On the non-existence of local Birkhoff coordinates for the focusing NLS equation

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Abstract

We prove that there exist potentials so that near them the focusing non-linear Schrödinger equation does not admit local Birkhoff coordinates. The proof is based on the construction of a local normal form of the linearization of the equation at such potentials.

1 Introduction

It is well known that the non-linear Schrödinger (NLS) equation

\[
\begin{align*}
\dot{\varphi}_1 &= i\varphi_{1xx} - 2i\varphi_1^2\varphi_2, \\
\varphi_2 &= -i\varphi_{2xx} + 2i\varphi_1\varphi_2^2,
\end{align*}
\]

(1)
on the torus \( T \equiv \mathbb{R}/\mathbb{Z} \) is a Hamiltonian PDE on the scale of Sobolev spaces \( H^s_c = H^s_c \times H^s_c \), \( s \geq 0 \), with Poisson bracket

\[
\{ F, G \}(\varphi) := -i \int_0^1 ((\partial_{\varphi_1} F)(\partial_{\varphi_2} G) - (\partial_{\varphi_1} G)(\partial_{\varphi_2} F)) \, dx,
\]

(2)

and Hamiltonian \( \mathcal{H} : H^1_c \to \mathbb{C} \), given by

\[
\mathcal{H}(\varphi) = \int_0^1 (\varphi_{1x}\varphi_{2x} + \varphi_1^2\varphi_2^2) \, dx, \quad \varphi = (\varphi_1, \varphi_2) \in H^1_c.
\]

(3)

Here, for any \( s \geq 0 \), \( H^s_c = H^s(\mathbb{T}, \mathbb{C}) \) denotes the Sobolev space of complex valued functions on \( \mathbb{T} \) and the Poisson bracket (2) is defined for functionals \( F \) and \( G \) on \( H^s_c \), provided that the pairing given by the integral in (2) is well-defined (cf. Section 2 for more details on these matters). The NLS phase space \( H^s_c \) is a direct sum of two reals subspaces \( H^s_r = H^s_c \oplus iH^s_r \) where

\[
H^s_r = \{ \varphi \in H^s_c \mid \varphi_2 = \overline{\varphi_1} \} \quad \text{and} \quad iH^s_r = \{ \varphi \in H^s_c \mid \varphi_2 = -\overline{\varphi_1} \}.
\]

The Hamiltonian vector field, corresponding to (2) and (3),

\[
X_{\mathcal{H}}(\varphi) = i(-\partial_{\varphi_2} \mathcal{H}, \partial_{\varphi_1} \mathcal{H}) = \left( i\varphi_{1xx} - 2i\varphi_1^2\varphi_2, -i\varphi_{2xx} + 2i\varphi_1\varphi_2^2 \right)
\]

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is tangent to the real subspaces $H^r_s$ and $iH^r_s$ (cf. Section 2) and for any $s \geq 0$ the restrictions
\[ X_H|_{H^2} : H^2 \rightarrow L^2_r \quad \text{and} \quad X_H|_{iH^2} : iH^2 \rightarrow iL^2_r \]
are real analytic maps (cf. Section 2). The vector field $X_H|_{H^2}$ corresponds to the defocusing NLS (dNLS) equation
\[ iu_t = -u_{xx} + 2|u|^2u \]
whereas $X_H|_{iH^2}$ corresponds to the focusing NLS (fNLS) equation
\[ iu_t = -u_{xx} - 2|u|^2u. \]
Both equations are known to be well-posed on the Sobolev space $H^s_r$ and respectively $iH^s_r$ for any $s \geq 0$. Moreover, they are integrable PDEs: the dNLS equation can be brought into Birkhoff normal form on the entire phase space $L^2_r$ (cf. [2]) whereas for the fNLS equation, an Arnold-Liouville type theorem has been established in [3]. The aim of this paper is to study the local properties of the vector field $X_H$ in small neighborhoods of the constant potentials
\[ \varphi_c(x) = (c, -\bar{c}) \in iL^2_r \cap C^\infty_r, \quad c \in \mathbb{C} \setminus \{0\}. \]
More specifically, for a given $c \in \mathbb{C}$, $c \neq 0$, consider the re-normalized NLS Hamiltonian
\[ H^c = H - 2|c|^2H_1 \]
where
\[ H_1(\varphi) = -\int_0^1 \varphi_1(x)\varphi_2(x) \, dx. \]
One of our main results is the following instance of an infinite dimensional version of Williamson classification theorem in finite dimensions [6].

**Theorem 1.1.** Assume that $c \in \mathbb{C}$ and $|c| \notin \pi\mathbb{Z}$. Then there exists a Darboux basis $\{\alpha_k, \beta_k\}_{k \in \mathbb{Z}}$ in $iL^2_r$ such that the Hessian $d^2_{\varphi_c} H^c$, when viewed as a quadratic form represented in this basis, takes the form
\[ d^2_{\varphi_c} H^c = 4|c|^2dp_0^2 - \sum_{0 < k < |c|} 4\pi k \sqrt{|c|^2 - \pi^2 k^2} (dp_k dq_k + dp_{-k} dq_{-k}) - \sum_{|k| > |c|} 4\pi |k| \sqrt{\pi^2 k^2 - |c|^2} (dp_k^2 + dq_k^2) \] (4)
where $\{(dp_k, dq_k)\}_{k \in \mathbb{Z}}$ are the dual coordinates.

We conjecture in Section 3 that Theorem 1.1 can be generalized to a small neighborhood of the constant potential $\varphi_c$. We refer to the end of Section 3 where the precise statement of such a generalization is given. An analog of Theorem 1.1 formulated in terms of the linearization of the Hamiltonian vector field $X_H$ at the constant potential $\varphi_c$ is formulated in Section 3 (see Theorem 3.2). As a consequence of these results, we obtain
Theorem 1.2. For any given \( c \in \mathbb{C} \) with \( |c| \notin \pi \mathbb{Z} \) and \( |c| > \pi \) the focusing NLS equation does not allow gauge invariant local Birkhoff coordinates in a neighborhood of the constant potential \( \varphi_c \).

We refer to Section 4 for the precise definition of gauge invariant local Birkhoff coordinates. In more general terms, Theorem 1.2 means that there is no neighborhood of the constant potential \( \varphi_c \) with \( |c| \notin \pi \mathbb{Z} \) and \( |c| > \pi \) where one can introduce action-angle coordinates for the fNLS equation so that the action variables commute with the Hamiltonian \( H_1 \). Note that a similar result could be obtained using the Bäcklund transform and the existence of a homoclinic orbit in a neighborhood of the constant potential \( \varphi_c \) (cf. [5]). Note however, that such a neighborhood of \( \varphi_c \) is not arbitrarily small since it contains the homoclinic solution of the fNLS equation.

Finally, note that the same results hold for the potentials \( \varphi_{c,k} := (c e^{2\pi ik x}, -c e^{-2\pi ik x}) \) where \( k \in \mathbb{Z} \) and \( c \in \mathbb{C} \) with \( |c| \notin \pi \mathbb{Z} \) and \( |c| > \pi \). The only difference is that the re-normalized Hamiltonian for these potentials is of the form \( H_{c,k} = H + \alpha H_1 + \beta H_2 \) where \( \alpha, \beta \in \mathbb{C} \) are specifically chosen constants depending on the choice of \( c \) and \( k \) and \( H_2 (\varphi) = i \int_0^1 \varphi_1(x) \varphi_2(x) \, dx \).

Organization of the paper: The paper is organized as follows: In Section 2 we introduce the basic notions related to the symplectic phase geometry of the NLS equation that are needed in this paper. Theorem 1.1 and related results are proven in Section 3. In Section 4 we prove Theorem 1.2.

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2 Set-up

1) The NLS phase space. It is well-known that the non-linear Schrödinger equation is a Hamiltonian system on the phase space \( L^2 := L^2_{\mathbb{C}} \times L^2_{\mathbb{C}} \), where \( L^2_{\mathbb{C}} = L^2(T, \mathbb{C}) \) is the space of square summable complex-valued functions on the torus \( T \). For any two elements \( f, g \in L^2_{\mathbb{C}} \), the Hilbert scalar product on \( L^2_{\mathbb{C}} \) is defined as \( \langle f, g \rangle_{L^2_{\mathbb{C}}} := \int_0^1 (f_1 \overline{g_1} + f_2 \overline{g_2}) \, dx \), where \( f = (f_1, f_2) \), \( g = (g_1, g_2) \), and \( \overline{g_1} \) and \( \overline{g_2} \) denote the complex conjugates of \( g_1 \) and \( g_2 \), respectively. In addition to the scalar product we will also need the non-degenerate pairing

\[
\langle f, g \rangle_{L^2} := \int_0^1 (f_1 g_1 + f_2 g_2) \, dx.
\]
The phase space $L^2_\mathbb{C}$ is

$$\omega(f, g) := -i \int_0^1 \det \begin{pmatrix} f_1 & g_1 \\ f_2 & g_2 \end{pmatrix} \, dx$$

(Note that $\omega(f, g)$ is not the Kähler form of the Hermitian scalar product $(\cdot, \cdot)_{L^2}$ on $L^2_\mathbb{C}$.) Consider also the scale of Sobolev spaces $H^s_\mathbb{C} := H^s_\mathbb{C} \times H^s_\mathbb{C}$ where $H^s_\mathbb{C} \equiv H^s(\mathbb{T}, \mathbb{C})$ is the Sobolev space of complex-valued distributions on $\mathbb{T}$ and $s \in \mathbb{R}$. For any given $s \in \mathbb{R}$ the pairing \((9)\) induces an isomorphism $i_s : (H^s_\mathbb{C})' \to H^{-s}_\mathbb{C}$ where $(H^s_\mathbb{C})'$ denotes the space of continuous linear functionals on $H^s_\mathbb{C}$. In this way, for any given $s \in \mathbb{R}$ the symplectic structure extends to a bounded bilinear map $\omega : H^s \times H^{-s} \to \mathbb{C}$. The $L^2$-gradient $\partial_\varphi F = (\partial_{\varphi_1} F, \partial_{\varphi_2} F)$ of a $C^1$-function $F : H^s \to \mathbb{C}$ at $\varphi \in L^2_\mathbb{C}$ is defined by $\partial_\varphi F := i_s(d_\varphi F) \in H^{-s}_\mathbb{C}$ where $d_\varphi F \in (H^s_\mathbb{C})'$ is the differential of $F$ at $\varphi \in L^2_\mathbb{C}$. In particular, the Hamiltonian vector field $X_F$ corresponding to a $C^1$-smooth function $F : H^s_\mathbb{C} \to \mathbb{C}$ at $\varphi \in H^s_\mathbb{C}$ defined by the relation $\omega(\cdot, X_F(\varphi)) = d_\varphi F(\cdot)$ is then given by

$$X_F(\varphi) = i( -\partial_{\varphi_2} F, \partial_{\varphi_1} F).$$

The vector field $X_F$ is a continuous map $X_F : H^s_\mathbb{C} \to H^{-s}_\mathbb{C}$.

**Remark 2.1.** Since $X_F : H^s_\mathbb{C} \to H^{-s}_\mathbb{C}$ we see that strictly speaking $X_F$ is a weak vector field on $H^{-s}_\mathbb{C}$. However, for the sake of convenience in this paper we will call such maps vector fields on $H^s_\mathbb{C}$.

**Remark 2.2.** The Poisson bracket of two $C^1$-smooth functions $F, G : H^s_\mathbb{C} \to \mathbb{C}$ is then given by

$$\{F, G\}(\varphi) := d_{\varphi} F(X_G) = -i \int_0^1 \left( (\partial_{\varphi_1} F)(\partial_{\varphi_2} G) - (\partial_{\varphi_1} G)(\partial_{\varphi_2} F) \right) \, dx$$

provided that the pairing given by the integral in \((8)\) is well-defined.

The Hamiltonian $\mathcal{H} : H^1_\mathbb{C} \to \mathbb{C}$ of the NLS equation is

$$\mathcal{H}(\varphi) := \int_0^1 (\varphi_{1x}\varphi_{2x} + \varphi_1^2 \varphi_2^2) \, dx.$$

By \((7)\) the corresponding Hamiltonian vector field is

$$X_{\mathcal{H}}(\varphi) = i( -\partial_{\varphi_2} \mathcal{H}, \partial_{\varphi_1} \mathcal{H}) = i( \varphi_{1xx} - 2\varphi_1^2 \varphi_2, -\varphi_{2xx} + 2\varphi_1 \varphi_2^2 ).$$

Clearly, $X_{\mathcal{H}} : H^2_\mathbb{C} \to L^2_\mathbb{C}$ is an analytic map. The NLS equation is then written as

$$\begin{cases} \dot{\varphi}_1 = i\varphi_{1xx} - 2\varphi_1^2 \varphi_2, \\
\dot{\varphi}_2 = -i\varphi_{2xx} + 2i\varphi_1 \varphi_2^2. \end{cases}$$

The phase space $L^2_\mathbb{C}$ has two real subspaces

$$L^2_1 := \{ \varphi \in L^2_\mathbb{C} \mid \varphi_2 = \varphi_1 \} \quad \text{and} \quad iL^2_1 := \{ \varphi \in L^2_\mathbb{C} \mid \varphi_2 = -\varphi_1 \}.$$
so that $L^2 = L^2 \oplus_R iL^2$. For any $s \in \mathbb{R}$ one also defines in a similar way the real subspaces $H^r_c$ and $iH^r_c$ in $H^s_c$ so that $H^r_c = H^s_c \oplus_R iH^s_c$. It follows from (9) that the Hamiltonian $\mathcal{H}$ is real valued when restricted to $H^1_r$ and $iH^1_r$. Moreover, one easily sees from (11) that the Hamiltonian vector field $X_{\mathcal{H}}$ is “tangent” to the real subspaces $H^s_r$ and $iH^s_r$ in $H^s_c$ so that

$$X_{\mathcal{H}}|_{H^s_r} : H^s_r \rightarrow L^2_r \quad \text{and} \quad X_{\mathcal{H}}|_{iH^s_r} : iH^s_r \rightarrow iL^2_r.$$ 

are well-defined, and hence real analytic maps. The vector field $X_{\mathcal{H}}|_{H^s_r}$ corresponds to the defocusing NLS equation and the vector field $X_{\mathcal{H}}|_{iH^s_r}$ corresponds to the focusing NLS equation. This is consistent with the fact that the restriction of the symplectic structure $\omega$ to $L^2_r$ and $iL^2_r$ is real valued. For the sake of convenience in what follows we drop the restriction symbols in $X_{\mathcal{H}}|_{H^s_r}$ and $X_{\mathcal{H}}|_{iH^s_r}$ and simply write $X_{\mathcal{H}}$ instead.

2) Constant potentials. For any given complex number $c \in \mathbb{C}$, $c \neq 0$, consider the constant potential

$$\varphi_c(x) := (c, -\overline{c}) \in iL^2_r \cap iC^\infty_r.$$ 

It follows from (10) that

$$X_{\mathcal{H}}(\varphi_c) = 2|c|^2 (c, \overline{c}). \quad (12)$$

Since this vector does not vanish we see that $\varphi_c$ is not a critical point of the NLS Hamiltonian (9) and hence $d\varphi_c \mathcal{H} \neq 0$ in $(H^1_c)'$.

3) The re-normalized Hamiltonian. In addition to the NLS Hamiltonian (9) consider the Hamiltonian

$$\mathcal{H}_1(\varphi) := -\int_0^1 \varphi_1(x) \varphi_2(x) \, dx. \quad (13)$$

Note that this is the first Hamiltonian appearing in the NLS hierarchy – see e.g. [2]. The corresponding Hamiltonian vector field is

$$X_{\mathcal{H}_1}(\varphi) = i(\varphi_1, -\varphi_2). \quad (14)$$

For any $s \in \mathbb{R}$ we have that $X_{\mathcal{H}_1} : H^s_c \rightarrow H^s_c$ and hence $X_{\mathcal{H}_1}$ is a (regular) vector field on $H^s_c$. This vector field is tangent to the real submanifolds $H^s_r$ and $iH^s_r$ and induces the following one-parameter group of diffeomorphisms of $iH^s_r$,

$$S^t : iH^s_r \rightarrow iH^s_r, \quad (\varphi_0^1, \varphi_0^2) \mapsto (\varphi_0^1 e^{it}, \varphi_0^2 e^{-it}). \quad (15)$$

The transformations (15) preserves the vector field $X_{\mathcal{H}}$, i.e. for any $t \in \mathbb{R}$ and for any $\varphi \in iH^2_r$

$$S^t(X_{\mathcal{H}}(\varphi)) = X_{\mathcal{H}}(S^t(\varphi)). \quad (16)$$

It follows from (12) and (14) that

$$X_{\mathcal{H}}(\varphi_c) = 2|c|^2 X_{\mathcal{H}_1}(\varphi_c). \quad (17)$$

We have the following

1In what follows we will restrict our attention to the real space $iH^s_r$, $s \in \mathbb{R}$. 

5
Lemma 2.1. Let \( c \in \mathbb{C} \setminus \{0\} \). Then one has:

(i) The re-normalized Hamiltonian \( \mathcal{H}^c : iH^2 \to \mathbb{R} \),

\[
\mathcal{H}^c(\varphi) := \mathcal{H}(\varphi) - 2|c|^2\mathcal{H}_1(\varphi), \quad \varphi \in iH^2;
\]

has a critical point at \( \varphi_c \).

(ii) The curve \( \gamma_c : \mathbb{R} \to iH^1 \),

\[
\gamma_c : t \mapsto (ce^{2|c|^2t}, -\overline{c}e^{-2|c|^2t}), \quad t \in \mathbb{R},
\]

is a solution of the NLS equation (11) with initial data at \( \varphi_c \). This is a time periodic solution with period \( \pi/|c|^2 \).

(iii) The range of the curve \( \gamma_c \) consists of critical points of the Hamiltonian \( \mathcal{H}^c \).

Proof of Lemma 2.1. Item (i) follows directly from (17). Since the symmetry (15) preserves both \( X_\mathcal{H} \) and \( X_{\mathcal{H}_1} \), we conclude from (17) that for any \( t \in \mathbb{R} \),

\[
X_\mathcal{H}(S^t(\varphi)) = 2|c|^2 X_{\mathcal{H}_1}(S^t(\varphi)).
\]

This together with the fact that \( S^t(\varphi)_c \) is the integral curve of \( X_\mathcal{H}_c \) with initial data at \( \varphi_c \) we conclude that \( \gamma(t) := S_{2|c|^2t}(\varphi_c) \) is an integral curve of \( X_\mathcal{H} \) with initial data at \( \varphi_c \). This proves item (ii). Item (iii) follows from (20).

3 The linearization of \( X_\mathcal{H}_c \) at \( \varphi_c \) and its normal form

In this Section we find the spectrum and the normal form of the linearized Hamiltonian vector field \( X_\mathcal{H}_c : iH^2 \to iL^2_\mathbb{R} \) at \( \varphi_c \in iH^2_\mathbb{C} \). In view of Lemma 2.1 the constant potential \( \varphi_c \) is a singular point of the vector field \( X_\mathcal{H}_c \), i.e. \( X_\mathcal{H}_c(\varphi_c) = 0 \). Moreover, by Lemma 2.1 (iii), the range of the periodic trajectory \( \gamma_c \) consists of singular points of \( X_\mathcal{H}_c \). It follows from (10) and (14) that for any \( \varphi \in iH^2_\mathbb{C} \),

\[
X_\mathcal{H}_c(\varphi) = i \left( \begin{array}{c} \varphi_{1xx} - 2|c|^2\varphi_1 - 2|c|^2\varphi_2 \\ -\varphi_{2xx} + 2|c|^2\varphi_2 + 2|c|^2\varphi_1 \end{array} \right).
\]

Hence, the linearized vector field \( \left. (dX_\mathcal{H}_c)\right|_{\varphi=\varphi_c} : iH^2_\mathbb{C} \to iL^2_\mathbb{R} \) is given by

\[
\left. (dX_\mathcal{H}_c)\right|_{\varphi=\varphi_c} \left( \begin{array}{c} \delta\varphi_1 \\ \delta\varphi_2 \end{array} \right) = i \left( \begin{array}{c} (\delta\varphi_1)_{xx} + 2|c|^2(\delta\varphi_1) - 2c^2(\delta\varphi_2) \\ -(\delta\varphi_2)_{xx} + 2c^2(\delta\varphi_2) - 2|c|^2(\delta\varphi_1) \end{array} \right) \tag{21}
\]

where \( (\delta\varphi_1, \delta\varphi_2) \in iH^2_\mathbb{C} \). Since the symmetry (13) preserves \( X_\mathcal{H}_c \) and since for any \( t \in \mathbb{R} \),

\[
S^t(\varphi_c) = (ce^{it}, -\overline{c}e^{-it}),
\]

the map \( S^t \) conjugates the operator (21) computed at \( \varphi_c \) with the one computed at \( \varphi_{c_t} \) with \( c_t := ce^{it} \). More specifically, one has the following commutative diagram

\[
\begin{array}{ccc}
iH^2_\mathbb{C} & \xrightarrow{L} & iL^2_\mathbb{R} \\
\mathcal{L} & \downarrow & \\
iH^2_\mathbb{C} & \xrightarrow{S^t} & iL^2_\mathbb{R}
\end{array}
\]

\[
\tag{22}
\]
where for simplicity of notation we denote
\[
L_c \equiv \left. (dX_{H_c}) \right|_{\varphi = \varphi_c}.
\]

By choosing \( t = -\arg(c) \) in the diagram above we obtain

**Lemma 3.1.** The operators \( L_c \) and \( L|_c \) are conjugate.

With this in mind, in what follows we will assume without loss of generality that \( c \) is real. In this case
\[
L_c = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{\partial^2}{\partial x^2} + 2ic^2 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \quad c \in \mathbb{R}.
\]  
(23)

For any \( k \in \mathbb{Z} \) consider the vectors
\[
\xi_k := \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2\pi ikx} \quad \text{and} \quad \eta_k := \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{2\pi ikx}.
\]  
(24)

The system of vectors \( \{ (\xi_k, \eta_k) \}_{k \in \mathbb{Z}} \) give an orthonormal basis in the complex Hilbert space \( L^2_c \) so that for any \( \varphi = (\varphi_1, \varphi_2) \in L^2_c \),
\[
\varphi = \sum_{k \in \mathbb{Z}} (z_k \xi_k + w_k \eta_k),
\]
where \( z_k := (\varphi_1)_k \) and \( w_k := (\varphi_2)_k \). Denote \( \ell^2_c := \ell^2(\mathbb{Z}, \mathbb{C}) \) is the space of square summable sequences of complex numbers. In this way, \( \{ (z_k, w_k) \}_{k \in \mathbb{Z}} \in \ell^2_c \) are coordinates in \( L^2_c \). In these coordinates, the real subspace \( iL^2_r \) is characterized by the condition that \( \forall k \in \mathbb{Z}, \ w_k = -(z_{-k}) \), and the real subspace \( L^2_r \) is characterized by the condition that \( \forall k \in \mathbb{Z}, \ w_k = (z_{-k}) \).

We denote the corresponding spaces of sequences respectively by \( i\ell^2_r \) and \( \ell^2_r \). It follows from (6) that for any \( k, l \in \mathbb{Z} \) one has
\[
\omega(\xi_k, \xi_l) = \omega(\eta_k, \eta_l) = 0 \quad \text{and} \quad \omega(\xi_k, \eta_{-l}) = -i\delta_{kl}
\]
where \( \delta_{kl} \) is the Kronecker delta. In addition to the vectors in (21) consider for \( k \in \mathbb{Z} \) the vectors
\[
\xi'_k := \frac{1}{\sqrt{2}}(\xi_k - \eta_{-k}) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{2\pi kix} \\ -e^{-2\pi kix} \end{pmatrix}, \quad \eta'_k := \frac{i}{\sqrt{2}}(\xi_k + \eta_{-k}) = \frac{i}{\sqrt{2}} \begin{pmatrix} e^{2\pi kix} \\ e^{-2\pi kix} \end{pmatrix}.
\]  
(25)

Note that the system of vectors \( \{ \xi'_k, \eta'_k \}_{k \in \mathbb{Z}} \) form an orthonormal basis in the real subspace \( iL^2_r \). In addition, this is a Darboux basis in \( iL^2_r \) with respect to the restriction of the symplectic structure \( \omega \) to \( iL^2_r \), i.e. for any \( k, l \in \mathbb{Z} \) one has
\[
\omega(\xi'_k, \xi'_l) = \omega(\eta'_k, \eta'_l) = 0 \quad \text{and} \quad \omega(\xi'_k, \eta'_l) = \delta_{kl}.
\]

Moreover, for any \( \varphi \in L^2_c \),
\[
\varphi = \sum_{k \in \mathbb{Z}} (x_k \xi'_k + y_k \eta'_k),
\]
Theorem 3.1. For any $iL$ is a decomposition of $L_\pi$ has a compact resolvent. In particular, the spectrum of $L$ properties:

Since $\{x_k, y_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $Z$, the complex linear space $V$ in the basis of $L$ consists of $c_0 \in \mathbb{C}$ given by $c_0 = \sum_{k=0}^\infty w_k z_k$. We have the following condition that $\{x_k, y_k\}_{k\in\mathbb{Z}} \in \ell^2(\mathbb{Z}, \mathbb{R}) \times \ell^2(\mathbb{Z}, \mathbb{R})$. For any $k \in \mathbb{Z}$,

$$x_k = \frac{1}{\sqrt{2}} (z_k - w_k) \quad \text{and} \quad y_k = \frac{1}{i\sqrt{2}} (z_k + w_k).$$

Finally, consider the 2-(complex)dimensional subspaces in $L_\pi$,

$$V_k^C := \text{span}_C \langle \xi_k, \eta_k \rangle, \quad k \in \mathbb{Z},$$

(26)

together with the 4-(real)dimensional symplectic subspaces in $iL_\pi$,

$$W_k^R := \text{span}_R \langle \xi'_k, \eta'_k, \xi'_{-k}, \eta'_{-k} \rangle, \quad k \in \mathbb{Z}_{\geq 1}.$$

(27)

and

$$W_0^R := \text{span}_R \langle \xi'_0, \eta'_0 \rangle.$$

(28)

It follows from (25) that for any $k \in \mathbb{Z}_{\geq 1}$,

$$W_k^R \otimes \mathbb{C} = V_k^C \oplus \mathbb{C} V_k^C \quad \text{and} \quad W_0^R \otimes \mathbb{C} = V_0^C.$$

(29)

Since $\{\xi_k, \eta_k\}_{k\in\mathbb{Z}}$ is an orthonormal basis of $iL_\pi$,

$$iL_\pi = \bigoplus_{k\in\mathbb{Z}_{\geq 0}} W_k^R$$

is a decomposition of $iL_\pi$ into $L^2$-orthogonal real subspaces. We have the following

Theorem 3.1. For any $c \in \mathbb{R}$, $c \notin \pi \mathbb{Z}$, the operator $\mathcal{L}_c \equiv (dX_{\mathcal{H}})_{|_{\varphi = \varphi_c}} : iH_\pi^2 \to iL_\pi^2$ has a compact resolvent. In particular, the spectrum of $\mathcal{L}_c$ is discrete and has the following properties:

(i) The spectrum of $\mathcal{L}_c$ consists of $\lambda_0 = 0$ and

$$\lambda_k = \begin{cases} 4\pi k \sqrt{|c|^2 - \pi^2 k^2}, & 0 < \pi |k| < |c|, \\ 4\pi ik \sqrt{\pi^2 k^2 - |c|^2}, & |k| > |c|, \end{cases}$$

(30)

for any integer $k \in \mathbb{Z} \setminus \{0\}$. The eigenvalue $\lambda_0$ has algebraic multiplicity two and geometric multiplicity one; for any $k \in \mathbb{Z} \setminus \{0\}$ the eigenvalue $\lambda_k$ has algebraic multiplicity two and geometric multiplicity two.

(ii) For any $k \in \mathbb{Z}$ the complex linear space $V_k^C$ (see (20)) is an invariant space of $\mathcal{L}_c$ in $L^2_\pi$ and

$$\mathcal{L}_c|_{V_k^C} = i \begin{pmatrix} 2|c|^2 - 4\pi^2 k^2 & -2|c|^2 \\ 2|c|^2 & 4\pi^2 k^2 - 2|c|^2 \end{pmatrix}$$

(31)

in the basis of $V_k^C$ given by $\xi_k$ and $\eta_k$. If $0 < \pi |k| < |c|$ the matrix (31) has two real eigenvalues $\pm 4\pi |k| \sqrt{|c|^2 - \pi^2 k^2}$ and if $|c| < |k| \pi$ it has two purely imaginary complex eigenvalues $\pm 4\pi |k| \sqrt{\pi^2 k^2 - |c|^2}$. 


(iii) For any $k \in \mathbb{Z}_{\geq 1}$ the real symplectic space $W^R_k$ (see (27)) is an invariant space of $\mathcal{L}_c$ in $iL^2_r$. When written in the basis $\{\xi_k, \eta_k, \xi_{-k}, \eta_{-k}\}$ of the complexification of $W^R_k$ the matrix representation of $\mathcal{L}_c|_{W^R_k}$ consists of two diagonal square blocks of the form (31). The real symplectic space $W^R_0$ is an invariant space of the operator $\mathcal{L}_c$ in $iL^2_r$ and

$$\mathcal{L}_c|_{W^R_0} = 2|c|^2 \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

when written in the basis $\{\xi_0, \eta_0\}$ of the complexification of $W^R_0$. Zero is a double eigenvalue of this matrix with geometric multiplicity one.

Proof of Theorem 3.1 The proof of this Theorem follows directly from the matrix representation of $\mathcal{L}_c$ when computed in the basis of $V^C_k$ given by the vectors $\xi_k$ and $\eta_k$. \hfill \Box

Theorem 3.1 implies that the linearized vector field $(dX_H)|_{\varphi = \varphi_c}$ has the following normal form.

**Theorem 3.2.** Assume that $c \in \mathbb{R}$ and $c \notin \pi \mathbb{Z}$. We have:

(i) The vectors $\alpha_0 := \xi'_0$ and $\beta_0 := \eta'_0$ form a Darboux basis of $W^R_0 \subseteq iL^2_r$ such that

$$(32) \quad \mathcal{L}_c|_{W^R_0} = 4|c|^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$ 

(ii) For any $k \in \mathbb{Z}_{\geq 1}$ there exists a Darboux basis $\{\alpha_k, \beta_k, \alpha_{-k}, \beta_{-k}\}$ in $W^R_k \subseteq iL^2_r$ such that for $0 < \pi k < |c|$,

$$(33) \quad \mathcal{L}_c|_{W^R_k} = 4\pi k \sqrt{|c|^2 - \pi^2 k^2} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix},$$

and for $\pi k > |c|$,

$$(34) \quad \mathcal{L}_c|_{W^R_k} = 4\pi k \sqrt{\pi^2 k^2 - |c|^2} \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$ 

In addition, one has the following uniform in $k \in \mathbb{Z}$ with $\pi k > |c|$ estimates

$$\begin{cases} 
    \alpha_k = \xi'_k + O(1/k^2), & \beta_k = \eta'_k + O(1/k^2), \\
    \alpha_{-k} = \xi'_{-k} + O(1/k^2), & \beta_{-k} = \eta'_{-k} + O(1/k^2),
\end{cases} \quad (35)$$

where $\{\xi'_k, \eta'_k\}_{k \in \mathbb{Z}}$ is the orthonormal Darboux basis (25) in $iL^2_r$.

Recall that $\mathfrak{h}^s(\mathbb{Z}, \mathbb{R})$ with $s \in \mathbb{R}$ denotes the Hilbert space of sequences of real numbers $(a_k)_{k \in \mathbb{Z}}$ so that $\sum_{k \in \mathbb{Z}} |\langle k \rangle^s a_k|^2 < \infty$ where $\langle k \rangle := \sqrt{1 + |k|^2}$. We will also need the Banach space $l^1_s(\mathbb{Z}, \mathbb{R})$ of sequences of real numbers $(a_k)_{k \in \mathbb{Z}}$ so that $\sum_{k \in \mathbb{Z}} |\langle k \rangle^s a_k| < \infty$.

---

2Here $\omega(\alpha_k, \beta_k) = \omega(\alpha_{-k}, \beta_{-k}) = 1$ while all other skew-symmetric products between these vectors vanish.
Remark 3.1. Note that the asymptotics \[35\] imply (see e.g. \[4\], Section 22.5) that for any \(s \in \mathbb{R}\) the system \(\{\alpha_k, \beta_k\}_{k \in \mathbb{Z}}\) is a basis in \(iH^s_r\) in the sense that for any \(\varphi \in iH^s_r\) there exists a unique sequence \(\{(p_k, q_k)\}_{k \in \mathbb{Z}}\) in \(h^s(Z, \mathbb{R}) \times h^s(Z, \mathbb{R})\) such that \(\varphi = \sum_{k \in \mathbb{Z}} (p_k \alpha_k + q_k \beta_k)\) where the series converges in \(iH^s_r\) and the mapping

\[iH^s_r \to h^s(Z, \mathbb{R}) \times h^s(Z, \mathbb{R}), \quad \varphi \mapsto \{(p_k, q_k)\}_{k \in \mathbb{Z}}\]

is an isomorphism. Note that \(\{\alpha_k, \beta_k\}_{k \in \mathbb{Z}}\) is a Darboux basis that is not an orthonormal basis in \(iL^2_r\).

Proof of Theorem 3.3. Item (i) follows by a direct computation in the basis of \(W^0_0\) provided by the vectors \(\alpha_0 := \xi'_0\) and \(\beta_0 := \eta'_0\) (see \[25\]). Towards proving item (ii), we first consider the case when \(\pi k > |c|\). Denote for simplicity

\[L_{\pm k} := L_c|_{V^C_{\pm k}}, \quad a_k := 4\pi^2 k^2 - 2|c|^2, \quad b := 2|c|^2,\]

and note that \(a_k^2 - b^2 = 16\pi^2 k^2 (\pi^2 k^2 - |c|^2) > 0\). Then, in view of \[31\],

\[L_k = i \begin{pmatrix} -a_k & -b \\ b & a_k \end{pmatrix} \quad \text{and} \quad L_{-k} = L_k\]

in the basis of \(V^C_m\) given by \(\{\xi_m, \eta_m\}\) for \(m = \pm k\). Denote, by \(x_k\) the positive square root of the quantity

\[x_k^2 = \sqrt{a_k^2 - b^2} (a_k + \sqrt{a_k^2 - b^2}).\]

It follows from Theorem 3.1 (ii) and \[31\] that

\[F_k := -\frac{1}{x_k} \left( a_k + \frac{-b}{\sqrt{a_k^2 - b^2}} \right) e^{-2\pi ikx} \quad (36)\]

and

\[F_{-k} := -\frac{1}{x_k} \left( a_k + \frac{-b}{\sqrt{a_k^2 - b^2}} \right) e^{2\pi ikx} \quad (37)\]

are linearly independent eigenfunctions of the restriction of \(L_c\) to the invariant space \(V^C_k \oplus V^C_{-k} = W^0_k \otimes \mathbb{C}\) with eigenvalue

\[\lambda_k = i \sqrt{a_k^2 - b^2} = 4\pi ik \sqrt{\pi^2 k^2 - |c|^2}.\]

The eigenfunctions have been normalized in a way convenient for our purposes. Denote by \(\sigma\) the complex conjugation in \(L^2_c\) corresponding to the real subspace \(iL^2_r\),

\[\sigma : L^2_c \to L^2_c, \quad (\varphi_1, \varphi_2) \mapsto (-\overline{\varphi_2}, -\overline{\varphi_1}). \quad (38)\]

Remark 3.2. One easily sees from \[38\] that \(\sigma|_{iL^2_r} = \text{id}_{iL^2_r}\) and \(\sigma|_{L^2_r} = -\text{id}_{L^2_r}\). This implies that for any \(\alpha, \beta \in iL^2_r\),

\[\sigma(\alpha + i\beta) = \alpha - i\beta. \quad (39)\]
Since the operator \( \mathcal{L}_c \) is real (i.e., \( \mathcal{L}_c : iH^2 \to iL^2 \)) and complex-linear, we conclude from (39) that if \( f \in H^2 \) is an eigenfunction of \( \mathcal{L}_c \) with eigenvalue \( \lambda \in \mathbb{C} \) then \( \sigma(f) \) is an eigenfunction of \( \mathcal{L}_c \) with eigenvalue \( \overline{\lambda} \). Moreover, one easily checks that for any \( f, g \in H^2 \),

\[
\omega(\mathcal{L}_c f, g) = -\omega(f, \mathcal{L}_c g),
\]

which is consistent with the fact that \( X_{\mathcal{L}_c} \) is a Hamiltonian vector field. Equation (40) implies that if \( f, g \in H^2 \) are eigenvectors of \( \mathcal{L}_c \) with the same eigenvalue \( \lambda \in \mathbb{C} \setminus \{0\} \) then they are isotropic, i.e., \( \omega(f, g) = 0 \).

Since \( L^2_c = iL^2 _c \oplus \mathbb{R} L^2 \) we have that

\[
F_k = \alpha_k + i\beta_k \quad \text{and} \quad F_{-k} = \alpha_{-k} + i\beta_{-k}
\]

where by (39)

\[
\alpha_{\pm k} := \frac{F_{\pm k} + \sigma(F_{\pm k})}{2} \quad \text{and} \quad \beta_{\pm k} := \frac{F_{\pm k} - \sigma(F_{\pm k})}{2i}
\]

are elements in \( iL^2_c \). Moreover, in view of Remark 3.2,

\[
G_k := \sigma(F_k) = \frac{1}{\sqrt{2}} \left( a_k + \sqrt{a_k^2 - b^2} \right) e^{2\pi ikx}
\]

and

\[
G_{-k} := \sigma(F_{-k}) = \frac{1}{\sqrt{2}} \left( a_k + \sqrt{a_k^2 - b^2} \right) e^{-2\pi ikx}
\]

are linearly independent eigenfunctions of the restriction of \( \mathcal{L}_c \) to the invariant space \( V^C_k \oplus V^C_{-k} = W^R_k \oplus \mathbb{C} \) with eigenvalue

\[
-\lambda_k = -i \sqrt{a_k^2 - b^2} = -4\pi ik \sqrt{\pi^2 k^2 - |c|^2}.
\]

It follows from (36), (37), (42), and (43) that

\[
\omega(F_k, G_k) = \omega(F_{-k}, G_{-k}) = -2i \quad \text{and} \quad \omega(F_k, G_{-k}) = \omega(F_{-k}, G_k) = 0
\]

while \( \omega(F_k, F_{-k}) = \omega(G_k, G_{-k}) = 0 \) in view of Remark 3.2. Hence,

\[
\omega(\alpha_k, \beta_k) = \frac{1}{4i} \omega(F_k + \sigma(F_k), F_k - \sigma(F_k)) = 1
\]

and similarly \( \omega(\alpha_{-k}, \beta_{-k}) = 1 \) whereas all other values of \( \omega \) evaluated at pairs of vectors from the set \( \{\alpha_k, \beta_k, \alpha_{-k}, \beta_{-k}\} \) vanish. This shows that the vectors \( \{\alpha_k, \beta_k, \alpha_{-k}, \beta_{-k}\} \) form a Darboux basis. The matrix representation (34) of \( \mathcal{L}_c \) in this basis then follows from (41) and the fact that \( F_k \) and \( F_{-k} \) are eigenfunctions of \( \mathcal{L}_c \) with eigenvalue \( i \sqrt{a_k^2 - b^2} \),

\[
\mathcal{L}_c(\alpha_k + i\beta_k) = i \sqrt{a_k^2 - b^2} (\alpha_k + i\beta_k) \quad \text{and} \quad \mathcal{L}_c(\alpha_{-k} + i\beta_{-k}) = i \sqrt{a_k^2 - b^2} (\alpha_{-k} + i\beta_{-k}).
\]

The asymptotic relations in (35) follow from the explicit formulas for \( F_{\pm k} \) and \( G_{\pm k} \) above together with \( \alpha_m = (F_m + G_m)/2 \), \( \beta_m = (F_m - G_m)/2i \) with \( m = \pm k \), and \( a_k = 4\pi^2 k^2 - |c|^2 \).
The case when $0 < \pi |k| < |c|$ is treated in a similar way. In fact, take $k \in \mathbb{Z}$ with $0 < \pi k < |c|$ and denote by $\kappa_k$ the branch of the square root of

$$\kappa_k^2 = \sqrt[4]{b^2 - a_k^2} \left( a_k - i \sqrt{b^2 - a_k^2} \right)$$

that lies in the fourth quadrant of the complex plane $\mathbb{C}$. It follows from Theorem 3.1 $(ii)$ and $(31)$ that

$$F_k := \frac{1}{\kappa_k} \left( a_k - i \sqrt{b^2 - a_k^2} \right) e^{2\pi i k x}$$

and

$$F_{-k} := \sigma (F_k) = - \frac{1}{\kappa_k} \left( a_k + i \sqrt{b^2 - a_k^2} \right) e^{-2\pi i k x}$$

are linearly independent eigenfunctions of the restriction of $L_c$ to the invariant space $V^C_k \oplus \mathbb{C} V^C_{-k} = W^R_k \otimes \mathbb{C}$ with eigenvalue

$$\lambda_k = \sqrt[4]{b^2 - a_k^2} = 4\pi k \sqrt{|c|^2 - \pi^2 k^2}.$$ 

By arguing in the same way as above, one sees that

$$G_k := - \frac{1}{\kappa_k} \left( a_k + i \sqrt{b^2 - a_k^2} \right) e^{2\pi i k x}$$

and

$$G_{-k} := \sigma (G_k) = \frac{1}{\kappa_k} \left( a_k - i \sqrt{b^2 - a_k^2} \right) e^{-2\pi i k x}$$

are linearly independent eigenfunctions of the restriction of $L_c$ to the invariant space $V^C_k \oplus \mathbb{C} V^C_{-k} = W^R_k \otimes \mathbb{C}$ with eigenvalue

$$-\lambda_k = - \sqrt[4]{b^2 - a_k^2} = -4\pi k \sqrt{|c|^2 - \pi^2 k^2}.$$ 

Since $L^2_c = i L^2_r \oplus_R L^2_r$ we have that

$$F_k = \alpha_k + i \alpha_{-k} \quad \text{and} \quad G_k = \beta_k + i \beta_{-k},$$

where

$$\alpha_{\pm k} := \frac{F_{\pm k} + \sigma (F_{\pm k})}{2} \quad \text{and} \quad \beta_{\pm k} := \frac{F_{\pm k} - \sigma (F_{\pm k})}{2i}$$

are elements in $i L^2_r$. By $(39)$ this implies that

$$F_{-k} = \alpha_k - i \alpha_{-k} \quad \text{and} \quad G_{-k} = \beta_k - i \beta_{-k}.$$ 

It follows from $(39)$, $(45)$, $(46)$, $(47)$, and $(48)$ that

$$\omega \left( F_k, G_k \right) = \omega \left( F_{-k}, G_{-k} \right) = 0 \quad \text{and} \quad \omega \left( F_k, G_{-k} \right) = \omega \left( F_{-k}, G_k \right) = 2$$ (51)
while \( \omega(F_k, F_{-k}) = \omega(G_k, G_{-k}) = 0 \) in view of Remark 3.2. This together with (49) and (50) implies that the vectors \( \{ \alpha_k, \beta_k, \alpha_{-k}, \beta_{-k} \} \) form a Darboux basis in \( W_k^R \). The matrix representation (53) of \( L_c \) in this basis then follows from (49), (50), and the fact that \( F_k \) and \( G_k \) given by (45) and (47) are eigenfunctions of \( L \) with (real) eigenvalues \( \pm \sqrt{b^2 - a_k^2} \).

\[
L_c(\alpha_k + i\alpha_{-k}) = \sqrt{b^2 - a_k^2}(\alpha_k + i\alpha_{-k}) \quad \text{and} \quad L_c(\beta_k + i\beta_{-k}) = -\sqrt{b^2 - a_k^2}(\beta_k + i\beta_{-k}).
\]

This completes the proof of Theorem 3.2. □

**Remark 3.3.** In fact, the canonical form (32), (33), and (34), of the restriction of the operator \( L_c \) to the invariant symplectic space \( W_k^R \) with \( k \geq 0 \) can be deduced from the description of the spectrum of \( L_c \) obtained in Theorem 3.2, (i), (ii), and the Williamson classification of linear Hamiltonian systems in \( \mathbb{R}^n \) (see [1, 8]). Instead of doing this, we choose to construct the normalizing Darboux basis directly. The reason is twofold: first, in this way we obtain explicit formulas for the normalizing basis, and second, we need the asymptotic relations (35) to conclude that the system of vectors \( \{ \alpha_0, \beta_0 \} \) together with \( \{ \alpha_{k}, \beta_{k}, \alpha_{-k}, \beta_{-k} \} \) \( k \geq 1 \), form a Darboux basis in \( \ell^2_k \) in the sense described in Remark 3.1.

In this way, as a consequence of Theorem 3.2 we obtain the following instance of an infinite dimensional version of the Williamson classification of linear Hamiltonian systems in \( \mathbb{R}^n \) ([6]).

**Theorem 3.3.** Assume that \( c \in \mathbb{R} \) and \( c \notin \pi \mathbb{Z} \). Then the Hessian \( \frac{\partial^2}{\partial \varphi^2} \mathcal{H}^c \), when viewed as a quadratic form represented in the Darboux basis \( \{ \alpha_k, \beta_k \} \) \( k \in \mathbb{Z} \), takes the form

\[
\frac{\partial^2}{\partial \varphi^2} \mathcal{H}^c = 4|c|^2 dp_0^2 - \sum_{0 < |k| < |c|} 4\pi k \sqrt{|c|^2 - \pi^2 k^2} (dp_k dq_k + dp_{-k} dq_{-k})
\]

\[
- \sum_{|k| > |c|} 4\pi |k| \sqrt{\pi^2 k^2 - |c|^2} (dp_k^2 + dq_k^2)
\]

(52)

where \( \{ (dp_k, dq_k) \} \) \( k \in \mathbb{Z} \) are the dual coordinates in this basis.

For any \( 0 < \pi k < |c| \) denote

\[
I_k := p_k q_k + p_{-k} q_{-k} \quad \text{and} \quad I_{-k} := p_k q_{-k} - p_{-k} q_k,
\]

(53)

and for \( \pi |k| > |c| \),

\[
I_k := (p_k^2 + q_k^2)/2
\]

whereas for \( k = 0 \)

\[
I_0 := p_0^2/2.
\]

Note that the functions in (53) are the commuting integrals characterizing the focus-focus singularity in the symplectic space \( \mathbb{R}^4 \) – see e.g. [8]. We conjecture that the following holds: There exists an open neighborhood \( U \) of \( \varphi \) in \( iL^2_k \), an open neighborhood \( V \) of zero in \( \ell^2_k(\mathbb{Z}, \mathbb{R}) \times \ell^2_k(\mathbb{Z}, \mathbb{R}) \), and a canonical real analytic diffeomorphism \( \Phi : U \rightarrow V \) such that for any \( s \geq 0 \),

\[
\Phi : U \cap iH^s_k \rightarrow V \cap (h^s(\mathbb{Z}, \mathbb{R}) \times h^s(\mathbb{Z}, \mathbb{R})), \quad \varphi \mapsto \{ (p_k, q_k) \} \kern1pt_{k \in \mathbb{Z}}.
\]
and for any \((p, q) \in V \cap (\mathfrak{h}^1(\mathbb{Z}, \mathbb{R}) \times \mathfrak{h}^1(\mathbb{Z}, \mathbb{R}))\),

\[ H^c \circ \Phi^{-1}(p, q) = H^c(\{I_k\}_{k \in \mathbb{Z}}), \]

where \(H^c : \ell^2_2(\mathbb{Z}, \mathbb{R}) \to \mathbb{R}\) is a real analytic map. We will discuss this conjecture in future work.

**4 Non-existence of local Birkhoff coordinates**

First, we will discuss the notion of local Birkhoff coordinates. Let \(\mathfrak{h}^s \equiv \mathfrak{h}^s(\mathbb{Z}, \mathbb{R})\) and \(\ell^2 \equiv \ell^2(\mathbb{Z}, \mathbb{R})\).

**Definition 1.** We say that the focusing NLS equation has local Birkhoff coordinates in a neighborhood of \(\varphi^s \in iH^2_s\) if there exist an open connected neighborhood \(U\) of \(\varphi^s\) in \(iL^2_s\), an open neighborhood \(V\) of \((p^s, q^s) \in \mathfrak{h}^2 \times \mathfrak{h}^2\) in \(\ell^2 \times \ell^2\), and a canonical \(C^2\)-diffeomorphism \(\Phi : U \to V\) such that for any \(0 \leq s \leq 2\),

\[ \Phi : U \cap iH^s_r \to V \cap (\mathfrak{h}^s \times \mathfrak{h}^s), \quad \varphi \mapsto \{(p_k, q_k)\}_{k \in \mathbb{Z}}, \]

is a \(C^2\)-diffeomorphism and for any \(k \in \mathbb{Z}\) the Poisson bracket \(\{I_k, H\}\), where \(H := H \circ \Phi^{-1}\) and \(I_k := (p^2_k + q^2_k)/2\), vanishes on \(V \cap (\mathfrak{h}^s \times \mathfrak{h}^s)\).

The map \(\Phi : U \to V\) being canonical means that

\[ (\Phi^{-1})^*\omega = \sum_{k \in \mathbb{Z}} dp_k \wedge dq_k \]  \hspace{1cm} (54)

where \(\Phi^{-1} : V \to U\) is the inverse of \(\Phi : U \to V\) and \(\omega\) is symplectic form \(6^6\) on \(iL^2_s\). Assume that the focusing NLS equation has local Birkhoff coordinates in a neighborhood of \(\varphi^s \in iH^2_s\). Then, for any \(k \in \mathbb{Z}\) and \(0 \leq s \leq 2\) consider the action variable

\[ I_k : U \cap iH^s_r \to \mathbb{R}, \quad I_k := I_k \circ \Phi. \]

Recall that \(\{S^t\}_{t \in \mathbb{R}}\) denotes the Hamiltonian flow,

\[ S^t : iH^s_r \to iH^s_r, \quad (\varphi_1, \varphi_2) \mapsto (\varphi_1 e^{it}, \varphi_2 e^{-it}), \]

generated by the Hamiltonian \(\mathcal{H}_1(\varphi) = -\int_0^1 \varphi_1(x)\varphi_2(x) \, dx\) (see \(13^c\)) from the standard NLS hierarchy (see e.g. \(2^4\)).

**Definition 2.** The local Birkhoff coordinates are called gauge invariant if for any \(k \in \mathbb{Z}\), \(0 \leq s \leq 2\), and for any \(\varphi \in U \cap iH^s_r\) and \(t \in \mathbb{R}\) such that \(S^t(\varphi) \in U \cap iH^s_r\) one has \(I_k(S^t(\varphi)) = I_k(\varphi)\).

**Remark 4.1.** The gauge invariance of local Birkhoff coordinates means that the Hamiltonian \(\mathcal{H}_1\) belongs to the Poisson algebra \(\mathcal{A}_I := \{F \in C^1(U, \mathbb{C}) \mid \{F, I_k\} = 0 \ \forall k \in \mathbb{Z}\}\) generated by the local action variables \(\{I_k\}_{k \in \mathbb{Z}}\). Note that, for example, \(\mathcal{H}_1\) belongs to the Poisson algebra generated by the functionals \(\{\Delta_\lambda\}_{\lambda \in \mathbb{C}}\) where \(\Delta_\lambda : iL^2_s \to \mathbb{C}\) is the discriminant \(\Delta_\lambda(\varphi) \equiv \Delta(\lambda, \varphi) := \text{tr} M(x, \lambda, \varphi)|_{x=1}\) and \(M(x, \lambda, \varphi)\) is the fundamental \(2 \times 2\)-matrix solution of the Zakharov-Shabat system (see e.g. \(3^3\)).
The main result of this Section is Theorem 1.2 stated in the Introduction which we recall for the convenience of the reader.

**Theorem 4.1.** For any given \( c \in \mathbb{C} \) with \( |c| \notin \pi \mathbb{Z} \) and \( |c| > \pi \) the focusing NLS equation does not admit gauge invariant local Birkhoff coordinates in a neighborhood of the constant potential \( \varphi_c \in iC^\infty_r \).

Consider the commutative diagram

\[
\begin{array}{ccc}
U \cap iH^2_r & \xrightarrow{X_{H^c}} & iL^2_r \\
\Phi \downarrow & & \downarrow \Phi_* \\
V \cap (\mathfrak{h}^2 \times \mathfrak{h}^2) & \xrightarrow{\tilde{X}_{H^c}} & \ell^2 \times \ell^2
\end{array}
\]

(55)

where \( X_{H^c} \) is the Hamiltonian vector field of the re-normalized Hamiltonian \( H^c \), \( \Phi_* \equiv (\partial \Phi)|_{\varphi = \varphi_c} \), and \( \tilde{X}_{H^c} \) is defined by the diagram. By linearizing the maps in this diagram at \( \varphi_c \) we obtain

\[
\begin{array}{ccc}
iH^2_r & \xrightarrow{\mathcal{L}_c} & iL^2_r \\
\Phi \downarrow & & \downarrow \Phi_* \\
i\mathfrak{h}^2 \times i\mathfrak{h}^2 & \xrightarrow{\tilde{\mathcal{L}}_c} & \ell^2 \times \ell^2
\end{array}
\]

(56)

where \( \mathcal{L}_c \) is the linearization of \( X_{H^c} \) at the critical point \( \varphi_c \) and \( \tilde{\mathcal{L}}_c \) is the linearization of \( \tilde{X}_{H^c} \) at the critical point \( (p^*, q^*) = \Phi(\varphi_c) \). In particular, we see that the (unbounded) linear operator \( \mathcal{L}_c \) on \( iL^2_r \) with domain \( iH^2_r \) is conjugated to the operator \( \tilde{\mathcal{L}}_c \) on \( \ell^2 \times \ell^2 \) with domain \( \mathfrak{h}^2 \times \mathfrak{h}^2 \). We have

**Lemma 4.1.** Assume that for a given \( c \in \mathbb{C} \) the focusing NLS equation has gauge invariant local Birkhoff coordinates in a neighborhood of the constant potential \( \varphi_c \). Then the spectrum of the operator \( \tilde{\mathcal{L}}_c \) is discrete and lies on the imaginary axis.

**Proof of Lemma 4.1.** Let \( \{(p_k, q_k)\}_{k \in \mathbb{Z}} \) be the local Birkhoff coordinates on \( V \cap (\ell^2 \times \ell^2) \) and let \( H^c := H^c \circ \Phi^{-1} : V \cap (\mathfrak{h}^2 \times \mathfrak{h}^2) \to \mathbb{R} \) be the Hamiltonian \( H^c \) in these coordinates. One easily concludes from (43) and (55) that in the open neighborhood \( V \cap (\mathfrak{h}^2 \times \mathfrak{h}^2) \) of the critical point \( z^\bullet := (p^*, q^*) \) one has

\[
\tilde{X}_{H^c} = X_{H^c} = \sum_{n \in \mathbb{Z}} \left( \frac{\partial H^c}{\partial p_n} \partial q_n - \frac{\partial H^c}{\partial q_n} \partial p_n \right).
\]

(57)

Since by Lemma 2.1 and (55), \( z^\bullet \) is a critical point of \( \tilde{X}_{H^c} \),

\[
\frac{\partial H^c}{\partial p_n} \bigg|_{z^\bullet} = \frac{\partial H^c}{\partial q_n} \bigg|_{z^\bullet} = 0
\]

(57)

for any \( n \in \mathbb{Z} \). In addition, we obtain that the operator \( \tilde{\mathcal{L}}_c : \mathfrak{h}^2 \times \mathfrak{h}^2 \to \ell^2 \times \ell^2 \) takes the form

\[
\tilde{\mathcal{L}}_c = d_{z^\bullet} \tilde{X}_{H^c}
\]

(58)

\[
= \sum_{n \in \mathbb{Z}} \partial q_n \otimes \sum_{l \in \mathbb{Z}} \left( \frac{\partial^2 H^c}{\partial p_n \partial p_l} dp_l + \frac{\partial^2 H^c}{\partial p_n \partial q_l} dq_l \right) \bigg|_{z^\bullet} \\
- \sum_{n \in \mathbb{Z}} \partial p_n \otimes \sum_{l \in \mathbb{Z}} \left( \frac{\partial^2 H^c}{\partial q_n \partial p_l} dp_l + \frac{\partial^2 H^c}{\partial q_n \partial q_l} dq_l \right) \bigg|_{z^\bullet}.
\]
Note that for any \( l \in \mathbb{Z} \),
\[
X_{t_l} = p_l \partial_{q_l} - q_l \partial_{p_l}.
\]
Since the local Birkhoff coordinates are assumed gauge invariant and since \( dH(X_{t_k}) = \{ H, I_k \} = 0 \) for any \( k \in \mathbb{Z} \), we obtain that for any \( l \in \mathbb{Z} \),
\[
0 = (dH^c)(X_{t_l}) = p_l \frac{\partial H^c}{\partial q_l} - q_l \frac{\partial H^c}{\partial p_l} \tag{59}
\]
in the open neighborhood \( V \cap (\mathbb{h}^2 \times \mathbb{h}^2) \) of \( z^* \). By taking the partial derivatives \( \partial_{p_n} \) and \( \partial_{q_n} \) of the equality above at \( z^* \) for \( n \in \mathbb{Z} \) we obtain, in view of (57), that for any \( n, l \in \mathbb{Z} \),
\[
\left( \frac{\partial^2 H^c}{\partial p_n \partial q_l} p_l - \frac{\partial^2 H^c}{\partial p_n \partial q_l} q_l \right)_{z^*} = 0 \quad \text{and} \quad \left( \frac{\partial^2 H^c}{\partial q_n \partial q_l} p_l - \frac{\partial^2 H^c}{\partial q_n \partial q_l} q_l \right)_{z^*} = 0. \tag{60}
\]
We split the set of indices \( \mathbb{Z} \) in the sum above into two subsets
\[
A := \{ l \in \mathbb{Z} \mid (p^*_l, q^*_l) \neq (0, 0) \} \quad \text{and} \quad B := \{ l \in \mathbb{Z} \mid (p^*_l, q^*_l) = (0, 0) \}.
\]
Note that for \( l \in B \) the relations (60) are trivial. More generally, by taking the partial derivatives \( \partial_{p_n} \) and \( \partial_{q_n} \) of (59) in \( V \cap (\mathbb{h}^2 \times \mathbb{h}^2) \) for \( n \neq l \) we see that for any \( l \in \mathbb{Z} \) and for any \( n \neq l \) we have
\[
\frac{\partial^2 H^c}{\partial p_n \partial q_l} p_l - \frac{\partial^2 H^c}{\partial p_n \partial q_l} q_l = 0 \quad \text{and} \quad \frac{\partial^2 H^c}{\partial q_n \partial q_l} p_l - \frac{\partial^2 H^c}{\partial q_n \partial q_l} q_l = 0 \tag{61}
\]
for any \((p, q) \in V \cap (\mathbb{h}^2 \times \mathbb{h}^2)\). This and Lemma 4.2 below, applied to \( I \) equal to \((p^2_l + q^2_l)/2\) and \( F \) equal to \( \frac{\partial H^c}{\partial p_n} \) and \( \frac{\partial H^c}{\partial q_n} \) respectively, implies that for any \( l \in B \) and for any \( n \neq l \),
\[
\left. \frac{\partial^2 H^c}{\partial p_n \partial q_l} \right|_{z^*} = 0 \quad \text{and} \quad \left. \frac{\partial^2 H^c}{\partial q_n \partial q_l} \right|_{z^*} = 0. \tag{62}
\]
By combining (62) with (58) we obtain
\[
\tilde{\mathcal{L}}_c = \sum_{n \in A} \partial_{q_n} \otimes \sum_{l \in A} \left( \frac{\partial^2 H^c}{\partial p_n \partial q_l} dp_l + \frac{\partial^2 H^c}{\partial p_n \partial q_l} dq_l \right)_{z^*} + \sum_{n \in B} (\partial_{p_n} \otimes \partial_{q_n}) \otimes \left( \frac{\partial^2 H^c}{\partial q_n \partial q_l} dp_l + \frac{\partial^2 H^c}{\partial q_n \partial q_l} dq_l \right)_{z^*} \tag{63}
\]
Since the local Birkhoff coordinates are assumed gauge invariant and since \( dH(X_{t_k}) = \{ H, I_k \} = 0 \) for any \( k \in \mathbb{Z} \), we conclude that the flow \( S^c_t \) of the vector field \( X_{t_k} \) preserves \( X_{H^c} \), that is for any \( t \in \mathbb{R} \) and for any \((p, q) \in V \cap (\mathbb{h}^2 \times \mathbb{h}^2)\) such that \( S^c_t(p, q) \in V \cap (\mathbb{h}^2 \times \mathbb{h}^2)\) we have the following commutative diagram
\[
\begin{array}{ccc}
V \cap (\mathbb{h}^2 \times \mathbb{h}^2) & \xrightarrow{X_{H^c}} & \ell^2 \times \ell^2 \\
S^c_t \downarrow & & \downarrow S^c_t \\
V \cap (\mathbb{h}^2 \times \mathbb{h}^2) & \xrightarrow{X_{H^c}} & \ell^2 \times \ell^2
\end{array} \tag{64}
\]
Remark 4.2. Note that for any $s \in \mathbb{R}$ and for any $k \in \mathbb{Z}$ we have that $X_{I_k} = p_k \partial_{q_k} - q_k \partial_{p_k}$ is a vector field in the proper sense (i.e. non-weak) on $\mathfrak{h}^* \times \mathfrak{h}^*$ and that $S_k^I : \mathfrak{h}^* \times \mathfrak{h}^* \rightarrow \mathfrak{h}^* \times \mathfrak{h}^*$ is a bounded linear map. In fact, if we introduce complex variables $z_k := p_k + i q_k$, $k \in \mathbb{Z}$, then

$$
(S_k^I(z))_l = \begin{cases} z_l, & l \neq k, \\
\ e^{-it z_k}, & l = k.
\end{cases}
$$

In particular, we see from (64) that for any $k \in \mathbb{Z}$ and for any $t \in \mathbb{R}$ near zero we have that $(d_{S^I_k(z)} \times X_{H^c}) \circ S_k^I = S_k^I \circ (d_z \times X_{H^c})$. For $k \in B$ we have $S_k^I(z^*) = z^*$ and hence, for any $t \in \mathbb{R}$ near zero,

$$
\tilde{\mathcal{L}}_c \circ S_k^I = S_k^I \circ \tilde{\mathcal{L}}_c.
$$

By taking the $t$-derivative at $t = 0$ we obtain that for any $k \in B$,

$$
[\tilde{\mathcal{L}}_c, d_z \times X_{I_k}] = 0 \quad \text{where} \quad d_z \times X_{I_k} = \partial_{q_k} \otimes dp_k - \partial_{p_k} \otimes dq_k.
$$

Formula (65) together with (64) and (60) then implies that

$$
\tilde{\mathcal{L}}_c = \sum_{n,k \in A} A_{nk} (X_{I_n}|z^*) \otimes (d_z \times I_k) + \sum_{n \in B} B_n (\partial_{q_n} \otimes dp_n - \partial_{p_n} \otimes dq_n) \quad (66)
$$

for some matrices $(A_{nk})_{n,k \in A}$ and $(B_n)_{n \in B}$ with constant elements. Note that in view of the commutative diagram (66) and Theorem 3.1(i) the unbounded operator $\tilde{\mathcal{L}}_c$ on $\ell^2 \times \ell^2$ with domain $\mathfrak{h}^2 \times \mathfrak{h}^2$ has a compact resolvent. In particular, it has discrete spectrum. Moreover, by Theorem 3.1(i), zero belongs to the spectrum of $\tilde{\mathcal{L}}_c$ and has geometric multiplicity one. Since, in view of (66), the vectors $X_{I_k}|z^*$, $k \in A$, are eigenvectors of $\tilde{\mathcal{L}}_c$ with eigenvalue zero, we conclude that $A$ consists of one element $A = \{ n_0 \}$ and that $B_n \neq 0$ for any $n \in \mathbb{Z} \setminus \{ n_0 \}$. Hence, the spectrum of $\tilde{\mathcal{L}}_c$ consists of $\{ \pm i B_n \}_{n \in \mathbb{Z} \setminus \{ n_0 \}}$ and zero, which has algebraic multiplicity two and geometric multiplicity one. This completes the proof of Lemma 4.1. \hfill \Box

Let $\{ (x, y) \}$ be the coordinates in $\mathbb{R}^2$ equipped with the canonical symplectic form $dx \wedge dy$ and let $I = (x^2 + y^2)/2$. The proof of the following Lemma is not complicated and thus omitted.

Lemma 4.2. If $F : \mathbb{R}^2 \rightarrow \mathbb{R}$ is a $C^1$-map such that $\{ F, I \} = 0$ in some open neighborhood of zero then $d_{(0,0)}F = 0$.

Now, we are ready to prove Theorem 4.1.

Proof of Theorem 4.1. Take $c \in C$ such that $|c| \notin \pi \mathbb{Z}$ and $|c| > \pi$, and assume that there exist gauge invariant local Birkhoff coordinates of the focusing NLS equation in a neighborhood of the constant potential $\varphi_c$. In view of Lemma 3.1 and Theorem 3.1(i) the spectrum of $\mathcal{L}_c$ on $iL^2_c$ is discrete and contains non-zero real eigenvalues. On the other side, by Lemma 4.1 the spectrum of $\tilde{\mathcal{L}}_c$ lies on the imaginary axis. This shows that the two operators are not conjugated and hence, contradicts the existence of local Birkhoff coordinates. \hfill \Box
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