoint. Occasionally, we will write \( \Sigma(\mathfrak{g} : a) = \Sigma \) and \( \Sigma(\mathfrak{g}_j : a) = \Sigma_j \). Since \( u_0 = \mathfrak{u}_{0,\mathbb{R}} \oplus \mathfrak{p}_{0,\mathbb{R}} \) is by assumption a compact Lie algebra, the even restricted roots \( \lambda \in \Sigma_0 \) are real on \( a_{\mathbb{R}} \). Let \( \mathfrak{g}^+_{0,\mathbb{R},a} = \mathfrak{g}_{0,\mathbb{R}} \cap \mathfrak{g}_{0,\mathbb{R}}^a \) for all \( \lambda \in \Sigma_0 \) and \( m_{0,\mathbb{R}} = \mathfrak{z}_{0,\mathbb{R}}(a_{\mathbb{R}}) \).

2.6. Let \( \Sigma^+ \subset \Sigma \) be a positive system, i.e. a subset such that \( \Sigma = \Sigma^+ \cup -\Sigma^+ \) and \( \Sigma \cap (\Sigma^+ + \Sigma^+) \subset \Sigma^+ \). Let \( \Sigma^+_j = \Sigma_j \cap \Sigma^+ \). Then \( \Sigma^+_0 = \Sigma_0 \cap \Sigma^+ \) is a positive system of the root system \( \Sigma_0 \). Set
\[
n = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_\lambda^a \quad \text{and} \quad n_j = \mathfrak{g}_j \cap n.
\]
By the assumptions on \( \Sigma^+ \), \( n = n_0 \oplus m_1 \) is an \( a \)-invariant subsuperalgebra. Moreover, \( n_{0,\mathbb{R}} = \mathfrak{g}_{0,\mathbb{R}} \cap n \) (which is a real form of \( n_0 \)) is an \( a_{\mathbb{R}} \)-invariant nilpotent subalgebra of \( \mathfrak{g}_{0,\mathbb{R}} \). Since the roots in \( \Sigma_0 \) are real on \( a_{\mathbb{R}} \),
\[
n_{0,\mathbb{R}} = \bigoplus_{\lambda \in \Sigma^+} \mathfrak{g}_{0,\mathbb{R},a}^\lambda.
\]

By [KR71 Proposition 5], we have \( \dim m_0 - \dim a = \dim \mathfrak{k} - \dim p_0 \). We also need the corresponding result for the odd part of \( g \), and this is the content of the following lemma.

**Lemma 2.7.** Let \( x \in p_0 \). Then
\[
\dim \mathfrak{z}_1(x) - \dim \mathfrak{z}_{p_1}(x) = \dim \mathfrak{t}_1 - \dim p_1.
\]

**Proof.** We proceed exactly as in [KR71 proof of Proposition 5]. Choose a \( g \)- and \( \theta \)-invariant non-degenerate even supersymmetric form \( b \) on \( g \). Then, certainly \( \mathfrak{z}_{g_1}(x) = \mathfrak{z}_1(x) \oplus \mathfrak{z}_{p_1}(x) \), and \( g_1/\mathfrak{z}_{g_1}(x) \) carries a symmetric form \( b_x \), induced by
\[
b_x(y, z) = b(x, [z, y]) = b([x, z], y) \quad \text{for all} \quad y, z \in g_1.
\]
Clearly, \( b_x \) is non-degenerate on \( g_1/\mathfrak{z}_{g_1}(x) \).

We have \( g_1/\mathfrak{z}_{g_1}(x) = t_1/\mathfrak{z}_{t_1}(x) \oplus p_1/\mathfrak{z}_{p_1}(x) \), and both summands are \( b_x \)-totally isotropic, seeing that \( b(t_1, p_1) = 0 \). As a well-known consequence of Witt’s cancellation theorem, the dimension of \( b_x \)-totally isotropic subspaces does not exceed \( \frac{1}{2} \dim g_1/\mathfrak{z}_{g_1}(x) \). Therefore,
\[
\dim t_1 - \dim \mathfrak{z}_{t_1}(x) = \dim t_1/\mathfrak{z}_{t_1}(x) = \dim p_1/\mathfrak{z}_{p_1}(x) = \dim p_1 - \dim \mathfrak{z}_{p_1}(x)
\]
which proves the assertion. \( \square \)

**Proposition 2.8.** Let \( (g, k, \theta) \) be a reductive symmetric suppair of even type, \( g_{0,\mathbb{R}} \) a non-compact cs form, \( a_{\mathbb{R}} \) a real even Cartan subspace, \( a = a_{\mathbb{R}} \otimes \mathbb{C} \), and \( n \) the nilpotent subalgebra for some positive system of \( \Sigma(\mathfrak{g} : a) \). Then
\[
\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n} \quad \text{and} \quad \mathfrak{g}_{0,\mathbb{R}} = \mathfrak{k}_{0,\mathbb{R}} \oplus \mathfrak{a}_{\mathbb{R}} \oplus \mathfrak{n}_{0,\mathbb{R}}.
\]
We call these the Iwasawa decompositions of \( g \) and \( g_{0,\mathbb{R}} \), respectively.
Proof. The subspace $\theta n \cap n \subset g$ is $\alpha$-stable; if it were non-zero, it would contain a non-zero joint $\alpha$-eigenvector. Since this is impossible, $\theta n \cap n = 0$. Hence, the intersection $\mathfrak{k} \cap (\mathfrak{a} \oplus \mathfrak{n}) = \mathfrak{k} \cap \mathfrak{n} = 0$, because $\mathfrak{k}$ is $\theta$-fixed. Since $g = \mathfrak{k} \oplus \mathfrak{p}$, the point is to show $\dim \mathfrak{a} + \dim \mathfrak{n} = \dim \mathfrak{p}$. Now,

$$\dim \mathfrak{k} + \dim \mathfrak{p} = \dim g = \dim \mathfrak{m} + \dim \mathfrak{a} + 2 \dim \mathfrak{n}.$$ 

Thus,

$$2 \cdot (\dim \mathfrak{a} + \dim \mathfrak{n}) = \dim \mathfrak{k} - \dim \mathfrak{m} + \dim \mathfrak{a} + \dim \mathfrak{p} = 2 \cdot \dim \mathfrak{p},$$

where the last equation follows from the remark in [2,4] and Lemma [27]. Hence the assertion. □

2.3. The global Iwasawa decomposition. We need to globalise the Iwasawa decomposition. This requires appropriate $cs$ supergroup pairs.

**Definition 2.9.** Let $(\mathfrak{g}, \mathfrak{k}, \theta)$ be a symmetric superpair. A triple $(G_0, g, \theta)$ is a $cs$ supergroup pair (cf. Appendix [13]) where $G_0$ is connected is called a global $cs$ form of $(\mathfrak{g}, \mathfrak{k}, \theta)$ if the Lie algebra $g_{0,\mathbb{R}}$ of $G_0$ is a $cs$ form of $(\mathfrak{g}, \mathfrak{k})$, and if $\theta$ is an involutive automorphism of $G_0$ (denoted by the same letter as the given involution on $g$) whose differential is the restriction of $\theta$ to $g_{0,\mathbb{R}}$, such that

$$\text{Ad}(\theta(g)) = \theta \circ \text{Ad}(g) \circ \theta \in \text{End}(g) \quad \text{for all } g \in G_0.$$ 

A global $cs$ form $(G_0, g, \theta)$ of $(\mathfrak{g}, \mathfrak{k}, \theta)$ is called non-compact if $g_{0,\mathbb{R}}$ is a non-compact $cs$ form of $(\mathfrak{g}, \mathfrak{k}, \theta)$, and if $\text{Ad}_g(K_0)$ is compact, where $K_0$ denotes the analytic subgroup of $G_0$ generated by $\mathfrak{k}_{0,\mathbb{R}}$, and $\text{Ad}_g$ denotes the adjoint representation of $G_0$ on the Lie superalgebra $g$.

**Proposition 2.10.** Let $(\mathfrak{g}, \mathfrak{k}, \theta)$ be a strongly reductive symmetric superpair, and $g_{0,\mathbb{R}}$ a non-compact $cs$ form. Define $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}], \mathfrak{g}'_{0,\mathbb{R}} = [g_{0,\mathbb{R}}, g_{0,\mathbb{R}}] = g_{0,\mathbb{R}} \cap [g_0, g_0]$ and $Z_g = \exp_{g_0}(\mathfrak{j}(g) \cap g_{0,\mathbb{R}})$. There exists a non-compact global $cs$ form $(G_0, g, \theta)$ such that the following conditions hold:

(i). $g_{0,\mathbb{R}}$ is the Lie algebra of $G_0$;
(ii). the analytic subgroup $G'_0$ with Lie algebra $\mathfrak{g}' \cap g_{0,\mathbb{R}}$ is closed;
(iii). the analytic subgroup $G''_0$ with Lie algebra $g'_{0,\mathbb{R}}$ is closed;
(iv). $G'_0 = Z(G'_0) \cdot G'_0$;
(v). $G_0 = G'_0 \times Z_g$ is connected and simply connected;
(vi). $Z(G_0) = Z(G'_0) \times Z_g$.

We shall call $(G_0, g, \theta, G'_0, G''_0, Z_g)$ a standard global $cs$ form of $(\mathfrak{g}, \mathfrak{k})$.

Proof. Let $G'_0$ be the subgroup of $GL(g)$ generated by $e^{\text{ad}(x)}, x \in g_{0,\mathbb{R}}$. By assumption, $g_{0,\mathbb{R}}$ contains a maximal compactly embedded subalgebra which is ad-compact. By [HN10] Corollary 13.5.6 (e)), $G'$ is closed in $GL(g)$.

The Lie algebra of $G'_0$ is $g_{0,\mathbb{R}} \cap g'$ where $g' = [\mathfrak{g}, \mathfrak{g}]$. Moreover, we may define $\theta$ on $G'_0$ by $\theta(g) = \theta \circ g \circ \theta$. If $G''_0$ is the analytic subgroup of $G'_0$ whose Lie algebra is $g''_{0,\mathbb{R}} = g_{0,\mathbb{R}} \cap [g_0, g_0]$, then $G''_0$ is closed in $G'_0$; indeed, it may be identified, under the restriction of even automorphisms on $g$ to $g_0$, with the adjoint group of $g_{0,\mathbb{R}}$ which is closed in $GL(g_0)$ because $g_{0,\mathbb{R}}$ is semi-simple [HN10] Corollary 13.5.7). We remark that $G'_0 = Z(G'_0) \cdot G''_0$.

Let $Z_g$ be the connected and simply connected real Lie group whose Lie algebra is $\mathfrak{j}(g) \cap g_{0,\mathbb{R}}$, i.e. the additive group $\mathfrak{j}(g) \cap g_{0,\mathbb{R}}$. Then $\theta$ is defined in an obvious way on $Z_g$. If we set $G_0 = G'_0 \times Z_g$, then $Z(G_0) = Z(G'_0) \times Z_g$, and $\theta$ extends to
manifolds induced by multiplication in the opposite order. We define $A,H$ by $\exp$ Proposition 2.11. Similarly, let $\tilde{\phi}$ for all $f$ $k$, $a,n$ $\mathbb{Z}$ Let $\Psi$ be the inverse map.

Proof. Since $n_{0,\mathbb{R}} = [a,n_{0,\mathbb{R}}]$, $N_0 \subset G_0'$. Since $G_0'$ is connected and linear real reductive, we have the Iwasawa decomposition $G_0' = (K_0 \cap G_0')(A \cap G_0')N_0$. Let $Z_t = K_0 \cap Z_g$ and $Z_p = \exp p_{0,\mathbb{R}} \cap Z_g = A \cap Z_g$. We have $G_0 = G_0' \times Z_g$, so it follows that $(k,a,n) \mapsto \text{kan} : K_0 \times A_0 \times N_0 \rightarrow G_0$ is a diffeomorphism.

Let $a = a_0 \otimes \mathbb{C}$. From Proposition 2.8 and the Poincaré–Birkhoff–Witt theorem Sch79 Chapter I, § 3, Corollary 1 to Theorem 1], it follows that the multiplication of $\mathfrak{U}(g)$ induces an isomorphism of super-vector spaces $\mathfrak{U}(\mathfrak{t}) \otimes \mathfrak{U}(a) \otimes \mathfrak{U}(n) \rightarrow \mathfrak{U}(g)$. Let $\Psi$ be the inverse map.

To see that the morphism $\phi$ in the statement of the theorem is an isomorphism of $cs$ manifolds, it suffices by Corollary 2.11 to check that $\phi^*$ induces an isomorphism on the level the algebras of global sections. We have

$$(\phi^* f)(b \otimes c \otimes d; k,a,n) = f(\text{Ad}((an)^{-1})(b) \text{Ad}(n^{-1})(c)d; \text{kan}) = f(\text{Ad}((an)^{-1})(bc)d; \text{kan})$$

for all $f \in \Gamma(\mathcal{O}_G)$, $b \in \mathfrak{U}(\mathfrak{t})$, $c \in \mathfrak{U}(a)$, $d \in \mathfrak{U}(n)$, $k \in K_0$, $a \in A$, $n \in N_0$. We see that $\phi^{-1}$ is given by the formula

$$(\phi^{-1}*h)(u; \text{kan}) = h((\text{id} \otimes \text{id} \otimes \text{Ad}((an)^{-1}))(\Psi(\text{Ad}(an)(u))); k,a,n).$$

for all $h \in \Gamma(\mathcal{O}_{K \times A \times N})$, $u \in \mathfrak{U}(g)$, $k \in K_0$, $a \in A$, $n \in N_0$.

The following notation will be used repeatedly.

Definition 2.12. Let $\phi : K \times A \times N \rightarrow G$ be the Iwasawa isomorphism from Proposition 2.11. Similarly, let $\phi : N \times A \times K \rightarrow G$ be the isomorphism of $cs$ manifolds induced by multiplication in the opposite order. We define $A,H : G \rightarrow a$ by $\exp \circ A \circ \phi = p_2$ and $\exp \circ H \circ \phi = p_2$ where $\exp : a \rightarrow A$ is the exponential map $a_{0,\mathbb{R}} \rightarrow A$, considered as a morphism of $cs$ manifolds. Further, define $k,u : G \rightarrow K$ by $k \circ \phi = p_1$ and $u \circ \phi = p_1$, and $n : G \rightarrow N$ by $n \circ \phi = p_1$.

This is consistent with Helgason’s notation Hel84; we feel that this supplies sufficient justification for indulging in the multiple uses of the letter $A$. The intended meaning will always be clear from the context.
3. INTEGRAL FORMULÆ

In what follows, let us assume given a strongly reductive symmetric superpair \((\mathfrak{g}, \mathfrak{t})\) of even type, a standard non-compact cs form \((G_0, \mathfrak{g}, G'_0, G'_0, Z_0)\), a real even Cartan subspace \(\mathfrak{a}_N\) with complexification \(\mathfrak{a}\), and a positive system \(\Sigma^+\) of \(\Sigma = \Sigma(\mathfrak{g} : \mathfrak{a})\). We fix the notation from Proposition 2.11. Moreover, let \(m = \mathfrak{z}_g(\mathfrak{a})\), \(M_0 = Z_K(\mathfrak{a})\), and \(M = C(M_0, m)\).

We will liberally use the notions introduced in Appendix C. In particular, recall the notation \(\Gamma_c\) for compactly supported global sections, the absolute Berezinian, and the concepts of geometric and analytic unimodularity.

3.1. Integral formulæ for the Iwasawa decomposition.

3.1. Let \(AN = NA = C(AN_0, \mathfrak{a} \oplus \mathfrak{n}) = C(N_0A, \mathfrak{a} \oplus \mathfrak{n})\). This is a closed \(cs\) sub-supergroup of \(G\). Similarly, define \(MA\).

**Lemma 3.2.** The \(cs\) Lie supergroups \(G, G', K, K', M, A, N\) are geometrically and analytically unimodular. The \(cs\) manifolds \(G/K, G'/K', G/MA\), and \(G/A\) are geometrically and analytically unimodular as \(G\)-spaces.

**Proof.** All of the above statements follow by successive applications of Proposition C.11.

3.3. Let \(\lambda \in \mathfrak{a}^*\). We may define a function \(e^\lambda \in C^\infty(\mathfrak{a}_N, \mathbb{C})\) by setting \(e^\lambda(x) = e^{\lambda(x)}\) for all \(x \in \mathfrak{a}_N\). Define the linear form \(\varrho = \frac{1}{2} \text{str} \text{ad} \mid \mathfrak{a}\). We have \(\varrho = \varrho_0 - \varrho_1\) where \(2\varrho_j = \sum_{\lambda \in \Sigma^+} m_{\lambda,j} \cdot \lambda, m_{\lambda,j} = \dim \mathbb{C} \lambda_{\mathfrak{g}_j, \mathfrak{a}}\). Observe that \(\varrho_0\) is real on \(\mathfrak{a}_N\).

**Proposition 3.4.** The invariant absolute Berezinians on \(G, K, A, \) and \(N\) can be normalised such that the following equations hold simultaneously:

\[
\begin{align*}
(3.1) & \quad \int_N f |D(\mathcal{O})| = \int_{N \times A} m^* f \cdot p_2^* \log^*(e^{-2\mathcal{O}}) \mid Dn \mid da, \quad f \in \Gamma_c(\mathcal{O}_N), \\
(3.2) & \quad \int_{N \times A} f |D(\mathcal{O})| = \int_{A \times N} m^* f \cdot |Dn| da, \quad f \in \Gamma_c(\mathcal{O}_N), \\
(3.3) & \quad \int_G f |Dg| = \int_{N \times K} \tilde{\phi}^* f \cdot p_2^* \log^*(e^{-2\mathcal{O}}) \mid Dn \mid da \mid Dk \mid, \quad f \in \Gamma_c(\mathcal{O}_G), \\
(3.4) & \quad \int_G f |Dg| = \int_{K \times N \times A} \phi^* f \cdot p_2^* \log^*(e^{2\mathcal{O}}) \mid Dn \mid da \mid Dk \mid, \quad f \in \Gamma_c(\mathcal{O}_G), \\
(3.5) & \quad \int_G f |Dg| = \int_{K \times N \times A} m^{(2)} f \mid Dk \mid \mid Dn \mid da, \quad f \in \Gamma_c(\mathcal{O}_G).
\end{align*}
\]

**Proof.** Equation (3.1) follows directly from Proposition 2.11 (C.12), and Lemma C.15 (3.2) follows in the same way, using the nilpotency of \(n\). Now (3.3), and (3.5) follow in the same vein from (3.1) (resp. 3.2), (C.6), and the analytic unimodularity of \(G/K\) in Lemma 3.2. The invariant absolute Berezinians can be normalised to give all equations simultaneously by the remark following (C.6). Finally, (3.4) follows from (3.3) by applying the invariance of \(|Dg|\) under \(i^*\) (where \(i\) is the inversion of \(G\)).

**Corollary 3.5.** For any \(a \in A\) and \(F \in \Gamma_c(\mathcal{O}_N)\), we have

\[
\int_N (R^*_a F) |Dn| = e^{2\varrho(a)} \cdot \int_N (L^*_a F) |Dn|, \quad (3.6)
\]
**Proof.** Considering \((R^\circ nF)|_N \otimes \chi\) where \(\chi \in \mathcal{C}_c(A)\) with \(\int_A \chi(a) \, da = 1\), the result follows by comparing Equations (3.2) and (3.1).

In the proof of the following lemma, we use the language of generalised points, cf. Appendix A.3. This method will be used repeatedly.

**Lemma 3.6.** For all \(f \in \Gamma_c(O_K)\), we have

\[
p_{21}((id \times i)^* m^* u f \cdot (|Dk| \otimes 1)) = p_{21}(p_1^* f m^* A^* (e^{2\varphi}) \cdot (|Dk| \otimes 1)).
\]

**Proof.** Take \(\chi \in \Gamma_c(O_{N \times A})\) such that \(\int_{N \times A} \chi \cdot p_2^* \log e^{-2\varphi} \, |Dn| \, da = 1\), and define \(h \in \Gamma_c(O_G)\) by \(\tilde{\phi}^* h = p_1^* \chi \cdot p_1^* f\). Then, since \(G\) is analytically unimodular as a \(cs\) Lie supergroup,

\[
\int_K f \, |Dk| = \int_{N \times A \times K} \tilde{\phi}^* h \cdot p_2^* \log^*(e^{-2\varphi}) \, |Dn| \, da \, |Dk| = \int_G h \, |Dg| = p_{21}(m^* h \cdot (|Dg| \otimes 1)) \quad (\ast)
\]

Let \(U\) be a \(cs\) manifold. Take \(U\)-points \(n \in U \, N, \, a \in U \, A, \, k \in U \, K,\) and \(g \in U \, G\). Then

\[
nakg = nan(kg) \exp A(kg)u(kg) = nan(kg)a^{-1} \cdot a \exp A(kg) \cdot u(kg),
\]

where the three factors are \(U\)-points of \(N, \, A, \, \) and \(K\), respectively. Since \(U\) and \(a, \, k, \, g\) were arbitrary, by Yoneda’s Lemma, there exist morphisms \(f_1 : A \times K \times G \rightarrow N, \quad f_2 : K \times G \rightarrow A\) such that

\[
m \cdot (\tilde{\phi} \times \text{id}) = \tilde{\phi} \circ (m \circ (id \times f_1), \, m \circ (p_2 \times f_2), \, u \circ m \circ p_34).
\]

Therefore, \((\ast)\) equals

\[
p_M((m \circ (id \times f_1), \, m \circ (p_2 \times f_2), \, u \circ m \circ p_34))^* \tilde{\phi}^* h p_2^* \log^*(e^{-2\varphi}) (|Dn| \, da \, |Dk| \otimes 1))
\]

\[
= p_M((p_12, \, u \circ m \circ p_34))^* \tilde{\phi}^* h \cdot p_2^* \log^*(e^{-2\varphi}) \cdot p_34^* m^* A^* (e^{2\varphi}) \cdot (|Dn| \, da \, |Dk| \otimes 1))
\]

\[
= p_{21}(m^* (u^* f \cdot A^* (e^{2\varphi})) \cdot (|D| \otimes 1)).
\]

Here, we have used the fact that \(N\) and \(A\) are analytically unimodular \(cs\) Lie supergroups.

To arrive at our claim, we need to ‘invert’ \(u^*\). Thus, let \(U\) be a \(cs\) manifold and \(k \in U \, K, \, g \in U \, G\) be \(U\)-points. There exist unique \(n \in U \, N, \, a \in U \, A\) such that \(nau(kg) = nakg\), which gives \(u(kg)g^{-1} = k\). Hence, if \(h(k) = f(u(kg))\) for all \(k \in U \, K,\) then \(h(u(kg^{-1})) = f(k)\) for all \(k \in U \, K\). Since \(U\) and \(g\) were arbitrary, this implies

\[
p_{21}((id \times i)^* m^* u h \cdot (|Dk| \otimes 1)) = p_M(p_1^* h \cdot m^* A^* (e^{2\varphi})) \cdot (|Dk| \otimes 1))
\]

for all \(h \in \Gamma_c(O_K)\). \(\square\)

### 3.2. Joint eigen-superfunctions

From now on, we will assume \(j(\mathfrak{g}) = 0\), so that \(G = G'\), and \(K = K'\) has a compact base.

In this section, we introduce a family of joint eigen-superfunctions for \(\mathfrak{U}(\mathfrak{g})^f\), similar to the elementary spherical functions. It is known [Zh91] that \(\int_K 1 \, |Dk| = 0\) if \(\dim \mathfrak{t}_1 \neq 0\); thus, the generalisation of Harish-Chandra’s Eisenstein integral yields superfunctions which are non-zero, but not obviously so.\(^1\)

\(^1\)In order to see that the Eisenstein integral is non-zero, one needs to study its asymptotic behaviour as \(\lambda \rightarrow \infty\). This study will be the subject of a series of subsequent papers.
If we are however willing to sacrifice $K$-biinvariance, we may introduce, into the Eisenstein integral, an auxiliary superfunction $\psi$ such that $\int_K \psi |Dk| = 1$. As we will see presently, this defines a set of joint eigen-superfunctions which are obviously non-zero.

3.7. For $D \in \mathfrak{U}(g)$, we define $D_a \in \mathfrak{U}(a)$ by

$$D - D_a \in \mathfrak{t} \mathfrak{U}(g) + \mathfrak{U}(g)n.$$ 

Such a definition is possible due to the Poincaré–Birkhoff–Witt theorem.

The proof of the following lemma is standard, so we omit it.

**Lemma 3.8.** Let $D \in \mathfrak{U}(g)$, $D' \in \mathfrak{U}(g)^\mathfrak{t}$. Then $(DD')_a = D_aD'_a$.

3.9. For $D \in \mathfrak{U}(g)$, we define $\Gamma(D) = e^{-\varphi}D_a e^\varphi \in S(a)$. Recall that there is an algebra isomorphism $S(a) \cong \mathbb{C}[[a^*]] : \mathcal{D} \rightarrow \mathfrak{p}$ defined by $p(\mu) = (De^\mu)(0)$. In these terms, $\Gamma(D)(\mu) = D_a(\mu + \varphi)$.

By Lemma 3.12, we have an algebra homomorphism $\Gamma : \mathfrak{U}(g)^\mathfrak{t} \rightarrow S(a)$, called the Harish-Chandra homomorphism. (Not to be confused with the global sections functor $\Gamma$.) Obviously, we have $(\mathfrak{U}(g)^\mathfrak{t})^\mathfrak{a} \subset \ker \Gamma$. (The converse inclusion will be established below.)

3.10. In what follows, we fix $\psi \in \Gamma(O_K)$ such that $\int_K \psi |Dk| = 1$. Since $K$ is analytically unimodular as a cs Lie supergroup, we also have $\int_K \tilde{\psi} |Dk| = 1$; hence, we may and will assume that $\psi = \tilde{\psi}$.

Recall the notation from Definition 2.12. Then $\tilde{\psi} := k^* \psi \in \Gamma(O_G)$ extends $\psi$. Slightly abusing notation, we will write $\psi = \tilde{\psi}$. Due to the Iwasawa decomposition,

$$(3.7) \quad p_{\mathfrak{a}}(m^* \psi \cdot (|Dk| \otimes 1)) = p_{\mathfrak{a}}(p_1^* \psi \cdot (|Dk| \otimes 1)) = \int_K \psi |Dk| = 1 \in \Gamma(O_G).$$

For $f \in \Gamma(O_G)$, we define

$$p_K(f) = f_K = p_{\mathfrak{a}}(m^*(\psi f) \cdot (1 \otimes |Dk|)).$$

Then $f_K \in \Gamma(O_G)^K = \Gamma(O_{G/K})$ (cf. C.67). We let

$$(3.8) \quad \phi_K^\mu = (H^* e^{\varphi - \varphi})_K \quad \text{for all } \mu \in a^*.$$ 

Observe that $e^* \phi^K_K = 1$, so that $\phi^K_K \neq 0$. (Here, $e : * \rightarrow G$ is the unit of $G$.)

In the following, we denote the $r$-action of $\mathfrak{U}(g)$ on $\Gamma(O_G)$ (cf. Appendix B) by juxtaposition, i.e. $Df$ instead of $r_D f$.

**Proposition 3.11.** For all $D \in \mathfrak{U}(g)^\mathfrak{t}$, we have

$$(3.9) \quad D \phi^K_N = D_a(\lambda + \varphi) \cdot \phi^K_N = \Gamma(D)(\lambda) \cdot \phi^K_N.$$ 

In particular, $\phi^K_N$ is a joint eigenfunction of all $D \in \mathfrak{U}(g)^\mathfrak{t}$.

For the proof, we first note the following lemma.

**Lemma 3.12.** We have

$$\phi^K_N = p_{\mathfrak{a}}(p_1^* \psi \cdot m^*(e^{\lambda + \varphi}) \cdot (|Dk| \otimes 1)).$$

**Proof.** First, we establish some identities for the morphisms $A$, $H$, $u$, using higher points. Thus, let $U$ be a cs manifold and $k \in V K$, $g \in V G$ be $U$-points. There exists a unique $n \in V N$ such that $k(g) \exp H(g)n = g$. Then

$$g^{-1} = n^{-1} \exp(-H(g))k(g)^{-1} = n^{-1} \exp A(g)u(g).$$
Thus,

\((3.10)\) \[ H \circ i = -A \quad \text{and} \quad i \circ k = u \circ i. \]

Next, let the unique \(n \in U N\) such that \(kg^{-1} = n \exp(A(kg^{-1}))u(kg^{-1})\). Then

\[
\begin{align*}
  k &= n \exp(A(kg^{-1}))u(kg^{-1})g \\
  &= n \exp(A(kg^{-1})) n' \exp(A(u(kg^{-1})g))u(kg^{-1})g \\
  &= n' \exp(A(kg^{-1}) + A(u(kg^{-1})g))k
\end{align*}
\]

for some \(n' \in U N\), where we write \(n'' = n \exp(A(kg^{-1})) n' \exp(-A(kg^{-1})) \in U N\) and recall that \(u(u(kg^{-1})g) = k\) (an identity which we have, in fact, rederived).

But this implies \(A(kg^{-1}) + A(u(kg^{-1})g) = 0\). Since \(U, k, g\) were arbitrary, we conclude that

\((3.11)\) \[ A \circ m \circ (u \circ m \circ (id \times i), p_2) = -A \circ m \circ (id \times i) \]

as morphisms \(K \times G \to a\).

Now we can compute, using \((3.10)\), \(i^*|Dk| = |Dk|\), \((3.11)\), Lemma 3.6 and the ‘inversion’ of \(u\) in its proof,

\[
\phi^\psi_\lambda = p_{21}(m^*(\psi \cdot H^*e^{\lambda - \theta}) \cdot (1 \otimes |Dk|))
\]

\[
= p_{21}(\sigma^* m^* \psi \cdot (i \times i) m^* A^*e^{-\lambda + \theta} \cdot (|Dk| \otimes 1))
\]

\[
= p_{21}(\sigma^* m^* \psi \cdot (id \times i) m^* A^*e^{-\lambda + \theta} \cdot (|Dk| \otimes 1))
\]

\[
= p_{21}((i \times id)^* \sigma^* m^* \psi \cdot (u \circ m \circ (id \times i), p_2) m^* A^*e^{-\lambda - \theta} \cdot (|Dk| \otimes 1))
\]

\[
= p_{21}(p_{21}, i \circ u \circ m \circ m^* \psi = p_{21}^i \psi, \text{and, hence, our claim.}
\]

Proof of Proposition 3.11. In view of Lemma 3.12 it suffices to prove the equation \(DA^*e^{\lambda + \theta} = \Gamma(D)(\lambda) \cdot A^*e^{\lambda + \theta}\), but this is trivial.

3.3. The Harish-Chandra orbital integral.

Definition 3.13. The elements of the dense open subset of \(a\) given by the equation \(a' = a \setminus \bigcup_{\lambda \in \Sigma^I} \lambda^{-1}(0)\) are called algebraically super-regular. We also consider the sets \(a'' = a \setminus \bigcup_{\lambda \in \Sigma^I} \lambda^{-1}(2\pi i\mathbb{Z})\) and \(A' = \exp(a_\mathbb{R} \cap a'')\). The elements of the latter set are called analytically super-regular.

For \(\lambda \in \Sigma, j = 0, 1\), let \(m_{\lambda,j}\) be the multiplicity of \(\lambda\) in the \(a\)-module \(g_j\). For \(a = \exp(h) \in A'\), the function

\[(3.12)\] \[ D(a) = \frac{\prod_{\lambda \in \Sigma^I} |\sinh \frac{1}{2} \lambda(h)|^{m_{\lambda,0}}}{\prod_{\lambda \in \Sigma^I} |\sinh \frac{1}{2} \lambda(h)|^{m_{\lambda,1}}} = e^{e(h)} \frac{\prod_{\lambda \in \Sigma^I} 2^{m_{\lambda,0}}|1 - e^{-\lambda(h)}|^{m_{\lambda,0}}}{\prod_{\lambda \in \Sigma^I} 2^{m_{\lambda,1}}(1 - e^{-\lambda(h)})^{m_{\lambda,1}}} \]

where we set \(\sinh \lambda = \frac{1}{2}(e^\lambda - e^{-\lambda})\), is well-defined and non-zero.
3.14. Let \( c : G \times G \to G \) denotes conjugation on \( G \), i.e. \( c = m \circ (m, i \circ p_1) \).

We compute \( c \) explicitly in terms of the cs supergroup pair. To that end, fix an open subset \( U \subset G_0 \), \( f \in \mathcal{O}_G(U) \), \( u, v \in \mathfrak{u}(g) \), \( (g, h) \in m^{-1}(U) \). We write \( \Delta(u) = \sum_j u_j \otimes v_j \) and \( \Delta(v) = \sum_i w_i \otimes z_i \). Then (cf. Appendix B)
\[
c^* f(u \otimes v; g, h) = \sum_j (-1)^{|v_j||w_i|} m^* f(\text{Ad}(h^{-1})(u_j)w_i \otimes \text{Ad}(g)(S(v_j))\varepsilon(z_i); gh, g^{-1}) = \sum_j (-1)^{|v_j||v|} m^* f(\text{Ad}(h^{-1})(u_j)v \otimes \text{Ad}(g)(S(v_j)); gh, g^{-1}) = \sum_j (-1)^{|v_j||v|} f(\text{Ad}(g)(\text{Ad}(h^{-1})(u_j)v)S(v_j)); ghg^{-1}) .
\]

One may define
\[
c_a^* f(v; g) = (-1)^{|v||f|} c^* f(u \otimes v; 1, g) = (-1)^{|v||f|} \sum_j (-1)^{|v_j||v|} m^* f(\text{Ad}(g^{-1})(u_j)vS(v_j)); g) .
\]

Then \( c_a^* = -\ell_x - r_x \) for all \( x \in \mathfrak{g} \), by (B.1) and (C.1). In particular, any subspace of \( \Gamma(\mathcal{O}_G) \) invariant under \( \ell_g \) and \( r_g \) is invariant under \( c_{\mathfrak{u}(g)}^* \), and vice versa.

For \( g \in G_0 \), we set
\[
c_g^* f(v, h) = c^* f(1 \otimes v; g, h) = f(\text{Ad}(g)(v); ghg^{-1}) .
\]

From (B.3) and (C.2), one sees that \( c_{g^{-1}}^* f = \ell_g r_g f \). Observe further that the pair \( c_g = (h \mapsto ghg^{-1}, c_g^* \) is an automorphism of the cs Lie supergroup \( G \).

Lemma 3.15. For \( a \in A' \), \( \xi_a = m \circ (c_{a^{-1}}, i) \) is a cs manifold automorphism of \( N \).

Proof. On the level of the underlying manifolds, \( \xi \) is a diffeomorphism by [Hel84, Chapter I, § 5, Lemma 6.4]. We compute the tangent map \( T_n \xi \).

To that end, observe that
\[
(T_n c_{a^{-1}})(dL_n(x)) = dL_{a^{-1}na} \text{Ad}(a^{-1})(x) ,
\]
\[
(T_n i)(dL_n(x)) = dL_{a^{-1}} \text{Ad}(n)(x) .
\]

By (B.9), we find
\[
(T_n \xi)(dL_n(x)) = dL_{a^{-1}nan^{-1}} \text{Ad}(n)(\text{Ad}(a^{-1})(x) - x) .
\]

By assumption, \( \text{Ad}(a^{-1}) - id : n \to n \) is an isomorphism, so the statement follows from Proposition A.20 \( \square \)

Corollary 3.16. For all \( f \in \Gamma_c(\mathcal{O}_N) \), \( a \in A' \), we have
\[
\int_N f |Dn| = \frac{\prod_{\lambda \in \Sigma_0^+} |1 - e^{-\lambda \varepsilon}|^{m_{\lambda,0}}}{\prod_{\lambda \in \Sigma_1^+} (1 - e^{-\lambda \varepsilon})^{m_{\lambda,1}}} \cdot \int_N \xi_a^* f |Dn| .
\]

Proof. By Lemma 3.15, \( \xi_a \) is an isomorphism of cs manifolds. By the invariance of absolute Berezin integrals under isomorphisms (cf. Appendix C), the expression in (3.15) of the tangent map of \( \xi_a \), and the \( N \)-invariance of the absolute Berezinian \( |Dn| \) on \( N \), everything comes down to the equation
\[
|\text{Ber}|_{n}(\text{Ad}(n)(\text{Ad}(a^{-1}) - 1)) = \frac{\prod_{\lambda \in \Sigma_0^+} |1 - e^{-\lambda \varepsilon}|^{m_{\lambda,0}}}{\prod_{\lambda \in \Sigma_1^+} (1 - e^{-\lambda \varepsilon})^{m_{\lambda,1}}},
\]
which is immediate. \( \square \)
3.17. Let \( \pi : G \to G/A \) be the canonical projection. Let \( f \in \mathcal{O}_G(G_0) \) and \( a \in A' \). Define \( \tilde{f}_a \in \Gamma(\mathcal{O}_G) \) by \( \tilde{f}_a = (\text{id} \times a)^* c^* f \) where we consider \( a \) as a morphism \( * \to A \). This function is right \( A \)-invariant. In particular, there exists a unique \( f_a \in \Gamma(\mathcal{O}_{G/A}) \) such that \( \pi^* f_a = \tilde{f}_a \).

**Proposition 3.18.** Let \( a \in A' \). For any \( f \in \Gamma_c(\mathcal{O}_G) \), we have \( f_a \in \Gamma_c(\mathcal{O}_{G/A}) \). Moreover, there is a normalisation of the invariant absolute Berezinian on \( G/A \) (independent of \( a \) and \( f \)), such that

\[
D(a) \cdot \int_{G/A} f_a |D\hat{y}| = e^{\rho(a)} \cdot \int_{K \times N} L^*_c(1,a)^* c^* f |Dk| |Dn| .
\]

**Proof.** That \( f_a \) is compactly supported when considered as a superfunction on \( G/A \) follows from [Hei84, Chapter I, §5, Proposition 5.6]. Since \( G/A \) is analytically unimodular, we deduce from (3.5) and (C.9) that the invariant absolute Berezinian can be normalised such that

\[
\int_{G/A} h |D\hat{y}| = 2^{\dim n_1 - \dim n_0} \cdot \int_{K \times N} m^* \pi^* h |Dk| |Dn|
\]

for all \( h \in \Gamma_c(\mathcal{O}_{G/A}) \). Then, setting \( C = 2^{\dim n_1 - \dim n_0} \),

\[
\int_{G/A} f_a |D\hat{y}| = C \cdot \int_{K \times N} m^* (\text{id}, a)^* c^* f |Dk| |Dn|
\]

\[
= C \cdot \int_{K \times N} (\text{id} \times \xi_a)^* L^*_c(1,a)^* c^* f |Dk| |Dn|
\]

\[
= C \cdot \prod_{\lambda \in \Sigma^+} (1 - e^{-\lambda(a)})^{m_{\lambda,1}} \cdot \prod_{\lambda \in \Sigma^0} |1 - e^{-\lambda(a)}|^{m_{\lambda,0}} \cdot \int_{K \times N} L^*_c(1,a)^* c^* f |Dk| |Dn| ,
\]

by Corollary 3.16. The equation follows from (3.12). \( \square \)

**Definition 3.19.** Let \( f \in \Gamma_c(\mathcal{O}_G) \). We define, for all \( a \in A \), the Harish-Chandra weighted orbital integral

\[
F_f(a) = e^{\rho(\log a)} \cdot \int_{K \times N} L^*_c(1,a)^* c^* f |Dk| |Dn| .
\]

By Proposition 3.18 we have for all \( a \in A' \),

\[
F_f(a) = D(a) \cdot \int_{G/A} f_a |D\hat{y}| .
\]

4. **The Harish-Chandra isomorphism**

Let us retain the assumptions and notation from Section 3. We will at first generally assume that \( \mathfrak{j}(\mathfrak{g}) = 0 \) (exceptions to this rule will be expressly stated).

4.1. **Even Weyl group invariance.**

4.1. Recall the definition of the Harish-Chandra homomorphism \( \Gamma \). Denote by \( W_0 = W(\mathfrak{g}_0 : a) \) the even Weyl group.

**Proposition 4.2.** We have \( \Gamma(\mathfrak{U}(\mathfrak{g})^f) \subset S(a)^{W_0} \).
The proof requires a little preparation. The idea comes from Harish-Chandra’s proof of the corresponding fact: One integrates the joint eigen-superfunctions $\phi^\psi$ against an arbitrary function, expresses this as the Abel transform of a weighted orbital integral, and uses the invariance of the latter. Compared to the even case, an additional complication is the lack of a well-behaved ‘invariant integral over $K$’, and the ensuing occurrence of the auxiliary superfunction $\psi$.

**Proposition 4.3.** Let $f \in \Gamma_c(\mathcal{O}_G)$. Then

$$F_f(a^w) = F_f(a) \quad \text{for all } w \in W_0, \ a \in A.$$  

Here, $W_0 = N_{K_0}(A)/Z_{K_0}(A)$, and $a^w = kak^{-1}$ for any $k \in N_{K_0}(A)$ such that $w = kZ_{K_0}(A)$, and any $a \in A$.

**Proof.** By continuity, it suffices to check the equality on $A'$. Fix $a \in A'$ and $w = kZ_K(A) \in W_0$ where $k \in N_{K_0}(a)$. Since $m_{\lambda,1} = \dim \mathfrak{g}_\lambda^k_a$ is even for all $\lambda \in \Sigma^-$, Proposition 2.10 (v)), $D(a^w) = D(a)$. We claim that

$$\int_{G/A} f_{a^w} |D\dot{g}| = \int_{G/A} (c_k^*f)_a |D\dot{g}|.$$  

It will then follow that $F_f(a^w) = F_{c_k^*f}(a)$, by (3.17). Let us prove (*).

Clearly, $c_k$ induces a morphism $G/A \to G$. For all $h \in \Gamma_c(\mathcal{O}_G)$, define $h_A$ by the vector-valued integral $h_A = \int_A h \, d\alpha$. Then

$$\int_{G/A} h_A |D\dot{g}| = \int_G h |D\dot{g}| = \int_G c_k^*h |D\dot{g}| = \int_{G/A} (c_k^*h)_{MA} |D\dot{g}|.$$  

On the other hand, the measure $d\alpha$ is $Z_{K_0}(A)$-conjugation invariant. This implies $(c_k^*h)_A = c_k^*(h_A)$, and we conclude that $c_k^*|D\dot{g}| = |D\dot{g}|$. Hence,

$$\int_{G/A} f_{a^w} |D\dot{g}| = \int_{G/A} c_k^*(f_{kak^{-1}}) |D\dot{g}| = \int_{G/A} (c_k^*f)_a |D\dot{g}|,$$

so (*) holds, and we find that $F_f(a^w) = F_{c_k^*f}(a)$. On the other hand,

$$F_{c_k^*f}(a) = e^{\rho(\log a)} \cdot \int_{K \times N} L^*_{(k, a^*c^*f \, |Dk| \, |Dn|}$$

$$= e^{\rho(\log a)} \cdot \int_{K \times N} L^*_{(1, a^*c^*f \, |Dk| \, |Dn|} = F_f(a),$$

since $|Dk|$ is $K$-invariant. This proves the claim. \hfill $\Box$

**Lemma 4.4.** For any $\lambda \in \mathfrak{a}^*$ and $f \in \Gamma_c(\mathcal{O}_G)$, we have

$$\int_G \phi^\psi_\lambda \cdot f |D\dot{g}| = \int_A e^{\lambda(\log a)} F_f(a) \, d\alpha.$$  

**Proof.** We observe $|\text{Ber}|_n(\text{Ad}(a)) = e^{2\rho(\log a)}$ (because the linear form $\phi_0|_{\mathfrak{a}_0}$ is real). Thus, we compute, using the analytic unimodularity of $G$, $\psi = k^*(\psi|_K)$, (3.3), the
analytic unimodularity of $K$, and $i^*(\psi|_K) = \psi|_K$,
\[
\int_{G} \phi^\psi_\lambda f |Dg| = \int_{G \times K} m^* (H^*(e^{\lambda - \theta}) \cdot \psi) \cdot p^*_1 f |Dg| |Dk| \\
= \int_{G \times K} p^*_1 H^*(e^{\lambda - \theta}) \cdot p^*_1 k^* \psi \cdot (id \times i)^* m^* f |Dg| |Dk| \\
= \int_{A} e^{(\lambda + \theta)(log a)} \int_{K \times K} p^*_1 \psi \cdot L_{1,1}^*(i \times id \times i)^* m^* f |Dk_1||Dn| |Dk_2| da \\
= \int_{A} e^{(\lambda + \theta)(log a)} \int_{K \times K} (i \circ \psi_1, p_1)^* m^* \psi \cdot L_{1,1}^* e^* f |Dk_1||Dn| |Dk_2| \\
= \int_{A} e^{(\lambda + \theta)(log a)} \int_{K \times K} p_{11}((i \times id)^* m^* \psi \cdot (1 \otimes |Dk|)) \cdot L_{1,1}^* e^* f |Dk| |Dn| da .
\]
Now, the invariance of $|Dk|$ implies
\[
p_{11}((i \times id)^* m^* \psi \cdot (1 \otimes |Dk|)) = \int_{K} \psi |Dk| = 1 .
\]
This proves the assertion. \hfill \Box

Proof of Proposition \ref{prop:4.2}. Let $w \in W_0$ and $\mu \in a^*$. We have
\[
\int_{G} \phi^\psi_{w\lambda} \cdot f \ |Dg| = \int_{A} e^{w\lambda(log a)} \cdot F_f(a) da = \int_{A} e^{\lambda(log a)} \cdot F_f(a^{w}) da \\
= \int_{A} e^{\lambda(log a)} \cdot F_f(a) da = \int_{G} \phi^\psi_\lambda \cdot f \ |Dg|
\]
for all $f \in \Gamma_c(O_G)$, by Lemma \ref{lem:4.4} and Proposition \ref{prop:4.3}.

By Lemma \ref{lem:C.5} we conclude that $\phi^\psi_{w\lambda} = \phi^\psi_\lambda$. Then
\[
\Gamma(D)(w\lambda) \cdot \phi^\psi_\lambda = \Gamma(D)(w\lambda) \cdot \phi^\psi_{w\lambda} = D\phi^\psi_{w\lambda} = D\phi^\psi_\lambda = \Gamma(D)(\lambda) \cdot \phi^\psi_\lambda,
\]
by \ref{eq:3.1}. This proves the proposition, since $\phi^\psi_\lambda \neq 0$. \hfill \Box

4.2. Odd Weyl group invariance.

4.5. Recall the notation from \ref{section:2.4}. In addition, we let $\Sigma^0_1 = \{ \lambda \in \Sigma_1 \mid \langle \lambda, \lambda \rangle = 0 \}$ and $\Sigma^1_1 = \Sigma_1 \setminus \Sigma^0_1$, where $\langle \cdot, \cdot \rangle$ denotes the dual form of $b$ on $a^*$.

Fix $\lambda \in \Sigma^1_1$. We define $b^\theta(x, y) = b(x, \theta y)$ for all $x, y \in g$. The restriction of $b^\theta$ is a symplectic form on $g^\lambda_{1, a}$ \cite{AHZ10} Proposition 2.10, and this defines on $g^\lambda_{1, a}$ the structure of symplectic $m_0$-modules (where we recall that $m = \mathfrak{g}(a)$). Moreover, $\theta : g^\lambda_{1, a} \to g^\lambda_{1, a}$ is an $m_0$-equivariant symplectomorphism. Let $x_j, \tilde{x}_j \in g^\lambda_{1, a}$ be a $b^\theta$-symplectic basis, i.e.
\[
b^\theta(x_i, x_j) = b^\theta(\tilde{x}_i, \tilde{x}_j) = 0 , \ b^\theta(x_i, \tilde{x}_j) = 2\delta_{ij} .
\]
Let $g^\lambda_{1, a} \oplus g^1_{1, a} = \mathfrak{t}^\lambda_1 \oplus \mathfrak{p}^\lambda_1$ where $\mathfrak{t}^\lambda_1 \subset \mathfrak{t}_1$ and $\mathfrak{p}^\lambda_1 \subset \mathfrak{p}_1$. Set $x_j = y_j + z_j$ and $\tilde{x}_j = \tilde{y}_j + \tilde{z}_j$, according to this decomposition. Then
\[
b(y_i, y_j) = b(\tilde{y}_i, \tilde{y}_j) = b(z_i, z_j) = b(\tilde{z}_i, \tilde{z}_j) = 0 , \\
b(y_i, z_j) = b(\tilde{y}_i, \tilde{z}_j) = b(y_i, \tilde{z}_j) = b(\tilde{y}_i, z_j) = 0 , \\
b(y_i, \tilde{y}_j) = b(\tilde{z}_j, z_i) = \delta_{ij} ,
\]
for all $i, j$ and $h \in a$. Let $m^\lambda_0 = [\mathfrak{t}^\lambda_1, \mathfrak{t}^\lambda_1]$. 
Lemma 4.6. Retain the above notation, and define \( A_\lambda \in \mathfrak{a} \) by \( b(A_\lambda, h) = \lambda(h) \) for all \( h \in \mathfrak{a} \). We have the following equations.

(i). For all \( i, j \),
\[
[y_i, z_j] = [\bar{y}_i, \bar{z}_j] = 0, \quad [\bar{y}_i, z_j] = -[y_i, \bar{z}_j] = \delta_{ij} A_\lambda.
\]

(ii). For all \( i, j \),
\[
[y_i, y_j] = -[z_i, z_j], \quad [\bar{y}_i, \bar{y}_j] = -[\bar{z}_i, \bar{z}_j], \quad [y_i, \bar{y}_j] = -[z_i, \bar{z}_j].
\]

(iii). For all \( i, j, k \),
\[
\begin{align*}
[[y_i, y_j], y_k] &= [[\bar{y}_i, \bar{y}_j], \bar{y}_k] = 0, \\
[[y_i, y_j], \bar{y}_k] &= \langle \lambda, \lambda \rangle (\delta_{jk} y_i + \delta_{ik} y_j), \\
\end{align*}
\]
\[
[[y_i, \bar{y}_j], y_k] = -\langle \lambda, \lambda \rangle (\delta_{jk} \bar{y}_i + \delta_{ik} y_j), \\
[[y_i, \bar{y}_j], \bar{y}_k] &= \langle \lambda, \lambda \rangle \delta_{ik} \bar{y}_j.
\]

(iv). For all \( i, j, k \),
\[
\begin{align*}
[[y_i, y_j], z_k] &= [[\bar{y}_i, \bar{y}_j], \bar{z}_k] = 0, \\
[[y_i, y_j], \bar{z}_k] &= \langle \lambda, \lambda \rangle (\delta_{jk} z_i + \delta_{ik} z_j), \\
\end{align*}
\]
\[
[[y_i, \bar{y}_j], z_k] = -\langle \lambda, \lambda \rangle (\delta_{jk} \bar{z}_i + \delta_{ik} \bar{z}_j), \\
[[y_i, \bar{y}_j], \bar{z}_k] &= \langle \lambda, \lambda \rangle \delta_{ik} \bar{z}_j.
\]

Proof. In the following, to simplify notation, we let
\[
(y,z), (y',z'), (y'',z''), (y''',z''') \in \{ (y_j, z_j), (\bar{y}_j, \bar{z}_j) \mid \ell = 1, \ldots, \frac{1}{2} m_{1,\lambda} \}.
\]

We have \( [g^1_{1,\lambda}, g^1_{1,\lambda}] = 0 \). Applying this to \( x_i, x_j \) and taking \( \mathfrak{e} \)- and \( \mathfrak{p} \)-projections gives the equations
\[
[y, y'] = [z, z'], \quad [y, z'] = -[y', z].
\]

This proves (ii), and reduces (i) to the cases \( (y, y') \in \{ (y_i, y_j), (y_i, \bar{y}_j) \} \). Since \( 2\lambda \not\in \Sigma \), we have \( [y, z'] \in \mathfrak{p}_0 \cap (\mathfrak{a} \oplus g^2_{2,\lambda}) = \mathfrak{a} \). On the other hand, for \( h \in \mathfrak{a} \),
\[
b([y, z'], h) = -\lambda(h) b(y, y') = \begin{cases} 0 & (y, y') \in \{ (y_i, y_j), (y_i, \bar{y}_j) \}, \\
-\delta_{ij} b(A_\lambda, h) & (y, y') = (y_i, \bar{y}_j). \end{cases}
\]

The assertions in (i) follow by supersymmetry.

To prove (iii), we observe
\[
b([y, y'], [y'', y''']) = b([y, y'], [y'', y''']) = -b([y, y'], [z'', z''']) = -b([y, y'], [z'', z''']) = -b([y, y'], [z'', z''']).
\]

Since \( b(z'', z''') = b(y'', y''') \), (iii) follows from (iv).

Finally,
\[
[[y, y'], z''] = [y, [y', z'']] + [y', [y, z'']] =: (*)\)
\]

This is zero if \( (y, y', z'') \in \{ (y_i, y_j, z_k), (\bar{y}_i, \bar{y}_j, \bar{z}_k) \} \), as follows from (i). Next, if \( (y, y', z'') \in \{ (y_i, y_j, \bar{z}_k), (\bar{y}_i, \bar{y}_j, z_k) \} \), then, again by (i),
\[
(*) = \mp (\delta_{jk} [y, A_\lambda] + \delta_{ik} [y', A_\lambda]) = \pm \langle \lambda, \lambda \rangle (\delta_{jk} z + \delta_{ik} z')
\]

where the sign \( + \) occurs in the first case, and \( - \) in the second. Finally, if we have
\[
(y, y', z'') \in \{ (y_i, y_j, \bar{z}_k), (\bar{y}_i, \bar{y}_j, z_k) \},
\]

\[
(*) = [y, [y', z'']] = \pm \delta_{jk} [y, A_\lambda] = \mp \langle \lambda, \lambda \rangle \delta_{jk} z
\]

where the sign \( - \) occurs in the first case, and \( + \) in the second. This proves (iv), and as remarked, Assertion (iii) follows. \( \square \)
Corollary 4.7. The graded subalgebra of \( g \) generated by \( g_1(\lambda) := g_{1,\theta}^\lambda + g_{1,\theta}^{-\lambda} \) is exactly \( m_{\lambda}^0 \oplus \mathcal{C}A_{\lambda} \oplus g_1(\lambda) \). It is invariant under \( \theta \) and \( a \). In particular, \( m_{\lambda}^0 \oplus t_{\bar{1}}^\lambda \) is a graded subalgebra of \( \mathfrak{k} \), and it leaves \( a \oplus p_1^\lambda \) invariant. The radical of \( b \) on \( m_{0}^0 \oplus \mathcal{C}A_{\lambda} \oplus g_1(\lambda) \) is \( m_{\lambda}^0 \oplus \mathcal{C}A_{\lambda} \) if \( \lambda \in \Sigma_0 \), and \( 0 \) if \( \lambda \in \Sigma_1 \).

Proof. The last statement follows from part (iii) in Lemma 4.9 and its proof. The other statements are immediate from the lemma. \( \square \)

4.8. To show that the image of \( \Gamma \) is satisfies ‘odd Weyl group invariance’, we adopt and adapt a technique due to Lepowsky, cf. [GVSS].

Let \( n_{\lambda} = g_{1,\theta}^\lambda \), \( n_{\lambda} \) \( = \bigoplus_{\Sigma} g_{\mu}^\lambda \) and \( t_{\lambda} = m_{0}^0 \oplus t_{\bar{1}}^\lambda \). If \( \lambda \in \Sigma_0 \), we choose \( h_0 \in a \) such that \( \lambda(h_0) = 1 \) and \( b(h_0, h_0) = 0 \); in this case, we set \( a_{\lambda} = (h_0, A_{\lambda}) \). If, on the other hand, \( \lambda \in \Sigma_1 \), then we let \( a_{\lambda} = \mathcal{C}A_{\lambda} \). In any case, \( a_{\lambda} \) is \( b \)-non-degenerate, and \( m_{\lambda} := t_{\lambda} \oplus a_{\lambda} \oplus n_{\lambda} \) is a graded subalgebra of \( g \). Moreover, \( m_{\lambda} \) is \( \theta \)-invariant with \( p_{\lambda} := m_{\lambda} \cap p = a_{\lambda} \oplus p_{1}^\lambda \). We write \( a_{\lambda}^+ := a_{\lambda} \cap (a_{\lambda})^\perp \).

We denote by \( I_{\lambda} \subset S(a) \) and \( I_{\lambda, m_{\lambda}} \subset S(a_{\lambda}) \) the image of \( S(a \oplus p_{1}^\lambda) \) and \( S(p_{1}^\lambda) \), respectively, under the projection onto \( S(a) \) along \( a^+ S(p) \). We observe that since \( [a_{\lambda}^+, t_{\lambda}^\lambda] = [a_{\lambda}^+, p_{1}^\lambda] = 0 \), we have \( S(p_{1}^\lambda) S(a_{\lambda}^+) = S(a \oplus p_{1}^\lambda) \). Let \( \Gamma_{\lambda} \) denote the Harish-Chandra homomorphism for \( m_{\lambda} \). Whenever \( \lambda \in \Sigma_0 \), then we let \( J_{\lambda} = I_{\lambda} \) and \( J_{\lambda, m_{\lambda}} = I_{\lambda, m_{\lambda}} \). For \( \lambda \in \Sigma_1 \), we denote by \( J_{\lambda, m_{\lambda}} \) the sets \( J_{\lambda} = J_{\lambda, m_{\lambda}}(\mathcal{U}(m_{\lambda})) \) and \( J_{\lambda, m_{\lambda}} S(a_{\lambda}^+) \), respectively. Then we have \( J_{\lambda} = J_{\lambda, m_{\lambda}} S(a_{\lambda}^+) \) in any case.

Lemma 4.9. Assume that \( \Gamma_{\lambda} : \mathcal{U}(m_{\lambda}) \rightarrow J_{\lambda, m_{\lambda}} \). Then \( \mathcal{U}(\mathfrak{g})^\mathfrak{k} \subset J_{\lambda} \).

Proof. For any \( u \in \mathcal{U}(\mathfrak{g})^\mathfrak{k} \), there exists a unique \( u_{0} \in \mathcal{B}(S(a \oplus p_{1}^\lambda)) \) such that \( u \equiv u_{0} \) (mod \( n_{\lambda} \mathcal{U}( \mathfrak{g} ) + \mathcal{U}( \mathfrak{g} )\mathfrak{k} \)). (This follows from the Poincaré–Birkhoff-Witt theorem, applied to the vector space decomposition \( \mathfrak{g} = a \oplus p_{1}^\lambda \oplus n_{\lambda}^+ \oplus \mathfrak{k} \).) In particular, if \( u = \mathfrak{k}_{\lambda} \)-invariant, then so is \( u_{0} \). Moreover,

\[
\mathfrak{k}_{\lambda} - (u_{0})_{a} = u_{a} = u - u - u_{0} + u_{0} - (u_{0})_{a} \in n \mathcal{U}( \mathfrak{g} ) + \mathcal{U}( \mathfrak{g} )\mathfrak{k},
\]

so \( u_{a} = (u_{0})_{a} \).

Let \( u \in \mathcal{U}(\mathfrak{g})^\mathfrak{k} \). We have \( S(p_{1}^\lambda) S(a_{\lambda}) = S(a \oplus p_{1}^\lambda) \) and \( \mathcal{B}(S(p_{1}^\lambda)) \subset \mathcal{U}(m_{\lambda}) \). Hence, there exist \( v_{j} \in \mathcal{U}(m_{\lambda}) \) and \( w_{j} \in \mathcal{U}(a_{\lambda}^+) \) such that \( u_{0} = \sum_{j} w_{j} v_{j} \). For \( \mu \in a^* \), we have

\[
\Gamma(u)(\mu) = u_{a}(\mu + g) = (u_{0})_{a}(\mu + g) = \sum_{j} w_{j}(\mu + g)(v_{j})_{a}(\mu + \frac{1}{2}m_{1,\lambda})
\]

so we have proved our claim. \( \square \)

Lemma 4.10. The set \( I_{\lambda} \) is exactly the common domain of the differential operators \( D \) with rational coefficients, whose local expression at any super-regular \( \mu \in a^* \) is exactly \( \gamma_{\mu}(d) \), for some fixed but arbitrary \( d \in S(p_{1}^\lambda) \).

Proof. This is the main content of the discussion in [AHZ10], Section 3.2]. \( \square \)

Lemma 4.11. Let \( (\lambda, \lambda) = 0 \). For any \( z \in \mathbb{C} \), the automorphism \( p \mapsto p(\cdot + z\lambda) \) of \( S(a_{\lambda}) \) leaves \( I_{\lambda, m_{\lambda}} \) invariant.

Proof. For any fixed \( p \in I_{\lambda, m_{\lambda}} \), the statement is that for any \( k \), the polynomial map \( z \mapsto \partial_{\lambda}^{k} (p(\cdot + z\lambda)) = (A_{\lambda}^{k}) : \mathbb{C} \rightarrow S(a_{\lambda})/(A_{\lambda}^{k}) \) is zero. By degree considerations, it takes values in a finite-dimensional vector space. Any non-zero univariate vector-valued polynomial has but a finite set of zeros, so it suffices to prove that it vanishes
at integer $z$, and by a trivial induction, for $z = 1$. Thus, all we need to show is that $p \mapsto p(\cdot + \lambda)$ leaves $I_{\lambda, m_\lambda}$ invariant.

We have $a_\lambda = \langle h_0, A_\lambda \rangle_C$, and $\partial_{A_\lambda}(h_0^k A_\lambda^l) = kh_0^{k-1} A_\lambda^l$. By Lemma 4.10 and $ABZ[10]$ Theorem 3.25, $I_{\lambda, m_\lambda}$ is the common domain of the operators $A_\lambda^j \partial_{A_\lambda}$ where $j = 1, \ldots, q := \frac{1}{2}m_{1, \lambda}$. Clearly, the latter is spanned by $h_0^k A_\lambda^l$ where $k \in \mathbb{N}$ and $\ell \geq \min(k, q)$. The automorphism $p \mapsto p(\cdot + \lambda)$ maps $h_0^k A_\lambda^l$ to the element

$$(h_0 + 1)^k A_\lambda^l = \sum_{j=0}^{k} \binom{k}{j} h_0^j A_\lambda^l,$$

and therefore leaves $I_{\lambda, m_\lambda}$ invariant. \hfill \Box

**Lemma 4.12.** We have $\Gamma_{m_\lambda}(\mathfrak{U}(m_\lambda)^{f_\lambda}) \subset I_{\lambda, m_\lambda}$ for $\lambda \in \Sigma^\dagger_1$.

**Proof.** We have $\mathfrak{U}(m_\lambda) = \beta(S(p_\lambda)) \oplus \mathfrak{U}(\mathfrak{g})\mathfrak{f}$, so in view of Lemma 4.11 it suffices to prove that $\beta(p_\lambda) \in I_{\lambda, m_\lambda}$ for all $p \in S(p_\lambda)^{f_\lambda}$.

Let $k \in \mathbb{N}$, $\ell \geq \min(k, q)$ where $q = \frac{1}{2}m_{1, \lambda}$, and set

$$p_{k\ell} = \sum_{j=0}^{\min(k, q)} \binom{k}{j} h_0^{k-j} A_\lambda^{\ell-j} Z^j$$

where $Z = z_1 \hat{z}_1 + \cdots + z_q \hat{z}_q$. We have $\text{ad}(y_n)(Z) = A_\lambda z_n$ and $\text{ad}(\tilde{y}_n)(Z) = A_\lambda \tilde{z}_n$, so

$$\text{ad}(y_n)(p_{k\ell}) = \sum_{j=0}^{\min(k, q)} \left[ -(k-j) \binom{k}{j} h_0^{k-j} A_\lambda^{\ell-j} Z^j + j \binom{k}{j} h_0^{k-j} A_\lambda^{\ell-j+1} z_n Z^{j-1} \right]$$

Since $z_n Z^0 = 0$, we have $\text{ad}(y_n)(p_{k\ell}) = 0$. Similarly, $\text{ad}(\tilde{y}_n)(p_{k\ell}) = 0$. By definition, $f_\lambda$ is generated by $y_j$, $\tilde{y}_j$, so $p_{k\ell} \in S(p_\lambda)^{f_\lambda}$. Moreover, $p_{k\ell}(\mu) = (h_0^k A_\lambda^l)(\mu)$ for all $\mu \in \mathfrak{a}^*$, and this uniquely determines $p_{k\ell}$. It follows that $S(p_\lambda)^{f_\lambda}$ is spanned by the $p_{k\ell}$ where $k \geq 0$ and $\ell \geq \min(k, q)$.

Next, we remark that $[m_\lambda, m_\lambda] \cap p = CA_\lambda$. Hence, applying $\beta$ to $p_{k\ell}$ does not increase the $h_0$-degree. Since $A_\lambda$ is central in $m_\lambda$, $\beta$ does also not decrease the $A_\lambda$-degree. It follows that $\beta(p_{k\ell}) \in I_{\lambda, m_\lambda}$, and hence the assertion. \hfill \Box

4.13. For $\lambda \in \Sigma^\dagger_1$, we have defined $J_{\lambda, m_\lambda} = \Gamma(\mathfrak{U}(m_\lambda)^{f_\lambda})$. To prove our main result, we have to determine this set explicitly.

To that end, we will need a basic understanding of the relation between $\Gamma$ and the ‘restriction’ map on $S(p)$, and this will also be useful below. Thus, consider $\mathfrak{U}(\mathfrak{g})$ with its standard filtration, which we denote $F_p \mathfrak{U}(\mathfrak{g})$. The associated graded algebra $\text{gr} \mathfrak{U}(\mathfrak{g})$ has graded pieces $\text{gr}_p \mathfrak{U}(\mathfrak{g}) = F_p \mathfrak{U}(\mathfrak{g})/F_{p-1} \mathfrak{U}(\mathfrak{g})$ and is supercommutative. The canonical map $\mathfrak{g} \to F_1 \mathfrak{U}(\mathfrak{g})$ induces a canonical map $\mathfrak{g} \to \text{gr}_1 \mathfrak{U}(\mathfrak{g}) \subset \text{gr} \mathfrak{U}(\mathfrak{g})$, which in turn gives a canonical map $S(p) \to \text{gr} \mathfrak{U}(\mathfrak{g})$.

This map is an isomorphism, and we henceforth identify the algebras $S(\mathfrak{g})$ and $\text{gr} \mathfrak{U}(\mathfrak{g})$. Under this identification, it is known that $\beta : S(\mathfrak{g}) \to \mathfrak{U}(\mathfrak{g})$ equals the inverse of the canonical map $\mathfrak{U}(\mathfrak{g}) \to \text{gr} \mathfrak{U}(\mathfrak{g})$. Since $\mathfrak{U}(\mathfrak{g})\mathfrak{f}$ is a filtered subspace of $\mathfrak{U}(\mathfrak{g})$, $S(p) = \mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{g})\mathfrak{f}$ as filtered super-vector spaces, and the isomorphism $\beta : S(p) \to \mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{g})\mathfrak{f}$ is inverse to the canonical map $\mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{g})\mathfrak{f} \to \mathfrak{U}(\mathfrak{g})$.

**Lemma 4.14.** The map $\text{gr} \Gamma : S(p) \to S(\mathfrak{a})$ induced by $\Gamma : \mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{g})\mathfrak{f} \to \mathfrak{U}(\mathfrak{a})$, considered as a map of filtered super-vector spaces, is the projection along $\mathfrak{a}^* S(p)$. 
Proof. Denote the image of \( p \in S(p) \) in \( S(a) \) by \( \bar{p} \). Since \( 1 - \theta : n \to a^\pm \) is surjective, we have \( a^\pm \subset \mathfrak{t} \oplus n \), so that \( p - \bar{p} \in (n \oplus \mathfrak{t})S^{d-1}(p) \) for all \( p \in S^d(p) \).

The supersymmetrisation map \( \beta : S(a) \to \bar{\mathfrak{u}}(a) \) is the identity, and on the level of \( \mathfrak{g} \), for all \( p_j \in S^{d_j}(p) \), \( j = 1, 2 \), \( \beta(p_1p_2) - \beta(p_1)\beta(p_2) = \sum_{k<d_1+d_2} \beta(S^k(p)) \). It follows that

\[
\beta(p) - \bar{p} \in n\bar{\mathfrak{u}}(\mathfrak{g}) + \bar{\mathfrak{u}}(\mathfrak{g})\mathfrak{t} + \sum_{k<d} \beta(S^k(p)) \quad \text{for all } \quad p \in S^d(p) .
\]

Since \( \Gamma(n\bar{\mathfrak{u}}(\mathfrak{g}) + \bar{\mathfrak{u}}(\mathfrak{g})\mathfrak{t}) = 0 \), we conclude

\[
(4.1) \quad \Gamma(\beta(p)) - \bar{p} \in \bigoplus_{k<d} S^k(a) \quad \text{for all } \quad p \in S^d(p) .
\]

This proves the claim.

This proves the claim.

4.15. Let \( \lambda \in \Sigma_1 \), \( c = \langle \lambda, \lambda \rangle \neq 0 \), \( 2q = m_{1,\lambda} \). Choose some square root of \( c \) and set

\[ a = c^{-1}A_\lambda \, , \quad w_j = c^{-1/2}z_j \, , \quad \bar{w}_j = c^{-1/2}z_j \, , \quad v_j = c^{-1/2}y_j \, , \quad \bar{v}_j = c^{-1/2}y_j \, . \]

Then

\[
\begin{align*}
[a, w_j] & = v_j \, , \quad [a, v_j] = w_j \, , \quad [a, \bar{w}_j] = \bar{v}_j \, , \quad [a, \bar{v}_j] = \bar{w}_j \, , \\
[v_j, w_k] & = [\bar{v}_j, \bar{w}_k] = 0 \, , \quad [-v_j, \bar{w}_k] = [\bar{v}_j, w_k] = \delta_{jk}a .
\end{align*}
\]

Let \( \bar{J}_{\lambda, m_\lambda} \) be the subalgebra of \( \mathbb{C}[a] \) generated by \( a^2 - q^2 \) and \( a(a^2 - q^2)^g \).

**Lemma 4.16.** Let \( \lambda \in \Sigma_1 \), \( \langle \lambda, \lambda \rangle \neq 0 \). Then \( \Gamma_{m_\lambda}(\mathfrak{u}(m_\lambda)^{\lambda}) = \bar{J}_{\lambda, m_\lambda} \).

Proof. We abbreviate \( \Gamma = \Gamma_{m_\lambda} \). Since \( \mathfrak{u}(m_\lambda)^{\lambda} = \beta(S(p_\lambda)^{\lambda}) \oplus (\mathfrak{u}(m_\lambda)^{\lambda}) \), it is sufficient to prove that \( \Gamma \circ \beta \) maps \( S(p_\lambda)^{\lambda} \) onto \( \bar{J}_{\lambda, m_\lambda} \).

To that end, let \( W = w_1\bar{w}_1 + \cdots + w_q\bar{w}_q \). Then by [AHZ10] 4.7], \( S(p_\lambda)^{\lambda} \) is generated by the elements \( P_2 = a^2 + 2W \) and

\[
P_{2q+1} = a^{2q+1} + \sum_{k=1}^q (-1)^{\frac{1}{2}k(k+3)}a_{2q+1,k}a^{2(q-k)+1}W^k ,
\]

where

\[
a_{Nk} = \sum_{i=(k-N)_+}^{k-1} \left( -\frac{1}{2} \right)^i N \cdots (N - k + i + 1) \frac{(k - 1 + i)!}{(k - 1 - i)!} .
\]

In particular,

\[
S^m(p_\lambda)^{\lambda} = \begin{cases} \mathbb{C}P_{2k}^2 & m = 2k , \\ \mathbb{C}P_{2q+1}P_{2}^{k-q} & m = 2k + 1, k < q , \\ 0 & m = 2k + 1, k > q . \end{cases}
\]

In view of this statement, to see that \( \Gamma(\beta(S(p_\lambda)^{\lambda})) \subset \bar{J}_{\lambda, m_\lambda} \), it suffices to prove the following: For all \( k \in \mathbb{N} \) and \( \ell \leq k \), \( \Gamma \circ \beta \) maps \( P_{2k}^\ell \), and if \( \ell \geq q \), then also \( P_{2q+1}P_{2}^{k-q} \), to \( \bar{J}_{\lambda, m_\lambda} \). We will prove this assertion by induction on \( k \).

First, we consider \( L_2 = \beta(P_2) \). Define \( \tilde{W} = \tilde{v}_1w_1 + \cdots + \tilde{w}_qw_q \). Then we have \( L_2 = a^2 + W - \tilde{W} \). Since

\[
W + \tilde{W} = \sum_{j=1}^q [w_j, \bar{w}_j] \in \mathfrak{m}_0^\lambda \subset \mathfrak{t}_\lambda ,
\]

we have, modulo \( n\bar{\mathfrak{u}}(\mathfrak{g}) + \bar{\mathfrak{u}}(\mathfrak{g})\mathfrak{t} \),

\[
W - \tilde{W} \equiv 2W \equiv - \sum_{j=1}^q v_j\bar{w}_j \equiv - \sum_{j=1}^q [v_j, \bar{w}_j] = qa .
\]
where we have used $u_j = v_j + w_j \in n_\lambda$ and $[v_j, w_j] = -a$. Thus,

$$L_{2,a} = a^2 + 2qa = (a + q)^2 - q^2,$$

and since $\frac{1}{2} \text{str}_{n_\lambda} \text{ad} |_{a_\lambda} = -\frac{1}{2} m_{1,\lambda} = -q\lambda$,

$$\Gamma_\lambda(L_2) = a^2 - q^2 \in \tilde{J}_{\lambda,m_\lambda}.$$

Now, assume we have proved the assertion for $k - 1$, and we wish to prove it for $k$. We have $L^k_2 \in \mathfrak{U}(m_\lambda)^{\mathfrak{k}_\lambda}$, so $L^k_2 \equiv \beta(p) \pmod{\mathfrak{U}(m_\lambda)^{\mathfrak{k}_\lambda}}$ for some $p \in S(p_\lambda)^{\mathfrak{k}_\lambda}$. On the other hand, $\Gamma(L^k_2) = (a^2 - q^2)^k$ and the $a$-restriction of $P^k_2$ is $P^k_2 = a^{2k}$.

Now, the maps $\Gamma_\lambda \circ \beta$ and $p \mapsto \tilde{p}$ coincide to leading order by Lemma 444 so that $p' = P^k_2 - p \in \bigoplus_{j < 2k} S^j(p_\lambda)$. The polynomial $p'$ being $\mathfrak{k}_\lambda$-invariant, we obtain $\Gamma_\lambda(\beta(p')) \in J_{\lambda,m_\lambda}$ by the inductive assumption. Since we have shown that $\Gamma_\lambda(\beta(p)) = (a^2 - q^2)^k \in J_{\lambda,m_\lambda}$, it follows that $\Gamma_\lambda \circ \beta$ maps $P^k_2$ to $\tilde{J}_{\lambda,m_\lambda}$.

If $k \geq q$, then we need to show that $\Gamma_\lambda(P_{2q+1} P^{k-q}_2) \in J_{\lambda,m_\lambda}$, too. Similarly as above, we may take $p \in S(p_\lambda)$ such that $aL^k_2 \equiv \beta(p) \pmod{\mathfrak{U}(m_\lambda)^{\mathfrak{k}_\lambda}}$, but we will need a little argument to prove that $p$ is $\mathfrak{k}_\lambda$-invariant.

To that end, we compute for $(u, v, w) \in \{(u_j, v_j, w_j), (\tilde{u}_j, \tilde{v}_j, \tilde{w}_j)\}$, where we let $u_j = v_j + w_j$ and $\tilde{u}_j = \tilde{v}_j + \tilde{w}_j$,

$$[v, aL^2_2] = [v, a]L^2_2 = -wL^2_2 \equiv -uL^2_2 + vL^2_2 \equiv [v, L^2_2] = 0 \pmod{n \mathfrak{U}(g) + n \mathfrak{U}(g)^{\mathfrak{k}_\lambda}}.$$

Since $[v, aL^2_2] = \beta([v, p]) \pmod{\mathfrak{U}(g)^{\mathfrak{k}_\lambda}}$, this implies $\beta([v, p]) \in n_\lambda \mathfrak{U}(g_\lambda) \pm n_\lambda$.

Observe that $\theta(aL^2_2) = -aL^2_2$ since $\theta(L_2) = L_2$. Because $\theta(v) = v$, we find

$$-\beta([v, p]) = \theta([v, p]) \in n_\lambda \mathfrak{U}(g_\lambda) \cap n_\lambda \mathfrak{U}(g_\lambda) = 0.$$

Because $\beta$ is injective, this shows $[v, p] = 0$, and it follows that $p$ is $\mathfrak{k}_\lambda$-invariant.

Since $L^k_2$ is $\mathfrak{k}_\lambda$-invariant, Lemma 53 implies

$$\Gamma_\lambda(aL^k_2) = \Gamma_\lambda(a) \Gamma_\lambda(L^k_2) = (a - q)(a^2 - q^2)^k \in \tilde{J}_{\lambda,m_\lambda}.$$

Of course, $P_{2q+1} P^{q-k}_2 = a^{2k+1}$, and the same argument as above allows us to conclude by induction that $\Gamma_\lambda(\beta(P_{2q+1} P^{q-k}_2)) \in \tilde{J}_{\lambda,m_\lambda}$.

Moreover, if $k = q$, then $\Gamma_\lambda(\beta(P_{2q+1}))$ and $\Gamma_\lambda(aL^q_2) = \beta(p) \pmod{\mathfrak{U}(m_\lambda)^{\mathfrak{k}_\lambda}}$ coincide to leading order. Thus, if $p \in S(p_\lambda)^{\mathfrak{k}_\lambda}$ is such that $aL^k_2 \equiv \beta(p) \pmod{\mathfrak{U}(m_\lambda)^{\mathfrak{k}_\lambda}}$, then the difference $P_{2q+1} - p$ is, by the inductive assumption, a polynomial in $P_2$. Since $\theta(P_{2q+1}) = -P_{2q+1}, \theta(aL^q_2) = -aL^q_2, \theta(P_2) = P_2$, and \(\beta\) is \(\theta\)-equivariant, we find that $p = P_{2q+1} + \theta(p)$, so $\Gamma(p) = \Gamma(P_{2q+1}) = \Gamma(aL^q_2) = (a - q)(a^2 - q^2)^q$.

This finally proves our claim that $\Gamma_\lambda(\mathfrak{U}(m_\lambda)^{\mathfrak{k}_\lambda}) \subset \tilde{J}_{\lambda,m_\lambda}$. Since we have the identities $\Gamma_\lambda(\beta(P_2)) = a^2 - q^2$ and $\Gamma_\lambda(\beta(P_{2q+1})) = (a-q)(a^2-q^2)^q$, we have also proved the equality $\Gamma_\lambda(\mathfrak{U}(m_\lambda)^{\mathfrak{k}_\lambda}) = \tilde{J}_{\lambda,m_\lambda}$.

**Corollary 4.17.** Let $\lambda \in \Sigma^1$. Then $J_{\lambda,m_\lambda} = \mathbb{C}[a^2 - q^2, (a-q)(a^2-q^2)^q]$ where $\langle \lambda, \lambda \rangle \cdot a = A_\lambda$ and $2q = m_{1,\lambda}$.

### 4.3. The Harish-Chandra isomorphism.

4.18. We can now finally state and prove our main result. To that end, recall some notation. Let $(\mathfrak{g}, \mathfrak{t}, \theta)$ be a strongly reductive symmetric superpair of even type, and $\mathfrak{a}$ an even Cartan subspace, giving rise to the set $\Sigma$ of restricted roots. Denote by $W_0$ the Weyl group of $\Sigma_0$ and fix a positive system $\Sigma^+$. Let

$$I(\mathfrak{a}) = S(\mathfrak{a}) \cap \bigcap_{\lambda \in \Sigma_1} I_\lambda \quad \text{and} \quad J(\mathfrak{a}) = S(\mathfrak{a})^{W_0} \cap \bigcap_{\lambda \in \Sigma_1} J_\lambda$$
Theorem 4.19. The image of $\Gamma$ is $J(\mathfrak{a})$, and its kernel is $(\mathfrak{U}(\mathfrak{g}))^\mathfrak{t}$. Therefore, it induces an algebra isomorphism $D(X)_G \to J(\mathfrak{a})$ for any global cs form $(G, K)$ of $(\mathfrak{g}, \mathfrak{t})$, where $D(X)_G$ is the set $G$-invariant differential operators on $X = G/K$.

Proof. Since $\mathfrak{U}(\mathfrak{g})^\mathfrak{t} = S(\mathfrak{j}(\mathfrak{g})) \otimes \mathfrak{U}(\mathfrak{g'})^\mathfrak{t}$, where $\mathfrak{g}' = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{t}' = \mathfrak{g}' \cap \mathfrak{t}$, there is no restriction in assuming that $\mathfrak{j}(\mathfrak{g}) = 0$. By the existence of a standard non-compact global cs form (Proposition 2.11), we may apply Proposition 4.2. By Lemma 4.12 and Lemma 4.9, the image of $\Gamma$ is contained in $J(\mathfrak{a})$. (Note that the assumption of the latter lemma is trivially verified for $\lambda \in \Sigma^1_1$.)

Define $I_0 = S(\mathfrak{a})^{W_0} \cap \bigcap_{\lambda \in \Sigma^0_1} I_\lambda$. For $\lambda \in \Sigma^1_1$, we have, by Proposition 4.5 and Corollary 4.4,

$$I_\lambda = \mathbb{C}[a^2, a^{2q+1}] S(A^1_\lambda) \quad \text{and} \quad J_\lambda = \mathbb{C}[a^2 - q^2, (a - q)(a^2 - q^2)] S(A^1_\lambda)$$

where $\langle \lambda, \lambda \rangle \cdot a = A_{\lambda}$ and $2q = m_{1, \lambda}$.

Then $I_0$ and $J_\lambda$ are $\mathbb{Z}$-graded subspaces of $S(\mathfrak{a})$, whereas $J_\lambda$ is a filtered subspace with $\text{gr} J_\lambda = J_\lambda$. We have $I(\mathfrak{a}) = I_0 \cap \bigcap_{\lambda \in \Sigma^0_1} I_\lambda$ and $J(\mathfrak{a}) = I_0 \cap \bigcap_{\lambda \in \Sigma^1_1} J_\lambda$. For any filtered subspace $V$ of $S(\mathfrak{a})$, $\text{gr} V$ injects into $\text{gr} S(\mathfrak{a}) = S(\mathfrak{a})$. In this sense, one has

$$\text{gr} J(\mathfrak{a}) = I_0 \cap \text{gr} \bigcap_{\lambda \in \Sigma^1_1} J_\lambda \subset I_0 \cap \bigcap_{\lambda \in \Sigma^1_1} J_\lambda = I_0 \cap \bigcap_{\lambda \in \Sigma^1_1} I_\lambda = I(\mathfrak{a}) .$$

On the other hand, $\text{gr} \Gamma$ is the ‘restriction’ map $S(\mathfrak{p}) \to S(\mathfrak{a})$, so Theorem 3.25 implies $I(\mathfrak{a}) = \text{im} \text{gr} \Gamma \subset \text{gr} J(\mathfrak{a})$, i.e. $I(\mathfrak{a}) = \text{gr} J(\mathfrak{a})$.

Consider the filtered complex

$$C: \quad 0 \longrightarrow (\mathfrak{U}(\mathfrak{g}))^\mathfrak{t} \longrightarrow \mathfrak{U}(\mathfrak{g})^\mathfrak{t} \longrightarrow J(\mathfrak{a}) \longrightarrow 0 .$$

We wish to see that this is an exact sequence, i.e. $H(C) = 0$. Since the filtration on $C$ is bounded below and exhaustive, the spectral sequence ($E^r$) of the filtration converges to $H(C)$ [Weinstein94, Theorem 5.5.1]. By Lemma 4.14

$$E^0 = \text{gr} C: \quad 0 \longrightarrow (S(\mathfrak{g}))^\mathfrak{t} \longrightarrow S(\mathfrak{g})^\mathfrak{t} \longrightarrow R(\mathfrak{a}) \longrightarrow 0$$

where $R(S(\mathfrak{g}))^\mathfrak{t} = 0$ and on $S(\mathfrak{p})$, $R$ is the ‘restriction’ map $p \mapsto \tilde{p}$. (Observe that $S(\mathfrak{g})^\mathfrak{t} \subset S(\mathfrak{g}) = \text{gr} \mathfrak{U}(\mathfrak{g})$ identifies with $\text{gr} (\mathfrak{U}(\mathfrak{g}))^\mathfrak{t}$.)

By Theorem 3.25, $R$ is injective on $S(\mathfrak{p})^\mathfrak{t}$, with image $I(\mathfrak{a})$. Thus, $E^1 = H(\text{gr} C) = 0$, and this proves the theorem.

A. Appendix: The category of cs manifolds

A.1. Unlike their ungraded counterparts, simple complex Lie superalgebras do not in general possess real forms whose even part is compact [Serre83]. At first sight, this seems to make the generalisation of many aspects of harmonic analysis to the graded setting unfeasible. On the other hand, it is common in the physics community to work with real ‘variables’ and complex ‘Grassmann variables’, i.e. to choose a real form only of the even part.
Taken seriously, this leads to the category of $cs$ manifolds introduced by J. Bernstein [Ber96, DM99]. These are complex super-ringed spaces, locally isomorphic to real superdomains with complexified structure sheaves.

For ordinary real manifolds, complexifying the structure sheaves does not change anything. That is, one obtains a fully faithful embedding of real manifolds into $cs$ manifolds. However, for supermanifolds, matters do change, i.e. the embedding does not extend fully faithfully to real supermanifolds. However, this should not be seen as a defect, since it is precisely this fact that resolves some of the issues related to real structures in the super world. The following two reasons make $cs$ manifolds a useful tool for our purposes.

First, the usual theory of Berezin integration on real supermanifolds does not require the choice of an orientation in the odd coordinate directions, in particular, it does not require a real structure. This allows for the transposition of this theory to the category of $cs$ manifolds, as was observed by J. Bernstein [Ber96, DM99].

Second, the group objects in the category of $cs$ manifolds (which we call $cs$ Lie supergroups) can be described by linear actions of real Lie groups $G$ on complex Lie superalgebras satisfying suitable compatibility assumptions. This allowed, in Section 2, for the definition of compact and non-compact ‘$cs$ forms’.

The theory of $cs$ manifolds seems to have been largely ignored by the mathematical community, perhaps because its utility has not been fully appreciated. For this reason, we find it appropriate to briefly review the foundations. Most arguments will carry over from the theory of real supermanifolds, but there are a few subtle points. Let us mention two: the theory of linear $cs$ manifolds, and the theory of $cs$ Lie supergroups. In both cases, real structures on the even parts intervene in a non-trivial fashion.

A.1. Basic theory of $cs$ manifolds. We will use notions of ‘super’ mathematics. We refer the reader to [Lei80, Kos77, DM99].

**Definition A.2.** A complex super-ringed space is a pair $X = (X_0, \mathcal{O})$ where $X_0$ is a topological space and $\mathcal{O} = \mathcal{O}_X$ is a sheaf of complex super-commutative superalgebras. We define a subsheaf $\mathcal{N} \subset \mathcal{O}$ as $\mathcal{N}(U) = \mathcal{O}(U) \cdot \mathcal{O}(U)_1$, the ideal generated by the odd elements, for all open $U \subset X_0$. We denote $\pi_X : \mathcal{O} \to \mathcal{O}/\mathcal{N}$ the canonical sheaf morphism.

A morphism $\varphi : X \to Y$ of complex super-ringed spaces is a tuple $(f, f^*)$ where $f : X_0 \to Y_0$ is a continuous function and $f^* : \mathcal{O}_Y \to f_* \mathcal{O}_X$ is a morphism of sheaves of complex superalgebras (in particular, even and unital). One obtains a well-defined category of complex super-ringed spaces. Occasionally, when $f : X \to Y$ is a morphism of complex super-ringed spaces, we shall write $f : X_0 \to Y_0$ for the underlying continuous map, and $f^* : \mathcal{O}_Y \to f_* \mathcal{O}_X$ for the map of sheaves, thus slightly abusing the notation.

There is a natural functor from real super-ringed spaces to complex super-ringed spaces, given by the complexification of the structure sheaves. We shall denote its application by the subscript $\mathbb{C}$. This functor is neither full nor faithful. By complexification, we associate with the real supermanifolds $\mathbb{R}^{p|q} = (\mathbb{R}^p, C^\infty_{\mathbb{R}^p} \otimes \wedge (\mathbb{R}^q)^*)$ the complex super-ringed spaces $\mathbb{R}^{p|q}_{\mathbb{C}} = (\mathbb{R}^p, C^\infty_{\mathbb{R}^p} \otimes \wedge (\mathbb{C}^q)^*)$.

---

2The abbreviation $cs$ stands for ‘complex super’, but $cs$ manifolds are not the same thing as complex supermanifolds.
If $X = (X_0, \mathcal{O})$ is a complex super-ringed space, then an open subspace $U$ is a complex super-ringed space of the form $(U_0, \mathcal{O}_{|U_0})$, for some open subset $U_0 \subset X_0$. A cs domain of graded dimension $p|q$ (where $p, q \in \mathbb{N}$) is a complex super-ringed space isomorphic to an open subspace of $\mathbb{R}_C^{p|q}$; the cs domain whose underlying set is $\emptyset$ may be given any graded dimension.

Finally, a complex super-ringed space $X = (X_0, \mathcal{O})$ is called a cs manifold if $X_0$ is Hausdorff and second countable, and $X$ possesses a cover by open subspaces which are cs domains (whose graded dimensions may vary). Such open subspaces (and their underlying open sets) are called coordinate neighbourhoods. If the open cover may be chosen such that the coordinate neighbourhoods all possess the same graded dimension, then we say that $X$ is of (pure) graded dimension $p|q$. One obtains a category of cs manifolds as a full subcategory of complex super-ringed spaces. When no confusion is possible, we simply say that $X$ is a cs manifold, and denote its structure sheaf by $\mathcal{O}_X$ (and similarly for other letters of the alphabet). We will the global sections functor by $\Gamma$.

We apologise to the reader for the somewhat unfortunate parlance ‘cs manifold’; we point out that besides being extant terminology, obvious replacements such as ‘semi-real supermanifold’ would probably not constitute an improvement on the already rather laboured ‘super’ nomenclature. Moreover, although this is clearly not a consistent use of the abbreviation ‘cs’, we will often explicitly add the prefix ‘super’ in derived terminology (viz. ‘cs Lie supergroup’), since contracting this prefix seems to hide the true nature of the objects considered.

A.3. Let $X = (X_0, \mathcal{O})$ be a cs manifold. If $U \subset X_0$ is open, $f \in \mathcal{O}(U)$, and $x \in U$, then we define $f(x)$ to be the unique complex number $\alpha \in \mathbb{C}$ such that $(f - \alpha)_x$ is not invertible—here, $h_x$ denotes the germ of $h$ at $x$. Such a complex number exists uniquely, because $\mathcal{O}_x$ is isomorphic to $\mathcal{C}_{V,y}^{\infty} \otimes \Lambda(C)^{\ast}$ for some $p, q$, some open $V \subset \mathbb{R}^p$, and some $y \in V$, and is hence a local algebra. We write $\tilde{f} : U \rightarrow \mathbb{C}$ for the function $x \mapsto f(x)$.

We say that $f \in \mathcal{O}(U)$ takes values in any given set $A \subset \mathbb{C}$ whenever the function $\tilde{f}$ does. Hence, $f$ may take real values, etc. Moreover, there exists a canonical antilinear involution on $\mathcal{C}_{X_0,\mathbb{C}}^{\infty} = \mathcal{O}/\mathcal{N}$. We denote by $\mathcal{C}_{X_0,\mathbb{C}}^{\infty}$ the sheaf of fixed points for this involution, given by the elements that take real values.

Let $(f, f^\ast) : X \rightarrow Y$ be a morphism of cs manifolds, and $h \in \mathcal{O}(V)$ where $V \subset Y_0$ is open. Let $x \in f^{-1}(V)$. Since $(h - h(f(x)))_{f(x)}$ is not invertible, nor is $(f^\ast h - h(f(x)))_{x} = (f^\ast h)_x - h(f(x))$. By the definition of $f^\ast h(x)$, it follows that $f^\ast h(x) = h(f(x))$, i.e. $f^\ast h = \tilde{h} \circ f$. In particular, any morphism of cs manifolds is local, and the induced morphism $\mathcal{C}_{Y_0,\mathbb{C}}^{\infty} \rightarrow f_* \mathcal{C}_{X_0,\mathbb{C}}^{\infty}$ respects the canonical antilinear involutions; so it gives a morphism $(X_0, \mathcal{C}_{X_0,\mathbb{C}}^{\infty}) \rightarrow (Y_0, \mathcal{C}_{Y_0,\mathbb{C}}^{\infty})$ of real ringed spaces. This has the following two consequences.

**Proposition A.4.** For any cs manifold $X = (X_0, \mathcal{O})$, there exists on $X_0$ a unique structure of smooth manifold for which the morphism $\mathcal{C}_{X_0,\mathbb{C}}^{\infty} \rightarrow \mathcal{C}_{X_0,\mathbb{C}}$ which sends $h$ (mod $\mathcal{N}$) to the continuous function $\tilde{h}$ defines an isomorphism of $\mathcal{C}_{X_0,\mathbb{C}}^{\infty}$ with the sheaf of real-valued smooth functions on $X_0$.

**Proposition A.5.** The complexification functor from the category of real super-ring spaces to the category complex super-ringed spaces restricts to a fully faithful embedding of smooth real manifolds into cs manifolds.
A.6. Let $X = (X_0, \mathcal{O})$ be a cs manifold. Although $\mathcal{O}$ does not have a canonical $C^\infty_{X_0}$-module structure, we can use the sheaf $C^\infty_{X_0}$ to show that $\mathcal{O}$ is a fine and, in particular, a soft sheaf (see below). Let us remark that a posteriori, the choice of a partition of unity subordinate to some locally finite cover by coordinate neighbourhoods can be used to define a $C^\infty_{X_0}$-module structure, but this construction is non-canonical.

**Proposition A.7.** Let $X = (X_0, \mathcal{O})$ be a cs manifold. Then $\mathcal{O}$ is fine.

In the proof of this proposition, the following lemmata are crucial.

**Lemma A.8.** Let $U, V \subset X_0$ be open, $K \subset U$ compact, such that $\nabla \subset U$ and $U$ is a coordinate neighbourhood. There exists $f \in \mathcal{O}(U)$, $\text{supp} \ f \subset V$, whose germ at $K$ is $1$, and which on $U$ takes values in the set $[0, 1] \subset \mathbb{R}$.

**Proof.** For any coordinate neighbourhood $U$, there exists an even algebra isomorphism $\mathcal{O}(U) \cong C^\infty_{X_0}(U) \otimes \Lambda(\mathbb{C}^\ast)^\ast$ over the identity $U \to U$, and thus an embedding $C^\infty_{X_0}(U) \subset \mathcal{O}(U)$ as an even real subalgebra. Hence the claim. $\square$

**Lemma A.9.** Let $U \subset X_0$ be open and $f \in \mathcal{O}(U)$ with $\text{supp} \ f$ compact. There exists $h \in \Gamma(\mathcal{O})$ such that $h|_U = f$ and $h_x = 0$ for all $x \notin U$.

**Proof.** Let $V = X \setminus (\text{supp} \ f)$. By definition of the support, we have for all $x \in U \cap V$ that the germ $f_x = 0$. Then $f$ on $U$ and $0$ on $V$ define the data of a global section of $\mathcal{O}$, by the sheaf axiom. $\square$

**Proof of Proposition A.7.** Let $(U_\alpha)$ be a locally finite open cover of $X_0$. We need to construct $h_\alpha \in \Gamma(\mathcal{O})$, such that $\text{supp} \ h_\alpha \subset U_\alpha$ form a locally finite family of closed sets, and the locally finite sum $\sum_\alpha h_\alpha = 1$. Since $X_0$ is paracompact, passing to a locally finite refinement, we may assume that $U_\alpha$ are coordinate neighbourhoods. There are open subsets $V_\alpha, W_\alpha \subset X_0$ such that $\overline{W_\alpha} \subset V_\alpha$ and $\overline{V_\alpha} \subset U_\alpha$ are compact, and $X_0 = \bigcup_\alpha W_\alpha$.

By Lemma A.8 there exist $f_\alpha \in \mathcal{O}(U_\alpha)$ such that $\text{supp} \ f_\alpha \subset V_\alpha$, the germ of $f_\alpha$ at $W_\alpha$ is $1$, and $f_\alpha$ takes values in $[0, 1]$. Applying Lemma A.9 we extend $f_\alpha$ by zero to an element $f_\alpha \in \Gamma(\mathcal{O})$ which takes values in $[0, 1]$.

The sum $f = \sum_\alpha f_\alpha$ is locally finite and therefore exists in $\Gamma(\mathcal{O})$. Since $(W_\alpha)$ is a cover of $X_0$, $f$ takes positive values. In particular, $f$ is invertible in $\Gamma(\mathcal{O})$ (each germ being invertible). Define $h_\alpha = f^{-1} \cdot f_\alpha$. $\square$

**Remark A.10.** Proposition A.7 can be used to prove an analogue of Batchelor’s theorem for cs manifolds: any cs manifold $(X_0, \mathcal{O})$ is (non-canonically) isomorphic to $(X_0, \Lambda(E \otimes \mathbb{C})$ where $E$ is a locally free $C^\infty_{X_0}$-module.

We shall need fineness in the construction of a Berezin integral for cs manifolds. Another useful consequence is the following corollary.

**Corollary A.11.** Let $X, Y$ be cs manifolds, and assume given an even superalgebra morphism $\varphi : \mathcal{O}_Y(Y_0) \to \mathcal{O}_X(X_0)$ which sends real-valued superfunctions to real-valued superfunctions. Then there exists a unique morphism $(f, f^*) : X \to Y$ such that $\varphi = f^*$ on $\mathcal{O}_Y(Y_0)$.

**Remark A.12.** It is natural to introduce the following category $\text{Alg}_{\mathbb{C}}^\ast$: The objects are pairs $(A, A_0)$ where $A$ is a superalgebra over $\mathbb{C}$, $A_0$ is a real subalgebra of $A_0$ containing $J_0 = A_0 \cap J$ where $J = A \cdot A_1$ is the graded ideal generated by $A_1$, such that $A_0/J_0$ is a real form of $A_0/J_0 = A/J$; the morphisms of such pairs...
\( \phi : (A, A_{0,R}) \to (B, B_{0,R}) \) are even unital complex algebra morphisms \( A \to B \) such that \( \phi(A_{0,R}) \subset B_{0,R} \).

Corollary A.11 may then be rephrased as the statement that the global sections functor \( \Gamma \) is fully faithful from \( \text{cs manifolds} \to \text{Alg}_{\text{cs}} \); to be precise, \( \Gamma \) is defined on objects as \( \Gamma(Z) = (\mathcal{O}_Z(Z_0), \mathcal{O}_Z(Z_0)_{0,R}) \) where \( \mathcal{O}_Z(Z_0)_{0,R} \) consists of the real-valued even elements of \( \mathcal{O}_Z(Z_0) \). We call the objects of \( \text{Alg}_{\text{cs}} \) cs algebras.

**Definition A.13.** Let \( U \) be a cs domain of graded dimension \( p|q \), and suppose given a morphism \( (f, f^*) : U \to \mathbb{R}^{|p,q|}_C \) which factors through an isomorphism onto an open subspace of \( \mathbb{R}^{|p,q|}_C \).

We define \( x_j = f^*(pr_j) \in \mathcal{O}_U(U_0)_0 \), \( j = 1, \ldots, p \), and \( \xi_j = f^*(\theta_j) \in \mathcal{O}_U(U_0)_1 \), \( j = 1, \ldots, q \); here \( pr_j : \mathbb{R}^q \to \mathbb{R} \) are the coordinate projections, and \( \theta_j = pr_j \in (\mathbb{C}^q)^* \) denote the standard generators of \( \wedge (\mathbb{C}^q)^* \).

The collection \( (x, \xi) = (x_1, \ldots, x_p, \xi_1, \ldots, \xi_q) \) will be called a coordinate system for \( U \), and its entries will be called coordinates. Note that the \( x_j \) take real values; we denote the subset of \( \mathbb{R}^p \) consisting of all tuples \( (x_1(u), \ldots, x_p(u)) \), where \( u \in U_0 \), by \( x(U) = x(U_0) \). (Of course, the \( \xi_j \) also take real values, as does any odd superfunction.)

**Remark A.14.** Hohnhold [Hoh6] also considers complex coordinate systems. This is a useful concept when studying complex supermanifolds as cs manifolds (by forgetting the holomorphic structure). The following mapping condition can, however, only be formulated with real coordinate systems.

**Proposition A.15.** Let \( X \) be a cs manifold and \( U \) a cs domain of graded dimension \( p|q \). Let \( (x, \xi) \) be a coordinate system on \( U \), where \( y = (y_1, \ldots, y_p) \in \mathcal{O}_X(X_0)^p \) and \( \eta = (\eta_1, \ldots, \eta_q) \in \mathcal{O}_X(X_0)^q \), such that the map

\[
X \to \mathbb{C}^p : x \mapsto (y_1(x), \ldots, y_p(x))
\]

takes values in \( x(U) \subset \mathbb{R}^p \). (In particular, the \( y_j \) are real-valued.) Then there exists a unique morphism of cs manifolds, \( (f, f^*) : X \to U \), such that \( f^*(x_j) = y_j \) and \( f^*(\xi_j) = \eta_j \).

**Proof.** This follows as for real supermanifolds, cf. [Sch84 3.2].

**Definition A.16.** Let \( V = V_0 \oplus V_1 \) be a finite-dimensional complex super-vector space such that \( V_0 \) is equipped with a real form \( V_{0,R} \). Then we call \((V, V_{0,R})\) a cs vector space. Moreover, \((V, V_{0,R})\) defines a cs manifold \( (V_{0,R}, \mathcal{O}^\infty_{V_{0,R}} \otimes \wedge(V_1)^*) \) of graded dimension equal to the complex graded dimension of the super-vector space \( V \). By abuse of notation, we shall denote it by \( V \), and its structure sheaf by \( \mathcal{O}_V \).

We call \( V \) a linear cs manifold.

The structure sheaf of \( V \) comes with a natural \( \mathbb{Z} \)-grading, and its 0th degree part is exactly \( \mathcal{O}^\infty_{V_{0,R}} \). Hence, in this special case, the given real form \( V_{0,R} \) of \( V_0 \) is precisely what specifies the real-valued elements of \( \mathcal{O}_V \). For this reason, we find it justifiable to use the subscript \( \mathcal{O}_{R} \) both for the real form of \( V_0 \) and the sheaf of real-valued superfunctions (although only the former really defines a real form). We forewarn the reader that we will systematically indulge in this abuse of notation.

The set \( V^\ast \) of all complex linear forms on \( V \) is embedded as a graded subspace in \( \mathcal{O}_V(V_{0,R}) \). The following statement follows from Proposition A.15.
Corollary A.17. Let $X$ be a cs manifold and $V$ a linear cs manifold. For any even linear map $\phi : V^* \to \mathcal{O}_X(X_0)$ such that any $h \in \phi(V^*_0)$ takes real values, there exists a unique morphism $(f, f^*) : X \to V$ such that $f^*|_{V^*} = \phi$.

In other words, $V(X) = \text{Hom}(X, V)$, the set of $X$-points of $V$, is exactly

$$V(X) = (\mathcal{O}_X(X_0) \otimes V)_{0, R} = \mathcal{O}_X(X_0)_{0, R} \otimes \mathcal{O}_X(X_0)_{1} \otimes \mathcal{O}_V(V_1).$$

A.18. To finalise our disquisition on the fundamentals of cs manifolds, we discuss the existence of finite products in this category. If $V$ and $W$ are linear cs manifolds, then we define the even product $V \times W = (V_0 \times W_0) \oplus (V_1 \times W_1)$. The even part $V_0 \times W_0$ of $V \times W$ is endowed with the real form $V_{0, R} \times W_{0, R}$. We thus obtain a linear cs manifold which we denote again by $V \times W$. By Corollary A.17 there are morphisms $p_1 : V \times W \to V$ and $p_2 : V \times W \to W$ induced by the inclusions $V^* = (V_0^* \times 0) \oplus (V_1^* \times 0) \subset (V \times W)^* \supset (0 \times W_0^*) \oplus (0 \times W_1^*) = W^*$ and $(V \times W)^* \subset \mathcal{O}_{V \times W}(V_{0, R} \times W_{0, R})$. By the same token, $V \times W$ is the product of $V$ and $W$ in the category of cs manifolds.

Next, consider open subspaces $A \subset V$ and $B \subset W$. We define $A \times B$ to be the (unique) open subspace of $V \times W$ whose underlying set is $A_0 \times B_0 \subset V_{0, R} \times W_{0, R}$. The morphisms $p_1$ and $p_2$ defined above restrict to morphisms $A \times B \to A$ and $A \times B \to B$, respectively. By Proposition A.15 $A \times B$ is the product of $A$ and $B$ in the category of cs manifolds.

Given cs manifolds $X$ and $Y$, we define $\mathcal{O}_{X \times Y}$ to be the (up to canonical isomorphism) unique sheaf on $X_0 \times Y_0$ such that $\mathcal{O}_{X \times Y}|_{U_0 \times V_0} = \mathcal{O}_U \otimes \mathcal{O}_V$ for all coordinate neighbourhoods $U \subset X$ and $V \subset Y$. This determines a cs manifold $X \times Y$. The canonical morphisms $U \times V \to U$ and $U \times V \to V$ for all coordinate neighbourhoods $U$, $V$ determine morphisms $X \times Y \to X$ and $X \times Y \to Y$. As in the case of real supermanifolds, one shows that $X \times Y$ is the product of $X$ and $Y$ in the category of cs manifolds.

At this point we mention that for any cs manifold $X$, $\mathcal{O}_X$ carries a natural structure of a sheaf of nuclear Fréchet spaces. Indeed, for any coordinate neighbourhood $U$, $\mathcal{O}_X(U_0)$ is a nuclear Fréchet space for the standard topology induced from $C^\infty_{\mathcal{O}}(U_0) \otimes \bigwedge((\mathcal{O}_0)^*)$, and one may take locally convex projective limits with respect to the restriction maps (here the paracompactness of $X_0$ ensures that countable limits suffice). Using the nuclearity, one can show that for any cs manifolds $X$ and $Y$, and any open subsets $U_0 \subset X_0$ and $V_0 \subset Y_0$, the inclusion $\mathcal{O}_X(U) \otimes \mathcal{O}_Y(V) \to \mathcal{O}_{X \times Y}(U \times V)$ given by $f \otimes h \to p_1^* f \cdot p_2^* h$ induces an isomorphism on the completion of the tensor product w.r.t. any locally convex tensor product topology (for instance, one may take the projective tensor product topology).

A.2. An inverse function theorem for cs manifolds. We give an easy special case of the inverse function theorem that is useful in various situations.

Definition A.19. Let $X = (X_0, \mathcal{O})$ be a cs manifold. For $x \in X_0$, we define the tangent space $T_x X = \text{Der}(\mathcal{O}_x, \mathbb{C})$. If $X = X_0$ is an ordinary manifold, then $T_x X = T_x X_0 \otimes \mathbb{C}$. In general, we have $s\text{dim}_c T_x X = s\text{dim}_c \mathfrak{m}_x / \mathfrak{m}_x^2$ by standard arguments, where $\mathfrak{m}_x$ is the maximal ideal of the local algebra $\mathcal{O}_x$ and sdim denotes graded dimension.

Given a morphism $f : X \to Y$ of cs manifolds, we define for all $x \in X_0$ even $\mathbb{C}$-linear maps $T_x f : T_x X \to T_{f(x)} Y$, called tangent maps, by

$$(A.1) \quad (T_x f)(\xi)h = \xi(f^* h) \quad \text{for all} \quad \xi \in T_x X, \ h \in \mathcal{O}_{Y, f(x)}.$$
Clearly, one has, for any morphism \( g : Y \to Z \), the chain rule
\[
T_x(g \circ f) = T_{f(x)}g \circ T_xf.
\]

Our main application of tangent maps is the following proposition.

**Proposition A.20.** Let \( f : X \to Y \) be a morphism of cs manifolds. Then \( f \) is an isomorphism if and only if \( f : X_0 \to Y_0 \) is bijective, and for all \( x \in X_0 \),
\[
T_xf : T_xX \to T_{f(x)}Y
\]
is an isomorphism of vector spaces.

**Proof.** For real supermanifolds, this is [Kos77, Corollary to Theorem 2.16]. The case of cs manifolds can be treated analogously. \( \square \)

A.3. **The functor of points.** We recall the functor of points. We use it chiefly as a device to prove identities of morphisms, so we do not go very deep in our discussion.

A.21. Let \( \text{SMan}_{cs} \) denote the category of supermanifolds. For \( U, X \in \text{SMan}_{cs} \), let \( X(U) \) denote the set of morphisms \( U \to X \) in \( \text{SMan}_{cs} \). Elements \( x \in X(U) \) are called \( U \)-points, and one writes \( x \in_U X \). If \( * \) is the 0|0-dimensional cs manifold whose underlying topological space is a point, then \( X(*) = X_0 \) is the real manifold underlying \( X \). (Hence the terminology.)

A.22. The assignment \( X \mapsto X(-) \) extends in a natural way to a functor from \( \text{SMan}_{cs} \) to the category \( \text{[SMan}_{op cs}, \text{Set}] \) of set-valued functors on cs manifolds. By Yoneda’s lemma [ML98], this functor is fully faithful, and we call it the Yoneda embedding. In particular, to define morphisms of cs manifolds, it suffices to define morphisms of their images under the Yoneda embedding.

The Yoneda embedding commutes with finite products. It follows that it induces a fully faithful embedding of the category of cs Lie supergroups (see below) into the category \( \text{[SMan}_{op cs}, \text{Grp}] \) of group-valued functors on cs manifolds.

**Remark A.23.** A more complete discussion of the functor of points, and its applications to the theory Lie supergroups, is given in [DM99, §§ 2.8–11].

B. **Appendix: The category of cs Lie supergroups**

B.1. **The category of cs Lie supergroups.** In this section, we will use standard notions of Lie theory and of the theory of Lie superalgebras without explicit reference. Standard texts for the latter would be [Kac77, Sch79]. Useful references for the former could be [Kna02, HN10], but most Lie theory texts should be sufficient.

**Definition B.1.** A cs Lie supergroup is a group object in the category of cs manifolds. Thus, it is a tuple \((G, m, i, e)\) where \( G \) is a cs manifold and \( m : G \times G \to G \), \( i : G \to G \), and \( e : * \to G \) are morphisms of cs manifolds subject to the obvious axioms. (Here, \( * \) is the terminal object in the category of cs manifolds.) In particular, \( G_0 \) is a real Lie group.

A morphism \( f : G \to H \) of cs Lie supergroups is a morphism of group objects in the category of cs manifolds. I.e., it is a morphism of cs manifolds, and satisfies the equations
\[
\phi \circ m_G = m_H \circ (\phi \times \phi), \quad \phi \circ i_G = i_H \circ \phi, \quad \phi \circ e_G = e_H.
\]

There is an easy way to construct cs Lie supergroups, due to J.-L. Koszul [Kos83] in the real case, and it uses the following concept.
Definition B.2. A \( \text{cs supergroup pair} \) (a.k.a. \textit{super Harish-Chandra pair}) is actually a triple \((G_0, g, \text{Ad})\) subject to the following assumptions: \( G_0 \) is a \textit{real} Lie group with Lie algebra \( g_{0, \mathbb{R}} \); \( g = g_0 \oplus g_1 \) is a \textit{complex} Lie superalgebra such that \( g_{0, \mathbb{R}} \) is a real form of \( g_0 \); and \( \text{Ad} : G_0 \times g \to g \) is a smooth linear action of \( G_0 \) by even Lie superalgebra automorphisms which extends the adjoint action of \( G \) on \( g_{0, \mathbb{R}} \), and whose differential \( d\text{Ad} : g_{0, \mathbb{R}} \times g \to g \) is the restriction of the bracket \([\cdot, \cdot] : g \times g \to g\).

By abuse of notation, we will generally write \((G_0, g)\) for a \textit{cs supergroup pair} (hence the parlance). The action \( \text{Ad} = \text{Ad}_G \) is understood, although strictly speaking, it is not determined by \( G_0 \) and \( g \) unless \( G_0 \) is connected.

A \textit{morphism of cs supergroup pairs} is a pair \((f, df) : (G_0, g) \to (H_0, h)\) fulfilling the following conditions: \( f : G_0 \to H_0 \) is a morphism of real Lie groups; \( df : g \to h \) is an even morphism of complex Lie superalgebras extending the differential of \( f \); and \( df \) is \( G_0 \)-equivariant for the \( G_0 \)-action on \( h \) induced by \( f \), i.e.

\[
\text{Ad}_H(f(g))(df(x)) = df(\text{Ad}_G(g)(x)) \quad \text{for all } g \in G_0, x \in g.
\]

B.3. A simple but salient point about \textit{cs supergroup pairs} \((G_0, g)\) is that \( C^\infty_{G_0} \otimes \mathbb{C} \) is a sheaf of \( g_0 \)-modules. Indeed, if \( U \subset G_0 \) is open, \( x \in g_{0, \mathbb{R}} \), and \( f \in C^\infty(U, \mathbb{C}) \), then we may define

\[
(r_x f)(g) = \frac{d}{dt} f(g \exp(tx)) \bigg|_{t=0} \quad \text{for all } g \in U.
\]

By complex linear extension, this defines a \( g_0 \)-module structure on \( C^\infty(U, \mathbb{C}) \) which is compatible with the restriction maps of \( C^\infty_{G_0} \otimes \mathbb{C} \).

B.4. Let \( \phi : h \to g \) be a morphism of complex Lie superalgebras and \( V \) a graded \( h \)-module sheaf. Then \( \mathcal{U}(g) \) is an \( h \)-module \textit{via} \( x.v = \phi(x)v \) for all \( x \in h, v \in \mathcal{U}(g) \).

One defines the \textit{coinduced module sheaf} \( \text{Coind}_h^g(V) \) by

\[
\text{Coind}_h^g(V)(U) = \text{Hom}_{\mathcal{U}(h)}(\mathcal{U}(g), \mathcal{V}(U)).
\]

Here, \( \text{Hom} \) denotes \textit{inner} Hom, \textit{i.e.} linear maps are considered without a parity constraint. Then \( \text{Coind}_h^g(V) \) is a graded \( g \)-module sheaf with \( \mathcal{U}(g) \)-module structure

\[
(r_u f)(v) = (-1)^{|u||F|+|v|} f(uv)
\]

for all \( u, v \in \mathcal{U}(g) \), \( f \in \text{Coind}_h^g(V(U)) \). If \( V \) is purely even, then we may replace the sign \((-1)^{|u||F|+|v|} \) by \((-1)^{|u|}\). One is given a canonical \( h \)-equivariant morphism \( \text{Coind}_h^g(V) \to V \) by the assignment \( f \mapsto F(1) \).

If \( V \) is a sheaf of superalgebras and \( h \) acts by graded derivations, then we obtain on \( \text{Coind}_h^g(V) \) the structure of a sheaf of \( g \)-superalgebras. The multiplication is

\[
f \otimes f' \mapsto \mu \circ (f \otimes f') \circ \Delta
\]

where \( \mu \) is the multiplication of \( V \), and \( \Delta : \mathcal{U}(g) \to \mathcal{U}(g) \otimes \mathcal{U}(g) = \mathcal{U}(g \oplus g) \) is the coproduct of \( \mathcal{U}(g) \) (\textit{i.e.} the unique extension to an even unital algebra morphism of the map \( g \to \mathcal{U}(g) \otimes \mathcal{U}(g) : x \mapsto x \otimes 1 + 1 \otimes x \)). Let \( \varepsilon : \mathcal{U}(g) \to \mathbb{C} \) be the unique extension to an even unital algebra morphism of the zero map \( g \to \mathbb{C} \); then the unit of \( \text{Coind}_h^g(V) \) is \( \eta \circ \varepsilon : \mathcal{U}(g) \to \mathbb{C} \to V \) where \( \eta \) is the unit of \( V \).

B.5. Next, assume again that \((G_0, g)\) is a \textit{cs supergroup pair}. Define a superalgebra sheaf by \( \mathcal{O}_G = \text{Coind}_{g_0}^g(C^\infty_{G_0} \otimes \mathbb{C}) \). If \( \beta : S(g) \to \mathcal{U}(g) \) is the supersymmetrisation map, then the morphism \( \mathcal{O}_G \to C^\infty_G \otimes \Lambda g_1^* \) defined by \( f \mapsto (f \circ \beta) \mid_{\Lambda g_1} \) is an
isomorphism of superalgebra sheaves, since $\beta$ is a supercoalgebra isomorphism. In particular, $G = (G_0, \mathcal{O}_G)$ is a $cs$ manifold.

We note that the action of $\mathfrak{g}_0, \mathbb{R}$ on $\mathcal{O}_G$ integrates to an action of $G_0$ via

$$r_h f(u; g) = f(\text{Ad}(h^{-1})(u); gh)$$

for all $f \in \mathcal{O}_G(U), u \in \mathfrak{U}(\mathfrak{g}), g \in G_0, h \in g^{-1} U$. Here and in the following, we use the notation $f(u; g) = f(u)(g)$ for $f \in \mathcal{O}_G(U), u \in \mathfrak{U}(\mathfrak{g}),$ and $g \in U$.

**Proposition B.6.** Let $(G_0, \mathfrak{g})$ be a $cs$ supergroup pair. Let $m, i, e$ be the structure maps of the real Lie group $G_0$. We define $m^* : \mathcal{O}_G \rightarrow m_* \mathcal{O}_{G \times G}, i^* : \mathcal{O}_G \rightarrow i_* \mathcal{O}_G,$ and $e^* : \mathcal{O}_G \rightarrow \mathcal{O} \times 0$ via

$$(B.3) \quad m^*(u \otimes v; g, h) = f(\text{Ad}(h^{-1})(u)v; gh)$$

for all $f \in \mathcal{O}_G(U), u, v \in \mathfrak{U}(\mathfrak{g}), (g, h) \in m^{-1}(U)$;

$$(B.4) \quad i^*(g, h) = f(\text{Ad}(g)(S(u)); g^{-1})$$

for all $f \in \mathcal{O}_G(U), u \in \mathfrak{U}(\mathfrak{g}), g \in i^{-1}(U)$; and $e^* f = f(1, 1)$. Here, $S : \mathfrak{U}(\mathfrak{g}) \rightarrow \mathfrak{U}(\mathfrak{g})$ is determined by $S(1) = 1, S(x) = -x$ for all $x \in \mathfrak{g}$, and $S(uv) = (1)^{|u||v|} S(v)S(u)$ for all homogeneous $u, v \in \mathfrak{U}(\mathfrak{g})$. Then $C(G_0, \mathfrak{g}) = (G, (m, m^*), (i, i^*), (e, e^*))$ is a $cs$ Lie supergroup.

If $(f, df) : (G_0, \mathfrak{g}) \rightarrow (H_0, \mathfrak{h})$ is a morphism of $cs$ supergroup pairs, then the assignment $C(f, df) = (f, f^*),$ where $f^* : \mathcal{O}_H \rightarrow f_* \mathcal{O}_G$ is defined by

$$(f^* h)(u; g) = h(df(u); f(g)) \quad \text{for all} \quad h \in \mathcal{O}_H(U), u \in \mathfrak{U}(\mathfrak{g}), g \in f^{-1}(U),$$

gives a morphism of $cs$ Lie supergroups. Here, $df$ is extended uniquely to an even unital algebra morphism $\mathfrak{U}(\mathfrak{g}) \rightarrow \mathfrak{U}(\mathfrak{h})$. This defines a functor $C$ from $cs$ supergroup pairs to $cs$ Lie supergroups.

**Proof.** The details of the proof are somewhat tedious, and most of the calculations are straightforward, so we only indicate the salient points.

First, by Corollary A.11, it is sufficient to work on the level of global sections; this tidies matters up a little. One needs to check that $m^*$ is well-defined, i.e. that $m^* f$ is $(\mathfrak{g}_0 \oplus \mathfrak{g}_0)$-equivariant for any $f$; this follows from the fact that $G_0$ acts on $\mathfrak{g}$ by Lie superalgebra automorphisms. From $\Delta \otimes \text{id} \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta$ (the coassociativity relation), it follows that

$$(m, m^*) \circ ((m, m^*) \times \text{id}) = (m, m^*) \circ (\text{id} \times (m, m^*)) .$$

Next, the multiplication of $\mathcal{O}_G(G)$ extends through the canonical embedding $\Gamma(\mathcal{O}_G) \otimes \Gamma(\mathcal{O}_G) \rightarrow \Gamma(\mathcal{O}_{G \times G})$ to a unital algebra morphism which coincides with $\delta : \Gamma(\mathcal{O}_{G \times G}) \rightarrow \Gamma(\mathcal{O}_G),$ given by $\delta f(u; g) = \Delta(u; \delta(g))$ where $\delta(g) = (g, g)$. But then $(\delta, \delta^*) : G \rightarrow G \times G$ is the diagonal morphism in $cs$ manifolds given by the universal property of the product. The equations

$$(m, m^*) \circ ((i, i^*) \times \text{id}) \circ (\delta, \delta^*) = (e, e^*) \circ (\ast, 1) = (m, m^*) \circ (\text{id} \times (i, i^*)) \circ (\delta, \delta^*)$$

where $(\ast, 1) : G \rightarrow \ast$ is the canonical morphism to the terminal object, then follow from the fact that $S$ is an antipode for the bialgebra $\mathfrak{U}(\mathfrak{g})$. The latter fact means that $\mu \circ (S \otimes \text{id}) \circ \Delta = 1 = \mu \circ (\text{id} \otimes S) \circ \Delta$ where $\mu$ is the algebra multiplication of $\mathfrak{U}(\mathfrak{g}),$ and can be extracted from standard texts on Hopf algebras such as [Swe69], or the excellent introduction [Car06].

The remaining statements are easily substantiated. \qed
B.7. Let $G$ be a \textit{cs} Lie supergroup with structure morphisms $(m, m^*)$, $(i, i^*)$, and $(e, e^*)$. We denote $1 = e(*) \in G_0$ the neutral element of the underlying Lie group. It is clear that $e^* f = f(1)$ for all $f \in \Gamma(O_G)$. Let $\mathfrak{g} \subset \text{Hom}(\Gamma(O_G), \mathbb{C})$ consist of the graded derivations along $e^*$. \textit{I.e.}, $\mathfrak{g}$ is spanned by the homogeneous linear maps $x : \Gamma(O_G) \rightarrow \mathbb{C}$ which satisfy

$$x(f \cdot h) = x(f)h(1) + (-1)^{|x||f|}f(1)x(h)$$

for all homogeneous $f, h \in \Gamma(O_G)$.

For any $x \in \mathfrak{g}$, one defines by continuous linear extension from the algebraic tensor product a linear map $\text{id} \otimes x : \Gamma(O_{G \times G}) \rightarrow \Gamma(O_G)$. It is a graded derivation along $\text{id} \otimes e^*$. Then $L_x = (\text{id} \otimes x) \circ m^* \in \text{Der}(\Gamma(O_G))$ is a graded derivation (along the map $\text{id}$).

\textbf{Lemma B.8.} For any \textit{cs} Lie supergroup $G$, the map $\mathcal{L} : \mathfrak{g} \rightarrow \text{Der}(\Gamma(O_G))$ defines a bijection onto the set of graded derivations $d$ which satisfy

$$(p_1, m^*) \circ (\text{id} \otimes d) = (\text{id} \otimes d) \circ (p_1, m)^*$$

where $(p_1, m) : G \times G \rightarrow G \times G$.

\textit{Proof.} The map $\mathcal{L}$ is well-defined because of the associativity equation for $(m, m^*)$ which translates in particular to $(\text{id} \otimes m^*) \circ m^* = (m^* \otimes \text{id}) \circ m^*$. Then one recovers $x$ from $x = e^* \circ \mathcal{L}_{x}$, so $\mathcal{L}$ is injective. One may also use this equation to determine the image of $\mathcal{L}$. $\square$

B.9. Let $G$ be a \textit{cs} Lie supergroup. We shall call the graded derivations in the image of $\mathcal{L}$ \textit{left-invariant vector fields}. The set of left-invariant vector fields is easily shown to be a Lie subsuperalgebra of $\text{Der}(\Gamma(O_G))$ under the super-commutator bracket. In particular, $\mathfrak{g}$ has the structure of a finite-dimensional complex Lie superalgebra; we call $\mathfrak{g}$ the \textit{Lie superalgebra of} $G$.

Let $g \in G_0$. Then $g$ defines a morphism $\Gamma(O_G) \rightarrow \mathbb{C}$, and there is an even unital algebra morphism $g \otimes \text{id} \otimes g^{-1} : \Gamma(O_{G \times G \times G}) \rightarrow \Gamma(O_G)$. We define morphisms $m^{(2)} = m \circ (m \times \text{id})$ and $e_g^* = (g \otimes \text{id} \otimes g^{-1}) \circ m^{(2)}$, the \textit{conjugation} by $g$. Then it is natural to define $\text{Ad} : G_0 \times \mathfrak{g} \rightarrow \mathfrak{g}$ by $\text{Ad}(g)(x) = x \circ e_g^*$. One checks that this is a finite-dimensional continuous linear representation, and hence smooth. Straightforward calculations prove the following lemma.

\textbf{Lemma B.10.} Let $G$ be a \textit{cs} Lie supergroup with Lie superalgebra $\mathfrak{g}$ and underlying real Lie group $G_0$. With the adjoint action $\text{Ad}$, $(G_0, \mathfrak{g})$ is a \textit{cs} supergroup pair.

B.11. Let $G$ be a \textit{cs} Lie supergroup given by $G = C(G_0, \mathfrak{g})$ for a \textit{cs} supergroup pair $(G_0, \mathfrak{g})$. Then there is a canonical isomorphism $T_1 G \cong \mathfrak{g}$ which can be derived from Lemma B.8.

For any $g \in G_0$, we define morphisms of \textit{cs} manifolds $L_g, R_g : G \rightarrow G$ by taking the left (resp. right) translation by $g$ on the level of $G$, and setting

$$L_g^* f(u; h) = m^* f(1 \otimes u; g, h) = f(u; gh),$$

$$R_g^* f(u; h) = m^* f(u \otimes 1; h, g) = f(\text{Ad}(g^{-1})(u); hg),$$

for all $f \in \mathcal{O}_G(U)$, $u \in \mathfrak{m}(g)$, and $h \in g^{-1}U$ (resp. $h \in U g^{-1}$).

Clearly, $L_g, R_g$ are isomorphisms with inverses $L_{g^{-1}}$ and $R_{g^{-1}}$, respectively. In particular, if we write $dL_g = T_1 L_g$ and $dR_g = T_1 R_g$, then $dL_g, dR_g : \mathfrak{g} \rightarrow T_g G$ are isomorphisms, by Proposition A.20.
A direct calculation shows that
\[(B.6) \quad (T_{(g,h)}m)(dL_g(x), dL_h(y)) = dL_{gh}(\text{Ad}(h^{-1})(x) + y)\]
for all \(g, h \in G_0, x, y \in \mathfrak{g} \).

**Proposition B.12.** The functor \(C\) is an equivalence of categories from \(\text{cs supergroup pairs to cs Lie supergroups}\).

**Remark B.13.** In the real case, Proposition B.12 is due to B. Kostant [Kos77]. As remarked above, we use a construction due to J. L. Koszul [Kos83].

**Proof of Proposition B.12.** We need to prove that \(C\) is fully faithful and essentially surjective. We begin with the essential surjectivity.

To that end, let \(G\) be a cs Lie supergroup with Lie superalgebra \(\mathfrak{g}\) and underlying real Lie group \(G_0\). We define a morphism \(\phi : C(G_0, \mathfrak{g}) \to G, \phi = (\text{id}, \phi^*)\), by specifying \(\phi^* : \mathcal{O}_G \to \mathcal{O}' = \text{Coind}^{\mathfrak{g}_0}_{\mathfrak{g}}(C_{\mathfrak{g}}^\infty \otimes \mathbb{C})\) via
\[
(\phi^* f)(u; g) = (-1)^{|f||u|}(\mathcal{L}_u f)(g) \quad \text{for all} \quad f \in \mathcal{O}_G(U), u \in \mathfrak{U}(\mathfrak{g}), g \in U.
\]

Here, \(\mathcal{L}\) is the extension of \(\mathcal{L} : \mathfrak{g} \to \text{Der}(\mathcal{O}_G(U))\) to an even unital algebra morphism \(\mathfrak{U}(\mathfrak{g}) \to \text{End}(\mathcal{O}_G(U))\) (inner End). To see that \((\text{id}, \phi^*)\) is an isomorphism of cs manifolds, we apply Proposition A.20.

Indeed, for \(g \in G_0\), denote the maximal ideal of \(\mathcal{O}_{G,g}\) and \(\mathcal{O}'\) by \(\mathfrak{m}_g\) and \(\mathfrak{n}_g\), respectively. Then
\[
\mathfrak{n}_g = \{ h \in \mathcal{O}'_g \mid h(1; g) = 0 \}, \quad \mathfrak{n}^2_g = \{ h \in \mathcal{O}'_g \mid h(x; g) = 0 \text{ for all } x \in \mathfrak{g} \},
\]
as follows from the definition of the algebra structure. Then \(\phi^* \mathfrak{m}_g \subset \mathfrak{n}_g\). On the other hand, if \(h \in \mathcal{O}_g\) is not contained in \(\mathfrak{m}^2_g\), then \((\mathcal{L}_x h)(g) \neq 0\) for some \(x \in \mathfrak{g}\). Thus, \(\phi^*\) induces an injection \(\mathfrak{m}_g/\mathfrak{m}^2_g \to \mathfrak{n}_g/\mathfrak{n}^2_g\), and the tangent map \(T_g(\text{id}, \phi^*) : T_g(G_0, \mathcal{O}') = (\mathfrak{n}_g/\mathfrak{n}^2_g)^* \to (\mathfrak{m}_g/\mathfrak{m}^2_g)^* = T_g G\) is surjective. Since \(G\) and \((G_0, \mathcal{O}')\) have the same graded dimensions, Proposition A.20 implies that \((\text{id}, \phi^*)\) is an isomorphism of cs manifolds.

One can use the isomorphism \(\phi^*\) and Lemma B.8 to prove that \(C\) is fully faithful. We leave the details to the reader. \(\square\)

**Remark B.14.** The argument using the inverse function theorem in the proof of Proposition B.12 is due to E. G. Vishnyakova [Vis09].

By a result of H. Hohnhold [Hoh06, § 4.4, Proposition 12], there is a forgetful functor from complex supermanifolds to cs manifolds, and it is faithful. Moreover, there is a natural notion of complex supergroup pairs (modeled over complex Lie groups and complex Lie superalgebras), and the forgetful functor from complex to cs supergroup pairs which maps \((G_0, \mathfrak{g})\) to \((G_0, \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C})\), forgetting complex structure on \(G_0\), is also faithful.

If we apply the functor \(C\) to a cs supergroup pair which comes from a complex supergroup pair, then we obtain the cs Lie supergroup of a complex Lie supergroup. In particular, we have reproved the following result which is also due to E. G. Vishnyakova [Vis09].

**Corollary B.15.** The categories of complex Lie supergroups and of complex supergroup pairs are equivalent.
B.16. The general linear supergroup deserves a separate discussion in the framework of \( cs \) manifolds. If \( V \) is a \textit{real} super-vector space, then the real Lie supergroup \( \mathcal{G}L(V) \) can be complexified to a \( cs \) Lie supergroup \( \mathcal{G}L(V)_{\mathbb{C}} \); in particular, this is the case for \( \mathcal{G}L(p|q) = \mathcal{G}L(\mathbb{R}^{p|q}) \). For any real super-vector space \( V \), the \( cs \) supergroups \( \mathcal{G}L(V)_{\mathbb{C}} \) and \( C(GL(V_0) \times GL(V_1), End(V \otimes \mathbb{C})) \) are canonically isomorphic. Let us describe the \( cs \) supergroup \( \mathcal{G}L(V)_{\mathbb{C}} \) in the language of points.

To that end, assume more generally that \( (A, A_{0, \mathbb{R}}) \) be a supercommutative \( cs \) algebra (see Remark \( \ref{A12} \)), and \((V, V_{0, \mathbb{R}})\) be a \( cs \) vector space. The free \( A \)-module \( A \otimes V = A \otimes_{\mathbb{C}} V \), and \( End_{A}(A \otimes V) = A \otimes End(V) \). Let \( End_{A}(A \otimes V)_{\mathbb{R}} \) denote the following set of block matrices:

\[
\begin{pmatrix}
A_{0, \mathbb{R}} \otimes_{\mathbb{R}} End_{\mathbb{R}}(V_{0, \mathbb{R}}) & A_{1} \otimes_{\mathbb{C}} Hom_{\mathbb{C}}(V_{1}, V_{0}) \\
A_{1} \otimes_{\mathbb{C}} Hom_{\mathbb{C}}(V_{0}, V_{1}) & A_{0} \otimes_{\mathbb{C}} End_{\mathbb{C}}(V_{1})
\end{pmatrix}.
\]

Recall the notation \( J \) for the graded ideal of \( A \) generated by \( A_{1} \). Then \( J_{0} \) is a complex subspace of \( A_{0} \), so that

\[
J_{0} \otimes End_{\mathbb{R}}(V_{0, \mathbb{R}}) = (J_{0} \otimes_{\mathbb{C}} \mathbb{C}) \otimes_{\mathbb{R}} End(V_{0, \mathbb{R}}) = J_{0} \otimes \mathbb{C} \, End_{\mathbb{C}}(V_{0})
\]

because \( \mathbb{C} \otimes End_{\mathbb{R}}(V_{0, \mathbb{R}}) = End_{\mathbb{C}}(V_{0}) \). Since \( J_{0} \subset A_{0, \mathbb{R}} \), Equation \( \ref{B7} \) shows that the composition of block matrices leaves the set \( End_{A}(A \otimes V)_{\mathbb{R}} \) invariant, and turns it into an associative \( \mathbb{R} \)-algebra.

Let \( GL_{cs}(A \otimes V) \) denote its group of units. This is a subgroup of \( GL(A \otimes V) \), the group of units of \( End_{A}(A \otimes V) = End_{A}(A \otimes V)_{0} \). For \( (A, A_{0, \mathbb{R}}) = (\mathbb{C}, \mathbb{R}) \), we obtain \( GL_{cs}(\mathbb{R}|\mathbb{R}) \subset GL(\mathbb{R}|\mathbb{R}) \).

If \( Z \) is any \( cs \) manifold and \( V \) is now again a \textit{real} super-vector space, then the set \( \mathcal{G}L(V)_{\mathbb{C}}(Z) \) of \( Z \)-points of \( \mathcal{G}L(V)_{\mathbb{C}} \) is exactly \( GL_{cs}(\mathbb{C}|\mathbb{R})_{Z} \).

Any linear \( cs \) manifold is isomorphic to the complexification of a real linear supermanifold. However, this isomorphism is non-canonical. To describe linear actions of \( cs \) Lie supergroups, one therefore needs a generalisation of the general linear group which does not take a real structure on \( V_{1} \) into account. The description of the generalised points of the complexification of the real general linear supergroup indicates the correct definition.

B.17. Let \( (A, A_{0, \mathbb{R}}) \) be a supercommutative \( cs \) algebra, and \((V, V_{0, \mathbb{R}})\) a \( cs \) vector space. In general, there does not exist a real subalgebra \( B \) of \( End(A \otimes V) \), spanning over \( \mathbb{C} \), such that \( End_{cs}(A \otimes V)_{\mathbb{R}} = B_{0} \). However, for \((A, A_{0, \mathbb{R}}) = (\mathbb{C}, \mathbb{R})\), we can consider the set \( \text{End}_{cs}(V)(V) \) of all block matrices

\[
\begin{pmatrix}
End_{\mathbb{R}}(V_{0, \mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} & Hom_{\mathbb{C}}(V_{1}, V_{0}) \\
Hom_{\mathbb{C}}(V_{0}, V_{1}) & End_{\mathbb{C}}(V_{1}) \otimes_{\mathbb{R}} \mathbb{C}
\end{pmatrix}.
\]

Let \( \mu_{00} \) denote the inverse of the isomorphism \( End_{\mathbb{R}}(V_{0, \mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C} \rightarrow End_{\mathbb{C}}(V_{0}) \) of \( \mathbb{C} \)-algebras. Furthermore, define \( \mu_{ij} : Hom_{\mathbb{C}}(V_{i}, V_{j}) \otimes_{\mathbb{R}} \mathbb{C} \rightarrow Hom_{\mathbb{C}}(V_{i}, V_{j}) \), for \((i, j) \in (0, 1), (1, 0), \) by

\[
\mu_{ij}(a \otimes z) = z \cdot a \quad \text{for all} \quad a \in Hom_{\mathbb{C}}(V_{i}, V_{j}), \quad z \in \mathbb{C}.
\]

We obtain an algebra multiplication on \( \text{End}_{cs}(V) \) by

\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + \mu_{00}(bg) & \mu_{10}(a(f \otimes 1) + (b \otimes 1)h) \\ cf \otimes 1 + dh \end{pmatrix}
\]

for any \( a, e \in End_{\mathbb{R}}(V_{0, \mathbb{R}}) \otimes_{\mathbb{R}} \mathbb{C}, \) \( b, f \in Hom(V_{1}, V_{0}), \) \( c, g \in Hom(V_{0}, V_{1}) \), and any \( d, h \in End(V_{1}) \otimes_{\mathbb{R}} \mathbb{C} \). One checks that (with the obvious grading), this turns
End\_cs(V) into a complex superalgebra such that a real form of the even part is given by End\_cs(V)_{0,\mathbb{R}} = \text{End}_{cs}(V)_{\mathbb{R}} = \text{End}_{\mathbb{R}}(V)_{0,\mathbb{R}} \oplus \text{End}_{\mathbb{C}}(V)_{1}.

We define
\[ \mathcal{GL}(V, V)_{0,\mathbb{R}} = C(\text{GL}(V) \times \text{GL}(V), \text{End}_{cs}(V)) \]
where the adjoint action is defined by the conjugation of block matrices. As a cs manifold, \( \mathcal{GL}(V, V)_{0,\mathbb{R}} \) is the open subspace of the linear cs manifold associated with \((\text{End}_{\mathbb{R}}(V), \text{End}(V) \oplus \text{End}(V))\) whose underlying subset of \(\text{End}(V) \oplus \text{End}(V)\) is \(\text{GL}(V) \times \text{GL}(V)\). In particular, for \(A = O(Z)\) (\(Z\) being any cs manifold),
\[ \mathcal{GL}(V, V)_{0,\mathbb{R}}(Z) = GL_{cs}(A \otimes V) , \]
as follows from Corollary A.17.

**Definition B.18.** Let \(G\) be a cs Lie supergroup and \(X\) a cs manifold. A morphism \(\alpha : G \times X \to X\) is called an action if
\[ \alpha \circ (m \times \text{id}) = \alpha \circ (\text{id} \times \alpha) \quad \text{and} \quad \alpha \circ (e \times \text{id}) = \text{id} . \]
The action is called linear if \(X\) is the linear cs manifold associated to a cs vector space \((V, V)_{0,\mathbb{R}}\), and \(\alpha^*(V^*) \subset O_G(G)_{0} \otimes V^*\).

Combining Corollary B.18, Proposition A.15, Corollary A.17 and the formulae from Proposition B.6, we obtain the following statement.

**Proposition B.19.** Let \((V, V)_{0,\mathbb{R}}\) be a cs vector space, \((G, g)\) a cs supergroup pair, and \(G = C(G, g)\). The following data are in bijection.

(i). Linear actions \(\alpha : G \times V \to V\).

(ii). Even linear maps \(f : V^* \to O_G(G)_{0} \otimes V^*\) such that
\[ \alpha \circ (m^* \otimes \text{id}_{V^*}) \circ f = (\text{id}_{V^*} \otimes f) \circ f \quad \text{and} \quad (e^* \otimes \text{id}_{V^*}) \circ f = \text{id}_{V^*} . \]

(iii). Elements \(F \in \text{GL}_{cs}(O_G(G)_{0} \otimes V) \subset (O_G(G)_{0} \otimes \text{End}(V))_{0}\) such that
\[ (m^* \otimes \text{id}_V) \circ F = (\text{id}_V \otimes F) \circ F \quad \text{and} \quad (e^* \otimes \text{id}_V) \circ F = \text{id}_V . \]

(iv). Morphisms \(\varphi : G \to \mathcal{GL}(V, V)_{0,\mathbb{R}}\) of cs Lie supergroups.

(v). Morphisms \((\varphi, d\varphi) : (G, g) \to (GL(V)_{0,\mathbb{R}} \times GL(V)_{1}, \text{End}_{cs}(V))\) of cs supergroup pairs.

The data in (iii) and (v) are related by the equation
\[ F(u; g) = \varphi(g) \circ d\varphi(u) \in \text{End}(V) \quad \text{for all} \quad u \in \Omega(g), \quad g \in G_{0} . \]

**Proof.** Given an even linear map \(f : V^* \to O_G(G)_{0} \otimes V^*\), we obtain an element \(F \in (O_G(G)_{0} \otimes \text{End}(V))_{0}\). Then \(f(V^*_{0,\mathbb{R}})\) consists of real-valued superfunctions if and only \(F \in \text{End}_{cs}(O_G(G)_{0} \otimes V)_{R}\). Assume now that \(f = \alpha^*\) where \(\alpha : G \times V \to V\) is a linear action.

We recall now from A.18 that for any cs manifold, the global sections module of the structure sheaf is endowed with a natural nuclear Fréchet topology. Then, more explicitly, \(F\), considered as an even element of the tensor product \(O_G(G)_{0} \otimes \text{End}(V)\), is given by
\[ (\text{id} \otimes \xi)(F(x)) = f(\xi)(x) \quad \text{for all} \quad \xi \in V^*, \quad x \in V . \]

In fact, this equation can easily be extended to hold for \(v \in O_G(G)_{0} \otimes V\) (by extending \(F\) and \(f O_G(G)_{0}\)-linearly). Since \(\alpha\) is an action, we have the relation
\[(m^* \hat{\otimes} \text{id}) \circ f = (\text{id} \hat{\otimes} f) \circ f. \] Hence, we compute for all \( \mu \in V^*, \ x \in V, \)

\[
(id \hat{\otimes} \text{id} \hat{\otimes} \xi)((id \hat{\otimes} f)(F(x))) = (id \hat{\otimes} f(\xi))(F(v)) = (id \hat{\otimes} f)(f(\xi))(x)
\]

\[
= (m^* \hat{\otimes} \text{id})(f(\xi))(x) = m^*(f(\xi)(x))
\]

\[
= m^*((id \hat{\otimes} \xi)(F(x))) = (m^* \hat{\otimes} \xi)(F(x)) ,
\]

so \((id \hat{\otimes} F) \circ F = (m^* \hat{\otimes} \text{id}) \circ F. \)

Moreover,

\[
\xi((e^* \hat{\otimes} \text{id})(F(x))) = e^*(f(\xi)(x)) = (e^* \hat{\otimes} \text{id})(f(\xi))(x) = \xi(x) ,
\]

so \((e^* \hat{\otimes} \text{id}) \circ F = \text{id}. \)

Denote by \( \delta : G \to G \times G \) the diagonal; \( \delta^* \) is the algebra multiplication of \( \mathcal{O}_G(G_0). \) Moreover, \( F, \) considered as an element of \( \text{End}_{\mathcal{O}_G(G_0)}(\mathcal{O}_G(G_0) \otimes V), \) is \((id \hat{\otimes} F). \) The composite in the endomorphism ring of \( F \) and \((i^* \hat{\otimes} \text{id}) \circ F \) is the left hand side of the following equation:

\[
\delta^* \hat{\otimes} \text{id} \circ (id \hat{\otimes} i^* \hat{\otimes} \text{id}) \circ (id \hat{\otimes} F) \circ F = (\delta^* \circ (id \hat{\otimes} i^* \circ m^*) \hat{\otimes} \text{id}) \circ F
\]

\[
= (1 \cdot e^* \hat{\otimes} \text{id}) \circ F = 1 \circ \text{id} .
\]

This shows that \( F \) is left invertible; hence, it is invertible and thus an element of \( \text{GL}_{cs}(\mathcal{O}_G(G_0) \otimes V). \)

Let \( F \in \text{GL}_{cs}(\mathcal{O}_G(G_0) \otimes V) = \mathcal{GL}(V, V_{0, \mathbb{R}})(G) \)

satisfy

\[
(m^* \hat{\otimes} \text{id}) \circ F = (\text{id} \hat{\otimes} F) \circ F \quad \text{and} \quad (e^* \hat{\otimes} \text{id}) \circ F = \text{id} .
\]

By the same computation as above, the inverse in \( \text{GL}_{cs}(\mathcal{O}_G(G_0) \otimes V) \) of \( F \) is given by \( F^{-1} = (i^* \hat{\otimes} \text{id}) \circ F. \)

The element \( F \) represents a morphism \( \varphi : G \to \mathcal{GL}(V, V_{0, \mathbb{R}}) \) of \( cs \) manifolds. By Proposition A.13 \( \varphi \circ m \) is represented by \((m^* \hat{\otimes} \text{id}) \circ F. \) If \( m_0 \) is the multiplication morphism of \( \mathcal{GL}(V, V_{0, \mathbb{R}}), \) then \( m_0 \circ (\varphi \times \varphi) \) is represented by \((id \hat{\otimes} F) \circ F. \) Similarly, \((i^* \hat{\otimes} F) \) represents \( \varphi \circ i, \) and if \( i_0 \) denotes the inversion morphism of \( \mathcal{GL}(V, V_{0, \mathbb{R}}), \) then \( i_0 \circ F \) is represented by \( F^{-1}. \) Finally, \( \varphi \circ e \) is represented by \((e^* \hat{\otimes} \text{id}) \circ F, \) and the unit morphism \( e_0 \) of \( \mathcal{GL}(V, V_{0, \mathbb{R}}) \) is represented by \( \text{id}. \) These considerations show that \( \varphi \) is a morphism of \( cs \) Lie supergroups.

Morphisms of \( cs \) Lie supergroups and of \( cs \) supergroup pairs are in bijection. Given a morphism \( (\varphi, d\varphi) : (G_0, g) \to (\text{GL}(V_{0, \mathbb{R}}) \times \text{GL}(V_1), \text{End}_{cs}(V)), \) we may define a map \( f : V^* \to \mathcal{O}_G(G_0) \otimes V^* \subset \mathcal{O}_G \times V(G_0 \times V_{0, \mathbb{R}}) \) by

\[
f(\xi)(u; g)(x) = (\xi \circ \varphi(g) \circ d\varphi(u))(x)
\]

for all \( \xi \in V^*, \ x \in V, \ u \in \mathfrak{u}(g), \ g \in G_0. \)

Let \( \alpha : G \times V \to V \) be the morphism of \( cs \) manifolds which corresponds via Corollary A.17 to \( f. \) Then

\[
((m^* \hat{\otimes} \text{id}) \circ f)(\xi)(u \otimes v; g, h)(x) = f(\mu(\text{Ad}(h^{-1}))(u)v; gh)(x)
\]

\[
= (\xi \circ \varphi(gh) \circ d\varphi(\text{Ad}(h^{-1}))(u))(x)
\]

\[
= (\xi \circ \varphi(g) \circ d\varphi(u) \circ \varphi(h) \circ d\varphi(v))(x)
\]

\[
= f(\xi)(u; g)((\varphi(h) \circ d\varphi(v))(x))
\]

\[
= ((\text{id} \hat{\otimes} f) \circ f)(\xi)(u \otimes v; g, h)(x)
\]
Hence, \((m^* \otimes \text{id}) \circ f = (\text{id} \otimes f) \circ f\). By the uniqueness statement in Corollary \[A.17\] it follows that \(\alpha \circ (m \times \alpha) = \alpha \circ (\text{id} \times \alpha)\). Analogously, one proves that \(\alpha \circ (e \times \text{id}) = \text{id}\). We have proved the claim.

\[\square\]

C. Appendix: Berezin integration on \(cs\) manifolds

C.1. Berezin integral and absolute Berezin integral.

C.1. Let \((A,A_{0,\mathbb{R}})\) be a supercommutative \(cs\) algebra, and \((V,V_0,\mathbb{R})\) a \(cs\) vector space. The homomorphism \(\text{Ber} : \text{GL}(A \otimes V) \to A_0^x\) defined by

\[
\text{Ber} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \text{det}(a - bd^{-1}c)(\det d)^{-1} = \text{det}(d - ca^{-1}b)(\det a)^{-1}
\]

called the Berezinian. Observe the following: If \(\bar{A} = A/A \cdot A_1\), then a matrix as above is invertible if and only if \(a\) and \(d\) are, if and only if their images in \(\text{End}_A(A_0 \otimes V_j)\) are (where \(j = 0\) and \(j = 1\), respectively).

If \((A,A_{0,\mathbb{R}}) = \Gamma(Z) = (\mathcal{O}_Z(Z_0),\mathcal{O}_Z(Z_0)_{0,\mathbb{R}})\) for some supermanifold \(Z\), then one can define a further homomorphism \(|\text{Ber}| : \text{GL}_{cs}(A \otimes V) \to A_0^x\) by

\[
|\text{Ber}| \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \text{sgn} \det a \cdot \text{Ber} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right).
\]

Here, recall that \(\tilde{f}\) is the function underlying the superfunction \(f\), \(\text{sgn} z = |z|^{-1}z\) for \(z \in \mathbb{C}^x\), and \(\text{sgn} 0 = 0\). The homomorphism \(|\text{Ber}|\) is slightly non-standard, cf. [Vor91] where the notation \(\text{Ber}_{1,0}\) is used. Compare also [Sha88].

C.2. Let \(M\) be a free graded \(A\)-module of graded dimension \(p/q\). We define an \(A\)-module \(\text{Ber}_A(M) = \text{Ber}(M)\) of graded rank \(1|0\) (if \(q\) is even) resp. \(0|1\) (if \(q\) is odd) as follows. With a basis \(x_1, \ldots, x_p, \xi_1, \ldots, \xi_q\) where the \(x_j\) are even and the \(\xi_j\) are odd, we associate a distinguished basis \(D(x,\xi)\) of \(\text{Ber}(M)\), of parity \(\equiv q\) \((2)\).

If \(y_1, \ldots, y_p, \eta_1, \ldots, \eta_q\) is another such basis, related to \(x,\xi\) by

\[
y_i = \sum_j a_{ij}x_j + b_{ij}\xi_j \quad \text{and} \quad \eta_i = \sum_j c_{ij}x_j + d_{ij}\xi_j,
\]

where \(a_{ij}, d_{ij} \in A_0, b_{ij}, c_{ij} \in A_1\), then

\[
D(y,\eta) = \text{Ber} \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot D(x,\xi).
\]

Assume that we have \((A,A_{0,\mathbb{R}}) = \Gamma(Z)\) for some \(cs\) manifold \(Z\) (cf. Remark \[A.12\]); moreover, suppose that we are given a choice \(M_{0,\mathbb{R}} \subset M_0\) of a maximal proper \(A_{0,\mathbb{R}}\)-submodule such that \(M_0 = (M_{0,\mathbb{R}})_C\) and \(A_1 \cdot M_1 \subset M_{0,\mathbb{R}}\) where \(\langle \cdot \rangle_C\) denotes complex linear span. (Because \(A_1 \subset A_{0,\mathbb{R}}\), such submodules manifestly exist whenever \(A_0 \neq 0\) and \(M_0 \neq 0\).) We define \(A_\mathbb{R} = A_{0,\mathbb{R}} \oplus A_1\). This is a real graded subalgebra of \(A\).

We now define a free \(A\)-module \(|\text{Ber}|_{A,A_{0,\mathbb{R}}}(M,M_{0,\mathbb{R}})\) of graded rank \(1|0\) (if \(q\) is even) resp. \(0|1\) (if \(q\) is odd). With any graded basis \(x,\xi\) of the \(A_\mathbb{R}\)-module \(M_\mathbb{R} = M_{0,\mathbb{R}} \oplus M_1\), we associate a distinguished basis \(|D(x,\xi)|\).

If \(y,\eta\) is related to \(x,\xi\) as above, where now \(a_{ij}, d_{ij} \in A_{0,\mathbb{R}}, b_{ij}, c_{ij} \in A_1\), then we require

\[
|D(y,\eta)| = |\text{Ber}| \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) \cdot |D(x,\xi)|.
\]
C.3. Let \( X = (X_0, \mathcal{O}) \) be a \( cs \) manifold of graded dimension \( p|q \). If \( U \) is a coordinate neighbourhood and \( (x, \xi) \) is a coordinate system, then any superfunction \( f \in \mathcal{O}(U) \) may be written uniquely in the form

\[
f = \sum_I f_I(x_1, \ldots, x_p) \xi^I \quad \text{where} \quad f_I \in \mathcal{C}^\infty(x(U), \mathbb{C}) ,
\]

the sum extends over all \( I = (1 \leq i_1 < \cdots < i_k \leq q) \), and \( \xi^I = \xi_{i_1} \cdots \xi_{i_k} \). Here, \( h(x_1, \ldots, x_p) \), for \( h \in \mathcal{C}^\infty(x(U), \mathbb{C}) \), is to be understood in the following way: Let \( \varphi : U \to x(U) \subset \mathbb{R}^p \) be the morphism determined by \( \varphi^*(\mathfrak{pr}_j) = x_j \); then \( h(x_1, \ldots, x_p) = \varphi^* h \).

Define even derivations \( \frac{\partial}{\partial x_i} \) of \( \mathcal{O}(U) \) by

\[
\frac{\partial f}{\partial x_i} = \sum_I (\partial_i f_I)(x_1, \ldots, x_p) \xi^I .
\]

In particular, \( \frac{\partial}{\partial x_i} = \delta_{iI} \). Moreover, define odd derivations \( \frac{\partial}{\partial \xi_j} \) by

\[
\frac{\partial f}{\partial \xi_j} = \sum_I f_I(x_1, \ldots, x_p) \frac{\partial \xi^I}{\partial \xi_j} .
\]

Then \( \frac{\partial}{\partial x_i}, \frac{\partial}{\partial \xi_j} \) form an \( \mathcal{O}(U) \)-basis of \( \text{Der}(\mathcal{O}(U)) \). The \( \mathcal{O}(U) \)-dual \( \Omega^1_X(U) \) has the basis \( dx_1, \ldots, dx_p, d\xi_1, \ldots, d\xi_q \) where

\[
\langle dx_i, \frac{\partial}{\partial x_j} \rangle = \langle d\xi_i, \frac{\partial}{\partial \xi_j} \rangle = \delta_{ij} \quad \text{and} \quad \langle dx_i, \frac{\partial}{\partial \xi_j} \rangle = \langle d\xi_i, \frac{\partial}{\partial x_j} \rangle = 0 .
\]

Let \( \Omega^1_X(U)_{0,\mathbb{R}} \) be the \( \mathcal{O}(U)_{0,\mathbb{R}} \) submodule spanned by \( dx_1, \ldots, dx_p \). One sees easily that this submodule is in fact independent of the choice of basis. (The point is that the \( x_j \) are real-valued for any coordinate system.)

We set \( \text{Ber}_X(U) = \text{Ber}_{\mathcal{O}(U)}(\Omega^1_X(U)) \). Denote the associated sheaf of \( \mathcal{O} \)-modules by \( \text{Ber}_X \); it is called the Berezinian sheaf of \( X \). Similarly, set

\[
|\text{Ber}|_X(U) = |\text{Ber}|_{\mathcal{O}(U)}(\Omega^1_X(U)_0,\mathbb{R}) ,
\]

and let \( |\text{Ber}|_X \) be the associated sheaf, called the absolute Berezinian sheaf.

If \( f : X \to Y \) is an isomorphism of supermanifolds, then we define a sheaf morphism \( f^* : \text{Ber}_Y \to f_* \text{Ber}_X \) as follows. For any coordinate neighbourhood \( V \subset Y_0 \) and any coordinate system \( (y, \eta) \) on \( V \), we let

\[
f^* (h \cdot D(dy, d\eta)) = (f^* h) \cdot D(df^* y, df^* \eta) \quad \text{for all} \quad h \in \mathcal{O}_Y(V) .
\]

If \( (z, \zeta) \) is another coordinate system, then \( D(dz, d\zeta) = \text{Ber}(J) \cdot D(dy, d\eta) \) where the Jacobian is given by

\[
J^* = \begin{pmatrix}
\frac{\partial z}{\partial y} & \frac{\partial \zeta}{\partial y} \\
\frac{\partial z}{\partial \eta} & \frac{\partial \zeta}{\partial \eta}
\end{pmatrix} ,
\]

the superscript \( ^* \) denoting the super-transpose. It follows by the multiplicative property of the Berezinian that \( f^* \) gives a well-defined sheaf morphism.

One can now proceed in exactly the same way to define a morphism of sheaves \( f^* : |\text{Ber}|_Y \to f_* |\text{Ber}|_X \). What enters here crucially is that \( \frac{\partial}{\partial y} \) is real-valued.

**Definition C.4.** A \( cs \) manifold \( X \) is called evenly oriented if the underlying manifold is oriented. For an evenly oriented supermanifold of graded dimension \( p|q \), we define as follows an even linear presheaf morphism \( f_X : \Gamma_c(\text{Ber}_X) \to \mathbb{C}_{X_0} \) called the Berezin integral—here, \( \Gamma_c \) denotes the set of compactly supported sections.
Let \((U_\alpha)\) be a locally finite cover of \(X_0\) by coordinate neighbourhoods, \((x_\alpha, \xi_\alpha)\) coordinate systems where \(x_1^\alpha, \ldots, x_p^\alpha\) is are oriented coordinates systems of the underlying manifold \(X_0\), and \((\chi_\alpha)\) an \(\mathcal{O}\)-partition of unity subordinate to \((U_\alpha)\).

For any \(f \in \mathcal{O}_X(U_\alpha)\), we write \(f = \sum I f_I \cdot \xi^\alpha I\) where \(f_I = h_I(x_1^\alpha, \ldots, x_p^\alpha)\) for some \(h_I \in C^\infty(x^\alpha(U_\alpha), \mathbb{C})\). If \(f\) is compactly supported, let
\[
\int_{U_\alpha} D(d\xi^\alpha, d\xi^\alpha) \cdot f = (-1)^{pq} \sum I d\xi^\alpha \cdot f_{I,1,...,1} \in \mathbb{C},
\]
where \(d\xi^\alpha = d\xi_1^\alpha \wedge \cdots \wedge d\xi_p^\alpha\). Then, we define, for all \(\omega \in \Gamma_e(\text{Ber}_X)\),
\[
\int_X \omega = \sum_{\alpha} \int_{U_\alpha} \chi_\alpha \cdot \omega.
\]

The Berezin integral is well-defined, independent of all choices \([\text{Lei}80, \text{Theorem } 2.4.5]\). Moreover, if \(f : X \to Y\) is an isomorphism of even oriented \(cs\) manifolds preserving the orientations of the underlying manifolds, then
\[
\int_X f^* \omega = \int_Y \omega \quad \text{for all } \omega \in \Gamma_e(\text{Ber}_Y).
\]

In exactly the same way, we define a presheaf morphism \(\int_X : \Gamma_e(\text{Ber}_X) \to C_{X_0}\) for any \(\text{cs}\) manifold \(X\) (possibly without even orientation) by using the integral of densities \(\int_{Y_0} d\xi^\alpha \cdot f_{I,1,...,1}\). The resulting integral is invariant under all isomorphisms, irrespective of preservation or even existence of orientations.

We shall use the following lemma repeatedly.

**Lemma C.5.** Let \(Z\) be a \(\text{cs}\) manifold and \(f \in |\text{Ber}_Z(Z_0)|\). If \(\int_Z f \cdot h = 0\) for all \(h \in \Gamma_e(Z_0, \mathcal{O}_Z)\), then \(f = 0\).

**Proof.** We argue by contraposition. Then we may assume that \(Z = \mathbb{R}^{p|q}_{\mathbb{C}}\) as a \(\text{cs}\) manifold, and that \(f = \varphi \cdot |D(x, \xi)|\) where \(0 \in (\text{supp } \varphi)^0\) and \(x, \xi\) is the standard coordinate system.

Write \(\varphi = \sum I \varphi_I \xi^I\) with \(\varphi_I \in C^\infty(\mathbb{R}^p, \mathbb{C})\) (this is possible, since \(x\) are the standard coordinates). There is some multi-index \(I = (1 \leq i_1 < \cdots < i_m \leq q)\) such that \(\varphi_I(0) \neq 0\). Let \(J = (1 - i_1, \ldots, 1 - i_m)\) and \(b = \xi^J \cdot e^T_{\chi}\) where \(\chi : \mathbb{R}^p \to [0, 1]\) is smooth, of compact support, and satisfies \(\chi(0) = 1\). Thus
\[
\int_{\mathbb{R}^p} f \cdot h = \pm \int_{\mathbb{R}^q} \xi^{(1, \ldots, 1)} \cdot |\varphi_I|^2 \cdot \chi \cdot |D(x, \xi)| = \pm \int_{\mathbb{R}^p} |\varphi_I|^2 \cdot \chi \neq 0.
\]
This proves the lemma. \(\square\)

**C.6.** For any morphism \(\varphi : X \to Y\), and any open subset \(U \subset Y\), we consider the set \(\Gamma^\varphi_f(U, |\text{Ber}_X|)\) of all local sections \(\omega \in \Gamma(\varphi^{-1}(U), |\text{Ber}_X|)\) such that \(\varphi : \text{supp } \omega \to Y\) is a proper map. This defines a presheaf \(\Gamma^\varphi_f\) on \(Y\), the presheaf of sections compactly supported along the fibres of \(\varphi\).

Assume that \(\varphi\) is surjective and that its underlying map \(\varphi_0\) is surjective. Then there is a well-defined even presheaf morphism \(\varphi_1 : \Gamma_f^\varphi(U, |\text{Ber}_Y|) \to |\text{Ber}_Y|\) of (Berezin) integration along the fibres. Note that we give \(|\text{Ber}_X|\) the parity \(\varepsilon \varphi\) where \(\varepsilon = 0\) and \(\dim X = p|q\). This makes \(\varphi_1\) an even morphism. Two fundamental properties of \(\varphi_1\) are
\[
\text{supp } \varphi_1(\omega) \subset \varphi(\text{supp } \omega) \quad \text{and} \quad \varphi_1(\varphi^* (f) \cdot \omega) = f \cdot \varphi_1(\omega).
\]
Compare \([\text{AH}10, \text{Proposition } 5.7]\) for the definition of \(\varphi_1\).
In the case of a projection \( p_1 : Y \times F \to Y \), one may define more generally a fibre integration map \( p_1 : \Gamma^G_{G_0} (E \otimes |\text{Ber}|_F) \to \Gamma (E) \), for any sheaf \( E \) on \( Y \).

C.2. Invariant Berezin integration. In this section, we will review some results from [AH10] concerning invariant Berezin integration on homogenous cs manifolds, reformulating them in the language of cs supergroup pairs.

C.7. Let \( G \) be a cs Lie supergroup. As in [AH10], one can define the concept of a quotient by an action of \( G \) and show that quotients exist for free and proper \( G \)-actions.

In particular, assume that \( H \) is a closed cs subssupergroup. I.e., we have a morphism \( f : H \to G \) of cs Lie supergroups which is, on the level of spaces, a closed embedding of \( H_0 \) in \( G_0 \), and \( f^* : \mathcal{O}_G \to f_* \mathcal{O}_H \) is an epimorphism. (We will suppress \( f \) from the notation.) Then the quotient cs manifold \( G/H = (G_0/H_0, \mathcal{O}_{G/H}) \) exists. Here, for all open subsets \( U \subset G_0/H_0 \),

\[
\mathcal{O}_{G/H}(U) = \left\{ f \in \mathcal{O}_G(\pi^{-1}(U)) \mid m^* f = p_1^{-1} f \in \mathcal{O}_{G \times H}(\pi^{-1}(U) \times H_0) \right\},
\]

where \( (m, m^* : G \times H \to G) \) is the restriction of the multiplication morphism of \( G \). On the level of sheaves, the canonical morphism \( \pi : G \to G/H \) maps \( f \) to \( f = \pi^* f \) (i.e. \( \pi^* \) is the inclusion). We will often view \( \mathcal{O}_{G/H} \) as a subsheaf of \( \mathcal{O}_G \).

If \( G = C(G_0, \mathfrak{g}) \) where \( (G_0, \mathfrak{g}) \) is a cs supergroup pair, and \( H = C(H_0, \mathfrak{h}) \) where the cs supergroup pair is given by a closed subgroup \( H_0 \subset G_0 \) and a Lie subsuperalgebra \( \mathfrak{h} \subset \mathfrak{g} \), then we have

\[
\mathcal{O}_{G/H}(U) = \mathcal{O}_G(\pi^{-1}(U))^{H_0, \mathfrak{h}} = \overline{\text{Hom}}_{H_0, \mathfrak{h}} (\mathfrak{U}(\mathfrak{g})/\mathfrak{U}(\mathfrak{g}), C^\infty(\pi^{-1}(U)))
\]

for all open \( U \subset G_0/H_0 \). Here, the superscript \( H_0, \mathfrak{h} \) denotes simultaneous \( H_0 \)- and \( \mathfrak{h} \)-invariants, and we recall the definition of the actions from the equations (B.1) and (B.3).

Moreover, \( \mathcal{O}_{G/H} \) inherits actions by \( \mathfrak{g} \) and \( G_0 \) from \( \mathcal{O}_G \), namely

\[(\ell_a f)(v; g) = (-1)^{|f||u|} f(\text{Ad}(g^{-1})(S(u))v; g)\]

for all \( f \in \mathcal{O}_{G/H}(U) \), \( u, v \in \mathfrak{U}(\mathfrak{g}) \), \( g \in \pi^{-1}(U) \), and

\[(\ell_g f)(u; h) = f(u; g^{-1}h)\]

for all \( f \in \mathcal{O}_{G/H}(U) \), \( u \in \mathfrak{U}(\mathfrak{g}) \), \( g \in G_0, h \in g \cdot \pi^{-1}(U) \), respectively.

C.8. If \( G \) is of graded dimension \( p/q \), then for all open \( U \subset G_0/H_0 \), there are canonical isomorphisms

\[
\text{Ber}_{G/H}(U) \cong \left( \text{Hom}_{H_0, \mathfrak{h}} (\mathfrak{U}(\mathfrak{g}), C^\infty(\pi^{-1}(U))) \otimes \text{Ber}((\mathfrak{g}/\mathfrak{h})^*) \right)^{H_0, \mathfrak{h}},
\]

\[
|\text{Ber}|_{G/H}(U) \cong \left( \text{Hom}_{H_0, \mathfrak{h}} (\mathfrak{U}(\mathfrak{g}), C^\infty(\pi^{-1}(U))) \otimes |\text{Ber}|((\mathfrak{g}/\mathfrak{h})^*) \right)^{H_0, \mathfrak{h}}
\]

where we abbreviate

\[
|\text{Ber}|((\mathfrak{g}/\mathfrak{h})^*) = |\text{Ber}|_{C, R}((\mathfrak{g}/\mathfrak{h})^*, (\mathfrak{g}_0, \mathfrak{h}/\mathfrak{h}_0, R)^*)
\]

Here, we use the canonical isomorphism \( \text{Ber}_R(R \otimes V) = R \otimes \text{Ber}(V) \), and the characterisation of \( \text{Ber}_{G/H} \) from [AH10] Corollary 4.12]. (In this reference, we consider real supermanifolds, but as should be clear by now, everything goes through for cs manifolds with only minor changes.) For the absolute Berezinians, essentially the same argument goes through.
If, according to the above isomorphism, we consider the elements of $\mathcal{B}er_{G/H}(U)$ as maps $\mathcal{U}(\mathfrak{g}) \times \pi^{-1}(U) \to \mathcal{B}er((\mathfrak{g}/\mathfrak{h})^*)$, then the actions of $\mathfrak{g}$ and $G_0$ induced from $\mathcal{O}_{G/H}$ may be expressed simply by

$$ (\ell_uf)(v;g) = (-1)^{|f||u|}f(\operatorname{Ad}(g^{-1})(S(u))v;g) $$

for all $f \in \mathcal{B}er_{G/H}(U)$, $u, v \in \mathcal{U}(\mathfrak{g})$, $g \in \pi^{-1}(U)$, and

$$ (\ell_gf)(u;h) = f(u;g^{-1}h) $$

for all $f \in \mathcal{B}er_{G/H}(U)$, $u \in \mathcal{U}(\mathfrak{g})$, $g \in G_0$, $h \in g \cdot \pi^{-1}(U)$, respectively. The same holds for the absolute Berezinians.

C.9. With the aid of Proposition B.19 one sees that $\operatorname{Ad}_G : G \to \mathcal{G}\mathcal{C}(\mathfrak{g})$, the adjoint morphism, is represented by $\operatorname{Ad}_{G_0,\mathfrak{g}} \in \Gamma(\mathcal{O}_G) \otimes \mathbf{End}(\mathfrak{g})$, defined by

$$ \operatorname{Ad}_{G_0,\mathfrak{g}}(u;h) = \operatorname{Ad}(h) \circ \operatorname{ad}(u) \in \mathbf{End}(\mathfrak{g}) \quad \text{for all} \quad g \in G_0, \ u \in \mathcal{U}(\mathfrak{g}). $$

Similarly, $\operatorname{Ad}_{G,H} : H \to \mathbf{End}(\mathfrak{g})$ is represented by $\operatorname{Ad}_{G_0,\mathfrak{g}}|_H \in \Gamma(\mathcal{O}_H) \otimes \mathbf{End}(\mathfrak{g})$, given by

$$ \operatorname{Ad}_{G_0,\mathfrak{g}}|_H(u;h) = \operatorname{Ad}(h) \circ \operatorname{ad}(u) \in \mathbf{End}(\mathfrak{g}) \quad \text{for all} \quad h \in H_0, \ u \in \mathcal{U}(\mathfrak{h}). $$

Analogously, one has $\operatorname{Ad}_{H_0,\mathfrak{h}} \in \Gamma(\mathcal{O}_H) \otimes \mathbf{End}(\mathfrak{h})$.

By [AH10, Theorem 4.13], $\Gamma(\mathcal{B}er_{G/H})$ contains a non-zero $(G_0,\mathfrak{g})$-invariant element if and only if

$$ \mathcal{B}er(\operatorname{Ad}_{H_0,\mathfrak{h}}) = \mathcal{B}er(\operatorname{Ad}_{G_0,\mathfrak{g}}|_H) \in \Gamma(\mathcal{O}_H). $$

Here, $\mathcal{B}er(\operatorname{Ad}_{H_0,\mathfrak{h}}) \in \Gamma(\mathcal{O}_H)_{\mathfrak{h}}$ is given by

$$ \mathcal{B}er(\operatorname{Ad}_{H_0,\mathfrak{h}})(u;h) = \mathcal{B}er(\operatorname{Ad}(h)) \cdot \operatorname{str}_{\mathfrak{h}} \operatorname{ad}(u) \quad \text{for all} \quad u \in \mathcal{U}(\mathfrak{h}), \ h \in H_0. $$

Here, $\operatorname{str}_{\mathfrak{h}} \circ \operatorname{ad} : \mathcal{U}(\mathfrak{g}) \to \mathbb{C}$ is the unique extension to an even unital algebra morphism of the map $\mathfrak{g} \to \mathcal{U}(\mathfrak{g}) : x \mapsto \operatorname{str}_{\mathfrak{h}} \operatorname{ad}(x)$. A similar equation defines $\mathcal{B}er(\operatorname{Ad}_{G_0,\mathfrak{g}}|_H)$.

Such a non-zero $(G_0,\mathfrak{g})$-invariant element (if it exists), is unique up to scalar multiples. In this case, we say that the $G$-space $G/H$ is geometrically unimodular. We say that the cs Lie supergroup $G$ is geometrically unimodular if it is so as a $G \times G$-space (it always is as a $G$-space).

C.10. Similarly, $|\mathcal{B}er|_{G/H}$ possesses a non-zero $(G_0,\mathfrak{g})$-invariant global section if and only if

$$ |\mathcal{B}er|(\operatorname{Ad}_{H_0,\mathfrak{h}}) = |\mathcal{B}er|(\operatorname{Ad}_{G_0,\mathfrak{g}}|_H) \in \Gamma(\mathcal{O}_H). $$

Here, $|\mathcal{B}er|(\operatorname{Ad}_{H_0,\mathfrak{h}}) \in \Gamma(\mathcal{O}_H)_{\mathfrak{h}}$ is given by

$$ |\mathcal{B}er|(\operatorname{Ad}_{H_0,\mathfrak{h}})(u;h) = |\mathcal{B}er|(\operatorname{Ad}(h)) \cdot \operatorname{str}_{\mathfrak{h}} \operatorname{ad}(u) \quad \text{for all} \quad u \in \mathcal{U}(\mathfrak{h}), \ h \in H_0. $$

Whenever (C.7) is satisfied, we say that the $G$-space $G/H$ is analytically unimodular; we say that the cs Lie supergroup $G$ is analytically unimodular if it is as a $G \times G$-space (it always is as a $G$-space).

**Proposition C.11.** Let $(G_0,\mathfrak{g})$ and $(H_0,\mathfrak{h})$ be cs supergroup pairs where $H_0 \subset G_0$ is a closed subgroup and $\mathfrak{h} \subset \mathfrak{g}$ is a Lie subsuperalgebra. Let $G = C(G_0,\mathfrak{g})$ and $H = C(H_0,\mathfrak{h})$.

(i) If $G$ and $H$ are geometrically (analytically) unimodular, then so is the $G$-space $G/H$. 

(ii). If \( G \) and \( H \) are geometrically (analytically) unimodular, then so is the direct product \( G \times H \) of \( cs \) Lie supergroups.

(iii). If \( \mathfrak{g} \) is nilpotent (in particular, if it is Abelian) and \( G_0 \) is connected, then \( G \) is geometrically and analytically unimodular.

(iv). If \( \mathfrak{g} \) is strongly reductive (cf. Definition \( \text{[C.7]} \) and \( G_0 \) is connected, then \( G \) is geometrically and analytically unimodular.

(v). If \( \text{Ad}_g(G_0) \subset \text{GL}(\mathfrak{g}) \) is compact, then \( G \) is analytically and geometrically unimodular.

**Proof.** Statements (i) and (ii) are immediate from \( \text{[C.6]} \) and \( \text{[C.7]} \). To prove that \( G \) is geometrically unimodular, it will be sufficient to prove that \( \text{Ber}_g(\text{Ad}(g)) = 1 \) for all \( g \in G_0 \) and \( \text{str}_g \text{ad}(x) = 0 \) for all \( x \in \mathfrak{g} \). The latter condition is always verified for \( x \in \mathfrak{g}_1 \), for reasons of parity. Thus, it will suffice to prove the former, since the latter then follows by differentiation; conversely, when \( G_0 \) is connected, the latter condition for \( x \in \mathfrak{g}_0 \) implies the former. A similar reasoning holds true for the case of analytic unimodularity.

If \( \mathfrak{g} \) is nilpotent, then \( \text{tr}_{\mathfrak{g}_0} \text{ad}(x) = \text{tr}_{\mathfrak{g}_1} \text{ad}(x) = 0 \) for all \( x \in \mathfrak{g}_0 \); thus, \( G \) is geometrically and analytically unimodular. If \( \mathfrak{g} \) is strongly reductive, then one has \( \mathfrak{g} = \mathfrak{z}(\mathfrak{g}) \oplus \mathfrak{g}' \). We have \( \text{ad}(\mathfrak{z}(\mathfrak{g})) = 0 \). But \( \text{str}_g([a, b]) = 0 \) for any \( a, b \in \text{End}(\mathfrak{g}) \). In particular, \( \text{str}_g \text{ad}(\mathfrak{g}') = 0 \). Finally, if \( \text{Ad}_g(G_0) \) is compact, then \( |\text{Ber}(\text{Ad}_g(G_0)) \) and \( \text{Ber}(\text{Ad}_g(G_0)) \) are compact subgroups of \( (\mathbb{C}^\times, \cdot) \), and hence trivial. \( \square \)

**C.12.** If \( G/H \) is a geometrically unimodular \( G \)-space, then \( \text{Ber}((\mathfrak{g}/\mathfrak{h})^*) \) is a trivial \( H_0^* \) and \( \mathfrak{h} \)-module. Thus, in this case, for all open \( U \subset G_0/H_0 \),

\[
\text{Ber}_{G/H}(U) \cong \mathcal{O}_{G/H}(U) \otimes \text{Ber}((\mathfrak{g}/\mathfrak{h})^*) ;
\]

similarly for the absolute Berezinians if \( G/H \) is analytically unimodular.

By Equations \( \text{[C.3]} \) and \( \text{[C.4]} \) for the \((G_0, \mathfrak{g})\)-action, a non-zero invariant element of \( \Gamma(\text{Ber}_{G/H}) \) is necessarily of the form \( 1 \otimes \omega \) for a non-zero \( \omega \in \text{Ber}((\mathfrak{g}/\mathfrak{h})^*) \).

**C.13.** If \( f \in \Gamma_c(\mathcal{O}_G) \) and an invariant \( 0 \neq |D\gamma| \in \Gamma(|\text{Ber}|_G) \) is fixed, we define \( \int_G f \cdot |D\gamma| \) as the integral of the absolute Berezianin \( f \cdot |D\gamma| \). If \( G \) is unimodular, then \( i^*|D\gamma| \) is \( G \times G \)-invariant again, and hence proportional to \( |D\gamma| \). Since there exist compactly supported superfunctions \( f \) on \( G \) such that \( \int_G f \cdot |D\gamma| \neq 0 \) and \( i^*f = f \), we find that \( i^*|D\gamma| = |D\gamma| \) in this case.

If \( G/H \) is an analytically unimodular \( G \)-space, and \( |D\tilde{\gamma}| \) is a non-zero and invariant absolute Berezinian, then we define, for \( f \in \Gamma_c(\mathcal{O}_G) \), \( \int_{G/H} f \cdot |D\tilde{\gamma}| \) as the integral of the absolute Berezinian \( f \cdot |D\tilde{\gamma}| \).

Moreover, by \( \text{[AH10 Corollary 5.12]} \) (which holds analogously for \( cs \) Lie supergroups and absolute Berezinians), invariant absolute Berezinians can always be normalised such that

\[
\int_G f \cdot |D\gamma| = \int_{G/H} \text{p}_{1!} (m^*f \cdot (1 \otimes |D\mathfrak{h}|)) \cdot |D\tilde{\gamma}| \quad \text{for all} \quad f \in \Gamma_c(\mathcal{O}_G)
\]

where \( m : G \times H \rightarrow G \) is multiplication, and \( \text{p}_{1!} : G \times H \rightarrow G \) the first projection.

Let \( \alpha : G \times G/H \rightarrow G/H \) denote the action \( G \) on \( G/H \) induced by left multiplication, and \( i : G \rightarrow G \) the inversion. Then the invariance of the absolute Berezinian implies

\[
\text{p}_{1!}(\alpha^*f \cdot p_2^*\mathfrak{h} \cdot (1 \otimes |D\tilde{\gamma}|)) = \text{p}_{1!}(p_2^*f \cdot (i \times \text{id})^*\alpha^*\mathfrak{h} \cdot (1 \otimes |D\tilde{\gamma}|))
\]
for all \( f, h \in \Gamma_c(\mathcal{O}_{G/H}) \). In particular, if \( H' \) is a closed \( cs \) Lie supergroup, then
\[
\int_{H' \times G/H} \alpha^* f \cdot p_2^* f' |Dh| |Dg| = \int_{H' \times G/H} p_2^* f \cdot (i \times \text{id})^* \alpha^* f' |Dh| |Dg|
\]
for all \( f, f' \in \Gamma_c(\mathcal{O}_{G/H}) \).

C.14. Let \( U = C(U_0, u) \) be a \( cs \) Lie supergroup, and let \( M = C(M_0, m) \) and \( H = C(H_0, h) \) be closed subsupergroups such that the restriction of the multiplication morphism \( m : M \times H \to U \) defines an isomorphism onto an open subspace \( V \) of \( U \).

For instance, this is clearly the case if \( u = m \oplus h \) as super-vector spaces. Similarly as for [AH10 Proposition 5.16], we have, for a suitable normalisation of the invariant absolute Berezinians,
\[
\int_U f |Du| = \int_{M \times H} m^* f \cdot \frac{pr_2^* |\Ber(|\text{Ad}_{H_0,h}|)}{pr_2^* |\Ber(|\text{Ad}_{U_0,u}|H)|} |Dm| |Dh|
\]
for all \( f \in \Gamma_c(U_0, \mathfrak{O}_U) \) such that \( \text{supp} f \subset V \). In fact, by [AH10 Lemma 5.15], the correct normalisation is given by \( |Du| = 1 \otimes \omega_1 \otimes \omega_2, |Dm| = 1 \otimes \omega_1, |Dh| = 1 \otimes \omega_2 \) in (CS) where \( \omega_1 \otimes \omega_2 \in |\Ber(|m|) \otimes |\Ber(|h^*|) = |\Ber(|u|) \).

Equation (C.12) is formally quite similar to the Lie group case. But care is to be taken here: The product and inverse of functions are to be computed in the algebra \( O_{M \times H}(M_0 \times H_0) \), and not in the pointwise sense. We give an explicit formula in the following lemma.

**Lemma C.15.** Let \( U = C(U_0, u) \) be a \( cs \) Lie supergroup and \( H = C(H_0, h) \) a closed \( cs \) subsupergroup. Then
\[
(|\Ber(|\text{Ad}_{H_0,h}|)|\Ber(|\text{Ad}_{U_0,u}|H)^{-1})(u; h) = |\Ber|_{u/h}(\text{Ad}(h^{-1})) \text{str}_{u/h} \text{ad}(S(u))
\]
for all \( u \in \mathfrak{U}(h) \), \( h \in H_0 \).

**Proof.** We compute explicitly. Define superfunctions \( f \) and \( g \) on \( H \) by
\[
f(u; h) = \text{str}_{h} \text{ad}(u) \quad \text{and} \quad g(u; h) = \text{str}_{u} \text{ad}(S(u)).
\]

For \( x \in h \), we have \( \text{str}_{u} \text{ad}(x) = \text{str}_{u/h} \text{ad}(x) + \text{str}_{h} \text{ad}(x) \), so
\[
(f \cdot g)(x; h) = \mu \circ ((\text{str}_{h} \circ \text{ad}) \otimes (\text{str}_{u} \circ \text{ad} \circ S))(\Delta(x)) = \text{str}_{u/h} \text{ad}(S(x)).
\]

Next, let \( u' = xu \) where \( u' \in \mathfrak{U}(h) \) and \( x \in h \). We write \( s_h = \text{str}_{h} \circ \text{ad} \) and \( s_u = \text{str}_{u} \circ \text{ad} \). Then, writing \( \Delta(u) = \sum_j u_j \otimes v_j \), we deduce by induction
\[
(f \cdot g)(u'; h)
\]
\[
eq \mu((s_h(x) \otimes 1 + 1 \otimes s_u(S(x))) \cdot ((s_h \circ s_u \circ S) \circ \Delta)(u))
\]
\[
= \sum_j s_h(x)s_h(u_j)s_u(S(v_j)) + (-1)^{|x||u_j|} s_h(u_j)s_u(S(x))s_u(S(v_j))
\]
\[
= s_h(x) \sum_j s_h(u_j)s_u(S(v_j)) + (-1)^{|x||u|} \sum_j s_h(u_j)s_u(S(v_j))s_u(S(x))
\]
\[
= s_h(x)s_u/S(u) + (-1)^{|x||u|} s_u/s_h(S(u))s_u(S(x))
\]
\[
= s_u/s_h(S(xu)) + (1 - (-1)^{|x||u|}) \cdot s_h(x)s_u/S(u)
\]
If \( |x| = |u| = 1 \), then \( s_h(x) = s_u/s_h(S(u)) = 0 \). Finally, if \( |x| \neq |u| \) or \( |x| = |u| = 0 \), then \((-1)^{|x||u|} = 1 \). Thus, in any case,
\[
(f \cdot g)(u'; h) = s_u/S(u) = \text{str}_{u/h} \text{ad} S(u').
\]
The $H_0$-dependent parts of the superfunctions occurring in the assertion of the lemma are easily treated, and the claim follows. □

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Mathematisches Institut, Universität zu Köln, Weyertal 86–90, 50931 Köln, Germany

E-mail address: alldrigd@math.uni-koeln.de