We present a class of new black hole solutions in $D$-dimensional Lovelock gravity theory. The solutions have a form of direct product $M^m \times H^n$, where $D = m + n$, $H^n$ is a negative constant curvature space, and the solutions are characterized by two integration constants. When $m = 3$ and $4$, these solutions reduce to the exact black hole solutions recently found by Maeda and Dadhich in Gauss-Bonnet gravity theory.

We study thermodynamics of these black hole solutions. Although these black holes have a nonvanishing Hawking temperature, surprisingly, the mass of these solutions always vanishes. While the entropy also vanishes when $m$ is odd, it is a constant determined by Euler characteristic of $(m-2)$-dimensional cross section of black hole horizon when $m$ is even. We argue that the constant in the entropy should be thrown away. Namely, when $m$ is even, the entropy of these black holes also should vanish. We discuss the implications of these results.
I. INTRODUCTION

With the development of string theory, supergravity and brane world scenarios, over the past years, gravity theories have been widely studied in higher dimensions. In the low-energy approximation, Einstein general relativity naturally arises from string theories. As corrections from massive states of string theories and from loop expansions in string theories, some higher derivative curvature terms also appear in the low-energy effective action of string theories \[1, 2\]. Therefore it is of great interest to discuss potential roles of those higher derivative terms in various aspects, for example, in black hole physics and early universe. Indeed, there exist a lot of works on higher derivative gravity theories in the literature.

In this paper we focus on a class of special higher derivative gravity theory, namely, Lovelock gravity \[3\], which is a natural generalization of general relativity in higher dimensions in the sense that the equations of motion of Lovelock gravity do not contain more than second order derivatives with respect to metric, as the case of general relativity. The Lagrangian of \(D\)-dimensional Lovelock gravity consists of the dimensionally extended Euler densities

\[
\mathcal{L} = \sum_{k=0}^{p} c_k \mathcal{L}_k, \tag{1.1}
\]

where \(p \leq [(D-1)/2]\) ([\(N\] denotes the integral part of the number \(N\)), \(c_k\) are arbitrary constants with dimension of [Length]^{2k-2}, and \(\mathcal{L}_k\) are the Euler densities

\[
\mathcal{L}_k = \frac{1}{2k} \sqrt{-g} \delta_{\lambda_1 \ldots \lambda_k \sigma_1 \ldots \sigma_k}^{\mu_1 \nu_1 \ldots \mu_k \nu_k} R_{\lambda_1 \sigma_1}^{\lambda_k \sigma_k} \cdots R_{\mu_1 \nu_1}^{\mu_k \nu_k}. \tag{1.2}
\]

Here, the generalized delta is totally antisymmetric in both sets of indices. \(\mathcal{L}_0 = 1\), so the constant \(c_0\) is just the cosmological constant. \(\mathcal{L}_1\) gives us the usual curvature scalar term, and for simplicity, we set \(c_1 = 1\), while \(\mathcal{L}_2\) is just the Gauss-Bonnet term. The Gauss-Bonnet term is argued to appear in the low-energy action of heterotic string theory with a positive coefficient \[1\]. The equations of motion following from the Lagrangian (1.1) have the form

\[
G_{\mu \nu} = 0, \tag{1.3}
\]

where

\[
G^\mu_{\nu} = \sum_{k=0}^{p} \frac{1}{2^{k+1}} c_k \delta_{\nu \lambda_1 \ldots \lambda_k \sigma_1 \ldots \sigma_k}^{\mu \mu_1 \nu_1 \ldots \mu_k \nu_k} R_{\lambda_1 \sigma_1}^{\lambda_k \sigma_k} \cdots R_{\mu_1 \nu_1}^{\mu_k \nu_k}. \tag{1.3}
\]

Since the action of Lovelock gravity is the sum of the dimensionally extended Euler densities, there are no more than second order derivatives with respect to metric in its equations of
motion. Furthermore, the Lovelock gravity is shown to be free of ghost when expanded on a flat space, evading any problems with unitarity [1, 4].

Finding exact analytic solutions of any gravity theory is an issue of long-standing interest. Indeed, there exist a lot of works to discuss exact black hole solutions for Lovelock gravity in the literature. The static, spherically symmetric black hole solutions in the theory have been found in [4, 5, 6, 8, 9, 10] and discussed [11], and topological nontrivial black holes have been studied in [6, 8, 9, 10]. Some rotating solutions in Gauss-Bonnet theory have been studied in [12, 13]. However, a general rotating solution is still absent even in Gauss-Bonnet gravity. See also [14, 15] for some other extensions including perturbative AdS black hole solutions in the gravity theories with second order curvature corrections. For a nice review of black holes in Lovelock gravity, see [16].

Recently, Maeda and Dadhich presented a class of exact solutions in Gauss-Bonnet gravity [17, 18, 19, 20] (Some similar black hole solutions have been found in Codimension-2 brane world theory [21, 22]). They assumed the spacetime has a direct product structure (one is a four/three-dimensional spacetime, the other is a negative constant curvature space), and then split the equations of motion into two sets according to the direct product structure of spacetime. For a suitable choice of those coefficients in Gauss-Bonnet gravity so that the set of equations of motion for the four/three-dimensional part is trivially satisfied, the set of the equations of motion for the negative constant curvature space part then reduces to a single equation. Solving the latter yields a class of new exact analytic solutions for Gauss-Bonnet gravity. This class of the solutions has two integration constants; one is argued to be related to the mass of the solutions and the other behaves like a Maxwell charge. The Maxwell charge is called “Weyl charge” due to the existence of the extra negative constant curvature space. This class of solutions is quite different from the normal ones in the sense which will become clear shortly. Quantum properties of those black hole solutions have not yet been studied so far.

In this paper, we consider a general Lovelock theory instead of the Gauss-Bonnet theory, and seek for more general black hole solutions and study their thermodynamics. The outline of the paper is as follows. In Sec. II, we present a class of new black hole solutions in the general Lovelock theory, following [17, 18]. In Sec. III, we study thermodynamic properties of those black hole solutions. Sec. IV is devoted to conclusion and discussion.
II. GENERAL BLACK HOLE SOLUTIONS

A. Equations of Motion

Consider a $D(= m + n)$-dimensional spacetime $X$, locally homeomorphic to $\mathcal{M}^m \times \mathcal{N}^n$. We assume the metric of this spacetime has the form

$$ds^2 = g_{ij}dx^idx^j + r_0^2\gamma_{ab}dy^ady^b,$$  \hspace{1cm} (2.1)

where $g_{ij}dx^idx^j$ is the metric on $\mathcal{M}^m$ with the coordinates $\{x^i, i = 1, \cdots, m\}$, $r_0$ is a constant and $\gamma_{ab}dy^ady^b$ is the metric on the $n$-dimensional space $\mathcal{N}^n$ with the coordinates $\{y^a, a = 1, \cdots, n\}$. $\mathcal{N}^n$ is a constant curvature space with curvature $\bar{k} = \pm 1, 0$. It is easy to find that the components of the Riemann tensor for (2.1) have the form

$$R_{ijkl} = \bar{R}_{ijkl}, \hspace{0.5cm} R_{ij}^{\hspace{0.5cm}kl} = \bar{R}_{ij}^{\hspace{0.5cm}kl},$$

$$R_{abcd} = \bar{R}_{abcd}, \hspace{0.5cm} R_{ab}^{\hspace{0.5cm}cd} = \bar{R}_{ab}^{\hspace{0.5cm}cd},$$  \hspace{1cm} (2.2)

where $\bar{R}_{ijkl}$ and $\bar{R}_{abcd}$ denote the components of Riemann tensor on $\mathcal{M}^m$ and $\mathcal{N}^n$ respectively. For $\mathcal{N}^n$, we can write the $\bar{R}_{abcd}$ as

$$\bar{R}_{abcd} = \bar{k}r_0^2(\gamma_{ac}\gamma_{bd} - \gamma_{ad}\gamma_{bc}), \hspace{0.5cm} \bar{R}_{ab}^{\hspace{0.5cm}cd} = \frac{\bar{k}}{r_0^2}\delta_{ab}^{cd}. \hspace{1cm} (2.3)$$

According to the decomposition of the Riemann tensor (2.2), we can decompose the equations of motion (1.3) into $m$-dimensional and $n$-dimensional parts:

$$G^i_{\hspace{0.5cm}j} = \sum_{k=0}^{p} \frac{1}{2k+1} c_k^{\delta_{ij}^{\mu_1\cdots\mu_k\nu_1\cdots\nu_k}} R_{\lambda_1\cdots\mu_1\cdots\nu_1\cdots\nu_k}, \hspace{0.5cm} R_{\lambda \mu_1\cdots\mu_k\cdots\nu_1\cdots\nu_k} = 0,$$

$$G^a_{\hspace{0.5cm}b} = \sum_{k=0}^{p} \frac{1}{2k+1} c_k^{\delta_{ab}^{\mu_1\cdots\mu_k\nu_1\cdots\nu_k}} R_{\lambda_1\cdots\mu_1\cdots\nu_1\cdots\nu_k}, \hspace{0.5cm} R_{\lambda \mu_1\cdots\mu_k\cdots\nu_1\cdots\nu_k} = 0. \hspace{1cm} (2.4)$$

Other components (such as $G^i_{\hspace{0.5cm}a}$) automatically vanish. Since $i, j$ run only in the range $\{1, \cdots, m\}$, and $R_{ijkl}$ can appear in the products “$RR\cdots$” no more than $q = [(m-1)/2]$ times, we have

$$G^i_{\hspace{0.5cm}j} = \sum_{k=0}^{p} \frac{c_k}{2k+1} \delta_{ij}^{\mu_1\cdots\mu_k\nu_1\cdots\nu_k} R_{\lambda_1\cdots\mu_1\cdots\nu_1\cdots\nu_k},$$

$$= \sum_{k=t}^{q} \sum_{k=t}^{p} \binom{k}{t} \frac{c_k}{2k+1} \delta_{jm_{11}\cdots m_{1m}1\cdots e_k t f_1 \cdots f_{k-t}} R_{k_1 f_1 m_{11} \cdots m_{1m}1 \cdots e_k t f_1 \cdots f_{k-t}} R_{e_k t f_k t} \times R_{e_k t f_k t} \cdots R_{e_k t f_k t}. \hspace{1cm} (2.5)$$
Substituting Eqs. (2.2) and (2.3) into (2.5) and using the identity

$$\delta^{\mu_1 \cdots \mu_p-1 \nu_1 \cdots \nu_p} \delta^{\nu_1 \cdots \nu_p-1 - \mu_p} = 2 [r - (p - 1)] [r - (p - 2)] \delta^{\nu_1 \cdots \nu_p-2},$$

(2.6)

where \( r \) denotes the range of the index \( (r = m \text{ for } M^m \text{ and } r = n \text{ for } N^n) \), we have

$$g^i_{(t)j} = \sum_{t=0}^{q} \left[ \sum_{k=t}^{p} \left( \frac{k}{t} \right) c_k \frac{(D - m)!}{(D - m - 2(k - t))!} \left( \frac{k}{r_0^2} \right)^{k-t} \right] \bar{g}^i_{(t)j},$$

(2.7)

where

$$\bar{g}^i_{(t)j} = \frac{1}{2^{q+1}} \delta^{c_1l_1 \cdots c_{s-1}l_{s-1}} R_{k_1l_1} R_{k_2l_2} \cdots R_{k_{m_n}l_{m_n}}.$$  

(2.8)

Similarly, we have

$$g^a_b = \sum_{k=0}^{s} \frac{c_k}{2k+1} \delta^{\mu_1 \cdots \mu_p-1 \nu_1 \cdots \nu_p} R^{\lambda_1 \sigma_1} \cdots R^{\lambda_p \sigma_p} s_i^k \mu_k \nu_k = \sum_{t=0}^{s} \sum_{k=t}^{p} \left( \frac{k}{t} \right) c_k \frac{(D - m)!}{(D - m - 1 - 2(k - t))!} \left( \frac{k}{r_0^2} \right)^{k-t} R_{c_1t} \cdots R_{c_{m_n}t},$$

(2.9)

where \( s = \lfloor m/2 \rfloor \). Substituting Eqs. (2.2), (2.3) and (2.6) into the above equation, we can express it as

$$g^a_b = \frac{1}{2} \delta^a_b \left\{ \sum_{t=0}^{s} \sum_{k=t}^{p} \left( \frac{k}{t} \right) c_k \frac{(D - m - 1)!}{(D - m - 1 - 2(k - t))!} \left( \frac{k}{r_0^2} \right)^{k-t} \bar{L}_t \right\},$$

(2.10)

where

$$\bar{L}_t = \frac{1}{2} \delta^{c_1l_1 \cdots c_{m_n}l_{m_n}} R_{k_1l_1} R_{k_2l_2} \cdots R_{k_{m_n}l_{m_n}}.$$  

(2.11)

Note that \( g^a_b \) is always proportional to \( \delta^a_b \), which is a crucial point to our discussions below.

Let us note that if the following equations are satisfied

$$0 = A_t = \sum_{k=t}^{p} \left( \frac{k}{t} \right) c_k \frac{(D - m)!}{(D - m - 2(k - t))!} \left( \frac{k}{r_0^2} \right)^{k-t}, \quad t = 0, \ldots, q,$$

(2.12)

then the equations of motion (2.7) are always trivially satisfied. These are \((q + 1)\)-linear equations for \( c_0, \ldots, c_p \). Recall \( c_1 = 1 \) and if we consider the case with

$$p = q + 1 = \left\lfloor \frac{m - 1}{2} \right\rfloor + 1,$$

(2.13)

then the equations (2.12) indicate that \( c_k (k \neq 1) \) has a unique expression in terms of \((\bar{k}/r_0^2)\) and dimension \( D \). When \( \bar{k} = -1 \) (which implies that \( N^n \) is a negative constant curvature.
space. It will be denoted by $\mathcal{H}^n$ in the following discussion.) and $D \geq m + 2$, we find that all $c_k$ are positive. Some examples will be given soon.

Now we turn to the equations of $G_{ab}$. Due to the fact that $G_{ab}$ is proportional to $\delta_{ab}$, the equations $G_{ab} = 0$ reduce to a single equation

$$0 = \sum_{i=0}^{s} \alpha_i \bar{L}_i ,$$

where the coefficients $\alpha_i$'s are given by

$$\alpha_i = \sum_{k=i}^{p} \binom{k}{i} c_k \frac{(D - m - 1)!}{(D - m - 1 - 2(k - i))!} \left( \frac{k}{r_0^2} \right)^{k-i} ,$$

and $c_k$ are determined by solutions (2.12).

Let us further assume the $m$-dimensional metric $g_{ij}$ takes the form

$$g = -f(r)dt^2 + \frac{1}{f(r)}dr^2 + r^2 d\Sigma_{m-2}^2 ,$$

where $d\Sigma_{m-2}^2$ is the line element of $(m-2)$-dimensional surface with constant scalar curvature $(m-2)(m-3)\delta$. Without loss of generality, $\delta$ can be set to $\pm 1$ or zero. It is easy to find that the nonvanishing components of the Riemann tensor are

$$R^r_{\ tr} = -\frac{f''}{2}, \quad R^i_{\ tj} = R^i_{\ rj} = -\frac{f'}{2r}\delta^i_j, \quad R^{ij}_{\ kl} = \frac{\delta - f'}{r^2}\delta^{ij}_{\ kl} ,$$

where the prime stands for derivative with respect to $r$. The Euler density then has the form

$$\bar{L}_i = \frac{(m-2)!}{(m-2-2i)!} \left( \frac{\delta - f}{r^2} \right)^i + 4i \frac{(m-2)!}{(m-1-2i)!} \left( \frac{f'}{2r} \right) \left( \frac{\delta - f}{r^2} \right)^{i-1}$$

$$+ 2i \frac{(m-2)!}{(m-2i)!} \left( \frac{f''}{2r} \right) \left( \frac{\delta - f}{r^2} \right)^{i-1} + 4i(i-1) \frac{(m-2)!}{(m-2i)!} \left( \frac{f'}{2r} \right)^2 \left( \frac{\delta - f}{r^2} \right)^{i-2} .$$

Defining

$$F(r) = \frac{\delta - f(r)}{r^2} ,$$

$\bar{L}_i$ can be rewritten as

$$\bar{L}_i = \frac{(m-2)!}{(m-2i)!} \frac{1}{r^{m-2}} \left( r^m F(r) \right)'' .$$

Finally Eq. (2.14) becomes

$$0 = \sum_{i=0}^{s} \hat{\alpha}_i \left( r^m F(r) \right)'' ,$$
where
\[ \hat{\alpha}_i = \frac{(m - 2)!}{(m - 2i)!} \alpha_i. \] (2.22)

The solution to (2.21) is determined by the algebraic equation
\[ \sum_{i=0}^{s} \hat{\alpha}_i F(r)^i = \frac{M}{r^{m-1}} + \frac{Q}{r^m}, \] (2.23)

where \( M, Q \) are two integration constants. Naively one may think they are related to the mass and Weyl charge of the solution [17, 23, 24], respectively. But actually the integration constant \( M \) has nothing to do with the mass of the solution, which will be shown shortly.

The constant \( Q \) may be positive, zero and negative. Here some remarks are in order.

(i) Since \( G^a_b \sim \delta^a_b \), the equations \( G^a_b = 0 \) reduce to a single equation (2.14). This is very different from the normal case in Lovelock gravity, where \( G^a_b \) is not proportional to \( \delta^a_b \) even under a spherical symmetric assumption. For example, the equation \( G^t_t = 0 \) of Lovelock theory will give a first order differential equation like [5, 6]
\[ 0 = \sum_{i=0}^{s} c_i \left( r^{m-1} F(r)^i \right) '\] (2.24)
in the static, spherically symmetric case. In that case, there is only one integration constant, which is nothing but the mass of the solution [6]. In the present case, one has only one traceless-like equation, which is a second order differential equation. There is therefore one more integration constant \( Q \) in the present case.

(ii) If \( Q = 0 \), Eq. (2.23) is very similar to the corresponding one for static, spherically symmetric black hole solutions in Lovelock theory [5, 6]. However, there are two obvious differences: one is that here the coefficients \( \hat{\alpha}_i \) are all fixed by \( \bar{k}/r_0^2 \) and \( D \), while in the normal case, those coefficients are free parameters [5, 6]. We will show this below. The other is that in Eq. (2.23), the range of \( i \) is \([0, \cdots, s = [m/2]]\), while in the normal case, the range is \([0, \cdots, [(m-1)/2]]\). Therefore when \( m \) is even, \( i \) can take the value \([m/2]\), which will not appear in the normal case.

(iii) Since all \( \hat{\alpha}_0 \) are fixed by Eq. (2.12) and they are all positive constants, the solutions are not asymptotically flat, but asymptotically AdS spacetimes.
B. Black hole solution with $M = 0$

The spacetime (2.16) describes a black hole provided $f(r_+) = 0$ and $f(r) > 0$ with $r > r_+$. Here $r = r_+$ is called black hole horizon. We can see from (2.23) that even if the “mass” $M$ of the solution vanishes, black hole horizon can still exist. To show this, let us discuss the solution $F = F_0$ of Eq. (2.23) with $M = 0$:

$$
\sum_{i=0}^{s} \hat{\alpha}_i F_0^i = \frac{Q}{r^m}
$$

with horizons. Assume $f_0(\bar{r}) = 0$ at some positive $\bar{r}$, and we have

$$
\bar{r}^2 F_0(\bar{r}) = \delta \quad \text{or} \quad F_0(\bar{r}) = \frac{\delta}{\bar{r}^2}.
$$

That is, $\bar{r}$ must satisfy the following equation

$$
\bar{r}^m \sum_{i=0}^{s} \hat{\alpha}_i \left( \frac{\delta}{\bar{r}^2} \right)^i = Q.
$$

In order for the equation to hold, the constant $Q$ must satisfy some constraints. We will discuss these constraints in the cases of $\delta = 0$ and $\delta = \pm 1$, respectively.

(i). $\delta = 0$. This case is simple. In this case, only one term in (2.27) remains. We have

$$
\bar{r}^m \hat{\alpha}_0 = Q,
$$

Since $\hat{\alpha}_0 > 0$, this indicates a positive $\bar{r}$ exists provided $Q > 0$, and $\bar{r}$ is just the horizon radius $r_+$.

(ii). $\delta = \pm 1$. If $m$ is odd, one then has $s = (m - 1)/2$, and

$$
\bar{r}^m \hat{\alpha}_0 \pm \bar{r}^{m-2} \hat{\alpha}_1 + \cdots + (\pm 1)^s \bar{r} \hat{\alpha}_s = Q.
$$

Obviously, because all coefficients $\{\hat{\alpha}_0, \hat{\alpha}_1, \ldots, \hat{\alpha}_s\}$ are positive, and especially $\hat{\alpha}_0 > 0$, Eq. (2.29) has at least one positive root $\bar{r}$ if $Q > 0$. The black hole horizon $r_+$ is just the largest positive root of Eq. (2.29).

On the other hand, if $m$ is even, one then has $s = m/2$, and

$$
\bar{r}^m \hat{\alpha}_0 \pm \bar{r}^{m-2} \hat{\alpha}_1 + \cdots + (\pm 1)^s \bar{r}^2 \hat{\alpha}_{s-1} + (\pm 1)^s \hat{\alpha}_s = Q.
$$

From the theory of polynomial, the above equation has at least one negative root and one positive root if $\hat{\alpha}_0 [(\pm 1)^s \hat{\alpha}_s - Q] < 0$. Recall $\hat{\alpha}_0 > 0$, and this condition can be always satisfied if $Q > \hat{\alpha}_s$. Namely, black hole horizon exists in this case.

In summary, black hole horizon always exists provided $Q - \hat{\alpha}_s > 0$, even when the parameter $M$ vanishes.
C. Examples of exact solutions

To be more explicit, in this subsection, we give some simple examples of exact solutions given in (2.23).

1. The case of \( m = 3, p = 2, D \geq 5 \)

In this case, Eqs. (2.12) give

\[
c_0 = \frac{1}{2} (D^2 - 3D - 6)r_0^{-2}, \quad c_2 = \frac{1}{2(D - 3)(D - 4)} r_0^2.
\]

We then have

\[
\hat{\alpha}_0 = \frac{2(2D - 9)}{3(D - 3)} r_0^{-2}, \quad \hat{\alpha}_1 = \frac{2}{D - 3}.
\]

Equation (2.23) for \( F \) becomes

\[
\hat{\alpha}_0 + \hat{\alpha}_1 F = \frac{M}{r^2} + \frac{Q}{r^3},
\]

which has the solution

\[
f = \frac{1}{\hat{\alpha}_1} \left( -M - \frac{Q}{r} + \hat{\alpha}_0 r^2 \right).
\]

Here we have used the fact that the \((m-2)\)-dimensional constant curvature space is always Ricci flat for \( m = 3 \), i.e., \( \delta = 0 \). Since \( \hat{\alpha}_0 \) is always positive, this solution is just a BTZ black hole deformed by the additional charge \( Q \). This kind of solution has been obtained in [18].

2. The case of \( m = 4, p = 2, D \geq 6 \)

In this case, Eq. (2.12) give

\[
c_0 = \frac{1}{2} (D^2 - 5D - 2)r_0^{-2}, \quad c_2 = \frac{1}{2(D - 4)(D - 5)} r_0^2.
\]

We then have

\[
\hat{\alpha}_0 = \frac{2D - 11}{3(D - 4)} r_0^{-2}, \quad \hat{\alpha}_1 = \frac{2}{D - 4}, \quad \hat{\alpha}_2 = \frac{1}{(D - 4)(D - 5)} r_0^2.
\]

The equation for \( F \) becomes

\[
\hat{\alpha}_0 + \hat{\alpha}_1 F + \hat{\alpha}_2 F^2 = \frac{M}{r^3} + \frac{Q}{r^4},
\]
which has the solution

\[ F(r) = -\frac{\hat{\alpha}_1}{2\hat{\alpha}_2} \left( 1 \mp \sqrt{1 - \frac{4\hat{\alpha}_0\hat{\alpha}_2}{\hat{\alpha}_1^2} + \frac{4\hat{\alpha}_2 M}{\hat{\alpha}_1^2} r^3 + \frac{4\hat{\alpha}_2 Q}{\hat{\alpha}_1^2} r^4} \right), \]  

(2.38)

\[ f(r) = \delta + \frac{\hat{\alpha}_1 r^2}{2\hat{\alpha}_2} \left( 1 \mp \sqrt{1 - \frac{4\hat{\alpha}_0\hat{\alpha}_2}{\hat{\alpha}_1^2} + \frac{4\hat{\alpha}_2 M}{\hat{\alpha}_1^2} r^3 + \frac{4\hat{\alpha}_2 Q}{\hat{\alpha}_1^2} r^4} \right). \]  

(2.39)

This solution with two branches is just the one recently obtained by Maeda and Dadhich in [17]. It is easy to see that for \( D \geq 6 \), we have

\[ \hat{\alpha}_1^2 - 4\hat{\alpha}_0\hat{\alpha}_2 = \frac{4}{3(D - 4)(D - 5)} > 0, \]  

(2.40)

So the vacuum AdS solution \((M = Q = 0)\) always exists.

3. The case of \( m = 5, p = 3, D \geq 7 \)

Equations (2.12) in this case lead to

\[ c_0 = \frac{D^4 - 10D^3 + 11D^2 + 22D + 360}{3(D^2 - 7D + 4)} r_0^{-2}, \]

\[ c_2 = \frac{1}{(D^2 - 7D + 4)} r_0^2, \]

\[ c_3 = \frac{1}{3(D^2 - 7D + 4)(D - 5)(D - 6)} r_0^4. \]  

(2.41)

Then the corresponding \( \hat{\alpha} \) are

\[ \hat{\alpha}_0 = \frac{4(5D^2 - 67D + 225)}{5(D^2 - 7D + 4)(D - 5)} r_0^{-2}, \]

\[ \hat{\alpha}_1 = \frac{8(2D - 13)}{(D^2 - 7D + 4)(D - 5)}, \]

\[ \hat{\alpha}_2 = \frac{12}{(D^2 - 7D + 4)(D - 5)} r_0^2. \]  

(2.42)

The equation for \( F \) is still in second order and has the solution

\[ F(r) = -\frac{\hat{\alpha}_1}{2\hat{\alpha}_2} \left( 1 \mp \sqrt{1 - \frac{4\hat{\alpha}_0\hat{\alpha}_2}{\hat{\alpha}_1^2} + \frac{4\hat{\alpha}_2 M}{\hat{\alpha}_1^2} r^3 + \frac{4\hat{\alpha}_2 Q}{\hat{\alpha}_1^2} r^4} \right), \]  

(2.43)

\[ f(r) = \delta + \frac{\hat{\alpha}_1 r^2}{2\hat{\alpha}_2} \left( 1 \mp \sqrt{1 - \frac{4\hat{\alpha}_0\hat{\alpha}_2}{\hat{\alpha}_1^2} + \frac{4\hat{\alpha}_2 M}{\hat{\alpha}_1^2} r^3 + \frac{4\hat{\alpha}_2 Q}{\hat{\alpha}_1^2} r^4} \right). \]  

(2.44)
This solution is an example with the third order Lovelock term. Note that when \( m = 5 \), the solution also has two branches. In addition, it is also easy to see that the vacuum AdS solution exists, because

\[
\hat{\alpha}_1 - 4\hat{\alpha}_0\hat{\alpha}_1 = \frac{64(5D - 34)}{5(D^2 - 7D + 4)(D - 5)} > 0
\]

for \( D \geq 7 \).

## III. THERMODYNAMIC PROPERTIES OF BLACK HOLE SOLUTIONS

### A. Naive consideration: \( m \)-dimensional black holes

The black hole spacetime has a direct product form \( \mathcal{M}^m \times \mathcal{H}^n \), where \( \mathcal{H}^n \) is a negative constant curvature space with a constant radius \( r_0 \). From the point of view of usual Kaluza-Klein dimensional reduction, the thermodynamics for the whole spacetime is equivalent to that for \( m \)-dimensional black hole with redefined gravitational constant \( G_m = G_{m+n}/\text{Vol}(n) \). Here \( G_m \) and \( G_{m+n} \) are gravitational constants in \( m \) dimensions and \((m+n)\) dimensions, respectively, while \( \text{Vol}(n) \) is the volume of the constant curvature space \( \mathcal{H}^n \). In this subsection, we will discuss the black hole thermodynamics from the point of view of \( m \) dimensions.

Assume that black hole has a horizon at \( r_+ \), which is the largest positive root of \( f(r) = 0 \). The horizon radius then must satisfy

\[
F(r_+) = \frac{\delta}{r_+^2}, \quad \text{or} \quad r_+^2 F(r_+) = \delta. \tag{3.1}
\]

The Hawking temperature of the black hole can be easily calculated by Euclidean method. To avoid conical singularity at the horizon, the period of Euclidean time should be \( \beta = 4\pi/f'(r_+) \), and the Hawking temperature is just the inverse of the period. This way we get the temperature of the black hole

\[
\mathcal{T} = \frac{1}{\beta} = \frac{1}{4\pi} f'(r_+) = -\frac{1}{4\pi} \left( \frac{2\delta}{r_+} + r_+^2 F'(r_+) \right). \tag{3.2}
\]

To get the explicit form of the Hawking temperature in terms of black hole horizon, we have to give the expression of \( F' \). Taking derivative on both sides of Eq. (2.23) with respect to \( r \), one has

\[
F'(r_+) = -\frac{(m-1)Mr_+ + mQ}{r_+^{m+1} \sum_{i=1}^{s} i \hat{\alpha}_i \left( \frac{\delta}{r_+^{i-1}} \right)} \tag{3.3}
\]
On the other hand, from the equation (2.23), we have

$$M = -\frac{Q}{r_+} + r_+^{m-1} \sum_{i=0}^{s} \hat{\alpha}_i \left( \frac{\delta}{r_+^2} \right)^i.$$  \hspace{1cm} (3.4)

Thus we can express $F'(r_+)$ as

$$F'(r_+) = -\frac{Q + (m - 1)r_+^m \sum_{i=0}^{s} \hat{\alpha}_i \left( \frac{\delta}{r_+^2} \right)^i}{r_+^{m+1} \sum_{i=1}^{s} i \hat{\alpha}_i \left( \frac{\delta}{r_+^2} \right)^{i-1}},$$  \hspace{1cm} (3.5)

and the Hawking temperature has the form

$$T = \frac{1}{4\pi} \frac{1}{r_+} \left( \sum_{i=0}^{s} (m - 2i - 1)\hat{\alpha}_i \delta \left( \frac{\delta}{r_+^2} \right)^{i-1} + \frac{Q}{r_+^{m-2}} \right).$$  \hspace{1cm} (3.6)

Clearly, if we choose a suitable $Q$, the Hawking temperature may vanish. This case corresponds to the “extremal” black holes with vanishing Hawking temperature. For example, when $m = 4$, choosing

$$Q = -3\hat{\alpha}_0 r_+^4 - \hat{\alpha}_1 r_+^2 \delta + \hat{\alpha}_2 \delta^2$$  \hspace{1cm} (3.7)

will lead to a vanishing temperature.

To get the mass of the black hole, we expand the metric $g_{00}$ in the large $r$ limit, subtract the corresponding one for a suitable reference background solution $F_b$, and then read off the mass with the coefficient in front of some power of the radial coordinate $r$. Here we choose the vacuum AdS solutions with vanishing $M$ and $Q$ as the reference background, i.e.,

$$\sum_{i=0}^{s} \hat{\alpha}_i F_b^i = 0.$$  \hspace{1cm} (3.8)

For large $r$, we can expand $F$ as $F = F_b + \Delta F$ with the leading order correction $\Delta F$. We arrive at

$$\Delta F \sum_{i=0}^{s} i \hat{\alpha}_i F_b^{i-1} = \Delta F \hat{\alpha} = \frac{M}{r^{m-1}}.$$  \hspace{1cm} (3.9)

Here the constant $\hat{\alpha}$ is given by

$$\hat{\alpha} = \sum_{i=0}^{s} i \hat{\alpha}_i F_b^{i-1}.$$  \hspace{1cm} (3.10)

For solutions in some branch, $\hat{\alpha}$ may be negative. However, we only consider the cases with positive $\hat{\alpha}$ here. So we find the expansion of metric around the background as

$$g_{tt} - g_{(b)tt} = -f + f_b = r^2 \Delta F \approx \frac{M}{\hat{\alpha} r^{m-3}} = \frac{16\pi G_m 2\mathcal{R}}{(m - 2)V_{m-2}r^{m-3}}.$$  \hspace{1cm} (3.11)
Thus we find that the mass of the black hole $\mathcal{M}$ has a relation to $M$ as

$$\frac{M}{\hat{\alpha}} = \frac{16\pi G_m \mathcal{M}}{(m-2)V_{m-2}},$$

(3.12)

where $V_{m-2}$ is the volume of $(m-2)$-dimensional cross section of horizon surface. The mass can be expressed in terms of horizon radius $r_+$ and $Q$ as

$$\mathcal{M} = \left(\frac{m-2}{16\pi G_m \hat{\alpha}}\right) \frac{V_{m-2}}{r_+} \left(-\frac{Q}{r_+} + r_+^{m-1} \sum_{i=0}^{s} \hat{\alpha}_i \left(\frac{\delta}{r_+^2}\right)^{i-1}\right),$$

(3.13)

and its variation with respect to the horizon radius is

$$\left(\frac{\partial \mathcal{M}}{\partial r_+}\right)_Q = \left(\frac{m-2}{16\pi G_m \hat{\alpha}}\right) \frac{V_{m-2}}{r_+^m} \left(\sum_{i=0}^{s} (m-2i-1) \hat{\alpha}_i \left(\frac{\delta}{r_+^2}\right)^{i-1} + \frac{Q}{r_+^{m-2}}\right).$$

(3.14)

Since we are dealing with black holes in higher derivative gravity theory, the well-known area formula for black hole entropy breaks down. Let us try to obtain the black hole entropy by integrating the first law of the black hole thermodynamics

$$\mathcal{S} = \int \xi^{-1} d\mathcal{M} = \int_r^{r_+} \xi^{-1} \left(\frac{\partial \mathcal{M}}{\partial r_+}\right)_Q dr_+. $$

(3.15)

(i). When $m$ is even, $s$ takes the value $m/2$. The integral gives

$$\mathcal{S} = \int \xi^{-1} d\mathcal{M} = \int_r^{r_+} \xi^{-1} \left(\frac{\partial \mathcal{M}}{\partial r_+}\right)_Q dr_+ = \frac{V_{m-2}}{4G_m \hat{\alpha}} \left[\sum_{i=1}^{s-1} \frac{m-2}{m-2i} \hat{\alpha}_i \delta^{i-1} r_+^{m-2i} + s(s-1) \hat{\alpha}_s \delta^{s-1} \ln \left(r_+^2\right)\right] + \mathcal{S}_0. $$

(3.16)

The last term $\mathcal{S}_0$ is an integration constant. Note that here a logarithmic term appears, which comes from the fact that $s$ can take the value $m/2$.

(ii). When $m$ is odd, $s$ is $(m-1)/2$. In this case, the integral gives

$$\mathcal{S} = \int \xi^{-1} d\mathcal{M} = \int_r^{r_+} \xi^{-1} \left(\frac{\partial \mathcal{M}}{\partial r_+}\right)_Q dr_+ = \frac{V_{m-2}}{4G_m \hat{\alpha}} \left[\sum_{i=1}^{s} \frac{m-2}{m-2i} \hat{\alpha}_i \delta^{i-1} r_+^{m-2i}\right] + \mathcal{S}_0. $$

(3.17)

Here $\mathcal{S}_0$ is also an integration constant. Note that here the integration constant $\mathcal{S}_0$ should be set to zero because when the black hole horizon shrinks to zero, the entropy of the black hole should vanish \[10\]. However, the integration constant cannot be fixed by the same argument in the case of even $m$, due to the existence of the logarithmic term in the black
hole entropy. In addition, let us notice that when the black hole horizon is a Ricci flat surface, namely, \( \delta = 0 \), not only does the logarithmic term disappear in (3.16), but also both (3.16) and (3.17) give an entropy proportional to horizon area. This is also a general feature of black hole entropy in Lovelock gravity [6, 10].

The entropy expressions (3.16) and (3.17) look quite similar to the entropy formula of static, spherically symmetric black holes in Lovelock gravity [6], except for the logarithmic term in (3.16). The appearance of the logarithmic term is strange, although such a term appears in the entropy expressions of black holes in Horava-Lifshitz gravity theory [28], while the latter is not a full diffeomorphism invariant theory. For a diffeomorphism invariant gravity theory, Wald showed that black hole entropy is a Noether charge [25]; further a well-known entropy formula was developed [25, 26]. By Wald’s entropy formula, black hole entropy must be a function of horizon geometry and a logarithmic term will never appear in Wald’s entropy formula. This may cause suspicion whether the results given above are valid or not.

Let us notice that the above way to obtain black hole solution in \( m \) dimensions is quite different from the usual Kaluza-Klein dimensional reduction. In the usual Kaluza-Klein theory, with the assumption of direct product of the manifold \( \mathcal{M}^m \times \mathcal{H}^n \), one gets reduced action by integrating the total action over the extra space \( \mathcal{H}^n \). Certainly here too, with the assumption of direct product structure of the spacetime, one can get reduced action from the total action (1.1). This reduced action is an \( m \)-dimensional version of Lovelock gravity as given below in Eq. (3.38). We can get equations of motion (2.7) only for \( m \)-dimensional part but not the \( n \)-dimensional part by variation of this reduced action. Obviously these are nothing but the usual equations of motion of the Lovelock gravity in \( m \) dimensions with special coefficients \( A_i \)'s. For example, when \( m = 4 \), these equations are the Einstein equations with cosmological constant

\[
A_0 g_{ij} + A_1 E_{ij} = 0 , \tag{3.18}
\]

where \( E_{ij} \) is the Einstein tensor in four dimensions, and \( A_0 \) and \( A_1 \) are given in (2.12). Because our solutions are obtained for \( A_0 = A_1 = 0 \), Eqs. (3.18) have no information on our solutions. Therefore the field equations for \( m \)-dimensional part is trivially satisfied, while the nontrivial solutions come from the trace equation of gravitational field for the \( n \)-dimensional part, which is not obtained from the reduced action. So the reduced action certainly exists,
but it does not give solutions in this paper. If we naively omit the extra dimensions in our solutions, the corresponding $m$-dimensional local diffeomorphism-invariant “effective action” is absent, and we have to consider the whole $(m + n)$-dimensional theory. This is very different from the usual Kaluza-Klein theory in which the effective action is just the reduced action. As a result, we cannot simply use Wald’s entropy formula to get the entropy of the $m$-dimensional black hole. On the other hand, Lovelock theory is diffeomorphism invariant and Wald’s entropy formula is applicable in the whole $(m + n)$ dimensions. In the following subsections, we discuss thermodynamics of the black holes by Euclidean action and Wald’s entropy formula in $(m + n)$ dimensions, and find quite different and surprising results.

**B. Entropy of $(m + n)$-dimensional black holes**

From the viewpoint of the whole $D(= m + n)$ dimensions, to study thermodynamics of these black holes is straightforward. The temperature of the black holes is the same as the one in Eq. (3.2) because it is determined by the horizon geometry only. In Lovelock gravity, Wald’s entropy formula can be expressed as

$$S = \sum_{k=1}^{p} \frac{4\pi kc_k}{D-2} \int d^{D-2}x \mathcal{L}_{k-1}(\tilde{h}),$$

(3.19)

where $\mathcal{L}_{k-1}(\tilde{h})$ has the same form as (1.2) except that metric is replaced by $\tilde{h}$, which is the induced metric on the $(D-2)$-dimensional cross section of the horizon. The induced metric $\tilde{h}$ is

$$\tilde{h} = \tilde{g}_{ij}dz^i dz^j + r_0^2 \gamma_{ab} dy^a dy^b,$$

(3.20)

where $\tilde{g}_{ij}$ is the induced metric of the cross section of the horizon in the $m$-dimensional part.

By the similar procedure to get (2.7) and (2.10), we have

$$\mathcal{S} = \sum_{t=0}^{w} \sum_{k=t+1}^{p} \left\{ \left( \begin{array}{c} k-1 \\ t \end{array} \right) \frac{4\pi kc_k}{(D-m)!} \frac{(D-m)!}{(D-m-2(k-1-t))!} \left( \frac{k}{r_0^2} \right)^{k-t-1} \right\} \times \int d^{D-m}y r_0^{D-m} \sqrt{\gamma} \int d^{m-2}z \mathcal{L}_t(\tilde{q}),$$

(3.21)
where \( w = [(m - 2)/2] \). Define \( \ell = t + 1 \), we have

\[
\mathcal{S} = \sum_{\ell=1}^{w+1} \left\{ 4\pi \ell \left[ \sum_{k=\ell}^{p} \binom{k}{\ell} c_k \frac{(D - m)!}{(D - m - (k - \ell))!} \left( \frac{\bar{k}}{r_0^2} \right)^{k-\ell} \right] \right. \\
\times \left. \int d^{D-m} y r_0^{D-m} \sqrt{\gamma} \int d^{m-2} z \mathcal{L}_{\ell-1}(\tilde{q}) \right\} .
\]

(3.22)

Thus the entropy can be expressed as

\[
\mathcal{S} = \sum_{\ell=1}^{w+1} \left\{ 4\pi \ell A_\ell \int d^{D-m} y r_0^{D-m} \sqrt{\gamma} \int d^{m-2} z \mathcal{L}_{\ell-1}(\tilde{q}) \right\} ,
\]

(3.23)

where \( A_\ell \) is defined in Eq. (2.12), from which we have

\[
A_\ell = 0, \quad \ell = 1, \ldots, [(m - 1)/2] .
\]

(3.24)

Thus we find from the entropy (3.23) that when \( m \) is odd, \( \mathcal{S} = 0 \), a vanishing entropy!

When \( m \) is even, the entropy is

\[
\mathcal{S} = 2\pi m \left( r_0^{D-m} \Omega_{D-m} \right) c_{m/2} \chi(\Sigma_{m-2}) .
\]

(3.25)

where \( \Omega_{D-m} \) is the volume of \( \mathcal{H}^n \), and \( \chi(\Sigma_{m-2}) \) is the integration of Euler characteristic on the \( (m - 2) \)-dimensional cross section of horizon surface, i.e.,

\[
\chi(\Sigma_{m-2}) = \int_{\Sigma_{m-2}} d^{m-2} z \mathcal{L}_{(m/2-1)}(\tilde{q}) .
\]

(3.26)

For the cross section of horizon surface \( \Sigma_{m-2} \), which is a constant curvature space, \( \chi(\Sigma_{m-2}) \) is constant, while in the case of \( \Sigma_{m-2} \) being a closed manifold, \( \Sigma_{m-2} \) need not be a constant curvature space, and in that case, \( \chi(\Sigma_{m-2}) \) is the Euler number of \( \Sigma_{m-2} \) up to a constant factor. For example, when \( m = 4, D = 6 \), we have

\[
\mathcal{S} = 64\pi^2 \cdot \delta \cdot c_2 \cdot (r_0^2 \Omega_2) .
\]

(3.27)

Clearly we see that when \( \Sigma_2 \) is a Ricci flat space, i.e. \( \delta = 0 \), the constant entropy vanishes. Here the constant means that it is independent of the horizon radius \( r_+ \) and charge \( Q \).

Now we argue that the constant entropy is meaningless for black hole thermodynamics and should be dropped. One simple reason is that when the cross section of horizon surface \( \sigma_{m-2} \) is a negative constant curvature space, \( \chi \) is negative, giving a negative entropy which does not make sense in thermodynamics. Another reason is provided by the following example.
Consider a four-dimensional Schwarzschild black hole solution. In this case, the Euler density is the Gauss-Bonnet term. If one considers the contribution of the Gauss-Bonnet term to the black hole entropy, besides the usual area entropy, one has an additional constant \( \mathcal{G} = 64\pi^2 c_2 \) from (3.25), where \( c_2 \) is the Gauss-Bonnet coefficient. Such a constant term remains even when the black hole horizon goes to zero. Both of these clearly indicate that the constant entropy from the horizon topology should be dropped when black hole thermodynamics is concerned.

Let us illustrate these discussions by two examples of \( m = 3 \) and 4.

1. The case of \( m = 3, \ p = 2, \ D \geq 5 \)

In this case, the metric \( \tilde{h} \) is very simple, which is just the metric of the constant curvature space \( \mathcal{H}^n \) plus the metric of a circle, i.e.,

\[
\tilde{h} = dz^2 + r_0^2 \gamma_{ab} dy^a dy^b, \tag{3.28}
\]

and Wald’s entropy (3.19) becomes

\[
\mathcal{G} = 4\pi \int d^{D-2}x + 8\pi c_2 \int d^{D-2}x R(\tilde{h}). \tag{3.29}
\]

Here we have set \( c_1 \) to unity as before, and \( R(\tilde{h}) \) is the scalar curvature of the metric \( \tilde{h} \). It is easy to find

\[
R(\tilde{h}) = (D - 3)(D - 4) \left( \frac{k}{r_0^2} \right), \tag{3.30}
\]

thus we have

\[
\mathcal{G} = 4\pi (r_0^{D-3} \Omega_{D-3}) V_1 \left[ 1 + 2c_2(D - 3)(D - 4) \left( \frac{k}{r_0^2} \right) \right], \tag{3.31}
\]

where \( V_1 \) is the volume of the circle. Considering \( k = -1 \), and the explicit relation among \( c_2, r_0 \) and \( D \) in Eq. (2.31), the entropy identically vanishes.

Actually, for \( m = 3 \), we have \( w = 0 \), so \( \ell \) in Eq. (3.23) can take the value 1 only. Since \( A_1 = 0 \) by Eq. (3.24), we reach a vanishing entropy from (3.23), as we have just shown above.
2. The case of $m = 4$, $p = 2$, $D \geq 6$

In this case, the metric $	ilde{h}$ consists of the metric of the constant curvature space $\mathcal{H}^n$ and the metric of a 2-dimensional constant curvature space $\Sigma_2$, i.e.,

$$\tilde{h} = \tilde{q}_{ij}dz^i dz^j + r_0^2 \gamma_{ab} dy^a dy^b,$$

and Wald’s entropy is the same as (3.29). Now, the scalar curvature $R(\tilde{h})$ becomes

$$R(\tilde{h}) = R(\tilde{q}) + (D - 4)(D - 5) \left( \frac{k}{r_0^2} \right),$$

(3.33)

where $R(\tilde{q})$ is the scalar curvature of the metric $\tilde{q}$. Thus the entropy (3.22) has the form

$$\mathcal{S} = 4\pi r_0^{D-4} \Omega_{D-4} V_2 \left[ 1 + 2c_2(D - 4)(D - 5) \left( \frac{k}{r_0^2} \right) \right] + 8\pi c_2 r_0^{D-4} \Omega_{D-4} \int \sqrt{\tilde{q}} d^2 z R(\tilde{q}).$$

(3.34)

Using the explicit relation among $c_2$, $r_0$ and $D$ in equation (2.35), and $\bar{k} = -1$, we see that only the last term remains

$$\mathcal{S} = 8\pi c_2 r_0^{D-4} \Omega_{D-4} \int \sqrt{\tilde{q}} d^2 z R(\tilde{q}).$$

(3.35)

In other words, in this case, the entropy is totally determined by the integration of the Euler characteristic $R(\tilde{q})$ on $\Sigma_2$, in agreement with (3.25). Note that the the two-dimensional induced horizon $\tilde{q}$ is a constant curvature space with scalar curvature $2\delta$. The entropy can be further expressed as

$$\mathcal{S} = 64\pi^2 \cdot \delta \cdot c_2 \cdot (r_0^{D-4} \Omega_{D-4}).$$

(3.36)

In fact, in the case of $m = 4$, one has $w = 1$, and $\ell$ in (3.23) can take values 1 and 2. We have $A_1 = 0$ from (3.24), while $A_2 = c_2$. By the general entropy expression (3.23), we arrive at the same result as Eq. (3.35). As argued above, the constant entropy does not make sense in black hole thermodynamics, we should drop it and conclude that the physical entropy is zero.

C. Mass and Euclidean action of $m + n$ dimensional black holes

In this subsection we show another surprising result that the mass of these black holes also vanishes. To do this, we employ the Euclidean approach to black hole thermodynamics. The Euclidean action $I_E$ of the black holes includes two parts, the bulk and boundary parts,

$$I_E = I + B,$$

(3.37)
where $I$ is the bulk action, while $B$ denotes the boundary term. The bulk part is given by

$$I = - \left( r_0^{D-m} \Omega_{D-m} \right) \left\{ \sum_{t=0}^{s} A_t \int \sqrt{gd^m} x \tilde{L}_t \right\},$$

(3.38)

where $\tilde{L}_t$ is given by (2.11). With the metric (2.16), we get

$$I = - \left( r_0^{D-m} \Omega_{D-m} V_{m-2} \right) \left\{ \sum_{t=0}^{s} A_t \frac{(m-2)!}{(m-2t)!} \int d\tau dr \left( r^m F^t \right) \right\},$$

(3.39)

where $V_{m-2}$ is the volume of the $\Sigma_{m-2}$ with unit radius. In general, the boundary term $B$ is a little bit complicated. For simplicity, we only consider here the case that the highest derivative term is the Gauss-Bonnet term, i.e., we deal with the cases with $m = 3$ and $m = 4$. In that case, the boundary term is given by [29, 30]

$$B = -2 \int_{\partial X} d^{D-1}v \sqrt{h} \left[ K + 2 c_2 (J - 2 E_{\mu\nu} K^{\mu\nu}) \right],$$

(3.40)

where we have set $c_1 = 1$ and $h$ is the induced metric on a timelike boundary $\partial X$. The tensor $K^{\mu\nu}$ is the extrinsic curvature of the boundary, and $K$ is its trace. We denote by $J$ the trace of the tensor

$$J_{\mu\nu} = \frac{1}{3} \left( 2 KK_{\mu\lambda} K^\lambda_{\nu} + K_{\lambda\sigma} K^{\lambda\sigma} K_{\mu\nu} - 2 K_{\mu\lambda} K^{\lambda\sigma} K_{\sigma\nu} - K^2 K_{\mu\nu} \right).$$

(3.41)

Tensor $E_{\mu\nu}$ is the Einstein tensor of the induced metric $h$. With this boundary term, the variation principle is well defined for the Gauss-Bonnet gravity.

In the following calculations, we consider a boundary $\partial X$ with a given $r \gg r_+$, and take the limit of $r \to \infty$ at the end of calculations.

1. The case of $m = 3$, $p = 2$, $D \geq 5$

The case of $m = 3$ is quite simple. Both the bulk and boundary terms identically vanish $I = B = 0$. In fact, in this case all $A_t$ are zero and therefore the bulk term (3.39) vanishes. For the boundary term $B$, after some calculation, it is not hard to find the tensor $J_{\mu\nu} = 0$, and

$$K = \frac{2f + rf'}{2r \sqrt{f}},$$

$$2c_2 (J - 2 E_{\mu\nu} K^{\mu\nu}) = -c_2 (D - 3)(D - 4) \left( \frac{2f + rf'}{r_0^2 \sqrt{f}} \right).$$

(3.42)
So the boundary term is given by

\[ B = -\beta r_0^{D-3} \Omega_{D-3} V_1 (2f + rf') \left[ 1 - \frac{2c_2(D-3)(D-4)}{r_0^2} \right], \tag{3.43} \]

where \( \beta \) is the period of Euclidean time. Again, considering the relation among \( c_2, r_0 \) and \( D \) in equation (2.31), the boundary term has no contribution to the Euclidean action. Thus the Euclidean action of the black hole solutions is always zero, which leads to the conclusion that the energy and entropy of the black holes always vanish.

In fact, when \( m \) is odd, since all \( A_t = 0 \), both the bulk and boundary terms always vanish. Thus the result with vanishing energy and entropy is universal for odd \( m \).

2. The case of \( m = 4, p = 2, D \geq 6 \)

When \( m = 4 \), the bulk action (3.39) reduces to

\[ I = 4c_2 \left( r_0^{D-4} \Omega_{D-4} V_2 \right) \beta \left[ f'(\delta - f) \right]_{r=\infty} - 4c_2 \left( r_0^{D-4} \Omega_{D-4} V_2 \right) \beta \left[ f'(\delta - f) \right]_{r=r_+}, \tag{3.44} \]

where \( \beta \) is the period of Euclidean time as before. It is easy to find the trace of the extrinsic curvature is given by

\[ K = \frac{4f + rf'}{2r\sqrt{f}}, \tag{3.45} \]

and after some calculations, we can obtain

\[ 2c_2(J - 2E_{\mu\nu}K^{\mu\nu}) = \frac{c_2}{r^3r_0^2\sqrt{f}} \left\{ rf' \left[ 2\delta r_0^2 - (D - 4)(D - 5)r^2 \right] - 2f \left[ 2(D - 4)(D - 5)r^2 - 4r_0^2 + rr_0^2f' \right] \right\}. \tag{3.46} \]

Therefore, the boundary term \( B \) is

\[ B = -2 \left( r_0^{D-4} \Omega_{D-4} V_2 \right) \beta \left[ 1 - \frac{2c_2(D-4)(D-5)}{r_0^2} \right] \left( 2rf + \frac{1}{2}r^2f' \right) + 2c_2f'(\delta - f) + \frac{8c_2}{r}f \bigg|_{r=\infty} \tag{3.47} \]

Note that the equation (2.35) or \( A_1 = 0 \) gives \( 1 - 2c_2(D - 4)(D - 5)/r_0^2 = 0 \). Thus the boundary term reduces to

\[ B = -4c_2 \left( r_0^{D-4} \Omega_{D-4} V_2 \right) \beta \left[ f'(\delta - f) + \frac{4}{r}f \right]_{r=\infty}. \tag{3.48} \]
We thus get the total action

\[ I_E = I + B = -4c_2 \left( r_0^{D-4} \Omega_{D-4} V_2 \right) \beta [f'(\delta - f)]_{r = r_+} - 16\beta c_2 \left( r_0^{D-4} \Omega_{D-4} V_2 \right) \frac{f}{r} \bigg|_{r = \infty}. \]  

(3.49)

For our solutions, the second term in the right hand side is divergent when \( r \to \infty \). This divergence can be removed by the background subtraction method. By subtracting the contribution from the reference background with vanishing \( M \) and \( Q \), we find that the second term does not make any contribution to the Euclidean action and only the first term remains.

Note that \( \beta = \frac{4\pi}{f'(r_+)} \) and \( f = 0 \) at the horizon. The first term can be expressed as

\[ I_E = -64\pi^2 \cdot \delta \cdot c_2 \cdot \left( r_0^{D-4} \Omega_{D-4} \right). \]  

(3.50)

This is a constant independent of temperature. Considering the relation between the Euclidean action \( I_E \) and free energy \( F \): \( I_E = \beta F \), we immediately see that the energy of the black holes always vanishes, while the Euclidean action (3.50) gives the constant entropy (3.25) found by Wald’s formula.

Thus by calculating Euclidean action and Wald’s entropy, we have shown that mass and entropy of these black hole solutions presented in the previous section vanish identically.

**IV. CONCLUSION AND DISCUSSION**

In this paper we have presented a class of black hole solutions in \((m+n)\)-dimensional Lovelock gravity. The black hole solutions have a direct product structure \( \mathcal{M}^m \times \mathcal{H}^n \), where \( \mathcal{H}^n \) is a negative constant curvature space with a constant radius. When \( m = 3 \) and \( 4 \), these solutions reduce to those recently found by Maeda and Dadhich [17, 18, 19, 20] in Gauss-Bonnet gravity. We have obtained these black hole solutions in a way as follows. We first decompose the equations of motion into two sets, one for \( m \)-dimensional part and the other for \( n \)-dimensional part. Then imposing constraints on the coefficients of higher curvature terms in Lovelock gravity so that the set of equations of motion for the \( m \)-dimensional part is trivially satisfied, we solve the trace equation for the \( n \)-dimensional part and obtain the black hole solutions. Since the trace equation is a second order differential equation, integrating the equation gives rise to two integration constants \( M \) and \( Q \).

We have tried to understand the physical meaning of the two integration constants by studying thermodynamics of these black hole solutions. The black holes we have found
are exact solutions in \((m + n)\)-dimensional Lovelock gravity theory. Naively considering the solution without the extra dimensions, it appeared that the mass were proportional to parameter \(M\) as in Eq. \((3.12)\). By using the first law, we then found that the entropy of the black hole would have logarithmic term when \(m\) is even. However, we have argued that this naive result is not valid for our solutions because the equations of motion for the \(m\)-dimensional part are trivially satisfied and our solutions come from the trace equation for the \(n\)-dimensional part. As a result the effective action for the \(m\)-dimensional part does not make any sense for the black hole solutions. We should consider the black hole solutions in the point of view of the whole \((m + n)\)-dimensional spacetime. It is not surprising because our theory is intrinsically \((m + n)\)-dimensional. We cannot naively neglect the extra dimensions as in the case of the usual Kaluza-Klein theory in which the thermodynamics in higher dimensions and lower dimensions are equivalent.

Then by employing Euclidean action approach to black hole thermodynamics and Wald’s entropy formula, we have found an astonishing result that both mass and entropy of these black holes always vanish identically although there exists a nonvanishing Hawking temperature for these black holes. Here it may be worth mentioning that when \(m\) is even, by Wald’s entropy formula, the black hole has a constant entropy coming from the topological structure of the black hole horizon. But we have argued that the constant entropy should be neglected from the point of view of black hole thermodynamics.

Black hole solutions with a nonvanishing temperature and always vanishing mass and entropy look strange. But such a situation has happened in a class of Lifshitz black holes in \(R^2\) gravity \([31]\). There these authors got the solution by adjusting the coupling constant of \(R^2\) term to a critical value. Note that the same thing happens here in the class of Lovelock black hole solutions since we have chosen a special set of coupling coefficients of higher derivative terms in order to get our solutions.

Let us now try to understand such a phenomenon that a black hole has a nonvanishing temperature, but vanishing mass and entropy. Recall that in the \(R^2\) gravity considered in \([31]\), the resulting Lifshitz black hole solution satisfied \(1 + 2\alpha R = 0\), where \(\alpha\) is the coefficient of the term \(R^2\), which is a crucial point to give the zero entropy of the black hole. Let us notice that in the \(R^2\) theory, the factor \(1 + 2\alpha R\) is nothing but the effective coupling constant for some polarized graviton. From the equations of motion, it is easy to see that the effective gravitational constant turns to be \(G_{eff} = G/(1 + 2\alpha R)\). As a result,
$1 + 2\alpha R = 0$ implies that the effective gravitational constant is divergent for the class of solutions with $1 + 2\alpha R = 0$. Wald’s entropy is equal to a quarter of the horizon area in units of the effective gravitational coupling $\mathcal{H}^n$. This is the reason why the entropy of the Lifshitz black hole has a vanishing entropy. Furthermore, because of $G_{\text{eff}} \to \infty$, from the point of view of background fluctuations, kinetic terms of those fluctuations always vanish, and only potential terms remain. This indicates that there is no dynamics for those fluctuations. In other words, there are no excitations of the background spacetime. This might be an interpretation why the black hole has no entropy. While the Hawking temperature (surface gravity) of a black hole is purely determined by black hole geometry in the sense that the Hawking temperature is just the inverse period of the Euclidean time of the black hole, the first law of thermodynamics enforces that a black hole has a vanishing mass (energy) if its entropy is zero.

Let us turn to the black hole spacetime discussed in the present paper. In fact the same happens here. The part of $\mathcal{H}^n$ is a trivial negative constant curvature space. The effective gravitational field equations for the part of $m$-dimensional black hole spacetime are trivially satisfied in the sense that those coefficients in front of some gravitational tensors like Einstein tensor are identically zero [see (2.7) and (2.12)]. To see this more clearly, one may refer to the simple case with Gauss-Bonnet gravity discussed in [17]. The vanishing coefficients are correspondent to the factor $1 + 2\alpha R$ discussed above for $R^2$ gravity. Therefore, due to the special reduction used to find the black hole solutions in the present paper, these effective coupling constants from the $m$-dimensional point of view identically vanish. In this sense, the effective gravitational constant $G_{\text{eff}}$ diverges as in the case of the $R^2$ gravity. Then the same story goes on as the case of the $R^2$ gravity and these black holes have vanishing entropy and mass.

If our arguments are true, our above discussions and those in Ref. [31] both have important consequence on our understanding of the microscopic degrees of freedom of black hole entropy. According to ‘t Hooft’s brick wall model [33], black hole entropy might come from statistical degrees of freedom of quantum fluctuations outside the black hole, namely if there is no such degrees of freedom of quantum fluctuations, there is no contribution to the entropy. The black hole entropy is not merely determined by the geometry of the horizon. In the examples discussed above, the Bekenstein-Hawking geometry entropy of the black holes always vanishes and the zero entropy is found to be closely related to the fact that the
effective gravitational coupling constants are infinity such that any fluctuations are forbidden, there are totally no physical degrees of freedom associated with quantum fluctuations. Thus our results provide evidence that black hole entropy comes from statistical degrees of freedom of quantum fluctuations around the black hole. No doubt, it is worthwhile to further investigate this interesting issue.

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