Exceptional orthogonal polynomials and the Darboux transformation

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Abstract
We adapt the notion of the Darboux transformation to the context of polynomial Sturm–Liouville problems. As an application, we characterize the recently described \( X_m \) Laguerre polynomials in terms of an isospectral Darboux transformation. We also show that the shape invariance of these new polynomial families is a direct consequence of the permutability property of the Darboux–Crum transformation.

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1. Introduction

The Darboux–Crum transformation [1, 2] is a well-known and powerful technique in quantum mechanics for generating new exactly solvable potentials from known ones [3–6]. A situation of particular interest arises when the Darboux transformation is used to construct new families of orthogonal polynomials from known ones. This is a natural development, since it is often the case that the bound states of an exactly solvable potential are polynomials after a change of independent variable and a rescaling of the wavefunction by a suitable non-vanishing weight function.

To set our results in context, it is necessary to recall the foundational theorem of Bochner [7] which states that if an infinite sequence of polynomials \( \{P_n(z)\}_{n=0}^{\infty} \) satisfies a second-order eigenvalue equation of the form

\[
p(x)P_n''(x) + q(x)P_n'(x) + r(x)P_n(x) = \lambda_n P_n(x), \quad n = 0, 1, 2, \ldots
\]

then \( p(x), q(x) \) and \( r(x) \) must be polynomials of degree 2, 1 and 0, respectively. In addition, if the \( \{P_n(x)\}_{n=0}^{\infty} \) sequence is an orthogonal polynomial system, then it has to be (up to an affine transformation) one of the classical orthogonal polynomial systems of Jacobi, Laguerre or Hermite [8–12].
In some recent papers [13–16], we have introduced the concept of an exceptional polynomial subspace and the closely related notion of exceptional orthogonal polynomials. Like the classical examples, these orthogonal polynomials are eigenfunctions of a second-order differential operator. However, what distinguishes our hypotheses from those made by Bochner is that the first eigenpolynomial of the sequence need not be of degree 0, even though the full set of eigenfunctions still forms a basis of the weighted $L^2$ space. We refer to these as $X_m$ polynomials, where $m \geq 1$, the degree of the first eigenpolynomial, represents the codimension of the corresponding polynomial flag. The situation we considered in [16] is that of complete orthogonal polynomial systems of codimension 1. For this $X_1$ case a full characterization of all Sturm–Liouville polynomial systems is available, thanks to the classification of codimension 1 exceptional polynomial subspaces performed in [15].

Shortly thereafter, a connection between exceptional orthogonal polynomials and the Darboux transformation was made by Quesne in [17]. The paper in question shows that the defining operators for the Laguerre and Jacobi-type $X_1$ polynomials are shape-invariant and also that they are related to the classical counterparts by an isospectral Darboux transformation. In a follow-up paper [18], Quesne writes down the first examples of $X_2$ exceptional polynomials, but in that paper it is not clear how to apply the construction to higher codimensions.

Subsequently, the question of higher codimension was successfully addressed by Odake and Sasaki; in [19–21] they constructed two distinct families of Laguerre and Jacobi-type $X_m$ orthogonal polynomials for arbitrarily large codimension $m$. In [22] they studied the properties of these polynomials and in particular proved that the corresponding potentials are shape-invariant. However, unlike Quesne’s $X_1$ and $X_2$ results there is no indication that the new potentials are related to their classical counterparts by an isospectral Darboux transformation.

In the present paper we extend Quesne’s results to higher codimensions and show that the $X_m$ families introduced by Odake et al are, indeed, related to classical orthogonal polynomials by means of an isospectral Darboux transformation. The key to our approach is the notion of an algebraic Darboux transformation, introduced and analyzed in [6, 23]. By construction, an algebraic Darboux transformation maps polynomials to polynomials, and is thus eminently suitable for constructing new families of orthogonal polynomials. Of course, care has to be exercised in order to characterize those cases in which the transformed eigenfunctions are not only polynomial but they also give rise to a well-defined Sturm–Liouville problem and are complete in the underlying weighted $L^2$ space. This approach allows us to simplify and to illuminate the formulas exhibited in [19–21].

The key to our analysis is the fact that a Darboux transformation that maps the standard flag to a polynomial flag of higher codimension only occurs at certain discrete factorization energies. Indeed, there are four families of these algebraic factorizations, only two of which give rise to isospectral transformations. This fact illuminates the origin of the two families of $X_m$ polynomials obtained by Odake and Sasaki.

Further, we adapt the Crum mechanism of iterated Darboux transformations to the polynomial context and show that the shape invariance of the $X_m$ families (already established in [22] by means of a direct calculation) follows naturally from and is better explained by the permutability property of the Darboux–Crum transformation. It is also worth noting that in our formulation of Darboux transformations the operators are written in the algebraic gauge and algebraic variable, rather than in the usual Schrödinger gauge. Of course, both approaches are equivalent, but the computations are much simpler in the algebraic gauge and some results would have otherwise been difficult to obtain.

Let us also mention that examples of Hermite-like orthogonal polynomials of higher codimension were already obtained some time ago via Darboux transformations in [24].
An instance of codimension 2 Hermite-like polynomials was also recently derived in [25]. However, in the above-mentioned papers, the codimension of the polynomial flag was not stable in the sense that the degree sequence of the constructed orthogonal polynomial family was \(0, m + 1, m + 2, \ldots\). The jump in degree is explained by the fact that the authors of [24] considered a state-adding, rather than an isospectral transformation. By contrast, the degree sequence of the polynomials created by an isospectral transformation is \(m, m + 1, m + 2, \ldots\); the codimension of the flag is stable. Let us also mention that recently exceptional orthogonal polynomials have found an application to mass-dependent potentials [26] and to quasi-exact solvability [27].

Our paper is organized as follows. In section 2, we define the notion of an \(m\)-orthogonal polynomial system arising from subspaces of higher codimension and introduce the notion of a polynomial Sturm–Liouville problem, corresponding to the case in which the eigenfunctions of the Sturm–Liouville operator are polynomials. In section 3, we carefully study the various cases in which the Darboux transformation preserves the polynomial character of the eigenfunctions; these are precisely the algebraic Darboux transformations introduced in [6]. Subsection 3.1 is devoted to the discussion of the notion of shape invariance in the context of operators with polynomial solutions. Subsection 3.2 proves a lemma on the covariance of the Darboux transformation [28] which is necessary for the proof of shape invariance in section 5. The results of section 3 are general and valid for any univariate OPS defined by a second-order differential equation. Having said this, it should be noted that presently the only known examples of an exceptional orthogonal polynomial with a stable codimension are the \(X_m\) Laguerre and Jacobi polynomials. In section 4, we apply these results to the case of the Sturm–Liouville problem defining the Laguerre polynomials and show precisely how the L1 and L2 families of codimension \(m\) Laguerre polynomials introduced in [21] fit into our general classification scheme. In section 5, we show that the exceptional operators are shape-invariant. This property is then exploited to derive raising and lowering operators for the exceptional Laguerre polynomials, whose recursive application leads to Rodrigues-type formulas.

The perspective taken in this paper is largely that of the formal calculus of differential operators. Previously, analytic aspects of the Darboux–Crum method for Sturm–Liouville systems were considered in [29]. Our approach is different in that we focus on algebraic properties and exact solutions. In a subsequent paper [30] we shall study the codimension \(m\) Laguerre polynomial system in a functional analytic setting by giving a spectral theoretic characterization of the codimension \(m\) Laguerre polynomial system in the context of Sturm–Liouville theory. A detailed analysis of the asymptotic properties of the zeros of these polynomials will be given. We shall also further develop some of the key formal properties of these polynomials that result from the factorization and shape invariance of their defining operators, namely their orthogonality properties, Rodrigues-type formulas and generating functions. Finally we mention that this entire analysis can also be carried out in the case of Jacobi polynomials.

2. Preliminaries

We will say that a differential operator
\[
T(y) = p(x) y'' + q(x) y' + r(x) y, \quad y = y(x)
\]
(2)
is exactly solvable by polynomials if it admits infinitely many polynomial solutions \(y_j\) of the eigenvalue equation:
\[
T(y_j) = \lambda_j y_j.
\]
(3)
In this setting, it will be convenient to refer to the $y_j$ solutions as eigenpolynomials and to order them by degree
\[ \deg y_j < \deg y_{j+1}, \quad j = 1, 2, \ldots. \]
Further, we say that a sequence of polynomials $\{y_j\}_{j=1}^\infty$ has codimension $m$ if
\[ \deg y_j = m + j - 1, \quad j = 1, 2, \ldots. \]  
(4)

We say that $T(y)$ is primitive, if the eigenpolynomials do not possess a common root (real or complex). Let us also note that if a second-order operator (2) admits three linearly independent eigenpolynomials, then, by Cramer’s rule, $p(x), q(x), r(x)$ are, necessarily, rational functions.

Let $I = (x_1, x_2)$ be an open interval (bounded, unbounded or semi-bounded) and let $W dx$ be a positive measure on $I$ with finite moments of all orders. We say that a sequence of real polynomials $\{y_j\}_{j=1}^\infty$ forms an orthogonal polynomial system (OPS for short) if the polynomials constitute an orthogonal basis of the Hilbert space $L^2(I, W dx)$. If (4) holds, we will say that the OPS has codimension $m$.

The following definition encapsulates the notion of a system of orthogonal polynomials defined by a second-order differential equation. Consider a boundary value problem
\[ -(Py')' + Ry = \lambda Wy \]  
(5)
\[ \lim_{x \to x_i} (Py'u - Pu'y)(x) = 0, \quad i = 1, 2, \ldots \]  
(6)
where $P(x), W(x) > 0$ on the interval $I = (x_1, x_2)$, and where $u(x)$ is a fixed polynomial solution of (5). We speak of a polynomial Sturm–Liouville problem (PSLP) if the resulting spectral problem is self-adjoint, pure-point and if all eigenfunctions are polynomial. We speak of an $m$-PSLP if the eigenpolynomials satisfy (4). If $m = 0$, then we recover the classical orthogonal polynomials, the totality of which is delineated by Bochner’s theorem. For $m > 0$, Bochner’s theorem no longer applies and we encounter a generalized class of polynomials; we name these exceptional or $X_m$ polynomials.

Given a PSLP, the operator $T(y) = W^{-1}(Py')' - W^{-1}Ry$ is exactly solvable by polynomials. Letting $p(x), q(x), r(x)$ be the rational coefficients of $T(y)$ as in (2), we have
\[ P(x) = \exp \left( \int x q/p \right), \]  
(7)
\[ W(x) = (P/p)(x), \]  
(8)
\[ R(x) = -(rW)(x). \]  
(9)
Hence, for a PSLP, $P(x), R(x), W(x)$ belong to the quasi-rational class [31], meaning that their logarithmic derivative is a rational function.

Conversely, given an operator $T(y)$ exactly solvable by polynomials and an interval $I = (x_1, x_2)$ we formulate a PLS (5) by employing (7)–(9) as definitions, and by adjoining the following assumptions:

(i) $P(x), W(x)$ are continuous and positive on $I$,
(ii) $W dx$ has finite moments, i.e. $\int_I x^n W(x) dx < \infty, n = 0, 1, 2, \ldots$,
(iii) $\lim_{x \to x_i} P(x)x^n = 0, i = 1, 2, n = 0, 1, 2 \ldots$ and
(iv) the eigenpolynomials of $T(y)$ are dense in the Hilbert space $L^2(I, W dx)$.
These definitions and assumptions (i) and (ii) imply Green’s formula:
\[
\int_{x_1}^{x_2} T(f) g W \, dx - \int_{x_1}^{x_2} T(g) f W \, dx = P(f' g - f g') \bigg|_{x_1}^{x_2}.
\] (10)

By (iii) if \( f(x), g(x) \) are polynomials, then the right-hand side is zero. If \( f \) and \( g \) are eigenpolynomials of \( T(y) \) with unequal eigenvalues, then necessarily, they are orthogonal in \( L^2(I, W \, dx) \). Finally, by (iv) the eigenpolynomials of \( T(y) \) are complete in \( L^2(I, W \, dx) \), and hence satisfy the definition of an OPS.

We now describe a construction that systematically generates polynomial Sturm–Liouville systems of arbitrarily large codimension \( m \geq 0 \).

3. The Darboux transformation

Let \( T(y) \) be a differential operator (2) with rational coefficients. We speak of a rational factorization if
\[
T - \lambda_0 = BA,
\] (11)
where \( A(y), B(y) \) are first-order operators with rational coefficients and where \( \lambda_0 \) is a constant. Let us write
\[
A(y) = b(y' - wy),
\] (12)
\[
B(y) = \hat{b}(y' - \hat{w}y),
\] (13)
where \( w(x), \hat{w}(x), b(x), \hat{b}(x) \) are all rational functions. Given a rational factorization we introduce the partner operator
\[
\hat{T} = AB + \lambda_0
\] (14)
whose explicit form is
\[
\hat{T}(y) = py'' + \hat{q}y' + \hat{r}y,
\] (15)
where
\[
\hat{q} = q + p' - 2pb'/b
\] (16)
\[
\hat{r} = -p(\hat{w}' + \hat{w}^2) - \hat{q}\hat{w} + \lambda_0.
\] (17)

We will refer to
\[
\phi(x) = \exp \left( \int x w \right)
\]
as a factorization function (quasi-rational) and to \( b(x) \) as the factorization gauge (rational). The former satisfies
\[
T(\phi) = \lambda_0 \phi.
\] (18)
Equivalently,
\[
w(x) = \phi'(x)/\phi(x)
\] (19)
is a rational solution of the following Ricatti-like equation:
\[
p(w' + w^2) + qw + r = \lambda_0.
\] (20)
For a fixed $T(y)$, a rational factorization is fully determined by a quasi-rational factorization function and a rational factorization gauge. Indeed, given $p, q, r, w, b$ relation (11) gives us

\begin{align*}
\hat{b} &= \frac{p}{b}, \\
\hat{w} &= -w - \frac{q}{p} + \frac{b'}{b}.
\end{align*}

(21)

(22)

The choice of $b(x)$ determines the gauge of the partner operator. Consider two factorization gauges $b_1(x), b_2(x)$ and let $\hat{T}_1(y), \hat{T}_2(y)$ be the corresponding partner operators. Then,

\[ \hat{T}_2 = \mu^{-1} \hat{T}_1 \mu, \]

where

\[ \mu(x) = \frac{b_1(x)}{b_2(x)}. \]

The above construction of the partner operator is symmetric with respect to the interchange of the hatted and unhatted variables. Letting $P(x)$ be as in (7), and setting

\[ \hat{\phi}(x) = \exp \left( \int \hat{w} \right) = \frac{1}{(P/b)(x)\phi(x)}, \]

we have

\[ \hat{T}(\hat{\phi}) = \lambda_0 \hat{\phi}. \]

We also have

\begin{align*}
\hat{P}/b &= \hat{P}/\hat{b}, \\
\hat{b}b &= p, \\
\hat{q}/p - \hat{b}'/\hat{b} &= q/p - b'/b.
\end{align*}

(24)

(25)

(26)

Thus, starting with $\hat{T}$ and taking $\hat{\phi}$ as the factorization function and $\hat{b}(x)$ as the factorization gauge, we recover $T(y)$.

Next, suppose that $T$ is an operator exactly solvable by polynomials with eigenpolynomials $\{y_j\}$. If $\mu(x)$ is a polynomial, then $\mu T \mu^{-1}$ is also exactly solvable by polynomials, with eigenpolynomials $\{\mu y_j\}$. Therefore, we can fix the gauge of such an operator by requiring that the eigenpolynomials are primitive (no common roots).

By construction, partner operators obey the following intertwining relations:

\[ \hat{T}A = AT, \quad B\hat{T} = TB. \]

(27)

Hence, if $T$ is exactly solvable by polynomials with eigenpolynomials $\{y_j\}$, then $\{A(y_j)\}$ are eigenpolynomials of the partner operator $\hat{T}$ with the same eigenvalues. By inspection of (12), with the appropriate choice of $b(x)$, the $A(y_j)$ are polynomials. Hence, if $T$ is exactly solvable by polynomials, then so is $\hat{T}$. Furthermore, the requirement that the eigenpolynomials of $\hat{T}$ be primitive fixes $b(x)$ up to a choice of a scalar multiple. In many cases, such as the factorization shown in (67)–(69), it will suffice to take $b(x)$ to be the denominator of $w(x)$. However, there are other cases, such as the factorization shown in (128)–(131), where $b(x)$ must be a rational function.

The duality between $T$ and $\hat{T}$ has another aspect. Let $W(x)$ be as in (8) and let $\hat{W}(x)$ be analogously defined. Hence, by equations (24) and (25),

\[ \hat{W} = \frac{P}{b^2} = \frac{pW}{b^2}. \]

(28)
Consequently, $A$ and $-B$ are formally adjoint relative to these measures:

$$
\int_{x_1}^{x_2} A(f)g \hat{W} \, dx + \int_{x_1}^{x_2} B(g)f \hat{W} \, dx = (P/b)f g \bigg|_{x_1}^{x_2}.
$$

(29)

If the above RHS vanishes for polynomial $f, g$, then $A$ and $-B$, with suitably defined domains, give rise to adjoint operators in the rigorous sense of densely defined linear operators on Hilbert spaces $L^2(I, W \, dx)$ and $L^2(I, \hat{W} \, dx)$, respectively.

Darboux transformations can be classified into three types as far as their spectral properties are concerned [5, 32]: state-deleting, state-adding or isospectral.

(i) **State-deleting transformation.** In this case the factorizing function $\phi(x)$ satisfies $\phi \in L^2(I, W \, dx)$ and the value of the spectral parameter, $\lambda_0$, is the maximum⁴ of the spectrum of $T$.

(ii) **State-adding transformation.** In this case the partner factorizing function $\hat{\phi}(x)$ satisfies $\hat{\phi} \in L^2(I, \hat{W} \, dx)$ and the spectral value $\lambda_0$ must be above the maximum of the spectrum of $T$. Equivalently, from (23) and (28) it follows that

$$
\hat{\phi} \in L^2(I, \hat{W} \, dx) \Leftrightarrow \frac{b^{1/2}}{P} \phi^{-1} \in L^2(I, W \, dx),
$$

so it is clear that the spectral properties of the transformation only depend on the choice of $\phi$, not on the choice of gauge $b(x)$.

(iii) **Isospectral transformation.** In this case $\phi \notin L^2(I, W \, dx)$, $\hat{\phi} \notin L^2(I, \hat{W} \, dx)$ and the factorizing spectral value $\lambda_0$ must be above the maximum of the spectrum of $T$.

In the context of algebraic Darboux transformations discussed in this paper, if we assume that both $T$ and $\hat{T}$ are PSLPs, the above spectral characterization can be particularized to purely algebraic conditions.

(i) A state-deleting transformation corresponds to $\phi = y_1$, the first eigenpolynomial of $T$.

(ii) A state-adding transformation corresponds to $\hat{\phi}$ (as defined by (23)) being a polynomial.

(iii) Isospectral transformations correspond to neither $\phi$ nor $\hat{\phi}$ being polynomials.

The above conditions can be explicitly verified on the factorizations performed in sections 4 and 5. For example, equations (67)–(69) show an isospectral factorization; neither of the factorizing functions is a polynomial. By contrast, equations (128)–(131) show a state-deleting/state-adding factorization; one of the factorizing functions is a polynomial, while the partner factorization function is not.

State-adding and state-deleting factorizations are dual notions, in the sense that if the factorization of $T$ is state-deleting, then the factorization of $\hat{T}$ is state-adding and vice versa.

As we already pointed out, the eigenpolynomials $\{y_j\}$ and $\{\hat{y}_j\}$ constitute orthogonal polynomial systems relative to $L^2(I, W \, dx)$ and $L^2(I, \hat{W} \, dx)$, respectively. The adjoint relation between $A$ and $B$ allows us to compare the $L^2$ norms of the two families. Indeed, by (11), (3) and (29),

$$
\int_I (A(y_j))^2 \hat{W} \, dx = - \int_I B(A(y_j))y_j W \, dx = (\lambda_0 - \lambda_j) \int_I y_j^2 W \, dx.
$$

(30)

⁴ Note that, as opposed to the usual convention in Schrödinger operators where the spectrum is bounded from below, in this paper the spectrum of all Sturm–Liouville problems is bounded from above. The eigenfunction corresponding to the maximum of the spectrum corresponds therefore to the ground state.
3.1. Shape invariance

Let $\kappa \in K$ be a parameter index set and let

$$T_\kappa(y) = p(x)y'' + q_\kappa(x)y' + r_\kappa(x)y, \quad \kappa \in K,$$

be a family of operators that are exactly solvable by polynomials. If this family is closed with respect to the state-deleting Darboux transformation, we speak of shape-invariant operators. To be more precise, let $\pi_\kappa(x) = \pi_{\kappa,1}(x)$ be the corresponding ground-state eigenpolynomial. Without loss of generality, we assume that the corresponding spectral value is zero, and let

$$T_\kappa = B_\kappa A_\kappa, \quad A_\kappa(\pi_\kappa) = 0$$

be the corresponding factorization. Shape invariance means that there exists a one-to-one map $h : K \rightarrow K$ and real constants $\lambda_\kappa$ such that

$$T_h(\kappa) = A_\kappa B_\kappa + \lambda_\kappa.$$  

Hence, there exist constants $\alpha_{\kappa,j}, \beta_{\kappa,j}$ such that

$$y_{h(\kappa),j} - 1 = \alpha_{\kappa,j} A_\kappa(y_{\kappa,j}), \quad j \geq 2,$$

$$y_{\kappa,j+1} = \beta_{\kappa,j+1} B_\kappa(y_{h(\kappa),j}), \quad j \geq 1,$$

$$\beta_{\kappa,j} \alpha_{\kappa,j} = \lambda_{\kappa,j},$$

$$\lambda_{h(\kappa),j} = \lambda_{\kappa,j+1} + \lambda_\kappa.$$

In accordance with (7), define

$$P_\kappa(x) = \exp \left( \int^x q_\kappa / p \right).$$

Let $b_\kappa(x)$ denote the shape-invariant factorization gauge, i.e.

$$A_\kappa(y) = (b_\kappa / \pi_\kappa) W(\pi_\kappa, y),$$

where

$$W(f, g) = fg' - f'g.$$  

Equation (16) implies the following necessary condition:

$$p P_\kappa / P_{h(\kappa)} = b_\kappa^2.$$  

This is a rather strong constraint, because the left-hand side is a product of quasi-rational functions, while the right-hand side is a rational squared.

3.2. Covariant factorization

Next, we introduce the notion of a covariant isospectral factorization. Let $T_\kappa(y)$ be a shape-invariant family of operators, as above. Let us assume that there exists an indexed family of isospectral factorization functions $\phi_\kappa(x)$. This assumption implies the existence of a corresponding isospectral factorization:

$$T_\kappa = B_\kappa A_\kappa + \tilde{\lambda}_\kappa,$$

$$A_\kappa(\phi_\kappa) = 0.$$  

5 In concrete examples, the parameter $\kappa$ represents one or more scalars, e.g. when treating Laguerre-type polynomials $\kappa$ is a single real number and will be replaced with the symbol $k$.

6 It is worth recalling that the operators $B_\kappa, A_\kappa$ are fixed up to a choice of scalar multiple by the requirement that the transformed flag is primitive, i.e. does not contain any common roots.
Let
\[
\hat{T}_\kappa = \tilde{A}_\kappa \tilde{B}_\kappa + \tilde{\lambda}_\kappa,
\]
be the partner operator and partner factorization function, respectively. Next, we make the following definition: the factorization with respect to \(\phi_\kappa\) is covariant if
\[
A_\kappa(\phi_\kappa) \propto \phi_{h(\kappa)}.
\] (44)
We have arrived at the main result of the present subsection: a simple but useful test for the covariance of factorization functions. First, we require one more definition. Let us say that an operator exactly solvable by polynomials is formally non-degenerate if for every \(\lambda \in \mathbb{R}\) there exists at most one linearly independent quasi-rational solution of \(T(y) = \lambda y\).

**Lemma 3.1.** Suppose that \(\phi_\kappa(x)\) is continuous with respect to \(\kappa \in K\) and formally non-degenerate for generic values of \(\kappa\). Furthermore, suppose that
\[
\tilde{\lambda}_{h(\kappa)} = \tilde{\lambda}_\kappa + \lambda_\kappa.
\] (45)
Then, the factorization with respect to \(\phi_\kappa\) is covariant.

**Proof.** By (32), (33) and (45),
\[
T_{h(\kappa)}(A_\kappa(\phi_\kappa)) = A_\kappa(T_\kappa(\phi_\kappa)) + \lambda_\kappa A_\kappa(\phi_\kappa)
\]
\[
= (\lambda_\kappa + \tilde{\lambda}_\kappa)A_\kappa(\phi_\kappa)
\]
\[
= \tilde{\lambda}_{h(\kappa)}A_\kappa(\phi_\kappa).
\] (48)
Since \(\phi_\kappa\) is quasi-rational and since \(A_\kappa\) has rational coefficients, \(A_\kappa(\phi_\kappa)\) is also quasi-rational. Hence, (44) holds for generic \(\kappa\). Therefore, it holds for all \(\kappa\). \(\square\)

4. Laguerre polynomials

The classical Laguerre operator is exactly solvable by polynomials
\[
L_k(y) := xy'' + (k + 1 - x)y', \quad k \in \mathbb{R}.
\] (49)
The associated Laguerre polynomials, \(L_{k,n}(x)\), are the corresponding eigenpolynomials
\[
L_k(L_{k,n}) = -nL_{k,n},
\] (50)
normalized by the condition
\[
L_{k,n}(x) = \frac{(-1)^n}{n!}x^n + \text{lower degree terms}.
\]
The classical Laguerre operator is shape-invariant by virtue of the following factorization:
\[
L_k = B_k A_k
\] (51)
\[
L_{k+1} = A_k B_k + 1,
\] (52)
where
\[
A_k(y) = y'
\] (53)
\[
B_k(y) = xy' + (k + 1 - x)y
\] (54)
For \(k > -1\), the resulting polynomials are orthogonal relative to the weight
\[
W_k(x) = x^k e^{-x}, \quad x \in (0, \infty),
\] (55)
and can be realized as solutions of a spectral problem [33, 34]. The corresponding $L^2$ norms are given by
\[ \int_0^\infty L_{k,n}^2 W_k \, dx = \Gamma(n + k + 1)/n! \, . \] (56)

The quasi-rational solutions of $L_k(y) = \lambda y$ are known [35, section 6.1]:
\begin{align*}
\phi_1(x) &= L_{k,m}(x), & \lambda_0 &= -m \\
\phi_2(x) &= x^{-k} L_{-k,m}(x), & \lambda_0 &= k - m \\
\phi_3(x) &= e^x L_{k,m}(-x), & \lambda_0 &= k + m \\
\phi_4(x) &= x^{-k} e^x L_{-k,m}(-x), & \lambda_0 &= m + 1,
\end{align*}
(57) (58) (59) (60)

where $m = 0, 1, 2, \ldots$. The corresponding factorizations were analyzed in [6]. Of these, $\phi_1$ with $m = 0$ corresponds to a state-deleting transformation and underlies the shape invariance of the classical Laguerre operator and the corresponding Rodrigues formula. For $m > 0$, the $\phi_1$ factorization functions yield singular operators and hence do not yield novel orthogonal polynomials. The $\phi_2, \phi_3$ family results in a state-adding transformation. The resulting orthogonal polynomials do not satisfy condition (4); such factorizations were discussed in [23]. The type 2 and 3 factorizations $\phi_2, \phi_3$ result in novel orthogonal polynomials, although for $\phi_3$ it is necessary to assume that $k > m$ (this is explained below). These families correspond, respectively, to the type L1, L2 Laguerre polynomials of [21].

Let us consider these two families of factorizations on a case-by-case basis. The derivations that follow depend in an elementary fashion on the following well-known identities of the Laguerre polynomials. We will apply them below without further comment:
\begin{align*}
L_{k,0}(x) &= 1 \\
L_{k,n}(x) &= 0, & n \leq -1, \\
nL_{k,n}(x) + (x - 2n - k + 1)L_{k,n-1}(x) + (n + k - 1)L_{k,n-2}(x) &= 0, \\
L_{k,n}^\prime(x) &= -L_{k+1,n-1}(x), \\
L_{k,n}(x) &= L_{k+1,n}(x) - L_{k+1,n-1}(x).
\end{align*}
(61) (62) (63) (64) (65)

4.1. The L1 family

Fix an integer $m \geq 1$ and a real $k > 0$. Take $\phi_3(x)$ as the factorization function and
\[ \xi_{k,m}(x) = L_{k,m}(-x) \] (66)
as the factorization gauge. Applying (12), (13), (21) and (22), the resulting factorization is
\[ L_k = B_{k,m}^1 A_{k,m}^1 + k + m + 1, \] (67)
where
\begin{align*}
A_{k,m}^1(y) &= \xi_{k,m} y' - \xi_{k+1,m} y \\
B_{k,m}^1(y) &= (xy' + (1 + k)y) / \xi_{k,m}.
\end{align*}
(68) (69)
The partner factorization function is $\tilde{\phi}(x) = x^{-1-k}$. Let us define
\[ L_{k,m}^1 = A_{k-1,m}^1 B_{k-1,m}^1 + k + m \]  
\[ L_{k,m}^1(y) = xy'' + (k + 1 - x)y' + my - 2\rho_{k-1,m}(xy' + ky), \]  
where
\[ \rho_{k,m} = \xi_{k,m}^2/\xi_{k,m} = \xi_{k+1,m-1}/\xi_{k,m}. \]  
On the basis of the above factorization, we define type I exceptional Laguerre polynomials to be
\[ L_{k,m,n}^I = -A_{k-1,m}^1(L_{k-1,n-m}) \]
\[ = \xi_{k,m}L_{k-1,n-m} + \xi_{k-1,m}L_{k,n-m}, \quad n \geq m. \]  
By construction, these polynomials satisfy
\[ L_{k,m}^1(L_{k,m,n}^I) = (m - n)L_{k,m,n}^I, \quad n \geq m. \]  
By (10) and (28) the sequence \( \{L_{k,m,n}^I\}_{n=m}^\infty \) constitutes an \( m \)-OPS relative to the weight
\[ W_{k,m}(x) = x^k e^{-x}/\xi_{k,m}^2, \quad x \in (0, \infty). \]  
By [36, section 6.73], the zeros of \( \xi_{k+1,m-1} \) are strictly negative, and hence \( W_{k,m}^I \) has finite moments of all orders. Using (30) and (56), we obtain
\[ \int_0^\infty (L_{k,m,n}^I)^2 W_{k,m}^I dx = (k + n)\Gamma(k + n - m)/(n - m)! \]  
For \( m = 0 \) and \( k > -1 \), the above definitions reduce to their classical counterparts, to wit,
\[ L_{k,0}^I = L_k \]  
\[ L_{k,0,n}^I = L_{k,n}, \]  
\[ W_{k,0}^I(x) = x^k e^{-x}. \]  

4.2. The \( L_2 \) family

Fix an integer \( m \geq 1 \) and a real \( k > m - 1 \), and take \( \phi_2(x) \) as the factorization function. Set
\[ \eta_{k,m}(x) = L_{k,m}(x); \]  
\[ \eta_{k,m}(x) = L_{k,m}(x); \]  
\[ x\eta_{k,m} \]  
\[ \text{as the factorization gauge. The resulting factorization is} \]
\[ L_k = B_{k,m}^{II} A_{k,m}^{II} + (k - m), \]  
where
\[ A_{k,m}^{II}(y) = x\eta_{k,m}y' + (k - m)\eta_{k+1,m}y \]  
\[ B_{k,m}^{II}(y) = (y' - y)/\eta_{k,m}. \]  
The partner factorization function is \( \hat{\phi}(x) = e^x \). Based on this factorization, we define
\[ L_{k,m}^{II} = A_{k+1,m}^{II} B_{k+1,m}^{II} + (k + 1 - m) \]  
\[ L_{k,m}^{II}(y) = xy'' + (k + 1 - x)y' - my + 2x\sigma_{k+1,m}(y' - y), \]
where
\[ \sigma_{k,m} = -\eta'_{k,m}/\eta_{k,m} = \eta_{k-1,m-1}/\eta_{k,m}. \]  
(87)

We now define the type II \( X_m \) Laguerre polynomials to be
\[ L_{k,m,n}^{\text{II}} = -A_{k+1,m}(L_{k+1,n-m}), \quad n \geq m \]  
(88)
\[ = x \eta_{k+1,m} L_{k+2,n-m-1} + (m - k - 1) \eta_{k+2,m} L_{k+1,n-m}. \]  
(89)

By construction, these polynomials satisfy
\[ L_{k,m,n}^{\text{II}}(L_{k,m,n}^{\text{II}}) = (m - n) L_{k,m,n}^{\text{II}}, \quad n \geq m. \]  
(90)

Thus, the sequence \( \{L_{k,m,n}^{\text{II}}\}_{n=m}^{\infty} \) constitutes an \( m \)-OPS relative to the weight
\[ W_{k,m}^{\text{II}}(x) = x^k e^{-x/\eta_{k+1,m}}, \quad x \in (0, \infty). \]  
(91)

Since \( k + 1 > m \), by [36, section 6.73], the zeroes of \( \eta_{k+1,m}(x) \) are either negative or complex. Hence, \( W_{k,m}^{\text{II}}(x) \) also has finite moments of all orders. Using (56), we also have
\[ \int_0^\infty (L_{k,m,n}^{\text{II}})^2 W_{k,m}^{\text{II}} dx = \frac{(1 + k + n - 2m)}{(n - m)!} \Gamma(2 + k + n - m). \]  
(92)

As above, for \( m = 0 \) and \( k > -1 \), the above definitions reduce to their classical counterparts, albeit the polynomials have a different normalization:
\[ L_{k,0,n}^{\text{II}} = -(k + 1 + n)L_{k,n}. \]  
(93)

The proof that the sets \( \{L_{k,m,n}^{\text{I}}\}_{n=m}^{\infty} \) and \( \{L_{k,m,n}^{\text{II}}\}_{n=m}^{\infty} \) span dense subspaces of the Hilbert spaces \( L^2([0, \infty), W_{k,m}^{\text{I}} dx) \) and \( L^2([0, \infty), W_{k,m}^{\text{II}} dx) \) will be given in a forthcoming publication [30].

5. Shape invariance of the exceptional operators

In this section we prove that above-defined \( X_m \) operators are shape-invariant. The explanation for this remarkable fact is the commutativity/permutability of iterated Darboux transformations, also known as the Darboux–Crum transformation. The relevant details are reviewed below.

Let \( T_0(y) = T(y) \) be a given operator exactly solvable by polynomials, and let \( \phi_1(x), \ldots, \phi_n(x) \) be quasi-rational factorization functions. Let \( T_1(y) = \hat{T}(y) \) be the partner operator corresponding to \( \phi_1 \). By construction, \( A_1(\phi_2) \) is a quasi-rational factorization function for \( T_1(y) \). Let \( T_2(y) = \hat{T}_1(y) \) be the corresponding partner operator. Continue in like fashion. We arrive at the following chain of factorizations:
\[ T_0 = B_1 A_1 + \lambda_1 \]  
(94)
\[ T_j = A_j B_j + \lambda_j, \quad j = 1, \ldots, n - 1, \]  
(95)
\[ = B_{j+1} A_{j+1} + \lambda_{j+1} \]  
(96)
\[ T_n = A_n B_n + \lambda_n, \]  
(97)

where
\[ (A_j \cdots A_2 A_1)(\phi_j) = 0, \quad j = 1, 2, \ldots, n. \]  
(98)

In the end, we obtain the following intertwining relations:
\[ T_0 B = B T_n, \quad \text{where} \quad B = B_1 B_2 \cdots B_n \quad (99) \]
\[ \mathcal{A} T_0 = T_n \mathcal{A}, \quad \text{where} \quad \mathcal{A} = A_n \cdots A_2 A_1. \quad (100) \]

By construction,
\[ \mathcal{A}(\phi_j) = 0, \quad j = 1, 2, \ldots, n. \quad (101) \]

Hence,
\[ \mathcal{A}(y) = b(x) W(\phi_1, \ldots, \phi_n, y) / W(\phi_1, \ldots, \phi_n), \quad (102) \]
where \( b(x) \) is the higher-order rational factorization gauge, and where \( W \) denotes the Wronskian operator. As before, \( b(x) \) is uniquely determined (up to scalar multiple) by the requirement that the eigenpolynomials of \( T_n(y) \) do not possess a common root (this is the primitivity assumption.)

The key observation is that up to sign, the above definition of \( T_0 \) is independent of the order of the factorization functions. Let us exploit this commutativity to prove that the above-defined \( X_m \) polynomials are shape-invariant. To do so, we consider a certain two-step factorization.

Let \( T_\kappa(y) \) be a family of shape-invariant operators (31), each one admitting an infinite sequence of eigenpolynomials \( y_\kappa,j \). Let \( \pi_\kappa(x) = y_\kappa,1(x) \) denote the corresponding lowest degree eigenpolynomials. Let (32) and (33) be the corresponding factorizations, where without loss of generality the factorization eigenvalue is set to zero.

Next, let \( \phi_\kappa(x) \) be a quasi-rational factorization function and let \( \hat{T}_\kappa(y) \) be the corresponding family of operators as per (43). Suppose that the corresponding factorization is covariant and isospectral as per (44). We claim that this family is also shape-invariant. Let
\[ \hat{T}_\kappa = \hat{B}_\kappa \hat{A}_\kappa, \quad \hat{A}_\kappa(\tilde{\pi}_\kappa) = 0 \quad (103) \]
be the ground-state factorization of the partner operator, where
\[ \tilde{\pi}_\kappa = \hat{A}_\kappa(\pi_\kappa) \quad (104) \]
is the new ground-state polynomial. Our claim is that
\[ \hat{T}_h(\kappa) = \hat{B}_\kappa \hat{A}_\kappa + \lambda_\kappa. \quad (105) \]

For convenience, let us set
\[ T_\kappa = \hat{A}_\kappa \hat{B}_\kappa. \quad (106) \]
The second-order intertwining relation is
\[ \mathcal{A}_\kappa T_\kappa = T_\kappa \mathcal{A}_\kappa, \quad (107) \]
where
\[ \mathcal{A}_\kappa(y) = b(x) W(\pi_\kappa, \phi_\kappa, y) / W(\pi_\kappa, \phi_\kappa), \quad (108) \]
and where \( b(x) \) is the second-order rational factorization gauge whose form is not relevant to our argument. There are two ways to factorize \( \mathcal{A}_\kappa \), the second-order intertwiner:
\[ \mathcal{A}_\kappa = \hat{A}_\kappa \tilde{A}_\kappa \quad (109) \]
\[ \mathcal{A}_\kappa = \hat{A}_{\kappa(\kappa)} \hat{A}_\kappa. \quad (110) \]
The second equation is true because by (44) we have
\[ \hat{A}_{\kappa(\kappa)}(\kappa(\phi_\kappa)) \propto \hat{A}_{\kappa(\kappa)}(\phi_{k(\kappa)}) = 0. \quad (111) \]
Hence,
\( A \kappa T \kappa = \tilde{\lambda}_h(\kappa) A \kappa T \kappa \)  
\( = \tilde{\lambda}_h(\kappa)(T_{h(\kappa)} - \lambda_\kappa)A \kappa \)  
\( = (\tilde{T}_{h(\kappa)} - \lambda_\kappa)A \kappa. \)  
\( \)  
Hence, by equation (107),  
\( \tilde{T}_\kappa A \kappa = (\tilde{T}_{h(\kappa)} - \lambda_\kappa)A \kappa. \)  
\( \)  
The ring of differential operators with rational coefficients has no zero divisors. Therefore, the desired relation (105) follows.

Next, let us illustrate the above result by explicitly showing the shape-invariant factorization for the type I exceptional Laguerre polynomials defined in the preceding section. The index set consists of real \( k > -1. \) Let us set
\( \pi_k(x) = 1, \)  
\( h(k) = k + 1, \)  
\( \lambda_k = 1, \)  
\( T_k(y) = L_k(y). \)

The classical Laguerre polynomials are shape-invariant; relations (32) and (33) hold, as per (51)–(54). Let us fix an integer \( m \geq 0 \) and set
\( \phi_k(x) = e^{\xi_k}x^{\kappa,m}, \)  
\( \tilde{\lambda}_k(y) = A_{\kappa,m}^{I}(y) \)  
\( \tilde{B}_k(y) = B_{\kappa,m}^{I}(y) \)  
\( \tilde{T}_k(y) = L_{I,k+1,m} \)  
\( \tilde{\lambda}_k = k + m. \)

These definitions realize a particular instance of the isospectral factorizations shown in (42) and (43). By inspection of (57)–(60), the operator \( L_k \) is formally non-degenerate for generic \( k. \) Equation (45) is satisfied, and hence by lemma 3.1 the isospectral factorization with respect to (120) is covariant. Therefore, the operators \( L_{I,k,m}^{I} \) are shape-invariant.

Next, we explicitly describe the ground-state factorization for \( L_{I,k,m}^{I} \) and verify the shape invariance property. To determine an explicit form for \( \tilde{\lambda}_k, \tilde{B}_k \) we make use of formula (41). Here,
\( \tilde{q}_k(x) = (k + 2 - x) - 2x\rho_{\kappa,m} \)  
\( \tilde{p}_k(x) = \exp \left( \int^{x} \tilde{q}_k/p \right) = e^{-x}x^{k+2}/\xi_{k,m}^2 \)  
\( p\tilde{p}_k/\tilde{p}_{k+1} = \xi_{k+1,m}^2/\xi_{k,m}^2. \)

In this way, we arrive at the shape-invariant factorization
\( L_{I,k,m}^{I} = \tilde{B}_{\kappa,m}^{I}A_{\kappa,m}^{I}, \)  
\( \tilde{A}_{\kappa,m}^{I}(\xi_{k,m}) = 0, \)  
\( L_{I,k+1,m}^{I} = \tilde{A}_{\kappa,m}^{I}\tilde{A}_{\kappa,m}^{I} + 1, \)  
\( \tilde{B}_{\kappa,m}^{I}(e^{x}x^{-1-k}\xi_{k-1,m}) = 0. \)
where

\[ \hat{A}^I_{k,m}(y) = \left( \xi_{k,m}/\xi_{k-1,m} \right)(y' - \rho_{k,m} y) \tag{130} \]

\[ \hat{B}^I_{k,m}(y) = \left( \xi_{k-1,m}/\xi_{k,m} \right)(xy' + (1 + k) y) - xy. \tag{131} \]

Thus, the type I polynomials obey the following lowering and raising relations:

\[ \hat{A}^I_{k,m}(L^I_{k,m,n}) = -L^I_{k+1,m,n-1}, \quad n \geq m. \tag{132} \]

\[ \hat{B}^I_{k,m}(L^I_{k+1,m,n}) = (n + 1 - m)L^I_{k,m,n+1}, \quad n \geq m. \tag{133} \]

It is useful to contrast the above compact formulas for the raising and lower operators to the analogous expressions found in [22].

In a similar fashion, we derive the following shape-invariant factorization for the type II polynomials. This time, we let

\[ \phi_k(x) = x^{-k}\eta_{k,m}, \tag{134} \]

\[ \tilde{\lambda}_k = k - m, \tag{135} \]

\[ \tilde{T}_k(y) = L^II_{k-1,m}, \tag{136} \]

\[ \tilde{\lambda}_k(y) = A^II_{k,m}, \tag{137} \]

\[ \tilde{B}_k(y) = B^II_{k,m}, \tag{138} \]

\[ \tilde{q}_k(x) = (k - x) + 2x\sigma_{k,m}, \tag{139} \]

\[ \tilde{P}_k(x) = \exp \left( \int x \tilde{q}_k/p \right) = e^{-x}x^k/\eta_{k,m}^2, \tag{140} \]

\[ p\tilde{P}_k/\tilde{P}_{k+1} = \eta_{k+1,m}/\eta_{k,m}^2. \tag{141} \]

Applying the formulas of section 3, we obtain the following shape-invariant factorization:

\[ L^II_{k,m} = \tilde{B}^II_{k,m} \tilde{A}^II_{k,m}, \quad \tilde{A}^II_{k,m}(\eta_{k+2,m}) = 0, \tag{142} \]

\[ L^II_{k+1,m} = \tilde{A}^II_{k+1,m} \tilde{B}^II_{k,m} + 1, \quad \tilde{B}^II_{k,m}(e^x x^{-1-k}\eta_{k+1,m}) = 0, \tag{143} \]

where

\[ \tilde{A}^II_{k,m}(y) = (\eta_{k+2,m}/\eta_{k+1,m})(y' + \sigma_{k+2,m} y) \tag{144} \]

\[ \tilde{B}^II_{k,m}(y) = (\eta_{k+1,m}/\eta_{k+2,m})(xy' + (1 + k - x)y) + (\eta_{k-1,m}/\eta_{k+2,m})xy. \tag{145} \]

The type II polynomials obey the same lowering and raising relations as in (132).

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